The Thompson-Higman monoids $M_{k,i}$: the $J$-order, the $D$-relation, and their complexity

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Abstract

The Thompson-Higman groups $G_{k,i}$ have a natural generalization to monoids, called $M_{k,i}$, and inverse monoids, called $Inv_{k,i}$. We study some structural features of $M_{k,i}$ and $Inv_{k,i}$ and investigate the computational complexity of related decision problems. The main interest of these monoids is their close connection with circuits and circuit complexity.

The maximal subgroups of $M_{1,1}$ and $Inv_{1,1}$ are isomorphic to the groups $G_{k,j}$ ($1 \leq j \leq k - 1$); so we rediscover all the Thompson-Higman groups within $M_{k,1}$.

Deciding the Green relations $\leq_J$ and $\equiv_D$ of $M_{k,1}$, when the inputs are words over a finite generating set of $M_{k,1}$, is in $P$.

When a circuit-like generating set is used for $M_{k,1}$ then deciding $\leq_J$ is $\text{coNP}$-complete (where $\text{DP}$ is the complexity class consisting of differences of sets in $\text{NP}$). The multiplier search problem for $\leq_J$ is $\text{xNP}$-$\text{search}$-complete, whereas the multiplier search problems of $\leq_R$ and $\leq_L$ are not in $\text{xNP}$-$\text{search}$ unless $\text{NP} = \text{coNP}$. We introduce the class of search problems $\text{xNP}$-$\text{search}$ as a slight generalization of $\text{NP}$-$\text{search}$.

Deciding $\equiv_D$ for $M_{k,1}$ when the inputs are words over a circuit-like generating set, is $\oplus_{k-1} \circ \text{NP}$-complete; for any $h \geq 2$, $\oplus_h \circ \text{NP}$ is a modular counting complexity class, whose verification problems are in $\text{NP}$. Related problems for partial circuits are the image size problem (which is $\# \circ \text{NP}$-complete), and the image size modulo $h$ problem (which is $\oplus_h \circ \text{NP}$-complete). For $Inv_{k,1}$ over a circuit-like generating set, deciding $\equiv_D$ is $\oplus_{k-1} \circ P$-complete. It is interesting that the little known complexity classes $\text{coDP}$ and $\oplus_{k-1} \circ \text{NP}$ play a central role in $M_{k,1}$.

1 Introduction

The Thompson-Higman groups $G_{k,i}$, introduced by Graham Higman in [20], can be generalized in a straightforward way to monoids, denoted $M_{k,i}$, and inverse monoids, denoted $Inv_{k,i}$. The generalization of $G_{k,1}$ to $M_{k,1}$ and $Inv_{k,1}$, was given in [4]. The definition of $M_{k,i}$ for $i \geq 2$ is a straightforward combination of the definitions of $M_{k,1}$ and $G_{k,i}$. In brief, $M_{k,i}$ consists of all maximally extended right ideal homomorphisms between right ideals of $BA^*$, where $A$ and $B$ are finite alphabets with $|A| = k \geq 2$ and $|B| = i \geq 1$. Detailed definitions of $M_{k,i}$ and $Inv_{k,i}$ appear below.

This paper is a continuation of our study of monoid generalizations of the Thompson-Higman groups. As in [4] [3], our motivations are the following: (1) The generalization of $G_{k,i}$ to a monoid or an inverse monoid is natural and straightforward; (2) the monoids $M_{k,i}$ and $Inv_{k,i}$ have interesting and surprising properties; (3) for certain infinite generating sets, the elements of $M_{2,1}$ are similar to circuits, with word-length polynomially equivalent to circuit-size.

The definition of $M_{k,i}$ requires some preliminary notions, most of which are familiar from formal language theory, or information theory, or algebra. Let $A$ and $B$ be finite alphabets with $|A| = k \geq 2$ and $|B| = i \geq 1$. By $A^*$ we denote the set of all words over $A$, including the empty word $\varepsilon$. A right ideal of $A^*$ is any set $R \subseteq A^*$ such that $R = RA^*$.
We consider the set $BA^*$, i.e., the set of all words of the form $b_2 x$ with $b_2 \in B$ and $x \in A^*$. Although $BA^*$ is not a monoid with respect to concatenation, we can nevertheless define the concept of a right ideal of $BA^*$: It is any set of the form $B_0 R$, where $B_0 \subseteq B$ and where $R \subseteq A^*$ is any right ideal of $A^*$. A right ideal $R$ is essential iff all right ideals of $BA^*$ intersect $R$. (We say that two sets $S_1$ and $S_2$ intersect iff $S_1 \cap S_2 \neq \emptyset$. ) For right ideals $R_2 \subseteq R_1 \subseteq BA^*$, we say that $R_2$ is essential in $R_1$ iff all the right ideals that intersect $R_2$ also intersect $R_1$. Two right ideals $R_2$ and $R_1$ of $BA^*$ are essentially equal iff every right ideal of $BA^*$ that intersects $R_2$ intersects $R_1$, and vice versa; in that case we write $R_2 =_{\text{ess}} R_1$. If $R_2 =_{\text{ess}} R_1$ then $R_2 =_{\text{ess}} R_1 \cap R_2$.

A prefix code in $BA^*$ is any set $P \subseteq BA^*$ such that no word in $P$ is a prefix of another word in $P$; hence, a prefix code of $BA^*$ is of the form $B_0 Q$ for some $B_0 \subseteq B$ and some prefix code $Q \subseteq A^*$. A set $P \subseteq BA^*$ is a maximal prefix code iff $P$ is a prefix code which is not a strict subset of any other prefix code in $BA^*$.

A right ideal homomorphism over $BA^*$ is a total surjective function $\varphi : R_1 \rightarrow R_2$ such that $R_1, R_2$ are right ideals of $BA^*$, and such that for all $r_1 \in R_1$ and all $x \in A^*$ : $\varphi(r_1 x) = \varphi(r_1) x$. A right ideal isomorphism over $BA^*$ is a homomorphism, as above, such that the domain $R_1$ and the image $R_2$ are essential ideals, and such that $\varphi$ is bijective. Two right ideal homomorphisms $\psi : Q_1 \rightarrow Q_2$ and $\varphi : R_1 \rightarrow R_2$ are essentially equal iff $Q_1 =_{\text{ess}} R_1$ and $\psi$ agrees with $\varphi$ on $Q_1 \cap R_1$; this implies that we also have $Q_2 =_{\text{ess}} R_2$.

Every right ideal homomorphism $\varphi$ over $BA^*$ has a unique maximal essentially equal extension to a right ideal homomorphism of $BA^*$ (which is denoted $\max(\varphi)$). This can be proved in the same way as for right ideal homomorphisms over $A^*$ (see Prop. 1.2 in [4] and Prop. 2.1 in [8]). When $\varphi$ is an isomorphism, $\max(\varphi)$ is also an isomorphism.

To define $G_{k,i}$ we let the underlying set consist of all maximally extended right ideal isomorphisms between essential right ideals of $BA^*$. The multiplication of $G_{k,i}$ is functional composition, followed by maximal extension (to a maximal right ideal isomorphism). This is similar to the definition of $G_{k,1}$ in [8]; a similar definition (with a different terminology) appears in [27]. We define the monoid $M_{k,i}$ by using maximally extended essentially equal right ideal homomorphisms between right ideals of $BA^*$. The multiplication is composition followed by maximal essentially equal extension. This is similar to the definition of $M_{k,1}$ in [4]. Along similar lines one can define $\Inv_{k,i}$, consisting of all maximally extended essentially equal right ideal isomorphisms between (not necessarily essential) right ideals of $BA^*$.

Comparing with the definition of $\Inv_{k,1}$ in [4]. We do not need to assume that the alphabets $A$ and $B$ are disjoint. We refer to Section 1 of [4], Section 1 of [3] for terminology that is not defined here.

Here are some nice facts about $G_{k,i}$ (discovered by Higman [20], see also [28]):

- If $i \equiv j \mod k - 1$ then $G_{k,i}$ and $G_{k,j}$ are isomorphic (Coroll. 2, page 12 in [20]). So, in the notation “$G_{k,i}$” we can (and will) always assume that $1 \leq i \leq k - 1$. We will show that this holds for $M_{k,i}$ too.
- By Theorem 6.4 in [20]: If $h \neq k$ then $G_{k,i}$ is not isomorphic to $G_{k,j}$ (for any $i, j$). Also, when $\gcd(k - 1, i) \neq \gcd(k - 1, j)$ then $G_{k,i}$ is not isomorphic to $G_{k,j}$. We will show that this holds for $M_{k,i}$ too. E.g., for all $1 \leq i \leq k - 2$, $G_{k,i}$ is not isomorphic to $G_{k,k - 1}$, and $M_{k,i}$ is not isomorphic to $M_{k,k - 1}$.
- However (Theorem 7.3 in [20]), if $d$ divides $k$ then $G_{k,i}$ is isomorphic to $G_{k,di}$ (where $di$ is taken mod $k - 1$). E.g., when $k$ is even, $G_{k,1}$ is isomorphic to $G_{k,2}$ and to $G_{k,k/2}$. Hence by all these observations: $G_{3,1} \not\cong G_{3,2}$, $G_{4,1} \cong G_{4,2} \not\cong G_{4,3}$.
- Every group $G_{k,i}$ is finitely presented. When $k$ is even, $G_{k,i}$ is a simple group, and when $k$ is odd, $G_{k,i}$ contains a simple subgroup of index 2.

We will show that the maximal subgroups of $M_{k,1}$ are isomorphic to the Higman groups $G_{k,i}$ (1 $\leq i \leq k - 1$). Thus, in $M_{k,1}$ we “rediscover” all the Higman groups $G_{k,i}$.

The monoid $M_{k,1}$ and the inverse monoid $\Inv_{k,1}$ are finitely generated [4].
Since $M_{k,1}$ acts partially on $A^*$, and in particular, $M_{2,1}$ acts partially on the set of all bit-strings \( \{0,1\}^* \), we can view the elements of $M_{k,1}$ as boolean functions. In order to formalize this connection between $M_{k,1}$ and combinational boolean circuits we will also use an infinite generating set for $M_{k,1}$, of the form $\Gamma \cup \tau$, where $\Gamma$ is any finite generating set of $M_{k,1}$, and $\tau$ consists of the letter position transpositions on strings. More precisely, $\tau = \{ \tau_{i,i+1} : i \geq 1 \}$, where $\tau_{i,i+1}(u x_i x_{i+1} v) = u x_{i+1} x_i v$, for all $u \in A^{i-1}$, $v \in A^*$, and $x_i, x_{i+1} \in A$.

Then, for every combinational circuit $C$ there is a word $w$ over $\Gamma \cup \tau$ such that: (1) the functions represented by $C$ and $w$ are the same, (2) $|w| \leq c \cdot |C|$ (for some constant $c$ which depends only on the choice of generators and gates). Here, $|C|$ is the size of the circuit $C$ (i.e., the number of gates, plus the number of wire crossings, plus the number of input or output ports), and $|w|$ is the length of the word $w$ over $\Gamma \cup \tau$; for this we define $|\tau_{i,i+1}| = i + 1$ and $|\gamma| = 1$ for all $\gamma \in \Gamma$.

Conversely, if a function $f : A^n \to A^n$ is represented by a word $w$ over $\Gamma \cup \tau$ then $f$ has a combinational circuit $C$ with $|C| \leq c \cdot |w|^2$ (for some constant $c$). See [5], Section 2.

We call a generating set of $M_{k,1}$ of the form $\Gamma \cup \tau$, as above, a circuit-like generating set.

The Green relations $\leq_J$, $\leq_L$, $\leq_R$, $\equiv_D$, and $\leq_H$ are classical concepts in the study of monoids (and semigroups), see e.g. [12] [17]. By definition, for any $u, v \in M$ (where $M$ is a monoid) we have: $u \leq_J v$ iff every ideal of $M$ containing $v$ also contains $u$; equivalently, $u \leq_J v$ iff there exist $x, y \in M$ such that $u = xvy$. Similarly, $u \leq_L v$ iff any left ideal of $M$ containing $v$ also contains $u$; equivalently, there exists $x \in M$ such that $u = vx$; the definition of $\leq_R$ is similar. By definition, $u \equiv_D v$ iff there exists $s \in M$ such that $u \equiv_R s \equiv_L v$; this is equivalent to the existence of $t \in M$ such that $u \equiv_L t \equiv_R v$.

The $H$-preorder is defined by $y \leq_H x$ iff $y \equiv_R x$ and $y \leq_L x$. For any pre-order $\leq_X$ we define the corresponding equivalence relation $\equiv_X$ by $y \equiv_X x$ iff $y \leq_X x$ and $x \leq_X y$.

In [3] we gave characterizations of $\leq_J$ and $\equiv_D$ in $M_{k,1}$. In [3] we characterized $\leq_L$ and $\leq_R$ in $M_{k,1}$, and we analyzed the computational complexity of deciding $\leq_L$ or $\leq_R$.

The main goal of this paper is to study the computational complexity of deciding $\equiv_D$ and $\leq_J$ in $M_{k,1}$. The problems of deciding whether $\psi \leq_J \varphi$, or deciding whether $\psi \equiv_D \varphi$, when $\psi$ and $\varphi$ are given by words over a finite generating set of $M_{k,1}$ (or of $Inv_{k,1}$), are in $P$. However, when the inputs $\psi$ and $\varphi$ are given by words over a circuit-like generating set, then deciding $\leq_J$ for $M_{k,1}$ is coNP-complete, and deciding $\equiv_D$ is $\oplus_{k-1}NP$-complete. The complexity class $DP$ (called “difference $P$”), introduced in [25], has not been used much in the literature; see Section 5 for details. The complexity class $\oplus_{k}NP$ (for a given $k \geq 2$) is a counting complexity class; it fits into a pattern that has appeared in the literature; but this particular class has never been studied; see Section 6 for details. There are related problems for circuits (see Sections 5 and 6) that are also complete for these unusual complexity classes. In addition, we study the complexity of some search problems associated with $\equiv_D$ and $\leq_J$ in $M_{k,1}$.

We characterize the complexity of deciding the Green relations of $Inv_{k,1}$. In particular, deciding whether $\psi \equiv_D \varphi$ when $\psi$ and $\varphi$ are given by words over $\Gamma_1 \cup \tau$ (where $\Gamma_1$ is a finite generating set of $Inv_{k,1}$), is $\oplus_{k-1}P$-complete. The class $\oplus_{h}P$ is a familiar counting complexity class. For details, see Section 7.

2 The maximal subgroups of $M_{k,1}$

We saw in [3] (Prop. 2.2 and Theorem 2.5) that $M_{k,1}$ has only one non-zero $J$-class, and that it has $k-1$ non-zero $D$-classes. These $D$-classes, denoted by $D_i$ for $i = 1, \ldots, k-1$, are given by

\[ D_i = \{ \varphi \in M_{k,1} : |imC(\varphi)| \equiv i \mod k-1 \}. \]

It is well known and easy to see that every subgroup of a semigroup is an $\equiv_H$-class, and that an $\equiv_H$-class $H$ is a group iff $H$ contains an idempotent. The $\equiv_H$-classes that contain an idempotent are the maximal subgroups of the semigroup, i.e., the subgroups that are not strictly contained in another
subgroup. It is well known and not hard to prove that all maximal subgroups of a same \(D\)-class are isomorphic (see e.g. [17] Prop. 2.1 and the remark that follows it, or [23] Coroll. 2.7).

We saw ([3] Prop. 2.1) that \(G_{k,1}\) is the group of units of \(M_{k,1}\) (i.e., the group of invertible elements). This implies that \(M_{h,1}\) is not isomorphic to \(M_{k,1}\) when \(h \neq k\) (since we know from [20] that \(G_{h,1}\) is not isomorphic to \(G_{k,1}\) when \(h \neq k\)). The fact that \(M_{k,1}\) has \(k - 1\) non-zero \(D\)-classes also implies \(M_{h,1} \not\cong M_{k,1}\) when \(h \neq k\).

The next theorem shows a very nice correspondence between the \(k - 1\) non-zero \(D\)-classes and the \(k - 1\) groups \(G_{k,i}\) \((1 \leq i \leq k - 1)\) that Higman introduced in [20]. It is surprising (at first) that all the \(G_{k,i}\) show up automatically in the structure of \(M_{k,1}\).

