We formulate the simplest minimal subtraction version for massive $\lambda \phi^4$ scalar fields with $O(N)$ symmetry for generic anisotropic Lifshitz spacetimes. We introduce the simplest geometric concepts to define it as a special manifold. Then we restrict ourselves to flat Euclidean spaces. An appropriate partial-$p$ operation is applied in the bare two-point vertex function diagrams, which separates the original diagram into a sum of two different integrals which are the coefficients of the corresponding polynomials in the mass and external momentum. Within the proposed method, the coefficient of the mass terms can be discarded and we obtain a minimal subtraction method almost identical to the same scheme in the massless theory in every external momentum/mass subspace. We restrict our demonstration of the method up to three-loop order in the two-point vertex part. We verify its consistency through a diagrammatic computation of static critical exponents, which validates the universality hypothesis.

Analytical methods in quantum field theory are rather compelling for at least three reasons. First, they provide a simple physical interpretation of the result. Second, a large number of intermediate steps might cancel out among each other producing a null effect. Third, we can figure out new insights that would be impossible otherwise using numerical methods. Specifically, static anisotropic $m$-axial Lifshitz critical behaviors were formulated as $\lambda \phi^4$ field theory, but their actual exact analytical solution in perturbation theory is still lacking.

The orthogonal approximation devised for this problem either in its massless or massive settings yields a systematic analytical solution to all orders in perturbation theory in momentum space. "Lifshitz space" and "generalized anisotropic Lifshitz spaces" were defined. This approximation in the massive framework to anisotropic $m$-axial Lifshitz points suggested naturally the concept of a "partial-$p$" operation defined in Lifshitz space(time)s from a particular Lifshitz scalar field theory using the BPHZ method. Nevertheless, a decisive explicit proof of how Lifshitz spacetimes come up in this context is still lacking. Can we give a truly spacetime description of Lifshitz type scalar quantum fields? Those theories are renormalizable. Can we find a simpler minimal subtraction method in order to achieve a maximal simplification in the renormalized theory?

In this Letter we discuss how generic anisotropic Lifshitz scalar quantum field theory (QFT) can be defined in a Euclidean flat space as a particular case from a metric tensor representing a special curved higher dimensional spacetime. We then introduce a partial-$p$ operation with many mass scales. We apply it in a particular type of one-particle irreducible (1PI) formalism and obtain a "masslesslike" massive minimal subtraction renormalization scheme, similarly to the recent method developed for pure $\phi^4$ theory. This simplified framework does not contain tadpole insertions in any 1PI vertex parts, therefore requiring a minimal number of diagrams. We discuss how the $m_n$ subspaces can be renormalized independently. We compute critical exponents diagrammatically that checks the consistency of our method. Whenever $m_n = 0$ for $n = 2, ..., L$ all the results here reduce to those from $\phi^4$.

Consider a bare scalar field Lagrangian density (Euclidean version) with $O(N)$ symmetry defined for the most general anisotropic Lifshitz static multicritical behavior (CECI model) whose action/(Helmholtz free energy) is:

$$S = \int \Pi_{n=1}^{n=L} d^{d+\sigma} x_n \left[ \frac{1}{2} \left( \nabla^2 - \sum_{n=2}^{L} m_n \right) \phi^2 + \sum_{n=2}^{L} \frac{\sigma_n}{2} \right] \times \left[ \nabla^2 m_n \phi^2 + \sum_{n=2}^{L} \frac{1}{2} \delta_{\sigma n} \phi^2 + \sum_{n=3}^{L-1} \frac{1}{2} \left( \nabla^2 + 1 \right) \phi^2 + \frac{1}{2} \lambda_n \phi^2 \right].$$

We particularize our discussion throughout to the cases $\delta_{0 n} = \tau_{n n'} = 0$. There are independent subspaces $(m_1, ..., m_L)$ (with coordinates $(x^{(1)}_{n 1}, ..., x^{(L)}_{n L})$ with $i_1 = 1, ..., m_1, ..., i_L = 1, ..., m_L$, respectively, such that $d = m_1 + ... + m_L$ is the space dimension. The above action has a purely quantum nature (for details see Ref.) and recall that in statistical mechanics the path integrals are weighted by $e^{-S}$.

In the terminology of Lifshitz quantum field theory the power of momentum is named $z$. In theory of membranes in quantum criticality, world-volume scalars (space vectors) are embedded in 26-dimensional target space with additional Lifshitz matter composed of extra terms with world-volume scalars presenting second derivatives (irrelevant deformations). The results for the classical
such features. For the time being it is sufficient to say that the basic aspects of the manifold structure which originates in the present work and we relegate these details to a future publication. Nevertheless, we will explain the most appropriate choice of the metric tensor with the prescriptions outlined above.

