Magnetic Vortices in the Abelian Higgs Model with Derivative Interactions

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We study the properties of a single magnetic vortex and magnetic vortex lattices in a
generalization of the Abelian Higgs model containing the simplest derivative interaction
that preserves the $U(1)$ gauge symmetry of the original model. The paper is motivated
by the study of finite isospin chiral perturbation theory in a uniform, external : since
pions are Goldstone bosons of QCD (due to chiral symmetry breaking by the QCD
vacuum), they interact through momentum dependent terms. We introduce a uniform
external magnetic field and find the asymptotic properties of single vortex solutions
and compare them to the well-known solutions of the standard Abelian Higgs Model.
Furthermore, we study the vortex lattice solutions near the upper critical field using the
method of “successive approximations”, which was originally used by Abrikosov in his
seminal paper on type-II superconductors. We find the vortex lattice structure, which
remains hexagonal as in the standard Abelian Higgs model, and condensation energy of
the vortex lattices relative to the normal vacuum (in a uniform magnetic field).

1. Introduction

Magnetic vortices are topological objects that quantize magnetic flux. They have
been studied extensively since type-II superconductors and vortices were first
proposed within the context of Ginzburg-Landau (GL) theory for metals by
Abrikosov. Their relevance and study have not been limited to metals but ex-
tended to particle physics including the Abelian Higgs model and electroweak theory
and quantum chromodynamics (QCD). Since GL theory and the Higgs model are
isomorphic, this is not entirely surprising. However, there are significant qualita-
tive differences between the standard GL theory (or the Abelian Higgs model) and
electroweak theory (or QCD at large magnetic fields). While magnetic vortices are
formed below a certain critical magnetic field, the opposite is true in the context of

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electroweak theory, where the electroweak vacuum becomes unstable to $W$-boson vortices above a critical magnetic field. Furthermore, it has also been suggested that the QCD vacuum may be populated by a color magnetic flux tube (also called the “Copenhagen vacuum”).

Currently, there is strong interest in the role of magnetic fields in systems described by QCD due to their relevance to magnetars, RHIC collisions, which may lead to the formation of quark-gluon plasma, the study of which is also relevant to the early formative stages of the universe after the Big Bang. Remarkably, it was found that the QCD vacuum at high enough magnetic fields leads to the condensation of $\rho$-mesons (similar to the electroweak theory), which form a vortex lattice that consists not only of charged mesons ($\rho_{\pm}$) but also the neutral meson ($\rho_0$). The possibility of neutral particle condensation was also explored in the context of a simple model, the “extended Abelian Higgs model” containing both complex and real scalar fields: it was show that the electromagnetically neutral, real fields when coupled to charged fields can potentially condense in a vortex phase through the coupling of the neutral field with the charged fields. Here, motivated by the presence of derivative interactions within $\chi$PT, we construct a simple generalization of the Abelian Higgs model to include derivative interactions and study its implications.

Additionally, there is also interest in vortices in the context of finite isospin chiral perturbation theory ($\chi$PT), which is the low-energy description of QCD with pions ($\pi_{\pm}, \pi_0$) as the relevant degrees of freedom. Son and Stephanov found that pions at finite isospin form a superfluid. Since the superfluid is electromagnetically charged, unsurprisingly they exhibit superconducting properties in the presence of an external magnetic field. It was shown in Ref. that pions condense into vortices with no neutral pion condensation ($\pi_0$) though this possibility isn’t fully ruled out except in single vortices. However, Ref. did not study the structure of the vortex lattice or condensation energy as a consequence of the derivative interactions of the Goldstone modes (i.e. pions) of QCD. The objective of this paper is to take steps to fill this intellectual gap by studying a generalization of the Abelian Higgs model to include derivative interactions and study its implications.

