ON THE KRULL DIMENSION OF THE DEFORMATION RING OF CURVES WITH AUTOMORPHISMS

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ABSTRACT. We reduce the study of the Krull dimension $d$ of the deformation ring of the functor of deformations of curves with automorphisms to the study of the tangent space of the deformation functor of a class of matrix representations of the $p$-part of the decomposition groups at wild ramified points, and we give a method in order to compute $d$.

1. Introduction

Let $X$ be a non-singular projective curve defined over the field $k$, and let $G$ be a fixed subgroup of the automorphism group of $X$. We will denote by $(X, G)$ the couple of the curve $X$ together with the group $G$.

A deformation of the couple $(X, G)$ over a local ring $A$ is a proper, smooth family of curves

$$\mathcal{X} \rightarrow \text{Spec}(A)$$

parametrized by the base scheme $\text{Spec}(A)$, together with a group homomorphism $G \rightarrow \text{Aut}_A(\mathcal{X})$ such that there is a $G$-equivariant isomorphism $\phi$ from the fibre over the closed point of $A$ to the original curve $X$:

$$\phi : \mathcal{X} \otimes_{\text{Spec}(A)} \text{Spec}(k) \rightarrow X.$$ 

Two deformations $\mathcal{X}_1, \mathcal{X}_2$ are considered to be equivalent if there is a $G$-equivariant isomorphism $\psi$, making the following diagram commutative:

$$\begin{array}{ccc}
\mathcal{X}_1 & \xrightarrow{\psi} & \mathcal{X}_2 \\
\downarrow & & \downarrow \\
\text{Spec } A & \xrightarrow{\phi} & \text{Spec } A
\end{array}$$

Let $\mathcal{C}$ denote the category of local Artin algebras over $k$. The global deformation functor is defined:

$$D_{\text{gl}} : \mathcal{C} \rightarrow \text{Sets}, \ A \mapsto \left\{ \begin{array}{l}
\text{Equivalence classes} \\
of \text{deformations of} \\
couples (X, G) \text{ over } A
\end{array} \right\}$$

The deformation functor $D_{\text{gl}}$ of non singular curves together with a subgroup of the automorphism group, admits a pro-representable hull $R$ as J. Bertin and A. Mézard proved using Schlessinger’s approach.

The Krull dimension of the hull is in general smaller than the dimension of the tangent space of the deformation functor; there are obstructions preventing infinitesimal deformations to be lifted.

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The study of obstructions, and therefore the study of the ring structure of the hull, is a difficult question but it has been done for ordinary curves by G. Cornelissen and F. Kato [6].

J. Bertin and A. Mézard proved a local-global principle [3, th. 3.3.4, cor. 3.3.5], that reduces the problem of computing the Krull dimension, to the computation of the dimension of a local deformation functor attached to each wild ramified point. The main difficulty for the study of obstructions, is that the obstructions to such liftings are reduced to obstructions of lifting representations from the groups $\text{Aut}_k[[t]]$ to $\text{Aut}_A[[t]]$, where $A$ is a local Artin algebra with maximal ideal $m$, such that $A/m = k$. The automorphism group of the ring of formal power series is a difficult object to study. In this paper we try to reduce the problem of these liftings to a similar lifting problem involving general linear groups.

In order to make the presentation and the calculations simpler, we will restrict ourselves to subgroups of the automorphism group that have order a power of $p$. Equivalently, for all decomposition groups at various points $P$ of the curve, we assume $G_0(P) = G_1(P)$, where $G_i(P)$ denotes the ramification filtration of the decomposition group at a ramified point.

We will give criteria depending on the Weierstrass semigroup structure on the wild ramified points, (Prop. [8] Cor. [10] that allow us to connect liftings of representations to $\text{Aut}_A[[t]]$ with liftings of representations of general linear groups and the deformation theory of such representations is better understood. If these conditions do not hold, then we can restrict ourselves to a subfunctor $D \subset D$ by posing the desired conditions as deformation conditions in the sense of B. Mazur [21, p.289]. Then the new subfunctor also has a hull $R_D$ and $\dim_k R_D \leq \dim_k R$.

We have to mention that our approach does not give us the ring structure of the hull, but provides us with a method to compute its Krull dimension.

The Krull dimension has been considered, by different methods, by R. Pries [23], but only for deformations that do not split the branch locus. The conditions we put are more general than the ones of Pries and can be applied to a wider class of deformations.

We begin our study by showing that there is a faithful representation

$$\rho : G_1(P) \to GL_n(k),$$

of the $p$-part of the decomposition group at a wild ramified point to a suitable general linear group, and we show how to relate the filtration of the ramification group to the radical decomposition of the algebra of lower triangular matrices. This allows us to describe the structure of $G_1(P)$ for some interesting examples. In particular we are able to prove that the representation in the case of ordinary curves is three dimensional.

Next we give conditions so that deformations over a local domain $A$, respect the flag of the Weierstrass subspaces and give rise to lifting of $\rho$ to a representation

$$\tilde{\rho} : G_1(P) \to GL_n(A),$$

that reduces to $\rho$ modulo $m$.

In order to perform such a construction we need deformations over local domains that can be extended to the generic fibre. Unobstructed deformations give rise to families over formal schemes that do not possess a generic fibre. We employ Artin’s algebraisation theorem in order to show that an extension of the family to the generic fibre is always possible. As a result, we can use (prop. [6] an algebraic
equivalence argument in order to compare Artin’s representations at wild ramified points in both the generic and the special fibre, obtaining a generalisation of a theorem of Bertin.

We introduce a deformation functor $F(\cdot)$ for deformations of matrix representations and we are able to relate these two deformation functors in proposition 4.8.

The deformation ring $R_D$ corresponding to the subfunctor $D$ may have nilpotents \([6, 4.4.1]\) and in general might not be irreducible. Factoring out the radical of $R_D$, we obtain a finitely generated $k$-algebra without nilpotents that corresponds to finite union of irreducible sets. In proposition 4.12 and corollary 4.13 we prove that all these algebraic sets have equal dimension, and this dimension is equal to the dimension of the tangent deformation functor $F(k[\epsilon])$. Moreover, we prove that infinitesimal deformations in $F(k[\epsilon])$ are unobstructed and this fact allows us to compute the desired Krull dimension.

Many authors have tried to study the “simplest possible” wild ramification. From the point of view of ramification filtrations the simplest wild ramification is the “moderate” wild ramification, i.e., when for all wild ramified points $P$ we have $G_i(P) = \{1\}$ for all $i \geq 2$. For instance ordinary curves have this property.

From the representation point of view the simplest possible wild ramification at a point $P$ is when the faithfull representation attached to this point is two dimensional. We can prove that this is the case for all curves of genus $g$ so that $p > 2g - 2$. One of the new results of this article is the following:

**Theorem 1.1.** If the faithfull representation attached to a wild ramification point is two dimensional, then the hull of the local deformation functor at this point is the ring of formal power series $k[[x_1, \ldots, x_s]]$, where $s = \log_p |G_1(P)|$.

We also try to illustrate our method by giving examples and by comparing our method with computation done by other authors. Namely we apply our method to the case of deformations of ordinary curves and as a final application, we show how the tools we have developed can be used in order to study deformations of the curves $y^p - y = \sum_{i=1}^{s-1} x^{p^i+1} + x^{p^s+1}$.

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2. **Representations**

Let $C$ be a nonsingular complete curve defined over an algebraically closed field of characteristic $p \geq 0$. Let $G$ be a subgroup of the automorphism group of $C$, and let $P$ be a wildly ramified point of $C$. We denote by $G_0(P)$ the decomposition subgroup of $G$, and by $G_1(P)$ the $p$-part of $G_0(P)$. The $p$-part of the decomposition group can be analysed in terms of the sequence of the $i$-th ramification groups \([26\) chap. IV]:

\[ G_1(P) \geq G_2(P) \geq \cdots \geq G_n(P) > \{1\}. \]

For every point $P$ of the curve $C$ of genus $g$ we consider the sequence of $k$-vector spaces

\[ k = L(0) = L(P) = \cdots = L((i-1)P) < L(iP) \leq \cdots \leq L((2g-1)P), \]
where

$$L(iP) := \{ f \in k(C) : \text{div}(f) + iP \geq 0 \} = H^0(X, L(iP)),$$

It is known that there are exactly $g$ pole numbers that are smaller or equal to $2g - 1$. If $g \geq 2$ then there is at least one of them not divisible by the characteristic.

**Lemma 2.1.** Let $m \leq 2g - 1$ be the smallest pole number not divisible by the characteristic. There is a faithful representation

$$\rho : G_1(P) \to \text{GL}(L(mP))$$

**Proof.** It is clear that the space $L(mP)$ is preserved by any automorphism in $G(P) = G_1(P)$. Let $f$ be a function with pole at $P$ of order $m$. We can write $f$ as $f = u \frac{1}{tm}$, where $u$ is a unit in the local ring $O_P$. Since $(m, p) = 1$, Hensel’s lemma implies that $u$ is an $m$-th power so the local uniformizer can be selected so that $f = \frac{1}{tm}$. Let $\sigma \in G_1(P)$ such that $\sigma(1/t^m) = 1/t^m$. Then $\sigma(t) = \zeta t$ where $\zeta$ is an $m$-th root of unity. Therefore, if $\sigma$ induces the trivial matrix in $\text{Aut} L(mP)$ and $\sigma$ is of order $p$, then $\zeta = 1$ since $(p, m) = 1$. □

The above lemma makes the $p$-part of the decomposition group $G_1(P)$ realizable as a finite algebraic subgroup of the linear group $GL_n(k)$. Moreover the flag of vector spaces $L(iP)$ for $i \leq m$ is preserved, so the representation matrices are upper triangular.

We assume that $m = m_0 > m_1 > \cdots > m_r = 0$, are the pole numbers less than $m$. Therefore, a basis for the vector space $L(mP)$ is given by

$$\{1, u_i \frac{1}{tm_i}, \frac{1}{tm} : \text{where } 1 < i < r, p \mid m_i \text{ and } u_i \text{ are units}\}$$

According to this basis, an element $\sigma \in G_1(P)$ acts on $L(mP)$ by

$$\sigma \frac{1}{tm} = \frac{1}{tm} + \sum_{i=1}^{r} c_i(\sigma) u_i \frac{1}{tm_i},$$

and equivalently it maps the local uniformizer $t$ to

$$\sigma(t) = \frac{\zeta t}{(1 + \sum_{i=1}^{r} c_i(\sigma) u_i t^{m-m_i})^{1/m}},$$

where $\zeta$ is an $m$-th root of one.

The above expression can be written in terms of a formal power series as:

$$\sigma(t) = \zeta t \left(1 + \sum_{\nu \geq 1} a_{\nu}(\sigma) t^{\nu}\right).$$

On the other hand the composition of the formal power series

$$f = \sum_{i \geq 0} a_i t^i \text{ and } g = \sum_{j \geq 0} b_j t^j$$

is written as $ta_0b_0 + \cdots$, so the automorphism $\sigma^{[G_1(P)]} = 1$ as it is given in (4) is $t\zeta^p + \cdots = t$ and since $m$ is prime to $p$, $\zeta = 1$ and (4) can be written as

$$\sigma(t) = t \left(1 + \sum_{\nu \geq 1} a_{\nu}(\sigma) t^{\nu}\right).$$

The above computation allows us to compute the “gaps” in the filtration of the group $G_1(P)$. 

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Proposition 2.2. Let \( P \) be a point on the curve \( C \) and let 
\[
\rho : G_1(P) \to \text{GL}_{\dim(L(mP))}(k)
\]
be the corresponding faithful representation we considered on lemma 2.1. Let \( m = m_0 > m_1 > \cdots > m_r = 0 \) be the pole numbers at \( P \) that are smaller than \( m \). For the filtration of \( G_1(P) \) we have 
\[
\text{if } G_i(P) > G_{i+1}(P) \text{ then } i + 1 = m - m_k + 1,
\]
for some pole number \( m_k \).

