Angular Momentum Imparted

To Test Particles by

Gravitational Waves

By

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I would like to dedicate this thesis to

my parents
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Chapter 1

GRAVITATIONAL WAVES

1.1 Introduction

Almost every student of science is familiar with the phenomenon of waves, such as water waves (ripples rolling across the ocean), sound waves (vibration in the air), etc. The above mentioned kinds of waves are easily observable. Electromagnetic waves are comparatively difficult to understand but not as difficult as gravitational waves. The existence of gravitational waves was disputed for a long time but is generally accepted now.

What are gravitational waves? How do they propagate? and what is their energy content? These questions are addressed in the first two chapters. In the third chapter the pseudo-Newtonian formalism and its extension is reviewed in general and the formula for the momentum imparted to test particles in arbitrary spacetimes is reviewed in particular. In chapter four the analysis of a paper claiming to determine the spin for gravitational waves is given, and
compared with the spin given by a geodesic analysis. It is demonstrated that the other claim is inconsistent. Finally in chapter five a summary of the work is given with the conclusion.

1.2 Idealization

The only way to come to grips with so complicated a subject as General Relativity is by idealization. Study one idealization after another. Build a catalogue of idealization, of their properties and of techniques for analyzing them and then arrive at a conclusion.

Let’s now see how can we idealize gravitational waves. As one idealizes “water waves” as small ripples of geometry rolling across the ocean so one gives the name “gravitational waves” to small ripples rolling across the space time. Both of the waves are idealizations. One cannot, with infinite accuracy, delineate at any moment which drops of water are in the waves and which are in the underlying ocean. Similarly one cannot tell precisely which parts of spacetime are in the ripples and which are in the cosmological backgrounds. One can almost do so; otherwise one would not speak of “waves”. Look at the ocean, the seascape is dominated by the waves. Changes occur at the surface of the ocean, which propagates obeying the following [1] wave equation:

$$\left(\frac{1}{g^2} \frac{\partial^4}{\partial t^4} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2}\right)\text{(height of the surface)} = 0. \quad (1.1)$$

Similarly gravitational waves are perturbations of spacetime. As I have already
stated that gravitational waves are ripples of geometry. So let me support my statement. Take a massive body and disturb it violently, the near field adjusts rapidly, but the far field must wait for the signal that the mass has moved to propagate to it at a finite speed “c”. Thus there is a travelling kink which falls off in strength with distance. Hence the gravitational waves are small ripples rolling across the spacetime.

Now get more sophisticated. Notice from a space ship the large-scale curvature of the ocean’s surface-curvature because the Earth is round. Curvature because the Earth, Sun and Moon pull the water. As waves propagate long distances, this curvature bends their fronts and changes slightly their simple wave equation. Spacetime is similar. Propagating through the universe, according to Einstein’s theory must be a complex pattern of small-scale ripples in the spacetime curvature, ripples produced by binary stars, by gravitational collapse, by explosion in galactic nuclei etc.

1.3 Linear approximation for the investigation of gravitational waves

We are discussing gravitational waves in the framework of General Relativity, a nonlinear field theory of gravity. Though many of the interesting consequences of General Relativity comes from its nonlinearity it is worthwhile to study its linear approximation. Linearization actually leads one to the gravitational waves.
What makes General Relativity nonlinear? As the field in General Relativity is the metric tensor, its appearance in the field equations non-linearly gives rise to the non-linearity of General Relativity. We cannot change the way the metric tensor enters into the curvature but we can write the curved spacetime metric as the flat spacetime metric tensor, $\eta_{\mu\nu}$, and an additional term, $h_{\mu\nu}$. We then require that $h_{\mu\nu}$ and its derivative occur only once in the field equations and higher powers be neglected[6].

\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \]  

(1.2)

Let

\[ g^{\mu\nu} = \eta^{\mu\nu} + f^{\mu\nu}, \]  

(1.3)

where $\eta^{\mu\nu}$ is the inverse of $\eta_{\mu\nu}$. Then by definition,

\[ g^{\mu\rho} g_{\rho\pi} = (\eta^{\mu\rho} + f^{\mu\rho})(\eta_{\rho\pi} + h_{\rho\pi}), \]  

(1.4)

Which implies that

\[ \delta^{\mu}_{\pi} = \delta^{\mu}_{\pi} + f^{\mu\rho} \eta_{\rho\pi} + \eta^{\mu\rho} h_{\rho\pi} + f^{\mu\rho} h_{\rho\pi}. \]  

(1.5)

Cancelling $\delta^{\mu}_{\pi}$ on both sides and multiplying through by $\eta^{\nu\pi}$ we get

\[ f^{\mu\nu} + \eta^{\mu\rho} \eta^{\nu\pi} h_{\rho\pi} + \eta^{\nu\pi} f^{\mu\rho} h_{\rho\pi} = 0. \]  

(1.6)
The last term is clearly quadratic in the difference between the curved and flat spacetime metric tensor. Thus to first order,

\[ f^{\mu\nu} = -\eta^{\mu\rho} \eta^{\nu\pi} h_{\rho\pi} + O(h^2). \]  
(1.7)

Using the flat spacetime metric tensor to raise and lower indices we can write

\[ f^{\mu\nu} = -h^{\mu\nu} + O(h^2). \]  
(1.8)

Using equation (1.3), equation (1.8) becomes

\[ g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + O(h^2) \]  
(1.9)

\[ \approx \eta^{\mu\nu} - h^{\mu\nu}. \]

Using this linearization and for the moment taking Cartesian coordinates, so that there are no derivatives of \(\eta^{\mu\nu}\), the Christoffel symbols linearize to:

\[
\left\{ \begin{array}{c}
\rho \\
\mu \\
\nu
\end{array} \right\} \approx \frac{1}{2} \eta^{\rho\pi}(h_{\mu\nu,\pi} + h_{\nu\pi,\mu} - h_{\mu\nu,\pi}). \]  
(1.10)

Clearly terms quadratic in the Christoffel symbols become quadratic in \(h\) and can be neglected compared with linear terms. Thus the linearized Ricci tensor is

\[ R_{\mu\nu} = \left\{ \begin{array}{c}
\rho \\
\mu \\
\nu
\end{array} \right\}_{,\rho} - \left\{ \begin{array}{c}
\rho \\
\mu \\
\rho
\end{array} \right\}_{,\nu} \]  
(1.11)
\[ \approx \frac{1}{2} [\eta^{\rho\pi} (h_{\mu\pi,\nu} + h_{\nu\pi,\mu} - h_{\mu\nu,\pi})]_{,\rho} \]  
(1.12)

\[ -\frac{1}{2} [\eta^{\rho\pi} (h_{\mu\pi,\rho} + h_{\nu\pi,\mu} - h_{\mu\nu,\pi})]_{,\nu} \]  
(1.13)

\[ = \frac{1}{2} \eta^{\rho\pi} (h_{\mu\pi,\rho} + h_{\nu\pi,\mu} - h_{\mu\nu,\rho} - h_{\rho\pi,\mu\nu}) \]  
(1.14)

A choice of coordinates can be made to have the first two and the last term in the brackets disappear. To see this first we note that we can rewrite the Ricci tensor as:

\[ R_{\mu\nu} \approx \frac{1}{2} (h_\rho^\mu - \frac{1}{2} h_\rho^\rho)_{,\rho\nu} + \frac{1}{2} (h_\rho^\nu - \frac{1}{2} h_\rho^\rho)_{,\mu\rho} \]

\[ -\frac{1}{2} \eta^{\rho\pi} h_{\mu\nu,\rho\pi}, \]  
(1.15)

where \( h = h_\rho^\mu \) then we can break the first bracket into two terms. Now consider an infinitesimal transformation

\[ x^\mu \rightarrow x'^\mu = x^\mu + \xi^\mu(x^\rho), \]  
(1.16)

so that the terms quadratic in \( h \) or \( \xi \) can be neglected. Since this is only a coordinate transformation, it must leave the metric invariant and hence

\[ ds^2 = g_{\mu\nu}(x^\rho)dx^\mu dx^\nu = g'_{\mu\nu}(x'^\rho)dx'^\mu dx'^\nu \]

\[ = g'_{\mu\nu}(x'^\rho)(dx^\mu + \xi^\mu_{,\alpha}dx^\alpha)(dx^\nu + \xi^\nu_{,\beta}dx^\beta). \]  
(1.17)

Using the linearization procedure it is easy to see that

\[ h'_{\mu\nu} \approx h_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu}. \]  
(1.18)
Thus we have

\[
(h'_\mu - \frac{1}{2} h \delta^\mu_\rho \xi) = (h'_\mu - \frac{1}{2} h' \delta^\mu_\rho) - \eta^{\nu \rho} \xi_{\nu,\rho} - \xi^\rho_{\nu} + \xi^\nu_{\rho} \delta^\rho_{\mu}.
\] (1.19)

Differentiating the above equation relative to \( x^\rho \) it is easy to see that the last two terms simply cancel and we have

\[
(h'_\mu - \frac{1}{2} h \delta^\mu_\rho \xi,\rho) = (h'_\mu - \frac{1}{2} h' \delta^\mu_\rho \xi,\rho) - \eta^{\nu \rho} \xi_{\nu,\rho}.
\] (1.20)

Taking the harmonic gauge condition

\[
\Box \xi^\mu = - \phi^\mu_{\mu,\rho},
\] (1.21)

we can make

\[
(h'_\mu - \frac{1}{2} h \delta^\mu_\rho \phi^\rho_{\mu,\rho} = \phi^\rho_{\mu,\rho} = 0.
\] (1.22)

It should be pointed out here that the use of Cartesian coordinates was not crucial, but merely for convenience. With any other coordinates we could have to introduce the corresponding flat spacetime Christoffel symbols. This has been avoided so as not cause confusion of notation. We now drop the primes and so obtain

\[
R_{\mu\nu} \approx - \frac{1}{2} \eta^{\rho \sigma} h_{\mu\nu,\rho\sigma} = - \frac{1}{2} \Box h_{\mu\nu}.
\] (1.23)

We can contract Equation (1.22) to obtain the Ricci scalar \( R = - \frac{1}{2} \Box h \) and hence combine them to obtain the Einstein tensor. Thus the Einstein field
equation becomes

\[ \Box \phi_{\mu\nu} = -2\kappa T_{\mu\nu}. \] (1.24)

Here \( T_{\mu\nu} \) is the stress energy tensor and \( \kappa \) is the proportionality constant. In regions where \( T_{\mu\nu} = 0 \), \( \phi_{\mu\nu} \) then satisfies the wave equation, Thus \( \phi_{\mu\nu} \) represent gravitational waves.

### 1.4 Plane wave solution in the linearized theory

The simplest of all solutions to the linearized equation \( h_{\mu\nu,\alpha}^\alpha = 0 \) is the monochromatic plane wave solution \[ h_{\mu\nu} = \Re[A_{\mu\nu} \exp(ik^{\alpha}x^\alpha)]. \] (1.25)

Here \( \Re \) means that one must take the real part in the bracket. While \( A_{\mu\nu} \) is the amplitude and \( k_{\mu} \) is the wave vector, satisfying

\[ k_\alpha k^\alpha = 0 \quad (k \text{ a null vector}), \] (1.26)

\[ A_{\mu\alpha} k^\alpha = 0 \quad (A \text{ orthogonal to } k). \]

This solution describes a wave with the frequency

\[ \omega = k^0 = \left( k_x^2 + k_y^2 + k_z^2 \right)^{1/2}, \]

which propagate with the speed of light in the direction \( (1/k^0)(k_x, k_y, k_z) \). At first sight the amplitude \( A_{\mu\nu} \) appears to have six independent components
(ten, less the four orthogonality constraints $A_{\mu\alpha}k^{\alpha} = 0$). But this cannot be right. The gravitational field has two dynamic degrees of freedom (not six).

The plane wave vector

$$\xi^{\mu} = -iC^{\mu} \exp(ik_{\alpha}x^{\alpha}), \quad (1.27)$$

with four arbitrary constants $C^{\mu}$, generates a gauge transformation that can change arbitrarily four of the six independent components of $A_{\mu\nu}$. One gets rid of this arbitrariness by choosing a specific gauge.

### 1.5 The transverse traceless (TT) gauge

Consider the four velocity $u$ through all the space time and impose the condition [1]

$$A_{\mu\nu}u^{\nu} = 0. \quad (1.28)$$

These are only three constraints on $A_{\mu\nu}$. Why not four? Because $A$ is orthogonal to $k$ i.e. $k^{\mu}(A_{\mu\nu}u^{\nu}) = 0$, as already mentioned. As a fourth constraint use a gauge transformation to set $A_{\mu}^{\mu} = 0$. So there are now four constraints in all, $A_{\mu\alpha}u^{\alpha} = A_{\mu\alpha}k^{\alpha} = A_{\alpha}^{\alpha} = 0$, on the ten components of amplitude. Thus the two remaining free components of $A_{\mu\nu}$ represents the two degrees of freedom in the plane gravitational wave.

It is useful to restate the four constraints, $A_{\mu\alpha}u^{\alpha} = A_{\mu\alpha}k^{\alpha} = A_{\alpha}^{\alpha} = 0$ in the Lorentz frame where $u^{0} = 1, u^{j} = 0$ and in a frame where $k^{\alpha}$ does not appear
explicitly:

\[ h_{\mu 0} = 0 , \text{ i.e only the spatial components } h_{jk} \text{ are non-zero} \quad (1.29) \]

\[ h_{kjj} = 0 \text{ i.e the spatial components are divergence free} \quad (1.30) \]

\[ h_{kk} = h_{\mu}^\mu = 0 \text{ i.e the spatial components are trace free.} \quad (1.31) \]

The gauge conditions are all linear in \( h_{\mu \nu} \) so an arbitrary wave will also satisfy the above gauge conditions. In this gauge only the \( h_{jk} \) are non-zero. So we need only to impose the six wave equations

\[ \Box h_{jk} = h_{jk,\alpha\alpha} = 0 \quad (1.32) \]

Now the definition. Any symmetric tensor satisfying the constraints (1.29), (1.30) and (1.31) is called a transverse traceless tensor. Why it is called transverse traceless? Because it is purely spatial \( (h_{\mu 0} = 0) \) and, if thought of as a wave is transverse to the direction of propagation \( (h_{ij,j} = h_{ij}k_j = 0) \) and traceless because \( h_{kk} = 0 \).

