Mutual influence of the supports stiffness and the first natural frequency at bending vibrations of a spring-hinged beam

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Abstract. This paper discusses the analytical finding of the required stiffness values of the supports, which would provide the given first natural frequency of bending vibrations for a spring-hinged beam. The solution is obtained through the support coefficient, the value of which, according to the theory of beam vibrations, connects the stiffness of the support of the beam and the value of its first vibration frequency. For this purpose, preliminary approximation of dependence of support coefficient values on a support stiffness is performed using quadratic analytical functions. In this case, the problem of determining the stiffness of the supports becomes inverse and is reduced to finding combinations of the supports stiffness values, at which there is a solution of the quadratic analytical functions for the required support coefficient and, accordingly, the required first beam frequency. Normalization of the obtained support coefficients is proposed for a more convenient estimation of their influence on the first frequency of beam vibration.

1. Introduction

In the theory of vibrations, free bending vibrations of a simple hinged beam are well studied [1-21]. However, in practice, a purely hinged joint is unattainable and elastic forces arise in the hinges, which we take into account in the form of spring stiffness values \( k_1 \) and \( k_2 \). These support stiffness values will affect the dynamic behavior of the beam, including its first natural bending frequency (figure 1).

![Figure 1. Free vibrations of a spring-hinged beam.](image)

In this case, the equation of the beam free vibrations has the form [1-10]:

\[
EJ_{\text{min}} \frac{\partial^4 y}{\partial x^4} + m \frac{\partial^2 y}{\partial t^2} = 0.
\]
where \( y = y(x) \) is the beam deflection; \( E \) is the elastic modulus; \( J_{\text{min}} \) is the minimum moment of inertia of the beam cross-section; \( m \) is the mass per beam length.

The solution for the first frequency of the beam free vibration according to [1-10] is defined as:

\[
f_1 = \frac{\alpha^2}{2\pi l^2} \sqrt{\frac{EJ_{\text{min}}}{m}}. \tag{2}
\]

where \( \alpha \) is the natural frequency dimensionless parameter, which depends on the boundary conditions; \( l \) is the length of the beam.

Reference literature [1-21] gives the values of support coefficients for common simple types of supports: hinge, fixed support, and its combinations. If the beam is supported in hinges with some given stiffness (figure 1), the support coefficient will be their function:

\[
\alpha = \alpha(k_1, k_2). \tag{3}
\]

Express the stiffness of the supports through the stiffness of the beam and get the relative stiffness:

\[
C_1 = k_1 \frac{l}{EJ_{\text{min}}}; \quad C_2 = k_2 \frac{l}{EJ_{\text{min}}}. \tag{4}
\]

Then the dependence of the support coefficient will be a function of their relative stiffness:

\[
\alpha = \alpha(C_1, C_2). \tag{5}
\]

The function (5) is very nonlinear and there are no exact analytical expressions for it, so the determination of the support coefficient in these cases is carried out by any numerical method. Tables with different relative stiffness values are given in known reference books, from which the desired support factor can be found. However, there are several problems. Firstly, these tables contain a very limited set of possible combinations of relative stiffness of supports, which makes their use difficult. Further, if intermediate values of relative stiffness are used, then high nonlinearity of dependence (5) leads to large errors when trying to linear interpolate table values. Finally, these tables sometimes contain erroneous values that are not always obvious to inexperienced designers.

The purpose of this work is the inverse problem for function (5), namely in determining the stiffness of supports \( C_i \), which will provide the required value of the support coefficient \( \alpha \). This corresponds to the function:

\[
f_1(C_1, C_2) = \alpha(C_1, C_2)^2 \cdot \frac{1}{2\pi l^2} \sqrt{\frac{EJ_{\text{min}}}{m}}. \tag{6}
\]

Solution is performed by analytical approach using quadratic functions.

2. Quadratic approximation of support coefficients values

A numerical solution [22,23] of equation (5) was made for more than 500 combinations of beam support stiffness values. Table 1 shows a reduced set of combinations of relative stiffness and their corresponding support factor values \( \alpha \).

| \( C_2 \) | 0  | 0.01 | 0.1 | 1   | 5   | 10  | 50  | 100 | 500 | 1000 | \( \infty \) |
|---------|----|------|-----|-----|-----|-----|-----|-----|-----|------|---------|
| 0       | 3.142 | 3.143 | 3.157 | 3.273 | 3.534 | 3.665 | 3.855 | 3.889 | 3.919 | 3.923 | 3.926 |
| 0.01    | 3.145 | 3.159 | 3.275 | 3.534 | 3.666 | 3.856 | 3.890 | 3.920 | 3.924 | 3.927 |
| 0.1     | 3.173 | 3.288 | 3.548 | 3.678 | 3.868 | 3.902 | 3.932 | 3.936 | 3.940 |
| 1       | 3.399 | 3.652 | 3.781 | 3.970 | 4.004 | 4.034 | 4.038 | 4.041 |
| 5       | 3.897 | 4.026 | 4.219 | 4.254 | 4.241 | 4.245 | 4.249 |  }
Insider each range, we approximate the obtained numerical values of the support coefficients $\alpha$ with quadratic functions using the least-squares method [23-25]. As a result, we obtain a solution for each zone in the form:

$$\alpha_i(C_1, C_2) = a_{i0} + a_{i1} \cdot C_1 + b_{i1} \cdot C_2 + a_{i2} \cdot C_1^2 + b_{i2} \cdot C_2^2.$$  

(7)

where $i$ is the zone number.