The paper is organized as follows: we begin in Section with the Lagrangian of the Abelian Higgs model with an additional derivative interaction that preserves the local $U(1)$ symmetry of the Abelian Higgs Model. We then study the asymptotic properties of single magnetic vortices in Section and generate numerical solutions, followed by the study of vortex lattices and their condensation energy in Section. Finally, in Section we present some concluding remarks.
2. Lagrangian

We begin with the Lagrangian for the Abelian Higgs Model with an additional derivative interaction

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi)^\dagger (D^\mu \phi) - V(|\phi|) + \ell^2 (D_\mu \phi)^\dagger (D^\mu \phi) \phi^\dagger \phi
\]

\[
V(|\phi|) = -m^2 |\phi|^2 + \frac{\lambda}{2} |\phi|^4 ,
\]

where \(D_\mu = \partial_\mu + ieA_\mu\). This convention for the covariant derivative assumes that the field \(\phi\) has charge \(+e\) which can be assumed to be positive with no loss of generality. Furthermore, the Lagrangian also consists of a derivative interaction term \((D_\mu \phi)^\dagger (D^\mu \phi) \phi^\dagger \phi\), which comes with a coupling constant that we denote by \(\ell^2\). Note that its mass dimension is \([\ell] = -1\). Also, we assume \(\ell^2 > 0\) – this is necessary to ensure that vortex solutions are stable. The Lagrangian reduces to the usual Abelian Higgs model Lagrangian in the limit \(\ell \to 0\). The derivative interaction has been chosen to preserve the original symmetries of the Abelian Higgs Model. The symmetry groups of the Lagrangian are a global \(U(1)\) symmetry:

\[
\phi \rightarrow e^{i\alpha} \phi
\]

with \(\alpha\) being spatially independent. However, if the \(\alpha\) is space-dependent, then the symmetry group is local \(U(1)\), i.e.

\[
\phi \rightarrow e^{i\alpha(x)} \phi
A_\mu \rightarrow A_\mu - \frac{1}{e} \partial_\mu \alpha(x) .
\]

The Lagrangian has an imaginary mass term, i.e. \(m^2 > 0\) and therefore the ground state spontaneously breaks the \(U(1)\) gauge symmetry leading to massive photons. The potential has two stationary points

\[
|\phi| = 0 \text{ and } |\phi| = \sqrt{\frac{m^2}{\lambda}} \equiv v ,
\]

and for \(m^2 > 0\), the vacuum of the theory has a non-zero vacuum expectation value. This results in the photons becoming massive through the Higgs mechanism. Expanding around the vacuum expectation value (vev) we find the follows masses for the scalar and gauge fields

\[
m_\phi = 2m
\]

\[
m_A = \sqrt{2em} \left(1 + \frac{\ell^2 m^2}{\lambda}\right) .
\]

Note that due to the derivative interaction which protects the local \(U(1)\) symmetry, the photon mass is different from that of the standard Abelian Higgs model. In the limit \(\ell \to 0\), we reproduce the photon mass in the Abelian Higgs model. The equations of motion for \(\phi\) and \(A^\mu\) are respectively

\[
D_\mu D^\mu \phi - m^2 \phi + \lambda |\phi|^2 \phi + \ell^2 D_\mu \left[(D^\mu \phi) \phi^\dagger \phi\right] = 0 ,
\]
and
\[ \partial_\mu F^{\mu\nu} = -j^\nu, \]  
where the electromagnetic current \( j^\nu \) is
\[ j^\nu = -ie \left[ \phi^\dagger D^\nu \phi - (D^\nu \phi)^\dagger \phi \right] [1 + \ell^2 \phi^\dagger \phi]. \]

3. Single Vortex

Using the Lorentz gauge, \( \partial_\mu A^\mu = 0 \), or equivalently the Landau gauge, since \( A^0 = 0 \), the equation of motion for \( A^\theta \) is
\[ -\partial_r \partial_r A^\theta = -j^\theta = ie \left[ \phi^\dagger D^\theta \phi - (D^\theta \phi)^\dagger \phi \right] [1 + \ell^2 \phi^\dagger \phi]. \]