Proof. By definition \( \sigma \in G_i(P) \) if and only if 
\[
\sigma(t) - t \in t^i k[[t]].
\]
Notice that there is at least one \( c_i(\sigma) \neq 0 \), because if all \( c_i(\sigma) = 0 \) for \( i = 1, \ldots, r \) then \( \sigma(1/t^m) = 1/t^m \) and \( \sigma \) is the identity. The valuation of the expression \( \sigma(t) - t \) can be explicitly computed:
\[
\sigma(t) - t = -\frac{1}{m} \sum_{i=1}^{r} c_i(\sigma) u_i t^{m_i} + \cdots,
\]
therefore
\[
v_i(\sigma(t) - t) = m - m_k + 1
\]
where \( k = \min\{i : c_i(\sigma) \neq 0\} \). The possible valuations are given by:
\[
m - m_1 + 1 \quad \text{if } c_1(\sigma) \neq 0
\]
\[
m - m_2 + 1 \quad \text{if } c_2(\sigma) \neq 0, c_i(\sigma) = 0 \text{ for } i < 2
\]
\[
\vdots
\]
\[
m - m_r + 1 \quad \text{if } c_r(\sigma) \neq 0, c_i(\sigma) = 0 \text{ for } i < r
\]
Assume that \( \sigma \in G_i(P) \) but \( \sigma \notin G_{i+1}(P) \), thus \( v(\sigma(t) - t) = i + 1 \) and this equals some \( m - m_k + 1 \). \( \square \)

Corollary 2.3. Every jump \( i \) in the ramification filtration, i.e., \( G_i > G_{i+1} \) is not divisible by \( p \).

Proof. By lemma 2.2 every gap in the ramification filtration is given as \( m - m_k \), where \( m \) is not divisible by \( p \) and \( m_k \) are divisible by \( p \) [20 IV. prop. 11]. \( \square \)

Examples:
1. The Fermat curves \( x^n + y^n + 1 = 0 \), where \( n - 1 = p^h \). The automorphisms of these curves where studied by H. W. Leopoldt in [13], even if \( n - 1 \) is not a power of the characteristic. Leopoldt constructed a basis for the space of holomorphic differentials of the curve and he was able to prove that for the points of the form \( P : (x, y) = (\zeta_{2n}, 0) \) where \( \zeta_{2n} \) is a \( 2n \)-root of one, we have the following sequence of \( k \)-vector spaces [13 Satz 4]:
\[
k = L(0P) = L(P) = \cdots = L((n-2)P) < L((n-1)P) < L(nP) \leq \cdots
\]
The interesting case for us (Hermitian Function Fields) appears when \( n - 1 \) is a power of the characteristic, so in this case the representation of the decomposition subgroup is of the form:
\[
\rho : G_0(P) \to \text{GL}(L(nP))
\]
with
\[
\sigma \mapsto \begin{pmatrix} 1 & 0 & 0 \\ \alpha & \chi & 0 \\ \gamma & \beta & \psi \\ 0 & 0 & 0 \end{pmatrix},
\]
According to proposition 4 the filtration of the decomposition subgroup is given by:

\[ G_0(P) > G_1(P) > G_2(P) = \cdots = G_n(P) > G_{n+1}(P) = \{1\}, \]

i.e., the gaps of the filtration are in \(0, 1, n\).

2. The curves \(x^n + x^m + 1 = 0\), where \(m \mid n\) and \(m - 1 = p^h\). The automorphism group of a nonsingular model of the above curve is studied by the author, in [15]. It is proved that for the points \(P : (x, y) = (\zeta_2^n, 0)\) we have the following sequence of vector spaces [15 eq. (4)]

\[ k = L(0P) = L(P) = \cdots = L((m-1)P) < L(mP) = L((m+1)P) \leq \cdots \]

Since \(m\) is not divisible by \(p\) we have the following representation

\[ G_0(P) \to GL(L(mP)), \]

sending

\[ \sigma \mapsto \begin{pmatrix} 1 & 0 \\ \alpha & \chi \end{pmatrix}. \]

Thus \(G_0(P)\) is the semidirect product of an elementary abelian group by a cyclic group of order prime to the characteristic. For the ramification filtration of \(G_0(P)\) we have

\[ G_0(P) > G_1(P) > G_2(P) = \cdots = G_m(P) > \{1\}. \]

3. Ordinary Curves. A curve is called ordinary if the \(p\)-rank of the Jacobian is equal to the genus of the curve. It is known that ordinary curves form a Zariski-dense set in the moduli space of curves of genus \(g\). For ordinary curves we have that \(G_2(P) = \{1\}\) [22], thus we have the following picture for the faithful representation \(\rho\) of the group \(G_1(P)\): There is a gap at \(G_1(P) > G_2(P) = \{1\}\), thus \(m_i = m - 1\) for some \(i\), and this \(i = 1\). In other words the pole numbers that are smaller or equal to \(m\) are \(\{m, m - 1\}\). This implies that if the genus \(g\) of \(X\) is \(g \geq 1\) then the representation has dimension 3, because otherwise, i.e., if the representation is two dimensional, we have the following sequence

\[ k = L(0P) = L(P) = \cdots L((m-1)P) < L(mP). \]

But \(m - 1\) is a pole number so \(m - 1 = 0\) and \(m = 1\), i.e., the Weierstrass semigroup is the semigroup of natural numbers, a contradiction, for \(g \geq 2\).

Moreover if \(c_1(\sigma) = 0\) then all \(c_i(\sigma) = 0\) for \(i > 1\), otherwise there will be more jumps at higher groups \(G_i\), and this is impossible. This proves that \(c_1(\sigma) = 0\) if and only if \(\sigma = 0\). By multiplying the representation matrices we can easily deduce that the map

\[ c_1 : G_1(P) \to k, \]

is a faithful homomorphism of the elementary abelian group \(G_1(P)\) into the additive group of \(k\).

The representation matrices are commuting, and by computation this implies that all elements \(\rho_{j+1,j}(\sigma)\), of the representation matrix, are of the form \(\lambda_j c_1(\sigma)\), and \(\lambda_j\) is independent of \(\sigma\).

Since \(c_1\) is faithful character, such that \(c_1(\sigma) = 0\) implies that \(\rho_i(\sigma) = 0\), for all \(i \neq j\), we can write \(\rho_{ij}(\sigma) = c_1(\sigma)a_{ij}(\sigma)\).
4. p-cyclic covers of the affine line. In this example we apply our computations to Artin-Schreier curves that are nonsingular models of the function field defined by:

\[ C_{t_1, \ldots, t_{s-1}} : W^p - W = \sum_{i=1}^{s-1} t_i X^{p^i+1} + X^p + 1. \]

These curves give extreme examples of automorphism groups and were studied by H. Stichtenoth [27] and C. Lehr, M. Matignon [17], N. Elkies [10], van der Geer and van der Vlugt [29].

There is only one ramified point in the cover \( C_{t_1, \ldots, t_{s-1}} \to \mathbb{P}^1 \), the point \( P \) that is over the point \( X = \infty \) of \( \mathbb{P}^1 \). The Weierstrass semigroup at \( P \) is computed by H. Stichtenoth [27] to be equal to \((p^s + 1)N + pN\). Thus, the smaller pole number that is not divisible by \( p \) is \( p^s + 1 \) and the Weierstrass semigroup up to \( p^s + 1 \) is computed to be

\[ 0, p, 2p, \ldots, \left\lfloor \frac{1+p^s}{p} \right\rfloor p, 1 + p^s. \]

One can prove that \( \left\lfloor \frac{1+p^s}{p} \right\rfloor p = p^s \). The remainder of the division of \( p^s + 1 \) by \( p \), is 1. According to proposition [27] the possible gaps at the ramification filtration are at the numbers \( 1 + kp, k = 0, \ldots, \left\lfloor \frac{1+p^s}{p} \right\rfloor = p^s - 1 \). The dimension \( \dim_k L((p^s + 1)P) \) is \( n = \left\lfloor \frac{1+p^s}{p} \right\rfloor + 2 = p^{s-1} + 2 \) and the representation of \( G_1(P) \) to \( L((p^s + 1)P) \) is given by an \( n \times n \) lower triangular matrix with 1 in the diagonal.

More precisely, if we choose the natural basis \( \{1, X, X^2, \ldots, X^{p^s-1}, W\} \) of \( L((p^s + 1)P) \) then the representation \( \rho \) is given by the matrix

\[
\rho(\sigma)_{ij} = \begin{cases} 
0 & \text{if } i < j \\
1 & \text{if } i = j \\
y^{i,j} & \text{if } i > j, i \neq p^{s-1} + 1 \\
b_j(y) & \text{if } i = p^{s-1} + 1 > j
\end{cases}
\]

where \( b_j(y) \) are the coefficients of the polynomial \( P_f(X, y) \), and \( y \) is a solution of \( Ad_f(Y) = 0 \) as defined in lemma 4.1 and definition 4.2 in [17].

3. Deforming Branch Points

Assume that we have a deformation \( \mathcal{X} \to \text{Spec} A \) over an Artin local ring, that admits a fibrewise action of the \( G \subset Aut(\mathcal{X}) \). In lemma [24] we have assigned to every wild ramified point \( P \) a representation of the decomposition group \( G(P) \) that corresponds to an upper triangular matrix. We will try to lift this representation to a representation of an upper triangular matrix with coefficients in \( A \). This will be accomplished by proving that we can fulfill the requirements of the following

**Proposition 3.1.** Consider a deformation \( \mathcal{X} \to \text{Spec} A \) of the curve \( X \) defined over a local integral domain \( A \), with \( G \subset Aut(\mathcal{X}) \) acting fibrewise on \( \mathcal{X} \), and let \( P \) be a wild ramified point on the special fibre of \( \mathcal{X} \). Suppose that there is a sequence of \( G \)-invariant invertible \( \mathcal{O}_{\mathcal{X}} \)-modules \( L_i \) so that the corresponding spaces of global sections \( L_i := \Gamma(\mathcal{X}, L_i) \) satisfy:

\[ L_1 \subset L_2 \subset \cdots \subset L_n, \]
and $L_i$ are finitely generated free $A$-modules and $L_i \otimes_A k = L(iP)$. Then the faithful representation defined in (3), can be lifted to a representation:

$$\rho_1 : G_1(P) \to GL_n(A),$$

such that $\rho_1(\sigma), \sigma \in G_1(P)$ is a lower triangular matrix with 1 at the diagonal.

**Proof.** Let $k = A/m$, be the residue field of the local ring $A$, modulo the maximal ideal $m$ of $A$. The $A$-module $L_i$ is a finitely generated free $A$-module, therefore $\dim_k(L_i \otimes_A k) = \text{rank}_A L_i$. Moreover, since $L_i$ are $G$-modules, there is a natural representation $\rho : G_1(P) \to GL_n(A)$. Since the flag $L_i$ of $A$ modules is preserved, the representation matrices of $\rho_1$ are lower triangular, and elements on the diagonal must have a multiplicative $p$-group structure, so they are units. $\square$

In subsection 3.1 we recall the theory of effective Cartier divisors and we define the relative ramification divisor. We also show that if exists a deformation $X \to \text{Spec}A$ over an integral domain $A$ that admits a fibrewise $G$-action exists then there is a constrain in the Artin representations at the special and the generic fibres expressed in proposition 3.4.

Subsection 3.2 is devoted to the problem of algebraisation. In order to construct the desired representation we will need deformations that possess generic fibres. The passage from deformations defined over formal schemes to deformations that defined over ordinary schemes is given by Artin’s algebraisation theorem 3.10 and in subsection 3.2 we check that Artin’s algebraisation theory can be applied.

In subsection 3.3 we try to construct the sequence $L_i$ of invertible $\mathcal{O}_X$ modules of proposition 3.1. For this we need Grauert theorem [12, III.12.9] and this is the reason we want our deformations to possess a generic fibre. We give a criterion 3.14 for the construction of the sequence $L_i$ and some lemmata that imply the truth of this criterion and depend only of the form of the ramification filtration of the special fibre. Unfortunately it seems that there are bad deformations and for them the construction of proposition 3.1 is not possible. The best think we can do is to show that the criterion 3.14 defines a deformation condition in the sense of Mazur, which in turn gives rise to a subfunctor that can be handled with our tools. Of course, this approach will lead to a lower bound of the desired Krull dimension.

### 3.1. Relative Ramification divisors

In this section we would like to address the following question: Let $P$ be a wild ramified point, of the special fibre of the deformation in study, with decomposition group $G(P)$. Is it possible to find a “horizontal divisor” $\tilde{P}$ that is invariant under the action of $G(P)$, so that the intersection of $\tilde{P}$ with the special fibre is the original point $P$?

Let $X \to \text{Spec}A$ be an $A$-curve, admitting a fibrewise action of the finite group $G$, where $A$ is a Noetherian local ring. Let $Y \to \text{Spec}A$ be the quotient $\text{Spec}A$-curve. A good notion of a horizontal divisor in this case can be given in terms of effective Cartier divisors. An effective Cartier divisor $D$ on $X \to \text{Spec}A$, is a closed subscheme $D \subset X$, such that $D$ is flat over $\text{Spec}A$, and the ideal sheaf $I(D) \subset \mathcal{O}_X$ is an invertible $\mathcal{O}_X$-module.

