On Cosmological Spacetime Structure and Symmetry: Manifold as a Lie Group, Spinor Structure and Symmetry Group, Minkowski Metric, and Unnecessariness of Double-Valued Representations

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Abstract

It is shown that cosmological spacetime manifold has the structure of a Lie group and a spinor space. This leads naturally to the Minkowski metric on tangent spaces and the Lorentzian metric on the manifold and makes it possible to dispense with double-valued representations.

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Introduction and Synopsis

The concept of spacetime symmetry plays a fundamental role in physics. In particular, the classification of elementary particles is based on irreducible representations of symmetry groups. When treating the concept, it is necessary to distinguish between the symmetry of spacetime manifold, $M$, per se and that of the manifold equipped with a metric, $(M, g)$.

In General Relativity (GR), which is a local theory, $M = M^4$ is a smooth four-dimensional manifold, and no specific global structure of $M$ is considered. Metric $g$, being a dynamical object, is not fixed. Therefore the symmetry group both of $M$ and of $(M, g)$ is that of diffeomorphisms of $M$.

A conventional approach to the particle classification is based on the Poincaré group and its subgroups $SO(3, 1)^\dagger = L^1_+$ (the restricted Lorentz group) and $SO(3)$ (the group of rotations in three-dimensional Euclidean space). In GR, the groups $SO(3)$ and $SO(3, 1)^\dagger$ figure as symmetry groups of the tangent space $M_p$ at a point $p \in M$ equipped with the Minkowski metric $\eta = \text{diag}(-1, 1, 1, 1)$, which is metric $g$ in inertial coordinates. Thus it is assumed that $g$ is a Lorentzian metric, i.e., a metric of signature $(-1, 1, 1, 1)$.

But the true (or proper) representations of $SO(3)$ and $SO(3, 1)^\dagger$ do not include half-integer representations, which are realized in nature. The situation is revised by introducing double-valued representations of $SO(3)$ and $SO(3, 1)^\dagger$ or by making use of the universal covering groups $[1] SU(2)$ and $SL(2, \mathbb{C})$ of $SO(3)$ and $SO(3, 1)^\dagger$, respectively, and their true representations. Another treatment in GR is based on a spin structure for $M$.

Now, $SO(3)$ and $SO(3, 1)^\dagger$ are symmetry groups of space and spacetime of Special Relativity (SR) rather than of GR. As to $SU(2)$ and $SL(2, \mathbb{C})$, they bear no relation to SR space and spacetime per se. Therefore it is natural to raise the problem of the symmetry of GR space and spacetime. Since metric is not fixed, the problem concerns the symmetry of spacetime manifold $M$.

We start from the cosmological spacetime manifold $[2]$, $M^\text{cosm} = T^\text{cosm} \times S^\text{cosm}$ where $T^\text{cosm}$ is cosmological time and $S^\text{cosm} = S^3$ (three-sphere) is cosmological space. A crucial point is that $S^3$ is diffeomorphic to $SU(2)$ $[3]$, so we put $S^\text{cosm} = SU(2)$. A symmetry group of $S^\text{cosm}$ is $SU(2)$ itself, which warrants the choice of a closed space.

The tangent space $M^\text{cosm}_p$, $p \in M^\text{cosm}$, is the direct sum of the tangent spaces $T^\text{cosm}_t$ and $S^\text{cosm}_s$, $M^\text{cosm}_p = T^\text{cosm}_t \oplus S^\text{cosm}_s$, $p = (t, s)$, $t \in T^\text{cosm}$, $s \in S^\text{cosm}$. We have $S^\text{cosm}_s = su(2)$ [the Lie algebra of the Lie group $SU(2)$]. A basis for $su(2)$ is $\{ \tau_j = i\sigma_j : j = 1, 2, 3 \}$ where $\sigma_j$ are the Pauli matrices. The $\tau_j$ are anti-Hermitian, $\tau_j^\dagger = -\tau_j$. A basis for $T^\text{cosm}_t$ is $cI_1$, $c \in \mathbb{C}$, so that a basis for $M^\text{cosm}_p$ is $\{ \tau_\mu : \mu = 0, 1, 2, 3 \}$, $\tau_0 = cI_2$. We invoke the invariance of the basis choice to put $c = i$, so that $\tau_0^\dagger = -\tau_0$, $\tau_0 = i\sigma_0$, $\sigma_0 = I_2$, and $\tau_\mu = i\sigma_\mu$. Thus $M^\text{cosm}_p = u(1) \oplus su(2)$ where $u(1)$ is the Lie algebra of the Lie group $U(1)$.

