Elliptic Quantum Group $U_{q,p}(\hat{sl}_2)$, Hopf Algebroid Structure and Elliptic Hypergeometric Series

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Abstract

We propose a new realization of the elliptic quantum group equipped with the $H$-Hopf algebroid structure on the basis of the elliptic algebra $U_{q,p}(\hat{sl}_2)$. The algebra $U_{q,p}(\hat{sl}_2)$ has a constructive definition in terms of the Drinfeld generators of the quantum affine algebra $U_q(\hat{sl}_2)$ and a Heisenberg algebra. This yields a systematic construction of both finite and infinite-dimensional dynamical representations and their parallel structures to $U_q(\hat{sl}_2)$. In particular we give a classification theorem of the finite-dimensional irreducible pseudo-highest weight representations stated in terms of an elliptic analogue of the Drinfeld polynomials.

We also investigate a structure of the tensor product of two evaluation representations and derive an elliptic analogue of the Clebsch-Gordan coefficients. We show that it is expressed by using the very-well-poised balanced elliptic hypergeometric series $_{12}V_{11}$.

1 Introduction

In this paper we revisit the elliptic algebra $U_{q,p}(\hat{sl}_2)$ and study its coalgebra structure. The algebra $U_{q,p}(\hat{sl}_2)$ was introduced in [34] as an elliptic analogue of the quantum affine algebra $U_q(\hat{sl}_2)$ in the Drinfeld realization [9]. It was realized in [25] that $U_{q,p}(\hat{sl}_2)$ is constructively defined by using the Drinfeld generators of $U_q(\hat{sl}_2)$ and the Heisenberg algebra $\{P,e^Q\}$. This construction was generalized to the elliptic algebra $U_{q,p}(\mathfrak{g})$ of all types of untwisted affine Lie algebras $\mathfrak{g}$ [25] and of the twisted type $A^{(2)}_2$ [32].

It was also realized that $U_{q,p}(\mathfrak{g})$ has an interesting relation to the deformed coset Virasoro/$W$ algebras [2,15,19,46]. Namely, the level one ($c = 1$) elliptic currents of $U_{q,p}(\mathfrak{g})$ are identified with the screening currents of the deformed $W$ algebras for $\mathfrak{g} = \hat{sl}_N$ [25,31,34] and for $\mathfrak{g} = A^{(2)}_2$ [32]. This observation led us to a conjecture that the elliptic currents $E_i(u)$ and $F_i(u)$ of $U_{q,p}(\mathfrak{g})$ ($i = 1,2,\cdots, \text{rank}\mathfrak{g}$) define the deformation of the Virasoro/$W$ algebra associated with the coset $\mathfrak{g} \oplus \mathfrak{g}/\mathfrak{g}$ [25,34].
A study of coalgebra structure on $U_{q,p}(\mathfrak{g})$ was far from straightforward. We constructed the $L$ operators in terms of the elliptic currents and derived the $RLL$ relation for the cases $\mathfrak{g} = \widehat{\mathfrak{sl}}_N$ [25, 31] and $\mathfrak{g} = A^{(2)}_2$ [32]. However it turned out that a naive Faddeev-Reshetikhin-Sklyanin-Takhtajan (FRST) construction [14, 47] does not work due to the dynamical shift appearing in the $R$ matrices. Instead we obtained a connection to the quasi-Hopf algebra $B_{q,\lambda}(\mathfrak{g})$ [25]. That is the isomorphism $U_{q,p}(\mathfrak{g}) \cong B_{q,\lambda}(\mathfrak{g}) \otimes \{P_i, e^{Q_i}\} (i = 1, 2, \cdots, \text{rank} \mathfrak{g})$ as an associative algebra, where $\{P_i, e^{Q_i}\}$ denotes a Heisenberg algebra.

The quasi-Hopf algebra $B_{q,\lambda}(\mathfrak{g})$ (the face type) was introduced by Jimbo, Konno, Odake and Shiraishi [24] motivated by the works of Drinfeld [10], Babelon, Bernard and Billey [3] and Frønsdal [20]. At the same time, we introduced the vertex type quasi-Hopf algebra $A_{q,p}(\widehat{\mathfrak{sl}}_N)$. Both $A_{q,p}(\widehat{\mathfrak{sl}}_N)$ and $B_{q,\lambda}(\mathfrak{g})$ are isomorphic to the corresponding quantum affine algebras $U_q(\mathfrak{g})$ as associative algebras, but their coalgebra structures are deformed from $U_q(\mathfrak{g})$ by the twistors $E(r)$ and $F(\lambda)$, respectively. By twisting the objects in $U_q(\mathfrak{g})$, such as the comultiplication, the universal $R$ matrices and the vertex operators, we can derive their quasi-Hopf algebra counterparts [24]. Then the relation $U_{q,p}(\mathfrak{g}) \cong B_{q,\lambda}(\mathfrak{g}) \otimes \{P_i, e^{Q_i}\}$ allows us to derive the $U_{q,p}(\mathfrak{g})$ counterparts from those of the quasi-Hopf algebra $B_{q,\lambda}(\mathfrak{g})$. Such a strategy led us to an extension of the algebraic analysis scheme of trigonometric solvable lattice models á la Jimbo and Miwa [23] to the face type elliptic models [25, 31–34].

However the tensor product with the Heisenberg algebras breaks down the quasi-Hopf algebra structure, so that $U_{q,p}(\mathfrak{g})$ is not a quasi-Hopf algebra. Moreover, the quasi-Hopf algebra itself has a disadvantage that its coalgebra structure is not suitable for a practical calculation due to a complication arising from the same twist procedure. This is a serious defect, for example, to develop the representation theory of the elliptic quantum groups and their harmonic analysis.

The aim of this paper is to show that a relevant coalgebra structure of $U_{q,p}(\mathfrak{g})$ is an $H$-Hopf algebroid and to formulate a new elliptic quantum group, which complements the disadvantage of the quasi-Hopf algebra. In this paper we consider the case $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$. The cases of other affine Lie algebra types will be discussed in future publications.

The $H$-Hopf algebroid was introduced by Etingof and Varchenko [12, 13] motivated by the works of Felder and Varchenko [16, 17]. Some additional structures were given by Koelink and Rosengren [29, 43]. A good review of this subject can be found in [53]. In [12, 30, 51], it was applied to a formulation of Felder’s elliptic quantum group $E_{\tau,\eta}(\mathfrak{sl}_2)$ by using the generalized FRST construction on the basis of the $RLL$ relation associated with the elliptic dynamical $R$ matrix. Another formulation of $E_{\tau,\eta}(\mathfrak{sl}_2)$ as a quasi-Hopf algebra was studied by Enriquez and Felder [11]. A similar Hopf algebroid structure was introduced by Lu [39] and Xu [55]. In [55],
Xu also studied the algebra $\mathcal{D} \otimes U_q(\mathfrak{g})$, where $\mathcal{D}$ denotes the algebra of meromorphic differential operators on $\mathfrak{h}^*$. His algebra is similar to $U_{q,p}(\mathfrak{g})$, but his $\mathfrak{g}$ is a finite-dimensional simple Lie algebra.

Our formulation is based on the fact that the $RLL$ relation for $U_{q,p}(\hat{\mathfrak{sl}}_2)$ obtained in [25] is identical to a central extension of the one for Felder’s elliptic quantum group. This enables us to apply the generalized FRST construction to our case with a modification due to a central extension. The main modification is that we use both the commuting subalgebra $H$ of $U_{q,p}(\hat{\mathfrak{sl}}_2)$ and the additive Abelian group $\bar{H}^* \subset H^*$ appropriately in the formulation. Here $H$ contains the central element $c$, whereas $\bar{H}$ does not. We consider the field of meromorphic functions on $H^*$, but we use $\bar{H}^*$ to define the bigrading structure of $U_{q,p}(\hat{\mathfrak{sl}}_2)$. As a result we obtain a new face type elliptic quantum group $U_{q,p}(\hat{\mathfrak{sl}}_2)$ as an $H$-Hopf algebroid, which is realized in terms of the Drinfeld generators and has the central extension. We also show that the coalgebra structure is enough simple for a practical calculations.

In comparison with the previous formulations [12,17,30,51], $U_{q,p}(\hat{\mathfrak{sl}}_2)$ has the advantage that it has a constructive definition in terms of the Drinfeld generators of $U_q(\hat{\mathfrak{sl}}_2)$. This allows a systematic derivation of both finite and infinite-dimensional representations of $U_{q,p}(\hat{\mathfrak{sl}}_2)$ from those of $U_q(\hat{\mathfrak{sl}}_2)$ and their parallel structures to $U_q(\hat{\mathfrak{sl}}_2)$. As an example, we study a classification theorem of the finite-dimensional irreducible pseudo-highest weight representations and make a statement in terms of an elliptic analogue of the Drinfeld polynomials. This gives an elliptic analogue of the works by Drinfeld [9] and by Chari and Pressley [6]. In addition, we investigate a submodule structure of the tensor product of two evaluation modules. We obtain the singular vectors explicitly and derive an elliptic analogue of the Clebsch-Gordan coefficients. We show that the coefficients are given by the terminating very-well-poised balanced elliptic hypergeometric series $_{12}V_{11}$, which was introduced by Frenkel and Turaev [18] on the basis of the work by Date, Jimbo, Kuniba, Miwa and Okado [8], and extensively studied by Spiridonov and Zhedanov [48, 49]. This provides the alternative to the representation theoretical derivation of $_{12}V_{11}$ by Koelink, van Norden and Rosengren [30]. In [30], $_{12}V_{11}$ was obtained as matrix elements of a co-representation of Felder’s elliptic quantum group.

In the separate paper [36], we discuss a free field representation of the infinite-dimensional highest weight representations of $U_{q,p}(\hat{\mathfrak{sl}}_2)$ and derive the vertex operators as intertwining operators of such $U_{q,p}(\hat{\mathfrak{sl}}_2)$-modules. The resultant vertex operators coincide with those obtained indirectly in [25] on the basis of the quasi-Hopf algebra structure of $B_{q,\lambda}(\hat{\mathfrak{sl}}_2)$. This indicates a consistency of our $H$-Hopf algebroid structure on $U_{q,p}(\hat{\mathfrak{sl}}_2)$ even in the case with non-zero central element. We hence establish the extension of the algebraic analysis scheme to the fusion RSOS
model on the basis of $U_{q,p}(\hat{\mathfrak{sl}}_2)$.

This paper is organized as follows. In Sect.2, we give a definition of the elliptic algebra $U_{q,p}(\hat{\mathfrak{sl}}_2)$ and review some results on the $RLL$ relation. In Sect.3, we recall the definition of $H$-Hopf algebroid from [12, 13, 29]. Then we define an $H$-Hopf algebroid structure on $U_{q,p}(\hat{\mathfrak{sl}}_2)$ and formulate it as an elliptic quantum group. In Sect.4, after a summary of basic facts on dynamical representations, we consider finite-dimensional representations of $U_{q,p}(\hat{\mathfrak{sl}}_2)$. In particular, we introduce an elliptic analogue of the Drinfeld polynomial and state a criterion for the finiteness of irreducible pseudo-highest weight representation of $U_q(\hat{\mathfrak{sl}}_2)$. We also investigate a submodule structure of the tensor product of two evaluation representations and derive an elliptic analogue of the Clebsch-Gordan coefficients. Sect.5 is devoted to a discussion on the trigonometric, non-affine and non-dynamical limits of the results. In Appendix A, we give a list of the commutation relations of the $L$ operator elements. Appendix B is devoted to a proof of Theorem 4.18.

2 The Elliptic Algebra $U_{q,p}(\hat{\mathfrak{sl}}_2)$

In this section, we give a definition of the elliptic algebra $U_{q,p}(\hat{\mathfrak{sl}}_2)$ in terms of the Drinfeld generators of the quantum affine algebra $U_q(\hat{\mathfrak{sl}}_2)$ and the Heisenberg algebra $\{P, c^Q\}$. We then recall some basic facts on $U_{q,p}(\hat{\mathfrak{sl}}_2)$ from [25, 34].

2.1 Quantum Affine Algebra $\mathbb{K}[U_q(\hat{\mathfrak{sl}}_2)]$

Throughout this paper we fix a complex number $q$ such that $q \neq 0$, $|q| < 1$.

Definition 2.1. [9] For a field $\mathbb{K}$, the quantum affine algebra $\mathbb{K}[U_q(\hat{\mathfrak{sl}}_2)]$ in the Drinfeld realization is an associative algebra over $\mathbb{K}$ generated by the standard Drinfeld generators $a_n$ ($n \in \mathbb{Z} \neq 0$), $x_n^\pm$ ($n \in \mathbb{Z}$), $h$, $c$ and $d$. The defining relations are given as follows.

\[
c: \text{ central ,}
\]
\[
[h, d] = 0, \quad [d, a_n] = na_n, \quad [d, x_n^\pm] = nx_n^\pm,
\]
\[
[h, a_n] = 0, \quad [h, x_n^\pm(z)] = \pm 2x_n^\pm(z),
\]
\[
[a_n, a_m] = \frac{[2n]_q [cn]_q}{n} q^{-c|n|} \delta_{n+m, 0},
\]
\[
[a_n, x^+(z)] = \frac{[2n]_q}{n} q^{-c|n|} z^n x^+(z),
\]
\[
[a_n, x^-(z)] = -\frac{[2n]_q}{n} z^n x^-(z),
\]
We set \[\bar{\Lambda} = \{\Lambda_1, \Lambda_2\},\]
\[\phi(q^{c/2}z) = q^{-h} \exp \left( (q - q^{-1}) \sum_{n > 0} a_n z^{-n} \right), \quad \varphi(q^{-c/2}z) = q^{-h} \exp \left( -(q - q^{-1}) \sum_{n > 0} a_n z^n \right).\]

We also denote by \(K[U_q(\hat{sl}_2)]\) the subalgebra of \(K[U_q(\hat{sl}_2)]\) generated by the same generators as \(K[U_q(\hat{sl}_2)]\) except for \(d\) excluded.

In the later sections, we use the symbols \(\psi_n\) and \(\phi_{-n}\) \((n \in \mathbb{Z}_{\geq 0})\) defined by
\[\psi(q^{c/2}z) = \sum_{n \geq 0} \psi_n z^{-n}, \quad \varphi(q^{c/2}z) = \sum_{n \geq 0} \phi_{-n} z^n.\]

Let \(\alpha_1\) and \(\bar{\Lambda}_1 = \frac{1}{2} \alpha_1\) be the simple root and the fundamental weight of \(\mathfrak{sl}(2, \mathbb{C})\), respectively. We set \(\mathfrak{h} = \mathbb{C}h\), \(Q = \mathbb{Z} \alpha_1\) and \(\mathfrak{h}^* = \mathbb{C} \bar{\Lambda}_1\). We denote by \(< , >\) the paring of \(\mathfrak{h}\) and \(\mathfrak{h}^*\) given by \(< \bar{\Lambda}_1, h > = 1.\)

### 2.2 Definition of the Elliptic Algebra \(U_{q,\alpha}(\hat{sl}_2)\)

Let \(r\) be a generic complex number. We set \(r^* = r - c\), \(p = q^{2r}\) and \(p^* = q^{2r^*}\). We define the Jacobi theta functions \([u]\) and \([u]^*\) by
\[\Theta_p(z) = (z; p)_\infty (p/z; p)_\infty (p; p)_\infty,\]
where
\[\sum_{n \geq 0} \psi_n z^{-n} = \frac{q^{\frac{1}{2}}}{(p, p)_\infty^3} \Theta_p(q^{2n}).\]

Setting \(p = e^{-\frac{2\pi i}{r^*}}, [u]\) satisfies the quasi-periodicity \([u + r] = -[u], [u + r\tau] = e^{-\pi i (2u/r + \tau)}[u].\)

Let \(\{P, Q\}\) be a Heisenberg algebra commuting with \(\mathbb{C}[U_q(\hat{sl}_2)]\) and satisfying
\[\{P, Q\} = -1.\] (2.1)

We set \(H = \mathbb{C} P \oplus \mathbb{C} r^*\) and \(H^* = \mathbb{C} Q \oplus \mathbb{C} \frac{\partial}{\partial r^*}\). We denote by the same symbol \(< , >\) as the above the pairing of \(H\) and \(H^*\) defined by
\[< Q, P > = 1 = \langle \frac{\partial}{\partial r^*}, r^* \rangle,\]
the others are zero. We regard \( H^* \oplus H \) as a Heisenberg algebra by

\[
[x, y] = \langle x, y \rangle.
\]

We also consider the Abelian group \( \bar{H}^* = \mathbb{Z}Q \). We have the isomorphism \( \phi : Q \to \bar{H}^* \) by \( n\alpha \mapsto nQ \). We denote by \( \mathbb{C}[\bar{H}^*] \) the group algebra over \( \mathbb{C} \) of \( \bar{H}^* \). We denote by \( e^\alpha \) the element of \( \mathbb{C}[\bar{H}^*] \) corresponding to \( \alpha \in \bar{H}^* \). These \( e^\alpha \) satisfy \( e^\alpha e^\beta = e^{\alpha + \beta} \) and \((e^\alpha)^{-1} = e^{-\alpha}\). In particular, \( e^0 = 1 \) is the identity element.

