A BLOW-UP CRITERION FOR THE 3D COMPRESSIBLE MAGNETOHYDRODYNAMICS IN TERMS OF DENSITY

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Abstract. We study an initial boundary value problem for the 3D magnetohydrodynamics (MHD) equations of compressible fluids in $\mathbb{R}^3$. We establish a blow-up criterion for the local strong solutions in terms of the density and magnetic field. Namely, if the density is away from vacuum ($\rho = 0$) and if the magnetic field is bounded above in terms of $L^\infty$-norm, then a local strong solution can be continued globally in time.

1. Introduction

We prove a blow-up criteria for the smooth solutions to the compressible magnetohydrodynamics (MHD) in three space dimensions (see Cabannes [1] for a more comprehensive discussion on the system):

$$\rho_t + \text{div}(\rho u) = 0, \quad (1.1)$$
$$\rho u_t + \text{div}(\rho u^2) + P(\rho)_{x_j} + \left(\frac{1}{2}|B|^2\right)_{x_j} - \text{div}(B^j B) = \mu \Delta u^j + \lambda \text{div} u_{x_j}, \quad (1.2)$$
$$B_t^j + \text{div}(B^j u - u^j B) = \nu \Delta B^j, \quad (1.3)$$
$$\text{div}B = 0. \quad (1.4)$$

Here $u = (u^1, u^2, u^3)$ and $B = (B^1, B^2, B^3)$ are functions of $x \in \mathbb{R}^3$ and $t \geq 0$ representing density, velocity and magnetic field; $P = P(\rho)$ is the pressure; $\varepsilon, \lambda, \nu$ are viscous constants. The system $(1.1)-(1.4)$ is solved subjected to some given initial data:

$$(\rho, u, B)(x, 0) = (\rho_0, u_0, B_0)(x). \quad (1.5)$$

The local existence of smooth solutions to the MHD system $(1.1)-(1.4)$ as well as the global existence of smooth solutions and weak solutions are studied by many mathematicians in decades, see [8], [9], [12], [10], [14]. When the initial data is taken to be close to a constant state in $H^3(\mathbb{R}^3)$, Kawashima [10] constructed global-in-time $H^3(\mathbb{R}^3)$-solutions. Later, Suen and Hoff [14] generalized Kawashima’s results to obtain global smooth solutions when the initial data is taken to be $H^3(\mathbb{R}^3)$ but only close to a constant state in $L^2(\mathbb{R}^3)$. The existence of global weak solutions to $(1.1)-(1.4)$ with large initial data was proved by Hu and Wang [8]-[9] and Sart [12] which are extensions of Lions-type weak solutions [8] for the Navier-Stokes system. With initial $L^2$-data close to a constant state, Suen and Hoff [14] generalized Hoff-type intermediate weak solutions [3]-[5] to obtain global solutions to the system $(1.1)-(1.4)$.

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On the other hand, the global existence of smooth solution to the MHD system (1.1)-(1.4) with arbitrary smooth data is still unknown. For the corresponding Navier-Stokes system, Z. Xin [16] proved that smooth solution will blow up in finite time in the whole space when the initial density has compact support, while Rozanova [11] showed similar results for rapidly decreasing initial density. Recently Fan-Jiang-Ou [2] established some blow-up criteria for the classical solutions to 3D compressible flows, which were further extended by Lu-Du-Yao [7] for MHD system. Rozanova [11] showed similar results for rapidly decreasing initial density. Recently the main goal of the present paper is to generalize the corresponding results of Sun-Wang-Zhang [15] to the MHD system (1.1)-(1.4). When the initial vacuum is allowed, Y. Sun, C. Wang and Z. Zhang obtained a blow-up criterion in terms of the upper bound of the density for the strong solution to the 3-D compressible Navier-Stokes equations. With the presence of magnetic field, we are able to obtain parallel results as in [15] except that we do not allow vacuum in the initial density.

We now give a precise formulation of our results. First concerning the assumptions on the parameters, we have

\begin{equation}
\varepsilon, \lambda, \nu > 0 \text{ and } \lambda < \varepsilon. \tag{1.7}
\end{equation}

For the initial data, we assume that

\[ \rho_0, u_0, B_0 \in H^3(\mathbb{R}^3) \text{ with } \inf(\rho_0) > 0 \text{ and } \text{div}(B_0) = 0 \tag{1.8} \]

and we also write

\[ C_0 = \|\rho_0 - \tilde{\rho}\|_{H^3}^2 + \|u_0\|_{H^3}^2 + \|B_0\|_{H^3}^2. \tag{1.9} \]

We make use of the following standard facts (see Ziemer [17], Theorem 2.1.4, Remark 2.4.3, and Theorem 2.4.4, for example). First, given \( r \in [2, 6] \) there is a constant \( C(r) \) such that for \( w \in H^1(\mathbb{R}^3) \),

\[ \|w\|_{L^r(\mathbb{R}^3)} \leq C(r) \left( \|w\|_{L^2(\mathbb{R}^3)}^{(6-r)/2r} \|\nabla w\|_{L^2(\mathbb{R}^3)}^{(3r-6)/2r} \right). \tag{1.10} \]

We denote the material derivative of a given function \( v \) by \( \dot{v} = v_t + \nabla v \cdot u \), and if \( X \) is a Banach space we will abbreviate \( X^3 \) by \( X \). Finally if \( I \subset [0, \infty) \) is an interval, \( C^1(I; X) \) will be the elements \( v \in C(I; X) \) such that the distribution derivative \( v_t \in D'(\mathbb{R}^3 \times \text{int} I) \) is realized as an element of \( C(I; X) \).

