Effect of Objective Function on Data-Driven Sparse Sensor Optimization

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Abstract

The selection problem of an optimal set of sensors estimating the snapshot of high-dimensional data is considered. The objective functions based on various criteria of optimal design are adopted to the greedy method: D-optimality, A-optimality, and E-optimality, which maximizes the determinant, minimize the trace of inverse, and maximize the minimum eigenvalue of the Fisher information matrix, respectively. First, the Fisher information matrix is derived depending on the numbers of latent state variables and sensors. Then, the unified formulation of the objective function based on A-optimality is introduced and proved to be submodular, which provides the lower bound on the performance of the greedy method. Next, the greedy methods based on D-, A-, and E-optimality are applied to randomly generated systems and a practical data set of global climates. The sensors selected by the D-optimality objective function works better than those by A- and E-optimality with regard to the determinant, trace of the inverse, and reconstruction error, while those by A-optimality works the best with regard to the minimum eigenvalue. On the other hand, the performance of sensors selected by the E-optimality objective function is worse for all indices and reconstruction error. This might be because of the lack of submodularity as proved in the paper. The results indicate that the greedy method based on D-optimality is the most suitable for high accurate reconstruction with low computational cost.

Index terms—Data-driven, sparse sensor optimization, greedy method, optimal experimental design.

1 Introduction

The development of an accurate and efficient model for estimation, prediction, and control of complex phenomena is an open challenge in various scientific and industrial fields. Recently, a vast amount of data, such as seismic data in earthquake phenomena...
and environmental data from remote platforms using satellites can be obtained owing to progress in measurement equipment and technology. However, these phenomena may involve multidimensional states with various timescales. It is impractical to process the full-state measurements in real time because they are computationally expensive toward the realization of fast state estimation for low-latency and high-bandwidth control.

In such a situation, dimensionality reduction is a promising approach. There are often a few dominant low-dimensional patterns, which may well explain the high-dimensional data, in many natural science systems. The singular value decomposition (SVD) provides a systematic way to determine a low-dimensional approximation of high-dimensional data based on the proper orthogonal decomposition (POD) \[1,2\]. Thus, SVD enables us to exploit the significant POD modes in the data for low-dimensional representations.

The sparse sensing is important as well as the dimensionality reduction, since the number of sensors is often limited because of the cost associated with placing sensors and computational constraints. Low-dimensionally approximated full states can be reconstructed from a small subset of measurements by sparse sensors. Thus, it is important to optimize the sensor placement to exploit the significant low-dimensional patterns based on efficient reduce-order models. This idea has been adopted by Manohar et al.\[3\], and the sparse-sensor-placement algorithm has been developed and discussed\[4,5\]. Furthermore, the sensor selection based on POD is a data-driven approach without the requirement for governing equations. Such a data-driven sensing generally needs to determine the optimal sensor locations from a large amount of sensor candidate. Hence, a fast greedy optimization method is required for high-performance computing or feedback control.

The optimal sensor selection problem is closely related to the optimal design of experiments, which provides small values of the variances of estimated parameters and predicted response\[6\]. Depending on the statistical criterion of optimal experimental design, the objective function of sensor selection problem is defined using the Fisher information matrix, which corresponds to the inverse of covariance matrix of estimator. The most important design criterion is that of D-optimality, in which the determinant of the Fisher information matrix is maximized. Here, ‘D’ in the name stands for ‘determinant.’ This criterion results in minimization of the volume of the confidence ellipsoid of the regression estimates. Joshi and Boyd have adopted D-optimality to a sensor selection problem and proposed to utilize a convex relaxation method to approximately solve D-optimal design problem\[7\]. Manohar et al.\[3\], have proposed the greedy method based on discrete-empirical-interpolation method (DEIM) and QR-DEIM (QDEIM) \[8,9\] that are methods in the framework of reduced-order modeling using sparse sampling points. Manohar et al. showed its advantage that it is significantly faster than convex optimization method\[3\]. Their greedy method optimizes the sensor position by QR-pivoting the row vector of the sensor-candidate or related matrix in previous study\[3\]. This greedy method was shown\[10\] to correspond to one by selecting the row vector the norm of which is the maximum and by eliminating its component from the rest of the matrix using a Gram-Schmidt procedure, though QR implementation is much faster in practical. More recently, Saito et al.\[11\] mathematically illustrated that the objective function adopted by Manohar et al.\[3\] corresponds to
the maximization of the D-optimality objective function when the number of sensors is less than that of state variables, and derived unified expression of the objective function based on D-optimality regardless of a number of sensors. Furthermore, they have successfully proposed the efficient greedy method based on D-optimality: a hybrid of greedy methods based on the QR decomposition and the straightforward maximization of the determinant in the cases of the number of sensors less and greater than that of state variables, respectively. The proposed method is confirmed to provide nearly optimal sensors as well as the convex approximation method, and also significantly reduces the computational cost compared to the convex approximation method and QR-based greedy method[11]. Thus, previous studies have provided valuable knowledge on the sensor selection problem in the context of D-optimality.

