Effective Hamiltonians from classical constraints: a new method for experimental realisation of fractional quantum Hall effects

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We show that model Hamiltonians of the Read-Rezayi series of the fractional quantum Hall effect can be realised by the von Neumann lattice of local potentials in the thermodynamic limit. Such von Neumann lattice is inspired from the classical reduced density matrix constraints introduced in [B. Yang, arXiv:1901.00047]. We show analytically that tuning local one-body potentials in the lattice can lead to enhanced specific few-body pseudopotentials important for realising exotic non-Abelian states. This new approach can potentially stabilise coveted non-Abelian quantum fluids including the Moore-Read state and the Fibonacci state in experiments.

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The fractional quantum Hall (FQH) systems are promising candidates for hosting anyonic and non-Abelian excitations that are topologically protected by strong interactions between electrons[1][2]. With these exotic properties, the relevant two-dimensional electron gas systems become robust for storing and processing quantum information, serving as the physical platform for topological quantum computers[3]. Experimentally, however, there has been no conclusive evidence for the non-Abelian statistics, even for the simplest non-Abelian FQH states: the Moore-Read (MR) state at half filling[4–10]. For universal quantum computation, the theoretically proposed Fibonacci FQH state[10, 11] at filling factor $\nu = 3/5$ is needed. While the MR state is constructed as the ground state of the model Hamiltonian with three-body interactions, the model Hamiltonian for the Fibonacci state requires a much more complex four-body interaction[2]. Many other interesting FQH states have been proposed theoretically with model Hamiltonians in the form of pseudopotentials, but the rich topological structures of these states are largely beyond the reach of experiments at the current stage.

One of the main challenges for non-Abelian and exotic FQH states is that the model Hamiltonians are highly artificial. While in realistic systems the electron-electron interaction is derived from the two-body Coulomb interaction, exact non-Abelian FQH states typically require three- or more body interactions. Numerical studies for small systems seem to indicate that it is possible for two-body realistic interactions to be adiabatically connected to some simple artificial model Hamiltonians based on wavefunction overlaps, and most works focused on the Moore-Read state[12, 14]. It is, however, difficult to deduce the topological properties of realistic systems in the thermodynamic limit. This is especially true for the non-Abelian states, in which the non-Abelian statistics are determined by degeneracies of elementary excitations. Thus the ground state incompressibility gap, as well as the low-lying excitation bandwidth, need to be carefully tuned in experiments[15, 16]. Unfortunately, experimental tuning of the electron-electron interaction is highly limited. There are also limited theoretical guidance on the optimal Coulomb based Hamiltonians that can mimic three or more body interactions[17, 18].

Recently, it was discovered that topological properties of many FQH phases (including both Abelian and non-Abelian candidates) can be uniquely determined by simple classical constraints of the reduced density matrix from the quantum fluids within a single Landau level[19, 20]. Such classical constraints lead to systematic truncation of the Hilbert space within a single LL, that do not have an exact implementation with a local quantum Hamiltonian. We term such reduced density matrix constraints as the local exclusion conditions (LECs). This new approach suggests that the quantum fluid needs to have a dominating energy gap for violating the LECs, for the relevant FQH phases to be robust. Model Hamiltonians applicable to some FQH phases are one way to guarantee the LECs, as evidenced by numerical computations. In principle, there could be other physical processes for the quantum fluids to obey LECs, beyond explicitly tuning of the electron-electron interactions.

In this Letter, we show that starting with the imposition of LECs at a single location with a local Hamiltonian $H_{0}$, a von Neumann lattice (vNL) of $H_{0}$ can lead to well-defined effective Hamiltonians. While such effective Hamiltonians do not strictly impose the LECs exactly, they are shown to be identical to the well-known model Hamiltonians with projection operators for the Read-Rezayi series. This leads to a new perspective in understanding the model Hamiltonians, not from explicit electron-electron interactions, but from a lattice of lo-
eral potentials punishing certain electron configurations. More importantly, the results imply that FQH states (including non-Abelian, exotic ones such as the Moore-Read state and the Fibonacci state) can be realised with a vNL of properly tuned local potentials.

