Decays of charged $B$–mesons into three charged leptons and a neutrino

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Abstract

In the framework of the Standard Model we present predictions for partial widths, double and single differential distributions, and forward–backward lepton asymmetries for four-leptonic decays $B^- \rightarrow \mu^+ \mu^- \bar{\nu}_e e^-$, $B^- \rightarrow e^+ e^- \bar{\nu}_\mu \mu^-$, $B^- \rightarrow \mu^+ \mu^- \bar{\nu}_\mu \mu^-$, and $B^- \rightarrow e^+ e^- \bar{\nu}_e e^-$. We consider the contributions of virtual photon emission from the light and heavy quarks of the $B^-$–meson, and we include bremsstrahlung of a virtual photon from the charged lepton in the final state. We use the model of vector meson dominance for calculation of virtual photon emission by the light quark of the $B^-$–meson and take into account the isotopic correction.

Introduction

Four-leptonic decays of $B$–mesons allow a precise test of Standard Model (SM) predictions in the higher orders of perturbation theory. At the same time these decays may be background processes to the helicity-suppressed ultra-rare decays $B_{d,s} \rightarrow \mu^+ \mu^-$, which are under study at the Large Hadron Collider (LHC) [1][2][3]. These studies are motivated by searches for Beyond the Standard Model physics.

Rare four-leptonic decays of $B$–mesons in the SM may be divided into two groups. The decays of the first group are forbidden at the tree level and occur through the higher order loop diagrams of perturbation theory – “penguin” and/or “box”. In this way of the SM includes flavor changing neutral currents (FCNC). An example of the first group of decays is the process $B_s \rightarrow e^+ e^- \mu^+ \mu^-$ and any other four-leptonic decays of neutral $B$–mesons. In the second group, in order to obtain the given multi-lepton final state, a number of tree level weak and electromagnetic processes are involved. Examples are the decay $B^- \rightarrow e^+ e^- \bar{\nu}_\mu \mu^-$ and analogous processes involving charged $B$–mesons. Both groups are studied at the LHC and potentially could be investigated at the Belle II experiment. Currently only upper limits for branching ratios of the decays $B_{d,s} \rightarrow \mu^+ \mu^- \mu^+ \mu^-$ and $B^- \rightarrow \mu^+ \bar{\nu}_\mu \mu^- \mu^-$ are available [4][5][6].

The experimental upper limits [4][5] for the decays $B_{d,s} \rightarrow \mu^+ \mu^- \mu^+ \mu^-$ are an order of magnitude higher than the corresponding theoretical predictions [7] and estimates [8]. The
situation with the decay \( B^- \rightarrow \mu^+ \bar{\nu}_\mu \mu^- \bar{\nu}_\mu \) is different. The experimental upper limit \[6\],

\[
\text{Br} \left( B^- \rightarrow \mu^+ \bar{\nu}_\mu \mu^- \bar{\nu}_\mu \right) < 0.16 \times 10^{-7},
\]

obtained with 95% confidence level (CL) is almost an order of magnitude lower than the theoretical predictions \[8, 9\]. We present here to more detailed calculation of the branching ratios of \( B^- \rightarrow \mu^+ \bar{\nu}_\mu \mu^- e^- \), \( B^- \rightarrow e^+e^- \bar{\nu}_\mu \mu^- \), \( B^- \rightarrow \mu^+ \bar{\nu}_\mu \mu^- \) and \( B^- \rightarrow e^+e^- \bar{\nu}_e e^- \), taking into account isotopic effects. Also in the phase space of the decays, a correction to non-zero lepton mass is considered. While this leads to better agreement between theory and experiment, some discrepancy remain. Special attention is given to the predictions of the behavior of differential distributions, e.g. forward–backward lepton asymmetries.

This article is organized as follows. In the “Introduction” we give a task description. In Section 1 we write the effective Hamiltonian and give definite the hadronic form factors. In Section 2 the common dependence of the decay amplitudes \( B^- \rightarrow \ell^+ \ell^- \bar{\nu}_\ell \ell^- \) on di-lepton 4-momenta is studied. Section 3 contains the exact formulae for amplitudes of the decay \( B^- \rightarrow \ell^+ \ell^- \bar{\nu}_\ell \ell^- \) for \( \ell \neq \ell' \), and Section 4 provides analogous formulae for \( \ell = \ell' \). In Section 5 we present numerical results for the decays of charged \( B^- \)–mesons into three charged leptons and \( \nu \) and discuss the precision of the predictions. The “Conclusion” contains the main outcome of the work. Some details of the four-leptonic decay kinematics are given in Appendix A.

### 1 Effective Hamiltonian and hadronic matrix elements

In terms of fundamental quark and lepton fields, the Hamiltonian for calculation of the amplitudes of four-lepton decays \( B^- \rightarrow \ell^+ \ell^- \bar{\nu}_\ell \ell^- \) has the form:

\[
\mathcal{H}_{\text{eff}}(x) = \mathcal{H}_W(x) + \mathcal{H}_{\text{em}}(x).
\]

The Hamiltonian of the transitions \( b \rightarrow uW^- \rightarrow u\ell^- \bar{\nu}_\ell \) is written as:

\[
\mathcal{H}_W(x) = -\frac{G_F}{\sqrt{2}} V_{ub}(\bar{u}(x) \gamma^\mu (1 - \gamma^5) b(x)) \left( \bar{\ell}(x) \gamma_\mu (1 - \gamma^5) \nu_\ell(x) \right) + \text{h.c.,}
\]

where \( u(x) \) and \( b(x) \) are quark fields, \( \ell(x) \) and \( \nu_\ell(x) \) are lepton fields, \( G_F \) is the Fermi constant, \( V_{ub} \) is the corresponding matrix element of the Cabibbo-Kobayashi-Maskawa (CKM) matrix, and the matrix \( \gamma^5 \) is defined as \( \gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3 \).

The Hamiltonian of the electromagnetic interaction has the form:

\[
\mathcal{H}_{\text{em}}(x) = -e \sum_f Q_f \left( \bar{f}(x) \gamma^\mu f(x) \right) A_\mu(x) = -j^\mu_{\text{em}}(x) A_\mu(x),
\]

where the unitary charge \( e = |e| \) is normalized by \( e^2 = 4\pi\alpha_{\text{em}} \); \( \alpha_{\text{em}} \approx 1/137 \), the fine structure constant, \( Q_f \) is the charge of the fermion of flavor \( f \) in units of the unitary charge, \( f(x) \) is the fermionic field of flavor \( f \), and \( A_\mu(x) \) is the four-potential of the electromagnetic field.
We define the following non-zero hadronic matrix elements, which are needed for the subsequent calculations:

\[
\begin{align*}
\langle 0 | \bar{u} \gamma^\mu \gamma^5 b | B^-(M_1, p) \rangle &= i f_{B^*} p^\mu, \\
\langle 0 | \bar{q} \gamma^\mu Q | V(M_V, k, \varepsilon) \rangle &= \varepsilon^\mu M_V f_V, \\
\langle V(M_2, q, \varepsilon) | \bar{u} \gamma_\mu b | B^-(M_1, p) \rangle &= \frac{2 V(k^2)}{M_1 + M_2} \varepsilon_{\mu\alpha\beta} \varepsilon^{*\nu} p^\alpha q^\beta, \\
\langle V(M_2, q, \varepsilon) | \bar{u} \gamma_\mu \gamma^5 b | B^-(M_1, p) \rangle &= i \varepsilon^{*\nu} \left[ (M_1 + M_2) A_1(k^2) g_{\mu\nu} - \frac{A_2(k^2)}{M_1 + M_2} (p + q)_\mu p_\nu - \frac{2M_2}{k^2} \left( A_3(k^2) - A_0(k^2) \right) k_\mu p_\nu \right], \\
\langle B^{*-}(M_{B^*}, k, \varepsilon) | \bar{b} \gamma_\mu b | B^-(M_1, p) \rangle &= \frac{2 V_b(q^2)}{M_1 + M_{B^*}} \varepsilon_{\mu\alpha\beta} \varepsilon^{*\nu} p^\alpha k^\beta,
\end{align*}
\]

where \( M_1 - B^- \) is the meson mass, \( p^\mu \) is the its four-momentum, \( M_{B^*} \) is the \( B^{*-} \)–meson mass, \( M_2 - \) mass of the light \((\rho(770) \text{ or } \omega(782))\) mesons, \( M_V = \{M_2, M_{B^*}\} \) are masses of the intermediate vector mesons, and \( \varepsilon^\mu \) are their polarizations. Four–vectors \( p^\mu, q^\mu, \) and \( k^\mu \) satisfy the conservation law: \( p^\mu = q^\mu + k^\mu \). The components of the fully antisymmetric tensor \( \varepsilon_{\mu\alpha\beta} \) are fixed by the condition \( \varepsilon^{0123} = -\varepsilon_{0123} = -1 \), and \( g_{\mu\nu} \) is the metric tensor in Minkovsky space with diag \( g_{\mu\nu} = (1, -1, -1, -1) \).

