In 70’s there was discovered a construction how to attach to some algebraic-geometric data an infinite-dimensional subspace in the space $k((z))$ of the Laurent power series. The construction was successfully used in the theory of integrable systems, particularly, for the KP and KdV equations [10, 19]. There were also found some applications to the moduli of algebraic curves [2, 3]. Now it is known as the Krichever correspondence or the Krichever map [2, 11, 1, 17, 4]. The original work by I. M. Krichever has also included commutative rings of differential operators as a third part of the correspondence.

The map we want to study here was first described in an explicit way by G. Segal and G. Wilson [19]. They have used an analytical version of the infinite dimensional Grassmanian introduced by M. Sato [18, 16]. In the sequel we consider a purely algebraic approach as developed in [11].

Let us just note that the core of the construction is an embedding of the affine coordinate ring on an algebraic curve into the field $k((z))$ corresponding to the power decompositions in a point at infinity (the details see below in section 2). In number theory this corresponds to an embedding of the ring of algebraic integers to the fields $\mathbb{C}$ or $\mathbb{R}$. The latter one is well known starting from the XIX-th century. The idea introduced by Krichever was to insert the local parameter $z$. This trick looking so simple enormously extends the area of the correspondence. It allows to consider all algebraic curves simultaneously.

But there still remained a hard restriction by the case of curves, so by dimension 1. Recently, it was pointed out by the author [13] that there are some connections between the theory of the KP-equations and the theory of $n$-dimensional local fields [13, 5]. From this point of view it becomes clear that the Krichever construction should have a generalization to the case of higher dimensions. This
generalization is suggested in the paper for the case of algebraic surfaces (see theorem 3 in section 2). A further generalization to the case of arbitrary dimension was recently proposed by D. V. Osipov [12].

Let us also note that the construction of the restricted adelic complex in section 1 is of an independent interest, also in arithmetics. It has already appeared in a description of vector bundles on algebraic surfaces [14].

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1 Adelic Complexes

We first discuss the adelic complexes for the case of dimension 1. Concerning a definition of the adelic notions we refer to [3], [9]. We also note that the sign \( \prod \) denotes the adelic product.

Let \( C \) be an projective algebraic curve over a field \( k \), \( P \) be a smooth point and \( \eta \) a general point on \( C \). Let \( \mathcal{F} \) be a torsion free coherent sheaf on \( C \).

**Proposition 1.** The following complexes are quasi-isomorphic:

i) adelic complex

\[ \mathcal{F}_\eta \oplus \prod_{x \in C} \hat{\mathcal{F}}_x \longrightarrow \prod_{x \in C} (\hat{\mathcal{F}}_x \otimes \hat{O}_x K_x) \]

ii) the complex

\[ W \oplus \hat{\mathcal{F}}_P \longrightarrow \hat{\mathcal{F}}_P \otimes \hat{O}_P K_P \]

where \( W = \Gamma(C \setminus P, \mathcal{F}) \subset \hat{\mathcal{F}}_\eta \).

**Proof** will be done in two steps. First, the adelic complex contains a trivial exact subcomplex

\[ \prod_{x \in U} \hat{\mathcal{F}}_x \longrightarrow \prod_{x \in U} \hat{\mathcal{F}}_x, \]

where \( U = C \setminus P \). The quotient-complex is equal to

\[ \mathcal{F}_\eta \oplus \hat{\mathcal{F}}_P \longrightarrow \prod_{x \in U} (\hat{\mathcal{F}}_x \otimes K_x) / \hat{\mathcal{F}}_x \oplus \hat{\mathcal{F}}_P. \]

It has a surjective homomorphism to the exact complex

\[ \mathcal{F}_\eta / W \longrightarrow \prod_{x \in U} (\hat{\mathcal{F}}_x \otimes K_x) / \hat{\mathcal{F}}_x. \]
The exactness of the complex is the strong approximation theorem for the curve \( C \) \( [3] \) [ch. II, \$3, corollary of prop. 9; ch. VII, \$8, prop. 2]. The kernel of this surjection will be the second complex from proposition.

Now we go to the case of dimension 2. Let \( X \) be a projective irreducible algebraic surface over a field \( k \), \( C \subset X \) be an irreducible projective curve, and \( P \in C \) be a smooth point on both \( C \) and \( X \). Let \( \mathcal{F} \) be a torsion free coherent sheaf on \( X \).

