AN INDEX FORMULA FOR HYPERSURFACES WHICH ADMIT ONLY GENERIC CORANK ONE SINGULARITIES

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Abstract. Let \( f : M^{2m} \to \mathbb{R}^{2m+1} \) be a hypersurface which allow only generic corank one singularities, and admits globally defined unit normal vector field \( \nu \). Then we show the existence of an index formula. For example, when \( m = 1 \) or \( m = 2 \), it holds that
\[
2 \deg(\nu) = \chi(M^+_2) - \chi(M^-_2) + \chi(A^+_3) - \chi(A^-_3) = \chi(M^+_3) - \chi(M^-_3) + \chi(A^+_4) - \chi(A^-_4),
\]
respectively, where \( M^+_{2m} \) (resp. \( M^-_{2m} \)) is the subset of \( M^{2m} \) at which the co-orientation of \( M^{2m} \) induced by \( \nu \) coincides (resp. does not coincide) with the orientation of \( M^{2m} \) for \( m = 1, 2 \), and \( \chi(A^+_{2k+1}) \) (resp. \( \chi(A^-_{2k+1}) \)) for \( k = 1, 2 \) is the Euler number of the set of positive (resp. negative) \( A_{2k+1} \)-singular points of \( f \). The formula for \( m = 1 \) is known. To prove the results, we prepare an index formula for corank one singularities of vector bundle homomorphisms on \( M^{2m} \). As its application, an index formula for Blaschke normal map of strictly convex hypersurfaces in \( \mathbb{R}^{2m+1} \) is also given.

1. Introduction

Let \( M^n \) be an oriented closed \( n \)-manifold, and \( \varphi : TM^n \to \mathcal{E} \) a homomorphism between the tangent bundle \( TM^n \) and an oriented vector bundle \( \mathcal{E} \) of rank \( n \) on \( M^n \). Suppose that \( \varphi \) allows only generic corank one singularities, namely it has only \( A_k \)-singularities. When \( n = 2m \) is an even number, the Euler characteristic \( \chi_{\mathcal{E}} \) of the vector bundle \( \mathcal{E} \) satisfies the following formula
\[
\chi_{\mathcal{E}} = \chi(M^+_n) - \chi(M^-_n) + \sum_{j=1}^{m} \left( \chi(A^+_{2j+1}) - \chi(A^-_{2j+1}) \right),
\]
where \( \chi(M^+_n) \) (resp. \( \chi(M^-_n) \)) is the Euler characteristic of the subset \( M^+_n \) (resp. \( M^-_n \)) of \( M^n \) at which the co-orientation induced by \( \varphi \) is (resp. is not) compatible with the orientation of \( TM^n \) (cf. [22]), the number \( \chi(A^+_{2j+1}) \) (resp. \( \chi(A^-_{2j+1}) \)) is the Euler characteristic of the set of positive (resp. negative) \( A_{2j+1} \)-points. In particular, \( \chi(A^+_{2m+1}) \) (resp. \( \chi(A^-_{2m+1}) \)) is equal to the number \( \#A^+_{2m+1} \) (resp. \( \#A^-_{2m+1} \)) of positive (resp. negative) \( A_{2m+1} \)-points (cf. Definition [22]). For example, the formulas for \( n = 2, 4 \) are given by
\[
\begin{align*}
\chi_{\mathcal{E}} &= \chi(M^+_2) - \chi(M^-_2) + \#A^+_3 - \#A^-_3, \\
\chi_{\mathcal{E}} &= \chi(M^+_4) - \chi(M^-_4) + \chi(A^+_5) - \chi(A^-_5) + \#A^+_5 - \#A^-_5.
\end{align*}
\]

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Moreover, when $\mathcal{E} = TM^n$ and $\varphi$ is the bundle homomorphism map, we get

\[2\chi(M^n) = \sum_{j=1}^{m} \left( \chi(A^+_{2j+1}) - \chi(A^-_{2j+1}) \right).\]

The formula (1.1) is an analogue of the Gauss Bonnet-type formula for closed surfaces given in \cite{10, 14, 16} and \cite{19}. In fact, the left hand side of (1.1) can be written in terms of the integration of the Pfaffian of curvature form of an arbitrarily given metric connection on $\mathcal{E}$. The assumption that $\varphi$ admits only $A_k$-singularities ($k \geq 2$) is meaningful, since one can generate a pair of connected $A_{2r+1}$-sets on the set of $A_{2r-1}$-points for each $r = 1, 2, \ldots$. Also, as pointed out in Saeki and Sakuma \cite{17}, any closed orientable 4-manifolds with vanishing signature admit $C^\infty$-maps into $R^4$ having only fold or cusp singularities. The $Z_2$-version of our formula (1.1) was given by Levine \cite{6}. If we set $\varphi$ the derivative of a Morin map $f : M^n \to N^n$, then we get a formula (1.1), which is proved by Nakai \cite{9} and Dutertre-Fukui \cite{4}. (In this paper, the numbering of Morin singularities is different from the usual one, see Remark 2.3.) So the formula (1.1) is a generalization of them. Our proof is independent of those in \cite{9} and \cite{4}. More precisely, we apply the Poincaré-Hopf index formula for sections of oriented vector bundles. (In \cite{9} and \cite{4}, Viro’s integral calculus \cite{18} is applied.) It should be remarked that index formulas in $Z_2$-coefficients for globally defined Morin maps $f : M^n \to N^p$ ($n \geq p$) are given by Fukuda \cite{5} and Saeki \cite{11}.

Our main purpose is to prove the following two assertions:

**Theorem 1.1.** Let $M^{2m}$ be a $2m$-manifold and $f : M^{2m} \to R^{2m+1}$ a wave front. Suppose that $f$ admits only $A_k$-singularities ($2 \leq k \leq 2m + 1$). Then the singular set of $f$ satisfies the identity

\[\deg(\hat{\varphi}) = \chi(M^{2m}) - \chi(M^{2m}) + \sum_{j=1}^{m} \left( \chi(A^+_{2j+1}) - \chi(A^-_{2j+1}) \right),\]

where $\deg(\hat{\varphi})$ is the degree of the Gauss map $\hat{\varphi} : M^{2m} \to S^{2m}$ induced by $f$, $\chi(M^{2m})$ (resp. $\chi(M^{2m})$) is the Euler characteristic of the subset $M_{2m}^+$ (resp. $M_{2m}^-$) of $M^{2m}$ at which the volume density function

\[\lambda := \det(f_{x_1}, \ldots, f_{x_{2m}}, \hat{\varphi})\]

is positive (resp. negative) for an oriented local coordinate system $(x_1, \ldots, x_{2m})$, where $f_{x_j} = \partial f/\partial x_j$.

This formula is not a direct consequence of the index formula for the Gauss map $\nu$. In fact, the singular set of $f$ does not coincide with that of its Gauss map in general. On the other hand, the following assertion is a hypersurface version of \cite{16} Theorem 3.2.

**Theorem 1.2.** Let $S^{2m}$ be the $2m$-sphere and $f : S^{2m} \to R^{2m+1}$ a strictly convex immersion. Suppose that the Blaschke normal map $\xi : S^{2m} \to R^{2m+1}$ (cf. (7.5)) admits only $A_k$-singularities for $2 \leq k \leq 2m + 1$. Then the singular set of $\xi$ satisfies the identity (1.2), where $M^{2m} = S^{2m}$ is the subset of $S^{2m}$ at which the determinant of the affine shape operator (called the affine Gauss-Kronecker curvature) is negative, and $A^+_{2j+1}$ (resp. $A^-_{2j+1}$) is the set of positive (resp. negative) $A_{2j+1}$-points of $\xi$ for each $j = 1, \ldots, m$. 

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To prove the formula, we apply our formula (1.2) by setting \( \varphi \) the derivative of the affine shape operator of \( f \) (see Section 7).

The paper is organized as follows: In Section 2 we give a precise definition of \( \mathcal{A}_k \)-singularities and show the well-definedness of the positivity and the negativity of odd order \( \mathcal{A}_{2k+1} \)-singular points. In Section 3 we define characteristic vector fields with respect to the homomorphism \( \varphi : TM^n \to \mathcal{E} \) and show the existence of such a vector field \( X \) defined on \( M^n \). It is well-known that the sum of all indices of zeros of a section \( Y \) of \( \mathcal{E} \) are equal to the Euler characteristic \( \chi_\mathcal{E} \) of the vector bundle \( \mathcal{E} \). Since the section \( Y := \varphi(X) \) of \( \mathcal{E} \) has finitely many zeros, it holds that

\[
\chi_\mathcal{E} = \sum_{p \in M^n \setminus \mathcal{A}_2} \text{ind}_p(Y) + \cdots + \sum_{p \in A_n \setminus A_{n+1}} \text{ind}_p(Y) + \sum_{p \in A_{n+1}} \text{ind}_p(Y). (1.4)
\]

Using this, we prove the formulas (1.1). However, for the sake of simplicity, we prove only for the cases \( n = 2, 4 \) in Sections 4 and 6. The general formulas can be easily followed by imitating these proofs. Finally, in Section 7 we prove Theorems 1.1 and 1.2. Several other applications are also given in Section 7.

2. Preliminaries

Let \( M^n \) be an oriented closed \( n \)-manifold and \( \varphi : TM^n \to \mathcal{E} \) a bundle homomorphism between the tangent bundle \( TM^n \) and an oriented vector bundle \( \mathcal{E} \) of rank \( n \). Then a point \( p \in M^n \) is called a singular point if the linear map \( \varphi_p : T_pM \to \mathcal{E}_p \) has a non-trivial kernel, where \( \mathcal{E}_p \) is the fiber of \( \mathcal{E} \) at \( p \). Since \( M^n \) is oriented, we can take a non-vanishing \( n \)-form \( \Omega \) defined on \( M^n \) which is compatible with respect to the orientation of \( M^n \). Furthermore, since \( \mathcal{E} \) is also oriented, there is a non-vanishing \( n \)-function \( \mu \) of the determinant line bundle of the dual \( \mathcal{E}^* \) of \( \mathcal{E} \) defined on \( M^n \) which is compatible with the orientation of \( \mathcal{E} \). Then there is a (unique) \( C^\infty \)-function \( \lambda : M^n \to \mathbb{R} \) such that

\[
\varphi^* \mu = \lambda \Omega, \tag{2.1}
\]

where \( \varphi^* \mu \) is the pull-back of \( \mu \) by \( \varphi \). We set

\[
M^n_+ := \{ p \in M^n ; \lambda(p) > 0 \}, \quad M^n_- := \{ p \in M^n ; \lambda(p) < 0 \}.
\]

A point \( p \in M^n \) is a singular point if and only if \( \lambda(p) = 0 \). A singular point \( p \in M^n \) is called nondegenerate if the exterior derivative \( d\lambda \) does not vanish at \( p \). The bundle homomorphism \( \varphi \) is called nondegenerate if all of the singular points are nondegenerate. If \( \varphi \) is nondegenerate, the singular set

\[
\Sigma^{n-1} := \{ p \in M^n ; \lambda(p) = 0 \}
\]

of \( \varphi \) is an embedded hypersurface of \( M^n \), which coincides with the boundary \( \partial M^n_+ = \partial M^n_- \).

