DISCRETRANZIZED LAPLACIANS ON AN INTERVAL
AND THEIR RENORMALIZATION GROUP

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ABSTRACT

The Laplace operator admits infinite self-adjoint extensions when considered on a segment of the real line. They have different domains of essential self-adjointness characterized by a suitable set of boundary conditions on the wave functions. In this paper we show how to recover these extensions by studying the continuum limit of certain discretized versions of the Laplace operator on a lattice. Associated to this limiting procedure, there is a renormalization flow in the finite dimensional parameter space describing the discretized operators. This flow is shown to have infinite fixed points, corresponding to the self-adjoint extensions characterized by scale invariant boundary conditions. The other extensions are recovered by looking at the other trajectories of the flow.
1. Introduction

Since the time of their introduction by Wilson [1], lattice field theories have proven to be a successful tool in the study of quantum field theories. In this approach, one replaces the space-time continuum (or rather, in the Hamiltonian version of the method [2], the space continuum) with a discrete lattice. Correspondingly, the action (or the Hamiltonian) describing the continuum theory is replaced by a discretized one. The spacing $a$ of the lattice then provides a natural cut-off for the theory and the continuum limit is recovered by taking $a$ to zero. This process is governed by the renormalization group, which dictates how the parameters on which the discretized action depends have to evolve with $a$ so that, for $a$ sufficiently small, the physics becomes insensitive to the presence of the lattice spacing; when this happens, one says that the continuum limit exists. Unfortunately, for most of the field theories of interest, the analysis of the renormalization group is too complicated to be carried out exactly and can be attacked only within perturbation theory.

In this paper we discuss a non-trivial illustration of the ideas of Wilson already in the context of quantum mechanics. The idea of renormalization has been applied already to quantum mechanics by other authors [3], [4], but in a different way from that pursued here. In particular, in [4], it is shown how the use of renormalization allows one to make physical sense of the inverse square potential, in one space dimension, by rendering finite the energy of the ground state, which would otherwise be infinitely negative.

What we consider here is simply a free particle confined to move on a finite interval. Its Hamiltonian is just a free Laplacian and the only non-trivial dynamical aspect of the system is the behaviour of the particle at the end-points. This behaviour is described, in mathematical terms, by a set of boundary conditions to be imposed on the wave functions. The most general choice of the boundary conditions compatible with the requisite of self-adjointness for the Hamiltonian is parametrized by a $U(2)$ matrix.

Upon discretisation, the interval is replaced by $N$ points and the Laplacian by the classic finite difference operator already used by Laplace. In the continuum limit, one recovers the self-adjoint extension of the Laplacian characterized by periodic boundary conditions for the wave functions. One can then wonder whether it is also possible to recover all the other self-adjoint extensions as continuum limits of some suitably modified finite difference operators. Indeed, after noticing that the classical finite difference operator is well defined only away from the end points of the interval, in order to define it everywhere, one is led to consider its most general symmetric extensions. The extended operators depend on 4 real parameters, which are the discrete analogue of the $U(2)$ matrix found earlier in the continuum. The continuum limit is then obtained by taking $N$ to infinity. The corresponding four-dimensional renormalization flow turns out to be exactly solvable and nevertheless non-trivial. We find and infinite number of fixed points which are related to the self-adjoint extensions of the Laplacian characterized by a set of scale-invariant boundary conditions. The other non-trivial trajectories, which in the limit approach the fixed points, correspond instead to the remaining choices of the boundary conditions.
conditions.

The paper is organized as follows. Section 2 reviews some basic facts about the self
adjoint extensions of the Laplacian on an interval. In Section 3 we introduce the finite
difference operators which represent the discretised version of the Laplacian with arbitrary
boundary conditions for an interval and in Section 4 we discuss the related eigenvalue
problem. The continuum limit is discussed in Section 5. Finally, the Appendix contains a
detailed analysis of the eigenvalue equations associated to the finite difference operators
introduced in Section 3.

2. Self-Adjoint extensions of the Laplacian on an interval.

Let us consider the interval of the real line \([0, 2\pi]\) and let \(L^2([0, 2\pi])\) be the Hilbert
space of square integrable functions over \([0, 2\pi]\) with inner product given by:
\[
\langle \phi, \psi \rangle = \int_0^{2\pi} dx \phi^* \psi.
\] (2.1)

Consider now the Laplacian operator: \(-\frac{d^2}{dx^2}\). As it stands it is only a formal differential
operator. In order to obtain a well defined operator in \(L^2([0, 2\pi])\) it is necessary to specify
the domain of functions on which it acts. As a starting domain, consider the set \(D^0\) of
functions which are twice differentiable and vanish at the end-points together with their
first derivatives:
\[
D^0 \equiv \{ \psi(x) : \psi(x) \in C^2[0, 2\pi] ; \psi(0) = \psi(2\pi) = \partial_x \psi(0) = \partial_x \psi(2\pi) = 0 \}.
\] (2.2)

Let us call \(\Delta^0\) the operator obtained by restricting the Laplacian to \(D^0\). \(\Delta^0\) is symmetric,
as a simple computation shows:
\[
\langle \Delta^0 \psi^0, \phi^0 \rangle = \langle \psi^0, \Delta^0 \phi^0 \rangle \quad \forall \phi^0, \psi^0 \in D^0.
\] (2.3)

Consider now the adjoint \(\Delta^0^\dagger\) of \(\Delta^0\). It is defined by the equation:
\[
\langle \Delta^0^\dagger \psi, \phi^0 \rangle = \langle \psi, \Delta^0 \phi^0 \rangle \quad \forall \phi^0, \psi^0 \in D^0.
\] (2.4)

The domain of \(\Delta^0^\dagger\) is defined to be the set \(D^0 \subset L^2([0, 2\pi])\) for which equation (2.4) is
fulfilled, for all \(\phi^0 \in D^0\). It is easily seen that
\[
D^0 \subset D^0^\dagger.
\] (2.5)

In fact, eq. (2.4) is satisfied by all twice differentiable functions \(\psi\), regardless of their
boundary values or of the boundary values of their derivatives. This shows that \(\Delta^0\),
though symmetric, is not self-adjoint (s.a.). In order to make it s.a. it is necessary to
extend its domain from $D_0$ to a larger one $D$. If it is possible to do it and in how many different ways, is established using the deficiency index theorem $[5]$. This theorem can be stated as follows. Let $A^0$ be a symmetric operator on some Hilbert space $\mathcal{H}$, with domain $D(A^0)$ and let $A^{0\dagger}$ be its adjoint. Let now $n_+, n_-$ be the number of linearly independent solutions of the equation

$$A^{0\dagger} \psi_j^{(\pm)} = \pm i \psi_j^{(\pm)}, \quad j = 1, \cdots, n_\pm.$$  \hspace{1cm} (2.6)

$n_+, n_-$ are called respectively positive and negative deficiency indices. Then, $A^0$ admits s.a. extensions if and only if $n_+ = n_- = n$ and they can be all obtained in the following way. Let $U \equiv \{U_{ij}\}$ be an arbitrary $n \times n$ unitary matrix. Consider now the domain:

$$D_U \equiv D_0 \oplus \operatorname{span}\{\psi^{(+)}_i + U_{ij}\psi^{(-)}_j\}. \hspace{1cm} (2.7)$$

The operators

$$A_U : D_U \rightarrow \mathcal{H}$$

$$A\psi = A^0\psi^0 + ic_j(\psi^{(+)}_j - U_{jk}\psi^{(-)}_k), \hspace{1cm} (2.8)$$

where

$$\psi = \psi^0 + c_j(\psi^{(+)}_j + U_{jk}\psi^{(-)}_k)$$

represent, for each $U$, a different extension of $A^0$ and it can be shown that they are all essentially s.a.. Moreover it can be proven that all possible s.a. extensions of $A^0$ can be obtained in this way, by taking different unitary matrices $U$ in equations (2.7),(2.8).

In the case of $\Delta^0$ it is easy to check that $n_+ = n_- = 2$. $\Delta^0$ then admits an $U(2)$ infinity of s.a. extensions, characterized by domains of the form (2.7). Now, it can be shown that, for all $U \in U(2)$ there exists a set of homogeneous linear boundary conditions (b.c.) which uniquely characterizes $D_U$:

$$D_U = \{ \psi : a_{11}^{U} \psi'(0) + a_{12}^{U} \psi'(2\pi) + a_{13}^{U} \psi(0) + a_{14}^{U} \psi(2\pi) = 0 \quad i = 1, 2 \}. \hspace{1cm} (2.9)$$

In eq. (2.3), $\tau$ denotes the derivative with respect to $x$. The set of all b.c. can be divided in six (in general overlapping) different classes, depending on the couple of quantities, chosen among $\psi(0), \psi(2\pi), \psi'(0)$ and $\psi'(2\pi)$, with respect to which (2.9) can be solved. We shall see that in order to get all b.c. it is in fact sufficient to consider only three cases, which we list hereafter.