Now we take the power series field \( \mathbb{F} = \mathbb{C}(\!(P, r^*)\!) \) as \( \mathbb{K} \) and consider the semi-direct product \( \mathbb{C}\text{-}\text{algebra} \ U_{q,p}(\hat{sl}_2) = \mathbb{F}[U_q(\hat{sl}_2)] \otimes_{\mathbb{C}} \mathbb{C}[\bar{H}^*] \) of \( \mathbb{F}[U_q(\hat{sl}_2)] \) and \( \mathbb{C}[\bar{H}^*] \). We impose the following relation. For \( \alpha \in \bar{H}^* \),

\[
e^\alpha f(P, r^*)e^{-\alpha} = f(P + <\alpha, P >, r^*).
\]

Then the multiplication of \( U_{q,p}(\hat{sl}_2) \) is defined by

\[
(f(P, r^*)a \otimes e^\alpha) \cdot (g(P, r^*)b \otimes e^\beta) = f(P, r^*)g(P + <\alpha, P >, r^*)ab \otimes e^{\alpha + \beta},
\]

\[
a, b \in \mathbb{C}[U_q(\hat{sl}_2)], \ f(P, r^*), g(P, r^*) \in \mathbb{F}, \ \alpha, \beta \in \bar{H}^*.
\]

Moreover, for \( f(P, r^*) \in \mathbb{F} \) we regard the object \( f(P + h, r^* + c) \) as the element of \( U_{q,p}(\hat{sl}_2) \) in the sense of completion. We then have the following relations.

\[
x^\pm(z)f(P + h, r^* + c) = f(P + h \mp 2, r^* + c)x^\pm(z), \quad (2.3)
\]

\[
[f(P + h, r^* + c), a_n] = 0, \quad [f(P + h, r^* + c), d] = 0. \quad (2.4)
\]

**Remark.** The relation (2.2) is automatically satisfied, if one takes the realization \( Q = \frac{\partial}{\partial P} \).

The following automorphism \( \phi_r \) of \( \mathbb{F}[U_q(\hat{sl}_2)] \) is the key to our “elliptic deformation” [25].

\[
c \mapsto c, \quad h \mapsto h, \quad d \mapsto d,
\]

\[
x^+(z) \mapsto u^+(z, p)x^+(z), \quad x^-(z) \mapsto x^-(z)u^-(z, p),
\]

\[
\psi(z) \mapsto u^+(q^{c/2}z, p)\psi(z)u^-(q^{-c/2}z, p),
\]

\[
\varphi(z) \mapsto u^+(q^{-c/2}z, p)\varphi(z)u^-(q^{c/2}z, p).
\]

Here we set

\[
u^+(z, p) = \exp \left( \sum_{n>0} \frac{1}{[r^n]_q}a_{-n}(q^rz)^n \right), \quad u^-(z, p) = \exp \left( -\sum_{n>0} \frac{1}{[r^n]_q}a_n(q^{r}z)^{-n} \right). \quad (2.5)
\]

We define the elliptic currents \( E(u), F(u) \) and \( K(u) \) in \( U_{q,p}(\hat{sl}_2)[[u]] \) as follows.
Definition 2.2 (Elliptic currents).

\[ E(u) = \phi_r(x^+(z))e^{2Qz}z^{-\frac{p+1}{r}}, \]
\[ F(u) = \phi_r(x^-(z))z^{\frac{p-h-1}{r}}, \]
\[ K(z) = \exp\left(\sum_{n>0} \frac{[n]_q}{2[n]_q[r^n]_q}a_n(q^n)^{z^n}\right) \exp\left(-\sum_{n>0} \frac{[n]_q}{2[n]_q[r^n]_q}a_nz^n\right) \]
\[ \times e^{Qz\frac{r}{4\pi r}(2P-1)+\frac{1}{4r^2}} \]
\[ \dot{d} = d - \frac{1}{4r^2}(P-1)(P+1) + \frac{1}{4r}(P+h-1)(P+h+1), \]

where we set \( z = q^{2u}, p = q^{2r}. \)

From (2.1) and Definition 2.1 we can derive the following relations.

Proposition 2.3.

\( c : \) central,
\[ [h, a_n] = 0, \quad [h, E(u)] = 2E(u), \quad [h, F(u)] = -2F(u), \]
\[ [\dot{d}, h] = 0, \quad [\dot{d}, a_n] = na_n, \]
\[ [\dot{d}, E(u)] = \left(-z \frac{\partial}{\partial z} - \frac{1}{r}\right) E(u), \quad [\dot{d}, F(u)] = \left(-z \frac{\partial}{\partial z} - \frac{1}{r}\right) F(u), \]
\[ [a_n, a_m] = \frac{2n_q}{q} [c|n]_q q^{-c|n}_q \delta_{n+m,0}, \]
\[ [a_n, E(u)] = \frac{2n_q}{q} [c|n]_q z^n E(u), \quad [a_n, F(u)] = -\frac{2n_q}{q} z^n F(u), \]
\[ E(u)E(v) = \frac{(u-v+1)^n}{(u-v-1)^n} E(v)E(u), \quad F(u)F(v) = \frac{(u-v-1)^n}{(u-v+1)^n} F(v)F(u), \]
\[ [E(u), F(v)] = \frac{1}{q-q^{-1}} \left(\delta \left(q^{-c}\frac{z}{w}\right) H^+(q^{-c/2}w) - \delta \left(q^{c}\frac{z}{w}\right) H^-(q^{-c/2}w)\right), \]

where \( z = q^{2u}, w = q^{2v}, \) and we set
\[ \dot{H}^\pm(z) = \kappa K\left(u \pm \frac{1}{2}(r - \frac{c}{2}) + \frac{1}{2}\right) K\left(u \pm \frac{1}{2}(r - \frac{c}{2}) - \frac{1}{2}\right), \quad (2.6) \]
\[ \kappa = \lim_{z\to q^{-2}} \frac{\xi(z;p^*,q)}{\xi(z;p,q)}, \quad \xi(z;p,q) = \frac{(q^2z;p,q^1)\infty(pq^2z;p,q^4)\infty}{(q^1z;p,q^4)\infty(pz;p,q^4)\infty}. \]

Moreover from (2.2) and (2.3), we obtain the following relations.

Proposition 2.4. For \( f(P) \in \mathbb{C}(\langle P \rangle), \)
\[ K(u)f(P) = f(P+1)K(u), \quad E(u)f(P) = f(P+2)E(u), \quad [F(u), f(P)] = 0, \]
\[ K(u)f(P+h) = f(P+h+1)K(u), \quad [E(u), f(P+h)] = 0, \quad F(u)f(P+h) = f(P+h+2)F(u). \]
In Sect. 3.2, we use the $L^\psi$ currents $RLL$. We also denote by $U_{q,p}(\hat{\mathfrak{sl}_2})$ the elliptic algebra $U_{q,p}(\hat{\mathfrak{sl}_2})$. We call a set $(\mathbb{F}[U_q(\mathfrak{sl}_2)] \otimes_{\mathbb{C}} \mathbb{C}[\hat{H}^*], \phi_r)$ the elliptic algebra $U_{q,p}(\hat{\mathfrak{sl}_2})$. We also denote by $U_{q,p}(\hat{\mathfrak{sl}_2})$ the subalgebra $\mathbb{F}[U_q(\mathfrak{sl}_2)] \otimes_{\mathbb{C}} \mathbb{C}[\hat{H}^*]$ of $U_{q,p}(\hat{\mathfrak{sl}_2})$.

The following relations are also useful.

**Proposition 2.6.**

$$K(u)K(v) = \rho(u-v)K(v)K(u),$$

$$K(u)E(v) = \frac{[u-v + \frac{1-r^*}{2}]}{[u-v + \frac{1+r^*}{2}]} E(v)K(u),$$

$$K(u)F(v) = \frac{[u-v + \frac{1+r^*}{2}]}{[u-v + \frac{1-r^*}{2}]} F(v)K(u),$$

$$H^+(u)H^-(v) = \frac{[u-v - 1 - \frac{s}{2}]}{[u-v + 1 + \frac{s}{2}]} [u-v + 1 + \frac{s}{2}]^* H^-(v)H^+(u),$$

$$H^\pm(u)H^\pm(v) = \frac{[u-v - 1]}{[u-v + 1]} [u-v + 1]^* H^\pm(v)H^\pm(u),$$

where

$$\rho(u) = \frac{\rho^{++}(u)}{\rho^{+}(u)}, \quad \rho^{++}(u) = \rho^{+}(u)|_{r\rightarrow r^*},$$

$$\rho^{+}(u) = z^{\frac{1}{2}} \frac{pq^2z}{\{pz\}\{pq^2z\}} \frac{\{z^{-1}\}\{q^4z^{-1}\}}{\{q^2z^{-1}\}^2}, \quad \{z\} = (z;p,q^4)_\infty.$$ 

Note that the function $\rho(u)$ satisfies

$$\rho(0) = 1, \quad \rho(1) = \frac{[1]^*}{[1]}, \quad \rho(u)\rho(-u) = 1, \quad \rho(u)\rho(u+1) = \frac{[u+1]^*}{[u]^*} \frac{[u]}{[u+1]}.$$ 

In addition, the following formulae indicate a direct construction of $H^\pm(u)$ from the Drinfeld currents $\psi(z)$ and $\varphi(z)$.

**Proposition 2.7.**

$$H^+(u) = \phi_r(\psi(z))e^{2Q}(q^r-c/2z)^{-\frac{r^*(P-1)+\frac{r}{2}}{r^*}},$$

$$H^-(u) = \phi_r(\varphi(z))e^{2Q}(q^{-r+c/2}z)^{-\frac{r^*(P-1)+\frac{r}{2}}{r^*}}.$$ 

### 2.3 The $RLL$-relation for $U_{q,p}(\hat{\mathfrak{sl}_2})$

Following [25], we summarize the results on the $L$ operator and the $RLL$-relation for $U_{q,p}(\hat{\mathfrak{sl}_2})$. In Sect. 3.2 we use the $L$ operator to define the $H$-Hopf algebraic structure of $U_{q,p}(\hat{\mathfrak{sl}_2})$.

We first define the half currents $E^+(u), F^+(u)$ and $K^+(u)$ as follows.
Definition 2.8 (Half currents).

\[
K^+(u) = K(u + \frac{1}{2}), \tag{2.7}
\]

\[
E^+(u) = a^* \oint_{C^{*}} E(u') \frac{[u - u' + c/2 - P + 1][1]}{[u - u' + c/2][P - 1]} \frac{dz'}{2\pi i z'}, \tag{2.8}
\]

\[
F^+(u) = a \oint_C F(u') \frac{[u - u' + P + h - 1][1]}{[u - u'][P + h - 1]} \frac{dz'}{2\pi i z'}. \tag{2.9}
\]

Here the contours are chosen such that

\[C^*: \ |p^*q^c z| < |z'| < |q^c z|, \quad C: \ |pz| < |z'| < |z|,
\]

and the constants \(a, a^*\) are chosen to satisfy

\[
a^*a[1]^*\kappa = 1. \quad q - q^{-1} = 1.
\]

The commutation relations for the elliptic currents in Propositions 2.3–2.6 yield the following relations for the half currents.

Proposition 2.9.

\[
K^+(u_1)K^+(u_2) = \rho(u)K^+(u_2)K^+(u_1), \tag{2.10}
\]

\[
K^+(u_1)E^+(u_2)K^+(u_1)^{-1} = E^+(u_2)\left[\frac{1 + u}{[u]}\right] - E^+(u_1)\left[\frac{P + u}{[u]}\right], \tag{2.11}
\]

\[
K^+(u_1)^{-1}F^+(u_2)K^+(u_1) = \left[\frac{1 + u}{[u]}\right] F^+(u_2) - \left[\frac{P + h - u}{[u]}\right] F^+(u_1), \tag{2.12}
\]

\[
\frac{[1 - u]^*[1]}{[u]} E^+(u_1)E^+(u_2) + \frac{[1 + u]^*[1]}{[u]} E^+(u_2)E^+(u_1)
\]

\[
= E^+(u_1)^2 \frac{[1]^*[P + 2 + u]}{[P - 2]^*[u]} + E^+(u_2)^2 \frac{[1]^*[P + 2 - u]}{[P - 2]^*[u]}, \tag{2.13}
\]

\[
\frac{[1 + u]}{[u]} F^+(u_1)F^+(u_2) + \frac{[1 - u]}{[u]} F^+(u_2)F^+(u_1)
\]

\[
= F^+(u_1)^2 \frac{[1]}{[P + h - 2]} \frac{[P + h - 2 - u]}{[u]} + F^+(u_2)^2 \frac{[1]}{[P + h - 2]} \frac{[P + h - 2 + u]}{[u]}, \tag{2.14}
\]

\[
[E^+(u_1), F^+(u_2)] = K^+(u_2 - 1)K^+(u_2)\left[\frac{P - 1 - u}{[u]}\right] - K^+(u_1)K^+(u_1 - 1)\left[\frac{1}{[u]}\right], \tag{2.15}
\]

where we set \(u = u_1 - u_2\).

We next define the \(L\)-operator \(\hat{L}^+(u) \in \text{End}(V) \otimes U_{q,p}(\hat{sl}_2)\) with \(V \cong \mathbb{C}^2\) as follows.
Definition 2.10 (L-operator).

\[
\hat{L}^+(u) = \begin{pmatrix}
1 & F^+(u) \\
0 & 1
\end{pmatrix} \begin{pmatrix}
K^+(u - 1) & 0 \\
0 & K^+(u)^{-1}
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
E^+(u) & 1
\end{pmatrix}.
\]  \hspace{1cm} (2.16)

Then the relations in Proposition 2.9 can be combined into the following single RLL relation.

Proposition 2.11. The \(\hat{L}^+(u)\) operator satisfies the following RLL relation.

\[
R^+(12)(u_1 - u_2, P + \hbar) \hat{L}^+(1)(u_1) \hat{L}^+(2)(u_2) = \hat{L}^+(2)(u_2) \hat{L}^+(1)(u_1) R^+*^{(12)}(u_1 - u_2, P),
\]  \hspace{1cm} (2.17)

where \(R^+(u, P + \hbar)\) and \(R^+*^*(u, P) = R^+(u, P)|_{r \to \ast r}\) denote the elliptic dynamical \(R\) matrices given by

\[
R^+(u, s) = \rho^+(u) \begin{pmatrix}
1 & b(u, s) & c(u, s) \\
c(u, s) & \bar{b}(u, s) & 1
\end{pmatrix}
\]  \hspace{1cm} (2.18)

with \(\rho^+(u)\) in Proposition 2.6 and

\[
b(u, s) = \frac{[s + 1][s - 1]}{[s]^2} \frac{[u]}{[1 + u]}, \quad c(u, s) = \frac{[1]}{[s]} \frac{[s + u]}{[1 + u]},
\]

\[
\bar{c}(u, s) = \frac{[1]}{[s]} \frac{[s - u]}{[1 + u]}, \quad \bar{b}(u, s) = \frac{[u]}{[1 + u]}. \]

One should note that the \(c = 0\) case of the RLL relation (2.17) is identical, up to a gauge transformation, with the one studied in the formulation of Felder’s elliptic quantum group in [12, 30, 51].

2.4 Connection to the Quasi-Hopf Algebra \(B_{q,\lambda}(\hat{sl}_2)\)

It is worth to remark a connection of \(U_{q,p}(\hat{sl}_2)\) to the quasi-Hopf algebra \(B_{q,\lambda}(\hat{sl}_2)\).

Let us define a new \(L\) operator \(L^+(u, P)\) by

\[
L^+(u, P) = \hat{L}^+(u) \begin{pmatrix}
e^{-Q} & 0 \\
0 & e^Q
\end{pmatrix}.
\]  \hspace{1cm} (2.19)

Then from Definitions 2.2 and 2.10 one finds that \(L^+(u, P)\) is independent of \(Q\). We hence regard \(L^+(u, P)\) as the operator in \(\mathbb{F}[U_q(\hat{sl}_2)]\) having \(P\) as a parameter. Substituting (2.19) into (2.17), we obtain the following statement.
Proposition 2.12. The operator $L^+(u, P)$ satisfies the following dynamical RLL relation.

\[
R^{+(12)}(u_1 - u_2, P + h)L^{+(1)}(u_1, P)L^{+(2)}(u_2, P + h^{(1)}) = L^{+(2)}(u_2, P)L^{+(1)}(u_1, P + h^{(2)})R^{+(12)}(u_1 - u_2, P).
\]

This $RLL$ relation is identified with the one for the quasi-Hopf algebra $B_{q,\lambda}(\hat{\mathfrak{sl}}_2)$ under the parametrization $\lambda = (r^* + 2)\Lambda_0 + (P + 1)\hat{\lambda}_1$ [24,25], where $\Lambda_0$ and $\Lambda_0 + \hat{\lambda}_1$ denote the fundamental weights of $\mathfrak{sl}(2, \mathbb{C})$. This is due to the fact that under this $\lambda$ the vector representation of the universal dynamical $R$ matrix $R^+(\lambda)$ of $B_{q,\lambda}(\hat{\mathfrak{sl}}_2)$ yields the elliptic dynamical $R$ matrix $R^{+(u, P)}$ [25,35]. Furthermore we have the isomorphism $B_{q,\lambda}(\hat{\mathfrak{sl}}_2) \cong F[U_q(\hat{\mathfrak{sl}}_2)]$ as an associative algebra. Combining these facts, we obtain the isomorphism $U_{q,p}(\hat{\mathfrak{sl}}_2) \cong B_{q,\lambda}(\hat{\mathfrak{sl}}_2) \otimes_{\mathbb{C}} \mathbb{C}[\hat{H}^*]$ with $\lambda = (r^* + 2)\Lambda_0 + (P + 1)\hat{\lambda}_1$ as a semi-direct product algebra.

