Superconformal Fixed Points with $E_n$ Global Symmetry

Joseph A. Minahan and Dennis Nemeschansky

Department of Physics and Astronomy
University of Southern California
Los Angeles, CA 90089-0484

Abstract

We obtain the elliptic curve and the Seiberg-Witten differential for an $N = 2$ superconformal field theory which has an $E_8$ global symmetry at the strong coupling point $\tau = e^{\pi i/3}$. The differential has 120 poles corresponding to half the charged states in the fundamental representation of $E_8$, with the other half living on the other sheet. Using this theory, we flow down to $E_7$, $E_6$ and $D_4$. A new feature is a $\lambda_{SW}$ for these theories based on their adjoint representations. We argue that these theories have different physics than those with $\lambda_{SW}$ built from the fundamental representations.

1 minahan@physics.usc.edu
2 dennisn@physics.usc.edu
1. Introduction

Four dimensional \( N = 2 \) superconformal theories with an unbroken \( U(1) \) gauge group are in one to one correspondence with Kodaira’s classification of toroidal singularities. There are 7 strong coupling conformal points, which have a global symmetry that is either \( A_0, A_1 \) or \( A_2, E_6, E_7 \) or \( E_8, D_4 \).

The \( D_4 \) case is of course the celebrated Seiberg-Witten result for \( SU(2) \) Super QCD with four hypermultiplets in the fundamental representation. The \( A_0, A_1 \) and \( A_2 \) cases can be derived from \( D_4 \) by taking appropriate limits for the masses in the theory.

The \( D_4 \) theory has associated with it an elliptic curve and a differential \( \lambda_{SW} \). The deformations of the curve are determined from the bare masses in the theory as well as the bare coupling. A crucial property of \( \lambda_{SW} \) is that it has poles in which the residues are linear combinations of the bare masses. Furthermore, it was shown in [1] that this is sufficient to completely determine the curve.

Of course, the \( E_N \) theories cannot be derived from the \( D_4 \) case. However, one can assume the existence of \( \lambda_{SW} \) for each of these theories, with the property that the residues of \( \lambda_{SW} \) are linear combinations of \( N \) mass parameters that determine the deformations of the elliptic curve. In [2], it was shown that this is enough to completely determine the curve for \( E_6 \). In this paper, we extend this analysis to the cases of \( E_8 \) and \( E_7 \).

We will also find an interesting surprise, namely, there exists other superconformal theories for \( E_6 \) and \( D_4 \) as well as \( E_7 \). We find that one can construct a Seiberg-Witten differential based on the adjoints of these groups. As it happens, the elliptic curve for the adjoint case is the same as the fundamental. But since \( \lambda_{SW} \) is different, the monodromies are different and hence the content of the physical states are different. Unlike a gauge symmetry, a global symmetry is a real symmetry of observable particles, and one can determine what representation these particles live in. A \( \lambda_{SW} \) constructed from the adjoint representation will necessarily lead to physical states living in the adjoint representation. This also has another interesting consequence for \( D_4 \), the adjoint case is invariant under \( SL(2, \mathbb{Z}) \), and not just a semi-direct product of \( SL(2, \mathbb{Z}) \) with \( SO(8) \) triality.

The surfaces that we describe are elliptic fibrations of del Pezzo surfaces. Such surfaces have appeared in the context of string theory [3,4,5] as well as in the study of 5 dimensional gauge theory [6,7,8]. It is hoped that the results presented here will be useful for \( F \)-theory considerations [9,10,11,12], such as the calculations of BPS masses.

In section 2 we discuss how one finds a set of rational curves for the case of \( E_8 \). In section 3 we discuss the derivation of the \( E_8 \) curve and compute \( \lambda_{SW} \). In section 4 we
discuss the flows to the $E_7$, $E_6$ and $D_4$ theories. In section 5 we present our conclusions. Most of the results are contained in the four appendices.

2. Lines, Parabolas and Perfect Squares

Consider the elliptic curves with an $E_n$ singularity

\[
\begin{align*}
E_6 : & \quad y^2 = x^3 - \rho^4 \\
E_7 : & \quad y^2 = x^3 - 2\rho^3 x \\
E_8 : & \quad y^2 = x^3 - 2\rho^5,
\end{align*}
\]

where we have chosen the factors of two for later convenience. These curves describe del Pezzo surfaces. These curves have relevant deformations, the number of which can be easily found by comparing the dimensions of $x$, $y$ and $\rho$. This number is $n$, the rank of the group. It is convenient to express these deformations in terms of the $SO(16)$ subgroup for $E_8$, $SO(12) \times SU(2)$ for $E_7$ and $SO(10) \times U(1)$ for $E_6$. Hence, for each of these cases, we have $n$ mass parameters that live in the Cartan subalgebra of these subgroups.

We also assume that there exists a Seiberg-Witten differential $\lambda_{SW}$ for each of these theories, which satisfies

\[
\frac{d\lambda_{SW}}{d\rho} \sim \frac{dx}{y}
\]

and which is allowed to have poles in the $x$ plane whose residues are linear combinations of the masses discussed in the previous paragraph. In order for this to happen, it must be true that at the positions of the poles, $y^2$ is a perfect square in terms of $\rho$ and the mass parameters.

In [1] it was shown that such poles can appear in the $D_4$ case at the positions

\[
x = \beta \rho + \theta
\]

where $\beta$ is a dimensionless quantity that depends on the bare coupling and $\theta$ depends on the mass parameters. The Seiberg-Witten differential will have four such poles on each sheet, and one is free to choose a vector, spinor or spinor bar representation for these poles.

In [2] it was shown that in the $E_6$ case, the positions of the poles also satisfy (2.3), except in this case, $\beta$ is proportional to the residue for the particular pole. There are 27 such poles, in one to one correspondence with the dimension of the representation.
As it turns out, the $E_7$ case also has poles described by (2.3). But an inspection of (2.1) shows that the poles for the $E_8$ case cannot have this form, since the curve has a $\rho^5$ piece and hence the leading term in $\rho$ would have an odd power if $x$ is linear in $\rho$. Obviously, $y^2$ cannot be a perfect square in this situation. Hence if a Seiberg-Witten differential is to exist, the poles would have to at least have the form

$$x = \gamma \rho^2 + \beta \rho + \theta,$$

in which case $y$ would be cubic in $\rho$.

There is another dilemma involving the poles for $E_8$, and it is related to the problem of the poles being given by parabolas and not lines in the $x-\rho$ plane. The fundamental representation for $E_8$ is its adjoint. Accordingly, if some mass parameters are taken to infinity, the curve should flow to an $E_7$ curve. The $E_8$ adjoint representation then flows to representations of the $E_7$ subgroup, which is comprised of the adjoint and two fundamentals. Hence, if $\lambda_{SW}$ exists for $E_8$, then there must also exist another $\lambda_{SW}$ for $E_7$. This argument can be extended to $E_6$ and $D_4$, that is, each of these theories has an $\lambda_{SW}$ with poles that transform in the adjoint representation for these groups. In all of these cases the poles are described by parabolas in the $x-\rho$ plane.

Since the dimension of $\lambda_{SW}$ is assumed to be one, in all cases, the dimension of $\rho^2/x$ is two, thus $\gamma$ is dimension negative two. As it will turn out $\gamma$ equals $8\pi^2/(\text{Res})^2$, where Res is the residue for the pole.

We can use homogeneous variables and express the curves in (2.1) as curves in the projective space $P^3$. Hence, the rational curves in (2.4) are of degree 3 in $P^3$. It is well known that the $E_6$ curve describes a cubic in $P^3$, which is isomorphic to a $P^2$ with 6 points blown up	extsuperscript{[14]}. A systematic counting of rational curves has been carried out for this case (with fixed moduli), where it was found that there are 72 distinct degree 3 curves (plus another 12 with arithmetic genus 1)	extsuperscript{[15]}. These 72 curves transform under the $E_6$ Weyl group. We will see that 72 poles do appear when flowing to the $E_6$ case, with $y$ a cubic in $\rho$, thus we see that the 72 poles that we have identified are precisely these curves. The 12 with arithmetic genus 1 are singlets under the Weyl group, and so their residues are zero.

For the $E_7$ and $E_8$ cases, the curves on the $P^3$ are isomorphic to del Pezzos constructed from $P^2$ with 7 and 8 points blown up. For the $E_8$ case, the degree 3 curve in $P^3$ maps to a curve of degree 1 on the del Pezzo, while for $E_7$, the degree 3 curve on $P^3$ maps to a degree 2 curve. This must be the case in order to match the counting of rational curves for these surfaces	extsuperscript{[14,4]}. 

3
The space of curves with degree higher than 1 have moduli \( [15, 4] \). Hence the pole positions for the \( E_6 \) and \( E_7 \) adjoints seem to be a special choice. The case of generic points in the moduli space is an interesting question and will be considered elsewhere [16].

