Hopf Algebraic Structure for Tagged Graphs and Topological Recursion

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Abstract

Using the shuffle structure of the graphs, we introduce a new kind of the Hopf algebraic structure for tagged graphs with, or without loops. Like a quantum group structure, its product is non-commutative. With the help of the Hopf algebraic structure, after taking account symmetry of the tagged graphs, we reconstruct the topological recursion on spectral curves proposed by B. Eynard and N. Orantin, which includes the one-loop equations of various matrix integrals as special cases.

Keywords: topological recursion, Hopf algebra, matrix integral.

1 Introduction

Correlation functions are basic quantities to be defined in a lot of physical systems. For a matrix model, the zero-dimensional quantum system, the correlation functions are given by various matrix integrals. As observed independently by G. ’t Hooft, E. Brezin and C. Itzykson etc \cite{16,14}, the integrals contain abundant useful information, which is important either in mathematics, or physical research and applications. The quantities of matrix

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integrals admit a topological expansion, in certain cases, they enumerate discrete colored surfaces for a given topology, while the well known Kontsevich integrals count spin intersection numbers. In appropriate double scaling limits, some integrals involve in a Liouville theory. Hence the study of various matrix integrals is one of the most important research fields in mathematical physics.

Various sorts of working approaches have been set up in the past few decades to deal with the matrix integrals, such as orthogonal polynomials method [15], loop equations [1] and topological string method [8, 9]. Among them, the loop equation method is one of the most effective ways at present in some sense. The loop equations are known as well as the Schwinger-Dyson equations for certain cases. The original form of the loop equations are emerged in the study of the Yang-Mills gauge theory [20, 21]. Then similar equations are obtained for cases of matrix model. In fact, in the matrix model cases, the loop equations are the functional identities for matrix integrals. The loop equations for the hermitian matrix integrals are given in [23]. For a long period of time, the loop equation is a key step in the study of matrix integrals in the double scaling limit. In order to solve the problems which are away from the double scaling limit, various kinds algorithmic methods for the calculating the higher genus contributions were suggested in the following decades. In practice, if the genus zero equation was solved, then the higher-genus equations were iteratively solved genus by genus by using the results of previous steps. Even though these algorithms were ingenious, nevertheless, only several lower order loop equations could be solved. It is very difficult to compute the results concretely for higher-genus cases.

A new type of loop equations was proposed by B. Eynard for one-hermitian matrix integrals [11]. After then, it was generalized to 2-matrix model case in the refs. [13, 5]. In contract to the original loop equation, the new version considers correlation functions on an algebraic curve given by the one-loop equation. The new approach combines elements which is depending on the algebraic curve. It allows us to compute the integrals by parts on the algebraic curve recursively, therefore the solutions have plentiful mathematical structures. The method can be applied to other problems which are irrelevant to matrix integrals as well. This implies that the structures of the solutions do not depend on the matrix integrals. In the ref. [14] B. Eynard and N. Orantin constructed the so called topological recursion structure for spectral algebraic curves. The topological recursion is a recursive defined identities, which associates a double indexed family differential forms ω_{g,n}, with the two integers g and n being non-negative. The ω_{g,n} are called the invariants of the spectral curve. Those invariants enjoy lots of fascinating properties, one could see ref. [14] for more details. If an algebraic curve is identical to the one-loop equation of certain matrix integrals, then these new constructed correlation functions coincide with the those given by matrix integral.

The expression for perturbative expansion in a quantum field theory (QFT) has a similar structure as the topological recursion. The perturbative expansion in QFT can be represented by the Feynman diagrams. There are two kinds of Feynman diagrams: tree diagrams and loop ones. In the Feynman diagrams approach for a quantum field
theory, the propagators are represented by lines, and in addition, interactions are represented by vertices. It was found that there was a Hopf algebraic structure hidden in the relations of divergences in the expansions of amplitudes \cite{17,18}. The Hopf algebra was graded according to their sub-divergences. Moreover, regulation and normalization of the divergences were compatible with the Hopf algebraic structure as well \cite{6,7}. In terms of algebraic language, these procedures were the Birkhoff decomposition, either the Schwinger-Dyson equation or the renormalization group equation can be restated in this terminology. With the help of the Hochschild cohomology and the Schwinger-Dyson equation, the relationships of the contributions of a single graph to the full correlation functions were established. In addition with the Hopf algebra, the computations of amplitudes can be simplified to a certain extent, even non-perturbative results could be obtained in some cases.

Similar to the perturbative expansion in quantum field theory, the topological recursion on an algebraic curve has a diagrammatic representation. The diagrammatic representation for the topological recursion is the Feynman diagram alike, although it is not a Feynman diagram. The main disparity between them is that, the diagrammatic representation of topological recursion is arrowed.

It is straightforward to ask the question, whether there is a Hopf algebraic structure on the topological recursion. Indeed, there is a Hopf algebraic structure on the tree diagrammatic representation with the help of the Loday-Ronco Hopf algebra \cite{10}. However, that structure cannot be extended to graphs with loops.

In this manuscript we present a Hopf algebraic structure on the topological recursion. Distinct from the Loday-Ronco Hopf algebra, we redefine the product and the coproduct on planar binary trees with tagged leaves, and a new Hopf algebra is formed, and more remarkably, the new Hopf algebraic structure can be generalized to graphs with loops. It seems that the numbers of the tagged diagrams are not the same as the diagrammatic representations of the topological recursion. However, if we consider the symmetries of the tagged diagrams, these graphs are just the diagrammatic representations of topological recursion. With the help of the new Hopf algebraic structure on the tagged graphs with loops, and in addition with the weighted map defined by B. Eynard and N. Orantin as well \cite{14}, we can obtain a Hopf algebraic structure on the topological recursion from its diagrammatic representation.

The manuscript is arranged as follows. In section 2, we introduce basic notations and briefly review the main results of Eynard-Orantin’s topological recursion in ref. \cite{14}. We briefly present its diagrammatic representation. In the next section, we recall the Loday-Ronco Hopf algebra defined on planar binary trees. In section 4, we prove the equivalence between the diagrammatic representation of the topological recursion and the graphs given in \cite{10}. The main part of the manuscript is in the following section. We give a new Hopf algebraic structure on planar binary trees. It is straightforward to extend our structures to the graphs with loops. Using the weighted map, we really get a Hopf algebraic structure on the topological recursion. In section 6, conclusions and further
discussions are followed.

2 Topological recursion and diagrammatic representations

The primary version of loop equations derived by B. Eynard gives an effective recursive way to solve formal one-hermitian matrix integrals [11], the solutions depend on the one-loop equation rather than formal matrix integrals. Then B. Eynard and N. Orantin generalized the results for one-loop equation to cases for algebraic curves, on which they constructed a sequence of differential forms, or geometric invariants named as correlation functions, furthermore a sequence of complex numbers regarded as partition functions by the topological recursion, respectively. Here, we review the basic notations and main results of [14], and for the limit of space, we only focus on the correlation functions. The argument can be applied to the cases of partition functions.

Let us consider a compact Riemann surface $\Sigma$ with genus $g$, $x$ and $y$ are meromorphic functions on $\Sigma$. For a local variable $p$, $dx(p) = 0$ gives the branch points $\{a_i | i = 1, \ldots, n\}$. Assume that all the branch points $\{a_i | i = 1, \ldots, n\}$ are simple.

On the Riemann surface $\Sigma$, basic ingredients in the correlation functions can be defined. Consider any point $p \in \Sigma$ and points $q, \overline{q}$ in the vicinity of a fixed branch point, such that $x(q) = x(\overline{q})$. The are two types of basic ingredients for the topological recursion: one is “vertex”, and another is the Bergmann kernel [14]. The “vertex” is defined as:

$$\omega(q) = (y(q) - y(\overline{q}))dx(q).$$

While the Bergmann kernel $B(p, q)$ is a symmetric 2-form on $\Sigma \times \Sigma$. There is exactly one double pole at $p = q$ with no residue. For a canonical basis of cycles $(A_I, B^i)$, such that $\int_{A_I} B = 0$.

