Yang-Mills Instantons Sitting on a Ricci-flat Worldspace of Double D4-brane

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Abstract

Thus far, there seem to be no complete criteria that can settle the issue as to what the correct generalization of the Dirac-Born-Infeld (DBI) action, describing the low-energy dynamics of the D-branes, to the non-abelian case would be. According to recent suggestions, one might pass the issue of world-volume solitons from abelian to non-abelian setting by considering the stack of multiple, coincident D-branes and use it as a guideline to construct or censor the relevant non-abelian version of the DBI action. In this spirit, here we are interested in the explicit construction of $SU(2)$ Yang-Mills (YM) instanton solutions in the background geometry of two coincident probe D4-brane worldspaces particularly when the metric of target spacetime in which the probe branes are embedded is given by the Ricci-flat, magnetic extremal 4-brane solution in type IIA supergravity theory with its worldspace metric being given by that of Taub-NUT and Eguchi-Hanson solutions, the two best-known gravitational instantons. And then we demonstrate that with this YM

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instanton-gravitational instanton configuration on the probe $D4$-brane world-volume, the energy of the probe branes attains its minimum value and hence enjoys stable state provided one employs the Tseytlin’s non-abelian DBI action for the description of multiple probe $D$-branes. In this way, we support the arguments in the literature in favor of Tseytlin’s proposal for the non-abelian DBI action.

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I. Introduction

With the recent advent of the physics of $D$-branes [1], the solitons in the non-perturbative spectrum of string theory, it has been realized that their low-energy dynamics can be properly described by the so-called, “Dirac-Born-Infeld (DBI)” action [1,2]. Since single branes are known to be described by the abelian DBI action, one might naturally expect that the multiple branes [3] would be by some non-abelian generalization of the DBI action. Unfortunately, however, there seem to be no complete criteria that can settle the issue as to what the correct generalization of the DBI action to the non-abelian case would be. To pass from the abelian version to the non-abelian one, the first obvious treatment to be done is to render the action group scalar by taking the trace over the group indices appearing in the non-abelian gauge field strength term. Indeed, different group trace operations have been proposed in the literature and they include ; $Tr$, the ordinary trace [1], $Str$, the symmetrized trace [4] and $(Str + i Atr)$, the combination of the symmetrized and the antisymmetrized trace [5]. Here, we will argue that the symmetrized trace operation put forward by Tseytlin stands out particularly in association with the “BPS” solitons that the $D$-branes may admit on their worldvolumes. Actually, it shall be demonstrated that Tseytlin’s choice of non-abelian DBI action, but no others, “knows” about the energy-minimizing worldvolume solitons. Since we shall take the BPS soliton on the $D$-brane worldvolume as a mean to argue the relevance of Tseytlin’s version of non-abelian DBI action, it seems appropriate to remind briefly the status of worldvolume solitons uncovered in the literature thus far. It has been found that the $D4$-brane worldvolume theories admit abelian instantons [6]; those of $D3$-brane admit abelian monopoles and dyons [7]; those of $D2$-brane admit abelian vortices [7,8]; and lastly, those of $D$-string admit kinks [9]. Then taking the abelian DBI action for the worldvolume field theory, it has been realized that the BPS-type condition for each case minimizes the energy of the corresponding worldvolume soliton. And another interesting point is that all of the above configurations can be obtained by starting from the $D4$-brane case and successively applying $T$-duality [10]. Thus it seems quite natural to pass this issue of worldvolume
solitons from abelian to non-abelian setting by considering instead the stack of multiple, coincident $D$-branes and use it as a guideline to construct or censor the non-abelian version of the DBI action and this is precisely what we would like to do in the present work. Then what is so special about Tseytlin’s suggestion for the non-abelian DBI action? As we shall see in a moment, for non-abelian gauge field configuration satisfying the (anti) self-dual field strength $F_{\alpha\beta} = \pm \tilde{F}_{\alpha\beta}$ (with $\alpha, \beta$ being the $D4$-brane worldspace indices), the determinant in the DBI action can be written as a “complete square” linearizing the DBI action and then turning it into that of usual Yang-Mills theory [10,11]. What is more, since we are dealing with static soliton configurations, the energy density of the $D4$-brane is just $H_{DBI} = -L_{DBI}$, which gets minimized if and only if $F_{\alpha\beta} = \pm \tilde{F}_{\alpha\beta}$ [6,10,11]. Thus the BPS condition, or the (anti) self-duality of non-abelian gauge theory instanton solution at once linearizes the otherwise highly non-linear DBI action and minimizes the energy of the $D4$-brane provided Tseytlin’s non-abelian DBI action is employed. In order to demonstrate that this procedure can indeed work, the question would be whether or not one can actually construct, say, $SU(2)$ Yang-Mills instanton solution having (anti) self-dual field strength in the background of $D4$-brane worldspace. Thus at this point, it might be relevant to remind what has been done and suggested to be done in the literature to answer to this question. In [10], it has been pointed out that a stack of two coincident $D4$-branes with flat worldvolume geometry admits worldvolume solitons corresponding to the standard BPST $SU(2)$ YM instanton solutions to the (anti) self-dual YM equation in the flat $D4$-brane worldspace. And then in [11], it was just argued that in a similar manner, a pile of two coincident $D4$-branes with $Ricci$-$flat$ worldvolume geometry may admit worldvolume solitons this time corresponding to the $SU(2)$ YM instanton solutions to the (anti) self-dual YM equation in the background of $Ricci$-$flat$ $D4$-brane worldspace described, say, by 4-dimensional gravitational instantons. In [11], however, they were not able to explicitly demonstrate the construction of such YM instanton solutions. Thus here in this work, we would like to demonstrate in an explicit manner that the worldvolume solitons of this sort, i.e., $SU(2)$ YM instanton solutions in the background of $Ricci$-$flat$ $D4$-brane worldspace with its metric being described by the Taub-
NUT or Eguchi-Hanson metric can actually be constructed. To be a little more precise, we shall be interested in the explicit construction of $SU(2)$ Yang-Mills instanton solutions in the background geometry of a stack of the two coincident probe $D4$-brane worldspaces particularly when the metric of target spacetime in which the probe branes are embedded is given by the Ricci-flat, magnetic extremal 4-brane solution in type IIA theory with its worldspace metric being given by that of Taub-NUT or Eguchi-Hanson solution, the two best-known gravitational instantons. And then we shall demonstrate that with this YM instanton-gravitational instanton configuration on the probe $D4$-brane worldvolume, the energy of the probe branes attains its minimum value and hence enjoys stable state provided one employs the Tseytlin’s non-abelian DBI action for the description of multiple probe $D$-branes. On the technical side, then, it might be relevant to describe how to generally construct the $SU(2)$ YM instantons as solutions to (anti) self-dual YM equations in the background of typical 4-dimensional gravitational instantons. Generally speaking, well below the Planck scale, the strength of gravity is negligibly small relative to those of particle physics interactions described by non-abelian gauge theories. Thus one might overlook the effects of gravity at the elementary particle physics scale. Nevertheless, as far as the topological aspect is concerned, gravity may have marked effects even at the level of elementary particle physics. Namely, the non-trivial topology of the gravitational field may play a role crucial enough to dictate the topological properties of, say, $SU(2)$ Yang-Mills (YM) gauge field [12] as has been pointed out long ago [13]. Being an issue of great physical interest and importance, quite a few serious study along this line have appeared in the literature but they were restricted to the background gravitational field with high degree of isometry such as the Euclideanized Schwarzschild geometry [13] or the Euclidean de Sitter space [14]. Even the works involving more general background spacetimes including gravitational instantons (GI) were mainly confined to the case of asymptotically-locally-Euclidean (ALE) spaces which is one particular such GI and employed rather indirect and mathematically-oriented solution generating methods such as the ADHM construction [15]. Recently, we [16] have proposed a “simply physical” and perhaps the most direct algorithm for generating the YM instanton
solutions in all species of known GI. Thus in the present work, we would like to employ this recently developed method to construct $SU(2)$ YM instanton solutions in the background of Ricci-flat $D4$-brane worldspace described by 4-dimensional gravitational instantons such as Taub-NUT (TN) or Eguchi-Hanson (EH) metric.

The rest of the paper is organized as follows. In sect.II, we shall present the Ricci-flat extremal $p$-brane solutions in supergravity (SUGRA) theories. Although their existence has been pointed out and the solution forms written down in [11], a careful and explicit derivation is hardly available in the literature.\footnote{A rather concise description of the derivation has been given by B. Janssen (JHEP 0001, 044 (2000)) and we were informed (private communication) that also it can be found in the PhD thesis of D. Brecher although the latter is not generally available.} Thus we provide the detailed derivation in this section as it plays the central role in this work. Sect.III is devoted to the introduction of different versions of the non-abelian DBI action and the detailed description of the features of Tseytlin’s action particularly in connection to the worldspace instanton of the $D4$-brane. Sect.IV is the main part of this work containing new ingredients. Namely, there we discuss the construction of $SU(2)$ instanton solutions in the background of TN and EH metrics representing the Ricci-flat $D4$-brane worldspace. Lastly in sect.V, we conclude with discussions and in the Appendix, we provide detailed analysis of the interesting nature of these YM instantons obtained in sect.IV.

II. Ricci-flat extremal $p$-brane solutions in supergravity theories

1. Construction of solutions

In order to derive and study the $M$-brane solutions in (the bosonic sector of) $D = 11$ SUGRA and $D$-brane solutions in $D = 10$ type IIA/IIB SUGRA conveniently in a single setting, we would like to consider a system in generally $D$-spacetime dimensions comprising the metric $G_{MN}$, a scalar (dilaton) field $\Phi$, and an $(n - 1)$-form antisymmetric R-R tensor...
gauge field $A_{[n-1]}$ with the associated field strength $F_{[n]}$ described by the action (our method here to construct the solutions closely follows that in [17])

$$S = \int d^{D}X \sqrt{G}[R - \frac{1}{2} \nabla_{M} \Phi \nabla^{M} \Phi - \frac{1}{2n!} e^{\hat{\phi} F_{[n]}^{2}}].$$

(1)

Upon extremizing this action with respect to $G_{MN}, \Phi$ and $A_{M_{1}...M_{n-1}}$, one would get a set of classical field equations of which the concrete forms shall be given later on. (Of course, it is known that the field equations of type IIB theory cannot be derived from a covariant action. Nevertheless, the field equations that result from the action given above are general ones in that they involve those of type IIB theory.) In order to practically solve these equations of motion, we would have to make a simplifying ansatz for solutions. As a simplest choice, one normally require translational symmetry along the $(p + 1) = (n - 1) \equiv d$-dimensional worldvolume of the $p$-brane configuration and isotropy in the directions “transverse” to this $p$-brane worldvolume, namely $(\text{Poincare})_{d} \times SO(D - d)$-symmetric solution ansatz.