3. The \( E_8 \) Case

In this section we explain the derivation of the \( E_8 \) curve. All other theories considered in this paper flow from this one. Since \( E_8 \) has a maximal \( SO(16) \) subgroup, we consider deformations described by eight mass parameters \( m_i \). We define the \( SO(16) \) invariants \( T_{2n} \) for \( n = 1..7 \), where

\[
T_{2n} = \sum_{0 < i_1 < i_2 < \ldots < i_n} m_{i_1}^2 \ldots m_{i_n}^2 \quad (3.1)
\]

There is also the invariant \( t_8 \),

\[
t_8 = \prod_{i=1}^{8} m_i. \quad (3.2)
\]

The \( E_8 \) curve should be expressible in terms of these independent \( SO(16) \) invariants.

Our method for deriving the curve is to turn on masses one by one, allowing for all possible terms in the curve consistent with \( r \)-symmetry, holomorphy and the remaining symmetries. This still leaves some ambiguity for the curve. However, the final terms can be nailed down by choosing \( y^2 \) to be a perfect square at the poles,

We choose \( x \) such that the curve is of the form

\[
y^2 = x^3 - fx - g. \quad (3.3)
\]

By turning on one mass, \( m_1 \), the symmetry of theory is broken to \( SO(14) \). Therefore, we should find a \( D_7 \) singularity as \( \rho \) approaches zero. Such a singularity satisfies \( g \sim \rho^3 \), and \( f \sim \rho^2 \), and has a discriminant \( \Delta = 4f^3 - 27g^2 \sim \rho^9 \). Up to a rescaling of \( m_1 \), one finds

\[
f = m_1^2 \rho^3 + \frac{m_1^8}{192} \rho^2 \]
\[
g = 2\rho^5 + \frac{m_1^6}{24} \rho^4 + \frac{m_1^{12}}{6912} \rho^3. \quad (3.4)
\]

We next assume the ansatz that there are poles at \( x = \gamma \rho^2 + \beta \rho + \theta \). For the curve in (3.3) and (3.4), there are two solutions for \( x \) where \( y^2 \) is a perfect square, \( x = -\rho^2/m_1^2 + \rho m_1^4/12 \) and \( x = -4\rho^2/m_1^2 - \rho m_1^4/24 \). At these points, \( y^2 \) is given by \( y^2 = \)
\[-\left(\rho^3 + 3\rho^2 m_1^6 / 8 / m_1^3\right)^2 \text{ and } y^2 = \left(8\rho^3 / m_1^3\right)^2 \text{ respectively. The form of these solutions is quite suggestive since the } E_8 \text{ adjoint has as an } SO(16) \text{ decomposition}

\[248 = 120 + 132.\] (3.5)

Hence, if we assume that \(\gamma = -\rho^2 / h_\alpha^2\), where \(h_\alpha\) is the charge under the Cartan subalgebra for a particular element of the representation, then we see that the first pole corresponds to an \(SO(16)\) adjoint and the second pole corresponds to an \(SO(16)\) spinor.

Therefore, to ease our search for the \(E_8\) curve we will assume that the contribution of an element of the representation to \(\lambda_{SW}\) is proportional to

\[
\frac{h_\alpha y_i}{x - x_\alpha} \frac{dx}{y} = \frac{\tilde{y}_\alpha}{h_\alpha^2 x - \tilde{x}_\alpha} \frac{dx}{y}
\] (3.6)

where \(\tilde{y}_\alpha = h_\alpha^3 y_i\), and

\[
\tilde{x}_\alpha = h_\alpha^2 x_\alpha = -\rho^2 + \tilde{\beta}_i \rho + \tilde{\theta}_i
\] (3.7)

and the terms \(\tilde{x}_\alpha\) and \(\tilde{y}_\alpha\) are polynomials in the masses and \(\rho\).

Turning on another mass \(m_2\) breaks the group down to \(SO(12)\), in which case we should choose the coefficients such that \(f \sim \rho^2\), \(g \sim \rho^3\) and \(\Delta \sim \rho^8\). This is not enough information to determine the terms in \(f\) and \(g\) and one must choose the coefficients such that \(y^2\) is a perfect square along a rational curve in the \(x - \rho\) plane. Assuming that the rational curve has the form in (3.7) is sufficient to determine the curve for nonzero \(m_1\) and \(m_2\). In fact, it is enough to only consider the pole with \(h_\alpha = m_1 + m_2\) to find \(f\) and \(g\).

With this \(f\) and \(g\), one then finds that the other poles corresponding to other elements of the representation are consistent.

One can keep on turning on masses until the generic deformation is obtained and the pole positions are determined. We won’t actually prove here that the results presented below are the unique solutions to the ansatz in (3.7)\(^3\).

The general idea for computing curves and pole positions is as follows. If \(x\) is chosen to be quadratic in \(\rho\), then \(y^2\) will be a sextic equation,

\[-y^2 = (h_\alpha)^{-6} \sum_{n=0}^{6} a_n \rho^n,\] (3.8)

\(^3\) The skeptical readers are invited to download a Mathematica file from [http://www.usc.edu/~minahan/Math/e8.ma](http://www.usc.edu/~minahan/Math/e8.ma) and see for themselves that the rational curves lead to perfect squares in \(y^2\) for this deformation of the \(E_8\) singular curve.
where the $a_n$ are polynomials in the $m_i$. In order that $y^2$ be a perfect square, $a_0$ and $a_6$ must be perfect squares of polynomials involving the $m_i$. The coefficient $a_6$ is always 1. The coefficient $a_0$ is more complicated. Requiring that it be a perfect square leads to a series of linear equations for the coefficients of the curve and for the pole positions. Once $a_0$ is found, then we look for an $a_1$ such that the square root of $a_0$ divides $a_1$. This then leads to more linear equations for the coefficients. Finally, we derive more linear equations by setting to zero the expression

$$
\left( \rho^3 + \frac{1}{2} a_5 \rho^2 + \frac{1}{2} \frac{a_1}{\sqrt{a_0}} + \sqrt{a_0} \right)^2 - \sum_{n=0}^{6} a_n \rho^n.
$$

(3.9)

This turns out to be sufficient for determining the complete curve and the poles.

The final results for the $E_8$ curve are presented in the appendix A. It is convenient to express the curve in terms of a different $SO(16)$ invariant $\tilde{T}_4 = T_2^2/4 - T_4$. This gets rid of most of the higher powers in $T_2$. By inspection, one sees that most of the generic terms in the curve actually have zero coefficient, which is a good thing, otherwise the expression for $g$ alone would have 341 terms instead of the much more manageable 71 terms.

One still has the freedom to shift $\rho$, removing the $\rho^4$ term in $g$. After this shift, $\rho \to \rho - (T_2 \tilde{T}_4/6 + T_6)$, the coefficients of $\rho$ in $f$ and $g$ we are left with are eight independent Casimirs of $E_8$ and hence form a natural basis for the entire set of $E_8$ casimirs.

We have also given the positions of the poles as well as the corresponding values for $y$ in the appendix. For an $SO(16)$ adjoint pole, the pole position should be expressible in terms of two masses, $m_i$ and $m_j$, and the casimirs for the unbroken $SO(12)$ orthogonal to $i$ and $j$, $W_n$ and $W_6$.

The residue of the spinor poles are given by $\frac{1}{2} \frac{1}{2 \sqrt{2} \pi i} \sum \pm m_i$, where the number of $-$ signs is even. The state with all $+$ signs has a residue that is proportional to the linear symmetric polynomial of the eight masses. Hence this pole position is expressible in terms of the symmetric polynomials $s_n$, where

$$
s_n = \sum_{0 < i_1 < i_2 < \ldots < i_n} m_{i_1} \ldots m_{i_n}. \tag{3.10}
$$

In order to show that this pole leads to a perfect square, we need the relation

$$
T_{2n} = s_n^2 + 2 \sum_{m=1}^{n} (-1)^m s_{n-m}s_{n+m}, \tag{3.11}
$$

6
where \( s_0 = 1 \) and \( s_p = 0 \) if \( p > 8 \). The poles for the rest of the spinor states can be found by changing an even number of the signs of \( m_i \) in the symmetric polynomials. Because the curve has \( t_8 \) dependence, there are no poles at the positions found by changing an odd number of the signs. Hence, only one spinor of \( SO(16) \) appears, as is expected.

There are some checks that we can do for our curve. We can let all the masses satisfy \( m_i = m \) and compute \( f \) and \( g \). In this case, we find that

\[
f \sim (\rho + 4m^6)^3 \quad g \sim (\rho + 4m^6)^5. \tag{3.12}
\]

This is the behavior for an \( E_7 \) singularity. A simple counting shows that this is sensible. The states with residues proportional to \( m_i - m_j \) are massless, as are the spinor states with an equal number of plus and minus signs. There are 56 states of the former type and 70 of the latter, leaving 126 massless states in the \( E_7 \) adjoint. Likewise, we can consider the case where \( m_i = m, \ i < 8 \) and \( m_8 = -m \). In this case there is an \( A_7 \) singularity. There are still 56 massless states coming from the adjoint, but now, none of the spinor states are massless since an odd number of \(-\) signs is not allowed. Hence the counting is consistent with the number of charged states in the adjoint of \( SO(8) \).

