Relating neutrino masses and mixings by discrete symmetries

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Abstract

Lepton mixing can originate from the breaking of a flavor symmetry in different ways in the neutrino and the charged lepton sector. We propose an extension of this framework which allows to connect the mixing parameters with masses, and more precisely, with certain types of degeneracy of the neutrino mass spectrum. We obtain relations between the mixing parameters for the cases of partial degeneracy, \( m_1 = m_2 \), and complete degeneracy, \( m_1 = m_2 = m_3 \). These relations determine also the Majorana phases. It is shown that relatively small corrections to these lowest order results can produce the required mass splitting and modify the mixing without significant changes of the other symmetry results.

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1 Introduction

Lepton mixing can be a consequence of the breaking of a flavor symmetry $G_f$ into different residual symmetries, $G_\nu$ and $G_\ell$, for the neutrino and charged lepton mass matrices respectively [1]. To a large extent, this approach was motivated by the peculiar values of the lepton mixing parameters that seemed to be well approximated by the so-called tri-bimaximal (TBM) mixing [2]. TBM turned out to be difficult to connect with the known ratios of masses of the charged leptons and neutrinos, and therefore, in this approach, the masses of neutrinos and charged leptons “decouple” from mixing. In other words, TBM implies the form invariance of the mass matrix - a situation in which mixing is determined by symmetry alone. Models that reproduced the TBM and other interesting mixing patterns were built along these lines [3, 4, 5, 6, 7]. Independent physics (additional symmetries) was assumed to be responsible for the hierarchy of masses of leptons. Indirect relations between mixing and mass spectrum appear in some specific models, as a result of model structure and particle content.

The separate description of mixing and mass hierarchies (ratios of masses) with different physics involved looks unsatisfactory. Indeed,

1. In general, masses and mixing have the same origins following formally from diagonalization of the mass matrices. They are generated by the same type of Yukawa couplings and so should be somehow related.

2. In the quark sector various relations between the mass ratios and mixing parameters have been uncovered with the Gatto-Sartori-Tonin relation [8] being the most appealing one. The latter can be explained by an abelian flavor symmetry, e.g., in the Froggatt-Nielsen approach [9]. There were some attempts to use discrete symmetries to produce relations between masses and mixing (see e.g. [10]). In a number of recent models with discrete flavor symmetries the sum rules for neutrino masses (linear relations between masses or their inverse quantities) have been realized (see e.g. [11]). However, these relations do not depend on mixing and turn out to be consequences of specific restricted model contents and vacuum alignments and do not follow from residual symmetries of the neutrino mass matrix.

3. The now established relatively large 1-3 mixing and the indications of significant deviations of the 2-3 mixing from maximal [12] rule out the exact TBM, and therefore cast doubts to the explanation of mixing separated from masses through nonabelian symmetries. At the same time, it was shown in [13, 14, 15, 16] that flavor symmetries can still accommodate the recent results on the 1-3 and 2-3 mixings.

In [14, 15] we proposed a formalism for “symmetry building” which relies on the aforementioned partial breaking of $G_f$ into two subgroups, $G_\ell$ and $G_\nu$. We used it
to obtain relations between the mixing parameters without explicit reference to any particular model. It was shown that consequences of symmetries for mixing, that is, relations between the mixing parameters or elements of mixing matrix (at least at the lowest order) can be obtained immediately once residual symmetries (transformations) of the neutrino and charged lepton mass matrices are known. These symmetry group relations can be obtained without model building and explicit construction of the mass matrices and their diagonalizations. Essentially, it is only assumed that the model is constructed and it leads to mass matrices with given symmetry properties. The relations in [14, 15] can be viewed as a tool of symmetry or/and model building. Once the required relations and the corresponding residual symmetries are identified, one can come back and construct the corresponding complete flavor symmetry and model. After a model is constructed one can compute corrections to zero order structures. In general, this formalism does not allow to compute the latter model-dependent corrections.

In this paper we further develop this formalism, which in [14, 15] was elaborated for mixing only, in order to include also neutrino masses. Consequently, we obtain relations between the mixing parameters and certain types of the neutrino mass spectrum.

The crucial point of the explanation of mixing decoupled from masses was to use, as $G_\nu$, the symmetry of a generic neutrino mass matrix $M_\nu$ with arbitrary eigenvalues. For Majorana neutrinos the maximal symmetry of $M_\nu$ is $Z_2 \times Z_2$, as can be seen immediately in the basis where the mass matrix is diagonal. This generic symmetry does not constrain the masses and therefore leads to the decoupling of mixing. Hence, in order for the symmetry to predict both masses and mixings, $G_\nu$ should be enlarged in such a way that invariance of the mass matrix is satisfied only for certain mass spectra.

In this paper, we focus on residual symmetries $G_\nu$ that lead to equalities of the neutrino masses. Indeed, unitary symmetry transformations can lead either to equalities of the masses or to zero values of masses. In this connection we will explore two possibilities: (i) two degenerate neutrinos, i.e., equality of two masses, (ii) three degenerate neutrinos. The first case can be considered as the lowest order approximation to spectra with both normal and inverted mass hierarchies. Then, corrections are required which lead to splitting between the masses. In the case of normal mass hierarchy with two vanishing masses, the corrections should generate the mass of at least the second neutrino.

The paper is organized as follows. In Sec. 2 we describe the model independent method for “symmetry building”, applied here to the case of specific neutrino mass spectrum. In Sec. 3 we consider the case of partial degeneracy (equality of two masses). We derive the relations between mixing parameters which also include the Majorana phases. In Sec. 5 we derive constrains on the mixing and phases in the case of completely degenerate spectrum. Discussion and conclusions are presented in Sec. 6.

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1 A complete scan of the groups with order less than 1536 and $G_\nu = Z_2 \times Z_2$ was performed in [17].

