Pre-big bang scenario and the WZW model

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Abstract

Extensive studies of pre-big bang scenarios for Bianchi-I type universe have been made, at various approximation levels. Knowing the solution of the equations for the post-big bang universe, the symmetries of the equations (“time reversal and scale dual transformations”) allow the study of pre-big bang solutions. However, the proposed solutions are unable to explain the actually observed acceleration of the expansion of the universe.

Calculating the $\beta$ equations for the Non-Linear Sigma model, at the first loop approximation and imposing conformal invariance at this level, lead to equations of motion that simply state that the curvature must be nil, which in turn allows the utilization of groups to solve the $\beta$ equations. This is what is done in the Weiss-Zumino-Witten (WZW) model.

In this article, I will show that using the WZW model on $SU_2$, some of the difficulties encountered in the determination of the pre and post big-bang solutions are eliminated. Combining the general results obtained with the $\Lambda CDM$ parameters lead to realistic solutions for the evolution of the universe, giving an explanation to the actually observed acceleration of the expansion in terms of the dilaton field $\phi(t)$.
I. INTRODUCTION

Extensive studies of pre-big bang scenarios for Bianchi-I type universe have been made at various approximation levels [1, 2] (and references herein). The gravitational (massless, bosonic) sector of the string action contains not only the metric, but also at least one more fundamental field, the dilation \( \phi \). The corresponding tree-level action lead to cosmological equations which have been established in the case where the (NS-NS) two form \( B_{\mu
u} = 0 \), but including the contribution of perfect fluid sources. These equations are invariant under “time reversal transformations”, but also under “scale dual transformations”. Knowing the solution of the equations for the post-big bang universe, these symmetries allow the study of pre-big bang solutions. The exact integration of the string cosmology equations in the fully isotropic case can be performed but lead to solutions which exhibit singularities in both the curvature and the dilaton kinetic energy. The solutions associated with the pre and post-big bang branches, being disconnected by a singularity, are not appropriate to describe the whole transition between the two regimes. Particular examples of regular solutions may be obtained, but one may also expect that the regularisation of the big bang singularity need also to introduce the effects of higher order loop and \( \alpha' \) corrections. Moreover, the proposed solutions are unable to explain the actually observed acceleration of the expansion of the universe.

The Polyakov action may be modified to incorporate the effect of massless excitation, and leads to the Non-Linear Sigma model [3, 4]. Calculating the \( \beta \) equations for the Non-Linear Sigma (NLS) model at the first loop approximation and imposing conformal invariance at this level, leads to equations of motion that simply state that the curvature must be nil, which in turn allows the utilization of group manifolds to solve the \( \beta \) equations. This is what is done in the Weiss-Zumino-Witten (WZW) model [5].

In this article, I will show how, using the WZW model on \( SU_2 \), some of the difficulties encountered in the determination of the pre and post big-bang solutions are eliminated. Combining the general results obtained with the \( \Lambda CDM \) parameters lead to realistic solutions for the evolution of the universe, giving an explanation to the actually observed acceleration of the expansion in terms of the dilaton field \( \phi(t) \).
II. THE WZW-MODEL ON $SU_2$

An element of $SU_2$ is parametrized by means of the well-known Euler angles $(\alpha, \beta, \gamma)$:

$$g = \exp (\alpha \tau_3) \exp (\beta \tau_2) \exp (\gamma \tau_3)$$

where the $\tau_i$ matrices are given in terms of Pauli matrices $\tau_k = \frac{\sigma_k}{2}$, and the Euler angles have values in the ranges $0 \leq \alpha < 2\pi$, $0 \leq \beta < \pi$, $0 \leq \gamma < 2\pi$.

Using the expressions for the Maurer-Cartan forms [6], one deduces the line element

$$ds^2 = d\alpha^2 + d\beta^2 + d\gamma^2 + 2d\alpha d\gamma \cos \beta$$

which determines the $g$-matrix

$$g = \begin{pmatrix}
1 & 0 & \cos \beta \\
0 & 1 & 0 \\
\cos \beta & 0 & 1
\end{pmatrix}$$

whose inverse is

$$g^{-1} = \frac{1}{\sin^2 \beta} \begin{pmatrix}
1 & 0 & -\cos \beta \\
0 & \sin^2 \beta & 0 \\
-\cos \beta & 0 & 1
\end{pmatrix}$$

From the elements of these matrix, one easily obtain the Christoffel symbols and finally the Ricci tensors elements and the curvature scalar

$$R = R_{\mu\nu} g^{\mu\nu} = \frac{3}{2}$$

A space of constant curvature is a conformally flat space, so that its Weyl tensor vanishes identically.