Accordingly, then, the spacetime coordinates can be split into two ranges

$$X^{M} = (x^{\mu}, y^{m})$$

(2)

where $x^{\mu}(\mu = 0, 1, ..., p = (d - 1))$ are the coordinates on the $p$-brane worldvolume and $y^{m}(m = (p + 1), ..., (D - 1))$ are the coordinates transverse to the worldvolume. Thus the worldvolume has $(p + 1) = d$ dimensions and the number of transverse directions is $(D - d) \equiv (d + 2)$. Of course this highly symmetric restriction on the solution ansatz can be somewhat relaxed in more generalized versions of the class of $p$-brane solutions and particularly here in the present work, we start by abandoning the Poincare-symmetry on the brane worldvolume and end by demonstrating that a simple, “Ricci-flat” $p$-brane solutions can actually be constructed. Namely, we employ the ansatz for the spacetime metric as

$$ds^{2} = G_{MN}dX^{M}dX^{N}$$

(3)

$$= e^{2A(x)} d^{2}x \mu d^{2}x \nu + e^{2B(x)} \delta_{mn} dy^{m} dy^{n}$$

$$= e^{2A(x)} \delta_{ab} e_{\mu}^{a}(x) e_{\nu}^{b}(x) d^{2}x \mu d^{2}x \nu + e^{2B(x)} dy^{m} dy^{m}$$

$$= \eta_{IJ} e^{I}(X) e^{J}(X)$$
where \( r = (y^m y^m)^{1/2} \) is the isotropic radial coordinate in the space transverse to the brane worldvolume and we introduced, to carry out later on the computation of the Ricci tensor components \( R_{MN} \) in an easier and elegant manner, the orthonormal basis or the vielbein \( e^I(X) = e^I_M dX^M = e^I_\mu dx^\mu + e^I_m dy^m \) given by

\[
e^I(X) = \{ e^a = e^A(r) \hat{e}_a(x) = e^A(r) \hat{e}_\mu(x) dx^\mu, \quad e^m = e^B(r) dy^m \} \tag{4}
\]

and to summarize, here \( M = (\mu, m) \) are coordinate basis indices and \( I = (a, m) \) are non-coordinate, orthonormal frame indices with \( a = 0, 1, ..., p = (d - 1) \) and \( m = (p + 1), ..., (D - 1) \) respectively. Since the metric components depend only on “\( r \)”, the \( SO(D-d) \)-symmetry in the transverse directions still remains. Then the corresponding ansätz for the scalar dilaton field is simply

\[
\Phi = \Phi(r). \tag{5}
\]

Finally, for the \((p + 1) = (n - 1)\)-form antisymmetric R-R tensor gauge field \( A_{M_1...M_{n-1}} \), two kinds of ansatz related by duality transformation are possible. Obviously, the first possibility is to choose \( A_{[n-1]} \) such that it “supports” a \((p + 1) = (n - 1)\)-dimensional worldvolume. To be a little more concrete, we naturally expect that \( A_{[n-1]} \) couples directly to the \((p + 1) = (n - 1)\)-dimensional worldvolume of the \( p \)-brane, just as the 1-form Maxwell gauge potential couples to the worldline of a charged particle as \( eA_\mu dx^\mu \) with \( e \) being the gauge charge. Then the “charge” of the \( p \)-brane here will be obtained via Gauss’ law from the surface integral involving the associated field strength \( F_{[n]} \). This first possible choice shall be referred to as “electric (or elementary)” ansätz and amounts to

\[
A_{\mu_1...\mu_{n-1}} = \epsilon_{\mu_1...\mu_{n-1}} e^C(r) \tag{6}
\]

with others being zero or in terms of its field strength

\[
F_{m\mu_1...\mu_{n-1}} = \epsilon_{\mu_1...\mu_{n-1}} \partial_m e^C(r) \tag{7}
\]

with all others zero. Here, \( \epsilon_{\mu_1...\mu_{n-1}} \) is generally the curved spacetime version of the totally antisymmetric tensor. We now turn to the second possibility for the choice of the ansätz
for $A_{[n-1]}$. This second possibility can be most conveniently expressed in terms of the field strength as it can be obtained by considering the Hodge dualized field strength $\tilde{F}_{[n]}$, which is a $(D - n) = (D - d - 1) = (D - p - 2)$-form. Since the duality transformation is involved, this second choice may be called “magnetic (or solitonic)” ansatz and amounts to

$$F_{m_1 m_2 \ldots m_{(D-n)}} = \lambda \epsilon_{m_1 m_2 \ldots m_{(D-n)}} \frac{y^I}{r^{(D-n+1)}}$$

(8)

where the undetermined parameter $\lambda$ in this ansatz is an integration constant representing the magnetic charge. Having constructed two kinds of ansatz for the R-R tensor gauge field, we now can write down the classical field equations in an explicit manner

$$R_{MN} = \frac{1}{2} \partial_M \Phi \partial_N \Phi + S_{MN} \quad \text{with}$$

$$S_{MN} = \frac{1}{2(n-1)!} e^{\tilde{\Phi}} \left[ F_{M \ldots N} F^{M \ldots N} - \frac{(n-1)}{n(D-2)} G_{MN} F^2 \right],$$

$$\frac{1}{\sqrt{G}} \partial_M [\sqrt{G} G^{MN} \partial_N \Phi] = \frac{\tilde{a}}{2n!} e^{\tilde{\Phi}} F^2,$$

$$\frac{1}{\sqrt{G}} \partial_{M_1} [\sqrt{G} e^{\tilde{\Phi}} F^{M_1 \ldots M_{(D-n)}}] = 0$$

for the electric ansatz and with

$$S_{MN} = \frac{1}{2(D-n-1)!} e^{\tilde{\Phi}} \left[ F_{M \ldots N} F^{M \ldots N} - \frac{(D-n-1)}{(D-n)(D-2)} G_{MN} F^2 \right],$$

$$\frac{1}{\sqrt{G}} \partial_M [\sqrt{G} G^{MN} \partial_N \Phi] = \frac{\tilde{a}}{2(D-n)!} e^{\tilde{\Phi}} F^2,$$

$$\frac{1}{\sqrt{G}} \partial_{M_1} [\sqrt{G} e^{\tilde{\Phi}} F^{M_1 \ldots M_{(D-n)}}] = 0$$

for the magnetic ansatz. Now the rest of the procedure to find electric/magnetic $p$-brane solutions consists simply of writing and solving the field equations given above in terms of the $SO(D - d)$-symmetric ansatz of the fields given. And central to this procedure is the computation of Ricci tensor components $R_{MN}$ which, as stated earlier, can be done most easily in the context of Riemann-Cartan formulation in which the line element is given in non-coordinate orthonormal basis as in eq.(4). Namely, one first obtains the spin connection 1-form $\omega^I_f = \omega^I_{M,f} dX^M$ via the Cartan’s 1st structure equation

$$de^I + \omega^I_f \wedge e^f = 0$$

(10)
and from them, one next calculates the Riemann curvature 2-form \( R^{IJ} = (1/2) R^{IJ}_{MN} dX^M \wedge dX^N \) via the Cartan’s 2nd structure equation

\[
R^{IJ} = d\omega^{IJ} + \omega^{IK} \wedge \omega^K_J
\]

to get, upon projecting back in the coordinate basis,

\[
R_{\mu\nu} = \hat{R}_{\mu\nu} - \gamma_{\mu\nu}(x)e^{2(A-B)} \left[ A'' + \frac{(\tilde{d} + 1)}{r} A' + d(A')^2 + \tilde{d} A'^2 \right],
\]

\[
R_{mn} = -\delta_{mn} \left[ B'' + dA'B' + \tilde{d}(B')^2 + \frac{(2\tilde{d} + 1)}{r} B' + \frac{d}{r} A' \right] - \frac{y^m y^n}{r^2} \left[ \tilde{d}B'' + dA'' - 2dA'B' + d(A')^2 - \tilde{d}(B')^2 - \frac{\tilde{d}}{r} B' - \frac{d}{r} A' \right],
\]

\[
R_{\mu m} = 0
\]

where prime denotes the derivative with respect to \( r = (y^m y^m)^{1/2} \) and in the first line, \( \hat{R}_{\mu\nu} \) denotes the Ricci tensor associated with the non-trivial metric on the brane \( \gamma_{\mu\nu}(x) \). Finally, the field equations written in terms of \( SO(D - d) \)-symmetric ansatz of the fields take the forms

\[
\hat{R}_{\mu\nu} - \gamma_{\mu\nu}(x)e^{2(A-B)} \left[ A'' + \frac{(\tilde{d} + 1)}{r} A' + d(A')^2 + \tilde{d} A'^2 \right] = -\gamma_{\mu\nu}(x)e^{2(A-B)} \frac{\tilde{d}}{2(D - 2)} S^2,
\]

\[
B'' + dA'B' + \tilde{d}(B')^2 + \frac{(2\tilde{d} + 1)}{r} B' + \frac{d}{r} A' = -\frac{d}{2(D - 2)} S^2,
\]

\[
\tilde{d}B'' + dA'' - 2dA'B' + d(A')^2 - \tilde{d}(B')^2 - \frac{\tilde{d}}{r} B' - \frac{d}{r} A' + \frac{1}{2}(\Phi')^2 = \frac{1}{2} S^2
\]

for the metric field equations and

\[
\Phi'' + (dA' + \tilde{d}B')\Phi' + \frac{(\tilde{d} + 1)}{r} \Phi' = \frac{1}{2} \zeta \tilde{a} S^2
\]

for the dilaton field equation. And here we denoted by \( S \) the quantity

\[
S \equiv \left[ e^{(\tilde{a}\Phi/2 - dA + C)} C' \right] \quad \text{(electric case :} \quad d = (n - 1), \quad \zeta = -1),
\]

\[
S \equiv \left[ \lambda e^{(\tilde{a}\Phi/2 - \tilde{d}B)} \frac{1}{r^{(d+1)}} \right] \quad \text{(magnetic case :} \quad \tilde{d} = (D - n - 1), \quad \zeta = 1).
\]

Lastly, we note that the remaining antisymmetric R-R tensor gauge field equation (i) for the electric case reads
\[
\left[ C'' + \frac{(\tilde{d} + 1)C'}{r} \right] + C'(C' + \tilde{a}\Phi') - dA' + \tilde{d}B' = 0,
\]

\[
e^{(\tilde{a}\Phi - dA + \tilde{d}B)}(\partial_m e^C)e_{\nu_1...\nu_{(n-2)}}\partial_{\mu_1}(d\varepsilon \gamma(x))\gamma^{\mu\nu}\gamma^{\mu_1\nu_1}...\gamma^{\mu_{n-2}\nu_{n-2}} = 0
\]

whereas (ii) for the magnetic case, the R-R tensor gauge field equation always holds regardless of the specific solution forms for the metric and the dilaton fields.

We now attempt to solve these coupled field equations. The field equations at hand, however, look quite formidable yet. To proceed any further, therefore, we need some hint from the requirements for supersymmetry preservation. Namely, we shall now refine the solution ansatz above by imposing the “linearity” condition

\[
dA' + \tilde{d}B' = 0.
\]

This condition, as one can readily see in the field equations above, amounts to eliminating \(B(r)\) in favor of \(A(r)\) in the metric and dilaton field equations where it is interesting, even at this early stage, to note that these field equations themselves require the “Ricci-flatness” of the \(p\)-brane worldvolume metric \(\gamma_{\mu\nu}(x)\). That is, due to the imposed condition \(dA' + \tilde{d}B' = 0\), in order for the first and second equations of the metric field equation in (13) to be consistent, it is required that \(\hat{R}_{\mu\nu} = 0\). Next, further refining our solution ansatz by setting

\[
\Phi' = \frac{\zeta\tilde{a}(D - 2)}{d}A'
\]

the remaining set of three coupled equations for \(A(r)\) and \(\Phi(r)\), two from eq.(13) and one in eq.(14), can be readily integrated to yield

\[
e^{2A} = \exp\left[ \frac{2\tilde{a}}{\zeta\tilde{a}(D - 2)}\Phi \right] = H^{-4\tilde{a}/\Delta(D-2)}(r),
\]

\[
e^{2B} = \exp\left[ -\frac{2d}{\zeta\tilde{a}(D - 2)}\Phi \right] = H^{4d/\Delta(D-2)}(r), \quad \text{with}
\]

\[
H(r) = 1 + \frac{k}{r^{\tilde{d}}}
\]

being the harmonic function as a solution to the Laplace equation in \((D - d) = (\tilde{d} + 2)\)-spacelike transverse dimensions. The integration constant \(k\) sets the mass scale of the
solution and has been taken to be positive to ensure the absence of naked singularities at finite $r$. And here as for the other integration constants, $\Phi(\infty)$ has been set to zero for simplicity while we have chosen $A(\infty) = 0 = B(\infty)$ so that the solution tends to flat space at transverse infinity $r \to \infty$. Also note that we defined a parameter $\Delta$ such that

$$\tilde{a}^2 = \Delta - \frac{2dd}{(D-2)}.$$  \hfill (20)

