ANYONS AND THE HOMFLY SKEIN ALGEBRA

SACHIN J. VALERA

ABSTRACT. We give an exposition of how the Kauffman bracket arises for certain systems of anyons, and do so outside the usual arena of Temperley-Lieb-Jones categories. This is further elucidated through the discussion of the Iwahori-Hecke algebra and its relation to modular tensor categories. We then proceed to classify the framed link-invariants associated to a system of self-dual anyons \( q \) with \( \sum x N_{xq}^\alpha \leq 2 \). In particular, we construct a trace on the HOMFLY skein algebra which can be expanded via gauge-invariant quantities, thereby generalising the case of the Kauffman bracket. Various examples are provided, and we deduce some interesting properties of these anyons along the way.

1. Introduction

Given a fusion space of \( n \) anyons (of charge) \( q \), the exchange matrices define a unitary representation of the braid group \( B_n \). This means the physical process described by a braid word \( b \) is the same as that described by a braid word \( b' \), where braids \( b \) and \( b' \) are related by braid isotopy. It follows that for \( q \) self-dual, the physical process described by a link spanned by (the worldlines of) \( n \) such anyons is preserved by type-II and III Reidemeister moves: this indicates that there should be an associated invariant of framed links. Moreover, since such links correspond to a physically measurable process, we expect to be able to expand this invariant in terms of gauge-independent quantities\(^1\). In this paper, we determine said invariant for all anyons \( q \) with fusion rules of the form \( q \otimes q = 1 \) or \( q \otimes q = 1 \oplus x \), and exploit the construction to deduce some useful properties of \( q \).

Section 1.1 clarifies some of the terminology and conventions used throughout the paper. We will repeatedly refer to the data calculated in [1] for various theories of anyons: this is further elaborated on in Section 1.2.

In Section 2, we begin with a discussion of what it means for a system of anyons to ‘span’ a link, noting that an arbitrary link can be realised by the worldlines of \( 2n \) anyons as the plat closure of a \( 2n \)-braid.

In Section 3, we discuss ‘Kauffman anyons’, covering the ideas of which the main work of this paper is a generalisation. These anyons are well-understood entities, and are commonly understood as Jones-Wenzl projectors: objects in Temperley-Lieb-Jones categories (discussed in the study of Jones-Kauffman theories) [2]. We avoid this formalism here. Section 3.1 reaps the definition of the Kauffman bracket and the Jones polynomial, arguing that the former should be pertinent to anyons satisfying certain conditions. A simple diagnostic aid is introduced in Proposition 3.4 for determining when said conditions are met. Section 3.2 then provides several examples, a few of which are presented in the context of topological quantum computation.

\(^1\)‘Gauge freedom’ here refers to a freedom in defining the basis elements of a fusion space. Of course, no physically observable quantity should depend on such a choice, and should thus be ‘gauge-invariant’.
Section 4 provides the background for the construction presented in Section 5, introducing the notion of a ‘Hecke anyon’: we motivate this handle by establishing a connection between representations of the Iwahori-Hecke algebra and unitary modular tensor categories in Section 4.1. Kauffman anyons are then revisited and shown to be a special case of Hecke anyons in Section 4.2, allowing us to further understand some of their properties.

Section 5 lifts the focus to the so-called Hecke anyons, presenting the main results of the paper which generalise the previous results pertaining to Kauffman anyons.

Section 5.1 follows the presentation of [3] in defining a trace on the skein algebra $H_n$, from which the HOMFLY polynomial is recovered (although we formulate our own proof for determining the basis of $H_n$ in Proposition 5.1, which is more algebraic in flavour). Section 5.2 introduces a slightly modified trace which assumes the role of the framed link-invariant associated to Hecke anyons (thus serving as an explicit analogue of the Kauffman bracket for Kauffman anyons). This is used to deduce some interesting properties of Hecke anyons. A selection of examples are provided in Section 5.3.

Finally, a few concluding questions are pondered in Section 6.

1.1. Some definitions. We will abbreviate (non-)Abelian anyons to (non-)Abelions.

**Definition 1.1.** We define a theory of anyons to be a fixed UMTC (unitary modular tensor category) modulo the symmetry $S \rightarrow -S$ (where $S$ is the $S$-matrix).

**Definition 1.2.** A Grothendieck class of fusion categories is a set of categories with mutually isomorphic fusion rules, and is sometimes referred to as a ‘fusion ring’. A model is a choice of ‘labels’ $\mathcal{L} = \{1, q_1, \ldots, q_{n-1}\}$ on the class, subject to the corresponding fusion coefficients $N_{XY}^Z \in \mathbb{N}_0$, where $X, Y, Z \in \mathcal{L}$. The trivial label (i.e. the vacuum) is written $1$. We only encounter multiplicity-free models in this paper, meaning $N_{XY}^Z \in \{0, 1\} \forall X, Y, Z$.

**Definition 1.3.** The type or charge $q$ of an anyon is its corresponding label, or simple object in the relevant UMTC. We abbreviate this to ‘an anyon $q$’. An anyon $q$ is self-dual if it is its own antiparticle $q^*$, whence we write $q = q^*$ and $N_{qq}^1 = 1$.

**Definition 1.4.** The quantity $\vartheta_q \in U(1)$ is called the topological spin of $q$, and has $\arg(\vartheta_q) \in [0, 2\pi)$.

**Definition 1.5.** The rank of a theory is the rank of its $S$-matrix (or the number of distinct labels). The rank of a self-dual anyon $q$ is defined to be $\sum_x N_{qq}^x \geq 1$, where equality only occurs for Abelions.

**Definition 1.6.** $\tilde{S}$ will denote the unnormalised $S$-matrix $\mathcal{D}S$, where $\mathcal{D}$ is the ‘total quantum dimension’ of the theory.

**Definition 1.7.** A fusion basis is a choice of the order in which anyons are fused. Given $n$ anyons on the plane, there are $C_{n-1}$ fusion bases (where $C_n$ is the $n^{th}$ Catalan number). The canonical fusion basis will mean the basis in which anyons are sequentially fused from left-to-right.

We adopt the ‘pessimistic’ convention in our illustrations: $(2+1)$-spacetime diagrams are drawn with time flowing downwards. The generators of the braid group $B_n$ are denoted by $\{\sigma_i\}_{i=1}^{n-1}$ (not to be confused with the Ising anyon $\sigma$), and the generators of the permutation group $S_n$ by $\{s_i\}_{i=1}^{n-1}$.
1.2. Data. When referring to a specific theory of anyons, we will use the data of 9 of the MTCs\(^2\) (Semion, Fibonacci, \(\mathbb{Z}_3\), Ising, \((A_1, 2)\), \((A_1, 5)_\frac{1}{2}\), \(\mathbb{Z}_4\), Toric code, \((A_1, 7)_\frac{1}{2}\)) listed in Section 5 of [1], but rename the Ising MTC to Ising\(_1\) and the \((A_1, 2)\) MTC to Ising\(_3\). These 9 MTCs are representatives of the 8 Grothendieck classes of non-trivial\(^3\) prime UMTCs of rank \(\leq 4\), each of which belong to a distinct class except for Ising\(_1\) and Ising\(_3\) which both belong to the Ising model (but are distinguished by Frobenius-Schur indicator \(\kappa = +1\) and \(\kappa = -1\) respectively). The Semion MTC belongs to the \(\mathbb{Z}_2\) fusion ring and also has \(\kappa = -1\) (solutions with positive Frobenius-Schur indicator are discarded here since they are non-anyonic i.e. describe statistical exchanges in 3D). No other self-dual anyons have Frobenius-Schur indicator \(-1\) in the listed data.

There exist UMTCs other than the representative(s) for each of these classes: a representative itself generates 4 prime UMTCs (2 if all of the data is real) via transformations (i) \((S, T) \rightarrow (-S, T)\) and (ii) \((S, T) \rightarrow (S^\dagger, T^\dagger)\) of the modular data, where (ii) corresponds to the conjugate theory.

There may also exist UMTCs with distinct modular data \((S', T')\) to the representative of their class e.g. for the Ising model, there are 16 (prime) UMTCs modulo \(S \rightarrow -S\) (i.e. 4 theories), and 2 theories modulo conjugacy (i.e. 2 sets of modular data modulo the transformations (i) & (ii)). For instance, the Ising model with \(\kappa = +1\) has theories Ising\(_1\), Ising\(_7\), Ising\(_9\) and Ising\(_{15}\) (with Ising\(_{15}\) = Ising\(_1^\dagger\) and Ising\(_9\) = Ising\(_{11}\) = Ising\(_7^\dagger\), where the subscript integer \(m\) indicates the value of the topological spin \(\vartheta = e^{\frac{im\pi}{8}}\) for the theory). The Ising model with \(\kappa = -1\) has theories Ising\(_3\), Ising\(_5\), Ising\(_{11}\) = Ising\(_3^\dagger\) and Ising\(_{13}\) = Ising\(_5^\dagger\). On the other hand, the Fibonacci model only has one modular datum (i.e. two mutually conjugate theories).

We will use \(\varphi\) to denote the golden ratio (the quantum dimension \(\frac{1 + \sqrt{5}}{2}\) of the Fibonacci anyon) where relevant.

2. Links and Worldlines

We begin by determining the circumstances under which the worldlines of a system of anyons will span a link. In this paper, we shall restrict ourselves to the case of each component of the link being spanned by the same type of anyon (in another sense, this means that all components of the link share the same colour). The setup is as follows:

1. Given a theory with a nontrivial self-dual anyon \(q\), initialise \(n\) such vacuum-pairs of particles. Denote the associated fusion space by \(V := V_{q^\otimes 2n}\).
2. Execute some desired braid \(b \in B_{2n}\) through a series of pairwise exchanges.
3. Fuse in any basis. All immediate fusion events (excluding fusion with the vacuum) must be annihilations. We denote this by the vacuum state \(|1\rangle \in V\).

\(^2\)We omit \((D_4, 1)\) which belongs to the same fusion ring as the Toric code MTC.

