Lowest Eigenvalues of Random Hamiltonians

J. J. Shen, 1 Y. M. Zhao, 1, 2, 3, 4 A. Arima, 5 and N. Yoshinaga 6

1Department of Physics, Shanghai Jiao Tong University, Shanghai 200240, China
2Center of Theoretical Nuclear Physics, National Laboratory of Heavy Ion Accelerator, Lanzhou 730000, China
3Nishina Center, Institute of Physical Chemical Research (RIKEN),
Hirosawa 2-1, Wako-shi, Saitama 351-0198, Japan
4CCAST, World Laboratory, P.O. Box 8730, Beijing 100080, China
5Science Museum, Japan Science Foundation, 2-1 Kitanomaru-koen, Chiyoda-ku, Tokyo 102-0091, Japan
6Department of Physics, Saitama University, Saitama 338-8570, Japan

(Dated: February 2, 2008)

In this paper we present results of the lowest eigenvalues of random Hamiltonians for both fermion and boson systems. We show that an empirical formula of evaluating the lowest eigenvalues of random Hamiltonians in terms of energy centroids and widths of eigenvalues are applicable to many different systems (except for \( d \) boson systems). We improve the accuracy of the formula by adding moments higher than two. We suggest another new formula to evaluate the lowest eigenvalues for random matrices with large dimensions (20-5000). These empirical formulas are shown to be applicable not only to the evaluation of the lowest energy but also to the evaluation of excited energies of systems under random two-body interactions.

PACS numbers: 21.10.Re, 21.10.Ev, 21.60. Cs

Keywords:

*yElectronic address: ymzhao@sjtu.edu.cn*
I. INTRODUCTION

The $I = 0^+$ ground state dominance for random Hamiltonians was discovered by Johnson et al. in 1998 \cite{1}. Many efforts have been devoted to understand this problem since then. See Refs. \cite{2, 3} and references therein for details.

Recently, Papenbrock and Weidenmueller considered fluctuations of and correlations between the $J$-dependent spectral widths \cite{4, 5}. By using such correlations they were able to approximately evaluate spin $I$ ground state probabilities for six nucleons in a single-$j$ ($j = 19/2$) shell under a two-body random ensemble (TBRE). Along this line, a very simple formula of the lowest energy of spin $I$ states based on energy centroid, spectral width and dimension of spin $I$ states, was presented in Ref. \cite{6} by the present authors. Our formula (see Eq. (12) of Ref. \cite{6}) was shown to be applicable to evaluate statistically the lowest energy of spin $I$ states, and proved to be very good in predicting spin $I$ ground state probabilities calculated by using random two-body interactions. We should note that the idea of evaluating the lowest eigenvalue based on energy centroids and spectral widths was suggested by Ratcliff \cite{7}, Vary et al. \cite{8}, and by Zuker and his collaborators \cite{9}. However, predicted results by formulas in Refs. \cite{4, 5, 7, 8, 9} are less accurate than the formula in Ref. \cite{6}.

Evaluation of the lowest eigenvalue is not only useful in studying regular structure of atomic nuclei in the presence of random interactions, but also a very common practice in many other fields. It is therefore the purpose of this article to revisit empirical formulas of evaluating the lowest eigenvalues under random Hamiltonians. In this paper we shall also suggest other empirical formulas of evaluating the lowest eigenvalues of random Hamiltonians and random matrices.

This paper is organized as follows. In Sec. II we review results of evaluating the lowest eigenvalues by using energy centroids and width. It is found that the results of single-$j$ shell and sd-boson systems follow the formula of Ref. \cite{6} (except $d$ bosons). In Sec. II we also improve such evaluations by adding higher moments for various systems. In Sec. III we investigate other four lowest eigenvalues. In Sec. IV we concentrate on discussions of $d$ bosons. The lowest eigenvalues of random matrices are discussed in the Appendix.

