Coherence in Substructural Categories

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Abstract

It is proved that MacLane’s coherence results for monoidal and symmetric monoidal categories can be extended to some other categories with multiplication; namely, to relevant, affine and cartesian categories. All results are formulated in terms of natural transformations equipped with “graphs” (g-natural transformations) and corresponding morphism theorems are given as consequences. Using these results, some basic relations between the free categories of these classes are obtained.

In [8] MacLane has shown that monoidal and symmetric monoidal categories are coherent, although the complete definition of the notion was given for the first time in [6]. Strictly keeping to that definition, we show that relevant, affine and cartesian categories are coherent. All the categories above we call substructural because they correspond to the minimal fragments of Associative Lambek’s Calculus, linear, relevant, BCK and intuitionistic logic that are sufficient to describe the underlying structural rules (see [10] to find about different aspects of substructural logics). We use equational axiomatizations of these categories, which originate from [1], rather than postulating the commutativity of certain diagrams. Of course, one who prefers diagram-chasing can easily convert these equations into commutative diagrams.

1 Substructural categories

By category with multiplication we mean a category $\mathcal{A}$ together with a bifunctor $\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ and a special object $I$. Categories with multiplication can be axiomatized by postulating the following equations between arrows

categorical equations

$(cat1)$ $f1_A = f = 1_Bf$ for all $f : A \to B$
$(cat2)$ $h(gf) = (hg)f$ for all $f, g, h \in \text{Mor}(\mathcal{A})$

functorial equations

$(\cdot) (g_1f_1 \cdot g_2f_2) = (g_1 \cdot g_2)(f_1 \cdot f_2)$
$(\cdot \cdot) 1_A \cdot 1_B = 1_{A \cdot B}$

A category with multiplication is monoidal if there are special arrows for all objects $A$, $B$ and $C$

\[
\begin{align*}
\sigma_A &: I \cdot A \to A \\
\delta_A &: A \cdot I \to A \\
\sigma'_A &: A \to I \cdot A \\
\delta'_A &: A \to A \cdot I \\
\overrightarrow{b}_{A,B,C} &: A \cdot (B \cdot C) \to (A \cdot B) \cdot C \\
\overrightarrow{b}'_{A,B,C} &: (A \cdot B) \cdot C \to A \cdot (B \cdot C)
\end{align*}
\]

and if it satisfies
\(\sigma\delta\)-equations

\((\sigma)\) For \(f : A \to B\), \(f\sigma_A = \sigma_B(1_B \cdot f)\).

\((\delta)\) For \(f : A \to B\), \(f\delta_A = \delta_B(f \cdot 1_I)\).

\((\sigma\sigma^i)\) \(\sigma_A\sigma^i_A = 1_A\), \(\sigma^i_A\sigma_A = 1_A\).

\((\delta\delta^i)\) \(\delta_A\delta^i_A = 1_A\), \(\delta^i_A\delta_A = 1_A\).

\((\sigma\delta)\) \(\sigma_1 = \delta_1\).

\(b\)-equations

\((b)\) For \(f : A \to D\), \(g : B \to E\) and \(h : C \to F\), \((f \cdot g) \cdot h\) \(= (f \cdot (g \cdot h))\).

\((bb)\) \(\sigma_{A,B,C} \sigma_{A,B,C} = 1_{A,B,C}\).

\((\sigma\delta)\) \(\sigma_{A,B} \cdot 1_{A,B} = 1_{A,B}\).

\((b5)\) \(\sigma_{A,B,C,D} \sigma_{A,B,C,D} = (\sigma_{A,B,C,D} \cdot 1_{D}) \sigma_{A,B,C,D}(1_A \cdot \sigma_{B,C,D})\)

A monoidal category is \textit{symmetric monoidal} if it has the special arrow

\(c_{A,B} : A \cdot B \to B \cdot A\)

for every pair \((A, B)\) of its objects, and if the following equations hold

\(c\)-equations

\((c)\) For \(f : A \to C\) and \(g : B \to D\), \((g \cdot f) c_{A,B} = c_{C,D}(f \cdot g)\)

\((cc)\) \(c_{B,A} \cdot c_{A,B} = 1_{A,B}\).

\((\sigma\delta c)\) \(\sigma_A c_{A,B} = \sigma_{A,B} = \delta_A\).

\((bc\delta)\) \(\sigma_{A,B,C} = (c_{A,B,C} \cdot 1_{D}) \sigma_{A,B,C}(1_A \cdot c_{B,C,D})\)

A symmetric monoidal category is \textit{relevant} if it has the special arrow

\(w_A : A \to A \cdot A\)

for every object \(A\), and if the following equations hold

\(w\)-equations

\((w)\) For \(f : A \to B\), \((f \cdot f) w_A = w_B f\).

\((\sigma\delta w)\) \(\sigma_{A,B} \sigma_{A,B} = 1_{A,B}\).

\((bw)\) \(\sigma_{A,A}(1_A \cdot w_A) = (w_A \cdot 1_A) w_A\).

\((cw)\) \(c_{A,A} w_A = w_A\).

\((bc\delta\delta)\) \(c_{A,B,C,D} = (c_{A,B,C,D} \cdot 1_{D}) c_{A,B,C,D}(1_A \cdot c_{B,C,D})\).

A symmetric monoidal category is \textit{affine} if it has the special arrow

\(k_A : A \to 1\)

for every object \(A\) and if the following equations hold

\(k\)-equations

\((k)\) For \(f : A \to B\), \(k_A = k_B \cdot f\).

\((1k)\) \(k_1 = 1_1\).

If a symmetric monoidal category is both relevant and affine (described in the same language) and if its arrows satisfy the following equations

\((\sigma kw)\) \(\sigma_A(k_A \cdot 1_A)w_A = 1_A\), \(\delta_A(1_A \cdot k_A)w_A = 1_A\)

then we say it is \textit{cartesian}.

We call this axiomatization of cartesian categories \textit{structural-equational}. It differs from the standard equational axiomatization (see \(\mathcal{C}\)) of these categories. The latter is based on the universality
of product and uses as primitive, arrows $1_A : A \to A$, $\pi_{A,B} : A \cdot B \to A$, $\pi'_{A,B} : A \cdot B \to B$ and $k_A : A \to I$ for all objects $A$ and $B$, and a partial binary operation on arrows $\langle, \rangle$, such that

$$f : C \to A \quad g : C \to B$$

$$\langle f, g \rangle : C \to A \cdot B$$

Equations that hold are the categorial equations plus

(E2) $f = k_A$, for every $f : A \to I$.

(E3a.) $\pi_{A,B} \langle f, g \rangle = f$, for $f : C \to A$ and $g : C \to B$.

(E3b.) $\pi'_{A,B} \langle f, g \rangle = g$, for $f : C \to A$ and $g : C \to B$.

(E3c.) $\langle \pi_{A,B} b, \pi'_{A,B} \rangle h = h$, for $h : C \to A \cdot B$.

To show that these two axiomatizations are extensionally equivalent we have to define

$$\pi_{A,B} = df \delta_A (1_A \cdot k_B), \quad \pi'_{A,B} = df \sigma_B (k_A \cdot 1_B),$$

and for $f : C \to A$ and $g : C \to B$

$$\langle f, g \rangle = df (f \cdot g) w_C,$$

in structural case. Then it is easy to show that (E2)–(E3c.) hold.

Conversely, if we start with the standard axiomatization, then we can define

$$\sigma_A = df \pi^1_A, \quad \sigma'_A = df (k_A, 1_A),$$

$$\delta_A = df \pi_A, \quad \delta'_A = df (1_A, k_A),$$

$$\delta^w_{A,B,C} = df \langle (\pi_{A,B,C}, \pi_{B,C} \pi'_{A,B,C}), \pi'_{B,C} \pi'_{A,B,C} \rangle$$

$$\delta^w_{A,B,C} = df \langle \pi_{A,B} \pi_{A,B,C}, (\pi'_{A,B} \pi_{A,B,C}, \pi'_{A,B,C}) \rangle,$$

$$c_{A,B} = df \langle \pi'_{A,B} \pi_{A,B}, w_A = df (1_A, 1_A),$$

and for $f : A \to C$ and $g : B \to D$

$$\langle f \cdot g \rangle = df \langle f \pi_{A,B} g, w'_{A,B} \rangle.$$

It is straightforward to prove that functorial, $\sigma \delta$, $b$, $c$, $w$, $k$-equations, as well as $(\sigma k w)$ and $(\delta k w)$ hold.

