ON INHOMOGENEOUS EXTENSION OF THUE-ROTH’S TYPE INEQUALITY WITH MOVING TARGETS

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Abstract. Let $\Gamma \subseteq \mathbb{Q}^\times$ be a finitely generated multiplicative group of algebraic numbers. Let $\delta, \beta \in \mathbb{Q}^\times$ be algebraic numbers with $\beta$ irrational. In this paper, we prove that there exist only finitely many triples $(u, q, p) \in \Gamma \times \mathbb{Z}^2$ with $d = [\mathbb{Q}(u) : \mathbb{Q}]$ such that $|\delta qu| > 1$ and
$$0 < |\delta qu + \beta - p| < \frac{1}{H(u)q^{d+\varepsilon}},$$
where $H(u)$ denotes the absolute Weil height. This is an inhomogeneous analogue of the main theorem in [2]. As an application of our result, we also prove a transcendence result, which states as follows: Let $\alpha > 1$ be a real number. Let $\beta$ be an algebraic irrational number and $\lambda$ be a non-zero real algebraic number. For a given real number $\varepsilon > 0$, if there are infinitely many natural numbers $n$ for which $||\lambda \alpha^n + \beta|| < 2^{-\varepsilon n}$ holds true, then $\alpha$ is transcendental, where $||x||$ denotes the distance from its nearest integer. When $\alpha$ and $\beta$ both are algebraic numbers satisfying same conditions, then a particular result of Kulkarni, Mavraki and Nguyen in [3] asserts that $\alpha^d$ is a Pisot number. When $\beta$ is an algebraic irrational, our result implies that no algebraic number $\alpha$ satisfies the inequality for infinitely many natural numbers $n$. Also, our result strengthens a result of Wagner and Ziegler [7].

1. Introduction

For a real number $x$, let $||x||$ denote the distance of $x$ to its nearest integer, given by
$$||x|| := \min \{|x - m| : m \in \mathbb{Z}\}.$$

It is interesting to understand the behaviour of $||\alpha^n||$ for a given real number $\alpha$ greater than 1. In this context, in 1957 Mahler [4] showed that for $\alpha \in \mathbb{Q}\setminus\mathbb{Z}$ with $\alpha > 1$ and $\varepsilon$ a positive real number, there are only finitely many $n \in \mathbb{N}$ satisfying $||\alpha^n|| < 2^{-\varepsilon n}$. The key ingredient in Mahler’s proof was the $p$-adic extension of Roth’s theorem established by Ridout [5]. Mahler also asked for which algebraic number $\alpha$ the above conclusion holds true.

In 2004, by ingenious applications of the Subspace Theorem, Corvaja and Zannier [2] proved a ‘Thue-Roth’ type inequality with ‘moving targets’. As an application of this result, they answered the question of Mahler and proved the following: let $\alpha > 1$ be a real algebraic number and $\varepsilon$ be a positive real number. Suppose that $||\alpha^n|| < 2^{-\varepsilon n}$ for infinitely many $n$. Then, there is some integer $d \geq 1$ such that the number $\alpha^d$ is a Pisot number. In particular $\alpha$ is an algebraic integer. We recall that a real algebraic integer $\alpha > 1$ is called a Pisot number, if the modulus value of all its Galois conjugates other than $\alpha$ lie inside the open unit disc.

In this paper, the main aim is to prove an inhomogeneous extension of Thue-Roth’s type inequality with moving targets in the same spirit as the result of Corvaja and Zannier in [2]. We prove the following.

Theorem 1.1. Let $\Gamma \subseteq \mathbb{Q}^\times$ be a finitely generated multiplicative group of algebraic numbers. Let $\delta$ be a non-zero algebraic number, $\beta \in (0, 1)$ be an algebraic irrational, and $\varepsilon > 0$ be a fixed real number. Then there exist only finitely many triples $(u, q, p) \in \Gamma \times \mathbb{Z}^2$ with $d = [\mathbb{Q}(u) : \mathbb{Q}]$ such that $|\delta qu| > 1$ and
$$0 < |\delta qu + \beta - p| < \frac{1}{H(u)q^{d+\varepsilon}}.$$
Recently in 2019, Kulkarni, Mavraki and Nguyen [3] generalized Mahler’s problem to an arbitrary linear recurrence sequence of the form \( \{Q_1(n)\alpha_1^n + \cdots + Q_k(n)\alpha_k^n : n \in \mathbb{N} \} \), where \( \alpha_i \)'s are non-zero algebraic numbers and \( Q_i(x) \in \overline{\mathbb{Q}}[x]\setminus\{0\} \). In a particular case, they proved the following inhomogeneous extension of the problem of Mahler: let \( \alpha > 1 \) be a real number, \( \beta \) be a real algebraic number and let \( \varepsilon \) be a positive real number. Suppose that \( ||\alpha^n + \beta|| < 2^{-\varepsilon n} \) for infinitely many \( n \). Then either \( \alpha \) is transcendental or there is an integer \( d \geq 1 \) such that \( \alpha^d \) is a Pisot number.

In the above result, if \( \beta \) is an integer and \( \alpha \) is an algebraic number such that \( \alpha^d \) is a Pisot number, then clearly there are infinitely many natural numbers \( n \) satisfying \( ||\alpha^d n + \beta|| < 2^{-\varepsilon n} \) for some \( \varepsilon > 0 \). Thus, we can conclude that the above assertion is best possible, if \( \beta \) is an integer. However, if \( \beta \) is an algebraic irrational, as an application of Theorem 1.1, we deduce the following surprising result.

**Theorem 1.2.** Let \( \alpha > 1 \) be a real number, \( \beta \) be an algebraic irrational and \( \lambda \) be a non-zero real algebraic number. For a given real number \( \varepsilon > 0 \), if there are infinitely many natural numbers \( n \) for which \( ||\lambda\alpha^n + \beta|| < 2^{-\varepsilon n} \) holds true, then \( \alpha \) is transcendental.

Note that Theorem 1.2 strengthens the main result of Wagner and Ziegler [7, Theorem 2].

