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KOLMOGOROV-BURGERS MODEL FOR STAR-FORMING TURBULENCE

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ABSTRACT

The process of star formation in interstellar molecular clouds is believed to be controlled by driven supersonic magnetohydrodynamic turbulence. We suggest that in the inertial range, such turbulence obeys the Kolmogorov law, while in the dissipative range, it behaves as Burgers turbulence developing shock singularities. On the base of the She-Lévêque analytical model, we then predict the velocity power spectrum in the inertial range to be $E_k \sim k^{-1.74}$. This result agrees well with recent numerical findings by Padoan & Nordlund and reproduces the observational Larson law $\langle u_i^2 \rangle \sim l^{0.74 - 0.76}$. The application of the model to more general dissipative structures with higher fractal dimensionality is discussed.

Subject headings: MHD — stars: formation — turbulence

1. INTRODUCTION

It was recently argued on both observational and numerical grounds that star-forming regions of interstellar molecular clouds are governed by supersonic and possibly super-Alfvénic turbulence (see, e.g., Padoan & Nordlund 1999, 2000—hereafter PN2000—and a review by Elmegreen 2001). The turbulence is driven on large scales by supernovae explosions and energy is then transferred to smaller scales via a turbulent cascade, forming a hierarchy of dense clumps. It is still unclear whether such turbulent fragmentation is crucial on small scales, where Jeans-unstable density cores collapse and stars are formed. However, it seems reasonable that at least at the initial stage of a clumpy structure formation, turbulent fragmentation is the definitive process. This assertion, stemming from the work by Larson (1981), was recently confirmed in the number of high-resolution numerical simulations (e.g., Padoan et al. 2000; Klein, Fisher, & McKee 2000; Klessen 2001a, 2001b; Geyer & Burkert 2001; Williams 2001; Mac Low et al. 2001).

Observations suggest that the Mach number of turbulent motion, $M$, can be greater than 10, and the Alfvenic Mach number, $M_a$, can be greater than 1 (e.g., Klessen 2001a; Williams 2001). Until recently, supersonic turbulence (both Navier-Stokes and MHD) has not received proper theoretical attention. In a series of papers, Porter, Woodward, & Pouquet (1998) analyzed numerically decaying turbulence with initial Mach number of the order of 1. It has been observed in large-resolution runs (up to $1024^3$) that the spectra of both the compressible and incompressible parts of the velocity field approximately follow the Kolmogorov value, $E_k \sim k^{-5/3}$. However, decaying turbulence is different from forced turbulence in many aspects. To mention just a few, we note that a supersonic motion forms shocks and quickly, on a crossing time, dissipates in decaying runs, while it can be sustained in forced ones. Also, it has been demonstrated by Smith, Mac Low, & Zuev (2000) and Smith, Mac Low, & Heitsch (2000) that, in a decaying case, most energy is dissipated in a large number of weak shocks, contrary to a forced case where the largest shocks dissipate most of energy. In the present paper, we consider supersonic, driven turbulent systems, stressing that they differ qualitatively from their subsonic, decaying counterparts.

In the last two years, there appeared a number of papers analyzing numerically forced supersonic turbulence, both with and without magnetic fields. Porter et al. (1999) investigated forced, nonmagnetized turbulence with Mach number of the order of 1 and observed no difference in power spectra with the unforced runs. However, when PN2000 and Padoan et al. (2000) simulated supersonic MHD turbulence ($M \sim 10, M_a \sim 3$), they found the velocity spectrum, $k^{-3}$, with approximate value $\beta = 1.8$. This spectrum is steeper than the Kolmogorov one, which indicates strong intermittency effects. Correspondingly, velocity fluctuations scale with distance according to $\langle u_i^2 \rangle \sim l^{3 - \beta}$. The steeper-than-Kolmogorov spectrum was linked to the supersonic nature of turbulence by Larson (1979, 1981) on observational grounds.

