On the Properties of the Compound Nodal Admittance Matrix of Polyphase Power Systems

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Abstract—Most techniques for power system analysis model the grid by exact electrical circuits. For instance, in power flow study, state estimation, and voltage stability assessment, the use of admittance parameters (i.e., the nodal admittance matrix) and hybrid parameters is common. Moreover, network reduction techniques (e.g., Kron reduction) are often applied to decrease the size of large grid models (i.e., with hundreds or thousands of state variables), thereby alleviating the computational burden. However, researchers normally disregard the fact that the applicability of these methods is not generally guaranteed. In reality, the nodal admittance must satisfy certain properties in order for hybrid parameters to exist and Kron reduction to be feasible. Recently, this problem was solved for the particular cases of monophase and balanced triphase grids. This paper investigates the general case of unbalanced polyphase grids. Firstly, conditions determining the rank of the so-called compound nodal admittance matrix and its diagonal subblocks are deduced from the characteristics of the electrical components and the network graph. Secondly, the implications of these findings concerning the feasibility of Kron reduction and the existence of hybrid parameters are discussed. In this regard, this paper provides a rigorous theoretical foundation for various applications in power system analysis.

Index Terms—Admittance parameters, hybrid parameters, Kron reduction, multiport networks, nodal admittance matrix, polyphase power systems, unbalanced power grids

I. INTRODUCTION

InherentlY, techniques for power system analysis need an exact analytical description of the grid. This description is normally deduced from an equivalent electrical circuit. For instance, in Power Flow Study (PFS) [1]–[3], State Estimation (SE) [4]–[6], and Voltage Stability Assessment (VSA) [7]–[9], the use of admittance parameters (i.e., the nodal admittance matrix) or hybrid parameters (i.e., hybrid parameters matrices) is a common practice. As the solution methods employed for these applications are computationally heavy (e.g., [10]–[14]), network reduction techniques, such as Kron reduction [15], are often applied in order to reduce the problem size. Thereby, the computational burden is decreased, and the execution speed is increased without the use of high-performance computers (e.g., [16]). However, neither the reducibility of the nodal admittance matrix nor the existence of hybrid parameters are guaranteed a priori. In order for this to be the case, the nodal admittance matrix has to satisfy certain properties (i.e., the corresponding diagonal subblocks have to be invertible).

Interestingly, most researchers and practitioners apparently ignore this fact. There exist a handful of publications which investigate the feasibility of Kron reduction (e.g., [17]) and the existence of hybrid parameters matrices (e.g., [18]–[20]), but their validity is limited. The former base their reasoning upon trivial cases (i.e., purely resistive/inductive monophase grids), and the latter establish feeble guarantees (i.e., hybrid parameters may exist for solely one partition of the nodes). Other works, which deal with triphase power flow, show that a subblock of the nodal admittance matrix, which is obtained by removing the rows and columns associated with one single node (i.e., the slack node), has full rank in practice [21], [22]. However, this finding cannot be generalized straightforwardly (i.e., for generic polyphase grids, or removal of several nodes).

Recently, the authors of this paper proved stronger properties for Kron reduction and hybrid parameters for monophase grids (see [23]). Two conditions were used in order to prove these properties: i) the connectivity of the network graph, and ii) the lossiness of the branch impedances. If these conditions are satisfied, then Kron reduction can be performed for any set of zero-injection nodes, and a hybrid parameters matrix can be constructed for any partition of the nodes. The theorems proven in [23] only apply to monophase grids, and polyphase grids that can be decomposed into decoupled monophase grids are discussed. However, this finding cannot be generalized straightforwardly (i.e., grids composed of elements whose impedance/admittance matrices are symmetric circulant), because they can be reduced to equivalent positive-sequence networks. Therefore, the generic case of unbalanced polyphase grids cannot be treated.

This paper develops the theory for the generic case, namely unbalanced polyphase grids. Since the method of symmetrical components cannot be applied, the grid has to be represented by a polyphase circuit, so that the electromagnetic coupling in between the phases can be accounted for properly. Therefore, the generalization to the polyphase case is actually non-trivial. More precisely, the electromagnetic coupling is modeled using compound electrical parameters [24]. It is argued that physical electrical components are represented by polyphase two-port equivalent circuits, whose compound electrical parameters are symmetric, invertible, and passive. Via mathematical derivation and physical reasoning, it is proven that the diagonal subblocks of the compound nodal admittance matrix have full rank if the network graph is weakly connected. Using this property, it is shown that the feasibility of Kron reduction and the existence of hybrid parameters are guaranteed under practical conditions. In that sense, this paper provides – for the first time in the literature – a rigorous theoretical foundation for the analysis of polyphase power systems, in particular for applications like PFS, SE, and VSA (cf. [25]–[28]).

The remainder of this paper is structured as follows: First, the basic theoretical foundations are laid in Sec. [I]. Thereupon, the properties of the compound nodal admittance matrix are developed in Sec. [II]. Afterwards, the implications with respect to Kron reduction and hybrid parameters matrices are deduced in Sec. [V]. Finally, the conclusions are drawn in Sec. [V].

