Controlling spontaneous emission of a two-level atom by hyperbolic metamaterials

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Abstract

Within the frame of quantum optics we analyze the properties of spontaneous emission of two-level atom in media with indefinite permittivity tensor where the geometry of the dispersion relation is characterized by an ellipsoid or a hyperboloid (hyperbolic medium). The decay rate is explicitly given with the orientation of the dipole transition matrix element taken into account. It indicates that for the ellipsoid case the intensity of the photons coupled into different modes can be tuned by changing the direction of the matrix element and for the hyperboloid case it is found that spontaneous emission in hyperbolic medium can be dramatically enhanced compared to the dielectric background. Moreover, spontaneous emission exhibit the strong directivity and get the maximum in the asymptote direction.
I. INTRODUCTION

Metamaterials is composed of the periodic dielectric element arrays where metal microstructure e.g. the split-ring resonators (SRRs) is included in each cell and used as an effective continuous medium in a narrow microwave frequency region [1-4] or in the near-visible light region [5]. The materials have the left-handed property for its negative permittivity and permeability simultaneously and cause to the refocusing and phase compensation [6], which provide a new manipulating space for the design of quantum optical devices. For instance, the suppression of spontaneous emission and superradiance over macroscopic distances in the left-handed materials (LHM) [7], the quantum interference enhancement between two spontaneous emission transitions with the LHM [8], long-lived entanglement between two distant atoms via the LHM [9]. On the other hand the effective permittivity or permeability of the metamaterials only holds for the specific lpolarization and it means that metamaterials is anisotropic as well as left-handed [10-12]. The anisotropy of metamaterials has also been found some new optical properties for example, the incident electric field $\mathbf{E}$ can couple to the magnetic resonance of the SRRs when the electromagnetic waves propagate perpendicular to the SRR plane [13] and it can be utilized to excite electron-spin resonance [14].

Since the elements of permittivity (permeability) tensor of the metamaterials can be negative or positive in different frequency range [10] we apply the term indefinite to anisotropic media in which not all of the principal components of the “$\varepsilon$” tensor has same sign. The geometry of the dispersion relation in the indefinite media can be characterized by an ellipsoid or a hyperboloid as shown in the inset to Fig.1. The medium with hyperboloid geometry of the dispersion relation is also called “hyperbolic medium” (HM). The HM has been recently found to be having many novel properties such as the superlens effect [15-17], slow-light effect [18] and the “big flash” of the photons in the HM by an optical metric signature phase transition [19]. Moreover, some interesting quantum optical properties (QOPs) of the HM has also been found experimentally [20] and theoretically [21,23], for instance, controlling spontaneous emission with the HM [20], broadband Purcell effect [21], the dipole radiation and its enhancement near the surface of the HM [22,23]. There are also some studies on the QOPs of the indefinite media where the emphasis is put on the left-handed property [24] and the singularity of the density of states [25].

The above theoretical analysis on the QOPs of the indefinite media are within frame-
work of the macroscopic electromagnetic wave theory and the atom is approximated as a dipole[8, 21–23] where the radiated power is exacted from the Green’s function of the system. However, the rate of spontaneous emission for a two-level atom in the indefinite media has not been investigated within the framework of the quantum optics. In this letter we will explicitly give the expression of the decay rate for a two-level atom in the indefinite media under the Weisskopf-Wigner approximation. It indicates that for the ellipsoid case the intensity of the photons coupled into different modes can be tuned by changing the direction of the matrix element. For the hyperboloid case it is found that spontaneous emission in the HM can be dramatically enhanced in comparison with the dielectric background, meanwhile, the spontaneous emission exhibit the strong directivity and get the maximum in the asymptote direction.