On the other hand, there are a variety of the criteria for optimal design. Two other criteria that have a statistical interpretation in terms of the information matrix are A- and E-optimality. In A-optimality, the trace of the inverse of the information matrix, which corresponds to the total variance of the parameter estimates, is minimized. It is equivalent to minimization of the ‘average’ variance and called A-optimality, where ‘A’ stands for ‘averaged.’ In E-optimality, the maximum eigenvalue of the information matrix is minimized, which minimizes the worst case variance of estimation error, where ‘E’ stands for ‘eigenvalue.’ Although the objective function based on these criterion have been introduced[6, 7], the formulation for the sensor selection in the underdetermined situation has not been derived, and the greedy method and its performance have not been investigated to our knowledge. On the other hand, for the linear dynamical system, several objective functions, which involve the determinant, trace, and eigenvalue of the controllability Gramian, have been adopted by Summers et al[12]. They have demonstrated the submodularity of the functions and compared the performance in terms of eigenvalue distribution. Similar to the linear dynamical system, it is also worth understanding the characteristics of objective functions for the reconstruction of the snapshots of systems. It would provide more fundamental knowledge for the sparse sensor selection problem.

The aim of this study is to obtain the insight into the objective functions suitable for the sparse sensor selection problem, especially focusing on the reconstruction of the snapshots of high-dimensional data. For this purpose, the greedy methods based on D-, A-, and E-optimality including the underdetermined situation are described and proposed, and the performance of the various greedy methods for the reconstruction of snapshots of systems is described. Here, the greedy method based on A- and E-optimality in the underdetermined situation is introduced for the first time. Firstly, the submodularity of objective functions is investigated. Next, the results of the greedy methods based on D-, A-, and E-optimality are shown for the randomly generated data, and then, the reconstruction error and computational cost are compared by reconstructing a data set of global ocean surface temperature.
2 Formulation and Algorithms for Sensor Selection Problem

We consider the linear system given by:

\[ y = HUz = Cz, \]

where \( y \in \mathbb{R}^p \), \( H \in \mathbb{R}^{p \times n} \), \( U \in \mathbb{R}^{n \times r} \), \( z \in \mathbb{R}^r \), and \( C \in \mathbb{R}^{p \times r} \) are the observation vector, the sparse sensor location matrix, and the sensor candidate matrix, the latent state vector, and the measurement matrix (\( C = HU \)), respectively. Here, the element corresponding to the sensor location is unity and the others are 0 in each row of \( H \). In addition, \( p \), \( n \), and \( r \) are the number of sensors, the number of spatial dimension, and the number of latent state variables. The system above represents the problem to choose \( p \) observations out of \( n \) sensor candidates for the estimation of the state variables. The various sensor selections can be expressed by changing \( H \) and by selecting row vectors as sensors from the sensor candidate matrix \( U \). The graphical image of the equation above is shown in Fig. 1.

The estimated parameters \( \hat{z} \) can be obtained by the pseudo inverse operation when the uniform independent Gaussian noises \( \mathcal{N}(0, \sigma^2 I) \) are imposed on the observation as follows:

\[ \hat{z} = C^+ y = \begin{cases} C^T (CC^T)^{-1} y, & p \leq r \\ (C^T C)^{-1} C^T y, & p > r \end{cases} \]

The covariance matrix of estimation error is expressed as follows:

\[ E[(z - \hat{z})(z - \hat{z})^T] = \begin{cases} (I - P_C)E[zz^T](I - P_C) + \sigma^2 C^T (CC^T)^{-2} C, & p \leq r, \\ \sigma^2 (C^T C)^{-1}, & p > r \end{cases} \]
where $P_C = C^\top (CC^\top)^{-1} C$ is the projection matrix onto the row vector space of $C$. Here, the estimation error in the case of $p < r$ is further considered in the observable subspace by transformation. The full singular value decomposition of $C$ is given as follows:

$$C = U_C \Sigma_C V_C^\top$$

$$= \begin{cases} 
U_C \begin{bmatrix} \Sigma_C & 0 \\ 0 & 0 \\ \end{bmatrix} \begin{bmatrix} \tilde{V}_C^\top \\
V_C^\top 
\end{bmatrix}, & p \leq r, \\
U_C \begin{bmatrix} \Sigma_C & 0 \\ 0 & 0 \\ \end{bmatrix} \tilde{V}_C^\top, & p > r. 
\end{cases} \quad (4)$$

The coordinate transform by $\zeta = \tilde{V}_C^\top z$ is considered, whereas $V_C$ is an orthonormal (or unitary) matrix and the amplitude of the error in the observable subspace does not change. This transformation only gives the observable components in the latent state vector. The error in $\zeta$ becomes as follows:

$$E\left[ (\zeta - \hat{\zeta})(\zeta - \hat{\zeta})^\top \right] = \begin{cases} 
\sigma^2 U_C^\top (CC^\top)^{-1} U_C, & p \leq r, \\
\sigma^2 \tilde{V}_C^\top (C^\top C)^{-1} \tilde{V}_C, & p > r. 
\end{cases} \quad (5)$$

Here, the only error covariance in the observable space is obtained in the case of $p \leq r$ and the amplitude of the error covariance matrix is given by $(CC^\top)^{-1}$ because row vectors in $\tilde{V}_C^\top$ are orthonormal to each other and their absolute value is unity. Here, the first term of (3) in the case of $p < r$ disappear in (5) because the term has only the components in the unobservable space. In the optimal design, the error covariance matrix and its inverse, the latter of which corresponds to the Fisher information matrix, are employed, and the optimality criteria are provided. Sensor selection problems can be defined based on the optimality criterion. There are a variety of optimal criteria, several of which are addressed hereafter.

