THE SPECIFIC HEAT OF A FERMI GAS AT LOW TEMPERATURE:
A closer look at the $T \ln T$ behavior

A. REBEI

Department of Physics
University of Wisconsin. Madison, WI 53706

W.N.G. HITCHON

Department of Electrical and Computer Engineering
University of Wisconsin. Madison, WI 53706

Abstract

At finite temperature, a Fermi gas can have states that hold simultaneously a particle and a hole with a finite probability. This gives rise to a new set of diagrams that are absent at zero temperature. The so called “anomalous” diagram is just one of the new diagrams. We have already studied the contribution of these new diagrams to the thermodynamic potential (Phys. Lett. A224, 127 (1996)). Here we continue that work and calculate their effect on the specific heat. We will also calculate the finite temperature contribution of the ring diagrams. We conclude that the $T \ln T$ behavior due to exchange gets canceled by the new contribution of the new diagrams and screening is not essential to solve this anomaly.

\footnotesize
\begin{itemize}
  \item \textsuperscript{1}rebei@math.wisc.edu
  \item \textsuperscript{2}hitchon@engr.wisc.edu
\end{itemize}
In this letter we calculate the specific heat of an electron gas at low temperature and high density. This calculation is not just an improvement over the Gell-Mann calculation [1]. It is fundamentally different. The Seitz formula used by Gell-Mann has been criticized in the literature and the relevance of screening has been questioned [2], [3]. We would like to stress that we are not arguing that the divergence of the inverse of the density of states at the Fermi level is not due to the long range nature of the Coulomb field. Here instead we are addressing the original problem of the $T \ln T$ behavior of the specific heat that was pointed out by Bardeen [4] (see also [5]) if exchange is included. In fact our calculation shows that there is no need to invoke the screening of the Coulomb potential to find a linear function for the specific heat at low temperature. We stress that the solution to this problem has apparently not been clarified yet! Clearly finite temperature calculations show that the problem is related to the Fermi distribution. In fact all of these calculations [6] give a finite answer for the specific heat with exchange included. Unfortunately, Gell-Mann in his paper did not settle this point, even though he cited Bardeen’s paper. This caused a confusion between two different types of logarithmic behaviors. One is due to the long range behavior of the Coulomb field while the other is due to the statistic of the Fermi particles. In this letter we show that the specific heat maintains its observed linear behavior at low temperature without one having to worry about screening. However as we showed in a previous letter [7] we have to include a new infinite set of diagrams. This new set does not contribute anything at zero temperature. The diagrams are merely due to the fact that at nonzero temperature there is a finite probability of finding a state that is occupied by a particle and a hole.

We start as usual from the Hamiltonian of the system,

$$H_0 = \int d^3x \left[ -\frac{1}{2} \Psi_\alpha^\dagger(x) \nabla^2 \Psi_\alpha(x) \right] + \frac{1}{2} \int d^3x d^3y \frac{1}{|\vec{x} - \vec{y}|} \Psi_\alpha^\dagger(x) \Psi_\beta^\dagger(y) \Psi_\beta(y) \Psi_\alpha(x)$$  

(1)

Here, we use units such that $\hbar = m = e = 1$. $\Psi(x)$ is a two-component electron field. $\alpha$ and $\beta$ are spin indices, i.e., $\alpha = 1$ for spin up and $\alpha = -1$ for spin down. Summation is implicit for repeated indices. The system is constrained by the condition
\[ \int d^3x \Psi_\alpha^\dagger(x) \Psi_\alpha(x) = N \] (2)

where \( N \) is the electron number operator which is constant. Because of this constraint, we prefer to work with the Hamiltonian

\[ H = H_0 - \mu N \] (3)