The LEC constraints – Let us first review the LEC constraints on the translationally invariant quantum Hall fluids, which is the only physical input needed for defining the topological properties of the FQH effects. The LECs are imposed on the reduced density matrix of the quantum fluid within a small droplet. It is denoted with a triplet of non-negative integers \( \hat{c} = \{ n, n_e, n_h \} \). Physically, it dictates for any small droplet containing \( n \) fluxes (thus with an area of \( 2n\pi l_B^2 \)), \( l_B \) being the magnetic length), a measurement in this droplet can never detect more than \( n_e \) number of electrons or \( n_h \) number of holes (unoccupied orbitals in a single LL). For example, \( \hat{c} = \{ 2, 1, 2 \} \) dictates that no more than one electron and no more than two holes can be detected in any circular droplet containing two fluxes. It gives the Laughlin state at filling factor \( \nu = 1/3 \) and topological orbital shift \( S_h = -2 \). Another example is \( \hat{c}_f = \{ 4, 3, 4 \} \), giving the Fibonacci state at \( \nu = 3/5 \) and \( S_h = -2 \).

The LECs, simple as they are, do not explicitly correspond exactly to any local Hamiltonians. On an infinite plane, we can easily define a local Hamiltonian at a single site (i.e. the origin) as follows:

\[
\mathcal{H}_0 = \sum_i |\psi_i\rangle \langle \psi_i| \tag{1}
\]

where each state \( |\psi_i\rangle \) consists of \( n \) orbitals of the symmetric gauge around the origin in a single LL, and the summation is over all such states that does not satisfy \( \hat{c} = \{ n, n_e, n_h \} \). Therefore Eq.\((1)\) imposes an energy punishment at a single location when the LEC is violated. Diagonalisation of Eq.\((1)\) in the sub-Hilbert space of translationally invariant states will obviously lead to the ground state of the corresponding FQH state. Within the full Hilbert space, however, we cannot obtain the FQH ground state because Eq.\((1)\) is not translationally invariant: the LEC is not imposed everywhere in the quantum Hall fluid by Eq.\((1)\).

The von Neumann lattice – To construct a translationally invariant analog of Eq.\((1)\), we briefly review the well-known vNL formalism in a single LL, and introduce the notations.\(^{21,23}\) Denoting the guiding center coordinates as \( R^x, R^y \) with the commutation relation \([R^x, R^y] = -il_B^2 \), these operators only have matrix elements within a single LL. We can construct the ladder operators \( b = (R^x - iR^y)/\sqrt{2}l_B \) and \([b, b^\dagger] = 1 \). Let us denote the single particle state centered at the origin as \( |n\rangle = 1/\sqrt{\text{Vol}(b)} \langle 0| \), with \( b|0\rangle = 0 \). Thus \( |0\rangle \) is the coherent state at the origin.

Given the commutation relation of \( R^x, R^y \), the magnetic translation operator is given by \( T^\hat{X} = \sum_i e^{iX_n R^x_i} \), where Einstein’s summation convention is adopted and the subscript \( i \) runs over all electrons in the system. The state \( |\hat{X}\rangle = T^\hat{X}|0\rangle \) is thus the coherent state centered at \( r^a = l_B^2 e^{a0} X_0 \) in the real space. The states \( |\hat{X}_i\rangle \) are not orthogonal, they nevertheless can form a complete basis with \( \hat{X} \) as a continuous variable. This basis is obviously over-complete, since the total number of linearly independent states in a single LL is \( A/(2\pi l_B^2) \), where \( A \) is the area of the sample. A complete basis of the coherent states can be formed from a square vNL with \( r^x = \sqrt{2\pi l_B} p, r^y = \sqrt{2\pi l_B} q \), where \( p, q \) are integers. Rewriting \( |\hat{X}\rangle = |p,q\rangle \), we thus have \( \sum_{p,q=-\infty}^{\infty} |p,q\rangle \langle p,q| = I \). This relationship also works if we replace \( |0\rangle \) with \( |n\rangle \) in the above analysis. We can thus form the vNL where each site contains more than one electron.

