Generalizations of Felder’s elliptic dynamical $r$-matrices associated with twisted loop algebras of self-dual Lie algebras

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Abstract

A dynamical $r$-matrix is associated with every self-dual Lie algebra $\mathcal{A}$ which is graded by finite-dimensional subspaces as $\mathcal{A} = \bigoplus_{n \in \mathbb{Z}} \mathcal{A}_n$, where $\mathcal{A}_n$ is dual to $\mathcal{A}_{-n}$ with respect to the invariant scalar product on $\mathcal{A}$, and $\mathcal{A}_0$ admits a nonempty open subset $\mathcal{A}_0$ for which $\text{ad} \kappa$ is invertible on $\mathcal{A}_n$ if $n \neq 0$ and $\kappa \in \mathcal{A}_0$. Examples are furnished by taking $\mathcal{A}$ to be an affine Lie algebra obtained from the central extension of a twisted loop algebra $\ell(\mathcal{G}, \mu)$ of a finite-dimensional self-dual Lie algebra $\mathcal{G}$. These $r$-matrices, $R : \mathcal{A}_0 \to \text{End}(\mathcal{A})$, yield generalizations of the basic trigonometric dynamical $r$-matrices that, according to Etingof and Varchenko, are associated with the Coxeter automorphisms of the simple Lie algebras, and are related to Felder’s elliptic $r$-matrices by evaluation homomorphisms of $\ell(\mathcal{G}, \mu)$ into $\mathcal{G}$. The spectral-parameter-dependent dynamical $r$-matrix that corresponds analogously to an arbitrary scalar-product-preserving finite order automorphism of a self-dual Lie algebra is here calculated explicitly.

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1 Introduction

The classical dynamical Yang-Baxter equation (CDYBE) introduced in its general form by Etingof and Varchenko [1] is a remarkable generalization of the CYBE. Currently we are witnessing intense research on the theory and the applications of the CDYBE to integrable systems [2, 3, 4]. For a review, see [5].

The aim of this paper is to study infinite-dimensional generalizations of a certain class of finite-dimensional classical dynamical $r$-matrices. Next we briefly recall these finite-dimensional $r$-matrices, which appear naturally in the chiral WZNW model (see e.g. [6] and references therein).

Let $A$ be a finite-dimensional complex Lie algebra equipped with a nondegenerate, symmetric, invariant bilinear form $\langle \cdot, \cdot \rangle$. Such a Lie algebra is called self-dual [7]. Consider a self-dual subalgebra $K \subset A$, on which $\langle \cdot, \cdot \rangle$ remains nondegenerate. Introduce the complex analytic functions $f$ and $F$ by

$$f : z \mapsto \frac{1}{2} \coth \frac{z}{2} - \frac{1}{z}, \quad F : z \mapsto \frac{1}{2} \coth \frac{z}{2}. \quad (1.1)$$

Suppose that $\mathcal{K}$ is a nonempty open subset of $K$ on which the operator valued function $R : \mathcal{K} \to \text{End}(A)$ is defined by

$$R(\kappa) := \begin{cases} f(\text{ad} \kappa) & \text{on } K \\ F(\text{ad} \kappa) & \text{on } K^\perp \end{cases} \quad \forall \kappa \in \mathcal{K}. \quad (1.2)$$

The decomposition $A = K + K^\perp$ is induced by $\langle \cdot, \cdot \rangle$. $R(\kappa)$ is a well defined linear operator on $A$ if and only if the spectrum of $\text{ad} \kappa$, acting on $A$, does not intersect $2\pi i \mathbb{Z}^*$, and $(\text{ad} \kappa)|_{K^\perp}$ is invertible. On $\mathcal{K} \subset K$ subject to these conditions, the following (modified) version of the CDYBE holds:

$$[RX, RY] - R([X, RY] + [RX, Y]) + \langle X, (\nabla R)Y \rangle + (\nabla_Y R)X - (\nabla_X R)Y = -\frac{1}{4}[X,Y], \quad \forall X, Y \in A. \quad (1.3)$$

Here the ‘dynamical variable’ $\kappa$ is suppressed for brevity, $\forall X \in A$ is decomposed as $X = X_K + X_{K^\perp}$, and

$$(\nabla_T R)(\kappa) := \frac{d}{dt} R(\kappa + tT)|_{t=0} \quad \forall T \in K, \quad \kappa \in \mathcal{K}, \quad (1.4)$$

$$\langle X, (\nabla R)(\kappa)Y \rangle := \sum_i K^i \langle X, (\nabla_K R)(\kappa)Y \rangle, \quad \forall X, Y \in A, \quad (1.5)$$

where $K_i$ and $K^i$ denote dual bases of $K$, $\langle K_i, K^j \rangle = \delta_i^j$. $R(\kappa)$ is antisymmetric, $\langle R(\kappa)X, Y \rangle = -\langle X, R(\kappa)Y \rangle$, and is $K$-equivariant in the sense that

$$(\nabla_{[T, \kappa]} R)(\kappa) = [\text{ad} T, R(\kappa)], \quad \forall T \in K, \kappa \in \mathcal{K}. \quad (1.6)$$

2The set of integers is denoted by $\mathbb{Z}$, $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$, and $\mathbb{N}$ denotes the positive integers.
These properties of $R$ have been established in this general setting in [6, 8]. In various special cases — in particular the case $\mathcal{K} = \mathcal{A}$ — they were proved earlier in [1, 9, 10]. If one introduces $r^\pm : \bar{\mathcal{K}} \to \mathcal{A} \otimes \mathcal{A}$ by

$$r^\pm(\kappa) := (R(\kappa)T_\alpha) \otimes T^\alpha \pm \frac{1}{2}T_\alpha \otimes T^\alpha,$$

(1.7)

where $\{T_\alpha\}$ and $\{T^\alpha\}$ are dual bases of $\mathcal{A}$, and uses the identification $\mathcal{K} \simeq \mathcal{K}^*$ induced by $\langle , \rangle$, then the above properties of $R$ become the CDYBE for $r^\pm$ with respect to the pair $\mathcal{K} \subset \mathcal{A}$ as defined in [1].

It is natural to suspect that whenever (1.2) is a well defined formula, the resulting $r$-matrix always satisfies (1.3). For this it is certainly not necessary to assume that $\mathcal{A}$ is finite dimensional. For example, Etingof and Varchenko [1] verified the CDYBE in the situation for which $\mathcal{A}$ is an affine Lie algebra based on a simple Lie algebra and $\mathcal{K} \subset \mathcal{A}$ is a Cartan subalgebra. Moreover, by applying evaluation homomorphisms to these $r$-matrices they recovered Felder’s celebrated spectral-parameter-dependent elliptic dynamical $r$-matrices [2]. Without presenting proofs, this construction was generalized in [8] to any affine Lie algebra, $\mathcal{A}(\mathcal{G}, \mu)$, defined by adding the derivation to the central extension of a twisted loop algebra, $\ell(\mathcal{G}, \mu)$, based on an appropriate automorphism, $\mu$, of a self-dual Lie algebra, $\mathcal{G}$. Namely, such an affine Lie algebra automatically comes equipped with the integral gradation associated with the powers of the loop parameter, and it can be shown that (1.2) provides a solution of (1.3) if one takes $\mathcal{K}$ to be the grade zero subalgebra in this gradation. In this paper, this solution will arise as a special case of a general theorem, which ensures the validity of (1.3) for (1.2) under the assumption that $\mathcal{K} = \mathcal{A}_0$ where $\mathcal{A} = \bigoplus_{n \in \mathbb{Z}} \mathcal{A}_n$ is graded by finite-dimensional subspaces and carries an invariant scalar product that is compatible with the grading in the sense that $\mathcal{A}_n \perp \mathcal{A}_m$ unless $(n + m) = 0$. Here $\mathbb{Z}$ is some abelian group, in our examples $\mathbb{Z} = \mathbb{Z}$. The precise statement, which is our first main result, is given by theorem 1 in section 2. We shall use this result to obtain dynamical $r$-matrices on the twisted loop algebras $\ell(\mathcal{G}, \mu)$ with the dynamical variable lying in the fixed point set $\mathcal{G}_0 \subset \mathcal{G}$ of the automorphism $\mu$ of $\mathcal{G}$. By means of evaluation homomorphisms, these $r$-matrices then yield spectral-parameter-dependent $\mathcal{G} \otimes \mathcal{G}$-valued dynamical $r$-matrices generalizing Felder’s elliptic $r$-matrices. The latter are recovered if $\mathcal{G}$ is taken to be a simple Lie algebra and $\mu$ a Coxeter automorphism, consistently with the derivation found in [1]. The existence of the above-mentioned family of elliptic dynamical $r$-matrices was announced in [8]. Our second main result is their derivation presented in section 3. See in particular proposition 2 and proposition 3 in subsection 3.3. We shall also find a relationship between the underlying $\ell(\mathcal{G}, \mu) \otimes \ell(\mathcal{G}, \mu)$-valued $r$-matrices with dynamical variables in $\mathcal{G}_0$, and certain $\mathcal{G} \otimes \mathcal{G}$-valued dynamical $r$-matrices on $\mathcal{G}_0$ introduced in [11]. This is contained in an appendix.

Before turning to the main text, the reader may consult the concluding section, where the results are summarized once more and some comments are offered on the possible applications of our dynamical $r$-matrices.
2 \( r \)-matrices on graded, self-dual Lie algebras

In this section we apply formula (1.2) to infinite-dimensional Lie algebras that are decomposed into finite-dimensional subspaces in such a way that the \( r \)-matrix leaves these subspaces invariant. The definition of the \( r \)-matrix on these subspaces will be given in terms of the well known holomorphic functional calculus of linear operators [12].

