Propagation processes of correlations of hard spheres

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Abstract. The paper develops an approach to the description of the evolution of correlations for many hard spheres based on a hierarchy of evolution equations for the cumulants of the probability distribution function governed by the Liouville equation. It is established that the constructed dynamics of correlations underlies the description of the evolution of the states of many hard spheres described by the BBGKY hierarchy for reduced distribution functions or the hierarchy of nonlinear evolution equations for reduced correlation functions. As an application of the developed approach to describing the evolution of the state of many hard spheres within the framework of dynamics of correlations, the challenges of the derivation of kinetic equations are discussed.

Key words: Liouville hierarchy, BBGKY hierarchy, kinetic equation, cumulant, correlation function.

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1 Introduction

Recently, mainly in connection with the problem of the rigorous derivation of kinetic equations from the underlying particle dynamics \([1]-[5]\), a number of papers \([6]-[16]\) have appeared discussing possible approaches to describing the evolution of the state of many-particle systems, in particular, many hard spheres.

As is known \([1]-[3]\), the evolution of the state of finitely many hard spheres is traditionally described by a probability distribution function governed by the Liouville equation. This article is developed an alternative approach to the description of the evolution of the state, which consists of the use of functions defined as cumulants of the mentioned probability distribution function. The cumulants of the probability distribution function are interpreted as correlations of the states of hard spheres, and further, the term correlation functions are used for them. The evolution of the correlation functions is governed by the Liouville hierarchy for hard spheres constructed in the article.

One more approach that allows describing the evolution of states of the systems both from a finite and from an infinite number of particles \([1]-[3]\) is to describe the state by means of a sequence of so-called reduced distribution functions (marginals \([5]-[8]\)) governed by the BBGKY
(Bogolyubov–Born–Green–Kirkwood–Yvon) hierarchy [1]-[5]. Note that this method is traditionally used in the rigorous derivation of the Boltzmann kinetic equation from the dynamics of hard spheres [17]-[20]. An alternative of the approach to such a description of a state is based on a sequence of functions determined by the cluster expansions of reduced distribution functions that are, by their cumulants. These functions are interpreted as the reduced correlation functions and they are governed by the corresponding hierarchy of the nonlinear evolution equations [21]-[40].

In this article, we develop an approach to the description of the evolution of a state by means of reduced distribution functions and reduced correlation functions, which is based on the dynamics of correlations in a system of hard spheres. It should be emphasized that the generating operators of solution expansions of the Cauchy problem of the corresponding hierarchies of evolution equations are induced by the generating operators of an expansion of the solution of the Cauchy problem of the Liouville hierarchy for correlation functions which are represented by means of the corresponding-order cumulants of the groups of operators of the Liouville equations [15], [22].

In the last section of the article, as an application of a developed approach to describing the evolution of the state of many hard spheres within the framework of the dynamics of correlations, we consider two issues related to the problem of the rigorous derivation of kinetic equations. One of them is a method for describing the evolution of the state of many hard spheres within the framework of the evolution of the state of a typical particle, described by a generalization of the Enskog equation [14], or, in other words, the fundamentals of describing the evolution of correlations by kinetic equations are discussed. We remark that the conventional approach to the mentioned problem is based on the construction of the Boltzmann–Grad asymptotic behavior [17]-[20] of a solution of the BBGKY hierarchy for reduced distribution functions represented as the series of the perturbation theory for initial data specified by a one-particle distribution function in the case of absence of correlations (so-called initial chaos) [1]-[5]. In this connection, we also consider a sketch of constructing the scaling asymptotics of the reduced correlation functions in the low-density limit.

2 Dynamics of correlations of a hard-sphere system

2.1 Preliminaries: dynamics of finitely many hard spheres

At the initial instant in the space $\mathbb{R}^3$ the state of a hard-sphere system of a non-fixed, i.e. arbitrary but finite average number of identical particles, is described by the sequence $D(0) = (1, D_1(0), \ldots, D_n(0), \ldots)$ of the probability distribution functions $D_n(0)$, $n \geq 1$ defined on the phase space $\mathbb{R}^{3n} \times (\mathbb{R}^3 \setminus W_n)$ of $n$ hard spheres. Each hard sphere of a diameter $\sigma > 0$ is characterized by the phase space coordinates $(q_i, p_i) \equiv x_i \in \mathbb{R}^3 \times \mathbb{R}^3$, $i \geq 1$ and for configurations in the space the following inequalities are satisfied: $|q_i - q_j| \geq \sigma$, $i \neq j \geq 1$, i.e. the set $W_n \equiv \{(q_1, \ldots, q_n) \in \mathbb{R}^{3n} | |q_i - q_j| < \sigma \text{ for at least one pair } (i, j) : i \neq j \in (1, \ldots, n)\}$ is the forbidden configuration set. The nonnegative functions $D_n(0) = D_n(0, x_1, \ldots, x_n)$, $n \geq 1$, that are symmetric with respect to permutations of the arguments $x_1, \ldots, x_n$, equal to zero on the set of forbidden configurations $W_n$ will be assumed to belong to the space of integrable functions $L^1_h \equiv L^1(\mathbb{R}^{3n} \times \mathbb{R}^{3n})$ equipped with the norm: $\|f_n\|_{L^1_h} = \int_{\mathbb{R}^{3n} \times \mathbb{R}^{3n}} dx_1 \ldots dx_n |f_n(x_1, \ldots, x_n)|$.

As is known [1]-[4], the evolution of all possible states of a hard-sphere system is described by
the sequence \( D(t) = (1, D_1(t), \ldots, D_n(t), \ldots) \) of the following probability distribution functions:

\[
D_n(t, x_1, \ldots, x_n) = \begin{cases} 
  D_n(0, X_1(-t, x_1, \ldots, x_n), \ldots, X_n(-t, x_1, \ldots, x_n)), & \text{if } (x_1, \ldots, x_n) \in (\mathbb{R}^{3n} \times (\mathbb{R}^{3n} \setminus \mathbb{W}_n)), \\
  0, & \text{if } (q_1, \ldots, q_n) \in \mathbb{W}_n,
\end{cases}
\]

(1)

where for \( t \in \mathbb{R} \) the function \( X_i(-t) \) is a phase space trajectory of the \( i \)th hard sphere constructed in \([1]\), which is defined almost everywhere on the phase space \( \mathbb{R}^{3n} \times (\mathbb{R}^{3n} \setminus \mathbb{W}_n) \), namely, beyond the set \( \mathbb{M}_n^0 \) of zero Lebesgue measure consisting of the phase space points specified by such initial data that multiple collisions can occur during the evolution, i.e. collisions of more than two particles, more than one two-particle collision at the same instant and infinite number of collisions within a finite time interval \([1, 20]\).

On the space \( L^1_n \equiv L^1(\mathbb{R}^{3n} \times \mathbb{R}^{3n}) \) one-parameter mapping \([1]\) generates an isometric strong continuous group of operators

\[
S_n(-t, 1, \ldots, n)D_n(0, x_1, \ldots, x_n) = D_n(t, x_1, \ldots, x_n).
\]

(2)

On the subspace of continuously differentiable functions with compact supports \( L^1_{n,0} \subset L^1_n \) for the group of operators \([2]\) the Duhamel equation holds

\[
S_n(-t, 1, \ldots, n) = \prod_{i=1}^n S_1(-t, i) + \int_0^t d\tau \prod_{i=1}^n S_1(\tau - t, i) \sum_{j_1, j_2=1}^n \mathcal{L}^*_{\text{int}}(j_1, j_2)S_n(-\tau, 1, \ldots, n) = \\
\prod_{i=1}^n S_1(-t, i) + \int_0^t d\tau S_n(\tau - t, 1, \ldots, n) \sum_{j_1, j_2=1}^n \mathcal{L}^*_{\text{int}}(j_1, j_2) \prod_{i=1}^n S_1(-\tau, i),
\]

where for \( t > 0 \) the operator \( \mathcal{L}^*_{\text{int}}(j_1, j_2) \) is defined by the formula

\[
\mathcal{L}^*_{\text{int}}(j_1, j_2)f_n = \sigma^2 \int_{\mathbb{S}^2_+} d\eta \langle \eta, (p_{j_1} - p_{j_2}) \rangle f_n(x_1, \ldots, p^*_{j_1}, q_{j_1}, \ldots, \rangle
\]

(3)

In definition \([3]\) the symbol \( \langle \cdot, \cdot \rangle \) means a scalar product, the symbol \( \delta \) denotes the Dirac measure, \( \mathbb{S}^2_+ = \{ \eta \in \mathbb{R}^3 \mid |\eta| = 1, \langle \eta, (p_1 - p_2) \rangle > 0 \} \) and the pre-collision momenta \( p^*_{j}, p^*_{j_2} \) are determined by the equalities:

\[
p^*_{j} = p_i - \eta \langle \eta, (p_i - p_j) \rangle, \\
p^*_{j_2} = p_j + \eta \langle \eta, (p_i - p_j) \rangle.
\]

In the case \( t < 0 \), the operator \( \mathcal{L}_{\text{int}}(j_1, j_2) \) is defined by the corresponding expression \([1]\).

Hence the infinitesimal generator \( \mathcal{L}^*_n \) of the group of operators \( S_n(t) \) has the following structure

\[
\mathcal{L}^*_n(1, \ldots, n)f_n = \sum_{j=1}^n \mathcal{L}^*(j)f_n + \sum_{j_1, j_2=1}^n \mathcal{L}^*_{\text{int}}(j_1, j_2)f_n,
\]

(4)
where the Liouville operator of free motion $L^*(j) = -(p_j, \frac{\partial}{\partial q_j})$ defined on the subspace $L_{n,0}^1 \subset L_n^1$ is denoted by the symbol $L^*(j)$.

If $D_n(0) \in L_n^1$, $n \geq 1$, the sequence of distribution functions defined by formula (1) is a unique solution of the Cauchy problem of the weak formulation of a sequence of evolution equations for the state known as the Liouville equation:

$$\frac{\partial}{\partial t} D_n(t) = L^*_n(1,\ldots,n)D_n(t), \quad n \geq 1. \quad (5)$$

$$D_n(t)|_{t=0} = D_n(0), \quad n \geq 1. \quad (6)$$

Thus, the traditional approach to the description of the evolution of all possible states of finitely many hard spheres is specified by the Cauchy problem for the sequence of Liouville equations.

