UNFREE GAUGE SYMMETRY IN THE HAMILTONIAN FORMALISM

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Abstract. The constrained Hamiltonian formalism is worked out for the theories where the gauge symmetry parameters are unfree, being restricted by differential equations. The Hamiltonian BFV-BRST embedding is elaborated for this class of gauge theories. The general formalism is exemplified by the linearized unimodular gravity.

1. Introduction

If the gauge variation of action identically vanishes under the condition that the gauge parameters obey differential equations, the gauge symmetry is said unfree. The general structure of unfree gauge symmetry algebra has been recently established in ref [1]. The extension of the BV (Batalin-Vilkovisky) formalism to unfree gauge symmetry is proposed in [2].

The most known example of unfree gauge symmetry is provided by unimodular gravity where the diffeomorphism parameters $\epsilon^\mu$ are constrained by transversality condition $\nabla_\mu \epsilon^\mu = 0$. For discussion of consequences of transversality condition in the unimodular gravity and various extensions, see [3], [4], [5], [6], [7], [8], [9], [10], [11] and references therein. More examples of unfree gauge symmetry can be found among multifarious higher spin field theories, see [12], [13], [14], [15].

While the phenomenon of unfree gauge symmetry is well-known in terms of Lagrangian formalism, it has been so far unclear how the unfree symmetry reveals in the corresponding Hamiltonian formalism. Even in well-studied models, like unimodular gravity, the transversality condition on the diffeomorphism parameters is not evident from the viewpoint of the Poisson algebra of Hamiltonian constraints. This problem is noticed in the literature, see [7] and references therein.

In this article, we work out the general Hamiltonian description of unfree gauge symmetry. In section 2, we list the basic features of Lagrangian description of general unfree gauge algebra as this is essential for constructing the Hamiltonian analogue. In section 3, we establish the general structure of unfree gauge symmetry in the constrained Hamiltonian formalism; in section 4, we construct corresponding Hamiltonian BRST (Becchi-Rouet-Stora-Tyutin) complex; the section 5 exemplifies the general formalism by the model of linearized unimodular gravity.
2. Generalities of unfree gauge symmetry in Lagrangian formalism

In the reference [1], it is noticed that the unfree gauge algebra is generated by four key ingredients: the action functional $S$; the generators of unfree gauge symmetry $\Gamma^i_\alpha$; the mass-shell completion functions $\tau_a$; the operators of gauge parameter constraints $\Gamma^a_\alpha$. The first two generating structures – the action, and the gauge symmetry generators – are the key ingredients of any gauge symmetry algebra, be the gauge parameters constrained, or not. The other two generating elements, $\tau_a$ and $\Gamma^a_\alpha$, are special for the unfree gauge symmetry. Let us non-rigorously explain their role in the dynamics, for a more systematic exposition we refer to [1], [2].

The main distinctive feature of theories with the unfree gauge symmetry is that the local quantities exist such that vanish on-shell while they are not spanned by the l.h.s. of Lagrangian equations (EoM’s). In other words, the generating set for the ideal of on-shell vanishing local functions is not exhausted by the l.h.s. of EoM’s, it includes some other quantities, denoted as $\tau_a$: 

$$T(\phi) \approx 0 \iff T(\phi) = \theta^i(\phi) \partial_i S(\phi) + \theta^a(\phi) \tau_a(\phi),$$

(1)

where $\approx$ means on-shell equality, and $\theta(\phi)$ are local. Here we use the DeWitt condensed notation.

The local quantities $\tau_a(\phi)$ are supposed independent, and they do not reduce to a linear combination of Lagrangian equations, $\tau_a \neq \theta^i \partial_i S$. The quantities $\tau_a$ are called the mass-shell completion functions. Examples of the completion functions can be found in [1], [2]. In general, modified Noether identities involve both Lagrangian equations and completion functions:

$$\Gamma^i_\alpha(\phi) \partial_i S(\phi) + \Gamma^a_\alpha(\phi) \tau_a(\phi) \equiv 0.\quad (2)$$

With appropriate regularity assumptions (see in [1], [2]), relations (1), (2) define the unfree gauge symmetry of the theory. In particular, the gauge variation of the fields,

$$\delta_\epsilon \phi^i = \Gamma^i_\alpha(\phi) \epsilon^\alpha,$$

(3)

is a symmetry of the action provided that the gauge parameters $\epsilon$ obey the equations

$$\Gamma^a_\alpha(\phi) \epsilon^\alpha = 0.\quad (4)$$

The operators of gauge parameter constraints $\Gamma^a_\alpha(\phi)$ are supposed independent. This means that the kernel of $\Gamma^a_\alpha(\phi)$ is, at maximum, finite dimensional,

$$\Gamma^a_\alpha(\phi) u_a = 0 \implies u_a \in M = \text{Ker} \Gamma^a, \quad \text{dim} M = n \in \mathbb{N}.$$

(5)
Here, $M$ is understood as a moduli space of the field theory. Given (2), (5), on shell $\tau_a \approx \Lambda_a, \Lambda_a \in M$. In principle, the modular parameters $\Lambda_a$ can be included into the definition of $\tau_a$, so the completion functions can be considered on-shell vanishing without loss of generality. Also notice that the relations $\tau_a \approx 0$ hold true for any solution of the EoM’s with corresponding modular parameter, though these relations are not differential consequences of EoM’s.