II. FOUNDATIONS

A. Numbers

Scalars are denoted by ordinary letters. The real and imaginary part of a complex scalar \( z \in \mathbb{C} \) are denoted by \( \Re\{z\} \) and \( \Im\{z\} \).
\[ \Im \{z\}, \text{respectively}. \] Thus, \( z \) can be expressed in rectangular coordinates as \( z = \Re \{z\} + j \Im \{z\} \). The complex conjugate of \( z \) is denoted by \( z^* \). The absolute value and the argument of \( z \) are denoted by \( |z| \) and \( \arg(z) \), respectively. Thus, \( z \) can be expressed in polar coordinates as \( z = |z| \angle \arg(z) \).

**B. Set Theory**

Sets are denoted by calligraphic letters. The cardinality of a set \( A \) is denoted by \(|A|\). The (set-theoretic) difference \( A \setminus B \) of two sets \( A \) and \( B \) is defined as
\[ A \setminus B := \{x \mid x \in A, x \notin B\} \] (1)

The Cartesian product \( A \times B \) of \( A \) and \( B \) is defined as
\[ A \times B := \{(a, b) \mid a \in A, b \in B\} \] (2)

A partition of a set \( A \) is a family of sets \( \{A_k \mid k \in \mathbb{K}\} \), where \( \mathbb{K} = \{1, \ldots, |\mathbb{K}|\} \) is an integer interval, for which
\[ A_k \subseteq A \quad \forall k \in \mathbb{K} \] (3)
\[ A_k \neq \emptyset \quad \forall k \in \mathbb{K} \] (4)
\[ A_k \cap A_l = \emptyset \quad \forall k, l \in \mathbb{K}, k \neq l \] (5)
\[ \bigcup_{k \in \mathbb{K}} A_k = A \] (6)

That is, the parts \( A_k \) are non-empty and disjoint subsets of \( A \), whose union is exhaustive. If \( A_k \subseteq A \ (\forall k \in \mathbb{K}) \), which means that \( \{A_k \mid k \in \mathbb{K}\} \neq \{A\} \), the partition is non-trivial.

**C. Linear Algebra**

Matrices and vectors are denoted by bold letters. Consider a matrix \( M \in \mathbb{C}^{[R \times [C]} \) and \( C \in \mathbb{C}^{[C \times [C]} \) be non-singular. Then, for any matrix \( M \in \mathbb{C}^{[R \times [C]} \), it holds that \( \text{rank}(AM) = \text{rank}(M) = \text{rank}(MB) \).

A non-singular complex matrix is called unitary if its inverse equals its conjugate transpose, that is \( M^{-1} = (M^*)^T \).

**Lemma 1.** \( \text{rank}(M^T) = \text{rank}(M) \forall M \in \mathbb{C}^{[R \times [C]} \).

**Lemma 2.** Let the matrices \( A \in \mathbb{C}^{[R \times [R]} \) and \( B \in \mathbb{C}^{[C \times [C]} \) be non-singular. Then, for any matrix \( M \in \mathbb{C}^{[R \times [C]} \), it holds that \( \text{rank}(AM) = \text{rank}(M) = \text{rank}(MB) \).

A real symmetric matrix is called positive definite if its inverse equals its conjugate transpose, that is \( M^{-1} = (M^*)^T \).

**Lemma 3.** Let \( M \in \mathbb{C}^{[R \times [R]} \) and \( M = M^T \). Then, \( M \) can be factorized as \( M = U^T D U \), where \( U \in \mathbb{C}^{[R \times [R]} \) is unitary, and \( D \in \mathbb{R}^{[R \times [R]} \) is non-negative diagonal (Autonne-Takagi factorization, see \([29]\)). If \( M \) is non-singular, then \( D \) is positive diagonal.

A real symmetric matrix is called positive definite if \( M > 0 \), or negative definite if \( M < 0 \), respectively, when
\[ M > 0 : \ x^T M x > 0 \ \forall x \neq 0 \] (7)
\[ M < 0 : \ x^T M x < 0 \ \forall x \neq 0 \] (8)

If the inequality is not strict, \( M \) is called positive semi-definite if \( M \succeq 0 \), or negative semi-definite if \( M \preceq 0 \).

**Lemma 4.** Let \( M \in \mathbb{C}^{[R \times [R]} \) and \( \Re \{M\} \succ 0 \). Then, \( M \) is non-singular and \( \Re \{M^{-1}\} > 0 \) (for proof, see \([30]\)).

**Lemma 5.** If \( M \in \mathbb{C}^{[R \times [R]} \) and \( \Im \{M\} \succ 0 \). Then, \( M \) is non-singular and \( \Im \{M^{-1}\} < 0 \) (for proof, see \([31]\)).

Let \( \{R_i \mid i \in I\} \) and \( \{C_j \mid j \in J\} \) be partitions of \( R \) and \( C \), where \( I := \{1, \ldots, |I|\} \) and \( J := \{1, \ldots, |J|\} \). The block formed by the intersection of the rows \( R_i \) and \( C_j \) is denoted by \( M_{ij} \). That is, \( M = \{M_{ij}\} \).

Let \( M_{ij} \succ 0 \) \( (j' \subseteq J, i' \subseteq I) \) be the submatrix consisting of the blocks \( M_{ij} \) \( (i, j \in I') \). Now, consider the particular case \( j' \subseteq J \setminus \{j\} \) and \( i' \subseteq I \setminus \{i\} \), and define
\[ M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : = \begin{bmatrix} M_{ij} \succ 0' & M_{ij} \times \{j'\} \\ M_{ij} \times \{i'\} & M_{ij} \end{bmatrix} \] (9)

If \( D \) is invertible, the Schur complement \( M/D \) of \( D \) in \( M \) is
\[ M/D := \begin{bmatrix} A - BD^{-1} C \end{bmatrix} \] (10)

The following properties hold (see \([32]\)).

