Gaussian states — or, more generally, Gaussian operators — play an important role in Quantum Optics and Quantum Information Science, both in discussions about conceptual issues and in practical applications. We describe, in a tutorial manner, a systematic operator method for first characterizing such states and then investigating their properties. The central numerical quantities are the covariance matrix that specifies the characteristic function of the state, and the closely related matrices associated with Wigner’s and Glauber’s phase space functions. For pedagogical reasons, we restrict the discussion to one-dimensional and two-dimensional Gaussian states, for which we provide illustrating and instructive examples.

Keywords: Gaussian states, Gaussian operators, positivity, separability, entanglement; EPR correlations

1. Introduction

Gaussian wave functions appeared already very early in the development of Quantum Mechanics, when Schrödinger explained the observed behavior of the linear harmonic oscillator in terms of the undulatory eigenfunctions of the corresponding differential equation.¹ Heisenberg, in his fundamental work of 1927 devoted to the uncertainty relation,² used a Gaussian wave function of the form (in historical notation)

\[
S(\eta, q) \propto e^{-\frac{(q-q')^2}{2\sigma^2}} - \frac{2\pi}{\sigma} \delta(q - q')
\]  

(1)
to show that the undulatory character of this wave function, combined with the Dirac-Jordan transformation theory, leads to the indeterminacy relations. Heisenberg used this Gaussian wave function as a probability amplitude — perhaps the first application of this kind — to exhibit, in his own words, that The more precisely the position is determined, the less precisely the momentum is known in this instant, and vice versa.

Gaussian statistical properties are fundamental to many applications in statistical physics. It is familiar that the classical theory of a Gaussian noise associated with Brownian motion is fully characterized by the covariance function of the phase space variables \( q \) and \( p \). The transition from classical phase space variables to canonical quantum position and momentum operators requires “quantization rules” for associating quantum operator functions \( F(q,p) \) with the classical numerical functions \( F(q,p) \).

In the framework of quantum noise associated with a one-dimensional harmonic oscillator, Gardiner was perhaps the first to discuss, in a textbook format, various phase space properties of the most general Gaussian operators. Such a systematic formalism is very useful for the discussion of Gaussian statistical properties of a single quantum system with one degree of freedom.

The remarkable role that is played by one-dimensional Gaussian wave functions in realizing the ultimate limit of Heisenberg’s uncertainty relation for a single position-momentum pair naturally invites generalizations to higher-dimensional systems. That, however, opened a path into the then-unexplored territory of quantum correlations and quantum entanglement and thus started a never ending story.

In their reasoning concerning the alleged incompleteness of quantum mechanics, Einstein, Podolsky, and Rosen (EPR) used in 1935 the following wave function for a system composed of two particles (in historical notation):

\[
\Psi(x_1,x_2) = \int_{-\infty}^{\infty} dp \ e^{(2\pi i / h)(x_1 - x_2 + x_0)p} .
\]

(2)

The fact that this function could be written as a sum of products of one factor each for the two particles,

\[
\Psi(x_1,x_2) = \int_{-\infty}^{\infty} dp \ \psi_p(x_1)u_p(x_2) ,
\]

(3)

leads to the intriguing concept of quantum entanglement, a term coined by Schrödinger in Refs. 5 and 6.\(^b\)

\(^a\)In Heisenberg’s more general context, this Gaussian wave function refers to an arbitrary one-dimensional system, be it a free particle, a harmonic oscillator, or something with other dynamical properties. For the sake of convenience, we shall invariably speak of harmonic oscillators, thinking in particular of those associated with modes of the quantized radiation field.

\(^b\)It was in Ref. 5 that the word entanglement, which refers to the non-separability of quantum states of composite systems, was given its familiar meaning within the context of quantum mechanics. For a brief history about the use of the English word entanglement and the German word Verschränkung see Ref. 7.
The entangled EPR wave function (2) is a singular function of the distance \( x_1 - x_2 \) and can be visualized as an infinitely sharp two-dimensional Gaussian wave function of the entangled two-party system. Soon after the publication of the EPR paper, Bohr pointed out\(^8\)\ that the two pairs of canonical variables \( q_1, p_1 \) and \( q_2, p_2 \) for the two particles of a composed system can be replaced by two new pairs of conjugate variables,

\[
Q_{1,2} = \frac{1}{\sqrt{2}} (q_1 \mp q_2), \quad P_{1,2} = \frac{1}{\sqrt{2}} (p_1 \mp p_2),
\]

(4)
each pair now referring to both particles. Bohr noted that, since they commute: \( Q_1 P_2 = P_2 Q_1 \), the observables \( Q_1 \) and \( P_2 \) can be assigned sharp values simultaneously, so that the wave function (2) represents a joint eigenstate of these commuting variables. This property is at the heart of quantum entanglement, unintentionally brought to light by EPR. Bell inequalities of some kind are violated for the EPR wave function, as can be demonstrated by using its Wigner representation.\(^9\)

Schrödinger, Heisenberg, and the EPR trio were dealing with systems described by wave functions, i.e., with pure quantum states. For a quantum system described by a Gaussian mixed statistical operator \( \rho \), the correspondence between classical Gaussian functions \( G(q_1, p_1, q_2, p_2) \) in two dimensions and the statistical operators \( \rho = G(q_1, p_1, q_2, p_2) \) should employ a “quantization rule” that preserves all the properties of a density matrix.

Recent applications of entangled two-mode squeezed states of light for quantum teleportation\(^10\) and other quantum information purposes\(^11\) have generated a lot of interest in the entangled properties of general mixed Gaussian states in quantum optics.\(^12\) It turns out that the concept of quantum entanglement, as defined by (3), has to be generalized when the system is not in a pure state. In the general case of a density operator, rather than a wave function, one uses the definition of quantum separability introduced by Werner.\(^13\) A general quantum density operator of a two-party system is separable if it is a convex sum of product states,

\[
\rho = \sum_k w_k \rho_1^{(k)} \rho_2^{(k)} \quad \text{with} \quad \sum_k w_k = 1 \quad \text{and} \quad w_k > 0,
\]

(5)
where \( \rho_1^{(k)} \) and \( \rho_2^{(k)} \) are statistical operators of the two subsystems in question.\(^5\)

The separability properties of general mixed states of a harmonic oscillator in two dimensions can be studied with the help of two different methods. The first uses the covariance matrix and the Heisenberg uncertainty relations.\(^14\) The second makes use of the criterion of positivity under partial transposition.\(^15\) These two approaches lead to essentially the same conclusions, while employing very different techniques.

\(^{5}\)Actually, one should only require that the given \( \rho \) can be approximated to any required accuracy by a sum of this kind with a finite number of terms, but we take the liberty to ignore mathematical details of such a more pedantic sort.
The objective of this tutorial is to review different operator and phase-space techniques that allows a systematic investigation of different properties of Gaussian operators of unit trace, referring, for instance, to a single mode or two modes of the radiation field. We discuss the properties of Gaussian operators with the aid of various techniques that are widely used in Quantum Optics. In particular, we provide a careful description of Gaussian operators using (i) the boson operator algebra, and (ii) phase-space descriptions based on the Wigner representation and the Glauber P-representation. We establish relations between these equivalent though different-in-form versions of Gaussian statistical operators. In Sec. 2 we gather the necessary mathematical tools used in this tutorial at the example of the general Gaussian operator of a one-dimensional harmonic oscillator. We provide a general parametrization of such states, discuss the positivity criteria and the so-called P-representability of positive Gaussian operators. At the end of Sec. 2 we illustrate the formalism with the example of a squeezed single-mode state of light. Then, in Sec. 3, we turn to two-party Gaussian states. The central part of this tutorial is Sec. 3.4 that deals with the separability of two-dimensional Gaussian states. We discuss the positivity criteria of such states and their separability conditions, thereby illustrating various approaches. At the end of Sec. 3 we provide a number of simple examples that are useful for studying and illustrating various physical effects in Quantum Optics and in Quantum Information Theory. We confine our discussion to two-party systems, but the methods presented in this tutorial can be extended to many-party systems. In concluding remarks we summarize and suggest some supplementary reading on the subject.

2. Gaussian States of a One-Dimensional Oscillator

2.1. Parameterizations

Any operator referring to a harmonic oscillator — position operator $q$, momentum operator $p$, both measured in natural units, so that $qp - pq = i$ — is a function of the familiar ladder operators

\[ a^\dagger = \frac{q - ip}{\sqrt{2}}, \quad a = \frac{q + ip}{\sqrt{2}}. \]

We can specify such an operator $G(a^\dagger, a)$ by its characteristic function $C(z^*, z)$,

\[ C(z^*, z) = \text{Tr} \left\{ e^{za^\dagger - z^*a} G(a^\dagger, a) \right\}, \]

which is a numerical function of the complex phase space variables

\[ z^* = \frac{q' - ip'}{\sqrt{2}}, \quad z = \frac{q' + ip'}{\sqrt{2}}. \]

Here, $q', p'$ are the cartesian coordinates of classical phase space as one knows them from Hamilton’s approach to classical mechanics or the Liouville formulation of statistical mechanics. A more compact way of writing (7) is

\[ C(z) = \text{Tr} \left\{ e^{-z^2Ea} G(a) \right\}. \]
where we introduce 2-component columns and rows in accordance with
\[ z = \begin{bmatrix} z \\ z^* \end{bmatrix}, \quad z^\dagger = [z^*, z], \quad a = \begin{bmatrix} a \\ a^\dagger \end{bmatrix}, \quad a^\dagger = [a^\dagger, a], \]
and meet the symplectic $2 \times 2$ matrix
\[ E = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \]
(11)

Reciprocally, we get the operator from its characteristic function by a phase space integration,
\[ G(a) = \int (dz) e^{z^\dagger E a} C(z), \]
(12)
which is essentially the Liouville trace of statistical mechanics and complements the quantum-mechanical trace of (9). The explicit form of the volume element $(dz)$ depends on the way phase space is parameterized. For the standard parameterization (8) one has
\[ (dz) = \frac{dq'}{2\pi} dp', \]
(13)
or $(dz) = \frac{1}{\pi} d\text{Re} \, z \, d\text{Im} \, z = \frac{1}{\pi} d^2 z$ in a popular notation. But sometimes the actual integrations are more easily performed with
\[ z = q', \quad z^* = -ip', \quad (dz) = \frac{dq' \, dp'}{2\pi}, \]
(14)
for instance. Note that we include the normalizing denominator of $2\pi$ in $(dz)$ which, in a rough manner of speaking, indicates that the integral in (12) counts one quantum state per phase space volume of $2\pi \hbar$.