Let \( A \in \text{End}(V) \) be a linear operator on a finite-dimensional complex vector space \( V \). Denote by \( \sigma_A \) the spectrum (set of eigenvalues) of \( A \). Consider a holomorphic complex function \( H \) defined on some open domain containing \( \sigma_A \), and take \( \Gamma \) to be a contour that lies in this domain and encircles each eigenvalue of \( A \) with orientation used in Cauchy’s theorem. Then the operator \( H(A) \in \text{End}(V) \) may be defined by

\[
H(A) := \frac{1}{2\pi i} \oint_{\Gamma} dz H(z)(zI_V - A)^{-1},
\]

(2.1)

where \( I_V \) is the identity operator on \( V \). This definition can be converted into an explicit formula by means of the Jordan decomposition of \( A \), which shows that \( H(A) \) only depends on the derivatives \( H^{(k)}(z_i) \) for \( z_i \in \sigma_A \) up to a finite order. For example, if \( Av = z_i v \) then \( H(A)v = H(z_i)v \) and for the constant function \( H_c(z) \equiv c \) one obtains \( H_c(A) = cI_V \). Furthermore, if the power series expansion \( H(z) = \sum_{k=0}^{\infty} c_k z^k \) is valid in a neighbourhood of \( \sigma_A \), then \( H(A) = \sum_{k=0}^{\infty} c_k A^k \). An important rule of this functional calculus is that if \( H_3 = H_1 H_2 \) on some admissible domain, then \( H_3(A) = H_1(A)H_2(A) \). See e.g. chapter VII of [12].

We now consider a complex Lie algebra \( \mathcal{A} \) equipped with a gradation based on some abelian group \( \mathcal{Z} \). We use the additive notation to denote the group operation on \( \mathcal{Z} \). The zero as a number and the unit element of \( \mathcal{Z} \) are both denoted simply by 0, but this should not lead to any confusion. We assume that as a linear space

\[
\mathcal{A} = \oplus_{n \in \mathcal{Z}} \mathcal{A}_n, \quad 0 \leq \dim(\mathcal{A}_n) < \infty, \quad \dim(\mathcal{A}_0) \neq 0,
\]

(2.2)

and

\[
[\mathcal{A}_m, \mathcal{A}_n] \subset \mathcal{A}_{m+n} \quad \forall m, n \in \mathcal{Z}.
\]

(2.3)

The elements of \( \mathcal{A} \) are finite linear combinations of the elements of the homogeneous subspaces, and we permit the possibility that \( \dim(\mathcal{A}_n) = 0 \) for some \( n \in \mathcal{Z} \). We further assume that \( \mathcal{A} \) has a nondegenerate, symmetric, invariant bilinear form \( \langle \ , \ \rangle : \mathcal{A} \times \mathcal{A} \rightarrow \mathbb{C} \), which is compatible with the gradation in the sense that

\[
\mathcal{A}_m \perp \mathcal{A}_n \quad \text{unless} \quad (m + n) = 0.
\]

(2.4)

This means that if \( (m + n) \neq 0 \) then \( \langle X, Y \rangle = 0 \) for any \( X \in \mathcal{A}_m, Y \in \mathcal{A}_n \), and the dual space of \( \mathcal{A}_n \) can be identified with \( \mathcal{A}_{-n} \) by means of the pairing given by \( \langle \ , \ \rangle \). In particular, \( \mathcal{A}_0 \) is a finite-dimensional self-dual subalgebra of \( \mathcal{A} \). Since \( [\mathcal{A}_0, \mathcal{A}_n] \subset \mathcal{A}_n \) and \( \mathcal{A}_n \) is finite dimensional, \( e^{\text{ad}_\kappa} \) is a well defined linear operator on \( \mathcal{A} \) for any \( \kappa \in \mathcal{A}_0 \). The invariance of the bilinear form, \( \langle [X, Y], Z \rangle + \langle Y, [X, Z] \rangle = 0, \forall X, Y, Z \in \mathcal{A} \), implies that \( \langle e^{\text{ad}_\kappa Y}, e^{\text{ad}_\kappa Z} \rangle = \langle Y, Z \rangle \) for any \( Y, Z \in \mathcal{A} \) and \( \kappa \in \mathcal{A}_0 \).
Now we wish to apply formula (1.2) to

\[ K := \mathcal{A}_0, \quad K^\perp = \bigoplus_{n \in \mathbb{Z} \setminus \{0\}} \mathcal{A}_n. \]  

(2.5)

For any \( \kappa \in K \) and \( n \in \mathbb{Z} \), introduce \((\text{ad} \, \kappa)_n := \text{ad} \, \kappa \mid_{\mathcal{A}_n}\) and let \( \sigma_n^\kappa \) denote the spectrum of this finite-dimensional linear operator \((\sigma_n^\kappa = \emptyset \text{ if } \dim(\mathcal{A}_n) = 0)\). Our crucial assumption is that there exists a nonempty, open subset \( \check{K} \subset K \) for which

\[ \sigma_n^\kappa \cap 2\pi i \mathbb{Z} = \emptyset \quad \forall n \neq 0 \quad \text{and} \quad \sigma_0^\kappa \cap 2\pi i \mathbb{Z}^* = \emptyset \quad \forall \kappa \in \check{K}, \]  

(2.6)

where \( \mathbb{Z} \) and \( \mathbb{Z}^* \) are the set of all integers, and nonzero integers, respectively. It is clear that if such a \( \check{K} \) exists, then there exists also a maximal one. If this assumption is satisfied, then we can define the map \( R : \check{K} \to \text{End}(\mathcal{A}) \) by requiring that the homogeneous subspaces \( \mathcal{A}_n \) be invariant with respect to \( R(\kappa) \) in such a way that \( \forall \kappa \in \check{K} \)

\[ R(\kappa) \mid_{\mathcal{A}_0} := f((\text{ad} \, \kappa)_0), \quad R(\kappa) \mid_{\mathcal{A}_n} := F((\text{ad} \, \kappa)_n) \quad \forall n \in \mathbb{Z} \setminus \{0\}. \]  

(2.7)

For \( n \in \mathbb{Z} \) for which \( \dim(\mathcal{A}_n) \neq 0 \), these finite-dimensional linear operators are given similarly to (2.1). The assumption (2.6) guarantees that the spectra \( \sigma_n^\kappa \) do not intersect the poles of the corresponding meromorphic functions \( f \) and \( F \) in (1.1), whereby \( R(\kappa) \) is well defined for \( \kappa \in \check{K} \).

Somewhat informally, we summarize (2.7) by saying that \( R(\kappa) \) equals \( f(\text{ad} \, \kappa) \) on \( K \) and \( F(\text{ad} \, \kappa) \) on \( K^\perp \).

**Theorem 1.** Let \( \mathcal{A} \) be a graded, self-dual, complex Lie algebra satisfying the assumptions given by (2.2)–(2.4). Take \( K := \mathcal{A}_0 \) and suppose the existence a nonempty, open domain \( \check{K} \subset K \) for which (2.6) holds. Then the \( r \)-matrix \( R : \check{K} \to \text{End}(\mathcal{A}) \) defined by (2.7) satisfies the CDYBE (1.3). Moreover, \( R(\kappa) \) is an antisymmetric operator \( \forall \kappa \in \check{K} \), and the \( K \)-equivariance condition (1.6) holds.

**Proof.** Since the CDYBE (1.3) is linear in \( X, Y \in \mathcal{A} \), it is enough to verify it case by case for all possible choices of homogeneous elements \( X \) and \( Y \). As a preparation, let us write the function \( F \) in (1.1) as

\[ F(z) = \frac{1}{2} \frac{Q_+(z)}{Q_-(z)} \quad \text{with} \quad Q_{\pm}(z) = e^z \pm e^{-z}, \]  

(2.8)

and define the linear operators \( Q_{\pm}(\kappa) \) on \( \mathcal{A} \) by

\[ Q_{\pm}(\kappa) = e^{K \pm} e^{-K} \quad \text{with} \quad K := \frac{1}{2} \text{ad} \, \kappa \quad \forall \kappa \in \check{K}. \]  

(2.9)

\( Q_{\pm}(\kappa) \) are well defined operators on \( \mathcal{A} \) since their restrictions to any \( \mathcal{A}_n \) are obviously well defined. It follows from the definitions of the domain \( \check{K} \) and that of \( R(\kappa) \) that \( Q_{-}(\kappa) \) is an invertible operator on \( \mathcal{A}_n \) for any \( n \neq 0 \) and that we have

\[ R(\kappa)Q_{-}(\kappa) = Q_{-}(\kappa)R(\kappa) = \frac{1}{2} Q_+(\kappa) \quad \text{on} \quad \mathcal{A}_n \quad \forall n \neq 0. \]  

(2.10)
We first consider the simplest case,
\[ X \in A_m, \quad Y \in A_n, \quad m \neq 0, \quad n \neq 0, \quad (m + n) \neq 0, \tag{2.11} \]
for which the derivative terms drop out from (1.3). Without loss of generality, we can now write
\[ X = Q_- (\kappa) \xi, \quad Y = Q_- (\kappa) \eta \tag{2.12} \]
with some \( \xi \in A_m, \eta \in A_n \). If we multiply (1.3) from the left by the invertible operator \( 4Q_- (\kappa) \) on \( A_{m+n} \), then by using (2.10) the required statement becomes
\[ Q_- (\kappa)[Q_- (\kappa) \xi, Q_- (\kappa) \eta] + Q_- (\kappa)[Q_+ (\kappa) \xi, Q_+ (\kappa) \eta]
- Q_+ (\kappa) \left( [Q_- (\kappa) \xi, Q_+ (\kappa) \eta] + [Q_+ (\kappa) \xi, Q_- (\kappa) \eta] \right) = 0. \tag{2.13} \]

We further spell out this equation by using that \( e^{\pm K} \) are Lie algebra automorphism, and thereby (2.13) is verified in a straightforward manner.

