Explicit Factorization of a Categorical Center

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Abstract

Given a braided fusion category $C$, it is well known that the natural map $C \boxtimes C^{\text{bop}} \to Z(C)$ from the square of $C$ to the (Drinfeld) categorical center $Z(C)$ is an equivalence if and only if $C$ is modular. However, it is not clear how to construct the inverse and the natural isomorphisms. In this work, we provide an explicit construction using insights from a specific quantum field theory, and explore how the equivalence fails for the degenerate cases.

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0 Introduction

Ubiquity of Symmetries

Symmetry has always been the central topic of pure mathematics. The reason of its ubiquity is straightforward: Any symmetrical object can be made much simpler by dividing out the extra information, and any object with its original symmetry divided out can be made more intuitive by recovering the extra information. It is therefore useful, practically and conceptually, to pass between both pictures.

Symmetries in Physics

During the mid-19th century, the use of symmetries entered and fundamentally changed theoretical physics. In particular, the advent of Maxwell’s equations and their further reduction by $U(1)$-symmetry provided profound insights leading us to the birth of Einstein’s relativity and quantum mechanics. Later
in the 20th century, the hidden symmetries of our universe were further exploited by gauge theorists, who provided the celebrated Standard Model.

Groups as Classical Symmetries
Mathematically, the backbone of symmetries in gauge theory was provided by groups (collections of symmetries). By understanding possible behaviors of groups, we can predict the phenomena in an object (e.g. the space-time, a statistical model, a molecule [Ser]. etc) whose symmetries are described by the group.

Higher Symmetries
However, while successful, groups do not describe all kinds of symmetries. The existence of the quantum groups is one of the precursor of such insufficiency. In general, we need higher symmetries, some of which are described by fusion categories, which play an important role in modern mathematics, physics (both theoretical and experimental) [Kon14], information theory and coding theory, quantum computing (both theoretical and architectural).... etc.

Fusion Categories as Higher Symmetries
In a nutshell, by viewing a finite group (and its group algebra) as a 1-categorical object, one can view a fusion category as an analogous 2-categorical object and a braided fusion category as an analogous 3-categorical object. As we climb the categorical ladder, more information about symmetries is preserved.

Fusion Categories in Quantum Field Theory
The current work is about the structure of a braided fusion category. While we have mentioned its usefulness and its abundance of applications, we wish to stress on a particular application in quantum field theory and mathematics.

The story started from knot theory. A knot is (an isotopy class of) an embedding of a circle in an Euclidean 3-space. While both a circle and an Euclidean 3-space are trivial, the embeddings are notoriously hard to study. As an illustrative example, try to tell if the following knots are isotopic to a trivial embedding [Och90].

Despite its playfulness, it is beyond a fun brain-teaser. In fact, knot theory relates deeply to theoretical physics [BM] and modern number theory. It is therefore of importance to develop a complete understanding towards knot theory, which remains to be a wildly open subject that currently keeps many mathematicians and physicists busy.
An Unexpected Invariant: Jones Polynomials

One of the tremendous breakthrough was made in the mid 80s by the Fields medalist Vaughan Jones. His work on the Jones polynomials unexpectedly \(^1\) connected knot theory and statistical mechanics. The mysterious Jones polynomials were later explained by Edward Witten in 1989 as a special case of a larger framework, the 3-dimensional Witten-Reshetikhin-Turaev (WRT) topological quantum field theory (TQFT), which is a quantized Chern-Simons theory in dimension 3.

Jones Polynomials as a Special Case of a 3D TQFT

Enter the braided fusion categories. They are the algebraic input of the WRT TQFTs. However, we need them to be modular (or called non-degenerate) in order to define the theory. The WRT theory is itself a vast framework that still keeps mathematicians busy, for its deep relationship with the theory of integer primes [Mor], (mock) modular forms [CFS19], topological quantum computations [Kit03].

A Larger Framework: Crane-Yetter 4D TQFT

Despite its vastness, there is a larger framework behind it, the 4-dimensional Crane-Yetter topological quantum field theory (CY). Several evidences \(^2\) have been found that WRT is the boundary theory of CY whenever the algebraic input is non-degenerate [BGM07] [KT21] [Tha21], making CY more flexible and potentially more powerful. Therefore, it is natural to understand how much more power the degeneracy gives to the CY model.

\[ \text{CY} \setminus \text{WRT} = ? \]

With the excision formula of CY, the degeneracy leads to the non-triviality of CY (in dimension 2) precisely because of the following statement.