Suppose \( \varphi \) is nondegenerate.

**Definition 2.1.** Let \( U \) be an open subset of \( M^n \). A function \( h : U \to \mathbb{R} \) is called a \( \varphi \)-function around \( p \in \Sigma^{n-1} \cap U \) if the following hold:

1. \( dh(p) \neq 0 \) and has the same sign of \( d\lambda(p) \), and
2. \( \Sigma^{n-1} \cap U \) is the level set \( h = 0 \) near \( p \).

For each \( \varphi \)-function, the well-known preparation theorem for \( C^\infty \)-functions yields that there exists a \( C^\infty \)-function \( \sigma \) on \( U \) such that

\[
h = e^\sigma \lambda. \tag{2.3}
\]
Of course, $\lambda$ itself is a $\varphi$-function. In the following discussion, we may replace $\lambda$ by an arbitrarily fixed $\varphi$-function.

Since $\varphi$ is nondegenerate, the kernel of $\varphi$ at each point $p \in \Sigma^{n-1}$ is of dimension 1. In particular, there exists a smooth vector field $\tilde{\eta}$ defined on a sufficiently small neighborhood $U_p(\subset M^n)$ such that the restriction

$$\eta := \tilde{\eta}|_{U_p \cap \Sigma^{n-1}}$$

has the property that $\eta_q$ belongs to the kernel of $\varphi_q$ for each $q \in U_p \cap \Sigma^{n-1}$. We call $\eta$ a null vector field and $\tilde{\eta}$ an extended null vector field (cf. [14, p. 733]). We set $\lambda^{(0)} := \lambda$,

$$\dot{\lambda}(= \lambda^{(1)}) := d\lambda(\tilde{\eta}), \quad \ddot{\lambda}(= \lambda^{(2)}) := d\dot{\lambda}(\tilde{\eta}), \quad \dddot{\lambda}(= \lambda^{(3)}) := d\ddot{\lambda}(\tilde{\eta})$$

and

$$(2.4) \quad \lambda^{(k+1)} := d\lambda^{(k)}(\tilde{\eta}) \quad (k = 0, 1, 2, \ldots)$$

inductively.

**Definition 2.2.** Let $\varphi: TM^n \to \mathcal{E}$ be a nondegenerate bundle homomorphism and $\Sigma^{n-1}$ its singular set. A point $p \in \Sigma^{n-1}$ is called an $A_2$-point if the extended null vector field $\tilde{\eta}$ is not a tangent vector of $\Sigma^{n-1}$ at $p$ (cf. [17, Definition 2.1]). Then the set of $A_2$-points is given by

$$(2.5) \quad A_2 := \{p \in \Sigma^{n-1} : \eta_p \notin T_p \Sigma^{n-1}\} = \{p \in \Sigma^{n-1} : \dot{\lambda}(p) \neq 0\}$$

A point $p \in \Sigma^{n-1} \setminus A_2$ is called $2$-nondegenerate if $d\dot{\lambda}$ does not vanish at $p$. In this case

$$(2.6) \quad \Sigma^{n-2} := \Sigma^{n-1} \setminus A_2 = \{q \in \Sigma^{n-1} : \dot{\lambda}(q) = 0\}$$

consists of a hypersurface in $\Sigma^{n-1}$ near $p$. A point $p \in \Sigma^{n-1}$ is called an $A_3$-point if it is not an $A_2$-point, but $\ddot{\lambda} = d\dot{\lambda}(\tilde{\eta})$ does not vanish at $p$. In this case, $p$ is 2-nondegenerate and the set of $A_3$-points near $p$ is given by (cf. [17] Definition 2.2)

$$(2.7) \quad A_3 := \{q \in \Sigma^{n-2} : \eta_q \notin T_q \Sigma^{n-2}\} = \{q \in \Sigma^{n-2} : \ddot{\lambda}(q) \neq 0\}.$$  

We define $A_{k+1}$-points ($k \geq 3$) inductively, assuming that $A_k$-points has been defined, as follows: Suppose that $p$ is a $k$-nondegenerate point. Then the set $\Sigma^{n-k}$ of $k$-nondegenerate points is an embedded $(n-k)$-submanifold of $M^n$ near $p$. Then $p$ is called a $(k+1)$-nondegenerate point, if $d\lambda^{(k-1)}$ does not vanish at $p$. By this induction, it holds that

$$\Sigma^{n-k} := \Sigma^{n-k+1} \setminus A_k = \{q \in \Sigma^{n-k+1} : \lambda^{(k-1)}(q) = 0\},$$

which is an $(n-k)$-dimensional submanifold of $M^n$. A $(k - 1)$-nondegenerate point $p \in \Sigma^{n-1}$ is called an $A_{k+1}$-point if it is not an $A_k$-point and $\lambda^{(k)} = d\lambda^{(k-1)}(\tilde{\eta})$ does not vanish at $p$. By definition, such a point $p$ is $k$-nondegenerate and the set of $A_{k+1}$-points has the following expression near $p$:

$$(2.8) \quad A_{k+1} := \{q \in \Sigma^{n-k+1} : \eta_q \notin T \Sigma^{n-k}\} = \{q \in \Sigma^{n-k+1} : \lambda^{(k-1)}(q) \neq 0\}.$$  

Our definition of $A_k$-points is independent of both the choice of extended null vector fields $\tilde{\eta}$ and the choice of $\varphi$-functions $\lambda$ (cf. [15]).
Remark 2.3. As seen in the following Examples 2.6 and 2.7, our definition of $A_k$-points gives a unified intrinsic treatment of singularities of Morin maps between the same dimensional manifolds and the $A_k$-singularities appeared in hypersurfaces in $\mathbb{R}^{n+1}$ at the same time. However, in this intrinsic treatment, usual $k$-th singularities for Morin maps and $A_{k+1}$-singularities for wave fronts are both regarded as $A_{k+1}$-singularities of bundle homomorphisms. In other words, the order of singularities of Morin maps is not synchronized with the order of singularities of the corresponding bundle homomorphisms. For example, a fold (i.e. a Morin-1-singularity) and a cusp (i.e. a Morin-2-singularity) induce an $A_2$-singular point and an $A_3$-singular point of bundle homomorphism, respectively.

The following assertion holds:

Fact 2.4 ([15, Theorem 2.4]). A point $p \in \Sigma^{n-1}$ is an $A_{k+1}$-point ($1 \leq k \leq n$) if and only if

1. $\lambda(p) = \lambda(p) = \cdots = \lambda^{(k-1)}(p) = 0$, $\lambda^{(k)}(p) \neq 0$,
2. and the Jacobian matrix of an $\mathbb{R}^k$-valued $C^\infty$-map $\Lambda := (\lambda, \lambda, \ldots, \lambda^{(k-1)})$ is of rank $k$ at $p$.

If $k = 1$, then $d\Lambda = d\lambda$ and the condition [2] of Fact 2.4 is automatically satisfied for $A_2$-points and $A_3$-points. In other words, the second condition of Fact 2.4 comes into effect only if $k \geq 3$. We denote by $A_k$ the set of $A_k$-points in $M^n$.

Definition 2.5. A nondegenerate bundle homomorphism $\varphi : TM^n \to \mathcal{E}$ is called a Morin homomorphism if the set of singular points of $\varphi$ consists of $A_k$-points for $k = 2, 3, \ldots, n + 1$.

This definition is motivated by the existence of the following two typical examples:

Example 2.6. Let $M^n$ and $N^n$ be oriented $n$-manifolds, and let $f : M^n \to N^n$ be a $C^\infty$-map having only Morin singularities. Then the differential $df$ of $f$ canonically induces a map

$$\varphi = df : TM^n \to \mathcal{E}_f := f^\ast TN^n,$$

which gives a Morin homomorphism (cf. Appendix of [14]). Let $\Omega_{M^n}$ and $\Omega_{N^n}$ be the fundamental $n$-forms of $M^n$ and $N^n$, respectively. Then there exists a $C^\infty$-function $\lambda$ on $M^n$ such that $f^\ast \omega_{N^n} = \lambda \omega_{M^n}$ holds. The set $M^n_\lambda$ (resp. $M^n_\lambda$) coincides with the set where $\lambda > 0$ ($\lambda < 0$). The sign of the $\varphi$-function $\lambda$ coincides with the sign of the Jacobian of $f$ with respect to oriented local coordinate systems of $M^n$ and $N^n$. In this case, $A_k$-singularities of the Morin map $f$ are $A_{k+1}$-singularities of the Morin homomorphism $\varphi = df$ (see Remark 2.3).

Example 2.7. Let $f : M^n \to \mathbb{R}^{n+1}$ be a wave front which admits only $A_{k+1}$-singularities ($k = 1, \ldots, n$). Suppose that $f$ is co-orientable, that is, there exists a globally defined unit normal vector field $\nu$ along $f$. Let $f^\ast \mathbb{R}^{n+1}$ be the pull-back of $\mathbb{R}^{n+1}$ by $f$, and consider the subbundle $\mathcal{E}_f$ of $f^\ast \mathbb{R}^{n+1}$ whose fiber $\mathcal{E}_p$ at $p \in M^n$ is the orthogonal complement of $\nu_p$. Then the differential $df$ of $f$ induces a bundle homomorphism

$$\varphi_f = df : TM^n \ni v \mapsto df(v) \in \mathcal{E}_f$$

called the first homomorphism of $f$ as in [15, Section 2], which gives a Morin homomorphism (cf. Appendix of [14]). Consider a function

$$\lambda := \det(f_{x_1}, \ldots, f_{x_n}, \nu),$$

where \( f_x := \partial f / \partial x_i \) (\( i = 1, \ldots, n \)) and \( (x_1, \ldots, x_n) \) is an oriented local coordinate system of \( M^n \). Then \( \lambda \) is a \( \varphi \)-function, and the set \( M^n_+ \) (resp. \( M^n_- \)) coincides with the set where \( \lambda > 0 \) (resp. \( \lambda < 0 \)).

3. Characteristic vector fields

We fix a Morin homomorphism \( \varphi : M^n \to \mathcal{E} \). Then

\[
\Sigma^{n-1} = A_2 \cup \cdots \cup A_{n+1}, \quad \cdots, \quad \Sigma^1 = A_n \cup A_{n+1}, \quad \Sigma^0 = A_{n+1}
\]

hold. Since \( M^n \) is orientable and closed, \( \Sigma^j \) are \( j \)-dimensional orientable and closed submanifolds of \( M^n \), for each \( j = 0, 1, \ldots, n - 1 \). We now fix a Riemannian metric \( ds^2 \) on \( M^n \). Then, we can take a normal vector field \( n \) globally defined on each hypersurface \( \Sigma^j \) embedded in \( \Sigma^{j-1} \) for \( j = 0, 1, \ldots, n - 1 \), where \( \Sigma^n := M^n \).