We focus on Hermitian operators $G(a) = [G(a)]^\dagger$ with characteristic functions that are of the Gaussian form
\[ C(z) = e^{-\frac{1}{2} z^\dagger C z} = e^{-\frac{1}{2} m z^2} - (n + \frac{1}{2}) z^* z - \frac{1}{2} m^* z^2 \]
(15)
with the $2 \times 2$ matrix $C$ given by
\[ C = \begin{bmatrix} n + \frac{1}{2} & m \\ m^* & n + \frac{1}{2} \end{bmatrix} = C^\dagger. \]
(16)
The hermiticity of $G$ implies $[C(z)]^* = C(-z)$, and vice versa; the Gaussian characteristic function (15) has this symmetry property because the parameter $n$ is real.

For a given $C(z)$, (15) does not specify the diagonal entries of $C$ uniquely, only their sum is determined. The symmetry of the standard form (16) of $C$ exploits this arbitrariness conveniently. A particularly useful way of looking at this proceeds from the response of $a$ and $z$ to transposition,
\[ a^T = [a, a^\dagger] = a^\dagger T, \quad z^T = [z, z^*] = z^\dagger T, \]
(17)
with
\[ T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \] (18)
so that
\[ (z^\dagger C z)^T = z^\dagger T C^T T z. \] (19)
This states that one can replace \( C \) by \( T C^T T \) in (15), or by any linear average of both, without changing the characteristic function \( C(z) \). It is therefore natural to enforce the symmetry
\[ C = T C^T T, \] (20)
and this establishes (16).

Since \( C(z) = 1 \) for \( z^* = 0, z = 0 \) in (15) we take for granted that \( G \) has unit trace, \( \text{Tr} \{ G \} = 1 \). This excludes Gaussian operators with infinite trace, but otherwise it is just a matter of convenient conventional normalization.

The absence of linear terms in the exponent indicates another convention: We assume that \( \text{Tr} \{ a G \} = 0, \text{Tr} \{ a^\dagger G \} = 0 \), which can always be arranged with a suitable unitary shift of \( a^\dagger \) and \( a \).

Upon expanding the characteristic function in powers of \( z^* \) and \( z \) one easily identifies the physical significance of the numerical parameters \( n \) and \( m \),
\[ n = \text{Tr} \{ a^\dagger a G \}, \]
\[ m = -\text{Tr} \{ a^2 G \}, \quad m^* = -\text{Tr} \{ a^\dagger^2 G \}, \] (21)
or, more compactly,
\[ \text{Tr} \{ a a^\dagger G \} = \text{ECE} + \frac{1}{2} \text{E} \] (22)
and
\[ C = \text{E} \text{Tr} \{ a a^\dagger G \} E - \frac{1}{2} \text{E} = \frac{1}{2} \text{E} \text{Tr} \left\{ [aa^\dagger + T(a a^\dagger)^T T] G \right\} E. \] (23)
These relations identify \( C \) as the covariance matrix of \( G \).

Anticipating that this will be of some relevance later, we note that a positive \( G \) can serve as a probability operator (alternatively called “state operator” or “density operator”). Then Heisenberg’s uncertainty relation requires
\[ \langle a^\dagger a \rangle \langle a a^\dagger \rangle \geq \langle a^\dagger^2 \rangle \langle a^2 \rangle \] (24)
so that \( n \) and \( m \) of a positive \( G \) must be such that
\[ n(n + 1) \geq m^* m \quad \text{or} \quad n \geq \sqrt{m^* m + \frac{1}{2}} - \frac{1}{2}. \] (25)
As a compact statement about the covariance matrix \( C \), this appears as
\[ C + \frac{1}{2} \text{E} \geq 0. \] (26)
Here we recall that one derivation of (24) simply exploits the positivity of operators of the form \(X^\dagger X\). Choose
\[
X = a^\dagger E x
\]
with
\[
x = \begin{bmatrix} x \\ y^* \end{bmatrix}
\]
so that
\[
\langle x^\dagger E a a^\dagger \rangle \geq 0
\]
has to hold for any \(x\). With (22), the recognition that (for \(G \geq 0\))
\[
E \langle a a^\dagger \rangle E = E \text{Tr} \{a a^\dagger G\} E = C + \frac{1}{2} E
\]
then establishes (26) as a necessary property of any positive \(G\).

In (12), \(G\) is expanded in the Weyl basis\(^d\) that consists of the unitary operators
\[
e^{z^\dagger E a} = e^{-a^\dagger E z}
\]
Equivalently, we can use the Hermitian Wigner basis\(^e\) for another expansion of the same \(G\). The Wigner basis comprises the operators
\[
2(-1)^{(a^\dagger - z^*)^2} = e^{-z^\dagger E a} 2(-1)^a e^{z^\dagger E a},
\]
which are obtained from the parity operator \((-1)^a^* a\) by unitary displacements;\(^f\)
the factor of 2 normalizes them to unit trace. This gives
\[
G(a) = \int (dz) 2(-1)^{(a^\dagger - z^*)^2} W(z),
\]
where
\[
W(z) = \text{Tr} \left\{2(-1)^{(a^\dagger - z^*)^2} G(a) \right\}
\]
\[
= \text{Tr} \left\{2(-1)^a G(a + z) \right\}
\]
is the Wigner function to \(G(a)\). A real Wigner function, \([W(z)]^* = W(z)\), is associated with a Hermitian operator, \([G(a)]^\dagger = G(a)\).

Since Fourier transformation relates the bases to each other,
\[
e^{z^\dagger E a} = \int (dz') e^{z^\dagger E z'} 2(-1)^a e^{z^\dagger E z'},
\]
\[
2(-1)^{(a^\dagger - z^*)^2} = \int (dz') e^{z^\dagger E z'} e^{-a^\dagger E z'}
\]
\(^d\)Concerning Weyl’s unitary operator basis, the seminal papers by Weyl (1927) and Schwinger (1960) are recommended reading.\(^19,20\) A recent textbook account is given in chapters 1.14–1.16 of Ref. 21.
\(^e\)Concerning Wigner functions, the seminal papers by Wigner (1932) and Moyal (1949) are recommended reading.\(^22,23\) and so are the more recent reviews by Tatarskii,\(^24\) by Balasz and Jennings.\(^25\) and by Hillery \emph{et al.},\(^26\) and also the textbook expositions by Scully and Zubairy,\(^27\) and Schleich.\(^28\)
\(^f\)Perhaps the first to note the intimate connection between the Wigner function and the parity operator was Royer;\(^29\) in the equivalent language of the Weyl quantization scheme the analogous observation was made a bit earlier by Grossmann.\(^30\) A systematic study from the viewpoint of operator bases is given in Ref. 31. — The appearance of the parity operator is central to experimental schemes for measuring Wigner functions directly.\(^32,33,34\)
the (real) Wigner function $W$ and the characteristic function $C$ are Fourier transforms of one another,

$$W(z) = \int (dz') e^{-z^\dagger E z'} C(z'),$$

$$C(z) = \int (dz') e^{-z^\dagger E z'} W(z'),$$

and, therefore, the Wigner function is also a Gaussian,

$$W(z) = \sqrt{\det W} e^{-\frac{1}{2} z^\dagger W z}$$

where

$$\text{Tr} \{ G \} = 1 \quad \text{is equivalent to} \quad \int (dz) W(z) = 1.$$ 

Note that $[C(z)]^* = C(-z)$ and $[W(z)]^* = W(z)$ imply each other.

Perhaps the simplest verification of (33) combines the Baker-Hausdorff identity

$$e^{z^\dagger E} e^{-z a^\dagger + z^* a} = e^{-z a^\dagger + z^* a} e^{-\frac{1}{2} z^\dagger z} = : e^{z^\dagger E} e^{-\frac{1}{2} z^\dagger z} :$$

and the normally-ordered form of the displaced parity operator,

$$e^{a^\dagger E z} e^{a^\dagger} = : e^{-(a^\dagger - z^\dagger)(a - z)} :,$$

with the basic Fourier-Gauss integral

$$\int (dz) e^{-\frac{1}{2} z^\dagger A z} e^{z^\dagger x} = \frac{1}{\sqrt{\det A}} e^{\frac{1}{2} x^T A^{-1} x},$$

valid for all matrices $A = T A^T T > 0$ and all columns $x$, whether $x^T T$ is simply related to $x^\dagger$ or not.

Upon using (37) in (12) or (38) in (31) we find the normally ordered form of $G(a)$,

$$G(a) = \sqrt{\det Q} : e^{-\frac{1}{2} a^\dagger Q a} :,$$

which is another Gaussian function. Fourier-Gauss integrals connect the various ways of writing $G(a)$ and, accordingly, the matrices $C$, $W$, and $Q$ must be simply related. Indeed, one finds

$$C = E W^{-1} E = E Q^{-1} E - \frac{1}{2} I,$$

$$W = E C^{-1} E = (Q^{-1} - \frac{1}{2} I)^{-1},$$

$$Q = E (C + \frac{1}{2} I)^{-1} E = (W^{-1} + \frac{1}{2} I)^{-1},$$

where $I$ is the $2 \times 2$ unit matrix. The symmetry property (20) of matrix $C$ is inherited by matrices $W$ and $Q$. 

