Surfaces immersed in Lie algebras associated with elliptic integrals

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Abstract
The objective of this work is to adapt the Fokas–Gel’fand immersion formula to ordinary differential equations written in the Lax representation. The formalism of generalized vector fields and their prolongation structure is employed to establish necessary and sufficient conditions for the existence and integration of immersion functions for surfaces in Lie algebras. As an example, a class of second-order, integrable, ordinary differential equations is considered and the most general solutions for the wavefunctions of the linear spectral problem are found. Several explicit examples of surfaces associated with Jacobian and $\wp$-Weierstrass elliptic functions are presented.

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(Some figures may appear in colour only in the online journal)

1. Introduction

The language of group theoretical methods for investigating soliton surfaces associated with integrable models is a very useful and adequate tool for studying the main features of various problems appearing in diverse areas of theoretical physics. Its main advantages appear when group analysis makes it possible to construct algorithms for finding certain classes of 2D surfaces without referring to any additional considerations but proceeding instead directly from the given model under consideration. A systematic method for constructing surfaces and their continuous deformations under various types of dynamics has been extensively developed by many authors (see e.g. [9, 10, 13, 14, 18]). A broad review of recent developments on integrable models and the geometry of associated surfaces can be found in several books [1, 2, 4, 7, 12, 21, 25] and references therein.

The methodological approach assumed in this work is based on the Fokas–Gel’fand approach initiated in [13] and further developed in [14]. Recently, the authors reformulated...
the analysis by providing a symmetry characterization of deformations of soliton surfaces using the formalism of generalized vector fields and their prolongation structure [15]. This approach allows for the systematic construction and analysis of surfaces associated with integrable models and, in the past, allowed for the identification of a new sufficiency condition for the integrated form of the surfaces to have the form proposed in [14]. The identification of necessary and sufficient conditions for the existence of 2D surfaces in terms of invariance conditions for generalized symmetries allows for the integration of the immersion functions explicitly. The results obtained for systems of partial differential equations (PDEs) were so promising that it is worthwhile to try to adapt this method and check its effectiveness for the case of ordinary differential equations (ODEs) associated with elliptic functions. This is, in short, the aim of this paper.

The crux of the matter is that we consider an ODE in the dependent function $u$ and the independent variable $x$ which can be written in the matrix Lax representation involving the differentiation of the Lax pair with respect to $x$ only [1, 11, 19, 26, 27]. It is possible to realize this type of Lax representation as the compatibility condition of some LSP for which an auxiliary variable, say $y$, has been introduced in the wavefunction $\Phi$. In this setting, the wavefunction $\Phi$ becomes a function of $\lambda, x, y, u$ and the derivatives of $u$ with respect to $x$. In this context, we obtain a group-valued immersion function $\Phi$ whose Gauss–Mainardi–Codazzi (GMC) equations are equivalent to a second-order ODE. We can then apply the Fokas–Gel’fand formula to obtain an algebra-valued immersion function $F$ whose GMC equations are equivalent to some infinitesimal deformation of the GMC for the group-valued immersion function $\Phi$. In this paper, we solve the LSP (equivalently the Gauss–Weingarten equations for the surface) and explicitly find the most general form of the wavefunction. We can then construct the explicit integrated form of the algebra-valued surface $F$ whenever the sufficiency conditions are satisfied. This fact is illustrated by an example of a class of surfaces related to elliptic functions. The main advantage of this procedure is that, in using the invariance criterion for the generalized symmetries in terms of their prolongation structure, it leads to simple formulas and allows us to write the explicit form of the soliton surfaces.

The plan of the paper is as follows. Section 2 contains a brief account of basic definitions and properties concerning the Fokas–Gel’fand formula for immersion functions in Lie algebras. We concentrate on the study of soliton surfaces associated with the Lax representation for ODEs related to elliptic functions. In section 3, we investigate the Lax pair and find the general solutions for its wavefunctions for these types of ODEs. These results are then used in section 4 to formulate soliton surfaces associated with conformal symmetry in the spectral parameter, gauge transformations of the wavefunction, and generalized symmetries of the associated integrable ODE and its LSP. Although the theory presented is robust enough to consider generalized symmetries, as an example we concentrate only on symmetries which depend linearly on $u_x$, and hence are evolutionary representatives of group transformations. In sections 5 and 6, we present examples of applications of our approach to the case of the Jacobian and $\mathcal{P}$-Weierstrass elliptic functions respectively and calculate their geometric characteristics. Section 7 contains final remarks concerning soliton surfaces associated with the elliptic functions, identifies some open questions on the subject and proposes some future developments.

2. Application of the Fokas–Gel’fand formula to ODEs in the Lax representation

In this paper, we construct soliton surfaces immersed in Lie algebras using the Fokas–Gel’fand formula for immersion [14], as formulated in [15], applied to ODEs. For this purpose, consider
an ODE
\[ \Delta[u] \equiv \Delta(x, u, u_x, u_{xx}, \ldots) = 0 \] (1)
which admits a Lax pair with potential matrices \( L(\lambda, [u]), M(\lambda, [u]) \) taking values in a Lie algebra \( \mathfrak{g} \) which satisfy
\[ D_xM + [M, L] = 0, \quad \text{whenever} \quad \Delta[u] = 0. \] (2)
Here, \( \lambda \) is the spectral parameter taking values in either \( \mathbb{C} \) or \( \mathbb{R} \). In what follows, we make use of the prolongation structure of vector fields as presented in the book by Olver [20]. For derivatives of \( u \), we use the standard notation \( u_t \) and \( u_j \) for the first and \( J \)th derivatives of \( u \) with respect to \( x \), respectively. For functions depending on the independent variable \( x \), dependent variable \( u \) and its derivatives, the following notation has been used:
\[ f[u] = f(x, u, u_x, u_{xx}, \ldots), \quad f(\lambda, [u]) = f(\lambda, x, u, u_x, u_{xx}, \ldots). \]
The total derivative in the direction of \( x \) takes the form
\[ D_x = \partial_x + u_t \frac{\partial}{\partial u_t} + u_{xx} \frac{\partial}{\partial u_{xx}} + \cdots. \] (3)
This Lax pair equation (2) can be regarded as the compatibility conditions of a linear spectral problem (LSP) for wavefunctions \( \Phi \) taking values in the Lie group \( G \) with independent variables \( x \) and an auxiliary variable \( y \), and spectral parameter \( \lambda \). For the purpose of symmetry analysis, we allow \( \Phi \) to also depend on the dependent variable \( u \) as well as its derivatives with respect to \( x \). The LSP is written as
\[ D_y \Phi(\lambda, y, [u]) = L(\lambda, [u]) \Phi(\lambda, y, [u]), \quad D_y \Phi(\lambda, y, [u]) = M(\lambda, [u]) \Phi(\lambda, y, [u]). \] (4)
Here, since the function \( u \) is independent of \( y \) (i.e. \( u_t = 0 \)), the total derivative in the direction \( y \) is given simply by
\[ D_y = \frac{\partial}{\partial y}. \]
Note that because \( L \) and \( M \) also do not depend on the auxiliary variable \( y \), the compatibility conditions for (4) are of the Lax form (2).

With such an LSP, we can then apply the results of [15] to construct soliton surfaces immersed in the Lie algebra \( \mathfrak{g} \). That is, suppose that \( L, M \in \mathfrak{g} \) and \( \Phi \in \mathfrak{g} \) satisfy the LSP (4) and its compatibility condition (2); then, there exists a \( \mathfrak{g} \)-valued function \( F \) with tangent vectors given by [14]
\[ D_yF \Phi^{-1}A \Phi, \quad D_yF \Phi^{-1}B \Phi \] (5)
for any \( \mathfrak{g} \)-valued functions \( A(\lambda, y, [u]) \) and \( B(\lambda, y, [u]) \) which satisfy
\[ D_yA - D_yB + [A, M] + [L, B] = 0. \] (6)
Whenever \( A \) and \( B \) are linearly independent, \( F \) is an immersion function for a 2D surface in the Lie algebra \( \mathfrak{g} \). As proved in [15], three linearly independent terms which satisfy (6) are
\[ A = a \frac{\partial}{\partial \lambda} L + D_yS + [S, L] + pr\tilde{\nu}_{\lambda}L \in \mathfrak{g}, \] (7)
\[ B = a \frac{\partial}{\partial \lambda} M + D_yS + [S, M] + pr\tilde{\nu}_{\lambda}M \in \mathfrak{g}, \] (8)
where \( a = a(\lambda) \in \mathbb{C}, S \) is an arbitrary \( \mathfrak{g} \)-valued function of \( \lambda, y, x \) as well as the function \( u \) and its derivatives, and \( \tilde{\nu}_{\lambda} \) is a generalized symmetry of (1). Further, the \( \mathfrak{g} \)-valued function \( F \)
can be explicitly integrated as
\[ F = a \Phi^{-1} \frac{\partial \Phi}{\partial \lambda} + \Phi^{-1} S \Phi + \Phi^{-1} \nu \phi \in \mathfrak{g}, \]  
(9)
as long as \( \nu \phi \) is a generalized symmetry of the LSP (4) as well as of the ODE (1).