The effective Hamiltonians – The generalisation of Eq.\((1)\) at a single site to the vNL is straightforward. Denoting \( |\psi_i\rangle_0 \) as a state of a circular droplet centered at the origin, we can apply the magnetic translation operator to get \( |\psi_i(\hat{X})\rangle = T^\hat{X}|\psi_i\rangle_0 \). The resulting state describes the circular droplet centered at \( r^a = l_B^2 e^{a0} X_0 \). In the thermodynamic limit with \( A \to \infty \) or \( l_B \to 0 \), the vNL effective Hamiltonian can thus be constructed as follows:

\[
\mathcal{H} = \int \frac{d^2x}{2\pi l_B^2} \sum_i |\psi_i(\hat{X})\rangle \langle \psi_i(\hat{X})| \tag{2}
\]

Due to the completeness of the vNL, Eq.\((2)\) is translationally invariant in a single LL. Since again \( \langle \psi_i(\hat{X}_1)|\psi_j(\hat{X}_2)\rangle \neq \delta_{ij}\delta(\hat{X}_1 - \hat{X}_2) \), Eq.\((2)\) does not necessarily imply that locally, the effective Hamiltonian is equivalent to \( \mathcal{H}_{\hat{X}} = \sum_i |\psi_i(\hat{X})\rangle \langle \psi_i(\hat{X})| \). While the LECs are not explicitly obeyed by Eq.\((2)\), the Hamiltonian does imply a greater energy punishment for states whose reduced density matrix violates the LECs.

We will now implement Eq.\((2)\) explicitly on spherical geometry, which allows us to focus on the bulk of the QH fluids without the complications from the presence of the boundary. It is also the geometry where topological indices such as topological shifts\(^{24,25}\) are well-defined even for finite systems. We fix the radius of the sphere to be \( R \), with a monopole strength of \( 2S \) placed at the center of the sphere. The thermodynamic limit is thus obtained by \( 2S \to \infty \), or equivalently \( l_B \to 0 \). The single particle wavefunctions on the sphere can be taken as spinors,\(^{26}\) thus magnetic translation on the sphere is given by a \( SU(2S+1) \) rotation operator:

\[
T_{\theta,\phi} = \sum_i e^{i\theta L_{z,i}} e^{i\phi L_{y,i}} \tag{3}
\]

\[
[\hat{L}_{a,i}, \hat{L}_{b,j}] = i\delta_{ij} e^{abc} \hat{L}_{c,i} \tag{4}
\]

Here \( a, b, c = x, y, z \) and the single particle states are given by \( \hat{L}_z|n\rangle = n|n\rangle \), with \( n = -S, -S + 1, \ldots, S - 1, S \). A rotationally invariant effective Hamiltonian on
the sphere, analogous to Eq. [2], is given as follows:

$$\mathcal{H}_m = \int_0^\pi \int_0^{2\pi} \frac{\sin \theta d\theta d\phi}{2m_l^2} \sum_i |\psi_i(\theta, \phi)\rangle \langle \psi_i(\theta, \phi)|$$

$$|\psi_i(\theta, \phi)\rangle = T_{\theta, \phi}|\psi_i(0, 0)\rangle$$

(6)

where $|\psi_i(0, 0)\rangle$ is the state describing a circular droplet centered at the north pole of the sphere.

As a simple illustration, we look at $\hat{c} = \{2, 1, 2\}$, which we know will lead to the Laughlin state at $\nu = 1/3$. The LEC dictates that for a circular droplet of two orbitals, the state with both orbitals occupied will be truncated. The projection Hamiltonian of Eq. [1] at the origin thus given by $\mathcal{H}_0 = |S, S - 1\rangle \langle S, S - 1| = |\psi_l(0, 0)\rangle \langle \psi_l(0, 0)|$. Enlisting the Wigner’s d-matrix $d_{m'm}(\theta) = \langle m'|e^{i\theta \Lambda} |m\rangle$, we have the following:

$$\mathcal{H}_l = \int_0^\pi \int_0^{2\pi} \frac{\sin \theta d\theta d\phi}{2m_l^2} |\psi_l(\theta, \phi)\rangle \langle \psi_l(\theta, \phi)|$$

$$= \sum_{m, n, m', n'} V^{m'n'}_{mn} c_m^\dagger c_{m'} c_{n} c_{n'}$$

(7)

$$V^{m'n'}_{mn} \sim \int_0^\pi \int_0^{2\pi} \int_0^{\pi} \sin \theta d\theta d\phi d\psi d\tilde{\psi} |\psi_l(\theta, \phi, \psi, \tilde{\psi})\rangle \langle \psi_l(\theta, \phi, \psi, \tilde{\psi})|$$

(8)

where $V^{m'n'}_{mn}$ is only non-vanishing for $m' + n' = m + n$ due to the integration over $\phi$, and $\sim$ in Eq. [8] indicates proper antisymmetrisation for fermions on the right hand side.

