Article

The Extinction of a Non-Autonomous Allelopathic Phytoplankton Model with Nonlinear Inter-Inhibition Terms and Feedback Controls

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Abstract: A non-autonomous allelopathic phytoplankton model with nonlinear inter-inhibition terms and feedback controls is studied in this paper. Based on the comparison theorem of differential equation, some sufficient conditions for the permanence of the system are obtained. We study the extinction of one of the species by using some suitable Lyapunov type extinction function. Our analyses extend those of Xie et al. (Extinction of a two species competitive system with nonlinear inter-inhibition terms and one toxin producing phytoplankton. Advances in Difference Equations, 2016, 2016, 258) and show that the feedback controls and toxic substances have no effect on the permanence of the system but play a crucial role on the extinction of the system. Some known results are extended.

Keywords: permanence; extinction; phytoplankton; feedback controls

1. Introduction

Recently, competition models with nonlinear inter-inhibition terms have been considered by many scholars ([1–7]). Wang, Liu and Li [1] considered the following competition system:

\[
\begin{align*}
    x_1'(t) &= x_1(t) \left( r_1(t) - a_1(t)x_1(t) - \frac{b_1(t)x_2(t)}{1+x_2(t)} \right), \\
    x_2'(t) &= x_2(t) \left( r_2(t) - \frac{b_2(t)x_1(t)}{1+x_1(t)} - a_2(t)x_2(t) \right),
\end{align*}
\]

where \(x_1(t), x_2(t)\) indicate the species \(x_1\) and \(x_2\) densities at time \(t\), respectively; \(r_i(t), i = 1, 2\) denote the net rates of production of two species; \(a_i(t), i = 1, 2\) are the rates of intraspecific competition of the species \(x_1\) and \(x_2\), respectively; \(b_i(t), i = 1, 2\) represent the interspecific competing rates. The nonlinear inter-inhibition terms \(\frac{b_1(t)x_2(t)}{1+x_2(t)}\) and \(\frac{b_2(t)x_1(t)}{1+x_1(t)}\) imply that for large phytoplankton density, the interspecific competing rate tends to a certain value. In other words, the interspecific competing rate will not increase infinitely with the increase of phytoplankton density, which could make us understand the real ecosystems deeper. For more information about the nonlinear inter-inhibition terms, see [8]. Based on differential inequality, the module containment theorem and constructing the Lyapunov function, Wang et al. [1] gave the sufficient conditions for the global asymptotic stability of system.

As we all know, phytoplankton is the primary producer in ocean and plays an important role in energy flow and nutrient cycling of marine ecosystems. In addition, phytoplankton can absorb
carbon dioxide for photosynthesis, which has a significant impact on the climate regulation. The
importance of phytoplankton to marine ecosystem has been widely recognized. Besides, many
authors attempted to explain the bloom phenomenon by different approaches, and find that toxic
phytoplankton certainly play an important role in the bloom phenomenon. Therefore, in recent
years, many scholars have studied the allelopathic toxic phytoplankton model ([4,5,7,9–20]). Rashi
Gupta [9] considered Holling type-II and Holling type-IV functional responses in a model of non-toxic
phytoplankton-toxic phytoplankton-zooplankton. He gave the the condition for diffusive instability of
a locally stable equilibrium of spatial and non-spatial model for one dimensional system. Based on
the work of Yue [4], recently, Xie et al. [5] further considered the effect of toxin on a non-autonomous
competitive phytoplankton system, written in the form as
\[
\begin{align*}
    x_1'(t) &= x_1(t)\left(r_1(t) - a_1(t)x_1(t) - \frac{b_1(t)x_2(t)}{1 + x_2(t)} - c_1(t)x_1(t)x_2(t)\right), \\
    x_2'(t) &= x_2(t)\left(r_2(t) - \frac{b_2(t)x_1(t)}{1 + x_1(t)} - a_2(t)x_2(t)\right),
\end{align*}
\]
where \(c_1(t)\) denotes the rate of toxic inhibition for the species \(x_1\) released by the second species.
The authors obtained the sufficient conditions for the extinction of a species and the global attractivity
of the other one. On the other hand, through experimental data of a experimental study on
two phytoplankton species, namely C. polylepis and H. triqueta, Sole et al. [10] found that the
allelopathic interaction using \(r_1x_1(t)^2x_2^2(t)\) is more suitable. M. Bandyopadhyay [11] proposed and
studied the following mathematical model of two competing phytoplankton species with allelopathic
interaction term:
\[
\begin{align*}
    x_1'(t) &= x_1(t)\left(r_1 - a_1x_1(t) - b_1x_2(t)\right) - \gamma x_1^2(t)x_2^2(t), \\
    x_2'(t) &= x_2(t)\left(r_2 - a_2x_2(t) - b_2x_1(t)\right).
\end{align*}
\]
Since the influence of human behavior on the ecosystems is more and more great, a large number
of precious species are facing extinction. It is important to know how to protect endangered species
and maintain the diversity of ecosystems. In ecology, we want to know that whether or not an
ecosystem can withstand those unpredictable disturbances. In the language of control variables, we
use feedback control variables to represent these unpredictable disturbances. In order to describe the
effect of people’s behavior, many researchers focused on the research of the systems with feedback
control variables ([7,15,21–26]). Muroya Y. [21] studied a Lotka-Volterra systems with infinite delays
and feedback controls, the authors applied a Lyapunov functional and established that the feedback
controls have no effect on the attractivity of a saturated equilibrium. Recently, Liu et al. [22] proposed
the following system with feedback controls:
\[
\begin{align*}
    x_1(n + 1) &= x_1(n)\exp\{r_1(n) - a_1(n)x_1(n) - \frac{b_1(n)x_2(n)}{1 + x_2(n)} - c_1(n)u_1(n)\}, \\
    x_2(n + 1) &= x_2(n)\exp\{r_2(n) - \frac{b_2(n)x_1(n)}{1 + x_1(n)} - a_2(n)x_2(n) - c_2(n)u_2(n)\}, \\
    \Delta u_1(n) &= -b_1(n)u_1(n) + d_1(n)x_1(n), \\
    \Delta u_2(n) &= -b_2(n)u_2(n) + d_2(n)x_2(n),
\end{align*}
\]
where \(\Delta u_i(n) = u_i(n + 1) - u_i(n)\), \(i = 1,2\) are the forward difference operators; \(u_i(n), i = 1,2\) denote
the feedback control variables. \(b_i(n), d_i(n)\) and \(c_i(n), i = 1,2\) are bounded positive almost periodic
sequences. Liu et al. [22] studied the existence and uniformly asymptotic stability of unique positive
almost periodic solution of system (4). Furthermore, based on a suitable Lyapunov function, Yu [7]
obtained the sufficient conditions for the extinction of one species.

