Derivation of Correction Terms to the Eikonal Expansions in the Abrasion-Ablation Model for the Light Ions in High Energy Collisions

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Abstract. The analytical abrasion-ablation model is widely used for the quantitative predictions of the neutron and light ion spectra from high energy collisions. The abrasion stage of the current model is based on the Glauber multiple scattering theory and applies the small angle approximation which assumes the longitudinal momentum transfer for the scattering amplitude to be small. However, for reactions with three or more particles in final state the validity of Glauber model is not clear. In this work we have extended the current abrasion model to include higher order terms to the first order Eikonal scattering using a perturbation approach. The perturbation approach uses an expansion of the Fourier Bessel series in the scattering amplitude in terms of Legendre polynomials, thus creating an infinite series. This expansion eliminates the small angle approximation used in the Glauber-type Eikonal approximation and hence extends it to larger angles and low energy regions. A comparison of this model with two and four correction terms respectively is made with the currently existing model for the derivations of angular distributions and double differential cross-sections for various light ion projectile target data sets.

1. Introduction

Analytical models for quantitative prediction of neutrons and light ion spectra from intermediate and high energy interactions present important insights in the assessment of radiation damage to astronauts and the spacecraft during long duration deep space missions. These models also provide a basis for theoretical modelling of production and transport of the light and heavy ion fragments in these reactions[1],[2]. The fundamental physics of production and transport of the secondary particles from nucleon-nucleus and nucleus-nucleus collisions at these energies have been widely described using the analytical abrasion-ablation model based on the Eikonal approximations developed from the Glauber multiple scattering theory [3],[4]. The abrasion-ablation model for the prediction of secondary particle production from high energy collisions of two relativistic heavy particles is basically a two stage process [5],[6]. The first stage, known as abrasion or the knockout process, involves a relativistic particle, projectile P, moving with some initial momentum p, colliding with the relatively stationary target T, and thus their overlapping volumes get sheared off. After the collision the remaining pieces of the target and the projectile are assumed to continue in their original trajectory with their pre-
collision velocities. The projectile pre-fragment, which is now in excited state, emits gamma rays and/or disintegrates into fragments and nucleons. This second step is known as the ablation process.

The abrasion process, formulated using the Glauber multiple scattering theory, applies the optical limit potential approximation to the nucleus-nucleus multiple scattering series and its further expansion in terms of the Eikonal differential approximation \cite{4}, \cite{7}, \cite{6}. The ablation process is based on the classical evaporation model by Weisskopf and Ewing and treats the statistical probability of particle emission from the excited nuclei after the abrasion stage as a competing process \cite{5}, \cite{8}. The Eikonal expansions in the Glauber model are treated in terms of small angle, high energy approximations and assume that the total Eikonal phase for the scattering is equal to the sum of individual scattering of the projectile off each target constituent \cite{5}, \cite{9}. It also limits the scattering strictly in the plane of the incident momentum. Although the Glauber model has been widely used, the validity of small angle approximations for systems with three or more particle in the final state and at lower energies is not clear \cite{10}. Also light ion production (z ≤ 2) is inherently a three dimensional problem and cannot be properly described by small angle approximation. There have been various different approaches to the correction of the Eikonal approximations but most derivations require a small angle approximation of some sort and there is no compelling argument showing the validity of the approximations at larger scattering angles and lower energies \cite{11}, \cite{12}, \cite{13}. In this work, we have used the Eikonal expansions developed by Wallace to include higher order correction terms to the phase shift operator in the calculation of the scattering amplitude \cite{14}. This is done by converting the partial wave series into a Fourier-Bessel expansion, based on an expansion of Legendre functions, thus creating an infinite series with the leading term in the phase shift operator being same as in the Glauber model, and additional terms being the correction terms \cite{8},\cite{14},\cite{12}. The expansion is exact in the sense that there is no small angle approximation made. Based on the partial wave representation of scattering amplitude, present work develops four higher order corrections terms to the phase shift operator, along with the Gaussian approximations to the nuclear densities used in the derivation of the optical potential for the light ion interactions \cite{1}. A comparison between contributions of two and four higher order correction terms and comparisons to the experimental data \cite{15} for various light ion data sets are presented.