In addition we have to prove that the “double translation” will take us to the same notions, i.e., to show that in the structural axiomatization the following equations hold

$$\sigma_A = \sigma_A (k_1, 1_A), \quad \sigma'_A = (k_A \cdot 1_A) w_A$$

$$\delta_A = \delta_A (1_A \cdot k_1), \quad \delta'_A = (1_A \cdot k_A) w_A$$

$$\widetilde{\delta}^w_{A,B,C} = \langle (\delta_A (1_A, k_B C), (\sigma_A (k_B 1_C) \sigma_B C (k_A 1_B C))) w_A (B_C) \rangle, \quad$$

$$\langle (\sigma_A (k_B C) \sigma_B C (k_A 1_B C)) w_A (B_C) \rangle$$

$$\widetilde{\delta}^w_{A,B,C} = \langle (\delta_A (1_A, k_B C), (\sigma_A (k_B 1_C) \sigma_B C (k_A 1_B C))) \rangle w_A (A_B C), \quad$$

$$\langle (\sigma_A (k_B C) \sigma_B C (k_A 1_B C)) \rangle w_A (A_B C)$$

$$c_{A,B} = \langle (\sigma_B (k_A 1_B)), (\delta_A (1_A, k_B C))) w_A B \rangle$$

$$w_A = \langle 1_A, 1_A \rangle w_A$$

$$f \cdot g = \langle (\delta_A (1_A, k_B)), (g \sigma_B (k_A 1_B))) w_A B \rangle, \quad f : A \to C, \quad g : B \to D$$

and in the standard axiomatization,

$$\pi_{A,B} = \pi_{A,B} (1_A \pi_{A,B} b, k_B \pi'_{A,B}), \quad \pi'_{A,B} = \pi_{A,B} (k_A \pi_{A,B} b, 1_B \pi'_{A,B})$$

$$\langle f, g \rangle = \langle f \pi_{C,C}, (g \pi_{C,C}) \rangle (1_C, 1_C), \quad f : C \to A, \quad g : C \to B.$$
2 G-natural transformations and transformational graphs

Let $\mathcal{A}$ be an arbitrary category and $F: \mathcal{A}^m \to \mathcal{A}$, $G: \mathcal{A}^n \to \mathcal{A}$, $m, n \geq 0$, two functors ($\mathcal{A}^0$ is trivial category). Let $\Gamma$ be a function from $\{1, \ldots, n\}$ to $\{1, \ldots, m\}$ called graph (if $n = 0$, then $\{1, \ldots, n\}$ is $\emptyset$). We say that an indexed family of morphisms from $\mathcal{A}$

$$\alpha = \{\alpha(A_1, \ldots, A_m) : F(A_1, \ldots, A_m) \to G(A_{\Gamma(1)}, \ldots, A_{\Gamma(n)}) \mid A_1, \ldots, A_m \in \mathcal{A}\}$$

is a g-natural transformation from $F$ to $G$ with the graph $\Gamma$, denoted by $\alpha : F \xrightarrow{\Gamma} G$, if for every $i$, $1 \leq i \leq m$, arbitrary $A_1, \ldots, A_i, A'_i, A_{i+1}, \ldots, A_m$ and $f : A_i \to A'_i$ from $\mathcal{A}$, the following diagram commutes

$$
\begin{array}{ccc}
F(A_1, \ldots, A_i, \ldots, A_m) & \xrightarrow{\alpha(A_1, \ldots, A_i, \ldots, A_m)} & G(A_{\Gamma(1)}, \ldots, A_{\Gamma(n)}) \\
| & & |
\downarrow F(1_{A_1}, \ldots, f_i, \ldots, 1_{A_m}) & \downarrow F(h_{\Gamma(1)}, \ldots, h_{\Gamma(n)}) & \downarrow F(h_{\Gamma(1)}, \ldots, h_{\Gamma(n)}) \\
F(A_1, \ldots, A'_i, \ldots, A_m) & \xrightarrow{\alpha(A_1, \ldots, A'_i, \ldots, A_m)} & G(A'_{\Gamma(1)}, \ldots, A'_{\Gamma(n)})
\end{array}
$$

where for $j \neq i$, $h_j = 1_{A_j}$, $A'_j \equiv A_j$ and for $i$, $h_i = f$, $A'_i \equiv A'_i$.

This definition follows the one given in ([4], page 94).

Example Let $\mathcal{A}$ be a relevant category. Denote by $w$ the indexed set

$$\{w_A : A \to A : A \in \mathcal{A}\}.$$ 

Then by the equation $(w)$, $w$ is a g-natural transformation from the identity functor $1_\mathcal{A} : \mathcal{A} \to \mathcal{A}$ to the multiplication functor $\cdot : \mathcal{A}^2 \to \mathcal{A}$ with the graph $\Gamma : \{1, 2\} \to \{1\}$, $\Gamma(1) = \Gamma(2) = 1$.

If $\alpha : F \xrightarrow{\Gamma} G$ for $F$, $G$ and $\Gamma$ as above, then it is easy to see that $\alpha$ is a classical natural transformation between $F$ and $G' : \mathcal{A}^m \to \mathcal{A}$ where

$$G'(A_1, \ldots, A_m) = \varphi G(A_{\Gamma(1)}, \ldots, A_{\Gamma(n)}),$$

$$G'(f_1, \ldots, f_m) = \varphi G(f_{\Gamma(1)}, \ldots, f_{\Gamma(n)}).$$

As in the classical case, g-natural transformations can be composed in the following way. Let $F : \mathcal{A}^m \to \mathcal{A}$, $G : \mathcal{A}^n \to \mathcal{A}$ and $H : \mathcal{A}^l \to \mathcal{A}$ be functors. Let $\alpha : F \xrightarrow{\Phi} G$ and $\beta : G \xrightarrow{\Psi} H$ for some graphs $\Phi : \{1, \ldots, n\} \to \{1, \ldots, m\}$ and $\Psi : \{1, \ldots, l\} \to \{1, \ldots, n\}$. We define its composition as

$$\beta \alpha = \varphi \{\beta \alpha(A_1, \ldots, A_m) = \beta(A_{\Phi(1)}, \ldots, A_{\Phi(n)}) \alpha(A_1, \ldots, A_m) \mid (A_1, \ldots, A_m) \in \mathcal{A}^m\}.$$

Then it is easy to prove (as in the case of classical natural transformations) that $\beta \alpha$ is a g-natural transformation from $F$ to $H$ with the graph $\Phi \Psi$ (the usual composition of functions $\Psi$ and $\Phi$).

Generalization of g-natural transformations to the case of several categories (here we have only $\mathcal{A}$) is not essential, and serves just to complicate the notation.

3 Canonical transformations in substructural categories

Throughout this section, $\mathcal{A}$ denotes an arbitrary substructural category.

Let $\mathcal{F}$ be a set of terms obtained from symbols $\Box$, $\sqcup$, $I$ and binary operation $\cdot$. Its elements we call shapes.

In a natural way, we define correspondence between shapes and functors of type $\mathcal{A}^n \to \mathcal{A}$ for some $n \geq 0$. 

4
1. Functor $1_A : A \to A$ corresponds to the term $\Box$.

2. Functor $I : A^0 \to A$, which maps the unique object from $A^0$ to the object $I$ from $A$, corresponds to the term $I$.

3. If $F : A^m \to A$ corresponds to the term $F$ and $G : A^n \to A$ corresponds to the term $G$, then the functor $H : A^{m+n} \to A$ such that for every $m+n$-tuple $(A_1, \ldots, A_{m+n})$ of $A$ objects, $H(A_1, \ldots, A_{m+n}) = GF(A_1, \ldots, A_m)G(A_{m+1}, \ldots, A_{m+n})$, and for every $m+n$-tuple $(f_1, \ldots, f_{m+n})$ of $A$ arrows, $H(f_1, \ldots, f_{m+n}) = GF(f_1, \ldots, f_m)G(f_{m+1}, \ldots, f_{m+n})$, corresponds to the term $F \cdot G$.

However, depending on the category $A$, two different shapes may define the same functor. From now on, if we say that a functor $F : A^n \to A$ is from $F$, that means it corresponds to some shape from $F$.

Functors from $F$ will serve as domains and codomains of canonical transformations we are going to introduce below.