As is well known, if the amount of the species is enough large, the continuous model is more
appropriate. But, to this day, still no scholar propose and study the continuous form of system (4) with
We can easily obtain the solution and prove the permanence of the system (5). In Section 3, we will discuss the extinction of one species.

Lemma 1.

Proof.

Lemma 2.

Lemma 3.

Recently, a few studies about the effect of feedback controls on allelopathic phytoplankton model have been carried out, it is worth noting that in this paper. Besides, the allelopathic interaction term is replaced by \( \gamma x_1(t)^2 x_2(t)^2 \) instead of \( \gamma x_1(t)^2 x_2(t) \). Our main objective is to study the effects of toxicity and feedback controls on the dynamics of the system.

The paper is organized as follows. In Section 2, we will state some necessary Lemmas and prove the permanence of the system (5). In Section 3, we will discuss the extinction of one species. Four examples together with their numeric simulations are present in Section 4, as we will show the feasibility of the main results. We give a a briefly discussion in the end of this paper.

2. Permanence

Given a continuous and bounded function \( f(t) \), let \( f^u \) and \( f^l \) denote \( \sup_{t \in R} f(t) \) and \( \inf_{t \in R} f(t) \), respectively. From the point of view of biology, we assume that \( x_i(0) > 0, u_i(0) > 0, i = 1, 2 \). We can easily obtain the solution \((x_1(t), x_2(t), u_1(t), u_2(t))\) passing through \((x_1(0), x_2(0), u_1(0), u_2(0))\) is positive.

Definition 1 ([27]).

1. Population \( x(t) \) is said to be permanent if there exist two constant \( M \) and \( m \) such that \( m = \lim \inf_{i \rightarrow +\infty} x(t) \leq \lim \sup_{i \rightarrow +\infty} x(t) \leq M \).
2. Population \( x(t) \) is said to be extinct if \( \lim_{i \rightarrow +\infty} x(t) = 0 \) almost surely.

Lemma 1.

1. If \( a > 0, b > 0 \) and \( x \geq b - ax \), when \( t \geq 0 \) and \( x(0) > 0 \), we have \( \lim \inf_{i \rightarrow +\infty} x(t) \geq \frac{b}{a} \).
2. If \( a > 0, b > 0 \) and \( x \geq b - ax \), when \( t \geq 0 \) and \( x(0) > 0 \), we have \( \lim \sup_{i \rightarrow +\infty} x(t) \leq \frac{b}{a} \).

Lemma 2.