\(^3\)The 2 trivial UMTCs \(S = \pm 1\) correspond to the unique rank 1 theory \(\mathcal{L} = \{1\}\).
The result is a \(k\)-link spanned by anyons \(q\), where \(1 \leq k \leq n\). Note that there are 4 possible fusion bases that could represent the final stage depicted in Figure 1. But generically for this setup, the nature of the pair-creations and annihilations are determined by a configuration of \(n\) nonintersecting caps and cups respectively.

Our goal is to span any link \(L\). By Alexander’s theorem, we know that \(L\) can be obtained as the Markov closure of some braid \(b \in B_n\). This can then be isotoped (using only type-II and III Reidemeister moves) into the plat closure of some \(b' \in B_{2n}\):

We thus fix the cap/cup configuration corresponding to the plat closure throughout the rest of this paper. Moreover, this allows us to choose the canonical fusion basis:

**Remark 2.1.** Given a fixed fusion basis of \(V\), note that the associated exchange matrices define a unitary representation \(\rho_V\) of \(B_{2n}\) (more details in Section 4).
3. Kauffman Anyons

3.1. Preliminaries. Recall that the Kauffman bracket of a link diagram $D$ is the Laurent polynomial $\langle D \rangle \in \mathbb{Z}[A \pm 1]$ determined by the following rules:

\begin{align*}
(3.1a) \quad \langle \bigcirc \bigcirc \rangle &= d \\
(3.1b) \quad \langle L \cup \bigcirc \bigcirc \rangle &= d\langle L \bigcirc \bigcirc \rangle \\
(3.1c) \quad \langle \bigcirc \bigotimes \bigcirc \bigotimes \rangle &= A \langle \bigcirc \bigotimes \bigotimes \bigcirc \rangle + A^{-1} \langle \bigotimes \bigotimes \bigcirc \bigotimes \rangle
\end{align*}

where $d = - (A^2 + A^{-2})$. This is an isotopy invariant up to type-I Reidemeister moves (twists evaluate as in Figure 4). It quickly follows that

\begin{equation}
(3.2) \quad X(D) = (-A^3)^{-w(D)} \langle D \rangle
\end{equation}

is a full isotopy-invariant, where $w(D)$ is the writhe of $D$. Therefore, if $D$ is some diagram for a link $L$, we may write $X(L)$. Up to reparametrisation and normalisation, $X(L)$ is the Jones polynomial and we shall refer to it as such in the sequel.

\begin{equation}
(3.3) \quad d_q = \frac{-2 \cos \left[ \frac{2}{3} \left( \arg(\vartheta_q) + \pi \right) \right]}{d}
\end{equation}

Remark 3.1. Naively, there is no immediate reason to expect the (mathematical) existence of anyons $q$ satisfying the (seemingly arbitrary) relation (3.3). Nonetheless, such anyons do exist (and with apparent ubiquity). Their presence is well explained by Temperley-Lieb-Jones categories in the context of Jones-Kauffman theories, which we do not describe here (see e.g. [2]). However, we offer some partial insight as to their occurrence in the discussion of Section 4.2.

In the literature, a prevalent convention is to set the evaluation (3.1a) to 1. Our calculations may consequently be greater by a factor of $d$ when compared to some other sources. We opt to fix this alternative convention as it is more suited to the physical context in which we are interested.

This is shown by promoting worldlines to worldribbons (e.g. sketched in [5]).
**Definition 3.2.** We call a self-dual anyon \( q \) satisfying (3.3) a Kauffman anyon.

We know that the physical process (described by a link) \( L \) spanned by self-dual anyons \( q \) will be the same for any continuous deformations of \( L \) up to twists. For a Kauffman anyon \( q \), we thus expect to be able to deduce the statistical phase and amplitude of process \( L \) (up to a probabilistic normalisation factor \( \zeta \)) by evaluating the Kauffman bracket\(^6\) \( \langle L \rangle_q \). For the setup we fixed in Section 2, we have

\[
\zeta = \zeta_{n,q} = \frac{\langle L \rangle_q}{\langle |1 \rangle \rho_V(b) |1 \rangle} = d_q^{2(n-1)}
\]

where \( L \) is the plat closure of \( b \in B_{2n} \) and \( \rho_V \) is as in Remark 2.1 (having fixed any fusion basis consistent with the plat closure e.g. the canonical one). Physically, the probability of process \( L \) occurring (where initialisation and braiding are strictly controlled) is given by that of the of the \( n \) immediate annihilations. In a completely undetermined fusion channel\(^7\), the probability of two \( q \)'s annihilating is \( d_q^{-2} \). If \( n-1 \) pairs fuse to the vacuum, we know that the final pair must also do so (by ‘conservation of charge’), whence (3.4) follows.

**Remark 3.3.** This physical interpretation of \( \langle L \rangle_q \) is well-defined, as it is a function of \( \vartheta_q \) which is a gauge-invariant quantity.

Finally, it is useful to have some elementary criterion that immediately tells us if an anyon isn’t Kauffman:

**Proposition 3.4.**

(a) An Abelian \( q \) is Kauffman if and only if \( \vartheta_q = \pm 1 \).

(b) A non-Abelian \( q \) is Kauffman only if \( d_q \in [\sqrt{2}, 2] \) and \( \arg(\vartheta_q) \in [\frac{\pi}{8}, \frac{7\pi}{8}] \).

**Proof.** Let \( d := d_q \) and \( z := (-\vartheta_q)^{\frac{3}{4}} \). By (3.3) we require \( d = -2Re(z) \), whence \( z = -\frac{1}{2}d + i\sqrt{\frac{1-d^2}{4}} \).

(a) \( d = 1 \iff z = e^{j\frac{3\pi}{4}}, e^{j\frac{4\pi}{4}} \iff \vartheta_q = \pm 1 \)

(b) For any non-Abelian we know that \( d \geq \sqrt{2} \), and since \( d = -2Re(z) \), (3.3) tells us \( d \leq 2 \). Now,

\[
d \in [\sqrt{2}, 2] \implies Re(z) \in [-\frac{1}{\sqrt{2}}, -1], \quad Im(z) \in [-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}] \\
\implies \arg(z) \in [\frac{3\pi}{4}, \frac{5\pi}{4}] \implies \arg(\vartheta_q) \in [\frac{\pi}{8}, \frac{7\pi}{8}]
\]

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\(^6\) As we have seen, the Kauffman bracket is sensitive to twists (i.e. type-I Reidemeister moves).

\(^7\) By a ‘completely undetermined fusion channel’ for anyons \( a \) and \( b \) with fusion rule \( a \otimes b = \bigoplus q_i \), we mean a channel for which the outcome can be any one of the \( q_i \). This means that (3.4) only holds under the assumption that at most one pair of anyons has not braided amongst the rest.
Figure 5. Write \( \vartheta_q = e^{i2\pi s_q} \) where \( s_q \in [0, 1) \cap \mathbb{Q} \) (rationality follows from Vafa’s theorem [6]). By Proposition 3.4, points \( \vartheta_q = \pm 1 \) correspond to Kauffman Abeliions, and Kauffman non-Abeliions can only lie at points \( s_q \in \left[ \frac{1}{16}, \frac{7}{16} \right] \) of the arc. We will call this the Kauffman arc.

3.2. Examples.

Example 3.5. (Some Kauffman anyons)

Using Definition 3.2 and Proposition 3.4, we conduct a search for Kauffman anyons on the 9 MTCs from [1] listed in Section 1.2.

(i) Semion MTC: Semion \( s \) is a self-dual Abelian (with \( \kappa_s = -1 \)), but \( \vartheta_s = i \) so it is not Kauffman.

(ii) Fibonacci MTC: Fibonacci anyon \( \tau \) is Kauffman:

\[
\vartheta_{\tau} = e^{i\frac{4\pi}{5}} \implies -2\text{Re}((\vartheta_{\tau})^2) = 2\cos\left(\frac{\pi}{5}\right) = \varphi = d_{\tau}
\]

(iii) \( \mathbb{Z}_3 \) MTC: \( \omega \) and \( \omega^* \) are distinct duals, so they cannot be Kauffman.

(iv) Ising_1 MTC: \( \psi \) and \( \sigma \) are both Kauffman: \( \psi \) is a fermion (\( \vartheta_\psi = -1 \)) and

\[
\vartheta_{\sigma} = e^{i\frac{3\pi}{4}} \implies -2\text{Re}((\vartheta_{\sigma})^2) = -2\cos\left(\frac{3\pi}{4}\right) = \sqrt{2} = d_{\sigma}
\]

(v) Ising_3 MTC: \( \psi \) is a fermion, but \( \sigma \) has \( \kappa_\sigma = -1 \) and is not Kauffman:

\[
\vartheta_{\sigma} = e^{i\frac{3\pi}{8}} \implies -2\text{Re}((\vartheta_{\sigma})^2) = -2\cos\left(\frac{11\pi}{12}\right) \neq d_{\sigma}
\]

(vi) \( (A_1, 5)_{\frac{1}{2}} \) MTC: \( \alpha \) and \( \beta \) are self-dual non-Abeliions. \( \beta \) is off-arc,\(^8\) but \( \alpha \) is Kauffman:

\[
\vartheta_{\alpha} = e^{i\frac{2\pi}{7}} \implies -2\text{Re}((\vartheta_{\alpha})^2) = -2\cos\left(\frac{6\pi}{7}\right) = 2\cos\left(\frac{\pi}{7}\right) = d_{\alpha}
\]

(vii) \( \mathbb{Z}_4 \) MTC: \( \sigma \) and \( \sigma^* \) are distinct duals, so they cannot be Kauffman. \( \epsilon \) is a fermion (Kauffman).

(viii) Toric code MTC: \( e, m \) and \( \epsilon \) are all Kauffman Abeliions.

(ix) \( (A_1, 7)_{\frac{1}{2}} \) MTC: \( \alpha, \omega \) and \( \rho \) are all self-dual non-Abeliions. \( \rho \) is off-arc and \( d_\omega > 2 \), but \( \alpha \) is Kauffman:

\[
\vartheta_{\alpha} = e^{i\frac{2\pi}{9}} \implies -2\text{Re}((\vartheta_{\alpha})^2) = -2\cos\left(\frac{10\pi}{9}\right) = 2\cos\left(\frac{\pi}{9}\right) = d_{\alpha}
\]

Remark 3.6. Note that for the Fibonacci, Ising_1 and Toric code MTC, all anyons are Kauffman. These are Jones-Kauffman theories.