The discussion of these issues would take us too far afield to deluge in detail. Furthermore, the derivation of Lifshitz terms with changing values of $z$ for massless scalars, spinors and gauge fields addresses the proton stability in the standard model (SM), whereas by adding a higher derivative scalar field of the gauge group explains the increase in the fermion masses in the SM. Unfortunately, only tree-level effects were discussed in $\sigma_n = 1$ in Eq. (11), but this affects the canonical dimensions of the coordinates in different subspaces. Indeed, if $[x^n_i] = M^{-1}$ (or $[p^n_i] = M$), in the $m_n$ subspace one has instead $[x^n_i] = M^{-\delta}$ (or $[p^n_i] = M^{\delta}$). This dimensional redefinition produces an interesting effect: the "dilution" of the dimensions contained in the competing subspaces.

We are now at a position to give a set-theoretic interpretation of the dimension in the competing subspaces. We use the definition given by Kolmogorov for the "box-counting dimension" of a set of points. (It gives the same result for the calculation of the dimension as that using the Hausdorff’s method. Kolmogorov’s method is, however, much simpler.) The definition is $D_0 = \frac{\log N(\delta)}{\log \frac{1}{\delta}}$ in the limit $\delta \to 0$. Here $N(\delta)$ is the maximal number of identical "boxes" with side $\delta$ needed to cover the entire set of points. The rigorous definition of the usual ternary Cantor set is in terms of the closed compact interval $[0, 1]$ of the real line. Since the interval $[0, 1]$ has a one-to-one correspondence to the ternary Cantor set and inherits its topology from the real line, the same topology is induced on the ternary Cantor set. We should relax the compact interval in our generalized Cantor sets to be defined below. Loosely speaking, the same topology of the real line is induced on each of them.

Let us apply Kolmogorov’s definition of the dimension to compute this object for all of them. We exemplify the "dilution" in the uniaxial case for $m_2$ with a generalized Cantor set, which is initially a line of length $l$. In the first iteration, it is divided into 4 pieces of equal length $\frac{l}{4}$ and we keep only the two disconnected intervals $[0, \frac{3}{4}] \cup [\frac{1}{4}, l]$. Iterating $p$ times using the same steps we conclude that $N = 2^p$ whereas $\delta = \frac{l}{2^p}$ which leads to $D_{02} = \frac{1}{p}$ in the limit $p \to \infty$. Let us call $F_{21}$ the set of intervals obtained in the first interaction, $F_{22}$ the set of intervals in the second interaction and so on. We can associate this subspace with the generalized Cantor set $L_2$, defined by $L_2 = \bigcap_{n=1}^{\infty} F_{2n}$. To see that this set has zero length, note that the total length withdrawn $(lw_2)$ from the interval $[0, l]$ is given by the expression $lw_2 = l \sum_{p=1}^{\infty} \frac{1}{2^p} = l$, therefore proving our assertion.

Consider $m_3$. In the first iteration, divide the interval by 27, and choose only the intervals $F_{31} = \left[\frac{8}{27}, \frac{2}{27}\right] \cup \left[\frac{17}{27}, \frac{20}{27}\right] \cup \left[\frac{26}{27}, l\right]$ and so forth, analogously to our above observations for the $m_2$ subspace. In the $p$-th iteration $N = 3^p$ and $\delta = \frac{l}{3^p}$. We obtain $D_{03} = \frac{1}{p}$ in the limit $p \to \infty$. The associated generalized Cantor set to this subspace is $L_3 = \bigcap_{n=1}^{\infty} F_{3n}$. Note that the total length withdrawn of the interval of length $l$ now reads $lw_3 = 8l \sum_{p=1}^{\infty} \frac{1}{3^p} = l$ which implies that $L_3$ also has...
zero length.

This can also be done for $m_n$ yielding $D_{0n} = \frac{1}{n^2}$, $n = 2, ..., L$, whose associated generalized Cantor space is $L_n = \bigcap_{\nu=1}^\infty F_{n\nu}$. In this case, $w_n = n(n^{-1} - 1)\sum_{\nu=1}^n \frac{1}{n^{-1} - 1} = l$ and $L_n$ has zero length as well (or zero Lebesgue measure). Note that Smith-Volterra-Cantor sets have positive Lebesgue measure. Since the new generalized Cantor sets above introduced share the zero Lebesgue measure property with the original ternary Cantor set but never reduce to it, we find appropriate to name them “Cantor-Leite” sets $L_n$.