2 Generic structure of the amplitudes for the decays \( B^- \rightarrow \ell^+ \ell^- \bar{\nu}_\ell \ell'^- \) with the zero lepton mass approximation

There are three main types of diagrams needed for description of the decays \( B^-(p) \rightarrow \gamma^*(q) W^-(k) \rightarrow \ell^+(k_1) \ell^-(k_2) \bar{\nu}_\ell(k_3) \ell'^-(k_4) \), when the flavor of lepton \( \ell \) is different from the flavor of lepton \( \ell' \). The first type arises in the situation when a virtual photon is emitted by light a \( u \)–quark (see Fig. (1)). The second type corresponds to the emission of a virtual photon from a \( b \)–quark (see Fig. (2)). The third type is related to bremsstrahlung, when a virtual photon is emitted by the lepton \( \ell'^- \) in the final state (see Fig. (3) below). The four–momenta \( q \) and \( k \) are defined in Appendix A.

The structure of the amplitude, corresponding to diagrams on Figs. (1), (2) and (3) may be presented as

\[
\mathcal{M}_{fi} (q^2, k^2) \sim \frac{1}{q^2} T^{\nu\mu(q, k)} j^{\nu}(k_2, k_1) J^{\mu}(k_4, k_3),
\]

where

\[
T^{\nu\mu(q, k)} = i \int d^4x e^{i(qx)} \langle 0 | j^{\nu}_e(x), \bar{u}(0) \gamma^\mu (1 - \gamma_5) b(0) | B^-(M_1, p) \rangle = T^{(u)}_{\nu\mu}(q, k) + T^{(b)}_{\nu\mu}(q, k) + T^{(brems)}_{\nu\mu}(q, k).
\]
The lepton currents are 
\[ j^\nu(k_2, k_1) = \left( \bar{\ell}(k_2) \gamma^\nu \ell(-k_1) \right) \quad \text{and} \quad J^\mu(k_4, k_3) = \left( \bar{\nu}(k_4) \gamma^\mu (1 - \gamma^5) \nu \ell(-k_3) \right). \]

In the amplitude \( \mathcal{M}_{fi}(q^2, k^2) \), the pole \( 1/q^2 \) of the photon propagator is evident. For calculations with \( q^2 \to 0 \) it is necessary to take into account non-zero lepton masses. This is done using the exact formula (31) for four-particle phase space, and by introducing an effective cut for some value \( q^2_{\text{min}} \). For the case when \( \ell = \mu \) for \( q^2_{\text{min}} \) it makes sense to choose the natural kinematical cut \( 4m_\mu^2 \).

For the case when \( \ell = e \) it is better to use the kinematical limits of an experimental device, which are definitely higher than \( 4m_e^2 \).

Tensor \( T_{\nu\mu}(q, k) \) satisfies the condition: \( q^\nu T_{\nu\mu}(q, k) = 0 \). According to this condition, and taking into account the result of Ref. [11], tensor \( T_{\nu\mu}(q, k) \) has the form:

\[
T_{\nu\mu}(q, k) = \epsilon_{\nu\mu qk} \frac{e a(q^2, k^2)}{M_1} - i \left( g_{\nu\mu} - \frac{q_\nu q_\mu}{q^2} \right) e M_1 b(q^2, k^2) - i e \left( k_\nu - \frac{(qk)}{q^2} q_\nu \right) \left( k_\mu \frac{2 d(q^2, k^2)}{M_1} - q_\mu \frac{2 c(q^2, k^2)}{M_1} \right) - i Q_{Bu} e f_{Bu} \frac{q_\nu k_\mu}{q^2},
\]

where \( Q_{Bu} = Q_b - Q_u = -1 \), is the electric charge of the \( B^- \)-meson in units of \( |e| \). The functions \( a(q^2, k^2), \ldots, d(q^2, k^2) \) are dimensionless form factors which depend on two variables, the squares of the transferred four-momenta, \( q^2 \) and \( k^2 \). From (5) it follows that \( d(0, 0) = Q_{Bu} f_{Bu}/M_1 \).

Using the equations of motion, in the limit of massless leptons, one can obtain the following generic structure for the amplitude \( \mathcal{M}_{fi} \):

\[
\mathcal{M}_{fi}(q^2, k^2) \sim \frac{e}{q^2} \left[ \epsilon_{\nu\mu qk} \frac{a(q^2, k^2)}{M_1} - i g_{\nu\mu} M_1 b(q^2, k^2) + i k_\nu q_\mu \frac{2 c(q^2, k^2)}{M_1} \right] j^\nu(k_2, k_1) J^\mu(k_4, k_3).
\]

The exact calculation of the form factors \( a(q^2, k^2), \ldots, c(q^2, k^2) \) is quite complicated. In the current work we will take into account only the leading singular factors to the corresponding form factors.
Let us start with a study of tensor \( T_{\nu\mu}^{(u)}(q, k) \), which describes the contribution of diagram from Fig. [1] to the tensor \( T_{\nu\mu}(q, k) \). The main contribution to the structure of tensor \( T_{\nu\mu}^{(u)}(q, k) \) is given by the lightest intermediate vector resonances that contain a \( u\bar{u} \)–pair. For such states tensor \( T_{\nu\mu}^{(u)}(q, k) \) has Breit–Wigner poles for variable \( q^2 \). Taking into account only the contributions from \( \rho^0(770) \) and \( \omega(782) \) mesons, we can write

\[
T_{\nu\mu}^{(u)}(q, k) \to \\
\to \sum_{i=p^0,\omega} \langle 0 | \bar{u}\gamma_{\mu}u | V(M_{2i}, q, \epsilon) \rangle \frac{e}{M_{2i}^2 - q^2 - iM_{2i}\Gamma_{2i}} \langle V(M_{2i}, q, \epsilon) | \bar{u}\gamma_{\mu}(1 - \gamma^5)b | B^{-}(M_1, p) \rangle ,
\]

where \( M_{2i} \) and \( \Gamma_{2i} \) are the masses and widths, respectively, of the intermediate vector resonances.

For the zero leptonic mass approximation, the range of values of the variable \( k^2 \) is \( 0 \lesssim k^2 \lesssim M_1^2 \). The closest pole in \( k^2 \) is related to the appearance of the intermediate vector state \( B^{-} \). As \( M_{B^{*-}} > M_1 \), this pole lies outside of the kinematically allowed range of the decay \( B^- \to \ell^+ \ell^- \bar{\nu}_\ell \ell^- \). The existence of the pole at the mass of the \( B^{*-} \)–meson is taken into account when choosing the pole parametrisation of the form factors of the transitions \( B \to \rho \) and \( B \to \omega \) [10]. For non-zero leptonic masses, \( m_\nu^2 \leq k^2 \lesssim (M_1 - 2m_\ell)^2 \). Hence all the remarks above on the poles of tensor \( T_{\nu\mu}^{(u)} \) for variable \( k^2 \) are still valid.

As the contribution from \( \rho^0 \) and \( \omega \) resonances is dominant, it is possible to use the following estimate for the branching ratio of \( B^- \to \mu^+ \mu^- \bar{\nu}_\ell \ell^- \):

\[
Br^{(u)}(B^- \to \mu^+ \mu^- \bar{\nu}_\ell \ell^-) \approx \sqrt{Br(B^- \to \rho^0 e^- \bar{\nu}_\ell) Br(\rho^0 \to \mu^+ \mu^-) + Br(B^- \to \omega e^- \bar{\nu}_\ell) Br(\omega \to \mu^+ \mu^-)} \approx 0.3 \times 10^{-7},
\]

where the necessary experimental values for the branching ratios are taken from [12]. The estimate [6] does not take into account the fact that the \( \rho^0(770) \)–meson is a wide resonance, i.e., in the case of the \( \rho^0(770) \)–meson, the naive factorization approximation should lead to a lower branching ratios. Also in estimate [6], the photon pole, which should also lead to lower results, is not taken into account. Does estimate [6] contradict the experimental upper limit in [1]? We do not think so, because we attribute to the factor of two 2 accuracy. But the estimate of [6] does point to the possibility that the minimum of the possible theoretical predictions may be above the experimental limit [6].

