Solution of QCD⊗QED coupled DGLAP equations at NLO

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Abstract

In this work, we present an analytical solution for QCD⊗QED coupled Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution equations at the leading order (LO) accuracy in QED and next-to-leading order (NLO) accuracy in perturbative QCD using double Laplace transform. This technique is applied to obtain the singlet, gluon and photon distribution functions and also the proton structure function. We also obtain contribution of photon in proton at LO and NLO at high energy and successfully compare the proton structure function with HERA data [1] and APFEL results [2]. Some comparisons also have been done for the singlet and gluon distribution functions with the MSTW results [3]. In addition, the contribution of photon distribution function inside the proton has been compared with results of MRST [4] and with the contribution of sea quark distribution functions which obtained by MSTW [3] and CTEQ6M [5].

I. Introduction

In quantum electrodynamic (QED), interactions of photon can be regarded as a structureless object since QED is an abelian gauge theory and photon has no self-interaction. The photon in an interaction can be fluctuate into a charged fermion and anti-fermion, because of Heisenberg uncertainty principle, and if one of fermions interacts with a gauge boson the photon reveals its parton structure. Fig. (1) shows scheme of deep inelastic scattering a photon with a gauge boson. In the deep inelastic scattering of electron-positron collider LEP and the electron-proton collider HERA are reported the main results on the structure of photon. Recent studies on the effect of Drell-Yan with high-mass in ATLAS have shown which the structure of photon or corrections of QED have effects on parton distribution functions [6, 7]. These results and discoveries improve our understanding about the internal structure of the proton and it can approximate theoretical activity to experimental data.

In Ref. [4], Martin and et al. showed that the photon distribution is larger than the $b$ quark distribution at $Q^2 = 20 GeV^2$ and also larger than the sea quarks at the highest values of $x$ inside the proton and neutron. So, it is interesting to study the photon distribution of the proton and neutron, to obtain these contributions at different scales can be used QCD⊗QED coupled DGLAP evolution equations. Recently, several methods have been proposed to solve the coupled DGLAP evolution equations as Laplace transform [8-12] and Mellin transform methods [13] and etc. The most appropriate and simplest of these methods is the laplace transform, because it simplifies the equations to simplest form. Block et al. in Ref. [8] showed that the NLO coupled DGLAP evolution equations, by using the double Laplace transform, can be solved and arrived to decoupled NLO evolved solutions.
In this method, the Laplace transforms are respect to $x$ and $Q^2$ and these transforms determine the singlet $F_s(x, Q^2)$ and gluon $G(x, Q^2)$ distribution functions directly, using as input $F_s^0(x) \equiv F_s(x, Q_{0}^2)$ and $G_0(x) \equiv G(x, Q_{0}^2)$ where $Q_{0}^2$ is initial scale. According to Ref. [8], it can be realized the accuracy of this method.

![GB]

Figure 1: Probing the structure of a quasi-real photon by a gauge boson (GB) in deep inelastic scattering.

In this work, we try to apply this method to solve of QCD⊗QED coupled DGLAP equations at LO and NLO QCD, since QED contributions have effects on the proton structure functions. Also, the individual singlet, gluon, photon distributions from starting distributions are analytically calculated using the double Laplace transform technique and extracted the photon distribution function in proton and proton structure function. By this method and QCD⊗QED coupled DGLAP equations, it can be obtained the singlet, gluon and photon distribution functions as follows

$$F_s^2(x, Q^2) = \mathcal{F}(F_s^0(x), G_0(x), M_0(x))$$

and

$$G(x, Q^2) = \mathcal{G}(F_s^0(x), G_0(x), M_0(x))$$

and

$$M(x, Q^2) = \mathcal{M}(F_s^0(x), G_0(x), M_0(x)),$$

which $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{M}$ are functions which can be obtained using splitting functions and $F_s^0(x, Q_{0}^2)$, $G(x, Q_{0}^2)$ and $M(x, Q_{0}^2)$, which are the singlet, gluon and photon distribution functions at initial scale respectively. By this method the singlet, gluon and photon distribution functions at arbitrary $Q^2$ can expressed as a convolution of these function at initial scale. The obtained results of the double Laplace transform show that the photon distribution function affect on value of singlet and gluon distribution functions inside the proton, however this effect is small, but these results show that the contribution of photon distribution function is larger than contributions of $b$ quark at middle $Q^2$ and high $x$. This contribution is significant in analogy with the contributions of sea quarks at high $x$. And also, these results show that photon distribution function at energy scales higher than initial scale depends on gluon distribution function at initial scale.

The remainder of this article is organized as follows: Section II involves an general solution for the decoupling of QCD⊗QED DGLAP evolution equations analytically at LO analyses with respect to the double Laplace transform method. In Section III, we use this method to calculate DGLAP evolution equations at NLO in QCD and LO in QED. In Section IV the proton distribution function have been obtained by Laplace transform, and some compression are presented between our results and available HERA data [1] and APFEL results [2]. In this section, the results gluon and singlet distribution compared with MSTW [3] and at the end of this section, we present most important part of this article, which is the photon distribution function and show comparison this distribution with sea quarks MSTW [3] and CTEQ6M [5]. In the last, we give our conclusions. In Appendix, we present results of a set of coefficients in Laplace s space, which are functions of splitting functions.
II. Master formula for LO corrections

The QCD×QED coupled DGLAP equations at LO for the evolution of parton distribution function, quarks, antiquarks, gluon and photon expressed as follows [4],

\[
\frac{\partial q(x, Q^2)}{\partial \ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi} \left\{ P_{qq}(x) \otimes q_i(x, Q^2) + P_{qg} \otimes g(x, Q^2) \right\} + \frac{\alpha_s(Q^2)}{2\pi} \left\{ e_i^2 \tilde{P}_{qq}(x) \otimes q_i(x, Q^2) + e_i^2 P_{q\gamma}(x) \otimes \gamma(x, Q^2) \right\},
\]

\(i,j\) are the Altarelli-Parisi splitting kernels at one loop correction, where the splitting functions \(P_{i,j}(x)\) are the Altarelli-Parisi splitting kernels at one loop correction, which are defined as follows,

\[
P_{qq}^{LO} = \frac{4}{3} \left( 1 + \frac{(1-x)^2}{x} \right), \quad P_{qg}^{LO} = \frac{1}{2} \left( x^2 + (1-x)^2 \right), \quad P_{gq}^{LO} = \frac{1}{2} \left( x^2 + (1-x)^2 \right), \quad P_{gg}^{LO} = \frac{1}{2} \left( x^2 + (1-x)^2 \right),
\]

where the splitting functions \(P_{i,j}(x)\) are the Altarelli-Parisi splitting kernels at one loop correction, which are defined as follows,

\[
P_{qq}^{LO} = \frac{4}{3} \left( 1 + \frac{(1-x)^2}{x} \right), \quad P_{qg}^{LO} = \frac{1}{2} \left( x^2 + (1-x)^2 \right), \quad P_{gq}^{LO} = \frac{1}{2} \left( x^2 + (1-x)^2 \right), \quad P_{gg}^{LO} = \frac{1}{2} \left( x^2 + (1-x)^2 \right),
\]

and

\[
\frac{\partial \bar{q}(x, Q^2)}{\partial \ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi} \left\{ P_{\bar{q}q}(x) \otimes \bar{q}_i(x, Q^2) + P_{\bar{q}g} \otimes g(x, Q^2) \right\}
\]

(2)

\[
\frac{\partial g(x, Q^2)}{\partial \ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi} \left\{ P_{gg}(x) \otimes \sum_i \left( q_i(x, Q^2) + \bar{q}_i(x, Q^2) \right) + P_{gq} \otimes g(x, Q^2) \right\}
\]

(3)

\[
\frac{\partial \gamma(x, Q^2)}{\partial \ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi} \left\{ P_{\gamma q}(x) \otimes \sum_i q_i(x, Q^2) + \bar{q}_i(x, Q^2) \right\} + P_{\gamma\gamma} \otimes \gamma(x, Q^2)
\]

