General symmetry operators of the asymmetric quantum Rabi model

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Abstract
The true level crossing in the asymmetric quantum Rabi model without any obvious symmetry can be exhibited in the energy spectrum if the qubit bias is a multiple of the cavity frequency, which should imply the existence of the hidden symmetry. In this work, within a Bogoliubov operator approach, we can readily derive the symmetry operators associated with the hidden symmetry hierarchically for arbitrary multiples. The symmetry operators for small multiples in the literature can be extremely easily reproduced in our general scheme. In addition, a general parity operator is defined through the symmetry operator, which naturally includes the well-known parity operator of the symmetric model. We believe that the present approach can be straightforwardly extended to other asymmetric Rabi models to find the relevant symmetry operators.

Keywords: asymmetric quantum Rabi model, symmetry operators, parity operators, Bogoliubov operator approach

1. Introduction
One of the simplest light–matter interaction models is the quantum Rabi model (QRM) [1, 2], which describes the interaction between a two-level system (qubit) and a single-mode cavity. It is a paradigmatic model in the conventional quantum optics that describes the cavity quantum electrodynamics (QED) systems [3]. Recently, the QRM has been realized in many solid-state devices, such as the circuit QED system [4, 5], trapped ions [6] and quantum dots [7], which can be described in the framework of an artificial qubit and a resonator coupling system [8–10], and thus continues to be the hot topic in many fields.

In contrast to the cavity QED systems, the static bias of the qubit is always present in these modern solid devices, resulting in the so-called asymmetric quantum Rabi model (AQRM).
The Hamiltonian of the AQRM reads

\[ H_{\text{AQRM}} = \frac{\Delta}{2} \sigma_z + \frac{\epsilon}{2} \sigma_x + \omega a^\dagger a + g (a^\dagger + a) \sigma_x, \]  

(1)

where the first two terms fully describe a qubit with the energy splitting \( \Delta \) and the static bias \( \epsilon \), \( \sigma_x, \sigma_z \) are the Pauli matrices, \( a^\dagger \) and \( a \) are the creation and annihilation operators with the cavity frequency \( \omega \), and \( g \) is the qubit–cavity coupling strength. With the zero bias parameter \( \epsilon = 0 \), the above model reduces to the standard QRM which possesses a discrete \( \mathbb{Z}_2 \)-symmetry, the two neighboring energy levels with opposite parities cross in the spectra leading to the double degenerate level crossing. For the AQRM Hamiltonian, the presence of the static bias breaks the \( \mathbb{Z}_2 \)-symmetry, and thus the AQRM Hamiltonian does not possess any obvious symmetry. In general, the resulting spectra show the avoided level crossings, instead of the true level crossings. Nevertheless, it does exhibit the phenomenon of energy level crossings if the bias parameter is multiple of the cavity frequency [11]. So the observed level crossings in the asymmetric model are certainly due to unknown hidden symmetries, which have attracted a lot of attention in the past decade [11–21]. On the other hand, since the AQRM is ubiquitous in the modern solid devices, many celebrated properties described in conventional quantum optics, where the static bias is usually lacking, would appear in the artificial superconducting qubit setups if the hidden symmetry is generated by manipulating the static bias. Therefore the hidden symmetry of the AQRM is of both fundamental and practical interest.

To uncover the true level crossings in the AQRM, one needs to resort to the analytical exact solutions to the models. Approximate analytical method usually does not give the true level crossings or results in artificial level crossings. Note that the generalized rotating-wave approximation would not result in the true level crossings in the AQRM [22]. Fortunately, the analytical exact solution of the AQRM has been found by Braak in the Bargmann space representation [23]. It was quickly reproduced by using the Bogoliubov operator approach (BOA) by Chen et al [24]. It was soon realized that the transcendental functions in references [23, 24] can be constructed in terms of the mathematically well-defined Heun confluent function [11]. The true level crossings in AQRM was just discussed in the Braak’s exact solutions. On the other hand, the BOA can be easily extended to the two-photon [24, 25] and two-mode [26] QRM, and solutions in terms of a G-function, which shares the common pole structure with Braak’s G-function for the one-photon QRM, are also found. Most recently, by this two-photon’s G-function, the present two authors successfully uncover the elusive level crossings in a subspace of the two-photon AQRM [19].

Recently, the hidden symmetry in AQRM was discussed based on the numerical calculation on the energy eigenstates [15]. Interestingly, the symmetry operators at small integer biases, i.e., \( \epsilon/\omega = 1 \) and 2, are rigorously derived in reference [16] by the expansion in original Fock space. The extensions to the various QRM’s have been performed within the same framework [17, 18]. Some interesting remarks on the hidden symmetry are also given in reference [20]. One can however note from reference [16] that the symmetry operators become much more complex for the further increasing integer biases within the Fock space approach. The practical approaches for a general expression of the symmetry operators at arbitrary large integer biases are still lacking, and would be very challenging within the framework of the Fock space [16, 20]. By the BOA scheme [24], the condition for the double degeneracy in both one- and two-photon AQRM’s can be acquired in an unified way [19]. In this work, we will propose a general scheme to find symmetry operators that are relevant to the hidden symmetry in the one-photon AQRM using the BOA.
The paper is structured as follows: in section 2, we describe the general scheme for symmetry operators in the AQRM in the framework of BOA. In section 3, we apply this scheme to AQRM for several integer biases, and all previous symmetry operators at low integer biases are recovered readily. Moreover, we can obtain the symmetry operators in arbitrary large biases without much effort. In section 4, a general parity operator of the AQRM is discussed within the BOA. A brief summary is given in the last section. The symmetry operator of the AQRM for a large integer bias, \( N = 5 \), is demonstrated in the appendix.