Second, let us consider the case for which
\[ X \in A_0, \quad Y \in A_n, \quad n \neq 0. \tag{2.14} \]
Then the derivative term \((\nabla_X R)(\kappa)(Y)\) appears in equation (1.3). To calculate this, we need the holomorphic complex function \( h \) given by
\[ h(z) := e^z - 1 \tag{2.15} \]
We recall (e.g. [13], page 35) that for a curve \( t \mapsto A(t) \) of finite-dimensional linear operators one has the identity
\[ \frac{d e^{\pm A(t)}}{dt} = \pm e^{\pm A(t)} h(\mp \text{ad}_{A(t)})(\dot{A}(t)), \quad \dot{A}(t) := \frac{dA(t)}{dt}. \tag{2.16} \]
The right hand side of the above equation is defined by means of the Taylor expansion of \( h \) around 0, and of course
\[ \text{ad}_{A(t)}^j(\dot{A}(t)) = [A(t), \text{ad}_{A(t)}^{j-1}(\dot{A}(t))], \quad j \in \mathbb{N}, \quad \text{ad}_{A(t)}^0(\dot{A}(t)) = \dot{A}(t). \tag{2.17} \]
In our case we consider the curve of linear operators on \( A_n \) given by
\[ t \mapsto \text{ad} \kappa + t(\text{ad} X). \tag{2.18} \]
Then (2.16) leads to the formula
\[ (\nabla_X e^{\pm K})(Y) = \pm \frac{1}{2} e^{\pm K}[h(\mp K)X, Y], \tag{2.19} \]
where \( K = \frac{1}{2} \text{ad} \kappa \). From this, by taking the derivative of the identity \( 2Q_- R = Q_+ \) on \( A_n \) along the curve (2.18) at \( t = 0 \), we obtain
\[ 4Q_- (\kappa)(\nabla_X R)(\kappa)Y = e^K [h(-K)X, Y - 2R(\kappa)Y] - e^{-K} [h(K)X, Y + 2R(\kappa)Y]. \tag{2.20} \]
On the other hand, for (2.14) the CDYBE (1.3) is equivalent to

$$
4Q_-(\kappa)(\nabla_X R)(\kappa)Y = Q_-(\kappa) [X, Y] + 4Q_-(\kappa) [R(\kappa) X, R(\kappa) Y] \\
-2Q_+(\kappa) ([X, R(\kappa) Y] + [R(\kappa) X, Y]).
$$

(2.21)

We fix $\kappa \in \tilde{\mathcal{K}}$ arbitrarily, and write $Y = Q_-(\kappa) \eta$ with some $\eta \in \mathcal{A}_n$. Then by a straightforward calculation, using that $e^{\pm K}$ are Lie algebra automorphisms and collecting terms, we obtain that the required equality of the right hand sides of the last two equations is equivalent to

$$
\left[ (e^K h(-K) + e^{-K} h(K) - e^K - e^{-K}) X, \eta \right] = 2 \left[ (e^{-K} R(\kappa) - e^K R(\kappa)) X, \eta \right].
$$

(2.22)

Here $R(\kappa) X = f(2K) X$ with (1.2), and the statement follows from the equality of the corresponding complex analytic functions, namely

$$
e^z \frac{1 - e^{-z}}{z} + e^{-z} \frac{e^z - 1}{z} - e^z - e^{-z} = e^{-z} \left( \coth z - \frac{1}{z} \right) - e^z \left( \coth z - \frac{1}{z} \right),
$$

(2.23)

which is checked in the obvious way.

The third case to deal with is that of $X \in \mathcal{A}_{-n}, Y \in \mathcal{A}_n$, $n \neq 0$, (2.24)

for which the derivative term $\langle X, (\nabla R)(\kappa) Y \rangle$ occurs in (1.3). At any fixed $\kappa \in \tilde{\mathcal{K}}$, we may write

$$
X = Q_-(\kappa) \xi, \quad Y = Q_-(\kappa) \eta
$$

(2.25)

with some $\xi \in \mathcal{A}_{-n}, \eta \in \mathcal{A}_n$. We introduce the holomorphic function

$$
z \mapsto g(z) := \frac{e^z - e^{-z}}{z},
$$

(2.26)

and define $g(K)$ by the Taylor series of $g(z)$ around $z = 0$. Then we can calculate that

$$
\langle X, (\nabla R)(\kappa) Y \rangle = \frac{1}{2} g(K)[\eta, \xi].
$$

(2.27)

To obtain this, note that

$$
\langle X, (\nabla R)(\kappa) Y \rangle = T^i \langle X, (\nabla_{T^i} R)(\kappa) Y \rangle
$$

(2.28)

with dual bases $T_i$ and $T^i$ of $\mathcal{A}_0$, where $(\nabla_{T^i} R)(\kappa) Y$ is determined by (2.20). By using these and the invariance of the scalar product of $\mathcal{A}$, it is not difficult to rewrite (2.28) in the form (2.27). As for the non-derivative terms in (1.3), with $X, Y$ in (2.25) we find

$$
[R(\kappa) X, R(\kappa) Y] - R(\kappa) ([X, R(\kappa) Y] + [R(\kappa) X, Y]) + \frac{1}{4} [X, Y] =
$$

$$
\frac{1}{2} \left( Q_+(\kappa) - 2R(\kappa) Q_-(\kappa) \right) [\xi, \eta].
$$

(2.29)
It is easy to check that the sum of the right hand sides of (2.27) and (2.29) is zero, which finishes the verification of the CDYBE (1.3) in the case (2.24).

The remaining case is that of \( X, Y \in A_0 \). Then the variable \( \kappa \) as well as all terms in (1.3) lie in the subalgebra \( A_0 \), and it is known \([1,3,10]\) that the formula \( \kappa \leftrightarrow f(\text{ad } \kappa) \) (1.2) defines a solution of the CDYBE on any finite-dimensional self-dual Lie algebra. This completes the verification of the CDYBE (1.3).

The antisymmetry of \( R(\kappa) \) follows from (2.7) since \( \text{ad } \kappa \) is antisymmetric by the invariance of \( \langle \ , \ \rangle \) and both \( f \) and \( F \) are odd functions. Finally, the equivariance property (1.6) is also verified from (2.7) by using that for any finite-dimensional linear operator given by (2.1) one has

\[
\langle \text{of} \rangle \text{ of (2.1)}
\]

Equation (2.30) is a completion of the algebraic tensor product containing the elements that are associated with the subalgebra \( A \) and it is known \([1,9,10]\) that the formula \( \kappa \rightarrow \text{ad } \kappa \) defines a solution of the CDYBE (1.3).

We conclude this section by describing the tensorial interpretation of the CDYBE (1.3) for the \( r \)-matrices of theorem 1. For this, consider dual bases \( T_i[n] \) and \( T^j[n] \) of \( A \) \((n \in \mathbb{Z}, i, j = 1, \ldots, \dim(A_n))\), which satisfy \( T_i[n] \in A_n \) and \( \langle T_i[n], T^j[n] \rangle = \delta_{m,-n}\delta_i^j \). Then introduce \( r^\pm : K \rightarrow A \otimes A \) by

\[
\tau^\pm(\kappa) := \sum_{n \in \mathbb{Z}} \sum_{i=1}^{\dim(A_n)} \left( (R(\kappa)T_i[n]) \otimes T^i[-n] \pm \frac{1}{2} T_i[n] \otimes T^i[-n] \right)
\]

In fact, as a consequence of the properties of \( R \) established in theorem 1, \( r^\pm \) satisfies the tensorial version of the CDYBE given by

\[
[r^\pm_{12}(\kappa), r^\pm_{13}(\kappa)] + [r^\pm_{12}(\kappa), r^\pm_{23}(\kappa)] + [r^\pm_{13}(\kappa), r^\pm_{23}(\kappa)]
+ T_j[0]^1 \frac{\partial}{\partial \kappa_j} r^\pm_{23}(\kappa) - T_j[0]^2 \frac{\partial}{\partial \kappa_j} r^\pm_{13}(\kappa) + T_j[0]^3 \frac{\partial}{\partial \kappa_j} r^\pm_{12}(\kappa) = 0, \quad s = \pm,
\]

where \( \kappa_j := \langle \kappa, T_j[0] \rangle \). Here the standard notations are used, \( T_j[0]^1 := T_j[0] \otimes 1 \otimes 1, r^\pm_{12} := r^\pm \otimes 1 \) etc. The expression on the left hand side of (2.32) belongs to a completion of \( A \otimes A \otimes A \); it has a unique expansion in the basis \( T_i[n_1] \otimes T_i[n_2] \otimes T_i[n_3] \) of \( A \otimes A \otimes A \). Similarly to the CYBE, the CDYBE (2.32) is compatible with homomorphisms of \( A \). This means that if \( \pi_i : A \rightarrow \mathcal{G}^i \) \((i = 1, 2, 3)\) are (possibly different) homomorphisms of \( A \) into (possibly different) Lie algebras \( \mathcal{G}^i \), then we can obtain a \( \mathcal{G}^1 \otimes \mathcal{G}^2 \otimes \mathcal{G}^3 \)-valued equation from (2.32) by the obvious application of the map \( \pi_1 \otimes \pi_2 \otimes \pi_3 \) to all objects on the left hand side of (2.32). More precisely, to take into account the unit element \( 1 \), here one uses the extensions of these Lie algebra homomorphisms to the corresponding universal enveloping algebras.

\[^3\text{Here } A \otimes A \text{ is a completion of the algebraic tensor product containing the elements that are associated with the linear operators on } A.\]
3 Applications to affine Lie algebras

Let \( \mathcal{G} \) be a finite-dimensional complex, self-dual Lie algebra equipped with an invariant ‘scalar product’ denoted as \( B : \mathcal{G} \times \mathcal{G} \to \mathbb{C} \). Suppose that \( \mu \) is a finite order automorphism of \( \mathcal{G} \) that preserves the bilinear form \( B \) and has nonzero fixed points\(^4\). With this data, one may associate the twisted loop algebra \( \ell(\mathcal{G}, \mu) \) and the affine Lie algebra \( A(\mathcal{G}, \mu) \) obtained by adding the natural derivation to the central extension of \( \ell(\mathcal{G}, \mu) \). We below show that theorem 1 is directly applicable to \( A(\mathcal{G}, \mu) \). Then we explain that the resulting dynamical \( r \)-matrices on \( A(\mathcal{G}, \mu) \) admit a reinterpretation as one-parameter families of \( r \)-matrices on \( \ell(\mathcal{G}, \mu) \). By applying evaluation homomorphisms to the corresponding \( \ell(\mathcal{G}, \mu) \otimes \ell(\mathcal{G}, \mu) \)-valued \( r \)-matrices, we finally derive spectral-parameter-dependent \( \mathcal{G} \otimes \mathcal{G} \)-valued dynamical \( r \)-matrices. These results were announced in \[8\] without presenting proofs, which are provided here.

