Non-Markovian quantum correlations of two-qubit system in equilibrium and nonequilibrium bosonic environments

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The quantum correlations of the two-qubit system are studied in different bosonic reservoirs by the tensor network algorithm. Both equilibrium and nonequilibrium scenarios are discussed. The non-Markov effect can improve the survival time of the quantum correlations significantly and weaken the decoherence effect. The non-Markovian dynamics with existing memory can lead to the entanglement rebirth in certain scenarios in contrast to the eventual entanglement decay or death under the memoryless Markovian case. In particular, the system in the sub-Ohmic reservoirs can reach steady state the quickest. The quantum correlations are more robust to the changes in the temperature or the temperature difference in the super-Ohmic reservoirs. At fixed times, we find that the memory can boost the correlations. However, more memory is not always better. Too much memory may also cause the decoherence.

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I. INTRODUCTION

In recent years, theoretical and experimental progresses manifest that the quantum correlations (including coherence\([1]\), entanglement\([2]\), quantum discord \([3]\) etc.) are essential resources for quantum information processing, such as quantum teleportation \([4]\), quantum cryptography \([5]\) and quantum dense coding \([6]\). The quantum correlations are not only the most primary resources for quantum information but also the cornerstone for the realization of quantum communication and quantum computer \([7]\). The physical system couples to the environment inevitably. This can cause the decoherence. The decoherence can lead to the disappearance of the quantum correlations. On the other hand, the environments can resume the quantum correlations or preserve it in certain conditions \([8–10]\). Therefore, it is very meaningful to study the quantum correlations in the open system to find the appropriate environments for implementing the quantum computing or other quantum information processing.

The open quantum system theory plays an important role in modern quantum mechanics. When the coupling between the system and environment is weak, while the memory time of the environment is short compared to the time scale of the evolution of the system, the Born-Markov approximation is applicable \([11]\). Although this approximation is very effective within its applicability, for most of the real systems in strong coupling regime or in the environment with long time correlations this approximation is rather limited and not justified. For instance, the behaviors of superconducting qubit in circuit QED system\([12]\), NV centres in diamonds \([13]\) and quantum dots in semiconductors\([14]\) all require a strong coupling description. The non-Markovian behaviors emerge inevitably in such strong coupling systems, and this has been experimentally demonstrated \([15–16]\). The memory effect of the non-Markovian dynamics may preserve the quantum correlations over a longer period of time, which opens up the potential for realizing quantum technologies \([17–19]\). The corresponding approaches go beyond the Markovian regime to the non-Markovian regime which include the Nakajima–Zwanzig projection operator equations \([11]\), time-convolutionless master equations \([11]\), Keldysh–Lindbald equations \([20]\) and reaction coordinate methods \([21]\). However, they are still limited to the weak coupling regimes. Other approaches hold for the strong coupling regime include the polaron transformed master equation \([22, 23]\), the hierarchical equations of motion (HEOM) method \([24]\) and the influence functional (IF) method \([25–28]\). We mainly focus on the IF method in this paper.

The IF integrates all the influences from the environment \([25]\). However, the cost of computing the IF without approximation is huge and the size of the IF scales exponentially with the number of the time steps. With finite memory approximation, Makri and Makarov showed that the path integral could be reformulated as a propagator of the augmented density tensor (ADT) which encodes the system’s history \([26–27]\). The influence functional can be assembled by a series of influence functions. The influence function quantifies how the system evolution at some time is influenced by the state of the system at the earlier time. This approach is called the Quasi-Adiabatic Path Integral (QUAPI) \([26, 27]\). Naturally, the IF can be described by MPO (matrix product operator) and the ADT can be efficiently represented and propagated in the form of a matrix product state (MPS) in the tensor network language \([29–33]\). The resulting TEMPO (time-evolving matrix product operators) method is numerically exact and was widely used to many studies, such as the optimal control of non-Markovian open quantum systems \([34]\), non-additive effects of the environments \([35]\), quantum heat statistics \([36]\) and the thermalization of the one dimensional many body system \([37]\). In fact, the TEMPO can be recast in the process tensor (PT) frame \([38]\), where the PT is a multi-linear map from the set of all possible control operation sequences in the lab on the system to the resulting output states and can be generally expressed as in MPO form \([39]\). Constructing the tensor network IF for the general dynamics can be found in \([40]\).