**Definition 1.** Let \( x \in C \). We let

\[
B_x(\mathcal{F}) = \bigcap_{D \neq C} ((\hat{\mathcal{F}}_x \otimes K_x) \cap (\hat{\mathcal{F}}_x \otimes \hat{\mathcal{O}}_{x,D})),
\]

where the intersection is done inside the group \( \hat{\mathcal{F}}_x \otimes K_x \),

\[
B_C(\mathcal{F}) = (\hat{\mathcal{F}}_C \otimes K_C) \cap \bigcap_{x \neq P} B_x,
\]

where the intersection is done inside \( \hat{\mathcal{F}}_x \otimes K_{x,C} \),

\[
A_C(\mathcal{F}) = B_C(\mathcal{F}) \cap \hat{\mathcal{F}}_C,
\]

\[
A(\mathcal{F}) = \hat{\mathcal{F}}_\eta \cap \bigcap_{x \in X-C} \hat{\mathcal{F}}_x.
\]

We will freely use the following shortcuts:

\[
\begin{align*}
K\hat{\mathcal{F}}_x &= \hat{\mathcal{F}}_x \otimes \hat{\mathcal{O}}_x K_x, \\
K\hat{\mathcal{F}}_D &= \hat{\mathcal{F}}_D \otimes \hat{\mathcal{O}}_D K_D, \\
\hat{\mathcal{F}}_{x,D} &= \hat{\mathcal{F}}_x \otimes \hat{\mathcal{O}}_x \mathcal{O}_{x,D}, \\
K\hat{\mathcal{F}}_{x,D} &= \hat{\mathcal{F}}_x \otimes \hat{\mathcal{O}}_x K_{x,D}.
\end{align*}
\]

Next, we need two lemmas connecting the adelic complexes on \( X \) and \( C \). They are the versions of the relative exact sequences, see [3], [6]. The curve \( C \) defines the following ideals:

\[
\begin{align*}
K_{x,C} &\supset \hat{\mathcal{O}}_{x,C} \supset \varphi^n_{x,C} \supset \ldots, \\
K_C &\supset \hat{\mathcal{O}}_C \supset \varphi^n_C \supset \varphi^{n+1}_C \supset \ldots, \\
K_x &\supset \hat{\mathcal{O}}_x \supset \varphi^n_x \ldots,
\end{align*}
\]

and \( \varphi_x = \hat{\mathcal{O}}_x \cap \varphi_{x,C} \).

**Lemma 1.** We assume that the curve \( C \) is a locally complete intersection. Let \( N_{X/C} \) be the normal sheaf for the curve \( C \) in \( X \). For all \( n \in \mathbb{Z} \) the maps

\[
\prod_{x \in C} \varphi^n_{x,C} \hat{\mathcal{F}}_{x,C}/\varphi^{n+1}_{x,C} \hat{\mathcal{F}}_{x,C} \longrightarrow A_{C,01}(\mathcal{F} \otimes \hat{N}_{X/C}^\otimes n),
\]
\[
\prod_{x,C} \varphi_x^n \hat{F}_x / \varphi_x^{n+1} \hat{F}_x \rightarrow A_{C,1}(\mathcal{F} \otimes \mathcal{N}_{X/C}^{\otimes n}),
\]
\[
\varphi^n_C \hat{F}_C / \varphi^{n+1}_C \hat{F}_C \rightarrow A_{C,0}(\mathcal{F} \otimes \mathcal{N}_{X/C}^{\otimes n}),
\]
are bijective.

In general, we have an exact sequence
\[
0 \rightarrow J^{n+1} \rightarrow J^n \rightarrow J^n|_C \rightarrow 0
\]
where \( J \subset \mathcal{O}_X \) is an ideal defining the curve \( C \). In our case \( J = \mathcal{O}_X(-C) \) and \( N_{X/C} = \mathcal{O}_X(C)|_C \). Thus the isomorphisms from the lemma are coming from the exact relative sequence
\[
0 \rightarrow A_X(\mathcal{F}(-(n + 1)C)) \rightarrow A_X(\mathcal{F}(-nC)) \rightarrow A_C(\mathcal{F}(-nC)|_C) \rightarrow 0.
\]

**Lemma 2.** Let \( P \in C \). For all \( n \in \mathbb{Z} \) the complex
\[
\varphi^n_C \hat{F}_C / \varphi^{n+1}_C \hat{F}_C \oplus \prod_{x \in C} \varphi^n_x \hat{F}_x / \varphi^{n+1}_x \hat{F}_x \rightarrow \prod_{x \in C} \varphi^n_{x,C} \hat{F}_{x,C} / \varphi^{n+1}_{x,C} \hat{F}_{x,C}
\]
is quasi-isomorphic to the complex
\[
(A_C(\mathcal{F}) \cap \varphi^n_C \hat{F}_C) / (A_C(\mathcal{F}) \cap \varphi^{n+1}_C \hat{F}_C) \oplus \varphi^n_P \hat{F}_P / \varphi^{n+1}_P \hat{F}_P \rightarrow \varphi^n_P \hat{F}_{P,C} / \varphi^{n+1}_P \hat{F}_{P,C}.
\]
This lemma is an extension of the proposition 1 above. The proves of the both lemmas are straightforward and we will skip them.

**Theorem 1.** Let \( X \) be a projective irreducible algebraic surface over a field \( k \), \( C \subset X \) be an irreducible projective curve, and \( P \in C \) be a smooth point on both \( C \) and \( X \). Let \( \mathcal{F} \) be a torsion free coherent sheaf on \( X \).

