Social Welfare Maximization and Conformism via Information Design in Linear-Quadratic-Gaussian Games

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Abstract

We consider linear-quadratic Gaussian (LQG) games in which players have quadratic payoffs that depend on the players’ actions and an unknown payoff-relevant state, and signals on the state that follow a Gaussian distribution conditional on the state realization. An information designer decides the fidelity of information revealed to the players in order to maximize the social welfare of the players or reduce the disagreement among players’ actions. Leveraging the semi-definiteness of the information design problem, we derive analytical solutions for these objectives under specific LQG games. We show that full information disclosure maximizes social welfare when there is a common payoff-relevant state, when there is strategic substitutability in the actions of players, or when the signals are public. Numerical results show that as strategic substitution increases, the value of the information disclosure increases. When the objective is to induce conformity among players’ actions, hiding information is optimal. Lastly, we consider the information design objective that is a weighted combination of social welfare and cohesiveness of players’ actions. We obtain an interval for the weights where full information disclosure is optimal under public signals for games with strategic substitutability. Numerical solutions show that the actual interval where full information disclosure is optimal gets close to the analytical interval obtained as substitution increases.

1 Introduction

In an incomplete information game, multiple players compete to maximize their individual payoffs that depend on the action of the player, other players’ actions and on the realization of an unknown state. Incomplete information games are used to model power allocation of users in wireless networks with unknown channel gains [1, 2, 3, 4], traffic flow in communication or transportation networks [5, 6, 7], oligopoly price competition [8, 9], consumer behavior in a demand response management scheme [10, 11], coordination of autonomous teams [12, 13], and currency attacks of investors [14, 15]. The information design problem refers to the determination of the information fidelity of the signals given to the players so that the induced actions of players maximize a system level objective. In this problem, we can envision the existence of an information designer that can provide the “best” information about the payoff-relevant unknown states to the players according to its objective. As per the above examples, the information designer can represent an entity such as a system designer overseeing the spectrum allocation, a market-maker, an independent

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system operator in the power grid, system designer, or the federal reserve. Depending on the context, the information designer may have different objectives such as maximizing social welfare, inducing consensus between players’ actions, or reducing the likelihood that players take a certain (undesired) action. In the absence of a literal information designer, the information design problem can be used to analyze the extent to which extra information to the players can influence the actions of players and affect a system level objective [16, 17].

We focus on the information design problem in linear quadratic Gaussian (LQG) games. In an LQG game, the players have quadratic payoff functions, and the state and the signals (types) come from a Gaussian distribution [18]. The quadratic payoff structure implies that there exists rational linear strategies which are mappings from local signals to actions. Under certain assumptions, the rational behavior in LQG games defined as the Bayesian Nash equilibrium (BNE) is unique. The linearity of BNE strategies allow the information design problem to be a semi-definite program (SDP) when the information designer’s objective is a quadratic function of the action profile of the players and the payoff-relevant states [19] (Section 2).

Building on the SDP formulation of the information design problem established in [19], we analyze optimal information structures when the system level objective is to maximize social welfare while maintaining a certain level of conformism in the actions of the players. Social welfare is defined as the sum of players’ payoffs. We measure the conformism of the players by the total deviation of the actions from the mean action. Minimizing the total deviation between the actions of players induces the actions of players to be close to each other. We represent the objective to maximize social welfare while maintaining conformism of the players by a weighted combination of the social welfare and the total deviation of actions objectives.

An information structure comprises signal transmission rules and the probability distribution from which signals are generated. Signals transmitted to players convey information about payoff relevant states. Privacy levels of information structures form a distinction between them. The information structure is public when all players receive a common signal. Otherwise, when the players receive individual signals, the signal structure is private. Another distinction is based on the fidelity of information carried by the signals. A signal can carry no, partial or full information. No information disclosure does not improve the prior information of the players about the payoff relevant state, while signals reveal the payoff relevant state under full information disclosure. A partial information disclosure is when the signals carry some information but do not fully reveal the payoff relevant state to the players.

When the information designer’s objective is to maximize social welfare, we show that full information disclosure is optimal if there is a common payoff state (Proposition 7), when there are symmetric interactions between the actions of players, or if we restrict feasible set to the set of public information structures (Proposition 9). In particular, when the interactions are strategically substitutable, i.e., a player’s incentive to increase its action decreases when others increase their actions, full information disclosure is optimal for welfare maximization (Theorem 1). Conversely, when the interactions are strategic complementary, i.e., a player’s incentive to increase its action increases when others increase their actions, full information disclosure is optimal for welfare maximization only if the complementarity is weak. These results follow the intuition that the designer would like to reveal as much information as possible when positively correlating the actions of players improves the system level objective, i.e., when the payoffs of players are aligned with the system-level objective [17].

When the objective is to minimize the deviation of players’ actions, we show that no information disclosure is optimal for any LQG game (Proposition 9). That is, players’ actions are closer to consensus when information is hidden from the players.

If the information designer aims to maximize social welfare while maintaining a level of conformism within the society, we identify a critical weight on the conformism term of the objective based on payoff coefficients below which full information disclosure is preferred to no information disclosure (Propositions 10).
and 11). That is, the benefit of revealing information outweighs the cost of deviation between players’ actions. Numerical solutions to the information design problem show the existence of private signal distributions that outperform both public (full and no) information disclosure schemes.

1.1 Related Literature

Information design is commonly used to induce desirable behavior in congestion games with a particular focus on routing or flow problems in transportation networks [20, 21, 22, 7]. Indeed, a well-established common result in these works is the diminishing value of extra information, and individual players and the system level being worse-off under full information compared to private information schemes. Besides congestion control, information design framework is used to identify key/central players in social networks with respect to the goals of the system designer [23, 24], to maximize the utility of the insured agents in a competitive insurance market [25], and to design public health warning policies against recurrent risks, e.g., pandemics, for public health agencies with different reputation levels [26]. Here, instead of focusing on a particular game, we study the class of LQG games that have broad applications in various fields including operations, autonomous teams, and smart grids.

In general, information design problem involves a maximization over all information structures (all possible distributions) which is algorithmically challenging to solve. In an effort to obtain analytical solutions, and thus insights about the effects of the information design on the system, current approaches make structural assumptions about the state space (finite, Gaussian), the system designer’s objective (depends on posterior belief induced or on the actions induced, or is a step function of the induced posterior mean), and the payoff of agents (posterior mean induced by a public signal)—see [27] for a more detailed discussion of different structural assumptions on information design problems. Similarly, we consider linear-quadratic utility functions for the agents, a quadratic objective for the system designer, and restrict the information structure to Gaussian signals as in [19]. This structure allows the representation of the information design problem as a tractable optimization problem, in particular an SDP with a quadratic objective and linear constraints. As per the discussion above, by focusing on particular objectives for the designer (social welfare and conformism), we provide analytical insights about the value of information and optimal information structures.

