Quasilinear Dirichlet Problems with Degenerated $p$-Laplacian and Convection Term

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Abstract: The paper develops a sub-supersolution approach for quasilinear elliptic equations driven by degenerated $p$-Laplacian and containing a convection term. The presence of the degenerated operator forces a substantial change to the functional setting of previous works. The existence and location of solutions through a sub-supersolution is established. The abstract result is applied to find nontrivial, nonnegative and bounded solutions.

Keywords: quasilinear elliptic problem; degenerated $p$-Laplacian; convection term; sub-supersolution; nonnegative solution

1. Introduction

In this paper, we study the following quasilinear elliptic problem

$$
\begin{aligned}
-\text{div}(a(x)|\nabla u|^{p-2}\nabla u) &= f(x, u, \nabla u) \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

(P)

on a bounded domain $\Omega \subset \mathbb{R}^N$ with $N \geq 2$ and $p \in (1, N)$. We assume that the boundary $\partial\Omega$ of $\Omega$ is locally Lipschitzian, i.e., each point of $\partial\Omega$ has a neighborhood whose intersection with $\partial\Omega$ is the graph of a Lipschitz continuous function. Throughout the text we denote by $|\cdot|$ and $\cdot \cdot$ the standard Euclidean norm and scalar product on $\mathbb{R}^N$, respectively. A main feature of the present work is that the leading part of the equation in (P) is the differential operator in divergence form $\text{div}(a(x)|\nabla u|^{p-2}\nabla u)$ known as the degenerated $p$-Laplacian with the weight $a \in L^1_{\text{loc}}(\Omega)$. It is supposed that the function $a$ is positive almost everywhere in $\Omega$ and that the following condition holds

$$a^{-s} \in L^1(\Omega) \quad \text{for some } s \in \left(\frac{N}{p}, +\infty\right) \cap \left[\frac{1}{p-1}, +\infty\right).$$

(1)

In the case where $a(x) \equiv 1$ we recover the ordinary $p$-Laplacian. Various examples of useful weights meeting the requirement (1) are given in [1]. For instance, it is obvious that defining $a(x) = \text{dist}(x, S)$ for $x \in \Omega$, with a nonempty closed subset $S$ of $\partial\Omega$, one obtains a function $a$ on $\Omega$ for which (1) holds true with any listed $s$.

The natural space associated with problem (P) is $W^{1,p}_0(\Omega)$ that is the closure of $C_0^\infty(\Omega)$ in the weighted Sobolev space $W^{1,p}(a, \Omega)$. In Section 2 we briefly survey the spaces $W^{1,p}_0(a, \Omega)$ and $W^{1,p}_0(a, \Omega)$. The (negative) degenerated $p$-Laplacian with the weight $a \in L^1_{\text{loc}}(\Omega)$ under condition (1) is defined on $W^{1,p}_0(a, \Omega)$ and takes values in the dual space $(W^{1,p}_0(a, \Omega))^\ast$.
Corresponding to the constant $s$ in (1) we set
\[ p_s = \frac{ps}{s+1} \]
and the Sobolev critical exponent $p^*_s = \frac{Np_s}{N-p_s}$ (we note that $1 \leq p_s < N$). There is a continuous embedding $W^{1,p}(a,\Omega) \hookrightarrow L^{p^*_s}(\Omega)$, so a continuous embedding $L^{p^*_s'}(\Omega) \hookrightarrow (W^{1,p}_0(a,\Omega))^*$, where $(p^*_s)'$ stands for the Hölder conjugate of $p^*_s$, i.e., $(p^*_s)' = \frac{p^*_s}{p^*_s-1}$. In order to handle problem (P) the idea is to arrange that the right-hand side $f(x,u,\nabla u)$ become an element of $L^{(p^*_s)'}(\Omega)$, which basically will be achieved through an adequate growth condition (see assumption (H)). We emphasize that the nonlinearity $f(x,u,\nabla u)$ depends on the solution $u$ and on its gradient $\nabla u$, which generally makes the variational methods be ineffective. Such a term $f(x,u,\nabla u)$ is often called convection. It is expressed by means of a function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ that is Carathéodory, i.e., $f(\cdot,t,\xi)$ is measurable for every $(t,\xi) \in \mathbb{R} \times \mathbb{R}^N$ and $f(x,\cdot,\cdot)$ is continuous for a.e. $x \in \Omega$.

The goal of our work is to build a systematical approach to problem (P) via the method of sub-supersolution. It is for the first time when the method of sub-supersolution is implemented for problem (P) involving the degenerated $p$-Laplacian and related convection. In this respect, the functional setting is adapted to the novel situation of degenerated operators relying in an essential way on the associated exponent $p_s$. For results on the method of sub-supersolution applied to problems exhibiting convection terms but not driven by degenerated differential operators we refer to [2–6].

By a (weak) solution to problem (P) we mean a function $u \in W^{1,p}_0(a,\Omega)$ such that $f(x,u,\nabla u) \in L^{(p^*_s)'}(\Omega)$ and
\[ \int_{\Omega} a(x)|\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla v(x) dx = \int_{\Omega} f(x,u(x),\nabla u(x))v(x) dx, \quad \forall v \in W^{1,p}_0(a,\Omega). \tag{2} \]

