Stable and decaying bound states on the naked Reissner-Nordström background

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Abstract

We present a quantum mechanical study of the bound states of a neutral scalar particle on the curved background described by the Reissner-Nordström (RN) spacetime that corresponds to a naked singularity. We show that there occurs both the metastable, i.e. decaying bound states and the stable ones. The corresponding energy spectra are calculated in the leading WKB approximation and the comparison with the RN black hole quasinormal mode spectrum is made. The metastable bound states on the naked singularity background turn out to be more long-living than the quasinormal modes for the black hole with the same mass.

1 Introduction

In general relativity, the probe with test particles and waves is an efficient tool to study the properties of spacetime near gravitating mass [1–6]. When using quantum test particles, one observes some features that are unexpected from the viewpoint of classical theory. In particular, it is well established that quantum test particles cannot form stable stationary bound states in the field of the Schwarzschild and Reissner-Nordström (RN) black holes [7]. Instead, there exist decaying quasistationary states known as the quasinormal modes (see Refs. [8, 9] for review).

However, if in the RN setup the gravitating mass $M$ and the electric charge $Q$ obey the superextremality condition, $Q > M$, the space-time geometry corresponds not to a black hole but to a naked singularity. At the classical level, this results in a repulsive potential barrier near the origin that prevents the absorption of neutral test particles [10, 11]. Therefore one may anticipate that the bound states are stable at the quantum level as well.

The main goal of this article is to verify the above proposition and to derive the energy spectrum of the bound states of a quantum test particle on the naked RN background.

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Probing the naked RN singularity, one faces two key problems. The first one concerns physical relevance. According to the cosmic censorship conjecture by Penrose [13], any singularity originated from gravitational collapse must be hidden inside an event horizon. Besides, since the RN singularity is electrically charged, it may be neutralized by spontaneous pair creation provided that charges are sufficiently high [14]. Nevertheless, one cannot exclude scenarios in which the RN geometry with $Q > M$ is an exterior solution connected with a regular interior solution with matter, for instance, the Friedmann solution in the model of friedmon [15]. Such models can be traced back to Dirac’s suggestion to describe leptons as charged shells [16]. Recall that it is $Q/M = 2.0 \cdot 10^{21}$ for the electron, so that the exterior geometry must correspond to the naked singularity.

The second problem is the non-uniqueness of the time evolution for test particles and waves on the naked RN background [5,17,18]. This means one has to specify an additional boundary condition at the singularity to obtain a fully unique dynamics. In the present work, we overcome this shortcoming by employing the WKB approach, which turns out to be insensitive to behavior in the vicinity of the singularity.

The work uses a simple quantum mechanical model. We consider a neutral scalar particle on the background geometry; the word “particle” may refer equally to the test scalar field perturbation. Both massless and massive cases are studied. This investigation expands our previous work [19], which was restricted to the stable bound states of a massive particle.

The plan of the article is as follows. In Section 2 we examine how particle wave functions behave in the vicinity of the singularity and advocate the WKB approximation. In Section 3 the effective potential is analyzed to distinguish between stable and decaying bound states. Section 4 is devoted to the calculation of the energy spectra through the WKB method. The energies obtained are then discussed and compared with the known quasinormal mode spectra for the RN black holes in Section 5. Finally, Section 6 presents our conclusion.

2 Behavior in the vicinity of a RN singularity

To start with, we establish the behavior of a particle wave function in the vicinity of a RN singularity and study how it is accommodated in the WKB approximation. The external RN geometry is described by the metric (in units with $G = \hbar = c = 1$)

$$ds^2 = Fdt^2 - F^{-1}dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2), \quad F = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}. \quad (1)$$

For concreteness, we assume that $Q > 0$.

The Klein-Gordon equation for a neutral scalar particle of mass $m$ on this background

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu}(\sqrt{-g} g^{\mu\nu} \frac{\partial \Psi}{\partial x^\nu}) + m^2 \Psi = 0$$

(2)
has stationary state solutions, which we decompose into the spherical harmonics

\[ \Psi(t, r, \theta, \phi) = \exp\left( -\frac{i\omega t}{\hbar} \right) Y_{lm}(\theta, \phi) \frac{1}{r} \sqrt{F(u)} \]  