(4)

where the splitting functions \(P_{i,j}(x)\) are the Altarelli-Parisi splitting kernels at one loop correction, which are defined as follows,

\[
P_{qq}^{LO} = \frac{4}{3} \left( 1 + \frac{(1-x)^2}{x} \right), \quad P_{qg}^{LO} = \frac{1}{2} \left( x^2 + (1-x)^2 \right), \quad P_{gq}^{LO} = \frac{1}{2} \left( x^2 + (1-x)^2 \right), \quad P_{gg}^{LO} = \frac{1}{2} \left( x^2 + (1-x)^2 \right),
\]

where \(C_F = \frac{4}{3}, \quad T_R = \frac{1}{2}, \quad n_f\) is the number of active quark flavours, QCD and QED running coupling constant at LO are as,

\[
\alpha_s^{LO}(Q^2) = \frac{4\pi}{\beta_0 \ln(Q^2/\Lambda^2)}, \quad \alpha_s(Q^2) = \frac{\alpha_s(\mu)}{1 - \frac{\alpha_s(\mu)}{8\pi} \ln(Q^2/\mu^2)}
\]

where \(\beta_0\) is the one loop (LO) correction to the QCD \(\beta\)-function and \(\alpha_s(1eV) = 1/137\). In the DGLAP evolution equations, we take \(n_f = 4\) and \(\Lambda = 192\) MeV for \(m_t^2 < Q^2 \leq m_b^2\) and \(n_f = 5\) and \(\Lambda = 146\) MeV for \(m_b^2 < Q^2\), these values have been used in CTEQ5L [14]. By summing Eqs.(1,2) and using \(P_2^0 = 5/18F^s\) approximation, we can write,

\[
\frac{\partial F^s(x, Q^2)}{\partial \ln Q^2} = \frac{\alpha_s^{LO}(Q^2)}{2\pi} \left\{ P_{qq}^{LO}(x) \otimes F^s(x, Q^2) + 2n_f P_{gq}^{LO} \otimes G(x, Q^2) \right\} + \frac{\alpha_s^{LO}(Q^2)}{2\pi} \left\{ \frac{5}{18} \tilde{P}_{qq}^{LO}(x) \otimes F^s(x, Q^2) + 2AP_{q\gamma}^{LO}(x) \otimes M(x, Q^2) \right\}
\]

(6)

\[
\frac{\partial G(x, Q^2)}{\partial \ln Q^2} = \frac{\alpha_s^{LO}(Q^2)}{2\pi} \left\{ P_{qq}^{LO}(x) \otimes F^s(x, Q^2) + P_{gq}^{LO} \otimes G(x, Q^2) \right\}
\]

(7)

\[
\frac{\partial M(x, Q^2)}{\partial \ln Q^2} = \frac{\alpha_s^{LO}(Q^2)}{2\pi} \left\{ \frac{5}{18} P_{q\gamma}^{LO}(x) \otimes F^s(x, Q^2) + P_{\gamma\gamma}^{LO} \otimes M(x, Q^2) \right\}
\]

(8)
To use of Laplace transform, we apply the variable change $x \equiv \exp(-v)$, $y \equiv \exp(-\omega)$, and also we define the following functions,

$$\hat{F}^s(v, Q^2) \equiv F^s(e^{-v}, Q^2), \quad \hat{G}(v) \equiv G(e^{-v}), \quad \hat{M}(v, Q^2) \equiv M(e^{-v}, Q^2).$$

Defining the Laplace transform method, we have,

$$f(s, Q^2) = \mathcal{L} \left[ \hat{F}^s(v, Q^2); s \right] = \int_0^\infty \hat{F}^s(v, Q^2)e^{-sv}dv$$

$$g(s, Q^2) = \mathcal{L} \left[ \hat{G}(v, Q^2); s \right] \quad \text{and} \quad m(s, Q^2) = \mathcal{L} \left[ \hat{M}(v, Q^2); s \right].$$

The Laplace transform converts Eqs. (6-8) into three coupled ordinary first order differential equations in $s$ space and these can be written as

$$\frac{\partial f}{\partial \ln Q^2}(s, Q^2) = \frac{\alpha_s^{\text{LO}}(Q^2)}{4\pi} \Phi_f^{\text{LO}}(s) f(s, Q^2) + \frac{\alpha_s^{\text{LO}}(Q^2)}{4\pi} \Theta_f^{\text{LO}}(s) g(s, Q^2) + \frac{\alpha_s(Q^2)}{4\pi} \Upsilon_f^{\text{LO}}(s) f(s, Q^2)$$

$$\frac{\partial g}{\partial \ln Q^2}(s, Q^2) = \frac{\alpha_s^{\text{LO}}(Q^2)}{4\pi} \Phi_g^{\text{LO}}(s) g(s, Q^2) + \frac{\alpha_s^{\text{LO}}(Q^2)}{4\pi} \Theta_g^{\text{LO}}(s) f(s, Q^2),$$

$$\frac{\partial m}{\partial \ln Q^2}(s, Q^2) = \frac{\alpha_s(Q^2)}{4\pi} \Omega_m^{\text{LO}}(s) m(s, Q^2) + \frac{\alpha_s(Q^2)}{4\pi} \Upsilon_m^{\text{LO}}(s) f(s, Q^2),$$

where coefficients $\Phi_f^{\text{LO}}$, $\Psi_f^{\text{LO}}$, $\Omega_f^{\text{LO}}$, and $\Upsilon_f^{\text{LO}}$ are the leading-order splitting functions at Laplace $s$ space by;

$$\Phi_f^{\text{LO}}(s) = 4 - \frac{8}{3} \left( \frac{1}{s+1} + \frac{1}{s+2} + 2(\psi(s+1) + \gamma_E) \right),$$

$$\Theta_f^{\text{LO}}(s) = 2nf \left( \frac{1}{s+1} - \frac{2}{s+2} + \frac{2}{s+3} \right),$$

$$\Upsilon_f^{\text{LO}}(s) = \frac{5}{24} \Phi_f^{\text{LO}}(s), \quad \Omega_f^{\text{LO}}(s) = \frac{A}{n_f} \Theta_f^{\text{LO}}(s),$$

$$\Phi_g^{\text{LO}}(s) = \frac{33 - 2nf}{3} + 12 \left( \frac{1}{s} - \frac{2}{s+1} + \frac{1}{s+2} - \frac{1}{s+3} - \psi(s+1) - \gamma_E \right),$$

$$\Theta_g^{\text{LO}}(s) = \frac{8}{3} \left( \frac{2}{s} - \frac{2}{s+1} + \frac{1}{s+2} \right),$$

$$\Omega_m^{\text{LO}}(s) = -\frac{4A}{3}, \quad \Upsilon_m^{\text{LO}}(s) = \frac{5}{24} \Theta_g^{\text{LO}}(s),$$

indeed $\psi(s)$ is the digamma function, $\gamma_E$ is Euler’s constant and $A = \sum_{i=1}^{n_f} e_i^2$. Introducing the new variable $\tau$ as $\frac{d\tau}{d\ln Q^2} = \frac{\alpha_s(Q^2)}{4\pi}$, the coupled first order differential equations in s-space can be rewritten

$$\frac{\partial f}{\partial \tau}(s, \tau) = \Phi_f^{\text{LO}}(s) f(s, \tau) + \Theta_f^{\text{LO}}(s) g(s, \tau) + \frac{\alpha_s(Q^2)}{\alpha_s^{\text{LO}}(Q^2)} \Upsilon_f^{\text{LO}}(s) f(s, \tau) + \frac{\alpha_s(Q^2)}{\alpha_s^{\text{LO}}(Q^2)} \Omega_f^{\text{LO}}(s) m(s, \tau),$$

$$\frac{\partial g}{\partial \tau}(s, \tau) = \Phi_g^{\text{LO}}(s) g(s, \tau) + \Theta_g^{\text{LO}}(s) f(s, \tau),$$

$$\frac{\partial m}{\partial \tau}(s, \tau) = \frac{\alpha_s(Q^2)}{\alpha_s^{\text{LO}}(Q^2)} \Omega_m^{\text{LO}}(s) l(s, \tau) + \frac{\alpha_s(Q^2)}{\alpha_s^{\text{LO}}(Q^2)} \Upsilon_m^{\text{LO}}(s) f(s, \tau).$$

