Dynamic localization of lattice electrons under time dependent electric and magnetic fields

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Abstract. Applying the method of characteristics leads to wavefunctions and dynamic localization conditions for electrons on the one dimensional lattice under perpendicular time dependent electric and magnetic fields. Such conditions proceed again in terms of sums of products of Bessel functions of the first kind. However, this time one deals with both the number of magnetic flux quantas times π and the quotients between the Bloch frequency and the ones characterizing competing fields. Tuning the phases of time dependent modulations leads to interesting frequency mixing effects providing an appreciable simplification of dynamic localization conditions one looks for. The understanding is that proceeding in this manner, the time dependent superposition mentioned above gets reduced effectively to the influence of individual ac-fields exhibiting mixed frequency quotients. Besides pure field limits and superpositions between uniform electric and time dependent magnetic fields, parity and periodicity effects have also been discussed.

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1. Introduction

The dynamic localization (DL) of electrons moving on the one dimensional (1D) lattice under the influence of a longitudinal time dependent (TD) electric field like \( E(t) = E_0 f(t) \) has received much interest since its discovery some 20 years ago [1-8]. This effect concerns the periodic return of the electron to the initially occupied site [1]. Accordingly, the mean square displacement (MSD) should remain bounded in time. In most cases which are of interest in practice the modulation function is periodic with period \( T \), but superpositions of several ac-fields can also be considered. The DL referred to above should then occur under selected conditions concerning the quotients between the field frequencies and the Bloch frequency \( \omega_B = E_0 e a / \hbar \), where \( a \) stands for the lattice spacing. Besides applications in several areas like high field and nonlinear effects [2,3], trapping in two level atoms [4], persistent THz emission [5,6], the generation of
higher harmonics [7] or the absolute negative conductance, the DL has been finally observed in the linear optical absorption coefficient of quantum dot superlattices [9]. It has also been found that the collapse of quasienergy bands is able to reflect the occurrence of DL [10-13]. Recent developments such as the DL of wave packets in barriers with TD parameters [14], the appearance and disappearance of resonant peaks in $I - V$ characteristics [15], or the influence of higher order neighbors [16], are worthy of being mentioned, too.

We have to realize that the influence of the uniform magnetic field on lattice electrons, such as exhibited by the celebrated Harper-equation [17-19], looks quite interesting in many respects. The same concerns several superpositions between electric and magnetic fields which have been studied during course [20-22]. However, a systematic study of wavefunctions and of DL effects produced by superpositions for which both electric and magnetic fields are TD seems of having been, to the best of our knowledge, overlooked. We shall then use this opportunity to discuss DL effects provided by the longitudinal TD electric field $\vec{E}(t) = (E(t), 0, 0)$ working in conjunction with a transversal magnetic field like $\vec{B}(t) = (0, 0, B(t))$, where $B(t) = B_0 g(t)$. To this aim we can start either by incorporating the electric field into the time dependent Harper-Hamiltonian [18], or from the decoupled limit of two parallel chains in electric and magnetic fields [21]. We have to realize that such systems can be converted one into another with the help of gauge transformations [23]. The former alternative is appropriate for Hall conductance studies. We shall then choose the latter alternative since it complies in a more suitable manner with the DL problem. This opens the way to the derivation of general DL conditions, but concrete realizations such as superpositions between uniform electric and time dependent magnetic fields, will also be discussed. In this context, we found that tuning the phases of TD modulations opens the way to interesting frequency mixing effects, which leads in turn to quickly tractable DL conditions proceeding again in terms of the zero’s of the zero-order Bessel function of the first kind. The understanding is that by virtue of the phase tuning, the TD superposition mentioned above is able to be reduced effectively to the influence of individual ac-fields characterized by mixed frequency quotients. Last but not at least we shall deal with parity and periodicity effects.