The effective Hamiltonian is a two-body interaction with matrix elements given by Eq. [9], and the integration can be carried out analytically [10]. The first main result is that we can show analytically that Eq. [9] is equivalent to the well-known model Hamiltonian $V_1^{3bdy}$ for the Laughlin state at $\nu = 1/3$, which is given by the first Haldane pseudopotential. Thus the vNL of local pseudopotentials in the thermodynamic limit gives the exact spectrum of the Laughlin quantum Hall fluids. Moreover, we conjecture the connection can be extended to the entire Read-Rezayi series. We have explicitly checked the case for the Moore-Read (MR) state with $\mathcal{H}_0 = |S, S - 1, S - 2\rangle \langle S, S - 1, S - 2| = |\psi_m(0, 0)\rangle \langle \psi_m(0, 0)|$ (from $\hat{c} = \{3, 2, 3\}$), where the effective Hamiltonian (denote as $\mathcal{H}_m$) is the same as Eq. [7] but with $|\psi_m(0, 0)\rangle$ replaced by $|\psi_m(0, 0)\rangle$. The case for the Fibonacci state with $\mathcal{H}_0 = |S, S - 1, S - 2, S - 3\rangle \langle S, S - 1, S - 2, S - 3| = |\psi_f(0, 0)\rangle \langle \psi_f(0, 0)|$ (from $\hat{c} = \{4, 3, 4\}$) gives the effective Hamiltonian (denote as $\mathcal{H}_f$) analogous to Eq. [7] with the replacement of $|\psi_f(0, 0)\rangle$ by $|\psi_f(0, 0)\rangle$.

In both cases, the effective Hamiltonians $\mathcal{H}_m, \mathcal{H}_f$ are equivalent to the model three-body Hamiltonian $V_1^{3bdy}$ for the MR state, and the model four-body Hamiltonian $V_0^{3bdy}$ for the Fibonacci state [37], respectively. The fact that one can derive exact model Hamiltonians from the LECs alone explains why all physical properties of the FQH fluids can be determined by the LECs, even though the LECs and the model Hamiltonians are not explicitly equivalent. The effective equivalence of the classical and the quantum approach is mainly because of the constraint of translational invariance, which is enforced in both approaches.

Tuning with anharmonic well lattice – The derivation of model Hamiltonians from the vNL not only reveals new perspectives on the physical nature of these Hamiltonians, it also naturally leads to a new way for the experimental realisation of exotic FQH fluids. If there are some mechanisms in mimicking Eq. [1] at a single location, e.g. with some local potential profile, then a proper lattice pattern of such local mechanisms in principle can realise the effective projection Hamiltonians, and thus the topological phases in a robust manner.

We now show that a local one-body anharmonic well with the width on the order of the magnetic length can be a viable way of tuning individual two- or more body pseudopotentials. With the plane geometry a potential well in the real space is given as follows:

$$V_0 = \frac{1}{2} \hbar \omega f \bigg( |r|^2 \bigg) \sim \frac{1}{2} \hbar \omega f \bigg( |R|^2 \bigg) = \sum_{k=0}^{n} \lambda_k |k\rangle \langle k|$$

(9)

Here we have $r^a = \tilde{R}^a + R^a$, with $\tilde{R}^a$ the cyclotron coordinates with only non-zero matrix elements between different LLs, and $f(x)$ is some function for the confining potential. In the limit of $\omega_c \gg \omega_r$, where $\omega_c$ is the cyclotron frequency, we ignore the dynamics involving higher LLs, leading to the effective Hamiltonian in Eq. [9] with single particle states $|k\rangle = 1/\sqrt{K!} (b^\dagger)^k |0\rangle$. The upper limit $n$ in the summation gives the range of the local potential, covering an area of $\sim 2\pi R^2$.