3.1 Application of theorem 1 to \( A(\mathcal{G}, \mu) \)

Any automorphism \( \mu \) of order \( N \), \( \mu^N = \text{id} \), gives rise to a decomposition of \( \mathcal{G} \) as

\[
\mathcal{G} = \bigoplus_{a \in \mathcal{E}_\mu} \mathcal{G}_a, \quad \mathcal{E}_\mu \subset \{0, 1, \ldots, (N - 1)\},
\]

(3.1)

with the eigensubspaces

\[
\mathcal{G}_a := \{ \xi \in \mathcal{G} \mid \mu(\xi) = \exp\left( \frac{ia2\pi}{N} \right) \xi \} \neq \{0\}.
\]

(3.2)

Since we assumed that \( B(\mu \xi, \mu \eta) = B(\xi, \eta) \) (\( \forall \xi, \eta \in \mathcal{G} \)), \( \mathcal{G}_a \) is perpendicular to \( \mathcal{G}_b \) with respect to the form \( B \) unless \( a + b = N \) or \( a = b = 0 \). This implies that if a nonzero \( a \) belongs to the index set \( \mathcal{E}_\mu \) then so does \( (N - a) \). We assume that \( 0 \in \mathcal{E}_\mu \), and thus \( \mathcal{G}_0 \neq \{0\} \) is a self-dual subalgebra of \( \mathcal{G} \).

The twisted (or untwisted if we choose \( \mu = \text{id} \)) loop algebra \( \ell(\mathcal{G}, \mu) \) is the subalgebra of \( \mathcal{G} \otimes \mathbb{C}[t, t^{-1}] \) generated by the elements of the form

\[
\xi^{n_a} := \xi \otimes t^{n_a} \quad \text{with} \quad \xi \in \mathcal{G}_a, \quad n_a = a + mN, \quad m \in \mathbb{Z},
\]

(3.3)

where \( t \) is a formal variable. The ‘affine Lie algebra’ \( A(\mathcal{G}, \mu) \) is then introduced as

\[
A(\mathcal{G}, \mu) := \ell(\mathcal{G}, \mu) \oplus \mathbb{C}d \oplus \mathbb{C}\hat{c}
\]

(3.4)

with the Lie bracket of its generators defined by

\[
[\xi^{n_a}, \eta^{p_b}] = [\xi, \eta]^{n_a + p_b} + n_a \delta_{n_a, -p_b} B(\xi, \eta) \hat{c}, \quad \forall \xi \in \mathcal{G}_a, \eta \in \mathcal{G}_b,
\]

(3.5)

\[
[d, \xi^{n_a}] = n_a \xi^{n_a}, \quad [\hat{c}, d] = [\hat{c}, \xi^{n_a}] = 0.
\]

(3.6)

\(^4\)The last two properties are automatic if \( \mathcal{G} \) is simple or \( \mu = \text{id} \), which are included as special cases.
\( \mathcal{A}(\mathcal{G}, \mu) \) is a self-dual Lie algebra as it carries the scalar product \( \langle \ , \ \rangle \) given by
\[
\langle \xi^n, \eta^p \rangle = \delta_{n_0, -p_0} B(\xi, \eta), \quad \langle \hat{c}, d \rangle = 1, \quad \langle d, \xi^n \rangle = \langle \hat{c}, \xi^n \rangle = 0.
\] (3.7)

We obtain a \( \mathbb{Z} \)-gradation of \( \mathcal{A}(\mathcal{G}, \mu) \) by the decomposition
\[
\mathcal{A}(\mathcal{G}, \mu) = \oplus_{n \in (\mathcal{E}_\mu + N \mathbb{Z})} \mathcal{A}(\mathcal{G}, \mu)_n = \oplus_{n \in \mathbb{Z}} \mathcal{A}(\mathcal{G}, \mu)_n,
\] (3.8)
where \( \mathcal{A}(\mathcal{G}, \mu)_n \) is the eigensubspace of \( ad \ d \) with eigenvalue \( n \) if \( n \in (\mathcal{E}_\mu + N \mathbb{Z}) \), and \( \mathcal{A}(\mathcal{G}, \mu)_n = \{0\} \) if \( n \notin (\mathcal{E}_\mu + N \mathbb{Z}) \). We need to introduce these zero subspaces for notational consistency, since \( (\mathcal{E}_\mu + N \mathbb{Z}) \) is not necessarily a group in general. This is also consistent with the fact that (3.5) gives zero if \( (n_\alpha + p_\beta) \notin (\mathcal{E}_\mu + N \mathbb{Z}) \). The gradation given by (3.8) clearly satisfies equations (2.2)–(2.4), where now \( \mathbb{Z} := \mathbb{Z} \). We below regard \( \mathcal{G}_0 \) as a subspace of \( \mathcal{A}(\mathcal{G}, \mu) \) by identifying \( \xi \in \mathcal{G}_0 \) with \( \xi \otimes t^0 \in \mathcal{A}(\mathcal{G}, \mu) \), whereby we can write
\[
\mathcal{A}(\mathcal{G}, \mu)_0 = \mathcal{G}_0 \oplus \mathbb{C}d \oplus \hat{c} \hat{\xi}.
\] (3.9)

Since we wish to apply theorem 1, we now set \( \mathcal{A} := \mathcal{A}(\mathcal{G}, \mu) \) and \( \mathcal{K} := \mathcal{A}(\mathcal{G}, \mu)_0 \). We parametrize the general element \( \kappa \in \mathcal{K} \) as
\[
\kappa = \omega + kd + l\hat{c}, \quad \omega \in \mathcal{G}_0, \quad k, l \in \mathbb{C}.
\] (3.10)

It follows from the above that formula (2.7) provides us with a dynamical \( r \)-matrix \( \hat{\mathcal{K}} : \hat{\mathcal{K}} \to \text{End}(\mathcal{A}) \) if we can find a nonempty, open domain \( \hat{\mathcal{K}} \subset \mathcal{K} \) whose elements satisfy the conditions given in (2.6). The point is that we can indeed find such a domain, and actually the maximal domain has the form
\[
\hat{\mathcal{K}} = \{ \kappa = \omega + kd + l\hat{c} \mid l \in \mathbb{C}, \ k \in (\mathbb{C} \setminus \mathbb{R}i), \ \omega \in \mathcal{B}_k \},
\] (3.11)
where \( \mathcal{B}_k \subset \mathcal{G}_0 \) is described as follows. Let \( \lambda_a \) denote an eigenvalue of the operator \( ad \omega|_{\mathcal{G}_0} \) associated with \( \omega \in \mathcal{G}_0 \). By definition, the subset \( \mathcal{B}_k \subset \mathcal{G}_0 \) consists of those \( \omega \in \mathcal{G}_0 \) whose eigenvalues satisfy the following conditions:
\[
(\lambda_0 + k(a + mN)) \notin 2\pi i \mathbb{Z} \quad \forall m \in \mathbb{Z}, \quad \forall a \in \mathcal{E}_\mu \setminus \{0\},
\] (3.12)
\[
\lambda_0 \notin 2\pi i \mathbb{Z}^* \quad \text{and} \quad (\lambda_0 + kmN) \notin 2\pi i \mathbb{Z} \quad \forall m \in \mathbb{Z}^*.
\] (3.13)

If we note that for \( \xi^n \) in (3.3) and \( \kappa \in \mathcal{K} \) written as in (3.10) one has
\[
(ad \kappa)(\xi^n) = kn_a \xi^{n_\alpha} + [\omega, \xi]^{n_\alpha},
\] (3.14)
then the conditions in (3.12) and (3.13) are recognized to be the translation of the condition in (2.6) to our case. The set \( \hat{\mathcal{K}} \) defined by these requirements obviously contains the elements of the form \( \kappa = kd + l\hat{c} \) for any \( k \in (\mathbb{C} \setminus i \mathbb{R}) \), \( l \in \mathbb{C} \), and therefore it is nonempty. It is not difficult to see that \( \hat{\mathcal{K}} \subset \mathcal{K} \) in (3.11) is an open subset, for which one needs \( k \) to have a nonzero real part, and \( \mathcal{B}_k \subset \mathcal{G}_0 \) is a nonempty open subset as well. For completeness, we present a proof of these statements in appendix A.
3.2 One-parameter family of $r$-matrices on $\ell(\mathcal{G}, \mu)$

We now reinterpret the dynamical $r$-matrices $R : \mathcal{K} \to \text{End}(\mathcal{A}(\mathcal{G}, \mu))$ constructed in subsection 3.1 as a family of $r$-matrices

$$R_k : \mathcal{B}_k \to \text{End}(\ell(\mathcal{G}, \mu)),$$

where the parameter $k$ varies in $(\mathbb{C} \setminus i\mathbb{R})$ and the $k$-dependent domain $\mathcal{B}_k \subset \mathcal{G}_0$ appears in (3.11). For any $\omega \in \mathcal{B}_k$, the operator $R_k(\omega)$ is given by

$$R_k(\omega)\eta := f(\text{ad}\omega)\eta, \quad R_k(\omega)\xi^{na} := F(kn_a + \text{ad}\omega)\xi^{na}$$

(3.16)

$\forall \eta \in \mathcal{G}_0 = \ell(\mathcal{G}, \mu)_0$ and $\forall \xi^{na} \in \ell(\mathcal{G}, \mu)_{n_a}$ with $n_a \neq 0$. In other words, by regarding $\ell(\mathcal{G}, \mu)$ as a subspace of $\mathcal{A}(\mathcal{G}, \mu)$, we have $R_k(\omega)X = R(\kappa)X$ for $X \in \ell(\mathcal{G}, \mu)$ and $\kappa \in \hat{\mathcal{K}}$.