Let $F : A^m \to A$, $G : A^n \to A$ and $H : A^l \to A$ be functors from $F$.

a) Denote by $1_F$ the indexed family $\{1(F(A_1, \ldots, A_m)) | (A_1, \ldots, A_m) \in A^m \}$ and let $\Gamma$ be the identity function from $\{1, \ldots, m\}$ to $\{1, \ldots, m\}$. It is easy to see that $1_F$ is a $g$-natural transformation from $F$ to $F$ with the graph $\Gamma$.

If $A$ is monoidal

b) Denote by $\sigma_F$ the indexed family $\{\sigma_F(A_1, \ldots, A_m) | (A_1, \ldots, A_m) \in A^m \}$ and let $\Gamma$ be as above. Then $\sigma_F : I \cdot F \to F$.

In a similar way we define $\sigma_F', \delta_F$ and $\delta_F'$.

c) Denote by $B_{F,G,H}$ the indexed family

$$\{B_{F(A_1, \ldots, A_m), G(A_{m+1}, \ldots, A_{m+n}), H(A_{m+n+1}, \ldots, A_{m+n+l})} | (A_1, \ldots, A_{m+n+l}) \in A^{m+n+l} \}$$

and let $\Gamma$ be the identity function from $\{1, \ldots, m+n+l\}$ to $\{1, \ldots, m+n+l\}$. Then $B_{F,G,H} : F \cdot (G \cdot H) \to (F \cdot G) \cdot H$.

In a similar way we define $B_{F,G,H}$.

If $A$ is symmetric monoidal
d) Denote by $c_{F,G}$ the indexed set $\{c_{F(A_1, \ldots, A_m), G(A_{m+1}, \ldots, A_{m+n})} | (A_1, \ldots, A_{m+n}) \in A^{m+n} \}$ and let $\Gamma$ be the function from $\{1, \ldots, m+n\}$ to $\{1, \ldots, m+n\}$ that satisfies $\Gamma(m+i) = i$ for $1 \leq i \leq n$ and $\Gamma(j) = n+j$ for $1 \leq j \leq m$. Then $c_{F,G} : F \cdot G \to G \cdot F$.

e) If $A$ is relevant category, we denote by $w_F$ the indexed family $\{w_{F(A_1, \ldots, A_m)} | (A_1, \ldots, A_m) \in A^m \}$. Let $\Gamma$ be the function from $\{1, \ldots, 2m\}$ to $\{1, \ldots, m\}$ defined as $\Gamma(i) = \Gamma(m+i) = i$ for $1 \leq i \leq m$. Then $w_F : F \to F$.

f) If $A$ is affine, we denote by $k_F$ the indexed family $\{k_{F(A_1, \ldots, A_m)} | (A_1, \ldots, A_m) \in A^m \}$. Let $\Gamma$ be the empty function from $\emptyset$ to $\{1, \ldots, m\}$. Then $k_F : F \to I$.

The $g$-natural transformations from above, which exist in the category $A$ constitute the class of $A$ primitive canonical transformations. If we declare $A$ is a monoidal category, its primitive canonical transformations are those from $a)$ to $c)$, though $A$ may have the structure of a cartesian category.

Let $F_1 : A^m \to A$, $F_2 : A^n \to A$, $G_1 : A^k \to A$, $G_2 : A^l \to A$ be functors from $F$, and $\alpha : F_1 \to G_1$ and $\beta : F_2 \to G_2$. Denote by $\alpha \cdot \beta$ the family

$$\{\alpha(A_1, \ldots, A_m) \cdot \beta(A_{m+1}, \ldots, A_{m+n}) | A_1, \ldots, A_{m+n} \in A \},$$

and let $\Gamma$ as a function from $\{1, \ldots, k+l\}$ to $\{1, \ldots, m+n\}$ satisfy the following:

$\Gamma(i) = \Phi(i)$ for $1 \leq i \leq k$ and $\Gamma(k+j) = m + \Psi(j)$ for $1 \leq j \leq l$. Then it is easy to see that
\(\alpha \cdot \beta : F_1 \cdot F_2 \xrightarrow{\Gamma} G_1 \cdot G_2.\)

Now, we can define canonical transformations in \(A\) as follows

1. Primitive canonical transformations from \(A\) are canonical transformations.

2. If \(\alpha : F_1 \xrightarrow{\Phi} G_1\) and \(\beta : F_2 \xrightarrow{\Psi} G_2\) are canonical transformations in \(A\), then \(\alpha \cdot \beta : F_1 \cdot F_2 \xrightarrow{\Phi \cdot \Psi} G_1 \cdot G_2\) is canonical.

3. If \(\alpha : F \xrightarrow{\Phi} G\) and \(\beta : G \xrightarrow{\Psi} H\) are canonical transformations in \(A\), then \(\beta \alpha : F \xrightarrow{\Phi \Psi} H\) is canonical.

It is easy to verify that symmetric monoidal canonical transformations have bijections as graphs, relevant canonical transformations have onto functions as graphs and affine canonical transformations have one-one functions as graphs. Now we can reformulate MacLane’s results from [8] in the following manner:

If \(A\) is monoidal or symmetric monoidal category and \(\alpha, \beta : F \xrightarrow{\Gamma} G\) are two canonical transformations (with the same graph \(\Gamma\)), then \(\alpha\) and \(\beta\) are the same indexed sets (i.e., the same functions from the sequences of objects to the morphisms of \(A\)).

This property of a category we call coherence. It completely follows the notion of coherence given in [8]. We can extend this definition to an arbitrary substructural category \(A\). Namely, we say that an arbitrary substructural category is coherent if for every pair of canonical transformations \(\alpha\) and \(\beta\) of the same type and with the same graph we have that \(\alpha = \beta\) as indexed sets.

4 Categories Mon, SyMon, Rel, Aff and Cart

Let \(\mathcal{M}\) be the category whose objects are monoidal categories and whose arrows are the monoidal structure preserving functors in the language given above. The equational axiomatization of monoidal categories enables us to distinguish a category from \(\mathcal{M}\) freely generated by a set of objects. Let \(P\) be an infinite linearly ordered set of objects, whose elements we call letters. We denote by \(\text{Mon}\) the free monoidal category generated by \(P\) whose construction is given below.

In the same way we will introduce the categories \(\text{SyMon}, \text{Rel}, \text{Aff}\) and \(\text{Cart}\), namely the free symmetric monoidal, relevant, affine and cartesian category generated by the same set \(P\) of objects.

The constructions of these categories are algebraic and the set of objects is always the set \(O\) of terms freely generated by \(P \cup \{I\}\) using the binary operation \(\cdot\).

**Primitive morphism-terms** are in the case of \(\text{Mon}\)

\[1_A : A \to A\]
\[\sigma_A : I \cdot A \to A\]
\[\delta_A : A \cdot I \to A\]
\[\overline{b}_{A,B,C} : A \cdot (B \cdot C) \to (A \cdot B) \cdot C\]
\[\overline{c}_{A,B} : A \cdot B \to B \cdot A\]

for all \(A, B, C \in O\).

**SyMon primitive morphism-terms** are those of \(\text{Mon}\) together with

\[c_{A,B} : A \cdot B \to B \cdot A\]

for every \(A, B \in O\).

**Rel primitive morphism-terms** are those of \(\text{SyMon}\) together with

\[w_A : A \to A \cdot A\]
for every $A \in \mathcal{O}$.

**Aff** primitive morphism-terms are those of **SyMon** together with

$$k_A : A \to I$$

for every $A \in \mathcal{O}$.

**Rel** and **Aff** primitive morphism-terms make the class of **Cart** primitive morphism-terms.

Morphism-terms are built from the primitive morphism-terms with the help of the binary operations of composition and multiplication.

Morphisms of the category **Mon** are equivalence classes of **Mon** morphism-terms modulo monoidal equations. Analogously, we define morphisms of other free categories mentioned above.

Let $\mathcal{C}$ be one among **Mon, SyMon, Rel, Aff** and **Cart**. We define a correspondence between the morphism-terms and canonical transformations of $\mathcal{C}$ in the following way

1. If $f : A \to B$ is primitive morphism-term, suppose it is of the form $1_{F(p_1, \ldots, p_m)}$ for some $F : \mathcal{C}^m \to \mathcal{C} \in \mathcal{F}$ and some, not necessarily distinct, letters $p_1, \ldots, p_m$. Then the canonical transformation $1_F : F \xrightarrow{\cdot} F$ where $\Gamma$ is the identity function on $\{1, \ldots, m\}$ corresponds to $f$. We proceed similarly in the remaining cases.

