New critical frontiers for the Potts and percolation models

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Abstract

We obtain the critical threshold for a host of Potts and percolation models on lattices having a structure which permits a duality consideration. The consideration generalizes the recently obtained thresholds of Scullard and Ziff for bond and site percolation on the martini and related lattices to the Potts model and to other lattices.

Key words: Critical frontier, Potts model, bond percolation, site percolation.
1 Introduction

The Potts model [1] has been in the forefront of active research for many years. Despite concerted efforts, however, very few exact results are known [2]. Unlike the Ising model for which the exact solution is known for all two-dimensional lattices, the relatively simple question of locating the critical frontier of the Potts model has been resolved only for the square, triangular, and honeycomb lattices [3, 4, 5]. The determination of the Potts critical frontier for other two-dimensional lattices has remained very much an open problem [6].

In two recent papers using a star-triangle relation and a dual transformation, Scullard [7] and Ziff [8] succeeded to determine the critical thresholds of site and bond percolation processes for several new two-dimensional lattices. As percolation problems are realized in the \(q = 1\) limit of the \(q\)-state Potts model [9, 10], the new percolation results suggest the possibility that similar thresholds can also be determined for the Potts model. In this paper we report this extension. We derive more generally the exact critical frontier of the Potts model for a large class of two-dimensional lattices including those considered in [7, 8], and obtain the corresponding percolation thresholds.

Consider a lattice having the structure shown in Fig. 1, where each shaded triangle denotes a network connected to its exterior through 3 spins \(\sigma_1, \sigma_2, \sigma_3\). It was established by Baxter, Temperley and Ashley [3] using an algebraic approach that this Potts model possesses a duality relation and a self-dual trajectory. A graphical proof of the duality relation was later given by Wu and Lin [11], and subsequently Wu and Zia [12] established rigorously that in the ferromagnetic regime of the parameter space the critical threshold is indeed the self-dual trajectory.

Specifically, write the Boltzmann factor for the shaded triangle as

\[
F(\sigma_1, \sigma_2, \sigma_3) = A + B_1 \delta_{23} + B_2 \delta_{31} + B_3 \delta_{12} + C \delta_{123} \tag{1}
\]

where \(\delta_{ij} = \delta_{\sigma_i, \sigma_j}\), \(\delta_{ijk} = \delta_{ij} \delta_{jk} \delta_{ki}\). Then the model possesses a duality relation in the parameter space \(\{A, B_1, B_2, B_3, C\}\). In the ferromagnetic regime

\[
B_1 + B_2 + B_3 + C > 0, \quad B_i + B_j + C > 0, \quad i \neq j \tag{2}
\]

the critical frontier of the Potts model is given by the self-dual trajectory

\[
qA - C = 0. \tag{3}
\]
By realizing the shaded network as a simple triangle, for example, one recovers from [3] the critical point for the Potts and bond percolation models on the square, triangular, and honeycomb lattices [5]. Another realization of the Boltzmann factor (1) is the random cluster model [9] with 2- and 3-site interactions [11]. The isotropic version of the random cluster model has been analyzed very recently by Chayes and Lei [13] who established on a rigorous ground the duality relation and the self-dual trajectory (3). Our new results concern with other realizations of (1).

2 The martini lattice

Consider the network shown in Fig. 2 as an instance of the shaded triangle in Fig. 1. This gives rise to the martini lattice shown in Fig. 3 [7, 8]. The Boltzmann factor for the network is

\[
F(\sigma_1, \sigma_2, \sigma_3) = \sum_{\{\sigma_4, \sigma_5, \sigma_6\}=1}^{q} \exp \left[ V_1 \delta_{14} + V_2 \delta_{25} + V_3 \delta_{36} + W_1 \delta_{56} + W_2 \delta_{46} + W_3 \delta_{45} + M \delta_{456} \right],
\]

(4)

where \(V_i\) and \(W_i\) are 2-site Potts interactions and \(M\) a 3-site interaction.

It is straightforward to cast (4) in the form of (1) [14, 15] to obtain

\[
A = v_1 v_2 v_3 + v_1 v_2 (q + w_1 + w_2) + v_2 v_3 (q + w_2 + w_3) + v_3 v_1 (q + w_3 + w_1) + (q + v_1 + v_2 + v_3) \left[ q^2 + q(w_1 + w_2 + w_3) + h \right]
\]

\[
B_i = v_j v_k \left[ h + (q + v_i) w_i \right], \quad i \neq j \neq k \neq j
\]

\[
C = v_1 v_2 v_3 h,
\]

(5)

where

\[
v_i = e^{V_i} - 1, \quad w_i = e^{W_i} - 1
\]

\[h = e^{M+W_1+W_2+W_3} - e^{W_1} - e^{W_2} - e^{W_3} + 2.
\]

(6)

As alluded to in the above, in the ferromagnetic regime \(W_i \geq 0, V_i \geq 0, M \geq 0\) satisfying (2), the critical frontier of this Potts model is the self-dual trajectory (3) which now reads

\[
q(q + v_1 + v_2 + v_3) \left[ q^2 + q(w_1 + w_2 + w_3) + h \right] + q \left[ v_1 v_2 v_3 + v_1 v_2 (w_1 + w_2 + q) + v_2 v_3 (w_2 + w_3 + q) + v_3 v_1 (w_3 + w_1 + q) \right] - v_1 v_2 v_3 h = 0.
\]

(7)
The critical frontier (7) is a new result for the Potts model.

