Graviton production from D-string recombination and annihilation

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Fundamental superstrings (F-strings) and D-strings may be produced at high temperature in the early Universe. Assuming that, we investigate if any of the instabilities present in systems of strings and branes can give rise to a phenomenologically interesting production of gravitons. We focus on D-strings and find that D-string recombination is a far too weak process for both astrophysical and cosmological sources. On the other hand if D-strings annihilate they mostly produce massive closed string remnants and a characteristic spectrum of gravitational modes is produced by the remnant decay, which may be phenomenologically interesting in the case these gravitational modes are massive and stable.

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I. INTRODUCTION

After the dismissal of cosmic string as candidate seeds for the formation of structure, they received a fair share of attention in the last few years from both the theoretical and observational perspective.

Scenarios of inflation involving branes have become popular recently [1,2], because of the appealing feature of providing a working model of inflation based on a fundamental theory on one side, and because they can represent a phenomenological handle for string theory on the other. In this kind of scenarios the production of D-strings as cosmic defects, by Kibble
mechanism, is a generic prediction\textsuperscript{3}, even if their (meta-)stability is not at all ensured\textsuperscript{4}.

In general, cosmic defects like monopoles, strings and walls can be problematic as their red-shift is slower than the one of radiation: if present in the early Universe and if cosmologically stable they tend to overclose the Universe, unless they are diluted by some release of entropy into the Universe after their creation. While this is certainly true for stable monopoles, whose annihilation rate is negligible for standard choice of parameters\textsuperscript{3}, in the standard scenario cosmic strings loose energy by chopping off small loops and finally reach \textit{scaling} solutions on which they make up a constant fraction of the energy density of the Universe during both the era of radiation and matter domination\textsuperscript{6}. This allows the density of long strings to be always roughly few per Hubble volume, while the small loops disappears by emitting gravitational radiation.

The situation is less clear for domain walls, as the existence of scaling solutions and the time required for the solution to fall into these kind of attractors in presently under investigation\textsuperscript{7}.

In this work we concentrate on cosmic strings. In particular we want to identify cosmic strings with the D-strings of string theory.

The production of lower dimensional defect can result for instance from Dp Dp-brane annihilation, which will produce D(p − 2n)-branes as remnants. In a type II theory Dp-branes are BPS only if p has a definite parity (even in IIA, odd in IIB), Dp-branes with the “wrong” dimensionality will decay immediately (on time scale given by the string scale).

Actually in type I string theory non BPS D0-branes can be stable, because the unstable mode is projected out by the orientation projection taking type IIB into type I. Still it has to be remember that the presence of a type I excitation spectrum is related to the presence of orientifolds. Even if the starting setup is endowed with a space filling orientifold, since some of the dimensions may get small during cosmological evolution, T-dualities over those directions getting smaller than the string length confine the orientifold hyperplane to subspace of strictly positive codimension, so the bulk physics, away from the orientifolds, will be again unoriented, i.e. of type IIA or IIB. Thus even taking a full type I configuration as a starting point, the general assumption that only the BPS Dp-branes are stable is justified on general grounds.

Strictly speaking BPS D-strings are present in type IIB and type I theories and not in type IIA, but we can consider Dp-branes with arbitrary p wrapped around (p − 1) dimensional
cycles to be D-strings from the 4 dimensional point of view, allowing a variety of values for the tension of the effective D-string. For instance, the tension of a D$p$-brane is $\mu_p \propto \alpha'^{(p+1)/2} g_s^{-1}$, (being $\alpha' \equiv l_s^2$ the square of the string fundamental length) and assuming that a D$(2 + n)$-brane wraps around a $n$ dimensional flat torus with radii $R_1, \ldots, R_n$ the effective tension $\mu$ of the resulting 4 dimensional D-string is

$$\mu = \frac{1}{2\pi l_s^2 g_s} \prod_{i=1}^{n} \frac{R_i}{l_s}.$$  

Moreover the internal dimensions do not need to be flat, but can be warped (à la Randall-Sundrum [8], for instance), so that their tension can be redshifted (or blue-shifted) by huge factors and in principle it can take any value.

One important and not completely settled issue is the stability one. The (effective) D-strings are generically created but for them to play a role in cosmology they must be stable, or at least metastable over cosmological time scale.

An instability already pointed out in [9] is the following. D-strings couple electrically to a 2-form, whose dual in 4 dimensions is a scalar, axion-like field. Once supersymmetry is broken the axion acquires a periodic potential with several degenerate minima, so the axion is expected to condense to different minima in different regions of space, separated by domain walls. Finite domain walls are bound by D-strings, and the domain wall tension, unless it is exceedingly small, will make D-strings shrink and collapse.

Another source of instability is breakage, which is present whenever systems of D$p$ and D$q$-branes are present at the same time. Let us define $\nu$ as the number of dimensions in which the D$p$ branes are extended and the D$q$ are not or vice-versa, i.e. the total number of Neumann-Dirichlet (ND) plus DN dimensions. If $\nu = 2$ the axion symmetry is Higgsed, there are no axion domain walls, so the branes can break. Strings can in general break or annihilate if they meet a brane with $\nu = 2$, if $\nu = 4$ or $\nu = 8$ the system is stable and supersymmetric, but we fall back into the domain wall problem when supersymmetry is broken (if $\nu = 6$ branes repel each others).

Since the branes come with orientation, meeting a brane with opposite orientation they can also annihilate, whereas if they have exactly the same orientation and $\nu = 0, 4, 8$ the full system is still BPS. Of course intermediate situation are possible, when branes meet at angle. A description of the microscopic mechanism leading to recombination is given in [10, 11].
The stability issue is thus far from settled but it can be solved in specific models. Here we will focus on the effects triggered by two of the aforementioned sources of instability, the meeting of a D-string at angles and the annihilation of D-strings. This kind of processes can be addressed in the context of string theory and the dynamic of the instability can be studied through the time evolution of a specific tachyonic mode of the system [12]. Since everything is coupled to gravity such instabilities can be expected to give rise to some gravitational radiation, which we study quantitatively in the present work. Modeling the string sources with an effective energy momentum tensor the amount of emitted gravitational radiation can be computed, but a negligible result is found.

Actually, rather than the emission by a time dependent dynamics we can consider the closed string remnant left by a brane-anti brane annihilation [13], whose decay can be an interesting source of gravitational radiation, as we will show. It is also worth noticing that no cosmic string has ever been observed directly and that the cosmic microwave background constrains cosmic string tension $\mu$ to be $G_N \mu < 10^{-7}$, where $G_N$ is the Newton gravitational constant.

The outline of the paper is the following. In sect. II the salient features of the analysis of [10] of the D string reconnection are reproposed and the GW emission in such a process is computed, then generalised to a cosmological setup like in [14]. In sect. III the annihilation process between a D string and a $\overline{D}$ string is analysed, finding again a negligible gravitational radiation emission. In sect. III B we study the gravitational radiation emitted by the decay of the massive closed strings which constitute the remnant of the D $\overline{D}$-brane annihilation and conclusions are finally drawn in sect. IV.

II. GRAVITATIONAL RADIATION FROM STRING-STRING RECONNECTION

The reconnection process of D-strings meeting at an angle and zero relative velocity has been studied in [10]. Adding a relative velocity between the strings boils down to introducing a non one (and non zero) probability of interaction [11, 15]. Stacks of parallel D-branes admit a low energy description in terms of a $U(N)$ super Yang Mills theory, where $N$ is the number of D-branes into play. The bosonic excitations of the F-strings stretched between the D-strings are gauge vectors (scalars) from the point of view
of the D-string Poincaré group, if polarised along (orthogonally to) the brane. The system is described by a 1+1 dimensional Yang Mills theory coupled to as many scalars as transverse direction. The scalar fields parametrise the displacements of the brane in the orthogonal directions.