Our interest in supersonic turbulence is also motivated by the argument of PN2000 that the spectral exponent $\beta$ may be directly related to the exponent of the mass distribution of collapsing cores, $N(m) \sim m^{-1 - \delta}$, as $\delta = 3/(4 - \beta)$. Without giving the details, we just note that this equation is obtained by considering a typical shock with the velocity jump of the order $\Delta u \sim l^{3 - \beta}/2$ and by using the relation of the density of the postshock gas with the shock strength. Here $l$ is the typical thickness of the dense postshock region. This suggests that the initial mass function (IMF) could be explained from the basic properties of turbulent fragmentation, without tunable parameters. The fact that supersonic MHD turbulence leads to sustaining of shock turbulence, to shock fragmentation, and to establishing a certain universal density distribution has also been recently demonstrated by Boldyrev & Brandenburg (2001) in a one-dimensional solvable Burgers model.

In this paper we present a theoretical model of driven supersonic turbulence incorporating both Kolmogorov and Burgers pictures in different parts of the phase space. We argue that due to the mostly solenoidal character of such turbulence, the characteristic times of energy cascade in the inertial interval scale as in the Kolmogorov turbulence, while the dissipative structures are completely different. Instead of filaments, as in an incompressible case, they can appear as sheets, which is more consistent with Burgers turbulence. In \S 2, we demonstrate that the standard She-Lévêque model (She & Lévêque 1994; She & Waymire 1995),
which links the most singular turbulent structures with turbulent spectra, has a solution corresponding to sheetlike dissipative structures, which reproduces the velocity power spectrum with exponent $\beta = 1.74$, close to the observational and numerical values. In § 3, we generalize the results to more realistic, fractal dissipative structures, which leads to better agreement with recent observational results. The Appendix provides a simple, although somewhat formal, derivation of the She-Lévêque model.

2. KOLMOGOROV-BURGERS MODEL OF SUPersonic TURBulence

At first sight, turbulence with small pressure should behave in the same way as Burgers turbulence (turbulence of a potential velocity field, the theory of which was substantially developed during the last few years (Polyakov 1995; Yakhot & Cheklov 1996; Boldyrev 1997; E et al. 1997; Gotof & Kraichnan 1998; E & Vanden Eijnden 1999; Verma 2000; Frisch & Beck 2000). However, this is true only in one- and two-dimensional cases; in a three-dimensional case, the behavior of a compressible fluid is qualitatively different from Burgers turbulence. The main difference is vorticity generation, an effect completely analogous to magnetic field generation existing in three dimensions and nonexisting in two dimensions. Indeed, the vorticity equation,

$$\partial_t \Omega + (\mathbf{u} \cdot \nabla)\Omega = - (\mathbf{V} \cdot \mathbf{u}) + (\mathbf{V} \cdot \mathbf{u}) \Omega = \nu \Delta \Omega,$$

where the vorticity is $\Omega = \mathbf{V} \times \mathbf{u}$, coincides with the induction equation for a magnetic field. Numerical experiments show that vorticity is generated quite effectively. In decaying turbulence with Mach numbers of the order of 1, simulated by Porter, Woodward, & Pouquet (1998), it was found that the turbulence was mostly solenoidal. If one decomposes the velocity field into the solenoidal part, $\mathbf{V} \cdot \mathbf{u} = 0$, and the compressible part, $\mathbf{V} \times \mathbf{u} = 0$, their ratio is observed to be $\gamma = (\partial_t^2 u^2)/(\partial_t^2 u^2) \sim 0.1$ in the inertial range. A pressure term ensuring incompressibility in subsonic turbulence turned out to be unimportant in supersonic dynamics: energy transfer over scales due to the pressure term was only 3%. The subsequent forced runs by Porter et al. (1999) revealed qualitatively the same results. In the case of forced turbulence with large Mach numbers ($M \sim 10$, $Ma \sim 3$) and a solenoidal large-scale force, simulations by PN2000 also demonstrated that in the inertial interval, this ratio is small, $\gamma < 0.2$. In these simulations, the isothermal equation of state was used to model fast gas cooling due to radiation. This result is not sensitive to the character of the external force, since compressible motion creates shocks and its divergent part decays faster than the solenoidal one (A. Nordlund 2001, private communication); however, it may be sensitive to the presence of magnetic field that is known to help generate vorticity (Vázquez-Semadeni, Passot, & Pouquet 1996).