Given a set of points $\{p_1, p_2, \ldots, p_k\}$ of the curve, if $J = \{i_1, i_2, \ldots, i_j\}$ is any a subset of $K = \{1, 2, \ldots, k\}$, denote $P_J = \{p_{i_1}, p_{i_2}, \ldots, p_{i_j}\}$. Then the $(k + 1)$-point correlation function to order $g$ has the following combinational structure:

$$W_{k+1}^{(g)}(p, P_K) = \text{Res}_{q \rightarrow a} K(q, p) \left( \sum_{m=0}^{g} \sum_{J \subset K} W_{|J|+1}^{(m)}(q, P_J) W_{k-|J|+1}^{(g-m)}(\overline{q}, P_{K/J}) + W_{k+2}^{(g-1)}(q, \overline{q}, P_K) \right)$$

where

$$K(q, p) = \frac{\int^q B(\xi, p)}{\omega(q)}$$

the integration path in $K(q, p)$ is chosen in the neighborhood of a branch point. The recursive basis are

$$W_k^{(g)} = 0 \quad \text{for } g < 0$$
It can be proved that the function \( W_{k+1}^{(g)}(p, p_1, \ldots, p_k) \) is symmetric with respect to its variables \([14]\).

The correlation functions, as well as the partition functions defined on an algebraic curve have graphic representations. B. Eynard and N. Orantin introduced a set of graphs \( \mathcal{G}_k^g(p, p_1, \ldots, p_k) \) (In \([14]\), it was denoted as \( \mathcal{G}_k^{g+1}(p, p_1, \ldots, p_k) \)). Pay your attention that the formulation is similar to the Feynman diagram expression in quantum field theories, however it is not the Feynman diagram. These graphs have two indices \( k, g, \) such that \( k \geq 0, g \geq 0 \) and \( k + 2g \geq 3 \). For a graphic representations, \( \mathcal{G}_k^g(p, p_1, \ldots, p_k) \) is a set of connected trivalent graphs which are subjected to the following three conditions:

(I), there are \( 2g + k - 1 \) vertices which are the trivalent vertices, there are \( k \) leaves which are the 1-valent vertices labelled with \( p_1, \ldots, p_k \), and there is one root which is the 1-valent vertex labelled by \( p \).

(II), there are \( 3g + 2k - 1 \) edges, among them \( 2g + k - 1 \) edges are arrowed, and \( k + g \) are non-arrowed. The arrowed edges form a skeleton tree with the root \( p \). The arrows from root to leaves induce a partial order \( \mathcal{R} \) of the vertices. Among the non-arrowed edges, there is \( k \) ones from a vertex to a leaf which are named as outlines, and the remaining \( g \) edges connect the pairs of vertices with the \( \mathcal{R} \) relations.

(III), for two edges, if one is arrowed, another one is non-arrowed, and they have a common vertex, furthermore, the non-arrowed edge links this vertex with its descendant. Then we set the arrowed edge to be the left child, and the non-arrowed one be the right child with labeling by a black dot in the graph.

We denote the set of all graphs of the correlation functions by \( \mathcal{G} = \bigcup_{k+2g \geq 3, k \geq 0, g \geq 0} \mathcal{G}_k^g(p, p_1, \ldots, p_k) \).

The graphs in \( \mathcal{G} \) are relevant to the correlation functions by the weighted map given by B. Eynard and N. Orantin [14]. The ingredients in the correlation functions are expressed as basic elements of the graphs, the weighted map \( \phi \) is defined as: \( \phi( p \quad \rightarrow \quad q ) = B(p, q) \); \( \phi( p \quad \rightarrow \quad \bullet \quad \rightarrow \quad q \quad \rightarrow \quad \bar{q} ) = K(p, q) \); and take a residue \( q \rightarrow \{ a_i \} \) at each vertex \( q \), and sum over all the branch points along the arrow from leaves to the root. Under the weighted \( \phi \) they proved the relationship of the graphs and the correlations [14]:

\[
W_{k+1}^{(g)}(p, p_1, \ldots, p_k) = \phi \left( \sum_{G \in \mathcal{G}_k^g(p, p_1, \ldots, p_k)} G \right) = \sum_{G \in \mathcal{G}_k^g(p, p_1, \ldots, p_k)} \phi(G) \tag{7}
\]

For a given vertex in \( \mathcal{G} \), if we exchange the left children with the right ones, the images under \( \phi \) are intact. This will be encoded into symmetry factors. By equation
Figure 1: Graphs in $\mathcal{G}_0^2$ and the $\bullet$ denotes the right child.

The elements of $W_{k+1}(p,p_1,\ldots,p_k)$ are one-to-one corresponding to the graphs in $\mathcal{G}_k^q(p,p_1,\ldots,p_k)$. Hence we will not distinguish them from now on. By the definition of $\mathcal{G}$, such diagrammatic representation $\mathcal{G}_0^2$ of $W_1^{(2)}$ contains the graphs given in Figure 1.

3 The Loday-Ronco Hopf algebra of planar binary trees

In this section, we review the Loday-Ronco Hopf algebra of planar binary trees. Details and proofs can be found in [19].

3.1 The Shuffle algebra over the symmetric groups

Let $S_n$ be a set of permutations of order $n$. It is a symmetric group acting on $\{1,\ldots,n\}$. The trivial group $S_0$ is nothing but the unit element 1. An element $\rho \in S_n$ is denoted by its image $(\rho(1)\cdots\rho(n))$. For a field $\mathbb{K}$, denoting the group algebra as $\mathbb{K}[S_n]$ which is generated by the $S_n$ over $\mathbb{K}$. Furthermore, we set the disjoint union of $S_n$ for $n \geq 0$ as $S_\infty$, hence

$$\mathbb{K}[S_\infty] = \bigoplus_{n\geq0}\mathbb{K}[S_n].$$

It is a graded vector space. An $(n,m)$–shuffle is defined as an element $\sigma \in S_{n+m}$ such that
\[ \sigma(1) < \sigma(2) < \cdots < \sigma(n) \\
\sigma(n+1) < \sigma(n+2) < \cdots < \sigma(n+m). \]  

(8)

The set of all \((n,m)\)-shuffles is denoted by \(Sh_{n,m}\). For any permutations \(\rho \in S_m\) and \(\sigma \in S_n\), the product \(\rho \times \sigma\) on \(S_\infty\) is the permutation in \(S_{n+m}\), which is obtained by limit the \(\rho\) action on the former \(m\) variables, and \(\sigma\) action on the remain \(n\) variables.

There exists a graded Hopf algebraic structure on \(K[S_\infty]\). For any \(x \in K[S_n]\), \(y \in K[S_m]\), the product is defined as

\[ x * y = \sum_{\alpha_{n,m} \in Sh_{n,m}} \alpha_{n,m} \circ (x \times y). \]  

(9)

For any one \(\sigma \in K[S_n]\), the coproduct can be defined as

\[ \Delta(\sigma) = \sum_{i=0}^{n} \sigma_i \otimes \sigma'_{n-i}, \]  

(10)

where \(\sigma_i \in K[S_i], \sigma'_{n-i} \in K[S_{n-i}]\) such that

\[ \sigma = (\sigma_i \times \sigma'_{n-i}) \circ \omega^{-1}, \]  

(11)

where \(\omega\) is a \((i,n-i)\)-shuffle. For a given element \(\sigma \in K[S_n]\) and a number \(i\), it can be proved that there exists a unique \((i,n-i)\)-shuffle \(\omega\) and \(\sigma_i \in K[S_i], \sigma_{n-i} \in K[S_{n-i}]\), such that the identity (11) is held [19]. By using the definition of product \(*\) and coproduct \(\Delta\), \(K[S_\infty]\) becomes a bialgebra, moreover, it is not only graded but also connected, therefore \(K[S_\infty]\) is a Hopf algebra [22].

3.2 Planar binary trees and Hopf algebra

A planar binary tree \(t\) is a special kind of the binary tree, which is an oriented graph on a plane without loops, all of its internal vertices are trivalent. Let \(|t|\) denote the degree of a planar binary tree \(t\), which is the number of trivalent vertices in \(t\). For every planar binary tree, there is a root, which is a particular edge with 1-valent vertex, the rest edges with 1-valent vertex are leaves. Obviously, the tree \(t\) has \(|t|+1\) leaves. Following the notation in [19], we denote the set of planar binary trees with \(n\) vertices by \(Y_n\). The disjoint union of \(Y_n\) for \(n \geq 0\) is denoted by \(Y_\infty\). Similarly, \(K[Y_\infty] = \oplus_{n \geq 0} K[Y_n]\).

