INTEGRABLE SYSTEMS WITH PAIRWISE INTERACTIONS AND FUNCTIONAL EQUATIONS

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Abstract

A new ansatz is presented for a Lax pair describing systems of particles on the line interacting via (possibly nonsymmetric) pairwise forces. Particular cases of this yield the known Lax pairs for the Calogero-Moser and Toda systems, as well as their relativistic generalisations. The ansatz leads to a system of functional equations. Several new functional equations are described and the general analytic solution to some of these is given. New integrable systems are described.
1 Introduction

Completely integrable systems arise in various diverse settings in both mathematics and physics and accordingly have been studied from many different points of view, a fact which underlies their importance and interest (see for example [23]). Within this area the study of Lax pairs, a zero curvature condition, plays an important role. The construction of such Lax pairs has followed many routes and this paper will further investigate the connection between functional equations and such zero curvature conditions. The essential idea in this approach is to reduce the constraints of the Lax pair $L, M$ implicit in $\dot{L} = [L, M]$ to that of a functional equation. Our study will broaden the ansatz for the Lax pair and correspondingly lead to a more general functional equation than hitherto studied. This enables us to understand many of the known integrable systems (and their corresponding functional equations) from a unified perspective. The symmetries of the functional equation we obtain are very large and this group relates distinct functional equations and their corresponding physical models.

We feel this connection between functional equations and completely integrable systems is part of a broader and less well understood aspect of the subject. Functional equations have of course a long and interesting history in connection with mathematical physics and touch upon many branches of mathematics [2, 3]. Novikov’s school for example considered the Hirzebruch genera associated with the index theorems of known elliptic operators and showed that these arose as solutions of functional equations. More recently Ochanine showed the string inspired Witten index could be described by Hirzebruch’s construction where the the functional equation was that appropriate to an elliptic function. These same functional equations arise (as we shall later see in more detail) in the context of completely integrable systems. The latter also appear in the study of conformal and string theories and this connection between string theory and finite dimensional completely integrable systems needs to be better understood. The functional equations and integrable systems we shall discuss arise naturally in investigations of the KP and KdV equations [1, 21].

To make matters concrete let us consider how such functional equations arise in the context of integrable systems of particles on the line. Here one starts with an ansatz for the matrices $L$ and $M$ of the Lax pair and seeks the restrictions necessary to obtain equations of motion of some desired form. These restrictions typically involve the study of functional equations. The Calogero-Moser [15] system provides the paradigm for this approach. Beginning with the ansatz (for $n \times n$ matrices)

$$L_{jk} = p_j \delta_{jk} + g \left( 1 - \delta_{jk} \right) A(q_j - q_k), \quad M_{jk} = g \left[ \delta_{jk} \sum_{l \neq j} B(q_j - q_l) - (1 - \delta_{jk}) C(q_j - q_k) \right]$$
one finds $\dot{L} = [L, M]$ yields the equations of motion for the Hamiltonian system ($n \geq 3$)

$$H = \frac{1}{2} \sum_j p_j^2 + g^2 \sum_{j<k} U(q_j - q_k) \quad U(x) = A(x)A(-x) + \text{constant}$$

provided $C(x) = -A'(x)$, and that $A(x)$ and $B(x)$ satisfy the functional equation

$$A(x)A'(y) - A(y)A'(x) = A(x + y)[B(x) - B(y)]. \quad (1)$$

The solutions to this functional equation may be expressed in terms of elliptic functions and their degenerations. Krichever used this functional equation in his proof of the ‘rigidity’ property of elliptic genera and it appears when discussing rational and pole solutions of the KP and KdV equations. Different starting ansatz lead to the relativistic Calogero-Moser systems, the Toda and relativistic Toda equations. Underlying the corresponding functional equations of each of these models lies the addition formula for elliptic functions. Further, in studying the quantum mechanics of these systems similar functional equations arise when factorising the ground state wave function.

In this paper we will introduce a new ansatz that includes the previous examples (together with their functional equations) as special cases. We shall be seeking Lax pairs that lead to equations of the form

$$\ddot{q}_j = \sum_{k \neq j} (a + b\dot{q}_j)(a + b\dot{q}_k)V_{jk}(q_j - q_k). \quad (2)$$

Here we are allowing the interaction $V_{jk}$ to in principle depend on the choice of pair $j, k$: when $V_{jk}$ is the same for all pairs we have a system of Calogero-Moser type while if $V_{jk}$ is the same for pairs $j, j \pm 1$ and zero otherwise we have a Toda system (of $A_n$ type). When the constant $b = 0$ we obtain the nonrelativistic systems and when $a = 0$ we obtain the relativistic systems previously examined. Indeed, because the interactions of the system only depend on coordinate differences, the shift $q \to q - a\tau t/b$ enables us without any loss of generality

1 The solution to this equation has been given by various authors with assumptions of even/oddness on the functions appearing or assumptions on the nature of $B$. The general solution was given in [6, 11]. The derivation we shall present later in fact yields the even/oddness assumptions of these earlier works.

2 When $b = 0$ then $V_{jk}$ is just $-\frac{d}{dq}U(q)$, where $U(q)$ is the potential energy of the system. When $b = 1$ and $a = 0$ then the Hamiltonian for the system is $H = \sum_j \cosh \theta_j V_j(q)$ where $\theta_j$ is the rapidity canonically conjugate to $q_j$ and $V_j(q) = \prod_{k \neq j} V_{jk}$. In this case corresponds to the flows of $S_\pm = \sum_j e^{\pm \theta_j} V_j(q)$. 

2
to set $a = 0$ when $b \neq 0$ and we shall do this where appropriate. Our first stage of generalisation then is to consider a matrix of pairwise (though in principle distinct) interactions and it is just this limitation to pairwise interactions $V_{jk} = V_{jk}(q_j - q_k)$ that enables us to derive functional equations of a given type. Yet rather than just a single functional equation our extension allowing different interactions $V_{jk}$ now leads to a system of functional equations, and this interplay of matrix relations and functional equations appears new. Although our ansatz by its very form must include the Calogero-Moser and Toda models, our approach shows how they may be unified by the study of one functional equation.

We find the functional equations needed to construct a Lax pair yielding the equations of motion (2) are of the form (for $b \neq 0$)

$$
\phi_1(x + y) = \frac{\phi_2(x)}{\phi_3(x)} \frac{\phi_2(y)}{\phi_3(y)}
$$

(3)

Elsewhere we have shown,

**Theorem 1** [3] The general analytic solution to the functional equation (3) is, up to a $G$ action given by (29-32), of the form

$$
\phi_1(x) = \frac{\Phi(x; \nu)}{\Phi(x; \mu)}, \quad \left(\frac{\phi_2(x)}{\phi_3(x)}\right) = \left(\frac{\Phi(x; \nu)}{\Phi(x; \nu')}\right) \quad \text{and} \quad \left(\frac{\phi_4(x)}{\phi_5(x)}\right) = \left(\frac{\Phi(x; \mu)}{\Phi(x; \mu')}\right).
$$

Here

$$
\Phi(x; \nu) \equiv \frac{\sigma(\nu - x)}{\sigma(\nu)\sigma(x)} e^{\zeta(x)}
$$

where $\sigma(x) = \sigma(x|\omega, \omega')$ and $\zeta(x) = \frac{\sigma(x')}{\sigma(x)}$ are the Weierstrass sigma and zeta functions.

The symmetries $G$ of (3) will be described in section three. The proof given in [3] is constructive and the transformations needed to obtain the solutions may be readily implemented.

The case $b = 0$ is more problematical. In this case we obtain functional equations of the form

$$
\phi_6(x + y) = \phi_1(x + y)(\phi_4(x) - \phi_5(y)) + \left|\begin{array}{cc}
\phi_2(x) & \phi_3(y) \\
\phi_4'(x) & \phi_5'(y)
\end{array}\right|.
$$

(4)

Certainly we may take the limit $b \to 0$ to our solutions of (3) (which, for example, will give the Calogero-Moser model as the nonrelativistic limit of the relativistic Calogero-Moser model) to obtain solutions of (4) but at present we don’t know
the general solution to (4). We can show however that known nonrelativistic models are solutions to this equation, together with new potentials such as

\[ V_{jk}(x) = a_j a_k \phi'(x). \]

When \( a_j = a_k \) this yields the usual type IV Calogero-Moser potential.

We remark that (3) and a suitably symmetrized form of (4) are particular cases of the functional equation

\[ \sum_{i=0}^{N} \phi_{3i}(x+y) \begin{vmatrix} \phi_{3i+1}(x) & \phi_{3i+1}(y) \\ \phi_{3i+2}(x) & \phi_{3i+2}(y) \end{vmatrix} = 0 \]  

(5)

with \( N = 1 \) in the case \( b \neq 0 \) and \( N = 2 \) in case \( b = 0 \). In the case \( \phi_{3i+2} = \phi'_{3i+1} \) Buchstaber and Krichever have discussed (3) in connection with functional equations satisfied by Baker-Akhiezer functions [14]. Dubrovin, Fokas and Santini [18] have also investigated integrable functional equations via algebraic geometry.

A further generalisation of the Calogero-Moser system has been to associate such an integrable system to the root system of an arbitrary semisimple Lie algebra [24, 25]. At this stage of generalisation we have essentially the Lax pairs associated with \( A_n \) type root systems. To incorporate more general root systems we may consider embedding \( \Omega : \mathbb{R}^n \to \mathbb{R}^N \) of our \( n \)-degrees of freedom into a larger space with the interactions \( V \) still of the given form. We will not present this generalisation here.

This paper then presents an ansatz for Lax pairs whose consistency yields equations of motion (2) and the corresponding functional equations (3) and (4). We shall show how various specialisations lead to the known systems and introduce some new ones. An outline of the paper is as follows. In section two we present the ansatz. For clarity of exposition we initially confine our attention to the case \( b \neq 0 \) returning to the \( b = 0 \) case in Section five. Here we determine in Theorem 2 the system of functional equations to be solved, reducing the nontrivial equations to be solved to the form (3). Section three describes the invariances of (3) and illustrates the solution of Theorem 1 as a means of introducing certain elliptic function identities useful in the sequel. Here we will apply the general analytic solution of this theorem to our system of equations (Corollary 1) and then as an example show how the relativistic example of Bruschi and Calogero [6] arises. In general our ansatz leads to a system of functional equations and Section four looks at the constraints imposed on the parameters of the solution to (3) by such a system. Theorem 3 determines these constraints and these are illustrated by the relativistic Calogero-Moser and Toda models. Further, we are able to characterise the relativistic Calogero-Moser model by a certain ‘generic’ property, Theorem 4. Section five returns to the
In an earlier version of this paper we proved Theorem 1. Subsequently we found a direct constructive proof which has been presented separately \[5\]. In revising the present paper accordingly we have also strengthened the results of section 4.