The following assertion holds, which will play a crucial role in proving our index formula:

**Proposition 3.1.** If \( k \) \((2 \leq k \leq n+1)\) is even, then the sign of the function \( \lambda^{(k)} \) as in (2.4) on the set \( \Sigma^{n-k} \) does not depend on the choice of the sign of the null vector field \( \eta \).

**Proof.** Even if we change the null vector field \( \eta \) to \( -\eta \), the sign of the function \( \lambda^{(k)} \) on the set \( \Sigma^{n-k} \) does not change, since \( k \) is even.

By the proposition above, the following two subsets of \( \Sigma^{n-k} \) are well-defined if \( k \) is even:

\[
\Sigma_{+}^{n-k} := \{ p \in \Sigma^{n-k} ; \lambda^{(k)}(p) > 0 \}, \quad \Sigma_{-}^{n-k} := \{ p \in \Sigma^{n-k} ; \lambda^{(k)}(p) < 0 \}.
\]

**Proposition 3.2.** Let \( k \) be an odd number, and \( p \) an \( A_{k+1} \)-point. Then the vector field \( \lambda^{(k)} \eta \) along \( \Sigma_{+}^{n-k} \) points towards the domain \( \Sigma_{+}^{n-k+1} \) at \( p \), while \( \Sigma_{-}^{n-k} \) does not change, since \( k \) is even.

**Proof.** We denote by \( ds^2_{n-k+1} \) the Riemannian metric of \( \Sigma_{+}^{n-k+1} \) induced by the Riemannian manifold \( (M^n, ds^2) \). Then the hypersurface \( \Sigma_{+}^{n-k} \) embedded in \( \Sigma_{+}^{n-k+1} \) can be characterized as the level set \( \lambda^{(k-1)}(p) = 0 \). Then we have that

\[
ds^2_{n-k+1}(\eta_p, \text{grad}(\lambda^{(k-1)}) \eta_p) = d\lambda^{(k-1)}(\eta_p) = \lambda^{(k)}(p),
\]

where “grad” denotes the gradient of the function with respect to the metric \( ds^2_{n-k+1} \). Thus \( ds^2_{n-k+1}(\lambda^{(k)} \eta, \text{grad}(\lambda^{(k-1)})) \) is positive at \( p \). Since \( \text{grad}(\lambda^{(k-1)}) \) gives the normal vector field along \( \Sigma_{-}^{n-k} \) pointing towards \( \Sigma_{+}^{n-k+1} \), the assertion is proven.

There is \( \pm \)-ambiguity in the choice of a normal vector \( n_{n-k} \) along the hypersurface \( \Sigma_{-}^{n-k} \) in \( \Sigma_{n-k+1} \). So, if \( k \) is odd, we assume that \( n_{n-k} \) points towards the direction \( \Sigma_{+}^{n-k+1} \).

**Definition 3.3.** Let \( \varphi : TM^n \to \mathcal{E} \) be a Morin homomorphism and \( p \) an \( A_{2k+1} \)-point. Since the sign of \( \lambda^{(2k)}(p) \) does not depend on the \( \pm \)-ambiguity of the choice of extended null vector field \( \tilde{\eta} \) (cf. Proposition 3.1), we call \( p \) a positive \( A_{2k+1} \)-point (resp. a negative \( A_{2k+1} \)-point) if \( \lambda^{(2k)}(p) \) is positive (resp. negative).

The set of positive (resp. negative) \( A_{2k+1} \)-points is denoted by \( A_{2k+1}^+ \) (resp. \( A_{2k+1}^- \)). Then

\[
A_{2k+1}^+ := \{ p \in A_{2k+1} ; \lambda^{(2k)}(p) > 0 \} = \Sigma_{+}^{n-2k} \setminus \Sigma_{-}^{n-2k-1},
\]

\[
A_{2k+1}^- := \{ p \in A_{2k+1} ; \lambda^{(2k)}(p) < 0 \} = \Sigma_{-}^{n-2k} \setminus \Sigma_{+}^{n-2k-1}.
\]
hold. If $n = 2$ and $f : M^2 \to \mathbb{R}^3$ is a wave front, then positive (resp. negative) $\mathcal{A}_3$-points as in Example 2.7 correspond to positive (resp. negative) swallowtails.

The Riemannian metric $ds^2$ on $M^n$ induces a geodesic distance function on each submanifold $\Sigma^j$ ($j = 0, 1, \ldots, n - 1$). For a positive number $\delta > 0$, we denote by $\mathcal{N}_\delta(\Sigma^j)$ the $\delta$-tubular neighborhood of $\Sigma^j$ in $\Sigma^{j+1}$, where $\Sigma^n := M^n$. Since our conclusion (1.1) does not depend on Riemannian metrics on $M^n$, we may also assume that our Riemannian metric $ds^2$ satisfies the following condition, by changing the metric if necessary:

(5) For each $k = 0, \ldots, n - 1$, the normal vector field $n_k$ is proportional to the null vector field $\eta$ on $\Sigma^k \setminus \mathcal{N}_\delta(\Sigma^{k-1})$, where $\delta > 0$ is a constant.

Let $X$ be a vector field of $M^n$ which vanishes at $p \in M^n$. Take a local coordinate system $(U; x_1, \ldots, x_n)$ at $p$ and write

$$X = \xi_1 \frac{\partial}{\partial x_1} + \cdots + \xi_n \frac{\partial}{\partial x_n}.$$ 

Then a zero $p$ of $X$ is called generic if the Jacobian of the map $U \ni x \mapsto (\xi_1(x), \cdots, \xi_n(x)) \in \mathbb{R}^n$ does not vanish at $x = p$. A vector field $X$ defined on $M^n$ is called generic if all its zeros are generic.

**Definition 3.4.** A $C^\infty$-vector field $X$ defined on $M^n$ is called a characteristic vector field for a Morin homomorphism $\varphi : TM^n \to \mathcal{E}$, if it satisfies the following four conditions.

(i) $X$ is a generic vector field on $M^n$ which does not vanish at each point of $\Sigma^{n-1}$.

(ii) For each $k = 0, \ldots, n - 1$, there exists a generic vector field $X_k$ along $\Sigma^k$ such that the zeros of $\varphi(X)$ on $\Sigma^k \setminus \Sigma^{k-1}$ coincide exactly with the zeros of $X_k$ on $\Sigma^k$. Moreover, $\varphi(X) = \varphi(X_k)$ holds on $\Sigma^k$. In particular, $\varphi(X)$ vanishes on $\mathcal{A}_{n+1}$-points.

(iii) For each $\mathcal{A}_{k+1}$-point $p$ (i.e. $p \in \Sigma^{n-k} \setminus \Sigma^{n-k-1}$) satisfying $\varphi(X_p) = 0$, there exists a neighborhood $U$ of $p$ of $M^n$ such that the restriction $X$ to $U \cap \Sigma^j$ is tangent to $\Sigma^j$ for $j = n - k + 1, \ldots, n$.

(iv) The function given by the inner product $ds^2(X, n_k)$ is non-negative along $\Sigma^k$ and positive on $\Sigma^k \setminus \mathcal{N}_\delta(\Sigma^{k-1})$ for $k = 0, 1, \ldots, n - 1$ and for sufficiently small $\delta > 0$.

In this section, we shall construct a characteristic vector field, which will play a crucial role in proving the formula (1.1) in the introduction:

**Proposition 3.5.** There exists a characteristic vector field defined on $M^n$ associated to a Morin homomorphism $\varphi$.

**Proof.** For each $p \in \mathcal{A}_{n+1}$, the 0-th normal vector $n_0(p) \in T_p \Sigma^1$ is a null vector at $p$. Applying Lemma $\text{A.1}$ in the appendix by setting $K = \Sigma^0$ and $M^1 = \Sigma^1$, there exists a generic vector field $X_1$ on $\Sigma^1$ such that $X_1(p) = n_0(p)$ for each $p \in \mathcal{A}_{n+1}$. Thus we can take a vector field $X_1$ on $\Sigma^1$ which does not vanish on $\Sigma^0 = A_{n+1}$. By definition, $Z(\varphi(X_1)) = \mathcal{A}_{n+1} \cup Z(X_1)$ holds, where $Z(\varphi(X_1))$ (resp. $Z(X_1)$) is the zeros of $\varphi(X_1)$ (resp. $X_1$).
We extend $X_1$ to a vector field $X_2$ on $\Sigma^2$. We consider a $\delta$-tubular neighborhood $N_\delta(\Sigma^1)$ in $\Sigma^2$. Since we can take $\delta$ to be sufficiently small, there exists a canonical diffeomorphism 
\[ \epsilon_2 : \Sigma^1 \times [-\delta, \delta] \to \overline{N_\delta(\Sigma^1)} \]
such that $s \mapsto \epsilon_2(q,s)$ is a normal geodesic with the arclength parameter starting from each $q \in \Sigma^1$ with the initial velocity vector $n_1$. Then 
\[ \tilde{n}_1(q, s) := \frac{\partial \epsilon_2(q, s)}{\partial s} \]
gives the unit vector field defined on $N_\delta(\Sigma^1)$ which is an extension of $n_1$. Let $\rho_1 : \Sigma^2 \to [0, 1]$ be a smooth function such that 
\[ \rho_1(q) = \begin{cases} 1 & \text{if } q \in N_\delta(\Sigma^0), \\ 0 & \text{if } q \not\in N_\delta(\Sigma^0), \end{cases} \]
and $\rho_1 \in (0, 1)$ on $N_{2\delta}(\Sigma^0) \setminus N_\delta(\Sigma^0)$. Let $\tilde{X}_1$ be the vector field on $N_\delta(\Sigma^1)$ via parallel transport of $X_1$ along each normal geodesic $s \mapsto \epsilon_2(p, s)$. We set 
\[ W_2(q, s) := \tilde{X}_1(q, s) + \left( s^2 \rho_1(q) + (1 - \rho_1(q)) \right) \tilde{n}_1(q, s), \]
which is a vector field on $N_\delta(\Sigma^1)$. Since 
\[ s^2 \rho_1 + (1 - \rho_1) \geq 0 \]
on $\Sigma^1 \setminus N_\delta(\Sigma^0)$, $W_2$ has no zeros on $N_\delta(\Sigma^1)$. Moreover, $\varphi(W_2) = \varphi(X_1)$ holds on $\Sigma^1$ because of the property (6).

Since $\Sigma^2$ is compact, we can apply Lemma [14] in the appendix by setting $K = \overline{N_\delta(\Sigma^1)}$ and $M^2 := \Sigma^2$, and get a generic vector field $X_2$ defined on $M^2$ such that $X_2$ coincides with $X_1$ on $\Sigma^1$. By definition, 
\[ Z(\varphi(X_2)) = A_{n+1} \cup Z(X_1) \cup Z(X_2) \]
holds. Moreover, $Z(X)$ does not meet $N_\delta(\Sigma^1)$, by our construction.

