An exponential Diophantine equation related to the difference between powers of two consecutive Balancing numbers

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Abstract

In this paper, we find all solutions of the exponential Diophantine equation $B_{n+1}^z - B_n^z = B_m$ in positive integer variables $(m, n, x)$, where $B_k$ is the $k$-th term of the Balancing sequence.

Keywords: Balancing numbers, Linear form in logarithms, reduction method.
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1. Introduction

The first definition of balancing numbers is essentially due to Finkelstein [3], although he called them numerical centers. A positive integer \( n \) is called a balancing number if

\[
1 + 2 + \cdots + (n - 1) = (n + 1) + (n + 2) + \cdots + (n + r)
\]

holds for some positive integer \( r \). Then \( r \) is called the balancer corresponding to the balancing number \( n \). For example, 6 and 35 are balancing numbers with balancers 2 and 14, respectively. The \( n \)-th term of the sequence of balancing numbers is denoted by \( B_n \). The balancing numbers satisfy the recurrence relation

\[
B_n = 6B_{n-1} - B_{n-2}, \quad \text{for all } n \geq 2,
\]

where the initial conditions are \( B_0 = 0 \) and \( B_1 = 1 \). Its first terms are

\[
0, 1, 6, 35, 204, 1189, 6930, 40391, 235416, 1372105, \ldots
\]

It is well-known that

\[
B_{n+1}^2 - B_n^2 = B_{2n+2}, \quad \text{for any } n \geq 0.
\]

In particular, this identity tells us that the difference between the square of two consecutive Balancing numbers is still a Balancing number. So, one can ask if this identity can be generalized?

Diophantine equations involving sum or difference of powers of two consecutive members of a given linear recurrent sequence \( \{U_n\}_{n \geq 1} \) were also considered in several papers. For example, in [5], Marques and Togbé proved that if \( s \geq 1 \) an integer such that \( F^s_m + F^s_{m+1} \) is a Fibonacci number for all sufficiently large \( m \), then \( s \in \{1, 2\} \). In [4], Luca and Oyono proved that there is no integer \( s \geq 3 \) such that the sum of \( s \)th powers of two consecutive Fibonacci numbers is a Fibonacci number. Later, their result has been extended in [8] to the generalized Fibonacci numbers and recently in [7] to the Pell sequence.

Here, we apply the same argument as in [4] to the Balancing sequence and prove the following:

**Theorem 1.1.** The only nonnegative integer solutions \((m, n, x)\) of the Diophantine equation

\[
B_{n+1}^x - B_n^x = B_m
\]

are \((m, n, x) = (2n + 2, n, 2), (1, 0, x), (0, n, 0)\).
2. Preliminary results

2.1. The Balancing sequences

Let \((\alpha, \beta) = (3 + 2\sqrt{2}, 3 - 2\sqrt{2})\) be the roots of the characteristic equation \(x^2 - 6x + 1 = 0\) of the Balancing sequence \((B_n)_{n \geq 0}\). The Binet formula for \(B_n\) is

\[
B_n = \frac{\alpha^n - \beta^n}{4\sqrt{2}}, \quad \text{for all } n \geq 0. \tag{2.1}
\]

This implies that the inequality

\[
\alpha^{n-2} \leq B_n \leq \alpha^{n-1} \tag{2.2}
\]

holds for all positive integers \(n\). It is easy to prove that

\[
\frac{B_n}{B_{n+1}} \leq \frac{5}{29} \tag{2.3}
\]

holds, for any \(n \geq 2\).

2.2. Linear forms in logarithms

For any non-zero algebraic number \(\gamma\) of degree \(d\) over \(\mathbb{Q}\), whose minimal polynomial over \(\mathbb{Z}\) is \(a \prod_{i=1}^{d} (X - \gamma^{(i)})\), we denote by

\[
h(\gamma) = \frac{1}{d} \left( \log |a| + \sum_{i=1}^{d} \log \max \left( 1, |\gamma^{(i)}| \right) \right)
\]

the usual absolute logarithmic height of \(\gamma\).