Now consider tensor \( T_{\nu\mu}^{(b)}(q, k) \), which is related to diagram from Fig. [2]. In the limit of massless leptons there are no poles for the variable \( q^2 \) in the kinematically allowed range \( 0 \leq q^2 \leq M_1^2 \) for the tensor \( T_{\nu\mu}^{(b)}(q, k) \). The closest pole outside the allowed range corresponds to the \( b\bar{b} \) quark composition. This is the \( \Upsilon(1S) \) meson, whose mass is almost two times higher than the mass of the \( B^- \)–meson. The dominant contribution to emission of a virtual photon by a heavy quark is described using the process \( B^- \to B^{*-} \gamma^* \). In this case

\[
T_{\nu\mu}^{(b)}(q, k) \to \\
\to \langle 0 | \bar{u}\gamma_{\mu}(1 - \gamma^5)b | B^{*-}(M_{B^*}, k, \epsilon) \rangle \frac{e}{M_{B^*}^2 - k^2} \langle B^{*-}(M_{B^*}, k, \epsilon) | \tilde{b}\gamma_{\nu}b | B^{-}(M_1, p) \rangle .
\]
Note that the imaginary addition $-iM_{B^*}\Gamma_{B^*}$ does not exist in the propagator, as $k^2 < M_{B^*}^2$, i.e., the pole of the $B^*$-meson is not reached. The contribution from the $\Upsilon(1S)$ is taken into account effectively when introducing pole parametrization for the form factor $V_b(q^2)$. For the variable $k^2$ in the kinematically allowed range, the tensor $T^{(b)}_{\nu\mu}(q, k)$ does not have any other poles.

Numerically the contribution of the process on the Fig. (2) to the branching ratio associated with the four-leptonic decay is suppressed comparing to the contribution of the process on the Fig. (1) by factor $(\Lambda/m_b)^2$, where $m_b \sim 5$ GeV, the mass of $b$-quark, and the parameter $\Lambda \approx 300 - 500$ MeV. This follows from the exact equations for the form factors of the rare leptonic radiative decays of $B$-mesons [13, 14]. Due to the interference between diagrams (1) and (2) near the photonic pole it is necessary, however to take into account the contribution of the diagram (2) to the full branching ratio.

The bremsstrahlung contribution is described by diagram on the Fig. (3). The bremsstrahlung amplitude has a single pole by $q^2$ from the photon propagator. Hence the tensor $T^{(brem)}_{\nu\mu}(q, k)$ does not have poles by $q^2$ and $k^2$. It is important to take into account the bremsstrahlung contribution near the pole by $q^2$, where the zero-mass approximation may not be fully correct.
This contribution should be calculated for non-zero lepton masses.

3 Formulae for the decay $B^- \rightarrow \ell^+ \ell^- \bar{\nu}_\ell \ell'$

Consider the decays $B^- \rightarrow \mu^+ \mu^- \bar{\nu}_e e^-$ and $B^- \rightarrow e^+ e^- \bar{\nu}_\mu \mu^-$, for the case when the lepton flavors in the final state are different. Generally these decays may be written as $B^- \rightarrow \ell^+ \ell^- \bar{\nu}_\ell \ell'$ for $\ell \neq \ell'$.

The contribution to the full decay amplitude $B^-(p) \rightarrow \ell^+(k_1) \ell^-(k_2) \bar{\nu}_\ell(k_3) \ell^-(k_4)$ from Fig. 1 may be calculated using the Vector Meson Dominance model (VMD) (see Fig. 4). Assuming $m_\ell = m_{\ell'} = 0$ and using the effective Hamiltonian (2), one finds, that for VMD the contribution from process (1) is described by diagram (4), and the corresponding amplitude may be written as:

$$M^{(u)}_{f_1} = \frac{A}{q^2} \left[ \sum_{i = \rho, \omega} \frac{I_i M_{2i} f_{V_i}}{q^2 - M_{2i}^2 + i \Gamma_{2i} M_{2i}} F^{(i)}_{\mu \nu}(k^2) \right] J^\mu(k_2, k_1) J^\nu(k_4, k_3), \quad (7)$$

where, using the motion equations,

$$F^{(i)}_{\mu \nu}(k^2) = \frac{2 V^{(i)}(k^2)}{M_1 + M_{2i}} \epsilon_{\mu \nu \rho \lambda} k_\rho - i (M_1 + M_{2i}) A^{(i)}_{1}(k^2) g_{\mu \nu} + 2i \frac{A^{(i)}_{2}(k^2)}{M_1 + M_{2i}} q_\mu k_\nu.$$

For the calculation of the resonances sum in the (7), only the contributions from the lightest $\rho^0$ and $\omega$ mesons, containing $u\bar{u}$–pairs, are taken into account. Because the $\rho^0$ and $\omega$ mesons are linear combinations of $u\bar{u}$ and $d\bar{d}$–pairs, in order to extract the contributions only $u\bar{u}$–pair alone, an isotopic coefficients $I_i$ are introduced. By definition

$$I_{\rho^0} = \langle \rho^0 | \bar{u} u \rangle = 1/\sqrt{2}$$
Figure 5: Diagram for the calculation of $\mathcal{M}_{f_i}^{(b)}$ (see the equation (8)) using the decay $B^- \rightarrow \mu^+ \mu^- \bar{\nu}_e e^-$ as an example.

Figure 6: Diagram for the calculation of the amplitude of the bremsstrahlung $\mathcal{M}_{f_i}^{(brem)}$ (see the equation (9)) for the decay $B^- \rightarrow \mu^+ \mu^- \bar{\nu}_e e^-$. 

and

$$I_\omega = \langle \omega | \bar{u} u \rangle = 1/\sqrt{2}.$$

The contribution from process (2) is given by diagram on Fig. (5), which is the cross–channel of the decay $B^* \rightarrow B\gamma^*$ of a heavy vector meson into a heavy pseudoscalar meson and a virtual photon, and is represented by

$$\mathcal{M}_{f_i}^{(b)} = \frac{2}{3} \frac{A}{q^2} \frac{M_{B^*} f_{B^*}}{k^2 - M_{B^*}^2} \frac{V_b(q^2)}{M_1 + M_{B^*}} \epsilon_{\mu\nu k q} \bar{J}^\nu(k_2, k_1) j^\mu(k_4, k_3).$$

There is no imaginary correction in the propagator, as $k^2 < M_{B^*}^2$.

Finally, the contribution of the bremsstrahlung process (3) of the virtual photon is described by diagram from Fig. (6). In the case when $m_\ell \neq 0$ and $m_{\ell'} \neq 0$, for the amplitude of the bremsstrahlung is:

$$\mathcal{M}_{f_i}^{(brem)} = \frac{A}{q^2} i f_{B_u} g_{\mu\nu} J^\nu(k_2, k_1) \bar{J}^\mu(k_4, k_3),$$
where

\[ \bar{J}^{\mu}(k_4, k_3) = J^{\mu}(k_4, k_3) + \frac{m_{\ell}}{(p - k_3)^2 - m_{\ell}^2} \left( \bar{\ell}'(k_4) \gamma^{\mu} (\hat{p} + m_{\ell}) (1 - \gamma^5) \nu_{\ell'}(-k_3) \right). \]

As \((2m_{\ell} + m_{\ell}')^2 \leq (p - k_3)^2 \leq M_1^2\), the second summand does not contain any poles in the whole kinematically allowed range. The second summand may be compatible with the first one only in the range where \((p - k_3)^2 \sim (2m_{\ell} + m_{\ell}')^2\). But this range is suppressed by the phase space (31) integration. For this the reason we assume that the bremsstrahlung amplitude may be written as

\[ \mathcal{M}_{fi}^{(\text{brems})} = \frac{A}{q^2} i f_{Bu} C_{\mu \nu \rho} \bar{J}^{\mu}(k_2, k_1) J^{\mu}(k_4, k_3). \] (9)

In formulas (7), (8) and (9) we denote \( A = \frac{G_F}{\sqrt{2}} 4 \pi \alpha_{em} V_{ub}. \)

The full decay amplitude \( B^- (p) \rightarrow \ell^+(k_1) \ell^- (k_2) \bar{\nu}_\ell (k_3) \ell^- (k_4) \) may be written as

\[ \mathcal{M}_{fi}^{(1234)} = \mathcal{M}_{fi}^{(a)} + \mathcal{M}_{fi}^{(b)} + \mathcal{M}_{fi}^{(\text{brems})} = \]

\[ = \frac{A}{q^2} \left( a(q^2, k^2) \frac{M_1^2}{x_{13}} - \frac{2 V_6 M_1^2}{x_{12} + M_{B^*}^2} + \sum_{i=\rho_1, \rho_2} \frac{I_i \tilde{M}_{2i} \hat{f}_{V_i}}{x_{12} - M_{2i}^2 + i \Gamma_{2i} M_{2i}} \right) \bar{J}^{\mu}(k_2, k_1) J^{\mu}(k_4, k_3). \] (10)