2. The derivation of the wavefunction

The Hamiltonian describing the electron on the 1D lattice under TD electric and magnetic fields specified above is given by [1,18,21]

$$
\mathcal{H} = \varepsilon_0 \sum_m |m><m| - e\alpha E_0 f(t) \sum_m |m><m| + 
\{V \sum_m \exp(-i\gamma/2)|m+1><m| + V \sum_m \exp(i\gamma/2)|m><m+1| \} 
\tag{1}
$$
where \( m \) is an integer ranging from \(-\infty\) to \( \infty \). One has
\[
\gamma = \gamma(t) = 2\pi \frac{\Phi}{\Phi_0} = 2\pi \beta_0 g(t)
\]  
(2)

where the transversal magnetic flux and the magnetic flux quantum are given by \( \Phi = B_0 a^2 g(t) \) and \( \Phi_0 = hc/e \), respectively. So, there is \( \beta_0 = eB_0 a^2 / hc \), which stands for the magnetic commensurability parameter. The constant on-site energy is denoted by \( \epsilon_0 = \hbar \omega_0 \), whereas \( V = \hbar U \) is responsible for the nearest neighbor hopping parameter. One deals, of course, with an orthonormalized Wannier-basis for which
\[
| m \rangle = \sum_m C_m(t) \exp(-i\omega_0 t) | m >
\]  
(3)

which leads in turn to the TD second-order discrete Schrödinger equation
\[
i \frac{\partial}{\partial t} C_m(t) = U \exp(-i\gamma/2)C_{m-1}(t) + U \exp(i\gamma/2)C_{m+1}(t) - m\omega_B f(t)C_m(t).
\]  
(4)

The next step is to apply the discrete Fourier-transform
\[
C_m(t) = \frac{1}{2\pi} \int_0^{2\pi} dk \exp(ik) \tilde{C}_k(t)
\]  
(5)

where \( k \) denotes the dimensionless wave number. Then (4) becomes
\[
\left[ \frac{\partial}{\partial t} + 2iU \cos \left( k + \frac{\gamma}{2} \right) + \omega_B \tilde{f}(t) \frac{\partial}{\partial k} \right] \tilde{C}_k(t) = 0
\]  
(6)

where
\[
\tilde{f}(t) = f(t) - \frac{1}{2\omega_B} \frac{d\gamma(t)}{dt} = f(t) - \frac{\pi \beta_0}{\omega_B} \frac{dg(t)}{dt}.
\]  
(7)

Now we have to remember that (4) and (6) have been discussed before when \( \gamma = 0 \) by resorting to the method of characteristics [1]. What then remains is to generalize these latter results towards incorporating \( \gamma(t) \) such as given by (2). This results in the solution
\[
\tilde{C}_k(t) = \exp \left[ -2iU \int_0^t dt' \cos \Omega_k(t, t') \right]
\]  
(8)

as it can be easily verified by direct computation. This time one has
\[
\Omega_k(t, t') = \cos \left[ k + \frac{\gamma}{2} - \omega_B (\tilde{\eta}(t) - \tilde{\eta}(t')) \right]
\]  
(9)

where
\[
\tilde{\eta}(t) = \int_0^t dt' \tilde{f}(t') = \eta(t) - \frac{1}{2\omega_B} (\gamma(t) - \gamma(0))
\]  
(10)
\[ \eta(t) = \int_0^t dt' f(t') \] (11)

which is well known in the description of electric field problems. Accordingly, one gets faced with the normalized wavefunction

\[ C_m(t) = \exp(-im\tilde{\psi}) J_m(2U | \tilde{Z}(t) |) \] (12)

by virtue of the well known properties of Bessel functions [24], where

\[ \tilde{\psi} = \tilde{\psi}(t) = \frac{\pi + \gamma(t)}{2} - \arg \tilde{Z}(t) \] (13)

and

\[ \tilde{Z}(t) = \int_0^t dt' \exp(-i\omega_B \tilde{\eta}(t')) \] (14)

which deserve further attention.

