A REMARK ON HIGHER CONGRUENCES FOR THE
NUMBER OF RATIONAL POINTS OF VARIETIES
DEFINED OVER A FINITE FIELD

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Abstract. We show that the \(\ell\)-adic cohomology of the mod \(p\)
reduction \(Y\) of a regular model of a smooth proper variety defined
over a local field, the cohomology of which is supported in codi-
mension \(\kappa\), can’t be Tate up to level \((\kappa - 1)\). As a consequence,
the number of rational points of \(Y\) can’t fulfill the natural relation
\(|Y(\mathbb{F}_q)| \equiv \sum_{i\geq 0} q^i \cdot b_{2i}(\bar{Y}) \mod q^n\).

Une remarque sur les congruences d’ordre supérieur
pour le nombre de points rationnels de variétés définies
sur un corps fini.

Résumé: Nous montrons que la cohomologie \(\ell\)-adique de la réduction
\(Y\) modulo \(p\) d’un modèle régulier d’une variété propre et lisse
définie sur un corps local, dont la cohomologie est supportée en
codimension \(\kappa \geq 1\), ne peut être de Tate jusqu’en niveau \((\kappa - 1)\). En
conséquence, le nombre de points rationnels de \(Y\) ne peut vérifier
la formule naturelle \(|Y(\mathbb{F}_q)| \equiv \sum_{i\geq 0} q^i \cdot b_{2i}(\bar{Y}) \mod q^n\).

Version française abrégée. Dans [5], Theorem 1.1, nous montrons
que si \(\mathcal{X}\) est un modèle régulier d’une variété \(X\) propre et lisse définie
sur un corps local de corps résiduel \(\mathbb{F}_q\), alors si la cohomologie \(\ell\)-adique
\(H^i(\bar{X})\) est supportée en codimension \(\geq 1\) pour \(i \geq 1\), le nombre de
points rationnels de sa réduction \(Y\) modulo \(p\) vérifie \(|Y(\mathbb{F}_q)| \equiv 1\) mod-
ulo \(q\). En fait, pour être plus précis, sous cette hypothèse, les valeurs
propres du Frobenius géométrique agissant sur la cohomologie \(\ell\)-adique
\(H^i(\bar{Y})\) de \(Y\) sont divisibles par \(q\) en tant qu’entiers algébriques. Le but
de cette note est de discuter une formulation en coniveau supérieur.
Une façon naturelle de généraliser la condition de coniveau \(\geq 1\) pour
\(i \geq 1\) est de supposer que \(H^i(\bar{X})/H^i(\bar{X})_{\text{alg}}\) est supportée en codi-
mension \(\kappa\), où \(H^i(\bar{X})_{\text{alg}}\) est nulle si \(i\) est impair et sinon est la partie
algébrique. Nous montrons cependant que cela n’implique pas que

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les valeurs propres du Frobenius géométrique sont divisibles par $q^\kappa$ en tant qu’entiers algébriques sur $H^i(Y)/H^i(Y)_{q^2}$, où $H^i(Y)_{q^2}$ est nulle si $i$ est impair et sinon est la partie sur laquelle le Frobenius agit par multiplication par $q^2$. En particulier la formule naturelle $|Y(F_q)| \equiv \sum_{i \geq 0} q^i \cdot b_2(\tilde{Y})$ modulo $q^\kappa$ n’est pas valable en général. Cette formulation est proposée par N. Fakhruddin dans [6] qui la montre sous certaines hypothèses pour une famille géométrique en égale caractéristique $p > 0$. Nous montrons en quoi ces hypothèses sont très fortes.

1. Introduction

In [5], Theorem 1.1, we show that if $X$ is a regular model of a smooth proper variety $X$ defined over a local field with finite residue field $\mathbb{F}_q$, then if $\ell$-adic cohomology $H^i(\tilde{X})$ is supported in codimension $\geq 1$ for $i \geq 1$, the number of rational points of its mod $p$ reduction $Y$ fulfills $|Y(\mathbb{F}_q)| \equiv 1$ modulo $q$. To be more precise, the assumption implies that the eigenvalues of the geometric Frobenius acting on $\ell$-adic cohomology $H^i(Y)$ of $Y$ are $q$-divisible algebraic integers. The proof relies on a version of Deligne’s integrality theorem [2], Corollaire 5.5.3 over local fields [3], Corollary 0.4. The goal of this note is to discuss a formulation in higher coniveau level. A natural generalization of the coniveau $\geq 1$ condition for $i \geq 1$ is to assume that $H^i(\tilde{X})/H^i(\tilde{X})_{\text{alg}}$ is supported in codimension $\geq \kappa$, where $H^i(\tilde{X})_{\text{alg}}$ is equal to 0 if $i$ is odd, else is the algebraic part of cohomology. This means that there is a codim $\geq \kappa$ subscheme $Z \subset X$ so that $H^i(\tilde{X}) \overset{\text{rest}=0}{\rightarrow} H^i(\tilde{X} \setminus \tilde{Z})/\text{Im}(H^i(\tilde{X})_{\text{alg}})$. Said differently, $H^i(\tilde{X}) = H^i(\tilde{X})_{\text{alg}}$ for $i \leq 2\kappa$, and $H^i_Z(\tilde{X}) \twoheadrightarrow H^i(\tilde{X})$ for $i \geq 2\kappa$.