Generally to do a calculation with high accuracy, we use the following expression for the $\frac{\alpha_s(Q^2)}{\alpha_s^{\text{LO}}(Q^2)}$ as

$$\frac{\alpha_s(Q^2)}{\alpha_s^{\text{LO}}(Q^2)} \approx a_{10} + a_{11} \exp(b_{11} \tau),$$

4
where the constants $a_{10}, a_{11}, b_{11}$ are found by fitting. To solve and decouple QCD \( \otimes \) QED DGLAP evolutions, we need transform Eqs. (20-22) from \( \tau \) space to \( u \) space by the Laplace transform which \( u \) is a parameter in this new space, therefore we have

\[
u F(s, u) - f_0(s) = \Phi_f^{LO}(s)f(s, u) + \Theta_f^{LO}(s)G(s, u) + \Omega_f^{LO}(s)(a_{10}M(s, u) + a_{11}M(s, u - b_{11})),
\]

\[
u G(s, u) - g_0(s) = \Phi_g^{LO}(s)G(s, u) + \Theta_g^{LO}(s)F(s, u),
\]

\[
u M(s, u) - m_0(s) = \Omega_m^{LO}(s)(a_{10}M(s, u) + a_{11}M(s, u - b_{11})) + \Omega_m^{LO}(s)(a_{10}F(s, u) + a_{11}F(s, u - b_{11})).
\]

where the Laplace transforms are as

\[
\begin{align*}
H(s, u) &= \mathcal{L}[h(s, \tau); u], \quad H(s, u - c) = \mathcal{L}[h(s, \tau)\exp(\epsilon\tau); u], \quad H(s, u + c) = \mathcal{L}[h(s, \tau)\exp(-\epsilon\tau); u].
\end{align*}
\]

The above equations can be easily solved by setting $a_{11} = 0$ in Eq. (23) which called as the first approximation of function \( \frac{a_1(Q^2)}{a_2^{LO}(Q^2)} \). This approximation, lead us to,

\[
\begin{align*}
\left(u - \Phi_f^{LO}(s) - \Omega_f^{LO}(s)a_{10}\right)F_1(s, u) - \Theta_f^{LO}(s)G_1(s, u) - \Omega_f^{LO}(s)a_{10}M_1(s, u) &= f_0(s), \\
\left(u - \Phi_g^{LO}(s)\right)G_1(s, u) + \Theta_g^{LO}(s)F_1(s, u) &= g_0(s), \\
\left(u - \Omega_m^{LO}(s)a_{10}\right)M_1(s, u) - \Omega_m^{LO}(s)a_{10}F_1(s, u) &= m_0(s).
\end{align*}
\]

One can solve these equations and obtain $F_1, G_1$ and $M_1$ distributions.

\[
\begin{align*}
F_1(s, u) &= A_1f_0(s) + A_2g_0(s) + A_3m_0(s), \\
G_1(s, u) &= B_1f_0(s) + B_2g_0(s) + B_3m_0(s), \\
M_1(s, u) &= C_1f_0(s) + C_2g_0(s) + C_3m_0(s).
\end{align*}
\]

The coefficients $A_1...C_3$ are given in Appendix. Using the inverse Laplace transform, one can transfer the obtained equations from \( u \) space to \( \tau \) space, the results clearly based on the input the singlet, gluon and photon distribution functions at \( Q_0^2 \). The inverse transforms of \( F_1(s, u), G_1(s, u), \) and \( M_1(s, u) \) denoted by \( f_1(s, \tau), g_1(s, \tau), \) and \( m_1(s, \tau) \) and therefore these functions are well defined and simple to calculate which can be expressed as

\[
\begin{align*}
f_1(s, \tau) &= l_{f1}(s, \tau)f_0(s) + l_{f2}(s, \tau)g_0(s) + l_{f3}(s, \tau)m_0(s), \\
g_1(s, \tau) &= l_{g1}(s, \tau)f_0(s) + l_{g2}(s, \tau)g_0(s) + l_{g3}(s, \tau)m_0(s), \\
m_1(s, \tau) &= l_{m1}(s, \tau)f_0(s) + l_{m2}(s, \tau)g_0(s) + l_{m3}(s, \tau)m_0(s),
\end{align*}
\]

where, the inverse Laplace transform of coefficients $A_i, B_i$ and $C_i$ in above equations from \( u \) space to \( \tau \) space defined as

\[
\begin{align*}
l_{f1}(s, \tau) &= \mathcal{L}^{-1}[A_i(s, u); \tau] = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} A_i(s, u)\exp(u\tau)du, \\
l_{g1}(s, \tau) &= \mathcal{L}^{-1}[B_i(s, u); \tau], \\
l_{m1}(s, \tau) &= \mathcal{L}^{-1}[C_i(s, u); \tau].
\end{align*}
\]

where $c$ is a real constant such that the integration contour lies to the right of all singularities of $A_i(s, u)$. We re-solve the Eqs. (24-26) to obtain the next approximation ($a_{11} \neq 0$) $F_2, G_2,$ and $M_2$ for $F, G, M$ and then repeat the process

\[
\begin{align*}
\left(u - \Phi_f^{LO}(s) - \Omega_f^{LO}(s)a_{10}\right)F_2(s, u) - \Theta_f^{LO}(s)G_2(s, u) - \Omega_f^{LO}(s)a_{10}M_2(s, u) &= f'_0(s),
\end{align*}
\]
where

\[ f_0'(s) = f_0(s) + a_{11} \left( \gamma_x^{\text{LO}} F_1(s, u - b_{11}) + \gamma_y^{\text{LO}} a_{10} T_{m}^{\text{LO}} F_2(s, u) \right), \]

\[ m_0'(s) = m_0(s) + a_{11} \left( \Omega_{m}^{\text{LO}} (s) M_1(s, u - b_{11}) + \gamma_x^{\text{LO}} (s) F_1(s, u - b_{11}) \right), \]

similar to the previous approximation \((a_{11} = 0)\), one can solve these equations and obtain the \( F_2, \ G \) and \( M \) as follows

\[ F_2(s, u) = A_1 f_0(s) + A_2 g_0(s) + A_3 m_0(s), \]

\[ G_2(s, u) = B_1 f_0(s) + B_2 g_0(s) + B_3 m_0(s), \]

\[ M_2(s, u) = C_1 f_0(s) + C_2 g_0(s) + C_3 m_0(s), \]

where \( A_1'...C_3' \) are given in Appendix

### III. Master formula for NLO corrections

The QCD\( \otimes \)QED coupled DGLAP equations at NLO approximation in QCD for the evolution of singlet, gluon and photon distribution function, which are combination of the QCD coupled DGLAP equations at NLO and the QED coupled DGLAP equations at LO approximation, using the convolution symbol \( \otimes \) can be written as,

\[
\frac{\partial F^s(x, Q^2)}{\partial \ln Q^2} = \frac{\alpha_s^{NLO}(Q^2)}{2\pi} \left[ \left( P_{qq}^{LO}(x) + \frac{\alpha_s^{NLO}(Q^2)}{2\pi} P_{gg}^{NLO}(x) \right) \otimes F^s(x, Q^2) + 2n_f \left( P_{gg}^{LO}(x) \right) \right] \\
+ \frac{\alpha_s^{NLO}(Q^2)}{2\pi} P_{gg}^{NLO}(x) G(x, Q^2),
\]

\[
\frac{\partial G(x, Q^2)}{\partial \ln Q^2} = \frac{\alpha_s^{NLO}(Q^2)}{2\pi} \left[ \left( P_{gg}^{LO}(x) + \frac{\alpha_s^{NLO}(Q^2)}{2\pi} P_{gg}^{NLO}(x) \right) \otimes F^s(x, Q^2) + \left( P_{qq}^{LO}(x) \right) \right] \\
+ \frac{\alpha_s^{NLO}(Q^2)}{2\pi} P_{gg}^{NLO}(x) G(x, Q^2),
\]

\[
\frac{\partial M(x, Q^2)}{\partial \ln Q^2} = \frac{\alpha_s^{NLO}(Q^2)}{2\pi} \left[ \frac{5}{18} P_{gg}^{LO}(x) \otimes F^s(x, Q^2) + P_{\gamma \gamma}^{LO} \otimes M(x, Q^2) \right],
\]