A. Almost parallel D-strings

1. Reconnection dynamics

Following the analysis of [10], let us consider a system made of 2 almost parallel D-strings at an angle $\gamma$. The low energy dynamics of the system is described by the action

$$S = -\frac{\mu}{2} (2\pi\alpha')^2 \text{Tr} \int dxdt \left( \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} D_\mu \Phi_i D^\mu \Phi^i \right),$$

where $\mu = 0, 1$ and $i \in \{2 \ldots d\}$ runs over the dimension orthogonal to $t$ and $x$.

In order to describe two D-strings at an angle $\gamma$ the previous action (1) has to be expanded around the background $\Phi_i^{(b)}(t), A_{\mu}^{(b)}$

$$\Phi_i^{(b)} = \delta_{i,2} qx \sigma^3, \quad A_{\mu}^{(b)} = 0,$$

with $q \equiv \tan \gamma/(\pi\alpha')$ and $\sigma^i$ the Pauli matrices which form the algebra of $SU(2)$, plus the fluctuations $\Phi^{(f)}, A_{\mu}^{(f)}$

$$\Phi_i^{(f)} = \delta_{i,2} \left( \xi(t, x) \sigma^3 + \phi(t, x) \sigma^1 \right), \quad A_{\mu}^{(f)} = \delta_{\mu,1} a(t, x) \sigma^2.$$

Figure 1: Two D-branes at an angle $\gamma$ before and after recombination.
To lowest non trivial order in interactions, the resulting Lagrangian density is,

\[ L \simeq \frac{1}{2} \dot{a}^2 + \frac{1}{2} \left( \dot{\phi}^2 + \dot{\xi}^2 \right) - \frac{1}{2} \left( \phi'^2 + \xi'^2 \right) - \frac{q^2}{2} - q \phi' (qx + \xi) + a \phi (q + \xi') - \frac{1}{2} a^2 (q^2 x^2 + 2qx) , \]  

(4)

and it takes to the equations of motion

\[ \ddot{a} + qx \phi' - q \phi + aq^2 x^2 = 0 , \]
\[ \ddot{\phi} - \phi'' - a' qx - 2aq = 0 , \]
\[ \ddot{\xi} - \xi'' + a' \phi + 2a \phi' + a^2 qx = 0 , \]  

(5)

where it is understood that a dot stands for a time derivative and a prime for a derivative with respect to \( x \).

We note that consistency of the equations requires that \( a, \phi \) are of the same order, while \( \xi \) is \( O(a^2) \), so that the first two equations of (5) can be solved for \( a \) and \( \phi \) and then the result plugged into the equation for \( \xi \). This justifies a posteriori the choice made in (4) to drop \( O(a^3) \) terms as well as \( O(a \xi^2) \).

For the coupled \( a, \phi \) system it is convenient to take the ansätze

\[ a(t, x) = \sum_{n>0} A_n e^{im \xi} a_n(x) , \]
\[ \phi(t, x) = \sum_{n>0} F_n e^{im \xi} a_n(x) , \]  

(6)

where the equation of motions force the condition \( m^2_n = (2n - 1)q \).

Both \( a \) and \( \phi \) have an unstable lowest lying mode, with a Gaussian profile, so that neglecting the positive mass squared mode the solution can be written as

\[ a(t, x) = A_0 e^{\sqrt{q}t} e^{-qx^2/2} = \phi(t, x) . \]  

(7)

The expression for \( \xi \) is slightly more complicated, but again its non trivial part is concentrated in an area of size \( q^{-1/2} \) around \( x = 0 \)

\[ \xi(t, x) = A_0^2 e^{2\sqrt{q}t} \frac{e}{2} \sqrt{\frac{\pi}{q}} \left[ e^{2\sqrt{q}x} \text{erf}(\sqrt{q}x + 1) + e^{-2\sqrt{q}x} \text{erf}(\sqrt{q}x - 1) - 2 \sinh(2\sqrt{q}x) \right] . \]  

(8)

In the previous (5) the integration constant of the solution to the homogenous part of the \( \xi \) equation is fixed by requiring that the \( \lim_{x \to \pm \infty} \xi = 0 \) independently of \( t \).

To give a geometrical interpretation we remind that the \( \Phi_i \)'s keep trace of the displacements of the D-strings in the orthogonal directions, their eigenvalues are proportional to
the locations of the D-branes in the $i$-th direction. In our case the distance $d_y$ along the $y$ direction between the 2 D-strings is $d_y = 2 \times 2\pi \alpha' \Phi_y$ where $\Phi_y$ is given by

$$\Phi_y = \frac{1}{2} \begin{pmatrix} qx + \xi & \phi_0(x) \\ \phi_0(x) & -qx - \xi \end{pmatrix} \to \frac{1}{2} \begin{pmatrix} \sqrt{q^2 x^2 + 2qx\xi + \phi_0^2} & 0 \\ 0 & \sqrt{q^2 x^2 + 2qx\xi + \phi_0^2} \end{pmatrix}, \quad (9)$$

where in the previous section a gauge transformation has been performed to diagonalise the $\phi$ field and only the first order in $\xi$ has been kept. The dynamics of the Yang Mills system tells the D-strings to recombine smoothly around the intersection point. This analysis neglects in the Lagrangian terms of the order $a^4$ and higher, thus it is valid for

$$A_0 e^{\sqrt{q} t} e^{-aq^2/2} < \sqrt{q}. \quad (10)$$

By a shift in time we can set $A_0 = \sqrt{q}$ (as we will do in the rest of the paper), so the solution (7) can be expected not to be valid for $t > 0$.

Here we neglected the effect of a relative velocity $v$ between the D-strings, which affects the probability of the reconnection taking place. It has been studied in [11], with the result that the probability $P \sim 1$ if $v \ll 1$, which takes to the condition

$$\sqrt{q} e^{\sqrt{q} t} \ll \sqrt{q}. \quad (11)$$

Again this restriction suggests that this picture is a good description only for negative times.

### 2. Energy momentum tensor

Having understood the dynamics of the recombination via the effective action of the lightest string modes, it is then possible to compute the energy momentum tensor of the system to estimate the amount of gravitational radiation emitted during the process. We take the definition

$$T^\mu_\nu = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^\mu_\nu}. \quad (12)$$

Explicitly on the solutions (7) the $t, x$ components of the energy momentum tensor are

$$T_\mu^\nu = \mu 2\pi^2 \alpha'^2 \begin{pmatrix} -q^2/2 - q\xi' & -q\xi \\ -q\xi & a^2 q - q\xi' - q^2/2 \end{pmatrix} \times [\delta (y - qx\pi\alpha') + \delta (y + qx\pi\alpha')] \delta(z), \quad (13)$$
where it is understood that the strings are immersed in 4-dimensional space-time.

In eq. (13) we recognise the constant contribution from the background configuration of the tilted D-strings, which is T-dual to a constant magnetic field on a D2-brane. The time dependent part is the source of the gravitational radiation. This energy momentum tensor is conserved, but it would not have been conserved without the inclusion of ξ in the ansatz (3), like in [10], where the solution considered have ξ = 0. This should not come as a surprise as forcing the solution (2) with ξ = 0 would imply an explicit breaking of Lorentz invariance, which would necessarily let no right to expect a conserved energy-momentum tensor. The inclusion of ξ turns the Lorentz symmetry breaking of the background into a spontaneous one, thus allowing the conservation of the energy momentum tensor.