We recall a local existence theorem for (1.1)-(1.4) by Kawashima [10], pg. 34–35 and pg. 52–53:

**Theorem 1.1 (Kawashima)** Assume that \( \varepsilon, \lambda, \nu \) are strictly positive and that the pressure \( P \) satisfies (1.6). Then given \( \tilde{\rho} > 0 \) and \( C_3 > 0 \), there is a positive time \( T \) depending on \( \tilde{\rho}, C_3 \) and the parameters \( \varepsilon, \lambda, \nu, P \) such that if the initial data \((\rho_0 - \tilde{\rho}, u_0, B_0)\) is given satisfying (1.8) and

\[ C_0 < C_3, \]

then there is a solution \((\rho - \tilde{\rho}, u, B)\) to (1.1)-(1.4) defined on \( \mathbb{R}^3 \times [0, T] \) satisfying

\[ \rho - \tilde{\rho} \in C([0, T]; H^3(\mathbb{R}^3)) \cap C^1([0, T]; H^2(\mathbb{R}^3)) \tag{1.11} \]

and

\[ u, B \in C([0, T]; H^3(\mathbb{R}^3)) \cap C^1([0, T]; H^1(\mathbb{R}^3)) \cap L^2([0, T]; H^4(\mathbb{R}^3)). \tag{1.12} \]
The following is the main result of this paper:

**Theorem 1.2** Assume that the system parameters satisfy (1.6)-(1.7). Given $\tilde{\rho} > 0$, suppose $(\rho_0 - \tilde{\rho}, u_0, B_0)$ satisfies (1.8). Assume that $(\rho - \tilde{\rho}, u, B)$ is the smooth solution as constructed in Theorem 1.1, and let $T^* \geq T$ be maximal existence time of the solution. If $T^* < \infty$, then we have

$$\lim_{t \to T^*} \|\rho\|_{L^\infty((0,t) \times \mathbb{R}^3)} + \|\rho^{-1}\|_{L^\infty((0,t) \times \mathbb{R}^3)} + \|B\|_{L^\infty((0,t) \times \mathbb{R}^3)} = +\infty.$$ 

The rest of the paper is organized as follows. We begin the proofs of Theorem 1.2 in section 2 with a number of a priori bounds for local-in-time smooth solutions. We make an important use of estimates on the Lamé operator $L$ which are mainly inspired by [3] and [15]. Finally in section 3 we prove Theorem 1.2 via a contradiction argument by deriving higher order $H^3$-bounds for smooth solutions.

### 2. A priori estimates

In this section we derive a priori estimates for the local solution $(\rho - \tilde{\rho}, u, B)$ on $[0, T]$ with $T \leq T^*$ as described by Theorem 1.1. Here $T^*$ is the maximal time of existence which is defined in the following sense:

**Definition** We call $T^* \in (0, \infty)$ to be the maximal time of existence of a smooth solution $(\rho - \tilde{\rho}, u, B)$ to (1.1)-(1.4) if for any $0 < T < T^*$, $(\rho - \tilde{\rho}, u, B)$ solves (1.1)-(1.4) in $[0, T] \times \mathbb{R}^3$ and satisfies (1.11)-(1.12); moreover, the conditions (1.11)-(1.12) fail to hold when $T = T^*$.

We will prove Theorem 1.2 using a contradiction argument. Therefore, for the sake of contradiction, we assume that

$$\|\rho\|_{L^\infty((0,T^*) \times \mathbb{R}^3)} + \|\rho^{-1}\|_{L^\infty((0,T^*) \times \mathbb{R}^3)} + \|B\|_{L^\infty((0,T^*) \times \mathbb{R}^3)} \leq C. \quad (2.1)$$

To facilitate our exposition, we first define some auxiliary functionals for $0 \leq t \leq T$:

$$A_1(t) = \sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} (|\nabla u|^2 + |\nabla B|^2) dx + \int_{0}^{t} \int_{\mathbb{R}^3} (|\dot{u}|^2 + |B_t|^2) dx ds,$$

$$A_2(t) = \sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} (|\dot{u}|^2 + |B_t|^2) dx + \int_{0}^{t} \int_{\mathbb{R}^3} (|\nabla \dot{u}|^2 + |\nabla B_t|^2) dx ds,$$

$$H(t) = \int_{0}^{t} \int_{\mathbb{R}^3} |\nabla u|^4 dx ds.$$

The following is the main theorem of this section:

**Theorem 2.1** Assume that the hypotheses and notations in Theorem 1.1 are in force. Given $C > 0$ and $\tilde{\rho} > 0$, assume further that $(\rho - \tilde{\rho}, u, B)$ satisfies (2.1).
Then there exists a positive number \( M \) which depends on \( C_0, C, T^* \) and the system parameters \( P, \varepsilon, \lambda, \nu \) such that, for \( 0 \leq t \leq T \leq T^* \),

\[
A_1(t) + A_2(t) \leq M.
\]

We prove Theorem 2.1 in a sequence of lemmas. We first derive the following lemma which gives estimates on the solutions of the Lamé operator \( L = \varepsilon \Delta + (\varepsilon + \lambda) \nabla \text{div} \). More detailed discussions can also be found in Sun-Wang-Zhang [15].

**Lemma 2.2** Consider the following equation:

\[
\varepsilon \Delta v + (\varepsilon + \lambda) \nabla \text{div}(v) = J,
\]

where \( v = (v^1, v^2, v^3) \), \( J = (J^1, J^2, J^3) \), \( x \in \mathbb{R}^3 \) and \( \varepsilon, \lambda > 0 \). Then for \( p \in (1, \infty) \), we have:

1. (2.4) if \( J \in W^{2,p}(\mathbb{R}^3) \), then \( \|D^2 v\|_{L^p} \leq \tilde{C} \|J\|_{L^p} \);
2. (2.5) if \( J = \nabla \phi \) with \( \phi \in W^{2,p}(\mathbb{R}^3) \), then \( \|\nabla v\|_{L^p} \leq \tilde{C} \|\phi\|_{L^p} \);
3. (2.6) if \( J = \nabla \text{div}(\phi) \) with \( \phi \in W^{2,p}(\mathbb{R}^3) \), then \( \|v\|_{L^p} \leq \tilde{C} \|\phi\|_{L^p} \).

Here \( \tilde{C} \) is a positive constant which depends only on \( \varepsilon, \lambda, p \).

**Proof.** A proof can be found in [15] pg. 39 and we omit the details here. \( \square \)

We proceed to the following a priori estimates which is the energy-balanced law.

**Lemma 2.3** Assume that the hypotheses and notations of Theorem 2.1 are in force. Then for any \( 0 \leq t \leq T \leq T^* \),

\[
\sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} ((\rho - \bar{\rho})^2 + \rho |u|^2 + |B|^2)dx + \int_0^t \int_{\mathbb{R}^3} (|\nabla u|^2 + |\nabla B|^2)dxds \leq M(C)C_0,
\]

where \( M(C) \) is a constant which depends on \( C \).