The special gauge in which \( h_{\mu \nu} \) reduces to its transverse traceless part is called the transverse traceless gauge. The conditions (1.29), (1.30) and (1.31) defining this gauge can be summarized as:

\[ h_{\mu \nu} = h_{\mu \nu}^{TT}. \quad (1.33) \]

Only pure waves can be reduced to TT gauge. In the TT-gauge the space components

\[ R_{j0k0} = R_{0j0k} = -R_{j00k} = -R_{0jk0}. \quad (1.34) \]
of the Riemannian curvature tensor have an especially simple form

\[ R_{j0k0} = -\frac{1}{2} h_{jk,00}^{TT}. \]  \hspace{1cm} (1.35)

As the curvature tensor is gauge invariant therefore \( h_{\mu\nu} \) cannot be reduced to still fewer components than it has in the TT-gauge.

### 1.6 Comparison with electromagnetic waves

A simple system consisting of two charges of equal magnitude but of opposite signs, each situated at a distance \( r/2 \) from the origin \( O \) (see Fig. 1.1), which is taken to lie on the line connecting the charges, is the simplest example of an electric dipole. The oscillation in the dipole generates electromagnetic waves.

Now we define the electric dipole moment of the pair of equal charges as the product of charge \( q \) and the separation \( r \),

\[ \mathbf{d} = qr \mathbf{e}, \] \hspace{1cm} (1.36)

where \( \mathbf{e} \) is the unit vector from negative to positive charge. Consider a system of charges \( q_a \) and let \( r_a \) be their radius vectors then

\[ \mathbf{d} = \sum_a q_a r_a, \] \hspace{1cm} (1.37)

is called the dipole moment of the system of charges.

Consider masses in an isolated, nearly Newtonian system, moving about each other. They emit radiations. For an order of magnitude estimate, one
Figure 1.1: A system consisting of four masses attached by springs, forming two dipoles. The oscillation in the system generate gravitational waves.
can apply the familiar radiation formula of electromagnetic theory, with the replacement $q^2 \rightarrow -m^2$ which converts the static Coulomb’s law into Newton’s law of attraction. Although it introduces a moderate error in numerical factor and changes angular distribution, but it gives an estimate of the total power radiated. In electromagnetic theory, electric dipole radiation dominates, with power output or luminosity “$L$" given by[1],

$$L_{e_{\text{lec dip}}} = \frac{2}{3} q^2 a^2.$$  \hspace{1cm} (1.38)

where $a$ is the acceleration in the dipole. Then using equation (1.36) we can write

$$L_{e_{\text{lec dip}}} = \frac{2}{3} d^2 .$$  \hspace{1cm} (1.39)

The gravitational analogue of the electric dipole moment is the mass dipole moment,

$$d = \sum A m_A x_A ,$$ \hspace{1cm} (1.40)

its first time-rate of change is the total momentum of the system,

$$\dot{d} = \sum A m_A x_A = P ,$$ \hspace{1cm} (1.41)

where $P$ is the momentum of the system. The time-rate of change of the mass dipole moment has to vanish because of the law of conservation of momentum, $\dot{d} = \dot{P} = 0$ therefore there can be no mass dipole radiation in gravitation physics.
The next strongest type of electromagnetic radiation are magnetic dipole and electric quadrupole. Magnetic dipole radiation is generated by the second time derivative of the magnetic moment, $\ddot{\mu}$. Here again the gravitational analogue is a constant of motion of the angular momentum i.e.,

$$\mu = \sum (\text{position of } A) \times (\text{current due to } A)$$

$$= \sum r_A \times (mV_A) = J = \text{constant} \quad (1.42)$$

Which shows that $\ddot{\mu} = 0$, so it can not radiate. Thus, there can be no gravitational dipole radiation of any sort. Physical example is a system of two masses attached by a spring, acts as an oscillating dipole.

1.6.1 Comparison with plane electromagnetic waves

Consider the metric [1] (with signature $(+,-,-,-)$)

$$ds^2 = L^2(u)(dx^2 + dy^2) - dv du.$$  \hspace{1cm} (1.43)

where $u = t - z$ and $v = t + z$ which is always flat. It satisfies the vacuum Einstein equations $R_{\mu\nu} = 0$ which implies that $L'' = 0$ (see Appendix 1 for the proof). Now if $L'' = 0$ the spacetime is static therefore cannot represent gravitational waves. In this metric the electromagnetic potential

$$A = A_\mu dx^\mu = A(u) dx$$ \hspace{1cm} (1.44)

satisfies the Maxwell equation for arbitrary $A(u)$. It represents an electromagnetic plane wave analogous to the gravitational plane wave. The only non-zero
components of this wave are

\[ F_{ux} = A' \quad \text{i.e.} \quad F_{tx} = -F_{zx} = A'. \tag{1.45} \]

So the wave propagates in the \( z \)-direction, the magnetic vector oscillates in the \( y \)-direction and electric vector in the \( x \)-direction. The only non-zero vector of the stress energy tensor is

\[ T_{uu} = (4\pi L^2)^{-1}(A')^2 \tag{1.46} \]

It can easily be verified that the Maxwell equation are satisfied by (1.44) in (1.43).

To make the metric acceptable, we need to impose the Einstein equations

\[ R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu}, \quad \kappa \text{ is generally } \frac{8\pi G}{c^4}, \text{but here we use gravitational units i.e. } G = c = 1 \text{ so it becomes } 8\pi. \]

As all the nonvanishing Ricci tensor components are the same and they read \(-\frac{2L''}{L}\), so the Einstein field equations becomes

\[ L'' + (4\pi T_{\mu\nu})L = 0. \]

This has exactly the form of the equation \( L'' + (\beta')^2 L = 0 \), (which will be discussed in detail in chapter 2) for gravitational plane wave.
Chapter 2

REALITY OF

GRAVITATIONAL WAVES

In the linearization procedure the stress energy tensor was taken to be zero, which creates a conceptual problem, namely the question of reality of gravitational waves. This will be discussed in detail in the first section of this chapter. For this purpose some exact gravitational wave solutions of the Einstein field equations are presented. Returning to the purpose of this chapter the energy contents of the waves are discussed. Finally as an additional but necessary, topic some sources of gravitational waves are discussed with in the scope of this work.
2.1 The conceptual problem

The exact gravitational waves being by definition, solutions of the vacuum field equations, have a zero stress energy tensor. This creates a conceptual problem. How can these solutions of the Einstein field equations represent waves if they carry no energy? Essentially, this problem arises because energy is not a well defined concept in General Relativity. For energy to be well defined in General Relativity the metric must have a time like isometry (Killing vector) so as to allow time translational invariance. This will not generally be true. Infact, spacetimes for which it is true are static whereas gravitational wave solutions must be non-static. Thus energy is not well defined for spacetimes containing gravitational waves. The question then is, how can we tell that these solutions really do behave as we would expect of the waves?

One way to answer this question: the very process of linearization provides the energy. The point is that if $\Box \phi_{\mu\nu} = 0$ exactly then $R_{\mu\nu} \approx 0$ to order $\hbar$ but $R_{\mu\nu} \neq 0$ to higher orders in $\hbar$. Thus $T_{\mu\nu} \approx 0$ only to order $\hbar$ but $T_{\mu\nu} \neq 0$ generally. Conversely if $T_{\mu\nu} = 0$ and hence $R_{\mu\nu} = 0$ exactly then $\Box \phi_{\mu\nu} \approx 0$ only to order $\hbar$. Thus we can expand $R_{\mu\nu}$ in powers of $\hbar$, retaining linear terms on the left side of the equation and transposing all higher powers to the right side. These higher order terms become an effective stress energy tensor and the linearized equations give the gravitational waves.
2.2 Some exact solutions of gravitational waves

So far we have obtained the wave equations for gravity by linearizing the Einstein field equations. In principle the solutions so obtained could be exact solutions of the vacuum Einstein field equations, of course there could be trivial static solutions which effectively satisfy the Laplace equation. But they do not represent moving waves so we are interested in the solutions which are non-static.

The first solution to be discovered was for cylindrical gravitational waves, by Einstein and Rosen in (1937) \[2\]. So first consider this:

2.2.1 Cylindrical gravitational wave solution

A cylindrically symmetric metric depending on two arbitrary functions \(\gamma\) and \(\psi\) of the time \(t\) and cylindrical radial coordinate \(\rho\), is

\[
ds^2 = e^{2(\gamma - \psi)}(dt^2 - d\rho^2) - e^{-2\psi}\rho^2 d\varphi^2 - e^{2\psi}dz^2.
\]

The metric tensor is

\[
g_{ab} = \begin{pmatrix}
e^{2(\gamma - \psi)} & 0 & 0 & 0 \\
0 & -e^{2(\gamma - \psi)} & 0 & 0 \\
0 & 0 & -\rho^2 e^{-2\psi} & 0 \\
0 & 0 & 0 & -e^{-2\psi}
\end{pmatrix}.
\]

Its inverse is
The non-zero Christoffel symbols are:

\[
g^{ab} = \begin{pmatrix}
e^{-2(\gamma - \psi)} & 0 & 0 & 0 \\
0 & -e^{-2(\gamma - \psi)} & 0 & 0 \\
0 & 0 & -\rho^2 e^{-2\psi} & 0 \\
0 & 0 & 0 & -e^{2\psi}
\end{pmatrix}
\]  \hspace{1cm} (2.3)

The non-zero Christoffel symbols are:

\[
\begin{align*}
\{ \{ 0 \} = \{ 0 \} = \{ 1 \} = \{ 1 \} &= \gamma' - \psi' ; \\
\{ 0 \} = \{ 0 \} = \{ 1 \} = \{ 1 \} &= \gamma' - \psi' ; \hspace{1cm} (2.4) \\
\{ 0 \} = \{ 0 \} = \{ 3 \} = \{ 3 \} &= \psi' e^{2(2\psi - \gamma)} ; \hspace{1cm} (2.5) \\
\{ 1 \} = \{ 1 \} = \{ 3 \} = \{ 3 \} &= -\psi' e^{2(2\psi - \gamma)} ; \hspace{1cm} (2.6) \\
\{ 2 \} = \{ 2 \} = \{ 2 \} = \{ 2 \} &= -(\psi' - 1/\rho) ; \hspace{1cm} (2.7) \\
\{ 3 \} = \{ 3 \} = \{ 3 \} = \{ 3 \} &= \psi' ; \hspace{1cm} (2.8)
\end{align*}
\]
\[
\begin{aligned}
\begin{cases}
\mu \\
\mu 0
\end{cases} = (\ln \sqrt{g})_0 = 2(\gamma - \psi') ; \\
\mu 0 \\
\mu \nu \nu
\end{cases} = (\ln \sqrt{g}),_0 = 2(\gamma - \psi') + 1/\rho .
\end{aligned}
\] (2.9)

\[
\begin{aligned}
\begin{cases}
\mu \\
\mu 1
\end{cases} = (\ln \sqrt{g}),_1 = 2(\gamma - \psi') + 1/\rho .
\end{aligned}
\] (2.10)

The "'" refers to differentiation with respect to \( \rho \) and the dot differentiation with respect \( t \).

The non vanishing components of the Ricci tensor are \( R_{00}, R_{11}, R_{22} \) and \( R_{33} \). Only four components are needed for the purpose. Using the Christoffel symbols listed above we obtain the following Ricci tensor components,

\[
R_{00} = \begin{cases}
\mu \\
0 \mu \\
\mu 0 \\
\mu \nu
\end{cases} - (\ln \sqrt{g}),_{00} + (\ln \sqrt{g}),_{\mu} \begin{cases}
\mu \\
0 \nu
\end{cases} - \begin{cases}
\mu \\
0 \nu
\end{cases} \begin{cases}
0 \\
\mu
\end{cases}
\begin{cases}
0 \\
0 \nu
\end{cases}
\begin{cases}
0 \\
\mu
\end{cases}
\begin{cases}
0 \\
0 \nu
\end{cases} - (\ln \sqrt{g}),_{00} + (\ln \sqrt{g}),_{0} \begin{cases}
0 \\
0
\end{cases}
\begin{cases}
0 \\
\mu
\end{cases}
\begin{cases}
0 \\
\mu
\end{cases} \begin{cases}
0 \\
0
\end{cases} - (\ln \sqrt{g}),_{1} \begin{cases}
1 \\
1
\end{cases} \begin{cases}
0 \\
0
\end{cases} - 2 \begin{cases}
0 \\
1
\end{cases} \begin{cases}
0 \\
0
\end{cases} \begin{cases}
0 \\
0
\end{cases} - 2 \begin{cases}
3 \\
0
\end{cases} \begin{cases}
2 \\
2
\end{cases} - \begin{cases}
0 \\
0
\end{cases} \begin{cases}
0 \\
3
\end{cases}
\end{aligned}
\] (2.11)
After simplification we get

\[
R_{00} = \gamma'' - \psi'' + \gamma'' - \psi'' - 2(\gamma' - \psi') + (\gamma' - \psi')^2 - 2(\gamma' - \psi')^2 \\
- (\gamma' - \psi')^2 - 2(\psi')^2 + \left[ \frac{1}{\rho} + 2(\gamma' - \psi') \right] (\gamma' - \psi') \\
= - (\gamma'' + \psi'' - \gamma'' + \psi'') + \frac{1}{\rho} (\gamma' - \psi') - 2(\psi')^2. \tag{2.13}
\]