A function $\underline{u} \in W^{1,p}(a,\Omega)$ is called a subsolution for problem (P) if $\underline{u} \leq 0$ on $\partial \Omega$ (in the sense of traces), $f(\cdot,\underline{u}(\cdot),\nabla \underline{u}(\cdot)) \in L^{(p^*_s)'}(\Omega)$ and
\[ \int_{\Omega} a(x)|\nabla \underline{u}(x)|^{p-2}\nabla \underline{u}(x) \cdot \nabla v(x) dx \leq \int_{\Omega} f(x,\underline{u}(x),\nabla \underline{u}(x))v(x) dx \tag{3} \]
for all $v \in W^{1,p}_0(a,\Omega)$, $v \geq 0$ a.e. in $\Omega$. Symmetrically, a function $\overline{u} \in W^{1,p}(a,\Omega)$ is called a supersolution for problem (P) if $\overline{u} \geq 0$ on $\partial \Omega$ (in the sense of traces), $f(\cdot,\overline{u}(\cdot),\nabla \overline{u}(\cdot)) \in L^{(p^*_s)'}(\Omega)$ and
\[ \int_{\Omega} a(x)|\nabla \overline{u}(x)|^{p-2}\nabla \overline{u}(x) \cdot \nabla v(x) dx \geq \int_{\Omega} f(x,\overline{u}(x),\nabla \overline{u}(x))v(x) dx \tag{4} \]
for all $v \in W^{1,p}_0(a,\Omega)$, $v \geq 0$ a.e. in $\Omega$. Corresponding to a subsolution $\underline{u}$ and a supersolution $\overline{u}$ with $\underline{u} \leq \overline{u}$ a.e. in $\Omega$ we can consider the ordered interval
\[ [\underline{u},\overline{u}] = \{ w \in W^{1,p}(a,\Omega) : \underline{u} \leq w \leq \overline{u} \}. \]

The following hypothesis for $f(x,s,\xi)$ is adapted to an ordered sub-supersolution $\underline{u} \leq \overline{u}$.

**Hypothesis 1.** Given an ordered sub-supersolution $\underline{u} \leq \overline{u}$ for problem (P), the Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies the growth condition
\[ |f(x,t,\xi)| \leq \sigma(x) + b|\xi|^{r'} \quad \text{for a.e.} \ x \in \Omega, \ \text{for all} \ t \in [\underline{u}(x),\overline{u}(x)], \ \xi \in \mathbb{R}^N, \]
with a function $\sigma \in L^{\frac{p_s}{p_s-1}}(\Omega)$ and constants $b > 0$ and $r \in (0,\frac{p_s}{(p^*_s)'})$. 
According to Hypothesis 1 we have
\[ f(x, u, \nabla u) \in L^{(p')'}(\Omega), \ \forall u \in [\underline{u}, \overline{u}], \]
thus the integrals in the definitions above exist since
\[ f(x, u, \nabla u)v \in L^1(\Omega), \ \forall u \in [\underline{u}, \overline{u}], \ v \in W_0^{1,p}(a, \Omega). \]

Under Hypothesis 1, our main result establishes the existence of a weak solution to problem (P) with the additional location property \( u \in [\underline{u}, \overline{u}] \). We stress that this location property represents a significant qualitative information for the solution giving actually a priori estimates for it. As an application we prove the existence of a nontrivial nonnegative solution for a class of problems of type (P). The applicability of the stated result is demonstrated by an example.

2. Preliminary Material

The notation \(|\Omega|\) stands for the Lebesgue measure of the bounded domain \( \Omega \) in \( \mathbb{R}^N \). In this section we discuss a few facts about the degenerated \( p \)-Laplacian entering problem (P). More details can be found in [1].

We note that (1) implies
\[ a^{-\frac{1}{p-1}} \in L^1(\Omega). \]

Indeed, it is seen that
\[
\int_{\Omega} a(x)^{-\frac{1}{p-1}} dx = \int_{\{ a(x) < 1 \}} a(x)^{-\frac{1}{p-1}} dx + \int_{\{ a(x) \geq 1 \}} a(x)^{-\frac{1}{p-1}} dx
\leq \int_{\{ a(x) < 1 \}} a(x)^{-\frac{s}{p-1}} dx + |\Omega| < \infty
\]
since according to (1) one has \( s \geq \frac{1}{p-1} \) and \( a^{-\frac{s}{p-1}} \in L^1(\Omega) \).

The weighted Sobolev space \( W^{1,p}(a, \Omega) \) consists of all the functions \( u \in L^p(\Omega) \) for which \( a^{\frac{1}{p}} |\nabla u| \in L^p(\Omega) \). It is endowed with the norm
\[ \| u \|_{1,p,a} = \left( \int_{\Omega} |u|^p dx + \int_{\Omega} a(x)|\nabla u|^p dx \right)^{\frac{1}{p}} \]
becoming a uniformly convex Banach space (due to the preceding property of the weight \( a(x) \), see ([1], [Theorem 1.3])), thus reflexive, that contains \( C_0^\infty(\Omega) \). The space \( W_0^{1,p}(a, \Omega) \) is the closure of \( C_0^\infty(\Omega) \) with respect to the norm \( \| \cdot \|_{1,p,a} \).

There is an extensive literature devoted to the weighted Sobolev spaces including embeddings and traces related to different boundary value problems (see, e.g., [1,7,8]). The results depend strongly on what type of weight is used, generally attempting reduction to nonweighted spaces. As described below, under assumption (1), we can embed the space \( W^{1,p}(a, \Omega) \) into the ordinary Sobolev space \( W^{1,p_s}(\Omega) \), hence automatically having the trace (note the boundary \( \partial \Omega \) is Lipschitz). This fact is needed in the definition of the sub-supersolution.