In the next section, we consider a specific form for the ODE which includes differential equations for elliptic functions. We then present explicitly the Lax pair and find the wavefunction for the associated LSP.

3. The Lax pair and its wavefunction for a second-order ODE

Consider a second-order, autonomous differential equation given by
\[ u_{xx} = \frac{1}{2} f'(u), \quad f'(u) = \frac{\partial}{\partial u} f(u) \]  
(10)
for some function \( f'(u) \). It is straightforward to see that (10) admits the first integral
\[ (u_x)^2 = f(u), \quad u_x = \epsilon \sqrt{f(u)}, \quad \epsilon^2 = 1, \]  
(11)
and its solutions satisfy
\[ \int \frac{du}{\epsilon \sqrt{f(u)}} = x - x_0, \quad x_0 \in \mathbb{R}. \]  
(12)
In the case that
\[ \frac{1}{\sqrt{f(u)}} = R(u, \sqrt{P(u)}), \]  
(13)
where \( R \) is a rational function of its arguments and \( P(u) \) is a polynomial of degree 3 or 4, then the function \( u \) which solves (10) is the inverse of an elliptic integral [3, 5, 6, 17]. In particular, this is the case when \( f(u) \) is a polynomial of degree 3 or 4.

In this section, we will construct a Lax pair for (10) in terms of matrix functions \( L, M \) taking values in \( \mathfrak{sl}(2, \mathbb{R}) \) which satisfy the Lax equation (2) and find solutions for the wavefunctions \( \phi \in \text{SL}(2, \mathbb{R}) \) which are solutions of the LSP (4). Let us make an assumption for the form of \( M \):
\[ M = \begin{bmatrix} u_x & m_{12} \\ u + \lambda & -u_x \end{bmatrix}, \quad \det(M) = -g(\lambda), \]  
(14)
where \( g(\lambda) \) is a rational function of \( \lambda \). In what follows, we call \( g(\lambda) \) the discriminant. From (14), we obtain
\[ \det(M) = -g(\lambda) = -u_x^2 - (u + \lambda)m_{12}, \]  
(15)
and making use of the first integral (11), we find that \( m_{12} \) has the form
\[ m_{12} = -\frac{f(u) - g(\lambda)}{u + \lambda}. \]  
(16)
Note that \( m_{12} \), and hence \( M \), are rational functions of \( \lambda \). Further, if \( f(u) \) is a polynomial in \( u \), then \( m_{12} \) will be a polynomial in \( u \) if and only if
\[ g(\lambda) = f(-\lambda). \]

Next, we solve for \( L \) written in the basis of \( \mathfrak{sl}(2, \mathbb{R}) \):
\[ L = \ell_1 e_2 + \ell_2 e_2 + \ell_3 e_3, \]  
(17)
where \( \ell_i, i = 1, 2, 3 \) are some functions of \( \lambda, x, u \) and \( u_x \). The matrices \( e_i \) written in this basis are
\[ e_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \]  
(18)
Applying the total derivative $D_x$ to $m_{12}$, we obtain

$$D_x(m_{12}) = \left(-\frac{f'(u)}{u+\lambda} + \frac{f(u) - g(\lambda)}{(u+\lambda)^2}\right) u_x.$$  \hfill (19)

The Lax representation (2) becomes

$$\begin{bmatrix}
  u_{xx} - \ell_1 \frac{f(u) - g(\lambda)}{u+\lambda} - \ell_2 (u + \lambda) & -\ell_1 \frac{f(u) - g(\lambda)}{u+\lambda} + 2\ell_2 u_x - 2\ell_3 \frac{f(u) - g(\lambda)}{u+\lambda} \\
  u_x - 2\ell_1 u_x + 2\ell_3 (u + \lambda) & -u_{xx} + \ell_1 \frac{f(u) - g(\lambda)}{u+\lambda} + \ell_2 (u + \lambda)
\end{bmatrix}$$

$$= \begin{bmatrix}
  0 & 0 \\
  0 & 0
\end{bmatrix},$$  \hfill (20)

with the choice of functions

$$\ell_1 = \frac{1}{2}, \quad \ell_2 = \frac{1}{2} \left(\frac{f'(u)}{u+\lambda} - \frac{f(u) - g(\lambda)}{(u+\lambda)^2}\right), \quad \ell_3 = 0.$$  \hfill (21)

Equation (20) becomes

$$\begin{bmatrix}
  u_{xx} - \frac{1}{2} f'(u) & 0 \\
  0 & -u_{xx} + \frac{1}{2} f'(u)
\end{bmatrix} = \begin{bmatrix}
  0 & 0 \\
  0 & 0
\end{bmatrix},$$  \hfill (22)

which is equivalent to the ODE (10). Thus, the matrices $L, M \in sl(2, \mathbb{R})$ of the form

$$L = \frac{1}{2} \begin{bmatrix}
  f(u) & f(u) - g(\lambda) \\
  u + \lambda & u + \lambda
\end{bmatrix}, \quad M = \begin{bmatrix}
  u_x & \frac{f(u) - g(\lambda)}{u+\lambda} \\
  u_x - 2\ell_3 & \ell_2 (u + \lambda)
\end{bmatrix}$$  \hfill (23)

satisfy the Lax equation (2). Note that in the case where the second-order ODE is either Jacobian elliptic or P-Weierstrass, this Lax representation coincides with the Lax pair for the KdV and mKdV equations respectively reduced to stationary states as in [11].

Next, we look for the most general solution of the wavefunction which satisfies the LSP (4). The components of the wavefunction $\Phi$ are denoted by

$$\Phi = \begin{bmatrix}
  \Phi_{11} & \Phi_{12} \\
  \Phi_{21} & \Phi_{22}
\end{bmatrix} \in SL(2, \mathbb{R}),$$  \hfill (24)

with solutions of the LSP (4)

$$\Phi_{11} = c_1 \phi_{1+} + c_2 \phi_{1-}, \quad \Phi_{12} = c_3 \phi_{1+} + c_4 \phi_{1-}, \quad c_i \in \mathbb{R}$$  \hfill (25)

$$\Phi_{21} = c_1 \phi_{2+} + c_2 \phi_{2-}, \quad \Phi_{22} = c_3 \phi_{2+} + c_4 \phi_{2-}, \quad i = 1, 2, 3, 4$$  \hfill (26)

and where

$$\phi_{1\pm} = \frac{\pm \sqrt{g(\lambda)} + u_x}{\sqrt{u+\lambda}} \Psi_{\pm},$$  \hfill (27)

$$\phi_{2\pm} = \sqrt{u+\lambda} \Psi_{\pm},$$  \hfill (28)

$$\Psi_{\pm} = \exp \left[ \pm \sqrt{-g(\lambda)} \left( y + \int \frac{dv}{2(u+\lambda)} \right) \right].$$  \hfill (29)

Here the choice of $\epsilon$ comes from (11). Note that, for the purpose of symmetry analysis, it is sometimes useful to express the integral appearing in (29) as

$$\int \frac{dx}{2(u+\lambda)} = \int \frac{du}{2(u+\lambda)u_x} = \int \frac{\epsilon du}{2(u+\lambda)\sqrt{f(u)}}$$  \hfill (30)
by invoking (11). This integral will be explored further for particular choices of \( f(u) \) in the following sections. The derivatives of the functions \( \Psi_{\pm} \) have the simple form

\[
D_{x}\Psi_{\pm} = \pm \sqrt{g(\lambda)} \Psi_{\pm}, \quad D_{y}\Psi_{\pm} = \pm \frac{\sqrt{g(\lambda)}}{2(u + \lambda)} \Psi_{\pm}.
\]