1. If \( a > 0, b > 0 \) and \( x \geq x(b - ax) \), when \( t \geq 0 \) and \( x(0) > 0 \), we have \( \lim \inf_{i \rightarrow +\infty} x(t) \geq \frac{b}{a} \).
2. If \( a > 0, b > 0 \) and \( x \leq x(b - ax) \), when \( t \geq 0 \) and \( x(0) > 0 \), we have \( \lim \sup_{i \rightarrow +\infty} x(t) \leq \frac{b}{a} \).

Lemma 3. Every positive solution \((x_1(t), x_2(t), u_1(t), u_2(t))^T\) of system (5) satisfies

\[
\limsup_{i \rightarrow +\infty} x_i(t) \leq \frac{M_i}{a_i} = M_i, \quad \limsup_{i \rightarrow +\infty} u_i(t) \leq \frac{N_i}{c_i} = N_i, \quad i = 1, 2. \tag{6}
\]

Proof. It follows from the first and second equation of system (5) yields

\[
x_i'(t) = x_i(t) \left( r_i(t) - a_i(t)x_i(t) \right) \leq x_i(t) \left( r_i^* - a_i^*x_i(t) \right), \quad i = 1, 2. \tag{7}
\]
According to Lemma 2 and differential inequality (7), we have
\[
\limsup_{t \to +\infty} x_i(t) \leq r_i^u \frac{a_i}{d_i} = M_i, \quad i = 1, 2.
\] (8)

From (8), there exists a $T_1 > 0$, such that for $t > T_1$ and any small positive constant $\varepsilon > 0$,
\[
x_i(t) \leq M_i + \varepsilon.
\] (9)

From the third and fourth equation of system (5) it follows that
\[
u_i'(t) = -e_i u_i(t) + d_i u_i(M_i + \varepsilon).
\] (10)

By applying Lemma 1 to differential inequality (10), we have
\[
\limsup_{t \to +\infty} u_i(t) \leq \frac{d_i}{e_i} (M_i + \varepsilon), \quad i = 1, 2.
\]

Setting $\varepsilon \to 0$ in above inequalities leads to
\[
\limsup_{t \to +\infty} u_i(t) \leq \frac{d_i}{e_i} M_i = \frac{d_i r_i^u}{e_i a_i} \overset{\text{def}}{=} N_i, \quad i = 1, 2.
\]

Theorem 1. Assume that
\[
r_1^u > b_1^u \frac{r_2^u}{a_2}, \quad r_2^u > b_2^u \frac{r_1^u}{a_1}
\] (11)

holds. Then, for any positive solution $(x_1(t), x_2(t), u_1(t), u_2(t))^T$ of the system (5), we have
\[
m_i \leq \liminf_{t \to +\infty} x_i(t) \leq \limsup_{t \to +\infty} x_i(t) \leq M_i, \quad n_i \leq \liminf_{t \to +\infty} u_i(t) \leq \limsup_{t \to +\infty} u_i(t) \leq N_i, \quad i = 1, 2.
\]
i.e., system (5) is permanent.

Remark 1. Theorem 1 shows that two kinds of phytoplankton can coexist under certain conditions. Besides, the conditions of Theorem 1 show that the feedback control variables and toxic substances do not effect on the permanence of the system.

Proof. From (5), for any small positive constant $\varepsilon > 0$, we may choose $\varepsilon$ small enough such that
\[
r_1^u > b_1^u \left( r_2^u \frac{a_2}{d_2} + \varepsilon \right) = b_1^u (M_2 + \varepsilon), \quad r_2^u > b_2^u \left( r_1^u \frac{a_1}{d_1} + \varepsilon \right) = b_2^u (M_1 + \varepsilon).
\] (12)