From (1) it is known that \( s \geq \frac{1}{p-1} \), so one has \( p_s \geq 1 \) and the continuous embedding
\[ W^{1,p}(a, \Omega) \hookrightarrow W^{1,p_s}(\Omega), \tag{5} \]
which is relation (1.22) in [1]. More precisely, observing that \( p > p_s \), through Holder’s inequality and (1) we get
\[
\int_{\Omega} |\nabla u|^{p_s} dx = \int_{\Omega} a^{-\frac{p_s}{p}} a^{\frac{p_s}{p}} |\nabla u|^{p_s} dx \leq \left( \int_{\Omega} a^{-s} dx \right)^{\frac{s}{p}} \left( \int_{\Omega} a |\nabla u|^{p} dx \right)^{\frac{s}{p}}
\]

for all \( u \in W^{1,p}(a, \Omega) \). As a consequence of the above inequality, we can endow \( W^{1,p}_0(a, \Omega) \) with an equivalent norm
\[
\| u \| = \left( \int_\Omega a(x)|\nabla u|^pdx \right)^{\frac{1}{p}}
\]
for which it holds
\[
\| u \|_{W^{1,p}_0(\Omega)} \leq \| a^{-\frac{1}{p}} \|_{L^1(\Omega)} \| u \|. \tag{6}
\]

The Sobolev embedding theorem ensures the continuous embedding \( W^{1,p}_0(\Omega) \hookrightarrow L^p(\Omega) \), with the critical exponent \( p^*_s = \frac{Np_s}{N-p_s} \) (note that \( 1 \leq p_s < N \)). Hence there exists a constant \( T_0 > 0 \) such that
\[
\| u \|_{L^p(\Omega)} \leq T_0 \| u \|_{W^{1,p}_0(\Omega)} \quad \forall u \in W^{1,p}_0(\Omega). \tag{7}
\]

The best embedding constant \( T_0 \) has been estimated by Talenti \cite{9} as follows
\[
T_0 \leq \pi^{-\frac{1}{2}}N^{-\frac{1}{p}} \left( \frac{p_s - 1}{N - p_s} \right)^{1 - \frac{1}{p}} \left( \frac{\Gamma \left( 1 + \frac{N}{p} \right) \Gamma(N)}{\Gamma \left( \frac{N}{p} \right) \Gamma \left( 1 + \frac{N - N}{p} \right)} \right)^\frac{1}{N},
\]
where \( \Gamma \) is the Euler function
\[
\Gamma(t) = \int_0^{+\infty} z^{t-1}e^{-z}dz, \forall t > 0.
\]

Moreover, by the Rellich–Kondrachov compact embedding theorem, if \( 1 \leq r < p_s^* \) then the embedding \( W^{1,p}_0(\Omega) \hookrightarrow L^r(\Omega) \) is compact.

By (7) and Hölder’s inequality we infer that
\[
\| u \|_{L^r(\Omega)} \leq T_0 |\Omega|^{\frac{p^*_r - r}{p^*_r}} \| u \|_{W^{1,p}_0(\Omega)} \tag{8}
\]
for every \( u \in W^{1,p}_0(\Omega) \) and \( r \in [1, p^*_s] \). Combining (6) and (8) we arrive at
\[
\| u \|_{L^r(\Omega)} \leq \kappa_r \| u \| \tag{9}
\]
for all \( u \in W^{1,p}_0(a, \Omega) \) and \( r \in [1, p^*_s] \), with the constant
\[
\kappa_r = T_0 |\Omega|^{\frac{p^*_r - r}{p^*_r}} \| a^{-\frac{1}{p}} \|_{L^1(\Omega)}.
\]

The (negative) degenerated \( p \)-Laplacian with the weight \( a \in L^1_{\text{loc}}(\Omega) \) satisfying condition (1) is the operator \( A : W^{1,p}_0(a, \Omega) \to (W^{1,p}_0(a, \Omega))^* \) defined by
\[
\langle A(u), v \rangle = \int_\Omega a(x)|\nabla u|^{p-2}\nabla u \cdot \nabla v dx, \quad \forall u, v \in W^{1,p}_0(a, \Omega). \tag{10}
\]

We readily check that the operator \( A \) in (10) is well defined noticing by means of Hölder’s inequality that for all \( u, v \in W^{1,p}_0(a, \Omega) \) it holds
\[
\left| \int_\Omega a(x)|\nabla u|^{p-2}\nabla u \cdot \nabla v dx \right| \leq \int_\Omega a(x)^{\frac{p-1}{p}} |\nabla u|^{p-1}a(x)\frac{1}{p} |\nabla v| dx \tag{11}
\leq \left( \int_\Omega a(x)|\nabla u|^{p} dx \right)^{\frac{p-1}{p}} \left( \int_\Omega a(x)|\nabla v|^{p} dx \right)^{\frac{1}{p}} < \infty.
\]
Important properties of the operator $A$ introduced in (10) are listed in the statement below.

**Proposition 1.** Assume that the measurable function $a : \Omega \to \mathbb{R}$ satisfies condition (1). Then the (negative) degenerated p-Laplacian $A : W^{1,p}_0(a, \Omega) \to (W^{1,p}_0(a, \Omega))^*$ defined by (10) has the following properties:

(i) $A$ is a bounded operator in the sense that it maps bounded sets to bounded sets;

(ii) $A$ is a coercive operator, i.e.,
$$\lim_{\|u\| \to \infty} \frac{\langle Au, u \rangle}{\|u\|} = +\infty;$$

(iii) $A$ is a strictly monotone operator, i.e.,
$$\langle Au - Av, u - v \rangle > 0, \quad u \neq v;$$

(iv) $A$ has the $S_+$ property meaning that any sequence $\{u_n\} \subset W^{1,p}_0(a, \Omega)$ that satisfies $u_n \rightharpoonup u$ in $W^{1,p}_0(a, \Omega)$ and
$$\limsup_{n \to \infty} \langle A(u_n), u_n - u \rangle \leq 0 \quad (12)$$
is strongly convergent.

**Proof.** (i) From (10) and (11) we infer that
$$|\langle Au, v \rangle| = \left| \int_\Omega a(x)|\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx \right| \leq \|u\|^{p-1}\|v\|, \quad \forall u, v \in W^{1,p}_0(a, \Omega).$$

We obtain
$$\|Au\|_{(W^{1,p}_0(a,\Omega))^*} = \sup_{v \in W^{1,p}_0(a,\Omega), \|v\| \leq 1} |\langle Au, v \rangle| \leq \|u\|^{p-1}, \quad \forall u \in W^{1,p}_0(a, \Omega),$$
whence $A$ is bounded.

