On a representation of the Verhulst logistic map

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ABSTRACT

One of the simplest polynomial recursions exhibiting chaotic behavior is the logistic map
\[ x_{n+1} = ax_n(1 - x_n) \]
with \( x_n, a \in \mathbb{Q} : x_n \in [0, 1] \forall n \in \mathbb{N} \) and \( a \in (0, 4] \), the discrete-time model of the differential growth introduced by Verhulst almost two centuries ago (Verhulst, 1838) [12]. Despite the importance of this discrete map for the field of nonlinear science, explicit solutions are known only for the special cases \( a = 2 \) and \( a = 4 \). In this article, we propose a representation of the Verhulst logistic map in terms of a finite power series in the map’s growth parameter \( a \) and initial value \( x_0 \), whose coefficients are given by the solution of a system of linear equations. Although the proposed representation cannot be viewed as a closed-form solution of the logistic map, it may help to reveal the sensitivity of the map on its initial value and, thus, could provide insights into the mathematical description of chaotic dynamics.

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1. Introduction

Let \( n \in \mathbb{N} \) and \( a, x_0 \in \mathbb{Q} \). The function
\[ p : [0, 1] \rightarrow \mathbb{Q} \]
with
\[ p(x_n) = ax_n(1 - x_n) \]
defines a discrete recursion
\[ x_{n+1} = p(x_n) \]
called the Verhulst logistic map [12]. It can be shown that \( x_n \in [0, 1] \forall n \) for \( a \in (0, 4] \) and \( x_0 \in [0, 1] \). Moreover, as the initial value \( x_0 \) determines all future values of the system, Eq. (3) defines a deterministic Markovian system which exhibits chaotic dynamics for all \( a_c < a \leq 4 \) with \( a_c \sim 3.569945672 \ldots \) defined as the edge of chaos.

Despite its simplicity, the logistic map (3) has served since its popularization some 40 years ago (see [6]) as a prototypical dynamical system exhibiting complex chaotic behavior, and must be viewed as one of the most influential recursive equations which helped to shape the field of nonlinear science (for a recent review, see [2]). However, only two explicit closed-form solutions in the parameter space considered here are known to date, namely for the special cases \( a = 2 \) and \( a = 4 \) [10,5], and the general case can only be treated numerically or statistically (e.g., see [4]). For \( a = 4 \), an approach utilizing invariants of associated difference equations and their embedding into a Hilbert space by using Bose operators was explored by Steeb and Hardy [11], and can be applied to higher-dimensional maps. Previous attempts to solve (3)

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explicitly for arbitrary $a$ include multi-dimensional functional integrals [9] and infinite-dimensional matrices [3,8], but did not provide a closed-form solution akin to those known for the aforementioned special cases. Recently, it was argued that such closed-form or smooth solutions cannot exist for generic values of $a$, except for $a$ even and nonzero [13]. However, when numerically exploring the “deviation” of the generic case from the known solution at $a = 4$ as a function of $a$ for any given $n$, a non-trivial yet smooth dependency can be observed [1], suggesting that at least a general solution smooth in $a$, albeit not necessarily closed-form, may exist.

In this article, we make use of an infinite-dimensional matrix operator acting on $\mathbb{Q}^\infty$ to describe the evolution of the logistic map from its initial state (Section 2). The explicit form of this operator is considered (Section 3), which effectively “linearizes” the discrete recursive quadratic map (3) by allowing for an explicit representation in terms of a finite power series in the map’s growth parameter $a$ and initial value $x_0$, with coefficients given in terms of the solution of an exponentially growing system of linear equations (Section 4). Although here also no simple closed-form solution is presented, the proposed representation might shed light on the nature of chaotic systems as well as their mathematical description.