To calculate above equations at NLO approximation, we consider the following expression for \( \frac{\alpha_{s}^{NLO}(Q^2)}{4\pi} \) and \( \frac{\alpha_{s}(Q^2)}{\alpha_{s}^{NLO}(Q^2)} \) as,

\[
\frac{\alpha_{s}(Q^2)}{\alpha_{s}^{NLO}(Q^2)} \approx a_{20} + a_{21} \exp(b_{21} \tau), \quad \frac{\alpha_{s}^{NLO}(Q^2)}{4\pi} \approx a_{30} + a_{31} \exp(-b_{31} \tau)
\]

as shown in Table (1), this expression includes an excellent accurate. The running coupling constant at NLO is defined as,

\[
\alpha_{s}^{NLO}(Q^2) = \frac{4\pi}{\beta_0 \ln \left( \frac{Q^2}{\Lambda^2} \right)} \left( 1 - \frac{\beta_1}{\beta_0} \ln \left( \frac{Q^2}{\Lambda^2} \right) \right),
\]

where \( \beta_1 = 102 - \frac{3\pi}{2} n_f \) is NLO correction to the QCD \( \beta \)-function. To solve Eqs. (47-49), we again need double Laplace transformation from \( x \) to \( s \) and \( \tau \) to \( u \) space. Therefore the solution of the differential-integral equations in Eqs. (47-49) can be converted to,

\[ u F^{NLO}(s, u) - f_0^{NLO}(s) = \Phi^{LO}(s) F^{NLO}(s, u) + \Phi^{LO}(s) G^{NLO}(s, u) + \]

\[ + \frac{\alpha_{s}^{NLO}(Q^2)}{4\pi} \left[ \left( P_{qq}^{LO}(x) + \frac{\alpha_{s}(Q^2)}{\alpha_{s}^{NLO}(Q^2)} P_{gg}^{NLO}(x) \right) \otimes F^s(x, Q^2) + \left( P_{gg}^{LO}(x) \right) \right] \\
+ \frac{\alpha_{s}(Q^2)}{\alpha_{s}^{NLO}(Q^2)} \left[ \frac{5}{18} P_{gg}^{LO}(x) \otimes F^s(x, Q^2) + P_{\gamma \gamma}^{LO} \otimes M(x, Q^2) \right].
\]
\( \Phi_f^{NLO}(s) \left( a_{30} F_{NLO}^{NLO}(s, u) + a_{31} F_{NLO}^{NLO}(s, u + b_{31}) \right) + \Theta_f^{NLO}(s) \left( a_{30} G(s, u) + a_{31} G^{NLO}(s, u + b_{31}) \right) + \\
\varepsilon_f^{NLO}(s) \left( a_{20} F_{NLO}^{NLO}(s, u) + a_{21} F_{NLO}^{NLO}(s, u - b_{21}) \right) + \Omega_f^{NLO}(s) \left( a_{20} M^{NLO}(s, u) + a_{21} M^{NLO}(s, u - b_{21}) \right), \\
\)  
\( (u G^{NLO}(s, u) - g_0(s) = \Phi_g^{NLO}(s) G^{NLO}(s, u) + \Theta_g^{NLO}(s) F^{NLO}(s, u) + \Theta_g^{NLO}(s)(a_{30} F_{NLO}^{NLO}(s, u) \\
\varepsilon_g^{NLO}(s)(a_{31} G^{NLO}(s, u) + a_{31} G^{NLO}(s, u + b_{31})) + \varepsilon_g^{NLO}(s)(a_{31} G^{NLO}(s, u) + a_{31} G^{NLO}(s, u + b_{31})), \\
\)  
\( u M^{NLO}(s, u) - m_0(s) = \Omega_m^{NLO}(s) \left( a_{20} M^{NLO}(s, u) + a_{21} M^{NLO}(s, u - b_{21}) \right) \\
+ \varepsilon_m^{NLO}(s)(a_{20} F(s, u) + a_{21} F(s, u - b_{21})). \\
\)  

The NLO splitting functions \( \Phi_f^{NLO}, \Theta_f^{NLO}, \Phi_g^{NLO}, \Theta_g^{NLO} \) can be easily obtained in \( s \) space using the NLO results derived in Refs. [15] [19] [17] as Ref. [18]. By setting \( a_{21} = 0 \) and \( a_{31} = 0 \) in Eqs. (49), the above equations can be obtained at first approximation as,

\[
\left( u - \phi_f^{NLO}(s) - a_{30}\phi_f^{NLO}(s) - \tau_f^{NLO}(s)a_{20} \right) G_1^{NLO}(s, u) - \left( \Theta_f^{NLO}(s) + a_{30}\Theta_f^{NLO}(s) \right) G_1^{NLO}(s, u) = f_0^{NLO}(s),
\]

\[
\left( u - \phi_g^{NLO}(s) - a_{30}\phi_g^{NLO}(s) \right) G_1^{NLO}(s, u) - \left( \Theta_g^{NLO}(s) + a_{30}\Theta_g^{NLO}(s) \right) F_1^{NLO}(s, u) = g_0^{NLO}(s),
\]

\[
\left( u - \Omega_m^{NLO}(s)a_{20} \right) M_1^{NLO}(s, u) - \tau_m^{NLO}(s)a_{20} F_1^{NLO}(s, u) = m_0(s).
\]

One can solve these equations and obtain \( F_1^{NLO}, G_1^{NLO} \) and \( M_1^{NLO} \) distributions, the results depend on the input singlet, gluon and photon distribution functions at initial scale,

\[
F_1^{NLO}(s, u) = A_1^{NLO} f_0^{NLO}(s) + A_2^{NLO} g_0^{NLO}(s) + A_3^{NLO} m_0(s),
\]

\[
G_1^{NLO}(s, u) = B_1^{NLO} f_0^{NLO}(s) + B_2^{NLO} g_0^{NLO}(s) + B_3^{NLO} m_0(s),
\]

\[
M_1^{NLO}(s, u) = C_1^{NLO} f_0^{NLO}(s) + C_2^{NLO} g_0^{NLO}(s) + C_3^{NLO} m_0(s),
\]