The benefit of choosing the notation \( \mathcal{M}_{fi}^{(1234)} \) for the full amplitude will be apparent while considering decays with identical charged fermions in the final state (see Section 4). Dimensionless functions \( a(q^2, k^2) \equiv a(x_{12}, x_{34}), b(q^2, k^2) \equiv b(x_{12}, x_{34}), \) and \( c(q^2, k^2) \equiv c(x_{12}, x_{34}) \) are defined as:

\[ a(x_{12}, x_{34}) = \frac{1}{3} \frac{\tilde{M}_{B^*} \hat{f}_{B^*}}{x_{34} - M_{B^*}^2} \frac{2 V_6 M_1^2}{x_{12} + M_{B^*}^2} + \sum_{i=\rho_1, \rho_2} \frac{I_i \tilde{M}_{2i} \hat{f}_{V_i}}{x_{12} - M_{2i}^2 + i \Gamma_{2i} M_{2i}} \frac{2 V^{(i)}(M_{2i}^2)}{1 + M_{2i}}; \]

\[ b(x_{12}, x_{34}) = -\hat{f}_{B_u} + \sum_{i=\rho_1, \rho_2} \frac{I_i \tilde{M}_{2i} \hat{f}_{V_i}}{x_{12} - M_{2i}^2 + i \Gamma_{2i} M_{2i}} \frac{1 + M_{2i}}{M_{2i}} A_{1}^{(i)}(M_{2i}^2 x_{34}); \]

\[ c(x_{12}, x_{34}) = \sum_{i=\rho_1, \rho_2} \frac{I_i \tilde{M}_{2i} \hat{f}_{V_i}}{x_{12} - M_{2i}^2 + i \Gamma_{2i} M_{2i}} \frac{A_{2}^{(i)}(M_{2i}^2 x_{34})}{1 + M_{2i}}, \] (11)

where the dimensionless variables \( x_{12} = q^2/M_{1}^2 \), and \( x_{34} = k^2/M_{2}^2 \) are defined in Appendix A, and the dimensionless constants are defined as \( \hat{f}_{B_u} = f_{B_u}/M_1, \hat{f}_{B^*} = f_{B^*}/M_1, \hat{f}_{V_i} = f_{V_i}/M_1, \tilde{M}_{2i} = M_{2i}/M_1, \hat{M}_{B^*} = M_{B^*}/M_1 \) and \( \Gamma_{2i} = \Gamma_{2i}/M_1 \). Form factors \( V_6(q^2), V^{(i)}(k^2), A_{1}^{(i)}(k^2), \) and \( A_{2}^{(i)}(k^2) \) are also dimensionless functions.

The differential branching ratio of the decay \( B^- \rightarrow \ell^+ \ell^- \bar{\nu}_\ell \ell^- \) is calculated as

\[ d\text{Br} \left( B^- \rightarrow \ell^+ \ell^- \bar{\nu}_\ell \ell^- \right) = \tau_B \frac{\sum_{s_1, s_2, s_3, s_4} \left| \mathcal{M}_{fi}^{(1234)} \right|^2}{2 M_1} d\Phi_{fi}^{(1234)} \] (12)
where \( \tau_{B^-} \) is the lifetime of the \( B^- \)–meson, four-particle phase space \( d\Phi_4^{(1234)} \) is defined by Equation (31), and the summation is performed over the spins of the final fermions. In formula (12) the integration over the angular variables \( y_{12}, y_{34}, \) and \( \varphi \) may be performed analytically. After the integration,

\[
\frac{d^3 \text{Br} \left( B^- \rightarrow \ell^+\ell^-\bar{\nu}_\ell\ell'^- \right)}{dx_{12} dx_{34}} = \tau_{B^-} \frac{G_F^2 M_1^5 \alpha_{em}^2 |V_{ub}|^2}{2^6 3^2 \pi^3} \sqrt{1 - \frac{4m^2_\ell}{x_{12}}} \left( 1 - \frac{\hat{m}^2_\ell}{x_{34}} \right) \tag{13}
\]

\[
\lambda^{1/2}(1, x_{12}, x_{34}) \left[ 2x_{12}x_{34} \lambda(1, x_{12}, x_{34}) \left| a(x_{12}, x_{34}) \right|^2 + \right.
\]

\[
+ \left( \lambda(1, x_{12}, x_{34}) + 12x_{12}x_{34} \right) \left| b(x_{12}, x_{34}) \right|^2 + \lambda^2(1, x_{12}, x_{34}) \left| c(x_{12}, x_{34}) \right|^2 +
\]

\[
+ 2\lambda(1, x_{12}, x_{34}) (x_{12} + x_{34} - 1) \text{Re} \left( b(x_{12}, x_{34}) c^*(x_{12}, x_{34}) \right) \right],
\]

which depends only on the dimensionless variables \( x_{12} \) and \( x_{34} \). Full integration by these variables may be performed only numerically.

Because in the decay of the \( B^- \)–meson, all the leptons in the final state are different, it makes sense to define two forward–backward leptonic asymmetries \( A_{FB}^{(B^-)}(x_{12}) \) and \( A_{FB}^{(B^-)}(x_{34}) \) as

\[
A_{FB}^{(B^-)}(x_{12}) = \int_0^1 d\cos \tilde{\theta}_{12} \frac{d^2 \Gamma \left( B^- \rightarrow \ell^+\ell^-\bar{\nu}_\ell\ell'^- \right)}{dx_{12} d\cos \tilde{\theta}_{12}} - \int_{-1}^0 d\cos \tilde{\theta}_{12} \frac{d^2 \Gamma \left( B^- \rightarrow \ell^+\ell^-\bar{\nu}_\ell\ell'^- \right)}{dx_{12} d\cos \tilde{\theta}_{12}} \tag{14}
\]

and

\[
A_{FB}^{(B^-)}(x_{34}) = \int_0^1 d\cos \tilde{\theta}_{34} \frac{d^2 \Gamma \left( B^- \rightarrow \ell^+\ell^-\bar{\nu}_\ell\ell'^- \right)}{dx_{34} d\cos \tilde{\theta}_{34}} - \int_{-1}^0 d\cos \tilde{\theta}_{34} \frac{d^2 \Gamma \left( B^- \rightarrow \ell^+\ell^-\bar{\nu}_\ell\ell'^- \right)}{dx_{34} d\cos \tilde{\theta}_{34}} \tag{15}
\]

where \( \tilde{\theta}_{12} \) is the angle between the propagation directions of the \( \ell^- \) and \( B^- \) in the rest frame of the \( \ell^+\ell^- \)–pair, and \( \tilde{\theta}_{34} \) is the angle between the propagation directions of \( \ell^- \) and \( B^- \) in the rest frame of the \( \ell^-\bar{\nu}_\ell \) pair. It is obvious that \( \tilde{\theta}_{12} = \pi - \theta_{12} \) and \( \tilde{\theta}_{34} = \pi - \theta_{34} \). Equations (14) and (15) are chosen such that they correspond to the notions of Ref. [11].

\section{Exact formulae for the decay \( B^- \rightarrow \ell^+\bar{\nu}_\ell \ell^- \ell'^- \)}

In practice, the muonic tracks are registered with a much higher efficiency at almost all contemporary experiments. That is why from the experimental point of view the decay \( B^- \rightarrow \mu^+\bar{\nu}_\mu \mu^-\mu^- \) is of the most interest. In this decay the final state contains two identical muons of negative charge. Hence the Fermi antisymmetry should be taken into account.

Consider the full amplitude of the decay \( B^-(p) \rightarrow \ell^+(k_1) \bar{\nu}_\ell(k_3) \ell^-(k_2) \ell^-(k_4) \). In the approximation of zero leptonic masses, the calculation below is applicable to the decay \( B^- \rightarrow \ell^+\bar{\nu}_\ell \ell^- \ell'^- \).
\[ \mu^+ \bar{\nu}_\mu \mu^- \bar{\mu}^- \] as well as to the decay \( B^- \rightarrow e^+ \bar{\nu}_e e^- e^- \). The full amplitude of the decay may be written as

\[ M_{fi}^{(\text{tot})} = M_{fi}^{(1234)} - M_{fi}^{(1432)}, \]

where the amplitude \( M_{fi}^{(1234)} \) is set by equation (10), and the amplitude \( M_{fi}^{(1432)} \) can be obtained from \( M_{fi}^{(1234)} \) by exchanging \( k_2 \leftrightarrow k_4 \). This leads to the necessity of replacing \( q_\mu \rightarrow \bar{q}_\mu, k_\mu \rightarrow \bar{k}_\mu, x_{12} \rightarrow x_{14}, \) and \( x_{34} \rightarrow x_{23} \) (see Appendix A) in the calculation of \( M_{fi}^{(1432)} \).