2. Operator representation of the logistic map recursion

Lemma 1. The logistic map (3) is equivalent to the recursive mapping

$$A : \mathbb{Q}^\infty \longrightarrow \mathbb{Q}^\infty$$

with

$$x_{n+1} = A \circ x_n,$$

$n \in \mathbb{N} : n \geq 0$, where $A$ denotes the infinite-dimensional matrix operator

$$a_{ij} = \begin{cases} (-1)^{i-j} \binom{i}{j} a^j & \forall i, j \in \mathbb{N} : i \geq 1, i \leq 2j \\ 0 & \text{otherwise}, \end{cases}$$

$a_{ij} \in \mathbb{Q}$, and

$$x_n = \begin{pmatrix} x_n \\ x_{n+1} \\ \vdots \end{pmatrix} \in \mathbb{Q}^\infty.$$

Proof. To show (6), we make use of the Carleman linearization [3,8]. To that end, consider successive powers of the logistic map (3):

$$x_{n+1} = ax_n(1 - x_n) = ax_n - ax_n^2,$$

$$x_{n+2} = a^2x_n(1 - x_n)^2 = a^2x_n^2 - 2a^3x_n^3 + a^3x_n^4,$$

$$\vdots$$

$$x_{n+m} = a^m x_m(1 - x_m)^m = a^m \sum_{k=0}^m (-1)^k \binom{m}{k} x_k^{m-k} + \sum_{k=m+1}^\infty \binom{m}{k} x_k^{m-k}.$$

Defining the vector $x_n$ according to (7), this system of nonlinear equations can be put into the form

$$x_{n+1,i} = \sum_{j=1}^\infty a_{ij} x_{n+1,j},$$

where $x_{n+1,i}$ denotes the $i$th component of $x_n$ and $a_{ij}$ the components of the matrix

$$A = \begin{pmatrix} a & -a & 0 & 0 & 0 \\ 0 & a^2 & -2a^2 & a^3 & 0 \\ 0 & 0 & a^4 & -3a^3 & 3a^3 & -a^3 \\ 0 & 0 & 0 & a^6 & -4a^4 & 6a^4 \\ 0 & 0 & 0 & 0 & a^8 & -5a^5 \\ \vdots \end{pmatrix}.$$
that is

\[ a_{ij} = \sum_{k=0}^{i} \binom{i}{k} (-1)^k \frac{i}{j-i} a^i \equiv \begin{cases} (-1)^{i-j} \frac{i}{j-i} a^i & \forall i, j \in \mathbb{N} : i \geq 1, i \leq j \leq 2i \\ 0 & \text{otherwise.} \end{cases} \]

The operator \( A \) is an infinite-dimensional strict upper triangular matrix with, for \( a \neq 0 \), non-vanishing entries for \( i \leq j \leq 2i \) (wedge-matrix) and non-zero diagonal elements, hence invertible and diagonalizable. Moreover, for any given \( i \) and finite \( n \), the sum in (8) will always terminate for \( j > 2i \).

With Lemma 1, we can now formulate

**Proposition 1.** The logistic map (3) takes the explicit form

\[ x_n = A^n \circ x_0, \]

\( n \in \mathbb{N} : n \geq 1 \), where \( A^n \) denotes the \( n \)th power of the matrix operator \( A \), i.e.

\[ (a^n)_{ij} = \begin{cases} (-1)^{i-j} \frac{i}{j-i} a^i & \text{for } n = 1 \\ (-1)^{i-j} a^i \sum_{l_1=i}^{2i} \sum_{l_2=1}^{2i} \cdots \sum_{l_{n-1}=l_{n-2}}^{2i} \prod_{p=1}^{n-2} (l_{p+1} - l_p) & \text{for } n \geq 2 \end{cases} \]

\( \forall i, j \in \mathbb{N} : i \geq 1, i \leq j \leq 2^i \). All other \( (a^n)_{ij} \) vanish.