By linearising general relativity over a flat background

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$

we have

$$\Box h_{\mu\nu} = 16\pi G N T_{\mu\nu},$$

where $h_{\mu\nu} \equiv h_{\mu\nu} - 1/2\eta_{\mu\nu}h^\rho_\rho$ has been defined and the condition $\partial_\mu h^\mu_\nu = 0$ has been imposed, consistently with the conservation of the energy momentum tensor. Following [16] the energy momentum tensor can be Fourier transformed both in space and time

$$\tilde{T}_{\mu\nu}(\omega, \vec{k}) = \int \tilde{T}_{\mu\nu}(t, \vec{x}) e^{i(\omega t - \vec{k} \cdot \vec{x})} dt d^3x,$$

(16)

to obtain the formal solution to (15) as

$$\tilde{\tilde{h}}_{\mu\nu}(t, \vec{x}) = 4G_N \int d^3x' \frac{1}{|\vec{x} - \vec{x}'|} \int \frac{d\omega}{2\pi} \frac{d^3k}{(2\pi)^3} \tilde{T}_{\mu\nu}(\omega, \vec{k}) e^{-i\omega(\tau - |\vec{x} - \vec{x}'|) - \vec{k} \cdot \vec{x}'}$$

$$\approx 2G_N \int \tilde{T}_{\mu\nu}(\omega, \vec{n}) e^{i\omega(t-x)} d\omega,$$

(17)

where $\vec{n}$ is the unit vector pointing into the direction of $\vec{x}$ and the local wave zone approximation $|\vec{x} - \vec{x}'| \approx x - \vec{x}' \cdot \vec{n}$ has been used, with $x \equiv |\vec{x}|$.

Given (13) and the explicit solution (7)

$$a(t, x) = \sqrt{q} e^{\sqrt{q} t} e^{-q x^2/2} \text{ for } t < 0$$

(18)

one can compute $\tilde{h}_{11}$, say, to be

$$\tilde{h}_{11}(t, \vec{x}) = \frac{4}{x} G_N \mu q^{3/2} \alpha^2 \pi^{3/2} \int d\omega \frac{1}{2\sqrt{q} + i\omega} e^{-i\omega(t-x)} \left[ e^{-\frac{2}{\sqrt{q}} \delta_+} + e^{-\frac{2}{\sqrt{q}} \delta_-} \right]$$

$$= \frac{4}{x} G_N \mu q^{3/2} \alpha^2 \pi^{5/2} e^{2\sqrt{q}(t-x)} f(t - x, \theta, \phi),$$

(19)
valid for $t < x$, where $f(t - x, \theta, \phi)$ takes account of the angular dependence of the radiation and its explicit form is

$$f(u, \theta, \phi) \equiv e^{\delta^2} \left[ 1 + \text{erf} \left( \frac{\sqrt{q}}{\delta_+} u + \delta_+ \right) \right] + e^{\delta^2} \left[ 1 + \text{erf} \left( \frac{\sqrt{q}}{\delta_-} u + \delta_- \right) \right], \quad (20)$$

where $\delta_{\pm} = | \sin \theta (\cos \phi \pm q \pi \alpha' \sin \phi) |$ and $\theta, \phi$ are the usual polar angles. Actually the details of the angular distribution are of little importance in view of the intensity of the radiation, as we will now show.

To compare with experimental sensitivity it is convenient to express the previous result in terms of the (double sided) spectral density $S_h(f)$ which for a short burst of (Fourier transformed) amplitude $\tilde{h}(f)$ and duration $\Delta t$ reduces to

$$S_h(f) \simeq \frac{\tilde{h}^2(f)}{\Delta t}. \quad (21)$$

From the computation (19) we can estimate a spectral density function, neglecting the angular dependence in the exponential,

$$S_h(f) \simeq 10^3 \left( \frac{G_N \mu}{x} \right)^2 \frac{q^{5/2} \alpha'^4}{x^2} \frac{e^{-\frac{x^2}{4q}}}{1 + \omega^2/(4q)} \simeq 10^{-126} \text{Hz}^{-1} (\tan \gamma)^{5/2} \frac{e^{-\frac{x^2}{4q}}}{1 + \omega^2/(4q)} \left( \frac{G_N \mu}{10^{-7}} \right)^2 \left( \frac{\alpha'}{\text{TeV}^{-1}} \right)^{3/2} \left( \frac{x}{1 \text{Mpc}} \right)^{-2}, \quad (22)$$

where redshift factors has been neglected. The typical signal frequency is $\simeq \sqrt{q}/(\pi (1 + z))$. For presently working experiments $S_h \gtrsim 10^{-42} \text{Hz}^{-1}$, so the result (22) is ridiculously small. This is due to the inherently microscopic origin of the dynamics, whereas to generate a consistent amount of radiation a coherent motion of a macroscopic mass is needed. Let us now investigate if a cosmological population of such recombining D-strings may help in enhancing the signal.

3. **Cosmological rate**

One can expect that several processes like the one described in the previous section take place in the Universe and that the signal resulting from the sum of all of them is highly enhanced with respect to the one found in (22).

We assume the Universe evolution is the standard one after cosmic string production, i.e. radiation domination followed by matter domination. Very sketchily, simulations of
the evolution of a network of strings show that no matter what is the initial density of
strings, within few Hubble times they reach a scaling solution characterised by few long open
strings per Hubble volume whose length stretches with the horizon, and a large number of
small closed strings [6]. A network of long open strings has a negative effective pressure
\( p_{\text{op}} = -\frac{\rho_{\text{op}}}{3} \), thus it redshifts more slowly than both radiation and non-relativistic matter,
but it does not overclose the Universe as open strings loose energy by chopping off small
loops which eventually decay by emission of gravitational radiation. At an effective level one
can write down a continuity equation which takes into account both the Universe expansion
and the energy lost into loops
\[
\dot{\rho}_{\text{op}} = -3H(\rho_{\text{op}} + p_{\text{op}}) - \frac{\lambda \rho_{\text{op}}}{L} = -2H \rho_{\text{op}} - \frac{\lambda \rho_{\text{op}}}{L},
\]
where \( 0 < \lambda < 1 \) is a number taking account of the possibility of a non-unity value of the
probability of reconnection. For a generic Hubble parameter \( H = \beta/t \) and \( \rho \propto \mu/L^2 \) eq. (23)
adopts the scaling solution
\[
L = \frac{\lambda}{2(1 - \beta)} t,
\]
allowing the string network to track the dominant energy source of the Universe and to
make up a constant fraction of it.

String loops have typical size \( l_{\text{cl}} \) and density \( n_{\text{cl}} \) roughly given by
\[
l_{\text{cl}} \simeq \alpha t, \quad n_{\text{cl}} \simeq \alpha^{-1} t^{-3}.
\]
The value of \( \alpha \) is not known, as an indication we can take the value given in [19] \( \alpha \simeq \frac{\Gamma}{G_N\mu} \)
where \( \Gamma \approx 50 \), so that it can well be that \( \alpha \ll 1 \) (we remind that smoothness of cosmic
microwave background requires \( G_N\mu < 10^{-7} \)). A relation similar to (23) between \( l_{\text{op}}, n_{\text{op}} \)
and \( t \) can be expected to hold for long, open strings with \( \alpha \simeq 1 \). Given (23) the number of
reconnection events between open strings per unit of spacetime volume can be estimated to be
\[
\nu_{\text{op-op}}(t) = n_{\text{op}}^2 v \sigma_{\text{op-op}} \simeq Pt^{-4},
\]
where the cross section \( \sigma_{\text{op-op}} \) has been estimated as \( \sigma_{\text{op-op}} \simeq P t_{\text{op}}^{2} \), with \( P \) the reconnection
probability, see (11), and relativistic velocities has been considered. Had we considered
instead the interaction between open and closed strings we would have obtained
\[
\nu_{\text{op-cl}}(t) = n_{\text{op}} n_{\text{cl}} v \sigma_{\text{op-cl}} \simeq P t^{-4} \alpha^{-1/3} ,
\]
which is slightly higher than \(\text{(26)}\), where it has been used that the average distance traveled by a string before meeting another one is \(\sim n_\text{cl}^{-1/3}\).