It is an easy consequence of theorem 1 that $R_k$ satisfies the operator version of the CDYBE for any fixed $k$:

$$[R_kX, R_kY] - R_k([X, R_kY] + [R_kX, Y]) + \langle X, (\nabla R_k)Y \rangle$$

$$+ (\nabla_{Y_0} R_k)X - (\nabla_{X_0} R_k)Y = -\frac{1}{4}[X, Y], \quad \forall X, Y \in \ell(\mathcal{G}, \mu).$$

(3.17)

Here the Lie brackets are evaluated in $\ell(\mathcal{G}, \mu)$, $X_0$ is the grade 0 part of $X$, and the scalar product on $\ell(\mathcal{G}, \mu)$ is given by the restriction of (3.7). This equation is verified by a simplified version of the calculation done in the proof of theorem 1, the simplification being that $\hat{c}$ has now been set to zero. It is also clear that $R_k : \mathcal{B}_k \to \text{End}(\ell(\mathcal{G}, \mu))$ is a $\mathcal{G}_0$-equivariant map in the natural sense.

For later purpose, we here introduce the shifted $r$-matrices

$$R_k^\pm := R_k \pm \frac{1}{2}I,$$

(3.18)

where $I$ is the identity operator on $\ell(\mathcal{G}, \mu)$. By using the scalar product, we associate with these operator valued maps the corresponding $\ell(\mathcal{G}, \mu) \otimes \ell(\mathcal{G}, \mu)$-valued maps. These are denoted respectively as

$$r^{k,\pm} : \mathcal{B}_k \to \ell(\mathcal{G}, \mu) \otimes \ell(\mathcal{G}, \mu).$$

(3.19)

By translating the CDYBE into tensorial terms, (3.17) becomes

$$[r_{12}^{k,s}(\omega), r_{13}^{k,s}(\omega)] + [r_{12}^{k,s}(\omega), r_{23}^{k,s}(\omega)] + [r_{13}^{k,s}(\omega), r_{23}^{k,s}(\omega)]$$

$$+ T_j^1 \frac{\partial}{\partial \omega_j} r_{23}^{k,s}(\omega) - T_j^2 \frac{\partial}{\partial \omega_j} r_{13}^{k,s}(\omega) + T_j^3 \frac{\partial}{\partial \omega_j} r_{12}^{k,s}(\omega) = 0, \quad s = \pm,$$

(3.20)

where $\omega_j := B(\omega, T_j)$ with a basis $T_j$ of $\mathcal{G}_0$. 

10
3.3 Spectral-parameter-dependent $r$-matrices

The loop algebra $\ell(G, \mu)$ admits an ‘evaluation homomorphism’ $\pi_v : \ell(G, \mu) \to G$ for any fixed $v \in \mathbb{C}^*$,

$$\pi_v : \xi \otimes t^n \mapsto v^n \xi \quad \forall (\xi \otimes t^n) \in \ell(G, \mu). \quad (3.21)$$

It is well known that spectral-parameter-dependent $G \otimes G$-valued $r$-matrices may be obtained by applying these homomorphisms to $\ell(G, \mu) \otimes \ell(G, \mu)$-valued $r$-matrices. In the context of dynamical $r$-matrices, Etingof and Varchenko [1] used this method to derive Felder’s elliptic dynamical $r$-matrices from the basic trigonometric dynamical $r$-matrices of the (untwisted) affine Kac-Moody Lie algebras. We here apply the same procedure to the general family of dynamical $r$-matrices introduced in eq. (3.19). As for the presentation below, we find it convenient to first provide a self-contained definition of the spectral-parameter-dependent $r$-matrices and show afterwards how they are obtained from the evaluation homomorphisms.

We start by collecting some meromorphic functions and identities that will be useful. Consider the standard theta function

$$\theta_1(z|\tau) := -\sum_{j \in \mathbb{Z}} \exp \left( \pi i (j + \frac{1}{2})^2 \tau + 2\pi i (j + \frac{1}{2})(z + \frac{1}{2}) \right), \quad \Im(\tau) > 0, \quad (3.22)$$

which is holomorphic on $\mathbb{C}$ and has simple zeros at the points of the lattice

$$\Omega := \mathbb{Z} + \tau \mathbb{Z}. \quad (3.23)$$

Recall that $\theta_1$ is odd in $z$ and satisfies

$$\theta_1(z + 1|\tau) = -\theta_1(z|\tau), \quad \theta_1(z + \tau|\tau) = -q^{-1}e^{-2\pi iz}\theta_1(z|\tau), \quad q := e^{\pi i \tau}. \quad (3.24)$$

Define now the function

$$\chi(w, z|\tau) := \frac{1}{2\pi i} \frac{\theta_1(\frac{w}{2\pi \tau} + z|\tau)\theta'_1(0|\tau)}{\theta_1(z|\tau)\theta_1(\frac{w}{2\pi \tau}|\tau)}. \quad (3.25)$$

This function is holomorphic in $w$ and in $z$ at the points

$$(w, z) \in (\mathbb{C} \setminus 2\pi i \Omega) \times (\mathbb{C} \setminus \Omega). \quad (3.26)$$

The following important identity holds:

$$\chi(w, z) = \frac{1}{2} \sum_{n \in \mathbb{Z}} e^{2\pi izn} \left[ 1 + \coth \left( \frac{w}{2} + \pi i \tau n \right) \right] \quad (3.27)$$

on the domain

$$D := \{(w, z) \mid w \in (\mathbb{C} \setminus 2\pi i \Omega), \quad -\Im(\tau) < \Im(z) < 0 \}. \quad (3.28)$$

All terms in the sum are holomorphic on $D$, the convergence is absolute at any fixed $(w, z) \in D$, and is uniform on compact subsets of $D$. The verification of (3.27) is a routine matter, example 13 on page 489 of [14] contains a closely related statement.

---

5 We have $\theta_1(z|\tau) = \vartheta_1(\pi z|\tau)$ with $\vartheta_1$ in [14].
We also need the functions
\[
\chi_a(w, z|\tau) := e^{2\pi i a \Omega_a} \left( \chi(w + 2\pi i \frac{a}{N} \tau, z|\tau) - \frac{\delta_{a,0}}{w} \right),
\]
(3.29)
where \(a \in \{0, 1, \ldots, (N-1)\}\) with some positive integer \(N\). The function \(\chi_a(w, z|\tau)\) is holomorphic in \(w\) and in \(z\) if \((w, z)\) belongs to the domain
\[
(C \setminus 2\pi i \Omega_a) \times (C \setminus \Omega) \quad \text{where} \quad \Omega_a := \left( \Omega - \frac{a}{N} \tau \right) \setminus \{0\}.
\]
(3.30)
By using the notation
\[
f_a(w) := \frac{1}{2} \left[ 1 + \coth \frac{w}{2} \right] - \frac{\delta_{a,0}}{w},
\]
(3.31)
we have the identity
\[
\chi_a(w, z|\tau) = e^{2\pi i a \Omega_a} \left( f_a(w + 2\pi i \frac{a}{N} \tau) + \frac{1}{2} \sum_{n \in \mathbb{Z}} e^{2\pi i zn} \left[ 1 + \coth \left( \frac{w}{2} + \pi i \frac{a}{N} \tau + \pi i \tau n \right) \right] \right),
\]
(3.32)
on the domain
\[
D_a := \{(w, z) \mid w \in (C \setminus 2\pi i \Omega_a), -\Im(\tau) < \Im(z) < 0 \}
\]
(3.33)
for any \(a \in \{0, 1, \ldots, (N-1)\}\). All terms in the sum are holomorphic on \(D_a\), the convergence is absolute at any \((w, z) \in D_a\), and is uniform on compact subsets of \(D_a\).

Let now \(\mu\) be an automorphism of \(G\) of order \(N\) as considered previously and fix \(\tau\) with \(\Im(\tau) > 0\). For any \(\omega \in G_0\) and \(a \in E_\mu\), let \(\sigma((\text{ad} \omega)_a)\) be the spectrum of the linear operator \((\text{ad} \omega)_a := \text{ad} \omega|_{\mathfrak{g}_a}\). Define \(B^\tau \subset G_0\) by
\[
B^\tau := \{ \omega \in G_0 \mid \sigma((\text{ad} \omega)_a) \cap 2\pi i \Omega_a = \emptyset \quad \forall a \in E_\mu \}.
\]
(3.34)
It is easy to verify that
\[
B^\tau = B_k \quad \text{if} \quad \tau = \frac{kN}{2\pi i},
\]
(3.35)
where \(B_k \subset G_0\) appears in (3.11). In particular, \(B^\tau\) is an open subset of \(G_0\) that contains the origin. By using the above notations, we now define the function \(R_\tau\) as
\[
R_\tau : B^\tau \times (C \setminus \Omega) \to \text{End}(G), \quad R_\tau(\omega, z)|_{G_a} := \chi_a((\text{ad} \omega)_a, z|\tau).
\]
(3.36)
It follows from the properties of the holomorphic functional calculus on Banach algebras [12] that \(R_\tau\) is well defined and is holomorphic in its variables. Next we introduce also the holomorphic function
\[
r^\tau : B^\tau \times (C \setminus \Omega) \to \mathcal{G} \otimes \mathcal{G}, \quad r^\tau(\omega, z) := B(T_\alpha, R_\tau(\omega, z)T_\beta)T^\alpha \otimes T^\beta,
\]
(3.37)
where \(T_\alpha, T^\beta\) are dual bases of \(G\). We now state one of our main results.
Proposition 2. The function $r^\tau$ introduced above satisfies the spectral-parameter-dependent version of the CDYBE:

$$
[r^\tau_{12}(\omega, z_{12}), r^\tau_{13}(\omega, z_{13})] + [r^\tau_{12}(\omega, z_{12}), r^\tau_{23}(\omega, z_{23})] + [r^\tau_{13}(\omega, z_{13}), r^\tau_{23}(\omega, z_{23})] + T^1_j \frac{\partial}{\partial \omega_j} r^\tau_{23}(\omega, z_{23}) - T^2_j \frac{\partial}{\partial \omega_j} r^\tau_{13}(\omega, z_{13}) + T^3_j \frac{\partial}{\partial \omega_j} r^\tau_{12}(\omega, z_{12}) = 0,
$$