Once we have the pole positions and the value of \( y^2 \) at these poles we can sum these contributions to the Seiberg-Witten differential. As we have already mentioned, if the masses are such that a residue is zero, because of the form of the sum, the corresponding term can still contribute to \( \lambda_{SW} \). However, an interesting feature occurs in this situation. By inspection of eqs. (A.3-A.7), one sees that if the residue is zero then \( \tilde{x}_\alpha \) divides \( \tilde{y}_\alpha \), leaving a term that is linear in \( \rho \) in front of \( dx/y \). Furthermore, the coefficient of \( \rho \) is the same for any state. It is also true that only the charged states are summed over, so it is unlikely that the sum of the poles is the complete Seiberg-Witten differential. However, the new piece should be invariant under the \( E_8 \) Weyl group and should be at most linear in \( \rho \).

Let us thus assume that \( \lambda_{SW} \) is given by

\[
\lambda_{SW} = \frac{1}{2\sqrt{2}\pi} (A \rho + BT_2^6 + CT_2T_4 + DT_6) \frac{dx}{y} + \frac{1}{2\sqrt{2}\pi i} \sum_{\alpha=1}^{120} \frac{\tilde{y}_\alpha}{h_\alpha^2 (x - \tilde{x}_\alpha)} \frac{dx}{y} \tag{3.13}
\]

where the sum is over half of the 240 charged states of the representation and \( h_\alpha/(2\sqrt{2}\pi i) \) is the residue of the state, normalized such that an \( SO(16) \) adjoint state has \( h_\alpha = \pm m_i \pm m_j \).
and the spinor state has $h_\alpha = \sum_i \pm m_i/2$, where the number of $-$ signs is even. The factor in front of the sum is chosen to match the normalization in [1]. The coefficients $A$, $B$ and $C$ are found by fixing

$$\frac{d\lambda_{SW}}{d\rho} = \frac{k}{\sqrt{2\pi}} \frac{dx}{y},$$

(3.14)

up to a total derivative, where $k$ is to be determined. Given this form the coefficients in $\lambda_{SW}$ can be derived by considering special values for the $m_i$. In particular, letting $m_1 = m, m_i = 0, i > 1$ determines $A$ and $k$, $m_1 = m_2 = m, m_i = 0 i > 2$ determines $B$ and $m_1 = m_2 = m_3 = m$ determines $C$. We find that $k = 30$, the Coxeter number for $E_8$ (half the index of the adjoint representation), and that $\lambda_{SW}$ is

$$\lambda_{SW} = \frac{1}{2\sqrt{2\pi}} (60\rho - T_2 T_4 + 6T_6) \frac{dx}{y} + \frac{1}{2\sqrt{2\pi i}} \sum_{\alpha=1}^{120} \tilde{y}_\alpha \frac{dx}{y}.$$  

(3.15)

Note that the piece without the poles is

$$\frac{1}{2\sqrt{2\pi}} \left(60\tilde{\rho} - \frac{1}{4} T_2^3\right) \frac{dx}{y}$$

where $\tilde{\rho}$ is the value of $\rho$ after shifting to remove the $\rho^4$ term in $g$. $T_2$ is invariant under the $E_8$ Weyl group, therefore, the entire term is invariant.

4. Flowing to the Other Cases

We can investigate the $E_7$ theory by taking two of the $E_8$ masses to infinity. Accordingly, let $m_7 = \Lambda - \phi/2$ and $m_8 = \Lambda + \phi/2$. These variables are the natural variables for the $E_7$ subgroup $SO(12) \times SU(2)$. We also rescale $x, y$ and $\rho$ by $x \to \Lambda^4 \rho, y \to \Lambda^6 y$ and $\rho \to \Lambda^2 \rho$. Plugging these new values into (A.1) and (A.2) and keeping the terms to leading order in $\Lambda$ gives the $E_7$ curve. The values of $f$ and $g$ are given in (B.1) and (B.2), where now the terms in the curve are given in terms of $D_6$ invariants. It is convenient to replace $T_2$ by $\tilde{T}_2 = T_2 - \phi^2$. In order to express the curve explicitly in terms of $E_7$ invariants, it is necessary to shift $\rho$ by $(\tilde{T}_2)^2/72 + T_4/6$, which removes the $\rho^2$ term in $f_{E_7}$.

The positions of the poles are also found by using these same scaling arguments. However, since the adjoint of $E_8$ decomposes under its $E_7 \times SU(2)$ subgroup to

$$248 = (133, 1) + (1, 3) + (56, 2)$$

(4.1)
some of these poles will flow to $E_7$ adjoints and others will flow to $E_7$ fundamentals. In fact, it is easy to check that for the fundamentals, the $\rho^2$ piece scales out of $x_\alpha$, leaving a linear relation between $x$ and $\rho$.

It should be possible to find a self-consistent $\lambda_{SW}$ for each representation. Clearly, the coefficient $k$ in (3.13) should split into a contribution of one $E_7$ adjoint and two fundamentals. Since $k$ is the Coxeter number, this picture is consistent with $k$ splitting into 18, the $E_7$ coxeter number, and two values of 6, half the index of the fundamental representation.

For the adjoint case, the $\lambda_{SW}$ is of the form

$$\lambda_{SW} = \frac{1}{2\sqrt{2}\pi}(A\rho + BT_2^2 + C\phi^2T_2 + D\phi^4 + ET_4)\frac{dx}{y} + \frac{1}{2\sqrt{2}\pi i} \sum_{\alpha=1}^{63} \tilde{y}_\alpha h_\alpha^2 x - \tilde{x}_\alpha dx,$$ (4.2)

where the sum is over half of the 126 charged states of the representation. The values for $\tilde{x}_\alpha$ and $\tilde{y}_\alpha$ are found in the appendix. In terms of the $SO(12) \times SU(2)$ subgroup, the residues of the poles are of the form $(\pm m_i \pm m_j)/(2\sqrt{2}\pi i)$, $\pm \phi/(2\sqrt{2}\pi i)$ or $\sum_j (\pm m_j \pm \phi/2)/(2\sqrt{2}\pi i)$, with an odd number of $-$ signs in front of the $m_i$ for the $SO(12)$ spinor. As in the $E_8$ case, the coefficients $A, B, C, D,$ and $E$ are chosen so that $d\lambda_{SW}/d\rho \sim dx/y$. Again, one can find these coefficients by choosing special values for the masses. The final result in this case is that

$$\lambda_{SW} = \frac{1}{2\sqrt{2}\pi} \left(9\rho + \frac{1}{8}(\tilde{T}_2)^2 + \frac{3}{2}T_4\right)\frac{dx}{y} + \frac{1}{2\sqrt{2}\pi i} \sum_{\alpha=1}^{63} \tilde{y}_\alpha h_\alpha^2 x - \tilde{x}_\alpha \frac{dx}{y},$$ (4.3)

and with $k = 18$, the Coxeter number for $E_7$. The residues for these poles are $(\pm m_i \pm \phi/2)/(2\sqrt{2}\pi i)$ or $\sum_j \pm m_i/(4\sqrt{2}\pi i)$ with an even number of $-$ signs for the spinors. The term without the poles is clearly proportional to the shifted value of $\rho$, and hence this term is clearly an $E_7$ invariant.

For the fundamental case, the positions of the poles are now linear in $\rho$ since the quadratic piece scaled out for these particular states. One can do an analysis similar to the adjoint case, with the result

$$\lambda_{SW} = \frac{1}{2\sqrt{2}\pi} \left(24\rho + \frac{1}{3}(\tilde{T}_2)^2 + \frac{3}{2}T_4 - 3 \left(\frac{2}{3}T_2 + \phi\right)^2\right)\frac{dx}{y} + \frac{1}{2\sqrt{2}\pi i} \sum_{\alpha=1}^{28} h_\alpha y_\alpha \frac{dx}{x - x_\alpha},$$ (4.4)

where the sum is over half of the 56 states. The value of $k$ is found to be $k = 6$. The term without poles in (4.4) is comprised of a shifted $\rho$, plus a piece that is proportional to the square of the $E_7$ casimir of weight 2, hence the entire term is an $E_7$ invariant.
A natural way to flow from the $E_7$ theory to the $E_6$ curve is under the scaling $m_6 = \Lambda$, $\phi = \Lambda - 6\lambda$, $x \to x\Lambda^2$, $y \to y\Lambda^3$, and $\rho \to \rho\Lambda$. The masses $m_i$ and $\lambda$ are the natural variables of the $SO(10) \times U(1)$ subgroup of $E_6$. Keeping only the leading orders in $\Lambda$ gives the values of $f$ and $g$ in (C.1) and (C.2).