### 2. Preliminaries

Let \( K \) be a number field which is a Galois extension over \( \mathbb{Q} \). Let \( M_K \) be the set of all places on \( K \) and \( M_\infty \) be the set of all archimedean places on \( K \). For each place \( w \in M_K \), let \( K_w \) denote the completion of the number field \( K \) with respect to \( w \) and \( d(w) = [K_w : \bar{\mathbb{Q}}_v] \), where \( v \) is the restriction of \( w \) to \( \mathbb{Q} \). For every \( w \in M_K \) whose restriction on \( \mathbb{Q} \) is \( v \) and \( \alpha \in K \), we define the normalized absolute value \( |\cdot|_w \) as follows:

\[
|\alpha|_w := |\text{Norm}_{K_w/\mathbb{Q}_v}(\alpha)|_{(\mathbb{R}/\mathbb{Z})}^{1/d(w)}.
\]  

Indeed if \( w \in M_\infty \), then there exists an automorphism \( \sigma \in \text{Gal}(K/\mathbb{Q}) \) of \( K \) such that for all \( x \in K \),

\[
|x|_w = |\sigma(x)|^{d(K)/[K:Q]},
\]

where \( d(K) = 1 \) if \( K \subset \mathbb{R} \) and \( d(K) = 2 \) otherwise.

Thus under the definition (2.1), the product formula \( \prod_{w \in M_K} |x|_w = 1 \) holds for any \( x \in K^\times \) and the absolute Weil height \( H(x) \) is defined as

\[
H(x) := \prod_{w \in M_K} \max\{1,|x|_w\}.
\]

One can see that this height is independent of the choice of the number field \( K \) containing \( x \).

For a vector \( \mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{K}^n \) and for a place \( w \in M_K \), the \( w \)-norm for \( \mathbf{x} \) denoted by \( ||\mathbf{x}||_w \) is given by

\[
||\mathbf{x}||_w := \max\{|x_1|_w, \ldots, |x_n|_w\}
\]

and the projective height, \( H(\mathbf{x}) \), is defined by

\[
H(\mathbf{x}) := \prod_{w \in M_K} ||\mathbf{x}||_w.
\]

For a finite set \( S \subset M_K \) of places on \( K \) which contains \( M_\infty \), the ring of \( S \)-integers, denoted by \( \mathcal{O}_S \), is defined as

\[
\mathcal{O}_S := \mathcal{O}_{K,S} = \{\alpha \in K : |\alpha|_v \leq 1 \text{ for all } v \notin S\}.
\]

The group of \( S \)-units in \( K \), denoted by \( \mathcal{O}_S^\times \), is the set of all invertible elements of \( \mathcal{O}_S \), defined as

\[
\mathcal{O}_S^\times := \{\alpha \in K : |\alpha|_v = 1 \text{ for all } v \notin S\}.
\]
Now we are ready to present a more general version of the Schmidt Subspace Theorem, which was formulated by Schlickewei and Evertse. For the reference, see [1, Chapter 7], [4, Chapter V, Theorem 1D] and [3, Page 16, Theorem II.2].

**Theorem 2.1.** (Schlickewei) Let \( K \) be an algebraic number field and \( m \geq 2 \) an integer. Let \( S \) be a finite set of places on \( K \) containing all the archimedean places. For each \( v \in S \), let \( L_{1,v}, \ldots, L_{m,v} \) be linearly independent linear forms in the variables \( X_1, \ldots, X_m \) with coefficients in \( K \). For any \( \varepsilon > 0 \), the set of solutions \( \mathbf{x} \in K^m \setminus \{0\} \) to the inequality

\[
\prod_{v \in S} \prod_{i=1}^m \frac{|L_{i,v}(\mathbf{x})|_v}{|\mathbf{x}|_v} \leq \frac{1}{H(\mathbf{x})^{m+\varepsilon}}
\]

is contained in finitely many proper subspaces of \( K^m \).

The following lemma, established in [2], is used at several places in the proof of the main result of [2].

**Lemma 2.2.** Let \( K \) be a number field which is Galois over \( \mathbb{Q} \) and \( S \) be a finite set of places, containing all the archimedean places. Let \( \sigma_1, \ldots, \sigma_n \) be distinct automorphisms of \( K \) for some integer \( n \geq 1 \) and let \( \lambda_1, \ldots, \lambda_n \) be non-zero elements of \( K \). Let \( \varepsilon > 0 \) be a positive real number and \( w \in S \) be a distinguished place. Let \( c > 0 \) be a real number and let \( E \subset O_K^\times \) be the set of solutions \( u \in O_K^\times \) of the inequality

\[
0 < |\lambda_1 \sigma_1(u) + \cdots + \lambda_n \sigma_n(u)|_w < c \max\{|\sigma_1(u)|_w, \ldots, |\sigma_n(u)|_w\} H(u)^{-\varepsilon}.
\]

If \( E \) is an infinite subset of \( O_K^\times \), then there exists a non-trivial linear relation of the form

\[
a_1 \sigma_1(u) + \cdots + a_n \sigma_n(u) = 0,
\]

with \( a_i \in K \) which holds for infinitely many elements \( u \in O_K^\times \).

A slight modification of Lemma 2.2 yields the following.

**Lemma 2.3.** Let \( K \) be a number field which is Galois over \( \mathbb{Q} \) and \( S \) be a finite set of places, containing all the archimedean places. Let \( \sigma_1, \ldots, \sigma_n \) be distinct automorphisms of \( K \) for some integer \( n \geq 1 \) and let \( \lambda_0, \lambda_1, \ldots, \lambda_n \) be non-zero elements of \( K \). Let \( \varepsilon > 0 \) be a positive real number and \( w \in S \) be a distinguished place. Let \( E \subset O_K^\times \times \mathbb{Z}\setminus\{0\} \) be the subset defined as

\[
E := \left\{ (u, q) \in O_K^\times \times \mathbb{Z}\setminus\{0\} : 0 < |\lambda_0 + \lambda_1 q \sigma_1(u) + \cdots + \lambda_n q \sigma_n(u)|_w < \frac{\max\{|q \sigma_1(u)|_w, \ldots, |q \sigma_n(u)|_w\}}{|q|^n H(u)^{-\varepsilon}} \right\}.
\]

If \( E \) is infinite subset of \( O_K^\times \times \mathbb{Z}\setminus\{0\} \), then there exists a non-trivial linear relation of the form

\[
a_1 \sigma_1(u) + \cdots + a_n \sigma_n(u) = 0,
\]

with \( a_i \in K \) which holds for infinitely many elements \( u \in O_K^\times \) along the pairs \( (u, q) \in E \).