With this notation, Matveev proved the following theorem (see [6]).

**Theorem 2.1.** Let \(\gamma_1, \ldots, \gamma_s\) be real algebraic numbers and let \(b_1, \ldots, b_s\) be nonzero rational integer numbers. Let \(D\) be the degree of the number field \(\mathbb{Q}(\gamma_1, \ldots, \gamma_s)\) over \(\mathbb{Q}\) and let \(A_j\) be positive real numbers satisfying

\[
A_j = \max\{Dh(\gamma_j), \log \gamma_j, 0.16\}, \quad \text{for } j = 1, \ldots, s.
\]

Assume that

\[
B \geq \max\{|b_1|, \ldots, |b_s|\}.
\]

If \(\gamma_1^{b_1} \cdots \gamma_s^{b_s} - 1 \neq 0\), then

\[
|\gamma_1^{b_1} \cdots \gamma_s^{b_s} - 1| \geq \exp(-1.4 \cdot 30^{s+3} \cdot s^{1.5} \cdot D^2(1 + \log D)(1 + \log B)A_1 \cdots A_s).
\]
2.3. Reduction algorithm

Lemma 2.2. Let $M$ be a positive integer, let $p/q$ be a convergent of the continued fraction expansion of the irrational $\gamma$ such that $q > 6M$, and let $A, B, \mu$ be some real numbers with $A > 0$ and $B > 1$. Let

$$\varepsilon = ||\mu q|| - M \cdot ||\gamma q||,$$

where $|| \cdot ||$ denotes the distance from the nearest integer. If $\varepsilon > 0$, then there is no solution of the inequality

$$0 < m\gamma - n + \mu < AB^{-k}$$

in positive integers $m, n$ and $k$ with

$$m \leq M \quad \text{and} \quad k \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

3. The proof of Theorem 1.1

3.1. An inequality for $x$ versus $m$ and $n$

The case $nx = 0$ is trivial so we assume that $n \geq 1$ and that $x \geq 1$. Observe that since $B_n < B_{n+1} - B_n < B_{n+1}$, the Diophantine equation (1.1) has no solution when $x = 1$.

When $n = 1$, we get $B_m = 6^x - 1$. In this case, we have that $m$ is odd. Thus, using the Binet formula (2.1), we obtained the following factorization

$$6^x = B_m + 1 = B_m + B_1 = B_{(m+1)/2}C_{(m-1)/2},$$

where $\{C_m\}_{m \geq 1}$ is the Lucas Balancing sequence given by the recurrence $C_m = 6C_{m-1} - C_{m-2}$ with initial conditions $C_0 = 2$, $C_1 = 6$. The Binet formula of the Lucas Balancing sequence is given by $C_n = \alpha^n + \beta^n$. This shows that the largest prime factor of $B_{(m+1)/2}$ is 3 and by Carmichael’s Primitive Divisor Theorem we conclude that $(m+1)/2 \leq 12$, so $m \leq 23$. Now, one checks all such $m$ and gets no additional solution with $n = 1$.

So, we can assume that $n \geq 2$ and $x \geq 3$. Therefore, we have

$$B_m = B_{n+1}^x - B_n^x \geq B_3^x - B_1^x = 215,$$

which implies that $m > 4$. Here, we use the same argument from [4] to bound $x$ in terms of $m$ and $n$. Since most of the details are similar, we only sketch the argument.

Using inequality (2.2), we get

$$\alpha^{m-1} > B_m = B_{n+1}^x - B_n^x \geq B_n^x > \alpha^{(n-2)x}$$
and
\[ \alpha^{m-2} < B_m = B_{n+1}^x - B_n^x < B_{n+1}^x < \alpha^{nx}. \]

Thus, we have
\[ (n-2)x + 1 < m < nx + 2. \]  \hspace{1cm} (3.1)

Estimate (3.1) is essential for our purpose.