2. General scheme for the symmetry operators in AQRM within BOA

To facilitate the BOA scheme, we write the Hamiltonian (1) in the matrix form after a unitary transformation \( \exp \left( \frac{i \pi}{4} \sigma_y \right) \) (unit is taken of \( \omega = 1 \))

\[
H'_{\text{AQRM}} = \begin{pmatrix}
    a^\dagger a + g (a^\dagger + a) + \frac{\epsilon}{2} & -\frac{\Delta}{2} \\
    -\frac{\Delta}{2} & a^\dagger a - g (a^\dagger + a) - \frac{\epsilon}{2}
\end{pmatrix}.
\]  

(2)

To remove the linear terms in \( a^\dagger (a) \) operator, we perform the following two Bogoliubov transformations

\[
a_+ = a + g, \quad a_- = a - g.
\]  

(3)

In the Bogoliubov operator \( a_+ (a_-) \), which is still creation (annihilation) operator, the Hamiltonian is expressed as

\[
H = \begin{pmatrix}
    a_\dagger_+ a_+ - \frac{g^2}{2} + \frac{\epsilon}{2} & -\frac{\Delta}{2} \\
    -\frac{\Delta}{2} & a_\dagger_- a_- - \frac{g^2}{2} - \frac{\epsilon}{2}
\end{pmatrix}.
\]  

(4)

The symmetry operator \( J \) should satisfy the commutation relation \([J, H] = 0\). Following [16, 20], we also write \( J \) as

\[
J = e^{i \pi a^\dagger a} Q.
\]  

(5)

The form of the symmetry operator is not necessarily unique [20], our goal is to find one concise form for it. Using \( e^{i \pi a^\dagger a} a = -a e^{i \pi a^\dagger a}, \ a e^{i \pi a^\dagger a} a^\dagger = -a^\dagger e^{i \pi a^\dagger a} \), we have

\[
QH = \tilde{H}Q.
\]  

(6)

where the Hamiltonian \( \tilde{H} \) reads

\[
\tilde{H} = \begin{pmatrix}
    a_\dagger_+ a_+ - \frac{g^2}{2} + \frac{\epsilon}{2} & -\frac{\Delta}{2} \\
    -\frac{\Delta}{2} & a_\dagger_- a_- - \frac{g^2}{2} - \frac{\epsilon}{2}
\end{pmatrix}.
\]

We now write operator \( Q \) explicitly as

\[
Q = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.
\]  

(7)
Note that the symmetry operator \( J \) in equation (5) is defined as self-adjoint \([16]\), that leads to \( Q = e^{i\xi a} Q^\dagger e^{-i\xi a} \). Thus, by equations (6) and (7), we immediately have

\[
X ( a_-, a_+^\dagger ) = e^{i\xi a} X ( a_-, a_+^\dagger ) e^{-i\xi a} = e^{i\xi a} X ( a_+, a_- ) e^{-i\xi a} = X ( -a_+, -a_- ) , \quad X = A, D ,
\]

(8)

similarly,

\[
B ( a_-, a_+^\dagger ) = C ( -a_+, -a_- ) ,
\]

(9)

where the matrix elements of \( Q \) include two Bogoliubov operators \( a_+^\dagger \) and \( a_- \). The four elements in equation (6) thus yield the following four equations

\[
\begin{align*}
A a_+^\dagger a_+ - a_-^\dagger a_- A + \frac{\Delta}{2} C - \frac{\Delta}{2} B &= 0 , \\
\Delta a_+^\dagger a_+ D - Da_-^\dagger a_- + \frac{\Delta}{2} C - \frac{\Delta}{2} B &= 0 , \\
\left[ B, a_+^\dagger a_- \right] - \epsilon B + \frac{\Delta}{2} D - \frac{\Delta}{2} A &= 0 , \\
\left[ a_+^\dagger a_+, C \right] - \epsilon C + \frac{\Delta}{2} D - \frac{\Delta}{2} A &= 0 .
\end{align*}
\]

(10)–(13)

The remaining task is to find the four operators \( A, B, C \) and \( D \) which satisfy the above four equations and then the symmetry operator \( J \) can be derived.

Equations (10)–(13) can be further reduced to two equations

\[
\begin{align*}
A a_+^\dagger a_+ - a_-^\dagger a_- A + \frac{\Delta}{2} C - \frac{\Delta}{2} B &= 0 , \\
\left[ B, a_+^\dagger a_- \right] - \left[ a_+^\dagger a_+, C \right] &= \epsilon ( B - C ) .
\end{align*}
\]

(14)–(15)

By the following general commutation relations

\[
\left[ ( a_- )^N, a_+^\dagger a_- \right] = N ( a_- )^N , \quad \left[ a_+^\dagger a_+, ( a_+^\dagger )^N \right] = N ( a_+^\dagger )^N ,
\]

for integer \( N \), we immediately know

\[
\left[ ( \pm a_- )^N, a_+^\dagger a_- \right] - \left[ a_+^\dagger a_+, ( \pm a_+^\dagger )^N \right] = N \left( ( \pm a_- )^N - ( \pm a_+^\dagger )^N \right) .
\]

Comparing both sides of equation (15) for the level crossing case at \( \epsilon = N \), we can see that the highest-order term in \( B \) is just \( ( \pm a_- )^N \) and \( C \) is \( ( \pm a_+^\dagger )^N \), because the highest-order \( N \) terms of \( [ B, a_+^\dagger a_- ] - \epsilon B \) and \( [ a_+^\dagger a_+, C ] - \epsilon C \) would vanish exactly. As proposed in \([20]\), it is also possible to have higher order terms, but for simplicity only the minimal order \( N \) case is considered here, as long as \([ J, H ] = 0 \) is satisfied. Thus instead of the operators \( a_+^\dagger, a_- \) in the original Fock space \([16]\), we can generally expand the four elements of the symmetry operator in terms of the normal Bogoliubov operator products of \( a_+^\dagger \) and \( a_- \) as
where $M = A, B, C$, and $D$. Note that $a_-$ and $a_+^i$ are not independent and they satisfy the commutation relation $[a_-, a_+^i] = 1$.

