We apply the Regge-Wheeler formalism to study the propagation of axial and polar gravitational waves in matter-filled Bianchi I universe. Assuming that the expansion scalar $\Theta$, of the background space-time, is proportional to the shear scalar $\sigma$, we have solved the background field equations in the presence of matter (found to behave like a stiff fluid). We then derive the linearised perturbation equations for both the axial and polar modes. The analytical solutions in vacuum spacetime could be determined in an earlier paper $[63]$ in a relatively straightforward manner. However, here we find that in the presence of matter they require more assumptions for their solution, and bear more involved forms. We use both analytical and numerical methods to find the solutions for the axial perturbations. As compared to the axial modes, the polar perturbation equations contain far more complicated couplings among the perturbing terms. Thus we have to apply suitable assumptions to derive the analytical solutions for some of the cases of polar perturbations. In both the axial and polar cases, the radial and temporal solutions for the perturbations separate out as product. We find that the axial waves are damped owing to the background anisotropy, and can deform only the azimuthal velocity of the fluid. In contrast, the polar waves must trigger perturbations in the energy density, the pressure as well as in the non-azimuthal components of the fluid velocity. Similar behaviour is exhibited by axial and polar waves propagating in the Kantowski-Sachs universe $[62]$.

KEYWORDS: Regge-Wheeler gauge, Bianchi I spacetime, Axial Gravitational waves, Polar Gravitational waves

I. INTRODUCTION

The detection of gravitational waves (GWs) have confirmed the last prediction of Einstein’s general theory of relativity. The geometry of a spacetime is related to its matter content through Einstein’s equations and GWs are obtained as their solutions under perturbations in the linearised approximation.

The present-day isotropic and homogeneous universe is described by the FLRW model. Although this model offers suitable explanation for the current state of the universe, certain observations indicate the existence of cosmological anisotropies. So anisotropic models and the evolution of perturbations in these backgrounds have gained importance. Of the various anisotropic but homogeneous models $[1]$, the Bianchi I (B-I) universe $[2]$ is the simplest one. The B-I model has different scale factors along different directions. In such a universe filled with matter obeying an equation-of-state (EoS) $p = \gamma \rho$, where the EoS parameter $\gamma < 1$, any initial anisotropy quickly dies away and eventually evolves into a FLRW universe $[3]$. The B-I spacetime is therefore a potential candidate for modelling the early universe.

Gravitational perturbations in homogeneous anisotropic universes (including B-I universes) have been extensively studied. Such investigations are of interest to understand how anisotropy influences the gravitational perturbations. Detection of primordial GWs can shed light on the nature of the early universe. Hu $[4]$ pointed out that perturbations in such universes can be treated as a first approximation to exact inhomogeneous anisotropic solutions in the chaotic cosmology, and the back-reaction of GWs may considerably change the dynamics of the early universe. He showed the decoupling of the two linear polarizations of the GWs. In contrast to the FLRW universe, where the two polarizations are decoupled, each being equivalent to a minimally coupled massless scalar field $[5]$, these become coupled in an expanding anisotropic universe $[6]$. The paper $[7]$ dealt with general perturbations of the B-I spacetime. Others $[8]$ introduced a non-perturbative exact formalism for GWs travelling through Bianchi I-VII spacetimes. The property obtained in $[8]$ was generalized in $[9]$ that there exist solutions in which the inhomogeneity initially dominates the structure of the cosmic singularity but later evolves into GWs propagating over the more homogeneous background.

Among the perturbation formalisms developed to study GWs, the Regge-Wheeler perturbation scheme is a relatively simple one. In their investigations on Schwarzschild black hole perturbations, Regge and Wheeler $[10]$ decomposed the perturbing elements in terms of spherical harmonics, and incorporated the Regge-Wheeler (RW) gauge in order to extract a single Schrödinger-type differential equation for the axial as well as polar perturbations. The solutions come out in the form of products of four factors, each being a function of only one coordinate ($t, r, \theta, \phi$). Subsequently, this procedure has been used in various articles $[11]-[17]$. Zerilli made corrections to the polar equation of $[10]$ in $[11]$ and studied the problem of a particle falling into a Schwarzschild black hole $[12]$. Vishveshwara $[13]$ used the RW gauge
in Kruskal coordinates to examine the stability of the Schwarzschild exterior metric. Metric perturbations have also been discussed in [18]. In Ref.[19], the solution to the Schwarzschild perturbations was found to describe an outgoing axial wave corresponding to a special case of Chandrasekhar’s perturbations. Anderson et al. derived an asymptotic gauge [20] to transform the metric from RW gauge to Chandrasekhar-Esposito gauge [21]. The exact axial solutions in the Schwarzschild background as calculated by Fiziev [22] are given by the confluent Heun’s functions. Much simpler solutions for RW equation have been reported later [23]. In the de Sitter spacetime [24], the set of perturbation equations, on using the RW formalism, reduces to a single second-order differential equation of the Heun-type for both electric and magnetic multipoles.

The RW procedure has been applied to the axial gravitational perturbations in the FLRW model by Malec and Wylezek [25]. Kulczycki and Malec [26] investigated the polar GWs which are found to cause perturbations in the density and non-azimuthal components of the velocity of the material medium, and are hence responsible for the evolution of matter inhomogeneities and anisotropies. On the other hand, since the initial data can be so adjusted as to decouple from matter, axial waves can perturb only the azimuthal velocity and trigger local cosmological rotation. In such spacetimes, if the initial profiles are not smooth, then the axial wave pulses bring about rotation of the radiation fluid, leading to the memory effect [27]. The propagation of axial and polar waves in FLRW universes using the RW formalism has been studied by Sharif and Siddiqua in the $f(R, T)$ gravity [28, 29] and by Salti et al. in Rastall gravity [30, 31]. In the Starobinsky model of $f(R)$ gravity, Ref. [33] derives analytical solutions to the axial perturbations in the radiation era and the de Sitter stage. Rostworowski [34] has suggested using the RW formalism to study perturbations, as an alternative to the standard Bardeen’s formalism of gauge-invariance [35]. A recent work [37] determines the equivalent of the RW equation, for generalized McVittie metric. Ref.[38] examines all possible (gauge-invariant) master functions and equations for the perturbations to vacuum spherically-symmetric spacetimes. Axial GWs in flat FRW universe and in the $f(R, T^\omega)$ gravity have been presented in [39].

Axial and polar perturbations have been investigated in the gauge-invariant framework in [40–42, 44]. With a $2 + 2$ decomposition of the metric, Gundlach and Martin-Garcia [40] analysed non-spherical perturbations of a spherically symmetric, time-dependent spacetime to study the generation of GWs. They have found three axial and seven polar gauge-invariant matter perturbations. Martel and Poisson [43] presented the Schwarzschild metric perturbations in the covariant, gauge-invariant formalism. The metric perturbation theory was analysed by Chandrasekhar [45] also.

Clarkson et al. [44] have produced a complete set of master equations for the LTB dust model. The decomposition of any perturbation into scalar, vector and tensor (SVT) modes, all evolving independently, which is feasible in the FLRW model, cannot be done in the LTB model, where the modes get coupled. However, they can be decoupled into two independent modes - axial (or odd) and polar (or even), classified according to the nature of their transformation on spherically symmetric surfaces. As non-trivial symmetric, transverse and trace-free rank-2 tensors cannot exist on $S^2$, further decomposition into tensor modes is not possible here. The master equation of the polar waves is numerically solved in [46]. Using the 1+1+2 covariant decomposition of spacetime, Keresztes et al. [48] have carried out a study of the perfect-fluid perturbations of the Kantowski-Sachs universe with vanishing vorticity and a positive cosmological constant and generalised the analysis for LRS class-II cosmologies in [49]. Although there arise four propagation equations for gravitational perturbations, it has been clarified in [48] that GWs possess two degrees of freedom, the ‘+’ and ‘×’ polarisations. However, in modified theories of gravity, there can be up to six polarisations of GWs, the additional degrees of freedom occurring due to scalar modes, whose presence or absence are observer-dependent [50]. These polarisations have been studied extensively in [51–60].

Several studies on axial and polar GWs employing the RW perturbation scheme have been performed with FLRW metric as the background [25–31, 34, 36]. But the same has not been widely discussed for an anisotropic spacetime. Recently, we have studied the propagation of GWs in Kantowski-Sachs background [62]. Our previous work on Bianchi I background also invoked RW gauge [63]. In these two articles, we have determined the solutions of the vacuum perturbation equations analytically. In the present paper, we aim to find complete analytical solutions when the B-I spacetime is no longer vacuous, but filled with matter, which is found to obey the stiff fluid EoS. Here the system of perturbation equations become more involved and require special conditions to be solved.

Our paper is organised as follows: Sec.II presents the Bianchi I background metric, the corresponding Einstein’s equations and solutions in terms of scale factors in the presence or absence of matter. In Sec.III, the Regge-Wheeler gauge is introduced. Sec.IV carries a note on the perturbed energy-momentum tensor and four-velocity of the fluid. In Sec.V, we concentrate on the axial GWs. The linearised field equations for the axially-perturbed background are derived in the presence of matter and then solved analytically under certain assumptions. The vacuum case is touched upon in brief. Sec.VI deals with the polar modes. The perturbation equations in the presence of matter (specifically stiff fluid), and in vacuum, followed by their analytical solutions in particular cases are obtained. We conclude with an analysis of our results and some remarks in Sec.VII. Throughout this paper, we will use geometrized units, i.e., $8\pi G = c = 1$, and overdots and primes to represent derivatives w.r.t. $t$ and $r$ respectively.
II. THE UNPERTURBED BACKGROUND METRIC AND FIELD EQUATIONS

Exact solutions for homogeneous spacetimes in GR belong to either Bianchi types or the Kantowski-Sachs model [64]. Among the LRS cosmologies of class II, the KS spacetime has positive curvature (2D scalar curvature), while those with zero and negative curvature are respectively the Bianchi I/ VII₀ (including flat Friedmann universes), and the Bianchi III models [48, 91]. The general form of the anisotropic line element can be written as [67]:

\[ ds^2 = dt^2 - a^2(t)dr^2 - b^2(t)d\theta^2 + f_K(\theta)d\phi^2, \] (1)

\( K \) being the spatial curvature index of the spacetime. For \( K = 0 \), \( f_K(\theta) = \theta^2 \), the universe is classified as Bianchi I. For \( K = 1 \), \( f_K(\theta) = \sin^2 \theta \), the model is closed and named after Kantowski-Sachs. For \( K = -1 \), \( f_K(\theta) = \sinh^2 \theta \), the model represents the semi-closed Bianchi III space-time.

This paper is devoted to the study of the propagation of GWs in matter-filed Bianchi I spacetime. To begin with, the corresponding line element is defined in spherical polar coordinates by:

\[ ds^2 = dt^2 - a^2(t)dr^2 - b^2(t)d\theta^2 - b^2(t)\theta^2 d\phi^2, \] (2)

where \( a(t) \) and \( b(t) \) are the scale factors representing the expansion along the parallel and the perpendicular to the radial direction respectively. Considering the spacetime to be filled with a perfect fluid having four-velocity \( u^\alpha \), energy density \( \rho \) and pressure \( p \), the energy-momentum tensor is given by:

\[ T_{\mu\nu} = (\rho + p)u_\mu u_\nu - pg_{\mu\nu}. \] (3)

The background field equations are obtained as:

\[ \frac{2\dot{a}\dot{b}}{ab} + \frac{\dot{b}^2}{b^2} = \rho_0, \quad -\left(\frac{2\ddot{b}}{b} + \frac{\dot{b}^2}{b^2}\right) = p_0, \quad -\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{\dot{b}^2}{b^2}\right) = p_0. \] (4)

Also, we note that the continuity equation for the B-I metric (2) is represented by:

\[ \dot{\rho}_0 + \left(\frac{\dot{a}}{a} + \frac{2\dot{b}}{b}\right)(\rho_0 + p_0) = 0. \] (5)

The subscript ‘0’ refers to the physical quantities (here, the energy density and pressure of the fluid) associated with the background metric (2). In order to solve these equations, we need an additional condition, which we discuss now.

A note on the assumption: \( a = b^n \)

The volume expansion \( \Theta \) and shear scalar \( \sigma \) for the metric (2) are found to be:

\[ \Theta = \frac{\dot{a}}{a} + \frac{2\dot{b}}{b}, \quad \text{and} \quad \sigma^2 = \frac{1}{3} \left(\frac{\dot{a}}{a} - \frac{\dot{b}}{b}\right)^2. \] (6)

It is known that a cosmological model remains anisotropic throughout its evolution if the ratio of the shear \( \sigma \) to the expansion \( \Theta \) is constant [69–73]. The expansion scalar being proportional to the shear scalar, one can assume that

\[ a = b^n, \] (7)

where \( n \) is an arbitrary real number and \( n \neq 0, 1 \) for non-trivial solutions. Substituting this in equation (6) gives

\[ \frac{\sigma^2}{\Theta^2} = \frac{1}{3} \left(\frac{n-1}{n+2}\right)^2. \] (8)

Clearly, this ratio is constant for any \( n \).
In view of the relation (7), the background equations (4) are solved by

\[ b(t) = [(n + 2)(k_1 t + k_2)]^{\frac{n}{n+2}}. \]  

(9)

\( k_1 \) and \( k_2 \) are the integration constants. This is equivalent to the corresponding expression determined by Shamir et al. in the \( f(R, T) \) theory [68]. Assuming \( b(t) = 0 \) at \( t = 0 \), \( k_2 \) vanishes and we are left with

\[ b(t) = [(n + 2)k_1 t]^{\frac{n}{n+2}}, \quad \text{and hence} \quad a(t) = [(n + 2)k_1 t]^{\frac{n}{n+2}}. \]  

(10)

Substituting these expressions for \( a(t) \) and \( b(t) \) in equations (4), we find that

\[ \rho_0(t) = \rho_0(t) = \frac{2n + 1}{[(n + 2)t]^{n+2}}, \]  

(11)

which implies that the matter content is a stiff fluid [68]. This expression also indicates that \( n = -1/2 \) for vacuum solutions where \( \rho_0(t) = \rho_0(t) = 0 \).

