FE approach for dynamic response of a functionally graded spinning shaft system containing a transverse fully open crack

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Abstract. Free vibration and stability analysis are studied for a rotor-disc-bearing system having a radially functionally graded (FG) shaft with a transversely fully open crack, based on finite element (FE) approach. Both internal viscous and hysteretic damping are incorporated in the FE model of FG cracked shaft using two nodded Timoshenko beam element having four degrees of freedom at each node. Material properties of the FG shaft are assumed temperature dependent and graded with different material law of gradation. Aluminium Oxide (Al2O3) and stainless steel (SS) are composed as FG material. Local flexibility coefficients (LFCs) are derived analytically as a function of crack size, power-law gradient index and temperature using Paris’s equation and Castigliano's theorem to compute the stiffness matrix at each instant in the FE analysis. Using the developed MATLAB code, the FE formulation and the cracked model are verified with the published results. Parametric studies are conducted to study the influences of different material gradient index, temperature gradient, crack size, internal damping, slenderness ratio and boundary conditions on the vibration responses of the FG cracked shaft system.

1. Introduction

FG materials (FGMs) are formed as highly thermo-mechanical resistance, multifunctional, non-homogeneous materials where thermomechanical material properties graded continuously and smoothly in a predefined way using different material gradations laws under thermal environment. Continuous changes in microstructure of FG materials offer superior material properties over conventional composite and metallic shaft, like high stiffness-to-weight ratio, stiffness and strength, and also used to reduce inter-laminar stresses, residual stress and delamination problem etc. Historically, FGM application was first noticed in 1984 during a space plane project in NAL of Japan [1]. After that FGMs applications amplified in automotive, aerospace, electronics, energy and biomedical sectors etc.

Many studies relating the field of FGMs structures have been reported during past few decades. Some of them are presented here. Basics, fabrication techniques, formations and applications of FGMs were reported by Miyamoto et al. [2]. Reddy and Chin [3] studied thermomechanical analyses of the FG plates and cylinders based of FE approach. Using FE analysis and Timoshenko beam theory (TBT), Shahba et al. [4] presented vibration and stability characteristics of an axially FG tapered beams and reported the importance of different parameters on the responses. Using TBT, Gayen and Roy [5] studied dynamic responses of an FG rotating shaft and reported the effect of gradient index on

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natural frequency and stability limit speed. Based on the TBT, dynamics of spinning FG shaft were presented by Boukhalifa [6].

Due to highly requirement of high power, high operating speed, shafts are the basic components in any machines. Therefore, it is required to determine dynamic behavior of the shaft system correctly, and many studies presented in this area. Papadopoulos [7] presented the review on cracked rotor for dynamics. Based on FE analysis, an improvement computational analysis of a rotating shaft with the considering Timoshenko beam elements were reported by Nelson and Mcvaugh [8] and later this work [8] extended by Zorzi and Nelson [9] with including the effect of both the shaft internal dampings (viscous and hysteretic) and same time Nelson [10] utilized TBT to derive the element matrices based on shape functions. Ku [11] studied on dynamics of a rotating shaft system, considering a three nodded C° Timoshenko beam elements with shaft internal damping effect. Coupled vibration and stability of a transverse cracked rotor were studied by Papadopoulos and Dimarogonas [12]. Huang et al. [13] presented dynamic responses of a cracked shaft and shown unstable region raises effectively with the addition of small amount of damping. Stability of a two cracked rotor system was presented by Sekhar and Dey [14] and reported influences of crack parameters, geometric parameters and internal damping on the instability speed. Gayen et al. [15] computed dynamics of an FG cracked rotor system using FE approach.

Even though so many research works studied in general on analysis of structure made of FGM and also many works are reported in stability and free vibration response for uncracked and cracked shaft made of homogeneous materials since long time. However, hardly very few works on stability and free vibration characteristics of cracked FG shaft systems are available under thermal environment. Therefore, this work aims to exhibit a FE model of a FG shaft system with fully open transverse crack, material properties (temperature dependent) graded by power law, or exponential law or sigmoid law, to study the influences of material gradient index, temperature gradient, crack size and crack location, internal viscous and hysteretic damping, slenderness ratio and boundary condition on various dynamic responses.

2. Theoretical Formulation for Material Properties of FGM

For a radially graded FG shaft, material properties $P$, under thermal environment is as [16],

$$P(T) = P_0 + P_1T + P_2T^2 + P_3T^3$$

where $P_0$, $P_1$, $P_2$, $P_3$ and $P_4$ are temperature ($T$ in Kelvin) coefficients of constituent materials.