We want now to relate the time variable \(t\) to the redshift factor \((1+z)\) as in \(\text{(14)}\). Let us consider the usual FRW metric

\[
d s^2 = -dt^2 + a^2(t) (dr^2 + r^2 d\Omega^2) \tag{28}
\]

the comoving distance \(r\) between an object emitting light at a time \(t(z)\), where \((1+z)\) denotes the redshift factor, and the observer at present time \(t_0\) is given by

\[
r(z) = \int_{t(z)}^{t_0} \frac{dt'}{a(t')} . \tag{29}
\]

Parameterising the scale factor as

\[
a = \begin{cases} 
a_{\text{eq}} & \text{if } t_i < t < t_{\text{eq}}, \\
a_0 \left(\frac{t}{t_0}\right)^{2/3} & \text{if } t_{\text{eq}} < t < t_0 ,
\end{cases} \tag{30}
\]

where \(t_i\) mark the epoch of the beginning of radiation domination and \(t_{\text{eq}}\) denotes the time of matter-radiation equality, the redshift-comoving distance relation is obtained

\[
r(z) = \frac{3t_0}{a_0} \times \left\{ \frac{(1+z_{\text{eq}})^{-1/2}}{(1+z)^{1/2}} \left[ (1+z_{\text{eq}})^{1/2} - 1 + \frac{2}{3} \frac{z - z_{\text{eq}}}{1+z} \right] \right\}
\quad \text{for } z_i > z > z_{\text{eq}}
\quad \text{and } z_{\text{eq}} > z > 0 , \tag{31}
\]

which, neglecting numerical factors, can be well interpolated by

\[
r(z) \simeq t_0 \frac{z}{a_0 (1+z)} . \tag{32}
\]

The previous formulae allow us to write the volume element as

\[
d V(z) = 4\pi a^3(t_0) r^2 dr \simeq 10^2 t_0^3 \frac{z^2}{(1+z)^{13/2}} \frac{1}{(1+z/z_{\text{eq}})^{1/2}} dz . \tag{33}
\]

Let us introduce \(N(z)\), the reconnection rate for events occurring at a given redshift \(z\). Its differential is given by

\[
d N = (1+z)^{-1} v(z) z \frac{dV}{dz} d\ln z \simeq \frac{10^2}{t_0^3} \frac{z^3 (1+z/z_{\text{eq}})^{3/2}}{(1+z)^{3/2}} d\ln z , \tag{34}
\]

where the time-redshift relation

\[
t(z) = t_0 (1+z)^{-3/2} (1+z/z_{\text{eq}})^{-1/2} \tag{35}
\]
has been used. The observer at \( r = 0 \) sees a burst with duration \( \Delta t_0 = \Delta t (1 + z) \), being \( \Delta t \) the proper time burst duration. By comparing \( (\Delta t_0)^{-1} \) with \( dN/d \ln z \) we can verify if the signals actually overlap to make a continuous background, or if they are resolved in time. Since \( N(z) \) is a growing function of \( z \), it turns out that for \( z_* > z \) the signal is made by a collection of bursts whereas signals originated at higher redshift than \( z_* \) combine into a stochastic background. Comparing (34) with \( \Delta t_0 \) one obtains

\[
z_* \simeq 10^{10} (\tan \gamma)^{1/8} \alpha^{1/4} \left( \frac{\alpha'^{1/2}}{\text{TeV}^{-1}} \right)^{-1/4}.
\]

(36)

The average over \( \gamma \) introduces a numerical factor \( \simeq 0.9 \). In the case of continuous background, i.e. for events generated at redshift \( z > z_* \), taking into account the cosmological redshift, the spectral density function can be estimated as

\[
S_h(f) = \int_{z_*}^{z_{in}} |\tilde{h}(f(1 + z'))|^2 \frac{dN(z')}{d \ln z'} d \ln z',
\]

(37)

where \( z_{in} \) stands for the red-shift of the set in of the scaling solution and \( \tilde{h}(f(1 + z)) \) stands for a sum over metric polarisation and average over directions \( \hat{n} \) of \( \tilde{h}_{\mu\nu}(f(1 + z), \hat{n}) \). Since \( dN(z)/d \ln z \propto z^3 \) at large \( z \), the integral in eq.(37) can be approximated by the value of the integrand at the upper limit of integration region

\[
S_h(f) = |\tilde{h}(f(1 + z_{in})))|^2 \frac{10^2}{t_0} \frac{z_{in}^3}{z_{eq}^{3/2}},
\]

(38)

where \( \tilde{h}(f) \) can be read from (22) with the aid of (21) after averaging over orientations. With respect to the signal from an astrophysical source given by (22) there is a factor \( 10^2 z_{in}^3/(t_0 z_{eq}^{3/2}) \) instead of \( \Delta t_0 \simeq q^{-1/2}(1 + z) \), leading to an enhancement for high enough redshift, but again the final result is negligible

\[
S_h(f) \simeq 10^{-144} \text{Hz}^{-1} e^{-\pi^2 f^2 \alpha' z_{in}^2} \left( \frac{G_N \mu}{10^{-7}} \right)^2 \left( \frac{\alpha'}{\text{TeV}^{-2}} \right)^2 \left( \frac{z_{in}}{10^{10}} \right)^3,
\]

(39)

where an average over D-string meeting angle \( \gamma \) has been performed and the distance \( x \) from the source has been expressed as a function of the red-shift according to (32). Moving to higher redshift brings competing effects: a higher distance from the source, see eq.(32) and a decreased integration volume, see eq.(33), both of which tend to suppress the signal, and increased event rate, see eq.(26), which tends to increase the signal and it is actually the dominant effect. We remind the relation between redshift \( z \) and temperature \( T \) during
standard cosmological evolution

\[ z \simeq 10^9 \left( \frac{T}{\text{MeV}} \right). \]  

(40)

Actually before the scaling solution sets in, the density can be higher than (25) and consequently the rate of string encounter higher than (26). One could consider the very extreme case of an initial density of cosmic D-strings

\[ n_{\text{ex}} \sim \mu^{-3/2} \alpha'^{-3} \]  

(41)

at the epoch of cosmic string creation (inflation ending), characterised by a redshift \( z_{\text{ex}} \). Such a density would correspond to a Hagedorn energy density \( \rho_h \sim \alpha'^{-2} \), which is generally a limiting energy density for open string systems. The corresponding rate of reconnection events per volume is

\[ \nu_{\text{op-op}}^{(\text{ex})} \sim \mu^{-2} \alpha'^{-4}, \]  

(42)

while the standard scaling solution is recovered in a few Hubble times. The highly nonstandard form of the D-brane tachyon action, which suppresses gradients of the field, allows us to assume a far higher density of defects than the standard Kibble argument would allow as we will see in the next section, see [18]. With this assumption we can reconsider the integral (37), noting that it will be dominated by the early epoch \( z' \sim z_{\text{ex}} \) thus giving, analogously to (39),

\[ S_h(f) \simeq 10^3 \frac{(G_N \mu)^2}{x^2(z_{\text{ex}})} \frac{\alpha'^2}{\alpha'^4} e^{-\pi^2 f^2 \alpha' z_{\text{ex}}^2} \frac{10^2 t_0^{-3} z_{\text{ex}}^{-5} \alpha'^{1/2}}{\mu^2 \alpha'^4} \simeq 10^{-91} \text{Hz}^{-1} e^{-\pi^2 f^2 \alpha' z_{\text{ex}}^2} \left( \frac{\alpha'}{\text{TeV}^{-2}} \right)^{-2} \left( \frac{z_{\text{ex}}}{10^{10}} \right)^{-5}. \]  

(43)

which is still way too small for detection. It has also to be noted that the Kibble density is smaller than the extremal density \( n_{\text{ex}} \) given in (41) only if

\[ \mu < G_N^{-1/3} t_s^{-4/3}. \]  

(44)