**Case 1.** $a_{ij}$ are such that eq. (2.3) can be solved with respect to $\phi(0)$ and $\phi(2\pi)$:

$$\begin{pmatrix} \phi'(0) \\ \phi'(2\pi) \end{pmatrix} = \begin{pmatrix} u & w \\ \zeta & v \end{pmatrix} \begin{pmatrix} \phi(0) \\ \phi(2\pi) \end{pmatrix}. \hspace{1cm} (2.10)$$

We show now that the coefficients $u, v, w$ and $\zeta$ in (2.10) have to fulfill certain conditions. If (2.10) is to define a s.a. extension of the Laplacian it must be that if $\psi$ is a $C^2$ function such that:

$$\langle -\psi'' , \phi \rangle = \langle \psi , -\phi'' \rangle \hspace{1cm} (2.11)$$

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for all $\phi$ fulfilling (2.10), $\psi$ also has to fulfill (2.10). Now we have:

$$\langle -\psi'', \phi \rangle - \langle \psi, -\phi'' \rangle = -\psi^* \phi'(2\pi) + \psi^* \phi'(0) =$$

$$= -\psi^*(2\pi)[\zeta \phi(0) + v\phi(2\pi)] + \psi^*(0)[u \phi(0) + w \phi(2\pi)] + \psi^*(2\pi) \phi(2\pi) - \psi^*(0) \phi(0) =$$

$$= [u \psi^*(0) - \zeta \psi^*(2\pi) - \psi^*(0)] \phi(0) + [w \psi^*(0) - v \psi^*(2\pi) + \psi^*(0)] \phi(2\pi).$$

(2.12)

Since now $\phi(0)$ and $\phi(2\pi)$ are arbitrary, $\psi$ must be such that:

$$\left( \begin{array}{c} \psi'(0) \\ \psi'(2\pi) \end{array} \right) = \left( \begin{array}{cc} u^* & -\zeta^* \\ -w^* & v^* \end{array} \right) \left( \begin{array}{c} \psi(0) \\ \psi(2\pi) \end{array} \right).$$

(2.13)

So, by comparing (2.12) with (2.10), we see that $\psi$ fulfills (2.10) too if and only if:

$$u = u^* , \quad v = v^* , \quad \zeta = -w^*$$

and so the most general b.c. of this form can be written as

$$\left( \begin{array}{c} \phi'(0) \\ \phi'(2\pi) \end{array} \right) = \left( \begin{array}{cc} u & w \\ -w^* & v \end{array} \right) \left( \begin{array}{c} \phi(0) \\ \phi(2\pi) \end{array} \right), \quad u \text{ and } v \text{ real.}$$

(2.14)

**Case 2.** $a_{ij}$ are such that (2.9) can be solved with respect to $\phi(0)$ and $\phi'(2\pi)$:

$$\left( \begin{array}{c} \phi'(0) \\ \phi'(2\pi) \end{array} \right) = \left( \begin{array}{cc} \tilde{u} & \tilde{w} \\ \tilde{\zeta} & \tilde{v} \end{array} \right) \left( \begin{array}{c} \phi(0) \\ \phi(2\pi) \end{array} \right).$$

(2.15)

It is obvious that, if $\tilde{v} \neq 0$, we can solve (2.15) with respect to $\phi'(2\pi)$ and the b.c. (2.13) can be rewritten in the form of case 1. Then, we set $\tilde{v} = 0$. A computation similar to that performed for case 1, shows that (2.15) define s.a. extensions if and only if $\tilde{w} = \tilde{\zeta}^*$ and $\tilde{u}$ is real:

$$\left( \begin{array}{c} \phi'(0) \\ \phi'(2\pi) \end{array} \right) = \left( \begin{array}{cc} \tilde{u} & \tilde{w} \\ \tilde{\zeta} & \tilde{v} \end{array} \right) \left( \begin{array}{c} \phi(0) \\ \phi(2\pi) \end{array} \right), \quad \tilde{u} \text{ real.}$$

(2.16)

**Case 3.** $a_{ij}$ are such that (2.9) can be solved with respect to $\phi'(0)$ and $\phi'(2\pi)$:

$$\left( \begin{array}{c} \phi(0) \\ \phi(2\pi) \end{array} \right) = \left( \begin{array}{cc} \hat{u} & \hat{w} \\ \hat{\zeta} & \hat{v} \end{array} \right) \left( \begin{array}{c} \phi'(0) \\ \phi'(2\pi) \end{array} \right).$$

(2.17)

Again a computation similar to that for case 1 shows that in order to define a s.a. extension the parameters $\hat{u}$ and $\hat{v}$ must be real, while $\hat{\zeta} = -\hat{w}^*$. Then we have:

$$\left( \begin{array}{c} \phi(0) \\ \phi(2\pi) \end{array} \right) = \left( \begin{array}{cc} \hat{u} & \hat{w} \\ -\hat{u}^* & \hat{v} \end{array} \right) \left( \begin{array}{c} \phi'(0) \\ \phi'(2\pi) \end{array} \right) \quad \hat{u} \text{ and } \hat{v} \text{ real.}$$

(2.18)

This set of b.c. is not already included in cases 1 and 2 if and only if

$$\hat{u}\hat{v} + |\hat{w}|^2 = 0 \quad \text{and} \quad \hat{u} = 0.$$
We are then left with:
\[
\begin{pmatrix}
  \phi(0) \\
  \phi(2\pi)
\end{pmatrix} =
\begin{pmatrix}
  0 & 0 \\
  0 & \hat{v}
\end{pmatrix}
\begin{pmatrix}
  \phi'(0) \\
  \phi'(2\pi)
\end{pmatrix} \quad \hat{v} \text{ real.}
\] (2.19)

It can be checked that the remaining possibilities lead to b.c. already included in the three cases we have examined. Then, the most general s.a. extension of the Laplacian on an interval is described by one of the above sets of b.c.

As we have seen earlier, the space of s.a. extensions of the Laplacian on an interval is a four dimensional manifold, precisely $U(2)$. Now, for the sake of simplicity and also because later it will allow us to draw three-dimensional pictures of the renormalization flow for the discretized Laplacians, we will restrict our attention to those extensions which are invariant under $PT$ symmetry ($P$ here stands for the Parity transformation which interchanges the end points $x = 0$ and $x = 2\pi$ while $T$ stands for Time-Reversal). Now under $P$ the wave function $\phi$ transforms as:
\[
(P\phi)(x) = \phi(2\pi - x)
\] (2.20)

while under $T$ we have:
\[
(T\phi)(x) = \phi^*(x)
\] (2.21)

and so under $PT$ we have:
\[
\tilde{\phi}(x) = (PT\phi)(x) = \phi^*(2\pi - x).
\] (2.22)

Let us check the constraints implied by the $PT$ symmetry on the b.c. examined earlier. **Case 1.** The $PT$ transformed wave function fulfills the b.c.:
\[
\begin{pmatrix}
  \tilde{\phi}'(0) \\
  \tilde{\phi}'(2\pi)
\end{pmatrix} =
\begin{pmatrix}
  -v & w \\
  -w^* & -u
\end{pmatrix}
\begin{pmatrix}
  \tilde{\phi}(0) \\
  \tilde{\phi}(2\pi)
\end{pmatrix},
\] (2.23)

and so $PT$ invariance requires $u = -v$. We then have b.c. of the form:
\[
\begin{pmatrix}
  \phi'(0) \\
  \phi'(2\pi)
\end{pmatrix} =
\begin{pmatrix}
  u & w \\
  -w^* & -u
\end{pmatrix}
\begin{pmatrix}
  \phi(0) \\
  \phi(2\pi)
\end{pmatrix}.
\] (2.24)

**Case 2.** A computation similar to that for case 1, shows that $PT$ invariance requires $|\tilde{w}|^2 = 1$. So we have:
\[
\begin{pmatrix}
  \phi'(0) \\
  \phi'(2\pi)
\end{pmatrix} =
\begin{pmatrix}
  \tilde{u} & e^{i\theta} \\
  e^{-i\theta} & 0
\end{pmatrix}
\begin{pmatrix}
  \phi(0) \\
  \phi(2\pi)
\end{pmatrix}.
\] (2.25)

**Case 3.** $PT$ invariance requires $\tilde{v} = 0$. Then we are left just with one set of b.c., those for an open line:
\[
\phi(0) = \phi(2\pi) = 0.
\] (2.26)