Then the radial wave function satisfies the equation

\[ \frac{d^2u}{dr^2} + \frac{1}{F(r)} \left[ \frac{1}{F(r)} \left( \omega^2 - \frac{Q^2 - M^2}{r^2} \right) - m^2 - \frac{l(l+1)}{r^2} \right] u = 0 \]  

where \( l = 0, 1, 2, \ldots \) is the orbital quantum number. In principle, this can be reduced to the confluent Heun equation [20], upon substituting

\[ u(r) = \exp\left( \sqrt{m^2 - \omega^2 r} \right) (M - \sqrt{M^2 - Q^2 - r})^\alpha (M + \sqrt{M^2 - Q^2 - r})^\beta y(r) \]  

with \( \alpha \) and \( \beta \) given by bulky expressions in terms of \( Q, M \) and \( m \). The so-obtained confluent Heun equation has singularities at

\[ r = M \pm \sqrt{M^2 - Q^2} \]  

(black-hole horizons) and \( r = \infty \). However, in the super-extreme case under consideration these singularities are shifted away from the real line because \( M^2 - Q^2 < 0 \). As a consequence, the reduction to the confluent Heun form does not simplify the problem.

It is more useful to note that the radial equation (4) holds invariant under rescaling

\[ Q \to \alpha Q, \quad M \to \alpha M, \quad m \to \frac{1}{\alpha} m, \quad \omega \to \frac{1}{\alpha} \omega, \quad R \to \alpha r \]  

and thus one of the three parameters \( Q, M \) and \( m \) can be fixed arbitrarily.

Close to the naked singularity at \( r = 0 \), the radial equation simplifies

\[ \frac{d^2u}{dr^2} - \frac{[1 + l(l+1)]Q^2 - M^2}{Q^4} u = 0, \quad r \to 0. \]  

If we now adopt the leading WKB approximation, choosing the radial wave function in the form \( u(r) \propto \exp(iS(r)) \), then in the super-extreme case we have \( S''(r) < 0 \) for small \( r \). Hence, the naked singularity is located in the classically forbidden region. As for the bound states we are interested in, this means that the wave function asymptotics is dominated by the term which decreases exponentially as \( r \) approaches 0. Note that for small \( r \) the condition of the WKB-method applicability \( |S''(r)/S'(r)| \ll 1 \) is still satisfied.

Nevertheless, the exact behavior of the wave function at \( r \to 0 \) remains undetermined. The reason is that in a pair of linearly independent solutions to Eq. (4) both solutions are locally normalized at \( r \to 0 \) and thus none of them can be discarded. As seen from Eq. (4), such a pair may easily be composed by taking a solution with \( u(0) = 0 \) and another one with \( u'(0)|_{r=0} = 0 \). In the general case, one should consider a linear combination of these solutions and employ the mixed boundary condition \( (u'(r) + a u(r))|_{r=0} = 0 \) with an arbitrary real parameter \( a \). Physically, the dependence of the dynamics on this parameter indicates that the naked singularity carries some additional degrees of freedom beyond those contained in the metric. It is said [5] that the RN singularity has ”hair”.

3
Mathematically, the ambiguity in choosing the boundary condition stems from the non-uniqueness of the self-adjoint extension to the wave operator on the naked RN background [5,17,18]. As a possible resolution, the Friedrichs extension was suggested which implies the choice \( a = 0 \) and is naturally connected with the quadratic form of energy [17,18]. It should be stressed that the application of the WKB method will permit us to bypass this problem at all because the method does not uses the value of the wave function in the origin.

In the end of this Section it is worthy to compare the above picture in terms of the areal radial coordinate \( r \) with that based on the usual tortoise coordinate \( x \) defined by \( dr = Fdx \). Adjusting the integration constant, one can always put the limit of \( r \to 0 \) in correspondence with \( x \to 0 \). Then in place of the radial equation (4) one obtains

\[
\frac{d^2 \phi}{dx^2} + \left[ \omega^2 - \left( m^2 + \frac{l(l+1)}{r^2} + \frac{2M}{r^3} - \frac{2Q^2}{r^4} \right) F \right] \phi = 0 \tag{7}
\]

where \( \phi(x) = u(r)/\sqrt{F(r)} \). In the limit of \( x \to 0 \) one has \( x \sim r^3/(3Q^2) \) and this equation reduces to the Schrödinger-type one with the inverse quadratic potential

\[
\frac{d^2 \phi}{dx^2} + \frac{2}{9x^2} \phi = 0. \tag{8}
\]

Then the textbook analysis [21] shows that the particle cannot fall on the origin because the numerical coefficient of the potential in Eq. (8) does not exceed its critical value 1/4. This is in agreement with our previous conclusion that the naked singularity is located in the classically forbidden region.