With the process considered so far one cannot obtain a large enough result, as an inherently microscopic mechanism is involved, which affects a portion of the string of size the order of the string length in a huge Universe. For the gravitational emission to give a sizeable effect we need it to be produced in a coherent motion of all its world volume. In sec. III we move to the study of systems of D-string D-string which annihilate, but before doing that we complete the analysis of recombination by describing two almost antiparallel D-strings, which is a system close to the D D-string case.
B. Almost antiparallel case

1. System dynamics

If the system of two D-strings is close to a brane-anti brane pair it can still be analysed through a Yang Mills low energy effective action, but a scalar tachyon has to be added to the spectrum, as it comes from the lowest lying excitations of the string stretched between the brane and the anti-brane \[10\]. The action is then

\[ S = -\mu (2\pi \alpha')^2 \int dtdx \left[ \text{Tr} \left( \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} D_{\mu}\phi^i D^\mu \Phi^i + \frac{1}{2} D_{\mu}T D^\mu T - \frac{m^2}{2} T^2 \right) \right], \quad (45) \]

with \( m^2 = -1/(2\alpha') \). Now the expansion has to be taken around the background (again \( q \equiv \tan \gamma/(\pi \alpha') \))

\[ \Phi_i^{(b)} = \delta_i, 2q x \sigma^3, \quad A_{\mu}^{(b)} = 0, \quad T^{(b)} = 0, \quad (46) \]

and the fluctuations

\[ \Phi_i^{(f)} = \delta_i, 2\xi(t, x) \sigma^3, \quad A_{\mu}^{(f)} = 0, \quad T^{(f)} = \tau(t, x) \sigma^2. \quad (47) \]

In terms of the ansätze (46) and (47) the Lagrangian, up to lowest order interaction, can be written as

\[ \mathcal{L} \propto \frac{1}{2} \dot{\tau}^2 - \frac{1}{2} \tau'^2 - \frac{1}{2} q^2 - \frac{1}{2} \left( q^2 x^2 + 2q x \xi \right) \tau^2 + \frac{1}{4\alpha'} \tau'^2 + \frac{1}{2} \dot{\xi}^2 - \frac{1}{2} \xi'^2. \quad (48) \]

The analysis follows straightforwardly as in the previous case to obtain the equations of motion

\[ \ddot{\tau} - \tau'' + q^2 x^2 \tau - \tau/(2\alpha') = 0, \]
\[ \ddot{\xi} - \xi'' + \tau^2 q x = 0. \quad (49) \]

Again the unstable mode solution is

\[ \tau(t, x) = C_0 e^{\beta t} e^{-q x^2/2}, \]
\[ \xi(t, x) = C_0 e^{2\beta t} \frac{e^{\sqrt{\pi}}}{\beta} \left[ \cosh(2\beta x) - \frac{1}{2} \left( e^{2\beta x} \text{erf}(1 + \beta x) + e^{-2\beta x} \text{erf}(-1 + \beta x) \right) \right], \quad (50) \]

where \( \beta \equiv \sqrt{1/(2\alpha') - q} \).
From the dynamics exposed in the previous section the energy momentum tensor can be estimated to be

$$T_{\mu\nu} = 2\mu\pi^2\alpha'^2 \begin{pmatrix} -q^2/4 - \beta^2\xi'/2 & -\beta^2\dot{\xi}/2 \\ -\beta^2\dot{\xi}/2 & \tau^2\beta^2/2 - \beta^2\xi'/2 - q^2/4 \end{pmatrix} \times \left[ \delta(y - qx\pi\alpha') + \delta(y + qx\pi\alpha') \right].$$

(51)

It is then straightforward in full analogy with sec. II A 2 to estimate the gravitational radiation, obtaining the same result.

III. STRING-ANTI STRING ANNIHILATION

We have seen that the recombination process of D-strings described in sec. II is not a phenomenologically interesting source of gravitational radiation. This was due to the smallness of the size of the interaction region, which was of the order of the string length. In general interesting sources of gravitational radiation are related to coherent motion of large and extended portions of mass.

We thus now turn to study a process in which two D-strings meet and annihilate, thus enabling a much larger reservoir of energy to be converted into gravitational radiation. Such systems have been studied in detail in a full string theory context for different configurations of the tachyon describing the instability [12].
In particular it has been shown that the tachyon can acquire different spatial profiles, triggering the decay of the original system into different final states. For instance a trivial tachyon profile in the space dimensions will lead the brane anti-brane system towards the so called \textit{tachyon matter} which will be discussed in sec.\ref{sec:third}, whereas a kink profile will let a Dp $\bar{D}p$-brane system decay into a D($p-1$)-brane, a vortex configuration into a D($p-2$)-brane and so on. We show next that in the case of a trivial spatial configuration, important conclusions can be drawn for the production of gravitational radiation.

\section*{A. Recombination and annihilation}

For ordinary phase transitions the presence of defects is related to the global configuration of some field triggering the transition. In an expanding Universe with Hubble rate $H$, the defect separation $d$ must be smaller than the Hubble length $H^{-1}$, as any super-horizon correlation would violate causality, thus leading to the lower bound of one defect per Hubble volume ($Hubble density$), and the actual number of defects per Hubble volume is as large as a few.

However the density of defects can be much larger than the Hubble one as this picture does not apply to the tachyon condensation, whose dynamics is described by the Lagrangian

$$S = -\mu \int d^{p+1}x V(T) \sqrt{1 + \partial_{\mu}T \partial_{\nu}T} g^{\mu\nu},$$

where the $V(T)$ is a symmetric, positive potential, which vanishes asymptotically for $T \to \pm\infty$ as $e^{-T}$ \footnote{\cite{21}} (it is not very important here the exact form of such potential).

As shown in \cite{18}, the non-standard form of the kinetic terms implies that once the tachyon has rolled to large values, gradients are exponentially suppressed, essentially eliminating the restoring force which would tend to eliminate gradients within a casual volume, thus allowing a higher defect density than the Kibble value.

We assume that the density $n_{ex}$ as in \cite{11} can be achieved, corresponding to an energy density of the order of the Hagedorn scale, which is a limiting energy density for open strings \cite{24}.

Actually in the evolution of an ordinary string network, the initial conditions are not crucial as the network reaches the scaling solution in a few Hubble time, long strings loose energy by chopping off small loops, which in standard simulations then looses energy by
emitting gravitational wave until disappearance. If we take the extremal case of Hagedorn energy density like in the end of sec. (II A 3) the long string inter-separation is \( d_{ex} \sim \mu^{1/2} \alpha' \) rather than the Kibble value \( d_K \sim H^{-1} \approx G_N^{-1/2} \mu^{-1} \), which is eventually reached after few Hubble times.

The high starting density of strings is anyway not enough in itself to produce a large amount of gravitational wave if the usual picture is assumed, i.e. that small loop radiates as in [14] via the formation of cusps and kinks. Again this is due to the smallness of the physical size of the radiating objects, in this case the small loops, whose typical size is some orders of magnitude smaller than the Hubble scale, see eq. (25), where \( H \sim \sqrt{G_N \rho} \approx l_P/l_s^2 \), taking for \( \rho \) the extremal Hagedorn density. In this picture the emission is originated in the early Universe, when the size of cosmic string is small as they had no time to stretch with cosmological expansion, so a small amount of radiation is produced. In [14] for instance, a typical signal close to experimental sensitivity at a frequency \( f \), with rate \( \dot{N} \), originated at redshift \( z \) is obtained from a kink in a cosmic loop of size \( l_{cl} \) roughly given by

\[
l_{cl} \approx 10^{20} \text{m} \left( 1 + z \right)^{-3/2} \left( 1 + z/z_{eq} \right)^{-1/2} \left( \frac{\alpha}{10^{-6}} \right),
\]

and the rate \( \dot{N} \) of such an event at a given frequency \( f \) is roughly given by

\[
\dot{N} = 2 \times 10^{-6} \text{year}^{-1} \frac{z^3 (1 + z/z_{eq})^{11/6}}{(1 + z)^{7/6}} \left( \frac{\alpha}{10^{-6}} \right)^{-8/3} \left( \frac{f}{\text{kHz}} \right)^{-2/3}.
\]

**B. Annihilation with spatially trivial tachyon profile**

So far we have dealt with an effective field theory description of D-string D-string annihilation, the excitations of the branes have been described as classical fields subject to ordinary second order differential equations.