We have $T^\text{cosm} = \mathbb{R}$ with the group $\mathbb{R} = \{ \text{real, } + \}$ and a universal covering homomorphism $\mathbb{R} \to U(1)$ $[4]$. Put $t = (T_0/2\pi)\theta$, so that $\mathbb{R} = \{ \theta, + \}$, $\theta \mapsto e^{i\theta}$; $M^\text{cosm} = \sum_{m=-\infty}^m M^\text{cosm}(n)$, $M^\text{cosm}(n) = U(1) \otimes SU(2)$, $M^\text{cosm}(n) \leftrightarrow \theta_n = \bar{\theta} + 2\pi n$, $0 \leq \bar{\theta} < 2\pi$. Thus we have arrived at the oscillating universe.

The basis $\{ \tau_\mu = i\sigma_\mu : \mu = 0, 1, 2, 3 \}$, leads in a standard way $[1], [5]$ to the Minkowski metric on $M^\text{cosm}_p$ and, hence, to the Lorentzian metric on $M^\text{cosm}$.

The $SU(2)$ group structure of $S^\text{cosm}$ gives rise to a spinor structure for the latter. (That structure should not be confused with the spin structure $[3], [5]$.) We have $SU(2) \ni a_{SU} \leftrightarrow$
z := (z^1, z^2)^{tr} \in Z_1 where \text{tr} stands for transpose, |z| := (|z^1|^2 + |z^2|^2)^{1/2} = 1, and Z_1 is the space of the spinors of three-dimensional space \cite{6}. Next we introduce the space of the spinors of four-dimensional space \cite{6} \( Z_+ = \{ \zeta = (\zeta^1, \zeta^2, \zeta^3, \zeta^4)^{tr} | |\zeta| > 0 \} \) and put \( \zeta \leftrightarrow (|\zeta|)^{-1}, z = \zeta/|\zeta| \in Z_1, \ |z| = e^{\theta} \). Thus \( Z_+ \ni \zeta \leftrightarrow (\theta, z) \leftrightarrow (t, s) = p \in M^{\text{cosm}}, \) so that \( M^{\text{cosm}} \) has the structure of the spinor space \( Z_+ \).

Consider operations of the group \( GL(2, \mathbb{C}) \) on \( Z_+ \). We have \( GL(2, \mathbb{C}) \ni a_{GL} = re^{i\varphi}a_{SL}, \ a_{SL} \in SL(2, \mathbb{C}) \). Operations of \( \{ e^{i\varphi}I_2 : \varphi \in \mathbb{R} \} \) are reduced to those of \( SU(2) \). Taking this into account, we introduce the group \( PL(2, \mathbb{C}) = \{ a_{GL} \in GL(2, \mathbb{C}) : \det a_{GL} = 1 \} \), \( PL(2, \mathbb{C}) \ni a_{PL} = ra_{SL}, \ r = (rt a_{PL})^{1/2} \). We have \( a_{SL} = a_{SU}e^{h}, \ h^\dagger = h, \ Trh = 0, \) so that \( a_{PL} = ra_{SL} = ra_{SU}e^{h} \); \( SU(2) \subset SL(2, \mathbb{C}) \subset PL(2, \mathbb{C}) \supset \{ rI_2 \} \).

Let us summarize: A symmetry group of \( M^{\text{cosm}} \) is \( PL(2, \mathbb{C}) \): \( \zeta \leftrightarrow \zeta' = a_{PL}\zeta; \ rI_2 \): time transformation (time shift), \( \theta \mapsto \theta, \ z' = z; \ a_{SU} \): space transformation (space rotation), \( \theta' = \theta, \ z \mapsto z' \); \( e^{h} \): spacetime transformation (boost), \( \theta \mapsto \theta', \ z \mapsto z' \); \( a_{SL} \): boost and space rotation.

Now that it is \( SU(2) \) and \( SL(2, \mathbb{C}) \), not \( SO(3) \) and \( SO(3, 1)^\dagger \), that are symmetry groups of spacetime manifold, there is no need for double-valued representations.