II. MODEL AND FORMULA

To deal with the our problem some theories of the quantum optics of dielectric media is needed. There are many schemes of the electromagnetic quantization for the different types of media. One of them is that the quantities characterizing the dielectric such as polarization field, magnetization field or some quantities of their combination are involved in the quantization procedure and the interaction between their quanta and the photon
is taken into account [26–28]. Another scheme is that the permittivity and permeability tensor are regarded as the parameters in the field equation with different gauges (gauge conditions) chosen for the different media [29–31]. In our case the medium is considered as anisotropic, homogenous, non-dispersive and lossless for simplicity. The quantization scheme in Ref. [31] is used where the medium is characterized by the constitutive equations

$$D(r) = \epsilon^{(1)}(r) \cdot E(r) + \epsilon^{(2)}(r) \cdot B(r); \quad H(r) = \mu^{(1)}(r) \cdot E(r) + \mu^{(2)}(r) \cdot B(r).$$

For our case $\epsilon^{(2)}(r) = \mu^{(1)}(r) = 0$; $\mu^{(2)}(r) = 1$ and

$$\epsilon^{(1)}(r) \equiv \epsilon = \epsilon_0 \begin{pmatrix} \epsilon_t & 0 & 0 \\ 0 & \epsilon_t & 0 \\ 0 & 0 & \epsilon_L \end{pmatrix}$$

where $\epsilon_L > 0$ and $\epsilon_t > 0$ for the uniaxial anisotropic medium or $\epsilon_L > 0$ and $\epsilon_t < 0$ for the HM. According to the Maxwell’s equations the dispersion relation for the medium characterized by Eq. (1) in principal axis coordinate system is expressed as:

$$\frac{k_x^2}{\epsilon_L} + \frac{k_x^2 + k_y^2}{\epsilon_t} = \left(\frac{\omega}{c}\right)^2$$

The geometry of the Eq. (2) in the K space represent a hyperboloid or an ellipsoid that depends on the sign of $\epsilon_t$ where $\epsilon_L > 0$ is assumed. Here we are interested in the decay rate for a two-level atom in the indefinite media. As the excitation of the quantum electromagnetic field the photon is emitted from an atom and subsequently coupled into the field modes permitted by the dielectric environment. In dielectric system the photon propagates in the fashion of the classical field modes of the system and inversely interact with the atom. In the spirit of Einstein’s original model the total energy of the electromagnetic field mode in the dielectric should be $\hbar \omega_k$ [32], which can be used to determine the field amplitude in the factor $g_k$. According to Ref. [31] the eigenvector field in the dielectric should satisfy the following gauge condition and the eigenequations:

$$\nabla \cdot \left[\epsilon(r) \cdot \bar{F}_k(r)\right] = 0$$

$$\nabla \times \nabla \times \bar{F}_k(r) = \omega_k^2 \epsilon(r) \cdot \bar{F}_k(r)$$

The quantized electric field can be expressed as $\hat{E}(r, t) = \sum_k \epsilon_k F_k(r) e^{-i\omega_k t} \hat{a}_k + H.c.$ and the
The total Hamiltonian of the photon and the atom under the rotating-wave approximation is

\[ \hat{H} = \sum_k \hbar \omega_k \hat{a}_k \hat{\sigma}_z + \frac{1}{2} \hbar \nu \hat{\sigma}_z \]

\[ + \hbar \sum_k g_k (\hat{\sigma}_- \hat{a}_k + \hat{\sigma}_+ \hat{a}_k^\dagger) \]

where \( \hat{\sigma}_z = |a\rangle\langle a| - |b\rangle\langle b| \), \( \hat{\sigma}_+ = |a\rangle\langle b| \), \( \hat{\sigma}_- = |b\rangle\langle a| \), \( |a\rangle \) and \( |b\rangle \) are the excited and ground states of the atom with eigenvalues \( E_a \) and \( E_b \).