Although the dimension of the different subspaces computed with our method gives basically the same result from Ref. [5], the latter did not associate this with a special set different from $R^{m_n}$, perhaps because the topology of the various generalized Cantor-Leite sets above described inherit the topology of $R^{m_n}$. However, in that work there are no extra dimensions: although this possibility was slightly mentioned there, all spatial dimensions belong to the $m_n$ subspace ($n = 2, 3, 4$).

Whenever $n^{np} \gg l$, the result consists of several disconnected continuous pieces but has fractal dimension. In this simple model, it can be interpreted as the attractive and repulsive combined effects of gravity in the extra dimensions in a close analogy with the CECI model in statistical mechanics. This competition “dilutes” them so to speak; see below. Even though every Cantor-Leite set inherits the topology of the real line, its local structure demands a different metric structure from that of $R$. Let $L_n^{m_n}$ be the set representing the $m_n$ competing subspace. We can perform integrals as if we were working in $R^{m_n}$, but clearly additional geometric structure is necessary since $K^{m_n}$ is naturally different from $R^{m_n}$. Some care must be exercised. All generalized Cantor sets discussed above $L_n$ were shown to have zero length. This does not mean that $K_n$ coincides with the empty set, since certainly several intervals were left in its construction. So the integrals we referred to above should be computed with the Riemann measure and the integrals over the real lines are well-defined. Thus the manifold corresponding to the extended space has the form $M^d = M_1^{4, m_1 - 1} \times L_2^{m_2} \times L_3^{m_3} \times \cdots \times L_L^{m_L}$.

Considering just flat competing subspaces, a realization of this spacetime manifold as a nontrivial metric space can be defined through its metric tensor whose components are generalizations of $g_{\mu_1 \nu_1}(x_{\mu_1})$ of the curved metric on the subspace $m_1$.

With a slight change of notation in the coordinates for the time being, the metric tensor defined on the $m_n$-dimensional subspace has the form

$$\tilde{g}_{i_1 i_2} = \delta_{i_1 i_2} (-1)^{n-1} \frac{1}{n} \sum_{r_1 s_1 = 1}^{m_n} \left( \frac{1}{\left(2n-2\right)^2} \right)_{\tilde{r}_i t} \tilde{t}_s.
$$

This metric has a double tensor structure, one coming from the Kronecker delta and the other coming from the $m_n^2$ dyadics components manifested by the products $\tilde{r}_i \tilde{t}_s$ ($\tilde{r}_i, \tilde{t}_s = \delta_{i s}$). Another important point is the appearance of the factors $(-1)^{n-1}$ in the metric pertaining to the $m_n$ competing subspace. They were introduced so that the signs of all higher derivative terms after integral by parts coincide with those from Eq. (1) when surface terms are discarded whenever we perform the identification $\sigma_n \equiv \sigma_m$.

Within the $m_1$ subspace the metric does not have to be diagonal. Its inverse for a curved space $g^{\mu_1 \nu_1}$ satisfies $g_{\mu_1 \nu_1} g^{\mu_1 \nu_2} = g^{\nu_1 \nu_2}$, $g_{\mu_1 \nu_1} = \delta_{\mu_1 \nu_1}$. It is worth emphasizing that the different subspaces do not mix which other. We also have to be careful to contract the dyadics in order to obtain a pure number.

The statistical mechanics model defined by Eq. (1) has a purely quantum nature (the exchange interactions, with competing effects due to different signs among farther neighbors in a generalized Ising model). In our QFT of a self interacting scalar field minimally coupled to a gravitational field in a higher dimensional space, we can think of the competing subspaces as resulting from gravitational and antigravitational interactions in extra dimensions (“dark matter” effects [20]). Thus something similar to quantum operators must arise.

This quantum structure of the extra dimensions can be partially implemented through the inverse metric. In order to achieve this, define $\tilde{g}^{i_1 i_2}$ as

$$\tilde{g}_{i_1 i_2} = \delta_{i_1 i_2} (-1)^{n-1} \frac{1}{n} \sum_{r_1 s_1 = 1}^{m_n} \left( \frac{1}{\left(2n-2\right)^2} \right)_{\tilde{r}_i t} \tilde{t}_s.
$$

Therefore, the matrix multiplication in ordinary spacetime turns into a new structure: the usual matrix multiplication in the $m_1$ subspace whereas this matrix multiplication includes the double internal product of the independent dyadics sector in the competing subspaces $m_n$ ($n = 2, ..., L$). Consequently this definition yields the identity $\tilde{g}^{MP} g_{PN} = \delta^M_N$.

In first quantized quantum mechanics, a classical quantity takes operator values. Our description corresponds to the semiclassical (first quantized theory) of the metric tensor along the extra directions (competing subspaces $m_n, n = 2, ..., L$). The subtle point is that only the contravariant metric tensor in the competing subspaces implements this idea. It is not surprising, since the origin of the Lagrangian (1) is purely quantum (the exchange couplings have no classical analogue). The (leading quantum) effect of attractive/repulsive competition provoked by the semi-classical gravitational field at short distances (UV) is the appearance of the competing subspaces.