However we show that this assumption does not imply that the eigenvalues of the geometric Frobenius acting on $H^i(Y)/H^i(Y)_{q^2}$ are divisible by $q^\kappa$-divisible algebraic integers, where $H^i(Y)_{q^2}$ is equal to 0 if $i$ is odd, else is the part of cohomology on which Frobenius acts by multiplication by $q^2$. In particular, the formula $|Y(\mathbb{F}_q)| \equiv \sum_{i \geq 0} q^i \cdot b_2(\tilde{Y})$ modulo $q^\kappa$ does not hold in general. This formulation was proposed in [6] by N. Fakhruddin, who shows it under certain assumptions in a geometric family in equal characteristic $p > 0$. We show how strong are those assumptions.

Our example consists of a Godeaux surface in characteristic 0. We take a reduction mod $p$ which is a cone over a smooth curve $C$ of higher degree. After desingularization of the mod $p$ reduction, $H^1(C)(-1)$
enters $H^3(\bar{Y})$, and this destroys the possibility of the $|Y(\mathbb{F}_q)| \equiv 1 + q \cdot b_2(\bar{Y})$ mod $q^2$ congruence.

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2. The example

Let us consider the Godeaux surface $X_0/\mathbb{Q}_p$ defined as the quotient of the Fermat quintic $F \subset \mathbb{P}^3_{\mathbb{Q}_p}$ of homogeneous equation $px_0^5 + x_1^5 + x_2^5 + x_3^5$ by the group $\mu_5$ acting via $\xi \cdot (x_i) = (\xi^i \cdot x_i)$. Here $p$ is prime to 5, and $\xi$ generates the group of 5-th roots of unity. As well known [1], V, 15 and VII, 11, $H^0(X_0, \Omega^1_{X_0}) = H^0(X_0, \Omega^2_{X_0})$ and by comparison of de Rham with étale cohomology, one obtains $H^1(\bar{X}_0) = H^3(\bar{X}_0) = 0$, $H^{2i}(\bar{X}_0) = H^{2i}_{\text{alg}}(\bar{X}_0)$ for $i = 0, 1, 2$. Let us assume we have a regular model $\mathcal{X} \to \text{Spec}(R)$ of $X_0$ over an extension $R \supset \mathbb{Z}_p$, with local field $K = \text{Frac}(R)$ and residue field $\mathbb{F}_q$. Thus the general fiber is $X = X_0 \times_{\mathbb{Q}_p} K$, and we denote by $Y$ the mod $p$ reduction over $\mathbb{F}_q$.

We use the computation in [5], sections 2 and 3. One has an exact sequence

$$H^i(\mathcal{X}^u) \to H^i(\bar{Y}) \xrightarrow{\text{sp}} H^i(X^u) \to H^i_{\text{Y}}(X^u)$$

(2.1)

where $^u$ means the pull back via the extension $K \subset K^u$ to the maximal unramified extension, and $^\text{sp}$ means the pull back via the extension to the algebraic closure. The sequence is equivariant with respect to the action of the geometric Frobenius $\text{Frob} \in \text{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ acting on $H^*(\bar{Y}), H^*_Y(X^u), H^*(X^u)$. One also has the exact sequence

$$0 \to H^1(I, H^{i-1}(\bar{X})) \to H^i(X^u) \to H^i(\bar{X})^I \to 0$$

(2.2)

where $I \subset \text{Gal}(\bar{K}/K)$ is the inertia group, with quotient $\text{Gal}(\bar{K}/K)/I = \text{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$. The sequence is equivariant with respect to the action of Frob. So using Gabber’s purity theorem [7], Theorem 2.1.1. as in [5], section 2, one obtains