**Proof.** Repeated application of the recursive map (5) yields

\[ x_n = A \circ A \circ \cdots \circ A \circ x_0. \]

As \( A \) is a matrix, these \( n \) successive applications of the operator \( A \) are equivalent to taking the \( n \)th power of \( A \), thus yielding (9). To show (10), one directly evaluates \( A^n \). For \( n = 1 \), the matrix operator (6) itself is obtained. For \( n \geq 2 \), one has

\[ (a^n)_{ij} = \sum_{l_1=1}^{2i} \cdots \sum_{l_{n-1}=1}^{2i} a_{il_1} a_{l_1l_2} \cdots a_{l_{n-1}l_{n-2}} \]

\[ = \sum_{l_1=1}^{2i} \cdots \sum_{l_{n-1}=1}^{2i} a_{il_1} a_{l_1l_2} \prod_{p=1}^{n-2} (l_{p+1} - l_p) \]

\[ = 2^i \sum_{l_1=1}^{2i} \cdots \sum_{l_{n-1}=1}^{2i} (-1)^{l_1+l_2+\cdots+l_{n-1}} \prod_{p=1}^{n-2} (l_{p+1} - l_p) \]

In the last step, the explicit form of \( a_{ij} \) was used. As successive terms in the first product will cancel, the last equation can be simplified, yielding (10). Moreover, due to the wedge-shape of \( a_{ij} \), this expression holds \( \forall i, j \in \mathbb{N} : i \geq 1, i \leq j \leq 2^i \), with all other \( (a^n)_{ij} \) being zero.

We note that Proposition 1 provides the explicit form of the original recursive map (5) in terms of a finite power series in the parameter \( a \), with minimum order \( ni \) and maximum order \( i + \sum_{p=1}^{n-1} 2^p i = 2^n - 1 \) for any given \( i \geq 1 \) and \( n \geq 1 \). However, the number of terms in this power series grows exponentially with \( n \), and a closed-form expression is made difficult due to the presence of \((n-1)\) nested sums over products of binomial coefficients.

### 3. Finite power series expansion of the operator representation

Proposition 1 can be used to represent the logistic map explicitly in terms of a finite power expansion in both the parameter \( a \) and initial value \( x_0 \). To that end, we introduce the following set of coefficients \( v_{ij}^{(n)} \in \mathbb{N} \):
The functions $V_{ij}^{(n)}$ are defined as:

$$V_{ij}^{(n)} := \sum_{l_1=1}^{2i} \sum_{l_2=i+1}^{2i+1} \cdots \sum_{l_{n-1}=l_{n-2}}^{2i} \left( i \atop l_1 - i \right) \left( l_{n-1} - i \right) \prod_{p=1}^{n-2} \left( l_p \atop l_{p+1} - l_p \right) \right)$$  \hspace{1cm} (11)

$$\forall n \in \mathbb{N} : n \geq 2, \; \forall i, j \in \mathbb{N} : i \geq 1, \; i \leq j \leq 2^n i \quad \text{and} \quad k \in \mathbb{N} : (n-1)i \leq k \leq (2^n - 2)i. \; \text{All other } V_{ij}^{(n)} \text{ are zero.}$$

The coefficients defined as such no longer depend on the parameter $a$, and thus yield

$$(a^n)_{ij} = (-1)^{j-i} a^i \sum_{k=(n-1)i}^{(2^n-2)i} V_{kij}^{(n)} a^k$$  \hspace{1cm} (12)

for the $n$th power ($n \geq 2$) of the matrix operator $A$. Interestingly, the coefficients $V_{ij}^{(n)}$ link the dynamics of the logistic map to a particular partitioning of integers, specifically the subset of partitions of a given integer $k$ into sums of $(n-1)$ integers $l_p, \; p \in [1, n-1]$ where $i \leq l_1 \leq 2i$ and $l_{p-1} \leq l_p \leq 2l_{p-1}$ for $p > 1$. As we will see below, the logistic map can be completely formulated in terms of $V_{ij}^{(n)}$.