For simplicity, a planar binary tree with levels is mentioned as a planar binary tree \(t\), in which every internal vertex is assigned to one of the horizontal lines. The horizontal lines are entangled from top to bottom by the numbers \(\{1, \cdots, |t|\}\). Hence, each vertex has a unique number. By this way, the level of the vertex attached to the root is \(|t|\). For
simplicity, we denoted them as $\tilde{Y}_n$, for the set of all planar binary trees which levels are $n$.

For a planar binary tree with levels, if we order its vertices from the left to right and from the top to bottom, then it gives a permutation $[10]$. The inverse is established in the same way. In other word, $\tilde{Y}_n$ is one-to-one corresponding to $S_n$ $[19]$. One can try out the proof in ref. $[19]$.

In order to define the Hopf algebra over $\mathbb{K}[Y_{\infty}]$, a grafting operator has not defined yet. In fact, for any two trees $t_1$ and $t_2$, the grafting $t_1 \vee t_2$ fuse into a new tree $t$ with the left branch $t_1$ and the right one $t_2$. Therefore $|t| = |t_1| + |t_2| + 1$. If we denote the one-leaf tree without trivalent vertices by $|t|$. And any picked tree can be expressed by two branches with a grafting operator.

The bijection $\tilde{Y}_n \rightarrow S_n$ and the forgetful map (forget the levels of vertices) $\tilde{Y}_n \rightarrow Y_n$ induce a linear map $\psi_n : S_n \rightarrow Y_n$. Hence it is defined from $\mathbb{K}[S_n]$ to $\mathbb{K}[Y_n]$ as well. The linear dual maps $\psi^*_n : \mathbb{K}[Y_n] \rightarrow \mathbb{K}[S_n]$ are inclusion maps for $n \geq 0$. For instance, if $t \in Y_n$, define the set $Z_t = \{ \sigma \in S_n | \psi_n(\sigma) = t \}$. One has $\psi^*_n(t) = \sum_{\sigma \in Z_t} \sigma$. Their disjoint union is a graded linear map $\psi^* : \mathbb{K}[Y_{\infty}] \rightarrow \mathbb{K}[S_{\infty}]$. J. L. Loday and M. O. Ronco proved that the image of $\psi^*$ is a Hopf subalgebra of $\mathbb{K}[S_{\infty}]$ $[19]$. The product $*$ on $\mathbb{K}[Y_{\infty}]$ satisfies the formula $[19]$

$$T * T' = T_1 \vee (T_2 * T') + (T * T'_1) \vee T'_2,$$

where $T$ and $T'$ are arbitrary trees in $\mathbb{K}[Y_{\infty}]$ such that $T = T_1 \vee T_2$ and $T' = T'_1 \vee T'_2$. The starting equation of the recursive series is

$$| * T = T = T * |.$$

The coproduct $\Delta$ on $\mathbb{K}[Y_{\infty}]$ surely satisfies a series of recursive formulas $[19]$. For a selected tree $T = T_1 \vee T_2 \in Y_{n+m+1}$ with $|T_1| = n$ and $|T_2| = m$, \n
$$\Delta(T) = \sum_{j,k} (T_{1,j} * T_{2,k}) \otimes (T'_{1,n-j} \vee T'_{2,m-k}) + T \otimes |,$$

where $\Delta(T_1) = \sum_j T_{1,j} \otimes T_{1,n-j}$ and $\Delta(T_2) = \sum_k T_{2,k} \otimes T'_{2,m-k}$.

The two operators product $*$ and coproduct $\Delta$ are obtained by restricting the Hopf algebra on $\mathbb{K}[S_{\infty}]$ to $\mathbb{K}[Y_{\infty}]$. J. L. Loday and M. O. Ronco proved they are internal in $\mathbb{K}[Y_{\infty}]$, so the image of $\psi^*$ is a Hopf subalgebra $[19]$.

4 Equivalence between two kinds of graphs

We have mentioned two kinds of graphs, one is the diagrammatic representation of the topological recursion, another one is the planar binary tree. If we connect two consecutive leaves on a planar binary tree, it will generates a graph with one loop. These graphs with loops satisfy the topological recursion formulas $[10]$, in fact, they are identical with the diagrammatic representation of the topological recursion.
If we consider the diagrammatic representation of genus 0 correlation functions on the topological recursion. For a fixed number \( n \) of internal vertices, the cardinality of \( \mathcal{G}_{n+2}^0 \) is given by the Catalan number \( C_n = \frac{(2n)!}{n!(n+1)!} \). It is equal to the dimension of \( \mathbb{K}[Y_n] \). Intuitively, there is a natural one-to-one correspondence between them [10]. The \( n + 2 \) points, genus 0 correlation function \( W_{n+2}^0(p, p_1, \ldots, p_{n+1}) \) satisfies the following equation

\[
\phi^{-1}(W_{n+2}^0(p, p_1, \ldots, p_{n+1})) = \sum_{t_i \in Y_n \text{ perm. of leaf labels } (p_1, \ldots, p_{n+1})} t_i = \sum_{p+q+1=n, \quad |t_1|=p, |t_2|=q} t_1 \cup t_2 
+ \text{perm. } \{p_1, \ldots, p_{n+1}\},
\]

(15)

where \( \phi^{-1} \) is the inverse of B. Eynard and N. Orantin’s weighted map \( \phi \). The proposition can be proved by the induction on \( n \). It is only to reorganize of the facts in [10].

If one denote by \( (Y_n)^g \) the set of distinct graphs with \( g \) loops, which are obtained by successively action on \( (i \leftrightarrow i+1) \) for \( g \) times on trees in \( Y_n \), then, similar to genus 0 case, there is the following equation

\[
\phi^{-1}(W_k^g(p, p_1, \ldots, p_{k-1})) = \sum_{t^g \in (Y_n)^g \text{ perm. } \{p_1, \ldots, p_{k-1}\}} t^g
= \sum_{i=0}^{g} \sum_{(t_1)^i \in (Y_n)^i, \quad (t_2)^{g-i} \in (Y_n)^{g-i}} (t_1)^i \cup (t_2)^{g-i} + \sum_{i=0}^{g-1} \sum_{(t_1)^{g-i-1} \in (Y_n)^{g-i-1}, \quad (t_2)^i \in (Y_n)^i} (t_1)^{g-1-i} \cup (t_2)^i
+ \text{perm. } \{p_1, \ldots, p_{k-1}\},
\]

(16)
in which \( n = 2g + k - 2 \), \( t = t_1 \cup t_2 \) and \( t_1 \cup t_2 \) means the identification between the rightmost leaf in \( t_1 \) to the leftmost leaf in \( t_2 \). Hence, \( (Y_n)^g = \mathcal{G}_{k-1}^g \).

In order to prove equation (16) is held, we start to set \( g = 1 \) and make the induction on the number \( n \), then make an induction on genus \( g \) case. The procedure is only to rearrange of the facts in [10]. Equations (15) and (16) imply that the planar trees with their contractions are identified with the diagrammatic representation of the topological recursion.

In order to obtain graphs with loops from planar binary trees, J. N. Esteves defined a contraction [10]. Given any planar binary tree \( t \in Y_n \), label its leaves from left to right by \( p_1, \ldots, p_{n+1} \) in turns. The action \( (i \leftrightarrow i+1) \) on \( t \) means attaching an edge to the consecutive leaves \( p_i \) and \( p_{i+1} \), then relabeling the remaining leaves by \( p_1, \ldots, p_{n-1} \). It should be noticed that the only action is the contraction for two the nearest neighbor leaves.

Based on this equivalence, the Hopf algebra on planar binary trees can gives a Hopf algebraic structure on the diagrammatic representations of correlation functions with genus 0. Unfortunately, the structure for the genus 0 functions cannot be generalized to
higher genus cases. In next section, a new Hopf algebra on planar binary trees will be
given, while this accompany algebraic structure can be naturally generalized to graphs
with loops.