**Proof.** Let \( G = G(\rho) \) be a functional defined by

\[
G(\rho) = \rho \int_0^\rho s^{-1} (P(s) - P(\bar{\rho}))ds.
\]

Multiplying the momentum equation (1.2) by \( u^j \), summing over \( j \), integrating and making use of the continuity equation (1.1), we get:

\[
\int_{\mathbb{R}^3} \left[ \frac{1}{2} \rho |u|^2 + G \right] dx \bigg|_0^t + \int_0^t \int_{\mathbb{R}^3} u \cdot \text{div} \left( \frac{1}{2} |B|^2 I_{3 \times 3} - BB^T \right) dxds + \int_0^t \int_{\mathbb{R}^3} \varepsilon |\nabla u|^2 + (\varepsilon + \lambda)(\text{div} u)^2 \right] dxds = 0. \tag{2.5}
\]

Similarly, we multiply the magnetic field equation (1.3) by \( B \) and integrate to get

\[
\int_{\mathbb{R}^3} \frac{1}{2} |B|^2 dx \bigg|_0^t + \int_0^t \int_{\mathbb{R}^3} B \cdot \text{div} (Bu^T - uB^T) dxds = -\nu \int_0^t \int_{\mathbb{R}^3} |\nabla B|^2 dxds. \tag{2.6}
\]
We then obtain (2.4) by adding (2.5) to (2.6) and using the fact that
\[
\int_0^t \int_{\mathbb{R}^3} \left[ u \cdot \text{div}\left( \frac{1}{2} |B|^2 I_{3x3} - BB^T \right) + B \cdot \text{div}(Bu^T - uB^T) \right] \, dx \, ds = 0.
\]

We obtain the following $L^4$ bounds for $u$ and $B$:

**Lemma 2.4** Assume that the hypotheses and notations of Theorem 2.1 are in force. Then for any $0 \leq t \leq T^*$,
\[
\int_{\mathbb{R}^3} (|u(x,t)|^4 + |B(x,t)|^4) \, dx \leq M. \tag{2.7}
\]

**Proof.** Multiply (1.2) by $2|u|^2u$ and integrate to obtain
\[
\frac{d}{dt} \int_{\mathbb{R}^3} p|u|^4 \, dx + \int_{\mathbb{R}^3} 2|u|^2 [\varepsilon|\nabla u|^2 + (\lambda + \varepsilon)(\text{div}(u))^2] \, dx
\]
\[
+ \int_{\mathbb{R}^3} 8 [\varepsilon|u|^2|\nabla (|u|)|^2 + (\varepsilon + \lambda)(\text{div}(u))|u|u \cdot \nabla (|u|)] \, dx
\]
\[
= 4 \int_{\mathbb{R}^3} (P(\rho) - P(\bar{\rho})) \text{div}(|u|^2 u) \, dx
\]
\[
+ \int_{\mathbb{R}^3} 2|B|^2 \text{div}(|u|^2 u) \, dx + \int_{\mathbb{R}^3} 4|u|^2 u \cdot \text{div}(BB^T) \, dx. \tag{2.8}
\]

The third term on the left side of (2.8) can be estimated from below by
\[
\int_{\mathbb{R}^3} 8|\varepsilon|u|^2|\nabla (|u|)|^2 + (\varepsilon + \lambda)(\text{div}(u))|u|u \cdot \nabla (|u|)] \, dx
\]
\[
\geq \int_{\mathbb{R}^3} 4|u|^2 \left[ \varepsilon|\nabla u|^2 + 2(\varepsilon - \frac{\varepsilon + \lambda}{2})|\nabla (|u|)|^2 \right] \, dx.
\]

By assumption (1.7) we have $\varepsilon < \lambda$, hence it implies
\[
\int_{\mathbb{R}^3} 8|\varepsilon|u|^2(\varepsilon - \frac{\varepsilon + \lambda}{2})|\nabla (|u|)|^2 \geq M \int_{\mathbb{R}^3} |u|^2|\nabla u|^2 \, dx. \tag{2.9}
\]

On the other hand, we multiply (1.3) by $4|B|^2B$ and integrate to get
\[
\frac{d}{dt} \int_{\mathbb{R}^3} |B|^2 \, dx + \int_{\mathbb{R}^3} 4\nu|B|^2|\nabla B|^2 \, dx + \int_{\mathbb{R}^3} 2\nu|B|^2|\nabla B|^2 \, dx
\]
\[
= - \int_{\mathbb{R}^3} |B|^2 B \cdot \text{div}(Bu^T - uB^T) \, dx. \tag{2.10}
\]

Adding (2.10) to (2.9) and integrate with respect to $t$, we get
\[
\left( \int_{\mathbb{R}^3} (|u|^4 + |B|^4) \, dx \right) + \int_0^t \int_{\mathbb{R}^3} (|u|^2|\nabla u|^2 + |B|^2|\nabla B|^2) \, dx \, ds
\]
\[
\leq M \left[ \int_0^t \int_{\mathbb{R}^3} 4(P(\rho) - P(\bar{\rho})) \text{div}(|u|^2 u) \, dx \, ds + \int_0^t \int_{\mathbb{R}^3} 2|B|^2 \text{div}(|u|^2 u) \, dx \, ds \right]
\]
\[
- M \left[ \int_0^t \int_{\mathbb{R}^3} 2|u|^2 u \cdot \text{div}(BB^T) \, dx \, ds + \int_0^t \int_{\mathbb{R}^3} |B|^2 B \cdot \text{div}(Bu^T - uB^T) \, dx \, ds \right]. \tag{2.11}
\]
Using the assumption (2.1), the right side of (2.11) can be bounded by
\[
\left[ \int_0^t \left( |P(\rho) - P(\hat{\rho})|^2 + |B|^4 + |B|^2 |\nabla B|^2 \right) dx ds \right]^{\frac{1}{2}} \left[ \int_0^t \left( |u|^2 + |B|^2 |\nabla B|^2 \right) dx ds \right]^{\frac{1}{2}} 
\leq M \left[ T^* C_0 + \int_0^t \left( |B|^4 + |u|^4 \right) dx ds \right]^{\frac{1}{2}} \left[ \int_0^t \left( |u|^2 + |B|^2 |\nabla B|^2 \right) dx ds \right]^{\frac{1}{2}}.
\]  
(2.12)

Using (2.12) on (2.11) and applying Cauchy Inequality, we get
\[
\int_0^t (|u|^4 + |B|^4) dx \leq M + \int_0^t (|u|^4 + |B|^4) dx ds,
\]
and (2.7) now follows by Gronwall’s inequality.