### 2.2 Correlation functions

An alternative approach to the description of states of a hard-sphere system of finitely many particles is given by means of functions determined by the cluster expansions of the probability distribution functions. They are called here as correlation functions (the cumulants of probability distribution functions). We introduce the sequence of correlation functions $g(t) = (g_1(t,x_1),\ldots,g_s(t,x_1,\ldots,x_s),\ldots)$ by means of cluster expansions of the probability distribution functions $D(t) = (D_1(t,x_1),\ldots,D_n(t,x_1,\ldots,x_n),\ldots)$, defined on the set of allowed configurations $\mathbb{R}^{3n} \setminus \mathbb{W}_n$ as follows:

$$D_n(t,x_1,\ldots,x_n) = g_n(t,x_1,\ldots,x_n) + \sum_{P : (x_1,\ldots,x_n) = \bigcup_i X_i, \quad X_i \subset \mathbb{P}} \prod_{X_i \subset \mathbb{P}} g_{|X_i|}(t,X_i), \quad n \geq 1, \quad (7)$$

where $\sum_{P : (x_1,\ldots,x_n) = \bigcup_i X_i, |P| > 1}$ is the sum over all possible partitions $P$ of the set of the arguments $(x_1,\ldots,x_n)$ into $|P| > 1$ nonempty mutually disjoint subsets $X_i \subset (x_1,\ldots,x_n)$.

On the set $\mathbb{R}^{3n} \setminus \mathbb{W}_n$ solutions of recursion relations (7) are given by the following expansions:

$$g_s(t,x_1,\ldots,x_s) = D_s(t,x_1,\ldots,x_s) + \sum_{P : (x_1,\ldots,x_s) = \bigcup_i X_i, \quad |P| > 1} (-1)^{|P| - 1} (|P| - 1)! \prod_{X_i \subset \mathbb{P}} D_{|X_i|}(t,X_i), \quad s \geq 1. \quad (8)$$

The structure of expansions (8) is such that the correlation functions can be treated as cumulants (semi-invariants) of the probability distribution functions (1).

Thus, correlation functions (8) are to enable to describe of the evolution of states of finitely many hard spheres by the equivalent method in comparison with probability distribution functions (1), namely within the framework of dynamics of correlations.

If the initial state is specified by the sequence $g(0) = (g_1^0(x_1),\ldots,g_s^0(x_1,\ldots,x_n),\ldots)$, of correlation functions $g_n^0 \in L_n^1$, $n \geq 1$, then the evolution of all possible states, i.e. the sequence $g(t) = (g_1(t,x_1),\ldots,g_s(t,x_1,\ldots,x_s),\ldots)$ of the correlation functions $g_s(t)$, $s \geq 1$, is determined by the following expansions:

$$g_s(t,x_1,\ldots,x_s) = \sum_{P : (x_1,\ldots,x_s) = \bigcup_j X_j} \mathfrak{A}_{|P|}(t,\{\hat{X}_1\},\ldots,\{\hat{X}_{|P|}\}) \prod_{X_j \subset \mathbb{P}} g_{|X_j|}^0(X_j), \quad s \geq 1, \quad (9)$$
where the symbol $\sum_{\mathcal{P}:(x_1,\ldots,x_s)=\bigcup_j X_j}$ is the sum over all possible partitions $\mathcal{P}$ of the set $(x_1,\ldots,x_s)$ into $|\mathcal{P}|$ nonempty mutually disjoint subsets $X_j$, the symbol $\hat{X}$ means the set of indexes of the set $X$ of phase space coordinates and the set $\{(\hat{X}_1),\ldots,\hat{X}_{|\mathcal{P}|}\}$ consists of elements that are subsets $\hat{X}_j \subset (1,\ldots,s)$, i.e. $|\{(\hat{X}_1),\ldots,\hat{X}_{|\mathcal{P}|}\}|=|\mathcal{P}|$. The generating operator $\mathfrak{A}_{|\mathcal{P}|}(t)$ in expansions (9) is the $|\mathcal{P}|$th-order cumulant of the groups of operators (2) which is defined by the expansion

$$\mathfrak{A}_{|\mathcal{P}|}(t, \{(\hat{X}_1),\ldots,\hat{X}_{|\mathcal{P}|}\}) \doteq \sum_{\mathcal{P}':((\hat{X}_1),\ldots,\hat{X}_{|\mathcal{P}|})=\bigcup_k Z_k} (-1)^{|\mathcal{P}'|-1}(|\mathcal{P}'|-1)! \prod_{Z_k \subset \mathcal{P}'} S_{\theta(Z_k)}(-t, \theta(Z_k)),$$

where $\theta$ is the declusterization mapping: $\theta((\hat{X}_1),\ldots,\hat{X}_{|\mathcal{P}|}) \doteq (1,\ldots,s)$. The simplest examples of correlation functions (9) are given as follows:

$$g_1(t,x_1) = \mathfrak{A}_1(t,1)g_0^1(x_1),$$
$$g_2(t,x_1,x_2) = \mathfrak{A}_1(t,\{1,2\})g_0^2(x_1,x_2) + \mathfrak{A}_2(t,1,2)g_0^1(x_1)g_0^1(x_2).$$

The structure of expansions (9) is established as a result of the permutation of the terms of cumulant expansions (8) for correlation functions and cluster expansions (7) for initial probability distribution functions (2). Thus, the cumulant origin of correlation functions induces the cumulant structure of their dynamics (9).

In particular, in the absence of correlations between hard spheres at the initial moment (initial state satisfying the chaos condition [1, 3]) the sequence of the initial correlation functions on allowed configurations has the form $g^{(c)}(0) = (0,g_0^1(x_1),0,\ldots,0,\ldots)$. In terms of a sequence of the probability distribution functions the chaos condition means that we start from an initial datum of the form $D^{(c)}(0) = (1,D_0^0(x_1),D_0^0(x_1)D_0^0(x_2)x_{R^2\setminus W_2},\ldots,\prod_{i=1}^n D_0^0(x_i)x_{R^{2s}\setminus W_s}^n,\ldots)$, where the function $x_{R^{2s}\setminus W_s}$ is the Heaviside step function of allowed configurations of $n$ hard spheres. In this case for $(x_1,\ldots,x_s) \in R^{3s} \times (R^{3s} \setminus W_s)$ expansions (9) are represented as follows:

$$g_s(t,x_1,\ldots,x_s) = \mathfrak{A}_s(t,1,\ldots,s) \prod_{i=1}^s g_0^1(x_i)x_{R^{2s}\setminus W_s}, \quad s \geq 1,$$

where the generating operator $\mathfrak{A}_s(t)$ of this expansion is the $s$th-order cumulant of groups of operators (2) defined by the expansion

$$\mathfrak{A}_s(t,1,\ldots,s) = \sum_{\mathcal{P}:(1,\ldots,s)=\bigcup_i X_i} (-1)^{|\mathcal{P}|-1}(|\mathcal{P}|-1)! \prod_{X_i \subset \mathcal{P}} S_{\theta(X_i)}(-t,X_i),$$

with notations accepted in formula (9). From the structure of series (11) it is clear that in case of the absence of correlations at the initial instant the correlations generated by the dynamics of a system of hard spheres are completely determined by cumulants (12) of the groups of operators of the Liouville equation (5).

We note that in the case of initial data $g^{(c)}(0)$ expansions (11) can be rewritten in another representation that explains their physical meaning. Indeed, for $n = 1$ we have

$$g_1(t,x_1) = \mathfrak{A}_1(t,1)g_0^1(x_1) = g_1^0(p_1,q_1-p_1t).$$
Then, according to formula (11) and the definition of the first-order cumulant $\mathfrak{A}_1(t) = S_1(-t)$, and its inverse group of operators $S_1^{-1}(-t) = S_1(t)$, we express the correlation functions $g_s(t)$, $s \geq 2$, in terms of the one-particle correlation function $g_1(t)$. Therefore, for $s \geq 2$ expansions (13) are represented in the following form:

$$
g_s(t, x_1, \ldots, x_s) = \hat{\mathfrak{A}}_s(t, 1, \ldots, s) \prod_{i=1}^{s} g_1(t, x_i), \quad s \geq 2,
$$

where $\hat{\mathfrak{A}}_s(t, 1, \ldots, s)$ is the $s$-order cumulant (12) of the scattering operators

$$
\hat{S}_n(t, 1, \ldots, n) \doteq S_n(-t, 1, \ldots, n) \mathcal{X}_{\mathbb{R}^{\mathfrak{m}} \setminus \mathfrak{w}_n} \prod_{i=1}^{n} S_1(t, i), \quad n \geq 1.
$$

On subspace $L_{0,n}^1 \subset L_1^1$ the generator of the scattering operator $\hat{S}_n(t, 1, \ldots, n)$ is determined by the operator: $\frac{d}{dt} \hat{S}_n(t, 1, \ldots, n) \big|_{t=0} = \sum_{j_1 < j_2 = 1}^n \mathcal{L}_{\text{int}}^* (j_1, j_2)$, where for $t \geq 0$ the operator $\mathcal{L}_{\text{int}}^* (j_1, j_2)$ is defined according to formula (3).

If $g_0^1 \in L_1^1$, $n \geq 1$, one-parameter mapping (9) generates strong continuous group of nonlinear operators

$$
\mathcal{G}(t; 1, \ldots, s \mid g(0)) \doteq g_s(t, x_1, \ldots, x_s), \tag{13}
$$

and it is bounded, and the following estimate holds: $\| \mathcal{G}(t; 1, \ldots, s \mid g) \|_{L_1^1} \leq s! c^n$, where $c \equiv \max(1, \max_{P: (1, \ldots, s) = \bigcup_i X_i} \| g|_{X_i} \|_{L_1^{\mathfrak{m}}_{X_i}})$. For $g_n \in L_{n,0}^1$, $n \geq 1$, the infinitesimal generator of this group of nonlinear operators has the following structure

$$
\mathcal{L}(1, \ldots, s \mid g) = \mathcal{L}_{\text{int}}^* (1, \ldots, s) g_s(x_1, \ldots, x_s) + \sum_{P: (x_1, \ldots, x_s) = X_1 \cup X_2} \sum_{i_1 \in X_1} \sum_{i_2 \in X_2} \mathcal{L}_{\text{int}}^* (i_1, i_2) g|_{X_1} (X_1) g|_{X_2} (X_2), \tag{14}
$$

where we used the notation adopted above in expansions (9).

