Proposal for Determining the Total Masses of Eccentric Binaries Using Signature of Periastron Advance in Gravitational Waves

Naoki Seto
Department of Earth and Space Science, Osaka University, Toyonaka 560-0043, Japan

We propose a new method for determining total masses of low frequency eccentric binaries (such as, neutron star binaries with orbital frequency $f \gtrsim 10^{-3}\text{Hz}$) from their gravitational waves. In this method we use the frequency shift caused by periastron advance, and it works even at low frequency band where chirp signal due to radiation reaction is difficult to be measured. It is shown that the total masses of several Galactic neutron star binaries might be measured accurately (within a few percent error) by LISA with operation period of $\sim 10$ years.

I. INTRODUCTION

The Galactic binaries such as close white dwarf binaries (CWDBs) or neutron star binaries (NBs) are promising targets of the Laser Interferometer Space Antenna (LISA, http://lisa.jpl.nasa.gov[1]). One of the important aim of gravitational-wave astronomy is to extract out information of sources that emit gravitational radiation. In the case of a Galactic binary we want to know masses of two stars, orbital parameters (semi-major axis, eccentricity, inclination), distance to the binary and so on. At the final in-spiral phase (e.g. the last three minutes of NBs) we can, in principle, determine various parameters by fitting time evolution of wave signal with post-Newtonian expansion[2]. But at LISA band many compact binaries evolve very slowly and situation is largely different.

It is observationally known that NBs (and some NS-WD binaries) with orbital period $\lesssim 1\text{day}$ have large eccentricities[3]. This is explained by the “kick” effect and instantaneous mass loss at birth of a neutron star, in contrast to CWDBs whose orbits are circulaized by strong tidal interaction during mass transfer phases. As the eccentricity $e$ decreases with increase of the orbital frequency due to radiation reaction[4], its effect is supposed to be negligible for most high-frequency sources that will be searched by ground-based detectors (TAMA300, GEO600, LIGO and VIRGO). But the eccentricity of NBs can be $e \sim 0.1$ at frequency $f \sim 10^{-3}\text{Hz}$ as expected from PSR B1913+16[5], and it might cause observable effects for LISA.

Frequency series of gravitational wave from an eccentric binary is affected by the periastron advance[6] (see Ref. [5] for radio observation of binary pulsars). Using this fact we might obtain information of binaries even at low frequency band where chirp signal would be difficult to be measured. In this Letter we propose a new method for determining binary parameters (mainly total mass), and investigate its prospect for studying Galactic NBs with LISA.

II. GRAVITATIONAL WAVE FROM ELLIPTICAL ORBIT

Let us study gravitational waves from elliptical orbits following Ref. [6] (see also [7]). With quadrupole formula of gravitational radiation[8] two polarization modes in TT-gauge are written as follows

$$h_x = -h_0 \cos \Theta \sum_n \left[ \frac{S_n - C_n}{2} \sin(2\pi f_n t + 2\Phi) + \frac{S_n + C_n}{2} \sin(2\pi f_n t - 2\Phi) \right],$$

$$h_+ = -\frac{1}{2} h_0 \sum_n \left\{ \sin^2 \Theta A_n \cos(2\pi f_n t) + (1 + \cos^2 \Theta) \times \right.$$  

$$\left. \left[ \frac{S_n - C_n}{2} \cos(2\pi f_n t + 2\Phi) + \frac{S_n + C_n}{2} \cos(2\pi f_n t - 2\Phi) \right] \right\},$$