2. If $f$ is of the form $f_1 \cdot f_2$ or $f_2 \cdot f_1$ and if the canonical transformations $\alpha_1$ and $\alpha_2$ correspond to the morphism-terms $f_1$ and $f_2$, then the canonical transformation $\alpha_1 \cdot \alpha_2$ respectively $\alpha_2 \cdot \alpha_1$ corresponds to the morphism-term $f$.

The graph of a transformation that corresponds to the morphism-term $f : A \to B$ we call simply a graph of $f$. It connects the occurrence of a letter in $A$ with a set (maybe empty) of occurrences of the same letter in $B$.

Let $\alpha$ be a term of a $\mathcal{C}$ canonical transformation of type $F \xrightarrow{\cdot} G$, for $F : \mathcal{C}^m \to \mathcal{C}$, $G : \mathcal{C}^n \to \mathcal{C} \in \mathcal{F}$. Let $p_1, p_2, \ldots, p_m$ be distinct letters from $P$. We call the morphism-term

$$\alpha(p_1, \ldots, p_m) : F(p_1, \ldots, p_m) \to G(p_{\Gamma(1)}, \ldots, p_{\Gamma(n)})$$

the representative of the transformation $\alpha$.

**Lemma 1** Let $\mathcal{A}$ be an arbitrary substructural category and $\mathcal{C}$ one of the free categories mentioned above which is of the same type as $\mathcal{A}$. Let $F$ and $G$ be from $\mathcal{F}$ and let $\alpha : F \xrightarrow{\cdot} G$ and $\beta : F \xrightarrow{\psi} G$ be in $\mathcal{A}$. Denote by $\alpha$ and $\beta$ canonical transformations in $\mathcal{C}$ defined by the same terms as $\alpha$ and $\beta$ respectively. Let $f : A \to B$ be the representative of $\alpha$ and $g : A \to B$ the representative of $\beta$. If $f = g$ in $\mathcal{C}$, then $\alpha = \beta$ in $\mathcal{A}$.

**Proof** Suppose that $f \equiv \alpha(p_1, \ldots, p_m) : F(p_1, \ldots, p_m) \to G(p_{\Phi(1)}, \ldots, p_{\Phi(n)})$ for some distinct letters $p_1, \ldots, p_m \in P$. Since $g$ is the representative equal to $f$, they share domains and codomains; hence $g \equiv \beta(p_1, \ldots, p_m) : F(p_1, \ldots, p_m) \to G(p_{\Psi(1)}, \ldots, p_{\Psi(n)})$, which implies that for every $1 \leq i \leq n$, $\Phi(i) = \Psi(i)$, and so $\Phi$ and $\Psi$ are equal graphs. Suppose that $f' \equiv \alpha(A_1, \ldots, A_m)$. By the assumption that $\mathcal{C}$ is free, there is a functor $U : \mathcal{C} \to \mathcal{A}$ that preserves the structure of $\mathcal{C}$ and that extends the mapping of the generators given by $p_1 \mapsto A_1$, $p_2 \mapsto A_2, \ldots, p_m \mapsto A_m$ (other generators are mapped arbitrarily). Then we have

$$\alpha \circ f' = \alpha(A_1, \ldots, A_k) = U(f) = U(g) = \beta(A_1, \ldots, A_k) \in \beta,$$

hence $\alpha \subseteq \beta$. In the same way we prove that $\beta \subseteq \alpha$.

**q.e.d.**
Our goal is to prove that every substructural category is coherent, and the following lemma will serve to reduce the problem to the case of the free category in the type. For an object $A \in \mathcal{O}$ we say that it is diversified if no letter occurs twice in it.

**Lemma 2** If for every pair of $\mathcal{C}$ morphism-terms $f, g : A \to B$, such that $A$ is diversified, holds that $f = g$, then every category of the same type as $\mathcal{C}$ is coherent.

**proof** Let $\mathcal{A}$ be an arbitrary substructural category of the same type as $\mathcal{C}$. Suppose that $F$ and $G$ are functors from $\mathcal{F}$ and $\alpha, \beta : F \Rightarrow G$ in $\mathcal{A}$. Let $f \equiv \alpha(p_1, \ldots, p_m) : A \to B$ and $g \equiv \beta(q_1, \ldots, q_n) : C \to D$ be the representatives of $\alpha$ and $\beta$ respectively, where $\alpha$ and $\beta$ are $\mathcal{C}$-canonical transformations defined by the same terms as $\alpha$ and $\beta$. Since they have the same graph, $A$ must be identical to $C$ and $B$ to $D$. By the definition of graph, it follows that $A$ is diversified and, by the assumption, $f$ is equal to $g$. Hence, by Lemma 1, we have $\alpha = \beta$.

$q.e.d.$

By the following series of definitions we introduce some auxiliary notions that will help us in proving our coherence results.

Denote by $\mathcal{P}\mathcal{F}$ the set of terms generated by the binary operation $\cdot$ from the elements of $P \cup \{\square, I\}$ (e.g. $($$\square \cdot p$$)$ $(I \cdot q)$ is in $\mathcal{P}\mathcal{F}$). As in the case of terms from $\mathcal{F}$, we can define in the same way the correspondence between terms from $\mathcal{P}\mathcal{F}$ and functors (with parameters) of the type $\mathcal{C}^n \to \mathcal{C}$ for some $n \geq 0$, where $\mathcal{C}$ is one of the free categories mentioned above.

A product term of $\mathcal{C}$ is a morphism-term defined recursively as follows

1. The primitive terms (if they exist in $\mathcal{C}$)
   
   $\sigma_Q, \sigma^1_Q, \delta_Q, \delta^1_Q, \overline{B}_{Q, S, R}, \overline{b}_{Q, S, R}, c_{Q, S}, w_Q, k_Q$.

   are product terms, called determining factors.

2. The terms $1_Q$ are product terms.

3. If $f$ is a product term, then $1_Q \cdot f$ and $f \cdot 1_Q$ are product terms.

The determining factor of a product term $f$, if it exists, is denoted by $d(f)$ (we call such a term structural product). A structural product $f$ is a b-product iff $d(f)$ is a b term, c-product iff $d(f)$ is a c term, and similarly for $\sigma$, $\sigma^1$, $\delta$, $\delta^1$, $k$ and $w$-products.

For a $w$-product-term we say that it is atomic if the index of its determining factor is a letter.

We say that an atomic $w$-product is left if there is not any 1 with the letter $p$ in the index, on the left of its determining factor $w_p$ (e.g. $(1_q \cdot r \cdot w_p) \cdot 1_p$ is the left atomic $w$-product, while $(1_p \cdot r \cdot w_p) \cdot 1_p$ is not left.)

For a $c$-product we say that it is atomic if the index of its determining factor is a pair of atoms (an atom is a letter or I).

We say that an atomic $c$-product is diversified if its determining factor is not of the form $c_{p, p}$ for some letter $p$.

For a $k$-product we say that it is atomic if the index of its determining factor is a letter.

We say that a composition of atomic $w$-products (k-products) is ordered, if a $w_{p, r}$-product (k$_p$-product) is to the right of $w_{q, r}$-product (k$_r$-product) in this composition iff the letter $p$ precedes the letter $q$ in the ordering of $P$. 

8
5 Coherence in Relevant categories

At the beginning of this section, we prove a lemma that states coherence for \( bw \) fragments of relevant categories.

**Lemma 3** Let \( F \) be from \( \mathcal{F} \) and let \( f : p \rightarrow F(p, \ldots, p) \), be a composition of atomic \( w \)-products. Then \( f \) is equal to a term of the form \( hg \) where \( g \) is a composition of left atomic \( w \)-products and \( h \) is a composition of \( b \)-products.

**proof** For the sake of clarity we introduce a tree that corresponds to \( f \), denoted by \( \tau_f \), in the following way.

If \( f \equiv w_p \), then \( \tau_f \) is

\[
\begin{array}{c}
\bullet \\
\end{array}
\]

If \( f \) is of the form \( G(w_p) f_1 \), where \( G \) is from \( \mathcal{P F} \), and if in the shape of \( G \), \( i - 1 \) letters \( p \) precede (from the left) the symbol \( \Box \), then \( \tau_f \) is obtained from \( \tau_{f_1} \) by forking the \( i \)-th leaf (from the left) and concatenating simple segments to remaining leaves.