Since $ECE$, $W$, $Q$ are functions of each other, these three matrices commute with one another. In fact, as long as we are dealing only with $2 \times 2$ matrices, Hermitian and with identical diagonal values, identities such as $ECE = C^{-1} \det C$ can be used to achieve a further simplification,

$$W = \frac{C}{\det C} ; \quad Q = \frac{C + \frac{1}{2}I}{\det(C + \frac{1}{2}I)},$$

but this is particular to $2 \times 2$ matrices and does not hold for the $4 \times 4$ matrices in Sec. 3.

In more explicit terms, we have

$$W = \left( \text{Tr} \{aa^\dagger G\} - \frac{1}{2}E \right)^{-1},$$

$$Q = \left( \text{Tr} \{aa^\dagger G\} + \frac{1}{2}(I - E) \right)^{-1},$$

or

$$W = \frac{1}{(n + \frac{1}{2})^2 - m^*m} \begin{bmatrix} n + \frac{1}{2} & m \\ m^* & n + \frac{1}{2} \end{bmatrix},$$

and

$$Q = \frac{1}{(n + 1)^2 - m^*m} \begin{bmatrix} n + 1 & m \\ m^* & n + 1 \end{bmatrix} \equiv \begin{bmatrix} 1 - \nu & \mu \\ \mu^* & 1 - \nu \end{bmatrix}. \quad (45)$$

They obey $(2I + W)(2I - Q) = 4I$, as they should, which one verifies easily by inspection.

It is time to note that the transitions from $C$ to $W$ and $Q$ are only possible if the respective Fourier integrals are not singular, which requires

$$C > 0 \quad (46)$$

or, explicitly,

$$n + \frac{1}{2} > \sqrt{m^*m} = |m|. \quad (47)$$

Values of $n$ and $m$ that violate this condition will, therefore, not be considered at all. The determinant of $C$ is then positive, and so are the determinants of $W$ and $Q$,

$$\det C = (n + \frac{1}{2})^2 - m^*m > 0,$$

$$\det W = (\det C)^{-1} > 0,$$

$$\det Q = \frac{1}{(n + 1)^2 - m^*m} > 0. \quad (48)$$

2.2. Positivity criteria

Owing to its simple Gaussian form, operator $G$ must be unitarily equivalent to the basic Gaussian $G_0$,

$$G_0 = (1 - g)g^{a^\dagger a} = (1 - g) : e^{-(1 - g)a^\dagger a} :, \quad (49)$$
where $-1 \leq g < 1$ ensures a finite trace. In other words, we have

$$G = U^\dagger G_0 U$$

with some unitary $U$ that effects a linear transformation on $a$ and $a^\dagger$, a *squeezing transformation* in the jargon of quantum optics. Its most general form is

$$U^\dagger a U = a e^{i\phi} \cosh \theta + a^\dagger e^{i\varphi} \sinh \theta ,$$
$$U^\dagger a^\dagger U = a^\dagger e^{-i\phi} \cosh \theta + a e^{-i\varphi} \sinh \theta$$

or, compactly,

$$U^\dagger a U = U^\dagger \begin{bmatrix} a \\ a^\dagger \end{bmatrix} U = U \begin{bmatrix} a \\ a^\dagger \end{bmatrix} = U a , \quad U^\dagger a^\dagger U = a^\dagger U^\dagger$$

with

$$U = \begin{bmatrix} e^{i\phi} \cosh \theta & e^{i\varphi} \sinh \theta \\ e^{-i\varphi} \sinh \theta & e^{-i\phi} \cosh \theta \end{bmatrix} ,$$

which is characterized by three real parameters: $\theta$, $\phi$, $\varphi$. In the present context, only the relative phase $\phi - \varphi$ enters, so that the initial parameters $n$ and $m$ determine $g$, $\theta$, and $\phi - \varphi$.

Note that the $2 \times 2$ matrix $U$ that is thus associated with the unitary operator $U$ is not a unitary matrix itself. Rather it obeys

$$U^\dagger E U = E \quad \text{or} \quad U^{-1} = E U^\dagger$$

(54)

to maintain the fundamental commutation relation

$$aa^\dagger - a^\dagger a = 1 \quad \text{or} \quad a^\dagger E a = -1 ,$$

(55)

and in addition

$$T U^T = U^\dagger T$$

(56)

must hold for consistency with $a^T = a^\dagger T$.

The resulting relation between the characteristic functions of $G_0$ and $G$ amounts to

$$C = U^\dagger C_0 U$$

(57)

and, as a consequence of (41) in conjunction with (54), we find

$$W = U^\dagger W_0 U$$

(58)

and

$$Q = \left[ \left( U^\dagger Q_0 U \right)^{-1} + \frac{\gamma}{\delta} E (I - U^\dagger U) E \right]^{-1} .$$

(59)
We remark that the statement (58) about the Wigner functions is, of course, consistent with the general observation that linear transformations on the operators $a, a^\dagger$ are reflected by exactly the same transformation on $z, z^*$ in $W$.

Further, we note that $Q \neq U^\dagger Q_0 U$ unless $U$ commutes with $E$, and this is as it should be: transformations with $EUE \neq U$ turn $a$ into a linear combination of $a$ and $a^\dagger$, so that the meaning of normal ordering is altered. The extra term in (59) takes just that into account.

Let us use the matrices $C$ and $C_0$ of the characteristic functions to find the relations between the initial parameters $n, m$ and the new parameters $g, \theta, \phi - \varphi$. With $C$ of (16) and 

$$C_0 = \frac{1}{2} \frac{1 + g}{1 - g},$$

(60) in (57) we have

$$n + \frac{1}{2} = \frac{1}{2} \frac{1 + g}{1 - g} \cosh(2\theta),$$

$$m = \frac{1}{2} \frac{1 + g}{1 - g} e^{-i(\phi - \varphi)} \sinh(2\theta),$$

(61)

and, in particular,

$$(n + \frac{1}{2})^2 - m^*m = \frac{1}{4} \left(\frac{1 + g}{1 - g}\right)^2.$$  

(62)

Thus $g$ is given by

$$g = \frac{\left[\left(n + \frac{1}{2}\right)^2 - m^*m\right]^\frac{1}{2} - \frac{1}{2}}{\left[\left(n + \frac{1}{2}\right)^2 - m^*m\right]^\frac{1}{2} + \frac{1}{2}},$$

(63)

and since the eigenvalues of $G$ are $(1 - g)g^k$ with $k = 0, 1, 2, \ldots$ we find that $G \geq 0$ requires $g \geq 0$, which in turn says that the argument of the square root must be at least $\frac{1}{4}$, and this is precisely the constraint (25). In other words: Condition (26) is both necessary and sufficient for the positivity of $G$.

These considerations are instructive, but they are not really needed if we just want to find the eigenvalues of $G$, for which purpose knowledge of $\theta$ and $\phi - \varphi$ is obsolete. A direct method proceeds from the observation that

$$\text{Tr} \left\{ G^2 \right\} = \sum_{k=0}^{\infty} \left[(1 - g)g^k\right]^2 = \frac{1 - g}{1 + g}$$

(64)

The statement is actually true for rather arbitrary linear similarity transformations, not just for linear unitary transformations, and it applies to multidimensional Wigner functions. Somewhat surprisingly, this important transformation property is not as widely known as it should be. Various special cases are demonstrated in Refs. 35, 31, and 36: linear unitary transformations for one degree of freedom; 35 linear similarity transformations (unitary or not) for one degree of freedom; 31 linear unitary transformations for many degrees of freedom. 36 And Ref. 37 deals with the general case of linear similarity transformations (unitary or not) for many degrees of freedom.
and employs the Wigner function for calculating this trace,
\[
\text{Tr} \{ G^2 \} = \int \text{d}z \ |W(z)|^2 = \frac{1}{2} \sqrt{\det W},
\]
(65)

In conjunction with (48) this reproduces (63).

Note that condition (65) follows from the general property of a density operator for which
\[
\text{Tr} \{ G^2 \} \leq 1 \quad \text{is equivalent to} \quad \int \text{d}z \ W(z)^2 \leq 1.
\]
(66)
The discussion above tells us that condition (66), which leads to
\[
\sqrt{\det W} \leq 2,
\]
(67)
is both necessary and sufficient for the positivity of \( G \) in one dimension. As we shall see in Sec. 3, however, this condition is not strong enough to guarantee the positivity of \( G \) in two and more dimensions.

We do not even need to know the eigenvalues of \( G \) if checking \( G \geq 0 \) is all that we are interested in. For, the normally ordered form (40) reads more explicitly
\[
G = \sqrt{\det Q} e^{-\frac{1}{2} \mu a^\dagger : e^{-\nu} a^\dagger a : e^{-\frac{1}{2} \mu^* a^2}}.
\]
(68)
This has the structure \( G = S^\dagger \bar{G} S \) with some \( S \) and
\[
\bar{G} = (1 - \nu) : e^{-(1 - \nu) a^\dagger a :} = (1 - \nu) \nu a^\dagger a;
\]
(69)
or
\[
\bar{G} = \sqrt{\det \bar{Q}} : e^{-\frac{1}{2} a^\dagger \bar{Q} a :}
\]
(70)
with
\[
\bar{Q} = \begin{bmatrix} 1 - \nu & 0 \\ 0 & 1 - \nu \end{bmatrix} = \frac{1}{2} (Q + E Q E).
\]
(71)
Consequently, \( G \geq 0 \) is ensured by \( \bar{G} \geq 0 \), and this just requires \( \nu \geq 0 \), or
\[
1 - \bar{Q} \geq 0.
\]
(72)
Now, upon recalling how \( \nu \) is related to \( n \) and \( m \) in (45),
\[
\nu = \frac{n(n+1) - m^* m}{(n+1)^2 - m^* m},
\]
(73)
we find, once more, that (26) is the positivity criterion.

Having found the value of \( g \), the unitary transformation of (50) can be identified. For this purpose we return to (57) and write it in the equivalent forms
\[
C E U^\dagger = U^\dagger C_0 E \quad \text{and} \quad U E C = E C_0 U,
\]
(74)
where
\[
E C_0 = \frac{1}{2} \left[ \begin{array}{cc} 1 + g & 1 \\ 1 - g & 1 \end{array} \right] = C_0 E.
\]
(75)
Therefore, the eigenvalues of the $2 \times 2$ matrices $EC$ and $CE$ are $\pm \frac{1}{2}(1 + g)/(1 - g)$, the columns of $U^\dagger$ are the respective eigencolumns of $CE$, and the rows of $U$ are the eigenrows of $EC$. With (61) and (63), it is a matter of inspection to verify these statements for $C$ of (16) and $U$ of (53).