### 2.3 The Liouville hierarchy

If $g_s^0 \in L_s^1$, $s \geq 1$, then for $t \in \mathbb{R}$ the sequence of correlation functions (13) defined on the set of allowed configurations is a unique solution of the Cauchy problem of the weak formulation of the Liouville hierarchy:

$$
\frac{\partial}{\partial t} g_s(t, x_1, \ldots, x_s) = \mathcal{L}_{\text{int}}^* (1, \ldots, s) g_s(t, x_1, \ldots, x_s) + \sum_{P: (x_1, \ldots, x_s) = X_1 \cup X_2} \sum_{i_1 \in X_1} \sum_{i_2 \in X_2} \mathcal{L}_{\text{int}}^* (i_1, i_2) g|_{X_1} (t, X_1) g|_{X_2} (t, X_2),
$$

$$
g_s(t, x_1, \ldots, x_s)|_{t=0} = g_s^0(x_1, \ldots, x_s), \quad s \geq 1, \tag{16}
$$

where $\sum_{P: (x_1, \ldots, x_s) = X_1 \cup X_2}$ is the sum over all possible partitions $P$ of the set $(x_1, \ldots, x_s)$ into two nonempty mutually disjoint subsets $X_1$ and $X_2$, the symbol $\hat{X}_i$ means the set of indexes of the set.
$X_i$ of phase space coordinates and the operator $L^*_s$ is defined on the subspace $L^1_0 \subset L^1$ by formula (4). It should be noted that the Liouville hierarchy (15) is the evolution recurrence equations set.

For $t \geq 0$ we give a few examples of recurrence equations set (15) for a system of hard spheres:

\[
\frac{\partial}{\partial t} g_1(t, x_1) = -\langle p_1, \frac{\partial}{\partial q_1} \rangle g_1(t, x_1),
\]

\[
\frac{\partial}{\partial t} g_2(t, x_1, x_2) = -\sum_{j=1}^{2} \langle p_j, \frac{\partial}{\partial q_j} \rangle g_2(t, x_1, x_2) + \sigma^2 \int d\eta \langle \eta, (p_1 - p_2) \rangle (g_2(t, q_1, p_1^*, q_2, p_2^*) \delta(q_1 - q_2 + \sigma \eta) - g_2(t, x_1, x_2) \delta(q_1 - q_2 - \sigma \eta)) + \sigma^2 \int d\eta \langle \eta, (p_1 - p_2) \rangle (g_1(t, q_1, p_1^*) g_1(t, q_2, p_2^*) \delta(q_1 - q_2 + \sigma \eta) - g_1(t, x_1) g_1(t, x_2) \delta(q_1 - q_2 - \sigma \eta)),
\]

where it was used notations accepted above in definition (3).

We note that because the Liouville hierarchy (15) is the recurrence evolution equations set, we can construct a solution of the Cauchy problem (15),(16), integrating each equation of the hierarchy as the inhomogeneous Liouville equation. For example, as a result of the integration of the first two equations of the Liouville hierarchy (15), we obtain the following equalities:

\[
g_1(t, x_1) = S_1(-t, 1) g_1^0(x_1),
\]

\[
g_2(t, 1, 2) = S_2(-t, 1, 2) g_1^0(x_1, x_2) + \int_0^t dt_1 S_2(t_1 - t, 1, 2) L^*_s \int (1, 2) S_1(-t_1, 1) S_1(-t_1, 2) g_1^0(x_1) g_1^0(x_2).
\]

Then for the corresponding term on the right-hand side of the second equality, an analog of the Duhamel equation holds

\[
\int_0^t dt_1 S_2(t_1 - t, 1, 2) L^*_s \int (1, 2) S_1(-t_1, 1) S_1(-t_1, 2) = - \int_0^t dt_1 \frac{d}{dt_1} (S_2(t_1 - t, 1, 2) S_1(-t_1, 1) S_1(-t_1, 2)) = S_2(-t, 1, 2) - S_1(-t, 1) S_1(-t, 2) = \mathcal{A}_2(t, 1, 2),
\]

where $\mathcal{A}_2(t)$ is the second-order cumulant (12) of groups of operators (2). As a result of similar transformations for $s > 2$, the solution of the Cauchy problem (15),(16), constructed by an iterative procedure, is represented in the form of expansions (9).

The following statement is true.
For \( t \in \mathbb{R} \) a unique solution of the Cauchy problem of the Liouville hierarchy (15), (16) is represented by a sequence of expansions (9). For \( g_0^0 \in L^1_{n,0} \subset L^1_n, n \geq 1 \), a sequence of functions (9) is a classical solution and for arbitrary initial data \( g_0^0 \in L^1_n, n \geq 1 \), one has a generalized solution.

The proof of the theorem is similar to the proof of the existence theorem for the BBGKY hierarchy in the space of sequences of integrable functions [1], [22]. Indeed, if the initial data is \( g_0^0 \in L^1_{n,0}, n \geq 1 \), then the infinitesimal generator of the group of nonlinear operators (13) coincides with the operator (14) and hence the Cauchy problem (15), (16) has a classical (strong) solution (9).

We remark that a steady solution of the Liouville hierarchy (15) is a sequence of the Ursell functions on the allowed configurations of hard spheres, i.e. it is the sequence \( g^{(eq)} = (0, e^{-\beta \frac{p^2}{2}}, 0, \ldots,) \), where \( \beta \) is a parameter inversely proportional to temperature.

Finally, we emphasize that the dynamics of correlations, that is, the fundamental equations (15) describing the evolution of correlations of states of hard spheres, can be used as a basis for describing the evolution of the state of both a finite and an infinite number of hard spheres instead of the Liouville equations for the state (5). Further, we establish that the constructed dynamics of correlation underlies the description of the dynamics of infinitely many hard-spheres governed by the BBGKY hierarchies for reduced distribution functions or reduced correlation functions.

### 3 Propagation of correlations in a hard-sphere system

#### 3.1 Evolution of states described by the dynamics of correlations: reduced distribution functions

As is known, an equivalent approach adapted to describing the evolution of states of systems of both finite and infinite number of hard spheres is to describe a state by means of a sequence of so-called reduced distribution functions (marginals) governed by the BBGKY hierarchy [1]. In what follows, we outline an approach to describing the evolution of a state using both reduced distribution functions and reduced correlation functions, based on the dynamics of correlations in a system of hard spheres governed by the Liouville hierarchy for correlation functions (15).

For a hard-sphere system of a non-fixed, i.e. arbitrary but finite average number of identical particles, the reduced distribution functions are defined by means of probability distribution functions as follows [1]:

\[
F_s(t, x_1, \ldots, x_s) \doteq (I, D(t))^{-1} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^n} dx_{s+1} \ldots dx_{s+n} D_{s+n}(t, x_1, \ldots, x_{s+n}),
\]

\( s \geq 1, \)

where the normalizing factor \((I, D(t)) \doteq \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^n} dx_1 \ldots dx_n D_n(t, x_1, \ldots, x_n)\) is a grand canonical partition function. The possibility of redefining of the reduced distribution functions naturally arises as a result of dividing the series in expression (17) by the series of the normalization factor.
A definition of reduced distribution functions equivalent to definition (17) is formulated on the basis of correlation functions (9) of a system of hard spheres by means of the following series expansion:

\[
F_s(t, x_1, \ldots, x_s) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^n} dx_{s+1} \ldots dx_{s+n} g_{1+n}(t, \{x_1, \ldots, x_s\}, x_{s+1}, \ldots, x_{s+n}), 
\]

\[s \geq 1,
\]

where on the set of allowed configurations \(\mathbb{R}^{3(s+n)} \setminus \mathbb{W}_{s+n}\) the correlation functions of clusters of hard spheres \(g_{1+n}(t), n \geq 0\), are determined by the expansions:

\[
g_{1+n}(t, \{x_1, \ldots, x_s\}, x_{s+1}, \ldots, x_{s+n}) = \sum_{P: \{(x_1, \ldots, x_s)\}, x_{s+1}, \ldots, x_{s+n} = \bigcup_i X_i} \mathcal{A}_{[P]}(t, \{\theta(\hat{X}_i)\}, \{\theta(\hat{X}_1)\}) \prod_{X_i \in P} g|_{X_i}(X_i), \quad n \geq 0.
\]

We remind that in expansions (19) the symbol \(\sum_{P: \{(x_1, \ldots, x_s)\}, x_{s+1}, \ldots, x_{s+n} = \bigcup_i X_i}\) means the sum over all possible partitions \(P\) of the set \(\{x_1, \ldots, x_s\}, x_{s+1}, \ldots, x_{s+n}\) into nonempty mutually disjoint subsets \(X_i\), and the generating operator \(\mathcal{A}_{[P]}(t)\) is the \(|P|th\)-order cumulant (10) of the groups of operators (2).

On allowed configurations the correlation functions of particle clusters in series (18), i.e. the functions \(g_{1+n}(t, \{x_1, \ldots, x_s\}, x_{s+1}, \ldots, x_{s+n}), n \geq 0\), are defined as solutions of generalized cluster expansions of a sequence of solutions of the Liouville equations (5):

\[
D_{s+n}(t, x_1, \ldots, x_{s+n}) = \sum_{P: \{(x_1, \ldots, x_s)\}, x_{s+1}, \ldots, x_{s+n} = \bigcup_i X_i} \prod_{X_i \in P} g|_{X_i}(t, X_i), \quad s \geq 1, \quad n \geq 0,
\]

namely,

\[
g_{1+n}(t, \{x_1, \ldots, x_s\}, x_{s+1}, \ldots, x_{s+n}) = \sum_{P: \{(x_1, \ldots, x_s)\}, x_{s+1}, \ldots, x_{s+n} = \bigcup_i X_i} (-1)^{|P|-1}(|P| - 1)! \prod_{X_i \in P} D_{\theta(X_i)}(t, \theta(X_i)), \quad s \geq 1, \quad n \geq 0,
\]

where \(\theta\) is the declustering mapping defined in formula (10), the probability distribution function \(D_{\theta(X_i)}(t, \theta(X_i))\) is solution (1) of the Liouville equations (5).