Now, what remains is to check if these metric and dilaton solutions obtained are really consistent with the R-R tensor gauge field equations. As we already noted earlier, (ii) for the magnetic case, the tensor gauge field equations always hold regardless of the specific solution forms for the metric and dilaton fields. And this implies that the supergravity field equations generally admit “Ricci-flat” magnetic $p$-brane solutions. Next, (i) for the electric case, the first of the set of two R-R tensor gauge field equations given in eq.(16) is actually implied by the metric and dilaton field equations and hence can be readily integrated to yield

$$e^C = \frac{2}{\sqrt{\Delta}} H^{-1}(r).$$  \hfill (21)

And it is straightforward to see that the second of the two R-R tensor gauge field equations trivially holds. Thus to conclude, both the electric and magnetic Ricci-flat $p$-branes turn out to be legitimate supergravity solutions. For $p \leq 2$, however, there are no non-trivial Ricci-flat $p$-brane solutions since a Ricci-flat manifold with spacetime dimensions less than or equal to 3 is necessarily flat.\footnote{We would like to thank D. Brecher (private communication) for pointing out an incorrect statement regarding the electric Ricci-flat $p$-brane solutions in the earlier version of the present work.}

Finally, we summarize the dilatonic $Ricci$-flat $p$-brane solutions of general supergravity theories as

$$ds^2 = H^{-4d/\Delta(D-2)}(r)[\gamma_{\mu\nu}(x)dx^\mu dx^\nu] + H^{4d/\Delta(D-2)}(r)[dr^2 + r^2 d\Omega^2_{(d+1)}],$$

$$e^\Phi(r) = H^{-2\tilde{a}/\zeta \Delta}(r) \quad \text{with} \quad \hat{R}_{\mu\nu}(\gamma) = 0, \quad H(r) = 1 + \frac{k}{r^d}.$$  \hfill (22)
where we introduced the spherical-polar coordinates for the transverse dimensions $dy^m dy^m = dr^2 + r^2 d\Omega^2_{(\tilde{d}+1)}$ with $d\Omega^2_{(\tilde{d}+1)}$ being the metric on unit $(\tilde{d}+1)$-sphere. And (i) for the electric case ($\zeta = -1$), the R-R tensor gauge field strength is given by

$$F_{m \mu_1 \ldots \mu_{n-1}} = \epsilon_{\mu_1 \ldots \mu_{n-1}} \partial_m \left[ \frac{2}{\sqrt{\Delta}} H^{-1}(r) \right]$$

whereas (ii) for the magnetic case ($\zeta = 1$), the R-R tensor gauge field strength is

$$F_{m_1 m_2 \ldots m_{(D-n)}} = \lambda \epsilon_{m_1 m_2 \ldots m_{(D-n)}} \frac{y^l}{r^{(D-n+1)}}$$

with $k = \frac{\sqrt{\Delta}}{2d} \lambda$. (24)

2. Applications

Consider now, as a particular example, the bosonic sector of $D = 11$ SUGRA represented by the action

$$S_{11} = \int d^{11}X \sqrt{G}[R - \frac{1}{2 \cdot 4!} F^{2}_{[4]}] - \frac{1}{6} \int A_{[3]} \wedge F_{[4]} \wedge F_{[4]}$$

where

$$A_{[3]} = \frac{1}{3!} A_{[MNP]} dX^M \wedge dX^N \wedge dX^P, \quad F_{[4]} = dA_{[3]}.$$ 

When we attempt to apply the results associated with the Ricci-flat dilatonic $p$-brane solutions obtained above to this $D = 11$ SUGRA system, we first need to note two particular points. The first is that, here, no scalar field is present. This follows from the supermultiplet structure of $D = 11$ SUGRA theory in which all fields are gauge fields. In lower dimensions, of course, scalars do emerge; namely the dilaton field in $D = 10$ type IIA SUGRA appears upon dimensional reduction from $D = 11$ SUGRA to $D = 10$. The absence of the scalar in $D = 11$ SUGRA can then be handled, in the context of our general discussion provided above, by simply identifying the dilaton coupling parameter “$\tilde{a}$” with zero so that the scalar may be consistently truncated. Namely, from $\tilde{a}^2 = \Delta - 2 \tilde{d} \tilde{d}/(D - 2)$, we identify $\Delta = 2 \cdot 3 \cdot 6/9 = 4$ for the $D = 11$ SUGRA case. The second point to note is the presence of AFF Chern-Simons term in the action. This term is required by $D = 11$ local supersymmetry with the coefficient precisely as given. Although we did not take into account the effects of this AFF term in our general discussion, this omission is not essential to the basic class of $p$-brane solutions we are considering.
Now, as an obvious example of Ricci-flat, magnetic $M$-brane solution to $D = 11$ SUGRA, we consider the $M5$-brane solution. Plugging $D = 11$, $\tilde{a} = 0$, $d = (p + 1) = 6$, $\tilde{d} = 3$, and $\Delta = 4$ in the general solution given in eqs.(22) and (24), it is given by (also see [11])

$$ds^2 = H^{-1/3}(r)[\gamma_{\mu\nu}(x)dx^\mu dx^\nu] + H^{2/3}(r)[dr^2 + r^2d\Omega_4^2],$$

$$H(r) = 1 + \frac{k}{r^3}, \quad \text{and}$$

$$F_{m_1m_2m_3m_4} = \epsilon_{m_1m_2m_3m_4n} \frac{3ky^n}{r^5}$$

where we used $\lambda = 2\tilde{d}k/\sqrt{\Delta} = 3k$ and $\mu, \nu = 0, \ldots, 5$ and $m, n = 6, \ldots, 10$. At first glance, it may look like that the metric/gauge solutions are singular at $r = 0$ just as we encountered for the Poincare-invariant $M5$-brane case. However, if one evaluates invariant components of the curvature tensor and the R-R gauge field strength, one readily finds that these invariants are non-singular at $r = 0$. Moreover, although the proper distance to the surface $r = 0$ along a $t = \text{const.}$ geodesic diverges, the surface $r = 0$ can be reached along null geodesics in finite affine parameter. This implies that the timelike hypersurface $r = 0$ is a horizon, i.e., a coordinate singularity. Besides, close inspection reveals that actually $r = 0$ is a “degenerate” horizon which is known to occur typically for extremal black holes when the inner and outer horizons coalesce. The near-horizon geometry of this $M5$-brane metric solution, however, fails to be that of $AdS_7 \times S_4$ due to the Ricci-flat nature of the worldvolume geometry. And the integration constant $k$ appearing in the Harmonic function $H(r)$ representing the mass scale of the solution can be explicitly determined for Poincare-invariant worldvolume metric solution $\gamma_{\mu\nu}(x) = \eta_{\mu\nu}$ in which case, $k = \pi Nl_{11}^3$ with $l_{11}$ denoting the 11-dimensional Planck length and $N$ the number of branes generally for stack of coincident branes.

We now turn to the bosonic sector of $D = 10$ type IIA/IIB SUGRA theories. It is well-recognized that $D = 10$ type IIA/IIB SUGRA theories can be thought of as the low energy effective theories of type IIA/IIB superstring theories. And in the application of our general $p$-brane solutions given earlier to this $D = 10$ type IIA/IIB SUGRA theories, solutions of particular interest are the “$\tilde{a} = 0$ subsets” for which the dilaton is set to zero or just a constant. Thus here, we just consider these “non-dilatonic” $p$-brane solutions.
Our first example is the Ricci-flat, magnetic $D3$-brane solution of type IIB theory. Plugging $D = 10$, $\tilde{a} = 0$, $d = (p + 1) = 4$, $\tilde{d} = 4$, and $\Delta = 4$ in eqs.(22) and (24), it is given by

$$ds^2 = H^{-1/2}(r)[\gamma_{\mu\nu}(x)dx^\mu dx^\nu] + H^{1/2}(r)[dr^2 + r^2d\Omega_5^2],$$

$$H(r) = 1 + \frac{k}{r^4}, \quad \text{and}$$

$$F_{m_1...m_5} = \epsilon_{m_1...m_5n} \frac{4ky^n}{r^6}$$

where we used $\lambda = 2\tilde{d}k/\sqrt{\Delta} = 4k$ and $\mu, \nu = 0, ...3$ and $m, n = 4, ...9$. The 5-form R-R tensor field strength here is self-dual. Again, the $r = 0$ timelike hypersurface is a harmless degenerate horizon and the integration constant $k$ appearing in the Harmonic function $H(r)$ can be explicitly determined for Poincare-invariant worldvolume metric solution $\gamma_{\mu\nu}(x) = \eta_{\mu\nu}$ in which case, $k = 4\pi g_s N l_s^2$ with $g_s$ being the string coupling constant and $l_s = \sqrt{\alpha'}$ the string length. And due to the generally Ricci-flat nature of the worldvolume metric, the near-horizon geometry of this $D3$-brane fails to be that of $AdS_5 \times S^5$.

Lastly, we consider the Ricci-flat, magnetic $D4$-brane solution of type IIA theory which is of our central concern in this work. Plugging $D = 10$, $\tilde{a} = 0$, $d = (p + 1) = 5$, $\tilde{d} = 3$, and $\Delta = 15/4$ in eqs.(22) and (24), it is given by

$$ds^2 = H^{-2/5}(r)[\gamma_{\mu\nu}(x)dx^\mu dx^\nu] + H^{2/3}(r)[dr^2 + r^2d\Omega_4^2],$$

$$H(r) = 1 + \frac{k}{r^3}, \quad \text{and}$$

$$F_{m_1...m_4} = \epsilon_{m_1...m_4n} \frac{12ky^n}{\sqrt{15}r^5}$$

where we used $\lambda = 2\tilde{d}k/\sqrt{\Delta} = 12k/\sqrt{15}$ and $\mu, \nu = 0, ...4$ and $m, n = 5, ...9$. $r = 0$ is again a degenerate horizon and for Poincare-invariant worldvolume metric solution, the integration constant $k$ can be explicitly determined to be $k \sim g_s N l_s^3$. Here, also the generally Ricci-flat nature of the worldvolume keeps the near-horizon geometry of this $D4$-brane from being that of $AdS_6 \times S^4$.

We now end with the remark on the asymptotic geometry of this $D4$-brane metric solution as $r \to \infty$ [11]. Far from the source 4-brane, or more precisely, at infinities in null directions, $H(r) \to 1$ and hence
\[ ds^2 \simeq \gamma_{\mu\nu}(x) dx^\mu dx^\nu + dy^m dy^m \]  
(29)

which is Ricci-flat as a whole. For instance, therefore, we may take the Ricci-flat worldvolume metric as that of (see also [11])

\[ R \times (\text{Taub} - \text{NUT}) : \quad ds^2 = [-dt^2 + d\hat{s}_{TN}^2] + dy^m dy^m, \]  
\[ R \times (\text{Eguchi} - \text{Hanson}) : \quad ds^2 = [-dt^2 + d\hat{s}_{EH}^2] + dy^m dy^m \]  
(30)

where \( d\hat{s}_{TN}^2 \) and \( d\hat{s}_{EH}^2 \) denote the Taub-NUT [18] and Eguchi-Hanson [19] “gravitational instantons” satisfying the Ricci-flat 4-dimensional worldspace metric and obeying (anti) self-dual worldspace Riemann curvature tensor.

Now, this completes the construction of Ricci-flat, extremal \( p \)-brane solutions in supergravity theories. Eventually, we shall be interested in the explicit construction of \( SU(2) \) Yang-Mills instanton solutions in the background geometry of a stack of two coincident probe \( D4 \)-brane worldspaces particularly when the metric of target spacetime in which the probe branes are embedded is given by the Ricci-flat, magnetic extremal 4-brane solution in type IIA theory with its worldspace metric being given by that of Taub-NUT or Eguchi-Hanson gravitational instanton. And then we shall demonstrate that with this YM instanton-gravitational instanton configuration on the probe \( D4 \)-branes’ worldvolume, the energy of the probe branes attains its minimum value and hence enjoys stable state provided one employs the Tseytlin’s non-abelian DBI action [4] for the description of multiple probe \( D \)-branes.