We now assume that $X_k$ ($k \geq 2$) has been already defined and give a method to construct the vector field $X_{k+1}$, as follows: Since $\delta$ can be taken to be sufficiently small, there exists a canonical diffeomorphism 
\[ \epsilon_{k+1} : \Sigma^k \times [-\delta, \delta] \to \overline{N_\delta(\Sigma^k)} \]
such that $t \mapsto \epsilon_{k+1}(q,t)$ is the normal geodesic of $\Sigma^{k+1}$ with arclength parameter starting from each $q \in \Sigma^k$ into the direction $n_k$. Then 
\[ \tilde{n}_k(q, s) := \frac{\partial \epsilon_{k+1}(q, s)}{\partial s} \]
gives the unit vector field defined on $N_\delta(\Sigma^k)$ as an extension of $n_k$. Let 
\[ \rho_k : \Sigma^k \to [0, 1] \]
be a smooth function such that 
\[ \rho_k(q) = \begin{cases} 1 & \text{if } q \in N_\delta(\Sigma^{k-1}), \\ 0 & \text{if } q \not\in N_{2\delta}(\Sigma^{k-1}), \end{cases} \]
and \( \rho_k \in (0, 1) \) on \( \mathcal{N}_\delta(\Sigma^{k-1}) \setminus \mathcal{N}_\delta(\Sigma^k) \). Let \( \tilde{X}_k \) be the vector field on \( \mathcal{N}_\delta(\Sigma^k) \) via parallel transport of \( X_k \) along each normal geodesic \( s \mapsto \epsilon_{k+1}(q, s) \). We set

\[
W_{k+1}(q, s) := \tilde{X}_k(q, s) + \left( s^2 \rho_k(q) + (1 - \rho_k(q)) \right) \tilde{n}_k(q, s),
\]

which is a vector field on \( \mathcal{N}_\delta(\Sigma^k) \). Then \( W_{k+1} \) has no zeros on \( \mathcal{N}_\delta(\Sigma^k) \) since \( X_k \) does not vanish on \( \mathcal{N}_\delta(\Sigma^{k-1}) \). We then apply Lemma [A.1] in the appendix by setting \( K = \mathcal{N}_\delta(\Sigma^k) \) and get a generic vector field \( X_{k+1} \) defined on \( \Sigma^{k+1} \) such that \( X \) coincides with \( X_k \) on \( \Sigma^k \). It can be easily checked that \( X := X_n \) is a desired characteristic vector field.

\[ \square \]

4. THE TWO DIMENSIONAL CASE

Here we prove formula (4.3) for \( n = 2 \). Although this formula was proved as a corollary of the Gauss-Bonnet formula in [16] and [17], our proof in this section is new.

Let \( X \) be a characteristic vector field associated to \( \varphi : TM^2 \to E \). Take a section \( Y \) of \( E \) as \( Y := \varphi(X) \). Then the following assertion holds:

**Proposition 4.1.** Let \( Z(Y) \), \( Z(X) \) and \( Z(X_1) \) be the set of zeros for \( Y \), \( X \) and \( X_1 \), respectively. Then it holds that

\[
(4.1) \quad Z(Y) \cap (M^2 \setminus \Sigma^1) = Z(X),
\]

\[
(4.2) \quad Z(Y) \cap (\Sigma^1 \setminus \Sigma^0) = Z(X_1) \subset A_2,
\]

\[
(4.3) \quad Z(Y) \cap \Sigma^0 = A_3.
\]

**Proof.** Since \( Y = \varphi(X) \), the property [i] implies that \( Z(X) \subset Z(Y) \). Since \( \varphi : T_pM^2 \to E_p \) is a linear isomorphism when \( p \in M^2 \setminus \Sigma^1 \), we have [4.1]. Since \( Z(X_1) \cap \Sigma^0 \) is an empty set, the property [iii] of characteristic vector field yields

\[
Z(Y) \cap (\Sigma^1 \setminus \Sigma^0) = Z(X_1).
\]

Since \( Y = \varphi(X_1) \) on \( \Sigma^1 \) and \( X_1 \) is proportional to a null vector \( n_0(p) \) on each \( A_3 \)-point \( p \), we can get [4.3]. \[ \square \]

When \( n = 2 \), (4.3) reduces to

\[
(4.4) \quad \chi_E = \sum_{p \in M^2 \setminus \Sigma^1} \text{ind}_p(Y) + \sum_{p \in A_2} \text{ind}_p(Y) + \sum_{p \in A_3} \text{ind}_p(Y).
\]

**Proposition 4.2.** The first term of the right-hand side of (4.4) satisfies

\[
(4.5) \quad \sum_{p \in M^2 \setminus \Sigma^1} \text{ind}_p(Y) = \chi(M_+^2) - \chi(M_-^2).
\]

**Proof.** We denote by \( \text{sgn}(\lambda(p)) \) the sign of the function \( \lambda \) at a point \( p \). Since \( \text{sgn}(\lambda(p)) = 1 \) (resp. \( \text{sgn}(\lambda(p)) = -1 \)) if \( \varphi_p : T_pM^2 \to E_p \) is orientation preserving (resp. orientation reversing), we have that

\[
\text{ind}_p(Y) = \text{sgn}(\lambda(p)) \text{ind}_p(X) \quad (p \in M^2 \setminus A_2).
\]

We set

\[
M_+^2(\delta) := M_+^2 \setminus \mathcal{N}_\delta(\Sigma^1), \quad M_-^2(\delta) := \overline{M_-^2 \setminus \mathcal{N}_\delta(\Sigma^1)} \quad (\delta > 0),
\]

where the overline means the closure operations. If we choose \( \delta \) sufficiently small, then \( Z(Y) \cap (M^2 \setminus \Sigma^1) \) is contained in \( M_+^2(\delta) \cup M_-^2(\delta) \) and \( M_-^2(\delta) \) (resp. \( M_+^2(\delta) \) has
the same homotopy type as $M^2_+$ (resp. $M^2_-$). In particular, the following identity holds
\begin{equation}
\sum_{p \in M^2_+ \setminus A_2} \text{ind}_p(Y) = \sum_{p \in M^2_+ (\delta)} \text{ind}_p(X) - \sum_{p \in M^2_+ (\lambda)} \text{ind}_p(X).
\end{equation}
Here, $-X$ (resp. $X$) is the outward vector of $\tilde{M}^2_+ (\delta)$ (resp. $\tilde{M}^2_+ (\lambda)$) by the property ([iv]) of Definition 3.4 of the characteristic vector field $X$. Since the operation $X \mapsto -X$ is orientation preserving, applying the Poincaré-Hopf index formula (cf. [8]), we have that
\[ \chi(M^2_-) = \chi(M^2_+(\delta)) = \sum_{p \in M^2_+(\delta)} \text{ind}_p(-X) = \sum_{p \in M^2_+(\delta)} \text{ind}_p(X) = \sum_{p \in M^2_+(\lambda)} \text{ind}_p(X). \]
Similarly, we can also show that
\[ \chi(M^2_-) = \sum_{p \in M^2_-} \text{ind}_p(X), \]
which proves the assertion. \hfill \Box

**Proposition 4.3.** The second term of the right-hand side of (4.4) satisfies
\begin{equation}
\sum_{p \in A_2} \text{ind}_p(Y) = 0.
\end{equation}

**Proof.** We fix $p$ on $Z(Y) \cap (\Sigma^1 \setminus \Sigma^0)$. Then $p$ is an $A_2$-point. Let $(U; x_1, x_2)$ be a local coordinate system of $M^2$ around $p$ which is compatible to the orientation of $M^2$ such that $\Sigma^1$ is the level set $x_1 = 0$ and
\[ \tilde{\eta} := \partial/\partial x_1 \]
gives an extended null vector field on $U$ (cf. (5.2)). By the well-known preparation theorem for $C^\infty$-functions, there exists a section $e_1$ of $E$ on $U$ such that $\varphi(\tilde{\eta}) = \lambda e_1$. On the other hand, we set $e_2 := \varphi(\partial/\partial x_2)$. Then $\{e_1, e_2\}$ gives a frame field of $E$ on $U$. In fact, since
\[ \mu \left( \varphi \left( \frac{\partial}{\partial x_1} \right), \varphi \left( \frac{\partial}{\partial x_2} \right) \right) \]
is a $\varphi$-function (cf. Definition 2.1), there exists a $C^\infty$-function $f$ such that (cf. (2.3))
\[ \lambda e f = \mu(\varphi(\partial/\partial x_1), \varphi(\partial/\partial x_2)) = \lambda \mu(e_1, e_2), \]
which implies that $\mu(e_1, e_2) \neq 0$. Now we set
\[ X = \xi_1 \frac{\partial}{\partial x_1} + \xi_2 \frac{\partial}{\partial x_2} \quad \text{and} \quad Y = \zeta_1 e_1 + \zeta_2 e_2. \]
Then it holds that
\[ \zeta_1 = \lambda \xi_1, \quad \zeta_2 = \xi_2. \]
Since $\lambda$ vanishes on $\Sigma^1$, we have $\lambda(p) = \lambda x_2 (p) = 0$, where $\lambda x_2 := \partial \lambda/\partial x_2$. In particular, $(\zeta_1)_{x_2} = \partial \zeta_1/\partial x_2 = 0$ holds at $p$. On the other hand, since $(\zeta_1)_{x_1} = \lambda x_1 \xi_1$ holds at $p$, we have that
\[ \text{sgn} \left( \det \begin{pmatrix} (\zeta_1)_{x_1} & (\zeta_1)_{x_2} \\ (\zeta_2)_{x_1} & (\zeta_2)_{x_2} \end{pmatrix} \right) = \text{sgn} \left( \lambda \xi_1 (\zeta_2)_{x_2} \right) \]
\[ = \text{sgn} \left( \lambda \xi_1 (\zeta_2)_{x_2} \right) = \text{ind}_p(X_1) \text{sgn} (\lambda \xi_1). \]
Here we used that $X_1 = \xi_2 (\partial / \partial x_2)$, which is the conclusion of (iii) of Definition 3.4. $\tilde{\eta} = \partial / \partial x_1$, and the fact that $\Sigma^1 = \{ x_1 = 0 \}$. Since $p \notin N_0(\Sigma^0)$ for sufficiently small $\delta$, the characteristic vector field $X$ points in the direction of $M^2_+ = \{ \lambda > 0 \}$ at $p$. So we have that

$$\text{sgn}(\xi_1) = \text{sgn}(\lambda)$$

holds at $p$. Thus $\dot{\lambda}(p) \xi_1(p) > 0$ and

$$\text{ind}_p(Y) = \text{ind}_p(X_1).$$

Since $Z(X_1) \subset A_2$, applying the Poincaré-Hopf index formula for the vector field $X_1$ on $\Sigma^1$, we get the assertion. \hfill \Box

By (iii), Propositions 4.2 and 4.3, the formula (1.1) follows immediately from the following assertion:

**Proposition 4.4.** Let $p$ be an arbitrarily given $A_3$-point. Then it holds that

$$\text{ind}_p(Y) = \begin{cases} 1 & (\text{if } p \in A_3^+), \\
-1 & (\text{if } p \in A_3^-). \end{cases}$$

**Proof.** We take a local coordinate system $(U; x_1, x_2)$ centered at $p$ such that $\partial / \partial x_2$ gives a null vector field along $\Sigma^1$. In particular, $(\partial / \partial x_2)_p \in T_p \Sigma^1$. We set $e_1 := \varphi(\partial / \partial x_1)$. Since $\varphi(\tilde{\eta}) (\tilde{\eta} := \partial / \partial x_2)$ vanishes on $\Sigma^1$, there exists a section $e_2$ of $\mathcal{E}$ on $U$ such that $\varphi(\partial / \partial x_2) = \varphi(\tilde{\eta}) = \lambda e_2$. Since $\mu(e_2, \varphi(\tilde{\eta})) = \mu(\varphi(\partial / \partial x_1), \varphi(\partial / \partial x_2))$ is a $\varphi$-function (see Definition 2.1), there exists a $C^\infty$-function $f$ such that (cf. (2.3))

$$\lambda e^f = \mu(e_1, \varphi(\tilde{\eta})) = \lambda \mu(e_1, e_2),$$

which implies that $e_1, e_2$ consists of a frame of $\mathcal{E}$ on $U$. We set

$$X = \xi_1 \frac{\partial}{\partial x_1} + \xi_2 \frac{\partial}{\partial x_2}.$$ 

By (iii) and (i) of Definition 3.4, $X_p \in T \Sigma^1$ and $X_p \neq 0$, and hence we have $\xi_1(p) = 0$ and $\xi_2(p) \neq 0$. We now set

$$(4.8) \quad Y = \xi_1 e_1 + \xi_2 e_2.$$ 

Then it holds that $\xi_1 = \xi_1$ and $\xi_2 = \lambda \xi_2$. By (iii) of Definition 3.4 $X$ is tangent to $\Sigma^1$ on a neighborhood of $p$. Since $\lambda$ vanishes along $\Sigma^1$, it holds that

$$0 = d\lambda(X) = \lambda x_1 \dot{\xi}_1 + \dot{\lambda} \xi_2$$

at $p$, where we used the fact that $\lambda x_2 = \dot{\lambda}$. Since $d\lambda(X)$ vanishes along $\Sigma^1$ and $\partial / \partial x_2 \in T \Sigma^1$, the fact $\xi_1(p) = \dot{\lambda}(p) = 0$ yields that

$$0 = \frac{d \lambda(X)}{\partial x_2} = \lambda x_1 x_2 \xi_1 + \lambda x_1 (\xi_1) x_2 + \dot{\lambda} \xi_2 + \dot{\lambda} (\xi_2) x_2$$

$$= \lambda x_1 (\xi_1) x_2 + \dot{\lambda} \xi_2$$

holds at $p$. Since $d \lambda(p) \neq 0$ and $\lambda x_2 = 0$, we can conclude that $\lambda x_1(p) \neq 0$. In particular, we have that

$$\frac{\xi_1 x_2(p)}{\lambda x_1(p)} = -\frac{\dot{\lambda}(p) \xi_2(p)}{\lambda x_1(p)}.$$
By using the facts $\lambda_{x_j}(p) = \dot{\lambda}(p) = 0$, we have that

$$\text{ind}_p(Y) = \text{sgn} \left( \det \begin{pmatrix} (\xi_1)_{x_1}(p) & (\xi_1)_{x_2}(p) \\ \lambda_{x_1}(p)\xi_2(p) & \xi_2(p)\lambda_{x_2}(p) \end{pmatrix} \right)$$

$$= \text{sgn} \left( \det \begin{pmatrix} (\xi_1)_{x_1}(p) & (\xi_1)_{x_2}(p) \\ \lambda_{x_1}(p)\xi_2(p) & 0 \end{pmatrix} \right)$$

$$= -\text{sgn} \left( \xi_2(p)\lambda_{x_1}(p) \left( -\frac{\dot{\lambda}(p)\xi_2(p)}{\lambda_{x_1}(p)} \right) \right) = \text{sgn} \left( \frac{\xi_2(p)^2\dot{\lambda}(p)}{\lambda_{x_1}(p)} \right).$$

Since the sign of an $\mathcal{A}_3$-point coincides with the sign of $\dot{\lambda}$, Proposition 1.4 is proved. □

5. A Key Lemma for Higher Dimensional Case

In this section, we prove the following assertion, which will be a key to proving our formula (1.1) for $n$-dimensional manifolds ($n \geq 3$).

**Theorem 5.1.** Let $X$ be a characteristic vector field associated to the bundle homomorphism $\varphi : TM^n \to \mathcal{E}$. Let $(U; x_1, \ldots, x_n)$ be a local coordinate system centered at an $\mathcal{A}_{k+1}$-point $p \in M^n$ ($k \geq 1$) satisfying the following properties:

(a) $(x_{j+1}, \ldots, x_n)$ gives a local coordinate system of the singular submanifold $\Sigma^{n-j}$ for each $j = 1, \ldots, k$, that is,

$$\Sigma^{n-j} \cap U = \{(x_1, \ldots, x_n); x_1 = \cdots = x_j = 0\},$$

(b) $\frac{\partial}{\partial x_{j+1}}, \ldots, \frac{\partial}{\partial x_n}$ spans the tangent space of $\Sigma^{n-j}$ for each $j = 1, \ldots, k$

at $p$,

(c) $\frac{\partial}{\partial x_k}$ coincides with an extended null vector field $\tilde{\eta}$ on $U$.

Suppose that $Y := \varphi(X)$ vanishes at $p$. Then $X$ has an expression

$$X = \xi_1 \frac{\partial}{\partial x_1} + \cdots + \xi_n \frac{\partial}{\partial x_n},$$

such that

$$\xi_k(p) \neq 0, \quad \xi_j(p) = 0 \quad (j = 1, \ldots, k - 1, k + 1, \ldots, n),$$

and we have

$$\text{ind}_p(Y) = \text{sgn} \left( \xi_k(p)^k\lambda^{(k)}(p) \right) \text{ind}_p(X_{n-k}).$$

**Remark 5.2.** The existence of the coordinate system $(x_1, \ldots, x_n)$ as in (5.1) is shown as follows: Firstly, we take a coordinate system $(y_1, \ldots, y_n)$ such that $(y_{j+1}, \ldots, y_n)$ gives a local coordinate (at $p$) of the singular submanifold $\Sigma^{n-j}$ for each $j = 1, \ldots, n - 1$. Let $\tilde{\eta}$ be an extended null vector field and $q \mapsto \psi_t(q)$ be the local 1-parameter group of transformations generated by $\tilde{\eta}$. Consider the map

$$(y_1, \ldots, y_k, \ldots, y_n) \mapsto \psi_t(y_1, \ldots, y_{k-1}, 0, y_k, \ldots, y_n),$$

which gives the desired local coordinate system.

**Proof of Theorem 5.1.** By $\varphi(X) = 0$ at $p$, the property (i) of the characteristic vector field $X$ and (b) we have (5.1). Moreover, the property (a) $X_{n-k}$ (as in Definition 3.4) is represented as

$$X_{n-k} = \xi_{k+1} \frac{\partial}{\partial x_{k+1}} + \cdots + \xi_n \frac{\partial}{\partial x_n}.$$
We set 
\[ e_j := \varphi(\partial/\partial x_j) \quad (j = 1, \ldots, k - 1, k + 1, \ldots, n). \]

By the well-known preparation theorem for \( C^\infty \)-functions, we can write \( \varphi(\partial/\partial x_k) = \lambda e_k \), where \( e_k \) is a local section defined on a neighborhood of \( p \). Since
\[ \lambda := \mu \left( \varphi \left( \frac{\partial}{\partial x_1} \right), \ldots, \varphi \left( \frac{\partial}{\partial x_n} \right) \right) = \lambda \mu(e_1, \ldots, e_n) \]
is a \( \varphi \)-function, we have that
\[ \mu(e_1, \ldots, e_n) \neq 0, \]
which implies that \( e_1, \ldots, e_n \) gives a frame on the vector bundle \( E \) around \( p \). So we can write
\[ (5.3) \quad Y = \zeta_1 e_1 + \cdots + \zeta_n e_n \quad \text{where} \quad \zeta_j = \begin{cases} \xi_j & (j \neq k) \\ \lambda \xi_k & (j = k). \end{cases} \]

We set
\[ J := \det(\zeta_{ij})_{i,j=1,\ldots,n}, \quad \zeta_{ij} := \frac{\partial \zeta_i}{\partial x_j}. \]

If \( J(p) \neq 0 \), it holds that
\[ (5.4) \quad \text{ind}_p(Y) = \text{sgn}(J(p)). \]

Since \( \lambda \) vanishes on \( \Sigma^{n-1} \), the property \([b]\) implies that
\[ (5.5) \quad \lambda(p) = \lambda_{x_2}(p) = \cdots = \lambda_{x_n}(p) = 0. \]

Moreover, since \( d\lambda(p) \neq 0 \) (in fact, \( p \) is a nondegenerate singular point), we have that
\[ (5.6) \quad \lambda_{x_1}(p) \neq 0. \]

Then the equality \( \zeta_k := \lambda \xi_k \) yields that
\[ (5.7) \quad (\zeta_k)_{x_1}(p) = \xi_k(p) \lambda_{x_1}(p) \neq 0, \quad (\zeta_k)_{x_2}(p) = \cdots = (\zeta_k)_{x_n}(p) = 0. \]

Consider the case that \( k \geq 2 \). Then by the property \([iii]\) in Definition 3.4, \( X \) is a tangent vector of \( \Sigma^{n-1} \). Since \( \lambda \) vanishes along \( \Sigma^{n-1} \), we have
\[ 0 = d\lambda(X) = \sum_{i=1}^n \xi_i \lambda_{x_i} \quad \text{on} \quad \Sigma^{n-1}. \]

Since \( \partial/\partial x_2, \ldots, \partial/\partial x_n \) are tangent vectors of \( \Sigma^{n-1} \) at \( p \), (5.5) and (5.1) yield that
\[ 0 = \frac{\partial d\lambda(X)}{\partial x_j} = \sum_{i=1}^n (\xi_{ij} \lambda_{x_i} + \xi_i \lambda_{x_i,x_j}) = \xi_1 \lambda_{x_1} + \xi_k \lambda_{x_k} - \xi_{1j} \lambda_{x_1} + \xi_{kj} \lambda_{x_j} \quad (j = 2, \ldots, n) \]
holds at \( p \), where \( \xi_{ij} := \partial \zeta_i / \partial x_j \) and we used the fact that
\[ \lambda_{x_j,x_k} = d\lambda_{x_j} \left( \frac{\partial}{\partial x_k} \right) = d\lambda_{x_j} (\tilde{\eta}) = \dot{\lambda}_{x_j}. \]