$$H^1(\bar{Y}) = 0,$$

and an equivariant exact sequence

$$0 \to H^0(Y^0)(-1) \to H^2(\bar{Y}) \to H^2(\bar{X})^I,$$

(2.4)

where $Y^0 = Y\setminus$ singular locus. Thus in particular, Frob acts via multiplication by $q$ on $H^2(\bar{Y})$. So via Grothendieck-Lefschetz trace formula [5] and the fact that $H^4(\bar{Y}) = \oplus \text{components} \mathbb{Q}_\ell(-2)$, we conclude

$$|Y(\mathbb{F}_q)| \equiv 1 + q \cdot b_2(\bar{Y}) - \text{Tr} \text{Frob}|H^3(\bar{Y}) \mod q^2.$$

(2.5)

The question becomes whether $H^3(\bar{Y})$ dies or not.
We now construct $\mathcal{X}$ and show $H^3(Y) \neq 0$ for this $\mathcal{X}$. The mod $p$ reduction in $\mathbb{P}^3_{\mathbb{F}_p}$ of the model $F \subset \mathbb{P}^3_{\mathbb{Z}_p}$ of $F$ defined by the same equation $px_0^5 + x_1^5 + x_2^5 + x_3^5$ is the cone over the Fermat curve $Q_{\mathbb{F}_p} \subset \mathbb{P}^2_{\mathbb{F}_p}$ of equation $x_0^5 + x_2^5 + x_3^5$. Then $\mu_5$ acquires one single fix point $(1 : 0 : 0 : 0) \in \mathbb{P}^3_{\mathbb{F}_p}$, which is the vertex of $\text{cone}(Q_{\mathbb{F}_p})$. We base change $\mathbb{Z}_p \subset R$ via $\pi^5 = p$ and denote by $k = \mathbb{F}_q$ the residue field and $K = \text{Frac}(R) \supset \mathbb{Q}_p$ the local field. So $F \times_{\mathbb{Z}_p} R \subset \mathbb{P}^3_R$ is defined by the equation $\pi^5 x_0^5 + x_1^5 + x_2^5 + x_3^5$. The $\mu_5$ operation is still defined by $\xi \cdot x_i = \xi^5 x_i$ and now the only fix point $x := (1 : 0 : 0 : 0) \in \mathbb{P}^3_{\mathbb{F}_q}$ is at the same time the only point in which $F \times_{\mathbb{Z}_p} R$ is not regular. The affine equation of $F \times_{\mathbb{Z}_p} R$ in $(\mathbb{A}^3_R, x_0 \neq 0)$ with coordinates $X_i = \frac{x_i}{x_0}$ on which $\mu_5$ acts via $\xi \cdot X_i = \xi^5 X_i$, is $\pi^5 + X_1^5 + X_2^5 + X_3^5$. We blow up the singularity $x$ to obtain $\sigma : F' \to F \times_{\mathbb{Z}_p} R$. Then $\sigma^{-1}(x)$ is isomorphic to the Fermat quintic $Z_2$ in $\mathbb{P}^3_{\mathbb{F}_q}$ of equation $X_0^5 + X_1^5 + X_2^5 + X_3^5$ with action $\xi \cdot X_i = \xi^i X_i$. Consequently, $\mu_5$ acts fix point free on $F'$ and the quotient $\mathcal{X}$, which is defined over $R$, is a regular model of $X = X_0 \times_{\mathbb{Q}_p} K := (F/\mu_5) \times_{\mathbb{Q}_p} K$. Furthermore, $\sigma^{-1}(F \times_{\mathbb{Z}_p} \mathbb{F}_q)$ is the union of two components, one being the blow up $Z_1$ in the vertex of $\text{cone}(Q_{\mathbb{F}_p} \times_{\mathbb{F}_p} \mathbb{F}_q)$, the other one being the Fermat quintic $Z_2$. Thus the mod $p$ fiber $Y$ of $\mathcal{X}$ has two components $S_i = Z_i/\mu_5$. They meet along $C = (Q_{\mathbb{F}_p} \times_{\mathbb{F}_p} \mathbb{F}_q)/\mu_5$. As $p \neq 5$, the covering $Q_{\mathbb{F}_p} \times_{\mathbb{F}_p} \mathbb{F}_q \to C$ is étale, and genus($C) = 2$.