II. EMPIRICAL FORMULA OF REF. \cite{6}

In this paper we take the same notations as our earlier work \cite{6}. For fermions in a single-$j$ shell

$$\hat{H} = \sum_{J=0, \text{even}}^{2j-1} \sqrt{2J+1} G_J \left[ A_J^\dagger \times \bar{A}^J \right]^0,$$

(1)

with $A_J^\dagger = \frac{1}{\sqrt{2}} \left[ a_J^\dagger a_J \right]^J$ and $\bar{A}^J = -\frac{1}{\sqrt{2}} \left[ a_J \bar{a}_J \right]^J$. Two-body matrix elements $G_J$’s are assumed to follow the TBRE, i.e., they are a set of random numbers with a distribution function

$$\rho(G_J) = \frac{1}{\sqrt{2\pi}} \exp(-G_J^2/2), J = 0, 2, ..., 2j - 1.$$

(2)

Matrix elements of $\hat{H}$ for spin-$I$ states can be expressed in terms of coefficients of fractional parentage (cfp’s):

$$H_{I\beta\gamma} = \langle j^n I\beta | \hat{H} | j^n I\gamma \rangle = \sum_J \alpha_{I\beta\gamma}^J G_J.$$

(3)

One can obtain eigenenergies of spin $I$ states by diagonalizing $H_{I\beta\gamma}$. Such definition can be easily generalized to many-$j$ shells without confusion.

In Ref. \cite{6}, a very simple formula of evaluating the lowest energy of $H_I$ was suggested as follows,

$$E_{I}^{(\text{min})} = \tilde{E}_I - \Phi(d_I) \sigma_I,$$

(4)

where $\Phi(d_I) = \sqrt{a \ln d_I + b}$, $\sigma_I$ is the square root of the second order moment of eigenvalues of all spin $I$ states, $d_I$ is dimension of spin $I$ states, and $a$ and $b$ were determined empirically to be 0.99 and 0.36, respectively. This formula was found to hold statistically.

Let us first investigate fermion systems. We take four fermions in a $j = 15/2$ shell, four fermions in a $j = 21/2$ shell, four fermions in a $j = 31/2$ shell, five fermions in a $j = 19/2$ shell, six fermions in a $j = 17/2$ shell, and four fermions in a two-$j$ ($j = 7/2, 5/2$) shell. The procedure is the same as in Ref. \cite{6}; first, $\Phi(d_I) = |\tilde{E}_I - E_{I}^{(\text{min})}|/\sigma_I$ are calculated for 1000 runs of the TBRE, and second, we calculate the average of $\Phi(d_I)$ for these 1000 runs. We
plot \((\Phi(d_I))^2\) as a function of \(\ln d_I\) in Fig. 1. One can see that \((\Phi(d_I))^2\) is close to linear correlation with \(\ln(d_I)\) for all these examples. Parameters \(a\) and \(b\) obtained by this procedure in each example are shown in Table I. They are close to those obtained in Ref. \[6\] \((a = 0.99, b = 0.36)\). The results of \(sd\) bosons are similar to those of fermions in a single-\(j\) shell or two-\(j\) shells, as shown in Fig. 1(b) and Table II. We also investigate results of a more complicated system: three valence protons and three valence neutrons in the \((2s_{1/2}, 1d_{3/2}, 0i_{11/2})\) shell, as shown in Fig. 2.

![Plot](image)

**FIG. 1.** (Color online) Factor \((\Phi(d_I))^2\) versus \(\ln(d_I)\) determined numerically for fermion and \(sd\)-boson systems. (a) fermion systems; (b) \(sd\)-boson systems. The dimension in these cases is 2-40.

**TABLE I.** \(a\) and \(b\) values for fermions in a single-\(j\) shell and two-\(j\) shells corresponding to (a)-(b) of Fig. 1(a).

| \((2j, n)\) | \((15, 4)\) | \((21, 4)\) | \((31, 4)\) | \((19, 5)\) | \((17, 6)\) | \((2j_1 = 5, 2j_2 = 7, n = 4)\) | Ref. \[6\] |
|------------|------------|------------|------------|------------|------------|----------------------------|----------|
| \(a\)      | 1.04 ± 0.03| 1.02 ± 0.01| 1.00 ± 0.02| 1.04 ± 0.03| 1.07 ± 0.02| 1.06 ± 0.02 | 0.99 |
| \(b\)      | 0.28 ± 0.04| 0.32 ± 0.03| 0.34 ± 0.04| 0.32 ± 0.08| 0.20 ± 0.06| 0.30 ± 0.04 | 0.36 |