On the other hand, \( \dot{\lambda} \) is identically zero on \( \Sigma^{n-2} \), so \([b]\) implies that
\[ (5.8) \quad \dot{\lambda}_{x_j} = 0 \quad (j \geq 3). \]
Hence we have
\[ \zeta_{ij} = \xi_{ij} = \begin{cases} 0 & \text{if } j \geq 3, \\ -\xi_k \lambda_{x_2}/\lambda_{x_1} & \text{if } j = 2. \end{cases} \]

Now we would like to show that
\[ \lambda^{(m-1)}(p) \neq 0, \quad \zeta_{mj} = \xi_{mj} = \begin{cases} 0 & \text{if } j \geq m + 2, \\ -\xi_k \lambda^{(m)}_{x_{m+1}}/\lambda^{(m-1)}_{x_m} & \text{if } j = m + 1 \end{cases} \]

for \( m = 1, \ldots, k - 1 \), by induction. We have already shown it for \( m = 1 \). We prove the assertion for \( m(\leq k - 1) \), assuming it is true up until \( m - 1 \). Since \( X \) is a tangent vector of \( \Sigma^{n-m} \) and \( \lambda^{(m-1)} \) is constant along \( \Sigma^{n-m} \), we have that
\[ 0 = d\lambda^{(m-1)}(X) = \sum_{i=1}^{n} \xi_i \lambda^{(m-1)}_i \quad \text{on } \Sigma^{n-m}, \]

since \( \lambda^{(m)} \) is identically zero on \( \Sigma^{n-m} \). On the other hand, since \( \lambda^{(m-1)}_{x_{j+1}}(p) \) vanishes for \( j \geq m + 1 \), the fact \( d\lambda^{(m-1)}(p) \neq 0 \) implies that \( \lambda^{(m-1)}_{x_m}(p) \neq 0 \). Differentiating (5.10) and using (5.1), we get the identity
\[ 0 = \frac{\partial d\lambda^{(m-1)}(X)}{\partial x_j} = \sum_{i=1}^{n} \left( \xi_{ij} \lambda^{(m-1)}_{x_i} + \xi_i \lambda^{(m-1)}_{x_{i,j}} \right) = \xi_{mj} \lambda^{(m-1)}_{x_m} + \xi_k \lambda^{(m)}_{x_{m+1}} = 0 \]

for \( j \geq m + 1 \).

Then, the fact \( 0 = \lambda^{(m)}_{x_j} \) \( (j \geq m + 2) \) reduces (5.12) to the following two identities
\[ \xi_{mj} \lambda^{(m-1)}_{x_m} = 0 \quad (j \geq m + 2), \]
\[ \xi_{mj} \lambda^{(m-1)}_{x_m} + \xi_k \lambda^{(m)}_{x_{m+1}} = 0, \]

which proves (5.9). In particular, when \( m = k - 1 \), it holds that
\[ \lambda^{(k-2)}(p) \neq 0, \quad \zeta_{k-1,j} = \xi_{k-1,j} = \begin{cases} 0 & \text{if } j \geq k + 1, \\ -\xi_k \lambda^{(k)}_{x_{k-1}}/\lambda^{(k-1)}_{x_k} & \text{if } j = k, \end{cases} \]

since \( \lambda^{(k-1)}_{x_k} = d\lambda^{(k-1)}(\tilde{y}) = \lambda^{(k)} \).

Now taking (5.2) into the account, we have that
\[
J = \det \left( \begin{array}{cccccccc}
\zeta_{1,1} & \zeta_{1,2} & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\zeta_{k-1,1} & \zeta_{k-1,2} & \cdots & \zeta_{k-1,k-2} & 0 & 0 & \cdots & 0 \\
\zeta_{k,1} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
* & * & \cdots & * & * & \zeta_{k+1,k+1} & \cdots & \zeta_{k+1,n} \\
\cdots & \cdots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots \\
* & * & \cdots & * & * & \zeta_{n,k+1} & \cdots & \zeta_{n,n} \\
\end{array} \right) = (-1)^{k-1} \zeta_{1,1} \zeta_{1,2} \cdots \zeta_{k-1,2} \det \left( \begin{array}{cccc}
\zeta_{k+1,k+1} & \cdots & \zeta_{k+1,n} \\
\vdots & \ddots & \vdots \\
\zeta_{n,k+1} & \cdots & \zeta_{n,n} \\
\end{array} \right). \]
Hence we have
\[
\text{ind}_p Y = \text{sgn}(J(p)) = (-1)^{k-1} \text{sgn}\left(\zeta_{k,1}\zeta_{1,2}\cdots\zeta_{k-1,2}\right) \text{ind}_p (X_{n-k})
\]
\[
= (-1)^{k-1} \text{ind}_p (X_{n-k}) \\
\times \text{sgn}\left(\frac{-\xi_k(p)\lambda_{x_1}(p) - \xi_k(p)\lambda_{x_2}(p)}{\lambda_{x_1}(p)} \cdots - \frac{-\xi_k(p)\lambda^{(k)}(p)}{\lambda_{x_{k-1}}^{(k-1)}(p)}\right)
\]
\[
= \text{ind}_p (X_{n-k}) \text{sgn}\left(\xi_k(p)^k\lambda^{(k)}(p)\right),
\]
which proves the assertion. \(\square\)

6. The four dimensional case

Let \(X\) be a characteristic vector field associated to a Morin homomorphism \(\varphi : TM^4 \to E\) as in the previous section. Take a section \(Y\) of \(E\) as
\(Y := \varphi(X)\).

We denote by \(Z(X)\) and \(Z(Y)\) the set of zeros of \(X\) and \(Y\), respectively. The following assertion can be proved like as in Proposition 4.1.

**Proposition 6.1.** Let \(Z(X_j)\) \((j = 0, 1, 2, 3)\) be the set of zeros for \(X_j\). Then it holds that

\[
\begin{align*}
(6.1) & \quad Z(Y) \cap (M^4 \setminus \Sigma^3) = Z(X), \\
(6.2) & \quad Z(Y) \cap (\Sigma^3 \setminus \Sigma^2) = Z(X_3), \\
(6.3) & \quad Z(Y) \cap (\Sigma^2 \setminus \Sigma^1) = Z(X_2), \\
(6.4) & \quad Z(Y) \cap (\Sigma^1) = Z(X_1), \\
(6.5) & \quad Z(Y) \cap (\Sigma^0) = A_5.
\end{align*}
\]

Like as for Proposition 4.2, one can prove the following assertion:

**Proposition 6.2.** The first term of the right-hand side of (1.4) in the introduction satisfies
\[
\sum_{p \in M^4 \setminus \Sigma^3} \text{ind}_p(Y) = \chi(M^4_+) - \chi(M^4_-).
\]

Now the formula (1.1) for the four dimensional case reduces to the following two propositions:

**Proposition 6.3.** The second and fourth terms of the right-hand side of (1.4) in the introduction satisfy
\[
\begin{align*}
(6.7) & \quad \sum_{p \in \Sigma^3 \setminus \Sigma^1} \text{ind}_p(Y) = 0, \\
(6.8) & \quad \sum_{p \in \Sigma^3 \setminus \Sigma^0} \text{ind}_p(Y) = 0.
\end{align*}
\]

**Proof.** We set \(j = 3\) (resp. \(j = 1\)) and fix a point \(p \in \Sigma^j \setminus \Sigma^{j-1}\) satisfying \(Y_p = 0\) arbitrarily. By the property (iii) in Definition 3.4, there exists a vector field \(X_j\) on
apply Theorem 5.1 by setting

\[ X := \frac{\partial}{\partial x_1} + \xi_2 \frac{\partial}{\partial x_2} + \xi_3 \frac{\partial}{\partial x_3} + \xi_4 \frac{\partial}{\partial x_4} \]

for a local coordinate system \((U; x_1, x_2, x_3, x_4)\) such that \(\eta := \partial / \partial x_{4-j}\) gives a null vector field. In this coordinate system, it holds that

\[ \text{ind}_p(Y) = \text{ind}_p(X_j) \text{ sgn} (\xi_{4-j}(p)^{4-j}) \text{ sgn} (\lambda^{(4-j)}(p)). \]

By \([\text{iv}]\) of Definition 3.4 it holds that

\[ \xi_k(p) = ds^2(X_p, n_j(p)) \geq 0. \]

Since \(k\) is odd and \(n_3\) is positively proportional to \(\lambda^{(k)}(p)\), \([\text{iv}]\) of Definition 3.4 yields that

\[ 0 \leq \text{sgn}(ds^2(\lambda^{(4-j)}(p)\eta_p, n_3(p))) = \text{sgn}(\lambda^{(4-j)}(p)). \]

Thus

\[ \text{ind}_p(Y) = \text{ind}_p(X_j) \]

holds. Since \(Z(X_j) = Z(Y) \cap (\Sigma^j \setminus \Sigma^{j-1})\) and \(\Sigma^j\) is odd dimensional, it holds that

\[ \sum_{p \in \Sigma^j \backslash \Sigma^{j-1}} \text{ind}_p(Y) = \sum_{p \in \Sigma^j} \text{ind}_p(X_j) = \chi(\Sigma^j) = 0. \]

\[ \square \]

**Proposition 6.4.** The third and fifth terms of the right-hand side of (1.4) in the introduction satisfy

\[ (6.9) \quad \sum_{p \in \Sigma^j \backslash \Sigma^1} \text{ind}_p(Y) = \chi(A^+_3) - \chi(A^-_3), \]

\[ (6.10) \quad \sum_{p \in \Sigma^0} \text{ind}_p(Y) = \chi(A^+_3) - \chi(A^-_3) = \#A^+_3 - \#A^-_3. \]