### 2.1 Objective function based on D-optimality

A D-optimal design maximizes the determinant of the Fisher information matrix. It is equivalent to minimizing the determinant of the error covariance matrix, resulting in minimizing the volume of the confidence ellipsoid of the regression estimates of the linear model parameters. Therefore, the problem can be expressed as the optimization problem

$$\text{maximize } f_D$$

$$f_D = \begin{cases} 
\det (CC^\top), & p \leq r, \\
\det (C^\top C), & p > r. 
\end{cases} \quad (6)$$

Basically, all the combinations of $p$ sensors out of $n$ sensor candidates should be searched by brute-force algorithm for the real-optimized solution of (6), which takes enormous computational time ($O(n!/(n-p)!/(p!) \approx O(n^p)$). Instead, greedy methods for the suboptimal solution by adding a sensor step by step has been devised. For D-optimal criterion, the objective function for the greedy method is demonstrated in the previous study[11].
2.2 Objective function based on A-optimality

An A-optimal design minimizes the mean square error in estimating the parameter $z$. Hence, the objective function is the trace of the error covariance matrix in A-optimal design. The sensor selection problem can be expressed as the optimization problem:

$$\min f_A$$

$$f_A = \begin{cases} 
\text{tr} \left( (CC^T)^{-1} \right), & p \leq r, \\
\text{tr} \left( (C^T C)^{-1} \right), & p > r.
\end{cases} \quad (7)$$

In the step-wise selection of the greedy method, the sensor selection of only the $k$th sensor is conducted in the $k$th step under the condition that the sensors up to $(k-1)$th are already determined. Let $S \subset \{1, 2, \ldots, n\}$ be a set of labels of selected sensors and $C_S$ be the corresponding sensor matrix. Namely, if $S = \{i_1, i_2, \ldots, i_k\}$, then $C_S$ is given by

$$C_S = C_k = \left[ \begin{array}{cccc} u_{i_1}^T & u_{i_2}^T & \cdots & u_{i_k}^T \end{array} \right]^T, \quad (8)$$

where $u_{i_k}$ is the corresponding row vector of the sensor-candidate matrix $U$. Therefore, the sensor index chosen in the $k$th step of the greedy method can be described as follows:

$$i_k = \arg \min_{i_k} f_{AG},$$

where

$$f_{AG} = \begin{cases} 
\text{tr} \left( (C_SC_k^T)^{-1} \right), & p \leq r, \\
\text{tr} \left( (C_k^T C_k)^{-1} \right), & p > r.
\end{cases} \quad (9)$$

In the case of $p \leq r$,

$$\text{tr} \left( (C_SC_k^T)^{-1} \right)$$

$$= \text{tr} \left[ \begin{bmatrix} C_{k-1} & u_{i_k} \\ u_{i_k}^T & I \end{bmatrix} \left( \begin{bmatrix} C_{k-1} & u_{i_k} \\ u_{i_k}^T & I \end{bmatrix} \right)^{-1} \right]$$

$$= \text{tr} \left( (C_{k-1}C_{k-1}^T)^{-1} \right)$$

$$+ \text{tr} \left[ \frac{(C_{k-1}C_{k-1}^T)^{-1} C_{k-1} u_{i_k} C_{k-1}^T u_{i_k}^T (C_{k-1}C_{k-1}^T)^{-1}}{u_{i_k^T} \left( I - C_{k-1} (C_{k-1}C_{k-1}^T)^{-1} C_{k-1} \right) u_{i_k}} \right]$$

$$+ \text{tr} \left[ \frac{1}{u_{i_k^T} \left( I - C_{k-1} (C_{k-1}C_{k-1}^T)^{-1} C_{k-1} \right) u_{i_k}} \right]. \quad (10)$$
Here, it is not necessary to evaluate the first term of (10) of the last equation since 
\( (C_{k-1}C_{k-1}^T)^{-1} \) is already determined in the \( k \)th step. Considering the cyclic property of 
trace, the greedy methods can be simply written as follows:

\[
i_k = \arg \min_{i} \frac{u_i C_{k-1}^T (C_{k-1} C_{k-1}^T)^{-2} C_{k-1} u_i + 1}{u_i (I - C_{k-1}^T (C_{k-1} C_{k-1}^T)^{-1} C_{k-1}) u_i^T}, \quad p \leq r. \tag{11}
\]