For single vortex solutions, which are cylindrically symmetric, we will parameterize the vortices as follows
\[ A_r = A_z = 0, \]
\[ A_\theta = A = \frac{n}{er} + \delta A, \]
\[ \phi(r, \theta) = w(r)e^{in\theta}, \]
with the electromagnetic current taking the form
\[ j_\theta = 2e^2|\phi|^2 \left( A - \frac{n}{er} \right) [1 + \ell^2|\phi|^2], \]
and the equation of motion for \( A \equiv A_\theta \) becomes
\[ \frac{\partial A}{\partial r^2} + \frac{1}{r} \frac{\partial A}{\partial r} - \frac{A}{r^2} = 2e^2|\phi|^2 \left( A - \frac{n}{er} \right) [1 + \ell^2|\phi|^2]. \]

In a region, where \( |\phi| = v \equiv \sqrt{m^2 + \frac{\lambda}{\ell^2}}, \) with \( v \) being the vacuum expectation value, the equation can be solved using the following parametrization
\[ A(r) = \frac{n}{er} + \delta A(r), \]
where \( \frac{n}{er} \) is the asymptotic behavior of the gauge field where the magnetic field is expected to be uniform and
\[ \delta A(r) \equiv \tilde{c}_A e^{-\sqrt{2\overline{\nu}}r} \left( \frac{1}{\sqrt{r}} + O \left( \frac{1}{r^{3/2}} \right) \right), \]
with \( \tilde{c}_A \) being an undetermined constant and \( \tilde{\nu} \) is defined as
\[ \tilde{\nu} \equiv v \sqrt{1 + \ell^2v^2}. \]

Note that \( \tilde{\nu} \) reduces to \( v \) in the absence of derivative interactions, i.e. \( \ell \to 0 \).

We find the asymptotic properties of the complex scalar field \( \phi \) using the parametrization
\[ \phi(r, \theta) = w(r)e^{i\chi(\theta)}, \]
where $\chi = n \theta$. This choice guarantees that the electromagnetic current vanishes on the boundary leading to the flux quantization condition

$$\int \vec{A} \cdot d\vec{l} = \int \vec{B} \cdot d\vec{S} = \frac{2n\pi}{e},$$

where the line integral is performed at infinity and $d\vec{S}$ is the differential area. The equation of motion in terms of $w(r)$ becomes

$$w''(r) + \frac{w'(r)}{r} - \left(\frac{n}{r} - eA(r)\right)^2 w(r) + m^2 w(r) - \lambda w(r)^3 + \ell^2 \left(\frac{n}{r} - 2eA(r)\right) \left(\frac{n}{r} - eA(r)\right) w(r)^3 = 0.$$  

Next, rewriting $w(r) = v + \delta \phi(r)$, we get a linearized equation for $\delta \phi(r)$,

$$(1 + \ell^2 v^2) \delta \phi''(r) + \left(1 + \ell^2 v^2\right) \frac{\delta \phi'(r)}{r} - 2v^2 \lambda \delta \phi(r) \approx \ell^2 2 e v^3 n \frac{\delta A(r)}{r}.$$  

It is straightforward to solve the linearized equation in the limit of large $r$, with final solution assuming the form

$$\delta \phi(r) = \delta \phi_h(r) + \delta \phi_p(r),$$

where the solution of the homogeneous equation is $\delta \phi_h(r)$ and the particular solution is $\delta \phi_p(r)$. We find that

$$\lim_{r \to \infty} \delta \phi(r) = c_\phi K_1(\nu r) = c_\phi e^{-\nu r} \left[1 + O\left(\frac{1}{r^{3/2}}\right)\right],$$