\(^8\)We will use ‘on-arc’ and ‘off-arc’ in the sense of the Kauffman arc (Figure 5).
Remark 3.7. Observe that $\sigma \in Ising_1$ resides on the right boundary of the Kauffman arc, and $\sigma \in Ising_7$ on the left boundary.

![Figure 6. The positively-oriented Hopf link $H^+$.

Example 3.8. (S-Matrix)
Consider $H^+$ (Figure 6). It is easy to show that $\langle H^+ \rangle = (2 - d^2)d$. We compute $\langle H^+ \rangle_q$ for all Kauffman anyons $q$ from Example 3.5.

(i) For any Abelian $q$ with $\vartheta_q = \pm 1$, we have $\langle H^+ \rangle_q = 1$.
(ii) For Fibonacci $\tau$, have $\langle H^+ \rangle_\tau = -\varphi^2 + 2\varphi = (-1 - \varphi) + \varphi = -1$
(iii) For $\sigma \in Ising_1$, have $\langle H^+ \rangle_\sigma = (2 - 2)\sqrt{2} = 0$
(iv) For $\alpha \in (A_1, 5)_{\frac{1}{2}}$, have $\langle H^+ \rangle_\alpha = -4\cos(\frac{2\pi}{7})\cos(\frac{\pi}{7})$
(v) For $\alpha \in (A_1, 7)_{\frac{1}{2}}$, have $\langle H^+ \rangle_\alpha = -4\cos(\frac{2\pi}{9})\cos(\frac{\pi}{9})$

Indeed,

$$\langle H^+ \rangle_q = (2 - d_q^2)d_q = \tilde{S}_{qq}, \text{ for a Kauffman anyon } q$$

Remark 3.9. Notice that no renormalisation factor $\zeta$ is required in (3.5), since fusion to the vacuum is a fixed detail of the $S$-matrix (i.e. there are no probabilistic considerations). Compare this to Example 3.10 where (3.4) is relevant.

We now look at two simple examples in the quantum computational context.

Example 3.10. (Physical Hopf link)

Take the Fibonacci qubit $|\Psi\rangle_{Fib} = \frac{1}{\varphi} |0\rangle + \sqrt{\frac{1}{\varphi}} |1\rangle \in V_{\tau\tau\tau}$ is the plat annihilation state. We may thus identify $|1\rangle$ and $|0\rangle$ for this simple example (Figure 8 (ii)).

| Figure 7. The Fibonacci qubit (canonical basis): let $|0\rangle$ and $|1\rangle$ correspond to outcomes $\mathbb{1}$ and $\tau$ respectively. The encoding space $V_{\tau\tau\tau}$ is enclosed by the blue dashed lines.

9$|\mathbb{1}\rangle$ is as defined in Section 2. Here, it belongs to the full fusion space of all 4 $\tau$ anyons and is the plat annihilation state. We may thus identify $|\mathbb{1}\rangle$ and $|0\rangle$ for this simple example (Figure 8 (ii)).
Figure 8. $b \in B_3$. (i) An arbitrary computation on the Fibonacci qubit in the canonical basis. (ii) The $|0\rangle$ state outcome of a computation is illustrated by the plat closure of $\iota(b) \in B_4$.

\[
R_{\tau\tau} = e^{-i\frac{4\pi}{5}} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{2\pi}{5}} \end{pmatrix} =: R, \quad F_{\tau\tau\tau} = \begin{pmatrix} \frac{1}{\varphi} & \frac{1}{\sqrt{\varphi}} \\ \frac{1}{\sqrt{\varphi}} & -\frac{1}{\varphi} \end{pmatrix} =: F (= F^{-1})
\]

Then,

\[
\rho_V(\sigma_1) = R, \quad \rho_V(\sigma_2) = F^{-1}RF
\]

\[
\rho_V(\sigma_2^2) = F^{-1}R^2F = e^{-i\frac{4\pi}{5}} \left( \begin{pmatrix} \frac{1}{\varphi} & \frac{1}{\sqrt{\varphi}} \\ \frac{1}{\sqrt{\varphi}} & -\frac{1}{\varphi} \end{pmatrix} \right) \left( e^{i\frac{4\pi}{5}} \cdot \frac{1}{\sqrt{\varphi}} \right) \cdot (-\frac{1}{\varphi})
\]

The Fibonacci anyons span the Hopf link $H^+$ for $b = \sigma_2^2$. We have,

\[
\langle 1 | \rho_V(\sigma_2^2) | 1 \rangle = \langle 0 | \rho_V(\sigma_2^2) | 0 \rangle = e^{-i\frac{4\pi}{5}} \left( \frac{1}{\varphi^2} + e^{i\frac{4\pi}{5}} \cdot \frac{1}{\varphi} \right) = \varphi^{-2} \cdot (-1)
\]

Note the renormalisation factor of $\zeta_{2,\tau} = d^2_\tau = \varphi^2$ relative to $\langle H^+ \rangle_\tau = \hat{S}_{\tau\tau} = -1$ (Example 3.8 (ii)), as expected per (3.4).

**Example 3.11. (Ising Trefoil)**

We now consider the left-handed trefoil knot $T_1$ spanned by 4 Ising anyons:

\[
R_{\sigma\sigma} = e^{-i\frac{\pi}{8}} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} =: R, \quad F_{\sigma\sigma\sigma} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} =: F (= F^{-1})
\]

Figure 9. The depicted knot $T_1$ is the plat closure of $\sigma_2 \sigma_1^{-1} \sigma_2 \in B_4$.

We can use the same setup as in Example 3.10 but switch out Fibonacci $\tau$ for Ising $\sigma$ and set $b = \sigma_2 \sigma_1^{-1} \sigma_2$. We have the Ising qubit $|\Psi\rangle_{Ising} = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \in V$, and
so,
\[ \rho_V(\sigma_2) = \frac{1}{\sqrt{2}} e^{i\frac{\pi}{4}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \implies \rho_V(\sigma_2^{-1}\sigma_2) = \frac{1}{2} e^{i\frac{3\pi}{8}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \]
whence
\[
\langle 1 | \rho_V(\sigma_2^{-1}\sigma_2) | 1 \rangle = \langle 0 | \rho_V(\sigma_2^{-1}\sigma_2) | 0 \rangle = \frac{1}{\sqrt{2}} e^{i\frac{5\pi}{8}}
\]
It is easy to show that
\[ \langle T_i \rangle = d[A^7 + A^{-1}(2 - d^2)] \]
and so
\[ \langle T_i \rangle |_\sigma = \sqrt{2} e^{i\frac{5\pi}{8}} \]
We have the expected renormalisation factor of \( \zeta_{2,\sigma} = d_\sigma^2 = 2 \) between (3.10) & (3.11).

**Example 3.12.** Suppose we have two links \( L_1 \) and \( L_2 \), with diagrams \( D_1 \) and \( D_2 \) which have \( k_1 \) and \( k_2 \) crossings respectively. Computing the Jones polynomial for each of the diagrams can be used to determine the inequivalence of the links. If \( k_1 \) and \( k_2 \) are sufficiently large, this task is classically intractable (since a link diagram with \( k \) crossings requires \( 2^k \) resolutions).

On a topological quantum computer, merely spanning the links with anyons transforms the complexity of the task to being parametrised by the size of the braid indices (instead of the number of crossings):

(i) Let \( D_i \) be the diagram for \( L_i \) realised by the plat closure of a braid \( b_i \in B_{2n_i} \). \( n_i \) is minimised (optimally, the braid index of \( L_i \)) and \( b_i \) is reduced.

(ii) Span \( D_i \) (as in Section 2) with Kauffman non-Abelions \( q \), fusing to the vacuum in the canonical basis with success probability \( (d_q^{-2})^{n_i-1} \in \left[ \frac{1}{4n_i-1}, \frac{1}{2n_i-1} \right] \).

(iii) The measured statistical phase and amplitude return \( \langle D_i \rangle |_q \) after renormalisation (3.4).

(iv) \( X(L_i) \) evaluated at a root of unity is recovered as \( \vartheta^{-w(b_i)} \langle D_i \rangle |_q \) (where \( w \) is the writhe).

Thus, the task is comparatively trivial here (for links with low braid indices). The task becomes intractable for higher braid indices: on average, we would expect the braid to have to be realised \( O(2^n) \) times before immediate fusion to the vacuum is achieved.

Note that the evaluation of \( X(L) \) at a root of unity is weaker than inequivalence-checking with \( A \) as a formal parameter: suppose \( X(L_i) = f_i(A) \) where \( f_1 \neq f_2 \). It is possible that \( A \) is a root of \( f_1 - f_2 \). If this occurs, we can repeat the procedure using distinct Kauffman anyons. If there is still no mismatch, we either have \( A \) is a root of \( f_1 - f_2 \) in all instances (which is unlikely), or the \( L_i \) have the same Jones polynomial. There are sophisticated quantum algorithms for approximating the Jones polynomial at roots of unity [7] ([2] for a concise outline).