Information design is one form of intervention mechanism to influence behavior of strategic agents [6, 28]. Other intervention mechanisms include providing financial incentives [6, 29, 30], system utility design [31, 32, 33, 34, 35], and nudging or player control during learning dynamics [36, 37, 38, 39, 40]. Financial incentives include taxations [6] or rewards [29] that directly modify the utility of the agents. System utility design is motivated by the use of game theoretic learning dynamics in technological multi-agent systems in which individual payoffs are assigned/coded by a system designer prior to the dynamics so that the learning dynamics converge to a desired operating conditions. Relevant to concepts in seeding in advertising and recommendation systems, nudging or player control primes a few individuals to follow the system designer’s will or intervene with the outcomes of a few individuals, so that the learning dynamics converge to desired equilibria. In contrast to these line of work, the information design framework aims to manage the uncertainties of players so that their expectation of their payoffs align with the objective of a system designer. In information design, the system designer does not control agents directly, rather it determines when and who to reveal information to, so that players’ evaluation of their payoffs lead to better outcomes from the system designer’s perspective. In this sense, there is a limit to the system designer’s capability to achieve its goal. This limit determines the value of information.
2 Information Design in Linear-Quadratic-Gaussian (LQG) Games

2.1 Linear-Quadratic-Gaussian Games

A linear-quadratic-Gaussian (LQG) game corresponds to an incomplete information game with quadratic payoff functions and Gaussian information structures. For a given set of players denoted with $N = \{1, ..., n\}$, each player $i \in N$ decides on his action $a_i \in A_i \equiv \mathbb{R}$ according to the payoff function,

$$u_i(a, \gamma) = -H_{i,i}a_i^2 - 2 \sum_{j \neq i} H_{i,j}a_i a_j + 2 \gamma_i a_i + d_i(a_{-i}, \gamma) \quad (1)$$

where $a \equiv (a_i)_{i \in N} \in A \equiv \mathbb{R}^n$ and $\gamma \equiv (\gamma_i)_{i \in N} \in \Gamma \equiv \mathbb{R}^n$ correspond to an action profile and a payoff state, respectively. The term $d_i(a_{-i}, \gamma)$ is an arbitrary function of the opponents’ actions $a_{-i} \equiv (a_j)_{j \neq i}$ and payoff state $\gamma$. We collect the coefficients of the quadratic payoff function in a matrix $H = [H_{ij}]_{n \times n}$.

Payoff state $\gamma$ follows a Gaussian distribution, i.e., $\gamma \sim \psi(\mu, \Sigma)$ where $\psi$ is a multivariate normal probability distribution with mean $\mu \in \mathbb{R}^n$ and covariance matrix $\Sigma$. Payoff structure of the game is defined as $G \equiv ((u_i)_{i \in N}, \psi)$. Each player $i \in N$ receives a private signal $\omega_i \in \Omega_i \equiv \mathbb{R}^{m_i}$ for some $m_i \in \mathbb{N}^+$. We define the information structure of the game $\zeta(\omega | \gamma)$ as the conditional distribution of $\omega \equiv (\omega_i)_{i \in N}$ given $\gamma$. We assume the joint distribution over the random variables $(\omega, \gamma)$ is Gaussian; thus, $\zeta$ is a Gaussian distribution.

Next, we provide two canonical examples of LQG games.

2.1.1 Bertrand competition

Firms determine the prices for their goods ($p_i$) facing a marginal cost of production ($\gamma_i$). The demand function is a function of the price of all the firms, $q_i = a - bp_i + \sum_{j \neq i} p_j$ with $a$ and $b$ positive constants. The payoff function of the firm $i$ is its profit given by its revenue $q_ip_i$ minus the cost of production $\gamma_iq_i$,

$$u_i(a, \gamma) = q_ip_i - \gamma_iq_i \quad (2)$$

The Bertrand competition model is analyzed in [8]. When firms choose their production quantity, instead of determining their price, and the price is determined by a linear inverse demand function, the LQG game is called a Cournot competition.

2.1.2 Beauty Contest Game

Payoff functions of players under common payoff state $\gamma$ representing, e.g., a stock value, are given by

$$u_i(a, \gamma) = -(1 - \beta)(a_i - \gamma)^2 - \beta(a_i - \bar{a}_{-i})^2 \quad (3)$$

where $\beta \in [0, 1]$ and $\bar{a}_{-i} = \sum_{j \neq i} a_j / (n - 1)$ represents the average action of other players. The first term in (3) denote the players’ urge for taking actions close to the payoff state $\gamma$. The second term accounts for players’ tendency towards taking actions in compliance with the rest of the population. The constant $\beta$ gauges the importance between the two terms. The payoff captures settings where the valuation of a good, e.g., stock, depends not just on the performance of the company but also on what other players think about its value [15].
2.2 Bayesian Nash and Correlated Equilibria

A strategy of player $i$ maps each possible value of the private signal $\omega_i \in \Omega_i$ to an action $s_i(\omega_i) \in A_i$, i.e., $s_i : \Omega_i \rightarrow A_i$. A strategy profile $s = (s_i)_{i \in N}$ is a Bayesian Nash equilibrium (BNE) with information structure $\zeta$, if it satisfies the following inequality

$$E_\zeta[u_i(s_i(\omega_i), s_{-i}, \gamma)|\omega_i] \geq E_\zeta[u_i(s_i', s_{-i}, \gamma)|\omega_i],$$

for all $a_i' \in A_i, \omega_i \in \Omega_i, i \in N$ where $s_{-i} = (s_j(\omega_j))_{j \neq i}$ is the equilibrium strategy of all the other players, and $E_\zeta$ is the expectation operator with respect to the distribution $\zeta$ and the prior on the payoff state $\psi$. The following result states a sufficient condition for having an unique BNE strategy, and provides a set of linear equations to determine the coefficients of the linear BNE strategy.

**Proposition 1** (Corollary 1, [18]). Suppose that $H + H^T$ and $\text{var}(\omega_i)$ are positive definite for each $i \in N$. Then LQG game has a unique Bayesian Nash equilibrium given by

$$s_i(\omega_i) = \bar{a}_i + b_i^T(\omega_i - E_\zeta[\omega_i])$$

for $i \in N$, where $b_1, \ldots, b_n$ are determined by the following systems of linear equations:

$$\sum_{j \in N} H_{i,j} \text{cov}(\omega_i, \omega_j)b_j = \text{cov}(\omega_i, \gamma_i)$$

for $i \in N$,

and $\text{cov}(\cdot, \cdot)$ represents the covariance between two random variables.

Assumptions of Proposition 1 guarantee existence and uniqueness of a linear strategy [5] that satisfy [1] with coefficients [6]. We assume the sufficient conditions above throughout the paper.

An action distribution represents the probability of observing an action profile $a \in A$ when agents follow a strategy profile $s$ under $\zeta$. We define the action distribution as $\phi(a|\gamma) = \sum_{\omega : s(\omega) = a} \zeta(\omega|\gamma)$. A Bayesian correlated equilibrium (BCE) is an action distribution in which no individual would profit by unilaterally deviating from selecting actions according to the given action distribution. The formal definition follows.