**Fact 0.1** Let \( C \) be a braided fusion category, and let \( F \) be the natural functor from the square of \( C \) to the (Drinfeld) categorical center

\[
C \boxtimes C \overset{\text{bop}}{\longrightarrow} Z(C)
\]

\[
X \boxtimes Y \mapsto (X \otimes Y, c_{-,X} \otimes (c^1)_{Y,-})
\]

where \( c_{-,x} \) denotes the braiding of \( C \). Then \( F \) is an equivalence of categories if and only if \( C \) is non-degenerate.

The degeneracy fails the equivalence and makes the algebra richer. But this also makes the TQFT stronger as it preserves more topological information.

It is then urgent to analyze how the degeneracy fails the equivalence. The main crux is to look at how \( F \) and its inverse fails to compose to functors that are equivalent to the identity functors. However, to the best of our knowledge, the explicit construction of the inverse map and the natural equivalences are not known. The main result provided in the current paper is an explicit construction of the inverse and the witnessing natural isomorphisms. Applications and explicit calculations will come in future work.

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\(^1\)This amazing discovery won him a Fields medal in 1990.

\(^2\)The full result has been long expected but left unwritten partly because a rigorous proof needs to establish an equivalence between \( \partial \text{CY} \) and \( \text{WRT} \) as 2-functors, which is quite technical.
Inspiration from Topology
The main observation of the current work is based on the following (surprising) fact.

**Fact 0.2** [KT21] Let $C$ be a premodular category and $\Sigma$ be the cylinder $S^1 \times I$. Then there is an equivalence of categories $\text{CY}_C(\Sigma) \simeq Z(C)$ where $Z(C)$ denotes the Drinfeld center.

This fact is interesting because it gives a simple topological interpretation of the grand idea - the Drinfeld center construction $^3$. Using this fact, we gained several insights about the Drinfeld center:

1. Besides the natural monoidal structure, there is another hidden tensor product for $Z(C)$, namely the reduced tensor product $\boxtimes$ [Tha20].
2. It admits natural generalizations to all (oriented punctured) surfaces, namely the categorical center of higher genera [Guu21].
3. It hints what an explicit inverse map of $F$ and the witnessing natural transformations would be. In hindsight, the explicit construction seems to be out-of-reach if one does not consider its topological picture provided by the Crane-Yetter model. This serves as the key idea of the paper.

1 Prerequisites

We fix an algebraically closed field $k$ with characteristic 0.

1.1 Premodular Categories

We will define (pre)modular categories assuming familiarity with a fusion category, a braided category, ribbon structure, and the (Drinfeld) categorical center. A complete and recommended source is [Eti+15]. For definitions written in a dictionary-style starting from “scratch” (additive categories and abelian categories), please refer to [Guu21, appendix]. Other useful sources are [BK02], [Kas95], [Tur10].

**Definition 1.1 (Braided Fusion Category)** A braided fusion category is a braided category whose underlying monoidal category is a fusion category.

**Definition 1.2 (Muger center)** Given a braided fusion category $C$ with braided structure $c_{-,-}^-,\star$, we say an object $X$ in $C$ is transparent (and otherwise opaque) if

$$c_{-,-}X \circ c_{X,-} = \text{id}_{X,-}. $$

We define the Muger center $\text{Mu}(C)$ of $C$ to be the full tensor subcategory of $C$ consisting of transparent objects. Note that in some other literature, the Muger center is also called a Muger centralizer or an $E_2$-center.

Recall that if $c_{-,-}^-,\star$ is a braided structure of a braided fusion category $C$, then $c_{-,-}^1$ is also a braided structure for the underlying fusion category. This produces an opposite braided fusion category, which we denote by $C^{\text{bop}}$. Directly by the definition of a (Drinfeld) categorical center, there is a tautological functor from $C \boxtimes C^{\text{bop}}$ to $Z(C)$.