For $M = \infty$ one retains only terms linear in $h$ and (7) reduces to the critical frontier $q^2 + q(v_1 + v_2 + v_3) = v_1 v_2 v_3$ of the honeycomb lattice. For $M = 0$, $V_1 = V_2 = V_3 = V$, $W_1 = W_2 = W_3 = W$, which is the isotropic model with pure 2-site interactions, (7) becomes

$$q(q+3v)(q^2+3qw+3w^2+w^3)+qv^2(v+6w+3q)-v^3(3w^2+w^3) = 0,$$

where $v = e^V - 1$, $w = e^W - 1$. For $w = v$ it reduces further to the expression

$$q^4 + 6q^2v + q^2v^2(15 + v) + qv^3(16 + 3v) - v^5(3 + v) = 0. \quad (9)$$

One variation of the martini lattice is the A lattice [7, 8] shown in Fig. 4(a) obtained from the martini lattice by setting $v_1 = \infty, v_2 = v_3 = v, w_1 = w_2 = w_3 = w$. This gives rise to the Potts critical frontier

$$q^3 + q^2(2v + 3w) + q(v^2 + 4vw + 3w^2 + w^3) - v^2(3w^2 + w^3) = 0, \quad A \text{ lattice,} \quad (10)$$

Another variation of the martini lattice is the B lattice [7, 8] shown in Fig. 4(b) obtained from the martini lattice by setting $v_2 = v_3 = \infty, v_1 = v, w_1 = w_2 = w_3 = w$. This leads to the Potts critical frontier

$$q^2 + q(v + 2w) - vw^2(3 + w) = 0, \quad B \text{ lattice.} \quad (11)$$

Both expressions (10) and (11) are new.

### 3 Percolation threshold

We now specialize the above results to percolation.

It is well-known that bond percolation is realized by taking the $q = 1$ limit of the $q$-state Potts model with 2-site interactions [9, 16]. For bond percolation on the martini lattice in Fig. 3, we set $q = 1$ and introduce bond occupation probabilities $x_i = 1 - e^{-V_i}, \ y_i = 1 - e^{-W_i}$. The percolation threshold (7) then assumes the form

$$x_1x_2(y_3 + y_1y_2 - y_1y_2y_3) + x_2x_3(y_1 + y_2y_3 - y_1y_2y_3) + x_3x_1(y_2 + y_3y_1 - y_1y_2y_3) - x_1x_2x_3(y_1y_2 + y_2y_3 + y_3y_1 - 2y_1y_2y_3) = 1 + (e^M - 1)(1 - x_1x_2 - x_2x_3 - x_3x_1 + x_1x_2x_3). \quad (12)$$
For isotropic bond percolation \( x_i = x, y_i = y \) and \( M = 0 \), this reduces to the threshold

\[
3x^2y(1 + y - y^2) - x^3y^2(3 - 2y) = 1, \tag{13}
\]

which is a result obtained in [8].

For bond percolation on the martini A and martini B lattices shown in Fig. 4, by setting \( y_i = y \) and \( x_i = x \) or \( x_i = 1 \) (for \( V_i = \infty \)) we obtain from (12) the thresholds

\[
2xy(1 + y - y^2) + x^2y(1 - y)^2 = 1, \quad A \text{ lattice},
\]

\[
(1 - y)^2(1 + y) - xy(2 - y) = 0, \quad B \text{ lattice}. \tag{14}
\]

For uniform bond occupation probability \( x = y = p \), (13) and (14) reduce to

\[
(2p^2 - 1)(p^4 - 3p^3 + 2p^2 + 1) = 0, \quad \text{martini lattice},
\]

\[
p^5 - 4p^4 + 3p^3 + 2p^2 = 1, \quad A \text{ lattice},
\]

\[
(1 - 2p)(1 + p - p^2) = 0, \quad B \text{ lattice}, \tag{15}
\]
yielding the thresholds \( p_c = 1/\sqrt{2}, 0.625457 \cdots \), and 1/2 respectively. These numbers have been reported in [7, 8]. Note that the thresholds (13) and (14) can also be deduced from (8), (10), and (11) by setting \( q = 1 \), \( v = x/(1 - x) \), \( w = y/(1 - y) \).

Consider next a correlated bond-site percolation process on the honeycomb lattice with edge occupation probabilities \( x_1, x_2, x_3 \) and alternate site occupation probabilities \( s \) and 1. Now the site percolation is realized in the \( q = 1 \) limit of the \( q \)-state Potts model with multi-site interactions [10]. Therefore, by setting \( y_i = 0 \) and \( s = 1 - e^{-M} \), we obtain from (12) the critical frontier for this site-bond percolation,

\[
s(x_1x_2 + x_2x_3 + x_3x_1 - x_1x_2x_3) = 1. \tag{16}
\]

The expression (16), which generalizes an early result due to Kondor [17] for \( x_1 = x_2 = x_3 \), is the central result of [7] derived from a star-triangle consideration. Here, it is deduced as the result of an application of our general formulation.