To further simplify notation, we also define a set of functions $V_{ij}^{(n)}(a) \in \mathbb{Q}$ according to

$$V_{ij}^{(n)}(a) := \sum_{k=(n-1)i}^{(2^n-2)i} V_{kij}^{(n)} a^k$$

$$= \sum_{l_1=1}^{2i} \sum_{l_2=i+1}^{2i+1} \cdots \sum_{l_{n-1}=l_{n-2}}^{2i} a^{\sum_{p=1}^{n-2} l_p} \left( i \atop l_1 - i \right) \left( l_{n-1} - i \right) \prod_{p=1}^{n-2} \left( l_p \atop l_{p+1} - l_p \right) \right)$$  \hspace{1cm} (13)

$$\forall n \in \mathbb{N} : n \geq 2 \quad \text{and} \quad \forall i, j \in \mathbb{N} : i \geq 1, \; i \leq j \leq 2^n i. \; \text{All other } V_{ij}^{(n)}(a) \text{ are identically zero.}$$

Functions defined as such depend explicitly on $a$, and yield

$$(a^n)_{ij} = (-1)^{j-i} V_{ij}^{(n)}(a) a^i$$  \hspace{1cm} (14)

for the $n$th power ($n \geq 2$) of the matrix operator $A$.

**Lemma 2.** The functions $V_{ij}^{(n)}(a)$ obey the recursive algebraic relations

$$V_{ij}^{(n+1)}(a) = \sum_{q=0}^{i} a^{i+q} \binom{i}{q} V_{i+q,j}^{(n)}(a)$$ \hspace{1cm} (15)

$$V_{ij}^{(n+1)}(a) = \sum_{q=i}^{2i} a^q \binom{q}{j-q} V_{q,j}^{(n)}(a)$$ \hspace{1cm} (16)

$$\forall n \in \mathbb{N} : n \geq 2 \quad \text{and} \quad \forall i, j \in \mathbb{N} : i \geq 1, \; i \leq j \leq 2^{n+1} i.$$

**Proof.** To show (15), we use the definition of the functions (13) for $(n+1)$ and sum over $l_1$. After changing the summation variable to $q = l_1 - i$, one obtains

$$V_{ij}^{(n+1)}(a) = \sum_{q=0}^{i} \left\{ \sum_{l_1=i+q}^{2i+q} \cdots \sum_{l_{n-1}=l_{n-2}}^{2i} a^{\sum_{p=1}^{n-2} l_p} \left( i \atop l_1 - i \right) \left( l_{n-1} - i \right) \prod_{p=1}^{n-2} \left( l_p \atop l_{p+1} - l_p \right) \right\}$$

$$= \sum_{q=0}^{i} a^{i+q} \left\{ \sum_{l_1=i+q}^{2i+q} \cdots \sum_{l_{n-1}=l_{n-2}}^{2i} a^{\sum_{p=1}^{n-2} l_p} \left( i \atop l_1 - i \right) \left( l_{n-1} - i \right) \prod_{p=1}^{n-2} \left( l_p \atop l_{p+1} - l_p \right) \right\}$$

$$= \sum_{q=0}^{i} a^{i+q} \binom{i}{q} V_{i+q,j}^{(n)}.$$
To show (16), we first argue that the \((n - 1)\) nested sums in (13) can be decoupled by changing the summation limits for each \(l_p\). Through simple inspection, one infers that the minimal value each \(l_p\) can take is \(i\), and the maximal value cannot exceed \(2^i\). With this, the \(l_p\)-relevant term in the argument of (13) is given by \(\sum_{l_p=1}^{2l_p-1} \binom{l_p}{l_p-l_p-1} \left( \frac{l_p}{l_p+1-l_p} \right)^n l_p\) with \(i \leq l_p-1 \leq 2^{2-i}n\). Given a \(l_p-1\), the argument of this term will vanish if \(l_p < l_p-1\), leaving the first non-vanishing term for \(l_p = l_p-1\) and all other terms with \(i \leq l_p < l_p-1\) zero. Similarly, given a \(l_p-1\), the argument will vanish for \(l_p > 2l_p-1\) due to \(\left( \frac{l_p}{l_p+1-l_p-1} \right) = \left( \frac{l_p-1}{l_p} \right)^n\), leaving the last non-vanishing term for \(l_p = 2l_p-1\) and all other terms with \(2l_p-1 < l_p \leq 2 \cdot 2^{2-i}n\) zero. With this, (13) takes for \((n + 1)\) the form