For example, if \( f \) is of the form

\[
(((1_p \cdot w_p) \cdot 1_{(p \cdot p) \cdot p})(1_{p \cdot p} \cdot (w_p \cdot 1_p))(1_{p \cdot p} \cdot w_p)(w_p \cdot 1_p)w_p
\]

then the corresponding tree is

\[
\begin{array}{c}
\bullet \\
\end{array}
\]

Denote by \( \Lambda \) the set of forking vertices in such a tree. Let \( k_\lambda \) be the number of right branches of these forking in the path from the vertex \( \lambda \) to the root. The complexity of the tree is measured by the number \( n_f \) defined as

\[
n_f = \sum_{\lambda \in \Lambda} k_\lambda
\]

In the example above \( n_f \) is 3. We prove the lemma by induction on \( n_f \).

If \( n_f = 0 \), then \( f \) is itself a composition of left atomic \( w \)-products.

If \( n_f > 0 \) and if there is no subtree of \( \tau_f \) of the form

\[
\begin{array}{c}
\bullet \\
\end{array}
\]

then by the functoriality of \( \cdot \) we obtain a term \( f' \) equal to \( f \) such that \( n_{f'} = n_f \), and there is a subtree of the above form in \( \tau_{f'} \). In the example, we obtain the term

\[
(((1_p \cdot w_p) \cdot 1_{(p \cdot p) \cdot p})(1_{p \cdot p} \cdot (w_p \cdot 1_p))(w_p \cdot 1_{p \cdot p})(1_{p \cdot p} \cdot w_p)w_p
\]

whose tree is

\[
\begin{array}{c}
\bullet \\
\end{array}
\]

\( n = 3 \)
By using the (bw) equality and the naturality of $b$-products we transform this term into the form $h_1 f_1$ where $f_1$ is a composition of atomic $w$-products with $n_f' = n_f - 1$, and $h_1$ is a composition of $b$-products. In our example we obtain the term

$$\overline{b}_{p(p_p),p,p,p,1}(((1_p \cdot 1_{p_p}) \cdot 1_p)(1_{p_p} \cdot 1_p) \cdot (1_{p_p} \cdot 1_p)(1_{p_p} \cdot 1_p)(1_{p_p} \cdot 1_p)(1_{p_p} \cdot 1_p)(1_{p_p} \cdot 1_p)(w_p \cdot 1_p)w_p,$$

and the tree corresponding to its initial part ($f_1$) is

```
 n = 2
```

By the induction hypothesis the term $f_1$ is equal to the term $h_2 g$ of the desired form, and therefore $f = h_1 h_2 g$ is such.

$q.e.d.$

In our example, the last term is transformed into

$$\overline{b}_{p(p_p),p,p,p,1}(((1_p \cdot 1_{p_p}) \cdot 1_p)(1_{p_p} \cdot 1_p) \cdot (1_{p_p} \cdot 1_p)(1_{p_p} \cdot 1_p)(1_{p_p} \cdot 1_p)(1_{p_p} \cdot 1_p)(1_{p_p} \cdot 1_p)(w_p \cdot 1_p)w_p,$$

and the tree corresponding to its initial part is

```
 n = 2
```

(we do this to obtain a subtree of the form  
and then by (bw) and (b) this term is transformed into

$$\overline{b}_{p(p_p),p,p,p,1}((\overline{b}_{p(p_p),p,p,p,1})((1_p \cdot 1_{p_p}) \cdot 1_p)(1_{p_p} \cdot 1_p) \cdot (1_{p_p} \cdot 1_p)(1_{p_p} \cdot 1_p)(1_{p_p} \cdot 1_p)(1_{p_p} \cdot 1_p)(w_p \cdot 1_p)w_p,$$

to whose initial part corresponds the tree

```
 n = 1
```

Then, again by (bw) and (b) the last term is transformed into the term

$$\overline{b}_{p(p_p),p,p,p,1}((\overline{b}_{p(p_p),p,p,p,1})(((1_p \cdot 1_{p_p}) \cdot 1_p)(1_{p_p} \cdot 1_p) \cdot (1_{p_p} \cdot 1_p)(1_{p_p} \cdot 1_p)(1_{p_p} \cdot 1_p)(1_{p_p} \cdot 1_p)(1_{p_p} \cdot 1_p)(w_p \cdot 1_p)w_p,$$

of the desired form, whose tree is of the form

```
 n = 0
```
Corollary Let $F : \text{Rel}^k \to \text{Rel}$ be from $\mathcal{F}$ and let $f : p \to F(p, \ldots, p)$, be a composition of atomic $w$-products. Then for every $i$, $1 \leq i \leq k-1$ there is a morphism term of the form

$$v((1_{p \cdots p} \cdot w_p) \cdot 1_{p \cdots p})^u$$

equal to $f$ (all products of $p$'s are associated to the left, i.e., $p \cdot p \cdot p$ means $(p \cdot p) \cdot p$), where $u : p \to (p \cdot p \cdots p \cdot p \cdots p)$ is a composition of atomic $w$-products, and $v$ is a composition of $b$-products.

proof By Lemma 3, there is a term of the form $h_1 g$ equal to $f$, where $g$ is a composition of left atomic $w$-products, and $h_1$ is a composition of $b$-products. Let

$$u : p \to (p \cdot p \cdots p)_{k-i-1}$$

be a composition of atomic $w$-products (such always exists). Again, by Lemma 3 there is a term of the form $h_2 g$ equal to the morphism-term

$$((1_{p \cdots p} \cdot w_p) \cdot 1_{p \cdots p})^u,$$

where $g$ is as before, and $h_2$ is a composition of $b$-products. Then,

$$f = h_1 g = h_1 h_2^{-1} h_2 g = h_1 h_2^{-1} ((1_{p \cdots p} \cdot w_p) \cdot 1_{p \cdots p})^u,$$

where $h_2^{-1}$ denotes a composition of $b$-products inverse to $h_2$ (by $(bb)$ equalities).

$q.e.d.$

This corollary will be of use in the proof of the main result of this section, which states that

Theorem 1 Every relevant category is coherent.

In the proof of the theorem, the following lemma, that gives the normal form of a morphism from $\text{Rel}$, is crucial.

Lemma 4 Let $h : A \to B$ be a morphism-term from $\text{Rel}$ with $A$ diversified. Then $h$ is equal to the morphism-term of the form $h'' h'$ where $h'$ is an ordered composition of atomic left $w$-products, and $h''$ is a composition of products with no $w$-products and with all $c$-products atomic diversified.

proof The transformation of the term $h$ is made in several steps. For the sake of clarity, we illustrate every step starting with the term

$$(1_q^c (w_p w_p)) c_{p,q}.$$ 

$1^\circ$ The term $h$ is equal to a composition of product terms. This follows from the functoriality of multiplication. In our example,

$$h = (1_q^c w_p w_p) (1_q^c w_p) c_{p,q}.$$ 

$2^\circ$ By $(bcw8)$ and $(\sigma \delta w)$, $h$ is equal to a term with all $w$-products atomic. In our example

$$h = t(1_q^c (w_p 1_p p)) (1_q^c (1_p^c w_p)) (1_q^c w_p) c_{p,q},$$

11
where
\[ t \equiv (1_q, \overrightarrow{B}_{p,p,p})(1_q(\overrightarrow{B}_{p,p,p} \cdot 1_p))(1_q(1_p \cdot c_{p,p} \cdot 1_p))(1_q(\overrightarrow{B}_{p,p,p} \cdot 1_p))(1_q \cdot B_p p, p, p) \]

3° By the multiplication functoriality and the naturality of \( \sigma, \delta, b, c \)-products, atomic \( w \)-products permute towards the right end, in order to obtain a term whose initial part (from the right) consists of atomic \( w \)-products and whose tail is a \( w \)-free composition of products. In our example this term is
\[ t^c_{(p,p) \cdot (p,p) \cdot q} ((w_p \cdot 1_p \cdot p)(1_q \cdot w_p \cdot 1_q)(w_p \cdot 1_q) \]

4° Using (be6), this term is equal to a term whose \( c \)-products are atomized. In the example, this atomization is not essential because its application to the unique nonatomic \( c \)-product \( c_{(p,p) \cdot (p,p) \cdot q} \) produces only diversified atomic \( c \)-products (see the following step), and therefore we write it in abbreviated form, which is enough for the further analysis:
\[ h = t_2(1_q((1_p \cdot c_{p,p} \cdot 1_p))t_1((w_p \cdot 1_p \cdot p)(1_q \cdot w_p \cdot 1_q)(w_p \cdot 1_q), \]

where \( t_2 \equiv (1_q \cdot B_{p,p,p})(1_q(\overrightarrow{B}_{p,p,p} \cdot 1_p)), \]

and \( t_1 \equiv (1_q(\overrightarrow{B}_{p,p,p} \cdot 1_p))(1_q \cdot B_p p, p, p, p)(c_{(p,p) \cdot (p,p) \cdot q})^* \)

(*) means a developed form with atomic \( c \)-products.)