Note also that the $c = 0$ case of (2.20) is identical to the one used in [16,17] to define Felder’s elliptic quantum group in its original form.

3 $H$-Hopf Algebroid

In this section, we introduce an $H$-Hopf algebroid structure into the elliptic algebra $U_{q,p}(\hat{\mathfrak{sl}}_2)$ and formulate $U_{q,p}(\hat{\mathfrak{sl}}_2)$ as an elliptic quantum group.

3.1 Definition of the $H$-Hopf Algebroid

Let us recall some basic facts on the $H$-Hopf algebroid following the works of Etingof and Varchenko [12,13] and of Koelink and Rosengren [29].

Let $A$ be a complex associative algebra, $H$ be a finite dimensional commutative subalgebra of $A$, and $M_{H^*}$ be the field of meromorphic functions on $H^*$ the dual space of $H$.

Definition 3.1 ($H$-algebra). An $H$-algebra is a complex associative algebra $A$ with 1, which is bigraded over $H^*$, $A = \bigoplus_{\alpha, \beta \in H^*} A_{\alpha \beta}$, and equipped with two algebra embeddings $\mu_l, \mu_r : M_{H^*} \to A_{00}$ (the left and right moment maps), such that

$$
\mu_l(\hat{f})a = a\mu_l(T_\alpha \hat{f}), \quad \mu_r(\hat{f})a = a\mu_r(T_\beta \hat{f}), \quad a \in A_{\alpha \beta}, \quad \hat{f} \in M_{H^*},
$$

where $T_\alpha$ denotes the automorphism $(T_\alpha \hat{f})(\lambda) = \hat{f}(\lambda + \alpha)$ of $M_{H^*}$.

Definition 3.2 ($H$-algebra homomorphism). An $H$-algebra homomorphism is an algebra homomorphism $\pi : A \to B$ between two $H$-algebras $A$ and $B$ preserving the bigrading and the moment maps, i.e. $\pi(A_{\alpha \beta}) \subseteq B_{\alpha \beta}$ for all $\alpha, \beta \in H^*$ and $\pi(\mu_i^A(\hat{f})) = \mu_i^B(\hat{f}), \pi(\mu_\alpha^A(\hat{f})) = \mu_\alpha^B(\hat{f})$. 
Let $A$ and $B$ be two $H$-algebras. The tensor product $A \tilde{\otimes} B$ is the $H^*$-bigraded vector space with
\[
(A \tilde{\otimes} B)_{\alpha\beta} = \bigoplus_{\gamma \in H^*} (A_{\alpha\gamma} \otimes_{M_{H^*}} B_{\beta\gamma}),
\]
where $\otimes_{M_{H^*}}$ denotes the usual tensor product modulo the following relation.
\[
\mu^A_r(\hat{f})a \otimes b = a \otimes \mu^B_r(\hat{f})b, \quad a \in A, b \in B, \hat{f} \in M_{H^*}. \tag{3.1}
\]
The tensor product $A \tilde{\otimes} B$ is again an $H$-algebra with the multiplication $(a \otimes b)(c \otimes d) = ac \otimes bd$ and the moment maps
\[
\mu^A_i \tilde{\otimes} B = \mu^A_i \otimes 1, \quad \mu^A \tilde{\otimes} B = 1 \otimes \mu^B_r.
\]

Let $\mathcal{D}$ be the algebra of automorphisms $M_{H^*} \rightarrow M_{H^*}$
\[
\mathcal{D} = \{ \sum_i \hat{f}_i \beta_i | \hat{f}_i \in M_{H^*}, \beta_i \in H^* \}. 
\]
Equipped with the bigrading $\mathcal{D}_{\alpha\beta} = \{ \hat{f} T_{\alpha} | \hat{f} \in M_{H^*}, \alpha \in H^* \}$, $\mathcal{D}_{\alpha\beta} = 0$ $(\alpha \neq \beta)$ and the moment maps $\mu^D_i, \mu^D_r : M_{H^*} \rightarrow \mathcal{D}_{00}$ defined by $\mu^D_i(\hat{f}) = \mu^D_r(\hat{f}) = \hat{f} T_{0}$, $\mathcal{D}$ is an $H$-algebra. For any $H$-algebra $A$, we have the canonical isomorphism as an $H$-algebra
\[
A \cong A \tilde{\otimes} \mathcal{D} \cong \mathcal{D} \tilde{\otimes} A \tag{3.2}
\]
by $a \cong a \tilde{\otimes} T_{-\alpha} \cong T_{-\alpha} \tilde{\otimes} a$ for all $a \in A_{\alpha\beta}$.

**Definition 3.3** ($H$-bialgebroid). An $H$-bialgebroid is an $H$-algebra $A$ equipped with two $H$-algebra homomorphisms $\Delta : A \rightarrow A \tilde{\otimes} A$ (the comultiplication) and $\varepsilon : A \rightarrow \mathcal{D}$ (the counit) such that
\[
(\Delta \tilde{\otimes} \text{id}) \circ \Delta = (\text{id} \tilde{\otimes} \Delta) \circ \Delta,
\]
\[
(\varepsilon \tilde{\otimes} \text{id}) \circ \Delta = \text{id} = (\text{id} \tilde{\otimes} \varepsilon) \circ \Delta,
\]
under the identification (3.2).

**Definition 3.4** ($H$-Hopf algebroid). An $H$-Hopf algebroid is an $H$-bialgebroid $A$ equipped with a $C$-linear map $S : A \rightarrow A$ (the antipode), such that
\[
S(\mu_r(\hat{f})a) = S(a)\mu_l(\hat{f}), \quad S(a\mu_l(\hat{f})) = \mu_r(\hat{f})S(a), \quad \forall a \in A, \hat{f} \in M_{H^*},
\]
\[
m \circ (\text{id} \tilde{\otimes} S) \circ \Delta(a) = \mu_l(\varepsilon(a)1), \quad \forall a \in A,
\]
\[
m \circ (S \tilde{\otimes} \text{id}) \circ \Delta(a) = \mu_r(T_{\alpha}(\varepsilon(a)1)), \quad \forall a \in A_{\alpha\beta},
\]
where $m : A \tilde{\otimes} A \rightarrow A$ denotes the multiplication and $\varepsilon(a)1$ is the result of applying the difference operator $\varepsilon(a)$ to the constant function $1 \in M_{H^*}$.
Remark. [29] Definition 3.4 yields that the antipode of an $H$-Hopf algebroid uniquely exists and gives the algebra antihomomorphism.

The $H$-algebra $D$ is an $H$-Hopf algebroid with $\Delta_D : D \to D \otimes D$, $\varepsilon_D : D \to D$, $S_D : D \to D$ defined by

$$
\Delta_D(\hat{f}T_\alpha) = \hat{f}T_\alpha \tilde{T}_\alpha,
$$
$$
\varepsilon_D = \text{id},
$$
$$
S_D(\hat{f}T_\alpha) = T_\alpha \hat{f} = (T_\alpha \hat{f})T_\alpha.
$$

3.2 $H$-Hopf Algebroid Structure on $U_{q,p}(\widehat{sl}_2)$

Now let us consider the elliptic algebra $U_{q,p}(\widehat{sl}_2)$. Using the isomorphism $\phi : Q \to \bar{H}^*$, we define the $\bar{H}^*$-bigrading structure of $U_{q,p} = U_{q,p}(\widehat{sl}_2)$ as follows.

$$
U_{q,p} = \bigoplus_{\alpha, \beta \in \bar{H}^*} (U_{q,p})_{\alpha \beta},
$$

$$(U_{q,p})_{\alpha \beta} = \left\{ x \in U_{q,p} \mid q^h x q^{-h} = q^{<\phi^{-1}(\alpha - \beta),h>}, q^p x q^{-p} = q^{<\beta,P>}x \right\}. \tag{3.3}
$$

Noting $<\phi^{-1}(\alpha), h> = <\alpha, P>$, we have

$$
q^{P+h}xq^{-(P+h)} = q^{<\alpha,P>\ x} \tag{3.4}
$$
for $x \in (U_{q,p})_{\alpha \beta}$.

Remark. The quantum affine algebra $U_q = \mathbb{F}[U_q(\widehat{sl}_2)]$ has the following natural grading over $\bar{H}^*$.

$$
U_q = \bigoplus_{\alpha \in \bar{H}^*} (U_q)_\alpha,
$$

$$(U_q)_\alpha = \left\{ x \in U_q \mid q^h x q^{-h} = q^{<\phi^{-1}(\alpha),h>\ x} \right\}.
$$

We then have

$$(U_{q,p})_{\alpha \beta} = (U_q)_{\alpha - \beta} \otimes \mathbb{C} e^{-\beta}.
$$

Next let us regard the elements $\hat{f} = f(P, r^*) \in \mathbb{F}$ as meromorphic functions on $H^*$ by

$$
\hat{f}(\mu) = f(<\mu, P>, <\mu, r^*>), \quad \mu \in H^*
$$
and consider the field of meromorphic functions $M_{H^*}$ on $H^*$

$$
M_{H^*} = \left\{ \hat{f} : H^* \to \mathbb{C} \mid \hat{f} = f(P, r^*) \in \mathbb{F} \right\}.
$$

We define two embeddings (the left and right moment maps) $\mu_l, \mu_r : M_{H^*} \to (U_{q,p})_{00}$ by

$$
\mu_l(\hat{f}) = f(P + h, r^* + c), \quad \mu_r(\hat{f}) = f(P, r^*). \tag{3.5}
$$

From (2.2) and (2.3), one can verify the following.
Proposition 3.5. For \( x \in (U_{q,p})_{\alpha\beta} \), we have

\[
\mu_l(\hat{f})x = f(P + h, r^*)x = xf(P + h + \alpha, P), \quad \mu_r(\hat{f})x = f(P + \beta, P), \quad r^* = x\mu_l(T_\alpha \hat{f}),
\]

where we regard \( T_\alpha = e^\alpha \in \mathbb{C}[\bar{H}^*] \) as the shift operator \( M_{H^*} \to M_{H^*} \)

\[
(T_\alpha \hat{f}) = e^\alpha f(P, r^*) e^{-\alpha} = f(P + \alpha, P), \quad r^*.
\]

Hereafter we abbreviate \( f(P + h, r^* + c) \) and \( f(P, r^*) \) as \( f(P + h) \) and \( f^*(P) \), respectively.

An important example of the elements in \( M_{H^*} \) is the elliptic dynamical \( R \) matrix elements

\[
(\hat{R}_u^{\pm})_{\epsilon_1 \epsilon_2}^{\epsilon_1' \epsilon_2'} = R^{\pm}(u, P)_{\epsilon_1 \epsilon_2}^{\epsilon_1' \epsilon_2'} \text{ in } \{2, 3\}, \quad \text{where } \epsilon_i, \epsilon_i' = +, - (i = 1, 2). \quad \text{We then have}
\]

\[
\mu_l((\hat{R}_u^{\pm})_{\epsilon_1 \epsilon_2}^{\epsilon_1' \epsilon_2'}) = R^{\pm}(u, P + h)_{\epsilon_1 \epsilon_2}^{\epsilon_1' \epsilon_2'}, \quad \mu_r((\hat{R}_u^{\pm})_{\epsilon_1 \epsilon_2}^{\epsilon_1' \epsilon_2'}) = R^{\pm}(u, P)_{\epsilon_1 \epsilon_2}^{\epsilon_1' \epsilon_2'}
\]

in the abbreviate notation.

Equipped with the bigrading structure \( \{3, 3\} \) and two moment maps \( \{3, 3\} \), the elliptic algebra \( U_{q,p}(\hat{sl}_2) \) is an \( H \)-algebra.

We also consider the \( H \)-algebra of the shift operators

\[
\mathcal{D} = \{ \sum_i \hat{f}_i T_{\alpha_i} | \hat{f}_i \in M_{H^*}, \alpha_i \in \bar{H}^* \},
\]

\[
\mathcal{D}_{\alpha_\alpha} = \{ \hat{f} T_{-\alpha} \}, \quad \mathcal{D}_{\alpha_\beta} = 0 (\alpha \neq \beta),
\]

\[
\mu_l^\mathcal{D}(\hat{f}) = \mu_r^\mathcal{D}(\hat{f}) = \hat{f} T_0 \quad \hat{f} \in M_{H^*}.
\]

Then we have the \( H \)-algebra isomorphism \( U_{q,p} \cong U_{q,p} \otimes \mathcal{D} \cong \mathcal{D} \otimes U_{q,p} \).

Now let us consider the \( H \)-Hopf algebroid structure on \( U_{q,p} \). It is conveniently given by the \( L \) operator \( \hat{L}^+(u) \). We shall write the entries of \( \hat{L}^+(u) \) as

\[
\hat{L}^+(u) = \begin{pmatrix} \hat{L}^+_{\pm\pm}(u) & \hat{L}^+_{\pm-}(u) \\ \hat{L}^+_{-\pm}(u) & \hat{L}^+_{-\pm}(u) \end{pmatrix}.
\]

According to the Gauß decomposition \( \{2, 16\} \), we have

\[
\hat{L}^+_{\pm\pm}(u) = K^+(u - 1) + F^+(u) K^+(u)^{-1} E^+(u), \quad \hat{L}^+_{\pm-}(u) = F^+(u) K^+(u)^{-1},
\]

\[
\hat{L}^+_{-\pm}(u) = K^+(u)^{-1} E^+(u), \quad \hat{L}^+_{-\pm}(u) = K^+(u)^{-1}.
\]

One finds

\[
\hat{L}^+_{\epsilon_1 \epsilon_2}(u) \in (U_{q,p})_{-\epsilon_1 Q, -\epsilon_2 Q}.
\]
It is also easy to check
\[
\begin{align*}
 f(P + h)\hat{L}^+_{\epsilon_1 \epsilon_2}(u) &= \hat{L}^+_{\epsilon_1 \epsilon_2}(u)f(P + h - \epsilon_1), \\
f^*(P)\hat{L}^+_{\epsilon_1 \epsilon_2}(u) &= \hat{L}^+_{\epsilon_1 \epsilon_2}(u)f^*(P - \epsilon_2).
\end{align*}
\]
(3.10)

We define two \(H\)-algebra homomorphisms, the co-unit \(\varepsilon : U_{q,p} \to \mathcal{D}\) and the co-multiplication \(\Delta : U_{q,p} \to U_{q,p}\hat{\otimes}U_{q,p}\) by
\[
\begin{align*}
\varepsilon(\hat{L}^+_{\epsilon_1 \epsilon_2}(u)) &= \delta_{\epsilon_1 \epsilon_2}T_{\epsilon_2}, \quad \varepsilon(e^Q) = e^Q, \quad (3.11) \\
\varepsilon(\mu_1(\hat{f})) &= \varepsilon(\mu_r(\hat{f})) = \hat{f}T_0, \quad (3.12) \\
\Delta(\hat{L}^+_{\epsilon_1 \epsilon_2}(u)) &= \sum_{\epsilon'} \hat{L}^+_{\epsilon_1 \epsilon'}(u)\otimes \hat{L}^+_{\epsilon' \epsilon_2}(u), \quad (3.13) \\
\Delta(e^Q) &= e^Q \otimes e^Q, \quad (3.14) \\
\Delta(\mu_1(\hat{f})) &= \mu_1(\hat{f})\otimes 1, \quad \Delta(\mu_r(\hat{f})) = 1\otimes \mu_r(\hat{f}). \quad (3.15)
\end{align*}
\]

In fact, one can check that \(\Delta\) preserves the relation (2.17). Noting (3.6) and the formula obtained from (3.1)
\[
f^*(u, P)a \otimes b = a \otimes f(u, P + h)b \quad a, b \in U_{q,p},
\]
we have
\[
\Delta(LHS) = \sum_{\epsilon_1', \epsilon_2'} \Delta(R^+(u, P + h)^{\epsilon_1 \epsilon_2}_{\epsilon_1' \epsilon_2'})\Delta(\hat{L}^+_{\epsilon_1 \epsilon_2}(u_1))\Delta(\hat{L}^+_{\epsilon_2 \epsilon_2'}(u_2))
\]
\[
= \sum_{\epsilon_1', \epsilon_2'} R^+(u, P + h)^{\epsilon_1 \epsilon_2}_{\epsilon_1' \epsilon_2'}\hat{L}^+_{\epsilon_1 \epsilon_2}(u_1)\hat{L}^+_{\epsilon_2 \epsilon_2'}(u_2)\hat{L}^+_{\epsilon_2 \epsilon_2'}(u_1)\hat{L}^+_{\epsilon_2 \epsilon_2'}(u_2)
\]
\[
= \sum_{\epsilon_1', \epsilon_2'} \hat{L}^+_{\epsilon_1 \epsilon_2}(u_2)\hat{L}^+_{\epsilon_1 \epsilon_2}(u_1)R^+(u, P)^{\epsilon_1 \epsilon_2}_{\epsilon_1' \epsilon_2'}\hat{L}^+_{\epsilon_2 \epsilon_2'}(u_1)\hat{L}^+_{\epsilon_2 \epsilon_2'}(u_2) + \hat{L}^+_{\epsilon_1 \epsilon_2}(u_2)\hat{L}^+_{\epsilon_1 \epsilon_2}(u_1)R^+(u, P)^{\epsilon_1 \epsilon_2}_{\epsilon_1' \epsilon_2'}\hat{L}^+_{\epsilon_2 \epsilon_2'}(u_1)\hat{L}^+_{\epsilon_2 \epsilon_2'}(u_2)
\]
\[
= \sum_{\epsilon_1', \epsilon_2'} R^+(u, P)^{\epsilon_1 \epsilon_2}_{\epsilon_1' \epsilon_2'}\hat{L}^+_{\epsilon_1 \epsilon_2}(u_2)\hat{L}^+_{\epsilon_1 \epsilon_2}(u_1)\hat{L}^+_{\epsilon_2 \epsilon_2'}(u_1)\hat{L}^+_{\epsilon_2 \epsilon_2'}(u_2) + \hat{L}^+_{\epsilon_1 \epsilon_2}(u_2)\hat{L}^+_{\epsilon_1 \epsilon_2}(u_1)R^+(u, P)^{\epsilon_1 \epsilon_2}_{\epsilon_1' \epsilon_2'}\hat{L}^+_{\epsilon_2 \epsilon_2'}(u_1)\hat{L}^+_{\epsilon_2 \epsilon_2'}(u_2)
\]
\[
= \sum_{\epsilon_1', \epsilon_2'} \hat{L}^+_{\epsilon_1 \epsilon_2}(u_2)\hat{L}^+_{\epsilon_1 \epsilon_2}(u_1)R^+(u, P)^{\epsilon_1 \epsilon_2}_{\epsilon_1' \epsilon_2'}\hat{L}^+_{\epsilon_2 \epsilon_2'}(u_1)\hat{L}^+_{\epsilon_2 \epsilon_2'}(u_2)
\]
\[
= \Delta(RHS).
\]

In the fourth line, we used the property
\[
R^+(u, P + \epsilon_1' + \epsilon_2')^{\epsilon_1 \epsilon_2}_{\epsilon_1' \epsilon_2'} = R^+(u, P)^{\epsilon_1 \epsilon_2}_{\epsilon_1' \epsilon_2'}
\]
Lemma 3.6. The maps $\varepsilon$ and $\Delta$ satisfy

\[(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta, \quad (3.17)\]
\[(\varepsilon \otimes \text{id}) \circ \Delta = \text{id} = (\text{id} \otimes \varepsilon) \circ \Delta. \quad (3.18)\]

Proof. Straight forward. \(\square\)

We also have the following formulae.