However, Eqs. (7) and (8) contain the singularity at \( x = 0 \) such that the condition of the WKB-method applicability breaks down for small \( x \). Therefore, to develop the WKB approximation in the subsequent sections, we employ the initial radial equation (4) in terms of the areal radial coordinate \( r \).

### 3 Effective potential

Now we shall establish possible types of the particle bound states on the naked RN background. To that end, let us analyze the shape of the effective potential \( V(r) \) introduced upon rewriting the radial equation (4) in shorthand notations

\[
\frac{d^2 u}{dr^2} + \frac{1}{F^2} \left( \omega^2 - V(r) \right) u = 0, \tag{9}
\]

where

\[
V(r) = \left( m^2 + \frac{l(l+1)}{r^2} \right) F + \frac{Q^2 - M^2}{r^4} = m^2 - \frac{2Mm^2}{r} + \frac{Q^2m^2 + \Lambda}{r^2} - \frac{2\Lambda M}{r^3} + \frac{Q^2\Lambda - M^2 + Q^2}{r^4} \tag{10}
\]
and we designated $\Lambda = l(l+1)$. Note that $V(r)$ is positive-defined because $F > 0$ for all positive $r$.

For a given particle energy $\omega$, the type of the bound state is regulated by the pattern of classically allowed and forbidden regions. From Eq. (9) one concludes that the classically allowed (forbidden) region is to be defined as a region in which the condition $\omega^2 - V(r) > 0$ ($\omega^2 - V(r) < 0$) holds.

Instead of solving those inequalities explicitly, we are looking for local extrema of $V(r)$. If $V(r)$ has a minimum at $r = r_{\text{min}}$ and no other extrema, then for $\omega$ close enough to $\sqrt{V(r_{\text{min}})}$ the classically allowed region consists of a single line segment and there stable bound states may exist. In this case we call $V(r)$ the single-well potential. If $V(r)$ has two or more extrema, then for certain $\omega$ the classically allowed region must be split into two subregions separated by the classically forbidden region. Because of the tunneling between the subregions, the bound states are now metastable, i.e. decaying. The corresponding potential will be referred to as the barrier-shaped one.

We are now in the position to deduce what type of the bound states occurs depending on the values of the particle mass $m$ and the singularity parameters $Q$ and $M$. Let us first study the simpler case of the massless particle and then turn to the massive case.

### 3.1 Case of $m = 0$

In this case the effective potential simplifies

$$V(r) = \frac{\Lambda}{r^2} - \frac{2\Lambda M}{r^3} + \frac{Q^2\Lambda - M^2 + Q^2}{r^4}. \quad \text{(11)}$$

Notice that for the S-waves ($l = 0 \Rightarrow \Lambda = l(l+1) = 0$) it is monotonically decreasing and thus only scattering states are present in the spectrum.

For $l \neq 0$ the formula

$$\left(\frac{Q}{M}\right)^2_{\text{cr}} = \frac{8 + 9\Lambda}{8 + 8\Lambda} \quad \text{(12)}$$

defines a critical charge-to-mass ratio such that for $Q/M > (Q/M)_{\text{cr}}$ the effective potential $V(r)$ has no extrema. If $1 < Q/M < (Q/M)_{\text{cr}}$, there must be one local minimum and one local maximum, so that the potential is barrier-shaped and there emerge the metastable bound states. Noticeably, in the limiting case of the extreme black hole, $Q/M \to 1$, the local minimum transforms in an event horizon.

From Eq. (12), one sees that the value $(Q/M)_{\text{cr}}$ increases with the growth of $\Lambda$, though not exceeding $\sqrt{9/8} \approx 1.0607$. Thus the metastable bound states may only exist in the narrow range $1 < Q/M < (Q/M)_{\text{cr}}$. The plots of $V(r)$ for $m = 0$, obtained with the values $Q/M = 1.01$ and $1.1$ inside and outside this range, respectively, are shown in Fig. 1.
3.2 Case of $m \neq 0$

We shall perform the same analysis as in the previous subsection but for the massive particle. For brevity, we set $M = 1$ in this subsection; the formulae with arbitrary $M$ can readily be obtained by rescaling (5).