These remarkable numerical observations lead us to a conjecture that the ratio $\gamma$ can be treated as a small parameter in the theory of three-dimensional compressible turbulence. We assume that in the inertial region, such turbulence is divergence-free with the Kolmogorov time of velocity fluctuation $t_1 \sim L^{1/3}$, where $L$ is the size of the fluctuation. Close to the dissipative range, shock structures start to play an important role in energy transfer and dissipation. The turbulence in this region thus inherits certain properties of Burgers turbulence. In the inertial region, the divergence-free turbulent motion looks like a pressureless, sheared flow threaded by thin, elongated shocks. In the dense regions inside the shocks, pressure and magnetic field are dynamically important, and these scales are thus not described by the presented model. Stars are formed as a result of further mass fragmentation that occurs inside these dense regions (clumps) due to gravitational instability. The process of fragmentation depends on a new equation of state in the clump, structure of the magnetic field, the mechanism of dissipation, etc. The smallest-scale $l_b$ described by our model can just be estimated as the scale of the shock thickness, i.e., the scale where pressure becomes important, $\langle u_t^2 \rangle = u_t^2 (l_b/L)^{3-1} \sim c^2$, where $c$ is the sound speed. This condition gives

$$l_b \sim L M_L^{2/3-1},$$

where $L$ is the largest available scale and $M_L$ is the Mach number at this scale. For a cloud of size $L = 50$ pc, the large-scale Mach number may reach $M_L = 30$. The inner scale obtained for $\beta = 1.7 \ldots 1.8$ is thus $l_b \sim 0.01$ pc, which agrees with observational ranges of scales (see, e.g., Ossenkopf & Mac Low 2000). In the present paper, we do not address scales smaller than $l_b$.

The theory allowing linkage between the most singular, dissipative structures of turbulence with its velocity spectrum was suggested by She & Lévêque (1994). This theory represents a turbulent cascade as an infinitely divisible log-Poisson process that has three input parameters. Two of these parameters are naive scaling exponents, $\Theta$ and $\Delta$, of the velocity field and of the “eddy-turnover time,” correspondingly: $u_t \sim l^\Theta$, $t_1 \sim l^\Delta$. The other parameter is the codimension, $C$, of the most singular dissipative structure. (Codimension is defined as dimension of space minus the dimension of the structure, $C = d - D$.) The objective of the theory is to predict the so-called structure functions of the velocity field, defined as

$$S_p(l) = \langle [u(x + l) - u(x)]^p \rangle \sim l^{p-C},$$

where $u$ is a component of the velocity field parallel or transverse to $l$. (According to the chosen component, the structure functions are called either longitudinal or transversal. It is believed that both scale in the same way, so we do not specify what component is assumed in eq. [3].) The velocity spectrum is a Fourier transform of the second-order structure function and is given by $E_k \sim k^{-1-\zeta(2)}$. If the turbulent cascade depended only on local eddy interactions, then the naive Kolmogorov scaling of structure functions would hold, $\zeta(p) = p/3$, and we would recover the energy distribution $E_k \sim k^{-5/3}$. Real turbulence is, however, intermittent, which means that its spectrum is not determined by the naive scaling. The She-Lévêque theory predicts the scaling function $\zeta(p)$ as