Denoting a state containing $n_c$ number of electrons as $|k_1 \cdots k_{n_c}\rangle$, with $k_i \leq n$, a feat that is possible so far for the Coulomb based electron-electron interaction. The vNL of each $|k_1 \cdots k_{n_c}\rangle$ is proportional to a linear combination of pseudopotentials. The coefficients of proportionality depend on the system
size and can be computed analytically. For example with \( n = 2 \), the corresponding vNL effective Hamiltonian is given by [40]:

\[
\mathcal{H} = \int \frac{d^2 r}{2\pi l_B^2} \lambda_{0,1}|0,1\rangle\langle 0,1| + \lambda_{0,2}|0,2\rangle\langle 0,2| + \lambda_{1,2}|1,2\rangle\langle 1,2| + \lambda_{0,1,2}|0,1,2\rangle\langle 0,1,2| \\
= \left( \lambda_{0,1} + 2\lambda_{0,2} + \frac{1}{2}\lambda_{1,2} \right) V^{2\text{bdy}}_1 + \frac{3}{2} \lambda_{1,2} V^{2\text{bdy}}_3 \\
+ \frac{4}{3} \lambda_{0,1,2} V^{3\text{bdy}}_3
\]

where we have ignored the uniform background. For \( n = 3 \), we have the following instead [40]:

\[
\mathcal{H} = \left( \frac{15}{2} \lambda_0 + \frac{9}{2} \lambda_1 + 4 \lambda_2 + 9 \lambda_3 \right) V^{2\text{bdy}}_1 \\
+ \left( \frac{3}{2} \lambda_0 + \frac{9}{2} \lambda_1 + 9 \lambda_2 + \frac{15}{2} \lambda_3 \right) V^{2\text{bdy}}_3 \\
+ \frac{15}{2} (\lambda_2 + \lambda_3) V^{2\text{bdy}}_5 \\
+ \left( 8 \lambda_0 + \frac{152}{27} \lambda_1 + \frac{116}{27} \lambda_2 + \frac{188}{27} \lambda_3 \right) V^{3\text{bdy}}_3 \\
+ \left( 16 \lambda_0 + \frac{16}{9} \lambda_1 + \frac{64}{9} \lambda_2 + \frac{64}{9} \lambda_3 \right) V^{3\text{bdy}}_5 \\
+ \frac{160}{27} (\lambda_1 + \lambda_2 + \lambda_3) V^{3\text{bdy}}_6 + 6 \lambda_{0,1,2,3} V^{4\text{bdy}}_6
\]  

(13)

Some comments are in order here. For \( n = 1 \) so that the local potential acts on the area of two magnetic fluxes, the vNL effectively adds a single two-body pseudopotential of \( V^{2\text{bdy}}_1 \) for any positive \( \lambda_0, \lambda_1 \). This seems to be the first viable way to tune a single pseudopotential in an experimental FQH system, leaving the rest of the pseudopotentials from electron-electron interaction intact. For \( n = 2 \) in Eq. (12), there are three tuning parameters \( \lambda_0, \lambda_1, \lambda_2 \). In principle we can realise any linear combinations of \( V^{2\text{bdy}}_1, V^{2\text{bdy}}_3, V^{3\text{bdy}}_3 \). In particular with \( \lambda_0 = 3\lambda_1 > 0, \lambda_2 = -\lambda_1 \), both the coefficients of \( V^{2\text{bdy}}_1, V^{2\text{bdy}}_3 \) vanishes, and Eq. (12) is the pure three-body interaction. Such vNL effectively adds only \( V^{3\text{bdy}}_3 \), which could be used to stabilise the Pfaffian state in higher LLs (see Fig. 1). For \( n = 3 \), however, one-body local potentials no longer provide enough degrees of freedom to tune all individual pseudopotentials. One can however tune the anharmonic confining potential to maximise \( V_4^{2\text{bdy}} \) in principle, or to design more complicated local potentials (beyond one-body), to enhance the incompressibility gap of the Fibonacci state.