Assume that the the surface \( X - C \) is affine. Then the following complexes are quasi-isomorphic:

i) the adelic complex
\[
\hat{\mathcal{F}}_\eta \oplus \prod_{D} \hat{\mathcal{F}}_D \oplus \prod_{x} \hat{\mathcal{F}}_x \rightarrow \prod_{D} (\hat{\mathcal{F}}_D \otimes K_D) \oplus \prod_{x} (\hat{\mathcal{F}}_x \otimes K_x) \oplus \prod_{x \in D} (\hat{\mathcal{F}}_x \otimes \hat{\mathcal{O}}_{x,D}) \rightarrow \prod_{x \in D} (\hat{\mathcal{F}}_x \otimes K_{x,D})
\]
for the sheaf \( \mathcal{F} \) and

ii) the complex
\[
A(\mathcal{F}) \oplus A_C(\mathcal{F}) \oplus \hat{\mathcal{F}}_P \rightarrow B_C(\mathcal{F}) \oplus B_P(\mathcal{F}) \oplus (\hat{\mathcal{F}}_P \otimes \hat{\mathcal{O}}_{P,C}) \rightarrow \hat{\mathcal{F}}_P \otimes K_{P,C}
\]
Proof will be divided into several steps. We will subsequently transform the adelic complex checking that every time we get a quasi-isomorphic complex.

**Step I.** Consider the diagram

$$
\begin{array}{cccc}
\prod_{D \neq C} \hat{F}_D + \prod_{x \in U} \hat{F}_x & \rightarrow & \prod_{D \neq C} \hat{F}_D + \prod_{x \in U} \hat{F}_x & \rightarrow \\
\downarrow & & \downarrow & \\
\hat{F}_\eta + \prod_{D} \hat{F}_D + \prod_{x \in C} \hat{F}_x & \rightarrow & \prod_{D} K\hat{F}_D \oplus \prod_{x} K\hat{F}_x & \rightarrow \\
\downarrow & & \downarrow & \\
\hat{F}_\eta + \prod_{D} \hat{F}_D + \prod_{x \in C} \hat{F}_x & \rightarrow & (\prod_{D \neq C} K\hat{F}_D/\hat{F}_D + \prod_{x \in D} \hat{F}_D) \oplus (\prod_{x \in U} K\hat{F}_x/\hat{F}_x + \prod_{x \in C} K\hat{F}_x) & + \\
\oplus \prod_{x \in D \neq C} \hat{F}_{x,D} & \rightarrow & \prod_{x \in D \neq C} \hat{F}_{x,D} & \\
\downarrow & & \downarrow & \\
\prod_{x \in D} \hat{F}_{x,D} & \rightarrow & \prod_{x \in D} K\hat{F}_{x,D} & \\
\downarrow & & \downarrow & \\
\prod_{x \in C} \hat{F}_{x,C} & \rightarrow & \prod_{x \in D \neq C} K\hat{F}_{x,D} + \prod_{x \in C} K\hat{F}_{x,C} & \\
\end{array}
$$

where $U = X - C$. The middle row is the full adelic complex and the first row is an exact subcomplex. The commutativity of the upper squares is obvious. The exactness follows from the trivial

**Lemma 3.** Let $f_{1,2} : A_{1,2} \rightarrow B$ be homomorphisms of abelian groups. The complex

$$
0 \rightarrow A_1 \oplus A_2 \rightarrow A_1 \oplus A_2 \oplus B \rightarrow B \rightarrow 0,
$$

where $(a_1 \oplus a_2) \mapsto (a_1 \oplus -a_2 \oplus -f(a_1) + f(a_2)), (a_1 \oplus a_2 \oplus b) \mapsto (f(a_1) + f(a_2) + b)$, is exact.

The third row in the diagram is a quotient-complex by the subcomplex and we conclude that it is quasi-isomorphic to the adelic complex.

**Step II.** We can make the same step with the adelic complex for the sheaf $F$ on the surface $U$. By assumption the surface $U$ is affine and we get an exact complex

$$
\hat{F}_\eta/A \rightarrow \prod_{D \neq C} (\hat{F}_D \otimes K_D)/\hat{F}_D \oplus \prod_{x \in U} (\hat{F}_x \otimes K_x)/\hat{F}_x \rightarrow \\
\prod_{x \in U} (\hat{F}_x \otimes K_{x,D})/(\hat{F}_x \otimes \hat{O}_{x,D}),
$$

where $A = \Gamma(U, F)$.

**Lemma 4.** The complex

$$
0 \rightarrow \prod_{x \in C} (\hat{F}_x \otimes K_x)/B_x(F) \rightarrow \prod_{x \in C} (\hat{F}_x \otimes K_{x,D})/(\hat{F}_x \otimes \hat{O}_{x,D}) \rightarrow 0
$$

is exact.
Proof. The injectivity follows directly from the definition of the ring $B_x$. The surjectivity is the local strong approximation around the point $X ∈ X$ (see [3][§1],[6][ch.4]).