Proposition 3.7.

\[
\varepsilon(q^h) = \varepsilon(q^c) = T_0, \quad (3.19)
\]
\[
\Delta(q^h) = q^h \otimes q^h, \quad \Delta(q^c) = q^c \otimes q^c, \quad (3.20)
\]
\[
\Delta \left( \frac{f(P, r^*)}{f(P + h, r^* + c)} \right) = \frac{f(P, r^*)}{f(P + h, r^* + c)} \otimes \frac{f(P, r^*)}{f(P + h, r^* + c)}. \quad (3.21)
\]

Proof. (3.19) follows from (3.5) and (3.12), whereas (3.20) follows from (3.5), (3.15) and (3.1).

For example,

\[
\Delta(q^h) = \Delta(q^P + h - P) = \Delta(q^P + h) \Delta(q^-P) = q^P + h \otimes q^-P = q^h \otimes q^h.
\]

To show (3.21) we use (3.15) and (3.1) as

\[
\text{LHS} = \Delta(\mu_r(\hat{f})) \Delta(\mu_l(\hat{f})^{-1}) = \mu_l(\hat{f})^{-1} \otimes \mu_r(\hat{f}) = \frac{f(P, r^*)}{f(P + h, r^* + c)} \otimes f(P, r^*) = \text{RHS}.
\]

\(\square\)

We next define an algebra antihomomorphism (the antipode) $S: U_{q,p} \to U_{q,p}$ by

\[
S(\hat{L}^+_{++}(u)) = \hat{L}^-_{+-}(u - 1), \quad S(\hat{L}^+_{+-}(u)) = -\frac{[P + h + 1]}{[P + h]} \hat{L}^+_{+-}(u - 1),
\]
\[
S(\hat{L}^+_{+-}(u)) = -\frac{[P]^*}{[P + 1]^*} \hat{L}^+_{+-}(u - 1), \quad S(\hat{L}^+_{++}(u)) = \frac{[P + h + 1][P]^*}{[P + h][P + 1]^*} \hat{L}^+_{++}(u - 1),
\]
\[
S(e^Q) = e^{-Q}, \quad S(\mu_r(\hat{f})) = \mu_l(\hat{f}), \quad S(\mu_l(\hat{f})) = \mu_r(\hat{f}).
\]

Note that $S$ preserves the $RLL$ relation (2.17). To show this, we use the relations in Proposition 2.9. Furthermore we have the following Lemma.

Lemma 3.8. The map $S$ satisfies

\[
m \circ (\text{id} \otimes S) \circ \Delta(x) = \mu_l(\varepsilon(x))1, \quad \forall x \in U_{q,p},
\]
\[
m \circ (S \otimes \text{id}) \circ \Delta(x) = \mu_r(T_\alpha(\varepsilon(x))1), \quad \forall x \in (U_{q,p})_{\alpha\beta}.
\]
Proof. We prove the first relation for \( x = \hat{L}_{++}(u) \). The other is similar. Using (3.8) and (3.10),

\[
\begin{align*}
LHS &= \hat{L}_{++}(u)\hat{L}_{--}(u-1) - \hat{L}_{+-}(u)\hat{L}_{-+}(u-1) \frac{[P-1]^*}{[P]^*} \\
    &= (K^+(u-1) + F^+(u)K^+(u)^{-1}E^+(u))K^+(u-1)^{-1} \\
    &\quad - F^+(u)K^+(u)^{-1}K^+(u-1)^{-1}E^+(u-1) \frac{[P-1]^*}{[P]^*} \\
    &= 1 = \mu_1(\varepsilon(\hat{L}_{++}(u))1).
\end{align*}
\]

In the last line, we used the relation (2.11) with the replacement \( u_1 \mapsto u-1, u_2 \mapsto u \) and \( u \mapsto -1 \).

From Lemmas 3.6 and 3.8 we have

**Theorem 3.9.** The \( H \)-algebra \( U_{q,p}(\hat{sl}_2) \) equipped with \((\Delta, \varepsilon, S)\) is an \( H \)-Hopf algebroid.

**Definition 3.10.** We call the \( H \)-Hopf algebroid \((U_{q,p}(\hat{sl}_2), H, M_{H*}, \mu_1, \mu_r, \Delta, \varepsilon, S)\) the elliptic quantum group \( U_{q,p}(\hat{sl}_2) \).

We also use the following comultiplication formulae for the half currents.

**Proposition 3.11.**

\[
\begin{align*}
\Delta(K^+(u)) &= K^+(u)\hat{\otimes}K^+(u) + \sum_{j=1}^{\infty} (-)^j E^+(u)^j K^+(u)\hat{\otimes}K^+(u)F^+(u)^j, \\
\Delta(E^+(u)) &= 1\hat{\otimes}E^+(u) + E^+(u)\hat{\otimes}K^+(u)K^+(u-1) \\
    &\quad + \sum_{j=1}^{\infty} (-)^j E^+(u)^j K^+(u)F^+(u)^j K^+(u-1), \\
\Delta(F^+(u)) &= F^+(u)\hat{\otimes}1 + K^+(u-1)K^+(u)\hat{\otimes}F^+(u) \\
    &\quad + \sum_{j=1}^{\infty} (-)^j K^+(u-1)E^+(u)^j F^+(u)^j K^+(u)\hat{\otimes}F^+(u)^j+1, \\
\Delta(H^\pm(u)) &= H^\pm(u)\hat{\otimes}H^\pm(u) \\
    &\quad + \sum_{j=1}^{\infty} (-)^j \left\{ \kappa K^+(u+C_\pm)E^+(u+C_\pm-1)^j K^+(u+C_\pm-1)\hat{\otimes}H^+(u)F^+(u+C_\pm-1)^j \\
    &\quad + E^+(u+C_\pm)^j H^+(u)\hat{\otimes}K^+(u+C_\pm)F^+(u+C_\pm)^j K^+(u+C_\pm-1) \right\} \\
    &\quad + \sum_{i,j=1}^{\infty} (-)^{i+j} \kappa E^+(u+C_\pm)^i K^+(u+C_\pm)^j E^+(u+C_\pm-1)^j K^+(u+C_\pm-1) \\
    &\quad \hat{\otimes}K^+(u+C_\pm)^i F^+(u+C_\pm)^j K^+(u+C_\pm-1)^j F^+(u+C_\pm-1)^j, \\
\end{align*}
\]

where \( C_\pm = -\frac{\eta}{2} \pm \left( \frac{\eta}{2} - \frac{\eta}{4} \right) \).

**Proof.** Use (3.13), (3.8) and (2.6) as well as \( \Delta(\kappa) = \kappa\hat{\otimes}\kappa \) obtained from (3.21).
4 Finite-Dimensional Representations

In this section, we discuss representations of the elliptic algebra $U_{q,p}' = U_{q,p}'(\mathfrak{sl}_2)$. The main results are the criterion for the finiteness of irreducible representations Theorem 4.11 and the submodule structure of the tensor product of two evaluation representations Theorem 4.17.

For brevity, we denote the entries of $\hat{L}^+(u)$ by

$$\hat{L}^+(u) = \begin{pmatrix} \alpha(u) & \beta(u) \\ \gamma(u) & \delta(u) \end{pmatrix}.$$  

4.1 Dynamical Representations

We introduce the concept of dynamical representation, i.e. representation as $H$-algebras [12, 13, 29]. We follow the definition given in [29]. We then give a construction of dynamical representations of $U_{q,p}'$. 

Let us consider a vector space $\hat{\mathcal{V}}$ over $\mathbb{F}$, which is $\mathfrak{h}$-diagonalizable,

$$\hat{\mathcal{V}} = \bigoplus_{\mu \in \mathfrak{h}^*} \hat{\mathcal{V}}_\mu, \quad \hat{\mathcal{V}}_\mu = \{v \in V \mid q^\mathfrak{h}v = q^\mu v \ (\mathfrak{h} \in \mathfrak{h})\}.$$

Let us define the $H$-algebra $\mathcal{D}_{H,\hat{\mathcal{V}}}$ of the $\mathbb{C}$-linear operators on $\hat{\mathcal{V}}$ by

$$\mathcal{D}_{H,\hat{\mathcal{V}}} = \bigoplus_{\alpha,\beta \in H^*} (\mathcal{D}_{H,\hat{\mathcal{V}}})_{\alpha\beta},$$

$$(\mathcal{D}_{H,\hat{\mathcal{V}}})_{\alpha\beta} = \left\{ X \in \text{End}_\mathbb{C} \hat{\mathcal{V}} \bigg| X(f^*(P)v) = f^*(P - <\beta, P>)X(v), \right.$$  

$$X(\hat{\mathcal{V}}_\mu) \subseteq \hat{\mathcal{V}}_{\mu + \phi^{-1}(\alpha - \beta)}, \ v \in \hat{\mathcal{V}}, \ f^*(P) \in \mathbb{F} \bigg\}.$$  

for $v \in \hat{\mathcal{V}}_\mu$. We follow the abbreviation mentioned below Proposition 3.5.

**Definition 4.1** (Dynamical representation). A dynamical representation of $U_{q,p}'$ on $\hat{\mathcal{V}}$ is an $H$-algebra homomorphism $\hat{\pi} : U_{q,p}' \to \mathcal{D}_{H,\hat{\mathcal{V}}}$. The dimension of the dynamical representation $(\hat{\pi}, \hat{\mathcal{V}})$ is $\dim_{\mathbb{F}} \hat{\mathcal{V}}$.

Let $(\hat{\pi}_1, \hat{\mathcal{V}}_1, \hat{\mathcal{W}}_1)$, $(\hat{\pi}_2, \hat{\mathcal{V}}_2, \hat{\mathcal{W}}_2)$ be two dynamical representations of $U_{q,p}'$. We define the tensor product $\hat{\mathcal{V}} \hat{\otimes} \hat{\mathcal{W}}$ by

$$\hat{\mathcal{V}} \hat{\otimes} \hat{\mathcal{W}} = \bigoplus_{\alpha \in \mathfrak{h}^*} (\hat{\mathcal{V}} \hat{\otimes} \hat{\mathcal{W}})_\alpha, \quad (\hat{\mathcal{V}} \hat{\otimes} \hat{\mathcal{W}})_\alpha = \bigoplus_{\beta \in \mathfrak{h}^*} \hat{\mathcal{V}}_\beta \hat{\otimes}_{M_{H^*}} \hat{\mathcal{W}}_{\alpha - \beta},$$

where $\hat{\otimes}_{M_{H^*}}$ denotes the usual tensor product modulo the relation

$$f^*(P)v \otimes w = v \otimes f(P + \nu)w \quad (4.1)$$

for $v \in \hat{\mathcal{V}}_\mu$.
for \( w \in \mathcal{W}_\nu \). The action of the scalar \( f^*(P) \in \mathbb{F} \) on the tensor space \( \mathcal{V} \otimes \mathcal{W} \) is defined as follows.

\[
f^*(P).(v \otimes w) = \Delta(\mu_r(\hat{f}))(v \otimes w) = v \otimes f^*(P)w.
\]

We have a natural \( H \)-algebra embedding \( \theta_{V,W} : \mathcal{D}_{H,V} \otimes \mathcal{D}_{H,W} \rightarrow \mathcal{D}_{H,V \otimes W} \) by \( \mathcal{X}_V \otimes \mathcal{X}_W \in (\mathcal{D}_{H,V}) \alpha \otimes (\mathcal{D}_{H,W}) \beta \mapsto \mathcal{X}_V \otimes \mathcal{X}_W \in (\mathcal{D}_{H,V \otimes W}) \alpha \beta \). Hence \( \theta_{V,W} \circ (\hat{\pi}_V \otimes \hat{\pi}_W) \circ \Delta : U'_\mathbb{F} \rightarrow \mathcal{D}_{H,V \otimes W} \) gives a dynamical representation of \( U'_\mathbb{F} \) on \( \mathcal{V} \otimes \mathcal{W} \).

Now let us consider a construction of dynamical representations of \( U'_\mathbb{F} \). Let \( V \) be an \( \mathfrak{h} \)-diagonalizable vector space over \( \mathbb{F} \). Let \( V_Q \) be a vector space over \( \mathbb{C} \), on which an action of \( e^Q \) is defined appropriately. Two important examples of \( V_Q \) are \( V_Q = \mathbb{C} \mathbb{I} \) and \( V_Q = \oplus_{n \in \mathbb{Z}} \mathbb{C} e^{nQ} \), where \( \mathbb{I} \) denotes the vacuum state satisfying \( e^Q \mathbb{I} = 1 \). Let us consider the vector space \( \mathcal{V} = V \otimes \mathbb{C} V_Q \), on which the actions of \( f^*(P) \in \mathbb{F} \) and \( e^Q \) are defined as follows.

\[
f^*(P).(v \otimes \xi) = f^*(P)v \otimes \xi,
\]

\[
e^Q.(f^*(P)v \otimes \xi) = f^*(P + 1)v \otimes e^Q\xi
\]

for \( f^*(P)v \otimes \xi \in V \otimes V_Q \). The following theorem shows a construction of dynamical representations.

**Theorem 4.2.** Let \( V, V_Q \) and \( \mathcal{V} \) be as in the above. Let \( (\pi_V : \mathbb{F}[U'_\mathbb{F}] \rightarrow \text{End}_\mathbb{F} V, V) \) be a representation of \( \mathbb{F}[U'_\mathbb{F}] \). Define a map \( \hat{\pi}_V = \pi_V \otimes \text{id} : U'_{\mathbb{F},p} = \mathbb{F}[U'_\mathbb{F}] \otimes \mathbb{C}[\mathfrak{h}^*] \rightarrow \text{End}_\mathbb{C} \mathcal{V} \) by

\[
\hat{\pi}_V(E(u)) = \pi_V(\phi_r(x^+(z)))e^{2Q}z^{-\frac{P-1}{\tau'}}z,
\]

\[
\hat{\pi}_V(F(u)) = \pi_V(\phi_r(x^-(z)))z^{\frac{P+\tau'h}{\tau'}-1},
\]

\[
\hat{\pi}_V(K(u)) = \exp \left( \sum_{n > 0} \frac{[n]_q}{[2n]_q [\tau^n_q]_q} \pi_V(a_n)(q^n z)^n \right) \exp \left( -\sum_{n > 0} \frac{[n]_q}{[2n]_q [\tau^n_q]_q} \pi_V(a_n)z^{-n} \right)
\]

\[
\times e^{Qz^{-\frac{2P}{\tau'}(2P-1)+\frac{\tau'}{\tau} \pi_V(h)}}.
\]

Then \( (\hat{\pi}_V, \mathcal{V}) \) is a dynamical representation of \( U'_{\mathbb{F},p} \) on \( \mathcal{V} \).

Through this paper we consider the dynamical representations obtained in this way.

### 4.2 Pseudo-highest Weight Representations

We define the concept of pseudo-highest weight representations and write down some basic results on them. Most of them are parallel to the trigonometric [6] and the rational [5] cases.

