ON GENERIC PROPERTIES OF FINITELY PRESENTED MONOIDS AND SEMIGROUPS

MARK KAMBITES
School of Mathematics, University of Manchester, Manchester M13 9PL, England.

Abstract. We study the generic properties of finitely presented monoids and semigroups. We show that for positive integers \( a > 1, k \) and \( m \), the generic \( a \)-generator \( k \)-relation monoid and semigroup (defined in any of several definite statistical senses) satisfy the small overlap condition \( C(m) \). It follows that the generic monoid is torsion-free and \( J \)-trivial and, by a recent result of the author, admits a linear time solution to its word problem and a regular language of unique normal forms for its elements. Moreover, the uniform word problem for finitely presented monoids is generically solvable in time linear in the word lengths and quadratic in the presentation size. We also prove some technical results about generic sets which may be of independent interest.

1. Introduction

Traditional complexity theory studies the time taken to solve a problem or execute an algorithm in the “worst case”, but for many problems the “worst case” arises very infrequently. Probably the best known example is Dantzig’s simplex method for linear programming [2], which has exponential worst case time complexity but in practice almost invariably terminates in linear time (see eg. [11]). Now over 60 years old, it remains the preferred choice for practical applications, even though there are now alternative algorithms with worst-case polynomial time complexity. Phenomena such as this motivated the development of average-case complexity [6], which measures, roughly speaking, the mean difficulty of a problem across instances, with respect to some measure. Average-case complexity has proved extremely helpful for obtaining a theoretical understanding of the “practical” difficulty of problems, especially within the class NP of problems admitting non-deterministic worst-case polynomial time solution.

Average-case analysis can also be applied outside NP, but it meets with a conceptual difficulty. For most applications, what matters is not so much the mean difficulty of a problem of across instances, but rather the typical difficulty of instances encountered in practice. As is well-known to statisticians, the mean value of a data set is not necessarily a guide to the typical values, since the former can be heavily skewed in one direction by a very small number of very extreme outliers. Likewise, the average-case complexity of a problem can be skewed upwards by a very small proportion of very difficult instances. Within NP worst cases are single exponential; this imposes a limit on the “extremeness” of outlying instances and hence their ability to distort the mean. Outside NP, however, the distortion can be much more dramatic,
with a tiny minority of extremely difficult instances potentially inflating the average-case complexity well beyond the complexity of the typical instance encountered in practice. This culminates in the extreme case of recursively unsolvable problems, whose average-case complexity is not defined at all, even though algorithms may exist to solve such problems efficiently for an overwhelming majority of cases [4].

The aim of generic-case complexity is directly to analyse the complexity of typical problem instances, as distinct from the average difficulty of problem instances. Rather than introducing a measure on the instance space, the key idea is the stratification of an instance space (or indeed any other set) into an infinite sequence of finite subsets. A subset \( X \) of the space is called generic if the proportion of elements in each finite set which belong to \( X \) approaches 1 as one moves along the sequence. The generic complexity is (very roughly speaking) the minimum complexity attainable on a generic set. Compared with the average-case approach, the key feature is that no single instance (indeed no finite set of instances), makes any contribution at all to the generic properties of the space. Generic-case complexity was introduced by group theorists [10], investigating the large stock of hard algorithmic problems which occur in the study of finitely generated infinite groups. It has proved especially useful in view of recent interest in the use of non-commutative groups as a basis for cryptographic systems [20], permitting for example a theoretical understanding of the success of the length-based attack [17] on the Shpilrain-Ushakov key establishment protocol based on the Thompson group [19].

The main aim of this paper is to study the generic properties of finitely presented monoids and semigroups, and hence to understand the generic-case complexity of uniform decision problems for monoids and semigroups. Our main results show that, with respect to a number of very natural stratifications, the generic finite monoid presentation (over a given alphabet and with a given number of generators) satisfies small overlap conditions in the sense introduced by Remmers [15, 16] (see also [7]). Small overlap conditions are natural semigroup-theoretic analogues of the small overlap conditions extensively used by combinatorial group theorists, and so our main result can be viewed as loosely analogous (although our objectives and hence our formalism are rather different) to the well-known fact, first asserted by Gromov [5] and proved in detail by Ol’shanskii [13], that the generic finitely presented group is word hyperbolic.

These results immediately tell us a great deal about the algebraic structure of the generic monoid. For example, we learn that it is \( J \)-trivial, and hence torsion-free with no non-trivial subgroups. Even more important, by recent results of the author [5], the uniform word problem for such presentations is solvable in (worst-case) time linear in the words lengths and quadratic in the presentation size. Since it can be checked in (worst-case) quadratic time whether a presentation satisfies a small overlap condition,

\footnote{For brevity, we use statements such as “the generic \( X \) has property \( Y \)” as shorthand for “there is a generic subset of the set of \( X \)’s, every member of which has property \( Y \)”. Of course the generic \( X \) truly “exists” only in the case that a single isomorphism type forms a generic subset of \( X \)’s; in this case the isomorphism type has all the ascribed properties, so the terminology is unambiguous!}
it follows that the uniform word problem for finitely presented monoids is generically solvable in (worst-case) time linear in the word lengths and quadratic in the presentation size. All of these results apply equally to semigroups without identity elements.

As already remarked, generic-case complexity has been developed by group theorists, and the literature is largely concerned with applications to advanced group theory; as a result, much of it is not readily accessible to non-algebraists. An additional objective of this article is to provide a gentle (although by no means comprehensive) introduction to generic sets and properties, and generic-case complexity, in a form fully intelligible to the reader without a specialist algebraic background. Monoid presentations are combinatorially simpler objects than group presentations, and most of our proofs are of an elementary combinatorial nature which should allow them double as detailed worked examples to give the reader a feel for the theory of generic-case complexity. The few places where we resort to more advanced algebraic notions are clearly delimited and self-contained, so that the bulk of the article can be understood without following these parts in detail.

In addition to this introduction, this article comprises four sections. Section 2 provides a gentle introduction to generic sets and generic-case complexity. In Section 3 we prove our main results about generic monoids and semigroups with respect to certain stratifications. In Section 4 we prove some technical results regarding the relationships between different stratifications; these may be of some independent interest; these are applied to show that our results about generic monoids apply regardless of which of several natural stratifications is chosen. Finally, Section 5 explores the consequences of our characterisations of generic monoids and semigroups, including the fact that the uniform word problems for finitely presented monoids and semigroups are generically solvable in time quadratic in the presentation lengths and linear in the word lengths.

2. Generic Properties and Generic-case Complexity

In this section we provide a brief introduction to generic sets and generic complexity. A more comprehensive treatment can be found in [4]. Our aim is to make the paper accessible to as wide an audience as possible, and so we endeavour to keep mathematical prerequisites to a minimum. However, we cannot avoid assuming some elementary familiarity with the theory of sets and sequences.

Let $S$ be a countably infinite set. A stratification of $S$ is an infinite sequence $S_1, S_2, \ldots, S_n, \ldots$ of finite subsets of $S$ whose union is $S$. The computationally-orientated reader may like to bear in mind the example where $S$ is the instance space for some problem, and $S_n$ is the set of instances of size $n$ for some suitable notion of size; however, we caution that in general the subsets $S_n$ need not be disjoint. We call the stratification spherical if the sets $S_n$ are pairwise disjoint $(S_i \cap S_j = \emptyset$ for all $i \neq j)$ and at the other extreme ascending if they form an ascending sequence under containment $(S_i \subseteq S_j$ for all $i < j)$.
Now let $X$ be a subset of $S$. We say that $X$ is \emph{generic} (with respect to the given stratification) if
\[ \lim_{n \to \infty} \frac{|X \cap S_n|}{|S_n|} = 1 \]  
(1)
The subset $X$ is called \emph{negligible} if $S \setminus X$ is generic, or equivalently, if the limit on the left-hand-side of (1) is defined and equal to 0. Intuitively, $X$ is generic if the probability that an instance of size $n$, chosen uniformly at random, lies in $X$ can be made arbitrarily close to 1 by choosing large enough $n$.