**Theorem 2.1** For every \(i\) \((1 \leq i \leq k - 1)\) we have: The maximal subgroups of the \(D\)-class \(D_i\) of \(M_{k,1}\) are isomorphic to the Thompson-Higman group \(G_{k,i}\).

**Proof.** In the \(D\)-class \(D_i\) we consider the idempotent \(\eta_i = \text{id}_{\{a_1, \ldots, a_i\}}\), i.e., the partial identity map that is defined on those (and only those) words that start with a letter in \(\{a_1, \ldots, a_i\}\). Since \(|\text{im}(\eta_i)| = i\) we have indeed \(\eta_i \in D_i\). Consider the set

\[ G_{\eta_i} = \{ \varphi \in M_{k,1} : \text{Dom}(\varphi) \text{ and } \text{im}(\varphi) \text{ are essential right subideals of } \{a_1, \ldots, a_i\}A^* \}. \]

The set \(G_{\eta_i}\) is a subgroup of \(M_{k,1}\), with identity element \(\eta_i\). Moreover, this group is isomorphic to \(G_{k,i}\); an isomorphism is obtained by replacing each \(b_jw \in BA^*\) \((1 \leq j \leq i)\) by \(a_jw \in AA^*\). Clearly, the subgroup \(G_{\eta_i}\) is contained in the \(H\)-class of \(\eta_i\).

Conversely, suppose \(\varphi \equiv_H \eta_i\). Then \(\varphi\) is injective with domain essentially equal to \(\{a_1, \ldots, a_i\}A^*\) (since \(\varphi \equiv_L \eta_i\), and by the characterization of \(\leq_L\) in Section 3.4 of [3]). And the image of \(\varphi\) is essentially equal to \(\{a_1, \ldots, a_i\}A^*\) (since \(\varphi \equiv_R \eta_i\), and by the characterization of \(\leq_R\) in Section 2 of [3]). It follows that \(\varphi \in G_{\eta_i}\), by the definition of \(G_{\eta_i}\). So \(G_{\eta_i}\) is the entire \(\equiv_H\)-class of \(\eta_i\), hence it is a maximal subgroup, in \(D_i\). Since all the maximal subgroups in the same \(\equiv_D\)-class \(D_i\) are isomorphic, every maximal subgroup of \(M_{k,1}\) is isomorphic to some \(G_{\eta_i}\) (which is itself isomorphic to \(G_{k,i}\)).

**3** **The Thompson-Higman monoids \(M_{k,i}\)**

In the Introduction we defined \(M_{k,i}\) by using two alphabets, \(A = \{a_1, \ldots, a_k\}\), and \(B_i = \{b_1, \ldots, b_i\}\). It follows from this definition that when \(1 \leq j \leq i\), \(M_{k,j}\) is a submonoid of \(M_{k,i}\) (not just up to isomorphism, but also as a subset).

The identity element of \(M_{k,i}\) can be described by the table \(\text{id}_{B_i} = \{(b, b) : b \in B_i\}\), and will be denoted by \(1\) (if \(k\) and \(i\) are clear from the context).

**Proposition 3.1** If \(s \equiv t \mod k - 1\) then \(M_{k,s} \cong M_{k,t}\).

**Proof.** It suffices to prove that for all \(n \geq k\) we have \(M_{k,n} \cong M_{k,n-(k-1)}\). We embed \(B_nA^*\) into \(B_{n-k+1}A^*\) by the map

\[ E : \begin{cases} b_i &\mapsto b_i \quad \text{for } i = 1, \ldots, n - k; \\ b_{i+n-k} &\mapsto b_{n-k+1}a_i \quad \text{for } i = 1, \ldots, k. \end{cases} \]

The image of this embedding is the essential right ideal \(B_{n-k}A^* \cup b_{n-k+1}AA^*\), which is an essential right sub-ideal of \(B_{n-k+1}A^*\). The embedding \(B_nA^* \hookrightarrow B_{n-k+1}A^*\) determines an embedding \(M_{k,n} \hookrightarrow M_{k,n-(k-1)}\) that we will also call \(E\). The embedding is surjective since it is the identity on the submonoid \(M_{k,n-(k-1)}\) of \(M_{k,n}\); hence the embedding is also a retract.

The embedding is a homomorphism: Consider any \(\psi, \varphi \in M_{k,1}\). After essential restrictions, if needed, we can assume that \(\varphi, \psi \in M_{k,n}\) have tables of the form \(\{(u_i, v_i) : i \in I\}\), respectively \(\{(v_j, w_j) : j \in J\}\), such that the set \(\{v_i : i \in I\} \cup \{v_j : j \in J\}\) is a prefix code. So the product \(\psi \varphi(.)\) is represented by the composition of these tables (without need to extend or restrict), i.e., \(\psi \varphi(.)\) has a
table \{(u_i, w_i) : i \in I \cap J\}. Then \(E(\psi) \cdot E(\varphi)(.)\) has a table \{(E(u_i), E(w_i)) : i \in I \cap J\}. On the other hand, by applying \(E\) to the table \{(u_i, w_i) : i \in I \cap J\} for \(\psi \varphi(.),\) we see that \{(E(u_i), E(w_i)) : i \in I \cap J\} is also a table for \(E(\psi \cdot \varphi)(.);\) hence, \(E(\psi \cdot \varphi)(.) = E(\psi) \cdot E(\varphi)(.). \) \(\square\)

From now on, when we write \(M_{k,i}\) we will always assume that \(1 \leq i \leq k - 1.\)

By definition, the group of units of a monoid \(M\) is the set of invertible elements of \(M;\) equivalently, the group of units is the maximal subgroup of \(M\) whose identity is the identity of the monoid.

**Proposition 3.2** The group of units of \(M_{k,i}\) and of \(\text{Inv}_{k,i}\) is \(G_{k,i}.\)

**Proof.** The proof is very similar to the proof of Prop. 2.1 in [4] (which shows that \(G_{k,1}\) is the group of units of \(M_{k,1}\)). \(\square\)

**Corollary 3.3** If \(G_{k,i} \not\cong G_{h,j}\) then \(M_{k,i} \not\cong M_{h,j}.\)

**Proof.** If two monoids have non-isomorphic groups of units then they are non-isomorphic. \(\square\)

**Theorem 3.4 (Green relations of \(M_{k,s}\)).** For all \(k \geq 2\) and \(s\) (with \(1 \leq s \leq k - 1\)) we have:

1. The monoids \(M_{k,s}\) and \(\text{Inv}_{k,s}\) are \(0\)-\(J\)-simple, i.e., they have only one non-zero \(J\)-class.
2. \(M_{k,s}\) and \(\text{Inv}_{k,s}\) are congruence-simple.
3. For all \(\psi, \varphi \in M_{k,s}\) (or \(\text{Inv}_{k,s}\)): \(\varphi \equiv_D \psi\) iff \(|\text{imC}(\varphi)| \equiv |\text{imC}(\psi)| \mod k - 1.\) Hence \(M_{k,s}\) and \(\text{Inv}_{k,s}\) have \(k - 1\) non-zero \(D\)-classes.
4. The \(\leq_R\) and \(\leq_C\) preorders for \(M_{k,s}\) have the same characterizations as for \(M_{k,1}\) (Theorems 2.1 and 3.32 in [3]).

**Proof.** (1) The proof of Prop. 2.2 in [4] can easily adapted to \(M_{k,s}\) and \(\text{Inv}_{k,s}\).

When \(\varphi \in M_{k,s}\) (or \(\in \text{Inv}_{k,s}\)) is not the empty map, there exist \(b_mx_0, b_0y_0 \in B^*\) such that \(b_ny_0 = \varphi(b_m x_0).\) Let \(P = \{p_1, \ldots, p_s\} \subset A^*\) be a prefix code (not necessarily maximal) with \(|P| = s.\) Then we have \(\varphi(b_m x_0 p_i) = b_n y_0 p_i\) for \(i = 1, \ldots, s.\) Let us define \(\alpha, \beta \in \text{Inv}_{k,s}\) by the tables \(\alpha = \{(b_i, b_m x_0 p_i) : i = 1, \ldots, s\}\) and \(\beta = \{(b_0 y_0 p_i, b_i) : i = 1, \ldots, s\}.\) Then \(\beta \varphi \alpha(.) = \{(b_i, b_i) : i = 1, \ldots, s\}.\) So every non-zero element of \(M_{k,s}\) (or \(\text{Inv}_{k,s}\)) is in the same \(J\)-class as the identity element.

(2) The proof of congruence-simplicity is exactly the same as the proof of Theorem 2.3 in [4].

(3) The proof of Theorem 2.5 in [4] works for \(M_{k,s}\) and \(\text{Inv}_{k,s}\) too. Proposition 2.4 in [4] remains unchanged, and Lemma 2.6 becomes:

For all finite alphabets \(A, B,\) and every integer \(i \geq 0\) there exists a maximal prefix code in \(BA^*\) of cardinality \(|B| + (|A| - 1)i.\) And every finite maximal prefix code in \(BA^*\) has cardinality \(|B| + (|A| - 1)i\) for some integer \(i \geq 0.\)

**Lemma 2.7** remains unchanged.

Let \(A = \{a_1, \ldots, a_k\}\) and \(B = \{b_1, \ldots, b_s\},\) be the finite alphabets used in the definition of \(M_{k,s}.\) The statement of Lemma 2.8 becomes:

1. For any \(m \geq k + s - 1,\) let \(i\) be the residue of \(m - (s - 1)\) modulo \(k - 1\) in the range \(2 \leq i \leq k,\) and let us write \(m = s - 1 + i + (k - 1)j,\) for some \(j \geq 0.\) Then there exists a prefix code \(Q_{i,j}\) of cardinality \(|Q_{i,j}| = m,\) such that \(\text{id}_{Q_{i,j}}\) is an essential restriction of \(\text{id}_{\{b_1, \ldots, b_{s-1}, a_1, \ldots, a_k\}}.\) Hence \(\text{id}_{Q_{i,j}} = \text{id}_{\{b_1, \ldots, b_{s-1}, a_1, \ldots, a_k\}}\) as elements of \(\text{Inv}_{k,s}.\)

2. In \(M_{k,s}\) and in \(\text{Inv}_{k,s}\) we have \(\text{id}_{\{b_1, \ldots, b_{s-1}, a_1\}} \equiv_D \text{id}_{\{b_1, \ldots, b_{s-1}, a_1, \ldots, a_k\}} = 1.\)

In the proof of Lemma 2.8(1), \(Q_{i,j}\) is replaced by \(Q'_{i,j} = \{b_1, \ldots, b_{s-1}\} \cup b_1 Q_{i,j}.\)

Lemmas 2.9 and 2.10 are unchanged.

In the final part of Theorem 2.5 we replace the end of the first paragraph by the following: In particular, when \(|Q_1| \equiv s - 1 + i \mod k - 1\) (with \(1 \leq i \leq k),\) then \(\varphi_1 \equiv_D \text{id}_{\{b_1, \ldots, b_{s-1}, a_1, \ldots, a_k\}}.\)

(4) The proofs of Theorems 2.1 and 3.32 in [3] are straightforwardly generalized to \(M_{k,s}.\) We will see later in Proposition 7.1 that (4) also holds for \(\text{Inv}_{k,1};\) and for \(\text{Inv}_{k,s}\) the proof is similar. \(\square\)
Proposition 3.5 The maximal subgroups of $M_{k,i}$ are isomorphic to $G_{k,j}$ for $j = 1, \ldots, k-1$, with $G_{k,j}$ being isomorphic to the maximal subgroup of the $D$-class $D_j = \{ \varphi \in M_{k,i} : |\text{im}C(\varphi)| \equiv j \mod k - 1 \}$. The same is true for $\text{Inv}_{k,i}$.

Proof. This is similar to the proof of Theorem 2.1 above. In the $D$-class $D_j$ we can pick, for example, the idempotent $\text{id}_{\{b_1, \ldots, b_j\}}$ if $1 \leq j < s$, and we can pick the idempotent $\text{id}_{\{b_1, \ldots, b_s, a_1, \ldots, b_a a_{j+1}\}}$ if $s \leq j \leq k - 1$. □

Since the Green relations $J, D, R, L$ of $M_{k,i}$ are quite similar to those of $M_{k,1}$, we will focus on $M_{k,1}$ from now on.

4 Complexity of $\leq J$ and $\equiv_D$ over a finite generating set

We are interested in the difficulty of checking on input $\psi, \varphi \in M_{k,1}$, whether $\psi \leq J \varphi$, or $\psi \equiv_D \varphi$, or $\psi \equiv J \varphi$. In [3] we addressed the question whether $\psi \leq R \varphi$, or $\psi \leq L \varphi$. We assume at first that $\psi, \varphi \in M_{k,1}$ are given either by tables, or by words over a chosen finite generating set $\Gamma$ of $M_{k,1}$. Recall that $M_{k,1}$ is finitely generated (Theorem 3.4 in [8]). For computational complexity it does not matter much which finite generating set of $M_{k,1}$ is used; finite changes in the generating set only lead to linear changes in the complexity.

Let $0$ denote the zero element of $M_{k,1}$ (represented by the empty map), and let $1$ denote the identity element of $M_{k,1}$ (represented by the identity map on $A^*$).

Checking whether $\psi \leq J \varphi$ is not difficult. Since $M_{k,1}$ is $0$-$J$-simple (Prop. 2.2 in [4]) we have: $\psi \leq J \varphi$ if $\varphi \neq 0$ or $\psi = 0$. Consider any element $\psi \in M_{k,1}$, given by a table or by a word over a chosen finite generating set $\Gamma$ of $M_{k,1}$. In order to check whether $\psi$ is equal to $0$, we calculate $\text{im}C(\psi)$, as an explicit list of words. If $\psi$ is given by a table, $\text{im}C(\psi)$ can be directly read from the table. If $\psi$ is given by a word over a finite generating set of $M_{k,1}$ we use Corollary 4.11 of [4] to find the list of elements of $\text{im}C(\psi)$ in polynomial time. To check whether $\psi = 0$ we now check whether $\text{im}C(\psi) = \varnothing$.

The relation $\psi \equiv_D \varphi$ can be checked in deterministic polynomial time, by using the characterization of $\equiv_D$ in Theorem 2.5 in [4] (which says that $\psi \equiv_D \varphi$ iff $|\text{im}C(\psi)| \equiv |\text{im}C(\varphi)| \mod k - 1$). We can compute $\text{im}C(\psi)$ and $\text{im}C(\varphi)$ as explicit lists of words, either from the table or by Corollary 4.11 of [4], in polynomial time.

This proves:

Proposition 4.1 The $\leq J$ decision problem and the $\equiv_D$ decision problem of $M_{k,1}$ are decidable in deterministic polynomial time, if inputs are given by tables or by words over a finite generating set.

In connection with the $\leq J$-relation we consider the multiplier search problem for $M_{k,1}$ over a finite generating set $\Gamma$. This problem is specified as follows:

Input: $\varphi, \psi \in M_{k,1}$, given by words over $\Gamma$.

Premise: $\psi \leq J \varphi$.

Search: Find some $\alpha, \beta \in M_{k,1}$, given by words over $\Gamma$, such that $\psi = \beta \varphi \alpha(\cdot)$.

Note that since the decision problem for $\leq J$ over a finite set of generators is in $\mathcal{P}$, the premise is easily checked, so this problem could be reformulated without a premise.

Proposition 4.2 The $\leq J$-relation multiplier search problem for $M_{k,1}$ is solvable in deterministic polynomial time, if inputs and output are given by words over a finite generating set.