\[
V_{ij}^{(n+1)}(a) = \sum_{l_1=1}^{2i} \sum_{l_2=i+1}^{2i} \cdots \sum_{l_{n-1}=1}^{2i} \sum_{l_n=l_{n-1}} a \sum_{p=1}^{n-1} l_p \left( l_1 - i \right) \left( l_n - l_{n-1} \right) \prod_{p=1}^{n-1} \left( l_{p+1} - l_p \right). 
\]

Performing the sum over \(l_n\) yields

\[
V_{ij}^{(n+1)}(a) = \sum_{q=i}^{2i} a^q \left( j - q \right) \left\{ \sum_{l_1=1}^{2i} \sum_{l_2=i+1}^{2i} \cdots \sum_{l_{n-1}=1}^{2i} \sum_{l_n=l_{n-1}} a \sum_{p=1}^{n-1} l_p \left( l_1 - i \right) \left( l_n - l_{n-1} \right) \prod_{p=1}^{n-2} \left( l_{p+1} - l_p \right) \right\},
\]

where we relabeled \(l_n \to q\). The term in the curly brackets is identical to \(V_{ij}^{(n)}(a)\) for \(j \to q\), thus proving (16). □

By utilizing the definition of the functions \(V_{ij}^{(n)}(a)\) in terms of \(V_{kij}^{(n)}\) given in Eq. (13), corresponding recursive relations between the coefficients can be found.

**Lemma 3.** The coefficients \(V_{kij}^{(n)}\) obey the recursive algebraic relations

\[
V_{kij}^{(n+1)} = \sum_{p=i}^{2^n-2p} \sum_{q=(n-1)p}^{p+i} \left( \frac{i}{p+i} \right) V_{qij}^{(n)}
\]

(17)

\[
V_{kij}^{(n+1)} = \sum_{p=i}^{2^n-2i} \sum_{q=0}^{2^n-i} \left( \frac{i+q}{j-q-i} \right) V_{n-i+1,q+i}^{(n)}
\]

(18)

\(\forall n \in \mathbb{N} : n \geq 2, \forall i, j \in \mathbb{N} : i \geq 1, i \leq j \leq 2^{n+1}i\) and \(k \in \mathbb{N} : ni \leq k \leq (2^{n+1} - 2)i\).

**Proof.** Both relations can be shown by inserting (13) on both sides of the relations given in Lemma 2, reordering the sums with respect to powers of \(a\), and comparing coefficients of the resulting finite power series in \(a\).

Specifically, inserting (13) into (15) yields

\[
\sum_{k=ni}^{2^{n+1}i} a^k V_{kij}^{(n+1)} = \sum_{q=i}^{2^n-2q} \sum_{k=(n-1)q}^{2^n} a^q k \left( \frac{i}{q-i} \right) V_{kij}^{(n)}
\]

\[
= \sum_{q=ni}^{2^{n+1}i} a^q \left\{ \sum_{k=(n-1)q}^{2^n} a^{-k} \left( \frac{i}{q-i} \right) V_{kij}^{(n)} \right\},
\]

where in the last step we collected on the right-hand side all terms proportional to \(a^l\), \(l \in [ni, (2^n+1)2i]\). Comparing the coefficients for terms proportional to a given power of \(a\) on both sides yields, after renaming the summation variables, relation (17).