Now, we rewrite equation (1.1) as
\[ \frac{\alpha^m}{4\sqrt{2}} - B_{n+1}^x = -B_n^x + \frac{\beta^m}{4\sqrt{2}}. \]  \hspace{1cm} (3.2)

Dividing both sides of equation (3.2) by \( B_{n+1}^x \), taking absolute value and using the inequality (2.3), we obtain
\[ \left| \alpha^m (4\sqrt{2})^{-1} B_{n+1}^{-x} - 1 \right| < 2 \left( \frac{B_n}{B_{n+1}} \right)^x < \frac{2}{5.8^x}. \]  \hspace{1cm} (3.3)

Put
\[ \Lambda_1 := \alpha^m (4\sqrt{2})^{-1} B_{n+1}^{-x} - 1. \]  \hspace{1cm} (3.4)

If \( \Lambda_1 = 0 \), we get \( \alpha^m = 4\sqrt{2} B_{n+1}^x \). Thus \( \alpha^{2m} \in \mathbb{Z} \), which is false for all positive integers \( m \), therefore \( \Lambda_1 \neq 0 \).

At this point, we will use Matveev’s theorem to get a lower bound for \( \Lambda_1 \). We set \( s := 3 \) and we take
\[ \gamma_1 := \alpha, \quad \gamma_2 := 4\sqrt{2}, \quad \gamma_3 := B_{n+1}, \quad b_1 := m, \quad b_2 := -1, \quad b_3 := -x. \]

Note that \( \gamma_1, \gamma_2, \gamma_3 \in \mathbb{Q}(\sqrt{2}) \), so we can take \( D := 2 \). Since \( h(\gamma_1) = (\log \alpha)/2 \), \( h(\gamma_2) = (\log 32)/2 \) and \( h(\gamma_3) = \log B_{n+1} < n \log \alpha \), we can take \( A_1 := \log \alpha \), \( A_2 := \log 32 \) and \( A_3 := 2n \log \alpha \). Finally, inequality (3.1) implies that \( m > (n-2)x \geq x \), thus we can take \( B := m \). We also have \( B := m \leq nx + 2 < (n+2)x \). Hence, Matveev’s theorem implies that
\[ \log |\Lambda_1| \geq -1.4 \times 30^6 \times 3^{1.5} \times 2^2 \times (1 + \log 2)(\log \alpha)(\log 32)(2n \log \alpha)(1 + \log m) \geq -2.1 \times 10^{13} n(1 + \log m). \]  \hspace{1cm} (3.5)

The inequalities (3.3), (3.4) and (3.5) give that
\[ x < 1.2 \times 10^{13} n(1 + \log m) < 2.1 \times 10^{13} n \log m, \]

where we used the fact that \( 1 + \log m < 1.7 \log m \), for all \( m \geq 5 \). Together with the fact that \( m < (n+2)x \), we get that
\[ x < 2.1 \times 10^{13} n \log((n+2)x). \]
3.2. Small values of $n$

Next, we treat the cases when $n \in [2, 37]$. In this case,

$$x < 2.1 \times 10^{13} n \log((n + 2)x) < 7.8 \times 10^{14} \log(46x)$$

so $x < 4 \times 10^{16}$.

Now, we take another look at $\Lambda_1$ given by expression (3.4). Put

$$\Gamma_1 := m \log \alpha - \log(4\sqrt{2}) - x \log B_{n+1}.$$ 

Thus, $\Lambda_1 = e^{\Gamma_1} - 1$. One sees that the right-hand side of (3.2) is a number in the interval $[-B_n^*, -B_n^* + 1]$. In particular, $\Lambda_1$ is negative, which implies that $\Gamma_1$ is negative. Thus,

$$0 < -\Gamma_1 < \frac{2}{5.8^x},$$

so

$$0 < x \left( \frac{\log B_{n+1}}{\log \alpha} \right) - m + \left( \frac{\log(4\sqrt{2})}{\log \alpha} \right) < \frac{2}{5.8^x \log \alpha}. \quad (3.6)$$

For us, inequality (3.6) is

$$0 < x \gamma - m + \mu < AB^{-x},$$

where

$$\gamma := \frac{\log B_{n+1}}{\log \alpha}, \quad \mu = \frac{\log(4\sqrt{2})}{\log \alpha}, \quad A = \frac{2}{\log \alpha}, \quad B = 5.8.$$ 

We take $M := 4 \times 10^{16}$.