2 The Ansatz

We shall now describe the ansatz, introducing our notation and illustrating some techniques useful in the reduction problem of Lax pairs to functional equations. Having presented the ansatz for our Lax pair we proceed to determine the restrictions on the functions that appear in this. The equations we find are a natural generalisation of those found in \[6\]. We will then seek the relevant functional equations to be solved in later sections.

We need a few definitions in order to specify our ansatz. Let \(\tau\) be the fixed vector \(\tau = (1,\ldots,1)\) and denote by \(X_d = \text{Diag}(X_1,\ldots,X_n)\) the injection \(\mathbb{R}^n \to \text{Mat}(n)\). Further let \(\mathcal{M}_n = \{A|A \in \text{Mat}(n), A_{ii} = 0, A_{ij} = A_{ij}(q_i - q_j)\}\) be a subset of matrix-valued functions of one variable. (One can also extend our analysis to the case of nonvanishing diagonal elements.) Note this set depends on a choice of coordinates \(\{e_i\}\) with respect to which we express our matrices and determine the coordinate projection \(q_i = (e_i,q)\). A change of basis results in a straightforward conjugation. We denote by \(e_j = (0,\ldots,1,\ldots,0)^T\) the \(j\)-th coordinate (column) vector. Thus \(e^T_j e_k = \delta_{jk}\), \(e^T_j A e_k = A_{jk}\) and \(X_d e_k = X_k e_k\).

Our ansatz for the Lax pair takes the form

\[
L(q) = \dot{q}_d + \sqrt{(a\tau + b\dot{q})_d} A \sqrt{(a\tau + b\dot{q})_d}\n
M(q) = (B,[a\tau + b\dot{q}])_d + \sqrt{(a\tau + b\dot{q})_d} C \sqrt{(a\tau + b\dot{q})_d}
\]

where \(A,B,C \in \mathcal{M}_n\). When \(a = 0\) the matrix \(\sqrt{q_d}\) corresponds to the diagonal matrix \(D\) of \[30\]. This Lax operator also possesses the symmetry

\[
L(q, b, a, A(q)) = l L(q/l, bl, a, l^{-1} A(l \cdot q/l)),
\]

which enables us to rescale \(b \neq 0\) to 1 and we shall later do this. It will be convenient to define

\[
G \equiv \frac{1}{b} \frac{\psi'}{\psi'} = B + \frac{b}{2} V \quad \text{and} \quad H = A' - C.
\]

We remark that there is no essential change if we take

\[
L(q) = \dot{q}_d + (a\tau + b\dot{q})_d^\epsilon A(a\tau + b\dot{q})_d^{1-\epsilon}
\]

and a similar modification to \(M\), for any value of \(\epsilon\). We choose the symmetrical value given. 

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\footnote{and a similar modification to \(M\), for any value of \(\epsilon\). We choose the symmetrical value given.}
Proof. By using the freedom to commute diagonal matrices we observe that
\[
\dot{L} = \sqrt{(a\tau + b\dot{q})_d} \left( \frac{\ddot{q}_d}{(a\tau + b\dot{q})_d} + b \left( \frac{\ddot{q}_d}{(a\tau + b\dot{q})_d} A + A \frac{\ddot{q}_d}{(a\tau + b\dot{q})_d} \right) + [\dot{q}_d, A'] \right) \sqrt{(a\tau + b\dot{q})_d}.
\]
Here \(\frac{dA}{dt} = \left( \frac{dA_{ij}}{dt} \right) = \left( [\dot{q}_i - \dot{q}_j, A']_{ij} \right) = [\dot{q}_d, A']\). The Lax equation \(\dot{L} = [L, M]\) consequently yields
\[
\frac{\ddot{q}_d}{(a\tau + b\dot{q})_d} + b \left( \frac{\ddot{q}_d}{(a\tau + b\dot{q})_d} A + A \frac{\ddot{q}_d}{(a\tau + b\dot{q})_d} \right) + [\dot{q}_d, A'] = [\dot{q}_d, C] + [A, (B, [a\tau + b\dot{q}]_d)] + A(a\tau + b\dot{q})_d C - C(a\tau + b\dot{q})_d A. \tag{13}
\]
To solve this equation we consider the diagonal and off-diagonal terms separately. The fact that \(A, B, C\) have vanishing diagonal terms results in
\[
\frac{\ddot{q}_d}{(a\tau + b\dot{q})_d} = \text{Diag} \left( A(a\tau + b\dot{q})_d C - C(a\tau + b\dot{q})_d A \right). \tag{14}
\]
This equation has several consequences. First, observe that if \(A\) and \(C\) have been determined then we obtain the sought after interaction,
\[
\ddot{q}_j = \sum_{k \neq j} (a + b\dot{q}_j)(a + b\dot{q}_k)V_{jk}(q_j - q_k) = (a + b\dot{q}_j) \left( A(a\tau + b\dot{q})_d C - C(a\tau + b\dot{q})_d A \right)_{jj}.
\]
That is

\[ V_{jk}(q_j - q_k) = A_{jk}C_{kj} - C_{jk}A_{kj} = \begin{vmatrix} A_{jk} & A_{kj} \\ C_{jk} & C_{kj} \end{vmatrix} = -V_{kj}(q_k - q_j). \] (15)

The second and crucial point is that (14) reduces the off-diagonal terms in our consistency equation (13) to a linear equation in \( a\tau \) and \( b\dot{q} \). Utilising our definitions of \( G \) and \( H \) we find the off-diagonal terms of (13) yield (for \( j \neq k \))

\[ \left( [\dot{q}_d, H] + [([G(a\tau + b\dot{q})]_d, A] + bA(V[a\tau + b\dot{q}]_d + C(a\tau + b\dot{q})_d A - A(a\tau + b\dot{q})_d C) \right)_{jk} = 0. \]

When \( b \neq 0 \) this equation reduces to considering

\[ \left( [\dot{q}_d, H] + b([G, \dot{q}_d], A] + b^2 A(V, \dot{q}_d)_d + bC\dot{q}_d A - bA\dot{q}_d C \right)_{jk} = 0 \] (16)

as the \( \tau \) terms of the previous equation are reproduced by taking \( \dot{q} \propto \tau \). We will consider separately the \( b = 0 \) situation which gives the equation

\[ \left( [\dot{q}_d, H] + a([B, \tau], A] + a[\tau], A] \right)_{jk} = 0. \] (17)

We may solve the linear equation (16) by taking particular choices for \( \dot{q} \). Substituting \( \dot{q} = e_k, \dot{q} = e_j \) and \( \dot{q} = e_m \) \( (m \neq j, k) \) in (16) yields the equations (9), (10) and (11) respectively.

We have thus shown how the Lax pair determined by the data \( A, B \) and \( C \) fixes the potential \( V \) via (15) and consequently matrices \( G \) and \( H \) such that (8), (11) and (11) must hold. It remains to show that \( V \) is also given by (12). First, if \( A_{jk} = 0 \) then by (7) and (8) we find \( C_{jk} = 0 \) and so by (14) \( V_{jk} = 0 \) as well; thus (12) holds in this case. Finally, if \( A_{jk} \neq 0 \), then upon adding (10) and (9) we obtain

\[ bV_{kj}(x) = G_{kj}(x) - G_{jk}(-x). \] (18)

Making use of the antisymmetry of \( V \) together with the definition of \( G \) gives as a consequence of (18) that \( B_{kj}(x) = B_{jk}(-x) \). Equation (12) now follows from (15) after making use of this symmetry and the expression

\[ C_{jk} = A'_{jk} - bA_{jk}G_{jk}, \] (19)

which follows from (14). Therefore we have established (12).

The converse of the theorem follows from our initial remarks that the matrices \( A \) and \( G \) together with the definition of \( V \) suffice to determine the Lax pair.

\[ \square \]
Our task therefore is to find matrices $A$ and $G$ for which (12) are satisfied. In order to understand this system of equations it is helpful to consider the consequences of an entry of $A$ either vanishing identically, or otherwise.

**Lemma 1** If $A_{jk} = 0$ for some $j, k$ then $H_{jk} = C_{jk} = V_{jk} = 0$. Further, for any $m \neq j, k$ for which $C_{jm}, C_{mk} \neq 0$ there exists a constant $a_{jmk}$ such that $A_{jm}(x) = a_{jmk}C_{jm}(x)$ and $A_{mk}(y) = a_{jmk}C_{mk}(y)$. If $a_{jmk} \neq 0$ then

$$A_{jm}(x) = \alpha_{jm} e^{x/a_{jmk}} \psi_{jm}(x), \quad A_{mk}(y) = \alpha_{mk} e^{y/a_{jmk}} \psi_{mk}(y).$$

(20)

for constants $\alpha_{jm}, \alpha_{mk}$.

**Proof.** The vanishing of $H_{jk}, C_{jk}$ and $V_{jk}$ is immediate. Further our assumption means that the right-hand side of (11) vanishes. Because $C_{jm}$ and $C_{mk}$ do not vanish identically, (19) entails that neither do $A_{jm}$ and $A_{mk}$. Thus the first row of the determinant must be proportional to the second and we have the second assertion of the lemma. Upon making use of (19) we obtain

$$A_{jm}(x) = a_{jmk}(A'_{jm}(x) - bA_{jm}(x)G_{jm}(x)) = a_{jmk}(A'_{jm}(x) - A_{jm}(x)\frac{\psi'_{jm}(x)}{\psi_{jm}(x)}).$$

When $a_{jmk} \neq 0$ this gives

$$\frac{A'_{jm}(x)}{A_{jm}(x)} = \frac{1}{a_{jmk}} + \frac{\psi'_{jm}(x)}{\psi_{jm}(x)},$$

and the first part of (20) follows upon integration; the expression for $A_{mk}(y)$ follows similarly.

\[\square\]

**Lemma 2** If $A_{jk} \neq 0$ for some $j, k$ then $V_{kj}(x) = G_{kj}(x) - G_{jk}(-x)$ and $B_{kj}(x) = B_{jk}(-x)$. Further, for any $m \neq j, k$:

1. if $A_{mk} = 0$, then $G_{jm} = G_{km} = c_1$, a constant;
2. if $A_{mk} \neq 0$ and $A_{jm} = 0$ then $G_{jm} = G_{mk} = c_2$, a constant;
3. if $A_{mk} \neq 0$ and $G_{jm} - G_{mk} \neq 0$ (and consequently $A_{jm} \neq 0$) then (11) may be written as

$$A_{jk} = \frac{\begin{vmatrix} A_{jm} & A_{mk} \\ C_{jm} & C_{mk} \end{vmatrix}}{G_{jm} - G_{mk}} = b A_{jm}A_{mk} + \frac{\begin{vmatrix} A_{jm} & A_{mk} \\ A'_{jm} & A'_{mk} \end{vmatrix}}{G_{jm} - G_{mk}}.$$  

(21)
Proof. We have already proven the first part of this lemma in our discussion of the theorem, (18). Now if \( A_{mk} = 0 \) for some \( m \), then \( V_{km} = 0 \) by (12) and so only the first term of (11) is nonvanishing. This means \( G_{jm} = G_{km} \) and as these are functions of different arguments, they must be constant.