**Lemma 6.** \( \det(M) = \det(D) \det(M/D) \).

**Lemma 7.** Let \( i \in I' \) and \( j \in J' \). Then
\[ (M/D)_{ij} = A_{ij} - B_i D^{-1} C_{ij} = \begin{bmatrix} A_{ij} & B_i \\ C_{ij} & D \end{bmatrix} / D \] (11)

The Kronecker product \( A \otimes B \) of two generic matrices \( A \) and \( B \) is a block matrix, whose blocks \( (A \otimes B)_{ij} \) are defined as the product of the corresponding element \( A_{ij} \) of \( A \) and \( B \).

\[ A \otimes B : (A \otimes B)_{ij} = A_{ij} B \] (12)

The following property holds (see \([33]\)).

**Lemma 8.** \( \text{rank}(A \otimes B) = \text{rank}(A) \cdot \text{rank}(B) \).

**D. Graph Theory**

A directed graph \( G = (V, E) \) consists of a set of vertices \( V (\mathcal{G}) := V \) and a set of directed edges \( E (\mathcal{G}) := E \), where
\[ E (\mathcal{G}) \subseteq \{(v, w) \mid v \in V (\mathcal{G}) \times V (\mathcal{G}) \mid v \neq w\} \] (13)

The sets of outgoing and incoming edges of a vertex \( v \in V (\mathcal{G}) \) are denoted by \( E_{\text{out}}(\mathcal{G}, v) \) and \( E_{\text{in}}(\mathcal{G}, v) \). Formally
\[ E_{\text{out}}(\mathcal{G}, v) := \{e \in E (\mathcal{G}) \mid e = (v, w), w \in V (\mathcal{G})\} \] (14)
\[ E_{\text{in}}(\mathcal{G}, v) := \{e \in E (\mathcal{G}) \mid e = (w, v), w \in V (\mathcal{G})\} \] (15)

A set of internal edges \( E_{\text{int}}(\mathcal{G}, W) \) with respect to \( V \subseteq V (\mathcal{G}) \) contains all directed edges that start and end in \( W \). So
\[ E_{\text{int}}(\mathcal{G}, W) := \{(u, v) \in E (\mathcal{G}) \mid u, v \in V \subseteq V (\mathcal{G})\} \] (16)

A cut-set \( E_{\text{cut}}(\mathcal{G}, W) \) with respect to \( W \subseteq V (\mathcal{G}) \) contains all directed edges that start in \( W \) and end in \( V (\mathcal{G}) \setminus W \). That is
\[ E_{\text{cut}}(\mathcal{G}, W) := \{(u, v) \in E (\mathcal{G}) \mid u \in W \subseteq V (\mathcal{G}) \setminus W \} \] (17)

The connectivity of the graph \( \mathcal{G} \) is given by its (edge-to-vertex) incidence matrix \( A_{\mathcal{G}} \), whose elements are given by (see \([34]\))
\[ A_{\mathcal{G}, iv} := \begin{cases} +1 & \text{if } e_i \in E_{\text{out}}(\mathcal{G}, v) \\ -1 & \text{if } e_i \in E_{\text{in}}(\mathcal{G}, v) \\ 0 & \text{otherwise} \end{cases} \] (18)

A directed graph \( \mathcal{G} \) is said to be weakly connected if there exists a connecting path (which need not respect the directivity of the edges) between any pair of vertices.

**Lemma 9.** If the directed graph \( \mathcal{G} \) is weakly connected, then \( \text{rank}(A_{\mathcal{G}}) = |V (\mathcal{G})| - 1 \) (for proof, see \([35]\)).
The directed graph $\Pi$ is described by the directed graph $G$. Hence, in a per-unit grid model, they are represented either by matters inside of electrical components, but not between them. Further, electromagnetic coupling only are linear and passive. Further, electromagnetic coupling only.

The grid consists of electrical components that hypothetically only. Let $N_E$ be the set of all polyphase nodes (i.e., physical and virtual). The vertices are

$$V := N \cup \mathcal{G}$$

(19)

The edges fall into two categories, namely polyphase branches and polyphase shunts. The former connect a pair of polyphase nodes, the latter a polyphase node and ground. Let $\mathcal{L}_\Pi \subseteq N \times N$ and $\mathcal{L}_T \subseteq N \times N_T$ be the polyphase branches associated with the II-section and T-section equivalent circuits, respectively. The set of all polyphase branches is $\mathcal{L} := \mathcal{L}_\Pi \cup \mathcal{L}_T$. Similarly, let $\mathcal{T}_E := N_E \times \mathcal{G}$ and $\mathcal{T}_T := N_T \times \mathcal{G}$ be the polyphase shunts associated with $N_E$ and $N_T$, respectively. Thus, the set of all polyphase shunts is $\mathcal{E} := \mathcal{T}_E \cup \mathcal{T}_T$. The edges are obtained as

$$\mathcal{E} := \mathcal{L} \cup \mathcal{G}$$

(20)

The polyphase branches are related to the longitudinal electrical parameters of the polyphase two-port equivalents. More precisely, every polyphase branch $\ell \in \mathcal{L}$ is associated with a compound branch impedance $Z_{\ell}$, which is given by

$$Z_{\ell} := \begin{cases} Z_{\Pi,(m,n)} & \text{if } \ell = (m,n) \in \mathcal{L}_\Pi \\ Z_{T,(n,x)} & \text{if } \ell = (n,x) \in \mathcal{L}_T \end{cases}$$