### III. Non-abelian DBI action which “knows” energy-minimizing worldvolume solitons

In general, the abelian Dirac-Born-Infeld (DBI) action describing the low energy dynamics of a single probe \( Dp \)-brane is given by [1]

\[ S_{DBI}^A = \frac{1}{(2\pi)^p(\alpha')^{p+1/2}} \int d^{p+1}x e^{-\Phi} \sqrt{\det [g_{\mu\nu} + 2\pi \alpha' (B + F)_{\mu\nu}]} \]  
(31)

where \( T_p \equiv (1/(2\pi)^p(\alpha')^{(p+1)/2}) \) is the brane tension and inside the square root, \( |\det [g_{\mu\nu} + 2\pi \alpha' (B + F)_{\mu\nu}]| \), i.e., its absolute value is understood. \( \Phi \) is the dilaton field and \( B_{\mu\nu} \) and \( g_{\mu\nu} \)
are the “pullbacks” of bulk NS-NS antisymmetric tensor gauge field and the target spacetime metric to the probe brane respectively in the sense, say, that

\[ g_{\mu\nu} = G_{MN} \frac{\partial X^M}{\partial x^\mu} \frac{\partial X^N}{\partial x^\nu} \bigg|_{brane} = G_{\mu\nu} + \partial_\mu \phi^m \partial_\nu \phi^m \]

(32)

where again \( \mu, \nu = 0, 1, ... p \) and \( m = (p+1), ...(D-1) \) denote the indices for longitudinal and transverse coordinates to the brane as before and we identified the “worldvolume scalars” as \( \phi^m \equiv y^m \) representing the excitations of the brane along directions transverse to the brane worldvolume. Hereafter, however, we shall only consider the case when no worldvolume scalars are excited, i.e., \( \phi^m = 0 \) for all \( m = (p+1), ...(D-1) \). Thus one may assume \( g_{\mu\nu} = G_{\mu\nu} \) throughout. \( F_{\mu\nu} \) is the \( U(1) \) gauge field strength living on the brane that represents the fluctuations of the brane along longitudinal directions. Thus it should not be confused with the R-R tensor gauge field strength in the previous section. Note also that in this expression for the DBI action above, we fixed the worldvolume reparametrization invariance by taking the physical “static” gauge in which the worldvolume parameters \( \{\sigma^\mu\} \) are identified with the first \((p + 1)\) target spacetime coordinates

\[ \sigma^\mu = x^\mu, \quad (\mu = 0, ... p). \]

(33)

We now consider the target spacetime with geometry given by that of the \( p \)-brane solution arising in various supergravity theories such as the \( D = 11 \) SUGRA or \( D = 10 \) type IIA/IIB (non-dilatonic) SUGRA. And particularly, consider the source \( p \)-brane solutions as the “Ricci-flat” solutions we discussed in the previous section given by

\[ ds^2 = G_{MN}dX^M dX^N \]

(34)

\[ = H^{-4d/\Delta(D-2)}(r)[\gamma_{\mu\nu}(x)dx^\mu dx^\nu] + H^{4d/\Delta(D-2)}(r)dy^m dy^m \]

with

\[ H(r) = 1 + \frac{k}{r^d}. \]

(35)

Namely, we consider a test probe \( Dp \)-brane in the background of Ricci-flat source \( p \)-brane geometry. Then the pullback of this target spacetime metric \( G_{MN} \) to the probe \( Dp \)-brane worldvolume is
\[ g_{\mu\nu} = G_{MN} \frac{\partial X^M}{\partial x^\mu} \frac{\partial X^N}{\partial x^\nu} = H^{-4\bar{d}/\Delta(D-2)}(r_0)\gamma_{\mu\nu}(x). \] (36)

Further, we may take a (non-coordinate) orthonormal basis for this \((p + 1)\)-dimensional \(Dp\)-brane worldvolume, i.e., \(e^a = e^a_\mu dx^\mu\) such that \(g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu\). Then the tetrad components of the worldvolume gauge field is

\[ F_{ab} = F_{\mu\nu} e^a_\mu e^b_\nu, \quad \text{or inversly} \quad F_{\mu\nu} = F_{ab} e^a_\mu e^b_\nu \] (37)

then obviously

\[ F^2 = F_{\mu\nu} F^{\mu\nu} = F_{ab} F^{ab}, \quad F \cdot \bar{F} = F_{\mu\nu} \bar{F}^{\mu\nu} = F_{ab} \bar{F}^{ab}, \]
\[ \bar{F}^2 = \bar{F}_{\mu\nu} \bar{F}^{\mu\nu} = \bar{F}_{ab} \bar{F}^{ab}, \quad \text{etc.} \] (38)

Particularly, therefore, if we turn off the dilaton \(\Phi(x)\) and the NS-NS gauge field \(B_{\mu\nu}(x)\), the DBI action, in terms of the tetrad components of the tensor fields involved, takes the form

\[ S_{DBI}^A = T_p \int d^{p+1}x \left[ \sqrt{g} - \sqrt{\det (g_{\mu\nu} + F_{\mu\nu})} \right] \] (39)
\[ = T_p \int d^{p+1}x \sqrt{g} \left[ 1 - \sqrt{\det (\eta_{ab} + F_{ab})} \right] \]

for a single probe \(Dp\)-brane with abelian worldvolume gauge field. Note that here we absorbed the factor \(2\pi\alpha'\) into the redefinition of \(F_{\mu\nu}\) and added a constant term corresponding to the Lagrangian density \(\sim T_p \sqrt{g}\) to the DBI action and used \(\det (g_{\mu\nu} + F_{\mu\nu}) = \det (e^a_\mu e^b_\nu) \det (\eta_{ab} + F_{ab}) = g \det (\eta_{ab} + F_{ab})\).

We now generalize our discussion to the case of a stack of \(N\)-coincident probe \(Dp\)-branes. Then following the argument of Witten [3], non-abelian \(SU(N)\) gauge theory should provide a good description of the relevant low energy dynamics of \(N\)-coincident \(Dp\)-branes and hence the non-abelian generalization of a multiple \(Dp\)-brane DBI action with \(SU(N)\) worldvolume gauge field may be taken as

\[ S_{DBI}^A = T_p \int d^{p+1}x \sqrt{g} \{ Tr, Str, (Str + i \ Atr) \} \left[ I - \sqrt{\det (\eta_{ab} + F_{ab})} \right] \] (40)
where $I$ is the unit $SU(N)$ matrix and now $F_{ab} = F_{ab}^{TA}$ with $T^A$ ($A = 1, 2, \ldots (N^2 - 1)$) being the $SU(N)$ generators. Of course, here, the “traces” are over the group indices to make the DBI action a group scalar and we defined

$$
Str(M_1 \ldots M_n) \equiv \frac{1}{n!} \sum_{\pi} Tr(M_{\pi(1)} \ldots M_{\pi(n)}),
$$

(41)

$$
Atr(M_1 \ldots M_n) \equiv \frac{1}{n!} \sum_{\pi} (-1)^{\pi} Tr(M_{\pi(1)} \ldots M_{\pi(n)}),
$$

and the factor “$i$” in $(Str + i Atr)$ is introduced since the basis of the group algebra is Hermitian. And among the different choices for the trace operations, $Tr[\ldots]$ can be found in Polchinski’s review article on D-branes [1], $Str[\ldots]$ was proposed by Tseytlin [4], and lastly $(Str + i Atr)[\ldots]$ has been suggested by Argyres and Nappi [5]. And in the most general sense, concerning the question as to what the correct generalization of the DBI action to the non-abelian case is (i.e., which specific trace operation one should take), the issue seems to be somewhat ambiguous. Nevertheless, here it will be argued that the $Str[\ldots]$ proposal put forward by Tseytlin is indeed singled out by noting that with this choice, the non-abelian DBI action “knows” about energy-minimizing BPS states, or worldvolume solitons.

The following argument is taken directly from [10,11] and as was pointed out there, it can be made explicit by taking the D4-brane for example. Note that for D4-brane case, the worldvolume is $(p + 1) = 5$-dimensional and hence the spacetime determinant of the DBI action can be computed as [10]

$$
|det (\eta_{ab} + F_{ab})| = -det (\eta_{ab} I + F_{ab})
$$

(42)

$$
= I + \frac{1}{2} F^2 + \frac{1}{3} F^3 - \frac{1}{4} [F^4 - \frac{1}{2} (F^2)^2] + \frac{1}{5} F^5 + \frac{1}{12} (F^2 F^3 + F^3 F^2)
$$

where $F^2 = F_{ab} F^{ab}$, $F^3 = F_{ab} F^{bc} F^c_a$, $F^4 = F_{ab} F^{bc} F_{cd} F^{da}$, and $F^5 = F_{ab} F^{bc} F_{cd} F^{de} F^e_a$. In the abelian case, all the odd powers of $F_{ab}$ vanish but this is not true for the case at hand. Besides, since the complete DBI action involves $[-det (\eta_{ab} I + F_{ab})]^{1/2}$, we need to evaluate the binomial expansion of above expression which would obviously results in an infinite series containing terms of both even and odd powers of $F_{ab}$. And the trace operation should be taken “after” this binomial series expansion. Then the important properties of the $Str$ and
Atr operations is that they pick out just the even and odd powers of \( F_{ab} \) respectively. This point is clear if one takes the \( SU(2) \) case for example and moreover, at least to the first few orders, the same will apply to the cross terms generated by the binomial expansion; that is, e.g., \( Str(F^2 F^3) = Str(F^2 F^5) = 0 \). Generally speaking, therefore, the non-abelian version of DBI action with \( \text{“} Str \text{”} \) can be written as a sum of even powers of \( F_{ab} \) alone, whereas that with \( \text{“} Tr \text{”} \) or \( \text{“} Atr \text{”} \) would involve odd powers of \( F_{ab} \) as well. Indeed this point was the motivation behind Tseytlin’s proposal. Since odd powers of \( F_{ab} \) can be written in terms of derivatives of \( F_{ab} \), i.e., \( F^3 \sim [F,F]F \sim [D,D]F \), they, Tseytlin thought, should not appear in non-abelian DBI theory action just as the abelian DBI action does not involve derivatives of the field strength. Thus Tseytlin was led to define the non-abelian DBI action in terms of \( \text{“} Str \text{”} \) operation alone so that it depends only on even powers of \( F_{ab} \). Having reviewed the nature of Tseytlin’s proposal for the non-abelian DBI action, we now provide an evidence that strongly supports Tseytlin’s action by invoking the argument concerning the energy-minimizing BPS states or worldvolume solitons. Here, in this work, we are particularly interested in the “instantons in \( D4 \)-branes” and thus we consider static configurations of \( D4 \)-branes with no worldvolume scalars being excited. Then \( F_{a0} = E_\alpha = 0 \) (where \( a, b = 0, 1, \ldots, p = 4 \) are worldvolume and \( \alpha, \beta = 1, \ldots, 4 \) are worldspace indices respectively) and hence

\[
- \det (\eta_{ab} I + F_{ab}) = - (\eta_{00}) \det (\eta_{a\beta} I + F_{a\beta}) \\
= 1 + \frac{1}{4} F_{a\beta} F_{a\beta} + \frac{1}{4} \tilde{F}_{a\beta} \tilde{F}_{a\beta} + \frac{1}{16} (F_{a\beta} \tilde{F}_{a\beta})^2 \\
= \left(1 \pm \frac{1}{4} F_{a\beta} \tilde{F}_{a\beta}\right)^2 + \frac{1}{4} \left(F_{a\beta} \mp \tilde{F}_{a\beta}\right)^2.
\]

(43)