We obtain estimates on the functional $A_1$ in terms of $H$:

**Lemma 2.5** Assume that the hypotheses and notations of Theorem 2.1 are in force. Then for any $0 \leq t \leq T \leq T^*$,
\[
A_1(t) \leq M[1 + H(t)].
\]  
(2.13)

**Proof.** We multiply (1.2) by $\hat{u}^j$, sum over $j$ and integrate to get
\[
\int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_0^t \int_{\mathbb{R}^3} \rho |\hat{u}|^2 dx ds 
\leq C_0 + \left[ \int_0^t \left( \int_{\mathbb{R}^3} \left( \frac{1}{2} |B|^2 - \hat{u} \cdot \text{div}(BB^T) \right) \right) + \int_0^t \int_{\mathbb{R}^3} |\nabla u|^3 dx ds. \right. \]  
(2.14)

Next we multiply (1.3) by $B_i$ and integrate,
\[
\int_{\mathbb{R}^3} |\nabla B|^2 dx + \int_0^t \int_{\mathbb{R}^3} |B|^2 dx ds \leq C_0 + \left[ \int_0^t \int_{\mathbb{R}^3} B_i \cdot \text{div}(uB^T - uT_B) dx ds \right]. \]  
(2.15)

Adding (2.14) and (2.15), we obtain
\[
\int_{\mathbb{R}^3} (|\nabla u|^2 + |\nabla B|^2) dx + \int_0^t \int_{\mathbb{R}^3} (\rho |\hat{u}|^2 + |B|^2) dx ds 
\leq C_0 + \int_0^t \int_{\mathbb{R}^3} |\nabla u|^3 dx ds + \int_0^t \int_{\mathbb{R}^3} (|\nabla B|^2 |B|^2 + |\nabla u|^2 |B|^2 + |\nabla B|^2 |u|^2) dx ds.
\]  
(2.16)

The second term on the right side of (2.16) is bounded by
\[
\left( \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 dx ds \right)^{\frac{1}{2}} \left( \int_0^t \int_{\mathbb{R}^3} |\nabla u|^4 dx ds \right)^{\frac{1}{2}} \leq C_0 + H(t),
\]
where the last inequality follows by Lemma 2.3. For the last integral on the right side of (2.16), using assumption (2.1), it can be bounded by \[
\int_0^t \int_{\mathbb{R}^3} |\nabla B|^2 |u|^2 dx ds + M \int_0^t \int_{\mathbb{R}^3} (|\nabla u|^2 + |\nabla B|^2) dx ds. \] So it remains to estimate \[
\int_0^t \int_{\mathbb{R}^3} |\nabla B|^2 |u|^2 dx ds.
\]

Recall from Lemma 2.4 that, for $0 \leq t \leq T \leq T^*$,
\[
\int_{\mathbb{R}^3} |u|^4 dx \leq M,
\]
Therefore, using (1.10),

\[
\int_0^t \int_{\mathbb{R}^3} |\nabla B|^2 |u|^2 \, dx \, ds \leq \int_0^t \left( \int_{\mathbb{R}^3} |\nabla B|^4 \, dx \right)^{\frac{1}{4}} \left( \int_{\mathbb{R}^3} |u|^4 \, dx \right)^{\frac{1}{4}} \, ds
\]

\[
\leq M \left( \int_0^t \int_{\mathbb{R}^3} |\nabla B|^2 \, dx \, ds \right)^{\frac{3}{4}} \left( \int_0^t \int_{\mathbb{R}^3} |D^2 B|^2 \, dx \, ds \right)^{\frac{1}{4}}
\]

\[
\leq MC^0_0 \left[ \int_0^t \int_{\mathbb{R}^3} \left( |B_t|^2 + |\nabla B|^2 |u|^2 + |\nabla u|^2 |B|^2 \right) \, dx \, ds \right]^{\frac{3}{4}},
\]

and by Cauchy inequality, we obtain

\[
\int_0^t \int_{\mathbb{R}^3} |\nabla B|^2 |u|^2 \, dx \, ds \leq MC^0_0 \left[ A_1(t) + C_0 + \int_0^t \int_{\mathbb{R}^3} |\nabla B|^2 |u|^2 \, dx \, ds \right]^{\frac{3}{4}},
\]

(2.17)

Applying (2.17) to (2.16) and absorbing terms, (2.13) follows.

We derive the following estimates on the effective viscous flux which were first described by Hoff [3] and later modified by Sun-Wang-Zhang [15].

**Lemma 2.6** Assume that the hypotheses and notations of Theorem 2.1 are in force. Then for any \(0 \leq t \leq T \leq T^*\),

\[
\sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} |\nabla w|^2 + \int_0^t \int_{\mathbb{R}^3} |w_t|^2 \, dx \, ds + \int_0^t \int_{\mathbb{R}^3} |D^2 w|^2 \, dx \, ds \leq M,
\]

(2.18)

where \(w = u - v\) with \(v\) satisfying:

\[
\varepsilon \Delta v + (\varepsilon + \lambda) \nabla \text{div}(v) = \nabla (P(\rho) - P(\tilde{\rho})).
\]

**Proof.** Using the momentum equation (1.2),

\[
\rho w_t - \varepsilon \Delta w - (\varepsilon + \lambda) \nabla \text{div}(w) = -\rho u \cdot \nabla u - \rho v_t - \nabla \left( \frac{1}{2} |B|^2 \right) + \text{div}(BB^T). \tag{2.19}
\]