For $\varepsilon > 0$ above, from Lemma 3 it follows that there exists $T_2 > 0$ such that for $t > T_2$,
\[
x_i(t) \leq M_i + \varepsilon, \quad u_i(t) \leq N_i + \varepsilon, \quad i = 1, 2.
\] (13)
From the first equation of system (5), we have
\[
x_1'(t) = x_1(t)(r_1(t) - a_1(t)x_1(t) - \frac{h_1(t)x_2(t)}{x_1(t)} - \gamma(t)x_1(t)\epsilon x_1(t) - c_1(t)u_1(t)) \\
\geq x_1(t)(r_1(t) - a_1(t)x_1(t) - b_1(t)x_1(t) - \gamma(t)x_1(t)x_1^2(t) - c_1(t)u_1(t)) \\
\geq x_1(t)(r_1 - a_1(M_1 + \epsilon) - b_1(M_2 + \epsilon) - \gamma(M_1 + \epsilon)(M_2 + \epsilon)^2 - c_1(N_1 + \epsilon))
\]
\[
= I_1^t x_1(t).
\]
Integrating the above differential inequality from \(s\) to \(t\), we have
\[
x_1(s) \leq x_1(t) \exp \left[ - I_1^t (t-s) \right].
\]
By the third equation of system (5), it follows
\[
u_1'(t) \leq -e_1^t u_1(t) + d_1^u x_1(t).
\]
According to Lemma 2.3 of [24] and inequality (15), integrating the above differential inequality from \(t_1(t_1 > T_2)\) to \(t\), we have
\[
u_1(t) \leq u_1(t_1) \exp \left[ - e_1^t (t - t_1) \right] + \int_{t_1}^{t} d_1^u x_1(s) \exp \left[ e_1^s (s - t) \right] ds, \\
\leq u_1(t_1) \exp \left[ - e_1^t (t - t_1) \right] + \int_{t_1}^{t} d_1^u x_1(t) \exp \left[ - I_1^t (t-s) \right] ds, \\
= u_1(t_1) \exp \left[ - e_1^t (t - t_1) \right] + d_1^u x_1(t) \frac{1}{T_1^t} \left( 1 - \exp \left[ - I_1^t (t-t_1) \right] \right), \\
\leq (N_1 + \epsilon) \exp \left[ - e_1^t (t - t_1) \right] + d_1^u x_1(t) \frac{1}{T_1^t} \left( 1 - \exp \left[ - I_1^t (t-t_1) \right] \right).
\]
There exists a \(T_1^t\) such that \(t - t_1 = T_1^t \geq T_1^t\), we have
\[
c_1^u(N_1 + \epsilon) \exp(-e_1^t T_1^t) < \frac{1}{2} (r_1^t - b_1^u(M_2 + \epsilon)),
\]
\[
u_1(t) \leq (N_1 + \epsilon) \exp(-e_1^t T_1^t) + d_1^u x_1(t) \frac{1}{T_1^t} \left( 1 - \exp(-I_1^t T_1^t) \right)
\]
\[
= (N_1 + \epsilon) \exp(-e_1^t T_1^t) + D_1^t x_1(t).
\]
where \(D_1^t = d_1^u \frac{1}{T_1^t} \left( 1 - \exp(-I_1^t T_1^t) \right)\).
By the first equation of system (5), we have
\[
x_1'(t) \geq x_1(t) \left[ r_1^t - a_1^u x_1(t) - b_1^u(M_2 + \epsilon) - \gamma u x_1(t)(M_2 + \epsilon)^2 \\
- c_1^u(N_1 + \epsilon) \exp(-e_1^t T_1^t) - c_1^d D_1 x_1(t) \right] \\
= x_1(t) \left[ r_1^t - b_1^u(M_2 + \epsilon) - c_1^u(N_1 + \epsilon) \exp(-e_1^t T_1^t) \\
- (a_1^u + \gamma u(M_2 + \epsilon)^2 + c_1^d D_1) x_1(t) \right].
\]
By applying Lemma 2 to the above differential inequality, it follows that
\[
\liminf_{t \to +\infty} x_1(t) \geq \frac{r_1^t - b_1^u(M_2 + \epsilon) - c_1^u(N_1 + \epsilon) \exp(-e_1^t T_1^t)}{a_1^u + \gamma u(M_2 + \epsilon)^2 + c_1^d D_1}.
\]
Setting $\varepsilon \to 0$ in this inequality leads to
\[
\liminf_{t \to +\infty} x_1(t) \geq \frac{r_1^t - b_2^t M_2 - c_2^t N_1 \exp(-e_1^t T_1^*)}{a_1^t + \gamma^u M_2^2 + c_1^u D_1} \overset{\text{def}}{=} m_1,
\] (20)
where
\[
D_1 = d_1^t \frac{1}{T_1}(1 - \exp(-l_1 T_1^*)),
\]
\[
l_1 = r_1^t - a_1^t M_1 - b_1^t M_2 - \gamma^u M_1 M_2^2 - c_1^u N_1.
\]
From the second equation of system (5) it follows that
\[
x_2'(t) \geq x_2(t) \left( r_2^t - b_2^t (M_1 + \varepsilon) - a_2^t (M_2 + \varepsilon) - c_2^t (N_2 + \varepsilon) \right) \overset{\text{def}}{=} I_2^* x_2(t).\] (21)
Integrating this inequality from $s$ to $t$, we get
\[
x_2(s) \leq x_2(t) \exp \left\{ - I_2^*(t-s) \right\}.\] (22)
By the fourth equation of system (5), we have
\[
u_2(t) \leq -e_2^t u_2(t) + d_2^u x_2(t).\] (23)
Integrating this inequality from $t_2$ to $t$, it follows
\[
u_2(t) \leq \nu_2(t_2) \exp \left\{ - e_2^t (t-t_2) \right\} + \int_{t_2}^t d_2^u x_2(s) \exp \left\{ e_2^t (s-t) \right\} ds,
\]
\[
\leq \nu_2(t_2) \exp \left\{ - e_2^t (t-t_2) \right\} + d_2^u x_2(t) \frac{1}{I_2^*} \left( 1 - \exp \left\{ - I_2^*(t-t_2) \right\} \right).\] (24)
From Lemma 3, we have
\[
u_2(t_2) \leq N_2 + \varepsilon, \quad t_2 > T_2.
\]
There exists a $T_2'$ such that $t - t_2 = T_2' \geq T_2^*$, we have
\[
c_2^u (N_2 + \varepsilon) \exp(-e_2^t T_2^*) < \frac{1}{2} (r_2^t - b_2^t (M_1 + \varepsilon)),
\]
\[
u_2(t) \leq (N_2 + \varepsilon) \exp(-e_2^t T_2^*) + D_2^u x_2(t),\] (25)
where $D_2^u = d_2^u \frac{1}{I_2^*} (1 - \exp(-l_1 T_2^*))$.
From the second equation of system (5), we have
\[
x_2'(t) \geq x_2(t) \left[ r_2^t - b_2^t (M_1 + \varepsilon) - a_2^t (N_2 + \varepsilon) \exp(-e_2^t T_2^*) \right.
\]
\[
\left. -(a_2^t + c_2^u D_2^u) x_2(t) \right].
\]
Similarly to the analysis of (19), we can obtain
\[
\liminf_{t \to +\infty} x_2(t) \geq \frac{r_2^t - b_2^t M_1 - c_2^u N_2 \exp(-e_2^t T_2^*)}{a_2^t + c_2^u D_2} \overset{\text{def}}{=} m_2,
\] (26)
where
\[
D_2 = d_2^u \frac{1}{I_2^*} (1 - \exp(-l_2 T_2^*)),
\]
\[ I_2 = r_2^1 - b_2^u M_1 - a_2^u M_2 - c_2^u N_2. \]