Using the least-squares method, we define the coefficients of approximating quadratic functions for each zone and summarize the results in the table 2.

| Zone | I | II | III |
|------|---|----|-----|
| Coefficient | $C_1$, $C_2=0...10$ | $C_1=0...10$, $C_2=10...100$ | $C_1=0...10$, $C_2=100...1000$ |
| $a_0$ | 3.1415 | 3.579 | 4.03 |
| $a_1$ | 0.1008 | 0.1008 | 0.0075 |
| $a_2$ | -0.005 | -0.00000435 | -0.00000435 |
| $b_1$ | 0.1008 | 0.0075 | 0.000125 |
| $b_2$ | -0.005 | -0.00000435 | -0.000000074 |

The obtained quadratic approximating functions (7) allow calculating the values of the support coefficient for equation (2) with an error of not more than 3.3% in the entire considered range of support stiffness. This is sufficient for the theory of beam vibrations, since the assumptions used in it lead to even greater deviations in the solution.

3. Squares of support coefficient values

In the equation of the frequency of free oscillations (6) of the hinge beam, the factor of supports is squared, so it makes sense to approximate the squares of their values. This will eliminate the accumulation of errors as a result of exponential operations. Using the analytical solution, we get the squares of the support coefficients and summarize them in the table 3.

| $C_2$ | 0 | 0.01 | 0.1 | 1 | 5 | 10 | 50 | 100 | 500 | 1000 | $\infty$ |
|-------|---|------|-----|---|---|----|-----|-----|-----|-----|-------|
| $C_1$ | 9.870 | 9.880 | 9.968 | 10.71 | 12.49 | 13.43 | 14.86 | 15.13 | 15.36 | 15.39 | 15.41 |
| 0.01 | 9.890 | 9.977 | 10.72 | 12.49 | 13.44 | 14.87 | 15.14 | 15.37 | 15.40 | 15.43 |
| 0.1 | 10.07 | 10.81 | 12.59 | 13.53 | 14.96 | 15.23 | 15.46 | 15.49 | 15.52 |
| 1 | 11.55 | 13.34 | 14.29 | 15.76 | 16.03 | 16.27 | 16.30 | 16.33 |
| 5 | 15.19 | 16.21 | 17.80 | 18.10 | 18.36 | 18.39 | 18.43 |
| 10 | 17.27 | 18.95 | 19.27 | 19.55 | 19.59 | 19.62 |
| 50 | 20.82 | 21.18 | 21.49 | 21.53 | 21.57 |
| Symmetrical | 20.82 | 21.18 | 21.49 | 21.53 | 21.57 |
| 100 | 21.54 | 21.86 | 21.91 | 21.95 |
| 500 | 22.20 | 22.24 | 22.28 |
| 1000 | 22.28 | 22.32 |
| $\infty$ | 22.36 |
The obtained values of the squares of the support coefficients are approximated using quadratic functions in the form (7). The coefficients of these functions are also determined by the least-squares method for each of the three stiffness zones and their values are shown in the table 4.

| Coefficient | Zone | I | II | III |
|-------------|------|---|----|-----|
|             | C₁   | C₁=0...10, | C₁=10...100, | C₁=100...1000, | C₁, C₂= 100...1000 |
| a₀          | 9.87 | 12.972 | 16.6 | 15.243 | 18.872 | 21.3 |
| a₁          | 0.736 | 0.736 | 0.0638 | 0.736 | 0.0638 | 0.00115 |
| a₂          | -0.0366 | -0.0366 | -0.0004 | -0.0366 | -0.0004 | -0.00000065 |
| b₁          | 0.736 | 0.0638 | 0.0638 | 0.00115 | 0.00115 | 0.00115 |
| b₂          | -0.0366 | -0.0004 | -0.0004 | -0.00000065 | -0.00000065 | -0.00000065 |

The obtained approximating functions allow calculating the squares of the support factor \( \alpha^2 \) for equation (2) with an error of not more than 3.5% in the entire considered range of support stiffness.

4. Determination of required supports stiffness

The use of quadratic approximating functions together with the division of the stiffness of supports into zones made it possible to analytically describe a complex nonlinear dependence (5) with sufficient accuracy for engineering calculations. At the same time, quadratic approximating functions allow solving the inverse problem of determining the stiffness of supports that provide the required support coefficients \( \alpha \) or its quadratic value \( \alpha^2 \).