A. The evolution of the universe:

We look for a coordinate system $\xi$ which diagonalizes the $g$ matrix at one point, having made the following transformation on the $g$ matrix:

$$g \to \text{diag}(-a_1^2, -a_2^2, -a_3^2) \times g$$
where the three constants only depend on $t$. To preserve the symmetric character of $g$, one must impose $a_1 = a_3 = a$ and one has the following transformation:

$$
\begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
0 & 1 & 0 \\
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}
\end{pmatrix}
\begin{pmatrix}
-a^2 & 0 & -a^2 \cos \beta \\
0 & -a^2 & 0 \\
-a^2 \cos \beta & 0 & -a^2
\end{pmatrix}
\begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
0 & 1 & 0 \\
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}
\end{pmatrix} =
\begin{pmatrix}
-a^2(1 - \cos \beta) & 0 & 0 \\
0 & -a^2 & 0 \\
0 & 0 & -a^2(1 + \cos \beta)
\end{pmatrix}
$$

(5)

In this coordinate system, the curvature scalar is given by the following constant:

$$R = -\frac{1}{a^2} - \frac{a}{2a^2}$$

(6)

As a consequence, the $\beta$-equations at the first loop in string perturbation theory, imposing that the theory is conformal invariant at this order, are satisfied.

To study the evolution of the universe, we consider a $(d + 1)$-dimensional space-time manifold, homogeneous but anisotropic, spatially flat, described by a diagonal metric of the Bianchi-I type.

$$g_{\mu\nu} = \text{diag}(1, -A_i^2 \delta_{ij}) \leftrightarrow \begin{cases}
A_1 = 2a^2 \sin^2 \beta/2 \\
A_2 = a^2 \\
A_3 = 2a^2 \cos^2 \beta/2
\end{cases}$$

(7)

$A_i$ being the scale factor in direction $i$.

We study the graviton-dilatons system, setting $B_{\mu\nu} = 0$, but including perfect fluid sources. Therefore $\phi = \phi(t)$, $T_{\mu\nu} = \text{diag}(\rho, -p_i \delta_i^j)$, $\rho = \rho(t)$, $p_i = p_i(t)$, $\sigma = \sigma(t)$, where $T_{\mu\nu}$ represents the tensor current density of the matter sources and $\sigma$ the scalar charge density.

We first calculate the components of the Christoffell connection $\Gamma$, and deduce the components of the Ricci tensor.

One obtains the following expressions, which apart from at constant, are the same as those calculated by M. Gasperini (see Eqns. 4.10 of [2])

$$R^0_0 = -\sum_i (\dot{H}_i + H_i^2)$$

$$R^i_i = -\dot{H}_i - H_i \left( \sum_k H_k \right) - \frac{1}{2} A_i^2$$

(8)
The associated scalar curvature is:

\[ R = - \sum_i \left(2 \dot{H}_i + H_i^2\right) - \left(\sum_i H_i\right)^2 - \frac{3}{2} A_2^2 \quad (9) \]

We retain also the following equations for the dilaton field:

\[ (\nabla \phi)^2 = \dot{\phi}^2, \quad \nabla^2 \phi = \ddot{\phi} + \dot{\phi} \sum_i H_i, \quad \nabla_0 \nabla^0 \phi = \ddot{\phi}, \quad \nabla_i \nabla^i \phi = \dot{\phi} H_i \delta_i^j \quad (10) \]

Setting \(2 \lambda_s^{d-1} = 1\), we obtain the Euler-Lagrange equation for the dilaton equation (analogous of Eqn 2.24 of [2]).

\[ \ddot{\phi} - \dot{\phi}^2 + 2 \dot{\phi} \sum_i H_i - \sum_i \left(2 \dot{H}_i + H_i^2\right) - \left(\sum_i H_i\right)^2 - \frac{3}{2} A_2^2 + V - \frac{\partial V}{\partial \phi} = \frac{1}{2} e^\phi \sigma \quad (11) \]

The \((0,0)\) component of (Eqn 2.24) in [2] gives

\[ \dot{\phi}^2 - 2 \ddot{\phi} \sum_i H_i + \left(\sum_i H_i\right)^2 - \sum_i (H_i^2) + \left(\frac{3}{2} A_2^2\right) - V = e^\phi \rho \quad (12) \]