We would like to assign to the cover of $A$-schemes $X \to Y$ a ramification divisor $D_{X/Y}$, such that the intersection of $D_{X/Y}$ with the fibres of the morphism $f : X \to Y$, corresponds to the usual notion of ramification divisor for coverings of $k$-curves.

Let $S = \text{Spec}A$, and $\Omega_{X/S}$, $\Omega_{Y/S}$ be the sheaves of relative differentials of $X$ over $S$ and $Y$ over $S$, respectively.
We begin by defining the ideal sheaf of the ramification divisor $D_{X/Y}$ as
\[ L(-R) = \Omega^{-1}_{X/S} \otimes_S f^* \Omega_{Y/S}. \]

We will first prove that the above defined divisor $D_{X/Y}$ is indeed an effective Cartier divisor.

We will use the following

**Criterion 3.2.** Let $S$ be locally Noetherian, $X$ a flat $S$-scheme of finite type, and $D \subseteq X$, a closed subscheme which is flat over $S$. Then $D$ is an effective Cartier divisor in $X/S$ if and only if, for all geometric points $\text{Spec} \ k \to S$ of $S$, the closed subscheme $D \otimes_S k$ of $X \otimes_S k$ is an effective Cartier divisor in $X \otimes_S k/k$.

**Proof.** [14, Cor. 1.1.5.2] \[ \square \]

**Proposition 3.3.** The ramification divisor $D_{X/Y}$ is an effective $A$-divisor.

**Proof.** We are interested in deformations of nonsingular curves. Since the base is a local ring and the special fibre is nonsingular, the deformation $X \to \text{Spec} \ A$ is smooth. (See the remark after the definition 3.35 p.142 in [19]). The smoothness of the curves $X \to S$, and $Y \to S$, implies that the sheaves $\Omega_{X/S}$ and $\Omega_{X/S}$ are $S$-flat, [19, cor. 2.6 p.222].

On the other hand the sheaf $\Omega_{Y, \text{Spec} A}$ is by [14] Prop. 1.1.5.1 $O_Y$-flat. Thus, $\pi^*(\Omega_{Y, \text{Spec} A})$ is $O_X$-flat and therefore $\text{Spec} A$-flat [12 Prop. 9.2]. The desired result follows by criterion 3.2. \[ \square \]

Let $A$ be a local $k$ algebra, that is a domain, and let $X \to \text{Spec} A$ be a deformation of the curve $X$, and $P \in X$, be a wild ramified point with decomposition group $G_0(P)$. Assume that the point $P$ appears in the ramification divisor of the covering of the special fibres $X \to X/G$, with multiplicity $d$, given by Hilbert’s formula [28, III.8.8].

We consider the effective Cartier divisor $D_P = \sum_{i=1}^\Lambda a_i P_i$, where $P_i$ denotes the irreducible components of the ramification divisor $R = R_{X/Y}$ that intersect the special fibre of $X$ at $P$.

We have the following picture:

![Diagram](image)

Two horizontal branch divisors can collapse to the same point in the special fibre. For instance this always happens if a deformation of curves from positive characteristic to characteristic zero with a wild ramification point is possible.
For a curve $X$ and a branch point $P$ of $X$ we will denote by $i_{G,P}$ the order function of the filtration of $G$ at $P$. The integer $i_{G,P}(\sigma)$ is equal to the multiplicity of $P \times P$ in the intersection of $\Delta \Gamma_\sigma$ in the relative $A$-surface $X \times X$, where $\Delta$ is the diagonal and $\Gamma_\sigma$ is the graph of $\sigma$ [26, p. 105]. Using an algebraic equivalence argument on $X \times_{\text{Spec} A} X$ we see the following generalisation of the result of J. Bertin [2]:

**Proposition 3.4.** Assume that $A$ is an integral domain, and let $X \to \text{Spec} A$ be a deformation of $X$. Let $P_i$, $i = 1, \ldots, s$ be the horizontal branch divisors that intersect at the special fibre, at point $P$, and let $P_i$ be the corresponding points on the generic fibre. For the Artin representations attached to the points $P, P_i$ we have:

$$ ar_P(\sigma) = \sum_{i=1}^s ar_{P_i}(\sigma). $$

**Remark** Consider the case of deformations of ordinary curves, together with a $p$-subgroup of the group of automorphisms. Then $|ar_P(\sigma)| = 2$ for all $\sigma \in G(P) = G_1(P), \sigma \neq 1$ [22]. On the other hand the ramification at the points of the generic fibre is also wild and implies that there is only one horizontal branch divisor extending every wild ramification point $P$.

### 3.2. Application of Algebraisation Theory

We would like to have our curves deformed over bases that have generic fibres. Formal deformation theory gives us information whether a curve can be defined over the formal spectrum of a complete ring, i.e. over

$$ \text{Spf} R = \{ P \in \text{Spec} R : P \text{ is open with respect to the } m_R - \text{adic topology}\}, $$

where $R$ is a complete domain with maximal ideal $m_R$. In order to extend the family over the generic fibre (the 0 ideal is not open) we have to use an algebraisation argument.

Let us denote by $\mathcal{C}$ the category of local Artin $k$-algebras and by $\hat{\mathcal{C}}$ the category of complete Noetherian local $k$-algebras. A covariant functor $F : \mathcal{C} \to \text{Sets}$ can be extended to a functor $\hat{F} : \hat{\mathcal{C}} \to \text{Sets}$ by defining

$$ \hat{F}(R) = \lim_{\leftarrow} F \left( \frac{R}{m_R^{n+1}} \right), $$

where $m_R$ is the maximal ideal of $R$ and $\frac{R}{m_R}$ is a local Artin $k$-algebra. On the other hand a functor $F : \hat{\mathcal{C}} \to \text{Sets}$ induces by reduction a functor $F|_{C} : \mathcal{C} \to \text{Sets}$. For any covariant functor $F : \hat{\mathcal{C}} \to \text{Sets}$ there is a canonical map

$$ F(R) \to \hat{F}(R) = \lim_{\leftarrow} F \left( \frac{R}{m_R^{n+1}} \right). $$

The above map is not in general a bijection. Let us denote by $h_R(\cdot) = \text{Hom}(R, \cdot)$. One can also prove [16, lemma 2.3] that $\hat{F}(R) \cong \hat{\text{Hom}}(h_R, F)$. A functor $F : \mathcal{C} \to \text{Sets}$ is called prorepresentable if there is an $R \in \text{Ob}(\mathcal{C})$ and $\hat{\xi} \in \hat{F}(R)$, that induces an isomorphism

$$ \hat{\xi} : h_R(A) \cong F(A). $$

A formal deformation of $\xi_0 \in F(k)$ is an element $\hat{\xi} \in \hat{F}(R)$, where $R \in \text{Ob}(\hat{\mathcal{C}})$.
Let $F$ be a functor $F : \hat{C} \to \text{Sets}$. We say that $F \mid C$ is effectively prorepresentable if $F \mid C$ is prorepresentable by $\xi \in \hat{F}(R)$, and this $\xi$ is the image of an element in $F(R)$, under the map $F(R) \to \hat{F}(R)$.

J. Bertin, A. Mézard [3], introduced at the wild ramified point $P$ the deformation functor:

(8) $D : C \to \text{Sets}, A \mapsto \begin{cases} \text{lifts } G \to \text{Aut}(A[[t]]) \, \text{of } \rho \text{ modulo conjugation with an element } \\ \text{of } \ker(\text{Aut}A[[t]] \to k[[t]]) \end{cases}$

Lemma 3.5. Let $D$ be the functor defined in (8). For every complete $k$-algebra $A$ the canonical map

(9) $D(A) \to \hat{D}(A)$.

is a bijection.

Proof. Let $A$ be a complete $k$-local algebra with maximal ideal $m_A$ and denote by $A_n$ the quotient $A/m^n_A$. We will show first that the map in (9) is surjective. Let $\rho_n : G \to \text{Aut}A_n[[t]]$ be a system of representatives of maps such that the following diagram

$$
\begin{array}{ccc}
A_{n+1} & \xrightarrow{\rho_{n+1}(g)} & A_{n+1} \\
\downarrow{\text{mod}^n_A} & & \downarrow{\text{mod}^n_A} \\
A_n & \xrightarrow{\rho_n(g)} & A_n
\end{array}
$$

is commutative.

In order to define $\rho : G \to \text{Aut}A[[t]]$ we have to define it on $t$. Let us write

$$
\rho_n(g)(t) = \sum_{i=0}^{\infty} a_{i,n}(g)t^i.
$$

The elements $\{a_{i,n}(g)\}_n$ form an inverse system and give rise to a limit element $a_i(g) \in A = \lim_{\leftarrow} A_n$ and to the desired extension

$$
\rho(g)(t) = \sum_{i=0}^{\infty} a_i(g)t^i.
$$

This proves that the canonical map in (9) is indeed surjective.

In order to prove that it is also injective we consider two representations $\rho_1, \rho_2 : G \to \text{Aut}A[[t]]$, such that for every $n$ there are isomorphisms $\gamma_n : A_n[[t]] \to A_n[[t]]$ that induce the identity on $k[[t]]$ such that

$$
\rho_{1,n} = \gamma_n\rho_{2,n}\gamma_n^{-1}.
$$

Arguing in the same way as in the proof of the subjectivity we see that the maps $\{\gamma_n\}_n$ give rise to a well defined isomorphism

$$
\gamma : A[[t]] \to A[[t]]
$$

that induces the identity map on $k[[t]]$ and makes $\rho_1, \rho_2$ equivalent. $\square$
Definition 3.6. We will say that the functor \( D : \hat{C} \to \text{Sets} \) is locally of finite presentation if and only if for every direct limit \( \lim_{\rightarrow} A_i \) of objects in \( \hat{C} \) the natural map

\[
\lim_{\rightarrow} D(A_i) \to D(\lim_{\rightarrow} A_i),
\]

is an isomorphism.

Definition 3.7. We will say that the functor \( D : \hat{C} \to \text{Sets} \) is coherent if there are two representable functors \( h_2, h_1 \) such that

\[
h_2 \to h_1 \to D \to 0.
\]

Lemma 3.8. Every coherent functor \( D : \hat{C} \to \text{Sets} \) is locally of finite presentation.

Proof. Since \( D \) is coherent there is an exact sequence

\[(10) \quad h_2 \to h_1 \to D \to 0,
\]

where \( h_1, h_2 \) are representable functors. Sequence \( (10) \) implies the following commutative diagram

\[
\begin{array}{ccc}
0 & \to & \text{Hom}(D(\dash), \lim A_i) \to \text{Hom}(h_1(\dash), \lim A_i) \to \text{Hom}(h_2(\dash), \lim A_i) \to 0 \\
\downarrow f & & \downarrow f_1 & & \downarrow f_2 \\
0 & \to & \lim \text{Hom}(D(\dash), A_i) \to \lim \text{Hom}(h_1(\dash), A_i) \to \lim \text{Hom}(h_2(\dash), A_i) \to 0
\end{array}
\]

where the last row is exact since we are considering direct limits in the category of sets. Now representable functors are locally of finite presentation therefore \( f_1, f_2 \) are isomorphisms and the commutativity of the diagram forces \( f \) to be also an isomorphism, i.e., the desired result. \( \square \)

Proposition 3.9. The global deformation functor \( D_{gl} \) of J. Bertin A. Mazur is coherent.

Proof. We will use the notation of [3, sect. 5]. Let \( M_{g,n,G} \) be the functor classifying equivalence of triples \([X/A, \phi, \theta]\), represented by the scheme \( M_{g,n}^G \). There is the following exact sequence of functors:

\[
N_{\text{GL}(\mathbb{Z}/n\mathbb{Z})}G(\dash) \xrightarrow{\alpha_1} M_{g,n,G}(\dash) \xrightarrow{\alpha_2} D_{gl}(\dash) \to 0,
\]

where \( N_{\text{GL}(\mathbb{Z}/n\mathbb{Z})}G \) denotes the normaliser of \( G \) in \( \text{GL}(\mathbb{Z}/n\mathbb{Z}) \) and \( N_{\text{GL}(\mathbb{Z}/n\mathbb{Z})}G \) the constant group scheme with fibre the group \( N_{\text{GL}(\mathbb{Z}/n\mathbb{Z})}G \).

Indeed, for a local ring \( A \), and a deformation \( d : X \to \text{Spec} A \) in \( D_{gl}(A) \), we can select a level structure \( \phi : X[n] \to (\mathbb{Z}/n\mathbb{Z})^2_{\text{Spec} A} \) that maps on \( d \) by forgetting the extra level structure. The group \( G \) can be considered as a subgroup of \( \text{GL}(\mathbb{Z}/n\mathbb{Z}) \) and we can select a map \( \theta : G \to \text{Aut} X \) such that

\[
(11) \quad \phi \circ \theta(\sigma)^{-1}[n] = \sigma \circ \phi.
\]

This proves the surjectivity of \( \alpha_2 \).