(31)

We will now check that the wavefunction \( \Phi \) as defined by (25)–(29) satisfies the LSP (4). In terms of the functions \( \phi_{\alpha,\pm} \), the differential equations (4) to be verified are reduced as follows:

\[
D_{x}(\phi_{1,\pm}) - \left( \frac{f'(u)}{u + \lambda} - \frac{f(u) - g(\lambda)}{(u + \lambda)^2} \right) \phi_{2,\pm} = 0,
\]

(32)

\[
D_{x}(\phi_{2,\pm}) - \frac{1}{2} \phi_{1,\pm} = 0,
\]

(33)

\[
D_{x}\phi_{1,\pm} - u_{\alpha}\phi_{1,\pm} + \frac{f(u) - g(\lambda)}{u + \lambda} \phi_{2,\pm} = 0,
\]

(34)

\[
D_{x}\phi_{2,\pm} + u_{\alpha}\phi_{2,\pm} - (u + \lambda)\phi_{1,\pm} = 0.
\]

(35)

We verify (32) by computing the following total derivative:

\[
D_{x}(\phi_{1,\pm}) = \frac{u_{\alpha}}{\sqrt{u + \lambda}} - \frac{1}{2} \left( \pm \sqrt{g(\lambda) + u_{\alpha}} \right) \phi_{1,\pm} + \pm \sqrt{g(\lambda)} \left( \frac{u_{\alpha}}{2(u + \lambda)} \phi_{2,\pm} \right) \Psi_{\pm}
\]

\[
= \left( \frac{u_{\alpha}}{\sqrt{u + \lambda}} \pm \frac{u_{\alpha}^2 + g(\lambda)}{(u + \lambda)^2} \phi_{2,\pm} \Psi_{\pm} \right)
\]

\[
= \left( \frac{f'(u)}{2(u + \lambda)} - \frac{f(u) + g(\lambda)}{(u + \lambda)^2} \right) \phi_{2,\pm}.
\]

Thus, we can directly observe that (32) is satisfied for the set of linearly independent solutions \( \{\phi_{1,\pm}, \phi_{2,\pm}\} \). Next, we verify (33) by computing the total derivative

\[
D_{x}(\phi_{2,\pm}) = \frac{u_{\alpha}}{2\sqrt{u + \lambda}} \Psi_{\pm} = \frac{1}{2} \phi_{1,\pm}.
\]

Thus, (33) holds. To verify (34), we note, using (31), that the dependence of \( \Phi \) on \( y \) is straightforward and so the computation is

\[
D_{x}\phi_{1,\pm} - u_{\alpha}\phi_{1,\pm} + \frac{f(u) - g(\lambda)}{u + \lambda} \phi_{2,\pm} = (\pm \sqrt{g(\lambda)} - u_{\alpha}) \phi_{1,\pm} + \frac{f(u) - g(\lambda)}{u + \lambda} \phi_{2,\pm}
\]

\[
= \left( \frac{g(\lambda) - u_{\alpha}^2}{\sqrt{u + \lambda}} + \frac{f(u) - g(\lambda)}{\sqrt{u + \lambda}} \right) \Psi_{\pm} = 0.
\]

Thus, (34) holds. The last equation to be verified is (35),

\[
D_{x}\phi_{2,\pm} + u_{\alpha}\phi_{2,\pm} - (u + \lambda)\phi_{1,\pm} = (\pm \sqrt{g(\lambda) + u_{\alpha}}) \phi_{2,\pm} - (u + \lambda) \phi_{1,\pm}
\]

\[
= \left( \pm \sqrt{g(\lambda) + u_{\alpha}} \sqrt{u + \lambda} - (u + \lambda)(\pm \sqrt{g(\lambda) + u_{\alpha}}) \sqrt{u + \lambda} \right) \Psi_{\pm}
\]

\[
= 0.
\]

We have thus proven that the set of linearly independent solutions \( \{\phi_{1,\pm}, \phi_{2,\pm}\} \) satisfies (32)–(35). Finally, we note that the requirement \( \det(\Phi) = 1 \) gives an algebraic constraint on the constants \( c_{\alpha} \):

\[
2\sqrt{g(\lambda)}(c_{2}c_{3} - c_{1}c_{4}) = 1.
\]
In the previous section, we assumed that the Lie algebra is $\mathfrak{sl}(2, \mathbb{R})$. In particular, we assumed that the dependent variable is a real function of a real variable, $x$, and the auxiliary variable, $y$, is real as well. On the other hand, we can consider the complexification of the independent variable $x$, for example, in the case of the $P$-Weierstrass function considered in section 6 which naturally extends to the complex plane. In this case, we can interpret the auxiliary variable $y$ as the complex conjugate of $x$ with the assumption that $u(x)$ is analytic in $x$. The ODE (1) along with the requirement that $u$ be analytic gives a set of real PDEs for the real and imaginary parts of $u$. The above construction of $L$ and $M$ still holds; only now they take values in $\mathfrak{sl}(2, \mathbb{C})$. The function $f$ is assumed to be a complex function of $u$. Here the spectral parameter $\lambda$ takes values in $\mathbb{C}$.

For the purposes of constructing surfaces generated by the wavefunction $\Phi_1$, we assume that $u$ is a real function and $x, y$ and $\lambda$ are also real. We choose the constants in $\Phi$ to be

$$c_1 = c_2 = \frac{1}{2}, \quad c_3 = -c_4 = -\frac{1}{2\sqrt{g(\lambda)}}$$

so that the wavefunction $\Phi$ simplifies to

$$\Phi = \begin{bmatrix}
\frac{1}{2}(\phi_{1+} + \phi_{1-}) & -\frac{1}{2\sqrt{g(\lambda)}}(\phi_{1+} - \phi_{1-}) \\
\frac{1}{2}(\phi_{2+} + \phi_{2-}) & -\frac{1}{2\sqrt{g(\lambda)}}(\phi_{2+} - \phi_{2-})
\end{bmatrix},$$

and $\Phi \in SL(2, \mathbb{R})$ for all values of $\lambda \in \mathbb{R}$. In particular, if $g(\lambda) < 0$, then

$$\phi_{\alpha-} = \phi_{\alpha+}, \quad \alpha = 1, 2$$

so that their sum will be real and their difference, divided by $\sqrt{g(\lambda)}$, will also be real.

Note that the wavefunction can be considered as a surface immersed in the Lie group $G$ whose GMC equations reduce to the considered ODE. Hence, the computations in this section give the most general solution of the corresponding Gauss–Weingarten equations for the $G$-valued immersion function $\Phi$.

### 4. The induced surfaces

For analytical descriptions of 2D surfaces, the matrices $A$ and $B$, given in equations (7) and (8), are assumed to be linearly independent. Using the form of the wavefunction $\Phi$ given by (24) or (36), we can construct a surface $F \in sl(2, \mathbb{R})$ whose GMC equations are infinitesimal deformations of the ODE. We also identify those cases where the Gauss–Weingarten equations can be solved explicitly and find the general integrated form of the $g$-valued immersion function $F$.

We construct three cases separately since the pair of matrices $A$ and $B$ corresponds to three types of symmetries. These are a conformal symmetry in the spectral parameter $\lambda$ (called the Sym–Tafel formula), gauge transformations of the wavefunction and the generalized symmetries of the ODE (10). These three types of symmetries yield different types of surfaces. Let us consider each type individually.

#### 4.1. Sym–Tafel formula for immersion

The first term in (7) and (8) corresponds to the Sym–Tafel formula for immersion which is given by [22, 23]

$$F_{ST} = a(\lambda) \Phi^{-1} \frac{\partial}{\partial \lambda} \Phi \in g,$$
where \(a(\lambda)\) is an arbitrary function of \(\lambda\). This surface \(F^{ST}\) has tangent vectors of the form

\[
D_x F^{ST} = \Phi^{-1} \left( \frac{\partial}{\partial \lambda} L \right) \Phi, \quad D_y F^{ST} = \Phi^{-1} \left( \frac{\partial}{\partial \lambda} M \right) \Phi. \tag{38}
\]

With \(L\) and \(M\) given by (23), the tangent vectors in (38) are linearly independent and so the function \(F^{ST}\) gives an immersion of a 2D surface in the Lie algebra \(\mathfrak{sl}(2, \mathbb{R})\).

