Constrained Differential Renormalization and Dimensional Reduction

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Abstract

We describe the equivalence at one loop between constrained differential renormalization and regularization by dimensional reduction in the MS scheme. To illustrate it, we reexamine the calculation of supergravity corrections to $(g-2)_l$.

1 Introduction

The most popular regularization method for perturbative calculations in gauge theories is dimensional regularization \[1\]. In conjunction with the minimal subtraction (MS) scheme, it leads to renormalized Green functions satisfying the Slavnov-Taylor identities. It has problems, however, in supersymmetric gauge theories, because invariance under supersymmetry transformations depends on

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the specific dimensionality of the objects involved. Hence, in general, dimensional regularization does not preserve supersymmetry. To improve the situation, Siegel proposed a modified version of the method \cite{1}, called regularization by dimensional reduction (or dimensional reduction, for short). It treats integral momenta (or space-time points) as $d$-dimensional vectors but takes all fields to be four-dimensional tensors or spinors. The relation between the four-dimensional and $d$-dimensional spaces is given by dimensional reduction from 4 to $d$ dimensions, \emph{i.e.}, the 4-dimensional space is decomposed into a direct sum of $d$- and $(4-d)$-dimensional subspaces in the following sense (we always work in Euclidean space) \cite{2}: the 4-dimensional metric-tensor $\delta_{\mu\nu}$ (satisfying the properties $\delta_{\mu\nu}\delta_{\nu\rho} = \delta_{\mu\rho}$ and $\delta_{\mu\mu} = 4$) and the $d$-dimensional one $\tilde{\delta}_{\mu\nu}$ (satisfying $\tilde{\delta}_{\mu\nu}\tilde{\delta}_{\nu\rho} = \tilde{\delta}_{\mu\rho}$ and $\tilde{\delta}_{\mu\mu} = d$) are related by

$$\delta_{\mu\nu}\tilde{\delta}_{\nu\rho} = \tilde{\delta}_{\mu\rho}. \tag{1}$$

Although this method is known to be inconsistent \cite{3}, the inconsistencies are in many cases under control and dimensional reduction is actually the preferred regularization method for explicit calculations in supersymmetric theories \cite{4}.

Differential renormalization is a position-space method that performs regularization and subtraction in one step by substituting ill-defined expressions by derivatives of well-defined ones \cite{6}. Recently, a new version aiming to preserve gauge invariance \cite{7} and supersymmetry \cite{8, 9} has been developed \cite{10, 11} and automatized \cite{12} at the one-loop level. This version, called constrained differential renormalization (CDR), is based on a set of rules that determine the renormalization of the Green functions. In Ref. \cite{12}, T. Hahn and one of the authors (M.P.V.) argued and explicitly checked that, to one loop, CDR and dimensional reduction in the MS scheme render the same results, up to a redefinition of the renormalization scales. Our purpose here is to discuss this in greater detail. In Section 2 we do it in position space. Its counterpart in momentum space is briefly discussed in Section 3. In Section 4, we illustrate the equivalence by comparing the calculation of the anomalous magnetic moment of a charged lepton in supergravity when CDR and dimensional reduction are employed. Finally, Section 5 is devoted to conclusions.

## 2 CDR and dimensional reduction in position space

CDR, as the usual differential renormalization method, is naturally formulated in position space. CDR renormalizes each Feynman graph by reducing it to a linear combination of basic functions (and their derivatives) which are then replaced by their renormalized expressions. The renormalization of the basic functions is fully determined by four rules to be described below, which are significant for the fulfillment of Ward identities. A generic (one-loop) basic

\footnote{In Ref. \cite{12} some modifications were proposed which make the scheme consistent at the price of breaking supersymmetry (at higher orders).}
function is a string of propagators, with a differential operator \( \mathcal{O} \) acting on the last one. \( \mathcal{O} \) is either the identity or a “product” of space-time derivatives. Basic functions with differential operators with contracted or uncontracted indices are considered independent, because it turns out that contraction of Lorentz indices does not commute with CDR. For this reason, to decompose a Feynman graph into basic functions one must simplify all the (Dirac) algebra and contract all Lorentz indices. Notice that reducing the renormalization of a Feynman graph to the renormalization of the basic functions is equivalent to linearity and compatibility of CDR with the Leibniz rule for the derivative of a product (which is used in the decomposition).