Similarly other non-vanishing components are,

\[
R_{11} = (\gamma'' - \psi'') - (\gamma'' - \psi'') + \frac{1}{\rho} (\gamma' + \psi') - 2(\psi')^2. \tag{2.14}
\]

\[
R_{22} = \rho^2 e^{-\gamma} (-\psi' + \psi'' + \frac{1}{\rho} \psi'). \tag{2.15}
\]

\[
R_{01} = \frac{1}{\rho} \gamma' - 2\psi' \psi'. \tag{2.16}
\]
So the Einstein field equations become

\[ (\gamma' - \psi') - (\gamma'' - \psi'') - \frac{1}{\rho}(\gamma' - \psi') + 2(\psi')^2 = 0. \]  \hspace{1cm} (2.17) 

\[ (\gamma' - \psi') + (\gamma'' - \psi'') + \frac{1}{\rho}(\gamma' + \psi') - 2(\psi'')^2 = 0. \]  \hspace{1cm} (2.18) 

\[ \rho^2 e^{-\gamma}(-\psi' + \psi'' + \frac{1}{\rho}\psi') = 0. \]  \hspace{1cm} (2.19) 

\[ \frac{1}{\rho}\gamma' - 2\psi' \psi' = 0. \]  \hspace{1cm} (2.20) 

Equation (2.19) is the usual cylindrical form of the wave. As this is a second order linear differential equation, the solution has two arbitrary constants, one corresponding to the ingoing cylindrical wave and the other corresponding to the outgoing. Retaining only the outgoing waves with amplitude \( A \) and frequency \( \omega \) we have

\[ \psi(t, \rho) = A[J_0(x) \cos(\omega t) + N_0(x) \sin(\omega t)]. \]  \hspace{1cm} (2.21) 

where \( x = \omega \rho \), \( J_0(x) \) and \( N_0(x) \) are the zero order Bessel and Neuman functions respectively. Add equation (2.17) and (2.18) to obtain
\[ \gamma' = \rho(\psi^2 + \psi'^2) \]  

Equations (2.20) and (2.22) gives the space and time derivative of \( \gamma(t, \rho) \) in terms of functions that are now known through equation (2.21). The only thing required is the integration with respect to each variable. The time integration is relatively easy and the space integration, though tedious, is in principle easy (using the standard formulae for the integrals of the Bessel and Neumann function). The resulting solution of \( \gamma \) is then

\[
\gamma(t, \rho) = \frac{1}{2} A^2 x \left\{ J_0(x) J_0'(x) + N_0(x) N_0'(x) + x [J_0''(x)^2 + N_0''(x)^2] \right. \\
+ [J_0(x) J_0'(x) - N_0(x) N_0'(x)] \cos(2\omega t) \\
+ [J_0(x) N_0'(x) + J_0'(x) N_0(x)] \sin(2\omega t) \left. \right\} - \frac{2}{\pi} A^2 \omega t. \tag{2.23}
\]

Where prime now refers to differentiation with respect to \( x \) not \( \rho \). Hence the metric

\[
ds^2 = e^{2(\gamma - \psi)} (dt^2 - d\rho^2) - e^{-2\psi} \rho^2 d\varphi^2 - e^{2\psi} dz^2. \tag{2.24}
\]

represents cylindrical gravitational wave with the above definition of \( \gamma \) and \( \psi \).

### 2.2.2 Plane gravitational wave solution

The solution which we are going to present here was discovered by Bondi and Robinson in 1957 [3]. We take a line element which incorporates the symmetries of a plane and represents a wave going in the x-direction.
\[ ds^2 = e^{2\Omega(u)}(dt^2 - dx^2) - u^2(e^{2\beta(u)}dy^2 - e^{-2\beta(u)}dz^2). \] (2.25)

Here all coefficients in the metric are functions of \((t-x)\) which are represented by \(u\). The \((x, y, z)\) are not the usual Cartesian coordinates but rectangular coordinates in a curved space-time. Thus the metric tensor of equation (2.25) is

\[
\begin{pmatrix}
  e^{2\Omega(u)} & 0 & 0 & 0 \\
  0 & -e^{2\Omega(u)} & 0 & 0 \\
  0 & 0 & -u^2e^{2\beta(u)} & 0 \\
  0 & 0 & 0 & -u^2e^{-2\beta(u)}
\end{pmatrix}.
\] (2.26)

Its inverse can be written as

\[
\begin{pmatrix}
  e^{-2\Omega(u)} & 0 & 0 & 0 \\
  0 & -e^{-2\Omega(u)} & 0 & 0 \\
  0 & 0 & -u^{-2}e^{-2\beta(u)} & 0 \\
  0 & 0 & 0 & -u^{-2}e^{2\beta(u)}
\end{pmatrix}.
\] (2.27)

Here it should be noted for convenience that

\[ \Omega_0 = \Omega' = -\Omega, \beta_0 = \beta' = -\beta. \] Here “\(‘\)” represents derivative with respect to \(u\).

\[
\begin{vmatrix}
  0 & 1 & 0 & 1 \\
  0 & 1 & 1 & 0 \\
  1 & 0 & 0 & 0 \\
  1 & 0 & 0 & 0
\end{vmatrix},
\] (2.28)
\[
\begin{align*}
&= - \begin{pmatrix}
1 \\
1 \\
1 
\end{pmatrix} = -\Omega'; \\
\begin{pmatrix}
0 \\
2 \\
0 
\end{pmatrix} &= \begin{pmatrix}
1 \\
2 \\
1 
\end{pmatrix} = u(u\beta' + 1)e^{2(\beta - \Omega)}; \\
\begin{pmatrix}
0 \\
3 \\
0 
\end{pmatrix} &= \begin{pmatrix}
1 \\
3 \\
1 
\end{pmatrix} = u(-u\beta' + 1)e^{-2(\beta + \Omega)}; \\
\begin{pmatrix}
2 \\
0 \\
2 
\end{pmatrix} &= \begin{pmatrix}
2 \\
2 \\
1 
\end{pmatrix} = \beta' + \frac{1}{u}; \\
\begin{pmatrix}
3 \\
0 \\
3 
\end{pmatrix} &= \begin{pmatrix}
3 \\
1 \\
3 
\end{pmatrix} = -\beta' + \frac{1}{u}; \\
\end{align*}
\]

The non zero Ricci tensor components are \( R_{00}, R_{01} \) and \( R_{11} \).

\[
R_{00} = \begin{pmatrix}
\mu \\
0 
\end{pmatrix} - (\ln |g|)_{00} + (\ln |g|)_{00} \begin{pmatrix}
\mu \\
0 
\end{pmatrix} = \begin{pmatrix}
\mu \\
0 
\end{pmatrix} + \begin{pmatrix}
\nu \\
0 
\end{pmatrix} = \begin{pmatrix}
\nu \\
0 
\end{pmatrix}.
\]

For the given metric (2.25) we have

\[
(\ln \sqrt{|g|}) = 2 \ln u + 2\Omega.
\]
\[
\ln \sqrt{|g|}' = 2(\Omega' + \frac{1}{u}) \quad (2.34)
\]

\[
(\ln \sqrt{|g|})'' = 2(\Omega'' - \frac{1}{u^2}) \quad (2.35)
\]

Now using the Christoffel symbols listed in equation (2.31) we get.

\[
R_{00} = 4\Omega'/u - 2\beta^2. \quad (2.36)
\]

Here the vacuum Einstein equation reduces to

\[
4\frac{\Omega'}{u} - 2\beta^2 = 0. \quad (2.37)
\]

It is an exact plane gravitational wave solution.

\[
R_{01} = -4\frac{\Omega'}{u} + 2\beta^2 = 0. \quad (2.38)
\]
\[ R_{11} = 4\frac{\Omega'}{u} - 2\beta'^2 = 0. \] (2.39)

Similarly equations (2.38) and (2.39) reduces to same value as equation (2.37), where other components of Ricci tensor are zero. Hence the only non-trivial exact solution of the plane gravitational wave is,

\[ \Omega'(u) = \frac{1}{2}u\beta'^2. \] (2.40)

### 2.3 Interaction of a particle with plane gravitational waves

Here it will be shown that gravitational waves carry energy following the method of Weber and Wheeler [3], considering the plane gravitational wave (2.25), which is discussed in the previous section.

As already mentioned, here we will show that plane gravitational waves carry energy. It can be shown by analyzing the motion of a particle which is initially at rest and interacts with the plane gravitational wave. Write the geodesic equation.

\[
\ddot{x}^\mu + \left\{ \begin{array}{c}
\mu \\
\nu \\
\rho
\end{array} \right\} \dot{x}^\nu \dot{x}^\rho = 0. \] (2.41)
Using the Christoffel symbols, listed in (2.31), and the usual summation convention over the repeated indices, equation (2.41) becomes:

\[
\frac{d^2x}{ds^2} + \Omega'\left(\frac{dt}{ds}\right)^2 + 2\Omega\frac{dt}{ds}\frac{dx}{ds} \Omega'\left(\frac{dx}{ds}\right)^2 + u(\beta' + 1)e^{2(\beta - \Omega)}\left(\frac{ds}{dy}\right)^2
\]

\[
+ u(-\beta' + 1)e^{-2(\beta + \Omega)}\left(\frac{dz}{ds}\right)^2 = 0.
\]

(2.42)

Here in the above equation (2.42) \(\frac{dt}{ds}\) is unknown. It can easily be found as follows. Dividing both sides of (2.25) by \(ds^2\) and simplifying, we get

\[
\frac{dt}{ds} = [\left(1 + \left(-2\Omega + 2\Omega^2 - \frac{4}{3}\Omega^3 + \ldots + \left(\frac{dx}{ds}\right)^2\right)\right]^{1/2}.
\]

(2.43)

Using the binomial expansion, we get the following after further simplification.

\[
\frac{dt}{ds} = 1 - \Omega + \Omega^2 - \frac{2}{3}\Omega^3 + \ldots + \left(\frac{dx}{ds}\right)^2 + \ldots
\]

(2.44)

Consider \(\Omega\) as a first order quantity and \(\beta\) a second order quantity. Let

\[
x = x(0) + x(1) + x(2) + \ldots
\]

(2.45)

Using these considerations and equation (2.44), equation (2.42) gives the following approximation equations.

\[
\frac{d^2x(0)}{ds^2} + 2\Omega'\frac{dt}{ds}\frac{dx(0)}{ds} = 0.
\]

(2.46)

\[
\frac{d^2x(1)}{ds^2} + 2\Omega'\frac{dt}{ds}\frac{dx(1)}{ds} - 2\Omega'\frac{dx(0)}{ds}\frac{dx(1)}{ds} = 0.
\]

(2.47)
\[
\frac{d^2 x_{(2)}}{ds^2} + 2\Omega'(\frac{dt}{ds} - \frac{dx_{(0)}}{ds}) \frac{dx_{(2)}}{ds} - 2\Omega'\left( \frac{dx_{(1)}}{ds} \right)^2 = 0. \tag{2.48}
\]

Equations (2.46), (2.47) and (2.48) are zero order, first order and second order approximation equations respectively. With the previously developed tools we can integrate the approximation equations. Here it should also be noted that as \( \Omega \) is a first order quantity, its derivative will also be a first order quantity. The result is:

i) The zero order approximation

\[
\frac{dx_{(0)}}{ds} = -2\Omega'(1 - \Omega)x_{(0)} + c_1, \tag{2.49}
\]

ii) The first order approximation

\[
\frac{dx_{(1)}}{ds} = -2c_2\Omega'x_{(1)} + c_3; \tag{2.50}
\]

iii) The second order approximation

\[
\frac{dx_{(2)}}{ds} = -2c_4\Omega'x_{(2)} + 2c_5\Omega's + c_6. \tag{2.51}
\]

Thus equations (2.49), (2.50) and (2.51) guarantee that a non-zero momentum is imparted to the particle after interacting with the plane gravitational wave. Hence gravitational waves carry energy.
2.4 Sources of gravitational waves

2.4.1 Spinning rod

A rod spinning about an axis perpendicular to its length is one of the first sources of gravitational radiation ever to have been considered [13].

Consider a steel beam of radius \( r = 1 \) meter length \( l = 20 \) meter, density \( \rho = 7.8gm/cm^3 \), mass \( M = 4.9 \times 10^8 \) gm and tensile strength \( t = 40,000lbft/in^2 \). Let the beam rotates about its middle, so it rotates end over end with an angular velocity \( \omega \) limited by the balance centrifugal force and tensile strength.

\[
\omega = \left( \frac{8t}{\rho l^2} \right)^{1/2} = 28\text{radians/sec} \tag{2.52}
\]

The internal power flow is

\[
L_{\text{internal}} = \left( \frac{1}{2} I \omega^2 \right) \omega = \frac{1}{28} M l^2 \omega^3
\approx 2 \times 10^8 \text{erg/sec} \approx 10^{-41} L_0 \text{ joule} \tag{2.53}
\]

where \( L_0 = c^5/G \). The order of magnitude of the power radiated is only \( L_{GW} \sim (10^{-41})^2 L_0 \sim 10^{-23} \text{erg/sec} \). Evidently the construction of a laboratory generator of gravitational wave is unattractive, as this is a very small quantity which cannot be detected without new engineering or new ideas. There are however a great variety of astrophysical sources of gravitational waves. We list some of them and then discuss them lightly without going into actual calculation.
2.4.2 Astrophysical sources of gravitational waves

Astrophysical sources of gravitational waves are the following:

1. (a) Pulsar

(b) Double star system;

(c) Gravitational collapse of a few solar mass star;

(d) Formation of a large black hole;

Now let me explain them.