On the other hand the annihilation process admits an exact description in terms of the fundamental string theory, i.e. in terms of the two-dimensional world-sheet theory, by adding to the standard world sheet action

\[
S_{ws} = -\frac{1}{4\pi} \int d\tau d\sigma \left( \frac{1}{\alpha'} \partial_a X^\mu \partial^a X^\nu g_{\mu\nu} + \bar{\psi}^\mu \rho^\alpha \partial_a \psi_\nu g_{\mu\nu} \right),
\]

a boundary term

\[
S_b \propto \lambda \int d\tau \psi^0 e^{X^0/(2l_s)} \otimes \sigma_1.
\]
The world-sheet (disk) partition function $Z_{ws}$, with the world sheet couplings interpreted as space-time fields, corresponds to the space time action $S$, given by
\[ S = Z_{ws} \propto \int d^p x \sqrt{-g} \int [dX^\mu] [d\psi^\mu] e^{-S_{ws} - S_b}. \] (57)

The boundary term describes the condensation of the tachyon to a profile of the type
\[ T = \lambda e^{t/(\sqrt{2}l_s)} \otimes \sigma_1. \] (58)

Giving an expectation value to the tachyon shifts the mass squared of the open string modes by an amount $\Delta m^2 \propto \langle T^2 \rangle$, eventually dropping them out of the physical spectrum as the tachyon increasing expectation value makes them more and more massive.

A D-brane in the free theory can be described by a specific boundary state $|B\rangle$, the $|B\rangle \rightarrow |B\rangle$ vacuum amplitude is written as the so called cylinder amplitude
\[ A_{BB} = \langle B | P | B \rangle, \] (59)
where $P$ is the closed string propagator. When the internal particle running in the propagator goes on shell the amplitude $A_{BB}$ picks an imaginary part. Once written the expansion of $|B\rangle$ into a set of complete states as
\[ |B\rangle = \sum_f U(\omega_f) |f\rangle, \] (60)
by the optical theorem we have
\[ \text{Im} A_{BB} = \sum_f \frac{1}{2\omega_f} |U(\omega_f)|^2, \] (61)
where $\omega_f$ is the energy of the state $f$. In the unperturbed theory, where the tachyon has vanishing expectation value, the brane boundary state can be written as
\[ |B\rangle = \mathcal{O} |\Omega, k\rangle, \] (62)
where $\mathcal{O}$ is an operator build out of fermionic and bosonic matter oscillators and ghost oscillators and $|\Omega, k\rangle$ is the Fock vacuum with momentum $k$. The specific form of the vacuum will not be needed here, as we shall simply quote known results, see [20, 21]. Taking the initial condition to be that of D-strings, the addition of the boundary term (56) turns $|B\rangle$ into $|B\rangle_T$, whose energy momentum tensor is obtained from (57) via (12)
\[ T_{\mu\nu} = 2\mu \text{ diag} (-1, g(t), 0, \ldots, 0), \] (63)
with
\[ g(t) = \frac{1}{1 + 2\pi^2 \lambda^2 e^{\sqrt{2}t/l_s}}. \] (64)

The boundary state \( |B\rangle_T \) at \( t \to -\infty \) corresponds to \( |B\rangle \), and at \( t \to \infty \) tends to a state without the original D-strings but endowed with the same amount of energy and vanishing pressure.

Applying the optical theorem as in (61) one can find, see [13, 25], that the expansion coefficient \( U_T(\omega_f) \) relative to \( |B\rangle_T \) turns out to be just the Fourier transform of \( g(t) \), which is
\[ |U_T(\omega_f)| = \frac{\pi/\sqrt{2}}{\sinh(\pi l_s \omega_f / \sqrt{2})}. \] (65)

The imaginary part of \( T\langle B|B\rangle_T \) is related to the average numerical density \( \bar{n} \) and energy density \( \bar{\rho} \) of particles emitted by
\[ \bar{n} = \sum_f \frac{1}{2\omega_f} |U(\omega_f)|^2, \] (66)
\[ \bar{\rho} = \sum_f \frac{1}{2} |U(\omega_f)|^2, \] (67)
where the sums run over a basis of closed string states, implicitly including both the sum over excitation level \( N \) of the closed string and the momenta of the spatial directions transverse to the brane. For large \( N \) the degeneracy of states \( \mathcal{G}(N) \) of a closed string in \( D \) dimensions is
\[ \mathcal{G}(N) = N^{-(D+1)/2} e^{4\pi \sqrt{D/6} \sqrt{N}}, \] (68)
and here we assume \( D = 10 \) from superstring theory. Given the degeneracy of the states and the mass formula for closed strings
\[ \alpha' m^2 = 4 \left( N - \frac{1}{2} \right), \] (69)
the exponential in eq. (65) and (68) combine to leave a power law spectrum for the radiated energy according to
\[ \bar{\rho} \propto \int dE \, E^{-p/2}, \] (70)
for Dp-branes, with a total emitted energy which is ultraviolet divergent for \( p \leq 2 \). Of course the emitted energy cannot be larger then the mass of the initial brane, so back-reaction is expected to set in and cut off the emission, with the result that the total radiated energy is finite and of the order of the initial mass of the brane. In particular cutting off the integral in eq. (70) at \( E_{\text{cut}} \sim \mu l_{\text{op}} \) would give all the energy into closed string modes, emitting most of their energy into very massive strings.

Actually eq.(70) seems to suggest that for \( p > 2 \) the amount of energy of the initial Dp-brane going into closed string modes is rather small. However the decay described by the boundary term \([56]\) is spatially homogeneous and it is not expected to realistically describe the decay process. Perturbations with a non trivial spatial profile can still be tachyonic and thus contribute the decay. We should expect the decay to take place rather inhomogeneously and the original Dp-brane can be thought of decomposing into a collection of small patches. In each of this small patches, disconnected one from the other, the behaviour of a Dp-brane is similar to that of a collection of D0-branes \([22]\).

C. Gravitational mode production

The problem of gravitational emission has now been traded from the study of annihilating D-branes to the one of decaying of a highly massive (hence highly excited) closed fundamental string. Two-body decays of F-strings have been studied in literature for both the bosonic string \([26]\) and the superstring \([27, 28]\). The decay of a closed string into \( n \) closed strings is a perturbative process whose amplitude \( \Gamma \propto g_c^n \), thus in the perturbative regime \( g_c \ll 1 \) the dominant process is two-body decay.