where $c_\phi$ is an arbitrary constant and $\nu$ is defined as

$$\nu = \sqrt{\frac{2v^2 \lambda}{1 + \ell^2 v^2}}.$$  

The particular solution on the other hand has the form

$$\lim_{r \to \infty} \delta \phi(r)_p = g_\phi e^{-\alpha r},$$

where

$$\alpha = 2\sqrt{2} e v \left(1 + \ell^2 v^2\right).$$

In Figs. 1 and 2, we plot single vortex solutions in a uniform magnetic field with flux $\frac{2n\pi}{e}$. The solid (blue) curve is the pion field, the red (dashed) curve is the gauge field and the black (dotted) line indicates a uniform magnetic field, i.e. $A = \frac{1}{r}$. Firstly, note that away from the vortex core ($r = 0$), the scalar field asymptotes to its vev and the magnetic field is uniform. Fig. 1 is a single vortex in the standard Abelian Higgs Model, i.e. $\ell = 0$ and Fig. 2 is a single vortex with the derivative interaction turned on. The gauge field asymptotes to $\frac{1}{e} r$ faster than when $\ell = 0$. This is as anticipated since the photon mass in the presence of derivative interactions is larger than in the standard Abelian Higgs model.
4. Vortex Lattice and Condensation Energy

In this section, we find the vortex lattice solutions and their corresponding free energies near the upper critical field, $B_{c2}$ and find the condensation (free) energy of these lattice. We define the condensation energy as

$$\mathcal{E} = \langle H \rangle - \frac{1}{2} B_{\text{ext}}^2,$$  \hspace{1cm} (25)
where \( \frac{1}{2} B_{\text{ext}}^2 \) is the free energy of the normal vacuum in the presence of a uniform magnetic field \( B_{\text{ext}} \) and \( \langle H \rangle \) is the expectation value of the time-dependent Hamiltonian (density), \( H \), which is defined as

\[
\langle H \rangle = \frac{1}{A_\perp} \int dx dy H ,
\]

(26)

where \( x \) and \( y \) are the coordinates of the transverse plane and we assume that the external magnetic field \( B_{\text{ext}} \) is pointing in the positive z-direction. The condensation (free) energy of the vortex lattice is defined relative to the normal vacuum in a uniform magnetic field. We use the method of successive approximation originally used by Abrikosov in Ref.\[3\] We will solve the equations of motion near the upper critical point, \( B_{c2} = \frac{m^2}{e} \), which is identical to the critical field in the standard version of the Abelian Higgs Model. We will work just below the critical field, \( B_{c2} \), i.e.

\[
\frac{B_{c2} - B_{\text{ext}}}{B_{c2}} \ll 1,
\]

(27)
a region where the condensate is expected to be small,

\[
\left| \frac{\phi}{v} \right| \ll 1 ,
\]

(28)

with \( v \) being the vacuum expectation value in the condensed phase, i.e. \( v > 0 \), defined in Eq. 4.

The method of “successive approximations” is effectively the following perturbative expansion\[\]

\[
B = B_0 + \epsilon \delta B + \cdots \\
\phi = \epsilon \phi_0 + \epsilon^3 \delta \phi + \cdots
\]

(29)

Similarly, the expansion for the gauge field, \( A_i \), that is consistent with the expansion for the magnetic field \( B \), is

\[
A_x = A_{x0} + \epsilon \delta A_x \\
A_y = A_{y0} + \epsilon \delta A_y \\
A_z = 0 .
\]

(30)

The \( \mathcal{O}(\epsilon) \) equation of motion is

\[
\left[ - (\partial_i + ieA_{0i})^2 - m^2 \right] \phi_0 = 0 ,
\]

(31)

which we rewrite in terms of

\[
\Pi_{0x} = -i \partial_x - eA_{0x} \\
\Pi_{0y} = -i \partial_y - eA_{0y} .
\]

(32)