For any small positive constant \( \varepsilon < \frac{1}{2} \min \{ m_1, m_2 \} \), from (20) and (26) it follows that there exists a \( T_3 > T_i' \), \( i = 1, 2 \), such that for \( t > T_3 \), we have

\[ x_i(t) \geq m_i - \varepsilon, \quad i = 1, 2. \]  \hspace{1cm} (27)

From the third and fourth equation of system (5) it follows that

\[ u'_i(t) \geq -e_i^u u_i(t) + d_i^l (m_i - \varepsilon), \quad i = 1, 2. \]  \hspace{1cm} (28)

From Lemma 1, we obtain

\[ \lim \inf_{t \to +\infty} u_i(t) \geq \frac{d_i^l (m_i - \varepsilon)}{e_i^u}. \]  \hspace{1cm} (29)

Setting \( \varepsilon \to 0 \) in this inequality leads to

\[ \lim \inf_{t \to +\infty} u_i(t) \geq \frac{d_i^l m_i}{e_i^u} \text{ def } n_i, \quad i = 1, 2. \]  \hspace{1cm} (30)

\[ \square \]

3. Extinction

\textbf{Theorem 2.} Assume that

\[ r_1^l > (1 + M_1) r_2^u a_1^u e_1^j + c_1^u d_1^u, \quad r_1^l > r_2^u \frac{b_1^u e_1^j}{a_2^l e_2^l + c_2^l d_2^l}, \]  \hspace{1cm} (31)

and

\[ \gamma^u < \min \left\{ \frac{1}{M_1 M_2} \left( r_1^l - (1 + M_1) r_2^u a_1^u e_1^j + c_1^u d_1^u, \quad r_1^l - r_2^u \frac{b_1^u e_1^j}{a_2^l e_2^l + c_2^l d_2^l} \right) \right\} \]  \hspace{1cm} (32)

hold, then the species \( x_1 \) is permanent and the species \( x_2 \) will be extinct, that is, for any positive solution \((x_1(t), x_2(t), u_1(t), u_2(t))\) of system (5),

\[ \lim_{t \to +\infty} x_2(t) = 0, \quad \lim_{t \to +\infty} u_2(t) = 0. \]

\textbf{Remark 2.} Theorem 2 gives the conditions for the permanence of nontoxic phytoplankton and the extinction of toxic phytoplankton. From Theorem 2, we known that lower rate of toxic production could not avoid the extinction of the second species.