In the case of \( p > r \),

\[
\text{tr} \left[ (C_k^T C_k)^{-1} \right] \\
= \text{tr} \left[ (C_{k-1} C_{k-1} + u_i u_i^T)^{-1} \right] \\
= \text{tr} \left[ (C_{k-1}^T C_{k-1})^{-1} \right] \\
- \text{tr} \left[ (C_{k-1}^T C_{k-1})^{-1} u_i^T \left( 1 + u_i (C_{k-1}^T C_{k-1})^{-1} u_i \right)^{-1} u_i (C_{k-1}^T C_{k-1})^{-1} \right]. \tag{12}
\]

Taking into account the facts that the first term of (12) does not contribute in the \( k \)th sensor selection, the greedy method can be again simply written as follows:

\[
i_k = \arg \min_{i} \frac{u_i (C_{k-1} C_{k-1}^T)^{-2} u_i^T}{1 + u_i (C_{k-1}^T C_{k-1})^{-1} u_i^T}, \quad p > r. \tag{13}
\]

In summary, the greedy method can be written as follows:

\[
i_k = \begin{cases} 
\arg \min_{i} \frac{u_i C_{k-1}^T (C_{k-1} C_{k-1}^T)^{-2} C_{k-1} u_i + 1}{u_i (I - C_{k-1}^T (C_{k-1} C_{k-1}^T)^{-1} C_{k-1}) u_i^T}, & p \leq r, \\
\arg \min_{i} \frac{u_i (C_{k-1} C_{k-1}^T)^{-2} u_i^T}{1 + u_i (C_{k-1}^T C_{k-1})^{-1} u_i^T}, & p > r.
\end{cases} \tag{14}
\]

### 2.3 Objective function based on E-optimality

An E-optimal design minimizes the worst case variance of estimation error, which 
corresponds to the maximum eigenvalue of the Fisher information matrix. Therefore, 
the sensor selection problem can be expressed as the optimization problem:

\[
\text{maximize} \ f_E \\
f_E = \begin{cases} 
\lambda_{\min} (CC^T), & p \leq r, \\
\lambda_{\min} (C^T C), & p > r.
\end{cases} \tag{15}
\]

The sensor index chosen in the \( k \)th step of the greedy method can be written as follows:

\[
i_k = \arg \max_{i} f_{EG},
\]
where

\[
 f_{EG} = \begin{cases} 
 \lambda_{\min} \left( C_k^t C_k \right), & p \leq r, \\
 \lambda_{\min} \left( C_k^t C_k^{-1} \right), & p > r. 
\end{cases}
\]

\[
= \begin{cases} 
 \lambda_{\min} \left( \begin{bmatrix} C_{k-1}^t & u_i \\ u_i^t & n \end{bmatrix} \right), & p \leq r, \\
 \lambda_{\min} \left( C_{k-1}^t C_{k-1} + u_i^t u_i \right), & p > r. 
\end{cases}
\] (16)

3 Submodularity and approximation rate

Here, the objective functions are mathematically redefined as set functions and their structural properties are focused on. Submodularity in the set functions especially plays an important role in combinatorial optimization and provides a lower bound of the greedy method.

A modular function has the property that each element of a subset gives an independent contribution to the function value. If the objective function is modular, the optimization problem is straightforwardly solved by the greedy method evaluating the objective function in a step-by-step manner. On the other hand, also for monotone increasing submodular functions, which is NP-hard, the greedy method can be utilized to obtain a solution that is a probably close to the optimal solution.

**Definition 3.1** (Submodularity). The function \( f : 2^{[1,2,\ldots,n]} \rightarrow \mathbb{R} \) is called submodular if for any \( S, T \subset \{1,2,\ldots,n\} \) with \( S \subset T \) and \( i \in \{1,2,\ldots,n\} \setminus T \), the function \( f \) satisfies

\[
 f(S \cup \{i\}) - f(S) \geq f(T \cup \{i\}) - f(T). 
\] (17)

**Definition 3.2** (Monotonicity). The function \( f : 2^{[1,2,\ldots,n]} \rightarrow \mathbb{R} \) is called monotone increasing if for any \( S, T \subset \{1,2,\ldots,n\} \) with \( S \subset T \) and \( i \in \{1,2,\ldots,n\} \setminus T \), the function \( f \) satisfies

\[
 S \subset T \Rightarrow f(S) \leq f(T). 
\] (18)

The performance of greedy method is guaranteed by a well-known lower bound when the objective function is monotone and submodular. Nemhauser et al. have proved that the following inequality holds [13]:

\[
 f(S_{\text{greedy}}) \geq \left( 1 - \left( 1 - \frac{1}{k} \right) \right)^k f(S_{\text{opt}}) \\
 \geq \left( 1 - \frac{1}{k} \right) f(S_{\text{opt}}) \\
 \geq 0.63 f(S_{\text{opt}}), 
\] (19)

where \( k \) denotes the number of sensors, \( S_{\text{opt}} \) is an optimal solution and \( S_{\text{greedy}} \) is the solution obtained from applying the greedy method.

We now evaluate the submodularity and monotonicity of objective functions introduced in previous section.
3.1 Objective function based on D-optimality

Saito et al. have derived the unified expression of the objective function in both cases of $p \leq r$ and $p > r$ for D-optimality. The objective function in (6) is redefined to be $\det \left( C^T C + \epsilon I \right)$, where $\epsilon$ is a sufficiently small number. It has been proved to be the monotone submodular function[11].