### 4.2. Surfaces associated with gauge symmetry

The second term in (7) and (8) corresponds to the gauge symmetry of the LSP (4). The surface \(F^S\) corresponding to this gauge term can be integrated explicitly as [9, 13, 14]

\[
F^S = \Phi^{-1} S(\lambda, y, [u]) \Phi \in \mathfrak{g} \tag{39}
\]

with tangent vectors

\[
D_x F^S = \Phi^{-1} (D_x S + [S, L]) \Phi, \quad D_y F^S = \Phi^{-1} (D_y S + [S, M]) \Phi. \tag{40}
\]

Again, for \(F\) to be an immersion, we require the linear independence of the tangent vectors.

Note that, for any surface \(P \in \mathfrak{sl}(2, \mathbb{R})\), \(P\) can be expressed as

\[
P = \Phi^{-1} S \Phi = F^S \tag{41}
\]

and hence \(F^S\) represents a completely arbitrary surface immersed in the Lie algebra. For any \(\mathfrak{sl}(2, \mathbb{R})\)-valued function \(S\), written in the basis (18) as

\[
S = s_1(\lambda, y, [u]) e_1 + s_2(\lambda, y, [u]) e_2 + s_3(\lambda, y, [u]) e_3, \tag{42}
\]

the surface \(F^S\) takes the form

\[
F^S = s_1 \left[ \frac{(u_+ + \sqrt{g}) \Psi_+ - (u_- - \sqrt{g}) \Psi_-}{4(u + \lambda)} (e_1 - e_2) \right]
\]

\[
+ \frac{(u_+ + \sqrt{g}) \Psi_+^2 - (u_- - \sqrt{g}) \Psi_-^2}{(u + \lambda) \sqrt{g}} e_3 \right]
\]

\[
+ s_2 \frac{(u + \lambda)}{4} \left[ -(\Psi_+ + \Psi_-)^2 e_1 + \frac{1}{g} (\Psi_+ - \Psi_-)^2 e_2 - \frac{1}{\sqrt{g}} (\Psi_+^2 - \Psi_-^2) e_3 \right]
\]

\[
+ s_3 \left[ \frac{1}{2} (\Psi_+ + \Psi_-) ((u_+ + \sqrt{g}) \Psi_+ - (u_- - \sqrt{g}) \Psi_-) e_1 \right.
\]

\[
+ \frac{(\Psi_+ - \Psi_-) ((u_+ + \sqrt{g}) \Psi_+ - (u_- - \sqrt{g}) \Psi_-)}{2g} e_2
\]

\[
- \frac{(u_+ + \sqrt{g}) \Psi_+^2 - (u_- - \sqrt{g}) \Psi_-^2}{2 \sqrt{g}} e_3 \right].
\]

We can interpret the surface given in the form (41) as an arbitrary surface immersed in this Lie algebra written in the frame defined by conjugation by the wavefunction \(\Phi\), an element of the Lie group \(SL(2, \mathbb{R})\).

### 4.3. Surfaces associated with generalized symmetries

The third term of (7) and (8) is associated with generalized symmetries of equation (86). That is, suppose that there exists a generalized vector field with the evolutionary representative

\[
\vec{v}_Q = Q[u] \frac{\partial}{\partial u} \tag{43}
\]
which is a generalized symmetry of (10) in the sense that

\[ \text{pr} \tilde{v}_Q \left( u \frac{1}{2} f'(u) \right) = 0, \quad \text{whenever } u \frac{1}{2} f'(u) = 0 \]  

(44)

holds, where we have used the standard definition of the prolongation of a generalized vector field given as in the book by Olver [20]:

\[ \text{pr} \tilde{v}_Q = Q \frac{\partial}{\partial u} + D_x Q \frac{\partial}{\partial u_x} + D_x^2 Q \frac{\partial}{\partial u_{xx}} + \cdots. \]  

(45)

The determining equations for \( Q \) are thus

\[ D_x^2 Q - \frac{1}{2} f''(u) Q = 0, \quad \text{whenever } u \frac{1}{2} f'(u) = 0. \]  

(46)

For such a generalized symmetry (43), there exists a surface immersed in the Lie algebra \( \mathfrak{g} \), say \( F^Q \), with tangent vectors [15]

\[ D_x F^Q = \Phi^{-1} \text{pr} \tilde{v}_Q L \Phi, \quad D_y F^Q = \Phi^{-1} \text{pr} \tilde{v}_Q M \Phi. \]  

(47)

Further, if the generalized symmetry is also a symmetry of the LSP in the sense that

\[ \text{pr} \tilde{v}_Q(D_x \Phi - L \Phi) = 0, \quad \text{whenever } D_x \Phi - L \Phi = 0 \]  

(48)

\[ \text{pr} \tilde{v}_Q(D_x \Phi - M \Phi) = 0, \quad \text{whenever } D_x \Phi - M \Phi = 0, \]  

(49)

then (47) can be integrated and the surface \( F^Q \) given, up to a constant matrix of integration in \( sl(2, \mathbb{R}) \) by [15]

\[ F^Q = \Phi^{-1} \text{pr} \tilde{v}_Q \Phi. \]  

(50)

The following characteristics, \( Q_i \), are solutions of the determining equation (46):

(i) \( Q_1 \equiv u_x \)

(ii) \( Q_2 \equiv u_x \int f(u)^{-\frac{1}{2}} \, du \)

(iii) \( Q_3 \equiv x u_x + \gamma u \), this is only in the special case \( f(u) = c_1 + c_2 u^\ell \) for \( \ell = 2(1 + 1/\gamma) \), \( \gamma, c_1, c_2 \in \mathbb{R} \).

We begin our consideration with \( Q_1 = u_x \). It can be observed directly that the differential equation (10) is invariant under translation in \( x \). Thus, the vector field

\[ \tilde{v}_u = u_x \frac{\partial}{\partial u} \]  

(51)

is a symmetry of (10). In fact, for any \( G(y, \lambda, [u]) \) which does not explicitly depend on \( x \), the prolongation of \( \tilde{v}_u \) acts as a total derivative:

\[ \text{pr} \tilde{v}_u (G(y, \lambda, [u]) = D_x(G(y, \lambda, [u])) \iff \frac{\partial}{\partial x} G(y, \lambda, [u]) = 0. \]  

(52)

Hence, we can see that

\[ \text{pr} \tilde{v}_u (u \frac{1}{2} f'(u) = D_x(u \frac{1}{2} f'(u) = 0, \quad \text{whenever } u \frac{1}{2} f'(u) = 0. \]  

(53)

Similarly, the wavefunctions and the potential matrices \( L \) and \( M \) do not depend explicitly on \( x \) and so

\[ \text{pr} \tilde{v}_u (D_x \Phi - L \Phi) = D_x(D_x \Phi - L \Phi) = 0, \quad \text{whenever } D_x \Phi - L \Phi = 0, \]  

(54)

\[ \text{pr} \tilde{v}_u (D_x \Phi - M \Phi) = D_x(D_x \Phi - M \Phi) = 0, \quad \text{whenever } D_x \Phi - M \Phi = 0. \]  

(55)

Thus, for an arbitrary function \( f(u) \), the vector field \( \tilde{v}_u \) is a symmetry of both (10) and the LSP (4) and so according to [15], there exists a surface defined by the immersion function

\[ F^{u_x} = \Phi^{-1} \text{pr} \tilde{v}_u \Phi = \Phi^{-1} D_x \Phi \]  

(56)
with tangent vectors
\[ D_s F^u = \Phi^{-1} (D_s L) \Phi, \quad D_s F^u = \Phi^{-1} (D_s M) \Phi. \] (57)