The properties $P$ (modulus of elasticity $E$ (in Pa), $K$ (in W/m K) is thermal conductivity, Poisson’s ratio $\nu$ and $\rho$ (in Kg/m$^3$) is density) for an FG shaft with temperature ($T$) effect is as [3],

$$P(y, T) = P_m(T) + \{P_r(T) - P_m(T)\}\{(y - R_m)/(R_c - R_m)\}^k, \quad R_c \leq y \leq R_m, \quad 0 \leq k \leq \infty$$

where $R_c$ and $R_m$ are the radius of the shaft at ceramic and metal side respectively, and $k$ is power law gradient index. In Equation (2), $k = 1$ denotes linear law of gradation and for others $k$ it is power law of gradation (P-FGM).

Material properties following to exponential law of gradation (E-FGM)

$$P(y, T) = P_m(T) e^{\lambda(y-R_m)}, \quad \lambda = \ln \{P_r(T)/P_m(T)\}/(R_c - R_m), \quad R_m \leq y \leq R_c$$

Material properties following to sigmoid law of gradation (S-FGM)

$$P(y, T) = P_m(T) + \{P_r(T) - P_m(T)\}\{(y - R_m)/(R_c - R_m)\}^n, \quad R_m \leq y \leq (R_m + R_c)/2$$

$$P(y, T) = P_r(T) + \{P_m(T) - P_r(T)\}\{(y - R_m)/(R_c - R_m)\}^n, \quad (R_m + R_c)/2 \leq y \leq R_c$$

where the sigmoid law of gradient index is $n$. 

2
For radially graded FG shaft, temperature profiles are computed using 1-D steady state heat conduction law with the end conditions at $y = R_m$, $T = T_m$ and at $y = R_e$, $T = T_e$ as

$$\frac{d}{dy} \left[ K(y) \frac{dT}{dy} \right] = 0$$

(5)

The Temperature profiles following P-FGM as

$$T(y) = T_m + (T_e - T_m) \left[ \sum_{j=0}^{\infty} \left( \frac{(-1)^j}{jk + 1} \left( \frac{K_{cm}}{K_m} \right)^j \left( \frac{y - R_m}{R_e - R_m} \right)^{j+1} \right) \right]$$

(6)

Temperature profiles following E-FGM as

$$T(y) = A + Be^{-\lambda(y-R_m)}$$

\(R_m \leq y \leq R_e\)

(7)

where $K_{cm} = K_e - K_m$, $A = T_m - \frac{(T_e - T_m)}{e^{-\lambda(R_e - R_m)} - 1}$, $B = \frac{(T_e - T_m)}{e^{-\lambda(R_e - R_m)} - 1}$ and $\lambda = \frac{1}{(R_e - R_m)} \ln \frac{K_e}{K_m}$

(8)

These graded properties are used for FE model of the shaft made of FGM.

**Figure 1.** The FG Rotor system with a transverse crack: (a) FE discretization (b) Orientation of Crack (c) Load (d) Cracked cross section

3. **FE Formulation for System Equation of Motion of an FG Cracked Shaft**

The system is considered with a FG shaft having a fully open transverse crack, bearings and rigid disks as shown in Figure 1.

3.3. **FG shaft element with a transverse crack**
The cracked FG shafts are discretized with FE and supported by bearings and shown in Figure 1(a), in which \( L \) total length, \( L_c \) element length, \( D \) diameter and \( \alpha \) crack depth located at distance \( L_c \) from the left end of the shaft. Figure 1(b) shows orientation of crack denoted by \( \theta \). Figure 1(c) shows that the shaft element subjected to shear forces \( P_1, P_2, P_3 \) and \( P_6 \) and bending moments \( P_4, P_5, P_7 \) and \( P_q \), rotating at speed \( \Omega \), \( v \) and \( \omega \), \( \beta \) and \( \gamma \) are the translation and rotational displacements of the shaft, respectively. The cracked cross section for \( \theta = 180^\circ \) is shown in Figure 1(d) and half-width of crack is \( b \).