where $\Theta$ represents direction of the orbital angular momentum (inclination) and $\Phi$ represents direction of the periastron in the orbital plane (we put $\Phi = 0$ when the periastron is on the plane determined by two vectors: the orbital angular momentum vector and the direction vector to the observer[6]). The amplitude $h_0$ is given as $h_0 = 4(2\pi)^{-2/3} G^{5/3} c^{-1} m_{\text{chirp}} r^{-1} (P_b)^{-2/3}$ ($m_{\text{chirp}}$: chirp mass, $r$: distance to the binary, $P_b$: orbital period from periastron to periastron). We have defined the frequency series $f_n \equiv P_b^{-1} n$. The angle $\Phi$ changes due to the periastron advance and we can denote it as $2\Phi = \delta f \times t$ by adjusting origin of the time coordinate $t$. Thus the periastron advance causes frequency shift $\delta f$ that is given in terms of the total mass $m_{\text{total}}$ and the eccentricity $e$ as
\[ \delta f = 6(2\pi)^{2/3}(P_0)^{-5/3}G^{2/3}c^{-2}m_{\text{total}}^{2/3}(1 - e^2)^{-1} \] (twice of the frequency for the periastron advance \(^8\)). The frequency \( f_n \) now splits into a triplet \( (f_n - \delta f, f_n, f_n + \delta f) \). Effect of the angle \( 2\Phi \) is related to rotation of a coordinate system and appears in the same form for every \( n \)-mode. This is an essential point. The simple replacement \( 2\Phi \rightarrow 2\delta f \times t \) into expressions \(^3\) and \(^6\) corresponds to an approximation that neglects terms of \( O(P_0 \delta f) \ll 1 \). The coefficients \( S_n \), \( C_n \) and \( A_n \) are given by the \( n \)-th Bessel function \( J_n(x) \) and its derivative \( J'_n(x) \) as

\[
A_n = J_n(ne), \quad S_n = -\frac{2(1 - e^2)^{1/2}}{e} J'_n(ne) + \frac{2n(1 - e^2)^{3/2}}{e^2} J_n(ne), \quad (3)
\]

\[
C_n = -\frac{2 - e^2}{e^2} J_n(ne) + \frac{2(1 - e^2)}{e} J'_n(ne). \quad (4)
\]

We can expand these coefficients around \( e = 0 \) (circular orbit) and find that only \( n \leq 3 \) modes have terms \( O(e^0) \) or \( O(e^1) \). They are given as \( S_1 = -\frac{4}{3}e + O(e^2) \), \( C_1 = -\frac{2}{3}e + O(e^2) \), \( A_1 = \frac{1}{3}e + O(e^3) \) for \( n = 1 \)-mode, \( S_2 = 1 - \frac{2}{3}e^2 + O(e^3) \), \( C_2 = 1 - \frac{2}{3}e^2 + O(e^3) \), \( A_2 = \frac{2}{3}e^2 + O(e^4) \) for \( n = 2 \)-mode, and \( S_3 = -\frac{4}{3}e + O(e^3) \), \( C_3 = -\frac{2}{3}e + O(e^3) \), \( A_3 = O(e^3) \) for \( n = 3 \)-mode. Our basic strategy is to compare \( O(e^0) \)-term of \( n = 2 \)-mode and \( O(e^1) \)-term of \( n = 3 \)-mode (as we see below, the eccentricity relevant for our analysis is expected to be small \( e \lesssim 0.1 \)). From these two frequencies \( f_2 - \delta f \) and \( f_3 - \delta f \) we obtain

\[
3(f_2 - \delta f)/2 - (f_3 - \delta f) = -\delta f/2,
\]

and thus the total mass \( m_{\text{total}} \) is estimated for a binary with small eccentricity \( e \) (note also we might determine the eccentricity \( e \) by comparing amplitudes of two waves). Amplitude of \( n = 1 \)-mode is smaller than \( n = 3 \)-mode, and furthermore the binary confusion noise would be larger at lower frequency \(^4\). Therefore we investigate prospect of our method by studying detectability of \( n = 3 \)-mode in stead of \( n = 1 \)-mode.

