Effect of disorder on a Pomeranchuk instability

A. F. Ho\textsuperscript{1} and A. J. Schofield\textsuperscript{2}

\textsuperscript{1} Department of Physics, Royal Holloway, University of London - Egham, Surrey TW20 0EX, UK, EU
\textsuperscript{2} School of Physics and Astronomy, University of Birmingham - Birmingham B15 2TT, UK, EU

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Abstract – We study the effect of weak and dilute disorder on the order parameter equation and transition temperature of a Pomeranchuk-type Fermi-surface instability using replica mean-field theory. We consider the example of a phase transition to a $d_{x^2-y^2}$ type Fermi surface distortion, and show that, in the regime where such a transition is second order, the transition temperature is reduced by disorder in essentially the same way as that for a $d$-wave superconductor. We argue that observing this disorder dependence of metal-to-metal transition is a useful indicator of a finite angular momentum Fermi surface distortion.

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Introduction. – In recent years a growing number of correlated phase systems have been found to exhibit thermodynamic phase transitions between metallic states. Examples include the 17K transition in URu$_2$Si$_2$ [1] and the transitions around the metamagnetic quantum critical endpoint of Sr$_3$Ru$_2$O$_7$ [2]. In contrast to the more familiar superconducting or magnetic instabilities, the order parameter which presumably develops at these transitions appears to be transparent or only weakly coupled to most experimental probes. Thus identifying the nature of this “dark” or “hidden” order is a challenging problem. Many years ago Pomeranchuk [3] found a condition for instabilities between metallic states characterized by Fermi surface shape distortions. It has been argued that this instability occurs in quantum Hall systems [4,5] and is the origin of the transitions in URu$_2$Si$_2$ [6] and Sr$_3$Ru$_2$O$_7$ [2]. However, the key question remains how to identify this sort of order—particularly if bulk changes are masked by domains formation. In this letter we calculate the form of the disorder dependence of the transition temperature and find it to have a characteristic signature of momentum space distortions of the metallic Fermi surface.

We are motivated by the empirical similarity in the way the mysterious phase in the bilayer ruthenate, Sr$_3$Ru$_2$O$_7$ [2], and the superconductivity in the related single-layer compound, Sr$_2$RuO$_4$ [7], are both eliminated with very low levels of disorder. In the latter case quantitative comparison of the strong disorder dependence of the superconducting transition temperature $T_c$ to the well-known form [8] has become the de-facto signature of non-zero orbital angular momentum pairing (in the absence of phase-sensitive methods). In contrast $s$-wave superconductors are insensitive to disorder [9]. Could a similar dependence be used to diagnose the Pomeranchuk transition?

In this letter we show how the Pomeranchuk instability is an analogue of non-$s$-wave superconductivity but in the particle-hole rather than particle-particle channel. We exploit this using a combination of standard methods to show that, not only might one similarly expect a sensitivity to disorder but that the precise form of the disorder dependence of $T_c$ is, under certain circumstances, identical to that of unconventional superconductors. This provides a quantitative test of the “dark order” metallic phase which parallels that now used for unconventional superconductivity. Moreover, since we show that the Pomeranchuk instability is strongly suppressed in the presence of weak disorder, our results may provide an explanation as to why this rather subtle metal-to-metal transition is not observed more often in nature.

Model. – To study effects of disorder in a simple model of a Pomeranchuk instability, we consider electrons on a two-dimensional (2D) tight-binding lattice with a quadrupolar interaction that has been studied extensively [10–13]

$$H_{\text{int}} = \sum_{\mathbf{k}, \mathbf{p}, \mathbf{q}} \frac{1}{2} V_q(\mathbf{k}, \mathbf{p}) \psi^\dagger(\mathbf{k}+\mathbf{q}) \psi(\mathbf{k}) \psi^\dagger(\mathbf{p}-\mathbf{q}) \psi(\mathbf{p}).$$

(1)
Here $\psi_k^\dagger = [\psi_k^\dagger \psi_k^\dagger]$ is the spinor creation operator. The interaction explicitly has angular momentum dependence: $V_q(k, p) = g_0 \delta_{k+q} \phi_p$, where the $d_{2−y^2}$ form factor $\phi_k = \cos k_x - \cos k_y$. Kee et al. [11] found that the $d_{xy}$ component of the quadrupolar interaction usually does not acquire an expectation value, which we thus drop.

We add a weak, dilute disorder potential that couples to the electron density $H_{\text{dis}} = \int d^3 x \hat{\phi}(x) \hat{\phi}(x)$ assuming, for simplicity, delta-correlated, static (quenched) disorder $Pr(\xi) \propto \exp - \int d^3 x \xi^2(x)/2D$. Here $D = 1/2\pi N_0 \tau$ is the strength of the disorder potential, with $N_0$ the density of state at the Fermi surface, and $\tau$ is the disorder scattering time. We do not here consider stronger or correlated disorder, where the disorder potential may couple directly to the Fermi surface distortion (Pomeranchuk) order parameter [14].