The correlation functions of particle clusters satisfy the Liouville hierarchy of evolution equations with the following generator

\[
\mathcal{L}(\{1, \ldots, s\}, s+1, \ldots, s+n \mid \mathcal{A}_{[Y]}g(t)) = \mathcal{L}_{s+n}^*(1, \ldots, s+n)g_{1+n}(t, X) + \sum_{P: X=X_1 \cup X_2} \sum_{i_1 \in \theta(\hat{X}_1)} \sum_{i_2 \in \theta(\hat{X}_2)} \mathcal{L}_{int}^*(i_1, i_2)g|_{X_1}(t, X_1)g|_{X_2}(t, X_2), \quad n \geq 0,
\]
where \( X \equiv (\{Y\}, x_{s+1}, \ldots, x_{s+n}) \equiv (\{x_1, \ldots, x_s\}, x_{s+1}, \ldots, x_{s+n}) \), the sequence of solutions of generalized cluster expansions (20) is denoted by means of the mapping
\[
(\partial\{Y\} g)_{n}(x_1, \ldots, x_n) \doteq g_{1+n}(\{x_1, \ldots, x_s\}, x_{s+1}, \ldots, x_{s+n}), \quad n \geq 0,
\]
and we also used the notations adopted above in expansion (9).

We note that on the allowed configurations correlation functions of hard-sphere clusters can be expressed through correlation functions of hard spheres (9) by the following relations:
\[
y_n(x_1, \ldots, x_n) = \sum_{\mathcal{Z} \in \mathcal{P} : \mathcal{Z}(\mathcal{X}_i) = \bigcup_i \mathcal{X}_i} (-1)^{|\mathcal{P}| - 1} (|\mathcal{P}| - 1)! \times \prod_{x_i \in \mathcal{P}} g_{|\mathcal{X}_i|}(t, x_i), \quad n \geq 0.
\]

In particular case \( n = 0 \), i.e. the correlation function of a cluster of the \( s \) hard spheres, these relations take the form
\[
y_{1+n}(t, \{x_1, \ldots, x_s\}, x_{s+1}, \ldots, x_{s+n}) = \sum_{\mathcal{P} : \mathcal{P}(\mathcal{X}_i) = \bigcup_i \mathcal{X}_i} \prod_{x_i \in \mathcal{P}} g_{|\mathcal{X}_i|}(t, x_i).
\]

As a consequence of these relations, for the initial state satisfying the chaos condition, from (19) the following generalization of expansions (11) holds:
\[
y_{s+n}(t, \{x_1, \ldots, x_s\}, x_{s+1}, \ldots, x_{s+n}) = \mathfrak{A}_{1+n}(t, \{1, \ldots, s\}, s + 1, \ldots, s + n) \prod_{i=1}^{s+n} g_{i}^0(x_i) \mathcal{X}^3_{\mathbb{Z}(s+n)} \setminus \mathbb{W}_{s+n}, \quad s \geq 1, n \geq 0.
\]

As we noted above, the possibility of the description of the evolution of a state based on the dynamics of correlations (18) occurs naturally in consequence of dividing the series of expressions (17) by the series of the normalizing factor. To provide evidence of this statement, we will introduce the necessary concepts and prove the validity of some auxiliary equalities.

On sequences of functions \( f, \tilde{f} \in L^1 \oplus_{n=0}^{\infty} L^1_n \) we define the following \(*\)-product (23)
\[
(f * \tilde{f})_n(x_1, \ldots, x_s) = \sum_{Z \subseteq \{x_1, \ldots, x_s\}} f_{|Z|}(Z) \tilde{f}_{s-|Z|}((x_1, \ldots, x_s) \setminus Z),
\]
where \( \sum_{Z \subseteq \{x_1, \ldots, x_s\}} \) is the sum over all subsets \( Z \) of the set \( \{x_1, \ldots, x_s\} \). Using the definition of the \(*\)-product (24), we introduce the mapping \( \text{Exp}_s \) and the inverse mapping \( \text{Ln}_s \) on sequences \( h = (0, h_1(x_1), \ldots, h_n(x_1, \ldots, x_n), \ldots) \) of functions \( h_n \in L^1_n \) by the expansions:
\[
(\text{Exp}_s h)_n(x_1, \ldots, x_s) = (1 + \sum_{n=1}^{\infty} \frac{h^n}{n!})_n(x_1, \ldots, x_s) =
\]
\[
\delta_{s,0} + \sum_{\mathcal{P} : \mathcal{P}(x_1, \ldots, x_s) = \bigcup_i \mathcal{X}_i} \prod_{x_i \in \mathcal{P}} h_{|\mathcal{X}_i|}(x_i),
\]
where we used the notations accepted in formula (7), \( \mathbb{I} = (1, 0, \ldots, 0, \ldots) \) and \( \delta_{s,0} \) is the Kronecker symbol, and respectively,

\[
(\mathbb{L} \mathbb{n}_s(\mathbb{I} + h))_s(x_1, \ldots, x_s) = \left( \sum_{n=1}^{\infty} (-1)^{n-1} \frac{h^{sn}}{n} \right)_s(x_1, \ldots, x_s) = 
\sum_{P: (x_1, \ldots, x_s) = \bigcup_i X_i} (-1)^{|P| - 1} (|P| - 1)! \prod_{X_i \in P} h_{|X_i|} (X_i).
\]

Therefore in terms of sequences of functions recursion relations (7) are rewritten in the form

\[ D(t) = \mathbb{E}_{\mathbb{n}_s} g(t), \]

where \( D(t) = \mathbb{I} + (0, D_1(t, x_1), \ldots, D_n(t, x_1, \ldots, x_n), \ldots) \). As a result, we get

\[ g(t) = \mathbb{L} \mathbb{n}_s D(t). \]

Thus, according to definition (24) of the *-product and mapping (26), in the component-wise form solutions of recursion relations (7) are represented by expansions (8).

For arbitrary \( f = (f_0, f_1, \ldots, f_n, \ldots) \in L^1 \) and the set \( Y \equiv (x_1, \ldots, x_s) \) we define the linear mapping \( \mathfrak{d}_Y : f \to \mathfrak{d}_Y f \), by the formula

\[
(\mathfrak{d}_Y f)_n (x_1, \ldots, x_n) \equiv f_{s+n} (x_1, \ldots, x_s, x_{s+1}, \ldots, x_{s+n}), \quad n \geq 0.
\]

For the set \( \{ Y \} \) consisting of the one element \( Y = (x_1, \ldots, x_s) \), we have, respectively

\[
(\mathfrak{d}_{\{ Y \}} f)_n (x_1, \ldots, x_n) \equiv f_{1+n} (\{ x_1, \ldots, x_s \}, x_{s+1}, \ldots, x_{s+n}), \quad n \geq 0.
\]

On sequences \( \mathfrak{d}_Y f \) and \( \mathfrak{d}_Y \tilde{f} \) we introduce the *-product

\[
(\mathfrak{d}_Y f \star \mathfrak{d}_Y \tilde{f})_{|X|} (X) \equiv \sum_{Z \subset X} f_{|Z|+|Y|} (Y, Z) \tilde{f}_{|X|\setminus|Z|+|Y'|} (Y', X \setminus Z),
\]

where \( X, Y, Y' \) are the sets, which characterize clusters of hard spheres, and \( \sum_{Z \subset X} \) is the sum over all subsets \( Z \) of the set \( X \). In particular case \( Y = \emptyset, Y' = \emptyset \), this definition reduces to definition of *-product (24).

Let us establish some properties of introduced mappings (24) and (28).

If \( f_n \in L^1_n, n \geq 1 \) for the sequences \( f = (0, f_1, \ldots, f_n, \ldots) \), according to definitions of mappings (24) and (28), the following equality holds

\[
\mathfrak{d}_{\{ Y \}} \mathbb{E}_{\mathbb{n}_s} f = \mathbb{E}_{\mathbb{n}_s} f \star \mathfrak{d}_{\{ Y \}} f,
\]

and for mapping (27) respectively

\[
\mathfrak{d}_Y \mathbb{E}_{\mathbb{n}_s} f = \mathbb{E}_{\mathbb{n}_s} f \star \sum_{P: Y = \bigcup_i X_i} \mathfrak{d}_{X_1} f \star \ldots \star \mathfrak{d}_{X_P} f,
\]

where \( \sum_{P: Y = \bigcup_i X_i} \) is the sum over all possible partitions \( P \) of the set \( Y \equiv (x_1, \ldots, x_s) \) into \( |P| \) nonempty mutually disjoint subsets \( X_i \subset Y \).
Hence in terms of mappings (27) and (28) generalized cluster expansions (20) take the form

$$d_Y D(t) = d_{\{Y\}} Exp_* g(t).$$ (30)

On sequences of functions $f \in L^1 = \oplus_{n=0}^\infty L_n$ we also define the analogue of the annihilation operator

$$(af)_n(x_1, \ldots, x_n) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx_{n+1} f_{n+1}(x_1, \ldots, x_n, x_{n+1}).$$ (31)

Then for sequences $f, \tilde{f} \in L^1$, the following equality holds

$$(e^a f * \tilde{f})_0 = (e^a f)_0 (e^a \tilde{f})_0.$$ (32)

where such a notation was used

$$(e^a f)_0 = \sum_{n=0}^\infty \frac{1}{n!} \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^n} dx_1 \ldots dx_n f_n(x_1, \ldots, x_n).$$ (33)

Now let us prove the equivalence of definition (17) of the reduced distribution functions and their definition (18) within the framework of the dynamics of correlations.

In terms of mapping (27) and notation (33) the definition of reduced distribution functions (17) is written as follows

$$F_s(t, x_1, \ldots, x_s) = (e^a D(t))_0 (e^a d_{\{Y\}} D(t))_0.$$ (34)

Using generalized cluster expansions (30), and as a consequence of equalities (29) and (32), we find

$$(e^a d_Y D(t))_0 = (e^a d_{\{Y\}} Exp_* g(t))_0 = (e^a Exp_* g(t) * d_{\{Y\}} g(t))_0 = (e^a Exp_* g(t))_0 (e^a d_{\{Y\}} g(t))_0.$$ (35)

Taking into account that, according to the particular case $Y = \emptyset$, of cluster expansions (20), the equality holds

$$(e^a Exp_* g(t))_0 = (e^a D(t))_0,$$

as a result, we establish the following representation for the reduced distribution functions

$$F_s(t, x_1, \ldots, x_s) = (e^a d_{\{Y\}} g(t))_0.$$ (36)

Therefore, in componentwise-form we obtain relation (18).