To study the shape of $V(r)$, we find its extrema that amounts to solving the cubic equation

$$\frac{dV}{dr} = \frac{2m^2}{r^5}(r^3 + ar^2 + br + c) = 0$$

with the coefficients

$$a = -Q^2 - \frac{\Lambda}{m^2}, \quad b = \frac{3\Lambda}{m^2}, \quad c = -\frac{2[(1 + \Lambda)Q^2 - 1]}{m^2}. \tag{14}$$

The number of real roots to this cubic equation is controlled by the sign of its determinant defined by

$$D = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2 \tag{15}$$

where

$$p = -\frac{a^2}{3} + b, \quad q = 2\left(\frac{a}{3}\right)^3 - \frac{ab}{3} + c. \tag{16}$$

There must be one, two and three real roots when $D > 0$, $D = 0$ and $D < 0$ respectively. Since the signs of the coefficients alternate, all these roots are positive and thus all local extrema of $V(r)$ are of physical meaning.

It is convenient to rewrite $D$ in terms of the deviation from the extreme charge...
value $\delta = Q^2 - 1$

\[
D = \frac{1}{108 m^8} \left[ (8 \delta - 1) \Lambda^4 + (-78 m^2 \delta + 6 m^2 + 24 m^2 \delta^2 + 8 \delta) \Lambda^3 
+ (24 m^4 \delta^3 + 15 m^4 - 84 m^2 \delta + 24 m^2 \delta^2 + 63 m^4 \delta^2 + 54 m^4 \delta) \Lambda^2 
+ (8 m^6 \delta^4 + 32 m^6 \delta + 48 m^6 \delta^2 + 32 m^6 \delta^3 + 8 m^6 + 24 m^4 \delta^3 + 156 m^4 \delta^2 
+ 132 m^4 \delta) \Lambda + 8 m^6 \delta + 24 m^6 \delta^2 + 24 m^6 \delta^3 + 8 m^6 \delta^4 + 108 m^4 \delta^2 \right]. \tag{17}
\]

For the super-extreme charge under consideration, we have $\delta > 0$. Then it becomes evident from (17) that the condition $D > 0$ always holds true for the S-states ($\Lambda = 0$) and, hence, the corresponding effective potential has one local extremum (minimum), thereby being single-well. Thus, in contrast to the case of $m = 0$, the particle with $m \neq 0$ may form stable bound S-states.

As an illustration, in Fig. 2 we offer plots of $V(r)$ obtained with $m = 0.1$, using $Q/M = 1.01$ and 1.1. In these plots the potential well where the bound states emerge lies below the rest energy level $\omega^2 = m^2$ depicted by the dotted line.

![Figure 2](image)

(a) (b)

Figure 2: The same as Fig. 1 but for the particle with $m = 0.1$.

As for the spectrum with $l \neq 0$, we have learnt that in the case of $m = 0$ it contains the metastable bound states provided the charge-to-mass ratio is close enough to its extreme value $Q/M = 1$. We now argue that in the case of $m \neq 0$ the situation is essentially the same. Indeed, for small $\delta$ the asymptotics of $D$ is given by

\[
D = \frac{\Lambda (8 m^2 - \Lambda) (\Lambda + m^2)^2}{108 m^8} + O(\delta), \tag{18}
\]

so that $D < 0$ if $\Lambda > 8 m^2$. In practice, the last condition holds for all $l \neq 0$, because $m \ll M = 1$ (the particle mass is supposed to be much less than the central object mass). The inequality $D < 0$ means that $dV/dr$ has three zeros and, as a consequence, $V(r)$ is the barrier-shaped potential with two minima and one maximum. This subcase may include the metastable bound states because one of the minima lies above the level of $\omega^2 = m^2$ (see Fig. 2a).
On the other hand, if $Q/M$ is far from being extreme, the metastable bound states are excluded. In the case of $m = 0$, the critical value (12) that provides the necessary and sufficient condition for their exclusion was derived. Now, for $m \neq 0$, we are able to deduce the sufficient but not necessary condition. To do this, we should establish for which values of $\delta$ the condition $D > 0$ holds, which guarantees that the effective potential is single-well.

Upon demanding that the coefficients in front of powers of $\Lambda$ in (17) be positive, one obtains several restrictions. The strongest one comes from the coefficient of the $\Lambda^2$-term which becomes positive-defined when $\delta > 7/2$. Thus we conclude that the condition $D > 0$ holds and, hence, the effective potential is single-well and the metastable bound states are excluded, provided

$$\frac{Q^2 - M^2}{M^2} > \frac{7}{2}, \quad \frac{Q}{M} > \frac{3\sqrt{2}}{2} \approx 2.121...$$  \hspace{1cm} (19)

It should be stressed that this is not an exact critical value and, in fact, $D > 0$ may hold at lesser values of $Q/M$ when $l$ is small enough. For example, if $Q/M = 1.1 < 2.121$ the effective potential turns out to be single-well for $l = 0, 1, 2$. In the case of $l = 0$ it can be seen from Fig. 2b whereas for $l = 1, 2$ the bottom of the well is located in the region of large $r$ outside the plot.