(ii) By (10) we have that
$$\langle Au, u \rangle = \int_\Omega a(x)|\nabla u|^{p} \, dx = \|u\|^p, \quad \forall u \in W^{1,p}_0(a, \Omega).$$

Taking into account that $p > 1$, it follows that the operator $A$ is coercive.

(iii) In view of the strict monotonicity of the mapping $\xi \mapsto \|\xi\|^{p-2}\xi$ on $\mathbb{R}^N$, it turns out
$$\langle Au - Av, u - v \rangle = \int_\Omega a(x) \left( |\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v \right) \cdot (\nabla u - \nabla v) \, dx > 0, \quad u \neq v,$$
so $A$ is a strictly monotone operator.

(iv) Let a sequence $\{u_n\} \subset W^{1,p}_0(a, \Omega)$ satisfy $u_n \rightharpoonup u$ in $W^{1,p}_0(a, \Omega)$ and (12). Using the monotonicity of the operator $A$ and (12) we have
$$\lim_{n \to \infty} \langle A(u_n) - A(u), u_n - u \rangle = 0.$$

Through Hölder’s inequality we obtain
≤ and a supersolution \( u \) with \( u \leq u \) and \( s \) related mappings. The cut-off function \( \pi \) possesses at least a solution \( u \in W^{1,p}_0(\Omega) \) from which we find that \( \lim_{n \to +\infty} \| u_n \| = \| u \| \). Due to the uniform convexity of \( W^{1,p}_0(\Omega) \) it follows that \( u_n \to u \) in \( W^{1,p}_0(\Omega) \), thus completing the proof. 

We also need the first eigenvalue \( \lambda_1 \) of the operator \( A : W^{1,p}_0(\Omega) \to (W^{1,p}_0(\Omega))^* \) in (10). Precisely, \( \lambda_1 > 0 \) is the least (positive) number for which the equation

\[
\begin{align*}
-\text{div}(a(x)|\nabla u|^{p-2}\nabla u) &= \lambda_1 |u|^{p-2}u & \text{in } \Omega \\
u &= 0 & \text{on } \partial\Omega
\end{align*}
\]  

(13)

admits a nontrivial solution called eigenfunction corresponding to the first eigenvalue \( \lambda_1 \). A solution to (13) is understood in the weak sense, i.e., \( u \in W^{1,p}_0(\Omega) \) satisfying

\[
\int_{\Omega} a(x)|\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla v(x)dx = \lambda_1 \int_{\Omega} |u(x)|^{p-2}u(x)v(x)dx, \forall v \in W^{1,p}_0(\Omega).
\]

It is known that there exists an eigenfunction \( u_1 \in W^{1,p}_0(\Omega) \) corresponding to the first eigenvalue \( \lambda_1 \) such that \( u_1(x) \geq 0 \) for a.e. \( x \in \Omega, u_1 \neq 0 \), and \( u_1 \in L^{\infty}(\Omega) \). For the proofs of these properties we refer to ([1], Chapter 3).

3. Main Results

Our main abstract result provides the existence of a solution to problem \((P)\) and its location within the ordered interval determined by a sub-supersolution.

**Theorem 1.** Let the weight \( a \in L^1_{\text{loc}}(\Omega) \) fulfill the requirement (1) and assume that the condition \((H)\) for a subsolution \( \underline{u} \) and a supersolution \( \overline{u} \) with \( \underline{u} \leq \overline{u} \) a.e. is satisfied. Then problem \((P)\) possesses at least a solution \( u \in W^{1,p}_0(\Omega) \) with the location property \( \underline{u} \leq u \leq \overline{u} \) for a.e. \( x \in \Omega \).

**Proof.** By means of the given sub-supersolution \( \underline{u} \leq \overline{u} \) for problem \((P)\), we introduce some related mappings. The cut-off function \( \pi : \Omega \times \mathbb{R} \to \mathbb{R} \) is defined by

\[
\pi(x,t) = \begin{cases} 
-\frac{(t - \overline{u}(x))^{r-s}}{r-s} & \text{if } t < \overline{u}(x) \\
0 & \text{if } \overline{u}(x) \leq t \leq \overline{u}(x) \\
\frac{(t - \underline{u}(x))^{r-s}}{r-s} & \text{if } t > \overline{u}(x),
\end{cases}
\]  

(14)

where \( s \) and \( r \) are the constants given in (1) and Hypothesis 1. Using (14) in conjunction with \( \underline{u}, \overline{u} \in L^p(\Omega) \) enables us to find that

\[
|\pi(x,t)| \leq c|t|^\frac{r-s}{r} + \varphi(x) \text{ for a.e. } x \in \Omega, \text{ all } t \in \mathbb{R},
\]  

(15)
with a constant $c > 0$ and a function $q \in L^{\frac{p}{p(r-1)}}(\Omega)$. Moreover, proceeding as in [4], we can establish that
\[
\int_{\Omega} \pi(x, u(x)) u(x) \, dx \geq b_1 \|u\|^p_{L^\frac{p}{p(r-1)}(\Omega)} - b_2 \quad \text{for all } u \in W^1_0(a, \Omega),
\]
with positive constants $b_1$ and $b_2$.

In view of (15), the Nemytskij operator $u \rightarrow \pi(\cdot, u(\cdot))$ generated by $\pi$ maps continuously $L^p(\Omega)$ to $L^{\frac{p}{p(r-1)}}(\Omega)$. Therefore, the mapping $\Pi : W^1_0(a, \Omega) \rightarrow (W^1_0(a, \Omega))^*$ defined by
\[
\langle \Pi(u), v \rangle = \int_{\Omega} \pi(x, u) v \, dx, \quad \forall u, v \in W^1_0(a, \Omega)
\]
is completely continuous. This is true because the inclusion $L^{\frac{p}{p(r-1)}}(\Omega) \subset (W^1_0(a, \Omega))^*$ is compact being the adjoint of the compact inclusion $W^1_0(a, \Omega) \subset L^{\frac{p}{p(r-1)}}(\Omega)$ (note that $\frac{p}{p(r-1)} < q^*$ owing to the assumption $r \in (0, \frac{p}{p(r-1)})$ in (H)).