Step III. Take the sum of the two complexes from step II. Then we have a map of the complex we got in the step I to this complex

$$\mathcal{F}_\eta \oplus \mathcal{F}_C \oplus \prod x \mathcal{F}_x \rightarrow (\prod_{D \neq C} K \mathcal{F}_D / \mathcal{F}_D \oplus K \mathcal{F}_C) \oplus (\prod_{x \in U} K \mathcal{F}_x / \mathcal{F}_x \oplus \prod_{x \in C} K \mathcal{F}_x) \oplus \prod_{x \in C} \hat{F}_x,C \rightarrow (\prod_{D \neq C} K \mathcal{F}_D / \mathcal{F}_D \oplus (\prod_{x \in U} K \mathcal{F}_x / \mathcal{F}_x \oplus \prod_{x \in C} K \mathcal{F}_x / B_x) \oplus \prod_{x \in C} K \mathcal{F}_x,C$$

For this map all the components which do not have arrows are mapped to zero. The diagram is commutative and the kernel of the map is equal to

$$A \oplus \hat{F}_C \oplus \prod_{x \in C} \hat{F}_x \rightarrow K \hat{F}_C \oplus \prod_{x \in C} B_x(F) \oplus \prod_{x \in C} K \hat{F}_x \rightarrow \prod_{x \in C} K \hat{F}_x,C.$$

We conclude that this complex is quasi-isomorphic to the adelic complex.

Step IV. Using the embedding $\mathcal{F}_x \rightarrow B_x(F)$ and lemma 3 we have an exact complex and it’s embedding into the complex of the step III:

$$\prod_{x \in C,P} \hat{F}_x \rightarrow \prod_{x \in C,P} B_x(F) \oplus \prod_{x \in C,P} \hat{F}_x \rightarrow A \oplus \hat{F}_C \oplus \prod_{x \in C} \hat{F}_x \rightarrow K \hat{F}_C \oplus \prod_{x \in C} B_x(F) \oplus \prod_{x \in C} \hat{F}_x,C \rightarrow \prod_{x \in C} B_x(F) \rightarrow \prod_{x \in C} K \hat{F}_x,C.$$

As a result we get the factor-complex

$$A \oplus \hat{F}_C \oplus \hat{F}_P \rightarrow K \hat{F}_C \oplus B_P(F) \oplus \prod_{x \in C,P} \hat{F}_x,C / \hat{F}_x \oplus \hat{F}_P,C \rightarrow \prod_{x \in C-P} K \hat{F}_x,C / B_x(F) \oplus K \hat{F}_P,C.$$

Step V. Now we need

Lemma 5 The complex

$$0 \rightarrow (\hat{F}_C \otimes K_C) / B_C(F) \rightarrow \prod_{x \in C-P} (\hat{F}_x \otimes K_{x,C}) / B_x(F) \rightarrow 0$$

is exact.
Proof. The injectivity is again the definition of the $B_C$ and the surjectivity follows from the strong approximation on the curve $C$ ([4]) and lemma 2 above.

As a corollary we have an isomorphism

$$\hat{\mathcal{F}}_C/A_C(\mathcal{F}) \xrightarrow{\cong} \prod_{x \in C-P} (\hat{\mathcal{F}}_x \otimes \hat{\mathcal{O}}_{x,C})/\hat{\mathcal{F}}_x,$$

where

$$A_C(\mathcal{F}) := B_C(\mathcal{F}) \cap \hat{\mathcal{F}}_C.$$

Combining the isomorphisms from the lemma and its corollary into a single complex of length 2, we get the diagram

$$\begin{array}{c}
A \oplus \hat{\mathcal{F}}_C \oplus \hat{\mathcal{F}}_P \rightarrow K\hat{\mathcal{F}}_C \oplus B_P(\mathcal{F}) \oplus (\prod_{x \in C-P} \hat{\mathcal{F}}_{x,C}/\hat{\mathcal{F}}_x \oplus \hat{\mathcal{F}}_{P,C}) \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
(0) \oplus \hat{\mathcal{F}}_C/A_C \oplus (0) \rightarrow K\hat{\mathcal{F}}_C/B_C \oplus (0) \oplus \prod_{x \in C-P} \hat{\mathcal{F}}_{x,C}/\hat{\mathcal{F}}_x \\
\rightarrow \prod_{x \in C-P} K\hat{\mathcal{F}}_{x,C}/B_x(\mathcal{F}) \oplus K\hat{\mathcal{F}}_{P,C} \\
\rightarrow \prod_{x \in C-P} K\hat{\mathcal{F}}_{x,C}/B_x(\mathcal{F}) \oplus (0)
\end{array}$$

The kernel of the map of the complexes is obviously equal to

$$A(\mathcal{F}) \oplus A_C(\mathcal{F}) \oplus \hat{\mathcal{F}}_P \rightarrow B_C(\mathcal{F}) \oplus B_P(\mathcal{F}) \oplus (\hat{\mathcal{F}}_P \otimes \hat{\mathcal{O}}_{P,C}) \rightarrow (\hat{\mathcal{F}}_P \otimes K_{P,C})$$

and we arrive to the conclusion of the theorem.