Assume \( A_{mk} \neq 0 \). Then employing (18) for these indices enables us to rewrite (11) as

\[
\bar{A}_{jk}(x + y) = \bar{A}_{jm}(x)\bar{A}_{mk}(y) - \psi_{jm}(x)\psi_{mk}(y) \begin{vmatrix} \bar{A}_{jm}(x) & \bar{A}_{mk}(y) \\ \bar{A}'_{jm}(x) & \bar{A}'_{mk}(y) \\ \psi_{jm}(x) & \psi_{mk}(y) \\ \psi'_{jm}(x) & \psi'_{mk}(y) \end{vmatrix},
\]

(23)

where \( \bar{A} = bA \). As we have already remarked, when \( b \neq 0 \) we may rescale so that \( b = 1 \) and for the remainder of this section and until section 5 we assume that we have done this.

**Lemma 3** Equation (21) may be rewritten in the form (3) where either

\[
A_{jk}(x + y) = \begin{vmatrix} A_{jm}(x) & A_{mk}(y) \\ A_{jm}(x) & A_{mk}(y) \\ 1/\psi_{jm}(x) & 1/\psi_{mk}(y) \\ 1/\psi_{mj}(x) & 1/\psi_{mk}(y) \end{vmatrix},
\]

(24)

or

\[
A_{jk}(x + y) = \begin{vmatrix} c_2 A_{mk}(x) & c_2 A_{mk}(y) \\ G_{mk}(x) & G_{mk}(y) \\ 1 & 1 \end{vmatrix},
\]

(25)

for some constant \( c_2 \) according to whether \( \begin{vmatrix} \psi_{jm}(x) & \psi_{jm}(y) \\ \psi_{mk}(x) & \psi_{mk}(y) \end{vmatrix} \neq 0 \) vanishes or not.

**Proof.** Adopting the shorthand \( \partial = \partial_j + \partial_k \) observe that

\[
\partial A_{jm}A_{mk} = -\begin{vmatrix} A_{jm} & A_{mk} \\ A'_{jm} & A'_{mk} \end{vmatrix}.
\]
Thus we may rewrite (23) as

$$A_{jk} = A_{jm}A_{mk} - \psi_{jm}\psi_{mk}\frac{\partial A_{jm}A_{mk}}{\partial \psi_{jm}\psi_{mk}}.$$  \hspace{1cm} (26)

Now because $A_{jk} = A_{jk}(x_j - x_k)$ we have $\partial A_{jk} = 0$. Thus applying $\partial$ to both sides of (26) shows

$$\frac{\partial A_{jm}A_{mk}}{\partial \psi_{jm}\psi_{mk}} = 0,$$

and consequently the ratio here is a function of $x_j - x_k$, i.e. $\frac{\partial A_{jm}A_{mk}}{\partial \psi_{jm}\psi_{mk}} = A_{jm}(x_j - x_k)$. If we set $x = x_j - x_m$ and $y = x_m - x_k$ then (26) takes the form

$$A_{jk}(x + y) = A_{jm}(x)A_{mk}(y) - \psi_{jm}(x)\psi_{mk}(y)A_{jm}(x + y).$$ \hspace{1cm} (27)

The left hand side of this equation is symmetric under the interchange of $x$ and $y$. Performing this interchange and subtracting the resulting equation from (27) shows

$$0 = \begin{vmatrix} A_{jm}(x) & A_{jm}(y) \\ A_{mk}(x) & A_{mk}(y) \end{vmatrix} - \begin{vmatrix} \psi_{jm}(x) & \psi_{jm}(y) \\ \psi_{mk}(x) & \psi_{mk}(y) \end{vmatrix} A_{jm}(x + y).$$ \hspace{1cm} (28)

Two possibilities now arise, each leading to an equation of the form (3). Suppose first that $\begin{vmatrix} \psi_{jm}(x) & \psi_{jm}(y) \\ \psi_{mk}(x) & \psi_{mk}(y) \end{vmatrix} \neq 0$. Then

$$A_{jm}(x + y) = \begin{vmatrix} A_{jm}(x) & A_{jm}(y) \\ A_{mk}(x) & A_{mk}(y) \\ \psi_{jm}(x) & \psi_{jm}(y) \\ \psi_{mk}(x) & \psi_{mk}(y) \end{vmatrix}.$$

Upon substituting this expression into (27) one obtains (24).

In the case when $\begin{vmatrix} \psi_{jm}(x) & \psi_{jm}(y) \\ \psi_{mk}(x) & \psi_{mk}(y) \end{vmatrix} = 0$ we have $\psi_{jm}(x) = c_1\psi_{mk}(x)$ for some constant $c_1$. Therefore $G_{jm}(x) = G_{mk}(x)$. Likewise from (28) we have that $A_{jm}(x) = c_2A_{mk}(x)$ and hence $C_{jm}(x) = c_2C_{mk}(x)$. Substituting these relations into (21) obtains (25), again of the stated form.

\hspace{1cm} $\square$

3 \quad The Functional Equation

Thus far we have reduced the consistency requirements for the Lax pair (6) to a functional equation of the form (3) and this section looks briefly at this equation. Particular cases of this equation have been described in the literature \cite{7, 26}.
Here we describe the invariances $G$ of (3). Theorem 1 gives a representative of each $G$ orbit on the solutions of (3) with a particularly nice form. For later calculations it is instructive to see how the stated solution satisfies (3), an exercise involving some elliptic function identities. We end the section by deriving the relativistic Calogero-Moser model found by Bruschi and Calogero [6].

First observe that a large group of symmetries $G$ of (3). Theorem 1 gives a representative of each $G$ orbit on the solutions of (3) with a particularly nice form. For later calculations it is instructive to see how the stated solution satisfies (3), an exercise involving some elliptic function identities. We end the section by deriving the relativistic Calogero-Moser model found by Bruschi and Calogero [6].

First observe that a large group of symmetries $G$ act on the solutions of (3).

The transformation

$$
\left( \phi_1(x), \left( \frac{\phi_2(x)}{\phi_3(x)}, \frac{\phi_4(x)}{\phi_5(x)} \right) \right) \rightarrow \left( e^{\lambda x} \phi_1(x), U \left( e^{-\lambda t x} \phi_2(x), e^{-\lambda t x} \phi_3(x) \right), V \left( e^{\lambda t x} \phi_4(x), e^{\lambda t x} \phi_5(x) \right) \right)
$$

clearly preserves (3) provided

$$
\lambda + \lambda' + \lambda'' = 0, \quad U, V \in GL_2, \quad \text{and} \quad \det U = c \det V.
$$

Further, (3) is also preserved by

$$
\left( \phi_1(x), \left( \frac{\phi_2(x)}{\phi_3(x)}, \frac{\phi_4(x)}{\phi_5(x)} \right) \right) \rightarrow \left( \frac{1}{\phi_1(x)}, \left( \frac{\phi_4(x)}{\phi_5(x)}, \frac{\phi_2(x)}{\phi_3(x)} \right) \right)
$$

and

$$
\left( \phi_1(x), \left( \frac{\phi_2(x)}{\phi_3(x)}, \frac{\phi_4(x)}{\phi_5(x)} \right) \right) \rightarrow \left( \phi_1(x), \left( \phi_2(x), f(x) \phi_3(x), f(x) \phi_5(x) \right) \right).
$$

These symmetries enable one to find a solution of (3) on each $G$ orbit with a particularly nice form. It is instructive to see how the stated solution satisfies (3). From the definition of the zeta function we have

$$
(\ln \Phi(x; \nu))' = -\zeta(\nu - x) - \zeta(x) + \zeta(\nu).
$$

Thus

$$
\begin{vmatrix}
\Phi(x; \nu) & \Phi(y; \nu) \\
\Phi(x; \nu)' & \Phi(y; \nu)' \\
\end{vmatrix} = \Phi(x; \nu) \Phi(y; \nu) \left[ (\ln \Phi(y; \nu))' - (\ln \Phi(x; \nu))' \right]
$$

$$
= \Phi(x; \nu) \Phi(y; \nu) \left[ \zeta(\nu - x) + \zeta(x) + \zeta(-y) + \zeta(y - \nu) \right].
$$

Upon using the definition of $\Phi$ the right hand side of this equation takes the form

$$
\Phi(x + y; \nu) \frac{\sigma(\nu - x) \sigma(\nu - y) \sigma(x + y)}{\sigma(\nu - x - y) \sigma(\nu) \sigma(x) \sigma(y)} \left[ \zeta(\nu - x) + \zeta(x) + \zeta(-y) + \zeta(y - \nu) \right].
$$

After noting the two identities [34]

$$
\zeta(x) + \zeta(y) + \zeta(z) - \zeta(x + y + z) = \frac{\sigma(x + y) \sigma(y + z) \sigma(z + x)}{\sigma(x) \sigma(y) \sigma(z) \sigma(x + y + z)}
$$

(34)
and

\[ \varphi(x) - \varphi(y) = \frac{\sigma(y - x)\sigma(y + x)}{\sigma^2(y)\sigma^2(x)} \]  

we find (33) simplifies to \( \Phi(x + y; \nu)[\varphi(x) - \varphi(y)] \), where \( \varphi(x) = -\zeta'(x) \) is the Weierstrass \( \varphi \)-function. Putting these together yields the addition formula

\[ \Phi(x + y; \nu) = \frac{\Phi(x; \nu) - \Phi(y; \nu) + \frac{1}{2}\varphi(x - y)}{\varphi(x) - \varphi(y)}, \]  

and consequently a solution of (3) with the stated form.

A consequence of Theorem 1 then is

**Corollary 1** If \( A_{jk} \) satisfies (24) then

\[ A_{jk}(x) = c_{jk} \frac{\Phi(x, \nu_{jk})}{\Phi(x, \mu_{jk})} e^{\lambda_{jk} x} \]  

for some constants \( c_{jk}, \nu_{jk}, \mu_{jk} \) and \( \lambda_{jk} \).