The renormalization of the basic functions is determined by four rules [10]:

1. **Differential reduction**: singular expressions are substituted by derivatives of regular ones. We distinguish two cases:

   (a) Functions with singular behaviour worse than logarithmic \( (\sim x^{-4}) \) are reduced to derivatives of logarithmically singular functions without introducing extra dimensionful constants.

   (b) Logarithmically singular functions are written as derivatives of regular functions. This requires introducing an arbitrary dimensionful constant.

2. **Formal integration by parts**: derivatives act formally by parts on test functions. In particular,

   \[
   [\partial F]^R = \partial F^R, \tag{2}
   \]

   where \( F \) is an arbitrary function and \( R \) stands for renormalized.

3. **Delta function renormalization rule**:

   \[
   [F(x, x_1, ..., x_n)\delta(x - y)]^R = [F(x, x_1, ..., x_n)]^R\delta(x - y). \tag{3}
   \]

4. The general validity of the *propagator equation*:

   \[
   [F(x, x_1, ..., x_n)(\Box^2 - m^2)\Delta_m(x)]^R = [F(x, x_1, ..., x_n)\delta(x)]^R, \tag{4}
   \]

   where \( \Delta_m(x) = \frac{1}{4\pi^2} mK_1(mx) \) and \( K_1 \) is a modified Bessel function.

Rule 1 reduces the "degree of singularity", connecting singular and regular expressions. Rule 2 is essential to make sense of rule 1, for otherwise the right-hand-side of it would not be a well-defined distribution. These two rules are the essential prescriptions of the method of differential renormalization. Forbidding the introduction of dimensionful scales outside logarithms, we completely fix the scheme.\footnote{This prescription simplifies calculations and the renormalization group equation. Nevertheless, in all cases we have studied (scalar and spinor QED and QCD) the inclusion of dimensionful constants outside logarithms does not spoil gauge invariance, as long as the other rules are respected.} Note that the last three rules are valid mathematical identities.
among tempered distributions when applied to a well-behaved enough function \( F \). The rules formally extend their range of applicability to arbitrary functions.

Rule 1 specifies the renormalization of any one-loop expression up to arbitrary local terms. The other rules lead to a system of algebraic equations for these local terms \([11]\). It turns out that a solution exists, and this solution is unique once an initial condition is given (apart from the requirement in rule 1a of not introducing extra dimensionful constants, which is also an initial condition):

\[
\left[ \Delta_0(x)^2 \right]^R = \left[ \frac{1}{4 \pi^2 x^2} \right]^R = - \frac{1}{(4 \pi^2)^2} \frac{1}{4} \log \frac{x^2 M^2}{x^2}.
\] (5)

This is the most general realization of rule 1b for \( \Delta_0(x)^2 \), and introduces the unique dimensionful constant of the whole process, \( M \), which has dimensions of mass and plays the role of renormalization group scale.

The decomposition of Feynman graphs into basic functions can be performed in both dimensional regularization and dimensional reduction in exactly the same way as we have described for CDR. Although in the dimensional methods this prescription is not necessary (for in \( d \) dimensions everything is well-defined), we shall assume that all Lorentz indices have been contracted before identifying the basic functions. These contain only \( d \)-dimensional objects both in dimensional regularization and in dimensional reduction. Indeed, although in the latter contractions with the 4-dimensional metric tensor are performed, Eq. 1 projects them into the \( d \)-dimensional subspace. Hence, the regulated basic functions are identical in these two methods. On the other hand, expressions dimensionally regulated satisfy rules 2 to 4 because they are well-defined distributions. They also satisfy rule 1a for the same reason (what agrees with the scaling property of \( d \)-dimensional integrals, which forbids the appearance of new dimensionful constants). A renormalization scale \( \mu \) is introduced to keep the coupling constant with a fixed dimension and appears only inside logarithms. Rule 1b is never needed because the (formal) degree of divergence is non-integer in the dimensional methods. Instead, the use of rule 1a in expressions which diverge logarithmically when \( \epsilon = \frac{4 - d}{2} \to 0 \), gives rise to poles of the form \( \frac{1}{\epsilon} \). In particular, the regularized value of \( \Delta_0(x)^2 \) is