1 Spacetime, symmetry groups, and double-valued representations: A conventional treatment

1.1 Euclidean space: \( SO(3) \) and \( SU(2) \)

The Euclidean space has \( SO(3) \) as its symmetry group. But the true representations of \( SO(3) \) do not include half-integer representations. In order to remedy the situation it is necessary to introduce the so-called double-valued representations of \( SO(3) \) or to make use of the representations of the universal covering group \( SU(2) \) \cite{1}, \cite{3}.

A homomorphism from \( SU(2) \) to \( SO(3) \) is constructed as follows \cite{3}. Introduce a standard basis for the Lie algebra \( su(2) \) of the Lie group \( SU(2) \):

\[ \{ \tau_j = i\sigma_j : j = 1, 2, 3 \}, \quad \tau_j^\dagger = -\tau_j, \quad \text{Tr} \tau_j = 0 \]  \hspace{1cm} (1.1.1)

where the \( \sigma_j \) are the Pauli matrices:

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]  \hspace{1cm} (1.1.2)

Let \( \vec{x} = \{ x^j : j = 1, 2, 3 \}, \ x_j = x^j, \) be a vector of the Euclidean space. Introduce the matrix

\[ x_{su} = x^j \tau_j = i \begin{pmatrix} x^3 & x^1 - ix^2 \\ x^1 + ix^2 & -x^3 \end{pmatrix}, \ x_{su} \in su(2), \ x_{su} \leftrightarrow \{ x^j \} \]  \hspace{1cm} (1.1.3)

We have

\[ x^j y_j = \frac{1}{2} [\det(x_{su} + y_{su}) - \det x_{su} - \det y_{su}], \ x^j x_j = \det x_{su} \]  \hspace{1cm} (1.1.4)

Consider a transformation

\[ \vec{x} \rightarrow \vec{x}' \]  \hspace{1cm} (1.1.5)
generated by a transformation

\[ x_{su} \mapsto x'_{su} = a_{SU}x_{su}a_{SU}^\dagger, \quad a_{SU} \in SU(2), \quad x'_{su} \in su(2) \]  

(1.1.6)

Since

\[ \det x'_{su} = \det x_{su}, \quad x'^{j}y_{j} = x^{j}y_{j} \]  

(1.1.7)

we have

\[ \overrightarrow{x} = a_{SO} \overrightarrow{x}, \quad a_{SO} \in SO(3), \quad \pm a_{SU} \leftrightarrow a_{SO} \]  

(1.1.8)

which gives rise to the double-valued representations of \( SO(3) \). In particular, a spinor of three-dimensional space is transformed into its negative when that space undergoes one complete rotation.

Consider [6]

\[ a_{SU}(\varphi) = \left( \begin{array}{cc} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2} \end{array} \right), \quad a_{SO}(\varphi) = \left( \begin{array}{ccc} \cos \varphi & -\sin \varphi & 0 \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{array} \right) \]  

(1.1.9)

with

\[ \pm a_{SU}(\varphi) \leftrightarrow a_{SO}(\varphi) \]  

(1.1.10)

We have

\[ a_{SU}(\varphi + 2\pi) = -a_{SU}(\varphi), \quad a_{SO}(\varphi + 2\pi) = a_{SO}(\varphi) \]  

(1.1.11)

so that

\[ (a_{SU}(\varphi), a_{SU}(\varphi + 2\pi)) \leftrightarrow a_{SO}(\varphi) \]  

(1.1.12)

In particular,

\[ a_{SO}(2\pi) = a_{SO}(0) = I_{3}, \quad a_{SU}(2\pi) = -a_{SU}(0) = -I_{2}, \quad a_{SU}(4\pi) = a_{SU}(0) = I_{2} \]  

(1.1.13)

1.2 Minkowski spacetime: \( SO(3,1)^\dagger \) and \( SL(2,\mathbb{C}) \)

In the case of the Minkowski spacetime, the situation is similar to that of the Euclidean space. A symmetry group is \( SO(3,1)^\dagger = L^+ \) (the restricted Lorentz group), the universal covering group is \( SL(2,\mathbb{C}) \). A homomorphism from \( SL(2,\mathbb{C}) \) to \( SO(3,1)^\dagger \) is constructed as follows [1], [5]. Introduce a standard basis for the Lie algebra \( u(2) \):

\[ \{ \tau_{\mu} = i\sigma_{\mu} : \mu = 0, 1, 2, 3 \}, \quad \tau_{\mu}^\dagger = -\tau_{\mu} \]  

(1.2.1)

\[ \sigma_{0} = I_{2} = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \]  