**Example** The relativistic example of Calogero and Bruschi arises as a particular case of our ansatz when \( a = 0, b = 1 \) and

\[ A_{jk}(x) = (1 - \delta_{jk})\alpha(x), \quad B_{jk}(x) = (1 - \delta_{jk})\beta(x), \quad C_{jk}(x) = (1 - \delta_{jk})\gamma(x). \]

In this case (35) takes the form

\[ \alpha(x + y) = \frac{\begin{vmatrix} \alpha(x) & \alpha(y) \\ \gamma(x) & \gamma(y) \end{vmatrix}}{\eta(x) \eta(y)}, \]  

where \( \eta(x) = \beta(x) + \frac{1}{2}\nu(x) \). Comparison of (38) with the general solution of (3) shows the solution to be given by

\[ \alpha(x) = \frac{\Phi(x; \nu)}{\Phi(x; \mu)}, \quad \eta(x) = -(\ln \Phi(x; \mu))^\prime = -\frac{1}{2} \frac{\varphi'(x) - \varphi'(|\mu|)}{\varphi(x) - \varphi(|\mu|)}. \]

In this case \( \alpha(x)\alpha(-x) = (\varphi(x) - \varphi(\nu))/(\varphi(x) - \varphi(\mu)) \) and so by (12)

\[ v(x) = \frac{\varphi'(x)}{\varphi(\mu) - \varphi(x)}; \]

and we have recovered\(^4\) the results of references [29] and [3].

\(^4\) We remark in passing that the the scaling of the elliptic function

\[ \varphi(t x|t \omega, t \omega') = t^{-2} \varphi(x|\omega, \omega') \]

is accounted for by the scaling of the Lax operator (7) and does not appear to give any new potentials.
4 Application of the Functional Equation

Thus far we have discussed the functional equation (3) in isolation while our application involves a system of such equations. In solving this system we encounter further constraints. To see how these arise consider (24). Our theorem says

\[ A_{jk}(x) = c_{jk} \frac{\Phi(x; \nu_{jk})}{\Phi(x; \mu_{jk})} e^{\lambda_{jk} x} \]

(for some constants \(c_{jk}, \nu_{jk}, \mu_{jk}\) and \(\lambda_{jk}\)) and further that

\[
\begin{pmatrix}
A_{jm}(x)/\psi_{jm}(x) \\
A_{mk}(x)/\psi_{mk}(x)
\end{pmatrix} = f(x) e^{-\lambda' x} U \begin{pmatrix}
\Phi(x, \nu_{jk}) \\
\Phi'(x, \nu_{jk})
\end{pmatrix},
\]

\[
\begin{pmatrix}
1/\psi_{jm}(x) \\
1/\psi_{mk}(x)
\end{pmatrix} = f(x) e^{\lambda'' x} V \begin{pmatrix}
\Phi(x, \mu_{jk}) \\
\Phi'(x, \mu_{jk})
\end{pmatrix},
\]

for an appropriate function \(f(x)\) and matrices \(U, V\) such that

\[ \lambda_{jk} + \lambda' + \lambda'' = 0 \quad \det U = c_{jk} \det V. \]

Now if \(A_{jm}\) and \(A_{mk}\) are also given by (37) we encounter restrictions on the possible parameters appearing:

**Theorem 3** Let \(A_{jk}, A_{jm}\) and \(A_{mk}\) have the form (37) and be related by (21). Then the constants determining these solutions are related by

\[ \nu_{jk} - \mu_{jk} = \nu_{jm} - \mu_{jm} = \nu_{mk} - \mu_{mk} \]

(39)

\[ \lambda_{jk} + \zeta(\nu_{jk}) - \zeta(\mu_{jk}) = \lambda_{jm} + \zeta(\nu_{jm}) - \zeta(\mu_{jm}) = \lambda_{mk} + \zeta(\nu_{mk}) - \zeta(\mu_{mk}) \]

(40)

and \(c_{jk} = \frac{\sigma(\nu_{jk})}{\sigma(\mu_{jk})} \tau_{jk}, c_{jm} = \frac{\sigma(\nu_{jm})}{\sigma(\mu_{jm})} \tau_{jm}, c_{mk} = \frac{\sigma(\nu_{mk})}{\sigma(\mu_{mk})} \tau_{mk}\) where the \(\tau\)'s satisfy

\[ \frac{\tau_{jk}}{\tau_{jm} \tau_{mk}} = \frac{\sigma(\nu_{jm} + \nu_{mk} - \nu_{jk})}{\sigma(\mu_{jm} + \mu_{mk} - \mu_{jk})}. \]

(41)

Letting \(\mu_{jk} - \nu_{jk} = c\) and \(\lambda_{jk} + \zeta(\nu_{jk}) - \zeta(\mu_{jk}) = \rho\) then

\[ A_{jk} = \tau_{jk} \frac{\sigma(\nu_{jk} - x)}{\sigma(c + \nu_{jk} - x)} e^{\rho x}, \]

(42)

and similarly

\[ A_{jm} = \tau_{jm} \frac{\sigma(\nu_{jm} - x)}{\sigma(c + \nu_{jm} - x)} e^{\rho x}, \quad A_{mk} = \tau_{mk} \frac{\sigma(\nu_{mk} - x)}{\sigma(c + \nu_{mk} - x)} e^{\rho x}. \]

Finally

\[ G_{jm}(x) = -\zeta(x - \mu_{jm}) + \zeta(x + \mu_{mk} - \mu_{jk}) + \text{const} \]

(43)

and similarly for \(G_{mk}(y)\) with the same constant appearing.
The proof of this theorem is rather lengthy, making repeated use of the elliptic function identities introduced in the previous section; it is given in appendix A. The first two relations (39) and (40) follow by equating poles and zeros amongst the various terms while the relation (41) comes from the determinantal constraint. The final constraint (43) arises by considering (22), which may be recast as
\[ G_{jm}(x) - G_{mk}(y) = \partial \ln \left( 1 - \frac{A_{jm}(x)A_{mk}(y)}{A_{jk}(x+y)} \right). \]
We remark that when \( A_{jm}(x) = c_2A_{mk}(x) \) (and so \( \nu_{jm} = \nu_{mk}, \mu_{jm} = \mu_{mk}, \lambda_{jm} = \lambda_{mk}, c_{jm} = c_2c_{mk} \)) several of these relations are immediately satisfied.

It is worth reflecting on this theorem. Given any three \( A_{jk}, A_{jm} \) and \( A_{mk} \) of the form (37) and connected via (21), the constants determining these functions are restricted. In particular, suppose every entry of the Lax matrix \( A \) is nonvanishing and \( G_{jm} - G_{mk} \neq 0 \) for every triple of distinct indices. Then the theorem holds for every triple \( A_{jk}, A_{jm} \) and \( A_{mk} \). Consideration of (43) shows that if this is to define a function \( G_{jm} \) then
\[ \mu_{jm} + \mu_{mk} - \mu_{jk} = \mu \tag{44} \]
for some fixed constant \( \mu \) and every distinct triple \( j, m, k \). Now (44) holds for every distinct triple if and only if (for each \( j, k \))
\[ \mu_{jk} = \mu + \mu_j - \mu_k \]
and similarly \( \nu_{jk} = \nu + \nu_j - \nu_k \). In this case \( \tau_{jk} = \sigma(\mu)/\sigma(\nu) \) and so
\[ A_{jk}(x) = \frac{\sigma(\mu)\sigma(\nu + \nu_j - \nu_k - x)}{\sigma(\nu)\sigma(\mu + \nu_j - \nu_k - x)} e^{\rho x}. \]
That is
\[ A_{jk}(x_j - x_k) = \frac{\Phi(x_j - \nu_j - (x_k - \nu_k); \nu)}{\Phi(x_j - \nu_j - (x_k - \nu_k); \mu)} e^{[\rho - \zeta(\nu) + \zeta(\mu)]x} \]
and we have recovered the relativistic Calogero-Moser model described in the last section. Now \( G_{jm} = G_{mk} \iff G_{jm} = G_{mk} = \text{constant} \). Thus we obtain the following ‘generic’ description of the relativistic Calogero-Moser model:

**Theorem 4** A Lax Pair of the form (2) for which the matrix \( A \) has no vanishing entries and for which the matrix \( G \) has nonconstant entries describes the relativistic Calogero-Moser model.

We have just considered the situation where every entry of \( A \) satisfying (24). We conclude the section by considering the opposite extreme where no entry does.
**Example** Here we adopt the ansatz

\[
A = \begin{pmatrix}
0 & a_1 & 0 & 0 & \ldots & 0 & 0 \\
1 & 0 & a_2 & 0 & 0 & 0 \\
1 & 1 & 0 & a_3 & 0 & 0 \\
\vdots \\
1 & 1 & 1 & 1 & 0 & a_{n-1} \\
1 & 1 & 1 & 1 & 1 & 0
\end{pmatrix}
\]

and where we shall determine the \(a_i's \neq 0\). We will work through the various cases determined by Lemma 1.

1. \(j < k - 1\). Then \(A_{jk} = 0\) and the only possible nonzero term in (11) is

\[
0 = \left| \begin{array}{cc}
A_{jj+1} & A_{j+1,j+2} \\
C_{jj+1} & C_{j+1,j+2}
\end{array} \right|.
\]

Thus for each \(j < n - 1\)

\[
\frac{C_{jj+1}}{A_{jj+1}} = \frac{C_{j+1,j+2}}{A_{j+1,j+2}} = \lambda
\]

and consequently for each \(j < n\)

\[
\frac{A'_{jj+1}}{A_{jj+1}} - G_{jj+1} = \lambda. \quad (45)
\]

2. \(j = k - 1\). In this case \(A_{jj+1} \neq 0\) and now if

(a) \(m < j\) then \(A_{mk} = 0\) and \(G_{jm} = G_{j+1m}\).

(b) \(j + 2 < m\) then \(G_{jm} = G_{j+1m}\).

