Estimating entropy rate from censored symbolic time series: a test for time-irreversibility

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Abstract.
In this work we introduce a method for estimating entropy rate and entropy production rate from finite symbolic time series. From the point of view of statistics, estimating entropy from a finite series can be interpreted as a problem of estimating parameters of a distribution with a censored or truncated sample. We use this point of view to give estimations of entropy rate and entropy production rate assuming that this is a parameter of a (limiting) distribution. The last statement is actually a consequence of the fact that the distribution of estimators coming from recurrence-time statistics comply with the central limit theorem. We test our method in a Markov chain model where these quantities can be exactly computed.

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1. Introduction

Entropy rate and entropy production rate are two quantities playing a central role in equilibrium and nonequilibrium statistical mechanics. On the one hand, entropy rate (also called Kolmogorov-Sinai entropy) is closely related to the thermodynamical entropy (see for instance [1, 2]) which is a central quantity in the context of equilibrium statistical mechanics. On the other hand, entropy production has a prominent role in the development of nonequilibrium statistical mechanics [3–5]. And as well as entropy rate, it has its rigorous definition in dynamical systems and stochastic processes (for complete details, see [6]). The entropy production rate quantifies, in some way, the degree of time-irreversibility of a given system from a microscopic point of view, which in turn states how much such a system is far from the thermodynamic equilibrium [4, 5, 7]. Moreover, time-irreversibility of certain dynamical processes in nature might be an important feature because it would imply the influence of nonlinear dynamics or non-Gaussian noise on the dynamics of the system [8]. All these characteristics of time-irreversibility has encouraged the study of this property in several different systems. For instance, in [9] it has been found that real DNA sequences would be spatially irreversible, a characteristic that has been explored aimed to understand the intriguing statistical features of current structure of the genome. The fact that DNA might be spatially irreversible has been used to propose a mechanism of noise-induced rectification of particle motion [10], a mechanism that would be important in the study of biological process involving the DNA transport. Determining the irreversibility of time series has also been the subject of intense research. For instance, in [8] it has been proposed a symbolic dynamics approach for determine time-irreversibility of time series. Another important study has been reported in [11], where the authors introduce a method for determining time-irreversibility of time series by using a visibility graph approach. The latter approach has also been used in [12] for understanding the time-reversibility of non-stationary process. The possibility of determine this temporal asymmetry has also lead to try to understand the dynamics of several processes beyond physical systems. For example, in [13] it has been explored the time-irreversibility of financial time-series a characteristic that could be used for ranking companies for optimal portfolio designs. In [14] it has been studied the time-irreversibility of human heartbeat time-series, relating this property to aging and disease of individuals. Moreover, time-irreversibility has also been used to understand several characteristics of classical music [15].

In the literature one can find many estimators of the entropy rate, directly from symbol sequences produced by natural phenomena as well as from dynamical systems, random sequences or even in natural languages taken from written texts. Perhaps, the most used method for entropy estimation is the empirical approach, by which one can estimate the probability of the symbols, by their empirical frequency of appearance in the sample and then, this is used to estimate the entropy rate using its definition (see for instance [16, 17]). One can find a lot of works in this direction trying to find better, unbiased and well-balanced estimators (see [18] and references therein). One can go further by asking for the consistency and fluctuation properties of these estimators. For instance in [19] and [20] there are explicit and rigorous fluctuation bounds under some mild additional assumptions for these so-called “Plug-In” estimators. On the other hand, but under the same empirical approach, there are also estimators for the relative empirical entropy as a measure of entropy production (see [21, 22]), this quantity, as we said before, is in some way a measure of the degree of irreversibility.
From another point of view, the problem of estimating the entropy of stationary processes has also been studied using the recurrence properties of the source. This is, another major technique used in the context of stationary ergodic processes on the space of infinite sequences, in areas such as information theory, probability theory and in the ergodic theory of dynamical systems (we refer the interested reader to [23] and the references there in). The basis of this approach is the Wyner-Ziv-Ornstein-Weiss theorem which establishes an almost sure asymptotic convergence for the logarithm of the recurrence time of a finite sample scaled by its length, to the entropy rate [23]. This result uses the Shannon-McMillan-Breiman theorem which in turn, can be thought as an ergodic theorem for the entropy. Under this approach it is possible to define estimators using the recurrence quantities, such as return time, hitting time, waiting time among others [24]. Moreover, it is possible to obtain very precise results on consistency and estimation of the fluctuations of these estimators because of the available results on the distribution of these quantities [25–27].

In the setting of Gibbs measures in the thermodynamic formalism, one can also find consistent estimators defined from the return, hitting, and waiting times for entropy in [28] and precise statements on their fluctuations such as the central limit theorem, large deviation bounds and fluctuation bounds [20, 28]. Similarly occurs within the study of the estimation of the entropy production rate. In the context of Markov chains applied to the quantification of the irreversibility or time-reversal asymmetry [7, 29], and in Gibbsian sources [30] as well as for their fluctuation properties [30, 31].

Nonetheless, for real systems, determining the value of the entropy rate and the entropy production rate is not a trivial task. This is because these quantities are defined as limiting values of the logarithm of the recurrence times, as the sample length goes to infinity. This is a fundamental limitation, since observations are always finite. So it is clear that instead of having the true value for the entropy or the entropy production, we obtain an approximation at a finite time. This makes us believe that there is a need to define estimators whose results are valid for finite samples from the point of view of the recurrence times.

The article is organized as follows. In Section 2 we give a summary of the asymptotic properties of the estimators bases on recurrence-time statistics. We also describe the method for estimating parameter of the normal distribution from a censored sample. In Section 3 we state the sampling schemes for estimating entropy rate and the reversed entropy rate using the recurrence-time statistics. We also describe the method that will be used for implementing the estimations from real data. In Section 4 we test the methodology established in Section 3 for estimating entropy rate and the reversed entropy rate in an irreversible three-state Markov chain. We compare our estimations with the exact values. Finally in Section 5 we give the main conclusions of our work.

2. Entropy rate and entropy production rate

2.1. Recurrence time statistics

Consider a finite set of symbols $A$ which we will refer to as alphabet. Let $X := \{X_n; n \in \mathbb{N}\}$ a discrete-valued stationary ergodic process generated by the law $\mathbb{P}$, whose realizations are infinite sequences of symbols taken from $A$, that is, the set of all possible realizations is a subset of $A^\mathbb{N}$. Here we denote by $x = x_1x_2x_3 \ldots$ an infinite
realization of the process \( X \). For a positive integer \( \ell \) we also denote by \( x_1^\ell \) the string of the first \( \ell \) symbols of the realization \( x \). We call a finite string \( a := a_1 a_2 a_3 \ldots a_\ell \) comprised of \( \ell \) symbols either \( \ell \)-word or \( \ell \)-block, we may use one or the other without making distinction. A \( \ell \)-word occurring at the \( k \)th site along a trajectory \( x \) will mean that \( x_{k+\ell-1}^k = a \). An alternative notation for indicating a \( \ell \)-word along a trajectory \( x \) at the \( k \)th site will also be denoted as \( x(k, k + \ell - 1) \).

Let us introduce the return time, the waiting time and the hitting time, which are the quantities of main interest in this study. Let us consider a finite string \( a_1^\ell \) made out of symbols in the alphabet \( A \). Given two independent realizations \( x \) and \( y \), let us denote by \( x_1^\ell \) and \( y_1^\ell \) be the finite strings of length \( \ell \) of \( x \) and \( y \) respectively. Then

\[
\rho_\ell := \rho_\ell(x) := \inf\{k > 1 : x_k^k = x_1^\ell\},
\]

\[
\omega_\ell := \omega_\ell(x, y) := \inf\{k \geq 1 : y_k^{k-\ell} = x_1^\ell\},
\]

\[
\tau_\ell := \tau_\ell(a_1^\ell, x) := \inf\{k \geq 1 : x_k^k = a_1^\ell\},
\]

define the return, waiting and hitting time respectively.