Note that, for any given set $X$, the limit on the left-hand-side of (1) may not be defined, and indeed for almost any stratification it is easy to construct a set $X$ for which it is not. The function
\[ X \mapsto \lim_{n \to \infty} \frac{|X \cap S_n|}{|S_n|} \]
is a finitely additive probability measure defined on those subsets of $X$ for which the limit converges, but it is typically \emph{not} a measure in the usual sense, since it lacks countable additivity. This fact is no accident: a countably additive measure on a countable set cannot assign 0-measure to all the singletons, but as we noted in the introduction, a key feature of the generic approach is that single instances are regarded as negligible. Nonetheless, the intuition that the generic sets are those of “full measure” can be helpful, and they satisfy many of the elementary properties of such sets. In particular, the reader can easily verify that if $X$ is generic and $X \subseteq Y$ then $Y$ is generic, while if $X$ and $Y$ are both generic then $X \cap Y$ is generic. Obvious dual statements hold for negligible sets.

Notice that, in our initial definition of generic sets, we have placed no requirements on the \emph{rate} of convergence of the left-hand-side of (1). Genericity is an asymptotic property, and if convergence is very slow then the asymptotic behaviour may not be reflected in “practical sized” instances. We call a set $X$ \emph{superpolynomially} generic/negligible if the appropriate limit converges faster than $1/n^p$ for every $p \in \mathbb{N}$, and \emph{exponentially} generic/negligible if it converges faster than $p^n$ for some $p \in (0, 1]$. (In the literature some authors use the term “strongly generic” for what we have called exponentially generic sets, while some use “strongly generic” to mean superpolynomially generic and “supergeneric” to mean exponentially generic. To avoid confusion, we shall avoid these terms in favour of less concise but more descriptive ones.)

We now turn our attention to the application of generic sets in computational complexity. This requires us to consider explicitly not just abstract algorithmic problems, but also also stratifications of instance spaces. We define a \emph{stratified problem} to be an algorithmic decision problem equipped with a stratification on its instance space. (We shall restrict our attention here to decision problems, but analogous definitions can be made for more general computational problems.)

Of course traditional complexity theory is implicitly concerned with stratified problems: to study the asymptotic complexity of a problem one requires a notion of the \emph{size} of each member of the instance space $S$. As we have
already remarked, this automatically induces a stratification given by setting \( S_n \) to be the set of all instances of size \( n \). We call this the \textit{input size stratification} for the problem. However, the dependence on stratification is much tighter in generic complexity theory than it is in traditional complexity theory – many authors discussing traditional complexity of algorithmic problems prefer to avoid detailed discussion of data encoding and hence of exact instance sizes; this is entirely reasonable since traditional complexity classes are largely insensitive to minor encoding issues. But for generic-case complexity, these issues can make a very big difference.

Note also that, while the input size stratification is a natural, canonical one to associate to any algorithmic problem, it is only one of many possible stratifications, and may not be the appropriate one for any given application. The ideal is rather to find a stratification which reflects the empirical distribution of problem instances, that is, the frequency with which they arise in practice in a particular application, and there is often no reason to suppose that this is strongly correlated with size.

Now let \( C \) be any class of decision problems (typically a complexity class of some kind). We say that a stratified problem \( \mathcal{P} \) is \textit{generically in} \( C \) if there exists a generic subset \( Y \) of the instance space such that

1. the membership problem for \( Y \) lies in \( C \); and
2. the problem \( \mathcal{P} \) restricted to \( Y \) lies in \( C \).

Intuitively, a stratified decision problem is generically in \( C \) if the decision problem admits a partial algorithm (that is, an algorithm which outputs “yes”, “no” or “don’t know”, and which in the former two cases is always correct) in \( C \), such that the probability of a “don’t know” is negligible. We write \( \text{Gen} C \) for the class of all stratified problems generically in \( C \).

Obvious examples are the class \( \text{GenP} \) of \textit{generically polynomial-time stratified problems} and \( \text{GenNP} \) of \textit{generically non-deterministic polynomial-time stratified problems}. Another interesting example is the class \( \text{GenBPP} \), which consists of stratified problems admitting a randomised polynomial-time algorithm with probability of error uniformly bounded away from \( 1/2 \) for every instance in some generic subset whose membership problem also lies in \( \text{BPP} \).

3. Generic Monoid Presentations

In this section we study the generic properties of finite monoid presentations. We begin with some basic definitions.

Let \( A \) be a finite alphabet (set of symbols). A \textit{word} over \( A \) is a finite sequence of zero or more elements from \( A \). The set of all words over \( A \) is denoted \( A^* \); under the operation of \textit{concatenation} it forms a monoid, called the \textit{free monoid} on \( A \). The length of a word \( w \in A^* \) is denoted \( |w| \). The unique \textit{empty word} of length \( 0 \) is denoted \( e \); it forms the identity element of the monoid \( A^* \). The set \( A^+ \setminus \{e\} \) of non-empty words forms a subsemigroup of \( A^* \), called the \textit{free semigroup on} \( A \).
A finite monoid presentation \( \langle A \mid R \rangle \) consists of a finite alphabet \( A \), together with a finite sequence \( R \subseteq A^* \times A^* \) of ordered pairs of words\(^2\). We say that \( u, v \in A^* \) are one-step equivalent if \( u = axb \) and \( v = ayb \) for some possibly empty words \( a, b \in A^* \) and relation \( (x, y) \in R \) or \( (y, x) \in R \). We say that \( u \) and \( v \) are equivalent, and write \( u \equiv_R v \) or just \( u \equiv v \), if there is a finite sequence of words beginning with \( u \) and ending with \( v \), each term of which but the last is one-step equivalent to its successor. Equivalence is clearly an equivalence relation; in fact it is the least equivalence relation containing \( R \) and compatible with the multiplication on \( R \). The equivalence classes form a monoid with multiplication well-defined by \( [u] \equiv [v] \equiv [uv] \); this is called the monoid presented by the presentation.

The word problem for a (fixed) monoid presentation \( \langle A \mid R \rangle \) is the algorithmic problem of, given as input two words \( u, v \in A^* \), deciding whether \( u \equiv_R v \). The uniform word problem for finitely presented monoids is the algorithmic problem of, given as input a monoid presentation \( \langle A \mid R \rangle \) and two words \( u, v \in A^* \), deciding whether \( u \equiv_R v \). It is well-known that there exist finite monoid presentations which the word problem is undecidable, and hence that the uniform word problem for finitely presented monoids is undecidable \[12, 14\]. More generally, if \( \mathcal{C} \) is a class of finite monoid presentations, then the uniform word problem for \( \mathcal{C} \) monoids is the algorithmic problem of, given as input a monoid presentation \( \langle A \mid R \rangle \) in \( \mathcal{C} \) and two words \( u, v \in A^* \), deciding whether \( u \equiv_R v \).