According to equation (9), we immediately have

$$B_{i,j} = C_{ij} (-1)^{i+j},$$

which will be very useful later. In this work, we chose the highest-order term in $B$ as $(a_-)^N$ and $C$ as $(-a_+^i)^N$. One may also chose $(-a_-)^N$ in $B$ and $(a_+^i)^N$ in $C$, and final symmetry operators are essentially the same. For convenience, we always use $B_{i,j}$ to describe the element $C$ by equation (17) in the following.

Inserting $A, B, D$ into equation (12) and comparing terms in $(a_+^i)^j$ and $a_-$, we have

$$\frac{\Delta}{2} (A_{i,j} - D_{i,j}) = (j - i - N)B_{i,j} + 2g (i + 1)B_{i+1,j}. \tag{18}$$

Inserting $A, B, C$ into equation (10) and $B, C, D$ into equation (11) respectively give

$$(j - i)A_{i,j} + 2gA_{i-1,j} + 2gA_{i,j-1} + 2g (j + 1)A_{i,j+1} + 2g (i + 1)A_{i+1,j} \nonumber$$

$$= \frac{\Delta}{2} (B_{i,j} - (-1)^{i+j}B_{ji}), \tag{19}$$

and

$$(i - j)D_{i,j} + 2gD_{i-1,j} + 2gD_{i,j-1} = \frac{\Delta}{2} (B_{i,j} - (-1)^{i+j}B_{ji}). \tag{20}$$

Canceling $A$ in equations (12) and (14), followed by substitution of $B, C$ and $D$ and comparing terms in $(a_+^i)^j$, we arrive at the following recursive relation

$$4g^2 (i + 2) (i + 1) B_{i+2,j} + 4g^2 (i + 1) (j + 1) B_{i+1,j+1} \nonumber$$

$$+ 2g (2j - 2i - 1 - N) (i + 1) B_{i+1,j} \nonumber$$

$$+ 2g (j + 1) (j + 1 - i - N) B_{i,j+1} + 4g^2 (i + 1) B_{i+1,j-1} \nonumber$$

$$+ [4g^2 i + (j - i) (j - i - N)] B_{i,j} \nonumber$$

$$+ 2g (j - i - 1 - N) B_{i-1,j} + 2g (j - i - 1 - N) B_{i,j+1} \nonumber$$

$$= \Delta [i - j] D_{i,j} - g (j + 1) D_{i,j+1} - g (i + 1) D_{i,j+1}. \tag{21}$$

Equations (20) and (21) are the crucial ones to determine the symmetry operators.

Similarly, in the previous study [20], by the expansion in the original Fock space, the recurrence relations to obtain the coefficients of the matrix elements in the symmetry operators have been given by their equations (7)–(10). Note from equation (16) that one term in the present BOA can consist of many terms in the Fock space expansion approach [20]. Due to the different expansion methods, the recursive process to compute the coefficients are different. Below, we
describe how to obtain all expressions of matrix elements recursively from \( i + j = N \) in detail. We will first derive the two matrix elements \( B \) and \( D \) in equations (20) and (21) completely and independently, then obtain \( C \) and \( A \) at the last stage.

To proceed, we begin with the highest order \( N \), and decrease the order step by step, then obtain the coefficients hierarchically.

(i) \( i + j = N + 1 \). Although \( M_{i,j} \) does not appear for \( i + j = N + 1 \) due to the limitation of the summation indexes in equation (16), it is still helpful to determine \( M_{i,j} \) for \( i + j = N \). In this case, equation (20) becomes

\[
Di_{i-1,N+1-i} + Di_{N-i-j} = 0. \tag{22}
\]

If \( i = 0 \), equation (22) gives \( D_{0,N} = 0 \), further for \( i = 1 \), we have \( D_{1,N-1} = 0 \). In this way, we find that for any value of \( i \)

\[
Di_{i,N-i} = 0. \tag{23}
\]

Analogously, equation (19) in this case also reads

\[
Ai_{i-1,N+1-i} + Ai_{N-i-j} = 0.
\]

we can also find that \( Ai_{N-i} = 0 \) for any \( i \). Thus \( Ai_{i-j} \) and \( Di_{i-j} \) vanish if \( i + j = N \). However, the other two elements \( Bi_{i-j} \) and \( Ci_{i-j} \) for \( i + j = N \) can be finite, as shown below.

(ii) \( i + j = N \). In this case, because \( Ai_{i-j} = Di_{i-j} = 0 \), equation (18) becomes

\[
iBi_{i,N-i} = 0.
\]

One immediately know that \( Bi_{i,N-i} \) can be non-zero only for \( i = 0 \), thus we set \( B_{0,N} = 1 \) for convenience, which does not influence the commutation relation \([J, H] = 0\), because each element \( M \) is linear in equations (10)–(13). Therefore we can generally write

\[
Bi_{i,N-i} = \delta_{i,0}. \tag{24}
\]

It is interesting to note that this result is consistent with the first inspection on the four matrix equations (10)–(13). But here it is derived rigorously and independently.

With the initial value for \( B \) in equation (24), equation (20) becomes

\[
Di_{i-1,N+1-i} + Di_{N-i-j} = \frac{\Delta}{4g} \left( \delta_{i,0} - (-1)^N \delta_{N-j} \right).
\]

For \( i = 0 \), we have \( D_{0,N-1} = \frac{\Delta}{4g} \), with iterations, we generally obtain

\[
Di_{i,N-i} = (-1)^i \frac{\Delta}{4g}, \tag{25}
\]

for any \( i \). Thus the elements \( Di_{i,j} \) for \( i + j = N - 1 \) are obtained. We stress here that the same procedure initiated from \( i = 0 \) to derive the remaining matrix elements by iterations is frequently employed in this work.

Now using the recursive relation equation (21), we have

\[
(1 - 2i) Bi_{i-1,N-j} - (1 + 2i) Bi_{i,N-i-1} = 0.
\]

For \( i = 0 \), \( B_{i,N} \) does not exist, one immediately know that \( B_{0,N-1} = 0 \). Increasing \( i \) one by one, in turn, we can have all

\[
Bi_{i,N-i} = 0, \tag{26}
\]
for any $i$, which gives $B_{i,j} = 0$ for $i + j = N - 1$.