In order to prove the exactness at the second factor we consider two triples \([X_i/A, \phi_i, \theta_i] \), \( i = 1, 2 \) that map on \( d \in D_{gl}(A) \). Thus \( X_1 \cong X_2 \) and there is an element \( a \in \text{GL}(\mathbb{Z}/n\mathbb{Z}) \) such that \( \phi_1 = a \circ \phi_2 \). But since both \( \phi_1, \phi_2 \) have to be \( \text{GL}(\mathbb{Z}/n\mathbb{Z}) \) invariant, i.e., equation \( (11) \) has to hold we obtain that:

\[
a \circ \sigma \circ \phi_2 = a \circ \phi_2 \circ \theta(\sigma)^{-1}[n] = \phi_1 \circ \theta(\sigma)^{-1}[n] = \sigma \circ \phi_1 = \sigma \circ a \circ \phi_2.
\]
for all \( \sigma \in G \). Therefore \( a \in N_{\text{GL}(\mathbb{Z}/n\mathbb{Z})}G \) and the desired result follows.

**Theorem 3.10 (Artin’s Criterion of Algebraisation).** Let \( F : (\text{Sch}/k)^0 \to \text{Sets} \) be a functor from the category of the formal schemes over the algebraically closed field \( k \) that is locally of finite presentation over \( k \). Let \( \xi_0 \in F(k) \), and suppose that an effective formal deformation \((R, \xi)\) exists, where \( R \) is a complete Noetherian local \( k \)-algebra and \( \xi \in F(R) \). Then there is a scheme \( X \) of finite type over \( k \) and \( x \in X \) a closed point with residue field \( k \), and \( \tilde{\xi} \in F(X) \), such that \((X, x, \tilde{\xi})\) is a versal deformation of \( \xi_0 \), such that \( \hat{\mathcal{O}}_{X,x} = R \), and
\[
\text{Spec}(R) \rightarrow \text{Spec}(\hat{\mathcal{O}}_{X,x}) \rightarrow X.
\]

**Proof.** [1, Th. 1.6] \( \square \)

**Corollary 3.11.** Every deformation \( X \to \text{Spf} R \) can be extended to a deformation \( X \to \text{Spec} R \).

**Proof.** By lemma 3.5 the local deformation functor attached to a wild ramification point, is effective, and since \( D_{\text{gl}} \) is the product of the local deformation functors at wild ramification points [3, 3.3.4], [6, 1.10.1] we have that \( D_{\text{gl}} \) is also effective. Moreover, lemma 3.8 and proposition 3.9 imply that \( D_{\text{gl}} \) is locally of finite presentation and the desired result follows by Artin’s criterion of algebraisation 3.10. \( \square \)

### 3.3. Construction of the \( A \)-free modules.

**Lemma 3.12.** Let \( S \) be any scheme and let \( X \to S \) be a deformation of \( X \), so that the special fibre is a nonsingular curve of genus \( g \geq 2 \). The relative curve \( X \to S \) is projective.

**Proof.** Observe that a deformation of a nonsingular curve of genus \( g \geq 2 \) is a stable curve. The desired result follows by [3, cor. p. 78]. \( \square \)

**Lemma 3.13.** Let \( X \to \text{Spec} A \) be a deformation over the local domain \( A \), let \( P \) be a wild ramified point on the special fibre and let \( T = \{ \bar{P}_i \}_{i=1,\ldots,s} \) be the set of horizontal branch divisors that restrict to \( P \) in the special fibre of \( X \), and let \( D \) be a \( G \)-invariant divisor supported on \( T \) of degree \( d \). There is a natural number \( a \) such that the map
\[
\Gamma(X, \mathcal{O}_X(aD)) \otimes_A k \to L(daP)
\]
is an isomorphism. Then \( \Gamma(X, \mathcal{O}_X(sD)) \) is a free \( A \)-module, and there is a free \( G \)-invariant \( A \)-module \( L \), such that \( L \otimes_A k = L(dP) \).

**Proof.** The divisor \( D \) is an effective Cartier divisor, and flatness of \( D \) over \( \text{Spec} A \) implies
\[
I(D) \otimes_A k \cong I(D \otimes_A k),
\]
i.e., the ideal sheaf of \( D \) restricted to the special fibre coincides with the ideal sheaf of the restriction of the divisor \( D \) to the special fibre [14, p. 7].

By inverting the ideal sheaves above we obtain
\[
I(D)^{-1} \otimes_A k \cong I(D \otimes_A k)^{-1} \Rightarrow \mathcal{L}(D) \otimes_A k \cong \mathcal{L}(D \otimes_A k).
\]

We would like to take global sections of the above two sheaves, in order to prove that
\[
\Gamma(X_s, \mathcal{L}(D)) \otimes_A k = \Gamma(X_s, \mathcal{L}(D \otimes k)) = L(dP).
\]
For all $i \geq 0$ there is a natural map \[ \Phi_i : H^i(X, \mathcal{L}(D)) \otimes k \to H^i(X_s, \mathcal{L}(D \otimes k)). \]

We are interested in global sections \textit{i.e.}, for the zero cohomology groups, but in general $\Phi_0$ can fail to be an isomorphism. Instead of looking at $D$ we will consider $aD$, where $a$ is a sufficiently large natural number. The degree of the divisor $aD$ remains the same in the special and in the generic fibre, since $H^0(X, \mathcal{L}(aD)/\mathcal{O}_X)$ is a free $A$-module of rank $\deg(aD)$, \[ \text{[13] 1.2.5}. \] Let $X_s$ and $X_\eta$ denote the special and the generic fibre of $X$ and let $K$ be the field of quotients of $A$. We will employ the Riemann-Roch theorem in both the special and the generic fibre and we can choose $a$ sufficiently big so that the index of speciality in both the generic and the special fibre is zero. Thus, the Riemann-Roch theorem implies:

\[ \dim_k H^0(X_s, \mathcal{L}(aD \otimes_A k)) = \dim_k H^0(X_\eta, \mathcal{L}(aD \otimes K)) = a \deg D + 1 - g. \]

Let $f : X \to \text{Spec} A$ be the structure map. By lemma \[ \text{[12] III.12.9}. \] the $A$-curve is projective and Grauert theorem \[ \text{[12] III.12.9}. \] implies that $H^0(f_* \mathcal{L}(aD))$ is a locally free sheaf on $\text{Spec} A$, and quasi-coherent by \[ \text{[12] III.8.6}. \] Since $A$ is a local ring we have that $H^0(X, \mathcal{L}(aD)) = H^0(\text{Spec} A, f_* \mathcal{L}(aD))$ is a free $A$-module.

Consider the $k$-subspace $L(dP) \subset L(adP)$. Since $D$ is $G$-invariant $L(dP)$ is also a $G$-invariant subspace of $L(adP)$. Let $\bar{x}_1, \ldots, \bar{x}_\ell$ be a basis of $L(dP)$ and let $x_1, \ldots, x_\ell \in H^0(X, \mathcal{L}(aD))$ that reduce to $\bar{x}_1$ modulo $m$.

The free submodule of $H^0(X, \mathcal{L}(aD))$ generated by $x_i$ is $G$-invariant and reduces to $L(dP)$ modulo $m$.

Let $T$ be as in lemma \[ \text{[12]}. \] Let $O(T)$ be the set of orbits of $T$ under the action of the group $G$, on $T$. The $A$-module $\mathcal{O}_X(D)$ is invariant under the action of $G$ if and only if, the divisor $D$ is of the form:

\[ D = \sum_{C \in O(T)} n_C \sum_{P \in C} P, \]

\textit{i.e.}, horizontal Cartier divisors that are in the same orbit of the action of $G$ must appear with the same weight in $D$.

Given a space $L(iP)$ we would like to construct a $G$-invariant divisor $D$ supported on $T$ that in turn will give a $G$-invariant $A$-module $\mathcal{O}_X(D)$. If $i = \sum_{C \in O(T)} n_C \# C$, where $n_C$ are non-negative integers, then we can consider the $G$-invariant divisor given in \[ \text{[12]}. \]

We have proved the following:

**Proposition 3.14.** If the semigroup $\sum_{C \in O(T)} n_C \# C$, $n_C \in \mathbb{N}$, contains the Weierstrass semigroup of the branch point $P$ of the special fibre, then the assumption of proposition \[ \text{[12]}. \] is satisfied and the representation can be lifted.

**Corollary 3.15.** Let $A$ be a local Artin algebra that is dominated by an integral local ring $R$, and suppose that the deformation $X \to \text{Spec} A$ can be lifted to a deformation with base $R$, such that the assumptions of proposition \[ \text{[12]}. \] are satisfied. The faithful representation defined in \[ \text{[3]}. \], can be lifted to a representation:

\[ \rho_1 : G_1(P) \to GL_n(A), \]

such that $\sigma(g)$ is an upper triangular matrix with 1 at the diagonal.
Proof. Suppose that we have a lift $\rho_1$ of the faithful representation defined in (3). There is a surjective map $R \to A$, with kernel an ideal $I$ of $R$. The desired result follows by considering the representation matrices modulo the ideal $I$. 

The following lemmata give us criteria, in order to prove that the assumption of proposition 3.14 is satisfied.

**Lemma 3.16.** If one orbit of $G$ acting on $T$ is a singleton, then the semigroup
\[ \sum_{C \in O(T)} n_C \# C, \quad n_C \in \mathbb{N}, \]
is the semigroup of natural numbers and the assumption of proposition 3.14 is satisfied.

**Lemma 3.17.** If $\# T \not\equiv 0 \mod p$ then there is at least one orbit that is a singleton.

Proof. If all orbits have more than one element then all orbits must have cardinality divisible by $p$, and since the set $T$ is the disjoint union of orbits it also must have cardinality divisible by $p$. □

**Lemma 3.18.** If $m$ is the first pole number that is not divisible by the characteristic, and $p \nmid m + 1$ then there is an orbit that consists of only one element.

Proof. By proposition 3.4 the Artin representation at the special fibre equals the sum of the Artin representations at the generic fibre. Thus if there is no singleton orbit then the sum of Artins representations at the generic fibre is divisible by $p$ and this contradicts corollary 2.3. □

If the assumptions of proposition 3.14 are not satisfied then we have to restrict to the “good deformations”. We will describe them, following B. Mazur in [21, p.289], by defining the notion of a deformation condition.

**Definition 3.19.** Let $X$ be an algebraic curve defined over the algebraically closed field $k$. We define the category $\mathcal{A}$ of deformations of curves over Artin local rings, whose objects are deformations $X \to \text{Spec} A$ of the initial curve $X$, together with a fibrewise action of the group $G$ on $X$, and the morphisms
\[
(X \to \text{Spec} A) \to (X' \to \text{Spec} A')
\]
are given by a local algebra homomorphism $A \to A'$ and an $\text{Spec} A'$-map $\phi : X \times_{\text{Spec} A} \text{Spec} A' \to X'$, making the following diagram commutative:
\[
\begin{array}{ccc}
\mathcal{X} & \xleftarrow{\phi} & \mathcal{X} \\
\downarrow & & \downarrow \\
\text{Spec} A & \xleftarrow{\phi} & \text{Spec} A'
\end{array}
\]
and moreover $\phi$ induces the identity on the special fibre $X$.

**Definition 3.20.** By a deformation condition on $\mathcal{A}$ we mean a full subcategory $\mathcal{DF}$ of $\mathcal{A}$ satisfying the following conditions:

1. For any morphism $(X, A) \to (X', A')$ if $(X, A) \in \text{Ob}(\mathcal{DA})$ then $(X', A') \in \text{Ob}(\mathcal{DA})$
(2) Let $A, B, C$ be artinian $k$-algebras fitting into a diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\alpha} & B \\
\downarrow \downarrow & & \downarrow \downarrow \\
C & \xleftarrow{\beta} & B
\end{array}
\]

We denote by $A \times C B$ the subring of $A \times B$ consisted of elements $(a, b)$ such that $\alpha(a) = \beta(b)$ \cite[p.270]{21}. Moreover there are two projections $p_A, p_B$ making the following diagram commutative:

\[
\begin{array}{ccc}
A \times C B & \xrightarrow{p_A} & A \\
\downarrow \downarrow & & \downarrow \downarrow \\
C & \xleftarrow{p_B} & B
\end{array}
\]

Consider an object $(\mathcal{X}, A \times C B)$ and let

\[
\mathcal{X}_A := \mathcal{X} \times \text{Spec}(A \times C B) \text{Spec}A \to \text{Spec}A
\]

and

\[
\mathcal{X}_B := \mathcal{X} \times \text{Spec}(A \times C B) \text{Spec}B \to \text{Spec}B
\]

be the fibre products with respect to the maps $p_A, p_B$. We ask that $(\mathcal{X}, A \times C B) \in \text{Ob}(\mathcal{D}A)$ if and only if both $\mathcal{X}_A, \mathcal{X}_B \in \text{Ob}(\mathcal{D}A)$.