The modified Noether identities (2) along with corresponding regularity assumptions lead to the compatibility conditions involving higher structures, which define the full unfree gauge symmetry algebra. This algebra is more general than the one with unconstrained gauge parameters. Corresponding gauge formalism is worked out in references \[1\], \[2\], we do not address it here, listing only the basic relations.

**Example.** Let us illustrate relations (2), (4), (5) in uncondensed notation by an example. Consider the theory of fields $\phi^i(x)$, where $i$ is a discrete index (it can be spinorial, tensorial, isotopic index), and $x^\mu$ are coordinates of space-time. Suppose the differential consequences of EoM’s can be linearly combined into the gradient of local quantity $\tau$, being the scalar function of fields and their first derivatives with respect to $x^\mu$:

$$\hat{\Gamma}_\mu^i(\phi, \partial \phi) \frac{\delta S[\phi]}{\delta \phi^i} + \partial_\mu \tau(\phi, \partial \phi) \equiv 0, \quad \tau(0, 0) \equiv 0,$$

where $\hat{\Gamma}_\mu^i$ is a matrix whose entries are linear differential operators with the field-depending coefficients. As the derivatives of $\tau$ vanish on-shell, it is an on-shell constant, $\tau(\phi, \partial \phi) \approx \Lambda = \text{const}$. The specific value of the constant $\Lambda$ is determined by the boundary conditions or asymptotics of fields, not by initial data. For example, if the fields are supposed vanishing at spacial infinity of the space-time, then $\Lambda = 0$ irrespectively to the initial data, as $\tau(0, 0) = 0$. So $\Lambda$ should be understood as a modular parameter rather than an integral of motion. This modular parameter can be included into definition of the completion function, so one can set $\tau \approx 0$ without restricting generality. Once the modified Noether identity (2) reads as (6) in this example, the gauge parameter should be the vector field $\epsilon^\mu(x)$. The gauge parameter constraint operator $\Gamma_\alpha^a$ of (2) is identified by (6) as $\partial_\mu$, so the equation (4) reduces to the transversality condition $\partial_\mu \epsilon^\mu = 0$ for the gauge parameter $\epsilon^\mu$. The unfree gauge transformations (3) are generated by the conjugate to the operator $\hat{\Gamma}$ involved in the modified Noether identity (2):

$$\delta_\epsilon \phi^i = \hat{\Gamma}_\mu^i \epsilon^\mu, \quad \partial_\mu \epsilon^\mu \equiv 0.$$
Given the modified Noether identity (6), the transformation (7) leaves the action invariant:

$$\delta \epsilon \approx \int dx \, \delta S[\phi] \approx \int dx \, \epsilon^\mu \hat{\Gamma}^i_{\mu}(\phi, \partial \phi) \frac{\delta S[\phi]}{\delta \phi^i} \approx \int dx \, (\partial_\mu \epsilon^\mu) \tau(\phi, \partial \phi) \approx 0.$$  

(8)

The unimodular gravity is covered by this example. The role of the modular parameter is played by the cosmological constant, while the completion function is the scalar curvature. Unfree gauge symmetries of some higher spin field theories, see [12], [13], [14], [15], also follow the pattern of this example, though the completion functions are tensors in these models, not scalars.

### 3. Unfree gauge symmetry transformations in the Hamiltonian formalism

Consider the action of Hamiltonian theory with primary constrains,

$$S = \int dt \left( p_i \dot{q}^i - H_T(q, p, \lambda) \right), \quad H_T(q, p, \lambda) = H(q, p) + \lambda^\alpha T_\alpha(q, p).$$  

(9)

The role of fields here is played by canonical variables $q^i, p_i$, and Lagrange multipliers $\lambda^\alpha$. In the previous section, the structures are described that define the unfree gauge symmetry for the general action. In this section, we detail these structures for the specific action (9) and find thereby the Hamiltonian form of unfree gauge symmetry transformations.

For the action (9), the EoM’s read

$$\frac{\delta S}{\delta p_i} \equiv \dot{q}^i - \{ q^i, H_T \} = 0, \quad \frac{\delta S}{\delta q^i} \equiv -\dot{p}_i + \{ p_i, H_T \} = 0; \quad \frac{\delta S}{\delta \lambda^\alpha} \equiv -T_\alpha(q, p) = 0.$$  

(10)

(11)

The constraints $T_\alpha(q, p)$ are supposed irreducible. At this point we accept an auxiliary assumption that the differential consequences of the equations do not fix $\lambda^\alpha$ as functions of $q, p$. Assuming that the Dirac conjecture holds true for theory (9), this means there are no second-class constraints.