We turn now to the eigenvalue problem for the $PT$ invariant s.a. extensions listed above. The general solution to the eigenvalue equation:
\[
-\phi'' = p^2 \phi
\] (2.27)
is of the form
\[ \phi_p = Ae^{ipx} + Be^{-ipx}. \] (2.28)
If \( p \) is real the function (2.28) is a plane wave solution and \( p \) represents the inverse wavelength, if \( p \) is imaginary (2.28) represents a bound state solution. An eigenvalue equation for \( p \) is obtained after imposing on (2.28) the b.c. (2.24), (2.25) or (2.26). We then get the following equations:

**Case 1.** The coefficients \( A \) and \( B \) have to fulfill the following linear eqs:
\[
A(u + we^{i2\pi p} - ip) + B(u + we^{-i2\pi p} + ip) = 0
\]
\[
A(ue^{i2\pi p} + w^* + ipe^{i2\pi p}) + B(ue^{-i2\pi p} + w^* - ipe^{-i2\pi p}) = 0.
\] (2.29)

The consistency condition for this system is the eigenvalue eq. for \( p \):
\[
(p^2 + |w|^2 - u^2)\sin 2\pi p - 2up\cos 2\pi p - (w + w^*)p = 0.
\] (2.30)

**Case 2.** The coefficients \( A \) and \( B \) have to fulfill now the following linear eqs:
\[
A(\tilde{u} + ipe^{i\theta}e^{i2\pi p} - ip) + B(\tilde{u} - ipe^{i\theta}e^{-i2\pi p} + ip) = 0
\]
\[
A(ue^{i2\pi p} - e^{-i\theta}) + B(e^{-i2\pi p} - e^{-i\theta}) = 0
\] (2.31)

and the consistency condition for this system gives the following eigenvalue equation for \( p \):
\[
2pcos 2\pi p + \tilde{u}\sin 2\pi p - 2pcos\theta = 0.
\] (2.32)

**Case 3.** The coefficients \( A \) and \( B \) have to fulfill the linear system:
\[
A + B = 0
\]
\[
Ae^{i2\pi p} + Be^{-i2\pi p} = 0.
\] (2.33)

The consistency condition now reads:
\[
\sin 2\pi p = 0.
\] (2.34)

Of particular interest among (2.24), (2.25) and (2.26) are those s.a. extensions characterized by a set of scale invariant b.c. As we shall see later they will correspond to the fixed points of the renormalization flow in the space of the lattice discretizations of the Laplacian on an interval. Among the b.c. considered in case 1, the only scale-invariant ones are:
\[
\phi'(0) = \phi'(2\pi) = 0.
\] (2.35)

These are the b.c. for an open line with “free” ends. Among the b.c. under case 2, the scale-invariant ones are:
\[
\phi(0) = e^{i\theta}\phi(2\pi)
\]
\[
\phi'(0) = e^{i\theta}\phi'(2\pi).
\] (2.36)
They can be interpreted as b.c. for a circle, threaded by a magnetic flux. Finally, the only scale-invariant b.c. of type 3 are:

\[ \phi(0) = \phi(2\pi) = 0. \] (2.37)

They describe an open line with completely reflecting end points.

3. Discretized Laplacians on an interval.

In the previous Section, we have seen that the Laplacian admits infinite self-adjoint extensions on the interval \([0, 2\pi]\), each distinguished by a different domain \(D_U\), characterized by a certain set of b.c. on the wave functions (2.9).

Suppose now we discretize the interval by dividing it in \(N\) subintervals of width \(2\pi/N\) and centered around the points \(x_i = (i - \frac{1}{2})\frac{2\pi}{N}, i = 1, \cdots, N\). Consider now the subspace \(H^N\) of stepwise functions \(\chi(x)\), which are constant in each of these subintervals:

\[ H^N \equiv \text{span} \{\chi^N(x) = \chi^N(x_i) \text{ for } (i - 1)\frac{2\pi}{N} < x < i\frac{2\pi}{N} \mid i = 1, \cdots, N\}. \] (3.1)

\(H^N\) is an \(N\)-dimensional subspace of \(L^2([0, 2\pi])\). It is clear that, in the limit \(N \to \infty\), \(H^N\) becomes dense in \(L^2([0, 2\pi])\). We can represent each function in \(H^N\) as an \(N\)-dimensional complex vector: \((\chi^N(i)) = (\chi^N(x_i))\).

Consider now the self-adjoint extension of the Laplacian \(\Delta_{\text{per}}\) acting on periodic wave functions:

\[ \phi(0) = \phi(2\pi) \quad \phi'(0) = \phi'(2\pi). \] (3.2)

(This extension falls under case 2, when \(\tilde{u} = 0\) and \(\tilde{w} = 1\).) As it is well known, this extension can be approximated with a finite difference operator acting on the space \(H^N\):

\[ (\Delta^N \chi^N)(i) = -2Z_N^2 [\chi^N(i + 1) + \chi^N(i - 1) - 2\chi^N(i)] \quad i = 1, \cdots, N. \] (3.3)

Here \(N + 1 \equiv 1\) and \(0 \equiv N\). \(Z_N^2\) is an overall normalization. Let us see how \(\Delta_{\text{per}}\) is recovered in the limit \(N \to \infty\). The eigenfunctions of \(\Delta_N\) are easily seen to be equal to:

\[ \chi^N_m(j) = \exp \left\{ im \left( \frac{2\pi}{N} j - \frac{\pi}{N} \right) \right\} = \exp(imx_j) \quad m = 0, \pm 1, \cdots, \pm \left[ \frac{N - 1}{2} \right]. \] (3.4)

where \([a]\) is equal to \(a\) if \(a\) is integer and to \(a - 1/2\) if \(a\) is half-integer. The corresponding eigenvalues are:

\[ \lambda^N_m = -2Z_N^2 \left( \cos \frac{2\pi m}{N} - 1 \right). \] (3.5)

The eigenfunctions are seen to be stepwise approximations of plane waves. The exponent \(m\) can be interpreted as an inverse wavelength (in units of the interval length). Now
the discretized operators \( \Delta^N \) converge to \( \Delta_{\text{per}} \) in the following sense. Let us order the eigenstates of \( \Delta_{\text{per}} \) according to their eigenvalues \( E_i \):

\[
\psi_i \quad E_i \leq E_{i+1} \quad i = 1, \ldots, \infty.
\] (3.6)

Let now \( r \) an arbitrary integer and consider the operators \( \Delta^N \), with \( N \geq r \). Let us order also their eigenfunctions \( \chi^N_m(x) \) according to their eigenvalues \( \lambda^N_m \):

\[
\chi^N_m(x) \quad \lambda^N_m \leq \lambda^N_{m+1}, \quad m = 1, \ldots, N.
\]

The claim is that the first \( r \) eigenfunctions \( \chi_1^N(x), \ldots, \chi_r^N(x) \) of \( \Delta^N \) converge pointwise to the first \( r \) eigenfunctions of \( \Delta_{\text{per}}, \psi_1(x), \ldots, \psi_r(x) \). Moreover it is possible to fix the constants \( Z_N \) so that, in the limit \( N \to \infty \), also the first \( r \) eigenvalues \( \lambda^N_m \) converge to the \( E_m \). These statements are easily checked in this example. For instance, if we choose \( Z_N = \frac{N}{2\pi} \), we see that:

\[
\lim_{N \to \infty} \lambda^N_m = \lim_{N \to \infty} -2 \left( \frac{N}{2\pi} \right)^2 \left( \cos \frac{2\pi m}{N} - 1 \right) = m^2 = E_m. \quad (3.7)
\]

The question that arises now is if we can recover all self-adjoint extensions of the Laplacian on the interval as continuum limits of finite difference operators similar to (3.3). We have seen in Sec. 1 that the s.a. extensions of the Laplacian on an interval all have different domains. On the contrary, any finite difference operator we can write will act on the entire subspace \( \mathcal{H}^N \) for every \( N \). It becomes then unclear how the continuum operators with their different domains can possibly emerge in the continuum limit. As we shall see in the sequel of this paper this is indeed the case. The operator (3.3) can be generalized in a natural way to give a family of finite difference operators containing four real parameters. These parameters will be subject to a renormalization procedure having \( N \), the number of subdivisions of the interval, as control parameter and this will give rise to a flow in the parameter space. The corresponding trajectories will converge towards certain fixed points. We shall see that each of these trajectories leads in the limit to a different s.a. extension of the Laplacian. We shall argue that all s.a. extensions can be recovered by simply looking at different trajectories.