\( ^{a}\text{This expansion is commonly used to study non-linear harmonic oscillators in classical mechanics.} \)
They satisfy the commutation relation \([\Pi_x, \Pi_y] = ieB_0\), where \(B_z\) is the magnetic field in the z-direction. Further, defining

\[
\Pi_0 = \Pi_{0x} + i\Pi_{0y} \\
\Pi_0^\dagger = \Pi_{0x} - i\Pi_{0y},
\]

we can rewrite the equation of motion as

\[
(\Pi_0^\dagger \Pi_0 + eB_0 - m^2)\phi_0 = 0.
\]

Since \([\Pi, \Pi^\dagger] = 2eB_0\), which now has a structure similar to that of the simple harmonic oscillator, we know that the dispersion relation is

\[
(2n + 1)eB_0 - m^2 = 0 \implies B_0 = \frac{m^2}{(2n + 1)e}.
\]

The maximum magnetic field, \(B_0\), is therefore,

\[
B_0 = \frac{m^2}{e} \equiv B_{c2}.
\]

Below this field and above the first critical field \(B_{c1}\), a phase of magnet vortices form in type-II superconductors. \(B_{c2}\) is the largest magnetic field that can be sustained by magnetic vortices, beyond which the repulsive interactions between the vortices becomes large enough such that it becomes energetically favorable to condense into the normal vacuum. Just below the critical field \(B_{c2}\), the vortex solutions satisfy \(\Pi_0\phi_0 = 0\). In terms of the complex variables \(z\) and \(\bar{z}\), which we define below

\[
\begin{align*}
  z &= x + iy, \quad \bar{z} = x - iy \\
  \partial &= \frac{1}{2}(\partial_x - i\partial_y), \quad \bar{\partial} = \frac{1}{2}(\partial_x + i\partial_y)
\end{align*}
\]

the equation of motion at leading order is

\[
\left(2\bar{\partial} + \frac{eB_0}{2}z\right)\phi_0 = 0,
\]

Working in the symmetric gauge with

\[
A_{0x} = -\frac{B_0}{2}y, \quad A_{0y} = \frac{B_0}{2}x,
\]

the ground state is a sum over the lowest Landau level,

\[
\phi_0 = \sum_{n=-\infty}^{\infty} C_n \phi_n(\nu, z, \bar{z})
\]

\[
\phi_n(\nu, z, \bar{z}) = e^{-\pi\nu^2n^2 - \pi\nu n(\frac{1}{2})} z^{\frac{2\pi}{\nu B_0} \nu n z},
\]

where \(L_{B_0}\) is the “reduced magnetic length”, \(L_{B_0} = \sqrt{\frac{2\pi}{eB_0}}\), \(n\) is an integer and \(\nu\) is a variational parameter. Since there are an infinite number of possible solutions, i.e. choice of a set of \(C_n\), we proceed by imposing a periodicity constraint such that \(C_n = C_{n+N}\). \(N\) determines the lattice structure. (For example, \(N = 1\) is a square
The result after averaging over the transverse plane is
\[ \langle -ie(\partial \delta A - \bar{\delta} \bar{A})|\phi_0|^2 + \lambda\langle|\phi_0|^4\rangle = 0 \].

Since
\[ -i(\partial \delta A - \bar{\delta} \bar{A}) = \delta B = B - B_{c2} \],
we get
\[ \mathcal{R} = \frac{\langle|\phi_0|^4\rangle}{\langle|\phi_0|^2\rangle} = \frac{e(B_{c2} - B_{ext}) - eB_1}{\lambda - e^2 + 2m^2\ell^2} \].
where
\[ B_1 = \langle |\varphi_0|^2 \rangle + \frac{1}{2} \ell^2 \langle |\varphi_0|^4 \rangle . \] (50)