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$^3$Such construction led to the construction of quantum groups, winning him a Fields medal in 1990.
Definition 1.3 (Tautological functor $F$) Given a braided fusion category $C$, there is a natural functor $C \boxtimes C^\text{bop} \xrightarrow{F} \mathcal{Z}(C)$ that maps each object $X \boxtimes Y$ to $(X \otimes Y, c_{-X} \otimes c_{Y}^{-1})$ and each morphism $(f \boxtimes g)$ to $(f \otimes g)$.

Definition 1.4 (Factorizable category) Given a braided fusion category $C$, if its tautological functor $F$ is an equivalence of categories, we say that $C$ is factorizable, and call any of its inverse functor $F^{-1}$ a factorization of the Drinfeld center $\mathcal{Z}(C)$.

Notice that the structure of $\mathcal{Z}(C)$ is in general pretty opaque. For example, even the fusion ring of $\mathcal{Z}(C)$ is hard to identify. Factorizability reduces the complexity of $\mathcal{Z}(C)$ to that of $C$.

Definition 1.5 (Premodular Category) A premodular category is a ribbon fusion category (equivalently, a braided fusion category equipped with a spherical structure).

Definition 1.6 (Complete set of simple objects) Let $C$ be a premodular category. By a complete set of simple objects $O(C)$ we mean a set $O(C) = \{i, j, \ldots\}$ of simple objects in $C$ that exhausts all simple types and that satisfies $(i \neq j) \Rightarrow (i \ncong j)$. Define its dual set to be

$$O(C)^* = \{i^* \mid i \in O(C)\},$$

where $i^*$ denotes the (left) dual object of $i$.

Notice that by the axiom of premodular category, any $O(C)$ is a finite set. From now on, we assume that any premodular category $C$ comes with a fixed complete set of simple objects $O(C)$.

Definition 1.7 (S-matrix) Let $C$ be a premodular category with the braided structure $c$. The $S$-matrix of $C$ is defined by

$$S := (s_{XY})_{X,Y \in O(C)}$$

where $s_{XY} = \text{Tr}(c_{Y}Xc_{X}Y) \in \mathbb{k}$, where $\text{Tr}$ denotes the (left) quantum trace that depends on the spherical structure of $C$.

Definition 1.8 (Modular Category) [Eti+15, p. 8.13.14] A modular category is a premodular category $C$ whose $S$-matrix is non-degenerate.

Fact 1.9 (Characterization of Modularity) [Eti+15, 8.20.12 and 8.19.3] The following conditions are equivalent for a premodular category $C$:

1. $C$ is modular.
2. $\text{Mu}(C) \simeq (\text{Vect})$
3. $C$ is factorizable.

Surprisingly, the fact indicates that modularity and factorizability are equivalent for premodular categories, so as a consequence modularity reduces the complexity of $\mathcal{Z}(C)$. While this is desirable from the algebraic point of view, it is not the case from the topological point of view: The power of the topological quantum field theory is largely reduced by modularity exactly due to this fact.
1.2 Graphical Calculus

We will use the technique of graphical calculus ([BK02] and [Kas95]) while dealing with premodular categories.

An advantage of this is that many equalities among morphisms can be proved graphically, thanks to the work of Reshetikhin and Turaev [Tur90] [BK02, Theorem 2.3.10]. For example, to prove

\[
\text{eval}_Y \circ c_{X,Y} \circ \text{coev}_Y = c_{X,Y},
\]

it suffices to establish an isotopy of ribbon tangles, which is a trivial task, and translate the procedure back into the equations in the syntactic equations. Such feature of graphical calculus provides sophisticated quantum link invariants (e.g. Jones polynomials). An interesting exercise left for the unconvinced reader is to turn all graphical equations in this paper into syntactic equations.

In the rest of the section, we provide some useful lemmas and notations for graphical calculus.

**Lemma 1.10** Let \( C \) be a premodular category with spherical structure \( \alpha \). Let \( X, Y \) be \( C \)-objects. Define a pairing of \( \mathbb{k} \)-linear spaces

\[
\text{Hom}_C(X, Y) \otimes \text{Hom}_C(Y, X) \rightarrow \mathbb{k}
\]

that sends \( \phi \otimes \psi \) to

\[
\text{Tr}(\psi \circ \phi) = \text{eval}_X \circ ((\alpha_X \circ \psi \circ \phi) \otimes 1_{X^*}) \circ \text{coev}_X \in \text{End}_C(1) \simeq \mathbb{k}.
\]

Then the pairing is nondegenerate by the semisimplicity of \( C \), identifying the linear space with its linear dual \( \text{Hom}_C(Y, X) \simeq \text{Hom}_C(X, Y)^* \).