For the second case, we can verify that \( Q_2 = u_1 \int f(u)^{-1} du \) solves the determining equation (46). The action of \( pr \tilde{v}_{Q_2} \) on the LSP is given by
\[ pr \tilde{v}_{Q_2}(D_s \Phi - M \Phi) = \frac{u_s}{\sqrt{u + \lambda}} \left[ \begin{array}{cc}
-(\Psi^+ + \Psi^-) & g(\lambda)^{-\frac{1}{2}}(\Psi^+ - \Psi^-) \\
0 & (\Psi^+ + \Psi^-)
\end{array} \right]. \] (58)

\[ pr \tilde{v}_{Q_2}(D_s \Phi - L \Phi) = \frac{u_s}{2(u + \lambda)} \left[ \begin{array}{cc}
-(\Psi^+ + \Psi^-) & g(\lambda)^{-\frac{1}{2}}(\Psi^+ - \Psi^-) \\
0 & (\Psi^+ + \Psi^-)
\end{array} \right]. \] (59)

Since these quantities do not vanish for all solutions of the LSP, the vector field \( \tilde{v}_{Q_3} \) is not a generalized symmetry of the LSP. Thus, while there exists an \( sl(2, \mathbb{R}) \)-valued immersion function \( F^{Q_3} \) with tangent vectors
\[ D_s F^{Q_3} = \Phi^{-1} (pr \tilde{v}_{Q_3}, L) \Phi, \quad D_s F^{Q_3} = \Phi^{-1} (pr \tilde{v}_{Q_3}, M) \Phi, \] (60)
the immersion function \( F^{Q_3} \) is not of the form given in (50).

Similarly, for the special case \( Q_1 \equiv xu_1 + y u \) when \( f(u) = c_1 + c_2 u^2 \), it is straightforward to verify that \( Q_1 \) satisfies the determining equation (46). However, its action on the LSP gives
\[ pr \tilde{v}_{Q_1}(D_s \Phi - M \Phi) = \frac{c_1(1 + \gamma)}{\sqrt{u + \lambda}} \left[ \begin{array}{cc}
-(\Psi^+ + \Psi^-) & g(\lambda)^{-\frac{1}{2}}(\Psi^+ - \Psi^-) \\
0 & (\Psi^+ + \Psi^-)
\end{array} \right] \] (61)

\[ pr \tilde{v}_{Q_1}(D_s \Phi - L \Phi) = \frac{c_1(1 + \gamma)}{2(u + \lambda)^2} \left[ \begin{array}{cc}
-(\Psi^+ + \Psi^-) & g(\lambda)^{-\frac{1}{2}}(\Psi^+ - \Psi^-) \\
0 & (\Psi^+ + \Psi^-)
\end{array} \right]. \] (62)

Thus, unless \( c_1 = 0 \) or \( \gamma = -1 \), \( \tilde{v}_{Q_1} \) is not a symmetry of the LSP. In the former case, the differential equation reduces to the degenerate case \( u_1^2 = c_2 u^2 \) and in the latter case \( u \) is linear in \( x \) since \( u_1^2 = c_1 + c_2 \).

Thus, the surface associated with the generalized vector field \( \tilde{v}_{Q_1} \) can be integrated explicitly and is given by (50), whereas the surfaces associated with the vectors \( \tilde{v}_{Q_2} \) and \( \tilde{v}_{Q_3} \) are only of the form (50) for special cases. However, for each generalized symmetry there exists a surface with tangent vectors given by (47) and we can study their geometric properties based on the tangent vectors to the surface. To this end, in the next section, we give a scalar product on the tangent spaces for the surfaces.

4.4. Induced metrics on the surfaces

Here, we would like to introduce two possible choices for an induced metric on the tangents to the surface \( F \in sl(2, \mathbb{R}) \). We take the basis for \( sl(2, \mathbb{R}) \) given by (18). A first choice for a metric would be to decompose the matrix in the basis \( \{e_1, e_2, e_3\} \) and then to use the standard Euclidean metric. That is, given \( X, Y \in sl(2, \mathbb{R}) \) with
\[ X = X^i e_i, \quad Y = Y^i e_i, \quad i = 1, 2, 3 \] (63)
the inner product and norm in the Euclidean space is defined by
\[ \langle X, Y \rangle = X^i Y^i, \quad ||X|| = \sqrt{X^i X^i}. \] (64)

This constitutes an inner product on the tangent surfaces for the surface \( F \in sl(2, \mathbb{R}) \).

We can also use a symmetric bilinear product defined using the Killing form \( B(X, Y) \). The main advantage of this form is that it is invariant under conjugation by the group and so, because of the form of the tangent vectors (5), the geometric quantities associated with the
surfaces will be independent of the wavefunction $\Phi \in SL(2, \mathbb{R})$. The Killing form on $sl(2, \mathbb{R})$ is given, up to a normalization factor, by (see e.g. [16])

$$B(X, Y) = \text{tr}(XY).$$

(65)

In terms of the basis $\{e_1, e_2, e_3\}$, the matrices $X, Y \in sl(2, \mathbb{R})$ and the Killing form $B(X, Y)$ can be represented as follows:

$$X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}, \quad Y = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}, \quad B(X, Y) = X^i b_{ij} Y^j,$$

(66)

with

$$b = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

(67)

where the symbol $\sim$ means equivalent up to conjugation. The Killing form has signature $(2, 1)$ and so induces a pseudo-Euclidean metric on the tangents to the surface given by the immersion function, $F \in sl(2, \mathbb{R})$. The surface defined via the immersion function $F$ with the Euclidean metric $\langle , \rangle$ is a Riemannian manifold while the surface with the Killing form $B$ is a pseudo-Riemannian manifold [8, 24].

As an example, for a general $f(u)$, the first fundamental form for the surfaces $F^{ST}$ and $F^{u}$ with the pseudo–Euclidean metric are given by

$$I_b(F^{ST}) = \left( \frac{2f - g}{(u + \lambda)^3} - \frac{f' - g'}{(u + \lambda)^2} \right) \text{dx} \text{dy} + 2 \left( \frac{g'}{v + \lambda} + \frac{f - g}{(u + \lambda)^2} \right) \text{dy}^2$$

and

$$I_b(F^{u}) = \left( \frac{ff''}{u + \lambda} - \frac{2ff'}{(u + \lambda)^2} + \frac{2f(f - g)}{(u + \lambda)^3} \right) \text{dx} \text{dy} + \left( \frac{f^2}{2} - \frac{2ff'}{(u + \lambda)} + \frac{2f(f - g)}{(u + \lambda)^2} \right) \text{dy}^2,$$

(68)

(69)

where for convenience, we denote

$$f = f(u), \quad g = g(\lambda), \quad f' = \frac{\partial}{\partial u} f(u), \quad g' = \frac{\partial}{\partial \lambda} g(\lambda).$$

In the Euclidean metric, the components of the metrics are significantly more complicated and can be found in the appendix. Similarly, the components of the metric for $F^{S}$, in both the Euclidean and pseudo-Euclidean metric, are too involved to write out in an illustrative fashion. However, they are directly computable from (43). In practice, the gauge term can be used to simplify the expressions for the surfaces.

5. Jacobian elliptic functions and associated surfaces

Consider the differential equation for the Jacobian elliptic functions

$$(u_x)^2 = (1 - u^2)(k_1 + k_2u^2),$$

(70)

or alternatively

$$u_{xx} = -2k_2u^3 + (k_2 - k_1)u.$$  

(71)

Solutions of (70) are given by Jacobian elliptic functions for the choice of constants given in table 1 with

$$k^2 + k = 1, \quad 0 \leq k, \quad k' \leq 1.$$  

(72)
In terms of the notation of section 3, the function \( f(u) \) takes the form

\[
f(u) = (1 - u^2)(k_1 + k_2 u^2).
\]

Choosing

\[
g(\lambda) = f(-\lambda) = (1 - \lambda^2)(k_1 + k_2 \lambda^2),
\]

the matrices \( L \) and \( M \) become

\[
M = \begin{bmatrix}
  u_x & (u - \lambda)(k_2 (u^2 + \lambda^2) + k_1 - k_2) \\
  u + \lambda & -u_x
\end{bmatrix} \in \text{sl}(2, \mathbb{R}),
\]

\[
L = \begin{bmatrix}
  0 & -3k_2u^2 + 2k_2u + k_1 - k_2 - k_2 \lambda^2 \\
  1 & 0
\end{bmatrix} \in \text{sl}(2, \mathbb{R}).
\]