where \( \mu \) is a Lagrangian multiplier. This calculation we present is mainly an improvement on calculations of the exchange energy and ring diagrams made by Isihara and coworkers \[2\] and others. The appearance of temperature dependent logarithmic terms in the internal energy found through various calculations are unacceptable. These terms will reappear again in the specific heat and will spoil the observed linear behaviour at low temperature. In the past these difficulties were not given close attention. They were either hidden in an effective mass or the expansion was not completely correct as we argue below. The history of this problem dates back to the ’30s when it was first pointed out by Bardeen \[4\]. A preliminary treatment of this work appeared in \[7\]. It was argued that a correct approximation to the internal energy of a many-particle system up to exchange, i.e., a Hartree-Fock approximation, will automatically include summing an infinite set of diagrams (figure 1). The first diagram in this set is known as the “anomalous” diagram \[6\]. Therefore any approximations must be made at the level of two-particle Green’s functions and not the single-particle wave functions as was done in most past treatments. This is exactly the reason for introducing two-point sources instead of the one-point source used in the usual effective potential method \[7\], \[9\]. We have seen that this enabled us to write an expression for the energy in terms of the true two-particle Green’s functions of the system. Hence for the two-body Coulomb potential a Hartree-Fock approximation is equivalent to writing the two-particle Green’s function in terms of the one-particle Green’s function as follows:

\[ G(x_1, x_2; x_3, x_4) = G(x_1, x_3)G(x_2, x_4) - G(x_1, x_4)G(x_2, x_3) \] (4)

This approximation is equivalent, as it is well known, to the usual Hartree-Fock approximation of the energy to order \( e^2 \) at zero temperature. At low temperature there is another contribution to order \( e^2 \) that comes from summing an infinite set of diagrams that were never treated before. The ring
diagrams treated in [1] are of order $e^4$ and therefore the new set cannot be ignored in any finite temperature treatment. The effective action method can be easily extended to the non-zero temperature case. In the presence of external sources, the grand partition function $Z$ of the system is given in this case by the following expression:

$$Z = \exp(-\beta \Omega) = \text{Tr} \exp \left( -\beta \left( \sum_{i} (H_{i} - \mu N_{i}) + j_{i} \phi_{i} + \frac{1}{2} \phi_{i} B_{i} \phi_{i} + \psi_{i} Q_{i} \psi_{i} \right) \right).$$ (5)

The trace is over space indices, internal indices and “imaginary time”. The imaginary time interval is $[0, \beta]$. The thermodynamic potential $\Omega$ should be considered as a functional of the sources $j(x, \tau), B(x, \tau; x', \tau')$ and $Q(x, \tau; x', \tau')$. The other parameters in the system are the variables $\mu, T$ and $V$. Now we define new variables by the following functional relations:

$$\frac{\delta \Omega}{\delta j} = \phi_{c},$$ (6)

$$\frac{\delta \Omega}{\delta B} = \Delta + \frac{1}{2} \phi_{c} B \phi_{c}$$ (7)

and

$$\frac{\delta \Omega}{\delta Q} = \rho.$$ (8)

The above equations are evaluated at $j = B = Q = 0$. To get the free energy $F[\rho, \phi_{c}, \Delta]$ we make the following triple Legendre transformation

$$F[\rho, \phi_{c}, \Delta] - \mu N = \Omega[j, B, Q] - j_{i} \phi_{i} - \frac{1}{2} \phi_{i} B_{i} \phi_{i} - \rho Q = \Delta B$$ (9)

To calculate $F$ perturbatively, we expand the exponent around the classical Hartree potential in a neutral background. The coefficients of this expansion are expressible in terms of Feynman diagrams. We keep only the first two terms of the expansion. The equations of motion of the variables $\rho, \phi_{c}$ and $\Delta$ are obtained by the standard way. The variables $\phi_{c}$ and $\Delta$ can be expressed as

a series expansion in $e^{2}$ with coefficients dependent only on $\rho$. Hence within
the above approximation, the expression for the thermodynamic potential has the following form:

\[ \Omega = \Omega_0 + \Omega_{ex} + \frac{1}{2} Tr \{ \ln [\delta(x - y) + e^2 \Re(x - y)] - e^2 \Re(x - y) \} - Tr \{ \ln [\delta(x - y) - e^2 \Im(x - y)] + e^2 \Im(x - y) \} \]