(c) \(m = j + 2\) then \(G_{jj+2} - G_{j+1,j+2} + V_{j+1,j+2} = 0\) whence upon using (12)

\[
G_{jj+2} - G_{j+1,j+2} = \frac{A'_{j+1,j+2}}{1 - A_{j+1,j+2}}. \quad (46)
\]

3. \(k < j\). Then \(A_{jk} = 1\). Now (22) becomes

\[
(1 - A'_{jm}A_{mk})(G_{jm} - G_{mk}) = \left| \begin{array}{cc}
A_{jm} & A_{mk} \\
A'_{jm} & A'_{mk}
\end{array} \right|.
\]

and we find

(a) if \(m < k - 1\) or \(k < j < m - 1\) that \(G_{jm} = G_{mk}\),
(b) \( m = k - 1 \)

\[
G_{jk-1} - G_{k-1k} = \frac{A'_{k-1k}}{1 - A_{k-1k}}
\]  
(47)

(c) \( j = m - 1 \)

\[
G_{jj+1} - G_{j+1k} = -\frac{A'_{jj+1}}{1 - A_{jj+1}}
\]  
(48)

with no constraints arising when \( k < m < j \). Further, when \( k < j + 1 \), we have \( V_{jk} = 0 = G_{jk} - G_{kj} \) and so

\[
G_{jk} = G_{kj} \quad |j - k| > 1.
\]  
(49)

Now case (2a) tells us each column of the matrix \( G \) is constant below the diagonal while (2b) tells us each column above the superdiagonal is constant. Combining this information with (49) enables us to parameterize \( G \) as

\[
G = \begin{pmatrix}
0 & g_1 & 0 & 0 & \ldots & 0 & 0 \\
1 & 0 & g_2 & 0 & 0 & 0 \\
1 & 1 & 0 & g_3 & 0 & 0 \\
\vdots \\
1 & 1 & 1 & 1 & 0 & g_{n-1} \\
1 & 1 & 1 & 1 & 1 & 0
\end{pmatrix} + d \begin{pmatrix}
0 & 1 & 1 & 1 & \ldots & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 \\
\vdots \\
1 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 0
\end{pmatrix}
\]

where \( d \) is a constant. Comparison with (45)-(48) shows we are left with two equations,

\[
g_j = \frac{-a'_j}{1 - a_j} \equiv V'_{jj+1} \quad \frac{a'_j}{a_j} - g_j = \lambda + d,
\]

with solution

\[
a_j(x) = 1 - \frac{1}{1 + c_{jj+1}e^{(\lambda + d)x}}
\]

and \( g_j(x) = -(\lambda + d)a_j(x) \). Here \( c_{jj+1} \) is a constant that may be removed by shifting the \( x \)'s.

We have recovered the relativistic Toda lattice of [30]. Our construction has given the Lax pair \((b = 1, a = 0) \ L = \sqrt{\dot{q}_d} (I + A) \sqrt{\dot{q}_d} \) and

\[
M = (B.\dot{q})_d + \sqrt{\dot{q}_d} C \sqrt{\dot{q}_d} = -d L + (d \sum_{j=1}^{n} \dot{q}_j I) + \begin{pmatrix}
\frac{1}{2}g_1\dot{q}_2 & 0 \\
0 & \frac{1}{2}g_1\dot{q}_1 + \frac{1}{2}g_2\dot{q}_3 \\
0 & \ldots
\end{pmatrix}
\]
Upon defining a conjugate Lax pair $L_N, M_N$ by 
\[ L = \sqrt{q_d} L_N 1/\sqrt{q_d} \quad \text{and} \quad M_N = 1/\sqrt{q_d}(M + dL - (d \sum_{j=1}^n \dot{q}_j)I)\sqrt{q_d} + \sqrt{q_d}/\sqrt{q_d} \]
we obtain the Lax pair
\[
L_N = \begin{pmatrix}
\dot{q}_1 & a_1 \dot{q}_2 & 0 & \ldots & 0 \\
\dot{q}_1 & \dot{q}_2 & a_2 \dot{q}_3 & 0 \\
\dot{q}_1 & \dot{q}_2 & \dot{q}_3 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\dot{q}_1 & \dot{q}_2 & \dot{q}_3 & a_{n-1} \dot{q}_n \\
\dot{q}_1 & \dot{q}_2 & \dot{q}_3 & \dot{q}_n 
\end{pmatrix},
\]
\[
M_N = \begin{pmatrix}
g_1 \dot{q}_2 & -g_1 \dot{q}_2 & 0 & \ldots & 0 \\
g_2 \dot{q}_3 & g_2 \dot{q}_3 & -g_2 \dot{q}_3 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & -g_{n-1} \dot{q}_n \\
0 & 0 & 0 & 0 & 0 
\end{pmatrix}
\]
When $\lambda = 1$ and $d = 0$ this is the Lax pair of [8].

5 The Nonrelativistic case.

Our discussion has thus far focussed on the $b \neq 0$ case of our ansatz for the Lax-pair and this has yielded the relativistic Calogero-Moser and Toda systems. We shall now consider the case $b = 0$ and see that this includes the nonrelativistic limits of these systems. The nonrelativistic dynamics is less constrained than the relativistic situation and this reflects itself in a more complicated functional equation which we have not been able to solve in general. In this section we shall first obtain the relevant equation and show how it encompasses the nonrelativistic Calogero-Moser and Toda systems as well as that of Buchstaber and Perelomov [13]. Our approach leads to a new derivation of the Calogero-Moser model in which we also determine the various symmetry properties of the functions entering the ansatz. Recalling that the diagonal entries of $A, B \in \mathcal{M}_n$ vanish we begin with:

**Theorem 5** Equation (17) yields the functional equation
\[
\sum_l A_{jk}(B_{jl} - B_{kl}) + A'_{jl}A_{lk} - A_{jl}A'_{lk} = 0.
\]
Further, each of the functions $\Psi_{jlk} \equiv A_{jk}(B_{jl} - B_{kl}) + A'_{jl}A_{lk} - A_{jl}A'_{lk}$ appearing as the terms of this sum, depend only on the combination $q_j - q_k$. Thus with $x = q_j - q_l$ and $y = q_l - q_k$ we have

$$\Psi_{jlk}(x + y) = A_{jk}(x + y)(B_{jl}(x) - B_{kl}(-y)) + A'_{jl}(x)A_{lk}(y) - A_{jl}(x)A'_{lk}(y).$$  \quad (51)$$

**Proof.** In order for (17) to remain true for all $\dot{q}$ we must have

$$H_{jk} = 0 \quad (52)$$

and therefore $C_{jk} = A'_{jk}$. In fact this is the $b \to 0$ limit of equations (9) and (10). With these simplifications (17) then becomes

$$A_{jk}(B_{jk} - B_{kj}) + \sum_{i \neq j, k} A_{jk}(B_{ji} - B_{ik}) + A'_{jl}A_{lk} - A_{jl}A'_{lk} = 0,$$

and with the conventions stated above we have (50).

Now the only dependence on $q_l$ ($l \neq j, k$) in (50) comes from the $l$-th term of the sum. Thus upon taking the derivative $\partial_l$ of this equation we see that

$$0 = \partial_l(A_{jk}(B_{jl} - B_{kl}) + A'_{jl}A_{lk} - A_{jl}A'_{lk}) = -(\partial_j + \partial_k)(A_{jk}(B_{jl} - B_{kl}) + A'_{jl}A_{lk} - A_{jl}A'_{lk})$$

from which we may conclude that $\Psi_{jlk} = \Psi_{jlk}(q_j - q_k)$ as stated. We remark that the $b \to 0$ limit of equation (11) divided by $b$ gives the quantity $\Psi_{jmk}$. \hfill $\square$

**Corollary 2** Solutions of (57) satisfy the functional equation (3) with $N = 2$.

**Proof.** Upon interchanging $x$ and $y$ in (51) and subtracting we obtain

$$A_{jk}(x + y) \begin{vmatrix} B_{jl}(x) + B_{kl}(-x) & B_{jl}(y) + B_{kl}(-y) \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} A'_{jl}(x) & A'_{jl}(y) \\ A_{lk}(x) & A_{lk}(y) \end{vmatrix} = 0 \quad (53)$$

Simple rearrangement of this gives equations of the form (54) or of (3) with $N = 2$. \hfill $\square$

Before further investigating (50) it is instructive to see how the nonrelativistic Calogero-Moser and Toda systems arise in this context and present a new example.
Example The Calogero-Moser reduction follows when we assume the functions $A_{jk}$ and $B_{jk}$ don't depend on the indices $j, k$. Upon setting $A_{jk}(q_j - q_k) = A(q_j - q_k)$ and $B_{jk}(q_j - q_k) = B(q_j - q_k)$ the function $\Psi_{jlk}$ defined above takes the form

$$\Psi_{jlk} \equiv A(q_j - q_k)(B(q_j - q_l) - B(q_l - q_l)) + A'(q_j - q_l)A(q_l - q_k) - A(q_j - q_l)A'(q_l - q_k).$$

Now the lemma asserts that $\Psi_{jlk}$ is independent of $q_l$ ($l \neq j, k$) and so (for example by setting $q_l = 0 = q_{l'}$) we see that $\Psi_{jlk} = \Psi_{jl'k}$ for $l, l' \neq j, k$. Thus (50) takes the form

$$0 = A(x + y)(B(x + y) - B(-x - y)) + (n - 2)(A(x + y)[B(x) - B(-y)] + A'(x)A(y) - A(x)A'(y)), \quad (56)$$

where $x = q_j$ and $y = -q_k$. Letting $B_o$ denote the odd part of $B$, then upon interchanging $x$ and $y$ in (56) and adding leads to

$$0 = A(x + y)(B_o(x + y) + \frac{(n - 2)}{2}[B_o(x) + B_o(y)]).$$

Now the functional equation ($n > 1$)

$$0 = B_o(x + y) + \frac{(n - 2)}{2}[B_o(x) + B_o(y)]$$

only has $B_o = 0$ as a solution and so if $A(x) \neq 0$ we may assume $B$ in (56) is an even function. In this case the leading term vanishes and we are left with

$$0 = A(x + y)[B(x) - B(y)] + A'(x)A(y) - A(x)A'(y)$$

which is the functional equation (11) obtained by Calogero [16] and that has been variously solved [16, 26, 27, 7, 11]. Observe that by setting $y = -x$ in this equation and using the fact that $B(x)$ is an even function we may deduce that $A(x)$ is an odd function and so we have obtained the symmetries of $A$ and $B$ normally [16, 26] imposed as constraints on the Lax-pair.