Multiply (2.19) by \(w_t\) and integrate,

\[
\int_{\mathbb{R}^3} \varepsilon |\nabla w|^2 \, dx \bigg|_0^t + \int_0^t \int_{\mathbb{R}^3} (\varepsilon + \lambda) |\text{div}w|^2 \, dx \, ds + \int_0^t \int_{\mathbb{R}^3} \rho |w_t|^2 \, dx \, ds
\]

\[
= \int_0^t \int_{\mathbb{R}^3} \left[ -\rho u \cdot \nabla u - \rho v_t - \nabla \left( \frac{1}{2} |B|^2 \right) + \text{div}(BB^T) \right] \cdot w_t \, dx \, ds. \tag{2.20}
\]

The first term on the right side of (2.20) can be estimated as follows:

\[
\int_0^t \int_{\mathbb{R}^3} (\varepsilon + \lambda) |\text{div}w|^2 \, dx \, ds \leq M \left[ \int_0^t \int_{\mathbb{R}^3} |u|^2 |\nabla u|^2 \, dx \, ds \right]^{\frac{1}{4}} \left[ \int_0^t \int_{\mathbb{R}^3} |w_t|^2 \, dx \, ds \right]^{\frac{3}{4}}.
\]

(2.21)
For the term $\int_0^t \int_{\mathbb{R}^3} |u|^2 |\nabla u|^2 \, dx \, ds$, using Lemma 2.4,

$$\int_0^t \int_{\mathbb{R}^3} |u|^2 |\nabla u|^2 \, dx \, ds \leq M \int_0^t \left( \int_{\mathbb{R}^3} |\nabla w|^4 \, dx \right)^{\frac{1}{4}} \left( \int_{\mathbb{R}^3} |D^2 w|^2 \, dx \right)^{\frac{1}{4}} \, ds \leq M \left[ \int_0^t \left( \int_{\mathbb{R}^3} |\nabla w|^2 \, dx \right)^{\frac{1}{4}} \left( \int_{\mathbb{R}^3} |D^2 w|^2 \, dx \right)^{\frac{1}{4}} \, ds \right]^{\frac{1}{2}} + M \int_0^t \left( \int_{\mathbb{R}^3} |\nabla v|^4 \, dx \right)^{\frac{1}{4}} \, ds \leq M \left( \int_0^t \int_{\mathbb{R}^3} |\nabla w|^2 \, dx \, ds \right)^{\frac{1}{4}} \left( \int_0^t \int_{\mathbb{R}^3} |D^2 w|^2 \, dx \, ds \right)^{\frac{1}{4}} + MC_1^2 T^{\frac{1}{2}}, \tag{2.22}$$

where the last inequality follows by Lemma 2.2 and assumption (2.1). Therefore (2.21) becomes

$$\int_0^t \int_{\mathbb{R}^3} (-\rho u \cdot \nabla u) \cdot w_t \, dx \, ds \leq M \left[ \left( \int_0^t \int_{\mathbb{R}^3} |\nabla w|^2 \, dx \, ds \right)^{\frac{1}{4}} \left( \int_0^t \int_{\mathbb{R}^3} |D^2 w|^2 \, dx \, ds \right)^{\frac{1}{4}} + MC_0^2 T^{\frac{1}{2}} \right] \left( \int_0^t \int_{\mathbb{R}^3} |w_t|^2 \, dx \, ds \right)^{\frac{1}{2}}. \tag{2.23}$$

The third and the fourth term on the right side of (2.20) are bounded by

$$\int_0^t \int_{\mathbb{R}^3} |\nabla B| |B| |w_t| \, dx \, ds \leq M \left( \int_0^t \int_{\mathbb{R}^3} |\nabla B|^2 \, dx \, ds \right)^{\frac{1}{2}} \left( \int_0^t \int_{\mathbb{R}^3} |w_t|^2 \, dx \, ds \right)^{\frac{1}{2}} \leq MC_0^4 \left( \int_0^t \int_{\mathbb{R}^3} |w_t|^2 \, dx \, ds \right)^{\frac{1}{2}}. \tag{2.24}$$

It remains to estimate the term $\int_0^t \int_{\mathbb{R}^3} -\rho v_t \cdot w_t \, dx \, ds$ on the right side of (2.20). By the definition of $v$ and $P(\rho)$, we have

$$\epsilon \Delta v_t + (\epsilon + \lambda) \nabla \div (v_t) = \nabla P(\rho)_t = \nabla \div (-P(\rho) u),$$

Hence we can apply Lemma 2.2 and Lemma 2.3 to get

$$\int_0^t \int_{\mathbb{R}^3} |v_t|^2 \, dx \, ds \leq \int_0^t \int_{\mathbb{R}^3} |P(\rho) u|^2 \, dx \, ds \leq MC_0,$$
and
\[
\int_{0}^{t} \int_{\mathbb{R}^3} -\rho v_1 \cdot w_t \, dx \, ds \leq M \left( \int_{0}^{t} \int_{\mathbb{R}^3} |w_t|^2 \, dx \, ds \right)^{1/2} \left( \int_{0}^{t} \int_{\mathbb{R}^3} |v_1|^2 \, dx \, ds \right)^{1/2} \\
\leq M \left( \int_{0}^{t} \int_{\mathbb{R}^3} |w_t|^2 \, dx \, ds \right)^{1/2}.
\] (2.25)

Using (2.23), (2.24) and (2.25) on (2.20),
\[
\int_{\mathbb{R}^3} \varepsilon |\nabla w|^2 \, dx \bigg|_{0}^{t} + \int_{0}^{t} \int_{\mathbb{R}^3} (\varepsilon + \lambda)|\text{div}(w)|^2 \, dx \, ds + \int_{0}^{t} \int_{\mathbb{R}^3} \rho |w_t|^2 \, dx \, ds \\
\leq M \left( \int_{0}^{t} \int_{\mathbb{R}^3} |\nabla w|^2 \, dx \, ds \right)^{1/2} \left( \int_{0}^{t} \int_{\mathbb{R}^3} |D^2 w_t|^2 \, dx \, ds \right)^{1/2} + M.
\] (2.26)