### III. DETECTABILITY

The frequency difference \( \delta f \) due to periastron advance is expressed as \(^8\)

\[
\delta f = \frac{1.2 \times 10^{-7}}{1 - e^2} \left( \frac{m_{\text{total}}}{2.8 M_\odot} \right)^{2/3} \left( \frac{f_3}{3 \times 10^{-5} \text{Hz}} \right)^{5/3} \text{Hz}, \quad (6)
\]

and the accumulated frequency difference (chirp signal) due to gravitational radiation reaction within observational period \( T_{\text{obs}} \) is given as \(^4\)

\[
(\dot{f})_{GW}T_{\text{obs}} = 3.0 \times 10^{-8} \left( \frac{m_{\text{chip}}}{1.2 M_\odot} \right)^{5/3} \left( \frac{f_2}{2 \times 10^{-5} \text{Hz}} \right)^{11/3} \left( 1 + \frac{73}{24}e^2 + \frac{37}{96}e^4 \right) \left( \frac{T_{\text{obs}}}{10 \text{yr}} \right) \text{Hz}. \quad (7)
\]

Thus at lower frequency the difference due to the periastron advance can take larger value. The estimation error (resolution) \( \Delta f \) for wave frequency \( f \) in matched filtering analysis is given by the Fisher information matrix of fitting parameters. For the set of three unknown parameters \( (f, \dot{f}, \text{initial phase}) \) of quasi-monochromatic wave \( (fT_{\text{obs}} \ll f) \) we obtain \(^4\)

\[
\Delta f = 4\sqrt{3}\pi^{-1}T_{\text{obs}}^{-1}SNR^{-1} = 6.6 \times 10^{-10} \left( \frac{T_{\text{obs}}}{10 \text{yr}} \right)^{-1} \left( \frac{SNR}{10} \right)^{-1} \text{Hz}, \quad (8)
\]

where \( SNR \) is signal to noise ratio of the wave. For the secular frequency modulation \( \dot{f} \) we have estimation error \( \Delta \dot{f} = 3\sqrt{3}\pi^{-1}T_{\text{obs}}^{-1}SNR^{-1} \). From equations \(^4\),\(^8\) and \(^4\) the frequency shift \( \delta f \) would be resolved

\[
\frac{\Delta f}{\delta f} \simeq 0.011(1 - e^2) \left( \frac{m_{\text{total}}}{2.8 M_\odot} \right)^{-2/3} \left( \frac{f_3}{3 \times 10^{-5} \text{Hz}} \right)^{-5/3} \left( \frac{SNR}{10} \right)^{-1} \left( \frac{T_{\text{obs}}}{10 \text{yr}} \right)^{-1}. \quad (9)
\]

\(^1\)When we include the direction and orientation of the source in fitting parameters, the estimation errors might be somewhat larger than our evaluation. We should also notice that these angular parameters are fixed well by \( f = 2 \)-mode and we might not need to fit them for determining frequency of \( f = 3 \)-mode.
Here (and hereafter) we consider contribution of the estimation error for $\delta f$ only from $n = 3$-mode, as $SNR$ of $n = 2$-mode would be larger. In the followings we estimate the number of Galactic NBs whose $n = 3$-mode is detected with $SNR > 10$.

First we discuss time evolution of the eccentricity parameter $e$. Using quadrupole formula for gravitational radiation the orbital semi-major radius $a$ is related to the eccentricity as 

$$a/a_i = (1 - e_i^2)/(1 - e^2) \left[ (1 + 121e_i^2/304)/(1 + 121e^2/304) \right]^{87/2299}$$

where $a_i$ and $e_i$ are their initial values. At $a/a_i < 1$ we have an useful approximation $e \sim (a/a_i(1 - e_i^2))^{19/12}$ $e_i = (f/f_i)^{-19/18} (1 - e^2)^{-19/12}e_i$ where Kepler law is used. For a given orbital period the eccentricity should have some distribution function $F(e; P_b)$ [10], but it is poorly known at present, as only several NBs have detected so far [3]. Here we use the binary pulsar PSR B1913+16 as a reference. This system has orbital period $P_b = 2.8 \times 10^4$sec and eccentricity $e = 0.62$ [3]. From above approximation its eccentricity evolves as

$$e \simeq 0.13(f_3/0.001Hz)^{-19/18}. \quad (10)$$

Hereafter we use this $e$-$f_3$ relation in our order estimation.