For any morphism $(\mathcal{X}, A) \to (\mathcal{X}', A')$ if $(\mathcal{X}', A') \in \text{Ob}(\mathcal{D}A)$ and $A \to A'$ is injective then $(\mathcal{X}, A) \in \text{Ob}(\mathcal{D}A)$

Suppose that we have a deformation condition $\mathcal{D}A$ of $A$. We define a subfunctor $\mathcal{D}$ of the global deformation functor $\mathcal{D}$ of Bertin-Mézard

\[
\mathcal{D} \subset \mathcal{D} : C \to \text{Sets},
\]

where $\mathcal{D}(A)$ contains the elements of $\mathcal{D}(A)$ that are objects of $\mathcal{D}A$.

**Proposition 3.21.** The subfunctor $\mathcal{D}$ satisfies the three first Schlessinger criteria and has a hull $R_\mathcal{D}$.

**Proof.** The deformation conditions imply that $\mathcal{D}$ is relatively representable \cite[p.278]{21}, and since the global deformation functor $\mathcal{D}$ satisfies the first three Schlessinger criteria \cite[p. 277]{21}, the functor $\mathcal{D}$ also satisfies them and therefore it has a hull. \qed

We are mainly interested in the deformation condition given in proposition \ref{3.14}.

**Lemma 3.22.** The condition given in \ref{3.14} defines a deformation condition.

**Proof.** In order to prove the desired result we notice that is enough to prove that if $d_1 : \mathcal{X}_1 \to \text{Spec}A_1$ are elements in $\mathcal{A}$ and $\phi : d_1 \to d_2$ then $d_1 \in \mathcal{D}A$ if and only if $d_2 \in \mathcal{D}A$.

Let $D$ be a $G$-invariant divisor of $\mathcal{X}_1$ such that $D \cap X = nP$. Then the divisor $\phi_*(D)$ is a $G$-invariant divisor on $\mathcal{X}_2$ such that $\phi_*(D) \cap X = nP$, since $\phi$ reduces to the identity on the special fibres.
If, on the other hand, \( D \) is a \( G \)-invariant divisor of \( X_2 \) such that \( D \cap X = nP \) then \( \phi^*(D) \) is a \( G \)-invariant divisor on \( X_1 \) such that \( \phi^*(D) \cap X = nP \), since \( \phi \) reduces to the identity on the special fibre.

\[ \square \]

4. Explicit Deformations of Matrix Representations

In this section we will employ the construction for universal deformation rings for matrix representations, explained by B. de Smit and H. W. Lenstra in [7]. Let \( G \) be a finite group with identity \( e \). We denote by \( k[G,n] \) the commutative \( k \)-algebra generated by \( X_{g}^{ij} \) for \( g \in G, 1 \leq j \leq i \leq n \), such that

\[
X_{e}^{ij} = \begin{cases} 
1 & \text{if } i = j \\
0 & \text{if } i \neq j 
\end{cases}
\]

(13)

\[
X_{g}^{ij} = \sum_{l=1}^{n} X_{g}^{il} X_{h}^{lj} \text{ for } g, h \in G \text{ and } 1 \leq i, j \leq n.
\]

and finally

\[
X_{g}^{ij} = 0 \text{ for } i < j \text{ and for all } g \in G.
\]

Let \( A \) be a \( k \)-algebra. Consider the multiplicative group \( L_n(A) < GL_n(A) \), of invertible lower triangular matrices with entries in \( A \), and 1 in the diagonal. We will focus on representations on \( L_n(A) \). For every \( k \)-algebra \( A \) we have a canonical bijection

\[
\text{Hom}_{k-\text{Alg}}(k[G,n], A) \cong \text{Hom}(G, L_n(A)),
\]

where a \( k \)-algebra homomorphism \( f : k[G,n] \rightarrow A \) corresponds to the group homomorphism \( \rho_f \) that sends \( g \in G \) to the matrix \( f(X_{g}^{ij}) \). The representation \( \rho : G \rightarrow L_n(k) \) corresponds to a homomorphism \( k[G,n] \rightarrow k \). Its kernel is a maximal ideal, which we denote by \( m_\rho \). We take the completion \( R(G) \) of \( k[G,n] \) at \( m_\rho \).

The canonical map \( k[G,n] \rightarrow R(G) \), gives rise to a map \( \rho_{R(G)} : G \rightarrow L_n(R(G)) \), such that the diagram:

\[
\begin{array}{ccc}
G & \xrightarrow{\rho_{R(G)}} & L_n(R(G)) \\
\downarrow & & \downarrow \\
G & \xrightarrow{\rho} & L_n(k)
\end{array}
\]

is commutative.

We consider the following functor from the category \( C \) of local Artin \( k \)-algebras to the category of sets

\[
F : A \in \text{Ob}(C) \mapsto \left\{ \begin{array}{l}
\text{liftings of } \rho : G \rightarrow L_n(k) \\
\text{to } \rho_A : G \rightarrow L_n(A) \text{ modulo conjugation by an element of } \ker(L_n(A) \rightarrow L_n(k))
\end{array} \right\}
\]

The ring \( R(G) \) defined above does not represent the deformation functor \( F \), since \( A \)-equivalent deformations may correspond to different maps in \( \text{Hom}(R(G), A) \).

If \( n = 2 \), i.e., in the case of a two dimensional representation, the conjugation action is trivial and \( F \) is representable by \( R(G) \).
In the deformation theory over fields of characteristic zero, if the representation is irreducible, one takes the closed subalgebra of $R(G)$ generated by the traces of elements in $\rho: G \to GL_n(R)$, and since characters distinguish equivalence classes of representations, this subalgebra represents the deformation functor.

We are working over fields of positive characteristic and this approach is not suitable: The representation can be chosen to be indecomposable but not irreducible. The theory of Brauer characters, does also not help very much, in the case of equicharacteristic deformations, since Brauer characters do not take values in $R(G)$.

We will avoid to answer whether the functor $F$ is representable; for our needs it is enough that there is a natural transformation from $Hom_{k-alg}(R(G), \cdot) \to F(\cdot)$.

Let $D : C \to Sets$ be the functor of J. Bertin, A. Mézard introduced in [5]. We will define a natural transformation of functors $\Psi : F \to D$, and using this natural transformation we will prove that the Krull dimension of the hull of the deformation functor $D$ equals the dimension $dim_k F(k[\epsilon])$, of the tangent space of the functor $F$.

Let $S = \{ t^n, n \in \mathbb{N} \}$ considered as a multiplicative system. We choose elements $F_i \in A[[t]]S^{-1}, i = 1, \ldots, m$ so that $F_i \equiv f_i \mod m_A$. The elements $F_i$ can be written as

$$F_i = \frac{u_i(t)}{t^{\lambda(i)}},$$

where $u_i(t)$ is a unit in $A[[t]]$ an $\lambda(i)$ is the pole order of the function $f_i$. Moreover we assume that $u_i(t)$ is a polynomial in $A[t]$.

An element in $F(A)$, defines a linear action on the free $A$-module generated by the elements $F_i$ given by

$$\sigma(F_i) = \sum_{\nu=0}^{i} \rho_{i\nu}(\sigma)F_{\nu}. \quad (14)$$

The above action is not necessary compatible with multiplication, i.e., $\sigma(ab)$ and $\sigma(a)\sigma(b)$ need not to be equal. We will give conditions so that the action is compatible with multiplication. In order to do this we have to determine the value of $\sigma(t)$ and expand to the elements of $A[[t]]$ so that $\sigma$ respects addition and multiplication. This definition of $\sigma(t)$ should give the same results on $F_i$ with (14).

In order to do such a computation, we will need a general version of Hensel lemma. We will follow the notation of Bourbaki [5 III 4.2]. We will say that a ring $R$ is linearly topologized, if there exists a fundamental system of neighbourhoods of 0 consisting of ideals of $R$. If the commutative ring $R$ is linear topologized, Hausdorff and complete we will say that the ring satisfies Hensel’s conditions. Let $(R, m)$ be an ordered pair, consisted of a ring $R$ and an ideal $m$ of $R$, so that $R$ satisfies Hensel’s conditions and the ideal $m$ is closed and the elements of $m$ are topologically nilpotent, i.e. for all $x \in m$ we have $\lim_{\nu \to \infty} x^\nu = 0$. Then the pair $(R, m)$ is said to satisfy Hensel’s conditions [5 III 4.5]. We will denote by $R\{X\}$ the subring of $R[[X]]$ consisted of elements $\sum_{i=0}^{\infty} a_i X^i$ so that $\lim_{i \to \infty} a_i = 0$.

We will need the following

**Lemma 4.1.** Let $R$ be a ring and $m$ an ideal of $R$ so that the ordered pair $(R, m)$ satisfies Hensel’s conditions. Let $f \in R\{X\}$, $a \in R$ end write $e = f'(a)$. If $f(a) \equiv 0 \mod m$, then there exists $b \in R$ such that $f(b) = 0$ and $b \equiv a \mod m$. If
b' is another element of R such that \( f(b') = 0 \) and \( b' \equiv a \mod m \) then \( e(b' - b) = 0 \). In particular, \( b \) is unique if \( e \) is not a divisor of zero in \( R \).

**Proof.** Corollary 1 [5 III 4.5]

**Lemma 4.2.** Let \( A \) be an Artin local ring with maximal ideal \( m_A \). We consider the ring \( A[[t]] \) so that the ideals \( \langle m_A^i, t^i A[[t]] \rangle \) form a fundamental system of neighbourhoods of 0. Then the ordered pair \( (A[[t]], m_A) \) satisfies Hensel’s conditions.

**Proof.** The ring \( A[[t]] \) is by construction linear topologized. The ideal \( m_A \) is an element of the the system of fundamental neighbourhoods of 0 so it is open and closed. Moreover the ring \( A[[t]] \) is Hausdorff and complete. \( \square \)

Let us consider the last row of (14):

\[
\sigma(F_m) = \sum_{\nu=1}^{m} \rho_{m\nu}(\sigma)F_{\nu}(t).
\]

Let \( Y \) be the desired value of the the lift of \( \sigma(t) \). Then the last equality can be written as

\[
\frac{u_m(Y)}{Y^m} = \sum_{\nu=1}^{m} \rho_{m\nu}(\sigma)F_{\nu}(t) \Rightarrow
\]

(15) \[ t^m u_m(Y) - Y^m \sum_{\nu=1}^{m} \rho_{m\nu}(\sigma)F_{\nu}(t)t^m = 0. \]

If we consider (15) modulo \( m_A \) then it as has solution the value \( \sigma(t) \in \text{Aut} k[[t]] \). We will use lemma 4.1 with \( a = \sigma(t) \). The derivative of (15) is computed

\[
e = t^m \left( \frac{\partial u_m(Y)}{\partial Y} - m Y^{m-1} \sum_{\nu=1}^{m} \rho_{m\nu}(\sigma)F_{\nu}(t) \right),
\]

and

\[
\sigma = t^m \left( \frac{\partial u_m(\sigma(t))}{\partial Y} - m \sigma(t)^{m-1} \sum_{\nu=1}^{m} \rho_{m\nu}(\sigma)F_{\nu}(t) \right).
\]

Observe that the zero divisors in \( A[[t]] \) have coefficients in the maximal ideal \( m_A \), therefore since \( e \mod m_A \) is computed to be \( -mt^m\sigma(t)^{m-2} \) and it is not a zero divisor, \( e \) is not a zero divisor in \( A[[t]] \) as well. Thus, lemma 4.1 implies that there is a unique solution \( b \) of (15) that reduces to \( \sigma(t) \) modulo \( m_A \). The value \( b \) is a candidate for the lift \( \tilde{\sigma}(t) \). We observe that the equations (14) give also information for the value of the lift of \( \sigma(t) \).

We say that the tuple \( (F_1, \ldots, F_m, \rho : G \to GL_n(A)) \) is compatible if the unique solution \( b \) of (15) satisfies the equations (14) for \( 1 \leq i < m \). Of course we know that \( (f_1, \ldots, f_m, \rho : G \to GL_n(k)) \) is compatible.