3.2 Objective function based on A-optimality

We now consider the trace of error covariance matrix in A-optimal design. Similar to the previous study[11], we first introduce the unified formulation for both cases in which the number of sensors is less than or equal to that of the modes and the number of sensors is greater than that of the modes.

Define a function $f_A : 2^{\{1,2,...,n\}} \to \mathbb{R}$ by

$$f_A(S) = -\text{tr} \left[ \left( C_S^T C_S + \epsilon I \right)^{-1} \right] + \frac{r}{\epsilon}.$$ (21)

for each $S \subset \{1,2,\ldots,n\}$. An offset term $r/\epsilon$ is added so that the value of $f$ for the empty set could be regarded as $f_A(\emptyset) = 0$.

**Proposition 1.** $f_A$ defined by (21) is submodular.

**Proof.** Take arbitrary $S, T \subset \{1,2,\ldots,n\}$ such that $S \subset T$. For simplicity of notation, let us set $A := C_S^T C_S + \epsilon I$ and $B := C_S^T C_T + u_i^T u_i + \epsilon I$. It follows from the definition
of $f_A$ that, for any $i \in \{1, 2, \ldots, n\} \setminus T$,

\[
f_A(S \cup \{i\}) - f(S)_A
= -\text{tr} \left[ (C^T_S C_S + u_i^T u_i + \epsilon I)^{-1} \right]
+ \text{tr} \left[ (C^T_S C_S + \epsilon I)^{-1} \right]
= \text{tr} (A^{-1} - B^{-1}).
\] (22)

Due to the positive definiteness of $A$, the value of (22) is positive for any $u_i \neq 0$. We next evaluate $f(T \cup \{i\}) - f(T)$. Since $S \subset T$, there exits a permutation matrix $P$ such that

\[
PC_T = \begin{bmatrix}
C_S \\
C_{T \setminus S}
\end{bmatrix}.
\] (23)

Hence, we have

\[
C^T_S C_T = (PC_T)^T P^T PC_T = C^T_S C_S + C^T_{T \setminus S} C_{T \setminus S},
\] (24)

where the fact $P^T P = I$ has been used. Direct computation together with (24) gives

\[
f(T \cup \{i\}) - f(T)
= -\text{tr} \left[ (C^T_T C_T + u_i^T u_i + \epsilon I)^{-1} \right]
+ \text{tr} \left[ (C^T_T C_T + \epsilon I)^{-1} \right]
= \text{tr} (A^{-1} - C^T_{T \setminus S} (I + C_{T \setminus S} A^{-1} C^T_{T \setminus S})^{-1} C_{T \setminus S} A^{-1})
- \text{tr} (B^{-1} - C^T_{T \setminus S} (I + C_{T \setminus S} B^{-1} C^T_{T \setminus S})^{-1} C_{T \setminus S} B^{-1}).
\] (25)

Therefore,

\[
f(S \cup \{i\}) - f(S) - f(T \cup \{i\}) + f(T)
= \text{tr} \left[ A^{-2} C^T_{T \setminus S} (I + C_{T \setminus S} A^{-1} C^T_{T \setminus S})^{-1} C_{T \setminus S} \right]
- \text{tr} \left[ B^{-2} C^T_{T \setminus S} (I + C_{T \setminus S} B^{-1} C^T_{T \setminus S})^{-1} C_{T \setminus S} \right]
\geq \text{tr} \left[ A^{-2} C^T_{T \setminus S} (I + C_{T \setminus S} A^{-1} C^T_{T \setminus S})^{-1} C_{T \setminus S} \right]
- \text{tr} \left[ B^{-2} C^T_{T \setminus S} (I + C_{T \setminus S} A^{-1} C^T_{T \setminus S})^{-1} C_{T \setminus S} \right]
= \text{tr} \left[ (A^{-2} - B^{-2}) C^T_{T \setminus S} (I + C_{T \setminus S} A^{-1} C^T_{T \setminus S})^{-1} C_{T \setminus S} \right],
\] (26)

where the fact that trace of a positive semidefinite matrix is nonnegative and the following semidefiniteness are adopted for the deviation of the inequality of the second to third equation.

\[
C^T_{T \setminus S} (I + C_{T \setminus S} A^{-1} C^T_{T \setminus S})^{-1} C_{T \setminus S}
- C^T_{T \setminus S} (I + C_{T \setminus S} B^{-1} C^T_{T \setminus S})^{-1} C_{T \setminus S} \geq 0.
\] (27)
Since
\[ A^{-2} - B^{-2} \geq 0 \] (28)
and
\[ C_{T,S}^T \left( I + C_{T,S} A^{-1} C_{T,S}^T \right)^{-1} C_{T,S} \geq 0, \] (29)
the trace of the product of a positive and non-negative semidefinite matrix is nonnegative. This completes the proof. □

**Proposition 2.** \( f_A \) defined by (21) is monotone increasing.

**Proof.** For given \( S, T \subset \{1, 2, \ldots, n\} \) with \( S \subset T \), it is clear that (22) holds also for any \( i \in T \setminus S \). Hence, \( f(S \cup \{i\}) - f(S) \geq 0 \). Similarly, we can show that \( f(S \cup \{i\} \cup \{j\}) - f(S) \geq 0 \) for \( j \in T \setminus (S \cup \{i\}) \). Repeated application of this argument yields \( f(S) \leq f(T) \), which is the desired conclusion. This completes the proof. □

### 3.3 Objective function based on E-optimality

We demonstrate the property of objective function based on E-optimality, given by (15). The objective function returns the minimum eigenvalue of the Fisher information matrix.