Define the Casimir element

\[
\omega_{X,Y} := \Sigma_i \phi_i \otimes \phi_i^\dagger \in \text{Hom}_C(X, Y) \otimes \text{Hom}_C(Y, X)
\]

where the \( \phi_i \)'s is any basis of the former multiplicant and the \( \phi_i^\dagger \)'s is its dual basis under the identification given in 1.10. Graphically, we use dummy variables \( \phi \) and \( \phi^\dagger \) as a short-hand notation:
Lemma 1.11 Let $C$ be a premodular category and $W$ be a $C$-object. Then

$$I_W = \Sigma_{i \in \mathcal{O}(C)} \Sigma_1 \text{dim}(i) \phi^1 \circ \phi_i,$$

where the $\phi_i$'s and the $\phi^1$'s form a pair of dual bases for the vector spaces $\text{Hom}_C(X, Y)$ and $\text{Hom}_C(Y, X)$ respectively, and $\text{dim}(i)$ denotes the (left) quantum trace of $\text{id}_i$.  

Notation 1.12 (regular color) Let $C$ be a premodular category and $O(C)$ a complete (up to isomorphism) set of simple objects of $C$. We use $\Omega$ in the graphics to represent the regular color $\oplus_{i \in \mathcal{O}(C)} \text{dim}(i) \text{id}_i : i \to i$. We also denote $\text{dim}(\Omega)$ by $\Sigma_{i \in \mathcal{O}(C)} \text{dim}(i)^2$, which is nonzero [ENO200505].

With this shorthand notation $\Omega$, we can present the lemma graphically by

Lemma 1.13 (Sliding lemma) Let $C$ be a premodular category. Then the following morphisms are all equal, where $\Omega$ is the shorthand notation given in 1.12.

Heuristically, the moral of this lemma is that $\Omega$ protects everything “inside” it by making it transparent.

Lemma 1.14 (Censorship of Opacity) [BK02, (3.1.19)] Let $C$ be a modular category, $i$ a simple $C$-object, and $\lambda = \text{dim}(\Omega) \delta_{i,1}$, we have the following equality.

$$\left( i \xrightarrow{\alpha} i \right) = \lambda \left( i \xrightarrow{i} \right)$$
2 Main Result

For any premodular category $\mathcal{C}$, we aim to construct a functor $Z(\mathcal{C}) \xrightarrow{G} \mathcal{C} \boxtimes \mathcal{C}^{\text{bop}}$, and prove it an inverse functor for $\mathcal{C} \boxtimes \mathcal{C}^{\text{bop}} F \xrightarrow{} Z(\mathcal{C})$ in the case that $\mathcal{C}$ is modular by constructing explicit natural isomorphisms.

Throughout this section, we fix a premodular category $\mathcal{C}$, fix a complete set of simple objects $O(\mathcal{C})$ and its dual $O(\mathcal{C})^\ast$ (1.6). With a $\mathcal{C}$-object $X$ fixed, $\text{Hom}_\mathcal{C}(X,i^\ast)$ is a finite dimensional vector space over $\mathcal{C}$ with a natural nondegenerate pairing 1.10 with $\text{Hom}_\mathcal{C}(i^\ast,X)$. Pick and fix an arbitrary basis $X[i] = \{\alpha_{i,1},\ldots,\alpha_{i,l_i}\}$ for the former space, and form its dual basis $X[i]^\ast = (\alpha_{l_i}^\ast \ldots, \alpha_{1}^\ast)$ in the latter. We will drop the super/subfix when there is little danger of confusion. We also identify $x$ and its (left) double dual $x^{\ast\ast}$ by the spherical structure of $\mathcal{C}$.