**Proof.** In order to prove this lemma, we shall apply Theorem 2.1 as in the proof [2, Lemma 1]. Without loss of generality, we can assume that

\[
|q \sigma_1(u)|_w = \max\{|q \sigma_1(u)|_w, \ldots, |q \sigma_n(u)|_w\}
\]

for all \((u, q) \in E\). For \( v \in S \), let us define \( n + 1 \) linear forms \( L_{v,0}, \ldots, L_{v,n} \) in \( n + 1 \) variables \( \mathbf{x} = (x_0, x_1, \ldots, x_n) \) as follows: Put \( L_{v,0}(x_0, x_1, \ldots, x_n) = X_0 \) and \( L_{v,1}(x_0, x_1, \ldots, x_n) = \lambda_0 x_0 + \lambda_1 x_1 + \cdots + \lambda_n x_n \). For \( 2 \leq i \leq n \), define \( L_{v,i}(x_0, x_1, \ldots, x_n) = x_i \). Also, for each \( v \neq w \in S \), and \( 0 \leq j \leq n \), we let \( L_{v,j}(x_0, x_1, \ldots, x_n) = x_j \). Take \( \mathbf{x} = (1, q \sigma_1(u), \ldots, q \sigma_n(u)) \in K^{n+1} \) and consider the product

\[
\prod_{v \in S} \prod_{i=0}^n |L_{v,i}(\mathbf{x})|_v.
\]

Using the fact that \( L_{v,j}(\mathbf{x}) = q \sigma_j(u) \) for \( 2 \leq j \leq n \) and that the \( \sigma_j(u) \) are \( S \)-units, by the product formula, we obtain

\[
\prod_{v \in S} \prod_{j=2}^n |L_{v,j}(\mathbf{x})|_v = \prod_{v \in S} \prod_{j=2}^n |q|_v \leq \prod_{v \in M_v} \prod_{j=2}^n |q|_v = |q|^{n-1}.
\]

(2.3)
Now we estimate \( \prod_{v \in S} \prod_{i=0}^{n} ||x||_{v} \):

\[
\prod_{v \in S} \prod_{i=0}^{n} ||x||_{v} = \prod_{i=0}^{n} \left( \prod_{v \in S} ||x||_{v} \right) \geq (H(x))^{n+1} = H^{n+1}(1, q_{\sigma_{1}}(u), \ldots, q_{\sigma_{n}}(u)) \tag{2.4}
\]

since \( ||x||_{v} \leq 1 \) for all \( v \) not in \( S \).

Since \( L_{v,0}(x) = 1 \) for all \( v \in S \) and by the product formula, we get

\[
\prod_{v \neq w \in S} |q_{\sigma_{1}}(u)|_{v} = \left( \prod_{v \neq w \in S} |q| \right) (|\sigma_{1}(u)|_{w})^{-1}.
\]

Thus from (2.2), (2.3) and (2.4), we obtain

\[
\prod_{v \in S} \prod_{i=0}^{n} \left| L_{v,i}(x) \right|_{v} \leq \frac{\lambda_{0} + \lambda_{1} q_{\sigma_{1}}(u) + \cdots + \lambda_{n} q_{\sigma_{n}}(u)|w|q^{n}}{|q_{\sigma_{1}}(u)|_{w}} \frac{1}{H^{n+1}(x)}
\]

\[
\leq \frac{\max\{|q_{\sigma_{1}}(u)|_{w}, \ldots, |q_{\sigma_{n}}(u)|_{w}\}}{|q_{\sigma_{1}}(u)|_{w}} \frac{1}{(|q|H(u))^{\varepsilon}} \frac{1}{H(x)^{n+1}},
\]

as \( (u, q) \in \mathcal{E} \). Using that

\[
|q_{\sigma_{1}}(u)|_{w} = \max\{|q_{\sigma_{1}}(u)|_{w}, \ldots, |q_{\sigma_{n}}(u)|_{w}\},
\]

we get

\[
\prod_{v \in S} \prod_{i=0}^{n} \left| L_{v,i}(x) \right|_{v} \leq \frac{1}{H(x)^{n+1}} \frac{1}{(|q|H(u))^{\varepsilon}}.
\]

Since the height of the vector \( x = (1, q_{\sigma_{1}}(u), \ldots, q_{\sigma_{n}}(u)) \) satisfies \( H(x) \leq |q|H(u)^{|\mathcal{K}:\mathcal{Q}|} = |q|H(u)^{n} \), the above estimate becomes

\[
\prod_{v \in S} \prod_{i=0}^{n} \left| L_{v,i}(x) \right|_{v} \leq \frac{1}{H(x)^{n+1}} \frac{1}{H(x)^{\varepsilon/|\mathcal{K}:\mathcal{Q}|}} = \frac{1}{H(x)^{n+1+\varepsilon/|\mathcal{K}:\mathcal{Q}|}}.
\]

Therefore by Theorem 2.1 there exists a non-trivial relation of the form

\[
a_{0} + a_{1} q_{\sigma_{1}}(u) + \cdots + a_{n} q_{\sigma_{n}}(u) = 0
\]

satisfied by infinitely many pairs \((u, q) \in \mathcal{E}\). In order to finish the proof, it is enough to prove the following claim.

**CLAIM.** There exists a non-trivial relation as \(2.5\) with \(a_{0} = 0\).

Assume that \(a_{0} \neq 0\). By rewriting the relation \(2.5\), we obtain

\[
a_{0} = -a_{1} q_{\sigma_{1}}(u) - \cdots - a_{n} q_{\sigma_{n}}(u) \iff 1 = -\frac{a_{1}}{a_{0}} q_{\sigma_{1}}(u) - \cdots - \frac{a_{n}}{a_{0}} q_{\sigma_{n}}(u).
\]

Since \(\frac{a_{1}}{a_{0}}, \ldots, \frac{a_{n}}{a_{0}}\) are not all zero, let \(a_{i_{1}}/a_{0}, \ldots, a_{i_{r}}/a_{0}\) be the non-zero elements among them. We enlarge our set \(S\), so that \(\frac{a_{i_{1}}}{a_{0}}, \ldots, \frac{a_{i_{r}}}{a_{0}} \in \mathcal{O}_{S}^{\times}\). Since \(\sigma_{i}(u) \in \mathcal{O}_{S}^{\times}\) for \(i = 1, \ldots, n\), from the relation \(2.0\), we conclude that \(q\) must be an \(S\)-unit.

Hence, by applying the \(S\)-unit equation theorem of Evertse and van der Poorten-Schlickewei [8, Theorem II.4] [8, Theorem II.4](see also [1], [6]) to the relation \(2.6\), there exists a non-trivial relation of the form

\[
a_{i_{1}} \sigma_{i_{1}}(u) + \cdots + a_{i_{r}} \sigma_{i_{r}}(u) = 0
\]

which holds for infinitely many values of \(u\) coming from the pairs \((u, q) \in \mathcal{E}\). This proves the claim and hence the lemma.
3. A Key lemma for the proof of Theorem 1.1

The following lemma is key to the proof of Theorem 1.1 and its proof is based on the Subspace Theorem along with the idea in [2], with various modifications.