Wyner and Ziv (see for instance [32]) proved that for a stationary ergodic process \( \frac{1}{\ell} \log \rho_\ell \) converges to the entropy rate in probability, and that for stationary ergodic Markov chains, \( \frac{1}{\ell} \log \omega_\ell \) also converges to the entropy rate \( h \), in probability. That is, these quantities grow exponentially fast with \( \ell \) and their limiting rate is equal to the entropy rate in probability. Later, Ornstein and Weiss in [33] showed that for stationary ergodic processes

\[
\lim_{\ell \to \infty} \frac{1}{\ell} \log \rho_\ell = h \quad \mathbb{P} - \text{a.s.}
\]

(4)

For the waiting time, it was proved by Shields (this can be found in [23]) that for stationary ergodic Markov chains one has,

\[
\lim_{\ell \to \infty} \frac{1}{\ell} \log \omega_\ell = h \quad \mathbb{P} \times \mathbb{P} - \text{a.s.}
\]

(5)

All these theorems are based on the Shannon-McMillan-Breiman theorem, which claims that \( -\frac{1}{\ell} \log \mathbb{P}(x_1^\ell) \) converges almost surely to the entropy rate \( h \), where \( x_1^\ell \) stands for the cylinder set \( [x_1^\ell] := \{z \in A^\mathbb{N} : z_1^\ell = x_1^\ell \} \). Furthermore, in [27] Kontoyiannis has obtained strong approximations for the recurrence and waiting times to the probability of a finite vector which in turn, let him obtain an almost sure convergence for the waiting time in \( \psi \)-mixing processes, extending previous results for Markov chains. He has also obtained an almost sure invariance principle for \( \log \rho_\ell \) and \( \log \omega_\ell \). This implies that these quantities satisfy a central limit theorem and a law of iterated logarithm [27].

In the same spirit, the work of Abadi and collaborators [24–26] shows very precise results for the approximation of the distribution of the hitting times (properly rescaled) to an exponential distribution under very mild mixing conditions for the process. Also very sharp bounds for the error term for this exponential distribution approximation. This enables to obtain precise results for the fluctuation of the entropy estimators using the hitting times [20, 28, 31].
2.2. Asymptotic behavior of estimators

We are interested in estimating the entropy and the entropy production rates, moreover we want to know that they have good properties of convergence and fluctuations of the estimators, because this will enable us to use our method.

Here, we are interested in estimators defined by recurrence times, for which one can find very precise asymptotic results in the sense of their fluctuations. It is known ([33]) that

$$\lim_{\ell \to \infty} \frac{1}{\ell} \log \rho_\ell = h,$$

converges almost surely in an ergodic process, thus one can use the recurrence time as an estimator of the entropy rate of the process. Furthermore, under the assumption of Gibbsianity, in [34], it is proved that the random variables \((\log \rho_\ell - \ell h) / \sqrt{\ell}\) converges in law to a normal distribution, when \(\ell\) tends to infinite.

In [28], the waiting and also return times are used as estimators, there it is proved that

$$\lim_{\ell \to \infty} \frac{1}{\ell} \log \omega_\ell(x, y) = h,$$

for \(\mathbb{P} \times \mathbb{P}\) almost every pair \((x, y)\), and where the distribution \(\mathbb{P}\) is a Gibbs measure. This is obtained from an approximation of the \(\frac{1}{\ell} \log \omega_\ell\) to the \(-\frac{1}{\ell} \log \mathbb{P}(x_1^\ell)\) which by the Shannon-McMillan-Breiman theorem, the second goes to the entropy almost surely. Also they proved the same log-normal fluctuations for the waiting times, i.e.

$$\lim_{\ell \to \infty} \mathbb{P} \times \mathbb{P} \left\{ \frac{\log \omega_\ell - \ell h}{\sigma \sqrt{\ell}} < t \right\} = \mathcal{N}(0, 1)(-\infty, t),$$

where in this case \(\sigma^2 = \lim_{\ell \to \infty} \frac{1}{\ell} \int (\log \omega_\ell - h)^2 d(\mathbb{P} \times \mathbb{P})\).

So, in the context of Gibbs measures one has the asymptotic normality for both the return times and the waiting times. This can be written in the context of exponential \(\phi\)-mixing processes as well. Moreover in [28] there are also a large deviations principle for both quantities (with some additional restrictions in the case of the return time). For the case of the hitting times one has to consider the possibility of really short returns for which the approximation changes (see [35] for the results for the short return distribution).

In the same context in [20] one can find fluctuation bounds for both, the plug-in estimators and for the waiting and the hitting time estimators. One of the main tools used there is the concentration inequalities that are valid for very general mixing processes. One can obtain non-asymptotic results, that is, upper bounds for the fluctuations of the entropy estimator valid for every \(n\), where \(n\) denotes the length of the sample.

Next, for the case of the entropy production rate estimation, in [30] two estimators of the entropy production were introduced. The entropy production was defined as a trajectory-valued function quantifying the degree of irreversibility of the process producing the samples, in the following way: let \(\mathbb{P}\) be the law of the process and let us denote by \(\mathbb{P}_r\) the law of the time-reversed process, then the entropy production rate is the relative entropy rate of the process with respect to the time-reversed one,

$$e_n = h(\mathbb{P} | \mathbb{P}_r) := \lim_{\ell \to \infty} \frac{H_\ell(\mathbb{P} | \mathbb{P}_r)}{\ell},$$

where
where
\[
H_\ell(P|P_r) := \sum_{x_1^\ell \in A^\ell} P([x_1^\ell]) \log \frac{P([x_1^\ell])}{P([x_1^\ell])},
\]
(10)
Here \(x_1^\ell\) stand for the word \(x_1^\ell\) reversed in order. The estimators defined in [30] using the hitting and waiting times are given as follows
\[
S_\tau^\ell(x) := \log \frac{\tau_{x_1^\ell}(x)}{\tau_{x_1^\ell}(x)},
\]
(11)
where \(\tau_{x_1^\ell}(x) := \inf\{k \geq 1 : x_k^{k+\ell} = x_1^\ell\}\). Notice that the estimator actually quantifies the logarithm of the first time the word \(x_1^\ell\) appears in the reversed sequence divided by the first return time of the first \(\ell\) symbols in \(x\). For the case of the estimator using the waiting time, one has in an analogous way that
\[
S_\omega^\ell(x, y) := \log \frac{\omega_{x_1^\ell}^\ell(x, y)}{\omega_{y}^\ell(x, y)},
\]
(12)
where \(\omega_{x_1^\ell}(y) := \tau_{x_1^\ell}(y)\) and \(\omega_{y}^\ell(x, y) := \tau_{x_1^\ell}(y)\). In the context of Gibbs measures, or exponential \(\psi\)-mixing, in [30] it has been studied the fluctuation properties of such estimators and it has been also proved the consistency, that is, \(P \times P\)-almost surely we have that,
\[
\lim_{\ell \to \infty} \frac{S_\omega^\ell}{\ell} = e_p,
\]
(13)
as well as, \(P\)-almost surely
\[
\lim_{\ell \to \infty} \frac{S_\omega^\ell}{\ell} = e_p.
\]
(14)
The asymptotic normality is also proved in that case the asymptotic variance of the estimator coincides with that of the entropy production. In the same reference the authors also obtain a large deviation principle for the waiting time estimator. Later in Ref. [31] the fluctuation bounds were obtained for the same estimators introduced in [30] under the same setting. This result is interesting from the practical point of view since it provides bounds that are valid for finite time and not only in the asymptotic sense.