The correctness of relation (18) can be shown in a similar fashion. □

We note that relation (17) links all coefficients \(V_{kij}^{(n+1)}\) at step \((n + 1)\) to a sum over coefficients \(V_{qij}^{(n)}\) with \(q \in [(n-1)i, (2^n - 2)i]\), \(p \in [i, 2i]\) and \(p + q = k\) at step \(n\). Equivalently, relation (18) is a recursive equation in \(n\) which links all coefficients \(V_{kij}^{(n+1)}\) at step \((n + 1)\) to a sum over coefficients \(V_{pqij}^{(n)}\) with \(p \in [(n-1)i, (2^n - 2)i]\), \(q \in [i, 2i]\) and \(p + q = k\) at step \(n\).

Introducing for simplicity of notation

\[
V_{kij}^{(n)} := (-1)^{n-1} V_{kij}^{(n)},
\]

we can now formulate
Proposition 2. The logistic map (3) is equivalent to the explicit finite power series

\[ x_n = a \sum_{j=1}^{2^n} \sum_{k=n-1}^{2^n-2} V_{kj}^{(n)} a^j x_0^k \]  
(20)

for \( n \in \mathbb{N} : n \geq 2 \), with the coefficients defined recursively in \( n \) through

\[ V_{kj}^{(n+1)} = \sum_{q=1}^{2^n} (-1)^{j-q} \binom{q}{j-q} V_{k-q,q}^{(n)} \]  
(21)

for all \( k \in [n, 2^{n+1} - 2] \) and \( j \in [1, 2^{n+1}] \) with the initial values

\[ V_{11}^{(2)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, V_{21}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \]
\[ V_{12}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, V_{22}^{(2)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \]
\[ V_{13}^{(2)} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, V_{23}^{(2)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \]
\[ V_{14}^{(2)} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, V_{24}^{(2)} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \]

Proof. Using the operator form (Proposition 1), we first observe that the first row in (9) yields the expression linear in \( x_n \), thus

\[ x_n = a \sum_{j=1}^{2^n} \sum_{k=n-1}^{2^n-2} V_{kj}^{(n)} a^j x_0^k \]

Here, we made use of the upper-triangular wedge-like structure of the operator matrix \( A \) and its powers in order to truncate the summation over \( j \). Inserting the definition of \( V_{kj}^{(n)} \), Eq. (19), yields (20).

The recursive form of the coefficients \( V_{kj}^{(n)} \) can be obtained from (18) for \( i = 1 \) and using (19). Finally, the initial values are deduced from definition (11) using \( i = 1 \).

Due to their original definition as sums over products of binomial coefficients, Eqs. (11) and (19), the coefficients \( V_{kj}^{(n)} \) are integers with rapidly growing absolute value for increasing \( n \). Moreover, the number of these coefficients for a given \( n \) is exponentially growing with \( n \), but the recurrence (21) is sufficient to calculate all \( 2^{n+1} (2^{n+1} - n - 1) \) coefficients \( V_{kj}^{(n+1)} \) from the \( 2^n (2^n - n) \) coefficients at step \( n \). To illustrate both points, we list in Table 1 all non-zero \( V_{kj}^{(n)} \) up to \( n = 4 \) and in Table 2 all coefficients for \( n = 5 \).

4. “Linearized” representation of the logistic map

Although the representation of the logistic map in Proposition 2 is explicit in terms of a finite power-series in \( a \) and \( x_0 \), the coefficients are given in form of a linear recursive relation with an exponentially growing number of terms for increasing \( n \). As the number of terms in this recursion depends on the step \( n \), classical methods, such as the generating function approach [14], cannot be employed to obtain an explicit closed-form expression for \( V_{kj}^{(n)} \). However, using the well-known non-trivial fixed-points of the original system

\[ x_0 = \frac{a - 1}{a} \]

(22)

for any given \( a \in (0, 4] \), we can represent the recursion (21) in terms of a system of linear equations. Relabeling \( V_{kj}^{(n)} \rightarrow \psi_{kj}^{(n)} \) with \( q = j - 1)(2^n - n) + k - n + 2, q \in [1, 2^n (2^n - n)] \), we have