2.1 Construction

**Definition 2.1 (The coupling morphism $\Gamma_{i,(X,\gamma)}$)** Let $(X, \gamma)$ be an object of $Z(\mathcal{C})$. For each $i \in O(\mathcal{C})$, define the $\mathcal{C}$-morphism $\Gamma_{i,(X,\gamma)}$ to be the product of $\frac{1}{\dim(i^{\ast}\boxtimes\Omega)}$ and the following morphism

$$i \xleftarrow{\Omega} \Gamma_{i,(X,\gamma)} \xrightarrow{i} X$$

By axiom, $\mathcal{C}$ is an abelian category, so there is a canonical object $I_{i,(X,\gamma)}$ and two canonical maps $(i \otimes X) \xrightarrow{\rightarrow} I_{i,(X,\gamma)}$ and $I_{i,(X,\gamma)} \xleftarrow{\subseteq} (i \otimes X)$ such that $\Gamma_{i,(X,\gamma)} = (\subseteq) \circ (\rightarrow)$. Both canonical maps depend on $i$ and $(X, \gamma)$. However for simplicity we often omit mentioning the dependence. Notice also that $\Gamma_{i,(X,\gamma)}^2 = \Gamma_{i,(X,\gamma)}$ by the tensoriality of $\gamma$ and the sliding lemma.

**Definition 2.2 (The functor $G$)** Define $G$ to be the functor that sends any object $(X, \gamma)$ in $Z(\mathcal{C})$ to

$$\bigoplus_{i \in O(\mathcal{C})} i^\ast \boxtimes I_{i,(X,\gamma)},$$

and any morphism $(X, \gamma) \xrightarrow{\phi} (Y, \beta)$ to

$$\bigoplus_{i} I_{i^\ast} \boxtimes (\rightarrow \circ \Gamma_{i,(Y,\beta)} \circ (I_{i} \otimes \phi) \circ \Gamma_{i,(X,\gamma)} \circ \subseteq).$$

The construction was motivated by the Crane-Yetter theory and the construction of the categorical center of higher genera [Guu21]. Notice what while its existence was known by (Frobenius-Perron) dimension argument [Eti+15], this construction is new and is expected to provide insight in the difference between two topological quantum field theories, the Witten-Reshetikhin-Turaev theory and the Crane-Yetter theory.
We will further construct four natural transformations

\[ 1 \overset{d}{\to} GF, \quad GF \overset{q}{\to} 1, \quad 1 \overset{b}{\to} FG, \quad FG \overset{p}{\to} 1. \]

and argue that they witness \( FG \simeq 1 \) and \( GF \simeq 1 \) where \( C \) is modular.

**Definition 2.3 (The transformations \( d \) and \( q \))** To construct the natural transformation \( 1 \overset{d}{\to} GF \), it suffices to construct a morphism in \( C \otimes C^{bop} \) for each object \( X \otimes Y \). We thus define

\[
d = d_{X \otimes Y} := \sum_{i \in O(C)} d_i := \sum_{i \in O(C)} \frac{1}{\sqrt{\dim(i)}} \sum_{k=1}^{\|X[i]\|} d_i(k),
\]

where \( d_i(k) \) denotes the following morphism:

\[
\begin{array}{ccc}
X & \xrightarrow{\alpha_{i,k}} & i^* \\
\otimes & \xleftarrow{\text{coev}} & \\
Y & \otimes & I_{i,F(X \otimes Y)} \\
\end{array}
\]

Similarly, define the natural transformation \( GF \overset{q}{\to} 1 \) to be the sum

\[
q = q_{X \otimes Y} := \sum_{i \in O(C)} q_i := \sum_{i \in O(C)} \frac{1}{\sqrt{\dim(i)}} \sum_{k=1}^{\|X[i]\|} q_i(k),
\]

where \( q_i(k) \) denotes the following morphism:

\[
\begin{array}{ccc}
i^* & \xrightarrow{\alpha^k_i} & X \\
\otimes & \xleftarrow{\text{coev}} & \\
Y & \otimes & I_{i,F(X \otimes Y)} \\
\end{array}
\]

Notice that while \( d_i(k) \) and \( q_i(k) \) depend on the choice \( X[i] \), the morphisms \( d_i \) and \( q_i \) do not.