(iii) $i + j = N - 1$. According to equations (21) and (25), equation (21) now is

$$iB_{i-1,N-i} + (i + 1)B_{i,N-i} = \frac{(-1)^i \Delta^2}{16g^2} (N - 2i - 1).$$

For $i = 0$, the first term is zero, we have $B_{0,N-2} = \frac{\Delta^2}{16g^2} (N - 1)$. Repeating the iteration by increasing $i$ one by one gives

$$B_{i,N-i} = (-1)^i \frac{\Delta^2}{16g^2} (N - i - 1). \quad (27)$$

By equation (26), equation (20) becomes

$$(2i + 1 - N)B_{i+1,N-i} + 2gD_{i+1,N-i-1} + 2gD_{i,N-i} = 0.$$  

Using equation (25), we have

$$2gD_{i-1,N-i} + 2gD_{i,N-i} = \frac{\Delta}{4g} (N - 2i - 1)(-1)^i.$$  

For $i = 0$, we have $D_{0,N-2} = \frac{\Delta}{8g} (N - 1)$ and all other elements $D_{i,j}$ for $i + j = N - 2$ can be obtained

$$D_{i,N-i} = (-1)^i \frac{\Delta}{16g^2} (N - i - 1)(i + 1). \quad (28)$$

Thus we have derived all $B_{i,j}$ and $D_{i,j}$ for $i + j = N - 2$.

(iv) $i + j = N - 2$. Using equations (24) and (26), equation (21) becomes

$$4g^2 (i + 1)B_{i+1,j-1} + [4g^2 i + (j - i)(j - i - N)] B_{i,j} + 2g (j - i + 1 - N)B_{i-1,j} + 2g (j - i - 1 - N)B_{i,j-1} = \Delta \left[ (i - j)D_{i,j} - g (j + 1)D_{i,j+1} - g (i + 1)D_{i+1,j} \right], \quad (29)$$

by equations (25), (27) and (28), we have

$$(1 + 2i)B_{i-1,N-2-i} + (3 + 2i)B_{i,N-i-3} = 0.$$  

By using the new recursive relation, we can get

$$B_{i,N-i-3} = 0,$$  

for arbitrary $i$ in the order of $N - 3$.

In this case, equation (20) here is

$$2gD_{i-1,N-2-i} + 2gD_{i,N-3-i} + (2 - N + 2i)D_{i,N-2-i} = \frac{\Delta}{2} \left( B_{i,N-2-i} - (-1)^{i+2} B_{N-2-i} \right).$$

For $i = 0$, using equations (27) and (28), we have

$$D_{0,N-3} = \frac{\Delta}{16g^3} (N - 2)(N - 1) + \frac{\Delta^3}{64g^3} (N - 2).$$
By iteration, we can get $D_{i,N-3-i}$ for any $i$ as

$$D_{i,N-3-i} = (-1)^i \left[ \frac{\Delta}{32g^2} (N-i-1)(i+2) + \frac{\Delta^3}{4!g^2} \right] \times (N-i-2)(i+1).$$

(v) $i + j = N - 3$. In this case, equation (21) gives

$$2g(N - 7 - 4i)(i+1)B_{i+1,N-3-i} - 4g(N - 2 - i)(1 + i)B_{i,N-2-i}$$

$$- 4g(1 + i)B_{i-1,N-3-i} - 2g(4 + 2i)B_{i,N-4-i}$$

$$= \Delta (2i - N + 3) D_{i,N-3-i} - \Delta g (N - 2 - i) D_{i,N-2-i}$$

$$- \Delta g (i+1) D_{i+1,N-3-i}.$$  

When $i = 0$, one can obtain

$$B_{0,N-4} = \left[ \frac{\Delta^4}{512g^4} - \frac{\Delta^2}{32g^2} + \frac{\Delta^2(N-1)}{128g^2} \right] (N-3)(N-2).$$

By iterations, we can summarize

$$(-1)^i B_{i,N-4-i}$$

$$= \frac{\Delta^2}{384g^2} (N-i-3)(N-i-2)(N-i-1)(i+1)(i+3)$$

$$+ \left( \frac{\Delta^4}{512g^4} - \frac{\Delta^2}{32g^2} \right) (N-i-3)(N-i-2)(i+1).$$

The elements $B_{ij}$ for $i + j = N - 4$ are obtained.

Equation (20) now becomes

$$(2i - N + 3) D_{i,N-3-i} + 2gD_{i-1,N-3-i} + 2gD_{i,N-4-i} = 0,$$

we can get

$$D_{i,N-4-i} = (-1)^i (i+1)(i+2) \frac{\Delta}{64g^2} (N-3-2i)(N-i-2)(N-i-1)$$

$$+ (-1)^i (i+1) \frac{\Delta^3}{128g^4} (N-3-2i)(N-i-2) - D_{i-1,N-3-i}.$$  

For $i = 0$ we have

$$D_{0,N-4} = \frac{\Delta}{32g^2} (N-3)(N-2)(N-1) + \frac{\Delta^3}{128g^4} (N-3)(N-2).$$

Repeating these procedures, all the coefficients for $i + j = N - 4$ can be readily obtained

$$D_{i,N-4-i} = (-1)^i \left[ \frac{\Delta}{192g^2} (N-i-1)(i+3) + \frac{\Delta^3}{256g^4} \right]$$

$$\times (N-i-3)(i+2)(N-i-2)(i+1),$$

(33)
which is the general expression of $D_{i,j}$ for $i + j = N - 4$.

Proceeding as the processes outlined above, we can further derive all coefficients $B_{i,j}$ and $D_{i,j}$. Actually we do not have to summarize the general expressions for the coefficients $X_{i,N-k-i}$ with $k$ as an integer, like equations (32) and (33). The most important thing is to derive all coefficients of the matrix elements one by one. In principle, for arbitrary integer biases $N$, we can obtain all expansion coefficients $B_{i,j}$ and $D_{i,j}$ initiated from $B_{0,N} = 1$. $C_{i,j}$ can be easily obtained through $C_{i,j} = B_{i,j}(-1)^{j+i}$, where we simply have $C_{N,0} = (-1)^N$. The operator $A_{i,j}$ can be obtained through equation (18) straightforwardly.