(1.2.2)

Let \( x = \{ x^{\mu} \}, \ x_{0} = -x^{0}, \ x_{j} = x^{j}, \) be a vector of the Minkowski spacetime. Introduce the matrix

\[ x_{u} = x^{\mu}\tau_{\mu} = i \left( \begin{array}{ccc} x^{0} + x^{3} & x^{1} - ix^{2} \\ x^{1} + ix^{2} & x^{0} - x^{3} \end{array} \right), \quad x_{u} \in u(2), \quad x_{u} \leftrightarrow x \]  

(1.2.3)

We have

\[ x^{\mu}y_{\mu} = \frac{1}{2}[\det(x_{u} + y_{u}) - \det x_{u} - \det y_{u}], \quad x^{\mu}x_{\mu} = \det x_{u} \]  

(1.2.4)
Consider a transformation
\[ x \mapsto x' \] (1.2.5)
generated by a transformation
\[ x_u \mapsto x_u' = a_{SL} x_u a_{SL}^\dagger, \quad a_{SL} \in SL(2, \mathbb{C}), \quad x_u' \in u(2) \] (1.2.6)
Since
\[ \det x'_u = \det x_u, \quad x'^\mu y'_\mu = x^\mu y_\mu \] (1.2.7)
we have
\[ x' = a_{L^+_+} x, \quad a_{L^+_+} \in L^+_+, \quad \pm a_{SL} \leftrightarrow a_{L^+_+} \] (1.2.8)
which gives rise to the double-valued representations of \( SO(3, 1)^\dagger = L^+_+ \). In particular, the Dirac field is transformed into its negative when three-dimensional space undergoes one complete rotation.

### 1.3 Drawback

A conventional invocation of the groups \( SU(2) \) and \( SL(2, \mathbb{C}) \) for the purpose of obtaining half-integer representations suffers from an obvious drawback: It has an unnatural character since both \( SU(2) \) and \( SL(2, \mathbb{C}) \) per se bear no relation to the Euclidean space and the Minkowski spacetime. In this connection we quote some authors:

[3]: “Note that in quantum mechanics, starting from the commutation relations of the angular momentum operators, we only know the Lie algebra \( so(3) \cong su(2) \). Experiments tell us that not only the rotation group \( SO(3) \) but also its universal covering group \( SU(2) \) are represented in nature: Some interference experiments with neutrons... do distinguish between \( g \) and \( -g \) in \( SU(2) \).”

[1]: “In application to physical systems possessing rotational symmetry, there is no \textit{a priori} reason to decide whether the double-valued representations occur in nature or not. In reality, we know that they do exist—all fermion (i.e. half-odd-integer spin) systems, such as electrons, protons, ... etc., are described by quantum mechanical wave functions that correspond to double-valued representations of \( SO(3) \). Of course, the single-valued representations are also realized in nature—they correspond to boson systems (with integer spin).”

[5]: “Here we examine the physicist’s somewhat paradoxical statement that the Dirac electron field is transformed into its negative when space undergoes one complete rotation.”

[8]: “It is sometimes said that the \( j \) half-integral representations of \( SU(2) \) are ‘double-valued’ representations of \( SO(3) \), allowed because of the nature of the measurement process. We prefer to think of it differently. The \( j \) half-integral representations are faithful representations of \( SU(2) \). Moreover, \( SO(3) \) is a homomorphic image of \( SU(2) \), and thus some of the representations of \( SU(2) \) are not representations of \( SO(3) \).

The heart of the matter is that we are accustomed to thinking in terms of either a geometric three-dimensional space \( \mathbb{R}_3 \) or a four-dimensional spacetime \( \mathbb{R}_4 \). All our concepts of geometry derive ultimately from the strong coupling of our senses (eyes) with photons (spin 1). We therefore interact strongly with the \( D^1 \) representations—the half-integral representations are unfamiliar. If our eyes were constructed to interact with \( j = 1/2 \) particles, we would see a peculiar twofold degeneracy in those properties of the universe depending on integral spin particles.”
2 Cosmological spacetime manifold