The Levi-Civita connection $\Gamma^M_{NP}$ vanishes for arbitrary $M, N, P \neq \mu_1, \nu_1, \rho_1$. Hereafter, our choice of flat Euclidean metric in the $m_1$ subspace implies $\Gamma^M_{NP} = 0$.

The masses are inversely proportional to the many independent correlation lengths $\xi_n$ in this multicritical behavior. (Henceforth, the Euclidean version of the metric
will interest us.) We choose to keep them different to tackle the problem purely from the field theory viewpoint. Anticipating future applications in the high energy (UV) regime, we discuss its renormalizability. Note that $\epsilon_L = 4 + \sum_{n=2}^{L} \frac{1}{n} m_n - d$, $m_1$ varies but the value of $m_n$ ($n \neq 1$) will be kept fixed throughout.

For the time being we consider only the primitively divergent vertex parts without worrying about graphs with tadpole insertions. How we achieve this, however, will be explained later during our explicit discussion of these vertex parts.

We begin with the integrals up to two-loop order in the $1PI$ four-point $\Gamma^{(4)}$ (and composed field $\Gamma^{(2,1)}$ since they are related with each other). Although not needed in what follows we write them with arbitrary external momenta in all subspaces in terms of a reference mass $\mu_n^*$. These integrals will not be of particular interest within the context of the present method. We just write down their $\epsilon_L$-expansions as $I_2(P_n) = \int \frac{d^n \bar{q}(n) q(n) d^{m_n} q_1(n) d^{m_n} q_2(n)}{(\sum_{n=1}^{L} q_1(n))^{2} + \mu_n^*} \prod_{n=1}^{L} \frac{1}{\Gamma(2 + \epsilon_L)}$ and $I_4 = \int \frac{2^{d_n + d^{m_n} q_1(n) d^{m_n} q_2(n)}}{(\sum_{n=1}^{L} q_1(n))^{2} + \mu_n^*} \prod_{n=1}^{L} \frac{1}{\Gamma(2 + \epsilon_L)}$, where $m_n = 1 - \frac{1}{2}(\psi(1) - \psi(2 - L) - \sum_{n=1}^{L} \frac{m_n}{2n})$ and $L(P_n, \mu_n^*) = \int_0^1 dx \ln \left[ \frac{\sum_{n=1}^{L} q_1(n)}{\mu_n^*} x(1 - x) + 1 \right]$. The conventions are detailed in Refs. [3, 5, 9].

The two- and three-loop graphs is identical to that in $\phi^4$ theory by the same token, consider the two- and three-loop contributions of the two-point vertex parts with arbitrary external momenta $p_n$ in arbitrary subspaces $m_n$ with the reference mass $\mu_n^*$. Their manipulation is the key ingredient in our method. Their Feynman integrals $I_3$ and $I_5$, respectively, can be written as

\[
I_3(p_n, \mu_n^*) = \int \frac{d^n \bar{q}(n) q(n) d^{m_n} q_1(n) d^{m_n} q_2(n)}{(\sum_{n=1}^{L} q_1(n))^{2} + \mu_n^*} \prod_{n=1}^{L} \frac{1}{\Gamma(2 + \epsilon_L)}
\]

\[
I_5(p_n, \mu_n^*) = \int \frac{d^n \bar{q}(n) q(n) d^{m_n} q_1(n) d^{m_n} q_2(n)}{(\sum_{n=1}^{L} q_1(n))^{2} + \mu_n^*} \prod_{n=1}^{L} \frac{1}{\Gamma(2 + \epsilon_L)}
\]

The "partial-p" operation [21] appropriate to our aim is written as

\[
1 = \frac{1}{d_{eff}} \left( \frac{\partial q_1(q_1)}{\partial q_1} + \sum_{n=2}^{L} \frac{1}{d_{eff}} \frac{\partial q_1(q_2)}{\partial q_1} \right),
\]

where $q_1(n)$ is each loop momentum of the $m_n$ subspace. This operation is applied to each loop diagram (the summation convention over repeated indices is implicit). Here $d_{eff} = d - \sum_{n=2}^{L} \frac{m_n}{2n} = 4 - \epsilon_L$. Note that no matter how many competing subspaces permitted by the problem, the effective dimension is always related to the critical dimension in the same way as in the usual $\phi^4$ with $\epsilon_L$ in the former replacing $\epsilon$ in the latter. The leading effect of quantum gravity in our quantum field is rather small: it only changes the critical dimension where the theory is renormalizable, manifesting itself in the perturbative expansion parameter and "detaches" the $m_1$ subspace from the others. (This was already suggested in Ref. [4].) The application of this operation to two- and three-loop graphs is identical to that in $\phi^4$ theory [5].