(10)

where the functions \( \Re(x - y) \) and \( \Im(x - y) \) are given by

\[ \Re(x - y) = \int_0^\beta d\tau \int d^3 z A'(y - z) \rho(x - y) \rho(z - x) \]

(11)

and

\[ \Im(x - y) = \int_0^\beta d\tau \int d^3 z A(x - z) \rho(z - x) \rho(y - z). \]

(12)

The term \( \Omega_0 \) is the free contribution, \( \Omega_{ex} \) is the usual first order exchange term, the third term represents the usual ring diagrams and the last term represents a new series of diagrams shown in figure 4. The function \( A(x - y) \) is the bare Coulomb potential and \( \rho(x - y) \) is the free one-particle Green’s function at finite temperature. The signs in front of the two \( Tr \ln \)-terms are determined by integrating over Bose fields and Fermi fields, respectively.

We have shown that the new diagrams provide a \( T \ln T \) term with the right weight to cancel the \( T \ln T \) term that appears in the exchange term. This term is beyond the reach of all previous treatments including those of Gell-Mann and Isihara. Hence the unphysical logarithmic behavior of the exchange energy gets cancelled by a consistent perturbative treatment of the many-body problem. The other important point we learned from this calculation is that this logarithmic term is not an immediate byproduct of the singular behavior of the Coulomb potential since these new diagrams are not singular like the ring diagrams. The new contribution at high density is given by

\[ \Gamma = \frac{e^2 V}{12 \pi \eta^2} k^4(\zeta + \frac{1}{2} \ln \eta). \]

(13)

The ring diagrams contribute a temperature dependent term of order \( e^4 \) as we will show next. The ring diagrams are a manifestation of the long range behavior of the Coulomb field. In fact, in our calculation, the ring diagrams are obtained by integrating over the Hartree field while the new
terms are obtained by integrating over the Fermi fields. We do not think that this point has been recognized in the literature. Obviously, these new diagrams have a nonzero contribution to the specific heat and must not be left out in any accurate calculation. Their contribution appears when we try to calculate $\mu$ in terms of the Fermi energy of the free electron gas. The fact that these new terms have an opposite contribution to the exchange energy can be easily deduced if we refer back to [7]. In fact, our method shows that for a many-body problem with a two-body potential, isolating the exchange energy is really an artificial effect. Reference [10] shows that the exchange energy term really comes from the ring diagrams by integrating over the Hartree potential. However, the new terms came by integrating the Fermi field degrees of freedom which have a statistical origin.

Now we continue evaluating the contribution of the ring diagrams at finite temperature. This contribution is given by

$$R = \frac{1}{2} Tr \{ \ln[\delta(x-y) + e^2 \mathcal{R}(x-y)] - e^2 \mathcal{R}(x-y) \}.$$  \hspace{1cm} (14)

In momentum representation, this is given by

$$R = \frac{V}{2\beta} \sum_{n=-\infty}^{n=\infty} \frac{4\pi}{(2\pi)^3} \int_0^\infty p^2 dp \left[ \ln\{1 + e^2 \mathcal{R}(p,\omega_n)\} - e^2 \mathcal{R}(p,\omega_n) \right]$$ \hspace{1cm} (15)

where $p$ represents momentum and $\omega_n = \frac{(2n+1)\pi}{\beta}, \ n = 0, \pm 1, \pm 2, \ldots$. The function $\mathcal{R}(p,\omega_n)$ is given by:

$$\mathcal{R}(p,\omega_n) = \frac{2}{(2\pi)^3 p^2} \int d\vec{q} \frac{\frac{1}{2}(\vec{p} + \vec{q})^2 - \frac{1}{2} \vec{q}^2}{\omega_n^2 + \left(\frac{1}{2}(\vec{p} + \vec{q})^2 - \frac{1}{2} \vec{q}^2\right)^2} (f(\vec{q}) - f(\vec{p} + \vec{q})).$$ \hspace{1cm} (16)