We first show by counterexample that the objective function fails to be submodular. If it is submodular, adding a sensor to a subset where a smaller number of sensors is selected gives a larger increment in the minimum eigenvalue than adding one to a subset where a larger number of sensors is selected. Let us set \( r = 3 \) and \( n = 10 \), and consider a sensor candidate matrix \( U \) defined by

\[
U = \begin{bmatrix}
  u_1 \\
  u_2 \\
  u_3 \\
  u_4 \\
  u_5 \\
  u_6 \\
\end{bmatrix} = \begin{bmatrix}
  0.2 & -0.1 & -0.2 \\
-0.5 & -0.1 & 0.2 \\
-0.2 & 0.3 & 0.2 \\
-0.5 & 0.3 & -0.3 \\
-0.4 & -0.3 & -0.4 \\
  0.3 & 0 & 0 \\
\end{bmatrix},
\]

Suppose the measurement matrices where 3, 4, and 5 sensors are selected given by

\[
C_3 = \left[ u_1^T \quad u_2^T \quad u_3^T \right]^T, \\
C_4 = \left[ u_1^T \quad u_2^T \quad u_3^T \quad u_4^T \right]^T, \\
C_5 = \left[ u_1^T \quad u_2^T \quad u_3^T \quad u_4^T \quad u_5^T \right]^T.
\]

Then, comparing the increment in the minimum eigenvalue by adding a sensor, we obtain

\[
\lambda_{\min}(C_{3p}) - \lambda_{\min}(C_3) > \lambda_{\min}(C_{4p}) - \lambda_{\min}(C_4), \tag{30}
\]
where

\[ C_{3p} = \begin{bmatrix} C_3 \\ u_5 \end{bmatrix}, C_{4p} = \begin{bmatrix} C_4 \\ u_5 \end{bmatrix}, \]

and

\[ \lambda_{\min}(C_{4p'}) - \lambda_{\min}(C_{4}) < \lambda_{\min}(C_{5p'}) - \lambda_{\min}(C_{5}), \] (31)

where

\[ C_{4p'} = \begin{bmatrix} C_4 \\ u_6 \end{bmatrix}, C_{5p'} = \begin{bmatrix} C_5 \\ u_6 \end{bmatrix}. \]

Since (30) and (31) correspond to the submodularity and supermodularity, respectively, the objective function based on E-optimality is neither submodular, or supermodular, or modular.

We mention the monotonicity of (15). There is a well-known theorem which shows the lower bound in the minimum eigenvalue by rank-one modification of Hermitian matrix [14]. According to the theorem, the objective function based on E-optimality is monotone increasing in the case of \( p > r \). Note that the minimum value of the eigenvalues of \( CC^T \) is not monotone in the case of \( p \leq r \) because a newly added eigenvalue into the system can be smaller or larger than the minimum eigenvalue of the previous subset.

4 Results and Discussions

In this section, the performance of sparse sensor selection methods based on different optimal criteria and schemes are evaluated. The results of numerical experiments on randomly generated systems and on a practical data set of global ocean surface temperature are illustrated. Four methods are compared: the D-optimality-based greedy (DG) method, the A-optimality-based greedy (AG) method, the E-optimality-based greedy (EG) method, and the D-optimality-based convex relaxation (DC) method listed in Table 1. The DG method which efficiently saves the computational time was proposed by Saito et al. [11]. In DG method, the sensors are determined by maximizing the determinant of (6) for each step in the case of \( p > r \), and on the other hand, the QR pivoting [3] is employed in the case of \( p \leq r \) since the QR implementation for the greedy method is much faster than the straightforward implementation of the greedy method maximizing [6]. Furthermore, the AG and EG methods are the greedy methods based on A- and E-optimality, respectively. Here, “eig” and “min” functions in Matlab are employed and the minimum eigenvalue is obtained in the EG method. The Matlab code is available on Github [15]. The DC method corresponds to the D-optimality-based convex relaxation method proposed by Joshi and Boyd. The numerical experiments are conducted under the computational environments listed in Table 2.
Table 1: Sensor selection methods investigated in this study

| Name | Optimality | Scheme |
|------|------------|--------|
| DG[11] | D | \( p \leq r \): Greedy base on QR \( p > r \): Greedy |
| AG | A | Greedy (Eq. (14)) |
| EG | E | Greedy (Eq. (16)) |
| DC[7] | D | Convex relaxation |