For the adjoint case, the Seiberg Witten differential is given by

$$\lambda_{SW} = \frac{1}{2\sqrt{2}\pi i} \sum_{\alpha=1}^{36} \frac{\tilde{y}_\alpha}{h^2_\alpha x - \tilde{x}_\alpha} \frac{dx}{y},$$

(4.5)

where the sum is over half of the charged states in the representation. The value for $k$ is found to be $k = 12$, the $E_6$ Coxeter number. The residues of these states are $(\pm m_i \pm m_j)/(2\sqrt{2}\pi i)$, and the spinors $\sum_j (\pm m_i/2 \pm 3\lambda)/(2\sqrt{2}\pi i)$, with an odd number of $-$ signs for $+3\lambda$ and an even number otherwise. Curiously, the sum over the poles is the complete differential. Unlike the $E_8$ and $E_7$ case, there is no extra piece linear in $\rho$.

The $E_6$ fundamental $\lambda_{SW}$ was given in [2] and was found to be

$$\lambda_{SW} = \frac{1}{2\sqrt{2}\pi} (18\rho + 18\lambda T_2 - 72\lambda^3) \frac{dx}{y} + \frac{1}{2\sqrt{2}\pi i} \sum_{\alpha=1}^{27} h_\alpha y_\alpha \frac{dx}{y},$$

(4.6)

where the sum is over the 27 charged states of the $E_6$ fundamental. Since this is a complex representation, the residues on the other sheet are part of the conjugate representation. Because of this, $k = 6$, which is the index for the fundamental representation instead of half the index. The residues are of the form $+4\lambda/(2\sqrt{2}\pi i)$, $(\pm m_i - 2\lambda)/(2\sqrt{2}\pi i)$ or $\sum_j (\pm m_j/2 + \lambda)/(2\sqrt{2}\pi i)$, where the number of $-$ signs is even.

Finally, we come to the $D_4$ case. This flow was discussed in [2]. The appropriate scaling is to let $x \to x\Lambda^2$, $y \to y\Lambda^3$ and to set $\rho = u\Lambda$, $\lambda = c_1\Lambda/6$ and $m_5 = -c_2\Lambda$, where $c_1$ and $c_2$ are the combination of theta functions defined in [1],

$$c_1 = \frac{1}{2} (\vartheta^4_3(\tau) + \vartheta^4_2(\tau)) \quad c_2 = \frac{1}{2} \vartheta^4_1(\tau).$$

(4.7)

Keeping only the leading powers in $\Lambda$, $f$ and $g$ reduce to the expressions in (D.1) and (D.2). The adjoint pole positions and values for $y$ at the poles are given in (D.3) and (D.4).

The Seiberg Witten differential for the adjoint representation is similar in form to the $E_6$ expression, with $\lambda_{SW}$ given by

$$\lambda_{SW} = \frac{1}{2\sqrt{2}\pi i} \sum_{\alpha=1}^{12} \frac{\tilde{y}_\alpha}{h^2_\alpha x - \tilde{x}_\alpha} \frac{dx}{y},$$

(4.8)

where the sum is over half of the 24 charged states in the representation. We also find that $k = 6$, the $SO(8)$ coxeter number. As in the case for $E_6$, the sum over poles is the entire $\lambda_{SW}$, there is no extra holomorphic piece.
5. Discussion

In this paper we have constructed superconformal theories with $E_n$ global symmetries. We have also constructed Seiberg-Witten differentials based on the adjoints of these groups, as well as $\lambda_{SW}$ for an $SO(8)$ global symmetry.

The natural question arises whether or not the theories are equivalent to theories where the Seiberg-Witten differential is constructed from the fundamental representation of these groups. At first one might think that they are equivalent since the elliptic curves are the same. So one immediately concludes that the coupling is the same if all parameters are the same. However, the masses of the BPS states are found from $\lambda_{SW}$ and here it seems that there could be differences. For instance, consider the $SO(8)$ case with all $m_i$ set to 0. Then $\lambda_{SW}$ looks identical for the vector, spinor or adjoint rep. However, the adjoint $\lambda_{SW}$ has a different normalization, so it would seem that all BPS states that one finds are 6 times heavier than those in the theory with a vector $\lambda_{SW}$, since $k = 6$ for the adjoint and $k = 1$ for the vector.

If we tried to divide by this factor of 6 to set the masses equal, then another problem arises when we turn on the $m_i$. Then we find that there are monodromies such that the coordinates $a$ or $a_D$ shift by $(\pm m_i \pm m_j)/(6\sqrt{2})$. No such shifts are possible for the vector $\lambda_{SW}$. So we must conclude that the theories are different.

Still, the behavior is surprising for $D_4$. The standard lore is that for each pole in $\lambda_{SW}$, there is an electric state with charge 1. Hence the electric states are transforming under the adjoint of $SO(8)$. But by triality, the magnetic and dyonic states also transform under the adjoint representation. We still find that the electric coupling runs to zero in the same fashion as in [1], even though the electric states are different. The resolution of this paradox must be that the monopoles and dyons somehow contribute to the $\beta$-function.

Acknowledgements: We thank Nick Warner for many helpful conversations. This research was supported in part by D.O.E. grant DE-FG03-84ER-40168.
Appendix A. $E_8$ results

For the curve of the form $y^2 = x^3 - fx - g$, the curve with $E_8$ symmetry has $f$ and $g$ given by

\[
f = \rho^3 T_2 + \rho^2 \left( 14 t_8 + \frac{(\tilde{T}_4)^2}{12} + T_8 \right) + \rho \left( 8 T_{14} - T_{12} T_2 - \frac{2 T_{10} \tilde{T}_4}{3} + \frac{11 t_8 T_2 \tilde{T}_4}{3} + 8 t_8 T_6 \right) - T_{12} T_8 - t_8 \tilde{T}_4 T_8 + \frac{T_{10}^2}{3} + 2 t_8 T_{12} - \frac{2 t_8 T_{10} T_2}{3} + \frac{t_8^2 T_2^2}{3} + 2 t_8^2 \tilde{T}_4 + 2 T_{14} T_2 \tilde{T}_4 + \frac{T_{12} (\tilde{T}_4)^2}{4} + \frac{t_8 (\tilde{T}_4)^3}{4} + 4 T_{14} T_6
\]

\[
g = 2 \rho^5 + \rho^4 \left( \frac{T_2 \tilde{T}_4}{6} + T_6 \right) + \rho^3 \left( -4 T_{12} + \frac{T_{10} T_2}{3} + \frac{5 t_8 T_2^2}{3} + \frac{22 t_8 \tilde{T}_4}{3} + \frac{4 (T_4)^3}{108} - \frac{T_4 T_8}{3} \right) - 2 T_{12} T_6 + \frac{10 t_8 \tilde{T}_4 T_6}{3} + \frac{T_{10} T_8}{3} + \frac{11 t_8 T_2 T_8}{3}
\]

\[
+ \rho \left( 8 t_8^3 + 2 T_{12}^2 - \frac{16 T_{10} T_{14}}{3} - \frac{T_{10} T_{12} T_2}{3} + \frac{4 t_8 T_{14} T_2}{3} + \frac{t_8 T_{12} T_2^2}{3} - \frac{2 T_{10}^2 \tilde{T}_4}{9} + \frac{10 t_8 T_{12} \tilde{T}_4}{3} - \frac{23 t_8 T_{10} T_2 \tilde{T}_4}{9} + \frac{25 t_8^2 T_2 \tilde{T}_2 \tilde{T}_4}{9} + \frac{10 t_8^2 (\tilde{T}_4)^2}{3} + \frac{5 T_{14} T_2 (\tilde{T}_4)^2}{6} - \frac{T_{14} (\tilde{T}_4)^3}{12} + \frac{T_{12} (\tilde{T}_4)^4}{24} - \frac{16 t_8 T_{10} T_6}{3} + \frac{16 t_8^2 T_2 T_6}{3} + \frac{8 T_{14} \tilde{T}_4 T_6}{3} - 8 t_8^2 T_8 + 2 T_{14} T_2 T_8 + \frac{T_{12} \tilde{T}_4 T_8}{3} - \frac{2 t_8 (\tilde{T}_4)^2 T_8}{3} + 2 t_8 T_8^2 \right)
\]

