Modified method of simplest equation and exact traveling wave solutions of a hyperbolic reaction-diffusion equation

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Abstract

We discuss a class of hyperbolic reaction-diffusion equations and apply the modified method of simplest equation in order to obtain an exact solution of an equation of this class (namely the equation that contains polynomial nonlinearity of fourth order). We use the equation of Bernoulli as a simplest equation and obtain traveling wave solution of a kink kind for the studied nonlinear reaction-diffusion equation.

1 Introduction

Differential equations arise in mathematical analysis of many problems from natural and social sciences [8], [11] - [17], [36], [43], [44], [66] - [68]. The reason for this is that the differential equations relate quantities with their changes and such relationships are frequently encountered in many disciplines such as meteorology, fluid mechanics, solid-state physics, plasma physics, ocean and atmospheric sciences, mathematical biology, chemistry, material science, etc. [10], [25], [42], [45], [48], [50], [63], [64]. The qualitative features and mechanisms of many phenomena and processes in the above-mentioned research areas can be studied by means of exact solutions of the model nonlinear differential equations. Examples are phenomena such as existence and change of different regimes of functioning of complex systems, spatial localization, transfer processes, etc. In addition the exact solutions can be used to test computer programs for numerical simulations. Because of all above exact solutions of nonlinear partial differential equations are studied very intensively
The nonlinear PDEs are integrable or nonintegrable. Well known methods exist for obtaining exact solutions of integrable nonlinear PDEs, e.g., the method of inverse scattering transform or the method of Hirota [2, 3, 21, 26, 46]. Many approaches for obtaining exact special solutions of nonintegrable nonlinear PDEs have been developed in the recent years, e.g., [18, 24, 30, 38, 53, 70, 71]. In this chapter we shall use a version of the method of simplest equation called modified method of simplest equation [30, 32, 33, 53, 56, 62]. The method of simplest equation uses a procedure analogous to the first step of the test for the Painlevé property [29, 31, 34]. In the modified method of simplest equation [54, 55, 57, 59, 60, 61] instead of this procedure (the procedure requires work in the space of complex numbers) one uses one or several balance equations. The modified method of simplest equation has shown its effectiveness on the basis of numerous applications, such as obtaining exact traveling wave solutions of generalized Swift-Hohenberg equation and generalized Rayleigh equation [56], generalized Degasperis-Procesi equation and b-equation [57], extended Korteweg-de Vries equations [59, 62], generalized Fisher equation, generalized Huxley equation [54], generalized Kuramoto-Sivashinsky equation, reaction-diffusion equation, reaction-telegraph equation [53], etc. [60, 61, 65].

Below we shall discuss hyperbolic reaction-diffusion equation of the kind

\[ \tau \frac{\partial^2 Q}{\partial t^2} + \frac{\partial Q}{\partial t} = D \frac{\partial^2 Q}{\partial x^2} + \sum_{i=1}^{n} \alpha_i Q^i, \]

where \( Q = Q(x, t) \), \( n \) is a natural number and \( \tau, D, \) and \( \alpha_i, i = 1, 2, ..., n \) are parameters. The difference between Eq. (1) and the classic nonlinear reaction-diffusion equation is in the term \( \tau \frac{\partial^2 Q}{\partial t^2} \). Equation of class (1) are known also as damped nonlinear Klein-Gordon equations [4, 7, 20, 22, 69]. We note that reaction-diffusion equations have many applications for describing different kinds of processes in physics, chemistry, biology, etc [9, 23, 72]. Traveling wave solutions of these equations are of special interest as they describe the motion of wave fronts or the motion of boundary between two different states existing in the studied system. Below we apply the modified method of simplest equation (described in Sect.2) for obtaining exact traveling solutions of nonlinear reaction-diffusion PDE with polynomial nonlinearity of fourth order (Sect3). The obtained waves are discussed in Sect. 3 and several concluding remarks are summarized in Sect.4.
2 The modified method of simplest equation

Below we shall apply the modified method of simplest equation. The current version of the methodology used by our research group is based on the possibility of use of more than one simplest equation [65]. We shall describe this version of the methodology and below we shall use the particular case when the solutions of the studied nonlinear PDE are obtained by use of a single simplest equation. The steps of the methodology are as follows.