Finally, we can write down the expectation value of the time-independent Hamiltonian (i.e. free energy) using the expression for \( R \) above, the equations of motion \( \varphi_0 \), which can be used to get rid of all terms containing derivatives and \( B \) found in Eq. 42
\[ \langle H \rangle = \frac{1}{2} \langle B^2 \rangle - \left( \frac{\lambda}{2} + m^2 \ell^2 \right) \langle |\varphi_0|^4 \rangle + O(|\varphi_0|^6) \]
\[ = \frac{1}{2} B_{\text{ext}}^2 - e(B_{c2} - B_{\text{ext}}) \langle |\varphi_0|^2 \rangle + \frac{e^2}{2} \langle |\varphi_0|^2 \rangle^2 \]
\[ + \frac{1}{2} \left( \lambda - e^2 + 2m^2 \ell^2 \right) \langle |\varphi_0|^4 \rangle + O(|\varphi_0|^6) , \] (51)
where we are ignoring all contributions containing more than four fields, which is valid if \( \phi_0 \ll v \). The (Gibbs free) energy can be minimized using the replacement
\[ \langle |\varphi_0|^4 \rangle \equiv \beta_A \langle |\varphi_0^2|^2 \rangle , \] (52)
where \( \beta_A \geq 1 \) is the Abrikosov ratio\(^{1–3} \) and subsequently minimizing with respect to \( \langle |\varphi_0|^2 \rangle \), followed by \( \beta_A \). We find that up to the order to which we are working the free energy is minimized by
\[ \langle |\varphi_0|^2 \rangle = \frac{e(B_{c2} - B_{\text{ext}})}{\beta_A (\lambda - e^2 + 2m^2 \ell^2) + e^2} + \cdots \] (53)
and the expectation value of the Hamiltonian for the lattice (free energy averaged over the transverse plane) given by
\[ \langle H \rangle = \frac{1}{2} B_{\text{ext}}^2 - \frac{e^2(B_{c2} - B_{\text{ext}})^2}{2e^2 + 2\beta_A (\lambda - e^2 + 2m^2 \ell^2)} + O((B_{c2} - B_{\text{ext}})^4) . \] (54)

Note that the free energy reduces to the standard result in the limit \( \ell \to 0 \). Furthermore, we see from the above expression that the free energy minimized by a vortex lattice with the smallest value of the Abrikosov ratio. The hexagonal lattice assume the smallest value of \( \beta_A = 1.1596 \ldots \), which has a corresponding value of \( \nu = \frac{3\sqrt{3}}{2} \) for the hexagonal lattice. As expected the free energy in the presence of the derivative interaction with \( \ell^2 > 0 \) is higher than without.

4.1. Vortex Lattice in an external field \( B_{\text{ext}} \)

The Abrikosov ratio that minimizes the free energy is that of a periodic hexagonal lattice with \( C_0 = C, C_1 = iC \) and \( C_n = C_{n+2} \). However, the solution of Eq. (40) with the hexagonal periodicity constraints only obeys the flux quantization condition
\[ B_{\text{ext}} \mid \vec{d}_1 \times \vec{d}_2 \mid = \frac{2\pi n}{e} , \] (55)
at the critical field, i.e. $B_{\text{ext}} = B_{c2}$ with $n = 1$ in each unit cell. Note that $\vec{d}_1 = \frac{L_B}{\nu} (0, 1)$ and $\vec{d}_2 = \frac{L_B}{\nu} \left( \frac{\sqrt{3}}{2}, \frac{1}{2} \right)$ are the lattice vectors of the periodic lattice with the reduced magnetic length $L_B = \sqrt{\frac{2\pi}{B}}$. In order to ensure the appropriate flux quantization condition, we make the change $L_B \to L_B$ in Eq. (40) with the guarantee that the vortex condensation energy we found remains unaffected since

$$
\langle \phi_0^\dagger \phi_0 \rangle = |C|^2 \sqrt{\frac{2\pi}{\nu}} \\
\langle (\phi_0^\dagger \phi_0)^2 \rangle = \frac{|C|^4}{4|\nu|} \left[ \theta_3(0, e^{-\pi \nu^2})^2 + 2\theta_3(\pi/2, e^{-\pi \nu^2}) \theta_3(0, e^{-\pi \nu^2}) - \theta_3(-\pi/2, e^{-\pi \nu^2})^2 \right] \\
= \frac{\beta_A |C|^4}{\sqrt{3}}
$$

(56)

where $\theta_3$ is a Jacobi Theta function. The condensation energy is a function of $\beta_A$ and $\langle \phi_0^\dagger \phi_0 \rangle$, the latter of which only depends on the quantity $C$ and the Abrikosov ration ($\beta_A$) is determined by the geometry of the vortex lattices.