Note, here, that \( \tilde{F}_{a\beta} = \epsilon_{a\beta\lambda\sigma} F^{\lambda\sigma} / 2 \) is the Hodge dual of \( F_{a\beta} \) with respect to the worldspace indices only. Therefore, Tseytlin’s non-abelian DBI action for the static configurations of a \( D4 \)-brane becomes

\[
L_{DBI} = T_4 \sqrt{g} Str [I - \sqrt{\det (\eta_{ab} + F_{ab})}] \\
= T_4 \sqrt{g} Str \left[ I - \sqrt{\left(1 \pm \frac{1}{4} F \cdot \tilde{F}\right)^2 + \frac{1}{4} |F \mp \tilde{F}|^2}\right]
\]

(44)
where to get the expression in the second line, use has been made of the symmetry properties of the “$\text{Str}$” operation. That is, since the symmetric trace will be taken only at the end, during the course of calculation, the matrices $F_{\alpha\beta} = (F^A_{\alpha\beta}T^A)$ can be treated as if they were “abelian” until the last at which point the non-commuting group generators can be re-inserted. And of course, this scheme can be justified only if we employ Tseytlin’s “$\text{Str}$” proposal in which one can effectively assume $AB = BA$. Obviously, therefore, one would never get the expression in the second line above for the non-abelian DBI action if he or she employed other trace operations such as “$\text{Tr}$” or “$\text{Atr}$”. Then what is so special about this last expression for the non-abelian DBI action? Notice that for the (anti) self-dual non-abelian gauge field configuration having $F_{\alpha\beta} = \pm \tilde{F}_{\alpha\beta}$, the determinant can be written as a “complete square” linearizing the DBI action and then turning it into that of usual Yang-Mills theory. What is more, since we are dealing with static configurations, the energy density of the $D4$-brane is just $H_{DBI} = -L_{DBI}$, which gets minimized if and only if $F_{\alpha\beta} = \pm \tilde{F}_{\alpha\beta}$. Thus the BPS condition, or the (anti) self-duality of non-abelian gauge theory instanton solution at once linearizes the otherwise highly non-linear DBI action and minimizes the energy of the $D4$-brane provided Tseytlin’s non-abelian DBI action is employed [10,11].

Before closing, a cautious comment might be relevant to eliminate a possible confusion. It might seem as if we simply put the non-abelian gauge field configuration having $F_{\alpha\beta} = \pm \tilde{F}_{\alpha\beta}$ by hand into $L_{DBI} = -H_{DBI}$ and then announce naively that, if one does so, the DBI action linearizes and the Hamiltonian gets minimized. What happens, however, is that the YM gauge field having $F_{\alpha\beta} = \pm \tilde{F}_{\alpha\beta}$, or the BPS condition, is indeed a solution to the Euler-Lagrange’s equation of motion that results from the DBI action provided one employs the Tseytlin’s action much as the same BPS condition automatically implies the linear gauge field equation in ordinary YM gauge theory. To see this, consider that generally the (non-abelian) gauge field $A_\mu$ ($\mu = 0,...4$) living on the worldvolume and the worldvolume scalar $\phi^m = y^m$ ($m = 5,...9$) represent the excitations of the probe $D4$-brane along directions longitudinal and transverse to the worldvolume. Thus in principle, both $A_\mu$ and $\phi^m$ are
to be dynamically determined by solving the Euler-Lagrange’s equation of motion that can be obtained by extremizing the DBI action. Now the expression for the DBI action for a probe $D4$-brane given in eq.(44) holds for the case with no worldvolume scalars and static configurations (provided the Tseytlin’s action is adopted) and manifestly it gets extremized for $F_{\alpha\beta} = \pm \tilde{F}_{\alpha\beta}$. Namely, the non-abelian gauge field configuration having the (anti) self-dual field strength is itself the saddle point of the DBI action satisfying the associated Euler-Lagrange’s equation of motion. Thus in order to demonstrate that this procedure can actually work, the actual question would be whether or not one can construct, say, $SU(2)$ Yang-Mills instanton solution having (anti) self-dual field strength in the background of Ricci-flat $D4$-brane worldspace with metric given, say, by 4-dimensional gravitational instantons. The answer is, as we shall see in the next section, “yes”.

IV. YM instantons in gravitational instanton backgrounds

Here in this section, eventually in order to construct the $SU(2)$ YM instanton solutions particularly in the background of TN and EH gravitational instantons, we begin by presenting a “simply physical” and hence perhaps the most direct algorithm for generating the YM instanton solutions practically in all species of known GI [16]. As we shall see in a moment, the essence of this method lies in writing the (anti) self-dual YM equation by employing truly relevant ansatz for the YM gauge connection and then directly solving it. After presenting the general formulation describing the algorithm, we shall apply the algorithm to the case of the TN and the EH metrics, the two best-known GI. As we shall discuss later on in the Appendix, interestingly the solutions to (anti) self-dual YM equation turn out to be the rather exotic type of instanton configurations which are everywhere non-singular having finite YM action but sharing some features with meron solutions [20] such as their typical structure and generally fractional topological charge values carried by them. Namely, the YM instanton solution that we shall discuss in the background of GI in this work exhibit characteristics which are mixture of those of typical instanton and typical meron. This seems remarkable since it is well-known that in flat spacetime, meron does not solve the 1st order
(anti) self-dual equation although it does the 2nd order YM field equation and is singular at its center and has divergent action.

In the loose sense, GI may be defined as a positive-definite metrics $g_{\mu\nu}$ on a complete and non-singular manifold satisfying the Euclidean Einstein equations and hence constituting the stationary points of the gravity action in Euclidean path integral for quantum gravity. There are several solutions to Euclidean Einstein equations that can fall into the category of the GI of this sort. But in the stricter sense [21], they are the metric solutions to the Euclidean Einstein equations having (anti) self-dual Riemann tensor

$$\tilde{R}_{abcd} = \frac{1}{2} \epsilon_{ab}^{\ e f} R_{e f c d} = \pm R_{abcd}$$  \hspace{1cm} (45)

(say, with indices written in non-coordinate orthonormal basis) and include only two families of solutions in a rigorous sense; the TN metric [18] and the EH instanton [19]. Recall that we are mainly interested in the explicit construction of SU(2) YM instanton solutions in the background of a probe $D4$-brane worldspace geometry particularly when the metric of target spacetime in which the probe brane is embedded is given by the Ricci-flat, magnetic extremal 4-brane solution in type IIA theory with its worldspace metric being given by that of TN or EH GI. Thus in this section, we shall be interested exclusively in the construction of YM instantons in the background of these two GI satisfying the rigorous definition.

We now begin with the action governing our system, i.e., the Einstein-Yang-Mills (EYM) theory given by

$$I_{EYM} = \int_M d^4x \sqrt{g} \left[ -\frac{1}{16\pi} R + \frac{1}{4g_c^2} F_{\mu\nu}^a F^{a\mu\nu} \right] - \int_{\partial M} d^3x \sqrt{h} \frac{1}{8\pi} K$$ \hspace{1cm} (46)

where $F_{\mu\nu}^a$ is the field strength of the YM gauge field $A_\mu^a$ with $a = 1, 2, 3$ being the SU(2) group index and $g_c$ being the gauge coupling constant. The Gibbons-Hawking term on the boundary $\partial M$ of the manifold $M$ is also added and $h$ is the metric induced on $\partial M$ and $K$ is the trace of the second fundamental form on $\partial M$. Then by extremizing this action with respect to the metric $g_{\mu\nu}$ and the YM gauge field $A_\mu^a$, one gets the following classical field equations respectively.
\[
R_{\mu
u} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu},
\]
\[
T_{\mu\nu} = \frac{1}{g_c^2} \left[ F_{\mu\alpha} F_{\nu}^{\alpha} - \frac{1}{4} g_{\mu\nu}(F_{\alpha\beta} F^{\alpha\beta}) \right],
\]
\[
D_\mu [\sqrt{g} F_a^{\mu\nu}] = 0, \quad D_\mu [\sqrt{g} \tilde{F}_a^{\mu\nu}] = 0
\]
where we added Bianchi identity in the last line and \( F_{\mu\nu} = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \epsilon^{abc} A_\mu^b A_\nu^c \), \( D_\mu = \partial_\mu \delta^{ac} + \epsilon^{abc} A_\mu^b \) and \( A_\mu = A_\mu^a (iT^a) \), \( F_{\mu\nu} = F_{\mu\nu}^a (-iT^a) \) with \( T^a = \tau^a/2 \) \( (a = 1, 2, 3) \) being the SU(2) generators and finally \( \tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu}^{\alpha\beta} F_{\alpha\beta} \) is the (Hodge) dual of the field strength tensor. We now seek solutions \( (g_{\mu\nu}, A_\mu^a) \) of the coupled EYM equations given above in Euclidean signature obeying the (anti) self-dual equation in the YM sector
\[
F^{\mu\nu} = g^{\mu\lambda} g^{\nu\sigma} F_{\lambda\sigma} = \pm \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}
\]
where \( \epsilon^{\mu\nu\alpha\beta} = \epsilon^{\mu\nu\alpha\beta}/\sqrt{g} \) is the curved spacetime version of totally antisymmetric tensor. As was noted in [13,14,16], in Euclidean signature, the YM energy-momentum tensor vanishes identically for YM fields satisfying this (anti) self-duality condition. This point is of central importance and can be illustrated briefly as follows. Under the Hodge dual transformation, \( F_{\mu\nu}^a \rightarrow \tilde{F}_{\mu\nu}^a \), the YM energy-momentum tensor \( T_{\mu\nu} \) given in eq.(47) above is invariant normally in Lorentzian signature. In Euclidean signature, however, its sign flips, i.e., \( \tilde{T}_{\mu\nu} = -T_{\mu\nu} \). As a result, for YM fields satisfying the (anti) self-dual equation in Euclidean signature such as the instanton solution, \( F_{\mu\nu}^a = \pm \tilde{F}_{\mu\nu}^a \), it follows that \( T_{\mu\nu} = -\tilde{T}_{\mu\nu} = -T_{\mu\nu} \), namely the YM energy-momentum tensor vanishes identically, \( T_{\mu\nu} = 0 \). This, then, indicates that the YM field now does not disturb the geometry while the geometry still does have effects on the YM field. Consequently the geometry, which is left intact by the YM field, effectively serves as a “background” spacetime which can be chosen somewhat at our will (as long as it satisfies the vacuum Einstein equation \( R_{\mu\nu} = 0 \)) and here in this work, we take it to be the gravitational instanton. Loosely speaking, all the typical GI, including TN metric and EH solution, possess the same topology \( R \times S^3 \) and similar metric structures. Of course in a stricter sense, their exact topologies can be distinguished, say, by different Euler numbers and H"{u}rlebeck signatures [21]. Particularly, in terms of the concise basis 1-forms,
the metrics of these GI can be written as [21,22]

\[ ds^2 = c_r^2 dr^2 + c_a^2 \left( \sigma_1^2 + \sigma_2^2 \right) + c_3^2 \sigma_3^2 \]

\[ = c_r^2 dr^2 + \sum_{a=1}^{3} c_a^2 (\sigma^a)^2 = e^A \otimes e^A \]  

(49)

where \( c_r = c_r(r) \), \( c_a = c_a(r) \), \( c_1 = c_2 \neq c_3 \) and the orthonormal basis 1-form \( e^A \) is given by

\[ e^A = \left\{ e^0 = c_r dr, \quad e^a = c_a \sigma^a \right\} \]  

(50)

and \( \{ \sigma^a \} (a = 1, 2, 3) \) are the left-invariant 1-forms satisfying the SU(2) Maurer-Cartan structure equation

\[ d\sigma^a = -\frac{1}{2} \epsilon^{abc} \sigma^b \wedge \sigma^c. \]  

(51)