(a) Gravitational radiation from a pulsar

Consider a highly dynamic astrophysical system. In particular take it to be a wildly rotating pulsar. If its mass is $M$ and its size is $R$ then by virial theorem its kinetic energy is $\sim M^2/R^2$. The characteristic time scale for mass to move from one side of the system to the other is

\[ T \sim R/(\text{mean velocity}) \sim R/(M/R)^{1/2} = (R^3/M)^{1/2} \]

(2.54)

The internal power flow is

\[ L_{\text{int}} \sim \frac{\text{kinetic energy}}{T} \sim (\frac{M^2}{R})(\frac{M^2}{R^3})^{1/2} \sim (M/R)^{5/2}. \]

(2.55)

The gravitational wave output is the square of this quantity or $L_{\text{GW}} \sim (M/R)^5 L_0$.

Clearly the maximum power output occurs when the system is near its gravitational radius, and because nothing, not even gravitational waves can escape
from inside the gravitational radius. The maximum value of the output is
\[ \sim L_0 = 3.63 \times 10^{59}\, \text{erg/ sec} \] regardless of the nature of the system.

(b) Double star system

It has been estimated that at least one-fifth of all the stars are binary systems. We will go into the details of how it happens so frequently because that is a topic of astrophysics and hydrodynamics.

Consider two stars of masses \( m_1 \) and \( m_2 \) revolving in a circular orbit about their common centre of gravity. For their circular frequency of revolution, we have the standard formula:

\[ \omega^2 = \frac{(m_1 + m_2)}{r^3} \text{ (geometrical units of mass and time)} \quad (2.56) \]

The calculated rate of loss of energy by radiation is [13].

\[ -\frac{dE}{dt} = \left(\frac{32}{5}\right)(m_1m_2/(m_1 + m_2))^2r^4\omega^6 \quad (2.57) \]

The following are important types of double star systems.

| Type name | \(-\left(\frac{dE}{dt}\right)_{\text{grav}}\) on Earth |
|-----------|-----------------------------------------------|
| \(\eta \cos\) | \(5.2 \times 10^{10}\, \text{erg/ sec} \quad 1.4 \times 10^{-29}\, \text{erg/cm}^2\, \text{sec}\) |
| \(\xi \beta\alpha\) | \(3.6 \times 10^{12}\, \text{erg/ sec} \quad 6.7 \times 10^{-28}\, \text{erg/cm}^2\, \text{sec}\) |
| \(Wuma\) | \(4.7 \times 10^{29}\, \text{erg/ sec} \quad 3.2 \times 10^{-13}\, \text{erg/cm}^2\, \text{sec}\) |

(c) Gravitational collapse of a few solar masses star

Collapse to form neutron stars or black holes in the mass range 1 to 10 \( M_\odot \) (\( M_\odot = 2 \times 10^{33}\, \text{gm} \) is a solar mass) will radiate waves in the frequency range 1 to 10 kHz with an amplitude that depends on how much symmetry there
is in the collapse. These collapses, at least sometimes, result in supernova explosions. The rate at which supernova occurs is relatively well known, but the fraction of collapse events that produce strong enough gravitational waves is not well known. The characteristic period of the waves is proportional to the light-travel time around the collapsed object, the dominant frequency scale is $1/M$. For sufficiently large $M$, the source will produce low frequency waves detectable in space.

(d) Formation of a giant black holes

Many astrophysicist believe that the most plausible explanation for quasars and active galactic nuclei is that they contain massive ($10^6 - 10^9 M_\odot$) black holes that accrete gas and stars to fuel their activity. There is growing evidence that even so called normal galaxies, like our own and Andromeda, contains black holes of modest size ($10^4 - 10^6 M_\odot$) in their nuclei. It is not clear how such holes form, but if they form by the rapid collapse of a cluster of stars or of a single supermassive star, then with a modest degree of non-symmetry in collapse, they could produce amplitudes $h \approx 10^{-16}$ to $10^{-18}$ meters in the low frequency range observable from space. If a detector has spectral noise density of $10^{-20} H_z^{-1/2}$ then such events could have signal to noise ratio ($S/N$) of as much as 1000. This strong signal would permit a detailed study of the event. If every galaxy has one such black hole formed in this way, then there could be one event per year in a galaxy. If no such events are seen, then either giant black holes don’t exist or they form much more gradually or with too much spherical symmetry.
Chapter 3

REVIEW OF THE
PSEUDO-NEWTONIAN
FORMALISM AND ITS
EXTENSION

In the first section a review of the Pseudo-Newtonian ($\psi N$) formalism is given. Section 2 provides a review of the extension of the $\psi N$ formalism. In section 3 the extended $\psi N$ ($e\psi N$) formalism is used to develop a formula for the momentum imparted to test particles in arbitrary spacetime. Finally this formula is applied to plane and cylindrical gravitational waves, both of which give very reasonable results.
### 3.1 The $\Psi N$ formalism

The $\Psi N$ formalism [7] is based on the observation that, whereas the gravitational force is not detectable in a freely falling frame (FFF), that is so only at a point. It is detectable over a finite spatial extent as the tidal force. It could be measured by an accelerometer, as shown in fig.3.1.

This accelerometer has a spring of length "$l$" which connects two masses. The spring ends in a needle which can move on the dial of the accelerometer to give a measure of the tension in the spring. Thus an observer in the FFF can observe the position of the spring by observing the moment of the needle on the dial. In Newtonian gravitation theory, the needle will show a zero position of the accelerometer in the absence of a central force. The force exerted by the source pulls the mass near it more than the mass further away. Thus the spring is stretched and the needle will move in the positive direction.

If the spring is compressed the needle will show a negative deflection otherwise it will show a positive deflection. This would occur if both discs and the source consisted of like electric charges. Hence the negative deflection corresponds to a repulsive source and the positive deflection corresponds to an attractive source. The strength of the source would be shown by the extent that the needle moved.

Mathematically the tidal acceleration is given by
Here \( R \) is the Riemann tensor, \( t \) a timelike killing vector, \( l \) is the spacelike separation vector representing accelerometer. Thus in geometrical terms tidal force is given by

\[
F^\mu = -m R^\mu_{\nu \rho \pi} t^\nu l^\rho t^\pi \tag{3.2}
\]

where \( m \) is the mass of the test particle, \( t^\nu \) the timelike vector tangent to the particles path and \( l^\rho \) the separation vector, which provides the observation of the tidal force. In the FFF \( t^\nu = f \delta_{0}^\nu \), \( f^2 = \frac{1}{g_{00}} \). Thus equation (3.2) becomes:

\[
F^\mu = -m f^2 R^\mu_{0 \rho 0} l^\rho. \tag{3.3}
\]

Regarding this as an eigenvalue equation, we get the maximum tidal force along the eigenvector of the matrix \( R^\mu_{0 \rho 0} \). Since \( l^\rho \) is a purely spacelike vector in the free fall frame, the maximum tidal force will be a purely space like vector. Thus we get

\[
\tilde{F}^i = l^j \gamma^i \tag{3.4}
\]

where “ ,” stands for covariant derivative. Thus \( \gamma^i \) is the relativistic analogue of the Newtonian gravitational force. Further this force is the gradient of a
Figure 3.1: An accelerometer to measure the force due to a gravitating source.

The force is measured by the instrument in terms of the tension in the spring and shown by motion of the needle on the dial.
scalar quantity \[7\]

\[\gamma^i = -\phi_i.\] (3.5)

Where an appropriate expression for \(\phi\) is

\[\phi = \frac{1}{2}(g_{00} - 1).\] (3.6)

When expressed in this way a Lorentz factor has to be introduced by hand. \(\gamma^i\) and \(\phi\) are called the \(\psi N\) force and potential respectively.

### 3.2 The \(e\psi N\) force

The quantity whose directional derivative along the accelerometer, placed along the principal direction, gives the extremised tidal force, which is zero in Minkowski space. Thus the \(e\psi N\) force, \(F^\mu\), satisfies the equation \[9\]

\[F^{\ast \mu} = l^\nu F^\mu_{,\nu};\] (3.7)

Where \(F^{\ast \mu}\) is the extremised tidal force. The fact that the zero component of the left side is zero does not guarantee that the zero component of \(F^\mu\) is zero.

The space could be made look flat in a small neighborhood of any point \(P\) in by a special choice of inertial coordinates. This locally inertial coordinate system has the metric of special relativity at some point \(P\) and so the Christoffel symbols \(\Gamma^\mu_{\nu\rho}\) are zero at \(P\). It is not, however, possible to make \(R_{\mu\nu\rho\sigma} = 0\) at \(P\) by any choice of coordinates, unless the space is flat in the neighborhood of
$P$. This is because $R_{\mu\nu\rho\tau}$ is a tensor; if it vanishes in one coordinate system it will do so in any other because of the transformation laws. Here we will restrict ourselves to Riemann normal coordinates spatially, but not temporally.

Equation (3.7) can be written in the space and time break up as

\[ l^i(F^0_{,i} + \Gamma^0_{ij} F^j) = 0 \]  
\[ (3.8) \]

\[ l^i(F^0_{,i} + \Gamma^0_{ij} F^j) = F^{*i} \]  
\[ (3.9) \]

A simultaneous solution of the above equations can be obtained by using Riemann normal coordinates for the spatial directions but not for the time coordinates. Thus the $e\psi N$ force four vector is (see Appendix 2 for the proof):

\[ F^0 = m([\ln A]_0 - \Gamma^0_{00} + \Gamma^i_{0j} \Gamma^j_{0i} / A] f^2, \]  
\[ (3.10) \]

\[ F^i = \Gamma^i_{00} f^2. \]  
\[ (3.11) \]

where $A = (\ln \sqrt{-g})_0$ \quad $g = \text{det}(g_{ij})$ For these, block diagonalized metrics

\[ \Gamma^0_{00} = \frac{1}{2} g^{00} g_{00,0} \]  
\[ \Gamma^i_{00} = -\frac{1}{2} g^{ij} g_{00,j} \]  
\[ \Gamma^i_{0j} = \frac{1}{2} g^{ij} g_{jk,0} \]

Thus, writing the covariant form of the $e\psi N$ force

\[ F_0 = m([\ln Af]_0 - g^{ij}_{00} g_{i,j,0} / 4A], \]  
\[ (3.12) \]
\[ F_i = m(\ln f)_{,i} \] (3.13)

It is worth considering the significance of the zero component of the \( e\psi N \) force, which is its major difference from the \( \psi N \) force. In special relativistic terms, which are relevant for discussing forces in a Minkowski space, the zero component of the four vector force corresponds to a proper rate of change of energy of the test particle. Further we know that, in general, an accelerated particle either radiates or absorbs energy according as \( \frac{dE}{dt} \) is less than or greater than zero. Thus \( F_0 \), should also correspond to energy-emission or absorption by the background spacetime. This point will be discussed further in the next chapter.

### 3.3 General formula for the momentum imparted to test particles in arbitrary spacetime

There has been a debate whether gravitational waves really exist [3,4]. To demonstrate the reality of gravitational waves Ehler and Kundt [4] considered a sphere of test particles in the path of plane fronted gravitational waves and showed that a constant momentum was imparted to the test particles. This was latter extended by Weber and Wheeler [3] for cylindrical gravitational waves. The \( \psi N \) force and \( e\psi N \) force are based on an operational procedure
embodying the same principle [6]. The proper time integral of the force four vector will be the momentum four vector. Here it will be verified that this procedure gives the Ehler-Kundt result for plane fronted gravitational waves. When it is applied to cylindrical gravitational waves it is found that the result so obtained is physically reasonable and gives an exact expression for the momentum imparted to test particles, corresponding to the approximation given by Weber and Wheeler.

From Equations (3.10) and (3.11) the momentum four vector, \( P_\mu \), is obtained as:

\[
P_\mu = \int F_\mu dt. \tag{3.14}
\]

Now we apply this formula to plane fronted gravitational waves and cylindrical gravitational waves.

### 3.3.1 Plane-fronted gravitational waves

The metric for Plane-fronted gravitational waves is

\[
ds^2 = dt^2 - dx^2 - L^2(t, x)\{\exp(2\beta(t, x))dy^2 + \exp(-2\beta(t, x))dz^2\}. \tag{3.15}
\]

Where \( L \) and \( \beta \) are arbitrary functions.

The metric tensor is
\[ g_{ab} = \begin{pmatrix} 
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -L^2e^{2\beta} & 0 \\
0 & 0 & 0 & -L^2e^{-2\beta} 
\end{pmatrix}. \]  

(3.16)

Its inverse is

\[ g^{ab} = \begin{pmatrix} 
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -L^{-2}e^{-2\beta} & 0 \\
0 & 0 & 0 & -L^{-2}e^{2\beta} 
\end{pmatrix}. \]  

(3.17)

To find the momentum four vector we need the force four vector. We calculate the force four vector. Here in this special case:

\[ g = \det(g_{ij}) = -L^4(t, x), \quad f = \frac{1}{\sqrt{g_{00}}} = 1 \]  

(3.18)

\[ \ln \sqrt{-g} = 2 \ln L \]  

(3.19)

which implies that

\[ Af = (\ln \sqrt{-g})_0 = 2\frac{\dot{L}}{L}. \]  

(3.20)

\[ (\ln(Af))_0 = \frac{\ddot{L}}{L} - \frac{\dot{L}}{L} \]  

(3.21)
\[ g_{ij} \delta g_{ij} = -8(\dot{\beta}^2 + \dot{L}^2). \]  

(3.22)

So from equations \((3.20), (3.21), (3.22)\) and \((3.10)\) we get

\[ F_0 = m(\ddot{L} + \dot{\beta}^2 L)/L. \]  

(3.23)

From the vacuum Einstein equations we have (already discussed in chapter 2)

\[ \ddot{L} + \dot{\beta}^2 L = 0, \]  

(3.24)

which implies that \(F_0 = 0\) that is the zero component of the force four-vector is zero. Also \(F_i = 0\) because \((\ln \sqrt{g_{00}}),_i = 0\).