It remains to estimate \( \int_{0}^{t} \int_{\mathbb{R}^3} |D^2 w_t|^2 \, dx \, ds \). We rearrange the terms in (2.19) to get
\[
\varepsilon \Delta w + (\varepsilon + \lambda) \text{div}(w) = \rho w_t + \rho \nabla u \cdot u + \rho v_t + \nabla \left( \frac{1}{2} |B|^2 \right) - \text{div}(BB^T),
\]
and so by Lemma 2.2,
\[
\int_{0}^{t} \int_{\mathbb{R}^3} |D^2 w_t|^2 \, dx \, ds \leq \int_{0}^{t} \int_{\mathbb{R}^3} (|\rho w_t|^2 + |\rho \nabla u \cdot u|^2 + |\rho v_t|^2 + |\nabla B|^2 |B|^2) \, dx \, ds \\
\leq M \left[ \int_{0}^{t} \int_{\mathbb{R}^3} |w_t|^2 \, dx \, ds + \left( \int_{0}^{t} \int_{\mathbb{R}^3} |\nabla w|^2 \, dx \, ds \right)^{1/2} \left( \int_{0}^{t} \int_{\mathbb{R}^3} |D^2 w_t|^2 \, dx \, ds \right)^{1/2} + 1 \right].
\]

Therefore
\[
\int_{0}^{t} \int_{\mathbb{R}^3} |D^2 w_t|^2 \, dx \, ds \leq M \left[ \int_{0}^{t} \int_{\mathbb{R}^3} |w_t|^2 \, dx \, ds + \int_{0}^{t} \int_{\mathbb{R}^3} |\nabla w|^2 \, dx \, ds + 1 \right],
\]
and we apply the above to (2.26) to conclude
\[
\int_{\mathbb{R}^3} \varepsilon |\nabla w|^2 \, dx \bigg|_{0}^{t} + \int_{0}^{t} \int_{\mathbb{R}^3} (\varepsilon + \lambda)|\text{div}(w)|^2 \, dx \, ds + \int_{0}^{t} \int_{\mathbb{R}^3} \rho |w_t|^2 \, dx \, ds \\
\leq M \left[ \int_{0}^{t} \int_{\mathbb{R}^3} |\nabla w|^2 \, dx \, ds + 1 \right],
\]
which (2.18) follows by Gronwall’s inequality. \(\square\)

We finally obtain an estimate on the functional \( A_2 \) which is sufficient to prove Theorem 2.1:

**Lemma 2.7** Assume that the hypotheses and notations of Theorem 2.1 are in force. Then for any \( 0 \leq t \leq T \leq T^* \),
\[
A_2(t) \leq M \left[ A_1(t) + H(t) + 1 \right]
\] (2.27)
Proof. Taking the convective derivative in the momentum equation (1.2), multiplying it by \( \dot{u}^j \), summing over \( j \) and integrating,

\[
\sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} |\dot{u}|^2 dx + \int_0^t \int_{\mathbb{R}^3} |\nabla \dot{u}|^2 dx ds \leq C_0 + H(t) + \int_0^t \int_{\mathbb{R}^3} |B|^2 (|B_t|^2 + |u|^2 |\nabla B|^2) dx ds. \tag{2.28}
\]

Next we differentiate the magnetic field equation (1.3) with respect to \( t \), multiply by \( B_t \) and integrate,

\[
\frac{1}{2} \int_{\mathbb{R}^3} |B_t|^2 dx \bigg|_0^t + \nu \int_0^t \int_{\mathbb{R}^3} |\nabla B_t|^2 dx ds = - \int_0^t \int_{\mathbb{R}^3} B_t \cdot [\text{div}(Bu^T - uB^T)] dx ds.
\]

Adding the above to (2.28) and absorbing terms,

\[
\sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} (|\dot{u}|^2 + |B_t|^2) dx + \int_0^t \int_{\mathbb{R}^3} (|\nabla \dot{u}|^2 + |\nabla B_t|^2) dx ds \leq M \left[A_1 + H + \int_0^t \int_{\mathbb{R}^3} |B|^2|u|^2(|\nabla u|^2 + |\nabla B|^2) dx ds\right]
+ M \int_0^t \int_{\mathbb{R}^3} (|B_t|^2|B_t|^2 + |B|^2|\dot{u}|^2 + |B_t|^2|u|^2) dx ds \tag{2.29}
\]

The third term on the right side of (2.29) is bounded by

\[
\int_0^t \int_{\mathbb{R}^3} |u|^2(|\nabla u|^2 + |\nabla B|^2) dx ds \leq \left[ \int_0^t \int_{\mathbb{R}^3} |u|^2|\nabla u|^2 dx ds + \int_0^t \int_{\mathbb{R}^3} |u|^2|\nabla B|^2 dx ds \right]
\leq M \left[1 + A_{1\frac{1}{2}}\right], \tag{2.30}
\]

where the last inequality follows by (2.17) and (2.22). The last term on the right side of (2.29) is bounded by

\[
\int_0^t \int_{\mathbb{R}^3} (|B_t|^2 + |\dot{u}|^2) dx ds + \int_0^t \int_{\mathbb{R}^3} |B_t|^2|u|^2 dx ds
\leq MA_1 + \int_0^t \left( \int_{\mathbb{R}^3} |B_t|^4 dx \right)^{\frac{1}{4}} \left( \int_{\mathbb{R}^3} |u|^4 dx \right)^{\frac{1}{4}} ds
\leq MA_1 + M \int_0^t \left( \int_{\mathbb{R}^3} |B_t|^2 dx \right)^{\frac{1}{4}} \left( \int_{\mathbb{R}^3} |\nabla B_t|^2 dx \right)^{\frac{1}{4}} ds
\leq MA_1 + M \left( \int_0^t \int_{\mathbb{R}^3} |B_t|^2 dx ds \right)^{\frac{1}{4}} \left( \int_0^t \int_{\mathbb{R}^3} |\nabla B_t|^2 dx ds \right)^{\frac{1}{4}}
\leq MA_1 + A_1^{\frac{1}{4}} A_{1\frac{1}{2}}. \tag{2.31}
\]

Using (2.30) and (2.31) on (2.29) and absorbing terms, (2.28) follows. \qed

proof of Theorem 2.1. Recall from Lemma 2.5 and 2.7 that

\[ A_1 \leq M [H + 1] \]
and

\[ A_2 \leq M [H + A_1]. \]

So it remains to estimate \( H \). Let \( w \) and \( v \) be as defined in Lemma 2.6. Then

\[ \int_{0}^{t} \int_{\mathbb{R}^3} |\nabla u|^4 \, dx \, ds \leq \int_{0}^{t} \int_{\mathbb{R}^3} |\nabla w|^4 \, dx \, ds + \int_{0}^{t} \int_{\mathbb{R}^3} |\nabla v|^4 \, dx \, ds \quad (2.32) \]

The second term on the right side of (2.32) is bounded by \( \int_{0}^{t} \int_{\mathbb{R}^3} |P(\rho) - P(\tilde{\rho})|^4 \, dx \, ds \).