(3.38)

where $z_{\alpha\beta} = (z_\alpha - z_\beta) \in (\mathbb{C}\setminus\Omega)$, $\omega \in B^\tau$, and $\omega_j := B(\omega, T_j)$ with a basis $T_j$ of $G_0$. Furthermore, $r^\tau$ has the properties

$$
\text{Res}_{z=0} r^\tau(\omega, z) = \frac{1}{2\pi i} T^\alpha \otimes T_\alpha,
$$

(3.39)

where $(r^\tau(\omega, z))^T := B(T_\alpha, R^\tau(\omega, z) T_\beta) T^\beta \otimes T^\alpha$ with dual bases $T_\alpha$, $T^\beta$ of $\mathcal{G}$, and

$$
\frac{d}{dx} r^\tau(e^{ad T x}(\omega), z)|_{x=0} = [T \otimes 1 + 1 \otimes T, r^\tau(\omega, z)] \quad \forall T \in \mathcal{G}_0.
$$

(3.40)

The statements in (3.39) follow immediately from the definition (3.36), (3.37) and the properties of the meromorphic functions $\chi_a$ in (3.29). For the first equality in (3.39), one can check that

$$
\text{Res}_{z=0} \chi_a(\omega, z|\tau) = \frac{1}{2\pi i}, \quad 0 \leq a < N.
$$

(3.41)

For the second statement, one uses the invariance of the scalar product $B$ on $\mathcal{G}$ and

$$
\chi_0(-w, z|\tau) = -\chi_0(w, -z|\tau), \quad \chi_a(-w, z|\tau) = -\chi_{N-a}(w, -z|\tau), \quad 0 < a < N.
$$

(3.42)

The $G_0$-equivariance property (3.40) is obvious from the definition of $r^\tau$. As for the CDYBE (3.38), it is consequence of the following result.

Proposition 3. The dynamical $r$-matrix $r^\tau$ given by (3.36), (3.37) results by evaluation homomorphism from the dynamical $r$-matrix $r^{k,+}$ in (3.19). More precisely, if we set

$$
\tau = \frac{kN}{2\pi i} \quad \text{and} \quad \frac{v_1}{v_2} = \exp(\frac{2\pi iz}{N}) \quad \text{with} \quad -\Im(\tau) < \Im(z) < 0,
$$

(3.43)

then the evaluation homomorphism (3.21) yields the relation

$$
(\pi_{v_1} \otimes \pi_{v_2})(r^{k,+}(\omega)) = r^\tau(\omega, z) \quad \forall \omega \in B_k = B^\tau.
$$

(3.44)

Proof. The left hand side of (3.44) gives only a formal infinite sum in general. Below we first calculate this sum, and then notice that it converges to the function on the right hand side of (3.44) if the variables satisfy (3.43).
Let $T_{a,j}$ and $T^j_a$ ($j = 1, \ldots, \text{dim}(G_a)$) denote bases of $G_a$ ($a \in E_\mu$) subject to the relations

$$
\langle T_{0,j}, T^l_0 \rangle = \delta^l_j, \quad \langle T_{a,j}, T^l_{N-a} \rangle = \delta^l_j, \quad \forall a \in E_\mu \setminus \{0\}.
$$

(3.45)

Introduce corresponding bases of $\ell(G, \mu)$:

$$
T_{a,j}[n_a] := T_{a,j} \otimes t_n^a, \quad T^j_a[n_a] := T^j_a \otimes t_n^a, \quad \forall a \in E_\mu, \ n_a \in (a + N\mathbb{Z}).
$$

(3.46)

By definition, we then have

$$
r^{k,+} (\omega) = \sum_{j,l=1}^{\text{dim}(G_0)} \sum_{n_0 \in N\mathbb{Z}} \langle T_{0,j}[-n_0], R_k^+ (\omega) T_{0,l}[n_0] \rangle \, T^j_0[n_0] \otimes T^l_0[-n_0]
$$

$$
+ \sum_{a \in E_\mu \setminus \{0\}} \sum_{j,l=1}^{\text{dim}(G_a)} \sum_{n_a \in (a + N\mathbb{Z})} \langle T_{N-a,j}[-n_a], R_k^+ (\omega) T_a[l][n_a] \rangle \, T^j_a[n_a] \otimes T^l_{N-a}[-n_a].
$$

(3.47)

By substituting the definition of $R_k^+ (\omega)$, (3.18) with (3.16), we obtain that

$$
\langle T_{N-a,j}[-n_a], R_k^+ (\omega) T_a[l][n_a] \rangle = B(T_{N-a,j}, (F(kn_a + \text{ad} \omega) + \frac{1}{2}) T_a[l])
$$

(3.48)

for $a \in E_\mu \setminus \{0\}$, and

$$
\langle T_{0,j}[-n_0], R_k^+ (\omega) T_{0,l}[n_0] \rangle = B(T_{0,j}, (F(kn_0 + \text{ad} \omega) + \frac{1}{2}) T_{0,l}), \quad n_0 \neq 0,
$$

(3.49)

$$
\langle T_{0,j}[0], R_k^+ (\omega) T_{0,l}[0] \rangle = B(T_{0,j}, (F(kn_0 + \text{ad} \omega) + \frac{1}{2}) T_{0,l}),
$$

(3.50)

where the functions $f$ and $F$ are given in (1.1). This implies that the left hand side of (3.44) can be written in the following form:

$$
(\pi_{v_1} \otimes \pi_{v_2})(r^{k,+} (\omega)) = \sum_{j,l=1}^{\text{dim}(G_0)} B(T_{0,j}, \psi_0((\text{ad} \omega)_0, z | k) T_{0,l}) \, T^j_0 \otimes T^l_0
$$

$$
+ \sum_{a \in E_\mu \setminus \{0\}} \sum_{j,l=1}^{\text{dim}(G_a)} B(T_{N-a,j}, \psi_a((\text{ad} \omega)_a, z | k) T_{a,l}) \, T^j_a \otimes T^l_{N-a}
$$

(3.51)

with

$$
\psi_a((\text{ad} \omega)_a, z | k) = \frac{\exp(2\pi izm)}{2} \sum_{m \in \mathbb{Z}} e^{2\pi izm} \left[ 1 + \coth \frac{kNm + ka + (\text{ad} \omega)_a}{2} \right], \quad a \neq 0,
$$

(3.52)

and

$$
\psi_0((\text{ad} \omega)_0, z | k) = \left[ \frac{1}{2} + f((\text{ad} \omega)_0) \right] + \frac{1}{2} \sum_{m \in \mathbb{Z}^*} e^{2\pi izm} \left[ 1 + \coth \frac{kNm + (\text{ad} \omega)_0}{2} \right].
$$

(3.53)
To obtain the $a \neq 0$ terms in (3.51) from (3.47), we used (3.48) and the parametrization
$$v_1 = \exp(\frac{2\pi i z}{N}),$$
whereby
$$\sum_{n_a \in (a+N \mathbb{Z})} \langle T_{N-a,j}[-n_a], R^+_k(\omega)T_{a,l}[n_a] \rangle (\pi_{v_1} \otimes \pi_{v_2}) (T^j_a[n_a] \otimes T^l_{N-a}[-n_a])$$
$$= \frac{1}{2} e^{2\pi i m} \sum_{m \in \mathbb{Z}} e^{2\pi i z m} B(T_{N-a,j}, [1 + 2F(ka + kmN + \text{ad} \omega)]T_{a,l}) T^j_a \otimes T^l_{N-a}$$
$$= B(T_{N-a,j}, \frac{1}{2} e^{2\pi i m} \sum_{m \in \mathbb{Z}} e^{2\pi i z m} [1 + 2F(ka + kmN + \text{ad} \omega)]T_{a,l}) T^j_a \otimes T^l_{N-a}. \quad (3.54)$$
This leads to (3.51) with (3.52) by inserting the definition of $F$ (1.1) and noting that $(\text{ad} \omega)_a T_{a,l} = (\text{ad} \omega)_a T_{a,l}$. The $a = 0$ term is dealt with in a similar manner.

Now we come to the main point. We notice that if on the right hand sides of (3.52) and (3.53) $(\text{ad} \omega)_a$ is replaced by a complex variable $w$ and one uses also $\tau = \frac{kN}{2\pi i}$, then these series become precisely identical with the corresponding series in (3.32), which are convergent on the domain $D_a$ (3.33) for any $a \in \mathcal{E}_\mu$. Since these are absolute convergent series and the convergence is uniform on compact subsets of $D_a$, it follows that the corresponding operator series in (3.52), (3.53) converge, too. Therefore, if
$$\tau = \frac{kN}{2\pi i}, \quad \omega \in B^\tau, \quad -\Im(\tau) < \Im(z) < 0, \quad (3.55)$$
then $\psi_a((\text{ad} \omega)_a, z|k) \in \text{End}(\mathcal{G}_a)$ is well defined by the corresponding series in (3.52), (3.53), and on this domain we obtain
$$\psi_a((\text{ad} \omega)_a, z|k) = \chi_a((\text{ad} \omega)_a, z|\tau), \quad \forall a \in \mathcal{E}_\mu. \quad (3.56)$$
If we now compare (3.51) with the definition of $r^\tau$ given by (3.36), (3.37), then (3.56) allows us to conclude that $(\pi_{v_1} \otimes \pi_{v_2})(r^{k_+}(\omega)) = r^\tau(\omega, z)$ holds indeed on the domain given by (3.43). Q.E.D.