**Lemma 4.3.** If the tuple \( (F_1, \ldots, F_m, \rho : G \to GL_n(A)) \) is compatible and \( Q \in L_n(A) \) then the tuple \( (Q F_1, \ldots, Q F_m, \rho : G \to QGL_n(A)Q^{-1}) \) is also compatible.

**Proof.** For every \( 1 \leq i \leq m \) we have \( Q F_i = \sum_{\mu=1}^{m} Q_{i\mu} F_\mu \). The system of equations (14) can be written in matrix form as

\[
\begin{pmatrix}
F_1(Y) \\
\vdots \\
F_m(Y)
\end{pmatrix}
= \rho(\sigma)
\begin{pmatrix}
F_1(t) \\
\vdots \\
F_m(t)
\end{pmatrix}.
\]
Thus,
\[
Q \cdot \begin{pmatrix}
F_1(Y) \\
\vdots \\
F_m(Y)
\end{pmatrix} = Q\rho(\sigma)Q^{-1} \cdot Q
\begin{pmatrix}
F_1(t) \\
\vdots \\
F_m(t)
\end{pmatrix},
\]
and the desired result follows. \(\square\)

**Lemma 4.4.** Let \(A\) be an artin local ring with maximal ideal \(m_A\) so that \(A/m_A = k\). Let \((F_1, \ldots, F_m, \rho : G \to L_n(A))\) be a compatible tuple, and for every \(\sigma \in G\) let \(\tilde{\sigma}\) denote the corresponding automorphism in \(\text{Aut}(A[[t]])\). For every \(\sigma_1, \sigma_2 \in G\) we have
\[
\tilde{\sigma}_1\tilde{\sigma}_2(t) = \sigma_2\sigma_1(t).
\]

**Proof.** The element \(b_{\sigma_2} = \tilde{\sigma}_2\) is a root of
\[
t^m u_m(Y) - t^m \sum_{\nu=1}^m \rho_{\mu\nu}(\sigma_2) F\nu(t).
\]
By applying \(\tilde{\sigma}_1\) to the above equation we obtain
\[
0 = \tilde{\sigma}_1(t) \left( u_m(\tilde{\sigma}_1(b_{\sigma_2})) - \tilde{\sigma}_1(b_{\sigma_2})^m \sum_{\nu=1}^m \rho_{\mu\nu}(\sigma_2) \tilde{\sigma}_1 F\nu(t) \right) =
\]
\[
= \tilde{\sigma}_1(t) \left( u_m(\tilde{\sigma}_1(b_{\sigma_2})) - \tilde{\sigma}_1(b_{\sigma_2})^m \sum_{\nu=1}^m \rho_{\mu\nu}(\sigma_2) \sum_{\mu=1}^m \rho_{\mu\nu}(\sigma_1) F\mu(t) \right) =
\]
\[
= \tilde{\sigma}_1(t) \left( u_m(\tilde{\sigma}_1(b_{\sigma_2})) - \tilde{\sigma}_1(b_{\sigma_2})^m \sum_{\nu=1}^m \rho_{\mu\nu}(\sigma_2\sigma_1) F\mu(t) \right).
\]
Since the element \(\tilde{\sigma}_1(t)\) is not a zero divisor we obtain that
\[
\tilde{\sigma}_1(\tilde{\sigma}_2(t))
\]
is the unique root of
\[
t^m u_m(Y) - t^m \sum_{\nu=1}^m \rho_{\mu\nu}(\sigma_2\sigma_1) F\nu(t).
\]
and the desired result follows. \(\square\)

**Definition 4.5.** We will say that the tuples \((F_1, \ldots, F_m, \rho : G \to GL_n(A))\) and \((F'_1, \ldots, F'_m, \rho' : G \to GL_n(A))\) are equivalent if there is an element \(Q \in L_n(A)\) so that
\[
(F'_1, \ldots, F'_m, \rho' : G \to GL_n(A)) = (QF_1, \ldots, QF_m, \rho : G \to QGL_n(A)Q^{-1})
\]
We define the functors
\[
\mathcal{F}_1 : A \in \text{Ob}(\mathcal{C}) \mapsto \left\{ \begin{array}{l}
\text{Set of tuples} \\
(F_1, \ldots, F_m, \rho : G \to GL_n(A))
\end{array} \right\}
\]
defined over \(A\)
and
\[
\mathcal{F} : A \in \text{Ob}(\mathcal{C}) \mapsto \left\{ \begin{array}{l}
\text{equivalence classes} \\
of tuples over A
\end{array} \right\}
\]
If \(g : B \to A\) is a morphism of local artin algebras and
\[
\Omega := (F_1, \ldots, F_m, \rho : G \to L_n(B)) \in \mathcal{F}_1(B)
\]

\text{Page 20}
Lemma 4.7. Give rise to a representable functor.

Proof. The set of compatibility conditions (14) give the following conditions on variables \(X\) as in [21], we have that the deformation conditions on \(\text{Hom}(\mathcal{F}(\mathcal{F}(B)) \to \mathcal{F}(A))\).

Observe that compatibility of equivalence classes of tuples is well defined by lemma 4.3.

**Proposition 4.6.** Compatibility of tuples is a deformation condition in both the functors \(\mathcal{F}, \mathcal{F}_1\).

**Proof.** Let us consider a pair \((A, \Omega)\), consist of an artin local ring \(A\) together with a tuple \(\Omega := (F_1, \ldots, F_m, \rho : G \to L_n(A))\). Let us consider the set of compatible tuples over \(A\) by \(F_1'(A) \subset \mathcal{F}_1(A)\). A morphism \(\phi : A \to A'\) of local artin algebras, induces the map \(\Omega \to F_1(\phi)(\Omega)\), on tuples. In order to have a deformation condition we should check that

1. For every local artin algebras \(A, A'\) and morphism \(f : A \to A'\), if \(\Omega \in \mathcal{F}_1'(A)\) then \(F_1(\phi)(\Omega) \in \mathcal{F}_1'(A')\).
2. Let \(A, B, C\) be local artin algebras and \(\alpha : A \to C\), \(\beta : B \to C\) be two morphisms of local artin algebras. We form the local artin algebra

\[ A \times_C B = \{(x, y) : x \in A, y \in B, \alpha(x) = \beta(y)\}, \]

equipped with coordinate addition, multiplication, and scalar \(k\)-multiplication. Let \(\Omega \in \mathcal{F}_1'(A \times_C B)\) we want the following

\[ \Omega \in \mathcal{F}_1'(A) \leftrightarrow F_1(\alpha)(\Omega) \in \mathcal{F}_1'(A) \text{ and } F_1(\beta)(\Omega) \in \mathcal{F}_1'(B) \]

3. For every local artin algebras \(A, A'\) and every injective map \(f : A \to A'\), we want the following

\[ \Omega \in \mathcal{F}_1'(A) \leftrightarrow F_1(\phi)(\Omega) \in \mathcal{F}_1'(A'). \]

Checking all this conditions is straightforward. A similar check can be done also for the functor \(\mathcal{F}\).

Consider the functor \(\text{Hom}_{k-\text{alg}}(R(G), \cdot)\). Every fixed selection \((F_1, \ldots, F_m)\), \(F_i \in A[[t]]S^{-1}, S := \{t^n, n \in \mathbb{N}\}\) gives rise to a natural map of \(\text{Hom}_{k-\text{alg}}(R(G), \cdot) \to \mathcal{F}_1\) and \(F(\cdot) \to \mathcal{F}\).

The compatibility of tuples gives rise to a set of deformation conditions (that depend on the basis selection \(F_1, \ldots, F_m\)) on \(\text{Hom}_{k-\text{alg}}(R(G), \cdot)\). By lemma on page 279 of [21], we have that the deformation conditions on \(\text{Hom}_{k-\text{alg}}(R(G), \cdot)\) give rise to a representable functor \(\text{Hom}(R(G)/I, \cdot)\), represented by the \(k\)-algebra \(R(G)/I, \cdot\), where \(I\) is a suitable ideal, that depends on the selection of \(F_1, \ldots, F_m\).

**Lemma 4.7.** The ideal \(I\) is an ideal that is generated by polynomials on the variables \(X^g_{i, \nu}\), \(g \in G, 1 \leq i < m\), i.e. the deformation condition does not involve the variables \(X^g_{m, \nu}\).

**Proof.** The set of compatibility conditions give the following conditions on \(R(G)\): For every \(g\) the unique solution \(Y\) of

\[ u_m(Y) = Y^m \sum_{\nu=1}^m X^g_{i, \nu} X^g_{i, \nu} \]
should satisfy the equations
\[ u_i(Y) = Y^i \sum_{\nu=1}^i X_{i\nu}^\theta F_\nu(t), \]
for every \( 1 \leq i < m \).

**Proposition 4.8.** There is a natural transformation \( \Psi \) from the functor \( F' \) to the functor \( D \).

**Proof.** Let \( A \in Ob(C) \) be a local Artin algebra with maximal ideal \( m_A \). Consider the \( n \)-dimensional \( k \)-vector space \( L(mP) \), together with the flag \( L(iP), i < m \) of vector spaces.

Every tuple in \( F(A) \) gives rise to a sequence \( L_i \) of free \( A \)-modules where
\[ L_i := A(F_1, \ldots, F_i), \]
so that \( L_i \otimes_A k = L(iP) \) and an action of \( G \) on them that is a lift of the action of \( G \) on the spaces \( L(iP) \).

Since the tuple \( \Omega := (F_1, \ldots, F_m, \rho : G \to L_n(A)) \) is compatible we get an automorphism \( \sigma(t) \) that defines an equivalence class \( \Phi(\Omega) \in D(A) \).

Let \( A, B \) be local artin algebras and consider a morphism \( g : B \to A \). For every tuple \( \Omega := (F_1, \ldots, F_m, \rho : G \to L_n(B)) \in F(B) \) the map \( F(g) \) gives the tuple \( F(g)(\Omega) \in F(A) \).

If \( \sigma \) denotes the automorphism of \( \text{Aut} B[[t]] \) that corresponds to \( \Omega \) then \( g(\sigma(t)) \) is the automorphism that corresponds to \( F(g)(\Omega) \).

**Example** Consider a functor that is represented by a complete ring that is not a domain. For example let \( R = k[[x_1, x_2]]/x_1^2x_2^2 \). Then the tangent space \( \text{Hom}(R, k[[\epsilon]]/\epsilon^2) \) is two dimensional generated by the maps \( \theta_j(x_j) = \delta_{ij}\epsilon \), but the arbitrary linear combination \( \hat{\theta} = 1 \lambda_1 \theta_1 + \lambda_2 \theta_2 \) could not lift to a homomorphism \( \hat{\theta} : R \to k[[\epsilon]] \), since \( \theta(x_1)^3\theta(x_2)^4 = 0 \) and \( k[[\epsilon]] \) is a domain.

In case \( R \) is a domain the following lemma shows that infinitesimal deformations are unobstructed:

**Lemma 4.9.** Let \( R \) be a complete local \( k \)-domain, and denote by \( m \) the maximal ideal of \( R \). Suppose that \( k = R/m \) is algebraically closed. Let \( A \) be an Artin local ring, and suppose that \( A = k[[a_1, \ldots, a_s]]/I \). Every element in \( \text{Hom}_k(R, A) \) can be lifted to \( \text{Hom}_k(R, k[[a_1, \ldots, a_s]]) \).

**Proof.** By Noether’s normalisation lemma [9 cor. 16.18] there are elements \( x_1, \ldots, x_d \in R \) such that \( R/k[[x_1, \ldots, x_d]] \) is a separating integral extension. A function \( g : k[[x_1, \ldots, x_d]] \to A \), given by \( g(x_i) = f_i(a_1, \ldots, a_s) \) can obviously extend to a function \( g : k[[x_1, \ldots, x_d]] \to k[[a_1, \ldots, a_s]] \).

Now if \( y \in R \), is an arbitrary element satisfying a separable equation \( \sum_{i=0}^n b_i y^i \), \( b_i \in k[[x_1, \ldots, x_n]] \), then Hensel’s lemma implies that the equation \( \sum_{i=0}^n g(b_i)T^i \) has a unique solution \( T \), extending the solution of \( \sum_{i=0}^n g(b_i)T^i \) of the equation taken modulo \( m_A \); recall that the field \( k \) is algebraically closed, and \( g(t) = t \) for all \( t \in k \). We define this unique solution to be the value of \( g \) at \( y \).