Now, we can finally write the vortex lattice solution in terms of $L_B$ and the periodic sum for the hexagonal lattice can be rewritten in terms of the third Jacobi (elliptic) theta function, $\theta_3$ as follows

$$
\phi_0(x, y) = \frac{C}{2\nu} \left[ e^{-\frac{\pi(x+i\nu y)}{L_B \nu}} \theta_3 \left( \frac{-\pi(x+i\nu y)}{2L_B \nu}, e^{-\frac{\pi}{\nu}} \right) \\
+ i e^{-\frac{\pi(x+i\nu y)}{L_B \nu}} \theta_3 \left( \frac{-\pi(x+i\nu L_B \nu)}{2L_B \nu}, e^{-\frac{\pi}{\nu}} \right) \right],
$$

(57)

where $\nu = \frac{4\sqrt{3}}{\sqrt{2}}$ for the hexagonal lattice that minimizes the free energy. Finally, we determine $C$ up to an arbitrary phase constant by minimizing the condensation energy. We get

$$
|C| = \sqrt{\frac{\sqrt{3}e(B_{c2} - B_{\text{ext}})}{\beta_A \left( \lambda - e^2 + 2m^2 \ell^2 \right) + e^2}}, \text{ for } B_{\text{ext}} \leq B_{c2} \ 
|C| = 0, \text{ for } B_{\text{ext}} \geq B_{c2}.
$$

(58)

Using the above result for $C$, we plot the vortex lattice. The vortex lattices as shown ($\phi_0^\dagger \phi$ plotted) in Fig. [3] We choose $m = 1$ such that distance scales, such that $x$ and $y$ shown in the plots are effectively measured in units of $m$. Also, we choose $\lambda = 1$. It is evident from the plots that changing $\ell^2$ does not affect the density but only the gradients and the maximum value of $|C|_{\text{max}}$. With increasing values of $\ell^2$, the gradient in $\phi$ becomes energetically expensive and therefore $\phi_0^\dagger \phi_0$ becomes flatter as the plots suggest. But note that the density of vortices is unaffected by a change in $\ell^2$. The density only depends on the external magnetic field ($B_{\text{ext}}$), which also affects $|C|$.
Fig. 3. Vortex Lattice Plot of $|\langle \phi^\dagger \phi \rangle|$ on the traverse-plane ($x - y$) of size $2 \times 2$. We use $m = 1$ to set the scale, $\lambda = 1$, $e = \sqrt{\frac{4\pi}{137}}$ and from left to right $m\ell^2 = 0, 0.2$, respectively. Note that the color shades are normalized differently for each plot but the contour lines have the same “normalization” across all the plots. We note that from left to right $|C| = 0.107213, 0.0904461$.

5. Conclusion

In this paper, we have found the asymptotic properties of single vortex solutions in the presence of an additional derivative interaction. Furthermore, we have also found the condensation energy and the corresponding vortex lattice structure associated with vortices that form near the upper critical field $B_{c2}$. One of the goals of this paper was to lay the groundwork to study vortex lattice solutions and the corresponding condensation energy in finite isospin chiral perturbation theory, where pions are Goldstone bosons and interact via momentum-dependent derivative interactions – the possibility of vortex condensation was first discussed in Ref. A detailed analysis of vortex lattice solutions in finite isospin chiral perturbation the-
ory will be done in a separate publication.