A stiff perfect fluid is characterised by its equation of state:

\[ p_0 = \rho_0. \]  

(12)

This is the extreme relativistic limit for a perfect fluid, when the speed of sound becomes equal to that of light. This was first suggested by Zel’dovich [75], who considered the early universe to be composed of a cold gas of baryons behaving like a stiff fluid. Its governing equations have the same characteristics as that of the gravitational field [76]. After the cosmic explosion, when the universe was characterised by high densities, the matter content could have been stiff [75, 77, 78]. Various aspects of Bianchi I spacetimes containing stiff fluid have been mentioned in [79], [80]. Since the stiff fluid in the FLRW universe exhibits faster decrease in density than radiation and matter, it plays a significant role in the early phase. This helped Dutta and Scherrer [81] to numerically compute the effect of the stiff fluid density on the primordial abundances of light elements. The propagation of GWs into flat, open and closed FLRW universes (ahead of the waves) filled with stiff fluid was investigated in [82]-[85]. Although GWs propagating through dust or through fluids with a realistic equation of state are somewhat problematic to examine, exact radiative solutions can be found if the medium is considered as a stiff fluid [82]-[86]. In certain models with self-interacting dark matter components, the self interaction is mediated by vector mesons via minimal coupling and its energy resembles a stiff fluid [87]. In [88], the effective equation-of-state of the Zel’dovich fluid is shown to start evolving from stiff nature, pass through a pressure-less state and eventually tend towards de Sitter epoch. The fluid when combined with decaying vacuum energy [89] yields the age of the universe in agreement with the observations.

For a stiff fluid, the continuity equation (5) reads as

\[ \dot{\rho}_0 + 2 \left( \frac{\dot{a}}{a} + \frac{2 \dot{b}}{b} \right) \rho_0 = 0. \]  

(13)

Ref. [90] mentions that a B-I model, defined in Cartesian coordinates in time-negative convention by:

\[ ds^2 = -dt^2 + t^{2p_1} dx^2 + t^{2p_2} dy^2 + t^{2p_3} dz^2, \]  

(14)

and characterised by a non-tilted perfect fluid with an equation of state: \( p_0 = (\gamma - 1)\rho_0, \gamma = \text{constant} \), will have Jacobs’ stiff perfect fluid solutions if the following relations hold:

\[ p_1 + p_2 + p_3 = 1, \quad p_1^2 + p_2^2 + p_3^2 < 1, \]

\[ \rho_0 = \frac{1}{2}(1 - p_1^2 - p_2^2 - p_3^2)t^{-2}, \quad \gamma = 2. \]  

(15)

Comparing the line elements (2) and (14), we can write

\[ a(t) \propto t^{p_1}, \quad b(t) \propto t^{p_2}, \quad p_2 = p_3. \]  

(16)

Now, it can be shown that the first two relations in (15) can be satisfied if \( p_1 = 1/2 \) and \( p_2 = 1/4 \). It follows that

\[ a \propto t^{1/2} \quad \text{and} \quad b \propto t^{1/4} \quad \Rightarrow n = 2. \]  

(17)
Also, using these values of \( p_1 \) and \( p_2 \) along with the other two relations in (15), we find

\[
\rho_0 = p_0 = \frac{1 - \frac{3}{2} \kappa}{2t^2} = \frac{5}{16t^2}.
\]  

(18)

Equating (11) and (18) gives

\[
n = 2 \quad \text{or} \quad n = 2/5.
\]

(19)

From the results obtained in (17) and (19), we can say that the solution \( n = 2 \) satisfies all the required conditions. Hence, from equation (10), we have

\[
b(t) = (4k_1t)^{\frac{4}{5}} = \kappa t^{\frac{4}{5}}, \quad a(t) = b(t)^2 = \kappa^2 t^{\frac{4}{5}}, \quad \kappa = (4k_1)^{\frac{1}{5}} = \text{constant}.
\]

(20)

\[Solutions\ for\ the\ background\ equations\ in\ vacuum\]

Solving the equations in (4) for the vacuum case, using the relation (7), we have

\[
b(t) = Kt^{2/3}, \quad \text{and} \quad a = b^{-1/2}.
\]

(21)

\(K\) is the integration constant for the vacuum solutions. The scale factors obtained in equations (20) and (21) are used in subsequent calculations.

III. THE PERTURBED METRIC IN THE REGGE-WHEELER GAUGE

Gravitational waves propagating in the Bianchi I background are represented by small perturbations \( h_{\mu\nu} \), and the perturbed metric is defined as:

\[
g_{\mu\nu} = g_{\mu\nu}^{(0)} + \epsilon h_{\mu\nu} + \mathcal{O}(\epsilon^2).
\]

(22)

Here, \( g_{\mu\nu}^{(0)} \) is the background metric (2). The smallness of the magnitude of the perturbations is indicated by the parameter \( \epsilon \), and all terms of \( \mathcal{O}(\epsilon^2) \) are to be neglected in the calculations.

As explained in [10], the components of the perturbation matrix transform differently under a rotation of the frame about the origin. For example, \( h_{00}, h_{01}, h_{11} \) transform like scalars, \( (h_{02}, h_{03}) \) and \( (h_{12}, h_{13}) \) transform like vectors, and \( h_{22}, h_{23} \) and \( h_{33} \) like a second-order tensor. These are decomposed in terms of spherical harmonics \( Y_{lm} \) (\( l \) is the angular momentum and \( m \) is its projection on the \( z \)-axis), and grouped according to odd or even parity. It is then found that the axial waves are given by three unknown functions of \( r \), and the polar waves by seven unknown functions. At this point, the Regge-Wheeler gauge is introduced to find the canonical form of the axial and polar waves. Subsequently, the \( t \) and \( r \) solutions separate out as product in the final expressions.

For odd (or axial) waves, the matrix \( h_{\mu\nu} \) has only two non-zero components[26] represented by:

\[
h_{t\phi} = h_0(t, r) \sin \theta (\partial_\theta Y) \quad \text{and} \quad h_{r\phi} = h_1(t, r) \sin \theta (\partial_\theta Y).
\]

(23)

Thus the background metric (2) in the presence of the axial perturbations is given by:

\[
ds^2 = dt^2 - a^2(t)dr^2 - b^2(t)d\theta^2 - b^2(t)\theta^2 d\phi^2 + 2eh_0(t, r) \sin \theta (\partial_\theta Y) dt d\phi + 2eh_1(t, r) \sin \theta (\partial_\theta Y) dr d\phi + \mathcal{O}(\epsilon^2).
\]

(24)

Here, the spherical harmonics \( Y_{lm}(\theta, \phi) \) are denoted by \( Y \), with \( m = 0 \). For all values of \( m \), the radial equation remains the same. Hence, \( m \) is chosen to be zero and the \( \phi \)-dependence of \( Y \) is removed [10]. For wavelike solutions, \( l \geq 2 \) [28]. Further, the spherical harmonics follow the relation:

\[
\partial_\theta \partial_\theta Y = -l(l+1)Y - \cot \theta (\partial_\theta Y).
\]

(25)

This equation looks like an eigenvalue equation of the Laplacian operator on \( S^2 \) by virtue of the spherically symmetric background [47], but with an additional term on the right hand side.
Coming to the even (or polar) waves, it is found that there are a number of non-zero components of $h_{\mu\nu}$ [10]. Adopting the Gerlach-Sengupta [42] formalism, which has later been developed by Gundlach and Martin-Garcia [40], Clarkson and others [44] have proposed the general form of the polar perturbations as:

$$h_{\mu\nu} = \begin{pmatrix} (\chi + \psi - 2\eta)Y & \zeta Y & 0 & 0 \\ \zeta Y & (\chi + \psi)Y & 0 & 0 \\ 0 & 0 & \psi Y & 0 \\ 0 & 0 & 0 & \psi Y \end{pmatrix}. \tag{26}$$

$\chi$, $\psi$, $\zeta$ and $\eta$ are all functions of $t$ and $r$, and equivalent to the gauge-invariant variables introduced in [40–42]. The matrices in (23) and (26) correspond to magnetic and electric multipoles respectively [24]. The polar perturbations for the BI background (2) are therefore given by:

$$ds^2 = \left[1 + e \left\{ \chi(t,r) + \psi(t,r) - 2\eta(t,r) \right\} Y \right] dt^2 + 2e\zeta(t,r)Y dt dr + \left[ -a^2(t) + e \left\{ \chi(t,r) + \psi(t,r) \right\} Y \right] dr^2 + [ -b^2(t) + e\psi(t,r)Y ] d\theta^2 + \left[ -b^2(t) + e\psi(t,r)Y \right] \theta^2 d\phi^2 + O(e^2). \tag{27}$$

The value of $\eta$ in the above equations may or may not be zero. $\eta(t,r)$ is non-zero in the $(0 - 0)$ element of the corresponding perturbation matrix for the LTB background [44]. The constraint $\eta = 0$ does not hold for the field equations in [44] while considering large-angle fluctuations, where $l = 0$ or 1. But $\eta$ must vanish in the FLRW background [26, 36]. Also, in [40, 41], for $l \geq 2$, one has $\eta = 0$. However, gauge-invariance is no longer valid [40] for the polar $l = 0$, and 1 cases. For $l = 1$, there exist no dipole tensorial spherical harmonics, and hence $\eta$ is no longer zero. Moreover, due to the missing tensorial components, the gauge-invariant variables defined for $l \geq 2$ become partially gauge-invariant, leaving one degree of freedom to be fixed [44, 47]. For $l = 0$, there are two degrees of freedom (see [41], Appendix A). Therefore, additional constraints are required for gauge fixing.

IV. THE PERTURBED ENERGY-MOMENTUM TENSOR

One has to consider the perturbations in the energy density and pressure of the fluid. These terms can be written (following the same in the FRLW case [26, 28, 29]) in the following way:

$$\rho = \rho_0(1 + e \cdot \Delta(t,r)Y) + O(e^2), \quad p = p_0(1 + e \cdot \Pi(t,r)Y) + O(e^2), \tag{28}$$

where $\Delta(t,r)$ and $\Pi(t,r)$ are the perturbations in the energy density and pressure respectively. Since an equation of state relates the background energy density $\rho_0$ and pressure $p_0$, $\Delta(t,r)$ and $\Pi(t,r)$ are also related to each other. Moreover, the four-velocity of the fluid has to be taken in account while incorporating perturbations. The fluid may or may not be co-moving with the unperturbed cosmological expansion of the universe. The perturbed components of the fluid four-velocity $u_\alpha = (u_0, u_1, u_2, u_3)$ are defined as [26]:

$$u_0 = \frac{2h_{00}^{(0)}}{2} + e h_{00} + O(e^2), \quad u_1 = e a(t) w(t,r) Y + O(e^2), \tag{29}$$
$$u_2 = e v(t,r) (\partial_0 Y) + O(e^2), \quad u_3 = e U(t,r) \sin \theta (\partial_0 Y) + O(e^2). \tag{30}$$

Here, $h_{00}$ is the $(0 - 0)$ element of the perturbation matrix. The four-velocity components obey the relation:

$$u_\mu u^\mu = 1 + O(e^2). \tag{31}$$

The non-zero elements of the perturbed energy-momentum tensor are listed below.

(i) For axial perturbations:

$$T_{tt} = \rho_0(1 + e \Delta Y), \tag{32}$$
$$T_{rr} = a^2 \rho_0(1 + e \Pi Y), \tag{33}$$
$$T_{\theta\theta} = b^2 \rho_0(1 + e \Pi Y), \tag{34}$$
$$T_{\phi\phi} = b^2 \theta^2 \rho_0(1 + e \Pi Y), \tag{35}$$
$$T_{tr} = (\rho_0 + p_0) e a w Y, \tag{36}$$
$$T_{t\theta} = (\rho_0 + p_0) e a w Y, \tag{37}$$
$$T_{t\phi} = [(\rho_0 + p_0) U - p_0 h_0] e \sin \theta (\partial_0 Y), \tag{38}$$
$$T_{r\phi} = -p_0 h_1 e \sin \theta (\partial_0 Y). \tag{39}$$
Now we derive the axial perturbation equations for Bianchi I background using the RW gauge. Subsequently, we seek analytical solutions of the equations to express the perturbing terms as products of functions of \( t \) and \( r \).