2. Abrasion-Ablation Model

2.1. Abrasion Formalism

The cross section for abrading ‘n’ projectile nucleons during a nuclear interaction is given by \cite{16}

\[
\sigma_n = \binom{A_p}{n} 2\pi \int \left[ 1 - P(b) \right]^{n} P(b)^{A_p - n} b db
\]

(1)

Where \( \binom{A_p}{n} \) is the binomial coefficient for the number of possible combinations of nucleons taken from an ensemble of \( A_p \) identical nucleons and \( A_p = A_p - n \). The total absorption cross section can thus be expressed as

\[
\sigma_{abs} = 2\pi \int \left[ 1 - P(b)^{A_p} \right] b db
\]

(2)
and is obtained by summing over all possible values of \( n \)

\[ \sigma_{abs} = \sum_{n=1}^{A_p} \sigma_n \]  

(3)

In equations (1) and (2), \( P(\vec{b}) \) is the nucleon non-removal probability as a function of impact parameter \( b \), given by

\[ P(\vec{b})^{A_p} = \exp \left[ -2 \text{Im} \chi(\vec{b}) \right] \]  

(4)

and thus

\[ P(\vec{b}) = \exp \left[ \frac{-2 \text{Im} \chi(\vec{b})}{A_p} \right] \]  

(5)

where the Eikonal phase function \( \chi(\vec{b}) \) in the Glauber model [4] is given by

\[ \chi(\vec{b}) = -\frac{1}{\hbar v} \int_{-\infty}^{\infty} V(\vec{r}) \, dz \]  

(6)

with \( V(\vec{r}) \) being the optical potential and \( v \) being the relative velocity of the incoming particle momentum in the z-direction. Hence \( 1 - P(\vec{b}) \) is the nucleon removal probability.

From equations (4) and (5), the abrasion cross section in equation (1) can be expressed as

\[ \sigma_n = \left( \frac{A_p}{n} \right) 2\pi \int \left[ 1 - \exp \left[ \frac{-2 \text{Im} \chi(\vec{b})}{A_p} \right] \right]^{A_p} \left( \exp \left[ \frac{-2 \text{Im} \chi(\vec{b})}{A_p} \right] \right)^{A_p} bdb \]  

(7)

Equation (7) treats all nucleons as identical objects. To differentiate between protons and neutrons, the above equation can be replaced by [17]

\[ \sigma_{nc} = \left( \frac{N_p}{n} \right) \left( \frac{Z_p}{z} \right) 2\pi \int \left[ 1 - \exp \left[ \frac{-2 \text{Im} \chi(\vec{b})}{A_p} \right] \right]^{N_p+n} \left( \exp \left[ \frac{-2 \text{Im} \chi(\vec{b})}{A_p} \right] \right)^{N_p+n} bdb \]  

(8)

where \( \sigma_{nc} \) is the cross section for abrading \( n \) out of \( N_p \) neutrons and \( z \) out of \( Z_p \) protons. For simplicity, it is implicitly assumed in the above expression that the neutron and proton distributions in the projectile are completely uncorrelated.

To obtain the momentum distributions of the projectile fragments, the nucleon momentum distributions in the rest frame of the projectile are given from the work by Haneishi and Fujita [18] as
\[ n(p) = n_0 \sum_{i=1}^{3} C_i \exp \left( \frac{-p_n^2}{2p_i^2} \right) \] (9)

where \( p_n \) is the momentum of the nucleon in the projectile rest frame, \( p_i \) is the momentum width parameter given in terms of the Fermi momentum \( k_F \) and \( n_0 \) is the normalization constant. The values for \( C_i \) and \( p_i \) are taken from Cucinotta et al [3] and are listed below.

| \( i \) | \( C_i \) | \( p_i \text{ MeV} / c \) |
|---|---|---|
| 1 | 1 | \( k_F \sqrt{(2/5)} \) |
| 2 | 0.03 | \( k_F \sqrt{(6/5)} \) |
| 3 | 0.0008 | 500 |

The values of the Fermi momentum are taken from experimental values listed in Cucinotta et al. [3] and its further extensions to other nuclei from [1].