Now suppose we have a fixed monoid presentation \( \langle A \mid R \rangle \). A relation word is a word which appears as one side of a relation in \( R \). A piece is a word which appears more than once as a factor in the relations, either as a factor of two different relation words, or as a factor of the same relation word in two different (but possibly overlapping) places. Let \( m \in \mathbb{N} \) be a positive integer. The presentation is said to satisfy \( C(m) \) if no relation word can be written as a product of strictly fewer than \( m \) pieces. Thus \( C(1) \) says that no relation word is empty; \( C(2) \) says that no relation word is a factor of another.

Definitions corresponding to all of those above can also be made for semigroups (without necessarily an identity element), by taking \( A^+ \) in place of \( A^* \) (in all places except the definition of one-step equivalence, where \( a \) and \( b \) must still be allowed to be empty).

Now fix an alphabet \( A \). To study generic properties of \( k \)-relation presentations over \( A \), we need a stratification on the (countable) set of all such. There are two obvious ways to define the size of a presentation, and hence two natural stratifications of the \( A \)-generated \( k \)-relation presentations. Firstly, one can take the size of the presentation to be the sum length of the relation words; this gives rise to the sum length stratification of presentations. Alternatively, one can define the size to be the length of the longest relation word; this results in the maximum relation stratification. Which choice is most natural depends on the application. For example, the sum length of a presentation is a good approximation to the space required to encode

\(^2\)The reader may think it more natural to consider a set of unordered pairs, but the definition we use simplifies the combinatorics in our analysis, and Theorem \[3\] will show that it makes no difference to the end results.
the presentation in the obvious way, and hence for computational applications seems the most natural. Intuitively, the sum length stratification lends greater weight to uneven distributions of the relation word lengths within a presentation; in particular, it results in a greater frequency of short words, which makes it seem less likely that small overlap conditions will hold. Nevertheless, it transpires that our main results hold for both stratifications, which may be regarded as some evidence of their “robustness”.

We emphasise that we are attempting here to stratify only the set of $A$-generated, $k$-relation semigroup presentations, where the alphabet $A$ and set of relations $k$ are fixed. There are, of course, also natural stratifications across all $A$-generated semigroup presentations, allowing the number of relations to vary. These typically lead to a high frequency of “short” relation words, which means that small overlap type conditions do not hold generically. However, it seems likely that, for at least some natural stratifications of this type, the word problem remains generically solvable for other reasons. This interesting issue will be studied further in a subsequent paper.

We shall need a couple of elementary definitions from combinatorics. Let $n$ and $k$ be non-negative integers. Recall that a composition of $n$ into $k$ is an ordered $k$-tuple of positive integers which sum to $n$, while a weak composition of $n$ into $k$ is an ordered $k$-tuple of non-negative integers which sum to $n$.

Having fixed the alphabet $A$, a $k$-relation monoid presentation of sum length $n$ is uniquely determined by its sequence of relation words; this in turn is uniquely determined by the concatenation in order of those words (a word in $A^n$) and the lengths of those words (a weak composition of $n$ into $2k$, called the shape of the presentation). Thus, $k$-relation monoid presentations of sum relation length $n$ are in a bijective correspondence with ordered pairs whose first component is a word of length $n$, and whose second component is a weak composition of $n$ into $2k$.

We shall need the following simple combinatorial lemma.

**Lemma 1.** Let $A$ be a finite alphabet and $c$ and $p$ be positive integers. The number of distinct words of length $c$ which admit factorisations as $x_1vy_1$ and as $x_2vy_2$ for some $x_1, x_2, y_1, y_2, v \in A^*$ with $|v| \geq p$ and $x_1 \neq x_2$ is bounded above by $c^2|A|^{c-p}$.

**Proof.** Clearly if a word admits such factorisations, then it admits such factorisations with $|v| = p$, so we may count only those words which admit such factorisations with $|v| = p$.

We claim, having fixed $A$, $c$ and $p$, any such word is uniquely determined by $x_1, y_1$ and the length of $x_2$. Clearly, there are fewer than $c^2$ ways to choose the lengths of $x_1$ and $x_2$; doing so also fixes the length of $y_1$, since we must have

$$|x_1| + |v| + |y_1| = |x_1| + p + |y_1| = c.$$ 

Now there are at most

$$|A|^{x_1+|y_1|} = |A|^{c-|v|} = |A|^{c-p}$$

ways to choose the words $x_1$ and $y_1$ with the given lengths, so proving the claim will suffice to prove the lemma.

Since $x_1$ and $x_2$ are distinct prefixes of the same word, their lengths cannot be equal. Suppose first that $x_1$ is longer than $x_2$ and write $v = v^{(1)} \ldots v^{(|v|)}$
and \( x_1 = x_1^{(1)} \ldots x_1^{(|x_1|)} \) with each \( v^{(i)} \) and \( x_1^{(i)} \) in \( A \). Then since \( x_1vy_1 = x_2vy_2 \) we have
\[
v^{(i)} = \begin{cases} x_1^{(|x_2|+i)} & \text{for } 1 \leq i \leq |x_1| - |x_2| \\ v^{(i-|x_1|+|x_2|)} & \text{for } |x_1| - |x_2| < i \leq |v| \end{cases}
\]
from which the claim follows.

If, on the other hand, \( x_1 \) is shorter than \( x_2 \) then we use the lengths of \( v \) and \( x_2 \) to deduce the length of \( y_2 \), whereupon a symmetric argument suffices to complete the proof. \( \square \)

**Proposition 1.** Let \( A \) be a finite alphabet, and \( n \) and \( r \) be positive integers, and fix a weak composition \( \sigma \) of \( n \) (into any number). Then the proportion of presentations of shape \( \sigma \) which have a piece of length \( r \) or more is bounded above by \( n^2|A|^{-r} \).

**Proof.** The set of presentations over \( A \) of shape \( \sigma \) is in 1:1 correspondence with the set \( A^n \) via the map which takes each presentation to the concatenation, in the obvious order, of its relation words. If the presentation has a piece of length \( r \) or more then the corresponding word will feature that piece as a factor in at least two different places. By Lemma 1 it follows that the number of presentations with a piece of length \( r \) or more is bounded above by \( n^2|A|^{n-r} \). The total number of such presentations in \( |A|^n \), so the proportion of presentations with the desired property is bounded above by \( n^2|A|^{-r} \) as required. \( \square \)

**Corollary 1.** Let \( A \) be a finite alphabet and \( k, n, m \) and \( K \) be positive integers with \( m \geq 2 \), and fix an weak composition \( \sigma \) of \( n \) into \( 2k \) such that no block has size less than. Then the proportion of presentations with alphabet \( A \) and shape \( \sigma \) which do not satisfy \( C(m) \) is bounded above by

\[
\frac{n^2}{|A|^{K/(m-1)}}
\]

**Proof.** If a presentation fails to satisfy \( C(m) \) then some relation word can be written as a product of \( m - 1 \) pieces. By assumption this relation word must have length at least \( K \), so one of the pieces must have length at least \( K/(m-1) \). The result is now immediately from Proposition 1. \( \square \)

Before proving the first of our main theorems, we will need an elementary combinatorial result concerning weak compositions; this will serve to bound the proportion of presentations which feature a “short” relation word.

**Lemma 2.** Let \( k \) be an integer, and \( f : \mathbb{N} \to \mathbb{N} \) be a function such that \( f(n)/n \) tends to zero as \( n \) tends to infinity. Then the proportion of weak compositions of \( n \) into \( k \) which feature a block of size \( f(n) \) or less tends to zero as \( n \) tends to infinity.