The program was developed in PARI/GP running with 200 digits. For the computations, if the first convergent such that $q > 6M$ does not satisfy the condition $\varepsilon > 0$, then we use the next convergent until we find the one that satisfies the condition. In one minute all the computations were done. In all cases, we obtained $x \leq 77$. A computer search with Maple revealed in less than one minute that there are no solutions to the equation (1.1) in the range $n \in [3, 37]$ and $x \in [3, 77]$.

3.3. An upper bound on $x$ in terms of $n$

From now on, we assume that $n \geq 38$. Recall from the previous section that

$$x < 2.1 \times 10^{13} n \log((n + 2)x). \quad (3.7)$$

Next, we give an upper bound on $x$ depending only on $n$. If

$$x \leq n + 2, \quad (3.8)$$

then we are through. Otherwise, that is if $n + 2 < x$, we then have

$$x < 2.1 \times 10^{13} n \log x^2 = 4.2 \times 10^{13} n \log x,$$
which can be rewritten as

$$\frac{x}{\log x} < 4.2 \times 10^{13} n.$$  \hspace{1cm} (3.9)

Using the fact that, for all \( A \geq 3 \)

$$\frac{x}{\log x} < A \quad \text{yields} \quad x < 2A \log A,$$

and the fact that \( \log(4.2 \times 10^{13} n) < 10 \log n \) holds for all \( n \geq 38 \), we get that

$$x < 2(4.2 \times 10^{13} n) \log((4.2 \times 10^{13} n))$$

$$< 8.4 \times 10^{13} n(10 \log n)$$

$$< 8.4 \times 10^{14} n \log n.$$

From (3.8) and (3.10), we conclude that the inequality

$$x < 8.4 \times 10^{14} n \log n$$  \hspace{1cm} (3.11)

holds.

### 3.4. An absolute upper bound on \( x \)

Let us look at the element

$$y := \frac{x}{\alpha^{2n}}.$$

The above inequality (3.11) implies that

$$y < \frac{8.4 \times 10^{14} n \log n}{\alpha^{2n}} < \frac{1}{\alpha^n},$$  \hspace{1cm} (3.12)

where the last inequality holds for any \( n \geq 23 \). In particular, \( y < \alpha^{-38} < 10^{-31} \).

We now write

$$B_n^x = \frac{\alpha^{nx}}{32^{x/2}} \left( 1 - \frac{1}{\alpha^{2n}} \right)^x$$

and

$$B_{n+1}^x = \frac{\alpha^{(n+1)x}}{32^{x/2}} \left( 1 - \frac{1}{\alpha^{2(n+1)}} \right)^x.$$

We have

$$0 < \left( 1 - \frac{1}{\alpha^{2n}} \right) < e^y < 1 + 2y,$$

because \( y < 10^{-31} \) is very small. The same inequality holds if we replace \( n \) by \( n + 1 \). Hence, we have that