### A. Perturbation equations in presence of matter:

Field equations for the axially perturbed metric (24) for a perfect fluid of energy density \( \rho_0 \) and pressure \( p_0 \), are:

\[
\begin{align*}
\frac{2\dot{a}}{ab} + \frac{\dot{b}^2}{b^2} &= \rho_0(1 + e\Delta Y), \\
-\left(\frac{2\dot{b}}{b} + \frac{\dot{b}^2}{b^2}\right) &= p_0(1 + e\Psi Y), \\
-\left(\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\dot{a}\dot{b}}{ab}\right) &= p_0(1 + e\Psi Y), \\
(\rho_0 + p_0)eawY &= 0, \\
(\rho_0 + p_0)e\theta(\dot{a}Y) &= 0,
\end{align*}
\]

\[
\frac{e}{2} \sin\theta(\dot{a}Y) \left[ \frac{h''_0}{a^2} - \frac{h'_0}{a^2} + \frac{2bh'_1}{a^2b} + \frac{2\dot{a}h_0}{a} + \frac{2\dot{b}h_0}{b} + \frac{2\dot{a}h_0}{ab} - \frac{h_0}{b^2} \{l(l+1)\} \right] = [(\rho_0 + p_0)U - p_0h_0] e \sin\theta(\dot{a}Y),
\]

\[
-\frac{e}{2} \sin\theta(\dot{a}Y) \left[ \dot{h}_1 - \dot{h}'_0 - \frac{\dot{a}h'_1}{a} + \frac{\dot{a}h''_0}{a^2} - \frac{2\dot{b}h'_0}{a} - \frac{2\dot{a}h_1}{a} - \frac{4\dot{b}h_1}{b} - \frac{2\dot{b}^2h_1}{b^2} + \frac{h_1}{b^2} \{l(l+1)\} \right] = -p_0\dot{h}_1 e \sin\theta(\dot{a}Y),
\]

\[
\frac{e}{2} \left( \dot{h}_0 + \frac{\dot{a}h_0}{a} - \frac{\dot{h}'_0}{a^2} \right) \left( \cos\theta(\dot{a}Y) - \frac{2 \sin\theta(\dot{a}Y)}{\theta} + \sin\theta(\dot{a}\theta Y) \right) = 0.
\]

After simplifying and using the background field equations (4), the above equations lead to the following relations:

\[
\begin{align*}
\Delta \cdot \rho_0 &= 0, \\
\Psi \cdot p_0 &= 0, \\
w(\rho_0 + p_0) &= 0, \\
v(\rho_0 + p_0) &= 0,
\end{align*}
\]

\[
\frac{h''_0}{a^2} - \frac{h'_0}{a^2} + \frac{2bh'_1}{a^2b} - \frac{h_0}{b^2} \{l(l+1)\} = 2U(\rho_0 + p_0),
\]

\[
\dot{h}_1 - \dot{h}'_0 - \frac{\dot{a}h'_1}{a} + \frac{\dot{a}h''_0}{a^2} - \frac{2\dot{b}h'_0}{a} - \frac{2\dot{a}h_1}{a} - \frac{4\dot{b}h_1}{b} - \frac{2\dot{b}^2h_1}{b^2} + \frac{h_1}{b^2} \{l(l+1)\} = 0,
\]

\[
\dot{h}_0 + \frac{\dot{a}h_0}{a} - \frac{\dot{h}'_0}{a^2} = 0.
\]

From equations (56) and (57), it can be concluded that \( \Delta = \Psi = 0 \) even in the presence of matter when \( \rho_0 \neq 0 \), \( \rho_0 \neq 0 \). Also, \( w = v = 0 \) if \( p_0 \neq -\rho_0 \), as deduced from equations (58) and (59). Thus axial waves do not perturb
the energy-density or pressure of the fluid. The only perturbation occurs in its azimuthal velocity $U$ as evident from equation (60). Since the background solutions determine the matter content to be a stiff fluid (11), i.e. $\rho_0 = p_0 \neq 0$, their perturbations $\Delta$ and $\Pi$ will be equal, the r.h.s. of equations (48) and (49), and hence equations (56) and (57) will be identical. We have assumed that $\cot \theta = 1/\theta$ in deriving the last terms of the equations (53) and (54). This condition reduces to $\tan \theta \simeq \theta$, which holds for small values of $\theta$.

B. Solutions to Perturbation equations in presence of matter:

To solve the perturbation equation (60), we eliminate $h_1(t, r)$ from (60) using equation (62), and obtain

$$\frac{h''_0}{a^2} - \frac{3b}{a} \frac{h_0}{a} + \frac{2b}{b} \frac{h''_0}{a^2} - \frac{a}{a} \frac{h_0}{a} + \frac{2a}{a} \frac{h_0}{b^2} \frac{h_0}{b^2} \{l(l + 1)\} = 2U(\rho_0 + p_0).$$

(63)

Using equation (20) and equating $p_0$ to $\rho_0$, the above equation reads

$$\frac{h''_0}{b^2} - \frac{4b}{b} \frac{h_0}{b} - \frac{2b^2}{b^2} \frac{h_0}{b} - \frac{2b^2}{b} \frac{h_0}{b} \{l(l + 1)\} = 4U \rho_0.$$

(64)

Now we write $h_0(t, r)$ in terms of a new quantity $Q(t, r)$ as

$$h_0(t, r) = r^\alpha(b(t))^\beta Q(t, r).$$

(65)

Here $\alpha$ and $\beta$ can assume integral or fractional values. Inserting this expression, substituting the expressions for $b(t)$ and $\rho_0(t)$ from (18) and (20), equation (64) becomes

$$-K^2 t^\beta r^\alpha \dot{Q} + K^2 \frac{t^\beta}{2} t^\alpha - 1 Q' + \left[\left(\frac{1}{2} \beta - 1\right) K^2 t^\beta (t^\beta - 1) r^\alpha \right] \dot{Q} + 2\alpha K^2 t^\beta t^\alpha - 1 Q' + \left[\frac{1}{16} \beta^2 + \frac{1}{4} \right] K^2 t^\beta (t^\beta - 2) r^\alpha - l(l + 1) K^2 - 2 t^\beta - 1 r^\alpha Q = \frac{5}{4 t^2} U.$$

(66)

This is an inhomogeneous wave equation in $Q(t, r)$ with a source term $U(t, r)$. This equation containing a single unknown variable $Q(t, r)$ may be termed as the equivalent of Regge-Wheeler equation in Bianchi I background. It contains additional terms in first-order derivatives of $Q(t, r)$ when compared to the RW equation (please see Eq.(87) in [97]) for Schwarzschild black hole perturbations.

Now, choosing $K = 1$ and $l = 2$, we are going to consider two sets of values of $\alpha$ and $\beta$.

Set I : $\alpha = \beta = 0$. In this case, equation (66) reduces to

$$-\dot{Q} + \frac{1}{t} Q'' - \frac{1}{t} \dot{Q} + \frac{1}{4 t^2} Q - \frac{6}{t^{1/2}} Q = \frac{5}{4 t^2} U.$$

(67)

Set II : $\alpha = 1$, $\beta = 2$. Here, equation (66) becomes

$$-\sqrt{t} r \ddot{Q} + \frac{r}{\sqrt{t}} Q' - \frac{2r}{\sqrt{t}} Q + \frac{2r}{\sqrt{t}} Q' - 6r Q = \frac{5}{4 t^2} U.$$

(68)

Finding an expression for $U$

In order to solve the equations (67) or (68), we require an explicit functional form of $U$ in terms of $t$ and $r$. We can find an expression for $U$ from the normalisation condition (31) satisfied by the perturbed four-velocity. We find:

$$u^0 = g^{00} u_0 + g^{03} u_3 = 1 + \frac{eU \sin \theta (\partial_b Y)}{e \theta \sin \theta (\partial_b Y)} = 1 + \frac{U}{h_0}, \quad u^3 = g^{33} u_3 + g^{30} u_0 = - \frac{eU \sin \theta (\partial_b Y)}{b^2 \theta^2} + \frac{1}{e \theta \sin \theta (\partial_b Y)}.$$

(69)

$$\therefore \quad u_\mu u^\mu = u_0 u^0 + u_3 u^3 = 1 + 2 \frac{U}{h_0} - \left( \frac{eU \sin \theta (\partial_b Y)}{b \theta} \right)^2.$$

(70)
Now we are going to consider two different cases of possible terms containing $e^2$.

**Case 1:**
Let us assume $O(e^2) \sim e^2$. Then comparing equation (70) with (31), we are left with

$$U = \pm \frac{b \theta}{\sin \theta (\partial_\theta Y)}.$$  \hspace{1cm} (71)

Now, with $l = 2$, $m = 0$, $Y_{20} = \frac{1}{4} \sqrt{\frac{5}{\pi}} (3 \cos^2 \theta - 1)$. Plugging in the expression for $b(t)$, we get

$$U = \frac{2 \sqrt{5/\pi} K t^{1/4} \theta}{15 \cos \theta \sin^2 \theta}.$$  \hspace{1cm} (72)

Setting different values of $\theta$, $U$ can be expressed as a function of $t$ only. Since $\theta$ must be chosen small, we take $\theta = 4^\circ = \pi/45$ radian, in which case we have

$$U = 7.5912K t^{1/4}.  \hspace{1cm} (73)$$

**Case 2:**
In this case, we assume $O(e^2) \sim e^2 r^2$, as it is the simplest expression containing $r$. Proceeding as above, we get

$$U = 7.5912K r t^{1/4}.  \hspace{1cm} (74)$$

For terms involving both $t$ and $r$, such as $e^2 r^2 t^2$, the analysis can be done in a similar way.

### Solutions in particular cases

Finally, we are in the last step of finding the analytical solutions for the axial modes. We do so by considering the two cases mentioned above, for both Set I and Set II. Here we want to mention that we have used Maple software in our analysis for the purpose of simplification. Although analytical solutions are available in all the cases, but there remains a multiplicative factor in each of these solutions, as a result of which it becomes difficult to interpret the results. Thus we have determined the final solutions numerically, which are presented in Discussions I.

#### Case 1: $U = 7.5912K t^{1/4}$

**Set I: $\alpha = \beta = 0$**

Inserting the expression of $U$ from equation (73), the perturbation equation (67) yields the solution:

$$Q(t, r) = N_1/D_1,  \hspace{1cm} (75)$$

where

$$N_1 = 79200 \left[ (t^2 - \frac{\sqrt{7}}{32}) \text{HG} \left[ 1 \right., \left. \frac{5}{6}, \frac{3}{2} \right], -\frac{8}{3} t^{3/2} \right] + \left( \frac{256}{275} t^{7/2} \right) \text{HG} \left[ 3 \right., \left. \frac{17}{6}, \frac{7}{2} \right] - \frac{8}{3} t^{3/2} \right] \left( C_1 \exp(\sqrt{C_4} r/2) + C_2 \exp(-\sqrt{C_4} r/2) \right) C_1 R_1$$

$$- \frac{16}{15} \left[ (t^{3/2}) \text{HG} \left[ 2 \right., \left. \frac{11}{6}, \frac{5}{2} \right], -\frac{8}{3} t^{3/2} \right] \sqrt{-t^{3/2}} \left( C_1 \exp(\sqrt{C_4} r/2) + C_2 \exp(-\sqrt{C_4} r/2) \right) C_1 R_1$$

$$- 115500 \left[ (t^{3/2}) \text{HG} \left[ 1 \right., \left. \frac{5}{6}, \frac{3}{2} \right], -\frac{8}{3} t^{3/2} \right] C_3 \sqrt{-t^{3/2}} + \text{Bessel} \left( \frac{2}{3}, 3 \right. \sqrt{6} \sqrt{-t^{3/2}} \right) C_2 t (-t^{3/2})^{1/6}$$

$$+ \text{Bessel} \left( \frac{4}{3}, \sqrt{6} \sqrt{-t^{3/2}} \right) C_3 (-t^{3/2})^{5/6} + \left( \frac{\sqrt{6}}{12} \right) \text{Bessel} \left( \frac{1}{3}, 3 \right. \sqrt{6} \sqrt{-t^{3/2}} \right) C_3 (-t^{3/2})^{1/3},$$

and

$$D_1 = 79200C_1 \sqrt{-t^{3/2}} \left[ (t^2 - \frac{\sqrt{7}}{32}) \text{HG} \left[ 1 \right., \left. \frac{5}{6}, \frac{3}{2} \right], -\frac{8}{3} t^{3/2} \right] + \left( \frac{256}{275} t^{7/2} \right) \text{HG} \left[ 3 \right., \left. \frac{17}{6}, \frac{7}{2} \right] - \frac{8}{3} t^{3/2} \right]$$

$$- \frac{16}{15} \left[ (t^{3/2}) \text{HG} \left[ 2 \right., \left. \frac{11}{6}, \frac{5}{2} \right], -\frac{8}{3} t^{3/2} \right].  \hspace{1cm} (76)$$
Here, $C_1$, $C_2$, $C_3$ and $C_4$ are integration constants. HG denotes hypergeometric series. BesselI functions are modified Bessel functions of first kind. The function $R_1(t)$ in $N_1$ is given by

$$R_1(t) = \text{DESol} \left( \left\{ \ddot{X}_1(t) + \frac{1}{t} \dot{X}_1(t) - \frac{1}{4t} C_4 X_1(t) - \frac{1}{4t^2} X_1(t) + \frac{6}{\sqrt{t}} X_1(t) \right\}, \{X_1(t)\} \right),$$

i.e., $R_1(t)$ represents the solutions of the second-order differential equation in $X_1(t)$:

$$\ddot{X}_1(t) + \frac{1}{t} \dot{X}_1(t) - \frac{1}{4t} C_4 X_1(t) - \frac{1}{4t^2} X_1(t) + \frac{6}{\sqrt{t}} X_1(t) = 0. \quad (77)$$

We have attempted to derive an approximate expression for $R_1$ in Discussions I.