Now, our primary objective is to identify completion functions (1) and gauge identities (2) for EoM’s (10), (11). We begin with applying the Dirac-Bergmann algorithm to equations (10), (11):

$$\dot{T}_\alpha \approx \{ T_\alpha, H_T \} \approx 0.$$  

(12)

Once the Lagrange multipliers are not defined by conservation of the primary constraints, the r.h.s. of the above relation should be a linear combination of the primary constraints and the secondary ones. Let us assume that the irreducible generating set $\{ \tau_\alpha(q, p) \}$ can be chosen for the
secondary constraints. This means the local differential operators $\Gamma, W$ exist such that
\[
\{ T_\alpha(q,p), H_T(q,p,\lambda) \} = W^\beta_\alpha(q,p,\lambda) T_\beta(q,p) + \Gamma^a_\alpha(q,p,\lambda) \tau_a(q,p).
\]
The irreducibility assumption of the secondary constraints $\tau_a$ is twofold. First, all the constraints should be independent:
\[
\theta^\alpha T_\alpha + \theta^a \tau_a = 0 \quad \Leftrightarrow \quad \theta^\alpha = A^{\alpha\beta} T_\beta + A^{\alpha a} \tau_a, \quad \theta^a = A^{ab} \tau_b - A^{\alpha a} T_\alpha,
\]
where $A^{ab} = -A^{ba}$, $A^{\alpha\beta} = -A^{\beta\alpha}$. Second, the kernel of the operator $\Gamma^a_\alpha$ should be, at most, finite dimensional, in the sense of relation (5). The only difference with the Lagrangian formalism is that $\Gamma$ in (13) involves derivatives only by space coordinates, while the Lagrangian counterpart can differentiate also by time.

If $\Gamma^a_\alpha$ admitted the dual differential operator $\tilde{\Gamma}^a_\alpha$ such that
\[
\tilde{\Gamma}^a_\alpha \Gamma^b_\alpha = \delta^b_a,
\]
then the secondary constraints $\tau_a$ would be the differential consequences of the original equations (10), (11). In the opposite case, $\tau_a$ reduce to the element of the kernel of the differential operator $\Gamma$. In this case $\tau$ are considered as completion functions, and hence the gauge symmetry should be unfree. Once the kernel is finite of $\Gamma^a_\alpha$, completion functions $\tau_a(q,p)$ can be redefined by adding modular parameters $\Lambda_a$ to make $\tau_a$ vanishing on shell:
\[
\Gamma^a_\alpha \tau_a = 0 \quad \Leftrightarrow \quad \tau_a = \Lambda_a, \quad \Lambda_a \in \text{Ker} \Gamma^a_\alpha; \quad \tau_a \mapsto \tau_a - \Lambda_a.
\]

Then $\tau_a$ still vanish on-shell and viewed as secondary constraints, though they are not differential consequences of the EoM’s, and hence the gauge symmetry should be unfree.

The next simplifying assumption is that no tertiary constraints appear. This means that the time derivatives of secondary constraints reduce on-shell to the combinations of themselves and the primary ones:
\[
\dot{\tau}_a \approx \{ \tau_a, H_T \} = W^\alpha_\alpha(q,p,\lambda) T_\alpha + \Gamma^b_\alpha(q,p,\lambda) \tau_b.
\]
Off shell, the time derivatives of primary and secondary constraints identically reduce to the linear combination of constraints and EoM’s (10):
\[
\{ T_\alpha, q^j \} \frac{\delta S}{\delta q^j} + \{ T_\alpha, p_j \} \frac{\delta S}{\delta p_j} + \left( \frac{\delta^\beta}{\delta \tau^\beta} - W^\beta_\alpha(q,p,\lambda) \right) \frac{\delta S}{\delta \lambda^\beta} + \Gamma^a_\alpha(q,p,\lambda) \tau_a \equiv 0;
\]
\[
\{ T_a, q^j \} \frac{\delta S}{\delta q^j} + \{ T_a, p_j \} \frac{\delta S}{\delta p_j} + \left( \frac{\delta^\beta}{\delta \tau^\beta} - W^\beta_\alpha(q,p,\lambda) \right) \frac{\delta S}{\delta \lambda^\beta} + \Gamma^a_\alpha(q,p,\lambda) \tau_a \equiv 0.
\]
Since the secondary constraints \( \tau_a \) are not the differential consequences of the primary ones (11), the above relations are modified gauge identities (2) rather than usual Noether identities between the variational equations. The identities (2) are equivalent to the unfree gauge symmetry (3) of the action, with the gauge parameters constrained by the equations (4). Given the gauge identities (18), (19), the unfree gauge transformations (3) for constrained Hamiltonian system read

\[
\delta_t O(q, p) = \{ O, T_a \} \epsilon^a + \{ O, \tau_a \} \epsilon^a, \quad \delta_t \lambda^a = \dot{\epsilon}^a + W_\beta^a(q, p, \lambda) \epsilon^\beta + W_a^a(q, p, \lambda) \epsilon^a. \quad (20)
\]

The constraints on the gauge parameters (4) are defined by the coefficients at \( \tau_a \) in the modified gauge identities (2). Given specific identities (18), (19), the constraints on gauge parameters read:

\[
\left( \delta_a^b \frac{d}{dt} + \Gamma_a^b(q, p, \lambda) \right) \epsilon^a + \Gamma_a^b(q, p, \lambda) \epsilon^a = 0. \quad (21)
\]

The unfree gauge transformations (20), (21) have been deduced above by using the gauge identities (18), (19) for the theory (9) with the involution relations (13), (17). By direct variation, one can verify that the action (9) is indeed invariant under the unfree gauge transformations (20), (21):

\[
\delta_t S = \int dt \left( - \delta_t q^i (\dot{p}_i - \{ p_i, H_T \}) + \delta_t p_i (\dot{q}^i - \{ q^i, H_T \}) - \delta_t \lambda^a T_a \right)
\equiv \int dt \left( - \dot{T}_a \epsilon^a - \dot{\tau}_a \epsilon^a + \{ T_a, H_T \} \epsilon^a + \{ \tau_a, H_T \} \dot{\epsilon}^a - (\dot{\epsilon}^a + W_\beta^a(q, p, \lambda) \epsilon^\beta + W_a^a(q, p, \lambda) \epsilon^a) T_a \right).
\]