In this paper, we utilize the process tensor-time evolving block decimation (PT-TEBD) algorithm \([33, 37, 41]\) to study the quantum correlations of the two-qubit system strongly coupled to the bosonic environment in both equilibrium and nonequilibrium scenarios. The non-Markovianess influences the dynamics of the correlations remarkably for all the baths we studied. The entanglement may give rebirth after sudden death due to the non-Markovian effect rather than eventual death in the dynamics under the memoryless conditions. The oscillations of the correlations are common in the different types of baths. The amplitudes of the correlation dynamics of the two qubits in the super-Ohmic are the largest and decay the slowest, showing the strongest memory effect, while those in the sub-Ohmic reach the steady value the quickest on the contrary. The higher temperature is, the faster is system arrives at the steady state. In the meanwhile, the system in the super-Ohmic baths are insensitive to the changes in the temperature or the temperature difference. We find that the reservoir with memory can boost the correlations. However, more memory is not necessarily better. Too much memory may sometimes also destroy the quantum correlations.

The organization of our paper is as follows. In section II, we introduce the model we study. Then we derive the influence functional and illustrate it in the tensor network language. We focus on the PT-TEBD algorithm to implement the system evolution. In section III, we introduce some quantum correlations and study them in different scenarios. At last, we draw a conclusion in section IV.
II. TENSOR NETWORK METHOD TO STUDY NON-MARKOVIAN DYNAMICS OF TWO-QUBIT SYSTEM IN THE ENVIRONMENT: MODEL AND THE EVOLUTION

We consider two interacting qubits coupled to their respective baths as shown in Fig. 1(a). The total Hamiltonian is given as

$$H_{total} = H_S + H_{SB} = H_S + H_B + H_I.$$  

(1)

The Hamiltonian of system is

$$H_S = \frac{\omega_1}{2} \sigma_1^z + \frac{\omega_2}{2} \sigma_2^z + J \sigma_1^x \sigma_2^x$$  

(2)

where $\sigma_i^1 = \sigma_i \otimes \mathbf{1}$ and $\sigma_i^2 = \mathbf{1} \otimes \sigma_i$, $\sigma_i$ is Pauli matrix. The energy gaps of the qubits are given as $\omega_1 = \omega_2 = 1$ and the coupling $J = 0.375$ measures the strength of the inter-qubit interaction. The Hamiltonian of the environment plus the interaction between the single qubit and corresponding interacting bath are given as

$$H_B^\alpha + H_I^\alpha = \sum_k \omega_k^\alpha \hat{a}_k^\alpha \hat{a}_k^{\alpha\dagger} + \sigma_2^\alpha \sum_k (g_k^\alpha \hat{a}_k^\alpha \hat{a}_k^{\alpha\dagger} + g_k^{\alpha\ast} \hat{a}_k^{\alpha\dagger} \hat{a}_k^\alpha)$$  

(3)

$g_k$ is coupling constant between qubit and environment. $\hat{a}_k (\hat{a}_k^{\dagger})$ is annihilation (create) operator of the bosonic environment. We assume that the system and baths are separable at initial time and the baths are initially in Gaussian states, e.g. thermal equilibrium at different temperatures $T_\alpha$. In the meanwhile, the two qubits are also separable initially and are both prepared in the ground state. The sketch of the model is shown in Fig. 1(a).

![Fig. 1.](image)

We work in Liouville space in the following calculation, i.e. the super-operators act on the vectorized density matrices. This can be illustrated as in Fig. 1(b). The red solid circle with one leg is called one rank tensor or vector in the tensor network language$^{[29]}$. The operator acts on the density matrix can be re-formated as

$$\rho = \sum_{ij} \rho_{ij} |i\rangle \langle j| \rightarrow |\rho\rangle\rangle = \sum_{ij} \rho_{ij} |i\rangle \otimes |j\rangle$$  

(4)

The trace is re-formated as $Tr\cdot = \sum_k \langle\langle k,k|\cdot = \langle\langle 1|\cdot$, where $\langle\langle k,k| = \langle k| \otimes \langle k|$ and $|1\rangle$ is the vectorized unity matrix. The operator acts on the density matrix can be re-formated as

$$\begin{cases} O\rho \rightarrow O^L|\rho\rangle = O \otimes \mathbf{1} |\rho\rangle \\ \rho O \rightarrow O^R|\rho\rangle = \mathbf{1} \otimes O^T |\rho\rangle \end{cases}$$  