For simplicity the Hamiltonian Eq. (5) can be expressed as the interaction picture

\[ \hat{H}_I = \hbar \sum_k \left[ g_k^* \hat{\sigma}_+ \hat{a}_k e^{i(\omega_0 - \omega_k)t} + g_k \hat{\sigma}_- \hat{a}_k^\dagger e^{-i(\omega_0 - \omega_k)t} \right] \]

where \( E_a - E_b = \hbar \omega_0 \). The state vector of the composite system of the photon and atom including the vacuum state \( |0\rangle \) is \( |\psi(t)\rangle = c_a |a, 0\rangle + \sum_k c_b,k |b, 1_k\rangle \). With the Weisskopf-Wigner approximation the decay rate of the atom in the anisotropic medium can be expressed as the following integration

\[ \Gamma = \frac{V}{(2\pi)^3} \int \left( |g_k(0)|^2 k^2 \frac{\partial \omega^{-1}}{\partial k} \right) \bigg|_{\omega=\omega_0} \sin \theta d\theta d\phi \]

where the dispersion relation of the modes is written as \( \omega = f(k, \theta, \phi) \) and \( k = f^{-1}(\omega, \theta, \phi) \). Here the eigen electric field is assumed as \( \vec{E}_k(r) = \vec{F}_k(r); \vec{H}_k = \frac{1}{\mu_0} \nabla \times \vec{E}_k \).

That a single photon is coupled into the eigen-mode require the total energy of the mode \( U = \frac{1}{2} \int (\vec{E}_k \cdot \vec{E}_k + \mu_0 |\vec{H}_k|^2) \) \( d^3x = \hbar \omega \), which determine the amplitude \( F_k(0) \) under the box normalization condition.

\section*{A. Spontaneous decay rate in the medium with dispersion geometry of ellipsoid}

In order to check the validity of the formulism, the case \( \epsilon_t > 0 \) is firstly considered where the medium become uniaxial anisotropic with an ellipsoidal dispersive geometry. In the medium there are usually two types of eigen-modes named as extraordinary wave and ordinary wave which accommodate the emitted photons form the atom. To this end we have to explore the amplitudes and the energy of the two modes in detail. The eigen electric field
is supposed as the plane wave $\vec{E}_k(r) = \vec{E}_{k0}e^{i\vec{k}\cdot r}e^{i\omega(t)}$ and $\vec{H}_k(r) = \vec{H}_{k0}e^{i\vec{k}\cdot r}e^{i\omega(t)}$ where the complex vector amplitudes are to be determined. Substitute the expression into the Maxwell Equations and after some algebra we get the two sets of dispersion and polarization relations

$$k = \frac{\omega}{c}\sqrt{\epsilon_t}$$  \hspace{1cm} (9)

with $E_z = 0$ and $k_xE_{k0x} + k_yE_{k0y} = 0$

$$\frac{k_x^2 + k_y^2}{\epsilon_L} + \frac{k_z^2}{\epsilon_t} = \left(\frac{\omega}{c}\right)^2$$  \hspace{1cm} (10)

with $H_z = 0$ and $\frac{E_{k0x}}{E_{k0y}} = \frac{k_x}{k_y}$

In the spherical coordinate system the dispersion relations can be uniformly written as: $\omega(k) = f(\theta)c k$, where $f(\theta) = \frac{1}{\sqrt{\epsilon_t}}$ for the transversal mode Eq(9), and $f(\theta) = \sqrt{\frac{\sin^2\theta}{\epsilon_L} + \frac{\cos^2\theta}{\epsilon_t}}$ for the longitudinal mode Eq(10). For the transverse mode the total energy in the volume $V$ and the relative amplitude are

$$U_T = \epsilon_0 (E_{k0}^T)^2 \epsilon_t V = \hbar \omega_0$$

$$E_{k0}^T = \sqrt{\frac{\hbar \omega_0}{V \epsilon_0 \epsilon_t}}$$  \hspace{1cm} (11)

For the longitudinal mode the corresponding quantities:

$$U_L = \frac{V \epsilon_0^2 \mu_0 \omega_0^2 (E_{k0}^L)^2}{k_x^2 \epsilon_L + \epsilon_t^2 (k_x^2 + k_y^2)} = \omega_0 \hbar$$

$$E_{k0}^L = \left[\frac{\omega_0 \hbar (\cos(2\theta) (\epsilon_L^2 - \epsilon_t^2) + \epsilon_L^2 + \epsilon_t^2)}{2 f^2 V \epsilon_0 \epsilon_L^2 \epsilon_t^2}\right]^{1/2}$$  \hspace{1cm} (12)