Upon substitution \( \{ T_a, H_T \}, \{ \tau_a, H_T \} \) from relations (13), (17), the variation reads

\[
\delta_t S = \int dt \left( (\dot{\epsilon}^a + \Gamma_a^b(q, p, \lambda) \epsilon^b + \Gamma_a^a(q, p, \lambda) \epsilon^a) \right) \tau_a - \frac{d}{dt} \left( T_a \epsilon^a + \tau_a \epsilon^a \right). \quad (22)
\]

Once the gauge parameters obey equations (21), the integrand reduces to the total derivative, so the action is indeed invariant under the unfree gauge variation (20), (21).

Let us discuss the constraints imposed on the gauge parameters \( \epsilon^a \) and \( \epsilon^a \) by equations (21). Equations (21) define \( \dot{\epsilon}^a \) in terms of \( \epsilon^a \). As the kernel of \( \Gamma_a^a \) is at maximum finite in the sense of relation (5), the time evolution of \( \epsilon^a \) is completely controlled by \( \epsilon^a \), while the latter parameters are unconstrained by the equations. As the equations (21) have the structure \( \dot{\epsilon}^a = f^a(\epsilon^b, \epsilon^c) \), they admit any initial data for \( \epsilon^a \), so these parameters are arbitrary at initial moment.

Alternatively, equations (21) can be considered as constraints imposed on the parameters \( \epsilon^a \), defining some of them in terms of the rest of \( \epsilon^a \) and \( \dot{\epsilon}^a, \epsilon^a \). If all the constraints (21) are explicitly resolved by excluding some of the gauge parameters \( \epsilon^a \), then the gauge transformations of canonical
variables (20) will include \( \dot{\epsilon}^a \), while the variation of \( \lambda^\alpha \) will involve \( \ddot{\epsilon}^a \). In this way, the first-order unfree gauge symmetry (20), (21) is replaced by the second-order gauge symmetry with unconstrained gauge parameters. If the spacial locality is not an issue, the constrained Hamiltonian equations (10), (11) always admit the unconstrained parametrization of gauge transformations with higher order time derivatives of gauge parameters [16]. Also notice that any linear system of local field equations admits unconstrained local parametrization of gauge symmetry, possibly with higher derivatives, though the transformations can be reducible [14]. So, all these facts lead to the conjecture that the unfree gauge symmetry can be always equivalently replaced by local higher order reducible gauge symmetry. This conjecture will be addressed elsewhere.

Now, let us discuss the issue of on-shell gauge invariants. The long-known wisdom of Hamiltonian constrained dynamics about the gauge invariants is that they should Poisson-commute on shell with all first-class constraints, both primary and secondary [17]. While unfree gauge transformations (20), (21) have been previously unknown, the gauge invariants turn out defined in the same way as with unconstrained gauge parameters. This fact can be seen from the above mentioned properties of equations (21). Let us explain that. Once any initial data for gauge parameters \( \epsilon^\alpha, \epsilon^a \) are admitted by the equations (21), the phase-space function \( O(q,p) \) cannot be invariant under the gauge transformation (20) unless it Poisson-commutes with primary and secondary constraints:

\[
\delta_c O(q,p) \approx 0 \iff \{O, T_\alpha\} \approx 0, \quad \{O, \tau_a\} \approx 0.
\]  

(23)

Also notice that the Lagrange multipliers cannot contribute to the on-shell invariants, as \( \delta_c \lambda^\alpha \) begins with \( \dot{\epsilon}^a \) (20), while the parameters \( \epsilon^\alpha \) are not constrained by equations (21).

Let us now detail involution relations (13), (17). As \( H_T = H(q,p) + \lambda^\alpha T_\alpha(q,p) \), the structure functions \( W(q,p,\lambda), \Gamma(q,p,\lambda) \) in (13), (17) are at most linear in \( \lambda^\alpha \):

\[
W_\beta^\alpha(q,p,\lambda) = V_\beta^\alpha(q,p) + U_{\alpha\gamma}^\beta(q,p)\lambda^\gamma, \quad \Gamma_\alpha(q,p,\lambda) = V_\alpha^\alpha(q,p) + U_{\alpha\gamma}^\alpha(q,p)\lambda^\gamma; \quad (24)
\]

\[
W_\alpha^a(q,p,\lambda) = V_\alpha^a(q,p) - U_{\gamma a}^\alpha(q,p)\lambda^\gamma, \quad \Gamma_\alpha(q,p,\lambda) = V_\alpha^b(q,p) - U_{\gamma a}^b(q,p)\lambda^\gamma. \quad (25)
\]