Let us generalize the finite difference operator (3.3) on an interval. Notice that, on an interval, formula (3.3) certainly makes sense for \( i = 2, \ldots, N-1 \), but becomes ill-defined at the endpoints, \( i = 1 \) and \( i = N \). If we then write \( \Delta^N \) as a matrix acting on the vector \( \chi^N(i) \) we get:

\[
\Delta^N \chi^N = -Z_N^2 \begin{pmatrix}
* & * & * & \cdots & * & * \\
1 & -2 & 1 & 0 & \cdots & 0 \\
0 & 1 & -2 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots \\
0 & \cdots & \cdots & 0 & 1 & -2 & 1 \\
* & * & \cdots & * & * & * & * 
\end{pmatrix} \begin{pmatrix}
\chi^N(1) \\
. \\
. \\
. \\
. \\
\chi^N(N) 
\end{pmatrix}. \quad (3.8)
\]
In (3.8) the first and the last rows of the matrix representing $\Delta^N$ are missing. Since there is no natural way of defining $\Delta^N$ at the end points, we will simply take for $\Delta^N$ the most general hermitian completion of its corresponding matrix and write:

$$\Delta^N = -Z_N^2 \begin{pmatrix} a & 1 & 0 & \cdots & \cdots & \cdots & c \\ 1 & -2 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & -2 & 1 \\ c^* & 0 & \cdots & \cdots & 0 & 1 & b \end{pmatrix} \begin{pmatrix} \chi^N(1) \\ \vdots \\ \vdots \\ \vdots \\ \chi^N(N) \end{pmatrix}.$$ (3.9)

Here $a$ and $b$ are arbitrary real parameters, while $c$ is an arbitrary complex number. Notice that the matrix elements containing the parameters only relate the wavefunction $\chi^N$ at the endpoints and seem therefore related to boundary conditions. We also remark that $a, b, c$ constitute a set of four real parameters, which is also the dimension of the set of $U(2)$ matrices that parametrize the s.a. extensions of the Laplacian.

4. The eigenvalue problem for $\Delta^N$

In this Section we study the eigenvalue equation for the matrix $\Delta$ (for simplicity of notation we will omit from now on the superscript $N$ from the symbols for the finite difference operators $\Delta^N$ or the wavefunctions $\chi^N(x)$):

$$\Delta \chi_\lambda \equiv -Z^2 \begin{pmatrix} a & 1 & 0 & \cdots & \cdots & \cdots & c \\ 1 & -2 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & -2 & 1 \\ c^* & 0 & \cdots & \cdots & 0 & 1 & b \end{pmatrix} \begin{pmatrix} \chi_\lambda(1) \\ \vdots \\ \vdots \\ \vdots \\ \chi_\lambda(N) \end{pmatrix}$$

$$= Z^2 \chi_\lambda \begin{pmatrix} \chi_\lambda(1) \\ \vdots \\ \vdots \\ \vdots \\ \chi_\lambda(N) \end{pmatrix}.$$ (4.1)

The eigenfunctions of $\Delta$ fall into 5 classes, depending on the eigenvalue $\lambda$. They are listed hereafter:

I) If $\lambda < 0$, we have:
$$\chi_\lambda(n) = Ae^{kn} + Be^{-kn},$$ (4.2)
where $\lambda = -2(coshk - 1), k > 0$.

II) If $\lambda = 0$, we have:
$$\chi_0(n) = A + Bn.$$ (4.3)
III) If $0 < \lambda < 4$, we have:

$$\chi_\lambda(n) = Ae^{ikn} + Be^{-ikn},$$

(4.4)

where $\lambda = -2(cosk - 1), 0 < k < \pi$.

IV) If $\lambda = 4$, we have:

$$\chi_4(n) = (-1)^n(A + Bn).$$

(4.5)

V) If $4 < \lambda < \infty$, we have:

$$\chi_\lambda(n) = A\xi^n + B\xi^{-n},$$

(4.6)

where $\lambda = -(\xi + \xi^{-1} - 2), \xi < -1$.

We prove the statement for case V, the proof being similar in the other cases. Let $\chi_\lambda$ be an eigenfunction with eigenvalue $\lambda > 4$. Let then $\xi$ be the unique real number smaller than -1 such that $\lambda = -(\xi + \xi^{-1} - 2)$ and define $A$ and $B$ to be such that:

$$\chi_\lambda(1) = A\xi + B\xi^{-1}$$

$$\chi_\lambda(2) = A\xi^2 + B\xi^{-2}.$$  

(4.7)

Notice that eq.ns (4.7) have a unique solution because

$$\det \begin{pmatrix} \xi & \xi^{-1} \\ \xi^2 & \xi^{-2} \end{pmatrix} = \xi^{-1} - \xi \neq 0.$$

By construction, $\chi_\lambda$ is of the form (4.6) for $n = 1, 2$. We now prove by induction that it must be so for all $n$. So, suppose the statement is true for $n \leq \pi$, with $\pi \geq 2$. Then, since $\chi_\lambda$ is an eigenfunction of $\Delta^N$, it must be that:

$$-\left[\chi_\lambda(\pi + 1) + \chi_\lambda(\pi - 1) - 2\chi_\lambda(\pi)\right] = \lambda\chi_\lambda(\pi)$$

or

$$\chi_\lambda(\pi + 1) = (\xi + \xi^{-1} - 2)(A\xi^\pi + B\xi^{-\pi}) - (A\xi^{\pi-1} + B\xi^{-(\pi-1)}) + 2(A\xi^\pi + B\xi^{-\pi}) =$$

$$= A\xi^{\pi+1} + B\xi^{-(\pi+1)},$$

(4.8)

which proves the statement.

The physical meaning of the solutions of type I, II and III is clear: they represent stepwise approximations of bound states, zero modes and plane waves respectively. As for the solutions of type IV and V we shall see in the next section that they “disappear” from the spectrum in the continuum limit and are unphysical. Notice also that the eigenfunctions of type I, III and V can all be written as:

$$\chi_\lambda(n) = Az^n + Bz^{-n}, \quad \lambda \neq 0, 4$$

(4.9)

where

$$\lambda = -(z + z^{-1} - 2).$$
\[ z \in C \equiv \mathbb{R} \cup \{e^{i\theta}, \ 0 < \theta < \pi\} \cup [1, \infty]. \]

Let us derive now the eigenvalue equation for the states of type I, III and V. Notice first the the wavefunctions (4.9) automatically satisfy the eigenvalue equation for \( n = 2, \ldots, N - 1 \). Then we just have to consider the endpoints \( n = 1 \) and \( n = N \). We find:

\[
A_N[(a + 2)z + cz^N - 1] + B_N[(a + 2)z^{-1} + cz^{-N} - 1] = 0
\]

\[
A_N[(b + 2)z^N + c^* z - z^{N+1}] + B_N[(b + 2)z^{-N} + c^* z^{-1} - z^{-(N+1)}] = 0. \tag{4.10}
\]

The condition of compatibility for these eqs. leads to the eigenvalue equation for \( z \):

\[
[\alpha(z^{N-1} - z^{-(N-1)}) + \beta(z^N - z^{-N}) + \gamma(z - z^{-1}) - (z^{N+1} - z^{-(N+1)}) = 0 \tag{4.11}
\]

where we have defined:

\[
\alpha \equiv |c|^2 - (a + 2)(b + 2)
\]

\[
\beta \equiv a + b + 4
\]

\[
\gamma \equiv 2(\Re c) . \tag{4.12}
\]

The real parameters \( \alpha, \beta \) and \( \gamma \) introduced here are independent on each other and are only subject to the condition:

\[
4\alpha + \beta^2 - \gamma^2 \geq 0. \tag{4.13}
\]

For the sake of simplicity we will restrict our analysis, as we did in the continuous case, only to the set of matrices \( \Delta \) which are left invariant by a \( PT \) transformation. Now, under \( PT \) the vector \( \chi(n) \) transforms as:

\[
(PT\chi)(n) = \chi^*(N - n + 1) \tag{4.14}
\]

and then in order for the matrix \( \Delta \) to be \( PT \) invariant it must be that \( a = b \):

\[
\Delta \equiv -Z^2 \begin{pmatrix}
    a & 1 & 0 & \cdots & \cdots & \cdots & c \\
    1 & -2 & 1 & 0 & \cdots & \cdots & 0 \\
    0 & 1 & -2 & 1 & 0 & \cdots & 0 \\
    \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
    0 & \cdots & \cdots & 0 & 1 & -2 & 1 \\
    c^* & 0 & \cdots & \cdots & 0 & 1 & a
\end{pmatrix}. \tag{4.15}
\]

A \( PT \) invariant \( \Delta \) is thus identified by three real constants and we can think of it as a point in \( \mathbb{R}^3 \). In the next Section we will turn to the study of the continuum limit for the finite difference operators (4.13) and show that in this limit we recover all \( PT \) invariant s.a. extensions of the Laplacian, eqs. (2.24), (2.25) and (2.26).
5. The continuum limit for $\Delta^N$.