**Proof.** It is well-known and easy to prove (see, for example, [1, Theorem 5.2]) that the number of weak compositions of \( n \) into \( k \) is given by

\[
C_k(n) = \frac{(n+k-1)!}{n!(k-1)!}
\]
Clearly, every partition of $n$ into $k$ featuring a block of size $f(n)$ or less can be obtained by refining a partition of $n$ into $k-1$, with the extra decomposition in one of $k(f(n)+1)$ places. Thus, the number of such partitions is bounded above by

$$k \cdot (f(n)+1) \cdot C'_{k-1}(n) = k \cdot (f(n)+1) \cdot \frac{(n+k-2)!}{n! \cdot (k-2)!}$$

Hence, the proportion of such partitions amongst all weak compositions of $n$ into $k$ is bounded above by

$$\frac{k \cdot (f(n)+1) \cdot C'_{k-1}(n)}{C'_k(n)} = k \cdot (f(n)+1) \cdot \frac{(n+k-2)! \cdot n! \cdot (k-1)!}{(n+k-1)! \cdot n! \cdot (k-2)!}$$

$$= \frac{k(k-1) \cdot (f(n)+1)}{n+k-1}$$

$$\leq k(k-1) \left( \frac{f(n)}{n} + \frac{1}{n+k-1} \right)$$

which clearly tends to zero as $n$ tends to infinity. \hfill \Box

We are now ready to prove our main theorem for the sum relation length stratification.

**Theorem 1.** Let $A$ be an alphabet of size at least 2, and $k$ and $m$ be positive integers. Then the set of $A$-generated, $k$-relation monoid presentations which satisfy the condition $C(m)$ is generic with respect to the sum length stratification.

**Proof.** Since $C(2)$ implies $C(1)$, we may clearly assume without loss of generality that $m \geq 2$. We need to show that the proportion of $A$-generated $k$-relation monoid presentations of length $n$ which fail to satisfy $C(m)$ tends to zero as $n$ tends to infinity.

For each $n$, let $P_n$ be the set of all weak compositions of $n$ into $k$, let $Q_n$ be the set of weak compositions of $n$ into $k$ featuring a block of size $3(m-1) \log_{|A|} n$ or less, and let $R_n = P_n \setminus Q_n$. By an application of Lemma 2 with the function $f : \mathbb{N} \to \mathbb{N}$ given by $f(n) = 3(m-1) \log_{|A|} n$, we see that the proportion $|Q_n|/|P_n|$ tends to 0 as $n$ tends to infinity.

For each weak composition $\sigma$, let $x_\sigma$ be the proportion of presentations of shape $\sigma$ which fail to satisfy $C(m)$. Note that by Corollary 1 we have

$$x_\sigma \leq \frac{n^2}{|A| \cdot K_\sigma/(m-1)}$$

where $K_\sigma$ denotes the smallest block size in $\sigma$. For each fixed $n$, there are clearly equally many ($|A|^n$ to be precise) presentations of each shape, so the proportion of presentations of length $n$ failing to satisfy $C(m)$ is just the
average over shapes σ of \( x_σ \), that is:

\[
\frac{1}{|P_n|} \left( \sum_{\sigma \in P_n} x_\sigma \right) = \frac{1}{|P_n|} \left( \sum_{\sigma \in Q_n} x_\sigma \right) + \frac{1}{|P_n|} \left( \sum_{\sigma \in R_n} x_\sigma \right)
\]

\[
\leq \frac{1}{|P_n|} \left( \sum_{\sigma \in Q_n} 1 \right) + \frac{1}{|P_n|} \left( \sum_{\sigma \in R_n} \frac{n^2}{|A|^{K_\sigma/(m-1)}} \right)
\]

\[
= \frac{|Q_n|}{|P_n|} + \frac{1}{|P_n|} \left( \sum_{\sigma \in R_n} \frac{n^2}{|A|^{K_\sigma/(m-1)}} \right).
\]

We have already observed that \( |Q_n|/|P_n| \) tends to zero as \( n \) tends to infinity. Moreover, by the definition of \( R_n \) we have \( K_\sigma > 3(m-1) \log |A| n \) for all \( \sigma \in R_n \) so that

\[
\frac{1}{|P_n|} \sum_{\sigma \in R_n} \frac{n^2}{|A|^{K_\sigma/(m-1)}} \leq \frac{1}{|P_n|} \sum_{\sigma \in R_n} \frac{n^2}{|A|^{(3(m-1) \log |A| n)/(m-1)}}
\]

\[
= \frac{|R_n|}{|P_n|} \frac{n^2}{|A|^{(3(m-1) \log |A| n)/(m-1)}}
\]

\[
= \frac{|R_n|}{|P_n|} \frac{n^2}{|A|^{|\log |A|/(n^3)}}
\]

\[
\leq \frac{n^2}{n^3}
\]

which tends to zero as required. □

An analysis of the proof shows, approximately speaking, that the proportion of presentations of \( A \) failing to satisfy any given small overlap condition goes to zero like \( \log |A| n / n \), which for practical purposes may be rather slow. The barrier to showing a faster convergence is the proportion of presentations featuring a “short” relation word \( \left( |Q_n|/|P_n| \right) \); this proportion really does seem to decrease very slowly, suggesting that for the sum length stratification, fast convergence to small overlap conditions is not possible. To obtain statements about the “superpolynomially generic monoid” or “exponentially generic monoid” with respect to the sum length stratification, one would require arguments which take detailed account of the “short” relation words.

Our next task is to prove that an equivalent result holds for the maximum length stratification. We begin with an analogue of Lemma 2, which will show that the frequency of presentations featuring a “small” relation word is again negligible. This time, because the number of presentations of each shape of maximum length \( k \) is not fixed, we must reason directly with presentations rather than just shapes. Having taken account of this, the result is easier and, as one might expect given our remarks above on the relative frequency of “short” relation words in this stratification, stronger.

**Lemma 3.** Let \( A \) be an alphabet of size at least 2, \( k \) be a non-negative integer, and \( f : \mathbb{N} \to \mathbb{N} \) be a function such that \( n - f(n) \) tends to infinity as
n tends to infinity. Then the proportion of A-generated k-relation presentations of maximum relation word length n which feature a relation word of length f(n) or less tends to zero as n tends to infinity. Moreover, if there exists a constant p > 0 such that n − f(n) > pn for sufficiently large n then the given proportion tends to zero exponentially fast.

Proof. Let X_n be the set of all presentations over A of maximum relation length n, let Y_n be the presentations in X_n which have a relation word of length f(n) or less, and let Z_n = X_n \ Y_n. The quantity we seek is thus the limit as n tends to infinity of |Y_n|/|X_n|. Let I = {1, ..., 2k} and define a map σ from I × X_n to the set of all presentations k-relation presentations of A, which takes (i, P) to the presentation obtained from P by removing n − f(n) characters from the end of the ith relation word, or replacing this relation word with the empty word if its length is less than n − f(n).

We claim that under the map σ, every presentation in Y_n has at least |A|^n−f(n) pre-images in I × X_n. Indeed, if Q ∈ Y_n then Q has some relation word (say the jth) of length less than f(n), say length p. Now for each of |A|^n−f(n) words w ∈ A^{n−f(n)} we can obtain from Q a presentation P_w ∈ X_n by appending w to the end of the jth relation word, and it is easily seen σ(j, P_w) = Q for all such w.