Thus the momentum four vector becomes \(P_\mu = \text{constant}\). Hence there is a constant energy and momentum imparted to the test particles. The constant here determines the strength of the wave. This exactly coincides with the Ehler-Kundt method in that they demonstrate that the test particles acquires a constant momentum and hence a constant energy from a plane gravitational wave.

### 3.3.2 Cylindrical gravitational waves

Consider the cylindrically symmetric metric \((2.1)\). We first calculate the force four vector. Here in this special case:

\[ g = \det(g_{ij}) = -e^{4(\gamma - \psi)}. \]  

(3.25)

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\[ \ln \sqrt{-g} = 2(\gamma - \psi). \] (3.26)

\[ A = 2(\gamma' - \psi'), \quad f = e^{\psi - \gamma}. \] (3.27)

\[ (\ln(Af))_0 = \frac{(\gamma' - \psi')}{(\gamma - \psi')} - (\gamma - \psi'). \] (3.28)

\[ g^{ij}_0 g_{ij,0} = -4(\gamma' - \psi')^2 - 8\psi^2. \] (3.29)

where a dot denotes differentiation with respect to \( t \).

The vacuum Einstein field equation gives [3].

\[ \psi(t, \rho) = AJ_0(\omega \rho) \cos(\omega t) + BN_0(\omega \rho) \sin(\omega t), \] (3.30)

where \( J_0(\omega \rho) \) and \( N_0(\omega \rho) \) are the zero order Basel and Neuman functions respectively. \( A \) and \( B \) are arbitrary constants corresponding to the strength of gravitational wave.

\[ \gamma(t, \rho) = \frac{1}{2} \omega \rho \{ (A^2J_0(\omega \rho)J'_0(\omega \rho) - B^2N_0(x)N'_0(x)) \cos(2\omega t) \]
\[ -AB[(J_0(\omega \rho)N'_0(\omega \rho) + J'_0(x)N_0(x))] \sin(2\omega t) - \]
\[ -2(J_0(\omega \rho)N'_0(\omega \rho) - J'_0(x)N_0(x))\omega t) \} \] (3.31)

Where prime now refers to differentiation with respect to \( \omega \rho \), \( \omega \) being the angular frequency.
Using equations (3.28), (3.29), (3.30) and (3.31) we get the zero component of the force four vector.

\[
F_0 = -m\omega\left\{AJ_0 \cos(\omega t) + BN_0 \sin(\omega t)\right\} - 2\rho\omega\left\{(A^2 J_0 J'_0 - B^2 N_0 N'_0) \cos(2\omega t) - AB(J_0 N'_0 + N_0 J'_0) \sin(2\omega t)\right\} + \frac{2[AJ_0 \sin(\omega t) - BN_0 \cos(\omega t)]^2}{AJ_0 \sin(\omega t)} - BN_0 \cos(\omega t) - 2\rho[AJ_0 \sin(\omega t) - BN_0 \cos(\omega t)]
\]

\[
[AJ'_0 \cos(\omega t) + BN'_0 \sin(\omega t)]\}.
\]

(3.32)

As \(\gamma\) and \(\psi\) are functions of \(t\) and \(\rho\) only so \(F_2\) and \(F_3\) are zero.

\[
F_1 = m(\ln f)_1 = -m\omega\left\{AJ'_0 \cos(\omega t) - BN'_0 \sin(\omega t)\right\} - \frac{1}{2}[(A^2 J_0 J'_0 - B^2 N_0 N'_0)] + \omega\rho(A^2 J_0 J'_0 - B^2 N_0 N'_0) \cos(2\omega t) - \frac{1}{2}AB[2(J_0 N'_0 + J'_0 N_0)] \sin(2\omega t)
\]

\[
-\frac{1}{2}AB[4(J_0 N'_0 - J'_0 N_0) + 2\omega\rho(J_0 N'_0 - J'_0 N_0)'] \omega t\}.
\]

(3.33)

The corresponding \(P_0\) and \(P_1\) are
\[ P_0 = -m \ln | AJ_0 \sin(\omega t) - BN_0 \cos(\omega t) | \] 
\[ + BN_0' \sin(\omega t) | \left( 1 + \frac{A^2 J_0 J_0' + B^2 N_0 N_0'}{\omega \rho (A J_0^2 + B N_0^2)} \right) \] 
\[ - \frac{AB(A J_0 N_0 N_0' + N_0 J_0' - A \omega \rho J_0 J_0' N_0 - N_0' J_0)}{\omega \rho (A^2 J_0^2 + B^2 N_0^2) \sqrt{1 - 4 \omega^2 \rho^2 (A^2 J_0^2 + B^2 N_0^2)}} \] 
\[ + f_1(\omega \rho). \] (3.34)

\[ P_1 = -m \{ [A J_0' \sin(\omega t) - B N_0' \cos(\omega t)] - \frac{1}{4} [(A^2 J_0 J_0' - B^2 N_0 N_0') \sin(2 \omega t)] - \frac{1}{2} AB[(J_0 N_0' + J_0' N_0)] \] 
\[ + \omega \rho (A^2 J_0^2 - B^2 N_0^2) \sin(2 \omega t) - \frac{1}{2} AB[(J_0 N_0' + J_0' N_0)] \] 
\[ \omega \rho (J_0 N_0' + J_0' N_0) \cos(2 \omega t) - AB \omega^2 \rho^2 [(J_0 N_0' + J_0' N_0)] \] 
\[ - \omega \rho (J_0 N_0' - J_0' N_0)] \} + f_2(\omega \rho). \] (3.35)

Where \( f_1 \) and \( f_2 \) are arbitrary constants of integration. Weber and Wheeler exclude solution that contain irregular Bessel function, \( N_0(\omega \rho) \), as not well defined at the origin. Taking the Weber-Wheeler solutions equations (3.34) and (3.35) reduces to

\[ P_0 = -m \ln | AJ_0 \sin(\omega t) | + (1 + A J_0 / \omega \rho J_0') \ln | 1 - 2 \omega \rho A J_0' \cos(\omega t) | \] 
\[ + f_1(\omega \rho). \] (3.36)
\[ P_1 = -m \{ AJ_0' \sin(\omega t) - \frac{1}{4} A^2 J_0 J_0' + \rho \omega (J_0 J_0')' \sin(2\omega t) + f_2(\omega \rho) \}. \] (3.37)

We see that the quantity \( P_1 \) given by equation (3.37) can be made zero for the large and small \( \rho \) limits by choosing \( f_2 \) equal to zero. This is physically reasonable expression for the momentum imparted to test particles by cylindrical gravitational waves. The quantity \( P_0 \) given by equation (3.36) remains finite for small \( \rho \) and can also be made finite for large \( \rho \) by choosing \( f_1 = -\ln(J_0) \).

However, there is a singularity at \( \omega t = n\pi \). This problem does not arise in the general expression given by equation (3.34). However, in that case there appears a term linear in time which creates interpretational problems. Also \( P_0 \) and \( P_1 \) becomes singular at \( \rho = 0 \) if \( B \neq 0 \).
Chapter 4

SPIN IMPARTED TO TEST PARTICLES BY GRAVITATIONAL WAVES

We can write the $e\psi N$ force four vector (discussed in section 3.2) as [7]

\[ F_0 = -U_0, \quad F_i = -V_i, \]  \hspace{1cm} (4.1)

where

\[ U = m \left[ \ln \left( \frac{Af}{B} \right) - \int \frac{g_{ij}^i g_{ij,0}}{4A} dt \right], \]  \hspace{1cm} (4.2)

\[ V = -m \ln(f), \]  \hspace{1cm} (4.3)

where B is a constant with units of time inverse so as to make $\frac{A}{B}$ dimensionless.

Here $V$ is the $e\psi N$ potential but there is no good interpretation of $U$. Thus there is a problem of interpretation of $F_0$. The $F_i$ was reinterpreted [8] as the
rate of change of the momentum imparted to test particles by the gravitational field, i.e. $F_i = \frac{dP_i}{d\tau}$ (where $\tau$ is the proper time). In this interpretation it would be natural to identify $F_0$ as $\frac{dP_0}{d\tau}$. The problem now is to interpret $P_0$ since one would normally take $P_0 = E = \sqrt{m^2 + P^i P_i g_{ij}}$ where $m$ is the mass of the test particle. Hence $\int F_0 dt$ cannot be this $P_0$. Sharif’s suggestion [11] for the interpretation of $P_0$, that it gives the spin angular momentum imparted to test rods, is given in the first section of this chapter, but in the same section it turns out to be inconsistent. To find an alternative check on its validity the geodesic [4] analysis for the angular momentum imparted to test particles by gravitational waves is undertaken in section 2. This formalism is applied to various cases in section 3.

4.1 Spin angular momentum imparted by gravitational waves

Sharif [11] considers a test rod of length $\lambda$ in the path of a gravitational wave whose preferred direction is given by $l^i$ in the preferred reference frame. He argues that the rod will acquire maximum angular momentum from the wave if it lies in the plane given by $e_{\rho jkl} l^\rho$, where $e_{\mu\nu\rho\pi}$ is a totally skew symmetric fourth rank tensor. Thus the spin vector will be given by

$$S^i = \frac{1}{2} e^{ijk\nu} e_{jkl} l^\nu P_\nu.$$  

(4.4)
For $i = 1$, 

$$S^1 = \frac{1}{2} e^{jk\nu} e_{jkl} l^l P_\nu$$

$$= \frac{1}{2}[e^{1230} e_{230} + e^{1320} e_{320}]l^l P_0$$

$$= P_0 l^1. \quad (4.5)$$

Similarly 

$$S^2 = P_0 l^2 \text{ and } S^3 = P_0 l^3. \quad (4.6)$$

So in the preferred direction the spin vector would be proportional to $l^i$ such that 

$$S^i = P_0 l^i. \quad (4.7)$$

The angular momentum imparted would be the magnitude of the spin vector. Thus the maximum angular momentum imparted to a test rod when it lies in the plane perpendicular to the preferred direction is: 

$$S = P_0 \lambda = m \lambda \int [(\ln(Af)_{,0} - g_{ij,0} g^{ij})/4A]dt. \quad (4.8)$$

Hence the physical significance of the zero component of the momentum four vector would be that it provides an expression for the spin imparted to a test rod in an arbitrary spacetime.

This formula was applied to plane and cylindrical gravitational waves to give the following results.
1. Plane gravitational waves.

Using metric (2.25) in equation (4.8) we get (for detailed calculations see section 3.2)

\[ P_0 = \text{constant.} \quad (4.9) \]

and thus the spin would also be constant.

2. Cylindrical gravitational waves.

Following the same procedure for the metric (2.1) we get (for detailed calculations see section 3.2)

\[ S = -m\lambda[(1 + AJ_0/\omega\rho J_0) \ln |1 - 2\omega\rho AJ_0/\omega J_0| + \text{constant}. \quad (4.10) \]

Notice that there can be no spin angular momentum imparted to test particles in a perfectly homogeneous and isotropic cosmological model [1]; its high degree of symmetry— in particular, spherical symmetry is incompatible with spin being imparted to test particles. However when we use Sharif’s formula for cosmological models, it gives exactly this error.

We give examples which had already partly been constructed by M. Sharif [14].

3. The Friedman model:

Consider the Friedman model, which is isotropic and homogeneous,

\[ ds^2 = dt^2 - a^2(t)[d\chi^2 + f^2_k(\chi)d\Omega^2], \quad (4.11) \]
where

\[ k = 1 \quad f_1(\chi) = \sin(\chi), \]

\[ k = 0 \quad f_0(\chi) = \chi, \]

\[ k = -1 \quad f_{-1}(\chi) = \sinh(\chi), \]

\( \chi \) being the hyperspherical angle and \( a(t) \) the scale parameter given by:

\[
\begin{align*}
    a(t) &= a_0(1 - \cos \eta)/2, \quad t = a_0(\eta - \sin \eta), \quad 0 \leq \eta \leq 2\pi, \quad k = 1 \\
    &= a_0^{1/3} t^{2/3}, \quad k = 0; \\
    &= a_0(\cosh \eta - 1)/2, \quad t = a_0(\sinh \eta - \eta)/2, \quad 0 \leq \eta \leq \infty, \quad k = -1;
\end{align*}
\]

(4.12)

Using the metric (4.11) the \( e\psi N \) force for the Friedmann models will be

\[ F_0 = -m \frac{a'}{a}, \quad (4.13) \]

and

\[ P_0 = \int F_0 dt = -m \ln(a'). \quad (4.14) \]

As \( S = m\lambda \int F_0 dt \). We get the spin for Friedmann models

\[ S = -m\lambda \ln(a'). \quad (4.15) \]

a. Closed Friedmann model

Using the metric (4.11) with \( k = 1 \) in equation (4.14) and (4.15) we get

\[ P_0 = m[\ln \sqrt{1 - \cos \eta} - \frac{3}{8} \cos \eta + \frac{1}{16} \cos^2 \eta + \ln \sqrt{2} + \frac{5}{16}], \quad (4.16) \]

and thus

\[ S = m\lambda[\ln \sqrt{1 - \cos \eta} - \frac{3}{8} \cos \eta + \frac{1}{16} \cos^2 \eta + \ln \sqrt{2} + \frac{5}{16}]. \quad (4.17) \]
b. Flat Friedmann model.

Again using the metric (4.11) with \( k = 0 \) in equation (4.14) and (4.15) we get

\[ P_0 = m \ln \left( \frac{\eta}{2} \right). \] (4.18)

And thus the spin for a flat Friedmann model is

\[ S = m \lambda \ln \left( \frac{\eta}{2} \right). \] (4.19)

c. Open Friedmann model.

Finally we obtain the spin for the open Friedmann model by the same procedure as the following

\[ P_0 = -m \ln \left| \frac{\sinh \eta}{\cosh \eta - 1} \right|. \] (4.20)

\[ S = -m \lambda \ln \left| \frac{\sinh \eta}{\cosh \eta - 1} \right|. \] (4.21)

Equations (4.17), (4.19) and (4.21) tells us that a non-zero spin is imparted by flat, open, and closed Friedmann models. Since no spin can be imparted thus Sharif’s interpretation of \( P_0 \) cannot be correct.