Since the correlation functions $g_{1+n}(t), n \geq 0$, are governed by the corresponding Liouville hierarchy for the cluster of hard spheres and hard spheres, the reduced distribution functions (18) are governed by the BBGKY hierarchy for hard spheres

$$\frac{\partial}{\partial t} F(t) = e^a \mathcal{L}(\cdot, \cdot | e^{-a} F(t)),$$ (34)
where the operator $L(\{\cdot, \cdot \} \mid f)$ is generator (21) of the Liouville hierarchy for a cluster of hard spheres and hard spheres. For a generator of this hierarchy of evolution equations takes place the following representation:

$$e^a L(\{\cdot, \cdot \} \mid e^{-a} F(t)) = e^a L^* e^{-a} F(t),$$

where the operator $L^* = \oplus_{n=0}^{\infty} L_n^*$ is a direct sum of the Liouville operators (4) and the operator $a$ is defined by formula (31). Due to the fact that pairwise collisions occur during the evolution, a generator of this hierarchy is reduced to the operator of such a structure [1]

$$e^a L^* e^{-a} = L^* + [a, L^*],$$

where the bracket $[\cdot, \cdot]$ is the commutator of operators. For $t \geq 0$ the collision part $[a, L^*]$ of a generator of the BBGKY hierarchy for hard spheres has the following explicit form

$$([a, L^*] F(t))_s(x_1, \ldots, x_s) = \sigma^2 \sum_{i=1}^{s} \int_{R^3 \times S^2} dp_{s+1} d\eta \langle \eta, (p_i - p_{s+1}) \rangle (F_{s+1}(t, x_1, \ldots, q_i, p^*_s, \ldots, x_s, q_i + \sigma \eta, p^*_s + 1)),$$

where the notations adopted above were used.

We note that the BBGKY hierarchy for hard spheres (34) was mathematically justified in paper [20] (see also [1]).

In consequence of definition (18) and the cumulant structure of representation of a solution (9) for the Liouville hierarchy (15), if initial state specified by the sequence of reduced distribution functions $F(0) = (1, F_0^0(x_1), \ldots, F_0^n(x_1, \ldots, x_n), \ldots)$, then the evolution of all possible states, i.e. the sequence of the reduced distribution functions $F_s(t), s \geq 1$, is determined by the following series expansions [22]:

$$F_s(t, x_1, \ldots, x_s) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(R^3 \times R^3)^n} dx_{s+1} \ldots dx_{s+n} \mathcal{A}_{1+n}(t, \{1, \ldots, s\},$$

$$s + 1, \ldots, s + n) F^0_{s+n}(x_1, \ldots, x_{s+n}), \quad s \geq 1,$$

where the generating operator of these series

$$\mathcal{A}_{1+n}(t, \{1, \ldots, s\}, s + 1, \ldots, s + n) = \sum_{P : (\{x_1, \ldots, x_s\}, x_{s+1}, \ldots, x_{s+n}) = \bigcup X_i} (-1)^{|P|-1} (|P| - 1)! \prod_{X_i \subset P} S_{\theta(X_i)}(-t, \theta(X_i))$$

is the $(1+n)th$-order cumulant (10) of the groups of operators (2) and the notations adopted above was used.

We remark that the representation (35) is directly established for the initial states satisfying the chaos condition due to the validity in this case of the representation (23) for the correlation functions of the hard-sphere cluster and of the hard spheres.
Consequently, as follows from the above, the cumulant structure of generating operators of expansions for correlation functions [9] or [19] induces the cumulant structure (36) of generating operators of series expansions for reduced distribution functions (35) or in other words, the evolution of the state of a system of an infinite number of hard spheres is governed by the dynamics of correlations on a microscopic scale.

Thus, we have established relation (18) between the reduced distribution functions and correlation functions.

3.2 Evolution of states described by the dynamics of correlations: reduced correlation functions

As is known, on a microscopic scale, the macroscopic characteristics of fluctuations of observables are directly determined by means of the reduced correlation functions (marginal or s-particle correlation functions [4], or cumulants of marginals [7], [8]). Assuming as a basis an alternative approach to the description of the evolution of states of a hard-sphere system within the framework of correlation functions (9), then the reduced correlation functions are defined by means of a solution of the Cauchy problem of the Liouville hierarchy (15), (16) as follows:

$$G_s(t, x_1, \ldots, x_s) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^n} dx_{s+1} \ldots dx_{s+n} g_{s+n}(t, x_1, \ldots, x_{s+n}), \quad s \geq 1,$$

where the generating function $g_{s+n}(t, x_1, \ldots, x_{s+n})$ is defined by expansion (9), or in terms of mapping (27) and notation (33) this definition takes the form

$$G_s(t, x_1, \ldots, x_s) = (e^a \partial_y g(t))_0,$$

or in terms of sequences of functions this expression has the form

$$G(t) = e^a g(t).$$

We emphasize that nth term of expansions (37) of the reduced correlation functions are determined by the $(s + n)th$-particle correlation function [9] in contrast with the expansions of reduced distribution functions (18) which are determined by the $(1 + n)th$-particle correlation function of clusters of hard spheres [19].

Such a representation for reduced correlation functions (37) can be derived as a result of the fact that the reduced correlation functions are cumulants of reduced distribution functions [18]. Indeed, traditionally reduced correlation functions are introduced by means of the cluster expansions of the reduced distribution functions similar to the cluster expansions of the probability distribution functions (7) and on the set of allowed configurations $\mathbb{R}^{3n} \setminus \mathbb{W}_n$ they have the form:

$$F_s(t, x_1, \ldots, x_s) = \sum_{P: (x_1, \ldots, x_s) = \bigcup_i X_i} \prod_{X_i \subset P} G_{|X_i|}(t, X_i), \quad s \geq 1,$$

where as above the symbol $\sum_{P: (x_1, \ldots, x_s) = \bigcup_i X_i}$ is the sum over all possible partitions P of the set $(x_1, \ldots, x_s)$ into $|P|$ nonempty mutually disjoint subsets $X_i \subset (x_1, \ldots, x_s)$. As a consequence
of this, the solution of recurrence relations (38) are represented through reduced distribution functions as follows:

\[ G_s(t, x_1, \ldots, x_s) = \sum_{P : (x_1, \ldots, x_s) = \bigcup_i X_i} (-1)^{|P|-1}(|P| - 1)! \prod_{X_i \in P} F_{[X_i]}(t, X_i), \quad s \geq 1. \] (39)

Functions (39) are interpreted as the functions which describe the correlations of hard-sphere states. The structure of expansions (39) is such that the reduced correlation functions are cumulants (semi-invariants) of the reduced distribution functions (35). Functions as follows:

\[ G_s(t, x_1, \ldots, x_s) = \sum_{P : (x_1, \ldots, x_s) = \bigcup_i X_i} (-1)^{|P|-1}(|P| - 1)! \prod_{X_i \in P} (e^{a\mathcal{Q}_{[X_i]}(t)} g(t)) = (e^{a\mathcal{Q}_Y(t)})_0. \]

Since the correlation functions \( g_{s+n}(t) \), \( n \geq 0 \), are governed by the Liouville hierarchy for hard spheres (15), the reduced correlation functions defined as (37) are governed by the hierarchy of nonlinear equations for hard spheres (the nonlinear BBGKY hierarchy):

\[ \frac{\partial}{\partial t} G_s(t, x_1, \ldots, x_s) = \mathcal{L}^*_s G_s(t, x_1, \ldots, x_s) + \sum_{P : (x_1, \ldots, x_s) = X_1 \cup X_2} \sum_{i_1 \in \tilde{X}_1} \sum_{i_2 \in \tilde{X}_2} \mathcal{L}^*_\text{int}(i_1, i_2) G_{[X_1]}(t, X_1) G_{[X_2]}(t, X_2) + \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx_{s+1} \left( \sum_{i=1}^s \mathcal{L}^*_\text{int}(i, s + 1) G_{s+1}(t, x_1, \ldots, x_{s+1}) + \sum_{P : (x_1, \ldots, x_{s+1}) = X_1 \cup X_2} \sum_{i \in \tilde{X}_1; s+1 \in \tilde{X}_2} \mathcal{L}^*_\text{int}(i, s + 1) G_{[X_1]}(t, X_1) G_{[X_2]}(t, X_2) \right), \]

\[ G_s(t, x_1, \ldots, x_s) \big|_{t=0} = G^0_s(x_1, \ldots, x_s), \quad s \geq 1, \] (41)

where the symbol \( \sum_{P : (x_1, \ldots, x_{s+1}) = X_1 \cup X_2} \) means the sum over all possible partitions of the set \( (x_1, \ldots, x_{s+1}) = X_1 \cup X_2 \), the sum over the index \( i \) which takes values from the subset \( \tilde{X}_1 \) provided that the index \( s + 1 \) belongs to the subset \( \tilde{X}_2 \) is denoted by \( \sum_{i \in \tilde{X}_1; s+1 \in \tilde{X}_2} \) and notations accepted in the Liouville hierarchy (15) are used.

A generator of this hierarchy of nonlinear evolution equations has the following structure:

\[ \frac{\partial}{\partial t} G(t) = e^a \mathcal{L}(\cdot | e^{-a} G(t)), \]

where the operator \( \mathcal{L}(\cdot | f) = \oplus_{n=0}^{\infty} \mathcal{L}(1, \ldots, n | f) \) is a direct sum of generators (14) of the Liouville hierarchy (15). Here are some component-wise examples of hierarchy (40):

\[ \frac{\partial}{\partial t} G_1(t, x_1) = \mathcal{L}^*_1(1) G_1(t, x_1) + \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx_2 \mathcal{L}^*_\text{int}(1, 2) (G_2(t, x_1, x_2) + G_1(t, x_1) G_1(t, x_2)), \]
\[
\frac{\partial}{\partial t} G_2(t, x_1, x_2) = \mathcal{L}^*_2(1, 2) G_2(t, x_1, x_2) + \mathcal{L}^*_\text{int}(1, 2) G_1(t, x_1) G_1(t, x_2) +
\int_{\mathbb{R}^3 \times \mathbb{R}^3} dx_3 \left( \sum_{i=1}^{2} \mathcal{L}^*_\text{int}(i, 3) \left( G_3(t, x_1, x_2, x_3) + G_2(t, x_1, x_2) G_1(t, x_3) \right) + \mathcal{L}^*_\text{int}(2, 3) G_2(t, x_1, x_3) G_1(t, x_2) + \mathcal{L}^*_\text{int}(1, 3) G_2(t, x_2, x_3) G_1(t, x_1) \right),
\]

where it was used notations accepted above in definition \([3]\).