Here we will use the approach for estimation entropy production rate defined in Ref. [7], since we want to compare with the exact results one has for Markov chains. In [7] it is shown that the entropy production rate can be obtained as the difference between the entropy rate and the \textit{reversed} entropy rate for Markov processes, for more general systems one can still have a definition of the entropy production in some analogous way (see for instance [5]). The reversed entropy rate is defined as the rate of entropy of the reversed process in time, i.e., as if we were estimating the entropy rate of the process evolving back in time. From the practical point of view, if we have a time series, the entropy production rate could be estimated as the difference between the entropy rate and the entropy rate estimated from the reversed time series. To implement the latter methodology using the recurrence time statistics, in Section 3 we will define the reversed recurrence times which will allow us to give estimations of the reversed entropy rate and eventually the corresponding estimations of the entropy production rate as a measure of time-irreversibility of the process. It is important to mention that the methodology here implemented can be still applied further than only Markov chains, but one expects to obtain results displaying the irreversibility as a consequence of the positivity of the entropy production, and not the exact results.
2.3. Parameter estimation from censored data for a normal distribution

Let us denote $\Theta_\ell$ the random variable whose realizations are estimations of the $\ell$-block entropy rate resulting from the recurrence-time statistics. To precise, $\Theta_\ell$ can be defined as

$$\Theta_\ell = \frac{1}{\ell} \log(T_\ell)$$

(15)

where $T_\ell$ can be the return, hitting, or waiting time random variable. As pointed out above, $\Theta_\ell$ satisfy a central limit theorem regardless the choice of the recurrence time statistics. This fact enables us to assume that $\Theta_\ell$ has a normal distribution, with a mean which we will denote by $h_\ell$ and a variance denoted by $\sigma^2_\ell$. As mentioned before, one of the problems arising in implementing this estimator for real time series is that the return time $T_\ell$ is censored from above by a finite value $T_c$. It is clear from eq. (15) that the latter implies that the random variable $\Theta_\ell$ becomes censored from above by a finite value $h_c := \log(T_c)/\ell$. Taking into account this observation, we can state our problem as follows: given a sample set $\{h_i : 1 \leq i \leq m\}$ of independent realizations of $\Theta_\ell$, we wish to estimate $h_\ell$ and $\sigma_\ell$ knowing that such a sample is censored from above by $h_c$.

It is important to remark that, since the realizations of $\Theta_\ell$ are censored from above by $h_c$, then any sample set $\mathcal{H} := \{h_i : 1 \leq i \leq m\}$ of (independent) realizations of $\Theta_\ell$ will contain realizations numerically undefined; i.e., for some realizations $h_i$ the most we know is that $h_i > h_c$. Despite this fact, as we will see below, this censored sample data will also be used in the estimation of $h_\ell$ and $\sigma_\ell$.

Let us assume that the total number of realizations numerically defined in the sample $\mathcal{H}$ is $k$ while which is clearly less than $m := |\mathcal{H}|$, the size of the sample. Then, the total number of realizations numerically undefined in the sample $\mathcal{H}$ is $m - k$. Since the realizations in the sample are assumed to be independent (a usual hypothesis in statistics) we have that $k$ is a realization of a random variable with binomial distribution. It is easy to see that the fraction $\hat{p} := k/m$ of numerically defined samples with respect to the total of realizations in $\mathcal{H}$ is an estimation of the parameter $p$ of the above-mentioned binomial distribution. As stated above, we assume that $\Theta_\ell$ has normal distribution, which implies that the parameter $p$ is given by,

$$p = \Phi \left( \frac{h_c - h_\ell}{\sigma_\ell} \right),$$

(16)

where $\Phi$ is the distribution function of a standard normal random variable, i.e.,

$$\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy.$$  

(17)

In Appendix A, following calculations from Ref. [36] we show that the parameters $h_\ell$ and $\sigma^2_\ell$ can be estimated by using the censored sample as follows:

$$\hat{h} = \bar{h} + \hat{\zeta} (h_c - \bar{h}),$$

(18)

$$\hat{\sigma}^2 = \bar{s}^2 + \hat{\zeta} (h_c - \bar{h})^2,$$

(19)

where $\bar{h}$ is the mean of the numerically defined samples and $\bar{s}^2$ the corresponding variance estimators, i.e.,

$$\bar{h} := \frac{1}{k} \sum_{i=1}^{k} h_i,$$

(20)
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\[ s^2 := \frac{1}{k} \sum_{i=1}^{k} (h_i - \bar{h})^2. \]  

(21)

Additionally \( \hat{\zeta} \) is defined as

\[ \hat{\zeta} \ := \frac{\phi(\hat{\xi})}{p\xi + \phi(\xi)} \]  

(22)

where \( \hat{\xi} \) is obtained by means of the normal distribution function as

\[ \hat{\xi} := \Phi^{-1}(\hat{\rho}). \]  

(23)

3. Sampling schemes for estimating entropy rate from recurrence-time statistics

Let us start by stating the problem we are concerned in. As we stated above, we are interested in estimating the entropy rate and the entropy production rate from an observed trajectory. The trajectory, in this context, stands for a finite-length symbolic sequence \( x = x_1 x_2 x_3 \ldots x_n \) which is assumed to be generated by some process with an unknown law \( \mathbb{P} \). As we saw in section 2 we have to assume that the process comply with the properties announced there, such as exponential \( \phi \)-mixing, in order for the central limit theorem to be valid. The next step is to obtain samples of the recurrence time statistics, i.e., we need to establish a protocol for extracting samples of hitting, return or waiting times from the sequence \( x \). This method for extracting samples is similar to the one introduced in Ref. [37] for estimating the symbolic complexity and particularly the topological entropy of a process. After that, we will state the corresponding estimator of the entropy rate and entropy production rate by using the fact that the observed samples might be censored.

3.1. Return time

Let us start by establishing the method for obtaining samples of the return-time. Given a sequence \( x \) of size \( 2n \), take two non-negative integers \( \ell \) and \( \Delta \) such that \( \ell < \Delta \ll n \). Then define the set \( \mathcal{M}_\ell \) as follows. First we associate to each word \( a_i \in \mathcal{M}_\ell \) the censored return time and the reversed return time as follows,

\[ \rho_\ell(x) := \inf\{t \geq 1 : x_{k+t} = a_i, t \leq n, \text{ given } a_i := x_{k+\ell-1}\}, \]  

(24)

\[ \rho_\ell(x) := \inf\{t \geq 1 : x_{k+t} = a_i, t \leq n, \text{ given } a_i := x_{k+\ell-1}\}. \]  

(25)

Figure 1. Selection of sample words.

Next we define the sample sets of return times \( \mathcal{R}_\ell \) and reversed return times \( \overline{\mathcal{R}_\ell} \) as follows. First we associate to each word \( a_i \in \mathcal{M}_\ell \) the censored return time and the reversed return time as follows,
Then $\mathcal{R}_\ell$ and $\overline{\mathcal{R}}_\ell$ are defined as

$$\mathcal{R}_\ell := \{ t \in \mathbb{N} : \rho_{\ell}^{(n)}(a) = t, a \in \mathcal{M}_\ell \},$$

$$\overline{\mathcal{R}}_\ell := \{ t \in \mathbb{N} : \rho_{\ell}^{(n)}(a) = t, a \in \mathcal{M}_\ell \}.$$  \hspace{1cm} (26)

It is necessary to stress the fact that the values in the above-defined sample set are not necessarily well defined. This is because the return-time function in eq. (24) is actually censored from above. We should notice that $\rho_{\ell}^{(n)}(a)$ can at most take the value $n$. This requirement is imposed by two reasons: on one hand we have that the return time cannot be arbitrarily large by the finiteness of the trajectory $x$. On the other hand, although it is possible that some sample values in $\mathcal{R}_\ell$ might be larger than $n$, it is not convenient for the statistics. Let us explain this last point with more detail.

If we take a sample word $a$ located at the $k$th site, its corresponding return-time can in principle be at most as large as $n - k - \ell$. This happens when the word $a$ occurs (by chance) at the $(n - k - \ell)$th site. Since all the sample words in $\mathcal{M}_\ell$ are located at different sites along $x$, it is clear that their corresponding return-time values have different upper bounds. Then, if we do not impose a homogeneous upper bound, the collection of return-time samples results in inhomogeneous censored data. As we have seen in section 2.3, having a homogeneous bound (homogeneous censored data) is crucial for implementing the corresponding estimators.