2 The possibility of the symmetry leading to vanishing masses has been considered in [13].
2 Symmetry relations for mixing and masses

We assume that neutrinos are Majorana particles. Working in the flavor basis, the mass terms of the lepton sector of the Lagrangian read

\[ \mathcal{L}_{\text{mass}} = \bar{E}_R m_\ell \ell_L + \frac{1}{2} \bar{\nu} jLM \nu_{\ell_L} + \text{h.c.}, \]  

(1)

where \( \nu_{\ell_L} \), \( \ell_L \) and \( E_R \) are the leptonic fields: \( \nu_{\ell_L} \equiv (\nu_e, \nu_\mu, \nu_\tau)_L \), \( \ell_L \equiv (e, \mu, \tau)_L \), and \( m_\ell \equiv \text{diag}\{m_e, m_\mu, m_\tau\} \). The flavor neutrino states are related to the mass eigenstates, \( \nu_L \equiv (\nu_1, \nu_2, \nu_3)_L \), by the Pontecorvo-Maki-Nakagawa-Sakata mixing matrix: \( \nu_{\ell_L} = U_{\text{PMNS}} \nu_L \). Then, the neutrino mass matrix in the flavor basis, \( M_{\nu U} \), can be expressed via the diagonal mass matrix of the neutrino mass eigenvalues, \( m_\nu \equiv \text{diag}\{m_1, m_2, m_3\} \), and \( U_{\text{PMNS}} \) as

\[ M_{\nu U} = U_{\text{PMNS}}^* m_\nu U_{\text{PMNS}}^\dagger. \]  

(2)

Let \( G_\ell \) and \( G_{\nu U} \) be the groups of symmetry transformations that leave invariant the charged lepton and neutrino mass terms in Eq. (1). The residual flavor symmetries in the charged lepton and neutrino sectors are in general finite subgroups of these: \( G_\ell \subset G_{\ell} \) and \( G_\nu \subset G_{\nu U} \). We proceed to identify \( G_\nu \) and \( G_\ell \) systematically.

For the charged leptons we have \( G_\ell \equiv U(1)^3 \) corresponding to the electron, muon and tau lepton numbers. As in our previous papers [14, 15], we assume that \( G_\ell = Z_m \). The fact that \( Z_m \) is generated by one element implies that it leads to a minimal number of constraints on the mixing matrix. Analysis of bigger groups, that lead to stricter conditions \( U_{\text{PMNS}} \), is beyond the scope of this paper.

A representation of \( G_\ell \) is given by the matrix \( T \) such that \( \ell \rightarrow T \ell_L \), \( E_R \rightarrow TE_R \),

\[ \ell \rightarrow T \ell_L, \quad E_R \rightarrow TE_R, \]  

(3)

where

\[ T \equiv \text{diag}\{e^{i\phi_e}, e^{i\phi_\mu}, e^{i\phi_\tau}\} \]  

(4)

and

\[ \phi_\alpha \equiv 2\pi k_\alpha/m_\alpha, \quad \alpha = e, \mu, \tau. \]  

(5)

Invariance of the charged lepton mass matrix \( m_\ell \) under \( T \) means that the following equality holds: \( T m_\ell T^\dagger = m_\ell \). According to Eqs. (1) and (5), \( T \) satisfies the condition \( T^m = I \). It is enough to consider a subgroup of \( SU(3) \) as the flavor group. This restriction simplifies the considerations without having any impact in the results of the paper. One can show that the additional \( U(1) \) of \( U(3) \) can be factored out and does not induce any constraint on mixing. Thus, we impose the equality

\[ \phi_e + \phi_\mu + \phi_\tau = 0, \]
or equivalently, $k_\tau = -k_\nu + k_\mu$, that ensures $\text{Det}[T] = 1$.

Considering now the symmetry group $G_\nu$ of the Majorana mass term of neutrinos, we explore the possibility of approximate degeneracy of the mass spectrum. In the neutrino mass basis the invariance of mass matrix under the transformation

$$\nu_L \rightarrow S\nu_L,$$

where $S$ belongs to the group $G_\nu$, means that

$$S^T m_\nu S = m_\nu.$$ (7)

If $S$ satisfies Eq. (7), the corresponding matrix $S_U$ that leaves $M_\nu U$ invariant, i.e.

$$S_U^T M_\nu U S_U = M_\nu U,$$

can be found by switching to the flavor basis:

$$S_U = U_{PMNS} S U_{PMNS}^\dagger.$$ (8)

Hence, the group $G_{\nu U}$ is obtained from $G_\nu$ by applying a similarity transformation with $U_{PMNS}$ to all elements of $G_\nu$. The residual symmetry $G_\nu$ is a discrete subgroup of $G_{\nu U}$.

Three cases can be distinguished for $G_\nu$ with increasing symmetry that corresponds to increasing degree of degeneracy of $m_\nu$.

A. No degeneracy: $G_\nu = Z_2 \otimes Z_2$.

The diagonal neutrino mass matrix $m_\nu$ with arbitrary eigenvalues is invariant under the transformations (9) with

$$S_1 = \text{diag}\{1, -1, -1\}, \quad S_2 = \text{diag}\{-1, 1, -1\},$$

and $S_3 = S_1 S_2$. This case was analyzed in [14, 15] and $G_{\nu U} = Z_2$ or $G_{\nu U} = Z_2 \otimes Z_2$ are possible.

B. 2 degenerate neutrinos: $G_{\nu U} = SO(2) \otimes Z_2$.

In addition to the matrices in Eq. (8), the neutrino mass matrix is invariant under rotations of the plane of mass degeneracy$^3$. Taking into account the measured

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$^3$A model for two degenerate neutrinos based on the group $D_N$ - a discrete subgroup of $O(2)$ - has been proposed in [19] where a relation which connects $\theta_{13}$, the electron mass and the CP-phases was obtained. The approach in [19] differs from ours in that $D_N$ is the symmetry of whole leptonic sector, introduced to explain small observed parameters which describe masses and mixing.
neutrino mass differences, the approximate degeneracy must be between \( m_1 \) and \( m_2 \). Therefore we take

\[
m_{\nu 3} \simeq \begin{pmatrix} m & m \\ m & m' \end{pmatrix},
\]

which is invariant under the transformation

\[
S \equiv S_\zeta = \begin{pmatrix} c_\zeta & -s_\zeta \\ s_\zeta & c_\zeta \end{pmatrix},
\]

where \( c_\zeta \equiv \cos \zeta, \ s_\zeta \equiv -\sin \zeta \). Analogously to the case of charged leptons, we impose \( G_\nu = Z_n \) so that \( S^n_{\zeta} = I \) and \( \zeta = 2\pi l/n \). In the flavor basis, Eq. (8), we must also have

\[
S^n_{\nu} = I.
\]

C. 3 degenerate neutrinos: \( G_\nu = SO(3) \).

In this case we have

\[
m_\nu \simeq mI,
\]

and \( S \) can be any orthogonal \( 3 \times 3 \) matrix. We note that, according to the Euler rotation theorem, any 3D rotation is a 1D rotation around a certain axis. Thus, any matrix \( S \) that is a symmetry of \( m_\nu \) can be written as

\[
S \equiv S_{\zeta O} = O S_\zeta O^T,
\]

where \( O \) is an orthogonal matrix. The Euler rotation theorem implies that a basis in the neutrino sector can be always selected such that the matrix \( S \) in this basis has the form \( S_\zeta \). Turning the argument around, imposing only a \( Z_n \) symmetry generated by \( S_{\zeta O} \) is not enough to force the three neutrinos to be degenerate. The most general mass matrix that is left invariant by a \( Z_n \) subgroup of \( O(3) \) has only two equal eigenvalues. Thus, if \( G_\nu \) imposes full degeneracy of the neutrino mass matrix, it must be one of the non-Abelian subgroups of \( O(3) \) with 3D representations, *i.e.*, \( A_4, S_4 \) or \( A_5^{\circ} \).