Set $\text{II}: \alpha = 1, \beta = 2$
With the same expression of $U$ (equation (73)), the solution of the equation (68) comes out as:

$$Q(t, r) = \frac{N_2}{D_2}, \quad (78)$$

where

$$N_2 = \left[ \left(t^{5/2} - \frac{t}{32}\right) \cdot \text{HG} \left( \left[1, \left[\frac{5}{6}, \frac{3}{2}\right], -\frac{8}{3} t^{3/2}\right] + \left(\frac{256}{275} t^{7/2}\right) \cdot \text{HG} \left( \left[3, \left[\frac{17}{6}, \frac{7}{2}\right], -\frac{8}{3} t^{3/2}\right] \right) \right) - \left(\frac{16}{15} t^{5/2}\right) \cdot \text{HG} \left( \left[2, \left[\frac{11}{6}, \frac{5}{2}\right], -\frac{8}{3} t^{3/2}\right] \right) \right] \cdot \text{HG} \left( \left[\frac{5}{3}, \left[\frac{1}{3}, \frac{1}{3}\right], -\frac{8}{3} t^{3/2}\right] \right) + \text{BesselI} \left( \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{8}{3} t^{3/2} \right) \cdot C_7 \left( -t^{3/2} \right)^{5/6} + \text{BesselI} \left( \frac{1}{3} \cdot \frac{4}{3} \cdot \frac{8}{3} t^{3/2} \right) \cdot C_7 \left( -t^{3/2} \right)^{1/3}, \quad (79)$$

$$D_2 = C_5 \cdot \sqrt{t} \cdot \sqrt{t^{3/2}} \left[ \left(t^2 - \frac{7}{32} \right) \cdot \text{HG} \left( \left[1, \left[\frac{5}{6}, \frac{3}{2}\right], -\frac{8}{3} t^{3/2}\right] \right) \right] + \left(\frac{256}{275} t^{7/2}\right) \cdot \text{HG} \left( \left[3, \left[\frac{17}{6}, \frac{7}{2}\right], -\frac{8}{3} t^{3/2}\right] \right) - \left(\frac{16}{15} t^{5/2}\right) \cdot \text{HG} \left( \left[2, \left[\frac{11}{6}, \frac{5}{2}\right], -\frac{8}{3} t^{3/2}\right] \right).$$

Just as before, HG represents hypergeometric series, BesselI are modified Bessel functions of first kind. $C_i$’s are all integration constants. The function $R_2(t)$ appearing in (79) is given by

$$R_2(t) = \text{DESol} \left( \left\{ \ddot{X}_2(t) + \frac{2}{t} \dot{X}_2(t) - \frac{1}{t} C_8 X_2(t) + \frac{6}{\sqrt{t}} X_2(t) \right\}, \{X_2(t)\} \right),$$

i.e., $R_2$ denotes the solutions of

$$\ddot{X}_2(t) + \frac{2}{t} \dot{X}_2(t) - \frac{1}{t} C_8 X_2(t) + \frac{6}{\sqrt{t}} X_2(t) = 0, \quad (80)$$

$X_2(t)$ being some unknown function of $t$.

**Case 2 :** $U = 7.5912 K t^{1/4}$

Set $\text{I}: \alpha = \beta = 0$
In this case, $U$ from equation (74) leads to the following solution of equation (67):

$$Q(t, r) = \frac{N_3}{D_3}, \quad (81)$$
where \( N_3 = 79200 \left[ \left( t^2 - \frac{\sqrt{t}}{32} \right) \text{HG} \left[ \frac{1}{3}, \frac{5}{6}, \frac{3}{2}, -\frac{8}{3} t^{3/2} \right] + \left( \frac{256}{275} t^{7/2} \right) \text{HG} \left[ \frac{3}{3}, \frac{17}{6}, \frac{7}{2}, -\frac{8}{3} t^{3/2} \right] \right. \)

\[- \left( \frac{16}{15} t^2 \right) \text{HG} \left[ \frac{2}{3}, \frac{11}{6}, \frac{5}{2}, -\frac{8}{3} t^{3/2} \right] \sqrt{-t^{3/2}} \left(c_1 \text{exp}(\sqrt{c_4} r/2) + c_2 \text{exp}(-\sqrt{c_4} r/2) \right) c_1 R_3 \]

\[-115500 r \left( t^{1/4} \right) \text{HG} \left[ \frac{1}{3}, \frac{5}{6}, \frac{3}{2}, -\frac{8}{3} t^{3/2} \right] c_1 \sqrt{-t^{3/2}} + \text{BesselI} \left( \frac{2}{3}, \frac{4}{3}, \sqrt{6 \sqrt{-t^{3/2}}} \right) c_2 t \left(-t^{3/2}\right)^{1/6} \]

\[+ \text{BesselI} \left( \frac{4}{3}, \frac{4}{3}, \sqrt{6 \sqrt{-t^{3/2}}} \right) c_3 \left(-t^{3/2}\right)^{5/6} + \left( \frac{\sqrt{6}}{12} \right) \text{BesselI} \left( \frac{1}{3}, \frac{4}{3}, \sqrt{6 \sqrt{-t^{3/2}}} \right) c_3 \left(-t^{3/2}\right)^{1/3}, \quad (82) \]

\[D_3 = 79200 c_1 \sqrt{-t^{3/2}} \left( t^2 - \frac{\sqrt{t}}{32} \right) \text{HG} \left[ \frac{1}{3}, \frac{5}{6}, \frac{3}{2}, -\frac{8}{3} t^{3/2} \right] - \left( \frac{16}{15} t^2 \right) \text{HG} \left[ \frac{3}{3}, \frac{17}{6}, \frac{7}{2}, -\frac{8}{3} t^{3/2} \right]. \]

Set \( \Pi : \alpha = 1, \beta = 2 \)

Proceeding as before, we find that equation (68) is now solved by:

\[Q(t, r) = N_4/D_4, \quad (83)\]

where \( N_4 = \left[ \left( t^{5/2} - \frac{t}{32} \right) \text{HG} \left[ \frac{1}{3}, \frac{5}{6}, \frac{3}{2}, -\frac{8}{3} t^{3/2} \right] + \left( \frac{256}{275} t^4 \right) \text{HG} \left[ \frac{3}{3}, \frac{17}{6}, \frac{7}{2}, -\frac{8}{3} t^{3/2} \right] \right. \)

\[- \left( \frac{16}{15} t^{5/2} \right) \text{HG} \left[ \frac{2}{3}, \frac{11}{6}, \frac{5}{2}, -\frac{8}{3} t^{3/2} \right] \sqrt{-t^{3/2}} \left(c_5 \text{sin}(c_8 r/2) + c_6 \text{cos}(c_8 r/2) \right) c_5 R_4 \]

\[-\left( \frac{35}{24} \right) \left( t^{3/4} \right) \text{HG} \left[ \frac{1}{3}, \frac{5}{6}, \frac{3}{2}, -\frac{8}{3} t^{3/2} \right] c_5 \sqrt{-t^{3/2}} + \text{BesselI} \left( \frac{2}{3}, \frac{4}{3}, \sqrt{6 \sqrt{-t^{3/2}}} \right) c_6 t \left(-t^{3/2}\right)^{1/6} \]

\[+ \text{BesselI} \left( \frac{4}{3}, \frac{4}{3}, \sqrt{6 \sqrt{-t^{3/2}}} \right) c_7 \left(-t^{3/2}\right)^{5/6} + \left( \frac{\sqrt{6}}{12} \right) \text{BesselI} \left( \frac{1}{3}, \frac{4}{3}, \sqrt{6 \sqrt{-t^{3/2}}} \right) c_7 \left(-t^{3/2}\right)^{1/3}, \quad (84) \]

\[D_4 = c_5 r \sqrt{t} \sqrt{-t^{3/2}} \left( t^2 - \frac{\sqrt{t}}{32} \right) \text{HG} \left[ \frac{1}{3}, \frac{5}{6}, \frac{3}{2}, -\frac{8}{3} t^{3/2} \right] - \left( \frac{16}{15} t^2 \right) \text{HG} \left[ \frac{3}{3}, \frac{17}{6}, \frac{7}{2}, -\frac{8}{3} t^{3/2} \right]. \]

In Eqns.(82) and (84) also, we use HG for hypergeometric series, and BesselI for modified Bessel functions of first kind. \( c_i, i = 1 \) to 8 are all integration constants. Also,

\[R_3(t) = \text{DESol} \left\{ \begin{array}{l} \dot{X}_3(t) + \frac{1}{t} X_3(t) - \frac{1}{4 t^2} c_4 X_3(t) - \frac{1}{4 t^2} X_3(t) + \frac{6}{\sqrt{t}} X_3(t) \\ \{X_3(t)} \end{array} \right\}, \]

and

\[R_4(t) = \text{DESol} \left\{ \begin{array}{l} \dot{X}_4(t) + \frac{2}{t} \dot{X}_4(t) - \frac{1}{t} c_8 X_4(t) + \frac{6}{\sqrt{t}} X_4(t) \\ \{X_4(t)} \end{array} \right\}, \]

where \( X_3(t) \) and \( X_4(t) \) are functions of \( t \), as obtained through Maple computation. The functions \( R_3 \) and \( R_4 \) refer to the respective solutions of the differential equations:

\[\dot{X}_3(t) + \frac{1}{t} X_3(t) - \frac{1}{4 t^2} c_4 X_3(t) - \frac{1}{4 t^2} X_3(t) + \frac{6}{\sqrt{t}} X_3(t) = 0, \quad (85) \]

\[\dot{X}_4(t) + \frac{2}{t} \dot{X}_4(t) - \frac{1}{t} c_8 X_4(t) + \frac{6}{\sqrt{t}} X_4(t) = 0. \quad (86) \]
Discussions I

We now look for the solutions $R_1$, $R_2$, $R_3$ and $R_4$ from equations (77)-(86). For different values of $U$ but the same $\alpha$ and $\beta$, the $Q$-solutions (See Eqs.(76) and(82), (79) and (84)) are found to be quite similar. Thus, for the ease of analysis, we will replace all $X_i(t)$’s by $X(t)$. Hence, we obtain two differential equations in $X(t)$ for the two sets of values of $\alpha$ and $\beta$, which differ only in the values of constants for different values of $U$. Setting the constants to unity, these equations (77) and (85), (80) and (77) reduce to:

\begin{align}
\dot{X}(t) + \frac{1}{t} \dot{X}(t) - \frac{1}{4t} X(t) - \frac{1}{4t^2} X(t) + \frac{6}{\sqrt{t}} X(t) &= 0, \\
\dot{X}(t) + \frac{2}{t} \dot{X}(t) - \frac{1}{t} X(t) + \frac{6}{\sqrt{t}} X(t) &= 0.
\end{align}

(87) (88)

These are solved numerically to determine the nature of $X(t)$. The solutions are shown in Figs.(1) and (2). The solutions of $X(t)$ are found to exhibit wave-like behaviour with slowly increasing amplitude at large $t$.

![X(t) in Eqn.(87) plotted against t.](image1)

![X(t) in Eqn.(88) plotted against t.](image2)

All the expressions of $Q$ are found to be highly complicated. We need to simplify them to determine the nature of $h_0$ and $h_1$. Let us elaborate the solution in Case 2, Set I. We choose this specific solution for the purpose of illustration due to two reasons. First of all, we can examine the effect of the radial coordinate $r$ appearing in the expression of $U$, which does not appear in Case I. Secondly, the ratio $Q(t, r) = N_3/D_3$ becomes much simpler than in the other cases. Thus we have

\begin{align}
Q &= [c_1 \exp(\sqrt{c_4} r/2) + c_2 \exp(-\sqrt{c_4} r/2)] R_4 - \left( \frac{35r}{24c_1} \times \frac{S_N}{S_D} \right),
\end{align}

(89)

where

\begin{align}
S_N &= \text{HG} \left[ \left[ 1, \left[ \frac{5}{6}, \frac{3}{2} \right], -\frac{8}{3} t^{3/2} \right] c_1 t^{3/4} \sqrt{-t^{3/2}} + \text{BesselI} \left[ \frac{2}{3}, \frac{4}{3} \sqrt{6} \sqrt{-t^{3/2}} \right] c_2 t (-t^{3/2})^{1/6} \\
&+ \text{BesselI} \left[ \frac{4}{3}, \frac{4}{3} \sqrt{6} \sqrt{-t^{3/2}} \right] c_3 (-t^{3/2})^{5/6} + \left( \frac{\sqrt{6}}{12} \right) \text{BesselI} \left[ \frac{1}{3}, \frac{4}{3} \sqrt{6} \sqrt{-t^{3/2}} \right] c_3 (-t^{3/2})^{1/3},
\end{align}