Hypothesis (H) and (5) imply that the Nemytskij operator $u \mapsto f(\cdot, u(\cdot), \nabla u(\cdot))$ maps continuously $\bar{u}, \bar{w} \in W^1_0(a, \Omega)$ to $L_{\neq}^p(\Omega)$ with $r \in (0, \frac{p}{p(r-1)})$. Composing the preceding Nemytskij operator with the inclusion $L^p(\Omega) \subset (W^1_0(a, \Omega))^*$, which is compact because it is the adjoint operator of the compact inclusion $W^1_0(a, \Omega) \subset L^p(\Omega)$ (note that $\frac{p}{p(r-1)} < q^*$ since $r \in (0, \frac{p}{p(r-1)})$ in (H)), we obtain a completely continuous mapping $N_f : \bar{u}, \bar{w} \mapsto (W^1_0(a, \Omega))^*$ given by
\[
\langle N_f(u), v \rangle = \int_{\Omega} f(x, u(x), \nabla u(x)) v(x) \, dx
\]
for all $u \in \bar{u}, \bar{w}$ and $v \in W^1_0(a, \Omega)$.

We also make use of the truncation operator $T : W^1_0(a, \Omega) \rightarrow W^1_0(a, \Omega)$ given by
\[
(Tu)(x) = \begin{cases} 
  w(x) & \text{if } u(x) < w(x) \\
  u(x) & \text{if } w(x) \leq u(x) \leq \bar{w}(x) \\
  \bar{w}(x) & \text{if } u(x) > \bar{w}(x)
\end{cases}
\]
for all $u \in W^1_0(a, \Omega)$ and a.e. $x \in \Omega$. It is a continuous and bounded mapping (in the sense that it maps bounded sets to bounded sets). Notice that its range lies in $\bar{u}, \bar{w}$, so $T$ can be composed with the operator $N_f$.

Now we consider for every $\lambda > 0$ the operator $A_\lambda : W^1_0(a, \Omega) \rightarrow (W^1_0(a, \Omega))^*$ defined by
\[
A_\lambda = A + \lambda \Pi - N_f \circ T.
\]
Explicitly, it reads as
\[
\langle A_\lambda(u), v \rangle = \int_{\Omega} a(x)|\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx + \lambda \int_{\Omega} \pi(x, u) v \, dx - \int_{\Omega} f(x, Tu, \nabla (Tu)) v \, dx \quad \text{for all } u, v \in W^1_0(a, \Omega).
\]

From Proposition 1(i) it is known that the operator $A : W^1_0(a, \Omega) \rightarrow (W^1_0(a, \Omega))^*$ is bounded, while the above comments demonstrate that the operators $\Pi, N_f$ and $T$ are all of them bounded. Therefore from (18) we infer that the operator $A_\lambda : W^1_0(a, \Omega) \rightarrow (W^1_0(a, \Omega))^*$ is bounded.
We claim that $A_\lambda : W^{1,p}_0(a, \Omega) \to (W^{1,p}_0(a, \Omega))^*$ is a pseudomonotone operator. In this respect, let a sequence $\{u_n\} \subset W^{1,p}_0(a, \Omega)$ satisfy $u_n \rightharpoonup u$ in $W^{1,p}_0(a, \Omega)$ and

$$\lim_{n \to \infty} \langle A_\lambda(u_n), u_n - u \rangle \leq 0.$$  \hfill (20)

The sequence $\{\Pi(u_n)\}$ is bounded in $L^{p/(p-1)}(\Omega)$, while $u_n \rightharpoonup u$ in $L^{p/(p-1)}(\Omega)$ by the compact embedding $W^{1,p}_0(a, \Omega) \subset L^{p/(p-1)}(\Omega)$, thus

$$\lim_{n \to \infty} (\Pi(u_n), u_n - u) = 0.$$  

The sequence $\{N_f \circ T(u_n)\}$ is bounded in $L^p(\Omega)$, while $u_n \rightharpoonup u$ in $L^{p/(p-1)}(\Omega)$ by the compact embedding $W^{1,p}_0(a, \Omega) \subset L^{p/(p-1)}(\Omega)$, producing

$$\lim_{n \to \infty} (N_f \circ T(u_n), u_n - u) = 0.$$  

Consequently, complying with (18), we see that (20) reduces to (12). This, in conjunction with the weak convergence $u_n \rightharpoonup u$, enables us to apply Proposition 1(iv) ensuring that the strong convergence $u_n \to u$ in $W^{1,p}_0(a, \Omega)$ holds.

From the strong convergence $a(\cdot)^{\frac{1}{p}} \nabla u_n(\cdot) \to a(\cdot)^{\frac{1}{p}} \nabla u(\cdot)$ in $(\mathcal{L}^p(\Omega))^N$ it follows the strong convergence $a(\cdot)^{\frac{p-1}{p}} |\nabla u_n(\cdot)|^{p-2} \nabla u_n(\cdot) \to a(\cdot)^{\frac{p-1}{p}} |\nabla u(\cdot)|^{p-2} \nabla u(\cdot)$ in $(\mathcal{L}^{p/(p-1)}(\Omega))^N$. This amounts to saying that $A_{u_n} \rightharpoonup A_{u}$ in $(W^{1,p}_0(a, \Omega))^*$ since

$$\langle A_{u_n}, v \rangle = \int_\Omega a(x)|\nabla u_n|^p \nabla u_n \cdot \nabla v dx \to \int_\Omega a(x)|\nabla u|^p \nabla u \cdot \nabla v dx = \langle A_{u}, v \rangle, \forall v \in W^{1,p}_0(a, \Omega).$$