While the diagonal part of the space component gives:

\[ \dot{H}_i - H_i \left(\dot{\phi} - \sum_k H_k\right) + \frac{1}{2} \frac{\partial V}{\partial \phi} - \frac{1}{2} \frac{1}{A_2^2} = \frac{1}{2} e^\phi \left(p_i - \sigma\right) \quad (13) \]

These equations are simplified, introducing the “shifted dilaton variable” \(\bar{\phi}\)

\[ \bar{\phi} = \phi - \ln \left(\prod_i a_i\right), \quad \dot{\bar{\phi}} = \dot{\phi} - \sum_i H_i \quad (14) \]

and the shifted variables for the fluid:

\[ \bar{\rho} = \rho \left(\prod_i a_i\right), \quad \bar{p} = \bar{p} \left(\prod_i a_i\right), \quad \bar{\sigma} = \sigma \left(\prod_i a_i\right) \]

then, we obtain the analogous of Eqns 4.39,4.40,4.41 in [2]:

\[ \dot{\bar{\phi}}^2 - \sum_i (H_i^2) - \frac{3}{2} A_2^2 - V = \bar{\rho} e^{\bar{\phi}} \]

\[ \dot{H}_i - H_i \dot{\bar{\phi}} + \frac{1}{2} \frac{\partial V}{\partial \bar{\phi}} - \frac{1}{2} \frac{1}{A_2^2} = \frac{1}{2} e^{\bar{\phi}} \left(\bar{p}_i - \bar{\sigma}\right) \]

\[ 2 \bar{\phi} - \dot{\bar{\phi}}^2 - \sum_i (H_i^2) - \frac{3}{2} A_2^2 + V - \frac{\partial V}{\partial \bar{\phi}} = \frac{1}{2} \bar{\sigma} e^{\bar{\phi}} \quad (15) \]
From these equations, we deduce the following conservation relation:

\[ \dot{\rho} + \sum_i H_i p_i = \frac{1}{2} \sigma \left( \dot{\phi} + \sum_i H_i \right) + e^{-\phi} \left[ \frac{\partial V}{\partial \phi} \sum_i H_i + \frac{3}{A_{22}} \dot{\phi} - \frac{1}{2A_{22}} \sum_i H_i - \frac{3}{2} \frac{d}{dt} \frac{1}{A_{22}} \right] \]

However \( V \) is not a scalar under general coordinates transformations, and it is impossible to define a potential which can be directly inserted as a scalar into the covariant action. However, it has been shown that the action and the corresponding equations of motion can be written in a generalized form which is invariant under general coordinates transformations using for the potential a non-local variable.

The result of the calculations is that the second Eqn. (15) is replaced by the simpler one (2.1.13) [2, 7]:

\[ \dot{H}_i - H_i \dot{\phi} - \frac{1}{4A_2^2} = \frac{1}{2} e^\phi \rho_i \] (16)

leading to the modified conservation equation:

\[ \dot{\rho} + \sum_i H_i p_i = \frac{1}{2} \sigma \dot{\phi} + e^{-\phi} \left[ \dot{\phi} 3 \frac{3}{2A_{22}} - \frac{1}{2A_{22}} \sum_i H_i - \frac{3}{2} \frac{d}{dt} \frac{1}{A_{22}} \right] \] (17)

Using these equations, we can now study models for the evolution of the Universe.

B. Models for the evolution of the Universe:

If we make the following “scale factor duality” transformation

\[ A_i \rightarrow A_i^{-1}, \phi \rightarrow \phi, \]

we see that we obtain another solution for the equations, but with a different curvature, due to the change in \( A_2 \).

We consider the homogeneous case where

\[ A_1 = A_2 = A_3 = A \] (18)

which implies that the propagation is performed on the surface \( \beta = \pi/2 \).

Instead of introducing a time parameter \( x \) as in [2] and get a full analytical result, we calculate a numerical solution of the differential equations of motion [8]. However the
relations between both methods are easily found. It has been shown by M. Gasperini and Veneziano [2, 7] that the cosmological equations are rigorously solved, in the homogeneous case and for the vacuum ($T_{\mu \nu} = 0 = \sigma$) by the solutions

$$A(t) = A_0 \left[ \frac{t}{t_0} + \left( 1 + \frac{t^2}{t_0^2} \right)^{1/2} \right]^{1/\sqrt{d}} \quad (19)$$

and

$$\bar{\phi} = -\frac{1}{2} \ln \left[ \sqrt{V_0 t_0} \left( 1 + \frac{t^2}{t_0^2} \right) \right]$$

Our equations (16) are slightly different with the presence of the factors $1/A^2$, leading to the following conditions, which must relate the scale factor to the sources properties:

$$\frac{1}{A^2} = -\frac{1}{3} e^{\bar{\phi} \bar{\sigma}} = -2 e^{\bar{\phi} \bar{p}} = -\frac{2}{3} e^{\bar{\phi} \bar{\rho}} \implies \bar{\rho} = 3 \bar{p}, \bar{\sigma} = 2 \bar{\rho} \quad (20)$$

This shows that the solutions (19) are even valid in the presence of matter, provided the conditions $\gamma = 1/3, \gamma_0 = 2$ are satisfied, i.e. in the case of a pure radiation field: this is an improvement over previous results.