**Lemma 3.1.** Let $K$ be a Galois extension over $\mathbb{Q}$ of degree $n$ and $k \subset K$ be a subfield of degree $d$ over $\mathbb{Q}$. Let $\delta, \beta$ be two non-zero elements of $K$ with $\beta$ irrational. Let $S$ be a finite set of places on $K$ containing all the archimedean places and let $\varepsilon > 0$ be a given real number. Let

$$
\mathcal{B} = \left\{ (u, q, p) \in (O_S^\times \cap k) \times \mathbb{Z}^2 : 0 < |\delta u - q| < \frac{1}{H(u)^\varepsilon q^{d+\varepsilon}} \right\}
$$

such that for each triple $(u, q, p) \in \mathcal{B}$, $|\delta u - q| > 1$. If $\mathcal{B}$ is infinite, then there exist a proper subfield $k' \subset k$, a non-zero element $\delta'$ in $k$ and an infinite subset $\mathcal{B}' \subset \mathcal{B}$ such that for all triples $(u, q, p) \in \mathcal{B}'$ we have $u/\delta' \in k'$.

**Proof.** Since $\mathcal{B}$ is an infinite set of solutions of (3.1), we first observe that we may assume that $H(u) \to \infty$.

Suppose that $H(u)$ is bounded. Then there exists an infinite subset $\mathcal{A}$ of $\mathcal{B}$ such that the number $u$ is constant for all elements in $A$, say $u_0$ for all triples $(u, q, p) \in \mathcal{A}$ and $q$ is unbounded along the set $\mathcal{A}$. Now we apply Theorem 2.1 to the field $\mathbb{Q}$ with the input $S = \{\infty\}$, linear forms $\left\{ L_{1,\infty}(x_1, x_2, x_3) = \delta u_0 x_1 + \beta x_2 - x_3, L_{i,\infty}(x_1, x_2, x_3) = x_i \right\}$ for $2 \leq i \leq 3$ and the points $(q, 1, p)$. From (3.1), we see that there is a $\eta > 0$ such that the inequality

$$
\sum_{i=1}^{3} |L_{i,\infty}(q, 1, p)|_\infty \leq \frac{1}{(\max\{|q|, |1|, |p|\})^\eta}
$$

holds for infinitely many triples $(q, 1, p) \in \mathbb{Z}^3$. Thus by Theorem 2.1 there exists a proper subspace of $\mathbb{Q}^3$ containing infinitely many triples $(q, 1, p)$, i.e., we have a non-trivial relation of the form

$$
a_0 + a_1 p + a_2 q = 0
$$

satisfied by infinitely many triples of the form $(q, 1, p)$. Since $a_i$’s are integers and $q \to \infty$ along the set $\mathcal{A}$, we conclude that $a_1 \neq 0$. By substituting the value of $p$ into the inequality (3.1) along the set $\mathcal{A}$, we get

$$
0 < |\delta u_0 + \beta + \left( \frac{a_0}{a_1} + \frac{a_2}{a_1} q \right)| \leq \frac{1}{H(u_0)^\varepsilon q^{d+\varepsilon}} \Leftrightarrow 0 < \left| \left( \frac{a_0}{a_1} + \frac{a_2}{a_1} \right) q + \beta + \frac{a_0}{a_1} \right| \leq \frac{1}{H(u_0)^\varepsilon q^{d+\varepsilon}},
$$

which is not true as $q \to \infty$. Therefore, we conclude that $H(u) \to \infty$ along the set $\mathcal{B}$.

Let $\mathcal{H} := \text{Gal}(K/k) \subset \text{Gal}(K/\mathbb{Q}) = \mathcal{G}$ be the subgroup of the Galois group $\mathcal{G}$ fixing $k$. Since $K$ is Galois over $\mathbb{Q}$, we have $K$ is Galois over $k$ and $|\mathcal{G}/\mathcal{H}| = d$. Therefore, among the $n$ embeddings of $K$, there are exactly $d$ embeddings $\sigma_1, \ldots, \sigma_d$, which are the representatives for the left cosets of $\mathcal{H}$ in $\mathcal{G}$ with $\sigma_1$ being the identity. More precisely,

$$
\mathcal{G}/\mathcal{H} = \{ \mathcal{H}, \sigma_2 \mathcal{H}, \ldots, \sigma_d \mathcal{H} \}.
$$

Each automorphism $\rho \in \text{Gal}(K/\mathbb{Q})$ defines an archimedean absolute value on $K$ by the formula

$$
|x|_\rho = |\rho^{-1}(x)|^{d(K)/[K: \mathbb{Q}]},
$$

where $|\cdot|$ denotes the usual complex absolute value and $d(K) = 1$ if $K \subset \mathbb{R}$ and $d(K) = 2$ otherwise. Let $\rho_1$ and $\rho_2$ be two distinct automorphism on $K$, which give rise to the same archimedean absolute values $v$ if and only if $\rho_1^{-1} \circ \rho_2$ is a complex conjugation. Then for each $\rho \in \text{Gal}(K/\mathbb{Q})$, by (3.2), we have

$$
|\delta u + \beta - p|^{d(K)/[K: \mathbb{Q}]} = |\rho(\delta) p(u) + \rho(\beta) - p|_\rho = |\rho(\delta) q p(u) + \rho(\beta) - p|_\rho. \quad (3.3)
$$