Remark 1. Sometimes we will call the complex from the theorem as the restricted adelic complex.

Lemma 6. Let $X$ be a projective irreducible variety over a field $k$ and $\mathcal{O}(1)$ be a very ample sheaf on $X$. Then

1. The following conditions are equivalent

   i) $X$ is a Cohen-Macaulay variety

   ii) for any locally free sheaf $\mathcal{F}$ on $X$ and $i < \dim(X)$ $H^i(X, \mathcal{F}(n)) = (0)$ for $n << 0$

2. If $X$ is normal of dimension $> 1$ then for any locally free sheaf $\mathcal{F}$ on $X$ $H^1(X, \mathcal{F}(n)) = (0)$ for $n << 0$

Proof see in [8][ch. III, Thm. 7.6, Cor. 7.8]. We only note that the last statement is known as the lemma of Enriques-Severi-Zariski. For dimension 2 every normal variety is Cohen-Macaulay and thus the second claim follows from the first one.
Proposition 2. Let $\mathcal{F}$ be a locally free coherent sheaf on the projective irreducible surface $X$.

Assume that the local rings of the $X$ are Cohen-Macaulay and the curve $C$ is a locally complete intersection. Then, inside the field $K_{P,C}$, we have

$$B_C(\mathcal{F}) \cap B_P(\mathcal{F}) = A(\mathcal{F}).$$

Proof will be done in several steps.

Step 1. If we know the proposition for a sheaf $\mathcal{F}$ then it is true for the sheaf $\mathcal{F}(nC)$ for any $n \in \mathbb{Z}$. Thus taking a twist by $\mathcal{O}(n)$ we can assume that $\deg_C(\mathcal{F}) < 0$.

Step 2. Now we show that $A_C(\mathcal{F}) \cap \hat{F}_P = (0)$. The filtrations from lemma 1 gives the corresponding filtration of the group $A_C(\mathcal{F})$. Lemma 2 implies that

$$\frac{(A_C(\mathcal{F}) \cap \hat{F}_P) \cap \varphi^n \hat{F}_P}{(A_C(\mathcal{F}) \cap \hat{F}_P) \cap \varphi^{n+1} \hat{F}_P} \cong \Gamma(C, \mathcal{F} \otimes \hat{N}^\otimes_{X/C}).$$

Since $\deg_C(\mathcal{F}) < 0$, $N_{X/C} = \mathcal{O}_X(C)|_C$ and $\deg_C(N_{X/C}) > 0$ we get that the last group is trivial.

Step 3. The next step is to prove the equality:

$$B_C(\mathcal{F}(-D)) \cap B_P(\mathcal{F}(-D)) = A(\mathcal{F}(-D)),$$

where $D$ is an sufficiently ample divisor on $X$ distinct from the curve $C$. By theorem 1 the cohomology of $\mathcal{F}_X(-D)$ can be computed from the complex

$$A(\mathcal{F}(-D)) \oplus A_C(\mathcal{F}(-D)) \oplus \hat{F}_P(-D) \to B_C(\mathcal{F}(-D)) \oplus B_P(\mathcal{F}(-D)) \oplus \hat{F}_{P,C}(-D)$$

$$\to K\mathcal{F}_{P,C}.$$ 

Now take $a_{01} \in B_C(\mathcal{F}(-D)), a_{02} \in B_P(\mathcal{F}(-D))$ such that $a_{01} + a_{02} = 0$. They define an element $(a_{01} \oplus a_{02} \oplus 0)$ in the middle component of the complex. By our condition for $D$ and the lemma 6 we have $H^1(X, \mathcal{F}_X(-D)) = (0)$ and thus there exist $a_0 \in A(\mathcal{F}(-D)), a_1 \in A_C(\mathcal{F}(-D)), a_2 \in \hat{F}_P(-D)$ such that $a_{01} = a_0 - a_1, a_{02} = a_2 - a_0, 0 = a_1 - a_2$.

By the second step $a_1 = a_2 \in (A_C(\mathcal{F}(-D)) \cap \hat{F}_P(-D)) \subset A_C(\mathcal{F}) \cap \hat{F}_P = (0)$ and, consequently, we have $a_{01}(= -a_{02}) \in A(\mathcal{F}(-D)).$

Step 4. The last step is to take two distinct divisors $D, D'$ such that $D \cap D' \subset C$. Since $C$ is a hyperplane section we can choose for $D, D'$ two hyperplane sections whose intersection belongs to $C$. Therefore their ideals in the ring $A(\mathcal{F})$ are relatively prime and

$$A(\mathcal{F}) = A(\mathcal{F}(-D)) + A(\mathcal{F}(-D')) \ni 1 = a + a', a \in A(\mathcal{F}(-D)), a' \in A(\mathcal{F}(-D')).$$

If now $b \in B_C(\mathcal{F}) \cap B_P(\mathcal{F})$, then $b = ba + ba'$, where $ba \in B_C(\mathcal{F}(-D)) \cap B_P(\mathcal{F}(-D)), ba' \in B_C(\mathcal{F}(-D')) \cap B_P(\mathcal{F}(-D'))$. We see that $b \in A(\mathcal{F})$ by the previous step.
Remark 2. The method we have used cannot be applied if our variety is not Cohen-Macaulay (by lemma 6 above). It would be interesting to know how to extend the result to the arbitrary surfaces $X$ and the sheaves $\mathcal{F}$ such that $\mathcal{F}$ are locally free outside $C$. The last condition is really necessary.