(21)

Similarly, the polyphase shunts are related to the transversal electrical parameters of the polyphase two-port equivalents. The aggregated shunt admittance $Y_{\Pi,n}$ resulting from the II-section equivalent circuits connected to the polyphase node $n \in N_E$ is given by

$$Y_{\Pi,n} := \sum_{(m,n) \in \mathcal{L}_\Pi} Y_{\Pi,n(m,n)} + \sum_{(m,n) \in \mathcal{L}_T} Y_{\Pi,n(m,n)}$$

(22)

Accordingly, the compound shunt admittance $Y_{t}$ associated with a polyphase shunt $t \in \mathcal{E}$ is given by

$$Y_{t} := \begin{cases} Y_{\Pi,n} & \text{if } t = (n,g) \in \mathcal{T}_E \\ Y_{T,x} & \text{if } t = (x,g) \in \mathcal{T}_T \end{cases}$$

(23)

With respect to the compound electrical parameters of the grid model, the following assumption is made:

**Hypothesis 3.** The compound branch impedances $Z_{\ell}$ defined by (21) are symmetric, invertible, and passive. That is

$$\forall \ell \in \mathcal{L} : \begin{bmatrix} Z_{\ell} = Z_{\ell}^T \\ \exists Y_{\ell} = Z_{\ell}^{-1} \\ \Re\{Y_{\ell}\} \geq 0 \end{bmatrix}$$

(24)

In particular, this implies $Z_{\ell} \neq 0$. Conversely, the compound shunt admittances $Y_{t}$ defined by (23) may be zero. If not, then they are also symmetric, invertible, and passive. That is

$$t \in \mathcal{E} \text{ with } Y_{t} \neq 0 : \begin{bmatrix} Y_{t} = Y_{t}^T \\ \exists Z_{t} = Y_{t}^{-1} \\ \Re\{Y_{t}\} \geq 0 \end{bmatrix}$$

(25)

In practice, the above-stated assumptions are valid for a broad variety of power system components, like transmission lines, transformers, and various FACTS devices. Further information about this subject is given in App. App:Equipment.

Let $V_{p,n}$ and $I_{n,p}$ be the phasors of the nodal voltage and the injected current in phase $p \in \mathcal{P}$ of the polyphase node $n \in N$. By definition, the nodal voltage is referenced to the ground node, and the injected current flows from the ground node into the corresponding terminal (see Fig. 2). Analogous quantities are defined for a polyphase node as a whole

$$V_{n} := \text{col}_{p \in \mathcal{P}}(V_{p,n})$$

(26)

$$I_{n} := \text{col}_{p \in \mathcal{P}}(I_{n,p})$$

(27)

\[ \text{Fig. 1. Polyphase two-port equivalent circuits of the components of the grid.} \]
By definition (28)–(29), the vectors blocks which correspond to the polyphase nodes of the grid.

\[
Y_n = \text{col}_{n \in \mathbb{N}}(V_n) \quad (28)
\]

\[
I = \text{col}_{n \in \mathbb{N}}(I_n) \quad (29)
\]

where the operator \(\text{col}\) constructs a (block) column vector.

The primitive compound branch admittance matrix \(Y_{\mathbb{C}}\) and the primitive compound shunt admittance matrix \(Y_T\) are

\[
Y_{\mathbb{C}} := \text{diag}_{\ell \in \mathbb{L}}(Y_{\ell}) \quad (30)
\]

\[
Y_T := \text{diag}_{t \in \mathcal{T}}(Y_t) \quad (31)
\]

where the operator \text{diag} constructs a (block) diagonal matrix. Let \(\mathcal{B} = (\mathbb{N}, \mathcal{L})\) represent the subgraph of \(\mathcal{G}\) comprising the branches only. Define its polyphase incidence matrix \(A^p_{\mathcal{B}}\) as

\[
A^p_{\mathcal{B}} := A_{\mathcal{B}} \otimes \text{diag}(I_{|\mathcal{P}|}) \quad (32)
\]

where \(1_{|\mathcal{P}|}\) is a vector of ones with length \(|\mathcal{P}|\). The compound nodal admittance matrix \(Y\), which relates \(I\) with \(V\) via

\[
I = YV \quad (33)
\]

(i.e., Ohm’s law) is given by (see [25], [35])

\[
Y = (A^p_{\mathcal{B}})^T Y_{\mathbb{C}} A^p_{\mathcal{B}} + Y_T \quad (34)
\]

By definition (28)–(29), the vectors \(V\) and \(I\) are composed of blocks which correspond to the polyphase nodes of the grid. Therefore, \(Y\) can be written in a block form as \(Y = (Y_{mn})\), where \(Y_{mn}\) relates \(I_m\) with \(V_n\) \((m, n \in \mathbb{N})\). As known from circuit theory, it holds that (see [35])

Lemma 10. \(t = (n, g) \in \mathcal{T} \) \( \implies \sum_{m \in \mathbb{N}} Y_{mn} = \sum_{m \in \mathbb{N}} Y_{mn} = Y_t\).