Table 2: Computational environments

| Processor information | Intel(R) Core(TM) i7-6800K @ 3.40 GHz |
|-----------------------|--------------------------------------|
| Random access memory  | 128 GB |
| System type           | 64 bit operating system x64 base processor |
| Operating system      | Windows 10 Pro Version: 1890 |
| Program code          | Matlab R2013a |

4.1 Performance on Random systems

First, randomly generated data are considered and the performance of the sparse sensor selection methods listed in Table 1 is evaluated. The values of indices utilized for optimal criteria, i.e. the determinant, the trace of the inverse, and the minimum eigenvalue of the Fisher information matrix are evaluated while the computational time to obtain sensors is also accounted. The random sensor-candidate matrices, \( U \in \mathbb{R}^{n \times r} \), are set where the component of the matrices is given by the Gaussian distribution of \( N(0, 1) \) with \( n = 2000 \) and \( r = 10 \).

Figures 2, 3, and 4 show the relationship between the number of sensors and three indices: the log determinant, the trace of the inverse, and the minimum eigenvalue of the Fisher information matrix, that is, \( CC^T = HUU^TH^T \) (\( p \leq r \)) and \( C^TC = U^TH^HU \) (\( p > r \)), respectively. The results of the DG, AG, EG, DC, and random selection methods are plotted together for comparison. The values plotted in Figs. 2, 3, and 4 are averages over 1000 random samples and normalized by those of the DG method since this study especially focuses on evaluating the methods based on A- and E-optimality in comparison with that based on D-optimality. Figure 2 reveals that the determinants obtained by the AG and EG methods are lower than those by the DG method in all cases. It indicates that the objective function based on D-optimality is the most suitable for the maximization of the determinants, as expected. Figure 2 also demonstrates that sensors selected by the AG method is only slightly inferior to those by the DG method in terms of maximizing the determinant. Figure 3 shows that the sensor selected by the AG method, which is dedicated to minimizing the trace of the inverse, works better for minimization of the trace of the inverse in the case of \( p \leq 11 \) than the those by the DG method. Note that, the trace of the inverse becomes lower for the sensors selected by the DG method than those by the AG method when \( p \) exceeds 11. It is mainly because the greedy method obtains not a combinatorial optimized solution but
a suboptimal one in a step-by-step manner. In addition, Fig. 4 shows that the sensors selected by the AG method has the minimum eigenvalues larger than those by the DG method in all cases. Accordingly, the AG method is more suitable for selection of the sensors which has the larger minimum eigenvalue. On the other hand, the sensors selected by the EG method does not show better performance on any indices than those by the DG and AG methods, especially as \( p \) increases. Although the EG method tries to maximizes the minimum eigenvalue, the minimum eigenvalues obtained by the EG method become smaller than those by the AG and DG methods in the case of \( p > 12 \). It might be due to the fact that the objective function based on E-optimality is not submodular; thus, the lower bound of the performance of the sensors selected by the EG method is not guaranteed, while those by the DG and AG methods have lower bound and are close to the optimal solution, as stated in previous section. Besides, one can see that the DC method tends to do well on the determinant; however, it does quite worse on the trace of the inverse and the minimum eigenvalue than the DG, AG, and EG methods. Although these characteristics are seemingly caused by the formulation using the convex relaxation, the cause requires further investigation.

Figure 5 illustrates the computational time required for the sensor selection using the DG, AG, EG, DC, and random selection methods, respectively. The computational time for the AG and EG methods gradually increases with increasing \( p \) because of the greedy method which chooses sensors in a step-by-step manner. The AG and EG methods need shorter time than the DC method. This indicates that all the greedy methods based on D-, A-, and E-optimality are superior to the convex relaxation method in terms of the computational time. On the other hand, the AG and EG methods require longer time than the DG method. It is mainly because the DG utilizes the QR decomposition in the case of \( p \leq r \). Although the AG method seems to be comparable to the DG method in terms of the indices of selected sensor quality, the DG method is more effective considering the computational cost.
| Number of sensors | DG | AG | EG | DC | Random |
|-------------------|----|----|----|----|--------|
| 0                 |    |    |    |    |        |
| 5                 |    |    |    |    |        |
| 10                |    |    |    |    |        |
| 15                |    |    |    |    |        |
| 20                |    |    |    |    |        |

Figure 3: Normalized trace of inverse of $CC^\top (p \leq r)$ or $C^\top C (p > r)$ vs. number of sensors for random systems

| Number of sensors | DG | AG | EG | DC | Random |
|-------------------|----|----|----|----|--------|
| 0                 |    |    |    |    |        |
| 5                 |    |    |    |    |        |
| 10                |    |    |    |    |        |
| 15                |    |    |    |    |        |
| 20                |    |    |    |    |        |

Figure 4: Normalized minimum eigenvalue of $CC^\top (p \leq r)$ or $C^\top C (p > r)$ vs. number of sensors for random systems
4.2 Performance on NOAA-SST problem