**Proof.** We set \(j = 2\) (resp. \(j = 0\)) and fix a point \(p \in \Sigma^j \setminus \Sigma^{j-1}\) satisfying \(Y_p = 0\) arbitrarily. By the property \([\text{iii}]\) of the characteristic vector field, there exists a vector field \(X_j\) on \(\Sigma^j\) such that \(Z(X_j) = Z(Y) \cap (\Sigma^j \setminus \Sigma^{j-1})\). Setting \(k = 0\) (resp. \(k = 2\)), we can apply Theorem 5.1 on putting

\[ X := \frac{\partial}{\partial x_1} + \xi_2 \frac{\partial}{\partial x_2} + \xi_3 \frac{\partial}{\partial x_3} + \xi_4 \frac{\partial}{\partial x_4} \]

for a local coordinate system \((U; x_1, x_2, x_3, x_4)\) such that \(\eta := \partial / \partial x_{4-j}\) gives a null vector field. Since \(j\) is even, it holds that

\[ \text{ind}_p(Y) = \text{ind}_p(X_j) \text{ sgn}(\xi_{4-j}(p))^{4-j} \text{ sgn} \lambda^{(4-j)}(p) \]

\[ = \text{ind}_p(X_j) \text{ sgn} \lambda^{(4-j)}(p) = \begin{cases} \text{ind}_p(X_j) & \text{if } p \in \Sigma^+_3, \\ -\text{ind}_p(X_j) & \text{if } p \in \Sigma^-_3. \end{cases} \]
We denote by $\tilde{\Sigma}_+^j(\delta)$ and $\tilde{\Sigma}_-^j(\delta)$ the closures of the sets $\Sigma_+^j \setminus N_\delta(\Sigma_-^{j-1})$ and $\Sigma_-^j \setminus N_\delta(\Sigma_-^{j-1})$, respectively. By (iv), $X_j$ points towards $\tilde{\Sigma}_+^j$. Since $j$ is even, the Poincaré-Hopf formula yields that

$$\chi(\tilde{\Sigma}_+^j) = \chi(\tilde{\Sigma}_+^j(\delta)) = \sum_{p \in \tilde{\Sigma}_+^j(\delta)} \text{ind}_p(\mathbf{-X}_j) = \sum_{p \in \tilde{\Sigma}_+^j(\delta)} \text{ind}_p(X_j) = \sum_{p \in \tilde{\Sigma}_+^j} \text{ind}_p(Y).$$

Similarly, we have that

$$\chi(\tilde{\Sigma}_-^j) = \chi(\tilde{\Sigma}_-^j(\delta)) = \sum_{p \in \tilde{\Sigma}_-^j(\delta)} \text{ind}_p(X_j) = -\sum_{p \in \tilde{\Sigma}_-^j(\delta)} \text{ind}_p(Y) = -\sum_{p \in \tilde{\Sigma}_-^j} \text{ind}_p(Y),$$

which proves the assertion.

Remark 6.5. When $M^n$ is odd dimensional, that is, $n = 2m - 1$, one can prove

$$\sum_{p \in \Sigma_-^{2i+1}} \text{ind}_p(Y) = \sum_{p \in \Sigma_-^{2i+1}} \text{ind}_p(-X) = -\text{ind}(\Sigma_-^{2i+1})$$

and

$$\sum_{p \in \Sigma_-^{2i+1}} \text{ind}_p(Y) = -\sum_{p \in \Sigma_-^{2i+1}} \text{ind}_p(X) = -\text{ind}(\Sigma_-^{2i+1})$$

like as in Proposition 6.3, since $X_{2i+1}$ is inward for $\Sigma_+^{2i+1}$ and outward for $\Sigma_-^{2i+1}$. Then we have

(6.11) \[\sum_{p \in \Sigma_-^{2i+1} \setminus \Sigma_-^i} \text{ind}_p(X) = -\chi(\Sigma_-^{2i+1}) - \chi(\Sigma_-^i) = \chi(\Sigma_-^{2i+1}) - \chi(\Sigma_-^i) = -\chi(\Sigma_-^{2i}).\]

On the other hand, we have

(6.12) \[\sum_{p \in \Sigma_-^{2i-1} \setminus \Sigma_-^{2i-1}} \text{ind}_p(Y) = \chi(\Sigma_-^{2i})\]

like as in Proposition 6.3. Hence (6.1) splits into $n$-sets of trivial identities. This is the reason why we restricted our attention to even dimensional case up to now.

7. Applications

In this section, we shall give several applications of our formula: Recall that a map $f : M^{2m} \to N^{2m}$ between $2m$-manifolds is called a Morin map if the corresponding bundle homomorphism $\varphi = df$ as in Example 2.6 admits only $A$-singularity for $k = 2, \ldots, 2m + 1$ (cf. Remark 2.3).

Theorem 7.1 ([9] and [13]). Let $M^{2m}$ and $N^{2m}$ be compact oriented $2m$-manifolds, and let $f : M^{2m} \to N^{2m}$ be a Morin map. Then it holds that

(7.1) \[\deg(f) \chi(N^{2m}) = \chi(M^{2m}_+) - \chi(M^{2m}_-) + \sum_{j=1}^m \chi(A_{2j+1}^+) - \chi(A_{2j+1}^-),\]

where $\deg(f)$ is the topological degree of the map $f$ and $M^{2m}_+$ (resp. $M^{2m}_-$) is the set of points at which the Jacobian of $f$ is positive (resp. negative).

This formula is a generalization of Quine’s formula [10] for Morin maps between 2-manifolds. It should be remarked that the numbering of Morin singularities are different from the usual one, see Remark 2.3. For example, fold (resp. cusp) singularity is an $A_2$-singularity (resp. an $A_3$-singularity) in (7.1).
Proof of Theorem 7.1. Let $E$ be the pull-back of the tangent bundle $TN^{2m}$ of $N^{2m}$ by $f$. Then the map $f$ induces a bundle-homomorphism $\varphi_f := df : TM^{2m} \to E$ as in Example 2.6. Since $f$ is a Morin map, $\varphi_f$ has only $A_k$-singular points, and then the formula follows from (1.1) using the fact that $\chi_E = \text{deg}(f)\chi(N^{2m})$. □

Remark 7.2. When $m = 2$, the stable singularities of $C^\infty$ maps between 4-manifolds consist of $A_k$-singular points, elliptic and hyperbolic umbilic points. Then a generalization of (7.1) for stable mapping $f$ is given in [4, Corollary 5.13] as follows:

\[(7.2) \quad (\text{deg } f)\chi(N^4) = \chi(M^4_+) - \chi(M^4_-) + \chi(A^+_3) - \chi(A^-_3) + \#H^+ - \#H^-,
\]

where $\#H^+$ (resp. $\#H^-$) is the number of positive (resp. negative) hyperbolic umbilic points. In fact, elliptic umbilic points do not affect the formula (7.2).

Next we give applications for immersed hypersurfaces in $R^{2m+1}$. Let $M^{2m}$ be a compact oriented $2m$-manifold and $f : M^{2m} \to R^{2m+1}$ a wave front. Suppose that there exists a globally defined unit normal vector field $\nu$ along $f$. Then it induces the Gauss map

\[(7.3) \quad \hat{\nu} : M^{2m} \longrightarrow S^{2m},
\]

and a family of wave fronts

\[f_t := f + t\hat{\nu} \quad (t \in R),
\]

each of which is called a parallel hypersurface of $f$. The Gauss map of $f_t$ is equal to $\hat{\nu}$.

Corollary 7.3. Let $M^{2m}$ be a compact oriented $2m$-manifold and $f : M^{2m} \to R^{2m+1}$ an immersion. Suppose that the Gauss map $\hat{\nu}$ is a Morin map. Then the singular set of $\hat{\nu}$ satisfies the identity (1.2) in the introduction, where $M^{2m}_-$ is the set of points at which the Gauss-Kronecker curvature of $f$ (i.e. the determinant of the shape operator) is negative.

This formula is a generalization of the Bleeker-Wilson formula [3] for Gauss maps of immersed surfaces in $R^{2m+1}$.

Proof of Corollary 7.3. We apply the formula (1.1) for the Gauss map $\hat{\nu}$ of the immersion $f$. Then we have that

\[2(\text{deg } \hat{\nu}) = \chi(M^{2m}_+) - \chi(M^{2m}_-) + \sum_{j=1}^{m} \left( \chi(A^{+}_{2j+1}) - \chi(A^{-}_{2j+1}) \right).
\]

Since $f$ is an immersion, it is well-known that $2(\text{deg } \nu)$ is equal to $\chi(M^{2m})$.

Next, we show that $M^{2m}_+$ (resp. $M^{2m}_-$) coincide with the set where the Gauss-Kronecker curvature is positive (resp. negative): Let $ds^2$ be the induced Riemannian metric on $M^{2m}$ by the immersion $f$, and let $e_1, \ldots, e_{2m}$ be an oriented local orthonormal frame field on $M^m$ such that

\[d\nu(e_j) = -\mu_j df(e_j) \quad (j = 1, \ldots, 2m),
\]
that is, $e_1, \ldots, e_{2m}$ are eigenvector fields of the shape operator of $f$, and $\mu_1, \ldots, \mu_{2m}$ are principal curvatures. Then we have that

$$
\lambda := \det (d\hat{\nu}(e_1), \ldots, d\hat{\nu}(e_{2m}), \hat{\nu}) = \prod_{j=1}^{2m} \mu_j = K,
$$

where $K := \mu_1 \cdots \mu_{2m}$ is the Gauss-Kronecker curvature of $f$. This $\lambda$ is positive (resp. negative) if and only if $K > 0$ (resp. $K < 0$), which proves the assertion. □

**Remark 7.4.** Applying to (7.2) the same argument, the statement for $m = 2$ of Corollary 7.3 is improved as follows: Let $f : M^4 \to R^5$ be an immersion. Suppose that the Gauss map $\hat{\nu}$ is a stable map. Then the singular set of $\hat{\nu}$ satisfies the identity

$$
2\chi(M^4) = \chi(S^2) - \chi(S^2) + \#A^+ + \#H^- - \#A^- + 2(\#H^- - \#H^+),
$$

where $\#H^+$ (resp. $\#H^-$) is the number of positive (resp. negative) hyperbolic umbilic points of $\hat{\nu}$.

**Proof of Theorem 1.1.** We apply our formula (1.1) for the bundle homomorphism $\varphi_f := df : TM^{2m} \to E$ as in Example 2.7. Then it is sufficient to show that $\chi_E$ is equal to $2 \deg(\nu)$. Let $\xi$ be a vector field on the 2$m$-sphere $S^{2m}$. By a parallel transport, $\xi(q) (q \in S^{2m})$ can be considered as a vectors in $E_p$ for $p \in \hat{\nu}^{-1}(q)$. Thus, $\xi$ induces a section $\tilde{\xi}$ of $E$ defined on $M^{2m}$. Then

$$
\chi_E = \sum_{p \in M^{2m}} \text{ind}_p(\tilde{\xi}) = \deg(\nu) \sum_{p \in S^{2m}} \text{ind}_p(\xi) = \deg(\nu) \chi(S^{2m}) = 2 \deg(\nu)
$$

holds, which proves the identity. □

Next, we give an application for parallel hypersurfaces for strictly convex hypersurfaces.

**Theorem 7.5.** Let $S^{2m}$ be the 2$m$-sphere and

$$
f : S^{2m} \to R^{2m+1}
$$

be a strictly convex immersion, that is, the Gauss map $\hat{\nu} : S^{2m} \to S^{2m}$ is a diffeomorphism. Let $t \in R$ be a value such that the parallel hypersurface

$$
f_t : S^{2m} \to R^{2m+1}
$$

has only $A_k$-singularities ($k = 2, \cdots, 2m + 1$). Then the singular set of $f_t$ satisfies the identity (1.2) and $1/K_t$ is a $\varphi_t$-function for $\varphi_t = df_t$ (cf. Definition 2.1), where $K_t$ is the Gauss-Kronecker curvature of $f_t$.