III. PROPERTIES

The properties of the compound nodal admittance matrix are developed in two steps. First, it is proven that the compound shunt admittances determine the rank of the whole matrix in Sec. III-A. Afterwards, it is shown that the compound branch impedances determine the rank of its diagonal blocks in Sec. III-B.
Hypothesis 3 implicates that the compound branch impedances $Z_{\ell} (\ell \in \mathcal{L})$ are symmetric and invertible. By construction

$$Z_{\ell} = \begin{cases} Z_{\ell} & \text{if } \ell = \ell' \\ Y_{\ell}^{-1} & \text{if } \ell = \ell' \in \mathcal{L}' \setminus \mathcal{L} \end{cases}$$ (44)

The claimed properties follow from (24) and (25), respectively.

Let $\mathcal{B}' := (\mathcal{N}', \mathcal{L}')$ be the analogon of $\mathcal{B}$ for the modified grid. Obviously, if $\mathcal{B}$ is weakly connected, $\mathcal{B}'$ is weakly connected, too. Clearly, the modified grid satisfies the conditions required for the application of Theorem 1. Observe that the compound shunt admittances from the polyphase nodes $\mathcal{N}'$ to the virtual ground node $\mathcal{G}'$ are zero by construction.

$$Y_{\ell'} = 0 \forall \ell' \in \mathcal{T}' := \mathcal{N}' \times \mathcal{G}'$$ (45)

Therefore, the first part of (35), which has already been proven, can be applied. Accordingly, the compound nodal admittance matrix $Y'$ of the modified grid has rank

$$\text{rank}(Y') = (|\mathcal{N}'| - 1)|\mathcal{P}| = |\mathcal{N}| |\mathcal{P}|$$ (46)

Let $y_{T}$ be the column vector composed of the compound shunt admittances $Y_{t} (t \in \mathcal{T})$. Namely

$$y_{T} := \text{col}_{t \in \mathcal{T}}(Y_{t})$$ (47)

Given that the voltages of the modified grid are referenced to the virtual ground node, Ohm’s law (33) reads as follows

$$\begin{bmatrix} I \\ I_{g} \end{bmatrix} = \begin{bmatrix} Y & -y_{T} \\ -y_{T}^{T} & \sum_{t \in \mathcal{T}} Y_{t} \end{bmatrix} \begin{bmatrix} V' \\ V_{g} \end{bmatrix}$$ (48)

That is, $Y'$ is given in block form as

$$Y' = \begin{bmatrix} Y & -y_{T} \\ -y_{T}^{T} & \sum_{t \in \mathcal{T}} Y_{t} \end{bmatrix}$$ (49)

It is known that elementary operations on the rows and columns of a matrix do not change its rank. Hence, one can add the first $|\mathcal{N}|$ block rows/columns of $Y'$ to the last block row/column without affecting its rank. Using Lemma 10, one finds

$$\begin{bmatrix} Y & -y_{T} \\ -y_{T}^{T} & \sum_{t \in \mathcal{T}} Y_{t} \end{bmatrix} \begin{bmatrix} \text{row}_{|\mathcal{N}|+1} \\ \text{col}_{|\mathcal{N}|+1} \end{bmatrix} = \begin{bmatrix} \text{row}_{|\mathcal{N}|+1} \\ \text{col}_{|\mathcal{N}|+1} \end{bmatrix}$$ (50)

$$\begin{bmatrix} Y & -y_{T} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \text{row}_{|\mathcal{N}|+1} \\ \text{col}_{|\mathcal{N}|+1} \end{bmatrix} = \begin{bmatrix} \text{row}_{|\mathcal{N}|+1} \\ \text{col}_{|\mathcal{N}|+1} \end{bmatrix}$$ (51)

It follows straightforward that

$$\text{rank}(Y) = \text{rank}(Y') = |\mathcal{N}| |\mathcal{P}|$$ (52)

which proves the claim.

B. Block Rank

**Theorem 2.** Suppose that Hypotheses 2, 3 hold. If the branch graph $\mathcal{B} = (\mathcal{N}, \mathcal{L})$ is weakly connected, and all the compound branch impedances $Z_{\ell} (\ell \in \mathcal{L})$ are strictly passive

$$\Re\{Z_{\ell}\} > 0 \forall \ell \in \mathcal{L}$$ (54)

then it follows that

$$\text{rank}(Y_{M \times M}) = |M| |\mathcal{P}| \forall M \subseteq \mathcal{N}$$ (55)

That is, every proper diagonal subblock of the compound nodal admittance matrix $Y$ has full rank.

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**Proof.** As known from circuit theory, $Y_{M \times M}$ relates $I_{M}$ and $V_{M}$ for $V_{N \setminus M} = 0$ (i.e., the polyphase nodes $N \setminus M$ are short-circuited). This is due to the assumption that the state of each electrical component is composed solely of ground-referenced nodal voltages. In that sense, $Y_{M \times M}$ can be regarded as the compound nodal admittance matrix of a fictional grid which is obtained by grounding the polyphase nodes $N \setminus M$. Therefore, the polyphase branches internal to $M$ (i.e., $E_{\text{int}}(\mathcal{B}, M)$) persist, whereas those interconnecting $M$ and $N \setminus M$ (i.e., $E_{\text{int}}(\mathcal{B}, M)$ and $E_{\text{cut}}(\mathcal{B}, N \setminus M)$) become polyphase shunts in the modified grid. The remaining branches are described by the graph

$$\mathcal{G} := (M, E_{\text{int}}(\mathcal{B}, M))$$ (56)