As it was shown in [14], the relations between mixing matrix elements follow from the condition that the symmetry transformations of the charged leptons and neutrinos in the flavor basis belong to the same discrete group \( G_f \). That is, the product

\[
W_U \equiv S_U T
\]

\footnote{For an early \( A_4 \) model predicting a nearly degenerate neutrino spectrum, see [20].}
must also belong to \( G_f \). Furthermore, since \( G_f \) is finite, there must exist an integer \( p \) such that
\[
W_U^p = (S_UT)^p = \mathbb{I}. \tag{15}
\]
The relations
\[
S^n = T^m = W_U^p = \mathbb{I} \tag{16}
\]
form a presentation of \( G_f \) and define the von Dyck group \( D(n,m,p) \).

Eq. (15) is a constraint on the mixing matrix \( [14] \). To see this, notice that the eigenvalues of \( W_U \) are three \( p \)-th roots of unity, \( \lambda_1^{(p)} \), \( \lambda_2^{(p)} \) and \( \lambda_3^{(p)} \). Defining
\[
a \equiv \text{Tr}[W_U] = \lambda_1^{(p)} + \lambda_2^{(p)} + \lambda_3^{(p)}
\]
we have from Eq. (15) that
\[
\text{Tr}[U_{PMNS}SU_{PMNS}^\dagger T] \equiv \text{Tr}[W] = a. \tag{17}
\]

Since the \( p \)-th roots of unity are a finite set, the RHS of this equation takes values from a finite set of known complex numbers. For known \( a \), and given \( S \) and \( T \), Eq. (17) is a complex condition that the entries of \( U_{PMNS} \) must satisfy.

Although we will proceed below in all generality, the case analysis of \( n \), \( m \) and \( p \) is significantly reduced after the following consideration. It is a known fact \([14]\) that in order for the von Dyck group to be finite, one of \( n \), \( m \) or \( p \) must be equal to 2. In \([14,15]\) we took \( n = 2 \) consistent with \( G_\nu = \mathbb{Z}_2 \). However, in order to enforce degeneracy in the neutrino mass matrix, it must be \( n \geq 3 \). Assuming also that all charged leptons have different charges under \( T \), i.e. \( m \geq 3 \), we obtain that due to the finiteness of the group it must be necessarily \( p = 2 \).

Eq. (15) can then be written as
\[
W_U^2 = (U_{PMNS}SU_{PMNS}^\dagger T)^2 = \mathbb{I},
\]
and the eigenvalues of \( W_U \) must be equal to +1 or −1. Moreover, taking into account that \( \det[W_U] = 1 \), the eigenvalues of \( W_U \) must be \( \{1, -1, -1\} \), if \( W_U \) is not trivial. Hence, we obtain
\[
a = \text{Tr}[W_U] = \text{Tr}[U_{PMNS}SU_{PMNS}^\dagger T] = -1. \tag{18}
\]
The condition in Eq. (18) is appropriate when a residual symmetry in the neutrino sector forces the neutrino mass matrix to be of the form B or C.

In what follows we find explicitly the constraints imposed on \( U_{PMNS} \) and compare them with experimental data. For \( U_{PMNS} \) we will use the standard parametrization given by
\[
U_{PMNS} = U_{23}(\theta_{23}) \Gamma_3 U_{13}(\theta_{13}) \Gamma_3^* U_{12}(\theta_{12}) \Gamma_M =
\]
\[
\left(
\begin{array}{ccc}
-c_{12} c_{13} & s_{12} c_{13} & e^{-i \delta} s_{13} \\
-s_{12} c_{23} - e^{i \delta} c_{12} s_{23} s_{13} & c_{12} c_{23} - e^{i \delta} s_{12} s_{23} s_{13} & s_{23} c_{13} \\
 s_{12} s_{23} - e^{i \delta} c_{12} c_{23} s_{13} & c_{12} s_{23} - e^{i \delta} s_{12} c_{23} s_{13} & c_{23} c_{13}
\end{array}
\right) \Gamma_M, \tag{19}
\]
\[
\left(
\begin{array}{ccc}
 e^{i \delta} & 0 & 0 \\
 0 & e^{i \delta} & 0 \\
 0 & 0 & 1
\end{array}
\right)
\]
\[
\left(
\begin{array}{ccc}
 s_{23} c_{13} & c_{23} c_{13} & e^{-i \delta} s_{13} \\
 c_{23} s_{13} & -s_{23} c_{13} & e^{i \delta} s_{13} \\
 s_{23} s_{13} & -c_{23} s_{13} & c_{23} c_{13}
\end{array}
\right) \Gamma_M, \tag{20}
\]
where \( U_{ij} \) are the matrices of rotations in the \( ij \)-planes on the angles \( \theta_{ij} \),

\[
\Gamma_\delta \equiv \text{diag}\{1, 1, e^{i\delta}\}, \quad \Gamma_M \equiv \text{diag}\{1, e^{i\kappa}, e^{i\chi}\},
\]

and \( c_{12} \equiv \cos \theta_{12}, \ s_{12} \equiv \sin \theta_{12}, \ etc.. \)

3 Constraints on mixing for the partially degenerate spectrum

For partially degenerate spectrum, the neutrino mass matrix and the corresponding symmetry are given by Eqs. (10) and (11) respectively, with \( \zeta = \frac{2\pi l}{n} \). The matrix \( m_\nu \) can be a good lowest order approximation to both normal and inverted mass hierarchies. Corrections could then produce small splitting between the degenerate states and modify mixing angles when needed.