2.1 Direct product manifold

Cosmological spacetime manifold, $M^{\text{cosm}}$, is the direct product manifold [2]:

$$M^{\text{cosm}} = T^{\text{cosm}} \times S^{\text{cosm}}, \quad M^{\text{cosm}} \ni p = (t, s), \quad t \in T^{\text{cosm}}, \quad s \in S^{\text{cosm}} \quad (2.1.1)$$

where $T^{\text{cosm}}$ is cosmological time and $S^{\text{cosm}}$ is cosmological space. Cosmological time

$$T^{\text{cosm}} = \mathbb{R} \quad (2.1.2)$$

Cosmological space is the three-sphere (the closed universe):

$$S^{\text{cosm}} = S^3 = \{ x_k : k = 1, 2, 3, 4, \sum_k x_k^2 = 1 \} \quad (2.1.3)$$

2.2 Tangent space

The tangent space of $M^{\text{cosm}}$ at a point $p \in M^{\text{cosm}}$, $M^{\text{cosm}}_p$, is the direct sum of the tangent spaces of $T^{\text{cosm}}$ and $S^{\text{cosm}}$:

$$M^{\text{cosm}}_p = T^{\text{cosm}}_t \oplus S^{\text{cosm}}_s, \quad p = (t, s) \quad (2.2.1)$$

3 Group structure: Manifold as a Lie group

3.1 Space as $SU(2)$

A crucial point is that $S^3$ is diffeomorphic to the Lie group $SU(2)$ [3]. We have

$$SU(2) \ni a_{SU} = \left( \begin{array}{cc} z^1 & -z^{2*} \\ z^2 & z^{1*} \end{array} \right), \quad |z^1|^2 + |z^2|^2 = 1 \quad (3.1.1)$$

Put

$$z^1 = x_1 + ix_2, \quad z^2 = x_3 + ix_4, \quad |z^1|^2 + |z^2|^2 = \sum_k x_k^2 = 1 \quad (3.1.2)$$

so that

$$SU(2) \ni a_{SU} \leftrightarrow (z^1, z^2) \leftrightarrow \{ x_k \} \in S^3 \quad (3.1.3)$$

So we put

$$S^{\text{cosm}} = SU(2) \quad (3.1.4)$$

A symmetry group of $SU(2)$ is $SU(2)$ itself—by a left action [3]:

$$a_{SU} \mapsto a'_{SU} = a_{SU} a_{SU} \quad (3.1.5)$$

Contemporary cosmology is faced with the problem of choice between a flat and a closed space. A conventional approach to the problem is based on continuous properties of the space [9], [10]. But a certain choice is possible only on the basis of a discrete characteristic of the latter. A symmetry group is just the one. Representations realized in nature are those of $SU(2)$, not only of $SO(3)$. Therefore it is the closed space that should be preferred.
3.2 Tangent space of space: \( su(2), \text{ Euclidean metric, } SO(3) \text{ and } SU(2) \)

The tangent space of \( SU(2) \) at the point \( I_2 \in SU(2) \) is \( su(2) \) [the Lie algebra of the Lie group \( SU(2) \)]:

\[
S^\text{cosm}_{I_2} = su(2)
\]  

(3.2.1)

Now the results of Subsection 1.1 follow for the tangent space—as a natural consequence of (3.2.1).

3.3 Tangent space of spacetime: \( u(1) \oplus su(2) \)

A basis for the tangent space \( T^\text{cosm}_t \) is

\[
cI_1 = c, \quad c \in \mathbb{C}
\]

(3.3.1)

so that a basis for \( M^\text{cosm}_p \) (2.2.1), \( p = (t, s = I_2) \), is

\[
\{cI_2, \tau_1, \tau_2, \tau_3\}, \quad \tau_j = i\sigma_j
\]

(3.3.2)

We have \( \tau_j^\dagger = -\tau_j \), and to retain this property under a change of the basis we put

\[
c = i
\]

(3.3.3)

so that

\[
T^\text{cosm}_t = u(1) \quad \text{(the Lie algebra of the Lie group } U(1) \text{)}
\]

(3.3.4)

and

\[
M^\text{cosm}_{p=(t,I_2)} = u(1) \oplus su(2)
\]

(3.3.5)

Now the results of Subsection 1.2 follow for the tangent space—as a natural consequence of (3.3.5).