Consequently, by definition each sample value in the sample sets $\mathcal{R}_\ell$ and $\overline{\mathcal{R}}_\ell$ are censored by the same value, which is $n$. In view of this reasoning, it is clear that some of the sample values in $\mathcal{R}_\ell$ and $\overline{\mathcal{R}}_\ell$ might be numerically undefined, in the sense that for some words $a \in \mathcal{M}_\ell$ the most we can know of $\rho_{\ell}^{(n)}(a)$ (or $\rho_{\ell}^{(n)}(\overline{a})$) is that it is larger than $n$. This is the case, for example, when the word $a \in \mathcal{M}_\ell$ is not found along all the trajectory $x$, except for its unique occurrence by which $a$ was included in the sample set $\mathcal{M}_\ell$. In Fig. 2 we give an illustrative description this fact.

![Figure 2](image)

Figure 2. Numerically defined and undefined return-time values. First we suppose that a sample word $a$ occurs at the $k$th site along a finite trajectory $x$ of length $2n$. In order to evaluate the function $\rho_{\ell}^{(n)}(a)$ we should look for the occurrence of $a$ along $x$ in the section that goes from the $(k+1)$th symbol to the $(k+n)$th symbol of $x$. Let us denote this section of the trajectory as $x(k+1, k+n)$.

(A) If $a$ is found in $x(k+1, k+n)$, then $\rho_{\ell}^{(n)}(a)$ is numerically well defined. (B) If $a$ is found in $x$ but not in the section $x(k+1, k+n)$ we consider that $\rho_{\ell}^{(n)}(a)$ is numerically undefined. (C) Finally, if we do not observe any other occurrence of $a$ in $x$ beyond the $(k+1)$th symbol, it is clear that $\rho_{\ell}^{(n)}(a)$ is numerically undefined.

Once we have the return-time sample set $\mathcal{R}_\ell$ we now proceed to state the estimator
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of the entropy rate and the entropy production rate. As we saw in section 2, if we take a return-time value \( t \) from the sample set \( R_\ell \), then the quantity \(-\log(t)/\ell\) can be observed as a realization of the block entropy \( S_\ell \) which in the limit of large \( \ell \) obey a central limit theorem. This fact enables us to implement the following hypothesis: we assume that for finite \( \ell \) the value \( \log(t)/\ell \) is a realization of a normal random variable with (unknown) mean \( h_\ell \) and variance \( \sigma^2_\ell \). Then the sample sets

\[
\mathcal{H}_\ell^p := \{ h = -\log(t)/\ell : t \in R_\ell \},
\]

\[
\overline{\mathcal{H}}_\ell^p := \{ h = -\log(t)/\ell : t \in \overline{R}_\ell \},
\]

which can be considered as sets of realizations of normal random variables censored from above by \( h_c := \log(n/2)/\ell \). Then the estimation procedure for the block entropy is essentially the same described in section 2.3. Here we just summarize the steps for performing the estimation of \( h_\ell \) for return-time statistics.

(i) First define the rate of numerically defined sample values \( \hat{p} := k/m \), where \( m \) is the total number of samples in \( \mathcal{H}_\ell^p \) and \( k \) is the number of numerically defined samples in \( \mathcal{H}_\ell^p \) (henceforth there are \( n - k \) numerically undefined samples in \( \mathcal{H}_\ell^p \)).

(ii) Denote by \( h_i \), for \( 1 \leq i \leq k \), the numerically defined samples in \( \mathcal{H}_\ell^p \). Then define the sample mean and variance as follows

\[
\bar{h} := \frac{1}{k} \sum_{i=1}^{k} h_i,
\]

\[
s^2 := \frac{1}{k} \sum_{i=1}^{k} (h_i - \bar{h})^2.
\]

(iii) Define the sample functions (see section 2.3 and appendix Appendix A for details)

\[
\hat{\zeta} := \frac{\phi(\hat{\xi})}{p\hat{\xi} + \phi(\hat{\xi})},
\]

\[
\hat{\xi} := \Phi^{-1}(\hat{p}).
\]

(iv) Finally, the estimations for the mean of the block entropy and its variance by means of the return-time estimator are given by

\[
h_\ell = \bar{h} + \hat{\zeta}(h_c - \bar{h}),
\]

\[
\hat{\sigma}^2_\ell = s^2 + \hat{\zeta}(h_c - \bar{h})^2.
\]

(v) Repeat all this procedure for the set \( \overline{R}_\ell \) in order to have an estimation of the reversed block entropy rate, which allows to have an estimation of the block entropy production rate just by taking the difference between the block entropy and the reversed block entropy [7].

3.2. Waiting time

The waiting-time estimator for the block entropy requires two distinct trajectories. As we stated above, we assumed that we only have a single trajectory (a single symbolic sequence). In order to have two “trajectories” of the process we split the original one in two equal-sized parts. Here is important to mention that since we are are under the assumption of sufficiently rapid mixing, the beginning of the second half of the sample is practically independent of the first half, provided that the size of the sample
is large, then by splitting the sample in two samples one may consider them as two independent trajectories. Then we collect \( m \) different words at random along one of such trajectories. This collection of \( \ell \)-words, which we denote by \( \mathcal{M}_\ell \), will now play the role of the set of sample words, in a similar way as it was introduced the set \( \mathcal{M}_\ell \) in section 3.1. A schematic representation of this sampling procedure is shown in Fig. 3.

![Figure 3](image)

The next step consists in defining the censored waiting time corresponding to each of the word in the sample \( \mathcal{M}_\ell \). Let us denote by \( \mathbf{x} \) the trajectory consisting of \( 2n \) symbols. Assuming that the sample of word were randomly collected from the segment of the string going from \( n+1 \) to \( 2n-\ell \), i.e. from the string \( \mathbf{x}(n+1, 2n-\ell) \). Then we define the censored waiting time and the censored reversed waiting time for \( \mathbf{a} \in \mathcal{M}_\ell \) as follows,

\[
\omega^{(n)}_\ell (\mathbf{a}, \mathbf{x}) := \inf \{ t \geq 1 : x_{t+\ell-1} = \mathbf{a}\}, \tag{36}
\]

\[
\omega^{(n)}_\ell (\mathbf{a}, \mathbf{x}) := \inf \{ t \geq 1 : x_{t+\ell-1} = \mathbf{a}\}. \tag{37}
\]

It is important to notice that the both, the waiting time and the reversed waiting time are bounded from above by \( n \), i.e., the sample waiting times are homogeneously censored by \( n \).

Actually, once we have established the scheme of sampling for the waiting time, the rest of the method is similar (almost the same) to the one described in section 3.1. Here we only summarize the main steps:

(i) Define the sets of waiting time samples and reversed waiting time samples as

\[
\mathcal{W}_\ell := \{ t \in \mathbb{N} : \omega^{(n)}_\ell (\mathbf{a}, \mathbf{x}) = t, \mathbf{a} \in \mathcal{M}_\ell \},
\]

\[
\overline{\mathcal{W}}_\ell := \{ t \in \mathbb{N} : \omega^{(n)}_\ell (\mathbf{a}, \mathbf{x}) = t, \mathbf{a} \in \mathcal{M}_\ell \}. \tag{39}
\]

(ii) From the sets of waiting-time samples define the sets of block entropy and reversed block entropy

\[
\mathcal{H}_\ell := \{ h = -\log(t)/\ell : t \in \mathcal{W}_\ell \}, \tag{40}
\]

\[
\overline{\mathcal{H}}_\ell := \{ h = -\log(t)/\ell : t \in \overline{\mathcal{W}}_\ell \}. \tag{41}
\]

(iii) Define the rate of numerically defined sample values as \( \hat{p} := k/m \), where \( m \) is the total number of samples in \( \mathcal{H}_\ell \) and \( k \) is the number of numerically defined samples in \( \mathcal{H}_\ell \) (henceforth there are \( n-k \) numerically undefined samples in \( \mathcal{H}_\ell \)).