They form a basis on the \( S^3 \) section of the geometry and hence can be represented in terms of 3-Euler angles \( 0 \leq \theta \leq \pi \), \( 0 \leq \phi \leq 2\pi \), and \( 0 \leq \psi \leq 4\pi \) parametrizing \( S^3 \) as

\[
\begin{align*}
\sigma^1 &= -\sin \psi d\theta + \cos \psi \sin \theta d\phi, \\
\sigma^2 &= \cos \psi d\theta + \sin \psi \sin \theta d\phi, \\
\sigma^3 &= -d\psi - \cos \theta d\phi.
\end{align*}
\]  

(52)

Now in order to construct exact YM instanton solutions in the background of these GI, we now choose the relevant ansätz for the YM gauge potential and the SU(2) gauge fixing. And in doing so, our general guideline is that the YM gauge field ansätz should be endowed with the symmetry inherited from that of the background geometry, the GI. Thus we first ask what kind of isometry these GI possess. As noted above, typical GI, including the TN and the EH metrics, possess the topology of \( R \times S^3 \). The geometrical structure of the \( S^3 \) section, however, is not that of perfectly “round” \( S^3 \) but rather, that of “squashed” \( S^3 \). In order to get a closer picture of this squashed \( S^3 \), we notice that the \( r \) =constant slices of these GI can be viewed as U(1) fibre bundles over \( S^2 \sim CP^1 \) with the line element

\[ d\Omega_3^2 = c_1^2 (\sigma_1^2 + \sigma_2^2) + c_3^2 \sigma_3^2 = c_1^2 d\Omega_2^2 + c_3^2 (d\psi + B)^2 \]  

(53)
where \( d\Omega_2^2 = (d\theta^2 + \sin^2 \theta d\phi^2) \) is the metric on unit \( S^2 \), the base manifold whose volume form \( \Omega_2 \) is given by \( \Omega_2 = dB \) as \( B = \cos \theta d\phi \) and \( \psi \) then is the coordinate on the \( U(1) \sim S^1 \) fibre manifold. Now then the fact that \( c_1 = c_2 \neq c_3 \) indicates that the geometry of this fibre bundle manifold is not that of round \( S^3 \) but that of squashed \( S^3 \) with the squashing factor given by \((c_3/c_1)\). And further, it is squashed along the \( U(1) \) fibre direction. Thus this failure for the geometry to be that of exactly round \( S^3 \) keeps us from writing down the associated ansatz for the YM gauge potential right away. Apparently, if the geometry were that of round \( S^3 \), one would write down the YM gauge field ansatz as \( A^a = f(r)\sigma^a \) [14,16] with \( \{\sigma^a\} \) being the left-invariant 1-forms introduced earlier. The rationale for this choice can be stated briefly as follows. First, since the \( r = \)constant sections of the background space have the geometry of round \( S^3 \) and hence possess the \( SO(4) \)-isometry, one would look for the \( SO(4) \)-invariant YM gauge connection ansatz as well. Next, noticing that both the \( r = \)constant sections of the frame manifold and the \( SU(2) \) YM group manifold possess the geometry of round \( S^3 \), one may naturally choose the left-invariant 1-forms \( \{\sigma^a\} \) as the “common” basis for both manifolds. Thus this YM gauge connection ansatz, \( A^a = f(r)\sigma^a \) can be thought of as a hedgehog-type ansatz where the group-frame index mixing is realized in a simple manner [14,16]. Then coming back to our present interest, namely the GI given in eq.(49), in \( r = \)constant sections, the \( SO(4) \)-isometry is partially broken down to that of \( SO(3) \) by the squashedness along the \( U(1) \) fibre direction to a degree set by the squashing factor \((c_3/c_1)\). Thus now our task became clearer and it is how to encode into the YM gauge connection ansatz this particular type of \( SO(4) \)-isometry breaking coming from the squashed \( S^3 \). Interestingly, a clue to this puzzle can be drawn from the work of Eguchi and Hanson [23] in which they constructed abelian instanton solution in Euclidean TN metric (namely the abelian gauge field with (anti)self-dual field strength with respect to this metric). To get right to the point, the working ansatz they employed for the abelian gauge field to yield (anti)self-dual field strength is to align the abelian gauge connection 1-form along the squashed direction, i.e., along the \( U(1) \) fibre direction, \( A = g(r)\sigma^3 \). This choice looks quite natural indeed. After all, realizing that embedding of a gauge field in a geometry with high
degree of isometry is itself an isometry (more precisely isotropy)-breaking action, it would be natural to put it along the direction in which part of the isometry is already broken. Finally therefore, putting these two pieces of observations carefully together, now we are in the position to suggest the relevant ansatz for the YM gauge connection 1-form in these GI and it is

$$A^a = f(r)\sigma^a + g(r)\delta^{a3}\sigma^3$$  \hspace{1cm} (54)

which obviously would need no more explanatory comments except that in this choice of the ansatz, it is implicitly understood that the gauge fixing $A_r = 0$ is taken. From this point on, the construction of the YM instanton solutions by solving the (anti)self-dual equation given in eq.(48) is straightforward. To sketch briefly the computational algorithm, first we obtain the YM field strength 2-form (in orthonormal basis) via exterior calculus (since the YM gauge connection ansatz is given in left-invariant 1-forms) as

$$F^1 = \frac{f'}{c_r c_1}(e^0 \wedge e^1) + \frac{f[(f - 1) + g]}{c_2 c_3}(e^2 \wedge e^3),$$

$$F^2 = \frac{f'}{c_r c_2}(e^0 \wedge e^2) + \frac{f[(f - 1) + g]}{c_3 c_1}(e^3 \wedge e^1),$$

$$F^3 = \frac{(f' + g')}{c_r c_3}(e^0 \wedge e^3) + \frac{[f(f - 1) - g]}{c_1 c_2}(e^1 \wedge e^2)$$  \hspace{1cm} (55)

from which we can read off the (anti)self-dual equation to be

$$\pm \frac{f'}{c_r c_1} = \frac{f[(f - 1) + g]}{c_2 c_3}, \quad \pm \frac{(f' + g')}{c_r c_3} = \frac{[f(f - 1) - g]}{c_1 c_2}$$  \hspace{1cm} (56)

where “+” for self-dual and “−” for anti-self-dual equation and we have only a set of two equations as $c_1 = c_2$. The specifics of different GI are characterized by particular choices of the orthonormal basis $e^A = \{e^0 = c_r dr, \ e^a = c_a \sigma^a\}$. Thus next, for each GI (i.e., for each choice of $e^A$), we solve the (anti) self-dual equation in (56) for ansatz functions $f(r)$ and $g(r)$ and finally from which the YM instanton solutions in eq.(54) and their (anti)self-dual field strength in eq.(55) can be obtained. We now present the solutions obtained by applying the algorithm introduced here to the two best-known GI, the TN and the EH metrics.

(I) YM instanton in Taub-NUT metric background

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The TN GI solution written in the metric form given in eq.(49) amounts to
\[ c_r = \frac{1}{2} \left[ \frac{r + m}{r - m} \right]^{1/2}, \quad c_1 = c_2 = \frac{1}{2} \left[ r^2 - m^2 \right]^{1/2}, \quad c_3 = m \left[ \frac{r - m}{r + m} \right]^{1/2}, \]
and it is a solution to Euclidean vacuum Einstein equation \( R_{\mu\nu} = 0 \) for \( r \geq m \) with self-dual Riemann tensor. The apparent singularity at \( r = m \) can be removed by a coordinate redefinition and is a ‘nut’ (in terminology of Gibbons and Hawking [22]) at which the isometry generated by the Killing vector \((\partial/\partial\psi)\) has a zero-dimensional fixed point set. And this TN instanton is an asymptotically-locally-flat (ALF) metric.

It turns out that only the anti-self-dual equation \( F^a = -\tilde{F}^a \) admits a non-trivial solution and it is \( A^a = (A^1, A^2, A^3) \) where
\[ A^1 = \pm 2 \frac{(r - m)^{1/2}}{(r + m)^{3/2}} e^1, \quad A^2 = \pm 2 \frac{(r - m)^{1/2}}{(r + m)^{3/2}} e^2, \quad A^3 = \frac{(r + 3m)(r - m)^{1/2}}{m} \left( \frac{r + m}{r + m} \right)^{3/2} e^3 \]
and \( F^a = (F^1, F^2, F^3) \) where
\[ F^1 = \pm \frac{8m}{(r + m)^3} \left( e^0 \wedge e^1 - e^2 \wedge e^3 \right), \quad F^2 = \pm \frac{8m}{(r + m)^3} \left( e^0 \wedge e^2 - e^3 \wedge e^1 \right), \]
\[ F^3 = \frac{16m}{(r + m)^3} \left( e^0 \wedge e^3 - e^1 \wedge e^2 \right). \]

It is interesting to note that this YM field strength and the Ricci tensor of the background TN GI are proportional as \( |F^a| = 2|R^0_a| \) except for opposite self-duality, i.e.,
\[ R^0_1 = -R^2_3 = \frac{4m}{(r + m)^3} \left( e^0 \wedge e^1 + e^2 \wedge e^3 \right), \quad R^0_2 = -R^3_1 = \frac{4m}{(r + m)^3} \left( e^0 \wedge e^2 + e^3 \wedge e^1 \right), \]
\[ R^0_3 = -R^1_2 = -\frac{8m}{(r + m)^3} \left( e^0 \wedge e^3 + e^1 \wedge e^2 \right). \]

(II) YM instanton in Eguchi-Hanson metric background

The EH GI solution amounts to
\[ c_r = \left[ 1 - \left( \frac{a}{r} \right)^4 \right]^{-1/2}, \quad c_1 = c_2 = \frac{1}{2} r, \quad c_3 = \frac{1}{2} r \left[ 1 - \left( \frac{a}{r} \right)^4 \right]^{1/2} \]
and again it is a solution to Euclidean vacuum Einstein equation \( R_{\mu\nu} = 0 \) for \( r \geq a \) with self-dual Riemann tensor. \( r = a \) is just a coordinate singularity that can be removed by a coordinate redefinition provided that now \( \psi \) is identified with period \( 2\pi \) rather than
$4\pi$ and is a ‘bolt’ (in terminology of Gibbons and Hawking [22]) where the action of the Killing field ($\partial/\partial\psi$) has a two-dimensional fixed point set. Besides, this EH instanton is an asymptotically-locally-Euclidean (ALE) metric.

In this time, only the self-dual equation $F^a = +\tilde{F}^a$ admits a non-trivial solution and it is $A^a = (A^1, A^2, A^3)$ where

$$A^1 = \pm \frac{2}{r} \left[ 1 - \left( \frac{a}{r} \right)^4 \right]^{1/2} e^1, \quad A^2 = \pm \frac{2}{r} \left[ 1 - \left( \frac{a}{r} \right)^4 \right]^{1/2} e^2, \quad A^3 = \frac{2}{r} \left[ 1 + \left( \frac{a}{r} \right)^4 \right]^{1/2} e^3 \quad (60)$$

and $F^a = (F^1, F^2, F^3)$ where

$$F^1 = \pm \frac{4}{r^2} \left( \frac{a}{r} \right)^4 \left( e^0 \wedge e^1 + e^2 \wedge e^3 \right), \quad F^2 = \pm \frac{4}{r^2} \left( \frac{a}{r} \right)^4 \left( e^0 \wedge e^2 + e^3 \wedge e^1 \right),$$
$$F^3 = -\frac{8}{r^2} \left( \frac{a}{r} \right)^4 \left( e^0 \wedge e^3 + e^1 \wedge e^2 \right). \quad (61)$$

Again it is interesting to realize that this YM field strength and the Ricci tensor of the background EH GI are proportional as $|F^a| = 2|R^0_a|$, i.e.,

$$R^0_1 = -R^2_3 = \frac{2}{r^2} \left( \frac{a}{r} \right)^4 \left( -e^0 \wedge e^1 + e^2 \wedge e^3 \right), \quad R^0_2 = -R^3_1 = \frac{2}{r^2} \left( \frac{a}{r} \right)^4 \left( -e^0 \wedge e^2 + e^3 \wedge e^1 \right),$$
$$R^0_3 = -R^1_2 = -\frac{4}{r^2} \left( \frac{a}{r} \right)^4 \left( -e^0 \wedge e^3 + e^1 \wedge e^2 \right). \quad (62)$$

The detailed analysis of the nature of these solutions to the (anti) self-dual YM equation in the background of TN and EH GI constructed thus far will be given in the Appendix.