In the principal axis spherical coordinate system there are three vectors to be identified. They are the vectorial transition matrix element $\vec{D}_{ab} = (\mathcal{D}_{ab}, \theta_0, \phi_0)$, the electric field vector $\vec{E}_{k0} = (E_{k0}, \theta_1, \phi_1)$ and the wave vector $\vec{k} = (k, \theta, \phi)$ which is shown in Fig. 11. The factor in Eq.(13) is explicitly given

$$|g_k|^2 = \frac{\mathcal{D}_{ab}^2 \epsilon_{k0}^2 \cos^2 \theta_{0,1}}{\hbar^2}$$  \hspace{1cm} (13)

where $\theta_{0,1}$ is the angle between $\vec{D}_{ab}$ and $\vec{E}_{k0}$. According to the geometrically relations of the vectors $\vec{D}_{ab}$ and $\vec{E}_{k0}$ and the gauge condition $\vec{k} \cdot \hat{e} \cdot \vec{E}_{k0} = 0$, we get the equations $\cos \theta_{0,1} = \sin \theta_0 \sin \phi_0 + \cos \theta_0 \cos \phi_1$ and $\epsilon_L \cos \theta_1 \cos \theta + \epsilon_t \sin \theta_1 \sin \theta \cos (\phi_1 - \phi) = 0$. In term of Eq.(9),Eq.(10), It is noted that $\phi_1 = \phi + \frac{\pi}{2}$ for the transversal mode and $\phi_1 = \phi$ for
the longitudinal mode. With these conditions we get the factor \( c_q = \cos^2 \theta_0,1 \) for the different modes

\[
\begin{align*}
    c_q^T &= \sin^2 \theta_0 \sin^2 (\phi_0 - \phi) \\
    c_q^L &= \frac{[\epsilon_l \cos \theta_0 - \epsilon_L \sin \theta_0 \cot \theta \cos (\phi_0 - \phi)]^2}{\epsilon_L^2 \cot^2 \theta + \epsilon_l^2}
\end{align*}
\] (14)

After the implementation of the Eq.(8) we get the decay rate for the two modes

\[
\begin{align*}
    \Gamma^T &= \frac{D_{ab}^2 \omega_0^3 \sqrt{\epsilon_l \sin^2 \theta_0}}{4 \pi c^3 \epsilon_0 \hbar} \\
    \Gamma^L &= \frac{D_{ab}^2 \omega_0^3 (-\cos (2\theta_0) (\epsilon_L - 4\epsilon_l) + \epsilon_L + 4\epsilon_l)}{24 \pi c^3 \epsilon_0 \hbar \sqrt{\epsilon_l}} \\
    \Gamma &= \frac{D_{ab}^2 \omega_0^3 (\cos (2\theta_0) (\epsilon_l - \epsilon_L) + \epsilon_L + 7\epsilon_l)}{24 \pi c^3 \epsilon_0 \hbar \sqrt{\epsilon_l}}
\end{align*}
\] (16a)