Let us consider the amplitude decay \( A \) of a specific initial closed string state \( \phi_N \) of mass \( M \approx 2\sqrt{N}/l_s \) into two given final states \( \phi_{N_1}, \phi_{N_2} \) of masses respectively \( m_{1,2} \approx 2\sqrt{N_{1,2}}/l_s \). The inclusive decay width of a generic fundamental string excited at mass level \( N \), into two generic final states at mass level \( N_1, N_2 \) is obtained by integrating the amplitude squared over the available final states, and averaging over initial states at mass level \( N \). We assume to have \( d = 3 + 1 \) large ordinary dimensions and \( d_c \) small ones, and a highly energetic (larger than the string scale) closed fundamental string which can both wrap and have momentum around the small dimensions. Phase space integration includes a sum over winding and momenta.
The differential decay width \(d \Gamma_{cl \rightarrow m_1,2}\) is then given by averaging over initial states and summing over final ones \([26, 27]\)

\[
d \Gamma_{cl \rightarrow m_1,2} = \frac{1}{2MG(N)} \sum_i |A|^2 d\Phi_{f_1,f_2} \simeq \frac{g_c^2}{2M} \frac{(N - N_1 - N_2)^2}{G(N_1)G(N_2)} \frac{G(N)}{G(N)} d\Phi_{f_1,f_2}. \tag{71}
\]

The two-body phase space in \(d = 3 + 1\) dimensions is

\[
d\Phi_{f_1,f_2}^{(3+1)} = \frac{k}{16\pi^2 M} d\Omega \tag{72}
\]

where \(k\) is the modulus of the momentum of the decay states and \(\Omega\) the solid angle. Considering for simplicity flat internal compact dimensions, the inclusion of momentum and winding modes in the mass formula \([69]\) adds a sum over momenta and windings in the compact directions. The closed string mass formula which generalises \([69]\) in the presence of discrete momenta and windings is

\[
\alpha' m^2 = 4 \left( N - \frac{1}{2} \right) + \sum_i \left( \alpha' n_i^2 \frac{R_i^2}{R^2} + w_i^2 \frac{R_i^2}{\alpha'} \right), \tag{73}
\]

which for large energies can be written as

\[
l_s m \simeq 2\sqrt{N} + \frac{1}{2} \sum_i \left( \frac{n_i^2 l_s}{R_i^2 m} + \frac{w_i^2 R_i^2}{l_s^3 m} \right). \tag{74}
\]

This allows to approximate the summations over \(n_i\) and \(w_i\) as Gaussian integrals in the case respectively \(m \gg l_s R_i^{-2}\) and \(m \gg R_i^2/l_s^3\). Using the asymptotic formula for the density of states \([68]\) the amplitude \([71]\) can be integrated to \([34]\)

\[
\Gamma_{cl \rightarrow m_1,2} \simeq \frac{2}{\pi M^{2-d_c}} \frac{(2\pi/a)^{d_c}}{l_s^{3D-d-2d_c}} e^{-a l_s E} \left( \frac{m_1 m_2}{M} \right)^{-(D-d_c+1)} \left( m_1 m_2 \right)^2 \left( \frac{2Em_1 m_2}{M} \right) \frac{d-d_c}{2}, \tag{75}
\]

where \(a = 4\pi/\sqrt{3}\), \(d_c\) the number of compact dimensions along which the closed strings have momenta and windings, the kinetic energy of the decay objects

\[
E \equiv M - m_1 - m_2 \simeq \frac{M}{2m_1 m_2} k^2 \tag{76}
\]

has been introduced and the approximate relation

\[
N - N_1 - N_2 \simeq \alpha' M \sqrt{k^2 + m_1^2} - m_1^2 \simeq \alpha' m_1 m_2 \tag{77}
\]

has been used.
Let us call $\Gamma$ the total rate of production of a string of mass $m$, which is obtained by summing $\Gamma$ over all values $N$. The sum over $N$ can then be traded for a continuous integral over $E$ with the substitution $\sum_{N} \rightarrow -\alpha' m_{2} dE$, obtaining

$$
\Gamma_{c_{l} \rightarrow m_{1,2}}^{(m)} \propto g_{c}^{2} \frac{(l_{s} M)^{D-d+1}}{m_{R}^{2}} \int E^{(d-3)/2} e^{-a_{l} s E} dE,
$$

(78)

where $m_{R} \equiv m(M - m)/M$. Since in an interval $\Delta N$ corresponds to $\alpha' m \Delta m/2$, the density of states per mass level is $\rho(m) = \alpha' m/2$, thus we can define the differential decay width per mass interval as

$$
\frac{d\Gamma_{c_{l} \rightarrow m_{1,2}}}{dm} = \Gamma_{c_{l} \rightarrow m_{1,2}}^{(m)} \rho(m) \propto g_{c}^{2} M (l_{s} M)^{D-d} (l_{s} m_{R})^{-(d-1)/2}.
$$

(79)

The differential decay rate (79) clearly shows that light strings are more easily produced than heavy ones. Thus we can safely assume that a consistent fraction of the initial rest mass is converted into massless closed strings, i.e. radiation.

We are then lead to consider the amplitude inclusive over $m_{2}$ and with $m_{1} = 0$ which is

$$
d\Gamma_{c_{l} \rightarrow \text{rad}} \simeq g_{c}^{2} \frac{e^{-a_{l} s k}}{1 - e^{-a_{l} s k} l_{s} M (l_{s} k)^{d-1}} dk,
$$

(80)

similar to a black body spectrum, leading to a total decay rate

$$
\Gamma_{c_{l} \rightarrow \text{rad}} \simeq g_{c}^{2} M.
$$

(81)

Thus in a time much shorter than the string scale a D-string and a $\overline{\text{D}}$-string decay efficiently into radiation.

If it is so one of the two decay objects can still be massless from the ten-dimensional point of view but it will be massive from the effective 4-dimensional theory. In this case the decay width (80) should have been modified by the inclusion of an additional integral

$$
d\Gamma_{c_{l} \rightarrow \text{KK}} = g_{c}^{2} \frac{e^{-a_{l} s \sqrt{k^{2}+m^{2}}}}{1 - e^{-a_{l} s \sqrt{k^{2}+m^{2}}} l_{s} M (l_{s} k)^{d-1}} dk \prod_{i} R_{l_{i}} d k_{i} \simeq g_{c}^{2} e^{-a_{l} s \sqrt{k^{2}+m^{2}} m_{d} c-1} d m.
$$

(82)

Then integration over the momenta in the compact dimensions have the effect of multiplying the differential decay width by a factor $\prod R_{l_{i}}/l_{s}$, for $R_{l_{i}} \gg l_{s}$, because of the opening up of new decay channels, and it does not modify the result (80) for $R_{l_{i}} \sim l_{s}$.

If two D-strings are not exactly anti-parallel but meet at an angle $2 \gamma$, as in fig. 2, the tachyon condensation makes the D-string disappear not over the entire world-volume of the
D-strings, but only as long as the D-strings are separated by a distance shorter than $2l_s$, which for strings meeting at angle $2\gamma$ happens for a region of size $L_x = 2l_s/\tan \gamma$, see fig. 3.

The energy density in gravitational waves $\rho_{gw}$ is related to the spectral density function by

$$G_N\rho_{gw} = \frac{\pi}{2} \int_0^{\infty} df f^2 S_h(f).$$

The energy a single annihilation of D-branes make available for radiation can be estimated as $\mu L_x$ which eventually expands spherically. From (80) and (83) one can then compute the strain due to a gravitational wave emitted at a distance $x$ from us (here we neglect red-shift for the moment, thus having $fl_s \ll 1$ at interesting frequencies)

$$S_h(f) \simeq \frac{2}{\pi} \frac{G_N \mu L_x \alpha'}{x^2} \frac{l_s f e^{-al_s f}}{1 - e^{-al_s f}} \simeq 10^{-108}\text{Hz}^{-1} \frac{e^{-al_s f}}{\tan \gamma} \left(\frac{G_N \mu}{10^{-7}}\right) \left(\frac{\alpha'}{\text{TeV}^{-2}}\right)^{3/2} \left(\frac{x}{\text{Mpc}}\right)^{-2}.$$ (84)