Taking the normalization of \( \Phi \) as in (36), the wavefunction takes the form

\[
\Phi = \begin{bmatrix}
  (\sqrt{g(\lambda)} - u_x)\Psi_+ - (\sqrt{g(\lambda)} + u_x)\Psi_- \\
  \sqrt{u + \lambda}(\Psi_+ + \Psi_-)
\end{bmatrix} \frac{2\sqrt{\lambda}}{2\sqrt{g(\lambda)}},
\]

\[
\Psi_\pm = \exp \left[ \sqrt{g(\lambda)} \left( y + \int \frac{dx}{2(u + \lambda)} \right) \right],
\]

where

\[
\int \frac{dx}{2(u + \lambda)} = \int \frac{\epsilon \, du}{2(u + \lambda)\sqrt{(1 - u^2)(k_1 + k_2 u^2)}},
\]

when \( u \) is a solution of (70). The integrated form of (79) is

\[
\int \frac{\epsilon \, du}{2(u + \lambda)\sqrt{(1 - u^2)(k_1 + k_2 u^2)} + c_0},
\]

where \( c_0 \) is a real integration constant and \( \Pi(u, a, b) \) is the normal elliptic integral of the third kind; see, e.g., [6]:

\[
\Pi(u, a^2, k) = \int_0^\infty \frac{dt}{(1 - \alpha^2 t^2)^{1/2} - t^2 + k^2 t^2}.
\]
The functions $\Psi_{\pm}$ are then given by

$$
\Psi_{\pm} = \exp \left[ \pm \sqrt{g(\lambda)} \left( y + \frac{\epsilon}{\lambda} \prod \left( \frac{1}{\lambda^2}, \frac{-k_2}{k_1} \right) + c_0 \right) \right] \\
\times \left[ \frac{2 \sqrt{g(\lambda)} \sqrt{(1-u^2)(k_1+2k_2u^2) + (k_2 - k_1 - 2k_2\lambda^2)u^2 + (k_3 - k_1)\lambda^2 + 2k_1}}}{2 \sqrt{g(\lambda)} \sqrt{(1-u^2)(k_1+2k_2u^2) - (k_2 - k_1 - 2k_2\lambda^2)u^2 - (k_3 - k_1)\lambda^2 - 2k_1}} \right]^{\mp 1}.
$$

In the graphs below, we choose the integration constant $c_0$ so that $\Psi_{\pm}(0,0) = 1$.

From the wavefunction $\Phi$, it is immediate to compute the analytical form of the surface generated by the terms described in section 4:

$$
F = \left( a(\lambda)\Phi^{-1} \frac{\partial}{\partial \lambda} \Phi + \Phi^{-1} S\Phi + b\Phi^{-1} D_1\Phi \right) \in sl(2, \mathbb{R}) \quad a(\lambda), b \in \mathbb{R}.
$$

Below, we have presented several graphs of the induced surfaces plotted with the help of MAPLE. The surfaces in figure 1 have constants chosen so that the discriminant is negative, $g(\lambda) < 0$, and so the surfaces behave like trigonometric functions whereas the surfaces in figure 2 have a positive discriminant, $g(\lambda) > 0$, and so have exponential-type behavior.
In the metric induced by the Killing form, the first fundamental form for the surface $F^{ST}$ is

$$ I_B(F^{ST}) = \frac{a(\lambda)^2}{2} (k^2(u - \lambda)) \, dx \, dy - \frac{a(\lambda)^2}{2} [k^2(u^2 - 2\lambda u + 3\lambda^2 - 2) + 1] \, dy^2 $$

and for $F^{u_2}$ is

$$ I_B(F^{u_2}) = -\frac{k^2b^2}{2} (3u - \lambda)(1 - u^2)(k^2u^2 + k^2) \, dx \, dy + \frac{b^2}{2} [k^4u^6 + 2u^5k^4\lambda - u^4k^4\lambda^2 - (4k^4\lambda - 2k^2\lambda^2)u^3 $$

$$ + (3k^2 + 2k^4\lambda^2 - 3k^4 - k^2\lambda^2)u^2 $$

$$ - (2k^2 - 2k^4\lambda)u - k^2\lambda^2 + 2k^4 + k^2\lambda^2 - 3k^2 + 1] \, dy^2. $$

In the Euclidean metric, the first fundamental form of the surfaces can be computed but the expressions are rather involved. We have included in the appendix the first fundamental form for general $f(u)$. However, a remarkable result is that the second fundamental form is quite simple and, up to a coordinate-independent factor, depends only on the constants $k$ and $\lambda$ and the function $u$. It is as follows:

$$ II_{Euc}(F^{ST}) = -\frac{1}{|F^{ST}_x \wedge F^{ST}_y|} \left( \frac{k^4}{2} (u - \lambda)^3 u^2 dx \, dy + (u^2 - \lambda^2)(u - \lambda)k^4 \, dx \, dy ight. $$

$$ - 2k^2(u - \lambda)(k^2u^2 - u - \lambda) + (k^2\lambda^2 - k^2 - 1)u + k^2\lambda^2 \right) \, dy^2 $$

and the Gaussian curvature is

$$ K(F^{ST}) = -\frac{k^6(u - \lambda)^3 u(2k^2u^2 - k^2 - 1)}{|F^{ST}_x \wedge F^{ST}_y|^4}. $$

Thus, we can see that the Gaussian curvature has zeros at $u = 0, \lambda$ and $k^2 - 1$ and is negative for large $u$.

We can also study the surfaces associated with generalized symmetries $Q_2 = u_2 \int f(u)^{-2} \, du$ and $Q_3 = xu_u + yu$, in the special case when $k_1 = -k_2$, $\gamma = 1$ and $f(u) = k_1(1 - u^4)$, but the immersion functions are not of the form (50) since the vector fields
\( \vec{v}_Q \) and \( \vec{v}_Q' \) are not generalized symmetries of the LSP. However, it is possible to study the geometric properties of these surfaces via their tangent vectors.

6. \( \mathcal{P} \)-Weierstrass elliptic function and associated surfaces

Consider the differential equation for the \( \mathcal{P} \)-Weierstrass elliptic function

\[
    u_x = -\sqrt{4u^3 - g_2 u - g_3}
\]

or alternatively

\[
    u_{xx} - 6u^2 + \frac{g_2}{2} = 0.
\]

In terms of the notation of section 3,

\[
    f(u) = 4u^3 - g_2 u - g_3,
\]

If we choose \( g(\lambda) = f(-\lambda) \), then the Lax pair \( L, M \) becomes

\[
    M = \begin{bmatrix} u_x & -4u^2 + 4\lambda u - 4\lambda^2 + g_2 \\ u + \lambda & -u_x \end{bmatrix} \in \text{sl}(2, \mathbb{R}),
\]

\[
    L = \frac{1}{2} \begin{bmatrix} 0 & 4\lambda - 2\lambda^2 \\ 1 & 0 \end{bmatrix} \in \text{sl}(2, \mathbb{R}).
\]

Taking the normalization of \( \Phi \) as in (36), the wavefunction takes the form given by (77) with \( \Psi_{\pm} \) as

\[
    \Psi_{\pm} = \exp \left[ \sqrt{g(\lambda)} \left( y + \int \frac{dx}{2(u + \lambda)} \right) \right].
\]

Again, for computing the action of the generalized vector field \( \vec{v}_Q \) on \( \Phi \), it is convenient to use the identity

\[
    \int \frac{dx}{2(u + \lambda)} = -\int \frac{du}{2(u + \lambda)\sqrt{4u^3 - g_2 u - g_3}}.
\]

The integrated form of (91) is

\[
    \int -\frac{du}{2(u + \lambda)\sqrt{4u^3 - g_2 u - g_3}} = -\frac{1}{2(\lambda - a_1)(2a_1 + a_2)} \Pi \left( \sqrt{\frac{u + a_1}{a_1 - a_2}} \frac{a_1 - a_2}{\lambda - a_1} \sqrt{\frac{a_1 - a_2}{2a_1 + a_2}} \right) + c_0,
\]

where, for simplicity, we have used the constants \( a_1, a_2 \) to represent the roots of the polynomial \( f(u) \) as in

\[
    f(u) = 4u^3 - g_2 u - g_3 = 4(u + a_1)(u + a_2)(u - a_1 - a_2).
\]

Here, \( c_0 \) is a real integration constant and \( \Pi \) is the solution of the elliptic integral defined by (81). In the graphs below, we choose the integration constant \( c_0 \) so that \( \Psi_{\pm}(0.5, 0) = 1 \).