References

1. A. A. Abrikosov, On the Magnetic properties of superconductors of the second group, Sov. Phys. JETP 5 (1957) 1174–1182, [Zh. Eksp. Teor. Fiz.32,1442(1957)].
2. A. Abrikosov, The magnetic properties of superconducting alloys, Journal of Physics and Chemistry of Solids 2 (3) (1957) 199–208.
3. A. A. Abrikosov, Fundamentals of the Theory of Metals, North Holland, Amsterdam, 1988.
4. H. B. Nielsen, P. Olesen, Vortex Line Models for Dual Strings, Nucl. Phys. B61 (1973) 45–61, [302(1973)].
5. J. Ambjorn, N. K. Nielsen, P. Olesen, Superconducting Cosmic Strings and W Condensation, Nucl. Phys. B310 (1988) 625–635.
6. M. N. Chernodub, Superconducting properties of vacuum in strong magnetic field, Int. J. Mod. Phys. D23 (6) (2014) 1430009.
7. B. J. Harrington, H. K. Shepard, Alteration of the Ground State by External Magnetic Fields. The Abelian Higgs Model, Nucl. Phys. B105 (1976) 527–537.
8. J. Ambjorn, P. Olesen, On the Formation of a Random Color Magnetic Quantum Liquid in QCD, Nucl. Phys. B170 (1980) 60–78.
9. J. Ambjorn, P. Olesen, A Color Magnetic Vortex Condensate in QCD, Nucl. Phys. B170 (1980) 265–282.
10. A. K. Harding, D. Lai, Physics of Strongly Magnetized Neutron Stars, Rept. Prog. Phys. 69 (2006) 2631.
11. D. Kharzeev, Parity violation in hot QCD: Why it can happen, and how to look for it, Phys. Lett. B633 (2006) 260–264.
12. D. E. Kharzeev, L. D. McLerran, H. J. Warringa, The Effects of topological charge change in heavy ion collisions: 'Event by event P and CP violation', Nucl. Phys. A803 (2008) 227–253.
13. K. Fukushima, D. E. Kharzeev, H. J. Warringa, The Chiral Magnetic Effect, Phys. Rev. D78 (2008) 074033.
14. M. N. Chernodub, J. Van Doorselaere, H. Verschelde, Electromagnetically superconducting phase of vacuum in strong magnetic field: structure of superconductor and superfluid vortex lattices in the ground state, Phys. Rev. D85 (2012) 045002.
15. M. N. Chernodub, J. Van Doorselaere, H. Verschelde, Spontaneous electromagnetic superconductivity and superfluidity of QCDxQED vacuum in strong magnetic field, PoS QNP2012 (2012) 109.
16. M. N. Chernodub, J. Van Doorselaere, H. Verschelde, Spontaneous electromagnetic superconductivity of vacuum induced by a strong magnetic field: QCD and electroweak theory, AIP Conf. Proc. 1492 (2012) 281–287.
17. D. T. Son, M. A. Stephanov, QCD at finite isospin density: From pion to quark - anti-quark condensation, Phys. Atom. Nucl. 64 (2001) 834–842, [Yad. Fiz.64,899(2001)].
18. D. T. Son, M. A. Stephanov, QCD at finite isospin density, Phys. Rev. Lett. 86 (2001) 592–595.
19. P. Adhikari, T. D. Cohen, J. Sakowitz, Finite Isospin Chiral Perturbation Theory...
20. P. Adhikari, Magnetic vortex lattices in finite isospin chiral perturbation theory (to appear).

21. P. Adhikari, J. Choi, Vortex solutions in the Abelian Higgs Model with a neutral scalar, Acta Phys. Polon. B48 (2017) 145. arXiv:1703.00961 doi:10.5506/APhysPolB.48.145

22. D. Saint-James, G. Sarma, E. Thomas, Type II Superconductivity, Commonwealth and International Library. Liberal Studies Divi, Elsevier Science & Technology, 1969.

23. D. Arovas, C. Wu, Lecture notes on superconductivity (a work in progress) (2016).