When \( \epsilon_L = \epsilon_l > 0 \), Eq.(16c) reduce as \( \frac{D_{ab}^2 \omega_0^3 \sqrt{\epsilon_l}}{3 \pi c^3 \epsilon_0 \hbar} = \Gamma(\epsilon_l) \), which give the decay rate of the atom in the homogeneous isotropic medium with permittivity \( \epsilon_l \). The decay rate relative to the vacuum \( \Gamma/\Gamma_0 = \tilde{\Gamma} \) in the case \( \epsilon_l = 2.5, \epsilon_L = 3.5 \) is shown in Fig.2 for the different modes. Form the Fig.2 it is found that when the matrix element vector \( \vec{D}_{ab} \) is parallel to the \( k_z \) axis(\( \theta_0 = 0 \)), the atom can get the maximal coupling with the longitudinal mode of the system where \( E_z \neq 0 \) and much more photons emitted form the atom is coupled into the mode. Meanwhile, the transverse mode get the no coupling with \( \vec{D}_{ab} \) for \( E_z = 0 \) and \( \tilde{\Gamma}_T = 0 \). In the case, \( \tilde{\Gamma} = \tilde{\Gamma}_L = 1.58 = \sqrt{\epsilon_l} \), which the anisotropic medium behave as the isotropic medium with permittivity \( \epsilon_l \) for the atom’s spontaneous emission. With the increase of \( \theta_0 \), \( \tilde{\Gamma}_T \) also increase due to the enhancement of the coupling with the transverse mode and \( \tilde{\Gamma}_L \) decrease due to the reduction of the coupling with the longitudinal mode. When \( \theta_0 = \frac{\pi}{2} \) where \( \vec{D}_{ab} \) is perpendicular to the \( k_z \) axis \( \tilde{\Gamma}_T \) get the maximum and \( \tilde{\Gamma}_L \) get the minimum. According to Eq.(16c) \( \Gamma \) get the maximum(\( \epsilon_l < \epsilon_L \)) or the minimum(\( \epsilon_l > \epsilon_L \)): \( \Gamma_m = \frac{D_{ab}^2 \omega_0^3 (2\epsilon_l + 6\epsilon_l)}{24 \pi c^3 \epsilon_0 \hbar \sqrt{\epsilon_l}} \) at \( \theta_0 = \frac{\pi}{2} \). These results can be used to control the intensity of the different modes from the spontaneous emission by tuning \( \theta_0 \).

B. Spontaneous decay rate in the medium with dispersion geometry of hyperboloid

For the case \( \epsilon_l < 0, \epsilon_L > 0 \), Eq.(1) indicates the hyperboloid geometry of the dispersion relation. Since \( \epsilon_l < 0 \), the branch \( k = \frac{\omega}{c \sqrt{\epsilon_l}} \) can only exist with evanescent wave. This
FIG. 2: The decay rate of atom in medium with $\varepsilon_t = 2.5, \varepsilon_L = 3.5$ relative to the case for the vacuum $\bar{\Gamma} = \Gamma / \Gamma_0$. Doted line: the decay rate for the transverse mode. Dashed line: the decay rate for the logitudinal mode. Solid line: the total decay rate.

The mode can not carry the energy away from the atom for the spontaneous emission and the contribution to the decay rate is ignored for simplicity. For the branch $\frac{k^2_t + k^2_L}{\varepsilon_L} + \frac{k^2_t}{\varepsilon_t} = \left(\frac{\omega}{c}\right)^2$ the decay rate $\Gamma$ is proportional to the following expression:

$$\Gamma = \beta \int_{-1}^{1} \frac{x^2 \varepsilon_L^2 \sin^2 \theta_0 - 2 (x^2 - 1) \varepsilon_t^2 \cos^2 \theta_0}{(x^2 - \varepsilon_u)^{5/2}} \, dx$$

where $x = \cos \theta$, $\varepsilon_u = \frac{\varepsilon_t}{\varepsilon_t - \varepsilon_L}$, $\beta = \frac{D^2 \omega^5 (\frac{\varepsilon_t + \varepsilon_L}{\varepsilon_t - \varepsilon_L})^{5/2}}{8 \pi c^3 \varepsilon_0 \hbar^2 \varepsilon_t^2}$.