Proof. If $\psi = 0$ we pick $\alpha = \beta = 0$. Let us assume now that $\psi \neq 0 \neq \varphi$. We can choose the multipliers $\alpha, \beta \in M_{k,1}$ as follows (as we did already in the proof of $0$-$J$-simplicity, i.e., Proposition 2.2 in [4]).
First, from \( \varphi \) (given by a word over \( \Gamma \)) we want to find some \( x_0, y_0 \in A^* \) such that \( y_0 = \varphi(x_0) \); we want to do this in deterministic polynomial time (as a function of \( |\varphi|_\Gamma \)). By Corollary 4.11 in [3] we find an explicit list of \( \text{imC}(\varphi) \) in polynomial time. In this list we pick any element \( y_0 \in \text{imC}(\varphi) \). From \( y_0 \) and the generator sequence for \( \varphi \) we can then find an element \( x_0 \in \varphi^{-1}(y_0) \) as follows. By Corollary 4.15 in [4] we can, in deterministic polynomial time, build a deterministic partial finite automaton that accepts the set \( \varphi^{-1}(y_0) \). By a search in this finite automaton we can (in deterministic polynomial time) find a word \( x_0 \) that is accepted by the automaton.

Now let \( \alpha = \{(\varepsilon, x_0)\} \) and \( \beta' = \{(y_0, \varepsilon)\} \). Since \( \alpha \) and \( \beta' \) have tables with one entry of polynomial length, we can (in polynomial time) find words over \( \Gamma \) that represent \( \alpha \), respectively \( \beta' \); for this we use Lemma 5.3 of [3] (which, in polynomial time, finds a word over \( \Gamma \) from a table).

Now we have \( \beta' \varphi \alpha(.) = 1 \). Hence \( \psi \beta' \varphi \alpha(.) = \psi \). Clearly, a word over \( \Gamma \) for \( \beta = \psi \beta' \) can be found in deterministic polynomial time, since we can find a word for \( \beta' \) in deterministic polynomial time.

In connection with the \( \equiv_D \)-relation we consider the \( D \)-pivot search problem for \( M_{k,1} \) over a finite generating set \( \Gamma \). This problem is specified as follows:

Input: \( \varphi, \psi \in M_{k,1} \), given by words over \( \Gamma \).
Premise: \( \varphi \equiv_D \psi \).
Search: Find an element \( \chi \in M_{k,1} \), given by a word over \( \Gamma \), such that \( \psi \equiv_R \chi \equiv_L \varphi \).

Note that since the decision problem for \( \equiv_D \) over a finite set of generators is in \( \mathcal{P} \), the premise is easily checked, so this problem can easily be transformed to an ordinary search problem, without premise.

**Proposition 4.3** The \( D \)-pivot search problem for the \( \equiv_D \)-relation of \( M_{k,1} \) is solvable in deterministic polynomial time, if inputs and output are given by words over a finite generating set.

**Proof.** As in the problem statement, let \( \varphi, \psi \in M_{k,1} \) with \( \varphi \equiv_D \psi \); so, \( |\text{imC}(\varphi)| \equiv |\text{imC}(\psi)| \mod k - 1 \). By Corollary 4.11 in [3], \( \text{imC}(\varphi) \) and \( \text{imC}(\psi) \) can be found in deterministic polynomial time (and hence they have polynomial size). In a polynomial number of steps, we can essentially restrict \( \varphi \) and \( \psi \) to \( \varphi' \), respectively \( \psi' \) such that \( |\text{imC}(\varphi')| = |\text{imC}(\psi')| \). We can obtain the restricted map \( \varphi' \) by taking \( \varphi' = \text{id}_{\text{imC}(\varphi')} \circ \varphi(.) \). Since \( |\text{imC}(\varphi')| \) is polynomially bounded in terms of \( |\varphi|_\Gamma \), the map \( \text{id}_{\text{imC}(\varphi')} \) has a polynomially bounded table, and hence a word over \( \Gamma \) can be found for \( \text{id}_{\text{imC}(\varphi')} \) in polynomial time (by Lemma 5.2 in [3]). Thus we obtain a word over \( \Gamma \) for \( \varphi' \) in polynomial time, and similarly for \( \psi' \). Let \( \alpha \) be any element of \( M_{k,1} \) that maps \( \text{imC}(\psi') \) bijectively onto \( \text{imC}(\varphi') \). Since \( \text{imC}(\psi') \) and \( \text{imC}(\varphi') \) can be explicitly listed in polynomial time, we can find a table (and hence a word over \( \Gamma \), by Lemma 5.2 in [3]) for \( \alpha \) in polynomial time. Then we have:

\[
\psi' \equiv_L \alpha \psi' \equiv_R \varphi'.
\]

The latter \( \equiv_R \) holds because \( \alpha \psi' \) is a map from \( \text{domC}(\psi') \) onto \( \text{imC}(\varphi') \), hence \( \alpha \psi' \) and \( \varphi' \) have the same image code (which implies \( \equiv_R \) by Theorem 2.1 of [3]). Thus, \( \alpha \psi' \) is a \( D \)-pivot. Since \( \alpha \) and \( \psi' \) can be found in deterministic polynomial time, we can find a word for this \( D \)-pivot in deterministic polynomial time. \( \Box \)

## 5 The complexity of \( \leq_J \) over the generating set \( \Gamma \cup \tau \)

We consider the \( \leq_J \) decision problem and the \( \leq_J \) multiplier search problem of \( M_{k,1} \) over the circuit-like generating set \( \Gamma \cup \tau \), where \( \Gamma \) is any chosen finite generating set of \( M_{k,1} \), and \( \tau = \{\tau_{i,i+1} : i \geq 1\} \). As we saw near the end of the Introduction, this generating set makes the elements of \( M_{k,1} \) similar to combinational circuits: Circuit-size becomes polynomially equivalent to the word-length [7] [5] [4] [6]. The word problem and the Green relations of \( M_{k,1} \) over \( \Gamma \) are in \( \mathcal{P} \). But over \( \Gamma \cup \tau \) the word problem
of $M_{k,1}$ is coNP-complete \cite{2}, the $\leq_R$ decision problem is $\Pi^P_2$-complete, and the $\leq_L$ decision problem is coNP-complete \cite{3}.

For complexity and word-length, finite changes in the generating set do not matter much; they only lead to linear changes in the complexity or the word-length. So, for a circuit-like generating set $\Gamma \cup \tau$ we can choose $\Gamma$ arbitrarily, provided that $\Gamma$ is finite and $\Gamma \cup \tau$ generates $M_{k,1}$.

5.1 The $\leq_J$ decision problem over $\Gamma \cup \tau$

Because of the 0-$J$-simplicity of the $J$-order we have to consider the following special word problem in $M_{k,1}$ over $\Gamma \cup \tau$.

**Input:** $\varphi \in M_{k,1}$, given by a word over the generating set $\Gamma \cup \tau$,

**Question (0 word problem):** Is $\varphi = 0$ as an element of $M_{k,1}$?

Recall that the word problem in $M_{k,1}$ over $\Gamma \cup \tau$ is coNP-complete (Theorem 4.12 in \cite{3}). In \cite{3} (Prop. 6.2) we proved the following:

**Proposition 5.1** The 0 word problem of $M_{k,1}$ over $\Gamma \cup \tau$ is coNP-complete.

**Proof.** We reduce the tautology problem for boolean formulas to the 0 word problem. Let $B$ be any boolean formula, with corresponding boolean function $\{0,1\}^m \to \{0,1\}$. We identify $\{0,1\}$ with $\{a_1, a_2\} \subseteq \{a_1, \ldots, a_k\} = A$. The function $B$ can be viewed as an element $\beta \in M_{k,1}$, represented by a word over $\Gamma \cup \tau$. The length of that word is linearly bounded by the size of the formula $B$ (by Prop. 2.4 in \cite{3}). In $M_{k,1}$ we consider the element $\text{id}_{0A^*}$ (i.e., the identity function restricted to $0A^*$), and we assume that some fixed representation of $\text{id}_{0A^*}$ by a word over $\Gamma$ has been chosen. We have:

$$\text{id}_{0A^*} \circ \beta(.) = 0 \text{ iff } \text{Im}(\beta) \subseteq 1A^*.$$  

The latter holds iff $B$ is a tautology. Thus we reduced the tautology problem for $B$ to the special word problem $\text{id}_{0A^*} \beta = 0$. Note that $\text{id}_{0A^*}$ is fixed, and independent of $B$.

It follows that the 0 word problem of $M_{k,1}$ over $\Gamma \cup \tau$ is coNP-hard for all $k \geq 2$. Moreover, since the word problem of $M_{k,1}$ over $\Gamma \cup \tau$ is in coNP (by Prop. 4.12 in \cite{2}), it follows that the 0 word problem is coNP-complete. \hfill \Box.

We can now characterize the complexity of the decision problem of the $J$-order of $M_{k,1}$ over $\Gamma \cup \tau$.

We will need the following complexity classes:

$$\text{DP} = \text{NP} \wedge \text{coNP} = \{L_1 \cap L_2 : L_1 \in \text{NP} \text{ and } L_2 \in \text{coNP}\} = \{N_1 - N_2 : N_1, N_2 \in \text{NP}\},$$

$$\text{coDP} = \text{NP} \vee \text{coNP} = \{L_1 \cup L_2 : L_1 \in \text{NP} \text{ and } L_2 \in \text{coNP}\}.$$  

In other words, DP consists of the set-differences between pairs of sets in NP. The class DP was introduced in \cite{25}, where several problems were proved to be DP-complete (see also pp. 92-95 in \cite{33}). In particular, the following problem, called Sat-and-unsat was given as an example of a DP-complete problem: The input consists of two boolean formulas $B_1$ and $B_2$, and the question is whether $B_1$ is satisfiable and $B_2$ is unsatisfiable. It follows immediately that the following problem is also DP-complete; the input is as before, and the question is whether $B_1$ is not a tautology and $B_2$ is a tautology. Hence, the following problem, which we call Nontaut-or-taut, is coDP-complete:

**Input:** Two boolean formulas $B_1$ and $B_2$.

**Question:** Is $B_1$ is not a tautology or is $B_2$ a tautology? (i.e., $(\forall x_1)B_1(x_1) \not\Rightarrow (\forall x_2)B_2(x_2)$)

The class coDP is closed under union and under polynomial-time disjunctive reduction, whereas DP is closed under intersection and under polynomial-time conjunctive reduction. The classes DP and coDP constitute the second level of the boolean hierarchy BH; for more information on DP and BH, see e.g. the survey \cite{9}.
Theorem 5.2  In \( M_{k,1} \) over the generating set \( \Gamma \cup \tau \) we have:

1. The \( \equiv_{\mathcal{J}} 0 \) decision problem is \( \text{coNP} \)-complete.
2. The \( \equiv_{\mathcal{J}} 1 \) decision problem is \( \text{NP} \)-complete.
3. The \( \equiv_{\mathcal{J}} \) and \( \leq_{\mathcal{J}} \) decision problems for \( M_{k,1} \) over \( \Gamma \cup \tau \) are \( \text{coDP} \)-complete (for the \( \equiv_{\mathcal{J}} \) decision problem, the \( \text{coDP} \)-completeness is with respect to polynomial-time disjunctive reductions).

Proof. (1) In any semigroup, \( \equiv_{\mathcal{J}} 0 \) is equivalent to \( = 0 \). We saw that the \( 0 \) word problem is \( \text{coNP} \)-complete (Prop. 5.1 above).

(2) By \( 0,\mathcal{J} \)-simplicity of \( M_{k,1} \), \( \varphi \equiv_{\mathcal{J}} 1 \) if \( \varphi \neq 0 \). So, the \( \equiv_{\mathcal{J}} 1 \) decision problem is equivalent to the negation of the \( 0 \) word problem, hence it is \( \text{NP} \)-complete.

(3) The \( \equiv_{\mathcal{J}} \)- and \( \leq_{\mathcal{J}} \)-decision problems are in \( \text{coNP} \lor \text{NP} \) because (by \( 0,\mathcal{J} \)-simplicity of \( M_{k,1} \)), \( \psi \leq_{\mathcal{J}} \varphi \) is equivalent to \( \psi = 0 \) or \( \varphi \neq 0 \) as elements of \( M_{k,1} \). The question whether \( \psi = 0 \) is in \( \text{coNP} \), and the question whether \( \varphi \neq 0 \) is in \( \text{NP} \).

Let us prove \( \text{coDP} \)-hardness of the \( \leq_{\mathcal{J}} \) decision problem. For boolean formulas \( B_1 \) and \( B_2 \) we have: \( B_1 \) is not a tautology or \( B_2 \) is a tautology if and only if \( \text{id}_{0,1} \circ \beta_1 = 0 \) or \( \text{id}_{0,1} \circ \beta_2 \neq 0 \), which is if and only if \( \text{id}_{0,1} \circ \beta_1 \leq_{\mathcal{J}} \text{id}_{0,1} \circ \beta_2 \). This reduces the \( \text{Nontaut-or-taut} \) problem to the \( \leq_{\mathcal{J}} \) decision problem.

The \( \equiv_{\mathcal{J}} \) decision problems are \( \text{coDP} \)-hard because the \( \leq_{\mathcal{J}} \) decision problem reduces to it by a polynomial-time disjunctive reduction: \( \psi \leq_{\mathcal{J}} \varphi \) if \( \psi \equiv_{\mathcal{J}} 0 \) or \( \psi \equiv_{\mathcal{J}} \varphi \). The class \( \text{coDP} \) is closed under union and under polynomial-time disjunctive reduction. \( \Box \).

5.2 The \( \leq_{\mathcal{J}} \) multiplier search problem

The multiplier search problem for \( M_{k,1} \) over \( \Gamma \cup \tau \) is specified as follows:

Input: \( \varphi, \psi \in M_{k,1} \), given by words over \( \Gamma \cup \tau \).

Premise: \( \psi \leq_{\mathcal{J}} \varphi \).

Search: Find some \( \alpha, \beta \in M_{k,1} \), expressed as words over \( \Gamma \cup \tau \), such that \( \psi = \beta \varphi \alpha(\cdot) \).

By \( 0,\mathcal{J} \)-simplicity of \( M_{k,1} \) the multiplier search problem is trivial when \( \psi \) or \( \varphi \) are \( 0 \). When \( \psi \) and \( \varphi \) are not \( 0 \), both will be \( \equiv_{\mathcal{J}} 1 \).

Therefore we consider the special multiplier search problem for \( \equiv_{\mathcal{J}} 1 \) in \( M_{k,1} \) over \( \Gamma \cup \tau \), specified as follows:

Input: \( \varphi \in M_{k,1} \), given by a word over \( \Gamma \cup \tau \).

Premise: \( 1 \equiv_{\mathcal{J}} \varphi \).

Search: Find one \( \alpha \) and one \( \beta \in M_{k,1} \), described by words over \( \Gamma \cup \tau \), such that \( \beta \varphi \alpha = 1 \).

When we have multipliers \( \alpha \) and \( \beta \) such that \( 1 = \beta \varphi \alpha \) then we can take the pair \( \psi \beta \), \( \alpha \) to obtain multipliers for \( \psi \leq_{\mathcal{J}} \varphi \).

We saw (Prop. 4.2) that the problem is solvable in deterministic polynomial time when \( M_{k,1} \) is taken over any finite generating set \( \Gamma \). Note that over \( \Gamma \cup \tau \), the premise (namely that \( 1 \equiv_{\mathcal{J}} \varphi \)) is non-trivial, being \( \text{NP} \)-complete.

See the Appendix for general information on search problems, the classes \( \text{NP}_{\text{search}} \) and \( \text{xNP}_{\text{search}} \), search reductions, and completeness.

Before we deal with the multiplier search problem for \( \leq_{\mathcal{J}} \) we will consider the domain element search problem of \( M_{k,1} \) over \( \Gamma \cup \tau \). The problem is specified as follows:

Input: \( \varphi \in M_{k,1} \), given by a word over \( \Gamma \cup \tau \).

Premise: \( \varphi \neq 0 \).

Search: Find an element \( x_0 \in \text{Dom}(\varphi) \).

The corresponding relation is \( \{ (\varphi, x_0) \in (\Gamma \cup \tau)^* \times A^* : \varphi(x_0) \neq \emptyset \} \).

A similar problem is the inverse image search problem of \( M_{k,1} \) over \( \Gamma \cup \tau \), specified as follows:

Input: \( y_0 \in A^* \), and \( \varphi \in M_{k,1} \), given by a word over \( \Gamma \cup \tau \).
Proposition 5.3 The domain element search problem and the inverse image search problem of $M_{k,1}$ over $\Gamma \cup \tau$ are $xNP$-search-complete.