Remark 3. This proposition is a version for the reduced adelic complex of the corresponding result for the full complex. Namely, $A_{X,01} \cap A_{X,02} = A_{X,0}$, see [3, ch.IV]. This should be generalized to arbitrary dimension $n$ in the following way.

Let $I, J \subset [0, 1, \ldots, n]$ and

$$A_{X,I}(\mathcal{F}) = \left( \prod_{\{\text{codim} \eta_0, \text{codim} \eta_1 \ldots \}} K_{\eta_0, \eta_1, \ldots} \right) \otimes \mathcal{F}_{\eta_0} \cap A_X(\mathcal{F}).$$

Then we have

$$A_{X,I}(\mathcal{F}) \cap A_{X,J}(\mathcal{F}) = A_{X,I \cap J}(\mathcal{F})$$

for a locally free $\mathcal{F}$ and a Cohen-Macaulay $X$.

Example. Let $X = \mathbb{P}^2 \supset C = \mathbb{P}^1 \supset P$. We introduce homogenous coordinates $(x_0 : x_1 : x_2)$ such that $C = (x_0 = 0); P = (x_0 = x_1 = 0$ and $U = X - C = \text{Spec} k[x,y]$ with $x = x_1/x_0, y = x_2/x_0$. Then $k(C) = k(y/x), x^{-1}$ is the last parameter for any two-dimensional local field $K_{Q,C}$ with $Q \neq P$. For local field $K_{P,C}$ we have

$$K_{P,C} = k((u))(t)), u = xy^{-1}, t = y^{-1}.$$

Then we can easily compute all the rings from the complex of theorem 1 for the sheaf $\mathcal{O}_X$.

$$\begin{align*}
B_P &= k[[u]](t)) \\
B_C &= k[u^{-1}][u^{-1}t]) \\
\hat{\mathcal{O}}_{P,C} &= k((u))[t] \\
A &= \Gamma(U, \mathcal{O}_X) = k[u^{-1}, t^{-1}] \\
A_C &= k[u^{-1}][u^{-1}t]) \\
\hat{\mathcal{O}}_P &= k[[u, t]]
\end{align*}$$

We can draw the subspaces as some subsets of the plane according to the supports of the elements of the subspaces (on the plane with coordinates $(i, j)$ for elements $u^i t^j \in K_{P,C}$. Then the first three subspaces $B_P, B_C, \hat{\mathcal{O}}_{P,C}$ will correspond to some halfplanes and the subspaces $A, A_C, \hat{\mathcal{O}}_P$ to the intersections of them.

2 Main Theorem

We need the following well known result.
Lemma 7. Let $X$ be an projective variety, $\mathcal{F}$ be a coherent sheaf on $X$ and $C$ be an ample divisor on $X$. If

$$S = \oplus_{n \geq 0} \Gamma(X, \mathcal{O}_X(nC)), \quad F = \oplus_{n \geq 0} \Gamma(X, \mathcal{F}(nC)), $$

then

$$X \cong \text{Proj}(S), \quad \mathcal{F} \cong \text{Proj}(F).$$

Proof. Let $mC$ be a very ample divisor, $S = \oplus_{n \geq 0} S_n$ and $S' = \oplus_{n \geq 0} S'_{nm}$. Then by [7][prop. 2.4.7]

$$\text{Proj}(S') \cong \text{Proj}(S).$$

The divisor $mC$ defines an embedding $i : X \to \mathbb{P}$ to a projective space such that $i^*\mathcal{O}_\mathbb{P}(1) = \mathcal{O}_X(mC)$. Let $\mathcal{J}_X \subset \mathcal{O}_\mathbb{P}$ be an ideal defined by $X$. If

$$I := \oplus_{n \geq 0} \Gamma(\mathbb{P}, \mathcal{J}_X(n)), \quad A := \oplus_{n \geq 0} \Gamma(\mathbb{P}, \mathcal{O}_\mathbb{P}(n)),$$

then $I \subset A$ and by [7][prop. 2.9.2]

$$\text{Proj}(A/I) \cong X.$$ 

We have an exact sequence of sheaves

$$0 \to \mathcal{J}_X(n) \to \mathcal{O}_\mathbb{P} \to \mathcal{O}_X(n) \to 0,$$

which implies the sequence

$$0 \to \bigoplus_{n \geq 0} \Gamma(\mathcal{J}_X(n)) \to \bigoplus_{n \geq 0} \Gamma(\mathcal{O}_\mathbb{P}(n)) \to \bigoplus_{n \geq 0} \Gamma(\mathbb{P}, \mathcal{O}_X(n)) \to \bigoplus_{n \geq 0} H^1(\mathbb{P}, \mathcal{J}_X(n)).$$

Here the last term is trivial for sufficiently large $n$. The first three terms are equal correspondingly to $I$, $A$ and $S'$. It means that the homogenous components of $A/I$ and $S' \supset A/I$ are equal for sufficiently big degrees.