From the wavefunction \( \Phi \), one can immediately compute the analytical form of the surface generated by the terms described in section 4:

\[
    F = \left( a(\lambda)\Phi^{-1} \frac{\partial}{\partial \lambda} \Phi + \Phi^{-1} S \Phi + b\Phi^{-1} D_i \Phi \right) \in \text{sl}(2, \mathbb{R}) \quad a(\lambda), b \in \mathbb{R}.
\]

In figure 3, we have included graphs of the induced surfaces for \( \mathcal{P} \)-Weierstrass functions with \( g_2 = 0, g_3 = 1 \), i.e. on an equianharmonic lattice [3]. Again, the surfaces exhibit periodic behavior when \( g(\lambda) < 0 \) and exponential behavior when \( g(\lambda) > 0 \).
Figure 3. Surfaces $F^{ST}$ for $u = \mathcal{P}(x, 0, 1)$ with $x \in [0.2, 3]$ and $y \in [-\pi/g(1), \pi/g(1)]$ for the negative discriminant cases and $x \in [-1, 1]$ and $y \in [-0.5, 0.5]$ for the positive discriminant cases. The axes indicate the components of the immersion function in the basis (18).

It is also possible to study the surfaces associated with generalized symmetries $Q_2 = u_s \int f(u)^{-2} du$ and $Q_3 = xu_s + \gamma u$, in the special case where $g_2 = 0$, $\gamma = 2$ and $f(u) = 4u^2 - g_3$. However, the immersion functions are not of the form (50) except in the latter case when $g_3$ is also equal to zero and the solutions of (86) are rational functions of $x$.

In the metric induced by the Killing form, the first fundamental form for the surface $F^{ST}$ is

$$\text{I}_\theta(F^{ST}) = -2a^2(\lambda)(dx \, dy + 3a^2(2\lambda - u) \, dy^2)$$

and for $F^{u*}$ is

$$\text{I}_\theta(F^{u*}) = 4b^2 \left[ (4u^2 - g_2u - g_3) \, dx \, dy 
+ (2u^4 + 8\lambda u^3 + g_2u^2 - 2(g_2\lambda - 2g_3)u - 2\lambda g_3 + \frac{1}{8}g_3^2) \, dy^2 \right].$$
Both fundamental forms depend on the solution $u$ and the spectral parameter $\lambda$. Note that, in both cases, the tangent vectors in the $x$-direction are null vectors in the pseudo-Euclidean metric.

7. Final remarks

In this paper, we have discussed certain classes of surfaces immersed in Lie algebras associated with elliptic integrals. The problem of constructing 2D surfaces has been studied recently for PDEs by the authors in [15] using a symmetry characterization of continuous deformations of soliton surfaces immersed in Lie algebras based on the formalism of generalized vector fields and their prolongation structure. The necessary and sufficient conditions for the existence of such surfaces in terms of the invariance conditions have been established for integrable PDEs. In this context, we have adapted the proposed procedure for integrable ODEs admitting a Lax representation (2) and shown, as in the PDE case, that the problem requires the examination of conformal symmetries in the spectral parameter, gauge transformations of the wavefunction and generalized symmetries of the associated model and its LSP. To perform this symmetry analysis, we have constructed a Lax pair for a second-order ODE which includes, among other cases, the differential equations for elliptic functions. Next, we solved the LSP and found explicitly the most general form of the wavefunction. We then constructed surfaces in $sl(2, \mathbb{R})$ by analytic methods for a general case where the right-hand side of ODE (11) is an arbitrary function $f(u)$. We were able to explicitly integrate this ODE in terms of elliptic integrals when $f(u)$ is a polynomial in $u$ of degree 3 or 4 and give explicit forms of the corresponding soliton surfaces for the Jacobian and P-Weierstrass elliptic functions. To this end, we have identified three generalized symmetries of the considered ODE and determined whether they are also symmetries of the LSP, that is, whether the immersion function $F$ can be explicitly integrated in the Fokas–Gel’fand form (9). Some geometric analysis of these surfaces has been performed by using an appropriate inner product which allows for the construction of Riemannian and pseudo-Riemannian manifolds. The elaborate procedure was applied to examples and we have given the first fundamental form for the surfaces as well as graphs of the surfaces for a range of parameters leading to diverse types of surfaces. As an example, we have computed the Gaussian curvatures in the Euclidean metric for the surfaces associated with Jacobian elliptic functions and showed that up to a coordinate-independent factor, they depend only on the constants $\lambda$ and $k$ and the given solution of the ODE.

Additional questions which could be asked involve the geometric properties, including global characteristics, of the surfaces associated with elliptic integrals. These geometric quantities, such as the Gaussian curvature, mean curvature, the Willmore functional and Euler–Poincaré characters, can be calculated but, in general, the expressions are rather involved. Hence, we omit them here. It is an open question as to whether the equations can be simplified, possibly by judicious choice of the linearly independent terms in the matrices $A$ and $B$ given by (7) and (8). Finally, another interesting avenue for future research could include the application of the methods presented here to other integrable equation ODEs, for example, hyperelliptic functions, which admit a Lax representation and to study their soliton surfaces.

We expect these results to have useful applications to the study of soliton surfaces for PDEs which admit group invariant solutions. For example, in the case of the KdV equation, the stationary states are elliptic functions and hence it would be of interest to consider the relation between the KdV surfaces obtained by the Fokas–Gel’fand formula and the surfaces associated with elliptic functions presented in this paper. That is, it would be interesting to study the KdV surfaces using the ODE surfaces as approximations (here the auxiliary variable
y becomes $t$). This analysis could have an impact on the approximation and stability analysis of the surfaces. This task will be undertaken in future work.

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**Appendix. Components of the first fundamental form for surfaces in the Euclidean metric**

The components of the first fundamental form for the surface induced via the Sym–Tafel formula for immersion, in the Euclidean metric, are given by

\[
\{F^ST_x, F^ST_y\} = \left(\frac{(f' - g')(u + \lambda) - 2(f - g)}{64(u + \lambda)^2 g(\lambda)}\right)^2 \\
\times (g^2 + g + 1)(\Psi_+^1 + \Psi_+^2) + 4(g^2 - 1)(\Psi_+^2 + \Psi_+^2) + 2(3g^2 - g + 3)),
\]

\[
\{F^ST_x, F^ST_y\} = \ \frac{2(f - g) - (f' - g')(u + \lambda)}{32(u + \lambda)^3 g(\lambda)^2} \\
- 2(\Psi_+^1 - 1) ((g^2 + g + 1)(1 + \Psi_+^2) + 2(g^2 - 1)\Psi_+^2)u_x \\
+ (g^2 + g + 1)(g'(u + \lambda) - 2g)(\Psi_+^1 + \Psi_+^2) \\
+ 4(g^2 - 1)(g'(u + \lambda) - g)(\Psi_+^2 + \Psi_+^2),
\]

\[
\{F^ST_y, F^ST_y\} = -\frac{1}{4g^4(u + \lambda)^2} (g^2 + g + 1)(g'(u + \lambda) - 2g)(1 + \Psi_+^2) \\
+ 2g'(g^2 - 1)(u + \lambda)\Psi_+^2 u_x \\
\times \left[ (g^2 + g + 1)(g'(u + \lambda) - 2g)^2 + 4g f'(u + \lambda)\right] \\
+ 4g' (g^2 - 1)(u + \lambda)(g'(u + \lambda) - 2g)(\Psi_+^1 + \Psi_+^2) \\
+ 2(u + \lambda)^2 (3g + 3g^2)g'^2 - 8g(g^2 - g + 1)(u + \lambda)g' \right].
\]

where, for convenience, we denote

\[
f' = \frac{\partial}{\partial \lambda} f(u), \quad g' = \frac{\partial}{\partial \lambda} g(\lambda).
\]