It is clear from the proof that (3.43) is necessary for (3.44); the series appearing in (3.52) and (3.53) do not converge if $z$ lies outside the strip in (3.43). Thus, by applying $\pi_{v_1} \otimes \pi_{v_2} \otimes \pi_{v_3}$ to the CDYBE (3.20), proposition 3 directly implies proposition 2 if $z_{12}, z_{13}, z_{23}$ all lie in this strip. However, by the holomorphicity of the function $r^\tau$, (3.38) is then necessarily valid for any $\omega, z$ for which $r^\tau$ is defined by eqs. (3.36), (3.37).

Of course, it is possible to calculate $(\pi_{v_1} \otimes \pi_{v_2})(r^{k_+}(\omega))$ as well on an appropriate domain of $v_1, v_2$. This is left as an exercise.

### 3.4 Recovering Felder’s $\tau$-matrices

In this subsection $\mathcal{G}$ is a complex simple Lie algebra, and we start by fixing a Cartan subalgebra and a corresponding set $\Phi^+$ of positive roots. We also choose root vectors $E_\alpha (\alpha \in \Phi)$ and dual
bases of the Cartan subalgebra, $H_i$ and $H^j$, normalized so that
\[ B(H_i, H^j) = \delta_i^j, \quad B(E_\alpha, E_{-\alpha}) = 1. \] (3.57)

If $\alpha_i \in \Phi^+$ are the simple roots, then there is a unique element, $J$, of the Cartan subalgebra for which
\[ \alpha_i(J) = 1 \quad \forall i = 1, \ldots, \text{rank}(G). \] (3.58)

Let $N$ be the largest eigenvalue of $(\text{ad} J)$ plus 1, i.e., the Coxeter number of $G$. We wish to show that the application of our preceding construction to the automorphism
\[ \mu := \exp(\frac{2\pi i}{N} \text{ad} J) \] (3.59)
provides an $r$-matrix that is equivalent to Felder’s solution of the CDYBE \cite{Felder}. The fixed point set $\mathcal{G}_0$ of this $\mu$ is the chosen Cartan subalgebra of $G$, and Felder’s $r$-matrix is in fact equivalent to
\[ S^\tau(\omega, z) := \frac{1}{2\pi i} \frac{\theta_i(z|\tau)}{\theta_1(z|\tau)} H_i \otimes H^i + \sum_{\alpha \in \Phi} \chi(\alpha(\omega), z|\tau) E_\alpha \otimes E_{-\alpha}. \] (3.60)

To be precise, Felder’s original $r$-matrix, $\mathcal{F}^\tau$, is given by $\mathcal{F}^\tau(\omega, z) := 2\pi i S^\tau(2\pi i \omega, z)$, which is a substitution that leaves the CDYBE invariant. Referring to the corresponding terms in (3.60), below we also write $S^\tau := S^\tau_{\text{Cartan}} + S^\tau_{\text{root}}$.

It is well known that $\mu$ (3.59) acts as a Coxeter element on a Cartan subalgebra which is ‘in opposition’ to the Cartan subalgebra $\mathcal{G}_0$ and that $\mathcal{A}(\mathcal{G}, \mu)$ with its natural gradation is isomorphic to the untwisted affine Lie algebra of $\mathcal{G}$ equipped with its principal gradation \cite{Kac}. In \cite{Kac} the homogeneous realization of the untwisted affine Lie algebra was used to recover Felder’s $r$-matrix with the aid of evaluation homomorphisms. The principal realization provided by $\mathcal{A}(\mathcal{G}, \mu)$ must of course give an equivalent result. It is enlightening to see how this works, and it also provides a useful check on our foregoing calculations.

By using the above notations, now we can spell out $r^\tau$ from (3.36), (3.37) explicitly as $r^\tau = r^\tau_{\text{Cartan}} + r^\tau_{\text{root}}$ with
\[ r^\tau_{\text{Cartan}}(\omega, z) = B(H_i, \chi_0(\text{ad} \omega, z|\tau) H_j) H^i \otimes H^j = \chi_0(0, z|\tau) H_i \otimes H^i. \] (3.61)

The second equality holds because $\chi_0(\text{ad} \omega, z|\tau) H_j = \chi_0(0, z|\tau) H_j$, which in turn follows from $(\text{ad} \omega) H_j = 0$. It is easy to compute that
\[ \chi_0(0, z|\tau) = \lim_{w \rightarrow 0} \chi_0(w, z|\tau) = \frac{1}{2\pi i} \frac{\theta_i(z|\tau)}{\theta_1(z|\tau)}. \] (3.62)

Thus the Cartan parts of $S^\tau$ and $r^\tau$ are equal, and are $\omega$-independent.

As for the root part, by using that $(\text{ad} \omega) E_\alpha = \alpha(\omega) E_\alpha$, the definitions give
\[ r^\tau_{\text{root}}(\omega, z) = \sum_{\alpha \in \Phi^+} e^{\frac{2\pi i \alpha(J)}{N} \tau} \chi(\alpha(\omega) + 2\pi i \frac{\alpha(J)}{N}, z|\tau) E_\alpha \otimes E_{-\alpha} \]
\[ + \sum_{\alpha \in \Phi^+} e^{\frac{2\pi i (N-\alpha(J))\tau}{N}} \chi(-\alpha(\omega) + 2\pi i \frac{N-\alpha(J)}{N}, z|\tau) E_{-\alpha} \otimes E_\alpha. \] (3.63)
Then we use the identity
\[ \chi(w + 2\pi i\tau, z|\tau) = e^{-2\pi iz}\chi(w, z|\tau), \] (3.64)
which permits us to rewrite \( r^\tau_{\text{root}} \) as
\[ r^\tau_{\text{root}}(\omega, z) = \sum_{\alpha \in \Phi} e^{2\pi i\frac{\alpha(J)z}{N}}\chi(\alpha(\omega) + 2\pi i\frac{\alpha(J)}{N}\tau, z|\tau)E_{\alpha} \otimes E_{-\alpha}. \] (3.65)

By comparing the above expressions of \( r^\tau \) and \( S^\tau \), we conclude that
\[ r^\tau(\omega, z) = \left( e^{\frac{2\pi i z_1 \text{ad} J}{N}} \otimes e^{\frac{2\pi i z_2 \text{ad} J}{N}} \right) S^\tau(\omega + 2\pi i\frac{\tau}{N}J, z|\tau) \text{ with } z = z_1 - z_2. \] (3.66)

If the dynamical variable \( \omega \) belongs to a Cartan subalgebra, \( \mathcal{G}_0 \), then the constant shifts of \( \omega \) and the similarity transformations by \( e^{z_1 \text{ad} H} \otimes e^{z_2 \text{ad} H} \) for any \( H \in \mathcal{G}_0 \), \( z_1 - z_2 = z \) map the solutions of the CDYBE to other solutions. In fact, these transformations are special cases of the gauge transformations considered in section 4.2 of [1].

In summary, we have shown that the solution of the CDYBE provided by proposition 2 in the principal case of \( \mu \) in (3.59) is gauge equivalent to Felder’s dynamical \( r \)-matrix in the sense of (3.66).

Recently generalizations of Felder’s \( r \)-matrices have been found [16] for which the dynamical variables belong to a subalgebra of a Cartan of a simple Lie algebra \( \mathcal{G} \). The subalgebra in question is the fixed point set of an outer automorphism of \( \mathcal{G} \) of finite order, and the \( r \)-matrices given by proposition 4.2 in [16] contain the same elliptic functions that appear in (3.36). These \( r \)-matrices are very likely to be gauge equivalent to those special cases of the \( r \)-matrices constructed in subsection 3.3 for which \( \mathcal{G} \) is simple and \( \mathcal{G}_0 \) is a contained in a Cartan subalgebra. The precise relationship will be described elsewhere.

### 4 Conclusion

The purpose of this paper has been to further develop the construction of dynamical \( r \)-matrices building mainly on the seminal paper [1] and our recent work [8]. Here our first main result is theorem 1, whereby a dynamical \( r \)-matrix is associated with any graded self-dual Lie algebra subject to the rather mild conditions in (2.2)–(2.4) and the strong spectral condition described in (2.6). Our second main result is the application of this construction to the general class of affine Lie algebras \( \mathcal{A}(\mathcal{G}, \mu) \) corresponding to the automorphisms of the finite-dimensional self-dual Lie algebras that preserve the scalar product and are of finite order. The resulting dynamical \( r \)-matrices are generalizations of the basic trigonometric dynamical \( r \)-matrices of [1], which are recovered if \( \mu \) is a Coxeter automorphism of a simple Lie algebra. Motivated by the derivation of Felder’s elliptic dynamical \( r \)-matrices [4] found in [1], we have also determined the spectral-parameter-dependent \( \mathcal{G} \otimes \mathcal{G} \)-valued dynamical \( r \)-matrices that correspond to the \( \mathcal{A}(\mathcal{G}, \mu) \otimes \mathcal{A}(\mathcal{G}, \mu) \)-valued \( r \)-matrices directly obtained from theorem 1. The result is given explicitly by proposition 2 and proposition 3 is subsection 3.3.
It is worth noting that the conditions of theorem 1 are satisfied also if $A$ is an arbitrary Kac-Moody Lie algebra associated with a symmetrizable generalized Cartan matrix, equipped with the principal gradation $[15]$. In this case one recovers the $r$-matrices given by equation (3.4) in [1]. It would be interesting to find applications of theorem 1 outside the aforementioned classes of Lie algebras. As candidates, we plan to examine the two-dimensional (toroidal) analogues of the affine Lie algebras [17].