Next we study the frequency and spatial distribution of the Galactic NBs. Assuming that the former is in steady state, we can evaluate the distribution function $dN/df$ using the coalescence rate $R_{NS}$ of Galactic NBs as

$$\frac{dN}{df} = R_{NS} \left( \frac{df}{dt} \right)^{-1} \sim 3.8 \times 10^4 \left( \frac{R_{NS}}{10^{-5}yr^{-1}} \right) \left( \frac{f_3}{10^{-3}Hz} \right)^{-11/3} Hz^{-1}. \quad (11)$$

Kalogera et al. [1] recently estimated the event rate as $R_{NS} \simeq 10^{-6} - 5 \times 10^{-4}yr^{-1}$ (see also [2]). For spatial distribution of Galactic NBs we use the standard exponential disk model $\rho(R, z) = \rho_0 \exp(-R/R_0)\exp(-|z|/z_0)$, where $(R, z)$ is the Galactic cylindrical coordinate [1][3]. We fix the radial scale length $R_0 = 3.5 kpc$ and the disk scale height $z_0 = 500 pc$, and assume that the solar system exists at the position $R = 8.5 kpc$ and $|z| = 30 pc$.

We have a relation

$$h_3(f_3) \simeq -9eh_2(2f_3/3)/4 \text{ between } n = 2\text{-mode and } n = 3\text{-mode (see eqs. (10) and (12)). The amplitude } h_3 \text{ of } n = 3\text{-mode is given as}$$

$$h_3 \simeq 4.4 \times 10^{-21} \left( \frac{m_{chirp}}{1.2 M_{\odot}} \right)^{5/3} \left( \frac{f_3}{3 \times 10^{-3}Hz} \right)^{2/3} \left( \frac{100 pc}{r} \right) \left( \frac{e}{0.1} \right), \quad (12)$$

where we have taken angular average with respect to orientation of sources [4]. From equations (10) and (12) we can estimate the distance to a binary whose $n = 3$-mode is detected with a given $SNR$,

$$r = 1800 \left( \frac{f_3}{3 \times 10^{-3}Hz} \right)^{-7/18} \left( \frac{h_{rms}(f_3, T_{obs})}{10^{-23}} \right)^{-1} \left( \frac{10}{SNR} \right) pc, \quad (13)$$

where $h_{rms}(f_3, T_{obs})$ is the noise spectrum. We use the angular averaged sensitivity (effectively a factor of $\sqrt{5}$ degradation [1][4]) to take rotation of LISA into account, and adopt the noise spectrum given in Ref. [15]. When the effective frequency bin $T_{obs}^{-1}$ is occupied by more than one Galactic CWDBs (more precisely, number of fitting parameters is larger than that of data), their confusion noise becomes important and the sensitivity $h_{rms}(f, T_{obs})$ would be significantly worse at lower frequency. Then binaries very close to the solar system would be only resolved. This transition occurs at frequency $f_t \simeq 1.6 \times 10^{-3}(T/10yr)^{-3/11}Hz$ for abundance of Galactic CWDBs estimated in Ref. [13]. As we see below, the prospect of our method depends strongly on the transition frequency and the position of $f_t$ is very important. In an extreme model that the Galactic halo MACHOs are WDs, the frequency $f_t$ can be close to $\sim 10^{-2}Hz$ [16] (see also [17]). If this is true, our method would be severely limited. Spatial filters that depends on the angular position of sources might work effectively in some cases [18]. At $f \gtrsim f_t$ the noise spectrum $h_{rms}(f, T_{obs})$ is mainly determined by the detector’s intrinsic noise and the confusion noise by extra-Galactic CWDBs that depends on cosmological evolution of binary systems [13][19]. At this frequency region the total noise spectrum behaves simply as $h_{rms}(f, T_{obs}) \propto T_{obs}^{-1/2}$. In the present analysis we only count NBs with frequency $f_2 = 2f_3/3 > f_t$.