2.3. $P$-representable positive Gaussian operators

For $g = 0$, the basic Gaussian $G_0$ of (49) is $:e^{-a^\dagger a} :$, the projector to the oscillator ground state. Unitary displacements turn it into $:e^{-(a^\dagger - z^*)(a - z)} :$, which project onto the coherent states, the eigenbras of $a^\dagger$ and eigenkets of $a$ with respective eigenvalues $z^*$ and $z$.

A positive Gaussian operator, $G > 0$, is said to be $P$-representable if one can write it as a mixture of coherent states,

$$G = \int (dz) :e^{-(a^\dagger - z^*)(a - z)} : \sqrt{\det P} e^{-\frac{1}{2}z^*Pz}$$

with $P > 0$. The limiting case of $G = :e^{-a^\dagger a} :$, when

$$\sqrt{\det P} e^{-\frac{1}{2}z^*Pz} \to \delta(z),$$

so that $P \to \infty$ in some sense, need not concern us too much. For the sake of notational simplicity, we exclude it by considering only $G > 0$, rather than $G \geq 0$.

For a $P$-representable Gaussian, we have

$$Q = (P^{-1} + I)^{-1},$$
$$W = (P^{-1} + \frac{1}{2}I)^{-1},$$
$$C = EP^{-1}E + \frac{1}{2}I,$$

and

$$P = E(C - \frac{1}{2}I)^{-1}E = (W^{-1} - \frac{1}{2}I)^{-1} = (Q^{-1} - I)^{-1}.$$  \hspace{1cm} (79)

So, a given $G$ is $P$-representable if

$$C - \frac{1}{2}I > 0, \quad W^{-1} - \frac{1}{2}I > 0, \quad Q^{-1} - I > 0,$$

which requires

$$n > |m|. \hspace{2cm} (80)$$

There are, therefore, positive Gaussians operators that are not $P$-representable, namely those with

$$|m| > n > \sqrt{m^*m + \frac{1}{4} - \frac{1}{2}}. \hspace{2cm} (82)$$

See (101) below for the definition of the two-dimensional Dirac delta function.
All positive Gaussian operators are, however, unitarily equivalent to a P-representable one, because $G_0$ of (49) is P-representable if $g > 0$, as

$$(1 - g)g^a a = \int (dz) : e^{-(a^\dagger - z^\ast)(a - z)} : \left(g^{-1} - 1\right) e^{-(g^{-1} - 1)z^\ast z}$$

shows explicitly. If $G > 0$ is P-representable, then the unitary transformation of (49) amounts to

$$P = \left[\left(U^\dagger P_0 U\right)^{-1} - \frac{1}{2} E (1 - U^\dagger U) E\right]^{-1},$$

where we see an extra term, similar to the one in (59), which reflects the injunction of normal ordering that is inherent in (76).

The unitary transformation of (50) that relates the given positive $G$ to the special P-representable $G_0$ of (49) and (83), is just one transformation of many, which are all such that $UGU^\dagger$ is a P-representable Gaussian. In terms of the respective characteristic functions, these transformations are such that

$$C = \left[\begin{array}{cc} n + \frac{1}{2} & m \\ m^\ast & n + \frac{1}{2} \end{array}\right] \rightarrow U^{-1} C U^{-1} = E U E C E U^{-1} = \left[\begin{array}{cc} N + \frac{1}{2} & M \\ M^\ast & N + \frac{1}{2} \end{array}\right]$$

with

$$N > |M|,$$

where $U$ is of the form (53) and obeys (54). For example, with $\text{Re}\left(me^{i\phi} - i\phi\right) = |m|$ ensured by choosing $\phi$ and $\varphi$ accordingly, we get

$$N + \frac{1}{2} = (n + \frac{1}{2}) \cosh(2\theta) - |m| \sinh(2\theta),$$

$$M = e^{i\phi + i\varphi} [m| \cosh(2\theta) - (n + \frac{1}{2}) \sinh(2\theta)],$$

and then (86), the condition that $UGU^\dagger$ is a P-representable Gaussian, amounts to

$$\frac{1}{2n + 1 - 2|m|} \leq e^{2\theta} \leq 2 + 2|m|. $$

Owing to (47), the lower bound is assuredly positive, and since (25) holds for a positive $G$, the upper bound is certainly larger than the lower one so that the range for $e^{2\theta}$ is not empty.

Any $\theta$ from this range will serve the purpose of relating the given $G$ to a P-representable one. In particular we note that $\theta = 0$ is permissible (of course) if $G$ itself is P-representable ($n > |m|$). The special transformation of (50), for which $UGU^\dagger = G_0$ and thus $M = 0$, obtains when $e^{2\theta}$ is the geometric mean of the bounds, so that

$$e^{4\theta_0} = \frac{n + \frac{1}{2} + |m|}{n + \frac{1}{2} - |m|}$$

identifies the $\theta$ value for this distinguished squeezing transformation.
2.4. Transposed Gaussians

As a preparation for a later discussion in the context of Gaussian states of two entangled harmonic oscillators, let us briefly discuss what happens when an operator transposition is done. First of all, we must note that transposing an operator is a representation-dependent operation. For a chosen basis of Hilbert state vectors, $|k\rangle$, the transpose $G^T$ of $G$ is defined by

$$\langle k|G^T|k'\rangle = \langle k'|G|k\rangle .$$

(90)

Clearly, if the $|k\rangle$’s are eigenkets of $G$, there is no difference between $G$ and $G^T$.

We shall consider two procedures based on the position and momentum representations associated with the eigenstates of $q = 2^{-\frac{i}{4}}(a^\dagger + a)$ and $p = 2^{-\frac{i}{4}}(ia^\dagger - ia)$, respectively. Transposition in the $q$-representation has this effect on products of a $q$-function and a $p$-function,

$$[f_1(q)f_2(p)]^T = f_2(-p)f_1(q),$$

(91)

and in the $p$-representation one gets

$$[f_1(q)f_2(p)]^T = f_2(p)f_1(-q).$$

(92)

We find the resulting transformation of $e^{za^\dagger - z^*a}$ by first noting that

$$e^{za^\dagger - z^*a} = e^{2^{-\frac{i}{4}}(z - z^*)q - 2^{-\frac{i}{4}}i(z + z^*)p} = e^{2^{-\frac{i}{4}}(z - z^*)q}e^{-2^{-\frac{i}{4}}i(z + z^*)p}e^{\frac{1}{4}(z - z^*)(z + z^*)},$$

so that

$$\begin{align*}
\left[e^{za^\dagger - z^*a}\right]^T &= e^{-2^{-\frac{i}{4}}i(z + z^*)p}e^{2^{-\frac{i}{4}}(z - z^*)q}e^{-\frac{1}{4}(z - z^*)(z + z^*)} \\
&= e^{2^{-\frac{i}{4}}(z - z^*)q + 2^{-\frac{i}{4}}i(z + z^*)p} \\
&= e^{za^\dagger - z^*a^\dagger}
\end{align*}$$

(94)

in the $q$-representation, whereas

$$\begin{align*}
\left[e^{za^\dagger - z^*a}\right]^T &= e^{-2^{-\frac{i}{4}}i(z + z^*)p}e^{-2^{-\frac{i}{4}}i(z - z^*)q}e^{-\frac{1}{4}(z - z^*)(z + z^*)} \\
&= e^{-2^{-\frac{i}{4}}i(z - z^*)q - 2^{-\frac{i}{4}}i(z + z^*)p} \\
&= e^{za^\dagger - za}
\end{align*}$$

(95)

in the $p$-representation. These are compactly summarized in

$$\begin{align*}
\left[e^{iEz}\right]^T &= \left[e^{-z^*Ea}\right]^T = e^{\pm a^\dagger TEz} = e^{a^\dagger ETz} \\
&= e^{\mp z^T Ea} = e^{\pm z^T Ea}
\end{align*}$$

(96)

where the upper sign refers to the $q$-representation, and the lower sign to the $p$-representation. Not surprisingly, we encounter the transposition matrix $T$ of (18).

In $C$ of (15) and $W$ of (35) we thus have

$$z \rightarrow \pm Tz, \quad z^\dagger \rightarrow \pm z^\dagger T,$$

(97)
and
\[ a \rightarrow \mp Ta, \quad a^\dagger \rightarrow \mp a^\dagger T \quad (98) \]
apply in the normally ordered form of (40). Since the sign is irrelevant for the Gaussians that we are concerned with, we have in both cases
\[ C \rightarrow CT, \quad W \rightarrow WT, \quad Q \rightarrow QT, \quad P \rightarrow PT, \quad (99) \]
which are, of course, consistent with (41) as well as (78) and (79).

Indeed, the net effect is simply \( m \leftrightarrow m^* \), and therefore \( G^T \) has the same eigenvalues as \( G \), so that \( G \) and \( G^T \) are unitarily equivalent. The situation is markedly different when a partial transposition is done on a Gaussian state of two harmonic oscillators.

2.5. Examples

We conclude this section on one-dimensional Gaussian operators by giving two explicit examples.