1. By means of appropriate ansätze (below we shall use a traveling-wave ansatz but in principle there can be one or several traveling-wave ansätze such as $\xi = \alpha x + \beta t; \zeta = \gamma x + \delta t, \ldots$ Other kinds of ansätze may be used too) the solved nonlinear partial differential equation is reduced to a differential equation $E$, containing derivatives of one or several functions

$$E[a(\xi), a_\xi, a_{\xi\xi}, \ldots, b(\zeta), b_\zeta, b_{\zeta\zeta}, \ldots] = 0$$

(2)

2. In order to make transition to the solution of the simplest equation we assume that any of the functions $a(\xi)$, $b(\zeta)$, etc., is a function of another function, i.e.

$$a(\xi) = G[f(\xi)]; \quad b(\zeta) = F[g(\zeta)]; \ldots$$

(3)

3. We note that the kind of the functions $F$, $G$, $\ldots$ is not prescribed. Often one uses a finite-series relationship, e.g.,

$$a(\xi) = \sum_{\mu_1=-\nu_1}^{\nu_2} q_{\mu_1} [f(\xi)]^{\mu_1}; \quad b(\zeta) = \sum_{\mu_2=-\nu_2}^{\nu_3} r_{\mu_2} [g(\zeta)]^{\mu_2}, \ldots$$

(4)

where $q_{\mu_1}, r_{\mu_2}, \ldots$ are coefficients. However other kinds of relationships may be used too. Below we shall work on the basis of relationships of kind (4).

4. The functions $f(\xi)$, $g(\zeta)$ are solutions of simpler ordinary differential equations called simplest equations. For several years the methodology of the modified method of simplest equation was based on use of one simplest equation. The new version of the methodology allows the use of more than one simplest equation. The idea for use of more than one simplest equation can be traced back two decades ago to the articles of Martinov and Vitanov [39, 40, 41].

5. Eq. (3) is substituted in Eq. (2) and let the result of this substitution be a polynomial containing $f(\xi), g(\zeta), \ldots$. Next we have to deal with the coefficients of this polynomial.
6. A balance procedure is applied that has to ensure that all of the coefficients of the obtained polynomial of $f(\xi)$ and $g(\zeta)$ contain more than one term. This procedure leads to one or several balance equations for some of the parameters of the solved equation and for some of the parameters of the solution. Especially the coefficients $\nu_i$ from Eq. (4) as well as the parameters connected to the order of nonlinearity of the simplest equations are terms in the balance equations. Note that the coefficients of all powers of the polynomials have to be balanced (and not only the coefficient of the largest power). This is why the extended balance may require more than one balance equation.

7. Eqs. (3) represent a candidate for solution of Eq. (2) if all coefficients of the obtained polynomial of are equal to 0. This condition leads to a system of nonlinear algebraic equations for the coefficients of the solved nonlinear PDE and for the coefficients of the solution. Any nontrivial solution of this algebraic system leads to a solution of the studied nonlinear partial differential equation. Usually the system of algebraic equations contains many equations that have to be solved by means of a computer algebra system.

Below we shall search a solution of the studied equation of the kind

$$Q(\xi) = \sum_{i=0}^{n} a_i [\phi(\xi)]^i, \quad \xi = x - vt$$

(5)

where $\phi(\xi)$ is a solution of the Bernoulli differential equation

$$\frac{d\phi}{d\xi} = a\phi(\xi) + b[\phi(\xi)]^k$$

(6)

where $k$ is a positive integer. We shall use the following solutions of the Bernoulli equation

$$\phi(\xi) = \sqrt[1-k]{\frac{ae^{a(k-1)(\xi+\xi_0)}}{1 - be^{a(k-1)(\xi+\xi_0)}}}, \quad \phi(\xi) = -\sqrt[1-k]{\frac{-ae^{a(k-1)(\xi+\xi_0)}}{1 + be^{a(k-1)(\xi+\xi_0)}}}$$