The analytical solution of square equation (7) is complicated by the presence of two unknowns: \( C_1 \) and \( C_2 \). For certainty, link these unknowns through a coefficient \( n \):

\[
C = C_1 = \frac{C_2}{n}.
\] (8)

Then equation (7) is reduced to solving the square equation at a given value of \( \alpha' \), which, depending on the problem to be solved, can be the factor of supports \( \alpha \) or its square value \( \alpha^2 \):

\[
(a_{12} + b_{12} \cdot n^2)C^2 + (a_{11} + b_{11} \cdot n)C + (a_{10} - \alpha') = 0.
\] (9)

The root of this equation, taking into account the signs, is the expression:

\[
C = \frac{-(a_{11} + b_{11} \cdot n) - \sqrt{(a_{11} + b_{11} \cdot n)^2 - 4(a_{12} + b_{12} \cdot n^2)(a_{10} - \alpha')}}{2(a_{12} + b_{12} \cdot n^2)}.
\] (10)

The solution (10) is valid for the problem of determining the supports stiffness that provides the required factor of supports \( \alpha \) or its quadratic value \( \alpha^2 \).

Depending on the problem to be solved, the coefficients from table 2 or table 4 are used, respectively. After determination of stiffness value \( C \), stiffness of supports according to equation (8) is defined as

\[
k_1 = C \cdot \frac{EJ_{\text{min}}}{l}, \quad k_2 = C \cdot \frac{EJ_{\text{min}}}{l} n.
\] (11)

Stiffness of supports (11) will provide the specified value of support coefficient and, therefore, the first frequency of natural vibrations of the beam.
5. Discussion
Approximating functions (7) with calculated coefficients allow obtaining support coefficients and their squares in the almost full range of stiffness. The error in determining the values is not more than 3.5% compared to the exact values [21, 22], which is quite permissible for beam structures. However, when solving the inverse problem of equation (10), the error of determining the required stiffness of the supports can be significantly higher due to the high nonlinearity of the dependence (5) and the accumulation of errors from the rounding of irrational numbers. Therefore, after determining the required stiffness of the supports, it is recommended to perform a check calculation of the value of the first natural frequency of the beam bending vibrations.

The proposed solution for a single-span hinge beam is also valid for a multi-span hinge beam, provided that all spans and stiffness of all spring-hinged supports are equal.

6. Conclusion
In this paper, the values of support coefficients for the spring-hinged beam were calculated at different combinations of the supports relative stiffness. According to the obtained data, the least-squares method was approximated with quadratic functions over three zones. The obtained analytical functions make it possible to determine the beam support coefficient at any combination of support stiffness with an error of not more than 3.5%. These analytical quadratic approximations also allow solving the inverse problem of finding the necessary stiffness of supports at the known required value of the support coefficient.

Thus, the obtained results make it possible to analytically determine the first frequency of natural vibration of a single-span beam at any combination of supports stiffness values and solve the inverse problem of finding the required supports stiffness values, which will provide the beam with the required frequency of vibration.

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References
[1] Lin Y K 1962 Int Journal of Mechanical Sciences 4 409-23
[2] Balachandran B 2009 Vibrations (Cengage Learning, Toronto, Canada)
[3] Benaroya H, Nagurka M and Han S 2017 Mechanical vibration (CRC Press, Taylor & Francis Group)
[4] Bottega W J 2006 Engineering vibrations (CRC Press, Taylor & Francis Group)
[5] Clough R E 1995 Dynamics of Structures (McGraw-Hill College, USA)
[6] Geradin M and Rixen D J 2015 Mechanical vibrations (John Wiley & Sons Ltd, UK)
[7] Hartog J P 1985 Mechanical vibrations (McGraw-Hill, New York)
[8] Hagedorn P 2007 Vibrations and waves in continuous mechanical systems (John Wiley & Sons Inc., NJ, USA)
[9] Inman D J 2014 Engineering vibration (Pearson Education, NJ, USA)
[10] Kelly S G 2007 Advanced vibration analysis (CRC Press, Taylor & Francis Group, LLC, USA)
[11] Jazar R N 2013 Advanced vibrations. A modern approach (Springer, NY, USA)
[12] Kelly S G 2012 Mechanical vibrations. Theory and applications (Cengage Learning, USA)
[13] Leissa A W 2011 Vibration of continuous systems (McGraw-Hill, New York)
[14] Meirovitch L 1967 Analytical methods in vibrations (The Macmillan company, New York, USA)
[15] Meirovitch L 2001 Fundamentals of vibrations (McGraw-Hill,Book Co, New York, USA)
[16] Rades M 2010 Mechanical vibrations II. Structural dynamic modeling (Printech Publisher, Turin)
[17] Rao S 2018 Mechanical vibrations (Pearson Education Limited, UK)
[18] Shabana A S 2019 Theory of vibration (Springer-Verlag New York, USA)
[19] Blekhman I I 2004 Selected topics in vibrational mechanics (World Scientific Publishing Co Pte
[20] Magnus K 1965 Vibrations (Blackie & Son, UK)
[21] Blevins R D 2016 Formulas for dynamics, acoustics and vibration (John Wiley & Sons, Ltd)
[22] Hibbeler R C 1975 J.Appl Mech 42 501-502
[23] Hamming R W 1987 Numerical Methods for Scientists and Engineers (Dover Publications)
[24] Chapra S and Canale R 2014 Numerical Methods for Engineers (McGraw-Hill Education)
[25] Burden R L 2015 Numerical Analysis (Cengage Learning)