3. Dynamic localization effects

Using (12) produces the MSD

\[ <m^2> = \sum_m m^2 | C_m(t) |^2 = 2U^2 | \tilde{Z}(t) |^2 \] (15)

which generalizes apparently (2.7) in [1] in terms of the substitution \( \eta(t) \rightarrow \tilde{\eta}(t) \). Choosing as an example usual modulations like

\[ f(t) = \cos(\omega_1 t) \] (16)

and

\[ g(t) = \sin(\omega_2 t) \] (17)

yields the characteristic function

\[ \tilde{Z}(t) = \int_0^t dt' \exp \left(-i\frac{\omega_B}{\omega_1} \sin(\omega_1 t') + i\pi \beta_0 \sin(\omega_2 t') \right) \] (18)

which can be rewritten equivalently as

\[ \tilde{Z}(t) = \sum_m \sum_n \exp(i\Omega_{m,n} t) \frac{\sin(\Omega_{m,n} t)}{\Omega_{m,n}} J_n \left( \frac{\omega_B}{\omega_1} \right) J_m(\pi \beta_0) \] (19)
by virtue of expansions characterizing generating functions of Bessel functions [24], where
\[ \Omega_{m,n} = \frac{1}{2} (m\omega_2 - n\omega_1) \] (20)
and \( q_j = \omega_B / \omega_j \ (j = 1, 2) \). The point is to decompose \( \widetilde{Z}(t) \) as
\[ \widetilde{Z}(t) = Q_1 t + Q_2(t) \] (21)
in which \( Q_2(t) \) oscillates with time. Having discriminated the linear term in \( t \) then produces the DL condition [1,13]
\[ Q_1 = 0 \] (22)
in which case the MSD remains bounded in time. On the other hand (19) shows that the discrimination of the linear term one looks for proceeds in terms of selected \( m \)- and \( n \)-values for which \( \Omega_{m,n} \to 0 \). To this aim let us assume that the frequencies \( \omega_1 \) and \( \omega_2 \) are commensurate. This amounts to deal with quotients like
\[ \frac{n}{m} = \frac{\omega_2}{\omega_1} = \frac{q_1}{q_2} = \frac{P}{Q} \] (23)
in which \( P \) and \( Q \) are mutually prime integers. We then have to realize that the DL condition characterizing specifically the present superposition of TD electric and magnetic fields is given in terms of sums of products of Bessel functions of the first kind like
\[ Q_1 = Q_1 (q_1, \pi \beta_0) = J_0 (q_1) J_0 (\pi \beta_0) + \sum_{l=1}^{\infty} p_l J_{P_l} (q_1) J_{Q_l} (\pi \beta_0) = 0 \] (24)
in which
\[ p_l = 1 + (-1)^{(P+Q)} \] (25)
and \( P = Q \omega_2 / \omega_1 \). Using, however, \( \tilde{g}(t) = \cos(\omega_2 t) \) instead of (17) produces the DL condition
\[ \tilde{Q}_1 (q_1, \pi \beta_0) = \left[ J_0 (q_1) J_0 (\pi \beta_0) + \sum_{l=1}^{\infty} \tilde{p}_l J_{P_l} (q_1) J_{Q_l} (\pi \beta_0) \right] = 0 \] (26)
which proceeds up to a phase factor like \( \exp(-i\pi \beta_0) \), where
\[ \tilde{p}_l = \exp \left( i \frac{\pi}{2} Ql \right) + \exp \left( -i \frac{\pi}{2} Ql \right) (-1)^{(P+Q)} . \] (27)
Such results indicate that contributions provided by electric and magnetic fields can be placed on the same footing. Of course, starting from \( f(t) = g(t) = \cos(\omega_1 t) \), one finds that the DL condition is still given by (26), but this time via \( P = Q = 1 \).