The direct and cross couple LFCs of FG shaft are obtained in the presence of crack, using Paris’s equations [17] and Castigliano’s theorem with the help of SIFs displacement \( u_i^e \) is as

\[
\frac{\partial u_i^e}{\partial \alpha} = \frac{1}{E(y,T)} \left[ \left( \sum_{i=1}^{4} K_{ii} \right)^2 + \left( \sum_{i=1}^{4} K_{i\alpha} \right)^2 \right] dy
\]

(9)

where \( K_{ii}, K_{i\alpha} \) and \( K_{\alpha\alpha} \) are SIFs for modes I, mode II, and mode III and load indices \( i = 1, 2, 3, 4 \).

For fully open crack \( \theta = 180^\circ \), the direct and cross couple LFCs are obtained as (ref. Figure 1(d))

\[
\frac{\partial u_i^e}{\partial \alpha} = \frac{1}{E(y,T)} \left[ \left( \sum_{i=1}^{4} K_{ii} \right)^2 + \left( \sum_{i=1}^{4} K_{i\alpha} \right)^2 \right] dy
\]

(10)

where \( \alpha_s = (\alpha - R) + \sqrt{R^2 - z^2} \) and \( b = \sqrt{R^2 - (R-\alpha)^2} \).

Using Equation (10), LFCs are determined by performing the integrations considering \( E(y,T) \) and \( \nu(y,T) \) as a function of \( T \) as well as \( y \), unlike homogeneous shaft these are considered as constants. Using Equation (10), LFCs are determined corresponding to \( \theta = 180^\circ \) and the local flexibility matrix due to crack \( C^e(y,T) \) is obtained applying energy method and details \( C^e(y,T) \) are taken from the published work [15].

Using cracked compliances matrix \( C^e(y,T) \), stiffness matrix due to crack \( K^e \) is obtained as

\[
K^e = \Pi^T C^e \Pi
\]

(11)

where \( \Pi \) is taken form the work [15].

The equations of motion (EOMs) for FG cracked shaft element expressed as

\[
\left( M_0^e + M_0^r \right) \ddot{p}(t) - \Omega \ G^e \ \dot{\phi}(t) + K^e \ p'(t) = f'(t)
\]

(12)

where \( M_0^e \) and \( M_0^r \), \( G^e \), \( K^e \), \( f'(t) \) and \( p'(t) \) are translational and rotary mass matrix, gyroscopic matrix, stiffness matrix, vector of external force and nodal displacement for the shaft element. The details of elemental matrices of Equation (12) shown in [8, 10 and 15].

3.2. Rigid Disk

The equations of motion for rigid disk element is as

\[
\left( M_0^d + M_0^r \right) \ddot{p}^d - \Omega G^d \dot{p}^d = f^d
\]

(13)

where the element matrices are same as in [8].

3.3. Bearing Model

Following the work [8] bearings are modelled and forces in each bearing are considered as
\[ C^b \ddot{p}^b + K^b p^b = f^b \]  

where \( C^b \), \( K^b \) and \( f^b \) are the damping, stiffness matrices and external force vector for bearing.

### 3.4. System Equations of Motion

The EOMs for the whole FG shaft system with shaft internal dampings (viscous \( \eta_c \) and hysteretic \( \eta_h \)) as used in [9] is as

\[
M \ddot{\mathbf{p}} + \left( \eta_c K + \Omega G \right) \dot{\mathbf{p}} + \left[ \frac{1 + \eta_h}{1 + \eta_h} K + \Omega \eta_c + \frac{\eta_h}{1 + \eta_h} K \omega \right] \mathbf{p} = \mathbf{f} \tag{15}
\]

The eigenvalue solution of Equation (15) is computed as \( \lambda_n = \xi_n \pm i \omega_n \), where \( \xi \), \( \omega \) and \( n \) are damping constant, natural whirl frequency and number of mode respectively. \( \delta_n = -2 \pi \xi_n / \omega_n \) is defined as logarithmic decrement and the stability threshold speed \( \Omega_n \) obtained at the operating speed for that \( \delta_n = 0 \).

### 4. Results and Discussions

In this work, an FG cracked shaft with \( D = 0.1 \) m and \( \theta = 180^\circ \) is used in the rotor-disk bearing system. Here, the constituents of the FG (SS/Al\(_2\)O\(_3\)) materials are considered based on the work of Reddy and Chin [3] with thermal environment. Twenty five (25) equal FEs are used to model the FG shaft and supported by rigid end bearing and with undamped bearing (stiffness coefficients \( K_{vv} = K_{ww} = 17.513 \) MN/m and \( K_{vw} = K_{ww} = 0 \)) at both ends. A centrally located disc with weight of 1.406 kg and polar and diametral moment of inertia of 0.002 kg-m\(^2\) and 0.0135 kg-m\(^2\) respectively, is considered.