The nucleon momentum distribution for abrading 1 through \( n \) projectile nucleons can now be given by

\[
\left( \frac{d^3 \sigma}{dp_n^3} \right)_{abr} = \left( \sum_{i=1}^{n} \sigma_{jn} \right) \left( n_0 \sum_{i=1}^{3} C_i \exp \left( \frac{-p_n^2}{2p_i^2} \right) \right) \] (10)

2.2. Ablation Formalism

In the ablation stage, the prefragment nuclei which are in the excited state, give up their excess energy by evaporating nucleons, light ion clusters and gamma rays to decay to the ground state [17], [19]. The ablation process is a complicated process requiring knowledge of prefragment thermalization, temperature dependence of fragments and also physics behind the evaporation model. The nucleon emission spectra from the ablation process is given by the Weisskopf-Ewing statistical decay model [8]. The probability function for emission of particle in the statistical decay model used here is from the works by Cucinotta et al. [3] and Kikuchi and Kawai [19] as

\[
P_j(j, E_i) = \frac{2 \mu_j g_j E_i \sigma_{CN} w_0 (E_i - E_j)}{\int_{0}^{E_j} P_j(j, E)dE} \] (11)

The secondary neutron momentum distribution for the ablation process can now be described in terms of the probability function as
\[
\left( \frac{d^3\sigma}{dp^3} \right)_{\text{abr}} = \sum_j \sigma_{abr} \left( A_j, Z_j, E_j^* \right) \mathbf{P}_j (j, k) \tag{12}
\]

And the total momentum distribution can be given as
\[
\left( \frac{d^3\sigma}{dp^3} \right)_{\text{total}} = \left( \frac{d^3\sigma}{dp^3} \right)_{\text{abr}} + \left( \frac{d^3\sigma}{dp^3} \right)_{\text{abl}} \tag{13}
\]

### 3. Derivation of Correction Terms

We will now look into derivation of higher order correction terms to the phase shift operator in order to eliminate the small angle approximation in the impact parameter representation of the scattering amplitude based on the works by Wallace [14], [20] and its further study by Waxman et al. [13].

The scattering amplitude can be expressed in terms of the sum of partial waves as
\[
f(\theta) = \frac{1}{2ik} \sum_{l=0}^{\infty} (2l + 1) P_l (\cos \theta) \left[ e^{i\phi} - 1 \right] \tag{14}
\]

Using the Glauber model [4], the scattering amplitude can be expressed in terms of an impact parameter representation using the small angle, forward scattering approximation as
\[
f(\theta) = ik \int_{0}^{\infty} J_0(qb) \left( 1 - \exp(i\chi(b)) \right) bdb \tag{15}
\]

with
\[
\chi(\hat{b}) = -\frac{m}{2k} \int_{-\infty}^{\infty} V(\vec{r})dz \tag{16}
\]

Wallace [14], [21] derives a relation between partial wave representation and impact parameter representation of the scattering amplitude using an expansion of Legendre Polynomials in terms of zero order Bessel functions as
\[
P_l(z) = \sum_{m=0}^{\infty} \frac{1}{(2m)!} \left( \frac{\partial}{\partial l} \right)^m b_m \left( \frac{1}{4} \frac{\partial}{\partial l} (2l + 1) \right) J_0 \left( (2l + 1) \left( \frac{1 - z}{2} \right)^{\frac{1}{2}} \right) \tag{17}
\]

where \( b_m(x) \) are generalized Bernoulli polynomials with \( b_0(x) = 1 \) and \( b_1(x) = -\frac{x}{6} \).

Using the above expansion, Wallace derives a relationship between the Eikonal phase factor and the phase shifts and thus derives an infinite series for the phase shift operator in terms of an impact parameter representation of the scattering amplitude. The first term in Wallace’s series is exactly the same as the Glauber model, while higher order terms are the correction terms to the phase function. A more detailed derivation is presented in [1].
Thus the total Eikonal phase as the sum of the zero order term and higher correction terms from Wallace’s model can be expressed as

\[
\chi(b) = \sum_n \frac{(-\mu)^{n+1}}{k(n+1)!} \left( \frac{b}{k} \frac{\partial}{\partial k} - \frac{\partial}{\partial b} \right) \left( \frac{1}{k} \right)^n \int_{-\infty}^{\infty} V^{n+1}(\vec{r}) \, dz
\]

and

\[
\chi_n(b) = \frac{(-\mu)^{n+1}}{k(n+1)!} \left( \frac{b}{k} \frac{\partial}{\partial k} - \frac{\partial}{\partial b} \right) \left( \frac{1}{k} \right)^n \int_{-\infty}^{\infty} V^{n+1}((b^2 + z^2)^{1/2}) \, dz
\]

with \( r = (b^2 + z^2)^{1/2} \).