(5)
The basis of the whole system is spanned by the product space connecting leg means contraction. The time evolution of the total system is governed by the Liouville operator \( \mathcal{L} = -i[\hat{H}, \cdot] \).

\[
\rho(t) = e^{\mathcal{L}t} \rho(0)
\]

We separate \( \mathcal{L}_{\text{total}} = \mathcal{L}_{\text{S}} + \mathcal{L}_{\text{SB}} \), where \( \mathcal{L}_{\text{S}} = -i[\hat{H}_s, \cdot] \) represents the pure system part and \( \mathcal{L}_{\text{SB}} \) represents the remaining part including both the interaction and environmental part. Discretize time as \( N \) uniform steps and perform the second-order Suzuki-Trotter splitting \([22]\) between the system Liouville operator and environmental Liouville operator

\[
e^{\mathcal{L}_{\text{total}}t} \approx \left[ e^{\mathcal{L}_{\text{S}}t} e^{\mathcal{L}_{\text{SB}}t} \right]^N + O(\delta t^3)
\]

The basis of the whole system is spanned by the product space \( |s_i, b_i\rangle = |s_1^i, s_2^i, b_1^i, b_2^i\rangle \) at \( i\)-th time step. The initial state can be represented by \( \rho_{\text{total}}(s_0, b_0) = \rho_0^s(s_0^1)\rho_0^s(s_0^2) \rho_0^b(b_0^1)\rho_0^b(b_0^2) = |s_0, b_0\rangle \). Trace out the degree of freedoms of the baths at both sides of the Eq.\( \ref{eq:8} \) and then insert \( \sum_i |s_i, b_i\rangle \langle s_i, b_i| \) at the \( ith \) time step

\[
\rho_s(s_N') = \text{Tr}_{b_N} \left[ \sum_{s_N, b_N} \sum_{s_{N-1}, b_{N-1}} \cdots \sum_{s_0, b_0} \sum_{s_0'} \sum_{b_0'} \langle s_N'| e^{\mathcal{L}_{\text{S}}t} |s_N, b_N\rangle \langle s_{N-1}| e^{\mathcal{L}_{\text{S}}t} |s_{N-1}, b_{N-1}\rangle \cdots \langle s_0'| e^{\mathcal{L}_{\text{S}}t} |s_0, b_0\rangle \right] (8)
\]

\[
\text{Tr}_{b_N} \left[ \langle s_N, b_N| e^{\mathcal{L}_{\text{S}}t} |s_{N-1}, b_{N-1}\rangle \langle s_{N-1}, b_{N-1}| e^{\mathcal{L}_{\text{S}}t} |s_{N-2}, b_{N-2}\rangle \cdots \langle s_0, b_0| \rho(s_0, b_0)\rangle \right] \]

where \( \mathcal{F}(s_1, s_1'; \ldots, s_N, s_N') \) is the influence functional\([23]\).

\[
\mathcal{F}(s_1, s_1'; \ldots, s_N, s_N') = \text{Tr}_{b_N} \left[ \langle s_N, b_N| e^{\mathcal{L}_{\text{S}}t} |s_{N-1}, b_{N-1}\rangle \langle s_{N-1}, b_{N-1}| e^{\mathcal{L}_{\text{S}}t} |s_{N-2}, b_{N-2}\rangle \cdots \langle s_0, b_0| \rho(s_0, b_0)\rangle \right] (9)
\]