Let us summarize our findings. For $l = 0$, the spectrum contains only the scattering states, if the particle is massless, and in addition the stable bound states, if $m \neq 0$. For $l \neq 0$, the metastable bound states emerge if the central object charge-to-mass ratio is close to its extreme value $Q/M = 1$. When $Q/M$ exceeds a certain critical value, the metastable bound states disappear and one observes the scattering states and the stable bound states.

4 Energy spectrum

In this Section we derive the energy spectrum of the particle (= test scalar field perturbation) in the leading WKB approximation.

Since the background geometry is naked rather than the black-hole one, our calculation will differ essentially from typical computations of the black-hole quasinormal modes. The latter imply that the solutions to the wave equation go as plane waves close to the horizon. Instead, in our case the solutions have to decrease exponentially deep into the classically forbidden region in the vicinity of the origin.

Actually, the barrier-shaped potentials that lead to the metastable bound states in our treatment (see Figs. 1a and 2a) resemble those of Gamow’s theory for $\alpha$-decay (Fig. 3). In turn, the single-well potential we obtained with $m \neq 0$ and $l = 0$ resembles the attractive Coulomb potential everywhere except for a narrow region close to the origin. Therefore we shall adopt the ordinary flat-space WKB formulae [21] with the
Langer modification for the centrifugal term, $\Lambda = l(l+1) \to (l+1/2)^2$, which is usual in the Coulomb-like problems [22][23].

Figure 3: Potential in the $\alpha$-decay theory.

4.1 Decaying bound states

Let us start with the metastable bound states. From the wave equation [9], it can be seen that the quasiclassical momenta $k(r)$ and $\kappa(r)$ for the classically allowed and forbidden regions respectively should be defined by

$$
    k(r) = \frac{1}{F} \sqrt{\omega^2 - V(r)}, \quad \kappa(r) = \frac{1}{F} \sqrt{V(r) - \omega^2}.
$$

For the metastable bound states the energy $\omega = \omega_R + i\omega_I$ is complex. In the lowest WKB approximation its real part is determined by the Bohr-Sommerfeld rule [21]

$$
    \int_{r_1}^{r_2} \frac{1}{F} \sqrt{\omega_R^2 - V(r)} dr = \pi(n + 1/2)
$$

where $r_{1,2}$ are the turning points at the ends of the potential well (see Fig. 3) and $n$ denotes the excitation number.

Further, we derive a Gamow-type formula for the imaginary part of $\omega$. Our derivation follows the conventional procedure [24]. First, multiplying the wave equation [9] by the complex conjugate wave function, we compose the expression $u'' \bar{u} - \bar{u}'' u = -(\omega^2 - \bar{\omega}^2) F^{-2} |u|^2$ and then integrate it to obtain

$$
    (u' \bar{u} - \bar{u}' u)|_{r=\infty} = -4i\omega_R \omega_I \int_0^\infty F^{-2} |u|^2 dr.
$$

Here we took into account that $(u' \bar{u} - \bar{u}' u)$ vanishes at $r = 0$ because $u$ must satisfy the boundary condition $u'(0) + au(0) = 0$ with real $a$ in order that the Klein-Gordon operator be symmetric.

Next, we resort to the well-known WKB formulae for the wave function asymptotics in the classically allowed region [21]

$$
    u \sim \frac{C_1}{2 \sqrt{k(r)}} \exp \left( i \int_{r_3}^r k(r) dr - i\pi/4 \right), \quad r > r_3,
$$
Using the first of these formulae, the left-hand side of (22) is estimated to be equal to \(i|C_1|^2/2\). The main contribution to the right-hand side of (22) comes from the region \(r_1 < r < r_2\). Approximating the squared cosine in this region by \(1/2\), we rewrite (22) as

\[
i|C_1|^2 = -4i\omega_R\omega_I|C_2|^2 \int_{r_1}^{r_2} \frac{dr}{2F^2k(r)}. \tag{25}\]

Applying the connection formula \[21\] \(C_2 = C_1 \exp(\int_{r_2}^{r_3} \kappa(r) dr)\), we end up with the requisite expression for the imaginary part of \(\omega_I\)

\[
\omega_I = -\exp \left( -2 \int_{r_2}^{r_3} \kappa(r) dr \right) \left[ 4\omega_R \int_{r_1}^{r_2} \frac{dr}{F^2k(r)} \right]^{-1}. \tag{25}\]

This formula as well as the WKB asymptotics \[23\] and \[24\] is valid if the barrier is nearly impenetrable. It means that the calculated value of \(\omega_I\), which determines the decay rate, has to obey the condition \(|\omega_I| \ll \omega_R\).