(7)

for the cases $b < 0$, $a > 0$ and $b > 0$, $a < 0$ respectively. Above $\xi_0$ is a constant of integration.
3 Studied hyperbolic reaction-diffusion equation and application of the method

Below we shall solve the equation

\[
\tau \frac{\partial^2 Q}{\partial t^2} + \frac{\partial Q}{\partial t} = D \frac{\partial^2 Q}{\partial x^2} + \sum_{i=1}^{4} \alpha_i Q^i. \tag{8}
\]

Reaction-diffusion equation of this kind (with polynomial nonlinearity of fourth order) was used to model the propagation of wave fronts in populations systems \cite{51, 52}. We apply the ansatz \(Q(\xi) = Q(x - vt)\) and then we substitute \(Q(\xi)\) by Eq.\((5)\) where \(n = 2\), i.e.,

\[
Q[\phi(\xi)] = a_0 + a_1 \phi(\xi) + a_2 \phi(\xi)^2. \tag{9}
\]

The balance procedure leads to a simplest equation of fourth order:

\[
\frac{d\phi(\xi)}{d\xi} = b_0 + b_1[\phi(\xi)] + b_2[\phi(\xi)]^2 + b_3[\phi(\xi)]^3 + b_4[\phi(\xi)]^4. \tag{10}
\]

Above quantities \(a_0, a_1, a_2, b_0, b_1, b_2, b_3\) and \(b_4\) are parameters. We note here that we shall use a particular case of this simplest equation where \(b_0 = b_2 = b_3 = 0\). The substitution of the traveling wave ansatz and Eqs.\((9), (10)\) in Eq.\((8)\) leads to the following system of 9 algebraic equations:

\[
\begin{align*}
10 (D - v^2 \tau) a_2 b_4^2 + \alpha_4 a_2^4 &= 0, \\
18 (D - v^2 \tau) a_2 b_3 b_4 &= 0, \\
4 \alpha_4 a_0 a_2^3 + \alpha_3 a_2^3 + (D - v^2 \tau) (8 a_2 b_3^2 + 16 a_2 b_2 b_4) &= 0, \\
(D - v^2 \tau) (14 a_2 b_3 b_4 + 14 a_2 b_2 b_3) + 2 v a_2 b_4 &= 0, \\
(D - v^2 \tau) (12 a_2 b_1 b_3 + 12 a_2 b_0 b_4 + 6 a_2 b_2^3) + (3 \alpha_3 a_0 + a_2 + 6 \alpha_4 a_0^2) a_2^2 + 2 v a_2 b_3 &= 0, \\
2 v a_2 b_2 + (D - v^2 \tau) (10 a_2 b_0 b_3 + 10 a_2 b_1 b_2) &= 0, \\
3 \alpha_3 a_0^2 a_2 + 2 \alpha_2 a_0 a_2 + 4 \alpha_4 a_0^3 a_2 + (D - v^2 \tau) (8 a_2 b_0 b_2 + 4 a_2 b_1^2) + 2 v a_2 b_1 + \alpha_1 a_2 &= 0, \\
2 v a_2 b_0 + 6 (D - v^2 \tau) a_2 b_0 b_1 &= 0, \\
2 (D - v^2 \tau) a_2 b_0^2 + \alpha_4 a_0^4 + \alpha_3 a_0^3 + \alpha_1 a_0 + \alpha_2 a_0^2 &= 0. \tag{11}
\end{align*}
\]
A nontrivial solution of the system (11) is:

\[
\begin{align*}
    b_1 &= \frac{[\alpha_3^3 (49\alpha_3^3 \tau - 640\alpha_2^2)]^{1/2}}{640\alpha_2^2 D^{1/2}} \\
    b_4 &= -\frac{7\alpha_3^2 \tau a_2 (49\alpha_3^3 \tau - 640\alpha_4^2)}{80\alpha_4^2 D \left( 320 + \frac{49\alpha_3^3 \tau - 320\alpha_4^2}{\alpha_4} \right)}^{1/2} \\
    b_0 &= b_2 = b_3 = 0, \\
    v &= \frac{\alpha_4^2 D^{1/2} \left( 320 + \frac{49\alpha_3^3 \tau - 320\alpha_4^2}{\alpha_4} \right)}{7\alpha_3 \tau [\alpha_3 (49\alpha_3^3 \tau - 540\alpha_4^2)]^{1/2}}, \\
    \alpha_2 &= \frac{3}{8} \alpha_4^2, \quad \alpha_4 = \frac{3}{64} \frac{\alpha_3^3}{\alpha_4^2}, \quad a_0 = -\frac{1}{4} \frac{\alpha_3}{\alpha_4}, \quad a_1 = 0
\end{align*}
\]
Then the solution of the simplest equation becomes \((k = 4)\)