Using the criteria of Schlesinger [25, 21] p.277] J.Bertin A. Mézard [8] were able to prove that the deformation \( D \) admits a hull. We have seen in [32] that the subfunctor \( D \) also admits a hull.
This means that there is a complete ring $\tilde{R}$ and a smooth map of functors $\mathcal{D}(\cdot) \to \text{Hom}_{k-\text{alg}}(\tilde{R}, \cdot)$, that induces an isomorphism on the tangent spaces. The hull $\tilde{R}$ might not be a domain. In order to study its dimension we factor out nilpotent elements obtaining $\tilde{R}/\sqrt{0}$, a ring that corresponds to a variety. We will study the rings of the irreducible components. We observe first that the rings $\tilde{R}$ and $\tilde{R}/\sqrt{0}$ have the same Krull dimension, and hence the Krull dimension of $\tilde{R}$ equals the maximum dimension of the rings that correspond to the irreducible components of $\tilde{R}/\sqrt{0}$.

Let $R_i$ be a ring, corresponding to an irreducible component of $\tilde{R}/\sqrt{0}$. There is an onto map $\tilde{R} \to R_i$, that gives rise to an injection

$$\text{Hom}_{k-\text{alg}}(R_i, \cdot) \to \text{Hom}_{k-\text{alg}}(\tilde{R}, \cdot).$$

We will use that fact that $R_i$ is a domain together with proposition 3.1 in order to construct a map $\text{Hom}_{k-\text{alg}}(R_i, \cdot) \to F(\cdot)$. We will need the following:

**Lemma 4.10.** Suppose that $R = R_i$ is one of the rings defined above. Let $A$ be an Artin local $k$-algebra and suppose that $A = k[[a_1, \ldots, a_s]]/I$. There is a noetherian complete local domain $k_1[[a_1, \ldots, a_s]]/I'$, a sequence of small extensions

\[
\begin{align*}
(16) \quad k[e]/(e^2) = A_0 \leftarrow A_1 \leftarrow \cdots \leftarrow A_i \leftarrow \cdots \leftarrow k[[a_1, \ldots, a_s]]/I',
\end{align*}
\]

such that $A = A_n$ for some natural $n$, and the following diagram is commutative:

\[
\begin{array}{ccc}
\mathcal{D}(A_{i+1}) & \longrightarrow & \text{Hom}(R, A_{i+1}) \\
\downarrow \mathcal{D}(\phi_{i+1}) & & \downarrow \text{Hom}(\phi_{i+1}) \\
\mathcal{D}(A_i) & \longrightarrow & \text{Hom}(R, A_i)
\end{array}
\]

Moreover, for every element $\gamma \in \text{Hom}(R, A)$ there are elements $\gamma_i$ in $\text{Hom}(R, A_i)$ and $\delta_i \in \mathcal{D}(A_i)$ such that $\text{Hom}(\phi_{i+1})(\gamma_{i+1}) = \gamma_i$ and $\mathcal{D}(\phi_{i+1})(\delta_{i+1}) = \delta_i$.

**Proof.** Let $A$ be an Artin local ring expressed as $A = k[[a_1, \ldots, a_s]]/I$. The existence of the sequence (16) is clear. Lemma 4.9 implies the existence of the sequence of elements $\gamma_i$ that are compatible with the maps $\text{Hom}(\phi_i)$. The existence of the elements $\delta_i$ compatible with the elements $\mathcal{D}(\phi)$ is implied by smoothness of the map $\mathcal{D}(\cdot) \to \text{Hom}(R, \cdot)$.

**Lemma 4.11.** There is a natural transformation $q : \text{Hom}_{k-\text{alg}}(R_i, \cdot) \to F'(\cdot)$.

**Proof.** Start with an element $\gamma$ in $\text{Hom}(R_i, A)$ for an Artin local ring $A$. Construct a sequence of small extensions and maps as in equation (16) of lemma 4.10. By lemma 4.10 there is a sequence of deformations $\delta_i \in \mathcal{D}(A_i)$ that leads to a deformation in $\mathcal{D}(k[[a_1, \ldots, a_s]]/I')$, where $k[[a_1, \ldots, a_s]]/I'$ is a noetherian local complete domain. By corollary 3.13 we obtain a deformation $X \to \text{Spec} k[[a_1, \ldots, a_s]]/I'$, and using corollary 3.13 we obtain an element $q(\gamma) \in F'(A)$.

We have the following picture:

\[
\begin{array}{ccc}
\text{Hom}_{k-\text{alg}}(R_i, \cdot) & \longrightarrow & \text{Hom}_{k-\text{alg}}(\tilde{R}, \cdot) \\
\downarrow q & & \downarrow \text{smooth} \\
F(\cdot) & \longrightarrow & F'(\cdot) \\
\downarrow & & \downarrow \\
\mathcal{D}(\cdot)
\end{array}
\]

\[23\]
Corollary 4.13. If the ring $\psi_D$ gives rise to a unique deformation in diagram (17) is commutative, the map $\psi$ induces an isomorphism between $\text{Im}(\psi)$ and $\text{dim}(\ell)$. Thus, $\psi$ is 1-1 and moreover $\text{Im}(\psi) = \text{Im}(\ell)$. \hfill $\blacksquare$

**Proposition 4.12.** The map $\psi$, induces an isomorphism between the $k$ vector spaces $\mathcal{F}'(k[\varepsilon])$ and $\text{Im}(\ell)$.

**Proof.** By construction $\ell$ induces an isomorphism between $\text{Hom}_{k-alg}(R_t,k[\varepsilon])$ and $\text{Im}(\ell)$. Let $v \in \text{Im}(\ell)$, we find the inverse $\ell^{-1}(v)$ and take $q(\ell^{-1}(v))$. Since the diagram (17) is commutative, the map $\psi$ is onto $\text{Im}(\ell)$.

We have seen that Hensel’s lemma implies that every deformation in $\mathcal{F}'(A)$ gives rise to a unique deformation in $\mathcal{D}(A)$. This proves that the map $\psi$ is 1-1 and moreover $\text{Im}(\psi) = \text{Im}(\ell)$. \hfill $\blacksquare$

**Corollary 4.13.** If the ring $\hat{R}/\sqrt{\mathcal{O}}$ is not a domain, then all rings $R_t$ that correspond to irreducible components of $\text{Spec}R$, have the same dimension, equal to $\text{dim}_k\mathcal{F}'(k[\varepsilon]) = \text{dim}\hat{R}$. Moreover every infinitesimal deformation in $\psi(\mathcal{F}'(k[\varepsilon]))$ is unobstructed.

**Proof.** We have shown that $\text{dim}_k\mathcal{F}'(k[\varepsilon]) = \text{dim}_k\text{Im}(\ell)$, and $\text{dim}_k\mathcal{F}'(k[\varepsilon])$ does not depend on $R_t$. On the other hand every infinitesimal deformation in $\psi(\mathcal{F}'(k[\varepsilon]))$, is the image of the representable functor $\text{Hom}_{k-alg}(R_t,\cdot)$, and this implies the desired result. \hfill $\blacksquare$

**Examples:** 1. **$n = 2$**. This is always the case if $p > 2g - 2$, because the first pole number is smaller than $2g - 2$. Automorphisms of curves with this property were studied by P. Roquette in [24].

The group $G_1(P)$ is an elementary Abelian group isomorphic to a subgroup of $\{ (a_{ij}), i,j = 1,2 : a_{11} = a_{22} = 1, a_{12} = 0 \}$.

We will check whether the assumptions of the lifting criterion of corollary 4.16 are satisfied. If the automorphism group $G$ is cyclic of order $p$, then every irreducible component of the ramification divisor is by construction $G$-invariant, and the orbits are singletons. If the group $G$ is elementary Abelian with more than one cyclic components of order $p$, and there are no singleton orbits, then all orbits have order divisible by $p$. By comparing the Artin representations at the special and generic fibres using proposition 5.3.1 we obtain that the order function of the ramification filtration is a multiple of $p$. Thus, $p|m + 1$, a contradiction. Thus the functors $\mathcal{D}, \mathcal{D}$ are equal.

Since the representation is two dimensional the conjugation action on the ring $R(G)$ is trivial, and the functor $F$ is representable by $R(G) := k[[t_1, \cdots, t_{\log_p(|G_1(P)|)}]]$.

The dimension of the tangent space of $F$ and the krull dimension of the hull is $\log_p(|G_1(P)|)$.

**Remark:** A comparison of this result with the computation of J. Bertin, A.Mezard for deformations of the cyclic group $\mathbb{Z}_p$ [3] Th. 5.3.3] gives us the nontrivial result: for the smallest $m$ such that $\text{dim} L(mP) = 2$, $m < p - 1$. 

2. \( n = 3 \). Ordinary Curves. We will use the tools developed so far in order to study ordinary curves with \( n = 3 \) and compare our results with the more general results of G. Cornelissen - F. Kato [7] (where the decomposition groups could have elements prime to \( p \) as well).

We will use the notation from example 3. of page 7. Let \( \rho_{ij}(g) \) be the representation matrix. We have \( \rho_{32}(g) = c_1(g) \) and \( \rho_{21}(g) = \lambda c_1(g) \).

We form the ring \( R(G) \), generated by \( X_{31}^{0}, X_{31}^{1}, X_{31}^{0} \). We observe that there are the relations \( X_{31}^{2} = \Lambda X_{31}^{0}, \Lambda \in R(G), \Lambda \equiv \lambda \mod m_{R(G)}. \) Since \( X_{32}^{g} = 0 \Rightarrow X_{31}^{g} = 0 \), \( X_{31}^{g} \in \langle X_{32}^{g} \rangle, \) i.e.,

\[
X_{31}^{g} = X_{32}^{g} f(X_{ij}^{g}).
\]

Thus the only independent elements are \( X_{32}^{g} \).

We will now study the image of \( \text{Hom}_{k-\text{alg}}(R(G), k[\epsilon]) \rightarrow F(k[\epsilon]), \) i.e., we will study representations to \( L_n(k[\epsilon]) \) modulo the conjugation action. Let

\[
\bar{\rho}^i(g) : G_1(P) \rightarrow L_n(k[\epsilon]) \quad i = 1, 2
\]

be two equivalent representations, with representation matrices

\[
\begin{pmatrix}
1 & 0 & 0 \\
\rho_{21}(g) & 1 & 0 \\
\rho_{31}(g) & \rho_{32}(g) & 1
\end{pmatrix} + \epsilon \begin{pmatrix}
0 & 0 & 0 \\
0 & a_{21}(g) & 0 \\
a_{31}(g) & a_{32}(g) & 0
\end{pmatrix}.
\]

Let \( Q = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & e & 1 \end{pmatrix} \) be a matrix that transforms by conjugation \( \bar{\rho}^1 \) to \( \bar{\rho}^2 \).

This is equivalent (recall that \( \epsilon^2 = 0 \)) to

\[
a_{21}^1(g) = a_{21}^2(g), \quad a_{32}^1(g) = a_{32}^2(g),
\]

and

\[
a_{31}^1(g) + b \rho_{21}(g) - a \rho_{32}(g) = a_{31}^2(g).
\]

We express equation (18) as

\[
X_{31}^{g} = X_{32}^{g} \left( a_0 + a_1 X_{31}^{g} + a_2 X_{31}^{2g} + \cdots \right),
\]

where \( a_i \) are polynomials in all variables except \( X_{31}^{g} \). The representations \( \bar{\rho}^i(g) \) are images of maps \( f_i \in \text{Hom}_{k-\text{alg}}(R(G), k[\epsilon]) \). By taking the functions \( f_i \) on \( (21) \) we compute (recall that \( \epsilon^2 = 0 \))

\[
a_{31}(g) \left( 1 - \rho_{32}(g) \sum_{i=1}^{\text{deg}(f)} a_i (\rho_{31}(g))^i \right) = a_{32}(g) \sum_{i=0}^{\text{deg}(f)} a_i \rho_{31}(g)^i.
\]

Therefore, if \( \frac{\partial f}{\partial X_{31}^{g}} (\rho_{31}(g_i)) \neq 1 \), then the value \( a_{31}^i \) can be computed from the values of \( a_{21}^0 \) and \( \rho_{ij}(g_i) \). In particular, \( a_{31}^3(g) = a_{31}^3(g) \). If on the other hand \( \frac{\partial f}{\partial X_{31}^{g}} (\rho_{31}(g_i)) = 1 \), then the value \( a_{31}^0 \) is independ of the values \( a_{21}^0 \) and \( \rho_{ij}(g_i) \). This, is not possible since the map \( c_1 \) defined in (3) considered on the generic fibre has to be faithfull.