Example The Toda reduction follows by assuming $B_{jk}$ and $\Psi_{jlk}$ vanish for all possible distinct indices. From this we deduce that

$$A_{jk}(x) = \alpha_{jk} e^{\lambda_{jk}x}$$

for some constants $\alpha_{jk}$ and $\lambda_{jk}$. The vanishing of $\Psi_{jlk}$ relates these constants (for $j \neq l \neq k$) by

$$(\lambda_{jl} - \lambda_{lk})\alpha_{jl}\alpha_{lk} = 0 \quad (57)$$
while the nonvanishing of an interaction $V_{jk}$ means (using (15)) that

$$(\lambda_{jk} - \lambda_{kj})\alpha_{jk}\alpha_{kj} \neq 0. \tag{58}$$

We thus seek solutions of (57) and (58). Observe that if $V_{jl}$ and $V_{lk}$ are both nonvanishing then $\lambda_{jl} = \lambda_{lk} \neq \lambda_{kl}$. Consequently if $V_{ij}$, $V_{jl}$ and $V_{lk}$ are nonvanishing then (for example) $V_{jk} = 0$. To see this suppose to the contrary and note that $\Psi_{ijk} = 0$ means $\lambda_{ij} = \lambda_{jk}$, while $\Psi_{jkl} = 0$ means $\lambda_{jk} = \lambda_{kl}$ and so $\lambda_{ij} = \lambda_{kl}$. However our previous observation shows $\lambda_{ij} = \ldots = \lambda_{jk} \neq \lambda_{kl}$, yielding a contradiction. Therefore $V_{jk} = 0$. Our ansatz means we can only achieve (possibly cyclic) chains of nonvanishing interactions, $V_{12} = V_{23} = \ldots = V_{n-1n}$ ($= V_{n1}$). Setting $\alpha_{i+1} = 1 = \alpha_{i+1}$ and $\lambda_{i+1} = -\lambda_{i+1}$ say, with all other $\alpha_{ij}$ and $\lambda_{ij}$ vanishing, yields the (periodic) Toda lattice. Equally the work of [20] shows how to obtain the Toda systems as limits of Calogero-Moser models.

**Example** By taking

$$A_{jk}(x) = \Phi(x; \nu) a_k, \quad B_{jk}(x) = \wp(x) a_k \tag{59}$$

where the $a_k$ are constants, we find

$$\Psi_{jkl}(x + y) = \Phi(x + y; \nu) (\wp(x) - \wp(y)) a_k a_l - \left| \begin{array}{cc} \Phi(x; \nu) & \Phi(y; \nu) \\ \Phi(x; \nu)' & \Phi(y; \nu)' \end{array} \right| a_k a_l = 0.$$

Here we have used (36) to show the vanishing of $\Psi_{jkl}$. By Theorem 5 we therfore have a new Lax pair associated with the potentials (2)

$$V_{jk}(x) = a_ja_k \wp'(x).$$

When $a_j = a_k$ this yields the usual type IV Calogero-Moser potential.

Although we cannot as yet solve (50) or (51) in general, we can say a little more according to whether $A_{jk}(B_{jl} - B_{kl}) = 0$ or $A_{jk} \neq 0$. For the first we note:

**Lemma 4** The functional equation

$$F(x + y) = \wp(x)\wp'(y) - \wp'(x)\wp(y) \tag{60}$$

has, up to symmetries, the solution $F(x) = c_1c_3 \exp(\lambda x), \wp(x) = c_1 \exp(\lambda x), \wp(x) = (c_2 + c_3x) \exp(\lambda x)$.

This equation is a particular case of the equation

$$f(x + y) + g(x - y) = \sum_{j=1}^{n} f_j(x)g_j(y)$$

which has a long history [3]; the solution, along standard lines, is given in Appendix B. In our context it yields
Corollary 3 If $A_{jk}(B_{jl} - B_{kl}) = 0$ the solution to (51) takes the form $\Psi_{jk}(x) = (c_1c_3 - c_2c_4)\exp(\lambda x)$, $A_{jl}(x) = (c_2 + c_3x)\exp(\lambda x)$ and $A_{lk}(y) = (c_1 + c_4y)\exp(\lambda y)$, where $c_3c_4 = 0$.

Remark: The Toda reduction was a particular example of this corollary. The various constants appearing are not all independent, but related by (50).

We now suppose that $A_{jk} \neq 0$. Upon setting

$$\Phi_{jk} = \sum_m A'_{jm}A_{mk} - A_{jm}A'_{mk}$$

for $j \neq k$ and $\Phi_{jj} = 0$ we may rewrite (50) as

$$\sum_m (B_{jm} - B_{km}) + \Phi_{jk} = 0. \quad (61)$$

Lemma 5 For each $j, k$ and $l$ for which $A_{jk}, A_{kl}$ and $A_{lj}$ are nonvanishing then (50) yields the functional equation

$$\Phi_{jk} + \Phi_{kl} + \Phi_{lj} = 0 \quad (62)$$

and consequently

$$\Phi_{jk} + \Phi_{kj} = 0 \quad (63)$$

Proof. This follows from (61) as

$$B_{jm} - B_{km} + B_{km} - B_{lm} + B_{lm} - B_{jm} = 0.$$

Example In the case when $n = 3$, equation (52) reduces to the functional equation of Buchstaber and Perelomov [13],

$$(f(x) + g(y) + h(z))^2 = F(x) + G(y) + H(z) \quad x + y + z = 0. \quad (64)$$

The equation is related to the factorization of a three-body quantum mechanical ground-state wavefunction [31, 32, 17]. In this case there is a unique $m \neq j, k$ such that

$$\Phi_{jk} = \frac{A'_{jm}A_{mk} - A_{jm}A'_{mk}}{A_{jk}}.$$

Set $x = q_2 - q_3, y = q_3 - q_1, z = q_1 - q_2$ (and so $x + y + z = 0$) and

$$f(x) = -A_{23}(x)A_{32}(-x), \quad g(y) = -A_{31}(y)A_{13}(-y), \quad h(z) = -A_{12}(z)A_{21}(-z).$$
Then
\[ \Phi_{21} + \Phi_{13} + \Phi_{32} = 0 \]
leads to
\[ A_{23}A_{31}(A_{13}A_{32}' - A_{13}'A_{32}) + A_{31}A_{12}(A_{21}A_{13}' - A_{21}'A_{13}) + A_{12}A_{23}(A_{32}A_{21}' - A_{32}'A_{21}) = 0 \]
and consequently
\[ g(y)A_{23}A_{32}' - f(x)A_{31}A_{13}' + h(z)A_{31}A_{13}' - g(y)A_{12}A_{21}' + f(x)A_{12}A_{21}' - h(z)A_{23}A_{32}' = 0. \] (65)
Similarly from \( \Phi_{12} + \Phi_{23} + \Phi_{31} = 0 \) we obtain
\[ f(x)A_{13}A_{31}' - g(y)A_{32}A_{23}' + h(z)A_{32}A_{23}' - f(x)A_{21}A_{12}' + g(y)A_{21}A_{12}' - h(z)A_{13}A_{31}' = 0. \] (66)
Now
\[ f'(x) = A_{23}A_{32}' - A_{32}A_{23}', \quad g'(y) = A_{31}A_{13}' - A_{13}A_{31}', \quad h'(z) = A_{12}A_{21}' - A_{21}A_{12}, \]
so upon adding (65) and (66) we obtain
\[ g(y)f'(x) - f(x)g'(y) + h(z)g'(y) - g(y)h'(z) + f(x)h'(z) - h(z)f'(x) = 0. \] (67)
Equation (67) may be rewritten as
\[
\begin{vmatrix}
1 & 1 & 1 \\
f(x) & g(y) & h(z) \\
f'(x) & g'(y) & h'(z)
\end{vmatrix} = 0, \quad x + y + z = 0
\] (68)
which is the equation studied by Buchstaber and Perelomov. The solutions to (68) are called nondegenerate if each of \( f(x), \ g(y) \) and \( h(z) \) have poles lying in some finite domain of the complex plane. Degenerate solutions may then be obtained from these. The nondegenerate solutions to (68) are given by
\[ f(x) = \alpha \varphi(x - a_1) + \beta, \quad g(y) = \alpha \varphi(y - a_2) + \beta, \quad h(z) = \alpha \varphi(z - a_3) + \beta, \] (69)
with \( a_1 + a_2 + a_3 = 0. \)
We now observe that, although we have not yet specified \( A_{12}, A_{23} \) and \( A_{31} \), we have in fact obtained the interactions in the present situation. We have
\[ V_{jk} = -(A_{jk}A_{kj})' \]
and so
\[ V_{jk}(q_j - q_k) = -\alpha \varphi'(q_j - q_k + a_j - a_k). \] (70)
Thus our functional equation determines the interaction for us. As for the Lax-pair we may simply choose $A_{jk}(x) = -A_{kj}(-x)$ or some other form that suits our purpose. Using the addition properties of the elliptic functions another choice for $A_{jk}(x)$ could be

$$A_{jk}(x) = \sqrt{\frac{\alpha\sigma(b - x + \lambda_j - \lambda_k)}{\sigma(b)\sigma(x - \lambda_j + \lambda_k)}}$$

where $\alpha(b) = -\beta$ and $a_i = \lambda_j - \lambda_k$ for cyclic $i, j, k$.

6 Discussion

This paper has introduced a new ansatz (6) for a Lax pair describing systems of particles on the line interacting via pairwise forces (2). Unlike existing ansatz we allow these forces to depend in principle on the particle pair, and so the one ansatz encompasses for example those of the Calogero-Moser and Toda systems within a unified framework. A consequence of allowing varying pairwise interactions is that the consistency equations for the Lax pair now become a system of functional equations rather than a single functional equation. Our approach has been to first study the constituent functional equations, of interest in their own right, and then to examine the contraints imposed by the system of which they are a part.

Two quite interesting functional equations (3), (4) arise in this manner. The first, which arises when $b \neq 0$, has a large group of symmetries acting on it and we have been able to give its general analytic solution with appropriate orbits corresponding to the relativistic Calogero-Moser and Toda interactions. It is this large symmetry group of (3) that enables us to relate previously distinct functional equations and different physical models. We remark that a particular case of this equation has recently arisen in the work [12] (see their equation (13) and Lemma 10) which examines the connection between functional equations and Dunkl operators.

Unfortunately we have not been able to say as much about the functional equation (4) or the associated system of equations when $b = 0$. Certainly the $b \to 0$ limit of our general solution yields a $b = 0$ solution, corresponding to an appropriate nonrelativistic limit, but the nonrelativistic equations are less rigid. Similarly we note that both (3), (55) as well as the functional equations satisfied by Baker-Akhiezer functions [14] are particular cases of (5).

The final step in our approach has been to examine the contraints imposed by the system of functional equations on the parameters appearing in the solutions to (3) and (4). The constraints for the relativistic system were encapsulated in Theorem 3. Although we have a conceptually straightforward unification of
various ansatz for Lax pairs, this stage of our approach is the most tedious as it can often involve case by case analysis. We plan to return to the equation (41) in a later work. Finally we have shown how the examples of the known relativistic and nonrelativistic Toda and Calogero-Moser models arise in our approach as well as introducing a new system.