Setting \( S = S_\zeta \) we have from (18) the symmetry relation

\[
\text{Tr}[SU_T] = \text{Tr}[U_{PMNS}S\zeta U_{PMNS}^\dagger T] = -1.
\]

Explicit computation of \( SU \) (see Eq. (8)) gives

\[
(S_U)_{\alpha\alpha} = c_\zeta + 2s_{\zeta/2}|U_{a_3}|^2 + i2s_\zeta \text{Im}(U_{a_2}U_{a_1}^*) , \quad \text{(22)}
\]

where \( s_{\zeta/2} \equiv \sin(\zeta/2) \). It is convenient to introduce the real and imaginary parts of \( (SU)_{\alpha\alpha} = R_\alpha + iI_\alpha \):

\[
R_\alpha = c_\zeta + 2s_{\zeta/2}|U_{a_3}|^2, \quad I_\alpha = 2s_\zeta \text{Im}(U_{a_2}U_{a_1}^*). \quad \text{(23)}
\]

Notice that the index 3 in the real part of \( (SU)_{\alpha\alpha} \) is related to the non-degenerate third mass eigenstate. According to Eq. (17), the trace of \( W_U \) equals

\[
\text{Tr}(W_U) = (S_U)_{ee}e^{i\phi_e} + (S_U)_{\mu\mu}e^{i\phi_\mu} + (S_U)_{\tau\tau}e^{i\phi_\tau} = a,
\]

and consequently, from the real and imaginary parts of this equation we obtain using Eq. (23)

\[
\sum_\alpha (R_\alpha \cos \phi_\alpha - I_\alpha \sin \phi_\alpha) = \text{Re}[a], \quad \alpha = e, \mu, \tau , \quad \text{(24)}
\]

\[
\sum_\alpha (R_\alpha \sin \phi_\alpha - I_\alpha \cos \phi_\alpha) = \text{Im}[a], \quad \alpha = e, \mu, \tau , \quad \text{(25)}
\]

These are the conditions imposed on mixing by the symmetry in the case of partial degeneracy. Explicit equations for the mixing angles and phases in Eq. (20) can be found by substituting \( R_\alpha \) and \( I_\alpha \) from Eq. (23) in Eqs. (24) and (25).
As an example, we consider the case in which $\phi_e = 0$, $\phi_\mu = -\phi_\tau \equiv \psi$, so that the charged lepton transformation matrix has the form

$$T = \text{diag}\{1, e^{i \psi}, e^{-i \psi}\}.$$  \hspace{1cm} (26)

Eqs. (24) and (25) are then reduced to

$$R_e + c_\psi(R_\mu + R_\tau) + s_\psi(I_\tau - I_\mu) = \text{Re}[a],$$

$$I_e + c_\psi(I_\mu + I_\tau) + s_\psi(R_\mu - R_\tau) = \text{Im}[a],$$

and inserting Eq. (23) in (27) and (28) we obtain

$$|U_{e3}|^2 - 2x \text{Im}[U_{\mu2}U_{\mu1}^* - U_{\tau2}U_{\tau1}^*] + x^2 = \frac{1 + \text{Re}[a]}{4s_\psi^2 s_\zeta/2},$$  \hspace{1cm} (29)

$$2\text{Im}[U_{e2}U_{e1}^*] - y (|U_{\tau3}|^2 - |U_{\mu3}|^2) = \frac{\text{Im}[a]}{s_\zeta}.$$  \hspace{1cm} (30)

Here we have introduced parameters

$$x \equiv \cot \frac{\psi}{2} \cot \frac{\zeta}{2}, \quad y \equiv \cot \frac{\psi}{2} \tan \frac{\zeta}{2},$$ \hspace{1cm} (31)

which depend only on the group properties. From Eq. (31), $xy = \cot^2 \frac{\psi}{2} = \cot^2 \frac{\zeta}{2}$, and consequently, $x$ and $y$ should have the same sign.

Eqs. (29, 30) can be immediately generalized to the cases in which the lepton that has zero charge under $T$ is the muon or the tau. The general equations are

$$|U_{\alpha3}|^2 - 2x \text{Im}[U_{\beta1}U_{\beta2}^* - U_{\gamma1}U_{\gamma2}^*] + x^2 = \frac{1 + \text{Re}[a]}{4s_\psi^2 s_\zeta/2},$$  \hspace{1cm} (32)

$$2\text{Im}[U_{\alpha1}U_{\alpha2}^*] - y (|U_{\gamma3}|^2 - |U_{\beta3}|^2) = \frac{\text{Im}[a]}{s_\zeta}.$$  \hspace{1cm} (33)

Here, $(\alpha, \beta, \gamma)$ is $(e, \mu, \tau)$ or any other combination with a cyclic permutation of these flavor indices. Eqs. (32), (33) correspond to the case $\alpha = e$. Notice that Eqs. (32) and (33) represent yet another generalization of the results of [14] which can be reproduced by setting $\zeta = \pi$.

When there is partial degeneracy in the neutrino mass matrix, only $a = -1$ (see Eq. (18)) leads to finite groups. The explicit conditions on mixing imposed by the symmetry are then reduced to

$$|U_{\alpha3}| \mp x = 2x \left( \text{Im}[U_{\beta1}U_{\beta2}^* - U_{\gamma1}U_{\gamma2}^*] \mp |U_{\alpha3}| \right),$$

$$2\text{Im}[U_{\alpha1}U_{\alpha2}^*] = y (|U_{\gamma3}|^2 - |U_{\beta3}|^2).$$
The set of solutions of Eqs. (34, 35) is very restricted. In order to show this, we will use the standard parametrization Eq. (20) for $|U_{ai}|^2$ and consider for definiteness the case $\alpha = e$. Notice nonetheless that our results do not lose generality since for any choice of $\alpha$ there exists a parametrization such that the equations have the form to be discussed below.

We distinguish two cases: $x > 0$ and $x < 0$ which imply $y > 0$ and $y < 0$ respectively. In the standard parametrization and for $|U_{e3}| = \sin \theta_{13} \neq 0$, Eqs. (34, 35) can be written as

$$
\begin{align*}
\left(\sin \theta_{13} \mp x\right)^2 &= 2x (A_1 + A_2), \\
\sin 2\theta_{12} \sin \kappa &= y \cos 2\theta_{23}.
\end{align*}
$$

Here $\kappa$ is the Majorana phase defined in Eq. (20) and the minus (plus) sign corresponds to $x > 0$ ($< 0$). $A_1$ and $A_2$ are given by

$$
\begin{align*}
A_1 &= -\frac{1}{2}(1 + \sin^2 \theta_{13}) \cos 2\theta_{23} \sin 2\theta_{12} \sin \kappa, \\
A_2 &= \sin \theta_{13} \left(\cos \kappa \sin \delta - \cos \delta \sin \kappa \cos 2\theta_{12}\right) \sin 2\theta_{23} \mp 1.
\end{align*}
$$