The differential branching ratio of the decay is given by

\[ d\text{Br} \left( B^- \rightarrow \ell^+ \bar{\nu}_\ell \ell^- \ell^- \right) = \frac{1}{2} \left\{ \tau_{B^-} \frac{\sum_{s_1, s_2, s_3, s_4} \left| M_{fi}^{(1234)} \right|^2}{2M_1} d\Phi_4^{(1234)} + \tau_{B^-} \frac{\sum_{s_1, s_2, s_3, s_4} \left| M_{fi}^{(1432)} \right|^2}{2M_1} d\Phi_4^{(1432)} - \tau_{B^-} \frac{\sum_{s_1, s_2, s_3, s_4} \left( M_{fi}^{(1234)} \right)^\dagger M_{fi}^{(1324)} + M_{fi}^{(1432)} \dagger M_{fi}^{(1234)} \right)}{2M_1} d\Phi_4^{(1234)} \right\}, \]

where \( d\Phi_4^{(1234)} \) and \( d\Phi_4^{(1432)} \) are set by equations (31) and (32). The common factor of 1/2 is due to Fermi antisymmetry.

The first and the second summands in (17) are equal. Hence for the branching ratio, it is possible to write

\[ \text{Br} \left( B^- \rightarrow \ell^+ \bar{\nu}_\ell \ell^- \ell^- \right) = \text{Br} \left( B^- \rightarrow \ell^+ \ell^- \bar{\nu}_\ell \ell^- \right) - \text{Br}_{\text{interf}} \left( B^- \rightarrow \ell^+ \bar{\nu}_\ell \ell^- \ell^- \right), \]

where

\[ \text{Br}_{\text{interf}} \left( B^- \rightarrow \ell^+ \bar{\nu}_\ell \ell^- \ell^- \right) = \frac{\tau_{B^-}}{4M_1} \int \sum_{s_1, s_2, s_3, s_4} \left( M_{fi}^{(1234)} \dagger M_{fi}^{(1324)} + M_{fi}^{(1432)} \dagger M_{fi}^{(1234)} \right) d\Phi_4^{(1234)}. \]

From (19) it follows that in the calculation of the interference contribution it is necessary to perform five-dimension of numerical integration. It is necessary to use the replacements (33) in the matrix element \( M_{fi}^{(1432)} \).

## 5 Numerical results

To calculate the branching ratio, differential distributions, and asymmetries, we use numerical values of the masses, lifetimes and decay widths of the pseudoscalar and vector mesons,
and matrix elements of the CKM matrix from Ref. [12]. The constants \( f_\rho(770) = 154 \text{ MeV} \) and \( f_\omega(782) = 46 \text{ MeV} \) were calculated in [13].

Suitable parametrizations of the hadronic formfactors [3], except the electromagnetic form factor \( V_b(q^2) \), were obtained in [10]. Using the generic formulae from [15, 16] it is possible to find the following parametrization for the form factor \( V_b(q^2) \), calculated in the framework of the Dispersion Quark Model:

\[
V_b(q^2) = \frac{1.044}{\left(1 - \frac{q^2}{M_T^2}\right)\left(1 - 0.81\frac{q^2}{M_T^2}\right)},
\]  

(20)

where \( M_T \) is mass of the \( \Upsilon(1S) \) meson. The same method allows us to obtain the values of the leptonic constants \( f_{B_u} = 191 \text{ MeV} \) and \( f_{B_s} = 183 \text{ MeV} \).

We now calculate the branching ratio of the decay \( B^- \to \mu^+\mu^-\bar{\nu}_e e^- \). The natural kinematical cut of the pole by \( x_{12} \) is \( x_{12 \text{min}} = (2m_\mu/M_1)^2 \approx 0.0016 \). In this case, the numerical integration of the equation [13] by \( x_{12} \) and \( x_{34} \) gives:

\[
\text{Br} \left( B^- \to \mu^+\mu^-\bar{\nu}_e e^- \right) \approx 0.6 \frac{\tau_{B^-}}{1.638 \times 10^{-12} \text{s}} \frac{|V_{ub}|^2}{1.55 \times 10^{-5}} \times 10^{-7}.
\]  

(21)

The value of the branching of the \( B^- \to \mu^+\mu^-\bar{\nu}_e e^- \) decay given in (21) is approximately two times less than the corresponding value of \( 1.3 \times 10^{-7} \) from Refs.[8, 9]. This difference is mostly due to the isotopic coefficients \( I_{\rho\phi} \) and \( I_\omega \) in (7), while decreases the contribution from the intermediate vector \( \rho(770) \) and \( \omega(782) \) resonances to the total branching ratio by a factor 2. This contribution is dominant, so the \( \text{Br} \left( B^- \to \mu^+\mu^-\bar{\nu}_e e^- \right) \) increase by almost the same factor. Also the mean value of \( V_{ub} \) is changed from \( 4.09 \times 10^{-3} \) [17] to \( 3.94 \times 10^{-3} \) [12]. A decrease of the branching by 10% is due to the use of the exact formula (31) for the phase space.

The result in equation (21) is compatible with the naive estimate of (6) up to an expected factor of two. The difference between the estimate of (6) and the exact calculation (21) is mostly due to the fact that the estimate does not take into account the pole contribution when \( x_{12} \to x_{12 \text{min}} \). The importance of the pole contribution becomes obvious when analysing the double differential distribution \( d^2 \text{Br} \left( B^- \to \mu^+\mu^-\bar{\nu}_e e^- \right) /dx_{12} dx_{34} \), which is presented in Fig. (7). The figure features the pole when \( x_{12} \to x_{12 \text{min}} = 4m_\mu^2/M_1^2 \) and the ridge of the narrow \( \omega(782) \) resonance, the contribution of which defines the maximum of the matrix element. The wide \( \rho(770) \)-meson also gives a significant contribution to the branching ratio, but in the distribution of Fig. (7) is not a prominent as the narrow \( \omega(782) \) resonance.

The uncertainty on the numerical value of the branching ratio of the decay \( B^- \to \mu^+\mu^-\bar{\nu}_e e^- \) depends on the uncertainty calculation of the hadronic form factors of the transitions \( B \to \rho(770) \) and \( B \to \omega(782) \), but does not exceed 20% [10]. Some uncertainty is related to the use of the Vector Meson Dominance model. This uncertainty is mostly due to the choice of a non-perturbative phase between the summands in the (7). In the VMD model this phase is equal to zero. If the relative phase between the contributions from the \( \rho(770) \) and \( \omega(782) \) mesons into the amplitude (7) becomes \( \pi \), then the numerical result in (21) may decrease to
0.2 \times 10^{-7}. This dependence points to the importance of a future model-independent study of non-perturbative and non-factorized contributions of the strong interaction to the amplitudes of the decays $B^- \to \ell^+\ell^- \bar{\nu}_\ell \ell'^-$. Similar issue of generation of additional relative phases between the contributions of different charmonia by nonfactorizable gluons was discussed in [18].

In the model used for the result of (21) the non-resonant contribution, which is not related to the tails from the $\rho^0(770)$ and $\omega(782)$ resonances, is not taken into account. This contribution may be estimated by using the results from Ref. [19], in this work the branching ratio of the decay $B \to \gamma\ell\nu$ was predicted, omitting the contributions from $\rho^0$ and $\omega$ resonances. An estimation of the non-resonant contribution gives

$$\text{Br}(B^- \to \mu^+\mu^-\bar{\nu}_e e^-)_{\text{NRC}} \sim \alpha_{\text{em}} \times \text{Br}(B \to \gamma\ell\nu)_{\text{Beneke}} \sim 0.1 \times 10^{-7},$$

which is about 15% of the value of the branching ratio of (21) and is comparable to the uncertainty of the form factors calculation. Note that numerically the contributions to (21) from the processes in Fig. (5) and Fig. (6), which were taken into account, are also comparable to the non-resonant contribution, which was not taken into account.