4.2 Stable bound states

Now let us turn to the stable bound states occurring in the case of the massive particle. If \(m \neq 0\), the effective potential (10) has the Coulomb tail \((V \propto -1/r)\), so that a Balmer-type approximate formula for the real bound-state energies \(\omega = \omega_R\) can be obtained.

To do this, we first compare the wave equation (9) for our effective potential and the ordinary Schrödinger equation for the Coulomb potential

\[
d^2\psi \over dr^2 + 2m \left( E + \frac{Ze^2}{r} - \frac{l(l+1)}{2mr^2} \right) \psi = 0. \tag{26}\]

Since at large \(r\) the factor \(F\) of the RN metrics (1) tends to 1, it becomes evident that the term \(2Mm^2/e^2/r\) in Eq. (9) corresponds to \(2mZe^2/r\) in Eq. (26). Then we can define the characteristic radius for our system, \(r_B = 1/(m^2M)\), corresponding to the Bohr radius, \(1/(mZe^2)\), for Eq. (26). Introducing now the dimensionless coordinate \(\rho = r/r_B\), the effective potential (10) with the Langer modification is rewritten as

\[
V(\rho) = m^2 - 2M^2m^4 \rho + \frac{M^2m^4(Q^2m^2 + (l + 1/2)^2)}{\rho^2} - 2(l + 1/2)^2M^4m^6 \rho^4 + \frac{M^4m^8\{Q^2[1 + (l + 1/2)^2] - M^2\}}{\rho^4}. \tag{27}\]

The last two terms in this expression contain higher powers of the particle mass \(m\) which is much less than the central object mass \(M\). Hence, we may neglect these terms provided that \(M\) is not much larger than the Planck mass (equal to 1 in our units). Thus we obtain the truncated potential

\[
V_{\text{trunc}}(\rho) = m^2 - 2M^2m^4 \rho + \frac{M^2m^4(Q^2m^2 + (l + 1/2)^2)}{\rho^2}. \tag{28}\]

\[10\]
serving as a good approximation to $V(\rho)$ in the classically allowed region. Using the same reasoning, we approximate $F = 1 - 2M^2m^2/\rho + M^2Q^2m^4/\rho^2$ by 1.

Within the above approximation, the Bohr-Sommerfeld integral (21) reduces to

$$\int_{\rho_1}^{\rho_2} \sqrt{\omega^2 - m^2 + \frac{2M^2m^4}{\rho} - \frac{M^2m^4[Q^2m^2 + (l + 1/2)^2]}{\rho^2}} \frac{d\rho}{m^2M} = \pi(n + 1/2)$$

(29)

where turning points $\rho_{1,2}$ are zeros of the expression under the square root. This integral is easily calculated, by applying the formula

$$\int_{\rho_1}^{\rho_2} \frac{\sqrt{(\rho - \rho_1)(\rho_2 - \rho)}}{\rho} d\rho = \pi \left( \frac{\rho_1 + \rho_2}{2} - \sqrt{\rho_1\rho_2} \right).$$

(30)

As a result, we get the explicit expression for the stable bound-state energies

$$\omega = m \left[ 1 - \frac{m^2M^2}{\left(n + 1/2 + \sqrt{(l + 1/2)^2 + Q^2m^2} \right)^2} \right]^{1/2}.$$  

(31)

It has essentially the same structure as the Balmer formula in the Coulomb problem, assuming that the values of the quantum numbers are high, $n, l \gg 1$, as usual in the WKB method. The only difference is that the coupling constant is now given by the product of masses, but not charges. Note that the quantization procedure that starts from the classical particle Hamiltonian has lead us to the same expression [19].

5 Numerical results

First we examine the spectrum of the metastable bound states for the massless neutral scalar particle in the field of naked RN singularity. It should be noticed that since the depth and the width of the corresponding potential well are finite, the number of the metastable bound states is limited. The energies of all the existing states with $l = 1, 2, 3$ calculated according to the WKB formulae (21) and (25) with $Q = 1.01$, $M = 1$ are presented in Table 1. That the states with larger excitation numbers $n$ do not exist was fixed by observing that the Bohr-Sommerfeld integral (21) cannot be saturated to its value $\pi(n + 1/2)$ even at the maximal allowed bound-state energy $\omega^2$ equal to the peak of the effective potential. As seen from Table 1, for higher $l$ the number of the existing bound states and their half-lives increase as the peak of the effective potential grows with $l$ (see Fig. 1a).