For each $v \in M_\infty$, let $\rho_v$ be an automorphism defining the valuation $v$ according to (3.2): $|\alpha|_v := |\alpha|_{\rho_v}$. Then the set $\{ \rho_v : v \in M_\infty \}$ denotes the left cosets of the subgroup generated by the complex conjugation in $\mathcal{G}$.
Denote by \( i : K \to \mathbb{C} \), the embedding given by \( \alpha \mapsto \bar{\alpha} \), the complex conjugation. Then for each \( j = 1, \ldots, d \), let
\[
S_j = \{ v \in M_{\infty} : \rho_v|_k = i \circ \sigma_j : k \to \mathbb{C} \}
\]
and hence \( S_1 \cup \ldots \cup S_d = M_{\infty} \). We keep this notation throughout the paper. Now we take the product of the terms in (3.3) where \( \rho \) runs through the set \( \{ \rho_v : v \in M_{\infty} \} \) to obtain
\[
\prod_{v \in M_{\infty}} |\rho_v(\delta)\rho_v(qu) + \rho_v(\beta) - p|_v = \prod_{j=1}^{d} \prod_{v \in S_j} |\rho_v(\delta)\sigma_j(qu) + \rho_v(\beta) - p|_v.
\]
(3.4)
By (3.3), we see that
\[
\prod_{v \in M_{\infty}} |\rho_v(\delta)\rho_v(qu) + \rho_v(\beta) - p|_v = \prod_{v \in M_{\infty}} |\delta qu + \beta - p|^{d(K)/[K:\mathbb{Q}]} = |\delta qu + \beta - p|^{\sum_{v \in M_{\infty}} d(K)/[K:\mathbb{Q}]}.
\]
From (3.4) and the formula \( \sum_{v \in M_{\infty}} d(K) = [K : \mathbb{Q}] \), it follows that
\[
\prod_{v \in M_{\infty}} |\rho_v(\delta)\sigma_j(qu) + \rho_v(\beta) - p|_v = |\delta qu + \beta - p|.
\]
(3.5)
Now, for each \( v \in S \), we define \( d + 2 \) linearly independent linear forms in \( d + 2 \) variables as follows: For \( j = 1, 2, \ldots, d \) and for \( v \in S_j \), let
\[
L_{v,0}(x_0, x_1, \ldots, x_{d+1}) = \rho_v(\beta)x_0 - x_1 + \rho_v(\delta)x_j + 1
\]
and for \( 2 \leq i \leq d + 1 \), put
\[
L_{v,i}(x_1, \ldots, x_{d+1}) = x_i.
\]
Also, for \( v \in S \setminus M_{\infty} \) and for \( 0 \leq i \leq d + 1 \), let
\[
L_{v,i}(x_1, \ldots, x_{d+1}) = x_i.
\]
Take points \( x \) in \( K^{d+2} \) as
\[
x = (1, p, q\sigma_1(u), \ldots, q\sigma_d(u)) \in K^{d+2}.
\]
In order to apply Theorem 2.1, we need to calculate the following quantity
\[
\prod_{v \in S} \prod_{i=0}^{d+1} \frac{|L_{v,i}(x)|_v}{||x||_v}.
\]
(3.6)
Using the fact that \( L_{v,i}(x) = q\sigma_i(u) \), for \( 2 \leq i \leq d + 1 \) and that the \( \sigma_j(u) \)'s are \( S \)-units, by the product formula, we obtain
\[
\prod_{v \in S} \prod_{i=2}^{d+1} |L_{v,i}(x)|_v = \prod_{v \in S} \prod_{i=2}^{d+1} |q|_v = \prod_{v \in M_{\infty}} \prod_{i=2}^{d+1} |q|_v = \prod_{v \in M_{\infty}} \sum_{v \in M_{\infty}} d(K)/[K:\mathbb{Q}] \leq |q|^d.
\]
(3.7)
Since \( ||x||_v \leq 1 \) for all \( v \notin S \), we estimate the denominators in (3.6) as
\[
\prod_{v \in S} \prod_{i=0}^{d+1} ||x||_v \geq \prod_{v \in M_K} \prod_{i=0}^{d+1} ||x||_v = \prod_{i=0}^{d+1} \left( \prod_{v \in M_K} ||x||_v \right) = \prod_{i=0}^{d+1} H(x) \geq H(x)^{d+2}.
\]
(3.8)
By (3.6), (3.7) and (3.8), it follows that
\[
\prod_{v \in S} \prod_{i=0}^{d+1} \frac{|L_{v,i}(x)|_v}{||x||_v} \leq \frac{1}{H(x)^{d+2}} |q|^d |\delta qu + \beta - p|.
\]
Thus, from (3.1), we have
\[ \prod_{v \in S} \prod_{i=0}^{d+1} \frac{|L_{v,i}(x)|}{||x||} \leq \frac{1}{H(x)^{d+2}} |q|^d \frac{1}{H^\varepsilon(u)} \frac{1}{|q|^{d+\varepsilon}} = \frac{1}{H(x)^{d+2}} \left( |q| H(u) \right)^\varepsilon. \]

Notice that
\[ H(x) = \prod_{v \in M_K} \max \{1, |p|, |q\sigma_1(u)|, \ldots, |q\sigma_d(u)| \} \leq \prod_{v \in S} \max \{1, |p|, |q\sigma_1(u)|, \ldots, |q\sigma_d(u)| \} \]
\[ \leq \prod_{v \in S} \max \{1, |p|, |q| \} \prod_{v \in S} \max \{1, |\sigma_1(u)|, \ldots, |\sigma_d(u)| \} \]
\[ \leq \max \{|p|, |q|\} \left( \prod_{v \in S} \max \{1, |\sigma_1(u)| \} \right) \cdots \left( \prod_{v \in S} \max \{1, |\sigma_d(u)| \} \right) = \max \{|p|, |q|\} H(u)^d. \]

By using the inequality $||x| - |y|| \leq |x - y|$ and since $H(u) \to \infty$ for $(u, q, p) \in \mathcal{B}$, from (3.1), we conclude that $|p| \leq |\delta q u + \beta| + 1$. Since $|u|^{1/2} \leq H(u)$, we get that
\[ |p| \leq |\delta q u + \beta| + 1 \leq |q| |\delta + \beta| H^d(u) + 1 \leq |q| H^{2d}(u) \]
for all but finitely many triples $(u, q, p) \in \mathcal{B}$. By combing both these inequalities, we obtain $H(x) \leq |q| H(u)^{3d}$, and hence $H(x)^{1/3d} \leq |q| H(u)$. Therefore, we get
\[ \prod_{v \in S} \prod_{i=0}^{d+1} \frac{|L_{v,i}(x)|}{||x||} \leq \frac{1}{H(x)^{d+2}} \left( |q| H(u) \right)^\varepsilon \leq \frac{1}{H(x)^{d+2+\varepsilon/3d}} = \frac{1}{H(x)^{d+2+\varepsilon}}, \]
for some $\varepsilon' > 0$ and for infinitely many tuples $(1, p, q\sigma_1(u), \ldots, q\sigma_d(u))$ along the triples $(u, q, p) \in \mathcal{B}$. By Theorem 2.1, there exists a proper subspace of $K^{d+2}$ containing infinitely many $x = (1, p, q\sigma_1(u), \ldots, q\sigma_d(u))$ along the triples $(u, q, p) \in \mathcal{B}$, i.e., we have a non-trivial linear relation of the form
\[ a_0 + a_1 p + b_1 q\sigma_1(u) + \cdots + b_d q\sigma_d(u) = 0, \quad a_i, b_j \in K, \quad (3.9) \]
satisfied by all the triples $(u, q, p) \in \mathcal{B}_1$ for an infinite subset of $\mathcal{B}_1 \subset \mathcal{B}$.

Under the hypotheses of the Main Theorem in [2], the authors established the existence of such a non-trivial linear relation with $a_0 = 0$. The present situation is slightly more complicated. As in [2], we will establish that there is a non-trivial linear relation as above with $a_0 = a_1 = 0$, and then we will conclude exactly as in [2].