(90)

\begin{align}
\dot{S}_N &= S_N / \sqrt{-t^{3/2}} = \text{HG} \left[ \left[ 1, \left[ \frac{5}{6}, \frac{3}{2} \right], -\frac{8}{3} t^{3/2} \right] c_1 t^{3/4} + \text{BesselI} \left[ \frac{2}{3}, \frac{4}{3} \sqrt{6} \sqrt{-t^{3/2}} \right] c_2 t (-t^{3/2})^{-1/3} \\
&+ \text{BesselI} \left[ \frac{4}{3}, \frac{4}{3} \sqrt{6} \sqrt{-t^{3/2}} \right] c_3 (-t^{3/2})^{1/3} + \left( \frac{\sqrt{6}}{12} \right) \text{BesselI} \left[ \frac{1}{3}, \frac{4}{3} \sqrt{6} \sqrt{-t^{3/2}} \right] c_3 (-t^{3/2})^{-1/3},
\end{align}

\begin{align}
S_D &= \left( t^2 - \frac{\sqrt{t}}{32} \right) \text{HG} \left[ \left[ 1, \left[ \frac{5}{6}, \frac{3}{2} \right], -\frac{8}{3} t^{3/2} \right] + \left( \frac{256}{275} t^{7/2} \right) \text{HG} \left[ \left[ 3, \left[ \frac{17}{6}, \frac{7}{2} \right], -\frac{8}{3} t^{3/2} \right] \\
&- \left( \frac{16}{15} t^{2} \right) \text{HG} \left[ \left[ 2, \left[ \frac{11}{6}, \frac{5}{2} \right], -\frac{8}{3} t^{3/2} \right] \right). \right.
\end{align}

(91)

Some of the terms occurring in the solutions of $Q$ contain imaginary parts. Only the real part of such terms have to be taken into account in order to get physically relevant solutions. Given the expression for $Q(t, r)$, one can find
Here, in Case 2, Set I, using equations (65) and (89), we get

\[ h_0(t, r) = r^0(b(t))^0 Q(t, r) = \text{Re} \left[ \left( c_1 \exp \left( \frac{\sqrt{c_4} r}{2} \right) + c_2 \exp \left( -\frac{\sqrt{c_4} r}{2} \right) \right) R_3(t) - \left( \frac{35 r}{24 c_1} \times \tilde{S}_N(t) \right) \right]. \]  

(92)

Hence, equation (62) yields

\[ h_1(t, r) = f_{sf}(t) + \kappa^4 t \int_{r_0}^r \left( h_0(t, r) + \frac{h_0(t, r)}{2t} \right) dr \]

\[ = f_{sf}(t) + \kappa^4 t \times \text{Re} \left[ \Lambda \left( c_1 e^{\Lambda r} - c_2 e^{-\Lambda r} \right) \hat{R}_3 - \frac{35}{24 c_1} \frac{d}{dt} \left( \frac{\tilde{S}_N}{S_D} \right) + \frac{1}{2t} \left\{ \Lambda \left( c_1 e^{\Lambda r} + c_2 e^{-\Lambda r} \right) \right\} R_3(t) - \frac{35 \tilde{S}_N}{24 c_1 S_D} \right]_0^r. \]

(93)

Here, \( \Lambda = \sqrt{c_4}/2 \). \( f_{sf}(t) \) is the arbitrary integration constant that can be chosen as zero as done in the last step. \( r_0 \) characterizes the initial hypersurface that generates GWs. The suffix ‘sf’ is used to denote the case of stiff fluid.

Analyzing the plots in Figs.(1) and (2), we may write an approximate trial solution of \( R_3(t) \) as \( R_3(t) = (0.05)t^{1/3} (\sin t + \cos t) \). Inserting this in equation (92), setting the values of the constants to unity and considering only the leading terms appearing in the series in (91), the 3-dimensional plot of \( h_0(t, r) \) is obtained as shown in Fig.(3). Similar analysis can be done for the remaining sets of axial solutions.

FIG. 3: The axial perturbation \( h_0(t, r) \) given by Eqn.(92) is plotted against \( t \) and \( r \).

C. In vacuum: Perturbation equations and Solutions

In the absence of matter, the axial perturbation equations become simpler to solve. This has been discussed in detail in our previous work \[63\]. The solution procedure is same as above. Only the relation between the scale factors, \( a(t) \) and \( b(t) \) is now defined by (21).

VI. POLAR PERTURBATIONS: EQUATIONS AND SOLUTIONS

We now discuss the polar perturbation equations and their analytical solutions for GWs propagating in B-I universe.
A. In presence of matter: Perturbation equations

We treat the cases \( \eta \neq 0 \) and \( \eta = 0 \) in separate paragraphs.

(i) Perturbation equations with \( \eta \neq 0 \)

In the presence of matter, the linearised field equations for the polar-perturbed metric (27) are obtained as follows:

\[
(t - t) \text{ equation :} \quad 4\theta \frac{e^{\psi Y}}{|-b^2 + e\psi Y|^3} \left\{ a^2 b^2 \left( (1 - a^2) (\chi + \psi) + 2a^2 \eta \right) \right\} \left\{ \left( a^2 b^2 \eta \right) \left( \theta \partial_\theta + (\partial_\theta \partial_\eta) \right) \right\} + eY \left\{ (4a^3 \partial^2 b - 2a^2 \partial b^4 + 2a^2 b^4 \partial^2 b)(\chi + \psi) - (4a^3 \partial^2 b + a^2 b^2 \partial^2 b) \right\} - 2(2a^4 b^4 b^2 + 4a^3 \partial b^4 b)(\chi + \psi) - (a^4 b^4 b + a^3 \partial b^4 b \psi + a^2 b^4 \psi'') + 2a^2 b^4 b' \right\} \right\} = \rho_0 [1 + e(\Delta + \chi + \psi - 2\eta) Y],
\]

\[
(r - r) \text{ equation :} \quad -4\theta \frac{e^{\psi Y}}{|-b^2 + e\psi Y|^3} \left\{ a^2 b^2 \left( (1 - a^2) (\chi + \psi) + 2a^2 \eta \right) \right\} \left\{ \left( a^2 b^4 \eta \right) \left( \theta \partial_\theta + (\partial_\theta \partial_\eta) \right) \right\} + eY \left\{ (a^4 b^2 \partial^2 b + 2a^2 b^4 \partial b^2 - 2a^2 b^4 \partial b)(\chi + \psi) + (4a^3 \partial b^4 b + 3a^2 b^2 \partial^2 b) \right\} - 2(2a^4 b^4 b^2 + 2a^4 b^4 b)(\chi + \psi) + a^4 b^4 b \left( \chi + \psi - 2\eta \right) - a^2 b^4 b \psi + a^4 b^2 \psi \right\} \right\} = p_0 [a^2 + e(a^2 \Pi - \chi - \psi) Y],
\]

\[
(\theta - \theta) \text{ equation :} \quad -\frac{4\theta}{|b^2 + e\psi Y|^3} \left\{ a^2 b^2 \left( (1 - a^2) (\chi + \psi) + 2a^2 \eta \right) \right\} \left\{ \left( a^2 b^4 \eta \right) \left( \theta \partial_\theta + (\partial_\theta \partial_\eta) \right) \right\} - a^2 \theta b^4 \left\{ a^2 b^4 \left( 1 - a^2 \right) (\chi + \psi) - 2a^4 b^2 \eta \right\} \left( \frac{1}{2} \theta \partial_\theta + (\partial_\theta \partial_\eta) \right) + (a \partial b^4 - a^2 b^4 - a^2 \partial b^2) \right\} - a^3 \theta b^4 \left\{ a^2 b^4 \left( (a^4 b^2 - a^2 b^2) (\chi + \psi) - 2a^4 b^2 \eta \right) \left( \frac{1}{2} \theta \partial_\theta + (\partial_\theta \partial_\eta) \right) + (a \partial b^4 - a^2 b^4 - a^2 \partial b^2) \right\} - a^3 \theta b^4 \left\{ a^2 b^4 \left( (a^4 b^2 - a^2 b^2) (\chi + \psi) - 2a^4 b^2 \eta \right) \left( \frac{1}{2} \theta \partial_\theta + (\partial_\theta \partial_\eta) \right) + (a \partial b^4 - a^2 b^4 - a^2 \partial b^2) \right\} - a^3 \theta b^4 \left\{ a^2 b^4 \left( (a^4 b^2 - a^2 b^2) (\chi + \psi) - 2a^4 b^2 \eta \right) \left( \frac{1}{2} \theta \partial_\theta + (\partial_\theta \partial_\eta) \right) + (a \partial b^4 - a^2 b^4 - a^2 \partial b^2) \right\} + 2(a^6 b^4 b^2 + 2a^6 b^4 b)(\chi + \psi) + a^6 b^4 b \left( \chi + \psi - 2\eta \right) - a^2 b^4 b \psi + a^4 b^2 \psi \right\} \right\} = p_0 [a^2 + e(a^2 \Pi - \chi - \psi) Y],
\]

\[
(\phi - \phi) \text{ equation :} \quad -\frac{4\theta^2}{|b^2 + e\psi Y|^3} \left\{ a^2 b^2 \left( (1 - a^2) (\chi + \psi) + 2a^2 \eta \right) \right\} \left\{ \left( a^2 b^4 \eta \right) \left( \theta \partial_\theta + (\partial_\theta \partial_\eta) \right) \right\} - a^3 \theta b^4 \left\{ a^2 b^4 \left( (a^4 b^2 - a^2 b^2) (\chi + \psi) - 2a^4 b^2 \eta \right) \left( \frac{1}{2} \theta \partial_\theta + (\partial_\theta \partial_\eta) \right) + (a \partial b^4 - a^2 b^4 - a^2 \partial b^2) \right\} - a^3 \theta b^4 \left\{ a^2 b^4 \left( (a^4 b^2 - a^2 b^2) (\chi + \psi) - 2a^4 b^2 \eta \right) \left( \frac{1}{2} \theta \partial_\theta + (\partial_\theta \partial_\eta) \right) + (a \partial b^4 - a^2 b^4 - a^2 \partial b^2) \right\} + 2(a^6 b^4 b^2 + 2a^6 b^4 b)(\chi + \psi) + a^6 b^4 b \left( \chi + \psi - 2\eta \right) - a^2 b^4 b \psi + a^4 b^2 \psi \right\} \right\} = p_0 [a^2 + e(a^2 \Pi - \chi - \psi) Y] \theta^2,
\]
\[ (t-r) \text{ equation: } \frac{ea^4b^4\theta}{[a^2 - eY](1 - a^2)(\chi + \psi) + 2a^2\eta]^{2\theta}} \left[ -\psi' + \left( \frac{\dot{a}}{a} + \frac{\dot{b}}{b} \right) \psi' \right] \]

\[ -bb(\chi' + \psi' - 2\eta') - 2b(\dot{b}\dot{b} + \dot{b}^2) \zeta \right) Y - \frac{1}{2} \left\{ (\partial_\theta \partial_\chi Y) + \frac{1}{\theta}(\partial_\theta Y) \right\} = e(\rho_0 + p_0)aw - p_0\zeta] Y, \]

\[ (t-\theta) \text{ equation: } \frac{eab(\partial_\theta Y)}{2[\dot{b}^2 + eY]^{2\theta}} \left[ (\dot{b}^2 - ab^3 + a^2\dot{b}^3) \right] \left[ -\psi' + \left( \frac{\dot{a}}{a} + \frac{\dot{b}}{b} \right) \psi' \right] = e(\rho_0 + p_0)v(\partial_\theta Y), \]

\[ (r-\theta) \text{ equation: } \frac{e(\partial_\theta Y)a^4b^4}{2[\dot{b}^2 + eY]^{2\theta}} \left[ -\psi' + \left( \frac{\dot{a}}{a} + \frac{\dot{b}}{b} \right) \psi' \right] = 0, \]

\[ (t-\phi) \text{ equation: } e(\rho_0 + p_0)U \sin \theta(\partial_\theta Y) = 0. \]