4. The Kasner model

Consider the Kasner model for a homogeneous anisotropic universe (near the cosmological singularity)

\[ ds^2 = dt^2 - t^{2p_1} dx^2 - t^{2p_2} dy^2 - t^{2p_3} dz^2 \] (4.22)

Here \( p_i \) are numbers such that \( p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1. \)
The metric tensor is

\[ g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -t^{2p_1} & 0 & 0 \\ 0 & 0 & -t^{2p_2} & 0 \\ 0 & 0 & 0 & -t^{2p_3} \end{pmatrix} \] (4.23)

Its inverse is

\[ g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -t^{-2p_1} & 0 & 0 \\ 0 & 0 & -t^{-2p_2} & 0 \\ 0 & 0 & 0 & -t^{-2p_3} \end{pmatrix} \] (4.24)

To find the momentum four vector we need the force four vector. We therefore calculate it. Now

\[ g = \det(g_{ij}) = -t^2, \quad f = \frac{1}{\sqrt{g_{00}}} = 1, \]

(4.25)

\[ \ln \sqrt{-g} = \ln t, \]

(4.26)

which implies that

\[ Af = (\ln \sqrt{-g})_0 = \frac{1}{t}, \]

(4.27)

\[ (\ln(Af))_0 = -\frac{1}{t}, \]

(4.28)

\[ g^{ij}_0g_{ij,0} = -\frac{4}{t^2}. \]

(4.29)

So from equations (4.28) and (4.29) we get

\[ F_0 = 0 \]

(4.30)
Hence from equation \((4.8)\) we get

\[ S = \text{constant.} \quad (4.31) \]

By physical consideration we could set \(S = 0\).

**5. The De Sitter universe (usual coordinates)**

Consider the metric

\[ ds^2 = (1 - r^2/D^2)dt^2 - (1 - r^2/D^2)^{-1}dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2), \quad (4.32) \]

The metric tensor is

\[ g_{\mu\nu} = \begin{pmatrix} (1 - r^2/D^2) & 0 & 0 & 0 \\ 0 & -(1 - r^2/D^2)^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2 \theta \end{pmatrix}. \quad (4.33) \]

Its inverse is

\[ g^{\mu\nu} = \begin{pmatrix} -D^2 & 0 & 0 & 0 \\ 0 & -D^2/r^2 & 0 & 0 \\ 0 & 0 & -1/r^2 & 0 \\ 0 & 0 & 0 & -1/r^2 \sin^2 \theta \end{pmatrix}. \quad (4.34) \]

Again proceeding on the same lines for the momentum four vector, we have

\[ g = \det(g_{ij}) = -r^4 \sin^2 \theta, \quad f = \frac{1}{\sqrt{g_{00}}} = \frac{1}{\sqrt{(1 - r^2/D^2)}}, \quad (4.35) \]

\[ \ln \sqrt{-g} = \ln(r^2 \sin \theta), \quad (4.36) \]

which implies that
\[ Af = \ln(\sqrt{-g}),_0 = 0, \quad (4.37) \]

\[(\ln(Af))_0 = 0, \quad (4.38)\]

\[ g^{ij}_{0}g_{ij,0} = 0. \quad (4.39) \]

So from equations (4.38), (4.39) we get

\[ F_0 = 0. \quad (4.40) \]

Hence from equation (4.8) we get

\[ S = \text{constant}. \quad (4.41) \]

Clearly, we would need to take \( S = 0 \) here.

6. The Lemaitre form of the De Sitter universe

Consider the empty space solution of the Einstein field equations with cosmological constant,

\[ ds^2 = dt^2 - a_0^2 e^{2(\Lambda/3)^{1/2}t} \left[ d\chi^2 + \chi^2 d\theta^2 + \chi^2 \sin^2 \theta d\phi^2 \right]. \quad (4.42) \]

The metric tensor is

\[
g_{\mu\nu} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -a_0^2 e^{2(\Lambda/3)^{1/2}t} & 0 & 0 \\
0 & 0 & -a_0^2 e^{2(\Lambda/3)^{1/2}t} \chi^2 & 0 \\
0 & 0 & 0 & -a_0^2 e^{2(\Lambda/3)^{1/2}t} \chi^2 \sin^2 \theta
\end{pmatrix}. \quad (4.43)\]
Its inverse is

\[
g^{\mu\nu} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 - \frac{1}{a_0^2 e^{2\sqrt{3} \sqrt{\Lambda} t}} & 0 & 0 \\
0 & 0 & - \frac{1}{a_0^2 e^{2\sqrt{3} \sqrt{\Lambda} \chi^2}} & 0 \\
0 & 0 & 0 & - \frac{1}{a_0^2 e^{2\sqrt{3} \sqrt{\Lambda} \chi^2 \sin^2 \theta}}
\end{pmatrix}.
\]

To find the momentum four vector we need the force four vector. We therefore calculate it. Now

\[
g = \det(g_{ij}) = -a_0^6 e^{2\sqrt{3} \sqrt{\Lambda} t} \chi^4 \sin^2 \theta, \quad f = \frac{1}{\sqrt{g_{00}}} = 1,
\]

\[
\ln \sqrt{-g} = 3 \ln a_0 + 2 \ln \chi + \sqrt{3} \sqrt{\Lambda} t + \frac{1}{2} \ln (-\cos \theta + 1) + \frac{1}{2} \ln (\cos \theta + 1),
\]

which implies that

\[
Af = (\ln \sqrt{-g})_{,0} = \sqrt{3} \sqrt{\Lambda},
\]

\[
(ln(Af))_{,0} = 0,
\]

\[
g_{00}^{ij}g_{ij,0} = -4\Lambda.
\]

So from equations (4.48), (4.49) and (3.12) we get

\[
F_0 = m \left( \frac{\Lambda}{3} \right)^{1/2}.
\]

Hence from equation (4.8) we get

\[
S = m\lambda \left( \frac{\Lambda}{3} \right)^{1/2} t + \text{constant}.
\]
This does not seem reasonable. Since the other form of the metric gives a different result, it is clear that the interpretation is not even internally consistent.

4.2 The geodesic analysis for angular momentum imparted to test particles by gravitational waves

In section 4.1 we have concluded that $S$ is not the spin angular momentum imparted to test particles by gravitational waves. Then the question arises, if $S$ is not the spin then what is the spin angular momentum. So in this section we use the geodesic analysis [4] to find the angular momentum imparted to test particles by gravitational waves. This formula is further applied to various cases.

Consider a time like congruence of the world lines (not necessarily geodetic) with tangent vector $u^a$. Decompose $u_{a;b}$ by means of the operator $h_{ab}$ projecting into the infinitesimal 3-space orthogonal to $u^a$ [4].

\[
  u_{a;b} = -\omega_{ab} + \sigma_{ab} + \frac{1}{3} \theta h_{ab} - u^a u_b. \tag{4.52}
\]

where

\[
  -\omega_{ab} \equiv u_{[a;b]} + u_{[a} u_{b]}, \quad u^a \equiv u_{a;b} u^b. \tag{4.53}
\]

\[
  \theta \equiv u^a_{\;;a}, \quad h_{ab} \equiv g_{ab} + u_a u_b \tag{4.54}
\]

\[
  \sigma_{ab} \equiv u_{(a;b)} + u_{(a} u_{b)} - \frac{1}{3} \theta h_{ab} \quad (\sigma^a_a = 0). \tag{4.55}
\]
For an observer along one of the world lines and using Fermi propagated axes, \( \omega_{ab} \) describe velocity of rotation, \( \sigma_{ab} \) shear and \( \theta \) describe expansion of the neighboring free particles. Since the \( \psi N \)–formalism uses the fermi-walker frame, it could be expected that the results of this analysis should be consistent with it. For our purpose only \( \omega_{ab} \) is needed. Choose the coordinates so that the tangent vector is \( u^a = f \delta^a_0 \). In the case \( g_{00} = 1 \), \( u^a = \delta^a_0 \). If \( g_{00} \neq 1 \), \( u^a = \frac{1}{\sqrt{g_{00}}} \delta^a_0 \) Thus, from equation (4.53) we have

\[
- \omega_{ab} = (u_{a,b} - u_{b,a})/2 + (u^a u_b - u^b u_a)/2.
\] (4.56)

The first and second component on the right hand side of the above equation vanishes. Thus we have

\[
- \omega_{ab} = (u^a u_b - u^b u_a)/2.
\] (4.57)

Further using equation (4.53) we have

\[
-2 \omega_{ab} = \begin{vmatrix}
0 & d & d \\
0 & c & 0 \\
0 & a & b
\end{vmatrix} u_d u^c u_b - \begin{vmatrix}
0 & d \\
0 & c \\
0 & a
\end{vmatrix} u_d u^c u_a
\] (4.58)

\[
= \begin{vmatrix}
0 & \delta^0_b \\
0 & \delta^0_a \\
a & b
\end{vmatrix}.
\]

Taking \( a \) to be \( i \) and \( b \) to be zero, we finally obtain the components of the spin vector:

\[
\omega_{i0} = -\frac{1}{2} \begin{vmatrix}
0 & \\
0 & i
\end{vmatrix}, \quad (i = 1, 2, 3)
\] (4.59)
This simple formula appears because $u^a = \delta^a_0$ only. This gives the angular momentum imparted to test particles by gravitational waves. We now apply this formula to different types of gravitational waves.

### 4.3 Applications

#### 4.3.1 Plane gravitational waves

For a metric $g_{\mu\nu}$ construct the new metric

$$\bar{g}_{\mu\nu} = \eta_{\mu\nu} + [g_{\mu\nu} - \eta_{\mu\nu}]u(s - s_0)$$

(4.60)

where $\eta_{\mu\nu}$ is the Minkowski spacetime and $u(s - s_0)$ is the step function defined as:

- $u(s - s_0) = 0$ when $s < s_0$
- $u(s - s_0) = 1$ when $s > s_0$.

According to this definition the metric tensor becomes:

$$\bar{g}_{\mu\nu} = 
\begin{pmatrix}
  \left[(e^{2\Omega(\alpha)} - 1)u + 1\right] & 0 & 0 & 0 \\
  0 & (1 - e^{2\Omega(\alpha)})u - 1 & 0 & 0 \\
  0 & 0 & (-\alpha^2 e^{2\beta(\alpha)} + 1)u - 1 & 0 \\
  0 & 0 & 0 & (-\alpha^2 e^{-2\beta(\alpha)} + 1)u - 1
\end{pmatrix}
$$

(4.61)

where $\alpha = t - x$. The inverse of this metric is
\[
\begin{pmatrix}
\frac{1}{(e^{2\Omega(\alpha)} - 1)u + 1} & 0 & 0 & 0 \\
0 & \frac{1}{-1 + (-e^{2\Omega(\alpha)} + 1)u} & 0 & 0 \\
0 & 0 & \frac{1}{-1 + (-\alpha^2e^{2\Omega(\alpha)} + 1)u} & 0 \\
0 & 0 & 0 & \frac{1}{-1 + (-\alpha^2e^{-2\Omega(\alpha)} + 1)u}
\end{pmatrix}.
\]

(4.62)

Now

\[
\begin{pmatrix}
0 \\
0 & 1
\end{pmatrix} = \frac{1}{2} g^{00} g_{00,1}, \quad \begin{pmatrix}
0 \\
0 & 2
\end{pmatrix} = \frac{1}{2} g^{00} g_{00,2}, \quad \begin{pmatrix}
0 \\
0 & 3
\end{pmatrix} = \frac{1}{2} g^{00} g_{00,3}.
\]

(4.63)

If we find \( g_{00,1}, g_{00,2} \) and \( g_{00,3} \) then we are done. Since \( g_{00} = (e^{2\Omega(\alpha)} - 1)u + 1 \).

\[
g_{00,1} = ((e^{2\Omega(\alpha)} - 1)u + 1)_{,1} = 2\Omega'(\alpha)e^{2\Omega(\alpha)}u - (e^{2\Omega(\alpha)} - 1)\frac{\partial s}{\partial x}\delta(s - s_0).
\]

(4.64)

Dividing equation (2.25) by \( ds^2 \) and keeping \( t, y \) and \( z \) fixed, we get:

\[
\frac{\partial s}{\partial x} = e^{\Omega(\alpha)}.
\]

(4.65)

Consequently we have

\[
g_{00,1} = -2\Omega'(\alpha)e^{2\Omega(\alpha)}u(s - s_0) + e^{\Omega(\alpha)}(e^{2\Omega(\alpha)} - 1)\delta(s - s_0).
\]

(4.66)

Further

\[
g_{00,2} = -(e^{2\Omega(\alpha)} - 1)\frac{\partial s}{\partial y}\delta(s - s_0).
\]

(4.67)
Again dividing equation (2.25) by \( ds^2 \) and keeping \( t, x \) and \( z \) fixed, we get

\[
\frac{\partial s}{\partial y} = \alpha e^{\beta(\alpha)}.
\] (4.68)

Putting this value of \( \frac{\partial s}{\partial y} \) into equation (4.67) we get

\[
g_{00,2} = -e^{\beta(\alpha)}(e^{2\Omega(\alpha)} - 1)\delta(s - s_0).
\] (4.69)

Also,

\[
g_{00,3} = -\alpha (e^{2\Omega(\alpha)} - 1) \frac{\partial s}{\partial z} \delta(s - s_0).
\] (4.70)

As in the previous cases, dividing equation (2.25) by \( ds^2 \) and keeping \( t, x \) and \( y \) fixed, we get

\[
\frac{\partial s}{\partial z} = \alpha e^{-\beta(\alpha)}.
\] (4.71)