#### 4.1. Validation

Since there are no published results available on free vibration and stability response for an FG spinning cracked shaft under thermal environment. Therefore, a homogeneous cracked shaft is considered for validating the FE formulation and the developed MATLAB code in two steps.

A steel shaft with \( D = 10.16 \) cm, \( L = 1.27 \) m, \( E = 206.80 \) GPa and \( \rho = 7833 \) kg/m\(^3\) are considered for validation the formulation. The influence of \( \eta_h \) on natural frequencies (dimensionless) \( \sigma_n = \sigma_n^0 = \rho AL^4 \omega^2 / EI \) is determined of a non-rotating simply supported shaft with varying shaft’s slenderness \( SR = R/2L \). Results in Table 1 are compared with Ku’s results [11] and show a well matched. Therefore the developed code and present formulation is validated.

| Modes (R/2L) | \( \eta_h = 0.000 \) | \( \eta_h = 0.001 \) | \( \eta_h = 0.002 \) |
|-------------|-----------------|-----------------|-----------------|
|             | Present          | Present          | Present          | Present          | Ku [11] | Ku [11] | Ku [11] |
| 1st         | 3.1300          | 3.1308          | 3.1316          | 3.1307          | 3.1309 |
|             | 3.1297          | 3.1301          | 3.1301          | 3.1307          | 3.1309 |
|             | 3.0452          | 3.0459          | 3.0467          | 3.0414          |         |
|             | 3.0403          | 3.0407          | 3.0407          | 3.0414          | 3.0414 |
| 2nd         | 2.9104          | 2.9112          | 2.9119          | 2.9006          | 2.9006 |
|             | 2.8995          | 2.8999          | 2.8999          | 2.9006          | 2.9006 |
|             | 6.2037          | 6.2052          | 6.2068          | 6.1943          | 6.1943 |
|             | 6.2018          | 6.1923          | 6.2068          | 6.1943          | 6.1943 |
|             | 5.7098          | 5.7112          | 5.7126          | 5.6490          | 5.6490 |
|             | 5.6540          | 5.6476          | 5.7126          | 5.6490          | 5.6490 |
|             | 5.1478          | 5.1491          | 5.1504          | 5.0560          | 5.0560 |
|             | 5.0537          | 5.0493          | 5.0493          | 5.0560          | 5.0560 |

A stainless steel shaft is used with \( D = 0.1 \) m, \( L = 1.25 \) m for validating the LFCs. The compliance in
dimensionless form $\overline{C}_{44}$ and $\overline{C}_{55}$ with crack size $\alpha/R$ are computed and compared with Chasalevris and Papadopoulos [18] and the results are listed in Table 2. It shows a good agreement to verify the crack formulation for direct and cross couple terms of compliances.

| $\alpha/R$ | Present | Ref [18] | % Error |
|------------|---------|----------|---------|
| 0.04       | 0.0000349 | 0.000035 | 0.285714 | 0.0029731 | 0.002973 | -0.00336 |
| 0.36       | 0.0747941 | 0.0747941 | 0.00134 | 0.5677802 | 0.567780 | 3.52E-05 |
| 0.68       | 0.8171641 | 0.8171641 | 1.22E-05 | 2.5852663 | 2.585270 | -0.00014 |
| 1.00       | 5.2663511 | 5.2663505 | -2.09E-05 | 7.7903871 | 7.790390 | 3.7E-05 |

Table 2. $\overline{C}_{44}$ and $\overline{C}_{55}$ with different values of $\alpha/R$

Table 3. Comparison of $\omega_{uc,c}^{\omega}$ with $\Delta T$ and material distributions for a non-spinning simply supported FG (SS/Al$_2$O$_3$) shaft for $R/2L=0.02$