(iv) Denote by \( h_i \), for \( 1 \leq i \leq k \), the numerically defined samples in \( \mathcal{H}_\ell \). Then define the sample mean and variance as follows

\[
\bar{h} := \frac{1}{k} \sum_{i=1}^{k} h_i. \tag{42}
\]
Estimating entropy rate

\[ s^2 := \frac{1}{k} \sum_{i=1}^{k} (h_i - \bar{h})^2. \]  
(43)

(v) Define the sample functions,

\[ \hat{\zeta} := \frac{\phi(\hat{\xi})}{p\hat{\xi} + \phi(\xi)} \]  
(44)

\[ \hat{\xi} := \Phi^{-1}(\hat{p}). \]  
(45)

(vi) Finally, the estimations for the mean of the block entropy and its variance by means of the return-time estimator are given by

\[ \hat{h}_\ell = \bar{h} + \hat{\zeta}(h_c - \bar{h}), \]  
(46)

\[ \hat{\sigma}^2 = s^2 + \hat{\zeta}(h_c - \bar{h})^2. \]  
(47)

(vii) Repeat steps (iii)-(vi) for the set \( \overline{H}_\ell \) in order to have an estimation of the reversed block entropy rate, which allows to have an estimation of the block entropy production rate just by taking the difference between the estimated block entropy and the reversed block entropy [7].

3.3. Hitting time

To implement the hitting-time estimator we should start by defining the sample sets of words which will be used to compute the hitting-time by looking for its occurrence along a given observed trajectory \( x \). In the case of the hitting-time, unfortunately it is not possible to construct such a word sample set. This is because the hitting-time estimator actually requires a set of words which should be drawn at random by using the process generating the observed trajectory \( x \). In practice, this is however impossible, because we do not known the actual process generating \( x \). We could still avoid this problem if the corresponding set of sample words is obtained by choosing at random \( \ell \)-words from another observed trajectory. However this was the very same method we used for collecting the sample words for the waiting-time estimator. Then, for practical purposes, the hitting-time and waiting-time method can be regarded as the same method from the statistical point of view.

4. Estimations tests

Now we will implement the above stated methods for estimating the block entropy and entropy production. First of all we will perform numerical simulations in order to implement a control test statistics which will be compared with the numerical experiments using the above-exposed methods. Thereafter we will test these methods using symbolic sequences having long-range correlations for which the entropy rate is known exactly.

In section 3 we established two methods for estimating block entropies by using either, the return-time statistics or the waiting-time statistics. These methods assume that we only have a single “trajectory” or, better said, symbolic sequence, obtained by making an observation of real life. Our purpose here is to test the estimators themselves, not the sampling methods. The latter means that we will implement the estimators (20) and (21) for both, the return-time and the waiting-time statistics, without referring to the sampling schemes mentioned in section 3. This is possible because we have access to an unlimited number of sequences, which are produced
Estimating entropy rate

Figure 4. Entropy rate and entropy production rate. (a) We show the behavior of the entropy rate and the time-reversed entropy rate as a function of the parameter $p$ using the exact formulas given in eqs. (49) and (50). (b) We display the behavior of the entropy production rate as a function of $p$ using the exact formula (51).

numerically with a three-states Markov chain. In this sense we have control of all of the parameters involved in the estimators, namely, the length of the block $\ell$, the entropy threshold $h_c$ (by which the recurrence-time samples are censored) and the sampling size $|H_\ell|$.

4.1. Finite-state Markov chain

For numerical purposes we consider a Markov chain whose set of states is defined as $A = \{0, 1, 2\}$. The corresponding stochastic matrix $P : A \times A \rightarrow [0, 1]$ is given by,

$$P = \begin{pmatrix} 0 & p & 1 - q \\ 1 - q & 0 & q \\ q & 1 - q & 0 \end{pmatrix},$$

(48)

where $q$ is a parameter such that $q \in [0, 1]$. It is easy to see that this matrix is doubly stochastic and the unique invariant probability vector $\pi = \pi P$ is given by $\pi = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Moreover, it is easy to compute the entropy rate and the time-reversed entropy for this system, giving,

$$h(q) = -q \log(q) - (1 - q) \log(1 - q),$$

(49)

$$\bar{h}(q) = -(1 - q) \log(q) - q \log(1 - q).$$

(50)

Additionally, it is also possible to see that the corresponding entropy production rate is given by

$$e_p(q) = (2q - 1) \log \left( \frac{q}{1 - q} \right).$$

(51)

The behavior of the entropy rate and entropy production rate can be observed in Figure 4. We will use this model to generate symbolic sequences for testing the estimators.

4.2. Statistical features of estimators for censored data

The first numerical experiment we perform is intended for displaying the statistical properties of the estimators without implementing the sampling schemes introduced
above. To this end we obtain a censored sample set of $5 \times 10^4$ return times from the several realizations of the three-state Markov chain. We obtained each return time as follows. First we initialize the Markov chain at the stationary state (i.e., we choose the first symbol at random using the stationary vector of the chain) and evolve the chain in time. This procedure generates a sequence which grows in time, say for instance $a_1, a_2, \ldots, a_t$. The evolution of the Markov chain will be stopped either, until the first $\ell$-word $a_1, a_2, \ldots, a_\ell$ equals the last one, i.e., $a_1, a_2, \ldots, a_t = a_{t-\ell+1}, a_{t-\ell+2}, \ldots, a_t$ or when the time $t - \ell + 1$ exceeds a given bound $T_c$. Then, the corresponding return time will be either, $\rho := t - \ell + 1$ or an undefined value $\rho > T_c$.

Once we have collected the sample set of return times $\{\rho_i\}$ we obtain a set of block entropy rate by means of the transformation

$$h_i = -\frac{\log(\rho_i)}{\ell},$$

if $\rho_i$ is numerically defined. Of course, we might obtain some numerically undefined sample block entropies $h_i > h_c$ because of the censored return times.

We also obtain a sample set of reversed entropy rates following the above-described procedure. We evolve in time the Markov chain and stop the evolution until the first $\ell$-word $a_1, a_2, \ldots, a_\ell$ is such that its reversal equals the last one, i.e., $a_\ell, a_{\ell-1}, \ldots, a_1 = a_{t-\ell+1}, a_{t-\ell+2}, \ldots, a_t$ or the time or until the time $t - \ell + 1$ has exceeded the upper bound $T_c$. The reversed return time will be $\rho = t - \ell + 1$ or numerically undefined if $t - \ell + 1$ has exceeded the upper bound $T_c$. Then we obtain the sample set $\{h_i\}$ by means of the transformation

$$h_i = -\frac{\log(\rho_i)}{\ell}.$$
and several values of $h_c$. Correspondingly, Figure 6 we show the histogram of relative frequencies of the reversed block entropy rate for $\ell = 10$, $q = 0.60$ and several values of $h_c$.

We can appreciate how the density of the block entropy rate is censored maintained $\ell$ fixed. If the value of $h_c$ is small, most of the samples are numerically undefined because they are censored from above. This is seen for instance, in Figure 5a, in which $h_c$ takes the smallest value for the displayed graphs. In this case, approximately a 25% of the samples are numerically defined resulting in the ‘partial’ histogram displayed in Figure 5a. In Figure 5b the value of $h_c$ increases which causes the ‘growing’ of the histogram. In the remaining graphs, from Figure 5c to Figure 5d, this tendency is clear, as we increase the value of $h_c$ the number of numerically defined samples grows thus completing gradually the corresponding histogram. Something similar occur.

**Figure 6.** Reversed return-time entropy density for $q = 0.60$ and $\ell = 10$. (a) $h_c = 0.48$, (b) $h_c = 0.57$, (c) $h_c = 0.66$, (d) $h_c = 0.75$, (e) $h_c = 0.84$, (f) $h_c = 1.02$.

![Figure 6](image)

**Figure 7.** Entropy estimated by means of the return-time statistics for the three-states Markov chain We show the histograms of the estimated entropy density for $q = 0.60$, $h_c = 1.155$ and (a) $\ell = 6$, (b) $\ell = 9$, (c) $\ell = 12$, (d) $\ell = 15$, (e) $\ell = 18$, (f) $\ell = 19$. We obtained the corresponding histograms using $5 \times 10^4$ sample words in each case.