2.5.1. Parity Operator

The first example is the parity operator that is used to form the Wigner basis. We have previously noted that the transition from \( C \) to \( W \) and \( Q \) is nonsingular if \( C > 0 \). Actually, the limit \( C \geq 0, C \neq 0 \) can be included with a bit of caution, and \( C = 0 \) is actually needed in the construction of the Wigner representation of the Gaussian operators (31). For this choice of \( C \), we obtain: \( Q = \frac{1}{2} I \), and accordingly the corresponding Wigner function of such an operator is given by a singular expression that can be normalized:
\[ W(z) = \delta(z) \quad \text{with} \quad \int(dz) W(z) = 1, \quad (100) \]

where
\[ \delta(z) \equiv 2\pi\delta(q')\delta(p'), \quad \int(dz) \delta(z)f(z) = f(z = \begin{bmatrix} 0 \\ 0 \end{bmatrix}) \quad (101) \]

for both parameterizations (8) and (14). Using this relation, we obtain from (31) that the resulting Gaussian operator
\[ G = 2 : e^{-2a^\dagger a} : = 2 (-1)^a a^\dagger \quad (102) \]
is twice the well known parity operator. This operator is non-positive and normalized to unit trace, \( \text{Tr} \{G\} = 1 \). Note that the Wigner function (100) is singular, normalized and positive, while the corresponding Gaussian operator is normalized but not positive. Upon unitarily shifting the parity operator by complex numbers \( z \), we reproduce all elements of the Wigner basis (30).
2.5.2. Pure Gaussian State

The second example is a Gaussian operator describing a pure quantum state. It follows from (65) that a Gaussian state is pure if
\[
\det C = (\det W)^{-1} = (n + \frac{1}{2})^2 - m^*m = \frac{1}{4}
\]
holds, which amounts to
\[
|m| = \sqrt{n(n+1)}.
\]
In this case the general formula (68) reduces to a projector
\[
G = |\mu\rangle\langle\mu| = \sqrt{\det Q} e^{-\frac{1}{2} \mu a^\dagger a} |0\rangle\langle 0| e^{-\frac{1}{2} \mu^* a^2},
\]
where
\[
|\mu| = \sqrt{n \over n+1}, \quad \det Q = {1 \over n+1} = 1 - \mu^*\mu,
\]
and
\[
|0\rangle\langle 0| = : e^{-a^\dagger a} :.
\]
projects on the 0th Fock state, the ground state of the harmonic oscillator: \(a|0\rangle = 0\).

In (104) we recognize the border case of (25), and thus of (24). This tells us that one can check the purity of a Gaussian state of a one-dimensional oscillator by measuring the expectation values of \(a^\dagger a\) and \(a^2\), and then verifying that the equal sign holds in (24).

From (105)–(107) we see that the ket vector of the pure Gaussian state has the form
\[
|\mu\rangle = \left(1 - \mu^*\mu\right)^{1/4} e^{-\frac{1}{2} \mu a^\dagger a} |0\rangle.
\]
Now assuming, for simplicity, that \(\mu\) is real and positive, \(\mu = |\mu| > 0\), we recognize in this expression a squeezed state of a one-dimensional harmonic oscillator, whose position wave function is
\[
\psi_\mu(q) = \langle q|\mu\rangle = \left(\frac{\kappa}{\pi}\right)^{1/4} e^{-\frac{1}{2} \kappa q^2}
\]
with
\[
\kappa = \frac{1 + \mu}{1 - \mu} = \frac{\sqrt{n+1} + n}{\sqrt{n+1} - \sqrt{n}}.
\]
For \(n = 0\), we have \(\mu = 0, \kappa = 1\); the wave function is then that of the ground state of the one-dimensional harmonic oscillator, as it should be.

The general pure squeezed state is never P-representable because the relation (81) is never satisfied when (104) holds. But as we have seen above, it can be unitarily transformed into the projector \(|0\rangle\langle 0|\) which is a P-representable Gaussian operator.
3. Gaussian States of a Two-Dimensional Oscillator

3.1. Parameterizations

Things look much the same except that the number of variables doubles. Thus now we have $a_1^\dagger, a_1$ and $z_1^\dagger, z_1$ for the first degree of freedom, as well as $a_2^\dagger, a_2$ and $z_2^\dagger, z_2$ for the second, so that

$$a = \begin{bmatrix} a_1 \\ a_1^\dagger \\ a_2 \\ a_2^\dagger \end{bmatrix}, \quad a^\dagger = \begin{bmatrix} a_1^\dagger, a_1, a_2^\dagger, a_2 \end{bmatrix}, \quad z = \begin{bmatrix} z_1 \\ z_1^\dagger \\ z_2 \\ z_2^\dagger \end{bmatrix}, \quad z^\dagger = [z_1^\dagger, z_1, z_2^\dagger, z_2]$$ (111)

are 4-component columns and rows, and $C, W, Q, P$ are $4 \times 4$-matrices. The number of integration variables doubles as well, of course, and

$$\int (dz) \ldots = \int (dz_1) \int (dz_2) \ldots$$ (112)

specifies how phase space integrals are to be understood. Relations (41), (78), and (79) remain valid with

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$ (113)

replacing the $2 \times 2$ version of $E$ in (11).

Linear unitary transformations of the form (50) now involve a $4 \times 4$ matrix $U$ for the transformation of $a$ and $a^\dagger$ as in (52), which continues to obey (54) and (56) with

$$T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$ (114)

now. With these replacements, the $4 \times 4$ matrices $C, W, Q, P$ transform according to (57), (58), (59), and (84), respectively.

The explicit parameterization of the matrix appearing in the Gaussian characteristic function is

$$C = \begin{bmatrix} m_1 + \frac{1}{2} & m_1 & m_s & m_c \\ m_1 & n_1 + \frac{1}{2} & m_c^* & m_s^* \\ m_s^* & m_c & n_2 + \frac{1}{2} & m_2 \\ m_c^* & m_s & m_2^* & n_2 + \frac{1}{2} \end{bmatrix}$$ (115)

and $C > 0$ is taken for granted again. Here, too, expanding

$$e^{-\frac{1}{2}z^\dagger C z} = \text{Tr} \left\{ e^{a^\dagger E z G} \right\} = \text{Tr} \left\{ e^{z_1^\dagger a_1^\dagger - z_1^\dagger a_1 z_2^\dagger a_2^\dagger - z_2^\dagger a_2 G} \right\}$$ (116)
in powers of \( z \) reveals the physical significance of the numerical parameters in \( C \), viz.

\[
\begin{align*}
    n_1 &= \text{Tr} \left\{ a_1^\dagger a_1 G \right\}, \quad m_1 = -\text{Tr} \left\{ a_1^2 G \right\}, \\
    n_2 &= \text{Tr} \left\{ a_2^\dagger a_2 G \right\}, \quad m_2 = -\text{Tr} \left\{ a_2^2 G \right\}, \\
    m_s &= \text{Tr} \left\{ a_1 a_2^\dagger G \right\}, \quad m_c = -\text{Tr} \left\{ a_1 a_2 G \right\},
\end{align*}
\]  

and positive two-dimensional Gaussians must obey (26) for analogous reasons as in the one-dimensional case of Sec. 2.

Partial transposition, in the \( z_1^* \), \( z_1 \) sector only, turns \( G \) into \( G^T_1 \). For \( C \), \( W \), \( Q \), and \( P \) the transition is as given in (99) with \( T \) replaced by

\[
T_1 = \begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]

We note that

\[
C \rightarrow \text{ECE} \quad \text{amounts to} \quad \begin{cases}
    m_1 \rightarrow -m_1 \\
    m_2 \rightarrow -m_2 \\
    m_c \rightarrow -m_c
\end{cases}
\]

(119)

with \( n_1, n_2, m_s \) unaffected, and

\[
C \rightarrow T_1 CT_1 \quad \text{amounts to} \quad \begin{cases}
    m_1 \rightarrow m_1^* \\
    m_s \rightarrow m_c^* \\
    m_c \rightarrow m_s^*
\end{cases}
\]

(120)

with \( n_1, n_2, m_2 \) unaffected; analogous statements apply to \( Q \rightarrow \text{EQE}, Q \rightarrow T_1 QT_1, \) \( \text{et cetera} \).

3.2. **Positivity criteria**

A first positivity check is the one that exploits that \( G \) must be unitarily equivalent to a (product of) basic Gaussian(s),

\[
G = U^\dagger (1 - g_1) a_1^\dagger a_1 (1 - g_2) a_2^\dagger a_2 U
\]

(121)

with some appropriate unitary \( U \). The positivity of \( G \) can be tested by checking whether \( g_1 \) and \( g_2 \) are positive, for which it suffices to see if the smaller one of the two is positive.\(^i\)

---

\(^i\)In marked contrast to the one-dimensional case, condition (66) alone does not guarantee the positivity of a two-dimensional Gaussian operator. As a counter example put \( g_1 = \frac{1}{2}, g_2 = -\frac{1}{2} \) in (121), so that the resulting Gaussian has negative eigenvalues while \( \text{Tr} \left\{ G \right\} = \text{Tr} \left\{ G^2 \right\} = 1 \). We are reminded here that the positivity of the statistical operator for a two-party system requires in particular the positivity of the reduced density operators, here characterized respectively by \( g_1 \) and \( g_2 \). In two dimensions, we also need to consider \( \text{Tr} \left\{ G^4 \right\} \) when verifying the positivity of \( G \).
Now, information about $g_1$ and $g_2$ is available in the traces
\[
\text{Tr} \{ G^2 \} = \frac{1 - g_1}{1 + g_1} \frac{1 - g_2}{1 + g_2},
\]
\[
\text{Tr} \{ G^4 \} = \frac{(1 - g_1)^4}{1 - g_1^4} \frac{(1 - g_2)^4}{1 - g_2^4}.
\]
(122)

The trace of $G^2$ can be expressed in terms of the $C$ matrix,
\[
\text{Tr} \{ G^2 \} = \frac{1}{\sqrt{\det(2C)}} = \frac{1}{4\sqrt{\det C}},
\]
and upon introducing $G \equiv G^2$ \[
\text{Tr} \{ G^2 \} = U^\dagger (1 - g_2) g_1 (1 - g_2^2) g_2 (1 - g_2^2) U,
\]
(124)

with matrix $\overline{C}$ in its characteristic function (see Sec. 3.3 below), we have
\[
\frac{\text{Tr} \{ G^4 \}}{(\text{Tr} \{ G^2 \})^2} = \frac{\text{Tr} \{ \overline{G}^2 \}}{4\sqrt{\det C}}.
\]
(125)

So, we extract the necessary information about $g_1$ and $g_2$ out of
\[
4\sqrt{\det C} = \frac{1 + g_1}{1 - g_1} \frac{1 + g_2}{1 - g_2} \quad \text{and} \quad 4\sqrt{\det C} = \frac{1 + g_1^2}{1 - g_1^2} \frac{1 + g_2^2}{1 - g_2^2}.
\]
(126)

A straightforward, yet somewhat tedious, calculation then establishes that $G \geq 0$ holds if both inequalities in
\[
\frac{1}{16} + 3 \det C \geq 4\sqrt{\det C} \sqrt{\det \overline{C}} \leq \frac{1}{8} + 2 \det C
\]
(127)
are obeyed, and only then. In deriving these two inequalities, it is taken for granted that
\[
-1 < g_1, g_2 < 1,
\]
(128)

which is not an actual limitation because $G^\dagger = G$ and $\text{Tr} \{ G^2 \} < \infty$ restrict the values of $g_1$ and $g_2$ to this range.