Proposition 3 (“Linearized” Representation). The logistic map (3) is equivalent to the explicit finite power series

\[ x_n = a \sum_{j=1}^{2^n} \sum_{k=n-1}^{2^n-2} \psi_{kj}^{(n)} a^j x_0^k \]  
(23)
Table 1
Coefficients \( V^{(n)} \) for \( n = \{2, 3, 4\} \). The gray boxes indicated the coefficients used to calculate \( V^{(5)}_{22,14} \) as an illustrative example of the recursive relation (21) (see Table 2).

| \( n \) | \( k \) | \( j \) |
|-------|------|------|
| 2     | 1    | -1   |
|       | 2    | -1   |
| 3     | 1    | -1   |
|       | 2    | -1   |
|       | 3    | -1   |
|       | 4    | -1   |
|       | 5    | -1   |
|       | 6    | -1   |
| 4     | 1    | -1   |
|       | 2    | -1   |
|       | 3    | -1   |
|       | 4    | -1   |
|       | 5    | -1   |
|       | 6    | -1   |
|       | 7    | -1   |
|       | 8    | -1   |
|       | 9    | -1   |
|       | 10   | -1   |
|       | 11   | -1   |
|       | 12   | -1   |
|       | 13   | -1   |
|       | 14   | -1   |

for \( n \in \mathbb{N} : n \geq 2 \), with the coefficients defined as the solution to the system of linear equations given by

\[
C_p^{(n)} = D_{pq}^{(n)} V_q^{(n)},
\]

where

\[
D_{pq}^{(n)} = a_p^{n+q-1 \mod (2^n-n)} \left( \frac{a_p - 1}{a_p} \right) \left( \frac{a_{p+1} - 1}{a_{p+1}} \right)^{\frac{q-1}{2^n-n} \mod (2^n-n)} + 1
\]

and

\[
C_p^{(n)} = a_p - 1
\]

for \( 2^n(2^n-n) \) different non-trivial fixed-points \( \{a_p \in \mathbb{Q}, a_p \in (0, 4]; p \in [1, 2^n(2^n-n)]\} \) of the logistic map.

**Proof.** Eqs. (24) and (25) follow straightforward by successively inserting fixed-points (22) for a chosen \( a_p \) into the left-hand and right-hand side of (23). \( \square \)

**Proposition 3** provides a fully linearized representation of the Verhulst logistic map on the expense of the size of the associated system of linear equations. However, although, in principle, (24) can be explicitly solved, it is of little use practically, especially for larger \( n \).

5. Conclusion

In this paper we have proposed a “linearized” representation of the Verhulst logistic map, a second order recursive relation exhibiting both periodic and chaotic behavior depending on its parameter \( a \). To that end, we first made use of the Carleman linearization and expresses the logistic map explicitly in terms of a matrix operator acting on an infinite-dimensional \( \mathbb{Q} \)-valued vector space (Proposition 1). The evolution of the logistic map is here given through successive powers of this matrix operator acting upon an initial state vector \( x_0 \).

Next, by using the explicit form of this operator, we expressed the logistic map explicitly in terms of a finite power series in the initial state value \( x_0 \) and the map’s parameter \( a \) (Proposition 2). Although the obtained expansion cannot be viewed as
a closed-form solution for generic $a$, it provides a finite representation, smooth in both the parameter $a$ and initial value $x_0$. The coefficients of this series $V^{(4)}_{k,j}$, defined in (19) together with (11), are $\mathbb{Z}$-valued numbers with rapidly growing absolute values involving a subset of partitions of natural numbers and obeying a set of recursive algebraic relations (Lemma 3).
Although this power series expansion is of little practical use for numerical calculations due to the exponentially growing number of coefficients with increasing step \( n \), it provides an insight into the nature of the original chaotic recursion. In particular, the order of the power series grows exponentially with step \( n \), thus demonstrating explicitly the sensitivity of the system to both its parameter \( a \) as well as initial condition \( x_0 \), the defining characteristic of chaotic systems. Moreover, for any given order in \( x_0 \) and \( a \) in the power series, the coefficients indirectly depend on \( n \). This effectively leads to a “mixing” of contribution of the various orders in the power expansion for successive steps, as illustrated in Tables 1 and 2 (gray boxes).