4. Optical Potential Calculation

Calculation of the phase function term mentioned in equations (19) and (20) requires knowledge of the optical potential. The optical potential term has been calculated in [2] for the nucleus-nucleus scattering and its further approximation in the context of high energy, using the one body Schrödinger equation. The optical potential from [2] can be expressed as

\[
V(\vec{r}) = \frac{2mA_p A_p}{N} \int d^3z \rho_T(\vec{z}) \int d^3y \rho_p(\vec{x} + \vec{y} + \vec{z}) \tilde{t}(e, \vec{y})
\]

with \( \rho_T(\vec{z}) \) and \( \rho_p(\vec{x} + \vec{y} + \vec{z}) \) being the target and projectile single particle nuclear densities respectively.

The two particle transition amplitude is expressed as [2]

\[
\tilde{t}(e, \vec{y}) = -\sqrt{\frac{2}{m}} \sigma(e) \left[ \alpha(e) + i \right] (2\pi B(e))^{1/2} \exp\left( -\frac{y^2}{2B(e)} \right)
\]

where
- \( e \) = energy in the two body center of mass frame,
- \( \sigma(e) \) = energy dependent total nucleon-nucleon cross section,
- \( \alpha(e) \) = energy dependent ratio of the real to the imaginary parts of the forward scattering amplitude,
- \( B(e) \) = slope parameter (related to the range of interaction)

The calculations for the terms in the two particle transition amplitude above are presented in detail in [1],[2]. The nuclear density distributions for the projectile and target nuclei are given by their ground state-single particle nuclear densities. These density distributions for the collision pair are calculated from their experimentally determined charge density distributions. For lighter nuclei (\( A \leq 20 \)), the nuclear charge distribution is assumed to have Harmonic Well (HW) form [22] which yields the single particle nuclear density for lighter nuclei as [2]
\[
\rho_A(\vec{r}) = \rho_0 \frac{a^3}{8s^3} \left[ 1 + \frac{3\gamma a^2}{2} - \frac{3\gamma a^2}{8s^3} + \frac{\gamma a^2 r^2}{16s^4} \right] \exp \left( -\frac{r^2}{4s^2} \right)
\]

(22)

with \( s^2 = \frac{a^2 - r^2}{4} \).

The values for charge parameters \( a \) and \( \gamma \) are taken from De Jager and De Vries [22]. \( r_p = 0.87 \text{ fm} \) being the proton root-mean-square charge radius and \( \rho_0 \) is the normalization constant.

The calculation of the optical potential requires integration of these single-particle nuclear densities. This leads to complicated integrations requiring both analytic and numerical solutions and yields many terms in the final solution [23]. This becomes even more challenging for the higher order terms. A simpler form, suitable for the present work, can be obtained by using a Gaussian approximation for the distributions. In this work, we developed simple Gaussian approximations to the nuclear densities. For lighter nuclei the HW distributions can be approximated using a Gaussian approximation as

\[
\rho_A(\vec{r}) = 0.66 \rho_0 \frac{a^3}{8s^3} \left[ 1.5 + \frac{3\gamma a^2}{2} - \frac{3\gamma a^2}{8s^3} + \frac{\gamma a^2 r^2}{16s^4} \right] \exp \left( -\frac{r^2}{4(1+0.52\gamma)s^2} \right)
\]

(23)

where, the normalization and charge parameters for equation (24) are same as the ones in equation (23). This allows for the eq. (24) to be written in simple form as

\[
\rho_A(\vec{r}) = C_i \exp(-D_i r^2)
\]

with \( i = P, T \) represent the projectile and target constituents respectively.

Using the single particle densities from equation (24) and applying them to calculate the optical potential in equation (21), we can now write the zero order term and the first four order correction terms to the phase function as

\[
\chi_0(b) = \frac{1}{2} \sqrt{\frac{\pi}{N}} M \exp\left(-Nb^2\right)
\]

(25)

\[
\chi_1(b) = \frac{1}{4k} \sqrt{\frac{\pi}{2N}} M^2 \left(4Nb^2 + 1\right) \exp\left(-2Nb^2\right)
\]

(26)

\[
\chi_2 = -\frac{1}{12k^2} \sqrt{\frac{\pi}{3N}} M^3 \left(-36b^4N^2\right) \exp\left(-3Nb^2\right)
\]

(27)

\[
\chi_3 = \frac{1}{48k^3} \sqrt{\frac{\pi}{4N}} M^4 \left(-24b^2N - 192b^4N^2 + 512b^6N^3 - 3\right) \exp\left(-4Nb^2\right)
\]