$$\zeta(p) = \Theta (1 - \Delta) p + C (1 - \Sigma^{\Theta p}),$$

where $\Sigma = 1 - \Delta/C$. For the original derivation, we refer the reader to the papers by She & Lévêque (1994), She & Waymire (1995), and Dubrulle (1994); more practical discussion can be found in Grauer, Krug, & Marliani (1994), Politano & Pouquet (1995), and Müller & Biskamp (2000). For completeness, we present a short derivation of the She-
Lévêque model in the Appendix. For three-dimensional incompressible turbulence, the naïve scaling exponents take the well-known Kolmogorov values $\Theta = \frac{1}{4}$ and $\Lambda = \frac{3}{4}$, and the dissipative structures are known to be filaments, so their codimension is $C = 2$. With these input parameters, equation (4) reproduces experimental results for incompressible Navier-Stokes turbulence with accuracy of several percent up to $p = 10$.

In our model of Kolmogorov–Burgers turbulence, the inertial range naïve scaling exponents are Kolmogorov ones, while the dissipative structures are quasi-one-dimensional shocks, which gives $C = 1$ and $\Sigma = \frac{1}{4}$. Equation (4) now reads

$$\zeta(p) = \frac{p}{9} + 1 - \left(\frac{1}{3}\right)^{p/3}.$$  (5)

This gives for the second-order structure function $\langle u^2 \rangle \sim l^{0.74}$, which reproduces the Larson law (Larson 1979, 1981), and the velocity power spectrum is given by $E_k \sim k^{-1.74}$, in a good agreement with numerical results by PN2000. The intermittency correction to the Kolmogorov scaling is even larger for the first-order structure function $\langle |u| \rangle \sim l^{0.42}$, which can be checked observationally or numerically in an easier way. Our analysis here is analogous to the analysis of *incompressible* MHD turbulence by Grauer, Krug, & Marliani (1994), Politano & Pouquet (1995), and also by Müller & Biskamp (2000), who noted that the most singular structures in such turbulence are microcurrent sheets. The sheetlike dissipative structures, together with the assumption that the energy cascade is given by the Kolmogorov rather than the Iroshnikov-Kraichnan mechanism, led the latter authors to the same prediction for the structure function scaling as our equation (5), which turned out to be in good agreement with numerical results. This indicates that both systems, though completely different, belong to the same class of universality, in agreement with the ideas put forward by Dubrulle (1994) and She & Waymire (1995).

3. GENERALIZATIONS

Our analysis relied considerably on sheetlike shock structures. Analogous considerations for the filament and core singularities would give $E_k \sim k^{-1.697}$ for filaments ($C = 2$) and $E_k \sim k^{-1.685}$ for cores ($C = 3$). All these spectra are steeper than the Kolmogorov one. Although the shocklike dissipative structures are clearly seen in simulations, the Mach number and the resolution are not large enough to make precise comparison with molecular clouds. As an important generalization of the theory, one can imagine that the dissipative structures are rather complicated on large scales and have dimensionality greater than two. This may be consistent with observed fractal structures of the density distribution, whose dimensionality is close to $D = 2.3$ (Larson 1992; Elmegreen & Elmegreen 2001; Chappell & Scalo 2001). Since the substantial part of dissipation occurs in shocks (see, e.g., Ostriker, Stone, & Gammie 2000), the dimension of the most singular dissipative structures may be close to $D = 2.3$ as well. (It would be interesting to check numerically the degree of correlation of the density field and the dissipation field.) Substitution of $C = 0.7$ in our equation (4) leads to the first order structure function $\langle |u| \rangle \sim l^{0.55}$ and to the energy spectrum $E_k \sim k^{-1.83}$. The results of recent observations (see, e.g., Brunt & Heyer 2001) seem to agree well with these predictions. However, more precise measurement of the structure functions scaling would be required to indicate what structures are most important. The intermittency correction to a scaling exponent of the first-order structure function is large enough to be detectable in numerical experiments, but higher-order structure functions are more difficult to measure. An attempt to infer such structure functions from observations was made by Miesch, Scalo, & Bally (1999) and Ossenkopf & Mac Low (2000), but the scaling was not established due to limited inertial ranges.