The corresponding assertion for a convex surface $f : S^2 \to R^3$ is given by Martinez-Maure [7] under the generic assumption that the Gaussian curvature is unbounded at the singular set of $f_t$, and proved in [17] for the general case. The above formula is a generalization of it.

**Proof of Theorem 7.5.** We apply Theorem 1.1 for the bundle homomorphism $\varphi_t = df_t : TS^{2m} \to E_{f_t}$. Since $f$ is convex, the Gauss map $\hat{\nu} : S^{2m} \to S^{2m}$ is of degree
On the other hand, family \( \{ K \} \) of the Gauss-Kronecker curvature implies that \( K \) one. Since \( M \) is an immersion, one can take the Riemannian metric \( d\sigma \) as in the statement of Theorem 1.1. Moreover, since \( \hat{\nu} \) is an immersion, one can take the Riemannian metric \( d\sigma^2 \) on \( S^{2m} \) as the pull-back of the canonical metric of \( S^{2m} \) by \( \hat{\nu} \), and let \( \{ e_1, \ldots, e_{2m} \} \) be an oriented local orthonormal frame field on \( S^{2m} \) with respect to \( d\sigma^2 \) such that

\[
df(e_j) = -(1/\mu_j)d\nu(e_j) \quad (j = 1, \ldots, 2m),
\]

that is, \( e_1, \ldots, e_{2m} \) are eigenvector fields of the shape operator of \( f \). Since

\[
df_t(e_j) = df(e_j) + t\,d\hat{\nu}(e_j) = -\left( \frac{1}{\mu_j} - t \right) d\hat{\nu}(e_j),
\]

the Gauss-Kronecker curvature \( K_t \) of \( f_t \) is expressed as

\[
K_t = \left( \prod_{j=1}^{2m} \left( \frac{1}{\mu_j} - t \right) \right)^{-1}.
\]

On the other hand,

\[
\lambda_t := \det(df_t(e_1), \ldots, df_t(e_{2m}), \hat{\nu})
\]

\[
= \left( \prod_{j=1}^{2m} \left( \frac{1}{\mu_j} - t \right) \right) \det(d\hat{\nu}(e_1), \ldots, d\hat{\nu}(e_{2m}), \hat{\nu})
= \frac{1}{K_t}
\]

which implies that \( K_t \) is a \( \varphi_t \)-function, where \( \varphi_t = df_t \).

Now we consider the singularities of vector fields on \( M^{2m} \). Let \( D \) be an arbitrary linear connection on \( M^{2m} \) and \( X \) a vector field defined on \( M^{2m} \). One can apply the formula (1.1) for the bundle homomorphism

\[
\varphi_X : TM^{2m} \ni v \mapsto D_vX \in TM^{2m}
\]

if \( \varphi_f \) admits only \( A_k \)-singularities and get the formula (1.2), where \( M^2 \) is the set of points where

\[
(D_{v_1}X, \ldots, D_{v_{2m}}X)
\]

forms a positive frame for a given locally defined positive frame \( v_1, \ldots, v_{2m} \) on \( T_pM^{2m} \). In [10], this map was introduced on a Riemannian 2-manifold, and we called the singular points of \( \varphi_X \) the irrotational points there. However, it would be better to call them the \( A_k \)-singular points of the vector field with respect to the connection \( D \). In fact, the singular set of \( \varphi_X \) has no relation to the rotations of the vector fields.

At the end of this section, we give an application for the Blaschke normal maps for strictly convex hypersurfaces: We fix a strictly convex immersion

\[
f : S^{2m} \rightarrow R^{2m+1}.
\]
Then there exists a unique vector field $\xi$ along $f$ satisfying the following two properties, which is called the affine normal vector field:

1. the linear map
   
   $$S : TS^{2m} \ni v \mapsto D_v \xi$$
   
   gives an endomorphism on $TS^{2m}$, that is, $S(v) := D_v \xi$ is tangent to $f(S^{2m})$ for each $v$, where $D$ is the canonical affine connection on $\mathbb{R}^{2m+1}$,

2. there exists a unique covariant symmetric tensor $h$ such that
   
   $$D_x df(Y) - h(X, Y)\xi$$
   
   gives a tangential vector field on $f(S^{2m})$ for any vector fields $X_1, \ldots, X_{2m}$ on $S^{2m}$. Since $f$ is strictly convex, $h$ is positive definite. Then the $2m$-form $\Omega$ defined by
   
   $$\Omega(X_1, \ldots, X_{2m}) := \det(df(X_1), \ldots, df(X_{2m}), \xi)$$
   
   coincides with the volume element associated to $h$, where $X_1, \ldots, X_{2m}$ are vector fields on $S^{2m}$ and “det” denotes the canonical volume form of $\mathbb{R}^{2m+1}$.

The vector field $\xi$ induces a map

$$\hat{\xi} : S^{2m} \ni p \mapsto \xi_p \in \mathbb{R}^{2m+1},$$

which is called the Blaschke normal map of $f$. The following assertion holds like as in the case of $m = 1$ (cf. [16] Lemme 3.1).

**Lemma 7.6.** The Blaschke normal map $\hat{\xi}$ gives a wave front.

**Proof.** Consider a non-zero section

$$L : S^{2m} \ni p \mapsto (\hat{\xi}_p, \nu_p) \in T^* \mathbb{R}^{2m+1} = \mathbb{R}^{2m+1} \times (\mathbb{R}^{2m+1})^*,$$

where $(\mathbb{R}^{2m+1})^*$ is the dual vector space of $\mathbb{R}^{2m+1}$, and $\nu : S^{2m} \to (\mathbb{R}^{2m+1})^*$ is the map defined by

$$\nu_p(\hat{\xi}_p) = 1, \quad \nu_p(df(T_p S^{2m})) = \{0\} \quad (p \in S^{2m}),$$

which is called the conormal map of $f$. By definition, $L$ induces an isotropic map of $S^{2m}$ into the projective cotangent bundle $P(T^* \mathbb{R}^{2m+1}) = \mathbb{R}^{2m+1} \times P^*(\mathbb{R}^{2m+1})$ with the canonical contact structure. Take a local coordinate system $(x_1, \ldots, x_{2m})$ of $S^{2m}$. Then we have that

$$\nu_{x_i}(f_{x_j}) = (D_{\partial/\partial x_i} \nu)(f_{x_j}) = \frac{\partial}{\partial x_i} \nu(f_{x_j}) - \nu(D_{\partial/\partial x_i} f_{x_j}),$$

$$= -\nu(D_{\partial/\partial x_i} f_{x_j}) - h \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) + h \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right),$$

$$= h \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) \quad (i, j = 1, \ldots, 2m).$$

Since $h$ is positive definite, one can show that $\nu_{x_1}, \ldots, \nu_{x_{2m}}$ are linearly independent. Moreover, since $\nu(T_p S^{2m}) = \{0\}$ for each $p \in S^{2m}$, $\nu, \nu_{x_1}, \ldots, \nu_{x_{2m}}$ is linearly independent. In particular, the map $L$ induces a Legendrian immersion, which proves the assertion. □
If the singular points of $\hat{\xi}$ consist only of $A_k$-singular points ($2 \leq k \leq 2m + 1$), the affine shape operator

$$S: TS^{2m} \ni v \mapsto -\nabla Dv \xi \in f^*T\mathbb{R}^{2m+1}$$

gives a Morin homomorphism. Applying (1.1) for $S$, we get Theorem 1.2.

Finally, we give an example which demonstrates Theorem 1.2: Consider a plane curve $\gamma(t) = (1 - 2\epsilon \sin t)(\sin t \cos t)$ ($-\pi/2 \leq t \leq \pi/2$), which lies on the upper-half plane and gives a convex curve if $0 \leq \epsilon < 1/4$. Rotate it along the horizontal axis, we get a rotationally symmetric strictly convex surface in $\mathbb{R}^3$. The left hand side of Figure 1 indicates the curve $\gamma$ for $\epsilon = 17/80$, and the right hand side of Figure 1 gives the profile curve of the Blaschke normal map $\xi$ of the surface for $\epsilon = 17/80$. As shown in Figure 1 (right), $\xi$ has no swallowtails (i.e. it has no $A_3$-singularities), and our formula implies that the Euler number $\chi(M_-)$ vanishes. In fact, the set $\xi(M_-)$ gives a cylindrical strip if one rotate the profile curve of $\xi$ along the horizontal axis.

**Appendix A. Extension of generic vector fields**

We prove the following assertion, which is needed to prove the existence of a characteristic vector field associated to a given Morin homomorphism:

**Lemma A.1.** Let $M^n$ be a compact manifold and $X$ a $C^\infty$-vector field defined on an open subset of $M^n$ containing a compact subset $K$ such that $X$ has no zeros on the boundary $\partial K$ of $K$. Then there exists a $C^\infty$-vector field $\hat{X}$ defined on $M^n$ such that $\hat{X}$ coincides with $X$ on $K$ and has only generic zeros on $M^n \setminus K$.

**Proof.** We may assume that $X$ is defined on a neighborhood $U$ of $K$. Take an open subset $V$ such that

$$K \subset V \subset \bar{V} \subset U,$$

where $\bar{V}$ is the closure of $V$. Taking $U$ sufficiently close to $K$, we may assume that $X$ has no zeros on $U \setminus K^0$, where $K^0$ (possibly empty) is the set of the interior points of $K$. Then we can take $C^\infty$-functions $\rho_j : M^n \to [0,1]$ $(j = 1,2)$ such that $\rho_1 = 1$ on $K$ (resp. $\rho_2 = 1$ on $\bar{V}$) and $\rho_1 = 0$ on $M^n \setminus V$ (resp. $\rho_2 = 1$ on $M \setminus U$). We set $\hat{X} := \rho_2 X$, which is a vector field on $M^n$. It is well-known that there exists a
sequence of generic vector fields \( \{ Z_j \}_{j=1,2,3,...} \) on \( M^n \) converging to \( \hat{X} \) with respect to the Whitney \( C^\infty \)-topology. We set

\[
\tilde{X}_j := \rho_1 \hat{X} + (1 - \rho_1) Z_j.
\]

Then \( \tilde{X}_j \) coincides with \( X \) on \( K \), because \( \rho_1 = \rho_2 = 1 \) on \( K \). Since \( \hat{X} \) has no zeros on the compact set \( \tilde{V} \setminus K^n \), \( \tilde{X}_j \) has a zero on \( p \in \tilde{V} \setminus K^n \) if \( \hat{X} = -\frac{1 - \rho_1}{\rho_1} Z_j \) holds at \( p \). This is impossible for sufficient large \( j \), since \( Z_j \to \hat{X} \) as \( j \to \infty \) and \( \rho_1 \in [0,1] \). Moreover, \( \tilde{X}_j \) coincides with \( Z_j \) on \( M^n \setminus V \), since \( \rho_1 = 0 \) on the complement of \( V \). Thus it has only generic zeros on \( M^n \setminus V \). In particular \( \tilde{X}_j \) has the desired property for sufficiently large \( j \). \( \square \)

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