In this Section we study the continuum limit of the finite difference operators $\Delta^N$. In the Appendix it is shown that at least $N - 2$ eigenstates of $\Delta$ are of the plane-wave type (eq. (4.4)). Let us rewrite them:

$$\chi_k(n) = A e^{i k n} + B e^{-i k n} = A e^{i 2\pi p n} + B e^{-i 2\pi p n} \equiv \chi_p(n), \quad 0 < k < \pi . \quad (5.1)$$

The number $p \equiv \frac{N k}{2\pi}$ represents the inverse wavelength of the state in units of the interval length. In the Appendix it is shown that the eigenvalues $k$ are distributed essentially uniformly in the interval $[0, \pi]$, which implies that the first values of $p$ are of the order of $1, 2, \cdots$. The lowest eigenstates then have wavelengths of the order of the length of the interval.

Now, our prescription for taking the continuum limit is the following. Assume we start with a certain set of values for $a$, $c$ and $N$, $(a_0, c_0; N_0)$, or equivalently with a set of values for $\alpha$, $\beta$, $\gamma$ and $N$, $(\alpha_0, \beta_0, \gamma_0; N_0)$. We then consider the eigenvalue equation determining the wavelengths of the “plane wave” states (4.4), namely eq. (4.11) with $z = e^{2\pi p/N}$:

$$\alpha_0 \sin 2\pi p \left(1 - \frac{1}{N_0}\right) + \beta_0 \sin 2\pi p + \gamma_0 \sin 2\pi p \frac{1}{N_0} = \sin 2\pi p \left(1 + \frac{1}{N_0}\right). \quad (5.2)$$

Let now:

$$p_0^1 < p_0^2 < \cdots < p_i^0 \quad i \leq N, \quad (5.3)$$

be its solutions (here we just list the eigenvalues and do not consider the existence of eventual degeneracies, which, as it is shown in the Appendix is an exceptional case). Our prescription for taking the continuum limit is then to vary $\alpha$, $\beta$ and $\gamma$ with $N$ so that the corresponding eigenvalue equation for the plane wave states:

$$\alpha(N) \sin 2\pi p \left(1 - \frac{1}{N}\right) + \beta(N) \sin 2\pi p + \gamma(N) \sin 2\pi p \frac{1}{N} = \sin 2\pi p \left(1 + \frac{1}{N}\right) \quad (5.4)$$

shares with (5.2) as many wavelengths as possible, starting from the longest one. We shall see that this prescription fixes completely the dependence of $\alpha$, $\beta$ and $\gamma$ on $N$. In general, it will be possible to hold constant only the first three inverse wavelengths, $p_1^0$, $p_2^0$, $p_3^0$, but there will be exceptional cases where we will have to consider the first four of them.

Having determined the evolution of the parameters $\alpha$, $\beta$, $\gamma$ with $N$, we have to assign now the renormalization condition for the overall constant $Z_N$ which appears in front of $\Delta^N$, eq. (4.13). This constant can be fixed by demanding that the eigenvalue $E_1 = Z_N^2 \lambda_1$ of the first plane-wave state (the one with inverse wavelength $p_1^0$) takes some fixed value, independent on $N$. Now, it can be verified, by studying the eigenvalue equation, that $p_1$ is always of the order of unity, which implies for $k_1$ a value of the order of $1/N$. Consequently, we have:

$$E_1 = Z_N^2 \lambda_1 = -2Z_N^2 (\cos k_1 - 1) \approx (Z_N k_1)^2 = \left(Z_N \frac{2\pi p_1}{N}\right)^2. \quad (5.5)$$
We see from this formula that $Z_1$ diverges in the continuum limit like $N$. Notice that the scaling behaviour of $Z_N$ implies that the eigenvalues of the states of type IV and V, which are always larger than or equal to $Z_N^24$, diverge in the limit $N \to \infty$ and disappear from the spectrum. This means that the states of type IV and V do not have physical meaning.

Our renormalization procedure generates a flow in the space of $PT$ invariant matrices $\Delta^N$, the control parameter being $N$. We will then be able to prove that, depending on the choice of the initial values $\alpha_0$, $\beta_0$, $\gamma_0$, the discretized Laplacians (4.15) converge in the continuum limit to any of the $PT$ invariant s.a. extensions discussed in Sec. 1. This will be achieved in two steps. First we will prove that the eigenvalue equations (5.4) converge in the limit $N \to \infty$ to those found for the $PT$ invariant s.a. of the Laplacian, eqs. (2.30), (2.32) and (2.34). The wavelength spectra of the finite difference operators $\Delta^N$ then converge to those of the s.a. extensions of the Laplacian. Then we prove that also the eigenfunctions $\chi^N$ converge, in the sense discussed earlier, to those of the continuous $PT$ invariant Laplacians. This will be done by showing that the linear systems (4.10) which determine $A_N$ and $B_N$ converge to the corresponding systems for the coefficients $A$ and $B$ of the continuum eigenfunctions.

We summarize here the qualitative features of the flow and anticipate what boundary conditions for the Laplace operator one gets in the continuum limit:

- We can describe the space of $PT$-invariant matrices $\Delta^N$ by means of the three parameters $a$, $b$, $c$ or, alternatively, by the real numbers $\alpha$, $\beta$, $\gamma$ defined in (4.12) and subject to condition (4.13). Hence the parameter space is:

$$
\mathcal{P} \equiv \{ (\alpha, \beta, \gamma) \in \mathbb{R}^3 : 4\alpha + \beta^2 - \gamma^2 \geq 0 \}.
$$

- The renormalization group has an infinite number of fixed points, belonging to the sets:

$$
L_1 \equiv (\alpha = -1, |\beta| \geq 2, \gamma = 0) \\
L_2 \equiv (\alpha = 1, \beta = 0, |\gamma| \leq 2).
$$

- If the initial data $\alpha_0$, $\beta_0$, $\gamma_0$ are chosen to belong to such lines, in the continuum limit, one recovers the s.a. extensions of the Laplacian on an interval which correspond to scale-invariant boundary conditions. In particular the point $P \equiv (-1, 2, 0) \in L_1$ corresponds to the s.a. extension (2.35), the other points on $L_1$ to the s.a. extensions (2.37) and the points on $L_2$ to (2.36).

- For all initial data not belonging to a certain surface $S(\alpha_0, \beta_0, \gamma_0, N_0) = 0$ in $\mathcal{P}$, the parameters $\alpha$, $\beta$, $\gamma$ follow trajectories that all approach the point $P$, for $N \to \infty$ (Figure 1). The continuum limit Laplace operators corresponding to such trajectories are the ones characterized by the non-scale invariant boundary conditions falling under Case 1.
• If the initial data are chosen to belong to the surface $S$, the trajectories approach a point on the line $L_2$ (Figure 2). The corresponding continuum limit Laplace operators are characterized by the non-scale invariant boundary conditions falling under Case 2.

Let us show now that our renormalization prescription is well posed and prove the above statements.

First, consider the case when $\alpha_0, \beta_0, \gamma_0$ belong to $L_1$. It is easy to verify that the eigenvalue equation (5.2) reduces to

$$\sin 2\pi p = 0 \quad 0 < p < \frac{N}{2}. \quad (5.6)$$

The eigenvalues are the integer and half-integer numbers $m$ less than $N/2$. Consequently the only effect of increasing $N$ is that of adding shorter wavelengths to the spectrum, without changing the preexisting ones: according to our prescription these points then represent fixed points of the flow. Let us look now at the behaviour of the eigenfunctions $\chi_m^N$. If $\beta_0 = 2$, we can easily verify, by expanding $z = \exp(i2\pi p/N)$ in powers of $1/N$ and then keeping only the leading terms in eqs. (4.10), that these equations reduce to those determining the coefficients $A$ and $B$ of the continuum eigenfunctions for the Laplacian with b.c. given by eq. (2.33). We can then conclude that the operators $\Delta^N$ associated to the fixed point $P = (-1,2,0)$ converge to the extension (2.35). If $\beta_0 \neq 2$, a similar computation shows that in the continuum limit the eigenfunctions of the discretized operators converge to those for the s.a. extension of the Laplacian with the b.c. given by eqs. (2.37).