Putting \( d\frac{\partial s}{\partial z} \) in equation (4.70) we get:

\[
g_{00,3} = -\alpha e^{-\beta(\alpha)}(e^{2\Omega(\alpha)} - 1)\delta(s - s_0).
\] (4.72)

Using equations (4.66), (4.69) and (4.72) in equation (4.63) we get:

\[
\left\{ \begin{array}{c}
0 \\
0 \end{array} \right\} = \frac{1}{2(e^{2\Omega(\alpha)} - 1)u + 1} \left[ -2\Omega'(\alpha)e^{2\Omega(\alpha)}u(s - s_0) + e^{\Omega(\alpha)}(e^{2\Omega(\alpha)} - 1)\delta(s - s_0) \right]
\] (4.73)

\[
\left\{ \begin{array}{c}
0 \\
0 \end{array} \right\} = \frac{1}{2(e^{2\Omega(\alpha)} - 1)u + 1} \left[ \alpha e^{\beta(\alpha)}(e^{2\Omega(\alpha)} - 1)\delta(s - s_0) \right],
\] (4.74)
\[
\begin{pmatrix}
0 \\ 0 \\ 3
\end{pmatrix} = \frac{1}{2(e^{2\Omega(\alpha)} - 1)u + 1} \left[ \alpha e^{-\beta(\alpha)}(e^{2\Omega(\alpha)} - 1)\delta(s - s_0) \right].
\]

(4.75)

Now using equations (4.73), (4.74) and (4.75) in equation (4.59) we get components of the spin vector

\[
\omega_{10} = -\frac{1}{4(e^{2\Omega(\alpha)} - 1)u + 1} \left[ -2\Omega'(\alpha)e^{2\Omega(\alpha)}u(s - s_0) + e^{\Omega(\alpha)}(e^{2\Omega(\alpha)} - 1)\delta(s - s_0) \right],
\]

(4.76)

\[
\omega_{20} = -\frac{1}{4(e^{2\Omega(\alpha)} - 1)u + 1} \left[ \alpha e^{\beta(\alpha)}(e^{2\Omega(\alpha)} - 1)\delta(s - s_0) \right],
\]

(4.77)

\[
\omega_{30} = -\frac{1}{4(e^{2\Omega(\alpha)} - 1)u + 1} \left[ \alpha e^{-\beta(\alpha)}(e^{2\Omega(\alpha)} - 1)\delta(s - s_0) \right].
\]

(4.78)

In equation (4.76) the first part in the brackets gives the spin imparted to test particles by the plane gravitational waves while the second part gives the spin of the wave itself.

### 4.3.2 Cylindrical gravitational waves

Now use the same construction for the metric (2.11) to get

\[
g_{00} = (e^{2(\gamma - \psi)} - 1)u(s - s_0) + 1,
\]

\[
g_{11} = -(e^{2(\gamma - \psi)} - 1)u(s - s_0) - 1,
\]

\[
g_{22} = -\rho^2 \left[ (e^{-2\psi} - 1)u(s - s_0) + 1 \right],
\]

69
\[ g_{33} = (e^{2\psi} - 1)u(s - s_0) + 1. \] (4.79)

The inverse of this metric is

\[ g^{\mu\nu} = \begin{pmatrix} g^{00} & 0 & 0 & 0 \\ 0 & g^{11} & 0 & 0 \\ 0 & 0 & g^{22} & 0 \\ 0 & 0 & 0 & g^{33} \end{pmatrix}, \] (4.80)

where

\[
\begin{align*}
g^{00} &= \frac{1}{(e^{2(\gamma - \psi)} - 1)u(s - s_0) + 1}, \\
g^{11} &= \frac{1}{(e^{2(\gamma - \psi)} - 1)u(s - s_0) - 1}, \\
g^{22} &= \frac{1}{-\rho^2[(e^{-2\psi} - 1)u(s - s_0) + 1]}, \\
g^{33} &= \frac{1}{-(e^{2\psi} - 1)u(s - s_0) - 1}.
\end{align*}
\]

In this case

\[ g_{00,1} = 2e^{2(\gamma - \psi)}(\gamma' - \psi')u(s - s_0) + (e^{2(\gamma - \psi)} - 1)\frac{\partial s}{\partial \rho} \delta(s - s_0). \] (4.81)

Dividing equation (2.1) by \( ds^2 \) and keeping \( t, \varphi, z \) fixed, we get

\[ 1 = e^{2(\gamma - \psi)} \left( \frac{\partial \rho}{\partial s} \right)^2, \] (4.82)

which implies that

\[ \frac{\partial s}{\partial \rho} = e^{(\gamma - \psi)}. \] (4.83)

Now putting \( \frac{\partial s}{\partial \rho} \) in equation (4.81) we get

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\[g_{00,1} = +2e^{2(\gamma - \psi)} \left( \gamma' - \psi' \right) u (s - s_0) + e^{\gamma - \psi} \left( e^{2(\gamma - \psi)} - 1 \right) \delta (s - s_0), \quad (4.84)\]

\[g_{00,2} = - \left[ \left( e^{2(\gamma - \psi)} - 1 \right) u + 1 \right]_2 \]
\[= - (e^{2(\gamma - \psi)} - 1) \frac{\partial s}{\partial \phi} \delta (s - s_0), \quad (4.85)\]

\[g_{00,3} = - (e^{2(\gamma - \psi)} - 1) \frac{\partial s}{\partial z} \delta (s - s_0). \quad (4.86)\]

Dividing equation \((2.1)\) by \(ds^2\) and keeping \(t, \rho, z\) fixed, we get

\[1 = e^{2\psi} \rho^2 \left( \frac{\partial y}{\partial s} \right)^2, \quad (4.87)\]

which implies that

\[\frac{\partial s}{\partial y} = \rho e^{-\psi}. \quad (4.88)\]

Now dividing equation \((2.1)\) by \(ds^2\) and keeping \(t, \rho, y\) fixed, we get

\[1 = e^{2\psi} \left( \frac{\partial z}{\partial s} \right)^2 \quad (4.89)\]

\[\frac{\partial s}{\partial z} = e^\psi. \quad (4.90)\]

Putting equation \((4.88)\) in to equation \((4.85)\) we get,

\[g_{00,2} = - (e^{2(\gamma - \psi)} - 1) \rho e^{-\psi} \delta (s - s_0). \quad (4.91)\]
Putting equation (4.90) in to equation (4.86) we get,

\[ g_{00,3} = -\left(e^{2(\gamma - \psi)} - 1\right)e^{\psi}\delta(s - s_0). \]  
(4.92)

Using equations (4.84), (4.91) and (4.92) in equation (4.63) we get:

\[
\begin{align*}
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} &= -\frac{1}{2(e^{2(\gamma - \psi)} - 1)u + 1} \left[2e^{2(\gamma - \psi)}(\gamma' - \psi')u + e^{\gamma - \psi}(e^{2(\gamma - \psi)} - 1)\delta(s - s_0)\right], \\
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} &= -\frac{1}{2(e^{2(\gamma - \psi)} - 1)u + 1} \left[(e^{2(\gamma - \psi)} - 1)\rho e^{-\psi}\delta(s - s_0)\right], \\
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} &= -\frac{1}{2(e^{2(\gamma - \psi)} - 1)u + 1} \left[(e^{2(\gamma - \psi)} - 1)e^{\psi}\delta(s - s_0)\right].
\end{align*}
\]  
(4.93, 4.94, 4.95)

Now using equations (4.93), (4.94) and (4.95) in equation (4.59) we get the components of the spin vector

\[
\begin{align*}
\omega_{10} &= \frac{1}{4(e^{2(\gamma - \psi)} - 1)u + 1} \left[2e^{2(\gamma - \psi)}(\gamma' - \psi')u + e^{\gamma - \psi}(e^{2(\gamma - \psi)} - 1)\delta(s - s_0)\right], \\
\omega_{20} &= \frac{1}{4(e^{2(\gamma - \psi)} - 1)u + 1} \left[(e^{2(\gamma - \psi)} - 1)\rho e^{-\psi}\delta(s - s_0)\right], \\
\omega_{30} &= \frac{1}{4(e^{2(\gamma - \psi)} - 1)u + 1} \left[(e^{2(\gamma - \psi)} - 1)e^{\psi}\delta(s - s_0)\right].
\end{align*}
\]  
(4.96, 4.97, 4.98)
4.3.3 The Friedmann model

Consider the Friedmann model. The metric for this model is given by equation (4.11) and the metric tensor is

\[
g_{\mu\nu} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -a^2(t) & 0 & 0 \\
0 & 0 & -a^2(t)f_k^2(\chi) & 0 \\
0 & 0 & 0 & -a^2(t)f_k^2(\chi) \sin^2 \theta \\
\end{pmatrix}.
\]

(4.99)

According to equation (4.59) the only thing we need is \(g_{00,1}, g_{00,2}, \) and \(g_{00,3} \).

As there are no wave fronts. Therefore we have \(g_{00} = 1 \). Here obviously all its derivatives will vanish. Hence

\[
\begin{pmatrix}
0 \\
0 1 \\
\end{pmatrix} = \begin{pmatrix}
0 \\
0 2 \\
\end{pmatrix} = \begin{pmatrix}
0 \\
0 3 \\
\end{pmatrix} = 0.
\]

(4.100)

Thus all components of spin vector are zero. i.e.

\[
\omega_{10} = \omega_{20} = \omega_{30} = 0.
\]

(4.101)

As we have stated earlier, there is no spin in an isotropic and homogeneous universe model, this analysis gives zero spin for this model as required.
4.3.4 The Kasner model

Consider the Kasner model. The metric for the Kasner model is given by equation (4.22) and the metric tensor is

$$g_{\mu \nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -t^{2p_1} & 0 & 0 \\ 0 & 0 & -t^{2p_2} & 0 \\ 0 & 0 & 0 & -t^{2p_3} \end{pmatrix}. \quad (4.102)$$

According to equation (4.59) the only thing we need is $g_{00,1}$, $g_{00,2}$, and $g_{00,3}$. As there is no “wave-front” so the original metric stands. From (4.102) we have $g_{00} = 1$. Here obviously all it’s derivatives will vanish. Hence

$$\begin{pmatrix} 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 & 3 \end{pmatrix} = 0. \quad (4.103)$$

Thus all components of spin vector are zero. i.e.

$$\omega_{10} = \omega_{20} = \omega_{30} = 0. \quad (4.104)$$

4.3.5 The De Sitter universe (usual coordinates)

The metric for this model is given by equation (4.32) and the metric tensor is

$$g_{\mu \nu} = \begin{pmatrix} (1 - r^2/D^2) & 0 & 0 & 0 \\ 0 & -(1 - r^2/D^2)^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2 \theta \end{pmatrix}. \quad (4.105)$$
As \( g_{00} \neq 1 \) so \( \omega_{i0} \) calculated in section 4.2 cannot work. Let \( u^a = f \delta^a_0 \). Here by definition

\[
u^a v^b g_{ab} = -1 = (u^0)^2 g_{00} \text{ or } u^0 = \frac{1}{\sqrt{g_{00}}} \quad (4.106)
\]

consequently we have

\[
u^a = \frac{1}{\sqrt{g_{00}}} \delta^a_0. \quad (4.107)
\]

Now we only need to calculate \( u_{|a,b|} \) and \( u_{|a} u_{|b|} \)

\[
u_{a,b} = \left( \sqrt{g_{00}} \delta^0_a \right)_b
\]

\[= \left( \ln \sqrt{g_{00}} \right)_1 \delta^0_a \delta^1_b. \quad (4.108)
\]

Hence we have

\[u_{|a,b|} = \left( \ln \sqrt{g_{00}} \right)_1 \delta^0_a \delta^1_b \quad (4.109)
\]

Similarly we get

\[
u_{|a} u_{|b|} = -\frac{1}{2} \left( \delta^0_b \delta^1_a - \delta^0_a \delta^1_b \right) \sqrt{g_{00}} \left\{ \begin{array}{l} 0 \\ 0 \\ 1 \end{array} \right\} \quad (4.110)
\]

From equation (4.109) and (4.110) the sum \( u_{|a,b|} + u_{|a} u_{|b|} \) for this metric vanishes and thus equation (4.53) implies that \( \omega_{ab} \) vanishes.
4.3.6 The Lemaitre form of the De Sitter universe

The metric for this model is given by (4.42). The metric tensor is

\[ g_{\mu\nu} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -a_0^2 e^{2\Lambda/3}t & 0 & 0 \\
0 & 0 & -a_0^2 e^{2\Lambda/3}t \chi^2 & 0 \\
0 & 0 & 0 & -a_0^2 e^{2\Lambda/3}t \chi^2 \sin^2 \theta
\end{pmatrix}. \] (4.111)

By the definition (4.60) of the metric tensor \( g_{00} = 1 \). Here obviously all its derivatives will vanish. Hence

\[ \begin{pmatrix}
0 \\
0 1
\end{pmatrix} = \begin{pmatrix}
0 \\
0 2
\end{pmatrix} = \begin{pmatrix}
0 \\
0 3
\end{pmatrix} = 0. \] (4.112)

Thus all components of spin vector are zero. i.e.

\[ \omega_{10} = \omega_{20} = \omega_{30} = 0. \] (4.113)

Thus we have got the same result for both the forms of the De Sitter universe which proves that the formalism is consistent.