(57) to obtain the next approximation \((a_{21} \neq 0, a_{31} \neq 0) F_2^{NLO}, G_2^{NLO}, \) and \( M_2^{NLO} \) for \( F, G, M \), we replace \( f_0^{NLO}, g_0^{NLO} \) and \( m_0^{NLO} \) with \( f_0^{NLO}, g_0^{NLO}, \) and \( m_0^{NLO} \) respectively in Eqs. (54-56). The method of calculating next approximation is as first approximation. \( f_0^{NLO}, g_0^{NLO} \) and \( m_0^{NLO} \) are as follows,

\[
f_0^{NLO}(s, u) = f_0^{NLO}(s) + a_{21} \left( \tau_f^{NLO}(s)(s, u - b_{21}) + \Omega_f^{NLO}(s)(s, u - b_{21}) \right)
\]

\[
+ a_{31} \left( \phi_f^{NLO}(s)(s, u + b_{31}) + \Theta_f^{NLO}(s)(s, u + b_{31}) \right) 
\]

\[
g_0^{NLO}(s, u) = g_0^{NLO}(s) + a_{31} \left( \phi_g^{NLO}(s)(s, u + b_{31}) + \Theta_g^{NLO}(s)(s, u + b_{31}) \right) 
\]

\[
m_0^{NLO}(s, u) = m_0(s) + a_{21} \left( \Omega_m^{NLO}(s)(s, u - b_{21}) + \tau_m^{NLO}(s)(s, u - b_{21}) \right),
\]

one can obtain the \( F_2^{NLO}, G_2^{NLO} \) and \( M_2^{NLO} \) from Eqs. (57-62) as follows

\[
F_2^{NLO}(s, u) = A_1^{NLO} f_0^{NLO}(s) + A_2^{NLO} g_0^{NLO}(s) + A_3^{NLO} m_0(s),
\]

\[
G_2^{NLO}(s, u) = B_1^{NLO} f_0^{NLO}(s) + B_2^{NLO} g_0^{NLO}(s) + B_3^{NLO} m_0(s),
\]

\[
M_2^{NLO}(s, u) = C_1^{NLO} f_0^{NLO}(s) + C_2^{NLO} g_0^{NLO}(s) + C_3^{NLO} m_0(s),
\]

where \( A_1^{NLO} \ldots C_3^{NLO} \) are given in Appendix. Using iterative solution of Eqs. (43-45) and Eqs. (63-65) and the inverse Laplace transform technique for back from \( u \) to \( \tau \) space, the following expressions for the singlet, gluon and photon distributions can be obtained as

\[
f^n(s, \tau) = k_{f1}(s, \tau)f_0^n(s) + k_{f2}(s, \tau)g_0^n(s) + k_{f3}(s, \tau)m_0(s),
\]

(66)
\[ g^n(s, \tau) = k_{g_1}^n(s, \tau)f_0^0(s) + k_{g_2}^n(s, \tau)g_0^0(s) + k_{g_3}^n(s, \tau)m_0(s), \]  
\[ m^n(s, \tau) = k_{m_1}^n(s, \tau)f_0^0(s) + k_{m_2}^n(s, \tau)g_0^0(s) + k_{m_3}^n(s, \tau)m_0(s), \]
\[ n = \text{LOorNLO} \]

where the coefficients at \( \tau \) space are,
\[ k_{f}^n(s, \tau) = \mathcal{L}^{-1} \left[ A_i^m(s, u); \tau \right], \]
\[ k_{g}^n(s, \tau) = \mathcal{L}^{-1} \left[ B_i^m(s, u); \tau \right], \]
\[ k_{m}^n(s, \tau) = \mathcal{L}^{-1} \left[ C_i^m(s, u); \tau \right], \quad i = 1, 2, 3 \quad \text{and} \quad n = \text{LOorNLO} \]

one can invert \( k_{j}^n(s) \) \( (j = f, g, m) \) from \( s \) space to \( v \) space using algorithms of Ref. [10], there is showed that the real basis of the method is in assuring that the inverse transforms can be calculated exactly to high orders (as defined there), even for function which diverge for \( s \to 0 \). We define their inverse Laplace as,
\[ K_{f}^n(v, \tau) = \mathcal{L}^{-1} \left[ k_{f}^n(s, \tau); v \right], \]
\[ K_{g}^n(v, \tau) = \mathcal{L}^{-1} \left[ k_{g}^n(s, \tau); v \right], \]
\[ K_{m}^n(v, \tau) = \mathcal{L}^{-1} \left[ k_{m}^n(s, \tau); v \right], \quad i = 1, 2, 3 \quad \text{and} \quad n = \text{LOorNLO} \]

so, we can write the solutions in \((v, \tau)\) space as the convolutions,
\[ \hat{F}^s n(v, \tau) = \int_0^v K_{f1}^n(v - w, \tau) \hat{F}^s n(w) dw + \int_0^v K_{f2}^n(v - w, \tau) \hat{G}^n_0(w) dw + \int_0^v K_{f3}^n(v - w, \tau) \hat{M}^n_0(w) dw, \]
\[ \hat{G}^n(v, \tau) = \int_0^v K_{g1}^n(v - w, \tau) \hat{F}^s n(w) dw + \int_0^v K_{g2}^n(v - w, \tau) \hat{G}^n_0(w) dw + \int_0^v K_{g3}^n(v - w, \tau) \hat{M}^n_0(w) dw, \]
\[ \hat{M}^n(v, \tau) = \int_0^v K_{m1}^n(v - w, \tau) \hat{F}^s n(w) dw + \int_0^v K_{m2}^n(v - w, \tau) \hat{G}^n_0(w) dw + \int_0^v K_{m3}^n(v - w, \tau) \hat{M}^n_0(w) dw. \quad n = \text{LOorNLO} \]

Finally, recalling the \( v \equiv \ln(1/x) \), one can convert the above solutions back into the usual space, Bjorken-\( x \) and virtuality \( Q^2 \). The \( Q^2 \) dependence of the solutions are performed by the \( \tau \) variable. Consequently, we can obtain the singlet, gluon and photon distribution as \( F^s(x, Q^2), G(x, Q^2) \) and \( M(x, Q^2) \). Eq. (73) indicates that photon contribution in \( Q^2 \)'s larger than initial scale depends on gluon contribution at initial condition.

IV. Results and Conclusion

In this section, we will present the results that have been obtained for the proton structure function and singlet, gluon and photon distribution functions by using the double Laplace transform technique to find an analytical solution for the QCD⊗ QED DGLAP evolution equations at LO and NLO QCD. Also we give an example to this approach and compare the proton structure function at NLO approximation with APFEL results, this function calculated with a suitable approximation from the singlet distribution function. In addition, the singlet and gluon distribution functions, obtained from Eqs. (71, 72) starting from the MSTW initial conditions at \( Q^2_0 = 1 \text{ GeV}^2 \) [3], are compared with NLO MSTW distributions and the photon distribution function with MRST [4]. Initial condition for the photon distribution function is gained at \( Q^2_0 = 1 \text{ GeV}^2 \) by [4]. In this method, it is written relations \( \frac{\alpha_s(Q^2)}{\alpha_s(Q^2)} \), \( \frac{\alpha_s(Q^2)}{\alpha_s^{NLO}(Q^2)} \) and \( \frac{\alpha_s^{NLO}(Q^2)}{4\pi} \) as summation a exponential term (which is a function of \( \tau \)
and a constant number, in Table (1) we showed $a_{10} ... b_{31}$ from $Q^2 = 3 \text{ GeV}^2$ to $8000 \text{ GeV}^2$ with a good fitting. In Figs. (2-4), it is compared the results of these functions with Eqs. (23,49). As these figures show, the used expansions are appropriate for the solution of DGLAP equations. Fig. (5) shows the results of the proton structure function at NLO approximation which is compared with MSTW and APFEL results at $Q^2 = 120, 1000$ and $8000 \text{ GeV}^2$ scales, in Figs. (6) we compared the our results at NLO QCD with HERA data, calculations show that NLO corrections are very close to the other results and data. Although the contribution of photon in the proton structure function is very small, but it is larger than contribution of $b$ quark at low $Q^2$ and large $x$ as shown in Ref. [3]. Figs. (7, 8) show the results singlet and gluon distribution functions at leading order and next-to-leading order obtained from Eqs. (71, 72) in $x$-space at LO and NLO in QCD for $Q^2$ values of $(20, 100, M_0^2 \text{ GeV}^2)$ and compare these results with MSTW. The singlet and gluon distributions are very similar to the NLO standard MSTW distributions, where $F^s_0$ and $G_0$ are the MSTW values at initial scale. Solid lines are the our results at LO (up) and NLO (down), and dash lines are MSTW values. In Fig. (9) is plotted the photon distribution function obtained from Eq. (73) in $x$-space in LO and NLO in QCD at different energy scales and it compared with MRST at $Q^2 = 20 \text{ GeV}^2$. This figure shows that the obtained results from present analysis based on Laplace transform technique are in good agreements with the ones obtained by MRST. To understand the effect of photon distribution function on the proton structure function, in Figs. (10,11) this distribution function is compared with the sea quarks distribution functions inside the proton which are presented by CETQ [5]. In Figs. (10) we showed different between the photon and $b$ quark distribution function, as it is seen, in low energy $Q^2 = 12, 25 \text{ GeV}^2$ the photon distribution is larger than $b$ quark but by increasing energy the photon distribution both in LO and NLO corrections decreases. It should be expressed that in range $x > 0.03$ and $Q^2 < 200 \text{ GeV}^2$ the photon distribution function has dominant contribution in comparison with $b$ quark. In Figs. (11), the sea distribution functions are larger than the photon distribution function but in high energy and $0.1 < x < 1$ the contribution of photon is significant in analogy with sea quarks.