3.4 Spacetime manifold as \( U(1) \otimes SU(2) \) and the oscillating universe

We have \( T^\text{cosm} = \mathbb{R} \) (2.1.2) with the additive group structure of \( \mathbb{R} \) and a universal covering homomorphism [4]

\[
\mathbb{R} \rightarrow U(1)
\]

(3.4.1)

Put

\[
T^\text{cosm} \ni t = \frac{T_\theta}{2\pi}
\]

(3.4.2)

so that

\[
\mathbb{R} = \{\theta, +\}, \quad \mathbb{R} \ni \theta \mapsto e^{i\theta} \in U(1)
\]

(3.4.3)

Now, taking into account (2.1.1), we put

\[
M^\text{cosm} = \sum_{n=-\infty}^{\infty} M^\text{cosm}(n), \quad M^\text{cosm}(n) \leftrightarrow \theta^{(n)} = \theta + 2\pi n, \quad 0 \leq \theta < 2\pi
\]

(3.4.4)

\[
M^\text{cosm}(n) = U(1) \otimes SU(2)
\]

(3.4.5)

Thus we have arrived at the oscillating universe with the Lie group structure (3.4.5) of spacetime manifold.
3.5 Tangent bundle structure

Let us turn our attention to a global structure of the set of tangent spaces of \( M^{\cosm(n)} \), i.e., of the tangent bundle. The Lie algebra structure is given by the space of left-invariant vector fields [4], which forms a Lie algebra under the commutator of vector fields. The Minkowski spacetime structure is given by the Minkowski frame bundle [3], [5], \( F_{\text{Minkowski}}(M^{\cosm(n)}) \). Since \( M^{\cosm(n)} \) is parallelizable [3], the Minkowski frame bundle is trivial:

\[
F_{\text{Minkowski}}(M^{\cosm(n)}) = M^{\cosm(n)} \times L^\uparrow \tag{3.5.1}
\]

A section is

\[
\sigma(p) = (p, \{\tau_\mu(p) : \mu = 0, 1, 2, 3\}) \tag{3.5.2}
\]

where

\[
\tau_\mu(p) = \Lambda^\nu_\mu(p) \tau_\nu = a_{SL}(p) \tau_\mu a_{SL}(p)^\dagger, \quad \Lambda(p) \in L^\uparrow, \quad a_{SL} \in SL(2, \mathbb{C})
\]

\[
\Lambda(p) \leftrightarrow \pm a_{SL}(p) \tag{3.5.3}
\]

and

\[
\tau_0(p) = \tau_0, \quad \tau_j(p) = R^j_l(p) \tau_l = a_{SU}(p) \tau_j a_{SU}(p)^\dagger, \quad R(p) \in SO(3), \quad a_{SU} \in SU(2)
\]

\[
R(p) \leftrightarrow \pm a_{SU}(p) \tag{3.5.4}
\]

In \( F_{\text{Minkowski}}(M^{\cosm(n)}) \), the fiber over \( p \in M^{\cosm(n)} \) is the set of all Minkowski bases of \( M^{\cosm(n)}_p \), i.e., bases with metric \( \text{diag}(-1, 1, 1, 1) \).

The above treatment gives rise to the spin structure for \( M^{\cosm(n)} \) [3], [5].

3.6 Lorentzian metric

In GR, the Minkowski metric \( \text{diag}(-1, 1, 1, 1) \) is a metric \( g \) in inertial coordinates. Although \( g \) is a dynamical object, which is not fixed, its signature is fixed. Thus we arrive at the conclusion that \( g \) should be a Lorentzian metric.

4 Spinor structure

4.1 Spinor structure of space: Spinors of three-dimensional space

The \( SU(2) \) group structure of cosmological space \( S^{\cosm} \) (3.1.4) gives rise to a spinor structure for \( S^{\cosm} \). (A spinor structure should not be confused with the spin structure.) Introduce the space of the spinors of three-dimensional space [6]:

\[
Z_1 = \{ z := (z^1, z^2)^{\text{tr}} : z^A \in \mathbb{C}, \quad |z| := (|z^1|^2 + |z^2|^2)^{1/2} = 1 \} \tag{4.1.1}
\]

where \( \text{tr} \) stands for transpose.

The space \( Z_1 \) is equipped with the Hermitian product [11]

\[
(z|z') = \sum_{A=1}^{1,2} z^A z'^A \tag{4.1.2}
\]
and with the invariant scalar product [11]
\[ zCz' = z^AC_ABz'^B, \quad A, B = 1, 2 \] (4.1.3)
where
\[ C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \] (4.1.4)

Both the Hermitian and invariant scalar products are invariant under transformations of the group SU(2):
\[ z \mapsto z' = a_{SU}z, \quad a_{SU} \in SU(2) \] (4.1.5)
\[ (a_{SU}z|a_{SU}z') = (z|z') \] (4.1.6)
\[ a_{SU}zC_{SU}z' = zCz' \] (4.1.7)

In view of (3.1.1), we have
\[ SU(2) \ni a_{SU} \leftrightarrow z \in Z_1 \] (4.1.8)
so that \( S^{\cosm} \) has the structure of \( Z_1 \).