**Summary and outlook** – We show that by forming a von Neumann lattice of local potentials in the plane of Hall manifold with strong magnetic field, the effective Hamiltonian does not break translational symmetry in the Hilbert space of a single LL. Instead, it is analytically equivalent to a linear combination of short range projection Hamiltonians, representing two- or few-body pseudopotential interactions. In general a periodic potential in a single LL splits the degeneracy of the single particle orbitals and forms sub-bands by explicitly breaking magnetic translational symmetry. These are cases where the lattice spacing is much larger than the magnetic length. Nontrivial physics arises in the form of Hofstadter states or quantum anomalous Hall insulators. Interestingly, with the vNL (i.e. lattice spacing \( \sim \sqrt{2\pi l_B} \)) the physics is fundamentally different, though in many cases equivalent to the usual fractional quantum Hall states purely from electron-electron interactions. Theoretically, we can now understand such pseudopotentials as equivalent to local projections in spatial dimensions, even if such projections are one-body in nature.

The construction of Eq. (2) is reminiscent of the projection operators in spin chains, e.g. the AKLT model, where translational invariance is achieved by projection operators on every pair of neighbouring spins. In both cases, while the quantum Hamiltonian cannot impose local truncation of Hilbert space exactly (for spin chains, each spin is shared by different projection operators, analogous to non-orthogonality of coherent states in vNL), the ground states of such Hamiltonians nevertheless do satisfy local truncation of the Hilbert space everywhere. This implies the LEC formalism can be generalised to many spin systems as well. It is important to note, however, the LECs in FQHE do not always lead to
quantum Hamiltonians of the same unique ground state. This is because LECs are insensitive to the coefficients in front of each projection operator (e.g., see Eq. (12)). For example, a unique FQH state\[19\] at $\nu = 3/7$ can be determined by $\hat{c} = \{2, 1, 2\} \lor \{6, 3, 6\}$, but the corresponding vNL Hamiltonians are not unique, forming a family of ground states described by the $S_3$ conformal field theory\[39\].

Experimentally, if it is technically possible to impose closely packed local potentials on the Hall bar and to accurately tune the potential profiles, the results in this work presents the exciting possibility of tuning individual PP (including three- or four-body PPs, etc), as well as the robust realisation of coveted non-Abelian FQH phases (e.g. the MR and the Fibonacci states). The lattice spacing needs to be on the order of $\sim 30\text{nm}$ with $B = 5T$, and $\sim 20\text{nm}$ with $B = 10T$. This is technically feasible with the formation of antidot arrays with the current e-beam processing technology. We would also like the local potentials to be overlapping: the range of the potential is larger than the potential spacing. It can be achieved if a spacer with proper thickness is synthesised between the antidot array and the Hall manifold. The accurate tuning of the local potentials could be more feasible in the cold atom systems, with much flexibility in tuning the entire photonic lattice\[35\][36]. The presence of imperfect von Neumann lattice formation, intrinsic disorder of the Hall manifold, and the boundary effects in realistic samples, could significantly impact the resulting effective Hamiltonians and the robustness of the FQH phases. These factors will be the focus of the future research.

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[40] More technical details can be found in the supplementary materials.
Supplementary Online Materials for “Effective Hamiltonians from classical constraints: a new method for experimental realisation of fractional quantum Hall effects”

In this supplementary material, we give more technical details on how the effective Hamiltonians can be constructed from the von Neumann lattice (vNL) of projection operators. For simplicity we use spherical geometry, and such Hamiltonians are in the following general form:

\[ \mathcal{H}_I = \int_0^\pi \int_0^{2\pi} \frac{\sin \theta d\theta d\phi}{2\pi l_B^2} |\psi(\theta, \phi)\rangle \langle \psi(\theta, \phi)| \]  