**TABLE II.** \(a\) and \(b\) values of \(sd\)-boson systems with \(n = 5, 6, 7, 8, 9,\) and 12, respectively. See Fig. 1(b).

| \(n\) | 5     | 6     | 7     | 8     | 9     | 12    |
|-------|-------|-------|-------|-------|-------|-------|
| \(a\) | 1.10 ± 0.02 | 1.09 ± 0.03 | 1.13 ± 0.05 | 1.08 ± 0.04 | 1.10 ± 0.05 | 1.05 ± 0.02 |
| \(b\) | 0.24 ± 0.03 | 0.25 ± 0.05 | 0.17 ± 0.08 | 0.26 ± 0.08 | 0.22 ± 0.11 | 0.30 ± 0.07 |
Fig. 2. (Color online) Factor $\Phi(d I)^2$ versus $\ln (d I)$ for three valence protons ($N_p = 3$) and three valence neutrons ($N_n = 3$) in the shell ($2s_1/2, 1d_3/2, 0i_{11/2}$). The line is plotted by using $\Phi(d I)^2 = a \ln d I + b$.

Eq. (4) was shown to be well applicable for evaluating spin $I$ g.s. probability (see Ref. [6] for details). However, it is not good enough for evaluation of the ground state energy in a reasonable precision. It is therefore desirable to improve the formula of Ref. [6].

Towards this goal, we consider higher orders of moment of the eigenenergies to compensate the deviation from Gaussian distribution for eigenvalues. In this paper we consider the cubic root of the third order of central moment, denoted by $\sigma_3$:

$$\sigma_3 = \left( \frac{1}{d I} \sum_{i=1}^{d I} \sum_{k=1}^{d I} H_{ki} \left( \sum_{j=1}^{d I} H_{ij} H_{jk} \right) \right) - 3 \frac{1}{d I^2} \left( \sum_{i=1}^{d I} \sum_{j=1}^{d I} H_{ij}^2 \right) \sum_{i=1}^{d I} H_{ii} + 2 \frac{1}{d I} \left( \sum_{i=1}^{d I} H_{ii} \right)^3 \right)^{1/3}. \tag{5}$$

Let us assume

$$E_{I}^{(\text{min})} = E_I - C_2\sigma_2 + C_3\sigma_3, \tag{6}$$

where $\sigma_2 = \sigma_f$ in Eq. (4). According to our numerical experiments (see Table III), the disagreement between predicted result of $E_I^{(\text{min})}$ by using Eq. (6) and that by diagonalizing $H_I$ can be reduced to about 1/2 on average, in comparison with our earlier formula Eq. (4). Unfortunately, we are not able to obtain a simple formula for $C_3$. The values of $C_2$ such obtained are very close to the value of $\Phi(d I)$.

| $d$ | 3 | 4 | 5 | 8 | 10 | 14 | 20 | 25 | 29 |
|-----|---|---|---|---|----|----|----|----|----|
| $I$ | 50 | 3 | 0 | 39 | 2 | 4 | 13 | 12 | 20 |
| $\epsilon_A$ | 0.40 | 0.46 | 0.18 | 0.28 | 0.33 | 0.30 | 0.13 | 0.15 | 0.17 |
| $\epsilon_B$ | 0.31 | 0.17 | 0.098 | 0.18 | 0.15 | 0.12 | 0.091 | 0.089 | 0.10 |
III. OTHER LOWEST ENERGIES

The success of evaluating the lowest energies encourages us to go further. Here we study eigenvalues of the first to the fourth excited states. Typical results are shown in Fig. 3, where we take the same form of Eq. (4). One can confirm here that Eq. (4) is also applicable to evaluate the excited energies for both fermions and \( sd \) boson systems while the values of \( a \) and \( b \) are different from the ground state. The results are summarized in Fig. 4 and Table IV.

FIG. 3. (Color online) \([\Phi(d_I)^2] \) versus \( \ln(d) \). (a) the first excited states for fermions. (a’ ) the first excited states for \( sd \) bosons. (b) the second excited states for fermions. (b’) the second excited states for \( sd \) bosons. (c) the third excited states for fermions. (c’) the third excited states for \( sd \) bosons. (d) the fourth excited states for fermions. (d’) the fourth excited states for \( sd \) bosons.