One remarks that both (24) and (26) are sensitive to the parity of \( P + Q \). So (24) becomes
\[ Q_1 = Q_1^{(-)} (q_1, \pi \beta_0; \xi_l) = J_0 (q_1) J_0 (\pi \beta_0) + 2 \sum_{l=1}^{\infty} \xi_l J_{2l} (q_1) J_{2l} (\pi \beta_0) = 0 \] (28)
when \( P + Q \) is an odd integer, but

\[
Q_1 = Q_1^{(+)} (q_1, \pi\beta_0; \xi_l) = J_0 (q_1) J_0 (\pi\beta_0) + 2 \sum_{l=1}^{\infty} \xi_l J_{Pl} (q_1) J_{Ql} (\pi\beta_0) = 0 \quad (29)
\]

when \( P + Q \) is even. So far we have to insert \( \xi_l = 1 \), but further generalizations are in order. Note that the presence of \( \xi_l \) in (28) and (29) has to be understood just as an insertion prescription working under the sum.

Proceeding in a similar manner one finds that (26) leads to

\[
\tilde{Q}_1^{(+)} (q_1, \pi\beta_0) = Q_1^{(+)} (q_1, \pi\beta_0; \cos(\pi Ql/2)) = 0 \quad (30)
\]

and

\[
\tilde{Q}_1^{(-)} (q_1, \pi\beta_0) = Q_1^{(-)} (q_1, \pi\beta_0; \cos(\pi Ql)) + i \Gamma_1 = 0 \quad (31)
\]

when \( P + Q \) is even and odd, respectively, where

\[
\Gamma_1 = \Gamma_1 (q_1, \pi\beta_0) = 2 \sum_{l=1}^{\infty} \sin \left( \frac{\pi}{2} (2l + 1) Q \right) J_{(2l+1)P} (q_1) J_{(2l+1)Q} (\pi\beta_0) . \quad (32)
\]

In the latter case the DL occurs when both real and imaginary parts of \( \tilde{Q}_1^{(-)} (q_1, \pi\beta_0) \) are zero, so that we have to account for twice harder computations. Such conditions are rather complex, so that when dealing with applications we have to resort to numerical studies. We can then say that looking for reasonable but tractable versions represents a quite desirable task.

4. Phase tuning and frequency mixing effects

A further point of interest is to account for the influence of phases on the DL. This opens the way to the derivation of reasonable DL conditions one looks for. For this purpose let us start from some appropriate modulations like

\[
f(t) = \cos(\omega_1 t + \theta_1) \quad (33)
\]

and

\[
g(t) = \cos(\omega_1 t + \theta_2) \quad (34)
\]

for which \( \omega_1 = \omega_2 \) and \( \theta_j \in [0, 2\pi] \). Repeating the same steps as before yields the DL condition

\[
\tilde{Q}_1 (R_1, R_2) = \sum_{m=-\infty}^{\infty} i J_m (R_1) J_m (R_2) = 0 \quad (35)
\]