Thus, we have 2k|X_n| = |I × X_n| ≥ |A|^n−f(n)|Y_n|, and so

\[
\frac{|Y_n|}{|X_n|} \leq \frac{2k}{|A|^n-f(n)}.
\]

Since n − f(n) tends to infinity with n, this clearly tends to zero. If moreover p > 0 is such that n − f(n) ≥ pn for n sufficiently large then we have

\[
\frac{|Y_n|}{|X_n|} \leq \frac{2k}{|A|^pn}
\]

so that the given quantity tends to zero exponentially fast. □

Corollary 2. Let A be an alphabet of size at least 2, k be a non-negative integer, and c a constant with 0 < c < 1. Then the proportion of A-generated, k-relation presentations of maximum relation word length n which feature a relation word of length cn tends to zero exponentially fast as n tends to infinity.

Proof. Define f : N → N by f(n) = cn, and choose p with

\[
0 < p < 1 - c.
\]

Then n − f(n) = (1 − c)n > pn for all n, so the result follows from Lemma 3. □

We are now ready to prove our main result for the maximum length stratification.

Theorem 2. Let A be an alphabet of size at least 2, and let k and m be positive integers. Then the set of A-generated, k-relation monoid presentations which satisfy C(m) is exponentially generic with respect to the maximum length stratification.
Proof. The structure of the proof is essentially the same as that for Theorem 2, but it is slightly complicated by the fact that the number of presentations of each shape for a given maximum relation word length \( n \) is not fixed. In addition, we must show that the rate of convergence is exponential. Once again, we assume without loss of generality that \( m \geq 2 \).

Let \( C_n \) be the total number of presentations over \( A \) of maximum relation word length \( n \). Let \( P_n \) be the set of all weak compositions of any integer into \( 2k \) with largest block size \( n \). Choose \( d \) with \( 0 < d < 1 \) and let \( Q_n \) be the set of all shapes in \( P_n \) with a word of length \( dn \) or less. Let \( R_n = P_n \setminus Q_n \). For each weak composition \( \sigma \in P_n \), let \( c_{\sigma} \) be the total number of presentations of shape \( \sigma \), and let \( x_{\sigma} \) be the proportion of presentations of shape \( \sigma \) which fail to satisfy \( C(m) \). For each shape \( \sigma \), by Corollary 1 we have

\[
x_{\sigma} \leq \frac{(n_{\sigma})^2}{|A|^{K_{\sigma}/(m-1)}}
\]

where \( n_{\sigma} \) is the total size of \( \sigma \) (that is, the sum of the block sizes of \( \sigma \), or the sum relation word length of a presentation of shape \( \sigma \)), and \( K_{\sigma} \) is the smallest block size in \( \sigma \). But \( \sigma \) has \( 2k \) blocks, none of which is larger than \( n \), so we must have \( n_{\sigma} \leq 2kn \), so that

\[
x_{\sigma} \leq \frac{(2kn)^2}{|A|^{K_{\sigma}/(m-1)}} = \frac{4k^2 n^2}{|A|^{K_{\sigma}/(m-1)}}.
\]

Now the proportion we seek is given by

\[
\frac{1}{C_n} \left( \sum_{\sigma \in P_n} c_{\sigma} x_{\sigma} \right) = \frac{1}{C_n} \left( \sum_{\sigma \in Q_n} c_{\sigma} x_{\sigma} \right) + \frac{1}{C_n} \left( \sum_{\sigma \in R_n} c_{\sigma} x_{\sigma} \right)
\]

\[
\leq \frac{1}{C_n} \left( \sum_{\sigma \in Q_n} c_{\sigma} \right) + \frac{1}{C_n} \left( \sum_{\sigma \in R_n} c_{\sigma} \frac{4k^2 n^2}{|A|^{K_{\sigma}/(m-1)}} \right).
\]

The first term in the last line is the proportion of presentations featuring a relation word of length \( dn \) or less; by Corollary 2, this tends to zero exponentially fast. Considering now the second term, by the definition of \( R_n \) we have that \( K_{\sigma} > dn \) for all \( \sigma \in R_n \) so that

\[
\frac{1}{C_n} \sum_{\sigma \in R_n} c_{\sigma} \frac{4k^2 n^2}{|A|^{K_{\sigma}/(m-1)}} \leq \frac{1}{C_n} \sum_{\sigma \in R_n} c_{\sigma} \frac{4k^2 n^2}{|A|^{dn/(m-1)}} \leq \frac{1}{C_n} \frac{4k^2 n^2}{|A|^{dn/(m-1)}} \left( \frac{\sum_{\sigma \in R_n} c_{\sigma}}{C_n} \right)
\]

\[
\leq \frac{4k^2 n^2}{(|A|^d/(m-1)) n^2}
\]

which since \( |A| \geq 2 \) and \( d > 0 \) clearly tends to zero exponentially fast. \( \square \)

4. Equivalence of Stratifications

It often happens that two stratifications (on the same set, or on related sets) are closely related, so that knowledge of the generic sets with respect to one yields corresponding information about the generic sets with respect to the other. In this section we establish some technical conditions under which
this holds, and use this to extend many of our earlier results to additional natural stratifications.

First, we consider the relationship between spherical and ascending stratifications. So far, we have seen examples only of spherical stratifications of instance spaces, but to each such stratification is associated an equally natural ascending stratification, the sets in the latter being unions of the sets in the former. The following proposition, which was first observed in [4] to be an easy consequence of the Stolz-Cesaro Theorem, says that the generic sets are independent of which of these stratifications is used (see [4] for a more detailed explanation).

**Proposition 2.** [4, Lemma 3.2] Let \( S_n \) be a spherical stratification of a set \( S \). Define a new stratification on \( S \) by

\[
B_n = \bigcup_{j=1}^{n} S_j.
\]

Then any set \( X \subseteq S \) is generic with respect to the stratification \( S_n \) if and only if it is generic with respect to the stratification \( B_n \).

We shall need the following elementary proposition, which essentially says that the restriction of a stratification to a generic set preserves generic sets.

**Lemma 4.** Let \( X \) be a stratified set, and \( X' \) a generic subset of \( X \). Then for any \( P \subseteq X \) we have

\[
\lim_{n \to \infty} \frac{|P \cap X_n|}{|X_n|} = \lim_{n \to \infty} \frac{|P \cap X_n \cap X'|}{|X_n \cap X'|}.
\]

**Proof.** First notice that, since \( X' \) is generic, we have

\[
\lim_{n \to \infty} \frac{|P \cap X_n \cap (X \setminus X')|}{|X_n|} = \lim_{n \to \infty} \frac{|(X \setminus X') \cap X_n|}{|X_n|} = 0 \quad (2)
\]