| Gradation law | Gradient index | $\Delta T=0$ K | $\Delta T=300$ K | $\Delta T=600$ K |
|---------------|----------------|----------------|----------------|----------------|
|               | No disc | With disc | No disc | With disc | No disc | With disc | No disc | With disc |
| P-FGM         | $k = 0.0$  | 2.86 | 2.75 | 2.80 | 2.70 | 2.78 | 2.68 |
|               | $k = 0.5$  | 2.34 | 2.27 | 2.29 | 2.23 | 2.24 | 2.18 |
|               | $k = 1.0$  | 2.12 | 2.07 | 2.07 | 2.02 | 2.01 | 1.96 |
|               | $k = 3.0$  | 1.84 | 1.80 | 1.79 | 1.76 | 1.71 | 1.68 |
|               | $k = 5.0$  | 1.76 | 1.72 | 1.71 | 1.68 | 1.63 | 1.60 |
|               | $k = 10.0$ | 1.69 | 1.66 | 1.64 | 1.61 | 1.55 | 1.52 |
| S-FGM         | $n = 0.0$ | 1.56 | 1.53 | 1.49 | 1.47 | 1.33 | 1.30 |
|               | $n = 0.5$ | 3.10 | 2.95 | 3.02 | 2.87 | 2.88 | 2.74 |
|               | $n = 1.0$ | 3.15 | 2.99 | 3.06 | 2.91 | 2.93 | 2.78 |
|               | $n = 3.0$ | 3.24 | 3.06 | 3.14 | 2.98 | 3.00 | 2.84 |
|               | $n = 5.0$ | 3.26 | 3.09 | 3.17 | 3.00 | 3.03 | 2.87 |
|               | $n = 10.0$ | 3.29 | 3.11 | 3.19 | 3.02 | 3.05 | 2.88 |
| E-FGM         | $n = 0.0$ | 3.30 | 3.11 | 3.20 | 3.03 | 3.06 | 2.89 |

4.2. Effects of $k$, $n$, $\Delta T$, $\alpha/R$ and support conditions on frequency response for a cracked FG shaft:

The importance of $k$ and $n$, temperature gradient $\Delta T$, crack depth $\alpha/R$ and end conditions (C, F and H are used for clamped, free and hinged, respectively), on dimensionless natural frequencies ($\omega_{uc,c}^{\omega} = \omega_{uc} \sqrt{\rho_d A E / E_{SS}}$, where superscripts uc and c denotes uncracked and cracked respectively) are studied of a non-spinning FG (SS/Al$_2$O$_3$) shaft for crack location $L_c/L=0.42$ and $R/2L=0.02$ with thermos-mechanical properties followed by gradation laws like P-FGM, E-FGM and S-FGM, results are listed in Table 3 - 5. Table 3 shows that increase in $k$ causes the decrease in the frequencies due to transition Al$_2$O$_3$ (lower density and higher $E$) to SS (higher density and lower $E$) and for S-FGM, frequencies increase with increase in gradient index $n$. Table 3 also shows that natural frequencies are decreased with a centrally located disc. The order of reduction in frequencies is E-FGM, S-FGM and P-FGM respectively for specific $\Delta T$, support conditions and material distributions. Table 4 shows that the variation in $\omega_{uc,c}^{\omega}$ is relatively smaller for higher $\Delta T$ as the
properties of material degrade with $\Delta T$ increases in radially graded cracked FG shaft. It also observed that the C-C and C-F boundary condition have the highest and lowest natural frequency respectively and also the reduction is more for higher mode of frequency. Table 5 shows that with the increase in depth of crack $\alpha/R$, $\sigma_{w}^{c}$ decrease due to the degradation of modified stiffness.

Table 4. Comparison of first $\sigma_{w}^{c}$ with $\Delta T$ and $k$ for a non-spinning FG (SS/Al$_2$O$_3$) shaft with different boundary conditions for $R/2L=0.02$, $\alpha/R=0.8$ and $L_c/L=0.42$

| $\Delta T$ | $\sigma_{w}^{c}$ | C-F | H-H | C-C |
|------------|-----------------|------------------|------------------|------------------|
|            | $k=0.5$ | $k=3.0$ | $k=0.5$ | $k=3.0$ | $k=0.5$ | $k=3.0$ |
| 0          | $\sigma_{1V,1H}^c$ | 0.83 | 0.65 | 2.27 | 1.80 | 5.01 | 3.97 |
|            | $\sigma_{1V}^c$ | 0.58 | 0.45 | 1.18 | 0.91 | 3.87 | 3.04 |
|            | $\sigma_{1H}^c$ | 0.79 | 0.62 | 2.04 | 1.60 | 4.77 | 3.74 |
| 300        | $\sigma_{1V,1H}^c$ | 0.81 | 0.64 | 2.23 | 1.76 | 4.91 | 3.87 |
|            | $\sigma_{1V}^c$ | 0.57 | 0.44 | 1.15 | 0.89 | 3.79 | 2.96 |
|            | $\sigma_{1H}^c$ | 0.78 | 0.61 | 2.00 | 1.56 | 4.67 | 3.65 |
| 600        | $\sigma_{1V,1H}^c$ | 0.80 | 0.61 | 2.18 | 1.68 | 4.80 | 3.70 |
|            | $\sigma_{1V}^c$ | 0.56 | 0.42 | 1.13 | 0.85 | 3.71 | 2.83 |
|            | $\sigma_{1H}^c$ | 0.76 | 0.58 | 1.96 | 1.49 | 4.56 | 3.48 |