Proof. The longest words in $\text{dom}C (\varphi)$ have length $\leq c \cdot |\varphi|_{\Gamma \cup \tau}$, for some constant $c$ (by Theorem 4.5 in [4]); the constant $c$ is the length of the longest word in the tables of the elements of $\Gamma$. Hence there exists $x_0 \in \text{dom}C (\varphi)$ with polynomial length, and in fact, all elements of $\text{dom}C (\varphi)$ have polynomial length. So, without loss of existence of solutions, we can consider the polynomially balanced subproblem

$$\{(\varphi, x_0) : \varphi (x_0) \neq \emptyset \text{ and } x_0 \in \text{dom}C (\varphi)\}.$$ 

By Prop. 5.5 in [3], we can verify in deterministic polynomial time whether $x_0 \in \text{dom}C (\varphi)$. Hence, this sub-problem is in $NP$-search.

In order to reduce the SATSEARCH problem to the domain element search problem of $M_{k,1}$ over $\Gamma \cup \tau$, we can view a boolean circuit $B$ as an element of $M_{k,1}$, given by a word over $\Gamma \cup \tau$. Then $x_0$ is an element of the domain of $id_{1(0,1)} \circ B (.)$ iff $x_0$ satisfies $B$.

Essentially the same proof works for the inverse image search problem. □

Proposition 5.4 The special multiplier search problem for $\equiv_J 1$ in $M_{k,1}$ over $\Gamma \cup \tau$ is $xNP$-search-complete. In particular, for any $\varphi \equiv_J 1$ there exist multipliers of polynomial word-length over $\Gamma \cup \tau$.

Proof. To show that the problem is in $xNP$-search we follow the proof of Prop. 4.2 above and of Proposition 2.2 in [4]. When $\varphi \neq 0$ there exists $x_0 \in \text{dom}C (\varphi)$; let $y_0 = \varphi (x_0)$. Since the longest words in $\text{dom}C (\varphi) \cup \text{im}C (\varphi)$ have length $\leq c \cdot |\varphi|_{\Gamma \cup \tau}$, for some constant $c$ (by Theorem 4.5 in [4]), $x_0$ and $y_0$ have polynomial length. Then $\beta \varphi = \{(\varepsilon, \varepsilon)\} = 1$, where $x = \{(\varepsilon, x_0)\}$ and $\beta = \{(y_0, \varepsilon)\}$. Thus, we take the sub-problem defined by the following relation:

$$\{(\varphi, \{(y_0, \varepsilon)\}, \{(\varepsilon, x_0)\}) : \varphi \in (\Gamma \cup \tau)^*, x_0 \in \text{dom}C (\varphi), y_0 = \varphi (x_0)\}.$$ 

We saw that $|x_0|, |y_0| \leq c \cdot |\varphi|_{\Gamma \cup \tau}$; hence this relation is polynomially balanced. The verification problem for this relation is in $P$: Indeed, we can check in deterministic polynomial time whether $x_0 \in \text{dom}C (\varphi)$ (by Prop. 5.5 in [3]). We can compute $\varphi (x_0)$ in deterministic polynomial time (by the proof of Theorem 4.12 in [4]), and compare $\varphi (x_0)$ with $y_0$.

Since $\alpha$ and $\beta$ have tables with one entry of polynomial size, we can (in polynomial time) find words over $\Gamma$ that represent $\alpha$, respectively $\beta$; for this we use Lemma 5.3 of [3] (which, in polynomial time, finds a word over $\Gamma$ from a table). So, $x_0$ and $y_0$ yield multipliers (expressed as strings over $\Gamma \cup \tau$) for $\varphi \equiv_J 1$.

To show $NP$-search-completeness we reduce the problem SATSEARCH to the $\equiv_J 1$ multiplier search problem over $\Gamma \cup \tau$. (See the Appendix for the definition of search reductions.) We construct an input-output reduction $(\rho_{in}, \rho_{sol})$ as follows. The function $\rho_{in}$ maps any boolean formula $B(x_1, \ldots, x_m)$ to $\text{id}_{1(0,1)} \circ B (.) \in M_{k,1}$; here, $\text{id}_{1(0,1)}$ is the partial identity with domain and image $1\{0,1\}^*$. From the boolean formula for $B$ we easily construct a word over $\Gamma \cup \tau$ for $B$; moreover, we can choose a fixed word over $\Gamma$ to represent $\text{id}_{1(0,1)}$. For all $t \in \{0,1\}^m$, $w \in \{0,1\}^*$:

$$\text{id}_{1(0,1)} \circ B (tw) = \begin{cases} 1w & \text{if } B(t) = 1, \\ \emptyset & \text{otherwise.} \end{cases}$$

We have $1 \equiv_J \text{id}_{1(0,1)} \circ B$ iff there are multipliers $\beta, \alpha$ such that $1 = \beta \circ \text{id}_{1(0,1)} \circ B \circ \alpha$. By what we saw, these multipliers can be chosen as follows: $\beta = \{(y_0, \varepsilon)\}$, $\alpha = \{(\varepsilon, x_0)\}$, with $x_0 \in \text{dom}C (\text{id}_{1(0,1)} \circ B)$, and $y_0 = \text{id}_{1(0,1)} \circ B (x_0)$. Moreover, $\text{dom}C (\text{id}_{1(0,1)} \circ B) = \{t \in \{0,1\}^m : B(t) = 1\}$, hence for $x_0 \in \text{dom}C (\text{id}_{1(0,1)} \circ B)$ we have $y_0 = 1$. 

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Therefore, the multiplier \( \alpha = \{(\varepsilon, x_0)\} \) determines a solution of \textsc{SatSearch}, by reading \( x_0 \) in the table of \( \alpha \). So, we simply define the map \( \rho_{sol} \) by \( \rho_{sol}(\beta, \alpha) = \alpha(\varepsilon) (= x_0) \).

Finally, to obtain a verification reduction we consider the map \( \rho_{ver} : (B, t) \mapsto (id_{1\{0,1\}} \circ B, \beta, \alpha) \), where \( \beta = \{(1, \varepsilon)\} \), and \( \alpha = \{(t, \varepsilon)\} \). Then \( \rho_{ver} \) reduces the verification problem “\( B(t) \equiv 1 \)” to the verification problem “\( \beta \circ id_{1\{0,1\}} \circ B \circ \alpha \equiv 1 \)”. Indeed, \( \beta \circ id_{1\{0,1\}} \circ B \circ \alpha = \{(\varepsilon, \varepsilon)\} = 1 \) iff \( \beta \circ id_{1\{0,1\}} \circ B(\alpha(\varepsilon)) = \beta \circ id_{1\{0,1\}} \circ B(t) = \varepsilon \), which holds when \( B(t) = 1 \), and does not hold when \( B(t) = 0 \). \( \square \)

5.3 The multiplier search problems for \( \leq_R \) and \( \leq_L \)

By definition, a left- (right-) inverse of an element \( x \) in a monoid \( M \) is an element \( t \in M \) such that \( tx = 1 \) (respectively \( xt = 1 \)). A left multiplier for \( \psi \leq_L \varphi \) in \( M \) is any \( \beta \in M \) such that \( \psi = \beta \varphi \). A right multiplier for \( \psi \leq_R \varphi \) in \( M \) is any \( \alpha \in M \) such that \( \psi = \varphi \alpha \).

Proposition 5.5. For \( M_{k,1} \) over \( \Gamma \cup \tau \) we have:

(1) The left multiplier search problem for \( \leq_L \), and the left-inverse search problem are not in xNPsearch (unless NP = coNP).

(2) The right multiplier search problem for \( \leq_R \), and the right-inverse search problem are not in xNPsearch (unless NP = coNP).

Proof. (1) If the problems were in xNPsearch we could guess a polynomial-size multiplier, and for some such guess the verification problem would be in \( P \) (by the definition of xNPsearch). Hence, the \( \leq_L \) and \( \equiv_L \) decision problems would be in \( NP \). However we saw in [3] (Section 6.2) that these two problems are coNP-complete. Hence, we would have NP = coNP, i.e., the polynomial hierarchy would collapse to level 1.

(2) If the search problems were in xNPsearch then (by the same reasoning as for the \( \leq_L \)-order) the \( \leq_R \) and \( \equiv_R \) decision problems would be in \( NP \). However, we saw in [3] (Section 2.2) that these two problems are \( \Pi^P_2 \)-complete. Hence we would have \( \Pi^P_2 = NP \), hence coNP = NP (since \( \Pi^P_2 \) contains coNP). \( \square \)

Note that we also saw in [3] (Section 5.3) that for \( M_{k,1} \) over \( \Gamma \cup \tau \) we have: Unless the polynomial hierarchy collapses to level 2, the \( \leq_R \)-multipliers and the right-inverses do not have polynomially bounded word-length.

6 The complexity of \( \equiv_D \) over the generating set \( \Gamma \cup \tau \)

Recall the characterization of the \( D \)-relation of \( M_{k,1} \): There are \( k - 1 \) non-0 \( D \)-classes, \( D_1, \ldots, D_{k-1} \); for any \( \varphi \in M_{k,1} \) we have \( \varphi \in D_i \) iff \( |imC(\varphi)| \equiv i \mod k - 1 \). Since \( M_{k,1} \) has only finitely many \( D \)-classes, the \( \equiv_D \)-decision problem is equivalent to the membership problems of these \( k \) \( D \)-classes.

The \( D \)-class \{0\} is special. Membership in the \( D \)-class \{0\} is the same thing as the 0 word problem, which is coNP-complete.

In order to characterize the complexity of the membership problem of a non-zero \( D \)-class \( D_i \) we need a somewhat exotic complexity class.

6.1 New counting complexity classes

Recall Valiant’s counting complexity class \#P (pronounced “number P”), consisting of all functions \( f_R : A^* \to \{0, 1\}^* \) of the form

\[
f_R(x) = \text{ binary representation of the number } |\{y \in A^* : (x, y) \in R\}|;
\]
here $R$ ranges over all predicates $R \subseteq A^* \times A^*$ such that the membership problem \"$(x, y) \in R$\" is in $P$ (deterministic polynomial time), and such that $R$ is polynomially balanced. A predicate $R$ is called \textit{polynomially balanced} iff there exists a polynomial $p$ such that for all $(x, y) \in R$: $|y| \leq p(|x|)$; see e.g. p. 181 in [24], and note that the definition of \"balanced\" is not symmetric in $x$ and $y$.

This can be generalized: In the above definition we replace $P$ by any complexity class $C$; then we obtain a counting class $\# \cdot C$, corresponding to polynomially balanced predicates whose membership problem is in $C$. For the history of these complexity classes, and in particular, for the reason why there is a dot in the notation, see \[29\] [19] [14]. The classes $\# \cdot \text{NP}$ and $\# \cdot \text{coNP}$ have been studied and, in particular, was proved that $\# \cdot \text{NP} = \# \cdot \text{coNP}$ iff $\text{NP} = \text{coNP}$ \[22\].

Another important counting class is $\oplus P$, introduced in \[26\] and \[16\]. More generally, $\oplus_{h,i} P$ consists of all sets $L_R$ of the form

$$L_R = \{ x \in A^* : |\{ y \in A^* : (x, y) \in R \}| \equiv i \mod h \};$$

where $R$ ranges over all predicates $R \subseteq A^* \times A^*$ such that the membership problem \"$(x, y) \in R$\" is in $P$, and such that $R$ is polynomially balanced. And $h, i$ are integers with $h \geq 2$. In that notation, $\oplus P$ is $\oplus_{2,1} P$. It was proved that if $\oplus P \subseteq \Sigma_2^P$ then the polynomial hierarchy collapses to $\Pi_2^P \cap \Sigma_2^P$ (due to Toda \[30\]; see also \[13\] pp. 334-340). The notation $\text{MOD}_h P$ was used in \[10\] \[2\] \[1\] for $\text{co} \oplus_{h,0} P \cup \text{MOD}_{h-1} P = (\bigcup_{h=1}^{h-1} \oplus_{h,j} P)$. See pp. 297-298 of \[18\] for some properties of $\oplus P$ and $\text{MOD}_h P$.

The class $\oplus_{h,i} P$ can be generalized to $\oplus_{h,i} \cdot C$ for any class $C$ of formal languages, and in particular to $\oplus_{h,i} \cdot \text{NP}$. The class $\oplus_{2,0} \cdot C$ was mentioned in \[19\]. For a predicate $R \subseteq A^* \times A^*$ and any $x_1, x_2 \in A^*$ we use the notation

$$(x_1)R = \{ x_2 \in A^* : (x_1, x_2) \in R \}, \text{ and } R(x_2) = \{ x_1 \in A^* : (x_1, x_2) \in R \}.$$  

For a predicate $R \subseteq A^* \times A^*$ we say that $R \in C$ iff the language $\{ xy \in A^* : bA^* : (x, y) \in R \}$ belongs to $C$, for some letter $b \notin A$.

\textbf{Definition 6.1} Let $h \geq 2$ and $i \geq 0$. A set $L \subseteq A^*$ belongs to $\oplus_{h,i} \cdot C$ iff there is a polynomially balanced predicate $R \in C$ such that for all $x \in A^*$:

$$x \in L \iff |(x)R| \equiv i \mod h.$$  

In that case we say that $L$ can be defined (in $\oplus_{h,i} \cdot C$) by the predicate $R$.

Note that (except when $h = 2$) this definition is unsymmetric for $x \in L$ versus $x \notin L$; so when $h > 2$, $\oplus_{h,i} \cdot C$ and $\text{co} \oplus_{h,i} \cdot C$ seem to be different (but this remains an open question). Obviously, if $i \equiv j \mod h$ then $\oplus_{h,i} \cdot C = \oplus_{h,j} \cdot C$.

\textbf{Lemma 6.2} Let $L$ be defined in $\oplus_{h,i} \cdot \text{NP}$ by a predicate $R$. Then $L$ can also be defined in $\oplus_{h,i+1} \cdot \text{NP}$ by a predicate $R'$, such that for all $x \in A^*$: $|(x)R'| = |(x)R| + 1$.

\textbf{Proof.} Let $L \in \oplus_{h,i} \cdot \text{NP}$ be defined by a polynomially balanced predicate $R \in \text{NP}$. Let us denote $\{(x, y) : x \in A^*\}$ by $\Delta$, and let us assume for the moment that $\Delta \cap R = \emptyset$. Then we have:

$$L = \{ x \in A^* : |\{ y \in A^* : (x, y) \in R \cup \Delta \}| \equiv i + 1 \mod h \}.$$  

Indeed, for any $x \in A^*$: $(x)(R \cup \Delta) = (x)R \cup \{ x \}$, and $x \notin (x)R$ since $\Delta \cap R = \emptyset$. So $|(x)(R \cup \Delta)| \equiv i + 1 \mod h$. The predicate $R \cup \Delta$ is in $\text{NP}$, and it is polynomially balanced. Thus, $L$ is also defined (in $\oplus_{h,i+1} \cdot \text{NP}$) by the predicate $R \cup \Delta$.

If $R$ does not satisfy $\Delta \cap R = \emptyset$, we consider $R' = \{(x, xya) \in A^* \times A^* : (x, y) \in R\}$, which is polynomially balanced and in $\text{NP}$, and satisfies $\Delta \cap R' = \emptyset$; here, $a_1$ is one of the letters of $A$. Moreover, $R'$ defines $L$ as an element of $\oplus_{h,i} \cdot \text{NP}$ since $|\{ y \in A^* : (x, y) \in R \}| = |\{ y \in A^* : (x, xya) \in R' \}| = \ldots$
Corollary 6.3  For all \( i, j \): \( \oplus_{h,i} \cdot \text{NP} = \oplus_{h,j} \cdot \text{NP} \). I.e., for any fixed \( h \geq 2 \), the classes \( \oplus_{h,i} \cdot \text{NP} \) are the same for all \( i \).

Therefore we will use the notation \( \oplus_{h} \cdot \text{NP} \) for each \( \oplus_{h,i} \cdot \text{NP} \).