Of course $S_h(f)$ cannot diverge for $\gamma \to 0$, as the naive estimation for $L_x$ does, but for small $\gamma$ one has to substitute $2l_s/\tan \gamma$ with $l_{op}$, the (open) cosmic string length. In this optimal case of perfect anti-alignment one has

$$S_h^{(op)}(f) \simeq \frac{2}{\pi} \frac{G_N \mu l_{op} \alpha'}{x^2} \frac{l_s f (1+z)e^{-al_s f(1+z)}}{1 - e^{-al_s f(1+z)}} \simeq 10^{-70}\text{Hz}^{-1} e^{-al_s f(1+z)} \frac{(1+z)^{1/2}}{z^2 (1+z/zeq)^{1/2}} \left(\frac{G_N \mu}{10^{-7}}\right) \left(\frac{\alpha'}{\text{TeV}^{-2}}\right),$$ (85)

where it has been used that $l_s f (1+z) \ll 1$, as a frequency $l_s^{-1}$ at an epoch characterised by a temperature $T = l_s^{-1}$, once red-shifted corresponds to 100GHz today, assuming standard adiabatic evolution in between.
Assuming that the radiation is produced by an annihilation event during cosmological evolution when the scaling solution has set in, the expected rate of such an event is

$$N = \frac{P}{t} \simeq \frac{P}{t_0} (1 + z)^{3/2} (1 + z/z_{eq})^{1/2}. \quad (86)$$

It is clear that here again we have to assume an extremal initial density of strings well above the Kibble density to hope to have a detectable signal. Let us then assume an initial density of annihilation \( \rho_{\text{ex}} \) at an epoch characterised by a red-shift \( z_{\text{ex}} \). Annihilation will thus take place in a region of volume \( \Delta V(z_{\text{ex}}) \), see eq.(33), and for a time duration

$$\Delta t(z_{\text{ex}}) = \left| \frac{dt}{dz} \right|_{z=z_{\text{ex}}} \Delta z \simeq \left| \frac{dt}{dz} \right|_{z=z_{\text{ex}}} z_{\text{ex}}$$

to give

$$S_h^{\text{ex}}(f) = \frac{2}{\pi f^2} G_N \int \frac{d \rho_{gw}}{df} \rho_{\text{ex}} \Delta V \Delta t = \frac{2 \rho_{\text{ex}}}{\pi f^2} \left(1 + z\right)^2 \times \frac{l_s f (1 + z) e^{-a_l f(1 + z)}}{1 - e^{-a_l f(1 + z)}} \times \frac{1}{\mu^2 \alpha'^4} \times \frac{t_{0.5}^4}{(1 + z)^6 (1 + z/z_{eq})^{3/2}} e^{-a_l f(1 + z)},$$  

where we have considered the optimal case \( L_x = l_{op} \). Since the extremal case of a Hagedorn energy density has been assumed, \( z_{\text{ex}} \) has to be high enough to accommodate such a high density phase. Assuming standard adiabatic expansion after this extremal density phase the relation between the red-shift factor \( z \) and the string scale is eq.(40)

$$z = 10^{15} \left( \frac{\alpha'}{\text{TeV}^{-2}} \right)^{-1/2}. \quad (88)$$

which takes the estimate \( S_h^{\text{ex}}(f) < 10^{-51} \text{Hz}^{-1} \left( \frac{\text{kHz}}{f} \right)^3 \). (89)

Substituting \( \rho_{\text{ex}} \) into \( S_h^{\text{ex}}(f) \) we can obtain

$$S_h(f) = 10^{-72} \text{Hz}^{-1} \left( \frac{\mu}{10^{-7} G_N} \right)^{-1/2} \left( \frac{\alpha'}{\text{TeV}^{-2}} \right)^{-5/2} \left( \frac{f}{10^{11} \text{Hz}} \right) e^{-a_l f(1 + z)}$$  

\( \mu \text{ eV} \)
In our toy model the spectral strength can saturate the bound only around 10 ÷ 100GHz, i.e. at the end of the spectrum. This is hardly surprising as most of the energy of the F-string decay is concentrated for frequencies \( f^{-1} \sim l_s(1 + z_{ex}) \).

On the other hand if Kaluza-Klein modes are created and if they are absolutely stable, they can rapidly overclose the Universe. In this case of extremal production their initial energy density is a non negligible fraction of the total energy density, so their decay is necessary otherwise they would start dominating as soon as they become non-relativistic. Their final abundance is then set by the annihilation cross section \[30\], which may not be Planck scale, as gravity can be strong much below the Planck scale in large extra dimension scenarios.

If instead we consider the standard scenario with the space-time rate of open string-open string encounters given by \[28\], we can estimate the fraction of the critical energy density going into the production of massive gravity modes as follows. Let us suppose that each encounter give rise to the production of massive modes for a total energy \( E_{KK} = \mu L_x \), thus giving an energy density per Hubble volume \( \rho_{KK} \simeq \mu L_x / t^3 \). By normalizing it by the critical energy density \( \rho_c = 3H^2/(8\pi G_N) \simeq (8\pi G_N t^2)^{-1} \) and integrating over the history of the Universe, using eq.\[35\], we have

\[
\Omega_{KK}(z_{nr}) \equiv \frac{\rho_{KK}(z_{nr})}{\rho_c(z_{nr})} = 32\pi^2 G_N \mu \frac{l_s}{t_0} \int_{z_{nr}}^{z_{in}} (1 + z')^{1/2} (1 + z'/z_{eq})^{1/2} \, dz \\
\simeq 10^{-50} \frac{z_{nr}^2}{z_{eq}} \left( \frac{G_N \mu}{10^{-7}} \right) \left( \frac{\alpha'}{\text{TeV}^{-2}} \right)^{1/2},
\]

where \( z_{in} \) denotes as in eq.\[37\] the time of the onset of the scaling solution. Note that the integral in \[91\] has been halted at \( z_{nr} \), the red-shift at which the Kaluza-Klein modes become non relativistic. For \( z_{nr} > z > z_{eq} \), \( \rho_{KK} \) keeps earning a factor of \( z \) over \( \rho_c \) thus resulting into a present fractional energy density \( \Omega_{KK} \equiv \Omega_{KK}(z = 0) \) given by

\[
\Omega_{KK} = \Omega_{KK}(z_{nr}) \frac{z_{nr}}{z_{eq}} \simeq 10^{-11} \left( \frac{G_N \mu}{10^{-7}} \right) \left( \frac{\alpha'}{\text{TeV}^{-2}} \right)^{-1}.
\]

where for simplicity we have assumed the mass of the Kaluza-Klein modes to be \( \alpha'^{-1/2} \), see eq.\[82\], and substituted the relation \[88\] for \( z = z_{in} = z_{nr} \).

This result can be phenomenologically relevant for a still moderate value of the string scale.
IV. CONCLUSIONS

Motivated by the recent revival in cosmic strings we have studied cosmic strings by modeling them as D-branes, the solitonic objects of string theory and computed the amount of gravitational waves emitted in recombination processes which can take place in the moderately early Universe. We have found that in general such processes are well below present sensitivities. Still the process leading to string annihilation can result into a production of gravity modes of fundamental strings which can be massive at an effective 4-D level and can thus have a cosmological effect. Such effect depends ultimately, in our simple model, on the string scale value, and it can be interesting even if the string scale is well below the Planck scale. From a more theoretical point of view, this mechanism represents a new way for producing out of equilibrium, very massive particles.

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The set of fluctuations $\phi_2 \propto \sigma^2$, $A_1 \propto \sigma^1$, $A_2 \propto \sigma^2$ and its Lagrangian is another copy of (4).

Here we have a limiting density different from (18), where the energy scale relative to the string tension and to the inverse distance are identified, as no Hagedorn limit.

This is actually the coefficient for the expansion over Neveu-Schwarz states, which is the interesting sector for gravitons.

We assume that the closed strings have enough energy to possess momenta and windings along the compact dimensions, if this were not the case the right hand side of (75) should be divided by $3^{1/4} \sqrt{s} m \frac{L_j}{2l_s}$ for each direction $i$ along which the momentum modes are not excited and $3^{1/4} \sqrt{s} m \frac{L_j}{2l_s}$ for each direction with unexcited winding modes.

Note that we obtain a dependence on $M$ and $m$ different from (27). This is basically because here we have summed over Kaluza-Klein momenta and winding modes of the decaying objects, thus obtaining the expected dependence of (79) on $m$ instead of the weird $m^{-\left(D-1+d_e\right)/2}$ of (27).