\[
\phi(\xi) = \left\{ \left[ \frac{\alpha_3^3 (49\alpha_3^3 \tau - 640\alpha_4^2)}{640\alpha_4 D^{1/2}} \right]^{1/2} \exp \left[ \frac{\alpha_3^3 (49\alpha_3^3 \tau - 640\alpha_4^2)}{640\alpha_4^2 D^{1/2}} \right] (\xi + \xi_0) \right\}^{1/3} \times \exp \left[ \frac{\alpha_4^3 D a_2 \left( 320 + \frac{49\alpha_3^3 \tau - 320\alpha_4^2}{\alpha_4^2} \right)}{\alpha_4^3 \tau (49\alpha_3^3 \tau - 640\alpha_4^2)} \right]^{1/2} \times \exp \left[ \frac{\alpha_3^3 (49\alpha_3^3 \tau - 640\alpha_4^2)}{640\alpha_4^2 D^{1/2}} \right] (\xi + \xi_0) \right\}^{1/3} ,
\]

\[
\phi(\xi) = \left\{ \left[ \frac{\alpha_3^3 (49\alpha_3^3 \tau - 640\alpha_4^2)}{640\alpha_4 D^{1/2}} \right]^{1/2} \exp \left[ \frac{\alpha_3^3 (49\alpha_3^3 \tau - 640\alpha_4^2)}{640\alpha_4^2 D^{1/2}} \right] (\xi + \xi_0) \right\}^{1/3} \times \exp \left[ \frac{\alpha_4^3 D a_2 \left( 320 + \frac{49\alpha_3^3 \tau - 320\alpha_4^2}{\alpha_4^2} \right)}{\alpha_4^3 \tau (49\alpha_3^3 \tau - 640\alpha_4^2)} \right]^{1/2} \times \exp \left[ \frac{\alpha_3^3 (49\alpha_3^3 \tau - 640\alpha_4^2)}{640\alpha_4^2 D^{1/2}} \right] (\xi + \xi_0) \right\}^{1/3} ,
\]

(13)
for the cases $b_4 < 0$, $b_1 > 0$ and $b_4 > 0$, $b_1 < 0$ respectively. Thus the solutions of Eq. (8) are

$$Q(\xi) = -\frac{1}{4} \alpha_3 + a_2$$

$$\times \left\{ \frac{[\alpha_3^3 (49 \alpha_3^3 \tau - 640 \alpha_4^2)]^{1/2}}{640 \alpha_4^2 D^{1/2}} \exp \left[ \frac{3 \alpha_3^3 (49 \alpha_3^3 \tau - 640 \alpha_4^2)}{640 \alpha_4^2 D^{1/2}} (\xi + \xi_0) \right] \right\}^{1/2}$$

$$\times \left[ \frac{7 \alpha_3^2 \tau a_2 (49 \alpha_3^3 \tau - 640 \alpha_4^2)}{80 \alpha_4^2 D \left( 320 + \frac{49 \alpha_3^3 \tau - 320 \alpha_4^2}{\alpha_4^2} \right) \right]^{-1/2}$$

$$\times \exp \left[ \frac{3 \alpha_3^3 (49 \alpha_3^3 \tau - 640 \alpha_4^2)}{640 \alpha_4^2 D^{1/2}} (\xi + \xi_0) \right] \right\}^{2/3},$$