![Figure 7](image)
On the other hand, if we keep $h_c$ constant and vary the block length $\ell$ we can appreciate the evolution of the histogram towards a normal-like distribution. We show this effect in Figure 7 for $q = 0.60$, $h_c = 1.155$ fixed. This is in agreement with the central limit theorem, as we have mentioned in previous sections. In Figure 7 we show the histograms for $\ell = 6, 9, 12, 15, 18$ and 19 (panels (a)–(f) respectively). We can see that for the lower value of $\ell$, the histogram is rather irregular, which means that the central limit theorem is not still manifested for the block entropy rate. We can also observe that increasing the block length, the histogram progressively evolve towards a bell-shaped distribution, which is reminiscent of the normal one. This shows that an estimation using our approach could be more accurate for large values of block lengths.

![Figure 8](image1.png)

**Figure 8.** Return-time entropy estimations as a function of $h_c$ for several values of $\ell$. Panel (a): The graphics shows the behavior of $\hat{h}$ as we increase the entropy threshold $h_c$ for $\ell = 6$ (filled squares), $\ell = 9$ (filled triangles), $\ell = 12$ (filled circles), $\ell = 15$ (X’s), $\ell = 18$ (stars), $\ell = 19$ (plus). Panel (b): it is shown the behavior of the estimated reversed entropy for the same parameter values used in panel (a).

![Figure 9](image2.png)

**Figure 9.** Waiting-time entropy estimations as a function of $h_c$ for several values of $\ell$. Panel (a): The graphics shows the behavior of $\hat{h}$ as we increase the entropy threshold $h_c$ for $\ell = 6$ (filled squares), $\ell = 9$ (filled triangles), $\ell = 12$ (filled circles), $\ell = 15$ (X’s), $\ell = 18$ (stars), $\ell = 19$ (plus). Panel (b): it is shown the behavior of the estimated reversed entropy for the same parameter values used in panel (a).
Once we have the sample set of block entropy rates we use the estimation procedure for censored data as described in Section 3. We perform this procedure for the entropy rates and reversed entropy rates obtained from the return-time and waiting-time statistics.

In Figure 8 we show the estimation of the block entropy rate and the reversed block entropy rate for the return-time statistics. In Figure 8a, the displayed curves (solid black lines) show the behavior of the estimation of the block entropy rate as a function of the censoring bound $h_c$ for several values of $\ell$. This figure exhibits two important features of our estimation technique. Firstly we notice that the estimation of the entropy rate has large fluctuations for small $h_c$. We can say that the smaller $h_c$ the larger statistical errors are observed. Secondly we should observe that the larger $\ell$ the better estimation. The latter can be inferred from the fact that the curve with the largest value of $\ell$ in Figure 8a is closest to the exact entropy rate (solid red line). A similar behavior occurs for the reversed block entropy rate estimations shown in Figure 8b.

For the waiting-time statistics an analogous effect occurs. In Figure 9 it is shown the curves for the estimations of the block entropy rate, in panel (a), and the reversed block entropy rate, in panel (b). As expected, the estimations for small values of censoring bound $h_c$ have large fluctuations which gradually decrease as $h_c$ is getting increased. This is clearly observed in Figure 9 because the black solid lines deviate largely from the exact value (solid red line) for small values of $h_c$. In concerning the value of $\ell$ it is clear that for the largest value of $\ell$ the estimation is closest to the exact entropy rate for $h_c$ large enough (see insets in Figure 9).

All these observations allows us to state that, for obtaining the best estimations (as far as possible within the present scheme) we should keep $h_c$ as large as possible. Similarly, in order for the central limit to be valid we should take the block length $\ell$ as large as possible.

4.3. Testing estimations in a single sequence obtained from a Markov chain

Now we turn our attention to the implementation of the estimations of block entropy rate using the schemes described in Section 3. For this purpose we first generate a single sequence of $N = 12 \times 10^6$ symbols by means of the three-states Markov chain. Then, we implement the sampling schemes for the return-time and waiting-time statistics. In each case we collect $10^5$ sample words which will correspond to $m = 10^5$ samples of block entropy rates and reversed block entropy rates. These sample sets contains both, numerically defined and undefined samples because of the censoring. In this case, the censoring bound for entropy rate $h_c$ is determined by

$$h_c := -\frac{\log(N/2)}{\ell}. \quad (53)$$

We should emphasize that in the present case we have control only on a single parameter, namely, the length $\ell$ of the block. Contrary to the above exposed numerical experiments, in this case $h_c$ is no longer a free parameter; it is actually determined by means of the length of the symbolic sequence $N$ and the length $\ell$ of the block. Consequently, changes in the values of $\ell$ implies changes in the value of $h_c$. The latter is important by two reasons: on one hand we have that, in order to assure the validity of the central limit theorem we should take $\ell$ as large as possible (actually, the entropy rate is obtained in the limit $\ell \to \infty$). On the other hand, it is desirable to have non-censored samples, i.e., it is convenient to have $h_c$ as large as possible. However, in
practice, we cannot comply with both requirements because of expression (53): the larger $\ell$ the shorter $h$ whenever the length $N$ of the symbolic sequence is maintained constant.

![Figure 10](image_url)

Figure 10. Estimation of block entropy rate as a function of $\ell$. Black lines stand for the estimated block entropy rate and red lines are the exact entropy rate. We show the curves corresponding to the Markov chain parameter $q = 0.50$ (solid lines), $q = 0.60$ (dotted lines), $q = 0.70$ (dashed lines), $q = 0.80$ (dotted–dashed lines) and $q = 0.90$ (double-dotted–dashed lines). (a) Block entropy rate estimations using the return time statistics. (b) Same as in (a) using waiting-time statistics. (c) Reversed block entropy rate estimations using the return-time statistics. (d) Same as in (c) using waiting-time statistics.

An important consequence of the latter consists in the fact that we cannot make $\ell$ as large as we want. Actually, the maximal block length $\ell$ that it is possible to analyze for entropy estimations is determined by the accuracy we would like to obtain. This is because for large $\ell$ we have a short censoring upper bound, implying that a few samples for block entropy rates are numerically defined. This clearly imply a loss of accuracy since, the less numerically defined samples, the larger variance of the estimators. This phenomenon can be observed in Figure 10 for several values of the parameter $q$ of the three-states Markov chain defined in Section 4.1.

In Figure 10a we show the estimation of block entropy rate as a function of $\ell$ using the return-time statistics. In such a figure, red lines show the exact value of the entropy rate given in eq. (49) and black lines correspond to estimations of the block entropy rate using the return-time statistics with the sampling scheme described in Section 3.1. Figure 10b we show the same as in Figure 10a but using the waiting-time statistics with the corresponding sampling scheme described in Section 3.1. Figures 10c and 10d show the corresponding curves for the reversed block entropy rate for the return-time and the waiting-time statistics respectively.

We should observe that the all the curves for the estimated block entropy rate
have a common behavior that we anticipated above: for small and large values of $\ell$ the estimated entropy deviates visibly from the exact value. Conversely, there is a special value of $\ell$ for which the estimation seems to be optimal. As we explained before, this phenomenon is a consequence of the fact that as the value of $\ell$ is getting increased, the number of numerically defined samples decreases because of the censoring. Then, we must state a criterium in order to have the optimal block length $\ell^*$ for which the estimation of entropy rate is optimal. The criterium for obtaining $\ell^*$ might not be unique and here we use a simple one. First of all we should have in mind that once the value of $\ell$ is chosen, the censoring bound is fixed according to eq. (53). This bound in turns determine the number of numerically defined samples; the shorter $h_c$, the lower number $k$ of numerically defined samples. Due to relationship (53) we can also say that the larger $\ell$, the lower number $k$ of numerically defined samples. A simple way to optimize this interplay between $\ell$ and $k = k(\ell)$ is taking the block length $\ell^*$ for which $k(\ell^*)$ is as close as possible to the half of the sample size $m$.