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A Proof of Theorem 2

In proving this theorem we shall consider the constraints imposed by (24) and (25) separately. In both cases we proceed by first finding relations amongst the constants appearing in (37) by equating the poles and zeros of the various terms given by our first theorem. This determines the functions $\psi_{jm}$ and $\psi_{mk}$ up to the action of the function $f(x)$ and an exponential. By comparing with (22) we then determine $G_{jm}$ and $G_{mk}$ up to constants.

A.1 Consistency for (24)

We begin with (24). In this case the theorem says

$$A_{jm}(x)/\psi_{jm}(x) = f(x)e^{-\lambda x}(u_{11}\Phi(x;\nu_{jk}) + u_{12}\Phi'(x;\nu_{jk}))$$

and

$$1/\psi_{jm}(x) = f(x)e^{\lambda x}(v_{11}\Phi(x;\mu_{jk}) + v_{12}\Phi'(x;\mu_{jk})).$$

The ratio of these two equations when

$$A_{jm}(x) = c_{jm}\frac{\Phi(x;\nu_{jm})}{\Phi(x;\mu_{jm})}e^{\lambda_{jm}x}$$

yields

$$A_{jm}(x) = c_{jm}\frac{\Phi(x;\nu_{jm})}{\Phi(x;\mu_{jm})}e^{\lambda_{jm}x} = \frac{u_{11}\Phi(x;\nu_{jk}) + u_{12}\Phi'(x;\nu_{jk})}{v_{11}\Phi(x;\mu_{jk}) + v_{12}\Phi'(x;\mu_{jk})}e^{\lambda_{jk}x}$$

$$= \frac{u_{12}\Phi'(x;\nu_{jk})}{v_{12}\Phi'(x;\mu_{jk})} \left[ \frac{u_{11}/u_{12} + \zeta(\nu_{jk}) - \zeta(x) - \zeta(\nu_{jk} - x)}{v_{11}/v_{12} + \zeta(\mu_{jk}) - \zeta(x) - \zeta(\mu_{jk} - x)} \right] e^{\lambda_{jk}x}. \quad (72)$$

(We will assume $u_{12}$ and $v_{12}$ are nonvanishing and later see that this is this case. Certainly by considering the behaviour of (72) as $x \to 0$ we see $u_{12} = 0 \Leftrightarrow v_{12} = 0 \Leftrightarrow A_{jm}(x) = c A_{jk}(x) e^{\lambda x}.$) Now the left hand side has a zero at $\nu_{jm}$ and pole
at \(\mu_{jm}\). Equating these with the right hand side (for \(\nu_{jm} \neq \nu_{jk}\) and \(\mu_{jm} \neq \mu_{jk}\) shows
\[
\frac{u_{11}}{u_{12}} = \zeta(\nu_{jk} - \nu_{jm}) - \zeta(\nu_{jk}) + \zeta(\nu_{jm})
\]
\[
\frac{v_{11}}{v_{12}} = \zeta(\mu_{jk} - \mu_{jm}) - \zeta(\mu_{jk}) + \zeta(\mu_{jm}).
\]
Thus, after making use of (34),
\[
\frac{u_{11}}{u_{12}} + \zeta(\nu_{jk}) - \zeta(x) - \zeta(\nu_{jk} - x) = \frac{\sigma(\nu_{jk})\sigma(x - \nu_{jm})\sigma(x - \nu_{jk} + \nu_{jm})}{\sigma(\nu_{jm})\sigma(\nu_{jk} - \nu_{jm})\sigma(x - \nu_{jk})\sigma(x)} \tag{73}
\]
and
\[
\frac{v_{11}}{v_{12}} + \zeta(\mu_{jk}) - \zeta(x) - \zeta(\mu_{jk} - x) = \frac{\sigma(\mu_{jk})\sigma(x - \mu_{jm})\sigma(x - \mu_{jk} + \mu_{jm})}{\sigma(\mu_{jm})\sigma(\mu_{jk} - \mu_{jm})\sigma(x - \mu_{jk})\sigma(x)} \tag{74}
\]
Utilising (73) and (74) in (72) now gives
\[
c_{jm}e^{(\lambda_{jm} + \zeta(\nu_{jm}) - \zeta(\mu_{jm}))x} = \frac{u_{12} \sigma(x - \nu_{jk} + \nu_{jm})\sigma(\mu_{jk} - \mu_{jm})}{v_{12} \sigma(x - \mu_{jk} + \mu_{jm})\sigma(v_{jk} - \nu_{jm})} e^{(\lambda_{jk} + \zeta(\nu_{jk}) - \zeta(\mu_{jk}))x}
\]
from which we deduce
\[
\nu_{jk} - \nu_{jm} = \mu_{jk} - \mu_{jm} \tag{75}
\]
\[
\lambda_{jm} + \zeta(\nu_{jm}) - \zeta(\mu_{jm}) = \lambda_{jk} + \zeta(\nu_{jk}) - \zeta(\mu_{jk}) \tag{76}
\]
\[
c_{jm}v_{12} = u_{12}. \tag{77}
\]
\[\]
From (76) and our expressions for \(\frac{u_{11}}{u_{12}}\) and \(\frac{v_{11}}{v_{12}}\) we find
\[
\lambda_{jm} - \lambda_{jk} = \frac{v_{11}}{v_{12}} - \frac{u_{11}}{u_{12}}. \tag{78}
\]
The same considerations now applied to \(A_{mk}(x)/\psi_{mk}(x)\) rather than \(A_{jm}(x)/\psi_{jm}(x)\) similarly show
\[
\frac{u_{21}}{u_{22}} = \zeta(\nu_{jk} - \nu_{mk}) - \zeta(\nu_{jk}) + \zeta(\nu_{mk})
\]
\[
\frac{v_{21}}{v_{22}} = \zeta(\mu_{jk} - \mu_{mk}) - \zeta(\mu_{jk}) + \zeta(\mu_{mk})
\]
\[
\nu_{jk} - \nu_{mk} = \mu_{jk} - \mu_{mk} \tag{79}
\]
\[
\lambda_{mk} + \zeta(\nu_{mk}) - \zeta(\mu_{mk}) = \lambda_{jk} + \zeta(\nu_{jk}) - \zeta(\mu_{jk}) \tag{80}
\]
\[
c_{mk}v_{22} = u_{22} \tag{81}
\]
\[
\lambda_{mk} - \lambda_{jk} = \frac{v_{21}}{v_{22}} - \frac{u_{21}}{u_{22}}. \tag{82}
\]
Combining these relations with (75) and (76) thus proves (39) and (40) for the case being examined.
We have yet to impose the constraint \( \det U = c_{jk} \det V \). Now

\[
c_{jk} = \frac{\det U}{\det V} = \frac{u_{12}u_{22}(\frac{u_{14}}{u_{12}} - \frac{u_{11}}{u_{22}})}{v_{12}v_{22}(\frac{v_{14}}{v_{12}} - \frac{v_{11}}{v_{22}})}
\]

\[
c_{jk} = c_{jm}c_{mk} \left( \frac{\zeta(\nu_{jk} - \nu_{jm}) - \zeta(\nu_{jk}) + \zeta(\nu_{jm}) - \zeta(\nu_{jm} - \nu_{mk}) + \zeta(\nu_{jm})}{\zeta(\mu_{jk} - \mu_{jm}) - \zeta(\mu_{jk}) + \zeta(\mu_{jm}) - \zeta(\mu_{jm} - \mu_{mk}) + \zeta(\mu_{jk})} \right)
\]

from which (41) follows. Substituting these results immediately yields (42).

After simplifying and again using (34) we obtain

\[
\frac{c_{jk}}{c_{jm}c_{mk}} = \frac{\sigma(\nu_{jk}) \sigma(\mu_{jm}) \sigma(\mu_{mk}) \sigma(\nu_{jm} + \nu_{mk} - \nu_{jk})}{\sigma(\mu_{jk}) \sigma(\nu_{jm}) \sigma(\nu_{jm}) \sigma(\mu_{jm} + \mu_{mk} - \mu_{jk})}.
\] (83)

At this stage we have obtained

\[
1/\psi_{jm}(x) = -v_{12} f(x)e^{[\lambda'' + \zeta(\mu_{jk})]x} \frac{\sigma(x - \mu_{jm})\sigma(x - \mu_{jm} + \mu_{jm})}{\sigma(x)\sigma(\mu_{jm})\sigma(\mu_{jm} - \mu_{jk})}
\]

with a similar expression holding for \( 1/\psi_{mk} \) and the desired expressions for \( A_{jm} \), \( A_{mk} \) and \( A_{jk} \).

The final constraints arise by considering (22) which may be recast as

\[
G_{jm}(x) - G_{mk}(y) = \partial \ln \left( 1 - \frac{A_{jm}(x)A_{mk}(y)}{A_{jk}(x + y)} \right).
\]

Now each side of this equation may be separately calculated and on comparison we find our last constraint. Using the form of \( A_{jm} \), \( A_{mk} \) and \( A_{jk} \) given by (42) we obtain for the left hand side

\[
\zeta(x + \mu_{mk} - \mu_{jk}) - \zeta(x - \mu_{jm}) - \zeta(y + \mu_{jm} - \mu_{jk}) + \zeta(y - \mu_{mk}),
\]

where substantial use has been made of (34). On the other hand, determining \( G_{jm}(x) \) directly from our expressions for \( \psi_{jm} \) yields

\[
G_{jm}(x) = -\zeta(x - \mu_{jm}) - \zeta(x + \mu_{jm} - \mu_{jk}) - F(x),
\]

where \( F(x) \) encodes the remaining functional dependence of \( \psi_{jm} \). Comparison shows

\[
F(x) = -\zeta(x + \mu_{jm} - \mu_{jk}) - \zeta(x + \mu_{mk} - \mu_{jk}) + const
\]

and so we have the final relation (43)

\[
G_{jm}(x) = -\zeta(x - \mu_{jm}) + \zeta(x + \mu_{mk} - \mu_{jk}) + const.
\]
We may for example choose
\[ f(x) = \frac{\sigma(\mu_{jk} - \mu_{jm})\sigma(\mu_{jk} - \mu_{mk})\sigma^2(x)}{\sigma(x - \mu_{jk} + \mu_{jm})\sigma(x - \mu_{jk} + \mu_{mk})}. \]

Still \( f(x) \) is only determined up to a constant multiple of an exponential. This gives
\[ \frac{1}{\psi_{jm}(x)} = v_{12} \frac{\sigma(x - \mu_{jm})\sigma(\mu_{jm} - \bar{\mu})}{\sigma(\mu_{jm})\sigma(x + \bar{\mu} - \mu_{jm})} e^{\kappa x} \]
where we have set \( \bar{\mu} = -\mu_{jk} + \mu_{jm} + \mu_{mk} \) and \( \kappa \) is an arbitrary constant. A similar expression holds for \( \frac{1}{\psi_{mk}(x)} \).