Lemma 6.4  Both \( \text{NP} \) and \( \text{coNP} \) are subsets of \( \oplus_{h} \cdot \text{NP} \). Moreover, every \( L_0 \) in \( \text{NP} \cup \text{coNP} \) can be defined (in \( \oplus_{h,1} \cdot \text{NP} \)) by a predicate \( R \) such that:

- \( x \in L_0 \) \iff \( |\{(y : (x, y) \in R)\} \equiv 1 \mod h \), and
- \( x \not\in L_0 \) \iff \( |\{(y : (x, y) \in R)\} \equiv 0 \mod h \).

Proof. Let \( L \in \text{NP} \), let \( x \in A^* \), and let \( \{u_1, \ldots, u_{h-1}\} \subset A^* \) be a fixed set of \( h-1 \) different non-empty words. Then we have

\[
\{y \in A^* : y = x \text{ and } x \in L\} = \begin{cases} \{x\} & \text{if } x \in L, \\ \emptyset & \text{if } x \not\in L. \end{cases}
\]

Similarly we have

\[
\{y \in A^* : y = x \text{ or } (y \in \{xu_1, \ldots, xu_{h-1}\} \text{ and } x \in L)\} = \begin{cases} \{x\} & \text{if } x \in \overline{L}, \\ \{x, xu_1, \ldots, xu_{h-1}\} & \text{if } x \in L. \end{cases}
\]

Hence,

\[
L = \{x \in A^* : |\{(y : y = x \text{ and } x \in L)\} \equiv 1 \mod h\},
\]

\[
\overline{L} = \{x \in A^* : |\{(y : y = x \text{ or } (y \in \{xu_1, \ldots, xu_{h-1}\} \text{ and } x \in L)\} \equiv 1 \mod h\}.
\]

The predicate \( R \) defined by \( (x, y) \in R \) \iff \( y = x \text{ and } x \in L \), belongs to \( \text{NP} \), and is polynomially balanced. Similarly, the predicate \( R' \) defined by \( (x, y) \in R' \) \iff \( y = x \text{ or } (y \in \{xu_1, \ldots, xu_{h-1}\} \text{ and } x \in L) \) belongs to \( \text{NP} \), and is polynomially balanced. So \( L \) and \( \overline{L} \) belong to \( \oplus_{h,1} \cdot \text{NP} \).

One sees immediately from the definition of the predicates \( R \) and \( R' \) that they have the following property: If \( x \not\in L \) then \( |\{(y : (x, y) \in R)\} \equiv 0 \mod h \); if \( x \not\in \overline{L} \) then \( |\{(y : (x, y) \in R')\} \equiv h \equiv 0 \mod h \). □

Lemma 6.4 inspires the following definition.

Definition 6.5  For any integer \( h \geq 2 \) and two disjoint sets \( S_1, S_2 \subset \{0, 1, \ldots, h-1\} \) we define the class \( \oplus_{h,S_1,S_2} \cdot \text{NP} \) as follows: \( L \subset A^* \) belongs to \( \oplus_{h,S_1,S_2} \cdot \text{NP} \) iff there exists a polynomially balanced predicate \( R \subset A^* \times A^* \) in \( \text{NP} \) such that for all \( x \in A^* \),

- \( x \in L \) \iff \( |(x)R| \in S_1 \mod h \), and
- \( x \not\in L \) \iff \( |(x)R| \in S_2 \mod h \).

We say then that \( L \) can be defined in \( \oplus_{h,S_1,S_2} \cdot \text{NP} \) by the predicate \( R \). In this notation the class \( \oplus_{h,i} \cdot \text{NP} \) is \( \oplus_{h,\{i\},\{j:j\neq i\}} \cdot \text{NP} \).

When \( S_1 = \{i\} \), \( S_2 = \{j\} \) with \( i \neq j \) we write \( \oplus_{h,i,j} \cdot \text{NP} \).

The second sentence of Lemma 6.4 says that \( \text{NP} \) and \( \text{coNP} \) are subclasses of \( \oplus_{h,1,0} \cdot \text{NP} \).

Clearly, \( \text{co} \oplus_{h,S_1,S_2} \cdot \text{NP} = \oplus_{h,S_1,S_2} \cdot \text{NP} \) (always assuming \( S_1 \cap S_2 = \emptyset \)).

By Lemma 6.2 \( \oplus_{h,i,j} \cdot \text{NP} = \oplus_{h,i+1,j+1} \cdot \text{NP} \), and \( \oplus_{h,S_1,S_2} \cdot \text{NP} = \oplus_{h,S_1+1,S_2+1} \cdot \text{NP} \). By definition, \( S_1 + 1 = \{i + 1 : i \in S_1\} \), and similarly for \( S_2 + 1 \); all numbers are taken modulo \( h \).
Lemma 6.6 Suppose $m$ is prime with $h$, and suppose that $L$ can be defined in $\oplus_{h,i,j}NP$ by a predicate $R$. Then $L$ can also be defined in $\oplus_{h,mi,mj}NP$ by a predicate $R'$ such that for all $x \in A^*$:

$$|\langle x \rangle R'| = m \cdot |\langle x \rangle R|.$$

Hence, $\oplus_{h,i,j}NP = \oplus_{h,mi,mj}NP$. (Here the numbers $mi$ and $mj$ are taken modulo $h$.)

Proof. By assumption, $x \in L$ iff $|\langle x \rangle R| \equiv i \bmod h$, and $x \not\in L$ iff $|\langle x \rangle R| \equiv j \bmod h$. We choose a prefix code $\{u_1, \ldots, u_m\} \subset A^*$ of size $m$, and for $s = 1, \ldots, m$ we let $R_s = \{x,u,sy : (y,x) \in R\}$. Then each $R_s$ is in $NP$ and polynomially balanced. Moreover, $L$ is also defined in $\oplus_{h,i,j}NP$ by $R_s$, since $|\langle x \rangle R_s| = |\langle x \rangle R_s|$. Let $R' = R_1 \cup \ldots \cup R_m$. Then $R'$ is also in $NP$ and it is polynomially balanced. Since $\{u_1, \ldots, u_m\}$ is a prefix code, we have $R_s \cap R_t = \emptyset$ when $s \neq t$. It follows that we have $|\langle x \rangle R'| = m \cdot |\langle x \rangle R|$ for all $x \in A^*$; and we have $x \in L$ iff $|\langle x \rangle R'| \equiv mi \bmod h$, and $x \not\in L$ iff $|\langle x \rangle R'\rangle \equiv mj \bmod h$.

Finally, when $m$ is prime with $h$ then $i \neq j \bmod h$ implies $mi \neq mj \bmod h$. Hence $\oplus_{h,i,j}NP = \oplus_{h,mi,mj}NP$. □

Corollary 6.7 For all $i, j$ such that $i - j$ is prime with $h$ we have: $\oplus_{h,i,j}NP = \oplus_{h,1,0}NP$.

Proof. By $h - j$ applications of Lemma 6.2 we obtain $\oplus_{h,i,j}NP = \oplus_{h,\ell,0}NP$, where $\ell = i - j \bmod h$. Since $\ell$ is prime with $h$, $\ell$ has a multiplicative inverse $\ell^{-1}$ modulo $h$, hence by Lemma 6.6 (with $m = \ell^{-1}$) we obtain $\oplus_{h,\ell,0}NP = \oplus_{h,1,0}NP$. □

Corollary 6.8 The class $\oplus_{h,1,0}NP$ is closed under complement, and contains $NP$ and $coNP$.

Proof. We noted already that $co\oplus_{h,1,0}NP = \oplus_{h,0,1}NP$. By Corollary 6.7, $\oplus_{h,0,1}NP = \oplus_{h,1,0}NP$. By Lemma 6.4 and by Definition 6.5, $\oplus_{h,1,0}NP$ contains $NP$ and $coNP$. □

Lemma 6.9 Suppose $L_1, L_2 \in \oplus_{h,1,0}NP$ can be defined (in $\oplus_{h,1,0}NP$) by predicates $R_1$, respectively $R_2$. Then $L_1$ and $L_2$ can also be defined (in $\oplus_{h,1,0}NP$) by predicates $R'_1$, respectively $R'_2$ such that $R'_1 \cap R'_2 = \emptyset$.

Proof. We choose a prefix code $\{u_1, u_2\} \subset A^*$, and for $i = 1, 2$ we let $R'_i = \{(x,u_iy) : (y,x) \in R\}$. Then $R'_1 \cap R'_2 = \emptyset$ since $\{u_1, u_2\}$ is a prefix code. Also, $R'_i$ is in $NP$ and is polynomially balanced. And $|\langle x \rangle R'_i| = |\langle x \rangle R_i|$. □

Corollary 6.10 For $h \geq 3$, if $L_1, L_2 \in \oplus_{h,1,0}NP$ then $L_1 \cap L_2 \in \oplus_hNP$.

Hence for $h \geq 3$, $DP \subseteq \oplus_hNP$.

Proof. Let $R_1, R_2$ be predicates (in $NP$) that describe $L_1$, respectively $L_2$ in $\oplus_{h,1,0}NP$. By Lemma 6.3 we can assume that $R_1 \cap R_2 = \emptyset$. So for all $x \in A^*$ : $(x)R_1 \cap (x)R_2 = \emptyset$, hence $|\langle x \rangle R_1 \cup (\langle x \rangle R_2| = |\langle x \rangle R_1| + |\langle x \rangle R_2|$.

Since $L_1, L_2 \in \oplus_{h,1,0}NP$ we have $|\langle x \rangle R_1| \equiv 0 \bmod h$, according as $x \in L_1$ or $x \not\in L_1$; and similarly for $L_2$. Therefore, $x \in L_1 \cup L_2$ iff $\langle x \rangle (R_1 \cup R_2) \equiv 1 \bmod h$; and $x \not\in L_1 \cup L_2$ iff $\langle x \rangle (R_1 \cup R_2) \equiv 0 \bmod h$. Hence, $L_1, L_2 \in \oplus_{h,1,0}NP$ implies $L_1 \cup L_2 \in \oplus_{h,1,0}NP$.

Replacing $L_1, L_2$ by their complements $\overline{L_1}, \overline{L_2}$, and using the fact that $\oplus_{h,1,0}NP$ is closed under complement, we also have: $L_1, L_2 \in \oplus_{h,1,0}NP$ implies $\overline{L_1}, \overline{L_2} \in \oplus_{h,1,0}NP$, which by the above implies $L_1 \cap L_2 \in \oplus_{h,1,0}NP$. Since $co\oplus_{h,1,0}NP = \oplus_{h,1,0}NP$, we have: $L_1 \cap L_2 \in \oplus_{h,0,1,2}NP$. By Lemma 6.9, $\oplus_{h,0,1,2}NP = \oplus_{h,1,0}NP$. Also, $\oplus_{h,1,0}NP \subseteq \oplus_{h}NP$. Hence, $L_1 \cap L_2 \in \oplus_{h}NP$.

Since we saw in Lemma 6.8 that $NP$ and $coNP$ are contained in $\oplus_{h,1,0}NP$, it follows that $DP$ is contained in $\oplus_{h}NP$. □
We will see next that $\oplus^h_1 \text{NP}$ and $\oplus^h_{1,0} \text{NP}$ have complete problems (with respect to polynomial-time many-to-one reduction). On the other hand BH and PH do not have complete problems – unless the these hierarchies collapse. It is also known that a collapse of BH implies a collapse of PH (Kadin and Chang [21, 11]). This shows that we have: Unless the polynomial hierarchy PH collapses, $\oplus^h_1 \text{NP}$ and $\oplus^h_{1,0} \text{NP}$ are both different from BH and different from PH.

Recall the $\exists$-quantified boolean formula problem, also called $\forall \exists \text{SAT}$. The input for $\forall \exists \text{SAT}$ is a fully quantified boolean formula $(\forall y_1, \ldots, y_n) (\exists x_1, \ldots, x_m) B(x_1, \ldots, x_m, y_1, \ldots, y_n)$, and the question is whether this formula is true. It is well known that $\forall \exists \text{SAT}$ is $\Pi^P_2$-complete. In a similar way, $\text{SAT}$ can be extended by any other quantifier sequence, which provides complete problems for the classes $\Pi^P_\ell$ and $\Sigma^P_\ell$ of the polynomial hierarchy PH. Another extension of $\text{SAT}$, called $\# \text{SAT}$ (“number sat”), is complete in the class $\#P$ for parsimonious many-to-one polynomial-time reductions (Valiant [31, 32]; see also Chapter 8 of [24] for the definition of these reductions). The problem $\# \text{SAT}$ is the function which maps any boolean formula $B(x_1, \ldots, x_m)$ to $|\{(b_1, \ldots, b_m) \in \{0,1\}^m : B(b_1, \ldots, b_m) = 1\}|$ (i.e., the number of satisfying truth-value assignments, this number being represented in binary). This was generalized by [12] to $\# \Pi^P_\ell \text{SAT}$ and $\# \Sigma^P_\ell \text{SAT}$ which are complete in $\# \cdot \Pi^P_\ell$, respectively $\# \cdot \Sigma^P_\ell$ (again for parsimonious many-to-one polynomial-time reductions).

In the context of $\oplus^h_1 \text{NP}$ we introduce the following extension of $\text{SAT}$, called $\oplus^h_1 \forall \exists \text{SAT}$.

**Input:** An existentially quantified boolean formula $(\exists x_1, \ldots, x_m)B(x_1, \ldots, x_m, y_1, \ldots, y_n)$ with free variables $y_1, \ldots, y_n$, where $B(x_1, \ldots, x_m, y_1, \ldots, y_n)$ is an ordinary boolean formula whose variables range over $\{0,1\}$.

**Question ($\oplus^h_1 \forall \exists \text{SAT}$-problem):** Does the following hold:

$$|\{(b_1, \ldots, b_n) \in \{0,1\}^n : (\exists x_1, \ldots, x_m) B(x_1, \ldots, x_m, b_1, \ldots, b_n)\}| \equiv 1 \mod h ?$$

In a similar way we define the problem $\oplus^h_{1,0} \forall \exists \text{SAT}$.

**Input:** $(\exists x_1, \ldots, x_m)B(x_1, \ldots, x_m, y_1, \ldots, y_n)$, as in $\oplus^h_1 \forall \exists \text{SAT}$.

**Question ($\oplus^h_{1,0} \forall \exists \text{SAT}$-problem):** Does the following hold:

$$|\{(b_1, \ldots, b_n) \in \{0,1\}^n : (\exists x_1, \ldots, x_m) B(x_1, \ldots, x_m, b_1, \ldots, b_n)\}| \equiv 1 \mod h ,$$

and

$$|\{\{b_1, \ldots, b_n) \in \{0,1\}^n : \text{not}(\exists x_1, \ldots, x_m) B(x_1, \ldots, x_m, b_1, \ldots, b_n)\}| \equiv 0 \mod h ?$$

The same parsimonious many-to-one polynomial-time reductions that prove completeness of $\# \text{SAT}$ and of $\# \Pi^P_\ell \text{SAT}$ yield the following:

*The problem $\oplus^h_1 \forall \exists \text{SAT}$ is $\oplus^h_1 \text{NP}$-complete, and the problem $\oplus^h_{1,0} \forall \exists \text{SAT}$ is $\oplus^h_{1,0} \text{NP}$-complete.*

### 6.2 The image size problem

In the following we will use combinational circuits (i.e., acyclic digital circuits, made from and, or, not, and fork gates). We will need to generalize these circuits to partial combinational circuits, simply by allowing a one-wire gate id1 which maps the boolean value 1 to 1, and is undefined on 0. We will denote “undefined” by $\bot$. When a gate has $\bot$ on one (or more) of its input wires, its output will be $\bot$. We also add the following rule about partial outputs of a circuit:

**Partial outputs rule:** If one or more output wires of a circuit receive the undefined value $\bot$ then the entire output of the circuit is viewed as undefined.

In other words, any string in $\{0,1,\bot\}^n$ containing at least one $\bot$ is equivalent to $\bot$; so, up to this equivalence, $\{0,1,\bot\}^n = \{0,1\}^n \cup \{\bot\}$. The inputs of a partial combinational circuit $C$ are the elements of $\{0,1\}^m$ for some $m$ (depending on $C$). For $x \in \{0,1\}^m$ the output belongs to $\{0,1\}^n \cup \{\bot\}$ for some $n$ (depending on $C$), and is denoted by $C(x)$. We denote the domain of $C$ by $\text{Dom}(C)$; it consists of the bitstrings in $\{0,1\}^m$ on which the output is defined. Hence, $\text{Dom}(C) = \{x \in \{0,1\}^m : C(x) \neq \bot\}$. The set of all outputs of $C$ (not counting $\bot$) is called the image of $C$ and is denoted by $\text{Im}(C)$; so, $\text{Im}(C) = \{C(x) \in \{0,1\}^n : x \in \{0,1\}^m\}$. 