The function $f(\vec{q})$ is the Fermi-Dirac function,

$$f(\vec{q}) = \frac{1}{1 + \exp[\beta(\vec{q}^2/2 - \mu)]}. \hspace{1cm} (17)$$

To evaluate $R$ we break up $\mathcal{R}(p,\omega_n)$ in two pieces. We write

$$\mathcal{R}(p,\omega_n) = F_1(p,\omega_n) + F_2(p,\omega_n)$$ \hspace{1cm} (18)
such that

\[ F_1(p, \omega_n) = \frac{2}{(2\pi)^3 p^2} \int d\vec{q} \frac{E_{\vec{p}+\vec{q}} - E_{\vec{q}}}{(E_{\vec{p}+\vec{q}} - E_{\vec{q}})^2 + \omega_n^2 f(q)} \]  

(19)

where \( E_{\vec{q}} = \frac{q^2}{2} \). The other function \( F_2(p, \omega_n) \) is similarly defined but with \( f(\vec{q}) \) replaced by \( f(\vec{p} + \vec{q}) \).

We start by evaluating the function \( F_1(p, \omega_n) \). Taking \( \vec{p} \) along the z-axis and integrating the angular coordinates, we get

\[ F_1(p, \omega_n) = \frac{1}{(2\pi)^2 p^3} \int_0^\infty q \, dq \, \ln\left[ \frac{\omega_n^2 + (pq + p^2/2)^2}{\omega_n^2 + (pq - p^2/2)^2} \right] \frac{1}{1 + \exp\left\{\beta(q^2/2 - \mu)\right\}}. \]

(20)

Now we rewrite this in the following form:

\[
\begin{align*}
F_1(p, \omega_n) &= \frac{1}{(2\pi)^2 p^3} \int_0^{k_F} q \, dq \ln\left[ \frac{\omega_n^2 + (pq + p^2/2)^2}{\omega_n^2 + (pq - p^2/2)^2} \right] \\
&\quad - \frac{1}{(2\pi)^2 p^3} \int_0^{k_F} q \, dq \ln\left[ \frac{\omega_n^2 + (pq + p^2/2)^2}{\omega_n^2 + (pq - p^2/2)^2} \right] \frac{1}{1 + \exp\{-\beta(q^2/2 - \mu)\}} \\
&\quad + \frac{1}{(2\pi)^2 p^3} \int_0^{k_F} q \, dq \ln\left[ \frac{\omega_n^2 + (pq + p^2/2)^2}{\omega_n^2 + (pq - p^2/2)^2} \right] \frac{1}{1 + \exp\{\beta(q^2/2 - \mu)\}}.
\end{align*}
\]

(21)

In the limit when \( \beta \to \infty \), the first integral that we isolated becomes simply part of the zero temperature contribution to \( R \). We denote this part by \( F_{1,0}(p, \omega_n) \). Next we set \( z = \beta(q^2/2 - \mu) \) and integrate over \( z \) instead of \( q \). Since we are interested in the low temperature limit, we get to leading order in \( 1/\beta \) that

\[
\begin{align*}
F_1(p, \omega_n) &= F_{1,0}(p, \omega_n) + \frac{\pi}{(2\pi)^3 p^3} \beta \int_0^\infty dz \frac{\ln\left[ \frac{\omega_n^2 + (p^2/2 + pk_F(1 + \frac{z}{2\eta}))^2}{\omega_n^2 + (p^2/2 - pk_F(1 + \frac{z}{2\eta}))^2} \right]}{1 + \exp z} \left\{ \ln\left[ \frac{\omega_n^2 + (p^2/2 + pk_F(1 + \frac{z}{2\eta}))^2}{\omega_n^2 + (p^2/2 - pk_F(1 + \frac{z}{2\eta}))^2} \right] \\
&\quad - \ln\left[ \frac{\omega_n^2 + (p^2/2 + pk_F(1 - \frac{z}{2\eta}))^2}{\omega_n^2 + (p^2/2 - pk_F(1 - \frac{z}{2\eta}))^2} \right] \right\}.
\end{align*}
\]