### A.2 Consistency for (25)

We next consider the constraints arising from the consistency of (22) when again \( A_{jk}, A_{jm} \) and \( A_{mk} \) are given by (37) but now \( A_{jm}(x) = c_2 A_{mk}(x) \). The latter of course means \( \nu_{jm} = \nu_{mk}, \mu_{jm} = \mu_{mk}, \lambda_{jm} = \lambda_{mk} \) and \( c_{jm} = c_2 c_{mk} \). We now must find a function \( f(x) \), matrices \( U, V \) and \( \lambda', \lambda'' \) such that
\[
\begin{pmatrix} c_2 A_{mk}(x) \\ C_{mk}(x) \end{pmatrix} = f(x)e^{-\lambda'x} U \begin{pmatrix} \Phi(x; \nu_{jk}) \\ \Phi'(x; \nu_{jk}) \end{pmatrix},
\]
\[
\begin{pmatrix} G_{mk}(x) \\ 1 \end{pmatrix} = f(x)e^{\lambda''x} V \begin{pmatrix} \Phi(x; \mu_{jk}) \\ \Phi'(x, \mu_{jk}) \end{pmatrix}.
\]

We proceed in much the same manner as in the previous case and accordingly we will be less detailed. Now
\[
\frac{c_2 A_{mk}(x)}{1} = \frac{e^{-(\lambda' + \lambda'')x} u_{11} \Phi(x; \nu_{jk}) + u_{12} \Phi'(x; \nu_{jk})}{v_{21} \Phi(x; \mu_{jk}) + v_{22} \Phi'(x; \mu_{jk})} = \frac{u_{12} \Phi(x; \nu_{jk})}{v_{22} \Phi(x; \mu_{jk})} \left[ \frac{u_{11}/u_{12} + \zeta(\nu_{jk}) - \zeta(x) - \zeta(\nu_{jk} - x)}{v_{21}/v_{22} + \zeta(\mu_{jk}) - \zeta(x) - \zeta(\mu_{jk} - x)} \right] e^{\lambda_{jk} x}.
\]

Again a comparison of zeros and poles shows
\[
\begin{align*}
\frac{u_{11}}{u_{12}} &= \zeta(\nu_{jk} - \nu_{mk}) - \zeta(\nu_{jk}) + \zeta(\nu_{mk}) \\
\frac{v_{21}}{v_{22}} &= \zeta(\mu_{jk} - \mu_{mk}) - \zeta(\mu_{jk}) + \zeta(\mu_{mk}) \\
\nu_{jk} - \nu_{mk} &= \mu_{jk} - \mu_{mk} \\
\lambda_{mk} + \zeta(\nu_{mk}) - \zeta(\mu_{mk}) &= \lambda_{jk} + \zeta(\nu_{jk}) - \zeta(\mu_{jk}) \\
c_2 c_{mk} v_{22} &= u_{12} \\
\lambda_{mk} - \lambda_{jk} &= \frac{v_{21}}{v_{22}} - \frac{u_{21}}{u_{22}}.
\end{align*}
\]
These are the exact analogues of our earlier equations. In addition we have using the definition of $C$ that

$$\frac{C_{mk}}{A_{mk}} = c_2 \frac{u_{21} \Phi(x; \nu_{jk}) + u_{22} \Phi'(x; \nu_{jk})}{u_{11} \Phi(x; \nu_{jk}) + u_{12} \Phi'(x; \nu_{jk})} = \frac{A'_{mk}}{A_{mk}} - G_{mk}$$

as well as

$$\frac{C_{mk}}{1} = \frac{v_{11} \Phi(x; \mu_{jk}) + v_{12} \Phi'(x; \mu_{jk})}{v_{21} \Phi(x; \mu_{jk}) + v_{22} \Phi'(x; \mu_{jk})}.$$ 

Upon using substituting the form of $A_{mk}$ these equations may be rearranged to give

$$c_2 \frac{u_{21} + u_{22}[\zeta(\nu_{jk}) - \zeta(x) - \zeta(\nu_{jk} - x)]}{u_{11} + u_{12}[\zeta(\nu_{mk}) - \zeta(x) - \zeta(\nu_{mk} - x)]} + \frac{v_{11} + v_{12}[\zeta(\mu_{jk}) - \zeta(x) - \zeta(\mu_{jk} - x)]}{v_{21} + v_{22}[\zeta(\mu_{mk}) - \zeta(x) - \zeta(\mu_{mk} - x)]} - \lambda_{mk}
= \frac{\sigma(x)\sigma(\mu_{mk} - \nu_{mk})\sigma(\mu_{mk} + \nu_{mk} - x)}{\sigma(\nu_{mk})\sigma(x - \nu_{mk})\sigma(\mu_{mk} - x)}.$$  \hspace{1cm} (84)

As $x \to 0$ we see

$$c_2 \frac{u_{22}}{u_{12}} + \frac{v_{12}}{v_{22}} - \lambda_{mk} = 0.$$

Upon making use of this and further simplifying (84) yields

$$c_2 \frac{u_{22}}{u_{12}} \left( \frac{u_{21}}{u_{22}} + \zeta(\nu_{jk}) - \zeta(\nu_{mk} - \nu_{mk} - \nu_{jk}) \right) \frac{\sigma(\nu_{mk})\sigma(\nu_{mk} - \nu_{mk})\sigma(x - \nu_{jk})}{\sigma(\nu_{mk})\sigma(x - \nu_{mk})\sigma(x + \nu_{mk} - \nu_{jk})} + \frac{v_{12}}{v_{22}} \left( \frac{v_{11}}{v_{12}} + \zeta(\mu_{jk}) - \zeta(\mu_{mk} - \mu_{mk} - \mu_{jk}) \right)\frac{\sigma(\mu_{mk})\sigma(\mu_{mk} - \mu_{mk})\sigma(x - \mu_{jk})}{\sigma(\mu_{mk})\sigma(x - \mu_{mk})\sigma(x + \mu_{mk} - \mu_{jk})}
= \frac{\sigma(\mu_{mk} - \nu_{mk})\sigma(\mu_{mk} + \nu_{mk} - x)}{\sigma(\nu_{mk})\sigma(x - \nu_{mk})\sigma(\mu_{mk})\sigma(x - \mu_{mk})}.$$

As $x \to \mu_{jk}$ we find

$$c_2 \frac{u_{22}}{u_{12}} \frac{u_{21}}{u_{22}} \frac{\sigma(\nu_{mk})\sigma(\nu_{mk} - \nu_{mk})}{\sigma(\nu_{mk})\sigma(2\nu_{mk} - \nu_{jk})} = -1$$

and as $x \to \nu_{jk}$ we obtain

$$\frac{v_{12}}{v_{22}} \left( \frac{v_{11}}{v_{12}} - \frac{v_{21}}{v_{22}} \right) \frac{\sigma(\mu_{mk})\sigma(\mu_{mk} - \mu_{mk})}{\sigma(\mu_{jk})\sigma(2\mu_{mk} - \mu_{jk})} = 1.$$

No further constraints are imposed on the matrices $U$ and $V$. The ratio of these last two equations (taking into account that $\det U = c_{jk} \det V$) then yields

$$\frac{c_{jk}}{c_2 c_{mk}^2} = \frac{\sigma(\nu_{jk})\sigma(\mu_{mk})\sigma(2\nu_{mk} - \nu_{jk})}{\sigma(\mu_{jk})\sigma(\nu_{mk})\sigma(2\mu_{mk} - \mu_{jk})}$$

which is just the $\nu_{jm} \to \nu_{mk}$ limit of our previous result and we again obtain the relations stated in the theorem by similar analysis.
B Proof of Lemma 4

Differentiating (60) with respect to $y$ yields
$$F'(x + y) = \phi(x)\psi''(y) - \phi'(x)\psi'(y),$$
and so upon letting $y = 0$ we must solve the two equations

\begin{align*}
F(x) &= \phi(x)\psi'(0) - \phi'(x)\psi(0), \\
F'(x) &= \phi(x)\psi''(0) - \phi'(x)\psi'(0).
\end{align*}

Now either $F(x) \not\equiv 0$ or $F(x) \equiv 0$ and $\phi(x) = c_1 \exp(\lambda x)$, and $\psi(x) = c_2 \exp(\lambda x)$. In the former case (perhaps by translating $y$ if necessary) either $\psi(0) \neq 0$ or $\psi'(0) \neq 0$ and we consider these two cases separately.

If $\psi(0) \neq 0$ then utilising the group of symmetries of the functional equation we may set $\psi(0) = 1$ and $\psi'(0) = 0$. From (85) and (86)

\begin{align*}
F(x) &= -\phi'(x) \\
F'(x) &= \psi''(0)\phi(x)
\end{align*}

and so
$$\phi''(x) = \lambda^2 \phi(x)$$
where $\lambda^2 = -\psi''(0)$. Thus $\phi(x) = \tilde{c}_1 \exp(\lambda x) + \tilde{c}_2 \exp(-\lambda x)$ and $F(x) = -\lambda(\tilde{c}_1 \exp(\lambda x) - \tilde{c}_2 \exp(-\lambda x))$. Therefore

\begin{align*}
F(x + y) &= -\lambda\tilde{c}_1 \exp \lambda(x + y) + \lambda\tilde{c}_2 \exp -\lambda(x + y) \\
&= \tilde{c}_1 \exp(\lambda x)(\psi'(y) - \lambda\psi(y)) + \tilde{c}_2 \exp(-\lambda x)(\psi'(y) + \lambda\psi(y))
\end{align*}

which upon rearranging yields

\begin{align*}
0 &= \tilde{c}_1 \exp(\lambda x)(\psi'(y) - \lambda\psi(y) + \lambda \exp(\lambda y)) + \tilde{c}_2 \exp(-\lambda x)(\psi'(y) + \lambda\psi(y) - \lambda \exp(\lambda y)).
\end{align*}

Thus either $\tilde{c}_1 = 0$ or $\tilde{c}_2 = 0$. The latter leads to
$$0 = \psi'(y) - \lambda\psi(y) + \lambda \exp(\lambda y)$$
and consequently (with the chosen initial conditions)
$$\psi(y) = (1 - \lambda y) \exp(\lambda y). \quad (87)$$

which is of the required form.

Finally, if $\psi(0) = 0$, we may set $\psi'(0) = 1$ and $\psi''(0) = 0$. Now we have $F(x) = \phi(x)$ and $F'(x) = -\phi'(x)$, whence $\phi(x) = c_1$ is a constant. In this case
$$\psi(y) = c_2 + y, \quad (88)$$
again of the required form. □
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