where

\[
R_1 = q_1 \cos(\theta_1) + \pi\beta_0 \sin(\theta_2) \quad (36)
\]
and
\[ R_2 = \pi \beta_0 \cos(\theta_2) - q_1 \sin(\theta_1) \]  \hspace{1cm} (37)
are responsible for mixing effects concerning electric and magnetic quotients mentioned above. Discriminating the \( n = 0 \)-term in (35) leads to the decomposition
\[ \tilde{Q}_1(R_1, R_2) = J_0(R_1)J_0(R_2) + \Delta \tilde{Q}_1 \]  \hspace{1cm} (38)
where
\[ \Delta \tilde{Q}_1 = 2 \sum_{n=1}^{\infty} \cos(\pi n/2) J_n(R_1)J_n(R_2) \]  \hspace{1cm} (39)
Fixing, for convenience, the \( \theta_1 \)-phase, one sees that the \( \theta_2 \)-phase can be tuned until \( R_2 = 0 \), in which case
\[ \cos(\theta_2) = \frac{q_1}{\pi \beta_0} \sin(\theta_1) \]  \hspace{1cm} (40)
Accordingly, (38) becomes, interestingly enough,
\[ \tilde{Q}_1(R_1, 0) = J_0(\tilde{R}_1) = 0 \]  \hspace{1cm} (41)
This shows that the celebrated DL condition derived before gets reproduced [1], now in terms of the mixed frequency quotient
\[ \tilde{R}_1 = q_1 \cos(\theta_1) + \pi \beta_0 \left[ 1 - \frac{q_1^2}{\pi^2 \beta_0^2} \sin^2(\theta_1) \right]^{1/2} \]  \hspace{1cm} (42)
relying effectively on the influence of a single ac-field. So one gets faced with the simplified DL condition \( \tilde{R}_1 = z_n \), where \( z = z_n \) \( (n = 1, 2, 3, \ldots) \) stands for the root of \( J_0(z) \). This condition can be rewritten equivalently as
\[ z_n^2 + 2z_nq_1 \sin(\theta_1) + q_1^2 = \pi^2 \beta_0^2 \]  \hspace{1cm} (43)
which is able to serve as an eigenvalue equation for \( \pi \beta_0 \) or \( q_1 \), respectively. However, the \( \theta_2 \)-phase can also be tuned so that \( \pi \beta_0 = 0 \), in which case the DL condition becomes
\[ \tilde{Q}_1(0, R_2) = J_0(\tilde{R}_2) = 0 \]  \hspace{1cm} (44)
instead of (41), where now the frequency quotient is given effectively by
\[ \tilde{R}_2 = -q_1 \sin(\theta_1) + \pi \beta_0 \left[ 1 - \frac{q_1^2}{\pi^2 \beta_0^2} \cos^2(\theta_1) \right]^{1/2} \]  \hspace{1cm} (45)
One sees immediately that \( \tilde{R}_2 = z_{n'} \) by virtue of (44), so that (43) is replaced by
\[ z_{n'}^2 - 2z_{n'}q_1 \cos(\theta_1) + q_1^2 = \pi^2 \beta_0^2 \]  \hspace{1cm} (46)
where, in general, \( n' \) differs from \( n \). We then have to account for two realizations of the DL condition, such as indicated by (43) and (46). One realizes that dealing with
\( \tilde{R}_1 \) and \( \tilde{R}_2 \) amounts to introduce effectively the influence of individual fields selected in accord with (42) and (45). However, we can choose \( \theta_1 \) such that

\[
\sin(\theta_1) + \cos(\theta_1) = 0
\]

which also means that \( \theta_1 \in [\pi/2, \pi] \) or \( \theta_1 \in [3\pi/2, 2\pi] \). The corresponding DL condition is then given by

\[
\tilde{R}_1 = \tilde{R}_2 = z_n
\]

where by now \( n' = n \), which represents an appreciable simplification. One realizes that (48) produces quantized flux values like

\[
\beta_0 = \beta_0(n) = \frac{1}{\pi} \left[ z_n^2 - 2z_nq_1 \cos(\theta_1) + q_1^2 \right]^{1/2}
\]

when starting from fixed \( q_1 \)-quotients. Such flux values exhibit unexpected limits like

\[
\pi \beta_0(n) \to z_n \pm q_1
\]

if \( \theta_1 \to \pi/2 \) and \( \theta_1 \to 3\pi/2 \), respectively.

5. Other concrete realizations

The superposition between a uniform electric field for which \( f(t) = 1 \) and a TD magnetic one deserves a little bit more attention. This time (10) becomes

\[
\tilde{\eta}(t) = t - \frac{\pi \beta_0}{\omega_B} (g(t) - g(0))
\]

so that

\[
\tilde{Z}(t) = \sum_m \exp(i\Omega_m t) \frac{\sin(\Omega_m t)}{\Omega_m} J_n(q_1) J_m(\pi \beta_0)
\]

where now

\[
\Omega_m = \frac{1}{2} (m\omega - \omega_B)
\]