(22)
Again this last integral over \(z\) can be separated into four integrals that can all be evaluated in the same way by elementary methods. To leading order in \(\beta\), we find

\[
F_1(p, \omega_n) = F_{1,0}(p, \omega_n) + \frac{1}{(2\pi)^2 pk_F} \frac{\pi^2}{12 \beta^2} \left( \frac{p/2 + k_F}{\omega_n^2 + (p^2/2 + pk_F)^2} + \frac{p/2 - k_F}{\omega_n^2 + (p^2/2 - pk_F)^2} \right)
\]  

(23)

The function \(F_2(p, \omega_n)\) is easily seen to be equal to \(F_1(p, \omega_n)\), hence to this approximation the function \(R(p, \omega_n)\) is equal to

\[
R(p, \omega_n) = 2F_{1,0}(p, \omega_n) + 2F_1(p, \omega_n).
\]  

(24)

Putting his back into \(R\), and expanding the logarithmic function in powers of \(1/\eta\), we get

\[
R = \frac{V}{2\beta (2\pi)^3} \sum_{n=\infty}^{n=-\infty} \int_0^\infty p^2 dp \ln \{1 + 2e^2 F_{1,0}(p, \omega_n)\} - 2e^2 F_{1,0}(p, \omega_n)
\]

\[
- \frac{V}{2\beta (2\pi)^3} \sum_{n=\infty}^{n=-\infty} \int_0^\infty p^2 dp \frac{2e^2 F_{1,0}(p, \omega_n)}{1 + 2e^2 F_{1,0}(p, \omega_n)} A(p, \omega_n)
\]  

(25)

where we have defined \(A(p, \omega_n)\) to be

\[
A(p, \omega_n) = \frac{1}{24 pk_F} \left( \frac{p/2 + k_F}{\omega_n^2 + (p^2/2 + pk_F)^2} + \frac{p/2 - k_F}{\omega_n^2 + (p^2/2 - pk_F)^2} \right)
\]  

(26)

The first integral in \(R\) will be denoted by \(R_0\) and the second one by \(R_\beta\). So far our approximation works for any density. Now we notice that \(F_{1,0}(p, \omega_n)\) is proportional to the inverse density. Hence for high densities the second integral can be expanded in powers of inverse densities. To lowest order, the second part can be easily evaluated. Letting \(x = \frac{p}{k_F}\) and \(y_n = \frac{\omega_n}{k_F}\), we have

\[
R_\beta = -\frac{e^4}{12(2\pi)^4 k_F^2 \beta^2} \sum_{n=\infty}^{n=-\infty} \int_0^\infty dx \frac{x/2 + 1}{x^2} \left( \frac{x/2 + 1}{y_n^2 + x^2(x/2 + 1)^2} + \frac{x/2 - 1}{y_n^2 + x^2(x/2 - 1)^2} \right)
\]

\[
\{x - y_n \arctan \left( \frac{x(x/2 - 1)}{y_n} \right) - y_n \arctan \left( \frac{x(x/2 - 1)}{y_n} \right) \}
\]

\[
+1/2 \ln \left[ \frac{y_n^2 + x^2(x/2 + 1)^2}{y_n^2 + x^2(x/2 - 1)^2} \right] + \frac{1}{8x^2} \left( 4y_n^2 - x^4 \right) \ln \left[ \frac{y_n^2 + x^2(x/2 + 1)^2}{y_n^2 + x^2(x/2 - 1)^2} \right]
\]  

(27)
In the limit that $\beta \to \infty$, the summation over $n$ can be changed to an integral over $y$ with measure $dy = \frac{2\pi}{\beta k_F}$. This integral has been evaluated numerically. It turns out that integrating over the variable $\frac{y}{x}$ instead of $y$ makes the integration more suitable from the numerical point of view. The numerical value of the integral

$$
I = \int_0^\infty \frac{dx}{x^2} \int_0^\infty dz K(x, z) P(x, z)
$$

is approximately equal to $-22.94$. In the above the functions $K(x, z)$ and $P(x, z)$ are given by

$$
K(x, z) = \frac{1 + x/2}{z^2 + (1 + x/2)^2} - \frac{1 - x/2}{z^2 + (1 - x/2)^2}
$$

and

$$
P(x, z) = 1 - z \arctan\left(\frac{1 + x/2}{z}\right) - z \arctan\left(\frac{1 - x/2}{z}\right)
+ \frac{1}{2x} \left(1 - x^2/4 + z^2\right) \ln\left[\frac{z^2 + (1 + x/2)^2}{z^2 + (1 - x/2)^2}\right].
$$