It should be also stressed that $B_{i,j}$ does not have to be zero for any $i + j = N - (2k + 1)$, but it indeed exactly vanishes for $i + j = N - 1$ and $N - 3$. We demonstrate this point in the appendix.

3. Symmetry operators within the BOA: demonstrations

In this section, we demonstrate the symmetry operators using the scheme outlined above. First, for $N = 0$, we can immediately have $B = C = 1$ from equation (24) and $A = D = 0$ by equation (23). One finds $Q^{(0)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, so the symmetry operator is

$$J^{(0)} = e^{i\pi a^\dagger a} \sigma_x,$$

which is just the parity operator in the symmetric QRM [16]. Next, we present the symmetry operator in AQRM for several integer biases.

**Symmetry operator for $N = 1$.** In this case, equation (24) gives $B_{0,1} = 1$. By equation (25) we have $D_{0,0} = \frac{\Delta^2}{16g^2}, C$ and $A$ can be then derived accordingly, therefore we readily have

$$Q^{(1)} = \begin{pmatrix} \Delta & a_- \\ -a_+ & \frac{\Delta}{4g} \end{pmatrix},$$

Note $J^{(1)} = e^{i\pi a^\dagger a} Q^{(1)}$, and the $J^{(1)}$ square is

$$(J^{(1)})^2 = H + \left( g^2 + \frac{\Delta^2}{16g^2} + \frac{1}{2} \right) I,$$

where $I$ is the $2 \times 2$ unit matrix. Equations (35) and (36) are the same as equations (57) and (59) in reference [16].

**Symmetry operator for $N = 2$.** From equations (24) and (27) we immediately have $B_{0,0} = \frac{\Delta^2}{16g^2}$, so the $B$ operator is

$$B = (a_-)^2 + \frac{\Delta^2}{16g^2}.$$

By equations (25) and (28), we have

$$D_{0,1} = \frac{\Delta}{4g}, \quad D_{1,0} = -D_{0,1}, \quad D_{0,0} = \frac{\Delta}{8g^2},$$

so we have
\[
D = \frac{\Delta}{4g} (a_+^\dagger - a_+) + \frac{\Delta^2}{8g^2}.
\]

Similarly, \(C\) and \(A\) are obtained immediately
\[
C = (a_+^\dagger)^2 + \frac{\Delta^2}{16g^2},
\]
\[
A = \frac{\Delta}{4g} (a_- - a_+^\dagger) - \frac{\Delta^2}{8g^2}.
\]

These elements are no other than those given by reference [16] (cf their equation (64)). One may note that only with a brief derivation, we can get the concise expression for the symmetry operators in BOA for \(N = 2\). We will show below that even for further high integer biases, we can derive all the elements for the symmetry operators without much more effort.

**Symmetry operator for \(N = 3\).** From equations (24) and (27) we immediately have
\[
B_{0,1} = \frac{\Delta^2}{8g^2}, \quad B_{1,0} = -\frac{\Delta^2}{16g^2},
\]
by which we can obtain
\[
B = (a_-)^3 + \frac{\Delta^2}{8g^2} a_- - \frac{\Delta^2}{16g^2} a_+^\dagger.
\]

While equations (25), (28) and (31) give
\[
D_{i,-i} = (-1)^i \frac{\Delta}{4g}, \quad i = 0, 1, 2
\]
\[
D_{0,1} = \frac{\Delta}{4g^2}, \quad D_{1,0} = -D_{0,1}, \quad D_{0,0} = \frac{\Delta^3 + 8\Delta}{64g^3},
\]
we then have
\[
D = \frac{\Delta}{4g} \left( a_+^\dagger a_+ \right)^i + \frac{\Delta^2}{8g^2} a_- - \frac{\Delta^2}{16g^2} a_+^\dagger.
\]

By the similar way, we have
\[
C = (-a_+^\dagger)^3 + \frac{\Delta^2}{16g^2} a_- - \frac{\Delta^2}{8g^2} a_+^\dagger,
\]
\[
A = -\frac{\Delta}{4g} (a_+^\dagger a_- - a_+^2) + \frac{\Delta}{4g^2} (a_+^\dagger a_-) + \frac{\Delta^3 + 8\Delta}{64g^3} = \frac{\Delta}{4g}.
\]

Very interestingly, these elements are just the same as those given in reference [20]. One can see that our expression is much more concise especially in the operators \(B\) and \(C\).

**Symmetry operator for \(N = 4\).** We can proceed with the further high integer bias, and can still easily get the symmetry operator. From equations (24), (27) and (32), for \(N = 4\),
\[
\begin{pmatrix}
B_{0,2}, B_{1,1}, B_{2,0}
\end{pmatrix} = \begin{pmatrix}
\Delta^2 & 0 & 0 \\
0 & \Delta^2 & 0 \\
0 & 0 & \Delta^2
\end{pmatrix},
\]
\[
B_{0,0} = \frac{12\Delta^2 - 16g^2\Delta + \Delta^4}{256g^4}.
\]
so we have

\[ B = (a_-)^4 + \frac{3\Delta^2}{16g^2}a_-^2 - \frac{\Delta^2}{8g^2}a_+^4a_- + \frac{\Delta^2}{16g^2}a_+^4 + \frac{12\Delta^2 - 16g^2\Delta_2 + \Delta^4}{256g^4}, \]

and by equations (25), (28), (31) and (33), we have

\[ D_{i,3-i} = (-1)^i \frac{\Delta}{4g} \quad i = 0, 1, 2, 3, \]

\[ D_{0,2} = \frac{3\Delta}{8g^2}, \quad D_{1,1} = -\frac{\Delta}{2g^2}, \quad D_{2,0} = D_{0,2}, \]

\[ D_{0,1} = \frac{\Delta(\Delta^2 + 12)}{32g^3}, \quad D_{1,0} = -D_{0,1}, \]

\[ D_{0,0} = \frac{\Delta^3 + 12\Delta}{64g^4}, \]

which give \( D \) as

\[
D = \frac{\Delta}{4g} \left( a_-^4 - a_+^4a_-^2 + a_+^4a_- + a_-^2 \right) + \frac{3\Delta}{8g^2} \left( a_+^2 + a_-^2 \right) - \frac{\Delta}{2g^2}a_+^4a_- \\
+ \frac{\Delta^3 + 12\Delta}{32g^3} \left( a_- - a_+^3 \right) + \frac{\Delta^3 + 12\Delta}{64g^4}.
\]