In conclusion, we obtained three analytical decoupled differential evolution equations for the singlet and gluon and photon distribution functions from the QCD⊗QED coupled DGLAP equations by using the double Laplace method at leading order and next-to-leading order. These equations are general and require only a knowledge of $F^s_0, G_0, M_0$ at the starting value $Q^2_0$ for the evolution. The most important parts of this paper was that photon distribution function at high energy scales depends on gluon distribution function at initial energy scale. Using the Laplace transform method, we obtained contribution of photon in proton at high energy and showed which this contribution is larger than $b$ heavy quark in specific area of $x$. We showed that the photon distribution function is comparable with sea quarks distribution functions of specially at high $x$ and middle $Q^2$. Also it is observed that the general solutions are in good agreement with available the experimental data and other parameterization models.

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Figure 2: Solid lines show the results of Eq. (23) and dash lines show the exact values of $\frac{\alpha_s(Q^2)}{\alpha_s^{LO}(Q^2)}$. 
Figure 3: Solid lines show the results of Eq. (49) (left) and dash lines show the exact values of $\frac{\alpha_s(Q^2)}{\alpha_s^{LO}(Q^2)}$. 
Figure 4: Solid lines show the results of Eq. (49) (right) and dash lines show the exact values of $\frac{\alpha_{\text{LO}}(Q^2)}{4\pi}$.
Figure 5: Solid lines show the results for the proton structure function at $Q^2 = 120, 1000, 8000 GeV^2$ scales in NLO, dash lines and symbols are the MSTW parameterization [3] and APFEL results [2] respectively.

Figure 6: Solid lines show the results for the proton structure function at $Q^2 = 120, 1000, 8000 GeV^2$ scales in NLO and symbols are HERA data [1].
Table 1: Parameter values $a_{10}, ..., b_{31}$ at different ranges of energy

|                | $3 \leq Q^2 < 20$ | $20 \leq Q^2 < 200$ | $200 \leq Q^2 < 1000$ | $1000 \leq Q^2 < 8000 GeV^2$ |
|----------------|-------------------|----------------------|------------------------|-------------------------------|
| $\frac{\alpha_s(Q^2)}{\alpha_s^{LO}(Q^2)}$ |                   |                      |                        |                               |
| $a_{10}$       | 0.0088            | 0.0101               | 0.0108                 | 0.0128                        |
| $a_{11}$       | 0.0109            | 0.0134               | 0.0141                 | 0.0141                        |
| $b_{11}$       | 16.00             | 11.93                | 11.45                  | 11.15                         |
| $\frac{\alpha_N(Q^2)}{\alpha_N^{LO}(Q^2)}$ |                   |                      |                        |                               |
| $a_{20}$       | 0.0091            | 0.010                | 0.0124                 | 0.0154                        |
| $a_{21}$       | 0.0175            | 0.0186               | 0.0187                 | 0.0205                        |
| $b_{21}$       | 17.12             | 13.10                | 12.57                  | 11.64                         |
| $\frac{\alpha_N^{LO}(Q^2)}{4\pi}$ |                   |                      |                        |                               |
| $a_{30}$       | 0.0015            | 0.0014               | 0.0014                 | 0.0014                        |
| $a_{31}$       | 0.0221            | 0.0228               | 0.0228                 | 0.0228                        |
| $b_{31}$       | 15.00             | 13.01                | 12.94                  | 12.90                         |
Figure 7: Solid lines show the results for singlet distribution function at $Q^2 = 20, 100, M_z^2 GeV^2$ scales in LO (up) and NLO (down) and dash lines are the MSTW parameterization [2].
Figure 8: Solid lines show the results for gluon distribution function at $Q^2 = 20, 100, M^2 GeV^2$ scales in LO (up) and NLO (down) and dash lines are the MSTW parameterization [3].
Figure 9: Solid and dot lines show the results for the photon distribution function at $Q^2 = 20, 200, 8000 \text{ GeV}^2$ scales in NLO and LO, respectively and dash line is the MRST parameterization \[\text{[4]}\].

Figure 10: Solid and dot lines show the results for the photon distribution function at $Q^2 = 12, 25, 60, 200 \text{ GeV}^2$ scales in NLO and LO, respectively and dash lines are $b$ quark distribution function presented by CETQ \[\text{[5]}\].
Figure 11: Solid and dot lines correspond to the results for photon distribution function at $Q^2 = 2, 100 \text{ GeV}^2$ scales in NLO and LO, respectively and other lines are the distribution functions of sea quarks presented by CETQ [5].

Appendix

The coefficients $A_1...C_3$ in Eqs. (31-33) are as follows

$$A_1(s,u) = \frac{R_1(s,u)R_2(s,u)}{H_1(s)},$$

$$A_2(s,u) = \frac{a_{10}\Omega_f(s)}{H_1(s,u)},$$

$$A_3(s,u) = \frac{R_3(s,u)\Theta_f(s)}{H_1(s,u)}$$

(A-1)

$$B_1(s,u) = \frac{R_3\Theta_g(s)}{H_1(s,u)},$$

$$B_2(s,u) = \frac{R_1(s,u)R_3 - a_{10}^2\Omega_f(s)\Upsilon_m(s)}{H_1(s,u)},$$

$$B_3(s,u) = \frac{a_{10}\Omega_f(s)\Theta_g(s)}{H_1(s,u)}$$

(A-2)

$$C_1(s,u) = \frac{a_{10}R_2(s,u)\Upsilon_m(s)}{H_1(s,u)},$$

$$C_2(s,u) = \frac{a_{10}\Theta_f(s)\Upsilon_m(s)}{H_1(s,u)},$$

$$C_3(s,u) = \frac{R_1(s,u)R_2(s,u) - \Theta_f\Theta_g}{H_1(s,u)},$$

(A-3)
and coefficients $A'_1$...$C'_3$ in Eqs. (43-45) are as follows

\[ A'_1(s, u) = \frac{1}{H_1(s, u)} \left[ R_2(s, u)R_3(s, u) + a_{11}R_2(s, u)R_3(s, u)\Omega_f(s)C_1(s, u - b_{11}) + a_{10}^2 R_2(s, u)\Omega_f(s) \right] \]

\[ \times \Upsilon_m(s)A_1(s, u - b_{11}) + a_{11}R_2(s, u)R_3(s, u)Y(s)A_1(s, u - b_{11}) + a_{10}^2 R_2(s, u)\Omega_f(s) \]

\[ A'_2(s, u) = \frac{1}{H_1(s, u)} \left[ R_3(s, u)\Omega_f(s) + a_{11}R_2(s, u)R_3(s, u)\Upsilon_f(s)A_2(s, u - b_{11}) + a_{10}^2 R_2(s, u)\Omega_f(s) \right] \]

\[ \times \Upsilon_m(s)A_2(s, u - b_{11}) + a_{11}R_2(s, u)R_3(s, u)\Omega_f(s)C_2(s, u - b_{11}) + a_{10}^2 R_2(s, u)\Omega_f\Omega_mC_2(s, u - b_{11}) \]