**Definition 1.** An action distribution $\phi$ under $\zeta$ is a BCE if and only if it satisfies

$$E_\phi[u_i((a_i, a_{-i}), \gamma)|a_i] \geq E_\phi[u_i((a_i', a_{-i}), \gamma)|a_i]$$

for all $a_i, a_i' \in A_i, i \in N$ where $E_\phi[\cdot|a_i]$ is the conditional expectation with respect to the action distribution $\phi$ and information structure $\zeta$ given action $a_i \in A_i$.

An equilibrium action distribution $\phi$, corresponding to a BNE strategy profile $s$ under $\zeta$, i.e., $\phi(a|\gamma) = \sum_{\omega : s(\omega) = a} \zeta(\omega|\gamma)$, satisfies [7] as stated in the following result.

**Proposition 2** (Corollary 2, [41]). An equilibrium action distribution is a BCE under any information structure. If a BCE corresponds to an equilibrium action distribution, a corresponding information structure exists.

Using Propositions 1 and 2, we can derive a necessary and sufficient condition for an action distribution comprised of jointly normally distributed action profile and payoff state.
Proposition 3. An action distribution $\phi$ comprised of jointly normally distributed action profile and a payoff state is a BCE if and only if the following conditions hold

$$E_\phi[a] = \bar{a}$$

(8)

$$\sum_{j \in N} H_{i,j} \text{cov}(a_i, a_j) = \text{cov}(a_i, \gamma_i).$$

(9)

Solution of (5) and (6) by $b_i = 1$ and $\bar{a}_i = E_\phi[\omega_i]$ constitute a necessary and sufficient condition for a BCE by Proposition 1. Conditions given in (8) and (9) corresponds to this solution; thus, Proposition 3 is established.

2.3 Information Design Problem

An information designer aims to optimize the expected value of an objective function $f(a, \gamma)$ that is quadratic in its arguments by deciding on an information structure $\zeta$ from a feasible region $Z$. Information designer will follow the timeline given below:

1. Selection of $\zeta \in Z$ and notification of all players about $\zeta$.
2. Realization of $\gamma$ and subsequent draw of signals $w_i, \forall i \in N$ from $\zeta(\omega, \gamma)$.
3. Players act according to BNE under $\zeta$.

The optimization problem of the information designer is as follows:

$$\max_{\zeta \in Z} E_\zeta[f(s, \gamma)]$$

(10)

where $s$ is the unique BNE under $\zeta$. Using the equilibrium action distribution $\phi$ under $\zeta$ and Proposition 2, we can replace the objective in (10) with $E_\phi[f(a, \gamma)]$. Thus, the information design problem in (10) is equivalent to the following optimization problem [19]:

$$\max_{\phi \in C(Z)} E_\phi[f(a, \gamma)]$$

(11)

where

$$C(Z) = \{\phi : \phi \text{ is the equilibrium action distribution under } \zeta \in Z\}.$$  

(12)

The solution to (11) and (12) determines a distribution $\zeta$ for signals conditional on the state $\gamma$. Upon realization of $\gamma$, information designer draws signals from $\zeta$ and shares them with the players. Note that we do not assume that the information designer knows the state realization. It just needs to be able to generate signals conditioned on the realization of the state.

When the objective function is quadratic, we can rewrite (11) as a Frobenius inner product of the coefficient matrix $F$ and the covariance matrix $X$ of $(a, \gamma)$ under $\phi \in C(Z)$—see [19] for details of the derivation. That is,

$$\max_{X \in X(Z)} F \cdot X := \max_{X \in X(Z)} \begin{bmatrix} \text{var}(a) & \text{cov}(a, \gamma) \\ \text{cov}(\gamma, a) & \text{var}(\gamma) \end{bmatrix}$$

(13)
where
\[ X(Z) = \{ X \in P_{+}^{2n} \} \quad \text{and} \quad F \cdot X := \sum_{i=1}^{2n} \sum_{j=1}^{2n} F_{i,j} X_{i,j}, \] (14)
and \( P_{+}^{2n} \) represents the set of all \( 2n \times 2n \) symmetric positive semi-definite matrices. In the objective (13), the variance matrix of actions \( \text{var}(a) \), and covariance matrix of the actions and the payoff state \( \text{cov}(\gamma, a) \) make up the decision variable \( X \). We can assume \( F_{2,2} \) is an \( n \times n \) zero matrix \( O \) because the variance of the payoff state \( \text{var}(\gamma) \) does not depend on the information structure \( \zeta \).

Next we define an important special case of the above problem where we restrict the information structure to public signals. That is, all players receive the same signal, and it is common knowledge that they will receive the same signal. A formal definition follows.

Lastly, we assign the payoff state covariance matrix \( \text{var}(\gamma) \) to the corresponding elements of \( X \),
\[ \text{var}(\gamma) = [\text{cov}(\gamma_i, \gamma_j)]_{n \times n} = [\var(a)_{i,j}]_{n \times n}. \] (16)
Together constraints (14)-(16) ensure that \( X \) is a feasible covariance matrix of an equilibrium action distribution. We state the information designer’s optimization problem using matrix notation:
\[ \max_{X \in X(Z^*)} \quad F \cdot X \] (17) s.t. \[ M_{k,l} \cdot X = \text{cov}(\gamma_k, \gamma_l), \quad \forall k, l \in N \quad \text{with} \quad k \leq l \] (18) \[ R_k \cdot X = 0 \quad \forall k \in \{1, \ldots, n\} \] (19) \[ X \in P_{+}^{2n} \] (20)
where \( M_{k,l} = [[M_{k,l}]_{i,j}]_{2n \times 2n} \in P_{+}^{2n} \) and \( R_k = [[R_k]_{i,j}]_{2n \times 2n} \in P_{+}^{2n} \) are defined as
\[
[M_{k,l}]_{i,j} = \begin{cases} 
1/2 & \text{if } k < l, i = n + k, j = n + l \\
1/2 & \text{if } k < l, i = n + l, j = n + k \\
1 & \text{if } k = l, i = n + k, j = n + l \\
0 & \text{otherwise},
\end{cases}
\]
\[
[R_k]_{i,j} = \begin{cases} 
H_{k,k} & \text{if } i = j = k, \\
H_{k,j}/2 & \text{if } i = k, 1 \leq j \leq n, j \neq k, \\
-1/2 & \text{if } i = k, j = n + k, \\
H_{k,i}/2 & \text{if } j = k, 1 \leq i \leq n, i \neq k \\
-1/2 & \text{if } j = k, i = n + k, \\
0 & \text{otherwise}.
\end{cases}
\]
Definition 2 (Public Information Structure). An information structure which has \( \omega_1 = \ldots = \omega_n \) with probability one is called a public information structure. \( Z^p \subseteq Z \) denotes the set of public information structures.

We define two important feasible solutions to (17) - (20) (no information and full information disclosure).