We begin by stating an analogue of the Poincaré-Birkhoff-Witt theorem for \( U'_q \).
Definition 4.3. Let \( H \) (resp. \( N_\pm \)) be the subalgebras of \( \mathbb{F}[U'_q, \hat{\mathfrak{sl}}_2] \) generated by \( c, h \) and \( a_k \) \((k \in \mathbb{Z}_{\neq 0})\) (resp. by \( x_n^\pm \) \((n \in \mathbb{Z})\)).

From Proposition 3.1 in [6] and a standard normal ordering procedure on the Heisenberg algebra, we have the following.

Theorem 4.4.

\[
U'_q = (N_- \otimes H \otimes N_+) \otimes \mathbb{C}[\bar{H}^*].
\]

Here the last \( \otimes \) should be understood as the semi-direct product.

The following indicates a characteristic feature of the finite-dimensional irreducible dynamical representation of \( U'_q \).

Theorem 4.5. Every finite-dimensional irreducible dynamical representation \((\hat{\pi}_V, \hat{V} = V \otimes V_Q)\) of \( U'_q \) contains a non-zero vector of the form \( \hat{\Omega} = \Omega \otimes 1 \), \( \Omega \in V \) such that

1) \( x_n^+ \hat{\Omega} = 0 \quad \forall n \in \mathbb{Z}, \)

2) \( \hat{\Omega} \) is a simultaneous eigenvector for the elements of \( H \),

3) \( e^Q \cdot \hat{\Omega} = \hat{\Omega}, \)

4) \( \hat{V} = U'_q \cdot \hat{\Omega}. \)

Furthermore \( q^c \) acts as 1 or \(-1\) on \( \hat{V} \).

Proof. Note that for each \( k \in \mathbb{Z}, \mathbb{C}\{x_k^+, x_k^-, q^k c^k\} \cong U_q(\mathfrak{sl}_2) \) is a subalgebra of \( U'_q(\hat{\mathfrak{sl}}_2) \). Then, concerning the action of the \( \mathbb{F}[U'_q(\hat{\mathfrak{sl}}_2)] \) part, the existence of a vector \( \hat{\Omega}' = \Omega \otimes \xi \in \hat{V} = V \otimes V_Q \) satisfying 1) and 2) follows from Proposition 3.2 in [6].

There are two types of \( \Omega \), the one depending on \( P \) and the other not. The latter case is simple. \( e^Q \) acts on \( \hat{\Omega}' \) as \( e^Q \cdot \hat{\Omega}' = \Omega \otimes e^Q \xi \). The finiteness and irreducibility of \( \hat{V} \) imply the existence of a unique non-zero vector \( \xi \) such that \( e^Q \xi = C \xi \) with a complex number \( C \neq 0 \).

Redefining \( \frac{1}{C} e^Q \) as \( e^Q \), we identify \( \xi \) with 1.

For \( \Omega \) depending on \( P \), let us write the \( P \) dependence explicitly as \( \hat{\Omega}'(P) = \Omega(P) \otimes \xi \). \( e^Q \) acts on \( \hat{\Omega}'(P) \) as \( e^Q \cdot \hat{\Omega}'(P) = \Omega(P+1) \otimes e^Q \xi \). The finiteness of \( \hat{V} \) implies that a finite number of vectors in \( \{\Omega(P+n) \mid n \in \mathbb{Z}\} \) are \( \mathbb{F} \)-linearly independent. Setting \( \hat{\Omega} = \sum_{n \in \mathbb{Z}} \hat{\Omega}(P+n) \otimes \xi \), we have \( e^Q \cdot \hat{\Omega} = \sum_{n \in \mathbb{Z}} \hat{\Omega}(P+n) \otimes e^Q \xi \). Then the same argument as the first case leads to \( \xi = 1 \), and we obtain \( \hat{\Omega} \) satisfying 3).

In both cases, Theorem 4.4 yields \( \hat{V} = U'_q \cdot \hat{\Omega} \). As for the action of \( q^c \) on \( \hat{V} \), the statement follows from Corollary 3.2 in [6]. \( \square \)
Remark. An example of the vector $\hat{\Omega}'$ independent of $P$ is $v_0' \otimes 1$ in Theorem 4.13 whereas the one depending on $P$ is $v^{(s)}$ in Theorem 4.17.

**Definition 4.6 (Elliptic loop algebra).** The elliptic loop algebra $U_{q,p}(L(sl_2))$ is the quotient of $U_{q,p}(\hat{sl}_2)$ by the two sided ideal generated by $c$.

Note $U_{q,p}(L(sl_2)) \cong F[U_q(L(sl_2))] \otimes \mathbb{C}[H^\ast]$, where $F[U_q(L(sl_2))]$ denotes the quantum loop algebra obtained as the quotient of $F[U_q(\hat{sl}_2)]$ by the two sided ideal generated by $c$ [6]. Note also that $U_{q,p}(L(sl_2))$ is an $H$-Hopf algebroid with the same $\mu_l, \mu_r, \Delta, \varepsilon, S$ as $U_{q,p}(\hat{sl}_2)$. Furthermore the $RLL$ relation for $U_{q,p}(L(sl_2))$ is given by (2.17) with replacing $R^{++}(u, P)$ with $R^+(u, P)$. It is identified with the one for Felder’s elliptic quantum group studied in [12, 30, 51]. Hence the corresponding $\hat{L}^+(u)$ in (2.16) gives a realization of Felder’s elliptic quantum group in terms of $U_{q,p}(L(sl_2))$.

Hereafter we consider dynamical representations of $U_{q,p}(L(sl_2))$.

**Definition 4.7 (Pseudo-highest weight representation).** A dynamical representation $(\hat{\pi}_V, \hat{\mathcal{V}} = V \otimes V_Q)$ of $U_{q,p}(L(sl_2))$ is said to be pseudo-highest weight, if there exists a vector (pseudo-highest weight vector) $\hat{\Omega} \in \hat{\mathcal{V}}$ such that $e_\Omega \cdot \hat{\Omega} = \hat{\Omega}$ and

1) $x_n^+ \hat{\Omega} = 0 \ (n \in \mathbb{Z})$

2) $\psi_n \cdot \hat{\Omega} = d_n^+ \hat{\Omega}, \quad \phi_{-n} \cdot \hat{\Omega} = d_{-n}^- \hat{\Omega} \ (n \in \mathbb{Z}_{\geq 0})$,

3) $\hat{\mathcal{V}} = U_{q,p}(L(sl_2)) \cdot \hat{\Omega}$,

with some complex numbers $d_{\pm n}$ satisfying $d_0^+ d_0^- = 1$. We call the set $d = \{d_{\pm n}\}_{n \in \mathbb{Z}_{\geq 0}}$ the pseudo-highest weight.

We can state the equivalent conditions in terms of the matrix elements of $\hat{L}^+(u)$.

**Theorem 4.8.** For a vector $\hat{\Omega} \in \hat{\mathcal{V}}$ satisfying $e_\Omega \cdot \hat{\Omega} = \hat{\Omega}$, the conditions 1) and 2) in Definition 4.7 are equivalent to the following.

i) $\gamma(u) \cdot \hat{\Omega} = 0 \ \forall u$,

ii) $q^\lambda \cdot \hat{\Omega} = q^\lambda \hat{\Omega} \ \exists \lambda \in \mathbb{C}$,

$\alpha(u) \cdot \hat{\Omega} = A(u) \hat{\Omega}, \quad \delta(u) \hat{\Omega} = D(u) \hat{\Omega}$

with some meromorphic functions $A(u)$ and $D(u)$ satisfying $D(u - 1)^{-1} = A(u)$ and

$$A(u) = z \sum_{m \in \mathbb{Z}, n \in \mathbb{Z}_{\geq 0}} A_{m,n} z^m p^n \quad A_{m,n} \in \mathbb{C}, \quad z = q^{2u}, \quad p = q^{2r}.$$ (4.2)
Proof. We show that \( i \) and \( i \) yield 1) and 2). Let us define \( e_n (n \in \mathbb{Z}) \) by

\[
\phi_r (x^+(z)) = \sum_{n \in \mathbb{Z}} e_n z^{-n}.
\]

From (2.8), we have [25]

\[
E^+(u) = e^{2Q} a^* [1] \sum_{n \in \mathbb{Z}} e_n \frac{1}{1 - q^{2(p-1)}} p^n z^{-n - \frac{p-1}{r}}.
\]

Here we used the following formula.

\[
\frac{[u + s]}{[u][s]} = - \sum_{n \in \mathbb{Z}} \frac{1}{1 - q^{-2sp^n}}.
\]

Then it follows from (3.8) that \( i \) is equivalent to \( e_n \Omega = 0 \) for all \( n \in \mathbb{Z} \).

Furthermore from the definition of \( x^+ (z) \) and (2.5), we have

\[
e_n = \sum_{k \in \mathbb{Z}_{\geq 0}} p_k \left( \frac{a - q^{4}}{|q|^2} \right) x^+_{n+k}.
\]

Here \( p_k(\{\alpha_l\}) \) denotes the Schur polynomial defined by

\[
\exp \left\{ \sum_{n \in \mathbb{Z}_{> 0}} \alpha_n z^n \right\} = \sum_{k \in \mathbb{Z}_{\geq 0}} p_k(\{\alpha_l\}) z^k.
\]

\( p_k(\{\alpha_l\}) \) has the following expression.

\[
p_k(\{\alpha_l\}) = \sum_{m_1 + 2m_2 + \ldots + km_k = k} \frac{\alpha_1^{m_1} \ldots \alpha_k^{m_k}}{m_1! \ldots m_k!}.
\]

Expanding \( p_k \left( \frac{a - q^{4}}{|q|^2} \right) \) as a power series in \( p = q^{2r} \), it follows that the condition \( e_n \Omega = 0 \) for all \( n \in \mathbb{Z} \) is equivalent to \( x^+_n \Omega = 0 \) for all \( n \in \mathbb{Z} \).

Similarly applying \( i \), we can evaluate \( H^+(u) \Omega \) as follows.

\[
(q^r z)^A u^+(z,p) \psi(z) u^-(z,p) \Omega = A(u+1) A(u) \Omega.
\]

Here we used Proposition 2.7 in the LHS, and (2.6) and (3.8) in the RHS. Note that due to (4.2) fractional powers of \( z \) in the both hand sides cancel out each other. Expanding the both sides as a Laurent series in \( z \) and a power series in \( p \), one finds that \( a_k (k \in \mathbb{Z}_{\neq 0}) \) are simultaneously diagonalized on \( \Omega \) and their eigenvalues are determined by the coefficients of the series in the right hand side. \( \square \)

**Definition 4.9 (Verma module).** Let \( d = \{d_{\pm n}\}_{n \in \mathbb{Z}_{\geq 0}} \) be any sequence of complex numbers. The Verma module \( M(d) \) is the quotient of \( U_{q,p}(L(\mathfrak{sl}_2)) \) by the left ideal generated by \( \{x^+_k (k \in \mathbb{Z}) \}, \psi_n - d^+_n \cdot 1, \phi_n - d^-_n \cdot 1 (n \in \mathbb{Z}_{\geq 0}), e^Q - 1 \).
Proposition 4.10. The Verma module $M(d)$ is a pseudo-highest weight representation of pseudo-highest weight $d$. Every pseudo-highest weight representation with pseudo-highest weight $d$ is isomorphic to a quotient of $M(d)$. Moreover $M(d)$ has a unique maximal proper submodule $N(d)$, and up to isomorphism, $M(d)/N(d)$ is the unique irreducible pseudo-highest weight module of $U_{q,p}(L(sl_2))$.

4.3 Elliptic Analogue of the Drinfeld Polynomials

We now consider a classification of finite-dimensional irreducible dynamical representations of $U_{q,p}(L(sl_2))$. We introduce a natural elliptic analogue of the Drinfeld polynomials.

Theorem 4.11. The irreducible pseudo-highest weight dynamical representation $(\hat{\pi}_V, \hat{V})$ of $U_{q,p}(L(sl_2))$ is finite-dimensional if and only if there exists an entire and quasi-periodic function $P_V(u)$ such that

$$H^\pm(u)\hat{\Omega} = c_V \frac{P_V(u+1)}{P_V(u)}\hat{\Omega},$$

$$P_V(u+r) = (-)^{\deg P} P_V(u),$$

$$P_V(u+r\tau) = (-)^{\deg P} e^{-\pi i \sum_{j=1}^{\deg P} \left( \frac{2(u-\alpha_j)}{r} + \tau \right)} P_V(u).$$

Here $\hat{\Omega}$ denotes the pseudo-highest weight vector in $\hat{V}$, and $\tau = -\frac{2\pi i}{\log p}$. The symbol $c_V$ denotes a constant given by

$$c_V = q^{\deg P} \prod_{j=1}^{\deg P} a_j^\frac{1}{r},$$

where $\deg P$ is a number of zeros of $P_V(u)$ in the fundamental parallelogram $(1, \tau) (= \text{the degree of the Drinfeld polynomial} P(z) = \lim_{r \to \infty} P_V(u), z = q^{2u})$, and $a_j = q^{2\alpha_j}$ with $\alpha_j$ being a zero of $P_V(u)$ in the fundamental parallelogram. The function $P_V(u)$ is unique up to a scalar multiple.

Proof of the “only if” part. From Theorem [4.5], $\hat{V}$ has the pseudo-highest weight vector $\hat{\Omega}$. From Theorem 3.4 in [6], there exists the Drinfeld polynomial $P(z) = \lim_{r \to \infty} P_V(u), z = q^{2u}$, and $a_j = q^{2\alpha_j}$ with $\alpha_j$ being a zero of $P_V(u)$ in the fundamental parallelogram. The function $P_V(u)$ is unique up to a scalar multiple.

Then using Proposition 2.7 and the formulae

$$u^+(z,p) = \prod_{l=0}^{\infty} q^h \varphi(q^{c/2} q^{2r(l+1)} z), \quad u^-(z,p) = \prod_{l=0}^{\infty} q^{-h} \psi(q^{c/2} q^{-2r(l+1)} z),$$

Here the first and second equalities are in the sense of the power series in $z$ and $z^{-1}$, respectively.
we obtain

\[ H^+(u)\hat{\Omega} = \left(q^r z\right)^{\frac{h}{2}} \prod_{l=0}^{\infty} q^{h\varphi(q^{2r(l+1)}z)} \cdot \psi(z) \cdot \prod_{l=0}^{\infty} q^{-h\psi(q^{-2r(l+1)}z)}\hat{\Omega} \]

\[ = \left(q^r z\right)^{\frac{h}{2}} \prod_{l=1}^{\infty} q^{\deg P} \frac{P(q^{-2q^{-2rl}z^{-1}})}{P(q^{-2rl}z^{-1})} \prod_{l=0}^{\infty} q^{\deg P} \frac{P(q^{-2q^{-2rl}z^{-1}})}{P(q^{2rl}z^{-1})}\hat{\Omega}. \]

Supposing that the Drinfeld polynomial \( P(z) \) is factorized as \( P(z) = \prod_{j=1}^{\deg P}(1 - a_j z) \), we have

\[ H^+(u)\hat{\Omega} = \left(q^r z\right)^{\frac{h}{2}} q^{\deg P} \prod_{j=1}^{\deg P} \frac{\Theta_{q^{2r}}(a_j/q^2 z)}{\Theta_{q^{2r}}(a_j/q^2 z)}\hat{\Omega} \]

\[ = q^{\frac{r-1}{r}\deg P} \prod_{j=1}^{\deg P} a_j^{\frac{1}{r}} [u + 1 - a_j]/[u - a_j]\hat{\Omega}. \]

This is the desired result with \( P_V(u) = \prod_{j=1}^{\deg P}[u - a_j] \). The quasi-periodicity of \( P_V(u) \) follows from the one of the theta function \([u]\).

The proof of the “if” part is given in the next subsection.

Remark. We can take \( c_V = 1 \) by the gauge transformation given from (2.11) in [25]. An example is given in Corollary 4.15.

**Proposition 4.12.** Let \( \hat{V} \) and \( \hat{W} \) be finite dimensional dynamical representations of \( U_{q,p}(L(\mathfrak{sl}_2)) \) and assume that the tensor product \( \hat{V} \otimes \hat{W} \) is irreducible. Let \( P_V(u), P_W(u) \) and \( P_{V \otimes W}(u) \) be the entire quasi-periodic function associated to \( \hat{V}, \hat{W} \) and \( \hat{V} \otimes \hat{W} \) in Theorem 4.11. Then

\[ P_{V \otimes W}(u) = P_V(u)P_W(u). \]

**Proof.** The statement follows from the comultiplication formulae for the half currents in Proposition 3.11.

**4.4 Evaluation Representations**

We consider an elliptic analogue of the evaluation representation of \( U_q(L(\mathfrak{sl}_2)) \) [6,22]. This is an important example of the finite-dimensional irreducible dynamical representation of \( U_{q,p}(L(\mathfrak{sl}_2)) \). Some formulae presented here were essentially obtained in [25]. Corollary 4.15 and Proposition 4.16 are new.

Let us consider the \( l + 1 \)-dimensional evaluation representation \( (\pi_{l,w}, V^{(l)}_w) \) of \( \mathbb{F}[U_q(L(\mathfrak{sl}_2))] \).