\[ A'_3(s, u) = \frac{1}{H_1(s, u)} \left[ a_{11}R_2(s, u)R_3(s, u)v_f(s)A_3(s, u - b_{11}) + a_{10}^2 R_2(s, u)\Omega_f(s) \right] \]

\[ \times \Upsilon_m(s)A_3(s, u - b_{11}) + a_{11}R_2(s, u)R_3(s, u)\Omega_f(s)C_3(s, u - b_{11}) + a_{10}^2 R_2(s, u)\Omega_f\Omega_mC_3(s, u - b_{11}) \]

\[ B'_1(s, u) = \frac{1}{H_1(s, u)} \left[ R_3(s, u)\Theta_g(s) + a_{11}R_3(s, u)\Theta_g(s)\Upsilon_f(m)A_1(s, u - b_{11}) + a_{10}^2 \Omega_f(s)\Theta_g(s) \right] \]

\[ \times \Upsilon_m(s)A_1(s, u - b_{11}) + a_{11}R_3(s, u)\Omega_f(s)\Theta_g(s)C_1(s, u - b_{11}) + a_{10}^2 \Omega_f(s)\Omega_m(s)\Theta_gC_1(s, u - b_{11}) \]

\[ B'_2(s, u) = \frac{1}{R_2(s, u)} + \frac{1}{H_1(s, u)} \left[ \frac{R_3(s, u)\Theta_f(s)\Theta_g(s)}{R_2(s, u)} + a_{11}R_3(s, u)\Theta_g(s)\Upsilon_f(s)A_2(s, u - b_{11}) + a_{10}^2 \Omega_f(s)\Theta_g(s) \right] \]

\[ \times \Upsilon_m(s)A_2(s, u - b_{11}) + a_{11}R_3(s, u)\Omega_f(s)\Omega_m(s)\Theta_gC_2(s, u - b_{11}) \]

\[ B'_3(s, u) = \frac{1}{H_1(s, u)} \left[ a_{10}\Omega_f(s)\Theta_g(s) + a_{11}R_3(s, u)\Theta_g(s)\Upsilon_f(s)A_3(s, u - b_{11}) + a_{10}^2 \Omega_f(s)\Theta_g(s) \right] \]

\[ \times \Upsilon_m(s)A_3(s, u - b_{11}) + a_{11}R_3(s, u)\Omega_f(s)\Theta_g(s)C_3(s, u - b_{11}) + a_{10}^2 \Omega_f(s)\Omega_m(s)\Theta_gC_3(s, u - b_{11}) \]
\[
\Omega_f(s)\Upsilon_m(s) + a_{10}^2 R_2(s,u)\Omega_f(s)\Upsilon_m^2(s)A_3(s,u - b_{11}) + a_{10}^3 R_2(s,u)\Omega_f\Omega_m \Upsilon m C_3(s,u - b_{11}) \]
\]

where

\[
H_1(s,u) = R_1(s,u) R_2(s,u) R_3(s,u) - R_3(s,u) \Theta_f(s) \Theta_g(s) - a_0^2 R_2(s,u) \Omega_f(s) \Upsilon_m(s)
\]

\[
R_1(s,u) = u - \Phi_f(s) - \Upsilon_f(s) a_{10}, \quad R_2(s,u) = u - \Phi_g(s), \quad R_3(s,u) = u - \Omega_m(s) a_{10}
\]

The coefficients \(A_1^{NLO} \ldots C_3^{NLO}\) in Eqs. (57-59) are as follow

\[
A_1^{NLO}(s,u) = \frac{Y_2(s,u) Z_2(s,u)}{H_2(s,u)}, \quad A_2^{NLO}(s,u) = \frac{X_2(s) Z_2(s,u)}{H_2(s,u)}, \quad A_3^{NLO}(s,u) = \frac{Y_2(s,u) X_3(s)}{H_2(s,u)}
\]

\[
B_1^{NLO}(s,u) = \frac{Y_2(s,u) Z_2(s,u)}{H_2(s,u)}, B_2^{NLO}(s,u) = \frac{X_2(s) Z_2(s,u) - X_3(s) Z_1(s)}{H_2(s,u)}, B_3^{NLO}(s,u) = \frac{X_3(s) Y_1(s)}{H_2(s,u)}
\]

\[
C_1^{NLO}(s,u) = \frac{Y_2(s,u) Z_1(s)}{H_2(s,u)}, C_2^{NLO}(s,u) = \frac{X_2(s) Z_1(s) - X_3(s) Z_1(s)}{H_2(s,u)}, C_3^{NLO}(s,u) = \frac{X_1(s,u) Y_2(s,u) - X_2(s) Y_1(s)}{H_2(s,u)}
\]

where

\[
H_2(s,u) = X_1(s,u) Y_2(s,u) Z_2(s,u) - X_2(s,u) Y_1(s) Z_2(s,u) - X_3(s) Y_2(s,u) Z_1(s)
\]

\[
X_1(s,u) = u - \Phi_f^L(s) - a_{30} \Phi_f^{NLO}(s) - a_{20} \Upsilon_f(s), \quad X_2(s) = \Theta_f^L(s) + a_{30} \Theta_f^{NLO}(s), \quad X_3(s) = a_{20} \Omega_f(s)
\]

\[
Y_1(s) = \Theta_g^L(s) + a_{30} \Theta_g^{NLO}(s) Y_2(s,u) = u - \Phi_g^L(s) - a_{30} \Phi_g^{NLO}(s)
\]

\[
Z_1(s) = a_{20} \Upsilon_m(s), \quad Z_2(s,u) = u - a_{20} \Omega_m(s)
\]

And the coefficients \(A_1^{NLO}(s,u) \ldots C_3^{NLO}(s,u)\) in Eqs. (63-65) are as follow

\[
A_1^{NLO}(s,u) = \frac{1}{H_2(s,u)} \left[ C_1^{NLO}(s,u - b_{21}) \Omega_m(s) + A_1^{NLO}(s,u - b_{12}) \Upsilon_m(s) \right] a_{21} X_3(s) Y_2(s,u)
\]

\[
+ \left[ B_1^{NLO}(s,u + b_{31}) \Phi_f(s) + A_1^{NLO}(s,u + b_{31}) \Theta_g^{NLO}(s) \right] a_{31} X_2(s) Z_2(s,u)
\]

\[
+ \left[ 1 + a_{21} C_1^{NLO}(s,u - b_{21}) \Omega_f(s) + a_{31} A_1^{NLO}(s,u + b_{31}) \Phi_f^{NLO}(s) + a_{31} B_1^{NLO}(s,u + b_{31}) \Theta_f^{NLO}(s) \\
+ a_{12} A_1^{NLO}(s,u - b_{12}) \Upsilon_f(s) \right] Y_2(s,u) Z_2(s,u)
\]

\[
A_2^{NLO} = \frac{1}{H_2(s,u)} \left[ C_3^{NLO}(s,u - b_{21}) \Omega_m(s) + A_2^{NLO}(s,u - b_{12}) \Upsilon_m(s) \right] a_{21} X_3(s) Y_2(s,u)
\]

\[
+ \left[ B_2^{NLO}(s,u + b_{31}) \Phi_f(s) + A_2^{NLO}(s,u + b_{31}) \Theta_g^{NLO}(s) \right] a_{31} X_2(s) Z_2(s,u)
\]

\[
+ \left[ 1 + a_{21} C_3^{NLO}(s,u - b_{21}) \Omega_f(s) + a_{31} A_2^{NLO}(s,u + b_{31}) \Phi_f^{NLO}(s) + a_{31} B_2^{NLO}(s,u + b_{31}) \Theta_f^{NLO}(s) \\
+ a_{12} A_2^{NLO}(s,u - b_{12}) \Upsilon_f(s) \right] Y_2(s,u) Z_2(s,u)
\]

\[
A_3^{NLO} = \frac{1}{H_2(s,u)} \left[ C_3^{NLO}(s,u - b_{21}) \Omega_m(s) + A_3^{NLO}(s,u - b_{12}) \Upsilon_m(s) \right] a_{21} X_3(s) Y_2(s,u)
\]
\[\begin{align*}
+ X_3(s) Y_2(s, u) &+ \left[ B_{3}^{NLO}(s, u + b_{31}) \Phi_f(s) + A_{3}^{NLO}(s, u + b_{31}) \Theta_g^{NLO}(s) \right] a_{31} X_2(s) Z_2(s, u) \\
+ \left[ a_{21} C_{3}^{NLO}(s, u - b_{21}) \Omega_f(s) + a_{31} A_{3}^{NLO}(s, u + b_{31}) \Phi_f^{NLO}(s) + a_{31} B_{3}^{NLO}(s, u + b_{31}) \Theta_f^{NLO}(s) \\
\quad + a_{12} A_{3}^{NLO}(s, u - b_{12}) \Upsilon_f(s) \right] Y_2(s, u) Z_2(s, u) &
\end{align*}\]