4. **Hecke Anyons**

4.1. **The Iwahori-Hecke algebra and unitary modular tensor categories.**

We have seen that UMTCs can induce representations of the braid group: these
are easily classified. Since the braid group is infinite\textsuperscript{10}, it has an infinite number of representations. One idea is to take a quotient of \( B_n \) to yield a finite group e.g.

\[
(4.1) \quad B_n / P B_n \cong S_n
\]

where \( P B_n \) is the pure braid group\textsuperscript{11} and \( S_n \) is the permutation group. However, representations of the generators of \( S_n \) have eigenvalues \( \pm 1 \), rendering them uninteresting from our perspective. The natural progression is to consider the group algebra \( R[\mathcal{B}_n] \) (where \( R \) is a commutative ring with identity) and quotient by the ideal \( Q(\sigma_i) \) generated by the quadratic \((\sigma_i - r_1)(\sigma_i - r_2)\), where \( r_1, r_2 \in \mathbb{R}^* \).

\[
(4.2) \quad R[\mathcal{B}_n] / Q(\sigma_i) \cong H_n
\]

This gives us the \((Iwahori-)Hecke algebra \( H_n \), which is a free \( R \)-module of rank \( n! \) \cite{[3, 4]}. We clearly have a presentation of \( H_n = H_n(r_1, r_2) \) given by

\[
(4.3a) \quad T_i T_j = T_j T_i, \quad |i - j| \geq 2 \quad \text{(far commutativity)}
\]

\[
(4.3b) \quad T_i T_j T_i = T_j T_i T_j, \quad |i - j| = 1 \quad \text{(braid relation)}
\]

\[
(4.3c) \quad (T_i - r_1)(T_i - r_2) = 0 \quad \text{(Hecke relation)}
\]

for generators \( \{T_i\}_{i=1}^{n-1} \). The Hecke relation tells us that

\[
(4.4) \quad T_i^{-1} = \frac{(r_1 + r_2) - T_i}{r_1 r_2}
\]

i.e. we can think of \( H_n(r_1, r_2) \) as \( R[\mathcal{B}_n] \) modulo the skein relation\textsuperscript{12} (4.5). We shall henceforth set \( R = \mathbb{C} \).

\[
(4.5) \quad \bigvee + r_1 r_2 \bigvee = (r_1 + r_2) \bigvee
\]

**How does \( H_n \) relate to UMTCs?** Consider the exchange matrices for the fusion space \( V_{q^n} =: V \) of \( n \) particles \( q \). In some fusion basis, these define a unitary representation \( \rho_V \) of \( B_n \), which for \( n \geq 2 \) can generically be written

\[
(4.6) \quad \rho_V(\sigma_i) = \bigoplus_j (\mathcal{F}_i)_j^{-1} R_{qq}(\mathcal{F}_i)_j
\]

where \( \mathcal{F}_i \) is a sequence of \( F \)-moves \( (j \) indexes the inputs and output of the relevant subsystem), and for at least one \( i \) we have \( (\mathcal{F}_i)_j = \delta_{ij} \) (since at least one pair of particles will be in a direct fusion channel). Now suppose \( R_{qq} =: R \) has at most 2 distinct eigenvalues \( r_1, r_2 \in U(1) \). It follows that,

\[
(R - r_1)(R - r_2) = 0 \quad \implies \quad R^2 - (r_1 + r_2)R + r_1 r_2 = 0
\]

\[
\implies \quad (\mathcal{F}_i)_j^{-1} R^2(\mathcal{F}_i)_j - (r_1 + r_2)(\mathcal{F}_i)_j^{-1} R(\mathcal{F}_i)_j + r_1 r_2 = 0
\]

\[
\implies \quad \bigoplus_j ((\mathcal{F}_i)_j^{-1} R(\mathcal{F}_i)_j)^2 - (r_1 + r_2)((\mathcal{F}_i)_j^{-1} R(\mathcal{F}_i)_j) + r_1 r_2 = 0
\]

whence \( \rho_V \) defines a representation of \( H_n \) for any such \( R \).

\textsuperscript{10}Of course, \( B_1 \) is trivial and \( B_2 \cong \mathbb{Z} \).

\textsuperscript{11}This is the kernel \( \langle \sigma_i^2 \rangle \) of the homomorphism \( \eta : B_n \to S_n \), where \( \eta(\sigma_i) = s_i \).

\textsuperscript{12}Note that \( H_n(\pm 1, \mp 1) \cong R[S_n] \). In this sense, \( H_n \) is a ‘deformation’ of \( R[S_n] \). This is intuitive since they are related by the change of ideal \( \langle \sigma_i^2 \rangle \mapsto \langle (\sigma_i - r_1)(\sigma_i - r_2) \rangle \) with respect to \( R[\mathcal{B}_n] \).
Remark 4.1. In particular, \( \rho_V \) will always be a unitary representation for fusion rules of the form \( q \otimes q = x \) and \( q \otimes q = x \oplus y \).

Remark 4.2. The same idea applies for \( R \) of rank \( k \geq 3 \) with \( q \otimes q = \bigoplus_{i=1}^{k} q_i \), apart from we consider the generalised Hecke algebras \( H(Q, n) \) which are obtained by defining the ideal \( Q(\sigma_i) \) of \( \mathbb{C}[B_n] \) to be generated by a polynomial of degree \( k \). E.g., for \( k = 3 \), we have the ‘cubic Hecke algebra’ [8, 9] with cubic Hecke relation \( \prod_{i=1}^{3} (T_i - r_i) = 0 \). These higher quotients are comparatively unwieldy and we do not study them here.

Definition 4.3. We define a Hecke anyon to be a self-dual anyon \( q \) such that \( R_{qq} \) has rank \( \leq 2 \). It follows that Hecke Abelions\(^{13}\) and Hecke non-Abelions have fusion rules of the form \( q \otimes q = 1 \) and \( q \otimes q = 1 \oplus x \) respectively.

By symmetry of parameters \( r_1 \) and \( r_2 \) (Eqs. 4.3c, 4.4, 4.5), their specific assignment of \( R \)-symbols does not matter. We henceforth set \( r_1 = R_{qq}^1 \) in any case.

Example 4.4. Consider (4.6) with \( V \) in the canonical basis.

(i) For a Hecke Abelion \( q \) and \( m \geq 1 \), we have the \( U(1) \) representation
\[
\rho_V(T_1) = R_{qq}^L, \quad \rho_V(T_{2m}) = (F_{qqq}^q)^{-1} R_{qq}^L F_{qqq}^q, \quad \rho_V(T_{2m+1}) = (F_{qqq}^1)^{-1} R_{qq}^L F_{qqq}^1
\]
(ii) For a Hecke non-Abelion \( q \otimes q = 1 \oplus x \), we have a \( U(2) \) representation where \( \rho_V(T_1) = R_{qq} \) and \( r_k(F_1)^j = 1 \) for \( j \neq (x, x) \), where \( i > 1 \).
(iii) Take a Hecke non-Abelion \( q \otimes q = 1 \oplus x \). Let \( V = V_{qqq}^q \subset V_{qq}^1 \). Then \( V \) is the encoding space for a ‘Hecke qubit’ \( |\Psi\rangle_{\text{Hecke}} = d_q^{-1} |0\rangle + d_q^{1/2} d_q^{-1} |1\rangle \), where
\[
\rho_V(T_1) = R_{qq}, \quad \rho_V(T_2) = (F_{qqq}^q)^{-1} R_{qq} F_{qqq}^q
\]
Most examples in this paper will be of this form. We have already encountered two such instances (Examples 3.10 & 3.11).

Example 4.5. Consider Fig. 10 for,

(i) A Hecke non-Abelion \( q \otimes q = 1 \oplus x \), where \( r_1 := R_{qq}^L \) and \( r_2 := R_{qq}^L \). For \( z = 1 \), this just reads \( r_1 = (r_1 + r_2) - r_1 r_2 (r_1)^{-1} \). For \( z = x \), swap the indices.
(ii) A Hecke Abelion \( q \). Let \( r = R_{qq}^L \). Then we have \( z = 1 \) and \( r = (r + r) - r^2 r^{-1} \).

Example 4.6. Consider \( \tilde{S}_{qq} \) for a Hecke non-Abelion \( q \). Applying (4.5),

\(^{13}\)By our definition, all self-dual Abelions are Hecke.
That is,
\[ \tilde{S}_{qq} = -r_1 r_2 d_q^2 + (r_1 + r_2) \vartheta_q d_q \]
which for a self-dual Abelion \( q \) becomes
\[ \tilde{S}_{qq} = 2 \kappa_q - r_2 \]

E.g. the Fibonacci anyon \( \tau \) is Hecke with \( \tilde{S}_{\tau \tau} = - e^{-i \frac{5 \pi}{4}} \varphi^2 + (e^{-i \frac{3 \pi}{4}} + e^{i \frac{3 \pi}{4}}) e^{i \frac{4 \pi}{5}} \varphi = -1 \), and the semion \( s \) is a self-dual Abelion with \( \tilde{S}_{ss} = - \frac{2}{-1} \).

### 4.2. Kauffman anyons revisited.

By the Kauffman relation (3.1a), we see that
\[ \tilde{S}_{qq} = -A - 2 \]
which in terms of (4.5) tells us that \( r_1 r_2 = -A - 2 \) and \( r_1 + r_2 = A - A^{-3} \), whence (4.3c) becomes
\[ r_1^2 - (A - A^{-3}) r_1 - A^{-2} = 0 \]
\[ \Rightarrow (r_1, r_2) = (-A^{-3}, A) \text{ or } (r_1, r_2) = (A, -A^{-3}) \]

In accordance with our conventions, we will fix the former solution\(^\text{14}\). This tells us that Hecke non-Abelions with \( R_{qq}^1 = -(R_{qq}^x)^{-3} \) are Kauffman. We will show that, in fact, all Kauffman anyons are Hecke.

Consider a braid \( b \in B_n \) spanned by \( n \) Kauffman anyons \( q \), whose \( 2n \) endpoints are connected (without intersections) by any closure \( \kappa \). We have seen that \( \langle \kappa(b) \rangle_q \) is well-defined. Note that,
\[ \langle \kappa(b) \rangle = (A \circ \kappa \circ \rho_J)(b) \]
where
\[ \rho_J : \mathbb{C}[B_n] \to TL_n(A) \]
\[ \sigma_i \mapsto A^{-1} U_i + A \]
is the Jones representation, \( TL_n(A) \) is the Temperley-Lieb algebra with presentation
\[
\begin{align*}
U_i U_j &= U_j U_i, & \text{if } |i - j| \geq 2 \\
U_i U_j U_i &= U_i, & \text{if } |i - j| = 1 \\
U_i^2 &= d U_i, & \text{if } |i - j| = 1 \\
U_i^2 &= d U_i, & \text{if } d = -(A^2 + A^{-2})
\end{align*}
\]
for generators \( \{U_i\}_{i=1}^{n-1} \), and \( A(D) = d^k \) for a diagram \( D \) containing \( k \) disjoint loops\(^\text{15}\). \( A, \kappa \) and \( \rho_J \) are all \( \mathbb{C} \)-linear.