Therefore the finite temperature contribution of the ring diagrams to leading order in $\frac{1}{\eta^2}$ and for high density is given by

$$
R = N(0.0622 \ln r_s - 0.142) + 0.0181 N(T) \mu \left(\frac{\alpha r_s(T)}{\eta}\right)^2
$$

Here $N$ is the total number of particles, which is a constant. All the other variables are defined as follows:

$$
\alpha = \left(\frac{4}{9\pi}\right)^{1/3},
\frac{r_s(T)}{a_0} = \frac{m e^2}{\alpha \hbar^2 k_F},
\frac{\hbar^2}{m e^2},
N(T) = \frac{\frac{4\pi}{3} r_s(T)^3 a_0^3}{V}.
$$
Note that here we did not resort to any artificial cutoff of the momentum in our calculation as was done by Gell-Mann [1] or Isihara [2]. The first author assumed $T = 0$, while the second author and coworkers carried out the integral up to $p \leq 2k_F$. In fact momenta larger than $2k_F$ contribute about 5 percent to the answer.

Our calculation of the specific heat starts by finding an expression for $\mu$ in terms of the Fermi energy at zero temperature. In other words, we expand $\mu$ in powers of $\frac{1}{\beta^2}$. We need to do this before any differentiation with respect to temperature, to get the free energy function which is by definition a function of $T$, $V$, and $N$. Here we will need to carry our expansion only to first order in $\frac{1}{\beta^2}$. This will at least give us the first nontrivial contribution beyond the ideal one. Hence in this case it will be enough to use for $\mu$ the equivalent free electron expression, that is we take,

$$\mu = \mu_0(1 - \frac{\pi^2}{12\eta_0^2}). \quad (33)$$

This implies that

$$N(T) = N(1 - \frac{\pi^2}{8\eta_0^2}). \quad (34)$$

However for $r_s(T)$, it is enough to take it equal to the corresponding zero temperature value $r_s$. The calculations that lead to the determination of the temperature dependent terms in the free energy are straightforward. The ring diagrams will contribute a term $R_T$, which in Rydbergs is given by

$$R_T = \frac{N}{\eta_0^2} \left[ (1 - \frac{\pi^2}{8})(0.0622 \ln r_s - 0.142) + \frac{22.90}{64(2\pi)^2} \right]. \quad (35)$$

Similarly, the “anomalous” diagrams contribute a term of the form

$$A_T = -\frac{3\pi N}{2\eta_0^2\alpha r_s}(\zeta + \frac{1}{2} \ln \eta_0). \quad (36)$$

9
The constant $\zeta$ is approximately equal to 8.0. Besides these, we have the usual contribution from a kinetic part and an exchange part. They are, respectively, given by

$$K_T = \frac{2.21 N 5\pi^2}{r_s^2 \eta_0^2} \quad (37)$$

and

$$E_T^{\text{exch}} = \frac{0.916 N \pi^2}{r_s} \frac{3\eta_0^2}{\eta_0^2} \left(1 + \frac{3a}{2\pi^2} + \frac{1}{2} \ln \eta_0 \right) \cdot \quad (38)$$

The constant $a$ has different values in the literature. For example Isihara and coworkers give it a value of approximately $-\frac{1}{2}$. Our calculation gives it a value of almost +2.0. Both values give similar answers to the specific heat. It will be seen below that the contribution from the new diagrams is much more important and hence any discrepancies in the values of $a$ will hardly matter.