Using this criterium we compute the optimal block length $\ell^*$ for several values of the parameter $q$ of the Markov chain. In Tables 3 and 4 we show the value $\ell^*$ for $q = 0.50$, $q = 0.60$, $q = 0.70$, $q = 0.80$, and $q = 0.90$ for the return-time and

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**Table 1.** Block entropy estimations using the return-time statistics.

| Parameters  | Estimations |
|-------------|-------------|
| $q$   | $\ell^*$ | $\hat{p}$ | $\hat{h}$ | $\Delta \hat{h}$ | $\Delta \hat{h}/h$ |
| 0.5  | 22 | 0.6066 | 0.692325 | 0.000822 | 0.001187 |
| 0.6  | 23 | 0.5488 | 0.699906 | 0.003106 | 0.004630 |
| 0.7  | 25 | 0.5118 | 0.607702 | 0.003162 | 0.005203 |
| 0.8  | 30 | 0.5880 | 0.496892 | 0.003510 | 0.007064 |
| 0.9  | 30 | 0.9250 | 0.328234 | 0.003151 | 0.009600 |

**Table 2.** Block entropy estimations using the waiting-time statistics.

| Parameters  | Estimations |
|-------------|-------------|
| $q$   | $\ell^*$ | $\hat{p}$ | $\hat{h}$ | $\Delta \hat{h}$ | $\Delta \hat{h}/h$ |
| 0.5  | 22 | 0.6096 | 0.691518 | 0.001630 | 0.002357 |
| 0.6  | 23 | 0.5460 | 0.670477 | 0.002535 | 0.003781 |
| 0.7  | 26 | 0.4590 | 0.609711 | 0.001153 | 0.001891 |
| 0.8  | 30 | 0.5926 | 0.396028 | 0.003153 | 0.006818 |
| 0.9  | 30 | 0.9242 | 0.333756 | 0.008673 | 0.025986 |

**Table 3.** Time-reversed block entropy estimations using the return-time statistics.

| Parameters  | Estimations |
|-------------|-------------|
| $q$   | $\ell^*$ | $\hat{p}$ | $\hat{h}$ | $\Delta \hat{h}$ | $\Delta \hat{h}/h$ |
| 0.5  | 22 | 0.6104 | 0.691449 | 0.001698 | 0.002456 |
| 0.6  | 21 | 0.4620 | 0.751255 | 0.002850 | 0.003794 |
| 0.7  | 17 | 0.4504 | 0.933973 | 0.015810 | 0.016928 |
| 0.8  | 12 | 0.5586 | 1.272741 | 0.059438 | 0.046701 |
| 0.9  | 8  | 0.4642 | 1.981793 | 0.101070 | 0.056999 |
Table 4. Time-reversed block entropy estimations using the waiting-time statistics.

| Parameters | Estimations |
|------------|-------------|
| $q$ | $\ell^*$ | $\hat{p}$ | $\hat{h}$ | $\Delta \hat{h}$ | $\Delta \hat{h}/h$ |
| 0.5 | 22 | 0.6094 | 0.692784 | 0.000363 | 0.000524 |
| 0.6 | 21 | 0.4612 | 0.751243 | 0.002861 | 0.003808 |
| 0.7 | 16 | 0.6494 | 0.938406 | 0.011377 | 0.012124 |
| 0.8 | 12 | 0.5286 | 1.285069 | 0.047112 | 0.046661 |
| 0.9 | 8 | 0.4472 | 1.999906 | 0.082957 | 0.041480 |

Table 5. Entropy production estimations from return and waiting time statistics.

| Parameters | Return | Waiting |
|------------|--------|---------|
| $q$ | $\Delta \epsilon_p$ | $\Delta \epsilon_p$ |
| 0.50 | -0.000876 | 0.000876 | 0.001266 | 0.001266 |
| 0.60 | 0.081349 | 0.000256 | 0.080766 | 0.000327 |
| 0.70 | 0.326271 | 0.012648 | 0.326965 | 0.010224 |
| 0.80 | 0.755849 | 0.055928 | 0.789039 | 0.042738 |
| 0.90 | 1.653559 | 0.104221 | 1.666150 | 0.091630 |

the waiting-time statistics respectively. We also show the corresponding values of the corresponding estimations of the block entropy rate and compare them with the corresponding exact values. We can appreciate from these tables that the relative error $\Delta \hat{h}/h$ (the relative difference between the estimation and the exact value) is maintained below 0.06. Moreover, for $q = 0.50$ and $q = 0.60$ the relative errors are even less than 1%. In Figure 11 we show both the block entropy rate (panel a) and the reversed block entropy rate (panel b) as a function of the parameter $q$. In such a figure the estimation corresponding to the return time and waiting time statistics are compared with the exact value. We can observe that the return time and waiting time statistics have approximately the same accuracy. From Figure 11 we can also observe an interesting behavior of the estimation, which is the fact that the larger entropy rate the larger deviation from the exact result. This effect can actually be explained as follows. First we should have in mind that the return time and waiting time is a measure of the recurrence of the system. This means that the entropy rate itself can in some way be interpreted as a measure of the recurrence per unit length of the word (this is a consequence of the fact that the logarithm is a one-to-one function). Thus becomes clear that the larger entropy rate the larger recurrence times in the system. Since all the samples are censored from above it is clear that a system having larger recurrence times will have larger errors in estimations. Therefore we have that a system with large entropy rate will exhibit large statistical errors in its estimations. Despite this effect we can observe in Figure 11 that the errors in the estimations are sufficiently small for practical applications.

Finally we show in Table 5 the entropy production of the system by taking the difference between the block entropy rate and the reversed block entropy rate, for both, the return and the waiting time statistics. It is important to remark that these recurrence statistics are consistent each other, having moderate deviations (statistical errors) when compared with the exact values.
Figure 11. Estimation of block entropy rate and reversed block entropy rate as a function of $p$. (a) It is shown the block entropy rate estimated from the return time statistics (black filled circles) and the waiting time statistics (red filled squares). We also show the corresponding exact values of the entropy rate (black solid line) of the system to compare these estimations. In panel (b) the same as in panel (a) for the reversed entropy rate.

5. Conclusions

The entropy rate is the limit of the block entropy rate when the block length goes to infinity. Attaining the limit of infinite block length is impossible in practice. Moreover, estimating block entropy rate from empiric measures would require a large amount of data even for moderately large block lengths. Estimating the block entropy rate for finite block length yet exhibit several difficulties in making the corresponding estimations. Particularly in this work we have shown that the finiteness of the observed trajectories results in errors that can be associated to censored samples of the entropy rate.

We have studied estimators for entropy rate defined from the recurrence time statistics; specifically the return-time and waiting-time estimators. Taking into account the problem of the finiteness of the observed trajectory, we made use of the theory of censored samples from statistics to obtain improved estimators for the entropy rate. Within this point of view we established a couple of sampling schemes for the return times and waiting times in order to implement the corresponding maximum likelihood estimators for censored normal distribution. The latter is justified by the assumption that entropy rate estimator comply with the central limit. These results show that there is some compromise between the length of the words used for the estimation and the size of the sample itself. This has to be considered in order to obtain the optimal estimation given a sample. The protocols we define are in some sense a new technique since we take advantage of combining the approach of the recurrence time statistics for estimating the entropy rate (and entropy production rate) and the existing tools for censored data statistics.

We would like to stress the importance of this approach, since it might be applicable for more general systems than Markov chains. The main assumptions we have made are applicable for more general dynamical and stochastic systems satisfying the mixing property up to certain degree of strength. Nevertheless, in the specific case of entropy production rate, using directly the protocol here defined, one would obtain estimators of certain indexes of irreversibility, instead of direct entropy production rate.
Finally, we have defined protocols applicable for time series coming from real data, and thus important for practical purposes, this is clearly because of finite size nature of real world time series.

Acknowledgements

Appendix A. Maximum Likelihood estimators for normal censored samples.

In this appendix we derive formulas (18) and (19) for estimating the mean and variance (respectively) of the entropy rate assuming normal distribution. These formulas essentially correspond to the maximum likelihood estimations of the mean and variance of a normal distribution with samples censored from above. Although the derivation of these estimators are found in Ref [36], we include the following calculations for the sake of completeness of the present work.