**Definition 2.4 (The transformations \( b \) and \( p \))** To construct the natural transformation \( 1 \overset{b}{\to} FG \), it suffices to construct a morphism in \( Z(C) \) from each object \( (X, \gamma) \) to \( FG(X, \gamma) = \bigoplus_{i \in O(C)} i^* \otimes \gamma \otimes X \). We thus define

\[
b = b_{(X, \gamma)} := \sum_{i \in O(C)} b_i
\]

where \( b_i \) denotes the product of \( \sqrt{\dim(i)} \) and the following morphism:
Similarly, define the natural transformation $F G \xrightarrow{p} 1$ to be the sum

$$p = p_{(X, \gamma)} := \sum_{i \in O(C)} p_i$$

where $p_i$ denotes the product of $\sqrt{\text{dim}(i)}$ and the following morphism:

$$i^* \otimes \rightarrow i^* \otimes \left\{ \begin{array}{c} X \otimes \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! 

It requires some effort to check that so defined transformations $b$ and $p$ are indeed morphisms in $Z(C)$. We prove that in the following lemma.

**Lemma 2.5** Given an $Z(C)$-object $(X, \gamma)$, the definition of the natural transformations $b = b_{(X, \gamma)}$ and $p = p_{(X, \gamma)}$ are indeed morphisms in $Z(C)$.

**Proof.** By the definition of $Z(C)$, it suffices to show that $b$ and $d$ respect the half-braidings $\gamma$ and $c \otimes c^{-1}$. We provide a graphical proof for this fact.

Since any proof for $b$ also works similarly for $p$, so we shall only prove for $b$. By definition of $Z(C)$, it suffices to prove the following equality (functorial in $Z \in \text{Obj}(C)$)

$$\begin{array}{ccc}
Z & \xrightarrow{i} & X \\
\downarrow & \searrow & \downarrow \searrow \\
\searrow & & \searrow \\
\downarrow & \searrow & \downarrow \searrow \\
Z & \xrightarrow{i} & X \\
\end{array}$$

However, this follows directly from the tensoriality of the half-braiding $\gamma$ and the sliding lemma 1.13.
2.2 Statement & Proof

We state our main theorem in this paper and will provide a proof after a few lemmas.

Theorem 2.6 (Main Theorem) If \( C \) is modular, then the functor \( G \) is a factorization of the Drinfeld center \( Z(C) \). More precisely, \( G \) is an inverse functor for \( F \) witnessed by the natural transformations \( b, d, p, q \). 

To prove the main theorem, we first observe an easy lemma. Note that the following lemma does not assume modularity.

Lemma 2.7 Let \( X \boxtimes Y \) be an object in \( C \boxtimes C^{\text{bop}} \). Then the morphism

\[
X \boxtimes Y \xrightarrow{q \circ d} X \boxtimes Y
\]

is equal to the identity morphism \( \text{id}_{X \boxtimes Y} \).

Proof. We prove the equality by direct computation.

\[
q \circ d = \left( \sum_i q_i \right) \circ \left( \sum_j d_j \right)
= \sum_i q_i \circ d_i
= \sum_i \sum_{k=1}^{\text{dim}(i)} \sum_{r=1}^{\text{dim}(i)} \frac{1}{\text{dim}(i)} q_i(k) \circ d_i(r)
= \sum_i \sum_{k=1}^{\text{dim}(i)} \frac{1}{\text{dim}(i)} q_i(k) \circ d_i(k)
= \frac{1}{\text{dim}(\Omega)} \sum_i \sum_{k=1}^{\text{dim}(i)} \frac{1}{\text{dim}(i)} \left( X \overset{\alpha_i}{\leftarrow} X \right)

= \text{id}_{X \boxtimes Y}
\]

The first pair of sums collapse to a single sum because \( \text{Hom}_C(i, j) \) is zero unless \( i = j \). The second pair of sums collapse to a single sum by the simplicity of \( i \) and the definition of the pairing between \( \text{Hom}_C(X, i^*) \) and \( \text{Hom}_C(i^*, X) \).