$$\max \left\{ \left| B_n^x - \frac{\alpha^{nx}}{32^{x/2}} \right|, \left| B_{n+1}^x - \frac{\alpha^{(n+1)x}}{32^{x/2}} \right| \right\} < \frac{2y\alpha^{(n+1)x}}{32^{x/2}}.$$
We now return to our equation (1.1) and rewrite it as
\[
\frac{\alpha_m - \beta_m}{4\sqrt{2}} = B_m = B_{n+1} - B_n = \frac{\alpha^{(n+1)x}}{32^{x/2}} - \frac{\alpha^{nx}}{32^{x/2}} + \left(B_{n+1} - \frac{\alpha^{(n+1)x}}{32^{x/2}}\right) - \left(B_n - \frac{\alpha^{nx}}{32^{x/2}}\right),
\]
or
\[
\left|\frac{\alpha_m}{32^{1/2}} - \frac{\alpha^{nx}}{32^{x/2}}(\alpha^x - 1)\right| = \left|\frac{\beta_m}{32^{1/2}} + \left(B_{n+1} - \frac{\alpha^{(n+1)x}}{32^{x/2}}\right) - \left(B_n - \frac{\alpha^{nx}}{32^{x/2}}\right)\right|
< \frac{1}{\alpha^m} + \left|B_{n+1} - \frac{\alpha^{(n+1)x}}{32^{x/2}}\right| + \left|B_n - \frac{\alpha^{nx}}{32^{x/2}}\right|
< \frac{1}{\alpha^m} + 2y\left(\frac{\alpha^{nx}(1 + \alpha^x)}{32^{x/2}}\right).
\]
Thus, multiplying both sides by \(\alpha^{-(n+1)x}32^{x/2}\), we obtain that
\[
\left|\alpha^{m-(n+1)x}32^{(x-1)/2} - (1 - \alpha^{-x})\right| < \frac{32^{x/2}}{\alpha^{m+(n+1)x}} + 2y(1 + \alpha^{-x})
< \frac{1}{2\alpha^n} + \frac{396y}{197} < \frac{3}{\alpha^n},
\]
where we used the fact that \(32^{x/2}/(\alpha^{(n+1)x}) \leq (4\sqrt{2}/\alpha^{38})^{x} < 1/2, m \geq (n-2)x \geq n\) and \(\alpha^x \geq \alpha^3 > 197\), as well as inequality (3.12). Hence, we conclude that
\[
\left|\alpha^{m-(n+1)x}32^{(x-1)/2} - 1\right| < \frac{1}{\alpha^x} + \frac{3}{\alpha^n} \leq \frac{4}{\alpha^l},
\]
where \(l := \min\{n, x\}\). We now set
\[
\Lambda_2 := \alpha^{m-(n+1)x}32^{(x-1)/2} - 1
\]
and observe that \(\Lambda_2 \neq 0\). Indeed, for if \(\Lambda_2 = 0\), then \(\alpha^{2((n+1)x-m)} = 32^{x-1} \in \mathbb{Z}\) which is possible only when \((n+1)x = m\). But if this were so, then we would get \(0 = \Lambda_2 = 32^{(x-1)/2} - 1\), which leads to the conclusion that \(x = 1\), which is not possible. Hence, \(\Lambda_2 \neq 0\). Next, let us notice that since \(x \geq 3\) and \(m \geq 38\), we have that
\[
|\Lambda_2| \leq \frac{1}{\alpha^3} + \frac{1}{\alpha^{38}} < \frac{1}{2},
\]
so that \(\alpha^{m-(n+1)x}32^{(x-1)/2} \in [1/2, 3/2]\). In particular,
\[
(n + 1)x - m < \frac{1}{\log \alpha} \left(\frac{(x - 1) \log 32}{2} + \log 2\right) < x \left(\frac{\log 32}{2\log \alpha}\right) < x
\]
and
\[
(n + 1)x - m > \frac{1}{\log \alpha} \left(\frac{(x - 1) \log 32}{2} - \log 2\right) > 0.9x - 1.4 > 0.
\]
We lower bound the left-hand side of inequality (3.15) using again Matveev’s theorem. We take
\[ s := 2, \gamma_1 := \alpha, \gamma_2 := 4\sqrt{2}, b_1 := m - (n + 1)x, b_2 := x - 1, \]
\[ D := 2, A_1 := \log \alpha, A_2 := \log 32, \text{ and } B := x. \]

We thus get that
\[ \log |\Lambda_2| > -1.4 \times 30^5 \times 2^{4.5} \times 2^2 (1 + \log 2)(\log \alpha)(\log 32)(1 + \log x). \] (3.19)

The inequalities (3.14) and (3.19) give
\[ l < 4 \times 10^{10} \log x. \]

Treating separately the case \( l = x \) and the case \( l = n \), following the argument in [4] we have that the upper bound
\[ x < 7 \times 10^{28} \]
always holds.