The right inequality in (127) amounts to
\[
g_1 g_2 \geq 0
\]
(129)

and the left to
\[
(g_1 + g_2) (1 + g_1) (1 + g_2) \geq (g_1 - g_2)^2.
\]
(130)

The first requires that both $g$’s are positive or both negative, but the second excludes the possibilities $g_1 \leq 0, g_2 < 0$ and $g_1 < 0, g_2 \leq 0$. So, taken together, the two inequalities of (127) imply $g_1, g_2 \geq 0$, that is: $G \geq 0$, indeed.

\footnote{Once $g_1$ and $g_2$ are available, a variety of unitarily invariant quantities can be computed, such as the von Neumann entropy of the mixed state represented by $G$.}
Alternatively, for a second positivity check we can apply the reasoning of (68)–(73) to
\[
Q = \begin{bmatrix}
1 - \nu_1 & \mu_1 & \mu_s & \mu_c \\
\mu_1^* & 1 - \nu_1 & \mu_c^* & \mu_s^* \\
\mu_s^* & \mu_c & 1 - \nu_2 & \mu_2 \\
\mu_c^* & \mu_s & \mu_2^* & 1 - \nu_2
\end{bmatrix},
\]
where now
\[
\tilde{Q} = \begin{bmatrix}
1 - \nu_1 & 0 & \mu_s & 0 \\
0 & 1 - \nu_1 & 0 & \mu_s^* \\
\mu_s^* & 0 & 1 - \nu_2 & 0 \\
0 & \mu_s & 0 & 1 - \nu_2
\end{bmatrix} = \frac{1}{2}(Q + EQE)
\]
shows up in the $4 \times 4$ version of (70). Since $\tilde{Q}$ commutes with $E$, we only need to consider unitary transformations with $UE = EU$ and thus $U^*U = I$ when diagonalizing $\tilde{G}$, so that the complication of the extra term in (59) is of no concern here. Therefore, the criterion (72) applies to the two-dimensional case as well, and we are asked to check if the eigenvalues of $\tilde{Q}$ exceed unity or not. These eigenvalues are
\[
1 - \frac{1}{2}(\nu_1 + \nu_2) \pm \sqrt{\left[\frac{1}{2}(\nu_1 - \nu_2)\right]^2 + \mu_s^* \mu_s},
\]
so that $G \geq 0$ if
\[
\nu_1 + \nu_2 \geq \sqrt{(\nu_1 - \nu_2)^2 + 4\mu_s^* \mu_s},
\]
or
\[
\nu_1 + \nu_2 \geq 0 \quad \text{and} \quad \nu_1 \nu_2 \geq \mu_s^* \mu_s
\]
and only then.

3.3. Squared Gaussians

The positivity criterion (127) needs knowledge of $C$, the $4 \times 4$-matrix appearing in the characteristic function of $G$. It is found by squaring
\[
G = \int (dz) e^{z^T E a - \frac{1}{2} z^T C z},
\]
recalling the Baker-Hausdorff relation et cetera, with the outcome
\[
C = \frac{1}{2} C + \frac{1}{4} E C^{-1} E = \frac{1}{2} C + \frac{1}{4} W.
\]
As a consequence, then
\[
\det C = \frac{1}{\frac{1}{2}} \det \left( C + \frac{1}{4} E C^{-1} E \right),
\]
and (127) reads more explicitly
\[
\frac{1}{16} + 3 \det C \geq \sqrt{(\det C) \det \left( C + \frac{1}{4} E C^{-1} E \right)} \leq \frac{1}{8} + 2 \det C.
\]
3.4. Separable and non-separable Gaussians

As a rule, the statistical operator \( \rho^{(1\&2)}(a) = \rho^{(1\&2)}(a_1, a_2) \) of a two-dimensional oscillator, be it of Gaussian shape or not, will be different from the product of the reduced statistical operators \( \rho^{(1)}(a_1), \rho^{(2)}(a_2) \) that one obtains by partial tracing, \[
\rho^{(1\&2)}(a) \neq \rho^{(1)}(a_1)\rho^{(2)}(a_2) \tag{140}
\]
with
\[
\rho^{(1)}(a_1) = \text{Tr}_2 \left\{ \rho^{(1\&2)}(a) \right\}, \quad \rho^{(2)}(a_2) = \text{Tr}_1 \left\{ \rho^{(1\&2)}(a) \right\}. \tag{141}
\]
But, as we mentioned in the Introduction, it could happen quite easily that \( \rho^{(1\&2)} \) is the convex sum of such products, in which case it represents a 
separable state, 
\[
\rho^{(1\&2)} \text{ is separable if } \rho^{(1\&2)} = \sum_k w_k \rho^{(1)}_k \rho^{(2)}_k \tag{142}
\]
with \( w_k > 0 \).

Whereas it can be rather difficult to decide if a given \( \rho^{(1\&2)} \) is separable, matters are remarkably simple for Gaussian states.

Positive Gaussians that are separable must have a positive partial transpose — this is Peres’s necessary criterion. \( ^{39} \) In the case of Gaussian states, it is also sufficient. \( ^{14,15,40} \)

Concerning the positivity of \( G^{T_1} \), please note that \( \det(T_1 C T_1) = \det C \) and that
\[
T_1 C T_1 = \frac{1}{2} T_1 \left( C + \frac{1}{4} T_1 E T_1 C^{-1} E T_1 \right) T_1
= \frac{1}{2} T_1 \left( C + \frac{1}{4} E T_1 C^{-1} E T_1 \right) T_1 \tag{143}
\]
so that a non-negative partial transpose, \( G^{T_1} \geq 0 \), is available only if
\[
\frac{1}{16} + 3 \det C \geq \sqrt{(\det C) \det \left( C + \frac{1}{4} E T_1 C^{-1} E T_1 \right)} \leq \frac{1}{8} + 2 \det C \tag{144}
\]
hold with
\[
E T_1 = T_1 E T_1 = \begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}. \tag{145}
\]

Probably simpler is the positivity criterion of \((131)–(135)\). When applied to \( G^{T_1} \), it asks whether the eigenvalues of
\[
\frac{1}{2} (T_1 Q T_1 + E T_1 Q T_1 E) = \begin{bmatrix}
1 - \nu_1 & 0 & \mu_c^* & 0 \\
0 & 1 - \nu_1 & 0 & \mu_c \\
\mu_c & 0 & 1 - \nu_2 & 0 \\
0 & \mu_c^* & 0 & 1 - \nu_2
\end{bmatrix} \tag{146}
\]
exceed unity. Therefore, if \( G > 0 \) so that the restrictions of \((135)\) are obeyed, \( G^{T_1} \) is positive if
\[
\nu_1 \nu_2 \geq \mu_c^* \mu_c. \tag{147}
\]
Gaussians that are P-representable,
\[ G = \int (dz) \sqrt{\det P} e^{-\frac{1}{2} z^t P z} : e^{-\frac{1}{2}(a_1^t - z_1^t)(a_1 - z_1)} e^{-\frac{1}{2}(a_2^t - z_2^t)(a_2 - z_2)} : \quad (148) \]
(with \( P > 0 \)), are separable by construction. Local unitary transformations, \( U_{\text{loc}} = U_1 U_2 \) with
\[
U_1 = \begin{bmatrix}
  e^{i\phi_1} \cosh \theta_1 & e^{i\phi_1} \sinh \theta_1 & 0 & 0 \\
  e^{-i\phi_1} \sinh \theta_1 & e^{-i\phi_1} \cosh \theta_1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix}
\]  
(149)
and
\[
U_2 = \begin{bmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & e^{i\phi_2} \cosh \theta_2 & e^{i\phi_2} \sinh \theta_2 \\
  0 & 0 & e^{-i\phi_2} \sinh \theta_2 & e^{-i\phi_2} \cosh \theta_2
\end{bmatrix},
\]
(150)

turn such a P-representable Gaussian into other separable Gaussians, which may or may not be P-representable themselves. All separable Gaussians can be constructed in this way, and as a consequence the Peres criterion is not only necessary, it is indeed sufficient.

Although this is reasonably obvious, the explicit construction of the unitary transformation that does the job is technically more involved than the analogous problem in one dimension that we treated in Sec. 2.3. We shall, therefore be content with the following general remarks, and present an explicit example in Sec. 3.5.4 below.

The separability of a Gaussian state can be investigated by studying the existence of a local transformation \( U_{\text{loc}} = U_1 U_2 \) that maps the general Gaussian characteristic function (115) into a P-representable Gaussian characteristic function
\[
U_{\text{loc}} : \mathbb{C} \to \mathbb{C}_P
\]
(151)
which has to satisfy the condition
\[
\mathbb{C}_P - \frac{1}{2} I > 0.
\]
(152)
This says that the four eigenvalues of \( \mathbb{C}_P - \frac{1}{2} I \) are strictly positive. As shown inRefs. 14 and 15, this is indeed equivalent to the separability condition (5).