Definition 3 (No information disclosure). No information disclosure refers to the case when there is no informative signal sent to the players. In this case, the equilibrium action of players is given by \( H^{-1} \mu \). The induced decision variable and the objective value is given by

\[
X = \begin{bmatrix} O \\ O \end{bmatrix} \quad \text{and} \quad F \cdot X = 0.
\] (21)

Definition 4 (Full information disclosure). The signals sent to the players reveal all elements of payoff state \( \gamma \) under full information disclosure. Equilibrium actions is given by \( H^{-1} \gamma \). The induced decision variable \( X \) and the objective value \( F \cdot X \) is given by

\[
X = \begin{bmatrix} H^{-1} \var(\gamma)(H^{-1})^T \\ \var(\gamma)(H^{-1})^T \\ \var(\gamma) \end{bmatrix}
\] (22)

and

\[
F \cdot X = F_H \cdot \var(\gamma)
\] (23)

where

\[
F_H = (H^{-1})^T (F_{11} + F_{12} H + H^T F_{21}) H^{-1}.
\] (24)

Now we state conditions for the optimality of no and full information disclosure solutions, when we restrict the information design problem to public information structures. Let \( \var(\gamma) = DD^T \) such that \( D \) is an \( n \times k \) matrix of rank \( k \) where \( k \) is the rank of \( \var(\gamma) \).

Proposition 4 ([19]). Assume \( D^T F_H D \neq O \) is negative semi-definite. Then, no information disclosure is optimal in \( Z^p \), and full information disclosure is not optimal in \( Z^* \).

Proposition 5 ([19]). Assume \( D^T F_H D \neq O \) is positive semi-definite. Then, full information disclosure is optimal in \( Z^p \), and no information disclosure is not optimal in \( Z^* \).

3 Social Welfare Maximization

Social welfare is the sum of individual utility functions, which is stated as follows:

\[
f(a, \gamma) = \sum_{i=1}^{n} u_i(a, \gamma) = \sum_{i=1}^{n} (-H_{i,i}a_i^2 - 2 \sum_{j \neq i} H_{i,j}a_i a_j + 2\gamma_i a_i + d_i(a_{-i}, \gamma)).
\] (25)

Given the objective, the coefficient matrix \( F \) is as follows,

\[
F_{i,j} = \begin{cases} -H_{i,i} & \text{if } i \leq n, \ j \leq n, \ i = j \\ -H_{i,j} & \text{if } i \leq n, \ j \leq n, \ i \neq j \\ 1 & \text{if } n \leq i \leq 2n, \ j \leq n, \ i - n = j \\ 1 & \text{if } n \leq j \leq 2n, \ i \leq n, \ i = j - n \\ 0 & \text{otherwise}, 
\end{cases}
\] (26)
which can also be written in the following form using $H$ and identity matrix as sub-matrices,

$$F = \begin{bmatrix} -H & I \\ I & O \end{bmatrix}.$$  \quad (27)

Our first result shows that full information disclosure will always be preferred to no information disclosure in social welfare maximization.

**Proposition 6.** If $H \succ 0$, then full information disclosure never performs worse than no information disclosure for maximizing social welfare objective.

**Proof.** No information disclosure has the objective value $F \bullet X = 0$ regardless of $F$. Objective value of full information disclosure is defined in (23) as $F \bullet X = F_H \bullet \text{var}(\gamma)$. Covariance matrices are in general positive semi-definite; therefore, $\text{var}(\gamma) \succeq 0$. If $F_H \succ 0$, then $F \bullet X = F_H \bullet \text{var}(\gamma) \geq 0$. We need $F_H = H^{-1} \succ 0$. If $H$ is positive definite, then $H^{-1}$ is positive definite. (Theorem 7.6.1, [42]). Thus, $H^{-1} \succ 0$ is equivalent to $H \succ 0$ which is given.

Note that the assumption of a positive definite payoff matrix $H$ automatically holds when the payoff is symmetric given the sufficient condition for the existence and uniqueness of a BNE in Proposition 1. The result implies that no information disclosure cannot be an optimal information structure for social welfare maximization.

Next, we show that full information disclosure is the optimal solution to the social welfare maximization problem for some important special cases.

### 3.1 Common Payoff State

We consider a scenario in which the payoff states are identical, i.e., $\gamma_1 = \gamma_2 = \ldots = \gamma_n$. In this setting we have the following result.

**Proposition 7.** Assume $H$ is symmetric and $\gamma_i = \gamma_j$, $\forall i, j \in N$. Then, full information disclosure is optimal for social welfare maximization objective.

**Proof.** First we note that the objective function $f$ in (27) is such that $F_{i,n+j} = 0$ for $\forall i, j \in N$ with $i \neq j$. Moreover, we have $F_{n+i,n+j} = 0$, $\forall i, j \in N$. Therefore,

$$F \bullet X = \sum_{i=1}^{n} \sum_{j=1}^{n} F_{i,j} \text{cov}(a_i, a_j) + 2 \sum_{i=1}^{n} F_{i,n+i} \text{cov}(a_i, \gamma_i).$$  \quad (28)

Using the BCE condition in [9] for the corresponding terms in (28), we obtain

$$F \bullet X = \sum_{i=1}^{n} \sum_{j=1}^{n} (F_{i,j} + 2F_{i,n+i}H_{i,j}) \text{cov}(a_i, a_j).$$  \quad (29)

Thus we have $F \bullet X = E \bullet \text{var}(a)$ where the coefficients of the matrix $E$ is given as

$$E_{i,j} = F_{i,j} + F_{i,i+n}H_{i,j} + F_{j,n+j}H_{j,i} \quad \forall i, j \in N.$$  \quad (30)

Substituting in the coefficients of the objective function in (27), we get that $E = H^T$. Since $H$ is symmetric, $E = H$. From Proposition 9 in [19], we have that if $E = \kappa H$ for some constant $\kappa > 0$, then full information disclosure is optimal under common payoff states. In our setting, the condition holds with $\kappa = 1$. \hfill \Box
Proposition \(7\) establishes that full information disclosure is the optimal information structure if the payoff state is common and \(H\) is symmetric. In the following example, we analyze the optimal information structure for social welfare maximization when the payoff states are correlated, but not necessarily common, and the game payoff matrix \(H\) is asymmetric.

**Example (Asymmetric payoffs and correlated payoff states):** We define an asymmetric payoff matrix \(H\) by determining its off-diagonal elements using a uniformly distributed random variable \(U_{i,j}\), for \(i, j \in N\) with range \([-1, 1]\),

\[
H_{i,j} = \begin{cases} 
4, & \text{if } i = j, \text{ where } i, j = 1, 2, \ldots, n \\
1 + cU_{i,j}, & \text{if } i \neq j, \text{ where } i, j = 1, 2, \ldots, n 
\end{cases}
\]  

(31)

where \(c \in [0, 1]\) is a constant determining the magnitude of the asymmetry.

Figure 1 shows the suboptimality of full information disclosure as we vary the correlation between payoff states \(\text{Corr}(\gamma_i, \gamma_j)\) between 0.5 and 1 under several values for the magnitude of asymmetry \(c \in \{0.2, 0.4, 0.6, 0.8, 1\}\). Note that when \(\text{Corr}(\gamma_i, \gamma_j) = 1\), there is a common payoff state. The loss with respect to the optimal information structure under full information disclosure increases with growing asymmetry and decreasing correlation. The rate of increase in the percentage difference with respect to decreasing correlation is larger when asymmetry is larger in magnitude, e.g., compare lines associated with \(c = 1\) and \(c = 0.2\).