\[
\mu^{2 \epsilon} \frac{\Gamma^2 \left( \frac{d}{2} - 1 \right)}{4^{2 \epsilon} \pi^{2 \epsilon}} x^{4 - 2d} \left[ \pi^2 \frac{1}{\epsilon} \delta^{(d)}(x) - \frac{1}{4} \log \left( \frac{x^2 \mu^2 \pi \gamma |E|^2}{x^2} \right) \right] + O(\epsilon),
\] (6)

where we have included the global factor \( \mu^{2 \epsilon} \) to have a dimensionless argument in the logarithm, expanded in \( \epsilon \) and used the \( d \)-dimensional equalities

\[
x^{-p} = \frac{\Box x^{-p+2}}{(-p+2)(d-p)}
\] (7)

and

\[
\Box \left[ \frac{\Gamma\left( \frac{d}{2} - 1 \right)}{4 \pi^{d/2}} x^{2-d} \right] = -\delta^{(d)}(x),
\] (8)
to rewrite it. \( \gamma_E = 1.781 \ldots \) is Euler’s constant.

Now, since the dimensionally regularized basic functions satisfy the CDR rules, they must also be a solution to the set of algebraic equations discussed above, but with the initial condition given by Eq. 1. This is true for each order of the Laurent series in \( \epsilon \). Therefore, substracting the \( \frac{1}{\epsilon} \) poles, which always multiply a local term, and taking the limit \( \epsilon \to 0 \) (i.e., using the MS scheme) one obtains renormalized basic functions that are \( \epsilon \)-independent solutions of the equations. In particular, the renormalization of \( \Delta_0(x) \) is given by Eq. 5 with

\[
M^2 = \mu^2 \pi \gamma_E \epsilon^2. \tag{9}
\]

Once the initial condition is completely fixed, the solution to the set of equations is unique, so it must be the same in CDR and in dimensional regularization or dimensional reduction. Summarizing, the renormalized basic functions in dimensional regularization, dimensional reduction and CDR are identical if the MS scheme is used for the former methods and Eq. 9 holds.

This does not mean that the renormalized Feynman diagrams are the same in the three methods, because in the dimensional ones the substra ction must be performed after multiplying by the coefficients outside the basic functions. Then, if these coefficients contain \( O(\epsilon) \) pieces, the structure of the Laurent series can be spoiled. Extra local \( O(\epsilon^0) \) terms are picked up from the \( \frac{1}{\epsilon} \) poles of the basic functions, and the final result does not, in general, coincide with the CDR one. However, this does not occur in the case of dimensional reduction because, if the decomposition of the diagram has been performed as in CDR, there are no \( O(\epsilon) \) pieces out of the basic functions. The reason is that all the coefficients are 4-dimensional and are never projected into \( d \) dimensions since all contractions with \( d \)-dimensional objects were already performed and included in the definition of the basic functions. In other words, with this decomposition all external indices are 4-dimensional (and all internal ones are \( d \)-dimensional). Therefore, renormalized Feynman diagrams in CDR and in dimensional reduction with MS coincide, if Eq. 9 holds. This is not true in dimensional regularization because the dimension \( d \) can appear explicitly outside the basic functions, for everything is considered \( d \)-dimensional. For example, \( \delta_{\mu\nu} = d \) can appear out of the basic functions. Dimensional regularization only coincides with dimensional reduction and CDR at the level of basic functions.