Suppose now that the initial point belongs to $L_2$. Eq. (5.2) reduces to:

$$2\cos 2\pi p = \gamma_0; \quad 0 < p < \frac{N}{2}. \quad (5.7)$$

Again the only effect of increasing $N$ is just that of adding shorter wavelengths to the spectrum and then all points of $L_2$ are fixed points. As to the eigenfunctions $\chi_p^N$, by looking at the limit of eqs. (4.10), we can check that they converge to those for the s.a. extension of the Laplacian having the b.c. (2.36), with $2\cos \theta = \gamma_0$.

We consider now the general case. Let $p_1, p_2$ and $p_3$ denote the first three eigenvalues of eq. (5.2) and suppose that the initial values of the parameters $\alpha_0, \beta_0, \gamma_0; N_0$ are such that the entire function of the complex plane:

$$h(x; p_j^0) \equiv \det \parallel \sin 2\pi p_j^0(1 - x), \sin 2\pi p_j^0, \sin 2\pi p_j^0 x \parallel \quad j = 1, 2, 3 \quad (5.8)$$

does not vanish identically in $x$ [if $h$ is identically zero if and only if at least two of the three eigenvalues $p_1, p_2, p_3$ are both integer or half-integer; this exceptional case will be examined later]. Since now $h(x; p_j^0)$ is an entire function of $x$, its zeros will be isolated points in the complex plane and this implies that there will be at most a finite number of them in the disc $|x| < 1/N_0$. Now, according to our renormalization prescription, the
running constants \( \alpha(N), \beta(N) \) and \( \gamma(N) \) will be chosen to be the solutions of the system of three linear equations:

\[
\alpha(N) \sin 2\pi p_j^0 \left( 1 - \frac{1}{N} \right) + \beta(N) \sin 2\pi p_j^0 + \gamma(N) \sin 2\pi p_j^0 \frac{1}{N} = \sin 2\pi p_j^0 \left( 1 + \frac{1}{N} \right). \tag{5.9}
\]

This system admits a unique solution for all \( N > N_0 \) [except possibly for a finite number of values of \( N \) such that \( 1/N \) is a zero of \( h(x) \)]. When this is the case we simply neglect these values of \( N \). This proves that our scheme is well posed.

We will now analyze the limiting behaviour of the solutions \( \alpha(N), \beta(N) \) and \( \gamma(N) \). For this purpose, we define two more entire functions of \( x \):

\[
f(x; p_j^0) \equiv \det \| \sin 2\pi p_j^0 (1 - x), \sin 2\pi p_j^0 (1 + x), \sin 2\pi p_j^0 x \| \tag{5.10}
\]

and

\[
g(x; p_j^0) \equiv \det \| \sin 2\pi p_j^0 (1 - x), \sin 2\pi p_j^0, \sin 2\pi p_j^0 (1 + x) \|; \tag{5.11}
\]

notice that \( f(x) \) is an odd function of \( x \), while \( g(x) \) is even and that \( f, g \) and \( h \) all vanish at \( x = 0 \). The solution of the system (5.9) can now be written, in terms of \( f, g \) and \( h \) as:

\[
\alpha(x; p_j^0) = -\frac{h(-x; p_j^0)}{h(x; p_j^0)}, \tag{5.12}
\]

\[
\beta(x; p_j^0) = \frac{f(x; p_j^0)}{h(x; p_j^0)}, \tag{5.13}
\]

\[
\gamma(x; p_j^0) = \frac{g(x; p_j^0)}{h(x; p_j^0)}. \tag{5.14}
\]

\( \alpha(x; p_j^0), \beta(x; p_j^0) \) and \( \gamma(x; p_j^0) \), being ratios of entire functions, are thus analytic functions of the complex plane having at most poles of finite order. We are now interested in the limits of these functions when \( x \to 0 \). Let us start from \( \alpha(x; p_j^0) \). Let \( \hat{n}(p_j^0) \) be the order of the first non vanishing derivative of \( h(x; p_j^0) \) for \( x = 0 \) (\( \hat{n} \) represents also the order of the zero of \( h(x) \) at the origin; a simple computation shows that \( \hat{n}(p_j^0) \) is always larger than 1). Use of the l’Hopital theorem then implies:

\[
\lim_{x \to 0} \alpha(x; p_j^0) = -\frac{\lim_{x \to 0} \frac{d^\hat{n}}{dx^\hat{n}} h(-x)}{\lim_{x \to 0} \frac{d}{dx} h(x)} = -(-1)^{\hat{n}(p_j^0)}. \tag{5.15}
\]

The limiting value of \( \alpha \) is then always equal to either 1 or -1. In order to study the limits of \( \beta \) and \( \gamma \) let us write the power expansions of \( h, f \) and \( g \) around \( x = 0 \):

\[
h(x; p_j^0) = \sum_{n=\hat{n}(p_j^0)}^\infty h_n(p_j^0) x^n \tag{5.16}
\]

\[
f(x; p_j^0) = \sum_{n=1}^\infty f_{2n}(p_j^0) x^{2n} \tag{5.17}
\]
\[ g(x; p_j^0) = \sum_{n=0}^{\infty} g_{n+1}(p_j^0) x^{2n+1}, \quad (5.18) \]

where \( h_n, f_n, g_n \) denote the derivative of order \( n \) with respect to \( x \) of \( h, f, g \) respectively.

In order to recover, in the continuum limit, all s.a. extensions of the Laplacian, it seems sufficient to consider only two cases \( \bar{n}(p_j^0) = 2, 3 \).

**Case a.** The initial data are such that \( \bar{n}(p_j^0) = 2 \). This is the most general case. \( h \) then has a zero of the second order at the origin. According to (5.15) we then have:

\[ \lim_{x \to 0} \alpha(x; p_j^0) = -1. \quad (5.19) \]

It is easy to check now that:

\[ f_2(p_j^0) = 2h_2(p_j^0) \]

and then use of the l’Hopital formula in eq. (5.13) implies:

\[ \lim_{x \to 0} \beta(x; p_j^0) = \frac{f_2(p_j^0)}{h_2(p_j^0)} = 2. \quad (5.20) \]

As for \( \gamma(x) \) it can be checked that \( g_2(p_j^0) \equiv 0 \) and then we find:

\[ \lim_{x \to 0} \gamma(x; p_j^0) = \frac{g_2(p_j^0)}{h_2(p_j^0)} = 0. \quad (5.21) \]

By putting together eqs. (5.19), (5.20) and (5.21), we see that:

\[
\text{if } \bar{n}(p_j^0) = 2 \quad \lim_{x \to 0} \langle \alpha(x), \beta(x), \gamma(x) \rangle = (-1, 2, 0) \equiv P. \quad (5.22)
\]

We saw earlier that the limiting point \( P \) is a fixed point of the flow and that the discretized operator associated with it converges in the continuum limit to the s.a. extension of the Laplacian characterized by the b.c. (2.35).

We show now that the eigenvalue equation (5.4) converges, for \( N \to \infty \), to an equation of the form (2.30). For this purpose, let us expand the r.h.s. of eqs. (5.12), (5.13) and (5.14) in power series of \( x = 1/N \):

\[
\alpha(x; p_j^0) = -1 + H_1(p_j^0) \frac{1}{N} + H_2(p_j^0) \left( \frac{1}{N} \right)^2 + \cdots \quad (5.23)
\]

\[
\beta(x; p_j^0) = 2 + F_1(p_j^0) \frac{1}{N} + F_2(p_j^0) \left( \frac{1}{N} \right)^2 + \cdots \quad (5.24)
\]

\[
\gamma(x; p_j^0) = G_1(p_j^0) \frac{1}{N} + G_2(p_j^0) \left( \frac{1}{N} \right)^2 + \cdots. \quad (5.25)
\]

If we substitute these expansions in (5.4) and keep only the leading terms in \( 1/N \) we get:

\[
\left[ \left( -2 + H_1(p_j^0) \frac{1}{N} + H_2(p_j^0) \frac{1}{N^2} \right) \cos \frac{2\pi p}{N} + 2 + F_1(p_j^0) \frac{1}{N} + F_2(p_j^0) \frac{1}{N^2} \right] \sin 2\pi p =
\]
\[ H_1(p^0_j) \frac{1}{N} \cos 2\pi p^0_j - G_1(p^0_j) \frac{1}{N} \frac{2\pi p^0_j}{N} . \] (5.26)

A simple computation shows that \( H_1(p^0_j) + F_1(p^0_j) \equiv 0 \) and then eq. (5.26) reduces to:

\[ [(2\pi p^0_j)^2 + H_2(p^0_j) + F_2(p^0_j)] \sin 2\pi p^0_j - 2\pi p^0_j [H_1(p^0_j) \cos 2\pi p^0_j - G_1(p^0_j)] = 0 \] (5.27)

This eq. has the same form as (2.30), if we identify:

\[ |w|^2 - u^2 = \frac{1}{(2\pi)^2} (H_2(p^0_j) + F_2(p^0_j)) \]

\[ 2u = \frac{1}{2\pi} H_1(p^0_j) \]

\[ w + w^* = -\frac{1}{2\pi} G_1(p^0_j) . \] (5.28)

Once these identifications are made, it is possible to check, by substituting the expansions (5.23), (5.24) and (5.25) into eqs. (4.10), that we recover, to the leading order in \( 1/N \), eqs. (2.29). This shows then that the continuum limit of our discrete operators coincides with the \( PT \) invariant s.a. extensions of the Laplacian contemplated under case 1.