Any \( n \)-dimensions array can be called \( n \) rank tensor in tensor network language, for example the density matrix as shown in Fig\( \ref{fig:1}\). For a general quantum state, its expansion coefficient under an orthonormal basis can be considered as a tensor. We can utilize tensor train decomposition to obtain its MPS form\([29, 30]\). With the help of the augmented matrix product state (aMPS \([37, 39, 40]\) as shown in Fig\( \ref{fig:1}\) – we formally construct MPS for the two-qubit system and the augmented legs measure the correlation between the system and the two baths. In our setting, the bond dimensions of the augmented legs are one which means that there is no correlation between the system and the two baths initially. The eq\( \ref{eq:3} \) is graphically illustrated in Fig\( \ref{fig:2} \). The network is composed by the total density matrix which is in the MPS form (the qubits and baths are represented by respective site), the pure
system Liouville tensor $\langle \langle s_i^t | e^{\frac{-\delta t}{2}} | s_i \rangle \rangle$ and environmental Liouville tensor $\langle \langle s_i, b_i | e^{\frac{-\delta t}{2}} | s_{i-1}^t, b_{i-1} \rangle \rangle$. The pure system Liouville tensor contains the interaction between two qubits. Therefore, it has four legs indexed by $s_1^i, s_2^i, s_1^i, s_2^i$. There is no interaction between two baths, so their action can be separated. The environment Liouville tensor for the single bath also has four legs indexed by $s_1^{i-1}, s_2^{i-1}, b_1^{i-1}, b_2^{i-1}$. The blue solid line in Fig.2(a) represents the degree of freedoms of the baths $b_i$ and these legs should only connect themselves. The black solid line in Fig.2(a) represents the degree of freedoms of the qubits $s_i$. We connect the same indexes from the MPS of the system according to the Eqn.8. The same indexes are connected which means summation. At last, we trace the degree of freedoms of the baths, where the trace cap is described by semi-circle in Fig.2(a). Also, the contraction of the influence functional produces the process tensor as shown in Fig.2(b), which is a multi linear map from the set of all possible control operation sequences in the lab on the system to the resulting output states[33]. The process tensor includes all the influences of the environments. We connect it to the pure system Liouville tensor according to the time order. The process tensor can be re-cycled. Finally, the evolution of the system can be performed by process tensor-time evolving block decimation (PT-TEBD)[33, 37, 41] as shown in Fig.2(c). We contract the network layer by layer from the bottom to the top.

The influence functional or the resulting process tensor can be constructed in a matrix product operator form (MPO)[31, 32, 39, 40]. When we choose the eigen-basis of the environment part as the computational basis, the influence functional for the single bath can be written as

$$\mathcal{F}(s_1, s_1'; \ldots s_N, s_N') = \exp \left(- \sum_{k=1}^{N} \sum_{k'=1}^{k} (s_k^+ - s_k^-)(\eta_{kk'} s_{k'}^+ - \eta_{kk'}^* s_{k'}^-) \right)$$

$$= \prod_{k=1}^{N} I_0(s_k^+) \prod_{k=1}^{N-1} I_1(s_{k+1}^+, s_k^+) \cdots \prod_{k=1}^{N} I_N(s_{N-1-N}^+, s_k^+)$$

where $I_m = \exp \left(- (s_{k+m}^+ - s_{k+m}^-)(\eta_{k+m,k} s_{k+m}^+ - \eta_{k+m,k}^* s_{k+m}^-) \right)$ and $|s^\pm\rangle \in \{|-1\rangle, |1\rangle\}$ of the eigen-basis of the $\sigma_x$. The
details expressions of the $\eta_{kk'}$ can be found in \cite{26, 27}. The $\eta_{kk'}$ is dependent on the spectrum density $J(\omega)$

$$J(\omega) = 2\alpha \frac{\omega^\zeta}{\omega^\zeta + 1} e^{-\frac{\omega}{\omega_c}}$$  \hspace{1cm} (11)

The $\omega_c$ is the frequency cutoff. When $\zeta = 1$, the reservoirs are Ohmic; when $\zeta < 1$, the reservoirs are sub-Ohmic; when $\zeta > 1$, the reservoirs are super-Ohmic. Under the sub-Ohmic spectrum, the lower frequencies $\omega < \omega_c$ dominate. Under the super-Ohmic spectrum the higher frequencies $\omega > \omega_c$ dominates. The low-frequency behavior is described by $J(\omega) \sim \omega^n$.

The Eq.\ref{eq:10} can be translated as MPO form, where $I_0$ is represented by two rank tensor and $I_{m>0}$ is represented by four rank tensor as shown in Fig.3a. The legs representing the same time points are connected. By the repetitive SVD decomposition and contraction as shown in Fig.3b, we can reshape the tensor in Fig.3a as the one in Fig.3c. Finally, we contract the tensor layer by layer from the bottom to the top in Fig.3c to derive the desired MPO form tensor in Fig.3d. In practice, one usually takes memory cutoff for saving computation cost. For a specific bath, its cutoff depends on its correlation property. The memory cutoff $\tau_c$ dictates how long the system histories are kept to capture the non-Markovianess. We make $I_m = I \otimes I$ once the $m \delta t$ exceeds the $\tau_c$.