In general, $\mathcal{G}$ is disconnected. However, since $\mathcal{B}$ is weakly connected as a whole, there exists a partition $\{M_{k} | k \in \mathcal{K}\}$ of $M$ so that the subgraphs $\mathcal{G}_{k}$ associated with $M_{k}$

$$\mathcal{G}_{k} := (M_{k}, E_{\text{int}}(\mathcal{B}, M_{k})), k \in \mathcal{K}$$ (57)

are weakly connected, and mutually disconnected (see Fig. 4). Therefore, $Y_{M \times M}$ is block diagonal. Namely

$$Y_{M \times M} = \text{diag}_{k \in \mathcal{K}}(Y_{M_{k} \times M_{k}})$$ (58)

In turn, $Y_{M_{k} \times M_{k}}$ can be interpreted as the compound nodal admittance matrix of a fictional grid, which is constructed by grounding the polyphase nodes $N \setminus M_{k}$. Define

$$\mathcal{L}_{k} := E(\mathcal{G}_{k}) = E_{\text{int}}(\mathcal{B}, M_{k}) \subset \mathcal{L}$$ (59)

$$\mathcal{T}_{k} := N \setminus M_{k} \times \mathcal{G} \subset \mathcal{T}$$ (60)

The compound branch impedances $\tilde{Z}_{t}$ and compound shunt admittances $\tilde{Y}_{t}$ of this fictional grid satisfy Hypothesis 3.
The grounding process does not affect the compound branch impedances. Therefore (see Fig. 5)

$$\ell \in L_k : Z_{\ell} = Z_{\ell}$$ (61)

Therefore, (24) of Hypothesis 3 obviously applies to $Z_{\ell}$, too. In contrast, the compound shunt admittances are modified, as some of the polyphase branches become polyphase shunts. The set $X_k$ containing these polyphase branches is given by

$$X_k := E_{\text{cut}}(\emptyset, M_k) \cup E_{\text{cut}}(\emptyset, N \setminus M_k)$$ (62)

Since $\emptyset$ is weakly connected, it holds that $X_k \neq \emptyset \forall k \in K$. Therefore, $\exists \ell \in X_k \forall k \in K$, which is a polyphase branch in the original grid, and a polyphase shunt in the modified grid. The compound branch impedances of the grounded polyphase branches contribute to the compound branch admittances of the modified grid. Namely (see Fig. 5)

$$\begin{bmatrix} t \in X_k, \quad t = (m, g) \end{bmatrix} : \tilde{Y}_t = Y_t + \sum_{\ell \in X_k} Y_{\ell} + \sum_{\ell \in X_k} Y_{\ell}$$ (63)

If the sums are empty, (25) of Hypothesis 3 obviously applies. Otherwise, it follows from Hypothesis 3 and Lemma 3 that $Y_t$ is symmetric and invertible, and has positive definite real part. Hence, (25) of Hypothesis 3 applies in this case, too.

As Hypothesis 3 holds, Theorem 1 can be applied. Due to the fact that $X_k \neq \emptyset \forall k \in K$, it follows that $\exists t \in X_k \forall k \in K$, for which $Y_t \neq 0$, even if $Y_t = 0 \forall t \in X_k$ (see Fig. 5). Thus

$$\text{rank}(Y_{M_k \times M_k}) = |M_k| \forall k \in K$$ (64)

Since $Y_{M \times M}$ is block diagonal with blocks $Y_{M_k \times M_k}$

$$\text{rank}(Y_{M \times M}) = \sum_{k \in K} \text{rank}(Y_{M_k \times M_k})$$

$$= |M| \sum_{k \in K} |M_k| = |M||M|$$ (66)

This proves the claim.

IV. IMPLICATIONS

Using the properties developed in the previous section, the findings of [23] can be extended to polyphase power systems. In Sec. IV-A it is is proven that Kron reduction is feasible for any subset of the polyphase nodes with zero current injections. In Sec. IV-B it is shown that hybrid parameters matrices exist for arbitrary partitions of the polyphase nodes.

A. Kron Reduction

Ohm’s law (33) establishes the link between injected current phasors and nodal voltage phasors through the grid. Obviously, the current injected into a polyphase node also depends on the phasors and nodal voltage phasors through the grid. Obviously, in Sec. IV-A it is shown that Kron reduction is feasible for any subset of the polyphase nodes with zero current injections. In Sec. IV-B it is shown that hybrid parameters matrices exist for arbitrary partitions of the polyphase nodes.

B. Kron Reduction

Ohm’s law (33) establishes the link between injected current phasors and nodal voltage phasors through the grid. Obviously, the current injected into a polyphase node also depends on the phasors and nodal voltage phasors through the grid. Obviously, in Sec. IV-A it is shown that Kron reduction is feasible for any subset of the polyphase nodes with zero current injections. In Sec. IV-B it is shown that hybrid parameters matrices exist for arbitrary partitions of the polyphase nodes.

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Therefore, $\hat{Y}_{M \times M}$ has full rank. Namely
\[
\text{rank}(\hat{Y}_{M \times M}) = |\mathcal{M}|\left|\mathcal{P}\right|
\] (82)
As $M \subseteq \mathcal{N}$ is arbitrary, this proves the second claim. □

Moreover, Corollary 1 itself has a fundamental implication.