V. Concluding remarks

In the present work, we were interested in the explicit construction of $SU(2)$ Yang-Mills instanton solutions in the background geometry of a stack of two coincident probe $D4$-brane worldspaces particularly when the metric of target spacetime in which the probe branes are embedded is given by the Ricci-flat, magnetic extremal 4-brane solution in type IIA theory with its worldspace metric being given by that of TN or EH gravitational instanton. This $D4$-brane worldvolume soliton configuration was of particular interest since with this YM instanton-gravitational instanton system on a probe $D4$-brane worldvolume, the energy
of the probe brane attains its minimum value and hence enjoys stable state provided one employs the Tseytlin’s non-abelian DBI action for the description of multiple probe D-branes. Here, for a pile of generally $N$ non-coincident Ricci-flat $D_4$-branes embedded in a target spacetime with metric given by that of TN or EH GI, it does not appear to be totally clear whether the metric induced on the probe branes’ worldspaces is that of multi-centered TN or EH GI or just that of a single TN or EH uniformly for each brane. If it were that of multi-centered TN or EH GI [11,24], then the $SU(N)$ instanton solutions constructed on them should be the multi-instanton solutions as well and would add more technical complexity to our consideration. Even in this case, however, if we confine our interest to the case of a stack of $N$ coincident $D_4$-branes, then things will become simpler. That is, for a stack of $N$ coincident $D_4$-branes, the centers of the $N$ GI would merge and as a result, the metric of the $N$ coincident $D_4$-branes’ worldspaces would coalesce to become that of a single GI. Therefore, as far as the case of the stack of two coincident Ricci-flat $D_4$-branes is concerned, the corresponding worldvolume soliton configuration would be properly described by the single $SU(2)$ YM instanton constructed on a single-centered TN or EH GI geometry background that we discussed in the previous section. Thus regarding the interesting observation that the BPS condition, or the (anti) self-duality of non-abelian gauge theory instanton solution at once linearizes the otherwise highly non-linear DBI action and minimizes the energy of the probe $D_4$-branes (provided Tseytlin’s non-abelian DBI action is employed), here in this work we have actually demonstrated that this procedure can actually work by constructing in an explicit manner the $SU(2)$ YM instanton solution having (anti) self-dual field strength in the background of Ricci-flat $D_4$-branes’ worldspaces with metrics given by two best-known 4-dimensional gravitational instantons, TN and EH metrics.

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Appendix : Analysis of the nature of solutions in sect.IV

In this Appendix, we would like to examine the nature of the solutions to (anti) self-dual YM equation in the background of TN and EH GI discussed in sect.IV. First, recall that the relevant ansatz for the YM gauge connection is of the form $A^a = f(r)\sigma^a + g(r)\delta^a \sigma^3$ in the TN and EH GI backgrounds with topology of $R \times (\text{squashed})S^3$. Here, however, the physical interpretation of the nature of YM gauge potential solutions $A^a$ is rather unclear when they are expressed in terms of these left-invariant 1-forms $\{\sigma^a\}$ or the orthonormal basis $e^A$ in eq.(50). Thus in order to get a better insight into the physical meaning of the structure of these YM connection ansatz, we now try to re-express the left-invariant 1-forms $\{\sigma^a\}$ forming a basis on $S^3$ in terms of more familiar Cartesian coordinate basis. And this can be achieved by first relating the polar coordinates $(r, \theta, \phi, \psi)$ to Cartesian $(t, x, y, z)$ coordinates (note, here, that $t$ is not the usual “time” but just another spacelike coordinate) given by [21]

$$x + iy = r \cos \frac{\theta}{2} \exp \left[\frac{i}{2}(\psi + \phi)\right], \quad z + it = r \sin \frac{\theta}{2} \exp \left[\frac{i}{2}(\psi - \phi)\right],$$

(63)

where $x^2 + y^2 + z^2 + t^2 = r^2$ which is the equation for $S^3$ with radius $r$. From this coordinate transformation law, one now can relate the non-coordinate basis to the Cartesian coordinate basis as

$$\begin{pmatrix} dr \\ r\sigma_x \\ r\sigma_y \\ r\sigma_z \end{pmatrix} = \frac{1}{r} \begin{pmatrix} x & y & z & t \\ -t & -z & y & x \\ z & -t & -x & y \\ -y & x & -t & z \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \\ dt \end{pmatrix}$$

(64)

where $\{\sigma_x = -\sigma^1/2, \ \sigma_y = -\sigma^2/2, \ \sigma_z = -\sigma^3/2\}$. Still, however, the meaning of YM gauge connection ansatz rewritten in terms of the Cartesian coordinate basis $dx^\mu = (dt, dx, dy, dz)$
as above does not look so apparent. Thus we next introduce the so-called \('tHooft tensor\) \([6,21]\) defined by

\[
\eta^{\mu \nu} = -\eta^{\nu \mu} = (\epsilon^{\alpha \mu \nu} + \frac{1}{2} \epsilon^{\alpha \beta \gamma} \epsilon_{\beta \gamma \mu \nu}).
\] (65)

Then the left-invariant 1-forms can be cast to a more concise form \(\sigma^a = 2\eta^a_{\mu \nu} (x^\nu / r^2) dx^\mu\).

Therefore, the YM instanton solution, in Cartesian coordinate basis, can be written as

\[
A^a = A^a_\mu dx^\mu = 2 \left[ f(r) + g(r) \delta^a_3 \right] \eta^a_{\mu \nu} \frac{x^\nu}{r^2} dx^\mu
\] (66)

in the background of TN and EH GI with topology of \(R \times (\text{squashed})S^3\). Now in order to appreciate the meaning of this structure, we go back to the flat space situation. As is well-known, the standard BPST \([12]\) SU(2) YM instanton solution in flat space takes the form \(A^a_\mu = 2\eta^a_{\mu \nu} [x^\nu / (r^2 + \lambda^2)]\) with \(\lambda\) being the size of the instanton while the meron solution which is another non-trivial solution to the second order YM field equation found long ago by De Alfaro, Fubini, and Furlan \([20]\) takes the form \(A^a_\mu = \eta^a_{\mu \nu} (x^\nu / r^2)\). Since the pure (vacuum) gauge having vanishing field strength is given by \(A^a_\mu = 2\eta^a_{\mu \nu} (x^\nu / r^2)\), the standard instanton solution interpolates between the trivial vacuum \(A^a_\mu = 0\) at \(r = 0\) and another vacuum represented by this pure gauge above at \(r \to \infty\) and the meron solution can be thought of as a “half a vacuum gauge”. Unlike the instanton solution, however, the meron solution only solves the second order YM field equation and fails to solve the first order (anti) self-dual equation. As is apparent from their structures given above, the meron is an unstable solution in that it is singular at its center \(r = 0\) and at \(r = \infty\) while the ordinary instanton solution exhibits no singular behavior. As was pointed out originally by De Alfaro et al. \([20]\), in contrast to instantons whose topological charge density is a smooth function of \(x\), the topological charge density of merons vanishes everywhere except at its center, i.e., the singular point, such that its volume integral is half unit of topological charge 1/2. And curiously enough, half-integer topological charge seems to be closely related to the confinement in the Schwinger model \([25]\). It is also amusing to note that a “time slice” through the origin, i.e., \(x_0 = 0\) of the meron configuration yields a \(SU(2)\) Wu-Yang monopole.
Lastly, the Euclidean meron action diverges logarithmically and perhaps needs some regularization whereas the standard YM instanton has finite action. Thus we are led to the conclusion that the YM instanton solution in typical GI backgrounds possess the structure of (curved space version of) meron at large (but finite) $r$. As is well-known, in flat spacetime meron does not solve the 1st order (anti) self-dual equation although it does the second order YM field equation. Thus in this sense, this result seems remarkable since it implies that in the GI backgrounds, the (anti) self-dual YM equation admits solutions which exhibit the configuration of meron solution at large $r$ in contrast to the flat spacetime case. And we only conjecture that when passing from the flat ($R^4$) to GI ($R \times S^3$) geometry, the closure of the topology of part of the manifold appears to turn the structure of the instanton solution from that of standard BPST into that of meron. The concrete form of the YM instanton solutions in each of these GI backgrounds written in terms of Cartesian coordinate basis as in eq.(66) will be given below after we comment on one more thing to which we now turn to.

Namely, we would like to investigate the topological charge values of these solutions. It has been pointed out in the literature that both in the background of Euclidean Schwarzschild geometry [13] and in the Euclidean de Sitter space [14], the (anti) instanton solutions have the Pontryagin index of $\nu[A] = \pm 1$ and hence give the contribution to the (saddle point approximation to) intervacua tunnelling amplitude of $\exp[-8\pi^2/g_s^2]$, which, interestingly, are the same as their flat space counterparts even though these curved space YM instanton solutions do not correspond to gauge transformations of any flat space instanton solution [12]. This unexpected and hence rather curious property, however, turns out not to persist in YM instantons in these GI backgrounds we studied here. In order to see this, we begin with the careful definition of the Pontryagin index or second Chern class in the presence of the non-trivial background geometry of GI.

Consider that we would like to find an index theorem for the manifold $(M)$ with boundary $(\partial M)$. Namely, we now need an extended version of index theorem with boundary. To this question, an appropriate answer has been provided by Atiyah, Patodi, and Singer (APS)
[26]. According to their extended version of index theorem, the total index, say, of a given geometry and of a gauge field receives contributions, in addition to that from the usual bulk term \((V(M))\), from a local boundary term \((S(\partial M))\) and from a non-local boundary term \((\xi(\partial M))\). The bulk term is the usual term appearing in the ordinary index theorem without boundary and involves the integral over \(M\) of terms quadratic in curvature tensor of the geometry and in field strength tensor of the gauge field. The local boundary term is given by the integral over \(\partial M\) of the Chern-Simons forms for both the geometry and the gauge field while the non-local boundary term is given by a constant times the “APS \(\eta\)-invariant” [21] of the boundary. And this last non-local boundary term becomes relevant and meaningful when Dirac spinor field is present and interacts with the geometry and the gauge field. Now specializing to the case at hand in which we are interested in the evaluation of the instanton number or the second Chern class of the YM gauge field alone, we only need to pick up the terms in the gauge sector in this APS index theorem which reads [21]

\[ \nu[A] = Ch_2(F) = -\frac{1}{8\pi^2} \int_{M=R\times S^3} tr(F \wedge F) - \int_{\partial M=S^3} tr(\alpha \wedge F)|_{r=r_0} \]  

(67)

where \(\alpha \equiv (A - A')\) is the “second fundamental form” at the boundary \(r = r_0\) and by definition [21] \(A'\) has only tangential components on the boundary \(\partial M = S^3\). Recall, however, that our choice of ansatz for the YM gauge connection involves the gauge fixing \(A_r = 0\) as we mentioned earlier. Namely, both \(A\) and \(A'\) possess only tangential components (with respect to the \(r = r_0\) boundary) at any \(r = r_0\) and hence \(\alpha \equiv (A - A') = 0\) identically there. As a result, even in the presence of the boundaries, the terms in the YM gauge sector in the APS index theorem remain the same as in the case of index theorem with no boundary, namely, only the bulk term survives in eq.(67) above. Thus what remains is just a straightforward computation of this bulk term and it becomes easier when performed in terms of orthonormal basis \(e^A = \{e^0 = c_r dr, \ e^a = c_a \sigma^a\}\), in which case,

\[ tr(F \wedge F) = \frac{1}{2} (F^a \wedge F^a) = \frac{1}{2} \left(\frac{1}{4}\right) \epsilon_{ABCD} F^a_{AB} F^a_{CD} \sqrt{g} d^4 x \]

\[ = (F^1_{01} F^1_{23} + F^2_{02} F^2_{31} + F^3_{03} F^3_{12}) \sqrt{g} d^4 x, \]  

(68)
\[ \int_{M = R \times S^3} d^4 \sqrt{g} = \int_R dr (c_r c_1 c_2 c_3) \int_0^{4\pi} d\psi \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta = 16\pi^2 \int_R dr (c_r c_1 c_2 c_3) \]

where we used \( \sqrt{g} = |\text{det}e| = c_r c_1 c_2 c_3 \sin \theta \). The period for the \( U(1) \) fibre coordinate \( \psi \) for the EH metric, however, is \( 2\pi \) rather than \( 4\pi \) to remove the bolt singularity at \( r = a \) as we mentioned earlier. This completes the description of the method for computing the topological charge of each solution.