There are three error sources when we perform the PT-TEBD program. The first type originates from the second-order Suzuki-Trotter split which causes the third and higher order error $O(\delta t^3)$. The second type error comes from the low rank matrix approximation when we derive the process tensor and perform time evolving block decimation. Suppose $X = U S V^\dagger$ which is the exact SVD decomposition. We throw away some lower singular value in $S$ and corresponding vectors in $U$ and $V$. The new $\hat{X} = \tilde{U} \tilde{S} \tilde{V}^\dagger$ and the error need to be $||\hat{X} - X||_2 < \epsilon \max S$. The third type is the memory cutoff, which depends on the baths property to retain the most non-Markovian time correlation. The procedure of the PT-TEBD is that we firstly construct process tensor for the baths, the PT then can be recycled on the different the initial states of the system. Thereafter, we perform the TEBD algorithm to realize the time evolution. So far, we can only derive the result at the $N-th$ time step. We are also interested in the intermediate time evolution. In this situation, we can trace out the latter time legs to obtain the information at the intermediate

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3.png}
\caption{(a) Graphically illustration of eq.\ref{eq:10} when $N = 3$; (b) SVD decomposition and then contraction (c) By the repetitive SVD decomposition and contraction, we can reshape the tensor in (a) as this one. (d) The final desired MPO form of the process tensor.}
\end{figure}
times as shown in Fig. 4. That means we use the trace cap on the legs of the system after the desired time step. This approach is permitted by the containment property of the process tensor, which says if \( k \geq k_0 \geq j_0 \geq j \), the process tensor \( T_{k_0,j_0} \) is contained in \( T_{k,j} \). The more information about the PT-TEBD algorithm can be found in [33, 37].

![Graphical illustration of how to derive the intermediate time evolution at the \( 2^{nd} \) time step when total time steps \( N = 3 \). The trace cap is \(|1\rangle\rangle\).](image)

III. RESULTS: NON-MARKOVIAN QUANTUM CORRELATIONS UNDER EQUILIBRIUM AND NONEQUILIBRIUM ENVIRONMENTS

A. The measures of the quantum correlations

In this subsection, we introduce certain important measures of the quantum correlations. Coherence, being at the heart of interference phenomena, plays a cornerstone role in quantum physics as it enables applications that are impossible within classical mechanics and can be measured as [1]

\[
\mathcal{C}_1 = \sum_{i \neq j} | \rho_{ij} | .
\]  

(12)

Quantum entanglement is a major resource for accomplishing quantum information processing tasks such as teleportation [4], quantum key distribution [5], and quantum computing [2] etc. Among many measures of entanglement of a two-qubit system, the concurrence is extensively used so far in many contexts. The concurrence of a two-qubit mixed state \( \rho \) is defined as [43]

\[
\mathcal{C} = \text{Max}(0,\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4),
\]  

(13)

where \( \lambda_i \) represents the square root of the \( i^{th} \) eigenvalue, in descending order of the matrix \( \tilde{\rho} \) with \( \tilde{\rho} = (\sigma_2 \otimes \sigma_2) \rho^T (\sigma_2 \otimes \sigma_2) \), while \( T \) denotes transposition.

Another important quantum correlation measure is the quantum discord [3]. It measures the non-classical correlation between two subsystems of a quantum system. The discord includes correlations that are due to quantum physical effects, but does not necessarily involve the concept of quantum entanglement. In fact it is a different type of quantum correlation than the entanglement because separable mixed states (that is, with no entanglement) can have non-zero quantum discord. Sometimes it is also identified as the measure of quantumness of the correlation functions. The geometric discord of a bipartite quantum state is defined as [44].
\[ D(\rho) = \min_{\rho_0 \in \Omega} ||\rho - \rho_0||^2 \]  

(14)

where \( \Omega \) denotes the set of zero-discord states and \( ||X - Y||^2 = Tr(X - Y)^2 \) is the square norm in the Hilbert-Schmidt space. It can be evaluated for an arbitrary two-qubit state. For any two qubit state the density matrix is given by the following expression:

\[
\rho_{AB} = \frac{1}{4}(I_a \otimes I_b + \sum_{i=1}^{3}(a_i \sigma_i \otimes I_b + I_a \otimes b_i \sigma_i) + \sum_{i,j=1}^{3} C_{ij} \sigma_i \otimes \sigma_j).
\]  

(15)

The geometric discord is given as [44]

\[ D(\rho) = \frac{1}{4}(||a||^2 + ||C||^2) - \lambda_{max} \]  

(16)

where \( \lambda_{max} \) is the maximum eigen-value of the \( aa^T + CC^T \). \( a \) is the vector composed by \( a_i \) and \( C \) is matrix composed by \( C_{ij} \).