And for \( \int_{0}^{t} \int_{\mathbb{R}^3} |\nabla w|^4 \, dx \, ds \), using (2.18),

\[ \int_{0}^{t} \int_{\mathbb{R}^3} |\nabla w|^4 \, dx \, ds \leq \left( \sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} |D^2_x w|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} |\nabla w|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{0}^{t} \int_{\mathbb{R}^3} |\nabla w|^2 \, dx \right)^{\frac{1}{2}} \leq M \left( \sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} |D^2_x w|^2 \, dx \right)^{\frac{1}{2}}. \]

Notice that, by rearranging the terms in (2.19),

\[ \varepsilon \Delta w + (\varepsilon + \lambda) \nabla \text{div}(\nu) = \rho \dot{u} + \nabla \left( \frac{1}{2} |B|^2 \right) - \text{div}(BB^T), \]

and so by Lemma 2.2,

\[ \int_{\mathbb{R}^3} |D^2_x w|^2 \, dx \leq \int_{\mathbb{R}^3} |\rho \dot{u}|^2 \, dx + \int_{\mathbb{R}^3} |\nabla \left( \frac{1}{2} |B|^2 \right) - \text{div}(BB^T)|^2 \, dx \leq M \left[ \int_{\mathbb{R}^3} |\dot{u}|^2 \, dx + \int_{\mathbb{R}^3} |\nabla B|^2 |B|^2 \, dx \right] \leq M [A_2 + A_1]. \]

Therefore we conclude that

\[ H \leq M [A_2 + A_1]^{\frac{1}{2}}, \]

and (2.2) follows.

\[ \square \]

3. Higher Order Estimates and Proof of Theorem 1.2

In this section we continue to obtain higher order estimates on the smooth local solution \((\rho - \tilde{\rho}, u, B)\) as described in section 2. Together with Theorem 2.1, we show that, under the assumption (2.1), the smooth local solution to (1.1)-(1.4) can be extended beyond the maximal time of existence \(T^*\) as defined in section 2, thereby contradicting the maximality of \(T^*\). The following is the main theorem of this section:

**Theorem 3.1** Assume that the hypotheses and notations in Theorem 2.1 are in force. Given \( C > 0 \) and \( \tilde{\rho} > 0 \), assume further that \((\rho - \tilde{\rho}, u, B)\) satisfies (2.1).
Then there exists a positive number $M'$ which depends on $C_0, C, T^*$ and the system parameters $P, \varepsilon, \lambda, \nu$ such that, for $0 \leq t \leq T \leq T^*$,
\[
\sup_{0 \leq s \leq t} \| (\rho - \hat{\rho}, u, B) \|_{H^1(\mathbb{R}^3)} + \int_0^t \| (u, B)(\cdot, s) \|_{H^1(\mathbb{R}^3)}^2 ds \leq M' \tag{3.1}
\]

Proof. We give the proof in a sequence of steps. Most of the details are reminiscent of Suen and Hoff [14] and we omit those which are identical to or nearly identical to arguments given in [14]. We first begin with the following estimates on the effective viscous flux $F$ and the vorticity matrix $\omega$:

**Step 1:** Define
\[
F = (2 \varepsilon + \lambda) \text{div}(u) - (P(\rho) - P(\hat{\rho})), \\
\omega = \omega^{j,k} = u^{j}_{x_k} - u^{k}_{x_j},
\]
Then for $q \in (1, \infty)$,
\[
\| \nabla u(\cdot, t) \|_{L^q} \leq M(q) \left[ \|F(\cdot, t)\|_{L^q} + \|\omega(\cdot, t)\|_{L^q} + \| (P(\rho) - P(\hat{\rho}))(\cdot, t) \|_{L^q} \right], \tag{3.2}
\]
\[
\| \nabla \omega(\cdot, t) \|_{L^q} \leq (q) M \left[ \| P(\rho)(\cdot, t) \|_{L^q} + \| \nabla B \cdot B(\cdot, t) \|_{L^q} \right], \tag{3.3}
\]
where $M(q)$ is a positive constant depending on $q$ and
\[
\sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} (|\nabla F|^2 + |\nabla \omega|^2) dx \leq M'. \tag{3.4}
\]

**proof of Step 1.** We give the proof of (3.2) as an example. Using the definition of $F$ and $\omega$,
\[
(2 \varepsilon + \lambda) \Delta u^j = F_{x_j}, \\
(2 \varepsilon + \lambda) \omega^{j,k}_{x_k} = (P(\rho) - P(\hat{\rho}))_{x_j}, \tag{3.5}
\]
Differentiating and taking the Fourier transform we then obtain
\[
(2 \varepsilon + \lambda) \hat{u}^j_{x_j}(y, t) = \frac{y_j y_l}{|y|^2} \hat{F}(y, t) + (2 \varepsilon + \lambda) \frac{y_k y_l}{|y|^2} \hat{\omega}^{j,k}_{x_k}(y, t) + \frac{y_k y_l}{|y|^2} \left( P - \hat{P} \right)(y, t)
\]
and (3.2) then follows immediately from the Marcinkiewicz multiplier theorem (Stein [13], pg. 96). Similarly, (3.3) can be proved by the same method. Also, by the definition of $F$, we have
\[
\Delta F = \text{div}(g), \tag{3.6}
\]
where $g^j = \rho \hat{u}^j + \left( \frac{1}{2} |B|^2 \right)_{x_j} - \text{div}(B^i B)$. So we have
\[
\sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} |\nabla F|^2 dx \leq \sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} |g|^2 dx \\
\leq \sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} (\rho |\hat{u}|^2 + |\nabla B|^2 |B|^2) dx \leq M',
\]
and similarly, $\sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} |\nabla \omega|^2 dx \leq M'$, which proves (3.4).