![Figure 1](image-url)

**Figure 1:** Percentage difference between optimal objective value and objective value of full information disclosure versus correlation between payoff states. We consider an asymmetric submodular game with payoffs given in (31). The loss with respect to the optimal information structure under full information disclosure increases with growing asymmetry and decreasing correlation.
3.2 Submodular and supermodular games

Next, we derive the optimal solution for a subset of submodular and supermodular games. A game is submodular if
\[
\frac{\partial^2 U_i}{\partial a_i \partial a_j} < 0, \forall i \neq j \in N.
\]
Otherwise, \[
\frac{\partial^2 U_i}{\partial a_i \partial a_j} > 0, \forall i \neq j \in N,
\]
the game is supermodular. In a quadratic game, these partial derivatives depend on the off-diagonal elements of the payoff matrix \(H\), i.e,
\[
\frac{\partial^2 U_i(a, \gamma)}{\partial a_i \partial a_j} = -2H_{i,j}.
\]
In a submodular game, a player’s marginal utility of “increasing” its action decreases with increases in other players’ actions. This strategic interaction between players’ actions and payoffs is known as strategic substitutability. In contrast, the payoff from increasing its action value increases as the actions of other players increase in a supermodular game. This strategic interaction between players’ actions and payoffs is known as strategic complementarity. Bertrand competition and beauty contest games are both examples of supermodular games. In Bertrand competition, if player \(j\) increases its price, the demand for player \(i\)’s product increases yielding a higher payoff for player \(i\). In the beauty contest game, an increase in the action of a player, increases the incentive for others to evaluate their evaluations.

We state our main result for submodular and supermodular games assuming the payoff matrix \(H\) has the following form
\[
H_{i,j} = \begin{cases} 
1 & \text{if } i = j; \ i,j \in \{1,2,..n\} \\
h & \text{if } i \neq j; \ i,j \in \{1,2,..n\} 
\end{cases}
\]
(33)

Theorem 1. Assume \(\sum_{i=1}^{n} \text{var}(\gamma_i) \geq 2h \sum_{i \neq j} \text{cov}(\gamma_i, \gamma_j)\) and \(H\) has the form in (33). Then, full information disclosure is optimal for the following games given the social welfare maximization objective:

1. A submodular game such that \(0 < h < 1\)
2. A supermodular game such that \(-\frac{1}{n-1} < h < 0\).

Proof. See Appendix for the proof.

Theorem shows that full information disclosure is optimal for welfare maximization when the game is submodular. That is, social welfare maximization objective is aligned with the incentives of players when increasing one player’s action reduces the incentive for other players to increase their actions. In contrast, when increasing one player’s action increases the incentive for other players to increase their actions, i.e., when we have a supermodular game, the optimality of full information disclosure is optimal as long as the effect of another players’ actions on a player’s action \(h\) is small. Indeed, full information disclosure ceases to be optimal in supermodular games as the number of players increases.

Another sufficient condition for optimality of full information disclosure in Theorem is the diagonal dominance of the covariance matrix of the payoff state. In the following numerical example, we identify that the full information disclosure remains optimal even when the diagonal dominance assumption does not hold in symmetric submodular/supermodular games.

Example (Relaxing the diagonal dominance of \(\text{var}(\gamma)\)): We consider a symmetric submodular game among \(n = 4\) players with payoff coefficient matrix \(H\) given in (33). We assume \(\text{var}(\gamma)\) has the following form.
\[
\text{var}(\gamma)_{(i,j)} = \begin{cases} 
v, & \text{if } i = j; \ i,j = 1,2,3,4 \\
0.2, & \text{if } i \neq j; \ i,j = 1,2,3,4. 
\end{cases}
\]
(34)
When we compare the social welfare value under full information disclosure solution (22) with the optimal solution to the information design problem in (17)-(20), we find that they are identical for all values of \( v \in [0.4, 0.48] \). In this interval of \( v \), the diagonal dominance assumption is not satisfied. We considered the off-diagonal elements of the payoff matrix values \( h \in \{-0.3, -0.2, -0.1, 0.2, 0.4, 0.6, 0.8\} \) for Fig. 2. This example suggests that full information disclosure remains optimal even when the diagonal assumption is not satisfied for both submodular and supermodular games. Fig. 2 shows the increasing gap between the social welfare values under full and no information disclosure. Indeed, as the dependence of the payoffs on other players’ actions, i.e., \(|h|\), increases, objective value increases. This means the value of revealing information becomes more important as strategic complementarity or substitutability increase.

3.3 Public information structures

Restricting the information structure to public signals, we establish that full information disclosure is optimal for social welfare maximization.

**Proposition 8.** Assume \( H \) is positive definite and consider \( Z^p \) as the feasible set is the set of public information structures. Then, full information disclosure maximizes social welfare.

**Proof.** \( H \) is positive definite, thus \( H^{-1} \) is positive definite by ([32], Theorem 7.6.1). Therefore, \( K^T F_H K \neq 0 \) is positive definite for any matrix \( K \) where

\[
F_H = (H^{-1})^T (H + IH + H^T I) H^{-1} = H^{-1}.
\]

Thus, the result follows from Proposition 5. \( \square \)
Together with the previous results in this section, Proposition 8 implies that in scenarios where full information disclosure is not optimal, e.g., for supermodular games, the optimal information structure has to include private signals.

4 Inducing conformism

We consider a scenario in which the information designer would like to make the actions of the players as close to each other as possible, i.e., conform to widely accepted behavior. We represent the designer’s objective using the sum of squared deviation between players’ actions and mean action $\bar{a}$,

$$f(a, \gamma) = -\sum_{i=1}^{n} (a_i - \bar{a})^2,$$

where $\bar{a} = \frac{1}{n} \sum_{i=1}^{n} a_i$. (35)

This objective is relevant in scenarios where deviation of the players’ actions can cost resources, e.g., in an energy distribution setting [43].

We can represent the objective function above as the Frobenius product of a coefficients matrix $F$ and the covariance matrix $X$, i.e., $F \cdot X$, where

$$F_{i,j} = \begin{cases} 
(1 - n)/n & \text{if } i = j, i \leq n \\
1/n & \text{if } i \neq j, i \leq n, j \leq n \\
0 & \text{otherwise.}
\end{cases} \quad (36)$$

See Lemma 1 in the Appendix for the derivation of the above matrix. Next we show that no information disclosure is an optimal information structure that minimizes (35).

**Proposition 9.** No information disclosure is a maximizer of the objective function in (35).

**Proof.** When we check eigenvalues of $F$, we see it has $n - 1$ eigenvalues with value -1 and $n + 1$ eigenvalues with value of 0. Thus, $F$ is negative semi-definite. We know $X$ is positive semi-definite. We deduce that $F \cdot X \leq 0$. Objective value of no information disclosure is 0 by (21); thus, no information disclosure is optimal.

Proposition 9 implies that the information designer induces the maximum similarity between players’ actions by not revealing any information to the players. Broadly, hiding information from players is optimal when there is a conflict between the utility functions of the players and the information designer’s objective.