3.5. Examples
We conclude this section on two-dimensional Gaussian operators by presenting a couple of examples, mostly generalizations of the historical EPR Gaussian state.
3.5.1. Pure Gaussian state

We begin our examples with the simplest case of a pure Gaussian state of a two-dimensional harmonic oscillator. For pure states, we have $\text{Tr} \{G^2\} = 1$, and so (123) implies that the Gaussian characteristic function must have a $C$ matrix (115) with

$$
\det C = \frac{1}{16}.
$$

(153)

For convenience, we rewrite (115) in the block form

$$
C = \begin{bmatrix} C_{11} & C_{12} \\
C_{21} & C_{22} \end{bmatrix}.
$$

(154)

For such pure states, the criterion for separability is simple, inasmuch as it is enough to study the properties of $C_{11}$ or $C_{22}$, which correspond to the reduced one-dimensional parts of the two-dimensional harmonic oscillator. If both $C_{11}$ or $C_{22}$ describe pure states, we say that the joint state is not entangled. This amounts to

$$
\det C_{11} = \frac{1}{4} \quad \text{or} \quad \det C_{22} = \frac{1}{4}.
$$

(155)

For Gaussian pure states this separability condition is much simpler than the general condition of Sec. 3.4 for mixed Gaussian states.

If we restrict the parameters that characterize a pure Gaussian state to real numbers only, and continue to ignore linear shifts, such a state is described by a wave function of the form

$$
\Psi(q) = \left( \frac{\det D}{\pi^2} \right)^{\frac{1}{4}} e^{-\frac{1}{2}q^{\dagger}Dq}
$$

(156)

in the position representation ($q = [q_1, q_2]^\dagger$). Here we meet the $2 \times 2$ matrix

$$
D = D^\dagger = \begin{bmatrix} \alpha & \gamma \\
\gamma & \beta \end{bmatrix} \quad \text{with} \quad \det D = \alpha\beta - \gamma^2,
$$

(157)

and $D > 0$, that is

$$
\alpha + \beta > \sqrt{(\alpha - \beta)^2 + 4\gamma^2},
$$

(158)

is needed for the normalizability of the Gaussian wave function (156).

Then, the $C$ matrix of the resulting Gaussian characteristic function (154) has the blocks

$$
C_{11} = \begin{bmatrix} \frac{\alpha}{4} + \frac{\beta}{4 \det D} & \frac{\alpha}{4} - \frac{\beta}{4 \det D} \\
\frac{\alpha}{4} - \frac{\beta}{4 \det D} & \frac{\alpha}{4} + \frac{\beta}{4 \det D} \end{bmatrix}.
$$
\[ C_{22} = \begin{bmatrix} \frac{\beta}{4} + \frac{\alpha}{4 \det D} & \frac{\beta}{4} - \frac{\alpha}{4 \det D} \\ \frac{\beta}{4} - \frac{\alpha}{4 \det D} & \frac{\beta}{4} + \frac{\alpha}{4 \det D} \end{bmatrix} \]

\[ C_{12} = C_{21} = \frac{\gamma}{4} \begin{bmatrix} 1 - \frac{1}{\det D} & 1 + \frac{1}{\det D} \\ 1 + \frac{1}{\det D} & 1 - \frac{1}{\det D} \end{bmatrix} \]  

(159)

It is easy to check that a \( C \) matrix with these blocks satisfies the purity condition (153).

The separability condition \( \det C_{11} = \frac{1}{\gamma^2} \) amounts to \( \det D = \alpha \beta \), so that the state is separable (not entangled) if and only if \( \gamma = 0 \), as it should be expected for pure states. For \( \gamma \neq 0 \), all pure states represented by the Gaussian wave function (156) are entangled.

This wave function defines a Gaussian operator that is a projector,

\[ G = |\Psi\rangle\langle \Psi| \] 

(160)

where

\[ |\Psi\rangle = (\det Q)^{\frac{1}{4}} e^{-\frac{1}{2} \mu_1 a_1^\dagger a_2^\dagger} e^{-\frac{1}{2} \mu_2 a_2^\dagger a_2^\dagger} e^{-\mu_c a_1^\dagger a_2^\dagger} |0,0\rangle \] 

(161)

with

\[ \mu_1 = \frac{\det D + \alpha - \beta - 1}{\det D + \alpha + \beta + 1}, \]

\[ \mu_2 = \frac{\det D - \alpha + \beta - 1}{\det D + \alpha + \beta + 1}, \]

\[ \mu_c = \frac{2\gamma}{\det D + \alpha + \beta + 1} \] 

(162)

being the non-zero parameters in (131), and

\[ \det Q = \frac{16 \det D}{(\det D + \alpha + \beta + 1)^2} \] 

(163)

follows. Therefore, such a Gaussian wave function corresponds to two squeezed one-dimensional harmonic oscillators, correlated with a strength that is characterized by the parameter \( \gamma \).

As a further simplification, and also as an important physical application related to the EPR wave function (2), let us investigate a correlated state that is not squeezed. We obtain such a state by choosing the parameters in accordance with

\[ \alpha = \beta \quad \text{and} \quad \det D = 1, \quad \text{so that} \quad \gamma^2 = \alpha^2 - 1. \] 

(164)

Such a Gaussian state is characterized by a single real parameter, for which \( \pi \) in

\[ \alpha = \beta = 1 + 2\pi \quad \text{and} \quad |\gamma| = 2\sqrt{\pi(\pi + 1)} \] 

(165)
is a convenient choice. Upon opting for $\gamma = -|\gamma|$, the wave function (156) is
\[
\Psi(q) = \pi^{-\frac{1}{2}} \exp \left( -\left(\bar{\pi} + \frac{1}{2}\right)(q_1^2 + q_2^2) + 2\sqrt{\bar{\pi}(\bar{\pi} + 1)} q_1 q_2 \right).
\]
For large $\bar{\pi}$ it reduces to
\[
\Psi(q) \sim e^{-\bar{\pi}(q_1 - q_2)^2}
\]
and becomes
\[
\Psi(q) \sim \delta(q_1 - q_2)
\]
in the limit $\bar{\pi} \to \infty$. We recognize here the famous historical wave function (2) of two correlated particles introduced by the EPR trio. This wave function cannot be normalized, and thus represents an overidealized situation. By contrast, the wave function (166) is a smoothed, normalized version of (2) and refers to a real physical situation. For illustration, we show this smoothed wave function in Fig. 1 for two different values of $\bar{\pi}$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{smoothed_EPR_wave_function}
\caption{Contour plot of the smoothed EPR wave function (166) for $\bar{\pi} = 0$ (not entangled, on the left) and $\bar{\pi} = 1$ (entangled, on the right).}
\end{figure}

### 3.5.2. Bell states

For entangled spin-$\frac{1}{2}$ states, a special role is played by the entangled basis of the four Bell states. For the two-dimensional harmonic oscillator it is possible to introduce a continuous-variable generalization of Bell’s states. Such a complete set of states consists of particular Gaussian operators.

Following the example (100) of the singular Wigner function for a one-dimensional harmonic oscillator, we write for the two-dimensional case a singular Wigner function in the form of a product of two two-dimensional Dirac delta functions,
\[
W(z) = \delta(z_1 - z_0)\delta(z_2 + z_0^*) \quad \text{with} \quad z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \quad \text{and} \quad z_0 = \begin{bmatrix} z_0^* \\ -z_0 \end{bmatrix},
\]

\[
W(z) = \delta(z_1 - z_0)\delta(z_2 + z_0^*) \quad \text{with} \quad z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \quad \text{and} \quad z_0 = \begin{bmatrix} z_0^* \\ -z_0 \end{bmatrix},
\]
where the two-dimensional δ-function \( \delta(z) \) is as defined in (101) and \( z_0 \) is an arbitrary complex number. The corresponding Gaussian operator,
\[
G(z_0) = e^{-a_1^† a_1 - a_2^† a_2 + a_1 a_2 + a_1^† a_2^† + z_0(a_2 - a_1^†) + z_0^*(a_2^† - a_1)} \langle z_0 \rangle \langle z_0 |,
\]
projects on the “continuous Bell state” specified by the ket
\[
\langle z_0 | = e^{-\frac{1}{2}|z_0|^2 + z_0^2 a_2^† - z_0 a_1^† + a_1^† a_2^†} |0,0\rangle = e^{\frac{1}{2}|z_0|^2 + (a_1^† - z_0^*) (a_2^† + z_0)} |0,0\rangle,
\]
which obtains from \( |z_0 = [0,0]^T \rangle \) by a simultaneous unitary shift of \( a_1 \) by \( z_0 \) and \( a_2 \) by \( -z_0^* \), respectively. The completeness relation
\[
\int (dz_0) \langle z_0 | \langle z_0 | = 1
\]
is easily demonstrated with the aid of the Fourier–Gauss formula (39).

### 3.5.3. Mixed EPR states

In this example the matrix \( C \) of the Gaussian characteristic function is of the form
\[
C = \begin{bmatrix}
  n + \frac{1}{2} & 0 & 0 & m_c \\
  0 & n + \frac{1}{2} & m_c & 0 \\
  0 & m_c & n + \frac{1}{2} & 0 \\
  m_c & 0 & 0 & n + \frac{1}{2}
\end{bmatrix},
\]
so that \( n (= n_1 = n_2) \) and \( m_c \) are the only non-zero parameters in (115) and, in addition, we take \( m_c \) to be real. The corresponding \( Q \) matrix of (131) is
\[
Q = \frac{1}{(n+1)^2 - m_c^2} \begin{bmatrix}
  n + 1 & 0 & 0 & m_c \\
  0 & n + 1 & m_c & 0 \\
  0 & m_c & n + 1 & 0 \\
  m_c & 0 & 0 & n + 1
\end{bmatrix},
\]
and the positivity criterion (135) amounts to
\[
n(n+1) \geq m_c^2.
\]
This is less stringent than the condition for separability,
\[
n \geq |m_c|,
\]
that follows from (147). These matters are illustrated in Fig. 2.