3 CDR and dimensional reduction in momentum space

Obviously, if CDR and dimensional reduction give the same renormalized amplitudes in position space, they do too in momentum space, because the Fourier transforms of well-defined distributions are uniquely determined. The decomposition into basic functions is performed exactly as in position space, the basic functions corresponding now to (tensor) basic integrals of a set of internal momenta times a product of propagators. This can be done because the linearity
of CDR in position space (together with rule 3) implies linearity in momentum space. Then, one just has to substitute divergent basic integrals by the Fourier transforms of the corresponding renormalized expressions in position space. In the case of dimensional reduction this is the same as doing the full calculation in momentum space. The Fourier transform of the initial condition Eq. 5 is

$$\left[ \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2(k-p)^2} \right]^R = \frac{1}{16\pi^2} \log \frac{\bar{M}^2}{p^2},$$

(10)

where $\bar{M} = 2M/\gamma_E$. The relation between this $\bar{M}$ and $\mu$ is given by

$$\bar{M}^2 = \frac{\mu^2 4\pi}{\gamma_E} e^2 = \bar{\mu}^2 e^2,$$

(11)

where $\bar{\mu}$ is the renormalization scale of the $\overline{\text{MS}}$ scheme. This is the relation found in Ref. [12].

4 A physical example: ($g-2)_\ell$ in supergravity

The calculation of the anomalous magnetic moment of a charged lepton, ($g-2)_\ell$, in supergravity is a convenient place to compare CDR with dimensional regularization and dimensional reduction. First, ($g-2)_\ell$ is an observable; second, it is power counting divergent (and hence regularization dependent); and third, supersymmetry requires it to vanish [14]. The diagrams giving $\epsilon \kappa^2$ corrections, where $\epsilon$ is the electric charge and $\kappa^2 = 8\pi G_N$, with $G_N$ Newton’s constant, are depicted in Fig. 1. This calculation has been discussed in detail several times [13, 14, 15, 16, 17]. Table 1 gathers the different contributions in dimensional regularization, dimensional reduction and CDR. Although the contribution of each diagram diverges, the sum of all diagrams where a graviton (D1-D5) is interchanged is finite [19] (in all methods), as is the sum of those with
\[
D1 \quad \frac{1}{6} \left( \frac{1}{\epsilon} + \log \frac{\bar{\mu}^2}{m^2} \right) - \frac{61}{36}
\]

\[
D2+D3 \quad \frac{1}{6} \left( \frac{1}{\epsilon} + \log \frac{\bar{\mu}^2}{m^2} \right) - \frac{29}{18}
\]

\[
D4+D5 \quad \frac{1}{6} \left( \frac{1}{\epsilon} + \log \frac{\bar{\mu}^2}{m^2} \right) - \frac{23}{18}
\]

\[
D6 \quad \frac{1}{6} \left( \frac{1}{\epsilon} + \log \frac{\bar{\mu}^2}{m^2} \right) - \frac{55}{18}
\]

\[
D7+D8 \quad \frac{1}{6} \left( \frac{1}{\epsilon} + \log \frac{\bar{\mu}^2}{m^2} \right) - \frac{37}{18}
\]

\[
D9+D10 \quad \frac{1}{6} \left( \frac{1}{\epsilon} + \log \frac{\bar{\mu}^2}{m^2} \right) - \frac{13}{9}
\]

\[
D1+D2+D3 \quad \frac{7}{4}
\]

\[
D6+D7+D8+D9+D10 \quad \frac{1}{2}
\]

\[
Graviton \quad \frac{5}{4}
\]

\[
Gravitino \quad 0
\]