It seems that the approximation of using only the contributions from the lightest $\rho(770)$ and $\omega(782)$ resonances, which is used in this work, is not applicable if the branching ratio of the decay $B^- \to \mu^+\mu^-\bar{\nu}_e e^-$ will be measured in the range of $\sqrt{q^2} > 1$ GeV. In this range it is necessary to take into account the contributions from the $\omega(1420)$, $\rho(1450)$, $\omega(1650)$, and $\rho(1700)$ resonances. These contributions should not affect the branching ratio of the decay $B^- \to \mu^+\mu^-\bar{\nu}_e e^-$ for $\sqrt{q^2} \leq 1$ GeV but will define the behavior in the range $\sqrt{q^2} > 1$ GeV. However in the experimental procedure [6], the variable $\sqrt{q^2}$ is chosen to be less than 980 MeV, in order to remove a potential background from the decay $\phi \to \ell^+\ell^-$. So the experimental data are available only in the range of applicability of the current work. This fact allows as to exclude from consideration resonances heavier than the $\rho^0(770)$ and $\omega(782)$.

We calculate the branching ratio of the decay $B^- \to e^+e^-\bar{\nu}_\mu \mu^-$. Formal integration in the
range around the photon pole by \( x_{12} \) leads to the rough dependence of the branching on \( x_{12 \text{ min}} \):

\[
\text{Br} \sim \int \frac{dx_{12}}{x_{12}^2} \sim \frac{1}{x_{12 \text{ min}}}.
\]

If we choose \( x_{12 \text{ min}} = (2m_e/M_1)^2 \), then by the order of magnitude

\[
\text{Br}(B^- \to e^+e^-\bar{\nu}_\mu \mu^-) \sim \left(\frac{m_\mu}{m_e}\right)^2 \text{Br}(B^- \to \mu^+\mu^-\bar{\nu}_e e^-) \sim 10^4 \text{Br}(B^- \to \mu^+\mu^-\bar{\nu}_e e^-).
\]

Because the efficiency of detection of the muonic pairs for \( \sqrt{q^2} \) below 80–100 MeV is low, this range is not suitable for an experimental observation. On the other hand, if we choose \( x_{12 \text{ min}} = (\Lambda/M_1)^2 = 0.0002 \) for \( \Lambda = 80 \text{ MeV} \), then

\[
\text{Br}(B^- \to e^+e^-\bar{\nu}_\mu \mu^-) \bigg|_{x_{12 \text{ min}}=0.0002} \approx 3.0 \frac{\tau_{B^-}}{1.638 \times 10^{-12} \text{ s}} \frac{|V_{ub}|^2}{1.55 \times 10^{-5}} \times 10^{-7}. \tag{22}
\]

The \( \text{Br}(B^- \to e^+e^-\bar{\nu}_\mu \mu^-) \) will decrease with increasing \( x_{12 \text{ min}} \).

The decays \( B^- \to \mu^+\mu^-\bar{\nu}_e e^- \) and \( B^- \to e^+e^-\bar{\nu}_\mu \mu^- \) may be suitable for tests of the hypothesis of leptonic universality, if one measures the branching ratio for the fixed value of \( x_{12} > (2m_\mu/M_1)^2 = 0.0016 \). If leptonic universality holds, then the condition:

\[
\frac{\text{Br}(B^- \to \mu^+\mu^-\bar{\nu}_e e^-)}{\text{Br}(B^- \to e^+e^-\bar{\nu}_\mu \mu^-)} \bigg|_{x_{12} > 0.0016} = 1. \tag{23}
\]

If the hints for \([20, 21, 22, 23, 24]\) violation of the leptonic universality are true, then the equation (23) may be violated.

We consider predictions for the branching ratio of the decay \( B^- \to \mu^+\bar{\nu}_\mu \mu^- \), which is the more suitable for experimental observation \([7]\), as the efficiency of muon detection is higher than the efficiency of electron detection. Numerical integration of the interference contribution \([19]\) for \( x_{12 \text{ min}} = (2m_\mu/M_1)^2 = 0.0016 \) gives

\[
\text{Br}_{\text{int}}(B^- \to \mu^+\bar{\nu}_\mu \mu^-) \approx -0.1 \frac{\tau_{B^-}}{1.638 \times 10^{-12} \text{ s}} \frac{|V_{ub}|^2}{1.55 \times 10^{-5}} \times 10^{-7},
\]

which is comparable due to uncertainty of the strong non-perturbative effects, the contributions from equations \([5]\) and \([6]\) and the result with the non-resonant contribution omitted. So we may state that in the limit of massless leptons, with a 30% precision from equations \([18]\) and \([21]\), it follows that:

\[
\text{Br}(B^- \to \mu^+\bar{\nu}_\mu \mu^-) \approx 0.7 \frac{\tau_{B^-}}{1.638 \times 10^{-12} \text{ s}} \frac{|V_{ub}|^2}{1.55 \times 10^{-5}} \times 10^{-7}. \tag{24}
\]

This is obtained for \( x_{12 \text{ min}} = (2m_\mu/M_1)^2 = 0.0016 \). This prediction is almost four times higher than the experimental upper limit \([1]\), obtained in Ref. \([6]\). What may explain the discrepancy between the experimental result and the theoretical prediction? First, there is quite high uncertainty of the theoretical prediction \([24]\). Second, the value of \( \text{Br}(B^- \to \mu^+\bar{\nu}_\mu \mu^-) \) depends on the relative phase between the contributions of the \( \rho(770) \) and \( \omega(782) \) resonances. In the framework of VMD it is zero, however various non-perturbative contributions may lead
Figure 8: Normalized differential distributions $\frac{1}{\Gamma} \frac{d\Gamma}{dx_{12}}$ for the decays a) $B^{-} \to \mu^{+}\mu^{-}\bar{\nu}_{e}e^{-}$ and b) $B^{-} \to \mu^{+}\bar{\nu}_{\mu}\mu^{-}\mu^{-}$, obtained by integration by $dx_{34}dy_{12}dy_{34}d\varphi$ of equations (12) and (17) respectively.

to non-zero value. All the other contribution, which were omitted in the current work could not significantly influence the numerical result of equation (24). It seems unlikely that the discrepancy between the prediction and measured result may be attributed to Beyond the Standard Model physics.

The decays $B^{-} \to e^{+}\bar{\nu}_{e}e^{-}e^{-}$ and $B^{-} \to \mu^{+}\bar{\nu}_{\mu}\mu^{-}\mu^{-}$ allow us to introduce yet another test for lepton universality:

$$\frac{\text{Br}(B^{-} \to \mu^{+}\mu^{-}\bar{\nu}_{e}e^{-})}{\text{Br}(B^{-} \to e^{+}\bar{\nu}_{e}e^{-}e^{-})} \bigg|_{x_{12} > 0.0016, \; x_{14} > 0.0016} = 1.$$

(25)

We consider single differential distributions for the decays $B^{-} \to \mu^{+}\mu^{-}\bar{\nu}_{e}e^{-}$ and $B^{-} \to \mu^{+}\bar{\nu}_{\mu}\mu^{-}\mu^{-}$. One-dimensional projections of the double differential distribution $\frac{d^{2}\text{Br}}{dx_{12}dx_{34}}$ by $x_{12}$ and $x_{34}$ are given in Fig. (8) and (9) respectively. The distributions by $x_{12}$ are given in the range [0, 0.04], which corresponds to the area of applicability of the model. Fig. (8) features a photon pole for $x_{12} \to x_{12 \text{min}} = (2m_{\mu}/M_{1})^{2} = 0.0016$ and a peak from the $\omega(782)$ resonance for $x_{12} \to (M_{\omega}/M_{1})^{2} \approx 0.023$. Due to the fact that the $\rho^{0}(770)$ meson has a width of about 150 MeV, the contribution from this meson in Fig. (8) appears as a wide background to the narrow peak of the $\omega(782)$ resonance. The distributions by $x_{34}$ in Fig. (9) does not have poles, in agreement with the analysis from Section 2 and demonstrates the importance of taking into account the Fermi antisymmetry in the decay $B^{-} \to \mu^{+}\bar{\nu}_{\mu}\mu^{-}\mu^{-}$, because due to the additional contribution from Fermi antisymmetry the shapes of the distributions by $x_{34}$ in the decays $B^{-} \to \mu^{+}\mu^{-}\bar{\nu}_{e}e^{-}$ and $B^{-} \to \mu^{+}\bar{\nu}_{\mu}\mu^{-}\mu^{-}$ are significantly different. An analogous difference may be seen in the distributions by $y_{12} = \cos\theta_{12}$ and $y_{34} = \cos\theta_{34}$, which are presented in Fig. (10) and (11) respectively. The definition of angular variables $y_{12}$ and $y_{34}$ is given in Appendix A.