4.3.7 The Gödel universe model

Consider the Gödel (spinning) universe model

\[ ds^2 = dt^2 - dx^2 - dy^2 + 2m(x) dt dz - l(x) dz^2, \] (4.114)
where \( m(x) = Ae^{ax} \) and \( l(x) = A^2 \left( \frac{a^2}{b^2} - 1 \right) e^{2ax} \). The metric tensor is
\[
\begin{bmatrix}
1 & 0 & 0 & m(x) \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
m(x) & 0 & 0 & -l(x)
\end{bmatrix}
\]  
(4.115)
Its inverse is
\[
\begin{bmatrix}
l(x) & 0 & 0 & m(x) \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
m(x) & 0 & 0 & -l(x)
\end{bmatrix}
\]  
(4.116)
According to equation (4.59) the only thing we need are the following three Christoffel symbols:
\[
\left\{ \begin{array}{c}
0 \\
0 1 \\
0 2
\end{array} \right\} \quad \left\{ \begin{array}{c}
0 \\
0 2 \\
0 3
\end{array} \right\} \quad \left\{ \begin{array}{c}
0 \\
0 1 \\
0 3
\end{array} \right\}
\]
\[
\left\{ \begin{array}{c}
0 \\
0 1 \\
0 2
\end{array} \right\} = \frac{1}{2} g^{03} g_{03,1}
\]
\[
= \frac{1}{2} \frac{m(x)}{l(x) + m^2(x)} (Ae^{ax})_{,1}
\]
\[
= \frac{1}{2} \frac{am^2(x)}{l(x) + m^2(x)},
\]
(4.117)
\[
\left\{ \begin{array}{c}
0 \\
0 2 \\
0 3
\end{array} \right\} = \left\{ \begin{array}{c}
0 \\
0 3
\end{array} \right\} = 0.
\]
(4.118)
Thus all components of the spin vector are
\[
\omega_{10} = \frac{1}{4} \frac{am^2(x)}{l(x) + m^2(x)}
\]
\[
\omega_{20} = \omega_{30} = 0.
\]
(4.119)
Chapter 5

SUMMARY AND

CONCLUSION

Linearized General Relativity predicts gravitational waves. These waves are analogous to electromagnetic waves. There may be different types of gravitational waves i.e. plane and cylindrical gravitational waves etc. This background was reviewed in chapter 1. Work on plane and cylindrical gravitational waves [3] has been helpful in understanding them further. The question of the reality of gravitational waves was discussed in chapter 2. There we used Weber-Wheeler [3] method for a particle in the path of a plane gravitational wave and obtained the standard constant momentum imparted to the particle. Some astrophysical sources of gravitational waves were also discussed. In chapter 3 the work of Qadir and Sharif [8] in which a general formula is developed for the momentum imparted to test particles by gravitational waves in arbitrary spacetime, was reviewed. In this paper the problem of the identification of
their zero component of the momentum four vector was mentioned. Sharif [11] gave a suggestion that it could be interpreted as the spin imparted to a test rod in an arbitrary spacetime. This suggestion was reviewed in chapter 4. Further analysis by Sharif [14] was shown here to provide a counterexample for this suggestion i.e. it gives a non zero spin for an isotropic and homogeneous universe model. Further, when the example of the De Sitter universe was considered, the interpretation gave different results for the static and Lemaitre forms. This proved that the interpretation is not even internally consistent. As such $P_0$ cannot be interpreted as the spin imparted to test rods. A geodesic analysis [4] was used in section 4.2 to evaluate the spin imparted to test particles in various cases. In the cases of plane and cylindrical gravitational waves we got very reasonable results. As required, the Friedmann, the Kasner and De Sitter models do not impart spin to test particles according to this analysis. Further we got the same results for both forms of the De Sitter universe, which confirms the validity of our analysis. Finally, the Gödel universe model gave a non-zero (non-constant) angular momentum, precisely as it should.

If $P_0$ does not provide an expression for the spin imparted to test rods in an arbitrary spacetime then what is its correct interpretation? Consider $E = \sqrt{P^i P^j g_{ij} + m^2}$ and $P_0 = \int F_0 dt$. Define the difference between the two as $\Delta E = P^0 - E$ for some of the spacetimes considered earlier. By definition the usual energy $E$ associated with a momentum $P^i$ is given by

$$P^\mu P^\nu g_{\mu\nu} = m^2 = (E^2) g_{00} + P^i P^j g_{ij}. \quad (5.1)$$
Hence

\[ E = \frac{1}{\sqrt{g_{00}}} \left( m^2 - P^i P^j g_{ij} \right)^{1/2}. \]  

(5.2)

Now, we want to compare this with

\[ P^0 = g^{00} \int F_0 dt. \]  

(5.3)

The energy difference, \( \Delta E \), is then

\[ \Delta E = g^{00} \int F_0 dt - \frac{1}{\sqrt{g_{00}}} \left( m^2 - P^i P^j g_{ij} \right)^{1/2}. \]  

(5.4)

When \( F_0 \) is zero we get no further understanding from \( \Delta E \). As such we will not compute these cases. Again for Gödel universe we have not calculated \( F_0 \) as the metric is not in the block diagonalized form (\( g_{0i} = 0 \)) and a gauge transformation would need to be made before it could be applied. Therefore we only work out \( \Delta E \) for the cases (a) cylindrical gravitational waves; (b) the Friedmann models and (c) the Lemaitre form of the De Sitter universe. Finally we will discuss the consequences of these results.

(a) Cylindrical gravitational waves

Consider the metric given by equation (2.1) and use the non-zero components of momentum four vector given by equations (3.36) and (3.37) to get

\[ E = m e^{-(\gamma-\psi)} \left( e^{-2(\gamma-\psi)} \left[ AJ'_0 \sin(\omega t) - \frac{1}{4} A^2 J_0 J'_0 + \rho \omega (J_0 J'_0)' \sin(2\omega t) \right]^2 + 1 \right)^{1/2}. \]  

(5.5)

Consequently we have

\[ \Delta E = -me^{-(\gamma-\psi)} [(1-e^{-2(\gamma-\psi)}(AJ'_0 \sin(\omega t)-\frac{1}{4} A^2 J_0 J'_0 + \rho \omega (J_0 J'_0)' \sin(2\omega t))^2)^{1/2}}\]
\[ + e^{-(\gamma-\psi)}(\ln |AJ_0 \sin(\omega t)| + (1 + AJ_0/\omega \rho J'_0) \ln |1 - 2\omega \rho AJ'_0 \cos(\omega t)|)] \]

(5.6)

(b) The Friedmann models

Since \( P^1, P^2 \) and \( P^3 \) are zero for the Friedmann models, from equations (4.11), (4.17), (4.18) and (4.20) we get the energy difference for the three models:

\[ \Delta E_{k=1} = m \ln \sqrt{1 - \cos \eta} - \frac{3}{8} \cos \eta + \frac{1}{16} \cos^2 \eta + \ln \sqrt{2 + \frac{5}{16}}; \]  

(5.7)

\[ \Delta E_{k=0} = m \ln \left( \frac{\eta}{2} \right); \]  

(5.8)

\[ \Delta E_{k=-1} = m \ln \frac{\cosh \eta - 1}{\sinh \eta}. \]  

(5.9)

These are plotted in Fig. (5.1), (5.2) and (5.3) and discussed shortly.

(c) The Lemaitre form of the De Sitter universe

Again \( P^1, P^2 \) and \( P^3 \) are zero for the metric given by equation (4.42). Thus using equation (4.51) we get

\[ \Delta E = m \sqrt{\Lambda/3t}. \]

(5.10)

as shown in Fig (5.4).

The three expressions for the Friedmann models have the same asymptotic behavior for sufficiently small values of \( \eta \), namely \( \Delta E \sim m \ln \eta \). For \( k = \pm 1 \) there is a correction term \( \sim \pm \frac{\eta^2}{12} \). We have inserted a constant term for
$k = +1$ so that the expressions for the three should match up to the zero
order terms. At the phase of maximum expansion of the closed model we get
$\Delta E = m(\ln 2 + \frac{3}{4})$. We could equally well, have set $\Delta E = 0$ at the phase of
maximum expansion and had a difference for it from the other two cases for
small $\eta$ (in the constant term). Note that for all the three models $\Delta E$ diverges
as $\eta \to 0$ and it also diverges as $\eta \to 2\pi$ for the closed model. For $k = +1$
we have chosen to display the constant term so that $\Delta E = 0$ at $\eta = \pi$. The
proposal does not seem inconsistent, but still needs further discussion.
Figure 5.1: Fig 5.1: $\Delta E$ for the closed Friedmann model. This becomes infinite at the bang and the crunch and becomes zero at the phase of maximum expansion.
Figure 5.2: Fig 5.2: $\Delta E$ for the flat Friedmann model, again $m$ is taken to be unity. $\Delta E$ diverges at $\eta = 0$.

Figure 5.3: Fig 5.3: $\Delta E$ for the open Friedmann model which diverges at $\eta = 0$.  

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Figure 5.4: Fig 5.4: $\Delta E$ for the Lemaitre form of the De Sitter universe. Here $\Lambda$ is taken to be unity.

**APPENDIX 1**

Consider the metric [1] (with signature $(+, -, -, -)$)

$$ds^2 = L^2(u)(dx^2 - dy^2) - dudv$$  \hspace{1cm} (5.11)

where $u = t - z$ and $v = t + z$ which is always flat.

The metric tensor is.

$$g_{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -L^2(u) & 0 & 0 \\ 0 & 0 & -L^2(u) & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$  \hspace{1cm} (5.12)

Its inverse is.
\[ g^{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -L^{-2}(u) & 0 & 0 \\ 0 & 0 & -L^{-2}(u) & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \]  \hspace{1cm} (5.13)

The non vanishing Christoffel symbols are:

\[
\begin{array}{c}
\{ \{1 \} = \{ 2 \} = \{ 1 \} = L'(u) / L(u). \\
\{ \{1 0 \} = \{ 2 0 \} = \{ 0 1 \} = L'(u) / L(u). \\
\{ \{0 \} = \{ 3 \} = L(u) L'(u). \\
\{ \{1 1 \} = \{ 1 1 \} = L(u) L'(u). \\
\{ \{1 2 \} = \{ 2 \} = L'(u) / L(u). \\
\{ \{2 2 \} = \{ 3 \} = L(u) L'(u). \\
\{ \{3 2 \} = \{ 1 \} = \{ 2 \} = L'(u) / L(u). \\
\end{array}
\]  \hspace{1cm} (5.14)

\[
\begin{array}{c}
\{ \{1 \} = \{ 2 \} = \{ 1 \} = L'(u) / L(u). \\
\{ \{1 0 \} = \{ 2 0 \} = \{ 0 1 \} = L'(u) / L(u). \\
\{ \{0 \} = \{ 3 \} = L(u) L'(u). \\
\{ \{1 1 \} = \{ 1 1 \} = L(u) L'(u). \\
\{ \{1 2 \} = \{ 2 \} = L'(u) / L(u). \\
\{ \{2 2 \} = \{ 3 \} = L(u) L'(u). \\
\{ \{3 2 \} = \{ 1 \} = \{ 2 \} = L'(u) / L(u). \\
\end{array}
\]  \hspace{1cm} (5.15)

The primes refers to differentiation with respect to \( u \). The nonvanishing components of the Ricci tensor are \( R_{00} \), \( R_{03} \), \( R_{30} \) and \( R_{33} \). Using Christoffel symbols listed above we obtain the following Ricci tensor components.
\[ R_{00} = R_{33} = \frac{-2L''(u)}{L(u)}. \]  
\hspace{1cm} (5.19)

\[ R_{30} = R_{03} = \frac{2L''(u)}{L(u)}. \]  
\hspace{1cm} (5.20)

From equations (A1.9) and (A1.10) the required result follows.
APPENDIX 2

We shall prove here that

$$F^0 = m[(\ln A)_0 - \Gamma^0_{00} + \Gamma^i_{0j} \Gamma^j_{0i}/A] f^2,$$  \hspace{1cm} (5.21)

and

$$F^i = \Gamma^i_{00} f^2$$ \hspace{1cm} (5.22)

are the solutions of the following equations.

$$l^i(F^0_j + \Gamma^0_{ij} F^j) = 0,$$ \hspace{1cm} (5.23)

$$l^i(F^i_j + \Gamma^i_{0j} F^0) = F^{*i}.$$ \hspace{1cm} (5.24)

where

$$F^{*i} = m(\Gamma^i_{00,j} - \Gamma^i_{0j,0} + \Gamma^i_{0j} \Gamma^0_{00} - \Gamma^i_{0k} \Gamma^k_{00}) f^2 l^j.$$  \hspace{1cm} (5.25)

For the verification of equation (A2.3) we note that

$$A = \Gamma^i_{0i}, \quad f^2 = \frac{1}{g_{00}}.$$ \hspace{1cm} (5.26)

Thus

$$(\ln A)_0 = g^{ij}_{0j}/g^{ij} + g_{ij,00}/g_{ij,0}.$$ \hspace{1cm} (5.27)

Here we will make use of the Riemann normal coordinates (RNCs) for spatial directions. We have

$$g_{0i} = 0$$ \hspace{1cm} (5.28)

$$g_{\mu\nu,0} \neq 0, \quad g_{\mu\nu,i} = 0 = g_{\mu\nu,0i}.$$ \hspace{1cm} (5.29)
Using this approximation we can write

\[(\ln A)_{0i} = 0 = \Gamma_{00,i}^0 - \Gamma_{0i}^i\Gamma_{0i}^i. \quad (5.30)\]

Now

\[
F_{0}^{0,i} = m[(\ln A)_{0i}^0 - \Gamma_{00,i}^0 + (\Gamma_{0j}^i \Gamma_{0i}^j / A)_{,i}]f^2
+ m[(\ln A)_0^0 - \Gamma_{00}^0 + \Gamma_{0i}^i \Gamma_{0i}^j / A](f^2)_{,i}. \quad (5.31)
\]

Using RNCs the first, second, third and the last term on the right hand side vanishes. Therefore

\[F_{0}^{0,i} = 0 \quad (5.32)\]

Thus equation (A2.3) is satisfied. Equation (A2.4) can directly be obtained just by replacing the values of \(F^0\) and \(F^i\) from equations (A2.1) and (A2.2).

Hence \(F^0\) and \(F^i\) given by equations (A2.1) and (A2.2) are the solutions of equations (A2.3) and (A2.4).
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