\[
+ \frac{2 T_{10}^3}{27} + \frac{2 t_8 T_{10} T_{12}}{3} + 4 t_8^2 T_{14} - \frac{2 t_8 T_{10}^2 T_2}{9} - \frac{2 t_8^2 T_{12} T_2}{3} + \frac{2 t_8^2 T_{10} T_2^2}{9} - \frac{2 t_8^3 T_2^3}{27} + \frac{2 t_8^2 T_{10} \tilde{T}_4}{3} - \frac{2 t_8^3 T_2 \tilde{T}_4}{3} + \frac{T_{12}^2 T_2 \tilde{T}_4}{2} - \frac{4 T_{10} T_{14} T_2 \tilde{T}_4}{3} + \frac{4 t_8 T_{14} T_2^2 \tilde{T}_4}{3} + \frac{T_{10} T_{12} (\tilde{T}_4)^2}{12} + t_8 T_{14} (\tilde{T}_4)^2 + \frac{11 t_8 T_{12} T_2 (\tilde{T}_4)^2}{12} + \frac{t_8 T_{10} (\tilde{T}_4)^3}{3} + \frac{5 t_8^2 T_2 (\tilde{T}_4)^3}{12} + \frac{T_{14} (\tilde{T}_4)^4}{16} + \frac{T_{12}^2 T_6}{3} - \frac{8 T_{10} T_{14} T_6}{3} + \frac{8 t_8 T_{14} T_2 T_6}{3} + 2 t_8 T_{12} \tilde{T}_4 T_6 + t_8^2 (\tilde{T}_4)^2 T_6 - \frac{T_{10} T_{12} T_8}{3} - 4 t_8 T_{14} T_8 + \frac{t_8 T_{12} T_2 T_8}{3} - \frac{t_8 T_{10} \tilde{T}_4 T_8}{3} + \frac{t_8^2 T_2 \tilde{T}_4 T_8}{3} - \frac{T_{14} (\tilde{T}_4)^2 T_8}{2} + T_{14} T_8^2
\]

(A.2)

where the $SO(16)$ invariants $T_n$, $t_8$ and $\tilde{T}_4$ are described in the text.
The 240 poles occur at the 112 charged states in the adjoint of $SO(16)$ and the 128 states of a spinor representation. In finding the pole positions, it is convenient to expand the expression in powers of $h_\alpha$, where $h_\alpha/(2\sqrt{2}\pi i)$ is the residue for that pole. For the adjoint case we find that $\tilde{x}_{ij} = h_{ij}^2 x_{ij}$ satisfies

$$\tilde{x}_{ij} = -\left(\frac{\rho}{2} + m_im_j \tilde{W}_4 - w_6\right)^2 + h_{ij}^2 \left(\frac{1}{3}\rho \tilde{W}_4 - w_6 \tilde{W}_4 + \frac{2}{3} W_{10}\right)$$

$$+ m_im_j \left(\frac{2}{3} W_8 - \frac{1}{6}\rho W_2 + \frac{5}{6} W_2 w_6\right) + (m_im_j)^2 \left(\frac{4}{3} w_6 + \frac{2}{3} W_6 + \frac{1}{4} W_2 \tilde{W}_4\right)$$

$$- h_{ij}^4 \left(\frac{1}{2} W_2 w_6 + \frac{1}{6} \rho W_2 + \frac{1}{3} W_8 + \frac{1}{12} m_im_j (29w_6 - 5\rho) + \frac{1}{16} (m_im_j)^2 (W_2^2 + 2\tilde{W}_4)\right)$$

$$+ h_{ij}^6 \left(\frac{3}{4} w_6 + \frac{1}{12} \rho + \frac{1}{16} (m_im_j)^2 W_2\right) - \frac{1}{64} h_{ij}^8 (m_im_j)^2$$

(A.3)

where $h_{ij} = (m_i + m_j)$ and the variables $W_n$ satisfy

$$W_n = \sum_{\substack{i_1 < \ldots < i_n \atop i \neq i_k \neq j}} m_{i_1}^2 \ldots m_{i_n}^2$$

$$\tilde{W}_4 = \frac{1}{4} W_2^2 - W_4$$

$$w_6 = t_8/(m_im_j)$$

(A.4)

We could also choose to change the sign of $m_j$ in these expressions. When $x = x_{ij}$ then $y^2$ is a perfect square, with $\sqrt{-y_{ij}^2}$ given by
\[
\sqrt{-y^2_{ij}} = (\rho - w_6 + \frac{1}{2} m_i m_j \tilde{W}_4)^3 - \frac{1}{8} h_{ij}^2 \left( \rho - w_6 + \frac{1}{2} m_i m_j \tilde{W}_4 \right) \times \\
\times \left( 4\rho \tilde{W}_4 + 8W_{10} - 12w_6 \tilde{W}_4 + 2m_i m_j(4W_8 + 5W_2 w_6 - \rho W_2) + (m_i m_j)^2(3W_2 \tilde{W}_4 + 8W_6 + 16w_6) \right) \\
+ \frac{1}{32} h_{ij}^4 \left( 16W_{10} \tilde{W}_4 - 8w_6(\tilde{W}_4)^2 - 8\rho^2 W_2 - 32w_6 W_6 - 24W_2 w_6^2 \right. \\
\left. + m_i m_j(12\rho^2 + 8\rho W_2 \tilde{W}_4 + 32\rho W_6 - 56\rho w_6 - 84w_6^2 - 24W_2 \tilde{W}_4 w_6 - 24W_2 W_{10} - 96w_6 W_6 \right. \\
\left. + (m_i m_j)^2(-4\tilde{W}_4 \rho - 2W_2^2 \rho + 20\tilde{W}_4 w_6 - 8W_2 W_6 - 64W_{10} - 6W_2^2 w_6) \right. \\
\left. + (m_i m_j)^3(3W_2^2 \tilde{W}_4 - 32w_6 + 8W_2 W_6 + 3(\tilde{W}_4)^2) \right) \\
+ \frac{1}{8} h_{ij}^6 \left( 3\rho^2 + 12\rho w_6 + 9w_6^2 + 2W_2 \tilde{W}_4 w_6 + 8w_6 W_6 + 2W_2 W_{10} \right) \\
+ \frac{1}{2} m_i m_j(-\rho W_2^2 - 2\rho \tilde{W}_4 + 3W_2^2 w_6 + 6\tilde{W}_4 w_6 + 30W_{10}) + \frac{1}{2} (m_i m_j)^2(3\rho W_2 + 7W_2 w_6 - 6W_8) \\
+ \frac{1}{8} (m_i m_j)^3(48w_6 - 8W_6 - W_2^3 - 6W_2 \tilde{W}_4) \\
+ \frac{1}{128} h_{ij}^8 \left( -48W_{10} - 8W_2^2 w_6 - 16\tilde{W}_4 w_6 + 8m_i m_j(\rho W_2 - 3W_2 w_6) - 2(m_i m_j)^2(5\rho + 19w_6) \right. \\
\left. + 3(m_i m_j)^3(W_2^2 + \tilde{W}_4) \right) \\
+ \frac{1}{256} h_{ij}^{10} (16W_2 w_6 - 4m_i m_j(\rho - 3w_6) - 3(m_i m_j)^3 W_2) - \frac{1}{512} h_{ij}^{12} (8w_6 - (m_i m_j)^3) \tag{A.5}
\]

The spinor poles are found at \( \tilde{x}_{sp} = h_{sp}^2 x_{sp} \), where \( h_{sp} = s_1/2 \) and

\[
\tilde{x}_{sp} = -\tilde{\rho}^2 - s_1 s_5 \tilde{\rho} + \frac{1}{12} s_1^2 (4\tilde{\rho}s_4 - s_5^2 - 8s_3 s_7) + \frac{1}{12} s_1^3 (-\tilde{\rho}s_3 + 2s_3 s_6 + 6s_2 s_7) \\
+ \frac{1}{24} s_1^4 (\tilde{\rho}s_2 - 2s_2 s_6 - 4s_8) - \frac{1}{8} s_1^5 s_7 - \frac{1}{96} s_1^6 (\tilde{\rho} - 2s_6). \tag{A.6}
\]

where \( s_n \) is the symmetric polynomial for eight masses of order \( n \), linear in each \( m_i \), and with an even number of \( m_i \) replaced with \( -m_i \). The term \( \tilde{\rho} \) is \( \tilde{\rho} = \rho - s_6 \). At the poles, \( y^2 \) is a perfect square, with \( \tilde{y}_{sp} = y_{sp} h_{sp}^3 \) satisfying
\[
\sqrt{-\tilde{y}_{2p}^2} = \tilde{\rho}^3 + \frac{2}{3} s_1 s_5 \tilde{\rho}^2 - \frac{1}{2} s_1^2 \tilde{\rho} (\tilde{\rho}s_4 - 2s_3 s_7 - s_5^2)
+ \frac{1}{8} s_1^3 (\tilde{\rho}^2 s_3 - 2\tilde{\rho}(s_3 s_6 + 3s_2 s_7 + s_4 s_5) + 4s_3 s_5 s_7)
+ \frac{1}{8} s_1^4 (\tilde{\rho}(s_3 s_5 + 2s_2 s_6 - 2s_8) - 2s_3 s_4 s_7 - 2s_2 s_5 s_7 - 2s_3^2 s_8 - 2s_7^2)
+ \frac{1}{16} s_1^5 (\tilde{\rho}(4s_7 - s_2 s_5) + 6s_2 s_3 s_8 + 2s_2 s_4 s_7 + 2s_6 s_7 - 2s_5 s_8 + 2s_3^2 s_7)
+ \frac{1}{16} s_1^6 (-\tilde{\rho}s_6 - 2s_2^2 s_8 + s_5 s_7 - 2s_2 s_3 s_7) + \frac{1}{64} s_1^7 (\tilde{\rho}s_5 + 2s_2^2 s_7 - 2s_4 s_7 - 6s_3 s_8)
+ \frac{1}{32} s_1^8 (2s_2 s_8 + s_3 s_7) - \frac{1}{64} s_1^9 s_2 s_7 - \frac{1}{128} s_1^{10} s_8 + \frac{1}{256} s_1^{11} s_7
\]  
(A.7)

Notice that in (A.3)-(A.7), when \(h_{\alpha}\) is zero, then \(\tilde{x}_\alpha\) divides \(\tilde{y}_\alpha\).