### 3.5. Reducing the bound on \( x \)

Next, we take
\[ \Gamma_2 := (x - 1) \log(4\sqrt{2}) - ((n + 1)x - m) \log \alpha. \]

Observe that \( \Lambda_2 = e^{\Gamma_2} - 1 \), where \( \Lambda_2 \) is given by (3.15). Since \( |\Lambda_2| < \frac{1}{2} \), we have that \( e^{||\Gamma_2|} < 2 \). Hence,
\[ |\Gamma_2| \leq e^{||\Gamma_2|} |e^{\Gamma_2} - 1| < 2 |\Lambda_2| < \frac{2}{\alpha^x} + \frac{6}{\alpha^n}. \]

This leads to
\[ \left| \log(4\sqrt{2}) - \frac{(n + 1)x - m}{x - 1} \right| < \frac{1}{(x - 1) \log \alpha} \left( \frac{2}{\alpha^x} + \frac{6}{\alpha^n} \right). \] (3.20)

Assume next that \( x > 100 \). Then \( \alpha^x > \alpha^{100} > 10^{33} > 10^4x \). Hence, we get that
\[ \frac{1}{(x - 1) \log \alpha} \left( \frac{2}{\alpha^x} + \frac{6}{\alpha^n} \right) < \frac{8}{x(x - 1)10^4 \log \alpha} < \frac{1}{2200(x - 1)^2}. \] (3.21)

Estimates (3.20) and (3.21) lead to
\[ \left| \log(4\sqrt{2}) - \frac{(n + 1)x - m}{x - 1} \right| < \frac{1}{2200(x - 1)^2}. \] (3.22)
By a criterion of Legendre, inequality (3.22) implies that the rational number \((\frac{nx+1}{x-1})\) is a convergent to \(\gamma := \frac{\log(4\sqrt{2})}{\log \alpha}\). Let

\([a_0, a_1, a_2, a_3, a_4, a_5, a_6, \ldots] = [0, 1, 57, 1, 234, 2, 1, \ldots]\)

be the continued fraction of \(\gamma\), and let \(p_k/q_k\) be its \(k\)th convergent. Assume that \((\frac{nx+1}{x-1}) = p_k/q_k\) for some \(k\). Then, \(x-1 = dq_k\) for some positive integer \(d\), which in fact is the greatest common divisor of \((n+1)x - m\) and \(x-1\). We have the inequality

\[q_{54} > 7 \times 10^{28} > x-1.\]

Thus, \(k \in \{0, \ldots, 53\}\). Furthermore, \(a_k \leq 234\) for all \(k = 0, 1, \ldots, 53\). From the known properties of the continued fraction, we have that

\[\left| \gamma - \frac{n+1}{x-1} \right| = \left| \gamma - \frac{p_k}{q_k} \right| > \frac{1}{(a_k + 2)q_k^2} \geq \frac{d^2}{236(x-1)^2} \geq \frac{1}{236(x-1)^2},\]

which contradicts inequality (3.22). Hence, \(x \leq 100\).

3.6. The final step

To finish, we go back to inequality (3.13) and rewrite it as

\[\left| \alpha^n(n+1)x - m \right| < \frac{3}{\alpha^n(1-\alpha^{-x})} < \frac{4}{\alpha^n}.\]

Recall that \(x \in [3, 100]\) and from inequalities (3.17) and (3.18), we have that

\[0.9x - 1.4 < (n+1)x - m < x.\]

Put \(t := (n+1)x - m\). We computed all the numbers \(|\alpha^t32^{(x-1)/2}(1+\alpha^{-x})^{-1} - 1|\) for all \(x \in [3, 100]\) and all \(t \in [(0.9x - 1.4), [x]]\). None of them ended up being zero and the smallest of these numbers is \(>10^{-1}\). Thus, \(1/10 < 3/\alpha^n\), or \(\alpha^n < 30\), so \(n \leq 3\) which is false.

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