**Claim 1.** At least one of the $b_j$’s is non-zero in the relation (3.9).

If not, suppose $b_i = 0$ for all $1 \leq i \leq d$. Then from (3.8), we have
\[ 0 \neq p = -\frac{a_0}{a_1} \in K. \quad (3.10) \]

We deduce from (3.1) and (3.10) that
\[ 0 < |\delta q u + \beta + \frac{a_0}{a_1}| < \frac{1}{H(u)^\varepsilon q^{d+\varepsilon}} \quad (3.11) \]
holds for infinitely many pairs $(u, q)$ along the set $\mathcal{B}_1$. Since $\beta$ is an irrational, from (3.10) we have $\beta + \frac{a_0}{a_1} \neq 0$. We then apply Theorem 2.1 with $S$ being the finite set composed of the archimedean places on $K$, the linear forms $L_{v,1}(x_1, x_2) = (\beta + \frac{a_0}{a_1})x_1 + \delta x_2$, $L_{v,2}(x_1, x_2) = x_1$ for $v \in S$, and the pairs $(1, qu) \in K^2$. Thus by Theorem 2.1 we get a non-trivial relation of the form
\[ c_0 + c_1 qu = 0, \]
which holds for infinitely many pairs $(u, q)$ along the set $\mathcal{B}_1$. This implies that $qu$ is a constant for infinitely many pairs $(u, q) \in \mathcal{B}_1$. However, this violates the inequality (3.11) because $H(u) \to \infty$ as we vary $(u, q)$ in $\mathcal{B}_1$. Therefore we conclude that at least one of the $b_j$’s is non-zero in the relation (3.9).
Claim 2. There exists a non-trivial relation as (3.9) with \( a_0 = a_1 = 0 \).

Suppose that \( a_0 \neq 0 \). Then by re-writing the relation (3.9), we obtain

\[
\beta = -\beta \left( \frac{a_1}{a_0}p + \frac{b_1}{a_0}q\sigma_1(u) + \cdots + \frac{b_d}{a_0}q\sigma_d(u) \right).
\]

(3.12)

Substituting the value of \( \beta \) from (3.12) in (3.1), we get

\[
0 < \left| \delta qu - (\beta a_1/a_0 + 1)p - \beta \left( \frac{b_1}{a_0}q\sigma_1(u) + \cdots + \frac{b_d}{a_0}q\sigma_d(u) \right) \right| < \frac{1}{H(u)^{\varepsilon}} q^{d+\varepsilon}.
\]

(3.13)

The rest of the proof of this claim divided into two cases, according to \( \beta a_1/a_0 + 1 \) is 0 or not.

Case 1. \( \beta a_1/a_0 + 1 = 0 \).

In this case, the relation (3.12) can be written as

\[
\beta - p = - \left( \frac{b_1}{a_1}q\sigma_1(u) + \cdots + \frac{b_d}{a_1}q\sigma_d(u) \right).
\]

(3.14)

Since \( K \) over \( \mathbb{Q} \) is Galois and \( \beta \) is an algebraic irrational, there exists an automorphism \( \rho_0 \in \text{Gal}(K/\mathbb{Q}) \) such that \( \rho_0(\beta) \neq \beta \). By applying the automorphism \( \rho_0 \) on both sides of the equality (3.14), we get

\[
\rho_0(\beta) - p = - \left[ \rho_0 \left( \frac{b_1}{a_1} \right) q\rho_0 \circ \sigma_1(u) + \cdots + \rho_0 \left( \frac{b_d}{a_1} \right) q\rho_0 \circ \sigma_d(u) \right] = - \left[ \rho_0 \left( \frac{b_1}{a_1} \right) q\sigma_{1,0}(u) + \cdots + \rho_0 \left( \frac{b_d}{a_1} \right) q\sigma_{d,0}(u) \right],
\]

as the restriction of \( \rho_0 \) on \( k \) belongs to \( \{\sigma_1, \ldots, \sigma_d\} \) and hence \( \sigma_{i,0} \in \{\sigma_1, \ldots, \sigma_d\} \) for \( 1 \leq i \leq d \). Now by subtracting this equality from (3.14), we obtain

\[
0 \neq \rho_0(\beta) - \beta = c_1q\sigma_1(u) + \cdots + c_dq\sigma_d(u) := \gamma, \quad c_i \in K
\]

for all the pairs \((q, u)\) along the triples \((u, q, p) \in B_1\) with \( \gamma = \rho_0(\beta) - \beta \). We can easily see that in this relation at least one of \( c_i \)'s is non-zero, say \( c_{i_1}, \ldots, c_{i_s} \) are non-zero elements among them, where \( \{i_1, \ldots, i_s\} \subset \{1, \ldots, d\} \). Dividing this equality by \( \gamma \), we get the non-trivial relation of the kind

\[
1 = \left( \frac{c_{i_1}}{\gamma} q\sigma_{i_1}(u) + \cdots + \frac{c_{i_s}}{\gamma} q\sigma_{i_s}(u) \right).
\]

(3.15)

As we have seen in the proof of Lemma 2.3, we enlarge our set \( S \) so that \( \frac{c_{i_1}}{\gamma}, \ldots, \frac{c_{i_s}}{\gamma} \in \mathcal{O}_S^\times \). Thus from the relation (3.15), we also conclude that \( q \in \mathcal{O}_S^\times \). We can apply the \( S \)-unit equation theorem of Evertse and van der Poorten-Schlickewei [8, Theorem II.4] to the relation (3.15), which entails that there exists a non-trivial relation of the kind

\[
b_{i_1}\sigma_{i_1}(u) + \cdots + b_{i_s}\sigma_{i_s}(u) = 0
\]

holds for infinitely many \( u \) coming from the triples \((u, q, p) \in B_1\) for an infinite subset \( B_1 \subset B \).

Case 2. \( \beta a_1/a_0 + 1 \neq 0 \).