Here \( V^{(l)} = \otimes_{m=0}^{l} v^l_m, \quad V^{(l)}_w = V^{(l)} \otimes \mathbb{C}[w, w^{-1}] \), and we define operators \( h, S_\pm \) on \( V^{(l)} \) by

\[ hv^l_m = (l - 2m)v^l_m, \quad S^\pm v^l_m = v^l_{m \mp 1}, \quad v^l_m = 0 \quad \text{for} \quad m < 0, \quad m > l. \]
The action of the Drinfeld generators on $V^{(l)}_w$ is given as follows.

\[
\pi_{l,w}(a_n) = \frac{w^n}{n} \frac{1}{q - q^{-1}} ((q^n + q^{-n}) q^{nh} - (q^{(l+1)n} + q^{-(l+1)n})) ,
\]

\[
\pi_{l,w}(x^\pm(z)) = S^\pm \left[ \frac{\pm h + l + 2}{2} \right] q \delta \left( q^{h \pm \frac{w}{z}} \right) .
\] (4.3)

Applying this to Proposition 4.2 and noting Definition 2.8, we obtain the following theorem.

**Theorem 4.13.** Let $\hat{V}^{(l)}(w) = V^{(l)}(w) \otimes \mathbb{C}1$ be the vector space, on which $e^Q$ acts as

\[
e^Q(f(P)v \otimes 1) = f(P + 1)v \otimes 1.
\]

The image of the half currents by the map $\pi_{l,w} = \pi_{l,w} \otimes \text{id}$ on $U_{q,p}(L(sl_2)) \cong \mathbb{F}[U_q(L(sl_2))] \otimes \mathbb{C}[\hat{H}^+]$ is given, up to fractional powers of $z, w$ and $q$, by

\[
\hat{\pi}_{l,w}(K^+(u)) = - \frac{\varphi_l(u - v)}{[u - v - \frac{h}{2}]} e^Q,
\]

\[
\hat{\pi}_{l,w}(E^+(u)) = - e^Q S^+ \frac{[u - v - \frac{h+1}{2} - P] [l + h + 2]}{[u - v - \frac{h+1}{2}] P} e^Q,
\]

\[
\hat{\pi}_{l,w}(F^+(u)) = S^+ \frac{[u - v + \frac{h+1}{2} + P] [l - h + 2]}{[u - v - \frac{h-1}{2}] P + h - 1} ,
\]

\[
\hat{\pi}_{l,w}(H^\pm(u)) = \frac{[u - v - \frac{h+1}{2}] [u - v + \frac{l+1}{2}]}{[u - v - \frac{h+1}{2}] [u - v - \frac{h-1}{2}]} e^{2Q} ,
\]

where $z = q^{2u}$, $w = q^{2v}$, and

\[
\varphi_l(u) = - z^{-\frac{1}{2}} \rho_{k,l}^+(z, p)^{-1} \frac{[u + l + 1]}{2} ,
\]

\[
\rho_{k,l}^+(z, p) = q^\frac{1}{2} \frac{\{ pq^{k-l-1} z \} \{ pq^{k-l+1} z \} \{ q^{k+l+2} / z \} \{ q^{-k-l+2} / z \}}{\{ pq^{k+l+2} z \} \{ pq^{k-l+1} z \} \{ q^{k+1+2} / z \} \{ q^{-k-l+2} / z \}}.
\]

Furthermore $(\hat{\pi}_{l,w}, \hat{V}^{(l)}(w))$ is the $l + 1$-dimensional irreducible dynamical representation of $U_{q,p}(L(sl_2))$ with the pseudo-highest weight vector $v_0^l \otimes 1$.

**Proof.** One can directly check that $\hat{\pi}_{l,w}(K^+(u)), \hat{\pi}_{l,w}(E^+(u))$ and $\hat{\pi}_{l,w}(F^+(u))$ satisfy the relations in Theorem 2.9. In the process, we use the formula

\[
\varphi_l(u) \varphi_l(u - 1) = [u - \frac{l+1}{2}] [u + \frac{l+1}{2}] .
\] (4.4)

\[
\square
\]

From Definition 2.10, we obtain the image of the matrix elements of $\hat{L}^+(u)$ as follows.
Theorem 4.14.

\[
\tilde{\pi}_{l,w}(\alpha(u)) = -\frac{u-v + \frac{h+1}{2}}{\varphi(l)(u-v)[P + h + 1]} \cdot \frac{e^{Q}(|P - \frac{l-h}{2}|) + \frac{h+2}{2}}{\varphi(l)(u-v)[P + h + 1]} e^{Q},
\]

\[
\tilde{\pi}_{l,w}(\beta(u)) = -S^{-1} \frac{u-v + \frac{h+1}{2}}{\varphi(l)(u-v)[P + h - 1]} e^{-Q},
\]

\[
\tilde{\pi}_{l,w}(\gamma(u)) = S^{+} \frac{u-v - \frac{h+1}{2}}{\varphi(l)(u-v)[P]} e^{Q},
\]

\[
\tilde{\pi}_{l,w}(\delta(u)) = -\frac{u-v - \frac{h+1}{2}}{\varphi(l)(u-v)} e^{-Q}.
\]

Corollary 4.15. The elliptic analogue of the Drinfeld polynomial associated to \(\bar{V}^{(1)}(q^{2\tau})\) is given by

\[
P_{l,v}(u) = [u-v - \frac{l-1}{2}][u-v - \frac{l-1}{2} + 1] \cdots [u-v + \frac{l-1}{2}].
\]

Proof. Noting \(\delta(u) = K^{+}(u)^{-1}\), from (2.6), (2.7) and Theorem 4.14 we obtain

\[
\tilde{\pi}_{l,w}(H^{\pm}(u))(v_{0}^{l} \otimes 1) = \frac{u-v + \frac{l+1}{2}}{[u-v - \frac{l-1}{2}]} \cdot v_{0}^{l} \otimes 1.
\]

Then the entireness and the quasi-periodicity of \(P_{l,v}(u)\) yield the desired result. \(\square\)

Note that the zeros of \(P_{l,v}(u)\) coincides with those of the Drinfeld polynomial corresponding to the evaluation representation \(V^{(1)}(q^{2\tau})\) of \(U_{q}(L(sl_{2}))\) modulo \(\mathbb{Z}r + \mathbb{Z}r\). Note also that we have no \(c_{V}\) factor in (4.3) due to the remark below Theorem 4.11.

Proof of the “if” part of Theorem 4.11. Let \(P_{V}(u)\) be any entire quasi-periodic function satisfying the conditions in the Theorem 4.11 and let its zeros in the fundamental parallelogram \((1, \tau)\) be \(\alpha_{1}, \ldots, \alpha_{r}\). From Theorem 4.8, \(P_{V}(u)\) determines the set of eigenvalues \(d\) of \(\psi_{k}\) and \(\phi_{-k}\) \((k \in \mathbb{Z}_{>0})\) uniquely. Consider the representation \(\bar{V} = \bar{V}^{(1)}(q^{2\alpha_{1}}) \otimes \cdots \otimes \bar{V}^{(1)}(q^{2\alpha_{1}})\). Let \(v_{0}^{l}(i) = v_{0}^{l} \otimes 1\) denote the pseudo-highest weight vector in \(\bar{V}^{(1)}(q^{2\alpha_{1}})\) and set \(\widehat{\Omega} = v_{0}^{l}(1) \otimes \cdots \otimes v_{0}^{l}(r)\). Then up to a scalar multiple, \(\widehat{\Omega}\) is a unique pseudo-highest weight vector such that \(q^{h} \cdot \widehat{\Omega} = q^{r} \cdot \widehat{\Omega}\). Let us consider the submodule \(\bar{V}' = U_{q,\bar{V}}(L(sl_{2})) \cdot \widehat{\Omega}\) of \(\bar{V}\). \(\bar{V}'\) has a unique maximal submodule \(\bar{V}''\). Then the quotient module \(\bar{V}' / \bar{V}''\) is irreducible, and from Corollary 4.15 and Theorem 4.12, \(\bar{V}' / \bar{V}''\) has the entire quasi-periodic function given by

\[
\tilde{\bar{P}}_{V}(u) = \prod_{j=1}^{r} [u - \alpha_{j}] - \alpha_{j}.
\]

\(\tilde{\bar{P}}_{V}(u)\) has the same quasi-periodicity and zeros as \(P_{V}(u)\). Hence \(\tilde{\bar{P}}_{V}(u)\) coincides with \(P_{V}(u)\) up to a scalar multiple. \(\square\)

The following Proposition indicates a consistency of our construction of \(\tilde{\pi}_{l,v}\) and the fusion construction of the dynamical \(R\) matrices (=face type Boltzmann weights).
Proposition 4.16. Let us define the matrix elements of \( \hat{\pi}_{l,w}(\hat{L}^+_{\epsilon_1 \epsilon_2}(u)) \) by
\[
\hat{\pi}_{l,w}(\hat{L}^+_{\epsilon_1 \epsilon_2}(u))v^l_m = \sum_{m' = 0}^l (\hat{L}^+_{\epsilon_1 \epsilon_2}(u))_{\mu_m' \mu_m} v^l_{m'},
\]
where \( \mu_m = l - 2m \). Then we have
\[
(\hat{L}^+_{\epsilon_1 \epsilon_2}(u))_{\mu_m' \mu_m} = R^+_{1l}(u - v, P)^{\epsilon_2 \mu_m}.
\]
Here \( R^+_{1l}(u - v, P) \) is the \( \mathcal{R} \) matrix from (C.17) in [25]. The case \( l = 1 \), \( R^+_{1l}(u - v, P) \) coincides with the image \((\pi_{1, z} \otimes \pi_{1, w})\) of the universal \( \mathcal{R}^+ \) \( (\lambda) [24] \) given in \( (2.18) \). The case \( l > 1 \), \( R^+_{1l}(u - v, P) \) coincides with the \( \mathcal{R} \) matrix obtained by fusing \( R^+_{11}(u - v, P) \) \( l \)-times. In particular the matrix element \( R^+_{1l}(u - v, P)^{\epsilon_\nu \mu'} \) is gauge equivalent to the fusion face weight \( W_{1l}(P + \epsilon', P + \epsilon' + \mu, P + \mu, P | u - v) \) from (4) in [7].

4.5 Tensor Product Representations

In this subsection, we investigate a submodule structure of the tensor product space \( \hat{V}^{(l_1)}(q^{2a}) \otimes \hat{V}^{(l_2)}(q^{2b}) \) and derive an elliptic analogue of the Clebsch-Gordan coefficients. We abbreviate the pseudo-highest weight vectors \( v^l_0 \otimes 1 \) and \( v^l_0 \otimes 1 \) of \( \hat{V}^{(l_1)}(q^{2a}) \) and \( \hat{V}^{(l_2)}(q^{2b}) \) as \( v^l_0 \) and \( v^l_0 \), respectively.

Theorem 4.17. There exists a vector \( v^{(s)} \in \hat{V}^{(l_1)}(q^{2a}) \otimes \hat{V}^{(l_2)}(q^{2b}) \) satisfying the conditions 1) \( \sim \) 3) in the below, if and only if \( b - a = \frac{l_1 + l_2 - 2s}{2} + 1 \) \( (s = 0, 1, \cdots, \min \{l_1, l_2\}) \).

1) \( q^h v^{(s)} = q^{l_1 + l_2 - 2s} v^{(s)} \) \( (h \in H) \),
2) \( \Delta(\gamma(u)).v^{(s)} = 0 \) \( \forall u \),
3) \( \Delta(\alpha(u)).v^{(s)} = A(u) v^{(s)} \), \( \Delta(\delta(u)).v^{(s)} = D(u) v^{(s)} \) \( \forall u \),

where
\[
A(u) = \frac{[u - a - \frac{l_1 + 1}{2}][u - a + \frac{l_1 + 1}{2}]}{\varphi_{l_1}(u - a) \varphi_{l_2}(u - b)},
\]
\[
D(u) = \frac{[u - a - \frac{l_1 - 1}{2} + s][u - a - \frac{l_1 - 1}{2} - l_2 + s - 1]}{\varphi_{l_1}(u - a) \varphi_{l_2}(u - b)}.
\]

Explicitly, the vector \( v^{(s)} \) is given by
\[
v^{(s)} = \sum_{m_1 = 0}^s C^s_{m_1}(P) v^l_{m_1} \otimes v^l_{s - m_1},
\]
\[
C^s_{m_1}(P) = C^s_0 \frac{[P - l_2 + s - m_1][l_2 - s + 1][m_1]}{[P + 1][s - m_1][l_1][m_1]}.
\]
Proof. We solve the conditions 1) ~ 3). The first condition yields

\[ v^{(s)} = \sum_{m_1=0}^{s} C_{m_1}^s (u, P) v^{l_1}_{m_1} \otimes v^{l_2}_{s-m_1} \]  

(4.7)

with unknown coefficients \( C_{m_1}^s (u, P) \).

By using 2) and (3.10), we obtain

\[
\Delta(\gamma(u))v^{(s)} = \sum_{m_1} \left\{ C_{m_1}^s (u, P + 1)\gamma(u)v^{l_1}_{m_1} \otimes \alpha(u)v^{l_2}_{s-m_1} + C_{m_1}^s (u, P - 1)\delta(u)v^{l_1}_{m_1} \otimes \gamma(u)v^{l_2}_{s-m_1} \right\} = 0.
\]

Apply Theorem 4.14 and move all the coefficients in the second tensor space to the first one by using the following formula obtained from (4.1)

\[
v^{s}f(u, P)v^{l_2}_{m_2} = v^{s}f(u, P + h - (l_2 - 2m_2))v^{l_2}_{m_2} = f(u, P - (l_2 - 2m_2))v^{s}v^{l_2}_{m_2}. 
\]

(4.8)

Here one should note \( f^s(u, P) = f(u, P) \), i.e. \( c = 0 \), in the evaluation representations. We thus obtain the following recursion relation.

\[
C_{m_1}^s (u, P) = -C_{m_1-1}^s (u, P - 2) \frac{[u - a - \frac{l_1 + 1}{2} + m_1][u - b + \frac{l_2 - 2s + 1}{2} + m_1 + 1 - P]}{[u - a - \frac{l_1 + 1}{2} + m_1 + 1 - P][u - b + \frac{l_2 - 2s + 1}{2} + m_1]} \times \frac{[l_2 - s + m_1][P + 1]}{[l_1 + 1 - m_1][P - l_2 + s - 1 - m_1][P + s - m_1]}.
\]

(4.9)

Then one finds that all the \( u \) dependent factors cancel out each other, if and only if \( b - a = \frac{l_1 + l_2 - 2s}{2} + 1 \) (\( s = 0, 1, \ldots, \min \{l_1, l_2\} \)). We hence obtain

\[
C_{m_1}^s (P) = -C_{m_1-1}^s (P - 2) \frac{[l_2 - s + m_1][P + 1]}{[l_1 + 1 - m_1][P - l_2 + s - 1 - m_1][P + s - m_1]}.
\]

(4.10)

Here we rewrote \( C_{m_1}^s (u, P) \) as \( C_{m_1}^s (P) \). Solving this, we obtain

\[
C_{m_1}^s (P) = C_0^s (P - 2m_1) \frac{[l_2 - s + 1]m_1[P - 2m_1 + 1]_{m_1}}{[-l_1]m_1[P - l_2 + s - 2m_1]m_1[P + s - 2m_1 + 1]_{m_1}}.
\]

(4.11)

Here \([u]_m\) denotes the elliptic shifted factorial

\[
[u]_m = [u][u+1] \cdots [u+m-1].
\]
Finally the third condition yields, for $\delta(u)$,

$$
\Delta(\delta(u))v^{(s)} = \sum_{m_1 = 0}^{s} C_{m_1}^{s}(P + 1)\gamma(u)v_{m_1}^{l_1} \otimes \beta(u)v_{s-m_1}^{l_2} + \sum_{m_1 = 0}^{s} C_{m_1}^{s}(P - 1)\delta(u)v_{m_1}^{l_1} \otimes \delta(u)v_{s-m_1}^{l_2}
$$

$$
= \sum_{m_1 = 0}^{s} C_{m_1}^{s}(P - 1)\varphi_{l_1}(u-a)\varphi_{l_2}(u-b)[P - l_2 + s - m_1 - 1][P + s - m_1]
$$

$$
\times \left( [P - l_2 + s - m_1 - 1][P + s - m_1][u - a - \frac{l_1 - 2m_1 - 1}{2}][u - b - \frac{l_2 - 2(s - m_1) - 1}{2}] + [s - m_1][l_2 - s + m_1 + 1][u - a - \frac{l_1 - 2m_1 - 1}{2} - P][u - b - \frac{l_2 - 2(s - m_1) - 1}{2} + P] \right) v_{m_1}^{l_1} \otimes v_{s-m_1}^{l_2}.
$$

In the second equality, we used (4.10). By using the theta function identity

$$
[u + x][u - x][v + y][v - y] - [u + y][u - y][v + x][v - x] = [x - y][x + y][u + v][u - v],
$$

we obtain

$$
\Delta(\delta(u))v^{(s)} = D(u) \sum_{m_1 = 0}^{s} C_{m_1}^{s}(P - 1) \frac{[P][P - l_2 + 2s - 2m_1 - 1]}{[P - l_2 + s - m_1 - 1][P + s - m_1]} v_{m_1}^{l_1} \otimes v_{s-m_1}^{l_2}
$$

with $D(u)$ appearing in the statement of the theorem. Then the necessary and sufficient condition that $\Delta(\delta(u))$ is diagonalized on $v^{(s)}$ with the eigen value $D(u)$ is

$$
C_{m_1}^{s}(P) = C_{m_1}^{s}(P - 1) \frac{[P][P - l_2 + 2s - 2m_1 - 1]}{[P - l_2 + s - m_1 - 1][P + s - m_1]}.
$$