We apply the partial-p operation first on $I_3$. We find the following expression

\[
I_3(p_n, \mu_n^*) = - \frac{1}{d_{eff}} \left[ 3 \mu_n^* 2n A(p_n, \mu_n^*) + B(p_n, \mu_n^*) \right],
\]

\[
A(p_n, \mu_n^*) = \int \frac{d^n \bar{q}(n) q(n) d^{m_n} q_1(n) d^{m_n} q_2(n)}{(\sum_{n=1}^{L} q_1(n))^{2} + \mu_n^*} \prod_{n=1}^{L} \frac{1}{\Gamma(2 + \epsilon_L)}
\]

\[
B(p_n, \mu_n^*) = \int \frac{2^{d_n + d^{m_n} q_1(n) d^{m_n} q_2(n)}(q_1(n) + q_2(n) + p_n)}{(\sum_{n=1}^{L} q_1(n))^{2} + \mu_n^*} \prod_{n=1}^{L} \frac{1}{\Gamma(2 + \epsilon_L)}
\]

One subdiagram of the integral $A_n$ can be solved using the orthogonal approximation [3, 5, 9]. By integrating first over the quadratic momenta and using another Feynman parameter, we get to

\[
A(p_n, \mu_n^*) = \frac{1}{2} \frac{1}{d_{eff}} \left[ 3 \mu_n^* 2n A(p_n, \mu_n^*) + B(p_n, \mu_n^*) \right]
\]

\[
\int_0^1 dx \ln \left[ (1 - x)^{-\frac{1}{2}} \int_0^1 dy y^{-\frac{1}{2}} (1 - y) \int \frac{2^{d_n + d^{m_n} q_1(n) d^{m_n} q_2(n)}}{(\sum_{n=1}^{L} q_1(n))^{2} + \mu_n^*} \prod_{n=1}^{L} \frac{1}{\Gamma(2 + \epsilon_L)} \right]
\]

Now we apply the parametric dissociation transform (PDT), namely we get rid of all the external momenta in the $A(p_n, \mu_n^*)$ integral by setting $y = 0$ in the integrand of the last loop integral [7]. We then find $A(p_n, \mu_n^*)_{PDT} = \frac{d_{eff}}{2} \left[ 1 + \epsilon_L (2h_{mL} - \frac{3}{2}) + \epsilon_L^2 (2h_{mL}^2 - 3h_{mL} + \frac{3}{4}) \right]$. As it is going to be shown in a moment, the diagrams left in the two-point vertex part must be subtracted from their values at zero
external momenta. Employing the PDT in the zero-moment analogue $A(p(n) = 0, \mu_0^*)$ we find $(A(p(n), \mu_0^* - A(0, \mu_0^*))_{PDT} = 0$. The remaining integral can be shown to be given by $B(p(n), \mu_0^*) = \frac{\mu_0^{x_2+2}L}{8e_L} \sum_{\ell=1}^{L} \hat{P}^{2n}_{\ell} \left[ 1 + (2h_{mL} - \frac{4}{3})e_L - 2e_L \tilde{L}_3(p(n), \mu_0^*) \right]$, where $\tilde{L}_3(p(n), \mu_0^*) = \int_0^1 dx \int_0^1 dy (1-y) \ln \left[ y^{-1} \left\{ \frac{\sum_{\ell=1}^{L} \hat{P}^{2n}_{\ell}}{\mu_0^{x_2+2}} \right\} + 1 + y + \frac{y}{x(1-x)} \right]$. Altogether, we find the result $(I_3(p(n), \mu_0^*) - I_3(0, \mu_0^*))_{PDT} = \frac{\mu_0^{x_2+2}L}{8e_L} \sum_{\ell=1}^{L} \hat{P}^{2n}_{\ell} \left[ 1 + (2h_{mL} - \frac{4}{3})e_L - 2e_L \tilde{L}_3(p(n), \mu_0^*) \right]$. We can define the restriction of all integrals to a given mass/momentum subspace by writing

$$ (I_{3n}(p(n), \mu_0^*) - I_{3n}(0, \mu_0^*))_{PDT} = -\frac{\mu_0^{x_2+1L}}{8e_L} \sum_{\ell=1}^{L} \hat{P}^{2n}_{\ell} \left[ 1 + (2h_{mL} - \frac{4}{3})e_L - 2e_L \tilde{L}_3(p(n), \mu_0^*) \right],$$