The normalization sequence for $Y$ yields a Frob equivariant exact sequence
\begin{equation}
H^3(Y) \to H^3(S_1) \oplus H^3(S_2) \to 0.
\end{equation}
On the other hand, one has
\begin{equation}
H^1(C)(-1) = H^1(Q_{\mathbb{F}_p})^\mu_5(-1) = H^3(\bar{Z}_1 \setminus Q_{\mathbb{F}_p})^\mu_5 = H^3(\bar{Z}_1)^{\mu_5} = H^3(S_1).
\end{equation}
Thus
\begin{equation}
H^3(Y) \to H^1(C)(-1) \neq 0
\end{equation}
which shows $H^3(Y) \neq 0$.

3. Discussion

3.1. Higher dimension. One can produce examples as above in all dimensions by taking the product $\mathcal{X} \times_R \mathbb{P}^n$, which is still regular. Then $H^i(X \times_K \mathbb{P}^n)/H^i(X \times_K \mathbb{P}^n)_{\text{alg}} = 0$ for all $i$, while $H^{3+2j}(Y \times_{\mathbb{F}_q} \mathbb{P}^n) \neq 0$ for all $j \geq 0$. 
3.2. Motivic condition. From \([2.1]\), using \([2.2]\) and applying \([3]\), Corollary 0.4 to the eigenvalues of \(H^i(X^u)\), we see immediately that the eigenvalues of Frob on \(H^i(\bar{Y})\) fulfill

\[
\text{sp}^u \text{ injective } + N^\kappa(H^*(\bar{X})/H^*(\bar{X})_{\text{alg}}) = (H^*(\bar{X})/H^*(\bar{X})_{\text{alg}})(3.1)
\]

\[\Rightarrow \text{eigenvalues Frob}_{|H^i(\bar{Y})} = \begin{cases} 0 & i < 2\kappa \text{ } i \text{ odd} \\ q^k & i \leq 2\kappa \text{ } i \text{ even} \\ \in q^\kappa \cdot \mathbb{Z} & i \geq 2\kappa. \end{cases}\]

Here \(N^\kappa\) is the coniveau filtration as explained in the Introduction. In \([6]\), N. Fakhruddin analyzes the motivic conditions for a family \(f: \mathbb{X} \rightarrow S\) defined over a finite field \(k\), with \(S, \mathbb{X}\) smooth, to have the property that a singular fiber \(Y\) over a closed point \(s\) with residue field \(F_q \supset k\) fulfills the property \(|Y(F_q)| = \sum_{i\geq0}(-1)^i q^i \cdot b_{2i}(\bar{Y})\) modulo \(q^\kappa\). More precisely, he studies the motivic conditions in a geometric family forcing the eigenvalue behavior described in (3.1). He singles out three conditions. We explain them and analyze the consequences they have on the completion \(\mathbb{X} = \mathbb{X} \times_S R\) at \(s\) of the family \(f\). Here \(R\) is the completion of the equal characteristic ring of functions at \(s \in S\). Surely, as in \([4]\), the first one is base change for the Chow groups \(CH_i(\bar{X}), i \leq (\kappa - 1)\). We know by Bloch’s type argument that this implies the coniveau condition in level \(\kappa\) on \(H^*(\bar{X})/H^*(\bar{X})_{\text{alg}}\), but we are extremely far of understanding that this is equivalent to it, as predicted by the general Bloch-Beilinson conjectures. The second one is that \(R^i f_* \mathbb{Q}_\ell\) are constant local systems. This is to say that the specialization map \(H^i(\bar{Y}) \rightarrow H^i(\bar{X})\) is an isomorphism, which in particular forces \(\text{sp}^u\) to be injective, but is stronger than this. So we see that those two conditions imply the weaker cohomological conditions in (3.1) which already force the eigenvalue conclusion on \(H^i(\bar{Y})\). The third condition says that the Chow groups \(CH_i(\bar{Y}), i \leq (\kappa - 1)\), are hit by specialization. This should translate into the condition \(\text{sp}^u\) injective above, which is then a consequence of the cohomological consequence of the condition forcing \(R^i f_* \mathbb{Q}_\ell\) being a constant local system.

At any rate, even if, as explained above, the conditions developed in \([6]\) are far from sharpness, they tacitly raise the question of a finer formulation, and are a motivation for this note.

3.3. Formula. It is of course extremely rare that one can check motivic conditions. It is in the rule easier to control cohomological conditions, and (3.1) gives conditions for a good behavior of rational points on \(Y\). However, the condition \(\text{sp}^u\) injective is very nongeometric and likely
very nonnatural as well. It would be better to understand a finer condition on the contribution of $H^i_Y(\mathcal{X}^u)$ in $H^i(Y)$ via the sequence (2.1).

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