For comparison, in Table 2 we list the first three quasinormal energies (frequencies) of the scalar field on the extreme RN black-hole background with $Q = M = 1$ that were computed in Ref. [25].

The striking difference between the results in Tables 1 and 2 can be ascribed to the fact that the event horizon is absent in the naked singularity case and thus
Table 1: Energies of the metastable bound states on the naked RN background calculated with $Q = 1.01$, $M = 1$ and $m = 0$.

| $l$ | $\omega_{WKB}$ | $\sqrt{V_{\text{min}}}$ | $\sqrt{V_{\text{max}}}$ |
|-----|-----------------|--------------------------|--------------------------|
| $n = 0$ | $0.294636 - i 0.263 \times 10^{-3}$ | $0.241152$ | $0.380747$ |

$l = 2$

| $n = 0$ | $0.423194 - i 0.164 \times 10^{-5}$ | $0.364334$ | $0.632606$ |
| $n = 1$ | $0.537046 - i 0.227 \times 10^{-3}$ | $0.423194$ | $0.709848$ |

$l = 3$

| $n = 0$ | $0.555002 - i 0.969 \times 10^{-10}$ | $0.494094$ | $0.884899$ |
| $n = 1$ | $0.673735 - i 0.236 \times 10^{-5}$ | $0.537046$ | $0.884745$ |
| $n = 2$ | $0.786160 - i 0.211 \times 10^{-3}$ | $0.62609$ | $0.912509$ |
| $n = 3$ | $0.884745 - i 0.523 \times 10^{-2}$ | $0.786160$ | $0.962509$ |

Table 2: Quasinormal energies of the extreme RN black hole with $Q = M = 1$ and $m = 0$ taken from Ref. [25]

| $l$ | $\omega_{WKB}$ | $\sqrt{V_{\text{min}}}$ | $\sqrt{V_{\text{max}}}$ |
|-----|-----------------|--------------------------|--------------------------|
| $n = 0$ | $0.37764 - i 0.08936$ | $0.34392$ | $0.625$ |
| $n = 1$ | $0.34392 - i 0.27828$ | $0.29661$ | $0.45750$ |
| $n = 2$ | $0.29661 - i 0.48145$ | $0.26077$ | $0.38074$ |
| $n = 3$ | $0.26077 - i 0.625$ | $0.236$ | $0.38074$ |
the metastable bound states and the quasinormal modes are of different nature. As was discussed in the previous Section, the setup for the naked RN bound states resembles that of Gamow’s theory for α-decay. In this setup the most longliving states are concentrated near the bottom of the effective potential well \( \omega^2 \simeq \min[V(r)] \).

This can be readily checked by comparing the calculated energies \( \omega \) with the values \( V_{\text{min}} = \min[V(r)] \) which are also shown in Table 1. In turn, the black-hole quasinormal modes are, roughly speaking, the scattering resonances travelling both to \( r = 0 \) and to \( r = \infty \). Thus they have much smaller half-lives and survive for energies close to the peak of the effective potential \( \omega^2 \simeq \max[V(r)] \) (see Table 2).

For the naked RN singularity, the quasinormal modes of the latter type were calculated in the recent work [26]. These authors postulate the Dirichlet boundary condition that rules out one of the linearly independent solutions to the Klein-Gordon equation. In the present work we do not specify the boundary condition because it is not needed in the WKB approximation. Moreover, for the metastable bound states we consider the point \( r = 0 \) is located deep into the classically forbidden region and thus the boundary condition cannot affect these states substantially.

Table 3: Energies of the metastable bound states on the naked RN background calculated with \( Q = 1.01, M = 1 \) and \( m = 0.1 \).

| \( n \) | \( \omega_{\text{WKB}} \) | \( \sqrt{V_{\text{min}}} \) | \( \sqrt{V_{\text{max}}} \) |
|-----|-----------------|---------------|---------------|
| 0   | 0.295156 - i 0.221 \times 10^{-3} | 0.241591 | 0.383910 |
| 1   | 0.423496 - i 0.142 \times 10^{-5} | 0.364614 | 0.634529 |
| 2   | 0.537417 - i 0.209 \times 10^{-3} | |
| 3   | 0.555216 - i 0.869 \times 10^{-10} | 0.494299 | 0.886277 |

Next we investigate the case of the massive particle. In Table 3 the energies of all the states with \( l = 1, 2, 3 \) computed using \( Q = 1.01, M = 1 \) and \( m = 0.1 \) are listed. Also, in Fig. 4 the imaginary part of \( \omega \) is plotted versus the real part of \( \omega \) for different values of the particle mass \( m \). From Tables 1, 3 and Fig. 4, we see that the real part of \( \omega \), i.e. the oscillation frequency, grows with increase of \( m \), while the imaginary part of \( \omega \), representing the decay rate, falls down. Interestingly, the same behavior
was observed for the quasinormal modes of the scalar field on the sub-extreme RN black-hole background in Ref. [27].