Solving this, we obtain

$$
C_{m_1}^{s}(P) = C_{m_1}^{s} \frac{[P - l_2 + 2s - m_1]}{[P + 1]_{s-m_1}}.
$$

(4.12)

Here $C_{m_1}^{s}$ is a coefficient independent of $P$, which can be determined by substituting (4.12) into (4.10). We hence obtain $C_{m_1}^{s}(P)$ in the form in (4.16). We can check that for $\Delta(\alpha(u))$, the similar argument leads to the same result (4.16).

\begin{remark}
A similar statement was obtained in [17].
\end{remark}

By applying $\beta(u)$ on $v^{(s)}$ repeatedly, we can compute the other weight vectors as follows.
Theorem 4.18. Setting \( l = l_1 + l_2 - 2s \), we have for \( 0 \leq m \leq l \)

\[
\Delta(\beta(u)\beta(u+1)\cdots\beta(u+m-1))v^{(s)} \nonumber
\]

\[
= \prod_{i=1}^{m} \phi_1(u-a+i-1)\phi_2(u-a-\frac{l}{2}+i-1) \times \sum_{k=\max(0,s-m-l_2)}^{\min(l_1,s+m)} (-)^k C_0^s [P+m-2k-l_2+s] \frac{u-a+l_1+1}{2} [u-a-l+\frac{l_1-1}{2}+m-k]_{m-k} \times [-u+a-l_1-\frac{l_1-1}{2}-m+k-P]_k [-u+a+l-\frac{l_1-1}{2}-m+1]_k \times \frac{(-)^k \beta }{[-u+a+l-\frac{l_1-1}{2}-m+1]_k} \times_{12} V_{11} \left( P+m-2k; -k, -s, P-k, l_2-s+1, -u+a-\frac{l_1-1}{2}, u-a-l+\frac{l_1-1}{2}+2m-2k+P, P+m-2k+l_1+1 \right) v^{(s)}_l \otimes v^{(s)}_{m+s-k}.
\]

Moreover, we have for \( l < m \)

\[
\Delta(\beta(u)\beta(u+1)\cdots\beta(u+m-1))v^{(s)} = 0.
\]

Here \(_{12}V_{11}\) denotes the very-well-poised balanced elliptic hypergeometric series defined by

\[
_{s+1}V_s(u_0; u_1, \cdots, u_{s-4}) = \sum_{j=0}^{s} \frac{[u_0+2j]}{[u_0]} \prod_{i=0}^{s-4} \frac{[u_i+j]}{[u_0+1-u_i+j]}
\]

with the balancing condition

\[
\sum_{i=1}^{s-4} u_i = \frac{s-7}{2} + \frac{s-5}{2} u_0.
\]

We give the proof in Appendix B.

The vector \( v^{(s)} \) obtained in Theorem 4.17 depends on the dynamical parameter \( P \). Let us write its \( P \) dependence explicitly as \( v^{(s)}(P) \). It satisfies \( e^Q. v^{(s)}(P) = v^{(s)}(P+1) \). Setting \( \tilde{v}^{(s)} = \sum_{n \in \mathbb{Z}} v^{(s)}(P+n) \), we have \( e^Q. \tilde{v}^{(s)} = \tilde{v}^{(s)} \). Hence \( \tilde{v}^{(s)} \) is a pseudo-highest weight vector. Note that \( \tilde{v}^{(s)} \) is a vector in the \( F \)-linear space spanned by the vectors \( v^{(s)}_{l_1} \otimes v^{(s)}_{l_2} \) \((0 \leq l_1 \leq s)\). Let us consider the pseudo-highest weight \( U_{q,P}(L(s_{l_2})) \)-module \( W^{(s)} \) generated by \( \tilde{v}^{(s)} \).

Theorem 4.19. If \( b-a = \frac{l_1+l_2-2s}{2} + 1 \quad \text{and} \quad \min(l_1, l_2) < s \leq \min(l_1, l_2) \), the pseudo-highest weight \( U_{q,P}(L(s_{l_2})) \)-module \( W^{(s)} \) is a unique proper submodule of \( \tilde{V} = \tilde{V}^{(l_1)}(q^{2a}) \otimes \tilde{V}^{(l_2)}(q^{2b}) \). Moreover we have

\[
W^{(s)} \cong \tilde{V}^{(l_1-s)}(q^{2(a-\frac{s}{2})}) \otimes \tilde{V}^{(l_2-s)}(q^{2(b+\frac{s}{2})}),
\]

\[
\tilde{V}/W^{(s)} \cong \tilde{V}^{(s-1)}(q^{2(a+\frac{l_1+1}{2})}) \otimes \tilde{V}^{(l_1+l_2-s+1)}(q^{2(b+\frac{l_2-s+1}{2})}).
\]
\textbf{Proof.} From Theorem 3.11, it is enough to show that the entire quasi-periodic functions associated with the representations in the both sides coincide with each other under the condition \( b - a = \frac{l_1+l_2-2a}{2} + 1 \) for \( 0 < s \leq \min(l_1,l_2) \). In fact, one can evaluate the action of the operators \( H^\pm(u) \) on the highest weight vectors \( \tilde{v}^{(s)} \) and \( v_0^{l_1-s} \otimes v_0^{l_2-s} \) of \( W^{(s)} \) and \( \hat{V}(l_1-s)(q^{-2(a-\frac{l_1}{2})}) \otimes \hat{V}(l_2-s)(q^{2(b+\frac{l_2}{2})}) \), respectively, and find that the eigen values coincide with each other. They are given by

\[
\frac{[u - a - \frac{l_1+1}{2}][u - a + \frac{l_1+1}{2}]}{[u - a - \frac{l_2}{2} + s][u - a - \frac{l_1+1}{2} - 1 + s]}. 
\]

Similarly, the eigen values of \( H^\pm(u) \) on the highest weight vectors \( v_0^{l_1} \otimes v_0^{l_2} \) and \( v_0^{s-1} \otimes v_0^{l_1+l_2-s+1} \) of \( \hat{V}/W^{(s)} \) and \( \hat{V}(s-1)(q^{2(a+\frac{l_1-1}{2})}) \otimes \hat{V}(l_1+l_2-s+1)(q^{2(b-\frac{l_1+1}{2})}) \), respectively, coincide and are given by

\[
\frac{[u - a + \frac{l_1+1}{2}][u - b + \frac{l_2+1}{2}]}{[u - a - \frac{l_1-1}{2}][u - b - \frac{l_2-1}{2}]}.
\]

\( \square \)

\textbf{Remark.} A similar statement was presented in [17] without proof.

\section{Discussions}

In this section, we consider the limits trigonometric \( r \to \infty \), non-affine \( u \to \infty \) and non-dynamical \( P \to \infty \) of the results and make some remarks on their algebraic structures and relations.

In the limits, the elliptic dynamical \( R \) matrix degenerates as follows.

\[
R^+(u,P) \xrightarrow{r \to \infty} R^+_{\text{trig.}}(u,P) \xrightarrow{u \to \infty} R^+(P) \xrightarrow{P \to \infty} R^+, \quad R^+(u) \xrightarrow{u \to \infty} R^+,
\]

where setting \( x = q^{2P} \) we have

\[
R^+_{\text{trig.}}(u,P) = \rho^+_{\text{trig.}}(z) \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{(1-q^2)(1-q^{-2})}{(1-x)^2}b(z) & \frac{1-x}{1-x}c(z) & 0 \\
0 & \frac{1-xz}{1-x}zc(z) & b(z) & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

\[
R^+(P) = q^{1/2} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{q(1-q^2)(1-q^{-2})}{(1-x)^2} & \frac{1-x^2}{1-x} & 0 \\
0 & -\frac{x(1-q^2)}{1-x} & q & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad R^+(u) = \rho^+_{\text{trig.}}(z) \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & b(z) & c(z) & 0 \\
0 & zc(z) & b(z) & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

31
\[ R^+ = q^{1/2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & q & 1 - q^2 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \]

\[ \rho_{\text{trig.}}^+(z) = q^{1/2} \frac{(z^{-1}; q^4)_\infty (q^4 z^{-1}; q^4)_\infty}{(q^2 z^{-1}; q^4)_\infty^2}, \]

\[ b(z) = \frac{q(1 - z)}{1 - q^2 z}, \quad c(z) = \frac{1 - q^2}{1 - q^2 z}. \]

Correspondingly naive degeneration limits of the RLL relation \((2.17)\) imply the following diagram of quantum algebras.

\[
\begin{array}{ccccccc}
U_{q,p}(\widehat{\mathfrak{sl}}_2) & \xrightarrow{r \to \infty} & U_{q,x}(\widehat{\mathfrak{sl}}_2) & \xrightarrow{\bar{u} \to \infty} & U_{q,x}(\mathfrak{sl}_2) & \xrightarrow{\bar{P} \to \infty} & U_q(\mathfrak{sl}_2), \\
& \downarrow{P \to \infty} & & \downarrow{\bar{u} \to \infty} & & & (5.1)
\end{array}
\]

Here \(U_{q,x}(\widehat{\mathfrak{sl}}_2)\) is the dynamical quantum affine algebra suggested in [1]. However neither its generators nor the \(L\) operators has yet been given explicitly. Let us make some speculations on it. The \(U_{q,x}(\widehat{\mathfrak{sl}}_2)\) should be a semi-direct product \(\mathbb{C}\)-algebra isomorphic to \(\mathbb{F}[U_q(\widehat{\mathfrak{sl}}_2)] \otimes \mathbb{C}[\widehat{H}^*]\) and characterized by the RLL relation of the type \((2.17)\) associated with \(R^+_{\text{trig.}}(u, P)\). In fact, the \(\widehat{L}^+(u)\) operator as well as the half currents in Definition 2.8 do survive in the trigonometric limit. The \(H\)-Hopf algebroid structure of \(U_{q,p}(\widehat{\mathfrak{sl}}_2)\) also survives in the limit. We hence expect that \(U_{q,x}(\widehat{\mathfrak{sl}}_2)\) is an \(H\)-Hopf algebroid. We will discuss this subject in elsewhere.

\(U_{q,x}(\mathfrak{sl}_2)\) denotes the dynamical quantum algebra introduced by Babelon [4], and \(U_q(\widehat{\mathfrak{sl}}_2)\), \(U_q(\mathfrak{sl}_2)\) are the standard quantum affine and non-affine algebras by Drinfeld-Jimbo, respectively.

The quasi-Hopf algebra structure of \(U_{q,x}(\mathfrak{sl}_2)\) was studied in [3], whereas the generalized FRST formulation and the \(H\)-Hopf algebroid structure were discussed in [12, 13, 29]. The FRST formulation and Hopf algebra structure of \(U_q(\widehat{\mathfrak{sl}}_2)\) and \(U_q(\mathfrak{sl}_2)\) were given in [41] and [14], respectively.

Concerning the FRST formulations, we should remark that in the elliptic algebra \(U_{q,p}(\widehat{\mathfrak{sl}}_2)\) as well as in \(B_{q,\lambda}(\mathfrak{sl}_2)\) the \(L\) operators \(\widehat{L}^+(u)\) and \(\widehat{L}^-(u)\) are not independent [24]. This is also true for its trigonometric and non-affine limits. However this is not true after the non-dynamical limit \(P \to \infty\), so that we need two \(L\) operators \(L^+\) and \(L^-\) for \(U_q(\widehat{\mathfrak{sl}}_2)\) and \(U_q(\mathfrak{sl}_2)\).

It is also interesting to see the limits of \(_{12}V_{11}\) obtained in [4, 13]. Corresponding to the upper
series in (5.1), we find the following.

\[ 12V_{11} \left( P + m - 2k; -s, -k, P - k, l_2 - s + 1, -u + a - \frac{l_1 - 1}{2}, \right. \]
\[ \left. u - a - l + \frac{l_1 - 1}{2} + 2m - 2k + P, P + m - 2k + l_1 + 1 \right) \]
\[
\overset{r \to \infty}{\longrightarrow} 10W_9 \left( q^{2(P+m-2k)}; q^{-2s}, q^{-2k}, q^{2(P-k)}, q^{2(l_2-s+1)}, q^{2(-u+a-l)} \right. \]
\[ \left. q^{2(u-a-l+\frac{l_1 - 1}{2}+2m-2k+P)}, q^{2(P+m-2k+l_1+1)}; q^2, q^2 \right) \]
\[
\overset{u \to \infty}{\longrightarrow} 8W_7 \left( q^{2(P+m-2k)}; q^{-2s}, q^{-2k}, q^{2(P-k)}, q^{2(l_2-s+1)}, q^{2(P+m-2k+l_1+1)}; q^2, q^{2(l-m)} \right) \]
\[ = \frac{(q^{2(P+m-k+1)}, q^{2(m+1)}, q^{2(P+m-k-l_2+s)}, q^{2(k-l_1)}; q^2)_s q^{-2sk}}{(q^{2(P+m-k+1)}, q^{2(m+1)}, q^{2(P+m-2k-l_2+s)}, q^{2(l_1)}; q^2)_s q^{-2sk}} \]
\[ \times 4\phi_3 \left( q^{-2s}, q^{-2k}, q^{-2(P+m-k+s)}, q^{2(l-m+1)} \right. \]
\[ \left. q^{-2(s+m)}, q^{-2(P+m-k+l_1-l-1)}, q^{2(l_1+s-k)}; q^2, q^2 \right) \]
\[
\overset{P \to \infty}{\longrightarrow} \frac{(q^{2(m+1)}; q^2)_s}{(q^{2(m-k+1)}; q^2)_s} 3\phi_2 \left( q^{-2s}, q^{-2k}, q^{-2(s+l+1)} \right. \]
\[ \left. q^{-2(s+m)}, q^{-2l_1}; q^2, q^2 \right) \right), \tag{5.2} \]

where
\[
(a_1, a_2, \cdots, a_m; q^2)_s = \prod_{i=1}^{m} (a_i; q^2)_s,
\]
\[
(a; q^2)_s = (1 - a)(1 - aq^2) \cdots (1 - aq^{2(s-1)}).
\]

Here we followed the notations in [21]. After the second limit, we used the transformation formula from (2.17) in [29] whereas after the third limit, the formula from (3.2.2) in [21]. As shown by Rosengren [44], \(10W_9\) in the above gives a system of biorthogonal functions identical to the one obtained by Wilson [54]. The \(4\phi_3\) part is identified with the \(q\)-Racah polynomial, and the \(3\phi_2\) part with the \(q\)-Hahn polynomial.

A representation theoretical derivation of \(3\phi_2\) or the \(q\)-Clebsch-Gordan coefficients was done on the basis of \(U_q(\mathfrak{sl}_2)\) in [27, 28, 52]. The case of \(8W_7\) or the \(q\)-Racah polynomials, or Askey-Wilson polynomial, has an interesting history. It was first done on the basis of the quantum group \(SU_q(2)\) with considering the so-called twisted primitive element in [38] and [40]. Later an alternative derivation was carried out on the basis of the co-representations of \(U_q(x)\) in [29]. The relation between these two derivations can be found in [42] and [50]. As for the case \(12V_{11}\), a representation theoretical derivation was first done in [30] on the basis of co-representations of Felder’s elliptic quantum group. In this paper we have given an alternative derivation on the basis of representations of \(U_q(\mathfrak{sl}_2)\). Comparing (5.1) and (5.2), we conjecture that Wilson’s biorthogonal functions \(10W_9\) can be derived similarly on the basis of \(U_q(x)\).
Moreover it is instructive to note that reading the diagram (5.2) in inverse direction the dynamical parameter $P$ modifies the $q$-$3j$-symbols ($3\phi_2$ or the Clebsch-Gordan coefficients) into the $q$-$6j$-symbols ($4\phi_3$ or $q$-Racah polynomials), the affinization parameter $u$ modifies the orthogonal polynomials ($q$-Racah, $q$-Hahn polynomials) into the biorthogonal functions (Wilson’s biorthogonal function). As a result $12V_{11}$ is an elliptic analogue of the $q$-$6j$-symbol and is biorthogonal. We think that this observation should become a guiding principle in choosing a suitable type of quantum groups, such as dynamical or non-dynamical, affine or non-affine, in a derivation of elliptic analogues of the $q$-special functions.

It is also interesting to note that the above degeneration diagram of $12V_{11}$ coincides with the one of the hypergeometric type special solutions of the discrete Painlevé equations [26] corresponding to the following degeneration of affine Weyl group symmetries [45].

$$E_8^{(1)} \rightarrow E_7^{(1)} \rightarrow E_6^{(1)} \rightarrow D_5^{(1)} \rightarrow \cdots.$$  

It should be interesting if one could find a direct connection between this diagram or the discrete Painlevé equations themselves and the quantum groups in (5.1).