(1) YM instanton in Taub-NUT metric background

In terms of the ansatz functions \( f(r) \) and \( g(r) \) for the YM gauge connection in GI backgrounds given in eq.(54), the standard instanton solutions in TN metric amount to

\[
\begin{align*}
    f(r) &= \left( \frac{r - m}{r + m} \right), \quad g(r) = \left( \frac{2m}{r + m} \right) \left( \frac{r - m}{r + m} \right), \\
    f(r) &= -\left( \frac{r - m}{r + m} \right), \quad g(r) = 2 \left( \frac{r + 2m}{r + m} \right) \left( \frac{r - m}{r + m} \right)
\end{align*}
\]

for self-dual and anti-self-dual YM equations respectively. Therefore, when expressed in Cartesian coordinate basis as in eq.(66), the solutions take the forms

\[
\begin{align*}
    A_\mu^a &= 2 \left( \frac{r - m}{r + m} \right) \left[ 1 + \left( \frac{2m}{r + m} \right) \delta^{a3} \right] \eta_{a\nu}^a x^\nu, \\
    A_\mu^a &= 2 \left( \frac{r - m}{r + m} \right) \left[ -1 + 2 \left( \frac{r + 2m}{r + m} \right) \delta^{a3} \right] \eta_{a\nu}^a x^\nu
\end{align*}
\]

for self-dual and anti-self-dual case respectively. Some comments regarding the features of these solutions are now in order. i) They appear to be singular at the center \( r = 0 \) but it should not be a problem as \( r \geq m \) for the background TN metric and hence the point \( r = 0 \) is absent. ii) It is interesting to note that the solutions become vacuum gauge \( A_\mu^a = 0 \) at the boundary \( r = m \) which has the topology of \( S^3 \). iii) For \( r \to \infty \), the solutions asymptote to another vacuum gauge \( |A_\mu^a| = 2 \eta_{a\mu}^a (x^\nu / r^2) \).

We now turn to the computation of the topological charge, i.e., the Pontryagin index of these YM solution. The relevant quantities involved in this computation are the ones in eq.(68) and they, for the case at hand, are
\[(c_1c_2c_3) = \frac{m}{8}(r^2 - m^2), \]  
\[F^\alpha_{\mu\nu}F^{\alpha\mu\nu} = 4(F^1_{01}F^1_{23} + F^2_{02}F^2_{31} + F^3_{03}F^3_{12}) = -24\frac{(8m)^2}{(r+m)^6}. \]

Thus we have
\[\nu[A] = \left(\frac{-1}{32\pi^2}\right) 16\pi^2 \int_m^\infty d\frac{m}{8}(r^2 - m^2) \left[ -24\frac{(8m)^2}{(r+m)^6} \right] = 1. \]

(2) YM instanton in Eguchi-Hanson metric background

The standard instanton solutions in EH metric amount to
\[f(r) = \left[ 1 - \left(\frac{a}{r}\right)^4 \right]^{1/2}, \quad g(r) = \left[ 1 + \left(\frac{a}{r}\right)^4 \right] - \left[ 1 - \left(\frac{a}{r}\right)^4 \right]^{1/2}, \]
\[f(r) = -\left[ 1 - \left(\frac{a}{r}\right)^4 \right]^{1/2}, \quad g(r) = \left[ 1 + \left(\frac{a}{r}\right)^4 \right] + \left[ 1 - \left(\frac{a}{r}\right)^4 \right]^{1/2}, \]
for self-dual and anti-self-dual YM equations respectively. Thus in Cartesian coordinate basis, the solutions take the forms
\[A^\alpha_{\mu} = 2 \left\{ \left[ 1 - \left(\frac{a}{r}\right)^4 \right]^{1/2} + \left[ 1 + \left(\frac{a}{r}\right)^4 \right] - \left[ 1 - \left(\frac{a}{r}\right)^4 \right]^{1/2} \right\} \eta^a_{\mu\nu} \frac{x^\nu}{r^2}, \]
\[A^\alpha_{\mu} = 2 \left\{ -\left[ 1 - \left(\frac{a}{r}\right)^4 \right]^{1/2} + \left[ 1 + \left(\frac{a}{r}\right)^4 \right] + \left[ 1 - \left(\frac{a}{r}\right)^4 \right]^{1/2} \right\} \eta^a_{\mu\nu} \frac{x^\nu}{r^2} \]
for self-dual and anti-self-dual cases respectively. Some comments regarding the features of these solutions are now in order. i) Again, they appear to be singular at the center \(r = 0\) but it should not be a problem as \(r \geq a\) for the background EH metric and hence the point \(r = 0\) is absent. ii) The solutions become \(A^\alpha_{\mu} = 4\eta^a_{\mu\nu} \delta^{a3} (x^\nu / r^2)\) at the boundary \(r = a\) which has the topology of \(S^3 / Z_2\). iii) For \(r \to \infty\), the solutions asymptote to the vacuum gauge \(|A^\alpha_{\mu}| = 2\eta^a_{\mu\nu} (x^\nu / r^2)\).

We turn now to the computation of the Pontryagin index of these YM solution. For the case at hand, the relevant quantities involved in this computation are
\[(c_1c_2c_3) = \frac{1}{8} r^3, \quad F^\alpha_{\mu\nu}F^{\alpha\mu\nu} = 24 \left(\frac{4a^4}{r^6}\right)^2. \]
Thus we have
\[
\nu[A] = \left(\frac{-1}{32\pi^2}\right) 8\pi^2 \int_a^\infty dr \frac{1}{r^3} \left[ 24 \left(\frac{4a^4}{r^6}\right)^2 \right] = -\frac{3}{2}
\] (76)

where we set the range for the $U(1)$ fibre coordinate as $0 \leq \psi \leq 2\pi$ rather than $0 \leq \psi \leq 4\pi$ for the reason stated earlier. Note particularly that it is precisely this point that renders the Pontryagin index of this solution fractional because otherwise, it would come out as $-3$ instead.

Let us now discuss the behavior of these solutions as $r \to 0$ once again to stress that they really do not exhibit singular behaviors there. For TN and EH instantons, the ranges for radial coordinates are $m \leq r < \infty$ and $a \leq r < \infty$, respectively. Since the point $r = 0$ is absent in these manifolds, the solutions in these GI are everywhere regular. At large but finite $r$, on the other hand, the solutions appear to take the structure close to that of meron solution in flat space. Another interesting point worthy of note is that the solution in TN background exhibits a generic property of the instanton solution in that it does interpolate between a trivial vacuum at $r = m$ and another vacuum (pure gauge) at $r \to \infty$. Namely, the solution in TN background appears to exhibit features of both meron such as their large $r$ behavior and instanton such as interpolating configuration between two vacua. Next, we analyze the meaning of the topological charge values of the solutions. It is remarkable that generally the solutions seem to carry fractional topological charge values. Here, however, the solution in EH metric background carries the half-integer Pontryagin index actually because the range for the $U(1)$ fibre coordinate is $0 \leq \psi \leq 2\pi$ and hence the boundary of EH space is $S^3/Z_2$. To summarize, the solution in TN background particularly displays features generic in the standard instanton while in the case of that in EH background, such generic features of the instanton is somewhat obscured by meron-type natures. There, however, is one obvious consensus. Both solutions in these GI backgrounds are non-singular at their centers and have finite Euclidean YM action. And this last point allows us to suspect that these solutions are more like instantons in their generic nature although looks rather like merons in their structures.
References

[1] J. Polchinski, *TASI Lectures on D-branes*, [hep-th/9611050](https://arxiv.org/abs/hep-th/9611050).

[2] See for instance, K. G. Savvidy, *Born-Infeld Action in String Theory*, [hep-th/9906073](https://arxiv.org/abs/hep-th/9906073) and references therein.

[3] E. Witten, Nucl. Phys. **B460**, 335 (1996).

[4] A. A. Tseytlin, Nucl. Phys. **B501**, 41 (1997).

[5] P. C. Argyres and C. R. Nappi, Nucl. Phys. **B330**, 151 (1990).

[6] J. P. Gauntlett, J. Gomis, and P. K. Townsend, JHEP **01**, 003 (1998); G. W. Gibbons and K. Hashimoto, JHEP **09**, 013 (2000).

[7] G. W. Gibbons, Nucl. Phys. **B514**, 603 (1998).

[8] C. G. Callan and J. Maldacena, Nucl. Phys. **B513**, 198 (1998).

[9] K. Dasgupta and S. Mukhi, Phys. Lett. **B423**, 261 (1998).

[10] D. Brecher, Phys. Lett. **B442**, 117 (1998).

[11] D. Brecher and M. J. Perry, Nucl. Phys. **B566**, 151 (2000).

[12] A. A. Belavin, A. M. Polyakov, A. S. Schwarz, and Yu. S. Tyupkin, Phys. Lett. **B59**, 85 (1975); G. ‘tHooft, Phys. Rev. Lett. **37**, 8 (1976).

[13] J. M. Charap and M. J. Duff, Phys. Lett. **B69**, 445 (1977); *ibid* **B71**, 219 (1977).

[14] H. Kim and S. K. Kim, Nuovo Cim. **B114**, 207 (1999) and references therein.

[15] M. Atiyah, V. Drinfeld, N. Hitchin, and Y. Manin, Phys. Lett. **A65**, 185 (1987); P. B. Kronheimer and H. Nakajima, Math. Ann. **288**, 263 (1990).

[16] H. Kim and Y. Yoon, Phys. Lett. **B495**, 169 (2000) ([hep-th/0002153](https://arxiv.org/abs/hep-th/0002153)); H. Kim and Y. Yoon, [hep-th/0012055](https://arxiv.org/abs/hep-th/0012055).
[17] K. S. Stelle, ICTP Lectures, hep-th/9803110.

[18] A. Taub, Ann. Math. 53, 472 (1951); E. Newman, L. Tamburino, and T. Unti, J. Math. Phys. 4, 915 (1963); S. W. Hawking, Phys. Lett. A60, 81 (1977).

[19] T. Eguchi and A. J. Hanson, Phys. Lett. B74, 249 (1978).

[20] V. De Alfaro, S. Fubini, and G. Furlan, Phys. Lett. B65, 163 (1976).

[21] T. Eguchi, P. B. Gilkey, and A. J. Hanson, Phys. Rep. 66, 213 (1980).

[22] G. W. Gibbons and C. N. Pope, Commun. Math. Phys. 66, 267 (1979); G. W. Gibbons and S. W. Hawking, ibid, 66, 291 (1979).

[23] T. Eguchi and A. J. Hanson, Ann Phys. 120, 82 (1979).

[24] S. W. Hawking, Phys. Lett. A60, 81 (1977); G. W. Gibbons and S. W. Hawking, Phys. Lett. B78, 430 (1978).

[25] C. G. Callan, R. Dashen, and D. J. Gross, Phys. Rev. D17, 2717 (1978).

[26] M. F. Atiyah, V. K. Patodi, and I. M. Singer, Bull. London Math. Soc. 5, 229 (1973); Proc. Camb. Philos. Soc. 77, 43 (1975); ibid. 78, 405 (1975); ibid. 79, 71 (1976).