Substituting $\cos 2\theta_{23}$ from Eq. (37) into Eq. (38) we obtain

$$
A_1 = -\frac{1}{2y}(1 + \sin^2 \theta_{13}) \sin^2 2\theta_{12} \sin^2 \kappa,
$$

so that $A_1 \leq 0$ for $x > 0$ and $A_1 \geq 0$ for $x < 0$ (recall that $x$ and $y$ have the same sign). Since

$$
|\cos \kappa \sin \delta - \cos \delta \sin \kappa \cos 2\theta_{12}| \leq 1,
$$

from Eq. (39) we have $A_2 \leq 0$ for $x > 0$ and $A_2 \geq 0$ for $x < 0$. We can combine these two results:

$$
\begin{align*}
A_i &\leq 0, & x > 0, \\
A_i &\geq 0, & x < 0, & (i = 1, 2).
\end{align*}
$$

Let us consider the case $x > 0$ and therefore $y > 0$. Since both $A_i$ are negative, see Eq. (41), the equality in Eq. (36) can be satisfied only if

$$
\sin \theta_{13} = x, \quad A_1 = 0, \quad A_2 = 0.
$$

Then, according to Eq. (40), there are two types of solutions of equation $A_1 = 0$:

1. $\kappa = 0$. In this case, we find from Eq. (39), $A_2 = \sin \theta_{13}(\sin \delta \sin 2\theta_{23} - 1) = 0$ which gives $\delta = \pi/2$ and $\theta_{23} = \pi/4$. There are no constraints on $\theta_{12}$.
2. $\theta_{12} = 0$. Then, from Eq. (39), we have $A_2 = \sin \theta_{13} [\sin(\delta - \kappa) \sin 2\theta_{23} - 1] = 0$ which is satisfied, if $\theta_{23} = \pi/4$ and $\delta - \kappa = \pi/2$. Now only one combination of the two phases is determined.

In particular, we stress that there exists a mixing matrix that satisfies the constraints for every value of $\theta_{12}$ which is in this sense undetermined.

Similarly one can consider the case of $x < 0$ which leads to $\sin \theta_{13} = -x$ and changes the signs of the phases. So, for both signs of $x$ the first class of solutions can be written in the following way:

$$\sin \theta_{13} = \pm x, \quad \theta_{23} = \frac{\pi}{4}, \quad \delta = \pm \frac{\pi}{2}, \quad \kappa = 0,$$

with $\theta_{12}$ and the second Majorana phase $\chi$ are undetermined. This type of solution can provide a good first approximation to the mixing matrix.

On the other hand, the second type is given by

$$\sin \theta_{13} = \pm x, \quad \theta_{23} = \frac{\pi}{4}, \quad \theta_{12} = 0, \quad \delta - \kappa = \pm \frac{\pi}{2}.$$  

Clearly the vanishing value of $\theta_{12}$ contradicts observation. However, since $m_1 = m_2$, small perturbations which produce splitting can also generate large 1-2 mixing (see Section 4).

There is simple interpretation of the two cases considered above in terms of the PMNS matrix in Eq. (24) and the neutrino mass matrix in the flavor basis, Eq. (2). If $\kappa = 0$, then $\Gamma_M = \text{diag}(1, 1, e^{i\chi})$, so that $U_{12}$ and $\Gamma_M$ commute. After permutation of these matrices in Eq. (2) $U_{12}$ turns out to be attached to the degenerate mass matrix. Consequently, it can be omitted and we obtain

$$U_{PMNS} = U_{23} \Gamma_8 U_{13} \Gamma_\delta^* \Gamma_M.$$  

Obviously, the same result Eq. (45) can be obtained for the second case when $\theta_{12} = 0$. Thus, the two solutions, we have found, correspond to the elimination of $U_{12}$ matrix.

The two types of solution can be represented in the form of immediate relations between the mixing parameters and mass ratios. In the first case, Eq. (43) we have

$$\sin^2 2\theta_{23} = \pm \sin \delta = \cos \kappa = \frac{m_1}{m_2};$$

and in the second one

$$\sin^2 2\theta_{23} = \cos 2\theta_{12} = \pm \sin(\delta - \kappa) = \frac{m_1}{m_2}.$$  

In both cases $\theta_{13}$ is not related to masses. Although the relations Eq. (46) and Eq. (47) are obtained post factum their appearance is not accidental: symmetry which led to
certain values of mixing angles and phases encodes information about masses (mass spectrum).

It is not hard to understand why $\theta_{12}$ should be undetermined in the solution of type 1. Indeed, for partially degenerate spectrum we have additional freedom to perform an arbitrary rotation in the 1-2 plane, $O_{12} = O_{12}(\theta)$. In this case the mixing matrix has general form

$$U_{PMNS} = U_{23} \Gamma_3 U_{13} \Gamma_3^* U_{12} \Gamma_M O_{12}. $$

If $\kappa = 0$, as is the case in the type 1 solution, then $\Gamma_M = \text{diag}\{1, 1, e^{i\chi}\}$, so that $\Gamma_M O_{12} = O_{12} \Gamma_M$. Consequently, the additional 1-2 rotation is reduced to

$$U_{12}(\theta_{12}) \rightarrow U_{12}(\theta_{12} + \theta),$$

where $\theta$ is arbitrary and therefore the 1-2 mixing is undetermined. On the other hand, for the type 2 solution $\kappa$ can be different from zero. Thus, the freedom to redefine $\theta_{12}$ no longer exists and it is natural that a precise value for $\theta_{12}$ is predicted, $\theta_{12} = 0$ in this case.

The solutions we obtained correspond to zero values of the left and right handed parts of Eqs. (34) and (35) separately. They can be written in a parameterization independent form as

$$|U_{\alpha 3}| = \pm x, \quad (48)$$

$$\text{Im} \left[ U_{\beta 1} U_{\gamma 2}^* - U_{\gamma 1} U_{\gamma 2}^* \right] = \pm x, \quad (49)$$

$$\text{Im} \left[ U_{\alpha 1} U_{\alpha 2}^* \right] = 0, \quad (50)$$

$$|U_{\beta 3}|^2 = |U_{\gamma 3}|^2, \quad (51)$$

being valid for any choice of flavor index $\alpha$. The relation Eq. (51) leaves only $\alpha = e$ for a plausible explanation of the experimental data.

Let us compute the group parameter $x$ which determines $\theta_{13}$ (see Eq. (43)). The combinations of numbers $(m, n) = (3, 3), (3, 4), (4, 3), (3, 5)$ and $(5, 3)$, which determine the angles $\psi = 2\pi k/m$ and $\zeta = 2\pi l/n$, exhaust the finite von Dyck groups. We find that the experimental value of $\theta_{13}$ is best approximated by choosing $(m, n) = (5, 3)$ or $(3, 5)$ which corresponds to the group $A_5$. In this case, for $k = 2, l = 1$ we obtain

$$\sin \theta_{13} = \cot \frac{\pi}{3} \cot \frac{2\pi}{5} = \sqrt{\frac{1}{3} \left(1 - \frac{2}{\sqrt{5}}\right)} \simeq 0.187. \quad (52)$$

This value, $\theta_{13} \approx 11^\circ$, is a good first approximation to the measured one [12].