Again, from the strong convergence $a(\cdot)^{\frac{1}{p}} \nabla u_n(\cdot) \to a(\cdot)^{\frac{1}{p}} \nabla u(\cdot)$ in $(\mathcal{L}^p(\Omega))^N$ we infer that

$$\langle A_{u_n}, u_n \rangle = \int_\Omega a(x)|\nabla u_n|^p dx \to \int_\Omega a(x)|\nabla u|^p dx = \langle A_{u}, u \rangle$$

as $n \to \infty$. Taking into account the continuity of the mappings $\Pi$ and $N_f \circ T$, we have

$$\langle A_\lambda u_n, v \rangle \to \langle A_\lambda u, v \rangle, \forall v \in W^{1,p}_0(a, \Omega),$$

and

$$\langle A_\lambda u_n, u_n \rangle \to \langle A_\lambda u, u \rangle$$

as $n \to \infty$, for every $\lambda > 0$. We can conclude that $A_\lambda : W^{1,p}_0(a, \Omega) \to (W^{1,p}_0(a, \Omega))^*$ is a pseudomonotone operator (see, e.g., ([2], Definition 2.97)).

The next step in the proof is to show that the operator $A_\lambda : W^{1,p}_0(a, \Omega) \to (W^{1,p}_0(a, \Omega))^*$ is coercive provided $\lambda > 0$ is large enough. Taking advantage of the fact that $T u \in [\mathcal{W}, \mathcal{W}]$ whenever $u \in W^{1,p}_0(a, \Omega)$, let us note by (16), (19) and Hypothesis 1 that

$$\langle A_\lambda(u), u \rangle = \langle A(u), u \rangle + \lambda \int_{\Omega} \pi(x, u) u dx - \int_{\Omega} f(x, T u, \nabla (T u)) u dx \geq \|u\|^p + \lambda(b_1 \|u\|_{L^{p/(p-1)}(\Omega)}^{p/(p-1)} - b_2) - \|\sigma\|_{L^{p/(p-1)}(\Omega)} \|u\|_{L^{p/(p-1)}(\Omega)}^{p/(p-1)} - b \int_{\Omega} |\nabla (T u)|' |u| dx \quad (21)$$
for all $u \in W^{1,p}_0(a, \Omega)$. Now we estimate the last term in (21) based on the fact that by (5) we know that $\nabla u \in (L^p(\Omega))^N$, and so $\nabla (Tu) \in (L^p(\Omega))^N$. Using the definition of $Tu$ in (17), Hölder’s inequality and the continuous embedding in (9) it turns out that

$$
\int_{\Omega} |\nabla (Tu)|^p |u| dx = \int_{\{u \leq \pi\}} |\nabla u|^p |u| dx + \int_{\{u \leq \pi\}} |\nabla u|^p |u| dx + \int_{\{u > \pi\}} |\nabla u|^p |u| dx \leq \int_{\Omega} |\nabla u|^p |u| dx + c_1 ||u||_p \forall u \in W^{1,p}_0(a, \Omega),
$$

with a constant $c_1 > 0$. We can insert the preceding inequality in (21) to derive

$$
\langle A_{\lambda}(u), u \rangle \geq ||u||^p + \lambda (b_1 ||u||^{p_1}_{L^{p_1}(\Omega)} - b_2) - c_2 ||u|| - b \int_{\Omega} |\nabla u|^p |u| dx,
$$

with a constant $c_2 > 0$. The Hölder’s and Young’s inequalities in conjunction with embedding (5) imply

$$
\int_{\Omega} |\nabla u|^p |u| dx \leq ||u||^{p_1}_{L^{p_1}(\Omega)} ||u||^{p_2}_{L^{p_2}(\Omega)} \leq c_3 ||u||^{p_1} + c_4 ||u||^{p_2}_{L^{p_2}(\Omega)},
$$

with constants $c_3 > 0$ and $c_4 > 0$. Then (22) entails

$$
\langle A_{\lambda}(u), u \rangle \geq ||u||^p + \lambda (b_1 ||u||^{p_1}_{L^{p_1}(\Omega)} - b_2) - c_2 ||u|| - b(c_3 ||u||^{p_1} + c_4 ||u||^{p_2}_{L^{p_2}(\Omega)})
$$

for all $u \in W^{1,p}_0(a, \Omega)$. Recalling from (16) that $b_1 > 0$, we can choose $\lambda > 0$ so large to have $\lambda b_1 > b c_4$. Hence due to $p > p_3 \geq 1$ (see (1)), (23) yields the coercivity of $A_{\lambda}$, i.e.,

$$
\lim_{||u|| \to +\infty} \frac{\langle A_{\lambda}(u), u \rangle}{||u||} = +\infty.
$$

We have shown that the nonlinear operator $A_{\lambda} : W^{1,p}_0(a, \Omega) \to (W^{1,p}_0(a, \Omega))^\ast$ is bounded, pseudomonotone and coercive provided $\lambda > 0$ is sufficiently large. Therefore, for such an $A_{\lambda}$ we can apply the main theorem of pseudomonotone operators (see, e.g., ([2], Theorem 2.99)) ensuring that there exists a solution $u \in W^{1,p}_0(a, \Omega)$ to the equation

$$
A_{\lambda}(u) = 0.
$$

Fix an admissible $\lambda > 0$ as pointed out above. We are going to prove that $u \in W^{1,p}_0(a, \Omega)$ resolving (24) is a weak solution of the original problem $(P)$, which means that (2) is satisfied. To this end, notice that (19) and (24) yield

$$
\int_{\Omega} a(x)|\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla v(x) dx + \lambda \int_{\Omega} \pi(x, u) v dx = \int_{\Omega} f(x, Tu, \nabla (Tu)) v dx \text{ for all } v \in W^{1,p}_0(a, \Omega).
$$