By [7][prop. 2.9.1]

$$\text{Proj}(A/I) \cong \text{Proj}(S'),$$

and combining everything together we get the statement of the lemma. The statement concerning the sheaf $\mathcal{F}$ can be proved along the same line.

Let us first explain the Krichever correspondence for dimension 1.

Definition 2.

$$\mathcal{M}_1 := \{C, P, z, \mathcal{F}, e_P\}$$

$C$ projective irreducible curve /$k$

$P \in C$ a smooth point

$z$ formal local parameter at $P$

$\mathcal{F}$ torsion free rank $r$ sheaf on $C$

$e_P$ a trivialization of $\mathcal{F}$ at $P$
Independently, we have the field \( K = k((z)) \) of Laurent power series with filtration \( K(n) = z^n k[[z]] \). Let \( K_1 := K(0) \). If \( V = k((z))^\oplus r \) then \( V(n) = K(n)^\oplus r \) and \( V_1 := V(0) \).

**Theorem 2** [11]. There exists a canonical map

\[
\Phi_1 : \mathcal{M}_1 \longrightarrow \{\text{vector subspaces } A \subset K, W \subset V\}
\]

such that

i) the cohomology of complexes

\[
A \oplus K_1 \longrightarrow K, \ W \oplus V_1 \longrightarrow V
\]

are isomorphic to \( H^\cdot(C, \mathcal{O}_C) \) and \( H^\cdot(C, \mathcal{F}) \), respectively

ii) if \((A, W) \in \text{Im } \Phi_1\) then \( A \cdot A \subset A, A \cdot W \subset W\),

iii) if \( m, m' \in \mathcal{M}_1 \) and \( \Phi_1(m) = \Phi_1(m') \) then \( m \) is isomorphic to \( m' \)

**Proof.** If \( m = (C, P, z, \mathcal{F}, e_P) \in \mathcal{M}_1 \) then we put

\[
A := \Gamma(C - P, \mathcal{O}_C),
\]

\[
W := \Gamma(C - P, \mathcal{F}).
\]

Also we have

\[
\mathcal{O}_P = k[[z]], \ K_P = k((z)), \quad \mathcal{F}_P = \mathcal{O}_P e_P = \mathcal{O}_P^\oplus r, \quad \mathcal{F}_P = \mathcal{O}_P^\oplus r.
\]

This defines the point \( \Phi_1(m) \in \mathcal{M}_1 \). Indeed, for the subspace \( W \) we have the following canonical identifications

\[
\Gamma(C - P, \mathcal{F}) \subset \mathcal{F}_P \otimes \mathcal{O}_P, \quad K_P = \mathcal{F}_P \otimes K_P = \mathcal{O}_P^\oplus r \otimes K_P = k((z))^\oplus r.
\]

The same works for the subspace \( A \).

The property ii) is obvious, the property i) follows from the proposition 1. To get iii) let us start with a point \( \Phi_1(m) = (A, W) \). The standard valuation on \( K \) gives us increasing filtrations \( A(n) = A \cap K(n) \) and \( W(n) = W \cap V(n) \) on the spaces \( A \) and \( W \). Then we have

\[
C - P = \text{Spec}(A), \quad C = \text{Proj}(\oplus_n A(n)), \quad \mathcal{F} = \text{Proj}(\oplus_n W(n)),
\]

by lemma 6. Thus we can reconstruct the quintuple \( m \) from the point \( \Phi_1(m) \).

**Remark 4.** It is possible to replace the ground field \( k \) in the Krichever construction by an arbitrary scheme \( S \), see [17].
Now we move to the case of algebraic surfaces. The corresponding data has the following

**Definition 3.**

\[ \mathcal{M}_2 := \{X, C, P, (z_1, z_2), \mathcal{F}, e_P\} \]

- \( \mathcal{M}_2 \): projective irreducible surface over \( k \)
- \( C \subset X \): projective irreducible curve over \( k \)
- \( P \in C \): a smooth point on \( X \) and \( C \)
- \( z_1, z_2 \): formal local parameter at \( P \) such that \( (z_2 = 0) = C \) near \( P \)
- \( \mathcal{F} \): torsion free rank \( r \) sheaf on \( X \)
- \( e_P \): a trivialization of \( \mathcal{F} \) at \( P \)

Then we have

\[ \hat{\mathcal{O}}_{X,P} = k[[z_1, z_2]], \quad K_{P,C} = k((z_1))((z_2)) \]

\[ \hat{\mathcal{F}}_P = \hat{\mathcal{O}}_P e_P = \hat{\mathcal{O}}^{\oplus r}_P. \]

For the field \( K = k((z_1))((z_2)) \) we have the following filtrations and subspaces:

\[ K_{02} = k[[z_1]]((z_2)), \quad K_{12} = k((z_1))[[z_2]], \quad K(n) = z_2^n K_{12}. \]

Taking the direct sums we introduce the subspaces \( V_{02}, V_{12}, V(n) \) of the space \( V = K^{\oplus r} \).