By varying \( \alpha_0, \beta_0, \gamma_0 \) or equivalently \( p^0_j \) we can get essentially all s.a. extensions of type 1. Consider in fact eq. (2.30). Our prescription for the continuum limit is such that \( p^0_j \) remain eigenvalues also in that limit and then they are roots of eq. (2.30), with \( u \) and \( w \) given by eqs. (5.28). Now, conversely, it can be checked that, if we start from eq. (2.30) and try determine the coefficients \( u \) and \( w \) such that \( p^0_j \) are solutions, the answer is in general unique and is given again by eqs. (5.28). So, assuming that by varying \( \alpha_0, \beta_0, \gamma_0 \) we can get all the triples \( p^0_j \) that can be obtained from eq. (2.30) by looking at its first three solutions, we can conclude that all of s.a. extensions of type 1 can be recovered.

There are a only few exceptional cases that are left: they correspond to the case when two or more among the \( p^0_j \) are integers or half-integers. These last extensions will be found by analyzing the case of \( \alpha_0, \beta_0, \gamma_0 \) such that \( h(x; p^0_j) \equiv 0 \).

**Case b.** Suppose that the initial data \( (\alpha_0, \beta_0, \gamma_0; N_0) \) are such that \( \tilde{n}(p^0_j) = 3 \). This is the same as saying that:

\[ h_2(p^0_j) = \det \| -2\pi p^0_j \cos 2\pi p^0_j, \sin 2\pi p^0_j, 2\pi p^0_j \| = 0 \] (5.29)

and

\[ h_3(p^0_j) = \frac{1}{2} \det \| -(2\pi p^0_j)^2 \sin 2\pi p^0_j, \sin 2\pi p^0_j, 2\pi p^0_j \| \neq 0 . \] (5.30)

These eqs. specify a surface \( S = S(\alpha_0, \beta_0, \gamma_0; N_0) = 0 \) in the parameter space. Now eq. (5.29) implies then that there exist three real numbers \( \mu, \nu \) and \( \rho \) such that:

\[ \mu 2\pi p^0_j \cos 2\pi p^0_j + \nu \sin 2\pi p^0_j + \rho 2\pi p^0_j = 0 . \] (5.31)

Moreover eq. (5.30) implies that \( \mu \neq 0. \) \( h(x) \) now has a zero of the third order at \( x = 0 \) and eq. (5.13) implies that:

\[ \lim_{x \to 0} \alpha(x; p^0_j) = 1 . \] (5.32)
A simple computation also shows that:

$$\lim_{x \to 0} \beta(x; p_j^0) = 0$$  \hspace{1cm} (5.33)$$

$$\lim_{x \to 0} \gamma(x; p_j^0) = -2\frac{\rho}{\mu}.$$  \hspace{1cm} (5.34)

Condition (4.13) then implies that $|\rho/\mu| \leq 1$.

This computation shows that the trajectories starting from a point of $S$ will approach the point $(\alpha, \beta, \gamma) = (1, 0, -2\rho/\mu)$ belonging to the set $L_2$. We showed earlier that the points of this set represent fixed points and that in the continuum limit they correspond to the s.a. extensions of type 2, characterized by the b.c. (2.36). If we now expand the r.h.s. of eqs. (5.12), (5.13), (5.14) around $x = 0$ we find:

$$\alpha(x; p_j^0) = 1 + \tilde{H}_1(p_j^0) \left( \frac{1}{N} \right)^2 + \cdots$$  \hspace{1cm} (5.35)$$

$$\beta(x; p_j^0) = \tilde{F}_1(p_j^0) \left( \frac{1}{N} \right)^2 + \cdots$$  \hspace{1cm} (5.36)$$

$$\gamma(x; p_j^0) = -2\frac{\rho}{\mu} + \tilde{G}_1(p_j^0) \left( \frac{1}{N} \right) + \cdots.$$  \hspace{1cm} (5.37)

Notice that these expansions do not coincide with those computed earlier, under case 1. If we substitute these expansions in the eigenvalue eq. (5.4) and keep only the leading order terms in $1/N$ we get the limiting eigenvalue eq.:

$$[\tilde{H}_1(p_j^0) + \tilde{F}_1(p_j^0)] \sin 2\pi p - 4\pi p \left[ \cos 2\pi p + \frac{\rho}{\mu} \right] = 0.$$  \hspace{1cm} (5.38)

As a consequence of eq. (5.31), we have the following relations:

$$\tilde{H}_1(p_j^0) = -\frac{2\nu}{3\mu}$$

$$\tilde{F}_1(p_j^0) = -\frac{\nu}{3\mu}$$

$$\tilde{G}_1(p_j^0) = \frac{2\rho \nu}{3\mu^2}$$  \hspace{1cm} (5.39)

and then eq. (5.38) can be rewritten as:

$$\frac{\nu}{\mu} \sin 2\pi p + 4\pi p \left[ \cos 2\pi p + \rho \right] = 0 \hspace{1cm} \mu \neq 0$$  \hspace{1cm} (5.40)

which is of the same form as eq. (2.32), if we identify:

$$\tilde{u} = \frac{\nu}{2\pi \mu}$$  \hspace{1cm} (5.41)
Using these identifications, we will now prove that, in the continuum limit, the linear system (4.10) is equivalent to its analogue, eqs. (2.31), for the $PT$ invariant extensions contemplated under case 2. Notice first that, to the zeroth order in $1/N$, eqs. (4.10) are both equivalent to the following eq.:

$$A(e^{i\theta}e^{i2\pi p} - 1) + B(e^{i\theta}e^{-i2\pi p} - 1) = 0$$  \hspace{1cm} (5.43)

which is clearly equivalent to the second of eqs. (2.31). If we now subtract the second of eqs. (4.10) from the first and expand the difference to the first order in $1/N$ and use eqs. (5.39), we find:

$$A\left(\frac{\nu}{2\pi\mu} + ipe^{-i\pi e^{i2\pi p}} - ip\right) + B\left(\frac{\nu}{2\pi\mu} - ipe^{-i\pi e^{-i2\pi p}} + ip\right) = 0$$  \hspace{1cm} (5.44)

which, in view of eqs. (5.41) and (5.42), is equivalent to the first of eqs. (2.31).

We now turn to the exceptional case, when the initial data $(\alpha_0, \beta_0, \gamma_0; N_0)$ are such that $h(x; p^0_l) \equiv 0$. This case occurs if and only if at least two of the eigenvalues $p^0_l$ are both integer or half-integer and this happens if and only if $\alpha + 1 + \gamma = 0$. In this case, $p^0_1$ and $p^0_3$ are actually half-integer, while $p^0_2$ is neither integer nor half-integer. The plane of equation $\alpha + 1 + \gamma = 0$ is then invariant under the renormalization flow, but in order to completely fix the evolution of the running constants $\alpha(N)$, $\beta(N)$ and $\gamma(N)$ we have to consider also the fourth eigenvalue $p^0_4$, which again is neither integer nor half-integer. If we use the relation $\alpha + 1 + \gamma = 0$ to eliminate $\gamma$, the evolution of $\alpha$ and $\beta$ is then determined by the equations:

$$\alpha(N)[\sin 2\pi p^0_1(1 - x) - \sin 2\pi p^0_1 x] + \beta(N)\sin 2\pi p^0_l = \sin 2\pi p^0_l(1 + x) + \sin 2\pi p^0_l x \hspace{1cm} l = 2,4$$  \hspace{1cm} (5.45)

(in what follows we will not show again the range of values for the index $l$; it will be understood that $l = 2,4$). We can apply to these eqs. the same methods as those used for eqs. (3.44), and the conclusions are similar. Therefore we will be brief here. We introduce the entire functions:

$$s(x; p^0_l) \equiv \text{det} \parallel \sin 2\pi p^0_l(1 - x) - \sin 2\pi p^0_l x, \sin 2\pi p^0_l \parallel$$  \hspace{1cm} (5.46)

$$t(x; p^0_l) \equiv \text{det} \parallel \sin 2\pi p^0_l(1 - x) - \sin 2\pi p^0_l x, \sin 2\pi p^0_l(1 + x) + \sin 2\pi p^0_l x \parallel.$$  \hspace{1cm} (5.47)

Notice that both these functions vanish for $x = 0$ and that $t(x; p^0_l)$ is an odd function of $x$. Using these functions we can write the solution to eqs. (5.44) as:

$$\alpha(x; p^0_l) = \frac{s(-x; p^0_l)}{s(x; p^0_l)},$$  \hspace{1cm} (5.48)

$$\beta(x; p^0_l) = \frac{t(x; p^0_l)}{s(x; p^0_l)}.$$  \hspace{1cm} (5.49)
Let now $\tilde{r}(p^0_l)$ be the order of the first non-vanishing derivative of $s(x)$ for $x = 0$. Use of l'Hopital theorem implies:

$$\lim_{x \to 0} \alpha(x; p^0_l) = (-1)^{\tilde{r}(p^0_l)}. \quad (5.50)$$

As before, it is sufficient to consider two cases.