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To get closer to the $\equiv_D$-decision problem of $M_{k,1}$ over $\Gamma \cup \tau$, we introduce the following problems, called the image size problem for partial combinational circuits and the modular image size problem for partial combinational circuits.

Input: A partial combinational circuit $C$.

Output (image size problem): The binary representation of the number $|\text{Im}(C)|$ (i.e., the number of non-$\bot$ outputs; the outcome $\bot$, if it occurs, is not counted as an output).

Question (mod $h$ image size problem, for fixed $h \geq 2$): $|\text{Im}(C)| \equiv 1 \mod h$?

Finally, in relation to the $\equiv_D$-decision problem we introduce the modular image size problem of $M_{k,1}$ over $\Gamma \cup \tau$:

Input: $\varphi \in M_{k,1}$, given by a word over $\Gamma \cup \tau$.

Question: $|\text{im}(C_{\varphi})| \equiv 1 \mod k - 1$?

The number $|\text{im}(C_{\varphi})|$ depends on the right ideal homomorphism that is chosen to represent $\varphi$; however, $|\text{im}(C_{\varphi})| \mod k - 1$ does not depend the choice of representative (Prop. 2.4 in [4]); i.e., $|\text{im}(C_{\varphi})| \mod k - 1$ is an invariant of $\varphi$ as an element of $M_{k,1}$. We only consider the modular image size problem of $M_{k,1}$ when $k \geq 3$. Recall that $M_{2,1}$ has only one non-zero $D$-class.

**Theorem 6.11**

1. The image size problem for partial combinational circuits is $\# \cdot \text{NP}$-complete.
2. The mod $h$ image size problem for partial combinational circuits (for $h \geq 2$) is $\oplus_h \text{NP}$-complete.
3. For $k \geq 3$, the modular image size problem of $M_{k,1}$ over $\Gamma \cup \tau$ is $\oplus_{k-1} \text{NP}$-complete.

**Proof.** (1) To prove that the image size problem is in $\# \cdot \text{NP}$ we consider the predicate $R$ defined by

$$(C, y) \in R \iff (\exists x \in \text{Dom}(C))(C(x) = y),$$

where $C$ ranges over all partial combinational circuits. Clearly, the membership problem of $R$ is in $\text{NP}$, and $R$ is polynomially balanced (in fact, $y \leq |C|$ since the output ports of $C$ are counted in the size of $C$). Then we have $\{y : (C, y) \in R\} = \text{Im}(C)$, hence the function $C \mapsto |\text{Im}(C)|$ is in $\# \cdot \text{NP}$.

To prove $\# \cdot \text{NP}$-hardness we will reduce $\#\text{\text{SAT}}$ to the image size problem. Let $B(x_1, x_2)$ be a boolean formula where $x_1$ is a sequence of $m$ boolean variables, and $x_2$ is a sequence of $n$ boolean variables. We map $B$ to a partial combinational circuit $C_{B,m,n}$ with partial input-output function defined by

$$(x_1, x_2) \mapsto C_{B,m,n}(x_1, x_2) = \begin{cases} x_2 \text{ if } B(x_1, x_2) = 1, \\ \bot \text{ if } B(x_1, x_2) = 0. \end{cases}$$

From the formula for $B(x_1, x_2)$ one can easily construct a partial combinational circuit for $C_{B,m,n}$. Moreover, $\text{Im}(C_{B,m,n}) = \{x_2 : (\exists x_1)B(x_1, x_2)\}$, hence the reduction is a parsimonious reduction from the function $[B \mapsto \{x_2 : (\exists x_1)B(x_1, x_2)\}]$ to the function $[C_{B,m,n} \mapsto \text{Im}(C_{B,m,n})]$.

(2) Membership in $\oplus_h \text{NP}$ is proved as in (1). The reduction in (1) also yields a parsimonious reduction of $\oplus_h \text{\text{SAT}}$ to the mod $h$ image size problem. This shows $\oplus_h \text{NP}$-hardness.

(3) To prove that the modular image size problem of $M_{k,1}$ is in $\oplus_{k-1} \text{NP}$ we consider the predicate $R$ defined by

$$(\varphi, y) \in R \iff y \in \text{im}(C_{\varphi}).$$

Here, $\varphi$ is expressed by a word over $\Gamma \cup \tau$, where each $\tau_{i-1,i} \in \tau$ has length $|\tau_{i-1,i}| = i$.

The predicate $R$ is in $\text{NP}$; see Prop. 4.9 about the image membership problem in [3]. The predicate $R$ is also polynomially balanced. In fact, for $y \in \text{im}(C_{\varphi})$ we have by Theorem 4.5(2) in [3]: $|y| \leq c \cdot |\varphi|_{\Gamma \cup \tau}$ (for some constant $c > 0$), since we have $|\tau_{i-1,i}| = i$.

Hardness follows from (2), since partial combinational circuits are special cases of elements of $M_{k,1}$ expressed over $\Gamma \cup \tau$. \(\square\)
Remark. The proof of (1) above shows why partial circuits were introduced: For an ordinary (total) circuit \( C \) the image size is never 0, whereas the set \( \{ x_2 : (\exists x_1)B(x_1, x_2) \} \) can be empty. So there is no parsimonious reduction from \( \#\text{SAT} \) to the image-size problem of total circuits.

For comparison, the domain size problem for partial combinational circuits (i.e., the function \( C \mapsto |\{ x : x \in \text{Dom}(C) \}| \)) is \( \#P \)-complete. Indeed, we can map any boolean formula \( B \) to a partial combinational circuit \( C_B \) which (on input \( x \)) outputs \( \bot \) when \( B(x) = 0 \), and outputs \( 1 \) when \( B(x) = 1 \). Then the domain of the partial circuit \( C_B \) consists of the satisfying truth values of \( B \), so this is a parsimonious reduction of \( \#\text{SAT} \) to the domain size problem. Moreover, the domain size problem is in \( \#P \). Indeed, the predicate \( \{(x, C) : x \in \text{Dom}(C)\} \) (where \( x \in \{0, 1\}^* \) and \( C \) ranges over partial combinational circuits) is in \( P \) since a circuit can be evaluated quickly on a given input.

For a fixed \( h \geq 2 \) and \( 0 \leq i \leq h - 1 \) we can also consider the modular domain size problem for partial combinational circuits; for a circuit \( C \), the question is whether \( |\text{Dom}(C)| \equiv 1 \mod h \). As above one proves that this problem is \( \oplus h P \)-complete.

Similarly, for \( 1 \leq i \leq k - 1 \) we have the modular domain code size problem in \( M_{k,1} \); for \( \varphi \in M_{k,1} \),

given by a word over \( \Gamma \cup \tau \), the question is whether \( |\text{domC}(\varphi)| \equiv i \mod k - 1 \). This problem is \( \oplus_{k-1} P \)-complete.

### 6.3 The complexity of \( \equiv_D \) over \( \Gamma \cup \tau \)

Recall that \( M_{k,1} \) has \( k - 1 \) non-zero \( D \)-classes \( D_i = \{ \varphi : |\text{imC}(\varphi)| \equiv i \mod k - 1 \} \), for \( 1 \leq i \leq k - 1 \).

**Theorem 6.12** Let \( k \geq 3 \) and \( 1 \leq i \leq k - 1 \). The membership problem of the \( D \)-class \( D_i \) of \( M_{k,1} \) over \( \Gamma \cup \tau \) is \( \oplus_{k-1} \cdot \text{NP-complete} \).

**Proof.** Checking whether an element is not \( \equiv_D \ 0 \) is in \( \text{NP} \) (by Prop. 5.1), and \( \text{NP} \) is contained in \( \oplus_{k-1} \cdot \text{NP} \). Checking whether a non-zero element is in \( D_i \) is \( \oplus_{k-1} \cdot \text{NP-complete} \) by Theorem 6.11 (3), and by the fact that for a non-zero element \( \varphi \in M_{k,1} \) we have \( \varphi \in D_i \) iff \( |\text{imC}(\varphi)| \equiv i \mod k - 1 \) (Theorem 2.5 in [4]). \( \square \)

**Remark.** The \( \equiv_D \)-decision problem of \( M_{2,1} \) over \( \Gamma \cup \tau \) is \( \text{coDP-complete} \), with respect to polynomial-time disjunctive reduction. Indeed, in \( M_{2,1} \), \( \equiv_D \) and \( \equiv_{\mathcal{J}} \) are the same (Theorem 2.5 in [4]), and we saw in Prop. 5.2 that the \( \equiv_{\mathcal{J}} \)-decision problem is \( \text{coDP-complete} \).

Earlier we considered the \( D \)-pivot search problem of \( M_{k,1} \), and we proved that it is in \( P \) when inputs are expressed over a finite generating set \( \Gamma \) of \( M_{k,1} \). Over circuit-like generating sets \( \Gamma \cup \tau \) we have the following.

**Theorem 6.13** The \( D \)-pivots of \( M_{3,1} \) do not have polynomially bounded word-length over \( \Gamma \cup \tau \), unless the polynomial hierarchy \( \text{PH} \) collapses. More precisely, suppose there is a polynomial \( p(.) \) such that for all \( \psi, \varphi \in M_{3,1} \) we have: \( \psi \equiv_D \varphi \) implies that there is a \( D \)-pivot \( \chi \) with \( |\chi|_{\Gamma \cup \tau} \leq p(|\psi|_{\Gamma \cup \tau} + |\varphi|_{\Gamma \cup \tau}) \); then \( \text{PH} \) collapses to \( \Pi^p_4 \cap \Sigma^p_4 \).

**Proof.** We proved in [3] that the \( \equiv_{\mathcal{R}} \)- and \( \equiv_{\mathcal{L}} \)-decision problems are in \( \Pi^p_2 \). If \( D \)-pivots had polynomially bounded word-length over \( \Gamma \cup \tau \) then the \( \equiv_D \)-decision problem would be in \( \Sigma^p_3 \), by just guessing a pivot \( \chi \) in nondeterministic polynomial time, and checking whether \( \psi \equiv_{\mathcal{L}} \chi \equiv_{\mathcal{R}} \varphi \) (which is in \( \Pi^p_2 \)). However, the \( \equiv_D \)-decision problem is complete in \( \oplus_2 \cdot \text{NP} \), hence \( \oplus_2 \cdot \text{NP} \) would be in \( \Sigma^p_3 \); hence \( \oplus_P \) would be contained in \( \Sigma^p_3 \). By [30], \( \oplus_P \subseteq \Sigma^p_3 \) implies that \( \text{PH} \) collapses to \( \Pi^p_4 \cap \Sigma^p_4 \). \( \square \)

For \( M_{k,1} \) with \( k > 3 \), Theorem 6.13 probably also holds, but the \( \text{mod} h \) version of Toda’s theorem (for \( h > 2 \)) has not been checked.

Case \( k = 2 \): We leave it as an open question whether Theorem 6.13 holds for \( M_{2,1} \). All non-zero elements of \( M_{2,1} \) are \( D \)-equivalent, so here pivots always exist.
7 The Green relations of \( \text{Inv}_{k,1} \) and their complexity

We saw in \([4]\) that \( \text{Inv}_{k,1} \) is an inverse monoid, i.e., for every \( \alpha \in M \) there exists one and only one \( \alpha' \in M \) such that \( \alpha \alpha' = \alpha \) and \( \alpha' \alpha = \alpha' \); the element \( \alpha' \) is called the inverse of \( \alpha \). Some elementary facts about inverse monoids: For all \( \alpha, \beta \in M : (\alpha \cdot \beta)' = \beta' \cdot \alpha' \). For all \( \alpha, \beta \in M : \beta \leq_{\mathcal{R}} \alpha \) iff \( \alpha' \leq_{\mathcal{L}} \beta' \); similarly, \( \beta \leq_{\mathcal{L}} \alpha \) iff \( \alpha' \leq_{\mathcal{R}} \beta' \).

**Proposition 7.1 (The Green relations of \( \text{Inv}_{k,1} \)).**

1. The Green relations \( \leq_{\mathcal{J}}, \equiv_{\mathcal{D}}, \leq_{\mathcal{R}}, \leq_{\mathcal{L}} \) of \( \text{Inv}_{k,1} \) are the restrictions of the corresponding Green relations of \( M_{k,1} \).
2. \( \text{Inv}_{k,1} \) is a union of \( \equiv_{\mathcal{L}} \)-classes of \( M_{k,1} \). In other words, if an \( \mathcal{L} \)-class of \( M_{k,1} \) intersects \( \text{Inv}_{k,1} \) then this entire \( \mathcal{L} \)-class is contained in \( \text{Inv}_{k,1} \).
3. Every \( \equiv_{\mathcal{R}} \)-class of \( M_{k,1} \) intersects \( \text{Inv}_{k,1} \).
4. Let \( \Gamma_I \) be a finite generating set of \( \text{Inv}_{k,1} \), and let us assume that \( \Gamma_I \) is closed under inverse. Then for every \( \varphi \in \text{Inv}_{k,1} \) we have \( |\varphi|_{\Gamma_I} = |\varphi^{-1}|_{\Gamma_I} \) and \( |\varphi|_{\Gamma_I \cup \tau} = |\varphi^{-1}|_{\Gamma_I \cup \tau} \).

**Proof.** Let us use \( \mathcal{L}(M_{k,1}) \) to indicate the \( \mathcal{L} \) relations of \( M_{k,1} \), and similarly for \( \mathcal{R}(M_{k,1}) \).

1. This is Lemma 2.9 in \([4]\).
2. For \( \varphi \in M_{k,1} \) we have: \( \varphi \in \text{Inv}_{k,1} \) iff \( \text{part}(\varphi) \) is the identity congruence on \( \text{Dom}(\varphi) \). By the characterization of \( \leq_{\mathcal{L}}(M_{k,1}) \) in \( M_{k,1} \) (Theorem 3.32 in the arXiv version of \([3]\)), this implies that every element in the \( \mathcal{L}(M_{k,1}) \)-class of \( \varphi \) has the identity congruence as its partition. Hence the whole \( \mathcal{L}(M_{k,1}) \)-class is contained in \( \text{Inv}_{k,1} \).
3. Let \( \varphi : P \to Q \) be a table for an element of \( M_{k,1} \), where \( P \) and \( Q \) are finite prefix codes. Then \( \text{id}_Q \) belongs to the \( \mathcal{R}(M_{k,1}) \)-class of \( \varphi \) (by Theorem 2.1 in \([3]\)), and \( \text{id}_Q \) obviously belongs to \( \text{Inv}_{k,1} \).
4. This is straightforward. \( \Box \)

**Proposition 7.2** The decision problems for the Green relations \( \leq_{\mathcal{J}}, \equiv_{\mathcal{D}}, \leq_{\mathcal{R}}, \leq_{\mathcal{L}} \) of \( \text{Inv}_{k,1} \) are in \( \text{P} \) when the inputs are given by words over a finite generating set of \( \text{Inv}_{k,1} \).

**Proof.** Since \( \text{Inv}_{k,1} \) is a finitely generated submonoid of \( M_{k,1} \), this is a consequence of the corresponding result for \( M_{k,1} \) (Proposition 4.1 above and Theorems 5.1 and 6.1 in \([3]\)). \( \Box \)

As a consequence of Prop. 7.1(4) and the elementary facts about inverse monoids mentioned before Prop. 7.1, the \( \leq_{\mathcal{R}} \) decision problem and the \( \leq_{\mathcal{L}} \) decision problem of \( \text{Inv}_{k,1} \) (over a circuit-like alphabet \( \Gamma_I \cup \tau \)) can be reduced to each other and have the same computational complexity.

Let \( \Gamma_I \) be a finite generating set of \( \text{Inv}_{k,1} \); we can assume that \( \Gamma_I \) is closed under inverse (since this is only a finite change in the generating set). The 0 word problem of \( \text{Inv}_{k,1} \) over \( \Gamma_I \cup \tau \) is specified as follows.