### 4.2 Spinor structure of spacetime: Spinors of four-dimensional space

Now we introduce the space of the spinors of four-dimensional space [6]
\[ Z_+ = \{ \zeta := (\zeta^1, \zeta^2)^{tr} : \zeta^A \in \mathbb{C}, \, |\zeta| := (|\zeta^1|^2 + |\zeta^2|^2)^{1/2} > 0 \} \] (4.2.1)
and the space of dotted spinors [6] or cospinors [12]
\[ \dot{Z}_+ = \{ \dot{\zeta} := (\dot{\zeta}^1, \dot{\zeta}^2)^{tr} : \dot{\zeta}^\dot{A} \in \mathbb{C}, \, |\dot{\zeta}| > 0 \} \] (4.2.2)
(In the physical literature, \( \zeta \) and \( \dot{\zeta} \) are called a right-handed and a left-handed Weyl spinors, respectively [13].)

In addition to the Hermitian product
\[ (\eta|\xi) = \sum_{A}^{1,2} \eta^A*\xi^A \] (4.2.3)
and the invariant scalar product
\[ \eta C\xi = \eta^A C_{AB} \xi^B \] (4.2.4)
there is the scalar product [12]
\[ (\dot{\eta}|\xi) \in \mathbb{C} \] (4.2.5)
which is nondegenerate in both \( \xi \) and \( \dot{\eta} \) and linear in \( \xi \) and antilinear in \( \dot{\eta} \):
\[ (\eta|c_1\xi + c_2\zeta) = c_1(\eta|\xi) + c_2(\eta|\zeta), \quad (c_1\dot{\eta} + c_2\dot{\zeta}|\xi) = c_1^*(\dot{\eta}|\xi) + c_2^*(\dot{\zeta}|\xi), \quad c_k \in \mathbb{C} \] (4.2.6)
The scalar product is invariant under these transformations:
\[ \xi \mapsto \xi' = a_{GL}\xi \leftrightarrow \dot{\xi} \mapsto \dot{\xi}' = (a_{GL}^\dagger)^{-1}\dot{\xi}, \quad a_{GL} \in GL(2, \mathbb{C}) \] (4.2.7)
The invariant scalar product is invariant under transformations of the group $SL(2, \mathbb{C})$:

$$\xi \mapsto \xi' = a_{SL}\xi, \quad a_{SL} \in SL(2, \mathbb{C})$$  \hspace{1cm} (4.2.8)

The Hermitian product is invariant under transformations of the group $SU(2)$:

$$\xi \mapsto \xi' = a_{SU}\xi, \quad a_{SU} \in SU(2)$$  \hspace{1cm} (4.2.9)

Now introduce the correspondence

$$\zeta \leftrightarrow (|\zeta|, z), \quad z = \zeta/|\zeta| \in Z_1$$  \hspace{1cm} (4.2.10)

and put

$$|\zeta| = e^\theta, \quad \zeta \leftrightarrow (\theta, z)$$  \hspace{1cm} (4.2.11)

Thus we have

$$Z_+ \ni \zeta \leftrightarrow (\theta, z) \leftrightarrow (t, s) = p \in M^{\text{cosp}}$$  \hspace{1cm} (4.2.12)

so that $M^{\text{cosp}}$ has the structure of the spinor space $Z_+$. Specifically,

$$|\zeta| \leftrightarrow \theta \leftrightarrow t \in T^{\text{cosp}}: \text{ time}$$  \hspace{1cm} (4.2.13)

$$z \leftrightarrow a_{SU} \leftrightarrow s \in S^{\text{cosp}}: \text{ space}$$  \hspace{1cm} (4.2.14)

$$\zeta \leftrightarrow (\theta, z) \leftrightarrow (t, s) \in T^{\text{cosp}} \times S^{\text{cosp}}: \text{ spacetime}$$  \hspace{1cm} (4.2.15)