Let $\Theta$ be a random variable normally distributed with mean $h$ and variance $\sigma^2$. Let $H := \{h_i : 1 \leq i \leq m\}$ be a sample of independent realizations of $\Theta$ censored from above, i.e., a given sample $h_i$ is either, numerically well defined in the sense that it has a specific numerical value, or numerically undefined in the sense that the most we know is that $h_i$ has a value that has exceeded a censoring threshold that will be denoted by $h_c$. We order the sample set in such a way that the first $k$ ($k \leq m$) samples are numerically defined, i.e. $h_i$ is numerically defined for $1 \leq i \leq k$ and numerically undefined if $k + 1 \leq i \leq m$. Let us also denote by $\hat{p} := k/m$ as the fraction of numerically defined samples in the whole sample set $H$. It is clear that $\hat{p}$ is an estimation of the probability $p$ that the sample be below the censoring threshold, or in other words, the probability that a sample be numerically defined. Since $\Theta$ is assumed to be normal we have that

$$ p := \Phi\left(\frac{h_c - h}{\sigma}\right), $$

where $\Phi$ is the distribution function of a standard normal random variable, i.e.,

$$ \Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy. $$

Next, the likelihood function, which can be interpreted as the probability of the occurrence of the collected samples, is given by

$$ L(h, \sigma^2; H) := \left( \prod_{i=1}^{k} \left( \frac{\Delta h}{\sigma} \phi\left(\frac{h_i - h}{\sigma}\right) \right) \right) (1 - p)^{m-k}, $$

where we denoted by $\phi$ the probability density function of the standard normal distribution, $\phi(x) := e^{-x^2/2}/\sqrt{2\pi}$. It is not hard to see that the logarithm of the likelihood function (some times also called loglikelihood function) can be written as

$$ \log L(h, \sigma^2; H) := \left( -\sum_{i=1}^{k} \frac{(h_i - h)^2}{2\sigma^2} \right) + k \log\left( \frac{\Delta h}{\sqrt{2\pi}} \right) $$

$$ - \frac{k}{2} \log(\sigma^2) + (m-k) \log\left(1 - \Phi\left(\frac{h_c - h}{\sigma}\right) \right). $$
Now, in order to obtain the maximum likelihood estimations we need to maximize the loglikelihood function with respect to the parameters \( h \) and \( \sigma^2 \). After some calculations it is possible to see that the first derivatives of \( \log L \) with respect to \( h \) and \( \sigma^2 \) are given by

\[
\frac{\partial \log L}{\partial h} = \sum_{i=1}^{k} \frac{h_i - h}{\sigma^2} + \left( \frac{m-k}{\sigma} \right) \frac{\phi \left( \frac{h-c-h}{\sigma} \right)}{1 - \Phi \left( \frac{h-c-h}{\sigma} \right)},
\]

(A.5)

\[
\frac{\partial \log L}{\partial \sigma^2} = \sum_{i=1}^{k} \frac{(h_i - h)^2}{2\sigma^4} - \frac{k}{2\sigma^2} + (m-k) \left( \frac{h_i - h}{2\sigma^3} \right) \frac{\phi \left( \frac{h_c-h}{\sigma} \right)}{1 - \Phi \left( \frac{h_c-h}{\sigma} \right)}.
\]

(A.6)

To maximize \( \log L \) we have to equate to zero the above partial derivatives. The solutions will correspond to the maximum likelihood estimations for the mean and variance of the distribution. Then we can write,

\[
\sum_{i=1}^{k} \frac{h_i - h}{\sigma^2} + \left( \frac{m-k}{\sigma} \right) \frac{\phi \left( \frac{h_c-h}{\sigma} \right)}{1 - \Phi \left( \frac{h_c-h}{\sigma} \right)} = 0,
\]

(A.7)

\[
\sum_{i=1}^{k} \frac{(h_i - h)^2}{2\sigma^4} - \frac{k}{2\sigma^2} + (m-k) \left( \frac{h_i - h}{2\sigma^3} \right) \frac{\phi \left( \frac{h_c-h}{\sigma} \right)}{1 - \Phi \left( \frac{h_c-h}{\sigma} \right)} = 0.
\]

(A.8)

For further calculations it is important to have a short-hand notation, then we define the following quantities. First we denote by \( \bar{h} \) and \( s^2 \) the sample mean and variance, respectively, as

\[
\bar{h} := \frac{1}{k} \sum_{i=1}^{k} h_i,
\]

(A.8)

\[
s^2 := \frac{1}{k} \sum_{i=1}^{k} (h_i - \bar{h})^2.
\]

(A.9)

Next we denote by \( \xi \) the following,

\[
\xi := \frac{h_c-h}{\sigma}.
\]

(A.10)

In terms of the above quantities it is possible to see that equations (A.7) and (A.7) can be rewritten as

\[
\frac{\bar{h} - h}{\sigma} + \left( \frac{1-p}{p} \right) \frac{\phi (\xi)}{1 - \Phi (\xi)} = 0,
\]

(A.11)

\[
s^2 + (\bar{h} - h)^2 - \frac{1}{2} \left( \frac{(1-p)\xi}{2p} \right) \frac{\phi (\xi)}{1 - \Phi (\xi)} = 0.
\]

(A.12)

Equations (A.11) and (A.12) can be further simplified as follows. First let us denote by \( \Omega \) the combination

\[
\Omega := \left( \frac{1-p}{p} \right) \frac{\phi (\xi)}{1 - \Phi (\xi)}.
\]

(A.13)

Then equations (A.11) and (A.12) can be rewritten as

\[
h - \bar{h} = \sigma \Omega
\]

(A.14)

\[
s^2 + (\bar{h} - h)^2 = \sigma^2 [1 - \xi \Omega].
\]

(A.15)
Now we can use equation (A.11) into equation (A.12) to eliminate the dependence on \( h - \bar{h} \). This results in
\[
s^2 = \sigma^2 \left[ 1 - \xi \Omega - \Omega^2 \right],
\]  
(A.16)
or, equivalently, as
\[
\sigma^2 = s^2 + \sigma^2 \left[ \xi \Omega + \Omega^2 \right],
\]  
(A.17)
On the other hand, recalling the definition of \( \xi \), we can write \( h = h_c - \sigma \xi \). Using this identity into equation (A.11) we obtain
\[
\sigma = \frac{h_c - \bar{h}}{(\xi + \Omega)}.  
\]  
(A.18)
Using the last identity into equation (A.17) we obtain
\[
\sigma^2 = s^2 + \frac{\xi \Omega + \Omega^2}{(\xi + \Omega)^2} \left( h_c - \bar{h} \right)^2 = s^2 + \frac{\Omega}{\xi + \Omega} \left( h_c - \bar{h} \right)^2 .
\]  
(A.19)
The identity (A.18) can also be used into equation (A.14) which results in
\[
h = \bar{h} + \frac{\Omega}{(\xi + \Omega)} \left( h_c - \bar{h} \right) .
\]  
(A.20)
Next, we denote by \( \zeta \) the combination \( \Omega/(\xi + \Omega) \), a quantity which appears in the expression for \( \sigma^2 \) and \( h \). Recalling that \( p = \Phi(\xi) \), which is a consequence of expressions (A.1) and (A.10), we can see that \( \Omega = \phi(\xi)/p \). Then, some algebraic manipulations show that
\[
\zeta := \frac{\Omega}{\xi + \Omega} = \frac{\phi(\xi)/p}{\xi + \phi(\xi)/p} .
\]  
(A.21)
or, equivalently,
\[
\zeta = \frac{\phi(\xi)}{p \xi + \phi(\xi)} .
\]  
(A.22)
In terms of \( \zeta \) we have that
\[
h = \bar{h} + \zeta \left( h_c - \bar{h} \right) ,
\]  
(A.23)
\[
\sigma^2 = s^2 + \zeta \left( h_c - \bar{h} \right)^2 .
\]  
(A.24)
Equations (A.22), (A.23), and (A.24) are the expressions anticipated in section 2.3.

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