$$Q(\xi) = -\frac{1}{4} \alpha_3 + a_2$$

$$\times \left\{ \frac{[\alpha_3^3 (49 \alpha_3^3 \tau - 640 \alpha_4^2)]^{1/2}}{640 \alpha_4^2 D^{1/2}} \exp \left[ \frac{3 \alpha_3^3 (49 \alpha_3^3 \tau - 640 \alpha_4^2)}{640 \alpha_4^2 D^{1/2}} (\xi + \xi_0) \right] \right\}^{1/2}$$

$$\times \left[ \frac{7 \alpha_3^2 \tau a_2 (49 \alpha_3^3 \tau - 640 \alpha_4^2)}{80 \alpha_4^2 D \left( 320 + \frac{49 \alpha_3^3 \tau - 320 \alpha_4^2}{\alpha_4^2} \right) \right]^{-1/2}$$

$$\times \exp \left[ \frac{3 \alpha_3^3 (49 \alpha_3^3 \tau - 640 \alpha_4^2)}{640 \alpha_4^2 D^{1/2}} (\xi + \xi_0) \right] \right\}^{2/3}$$

for the cases $b_4 < 0$, $b_1 > 0$ and $b_4 > 0$, $b_1 < 0$ respectively. The obtained solutions (14) describe kink waves. Several of the waves are shown in Figs. 1-3. The parameters of the solutions are the same except the parameter $\alpha_3$ that has different values for the three kinks. As one can observe the decrease of the value of the parameter $\alpha_3$ leads to: (i) change of the values of $Q$ (from negative to positive); (ii) decrease of the width of the ink, and (iii) increase of the amplitude of the kink. Similar effects can be observed also in the case when the values of other parameters of the solution are varied.
Figure 1: Solution of equation (14). The values of parameters are: $\tau = 20$; $D = 2$; $a_2 = 1$; $\alpha_3 = 1$; $\alpha_4 = 1$; $\xi_0 = 0$. $w = \xi + \xi_0$.

Figure 2: Solution of equation (14). The values of parameters are: $\tau = 20$; $D = 2$; $a_2 = 1$; $\alpha_3 = -5$; $\alpha_4 = 1$; $\xi_0 = 0$. $w = \xi + \xi_0$. 
Figure 3: Solution of equation (14). The values of parameters are: $\tau = 20$; $D = 2$; $a_2 = 1$; $\alpha_3 = -1/2$; $\alpha_4 = 1$; $\xi_0 = 0$. $w = \xi + \xi_0$.

4 Concluding remarks

In this chapter we have discussed the nonlinear hyperbolic reaction-diffusion equation (8). It can be related to the nonlinear reaction-diffusion equation that was used to model systems from population dynamics [51, 52]. We note that:

- The obtained solutions of the hyperbolic reaction-diffusion equation do not contain as particular cases the kink solutions of the reaction-diffusion equation discussed in [51, 52]. This is easily seen from the relationship for $b_4$ in Eqs. (12). If we set there $\tau = 0$ then $b_4 = 0$ and we cannot construct a kink solution of the kind $Q(\xi) = b_1 + b_4 \phi(\xi)^2$.

- Figs. 4 and 5 show the influence of increasing values of the parameter $\tau$ on the obtained kink solutions of the nonlinear hyperbolic reaction-diffusion equation. As it can be seen from the figures the influence of the increasing value of $\tau$ on the kink profile is: (i) to decrease the amplitude of the kink, and (ii) to make the transition between the areas of lower and higher values of the kink more concentrated (i.e. this transition happens in the smaller interval of values of $w$).

- The exact solution of the studied nonlinear partial differential equation was obtained by means of the modified method of simplest equation.
Figure 4: Solution of equation (14). The values of parameters are: $\tau = 10$; $D = 2$; $a_2 = 1$; $\alpha_3 = -1/2$; $\alpha_4 = 1$; $\xi_0 = 0$. $w = \xi + \xi_0$.

Figure 5: Solution of equation (14). The values of parameters are: $\tau = 1000$; $D = 2$; $a_2 = 1$; $\alpha_3 = -1/2$; $\alpha_4 = 1$; $\xi_0 = 0$. $w = \xi + \xi_0$. 
We have shown that this method is an effective method for obtaining particular exact solutions of nonlinear partial differential equations that do not belong to the class of integrable equations.

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