By introducing uniform notation for primary and secondary constraints \( T_A = (T_\alpha, \tau_a) \), \( A = (\alpha, a) \), and accounting for (24), (25), the involution relations (13), (17) are brought to the following form:

\[
\{T_A(q,p), H(q,p)\} = V_A^B(q,p)T_A(q,p), \quad \{T_A(q,p), T_B(q,p)\} = U_{AB}^C(q,p)T_C(q,p). \quad (26)
\]
The above involution relations include both primary and secondary constraints on an equal footing and merely correspond to a general first-class system. These relations, per se, do not reveal any indication of the equations imposed on the gauge parameters $\tau_a$. At the level of action (9), however, the differences exist as the primary constraints are included into the action with the Lagrange multipliers, while the secondary ones are not. It is the asymmetry which leads to equations on gauge parameters (21). With this regard, we mention the long-known idea that the secondary first-class constraints $\tau_a$ can be included into the action with their own Lagrange multipliers $\lambda^a$,

$$S[q,p,\lambda] = \int dt \left( p_i \dot{q}^i - H_T \right), \quad H_T = H(q,p) + \lambda^A T_A(q,p),$$  \hspace{1cm} (27)

where $\lambda^A = (\lambda^\alpha, \lambda^a)$. If we begin with this action, it will have the usual first-order gauge symmetry,

$$\delta_\epsilon O(q,p) = \{O(q,p), T_A(q,p)\} \epsilon^A; \quad \delta_\epsilon \lambda^A = \dot{\epsilon}^A + (V^A_B - U^A_{CB} \lambda^C) \epsilon^B,$$  \hspace{1cm} (28)

with unconstrained gauge parameters $\epsilon^A = (\epsilon^\alpha, \epsilon^a)$. The introduced multipliers $\lambda^a$ can be considered as “compensatory fields” to the constraints on gauge parameters (21) in the theory with original action (9). The gauge invariants (23) of the unfree gauge symmetry (20), (21) obviously coincide with the invariants of the transformations (28). At the level of action, however, there may be a subtle difference between the theory (9) with unfree gauge symmetry and the corresponding theory with unconstrained gauge symmetry and compensatory fields (27). The matter is that the modular parameters $\Lambda_a$ (16) do not contribute to the gauge transformations nor they are explicitly involved in the original action (9). Action (27) involves compensatory fields $\lambda_a$ and secondary constrains $\tau_a$, while the latter explicitly include modular parameters $\Lambda_a$ (16). So, the action (27) describes the dynamics with fixed values of modular parameters, while the original action encompasses the dynamics with entire moduli space (5), (16). In the case of gravity, for example, it would be the difference between the action of unimodular gravity which encompasses dynamics with any value of cosmological constant and the Einstein’s action with fixed $\Lambda$. The role of compensatory field is played in this case by lapse function, or equivalently by $\text{det } g$.

4. Hamiltonian BFV-BRST formalism

In the BFV (Batalin-Fradkin-Vilkovisky) theory, the gauge invariants are represented by zero ghost number BRST cohomology classes in the so-called minimal sector of the ghost extended phase space. For the basics of the formalism, we refer to the textbook [18]. As we have seen
in section 3, the gauge invariants of unfree gauge symmetry (20), (21) should Poisson-commute on-shell to all the constraints (23). The involution relations of primary and secondary constraints define a general first-class constraint algebra (26), which does not reveal any specifics related to the equations imposed on gauge parameters (21). This means that in the minimal sector, the Hamiltonian BRST formalism is constructed along the usual lines of the BFV method, while the specifics of the unfree gauge symmetry is accounted for by the non-minimal sector.

To begin with the Hamiltonian BRST embedding of the theory, we briefly describe the minimal ghost sector in the BFV formalism. Every first-class constraint $T_A$ be it primary, or secondary, is assigned with a pair of canonically conjugate ghosts with usual ghost numbers

$$\text{gh} C^A = -\text{gh} \bar{P}_A = 1, \quad \{C^A, P_B\} = \delta^A_B. \quad (29)$$

The BRST charge in the minimal sector is defined as

$$Q_{\text{min}}(q, p, C, \bar{P}) = C^A T_A + \ldots, \quad \text{gh} Q_{\text{min}} = 1, \quad \{Q_{\text{min}}, Q_{\text{min}}\} = 0, \quad (30)$$

where $\ldots$ mean $\bar{P}$-depending terms that are iteratively defined by the equation $\{Q_{\text{min}}, Q_{\text{min}}\} = 0$. Any gauge invariant $O(q, p)$ (23), including $H$, is extended by ghosts to become BRST-invariant:

$$H(q, p) \mapsto \mathcal{H}(q, p, C, \bar{P}) = H + \ldots, \quad \text{gh} \mathcal{H} = 0, \quad \{Q_{\text{min}}, \mathcal{H}\} = 0. \quad (31)$$

Let us discuss the non-minimal sector that is needed for the gauge fixing. The original action (9) and unfree gauge transformations (20), (21) involve the Lagrange multipliers to the primary constraints only. The number of independent gauge parameters (if they could be extracted by resolving the equations (21) as explained in section 3) should be equal to the number of primary constraints. Hence, the same number of independent conditions should be imposed for gauge fixing. Therefore, the non-minimal sector of the theory includes the Lagrange multipliers $\lambda^\alpha$ to the primary constraints, and the Lagrange multipliers $\pi_\alpha$ to the independent relativistic gauge conditions $\dot{\lambda}^\alpha - \chi^\alpha(q, p) = 0$. The corresponding canonical ghost pair is introduced for every pair of the Lagrange multipliers, so the complete non-minimal sector reads

$$\text{gh} \lambda^\alpha = \text{gh} \pi_\alpha = 0, \quad \text{gh} P_\alpha = -\text{gh} C_\alpha = 1, \quad \{\lambda^\alpha, \pi_\beta\} = \{P^\alpha, C_\beta\} = \delta^\alpha_\beta. \quad (32)$$

Given the extended set of variables, the complete BRST charge reads

$$Q = Q_{\text{min}} + \pi_\alpha P^\alpha. \quad (33)$$
With this charge, the gauge-fixed BRST invariant Hamiltonian is defined in the usual way,\n\[ H_{\Psi} = \mathcal{H} + \{ Q, \Psi \} , \quad \Psi = \bar{C}_\alpha \chi^\alpha + \lambda^\alpha \bar{P}_\alpha . \] (34)

The partition function \( Z_\Psi \), being defined by the Hamiltonian \( H_{\Psi} \), does not depend on the choice of gauge conditions included in \( \Psi \) due to usual reasons of the Hamiltonian BRST formalism [18].