5 Welfare Maximization vs. Inducing Conformism

We consider the following optimization problem

$$\max_{X \in X(\mathcal{Z})} \left( (1 - \lambda)F_{sw} + \lambda F_{ssd} \right) \cdot X, \quad \lambda \in [0, 1] \quad (37)$$

where $F_{sw}$ and $F_{ssd}$ refers to objective coefficient matrices for social welfare objective [26] and sum of squared deviations [36], respectively. $F := (1 - \lambda)F_{sw} + \lambda F_{ssd}$ is in the form below

$$F = \begin{bmatrix} Y & (1 - \lambda)I \\
(1 - \lambda)I & O \end{bmatrix}, \quad (38)$$
where the elements of $Y$ are defined as

$$Y_{i,j} = \begin{cases} \frac{\lambda(1-n)}{n} - (1 - \lambda)H_{i,j} & \text{if } i = j, \text{ for } i, j \in N \\ \frac{\lambda}{n} - (1 - \lambda)H_{i,j} & \text{if } i \neq j, \text{ for } i, j \in N. \end{cases}$$ (39)

In (37), the designer has a trade-off between maximizing welfare and inducing conformism among the actions of the players in the system. The constant $\lambda$ quantifies the importance of coherence between players’ actions. Our theoretical results for this multi-objective maximization problem in (37) focus on submodular games in which the payoff coefficients matrix has the form in (33) with coefficient $h \in (0,1)$. Common payoff state: We consider identical payoff states, i.e., $\gamma_1 = \gamma_2 = \cdots = \gamma_n$.

**Proposition 10.** Assume $H$ has the form of (33) with $h \in (0,1)$ and $\gamma_i = \gamma_j, \forall i, j \in N$, and $\lambda < \frac{1-h}{2-h}$ and $\lambda \in (0,1)$.

Then, no information disclosure is not optimal for the problem defined in (37).

**Proof.** We first write down $E$ as defined in (30) for this objective.

$$E_{i,j} = \begin{cases} \frac{\lambda(1-n)}{n} + 1 - \lambda & \text{if } i = j \\ \frac{\lambda}{n} + (1 - \lambda)h & \text{if } i \neq j \end{cases}$$ (41)

First eigenvalue of $E$ is equal to $[(n-1)h + 1](1 - \lambda)$. The rest of the eigenvalues of $E$ are equal to $-\lambda + (1 - \lambda)(1 - h)$. $E$ is positive definite if both eigenvalues are greater than zero. The assumptions in (40) ensure that $E$ is positive definite. If $E$ is positive definite, then the objective value $E \cdot X_{11} = E \cdot \text{var}(a) \succeq 0$. Thus, no information disclosure is not optimal. □

**Proposition 10** specifies the threshold below which social welfare maximization objective dominates the objective to unify players’ actions by showing that hiding information cannot be optimal if the constant $\lambda$ is below some threshold. As $h \to 0^+$, we have the region of no optimality increase to $\lambda \in (0,0.5)$. The region where no information disclosure is not optimal shrinks as strategic substitutability ($h$) increases. Indeed we have the region of no optimality given by $\lambda \in (0, \frac{1}{1+\epsilon})$ when $h = 1-\epsilon$ for small $\epsilon > 0$. That is the dominance of welfare maximization in dictating the information designer disappears as strategic substitutability increases.

**Public information structures:** We restrict our attention to the set of public information structures $Z_p \subseteq Z$ in which each player receives the same signal.

**Proposition 11.** Assume $H$ has the form in (33) with $h \in (0,1)$, feasible set is public information structures $Z_p$, and $\lambda < \frac{1-h}{2-h}$ and $\lambda \in (0,1)$.

Then, full information disclosure is optimal for the problem given in (37).

**Proof.** We start by writing out $F_H$ as defined in (24)

$$F_H = (H^{-1})^T(F_{11} + F_{12}H + H^TF_{21})H^{-1}$$ (43)

$$= (H^{-1})^T(Y + (1 - \lambda)IH + H^T(1 - \lambda)I)H^{-1}$$ (44)

$$= (H^{-1})^T(Y + 2(1 - \lambda)H)H^{-1}. \quad (45)$$

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\( F_H > 0 \) is equivalent to \( Y + 2(1 - \lambda)H > 0 \) from (45). Let \( T = Y + 2(1 - \lambda)H \). Thus,

\[
T_{i,j} = \begin{cases} 
\lambda (1 - n) + (1 - \lambda) & \text{if } i = j \\
\frac{\lambda}{n} + (1 - \lambda)h & \text{if } i \neq j 
\end{cases}
\]

for all \( i, j \in N \). We need the conditions below for \( T > 0 \). \( T \) is the same as \( E \) defined in (41). Therefore, the first eigenvalue of \( T \) is \( [(n-1)h+1](1-\lambda) \) and the rest of the eigenvalues of \( T \) are equal to \(-\lambda + (1-\lambda)(1-h)\). The matrix \( T \) is positive definite if both of the eigenvalues are greater than zero. The conditions in (42) ensure that \( T \) is positive definite. Thus by Proposition 5, full information disclosure is optimal for public information structures.

Proposition 11 determines the threshold of the constant \( \lambda \) below which social welfare maximization is dominant under public information structures. As shown in Proposition 8, full information disclosure is optimal if the objective is social welfare maximization and feasible set is the set of public information structures. Similarly, full information disclosure is optimal for the multi-objective case if social welfare maximization objective is important enough.

It is worth noting that the conditions we seek in Proposition 10 and Proposition 11 are the same. This stems from the fact that we seek the same matrix (see \( E \) in (41) and \( T \) in (46)) to be positive definite. Along the lines of discussion following Proposition 10, we have the region of optimality for full information shrink as strategic substitutability increases.

We note that the threshold for \( \lambda \) below which full information disclosure is optimal in (42) is a sufficient condition. We assess the strictness of this threshold and the optimality of no and full information disclosures for the class of general information structures in a numerical example.

**Numerical example:** We consider a symmetric game with the payoff matrix \( H \) having the form in (33) among \( n = 4 \) players. We consider \( h = 0.25, 0.5 \) and 0.75 and define the covariance matrix of payoff states as \( \text{var}(\gamma) \) be as follows

\[
\text{var}(\gamma)_{(i,j)} = \begin{cases} 
4, & \text{if } i = j; i, j = 1, 2, 3, 4 \\
1, & \text{if } i \neq j; i, j = 1, 2, 3, 4.
\end{cases}
\]

According to Proposition 11, \( \lambda \) region for the optimality of full information disclosure is given by \( \{(0, \frac{1}{2}), (0, \frac{1}{4}), (0, \frac{1}{8})\} \) for \( h \in \{0.25, 0.5, 0.75\} \), respectively. These \( \lambda \) thresholds are marked by purple dashed line in Figure 3. Black star shows the intersection point of full information disclosure and no information disclosure. Figure 3 shows that the region of \( \lambda \) for the optimality of full information under public information structures is larger in the example than the theoretical regions. The gap between the optimality region guaranteed by Proposition 11 and real region of optimality decreases as \( h \) increases.

Figure 4 shows that as supermodularity increases, that is as \( h \) decreases, objective function of full information disclosure approaches to the optimal objective function. Also, full information disclosure becomes the optimal solution for a wider range of \( \lambda \) values under public information structures as supermodularity increases.