5 The Hopf algebra of topological recursion

It is known that, the Loday-Ronco Hopf algebra on planar binary trees cannot be extended
to graphs with loops generated by contractions, in the main, the operator as of the Loday-
Ronco Hopf algebra keep the number of vertices, meanwhile they change the number of
leaves, and the product and coproduct are not well defined on the graphs with no leaves.

In this section, we give a new form of Hopf algebra on graph with tagged leaves,
it can be well defined on graphs with loops after leaves contractions as well. In this
way, the planar binary trees is a special case of the graph with tagged leaves without
any contractions. Based on the equivalence of the diagrammatic representation of the
topological recursion and the graph with tagged leaves, as well as using the weight map,
in fact, the new Hopf algebra gives a Hopf algebraic structure on the topological recursion.

In the following, we will give a bracket representation of planar binary trees with
tagged leaves in subsection 5.1. In the bracket representation, a new Hopf algebra is
defined in the next subsection. Finally, we generalize it to graphs with tagged leaves and
loops, and derive the relationship between coproduct and the topological recursion.

5.1 Bracket representations of planar binary trees

At first, In this subsection, we consider the case of planar binary tree, a special kind of
graph without loop. Let \( t \) be a planar binary tree with tagged leaves in \( Y_n \). Its leaves are
labelled by \( 1, \cdots, n+1 \) (or other distinguishable labels) from left to right. We can denote
\( t \) by a string of numbers with apex angles. We list the numbers \( 1 \cdots n+1 \) in succession,
then add an apex angle for two adjacent numbers if their corresponding leaves are on
the same branch. If regarding an apex angle as a leaf, we keep on the way to add apex
angles till that all the numbers are in one apex angle. For example in Figure 2 the tree is
\( \langle\langle 12 \rangle\langle 3\langle 45 \rangle \rangle \rangle \).

Obviously, any planar binary tree has a unique expression with apex angles. We call
these expressions as the bracket representations of planar binary trees. Actually, it can
be regarded as a dual representation of the permutation group representation. In fact,
from the permutation of a tree \( t \), we can get the expression with apex angles directly. If
we denote \( t = (i_1 \cdots i_k) \), its expression with apex angles can be achieved in the following
procedure. We list \( p_1 \cdots p_{k+1} \) in succession, with corresponding to their leaf labels. Then
any interval between two labels in \( p_1 \cdots p_{k+1} \) corresponds a certain number in \( (i_1 \cdots i_k) \)
from left to right. Then we add \( k \) apex angles in an arrangement in accordance with the
order of the natural numbers in \( (i_1 \cdots i_k) \). An apex angle which has added is viewed as a
new leaf while adding the next pair apex angles.
For an example, assume that $t = (231)$. It has four leaves denoted by $p_1, p_2, p_3, p_4$ from left to right. The number 1 in (231) corresponds to the interval between $p_3$ and $p_4$, then we add the first apex angle on $p_3p_4$, i.e. $p_1p_2(p_3p_4)$. The number 2 in (231) corresponds to the interval between $p_1$ and $p_2$, hence we add the second apex angle to be $p_1p_2(p_3p_4)$. At last, the number 3 is matched with the interval between $p_2$ and $p_3$, i.e. $p_1p_2$ and $p_3p_4$. We add the third apex angle to become $p_1p_2(p_3p_4)$.

Permutations are one-to-one correspondence to planar binary trees with levels, expressions with apex angles are bijective with planar binary trees. Hence, one permutation has a unique expression with apex angles. But one expression with apex angles can correspond to distinct permutations in general. An example, permutations (231) and (132) have the same expression $⟨⟨12⟩(34)⟩$.

Let $X_n (n \geq 2)$ be a set of digital expressions with apex angles matching to the following conditions: in any pair of apex angles, there are two elements, a number and a pair of apex angle in a whole viewed as an element; and the whole expression is in a pair of apex angles. Particularly, $X_0 = \{\emptyset\}; X_1 = \{(1)\}; X_2 = \{(12)\}$. Among them, (1) represents a planar binary tree without trivalent vertices |, (12) stands a planar binary tree with two leaves, and one trivalent vertex. Graphs with different symbol system have the relationship $X_{n+1} = Y_n = \mathcal{G}_n$.

If we indicate $\mathbb{K}[X_n]$ the vector space generated by $X_n$ over field $\mathbb{K}$. The disjoint union of the sets $X_n$ for $n \geq 0$ is stated as $X_\infty$. Hence,

$$\mathbb{K}[X_\infty] = \oplus_{n \geq 0} \mathbb{K}[X_n].$$

(17)

It should be noticed that $\mathbb{K}[X_\infty]$ is not identical to $\mathbb{K}[Y_\infty]$, although both they are generated through planar binary trees. In $\mathbb{K}[X_\infty]$, $\emptyset$ is included as a generator. But $\mathbb{K}[Y_\infty]$ do not include $\emptyset$, this means that

$$\mathbb{K}[X_\infty] = \mathbb{K}[Y_\infty] \oplus \mathbb{K}[\emptyset].$$

(18)
In the next subsection, we will prove that there is a Hopf algebraic structure on $\mathbb{K}[X_\infty]$, which is different from the known Loday-Ronco structure on $\mathbb{K}[Y_\infty]$.

### 5.2 Hopf algebraic structure for diagrams without loops

Operators (9) and (10), or the expressions in (12) and (14), they surely keep the numbers of trivalent vertices on planar binary trees. However, the story for leaf is varied. The trees in the right sides of equations (9) or (12) are one leaf less than the ones in the left, on the contrary, the sum on leaves of trees in the right of equations (10) or (14) are one more than that in the left.

In this subsection, we will introduce a new Hopf algebraic structure on $\mathbb{K}[X_\infty]$. The operators are different from the Loday-Ronco action, they keep the number of leaves, by the way, the product increases one trivalent vertex, and while the coproduct reduces one trivalent vertex. More remarkable, these definitions can be generalized to graphs with loops in a straightforward way.

**Definition 5.1.** For $\rho = \langle \rho_1 \rho_2 \rangle \in \mathbb{K}[X_n]$ and $\tau = \langle \tau_1 \tau_2 \rangle \in \mathbb{K}[X_m]$, then their product is defined as

\[
\rho \ast \tau = \langle \rho \tau \rangle + \langle \rho \ast \tau_1 \tau_2 \rangle + \langle \rho_1 \rho_2 \ast \tau \rangle,
\]

where $\rho_1, \rho_2, \tau_1, \tau_2 \neq \emptyset$. For the righthand side, we add $n$ to the numbers in $\tau$ for the self-consistency (if the leaves in $\tau$ are labeled by numbers). For any element $\sigma \in \mathbb{K}[X_\infty]$, there are

\[
\sigma \ast \langle 1 \rangle = \langle \sigma 1 \rangle + \langle \sigma_1 1 \rangle,
\]

\[
\langle 1 \rangle \ast \sigma = \langle 1 \sigma \rangle + \langle (1 \sigma_1) \sigma_2 \rangle,
\]

and

\[
\emptyset \ast \sigma = \sigma = \sigma \ast \emptyset.
\]

Some examples are listed below.

\[
\emptyset \ast \langle 1 \rangle = \langle 1 \rangle = \langle 1 \rangle \ast \emptyset;
\]

\[
\langle 1 \rangle \ast \langle 12 \rangle = \langle 1 \langle 23 \rangle \rangle + \langle 1 \rangle \langle 3 \rangle ;
\]

In terms of diagrams, it is
Similarly

\[
\langle 12 \rangle \star \langle 1 \rangle = \langle 1\langle 23 \rangle \rangle + \langle \langle 1 \rangle \langle 12 \rangle \rangle;
\]

\[
\langle \langle 12 \rangle \rangle \star \langle 1 \rangle = \langle \langle \langle 1 \rangle \langle 2 \rangle \langle 3 \rangle \rangle \rangle + \langle \langle 1 \rangle \langle 2 \rangle \langle 4 \rangle \rangle;
\]

\[
\langle 1 \rangle \star \langle \langle 12 \rangle \rangle = \langle \langle \langle 1 \rangle \langle 2 \rangle \langle 3 \rangle \rangle \rangle + \langle \langle 1 \rangle \langle 2 \rangle \langle 3 \rangle \langle 4 \rangle \rangle + \langle \langle 1 \rangle \langle 2 \rangle \langle 4 \rangle \rangle + \langle \langle 1 \rangle \langle 3 \rangle \rangle + \langle \langle 1 \rangle \langle 4 \rangle \rangle;
\]

The above two expressions imply that the product $\star$ is non-commutative. The non-commutate is the essence property of a quantum group. The product expression for two simplest vertices is

\[
\langle 12 \rangle \star \langle 12 \rangle = \langle \langle 1 \rangle \langle 2 \rangle \langle 3 \rangle \rangle + \langle \langle \langle 1 \rangle \langle 2 \rangle \langle 3 \rangle \rangle \rangle + \langle \langle 1 \rangle \langle 2 \rangle \langle 4 \rangle \rangle + \langle \langle 1 \rangle \langle 3 \rangle \rangle + \langle \langle 1 \rangle \langle 4 \rangle \rangle + \langle 1 \rangle \langle 2 \rangle \langle 3 \rangle \langle 4 \rangle \rangle + \langle 1 \rangle \langle 2 \rangle \langle 3 \rangle \langle 4 \rangle \rangle;\]
It is easy to verify that the sums of leaves in two sides of the equations are identical, but the number of vertices is changed.