(S1)

where \( \theta \) and \( \phi \) are the polar and azimuth angle respectively, and \( l_B \) is the magnetic length. The single particle orbitals are indexed by \( -S \leq k \leq S \), where \( k = S \) corresponds to the state at the north pole, \( k = -S \) corresponds to the state at the south pole. In the thermodynamic limit \( S \to \infty \), we can always keep the size of the sphere fixed, while taking \( l_B \to 0 \). Eq.\( \text{(S1)} \) thus gives a vNL of the projection operator in the thermodynamic limit, where the lattice spacing is \( \sqrt{2\pi l_B} \). Defining \( T_{\theta,\phi} = \sum_i e^{\imath \theta L_z} e^{\imath \phi L_y} \) with \( [L_{a_1}, L_{b_2}] = i\delta_{ij} e^{abc} L_{c,ij} \), the projection operator located at \( (\theta, \phi) \) is given by

\[ |\psi(\theta, \phi)\rangle \langle \psi(\theta, \phi)| = T_{\theta,\phi} |\psi(0,0)\rangle \langle \psi(0,0)| \]  

(S2)

where \( |\psi(0,0)\rangle \langle \psi(0,0)| \) is the projection at the north pole. Let \( |\psi(0,0)\rangle \) describes a droplet at the north pole containing \( n \) orbitals and \( n_e \) electrons, so that we rewrite \( |\psi(0,0)\rangle = |S - k_1, \ldots, S - k_{n_e}\rangle \), \( 0 \leq k_i < n \). In this case Eq.\( \text{(S1)} \) is an \( n_e \)-body interaction that can be rewritten as follows:

\[ \mathcal{H} = \sum_{a_1, \ldots, a_n, a'_1, \ldots, a'_{n_e}} V_{a_1a'_1 \ldots a_n a'_n} c_{a_1} \ldots c_{a_n} c_{a'_1} \ldots c_{a'_{n_e}} \]  

(S3)

Here \( -S \leq a_i, a'_i \leq S \). To compute the matrix elements in Eq.\( \text{(S3)} \), we take \( n_e = 2 \) and \( |\psi(0,0)\rangle = |S, S - 3\rangle \) as an example. Using the Wigner’s d matrix we have the following expression:

\[ a_{a,S}^S = \langle a| e^{iL_z \theta} e^{iL_y \theta} |S\rangle = e^{-ia\phi} \sqrt{C_{S-a}^S} \left( \cos \frac{\theta}{2} \right)^{S+a} \left( \sin \frac{\theta}{2} \right)^{S-a} \]  

(S4)

\[ a_{a,S-3}^S = \langle a| e^{iL_z \theta} e^{iL_y \theta} |S-3\rangle = e^{-ia\phi} \sqrt{\frac{(2S-2)(2S-1)2S}{6C_{S-a}^S}} \left( \cos \frac{\theta}{2} \right)^{S+a-3} \left( \sin \frac{\theta}{2} \right)^{S-a-3} \]  

\[ \left( -C_{S-3-a}^{S-3} \left( \sin \frac{\theta}{2} \right)^6 + 3C_{S-3-a}^{S-3} \frac{1}{S-a-1} \left( \cos \frac{\theta}{2} \right)^4 \left( \sin \frac{\theta}{2} \right)^4 - 3C_{S-3-a}^{S-3} \frac{1}{S-a-2} \left( \cos \frac{\theta}{2} \right)^4 \left( \sin \frac{\theta}{2} \right)^2 + C_{S-3-a}^{S-3} \left( \cos \frac{\theta}{2} \right)^6 \right) \]  

(S5)

To find the matrix elements of Eq.\( \text{(S4)} \), we note the integration over the azimuth angle \( \phi \) gives \( \langle a'_1, a'_2|H|a_1, a_2\rangle \sim \delta(a_1 + a_2 - a'_1 - a'_2) \), as expected. The integration over the polar angle \( \theta \) can be done analytically for each set of integers \( a_1, a_2, a'_1, a'_2 \) in Mathematica, though the final expression is long and complicated.

To show \( V_{a_1a_2'a_2}' \) is equivalent to a linear combination of the two-body interactions with Haldane pseudopotentials \( V_{1}^{2\text{bdy}} \) and \( V_{3}^{2\text{bdy}} \), we note that the pseudopotential interaction \( V_{k}^{2\text{bdy}} \) effectively projects into the Hilbert space of two electrons when the relative \( L_z \) angular momentum is \( k \). For the two-electron state on the sphere as the eigenstates of \( L_z = L_{z,1} + L_{z,2} \) and \( \mathbf{L}^2 = L_{1}^2 + L_{2}^2 + 2 \mathbf{L}_1 \cdot \mathbf{L}_2 \), we can label the states as \( |M, J\rangle \) with \( L_z|M, J\rangle = M|M, J\rangle, \mathbf{L}^2|M, J\rangle = J(J+1)|M, J\rangle \). The matrix elements of \( V_{k}^{2\text{bdy}} \) is given by:

\[ \langle M, J|V_{k}^{2\text{bdy}}|M', J'\rangle = \delta_{J,J'} \delta_{M,M'} \delta_{2J-M,0} \]  