Now we can estimate the number of NBs whose $n = 3$-mode is detectable by LISA with $SNR > 10$. In figure 1 we show the results for observational period $T_{obs} = 1yr$ and $T_{obs} = 10yr$. The total number (integrated in frequency space) becomes $0.11(R_{NS}/10^{-5}yr)$ for 1yr observation and $3.7(R_{NS}/10^{-5}yr)$ for 10yr. Thus our method would be effective for a long (but realistic) observational period [1]. For these NBs ($T_{obs} = 10yr$, $SNR \leq 10$ and $f_2 > f_t$) the total masses $m_{total}$ would be resolved within a few percent accuracy (see eq. (13)). In figure 1 we also plot all the Galactic NBs (dashed-line). Note that our method works well for Galactic NBs with $f > f_t$ in the case of $T_{obs} = 10yr$. 

3
The estimation error for the overall amplitude of wave becomes \( \sim 0.15(SNR/10)^{-1} \) in the LISA band (see table 4.5 of [1]). Thus the eccentricity parameter \( e \) for a detected NB (with \( SNR > 10 \) for \( n = 3 \)-mode) would be measured within \( \sim 15\% \) accuracy. It has been observationally clarified that the mass of a neutron star is \( \sim 1.4M_\odot \) [20] (that of an ordinal white dwarf is \( \lesssim 1.2M_\odot \)). If most of an eccentric binaries with \( m_{\text{total}} \lesssim 2.8M_\odot \) and \( f \gtrsim 10^{-3} \)Hz are either NS+NS or NS-WD binary [13], we can obtain further information of such binaries detected by our method in the following manner. One of the two stars would be a neutron star with mass \( \sim 1.4M_\odot \), and we can estimate the chirp mass of the system. Then the distance \( r \) to the binary is obtained as proposed by Schutz [21].

Using a similar method based on the expression \( \Delta f \) given just after equation (8), we have also estimated the number of Galactic NBs whose chirp signal \( f_{GW} \) for \( n = 2 \)-mode due to gravitational radiation are measured within \( 5\% \) accuracy (5 \times 3/5 \sim 3\% for the chirp mass eq. (1)) [1]. As shown in figure 1, all Galactic NBs with \( f_2 \gtrsim 4 \times 10^{-3} \)Hz satisfies this observational criteria for \( T_{\text{obs}} = 10\text{yr} \). There might be \( \sim 1(R_{NS}/10^{-5}\text{yr}^{-1}) \) NBs whose individual masses can be determined accurately from observed \( m_{\text{total}} \) and \( m_{\text{chirp}} \).

One might wonder whether two gravitational waves from different binaries (mainly CWDBs) are confused as \( n = 2 \) and \( n = 3 \)-modes of a same eccentric source. Frequency distribution of Galactic close white dwarf binaries (CWDBs) is estimated as \( dN/df \sim 1.4 \times 10^3(f/10^{-3}\text{Hz})^{-11/3}\text{Hz}^{-1} \) [15]. We can specify the direction of a monochromatic source with estimation error \( \lesssim 10^{-2}\text{sr} \) for frequency \( f \gtrsim 2 \times 10^{-3} \)Hz (in the case of signal to noise ratio:10, table 4.5 of Ref. [1]) using annual modulation of gravitational wave due to motion of LISA [22]. The mean frequency interval \( \langle \delta f \rangle_{\text{int}} \) for binaries within a box of \( \sim 10^{-2}\text{sr} \) is given as \( \langle \delta f \rangle_{\text{int}} \sim (10^{-2}/4\pi)^{-1}(dN/df)^{-1} \sim 6 \times 10^{-5}(f/4/10^{-3}\text{Hz})^{11/3}\text{Hz} \). The typical frequency difference due periastron advance \( \delta f \) (eq. (6)) is much smaller than this interval and coincidence of miss-identification would be \( \langle \delta f \rangle/\langle \delta f \rangle_{\text{int}} \sim 2 \times 10^{-3} \). The total number of Galactic CWDBs within frequency \( 2 \times 10^{-3}\text{Hz} < f_2 < 4 \times 10^{-3}\text{Hz} \) is estimated as \( \sim 4 \times 10^4 \). Thus number of miss-identified CWDB pairs would be at most \( \sim 80 \). The estimation error of LISA for the direction of the orbital angular momentum is \( \sim 10^{-3}\text{sr} \ (SNR = 10, \text{table 4.5 of Ref. [1]}) \). Thus there would be only \( \sim 80 \times 10^{-3}/4\pi \sim 1 \) miss-identified CWDBs. It would be also possible to distinguish NBs using the estimated total mass itself, as most CWDBs have total masses smaller than \( 2.4M_\odot \) [13].