Similarly, \( C \) and \( A \) can be also obtained straightforwardly,

\[
C = \left(-a_+^4\right) + \frac{3\Delta^2}{16g^2}a_+^4 - \frac{\Delta^2}{8g^2}a_+^4a_- + \frac{\Delta^2}{16g^2}a_-^2 + \frac{12\Delta^2 - 16g^2\Delta_2 + \Delta^4}{256g^4},
\]

\[
A = \frac{\Delta}{4g} \left(-a_+^4 + a_-^4 + a_+^2a_-^2 - a_+^4a_-^2 \right) - \frac{3\Delta}{8g^2} \left( a_+^2 + a_-^2 \right) + \frac{\Delta}{2g^2}a_+^4a_- \\
+ \left( \frac{\Delta^3 + 12\Delta}{32g^3} - \frac{\Delta}{2g} \right) \left( a_- - a_+^3 \right) - \frac{\Delta^3 + 12\Delta - 32g^2\Delta}{64g^4}.
\]

One may be impressed deeply by that these elements would be very complicated if expanding in the original Fock space. One can see that the number of the expansion terms in the Fock space will increase rapidly with \( N \) with different coefficients, e.g., \((a - g)^N\) and \((a^i)^N\). In the present BOA scheme, many terms with the same power share the same coefficients, such as the dominant terms \((i + j = N - 1)\) in \( D \) (also in \( A \)), cf. equation (25), which is independent of \( N \), so the symmetry operators can be written in a compact way by the Bogoliubov operators, no matter how large \( N \) is. For the same values of \( i + j < N - 1 \), the coefficients also possess simple relations, which remarkably reduce the complexity of the expression for the symmetry operators in BOA. Moreover, we can obtain the symmetry operators only by simple algebra, as described in section 2. To demonstrate the universality of this approach further, we present the symmetry operator for \( N = 5 \) in the appendix as an additional example. It should be stressed here that there is no limit on the value of \( N \) in the present scheme.
4. General parity operators of the AQRM

We construct the parity operator of the AQRM at integer biases through the obtained symmetry operators in this section. Generally, if the symmetry operator $J$ is a function of the Hamiltonian, $J = f(H)$, the commutation relation $[J, H] = 0$ should trivially hold. Inspecting the parity operator of the isotropic QRM in equation (34), one may note $[J^{(0)}]^2 = I$ and $J^{(0)}$ cannot be expressed in $H$ in a simple way. Thus, the symmetry operator of the one-photon AQRM is generally assumed to satisfy the following form (cf [16])

$$J^2 = \sum_{n=0}^{N} x_n H^n,$$

which also ensures $[J, H] = 0$ for $\epsilon = N \neq 0$. Note that even for $\epsilon = 0, N = 0$, equation (37) becomes $J^2 = x_0 H^0 I$, consistent with the usual definition of the parity operator of the QRM.

According to equations (5) and (6), we know

$$J^2 = \left( \begin{array}{cc} A^I A + C^I C & A^I B + C^I D \\ B^I A + D^I C & B^I B + D^I D \end{array} \right),$$

where all elements are known. For simplicity, we only focus on the second diagonal element, namely

$$(J^2)_{2,2} = \left( \begin{array}{c} (a^-)_N^N + \sum_{i+j \leq N-2} B_{i,j} (a^-)_i^j (a^+)_i^j \\ (a^-)_0^N + \sum_{i+j \leq N-2} B_{i,j} (a^-)_i^j (a^+)_i^j \end{array} \right) + \sum_{i=0}^{i+j \leq N-1} D_{i,j} (a^-)_i^j (a^+)_i^j (a^-)_0^j.$$  \quad (38)

For convenience, it can be reexpressed only with $a^I$ and $a_-$ as

$$(J^2)_{2,2} = \left( \begin{array}{c} (a^-)_N^N + \sum_{i,j=0}^{i+j \leq N-2} \sum_{n=0}^{i+j} B_{i,j} z^n (a^-)_i^j (a^+)_i^j \\ \sum_{i,j=0}^{i+j \leq N-2} \sum_{n=0}^{i+j} B_{i,j} z^n (a_-)_i^j (a^+)_i^j \end{array} \right) + \sum_{i,j=0}^{i+j \leq N-1} D_{i,j} (a^-)_i^j (a^+)_i^j (a^+)_0^j.$$  \quad (38)

where $z^n = \frac{\mathcal{d}}{\mathcal{d}(2g)^n}$. In principle, we can write

$$(J^2)_{2,2} = \sum_{i=0}^{N} J_{i,j} (a^-)_i^j (a^-)_0^j.$$
On the other hand, we can also rewrite (2) as

\[(H^n)_{2,2} = \sum_{i,j=0}^{n} y^{(n)}_{ij} \left( a^\dagger_i \right) \left( a_{-j} \right). \]

By equation (37) we have

\[\sum_{i,j}^{N} J_{i,j} \left( a^\dagger_i \right) \left( a_{-j} \right) = \sum_{n=0}^{N} x_n \left( \sum_{i,j=0}^{n} y^{(n)}_{ij} \left( a^\dagger_i \right) \left( a_{-j} \right) \right). \]  \hfill (39)

Since all the coefficients \(J_{i,j}\) and \(y^{(n)}_{ij}\) are known, one can in principle get the coefficients \(x_n\) in equation (37).