Even though the product $\star$ on $\mathbb{K}[X_n]$ and the product $\ast$ on $\mathbb{K}[Y_n]$ seem to satisfy similar formulas. However, indeed the product $\star$ is very different from $\ast$. Of course, for the case $\mathbb{K}[Y_\infty]$, they are linked by the following identity

**Proposition 5.1.** For arbitrary two planar binary trees $\rho, \sigma \in \mathbb{K}[Y_\infty]$, the following identity is preserved

$$\rho \ast \sigma = \rho \ast (1) \ast \sigma,$$

where $(1)$ is the planar binary tree with one trivalent vertex in permutation representation.

**Proof.** The proof is carried out inductively on the number of vertices $k = |\rho| + |\sigma|$. If $k = 0$, we have $\rho = |$ and $\sigma = |$, then the righthand side of equation (20) is equal to the left one, either side is a planar tree with one trivalent vertex.

If $k = 1$, the cases are, either $\rho = |$, $\sigma = (1)$ or $\rho = (1)$, $\sigma = |$. It is straightforward to check that, in each case, any handsie of equation (20) is identical to $\langle\langle 12 \rangle 3 + 1 \langle 23 \rangle \rangle$.

Suppose the proposition is verified whenever $k < n$. Assume that for element $\rho = \rho_1 \lor \rho_2, \sigma = \sigma_1 \lor \sigma_2$ are arbitrary planar binary trees. If $k = n$, then the right side of the equation (20) is

$$\rho \ast (1) \ast \sigma = ((\rho \lor |) \lor |) \ast \sigma + (\rho_1 \lor (\rho_2 \ast (1))) \ast \sigma$$

$$= ((\rho \lor |) \ast \sigma_1) \lor \sigma_2 + \rho \lor \sigma + \rho_1 \lor (\rho_2 \ast (1) \ast \sigma)$$

$$+ ((\rho_1 \lor (\rho_2 \ast (1))) \ast \sigma_1) \lor \sigma_2$$

$$= (\rho \lor (1) \ast \sigma_1) \lor \sigma_2 + \rho \lor \sigma + \rho_1 \lor (\rho_1 \ast (1) \ast \sigma)$$

$$= \langle\langle \rho \ast \sigma_1 \rangle \rangle \lor \langle\langle \rho \sigma \rangle \rangle + \langle\langle \rho_1 (\rho_2 \ast \sigma) \rangle \rangle$$

$$= \rho \ast \sigma.$$
The first two equalities follow from (12), the fourth equality is governed by the induction. Therefore, we have proved equation (20).

In the above proposition, \( \rho = \emptyset \) or \( \sigma = \emptyset \) is not allowed, for that case there is no definition for \( \ast \). Now we assume that \( \rho, \sigma, \tau \) are two any chosen planar binary trees in \( \mathbb{K}[Y_\infty] \). Due to the Proposition (5.1), one has

\[
\rho \ast (\tau \ast \sigma) = (\rho \ast (1) \ast (\tau \ast (1) \ast \sigma)) = (\rho \ast (1) \ast \tau) \ast (1) \ast \sigma = (\rho \ast \tau) \ast \sigma
\]

If any one of \( \rho, \sigma \) or \( \tau \) is set to be \( \emptyset \), obviously \( \rho \ast (\tau \ast \sigma) = (\rho \ast \tau) \ast \sigma \). In other words, the product \( \ast \) is associative.

We define the unit map \( u \) such that \( u(1_{\mathbb{K}}) = \emptyset \) and \( u(\lambda) = \lambda 1_{\mathbb{K}} \) (for any \( \lambda \in \mathbb{K} \)). By the definition, it is easy to check on \( \mathbb{K}[X_\infty] \) that

\[
\ast \circ (u \otimes \text{id}) = \text{id} = \ast \circ (\text{id} \otimes u).
\]

Obviously, \( (\mathbb{K}[X_\infty], \ast, u) \) is an associative algebra.

In order to match the definition of a Hopf algebra, it is required to introduce a coproduct operator on \( \mathbb{K}[X_\infty] \).

**Definition 5.2.** For any element \( \rho \in \mathbb{K}[X_n] \) on a planar binary tree, assume that \( N \) is the set of leaf labels. Then the coproduct of \( \rho \) are held with respect to the following equation

\[
\delta(\rho) = \sum_{J \subset N} \rho_J(J) \otimes \rho_{n-J}(N/J),
\]

where \( \rho_J(J) \) is a subgraph with \( j = |J| \) leaves labelled with \( J \), and \( \rho_{n-J}(N/J) \) is a subgraph with \( n-j \) leaves labelled with \( N-J \). These leaves keep on their relationships in \( \rho \). Particularly, \( \delta(\emptyset) = \emptyset \otimes \emptyset, \delta(\langle 1 \rangle) = \emptyset \otimes \langle 1 \rangle + \langle 1 \rangle \otimes \emptyset \).

It is evident that

\[
\{\rho_J|J \subset N\} = \{\rho_{N/J}|J \subset N\}.
\]

It is clearly that, \( \delta \) and \( \Delta \), the definitions of two kinds coproduct are different. Simple examples are arranged below to illustrate equation (22).

\[
\delta(12) = \emptyset \otimes \langle 12 \rangle + \langle 1 \rangle \otimes \langle 2 \rangle + \langle 2 \rangle \otimes \langle 1 \rangle + \langle 12 \rangle \otimes \emptyset;
\]
or in terms of graphs

\[
\delta = \emptyset \otimes + \otimes + \otimes
\]

For less trivial examples

\[
\delta((\langle 12 \rangle 3)) = \emptyset \otimes (\langle 12 \rangle 3) + (1) \otimes (23) + (2) \otimes (13) + (3) \otimes (12) + (12) \otimes (3) + (13) \otimes (23) + (13) \otimes (1)+ (\langle 12 \rangle 3) \otimes \emptyset;
\]

\[
\delta((\langle 1 \rangle 23)) = \emptyset \otimes (\langle 1 \rangle 23) + (1) \otimes (23) + (2) \otimes (13) + (3) \otimes (12) + (12) \otimes (3) + (13) \otimes (2) + (23) \otimes (1) + (\langle 1 \rangle 23) \otimes \emptyset.
\]
and equivalently expressed through graphs

\[
\delta = \emptyset \otimes + \oplus + \oplus \oplus \oplus + \oplus
\]

It is clear that the numbers of leaves on the left and right sides are intact for each equation, while the numbers of vertices are changed. Using the definition of the coproduct \(\delta\), for any given tree \(\rho\), one has

\[
(id \otimes \delta) \circ \delta \rho(K) = (id \otimes \delta) \circ \sum_{J \subset K} \rho_{|J|}(J) \otimes \rho_{k-|J|}(K/J) = \sum_{J \subset K} \rho_{|J|}(J) \otimes \left( \sum_{I \subset K/J} \rho_{|I|}(I) \otimes \rho_{k-|I|-|J|}(K/(J \cup I)) \right)
\]

and

\[
(\delta \otimes id) \circ \delta \rho(K) = (\delta \otimes id) \circ \sum_{J \subset K} \rho_{|J|}(J) \otimes \rho_{k-|J|}(K/J) = \sum_{J \subset K} \sum_{I \subset J} \rho_{|I|}(I) \otimes \rho_{|J|-|I|}(J/I) \otimes \rho_{k-|J|}(K/J)
\]
Hence, for any chosen \( \rho \in \mathbb{K}[X] \), the identity
\[
(id \otimes \delta) \circ \delta(\rho) = (\delta \otimes id) \circ \delta(\rho).
\]
is held. Now, we can introduce a counit map \( \varepsilon \in \mathbb{K}[X] \). For any element \( \sigma \in \mathbb{K}[X] \), we set
\[
\varepsilon(\sigma) = \begin{cases} 
1_\mathbb{K} & \text{if } \sigma = \emptyset \\
0 & \text{otherwise}.
\end{cases}
\]
By equation (25), we can easily get \((\varepsilon \otimes \varepsilon) \circ \delta = id = (id \otimes \varepsilon) \circ \delta\). Therefore, \((\mathbb{K}[X], \delta, \varepsilon)\) is a counital coalgebra.