\textbf{Proof.} Condition (31) is equivalent to

\[ \frac{c_1^u}{e_1^u} < \frac{r_1^l b_1^u}{(1 + M_1) r_2^u a_1^u} - \frac{a_1^u}{d_1^l} \quad \text{ and } \quad \frac{c_1^u}{e_2^l} > \frac{r_2^u b_1^u}{r_1^l a_1^l} - \frac{a_1^u}{d_1^l}. \]  \hspace{1cm} (33)
From (32) and (33), there exist positive constants \( a, b, \delta_1, \delta_2 \) and enough small positive \( \varepsilon \) such that

\[
\frac{r_1^t}{r_2^t} \geq \frac{\beta}{\alpha}, \quad \frac{c_1\alpha}{c_1} < \frac{\delta_1}{\alpha} < \frac{\beta b_1 - (1 + M_1 + \varepsilon) \alpha a d_1^u}{(1 + M_1 + \varepsilon) \alpha d_1^u} < \frac{r_1^t b_1^u}{(1 + M_1 + \varepsilon) r_2^t d_2^u} - \frac{a_1^u}{d_1^u},
\]

\[
\frac{c_2\alpha}{c_1} > \frac{\delta_2}{\beta} > \frac{ab_1^u - \beta a_2^u}{\beta d_2^u} > \frac{b_1^u b_1^u}{b_1^u d_2^u} - \frac{a_2^u}{d_2^u},
\]

\[
(1 + M_1 + \varepsilon) (a_1^u c_1^u + c_1^u d_1^u) < \frac{\beta}{\alpha} = \frac{\beta}{\alpha} - \frac{\gamma(M_1 + M_1 + \varepsilon)(M_2 + \varepsilon)^2}{r_2^t},
\]

\[
\frac{b_1^u d_2^u}{a_2^u d_1^u + c_2^u d_2^u} < \frac{\beta}{\alpha} < \frac{r_1^t - \gamma(M_1 + \varepsilon)(M_2 + \varepsilon)^2}{r_2^t}.
\]

That is

\[
ac_1^u - \delta c_1^u < 0, \quad \delta c_2^u - \beta c_2^u < 0,
\]

\[
\alpha a_1^u < 1 + M_1 + \epsilon < 0, \quad \alpha b_1^u - \beta a_2 - \delta b_2 < 0,
\]

\[
-\alpha r_1^t + \beta r_2^t + \alpha \gamma(M_1 + \varepsilon)(M_1 + \varepsilon)^2 = -\xi_1 < 0.
\]

Let \((x_1(t), x_2(t), u_1(t), u_2(t))^T\) be a positive solution of system (5). For above \( \varepsilon \), from Lemma 2, there exists a enough large \( T_4 \), such that

\[
x_i(t) < M_1 + \varepsilon, \quad u_i(t) < N_1 + \varepsilon, \quad t \geq T_4, \quad i = 1, 2.
\]

Let

\[
V_1(t) = x_1^{-u}(t) x_2^\varepsilon(t) \exp \left( \delta_1 u_1(t) - \delta_2 u_2(t) \right).
\]

Calculating the derivative of \( V_1(t) \), from (35), for \( t \geq T_4 \), we can obtain

\[
D^+ V_1(t) = V_1(t) \left( (-\alpha r_1(t) + \beta r_2(t)) + (aa_1(t) - \frac{\beta b_2(t)}{1 + x_1(t)} + \delta_1 c_1(t)) x_1(t) \right.
\]

\[
\left. + \left( \frac{ab_1(t)}{1 + x_1(t)} - \beta a_2(t) - \delta_2 c_2(t) \right) x_2(t) + (ac_1(t) - \delta_1 c_1(t)) u_1(t) \right)
\]

\[
+ (-\beta c_2(t) + \delta_2 c_2(t)) u_2(t) + \alpha \gamma(t) x_1(t) x_2(t)) \right]
\]

\[
\leq V_1(t) \left( (-\alpha r_1^t + \beta r_2^t) + (aa_1^u - \frac{\beta b_2^u}{1 + (M_1 + \varepsilon)} + \delta_1 d_1^u) x_1(t) \right.
\]

\[
\left. + (ab_1^u - \beta a_2 - \delta_2 d_2^u) x_2(t) + (ac_1^u - \delta_1 c_1^u) u_1(t) \right)
\]

\[
+ (-\beta c_2^u + \delta_2 c_2^u) u_2(t) + \alpha \gamma^u(M_1 + \varepsilon)(M_1 + \varepsilon)^2 \right].
\]

From inequalities (34), we obtain

\[
V_1'(t) \leq -\xi_1 V_1(t).
\]

Integrating the above inequality from \( T_4 \) to \( t(\geq T_4) \), we have

\[
V_1(t) \leq V_1(T_4) \exp \left( -\xi_1(t - T_4) \right).
\]