Figure 4: Real and imaginary parts of the metastable bound-state energies for \( n = 0, l = 1 \) using (a) \( M = 1, Q = 1.01 \), (b) \( M = 1, Q = 1.02 \). The squares left to right present the results obtained with \( m = 0, 0.05, 0.1, 0.15, 0.2 \) respectively.

Now let analyze the spectrum of the stable bound states. In Table 4 we list the energies \( (\omega/m)_{WKB} \) of the ground and the first two excited S-states \((l = 0)\) computed according to the WKB formula (31) and also the energies \( (\omega/m)_{num} \) obtained by direct numerical integration of the Klein-Gordon equation (4) with the Dirichlet boundary condition \( u(0) = 0 \). The calculation was made using \( Q = 1.1, M = 1 \) and various \( m \). From Table 4 we see that the binding energy \( \omega_{\text{bind}} = -(\omega - m) \) increases as the particle mass \( m \) increases. However, this binding energy is more than two orders of magnitude smaller than the metastable bound-state energy \( \omega_R \) calculated with the same values of \( m \) and \( M \) (see Table 3). It means that transitions between levels of the stable states due to an external perturbation would have much lower frequencies than the those due to the decay of the metastable bound states.

| \( n \) | \( m = 0.05 \) | \( m = 0.1 \) | \( m = 0.15 \) |
|---|---|---|---|
| | \( \omega/m \) \(_{WKB}\) | \( \omega/m \) \(_{num}\) | \( \omega/m \) \(_{WKB}\) | \( \omega/m \) \(_{num}\) | \( \omega/m \) \(_{WKB}\) | \( \omega/m \) \(_{num}\) |
| 0 | 0.998757 | 0.998733 | 0.995105 | 0.994710 | 0.989266 | 0.987068 |
| 1 | 0.999688 | 0.998685 | 0.998764 | 0.998701 | 0.997257 | 0.996906 |
| 2 | 0.999861 | 0.999860 | 0.999449 | 0.999429 | 0.998771 | 0.998661 |

6 Conclusion

In this work we have studied the states of neutral scalar particles in the field of the naked RN singularity. It has been established that their energy spectrum is
qualitatively different from the quasinormal mode spectrum for the RN black hole. The conditions for the bound states to be formed have been found and the possible types of these states have been examined.

The first possible type is the metastable, i.e. decaying bound states. They occur provided that the charge-to-mass ratio for the central object, $Q/M$, is close to its extreme value $Q/M = 1$. If the particle is massless, this is expressed by the inequality $1 < (Q/M)^2 < [8 + 9l(l + 1)]/[8 + 8l(l + 1)]$ where $l$ is the orbital number.

Using the WKB method, we have calculated the metastable bound-state spectrum which shows that there is no continuous transition in energies between the RN naked singularity and the extremal RN black hole cases. The metastable bound states on the naked RN background with $Q/M > 1$ have much larger live-times than the quasinormal modes for $Q/M = 1$. This is not surprising because for $Q/M > 1$ there exists no event horizon to fall on, so that the metastable states can decay only at the expense of the particle escape to spatial infinity upon tunneling through the wide potential barrier.

As the second possible type of particle states, we identify the stable bound states. It should be stressed that these states have no analog in the case of the RN black hole and can only be formed by massive particles in the field of the central object with the sufficiently high charge-to-mass ratio. Their energy spectrum, calculated by means of the WKB method, has the structure similar to that of the Coulomb spectrum.

Energy gaps between the stable bound states, i.e. transition frequencies, are several orders of magnitude smaller than the frequencies for the metastable bound states obtained with the same masses of the particle and the central object. Thus the latter states are more favored to be ever found in experiment. Nevertheless, the stable bound states may manifest themselves in a different way, by making up a scalar condensate around the central object. Then the question arises as to whether such a condensate can provide mass needed to screen the RN singularity. Answering this question requires a self-consistent model for interacting scalar and gravitational fields which may be constructed in a future work.

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