(10)

with $\tilde{L}_3(p(n), \mu_0^*) = \int_0^1 dx \int_0^1 dy (1-y) \ln \left[ y^{-1} \left\{ \frac{\sum_{\ell=1}^{L} \hat{P}^{2n}_{\ell}}{\mu_0^{x_2+2}} \right\} + 1 + y + \frac{y}{x(1-x)} \right]$. A similar restriction for the four-point and composite vertex part is to work with the restricted integral $\tilde{L}(P(n), \mu_0^*) = \int_0^1 dx \int_0^1 \sum_{\ell=1}^{L} \hat{P}^{2n}_{\ell} (x(1-x) + 1)$. We will use the restriction definition extensively in our diagrammatic expansion, since it has no effect in the PDT.

We apply now the partial-$p$ operator in the integral $I_5(p(n), \mu_0^*)$. We find

$$ I_5(p(n), \mu_0^*) = -\sum_{\ell=1}^{L} \hat{P}^{2n}_{\ell} \left[ C_1(p(n), \mu_0^*) + 4C_2(p(n), \mu_0^*) \right] + 2D(p(n), \mu_0^*),$$

(11a)

$$ C_1(p(n), \mu_0^*) = \int \frac{\mu_0^{x_2+2}}{\left\{ \sum_{\ell=1}^{L} \hat{P}^{2n}_{\ell} + \mu_0^2 \right\}^2} \left[ \frac{1}{\left\{ \sum_{\ell=1}^{L} \hat{P}^{2n}_{\ell} \right\} + \mu_0^2} \right] \times \left[ \frac{1}{\left\{ \sum_{\ell=1}^{L} \hat{P}^{2n}_{\ell} \right\} + \mu_0^2} \right],$$

(11b)

$$ C_2(p(n), \mu_0^*) = \int \frac{\mu_0^{x_2+2}}{\left\{ \sum_{\ell=1}^{L} \hat{P}^{2n}_{\ell} + \mu_0^2 \right\}^2} \left[ \frac{1}{\left\{ \sum_{\ell=1}^{L} \hat{P}^{2n}_{\ell} \right\} + \mu_0^2} \right] \times \left[ \frac{1}{\left\{ \sum_{\ell=1}^{L} \hat{P}^{2n}_{\ell} \right\} + \mu_0^2} \right],$$

(11c)

$$ D(p(n), \mu_0^*) = \int \frac{\mu_0^{x_2+2}}{\left\{ \sum_{\ell=1}^{L} \hat{P}^{2n}_{\ell} + \mu_0^2 \right\}^2} \left[ \frac{1}{\left\{ \sum_{\ell=1}^{L} \hat{P}^{2n}_{\ell} \right\} + \mu_0^2} \right] \times \left[ \frac{1}{\left\{ \sum_{\ell=1}^{L} \hat{P}^{2n}_{\ell} \right\} + \mu_0^2} \right].$$

(11d)

By applying the PDT with the same set of parameters appropriate to the three-loop order integrals $C_1(p(n), \mu_0^*)$, we find $C_1(p(n), \mu_0^*) + 4C_2(p(n), \mu_0^*) = -\frac{\mu_0^{x_2+2}}{8e_L} \sum_{\ell=1}^{L} \hat{P}^{2n}_{\ell} \left[ 1 + (3h_{mL} - \frac{4}{3})e_L - \frac{2}{3} \left( 13h_{mL}^2 + 32h_{mL} + 25 + \frac{2}{3} \psi(2 - \sum_{\ell=1}^{L} \hat{P}^{2n}_{\ell} + \mu_0^2) \right) \right]$. When this contribution is subtracted from its value at $p(n) = 0$ using PDT, it vanishes. The integral $D(p(n), \mu_0^*)$ can be computed and what is left from the original integral $I_5$ after PDT. Therefore, using the restriction aforementioned we obtain the result

$$ (I_{5n}(p(n), \mu_0^*) - I_{5n}(0, \mu_0^*))_{PDT} = -\frac{\mu_0^{x_2+2}}{8e_L} \sum_{\ell=1}^{L} \hat{P}^{2n}_{\ell} \left[ 1 + 3(2h_{mL} - \frac{4}{3})e_L - 3e_L \tilde{L}_3(p(n), \mu_0^*) \right].$$

(12)

Next, we define the 1PI primittively divergent vertex parts in each subspace. Define the three-loop bare mass in the $m_n$ subspace by writing $\mu_0^{2n} = \Gamma_2^{(n)}(p(n) = 0, \mu_0, \Lambda_0, \Lambda_n)$, which can be inverted to express all the vertex parts in terms of $\mu_0$ in the diagrammatic expansion and getting rid, at the same time, of all tadpole insertions in the vertex parts which can be multiplicatively renormalized (see Ref. [22] for the pure $\phi^4$ theory). We shall need only the restrictions of the results above to the subspace $m_n$. From now on we drop the subscript PDT.