It follows from (35) that

\[
V_1(T_4) = x_1^{-u}(T_4) x_2^\varepsilon(T_4) \exp \left( \delta_1 u_1(T_4) - \delta_2 u_2(T_4) \right) < +\infty.
\]

\[
V_1(t) = x_1^{-u}(t) x_2^\varepsilon(t) \exp \left( \delta_1 u_1(t) - \delta_2 u_2(t) \right) \geq (M_1 + \varepsilon)^{-u}(t) \exp \left( -\delta_2(N_2 + \varepsilon) \right).
\]
Combining inequalities (38) and (39), we have

\[ x_2(t) \leq C \exp \left( -\frac{\xi_1}{\beta} (t - T_4) \right), \]

where

\[ C = (M_1 + \epsilon) \xi \exp \left( \frac{\delta_2}{\beta} (N_2 + \epsilon) \right) V_1(T_4) \xi. \]

Hence we obtain that

\[ \lim_{t \to +\infty} x_2(t) = 0. \quad (40) \]

And so, \( \forall \epsilon > 0, \exists T_5 > T_4 \), such that \( x_2(t) < \epsilon \) for all \( t > T_5 \). From the fourth equation of system (5), we have

\[ u_2'(t) \leq -d_1^2 u_2(t) + d_2^2 \epsilon. \quad (41) \]

From Lemma 1, we obtain

\[ \lim_{t \to +\infty} u_2(t) \leq \limsup_{t \to +\infty} u_2(t) \leq \frac{d_2^2 \epsilon}{c_2}. \]

Setting \( \epsilon \to 0 \) leads to

\[ \lim_{t \to +\infty} u_2(t) = 0. \quad (42) \]

By using the analysis technique of [24], one could show that under the conditions of Theorem 2, the first species of system (5) is permanent. We omit the detail here. This ends the proof of Theorem 2. \( \square \)

**Theorem 3.** Assumes that

\[ r_1^u < r_2 \frac{d_1^1 e_1^1 + c_1^1 d_1^1}{b_2^2 e_1^1}, \quad r_1^u < \frac{1}{1 + M_2} r_2 \frac{b_2^2 e_1^1}{d_2^2 e_2^1 + c_2^2 d_2^1} \quad (43) \]

hold, then the species \( x_1 \) will be extinct and the species \( x_2 \) is permanent, that is, for any positive solution \((x_1(t), x_2(t), u_1(t), u_2(t))^T\) of system (5),

\[ \lim_{t \to +\infty} x_1(t) = 0, \quad \lim_{t \to +\infty} u_1(t) = 0. \]

**Proof.** The proof of Theorem 3 is similar to Theorem 2, which we omit here. \( \square \)

**Remark 3.** Theorem 3 gives the conditions for the permanence of toxic phytoplankton and the extinction of nontoxic phytoplankton. Besides, when \( c_i = 0, i = 1, 2 \), Theorem 1 obtained by Xie and Xue et al. [5] are the corollary of Theorem 3, which extends the results of Xie and Xue et al. [5] and reveal that by choosing suitable feedback control variables, the extinction property of system still contains.

**4. Example**

**Example 1.** Consider the following equations

\[
\begin{align*}
x_1'(t) &= x_1 \left( 6 - (3.2 + 0.2 \sin t)x_1 - \frac{0.5x_1}{1 + x_2} - 0.005x_1x_2^2 - 0.3u_1 \right), \\
x_2'(t) &= x_2 \left( 12.05 - 0.05 \cos t - \frac{5x_1}{1 + x_1} - (3.5 + 0.5 \sin t)x_2 - 0.3u_2 \right), \\
u_1'(t) &= -(0.8 + 0.2 \sin t)u_1 + 0.5x_1, \\
u_2'(t) &= -(0.8 + 0.2 \sin t)u_2 + 0.2x_2.
\end{align*}
\]
Corresponding to system (44), one has
\[ r_1 = 6 > b_1 \frac{r_1}{a_1} \approx 2.02, \quad r_2 = 12 > b_2 \frac{r_2}{a_2} = 10. \]

Clearly, condition (11) are satisfied, from Theorem 1, we know that the system (44) is permanent. Figure 1 shows the dynamic behaviors of system (44) which is consistent with the conclusion obtained above.