| Diagram | Dimensional regularization | Dimensional reduction | CDR |
|---------|--------------------------|----------------------|-----|
| D1      | -1/6 \left( \frac{1}{\epsilon} + \log \frac{\bar{\mu}^2}{m^2} \right) - 61/36 | -1/6 \left( \frac{1}{\epsilon} + \log \frac{\bar{\mu}^2}{m^2} \right) - 29/18 | -1/6 \log \frac{\bar{M}^2}{m^2} - 23/18 |
| D2+D3   | -1/6 \left( \frac{1}{\epsilon} + \log \frac{\bar{\mu}^2}{m^2} \right) - 29/18 | -1/6 \left( \frac{1}{\epsilon} + \log \frac{\bar{\mu}^2}{m^2} \right) - 29/18 | 2 \log \frac{\bar{M}^2}{m^2} + 2 |
| D4+D5   | -1/6 \left( \frac{1}{\epsilon} + \log \frac{\bar{\mu}^2}{m^2} \right) - 23/18 | 2 \left( \frac{1}{\epsilon} + \log \frac{\bar{\mu}^2}{m^2} \right) + 6 | 2 \log \frac{\bar{M}^2}{m^2} |
| Graviton (D1+D2+D3 +D4+D5) | 7/4 | 1/2 | 1/2 |
| D6+D7+D8+D9+D10 | -1/6 \left( \frac{1}{\epsilon} + \log \frac{\bar{\mu}^2}{m^2} \right) - 55/18 | 2 \left( \frac{1}{\epsilon} + \log \frac{\bar{\mu}^2}{m^2} \right) + 2 | 2 \log \frac{\bar{M}^2}{m^2} |
| Gravitino (D6+D7+D8 +D9+D10) | -1/6 \left( \frac{1}{\epsilon} + \log \frac{\bar{\mu}^2}{m^2} \right) - 37/18 | 2 \left( \frac{1}{\epsilon} + \log \frac{\bar{\mu}^2}{m^2} \right) + 2 | 2 \log \frac{\bar{M}^2}{m^2} |

Table 1: Contributions of the diagrams in Fig. 1 to \((g^2 - 2/2)\), in units of \(G_N m^2\pi\), obtained with dimensional regularization, dimensional reduction and CDR.

a gravitino interchange (D6-D10), and hence the total sum. Whereas dimensional regularization breaks supersymmetry and gives a non-zero result [13, 14], a vanishing correction is obtained both in dimensional reduction [15, 16] and in CDR [8, 9]. We see that CDR and dimensional reduction in MS do give the same results for each diagram if the renormalization scales are related by Eq. 11. The total graviton (gravitino) contribution being finite, it is identical in both methods. (In Ref. 13 the scale-independent parts of the CDR result have errors due to the omission of a local term in one basic function, but the total graviton and gravitino contributions are correct because that local term cancels in the sums.)

5 Conclusions

We have discussed the one-loop equivalence of CDR and dimensional reduction in the MS (MS) scheme. The result also applies in the presence of anomalies, for both methods can be used with the same computing rules in position (momentum) space: Feynman diagrams are decomposed completely into basic functions (integrals), doing all the algebra in 4 dimensions, and then the singular (diver-
gent) expressions are replaced by the renormalized ones. In the two methods, chiral anomalies appear as ambiguities in the writing of the external tensors: it is possible to add pieces which vanish in 4 dimensions but change the decomposition into basic functions (integrals), and this can affect the final result due to the non-commutation of renormalization with contraction of Lorentz indices. In dimensional reduction this can be also understood as the fact that these pieces are projected into $d$ dimensions, where they do not vanish any longer. In Ref. [20] we showed how the right ABJ anomaly [21] was recovered in the context of CDR and checked that a democratic treatment of the traces located all the anomaly in the chiral current. Exactly the same applies to dimensional reduction.

CDR has been only developed at the one-loop level, but an extension of the method to higher orders, based on the same rules [1-4] or their extension, is in principle possible. It does not follow from our discussion that such a method should be equivalent to dimensional reduction. On the one hand, dimensional reduction might not obey the extended rules; on the other, the mere presence of subdivergencies changes the simple procedure discussed here. In the best of the worlds, the extended CDR would preserve gauge invariance and supersymmetry, and not suffer from inconsistencies as the ones in dimensional reduction.

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