Remembering that $F(T, V, N) = \Omega + \mu N$, we can easily find the contribution of all of the above terms to the temperature dependent term, $F_T$, of the free energy $F$ using standard techniques [11]. Finally, we can now get an estimate for the specific heat,

$$C_V = -\frac{\partial^2 F}{\partial T^2} \quad (39)$$

such that

$$F(T, V, N) = F_0(T, V, N) + \frac{k^2 T^2}{\mu_0^2} \left( -\frac{20.19}{r_s} - 0.077 \ln r_s + 0.184 \right) \quad (40)$$

and $F_0$ is the free energy of a free electron gas. Therefore, if $C_V^0$ is the specific heat for the ideal Fermi gas, we get for the specific heat at low temperatures and high densities the expression,

$$\frac{C_V}{C_V^0} = 1.00 + 0.185 r_s - r_s^2 (0.0017 - 0.0007 \ln r_s) \quad (41)$$
Some comments on this expression are called for. First, without the new effect of the anomalous diagrams, the second term would have been negative besides the $\ln T$ dependency. Second, for sodium, which has $r_s = 3.93$, the ratio $C_V/C^0_V = 1.71$ compared to the experimental value of 1.26. This is not a bad outcome considering that our calculation is strictly valid for only small $r_s$. Gell-Mann’s computation gives 0.48 as an answer in this case. Going back to Eq.(22), it is easily seen that had we kept the next term in our approximation, we could easily improve our answer. Besides the numerical answer, it is important to realize that screening has little effect on the specific heat and need not be included to get a finite and reasonable answer for the specific heat. Exchange energy is essentially of statistical origin. At nonzero temperature a point in phase space can simultaneously hold a particle and a hole and hence the statistics of holes becomes important as well and it is manifested in the appearance of this new type of diagrams. The name “anomalous” is therefore misleading. In fact, they should be called particle-hole exchange diagrams. This name shows that this kind of exchange is not allowed at zero temperature.

We end this letter by stressing the fact this expansion is not an expansion in $e^2$, and that what this method considers exchange is not the usual meaning of the word. The usual first order exchange, the ring diagrams and the new particle-hole exchange diagrams are all part of the first correction term to the energy of an interacting Fermi gas. The second order exchange diagram and the ring diagrams with exchange appear in the next term of the expansion. The third term in this expansion contains among other things the ladder diagrams. These diagrams are important for metal densities. We hope to include all these effects in another communication.

Finally, we would like to mention that this expansion with two-point functions was also recently used in the study of the anharmonic oscillator at low temperature. Here it was shown that effective actions with two-point functions give better results than effective actions with one-point functions.
References

[1] M. Gell-Mann, Phys.Rev.106 , 369 (1957)

[2] A.Isihara, Condensed Matter Physics. Oxford,1991. A. Isihara and D. Y. Kojima, Physica 77, 469 (1974). A. Isihara and D. Y. Kojima, Z.Phys. 21, 33 (1975). A.Isihara and D.Y.Kojima, Z.Physik 21 , 33 (1975)

[3] W. D. Kraeft and W. Stolzmann, Phys. Lett.A 56, 41 (1976)

[4] J. Bardeen, Phys. Rev. 50, 1098 (1936)

[5] E. Wigner, Phys. Rev. 46, 1002 (1934); Trans. Faraday Soc. 34, 678 (1938)

[6] B.Horovitz and R.Thieberger, Physica 71 , 99 (1974). I. Yokota, J. Phys. Soc. Japan 4, 82 (1949). A. B. Lidiard,Proc. Phys. Soc.A64, 814 (1951). E. Wohlfarth, Phil. Mag.41, 534 (1950)

[7] A. Rebei and W.N.G. Hitchon, Phys. Lett. A224, 127 (1996)

[8] K. Kohn and J.M. Luttinger, Phys. Rev. 118, 41 (1960)

[9] C.de Dominicis and P.C.Martin, J.Math.Phys. 5, 14 (1964)

[10] A. Rebei and W. N. G. Hitchon, in preparation

[11] A. L. Fetter and J. D. Walecka Quantum Theory of Many-Particle Systems. McGraw-Hill, 1971.

[12] A. Okopiska, Int. J. Mod. Phys.A 12, 585 (1997),hep-th/9704038
Figure 1: A new infinite set of “anomalous” diagrams. In this approximation, the solid line corresponds to the Fermi propagator and the dashed line corresponds to the Coulomb field.