Example 2. Consider the following equations
\begin{align*}
x_1'(t) &= x_1 \left( 6 - (2.5 + 0.5 \sin t)x_2 - \frac{0.5 x_1 x_2^2}{1 + x_1} - 0.00005 x_1^2 - 0.3 u_1 \right), \\
x_2'(t) &= x_2 \left( 0.95 - 0.05 \cos t - \frac{5 x_1}{1 + x_1} - 3 x_2 - 0.3 u_2 \right), \\
u_1'(t) &= -(0.8 - 0.2 \sin t) u_1 + 5 x_1, \\
u_2'(t) &= -(0.8 - 0.2 \sin t) u_2 + 2 x_2.
\end{align*}

By calculation, one has
\[ M_1 = r_1 \frac{a_1 b_1}{c_1} = 3, \quad M_2 = r_2 \frac{a_2 b_2}{c_2} = \frac{1}{3}, \]
\[ (1 + M_1) r_2 \frac{a_2 c_1 + c_2 d_1}{b_2 c_1} = 4.4, \quad r_2 \frac{b_1 c_2}{a_2 c_2 + c_2 d_2} = \frac{5}{36}, \]
\[ \frac{1}{M_1 M_2} \left( r_1 - (1 + M_1) r_2 \frac{a_1 c_1 + c_1 d_1}{b_2 c_1} \right) = 4.8, \]
\[ \frac{1}{M_1 M_2} \left( r_1 - r_2 \frac{b_1 c_2}{a_2 c_2 + c_2 d_2} \right) = \frac{211}{12}. \]

We assume that \( \gamma = 0.00005 \), clearly, conditions (31) and (32) are satisfied, from Theorem 2, we know that the first species is permanent and the rest of species is driven to extinction. Figure 2 shows the dynamic behaviors of system (45) which is consistent with the conclusion obtained above.
Figure 2. Dynamic behaviors of the solution \((x_1(t), x_2(t), u_1(t), u_2(t))^T\) of system (45) with the initial conditions \((x_1(0), x_2(0), u_1(0), u_2(0)) = (9, 13, 7, 11.5)^T\), \((0.5, 1.5, 6)^T\) and \((3, 7, 4, 9)^T\), respectively.

Example 3. Consider the following equations

\[
\begin{align*}
    x_1'(t) &= x_1 \left(1 - (3.2 + 0.2 \sin t)x_1 - \frac{5x_2}{1 + x_1^2} - 0.00005x_1x_2^2 - 0.3u_1\right), \\
    x_2'(t) &= x_2 \left(1.55 - 0.05 \cos t - \frac{1.5x_1}{1 + x_1} - 0.4x_2 - 0.3u_2\right), \\
    u_1'(t) &= -(0.8 - 0.2 \sin t)u_1 + 5x_1, \\
    u_2'(t) &= -(0.8 - 0.2 \sin t)u_2 + 2x_2.
\end{align*}
\]

By calculation, one has

\[
\begin{align*}
    r_2^l &- a_1^l e_1^l + c_1^l d_1^l = 4.5, \\
    r_2^u &- \frac{1}{1 + M_2} - \frac{b_1^l e_2^l}{a_2^l e_2^l + c_2^l d_2^u} \approx 1.071.
\end{align*}
\]

Clearly, \(r_2^l < 4.5, r_2^u < 1.071\), condition (43) are satisfied, from Theorem 3, we know that the second species is permanent and the rest of species is driven to extinction.

Figure 3 shows the dynamic behaviors of system (46) is consistent with the conclusion obtained above.

Figure 3. Dynamic behaviors of the solution \((x_1(t), x_2(t), u_1(t), u_2(t))^T\) of system (4.3) with the initial conditions \((x_1(0), x_2(0), u_1(0), u_2(0)) = (3, 2, 6, 4)^T\), \((1, 0.5, 4, 8)^T\) and \((2, 1, 5, 6)^T\), respectively.
5. Conclusions

(1) In this paper, we consider a non-autonomous allelopathic phytoplankton model with nonlinear inter-inhibition terms and feedback controls, i.e., Equation (5), The difference from the model in [5] is that we consider two feedback control variables \( u_i(t), i = 1, 2 \) and the allelopathic interaction term is replaced by \( \gamma x_1(t)x_2(t) \) instead of \( \gamma x_1(t)x_2(t) \). We further investigate the influence of feedback control variables and toxic substances on the dynamic behaviors of system (5).

(2) Theorem 2 and 3 show that the feedback control variables and toxic substances play an important role on the extinction of system (5). Despite the second species could produce toxic, but lower rate of toxic production could not avoid the extinction of the second species. The conditions of Theorem 1 show that the feedback control variables and toxic substances do not effect on the permanence of the system.

(3) Moreover, when \( c_i = 0, i = 1, 2 \), moldel (5) becomes (2), we can easily find that Theorems 2.1 and 2.5 obtained by Xie and Xue et al. [5] are the corollary of Theorem 2 and 3, which extends the results of Xie and Xue et al. [5]. When \( c_i = 0, i = 1, 2, \gamma = 0 \), moldel (5) becomes (1), we can easily find that Theorem 1 and 2 obtained by Yu [18] are the corollary of Theorem 2 and 3, which extends the results of Yu [18].

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