\(^\text{14}\)Recall that this is w.l.o.g by symmetry.
\(^\text{15}\)Observe that \( \kappa \) and \( \rho_J \) commute if we relax the restriction of the domain.
It follows that the unitary representation $\rho_V$ of $B_n$ induced by a fusion space $V = V_{q^\otimes n}$ of Kauffman anyons $q$ is precisely the unitary Jones representation $\rho_j$.

**Proposition 4.7.** $[2]$ $\rho_j$ is unitary if and only if $|A| = 1$ and $U_i = U_i^\dagger$.

**Proof.**

\begin{align*}
14 \quad \rho_j(\sigma_i)\rho_j(\sigma_i) &= 1 \iff |A|^2 + \left(\frac{A}{|A|}\right)^{-2}U_i + \left(\frac{A}{|A|}\right)^2U_i^\dagger + \frac{1}{|A|^2}U_i^\dagger U_i = 1 \\
\text{Write } U_i^\dagger &= \lambda + u_i, \text{ where } u_i \in \text{span}_\mathbb{C} < U_1, \cdots, U_{n-1} > \text{ and } \lambda \in \mathbb{C}. \text{ Then,} \\
U_i^\dagger U_i &= \lambda U_i + u_i U_i^\dagger \implies \lambda^2 U_i + U_i^\dagger u_i^\dagger = |\lambda|^2 + \lambda^* u_i + \lambda u_i^* + u_i u_i^* \\
\text{Since } U_i^\dagger U_i \text{ is invariant under } \dagger, \text{ we must have } \lambda = 0. \text{ This allows us to equate coefficients in (4.14) to get } |A| = 1 \text{ and } A^{-2}U_i + A^2U_i^\dagger + U_i^\dagger U_i = 0, \text{ whence} \\
U_i^\dagger &= \frac{-A^{-2}U_i}{U_i + A^2} = U_i
\end{align*}

By Proposition 4.7, unitarity tells us that $U_i$ is Hermitian, whence

\begin{align*}
A^{-1}\rho_j(\sigma_i) - A^{-2} &= A\rho_j(\sigma_i) - A^2 \\
\iff (\rho_j(\sigma_i))^2 - (A - A^{-3})\rho_j(\sigma_i) - A^{-2} &= 0 \\
\iff (\rho_j(\sigma_i) + A^{-3})(\rho_j(\sigma_i) - A) &= 0
\end{align*}

That is, *unitary Jones representations are unitary representations of $H_n(-A^{-3}, A)$*. Indeed, we have the following commutative diagram of homomorphisms:

\begin{align*}
\mathbb{C}[B_n] \xrightarrow{\phi} H_n(r_1, r_2) \\
\rho_j \downarrow \quad \downarrow \xi \\
TL_n(A) \xrightarrow{\xi} H_n(r_1, r_2)
\end{align*}

**Figure 12.** A commutative triangle of algebra homomorphisms.

where $\phi(\sigma_i) = T_i$, and $\ker(\phi) = Q(\sigma_i)$ as in (4.2). For $\xi$, we set $\xi(T_i) = A^{-1}U_i + A$.

By way of $\rho_j$, we know that $\xi$ preserves far commutativity and the braid relation.

Checking the Hecke relation, we see that

\begin{align*}
\xi(T_i + r_1r_2T_i^{-1}) &= (r_1 + r_2)\xi(1) \\
\iff (r_1, r_2) &= (-A^{-3}, A) \quad \text{or} \quad (A, -A^{-3})
\end{align*}
In keeping with our conventions, we choose $\xi$ to be defined by

$$\xi : \begin{cases} T_i \mapsto A^{-1}U_i + A \\ r_1 \mapsto -A^{-3} \\ r_2 \mapsto A \end{cases}$$

(4.18)

A good sanity check is to verify that $\xi(T_i^2) = d\xi(T_i)$. Finally, it is easy to see that,

$$\xi(\phi)\sigma_i^{\pm 1} = \rho_q(\sigma_i^{\pm 1})$$

(4.19)



\textbf{Corollary 4.8.} All Kauffman anyons are Hecke.

\textit{Proof.} We know that $\rho_V = \rho_q$ for Kauffman anyons. Furthermore, this $\rho_q$ is unitary and is thus a representation of $H_n$. From Section 4.1, we know that such $\rho_V$ arises precisely for a non-Abelion $q$ with $r^k(R_{qq}) = 2$. Since $q$ is self-dual here, it is Hecke. Trivially, all Kauffman Abelions are Hecke (as all self-dual Abelions are Hecke).

We have shown that for a Kauffman non-Abelion $q$, we always have\(^\text{16}\)

$$R_{qq} = \begin{pmatrix} R_{qq}^1 \\ 0 \\ R_{qq}^x \end{pmatrix} = \begin{pmatrix} -A^{-3} & 0 \\ 0 & A \end{pmatrix}$$

whence

$$R_{qq}^x = (-R_{qq}^1)^{-\frac{1}{2}} = \left(-q^*\right)^{-\frac{1}{2}}$$

\textbf{Corollary 4.9.} For a Kauffman non-Abelion $q \otimes q = 1 \oplus x$, we have

(i) $\kappa_q = +1$.

(ii) $d_q = -2\cos(2\arg(R_{qq}^x))$

(iii) $\arg(R_{qq}^1) \in \left[-\frac{7\pi}{8}, -\frac{\pi}{8}\right]$ , $\arg(R_{qq}^x) \in \left[\frac{3\pi}{8}, \frac{5\pi}{8}\right] \cup \left[-\frac{5\pi}{8}, -\frac{3\pi}{8}\right]$

\textit{Proof.}

(i) $R_{qq}^1 = \kappa_q q^*$ and $\vartheta^* = -A^{-3}$ (Fig. 4 (ii)). Thus, $\kappa_q = 1$ by (4.20).

(ii) Follows from $d = -(A^2 + A^{-2})$ and (4.20).

(iii) By (i), $R_{qq}^1 = \vartheta_q^*$ whence the Kauffman arc implies $\arg(R_{qq}^1) \in \left[-\frac{7\pi}{8}, -\frac{\pi}{8}\right]$.

Then use $\arg(R_{qq}^x) \in [0, 2\pi]$, $d_q \in [\sqrt{2}, 2]$ and (ii).

\textbf{Corollary 4.10.} Consider the Jones representation for a system of Kauffman non-Abelions $q$. We have $(tr(U_i)|q)/d_q = \beta_i$, where $\beta_i$ counts the blocks in $\rho_q(\sigma_i)$.

\textit{Proof.} There exists at least one $k$ such that $\rho_q(\sigma_k) = R_{qq}$. For such $k$,

$$U_k = AR_{qq} - A^2I_2$$

(4.22)

$$\Rightarrow tr(U_k) = A(-A^{-3} + A) - 2A^2 = -A^2 - A^{-2} = d_q$$

where $I_2$ is the $(2 \times 2)$ identity matrix. By (4.6), for generic $\sigma_i$ we have

$$\rho_q(\sigma_i) = \bigoplus_{j=1}^{\beta_i} (F_i)^{-1}R_{qq}(F_i)_j$$

(4.23)

$$\Rightarrow U_i = \bigoplus_{j} [A((F_i)^{-1}R_{qq}(F_i)_j] - A^2I_2] = \bigoplus_{j} (F_i)^{-1}U_k(F_i)_j$$

\(^{16}\)See [2] for an alternative derivation of (4.20) for Jones-Kauffman theories.
whence
\[
tr(U_i) = \sum_j tr(U_k) = \beta_i d_q
\]

We can find a matrix expression for \(U_i\). For \(k\) such that \(\rho_3(\sigma_k) = R_{qq}\), (4.20) and (4.22) imply
\[
U_k = \begin{pmatrix}
d_q & 0 \\
0 & 0
\end{pmatrix}
\]

From Example 4.4 (ii), we know that \(rk((\mathcal{F}_i)_j) = 1\) for \(j\) such that \((\mathcal{F}_i)_j \neq F_{qqq}^q\). For such \(j\), \((\mathcal{F}_i)_j^{-1}U_k(\mathcal{F}_i)_j = U_k\). Now suppose \((\mathcal{F}_i)_j = F_{qqq}^q =: F\). By symmetry of the charges, the unitary matrix \(F\) is Hermitian, and so in the appropriate gauge, \(F\) will be real-symmetric and orthogonal. We fix this gauge and write,
\[
F = \begin{pmatrix}
a & c \\
c & b
\end{pmatrix}, \quad a, b, c \in \mathbb{R}
\]

where \(a = F_{11} = d_q^{-1}, b = F_{xx}, c = F_{1x} = F_{x1}, a^2 + c^2 = 1\) and \((a + b)c = 0\). By (4.25), we have
\[
F^{-1}U_kF = d_q \begin{pmatrix}
a^2 & ac \\
ac & c^2
\end{pmatrix}
\]

Note that (4.27) also holds for \((\mathcal{F}_i)_j \neq F\) by setting \(a = 1\) and \(c = 0\) for such \(j\) (i.e. when \(j \neq (q,q)\)). So by (4.23), it follows that
\[
U_i = d_q \bigoplus_{j=1}^{\beta_i} \begin{pmatrix}
a_j^2 & a_j c_j \\
a_j c_j & c_j^2
\end{pmatrix}
\]

where \((a_j, c_j) = (1, 0)\) for \(j\) such that \((\mathcal{F}_i)_j \neq F\), and \((a_j, c_j) = (a, c)\) for \(j = (q,q)\) (i.e. when \((\mathcal{F}_i)_j = F\)). Note that \(a_j^2 + c_j^2 = 1\) and \((a_j + b_j)c_j = 0\) for all \(j\). It is easy to check that the \(U_i\) satisfy (4.13a)-(4.13c), and that \(tr(U_i)\) agrees with (4.24).