Now

\[
\lim_{n \to \infty} \frac{|P \cap X_n \cap X'|}{|X_n \cap X'|} = \lim_{n \to \infty} \frac{|P \cap X_n \cap X'|}{|X_n|} \cdot \frac{|X_n|}{|X_n \cap X'|} = \left( \lim_{n \to \infty} \frac{|P \cap X_n \cap X'|}{|X_n|} \right) \left( \lim_{n \to \infty} \frac{|X_n \cap X'|}{|X_n|} \right)^{-1}.
\]

\[
= \left( \lim_{n \to \infty} \frac{|P \cap X_n \cap X'|}{|X_n|} \right) \left( \lim_{n \to \infty} \frac{|X_n \cap X'|}{|X_n|} \right)^{-1} \quad (\text{since } X' \text{ is generic})
\]

\[
= \left( \lim_{n \to \infty} \frac{|P \cap X_n \cap X'|}{|X_n|} \right) + 0
\]

\[
= \left( \lim_{n \to \infty} \frac{|P \cap X_n \cap X'|}{|X_n|} \right) + \left( \lim_{n \to \infty} \frac{|P \cap X_n \cap (X \setminus X')|}{|X_n|} \right) \quad \text{(by (2))}
\]

\[
= \lim_{n \to \infty} \frac{|P \cap X_n \cap X'|}{|X_n|} + \lim_{n \to \infty} \frac{|P \cap X_n \cap (X \setminus X')|}{|X_n|}
\]

\[
= \lim_{n \to \infty} \frac{|P \cap X_n|}{|X_n|}
\]

as required. \( \Box \)
Next, we introduce a very useful sufficient condition for a map between stratified sets to preserve generic sets. To do so, we need some terminology. Let $X$ and $Y$ be stratified sets, $X' \subseteq X$ and $Y' \subseteq Y$, and $f : X' \rightarrow Y'$ a map. Then $f$ is called stratification-preserving if for every $x \in X'$ and $n \in \mathbb{N}$ we have $x \in X_n$ if and only if $f(x) \in Y_n$. If $P \subseteq X$ then $f$ is said to respect $P$ if $f(P \cap X')$ and $f((X \setminus P) \cap X')$ are disjoint, that is, if whenever $x_1, x_2 \in X'$ are such that $f(x_1) = f(x_2)$ we have either $x_1, x_2 \in P$ or $x_1, x_2 \notin P$. Recall that the fibre size of $f$ at a point $y \in Y'$ is the cardinality of the set of elements $x \in X'$ such that $f(x) = y$. The map $f$ is called bounded-to-one if there is a finite upper bound on its fibre sizes.

**Theorem 3.** Let $X$ and $Y$ be stratified sets, $X' \subseteq X$ and $Y' \subseteq Y$ be generic subsets of $X$ and $Y$ respectively, $d \in \mathbb{N}$ and $f : X' \rightarrow Y'$ a surjective, stratification-preserving map, such that for every $n \in \mathbb{N}$ there exists $k_n \in \mathbb{N}$ such that the fibre sizes of $f$ at points in $X_n \cap X'$ all lie between $k_n$ and $dk_n$. Then for any set $P \subseteq X$ we have

(i) 
\[
\frac{1}{d} \lim_{n \rightarrow \infty} \frac{|f(P \cap X') \cap Y_n|}{|Y_n|} \leq \lim_{n \rightarrow \infty} \frac{|P \cap X_n|}{|X_n|} \leq d \lim_{n \rightarrow \infty} \frac{|f(P \cap X') \cap Y_n|}{|Y_n|}
\]

wherever both limits are defined;

(ii) 
\[
\frac{1}{d} \lim_{n \rightarrow \infty} \frac{|P \cap X_n|}{|X_n|} \leq \lim_{n \rightarrow \infty} \frac{|f(P \cap X') \cap Y_n|}{|Y_n|} \leq d \lim_{n \rightarrow \infty} \frac{|P \cap X_n|}{|X_n|}
\]

wherever both limits are defined;

(iii) $P$ is negligible in $X$ if and only if $f(P \cap X')$ is negligible in $Y$;

(iv) If $P$ is generic in $X$ then $f(P \cap X')$ is generic in $Y$;

(v) If $d = 1$ and $f(P \cap X')$ is generic in $Y$ then $P$ is generic in $X$; and

(vi) If $f$ respects $P$ and $f(P \cap X')$ is generic in $Y$ then $P$ is generic in $X$.

Before proving Theorem 3 we emphasise that parts (i) and (ii) do not guarantee that one of the limits involved is defined exactly if the other is defined. If one of the sequences converges to some value $c$, then only in the case $c = 0$ can we be certain that the other will converge. If $c \neq 0$ then the other may fail to converge, although one can easily show that it will eventually be constrained to vary within the range $[d^{-1}c, dc]$. We now turn to proving Theorem 3.

**Proof.** By the bounds on the fibre sizes of $f$ we clearly have
\[
|f(P \cap X' \cap X_n)| \leq |P \cap X' \cap X_n| \leq d|f(P \cap X' \cap X_n)|
\]
and
\[
|f(X' \cap X_n)| \leq |X' \cap X_n| \leq d|f(X' \cap X_n)|
\]
for all $n \in \mathbb{N}$. It follows from the fact that $f$ is surjective and stratification-preserving that $f(X' \cap X_n) = Y' \cap Y_n$ and $f(P \cap X' \cap X_n) = f(P \cap X') \cap Y_n$, so the above inequalities become
\[
|f(P \cap X') \cap Y_n| \leq |P \cap X' \cap X_n| \leq d|f(P \cap X') \cap Y_n|
\]
and

\[ |Y' \cap Y_n| \leq |X' \cap X_n| \leq d|Y' \cap Y_n| \]

respectively. Now combining these yields

\[
\frac{1}{d} \frac{|P \cap X' \cap Y_n|}{|Y' \cap Y_n|} \leq \frac{|P \cap X_n \cap X'|}{|X_n \cap X'|} \leq \frac{d|f(P \cap X') \cap Y_n|}{|Y' \cap Y_n|}.
\] (3)

It follows also that

\[
\frac{1}{d} \frac{|P \cap X_n \cap X'|}{|X_n \cap X'|} \leq \frac{|f(P \cap X') \cap Y_n|}{|Y_n \cap Y'|} \leq \frac{d|P \cap X_n \cap X'|}{|X_n \cap X'|}
\] (4)

where the left-hand [respectively, right-hand] inequality is obtained by dividing [multiplying] both sides of the right-hand [left-hand] inequality in (3) by \(d\).

Now since \(X'\) and \(Y'\) are generic in \(X\) and \(Y\) respectively, Lemma 4 gives

\[
\lim_{n \to \infty} \frac{|P \cap X_n|}{|X_n|} = \lim_{n \to \infty} \frac{|P \cap X' \cap X_n|}{|X_n \cap X'|}
\]

and

\[
\lim_{n \to \infty} \frac{|f(P \cap X') \cap Y_n|}{|Y_n|} = \lim_{n \to \infty} \frac{|f(P \cap X' \cap Y_n \cap Y')|}{|Y_n \cap Y'|} = \lim_{n \to \infty} \frac{|f(P \cap X') \cap Y_n|}{|Y_n \cap Y'|}
\]

where the second equality on the second line holds because \(f(P \cap X') \subseteq Y'\). It is now clear that parts (i) and (ii) follow from (3) and (4) respectively.

If \(f(P \cap X')\) is negligible in \(Y\) then the left and right-hand sides of (i) converge to 0, from which it follows that the middle expression converges to 0, and so \(P\) is negligible. Conversely, if \(P\) is negligible then exactly the same argument applies with (ii) in place of (i) to show that \(f(P \cap X')\) is negligible. This proves part (iii).

If \(P\) is generic in \(X\) then \(X \setminus P\) is negligible in \(X\), so by part (iii), \(f((X \setminus P) \cap X')\) is negligible in \(Y\). But by surjectivity, we must have

\[ Y' \setminus f(P \cap X') \subseteq f((X \setminus P) \cap X') \]

so that \(Y' \setminus f(P \cap X')\) is negligible in \(Y\). Since \(Y'\) is generic in \(Y\) and generic sets are closed under intersection, it follows that

\[ Y \setminus f(P \cap X') = (Y' \setminus f(P \cap X')) \cup (Y \setminus Y') \]

is negligible in \(Y\), so that \(f(P \cap X')\) is generic in \(Y\) as required to prove part (iv).

If \(d = 1\) and \(f(P \cap X')\) is generic in \(Y\) then it is immediate from part (i) that \(P\) is generic in \(X\), so that part (v) holds.