(S7)

Thus with \( n_e = 2, |\psi(0,0)\rangle = |S, S - 3\rangle \), from Eq.\( \text{(S1)} \) the effective Hamiltonian is given by:

\[ H^{(0,3)} = \sum_{a_1a_2a'_1a'_2} V_{a_1a_2'a_2'} c_{a_1}^\dagger c_{a'_1}^\dagger c_{a_2} \]  

(S8)
where the upper indices of $H^{(k_1,\cdots,k_{n_e})}$ indicates the projection operator in the Hamiltonian is given by $|\psi(0,0)\rangle = |S-k_1,\cdots,S-k_{n_e}\rangle$. The matrix elements $V^{a_1\sigma_1}_{a_2\sigma_2}$ can be expressed in terms of Eq.(S5) and Eq.(S6). We can thus analytically show that:

$$\lim_{S \to \infty} \langle 2S, 2S - 1 | H | 2S - 1, 2S \rangle = \lim_{S \to \infty} \frac{36 (2S + 1) (S - 1)}{16S^2 - 16S + 3} = \frac{9}{2} \quad (S9)$$

$$\lim_{S \to \infty} \langle 2S, 2S - 3 | H | 2S - 3, 2S \rangle = \lim_{S \to \infty} \frac{3 (2S + 1) \left( 15 + 30 \sqrt{S (S - 1)} + 4S \left( -10 + 7S - 3 \sqrt{S (S - 1)} \right) \right)}{(4S - 1)(4S - 3)(4S - 5)}$$

$$= \frac{3}{2} \quad (S10)$$

All other matrix elements relevant to pseudopotentials are zero. This allows us to write down the following exact relationship:

$$H^{(0,3)} = \frac{9}{2} V_1^{2b,dy} + \frac{3}{2} V_3^{2b,dy} \quad (S11)$$

For general projection operator with $|\psi(0,0)\rangle = |S-k_1,\cdots,S-k_{n_e}\rangle$, the expansion of the vNL effective Hamiltonian into a linear combination of pseudopotentials can be computed similarly. We will not repeat the detailed computation here, but to list a number of results that are used in the main text, which are all valid in the thermodynamic limit with $S \to \infty$:

$H^{(0,1)} = V_1^{2b,dy}$

$H^{(0,2)} = 2V_1^{2b,dy}$

$H^{(1,2)} = \frac{3}{2} V_3^{2b,dy}$

$H^{(0,1,2)} = \frac{4}{3} V_3^{3b,dy}$

$H^{(1,3)} = 3V_1^{2b,dy} + 3V_3^{2b,dy}$

$H^{(2,3)} = \frac{3}{2} V_1^{2b,dy} + 3V_3^{2b,dy} + \frac{15}{2} V_5^{2b,dy}$

$H^{(0,1,3)} = 4V_3^{3b,dy}$

$H^{(0,2,3)} = \frac{8}{3} V_3^{3b,dy} + \frac{16}{3} V_5^{3b,dy}$

$H^{(1,2,3)} = \frac{8}{27} V_3^{3b,dy} + \frac{16}{9} V_5^{3b,dy} + \frac{160}{27} V_6^{3b,dy}$

$H^{(0,1,2,3)} = 6V_6^{4b,dy}$

(S12) \hspace{1cm} (S13) \hspace{1cm} (S14) \hspace{1cm} (S15) \hspace{1cm} (S16) \hspace{1cm} (S17) \hspace{1cm} (S18) \hspace{1cm} (S19) \hspace{1cm} (S20) \hspace{1cm} (S21)

Various vNL effective Hamiltonians with local anharmonic well as mentioned in the main text are obtained as linear combinations of the expressions listed above.