Let us summarize this Letter. We have proposed a new method to determine the total masses of low frequency eccentric binaries form their gravitational waves. We use effects of periastron advance on the frequency space of gravitational wave. Our method would be effective for Galactic NBs \( (f \gtrsim 2 \times 10^{-3}\text{Hz}) \) with a long term \((\sim 10\text{yr})\) operation of LISA.

ACKNOWLEDGMENTS

N.S. would like to thank Takashi Nakamura for enlightening suggestions. He also thanks Kunihito Ioka, Misao Sasaki, and Hideyuki Tagoshi for helpful comments. This work was supported in part by Grant-in-Aid of Scientific Research of the Ministry of Education, Culture, Sports, Science and Technology No. 0001416.

[1] P. L. Bender et al., LISA Pre-Phase A Report, Second edition, July 1998.
[2] C. Cutler et al., Phys. Rev. Lett. 70, 2984 (1993); L. Blanchet, et al., Phys. Rev. Lett. 74, 3515 (1995); C. Cutler and E. E. Flanagan, Phys. Rev. D 50, 2658 (1994).
[3] G. E. Brown, C.-H. Lee, S. F. P. Zwart, and H. A. Bethe, Astrophys. J. 547, 345 (2001).
[4] P. C. Peters and J. Mathews, Phys. Rev. 131, B435 (1966); P. C. Peters, Phys. Rev. 136, B1224 (1964).
[5] R. Blandford, and S. A. Teukolsky, Astrophys. J. 198, L27 (1975); J. H. Taylor and J. M. Weisberg, Astrophys. J. 345, 434 (1989).
[6] C. Moreno-Garrido, et al., Mon. Not. Roy. Astron. Soc. 274, 115 (1995).
[7] H. Wahlquist, Gen. Rel. Grav. 19, 1101 (1987); C. Moreno-Garrido, et al., Mon. Not. Roy. Astron. Soc. 266, 16 (1994); K. Martel and E. Poisson, Phys. Rev. D 60, 124008 (1999); V. Pierro, et al., Mon. Not. Roy. Astron. Soc. 325, 358 (2001).
[8] C. W. Misner, K. S. Thorne, and J. W. Wheeler, Gravitation, (Freeman, San Francisco, 1973).
[9] N. Seto, preprint (2001).
[10] J. Buitrago, et al., Mon. Not. Roy. Astron. Soc. 268, 841 (1994); V. B. Ignatiev, A. G. Kuranov, K. A. Postnov and M. E. Prokhorov, astro-ph/0106299; A. V. Gusev, et al., astro-ph/0111066.
[11] V. Kalogera, R. Narayan, D. N. Spergel and J. H. Taylor, Astrophys. J. 556, 340 (2001).
FIG. 1. Various distributions of Galactic NBs within frequency bins $f_0 \leq f_2 < 1.26 f_0$. The dashed-line represents all Galactic NBs. The filled triangles are the number of NBs whose gravitational wave of $n = 3$-mode are detected with $SNR \geq 10$, and open triangles are those whose chirp signals ($\dot{f}_{GW}$) due to radiation reaction are measured within 5% accuracy. We take the coalescence rate of Galactic NBs at $R_{NS} = 10^{-7} \text{yr}^{-1}$.