We proceed by comparing $u$ with the subsolution $\underline{u}$ and supersolution $\bar{u}$ postulated in assumption (H). We claim that $u \leq \bar{u}$ a.e. in $\Omega$. Towards this, it can be readily checked that $(u - \bar{u})^+ = \max\{u - \bar{u}, 0\} \in W^{1,p}_0(a, \Omega)$, where the condition $\bar{u} \geq 0$ on $\partial \Omega$ in the sense of traces is essentially used. Thus, we can insert $v = (u - \bar{u})^+$ in (25) and (4) which gives

$$
\int_{\Omega} a(x)|\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla (u - \bar{u})^+(x) dx + \lambda \int_{\Omega} \pi(x, u(x)) (u - \bar{u})^+(x) dx = \int_{\Omega} f(x, Tu(x), \nabla (Tu(x))(u - \bar{u})^+(x) dx
$$

(26)
and
\[
\int_{\Omega} a(x)|\nabla \pi(x)|^{p-2} \nabla \pi(x) \cdot \nabla (u - \pi)^+ (x) dx \geq \int_{\Omega} f(x, \pi(x), \nabla \pi(x))(u - \pi)^+(x) dx. \quad (27)
\]

From (26) and (27), by subtraction we are led to
\[
\int_{\Omega} a(x)\left(\nabla u(x)|^{p-2} \nabla u(x) - |\nabla \pi(x)|^{p-2} \nabla \pi(x)\right) \cdot \nabla (u - \pi)^+ (x) dx + \lambda \int_{\Omega} \pi(x, u(x))(u - \pi)^+(x) dx
\]
\[
\leq \int_{\Omega} \left(f(x, Tu(x), \nabla (Tu))(x) - f(x, \pi(x), \nabla \pi(x))\right)(u - \pi)^+(x) dx.
\]

By (14), (17), and the preceding inequality we get
\[
\int_{\{u > \pi\}} a(x)\left(\nabla u(x)|^{p-2} \nabla u(x) - |\nabla \pi(x)|^{p-2} \nabla \pi(x)\right) \cdot \nabla (u - \pi)^+(x) dx + \lambda \int_{\{u > \pi\}} (u - \pi)(x) \frac{\pi}{\pi - \pi^+} dx
\]
\[
\leq \int_{\{u > \pi\}} \left(f(x, Tu, \nabla (Tu)) - f(x, \pi, \nabla \pi)\right)(u - \pi)^+ dx = 0.
\]

Since the function \(a(x)\) is positive almost everywhere in \(\Omega\) and the mapping \(\xi \mapsto |\xi|^{p-2}\xi\) on \(\mathbb{R}^N\) is monotone, we arrive at
\[
\int_{\{u > \pi\}} (u - \pi)(x) \frac{\pi}{\pi - \pi^+} dx \leq 0.
\]

Therefore, the Lebesgue measure of the set \(\{u > \pi\}\) is zero, i.e., \(u \leq \pi\) a.e. in \(\Omega\).

Similarly, we can prove that \(u \leq u^\#\) a.e. in \(\Omega\). Specifically, relying on the condition \(u^\# \leq 0\) on \(\partial \Omega\) (in the sense of traces), it holds \((u-u)^+ = \max\{u-u^\#, 0\} \in W_0^{1,p}(a, \Omega)\), which allows us to test (25) and (3) with \(v = (u-u)^+ \in W_0^{1,p}(a, \Omega)\). This results in
\[
\int_{\Omega} a(x)|\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla (u - u^+)^+(x) dx + \lambda \int_{\Omega} \pi(x, u(x))(u - u^+)(x) dx
\]
\[
= \int_{\Omega} f(x, Tu(x), \nabla (Tu)(x))(u - u^+)(x) dx
\]
\[
(28)
\]

and
\[
\int_{\Omega} a(x)|\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla (u - u^+)^+(x) dx \leq \int_{\Omega} f(x, u(x), \nabla u(x))(u - u^+)(x) dx. \quad (29)
\]

Arguing as before, we deduce from (28), (29), (14), and (17) the following estimate
\[
\int_{\{u > u^\#\}} a(x)\left(\nabla u(x)|^{p-2} \nabla u(x) - |\nabla u(x)|^{p-2} \nabla u(x)\right) \cdot \nabla (u - u^\#)(x) dx + \lambda \int_{\{u > u^\#\}} (u - u^\#)(x) \frac{u^\#}{\pi - \pi^+} dx
\]
\[
\leq \int_{\Omega} \left(f(x, u, \nabla u) - f(x, Tu, \nabla (Tu))(u - u^\#)^+ dx
\]
\[
= \int_{\{u > u^\#\}} \left(f(x, u, \nabla u) - f(x, Tu, \nabla (Tu))(u - u^\#)^+ dx = 0.
\]

At this point, the positivity of the function \(a(x)\) on \(\Omega\) and the monotonicity of the mapping \(\xi \mapsto |\xi|^{p-2}\xi\) on \(\mathbb{R}^N\) confirm that
\[
\int_{\{u > u^\#\}} (u(x) - u(x)) \frac{u^\#}{\pi - \pi^+} dx \leq 0,
\]
from which we can readily derive that \(u \leq u^\#\) a.e in \(\Omega\).