(28)

\[
\chi_4 = \frac{1}{240k^4} \sqrt{\frac{\pi}{5N}} M^5 \left(8000b^6N^3 - 10000b^8N^4\right) \exp\left(-5Nb^2\right)
\]

(29)

with \( \chi(b) = \chi_0(b) + \chi_1(b) + \chi_2(b) + \chi_3(b) + \chi_4(b) + \ldots \)

\[
M = \pi^3 A_P A_T C_P C_P \sigma(e) \left[ \alpha(e) + i \right] \left( 2\pi B(e) \right)^{\frac{3}{2}} \left( D_P + \frac{1}{2B(e)} \right)^{\frac{3}{2}} \left( D_T + D_P = \frac{D_P^2}{D_P + \frac{1}{2B(e)}} \right)^{\frac{3}{2}}
\]

(30)
\[ N = \left\{ \begin{array}{l} D_p - \frac{D_p^2}{(D_p + \frac{1}{2B(e)})} \\ D_p + D_p - \frac{D_p^2}{(D_p + \frac{1}{2B(e)})} \end{array} \right\}^2 \] (31)

It is evident that this point that the zero order term is exactly same as the Glauber model and higher order terms are the correction terms.

5. Results and Conclusion

The study of contributions of the higher order terms in the calculation of the total abrasion cross section for lighter ions \((A \leq 20)\) shows that the first two correction terms yield significant contributions towards the total abrasion cross sections, especially at lower energies. The correction terms became smaller with increasing energy. The contributions of third and fourth correction terms at higher energies were negligible and did not have any significant impact on the total abrasion cross sections. The amplitude of the correction terms was significantly higher at lower energies, especially for energies below 200 MeV/nucleon. These trends in the comparisons of the double differential cross sections for different projectile target data sets at different energies and scattering angles will be seen to agree with these calculated results of the total abrasion cross sections.

Comparisons of the present work with experimental data from Nakamura and Heilbronn [15] have been performed. Since the contributions from the third and fourth order terms were not significant, only the first two terms have been used in the present comparisons with the experimental data. The comparisons show that the double differential cross sections increase with the addition of correction terms. There was an improvement in agreement of the predictions to the experimental data at the smaller scattering angles. At angles larger than 30 degrees, the current work under predicts the cross sections. Also, some of the disagreement for cross sections below the incident beam energy are due to the lack of the isobar formation and decay channel in present work. Overall there was a significant increase by adding two higher order terms, while third and fourth order terms did not contribute to the cross sections.

| Energy (MeV/Nucleon) | \(\sum_{i=0}^{X_i} \sigma_i\) (mb) | \(\sum_{i=0}^{X_i} \sigma_i\) (mb) | \(\sum_{i=0}^{X_i} \sigma_i\) (mb) | \(\sum_{i=0}^{X_i} \sigma_i\) (mb) | \(\sum_{i=0}^{X_i} \sigma_i\) (mb) |
|----------------------|----------------|----------------|----------------|----------------|----------------|
| 100                  | 1476.63        | 2259.92        | 2577.92        | 2635.33        | 2633.68        |
| 200                  | 1259.47        | 1604.47        | 1654.65        | 1658.55        | 1658.55        |
| 300                  | 1251.13        | 1529.00        | 1559.52        | 1561.36        | 1561.36        |
| 500                  | 1329.27        | 1604.52        | 1630.31        | 1631.64        | 1631.64        |
| 1000                 | 1395.25        | 1623.86        | 1638.73        | 1639.27        | 1639.27        |
| 3000                 | 1343.94        | 1421.68        | 1423.27        | 1423.29        | 1423.29        |
| 5000                 | 1346.69        | 1390.02        | 1390.49        | 1390.50        | 1390.50        |
| 10000                | 1348.77        | 1364.36        | 1364.43        | 1364.43        | 1364.43        |
Figure 1: Comparison of neutron differential cross sections from the abrasion process for 400 MeV/nucleon $^{14}$N on $^{12}$C at 40° scattering angle for 0, 2 and 4 correction terms.

Figure 2: Neutron DD cross section for 290 MeV/Nucleon $^{12}$C beam colliding on $^{12}$C target at 5, 10, 20 and 60 degrees (top left, top right, bottom left, bottom right respectively)
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