**B. Ohmic reservoirs**

We study the quantum correlations in the Ohmic reservoirs under different temperatures in this subsection. The memory cutoff is 40 steps and the time interval is \( \delta t = 0.2 \) for each step. This setting contains enough non-Markovian effect. The truncation error for deriving the process tensor is \( \xi = 10^{-5} \) and is \( \epsilon = 10^{-6} \) for performing TEBD in this study. In Fig.5(a)∼(c), we plot the quantum correlations in the Ohmic baths under different temperatures. They all oscillate and the amplitudes decay until the correlations reach certain constants. The oscillation is very common in non-Markovian environment[10, 19]. There are no further new phenomena even though we prolong the evolution time. The concurrence vanishes in the higher temperature zone but survives forever in the low temperature regime. The entanglement will suddenly die and then reappear again for the higher temperature. Sudden deaths and rebirths of entanglement have been found in different physical models (see, for instance, Ref.[17, 45, 46]). The geometry discord and the coherence can even survive at higher temperatures. More than that, the geometry discord and the coherence vary non-monotonically with the temperature at certain times. We also plot the quantum correlations under the Markovian approximation (that means we only keep one step memory in the evolution) as a contrast. In the Fig.5(d)∼(f), the oscillations are weaker compared to the non-Markovian case. The most harvesting quantum correlations under the memoryless approximation are less than those without the approximation. Therefore, the temporal correlations or the memories can boost and maintain the quantum correlations under certain conditions.

We also plot the quantum correlations at the nonequilibrium as shown in figure Fig.6. The temperature of one of two bathes is 0.01 while the temperature of another bath increases from 0.01. Compared to the equilibrium case, the trends are similar. The geometry discord and the coherence vary non-monotonically with the temperature differences at certain times.

**C. sub-Ohmic reservoirs**

We now study the quantum correlations in the sub-Ohmic reservoirs with different temperatures in this subsection. The memory cutoff is 50 steps and the time interval is \( \delta t = 0.2 \) for each step. In contrast to the Ohmic case, the quantum correlations in the sub-Ohmic bathes oscillate with less amplitudes and the amplitudes decay more rapidly as shown in Fig.7(a)∼(c). In the meanwhile, the harvesting quantum correlations are hard to retain when the environment loses memory as shown in Fig.7(d)∼(f). The geometry discord and the coherence also vary non-monotonically with the temperature at certain times.
FIG. 5. Equilibrium quantum correlations in the Ohmic reservoirs with different temperatures (a)∼(c) for non-Markovian evolution and (d)∼(f) for Markovian evolution. Other parameters $\delta t = 0.2$ for both the non-Markovian evolution and the Markovian evolution, $\zeta = 1, \alpha = 0.1, \omega_c = 4$.

FIG. 6. Nonequilibrium quantum correlations in the Ohmic reservoirs under different temperatures (a)∼(c) for non-Markovian evolution and (d)∼(f) for Markovian evolution. The temperature of the one of two bathes is 0.01, another bath increases from 0.01. Other parameters $\delta t = 0.2$ for the non-Markovian evolution, $\delta t = 0.05$ for the Markovian evolution, $\zeta = 1, \alpha = 0.32, \omega_c = 4$. 
In the nonequilibrium scenario in Fig. 8(a), the entanglement can still survive in the long time at larger temperature difference. And the larger temperature difference can also boost the correlations in Fig. 8(b, c). The effect of the nonequilibrium is, however, weak for the case under the memoryless approximation. The system reaches the steady state faster when the baths are at higher temperatures for both the Ohmic and sub-Ohmic reservoirs.
D. super-Ohmic reservoirs

We now study the quantum correlations in the super-Ohmic baths under different temperatures in this subsection. The memory cutoff is 40 steps and the time interval is $\delta t = 0.025$ for each step. As shown in Figure 9, the oscillations of the quantum correlations are more significant and they damp more slowly compared to the previous two cases. The entanglement can even give rebirth at high temperatures in Figure 9(a). The geometric discord and coherence are not sensitive to the temperature compared to the two previous cases in Figures 9(b,c). The quantum correlations of the two-qubit system in the super-Ohmic baths can reach the largest values among the three types of baths. Under the Markovian approximation, the quantum correlations seem to be not sensitive to the temperature. The dynamics of the quantum correlations under the memoryless approximation also decays more slowly compared to the previous two cases. The super-Ohmic environment has the strongest memory effect. This is consistent with [47].