It is quite complicated and tedious to provide all coefficients \(x_n\) in detail, here we only confine us to show some highest order coefficients. Note that

\[(H^2)_{2,2} = \left( a^\dagger \right)^N \left( a_{-} \right)^N + \frac{N \Delta^2}{16 g^2} \left( a^\dagger \right)^{N-1} \left( a_{-} \right)^{N-1}
+ \sum_{i,j=0}^{N-2} J_{i,j} \left( a^\dagger_i \right) \left( a_{-j} \right), \]

and

\[(H^N)_{2,2} = \left( a^\dagger \right)^N \left( a_{-} \right)^N - N \left( g^2 + \frac{1}{2} \right) \left( a^\dagger \right)^{N-1} \left( a_{-} \right)^{N-1}
+ \sum_{i,j=0}^{N-2} y^{(N)}_{ij} \left( a^\dagger_i \right) \left( a_{-j} \right), \]

\[(H^{N-1})_{2,2} = \left( a^\dagger \right)^{N-1} \left( a_{-} \right)^{N-1} - (N - 1) \left( g^2 + 1 \right) \left( a^\dagger \right)^{N-2} \left( a_{-} \right)^{N-2}
+ \sum_{i,j=0}^{N-3} y^{(N-1)}_{ij} \left( a^\dagger_i \right) \left( a_{-j} \right). \]

By equation (39), we can readily find

\[x_N = 1, \quad x_{N-1} = N \left( g^2 + \frac{1}{2} + \frac{\Delta^2}{16 g^2} \right), \]

which leads to

\[J^2 = H^N + N \left( g^2 + \frac{1}{2} + \frac{\Delta^2}{16 g^2} \right) H^{N-1} + \cdots. \]

Interestingly, one can find that the first two terms in the above general expression are the same as those for \(\epsilon = 1, 2, 3\) in references [16, 20]. Even for \(J^{(0)}\) in equation (34) for \(\epsilon = 0\), they can also be applied, i.e., \([J^{(0)}]^2 = I\). A mathematical proof to the general assumption (37) for arbitrary \(N\) has been discussed at length in references [16, 20, 21], and it has been confirmed by the concrete examples at low integer biases. However, a general expression of the operator
$J^2$ has yet to be given. We propose a way using BOA to obtain the coefficients for arbitrary order of $H$. In particular, the coefficients are given recursively starting from the highest degree, analogous to the derivation processes of the elements of matrix $Q$. It is worthwhile to explore a more efficient and simple method to obtain the general expression for $J^2$.

Based on equation (37), for given $\Delta$ and $g$, in any eigenstate $|\psi\rangle$, we have

$$[J^{(N)}]^2 |\psi\rangle = \sum_{n=0}^{N} x_n E^n |\psi\rangle, \quad (40)$$

it follows that the eigenvalue of $J^2$ is energy dependent. To construct a general parity operator $\Pi^{(N)}$ of the AQRM with eigenvalues independent of the energy, we can define the parity operator via the symmetry operator as

$$\Pi^{(N)} = \frac{J^{(N)}}{\sqrt{\sum_{n=0}^{N} x_n H^n}} \quad (41)$$

Its eigenvalue is $\pm 1$, still implying a $\mathbb{Z}_2$-symmetry in the AQRM at integer biases [16]. Nevertheless, equation (41) might be invalid if the denominator vanishes [21], which requires further discussions. For $N = 0$, $\Pi^{(0)}$ is exactly the same as $J^{(0)}$, so the well known parity operator of the symmetric QRM is also naturally included in the general definition (41).

5. Conclusion

In this work, we have developed a BOA scheme to derive the coefficients of the matrix elements of the symmetry operator in the AQRM systematically, and the derivation is much more concise than the previous approach using the expansion in the original Fock space. Because the Bogoliubov operators are actually expressed linearly in terms of the original operator, although the previous method based on the expansions of the elements in $2 \times 2$ matrix in the Fock space is rather complicated, it still works for small integer biases. With further large biases, it is difficult to obtain the symmetry operators in the Fock space. For small integer biases, such as $\epsilon = 1, 2$, the hidden symmetry can be very easily uncovered within the BOA. Moreover its simple form in BOA renders it somehow obvious or not so ‘hidden’.

In addition, through the symmetry operator, a general parity operator is defined with eigenvalue $\pm 1$, still implying a $\mathbb{Z}_2$-symmetry of the AQRM for integer biases. The standard parity operator in the symmetric QRM is naturally included in this general definition.

We believe that this general scheme within BOA can be easily extended to the other more complicated light–matter interaction system, such as the anisotropic AQRM [17, 27–29], two-photon and two-mode AQRMs [19, 25, 26], and the finite size asymmetric Dicke model [30, 31].

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Data availability statement

No new data were created or analysed in this study.

Appendix. Symmetry operator of AQRM for \( N = 5 \)

Following the approach described in section 2, we can derive the four elements of the symmetry operator for \( N = 5 \). From equations (27) and (32) at \( N = 5 \), we have

\[
B_{0,3} = \frac{\Delta^2}{4g^2}, \quad B_{1,2} = -\frac{3\Delta^2}{16g^2}, \quad B_{2,1} = \frac{\Delta^2}{8g^2}, \quad B_{3,0} = -\frac{\Delta^2}{16g^2},
\]

\[
B_{0,1} = \frac{3\Delta^2}{16g^4} + \frac{3\Delta^2 (\Delta^2 - 16g^2)}{256g^4}, \quad B_{1,0} = \frac{\Delta^2 (16g^2 - \Delta^2 - 16)}{128g^4}.
\]

By equations (25), (28), (31) and (33), we have

\[
D_{i,4-i} = (-1)^i \frac{\Delta}{4g}, \quad i = 0, 1, 2, 3, 4,
\]

\[
D_{0,3} = \frac{\Delta}{2g^2}, \quad D_{1,2} = -\frac{3\Delta}{4g^2}, \quad D_{2,1} = -D_{1,2}, \quad D_{3,0} = -D_{0,3},
\]

\[
D_{0,2} = \frac{3\Delta (\Delta^2 + 16)}{64g^4}, \quad D_{1,1} = -\frac{\Delta (\Delta^2 + 18)}{16g^3}, \quad D_{2,0} = D_{0,2},
\]

\[
D_{1,0} = -\frac{3\Delta (\Delta^2 + 16)}{64g^4}, \quad D_{0,1} = -D_{1,0}.
\]

So far, we have not completely derived all elements for \( B \) and \( D \), because we have not discussed the case of \( i + j = N - 4 \) in the main text. We actually only need the elements, and do not have to derive the general formulae like those in the main text.