The assumption: \( \cot \theta = 1/\theta \) is applied to equations (94)-(98) also. As in the axial perturbation equations, this condition implies that \( \tan \theta \approx \theta \), hence \( \theta \) must be small. Neglecting all terms containing second or higher orders of \( e \) in the expansions appearing here, and using the background field equations (4), and the relation (25), the above set of equations lead to the following:

\[ w = \frac{1}{a^2b^2(\rho_0 + p_0)} \left[ -\frac{1}{2} \left( \chi + \psi' + \frac{\dot{a}}{a} \frac{\dot{b}}{b} \right) \right], \]

\[ v = \frac{1}{2(\rho_0 + p_0)} \left[ \left( \frac{\dot{a}}{a} + \frac{\dot{b}}{b} \right) \chi + \left( \frac{\dot{a}}{a^2} + \frac{\dot{b}}{a^2b} + \frac{\dot{b}}{b} \right) \psi \right. \]

\[ -2 \left( \frac{\dot{a}}{a} + \frac{\dot{b}}{b} \right) \eta + \left( \frac{1}{a^2} + \frac{1}{b^2} \right) \psi' - \frac{1}{a^2} \zeta', \]

\[ U = 0, \]

\[ \dot{a} + \dot{\zeta} - \chi' + \left( \frac{1}{b^2} - 1 \right) \psi' + 2\eta' = 0, \]

\[ \Delta = \frac{1}{a^4b^4\rho_0} \left[ \left\{ 2a\dot{a}\dot{b}^2 - 2a^3\dot{a}\dot{b}\dot{b} + a^2\dot{b}^2 \right\} \chi + \left\{ 2b\dot{a}\dot{b}\dot{b} + 2a^3\dot{a}\dot{b} + 2a^2\dot{b}^2 - 2a^4b^2 \right\} \psi \right. \]

\[ + 2(2a^3\dot{a}\dot{b}\dot{b} + a^4b^2) \eta - a^2\dot{b}^2 \chi - \left\{ a^6\dot{b}^2 + a^4\dot{b}\dot{b}^2 + a^2\dot{b}^4 \right\} \psi' + a^2b^4\psi'' - 2a^2\dot{b}^2 \psi' \]

\[ \Pi = \frac{1}{a^4b^4\rho_0} \left[ \left\{ a^{6}b^{4}b^{2} + a^{6}b^{3}b^{2} - \frac{1}{2}(l+1) \cdot a^{2}b^{4} \right\} \chi + \left\{ a^{6}b^{4}b^{2} - 2a^{6}b^{3}b + 2a^{6}b^{5}b \right\} \psi \right. \]

\[ - \frac{1}{2}(l+1) \cdot a^{6}b^{3}(b^{2} - 1) \psi' + 2 \left\{ a^{6}b^{4}b^{2} - 2a^{6}b^{5}b + \frac{1}{2}(l+1) \cdot a^{4}b^{4} \right\} \eta \]

\[ + a^{6}b^{5}b^{2}\chi + (a^{6}b^{5}b - a^{6}b^{3}b)\psi' - 2a^{6}b^{5}\eta' + a^{6}b^{4}\psi' \]. \]
\[ p_0 b^2 \Pi = \frac{1}{a b^2} \left[ (a^2 b^4 - a a b^4 + a b^3 b + a^3 a b^4 + a^3 a b^4 b - a a b^3 b)(\chi + \psi) + (a^4 b^2 - a b^3 b - a^3 a b^4 \psi - 2(a^4 b^3 b + a^3 a b^4 + a^3 a b^3 b) \eta + (a^4 b^2 + \frac{1}{2} a^3 a b^4 + a^3 a b^3 b)(\chi + \psi) \right] \\
+\left( -a^4 b b + \frac{1}{2} a^2 a b^2 b \right) \psi - (a^4 b^3 b + a^3 a b^4) \eta - a^2 b^3 b \zeta' + \frac{1}{2} a^2 b^4 (\chi + \psi + \chi'' + \psi''') + \frac{1}{2} a^4 b^2 \psi - \frac{1}{2} a^2 b^2 \psi'' - a^2 b^4 \psi'' - a^2 b^4 \zeta' + \frac{1}{2} \{ a^2 b^2 (a^2 - 1)(\chi + \psi) - 2a^4 b^2 \eta \} \frac{1}{b} (\partial_\theta Y) \right], \tag{108} \]

\[ p_0 b^2 \Pi = \frac{1}{a b^2} \left[ (a^2 b^4 - a a b^4 + a b^3 b + a^3 a b^4 + a^3 a b^3 b - a a b^3 b)(\chi + \psi) + (a^4 b^2 - a b^3 b - a^3 a b^4 \psi - 2(a^4 b^3 b + a^3 a b^4 + a^3 a b^3 b) \eta + (a^4 b^2 + \frac{1}{2} a^3 a b^4 + a^3 a b^3 b)(\chi + \psi) \right] \\
+\left( -a^4 b b + \frac{1}{2} a^2 a b^2 b \right) \psi - (a^4 b^3 b + a^3 a b^4) \eta - a^2 b^3 b \zeta' + \frac{1}{2} a^2 b^4 (\chi + \psi + \chi'' + \psi''') + \frac{1}{2} a^4 b^2 \psi - \frac{1}{2} a^2 b^2 \psi'' - a^2 b^4 \psi'' - a^2 b^4 \zeta' + \frac{1}{2} \{ a^2 b^2 (a^2 - 1)(\chi + \psi) - 2a^4 b^2 \eta \} \frac{1}{b} (\partial_\theta Y) \right]. \tag{109} \]

Adding equations (108) and (109) and substituting equation (107), we arrive at

\[ \left[ a^2 b^4 - a a b^4 - a^4 b^3 b + a^3 a b^4 + a^3 a b^3 b - a a b^3 b - a^4 b^2 \right] \chi + \left[ -a a b^4 + a^4 b^3 b + a^3 a b^4 + a^3 a b^3 b - a^3 a b^3 b - a^4 b^2 \right] \eta + \frac{1}{2} a^2 a b^4 - \frac{1}{2} a^2 b^3 b - a a b^4 \right] \psi + \left[ 2a^4 b^3 b - 2a^3 a b^4 - 2a^3 a b^3 b + 2a^4 b^2 b^2 - \frac{l(l+1)}{2} \cdot a^2 b^2 \right] \eta \\
+\left( a^4 b^3 b - a^3 a b^4 \right) \eta - a^2 b^3 b \zeta' + \frac{1}{2} a^2 b^4 (\chi + \chi'') + \left( \frac{1}{2} a^2 b^4 - \frac{1}{2} a^4 b^2 \right) \psi + \left( \frac{1}{2} a^2 b^4 - \frac{1}{2} a^4 b^2 \right) \psi'' - a^2 b^4 \eta'' - a^2 b^4 \zeta' = 0. \tag{110} \]

Equations (102) and (103) determine the perturbation of the two components of the fluid four-velocity. From equation (104), one finds that the third component, i.e. the azimuthal velocity \( U \) is zero in case of polar waves. These properties distinguish polar GWs from the axial GWs. Moreover, the polar perturbations are marked in the energy density and pressure as obtained in equations (106) and (107) respectively. Equations (105) and (110) do not explicitly depend on matter. Following Clarkson et al. [44], the equations (105), (107) and (110), together with the constraint equation \( \eta = 0 \) are the three master equations governing the evolution of the perturbing variables. Equations (102),(103) and (106) are the constraint equations. Also, if we suppose that all the matter perturbations become zero, then inserting \( \Pi = \Delta = w = v = 0 \) in equations (102), (103), (106) and (107) yields \( \chi = \psi = \zeta = 0 \), i.e. the polar perturbations vanish. Hence it can be concluded that polar GWs must bring about matter inhomogeneities and anisotropies in the background they travel through. As one can see from the set of equations (102)-(110), the polar perturbation variables are heavily coupled to one another. Therefore, the Zerilli equation in a single variable cannot be extracted. However, we may identify the Zerilli equation from (110) under suitable assumptions. We elaborate this point in Discussions II while analysing the Case 4 of the solutions for vacuum perturbations.

\[ \text{(ii) Perturbation equations with } \eta = 0 \]

When \( \eta = 0 \), the perturbation equations (102) - (107) and (110) become a little simpler. We will use this constraint to solve the equations analytically.

\[ \text{The particular case of stiff fluid} \]

We now concentrate on the stiff fluid as the matter content of the Bianchi I universe. The fluid pressure and energy density being equal, the polar perturbation equations with both zero or non-zero value of \( \eta \) are slightly more simplified.
(a) Perturbation equations with $\eta \neq 0$

The polar-perturbed equations (102)-(106) in the presence of a stiff fluid get reduced to:

$$w = \frac{2}{5\kappa^4} \left[ -2tl(l+1)\zeta + \kappa^2 t^{1/2} \chi' + \left( \kappa^2 t^{1/2} - 3 \right) \psi' - 2\kappa^2 t^{1/2} \eta' + 4t\psi'' \right] + \left( 4 \right)$$

$$v = \frac{1}{5\kappa^4} \left[ (3\kappa^4 t - 3)\chi + (3\kappa^4 t - 3 - 2\kappa^2 t^{1/2})\psi - 6\kappa^4 t\eta + 4t\dot{\chi} + (4t + 4\kappa^2 t^{3/2})\psi - 4t\zeta' \right],$$

$$U = 0,$$

$$\frac{1}{2t} \dot{\zeta} + \dot{\zeta} - \chi' + \left( \frac{1}{\kappa^2 t^{1/2}} - 1 \right) \psi' + 2\eta' = 0,$$

$$\Delta = \frac{2}{5\kappa^4} \left[ \left\{ \frac{2}{t} - \frac{5\kappa^4}{2} - \frac{4l^{1/2}/l(l+1)}{\kappa^2} \right\} \chi + \left\{ \frac{2}{t} + \frac{3\kappa^2}{l^{1/2}} - \frac{5\kappa^4}{2} - 4 \left( \frac{l^{1/2}}{\kappa^2} + t \right) l(l+1) \right\} \psi - 5\kappa^4 \eta \right.$$}

$$-2\ddot{\chi} - \left\{ 2 + 6\kappa^2 t^{1/2} \right\} \dot{\psi} + \frac{8t^{1/2}}{\kappa^2} \psi'' + 4\zeta'.$$

Since $p_0 = \rho_0$, their respective perturbations $\Delta$ and $\Pi$ must be equated. Hence, inserting $a = b^2$ in equations (107) and (110), replacing $\Pi$ by $\Delta$ and plugging in the expression for $\Delta$ from equation (115), we get respectively:

$$\frac{4}{5\kappa^4} \left[ \left\{ \frac{2}{t} + (2\kappa^2 \frac{t^{3/2} - 2l^{1/2}}{\kappa^2} \right\} l(l+1) \right\} \chi + \left\{ \frac{2}{t} + (2\kappa^2 \frac{t^{3/2} - 4t - 2l^{1/2}}{\kappa^2} \right\} l(l+1) \right\} \psi - 4\kappa^2 \frac{t^{3/2} l(l+1) \eta}{(l+1)}.$$
Following [44], it can be said that for polar perturbations to Bianchi I background, equations (112), (114), (116), (117) and (119) hold for \( l \geq 1 \), and the rest, (111), (115), (118) and (120), for \( l \geq 0 \).

\[ \text{(b) Perturbation equations with } \eta = 0 \]

For \( l \geq 2 \), the above equations will hold with the terms containing \( \eta \) set to zero.

\[ \text{B. In presence of matter: Solutions to Perturbation equations} \]

As evident from the set of polar perturbation equations, in contrast to the axial perturbation equations, the polar solutions cannot be derived straight away. The equation (114) containing the three perturbation elements does not help in simplifying the remaining equations and extracting a differential equation in a single variable. The above equations are highly complicated even when \( \eta = 0 \), and cannot be solved analytically without certain simplifying assumptions. We attempt to solve equation (117). Choosing \( \eta = 0 \), \( l = 2 \), \( \kappa = 1 \), and assuming \( \psi(t,r) = 0 \), and \( \zeta(t,r) = q_{sf}\chi(t,r) \), \( q_{sf} \) being an arbitrary constant, the equation reduces to

\[
\left( 12t^{3/2} - 12t^{1/2} + \frac{1}{t} \right) \chi - (1 + t)\dot{\chi} + 2q_{sf}\chi' = 0. \tag{121}
\]

The corresponding solution is obtained as:

\[
\chi(t,r) = \mathcal{F}(r + 2q_{sf} \ln(1+t)) \left( \frac{t}{1+t} \right) \exp \{ 8t^{3/2} + 48 \arctan t^{1/2} - 48t^{1/2} \}. \tag{122}
\]

Here, \( \mathcal{F} \) is an undetermined function of both \( t \) and \( r \). But this does not conform to the Regge-Wheeler scheme where the function of \( t \) and that of \( r \) must separate out as product in the solutions of the perturbation equations.

\[ \text{C. In vacuum: Perturbation equations} \]

In the absence of matter, the set of polar perturbation equations (102)-(110) with \( \eta = 0 \) are reduced to the following:

\[
-\frac{l(l+1)}{2} \zeta + b\dot{b}\chi' + \left( -\frac{\dot{a}}{a} - \frac{\dot{b}}{b} + \ddot{b} \right) \psi' + \dot{\psi} = 0, \tag{123}
\]

\[
\left( -\frac{\dot{a}}{a^3} + \frac{\dot{a}}{a} - \frac{\dot{b}}{a^2b} + \frac{\ddot{b}}{b} \right) \chi + \frac{1}{a^2} \chi + \left( -\frac{\dot{a}}{a^3} + \frac{\dot{a}}{a} - \frac{\dot{b}}{a^2b} + \frac{\ddot{b}}{b} - \frac{2\dot{b}}{b^3} \right) \psi + \left( \frac{1}{a^2} + \frac{1}{b^2} \right) \dot{\psi} - \frac{1}{a^2} \zeta' = 0, \tag{124}
\]

\[
\frac{\dot{a}}{a} \zeta + \dot{\zeta} - b\dot{b} - \left( \frac{1}{b^2} - 1 \right) \psi' = 0, \tag{125}
\]

\[
\left\{ \frac{2\dot{a}\dot{b}}{a} - \frac{l(l+1)}{2} \right\} \chi + \left\{ \frac{2\dot{a}\dot{b}}{a} - \frac{2\dot{a}\dot{b}}{b} - \frac{l(l+1)}{2} \left( 1 + \frac{a^2}{b^2} \right) \right\} \psi - b\dot{b}\dot{\chi} \left\{ 2\dot{b} + a\dot{a} + \frac{a^2}{b} \right\} \psi + \psi'' + 2b\dot{b}\zeta' = 0, \tag{126}
\]

\[
\left\{ -\frac{l(l+1)}{2} \right\} \chi + \frac{\dot{b}^2}{b^2} - \frac{l(l+1)}{2} \left( 1 - \frac{1}{b^2} \right) \psi + b\dot{b}\dot{\chi} + \left( \frac{b\ddot{b}}{b} - \frac{\dot{b}}{b} \right) \dot{\psi} + \dot{\psi} = 0, \tag{127}
\]

\[
(\dot{a}^2b^4 + a^{-3}b^3\dot{b})(\chi + \psi) + (a^4b^2 + a^{-3}\dot{a}\dot{b})(\psi + (-a\dot{a}b^2 + \frac{1}{2}a^4\dot{b}^2b^2 + \frac{1}{2}a^4\ddot{a}b^4 + \frac{1}{2}a^3b^3\ddot{b})(\chi + \psi) + \frac{1}{2}a^3\ddot{b}b^2 - a^4\dot{b}b)\psi') + \frac{1}{2}a^2b^4(\chi + \psi' + \psi'') + \frac{1}{2}a^2b^2\psi' - \frac{1}{2}a^2b^2\psi'' - a^2\dot{b}\dot{c}' + a^2\dot{c}' + \frac{1}{2}a^2b^2(b^2 - 1)(\chi + \psi)'(\psi') = 0, \tag{128}
\]
\[(\dot{a}b^4 + a^2b^3b)(\chi + \psi) + (a^4b^2 + a^3ab^2 + (-a\dot{b}b^4 + \frac{1}{2}a^3b\dot{b}^3b + \frac{1}{2}a^2b^3b)(\chi + \dot{\psi}) + \frac{1}{2}a^3b^2 - a^4b)\psi + 1/2a^2b^3(\chi + \dot{\psi} + \chi'' + \psi'') + \frac{1}{2}a^4b^2\psi - \frac{1}{2}a^3b^3\psi' - a^2b^4\psi' + \frac{1}{2a^2b^2}(a^2 - 1)(\chi + \psi)(\partial b\partial b Y) = 0.\] (129)