Note that the Gaussian operator specified by (173) represents a mixed state in general. Only if
\[
\det C = [(n + \frac{1}{2})^2 - m_c^2]^2 = \frac{1}{16}
\]
Fig. 2. Criteria for positivity and separability for the mixed EPR state specified by (174). In this plot of \( n \) versus \( m_c \), the states associated with points on and above the dashed line are separable, whereas points on and above the solid line mark positive Gaussians, with pure states obtaining for \( n, m_c \) values on the solid line.

holds, the state is a pure state, which is the border case of the positivity criterion (175). With the identification \( \gamma = 2m_c \) and \( n = \pi \), we then recover the EPR pure state of Sec. 3.5.1, as we should.

We further note that the remarkably simple separability condition (176) can be expressed in terms of Bohr’s variables (4) as

\[
\langle P_1^2 \rangle + \langle Q_2^2 \rangle \geq 1 \quad \text{and} \quad \langle P_2^2 \rangle + \langle Q_1^2 \rangle \geq 1.
\]  

(178)

These relations have important physical implications when the separability of the EPR state is put to an experimental test. It is then enough to measure the variances of the Bohr variables and check if the conditions (178) are satisfied. One should remember, however, that (178) applies only to the very special state characterized by the \( C \) matrix of (173). As we will see in the next examples, Gaussian states different from this generalization of the EPR state will lead to a separability condition that will be different in form.

3.5.4. Anti-EPR states

We generalize the previous example by adding anti-EPR terms to the matrix \( C \) of the Gaussian characteristic function in (173), so that now \( C \) is of the form

\[
C = \begin{bmatrix}
  n + \frac{1}{2} & 0 & m_s & m_c \\
  0 & n + \frac{1}{2} & m_e & m_s \\
  m_s & m_e & n + \frac{1}{2} & 0 \\
  m_c & m_s & 0 & n + \frac{1}{2}
\end{bmatrix}
\]  

(179)

and is characterized by the three real parameters \( n, m_c, \) and \( m_s \). According to (120), the partially transposed Gaussian has \( m_c \) and \( m_s \) interchanged, and because of this property we call the \( m_s \) terms the anti-EPR contribution. The \( Q \) matrix
(131) of such a squeezed Gaussian operator has these non-zero elements:

\[ \nu_1 = \nu_2 = 1 - \frac{(n+1)((n+1)^2 - m_s^2 - m_c^2)}{d}, \quad \mu_1 = \mu_2 = -\frac{m_s m_c (n+1)}{d}, \]

\[ \mu_s = -m_s \frac{(n+1)^2 - m_s^2 + m_c^2}{d}, \quad \mu_c = m_c \frac{(n+1)^2 + m_s^2 - m_c^2}{d}, \]

where \( d = ((n+1)^2 - (m_s + m_c)^2)((n+1)^2 - (m_c - m_s)^2) \). As a consequence, the positivity condition (135) reads here

\[ n(n+1) - 2m_s(n + \frac{1}{2}) + m_s^2 - m_c^2 \geq 0, \]

and the separability criterion (147) amounts to

\[ n(n+1) - 2m_c(n + \frac{1}{2}) + m_c^2 - m_s^2 \geq 0. \]

For \( m_s = 0 \), we recognize the particular case of the mixed EPR state discussed in Sec. 3.5.3.

In Fig. 3, we depict the regions for the parameters \( n \) and \( m_c \) for which the anti-EPR Gaussian states are non-separable, for two ratios of \( m_s \) to \( m_c \). These regions are bounded by the curves defined by the positivity condition (181) and the separability criterion (182). Note that the right plot corresponds to \( m_s = m_c \), so that there the region of interest has no area at all, which means that such a state is never non-separable.

We conclude this example by the explicit construction of the local transformation that maps the Gaussian state associated with (179) onto a Gaussian P-representable state, that is

\[ U_{\text{loc}}(\theta) : \mathbb{C} \rightarrow \mathbb{C}_P(\theta) = \begin{bmatrix} N + \frac{1}{2} & M & M_s & M_c \\ M & M + \frac{1}{2} & M_c & M_s \\ M_s & M_c & N + \frac{1}{2} & M \\ M_c & M_s & M & N + \frac{1}{2} \end{bmatrix} \]

(183)
in terms of the $C$ matrix of (179). Because we have selected real parameters in $C$, the local transformation is labeled by a single parameter $\theta$. The transformed matrix $C_P$ belongs to a P-representable Gaussian operator if the conditions

$$N \pm M \geq |M_c \pm M_s|$$  \hspace{1cm} (184)

are obeyed. Here we have

$$N \pm M = \left( n + \frac{1}{2} \right) e^{\mp 2\theta} - \frac{1}{2},$$

$$|M_c \pm M_s| = e^{\mp 2\theta} |m_c \pm m_s|.$$  \hspace{1cm} (185)

The lower bounds in (184) are achieved if

$$2M = |M_c + M_s| - |M_c - M_s|,$$

$$2N = |M_c + M_s| + |M_c - M_s|.$$  \hspace{1cm} (186)

The first equality implies

$$(n + \frac{1}{2})(e^{-2\theta} - e^{2\theta}) = |m_c + m_s| e^{-2\theta} - |m_c - m_s| e^{2\theta}$$  \hspace{1cm} (187)

or, more explicitly,

$$e^{4\theta} = \frac{n + \frac{1}{2} - |m_c + m_s|}{n + \frac{1}{2} - |m_c - m_s|}.$$  \hspace{1cm} (188)

for the transformation parameter $\theta$. We see that for the two special cases of the EPR state ($m_s = 0$) and the anti-EPR case ($m_c = 0$), we have $\theta = 0$, and no local transformation of the Gaussian characteristic function (179) is required. In these two cases the P-representability can be studied directly by using (179).

More generally, the conditions for P-representability that follow from (184) and (188) are

$$(n + \frac{1}{2}) - \frac{1}{2} e^{-2\theta} \geq |m_c + m_s|,$$

$$(n + \frac{1}{2}) - \frac{1}{2} e^{2\theta} \geq |m_c - m_s|.$$  \hspace{1cm} (189)

They are easily seen to be equivalent to the separability criterion (182).

3.5.5. Squeezed EPR states

As the last example of this tutorial we consider squeezed EPR states. Their $C$ matrix (115) is specified by the three real and positive parameters $n, m_c, m$ and is of the form

$$C = \begin{bmatrix} n + \frac{1}{2} & m & 0 & m_c \\ m & n + \frac{1}{2} & m_c & 0 \\ 0 & m_c & n + \frac{1}{2} & m \\ m_c & 0 & m & n + \frac{1}{2} \end{bmatrix}.$$  \hspace{1cm} (190)
Note that this matrix differs from the one in (173) by the additional squeezed terms associated with $m = m_1 = m_2$. The $Q$ matrix (131) of such a squeezed Gaussian operator has the non-zero elements

$$
\nu_1 = \nu_2 = 1 - \frac{(n + 1)[(n + 1)^2 - m^2 - m_c^2]}{D}, \quad \mu_1 = \mu_2 = \frac{m(n + 1)^2 - m^2 + m_c^2}{D},
$$

$$
\mu_s = \frac{2mm_c(n + 1)}{D}, \quad \mu_c = m_c(n + 1)^2 - m_c^2 + m^2
$$

(191)

with $D = [(n + 1)^2 - (m + m_c)^2][(n + 1)^2 - (m - m_c)^2]$.

Here the positivity condition (135) requires

$$n(n + 1) - (m_c + m)^2 \geq 0 \quad (192)$$

and, according to the separability criterion (147), the squeezed EPR state is separable if

$$n(n + 1) - 2m_c(n + \frac{1}{2}) + m_c^2 - m^2 \geq 0 \quad (193)$$

holds. For $m = 0$, we recognize the particular case of the mixed EPR state discussed in Sec. 3.5.3.

In Fig. 4, we depict the regions for the parameters $n$ and $m_c$ for which the anti-EPR Gaussian states are non-separable, for two ratios of $m$ to $m_c$. These regions are bounded by the curves defined by the positivity condition (192) and the separability criterion (193).

![Fig. 4. The positivity and separability criteria (192) and (193) bound the region of non-separable squeezed EPR states (solid and dashed curves, respectively). The left plot corresponds to $m = \frac{1}{2}m_c$, the right plot to $m = m_c$.](image)

4. Concluding remarks

The main goal of this tutorial is to provide, for a reader trained in Quantum Optics or in Quantum Information Theory, a general and detailed description of various
mathematical tools used in the description of one-party and two-party Gaussian states. For such states we presented a general parametrization of Gaussian states and discussed the positivity criteria and the so-called P-representability of positive Gaussian statistical operators. With the help of these techniques we showed that it is possible to address the problem of quantum separability of two-party Gaussian states using different-in-form, but mathematically equivalent, methods based on the Heisenberg uncertainty relations, partial transposition, or the P-representability. The tutorial includes a number of simple, instructive examples that illustrate the techniques and methods.

The one-dimensional Gaussian operators discussed in Sec. 2 have been widely used in various application of squeezed states of light and in the theory of optical Gaussian beams involving linear optical elements. For such one-dimensional problems an important issue dealt with in the literature is the degree of nonclassicality of one-mode Gaussian states of the quantum electromagnetic field. A general discussion of this problem has been given in Ref. 41. In a different approach the nonclassicality has been investigated with the aid of the Uhlmann fidelity of two single-mode Gaussian states.42

Owing to their simplicity, Gaussian states have found many applications in Quantum Information Theory. Examples of recent achievements that involve Gaussian states include important problems such as entanglement key distribution,43 quantum error correction,44 entanglement distillation,45 and quantum cloning.46 The separability and distillability of two-party Gaussian states are reviewed in Ref. 47. The problem of quantum separability of three-party Gaussian states and the entanglement criteria for all bipartite Gaussian states have been addressed as well.48 A number of issues related to the bound entangled Gaussian states,49 and the optical generation of entangled Gaussian states50 have also been addressed in the published literature.

Our tutorial is confined to Gaussian states. The reader may wish to consult Ref. 51, a “primer” on general composite quantum systems involving entangled states in discrete, finite-dimensional Hilbert spaces.

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