**Theorem 3.** Let \( C \) be a hyperplane section on the surface \( X \). Then there exists a canonical map

\[ \Phi_2 : \mathcal{M}_2 \longrightarrow \{ \text{vector subspaces } B \subset K, W \subset V \} \]

such that

**i)** for all \( n \) the complexes

\[
\begin{align*}
\frac{B \cap K(n)}{B \cap K(n+1)} \oplus \frac{K_{02} \cap K(n)}{K_{02} \cap K(n+1)} & \rightarrow \frac{K(n)}{K(n+1)} \\
\frac{W \cap V(n)}{W \cap V(n+1)} \oplus \frac{V_{02} \cap V(n)}{V_{02} \cap V(n+1)} & \rightarrow \frac{V(n)}{V(n+1)}
\end{align*}
\]

are Fredholm of index \( \chi(C, \mathcal{O}_C) + nC.C \) and \( \chi(C, \mathcal{F}|_C) + nC.C \), respectively

**ii)** the cohomology of complexes

\[
\begin{align*}
(B \cap K_{02}) \oplus (B \cap K_{12}) \oplus (K_{02} \cap K_{12}) & \rightarrow B \oplus K_{02} \oplus K_{12} \rightarrow K
\end{align*}
\]
are isomorphic to \( H(X, \mathcal{O}_X) \) and \( H(X, \mathcal{F}) \), respectively

(iii) if \( (B, W) \in \text{Im } \Phi_2 \) then \( B \cdot B \subset B, B \cdot W \subset W \)

(iv) for all \( n \) the map

\[
(C, P, z_1|_C, \mathcal{F}(nC)|_C, e_P(n)|_C) \mapsto \left( \begin{array}{c} B \cap K(n) \\ B \cap K(n + 1) \end{array} \right) \subset \left( \begin{array}{c} K(n) \\ K(n + 1) \end{array} \right) = k((z_1)),
\]

\[
\left( \begin{array}{c} W \cap V(n) \\ W \cap V(n + 1) \end{array} \right) \subset \left( \begin{array}{c} V(n) \\ V(n + 1) \end{array} \right) = k((z_1))^{\oplus r}
\]

coincides with the map \( \Phi_1 \).

(v) let the sheaf \( \mathcal{F} \) be locally free and the surface \( X \) be Cohen-Macaulay. If \( m, m' \in \mathcal{M}_1 \) and \( \Phi_2(m) = \Phi_2(m') \) then \( m \) is isomorphic to \( m' \)

**Proof.** If \( m = (X, C, P, (z_1, z_2), \mathcal{F}, e_P) \in \mathcal{M}_2 \) then to define the map \( \Phi_2 \) we put

\[
B = B_C(\mathcal{O}_X),
\]

\[
W = B_C(\mathcal{F}),
\]

\[
\Phi_2(m) = (B, W).
\]

Since we have the local coordinates \( z_{1,2} \) and the trivialization \( e_P \) the subspaces \( B \) and \( W \) will belong to the space \( k((z_1))(z_2) \) exactly as in the case of dimension 1 considered above.

We note that our condition on the curve \( C \) implies that \( C \) is a Cartier divisor and the surface \( X - C \) is affine.

The property (i) follows from lemma 2, the property (ii) follows from theorem 1. The property (iii) is trivial again, to get (iv) one needs again to apply lemma 2 and to get (v) it is enough to use proposition 2 and lemma 7. They show that given a point \( (B, W) \in \mathcal{M}_2 \) such that \( (B, W) = \Phi_2(m) \) we can reconstruct the data \( m \) up to an isomorphism.

**Remark 5.** The property (v) of the theorem cannot be extended to the arbitrary torsion free sheaves on \( X \). We certainly cannot reconstruct such sheaf if it is not locally free outside \( C \). Indeed, if \( \mathcal{F}, \mathcal{F}' \) are two sheaves and there is a monomorphism \( \mathcal{F}' \rightarrow \mathcal{F} \) such that \( \mathcal{F}/\mathcal{F}' \) has support in \( X - C \) then the restricted adelic complexes for the sheaves \( \mathcal{F}, \mathcal{F}' \) are isomorphic.

**Remark 6.** A definition of the map \( \Phi_n \) for all \( n \) was suggested in [12]. It has the properties that correspond to the properties (i) - (v) of the theorem.

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