**Acknowledgments**

The author would like to thank Michio Jimbo, Anatol Kirillov, Atsushi Nakayashiki, Masatoshi Noumi, Hjalmar Rosengren and Tadashi Shima for stimulating discussions and valuable suggestions. He is also grateful to Tetsuo Deguchi, Jonas Hartwig, Masahiko Ito, Masaki Kashiwara, Christian Korf, Barry McCoy, Tetsuji Miwa, Tomoki Nakanishi, Masato Okado, Vitaly Tarasov and Yasuhiko Yamada for their interests and useful conversations. He also thank Hjalmar Rosengren for his kind hospitality during a stay in Charmers University of Technology and Göteborg University. This work is supported by the Grant-in-Aid for Scientific Research (C)19540033, JSPS Japan.
A The RLL relation (2.17) at $c = 0$

We write down the RLL relation (2.17) in terms of the matrix elements of $\hat{L}^+(u)$ in the case $c = 0$.

\begin{align*}
[\alpha(u_1), \alpha(u_2)] &= 0, \quad [\delta(u_1), \delta(u_2)] = 0, \quad \text{(A.1)} \\
[\beta(u_1), \beta(u_2)] &= 0, \quad [\gamma(u_1), \gamma(u_2)] = 0, \quad \text{(A.2)} \\
\alpha(u_1)\beta(u_2) &= \tilde{c}(u, P)\alpha(u_2)\beta(u_1) + b(u, P)\beta(u_2)\alpha(u_1), \quad \text{(A.3)} \\
\beta(u_1)\alpha(u_2) &= \tilde{b}(u, P)\alpha(u_2)\beta(u_1) + c(u, P)\beta(u_2)\alpha(u_1), \quad \text{(A.4)} \\
\gamma(u_1)\delta(u_2) &= \tilde{c}(u, P)\gamma(u_2)\delta(u_1) + b(u, P)\delta(u_2)\gamma(u_1), \quad \text{(A.5)} \\
\delta(u_1)\gamma(u_2) &= \tilde{b}(u, P)\gamma(u_2)\delta(u_1) + c(u, P)\delta(u_2)\gamma(u_1), \quad \text{(A.6)} \\
c(u, P + h)\gamma(u_1)\alpha(u_2) + b(u, P + h)\alpha(u_1)\gamma(u_2) &= \gamma(u_2)\alpha(u_1), \quad \text{(A.7)} \\
\tilde{b}(u, P + h)\gamma(u_1)\alpha(u_2) + \tilde{c}(u, P + h)\alpha(u_1)\gamma(u_2) &= \alpha(u_2)\gamma(u_1), \quad \text{(A.8)} \\
c(u, P + h)\delta(u_1)\beta(u_2) + b(u, P + h)\beta(u_1)\delta(u_2) &= \delta(u_2)\beta(u_1), \quad \text{(A.9)} \\
\tilde{b}(u, P + h)\delta(u_1)\beta(u_2) + \tilde{c}(u, P + h)\beta(u_1)\delta(u_2) &= \beta(u_2)\delta(u_1), \quad \text{(A.10)} \\
c(u, P + h)\gamma(u_1)\beta(u_2) + b(u, P + h)\alpha(u_1)\delta(u_2) \\
&= \tilde{c}(u, P)\gamma(u_2)\beta(u_1) + b(u, P)\delta(u_2)\alpha(u_1), \quad \text{(A.11)} \\
\tilde{b}(u, P + h)\gamma(u_1)\beta(u_2) + \tilde{c}(u, P + h)\alpha(u_1)\delta(u_2) \\
&= b(u, P)\beta(u_2)\gamma(u_1) + \tilde{c}(u, P)\alpha(u_2)\delta(u_1), \quad \text{(A.12)} \\
b(u, P + h)\beta(u_1)\gamma(u_2) + c(u, P + h)\delta(u_1)\alpha(u_2) \\
&= \tilde{b}(u, P)\gamma(u_2)\beta(u_1) + c(u, P)\delta(u_2)\alpha(u_1), \quad \text{(A.13)} \\
\tilde{c}(u, P + h)\beta(u_1)\gamma(u_2) + \tilde{b}(u, P + h)\delta(u_1)\alpha(u_2) \\
&= c(u, P)\beta(u_2)\gamma(u_1) + \tilde{b}(u, P)\alpha(u_2)\delta(u_1). \quad \text{(A.14)}
\end{align*}

B Proof of Theorem 4.18

In order to prove the theorem, we need the following four Lemmas.

Lemma B.1. For $c = 0$,

$$
\alpha(u)\beta(v_1) \cdots \beta(v_l) = \frac{[P + 1][P - l][u - v_1]}{[P][P - l + 1][u - v_1 + 1]} \beta(v_1) \cdots \beta(v_l)\alpha(u) \\
+ \sum_{k=1}^{l} \frac{[P + 1][P - k + 1 - u + v_k]}{[P][u - v_1 + 1][P - k + 2]} \beta(v_1) \cdots \alpha(v_k) \cdots \beta(v_l)\beta(u).
$$
Proof. Use \(\text{A.3}\) repeatedly.

Lemma B.2.

\[
\Delta(\beta(u)\beta(u + 1) \cdots \beta(u + m - 1)) = \sum_{j=0}^{m} D_j^m(P)\alpha(u + m - 1) \cdots \alpha(u + m - j)\beta(u + m - j - 1) \cdots \beta(u) \alpha(u + m) \\
\otimes \delta(u) \cdots \delta(u + m - j - 1)\beta(u + m - j) \cdots \beta(u + m - 1)\beta(u + m),
\]

where

\[
D_j^m(P) = \frac{[1]_m}{[1]_j[1]_{m-j}} \frac{[P][P - m + 2j]}{[P + j][P - m + j]}, \quad m \in \mathbb{Z}_{\geq 0}.
\]

Proof. We prove the statement by induction on \(m\). The case \(m = 1\) is just the comultiplication formula for \(\beta(u)\). Assume that the statement is true for \(m\). Then

\[
\Delta(\beta(u)\beta(u + 1) \cdots \beta(u + m - 1))\Delta(\beta(u + m)) = \sum_{j=0}^{m} D_j^m(P)\alpha(u + m - 1) \cdots \alpha(u + m - j)\beta(u + m - j - 1) \cdots \beta(u) \alpha(u + m) \\
\otimes \delta(u) \cdots \delta(u + m - j - 1)\beta(u + m - j) \cdots \beta(u + m - 1)\delta(u + m) \\
+ \sum_{j=0}^{m} D_j^m(P)\alpha(u + m - 1) \cdots \alpha(u + m - j)\beta(u + m - j - 1) \cdots \beta(u)\beta(u + m) \\
\otimes \delta(u) \cdots \delta(u + m - j - 1)\beta(u + m - j) \cdots \beta(u + m - 1)\delta(u + m) \\
= \alpha(u + m - 1) \cdots \alpha(u)\alpha(u + m) + \beta(u + m - 1) \cdots \beta(u)\beta(u + m) \\
+ \sum_{j=1}^{m} \left\{ D_{j-1}^m(P)\alpha(u + m - 1) \cdots \alpha(u + m - j + 1)\beta(u + m - j) \cdots \beta(u)\alpha(u + m) \\
+ \frac{[P + j - m]}{[P + 2j - m]} D_j^m(P)\alpha(u + m - 1) \cdots \alpha(u + m - j)\beta(u + m - j - 1) \cdots \beta(u)\beta(u + m) \right\} \\
\otimes \delta(u) \cdots \delta(u + m - j - 1)\beta(u + m - j) \cdots \beta(u + m - 1)\delta(u + m).
\]

To obtain the second equality we used the property of \(\otimes\) in \(\text{A.16}\) with putting \(c = 0\) and the following relation obtained from \(\text{A.10}\) with putting \(u_1 = v\) and \(u_2 = v + 1\)

\[
\delta(v)\beta(v + 1) = \frac{[P + h + 1]}{[P + h]} \beta(v)\delta(v + 1).
\]

Therefore we need to show

\[
D_{j+1}^{m+1}(P - j + 1)\alpha(u + m)\beta(u + m - j) \cdots \beta(u) \\
= D_{j-1}^m(P - j + 1)\beta(u + m - j) \cdots \beta(u)\alpha(u + m) \\
+ \frac{[P - m + 1]}{[P + j - m + 1]} D_j^m(P - j + 1)\alpha(u + m - j)\beta(u + m - j - 1) \cdots \beta(u)\beta(u + m) \quad \text{(B.2)}
\]

36
for $j = 1, 2, \cdots, m$.

Specializing $l \to m - j, u \to u + m - j, v_k \to u + k - 1 (1 \leq k \leq m - j)$ in Lemma 3.1, we have

$$
\alpha(u + m - j)\beta(u + m - j - 1) \cdots \beta(u) = \sum_{k=1}^{m-j+1} \frac{[P+1][P-m+j][1]}{[P][m-j+1][P-k+2]} \beta(u) \cdots \alpha(u+k-1) \cdots \beta(u + m - j).
$$

Substituting this into the second term in the RHS of (B.2), we obtain

$$
D_j^{m+1}(P - j + 1)\alpha(u + m)\beta(u + m - j) \cdots \beta(u)
+ D_j^m(P - j + 1) \sum_{k=1}^{m-j+1} \frac{[P+1][P-m+j][1]}{[P+j-m+1][P][m-j+1][P-k+2]} \beta(u) \cdots \alpha(u+k-1) \cdots \beta(u + m - j)\beta(u + m).
$$

Similarly, specializing $l \to m - j + 1, u \to u + m, v_k \to u + k - 1 (1 \leq k \leq m - j + 1)$ in Lemma 3.1, we obtain

$$
\alpha(u + m)\beta(u + m - j) \cdots \beta(u)
= \sum_{k=1}^{m-j+1} \frac{[P+1][P-m+j][1]}{[P][m+1][P-k+2]} \beta(u) \cdots \alpha(u+k-1) \cdots \beta(u + m - j)\beta(u + m).
$$

We compare this with (B.3). These two relations coincide with each other if and only if

$$
\frac{D_j^{m+1}(P - j + 1)}{D_j^m(P - j + 1)} = \frac{[P+1][P-m+j-1][1]}{[P][m+1][P-k+2]},
\frac{\overline{D}_j^m(P - j + 1)}{\overline{D}_j^{m+1}(P - j + 1)} = \frac{[P+1][P-m+j][1]}{[P][m+1][P-k+2]}.
$$

Therefore we obtain

$$
\frac{D_j^m(P)}{D_{j-1}^m(P)} = \frac{[P+1][P-m-j][1]}{[P-m+j][P-m+2(j-1)][P+j][j]}.
$$

Solving this with the initial condition $D_0^m(P) = 1$, we obtain (B.1).

**Lemma B.3.** For $t^j_{m_1} \in \hat{V}(l_1)(q^{2a})$, we have

$$
\alpha(u + m - 1) \cdots \alpha(u + m - j)\beta(u + m - j - 1) \cdots \beta(u)t^j_{m_1}
= (-)^{m+1} \sum_{k=1}^{m+1} \frac{[u-a+\frac{l_1+1}{2}]m-k[P-k][l_1-2k+1][u-a-m-\frac{l_1-1}{2}]}{\prod_{i=1}^{m}[u-a+i-1]} \frac{P}{\prod_{i=1}^{m}[P-l_1-2k+1]} \cdot t^j_{m_1}.
$$
Here we set \( k = m_1 + m - j \).

**Proof.** Applying Theorem 4.14 we evaluate the LHS as

\[
\text{LHS} = (-)^m \prod_{i=1}^{j} \frac{u - a + m - i + \frac{h+1}{2}}{\varphi_i(u - a + m - i)(P + i - 1 - \frac{h}{2})[P + i - 1 + \frac{h+2}{2}]} \varphi_i(u - a + m - i)(P + i - 1 + \frac{h+2}{2}) \\
\times \prod_{i=1}^{m-j} \frac{u - a + m + \frac{h-1}{2} + P - i + 1 - \frac{h-2}{2} - i}{\varphi_i(u - a + m - j - i)(P + h + j + i)} \quad \text{for } k \neq 0.
\]

Then using the following formulae, we obtain the desired result. For \( b, k, m_1, k - m_1 \in \mathbb{Z}_{\geq 0} \),

\[
[a]_{m_1+b} = [a]_b[a + b]_{m_1},
\]

\[
[a - b]_{m_1+b} = (-)^b[-a + 1]_b[a]_{m_1},
\]

\[
[a + k]_{k-m_1} = (-)^m \frac{[a + k]_k}{[-a - 2k + 1]_m}.
\]

**Lemma B.4.** For \( v_{s-m_1}^{l_2} \in \tilde{\mathcal{V}}(l_2)(q^{2b}) \), we have

\[
v_{k}^{l_1} \otimes \delta(u) \cdots \delta(u + m - j - 1) \beta(u + m - j) \cdots \beta(u + m - 1) \quad v_{s-m_1}^{l_2}
\]

\[
= (-)^{m+k} \left[ \prod_{i=1}^{j} \frac{u - b + \frac{h-1}{2} + m + s + 1 - k + P}{\varphi_i(u - b + i - 1)} \right] \left[ \prod_{i=1}^{m-k} \frac{u - b + \frac{h-1}{2} + m + s + 1 - k + P}{\varphi_i(u - b + m - j + i - 1)} \right] \quad v_{s-m_1+2j}^{l_2}
\]

Here we set \( k = m_1 + m - j \).

**Proof.** Applying Theorem 4.14, we evaluate the LHS as

\[
v_{k}^{l_1} \otimes (-)^{m} \left[ \prod_{i=1}^{j} \frac{u - b + i - 1 - \frac{h-1}{2}}{\varphi_i(u - b + i - 1)} \right] \left[ \prod_{i=1}^{m-k} \frac{u - b + \frac{h-1}{2} + P + i}{\varphi_i(u - b + m - j - i - 1)} \right] \quad v_{s-m_1}^{l_2}
\]

\[
= v_{k}^{l_1} \otimes (-)^{m+k} \left[ \prod_{i=1}^{j} \frac{u - b + \frac{h-1}{2} + m + s - k}{\varphi_i(u - b + i - 1)} \right] \left[ \prod_{i=1}^{m-k} \frac{u - b + \frac{h-1}{2} + m + s + P + h + 1 - k}{\varphi_i(u - b + m - j - i)} \right] \quad v_{s-m_1}^{l_2}
\]

Then the statement follows from (4.8) and

\[
[a - k]_{k-m_1} = (-)^{m+1} \frac{[-a + 1]_k}{[-a + 1]_{m_1}},
\]

\[
[a + m_1 - k]_{m_1+m-k} = \frac{[a - k]_{m+k}[a + m - 2k]_{m_1}}{[a - k]_{m_1}}, \quad k, m_1 \in \mathbb{Z}_{\geq 0}, m \geq k \geq m_1. \quad \square
\]
Proof of the first statement in Theorem 4.18. By using Lemma [B.2 and [B.7], we obtain

\[
\text{LHS} = \sum_{m_1=0}^{s} \sum_{j=0}^{m} D_j^m(P) C_{m_1}^s (P - m + 2j)\alpha(u + m - 1) \cdots \alpha(u + j) \beta(u + m - j - 1) \cdots \beta(u) v_{m_1}^{l_1} \\
\sim \delta(u) \cdots \delta(u + m - j - 1) \beta(u + m - j) \cdots \beta(u + m - 1) v_{s-m_1}^{l_2}.
\]

Then using (1.6) and Lemma [B.3, B.4] and change the summation variable from \( j \) to \( k \) by \( k = m_1 + m - j \), we obtain the desired result. In the process, \( 12V_{11} \) is identified with the part associated with the summation with respect to \( m_1 \) over \( \max(0, k - m) \leq m_1 \leq \min(k, s) \). There we also manipulate (1.6) by the formula

\[
\frac{[P - l_2 + s - m_1]_{s-m_1}}{[P + 1]_{s-m_1}} = \frac{[P - l_2 + s - 2m_1][P - 2m_1 + 1]_{2m_1}}{[P - 2m_1 + 1][P - l_2 + s - 2m_1][P + s - 2m_1 + 1]_{m_1}}.
\]

Proof of the second statement. Let us set \( m = l_1 + l_2 + 2s + 1 \). Then \( l_1 - s + 1 \leq k \leq l_1 \). We show that \( 12V_{11} \) in part (1.13) vanishes for \( k = l_1 - s + n \) \((n = 1, 2, \cdots, s)\). In fact, substituting \( m = l_1 + l_2 + 2s + 1 \) and \( k = l_1 - s + n \), we find that the \( 12V_{11} \) part is reduced to

\[
\sum_{m_1=0}^{s} \frac{[P - l_1 + l_2 + 1 - 2n + 2m_1]}{[P - l_1 + l_2 + 1 - 2n]} \times \frac{[P - l_1 + l_2 + 2s - 2n + 1]}{[1 - n]_{s} \cdots [P - l_2 - 2s + 1]_{s}} \frac{[P - l_2 + 2s - 2n + 1]}{[P - l_2 + 2s - 2n + 2]_{s}} [P - l_2 + 2s + 2n]_{s}.
\]

In the last line we used the Jackson-Frenkel-Turaev summation formula [18]

\[
10V_{10}(\beta - \gamma - s; -s, \alpha - \gamma, -\alpha - \gamma + 1 - s, \beta + \delta, \beta - \delta) = \frac{[\gamma - \beta, \gamma + \beta, \alpha + \delta, \alpha - \delta]_{s}}{[\alpha - \beta, \alpha + \beta, \gamma + \delta, \gamma - \delta]_{s}}.
\]

(B.5) vanishes for \( n = 1, 2, \cdots, s \).

\[
\square
\]

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