As another example we take $(m, n) = (3, 3)$ with $k = 1$ and $l = 1$. This corresponds to an $A_4$ group and leads to

$$\sin \theta_{13} = \cot^2 \frac{\pi}{3} = \frac{1}{3} \quad (53)$$

which has interesting theoretical implications for the case of complete degeneracy, as we see in Sec. 5 despite being rather far from the experimental value.
4 Corrections to symmetry results

The mixing and mass splitting we have obtained in the previous sections as consequences of symmetry do not agree with experimental data. In particular, the 1-2 mass splitting is zero, the 1-2 mixing is zero or undefined and the 2-3 mixing is maximal which is somewhat disfavored by present data. In what follows we will show that actually, the obtained mass and mixing patterns can be considered as zero order structures. For this we prove that relatively small corrections to the neutrino mass matrix (and not to the mixing) can fix the problems listed above.

For definiteness we will consider the effect of a perturbation on the first solution, Eqs. (43), (45) - the second solution can be considered similarly. In this case the mixing matrix is given by

\[
U_{PMNS}^0 = U_{23}(45^\circ) \Gamma_{\pi/2} U_{13} \Gamma_{-\pi/2} = \frac{1}{\sqrt{2}} \begin{pmatrix}
\sqrt{2} c_{13} & 0 & -i \sqrt{2} s_{13} \\
-is_{13} & 1 & c_{13} \\
is_{13} & -1 & c_{13}
\end{pmatrix}.
\] (54)

In what follows, it will be convenient to consider the Majorana phase attached to the third mass eigenvalue. That is, we start with a zeroth order mass matrix:

\[
m_\nu^0 \equiv \text{diag}\{m, m, m'_{\chi}\}
\] (55)

with

\[
m'_{\chi} \equiv m' e^{-i2\chi}.
\]

Let us introduce a perturbation of Eq. (55):

\[
m_\nu = m_\nu^0 + \delta m_\nu,
\]

where \(\delta m_\nu\) is assumed to take a simple form

\[
\delta m_\nu \equiv \text{diag}(0, \epsilon, 0).
\]

This perturbation yields the 1-2 mass splitting

\[
\Delta m_{21}^2 = 2m\epsilon + \epsilon^2 \epsilon
\] (56)

and makes the 1-2 rotation physical. Using Eqs. (2) and (51) we obtain in the flavor basis:

\[
m_\nu = m_\nu^0 + \delta m_\nu,
\] (57)

where the zeroth order matrix \(m_\nu^0\)

\[
m_\nu^0 = U_{PMNS}^{0*} m_d U_{PMNS}^{0\dagger} = m \begin{pmatrix}
1 - s_{13}^2 (1 + r) & i \frac{1}{2} s_{13} c_{13} (1 + r) & i \frac{1}{2} s_{13} c_{13} (1 + r) \\
... & \frac{c_{13}^2}{2} (1 + r) & \frac{c_{13}^2}{2} (1 + r) - 1 \\
... & ... & \frac{c_{13}^2}{2} (1 + r)
\end{pmatrix}.
\] (58)
The matrix in Eq. (58) has the features that correspond to maximal $\theta_{23}$ and vanishing $\theta_{12}$, i.e., $m^0_{ee} = m^0_{e\tau}$ and $m^0_{\mu\mu} = m^0_{\mu\tau}$. The partial degeneracy is encoded in a more complicated relation between the elements: $m^0_{\mu\mu}(m^0_{ee} - m^0_{\mu\mu} + m^0_{\mu\tau}) = m^0_{e\tau}$. Violation of these equalities leads to generation of the 1-2 mixing and splitting as well as deviation of the 2-3 mixing from maximal.

The matrix of corrections $\delta m_\nu$ in Eq. (57) can be written as

$$\delta m_\nu = \epsilon V \times V^T,$$

where $V$ is the second column of the PMNS matrix:

$$V^T \equiv \left\{ s_{12} c_{13}, \frac{1}{\sqrt{2}} (c_{12} + i s_{12} s_{13}), \frac{1}{\sqrt{2}} (c_{12} - i s_{12} s_{13}) \right\}.$$

Here we left all the parameters unchanged except for the introduction of nonzero 1-2 mixing. From Eq. (56) we obtain

$$\epsilon = \sqrt{\Delta m^2_{21} + m^2 - m}.$$

In the case of strong normal mass hierarchy $m \ll \epsilon$ and $\epsilon = \sqrt{\Delta m^2_{21}}$. On the other hand, for strongly degenerate spectrum we have $\epsilon = \Delta m^2_{31}/2m$. The latter expression is also obtained in the case of strong inverted mass hierarchy when $m \approx \sqrt{\Delta m^2_{31}}$. In this case

$$\frac{\epsilon}{m} \approx \frac{\Delta m^2_{21}}{2 \Delta m^2_{31}} \approx 1.6 \cdot 10^{-2}.$$

Comparing the zeroth order values of the elements of the mass matrix, Eq. (58), with the corrections in Eq. (60) we arrive at the following conclusions:

1. For the $ee$-element, $\delta m_{ee}/m^0_{ee} \approx 2$ for the normal mass hierarchy when $m \ll \epsilon$. The ratio goes below 0.4 when $m^2 \geq \Delta m^2_{21}$.

2. For the off-diagonal elements in the case of normal mass hierarchy we obtain $\delta m_{e\mu}/m^0_{e\mu} \sim 1/2 s_{13}$, $\sqrt{\Delta m^2_{31}/\Delta m^2_{31}} \sim 0.4$ for $m \ll \epsilon$. If $m^2 \geq \Delta m^2_{21}$, the ratio is less than 0.15.

3. In the case of inverted mass hierarchy for $r \ll 1$ the corrections equal $\delta m_{ee}/m^0_{ee} \sim \epsilon/3m \approx 5 \cdot 10^{-3}$ and $\delta m_{e\mu}/m^0_{e\mu} = \epsilon/(m \sin 2 \theta_{13}) \approx 5 \cdot 10^{-2}$. The corrections for the elements of the $\mu - \tau$ block are of the order $\epsilon/m$. 

14
Thus, except for the $ee$-elements in the case of strong normal mass hierarchy the relative corrections to the mass matrix required to generate 1-2 mass splitting and 1-2 mixing are small: less than 0.2. At the same time, other parameters - the masses, 1-2 and 1-3 mixing and the CP-phase - can remain unchanged. The latter however implies correlations among the corrections to different elements of mass matrix which might be difficult to achieve.