The curves representing $H(t)$ and $\bar{\phi}$ are similar to those of [2] (Fig. 4.7) with a bell-like shape for the curvature and the dilaton kinetic energy. They show a pre-big bang inflationary evolution, followed by a decelerated expansion.

These solutions exhibit no possible acceleration of the universe expansion, which is actually observed. This demonstrates that such an acceleration is due to the presence of another kind of field, such as dark energy.

The most recent experimental results on the observation of an acceleration of the actual expansion of the universe is found in [10]. From the results published in this report, we can perform a fitting of the continuous curve of $\dot{A}(t)$ represented on Fig. 21 of [10]. For that, we must first numerically solve the first order differential equation which determine $A(t)$:

$$\dot{A}(t) = A(t) \times H_0 (\Omega_\Lambda + \Omega_M (1 + z)^3 + (1 - \Omega_\Lambda - \Omega_M) (1 + z^2))^{1/2} \quad (21)$$

Performing this operation is not easy, because this is a stiff equation which must be solved by efficient algorithms. This is done using the Mathematica package [8]. Concerning the variables, we will use either $t$ or the redshift parameter $z$, which taking arbitrarily $A(0) = 1$, are related by

$$1 + z(t) = \frac{1}{y(t)}, \quad y(t) = A(t)$$
A numerical fitting of $z(t)$, Taking arbitrarily $A(0) = 1$, is:

$$1/A(t) = 1.0 - 70.62831490791527t + 3740.638505154311t^2$$
$$-79103.96903009895t^3 - 1.578335812532445 \times 10^7 t^4$$
$$-6.744184076495469 \times 10^6 t^5 + 8.13610441644515 \times 10^{11} t^6$$
$$+1.1127413276575105 \times 10^{14} t^7 - 5.506299532617992 \times 10^{15} t^8$$
$$-8.075146184886697 \times 10^{17} t^9$$  \hspace{1cm} (22)

valid in the interval $-0.011 \leq t \leq 0.008$, $t = 0$ being the present epoch. In fact the unit for $t$ is very large: relatively to the present epoch, $\Delta z = 0.5$ corresponds to $\Delta t = 0.0066$ s. If we consider that the expansion of the universe began to accelerate at $\Delta T = 5 My$ (cosmic time)\[9\], it follows $\Delta T / \Delta t = 2.38 \times 10^{19}$. So the time interval between $t_1 = -0.011$ and $t_2 = 0.008$ is $\Delta T = 14.35 My$ and covers the whole life of the universe.

$A(t)$ is an increasing function of $t$. The curve $\dot{A}(t)$, when expressed in terms of the redshift parameter $z(t)$, reproduces the continuous curve of Fig. 20 \[10\].

The equation which determines $\ddot{\phi}(t)$ is deduced from [15]:

$$\dddot{\phi} + \left(1 - \frac{\gamma_0}{2}\right) \frac{1}{\gamma} \dot{\phi} - \frac{\partial V}{\partial \phi} = dH^2 + \left(1 - \frac{\gamma_0}{2}\right) \frac{1}{\gamma} \dot{H} + \frac{1}{2A^2} \left[3 - \frac{1}{2\gamma} \left(1 - \frac{\gamma_0}{2}\right)\right]$$  \hspace{1cm} (23)

Now, the solutions that we have adopted for $A(t)$, $\dot{\phi}(t)$, $V(\dot{\phi})$ are not rigorous solutions of [15]. To get a rigorous solution, we must add a correction $\delta V$ to the potential to compensate for the other components of the cosmic field, such that

$$-\frac{3}{2A^2} - \delta V = e^{\ddot{\phi}}(\bar{\rho} - \bar{\rho}_r)$$
$$-\frac{1}{4A^2} = \frac{1}{2} e^{\ddot{\phi}}(\bar{\rho} - \bar{\rho}_r) = \frac{1}{2} e^{\ddot{\phi}} \gamma(\bar{\rho} - \bar{\rho}_r)$$
$$-\frac{3}{2A^2} + \delta V - \frac{\partial}{\partial \phi} (\delta V) = \frac{1}{2} e^{\ddot{\phi}}(\bar{\sigma} - \bar{\sigma}_r)$$  \hspace{1cm} (24)