As pointed out by Scullard [7] and Ziff [8], the expression (16) also gives the threshold for site percolation on the martini lattice of Fig. 3, where \( x_1, x_2, x_3 \) are occupation probabilities of the three sites around
a triangle and $s$ is the occupation probability of the site at the center of the $Y$. For uniform occupation probability $x_1 = x_2 = x_3 = s$, (16) yields the threshold $s_c = 0.764826 \cdots$ for site percolation on the martini lattice.

Setting $x_3 = 1$ in (16) we obtain the threshold for site percolation on the martini A lattice of Fig. 4(a) as

$$s(x_1 + x_2) = 1, \quad \text{site percolation – A lattice} \quad (17)$$

where $x_1, x_2$ are occupation probabilities of the 3-coordinated sites and $s$ the occupation probability of the 4-coordinated sites. For uniform occupation probability $x_1 = x_2 = s$, (17) yields the threshold $s_c = 1/\sqrt{2}$ for site percolation on the A lattice. Likewise setting $x_2 = x_3 = 1$ in (16), we obtain the threshold for site percolation on the martini B lattice of Fig. 4(b),

$$s(1 + x) = 1, \quad \text{site percolation – B lattice} \quad (18)$$

where $x = x_1$ and $s$ are, respectively, the occupation probabilities of the 5-coordinated sites and 3-coordinated sites. For uniform occupation probability $x = s$, (18) yields the threshold $s_c = (\sqrt{5} - 1)/2$ for site percolation on the B lattice. These results have been reported in [7, 8].

4 Other lattices

As another example of our formulation, consider the Potts model on the lattice in Fig. 5 with pure 2-site interactions $U, V, W \geq 0$. Writing $u = e^U - 1$, $v = e^V - 1$, $w = e^W - 1$, we obtain after a little algebra the Boltzmann factor with

$$A = v^3 + 3v^2(q + 2w) + (3v + q)(q^2 + 3qw + 3w^2 + w^3)$$
$$B = uA + v^2\big[3w^2 + w^3 + (q + v)w\big]$$
$$C = u^2(u + 3)A + 3uv^2(u + 1)(u + 2)\big[3w^2 + w^3 + (q + v)w\big]$$
$$+ (u + 1)^3v^3(3w^2 + w^3). \quad (19)$$

The critical frontier is again the self-dual trajectory $qA - C = 0$.

The resulting self-dual trajectory assumes a simpler form for the percolation problem. For bond percolation we set $q = 1$, $u = z/(1 -$
\( z, v = x/(1 - x), w = y/(1 - y) \) where \( x, y, x \) are the respective bond occupation probabilities shown in Fig. 5. This yields the bond percolation critical threshold

\[
1 - 3z + z^3 - (1 - z^2)\left[3x^2y(1 + y - y^2)(1 + z) + x^3y^2(3 - 2y)(1 + 2z)\right] = 0. \tag{20}
\]

Setting \( z = 0 \) in (20) it reduces to the bond percolation threshold \( 13 \) of the martini lattice. Setting \( y = 1 \) (20) gives the bond percolation threshold

\[
1 - 3z + z^3 - (1 - z^2)\left[3x^2(1 + z) - x^3(1 + 2z)\right] = 0 \tag{21}
\]

for the dual of the martini lattice, which is the lattice in Fig. 5 with all small triangles shrunk into single points.

For uniform bond percolation probabilities \( x = y = z = p \), (20) becomes

\[
1 - 3p - 2p^3 + 12p^5 - 5p^6 - 15p^7 + 15p^8 - 4p^9 = 0 \tag{22}
\]

yielding the threshold \( p_c = 0.321808 \cdots \). Compared with the threshold \( p_c = 0.707106 \cdots \) for the martini lattice, it confirms the expectation that percolation threshold decreases as the lattice becomes more connected.

**Summary and acknowledgment**

In summary, we have shown that the critical frontier of a host of Potts models with 2- and multi-site interactions on lattices having the structure depicted in Fig. 1 can be explicitly determined. The resulting critical frontier assumes the very simple form \( qA - C = 0 \), where \( A \) and \( C \) are parameters defined in (1). The corresponding threshold for bond and/or site percolation are next deduced by setting \( q = 1 \). Specializations of our formulation to the martini, the A, B, and other lattices are presented.

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Figure captions

Fig. 1. The structure of a lattice possessing a duality relation.

Fig. 2. The realization of Fig. 1 for the martini lattice.

Fig. 3. The martini lattice.

Fig. 4. (a) The martini-A lattice. (b) The martini-B lattice.

Fig. 5. A lattice with Potts interactions $U, V, W$. Labels shown are the corresponding bond percolation probabilities $x = 1 - e^{-V}$, $y = 1 - e^{-W}$, $z = 1 - e^{-U}$. 
Fig. 2
Fig. 3
Fig. 4