FIG. 4. Comparison of average \( \Phi(d_I)^2 \) versus \( \ln(d) \). In the figure, “g.s.” means the ground state, “1st” means the
first excited state, “2nd” means the second excited state, “3rd” means the third excited state, and “4th” means the fourth excited state.

It is worthy to note that our empirical formulas are based on calculations with finite dimensions (dimension is less than $10^4$). According to the linear correlation between $(\Phi(d_I))^2$ and $\ln(d_I)$ shown in Fig. 4, the straight line corresponding to ground states and that corresponding to excited states (e.g., the 1st excited states) seem to cut across each other. Such an intersection does not occur, because the g.s. energy always corresponds a larger $\Phi$ value. In other words, the linear correlation between $(\Phi(d_I))^2$ and $\ln(d_I)$ is valid when $d_I$ is smaller than $10^4$. For larger $d_I$, the results should be further investigated.

**TABLE IV.** $a$ and $b$ values for fermion systems and sd-boson systems with eigenvalues of the ground state and the first to the fourth excited states corresponding to Fig. 4, respectively.

| State | g.s. $a$ | 1st $a$ | 2nd $a$ | 3rd $a$ | 4th $a$ | 1st $b$ | 2nd $b$ | 3rd $b$ | 4th $b$ |
|-------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| $a$   | $1.04 \pm 0.01$ | $1.18 \pm 0.01$ | $1.21 \pm 0.01$ | $1.23 \pm 0.01$ | $1.25 \pm 0.01$ | $0.30 \pm 0.03$ | $-1.42 \pm 0.02$ | $-2.14 \pm 0.03$ | $-2.64 \pm 0.02$ | $-3.03 \pm 0.02$ |
| $b$   | $1.04 \pm 0.01$ | $1.18 \pm 0.01$ | $1.21 \pm 0.01$ | $1.23 \pm 0.01$ | $1.25 \pm 0.01$ | $0.30 \pm 0.03$ | $-1.42 \pm 0.02$ | $-2.14 \pm 0.03$ | $-2.64 \pm 0.02$ | $-3.03 \pm 0.02$ |

IV. D-BOSON SYSTEMS

In this Section we study very simple systems, $d$ bosons, for which eigen-energies are linear combinations of two-body matrix elements. The two-body Hamiltonian of a $d$-boson system is given by

$$H_d = H_0 + \sum_{l} \frac{1}{2} \sqrt{2l+1} c_l \left[ (d^\dagger \times d^\dagger)^l \times (\bar{d} \times \bar{d})^l \right]_0 .$$

From Eqs. (2.79) and (2.82) of Ref. [10], we have

$$E = E_0 + \frac{1}{14} (4c_2 + 3c_4) n_d (n_d - 1) + \frac{1}{70} (7c_0 - 10c_2 + 3c_4) [n_d (n_d + 3) - v(v + 3)]$$
$$+ \frac{1}{14} (-c_2 + c_4) [I(I+1) - 6n_d] .$$

For states with given $I$, the state with $v$ being maximum value or minimum (depending on the sign of $\beta' = \frac{1}{70} (7c_0 - 10c_2 + 3c_4)$) is the lowest among all eigenvalues. For given $n_d$ and $I$,

$$E_{Iv} = E(n_d, I) - \frac{1}{70} (7c_0 - 10c_2 + 3c_4) v(v + 3) ,$$

where $E(n_d, I)$ is the same for all spin $I$ states.

Our results of $\Phi(d_I)$ versus $\ln(d_I)$ are presented in Fig. 5. There seems no simple relationship between $\Phi$ and $\ln(d_I)$. Apparently, there is systematic deviation from linear correlation between $(\Phi(d_I))^2$ and $\ln(d_I)$. The value of $\Phi(d_I)$ seems to saturate when the dimension goes to infinity.
Fig. 5. Phenomenological factors $[\Phi(d_I)]^2$ versus $\ln(d)$ for $d$ bosons with different boson numbers. The line is plotted by assuming $[\Phi(d_I)]^2 = 0.99 \ln(d) + 0.36$. (a) $n_d = 9$, (b) $n_d = 18$, (c) $n_d = 36$, (d) $n_d = 72$, (e) $n_d = 108$, (f) $n_d = 180$.