**Step 2:** The velocity gradient satisfies the following bound
\[
\int_0^t \| \nabla u(\cdot, t) \|_{L^\infty} ds \leq M'.
\]

**proof of Step 2.** The proof is identical to Suen and Hoff [14] pg. 51–53, and we omit the details here.

Step 3: We further obtain

$$||D^2_\tau u(\cdot, t)||_{L^2} \leq M' \left[ ||\rho u(\cdot, t)||_{L^2} + ||\nabla B \cdot B(\cdot, t)||_{L^2} + ||\nabla P(\cdot, t)||_{L^2} \right], \quad (3.7)$$

$$||D^3_\tau u(\cdot, t)||_{L^2} \leq M' \left[ ||\nabla \rho \cdot \hat{u}(\cdot, t)||_{L^2} + ||\nabla \hat{u}(\cdot, t)||_{L^2} + ||B \cdot D^2_\tau B(\cdot, t)||_{L^2} \right] + M' \left[ ||\nabla B||^2(\cdot, t)||_{L^2} + ||D^2_\tau P(\cdot, t)||_{L^2} \right]. \quad (3.8)$$

proof of Step 3. These follow immediately from the momentum equation (1.2) and the ellipticity of the Lamé operator $\varepsilon \Delta + (\varepsilon + \lambda) \nabla \text{div}$. □

Step 4: The following $H^2$-bound for density holds

$$\sup_{0 \leq s \leq t} ||(\rho - \hat{\rho})(\cdot, s)||_{H^2} \leq M'. \quad (3.9)$$

proof of Step 4. We take the spatial gradient of the mass equation (1.1), multiply by $\nabla \rho$ and integrate by parts to obtain

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^3} |\nabla \rho|^2 \, dx \leq M' \left[ \int_{\mathbb{R}^3} |\nabla \rho|^2 \, dx + \int_{\mathbb{R}^3} |D^2_\tau u|^2 \, dx \right] \quad (3.10)$$

From (3.7),

$$\int_0^t \int_{\mathbb{R}^3} |D^2_\tau u|^2 \, dx \, ds \leq \int_0^t \int_{\mathbb{R}^3} (|\dot{u}|^2 + |\nabla B \cdot B|^2 + |\nabla \rho|^2) \, dx \, ds \leq M' + \int_0^t \int_{\mathbb{R}^3} |\nabla \rho|^2 \, dx \, ds.$$

Applying the above to (3.10) and using the result of Step 2,

$$\sup_{0 \leq s \leq t} ||\nabla \rho(\cdot, s)||_{L^2} \leq M'.$$

By similar argument, we can show that $\sup_{0 \leq s \leq t} |||D^2_\tau \rho(\cdot, s)|||_{L^2} \leq M'$ and (3.9) follows. □

Step 5: The velocity and magnetic field satisfy

$$\sup_{0 \leq s \leq t} \left( ||u(\cdot, s)||_{H^3} + ||B(\cdot, s)||_{H^3} \right) \leq M'. \quad (3.11)$$

proof of Step 5. Define the forward difference of quotient $D^h_t$ by

$$D^h_t(f)(t) = (f(t + h) - f(t))h^{-1}$$

and let $E^j = D^h_t(u^j) + u \cdot \nabla u^j$. By differentiating the momentum equation, we obtain

$$\int_{\mathbb{R}^3} \rho|E^j_{x_j}|^2 \, dx + \int_0^t \int_{\mathbb{R}^3} \left( |\nabla E^j_{x_j}|^2 + |D^h_t(\text{div}(u_{x_j})) + u \cdot \nabla(\text{div}(u_{x_j}))|^2 \right) \, dx \, ds \leq M' + \int_0^t \int_{\mathbb{R}^3} |\nabla E|^2 \, dx \, ds + O(h),$$

where $O(h) \to 0$ as $h \to 0$. Therefore by taking $h \to 0$,

$$\sup_{0 \leq s \leq t} ||\nabla \dot{u}(\cdot, s)||_{L^2} + \int_0^t \int_{\mathbb{R}^3} |D^2_\tau \dot{u}|^2 \, dx \, ds \leq M'.$$

The bound for $\nabla B_t$ can be derived in an exactly same way. □
Step 6: Finally we have the following bounds
\[
\int_0^t \int_{\mathbb{R}^3} \left( |D_x^4 u|^2 + |D_x^4 B|^2 \right) dx ds \leq M' \left[ 1 + \int_0^t \int_{\mathbb{R}^3} |D_x^3 \rho|^2 dx ds \right], \tag{3.12}
\]
\[
\sup_{0 \leq s \leq t} \left( ||D_x^3 \rho(\cdot, s)||_{L^2} + ||D_x^3 B(\cdot, s)||_{L^2} \right) + \int_0^t \int_{\mathbb{R}^3} |D_x^3 u|^2 dx ds \leq M'. \tag{3.13}
\]

Proof of Step 6. For (3.12), it can be obtained by differentiating (1.2) and (1.3) twice with respect to space, expressing the fourth derivatives of \( u \) and \( B \) in the terms second derivatives of \( \dot{u}, B_t, \nabla \rho \) and lower order terms, and applying the bounds in (2.1) and (3.11).

For (3.13), it can be obtained by applying two space derivatives and one spatial difference operator \( D_x^h \) defined by
\[
D_x^h(f)(t) = (f(x + h e_j) - f(x))h^{-1}
\]
such that
\[
\int_{\mathbb{R}^3} |D_x^h D_x D_x D_x \rho|^2 dx \leq M' + \int_0^t \int_{\mathbb{R}^3} \left( |D_x^4 u|^2 + |D_x^h D_x D_x D_x \rho|^2 \right) dx ds
\]
\[
\leq M' + \int_0^t \int_{\mathbb{R}^3} |D_x^3 \rho|^2 dx ds.
\]
Taking \( h \to 0 \) and applying Gronwall’s inequality, we obtain the required bound for the term ||\( D_x^3 \rho(\cdot, s) ||_{L^2} ||.

Proof of Theorem 1.2. Using Theorem 3.1, we can apply an open-closed argument on the time interval which is identical to the one given in Suen and Hoff \[14\] pp. 31 to extend the local solution \( (\rho - \tilde{\rho}, u, B) \) beyond \( T^* \), which contradicts the maximality of \( T^* \). Therefore the assumption (2.1) does not hold and this completes the proof of Theorem 1.2.

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