**Case a:** $\tilde{r}(p^0_l) = 1$. A simple computation analogous to that done earlier shows that:

$$\lim_{x \to 0} (\alpha(x; p^0_l), \beta(x; p^0_l), \gamma(x; p^0_l)) = (-1, 2, 0). \quad (5.51)$$

Moreover, the eigenvalue equation, eq. (4.11) and the linear system determining $A$ and $B$, eqs. (4.10), reduce in the continuum limit to the corresponding equations for the Laplacian with b.c. of type 1.

**Case b:** $\tilde{r}(p^0_l) = 2$. This case occurs if and only if:

$$s_1(p^0_l) = \left. \frac{ds(x; p^0_l)}{dx} \right|_{x=0} = -det \| 2\pi p^0_l (cos 2\pi p^0_l + 1), sin 2\pi p^0_l \| =$$

$$- 4cos p^0_l cos p^0_l det \| 2\pi p^0_l cos p^0_l, sin p^0_l \| = 0. \quad (5.52)$$

This conditions is fulfilled if and only if there exist real numbers $\sigma$ and $\tau$ such that:

$$2\pi \sigma p^0_l sin p^0_l + \tau cos p^0_l = 0. \quad (5.53)$$

Computations similar to the previous ones show that:

$$\lim_{x \to 0} (\alpha(x; p^0_l), \beta(x; p^0_l), \gamma(x; p^0_l)) = (1, 0, -2). \quad (5.54)$$

The limits of the eigenvalue eq. (4.11) and of the linear system (4.10) coincide now with those for the Laplacian with b.c. of type 2.
A. Study of the eigenvalue equation for $\Delta^N$

In this Appendix we analyze the eigenvalue equation for the finite difference operators $\Delta^N$. We start by showing that at least $N - 2$ eigenfunctions are of type III, (eq. (4.4)), namely of the plane-wave type. Consider the eigenvalue eq. (4.11), for $z = e^{ik}$ with $0 < k < \pi$:

$$\alpha \sin(k(N-1)) + \beta \sin(N) + \gamma \sin(k) - \sin(k(N+1)) = 0. \quad (A.1)$$

It is convenient to expand the sines in eq. (A.1) and rewrite it as:

$$[(\alpha - 1)\cos + \beta] \sin(N) = [(\alpha + 1)\cos - \gamma] \sin(k). \quad (A.2)$$

There are now several cases. Here we will examine only some of them, the analysis being similar in the other cases. First, there are some exceptional cases.

**Case a:** $\alpha = -1$ and $\gamma = 0$. Condition (4.13) then implies $|\beta| \geq 2$. These points all belong to the line $L_1$. In this exceptional case eq. (A.1) simplifies to

$$\sin(Nk) = \sin(2\pi p) = 0. \quad (A.3)$$

The eigenvalues $p$ are the integers and the half-integers less than $N/2$ and there are $N - 1$ of them. It can be checked that $N/2$ is also an eigenvalue.

**Case b:** $\alpha = 1$ and $\beta = 0$. Condition (4.13) implies $|\gamma| \leq 2$. These points all belong to the line $L_2$. In this other exceptional case eq. (A.1) reduces to:

$$2 \cos(N) = \cos(2\pi p) = \gamma. \quad (A.4)$$

**Case c:** $\alpha + 1 \pm \gamma = 0$; $\alpha - 1 + \beta \neq 0$ and $\beta$ and $\gamma$ are not such that we are in cases (a) and (b). Now $kN = (2m - 1)\pi$, $m = 1, \ldots, [N/2]$ are solutions to (A.2). The remaining solutions are the roots of the equation:

$$\frac{(\alpha - 1)\cos + \beta}{\sin} = (\alpha + 1)\cotg \frac{kN}{2}. \quad (A.5)$$

This equation always has exactly one solution in each interval $\frac{\pi}{N}(2m - 1) < k < \frac{\pi}{N}(2m + 1)$, $m = 1, \ldots, [N/2]$ for $N$ odd and $m = 1, \ldots, N/2 - 1$ for $N$ even. These solutions are neither integer nor half-integer. All in all we have then $N - 1$ plane wave solutions.

**Case d:** $\alpha + 1 \pm \gamma \neq 0$ (and the values of $\alpha$, $\beta$ and $\gamma$ are such that we are not in case (b)). This is the most general case. Now $p$ cannot be integer or half-integer and then eq. (A.1) is equivalent to:

$$\frac{(\alpha - 1)\cos(k) + \beta}{\sin} = (\alpha + 1)\cos(kN) - \gamma. \quad (A.6)$$
There are now several subcases, depending on the relative signs of $\alpha - 1 \pm \beta$ and $\alpha + 1 \pm \gamma$. We will discuss here only two subcases, the analysis being similar in the others. So suppose that:

$$\beta > |\alpha - 1| \quad |\gamma| < \alpha + 1.$$  \hfill (A.7)

Condition (4.13) is then always satisfied. The l.h.s. of eq. (A.2) is then always positive for $0 < k < \pi$ and diverges both at $k = 0$ and $k = \pi$. The r.h.s. is a periodic function with period $2\pi/N$. Its qualitative behaviour is the same as that of the function $\cotg nkN$ and it is easy to verify that there is exactly one solution in each of the intervals $\frac{k}{N}n < k < \frac{k}{N}(n + 1), \ n = 1, \cdots, N - 1$. There are then at least $N - 1$ plane wave solutions.

It is easily verified that there is also one solution of type I, namely a bound state. Since we have managed to find $N$ eigenvalues, it also follows that they are all non-degenerate.

Consider now the case:

$$\beta > |\alpha - 1| \quad \gamma > |\alpha + 1|,$$  \hfill (A.8)

where $\alpha$, $\beta$ and $\gamma$ have to fulfill condition (4.13). The l.h.s. has the same behaviour as in the previous subcase, but the r.h.s. has the qualitative behaviour of the function $-\gamma/sinkN$. It can be checked that the conditions (4.13) ensures that the minimum value of the l.h.s. is larger than the minimum value of the r.h.s.. It is then easy to verify that there are exactly two distinct eigenvalues in each of the subintervals $\frac{k}{N}(2m - 1) < k < \frac{k}{N}2m, \ m = 1, \cdots \lfloor N/2 \rfloor - 1$. Moreover, if $N$ is even there is one more solution in the interval $\frac{k}{N}(N - 1) < k < \frac{k}{N}$. For all $N$ there are then $N - 1$ solutions.

Again, it can be checked that there exists one solution of type I. As before, these eigenvalues are then all non-degenerate.

We show now that eq. (A.2) admits degenerate solutions only if $\alpha, \beta, \gamma$ belong to a certain curve. From eqs. (4.10), with $a = b$ and $z = e^{ik}$, we see that $k$ is a degenerate eigenvalue if and only if the coefficients of $A_N$ and $B_N$ all vanish simultaneously. This condition is equivalent to:

$$(a + 2)e^{ik} + ce^{ink} = 1$$
$$= (a + 2)e^{-ik} + ce^{-ink} = 1.$$  \hfill (A.9)

This system is compatible if and only if $\sin(N - 1)k \neq 0$. Then its solution is:

$$a + 2 = \frac{\beta}{2} = \frac{\sin Nk}{\sin k(N - 1)}$$
$$c = -\frac{\sin k}{\sin k(N - 1)}.$$  \hfill (A.10)

When $k$ is varied between 0 and $\pi$ in eqs. (A.10) $\alpha, \beta, \gamma$ describe a curve: this is then the set of values of the parameters for which some of the eigenvalues of eq. (A.1) can be degenerate.

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Fig. 1. Some examples of trajectories approaching the point $P$. 
Fig. 2. Some examples of trajectories approaching the line $L_2$. 