Based on the enclosure property \(u^\# \leq u \leq \pi\) a.e. in \(\Omega\), it follows through (17) that \(T(u) = u\) and through (14) that \(\Pi(u) = 0\). As a result, (25) takes the form of (2), thus the proof is complete. \(\square\)

Now we present an application of Theorem 1 describing how the existence of a nontrivial nonnegative solution can be established by effectively determining a sub-supersolution. In the sequel, by \(\lambda_1\) we denote the first eigenvalue of problem (13) (see Section 2).
Theorem 2. Let the weight $a \in L^1_{\text{loc}}(\Omega)$ fulfill the requirement (1). Assume that the Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies the conditions:

(i) there is a constant $\mu > 0$ such that

$$\lambda_1^{p-1} \leq f(x, t, \xi) \text{ for a.e. } x \in \Omega, \text{ all } t \in [0, \mu], \xi \in \mathbb{R}^N;$$

(ii) there is a constant $C > 0$ such that

$$f(x, C, 0) \leq 0 \text{ for a.e. } x \in \Omega;$$

(iii) there is a function $\sigma \in L^p_{\text{loc}}(\Omega)$ and constants $b > 0$ and $r \in (0, \frac{p}{p-1})$ such that

$$|f(x, t, \xi)| \leq \sigma(x) + b|\xi|^r \text{ for a.e. } x \in \Omega, \text{ all } t \in [0, C], \xi \in \mathbb{R}^N.$$

Then problem (P) has a nontrivial, nonnegative and bounded weak solution $u \in W^{1,p}_0(a, \Omega)$ satisfying the estimate $u \leq C$.

Proof. Our goal is to apply Theorem 1 by constructing an appropriate sub-supersolution. In order to determine a subsolution, we use an eigenfunction $u_1 \in W^{1,p}_0(a, \Omega)$ corresponding to the first eigenvalue $\lambda_1$ of problem (13) with the properties $u_1(x) \geq 0$ for a.e. $x \in \Omega$, $u_1 \not\equiv 0$, and $u_1 \in L^\infty(\Omega)$ as mentioned in Section 2. Then we choose an $\varepsilon > 0$ sufficiently small to verify

$$\varepsilon u_1(x) \leq \mu \text{ for a.e. } x \in \Omega, \quad (30)$$

where $\mu$ is the positive constant postulated in assumption (i). Then assumption (ii) implies

$$\lambda_1(\varepsilon u_1)^{p-1} \leq f(x, \varepsilon u_1, \nabla(\varepsilon u_1)) \text{ for a.e. } x \in \Omega. \quad (31)$$

For a possibly smaller $\varepsilon > 0$ we can suppose

$$\varepsilon u_1(x) \leq C \text{ for a.e. } x \in \Omega, \quad (32)$$

with $C > 0$ in assumption (iii).

Let us fix an $\varepsilon > 0$ for which (30) and (32) are fulfilled. We claim that $\underline{u} = \varepsilon u_1$ is a subsolution to problem (P). Indeed, by (13) with $u_1$ in place of $u$ and (31) we note that

$$\int_{\Omega} a(x)|\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla v(x)dx = \varepsilon^{p-1}\lambda_1 \int_{\Omega} u_1(x)^{p-1}v(x)dx$$

$$\leq \int_{\Omega} f(x, \varepsilon u_1(x), \nabla(\varepsilon u_1)(x))v(x)dx = \int_{\Omega} f(x, u(x), \nabla u(x))v(x)dx$$

for all $v \in W^{1,p}_0(a, \Omega)$, $v \geq 0$ a.e. in $\Omega$, thereby proving the claim.

Next we claim that the constant function $\overline{u} = C$, with $C > 0$ in assumption (iii), is a supersolution to problem (P). Accordingly, from assumption (iii) we find that

$$\int_{\Omega} a(x)|\nabla \overline{u}(x)|^{p-2}\nabla \overline{u}(x) \cdot \nabla v(x)dx = 0 \geq \int_{\Omega} f(x, C, 0)v(x)dx = \int_{\Omega} f(x, \overline{u}(x), \nabla \overline{u}(x))v(x)dx$$

for all $v \in W^{1,p}_0(a, \Omega)$, $v \geq 0$ a.e. in $\Omega$, which proves the claim.

It is clear from (32) that $\underline{u}(x) \leq \overline{u}(x)$ for a.e. in $\Omega$. Assumption (iii) ensures that the growth condition required in Hypothesis 1 of Theorem 1 holds true. Therefore, all the hypotheses of Theorem 1 are verified, which permits the conclusion that there exists a solution $u \in W^{1,p}_0(a, \Omega)$ of problem (P) within the ordered interval $[\underline{u}, \overline{u}]$. Since the function $\overline{u} = \varepsilon u_1$ is nontrivial and nonnegative, and $u \geq \underline{u}$, we have that $u$ is nontrivial and
nonnegative, whereas \( u \in [\underline{u}, \overline{u}] \) renders the boundedness of \( u \) and the a priori estimate \( u \leq C \). The proof is complete. \( \square \)

We end the paper with a simple example for which Theorem 2 applies.

**Example 1.** Fix a positive weight \( a \in L^1_{\text{loc}}(\Omega) \) with the property (1). Let the function \( f: \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R} \) be defined by

\[
f(x,t,\xi) = \begin{cases} 
0 & \text{if } t < 0 \\
 t^{p-1}(\rho(x) + |\xi|^r) & \text{if } 0 \leq t \leq 1 \\
 (2-t)(\rho(x) + |\xi|^r) & \text{if } t > 1,
\end{cases}
\]

with some \( r \in [1, \frac{p}{p^*}] \) and \( \rho \in L^\infty(\Omega) \) satisfying \( \rho(x) \geq \lambda_1 \) for a.e. \( x \in \Omega \). It follows that \( f \) is a Carathéodory function for which conditions (j) – (jjj) in Theorem 2 are verified. Precisely, condition (j) holds with \( \mu = 1 \) because \( \rho(x) \geq \lambda_1 \), condition (jj) holds with \( C = 2 \), and condition (jjj) is fulfilled with the given \( r \). Hence Theorem 2 applies to problem (P) whose equation has the right-hand side expressed with the function \( f(x,t,\xi) \) given above.

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