Another important question is the relation of the obtained spectrum to the initial mass distribution function. The consideration of PN2000 was based on an implicit assumption of the mean-field approximation, while our explanation of the observed steeper-than-Kolmogorov spectrum is essentially based on intermittency effects. In the presence of strong fluctuations, this relation may be modified, also acquiring intermittency corrections.

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**APPENDIX**

**SHE-LÉVÊQUE MODEL**

The She-Lévêque (SL) model of turbulence can be simply derived in the following way, pointed out by Dubrulle (1994). Suppose that the turbulent energy cascade occurs over a hierarchy of eddies, or velocity fluctuations of different sizes and strengths. Let us divide the volume of the box into volumes of size $l_0$, and let us introduce the rate of energy dissipation, $e_i$, inside each of these boxes. In a steady state, the averaged overall box’s rate of energy dissipation is the same as the rate of energy transfer over scales and is the same as the rate of energy input at the largest scale. Following Kolmogorov, we assume that it is constant.

To model the turbulence cascade, let us divide each box into smaller boxes, with size $l_0 \Gamma$, where $\Gamma < 1$, then divide the newly formed boxes using the same scale factor $\Gamma$, and so on. Let us concentrate on the cascade emerging from division of one box of size $l$ as shown in Figure 1. Consider the newly formed boxes of size $l' = \Gamma l$. We now need to prescribe the rates of energy dissipation inside these new boxes. Assume that fraction $\alpha$ of the new boxes has the energy dissipation rate $e'_1 = \beta_1 e_i$, while the rest
of the boxes, whose fraction is \(1 - x\), dissipate energy with the rate \(\epsilon_x' = \beta_1 \epsilon_1\). Due to energy flux conservation, we must have \(\beta_1 x + \beta_2 (1 - x) = 1\). We also assume that the division happens completely randomly, i.e., at each division our fixed point of observation can find itself inside any of the newly formed boxes with equal probability.

After some large number of divisions, \(n\), we reach the scale \(l = l_0 \Gamma^n\). One can show that at this scale, \(\langle \epsilon_I' \rangle \sim l^{-p}\), where the averaging is performed over the whole box. The scaling exponent \(\tau\) is given by the equation

\[
\tau(p) = \log(W^n)/\log \Gamma,
\]

where \(W\) takes the value \(\beta_1\) with probability \(x\) and the value \(\beta_2\) with probability \(1 - x\). We now assume that \(x\) is small and that the cascade proceeds by small steps as well, \(\Gamma = 1 - x/C\). Using equation (6), it is now easy to obtain that \(\tau(p) = C(\beta_1 - 1)p + C(1 - \beta_2)^p\), where \(C\) is, by construction, a codimension of the structure which is covered only by boxes \(\beta_2\) at each step; since \(\beta_1 < 1\) and \(\beta_2 > 1\), this is the most intense, most singular structure. To see that \(C\) is a codimension of the structure covered only by boxes "\(\beta_2\)" on each step, we note that the number of such boxes at step \(n\) is \(N_n = N_0 [(1 - x)/\Gamma^3]^n\), while the box size is \(l_n = l_0 \Gamma^n\). By the definition of the fractal dimension, we now obtain

\[
D = - \lim_{n \to \infty} \log(N_n)/\log(l_n/l_0) = 3 - C,
\]

where we used the fact that \(x\) is small. Using the Kolmogorov relation \(\epsilon_I \approx u_I^3/l\), one now gets for the velocity structure functions

\[
\zeta(p) = [1 - C(1 - \beta_1)]p/3 + C(1 - \beta_1^{p/3}).
\]

In this equation, \(\beta_1\) plays the role of \(\Sigma\), the intermittency parameter, in the SL model. In the original SL derivation, the additional assumption about the nonintermittent scaling of the time of velocity fluctuations near the most singular dissipative structure was made, which gave \(\Sigma = 1 - \Delta/C\). Our derivation allowed us to establish the structure of equation (4) without any additional assumptions, and therefore, \(\beta_1\) is left undetermined. In principle, it can be found from experiments or numerical simulations.