By (3.13), we have

\[
0 < \left| \delta qu - \left( \frac{\beta a_1}{a_0} + 1 \right)p - \beta \left( \frac{b_1}{a_0}q\sigma_1(u) + \cdots + \frac{b_d}{a_0}q\sigma_d(u) \right) \right| < \frac{1}{H(u)^{\varepsilon}} q^{d+\varepsilon}.
\]

(3.16)
We follow the similar procedure to the inequality (3.16) as we have seen in the beginning of this lemma to get the following
\[
\prod_{j=1}^{d} \prod_{v \in S_j} \rho_v \left( \delta - \beta b_1/a_0 \right) q_{\sigma_{v(1)}(u)} - \rho_v \left( \beta a_1/a_0 + 1 \right) p - \rho_v \left( \beta b_2/a_0 \right) q_{\sigma_{v(2)}(u)} - \cdots - \rho_v \left( \beta b_d/a_0 \right) q_{\sigma_{v(d)}(u)} \right|_v
\]
\[
= \left| \left( \delta - \beta b_1/a_0 \right) q_{\sigma_{1}(u)} - (\beta a_1/a_0 + 1)p - \beta \left( \frac{b_2}{a_0} q_{\sigma_{2}(u)} + \cdots + \frac{b_d}{a_0} q_{\sigma_{d}(u)} \right) \right| < \frac{1}{H(u)\varepsilon q^{d+\varepsilon}},
\]
where for each \( v \in M_\infty \) and \( j = 1, 2, \ldots, d \), we have set \( \rho_v \circ \sigma_j = \sigma_{v(j)} \) on the field \( K \) and \( \{v(1), \ldots, v(d)\} \) is a permutation of \( \{1, \ldots, d\} \). Now for each \( v \in S \), we define \( d+1 \) linearly independent linear forms in \( d+1 \) variables as follows: for \( j = 1, \ldots, d \) and for each \( v \in S_j \) define
\[
L_{v,0}(x_0, x_1, \ldots, x_d) = -\rho_v \left( \frac{\beta a_1}{a_0} + 1 \right) x_0 + \rho_v \left( \delta - \frac{b_1}{a_0} \right) x_{v(1)} - \rho_v \left( \frac{\beta b_2}{a_0} \right) x_{v(2)} - \cdots - \rho_v \left( \frac{\beta b_d}{a_0} \right) x_{v(d)}
\]
and for \( 1 \leq i \leq d+1 \), define \( L_{v,i}(x_1, \ldots, x_{d+1}) = x_i \). Also for \( v \in S \cap M_\infty \) and \( 0 \leq i \leq d+1 \), let \( L_{v,i}(x_1, \ldots, x_{d+1}) = x_i \). Since in this case \( \frac{\beta a_1}{a_0} + 1 \) is non-zero, we see that the linear forms \( L_{v,0}, \ldots, L_{v,d} \) are linearly independent for each \( v \in S \). Finally, let \( x \) be the point in \( K^{d+1} \), which is of the form
\[
(x, q_{\sigma_{1}(u)}, \ldots, q_{\sigma_{d}(u)}).
\]
Then by using Theorem 2.1 similar to the first part of this lemma, we get a non-trivial relation of the form
\[
a_1 p + b_1 q_{\sigma_{1}(u)} + \cdots + b_d q_{\sigma_{d}(u)} = 0.
\]
Now we prove that there exists a relation with \( a_1 = 0 \). In order to prove this, we follow the similar method as in [2, Lemma 3, Claim] together with Lemma 2.3. If \( a_1 = 0 \), then we have
\[
p = -\frac{b_1}{a_1} q_{\sigma_{1}(u)} - \cdots - \frac{b_d}{a_1} q_{\sigma_{d}(u)}.
\]
(3.17)
First suppose that \( \sigma_j \left( \frac{b_j}{a_1} \right) \neq \frac{b_j}{a_1} \) for some \( j \) with \( 2 \leq j \leq d \). By applying the automorphism \( \sigma_j \) on both sides of (3.17) and subtracting it from (3.17), we obtain a non-trivial relation of the form
\[
b_1 q_{\sigma_{1}(u)} + \cdots + b_d q_{\sigma_{d}(u)} = 0, \quad \text{with} \quad b_i \in K.
\]
We now assume that \( \frac{b_j}{a_1} = \sigma_j \left( \frac{b_j}{a_1} \right) \) for all \( 2 \leq j \leq d \).

Note that \( b_1 \neq 0 \). If not, then \( 0 = \sigma_j(b_1/a_1) = b_j/a_1 \) for every \( j \). Hence \( b_i = 0 \) for all \( i \), which contradicts Claim 1. Therefore, we can assume that \( b_1 \neq 0 \). By putting \( \lambda = -b_1/a_1 \), we re-write (3.17) as
\[
p = -q(\sigma_1(\lambda)q_{\sigma_{1}(u)} + \cdots + \sigma_{d}(\lambda)q_{\sigma_{d}(u)}).
\]
(3.18)
Since \( b_j \in K \), it may happen that \( \lambda \) does not belong to \( K \). If \( \lambda \notin K \), then there exists an automorphism \( \tau \in H \) with \( \tau(\lambda) \neq \lambda \). By applying the automorphism \( \tau \) on both sides of (3.18) to eliminate \( p \), we obtain the linear relation
\[
(\lambda - \tau(\lambda))q_{\sigma_{1}(u)} + q \sum_{i=2}^{d} (\sigma_{i}(\lambda)q_{\sigma_{i}(u)} - \tau \circ \sigma_{i}(u)) = 0.
\]
Note that \( \tau \circ \sigma_{j} \) coincides on \( k \) with some \( \sigma_{i} \) and since \( \tau \in H \) and \( \sigma_{2}, \ldots, \sigma_{d} \notin H \), none of the \( \tau \circ \sigma_{j} \) with \( j \geq 2 \) belongs in \( H \). Hence the above relation can be viewed as a linear combination of \( \sigma_{i}(u) \)'s with the property that the coefficient of \( \sigma_{1}(u) \) will remain \( \lambda - \tau(\lambda) \) and which is non-zero. Therefore, we obtain the required non-trivial relation among \( \sigma_{i}(u) \) as desired.

Hence, we can assume that \( \lambda \in k \) and substitute value of \( p \) from (3.18) into (3.11), we get that
\[
0 < |\beta + (\lambda - \delta)q_{\sigma_{1}(u)} + q_{\sigma_{2}(\lambda)}q_{\sigma_{2}(u)} + \cdots + q_{\sigma_{d}(\lambda)}q_{\sigma_{d}(u)}| < \frac{1}{H^\varepsilon(u)q^{d+\varepsilon}}
\]
(3.19)
holds for infinitely many pairs \((u, q)\) along the triples \((u, q, p) \in B\).
If \( \lambda = \delta \), then by (3.19), we have
\[
0 < -\beta + q\sigma_2(\lambda)\sigma_2(u) + \cdots + q\sigma_d(\lambda)\sigma_d(u) < \frac{1}{H^e(u)q^{d+\varepsilon}}.
\] (3.20)

If \( \max\{|\sigma_2(q\lambda u)|, \ldots, |\sigma_d(q\lambda u)|\} < \frac{\beta}{2d} \) for all pairs \((q, u)\) satisfying (3.20), then, we get
\[
-\beta + q\sigma_2(\lambda)\sigma_2(u) + \cdots + q\sigma_d(\lambda)\sigma_d(u) \geq \frac{\beta}{2}.
\]
Therefore by (3.20), we have
\[
\frac{\beta}{2} \leq -\beta + q\sigma_2(\lambda)\sigma_2(u) + \cdots + q\sigma_d(\lambda)\sigma_d(u) < \frac{1}{H^e(u)q^{d+\varepsilon}}.
\] (3.21)