We identify the set of representations \( G_1(P) \rightarrow L_n(k[\epsilon]) \), with the set of homomorphisms of additive groups \( \text{Hom}(G_1(P), k) \) which is of dimension \( \log_p |G_1(P)| \). By (20) we have \( 0 = b \rho_{21}(g) - a \rho_{32}(g) = (b \lambda - a) \rho_{32}(g) \), and since \( a, b \) are arbitrary this means that \( \rho_{32}(g) = 0 \) and the tangent space of \( F(k[\epsilon]) \) is identified...
by $\text{Hom}(G_1(P), k)/\rho_{32}(g)$, i.e. the group homomorphism $\rho_{32}(g) : G_1(P) \to k$ is considered to be zero. This is a space of dimension $\log_p |G_1(P)| - 1$, and this result coincides with the result of G. Cornelissen, F. Kato.

3. $p$-cyclic covers of the affine line. We consider in this case deformations of the curve defined in example 4. of page 7. So we consider the curve $X : w^p - w = x^{p^i+1}$, and let $G$ denote the $p$-part of the automorphism group of $X$. In this section we will consider deformations $\mathcal{X} \to \text{Spec} A$ of the couple $(X, G)$, where $A$ is a local integral domain.

Consider the Galois group $H' := \text{Gal}(X/\mathbb{F}_p) = \mathbb{Z}/p\mathbb{Z}$, a cyclic group of order $p$. Denote by $\text{Def}_{G}(X), \text{Def}_{G/H'}(X)$ the global deformation functors of the curve $X$ with respect to the groups $G, H'$ respectively. The group $G$ acts freely on the completion of the generic fibre of $X \to \text{Spec} A$, and J. Bertin, A. Mézard in [4] proved that there is a well defined map $\text{ind} : \text{Def}_{G}(X) \to \text{Def}_{G/H'}(X/H')$. That means that $\text{ind}(X/H' \to \text{Spec} A) \in \text{Def}_{G/H'}(X/H')$ is a deformation of $X/H' = \mathbb{P}^1$. But such a deformation is trivial i.e. $X/H' = \mathbb{P}^1_A \to A$. Therefore, the generic fibre $X_0 \to \text{Spec} \text{Quo}(A) = \text{Spec}k$, is a cyclic extension of $\mathbb{P}^1$, and is given in terms of an Artin-Schreier extension: there are functions $W, X$ on the generic fibre of $X$, such that

$$W^p - W = f(X),$$

and $f(X) \in k(X)$. Let us write $f(x) = \prod f_i / \prod g_i$, where $f_i, g_i$ are irreducible polynomials in $k[x]$ and since $k$ is assumed algebraically closed all are of degree $\leq 1$. The places ramified in the cover $X_0 \to \mathbb{P}^1$ correspond to the poles of the function $f$. If there are polynomials $g_i$ of degree 1, then there should be polynomials $f_i$ so that $g_i = f_i \mod m_A$, since at the special fibre only $\infty$ is ramified. But this situation does not give as a proper relative curve, since any point on the generic fibre gives rise to a unique horizontal divisor intersecting the special fibre at a unique point.

Thus, by deforming the curve $X$ we deform the polynomial $f_0(x) = x^{p^i+1}$, and since the genus of every fibre, that is depended on the degree of $f$, has to remain constant, we deform $f$ by adding lower degree terms $a_i X^i, i < p^i + 1$, such that $a_i = 0 \mod m_A$.

Let $G_1(\infty)$ be the $p$-part of the decomposition group at $\infty$. According to [17 prop. 8.1] we have $A_{df^{p^i+1}}(Y) = Y + Y^{p^i},$ and $A_{df}(Y) = \sum_{0 \leq \lambda \leq s} t^p_{\lambda} Y^{p^i} Y^p$; thus the group $H := G_1(\infty)/\text{Gal}(X/\mathbb{F}_p)$ is an elementary abelian group of order $p^{2s}$. The group $G_1(\infty)$ is an extraspecial group, i.e., an extension of a $p$-cyclic group by an elementary abelian group. For the center we have $Z(G) = \text{Gal}(X/\mathbb{F}_p) = \mathbb{Z}/p\mathbb{Z}$. Let us fix elements $B = \{ g_i, i = 1, \ldots, 2s \}$ in $G_1(\infty)$ such that their image in $H$ form a basis of $H$ as an $\mathbb{Z}/p\mathbb{Z}$-module of rank $2s$. The elements $g_i$ generate $G_1(\infty)$. Indeed, the group generated by $g_i$’s is of index 1 or $p$ in $G_1(\infty)$ and by Sylow theorems normal. If the index is $p$ then the quotient is abelian, therefore $[G_1(\infty), G_1(\infty)] \subset \langle g_i \rangle$. This implies that $Z(G_1(\infty)) \subset \langle g_i \rangle$, a contradiction [20].

To each $g_i$ we associate the variables $X_{ij}^p, i > j$. Let $X$ be a function on $X$ that reduces to $x$ on the special fibre. Instead of studying arbitrary liftings in $L_{s+1}(A)$ and then computing the effect of taking the conjugation equivalence on the Krull dimension, we will choose a suitable basis on $L_1$. We choose the elements $\{ 1, X^p, X^{p^2}, \ldots, X^{p^{s+1}}, W \}$ as a basis for the free $A$ modules $L_1$ of proposition 3.1.
An element $g_i \in B$ corresponds to the automorphism given by

$$X \mapsto X + X_{21}^g, W \mapsto W + f_{g_i}(X),$$

where

$$f_{g_i}(X) = \sum_{\nu=1}^{s-1} X_{s+1,\nu}^g X^{p^\nu}.$$ 

The elements $X_{i,\nu}^g$ for $2 \leq \nu \leq s$, $1 \leq \mu \leq s$ are expressed in terms of $X_{21}^g$ by expanding $(X + X_{21}^g)^\mu$. We also observe that the variables $X_{s+1,\nu}^g$ are also depended to $X_{21}^g$ in the following way: Let $\Delta(f)(X,Y) := f(X + Y) - f(X) - f(Y)$. Then, according to [17, section 5] we know that

$$\Delta(f)(X,Y) = R(X,Y) + (\text{Id} - F)P_f(X,Y),$$

where $P_f(X,Y)$ is uniquely characterised by

$$P_f(X,Y) = (\text{Id} + F + \cdots + F^n)\Delta(f) \operatorname{mod}(X^{p^s-1} + 1),$$

for any $n > s$. Moreover, to every root $y$ of $Ad_f(Y)$ corresponds an automorphism $X \mapsto X + y$, $W \mapsto W + P_f(X,y)$. These results combined with (22) express the variables $X_{s+1,\nu}^g$ as functions of $X_{21}^g$, since $f_{g_i}(X) = \sum_{\nu=1}^{s-1} X_{s+1,\nu}^g X^{p^\nu} = P_f(X,X_{21}^g).$

The desired deformation ring is given as a quotient $k[X_{21}^g : 1 \leq i \leq 2s]/I$, where $I$ is an ideal generated by all relations among the elements $X_{21}^g$.

Moreover for the the polynomial $Ad_f(Y) = \sum_{\nu=0}^{2s} a_{\nu} Y^{p^\nu}$ we have

$$Ad_f(Y) = \sum_{0 \leq \lambda \leq s} R_{\lambda}^{p^\nu} Y^{p^\nu} + R_{\lambda}^{p^{s+1}} Y^{p^{s+1}}.$$ 

This means that the coefficients $a_{\lambda}$ must obey the following relation:

$$a_{s+\lambda} = a_{s-\lambda}$$

for all $\lambda = 1, \ldots, s$. Since $X_{21}^g$ are all roots of $Ad_f(Y)$ we see that the condition on coefficients implies conditions on $X_{21}^g$. In order to describe them we need to recall some theory on additive polynomials. Set

$$\Delta(x_1, \ldots, x_r) = \det \begin{pmatrix} x_1 & \cdots & x_r \\ x_1^p & \cdots & x_r^p \\ \vdots & \vdots & \vdots \\ x_1^{p^{s-1}} & \cdots & x_r^{p^{s-1}} \end{pmatrix}.$$ 

We will need the following

**Proposition 4.14.** A polynomial is additive if and only if the set of its roots is an $\mathbb{F}_p$-vector space. If $x_1, \ldots, x_r$ form a basis of an $\mathbb{F}_p$-vector space then a monic additive polynomial with these roots is of the form:

$$f(Y) = \frac{\Delta(x_1, \ldots, x_r, Y)}{\Delta(x_1, \ldots, x_r)}.$$ 

**Proof.** [11] th. 1.2.1,lemma 1.3.6] □
The additive polynomial \( Ad_f(Y) \) is given by

\[
Ad_f(Y) = c \cdot \frac{\Delta(X_{21}^{g_1}, \ldots, X_{21}^{g_{2s}})}{\Delta(X_{21}^{g_1}, \ldots, X_{21}^{g_{2s}})},
\]

where \( c \neq 0 \). For the coefficients \( a_i \) of \( Ad_f(Y) \) we have the following formulas:

\[
a_{2s} = c \cdot \frac{\Delta(X_{21}^{g_1}, \ldots, X_{21}^{g_{2s}})}{\Delta(X_{21}^{g_1}, \ldots, X_{21}^{g_{2s}})} = c.
\]

\[
a_0 = c \cdot \frac{\Delta(X_{21}^{g_1}, \ldots, X_{21}^{g_{2s}})^p}{\Delta(X_{21}^{g_1}, \ldots, X_{21}^{g_{2s}})} = c \cdot \Delta(X_{21}^{g_1}, \ldots, X_{21}^{g_{2s}})^{p-1}.
\]

The condition \( a_{2s} = a_0^p \), implies that

\[
(24) \quad c = c^p \Delta(X_{21}^{g_1}, \ldots, X_{21}^{g_{2s}})^{p(p-1)}.
\]

For \( 0 < \lambda < s \), equation \( a_{s+\lambda} = a_{s-\lambda} \) implies

\[
(25) \quad c = c^p \Delta(X_{21}^{g_1}, \ldots, X_{21}^{g_{2s}})^{p^s(p-1)}.
\]

We are now going to look at the problem of finding the Krull dimension from the infinitesimal point of view. Write \( X_{21}^2 \equiv x_i + \epsilon A_i \), where \( x_i \) is a basis for the vector space of the roots of \( Ad_f(Y) \) \( \text{mod} = Y^{p^{2s}} + Y \). According to Lemma 1.3.3, we have to compute the dimension of the vector space in the coordinates \( A_i \). Observe that since \( p > 2 \) for every \( a, b \in k \) we have \((a + b)^p = a^p + c^p b^p = a^p \text{mod}^2 \). Therefore (21) does not give us any information, while (25) is transformed to

\[
\begin{pmatrix}
x_1 + \epsilon A_1 & \ldots & x_{2s} + \epsilon A_{2s} \\
\vdots & \ddots & \vdots \\
x_1^{p^s} & \ldots & x_{2s}^{p^s} \\
\vdots & \ddots & \vdots \\
x_1^{p^s+\lambda-1} & \ldots & x_{2s}^{p^s+\lambda-1} \\
\vdots & \ddots & \vdots \\
x_1^{p^s+\lambda} & \ldots & x_{2s}^{p^s+\lambda}
\end{pmatrix}
= 0 \Rightarrow
\begin{pmatrix}
A_1 & \ldots & A_{2s} \\
\vdots & \ddots & \vdots \\
x_1^{p^s} & \ldots & x_{2s}^{p^s} \\
\vdots & \ddots & \vdots \\
x_1^{p^s+\lambda-1} & \ldots & x_{2s}^{p^s+\lambda-1} \\
\vdots & \ddots & \vdots \\
x_1^{p^s+\lambda} & \ldots & x_{2s}^{p^s+\lambda}
\end{pmatrix}
= 0
\]

The later equation gives a linear equation in \( A_1, \ldots, A_{2s} \) that has as solution space a \( 2s - 1 \) dimensional subspace. Let us denote this subspace by \( V_\lambda \). Observe that all vectors \( v_\nu := (x_1^{\nu 1}, \ldots, x_{2s}^{\nu}) \) for \( \nu \neq s + \lambda \) are in \( V_\lambda \). The vectors \( (x_1^{\nu 1}, \ldots, x_{2s}^{\nu}) \) are linear independent since \( \Delta(x_1, \ldots, x_{2s}) = 1 \neq 0 \) (Lemma 1.3.3]. Therefore

\[
V_\lambda = \langle v_\nu : 1 \leq \nu \leq 2s, \nu \neq \lambda + s \rangle.
\]
All conditions we obtain from all comparisons of coefficients form the vector space
\[ V := \bigcap_{1 \leq \lambda \leq s} V_\lambda, \]
that is clearly a vector space of dimension \( s \). This is the desired Krull dimension.

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