**Observation 1.** Kron reduction preserves the property which enables its applicability in the first place (i.e., that every proper diagonal subblock of a compound nodal admittance matrix has full rank). Therefore, if $\mathcal{N}$ is partitioned as $\{\mathcal{N}_k\, | \ k \in \mathbb{K}\}$, the parts $\mathcal{N}_k$ can be reduced one after another, and the (partially or fully) reduced compound nodal admittance matrices obtained after every step of the reduction also exhibit the rank property.

Performing the reduction sequentially rather than “en bloc” is beneficial in terms of computational burden, because the Schur complement (10) requires a matrix inversion. This operation is computationally expensive, and scales poorly with problem size (even if the inverse is not computed explicitly).

**B. Hybrid Parameters**

Evidently, the circuit equations are in admittance form. Namely, the injected current and nodal voltage phasors appear in separate vectors, which are linked by the nodal admittance matrix $Y$. In power system analysis, it is often more convenient to write the circuit equations in hybrid form (if this is feasible). The corresponding system of linear equations features vectors composed of both voltage and current phasors, and is described by a so-called hybrid parameters matrix $H$. In this context, observe that there is no guarantee for the existence of a hybrid representation. Whether a suitable matrix $H$ exists, depends both on the nodal admittance matrix $Y$ of the grid and on the partition of the nodes underlying the hybrid representation.

Various researchers have treated the subject of hybrid parameters matrices for monophase grids. Some authors plainly describe how a hybrid parameters matrix can be built, provided that it exists at all (e.g., [38]–[40]). Others do provide criteria for the existence of hybrid parameters matrices, but only for some (i.e., at least one) partition of the nodes (e.g., [18]–[20]). One recent work establishes a criterion for arbitrary partitions of the nodes [23]. All of the aforementioned works only study the monophase case. Accordingly, those results may apply to balanced triphase grids (respectively, their equivalent positive-sequence networks), but not to generic unbalanced polyphase grids. In contrast, Corollary 2 ensures the existence of hybrid parameters matrices for arbitrary partitions of the polyphase nodes of a generic polyphase grid.

**Corollary 2.** Suppose that Theorem 2 applies. Let $\mathcal{M} \subseteq \mathcal{N}$ be non-empty, that is $M$ and $M_{\mathbb{K}} := \mathcal{N} \setminus M$ form a partition of $\mathcal{N}$. Then, there exists a compound hybrid parameters matrix $H$, and the grid is described by the hybrid multiport equations
\[
\begin{bmatrix}
    I_{M_{\mathbb{K}}} \\
    V_M
\end{bmatrix} = H \begin{bmatrix}
    V_{M_{\mathbb{K}}} \\
    I_M
\end{bmatrix}
\] (83)
The blocks of $H$ are given as follows
\[
H_{M \times M} = Y^{-1}_{M \times M}
\] (84)
\[
H_{M \times M_{\mathbb{K}}} = -Y^{-1}_{M \times M} Y_{M \times M_{\mathbb{K}}}
\] (85)
\[
H_{M_{\mathbb{K}} \times M} = Y_{M_{\mathbb{K}} \times M} Y^{-1}_{M \times M}
\] (86)
\[
H_{M_{\mathbb{K}} \times M_{\mathbb{K}}} = Y_{M_{\mathbb{K}} \times M_{\mathbb{K}}}
\] (87)

**Proof.** Write Ohm’s law for $I_M$
\[
I_M = Y_{M \times M_{\mathbb{K}}} V_{M_{\mathbb{K}}} + Y_{M \times M} V_M
\] (88)
By Theorem 2, $Y_{M \times M}$ has full rank, and is hence invertible. Define $Z_{M \times M} := Y^{-1}_{M \times M}$, and solve for $V_M$
\[
V_M = Y^{-1}_{M \times M} (I_M - Y_{M \times M_{\mathbb{K}}} V_{M_{\mathbb{K}}})
\] (89)
\[
= H_{M \times M} I_M + H_{M \times M_{\mathbb{K}}} V_{M_{\mathbb{K}}}
\] (90)
as claimed in [84]–[85]. Write Ohm’s law for $I_{M_{\mathbb{K}}}$, and substitute the above formula for $V_M$. This gives
\[
I_{M_{\mathbb{K}}} = Y_{M_{\mathbb{K}} \times M_{\mathbb{K}}} V_{M_{\mathbb{K}}} - Y_{M_{\mathbb{K}} \times M} V_M
\] (91)
\[
= Y_{M_{\mathbb{K}} \times M} Y^{-1}_{M \times M} I_M
\] (92)
\[
= +Y_{M_{\mathbb{K}} \times M} V_{M_{\mathbb{K}}}
\] (93)
as claimed in [86]–[87]. □

It is worth noting that compound hybrid parameters matrices also exist for Kron-reduced grids.

**Observation 2.** The existence of compound hybrid parameters matrices is based on the same rank property as the feasibility of Kron reduction. Since Kron reduction preserves this property (recall Observation 1), compound hybrid parameters matrices can be obtained from unreduced, partially reduced, and fully reduced compound nodal admittance matrices. In this regard, $Y$ can be replaced by $\hat{Y}$ in Corollary 2.

**V. CONCLUSIONS**

This paper examined the properties of the compound nodal admittance matrix of polyphase power systems, and illustrated their implications for power system analysis. Using the concept of compound electrical parameters, and exploiting the physical characteristics of electrical components, rank properties for the compound nodal admittance matrix and its diagonal subblocks were deduced. Notably, it was proven that the diagonal blocks have full rank if the grid is connected and lossy. Based on these findings, it was shown that the feasibility of Kron reduction and the existence of hybrid parameters are guaranteed in practice. Thus, this paper provided a rigorous theoretical foundation for the analysis of generic polyphase power systems.