If \( G(0) = (1, G^0_1(x_1), \ldots, G^0_s(x_1, \ldots, x_s), \ldots) \) is a sequence of reduced correlation functions at initial instant, then by means of mappings \([13]\) the evolution of all possible states, i.e. the sequence of the reduced correlation functions \( G_s(t), s \geq 1 \), is determined by the following series expansions:

\[
G_s(t, x_1, \ldots, x_s) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^n} dx_{s+1} \ldots dx_{s+n} \mathfrak{A}_{1+n}(t; \{1, \ldots, s\}, s+1, \ldots, s+n | G(0)), \quad s \geq 1,
\]

where the generating operator \( \mathfrak{A}_{1+n}(t; \{1, \ldots, s\}, s+1, \ldots, s+n | G(0)) \) of this series is the \((1+n)th\)-order cumulant of groups of nonlinear operators \([9]\):

\[
\mathfrak{A}_{1+n}(t; \{1, \ldots, s\}, s+1, \ldots, s+n | G(0)) = \sum_{P: \{1, \ldots, s, s+1, \ldots, s+n\} = \bigcup_k X_k} (-1)^{|P|-1} (|P|-1)! G(t; \theta(X_1) | \ldots G(t; \theta(X_{|P|}) | G(0)) \ldots),
\]

\( n \geq 0, \)

and where the composition of mappings \([9]\) of the corresponding noninteracting groups of particles was denoted by \( G(t; \theta(X_1) | \ldots G(t; \theta(X_{|P|}) | G(0)) \ldots) \), for example,

\[
G(t; 1 | G(t; 2 | G(0))) = \mathfrak{A}_1(t, 1) \mathfrak{A}_1(t, 2) G^0_2(x_1, x_2),
\]

\[
G(t; 1, 2 | G(t; 3 | G(0))) = \mathfrak{A}_1(t, 1, 2) \mathfrak{A}_1(t, 3) G^0_3(x_1, x_2, x_3) + \mathfrak{A}_2(t, 1, 2) \mathfrak{A}_1(t, 3) \left( G^0_1(x_1) G^0_2(x_2, x_3) + G^0_2(x_2) G^0_1(x_1, x_3) \right).
\]

We will adduce examples of expansions \([13]\). The first order cumulant of the groups of nonlinear operators \([9]\) is the group of these nonlinear operators

\[
\mathfrak{A}_1(t; \{1, \ldots, s\} | G(0)) = G(t; 1, \ldots, s | G(0)).
\]

In case of \( s = 2 \) the second order cumulant of nonlinear operators \([9]\) has the structure

\[
\mathfrak{A}_{1+1}(t; \{1, 2\}, 3 | G(0)) = G(t; 1, 2, 3 | G(0)) - G(t; 1, 2 | G(t; 3 | G(0))) = \mathfrak{A}_{1+1}(t; \{1, 2\}, 3) G^0_3(1, 2, 3) + \left( (\mathfrak{A}_{1+1}(t; \{1, 2\}, 3) - \mathfrak{A}_2(t, 2, 3) \mathfrak{A}_1(t, 1) \right) G^0_1(x_1) G^0_2(x_2, x_3) + \left( (\mathfrak{A}_{1+1}(t; \{1, 2\}, 3) - \mathfrak{A}_2(t, 1, 3) \mathfrak{A}_1(t, 2) \right) G^0_1(x_2) G^0_2(x_1, x_3) + \mathfrak{A}_{1+1}(t; \{1, 2\}, 3) G^0_1(x_3) G^0_2(x_1, x_2) + \mathfrak{A}_3(t, 1, 2, 3) G^0_1(x_1) G^0_1(x_2) G^0_1(x_3),
\]
where the operator
\[ \mathcal{A}_3(t, 1, 2, 3) = \mathcal{A}_{1+1}(t, \{1, 2\}, 3) - \mathcal{A}_2(t, 2, 3)\mathcal{A}_1(t, 1) - \mathcal{A}_2(t, 1, 3)\mathcal{A}_1(t, 2) \]
is the third-order cumulant \((12)\) of groups of operators \((2)\) of a system of hard spheres.

If \( G(0) \in \oplus_{n=0}^{\infty} L^n_{1} \), then provided that \( \max_n \| G^n_n \|_{L^n_{1}} < (2e^3 - 1)^{-1} \), for \( t \in \mathbb{R} \) the sequence of reduced correlation functions \((42)\) is a unique solution of the Cauchy problem of the nonlinear BBGKY hierarchy \((40),(41)\) for hard spheres.

In the particular case of the initial state specified by the sequence of reduced correlation functions \( G((c)) = (0, G^0_1, 0, \ldots, 0, \ldots) \) on the allowed configurations, that is, in the absence of correlations between hard spheres at the initial moment of time \([1, 19]\), according to definition \((43)\) of the generating operators, reduced correlation functions \((42)\) are represented by the following series expansions:
\[ G_s(t, x_1, \ldots, x_s) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^n} dx_{s+1} \ldots dx_{s+n} \mathcal{A}_{s+n}(t; 1, \ldots, s + n) \prod_{i=1}^{s+n} G^0_i(x_i) \mathcal{X}_{(s+n)\mathbb{R}^3 \setminus W_{s+n}}, \quad s \geq 1, \]
where the generating operator \( \mathcal{A}_{s+n}(t) \) is the \((s + n)th\)-order cumulant \((12)\) of the groups of operators \((2)\) of the Liouville equations \((5)\).

We emphasize that in the absence of correlations of states of hard spheres on allowed configurations at the initial moment of time, the generators of expansions into a series of reduced correlation functions \((44)\) and reduced distribution functions \((35)\) differ only in the order of cumulants of groups of operators of hard spheres. Therefore, by means of such reduced distribution functions or reduced correlation functions, the process of creating correlations in a system of hard spheres is described.

We note that the reduced correlation functions give an equivalent approach to the description of the evolution of states of many hard spheres along with the reduced distribution functions. Indeed, the macroscopic characteristics of fluctuations of observables are directly determined by the reduced correlation functions on the microscopic scale \([21]\) for example, the functional of the dispersion of an additive-type observable, i.e. the sequence \( A^{(1)} = (0, a_1(x_1), \ldots, \sum_{i_1=1}^{n} a_1(x_{i_1}), \ldots) \), is represented by the formula
\[ \langle (A^{(1)} - \langle A^{(1)} \rangle)^2 \rangle(t) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx_1 (a_1^2(x_1) - \langle A^{(1)} \rangle^2(t)) G_1(t, x_1) + \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^2} dx_1 dx_2 a_1(x_1) a_1(x_2) G_2(t, x_1, x_2), \]
where
\[ \langle A^{(1)} \rangle(t) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx_1 a_1(x_1) G_1(t, x_1) \]
is the mean value functional of an additive-type observable.
4 The description of correlations by means of kinetic equations

4.1 The non-Markovian Enskog equation

Now we will consider an approach to the description of the state evolution by means of the state of a typical particle of a system of many hard spheres, or in other words, foundations are overviewed of describing the evolution of a state by kinetic equations.

Let the initial state specified by a one-particle reduced correlation function, namely, the initial state specified by a sequence of reduced correlation functions satisfying a chaos property stated above, i.e. by the sequence \( G^{(c)} = (G_0, G_1^0, 0, \ldots, 0, \ldots) \) on the allowed configurations. We remark that such an assumption about initial states is intrinsic in the contemporary kinetic theory of many-particle systems [1], [2].

Since the initial data \( G^{(c)} \) is completely specified by a one-particle correlation function, the Cauchy problem of the nonlinear BBGKY hierarchy (40),(41) for hard spheres is not a completely well-defined Cauchy problem, because the initial data is not independent for every unknown function governed by the hierarchy of mentioned evolution equations. As a result, it becomes possible to reformulate such a Cauchy problem as a new Cauchy problem for a one-particle correlation function with independent initial data and explicitly defined functionals of the solution of this Cauchy problem.

We formulate such a restated Cauchy problem and the sequence of the suitable functionals. The following statement is true.

In the case of the initial state \( G^{(c)} \) specified by a one-particle correlation function the evolution that described by means of the sequence \( G(t) = (G_0, G_1(t), \ldots, G_s(t), \ldots) \) of reduced correlation functions (12), is also be described by the sequence \( G(t \mid G_1(t)) = (G_0, G_1(t), \ldots, G_s(t \mid G_1(t)), \ldots) \) of the reduced (marginal) correlation functionals: \( G_s(t, x_1, \ldots, x_s \mid G_1(t)), s \geq 2 \), with respect to the one-particle correlation function \( G_1(t) \) governed by the non-Markovian Enskog kinetic equation.