Consider \( Z_\Psi \) for the simplest case when the Hamiltonian \( H_{\Psi} \) is at most squared in ghost variables. The path integral for the partition function reads:
\[ Z_\Psi = \int [\mathcal{D} \varphi] \exp \left\{ \frac{i}{\hbar} \int \left( \dot{p} q - H(q, p) - \chi^\alpha T_\alpha + \pi_\alpha (\lambda^\alpha - \chi^\alpha) + \bar{P}_a (\dot{C}^a + \Gamma_b^a C^b + \Gamma^a C^a) \right. \right. \]
\[ - \left. \left. \bar{C}^a \left( \{(\chi^\alpha, T_\beta) C^\beta + \{\chi^\alpha, \tau_\beta) C^\beta \} + \bar{P}_a (\dot{C}^a + W^a C^\beta + W^a C^a) + P^a (\bar{P}_a + \dot{C}_a) \right) \right) \right\} , \] (35)

where \( \varphi = \{ q, p, \chi^\alpha, \pi_\alpha, C^a, \bar{P}_a, \bar{C}_a \} \). The integral by \( P^a \) results in \( \delta (\bar{P}_a + \dot{C}_a) \), which removes the integral over \( \bar{P}_a \). The result reads:
\[ Z_\Psi = \int [\mathcal{D} \varphi'] \exp \left\{ \frac{i}{\hbar} \int \left( \dot{p} q - H(q, p) - \chi^\alpha T_\alpha + \pi_\alpha (\lambda^\alpha - \chi^\alpha) + \bar{P}_a (\dot{C}^a + \Gamma_b^a C^b + \Gamma^a C^a) \right. \right. \]
\[ - \left. \left. \bar{C}^a \left( \{(\chi^\alpha, T_\beta) C^\beta + \{\chi^\alpha, \tau_\beta) C^\beta \} - \dot{\bar{C}}_a (\dot{\bar{C}}^a + W^a C^\beta + W^a C^a) \right) \right) \right\} , \] (36)

where \( \varphi' = \{ q, p, \chi^\alpha, \pi_\alpha, C^a, \bar{P}_a, \bar{C}_a \} \). The integral over anti-ghosts \( \bar{P}_a \) would enforce constraints \( \dot{C}^a + \Gamma_b^a C^b + \Gamma^a C^a = 0 \). This is quite a natural phenomenon: once gauge variations (20) induced by primary and secondary constraints are unfree, being restricted by equations (21), the ghosts should obey the same conditions as the gauge parameters do. The constraint on ghosts is the cornerstone for the extension of the BV formalism for the theories with unfree gauge symmetry [11, 2]. Here, we see that they naturally arise from the Hamiltonian BFV-BRST quantization.

5. Example: Linearized Unimodular Gravity

Consider the action of unimodular gravity linearized in the vicinity of Minkowski space background
\[ S = \frac{1}{4} \int d^4 x \left( \partial^\mu h_{\rho\bar{\rho}} \partial^{\mu} h^{\rho\bar{\rho}} - 2 \partial^\mu h_{\rho\bar{\rho}} \partial^{\mu} h^{\rho\bar{\rho}} \right) , \quad \eta^{\bar{\alpha} \bar{\beta}} h_{\bar{\alpha} \bar{\beta}} = 0 , \] (37)

where \( \bar{\alpha} = 0, 1, 2, 3 \), \( \eta_{\bar{\alpha} \bar{\beta}} = \text{diag}(1, -1, -1, -1) \). Gauge identity (2) for (37) reads:
\[ 2 \partial_{\bar{\alpha}} \frac{\delta S}{\delta h_{\bar{\alpha} \bar{\beta}}} - \partial^{\bar{\beta}} \tau \equiv 0 , \quad \tau = \frac{1}{2} \left( \partial^\mu \partial_{\mu} h^{\bar{\alpha} \bar{\beta}} \right) , \] (38)
cf. (6). Once \( \partial^{\bar{\beta}} \tau \approx 0 \), \( \tau \) is a constant on-shell, so we have \( \tau - \Lambda \approx 0 \), where specific value of the constant \( \Lambda \) is determined by the asymptotics of \( h \), not by Cauchy data. If the boundary conditions
admit the growing solutions, then \( \Lambda \) can be non-vanishing. In particular, there is a solution,
\[
h_{\alpha\beta} = h_{\alpha\beta}^{(0)} + \Lambda \left( x_\alpha x_\beta - \frac{\eta_{\alpha\beta}}{4} x^2 \right),
\]
with \( \tau(h) = \Lambda \neq 0 \), where \( h_{\alpha\beta}^{(0)} \) is any solution vanishing at infinity. Minkowski space solutions \((39)\) approximate, in a sense, the solutions of unimodular gravity with (Anti-)de Sitter asymptotics. The higher spin analogues \([12], [13]\) admit similar solutions. For higher spins, this may be even more essential because the cosmological constant plays the role of interaction parameter for \( s > 2 \).