5 Symmetry group

5.1 Operations of $GL(2, \mathbb{C})$

Consider operations of the group $GL(2, \mathbb{C})$ on the spinor space $Z_+$. We have

$$GL(2, \mathbb{C}) \ni a_{GL} = re^{i\varphi}a_{SL}, \quad a_{SL} \in SL(2, \mathbb{C})$$  \hspace{1cm} (5.1.1)

so that

$$a_{GL}\zeta = re^{i\varphi}a_{SL}\zeta, \quad a_{SL}\zeta = \overline{\zeta} = |\zeta| z, \quad a_{GL}\zeta = r|\zeta|e^{i\varphi z}$$  \hspace{1cm} (5.1.2)

Now

$$z \leftrightarrow a_{SU} \in SU(2)$$  \hspace{1cm} (5.1.3)

Consider

$$a'_{SU} \begin{pmatrix} z_1 & -z_2^* \\ z_2 & z_1^* \end{pmatrix} = \begin{pmatrix} e^{i\varphi}z_1 & -(e^{i\varphi}z_2)^* \\ e^{i\varphi}z_2 & (e^{i\varphi}z_1)^* \end{pmatrix} = a_{SU}'$$  \hspace{1cm} (5.1.4)

i.e.,

$$a_{SU}'a_{SU} = a_{SU}', \quad a_{SU}' = a_{SU}^{-1}(a_{SU})^{-1}$$  \hspace{1cm} (5.1.5)

We have

$$a'_{SU} \begin{pmatrix} z_1^2 & z_2 \\ z_2 & z_1 \end{pmatrix} = \begin{pmatrix} e^{i\varphi}z_1^2 & e^{i\varphi}z_2 \\ e^{i\varphi}z_2 & e^{i\varphi}z_1 \end{pmatrix}, \quad a_{SU}'z = e^{i\varphi}z$$  \hspace{1cm} (5.1.6)

Thus operations of $\{e^{i\varphi}I_2\}$ are reduced to those of $SU(2)$. (See also [6].)
5.2 \( PL(2, \mathbb{C}) \): Symmetry group of manifold

Taking into account the results proved above, introduce the group

\[
PL(2, \mathbb{C}) = \{a_{GL} \in GL(2, \mathbb{C}) : \det a_{GL} > 0\}
\]  

(5.2.1)

We have

\[
PL(2, \mathbb{C}) \ni a_{PL} = ra_{SL}, \ r = (\det a_{PL})^{1/2}, \ a_{SL} \in SL(2, \mathbb{C})
\]  

(5.2.2)

A symmetry group of \( M^{\text{cosm}} \) is \( PL(2, \mathbb{C}) \):

\[
\zeta \mapsto \zeta' = a_{PL}\zeta
\]  

(5.2.3)

5.3 Operations of subgroups of \( PL(2, \mathbb{C}) \)

We have [7]

\[
a_{SL} = a_{SU}e^h, \ h^\dagger = h, \ \text{Tr} h = 0
\]  

(5.3.1)

so that

\[
a_{PL} = ra_{SL} = ra_{SU}e^h
\]  

(5.3.2)

Consider operations of subgroups of \( PL(2, \mathbb{C}) \)

\[
SU(2) \subset SL(2, \mathbb{C}) \subset PL(2, \mathbb{C}) \supset \{rI_2\}
\]  

(5.3.3)

on \( M^{\text{cosm}} \):

\[
rI_2 : \text{time transformation (time shift), } \theta \mapsto \theta', \ z' = z
\]  

(5.3.4)

\[
a_{SU} : \text{space transformation (space rotation), } \theta' = \theta, \ z \mapsto z'
\]  

(5.3.5)

\[
e^h : \text{spacetime transformation (boost), } \theta' = \theta, \ z \mapsto z'
\]  

(5.3.6)

\[
T \times S \rightarrow T' \times S'
\]

\( T^{\text{cosm}} \times S^{\text{cosm}} \) is distinguished in connection with dynamics

\[
a_{SL} : \text{boost and space rotation}
\]  

(5.3.7)

\[
a_{PL} : \text{boost, space rotation, and time shift}
\]  

(5.3.8)

5.4 Farewell to double-valued representations

Since it is \( SU(2) \) and \( SL(2, \mathbb{C}) \), not \( SO(3) \) and \( SO(3, 1)^{\uparrow} = L_+^{\uparrow} \), that are symmetry groups of the spacetime manifold, there is no need for double-valued representations. It is single-valued, or true representations that are realized in nature.

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