The final representation (Proposition 3) makes use of the fixed-points of the logistic map, leading to a formal representation of the coefficients \( V_{kj}^{(n)} \) in terms of solutions of a system of linear equations (24). Although, in principle, a solution to this system can be found, it is of little or no practical interest due to the size of the system. However, this representation can be viewed as an effective “linearization” of the chaotic system in question, a linearization achieved at the expense of an exponentially growing size of the linear system in \( V_{kj}^{(n)} \).

Although the proposed representation can be viewed as an explicit form of the Verhulst logistic map, the prospects for numerical application are challenging. Numerical evaluation of this representation will necessarily involve either calculating recursively an exponentially growing number of coefficients, calculating the \( n \)th power of a infinite-dimensional matrix, or solving an exponentially growing system of linear equations. However, while the “classical” double-precision implementation of the logistic map can be viewed with algorithmic complexity \( O(n) \) in time and \( O(1) \) in memory, we note that in order to avoid round-off errors in any practical implementation [7, 15], arbitrary precision methods must be employed, thus implicating exponential complexity into the problem. The representation presented here has the full precision of an analytic expression, hence allowing to evaluate the logistic map, in principle, to arbitrary precision. Moreover, we hope that this representation sheds some light on the nature of chaotic systems, and potentially paves the way for a discrete mathematics of large numbers which might be more suitable for describing nonlinear or even chaotic systems.

We note, finally, that the proposed representation of the Verhulst logistic map is applicable to general polynomial recursions, thus potentially allowing for a formulation of such maps within a unified mathematical framework.

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References

[1] A representative plot illustrating this point as well as Mathematica for exploration of the logistic map script can be downloaded at: http://newscienceportal.com/MLR/publications/A31/A31.php.
[2] M. Ausloos, M. Dirickx, The Logistic Map and the Route to Chaos, Springer, 2006.
[3] K. Kowalski, W.-H. Steeb, Nonlinear Dynamical Systems and Carleman Linearization, World Scientific, 1991.
[4] M. Little, D. Heesch, Chaotic root-finding for a small class of polynomials, J. Difference Equ. Appl. 10 (2004) 949–953.
[5] E. Lorenz, The problem of deducing the climate from the governing equations, Tellus 16 (1964) 1–11.
[6] R.M. May, Simple mathematical models with very complicated dynamics, Nature 261 (1976) 459–467.
[7] J.A. Oteo, J. Ros, Double precision errors in the logistic map: statistical study and dynamical interpretation, Phys. Rev. E 76 (2007) 036214.
[8] S. Rabinovich, G. Berkolaiko, S. Buldyrev, A. Shehter, S. Havlin, “Logistic map”: an analytical solution, Physica A 218 (1995) 457–460.
[9] S. Rabinovich, V. Malysutin, S. Havlin, An explicit solution for the logistic map, Physica A 264 (1999) 222–225.
[10] E. Schröder, Über iterierte Funktionen, Math. Ann. 3 (1870) 296–322.
[11] W.-H. Steeb, Y. Hardy, Chaotic maps, invariants,bose operators, and coherent states, Internat. J. Theoret. Phys. 42 (2003) 85–88.
[12] P.-F. Verhulst, Notice sur la loi que la population poursuit dans son accroissement, Corresp. Math. Phys. 10 (1838) 113–121.
[13] E.W. Weisstein, Logistic Map, From MathWorld—A Wolfram Web Resource. http://mathworld.wolfram.com/LogisticMap.html.
[14] H.S. Wilf, Generatingfunctionology, Academic Press, 1994.
[15] M. Yabuki, T. Tsuchiya, Double precision computation of the logistic map depends on computational modes of the floating-point processing unit, arXiv:1305.3128v1 [nlin.CD].