**Input:** \( \varphi \in \text{Inv}_{k,1} \), given by a word over the generating set \( \Gamma_I \cup \tau \).

**Question:** Is \( \varphi = 0 \) as an element of \( \text{Inv}_{k,1} \)?

**Theorem 7.3** The 0 word problem of \( \text{Inv}_{k,1} \) over \( \Gamma_I \cup \tau \) is coNP-complete.

**Proof.** The problem is in coNP, for the same reason as the 0 word problem of \( M_{k,1} \) over \( \Gamma \cup \tau \) is in coNP (Prop. 6.2 in \([3]\)).

To show coNP-hardness we reduce the tautology problem for boolean formulas to the 0 word problem of \( \text{Inv}_{k,1} \). This is done in two steps; in the first step we work over the alphabet \( \{0, 1\} \), rather than over \( A = \{a_1, \ldots, a_k\} \), and (if \( k > 2 \)) in the second step we use \( A \).

Let \( \Gamma_{G,k} \) and \( \Gamma_{G,k} \) be finite generating sets for, respectively, \( \text{Inv}_{k,1} \) and \( G_{k,1} \). We can assume that \( \Gamma_{G,k} \subset \Gamma_{I,k} \) (since only finite changes are needed to achieve this).
Step 1. We will reduce the tautology problem for boolean formulas to the 0 word problem of $Inv_{2,1}$ (over $\Gamma_{I,2} \cup \tau$). Let $B(x_1, \ldots, x_m)$ be any boolean formula; it defines a map $B : \{0,1\}^m \rightarrow \{0,1\}$. By Theorem 4.1 in [5], we map $B$ (given by a boolean formula or a circuit) to an element $\Phi_B \in G_{2,1}$ (given by a word over $\Gamma_{G,2} \cup \tau$), such that for all $x \in \{0,1\}^m$:
$$\Phi_B(0x) = 0 B(x) x.$$ By Theorem 4.1 in [5], this mapping from a formula $B$ to word for $\Phi_B$ can be computed in deterministic polynomial time. Now we have:

$B$ is a tautology \iff $id_{\{0,1\}^*} \circ \Phi_B \circ id_{\{0,1\}^*} = 0$.

Since $\Gamma_{G,2} \subset \Gamma_{I,2}$, $\Phi_B$ is automatically over $\Gamma_{I,2} \cup \tau$. Also, the partial identities $id_{\{0,1\}^*}$ and $id_{\{0,1\}^*}$ are fixed elements of $Inv_{2,1}$ and they can be represented by fixed words over $\Gamma_{I,2}$. So the map $B \mapsto id_{\{0,1\}^*} \circ \Phi_B \circ id_{\{0,1\}^*}$ reduces the tautology problem for boolean formulas to the 0 word problem of $Inv_{2,1}$ (over $\Gamma_{I,2} \cup \tau$).

Step 2. We reduce the 0 word problem of $Inv_{2,1}$ (over $\Gamma_{I,2} \cup \tau$) to the 0 word problem of $Inv_{k,1}$ (over $\Gamma_{I,k} \cup \tau$), for any $k \geq 2$. Let $\psi \in Inv_{2,1}$, and let $\ell(\psi) = \max \{|z| : z \in domC(\psi) \cup imC(\psi)|$. Suppose $\psi$ is given by a word $w$ over $\Gamma_{I,2} \cup \tau$.

Let $\gamma \in \Gamma_{I,2}$ be any generator, and let us take a table $P \rightarrow Q$ be for $\gamma$, where $P, Q \subset \{0,1\}^*$ are finite prefix codes. We view $\gamma$ as an element $\gamma^A$ of $Inv_{k,1}$ by taking the table $P \rightarrow Q$ as a table over $A = \{a_1, \ldots, a_k\}$, by identifying $\{0,1\}$ with $\{a_1, a_2\} \subseteq A$. Let $\Gamma_{I,2} = \{\gamma^A : \gamma \in \Gamma_{I,2}\}$. Since $\Gamma_{I,2}^A$ is finite we can assume that $\Gamma_{I,2}^A \subset \Gamma_{I,k}$ (since only finite changes are needed to achieve this).

Let $W$ be the word over $\Gamma_{I,2} \cup \tau$ obtained by replacing each generator $\gamma \in \Gamma_{I,2}$ by the corresponding $\gamma^A$; elements of $\tau$ are not changed (except that they now act on $A^*$, rather than just $\{0,1\}^*$). Let $\Psi \in Inv_{k,1}$ be the element of $Inv_{k,1}$ represented by $W$. For $\psi \in Inv_{2,1}$, $\psi(z)$ is undefined when $z \not\in \{a_1, a_2\}$. For a prefix code $P \subset A^*$ we abbreviate $id_{P^{A^*}}$ to $id_P$. Then we have:

Claim. For all $z \in A^{\ell(\psi)}$: $\psi(z) = \Psi \circ id_{\{a_1, a_2\}^{\ell(\psi)}}(z).

Moreover, $\psi = 0$ as an element of $Inv_{2,1}$ if $\Psi \circ id_{\{a_1, a_2\}^{\ell(\psi)}}(z) = 0$ as an element of $Inv_{k,1}$.

Indeed, both sides of the equality are undefined on $A^{\ell(\psi)} \setminus \{a_1, a_2\}^{\ell(\psi)}$. For $z \in \{a_1, a_2\}^{\ell(\psi)} (= \{0,1\}^{\ell(\psi)})$ we have: $id_{\{a_1, a_2\}^{\ell(\psi)}}(z) = z$ and $\Psi(z) = \psi(z)$. Moreover, both $domC(\psi)$ and $domC(\Psi \circ id_{\{a_1, a_2\}^{\ell(\psi)}})$ are subsets of $A^{\leq \ell(\psi)}$. It follows that $\psi = 0$ in $Inv_{2,1}$ if $\Psi \circ id_{\{a_1, a_2\}^{\ell(\psi)}} = 0$ in $Inv_{2,1}$.

[This proves the Claim.]

One easily verifies that

$\text{id}_{\{a_1, a_2\}^{\ell(\psi)}} = \tau_{\ell(\psi), 1} \circ id_{\{a_1, a_2\}} \circ \tau_{\ell(\psi), 1} \circ \cdots \circ \tau_{j, 1} \circ id_{\{a_1, a_2\}} \circ \tau_{j, 1} \circ \cdots \circ \tau_{2, 1} \circ id_{\{a_1, a_2\}} \circ \tau_{2, 1}(\cdot)$.

Hence, the word-length of $\text{id}_{\{a_1, a_2\}^{\ell(\psi)}}$ over $\Gamma_{I,k} \cup \tau$ is polynomially bounded (in terms of $|\psi|_{\Gamma_{I,2} \cup \tau}$).

Thus the map from $\psi \in Inv_{2,1}$ (given by a word over $\Gamma_{I,2} \cup \tau$) to $\Psi \circ id_{\{a_1, a_2\}^{\ell(\psi)}} \in Inv_{k,1}$ (given by a word over $\Gamma_{I,k} \cup \tau$) is polynomial-time computable. Hence this map is a polynomial-time reduction from the 0 word problem of $Inv_{2,1}$ to the 0 word problem of $Inv_{k,1}$.

\textbf{Theorem 7.4 (The $R$ and $L$ decision problems).} The $\leq_L$ and $\leq_R$ decision problems of $Inv_{k,1}$ over $\Gamma_I \cup \tau$ are each coNP-complete.

\textbf{Proof.} The $\leq_L$ decision problem is in coNP for $M_{k,1}$ (by Theorem 6.7 in [3]), hence (by Prop. 7.1(1)) it is in coNP for $Inv_{k,1}$ too. In $Inv_{k,1}$, the $\leq_R$ decision problem reduces to the $\leq_L$ decision problem by Prop. 7.1(4), so the $\leq_R$ decision problem of $Inv_{k,1}$ is in coNP.

The $\leq_L$ and the $\leq_R$ decision problems are coNP-hard by Theorem 7.3 since $\varphi \leq_L 0$ if $\varphi = 0$ (and similarly for $\leq_R$).

\textbf{Theorem 7.5 (The $J$ decision problem).} The $\leq_J$ and the $\equiv_J$ decision problems of $Inv_{k,1}$ over $\Gamma_I \cup \tau$ are coNP-complete (for the $\equiv_J$ decision problems the coNP-completeness is with respect to polynomial-time disjunctive reduction).
Proof. The proof is the same as for $M_{k,1}$ (Theorem 5.2). For $\phi, \psi \in Inv_{k,1}$ we have $\psi \leq_{\mathcal{J}} \phi$ iff $\psi = 0$ or $\phi \neq 0$. The result follows since the 0 word problem is coNP-complete. Moreover, the $\leq_{\mathcal{J}}$ decision problem reduces to the $\equiv_{\mathcal{J}}$ decision problem by a polynomial-time disjunctive reduction, since $\psi \leq_{\mathcal{J}} \phi$ if and only if $\psi \equiv_{\mathcal{J}} 0$ or $\phi \equiv_{\mathcal{J}} \phi$. □

We will prove next that the membership problem of a non-zero $D$-class is easier for $Inv_{k,1}$ than for $M_{k,1}$ (seen in Theorem 6.12), if $\oplus_{k-1}P$ is different from $\oplus_{k-1}\bullet NP$. The class $\oplus_{h,j}P$ (for integers $h \geq 2$ and $0 \leq i \leq h - 1$) was defined at the beginning of Section 6.1. Just as for $\oplus_{h,j}P$, we can prove that $\oplus_{h,j}P = \oplus_{h,j}P$ for all $i, j$; therefore we denote $\oplus_{h,j}P$ by $\oplus_{h,j}P$ for every $i$.

**Theorem 7.6 (Complexity of $D$).** Let $k \geq 3$ and $1 \leq i \leq k - 1$. The membership problem of the $D$-class $D_i$ of $Inv_{k,1}$ over $\Gamma_I \cup \tau$ is $\oplus_{k-1}P$-complete.

Proof. Let $\phi \in Inv_{k,1}$ be given by a word over $\Gamma_I \cup \tau$. By injectiveness, $|\text{imC}(\phi)| = |\text{domC}(\phi)|$. Hence, by the characterization of the $D$ relation (Theorem 2.5 in [4]) and by the fact that this characterization applies to $Inv_{k,1}$ as well (Prop. 7.1(1)), the $D$-class $D_i$ of $Inv_{k,1}$ satisfies

$$D_i = \{ \phi \in Inv_{k,1} : |\text{domC}(\phi)| \equiv i \text{ mod } k - 1 \}.$$

Since the predicate $R = \{ (x, \phi) \in A^* \times (\Gamma_I \cup \tau)^* : x \in \text{domC}(\phi) \}$ is in P (by Prop. 5.6(1) in [4]), we conclude that the membership problem of $D_i$ is in $\oplus_{k-1}P$.

To show that the membership problem of $D_i$ is $\oplus_{k-1}P$-hard we will reduce $\oplus_{k-1}\text{SAT}$ to it by a polynomial-time parsimonious reduction. The input to $\oplus_{k-1}\text{SAT}$ is any boolean formula $B(x_1, \ldots, x_m)$; this formula defines a boolean function $B : \{0,1\}^m \rightarrow \{0,1\}$.

Let $\Gamma_{I,k}$ and $\Gamma_{G,k}$ be finite generating sets for, respectively, $Inv_{k,1}$ and $G_{k,1}$. We can assume that $\Gamma_{G,k} \subset \Gamma_{I,k}$ (since this can be achieved by finite changes). We build the reduction in two steps, the first for $k = 2$, the second for $k > 2$.

**Step 1.** As in the proof of Theorem 7.3 we map $B$ to the element $\Phi_B \in G_{2,1}$ (given by a word over $\Gamma_{G,2} \cup \tau$) such that for all $x \in \{0,1\}^m$:

$$\Phi_B(0x) = 0 B(x) x.$$

By Theorem 4.1 in [5], this mapping from a formula $B$ to word for $\Phi_B$ can be computed in deterministic polynomial time. Now we consider the element $\varphi_B \in Inv_{2,1}$ (given by a word over $\Gamma_{I,2} \cup \tau$), defined by

$$\varphi_B(\cdot) = \text{id}_{0(0,1)^*} \circ \Phi_B \circ \text{id}_{0(0,1)^*}(\cdot).$$

Then,

$$|\text{domC}(\varphi_B)| = 0 \{ x \in \{0,1\}^m : B(x) = 1 \} \subseteq \{0,1\}^{m+1} \text{ and }$$

$$|\text{imC}(\varphi_B)| = 0 \{ x \in \{0,1\}^m : B(x) = 1 \} \subseteq \{0,1\}^{m+2}.$$

Hence,

$$|\text{imC}(\varphi_B)| = |\{ x \in \{0,1\}^m : B(x) = 1 \}|.$$

Moreover, the partial identities $\text{id}_{0(0,1)^*}$ and $\text{id}_{0(0,1)^*}$ can be given by fixed words over $\Gamma_{I,2}$. So the map which sends a formula for $B$ to a word that represents $\varphi_B$ (over $\Gamma_{I,2} \cup \tau$) is polynomial-time computable, and it is parsimonious (in the sense that the image code size of $\varphi_B$ is the number of satisfying truth-value assignments of $B$).

**Step 2.** We identify $\{0,1\}$ with $\{a_1, a_2\} \subseteq A$. We will map a word representing $\varphi_B$ (over $\Gamma_{I,2} \cup \tau$) to a word over $\Gamma_{I,k} \cup \tau$, representing an element $\psi_B \in Inv_{k,1}$; this map should be polynomial-time computable, and it should be parsimonious in the sense that $|\text{imC}(\varphi_B)| = |\text{imC}(\psi_B)|$.

Let $w \in (\Gamma_{I,2} \cup \tau)^*$ be a word that represents $\varphi_B$. Let $\gamma \in \Gamma_{I,2}$ be any generator, and let $P \rightarrow Q$ be a table for $\gamma$, where $P, Q \subseteq \{0,1\}^*$ are finite prefix codes. We view $\gamma$ as an element $\gamma_A$ of $Inv_{k,1}$ by taking the table $P \rightarrow Q$ as a table over $A = \{a_1, \ldots, a_k\}$, by identifying $\{0,1\}$ with $\{a_1, a_2\} \subseteq A$. Let $\Gamma_{I,2}^A = \{ \gamma_A : \gamma \in \Gamma_{I,2} \}$. Since $\Gamma_{I,2}^A$ is finite we can assume that $\Gamma_{I,2}^A \subset \Gamma_{I,k}$.

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Let \( W \) be the word over \( \Gamma_{I,2} \cup \tau \) obtained from \( w \) by replacing each generator \( \gamma \in \Gamma_{I,2} \) by the corresponding \( \gamma^j \); elements of \( \tau \) are not changed (except that they now act on \( A^* \)). Let \( \Phi_B \) be the element of \( \text{Inv}_{k,1} \) represented by \( W \). For \( \varphi_B \in \text{Inv}_{k,1} \), \( \varphi_B(z) \) is undefined when \( z \notin \{a_1,a_2\}^* \). We have:

Claim. For all \( z \in A^{m+1} : \varphi_B(z) = \Phi_B \circ \text{id}_{\{a_1,a_2\}^{m+1}}(z) \). Moreover, after a restriction,

\[
\text{imC}(\varphi_B) = \text{imC}(\Phi_B \circ \text{id}_{\{a_1,a_2\}^{m+1}}) \subseteq a_1a_2\{a_1,a_2\}^m \subseteq \{a_1,a_2\}^{m+2}.
\]

Proof of the Claim: By the definition of \( \varphi_B \) we have \( \text{domC}(\varphi_B) \subseteq \{a_1,a_2\}^m \). Both sides of the equality are undefined on \( A^{m+1} - \{a_1,a_2\}^{m+1} \). For \( z \in \{a_1,a_2\}^{m+1} \) we have: \( \text{id}_{\{a_1,a_2\}^{m+1}}(z) = z \) and \( \Phi_B(z) = \varphi_B(z) \) (since \( \Phi_B \) and \( \varphi_B \) agree on \( \{a_1,a_2\}^* \)). [This proves the Claim.]