**Corollary 4.11.** For a Kauffman non-Abelian \(q\) and an appropriate choice of gauge,
\[
F_{qqq}^q = \begin{pmatrix}
1 & \sqrt{d_q^2 - 1} \\
\sqrt{d_q^2 - 1} & d_q
\end{pmatrix}
\]

**Proof.** Fix the gauge as for (4.26). By the above,
\[
\rho_3(\sigma_i) = A^{-1}d_q \bigoplus_j \begin{pmatrix}
a_j^2 & a_j c_j \\
a_j c_j & c_j^2
\end{pmatrix} + A \bigoplus_j I_2
\]

\[
= \begin{pmatrix}
A^{-1}d_qa^2 + A & A^{-1}d_qac \\
A^{-1}d_qac & A^{-1}d_qc^2 + A
\end{pmatrix} \bigoplus_{j \neq (q,q)} \begin{pmatrix}
A^{-1}d_q + A & 0 \\
0 & A
\end{pmatrix}
\]
But using (4.6), we may also write
\[
\rho_q(\sigma_i) = \bigoplus_j \left( \begin{array}{cc} a_j & c_j \\ c_j & b_j \end{array} \right) \left( \begin{array}{cc} -A^{-3}a^2 + Ac^2 & -A^{-3}ac + Abc \\ -A^{-3}ac + Abc & -A^{-3}c^2 + Ab^2 \end{array} \right) \oplus \bigoplus_{j \notin (q,q)} \left( \begin{array}{cc} -A^{-3} & 0 \\ 0 & A \end{array} \right)
\]
(4.31)
Equating the top-left element of the block \(j = (q,q)\) in (4.30) and (4.31), we find
\[
c^2 = \frac{d_q^2 - 1}{d_q^2}
\]
(4.32)
The resulting choice of sign can be attributed to a choice of gauge, so we fix the positive root. Lastly, \(b = -1/d_q\), as \((a + b) c = 0\) (where \(c \neq 0\) since \(q\) is a non-Abelian).

\[\square\]

**Remark 4.12.** Remarkably for a Kauffman anyon \(q\), the topological spin \(\vartheta_q\) characterises all of its pertinent data: \(R_{qq}^x = \vartheta_q^*\), the quantum dimension \(d_q\) is deduced from (3.3), and then the \(F\)-matrix from (4.29) (and \(\hat{S}_{qq}\) via (3.5)). Finally, we can calculate \(R_{qq}^x\) using (4.21) and Corollary 4.9(ii).

**Remark 4.13.** We will call Hecke anyons which are not Kauffman, HNK anyons. The Kauffman arc tells us that for every theory with a Kauffman anyon, there exists a conjugate theory with an HNK anyon. By Corollary 4.9(i), we also know that Hecke anyons \(q\) with \(\varpi_q = -1\) are HNK anyons (e.g. the semion \(s\) is an HNK Abelian).

5. ANYONS AND THE HOMFLY SKEIN ALGEBRA

5.1. Preliminaries. Let \(\mathcal{H}_n\) be the toroidal skein algebra obtained through the Markov closure of \(H_n\): this is the vector space of \(\mathbb{C}\)-linear combinations of closed \(n\)-braids modulo type-II and III Reidemeister moves and the skein relation (4.5).

What is a basis for \(\mathcal{H}_n\)? We know that \(H_n\) has a basis \(\{T_w\}_{w \in S_n}\), where for a reduced expression \(w = s_{i_1} \cdots s_{i_r}\), we write \(T_w = T_{i_1} \cdots T_{i_r}\) [3, 4]. For each \(w \in S_n\) with such a reduced expression, we say this lifts to \(\sigma_w = \sigma_{i_1} \cdots \sigma_{i_r} \in B_n\). Recall the isomorphism \(\hat{\phi} : \mathbb{C}[B_n]/Q(\sigma) \rightarrow H_n\). The basis \(\{T_w\}\) can be thought of as the image of all minimal positive braids \(\{\sigma_w\}\) under \(\hat{\phi}\).

Consider the linear map \(\pi : H_n \rightarrow \mathcal{H}_n\) given by Markov closure. Clearly, \(\mathcal{H}_n\) has the same basis as \(H_n\) modulo Markov closure. It thus remains to determine when \(\pi(T_w) = \pi(T_{w'})\). By Markov’s theorem, this reduces to determining the conjugacy classes amongst the positive braids \(\{\sigma_w\}\).

**Proposition 5.1.** \(\mathcal{H}_n\) has a basis given by the image of braids \(b_\lambda\) under \(\pi \circ \hat{\phi}\), where \(b_\lambda = b_{(\lambda_1)} \cdots b_{(\lambda_k)}\) with \(b_{(m)} = \sigma_{m-1} \cdots \sigma_1\) given a partition \(\lambda = (\lambda_1, \ldots, \lambda_k)\) of \(n\).

**Proof.** Recall the homomorphism \(\gamma : B_n \rightarrow S_n\), where \(\ker(\gamma) = PB_n\). For \(w\) reduced, we have \(\gamma(\sigma_w) = w\). Suppose \(\sigma_w = b_\sigma w b^{-1}, b \in B_n\). Then \(w = \gamma(b)w'\gamma(b)^{-1}\). Hence, \(\pi(T_w) = \pi(T_{w'}) \iff w\) is conjugate to \(w'\). The conjugacy classes of \(S_n\) are given by its cycle types. A convenient choice of representative for a \(\lambda\)-cycle is given by \(s_\lambda \in S_n\) (which lifts to the minimal positive braid \(b_\lambda\) as defined above). Since the \(T_i\) are indexed by \(w \in S_n\), it follows that a basis of \(\mathcal{H}_n\) is given by the image of \(\{b_\lambda\}\) under \(\pi \circ \hat{\phi}\). \[\square\]
Thus, \( \dim(\mathcal{H}_n) = p(n) \) (where \( p(n) \) is the \( n \)th partition number), and \( \pi \) is a linear projector with \( \text{null}(\pi) = n! - p(n) \). We shall henceforth write the basis of \( \mathcal{H}_n \) as \( \{b_\lambda\} \) (and implicitly assume the image under \( \tilde{\phi} \) or \( \pi \circ \tilde{\phi} \) where appropriate). Any closed \( n \)-braid in \( \mathcal{H}_n \) may be parsed into the \( \mathbb{C} \)-span of \( \{b_\lambda\} \) (e.g. using the algorithm formulated in Theorem 5.1 of [3]).

The HOMFLY invariant is readily fashioned from the Hecke algebra. An Ocneanu trace is a \( \mathbb{C} \)-linear map \( tr : H_n \rightarrow \mathbb{C} \) characterised by

\[
\begin{align*}
(5.1a) & \quad tr(ab) = tr(ba) \\
(5.1b) & \quad tr(b) = tr(T_n \iota(b)) \\
(5.1c) & \quad tr(b) = tr(T_n^{-1} \iota(b))
\end{align*}
\]

where \( \iota : H_n \rightarrow H_{n+1} \), with \( a, b \in H_n \) for all \( n \geq 1 \). By Markov’s theorem, this trace is clearly defined such that it yields a link-invariant for a link \( L \) obtained from the Markov closure of a braid \( b \) (since (5.1a) corresponds to invariance under type-I Markov moves, and (5.1b) & (5.1c) invariance under type-II Markov moves). The argument of this trace will thus often be presented as a braid (though we implicitly assume its image in the Hecke algebra). From (5.1b) & (5.1c), we deduce that

\[
\begin{align*}
(5.1c) & \quad \frac{1}{r_1 r_2} [(r_1 + r_2) tr(\iota(b)) - tr(T_n \iota(b))] \\
& \quad \stackrel{(5.1b)}{=} \frac{1}{r_1 r_2} [(r_1 + r_2) tr(\iota(b)) - tr(b)]
\end{align*}
\]

and so,

\[
(5.2) \quad tr(\iota(b)) = \frac{1 + r_1 r_2}{r_1 + r_2} tr(b)
\]

By (5.1a), we know that this trace is defined through its action on the basis of \( H_n/[\cdot, \cdot] \cong \mathcal{H}_n \). It thus suffices to consider \( tr(b_\lambda) \). Suppose \( b_\lambda = b_{(\lambda_1)} \sqcup \cdots \sqcup b_{(\lambda_k)} \). By application of type-II Markov moves, we get

\[
(5.3) \quad tr(b_\lambda) = tr\left( \bigcup_{i=1}^{k} \text{id}_1 \right)
\]

where \( H_1 = \{\text{id}_1\} \). By (5.2), we get

\[
(5.4) \quad tr(b_\lambda) = \left( \frac{1 + r_1 r_2}{r_1 + r_2} \right)^{k-1} tr(\text{id}_1)
\]

noting that \( k \) is the number of components\(^{17} \) in the closure of \( b_\lambda \). It is easy to check that (5.4) is an Ocneanu trace, whence the trace exists and is unique upto the factor \( tr(\text{id}_1) \in \mathbb{C} \).

**Remark 5.2.** The HOMFLY(-PT) polynomial \( P_L(r_1, r_2) \) of a link \( L \) is given by \( tr(b) \) for any braid \( b \) whose Markov closure is \( L \). For this reason, (4.5) is often referred to as the HOMFLY skein relation, and \( \mathcal{H}_n \) the HOMFLY skein algebra (of the torus). Clearly, the Ocneanu trace can also be defined as a trace on this skein algebra.