Now \(\mathbb{K}[X]\) is an algebra as well as a coalgebra. For our target, the algebraic operators and the coalgebraic operators should form a bialgebra.

**Proposition 5.2.** \((\mathbb{K}[X], \star, \rho, \delta, \varepsilon)\) is a bialgebra.

**Proof.** We just need to prove \(\star\) is a morphism of coalgebra
\[
\delta \circ \star = (\star \otimes \star) \circ \tau_{2,3} \circ (\delta \otimes \delta)
\]
and
\[
\varepsilon \otimes \varepsilon = \star \circ (\varepsilon \otimes \varepsilon) = \varepsilon \circ \star.
\]
By using of the definition of \(\varepsilon\), equation (27) is obvious. Now we pay attention on equation (26). Let \(K, L\) be sets of labels and \(|K| = k, |L| = l\). For any two picked out trees \(\rho(K) = (\rho_a \rho_b), \sigma(L) = (\sigma_a \sigma_b)\) in \(X\), the left side of equation (26) is
\[
\delta(\rho \star \sigma) = \delta(\rho \sigma) + \delta(\rho_a(\rho_b \star \sigma)) + \delta((\rho \star \sigma_a) \sigma_b).
\]
The right side of equation (26) is equal to
\[
\star \otimes \star \circ \tau_{23} \circ \delta \otimes \delta(\rho \otimes \sigma)
= \star \otimes \star \circ \tau_{23} \circ (\sum_{I \subseteq K} \rho_{|I|} \otimes \rho_{k-|I|}) \otimes (\sum_{J \subseteq L} \sigma_{|J|} \otimes \sigma_{l-|J|})
= \sum_{I \subseteq K} \sum_{J \subseteq L} \sum \otimes \star \circ (\rho_{|I|} \otimes \sigma_{|J|}) \otimes (\rho_{k-|I|} \otimes \sigma_{l-|J|})
= \sum_{I \subseteq K} \sum_{J \subseteq L} \langle (\rho_{|I|} \sigma_{|J|}) \rangle + \langle \rho_a|_{I_1}|(\rho_b|_{J_2}| \sigma_{|J|}) \rangle + \langle (\rho_{|I|} \sigma_a|_{J_1}|) \sigma_{b|J_2|} \rangle \otimes \langle \rho_{k-|I|} \sigma_{l-|J|} \rangle
= \delta[\langle \rho \sigma \rangle + \langle \rho_a(\rho_b \star \sigma) \rangle + \langle (\rho \star \sigma_a) \sigma_b \rangle].
\]
where \(I_1 \cup I_2 = I; I_3 \cup I_4 = K / I; J_1 \cup J_2 = J; J_3 \cup J_4 = L / J\). That equation (26) is held. \(\square\)

In fact, \(\mathbb{K}[X]\) is a graded bialgebra. Using the definition, we know \(\mathbb{K}[X] = \sum_{n \in \mathbb{N}_0} L^n\), where \(L^0 = \{\emptyset\}, L^1 = \{(1)\}, L^2 = X_1, \ldots, L^n = X_{n-1} \cdots\). Since \(L^n\) is one dimensional vector space generated by \(\emptyset\), then \(\mathbb{K}[X]\) is a connected bialgebra. In term of another word, \((\mathbb{K}[X], \star, \rho, \delta, \varepsilon)\) is a Hopf algebra.
5.3 Hopf algebra for graphs with loops

In this subsection, we derive the Hopf algebra on graphs with loops. With an action of the contraction operators, a planar binary tree generates planar binary graph with loops. The new Hopf algebra introduced at moment on planar binary trees can be generalized to graphs with loops.

Let $\bar{X}_n$ mark the set of graphs obtained from $X_n$ by making all possible contractions $(i \leftrightarrow i+1)$ on near leaves. It is obvious that $\bar{X}_0 = X_0, \bar{X}_1 = X_1, X_n \subset \bar{X}_n (n > 1)$. And mark again $\bar{X}_g^n$ the set of trees contracted all possible $g$ ($g \leq \lfloor \frac{n}{2} \rfloor$) pairs with the nearest leaves of trees in $\bar{X}_n$. From the section 4, one has $\bar{X}_g^n = Y_{g-2}^n = G_{g-1}^n$. With the notations introduced, $K[\bar{X}_n]$ is the vector space generated by $\bar{X}_n$ over $K$. The disjoint union of $\bar{X}_n$ ($n \geq 0$) is $\bar{X}_\infty = \cup_{n \in \mathbb{N}_0} \bar{X}_n$, and its vector space over $K$ is $K[\bar{X}_\infty]$. By the definition, $K[X_\infty] \subset K[\bar{X}_\infty]$. Now we extend the structure $(K[X_\infty], \ast, u, \delta, \varepsilon)$ to $K[\bar{X}_\infty]$.

Now we define the product and the unit map on $K[\bar{X}_\infty]$. The operator $\ast$ on $K[\bar{X}_\infty]$ does not effect the adjacent relationships between leaves on planar binary trees. In other words, the contraction operator keeps the adjacent positions invariant. Hence the contraction is commutative with the product $\ast$. Therefore, we can define the product and the unit map on $K[\bar{X}_\infty]$ as following.

\[ \ast|_{K[X_\infty]} = \sum_i (i \leftrightarrow i+1) \circ \ast|_{K[X_\infty]} ; \]
\[ u|_{K[X_\infty]} = \sum_i (i \leftrightarrow i+1) \circ u|_{K[X_\infty]} . \] (28)

Evidently, $(K[\bar{X}_\infty], \ast, u)$ is a unital algebra as expected.

Following the process to put the product operator on $K[\bar{X}_\infty]$, we can introduce the coproduct operator and the counit map by their commutation with the contraction operator.

\[ \delta|_{K[X_\infty]} = \sum_i (i \leftrightarrow i+1) \circ \delta|_{K[X_\infty]} ; \]
\[ \varepsilon|_{K[X_\infty]} = \sum_i (i \leftrightarrow i+1) \circ \varepsilon|_{K[X_\infty]} . \] (29)

It gives $\delta(\emptyset) = \emptyset \otimes \emptyset, \delta(1) = \emptyset \otimes \langle 1 \rangle + \langle 1 \rangle \otimes \emptyset$.