Detectability of the multi-lepton decays of the $B$–mesons with a neutrino in the final state may be linked to the distributions by normalized invariant mass of the charged leptons. The
square of the corresponding mass is defined as:

\[ x_{124} = \frac{(k_1 + k_2 + k_4)^2}{M_1^2}, \]  

where the \( k_i \) are four-momenta of charged leptons in the final state. The distributions by \( x_{124} \) are presented in Fig. 12. One can see from the figure that the shape of the distribution by \( x_{124} \) is not very sensitive to the procedure of Fermi antisymmetrization.

It is well known that forward–backward lepton asymmetries are very sensitive to BSM physics. For the decay \( B^- \to \mu^+\mu^-\bar{\nu}_\mu e^- \) it is possible to define forward–backward lepton asymmetries \( A_{FB}^{(B^-)}(x_{12}) \) and \( A_{FB}^{(B^-)}(x_{34}) \) according to equations (14) and (15). These asymmetries are shown in Fig. 13. The asymmetry \( A_{FB}^{(B^-)}(x_{12}) \) is shown only for the interval \( x_{12} \in [0, 0.04] \), which corresponds to the area of applicability of the current model. In this interval, excluding the area of the \( \omega(782) \) resonance, the contributions to \( A_{FB}^{(B^-)}(x_{12}) \) come from electromagnetic and strong processes; thus this asymmetry is close to zero in almost all of the
Figure 11: Normalized differential distributions  \( \frac{1}{\Gamma} \frac{d\Gamma}{dy_{34}} \) for the decays a) \( B^- \to \mu^+ \mu^- \bar{\nu}_e e^- \) and b) \( B^- \to \mu^+ \bar{\nu}_\mu \mu^- \mu^- \), obtained by integration by \( dx_{12} \, dx_{34} \, dy_{12} \, d\phi \) of equations (12) and (17) respectively.

Figure 12: Normalized differential distributions  \( \frac{1}{\Gamma} \frac{d\Gamma}{dx_{124}} \) by invariant mass of all of the charged leptons in the final state for the decays a) \( B^- \to \mu^+ \mu^- \bar{\nu}_e e^- \) and b) \( B^- \to \mu^+ \bar{\nu}_\mu \mu^- \mu^- \).

considered range. The shape of the asymmetry \( A^{(B^-)}_{FB}(x_{34}) \) is very similar to the shape of the asymmetries in three-body semileptonic decays of \( B^- \)-mesons.

One cannot to study forward–backward lepton asymmetries in the decay \( B^- \to \mu^+ \bar{\nu}_\mu \mu^- \mu^- \), as in this case there are two identical negative muons in the final state. Experimentally it is not possible to distinguish which of the negatively charged muons should be attributed to the \( \mu^+ \mu^- \)-pair, and which to the \( \bar{\nu}_\mu \mu^- \)-pair.

All the above that is related to the differential distributions for the decays \( B^- \to \mu^+ \mu^- \bar{\nu}_e e^- \) and \( B^- \to \mu^+ \bar{\nu}_\mu \mu^- \mu^- \) is also related to the differential distributions for the decays \( B^- \to e^+ e^- \bar{\nu}_\mu \mu^- \) and \( B^- \to e^+ \bar{\nu}_e e^- e^- \). In this model the lepton universality holds, so the differential distributions of the two latter decays are not needed.
Figure 13: Forward–backward lepton asymmetries a) $A_{FB}^{(B^-)}(x_{12})$ and b) $A_{FB}^{(B^-)}(x_{34})$ for the decay $B^- \rightarrow \mu^+\mu^-\bar{\nu}_e e^-$, calculated using equations (14) and (15) respectively.

Conclusion

In the present work,

- theoretical predictions for the branching ratios of the decays $B^- \rightarrow \mu^+\mu^-\bar{\nu}_e e^-$ and $B^- \rightarrow \mu^+\bar{\nu}_\mu \mu^-\mu^-$ are obtained in the framework of Standard Model:
  \[ \text{Br} \left( B^- \rightarrow \mu^+\mu^-\bar{\nu}_e e^- \right) \approx \frac{0.6 \times 10^{-7}}{} \]
  and
  \[ \text{Br} \left( B^- \rightarrow \mu^+\bar{\nu}_\mu \mu^-\mu^- \right) \approx \frac{0.7 \times 10^{-7}}{} \],
  and uncertainties for every prediction are discussed;
- the difference between the obtained predictions and the predictions from Ref.[8] is discussed, as well as the compatibility with the recent experimental result [6] by the LHCb collaboration;
- the possibility to test the hypothesis of lepton universality in rare four-leptonic decays of $B$–mesons with three charged leptons in the final state is analysed;
- double and single differential distributions for the decays $B^- \rightarrow \mu^+\mu^-\bar{\nu}_e e^-$ and $B^- \rightarrow \mu^+\bar{\nu}_\mu \mu^-\mu^-$ are considered, and some recommendations for searches for Beyond the Standard Model physics in these decays are given.

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A  Kinematics of four-lepton decays

Denote the four-momenta of the final leptons in four-leptonic decays of $B$–mesons as $k_i$, $i = \{1, 2, 3, 4\}$. Let

$$q = k_1 + k_2; \quad k = k_3 + k_4; \quad \bar{q} = k_1 + k_4; \quad \bar{k} = k_2 + k_3; \quad p = k_1 + k_2 + k_3 + k_4,$$

where $p$ is the four-momentum of the $B$–meson and $p^2 = M_1^2$. For the calculations below it is suitable to use the dimensionless variables:

$$x_{12} = \frac{q^2}{M_1^2}, \quad x_{34} = \frac{k^2}{M_1^2}, \quad x_{14} = \frac{\bar{q}^2}{M_1^2}, \quad x_{23} = \frac{\bar{k}^2}{M_1^2}.$$

By common notation, $x_{ij} = (k_i + k_j)^2/M_1^2$. Hence $x_{ij} = x_{ji}$. The leptons may be considered as massless in almost all of the calculations of the present work, i.e., $k_i^2 = 0$. However during the calculation of the bremsstrahlung contribution in the area $q^2 \sim 4m_\ell^2$, where $m_\ell$ is the mass of any of the charged leptons of the $\ell^+\ell^-$ pair, it is necessary to take into account the dependence of the bremsstrahlung matrix element and phase space on the value of $m_\ell$.

From the conservation law of four-momentum that in the zero-mass limit the variables $x_{ij}$ are linked by

$$x_{12} + x_{13} + x_{14} + x_{23} + x_{24} + x_{34} = 1. \quad (27)$$

Let us find the intervals for $x_{ij}$ using the inequality $(p_1 p_2) \geq \sqrt{p_1^2 p_2^2}$; then any $x_{ij} \geq 0$. On the other hand,

$$1 = \frac{p^2}{M_1^2} = \frac{(q + k)^2}{M_1^2} \geq \left(\sqrt{\frac{q^2 + k^2}{M_1^2}}\right)^2 = \left(\sqrt{x_{12} + x_{34}}\right)^2.$$

As $0 \leq x_{34}$, then $x_{12} \leq 1$, so $x_{12} \in [0, 1]$. The upper limit of the variable $x_{34}$ depends on the value of $x_{12}$:

$$x_{34} = \frac{(p - q)^2}{M_1^2} \leq \left(\frac{M_1 - \sqrt{q^2}}{M_1^2}\right)^2 = (1 - \sqrt{x_{12}})^2.$$

Thus for a fixed value of $x_{12}$ the variable $x_{34} \in \left[0, (1 - \sqrt{x_{12}})^2\right]$. 

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For the pair $x_{14}$ and $x_{23}$, the analogous condition holds: $x_{14} \in [0, 1]$ and for a fixed $x_{14}$, $x_{23} \in \left[0, (1 - \sqrt{x_{14}})^2\right]$.