**Appendix B. E\(_7\) Results**

The \(E_7\) curve is derived from the \(E_8\) curve by setting \(m_7 = \Lambda - \phi/2\) and \(m_8 = \Lambda + \phi/2\) and then taking the limit \(\Lambda \rightarrow \infty\). Scaling the other variables as described in the text, and keeping only the leading order terms, one finds

\[
f_{E_7} = 2 \rho^3 + \rho^2 \left( \frac{\tilde{T}_2}{12} \right) + \rho \left( 8 \phi^2 t_6 + \frac{2 \tilde{T}_2 t_6}{3} + \frac{2 \tilde{T}_2 T_6}{3} - 2 T_8 \right) \]  
(B.1)

and

\[
+ 4 \phi^2 T_{10} - \frac{(\tilde{T}_2)^3 t_6}{4} + \tilde{T}_2 T_4 t_6 + \frac{4 t_6}{3} - \frac{4 t_6 T_6}{3} + \frac{T_6^2}{3} + \frac{(\tilde{T}_2)^2 T_8}{4} - T_4 T_8
\]
\[ g_{E_7} = \rho^4 \left( \phi^2 + \frac{2 \tilde{T}_2}{3} \right) + \rho^3 \left( \frac{(-\tilde{T}_2)^3}{108} + \frac{\tilde{T}_2 T_4}{3} + \frac{20 t_6}{3} + \frac{2 T_6}{3} \right) \]
\[ + \rho^2 \left( 4 T_{10} - \frac{10 \phi^2 \tilde{T}_2 t_6}{3} - \frac{29 (\tilde{T}_2)^2 t_6}{18} + \frac{22 T_4 t_6}{3} + \frac{5 (\tilde{T}_2)^2 T_6}{36} + \frac{T_4 T_6}{3} \right) \]
\[ - 2 \phi^2 T_8 - \frac{2 \tilde{T}_2 T_8}{3} \]
\[ + \rho \left( \frac{-8 \phi^2 \tilde{T}_2 T_{10}}{3} - (\tilde{T}_2)^2 T_{10} + 4 T_{10} T_4 + \frac{(\tilde{T}_2)^4 t_6}{24} - \frac{2 (\tilde{T}_2)^2 T_4 t_6}{3} \right) \]
\[ + 2 T_4^2 t_6 + \frac{32 \phi^2 t_6^2}{3} - \frac{4 \tilde{T}_2 t_6^2}{9} - \frac{16 \phi^2 t_6 T_6}{3} - \frac{2 \tilde{T}_2 t_6 T_6}{9} \]
\[ + \frac{2 \tilde{T}_2 T_6^2}{9} + \frac{(\tilde{T}_2)^3 T_8}{12} - \frac{\tilde{T}_2 T_4 T_8}{3} + \frac{4 t_6 T_8}{3} - \frac{2 T_6 T_8}{3} \] (B.2)
\[ + \frac{(\tilde{T}_2)^4 T_{10}}{16} - \frac{(\tilde{T}_2)^2 T_{10} T_4}{2} + T_{10} T_4^2 + \frac{16 \phi^2 T_{10} t_6}{3} + \phi^2 (\tilde{T}_2)^2 t_6^2 \]
\[ + \frac{(\tilde{T}_2)^3 t_6^2}{6} - \frac{2 \tilde{T}_2 T_4 t_6^2}{3} - \frac{16 t_6^3}{27} - \frac{8 \phi^2 T_{10} T_6}{3} - \frac{(\tilde{T}_2)^3 t_6 T_6}{12} \]
\[ + \frac{\tilde{T}_2 T_4 t_6 T_6}{3} + \frac{8 t_6^2 T_6}{9} - \frac{4 t_6 T_6^2}{9} + \frac{2 T_6^3}{27} - \frac{2 \phi^2 \tilde{T}_2 t_6 T_8}{6} - \frac{(\tilde{T}_2)^2 t_6 T_8}{6} \]
\[ + \frac{2 T_4 t_6 T_8}{3} + \frac{(\tilde{T}_2)^2 T_6 T_8}{12} - \frac{T_4 T_6 T_8}{3} + \phi^2 T_8^2 \]

where \( \tilde{T}_2 = T_2 - \phi^2 \). The variables \( T_n \) now refer to invariants of \( SO(12) \). \( t_6 \) is the product of six masses.

The \( E_8 \) representation splits into an adjoint and two fundamentals of \( E_7 \). The \( E_7 \) adjoint is made up of an \( SO(12) \) adjoint, spinor and singlet, where the spinor and singlet are charged under the \( SU(2) \). The poles for the \( SO(12) \) adjoint, whose residues are \( h_{ij} = m_i + m_j \) (plus sign permutations) divided by the usual factor of \( 2\sqrt{2}\pi i \), satisfy

\[
\tilde{x}_{ij,7} = - \left( \rho - w_4 - \frac{1}{2} m_i m_j \tilde{W}_2 \right)^2 + h_{ij}^2 \left( -\frac{1}{3} \rho \tilde{W}_2 + \tilde{W}_2 w_4 + \frac{2}{3} W_6 + \frac{1}{3} m_i m_j (\rho - 2 W_4 + 5 w_4) \right.
\]
\[ + \frac{1}{6} (m_i m_j)^2 (\tilde{W}_2 + 4 \phi^2) \bigg) + h_{ij}^4 \left( -\frac{1}{3} \rho - \frac{1}{3} W_4 - w_4 - \frac{1}{4} (m_i m_j)^2 \right) \], (B.3)

where \( W_n \) and \( w_4 \) refer to the \( SO(8) \) invariants transverse to the \( i \) and \( j \) directions and
where $\tilde{W}_2 = W_2 - \phi^2$. The $\tilde{y}_{ij}$ for these poles are

\[
\sqrt{-\tilde{y}^2_{ij}} = \left( \rho - w_4 - \frac{1}{2} m_i m_j \tilde{W}_2 \right)^3 - \frac{1}{4} h^2_{ij} \left( \rho - w_4 - \frac{1}{2} m_i m_j \tilde{W}_2 \right) \times \\
\left( -2 \rho \tilde{W}_2 + 6 \tilde{W}_2 w_4 + 4 W_6 + m_i m_j (-2 \rho + 4 W_4 + 10 w_4) + (m_i m_j)^2 (\tilde{W}_2 + 4 \phi^2) \right) \\
- \frac{1}{8} h^4_{ij} \left( 4 \rho^2 + 8 w_4 W_4 + 12 w_4^2 + 4 \tilde{W}_2 W_6 + 2 \tilde{W}_2^2 w_4 \\
+ 4 m_i m_j (-\rho \tilde{W}_2 - 2 \rho \phi^2 + 3 \tilde{W}_2 w_4 + 6 \phi^2 w_4 + 3 W_6) \\
+ 2 (m_i m_j)^2 (\rho^2 + 3 w_4 + W_4) - (m_i m_j)^3 (\tilde{W}_2 + 4 \phi^2) \right) \\
= \frac{1}{8} h^6_{ij} \left( 4 w_4 \tilde{W}_2 + 8 w_4 \phi^2 + 4 W_6 + 2 m_i m_j (-\rho + 3 w_4) - (m_i m_j)^3 \right) - \frac{1}{4} h^8_{ij} w_4.
\]

(B.4)

The $SO(12)$ singlet pole position, $x_7 = \tilde{x}_7/h^2$, where $h = \phi$, is given by

\[
\tilde{x}_7 = - \left( \rho - \frac{1}{8} (\tilde{T}_2)^2 + \frac{1}{2} T_4 \right)^2 + \frac{1}{3} \phi^2 (\rho \tilde{T}_2 + T_6 - 2t_6)
\]

(B.5)

and $\tilde{y}_7$ is

\[
\sqrt{-\tilde{y}^2_7} = \left( \rho - \frac{1}{8} (\tilde{T}_2)^2 + \frac{1}{2} T_4 \right)^3 - \phi^2 \left( \rho - \frac{1}{8} (\tilde{T}_2)^2 + \frac{1}{2} T_4 \right) (\rho \tilde{T}_2 + T_6 - 2t_6) - \phi^4 (\rho^2 + \tilde{T}_2 t_6 - T_8).
\]