Given any particular planar binary graph with loops, we can realize its coproduct following a simple procedure. As an example, now we carry out $\delta(\langle 1 \rangle)$. The first step, we deal with the coproduct on the tree $\langle 1 \rangle$,

\[ \delta(\langle 1 \rangle) = \emptyset \otimes \langle 1 \rangle + \langle 1 \rangle \otimes (1) + (1) \otimes \langle 1 \rangle \otimes \emptyset . \]

Then, we perform the same contraction as on the primary tree, that links leaves 1, 2 by a line. In another word, we have

\[ \delta(\langle 1 \rangle) = \emptyset \otimes \langle 1 \rangle + \langle 1 \rangle \otimes (1) + (1) \otimes (1) + (1) \otimes \emptyset . \]
where two leaves with the same label under a bar imply that they are joined by a line to make a loop. Of course, it is trivial that \( \langle 1 \rangle \otimes \langle 1 \rangle = \langle 1 \rangle \otimes \emptyset \). The graphic representation for the coproduct on the simplest diagram with one loop is

\[
\delta = \emptyset \otimes + \emptyset \otimes + \emptyset \otimes
\]

Now we deal with \( \delta \langle 1\langle 23 \rangle \rangle \), a less simpler example. At first, we expand the coproduct of the tree with no contractions.

\[
\delta(\langle 1\langle 23 \rangle \rangle) = \emptyset \otimes \langle 1\langle 23 \rangle \rangle + \langle 1 \rangle \otimes \langle 23 \rangle + \langle 2 \rangle \otimes \langle 13 \rangle + \langle 3 \rangle \otimes \langle 12 \rangle + \langle 12 \rangle \otimes \langle 3 \rangle \\
+ \langle 13 \rangle \otimes \langle 2 \rangle + \langle 23 \rangle \otimes \langle 1 \rangle + \langle 1\langle 23 \rangle \rangle \otimes \emptyset
\]

and,

\[
\delta \langle 1\langle 23 \rangle \rangle = \emptyset \otimes \langle 1\langle 22 \rangle \rangle + \langle 1 \rangle \otimes \langle 22 \rangle + 2\langle 2 \rangle \otimes \langle 12 \rangle + 2\langle 12 \rangle \otimes \langle 2 \rangle + \langle 22 \rangle \otimes \langle 1 \rangle \\
+ \langle 1\langle 22 \rangle \rangle \otimes \emptyset
\]
It is easy to verify that \((\mathbb{K}[\bar{X}_\infty], \delta, \varepsilon)\) is a counital algebra. Next, we will prove that it is a Hopf algebra.

**Proposition 5.3.** \((\mathbb{K}[\bar{X}_\infty], \star, u, \delta, \varepsilon)\) is a Hopf algebra.

**Proof.** At first, we prove \((\mathbb{K}[\bar{X}_\infty], \star, u, \delta, \varepsilon)\) forms a bialgebra. As we have announced \(\mathbb{K}[\bar{X}_\infty]\) is graded, as well as connected, in the other word, \((\mathbb{K}[\bar{X}_\infty], \star, u, \delta, \varepsilon)\) is a Hopf algebra. Hence, it is sufficient to prove that equation (26) is made sense on \(\mathbb{K}[\bar{X}_\infty]\).

For the leaves are unlabelled, the operator \((i \leftrightarrow i+1)\) is commutative with the permutations \(\tau_{23}\). For any given two trees \(\bar{\rho}, \bar{\sigma} \in \mathbb{K}[\bar{X}_\infty]\), and the corresponding primary trees \(\rho = (\rho_1 \rho_2), \sigma = (\sigma_1 \sigma_2) \in \mathbb{K}[X_\infty]\).

\[
\delta \langle \bar{\rho} \star \bar{\sigma} \rangle = \delta \circ \left( \sum_{\rho, \delta} (i \leftrightarrow i+1) \right) \circ (\rho \star \delta) 
= \sum_{\rho, \delta} (i \leftrightarrow i+1) \circ \delta (\rho \star \delta)
\]

On the other hand

\[
\star \otimes \star \circ \tau_{23} \circ \delta \otimes \delta (\bar{\rho} \otimes \bar{\sigma}) = \star \otimes \star \circ \tau_{23} \circ \delta \otimes \delta \left( \sum_{\rho, \delta} (i \leftrightarrow i+1) \right) \circ (\rho \star \delta) = \sum_{\rho, \delta} (i \leftrightarrow i+1) \circ \star \otimes \star \circ \tau_{23} \circ \delta \otimes \delta (\rho \otimes \sigma)
\]

Using the equality (26), one gets

\[
\delta \langle \bar{\rho} \star \bar{\sigma} \rangle = \star \otimes \star \circ \tau_{23} \circ \delta \otimes \delta (\bar{\rho} \otimes \bar{\sigma}).
\]

\[\square\]

Immediately, using the structure on on \(\mathbb{K}[\bar{X}_\infty]\), we can dress the Hopf algebra structure on the topological recursion by using the weighted map \(\phi\). The Hopf algebra on diagrammatic representations of the correlation functions is one of the results of the Hopf algebra structure on \(\mathbb{K}[\bar{X}_\infty]\).

With the help of Hopf algebra on \(\mathbb{K}[\bar{X}_\infty]\), immediately, as a corollary, we achieve a more concise result in contrast to that in [10]. The formula can be expressed as the following theorem.
Corollary 5.1. The diagrammatic representation of the correlation function $W^g_k(p, p_1, \ldots, p_{k-1})$ is given by summing over all permutations $k - 1$ labelled leaves for graphs in $(X_{k-1})^g$,

$$
\phi^{-1}(W^g_k(p, p_1, \ldots, p_{k-1})) = \sum_{t^g \in (X_{k-1})^g \text{perm. } \{p_1, \ldots, p_{k-1}\}} t^g,
$$

in which $(X_{k-1})^g$ is the set of graphs obtained by the product $\star$ of $(2g + k - 1) \langle 1 \rangle$ with all possible $g$ pairs of the nearest leaves identifications,

$$
X^g_{k-1} = \{ \langle 1 \rangle * \langle 1 \rangle * \cdots * \langle 1 \rangle \mid \text{all possible contractions of } g \text{ pairs of the nearest leaves} \}.
$$

Proof. It is easy to see that $X^g_{k-1}$ is identified with $(Y^{2g+k-2})^g$ in [10]. Using the result in [10], we know that equation (32) is sured. For any element $\rho \in \mathbb{K}[X_\infty]$, it is straightforward $\rho \star \langle 1 \rangle = \rho \star \langle 12 \rangle$.

Hence, equation (33) is followed directly the Theorem 2 in [10]. It should be noticed that the number of $\langle 1 \rangle$ in equation (33) is one more than the number of (1) in Theorem 2 in [10].

5.4 Antipode

For a graded and connected Hopf algebra, there is a canonical antipode $S$ governed by the following expression:

$$
\ast \circ (S \otimes Id) \circ \delta = u \circ \varepsilon.
$$

For any element $\rho \in \mathbb{K}[X_\infty]$, we have

$$
S(\rho) = -\rho - \sum S(\rho_1) * \rho_2,
$$

where in the Sweedler notation

$$
\delta \rho = \sum \rho_1 \otimes \rho_2
$$

is assumed. It is easy to see that

- $S(\emptyset) = \emptyset$;
- $S(\langle 1 \rangle) = -\langle 1 \rangle$;
- $S(\langle 12 \rangle) = \langle 21 \rangle$;
- $S(\langle 123 \rangle) = \langle 1 \langle 23 \rangle \rangle - \langle 32 \rangle \langle 1 \rangle - \langle 3 \langle 21 \rangle \rangle$.
With the help of the identity
\[ S \circ \star = \star \circ \tau \circ (S \otimes S), \]
here the \( \tau \) is a permutation. Therefore, we have
\[ S((1) \star (2) \star \cdots \star (n)) = (-1)^n (n) \star (n-1) \star \cdots \star (1). \] (35)
And then the weighted map \( \phi^{-1}S \) induced by the antipode on the vector space of the correlation functions implies
\[ (\phi^{-1}S)W_{n+1}^g(p, p_1, \cdots, p_n) = (-1)^n W_{n+1}^g(p, p_1, \cdots, p_n). \] (36)
As the action \( (i \leftrightarrow i+1) \) is commutative with all of the the product \( \star \), the coproduct \( \delta \), the unit map \( u \) as well as the counit map \( \varepsilon \). Hence, the contraction operator is also commutative with the antipode \( S \). As an example
\[ S((1 \leftrightarrow 2)(12)) = (1 \leftrightarrow 2)S((12)) = (1 \leftrightarrow 2)(21). \]
Therefore, we have
\[ (\phi^{-1}S)W_{n+1}^g(p, p_1, \cdots, p_n) = (-1)^{n+2g} W_{n+1}^g(p, p_1, \cdots, p_n). \]

5.5 Coproduct and topological recursion

At first sight, it seems that the expression of coproduct \( \delta \) defined on the tagged graphs by Equations (22), (29) and the topological recursion are very different. The main discrepancy between them is that the numbers of their diagram representation are different. It looks like that this unmatched point cannot be eliminated. However, after a careful inspection, in fact, they share numerous common properties. The unmatched point will be removed by considering the symmetry of the the tagged graphs. If we tie with coefficients to terms of coproduct for the tagged graphs, the topological recursion can be very naturally reconstructed.