We observe that except for the extreme values of the constant \( \lambda \), there exist private signal structures that perform better than no and full information disclosure. When \( \lambda \) approaches to zero, objective value of full information disclosure converges to optimal value under general information structures. When \( \lambda \) approaches to 1, objective value of no information disclosure converges to optimal value under general information structures.
Figure 3: Objective values of different solutions versus weight of sum of squared deviations for multi-objective case. The game is symmetric submodular with payoffs given in (33). Covariance matrix of payoff distribution is given in (47). The gap between the optimality region guaranteed by Proposition 11 and real region of optimality decreases as $h$ increases.

Figure 4: Objective values of different solutions versus weight of sum of squared deviations for multi-objective case. The game is symmetric supermodular with payoffs given in (33). Covariance matrix of payoff distribution is given in (47). As we move towards greater supermodularity (lower $h$), objective function of full information disclosure approaches to the optimal objective function.
6 Conclusions

We analyzed information design problem for LQG games under social welfare maximization and minimization of action deviation objectives. We showed that full information disclosure is an optimal solution for welfare maximization if there are common payoff states, specific submodularity or supermodularity in the game, or when we restrict the information design problem to public signals. For minimization of the deviation between players’ actions, we showed that no information disclosure is optimal in general. These results follow the intuition that if the objectives of the information designer and the payoffs of players are in conflict, information designer should blur or hide the information, and if their objectives align, the information designer should reveal information.

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A Payoff Coefficients Matrix for the Minimum Deviation Between Players’ Actions

Lemma 1 (Minimum deviation between players’ actions). The expected value of \( f(a, \gamma) \) for minimum deviation between players’ actions can be written as following.

\[
E_{\phi}[f(a, \gamma)] = \sum_{i=1}^{n} \frac{1}{n} \text{var}(a_i) + \frac{2}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \text{cov}(a_i, a_j)
\]  

Proof.

\[
E_{\phi}[f(a, \gamma)] = \sum_{i=1}^{n} E[a_i] + \frac{2}{n} \sum_{i=1}^{n} \sum_{k=1}^{n} E[a_i a_k] - \frac{1}{n} \sum_{i=1}^{n} E[a_i^2] - \frac{2}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} E[a_i a_j]
\]  

\[
= \sum_{i=1}^{n} \frac{1}{n} E[a_i^2] + \frac{2}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} E[a_i a_j]
\]  

Because \( E[a] \) is constant, we can write (51) as below.

\[
E_{\phi}[f(a, \gamma)] = \sum_{i=1}^{n} \frac{1}{n} \text{var}(a_i) + \frac{2}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \text{cov}(a_i, a_j)
\]

Note: In inner expectations, \( \phi \) subscript is dropped.

B Proof of Theorem 1

Proof. We verify that the full information disclosure solution satisfies the KKT conditions. Let \( \bar{X} \in P_{2n} \) denote the solution in (21) under full information disclosure. Primal feasibility conditions stated in (18), (19) and (20) are satisfied by full information disclosure. We denote the associated dual variables by \( \bar{\mu} \in \mathbb{R}^{n(n+1)/2}, \bar{\lambda} \in \mathbb{R}^n \) and \( \bar{\Gamma} \). Next we state the rest of the KKT conditions, i.e., dual feasibility, first order optimality and complementary slackness conditions respectively follow.

\[
\bar{\Gamma} \in P_{2n}^+, \quad F + \sum_{k=1}^{n} \bar{\lambda}_k R_k + \sum_{k=1}^{n} \sum_{l=1}^{k} \bar{\mu}_{(n-1)k+l} M_{k,l} + \bar{\Gamma} = 0,
\]

\[
\bar{X} \cdot \bar{\Gamma} = 0.
\]

We check whether the above KKT conditions are satisfied by \( \bar{X} \). We start by substituting the equality for the dual variable \( \bar{\Gamma} \) in (54) into (55) to get the following condition,

\[
\bar{X} \cdot \bar{\Gamma} = \begin{bmatrix} H^{-1} \text{var}(\gamma)(H^{-1})^T & H^{-1} \text{var}(\gamma) \\ \text{var}(\gamma)(H^{-1})^T & \text{var}(\gamma) \end{bmatrix} \cdot \begin{bmatrix} H & -I \\ -I & 0 \end{bmatrix} - \sum_{k=1}^{n} \bar{\lambda}_k R_k - \sum_{k=1}^{n} \sum_{l=1}^{k} \bar{\mu}_{(n-1)k+l} M_{k,l} = 0.
\]
If the condition above is satisfied, then both (54) and (55) are also satisfied.

We look for a uniform dual variable \( \lambda \), i.e., \( \lambda_k = \lambda, \forall k \in N \) where \( \lambda \in \mathbb{R} \), that satisfies (56). Define \( \Xi = -\sum_{k=1}^{n} \sum_{l=1}^{\lambda} \mathcal{P}_{(n-1)k+l} M_{k,l} \) in matrix form. We also assume \( \Xi = \mu I, \mu > 0 \). Thus, we have

\[
\mathcal{X} \cdot \Gamma = \begin{bmatrix}
H^{-1} \text{var}(\gamma) (H^{-1})^T & H^{-1} \text{var}(\gamma) \text{var}(\gamma) (H^{-1})^T & \text{var}(\gamma) (H^{-1})^T & \text{var}(\gamma) \\
(1 - \lambda) H & (1 - \lambda) I & \Xi
\end{bmatrix} = 0.
\]

(57)

The dual feasibility condition in (53) requires the dual variables \( \Gamma \) to be positive semi-definite. We will utilize Schur complement to analyze positive definiteness of \( \Gamma \). A strict version of dual feasibility condition \( \Gamma \gg 0 \) is satisfied if and only if \( \Xi \gg 0 \) and Schur complement \( \Gamma / \Xi \) of block matrix \( \Xi \) of matrix \( \Gamma \) is positive definite where

\[
\Gamma / \Xi = (1 - \lambda) H - (\frac{1}{2} - 1)^2 I / \mu.
\]

Thus,

\[
\Gamma / \Xi = \begin{cases}
(1 - \lambda) - (\frac{1}{2} - 1)^2 / \mu & \text{if } i = j; i, j \in N \\
(1 - \lambda) h & \text{if } i \neq j; i, j \in N.
\end{cases}
\]

(59)

Sum of each row of \( \Gamma / \Xi \) is \( (1 - \lambda) - (\frac{1}{2} - 1)^2 / \mu + (n - 1)(1 - \lambda) h \). This is the first eigenvalue of \( \Gamma / \Xi \). Rest of the eigenvalues of \( \Gamma / \Xi \) are equal to

\[
(1 - \lambda)(1 + h) - (\frac{1}{2} - 1)^2 / \mu.
\]

(60)

Dual variable \( \mu \) is the free variable in these eigenvalues. We need these eigenvalues to be positive, i.e.,

\[
\mu > \max\{ (\frac{1}{2} - 1)^2 / (1 - \lambda)(1 + h), 0 \}.
\]

(61)

First term on the right hand side of (61) ensures that \( \Gamma / \Xi \) is positive definite and \( \mu > 0 \) ensures that \( \Xi \) is positive definite. Thus \( \Gamma \) is positive definite if \( \mu \) satisfies (61).