The bare primittively vertex parts after this maneuver can be written as

$$ \Gamma_2^{(n)}(p(n), p_2(n), \mu_0, \Lambda_n, \Lambda_0) = \frac{2n}{18} \left[ 1 + 3(2h_{mL} - \frac{4}{3})e_L - 3e_L \tilde{L}_3(p(n), \mu_0^*) \right],$$

(13a)

$$ \lambda_0^{2n}(N + 2) \left[ I_{2n}(p_1(n) + p_2(n), \mu_0) + 2\text{perms.} \right] + \lambda_0^{2n}(N + 2) \left[ I_{2n}(p_1(n) + p_2(n), \mu_0) + 2\text{perms.} \right],$$

(13b)

$$ \lambda_0^{2n}(N + 2) \left[ I_{2n}(p_1(n) + p_2(n), \mu_0) + 2\text{perms.} \right] + \lambda_0^{2n}(N + 2) \left[ I_{2n}(p_1(n) + p_2(n), \mu_0) + 2\text{perms.} \right],$$

(13c)

The definitions $\lambda_0 = u_0 \mu_0^{\kappa}$ and $g_n = u_n \mu_n^{\kappa}$ in terms of the dimensionless bare ($u_0$) and renormalized ($u_n$) coupling constants will be useful in what follows. The dependence of the integrals in the overall mass scales $\mu_n$ is offset by the contribution of the bare (and renormalized) coupling constants. This occurs not only in the perturbative primittively divergent vertex parts, but also in all vertex parts that can be renormalized multiplicatively, resulting in a perturbative expansion only in the dimensionless coupling constants.

Multiplicative renormalizability is the statement that if the primittively divergent vertex are renormalized multiplicatively in a given loop order, then all multiplicatively renormalized vertex part with arbitrary number of composite fields are related to
the bare ones through $\Gamma^{(N,L)}_{R(n)}(p_i(n);Q_i(n);m_n,u_n) = Z_{\phi(n)}^{\bar{Z}} Z_{\phi(n)}^L \Gamma^{(N,L)}_{R(n)}(p_i(n);Q_i(n);m_n,u_n)$, where $m_n$ are the renormalized masses. First we renormalize the primi-
tively divergent parts utilizing this definition. We demand that

\[ \begin{align*}
\Gamma^{(2)}_{R(n)}(p_1(n),m_n,u_n) &= Z_{\phi(n)}^{\bar{Z}} \Gamma^{(2)}_{R(n)}(p_1(n),m_n,u_n), \\
\Gamma^{(4)}_{R(n)}(p_1(n),m_n,u_n) &= Z_{\phi(n)}^{\bar{Z}} \Gamma^{(4)}_{R(n)}(p_1(n),m_n,u_n), \\
\Gamma^{(2,1)}_{R(n)}(p_1(n),p_2(n),Q_\mu(n),m_n,u_n) &= Z_{\phi(n)}^{\bar{Z}} \times \\
\Gamma^{(2,1)}_{(n)}(p_1(n),p_2(n),Q(n),m_n,u_n),
\end{align*} \]

are finite by minimal subtraction $(MS)$ of the dimension-
\alpional poles $Z_{\phi(n)}^{\bar{Z}} \equiv Z_{\phi(n)}Z_{\bar{Z}_{\phi(n)}}$. Then, we perform the expansion of these functions in terms of the dimen-
\sionless couplings $u_n$ as

\[ u_n = u_n(1 + a_1 u_n + a_2 u_n^2), \]

\[ Z_{\phi(n)} = 1 + b_2 u_n^3 + b_3 u_n^3, \]

\[ Z_{\bar{Z}_{\phi(n)}} = 1 + c_1 u_n + c_2 u_n^2. \]

By defining the three-loop renormalized masses $m_n^3 = Z_{\phi(n)}^{\bar{Z}} u_n^2$ we can find all coefficients using the diagram-
f\atic expansion: $b_2$ from the singular part of the two-
loop contribution of $\Gamma^{(2)}_{R(n)}$ and $a_1, c_1$ from the one-loop contri-
bution from $\Gamma^{(4)}_{R(n)}(\Gamma^{(2,1)}_{R(n)})$. Now $b_3$ can be de-
n\alpred and proves that the coefficient of the integrals $\bar{L}_3(p_1(n),\mu_n)$ vanishes; $a_2$ and $c_2$ are then determined with the cancellation of the coefficient of the integrals $\bar{L}(p_1(n),\mu_n)$. We then find:

\[ u_0 = u_n \left(1 + \frac{(N+8)}{6\epsilon_L} u_n + \left(\frac{(N+8)^2}{24\epsilon_L} - \frac{(3N+14)}{24\epsilon_L}\right) u_n^2\right), \]

\[ Z_{\phi(n)} = 1 + \frac{(N+2)}{6\epsilon_L} u_n^2 + \left(\frac{(N+2)(N+5)}{30\epsilon_L} - \frac{(N+2)}{24\epsilon_L}\right) u_n^2, \]

\[ Z_{\phi(n)} = 1 - \frac{(N+2)}{144\epsilon_L} u_n^2 + \left(\frac{(N+2)(N+8)}{1296\epsilon_L} + \frac{N+2}{5184\epsilon_L}\right) u_n^2. \]

The method can employed in the calculation of critical exponents using the Callan-Symanzik $(CS)$ method [23], since in the Lagrangian $(1)$, the mass $m_n$ is proportional to $|T - T_c|^k$, with $m_n \neq 0$ but still in the critical region. By applying the operator $m_n \frac{\partial}{\partial m_n}$ on $\Gamma^{(N,L)}_{R(n)}$, we find

\[ \left[m_n \frac{\partial}{\partial m_n} + \beta_n(u_n) \right] \Gamma^{(N,L)}_{R(n)}(p_i(n),m_n,u_n) - \left(2 - \gamma_{\phi(n)}(0)\right) m_n^2 \Gamma^{(N,L+1)}_{R(n)}(p_i(n),m_n,u_n). \]

The Wilson functions

\[ \beta_n(u_n) = -n\epsilon_L \left(\frac{\partial u_n}{\partial u}\right)^{-1} = n u_n \left[-\epsilon_L + \frac{(N+8)}{6}\right], \]

has a nontrivial (repulsive) UV fixed point $(\beta_n(u_n) = 0)$ at $u_n = -\frac{6\epsilon_L}{N+8} \left[1 + \frac{3(N+14)}{(N+8)^2}\epsilon_L \right]$. The func-
tions $\gamma_{\phi(n)}(u_n) = \left[\beta_n(u_n) \frac{\partial Z_{\phi(n)}}{\partial u_n}\right] = n \left[\left(N+2\right)/(128u_n^3)\right]$ computed at the fixed point yields $\eta_n$ up to three-loop order. It is given by

\[ \eta_n = \frac{n(N+2)}{2(N+8)^2} \left[1 + \left(6(N+14)/(N+8)^2 - \frac{1}{4}\right)\epsilon_L \right]. \]

The functions $\tilde{\eta}_{\phi(n)}(u_n) = -\left[\frac{\partial^2 Z_{\phi(n)}}{\partial u_n^2}\right] = n u_n \left(\frac{N+2}{8\epsilon_L} - \frac{1}{4}\right)$ computed at the fixed point together with the identity $\nu_n^{-1} = 2n - \tilde{\eta}_{\phi(n)}(u_n) - \eta_n$ lead to the correlation length exponents

\[ \nu_n = \frac{(N+2)}{4n(N+8)} + \frac{(N+2)(N^2 + 23N + 60)\epsilon_L^2}{8n(N+8)^2}. \]

A detailed discussion will be reported in the near future.

When the $m_1$ subspace is extended to Minkowski spacetime, the massive quantum scalar fields in Lif-
\shitz spacetimes interact with the competing directions through the leading order quantum gravity effect. The subspace $m_1 < 4$ can have its diffeomorphism invariance restored in $S_\text{QFT}$ so that general relativity can be formulated on it as explained. Thus an observer living on the $m_1$ subspace would feel a gravitational field weakly disturbed by the extra dimensions, having the impression that they are not there. The other terms with higher momentu
\m powers in the "extra" directions break Lorentz invariance only in the "bulk" $(d$ dimensions). The exponents corresponding to the power of momenta in the subspace are fixed. The only way to come back to the original $\phi^4$ is to set $m_n = 0$, what corresponds to a mas-
sive quantum scalar field in a classical Minkowski/GR $(d = 4)$ background.
To summarize, we have shown for the first time the equivalence of the CECI model with a scalar quantum field theory in a Lifshitz spacetime. Even its flat version already includes the first corrections of quantum gravity emerging from the competition subspaces. In addition, the quantum theory of the massive scalar field through the usage of the partial-\(p\) operation along with the PDT, etc., results in a masslesslike massive method of minimal subtraction. It is quite simple, although different from the massless one \[26\]. We computed critical exponents as an application of the method. Massive nonlinear sigma models with competing interactions studied using the \(\beta\) expansion \[27\] could be complemented using the present minimal subtraction.

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