5° Suppose that the tail (\( w \)-free composition of products) from the last term is in the form \( t_2 F(c_{p,p})t_1 \), where \( F : \text{Rel} \rightarrow \text{Rel} \) is from \( \mathcal{P}F \) and all the \( c \)-products in \( t_1 \) are atomic diversified. Let \( F(c_{p,p})t_1 \) be of the type \( G(p, p) \rightarrow F(p, p) \) for some \( G : \text{Rel}^2 \rightarrow \text{Rel} \) from \( \mathcal{P}F \), where the left \( p \) from \( F(p, p) \) is mapped to the right \( p \) from \( G(p, p) \) by the graph of \( F(c_{p,p})t_1 \), and the right \( p \) from \( F(p, p) \) is mapped to the left \( p \) from \( F(p, p) \) by the same graph. By the assumption that \( A \) is diversified, that all \( c \)-products in \( t_1 \) are atomic diversified and the initial part consists of atomic \( w \)-products only, we have that two emphasized letters \( p \) in \( G(p, p) \) occur consecutively (not necessarily in the form \( (p, p) \)).

Using the functoriality of \( \cdot \), we can push all \( w_p \) products to the (left) end of an initial part consisting of \( w \)-products only. Now, we apply the corollary of Lemma 3, to show that such an initial part is equal to a term of the form \( i_2 H(w_p)i_1 \), where \( i_1 \) is a composition of atomic \( w \)-products, \( H : \text{Rel} \rightarrow \text{Rel} \) is from \( \mathcal{P}F \), \( i_2 \) is a composition of \( b \)-products, the term \( i_2 H(w_p) \) is of the type \( H(p) \rightarrow G(p, p) \) and its graph maps both distinguished \( p \)'s from \( G(p, p) \) to the distinguished \( p \) in \( H(p) \). (Assume that distinguished \( (p, p) \) in \( G(p, p) \) are \( i \)-th and \( i + 1 \)-th occurrence of the letter \( p \) and just apply the corollary of Lemma 3.)

In our example
\[ h = t_2(1_q((1_p \cdot c_{p,p} \cdot 1_p))t_1 i_2((1_p \cdot w_p \cdot 1_p \cdot 1_q)i_1, \]

where \( i_1 \equiv ((w_p \cdot 1_p \cdot 1_q)(w_p \cdot 1_q), i_2 \equiv (B_{p,p,p} \cdot 1_q)(\overrightarrow{B}_{p,p,p} \cdot 1_p \cdot 1_q), \]

and \( F \equiv (q \cdot (p, p) \cdot p), G \equiv ((p, p) \cdot (p, p))q, H \equiv ((p, p) \cdot p)q. \)

The \textbf{SyMon} terms \( F(c_{p,p})t_1 i_2 \) and \( t_1 i_2 H(c_{p,p}) \) are of the same type. Denote by \( \alpha \) and \( \beta \) canonical transformations in \textbf{SyMon} corresponding to these terms. It is easy to see that \( \alpha \) and \( \beta \) have the same graphs (the graph of \( H(c_{p,p}) \) “commutes” in composition with the graph of \( t_1 i_2 \) and is transformed into the graph of \( F(c_{p,p}) \)). By MacLane’s coherence for symmetric monoidal categories, we have that \( \alpha = \beta \). From the property that every canonical transformation of \textbf{SyMon} contains at most one morphism of a certain type (this is because the set \( O \) of its objects is freely generated by \( P \cup \{I\} \)), we conclude that \( F(c_{p,p})t_1 i_2 = t_1 i_2 H(c_{p,p}) \) holds in \textbf{SyMon}. Now, because all the \textbf{SyMon}-equalities hold in \textbf{Rel}, these terms are equal in \textbf{Rel} too.

Therefore, \( h \) is equal to a term of the form \( t_3 t_1 i_2 H(c_{p,p}) H(w_p)i_1 \), which is by (cw), equal to \( t_2 t_1 i_2 H(w_p)i_1 \). Repeating this procedure we can eliminate non diversified \( c \) products occurring in \( t_2 \) obtaining a term equal to \( h \) whose initial part consists of atomic \( w \)-products and whose tail is
a \textbf{w}-free composition of products, whose \textbf{c}-products are atomic diversified. In the example, this procedure includes the following steps

\[
t_2t_1\(((\mathbf{1}_p\cdot\mathbf{w}_p)\cdot\mathbf{1}_p)\cdot\mathbf{1}_q)\cdot\mathbf{1}_1 = \\
t_2t_1((\mathbf{1}_p\cdot\mathbf{w}_p)\cdot\mathbf{1}_p)\cdot\mathbf{1}_q(\mathbf{1}_p\cdot\mathbf{w}_p)\cdot\mathbf{1}_q\cdot\mathbf{1}_1 = \\
t_2t_1((\mathbf{1}_p\cdot\mathbf{w}_p)\cdot\mathbf{1}_p)\cdot\mathbf{1}_q,\]

where \(t_2 = (1_q(\overline{\mathbf{B}}_{p,p,p,p}1_p))q(\overline{\mathbf{B}}_{p,p,p,p}1_p)).\)

6\textdegree \ By Lemma 3 and the functoriality of multiplication, this term is equal to the one whose initial part is an ordered composition of atomic left \textbf{w}-products.

\[g.e.d.\]

In the example, we have

\[
(((\mathbf{1}_p\cdot\mathbf{w}_p)\cdot\mathbf{1}_p)\cdot\mathbf{1}_q)\cdot\mathbf{1}_1 = \\
(((\mathbf{1}_p\cdot\mathbf{w}_p)\cdot\mathbf{1}_p)\cdot\mathbf{1}_q)(\mathbf{w}_p\cdot\mathbf{1}_q) = \\
((\overline{\mathbf{B}}_{p,p,p,p}1_p)\cdot\mathbf{1}_q)(((\mathbf{w}_p\cdot\mathbf{1}_p)\cdot\mathbf{1}_q)(\mathbf{w}_p\cdot\mathbf{1}_q).)
\]

Lemma 5 \ Let \(f,g : A \to B\) be two morphism-terms in \textbf{Rel} with \(A\) diversified. Then \(f = g\) in \textbf{Rel}.

\textbf{proof} \ By Lemma 4, \(f = f''f'\) and \(g = g''g'\), with \(f',f'',g',g''\) of the given form. The terms \(f'\) and \(g'\) are completely determined by the codomain \(B\) (the number of occurrences of each letter in \(B\) determines \(f'\) and \(g'\)) and therefore \(f'\) and \(g'\) are identical. Suppose its type is \(A \to A'\). Then the terms \(f''\) and \(g''\) are of the same type \(A' \to B\). Let \(\alpha\) and \(\beta\) be the \textbf{SyMon} canonical transformations corresponding to these terms. Taking \(\Gamma_\alpha\) and \(\Gamma_\beta\) (the graphs of these transformations) as connections that connect an occurrence of a letter in \(B\) with occurrence of the same letter in \(A'\) and since all \textbf{c}-products in \(f''\) and \(g''\) are atomic diversified, they must connect the first (from the left) occurrence of one letter in \(B\) with the first (again from the left) occurrence of the same letter in \(A'\), the second with the second etc. This means that \(\Gamma_\alpha\) and \(\Gamma_\beta\) are the same graphs and by coherence in symmetric monoidal categories, \(\alpha\) and \(\beta\) are the same canonical transformations in \textbf{SyMon}. Therefore, \(f''\) and \(g''\) are equal in \textbf{SyMon}, and since all symmetric monoidal equalities hold in \textbf{Rel}, they are equal in \textbf{Rel} too. Hence, \(f = g\) in \textbf{Rel}.

\[g.e.d.\]

Now Theorem 1 follows from lemmata 2 and 5.