---

\(^{17}\)Or alternatively, the number of disjoint cycles in the permutation \( s_\lambda \).
5.2. A modified trace. We previously saw that the Kauffman bracket returns the statistical phase and amplitude (upto a normalisation factor) associated to a link spanned by Kauffman anyons. We will find an explicit analogue for Hecke anyons. The ribbon structure of a Hecke anyon can be captured through a slight modification of the Ocneanu trace:

\[(5.5a) \quad \text{tr}(ab) = \text{tr}(ba)\]

\[(5.5b) \quad \varkappa r_1^* \text{tr}(b) = \text{tr}(T_n \varepsilon(b))\]

\[(5.5c) \quad \varkappa r_1 \text{tr}(b) = \text{tr}(T_n^{-1} \varepsilon(b))\]

where \(\varepsilon : H_n \to H_{n+1}\), with \(a, b \in H_n(r_1, r_2)\) and \(\varkappa \in \{\pm 1\}\) for all \(n \geq 1\). Now, (5.5b) and (5.5c) imply

\[(5.6) \quad \text{tr}(\varepsilon(b)) = \varkappa \left(\frac{r_1^* + r_2^2 r_2^*}{r_1 + r_2}\right) \text{tr}(b)\]

For \(b_{\lambda} = b_{(\lambda_1)} \sqcup \cdots \sqcup b_{(\lambda_k)}\), repeated application of (5.5b) gives

\[(5.7) \quad \text{tr}(b_{\lambda}) = \prod_{i=1}^{k} (\varkappa r_1^*)^{\lambda_i - 1} \text{tr}\left(\bigcup_{i=1}^{k} \text{id}_1\right) = (\varkappa r_1^*)^{n-k} \text{tr}\left(\bigcup_{i=1}^{k} \text{id}_1\right)\]

By (5.6),

\[(5.8) \quad \text{tr}\left(\bigcup_{i=1}^{k} \text{id}_1\right) = \varkappa^{k-1} \left(\frac{r_1^* + r_2^2 r_2^*}{r_1 + r_2}\right)^{k-1} \text{tr}(\text{id}_1)\]

whence

\[(5.9) \quad \text{tr}(b_{\lambda}) = \varkappa^{n-1} (r_1^*)^{n-k} \left(\frac{r_1^* + r_2^2 r_2^*}{r_1 + r_2}\right)^{k-1} \text{tr}(\text{id}_1)\]

**Theorem 5.3.** The trace \(\text{tr}\) exists and is unique.

*Proof.* Existence is given by (5.9), which clearly satisfies (5.5a) since it is defined through its action on the basis of \(\mathcal{H}_n \cong H_n/\{\cdot, \cdot\}\). Next, note that \(T_n \varepsilon(b)\) is a basis element \(b_{\mathcal{H}}\) of \(\mathcal{H}_{n+1}\), and so we can apply (5.9). The number of components in the closure of \(b_{\mathcal{H}}\) is the same as for \(b_{\lambda}\), and so \(\text{tr}(b_{\mathcal{H}}) = \varkappa r_1^* \text{tr}(b_{\lambda})\) (i.e. (5.9) satisfies (5.5b)). Finally,

\[
\text{tr}(T_n^{-1} \varepsilon(b_{\lambda})) = \frac{1}{r_1 r_2^*} [(r_1 + r_2) \text{tr}(\varepsilon(b_{\lambda})) - \text{tr}(T_n \varepsilon(b_{\lambda}))]
\]

\[
\overset{(5.5b),(5.6)}{=} \frac{\varkappa (r_1^* + r_2^2 r_2^*) - r_1^*}{r_1 r_2^*} = \varkappa r_1
\]

and so (5.9) satisfies (5.5c). Thus, \(\text{tr}\) is unique (upto the value of \(\text{tr}(\text{id}_1) \in \mathbb{C}\)). \(\square\)

Like the Kauffman bracket, \(\text{tr}\) is clearly an invariant of framed links\(^{18}\). In the context of a Hecke anyon \(q\), we will write \(\text{tr} = \text{tr}_q\). Here, \(\varkappa = \varkappa_q\). Furthermore, \(\text{tr}_q\) coincides with the quantum trace\(^{19}\), whence \(\text{tr}_q(\text{id}_1) = d_q\). As before, we associate

\[(5.10) \quad r_1 = R_{qq}^1 = \varkappa_q \theta_q^* , \quad r_2 = R_{qq}^2\]

\(^{18}\)That is, \(\text{tr}(b)\) is an invariant of the framed link arising from the Markov closure of \(b\) (as opposed to the trace constructed for the HOMFLY polynomial in Section 5.1, which is a full link-invariant).

\(^{19}\)As defined for spherical fusion categories. This is illustrated by the Markov closure.
for a Hecke non-Abelion $q \otimes q = 1 \oplus x$. And for a Hecke Abelian, this becomes

$$ r = r_1 = r_2 = R^1_{qq} = \varkappa_q^* q $$

Note that $n - k$ is the number of strands in $b_\lambda$ minus the number of components in the Markov closure of $b_\lambda$. We write

$$ \text{tr}_q(b_\lambda) = \varkappa_q^{n - 1} d_q(r_1^*)^{n - k} \left( \frac{r_1^* + r_1^* r_2}{r_1 + r_2} \right)^{k - 1} $$

For a Hecke Abelian, this simplifies to

$$ \text{tr}_q(b_\lambda) = \varkappa_q^{n - 1} (r_1^*)^{n - k} \left( \frac{r_1^* + r_3}{2r} \right)^{k - 1} $$

So for a Hecke anyon $q$, we have

$$ \text{tr}_q(b) = \zeta_{n,q} \langle 1 | \rho_V(b') | 1 \rangle $$

where $b' \in B_{2n}$ is the braid whose plat closure yields the same framed link as the Markov closure of $b \in B_n$. $\rho_V$ is the unitary representation of $H_{2n}$ induced by the fusion space $V = V_{q^\otimes 2n}$, and $\zeta_{n,q} = d_q^{2(n-1)}$ by the same reasoning as for (3.4).

**Remark 5.4.** Note that the quantity $\text{tr}_q(b)$ is physically well-defined, since it is a function of $r_1$ and $r_2$ which are gauge-invariant quantities.

**Remark 5.5. (Caveat)** For Hecke anyons with $\varkappa_q = -1$, we have to account for deformations of the worldlines that give rise to consecutive maxima and minima (such that the anyon moves forwards, backwards and then forwards along the time-axis), as these induce a $-1$ phase evolution: this geometric dependence is not captured by $\text{tr}_q$. However, the physical occurrence of such deformations would require the creation of a virtual pair of anyons, the likelihood of which is exponentially suppressed for a sufficiently gapped Hamiltonian.

**Corollary 5.6.** For a Hecke anyon $q$, the quantum dimension is given by

$$ d_q = \varkappa_q \left( \frac{r_1^* + r_1^* r_2}{r_1 + r_2} \right) $$

whence for a Hecke non-Abelian, we have

$$ R^x_{qq} = \frac{d_q - \varkappa_q d_q\vartheta_q^*}{d_q - \varkappa_q \vartheta_q^{-2}} $$

**Proof.** Since $\text{tr}_q$ coincides with the quantum trace, $\text{tr}_q(\iota(b)) = d_q \text{tr}_q(b)$. Eq. (5.15) follows by (5.6), and (5.16) simply by rearranging. \qed

So as expected, (5.12) may be written

$$ \text{tr}_q(b_\lambda) = d_q^n (\varkappa_q r_1^*)^{n-k} = d_q \vartheta_q^{n-k} $$

It is easy to check that (3.3) and (4.21) are recovered from (5.15) and (5.16) respectively for $q$ Kauffman. Equation (5.15) may also be written

$$ d_q = \varkappa_q \left( \frac{\cos(2u) + \cos(u + v)}{1 + \cos(u - v)} \right) $$

where $(u, v) = (\arg(r_1), \arg(r_2))$ and $d_q \in \{1\} \cup [\sqrt{2}, \infty)$.

**Corollary 5.7.** The topological spin of a self-dual Abelian is a $4^{th}$ root of unity.
Proof. For a self-dual Abelian, (5.15) becomes

\[(5.19)\]
\[
d_q = \varpi_q \left( \frac{r^* + r^3}{2r} \right) = 1
\]
whose solution gives \( r^4 - 2\varpi_q r^2 + 1 = 0 \iff r^2 = \varpi_q = \pm 1. \)

**Remark 5.8.** 20 This tells us that all self-dual Abelianons \( q \) have \( R_{qq} = \vartheta_q = \pm 1, \pm i \).

All anyons with \( \vartheta_q = \pm 1 \) are Kauffman Abelianons (and have \( \varpi_q = +1 \)). All other self-dual Abelianons are HNK Abelianons with \( \vartheta_q = \pm i \) (and have \( \varpi_q = -1 \)).

We also have the obvious analogue of Corollary 4.11 for Hecke anyons:

**Corollary 5.9.** For a Hecke non-Abelianon \( q \) and an appropriate choice of gauge,

\[(5.20)\]
\[
F_{qqq}^q = \varpi_q \left( \begin{pmatrix}
\frac{1}{d_q} & \frac{\sqrt{d_q^2 - 1}}{d_q} \\
\frac{\sqrt{d_q^2 - 1}}{d_q} & -\frac{1}{d_q}
\end{pmatrix}
\right)
\]

Proof. We adopt the same notation and setup as in Corollary 4.11, but here we have \( a = F_{11} = \varpi_q d_q^{-1} \), and write \( R_{qq} = \text{diag}(r_1, r_2) \). Given the fusion space \( V = V_{q^\otimes n} \), we know from (4.6) that we have the unitary representation \( \rho_V \) of \( B_n \),

\[(5.21)\]
\[
\rho_V(\sigma_i) = \bigoplus_j \begin{pmatrix} a_j & c_j \\ c_j & b_j \end{pmatrix} \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} \begin{pmatrix} a_j & c_j \\ c_j & b_j \end{pmatrix}
\]
\[
= \begin{pmatrix} a^2 r_1 + c^2 r_2 & c(a r_1 + b r_2) \\ c(a r_1 + b r_2) & c^2 r_1 + b^2 r_2 \end{pmatrix} \bigoplus \bigoplus_{j \neq (q,q)} \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}
\]

But we may also write this representation

\[(5.22)\]
\[
\rho_V : B_n \to H_n(r_1, r_2) \\
\sigma_i \mapsto T_i
\]

which by (4.4) gives

\[(5.23)\]
\[
\rho_V(\sigma_i) = (r_1 + r_2) \rho_V(1) - r_1 r_2 \rho_V^{-1}(\sigma_i)
\]

whence we may write (using (4.6)),

\[(5.24)\]
\[
\rho_V(\sigma_i) = \bigoplus_j \left[ (r_1 + r_2) I_2 - r_i r_j \begin{pmatrix} a_j & c_j \\ c_j & b_j \end{pmatrix} \begin{pmatrix} r_1^* & 0 \\ 0 & r_2^* \end{pmatrix} \begin{pmatrix} a_j & c_j \\ c_j & b_j \end{pmatrix} \right]
\]
\[
= \begin{pmatrix} r_1 + r_2 - a^2 r_2 - c^2 r_1 & -c(a r_2 + b r_1) \\ -c(a r_2 + b r_1) & r_1 + r_2 - b^2 r_1 - c^2 r_2 \end{pmatrix} \bigoplus \bigoplus_{j \neq (q,q)} \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}
\]