Since \( H(u) \to \infty \) along infinitely many pairs \((u, q)\) satisfy (3.20) and \( \beta \) is non-zero, we see that the inequality (3.21) can have only finitely many solutions in \((q, u)\), a contradiction. Therefore we must have
\[
\max\{|\sigma_2(q\lambda u)|, \ldots, |\sigma_d(q\lambda u)|\} \geq \frac{\beta}{2d}
\]
holds for all but finitely many pairs \((q, u)\) satisfying (3.20). Thus from (3.20), we conclude that
\[
0 < -\beta + q\sigma_2(\lambda)\sigma_2(u) + \cdots + q\sigma_d(\lambda)\sigma_d(u) < \frac{1}{H^e(u)q^{d+\varepsilon}} < \frac{C\max\{|\sigma_2(u)|, \ldots, |\sigma_d(u)|\}}{H^e(u)q^{d+\varepsilon}},
\]
where \( C = \frac{2d\max\{|\sigma_2(\lambda)|, \ldots, |\sigma_d(\lambda)|\}}{\min(1, |\beta|)} \). Hence by Lemma 2.3 we get a non-trivial relation as desired.

Now we assume that \( \lambda \neq \delta \). In this case the term \((\lambda - \delta)q\sigma_1(u)\) does appear in (3.19). By applying Lemma 2.3 with the distinguished place \( v \) as in the case \( \lambda = \delta \) and with the inputs \( n = d, \lambda_1 = (\lambda - \delta) \) and \( \lambda_i = \sigma_i(\lambda) \) for \( i = 2, \ldots, d \) we conclude the same as in the case \( \delta = \lambda \).

Thus by combining all the cases, we obtain a non-trivial relation of the form
\[
b_1\sigma_1(u) + \cdots + b_d\sigma_d(u) = 0, \quad b_i \in K
\]
for infinitely many \( u \) along the triples \((u, q, p) \in B \). This proves our Claim 2. We then conclude the proof of the theorem exactly as in [2, Lemma 3].

\[\square\]

4. Proofs

Proof of Theorem 1.1. Since \( \Gamma \) is a finitely generated multiplicative subgroup of \( \overline{\mathbb{Q}}^\times \), by enlarging \( \Gamma \) if necessary, we can reduce to the situation where \( \Gamma \subset \overline{\mathbb{Q}}^\times \) is the group of \( S \)-units, namely,
\[
\Gamma = \mathcal{O}_S^\times = \{u \in K : \prod_{v \in S} |u|_v = 1\}
\]
of a suitable Galois extension \( K \) over \( \mathbb{Q} \) containing \( \delta, \beta \) and for a suitable finite set \( S \) of places of \( K \) containing all the archimedean places. Also, \( S \) is stable under Galois conjugation.

Suppose that the conclusion of Theorem 1.1 is not true. Then there exists an infinite subset \( B \subset \Gamma \times \mathbb{Z}^2 \) of solutions \((u, q, p)\) to the inequality (1.1). Then inductively, we construct a sequence \( \{\delta_i\}_{i=0}^\infty \) of elements of \( K \), an infinite decreasing chain \( B_i \) of an infinite subset of \( B \) and an infinite strictly decreasing chain \( k_i \) of subfields of \( K \) with the following properties:

For each integer \( n \geq 0 \), \( B_n \subset (k_n \times \mathbb{Z}^2) \cap B_{n-1}, k_n \subset k_{n-1}, k_n \neq k_{n-1} \) and for all but finitely many triples \((u, q, p) \in B_n \) satisfying
\[
|\delta_0 \cdots \delta_n q u + \beta - p| < \frac{1}{H^e(u)^{e(n+1)}q^{d+\varepsilon}}.
\] (4.1)

If such a sequence exists, then we eventually get a contradiction to the fact that the number field \( K \) does not admit an infinite strictly decreasing chain of subfields. Thus in order to complete the proof of the theorem, it is enough to construct such a sequence.
We proceed our construction by applying induction on $n$: for $n = 0$, put $\delta_0 = \delta$, $k_0 = K$ and $B_0 = B$, and we are done in this case, since by our supposition the inequality
\[ |\delta_0 q u + \beta - p| < \frac{1}{H(u)q^{d+\varepsilon}} \]
has infinitely many solutions in triples $(u, q, p)$. Then by the induction hypothesis, we assume that $\delta_n$, $k_n$ and $B_n$ exist for an integer $n \geq 0$ such that \((4.1)\) holds. Then by Lemma 3.1 to the choices $\delta = \delta_0 \delta_1 \cdots \delta_n$ and $k = k_n$, we obtain an element $\delta_{n+1} \in k_n$, a proper subfield $k_{n+1}$ of $k_n$ and an infinite set $B_{n+1} \subset B_n$ such that all triples $(u, q, p) \in B_{n+1}$ satisfy $u = \delta_{n+1} u'$ with $u' \in k_{n+1}$. Since $u' \in K$, $H'(\delta_{n+1} u') \geq H'((\delta')^{-1} H(u'))$, we have in particular that for almost all $u' \in K$, $H(\delta_{n+1} u') \geq H + \frac{1}{(n+1)}(u')$. Therefore by replacing $u$ by $\delta_{n+1} u'$, for all but finitely many triples $(u', q, p) \in B_{n+1}$, we have the following inequality
\[ |\delta_0 \delta_1 \cdots \delta_n q \delta_{n+1} u' + \beta - p| < \frac{1}{H(u')(u+2)q^{d+\varepsilon}}. \]
The proof of the theorem is now complete by the induction. \qed

Proof of Theorem 1.2. Suppose that $\alpha$ is an algebraic number. Since $|\alpha| > 1$, we have $|\lambda \alpha^n| > 1$ for all large enough integers $n$. Choose $\varepsilon' > 0$ such that $\varepsilon' < \varepsilon \log 2/ \log H(\alpha)$. Then we get
\[ 0 < \|\lambda \alpha^n + \beta\| < H(\alpha^n)^{-\varepsilon'} \]
holds true for infinitely many natural numbers $n$. On the other hand, by taking $\Gamma$ to be the subgroup generated by $\alpha$ and $q = 1$, $\delta = \lambda$ and $u = \alpha^n$, we see that the hypothesis of Theorem 1.1 is satisfied, but not the assertion, which is a contradiction. Thus $\alpha$ must be a transcendental number and hence the theorem. \qed

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