6 \ Coherence in Affine Categories

As in the case of relevant categories, first we prove a lemma about representation of \textbf{Aff} morphism terms.

Lemma 6 \ Every \textbf{Aff} morphism-term is equal to a term of the form \(h_2h_1\), where \(h_1\) is an ordered composition of atomic \textbf{k}-products, and no \textbf{k}-product occurs in \(h_2\).

\textbf{proof} \ As in the proof of Lemma 4, we transform the \textbf{Aff} morphism-term \(h\) in several steps.

1\textdegree \ By the functoriality of multiplication, \(h\) is equal to a composition of product terms.

2\textdegree \ By the equalities \((\textbf{1}\textbf{k})\) and \(\textbf{k}_{A:B} = \sigma(I\cdot\textbf{k}_B)(\textbf{k}_A\cdot\textbf{1}_B)\), which is derivable from \((\textbf{k})\) and \((\textbf{1}\textbf{k})\),
this term is equal to a composition of products with all \( k \)-products atomic.

3° By the naturality of \( \sigma, \delta, b, c \)-products and functoriality of multiplication, atomic \( k \)-products permute to the right in order to obtain a term whose initial part consists of all \( k \)-products present in the term.

4° By the functoriality of multiplication, we can order this initial part to obtain a term in the desired form, which is equal to \( h \).

\[ q.e.d. \]

**Lemma 7** Let \( f, g : A \to B \) be two morphism terms from \( \text{Aff} \) with \( A \) diversified. Then \( f = g \) in \( \text{Aff} \).

**proof** By Lemma 6, there are terms \( f', f'', g', g'' \) such that \( f = f'' f' \) and \( g = g'' g' \), where \( f' \) and \( g' \) are ordered compositions of atomic \( k \)-products, and \( f'' \) and \( g'' \) are \( \text{SyMon} \) terms. The objects \( A \) and \( B \) (the letters occurring in \( A \) not in \( B \)) completely determine terms \( f' \) and \( g' \), hence they are identical morphism terms, suppose of the type \( A \to A' \). Therefore \( f'' \) and \( g'' \) are of the type \( A' \to B \). By the assumption concerning \( A \) it follows that \( A' \) and \( B \) are diversified. Let \( \alpha \) and \( \beta \) be the canonical transformations in \( \text{SyMon} \) corresponding to \( f'' \) and \( g'' \) respectively. Taking \( \Gamma_\alpha \) and \( \Gamma_\beta \) (their graphs) as connections between letter occurrences in \( B \) with letter occurrences in \( A' \), since they connect a letter occurrence with an occurrence of the same letter, and by the assumption about \( A' \) and \( B \), we must have that \( \Gamma_\alpha = \Gamma_\beta \). This implies, by the coherence in symmetric monoidal categories, that \( \alpha = \beta \), which has as consequence that \( f'' = g'' \) in \( \text{SyMon} \), hence in \( \text{Aff} \). We conclude that \( f = f'' f' = g'' g' = g \).

\[ q.e.d. \]

From lemmata 2 and 7 it follows that

**Theorem 2** Every affine category is coherent.

## 7 Coherence in Cartesian Categories

The coherence in cartesian categories is not a new result. For the first time it was mentioned in [3] and more recently in [4] and [5]. Since we would like to keep to the definition of coherence given above, we give another proof of this result here.

One could expect that the proof of the coherence in cartesian categories will follow the proofs given in the last two sections. However, this method turns out to be too complicated and we use the standard equational axiomatization of cartesian categories to avoid this.

Denote by \( \mathcal{P} \) the set of translations of morphism terms from \( \text{Cart} \) into the language of standard axiomatization (see Section 1).

**Distributed terms** form the smallest class of morphism terms from \( \mathcal{P} \) that satisfies:

1. For all \( \text{Cart} \) objects \( A, B, C, D, E \), the term \( 1_A \) as well as well-founded compositions of \( \pi_{A,B}, \pi_{C,D}, k_E \) are in the class and we call them \( \text{compat} \).

2. If \( f : C \to A \) and \( g : C \to B \) are in the class then \( (f, g) \) is in the class.

The following corresponds to the notion of the expanded normal form of a natural deduction proof.

A distributed term is **atomic** if every compat in this term has an atomic codomain (letter or 1).

**Lemma 8** Every morphism-term from \( \mathcal{P} \) is equal to an atomic distributed term.
proof First, we show by induction on complexity of \( f \) from \( P \) that it is equal to a distributed term.

1° If \( f \) is \( 1_A, \pi_{A,B}, \pi'_{A,B} \) or \( k_A \), then it is compat and therefore distributed.

2° a) Suppose that \( f \) is of the form \( \langle g, h \rangle \). By the induction hypothesis, \( g \) and \( h \) are equal to distributed terms \( g' \) and \( h' \); hence \( f \) is equal to the distributed term \( \langle g', h' \rangle \).

b) Suppose that \( f \) is of the form \( hg \). Then by the induction hypothesis, \( g \) and \( h \) are equal to distributed terms \( g_1 \) and \( h_1 \). Suppose that every composition of lower complexity than \( h_1g_1 \), of distributed terms is equal to a distributed term (if \( g_1 \) and \( h_1 \) are primitive, then its composition is compat and therefore distributed). There are three possibilities.

i) If \( g_1 \) and \( h_1 \) are compat, then \( h_1g_1 \) is compat too, hence \( f \) is equal to the distributed term \( h_1g_1 \).

ii) If \( h_1 \) is of the form \( \langle j, l \rangle \), for \( j \) and \( l \) distributed, then \( f = \langle j, l \rangle g_1 = \langle jg_1, lg_1 \rangle \). The terms \( jg_1 \) and \( lg_1 \) are of lower complexity than \( h_1g_1 \), and by assumption they are equal to some distributed terms; hence \( f \) is equal to a distributed term.

iii) Suppose that \( h_1 \) is compat and \( g_1 \) is of the form \( \langle j, l \rangle \). If \( h_1 \equiv 1 \), then \( f \) is equal to the distributed term \( g_1 \). If \( h_1 \equiv h_2 \pi \), then \( f = h_2 \pi \langle j, l \rangle = h_2j \), where \( h_2 \) and \( j \) are distributed and \( h_2j \) is of lower complexity than \( h_1g_1 \); therefore it is equal to a distributed term. The case when \( h_1 \equiv h_2 \pi' \) is analogous. If \( h_1 \equiv h_2 k \), then \( f = h_2 k \langle j, l \rangle = h_2 k \). The last term is compat, hence distributed.

This is the end of the induction. It follows that every \( f \) from \( P \) is equal to a distributed term \( f_1 \).

For every nonatomic compat \( h \) in \( f_1 \), using the equality \( h = \langle \pi_{A,B} h, \pi'_{A,B} h \rangle \) for \( h : C \rightarrow A \cdot B \), we can find a distributed term equal to \( h \), such that every compat in it has a codomain of lower complexity than \( h \). Substituting this term for \( h \) and repeating the procedure we obtain an atomic distributed term equal to \( f_1 \) and therefore to \( f \).

\textit{q.e.d.}

Lemma 9 If \( f, g : A \rightarrow B \) are two morphism-terms from \( P \) with \( A \) diversified, and \( B \) an atom, then \( f = g \).

proof If \( B \equiv 1 \), then because it is terminal in \textit{Cart}, we have that \( f = g \). Suppose that \( B \equiv q \) for \( q \) a letter. By the previous lemma \( f \) and \( g \) are equal to distributed terms \( f_1 \) and \( g_1 \), which are compat by the assumption that codomain is \( q \). Keeping in mind that there is no morphism in \textit{Cart} of the type \( A \rightarrow q \) such that \( q \) doesn’t occur in \( A \), we prove the lemma by induction on the complexity of the domain \( A \).

1° If \( A \) is an atom then \( f_1 \equiv g_1 \equiv 1_q \).

2° Suppose that \( A \equiv A_1 \cdot A_2 \). Then, by the assumption, \( q \) occurs either in \( A_1 \) or in \( A_2 \). Suppose it occurs in \( A_1 \). Then we must have that \( f_1 \equiv f_2 \pi \) and \( g_1 \equiv g_2 \pi \) for some compat \( f_2, g_2 : A_1 \rightarrow q \). By the induction hypothesis, \( f_2 = g_2 \) holds, and therefore \( f = f_1 = f_2 \pi = g_2 \pi = g_1 = g \). We prove analogously the case when \( q \) occurs only in \( A_2 \).