Adding equations (128) and (129), and simplifying gives
\[
\left\{b\ddot{b} + \frac{a^2b^2}{a^2} - \frac{l(l+1)}{4}(a^2 - 1)\right\}\chi + \left\{a\ddot{a} + \ddot{b} + a^2b^2 + a^2b^2 - \frac{l(l+1)}{4}(a^2 - 1)\right\}\psi + \left\{-\dot{a}a + \frac{1}{a} + \frac{1}{2}a\dot{b}^2 + \frac{1}{2}a^2b\dot{b}^2\right\} + \left\{\frac{1}{2}b^2(\chi + \chi'') + \frac{1}{2}(a^2 + b^2)\dot{\psi} + \frac{1}{2}(b^2 - 1)\psi' - b\dot{b}^2 - \dot{b}^2\dot{\psi} = 0.\] (130)

D. In vacuum: Solutions to Perturbation equations

In order to eliminate the \(\dot{\psi}'\)-term from equation (130), equation (125) is differentiated w.r.t. \(r\), which yields
\[
\dot{\zeta}' = -\frac{\dot{a}}{a}\zeta' - \chi'' + \left(\frac{1}{b^2} - 1\right)\psi',
\] (131)

and equation (130) now reads as
\[
\left\{b\ddot{b} + \frac{a^2b^2}{a^2} - \frac{l(l+1)}{4}(a^2 - 1)\right\}\chi + \left\{a\ddot{a} + \ddot{b} + a^2b^2 + a^2b^2 - \frac{l(l+1)}{4}(a^2 - 1)\right\}\psi + \left\{-\dot{a}a + \frac{1}{a} + \frac{1}{2}a\dot{b}^2 + \frac{1}{2}a^2b\dot{b}^2\right\} + \left\{\frac{1}{2}b^2(\chi + \chi'') + \frac{1}{2}(a^2 + b^2)\dot{\psi} - \frac{1}{2}(b^2 - 1)\psi' + \frac{1}{2a^2b^2}\right\} = 0.
\] (132)

This equation (132), having both the \(\ddot{\zeta}\) and \(\psi''\) terms, resembles a wave equation in \(\chi(t, r)\) and \(\psi(t, r)\). One can see that unlike the axial case, the solutions for polar modes cannot be derived easily even in the vacuum case for \(l \geq 2\). However, we may proceed further only if, after inserting \(l = 2\) and substituting equation (21), we make certain assumptions, such as neglecting the perturbation due to \(\chi(t, r)\) (or \(\psi(t, r)\)), or assuming \(\zeta(t, r)\) to be constant, or assuming \(\chi(t, r)\) and \(\psi(t, r)\) to be proportional to each other. Before proceeding with these assumptions, we write down the polar perturbation equations which will be used in the subsequent analysis:

\[
-3\chi + \frac{2}{3}K^2t^{1/3}\chi' + \left(\frac{2}{3}K^2t^{1/3} - \frac{1}{3t}\right)\psi' + \psi' = 0,
\] (133)

\[
-\frac{1}{3t}\chi + \chi' + \left(\frac{1}{K^2t^{4/3}} - 1\right)\psi' = 0
\] (134)

and

\[
-3\chi + \left(\frac{4}{9b^2} + \frac{3}{K^2t^{4/3}} - 3\right)\psi + \frac{2}{3}K^2t^{1/3}\chi + \left(\frac{2}{3}K^2t^{1/3} - \frac{2}{3t}\right)\psi + \psi' = 0.
\] (135)

We now attempt to analyse these equations in particular cases.

Case 1:
Let us assume that \(\chi(t, r)\) and \(\psi(t, r)\) are proportional to each other, i.e.
\[
\psi(t, r) = q\chi(t, r).
\] (136)
$q$ being a constant, then equation (135) leads to:

$$
\left(-3 - 3q + \frac{4q}{9t^2} + \frac{3q}{K^2t^{4/3}}\right) \chi + \left(\frac{2}{3}K^2t^{1/3} + \frac{2}{3}qK^2t^{1/3} - \frac{2q}{3t}\right) \tilde{\chi} + q\tilde{\chi} = 0.
$$

(137)

Its solution is given by

$$
\chi(t, r) = t^{1/3} \exp\left(\frac{27t^{2/3}}{4K^2}\right) (tF_1(r)B_1 + F_2(r)B_2),
$$

(138)

where

$$
B_1 = \text{HeunB}\left(\frac{3}{2}, \frac{54q^2(1 + q)}{(-2q(1 + q))^{3/2}K^3}, -\frac{4(1 + q)K^6 - 729q}{8K^6(1 + q)}, -\frac{27q}{K^3\sqrt{-2q(1 + q)}}, \frac{\sqrt{-2q(1 + q)Kt^{2/3}}}{2q}\right),
$$

$$
B_2 = \text{HeunB}\left(-\frac{3}{2}, \frac{54q^2(1 + q)}{(-2q(1 + q))^{3/2}K^3}, -\frac{4(1 + q)K^6 - 729q}{8K^6(1 + q)}, -\frac{27q}{K^3\sqrt{-2q(1 + q)}}, \frac{\sqrt{-2q(1 + q)Kt^{2/3}}}{2q}\right).
$$

(139)

$B_1$ and $B_2$ are two different biconfluent Heun’s functions, and $F_1(r)$ and $F_2(r)$ are undetermined functions of $r$. Hence, equation (136) gives

$$
\psi(t, r) = qt^{1/3} \exp\left(\frac{27t^{2/3}}{4K^2}\right) (tF_1(r)B_1 + F_2(r)B_2).
$$

(140)

Moreover, from equation (133), we get

$$
\zeta(t, r) = \frac{1}{9K^2} \exp\left(\frac{27t^{2/3}}{4K^2}\right) \left[\frac{2}{3^{5/3}} \left\{(1 + q)K^4t^{10/3} + \frac{27}{4}qK^{2/3}\right\} B_1 F'_1 + 2 \left\{(1 + q)K^4t^{2/3} + \frac{27}{4}q\right\} B_2 F'_2 + K^3\sqrt{-2q(1 + q)} \left\{t\tilde{B}_1 F'_1 + \tilde{B}_2 F'_2\right\}\right].
$$

(141)

The tilde over B’s denotes the $z$-derivative of the biconfluent Heun’s function, $\text{HeunB}(\alpha, \beta, \gamma, \delta, z)$.

**Case 2:**

Putting $\chi(t, r) = 0$, equation (135) reduces to:

$$
\left(-3 + \frac{4}{9t^2} + \frac{3q}{K^2t^{4/3}}\right) \psi + \left(\frac{2}{3}K^2t^{1/3} - \frac{2}{3t}\right) \psi + \tilde{\psi} = 0,
$$

(142)

which is solved by

$$
\psi(t, r) = t^{1/3} \exp\left(-\frac{t^{2/3}(K^4t^{2/3} + 27)}{4K^2}\right) (tF_3(r)B_3 + F_4(r)B_4),
$$

(143)

with

$$
B_3 = \text{HeunB}\left(\frac{3}{2}, \frac{27\sqrt{2}}{2K^3}, \frac{(4K^6 + 729)}{8K^6}, -\frac{27\sqrt{2}}{2K^3}, \frac{\sqrt{2}Kt^{2/3}}{2}\right),
$$

$$
B_4 = \text{HeunB}\left(-\frac{3}{2}, \frac{27\sqrt{2}}{2K^3}, \frac{(4K^6 + 729)}{8K^6}, -\frac{27\sqrt{2}}{2K^3}, \frac{\sqrt{2}Kt^{2/3}}{2}\right).
$$

(144)

(145)

Inserting this in equation (133) gives

$$
\zeta(t, r) = \frac{1}{18K^2t^{5/3}} \exp\left(-\frac{t^{2/3}(K^4t^{2/3} + 27)}{4K^2}\right) \left\{6K^2t^2 - 27t^{8/3}\right\} B_3 F'_3 - 27t^{8/3}B_4 F'_4 + 2\sqrt{2}K^3 \left\{t^{8/3}B_3 F'_3 + t^{5/3}B_4 F'_4\right\}.
$$

(146)
Here $F_3(r)$ and $F_4(r)$ are undetermined functions of $r$. As in the previous case, two different biconfluent Heun’s functions represented by $B_3$ and $B_4$ appear in the $t$-solution. The tilde over ‘$B$’ s indicates the respective $z$-derivatives.

**Case 3:**
Now we set $\psi(t,r)$ and its derivatives to zero so that equation (135) becomes:

$$-3\chi + \frac{2}{3}K^2t^{1/3}\dot{\chi} = 0.$$  \hfill (147)

In its solution:

$$\chi(t,r) = F_5(r) \exp \left[ \frac{27t^{2/3}}{4K^2} \right],$$  \hfill (148)

$F_5(r)$ remains undetermined. Subsequently, we obtain

$$\zeta(t,r) = \frac{2}{9}K^2t^{1/3} \exp \left[ \frac{27t^{2/3}}{4K^2} \right] F_6'.$$  \hfill (149)

**Case 4:**
Lastly, we choose $\zeta(t,r) = \text{constant} = 1$. This choice eliminates the derivatives of $\zeta$ from the perturbation equations. We will show in Discussions II that equation (132) under this particular assumption reduces to an equivalent of the Zerilli equation. From equation (134), we get

$$\dot{\chi} = -\frac{1}{3t} + \left( \frac{1}{K^2t^{2/3}} - 1 \right) \psi.$$  \hfill (150)

Integrating w.r.t. $r$ and setting the integration constant to zero, equation (150) gives

$$\chi(t,r) = -\frac{r}{3t} + \left( \frac{1}{K^2t^{2/3}} - 1 \right) \psi(t,r),$$  \hfill (151)

which when substituted in equation (135) yields an equation in $\psi(t,r)$ only:

$$\ddot{\psi} - \frac{4}{9t^2} \psi + \frac{2}{9} \frac{K^2r}{t^{5/3}} + \frac{r}{t} = 0.$$  \hfill (152)

Hence, we arrive at

$$\psi(t,r) = F_6(r)t^{4/3} + F_7(r)t^{-1/3} + \frac{1}{12}rt^{1/3}(27t^{2/3} + 4K^2),$$  \hfill (153)

$$\chi(t,r) = -\frac{r}{3t} + \left( \frac{1}{K^2t^{2/3}} - 1 \right) \left[ F_6(r)t^{4/3} + F_7(r)t^{-1/3} + \frac{1}{12}rt^{1/3}(27t^{2/3} + 4K^2) \right].$$  \hfill (154)

Here also, we obtain $F_6(r)$ and $F_7(r)$ as undetermined functions of $r$.

*Discussions II*

In all the cases discussed here, we find that the radial part of the solutions remain undetermined. However, on inspection, it can be shown from the set of perturbation equations that

$$\zeta(t,r) \propto R'_\mu(r) \quad \text{or} \quad \zeta(t,r) \propto \int R_\mu(r)dr, \quad \mu = \chi, \psi.$$  \hfill (155)
\( \mathcal{R}(r) \) is the radial part of the perturbations \( \chi(t, r) \) and \( \psi(t, r) \). This indicates that \( \mathcal{R}(r) \) will behave sinusoidally. Hence, one can express it in the form:

\[
\mathcal{R}_\mu(r) = \lambda_1 \sin(Mr) + \lambda_2 \cos(Mr).
\]  
(156)

Subsequently,

\[
|\mathcal{R}_\zeta(r)| = \lambda_3 [\lambda_1 \cos(Mr) - \lambda_2 \sin(Mr)].
\]  
(157)

Here, \( \lambda_1, \lambda_2, \lambda_3 \) and \( M \) are constants. Since the \( t \) and \( r \)-solutions separate as a product in the Regge-Wheeler formalism, the polar perturbation terms can be split as:

\[
\nu(t, r) = T_\nu(t)R_\nu(r), \quad \nu = \chi, \psi, \zeta, 
\]  
(158)

the \( T_\nu(t) \) solutions being explicitly obtained in equations (138)-(141), (143)-(146), (148)-(149), (153)-(154), and the \( R_\nu(r) \) solutions in (156) and (157). However, in Case 4, where \( \zeta(t, r) = \text{constant} = 1 \), the solution has a slightly different nature. In addition to the terms containing undetermined functions of \( r \), another term involving only \( r \) multiplied by some powers of \( t \) appears here. The temporal part of the solution is much simpler as compared to that in the other cases. Inserting suitable values for the constants, we have generated the 3-dimensional plots (Figs.(4)-(7)) of the perturbations for each set of the polar solutions.