Given the gauge identity \((38)\), which involves the completion function \( \tau(h) \), the action \((37)\) should enjoy unfree gauge symmetry. It does, in full accordance with the general prescription \((3), (4)\):
\[
\delta \epsilon h_{\alpha\beta} = \partial_{\alpha} \epsilon_{\beta} + \partial_{\beta} \epsilon_{\alpha} - \frac{1}{2} \eta_{\alpha\beta} \partial_{\gamma} \epsilon_{\gamma}, \quad \delta S \equiv \int d^4 x \partial_{\alpha} \epsilon^{\alpha} \tau,
\]
cf. \((8)\). So, the action is gauge invariant off-shell under the condition \( \partial_{\alpha} \epsilon^{\alpha} = 0 \).

By Legendre transform of \((37)\), we get the Hamiltonian action
\[
S[h, \Pi, \lambda] = \int d^4 x \left( \Pi^{\alpha\beta} \dot{h}_{\alpha\beta} - H(h, \Pi) - \lambda^{\alpha} T_{\alpha}(\Pi) \right),
\]
\[
H = \Pi^{\alpha\beta} \Pi_{\alpha\beta} - \frac{1}{2} \Pi^2 + \frac{1}{2} \left( 2 \partial_{\alpha} h_{\beta\gamma} \partial^{\alpha} h_{\beta\gamma} - \partial_{\alpha} h_{\beta\gamma} \partial^{\alpha} h_{\beta\gamma} - \partial_{\alpha} h_{\beta\gamma} \partial^{\alpha} h_{\beta\gamma} \right), \quad T_{\alpha} = -2 \partial_{\gamma} \Pi^{\gamma \alpha}.
\]
where \( \alpha, \beta = 1, 2, 3 \), \( \eta_{\alpha\beta} = -\delta_{\alpha\beta}, \ h = \eta^{\alpha\beta} h_{\alpha\beta}, \ \Pi = \eta_{\alpha\beta} \Pi^{\alpha\beta}, \ \lambda^{\alpha} = h^{0\alpha} \). Conservation of primary constraints \( T_{\alpha} \) leads to the secondary constraint
\[
\hat{T}_{\alpha} = \{T_{\alpha}, H\} = -\partial_{\alpha} \tau_0 = 0, \quad \tau_0 = \partial_{\beta} \partial_{\gamma} h^{\beta\gamma} - \partial_{\gamma} \partial^{\gamma} h.
\]
Once \( \partial_{\alpha} \tau_0 \approx 0 \), hence \( \tau_0 - \Lambda_0 \approx 0 \), where the constant \( \Lambda_0 \) is determined by asymptotics of \( h \) at infinity. Secondary constraint \( \tau_0 \) in \((43)\) will coincide with completion function \( \tau \) in Lagrangian formalism \((38)\) if the the second time derivatives are excluded from \( \partial_{\mu} \partial_{\nu} h^{\mu\nu} \) by using Lagrangian equations. Involution relations \((43)\) correspond to the spacial components of gauge identity \((38)\). The secondary constraint conserves by virtue of the primary ones:
\[
\hat{\tau}_0 = \{\tau_0, H\} = -\partial^{\alpha} T_{\alpha}.
\]
This relation corresponds to the time component of gauge identity \((38)\). All the constraints Poisson-commute to each other. The general involution relations \((18), (19)\) define unfree gauge transformations in Hamiltonian formalism by the rule \((20), (21)\). Substituting specific constraints and structure coefficients of involution relations of the unimodular gravity \([12], [13], [14] \) into
the general recipe (20), (21), we arrive at the unfree gauge symmetry of this theory:

\[ \delta \epsilon_{\alpha\beta} = \partial_\alpha \epsilon_\beta + \partial_\beta \epsilon_\alpha, \quad \delta \Pi^{\alpha\beta} = -\partial^\alpha \partial^\beta \epsilon^0 + \eta^{\alpha\beta} \partial_\gamma \epsilon^\gamma, \quad \delta \lambda^\alpha = \dot{\epsilon}^\alpha + \partial^\alpha \epsilon^0, \quad (45) \]

\[ \dot{\epsilon}^0 + \partial_\alpha \epsilon^\alpha = 0. \quad (46) \]

This symmetry can be verified by direct computation. Variation (45) of action (41) reads

\[ \delta \epsilon S \equiv \int d^4 x \left( (\dot{\epsilon}^0 + \partial_\alpha \epsilon^\alpha) \tau_0 - \partial_0 (T_\alpha \epsilon^\alpha + \tau_0 \epsilon^0) \right). \quad (47) \]

It is a symmetry indeed once \( \epsilon^0 \) obeys equation (46). As we see, the general procedure of Section 4 identifies the linearized transverse diffeomorphism (45), (46) as the gauge symmetry of Hamiltonian action (41), (42).