From which results:

$$-\frac{1}{A^2} = 2 e^{\ddot{\phi}} \gamma(\bar{\rho} - \bar{\rho}_r)$$
$$\delta V = e^{\ddot{\phi}} (3\gamma - 1)(\bar{\rho} - \bar{\rho}_r)$$
$$\frac{\partial}{\partial \phi} (\delta V) = e^{\ddot{\phi}} \left(6\gamma - 1 - \frac{\gamma_0}{2}\right)(\bar{\rho} - \bar{\rho}_r)$$  \hspace{1cm} (25)
It follows that Eqn (23) is now
\[ \ddot{\phi} + \left(1 - \frac{\gamma_0}{2}\right) \frac{1}{\gamma} \dot{H} \dot{\phi} = dH^2 + \left(1 - \frac{\gamma_0}{2}\right) \frac{1}{\gamma} \dot{H} - e^{\phi/2} (\rho - \bar{\rho}_r) \]
\[ = dH^2 + \left(1 - \frac{\gamma_0}{2}\right) \frac{1}{\gamma} \dot{H} + \frac{1}{4} \gamma_0 \left(\frac{26}{\gamma_0}\right) \]
(26)

with
\[ \gamma_0 = 2, \gamma = \frac{1}{3} \]

With these values for the two constant, Eqn (26) takes the remarkable simple form
\[ \ddot{\phi} = dH^2 + \frac{1}{4} (\gamma_0/\gamma) \]
(27)

We have the following values for the density [10]:
\[ \rho = \Omega_M (1 + z)^3 + \Omega_\Lambda + (1 - \Omega_\Lambda - \Omega_M)(1 + z)^2 \]

A plot of the density as a function of \( z \) is given in Fig. [1].

We make a numerical integration of Eqn. (26), we need to evaluate \( H(t) \) and \( \dot{H}(t) \) which are given by the following expressions:
\[ H(t) = \frac{\dot{A}(t)}{A(t)}, \quad \dot{H}(t) = \left(\frac{\dot{A}(t)}{A(t)}\right)^2 - \left(\frac{\dot{A}(t)}{A(t)}\right) \]

\[ 1/A(t) \) being given by (22).

We have also:
\[ H(z) = 69.9868 + 28.7795 \times z + 22.612 \times z^2 - 2.45536 \times z^3 \]

We obtain the curve given in Fig. [2] showing the evolution of \( \dot{\phi}(t) \) with the initial condition \( \dot{\phi}(t = -0.0066s) = 0 \), fixing the beginning of an accelerating expansion at \( T = -5My \). \( \dot{\phi}(t) \), correlating to the evolution of the expansion, is decreasing during the decelerating expansion, and then increasing during the accelerating expansion phase.
Figure 1: The density $\rho$ as a function of $z$

Figure 2: The dilaton $\tilde{\phi}(t)$
Figure 3: The global solution for the dilaton field $\phi(t)$.

**Matching of the solutions:** To get a full description of the universe evolution, we must first propagate Gasperini solution as soon as the radiation field dominates,- as we have seen that the bell-shaped GV-solution is still valid. For later time, we must use the $\Lambda CDM$ solution, matching both at the junction point.

This is done in the following way.

The GV-solution (Gasperini-Veneziano) is

$$H(t) = \left(d \times (t^2 + t_1^2)\right)^{-0.5}, \quad \dot{\phi}(t) = -t/(t^2 + t_1^2)$$

Matching $H(\Lambda CDM)$ and $H(GV)$ at $t = -0.006$ fixes $t_1 = 0.000678617$. It results that $\ddot{\phi}(t = -0.006) = 166.667$. Taking $\ddot{\phi}(-0.006) = 0$ we get the curve given in Fig. 3 showing the increasing of $\phi(t)$.

**III. CONCLUSION.**

In this article, I have developped models for the evolution of the universe, using the Weiss-Zumino-Witten method on the $SU_2$ group. Evolution equations for the fields variables have
been established in the simplest case where the graviton-dilaton system is described by the dilaton field $\phi(t)$. Solution are numerically computed, taking care of the stiffness of the equations. These solutions describe well the actual acceleration of the expansion which, according to the most present measurements began around 5 billion years ago. So this simple models show the ability of string theory, to describe the actually observed evolution of the universe, giving an interpretation in terms of the dilaton field.

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