Another interesting problem is to find applications of the generalizations of Felder’s $r$-matrices provided by proposition 2 in integrable systems. In this respect, it appears promising to seek for generalized Calogero-Moser type systems, since certain spin Calogero-Moser systems are known to be closely related to Felder’s $r$-matrices [3, 4]. A different possibility is to uncover these $r$-matrices in the framework of generalized WZNW models, such as those introduced recently in [18]. This approach would require generalizing the results in [10] about the exchange $r$-matrices of the usual WZNW model. We wish to pursue this line of research in the future.

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A The maximal open domain $\mathcal{K} \subset A(G, \mu)_0$

In this appendix we show that if $A = A(G, \mu)$, then the maximal, nonempty, open domain on which the $r$-matrix of theorem 1 can be defined is given by $\mathcal{K}$ in (3.11), where $\omega \in B_k$ is subject to the conditions in (3.12) and (3.13).

In general, the elements of the domain $\mathcal{K} \subset A_0$ must satisfy the spectral conditions (2.7). If $A = A(G, \mu)$ and $\kappa \in \mathcal{K}$ is parametrized as in (3.10), then these conditions are explicitly given by (3.12) and (3.13), where $\lambda_\omega$ is an arbitrary eigenvalue of ad $\omega|G_a$. Since $\lambda_0 = 0$ is always one of the eigenvalues, the second condition in (3.13) implies that $k \neq 2\pi i m/n$ for any $n \in \mathbb{Z}$, $m \in \mathbb{Z}^*$. As $\mathcal{K}$ must be an open subset of $\mathcal{K}$, it follows that $k \in (\mathbb{C} \setminus i\mathbb{R})$ for any admissible $\kappa = \omega + kd + l\hat{c}$. Note that $\mathcal{K} \neq \emptyset$, since e.g. the elements of the form $\kappa = kd + l\hat{c}$ in (3.11) satisfy the conditions (3.12), (3.13). Hence we only have to show that (3.11) subject to these conditions is an open subset of $\mathcal{K}$.

If $\lambda_\omega$ is an arbitrary eigenvalue of ad $\omega$ on $G_a$ and $k \in (\mathbb{C} \setminus i\mathbb{R})$, then let us consider the real line in $\mathbb{C}$ defined by

$$L_{\lambda_\omega, k}(t) = \lambda_\omega + kt, \quad \forall t \in \mathbb{R}. \quad (A.1)$$

This line intersects the imaginary axis for $t = t_{\lambda_\omega, k}$ at the point $P_{\lambda_\omega, k} = L_{\lambda_\omega, k}(t_{\lambda_\omega, k})$,

$$t_{\lambda_\omega, k} = -\Re(\lambda_\omega)/\Re(k), \quad P_{\lambda_\omega, k} = \lambda_\omega - k\Re(\lambda_\omega)/\Re(k). \quad (A.2)$$
Now the condition in (3.12) can be reformulated as follows:
\[ P_{\lambda_a,k} \notin 2\pi i\mathbb{Z} \quad \text{or} \quad t_{\lambda_a,k} \notin (a + N\mathbb{Z}), \quad \forall a \in \mathcal{E}_\mu \setminus \{0\}. \quad (A.3) \]
This can be further reformulated as the requirement
\[
\left| e^{P_{\lambda_a,k}} - 1 \right|^2 + \left| e^{\frac{2\pi i}{N}(t_{\lambda_a,k} - a)} - 1 \right|^2 \neq 0.
\] (A.4)

It is also useful to rephrase the second condition in (3.13) as
\[ P_{\lambda_0,k} \notin 2\pi i\mathbb{Z} \quad \text{or} \quad t_{\lambda_0,k} \notin N\mathbb{Z}^*. \quad (A.5) \]
Let \( T : \mathbb{C} \to \mathbb{C} \) be an arbitrary continuous function, which is zero precisely on \( N\mathbb{Z}^* \). (For example, we may use \( T(z) = z^{-1} \sin(N^{-1}\pi z) \).) Then (A.5) is equivalent to
\[
\left| e^{P_{\lambda_0,k}} - 1 \right|^2 + |T(t_{\lambda_0,k})|^2 \neq 0.
\] (A.6)

Since the left hand sides of (A.4) and (A.6) are given by continuous functions of \( k \) and the \( \lambda_a \), it follows that these inequalities are stable with respect to small variations of \( k \) and the \( \lambda_a \). The same is true for the first condition \( \lambda_0 \notin 2\pi i\mathbb{Z}^* \) in (3.13). The statement that \( \mathcal{K} \subset \mathcal{K} \) and \( \mathcal{B}_k \subset \mathcal{G}_0 \) subject to (3.11), (3.12), (3.13) are open subsets follows from this observation by taking into account that the position of the eigenvalues of \( \text{ad} \omega \) varies continuously with \( \omega \in \mathcal{G}_0 \).

This means that by choosing \( \omega \) near enough to say \( \omega^* \), any eigenvalue of \( \text{ad} \omega \) can be taken to be arbitrarily close to some eigenvalue of \( \text{ad} \omega^* \).

\section{B A remark on some finite-dimensional \( r \)-matrices}

We here describe some finite-dimensional dynamical \( r \)-matrices, which were first considered in the appendix of [11], and point out a relationship between these and the infinite-dimensional \( r \)-matrices described in subsection 3.2.

Let \( \mu \) be an automorphism of a self-dual Lie algebra of the same type as in section 3 and recall the decomposition in (3.1), (3.2). For any \( a \in \mathcal{E}_\mu \) and integer \( q \) specified below, introduce the meromorphic function \( f_{a,q} \) by

\[
f_{0,q}(w) := \frac{1}{2} \coth \frac{w}{2} - \frac{1}{w}, \quad f_{a,q}(w) := \frac{1}{2} \coth \frac{1}{2}(w + \frac{2\pi i}{N}qa) \quad \text{if} \quad a \neq 0. \quad (B.1)
\]

In order to guarantee that these functions are holomorphic in a neighbourhood of \( w = 0 \), we require the integer \( q \) to satisfy the conditions

\[
1 \leq q \leq (N - 1), \quad qa \notin N\mathbb{Z}^* \quad \forall a \in \mathcal{E}_\mu \setminus \{0\}. \quad (B.2)
\]

Then there exists a nonempty open domain \( \mathcal{G}_0 \subset \mathcal{G}_0 \), containing the origin, on which the map \( \rho_q : \mathcal{G}_0 \to \text{End}(\mathcal{G}) \) can be defined by

\[
\rho_q(\omega) \xi := f_{a,q}(\text{ad} \omega) \xi \quad \forall \xi \in \mathcal{G}_a, \quad \omega \in \mathcal{G}_0. \quad (B.3)
\]
It can be shown that $\rho_q$ satisfies the CDYBE (1.3), where $\mathcal{A}$ is replaced by $\mathcal{G}$ and $\mathcal{K}$ is taken to be $\mathcal{G}_0$. If $\mu = \text{id}$, then $\rho_q$ becomes the well-known canonical (or Alekseev-Meinrenken) dynamical $r$-matrix [1, 3, 10]. In the case $q = 1$, which always satisfies (B.2), $\rho_q$ has been introduced in [11], where it was proved that it solves the CDYBE. The proof given in [11] is very elegant and is very indirect. A direct proof in the case $\mu = \text{id}$ is written down in [19]. For general $\mu$ and $q$, a proof of the CDYBE for $\rho_q$ can be extracted from the following observation. If we let $k := \frac{2\pi i}{N}q$, then we have

$$\rho_q(\omega)\eta = R_k(\omega)\eta \quad \text{and} \quad (\rho_q(\omega)\xi)^{n_a} = R_k(\omega)\xi^{n_a}$$

(B.4)

for any $\eta \in \mathcal{G}_0$ and $\xi \in \mathcal{G}_a$, $a \neq 0$, $n_a \in (a + NZ)$, where $R_k$ refers to the formula (3.16). It should be stressed that this is a relationship purely at the level of formulas, since in the definition of the infinite-dimensional $r$-matrices in section 3 the imaginary values of $k$ were excluded for domain reasons. Nevertheless, it follows from this coincidence of formulas that essentially the same algebraic computation that proves the CDYBE (3.17) can be repeated to verify the CDYBE for $\rho_q$. We have also verified the CDYBE for $\rho_q$ by a direct calculation that proceeds analogously to the proof of our theorem 1.

In certain cases $\rho_q$ is equivalent to an $r$-matrix of the form in (1.2) by a shift of the dynamical variable. Namely, this happens if the automorphism $\mu$ can be written as

$$\mu = \exp\left(\frac{2\pi i}{N}\text{ad} M\right), \quad M \in \mathcal{G},$$

(B.5)

where $\text{ad} M$ is diagonalizable and the fixed point set $\mathcal{G}_0$ of $\mu$ satisfies

$$\mathcal{G}_0 = \text{Ker}(\text{ad} M).$$

(B.6)

In particular, by (B.3), $\mu$ is an inner automorphism of $\mathcal{G}$. If these assumptions hold, then we can define a new $r$-matrix $\tilde{\rho}_q$ by

$$\tilde{\rho}_q(\omega) := \rho_q(\omega - \frac{2\pi i}{N}qM),$$

(B.7)

and this $r$-matrix can be identified with $R$ in (1.2) by taking $\mathcal{A} := \mathcal{G}$ and $\mathcal{K} := \mathcal{G}_0$. The $\mathcal{G}_0$-equivariance property of the dynamical $r$-matrices is respected by the shift of the variable in (B.7) on account of (B.6).
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