It is noted that under the Weisskopf-Wigner approximation the integration Eq.(8) is actually calculated on the equal-frequency surface. For the ellipsoid Eq.(8) $\theta$ integrates over the range $[0, \pi]$, while for the hyperboloid $\theta$ integrates over the range $[\theta_a, \pi - \theta_a]$, as shown in Fig.3 where $\theta_a$ is the polar angle of the asymptote. In term of Eq.(17) there are two poles $\pm \sqrt{\varepsilon_u}$ the positions of which on the axis depend on the parameters $\varepsilon_t, \varepsilon_L$. When $\varepsilon_t > \varepsilon_L > 0$, $\varepsilon_u > 1$ and $\varepsilon_L > \varepsilon_t > 0, \varepsilon_u < 0$ the poles $\pm \sqrt{\varepsilon_u}$ are out of the range $[-1, 1]$ or on the imaginary axis. It enable Eq.(17) to be calculated and the result is given in Eq.(16). However, for the hyperboloid case where $\varepsilon_t < 0, \varepsilon_L > 0, 0 < \varepsilon_u < 1$ the poles lie in the range $[-1, 1]$ the integration diverges. This divergence is the manifestation of the change of the topology from the ellipsoid to the hyperboloid and it also indicate that hyperboloid of the dispersion relation is only the perfect effective medium approximation of some composite materials e.g. photonics crystal, metamaterials under the long wave-length limit. Therefore, some cutoff
methods have to be introduced for the calculation of Eq. (17) in the case \( \epsilon_t < 0, \epsilon_L > 0 \). To this end it is defined that \( x_a = \cos \theta_a = \sqrt{\epsilon_u} \) and \( \cos \theta_c = x_c \equiv \alpha x_a \), where \( \pm x_c \) is the new integration limits for Eq. (17) with condition \( x_c < x_a \). The integration range of the variable \( \theta \) for the cutoff is marked in the shadowed region in Fig. 3. When the coefficient \( \alpha = 1 \), the integration limit approach to the two polar poles \( \pm x_a \). Under the cutoff approximation we get the decay rate \( \Gamma \) as follows:

\[
\Gamma_H = \Gamma_0 \frac{\alpha^3 \left[ \epsilon_L^3 - \epsilon_L \cos^2 \theta_0 \left( -4 \epsilon_L \epsilon_t + \epsilon_L^2 + 6 \epsilon_t^2 \right) \right] - 6 \alpha \epsilon_L \epsilon_t \cos^2 \theta_0 \left( \epsilon_L - \epsilon_t \right)}{4 \left( \alpha^2 - 1 \right)^{3/2} \left( \epsilon_L - \epsilon_t \right)^2 \sqrt{\frac{\Gamma_H}{\epsilon_L - \epsilon_t}}} 
\]

where \( \Gamma_0 = \frac{\hbar^2 \epsilon_0^3}{3 \pi e^4 c} \)

In Fig. 4 we give the \( \tilde{\Gamma}_H = \Gamma / \Gamma_0 \) vs the parameter \( \alpha \) for the different \( \theta_0 \) with \( \epsilon_t = -2.5, \epsilon_L = 3.5 \). It is obviously found that \( \tilde{\Gamma}_H \) increases on the whole when \( \alpha \) approach 1. It is noted that when \( \alpha \) close 1 the \( \tilde{\Gamma}_H \) increases more sharply. It is because that more large \( \alpha \) is, more modes with high \( k \) vectors are involved in the spontaneous emission. In HM the modes with high \( k \) points usually have large density of states and more photons emitted from the atom can be accommodated. Moreover, since the photons is mainly coupled into the logitudial mode, \( \tilde{\Gamma}_H \) get more large value when the matrix element vector \( \vec{D}_{ab} \) get the small angle \( \theta_0 \) with \( z \) axis which enhance the coupling factor \( g_k \).
FIG. 4: The relative decay rate of atom in medium with $\epsilon_t = -2.5, \epsilon_L = 3.5$ relative to the case for the vacuum according to parameter $\alpha$ for different angles $\theta_0$

FIG. 5: The relative decay rate of atom in medium with $\epsilon_t = -2.5, \epsilon_L = 3.5$ to the case for the background $\epsilon_L = 3.5$ according to parameter $\alpha$ for different angles $\theta_0$