In the case of \( i + j = N - 4 \), for \( N = 5 \), we first select \( i = 0, j = 1 \), equation (21) gives

\[
8g^2B_{2,1} + 8g^2B_{1,2} - 8gB_{1,1} - 12gB_{0,2} + 4g^2B_{1,0} - 4B_{0,1} - 10gB_{0,0} = \Delta \left( -D_{0,1} - 2gD_{0,2} - gD_{1,1} \right).
\]

(A.1)

Only \( B_{0,0} \) is unknown, which can be given by

\[
B_{0,0} = \frac{\Delta^2}{16g^4}.
\]

Equation (20) now becomes

\[
(2i - 1)D_{i,1-i} + 2gD_{i-1,1-i} + 2gD_{i-i} = \frac{\Delta}{2} (B_{i,1-i} + B_{1-i})
\]

(A.3)

for \( i = 0 \), we have

\[
-D_{0,1} + 2gD_{0,0} = \frac{\Delta}{2} (B_{0,1} + B_{1,0}).
\]
Only $D_{0,0}$ is unknown and is given by
\[
D_{0,0} = \frac{\Delta \left( -16g^2\Delta^2 + \Delta^4 + 40\Delta^2 + 384 \right)}{1024g^5}.
\]

Finally we give
\[
A = \frac{\Delta}{4g} \left( a_{\uparrow}^4 + a_{\downarrow}^4 - a_{\uparrow}^3 a_{\downarrow}^1 - a_{\uparrow}^1 a_{\downarrow}^3 + \frac{3\Delta}{2g} \left( a_{\uparrow}^3 - a_{\downarrow}^3 \right) \right)
- \frac{3\Delta}{4g^2} \left( a_{\uparrow}^2 a_{\downarrow}^2 - a_{\downarrow}^4 a_{\uparrow}^1 \right) + \left( \frac{3\Delta}{64g^3} - \frac{3\Delta}{4g^2} \right) \left( a_{\uparrow}^2 + a_{\downarrow}^2 \right)
+ \frac{\Delta}{2g} \left( \frac{\Delta^3 + 18\Delta}{16g^3} \right) a_{\uparrow}^1 a_{\downarrow}^1 + \left( \frac{3\Delta}{64g^4} - \frac{3\Delta}{2g^2} \right) \left( a_{\uparrow} + a_{\downarrow} \right)
+ \frac{\Delta}{1024g^5} \left( 384 - 16\Delta^2 + 40\Delta^2 + \Delta^4 \right) + \frac{\Delta^3 + 18\Delta}{16g^3}.
\]
\[
B = a_{\downarrow}^5 - \frac{\Delta^2}{16g^2} a_{\uparrow}^3 a_{\downarrow}^3 + \frac{\Delta^2}{4g^2} a_{\downarrow}^5 - \frac{3\Delta^2}{8g^2} a_{\uparrow}^3 a_{\downarrow}^3 + \frac{\Delta^2}{16g^2} a_{\uparrow}^1 a_{\downarrow}^3
+ \frac{\Delta^2}{256g^4} \left( 2a_{\uparrow} - 3a_{\downarrow} \right) + \frac{\Delta^2}{16g^3}.
\]
\[
C = -a_{\uparrow}^5 + \frac{\Delta^2}{16g^2} a_{\downarrow}^3 - \frac{\Delta^2}{4g^2} a_{\uparrow}^3 + \frac{\Delta^2}{8g^2} a_{\uparrow}^3 a_{\downarrow}^3 + \frac{3\Delta^2}{16g^3} a_{\uparrow}^4 a_{\downarrow}
- \frac{\Delta^2}{256g^4} \left( 2a_{\downarrow} - 3a_{\uparrow} \right) + \frac{\Delta^2}{16g^3}.
\]
\[
D = \frac{\Delta}{4g} \left( a_{\uparrow}^4 + a_{\downarrow}^4 - a_{\uparrow}^3 a_{\downarrow}^1 - a_{\uparrow}^1 a_{\downarrow}^3 + \frac{3\Delta}{2g} \left( a_{\uparrow}^3 - a_{\downarrow}^3 \right) \right)
+ \frac{3\Delta}{4g^2} \left( a_{\uparrow}^2 a_{\downarrow}^2 - a_{\downarrow}^4 a_{\uparrow}^1 \right) + \left( \frac{3\Delta}{64g^3} - \frac{3\Delta}{4g^2} \right) \left( a_{\uparrow}^2 + a_{\downarrow}^2 \right)
- \frac{\Delta^3 + 18\Delta}{16g^3} a_{\uparrow}^1 a_{\downarrow}^1 + \left( \frac{3\Delta}{64g^4} - \frac{3\Delta}{2g^2} \right) \left( a_{\uparrow} + a_{\downarrow} \right)
+ \frac{\Delta}{1024g^5} \left( 384 - 16\Delta^2 + 40\Delta^2 + \Delta^4 \right).
\]

One can observe the finite $B_{0,0}$ and $C_{0,0}$, indicating that $B_{ij}$ and $C_{ij}$ for $i + j = N - (2k + 1)$ with $k$ an integer are not zero, but still satisfy $C_{ij} = B_{i,-(1)^{i+j}}$. Here $N = 5, i = j = 0, k = 2$.

For a given large bias $N$, we do not have to present the general formulae for each case, as shown above for $N = 5$. The only important thing is to find the elements of the symmetry operator for this $N$. The present BOA scheme is always concise and explicit even for large integer biases.
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