Consider the BFV construction for the model following the general prescription of Section 4. The ghosts of minimal sector are assigned to all the constraints (cf. (29)), while the non-minimal sector is assigned only to the primary constraints (see (32)). The BRST charge (30), (33) for the linearized unimodular gravity reads:

\[ Q = -2 C^\alpha \partial_\beta \Pi_\beta^\alpha + C^0 (\partial_\beta \partial_\gamma h^{\beta\gamma} - \partial_\gamma \partial^\gamma h - \Lambda_0) + \pi_\alpha P^\alpha. \quad (48) \]

Impose three independent gauge fixing conditions,

\[ \partial_\beta h^{\delta\alpha} \equiv \dot{\lambda}^\alpha - \chi^\alpha = 0, \quad \chi^\alpha = -\partial_\beta h^{\beta\alpha}. \quad (49) \]

Introduce gauge fermion \( \Psi = \bar{C}_\alpha \chi^\alpha + \lambda_\alpha \bar{P}^\alpha \), and define gauge-fixed Hamiltonian \( H_\Psi \) (51),

\[ H_\Psi = H(h, \Pi) - C^0 \partial^\alpha \bar{P}_\alpha - C^\alpha \partial_\alpha \bar{P}_0 + \partial_\beta \bar{C}_\alpha \partial^\beta C^\alpha + \partial_\beta \bar{C}_\alpha \partial^\alpha C^\beta - \pi_\alpha \partial_\beta h^{\beta\alpha} + \lambda^\alpha T_\alpha - P^\alpha \bar{P}_\alpha. \quad (50) \]

where \( H \) is the original Hamiltonian (42). For \( H_\Psi \) (50), partition function (35) reads

\[ Z_\Psi = \int \left[ \mathcal{D} \varphi \right] \exp \left\{ \frac{i}{\hbar} \int d^4 x \left( \Pi^{\alpha\beta} \dot{h}_{\alpha\beta} - H(h, \Pi) - \lambda^\alpha T_\alpha + \pi_\alpha (\dot{\lambda}^\alpha + \partial_\beta h^{\beta\alpha}) \right. \right. \]

\[ \left. + \bar{P}_0 (\dot{C}^0 + \partial_\alpha C^\alpha) + \bar{C}_\alpha (\partial_\beta \partial^\beta C^\alpha + \partial^\alpha \partial_\beta C^\beta) + \bar{P}_\alpha (\dot{C}^\alpha + \partial^\alpha C^0) + \bar{P}_\alpha (\dot{P}_\alpha + \dot{\bar{C}}_\alpha) \right\}, \quad (51) \]

where \( \varphi = \{ h_{\alpha\beta}, \Pi^{\alpha\beta}, \lambda^\alpha, \pi_\alpha, C^\alpha, \bar{P}_0, C^\alpha, \bar{P}_\alpha, P^\alpha, \bar{C}_\alpha \}; H \) and \( T_\alpha \) are the original Hamiltonian and primary constraints (12). Integrating over \( P^\alpha, \bar{P}_\alpha, \Pi^{\alpha\beta} \) we get Lagrangian representation for \( Z_\Psi \),

\[ Z_\Psi = \int \left[ \mathcal{D} \varphi' \right] \exp \left\{ \frac{i}{\hbar} \int d^4 x \left( \mathcal{L} + \pi_\alpha \partial_\beta h^{\beta\alpha} + \bar{P}_0 \partial_\alpha C^\alpha + \bar{C}_\alpha \square C^\alpha \right) \right\}, \quad \square = \partial_\mu \partial^\mu. \quad (52) \]
where $\phi' = \{h_{\alpha\beta}, \pi_\alpha, \bar{P}_0, C^a, \bar{C}_\alpha\}$, and $\mathcal{L}$ is the original Lagrangian. This representation of partition function has been deduced by Hamiltonian BFV-BRST quantization of the model. It appears to be a reasonable adjustment of Faddeev-Popov (FP) recipe to the case. Among the ghost terms, the first one represents constraint imposed on ghosts, with $\bar{P}_0$ being the Lagrange multiplier. The ghost constraint mirrors the transversality condition imposed on the diffeomorphisms (46). As the gauge parameters are unfree, it is natural to have the corresponding ghosts constrained. The FP term (52) is not Poincaré covariant because the gauge is fixed by independent condition (49), being 3d vector. If the gauge condition was a 4d vector, the vector components would have to be redundant, to avoid “over-rigid” gauge fixing. This would require some extra ghosts. In the covariant formalism, this issue is considered in [2], while the Hamiltonian analogue will be addressed elsewhere.

6. Concluding remarks

The field theories with unfree gauge symmetry represent a special class of models where the gauge parameters have to obey differential equations. Every known example of these theories (see [3]-[15] and references therein) admits an “almost equivalent” analogue without constraints on gauge parameters. The subtle difference is that the models with unfree gauge symmetry comprise dynamics with arbitrary modular parameters, which are involved as integration constants, while the analogues explicitly involve fixed modular parameters. The example of such a parameter is a cosmological constant in unimodular gravity. It is the distinction which is behind the constraints on gauge parameters (21). In terms of Hamiltonian formalism, these constraints on gauge parameters have been previously unknown even in the examples, not to mention the general theory. We have worked out the general Hamiltonian BFV-BRST formalism with a due account for the unfree gauge symmetry. As we see by examples, it corresponds well to the extension of BV method to the unfree gauge symmetry [2], though these two schemes do not mirror each other. In the BV scheme the equations on parameters are directly accounted for as constraints on the corresponding ghosts, while the Hamiltonian formalism accounts for the conditions (21) indirectly, by an adjusted structure of the non-minimal ghost sector.

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