We can rewrite the inverse of the matrix \( H \), i.e., \( H^{-1} \), as follows for \( n \geq 3 \)

\[
H_{i,j}^{-1} = \begin{cases}
(\frac{n-2}{n-1} h + \frac{1}{2} h) & \text{if } i = j; i, j \in N \\
(\frac{h}{(n-1)(n-2) h} + \frac{1}{2} h) & \text{if } i \neq j; i, j \in N.
\end{cases}
\]

(62)

Using (62), we write out the Frobenius product terms within (57) corresponding to each of the four sub-matrices. First Frobenius product in (57) can be written as following by using the distributive property of Frobenius inner product over matrix multiplication and substituting (62) for \( H^{-1} \), we have

\[
(H^{-1} \text{var}(\theta)(H^{-1})^T) \cdot ((1 - \lambda) H) = \left( \frac{(1 - \lambda)(n((2 - n)h - 1) + (n^2 - nh))}{(n - 1)h^2 - (n - 2)h - 1} \right)^2
\]

\[
* \sum_{i=1}^{n} \text{var}(\gamma_i) + 2h \sum_{i=1}^{n} \sum_{j=1}^{n} \text{cov}(\gamma_i, \gamma_j).
\]

(63)
Second Frobenius product corresponding to the off-diagonal elements can be written as

$$ (H^{-1} \text{var}(\gamma)) \bullet (\frac{\lambda}{2} - 1)I = (\text{var}(\gamma)(H^{-1})^T) \bullet (\frac{\lambda}{2} - 1)I $$

\begin{equation}
= (\lambda - 2)[(2 - n)\var{\gamma} + 2h\sum_{i \neq j} \cov(\gamma_i, \gamma_j)]
\end{equation}

\begin{equation}
= \frac{(\lambda - 2)[(2 - n)\var{\gamma} + 2h\sum_{i \neq j} \cov(\gamma_i, \gamma_j)]}{(n-1)h^2 - (n-2)h - 1}.
\end{equation}

The second equality \((65)\) comes from the fact that the product of non-diagonal elements of \(\text{var}(\gamma)(H^{-1})^T\) with the corresponding elements of the sub-matrix \((\frac{\lambda}{2} - 1)I\) is equal to zero. The remaining products are given as

$$ \text{var}(\gamma) \bullet \Xi = \mu \sum_{i=1}^{n} \text{var}(\gamma_i). $$

Combining the above terms, we can expand the equality \((57)\) as follows

\begin{equation}
\overline{X} \cdot \Gamma = n^2(1 - \lambda)^2 * \left( \sum_{i=1}^{n} \text{var}(\gamma_i) + h \sum_{i \neq j} \cov(\gamma_i, \gamma_j) \right)
+ 2(\lambda - 2)[(2 - n)\var{\gamma} + 2h\sum_{i \neq j} \cov(\gamma_i, \gamma_j)]
+ \mu \sum_{i=1}^{n} \text{var}(\gamma_i) = 0.
\end{equation}

Next we show that there exists at least one real root of \((67)\) with respect to \(\lambda\) with \(\mu\) satisfying \((61)\). If there is such a real root, there exists a \(\lambda \in \mathbb{R}\) satisfying the KKT conditions. Let \(\tau = \sum_{i=1}^{n} \var{\gamma_i}\) and \(\phi = 2 \sum_{i \neq j} \cov(\gamma_i, \gamma_j)\) to simplify the exposition.

We first consider the case \(\mu = \frac{(\lambda - 1)^2}{(1 - \lambda)(1 + h)} + \epsilon, \epsilon > 0\). In this case, \((67)\) becomes

\begin{equation}
\overline{X} \cdot \Gamma = n^2(1 - \lambda)^2(\tau + h\phi) + \frac{2(\lambda - 2)[(2 - n)\var{\gamma} + (2 - n)\tau + h\phi]}{(n-1)h^2 - (n-2)h - 1} + \frac{(\lambda - 1)^2}{(1 - \lambda)(1 + h)} + \epsilon \tau = 0.
\end{equation}

When we equalize denominators, \((68)\) becomes a cubic equation. The cubic equation with real coefficients always has at least one real root. Second, we consider the case \(\mu = \epsilon, \epsilon > 0\). In this case, \((67)\) becomes

\begin{equation}
\overline{X} \cdot \Gamma = n^2(1 - \lambda)^2 * (\tau + h\phi) + 2(\lambda - 2)[(2 - n)\var{\gamma} + h\phi] + \epsilon \tau = 0,
\end{equation}

where \(a, b\) and \(c\) are defined as

\begin{equation}
a = n^2(\tau + h\phi)
\end{equation}

\begin{equation}
b = -2n^2(\tau + h\phi) + 2[(2 - n)\var{\gamma} + h\phi]
\end{equation}

\begin{equation}
c = n^2(\tau + h\phi) - 4[(2 - n)\var{\gamma} - 4h\phi] + \epsilon \tau
\end{equation}

We want to show \(b^2 - 4ac > 0\), so that there exists a root \(\lambda_r > 1\). Thus, \(\lambda_r\) will satisfy \((61)\). It can be easily observed that \((n - 1)h^2 - (n - 2)h - 1 < 0\) for \(0 < h < 1\). Also, by our assumption \(\tau \geq h\phi\). Thus, we can
deduce that the discriminant is positive, i.e.,

\[
b^2 - 4ac = \left( -2n^2(\tau + h\phi) + 2\frac{((2 - n)h - 1)\tau + h\phi}{(n - 1)h^2 + (2 - n)h - 1} \right)^2 - 4n^4(\tau + h\phi)^2 - \frac{4n^2(\tau + h\phi)[-4((2 - n)h - 1)\tau - 4h\phi]}{(n - 1)h^2 + (2 - n)h - 1} + n^2(\tau + h\phi)\varepsilon \tau \]

\[
= \frac{8n^2(\tau + h\phi)[((2 - n)h - 1)\tau + h\phi]}{(n - 1)h^2 + (2 - n)h - 1} + 4\left( \frac{((2 - n)h - 1)\tau + h\phi}{(n - 1)h^2 + (2 - n)h - 1} \right)^2 + n^2(\tau + h\phi)\varepsilon \tau > 0.
\]

Therefore roots of \((69)\) are real. We also need to show at least one of roots of \((69)\) \(\lambda_r\) is such that \(\lambda_r > 1\); thus, \((61)\) is satisfied. We consider the larger root,

\[
\lambda_r = 1 - \frac{((2 - n)h - 1)\tau + h\phi}{n^2(\tau + h\phi)[(n - 1)h^2 + (2 - n)h - 1]} + \frac{\sqrt{b^2 - 4ac}}{2a} > 1. \tag{73}
\]

We know \(a > 0\). Also, it can be deduced that the third term \((73)\) is greater than the absolute value of the second term in \((73)\) from prior discussion. Thus, \(\lambda_r > 1\). This concludes the proof. \(\square\)