Lemma 2.9 Suppose \( C \) is modular. Let \( X \boxtimes Y \) be an object in \( C \boxtimes C^{\text{bop}} \). Then \( (d \circ q)_{X \boxtimes Y} \) is equal to the identity isomorphism \( \text{id}_{GF(X \boxtimes Y)} \).

\[
\text{Diagram:}
\]

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Proof. Recall that the image of $X \boxtimes Y$ under $\text{GF}$ is
\[
\bigoplus_{i \in \Omega(C)} i^* \boxtimes I_i,
\]
where $I_i$ denotes $I_{i, F(X \boxtimes Y)}$. We will prove the equality by direct computation.

\[
d \circ q = \left( \sum_i d_i \right) \circ \left( \sum_j q_j \right) \\
= \sum_i d_i \circ q_i \\
= \sum_i \sum_{k=1}^{|X[i]|} \sum_{r=1}^{|X[i]|} \frac{1}{\text{dim}(i)} d_i(r) \circ q_i(k) \\
= \sum_i \sum_{k=1}^{|X[i]|} \frac{1}{\text{dim}(i)} d_i(k) \circ q_i(k) \\
= \frac{1}{\text{dim}(\Omega)} \sum_i \sum_{k=1}^{|X[i]|} \left( \begin{array}{c} \\
\otimes \\
i \subseteq X \\
\alpha_{i, k} \\
\rightarrow Y \\
\end{array} \right) \\
= \text{id}_{\text{GF}(X \boxtimes Y)}
\]

The pairs of sums collapse as in the proof of the last lemma. The cut skeins are connected and protected by a $\Omega$-circle by lemma 1.14. ■

Lemma 2.11 Suppose $C$ is modular. Let $(X, \gamma)$ be an object in $Z(C)$. Then $(p \circ b)_{(X, \gamma)}$ is equal to the identity isomorphism $\text{id}_{(X, \gamma)}$. ☐

Proof. It is not hard to check that $(p \circ b)_{(X, \gamma)}$ is equal to the product of $\frac{1}{\text{dim}(\Omega)}$ and the following morphism:

\[
\begin{array}{c}
\Omega \\
\gamma \\
\end{array}
\]

By lemma 1.14, the horizontal $\Omega$ kills off all nontrivial components in the vertical $\Omega$ providing the desired equality. ■

Lemma 2.12 Suppose $C$ is modular. Let $(X, \gamma)$ be an object in $Z(C)$. Then $(b \circ p)_{(X, \gamma)}$ is equal to the identity isomorphism $\text{id}_{F(G(X, \gamma))}$. ☐

Proof. We prove the equality by direct computation.
The tensoriality of $\gamma$ allows the left circle to attach on the right; thus follows the first equality. The sliding lemma allows us to slide one strand to the background, and then again the tensoriality of $\gamma$ allows detachment; thus follows the second equality. Finally, we sear the two strands and use lemma 1.14 to smooth it out; thus follows the third equation.

3 Discussion & Prospect

Using the topological insight from the Crane-Yetter TQFT, we provided an explicit equivalence between $\mathbb{C} \boxtimes \mathbb{C}^\text{bop}$ and $Z(C)$ for modular categories $C$. With the same idea, we can also provide explicit equivalences (and witnessing natural isomorphisms) between the categorical centers of higher genera $Z_\Sigma(C)$ [Guu21] and $C^n$, where $\Sigma$ is an oriented surface with $n$ punctures. In particular, this provides an explicit equivalence between $C$ and the elliptic Drinfeld center $Z_{\text{el}}(C)$ [Tha19].

We stress again that this only works in the case where $C$ is modular. This happens for a good reason. Over modular categories, the Crane-Yetter theory is expected to trivialize to the Witten-Reshetikhin-Turaev theory by taking boundaries. It is interesting to investigate the situation where $C$ is not modular. In fact, this is the motivation of the current paper. We expect that by measuring how the adjoint functors $F$ and $G$ fail to be an inverse of the other, the difference between both theories will become clear, leading to a better understanding of the full power of the Crane-Yetter theory. Moreover, this will also help understand the structures of the categorical center of higher genera (note that the Drinfeld center is hard enough).

One way to attack this problem is to look for a general tool in category theory that measures the failure of the invertibility of a pair of adjoint functors. Unfortunately, the authors have not found such tools yet. Another way is to analyze how the invertibility fails for explicit non-modular categories. Such genuine premodular categories arise from (super) groups, crossed modules, and the even part of the semisimplification of $\text{Rep}(U_q(sl_2))$ [KO01].

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