![Figure 9](image-url)

**Figure 9.** Equilibrium quantum correlations in the super-Ohmic baths under different temperatures (a)–(c) for non-Markovian evolution and (d)–(f) for Markovian evolution. Other parameters $\delta t = 0.025$ for both the non-Markovian evolution and the Markovian evolution, $\zeta = 2, \alpha = 0.1, \omega_c = 4$.

As shown in Figure 10, in the nonequilibrium scenario, the dynamics of the correlations are similar to the equilibrium case. The dynamics of the correlations under the memoryless approximation are not sensitive to the temperature differences. In all cases, we find that the reservoirs with memory maintain the quantum correlations and show a weak decoherence effect.
FIG. 10. Nonequilibrium quantum correlations in the super-Ohmic baths under different temperatures (a)∼(c) for non-Markovian evolution and (d)∼(f) for Markovian evolution. The temperature of the one of the two baths is 0.01, the temperature of another bath increases from 0.01. Other parameters $\delta t = 0.025$ for both the non-Markovian evolution and the Markovian evolution, $\zeta = 2, \alpha = 0.1, \omega_c = 4$.

E. The effect of the nonequilibrium and the memory on quantum correlations

To see the influence of the nonequilibrium and the memory on quantum correlations, we plot the variations of the quantum correlations with respect to the memory cutoff and the temperature difference at the fixed long time in Fig.11. We also provide the two dimensional histogram view in Fig.12. Obviously the memory can improve the harvesting quantum correlations in all cases. In Fig.11, the entanglement can be awakened by more memory. However, the quantum correlations may also be reduced as shown in Fig.11. This shows that more memory is not necessarily always better for improving the quantum correlations. The entanglement varies with the temperature difference monotonically. The geometric discord and coherence may be amplified by the nonequilibriumness as shown in Fig.12.

IV. DISCUSSION

In conclusion, we use the numerically exact PT-TEMPO algorithm to study the quantum correlations within the two-qubit system in the different type reservoirs. In contrast to the system under the Markovian approximation, the memory influences the correlation dynamics remarkably. The quantum correlations not only can re-occur after the sudden death and oscillate but also can give rise to the correlations even after a long time period. We find that the environments with memory shows a weak decoherence effect. The memory can improve the harvesting quantum correlations while it may also reduce them depending on the lengths of the memories. For the different type baths, the behaviors of the quantum correlations are somewhat different. These are mainly embodied in the duration time and the amplitudes of the oscillations. Concretely, the quantum correlations of the two qubits oscillate the most lastingly in the super-Ohmic baths, while reach the steady the fastest in the sub-Ohmic baths. The super-Ohmic environment has the strongest memory effect. At fixed long time, the correlations can show non-monotonic behaviors with the varying temperatures or the temperature differences. Under certain conditions, the nonequilibrium can boost the quantum correlations. This suggests a way of using the environmental engineering to boost or maintain the quantum correlations. The quantum correlations within two-qubit system in the super-Ohmic baths are the most insensitive to the temperatures or the temperature differences. This robust property provide a possibility for large-scale quantum computing.
FIG. 11. Variations of the quantum correlations with respect to the memory cutoff and the temperature difference at the fixed time $t = 20$, (a) Ohmic, $\zeta = 1$; (b) sub-Ohmic, $\zeta = 0.6$; (c) super-Ohmic, $\zeta = 2$. The temperature of the one of the two baths is 0.01, the temperature of another bath increases from 0.01. The time step is $\delta t = 0.2$ for both Ohmic baths and sub-Ohmic and $\delta t = 0.1$ for the super-Ohmic baths.

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Appendix A: histogram view of Fig. 11

We provide the histogram view of Fig. 11 here. The parameters are the same.

FIG. 12. Variations of the quantum correlations with respect to the memory cutoff and the temperature difference at the fixed time $t = 20$, (a) Ohmic, $\zeta = 1$; (b) sub-Ohmic, $\zeta = 0.6$; (c) super-Ohmic, $\zeta = 2$. The temperature of the one of the two baths is 0.01, the temperature of another bath increases from 0.01. The time step is $\delta t = 0.2$ for both Ohmic baths and sub-Ohmic and $\delta t = 0.1$ for the super-Ohmic baths.
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