Indeed, in the case under consideration the reduced correlation functionals \( G_s(t \mid G_1(t)) \), \( s \geq 2 \), are represented with respect to the one-particle correlation function (44), i.e.

\[
G_1(t, x_1) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^n} dx_2 \ldots dx_{1+n} \mathfrak{A}_{1+n}(t, 1, \ldots, n + 1) \prod_{i=1}^{n+1} G_1^0(x_i) \mathcal{X}_{n+1}(t) \mathcal{W}_{n+1},
\]

by the following series expansions:

\[
G_s(t, x_1, \ldots, x_s \mid G_1(t)) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^n} dx_{s+1} \ldots dx_{s+n} \mathfrak{V}_{s+n}(t, 1, \ldots, s + n) \prod_{i=1}^{s+n} G_1(t, x_i), \quad s \geq 2.
\]

The generating operator \( \mathfrak{V}_{s+n}(t), n \geq 0 \), of the \((s+n)th\)-order of this series is determined by the
following expansion:

\[ \mathfrak{V}_{s+n}(t, 1, \ldots, s, s + 1, \ldots, s + n) = n! \sum_{k=0}^{n} \frac{(-1)^k}{n!} \sum_{n_1=1}^{n} \ldots \]

\[ \sum_{n_k=1}^{n-n_1-\ldots-n_{k-1}} \frac{1}{(n-n_1-\ldots-n_k)!} \hat{\mathfrak{A}}_{s+n-n_1-\ldots-n_k}(t, 1, \ldots, s+n-n_1-\ldots-n_k) \prod_{j=1}^{k} \sum_{D_j : Z_j = \bigcup_{l \in D_j} X_{l_j}, |D_j| \leq s+n-n_1-\ldots-n_j} \frac{1}{|D_j|!} \times \]

\[ \sum_{i_1 \neq \ldots \neq i_{|D_j|=1}} \prod_{l \in D_j \subset X_{l_j}} X_{l_j}(t, i_j, X_{l_j}), \]

where \( \sum_{D_j : Z_j = \bigcup_{l \in D_j} X_{l_j}} \) is the sum over all possible dissections of the linearly ordered set \( Z_j \equiv (s+n-n_1-\ldots-n_j+1, \ldots, s+n-n_1-\ldots-n_{j-1}) \) on no more than \( s+n-n_1-\ldots-n_j \) linearly ordered subsets, the \((s+n)\)th-order scattering cumulant is defined by the formula

\[ \hat{\mathfrak{A}}_{s+n}(t, 1, \ldots, s+n) \equiv \mathfrak{A}_{s+n}(t, 1, \ldots, s+n) \mathcal{X}_{\mathbb{R}^{s(s+n)} \setminus \mathbb{W}_{s+n}} \prod_{i=1}^{s+n} \mathfrak{A}_1^{-1}(t, i), \]

and notations accepted above were used.

We adduce simplest examples of generating operators (47):

\[ \mathfrak{V}_s(t, 1, \ldots, s) = \mathfrak{A}_s(t, 1, \ldots, s) \mathcal{X}_{\mathbb{R}^{3s} \setminus \mathbb{W}_s} \prod_{i=1}^{s} \mathfrak{A}_1^{-1}(t, i), \]

\[ \mathfrak{V}_{s+1}(t, 1, \ldots, s, s+1) = \mathfrak{A}_{s+1}(t, 1, \ldots, s+1) \mathcal{X}_{\mathbb{R}^{3(s+1)} \setminus \mathbb{W}_{s+1}} \prod_{i=1}^{s+1} \mathfrak{A}_1^{-1}(t, i) - \]

\[ \mathfrak{A}_s(t, 1, \ldots, s) \mathcal{X}_{\mathbb{R}^{3s} \setminus \mathbb{W}_s} \prod_{i=1}^{s} \mathfrak{A}_1^{-1}(t, i) \sum_{j=1}^{s} \mathfrak{A}_2(t, j, s+1) \mathcal{X}_{\mathbb{R}^6 \setminus \mathbb{W}_2} \mathfrak{A}_1^{-1}(t, j) \mathfrak{A}_1^{-1}(t, s+1). \]

A method of the construction of reduced correlation functionals (46) is based on the application of the so-called kinetic cluster expansions [14] to generating operators (12) of series (44). If \( ||G_1(t)||_{L^1(\mathbb{R}^3 \times \mathbb{R}^3)} < e^{-(3s+2)} \), for arbitrary \( t \in \mathbb{R} \) series (46) converges in the norm of the space \( L^1_s \) [14].

We note that in the case of initial state specified by a one-particle correlation function the reduced correlation functionals (46) describe all possible correlations generated by the dynamics of many hard spheres in terms of a one-particle correlation function.

If initial data \( G_1^0 \in L_1^1 \), then for arbitrary \( t \in \mathbb{R} \) one-particle correlation function (45) is a weak
solution of the Cauchy problem of the non-Markovian Enskog kinetic equation

\[
\frac{\partial}{\partial t} G_1(t, x_1) = \mathcal{L}^*(1)G_1(t, x_1) + \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx_2 \mathcal{L}_{\text{int}}^*(1, 2)G_1(t, x_1)G_1(t, x_2) + \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx_2 \mathcal{L}_{\text{int}}^*(1, 2)G_2(t, x_1, x_2 | G_1(t)),
\]

where the first part of the collision integral in equation (48) has the Boltzmann–Enskog structure, and the second part of the collision integral is determined in terms of the two-particle correlation functional represented by series expansion (46) and it describes all possible correlations which are created by hard-sphere dynamics and by the propagation of initial correlations related to the forbidden configurations.

By virtue of definitions (3), (4) of the generator of the non-Markovian Enskog equation (48), for \( t > 0 \) the kinetic equation gets such explicit form

\[
\frac{\partial}{\partial t} G_1(t, x_1) = -\langle p_1, \frac{\partial}{\partial q_1} \rangle G_1(t, x_1) + \sigma^2 \int_{\mathbb{R}^3 \times S_+^2} dp_2 d\eta \langle \eta, (p_1 - p_2) \rangle (G_1(t, p_1^*, q_1)G_1(t, p_2^*, q_1 - \sigma\eta, \sigma) - G_1(t, x_1)G_1(t, p_2, q_1 + \sigma\eta)) + \\
\sigma^2 \int_{\mathbb{R}^3 \times S_+^2} dp_2 d\eta \langle \eta, (p_1 - p_2) \rangle (G_2(t, p_1^*, q_1, p_2^*, q_1 - \sigma\eta | G_1(t)) - G_2(t, x_1, p_2, q_1 + \sigma\eta | G_1(t))).
\]

In the paper [14], similar statements were proved for the evolution of the state of a hard-sphere system described in terms of reduced distribution functions governed by the BBGKY hierarchy. We emphasize that the \( n \)th term of expansions (46) of the reduced correlation functionals are determined by the \( (s + n) \)th-order generating operator (47) in contradistinction to the expansions of reduced distribution functionals of the state constructed in [14] which are determined by the \( (1 + n) \)th-order generating operator (47).

Thus, for the initial state specified by a one-particle correlation function all possible states of a system of hard spheres can be described without any approximations within the framework of a one-particle correlation function governed by the non-Markovian kinetic equation (48), and a sequence of explicitly defined functionals (46) of its solution (45).

4.2 On the low-density approximation of hard-sphere correlations

The conventional philosophy of the description of kinetic evolution consists in the following. If the initial state is specified by a one-particle distribution function, then the evolution of the state can be effectively described in a suitable scaling limit [4], [17] by means of a one-particle distribution function governed by the nonlinear kinetic equation.

In the last decade, the Boltzmann–Grad limit (low-density scaling limit) [17], [18] of the reduced distribution functions constructed by means of the theory of perturbations was rigorously discussed.
in several papers, see for example [7], [11], [15] and references therein. In this subsection, we consider a scheme of constructing the scaling asymptotic behavior of reduced correlation functions \( [20] \) in the Boltzmann–Grad limit for initial data satisfying a chaos property stated above, namely, for the sequence of reduced correlation functions \( [44] \).

Let for \( t \geq 0 \) the operator \( \mathcal{L}^*_{\text{int}} \) in the dimensionless form of the hierarchy of evolution nonlinear equations \( [40] \) be scaled in such a way that

\[
\mathcal{L}^*_{\text{int}}(j_1, j_2)f_n = \epsilon^2 \int d\eta \langle \eta, (p_{j_1} - p_{j_2}) \rangle (f_n(x_1, \ldots, p^*_{j_1}, q_{j_1}, \ldots, p^*_{j_2}, q_{j_2}, \ldots, x_n) - f_n(x_1, \ldots, x_n) \delta(q_{j_1} - q_{j_2} + \epsilon \eta), \quad \quad (50)
\]

where \( \epsilon > 0 \) is a scaling parameter (the ratio of the diameter \( \sigma > 0 \) to the mean free path of hard spheres) and the notations similar to definition \( [3] \) are used. We will consider initial states of a hard-sphere system specified by the scaled one-particle correlation function \( G^0_{1, \epsilon}(x_1) \), such that:

\[
|G^0_{1, \epsilon}(x_1)| \leq e^{-\frac{\beta}{2} p^2}, \quad \beta > 0 \text{ is a parameter and } c < \infty \text{ is some constant. We emphasize that } \text{the states of a system of infinitely many hard spheres are described by sequences of functions bounded with respect to the configuration variables } [11] \text{ as is assumed above. Regarding the technical details of the scheme presented below for constructing the Boltzmann–Grad asymptotics of reduced correlation functions } [44], \text{ we refer to our article } [20].

Let us assume that the Boltzmann–Grad limit of the initial one-particle correlation function \( G^0_{1, \epsilon} \) exists in the sense of the weak convergence \( [20] \)

\[
w - \lim_{\epsilon \to 0} (\epsilon^2 G^0_{1, \epsilon}(x_1) - f_1^0(x_1)) = 0.
\]

If equality \( (50) \) holds for the initial one-particle correlation function, then for series expansion \( [44] \) the following equality is true

\[
w - \lim_{\epsilon \to 0} (\epsilon^2 G_1(t, x_1) - f_1(t, x_1)) = 0,
\]

where for some finite time interval the limit one-particle correlation function \( f_1(t, x_1) \) is represented by the series expansion

\[
f_1(t, x_1) = \sum_{n=0}^{\infty} \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^n} dx_2 \ldots dx_{1+n} S_1(t_1 - t, 1) \times \quad (51)
\]

\[
\mathcal{L}^*_{\text{int}}(1, 2) \prod_{j_1=1}^2 S_1(t_2 - t_1, j_1) \ldots \prod_{i_n=1}^n S_1(t_n - t_{n-1}, i_n) \times
\]

\[
\sum_{k_n=1}^n \mathcal{L}^*_{\text{int}}(k_n, n + 1) \prod_{j_1=1}^{n+1} S_1(-t_n, j_n) \prod_{i_1}^{n+1} f_1^0(x_i).
\]

In this series expansion for \( t \geq 0 \) the operator \( \mathcal{L}^*_{\text{int}}(j_1, j_2) \) is defined by the formula

\[
\mathcal{L}^*_{\text{int}}(j_1, j_2)f_n = \int d\eta \langle \eta, (p_{j_1} - p_{j_2}) \rangle (f_n(x_1, \ldots, p^*_{j_1}, q_{j_1}, \ldots, p^*_{j_2}, q_{j_2}, \ldots, x_n) - f_n(x_1, \ldots, x_n) \delta(q_{j_1} - q_{j_2}), \quad (50)
\]

\[
p^*_{j_2}, q_{j_2}, \ldots, x_n) - f_n(x_1, \ldots, x_n) \delta(q_{j_1} - q_{j_2}).
\]
where notations adopted in formula (3) are used.