Similarly, the components of the first fundamental form for the surface induced via generalized symmetries are

\[
\{F^Q_x, F^Q_x\} = \frac{Q^2(f''(u + \lambda)^2 - 2(u + \lambda)f' + 2(f - g))^2}{64(u + \lambda)^4 g^2} \\
\times [(g^2 + g + 1)(\Psi_+^1 + \Psi_+^2) + 4f(g^2 - 1)(\Psi_+^2 + \Psi_+^2) + 2(3g^2 - g + 3)],
\]

\[
\{F^Q_x, F^Q_y\} = \ \frac{Q(f''(u + \lambda)^2 - 2(u + \lambda)f' + 2(f - g))^2}{32(u + \lambda)^3 g^2} \\
\times [Qg(g^2 + 1) + 2\sqrt{D}(D_1 Q - Q u_x)((g^2 + g + 1)(\Psi_+^1 - \Psi_+^2) \\
+ 2(g^2 - 1)(\Psi_+^2 - \Psi_+^2) + (2D_1 Q u_x - Q f'(u + \lambda) + 2gQ) \\
\times (g^2 + g + 1)(\Psi_+^1 + \Psi_+^2) + 2(g^2 - 1)(\Psi_+^2 + \Psi_+^2) + 2(3g^2 - g + 3))].
\]
\[ \{F^Q_y, F^Q_{y'}\} = \frac{(g^2 + g + 1)}{16 g^2 (u + \lambda)} \left[ \left[ 4 \sqrt{g}(f'(u + \lambda) + 2g)u_i Q^2 + 2(D^2_i Q) u_i \right. \right. \\
- 2(f'(u + \lambda) + 2g + 2f)Q_i Q_j \left( \Psi^4_+ - \Psi^4_- \right) + \left. \left( (f'(u + \lambda) + 2g)^2 + 4gf \right) Q^2 \right. \\
- 4f'(u + \lambda) + 4g)D_i Q Q u_i + 4(f + g)(D_i Q) \left. \right) (\Psi^4_+ - \Psi^4_-) \right] \\
+ \frac{(g^2 - 1)}{4g^2 (u + \lambda)} \left[ 2 \sqrt{g}(f'Q^2 u_i + 4(u + \lambda)(D_i Q)^2 u_i \right. \\
- 2(f'(u + \lambda) + 2f)D_i Q Q \left( \Psi^4_+ - \Psi^4_- \right) - \left( 4(f'(u + \lambda) + g)D_i Q Q u_i \right. \\
+ (f'(u + \lambda) + 2g)f' Q^2 + (u + \lambda)f' (D_i Q)^2 \left( \Psi^4_+ - \Psi^4_- \right) \right. \\
- \left. \left. (3g^2 - g + 3)(f' D_i Q Q u_i + f (D_i Q)^2) + g(g^2 - g + 1)(D_i Q)^2 \right) \right] \\
+ \left( \frac{(3g^2 - g + 3)f'^2}{8g^2} + \frac{(g^2 - g + 1)f'}{2g(u + \lambda)} + \frac{(g^2 - g + 1)(g - f)}{2g(u + \lambda)^2} \right) Q^2. \quad (A.2) \]

For the vector field \( \vec{u} \), the components simplify to
\[ \{F^Q_x, F^Q_x'\} = \frac{f'(u + \lambda)^2 - 2(u + \lambda) f' + (f - g)}{64 (u + \lambda)^4 g^2} \left[ (g^2 + g + 1)(\Psi^4_+ - \Psi^4_-) + 4f(g^2 - 1)(\Psi^2_+ - \Psi^2_-) + 2(g^2 - g + 3) \right]. \]
\[ \{F^u_x, F^u_x'\} = \frac{f'(u + \lambda)^2 - 2(u + \lambda) f' + (f - g)}{32g^2 (u + \lambda)^3} \left[ (g^2 + g + 1)(\Psi^4_+ - \Psi^4_-) + 2(g^2 - 1)(\Psi^2_+ - \Psi^2_-) \right. \\
\times 2 \sqrt{f} \left( (g^2 + g + 1)(\Psi^4_+ - \Psi^4_-) + 2(g^2 - 1)(\Psi^2_+ - \Psi^2_-) \right) \right. \\
\left. \times \frac{1}{(u + \lambda)^3} g \right] \left( f'(u + \lambda)^2 - 4f'(u + \lambda) + 4f(f + g)(g^2 - g + 1)(\Psi^4_+ - \Psi^4_-) \right. \\
\left. - 2(f'(u + \lambda)^2 - 4f'(u + \lambda) + 4f(f - g))(g^2 - g + 1) \right]. \quad (A.3) \]

References

[1] Ablowitz M J and Clarkson P A 1991 Solitons, Nonlinear Evolution Equations and Inverse Scattering (Cambridge: Cambridge University Press)
[2] Ablowitz M J and Segur H 1991 Solitons and Inverse Scattering (Philadelphia, PA: SIAM)
[3] Abramowitz M and Stegun L A 1972 Handbook of Mathematical Functions (10th edn; New York: Dover)
[4] Bobenko A I and Esterle U 1985 Painlevé Equations in the Differential Geometry of Surfaces (Lecture Notes in Mathematics vol 1753) (Berlin: Springer)
[5] Briot C and Bouquet J C 1875 Théories des Fonctions Elliptiques 2nd edn (Paris: Gauthier-Villars)
[6] Byrd P and Friedmann M 1971 Handbook of Elliptic Integrals for Engineers and Scientists (New York: Springer)
[7] Calogero F and Degasperis A 1982 Spectral Transform and Solitons: Tools to Solve and Investigate Nonlinear Evolutionary Equations (New York: North-Holland)
[8] do Carmo M P 1992 Riemannian Geometry (Boston, MA: Birkhäuser)
[9] Cieśliński J 1997 A generalized formula for integrable classes of surfaces in Lie algebras J. Math. Phys. 38 4255–72
[10] Cieśliński J, Goldstein P and Szm A 1995 Isothermic surfaces in \( E^3 \) as soliton surfaces Phys. Lett. A 205 37–43
[11] Conte R and Musette M 2008 The Painlevé Handbook (Dordrecht: Springer)
[12] Faddeev L D and Takhtajan V E 1986 Hamiltonian Methods in the Theory of Solitons (Berlin: Springer)
[13] Fokas A S and Gel’fand I M 1996 Surfaces on Lie groups, on Lie algebras, and their integrability Commun. Math. Phys. 177 203–20
[14] Fokas A S, Gel’fand I M, Finkel F and Liu Q M 2000 A formula for constructing infinitely many surfaces on Lie algebras and integrable equations Sel. Math. 6 347–75
[15] Grundland A M and Post S 2011 Soliton surfaces associated with generalized symmetries of integrable equations J. Phys. A.: Math. Theor. 44 165203
[16] Helgason S 1978 Differential Geometry, Lie Groups and Symmetric Spaces (New York: Academic)
[17] Ince E L 1972 Ordinary Differential Equations (New York: Dover)
[18] Konopelchenko B 1996 Induced surfaces and their integrable dynamics Stud. Appl. Math. 96 9–51
[19] Lax P D 1968 Integrals of nonlinear equations and solitary waves Commun. Pure Appl. Math. 21 467–90
[20] Olver P J 1993 Applications of Lie Groups to Differential Equations 2nd edn (New York: Springer)
[21] Rogers C and Schief W K 2000 Backlund and Darboux Transformations: Geometry and Modern Applications in Soliton Theory (Cambridge: Cambridge University Press)
[22] Sym A 1985 Soliton surfaces and their applications (soliton geometry from spectral problems) Geometric Aspects of the Einstein Equations and Integrable Systems (Lecture Notes in Physics vol 239) ed R Martini (Berlin: Springer) pp 154–231
[23] Tafel J 1995 Surfaces in $\mathbb{R}^3$ with prescribed curvature J. Geom. Phys. 17 381–90
[24] Willmore T 1997 Riemannian Geometry (Oxford: Clarendon)
[25] Zakharov V E, Manakov S V, Novikov S P and Pitaevskii L P 1980 Soliton Theory: Inverse Scattering Method (Moscow: Nauka)
[26] Zakharov V E and Shabat A B 1974 Integration of nonlinear equations in mathematical physics by the method of inverse scattering: I Anal. Appl. 8 226–35
[27] Zakharov V E and Shabat A B 1978 Integration of nonlinear equations in mathematical physics by the method of inverse scattering: II Anal. Appl. 13 13–22