Finally, suppose that \(f\) respects \(P\) and that \(f(P \cap X')\) is generic in \(Y\). Since \(f\) is surjective we have

\[ Y' = f(X') = f((X \setminus P) \cap X') \cup f(P \cap X'). \]

Now since \(f\) respects \(P\), we know that \(f((X \setminus P) \cap X')\) and \(f(P \cap X')\) are disjoint, and since \(Y'\) is generic in \(Y\) is follows that

\[ f((X \setminus P) \cap X') = Y' \setminus f(P \cap X') \]

is negligible in \(Y\). But now by part (iii), we deduce that \(X \setminus P\) is negligible in \(X\), and hence that \(P\) is generic in \(X\), as required to prove part (vi). \(\square\)

A particularly useful special case is the following immediate corollary.
Corollary 3. Let $X$ and $Y$ be stratified sets, $X' \subseteq X$ and $Y' \subseteq Y$ be generic subsets of $X$ and $Y$ respectively, $f : X' \to Y'$ a surjective, stratification-preserving, bounded-to-one map. Then for any $P \subseteq X$ such that $f$ respects $P$, we have that $P$ is generic [respectively, negligible] in $X$ if and only if $f(P \cap X')$ is generic [negligible] in $Y$.

Next, we apply Theorem 3 to show that the generic properties of finitely presented semigroups are essentially governed by those of finitely presented monoids. Recall that if $S$ is a semigroup then $S^1$ denotes the monoid with set of elements $S \cup \{1\}$ where 1 is a new symbol not in $S$, and multiplication defined by

$$st = \begin{cases} 
\text{the } S\text{-product } st & \text{if } s, t \in S; \\
 s & \text{if } t = 1; \\
 t & \text{if } s = 1. 
\end{cases}$$

Theorem 4. Let $\mathcal{C}$ be a class of monoids, $A$ a finite alphabet and $k \in \mathbb{N}$. Then the generic $A$-generated $k$-relation monoid (with respect to either the sum length stratification or the maximum length stratification) belongs to $\mathcal{C}$ if and only if the generic $A$-generated $k$-relation semigroup $S$ (with respect to the corresponding stratification) is such that $S^1$ belongs to $\mathcal{C}$.

Proof. Let $X$ and $Y$ be the sets of $k$-relation monoid and semigroup presentations respectively over $A$. Suppose $X$ and $Y$ are equipped with either the sum length or the maximum length stratification. Let $P$ be the set of presentations in $X$ such that the monoid presented lies in $\mathcal{C}$, and let $Q$ be the set of presentations in $Y$ such that the semigroup $S$ presented is such that $S^1$ lies in $\mathcal{C}$.

Let $Y' = Y$ and let $X' = Y \subseteq X$ be the set of semigroup presentations viewed as a subset of the set of monoid presentations, that is, those monoid presentations in which no relation word is empty. By Lemma 2 (for the sum length stratification) or Lemma 3 (for the maximum length stratification) $X'$ is generic in $X$, and obviously $Y' = Y$ is generic in $Y$.

Define $f : X' = Y \to Y' = Y$ to be the identity function. Then $f$ is $1 : 1$, surjective onto $Y'$, and preserves the sum length and maximum length stratifications. Letting $d = 1$ and $k_n = 1$ for all $n$, we see that the conditions of Theorem 3 are satisfied, so $P$ is generic in $X$ if and only if $f(P \cap X')$ is generic in $Y$.

Since $f$ is the identity function on $X'$, a semigroup presentation $\mathcal{P} \in f(P \cap X')$ exactly if $\mathcal{P}$ interpreted as a monoid presentation lies in $P$. Since $\mathcal{P}$ has no empty relation words, it is easy to see that the monoid presented by $\mathcal{P}$ is isomorphic to $S^1$, where $S$ is the semigroup presented by $\mathcal{P}$. Thus, $\mathcal{P} \in f(P \cap X')$ if and only if $S^1 \in \mathcal{C}$, that is, if and only if $\mathcal{P} \in Q$. Hence, $f(P \cap X') = Q$, and so $P$ is generic in $X$ if and only if $Q$ is generic in $Y$, as required. \qed

Corollary 4. For every $m \geq 1$, $k \in \mathbb{N}$ and alphabet $A$ of size at least 2, the generic $A$-generated $k$-relation semigroup (with respect to either the sum length stratification or the maximum length stratification) satisfies the small overlap condition $C(m)$. 

An unordered monoid presentation consists of a set $A$ of generators and an (unordered) set $R$ of relations, each of which is an unordered pair of words from $A^*$. Equivalence of words is defined exactly as for ordered presentations (see Section 3), as are the sum length and maximum length stratifications on the sets of $A$-generated presentations with some fixed number $k$ of relations.

There is an obvious map from the ordered to the unordered presentations over a given alphabet $A$, which simply “forgets” the ordering of the relations and the ordering of the pair of words in each relation, and discards any “duplicate” relations. Unordered semigroup presentations can of course be defined analogously.

**Theorem 5.** Let $C$ be a class of monoids, $A$ an alphabet and $k$ a non-negative integer. Then the generic [negligible] $A$-generated $k$-relation monoid (with respect to either the sum length stratification or the maximum length stratification) belongs to $C$ if and only if the generic [respectively negligible] $a$-generator $k$-relation unordered monoid (with respect to the corresponding stratification) belongs to $C$. The corresponding statement for semigroups also holds.

**Proof.** We prove the result for monoids; that for semigroups can be proved in exactly the same way. Let $X$ be the set of ordered $k$-relation monoid presentations over $A$, and $Y$ the set of unordered $k$-relation monoid presentations over $A$. Let $P \subseteq X$ and $Q \subseteq Y$ be the sets of presentations in $X$ and $Y$ respectively such that the monoid presented belongs to $C$.

Let $X' \subseteq X$ be the set of ordered presentations which do not feature the same relation twice, or two relations of the form $(u, v)$ and $(v, u)$ for some distinct words $u$ and $v$. We have seen that $C(2)$ presentations do not feature the same relation word twice, so $X'$ certainly contains all the $C(2)$ presentations. It follows by Theorem 1 (for the sum relation length stratification) or Theorem 2 (for the maximum relation length stratification) that $X'$ is generic in $X$. Let $Y' = Y$; then certainly $Y'$ is generic in $Y$.

Define $f : X' \to Y' = Y$ to be the restriction to $X'$ of the obvious map described above from ordered to unordered presentations. It is clear from the definition of $X'$ that $f$ preserves the number of relations in the presentation and so really does define a map to $Y$, and moreover that this map is surjective. Since $f$ takes each ordered presentation to an unordered presentation of the same monoid, it is also obvious that $f$ respects $P$ and maps $P \cap X'$ onto $Q$. It is easily seen that $f$ preserves both the sum length and the maximum length stratifications. Moreover, $f$ clearly has fibre size bounded above by $k!2^k$. It follows that the conditions of Corollary 3 are satisfied, so that $P$ is generic in $X$ if and only if $f(P) = Q$ is generic in $Y$. \hfill \Box

We thus allow ourselves to speak of a generic monoid or generic semigroup, without worrying about whether the presentation is defined to have a set or a sequence of relations.
5. Properties of Generic Monoids and Semigroups

In this section we explore some of the consequences of our results for generic monoids and semigroups. Recall that a monoid or semigroup is called $J$-trivial if distinct elements always generate distinct principal ideals.