In this subsection, the results of application of the sensor selection methods to the practical problem are shown. A data set adopted is the NOAA OISST (NOAA-SST) V2 mean sea surface temperature set, comprising weekly global sea surface temperature measurements in the years between 1990 and 2000[16]. Note that, the dimensional reduction is considered for this problem in exactly the same way as in previous study[11]. The data matrix $X \in \mathbb{R}^{n \times m}$, which consists of $m$ snapshots with spatial dimension $n$, is decomposed into the left singular matrix $U \in \mathbb{R}^{n \times m}$, which shows the spatial POD modes, the diagonal matrix of singular values $S \in \mathbb{R}^{m \times m}$, and the right singular matrix $V \in \mathbb{R}^{m \times m}$, which shows temporal POD mode by economy SVD, corresponding to POD. Hence, applying the truncated SVD for a given rank $r$ gives the reduced-order modeling:

$$X \approx U_{1:r} S_{1:r} V_{1:r}^\top.$$ 

Then, the measurement matrix is described by $C = H U_{1:r}$, and the latent state variables becomes the POD mode amplitude: $Z = S_{1:r} V_{1:r}^\top$. Refer to the previous study[11] for detailed information. The data set of NOAA-SST have 520 snapshots on a $360 \times 180$ spatial grid, and the data truncated to the $r = 10$ POD modes is used in this study.

Figure 6 shows the sensor positions obtained by using the DG, AG, and EG methods, respectively, in the case of $p = 15$. Although the positions selected are slightly different depending on the method, similar positions are selected, especially in the case of the DG and AG methods. Figure 7 shows the relationship between the reconstruction error and the number of sensors, where the estimation error is defined to be the ratio of the difference between the reconstructed data and the full observation data to the full observation. The reconstruction error decreases as $p$ increases for sensors selected by all the methods. The sensors selected by the DG method tends to slightly reduce the error compared to those by the AG method, while the trend depends on the condition. The results above implies that the determinant and the trace of the inverse are important indices for accurate reconstruction. On the other hand, the sensors selected by the EG method have larger errors than those of the DG and AG methods. It is consistent with
the performance on the indices for the random systems. Figure 8 shows the compu-
tational time to obtain sensors. The qualitative characteristics in Fig. 8 are the same
as those in Fig. 5. Therefore, considering the reconstruction error and computational
cost comprehensively, D-optimality is the most suitable for the objective function of
the sensor selection using the greedy method, while A-optimality is more suitable than
E-optimality.

Figure 6: Sensor positions in the case of \( p = 15 \) for NOAA-SST sensor problem

Figure 7: Reconstruction error vs. number of sensors for NOAA-SST problem
5 Conclusions

The problem of choosing an optimal set of the sensors estimating the snapshot of high-dimensional data is considered. The greedy method for D-, A- and E-optimality of optimal designs are considered, where D-, A- and E-optimality correspond to the designs that maximize the determinant, minimize the trace of the inverse and maximize the minimum eigenvalue of the Fisher information matrix, respectively. First, depending on the number of latent state variables and sensors, the Fisher information matrix is derived using the measurement matrix. Then, the unified formulation of the objective function based on A-optimality is introduced and proved to be submodular, which provides the lower bound on the performance of the greedy method. On the other hand, we also show that the objective function based on E-optimality is neither submodular nor supermodular while it is monotone. Next, the greedy methods based on D-, A-, and E-optimalities are adopted to randomly generated systems and a practical data set of global climates. The greedy method based on D-optimality provides a set of sensors with higher determinant and lower trace of inverse of the Fisher information matrix, and that based on A-optimality provides the larger minimum eigenvalue of the matrix than the other methods in a plurality of cases. The reconstruction error slightly becomes lower by the sensors selected by the D-optimality-based greedy method than those by the A-optimality-based one. On the other hand, the greedy method based on E-optimality does not work better on any indices and reconstruction at all. It might be because the objective function based on E-optimality is not submodular, while those based on D- and A-optimality are submodular. Furthermore, the computational time of the greedy method based on D-optimality is shorter than the other methods. Considering the reconstruction error and computational cost, the greedy sensor selection method based on the D-optimality is the most suitable for the sensor selection from the many sensor candidates.

In addition, these results seem to be also applicable to the sampling point of the
reduced order model (ROM) using the DEIM or Q-DEIM framework (hereafter, ROM-DEIM). This is because the procedure employed in Q-DEIM, i.e. QR decomposition, corresponds to the optimization using D-optimality criterion for the number of sensors less than or equal to that of latent variable, as stated previously. Although the oversampling in DEIM framework is being discussed, we suggest that the oversampling in those framework should be straightforwardly considered using optimal design of experiment, as done in the present study. This implicates that the other optimal designs, e.g. A-optimal design or E-optimal design, also can be utilized for selecting sampling points for the ROM-DEIM framework. These applications seem to be interesting, while the results of this study suggest that the greedy method based on D-optimality is also expected to be suitable including oversampling strategy of the ROM-DEIM framework\cite{8, 17} which is partly demonstrated in Appendix of reference\cite{11}; but it is out of scope of this study. The applicability of the sensor selection methods in the present study to the sampling and oversampling strategy in the ROM-DEIM framework will be considered in future works.

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