The function $f_1(t)$ represented by series (51) is a weak solution of the Cauchy problem of the Boltzmann kinetic equation for hard spheres

$$\frac{\partial}{\partial t}f_1(t, x_1) = -\langle p_1, \frac{\partial}{\partial q_1} \rangle f_1(t, x_1) + \int_{\mathbb{R}^3 \times S^2_+} dp_2 d\eta \langle \eta, (p_1 - p_2) \rangle (f_1(t, p_1^*, q_1) f_1(t, p_2^*, q_1) - f_1(t, p_1, q_1) f_1(t, p_2, q_1)),$$

$$f_1(t, x_1) \big|_{t=0} = f_1^0(x_1).$$

The property of propagation of initial chaos in the of low-density limit is a consequence of the following equalities for reduced correlation functions (44):

$$w-\lim_{\epsilon \to 0} \epsilon^{2s} G_s(t, x_1, \ldots, x_s) = 0, \quad s \geq 2.$$

The proof of these statements (51) and (54) is based on the validity of the limit theorems for cumulants of asymptotically perturbed groups of operators (2) and the explicit structure of the generating operators of series expansions of reduced correlation functions (44).

Indeed, if $|f_s| \leq ce^{-\frac{\beta}{2} \sum_{i=1}^{s} p_i^2}$, then for arbitrary finite time interval for asymptotically perturbed first-order cumulant (12) of the groups of operators (2) the following equality takes place [1], [20]

$$w-\lim_{\epsilon \to 0} \left( S_s(-t, 1, \ldots, s) f_s - \prod_{j=1}^{s} S_1(-t, j) f_s \right) = 0.$$

Therefore, for the $(s + n)th$-order cumulant of asymptotically perturbed groups of operators (2) the equalities are true:

$$w-\lim_{\epsilon \to 0} \frac{1}{\epsilon^{2n}} \mathcal{A}_{s+n}(t, 1, \ldots, s + n) f_{s+n} = 0, \quad s \geq 2.$$

In conclusion, we note that in the articles [16], [15] some other approaches to the derivation of kinetic equations were developed, including the kinetic equations for the states with correlations at the initial instant. In particular, a new method of the description of the kinetic evolution of a system with hard-sphere collisions has been suggested in the paper [16]. Formalism is based on the description of evolution within the framework of observables which are governed by the dual Boltzmann hierarchy in the low-density limit.

### 5 Conclusion

This article dealt with the mathematical problems of the description of the evolution of many hard spheres based on various ways of describing their state, namely by means of functions describing the propagation of correlations. One of these approaches allows one to describe the evolution...
of both a finite and an infinite average number of hard spheres using reduced distribution functions \(35\) or reduced correlation functions \(12\), which are determined by the dynamics of correlations \(9\) of a hard-sphere system.

We note the importance of the description of the processes of the creation and propagation of correlations \[24\], in particular, it is related to the problem of the description of the memory effects in many-particle systems with collision dynamics.

It was established that the notion of cumulants \(10\) of the groups of operators \(2\) underlies non-perturbative expansions of solutions for the fundamental evolution equations describing the state evolution of a hard-sphere system, namely, of the Liouville hierarchy \(15\) for correlation functions, of the BBGKY hierarchy for reduced distribution functions and of the nonlinear BBGKY hierarchy \(40\) for reduced correlation functions, as well as it underlies the kinetic description of infinitely many hard spheres \(46\). We remark that for quantum many-particle systems the concept of cumulants of groups of operators is considered in the papers \[25\]- \[27\].

We emphasize that the structure of expansions for correlation functions \(9\), in which the generating operators are the cumulants of the corresponding order \(10\) of the groups of operators \(2\) of hard spheres, induces the cumulant structure of series expansions for reduced distribution functions \(35\), reduced correlation functions \(12\) and reduced correlation functionals \(46\). Thus, in fact, the dynamics of systems of infinitely many hard spheres is generated by the dynamics of correlations.

Above, one more approach to the description of the state evolution of a system of many hard spheres in terms of the state evolution of a typical particle was also studied. In other words, the origin of the collective behavior of a hard-sphere system on a microscopic scale was described by means of a one-particle correlation function that is determined by the non-Markovian Enskog kinetic equation \(48\). As already mentioned, one of the advantages of such an approach to the derivation of kinetic equations from underlying collisional dynamics consists of an opportunity to construct the kinetic equations with initial correlations, which makes it possible to describe the propagation of initial correlations in the Boltzmann–Grad limit \[15\]. Another advantage of this approach is related to the rigorous derivation of the Boltzmann equation with higher-order corrections to the main term of the Boltzmann–Grad asymptotics of collisional dynamics.

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References

[1] Cercignani, C., Gerasimenko, V., Petrina, D.: Many-Particle Dynamics and Kinetic Equations. Springer: The Netherlands (2012)

[2] Cercignani, C., Illner, R., Pulvirenti, M.: The Mathematical Theory of Dilute Gases. Springer-Verlag, New York (1994)

[3] Spohn, H.: Large Scale Dynamics of Interacting Particles. Springer-Verlag, New York (1991)

[4] Bogoliubov, N.N.: Problems of the dynamical theory in Statistical Physics, Gostechizdat, Moscow, (1946) (in Russian)

[5] Gallagher, I., Saint-Raymond, L., Texier, B.: From Newton to Boltzmann: Hard Spheres and Short-range Potentials. EMS Publ. House: Zürich Lectures in Advanced Mathematics (2014)
[6] Bodineau, T., Gallagher, I., Saint-Raymond, L., Simonella, S.: Statistical dynamics of a hard sphere gas: fluctuating Boltzmann equation and large deviations. arXiv:2008.10403 (2020)

[7] Bodineau, T., Gallagher, I., Saint-Raymond, L., Simonella, S.: Fluctuation theory in the Boltzmann–Grad Limit. J. Stat. Phys. 180, 873–895 (2020)

[8] Duerinckx, M., Saint-Raymond, L.: Lenard–Balescu correction to mean-field theory. Probab. Math. Phys. 2(1), 27–69 (2021) DOI: 10.2140/pmp.2021.2.27

[9] Duerinckx, M.: On the size of chaos via Glauber calculus in the classical mean-field dynamics. Commun. Math. Phys. 382, 613—653 (2021) DOI: 10.1007/s00220-021-03978-3

[10] Simonella, S.: Evolution of correlation functions in the hard sphere dynamics. J. Stat. Phys. 155(6), 1191–1221 (2014)

[11] Pulvirenti, M., Simonella, S.: Propagation of chaos and effective equations in kinetic theory: a brief survey. Mathematics and Mechanics of Complex Systems. 4(3-4), 255–274 (2016)

[12] Pulvirenti, M., Simonella, S.: The Boltzmann–Grad limit of a hard sphere system: analysis of the correlation error. Invent. Math. 207(3), 1135–1237 (2017)

[13] Pulvirenti, M., Simonella, S., Trushechkin, A.: Microscopic solutions of the Boltzmann-Enskog equation in the series representation. Kinetic and Rel. Mod. 11(4), 911–931 (2018)

[14] Gerasimenko, V.I., Gapyak, I.V.: Hard sphere dynamics and the Enskog equation. Kinet. and Relat. Models. 5(3), 459–484 (2012) DOI: 10.3934/krm.2012.5.459

[15] Gerasimenko, V.I., Gapyak, I.V.: Boltzmann–Grad asymptotic behavior of collisional dynamics. Reviews in Math. Phys. 33, 2130001, 32 (2021) DOI: 10.1142/S0129055X21300016

[16] Gerasimenko, V.I., Gapyak, I.V.: Low-density asymptotic behavior of observables of hard sphere fluids. Advances in Math. Phys., 2018, Article ID 6252919, (2018) DOI: 10.1155/2018/6252919

[17] Grad, H.: Principles of the kinetic theory of gases. in Handbuch der Physik, Springer, Berlin, 12, 205–294 (1958)

[18] Lanford, O.E.: Time evolution of large classical systems. in Dynamical systems, theory and applications, (ed. J. Moser), Springer–Verlag, Berlin. Lect. Notes in Phys. 38, 1–111 (1975)

[19] Spohn, H.: Kinetic equations from Hamiltonian dynamics: Markovian limits, Rev. Modern Phys. 52(3), 569–615 (1980)

[20] Petrina D.Ya., Gerasimenko, V.I.: Mathematical problems of the statistical mechanics of a hard-sphere system, Russ. Math. Surv. (Uspekhi Mat. Nauk), 5(3), 135–182 (1990) DOI: 10.1070/RM1990v045n03ABEH002360

[21] Bogolyubov, M.M.: Lectures on quantum statistics. Problems of statistical mechanics of quantum systems. Kyiv: Rad. Shcola (in Ukrainian) (1949)

[22] Gerasimenko, V.I., Ryabukha, T.V., Stashenko, M.O.: On the structure of expansions for the BBGKY hierarchy solutions, J. Phys. A: Math. Gen., 37, 9861–9872 (2004) DOI: 10.1088/0305-4470/37/42/002

[23] Ruelle, D.: Statistical Mechanics: Rigorous Results. W. A. Benjamin Advanced Bk Program, (1969)
[24] Prigogine, I.: Non-Equilibrium Statistical Mechanics. John Wiley & Sons, New York, (1962)

[25] Gerasimenko V.I., Shtyk, V.O.: Evolution of correlations of quantum many-particle systems. J. Stat. Mech. Theory Exp., 3, P03007 (2008) DOI: 10.1088/1742-5468/2008/03/P03007

[26] Gerasimenko V.I., Polishchuk, D.O.: Dynamics of correlations of Bose and Fermi particles. Math. Meth. Appl. Sci. 34(1), 76–93 (2011) DOI: 10.1002/mma.1336

[27] Gerasimenko, V.I.: Hierarchies of quantum evolution equations and dynamics of many-particle correlations. Int. J. Evol. Equ., 7(2), 109–163, (2012)