To explore the effect of the enhancement of the spontaneous emission we compare the decay rate $\Gamma_H$ with the decay rate $\Gamma_{\epsilon_L}$, the decay rate in the background medium with permittivity $\epsilon_L$. The relative decay rate $\tilde{\Gamma}_H = \Gamma_H/\Gamma_{\epsilon_L}$ corresponding to parameter $\alpha$ is given in Fig. 4. It is easily found that $\tilde{\Gamma}_H$ can be larger than one only if $\alpha$ is sufficiently large in spite of the difference of $\theta_0$, which means that the more like the strict hyperbolic medium the real materials behave, the more the spontaneous emission is enhanced relative to the background
medium. According to Eq. (18) there are $\lim_{\alpha \to 1} \tilde{\Gamma}_H = \infty$, which implicate that the perfect hyperbolic medium is the an idealization model of some real composite materials. Besides the enhancement of the spontaneous emission the directivity is also worthy of being noted. To this end the Eq. (17) can be rewritten as $\Gamma = \beta \int_0^\pi \gamma(\theta) d\theta$. Considering the divergence, a small imaginary part is added to $\epsilon_t$. Fig 6 gives the amplitude of the integrand $\gamma(\theta)$ with $\epsilon_t = -2.5 + 0.5i, \epsilon_L = 3.5$. There are two obvious peaks in the positions corresponding to the directions of the asymptotes. This strong directivity of the spontaneous emission has been noted experimentally and used to design the single gun [16, 34].

III. DISCUSSION

We explore the spontaneous emission of the two-level atom in the homogeneous anisotropic medium where the dispersion geometry exhibit as an ellipsoid or a hyperboloid and the corresponding decay rate $\Gamma$ in detail under the Weisskopf-Wigner approximation. For the ellipsoid case, there are two kinds of modes that contribute the decay rate $\Gamma$ and to some degree the medium with ellipsoid dispersion provides two type of mode space to accommodate the emitted photons. Moreover, the polar angle $\theta_0$ also be used to change the intensity of the photons coupled into the different modes. When $\epsilon_L = \epsilon_t$ the obtained formula Eq. (16c) reduce to the case of the isotropic medium, which justify the validity of our model.
When the above model is applied to the hyperboloid case the divergence is encountered and the cutoff for the integration variable $\theta$ (equivalently, the wave vector $K$) is introduced for the approximation of the real materials. Though the transverse radiating wave mode degenerates into the evanescent wave mode, the enhanced spontaneous emission relative to the background medium $\varepsilon_L$ can be obtained only if the dispersion geometry of the real materials is sufficiently close to the strict hyperboloid. It mean that more high $k$ modes which corresponds lager parameter $\alpha$ get involved in the coupling of the photons. In real world, the hyperbolic medium is used as the perfect model on the some materials with periodic micro-structure where band structure exhibit the approximative hyperbolic geometry in the small range of $k$, e.g. the hyperbolic metamaterials. The simplest of them is the one dimensional photonic crystal including the metal layer as shown in Fig. 7 together with the equifrequency contour. Generally, in the long wavelength limit the system can be approximated as the hyperbolic medium within the effective medium theory. It is expected that when the more small the dimension of the lattice $a$ is, the more close to the perfect hyperbolic medium the system is and more enhancement of the spontaneous emission can be achieved.

The physical mechanism of the enhancement on the spontaneous emission in HM can be understood from the perspective of density of states. According to the Fermi’s golden rule the one-to-many transition probability per unit of time depends not only on the matrix element but the density of final states $\rho(\omega)$ as well. When the geometry of physical dispersion relation of the medium changes form the ellipsoid to the hyperboloid, due to the change of the topological property the density of states diverges in the lossless continuous hyperbolic medium limit: $\rho(\omega) \approx \frac{K_{\text{cut}}^3}{12\pi^2} \left| \frac{\varepsilon_L}{\varepsilon_\ell} - \frac{1}{\varepsilon_\ell} \frac{d\varepsilon_\ell}{d\omega} \right|$ where $K_{\text{cut}}$ is the momentum cutoff\[19\]. $K_{\text{cut}}$
is defined by either metamaterial structure scale or by losses. It is the occurrence of the large $\rho$ that enable the transition probability to be increased dramatically. Even the loss and the dispersion of the $\epsilon(\omega)$ in the materials is taken into account the spontaneous emission is expected to be largely enhanced.

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