Table 5. Comparison of first $\sigma_{w}^{c}$ with $k$, $\Delta T$ and $\alpha/R$ for a non-spinning simply supported cracked FG (SS/Al$_2$O$_3$) shaft for $R/2L=0.02$ and $L_c/L=0.42$

| $\alpha/R$ | $\sigma_{w}^{c}$ | $\Delta T=0$ K | $\Delta T=300$ K | $\Delta T=600$ K |
|------------|-----------------|-----------------|-----------------|-----------------|
|            | $k=0.5$ | $k=3.0$ | $k=0.5$ | $k=3.0$ | $k=0.5$ | $k=3.0$ |
| 0.2        | $\sigma_{1V}^c$ | 2.23 | 1.75 | 2.21 | 1.75 | 2.20 | 1.74 |
|            | $\sigma_{1H}^c$ | 2.33 | 1.83 | 2.31 | 1.82 | 2.31 | 1.82 |
| 0.4        | $\sigma_{1V}^c$ | 1.91 | 1.50 | 1.89 | 1.48 | 1.87 | 1.46 |
|            | $\sigma_{1H}^c$ | 2.31 | 1.82 | 2.28 | 1.80 | 2.26 | 1.77 |
| 0.6        | $\sigma_{1V}^c$ | 1.53 | 1.19 | 1.51 | 1.17 | 1.48 | 1.14 |
|            | $\sigma_{1H}^c$ | 2.23 | 1.75 | 2.19 | 1.72 | 2.16 | 1.68 |
| 0.8        | $\sigma_{1V}^c$ | 1.18 | 0.91 | 1.15 | 0.89 | 1.13 | 0.85 |
|            | $\sigma_{1H}^c$ | 2.04 | 1.60 | 2.00 | 1.56 | 1.96 | 1.49 |
| 1.0        | $\sigma_{1V}^c$ | 0.86 | 0.67 | 0.84 | 0.65 | 0.82 | 0.61 |
|            | $\sigma_{1H}^c$ | 1.69 | 1.33 | 1.65 | 1.29 | 1.60 | 1.20 |

4.3. Effects of material gradient index, temperature gradient on whirling frequencies, critical speeds and threshold speeds for an FG cracked shaft system including

Influences of $\eta$, $k$, $\Delta T$ and $\alpha/R$ on critical speeds, threshold speeds and maximum real part of eigenvalues of an elastic rotor system having a FG shaft supported on bearing with stiffness coefficients $K_{ww}^b = K_{ww}^b = 17.513$ MN/m, $K_{ww}^b = K_{ww}^b = 0$ and with the same material properties and dimension stated earlier, and results are listed in Table 6 and presented in Figures 2(a-d). Table 6 show that the critical speeds and threshold speeds increases with the increase in $\Delta T$ and $\alpha/R$, and
also shows that all the backward whirling (BW) modes are stable for entire spin-speed studied, whereas for forward whirling (FW) modes instability threshold begins at spin speeds higher than the critical speeds. However, for the shaft material with $\eta_h$, all the BW modes are stable, while FW modes become unstable.

**Table 6.** Critical speed (rpm) and threshold speed (rpm) of the rotor system having FG (SS/Al$_2$O$_3$) shaft with $\Delta T$ and $\alpha/R$ for $k=0.5$, $R/2L=0.02$ and $L_c/L=0.42$