\( B_{1}^{NLO}(s) = \frac{a_{31}}{Y_2(s, u)} \left[ B_{1}^{NLO}(s, u + b_{31}) \Phi_f^{NLO}(s) + A_{1}^{NLO}(s, u + b_{31}) \Theta_g^{NLO}(s) \right] \)

\[\begin{align*}
+ \frac{1}{H_2(s, u)} \left[ C_{1}^{NLO}(s, u - b_{21}) \Omega_m(s) + A_{1}^{NLO}(s, u - b_{21}) \Upsilon_m(s) \right] a_{21} X_2(s) Y_1(s) &+ \left[ 1 + a_{21} C_{1}^{NLO}(s, u - b_{12}) \Omega_f(s) \\
\quad + a_{31} A_{1}^{NLO}(s, u + b_{31}) \Phi_f^{NLO}(s) + a_{31} B_{1}^{NLO}(s, u + b_{31}) \Theta_f^{NLO}(s) \\
+ a_{21} A_{1}^{NLO}(s, u - b_{21}) \Upsilon_f(s) + \frac{X_2(s)}{Y_2(s, u)} \left[ a_{31} B_{1}^{NO}(s, u + b_{31}) \Phi_f^{NO}(s) + a_{31} A_{1}^{NLO}(s, u + b_{31}) \Theta_g(s) \right] \right] Y_1(s) Z_2(s, u) \end{align*}\]

\( B_2^{NLO}(s) = \frac{1}{Y_2(s, u)} \left[ 1 + a_{31} B_{1}^{NO}(s, u + b_{31}) \Phi_f^{NO}(s) + a_{31} A_{2}^{NLO}(s, u + b_{31}) \Theta_g^{NLO}(s) \right] \)

\[\begin{align*}
+ \frac{1}{H_2(s, u)} \left[ 1 + C_{3}^{NLO}(s, u - b_{21}) \Omega_m(s) + A_{3}^{NLO}(s, u - b_{21}) \Upsilon_m(s) \right] a_{21} X_3(s) Y_1(s) &+ \left[ a_{21} C_{3}^{NLO}(s, u - b_{12}) \Omega_f(s) \\
\quad + a_{31} A_{3}^{NLO}(s, u + b_{31}) \Phi_f^{NLO}(s) + a_{31} B_{3}^{NLO}(s, u + b_{31}) \Theta_f^{NLO}(s) \\
+ a_{21} A_{3}^{NLO}(s, u - b_{21}) \Upsilon_f(s) + \frac{X_3(s)}{Z_2(s, u)} \left[ a_{31} B_{3}^{NO}(s, u + b_{31}) \Phi_f^{NO}(s) + a_{31} A_{3}^{NLO}(s, u + b_{31}) \Theta_g(s) \right] \right] Y_1(s) Z_2(s, u) \end{align*}\]

\( C_{1}^{NLO}(s) = \frac{a_{21}}{Z_2(s, u)} \left[ C_{1}^{NLO}(s, u - b_{21}) \Omega_m(s) + A_{1}^{NLO}(s, u - b_{21}) \Upsilon_m(s) \right] \)

\[\begin{align*}
+ \frac{1}{H_2(s, u)} \left[ 1 + B_{1}^{NLO}(s, u + b_{31}) \Phi_f^{NLO}(s) + A_{1}^{NLO}(s, u + b_{31}) \Theta_g(s) \right] a_{31} X_2(s) Z_1(s) &+ \left[ a_{21} C_{1}^{NLO}(s, u - b_{12}) \Omega_f(s) \\
\quad + a_{31} A_{1}^{NLO}(s, u + b_{31}) \Phi_f^{NLO}(s) + a_{31} B_{1}^{NLO}(s, u + b_{31}) \Theta_f^{NLO}(s) \\
+ a_{21} A_{1}^{NLO}(s, u - b_{21}) \Upsilon_f(s) + \frac{X_3(s)}{Z_2(s, u)} \left[ a_{21} C_{1}^{NO}(s, u - b_{21}) \Omega_m(s) + a_{21} A_{1}^{NLO}(s, u - b_{21}) \Upsilon_m(s) \right] \right] Y_2(s, u) Z_1(s) \end{align*}\]

\( C_{2}^{NLO}(s) = \frac{a_{21}}{Z_2(s, u)} \left[ C_{3}^{NLO}(s, u - b_{21}) \Omega_m(s) + A_{2}^{NLO}(s, u - b_{21}) \Upsilon_m(s) \right] \)

\[\begin{align*}
+ \frac{1}{H_2(s, u)} \left[ 1 + a_{31} B_{2}^{NLO}(s, u + b_{31}) \Phi_f^{NLO}(s) + a_{31} A_{2}^{NLO}(s, u + b_{31}) \Theta_g(s) \right] X_2(s) Z_1(s) &+ \left[ a_{21} C_{3}^{NLO}(s, u - b_{12}) \Omega_f(s) \\
\quad + a_{31} A_{2}^{NLO}(s, u + b_{31}) \Phi_f^{NLO}(s) + a_{31} B_{2}^{NLO}(s, u + b_{31}) \Theta_f^{NLO}(s) \\
+ a_{21} A_{2}^{NLO}(s, u - b_{21}) \Upsilon_f(s) + \frac{X_3(s)}{Z_2(s, u)} \left[ a_{21} C_{3}^{NO}(s, u - b_{21}) \Omega_m(s) + a_{21} A_{2}^{NLO}(s, u - b_{21}) \Upsilon_m(s) \right] \right] Y_2(s, u) Z_1(s) \end{align*}\]
\[ C_3^{NLO}(s) = \frac{1}{Z_2(s,u)} \left[ 1 + a_{21} C_3^{NLO}(s, u - b_{21}) \Omega_m(s) + a_{21} A_3^{NLO}(s, u - b_{21}) \Upsilon_m(s) \right] + \frac{1}{H_2(s,u)} \left[ a_{31} \left[ B_3^{NLO}(s, u + b_{31}) \Phi_f^{NLO}(s) + a_{31} A_3^{NLO}(s, u + b_{31}) \Theta_g(s) \right] X_2(s) Z_1(s) + a_{21} C_3^{NLO}(s, u - b_{12}) \Omega_f(s) \\
+ a_{31} A_3^{NLO}(s, u + b_{31}) \Phi_f^{NLO}(s) + a_{31} B_3^{NLO}(s, u + b_{31}) \Theta_f^{NLO}(s) + a_{21} A_3^{NLO}(s, u - b_{21}) \Upsilon_f(s) \right] \left[ 1 + a_{21} C_3^{NO}(s, u - b_{21}) \Omega_m(s) + a_{21} A_3^{NLO}(s, u - b_{21}) \Upsilon_m(s) \right] Y_2(s,u) Z_1(s) \]

(A-10)
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