(B.6)

Because of the manner in which $E_7$ was reached from $E_8$, the $SO(12)$ spinor that is part of the $E_7$ adjoint has an odd number of minus signs. The residue is $h_{sp}/(2\sqrt{2\pi}i)$, with $h_{sp} = (S_1 - \phi)/2$, and where $S_1$ is the sum of $SO(12)$ masses with an odd number of minus signs. It is straightforward to find the pole position from (A.6), with the result

\[
x_{sp,7} = - \tilde{\rho}^2 + 2 h_{sp} \tilde{\rho} S_3 - \frac{1}{3} h^2_{sp} (4 \tilde{\rho} S_2 + S_3^2 + 8 S_5 \phi) + \frac{2}{3} h^3_{sp} (\tilde{\rho} \phi + 2 S_4 \phi - 2 S_5)
\]

(B.7)

where the $S_n$ are symmetric polynomials in the $m_i$ but with an odd number of $m_i$ replaced by $-m_i$ and the term $\tilde{\rho}$ is $\tilde{\rho} = \rho + S_4$ The value for $y$ at such a pole, $\tilde{y}_{sp,7}$, is given by

\[
\sqrt{-\tilde{y}^2_{sp,7}} = \tilde{\rho}^3 - 3 h_{sp} \tilde{\rho}^2 S_3 + 2 h^2_{sp} \tilde{\rho} (\tilde{\rho} S_2 + S_3^2 + 2 S_5 \phi) - h^3_{sp} \left( \tilde{\rho}^2 \phi + 4 S_3 S_5 \phi \\
+ 2 \tilde{\rho} (S_2 S_3 + S_4 \phi - 2 S_5) \right) - 2 h^4_{sp} \left( \tilde{\rho}^2 - \tilde{\rho} S_3 \phi + 2 S_3 S_5 - 2 S_2 S_5 \phi - 2 S_6 \phi^2 \right) \\
+ 2 h^5_{sp} (\tilde{\rho} Q_3 + 2 S_2 S_5 + 2 S_6 \phi - 2 S_5 \phi^2) - 8 h^6_{sp} S_5 \phi - 4 h^7_{sp} S_5.
\]

(B.8)
The $E_7$ fundamental has an $SO(12)$ vector and the other spinor. The vector poles are at

\[
x_{i,7} = \rho \left( \frac{5 m_i^2}{3} + 2 m_i \phi + \frac{\phi^2}{3} - \frac{W_2}{3} \right) - \frac{m_i^6}{4} + \frac{2 m_i w_5}{3} - m_i^5 \phi + 2 w_5 \phi - \frac{3 m_i^4 \phi^2}{2} - m_i^3 \phi^3
\]

with

\[
y_{i,7}^2 = (\phi + 2 z_i) \rho^2 + \left( 2 w_5 + W_4 z_i - \frac{1}{4} ((\phi + z_i)^2 - W_2)^2 - z_i^2 (\phi + z_i) ((\phi + z_i)^2 - W_2) \right) \rho
\]

\[
+ \frac{1}{8} z_i^3 ((\phi + z_i)^2 - W_2)^3 - z_i^3 W_4 ((\phi + z_i)^2 - W_2) - w_5 ((\phi + z_i)^2 - R_2)^2 + 2 z_i^2 (\phi + z_i)) + W_6 \phi z_i^2 - W_8 \phi
\]

(B.9)

The spinor poles are at

\[
x_{sp,7} = \rho \left( \frac{s_1^2}{6} + \frac{2 s_2}{3} - \frac{\phi^2}{6} \right) - \frac{s_3^2}{3} - \frac{s_1^2 s_4}{2} + \frac{4 s_1 s_5}{3} - \frac{8 s_6}{3} + \frac{s_4 \phi^2}{2}
\]

(B.10)

where $s_n$ are symmetric polynomials with an even number of minus signs. The $y$ values at the poles are

\[
y_{sp,7}^2 = \rho^2 s_1 + \rho \left( + 2 s_5 + \frac{s_3 \phi^2}{2} - \frac{s_1^2 s_3}{2} \right) + \frac{s_1^2 s_3 s_4}{2} - s_1 s_4^2 + \frac{s_1^4 s_5}{4}
\]

\[- s_1^2 s_2 s_5 + 2 s_4 s_5 - s_1^3 s_6 + 4 s_1 s_2 s_6 - 4 s_3 s_6 - \frac{s_3 s_4 \phi^2}{2} - \frac{s_1^2 s_5 \phi^2}{2}
\]

\[+ s_2 s_5 \phi^2 + s_1 s_6 \phi^2 + \frac{s_5 \phi^4}{4}
\]

(B.12)

Notice that in (B.10) and (B.12), the coefficient in front of the $\rho^2$ term is twice $h_\alpha$.

Appendix C. $E_6$ Results

The $E_6$ curve is reached by letting $m_6 = \Lambda$ and $\phi = \Lambda - 6 \lambda$, while scaling $x \to x \Lambda^2$, $y \to y \Lambda^3$ and $\rho \to \rho \Lambda$. The resulting expressions for $f$ and $g$ are

\[
f_{E_6} = \rho^2 (12 \lambda^2 + T_2) + \rho (8 \lambda T_4 + 8 t_5)
\]

\[+ 36 \lambda^2 T_6 - T_2 T_6 + 4 T_8 + t_5 \frac{T_4^2}{3} - 432 \lambda^3 + 12 \lambda T_2 t_5
\]

(C.1)
\[ g_{E_6} = \rho^4 + \rho^3 (-16\lambda^3 + 4\lambda T_2) + \rho^2 \left( 20\lambda^2 T_4 + \frac{T_2 T_4}{3} - 40\lambda t_5 - 2T_6 \right) \]
\[ + \rho \left( \frac{8\lambda T_4^2}{3} + 864\lambda^4 t_5 - 96\lambda^2 T_2 t_5 + 2T_2^2 t_5 - \frac{16T_4 t_5}{3} + 144\lambda^3 T_6 \right) \]
\[ - 4\lambda T_2 T_6 - 32\lambda T_8 \] (C.2)
\[ + \frac{2T_4^3}{27} - 144\lambda^3 T_4 t_5 + 4\lambda T_2 T_4 t_5 + 144\lambda^2 t_5^2 + 12\lambda^2 T_4 T_6 - \frac{T_2 T_4 T_6}{3} \]
\[ - 24\lambda t_5 T_6 + T_6^2 + 1296\lambda^4 T_8 - 72\lambda^2 T_2 T_8 + T_2^2 T_8 - \frac{8T_4 T_8}{3} \]

The \( E_6 \) adjoint is made up of the \( SO(10) \) adjoint and two spinors. The spinors are charged under the \( U(1) \), with the separate spinor representations having opposite charges. The \( SO(10) \) adjoint poles are positioned at \( \tilde{x}_{ij,6} = x_{ij,6} h_{ij} \), where \( h_{ij} = m_i + m_j \) and

\[ \tilde{x}_{ij,6} = - (\rho - 6m_i m_j \lambda - w_3)^2 + \frac{2}{3} h_{ij}^2 \left( -6\lambda (\rho - 3w_3) + W_4 + m_i m_j W_2 + (m_i m_j)^2 \right) \]
\[ - \frac{1}{3} h_{ij}^4 W_2, \] (C.3)

where \( W_n \) and \( w_3 \) are the \( SO(6) \) invariants orthogonal to \( i \) and \( j \). At these points, \( \tilde{y}_{ij,6} \) is given by

\[ \sqrt{-\tilde{y}_{ij,6}^2} = (\rho - 6m_i m_j \lambda - w_3)^3 + h_{ij}^2 (\rho - 6m_i m_j \lambda - w_3) \times \]
\[ \times \left( 6\lambda (\rho - 3w_3) - W_4 - m_i m_j W_2 - (m_i m_j)^2 \right) \] (C.4)
\[ - h_{ij}^4 \left( w_3 (W_2 + 36\lambda^2) + 6\lambda W_4 - m_i m_j (\rho - 3w_3) \right) + h_{ij}^6 w_3 \]