Given any graph \( G_1 \in \mathcal{G}_k^g(p, p_1, \ldots, p_k) \), if we introduce \( \delta'G_1 = \delta G_1 - \emptyset \otimes G_1 - G_1 \otimes \emptyset \).
In other words,
\[ \delta'G_1(p, K) = \sum_{I \subseteq K} \rho_{1I} \otimes \tilde{\rho}_{1K/I}, \] (37)
where \( K = \{p_1, p_2, \cdots, p_k\} \). For simplicity, we collect the terms with equal leaves in Equation (37) and set \( T_i \otimes \tilde{T}_{k-i} = \sum_{I \subseteq K, |I|=i} \rho_{1I} \otimes \tilde{\rho}_{K/I} \) formally, then one has the following proposition.

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Proposition 5.4. If
\[ \sum_{G \in \mathcal{G}_k^g} \delta' G(p, K) = \sum_{i=1}^{k-1} T_i \otimes \bar{T}_{k-i}, \] (38)
then the following equation holds
\[ \phi(\sum_{i=1}^{i=k-1} a_i T_i \otimes \bar{T}_{k-i}) = \text{Res}_{q \rightarrow a} K_2(p; q)\sum_{I \subset K} W_{|I|+1}(q, I)W_{k-i+1}(\bar{q}, K/I) = W_{k+1}(p, K), \] (39)
where the coefficients \( a_i = \frac{C_i C_{k-i-1}}{C_{k-1}(k_i)} \), and \( C_i \) is the Catalan number.

Proof. As the labels in \( K \) are symmetrical, the graphs in \( T_i \otimes \bar{T}_{k-i} \) are also symmetrical with respect to labels in \( K \). By the definition of the coproduct \( \delta' \), there are much more the tagged graphs in \( T_i \otimes \bar{T}_{k-i} \) than the diagrammatic representations of the topological recursion \( W_{|I|+1}(q, I)W_{k-i+1}(\bar{q}, K/I) \). However, due to the symmetry, if we define \( a_i \) as the ratio of graphs in diagrammatic representations of the topological recursion \( W_{|I|+1}(q, I)W_{k-i+1}(\bar{q}, K/I) \) to the tagged graphs in \( T_i \otimes \bar{T}_{k-i} \). With the help of results in [11], one has
\[ a_i = \frac{\binom{k}{i} i!(k-i)!C_{i-1}C_{n-i-1}}{k!C_{k-1}(k_i)} = \frac{C_i C_{k-i-1}}{C_{k-1}(k_i)}. \]

Of course, the results can be extended to the correlation function. For arbitrary correlation function of order to genus \( g \), we have similar results.

Proposition 5.5. If
\[ \sum_{G \in \mathcal{G}_k^g} \delta' G(p, K) = \sum_{m=0}^{g} \sum_{i=1}^{i=k-1} T_i^m \otimes \bar{T}_{k-i}^m + T_{k+1}^{g-1}, \] (40)
then, the identity holds
\[ \phi(\sum_{m=0}^{g} \sum_{i=1}^{i=k-1} a_i^m T_i^m \otimes \bar{T}_{k-i}^m + b_{k+1}^g \tau_{k+1}^{g-1}) = \text{Res}_{q \rightarrow a} K_2(p; q)\sum_{m=0}^{g} \sum_{I \subset K} W_{|I|+1}^m(q, I)W_{k-i+1}^{m-g}(\bar{q}, K/I) + W_{k+2}^{g-1}(q, \bar{q}, K) \]
\[ = W_{k+1}^g(p, K), \] (41)
where the coefficients $a_{i}^{m} = \frac{s_{m}a_{m-i}^{m}((k-i)!)(\frac{2}{m}((g-m-1)+k-i))}{s_{g}^{k}(g-1)!k^{g-k}}$, while $b_{k+1}^{g-1} = \frac{g}{2^{g}(4^{g}-2^{g})}$, and $C_{i}$ is the Catalan number. The parameters $s_{m}$ for $m \geq 1$ obey [11]:

\[ s_{1} = 1, \quad s_{m} = 2(3m-4)s_{m-1} + \sum_{n=1}^{m-1} s_{n}s_{m-n}. \]

Proof. Following the same process as that in the proposition (5.4), we just list out the ratio of graphs on diagrammatic representation of the topological recursion to the tagged graphs with the coproduct.

\[
\frac{a_{i}^{m}}{b_{g-k+1}^{1}} = \frac{(k-i)!s_{m}s_{g-m}!(k-i)!4^{k}\left(\frac{2}{m}((m-1)+i)\right)\left(\frac{2}{g}((g-m-1)+k-i)\right)}{s_{g}^{k}4^{k}\left(\frac{2}{g}((g-1)+k)\right)}
\]

in which, the numerator counts the graphs on diagrammatic representation of the topological recursion $\sum_{I \subset K} W_{|I|+1}^{m}(q, I) \times W_{1}^{m-g} \times W_{|I|+1}^{m-g}(\bar{q}, K/I)$, the denominator counts the tagged graphs in the coproduct of $T_{k}^{g}$. Meanwhile

\[
b_{k+1}^{g-1} = \frac{s_{g}^{k}4^{k}\left(\frac{2}{g}((g-1)+k)\right)}{2^{g}(4^{g}-2^{g})s_{g}^{k}4^{k}\left(\frac{2}{g}((g-1)+k)\right)} = \frac{1}{2^{g}(4^{g}-2^{g})}.
\]

By the actual meaning of the ratio, $\frac{a_{i}^{m}}{b_{k+1}^{1}}$ are integers. That is obvious for $b_{k+1}^{g-1}$. \(\square\)

With these formula, we set up the concrete relationship between the Hopf algebraic structure defined on the tagged graphs and the topological recursion. Of course, the numbers of the tagged graphs are much more than the ones of the diagram represents for the topological recursion. This implies that the diagram represents for the topological recursion is very special case of the tagged graphs with a coproduct structure. The tagged graphs could be involved with more mathematical information and structures.

6 Conclusions and discussions

In this article, we propose a new Hopf algebraic structure on the tagged graphs with or without loops, the topological recursion on a spectral curve is reconstructed by considering
the symmetry of the the tagged graphs. In certain sense, the topological recursion on an arbitrary algebraic curve is a special case of the new version Hopf algebraic structure.

The formal hermitian matrix integrals are important models in mathematical physics, it is the zero-dimensional quantum field theory. Hence, it is naturally to consider that the Hopf algebraic structures on the Feynman graphs of quantum field theory obtained by A. Connes and D. Kreimer. The Connes-Kreimer Hopf algebra originates on trees diagrams as well, maybe it has intrinsic relationship with the Hopf algebra on topological recursion.

However, compared with the Connes-Kreimer Hopf algebra on the Feynman diagrams of quantum field theory, the operators here have completely different meaning for the properties of the graphs. As we explore in this paper, the new Hopf algebraic structure on the tagged graphs with loops is very different from the Loday-Ronco Hopf algebra. In a sense, the new Hopf algebra can be regarded as a dual version of the Loday-Ronco Hopf algebra. While it is known that, the Loday-Ronco Hopf algebra is isomorphic to the non-commutative Connes-Kreimer Hopf algebra [2]. Like a quantum group structure, the non-commutate is the essence property for the Hopf algebraic structure introduced in this paper. Therefore It is interesting to investigate the relationships between the new Hopf algebra and the non-commutative Connes-Kreimer Hopf algebra.

It is worth exploring the reasons of a Hopf algebra existence in the context of the topological recursions as well as the Feynman diagram perturbative expansion in quantum field theory, this is helpful to understand the phenomena of topological recursions. Whether the properties on topological recursions can be established in perturbative quantum field theories? There are a lot of problems remained to be clarified yet.

It needs to mention that, in this paper we limit to consider the topological recursion which is defined on algebraic curves with simple branch points. In this case, the recursion relations are represented by diagrams with trivalent vertices [13]. For more general topological recursions, in which there are vertices beyond three-valent vertices [3], maybe there is a Hopf algebraic structure, too. We hope to consider that problem in our future work.

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