\textit{q.e.d.}

Lemma 10 Let \( f, g : A \rightarrow B \) be two morphism-terms from \( P \) with \( A \) diversified. Then \( f = g \).

proof By Lemma 8, \( f \) and \( g \) are equal to atomic distributed terms \( f_1 \) and \( g_1 \). The proof follows by induction on the complexity of the codomain \( B \). If \( B \) is an atom, then by the previous lemma \( f = g \) holds. If \( B \) is not an atom, then neither \( f_1 \) nor \( g_1 \) are compat, and therefore \( f_1 \equiv \langle i, j \rangle \) and
\[ g \equiv (l, h), \text{ where } B = B_1 \cdot B_2. \] The terms \( i, l : A \to B_1 \) and \( j, h : A \to B_2 \) are atomic distributed and \( B_1, B_2 \) are of lower complexity than \( B \). Therefore, by the induction hypothesis, \( i = l \) and \( j = h \); hence \( f = g \).

\[ q.e.d. \]

**Corollary**  Let \( f, g : A \to B \) be morphism terms from \( \text{Cart} \) and let \( A \) be diversified. Then \( f = g \) in \( \text{Cart} \).

**proof**  Let \( f \) and \( g \) from \( P \) correspond to \( f \) and \( g \) respectively. By Lemma 10, \( f = g \) and by the extensional equivalence of these two axiomatizations we have that \( f = g \) in \( \text{Cart} \).

\[ q.e.d. \]

From this corollary and Lemma 2 it follows that

**Theorem 3**  Every cartesian category is coherent.

### 8 Some consequences of the coherence

Usually, in the literature, coherence is not related to matters concerning natural transformations, but to conditions that imply equality of morphisms of certain categories. In the previous sections, we have lemmata 3, 5 and 7 as examples of such an opinion. Now we prove some facts that can be of practical interest for substructural categories, especially for the free categories of each type.

In Section 4, a definition of canonical transformation that corresponds to a morphism term \( f \) of a free category \( C \) is given. From now on, a graph of this transformation will be called a graph of \( f \). By a straightforward induction on the complexity of \( f \) we can prove the following.

**Lemma 11**  If \( f \) and \( g \) are two equal morphism terms in \( C \), then the graph of \( f \) is identical to the graph of \( g \).

Denote by \( \text{Finord}^{op} \) the dual of the category whose objects are finite ordinals and whose arrows are mappings between them. The coherence of substructural categories together with Lemma 11 is equivalent to the fact that there exist embeddings of \( \text{SyMon}, \text{Rel}, \text{Aff} \) and \( \text{Cart} \) into \( \text{Finord}^{op} \) given by the “graph” functor \( G \), such that for every \( A \in \mathcal{O} \), \( G(A) \) is the number of occurrences of letters in \( A \) and \( G(f) \) is the graph of \( f \).

In the case of \( \text{Cart} \), this embedding is onto on morphisms, and if we restrict ourselves to one-one functions in \( \text{Finord}^{op} \), then the embedding of \( \text{Aff} \) in this category is also onto on morphisms. Similarly we obtain embeddings which are onto on morphisms of \( \text{Rel} \) and \( \text{SyMon} \) in \( \text{Finord}^{op} \) with restrictions to onto functions and bijections respectively. This is an alternative characterization of the coherence of substructural categories.

The results obtained in previous sections imply that the categories \( \text{Rel}, \text{Aff} \) and \( \text{Cart} \) are trivial in some sense. However, they are not preorders (in a preorder, there is at most one arrow between two objects) as the case is with \( \text{Mon} \), though lemmata 5, 7 and 10 come close to preordering. We can use the following consequence of coherence in these categories:

Let \( C \) be one of the mentioned free categories and let

\[
\begin{array}{c}
A \\
f \downarrow \\
g \\
\rightarrow \ \\
\rightarrow \ \\
B
\end{array}
\]
be one of its diagrams. It commutes iff the graphs of $f$ and $g$ are identical.

Together with freedom of $C$, this consequence can be of practical use because it transforms computations in algebra of morphism terms to the simple calculus of morphism graphs.

**Example** Suppose we want to simplify the term

$$((\delta_A(1_A \cdot \sigma_1)) \cdot (\delta_B \sigma_{B \cdot 1}))c_{A,1,1 \cdot B,1}(1_A \cdot c_{1,1 \cdot B,1})(\delta_A(\sigma^i_B \cdot \delta^i_1))$$

representing a morphism of some symmetric monoidal category $S$. The type of this term is $A \cdot (B \cdot 1) \rightarrow A \cdot B$. Consider the SyMon term

$$((\delta_p(1_p \cdot \sigma)) \cdot (\delta_q \sigma_{q \cdot 1}))c_{p,1,1 \cdot q,1}(1_p \cdot c_{1,1 \cdot q,1})(\delta^i_p(\sigma^i_q \cdot \delta^i_1)) : p \cdot (q \cdot 1) \rightarrow p \cdot q.$$  

Since its domain is diversified, it is enough to find a simple SyMon term of the same type. In this case, the term $1_p \cdot \delta_q$ is imposed. So, these two terms are equal in SyMon, and by the freedom of this category, the initial term is equal to $1_A \cdot \delta_B$ in $S$.

**Example** Prove the equality of the following Cart terms

$$(1_p \cdot c_{p,q \cdot p}) \cdot (1_p \cdot (c_{p,q \cdot p})(w_p \cdot 1_p)),$$

and

$$(1_p \cdot ((\delta_{q,p} \cdot B_{q,p,1}) \cdot 1_p)) \cdot (1_p \cdot (w_p \cdot 1_p)) \cdot (1_p \cdot (c_{p,p \cdot q})) \cdot (w_p \cdot 1_p)).$$

Their graphs are identical, which can be checked directly from the constructions of these terms, following the linkages between the occurrences of letters in domains and codomains of its primitive components.

Since these terms are of the same type, they are equal in Cart.

We conclude this section with a discussion about the hierarchy (in the sense of embeddability) in the set of free categories mentioned above. All these categories have the same set $O$ as the set of objects, and their morphism terms satisfy the following inclusions
where arrows stay for inclusions. The fact that equalities between morphism terms that hold in the “lower” category are also true in the “higher” one was used in this paper several times. To show that a “lower” category is a subcategory of those above, in the diagram, we have to show the following

**Lemma 12** Let $\mathcal{C}$ and $\mathcal{D}$ be categories from the above diagram such that $\mathcal{C}$ is below $\mathcal{D}$. If for $\mathcal{C}$ morphism terms $f$ and $g$, the equality $f = g$ holds in $\mathcal{D}$, then they are equal in $\mathcal{C}$, too.

**proof** Let $\Phi$ and $\Psi$ be the graphs of $f$ and $g$ respectively. Since $f = g$ in $\mathcal{D}$, by Lemma 11, the graphs $\Phi$ and $\Psi$ are identical. By the coherence of $\mathcal{C}$ (there is an unique morphism of a certain type in a canonical transformation of the free category $\mathcal{C}$) it follows that $f = g$ in $\mathcal{C}$.

$q.e.d.$

So, by the embedding $E$ such that $E(A) = A$ for every $A \in \mathcal{O}$, and $E([f]_\mathcal{C}) = [f]_\mathcal{D}$ where $[f]_\mathcal{C}$ is the equivalence class of the morphism term $f$ in $\mathcal{C}$, we have the following

**Theorem** The category $\text{Mon}$ is a subcategory of $\text{SyMon}$, $\text{Rel}$, $\text{Aff}$ and $\text{Cart}$. The category $\text{SyMon}$ is a subcategory of $\text{Rel}$, $\text{Aff}$ and $\text{Cart}$. The categories $\text{Rel}$ and $\text{Aff}$ are subcategories of $\text{Cart}$.

This theorem looks almost trivial, but it is of the same strength as the coherence theorems of substructural categories. An independent proof of this theorem immediately delivers MacLane’s coherence for monoidal and symmetric monoidal categories and the above coherence theorems for relevant and affine categories, from the simplest case of cartesian coherence.

Some other coherence applications in this spirit are given in [3], and the main result of [3] can be proven more easily by using this apparatus.

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