**Proposition 3.** Any $C(3)$ semigroup or monoid is torsion-free and $J$-trivial.

**Proof.** Let $S$ be a semigroup or monoid with a $C(3)$ presentation $\langle A \mid R \rangle$. By a result of Remmers [15], only finitely many words over the alphabet $A$ represent the same element of $S$.

Suppose first that $S$ is not $J$-trivial, and choose $a, b \in S$ be distinct elements generating the same ideal. Then in particular, $a$ is in the ideal generated by $b$, so we have $a = pbq$ for some $p, q \in S$. But also $b$ is in the ideal generated by $a$, so that and $b = ras = rpbqs$ for some $r, s \in S$. Now choose words $\hat{b}, \hat{p}, \hat{q}, \hat{r}, \hat{s} \in \langle A \rangle$ representing $b, p, q, r, s \in S$ respectively. Certainly at least one of $\hat{r}$ and $\hat{s}$ is non-empty, since otherwise we would have $r = s = 1$ so that $b = ras = a$. But now it is easily seen that $(\hat{r}\hat{p})^i\hat{b}(\hat{q}\hat{s})^i$ represents $b$ for every $i > 0$, contradicting Remmers’ result.

Similarly, suppose $a \in S$ is non-identity torsion element. Then there is a non-empty word $\hat{a} \in A$ representing $a$. But now it is easy to see that infinitely many powrs of $\hat{a}$ must represent the same element, again contradicting Remmers’ result.

□

Combining with our theorem we have the following.

**Theorem 6.** Let $A$ be an alphabet of size at least 2 and let $k$ be a positive integer. Then the monoid defined by the generic $A$-generated $k$-relation presentation (with respect to either the sum length stratification or the maximum length stratification) is non-trivial, torsion-free and $J$-trivial. In particular, it is not a group, an inverse monoid or a regular monoid. The corresponding statements for semigroups also hold.

**Proof.** By Theorem 1 (respectively Theorem 2 for the other stratification) the generic $A$-generated $k$-relation presentation satisfies $C(3)$, and so by Proposition 3 the semigroup presented is torsion-free and $J$-trivial. If it were trivial then every word over the alphabet would have to represent the identity, contradicting once more Remmers’ result mentioned in the proof of the previous proposition.

□

By a recent result of the author, the uniform word problem for $C(4)$ semigroups is solvable in time linear in the word lengths and polynomial in the presentation size [8, Theorem 2]. Hence, we obtain

**Theorem 7.** Let $A$ be an alphabet of size at least 2 and let $k$ be a positive integer. Then the generic $A$-generated $k$-relation presentation (with respect to either the sum length stratification or the maximum length stratification) has word problem solvable in linear time. The corresponding statement for semigroups also holds.

Since there is also an algorithm to decide, in (worst-case) polynomial time whether a given presentation satisfies the condition $C(4)$ [8, Corollary 5], we also obtain
Theorem 8. Let $A$ be an alphabet of size at least 2 and $k$ be a positive integer. Then the uniform word problem for $A$-generated, $k$-relation monoid presentations is generically solvable in polynomial time. The corresponding statement for semigroups also holds.

Further work of the author [9] has established a number of automata-theoretic properties of monoids which admit finite presentations satisfying the condition $C(4)$. It follows from Theorems 1 and 2 that the “generic” monoid and semigroup will enjoy all these properties. The following theorem summarises these properties; for brevity we omit definitions of terms; which can be found in [9].

Theorem 9. Let $A$ be an alphabet of size at least 2 and let $k$ be a positive integer. Then the monoid defined by the generic $A$-generated $k$-relation presentation (with respect to either the sum length stratification or the maximum length stratification) is rational in the sense of [18], asynchronous automatic and word hyperbolic in the sense of [3]. It also satisfies an analogue of Kleene’s theorem and has a boolean algebra of rational subsets and decidable rational subset membership problem.

Acknowledgements

This research was supported by an RCUK Academic Fellowship. The author would like to thank A. V. Borovik and V. N. Remeslennikov for their many suggestions; he also thanks the organisers and participants of the AIM Workshop on Generic Complexity, held in Palo Alto in August 2007, where he had many helpful conversations, and the American Institute of Mathematics for funding his attendance there.

References

[1] Miklós Bóna. A walk through combinatorics. World Scientific Publishing Co. Inc., River Edge, NJ, 2002.
[2] G. B. Dantzig. Maximization of a linear function of variables subject to linear inequalities. In Activity Analysis of Production and Allocation, Cowles Commission Monograph No. 13, pages 339–347. John Wiley & Sons Inc., New York, N. Y., 1951.
[3] A. Duncan and R. H. Gilman. Word hyperbolic semigroups. Math. Proc. Cambridge Philos. Soc., 136(3):513–524, 2004.
[4] R. Gilman, A. G. Miasnikov, A. D. Myasnikov, and A. Ushakov. Report on generic case complexity. Available online at www.acc.stevens.edu/Files/GC/gc_survey.pdf, 2007.
[5] M. Gromov. Hyperbolic groups. In Essays in Group Theory, volume 8 of Math. Sci. Res. Inst. Publ., pages 75–263. Springer, New York, 1987.
[6] Y. Gurevich. Average case complexity. In Automata, languages and programming (Madrid, 1991), volume 510 of Lecture Notes in Comput. Sci., pages 615–628. Springer, Berlin, 1991.
[7] P. M. Higgins. Techniques of semigroup theory. Oxford Science Publications. The Clarendon Press Oxford University Press, New York, 1992. With a foreword by G. B. Preston.
[8] M. Kambites. Small overlap monoids: the word problem. arXiv:0712.0250 [math.RA], 2007.
[9] M. Kambites. Small overlap monoids II: automatic structures and normal forms. arXiv:0806.3891 [math.RA], 2008.
[10] I. Kapovich, A. Myasnikov, P. Schupp, and V. Shpilrain. Generic-case complexity, decision problems in group theory, and random walks. *J. Algebra*, 264(2):665–694, 2003.

[11] V. Klee and G. J. Minty. How good is the simplex algorithm? In *Inequalities, III (Proc. Third Sympos., Univ. California, Los Angeles, Calif., 1969; dedicated to the memory of Theodore S. Motzkin)*, pages 159–175. Academic Press, New York, 1972.

[12] A. Markov. On the impossibility of certain algorithms in the theory of associative systems. *C. R. (Doklady) Acad. Sci. URSS (N.S.)*, 55:583–586, 1947.

[13] A. Yu. Ol’shanski˘ı. Almost every group is hyperbolic. *Internat. J. Algebra Comput.*, 2(1):1–17, 1992.

[14] E. L. Post. Recursive unsolvability of a problem of Thue. *J. Symbolic Logic*, 12:1–11, 1947.

[15] J. H. Remmers. *Some algorithmic problems for semigroups: a geometric approach*. PhD thesis, University of Michigan, 1971.

[16] J. H. Remmers. On the geometry of semigroup presentations. *Adv. in Math.*, 36(3):283–296, 1980.

[17] D. Ruinskiy, A. Shamir, and B. Tsaban. Length-based cryptanalysis: the case of Thompson’s group. *J. Math. Cryptot.*, 1:359–372, 2007.

[18] J. Sakarovitch. Easy multiplications I. The realm of Kleene’s theorem. *Inform. and Comput.*, 74:173–197, 1987.

[19] V. Shpilrain and A. Ushakov. Thompson’s group and public key cryptography. *arXiv:math/0505487v1 [math.GR]*, 2005.

[20] V. Shpilrain and G. Zapata. Combinatorial group theory and public key cryptography. *Appl. Algebra Engrg. Comm. Comput.*, 17(3-4):291–302, 2006.