The \( SO(10) \) spinors in the \( E_6 \) adjoint have poles at \( x h_{sp}^2 = \tilde{x}_{sp,6} \), where \( h_{sp} = S_1/2 + 3\lambda \) and

\[ x_{sp,6} = - \bar{\rho}^2 + 2h_{sp} \bar{\rho} S_2 + h_{sp}^2 \left( 8\bar{\rho} \lambda - 6\lambda S_3 - \frac{1}{3} S_2^2 - \frac{8}{3} S_4 \right) - \frac{2}{3} h_{sp}^3 (3\bar{\rho} - 2S_3) \] (C.5)

where the \( S_n \) are the symmetric polynomials over the five \( m_i \) of \( SO(10) \) but with an odd number of \( m_i \) replaced with \( -m_i \). The variable \( \bar{\rho} \) is \( \bar{\rho} = \rho + S_3 \). At these poles, \( \tilde{y}_{sp} = y h_{sp}^3 \) satisfies

\[ \sqrt{-\tilde{y}_{sp,6}^2} = \bar{\rho}^3 - 3h_{sp} S_2 \bar{\rho}^2 - 2h_{sp}^2 \bar{\rho} (6\bar{\rho} \lambda - S_2^2 - 2S_4) \]
\[ + h_{sp}^3 (\bar{\rho} (3\bar{\rho} - 2S_3) + 12\bar{\rho} \lambda S_2 - 4S_2 S_4) + 2h_{sp}^4 (-\bar{\rho} S_2 - 12\lambda S_4 + 2S_5). \] (C.6)
The $E_6$ fundamental poles were given in \[2\], which we repeat here for convenience. The $SO(10)$ vector pole positions and values for $y$ are

\[ x_{i,6} = \rho (-4 \lambda + 2 m_i) - 36 \lambda^2 m_i^2 + 12 \lambda m_i^3 - m_i^4 + 2 w_4 + \frac{2 m_i^2 W_2}{3} - \frac{W_4}{3} \quad (C.7) \]

\[ \sqrt{-y_{i,6}^2} = \rho^2 + \rho (-36 \lambda^2 m_i + 24 \lambda m_i^2 - 3 m_i^3 + m_i W_2) - 216 \lambda^3 m_i^3 + 108 \lambda^2 m_i^4 
- 18 \lambda m_i^5 + m_i^6 - 36 \lambda^2 w_4 + 12 \lambda m_i w_4 - 3 m_i^2 w_4 + 6 \lambda m_i^3 W_2
- m_i^4 W_2 + w_4 W_2 + m_i^2 W_4 - W_6 \quad (C.8) \]

where $W_n$ and $w_4$ are the $SO(8)$ invariants orthogonal to $i$. The residues are $h_i = m_i - 2\lambda$ divided by $2\sqrt{2\pi}i$. The spinor poles satisfy

\[ x_{sp',6} = \rho (2 \lambda + s_1) - \frac{s_2^2}{3} - 6 \lambda s_3 - \frac{s_1 s_3}{3} + \frac{4 s_4}{3} \quad (C.9) \]

\[ \sqrt{-y_{sp',6}^2} = \rho^2 + \rho (-6 \lambda s_2 - s_1 s_2) + 6 \lambda s_2 s_3 + s_1 s_2 s_3 - s_3^2 + 36 \lambda^2 s_4 
- s_1^2 s_4 - 12 \lambda s_5 + 2 s_1 s_5 \quad (C.10) \]

where $s_n$ are symmetric polynomials with an even number of $-$ signs. The residues are $(s_1/2 + \lambda)/(2\sqrt{2\pi}i)$. Finally, the $SO(10)$ singlet, with residue $4\lambda/(2\sqrt{2\pi}i)$ has poles at

\[ x_{s,6} = 8 \lambda \rho - 324 \lambda^4 + 18 \lambda^2 T_2 - \frac{T_2^2}{4} + \frac{2 T_4}{3} \quad (C.11) \]

with

\[ \sqrt{-y_{s,6}^2} = \rho^2 + 5832 \lambda^6 - 486 \lambda^4 T_2 + \frac{27 \lambda^2 T_2^2}{2} - \frac{T_2^3}{8} + \rho (-216 \lambda^3 + 6 \lambda T_2) 
- 18 \lambda^2 T_4 + \frac{T_2 T_4}{2} + 12 \lambda T_5 - T_6 \quad (C.12) \]

**Appendix D. $D_4$ Results**

For the $D_4$ case, the appropriate scaling is $\lambda = -c_1 \Lambda/6$, $m_5 = -c_2 \Lambda$, $\rho = u\Lambda$, $x \to x\Lambda^2$, and $y \to y\Lambda^3$, where $c_1$ and $c_2$ are defined in \[3\]. Keeping the leading order in $\Lambda$, the $f$ and $g$ terms reduce to

\[ f_{D_4} = \left( \frac{c_1^2}{3} + c_2^2 \right) u^2 - \frac{4 c_1 c_2^2 T_2}{3} u - 2 c_1^3 c_2 t_4 + 2 c_1 c_2^3 t_4 + \frac{c_2^4 T_2^2}{3} 
+ c_1^2 c_2^2 T_4 - c_2^4 T_4 \quad (D.1) \]
and

\[ g_{D4} = \left( \frac{2c_1^3}{27} - \frac{2c_1c_2^2}{3} \right) u^3 + \left( \frac{5c_1^2c_2^2T_2}{9} + \frac{c_2^4T_2}{3} \right) u^2 \]

\[ + \left( \frac{-2c_1^4c_2T_4}{3} + \frac{8c_1^2c_2^3T_4}{3} - 2c_2^5T_4 \right) \frac{4c_1c_2^4T_2^2}{9} - \frac{2c_1^3c_2^2T_4}{3} + \frac{2c_1c_2^4T_4}{3} \right) u \]

\[ - \frac{2c_1^3c_2^3T_4T_2}{3} + \frac{2c_1c_2^5T_4T_2}{3} + \frac{2c_2^6T_2^3}{27} + \frac{c_2^4T_2^2T_4}{3} - \frac{c_2^6T_2T_4}{3} \]

\[ + c_1^4c_2^2T_6 - 2c_1^2c_2^4T_6 + c_2^6T_6. \]  

(D.2)

The \( T_n \) and \( t_4 \) are \( SO(8) \) invariants. The pole positions for the \( SO(8) \) adjoint, with \( h_{ij} = \pm m_i \pm m_j \) satisfy

\[ \tilde{x}_{ij,4} = - (u + c_2w_2 + c_1m_im_j)^2 + \frac{2}{3}h_{ij}^2(c_1u + c_2^2W_2 + 3c_1c_2W_2 + c_2^2m_im_j) - \frac{1}{3}c_2^2h_{ij}^4, \]  

(D.3)

where \( W_2 \) and \( w_2 \) refer to the \( SO(4) \) invariants transverse to the \( i \) and \( j \) directions. The values of \( \sqrt{-\tilde{y}_{ij,4}^2} \) at these poles are

\[ \sqrt{-\tilde{y}_{ij,4}^2} = (u + c_2w_2 + c_1m_im_j)^3 - h_{ij}^2(u + c_2w_2 + c_1m_im_j)(c_1u + c_2^2W_2 + 3c_1c_2W_2 + c_2^2m_im_j) \]

\[ + h_{ij}^4c_2(c_1c_2W_2 + (c_1^2 + c_2^2)w_2). \]  

(D.4)
References

[1] N. Seiberg and E. Witten, *Monopoles, Duality and Chiral Symmetry Breaking in N = 2 Supersymmetric QCD*, hep-th/9408099, Nucl. Phys. B431 (1994) 484.

[2] J. Minahan and D. Nemeschansky, An N=2 Superconformal Fixed Point with $E_6$ Global Symmetry, hep-th/9608047.

[3] D. Morrison and C. Vafa, Compactifications of F-Theory on Calabi–Yau Threefolds, hep-th/9602114, hep-th/9603161.

[4] A. Klemm, P. Mayr and C. Vafa, BPS States of Exceptional Non-Critical Strings, hep-th/9607139.

[5] W. Lerche and N. Warner, Exceptional SW Geometry from ALE Fibrations, hep-th/9608183

[6] N. Seiberg, Five Dimensional SUSY Field Theories, Non-trivial Fixed Points and String Dynamics, hep-th/9608111.

[7] O. Ganor, Toroidal Compactification of Heterotic 6D Non-Critical Strings Down to Four Dimensions, hep-th/9608109.

[8] M. Douglas, S. Katz and C. Vafa, Small Instantons, del Pezzo Surfaces and Type I’ theory, hep-th/9609071.

[9] D. Morrison and N. Seiberg, Extremal Transitions and Five-Dimensional Supersymmetric Field Theories, hep-th/9609070.

[10] C. Vafa, *Evidence for F-Theory*, hep-th/9602022, Nucl. Phys. B469 (1996) 403.

[11] A. Sen, *F-Theory and Orientifolds*, hep-th/9605150.

[12] T. Banks, M. Douglas and N. Seiberg, Probing F-theory With Branes, hep-th/9605199.

[13] K. Dasgupta and S. Mukhi, *F-Theory at Constant Coupling*, hep-th/9606044.

[14] R. Hartshorne, *Algebraic Geometry*, (Springer, New York, 1974)

[15] C. Itzykson, *Int. Jour. Mod. Phys. B* 8 (1994), 3725.

[16] J. Minahan, D. Nemeschansky and N. Warner, to appear