**APPENDIX A**

**Power System Components**

**A. Transmission Lines**

Consider a transmission line with $|\mathcal{P}|$ phase conductors and one neutral conductor. Let $v(z, t)$ and $i(z, t)$ be the vectors of phase-to-neutral voltages and phase conductor currents, i.e.
\[
v(z, t) = \text{col}_{p \in \mathcal{P}} (v_p(z, t))
\] (94)
\[
i(z, t) = \text{col}_{p \in \mathcal{P}} (i_p(z, t))
\] (95)
where $z$ is the position along the line. If (i) the electromagnetic parameters of the line are state-independent, (ii) the conductors are parallel, and the perpendicular distance between any two of them is much shorter than the wavelength, (iii) the conductors have finite conductance, (iv) the electromagnetic field outside of the conductors produced by the charges and currents inside of them is purely transversal, and (v) the sum of the conductor currents is zero, Maxwell’s equations simplify to the so-called telegrapher’s equations (see [41])
\[
\partial_z v(z, t) = -\left( R' + L' \partial_t \right) i(z, t)
\] (96)
\[
\partial_t i(z, t) = -(G' + C' \partial_z) v(z, t)
\] (97)
If a transmission line is ideal and the ambient dielectric is uniform, it follows straightforward that the electrical parameters satisfy Hypothesis 3. When analyzing power systems operating at rated frequency, this holds irrespective of whether the polyphase transformer is represented by a -section equivalent circuit or inductors, which can be stepwise (dis)connected. This kind of devices are symmetrical w.r.t. the phases, and lossy. Hence, the compound electrical parameters satisfy Hypothesis 3 (see App. A-A), the compound electrical parameters (108) and (109)-(110) are symmetric, have positive definite real part, and are invertible (by Lemma 4). Accordingly, transmission lines equipped with series or shunt compensators satisfy Hypothesis 3.

B. Transformers

When analyzing power systems operating at rated frequency, transformers are represented by a T-section equivalent circuits, which correspond one-to-one to the composition of the devices (see [42]). Consider a polyphase transformer connecting two polyphase nodes, m, n Ξ N. Let m be its primary side, and n its secondary side. In this case, the parameters of the T-section equivalent circuit are given as:

\[ Z_{T,(m,n)} = R_{w,1} + jwL_{L,1} + jwB_{L,2} \]

The resistance matrices R_{w,1} and R_{w,2} represent the winding resistances, and the inductance matrices L_{L,1} and L_{L,2} the leakage inductances of the coils on the primary and secondary side, respectively. The former are positive diagonal, the latter are positive definite. The conductance matrix G_{h} represents the hysteresis losses, and the susctance matrix B_{m} the magnetization of the transformer’s core. The former is positive diagonal, the latter is positive definite. Therefore, according to Lemmata 4 & 5, Z_{T,(m,n)} and Y_{T,(m,n)} are invertible. This holds irrespectively of whether the polyphase transformer is built with one single multi-leg core or several separate cores. In the latter case, L_{L,1}, L_{L,2}, and B_{m} are diagonal, since separate cores are magnetically decoupled. In either case, the compound electrical parameters satisfy Hypothesis 3.

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C. FACTS Devices

There exist three families of FACTS devices, namely series compensators, shunt compensators, and combined series-and-shunt compensators (see [43]). Here, the two first cases are discussed. Let \( Z_{\Lambda,(m,n)} \), \( Y_{\Lambda,(m,n)} \) and \( Y_{\Lambda,(m,n)} \) be the compound electrical parameters describing a transmission line without compensation (see App. A-A). If a series compensator is installed, the compound branch impedance of the respective transmission line is altered. In the T-section equivalent circuit, this is reflected by adding the compound impedance \( Z_{\Gamma,(m,n)} \) of the compensator to the compound branch impedance of the transmission line (see Fig. 6a). Namely

\[ Z_{\Pi,(m,n)} = Z_{\Pi,(m,n)} + Z_{\Gamma,(m,n)} \]

If shunt compensators are installed, the compound admittances \( Y_{\Gamma,(m,n)} \) and \( Y_{\Gamma,(m,n)} \) of the compensators add to the compound shunt admittance of the transmission line (see Fig. 6b). Namely

\[ Y_{\Pi,(m,n)} = Y_{\Pi,(m,n)} + Y_{\Gamma,(m,n)} \]

Usually, such compensators are built from banks of capacitors or inductors, which can be stepwise (dis)connected. This kind of devices are symmetrical w.r.t. the phases, and lossy. Hence

\[
\begin{align*}
\text{series compensation:} & \quad Y_{\Gamma,(m,n)} = Y_{\Gamma,(m,n)}^T > 0 \\
\text{shunt compensation:} & \quad Y_{\Gamma,(m,n)} = Y_{\Gamma,(m,n)}^T > 0 \\
\end{align*}
\]

Since the compound electrical parameters of transmission lines satisfy Hypothesis 3 (see App. A-A), the compound electrical parameters (108) and (109)-(110) are symmetric, have positive definite real part, and are invertible (by Lemma 4). Accordingly, transmission lines equipped with series or shunt compensators satisfy Hypothesis 3.
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