Equating the top-left element of the block \( j = (q,q) \) in (5.21) and (5.24), we get

\[(5.25)\]
\[
c^2(r_1 + r_2) = (1 - a^2)(r_1 + r_2)
\]

noting \( r_1 \neq -r_2 \) since (5.18) diverges on the lines \( u = v + (2m + 1)\pi \ \forall m \in \mathbb{Z} \). The result follows by solving for \( c \) (choosing the appropriate root corresponds to a choice of gauge), \( b \) following as before. Corollary 4.11 is recovered as a special case.  \( \Box \)

\( ^{20} \)Note that this agrees with (5.18) for \( u = v, d_q = 1. \)
Finally, if we restrict $q$ to be Kauffman, we have $\kappa_q = +1$, and $(r_1, r_2) = (-A^{-3}, A)$. In this instance, (5.12) is
\[
\text{tr}_q(b_\lambda) = (-A^2 - A^{-2})(-A^3)^{n-k} \left( \frac{-A^3 + A^{-6} A}{A - A^{-3}} \right)^{k-1}
= (-A^3)^{n-k}(-A^2 - A^{-2})^k = \varphi_q^{n-k}d_q^k
\]
as expected. Indeed, $\mathcal{H}_n(-A^{-3}, A)$ is the Kauffman skein algebra and we have
\[
\text{tr}_q(b) = \langle b \rangle |_q, \quad b \in \mathcal{H}_n(-A^{-3}, A)
\]
Of course, if we let $\text{tr}'_q(b_\lambda) = (\kappa_q r_1)^{n-k}\text{tr}_q(b_\lambda)$, we recover the Jones polynomial:
\[
\text{tr}'_q(b) = X(b) |_q, \quad b \in \mathcal{H}_n(-A^{-3}, A)
\]

5.3. Examples. Let $\{L\}|_q$ denote $\text{tr}_q(b)$ for a braid $b$ whose Markov closure is $L$.

Example 5.10. (S-Matrix)
Recall the Hopf link $H^+$. For a Hecke anyon $q$, we have
\[
\{H^+\}|_q = \text{tr}_q \left( \begin{array}{c}
\begin{array}{cc}
& *
\
*
& 2
\end{array}
\end{array} \right) = (r_1 + r_2)\text{tr}_q \left( \begin{array}{c}
\begin{array}{cc}
& *
\
*
& 2
\end{array}
\end{array} \right) - r_1 r_2 \text{tr}_q \left( \begin{array}{c}
\begin{array}{c}

\end{array}
\end{array} \right)
= (r_1 + r_2)(r_1^* \varphi_q d_q) - r_1 r_2 \left[ \varphi_q d_q \left( \frac{r_1^* + r_1^* r_2}{r_1 + r_2} \right) \right]
\]
where in the second equality, we expand in terms of the basis $\{b_\lambda\}$, and in the third we use (5.12). Thus,
\[
\{H^+\}|_q = \varphi_q d_q \left[ r_1^*(r_1 + r_2) - r_1 r_2 \left( \frac{r_1^* + r_1^* r_2}{r_1 + r_2} \right) \right] = \tilde{S}_{qq}
\]
Note that there is no renormalisation factor $\zeta$ (Remark 3.9), and that (5.29) is consistent with (4.7) and (5.15). It is also easy to check that (3.5) is recovered for $q$ Kauffman. By (5.29) and Corollary 5.7, we see that
\[
\tilde{S}_{qq} = \frac{1}{2} \varphi_q (3 - r^4) = \varphi_q (= \pm 1), \quad \text{for a self-dual Abelianon } q
\]
We have determined $\tilde{S}_{qq}$ for all self-dual Abelianons, and have checked some examples for Kauffman non-Abelianons in Example 3.8. Note that $\sigma$ in $\text{Ising}_9$ and $\text{Ising}_{15}$ are HNK non-Abelianons (with $\varphi_{\sigma} = +1$). Anyons $\sigma$ in $\text{Ising}_3$, $\text{Ising}_5$, $\text{Ising}_{11}$ and $\text{Ising}_{13}$ are HNK non-Abelianons (with $\varphi_{\sigma} = -1$). In all cases, plugging in the values for $r_1$ and $r_2$ gives $\{H^+\}|_\sigma = 0$ as expected.

Example 5.11. (Ising Trefoil II)
Consider the right-handed trefoil knot $T_r$ spanned by 4 Ising anyons as follows:
Figure 13. The depicted knot $T_r$ is obtained from the plat closure of $\sigma_3^2 \in B_4$, and from the Markov closure of $\sigma_1^3 \in B_2$.

The setup is the same as in Examples 3.10 & 3.11. We will consider HNK anyons $\sigma$ for $\text{Ising}_9$, $\text{Ising}_{15}$ and $\text{Ising}_3$, and will denote the induced braid representation for the subsystem $V = V^{\sigma}_{\sigma\sigma}$ by $\rho_9$, $\rho_{15}$ and $\rho_3$ respectively. For each of these, we will denote the matrices $F^{\sigma}_{\sigma\sigma}$ and $R_{\sigma\sigma}$ by $F_k$ and $R_k$, where $k = 3, 9, 15$. Note that,

(5.31) $\rho_k(\sigma_2^2) = F_k^{-1} R_k^3 F_k$

We have,

(5.32) $R_9 = e^{i \frac{7\pi}{8}} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$, \quad $F_9 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ ($= F_9^{-1}$)

whence

(5.33) $\rho_9(\sigma_2^2) = \frac{1}{\sqrt{2}} e^{i \frac{3\pi}{8}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \implies \langle 1 | \rho_9(\sigma_2^2) | 1 \rangle = \frac{1}{\sqrt{2}} e^{i \frac{3\pi}{8}}$

We have $F_{15} = F_9$ and $R_{15} = -R_9^\dagger$, which tells us that

(5.34) $\rho_{15}(\sigma_2^2) = -(\rho_9(\sigma_2^3))^\dagger \implies \langle 1 | \rho_{15}(\sigma_2^3) | 1 \rangle = \frac{1}{\sqrt{2}} e^{i \frac{3\pi}{8}}$

Lastly, we have $F_3 = -F_{15}$ and $R_3 = iR_{15}$, which gives

(5.35) $\rho_3(\sigma_2^3) = (-1)^2 \cdot i^3 \cdot \rho_{15}(\sigma_2^3) \implies \langle 1 | \rho_3(\sigma_2^3) | 1 \rangle = \frac{1}{\sqrt{2}} e^{i \frac{\pi}{8}}$

Next, we calculate $\{T_r\}_\sigma$ for each of the theories:

$$
\{T_r\}_\sigma = \text{tr}_\sigma \begin{pmatrix} 
\begin{array}{c}
\vdots \\
\vdots
\end{array} 
\end{pmatrix} = (r_1 + r_2) \text{tr}_\sigma \begin{pmatrix} 
\begin{array}{c}
\vdots \\
\vdots
\end{array} 
\end{pmatrix} - r_1 r_2 \text{tr}_\sigma \begin{pmatrix} 
\begin{array}{c}
\vdots \\
\vdots
\end{array} 
\end{pmatrix} 
$$

where we used $\{H^+\}_\sigma = 0$. So for $\text{Ising}_9$, we have

(5.36) $\{T_r\}_\sigma = -\sqrt{2} (ie^{i \frac{7\pi}{8}}) = \sqrt{2} e^{i \frac{11\pi}{8}}$

while for $\text{Ising}_{15}$, we have

(5.37) $\{T_r\}_\sigma = -\sqrt{2} (-ie^{i \frac{\pi}{8}}) = \sqrt{2} e^{i \frac{9\pi}{8}}$
and for Ising$_3$, we have

\[
(T_{\sigma})|_{\sigma} = -(-\sqrt{2})(-ie^{i\frac{5\pi}{8}}) = \sqrt{2}e^{i\frac{\pi}{8}}
\]

Indeed, (5.33)-(5.35) agree with (5.36)-(5.38) respectively: as expected per (5.14), we have the amplitude discrepancy of $\zeta_{2,\sigma} = 2$.

6. Outlook

An interesting programme would be to extend this work to self-dual anyons of rank $k \geq 3$ (examples of such anyons for $k = 3$ include $\beta$ in $(A_1, 5)_{\frac{1}{2}}$ and $\omega$ in $(A_1, 7)_{\frac{1}{2}}$). One possible approach might be to do so via the study of Markov traces on towers of quotients of $\mathbb{C}[B_n]$ (Remark 4.2). However, the structure of these algebras fast becomes complex, and constructing a trace becomes accordingly difficult: even in the case of the cubic Hecke algebra ($k = 3$), $\dim_{\mathbb{C}} H(Q, n)$ is known to be infinite for $n \geq 6$ [8, 9].

**Question 6.1.** Do there exist any HNK non-Abelions $q$ with $\kappa_q = +1$ on the Kauffman arc? If there exist no such anyons, then Proposition 3.4(b) is necessary and sufficient for a Hecke anyon $q$ with $\kappa_q = +1$ to be Kauffman. If this were the case, a further question would be whether all HNK non-Abelions with $\kappa_q = +1$ are obtained as the conjugate theory of a Kauffman non-Abelion. The veracity of the latter would be refuted by the existence of a Hecke non-Abelion $q$ with $\vartheta_q \in (0, \frac{\pi}{8}) \cup (\frac{7\pi}{8}, \pi) \cup (\pi, \frac{9\pi}{8}) \cup (\frac{15\pi}{8}, 2\pi)$. 
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Selmer Center, Department of Informatics, University of Bergen, Norway
E-mail address: sachin.valera@uib.no