If generic corrections of order $\epsilon \sim 0.2m$ are introduced in the mass matrix, all the mass and mixing parameters will be modified. Let us prove that these modifications can be small. For this we will take the simple perturbation matrix

$$
\delta m_{dem} = \frac{\epsilon}{3} \Gamma \pi D \Gamma \pi = \frac{\epsilon}{3} \left( \begin{array}{ccc}
1 & 1 & -1 \\
\ldots & 1 & -1 \\
\ldots & \ldots & 1
\end{array} \right),
$$

where $D$ is the democratic matrix with all elements being 1 and $\Gamma \pi = \text{diag}(1, 1, -1)$.5

Let us compute the masses and mixing parameters for $m'_\nu = m_\nu^0 + \delta m_{dem}$, where $m_\nu^0$ is given in Eq. (58). For the mass eigenvalues $m'_i$ of $m'_\nu$, and neglecting contributions of order $s^2_{13}$, we obtain:

$$
m'_1 = m, \quad m'_2 = m + \frac{1}{3}(2 + c^2_{13}) \simeq m + \epsilon, \quad m'_3 \simeq m'.
$$

In order to find the corresponding mixing angles, we first make the zeroth order rotation in Eq. (15) which yields

$$
U_{PMNS}^{0T}(m_\nu^0 + \delta m_{dem})U_{PMNS}^{0} = \frac{\epsilon}{3} \left( \begin{array}{ccc}
c^2_{13} + 3m/\epsilon & \sqrt{2} c_{13} & -is_{13}c_{13} \\
\ldots & 2 + 3m/\epsilon & -i\sqrt{2}s_{13} \\
\ldots & \ldots & -s^2_{13} + 3mr/\epsilon
\end{array} \right).
$$

The matrix above can be subsequently diagonalized by a rotation

$$
U' = \Gamma_{\pi/2} U'_{13} U'_{23} U'_{12} \Gamma_{-\pi/2}.
$$

Up to order $s^2_{13}$ corrections and other small angles corrections, this gives $\sin^2 \theta'_{12} \approx 1/3$ in good agreement with data. Furthermore, if we assume for simplicity that $\chi = 0$ so that $r$ is real, and multiply Eq. (64) by $\Gamma_{\pi/2}$ which follows from Eq. (65) we obtain

$$
\tan \theta'_{13} \approx -\frac{\epsilon}{3m'} s_{13} c_{13} \leq s_{13} \frac{1}{3} \sqrt{\frac{\Delta m^2_{31}}{2\Delta m^2_{31}}} \simeq 0.05 s_{13},
$$

and

$$
\tan \theta'_{23} \approx -\frac{2s_{13}}{\sqrt{2}} \frac{\epsilon}{3m'} \simeq 0.02 \frac{\epsilon}{\sqrt{2}},
$$

5This matrix is close to the correction matrix in Eq. (60) and can be motivated by symmetry arguments.
i.e. less than 2%. Thus, the PMNS matrix including corrections can be written as

\[
U_{PMNS} = U_{PMNS}' = U_{23}(45^\circ)\Gamma_{\pi/2}U_{13}(\theta_{13} + \theta'_{13})U_{23}'U_{12}'\Gamma_{-\pi/2}
\]

\[
= U_{0}^{PMNS}(\theta_{13} + \theta'_{13})\Gamma_{\pi/2}U_{23}'U_{12}'\Gamma_{-\pi/2}.
\]  

The PMNS matrix is determined by Eq. (66) up to a phase matrix which can be attached from the right and we will use this to reduce the Eq. (66) to standard parametrization form. We can now compute the elements of the matrix in Eq. (66) explicitly and identify them with the elements of the mixing matrix in the standard parameterization (subscripts s). The \(e2\)-element equals \(c_{13}s_{12}^{s} = c_{13}s_{12}' - s_{13}c_{12}s_{23}'\), that is, the correction to the equality \(\theta_{12}' = \theta_{12}\) is of the order \(s_{13}s_{23}'\). In order to determine other angles and the Dirac CP phase it is enough to consider the third column of Eq. (66):

\[
(U_{PMNS})_{\alpha3} = (\rho e^{-i\pi/2}, \rho e^{i\xi}, \rho e^{-i\xi}),
\]  

where

\[
\rho = \frac{1}{\sqrt{2}}\sqrt{c_{13}^{2}c_{23}^{2} + s_{23}^{2}}, \quad \tan \xi = -\frac{1}{c_{13}}\tan \theta_{23}'
\]  

or \(\xi \approx -\theta_{23}'\). The phase of the \(U_{\mu3}\)-element in Eq. (67) can be removed (as it should be in the standard parametrization) by acting on the right hand side of Eq. (66) with the additional phase matrix \(\text{diag}(1, 1, e^{-i\xi})\). This means that the CP-phase is modified to \(\delta = \pi/2 + \xi\).

Thus, we have shown that a simple correction matrix can generate an acceptable 1-2 mixing, the required mass splitting and produces only small (few per cent) corrections to the other mixings and to the CP-violation phase.

5 Constraints on mixing for the completely degenerate spectrum

As we remarked in Sec. 2, \(M_{\nu U}\) can be forced to be completely degenerate, if a non-abelian discrete subgroup of \(O(3)\) with three-dimensional representations is imposed as \(G_\nu\). The possible groups are thus restricted to \(A_4, S_4\) and \(A_5\). These can be generated by two matrices: \(S_\zeta\) and \(P\) that satisfy

\[
S_\zeta^u = P^2 = (S_\zeta P)^r = I.
\]  

We take a basis for the neutrinos such that \(S_\zeta\) is given by Eq. (11). The second matrix, \(P\), can be represented as

\[
P = O^T P_D O,
\]

where

\[
P_D = \text{diag}\{1, -1, -1\},
\]
and \( O = O(\phi_{12}, \phi_{13}, \phi_{23}) \) is a generic orthogonal matrix of rotations on the angles \( \phi_{ij} \).