One easily verifies that

\[
id_{\{a_1,a_2\}^{m+1}} = \tau_{m+1,1} \circ \text{id}_{\{a_1,a_2\}} \circ \tau_{m+1,1,1} \circ \ldots \circ \tau_{j,1} \circ \text{id}_{\{a_1,a_2\}} \circ \tau_{j,1,1} \circ \ldots \circ \tau_{2,1} \circ \text{id}_{\{a_1,a_2\}} \circ \tau_{2,1,1}(...).
\]

Hence, the word-length of \( \text{id}_{\{a_1,a_2\}^{m+1}} \) over \( \Gamma_{I,k} \cup \tau \) is polynomially bounded. Thus the map from \( \varphi_B \in \text{Inv}_{2,1} \) (given by a word over \( \Gamma_{I,2} \cup \tau \)) to \( \Phi_B \circ \text{id}_{\{a_1,a_2\}^{m+1}} \in \text{Inv}_{k,1} \) (given by a word over \( \Gamma_{I,k} \cup \tau \)) is polynomial-time computable.

Since it follows from the Claim that \( |\text{imC}(\varphi_B)| = |\text{imC}(\Phi_B \circ \text{id}_{\{a_1,a_2\}^{m+1}})| \), the map from \( \varphi_B \) to \( \Phi_B \circ \text{id}_{\{a_1,a_2\}^{m+1}} \) is parsimonious.

Combining Step 1 and Step 2, we obtain a polynomial-time reduction from the a boolean formula \( B \) to an element \( \Phi_B \circ \text{id}_{\{a_1,a_2\}^{m+1}} \in \text{Inv}_{k,1} \) (given by a word over \( \Gamma_{I,k} \cup \tau \)). The reduction is parsimonious since \( |\{x \in \{0,1\}^m : B(x) = 1\}| = |\text{imC}(\Phi_B \circ \text{id}_{\{a_1,a_2\}^{m+1}})| \). Obviously, the latter equality holds modulo \( k - 1 \) too.

\[\square\]

**Theorem 7.7** The \( \mathcal{D} \)-pivots of \( \text{Inv}_{3,1} \) do not have polynomially bounded word-length over \( \Gamma_I \cup \tau \), unless the polynomial hierarchy \( \text{PH} \) collapses to \( \Sigma_3^p \cap \Pi_3^p \).

**Proof.** The proof is similar to the proof of Theorem 6.13. We saw that the \( \equiv_L \) and \( \equiv_R \) decision problems are in \( \text{coNP} \). If pivots always had polynomially bounded lengths, the \( \equiv_D \) decision problem would be in \( \Sigma_2^p \), by just guessing a pivot \( \chi \) in nondeterministic polynomial time, and checking whether \( \psi \equiv_L \chi \equiv_R \varphi \) (which is in \( \Pi_1^p \)). However, the \( \equiv_D \)-decision problem of \( \text{Inv}_{3,1} \) is \( \oplus P \)-complete, hence \( \oplus P \) would be contained in \( \Sigma_2^p \). By 30, \( \oplus P \subseteq \Sigma_2^p \) implies that \( \text{PH} \) collapses to \( \Pi_3^p \cap \Sigma_3^p \). \[\square\]

The remarks after the proof of Theorem 6.13 apply also to \( \text{Inv}_{k,1} \) when \( k \neq 3 \).

Finally, we consider the generalized word problem of \( \text{Inv}_{k,1} \) or \( G_{k,1} \) in \( M_{k,1} \) over a finite generating set \( \Gamma_M \) of \( M_{k,1} \) (or over a circuit-like generating set \( \Gamma_M \cup \tau \)). These generalized word problems over \( \Gamma_M \) or over \( \Gamma_M \cup \tau \), are specified as follows.

**Input:** \( \varphi \in M_{k,1} \), given by a word over \( \Gamma_M \) (or over \( \Gamma_M \cup \tau \)).

**Question (generalized word problem of \( \text{Inv}_{k,1} \) in \( M_{k,1} \))**: Is \( \varphi \) in \( \text{Inv}_{k,1} \) ?

**Question (generalized word problem of \( G_{k,1} \) in \( M_{k,1} \))**: Is \( \varphi \) in \( G_{k,1} \) ?

**Proposition 7.8** When inputs are given over a finite generating set of \( M_{k,1} \) the generalized word problems of \( \text{Inv}_{k,1} \) and of \( G_{k,1} \) in \( M_{k,1} \) are in \( P \).

**Proof.** We saw in 3, Sections 5.1 and 6.1, that for inputs over \( \Gamma \) we can check \( \equiv_R \) and \( \equiv_L \) in deterministic polynomial time. Since \( \varphi \in G_{k,1} \) iff \( \varphi \equiv_L 1 \) and \( \varphi \equiv_R 1 \), it follows that the generalized word problem of \( G_{k,1} \) in \( M_{k,1} \) over \( \Gamma \) is in \( P \).

To solve the generalized word problem of \( \text{Inv}_{k,1} \) in \( M_{k,1} \), observe that \( \varphi \in \text{Inv}_{k,1} \) iff \( \varphi \) is injective, which holds iff for every \( y \in \text{imC}(\varphi) \), \( |\varphi^{-1}(y)| = 1 \). Recall that when \( \varphi \) is given by a word over \( \Gamma \), we can compute \( \text{imC}(\varphi) \) as an explicit list of words, in deterministic polynomial time (Corollary 4.11 in 3). Also, for each \( y \in \text{imC}(\varphi) \) we can compute a finite-state automaton \( A_y \) accepting \( \varphi^{-1}(y) \); the
set $\varphi^{-1}(y)$ is finite and $A_y$ is acyclic, and reduced. We have $|\varphi^{-1}(y)| = 1$ iff every state in $A_y$ has out-degree 1, i.e., the graph of $A_y$ is a chain (with every edge labeled by one letter). Checking whether $A_y$ is a chain can be done in polynomial time, so the generalized word problem of $Inv_{k,1}$ in $M_{k,1}$ is in $P$. □

**Proposition 7.9** When inputs are given over a circuit-like generating set $\Gamma \cup \tau$ of $M_{k,1}$ the generalized word problem of $Inv_{k,1}$ in $M_{k,1}$ is coNP-complete.

**Proof.** An element $\varphi \in M_{k,1}$ belongs to $Inv_{k,1}$ iff $\varphi$ is injective. By Theorem 4.5 in [4], the length of the longest words in $\text{dom}C(\varphi)$ is $\leq c \cdot |\varphi|_{\Gamma \cup \tau}$, for some constant $c$. Hence, non-injectiveness of $\varphi$ can be decided in nondeterministic polynomial time by guessing two different words $x_1, x_2 \in \text{dom}C(\varphi)$ and checking that $\varphi(x_1) = \varphi(x_2)$. We know from Theorem 4.12 in [4] that $\varphi(x_1)$ and $\varphi(x_2)$ can be computed in deterministic polynomial time. Hence, the generalized word problem of $Inv_{k,1}$ in $M_{k,1}$ is in coNP.

In Prop. 6.5 in [3] it was proved that the injectiveness problem for combinational circuits is coNP-complete. Since combinational circuits can be represented by words over the circuit-like generating set $\Gamma \cup \tau$ of $M_{k,1}$, it follows that the generalized word problem of $Inv_{k,1}$ in $M_{k,1}$ is coNP-hard. □

**Open question:** What is the complexity of the generalized word problem of the Thompson-Higman group $G_{k,1}$ in $M_{k,1}$ when the input is a word over a circuit-like generating set $\Gamma_M \cup \tau$ of $M_{k,1}$?

We know that the problem is in $\Pi^p_2$, since the question whether $\varphi \equiv_L 1$ is in coNP (by Theorem 6.7 in [3]), and the question whether $\varphi \equiv_R 1$ is in $\Pi^p_2$.

We also know that the problem is coNP-hard. Indeed, Prop. 6.5 in [3] gives a polynomial-time reduction $B \mapsto F_B$ where $B$ is any boolean formula, and $F_B \in M_{k,1}$ (given by a word over $\Gamma_M \cup \tau$) is such that:

1. if $B$ is a tautology then $F_B = 1$ as an element of $M_{k,1}$;
2. if $B$ is not a tautology then $F_B$ is not injective.

Since $1 \in G_{k,1}$, whereas non-injective elements are not in $G_{k,1}$, this reduces the tautology problem (which is coNP-complete) to the generalized word problem of $G_{k,1}$ in $M_{k,1}$.

**Open question:** What is the distortion of $G_{k,1}$ (over $\Gamma_G \cup \tau$) within $M_{k,1}$ (over $\Gamma_M \cup \tau$)? Similarly, what is the distortion of $Inv_{k,1}$ in $M_{k,1}$ over circuit-like generating sets?

## 8 Appendix: Search problems, NPsearch, and xNPsearch

A **search problem** is a relation of the form $R \subseteq A^* \times B^*$ (where $A$ and $B$ are finite alphabets). We usually formulate the search problem $R$ in the following form.

**Input:** A string $x \in A^*$;

**Premise:** There exists $y \in B^*$ such that $(x, y) \in R$;

**Search:** Find one $y \in B^*$ such that $(x, y) \in R$.

A **premise problem** is a decision problem (or a search problem) which, in addition to an input and a question (or a requested output), also has a premise concerning the input. The premise, also called “pre-condition”, is an assumption about the input that any algorithm for the problem is allowed to use as a fact. The algorithm does not need to check whether the assumption actually holds for the given input and, indeed, we don’t care about the answer (or the output) when the premise does not hold. In the literature the word “premise” is often used for “premise” (although, according to the dictionaries of the English language, “premise” is more logical).

For $R \subseteq A^* \times B^*$, the domain of $R$ is $\text{Dom}(R) = \{x \in A^* : (\exists y \in B^*)[(x, y) \in R]\}$; in words, $\text{Dom}(R)$ is the set of inputs for which the search problem $R$ has a solution. The membership problem
of Dom(R) is called the decision problem associated with the search problem R. The membership problem of R is called the verification problem associated with R.

The best known search problem complexity class is NPsearch (called FNP or “function problems associated with NP” in [23]). The class NPsearch consists of all relations of the form R ⊆ A* × B* (where A and B are finite alphabets) such that (according to [24], pages 227-240):

1. the membership problem of R (i.e., the verification problem) belongs to P, i.e., there exists a deterministic polynomial-time algorithm which on input (x, y) ∈ A* × B* decides whether (x, y) ∈ R;
2. R is polynomially balanced; this means that there exists a polynomial p such that for all (x, y) ∈ R: |y| ≤ p(|x|).

When R is in NPsearch then the associated decision problem is in NP. The complementary decision problem (namely the task of answering “no” on input x iff there is no y such that (x, y) ∈ R) is in coNP. It is interesting to compare the class NPsearch also with #P, which consists of the functions that count the number of solutions of NPsearch problems.

By definition, a deterministic algorithm A solves the search problem R iff for every input x ∈ Dom(R), the algorithm A outputs an element y ∈ B* such that (x, y) ∈ R. No requirement is imposed on A when x ∉ Dom(R); however, if complexity bounds are known (or required) for A the above definition implies that A also, indirectly, determines whether x ∉ Dom(R), and we output “no” in that case. Probabilistic solutions of a search problem can also be defined. One way to do that is to say that a probabilistic algorithm A solves the search problem R iff for every x ∈ Dom(R): P(|y ∈ B* : y = A and (x, y) ∈ R|) ≥ c (where c is a constant, 0 < c < 1). No requirement is imposed on A when x ∉ Dom(R). But since R is in P, proposed false solutions can be ruled out.

Remark. The idea of solving a search problem by a deterministic algorithm explains why NPsearch was called FNP (where the “F” stands for “function”). However, it is better not to attach the word “function” to NPsearch because the problems in NPsearch are relations. Functions may play a role in special ways of solving a search problem; but other, non-functional, solutions of search problems are often considered too, e.g., probabilistic algorithms.

Following [24], page 229, we define the concept of a polynomial-time many-to-one search reduction from a search problem R1 ⊆ A1* × B1* to a search problem R2 ⊆ A2* × B2* as follows. Such a reduction is a triple of polynomial-time computable total functions ρin : A1* → A2*, ρsol : A1* × B1* → B2*, and ρver : A1* × B1* → A2* × B2* such that:

1. For all x ∈ Dom(R1) : ρin(x) ∈ Dom(R2).
2. For all x ∈ A1* and all y ∈ B2* : (ρin(x), y) ∈ R2 implies (x, ρsol(x, y)) ∈ R1.
3. For all (x, y) ∈ A1* × B1* : (x, y) ∈ R1 iff ρver(x, y) ∈ R2. Moreover, ρver is “polynomially balanced”, i.e., there is a polynomial p such that for all (x, y) ∈ A1* × B1* : if ρver(x, y) = (x, y) then |y| ≤ p(|x|).

In words, condition (1) says that if R1 has a solution for input x then R2 has a solution for input ρin(x). When R2 is total, i.e., Dom(R2) = A1*, then condition (1) holds automatically. Condition (2) means that every R2-solution y2 for input ρin(x1) yields an R1-solution ρsol(x1, y2) for input x1. Condition (3) means that the verification problem of R1 reduces to the verification problem of R2. As a consequence of condition (3), the class NPsearch is closed under polynomial-time many-to-one search reduction. (Condition (3) is usually omitted in the literature; however, the literature also claims that NPsearch is closed under search reduction, but this does not follow from conditions (1) and (2) alone.) The pair of maps (ρin, ρsol) is called the input-output reduction, and the map ρver is called the verification reduction.

There are well-known search problems that are closely related to NP but that don’t exactly fit into the class NPsearch. For example, the search version of integer linear programming is not polynomially balanced: for some inputs there are infinitely many solutions, of unbounded size, although there also
exist polynomially bounded solutions for every input that has a solution. Similar examples are certain versions of the Traveling Salesman problem, or finding solutions to certain equations (search version of problems on pp. 249-253 in [15]). We prove in Section 5.2 that the special multiplier search problem for \( \equiv \mathcal{J} \) 1 in \( M_{k,1} \) over \( \Gamma \cup \tau \) is another example. In the \( \equiv \mathcal{J} \) 1 multiplier search problem, when there are solutions then there are also solutions that are polynomially bounded and verifiable in deterministic polynomial time. But the general verification problem for \( \equiv \mathcal{J} \) 1 is not polynomially balanced. The main observation is that in a search problem we only want to find one solution for each input, so the difficulty of the general verification problem and the size of all solutions in general should not concern us.

Therefore we introduce the class xNPsearch (extended NP search), consisting of all relations of the form \( R \subseteq A^* \times B^* \) (where \( A \) and \( B \) are finite alphabets) such that there is a relation \( R_0 \subseteq R \) with the properties

1. \( R_0 \in \text{NPsearch} \),
2. \( \text{Dom}(R) = \text{Dom}(R_0) \).

When \( R \) is in xNPsearch then the associated decision problem is in \( \text{NP} \), just as for \( \text{NPsearch} \), since problems in \( \text{NPsearch} \) and xNPsearch have the same domains.

By definition, a search problem \( R \) is NPsearch-complete iff \( R \) is in NPsearch, and every problem in NPsearch can be reduced to \( R \) by a polynomial-time many-to-one search reduction. We say that \( R \) is xNPsearch-complete iff there is an NPsearch-complete problem \( R_0 \) such that \( R_0 \subseteq R \) and \( \text{Dom}(R_0) = \text{Dom}(R) \).

It follows that an xNPsearch-complete problem is in xNPsearch. And it follows that an NPsearch-complete problem is automatically xNPsearch-complete.

An example of an NPsearch-complete (hence xNPsearch-complete) problem is the following, called SatSearch; it is the relation \( \{(B,t) : B \) is a boolean formula with \( m \) variables, \( m > 0 \), \( t \in \{0,1\}^m \), and \( B(t) = 1 \} \). Equivalently, SatSearch is specified as follows:

**Input:** A boolean formula \( B(x_1,\ldots,x_m) \) (where \( m \) is part of the input, hence variable).

**Premise:** \( B(x_1,\ldots,x_m) \) is satisfiable.

**Search:** Find a satisfying truth-value assignment \( t \in \{0,1\}^m \) for \( B(x_1,\ldots,x_m) \).

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