The deviation from linear correlation between $\Phi(d_I)$ and $\ln d_I$ in Fig. 5 originates from the distribution of eigenenergies of $d$ bosons. The eigenvalues of other more complicated systems in this paper exhibit a Gaussian distribution [15], while that of $d$ bosons is close to a triangular ($I \geq n$) or trapezoidal ($I < n$) distribution, as shown in Fig. 6.
Fig. 6. Relative distribution of eigenvalues of \( d \) bosons. (a) \( n = 180, I = 240 \); (b) \( n = 180, I = 100 \); (c) \( n = 360, I = 460 \); (d) \( n = 360, I = 260 \).

V. SUMMARY AND DISCUSSION

To summarize, in this paper we have studied lowest eigenvalues of random Hamiltonians. First, we demonstrate that our semi-empirical formula suggested in Ref. \([5]\), \( E^{(\text{min})}_I = E_I - \Phi(d_I)\sigma_I \) with \( (\Phi(d_I))^2 = a\ln d + b \) works well for various examples (\( sd \) bosons, fermions in a single-j or many-j shells) with dimension ranging from 2 to 5000 (except for \( d \) boson systems for which there are systematic deviations). We also improve our formula by adding the third-order central moment.

Second, we investigate eigen-energies of the excited states including the first, second, third and fourth lowest energies. We find that the same formula with different parameters describes statistically very well to eigen energies.

Third, we study \( d \)-boson systems and discuss why there are systematic deviation from our statistical formula of the lowest eigenvalues.

In this paper we also study the lowest eigenvalues of random matrices in Appendix. We see that there is systematic deviation from Wigner’s semi-circle prediction \([10]\) when the dimension is not very large: when the dimension is less than 100, the lowest eigenvalue can be evaluated statistically by the same formula of Ref. \([8]\). We suggest another statistical formula of lowest eigenvalues of pure random matrices.

Acknowledgements: We would like to thank the National Natural Science Foundation of China for supporting this work under grants 10575070, 10675081. This work is also supported partly by the Research Foundation Doctoral Program of Higher Education of China under grant No. 20060248050, Scientific Research Foundation of Ministry of Education in China for Returned Scholars, the NCET-07-0557, and by Chinese Major State Basic Research Developing Program under Grant 2007CB815000.

Appendix A The lowest eigenvalues of random matrices

All matrix elements in this Appendix are given by Gaussian distributed random numbers. We change the dimension of matrices with dimension ranging from 2 to 3000. In Fig. 8(a), the dimension changes from 2 to 15. In this case, we see a similar result as in main text. There exists linear correlation between \( \Phi^2 \) and \( \ln d \) (see Fig. 8(a)), with \( a \) and \( b \) are very close to those in Sec. II. This means that Eq. (4) applies to general cases (statistically) when dimension of matrix is not very large.

When the dimension changes from 15 to 3000, we find that there is systematic deviation from linear correlation between \( \Phi^2 \) and \( \ln d \). We empirically obtain

\[- \Phi(d) = a\ln d/b + b : a = 1.59 \pm 0.03, b = -2.00 \pm 0.003. \tag{10}\]

as shown in Fig. 8(b). We see that Eq. (10) is more applicable than \( \Phi^2 = a\ln d + b \) when dimension of matrix becomes large. We note that the parameter \( b \) in Eq. (10) saturates at 2 in the large \( d \) limit. This saturation value is the predicted value of Wigner’s semi-circle theorem.
Fig. 7. (Color online) Phenomenological factors $\Phi(d)$ versus $\ln (d)$ for random matrices. (a) matrices with smaller dimension. Suppose $[\Phi]^2 = a \ln (d) + b$. We obtain $a = 0.97758 \pm 0.02426$, $b = 0.37201 \pm 0.04754$ by $\chi^2$ fitting. These values are very close to those in Ref.[6]; (b) matrices with large dimension ($d = 15 - 3000$). One can see a nice linear correlation between $-\Phi(d)$ and $\ln d/d$. We obtain that $a = 1.589 \pm 0.028$, $b = -2.001 \pm 0.003$ in Eq. (10).