In the charged lepton sector we take, as before, \( G_\ell = Z_m \). The generator \( T \) must now satisfy conditions like Eq. (15) with both \( S_U \) and \( P_U = U_{PMNS}P^U_{PMNS} \). Hence, the complete presentation for the flavor group \( G_f \) is given by

\[
S_U^m = T^m = P_U^2 = 1, \quad (S_U P_U)^r = (S_\zeta P)^r = 1,
\]

\[
(S_U T)^2 = (P_U T)^q = 1.
\]

Notice that this presentation does not guarantee that \( G_f \) is finite. Following the same argument exploited in case B, we obtain that Eqs. (72, 73) impose a set of conditions on matrices \( U_{PMNS} \) and \( O \):

\[
\text{Tr}[U_{PMNS} S_\zeta U^T_{PMNS} T] = -1, \tag{74}
\]

\[
\text{Tr}[O S_\zeta O^T P_D] = a_r, \tag{75}
\]

\[
\text{Tr}[U_{PMNS} O^T P_D O U^T_{PMNS} T] = a_q, \tag{76}
\]

where \( a_r(a_q) \) is the sum of three \( r \)-th (\( q \)-th) roots of unity. The solutions of Eq. (74), which coincides with condition of the previous case, are given in Eqs. (43), (44). Two other equations are new: Eq. (75) is the one for the matrix \( O \), instead of \( U_{PMNS} \), and it can be solved in a similar way. Using a parametrization for matrix \( O \) similar to Eq. (20) with vanishing CP phases we get

\[
\sin^2 \phi_{13} = \frac{a_r + 1}{2(1 + \cos \zeta)}, \tag{77}
\]

where \( \phi_{13} \) is the angle in \( O \) equivalent to \( \theta_{13} \) in Eq. (20). Substituting (43) and (77) in Eq. (75), we obtain a new equation for the remaining parameters of \( U_{PMNS} \) that either has no solution - and the group representation in question does not exist - or fixes the Majorana phase \( \chi \).

For the values of the parameters in Eqs. (43) and (52) the Eq. (76) has no solutions. For the pattern with the 1-3 mixing from Eq. (53) the Eq. (76) does have a solution if \( r = 3 \) and \( q = 3 \). that for \( r = 3 \) the group \( G_\nu = A_4 \). We obtain for the second Majorana phase, \( \chi \):

\[
\chi = \frac{3\pi}{2}. \tag{78}
\]

A few comments are in order. It is easy to check that \( T \) can be written as a combination of \( P_U \) and \( S_U \), so that it is not an independent generator. Since \( G_f = A_4 \), this theory corresponds to a case in which the flavor group \( G_f \) remains unbroken in the neutrino sector while it is broken to a \( Z_3 \) subgroup in the charged lepton sector.

Out of six parameters that appear in \( U_{PMNS} \) three are unphysical in the fully degenerate case [21]. This seems to be in contradiction with the fact that we have determined five parameters \( \{\theta_{13}, \theta_{23}, \delta, \kappa, \chi\} \) by means of the symmetry. Actually, some of these
parameters have been fixed, not by the symmetry but by our choice of basis. Indeed, in order to perform the analysis, we assumed that the group $G_f = A_4$ included the 1-2 rotation $S_\xi$ as one of the generators. However, for fully degenerate neutrinos, rotations around any axis could serve as symmetries of the neutrino mass matrix. Hence, if $S_U$, $P_U$, and $T$ satisfy Eqs. (74) and (75) for some $U_{PMNS}$, then also $T$ and the new matrices $S'_U$, $P'_U$ defined as

$$S'_U = V S_U V^T, \quad P'_U = V P_U V^T$$

satisfy Eqs. (74) and (75) for a mixing matrix $U'_{PMNS}$ given by

$$U'_{PMNS} = U_{PMNS} V^T.$$  \hfill (79)

Here $V$ is any orthogonal matrix. Thus, the mixing parameters found are written in basis-dependent form. One can only say that there exists a basis in which the $U_{PMNS}$ angles and phases have the values in Eqs. (13) and (78). In general, according to Eq. (80), three parameters are unphysical out of the six that appear in $U_{PMNS}$ in the standard parametrization.

The basis-independent physical quantities are combinations of the elements of $U_{PMNS}$ that are invariant under orthogonal rotations of the neutrino fields and the usual phase redefinitions of leptons. These functions are nothing else but the absolute values of the elements of the matrix $U = U_{PMNS}^T U_{PMNS}$. It is easy to see that since $U$ is symmetric and unitary, only 3 out of the 9 elements $|U_{ij}|$ are independent as expected according to the analysis above. Furthermore, the matrix $U$ is proportional to the mass matrix in the flavor basis which has physical meaning, e.g. its $ee$-element determines the amplitude on neutrinoless double-beta decay.

6 Conclusions

In this paper, we further developed the formalism of the “symmetry building” in such a way that it includes both mixing parameters and neutrino masses. More precisely, the formalism connects partially and completely degenerate neutrino spectra with the mixing angles and CP-phases. These are the only possibilities (along with zero mass) which can be obtained as consequences of the unitary residual symmetries.

The case of partial degeneracy, $m_1 = m_2$, follows when a $\mathbf{Z}_n$ subgroup of $SO(2)$ with $n \geq 3$ is preserved in the neutrino sector. It can be a good lowest order approximation to the spectrum of normal (inverted) mass hierarchy. This case is very restrictive, leading to 4 conditions on the mixing parameters. For $m_1 = m_2$ we have found two types of solutions with 4 mixing parameters fixed. Both solutions show maximal 2-3 mixing and 1-3 mixing determined directly by the group parameters. They differ by the values of the 1-2 mixing and CP-violation phases. The first solution has zero $\theta_{12}$, and one condition on the phases: $\delta - \kappa = \pi/2$. In the second solution, $\theta_{12}$ is undefined but both phases
are fixed: $\delta = \pi/2$ and $\kappa = 0$. In the case that gives the best approximation to the measured values, the symmetry group is $A_5$ and we obtain $\sin \theta_{13} = 0.187$.

These solutions should be considered as a lowest order approximation. Relatively small corrections can produce the mass splitting and fix $\theta_{12}$ in one case and generate $\theta_{12}$ in another. Corrections may also give better agreement of the 1-3 and 2-3 mixings with observations. We show that in the first case the corrections proportional to the “democratic” matrix can produce the 1-2 mass splitting and mixing in agreement with observations while giving rise to very small corrections to the other mixing parameters and CP-phases.

A completely degenerate spectrum is achieved if the residual symmetry in the neutrino sector is either $A_4$, $S_4$ or $A_5$. In this case, $U_{PMNS}$ has only 3 physical parameters all of which are determined by the symmetry. In our formalism, this is made explicit by the fact that, in a particular basis, all the angles and CP-phases of the mixing matrix - except for $\theta_{12}$ which remains undefined - are fixed.

The values of the charged lepton masses are not involved in this consideration. In fact, the inclusion of charged leptons may produce corrections which will make the scheme with degeneracy to be viable. At the same time, it will be probably difficult to immediately extend this consideration to the quark sector and treat two light families as being degenerate in the first approximation.

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