Consider the kinematics of the decay $B^-(p) \rightarrow \ell^+(k_1) \ell^-(k_2) \bar{\nu}_\ell(k_3) \ell^-(k_4)$, when the flavor of negatively charged lepton $\ell^-(k_2)$ is different from the flavor of the negatively charged lepton $\ell^-(k_4)$. Let the positively charged lepton have the momentum $k_1$, and let the antineutrino have the momentum $k_3$. We define an angle $\theta_{12}$ between the momentum of the positively charged lepton and the direction of the $B$–meson ($z$–axis) in the rest frame of the $\ell^+\ell^−$–pair, and another angle $\theta_{34}$ between the direction of the antineutrino and the direction of the $B$–meson ($z$–axis) in the rest frame of the $\ell^−\bar{\nu}_\ell$–pair. Then

$$y_{12} \equiv \cos \theta_{12} = \frac{1}{\lambda^{1/2}(1, x_{12}, x_{34})} \left(x_{23} + x_{24} - x_{13} - x_{14}\right),$$

$$y_{34} \equiv \cos \theta_{34} = \frac{1}{\lambda^{1/2}(1, x_{12}, x_{34})} \left(x_{14} + x_{24} - x_{13} - x_{23}\right),$$

where $\lambda(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc$, the triangle function. Angles $\theta_{12} \in [0, \pi]$ and $\theta_{34} \in [0, \pi]$. Hence $y_{12} \in [-1, 1]$ and $y_{34} \in [-1, 1]$. Angles are measured relative to $z$–axis. Also let us define an angle $\varphi \in [0, 2\pi]$ in the rest frame of the $B$–meson between the planes which are set by the pairs of vectors $(k_1, k_2)$ and $(k_3, k_4)$. Introduce a vector $a_1 = k_1 \times k_2$, perpendicular to the plane $(k_1, k_2)$, and vector $a_3 = k_4 \times k_3$, perpendicular to the plane $(k_3, k_4)$. Then

$$\cos \varphi = \frac{(a_1, a_3)}{|a_1||a_3|}.$$
Using the technique from Ref. [25], for \( \cos \varphi \) we can write:

\[
\cos \varphi = \frac{\det \begin{pmatrix} M^2_1 & (p_1 k) & (p_2 k) \\ (p_1 k) & (k_1 k) & (k_2 k) \\ (p_2 k) & (k_1 k) & (k_2 k) \end{pmatrix}}{\sqrt{\det \begin{pmatrix} M^2_1 & (p_1 k) & 0 \\ (p_1 k) & (k_1 k) & 0 \\ (p_2 k) & 0 & (k_2 k) \end{pmatrix} \det \begin{pmatrix} M^2_1 & (p_3 k) & (p_4 k) \\ (p_3 k) & (k_3 k) & 0 \\ (p_4 k) & 0 & (k_4 k) \end{pmatrix} \det \begin{pmatrix} M^2_1 & (p_3 k) & (p_4 k) \\ (p_3 k) & 0 & (k_3 k) \\ (p_4 k) & (k_3 k) & 0 \end{pmatrix}}}. \tag{29}
\]

Simplifying the (29) gives

\[
-2 \sqrt{x_{12} x_{34} (1 - y_{12}^2) (1 - y_{34}^2)} \cos \varphi + (1 - x_{12} - x_{34}) y_{12} = x_{13} - x_{14} - x_{23} + x_{24}. \tag{30}
\]

Four-particle phase space has the form:

\[
d\Phi_{4}^{(1234)} = M^4_1 \frac{dx_{12}}{2\pi} \frac{dx_{34}}{2\pi} d\Phi_{2}^{(qk)} d\Phi_{2}^{(12)} d\Phi_{2}^{(34)},
\]

where (assuming zero masses for leptons \( \ell^\pm \) and \( \ell'^- \) masses) we can write

\[
\begin{align*}
d\Phi_{2}^{(qk)} &= 2\pi \delta \left( q^2 - x_{12} M^2_1 \right) \frac{d^4q}{(2\pi)^4} 2\pi \delta \left( k^2 - x_{34} M^2_1 \right) \frac{d^4k}{(2\pi)^4} (2\pi)^4 \delta^4 (p - q - k), \\
d\Phi_{2}^{(12)} &= 2\pi \delta \left( k^2_1 - m^2_\ell \right) \frac{d^4k_1}{(2\pi)^4} 2\pi \delta \left( k^2_2 - m^2_\ell \right) \frac{d^4k_2}{(2\pi)^4} (2\pi)^4 \delta^4 (q - k_1 - k_2), \\
d\Phi_{2}^{(34)} &= 2\pi \delta \left( k^2_3 \right) \frac{d^4k_3}{(2\pi)^4} 2\pi \delta \left( k^2_4 - m^2_{\ell'} \right) \frac{d^4k_4}{(2\pi)^4} (2\pi)^4 \delta^4 (k - k_3 - k_4).
\end{align*}
\]

It is suitable to choose \( x_{12}, x_{34}, y_{12}, y_{34}, \) and \( \varphi \) as independent integration variables when calculating the four-body phase space. Then

\[
\Phi_{2}^{(qk)} = \frac{1}{2^{3/2} \pi} \lambda^{1/2} \left( 1, x_{12}, x_{34} \right);
\]

and

\[
d\Phi_{2}^{(12)} = \frac{1}{2^{4} \pi} \sqrt{1 - \frac{4\hat{m}_\ell^2}{x_{12}}} dy_{12}, \quad d\Phi_{2}^{(34)} = \frac{1}{2^{6} \pi^2} \left( 1 - \frac{4\hat{m}_{\ell'}^2}{x_{34}} \right) dy_{34} d\varphi.
\]

This gives

\[
\begin{align*}
d\Phi_{4}^{(1234)} &= \frac{M^4_1}{2^{14} \pi^6} \lambda^{1/2} \left( 1, x_{12}, x_{34} \right) \sqrt{1 - \frac{4\hat{m}_\ell^2}{x_{12}}} \left( 1 - \frac{4\hat{m}_{\ell'}^2}{x_{34}} \right) dx_{12} dx_{34} dy_{12} dy_{34} d\varphi, \tag{31}
\end{align*}
\]

where \( \hat{m}_\ell = m_\ell/M_1 \) and \( \hat{m}_{\ell'} = m_{\ell'}/M_1. \)

In the decay \( B^- (p) \rightarrow \ell^+ (k_1) \ell^- (k_2) \bar{\nu}_\ell (k_3) \ell^- (k_4) \) there are two identical leptons \( \ell^- (k_2) \) and \( \ell^- (k_4) \) in the final state, so Fermi antisymmetrization of the decay amplitude is necessary by four-momenta \( k_2 \) and \( k_4. \) We will need an additional formula to calculate of the branching ratio in this case for \( m_\ell \neq 0: \)

\[
\begin{align*}
d\Phi_{4}^{(1432)} &= \frac{M^4_1}{2^{14} \pi^6} \lambda^{1/2} \left( 1, x_{14}, x_{23} \right) \sqrt{1 - \frac{4\hat{m}_\ell^2}{x_{12}}} \left( 1 - \frac{4\hat{m}_{\ell'}^2}{x_{34}} \right) dx_{14} dx_{23} dy_{14} dy_{23} d\varphi, \tag{32}
\end{align*}
\]
where $\tilde{\phi}$ is the angle of planes $(k_1, k_4)$ and $(k_2, k_3)$, measured relative to plane $(k_1, k_4)$. The equation (32) may be obtained in a fully analogous way to (31). The $\cos \tilde{\phi}$ can be found by exchanging indices in equations (29) and (30) as $2 \leftrightarrow 4$. Also in order to perform numerical integration it is necessary to have all the definitions of $x_{ij}$ using the set of variables $x_{12}, x_{34}, y_{12}, y_{34},$ and $\phi$. From (27), (28), and (30), assuming zero lepton masses, we have:

\[
\begin{align*}
    x_{13} &= \frac{1}{4} \left( -2 \sqrt{x_{12} x_{34} (1 - y_{12}^2) (1 - y_{34}^2)} \cos \phi + (1 - x_{12} - x_{34}) y_{12} y_{34} - 
    \lambda^{1/2} (1, x_{12}, x_{34}) (y_{12} + y_{34}) + 1 - x_{12} - x_{34} \right); \\
    x_{14} &= \frac{1}{4} \left( 2 \sqrt{x_{12} x_{34} (1 - y_{12}^2) (1 - y_{34}^2)} \cos \phi - (1 - x_{12} - x_{34}) y_{12} y_{34} - 
    \lambda^{1/2} (1, x_{12}, x_{34}) (y_{12} - y_{34}) + 1 - x_{12} - x_{34} \right); \\
    x_{23} &= \frac{1}{4} \left( 2 \sqrt{x_{12} x_{34} (1 - y_{12}^2) (1 - y_{34}^2)} \cos \phi - (1 - x_{12} - x_{34}) y_{12} y_{34} + 
    \lambda^{1/2} (1, x_{12}, x_{34}) (y_{12} - y_{34}) + 1 - x_{12} - x_{34} \right); \\
    x_{24} &= \frac{1}{4} \left( -2 \sqrt{x_{12} x_{34} (1 - y_{12}^2) (1 - y_{34}^2)} \cos \phi + (1 - x_{12} - x_{34}) y_{12} y_{34} + 
    \lambda^{1/2} (1, x_{12}, x_{34}) (y_{12} + y_{34}) + 1 - x_{12} - x_{34} \right); \\
\end{align*}
\]

This paper use notations almost identical to the notations of Ref. [26], except for in case of the $y_{ij}$, which here have the opposite sign compared to Ref. [26].
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