One sees that \( \tilde{Z}(t) \) contains only oscillatory contributions in so far as \( \omega_B < \omega_2 \). This means in turn that \( Q_1 = 0 \), so that requirements needed for the onset of DL are fulfilled from the very beginning. Furthermore let us assume that

\[
\omega_B = \tilde{n} \omega_2
\]

if \( \omega_B \gg \omega_2 \), where \( \tilde{n} \) denotes a positive integer. Then the linear term gets discriminated again via \( \Omega_m \to 0 \), which leads to a rather special DL condition like

\[
Q_1 = Q_1^* (q_2, \pi \beta_0) = \theta (q_2 - 1) J_{\tilde{n}}(\pi \beta_0) = 0
\]

where \( \theta(x) \) stands for Heaviside’s function. One realizes that this time the DL proceeds selectively in terms of zero’s characterizing higher order Bessel functions for which
$\tilde{n} = 1, 2, 3, \ldots$, in accord with (32). Complementarily, the regime is ballistic when $J_n(\pi \beta_0) \neq 0$, but the DL gets restored again when the quotient $\omega_B/\omega_2$ is either pure rational or irrational.

Next let us account for pure electric and magnetic fields via $\beta_0 \to 0$ and $\omega_B \to 0$, respectively. In the first case one recovers the well-known DL condition [1]

$$Q_1(q_1) = J_0(q_1) = 0$$

(56)

In the second case one finds

$$Q_1(0, \pi \beta_0) = J_0(\pi \beta_0) = 0$$

(57)

which means that the DL occurs whenever the number of flux quanta times $\pi$ is a root of the zero order Bessel function, too. This proceeds in terms of dimensionless flux values centered around $\beta_0 = \beta_n \cong z_n/\pi$, where $J_0(z_n) = 0$. The approximation

$$\beta_{n+1} - \beta_n \cong 1$$

(58)

should also be mentioned, which indicates that the DL occurs periodically with unit dimensionless flux period. A such periodicity, which is reminiscent to Aharonov-Bohm oscillations, has also been remarked in superpositions between uniform magnetic and dc-ac electric fields [22].

6. Conclusions

The influence of TD electric and magnetic fields on the DL of electrons on the 1D lattice has been discussed systematically in terms of MSD’s remaining bounded in time. The main result is given by the rather general formula (24). A such formula is able to exhibit several concrete realizations and serves as a starting point for further developments. Indeed, tuning the phases results in interesting frequency mixing effects which provide controllable DL conditions working again in terms of the zero-order Bessel function, as shown by (41) and (44). For this purpose one resorts to mixed frequency quotients such as given by (42) and (45). Such quotients proceed effectively in terms of individual ac-fields. Related flux quantization rules have also been derived in accord with (49), now by starting from fixed values of the electric frequency quotient $q_1 = \omega_B/\omega_1$. Of course, (41) and (44) can also be discussed by starting from fixed $\pi \beta_0$-values, which leads in turn to the quantization of the electric field amplitude. We emphasize that such results can be readily generalized towards modulations for which $\omega_1 \neq \omega_2$. We have also to remark that (49) can be rewritten in terms of pertinent vectors as

$$\pi \vec{\beta}_0(n) + \vec{q}_1 = \vec{z}_n$$

(59)

with the understanding that $\theta_1$ stands for the angle between $\vec{z}_n$ and $\vec{q}_1$.

Pure fields limits can be readily performed, as shown by (56) and (57). One realizes that the first root in (57), namely the selected dimensionless magnetic flux

$$\beta_0 = \beta_1 \cong \frac{2.405}{\pi} \cong 0.76$$

(60)
deserves experimental verification in a close analogy with the confirmation of the DL concerning the "electric" quotient $\omega_B/\omega_1 \approx \omega \sim 2.405$ [9]. The superposition between a uniform electric field and a TD magnetic one has also been discussed. This is a rather special example for which the fields are accounted for in an asymmetric manner. Now the DL occurs in terms of (55), but when the Bloch frequency is quantized in accord with (54) only. Correspondingly, the electric field should be quantized itself by virtue of the rule

$$ E_0 = E_0^{(n)} = \tilde{n} \frac{\hbar \omega_2}{e a} $$

which can be viewed as being similar to (49). We can then say that results presented in this paper provide a deeper understanding of DL effects, with a special emphasis on the role of phase tuning effects. Applications in the design of nanoelectronic devices can also be invoked.

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