![Graphs showing perturbations](image)

**FIG. 4**: The vertical axes represent the polar perturbations in Case 1: (a) \( \chi(t, r) \) given by Eqn.(138), and (b) \( \zeta(t, r) \) given by Eqn.(141).

We can determine an order-of-magnitude estimate of the frequency of the polar waves to be lying approximately in the range 1000-2000 Hz from the plot of the temporal part of the \( \chi(t, r) \)-solution in Case 1 with \( q = K = 1 \). The strain generated by the GWs of a given frequency can be used to constrain the perturbation parameters just as we have discussed in the case of Kantowski-Sachs background (Sec.7 of [62]). Following the same procedure, we find that the strain in Bianchi I spacetime (with \( \theta = \pi/45 \)) is roughly four times the strain obtained in the Kantowski-Sachs spacetime [62]. The constraints on the unknown constants are found to be similar in both the spacetimes, so as to keep the magnitude of strain to be in agreement with the observational data: \( \sim 10^{-24} \) [95, 96] for the specified frequency range.

Now, from equation (132), we have

\[
\left( \frac{3}{2} - \frac{3}{2Kt^{2/3}} - \frac{K^2}{9t^{2/3}} \right) \chi + \left( \frac{3}{2} - \frac{3}{2Kt^{2/3}} + \frac{8}{9Kt^{8/3}} - \frac{K^2}{9t^{2/3}} \right) \psi + \left( \frac{2}{3} K^2 t^{1/3} + \frac{K}{6t^{1/3}} \right) \dot{\chi} + \left( \frac{2}{3} K^2 t^{1/3} + \frac{K}{6t^{1/3}} \right) \dot{\psi} - K^2 t^{1/3} \xi' + \frac{\dot{K}^2 t^{4/3}}{2} \ddot{\chi} + \frac{1}{2} \left( K^2 t^{4/3} + \frac{1}{Kt^{2/3}} \right) \ddot{\psi} - \frac{1}{2} K^2 t^{4/3} \chi'' - \frac{1}{2} (K^2 t^{4/3} - 1) \psi'' = 0.
\]  
(159)
Using the relation (151) between \(\chi(t, r)\) and \(\psi(t, r)\) when \(\zeta(t, r)\) is assumed to be unity, equation (159) reads as

\[
\frac{1}{2} \left(1 + \frac{1}{Kt^{2/3}} \right) \psi - \frac{2}{3} \left(1 + \frac{1}{Kt^{5/3}} \right) \dot{\psi} + \left(\frac{3}{2K^2 t^{4/3}} + \frac{2}{3K t^{8/3}} - \frac{3}{2K^3 t^2} + \frac{5}{9t^2}\right) \psi + \left\{\frac{Kr}{18t^{2/3}} + \frac{r}{2K t^{5/3}} - \frac{2K^2 r}{27t^{5/3}} - \frac{r}{2t}\right\} = 0. 
\] (160)

This equation involving only one polar perturbation variable can be said to be the analogue of the Zerilli equation for the Bianchi I background. The pre-factor of \(\psi(t, r)\) behaves as a potential. One can compare it with Eq.(101) of Ref.[97]. An additional \(\dot{\psi}\)-term is responsible for damping. However, the terms containing the second-order \(r\)-derivatives of \(\psi(t, r)\) disappear here. The equation is inhomogeneous due to the presence of the terms within the curly brackets.

**VII. CONCLUSIONS**

We have investigated the axial and polar perturbations to a matter-filled Bianchi I spacetime using the Regge-Wheeler gauge. To begin with, we derived the solutions of the background field equations. The relation \(a = b^n\) among the scale factors imply a matter content which behaves like a stiff perfect fluid. We have solved for the scale factors, the fluid pressure and energy density as explicit functions of time. The numerical value of \(n\) can be exactly determined.
for the stiff fluid as well as for the vacuum case from the conditions proposed by Jacobs [90]. The two different values of $n$ subsequently leads to somewhat different nature of the GW solutions in the presence and absence of matter. No such condition was available for determining Kantowski-Sachs background solutions, the $n$-value remaining arbitrary for the stiff fluid (we assumed $n = 1/2$), but was obtainable for the vacuum background [62].

Moving on to the perturbed metric, the axial and polar perturbation equations are treated separately. In either case, these equations yield the $t$ and $r$-solutions of the perturbing terms. The $\theta$-dependence is defined by the term $\sin \theta (\partial_b Y)$ in the axial case (23) and by $Y(\theta)$ in the polar case (26). In accordance with [10], the $\phi$-dependence has been removed at the beginning by choosing $m = 0$. Thus every perturbing element can be expressed by a product of four terms, each being a function of only one of the coordinates $t, r, \theta$ and $\phi$.

In Sec. V, we have dealt with the axial perturbations. First we derived the linearised field equations for the axially perturbed metric (24). Then we obtained the wave equation and the respective analytical solutions in terms of $Q(t, r)$, from which the axial perturbations $h_0(t, r)$ and $h_1(t, r)$ are evaluated. As pointed out earlier, equation (66) can be called the Regge-Wheeler equation in B-I background. In the wave equation (66), the pre-factor of $Q$ serves as an effective potential (analogous to Ref.[10], Ref.[13], Ref.[97]). Therefore, it is nothing but a wave equation in a hypersurface. We can infer that this damping originates from the anisotropy of the B-I background. It is known that unlike in FLRW universe [25, 28], damping terms appear in LTB background [44]. Such damping has been found to occur in Kantowski-Sachs spacetime [62] also. In all the cases analysed here ((75)-(84)), the solutions of $Q(t, r)$, and hence $h_0(t, r)$ and $h_1(t, r)$ in the matter-filled spacetime are very similar. They are in the form of combinations of hypergeometric and modified Bessel functions of first kind. To arrive at physical solutions, one has to consider the real part of the imaginary terms occurring here. When $\alpha$ (the power of $r$), is non-zero, as in Set II, irrespective of the azimuthal fluid velocity $U$ (Eqs.(79) and (84)), $r$ appears as a multiplicative factor in the denominator. For different $U$-values, but the same set of $\alpha$ and $\beta$ (Eqs.(76) and (82)), the difference is due to the presence of $r$ as a coefficient of the second term (sum of hypergeometric and Bessel functions) in the numerator of (82). Making assumptions for determining $U$, in particular the normalisation condition (31), is a crucial step in finding complete solutions for the perturbations $h_0(t, r)$ and $h_1(t, r)$. Its value depends explicitly on $\theta$ and $b(t)$, and sometimes on $r$ (See Eqs.(71), (73), (74)). In our work, we have considered $O(\epsilon^2) \sim \epsilon^2 r^2$. Sharif and his collaborators derived the expression for the azimuthal velocity from the solutions of the axial perturbations [28]. However, in B-I spacetime, the equations are not as simple as in [28] because of the different scale factors, and we assume possible expressions for $U$ in order to find the axial solutions. The solutions become easier to derive in absence of matter. The expressions for $h_0(t, r)$ and $h_1(t, r)$ are obtained in terms of Heun’s biconfluent functions (temporal part) and sinusoidal functions (radial part) (see Ref.[63]). Unlike in the case of matter-filled spacetime, the vacuum perturbation equations do not require additional assumptions and can be solved by the method of separation of variables.

We have used $l = 2$ for wave-like solutions in both the axial and polar cases. For $l = 0$ the spherical harmonics are characterised by spherical symmetry. However, $l \geq 0$ indicates deviation from spherical symmetry, which renders the quadrupole moment non-zero, and hence important in gravitational radiations. According to Clarkson et al. [44], scalars on $S^2$ can be expressed as a sum over polar modes, and higher-rank tensors as sums over both the polar and axial modes. We need to consider $l \geq 2$ in order to take into account the axial modes coming from the expansion of both vector and tensor functions. The value of $l$ determines the height of the effective potential barrier, given by the coefficient of $Q$ in the wave equation for axial modes [10]. A change in the value of $l$ brings about only a small change
in the Heun’s function, without affecting the radial solution [63]. Consequently, the expressions for \( h_0(t, r) \) and \( h_1(t, r) \) will slightly change. Although no axial perturbations with \( l = 0 \) are feasible [44], there exist polar perturbations for all values: \( l = 0, 1, 2 \ldots \). In the \((0, 0)\) element of the polar perturbation matrix (Eqn.(26)), a third term \( \eta(t, r) \) appears in addition to \( \chi(t, r) \) and \( \psi(t, r) \). Studies have shown that \( \eta \) vanishes in the FLRW background [26, 36]. But Clarkson et al. [44] remarked that when \( l = 0 \) or 1, i.e. for large angle fluctuations, the field equations do not yield \( \eta = 0 \). The assumption \( \eta = 0 \) itself acts as a constraint equation [44]. We have treated the zero and non-zero values of \( \eta \) in separate subsections.

The stiff fluid conditions have been used to simplify the polar perturbation equations in presence of matter. In this case, under certain assumptions, we have been able to extract a solution (Eqn.(122)) for the polar perturbation \( \chi(t, r) \). However, this solution carries an undefined function depending on both \( t \) and \( r \), i.e., the temporal and radial parts do not separate out as products and therefore cannot be considered as a valid solution in the context of RW gauge. Since the absence of matter further simplifies the perturbation equations, we have analysed the vacuum case in details. We find that the polar perturbation equations, even with \( \eta = 0 \), contain far more complicated couplings among the perturbing elements than the axial perturbations to the B-I background (and Kantowski-Sachs background) and also their FLRW counterparts. Unlike in the case of FLRW background [26, 29, 36], no perturbation equation in a single variable can be obtained in the B-I case, thereby complicating the derivation of analytical solutions for the polar modes. However, if we assume a constant value for the perturbation variable \( \chi(t, r) \), we can extract an equation in only \( \psi(t, r) \) (please see Eqn.(160)), which is comparable to the Zerilli equation [97] for polar perturbations. Following Clarkson et al. [44], we have solved the equations analytically in some particular cases, such as assuming \( \chi(t, r) \) and \( \psi(t, r) \) proportional to each other, or \( \chi(t, r) = \) constant, or switching off either \( \chi(t, r) \) or \( \psi(t, r) \). In each of these cases, the polar solutions may be expressed as a product of a radial function which is sinusoidal in nature, and a temporal function which is again a combination of biconfluent Heun’s functions and their derivatives (Eqns.(138)-(141), (143)-(146)) or contains an exponential function (Eqns.(148), (149)), or some powers of \( t \). In the last case, the polar solution is much simpler compared to the other three. The figures in (4)-(7) show the nature of these solutions. The combined effects of these complicated perturbing terms will result in the spacetime perturbations.

The effects of axial and polar GWs on the material medium, which are already established in the FLRW background [26, 28, 29, 33], have been found to hold in the Bianchi I background also. The azimuthal velocity of the fluid is perturbed by the axial modes only. The non-zero value of azimuthal velocity \( U \) indicates the cosmological rotation of the fluid induced by the propagating axial GWs. The remaining components are influenced by the polar GWs. Besides, the energy density and pressure undergo deformation due to the polar modes but remain unaffected in the presence of axial waves. We have shown that polar modes cannot exist if the matter perturbations vanish. Thus the polar GWs must be followed up by matter inhomogeneities and anisotropies. The same effects have been demonstrated in our studies on Kantowski-Sachs background [62]. Comparing the corresponding results in Bianchi I and Kantowski-Sachs spacetimes, we find that both the axial and polar GWs are much alike in nature. The nature of temporal and radial solutions of the axial perturbation equations in vacuum are very similar. Due to the difference in the term containing \( l \) in the wave equation, a minor difference appears only in the Heun’s function. The polar solutions in vacuum also closely resemble one another, except for differences in the numerical factors within the exponential and Heun’s functions. The assumption that cot \( \theta \equiv 1/\theta \), valid for small values of \( \theta \), has been made in B-I case, but is not required in the KS background. This is because of the nature of the line element of the background spacetime.

It has been shown through rigorous calculations in [44] that the gauge-invariant perturbation parameters in the inhomogeneous LTB model comprise of a cumbersome mixture of scalar, vector and tensor modes. Their couplings further complicate the evolution equations. Thus these gauge-invariants are much different from those in FLRW model. On the other hand, spherically symmetric spacetimes do not favour SVT decomposition [47]. Likewise, we observe that the comparison of the corresponding results in Bianchi I and FLRW spacetimes is non-trivial. Equating the two scale factors responsible for anisotropy is not sufficient to reproduce the results in the isotropic limit because we cannot extract the master equation in terms of a single variable. Only when considered in a general gauge chosen suitably as in [44], can the perturbations in the two models be matched.

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