REFERENCES

Boldyrev, S. 1997, Phys. Rev. E, 55, 6907
Boldyrev, S., & Brandenburg, A. 2001, in preparation
Brunt, C. M., & Heyer, H. H. 2002, ApJ, 566, 289
Chappell, D., & Scalo, J. 2001, ApJ, 551, 712
Dubrulle, B. 1994, Phys. Rev. Lett., 73, 959
E, W., Khanin, K., Mazel, A., & Sinai, Ya. 1997, Phys. Rev. Lett., 78, 1904
E, W., & Vanden Eijnden, E. 1999, Phys. Rev. Lett., 83, 2572
Elmegreen, B. G. 2001, in ASP Conf. Ser., From Darkness to Light, ed. T. Montmerle & P. Andre (San Francisco: ASP), in press
Elmegreen, B. G., & Elmegreen, D. M. 2001, AJ, 121, 1507
Frisch, U., & Beck, J. 2000, in Les Houches, New Trends in Turbulence, (Amsterdam: Elsevier), preprint (nlin. CD/0012033)
Geyer, M. P., & Burkert, A. 2001, IAU Symp. 207, Extragalactic Star Clusters, ed. E. Grebel et al. (San Francisco: ASP), in press
Gotol, T., & Kraichnan, R. H. 1998, Phys. Fluids, 10, 2859
Grauer, R., Krug, J., & Marliani, C. 1994, Phys. Lett. A, 195, 335
Heitsch, F., Mac Low, M.-M., & Klessen, R. S. 2000, 2001, ApJ, 547, 280
Klein, R. I., Fisher, R., & McKee, C. F. 2000, preprint (astro-ph/0007332)
Klessen, R. S. 2001a, ApJ, 566, 837—2001b, preprint (astro-ph/0106332)
Larson, R. B. 1979, MNRAS, 186, 479
———. 1981, MNRAS, 194, 809
———. 1992, MNRAS, 256, 641
Mac Low, M.-M., Balsara, D., Avillez, M. A., & Kim, J. 2001, preprint (astro-ph/0106509)
Miesch, M. S., Scalo, J., & Bally, J. 1999, ApJ, 524, 895
Müller, W.-C., & Biskamp, D. 2000, Phys. Rev. Lett., 84, 475
Ossenkopf, V., & Mac Low, M.-M. 2000, preprint (astro-ph/0012247)
Ostriker, E. C., Stone, J. M., & Gammie, C. F. 2001, ApJ, 546, 980
Padoan, P., & Nordlund, Å., & Rögnvaldsson, Ó. E., & Goodman, A. 2000, preprint (astro-ph/0011229)
———. 2000, preprint (astro-ph/0011465) (PN2000)
Politano, H., & Pouquet, A. 1995, Phys. Rev. E, 52, 636
Polyakov, A. M. 1995, Phys. Rev. E, 52, 6183
Porter, D., Pouquet, A., Sytine, I., & Woodward, P. 1999, Physica A, 263, 263
Porter, D. H., Woodward, P. R., & Pouquet, A. 1998, Phys. Fluids, 10, 237
She, Z.-S., & Leveque, E. 1994, Phys. Rev. Lett., 72, 336
She, Z.-S., & Waymire, E. C. 1995, Phys. Rev. Lett., 74, 252
Smith, M. D., Mac Low, M.-M., & Heitsch, F. 2000, A&A, 362, 333
Smith, M. D., Mac Low, M.-M., & Zwart, J. M. 2000, A&A, 356, 287
Vazquez-Semadeni, E., Passot, T., & Pouquet, A. 1996, ApJ, 473, 881
Verma, M. K. 2000, Physica, 277, 359
Williams, J. 2001, preprint (astro-ph/0105367)
Yakhot, V., & Chekhlov, A. 1996, Phys. Rev. Lett., 77, 3118