| $\alpha/R$ | Modes | Critical speed, $\Omega_c$ | $\eta_v=0.0002$ s | Threshold speed, $\Omega_h$ |
|------------|-------|---------------------------|------------------|---------------------------|
|            |       | $\Delta T=0$ K | $\Delta T=300$ K | $\Delta T=600$ K | $\Delta T=0$ K | $\Delta T=300$ K | $\Delta T=600$ K |
| 0.0        | 1 FW  | 6587                      | 6541             | 6492             | 6590                      | 6545             | 6495             |
|            | 2 FW  | 13522                     | 13506            | 13487            | 13525                     | 13510            | 13490            |
| 0.2        | 1 FW  | 6581                      | 6576             | 6559             | 6555                      | 6545             | 6535             |
|            | 2 FW  | 13525                     | 13520            | 13516            | 13520                     | 13515            | 13510            |
| 0.4        | 1 FW  | 6561                      | 6536             | 6516             | 6480                      | 6455             | 6435             |
|            | 2 FW  | 13524                     | 13514            | 13507            | 13475                     | 13465            | 13455            |
| 0.6        | 1 FW  | 6517                      | 6480             | 6447             | 6460                      | 5430             | 6400             |
|            | 2 FW  | 13522                     | 13509            | 13496            | 13365                     | 13355            | 13340            |
| 0.8        | 1 FW  | 6439                      | 6390             | 6337             | 6370                      | 6340             | 6310             |
|            | 2 FW  | 13521                     | 13503            | 13482            | 13125                     | 13110            | 13090            |
| 1.0        | 1 FW  | 6332                      | 6268             | 6188             | 5320                      | 5310             | 5295             |
|            | 2 FW  | 13521                     | 13499            | 13470            | 12705                     | 12685            | 12660            |

In the presence of crack on an FG shaft system having $k=3.0$, $\Delta T=0$ K, $\alpha/R=0.8$, $R/2L=0.02$ and $L_c/L=0.42$, variation in maximum real part of eigenvalues with $\eta_v$ are shown in **Figure 2(a)** and it is seen that with the increase in $\eta_v$ the real part increases but the $\Omega_h$ decreases. Similarly, for the cracked FG shaft system having $R/2L=0.02$, $\Delta T=0$ K, $\alpha/R=0.8$, $\eta_v=0.0002$ s and $L_c/L=0.42$, variations in maximum real part of eigenvalues with $k$ are shown in **Figure 2(b)**. It is also found that the maximum real part and threshold speeds increase with decrease in $k$ as amount of metallic portion decrease with increase in $k$. For the FG shaft with isotropic un-damped bearing condition for $k=3.0$, $\eta_v=0.0002$ s, $R/2L=0.02$, $\alpha/R=0.8$ and $L_c/L=0.42$, maximum real part with $\Delta T$ is shown in **Figure 2(c)** and it could be observed that the changes of threshold speeds as well as maximum real part are quite less for increase in $\Delta T$ for the FG cracked rotor system. Similarly, variation in maximum real part with different values of $\alpha/R$ are shown in **Figure 2(d)** for the FG shaft supported by isotropic un-damped bearing condition for $k=0.5$, $\eta_v=0.0002$ s, $R/2L=0.02$, $\Delta T=0$ K and $L_c/L=0.42$, and it could be observed that increase in $\alpha/R$ causes increase in maximum real part but threshold speeds decreases. Therefore, designing in FG shaft systems $k$, $\Delta T$, $\eta_v$ and $\alpha/R$ can play a great role to operate at high spinning speeds.
Figure 2. Variation in maximum real part of eigenvalues of the FG shaft system with the spin speed $\Omega$ for $L_i/L = 0.42$ and $R/2L = 0.02$ with different (a) $\eta_c$ (b) $k$ (c) $\Delta T$ (d) $\alpha/R$

5. Conclusions
The present work enables us the following important conclusions:

1. In the radially graded FG shaft under thermal environment, material’s properties have significant role to calculate local cracked compliances and various dynamic responses following P-FGM, E-FGM and S-FGM, respectively.
2. A parametric study has been conducted for better understanding the influences of gradient index, crack size, internal damping, end conditions and temperature gradient on natural frequency and stability threshold speeds of the cracked shaft made of FGM in using at higher spin speed.
3. It has been observed that for a cracked FG shaft system, with the increase in gradient index and thermal effects have a significant influenced to increase in natural frequencies, whirling speeds and threshold speeds. Therefore, designing an FG shaft for large $\Delta T$ and high $\Omega$ the power law plays an important role along with temperature.
4. It is observed the responses of the shaft made of stainless steel and FGMs that FG shaft able to operate at high spin speed than the stainless steel in using as machinery components.
5. Finally, this work recommended for various dynamics of shafts made of FGMs having more than one cracks with thermal environment.

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