Methods. – For quenched disorder, one needs to disorder-average the free energy instead of the partition function. One standard method that works also for interacting systems is the replica trick [15, 16], based on the identity: $\ln Z = \lim_{n \to 0} (Z^n - 1)/n$. Note that we have also derived the results presented using diagrammatic perturbation theory [17] but we found that the replica method makes the parallel with unconventional superconductivity explicit. The idea is to replicate $n$ copies of the partition function $Z$, disorder-average $Z^n$, and finally take the limit $n \to 0$ to get the disorder-averaged free energy. Since we have taken a simple Gaussian distribution for the disorder potential, this disorder-average is readily done to give the disorder-induced interaction 4-fermion term (in momentum representation)

$$S_{\text{dis}} = \frac{-1}{4\pi N_0 \tau} \sum_{\alpha, \beta} T^2 \sum_{n, m, k, k} \sum_{k_1, k_2, k_3, k_4} \delta_{k_1+k_3-k_2-k_4} \psi_n^\dagger (k_1) \psi_n^\dagger (k_2) \psi_m^\dagger (k_3) \psi_m^\dagger (k_4),$$

where the subscript $n, m$ on the electron operators denote the Matsubara frequencies $\omega_n, i\omega_m$, also $\sum_{n, m}$ refers to Matsubara frequencies summation, and $\alpha, \beta = 1, \ldots n$ are replica indices.

To derive a low-energy effective theory, we follow Belitz and Kirkpatrick [16] and consider the disorder-induced interaction with all momenta near the Fermi surface. There are three possible ways of pairing up the scattering: 1) is the small angle (or direct) scattering with $k_2 = k_1 + q$, 2) the large angle (or exchange) scattering with $k_4 = k_1 + q$, and 3) is the pair (or $2k_F$) scattering where $k_3 = k_1 + q$. The momentum transfer $q$ is now restricted to be small (with a cut-off much smaller than the Fermi momentum). It can be shown that type 1) only leads to a renormalization of the chemical potential and we drop it from now on. Type 3) couples to the superconducting order parameter, but not to the Pomeranchuk one, and we can show that this term does not have any effect on the Pomeranchuk order at mean field level so we neglect it. However, for a superconductor, the type 3) term generates a vertex correction that for an s-wave superconductor cancels the propagator correction due to type 2) term, thereby rendering it insensitive to disorder (Anderson’s theorem) [17]. Thus, the disorder-induced interaction is

$$S_{\text{dis}} = \frac{-1}{4\pi N_0 \tau} \sum_{\alpha, \beta} \sum_{n, m, k, p, q} T^2 \sum_{\alpha, \beta} \sum_{n, m, k, p, q} \psi_n^\dagger (k) \psi_n^\dagger (p) \psi_n (p + q) \psi_m (k + q).$$

The full low-energy effective action after disorder-averaging is thus $S = \sum_{\alpha} (S_0^\alpha + S_{\text{int}}^\alpha) + S_{\text{dis}}$ with

$$S_0^\alpha = \sum_k T \sum_n \psi_n^\dagger (k)(-i\omega_n + \epsilon_k) \psi_n (k),$$

$$S_{\text{int}}^\alpha = \sum_{n_1, n_2, m} \sum_{k, k'} T^3 \sum_{\alpha, \beta} \sum_{n, m} \psi_n^\dagger (k) \psi_m^\dagger (p) \psi_n (p + q) \psi_m (k + q).$$

To decouple the four-fermion interaction terms, we introduce the $Q$-matrix via essentially a Hubbard-Stratonovich transformation (generalizing ref. [18])

$${Q_{n_{\alpha}}^{\beta}} = \left[ \psi_n^\dagger (k) \right]_1 \left[ \psi_m (p) \right]_1,$$

where $\alpha, \beta = 1, \ldots n$.

Since the replica-$Q$-matrix method has already been comprehensively reviewed in refs. [16,18], we here only sketch out its application to the Pomeranchuk instability in the presence of weak quenched disorder. Assuming that at the saddle point, there is replica symmetry and spin symmetry, the homogeneous and un-retarded ansatz for the saddle-point of this action is

$$\left[ Q_{n_{\alpha}}^{\beta} \right]_1 = \delta_{\alpha, \beta} \delta_{n, m} \delta_{kp} \delta_{ij} Q_{nk}.$$

Note that only one function $Q_{nk}$ is needed here (unlike for magnets or superconductors), because both disorder and the quadrupolar interaction induces a self-energy $i\Lambda_{nk}$ that enters in the same way into the propagator renormalization. $\Lambda_{nk}$ is the Fourier transform dual of $Q_{nk}$ [18] and thus has the same structure as $Q_{nk}$.

With the ansatz eq. (7), the saddle point action becomes

$$S_{\text{sp}} = -\text{Tr} \ln [-i\omega_n + \epsilon_k + i\Lambda_{nk}] - 2iT \sum_{nk} \Lambda_{nk} Q_{nk}$$

$$+2 \sum_{kp} V_0(k, p) T^2 \sum_{nm} Q_{nk} Q_{nm}$$

$$+ \frac{1}{2\pi N_0 \tau} \sum_{nk} Q_{nk} Q_{np}.$$

Note that only the $q = 0$ component of $V_0(k, p)$, i.e. $V_0(k, p)$ matters, because of the assumption of spatial
homogeneity in the saddle point ansatz $Q_{n\mathbf{k}}$. Also we have assumed replica symmetry: we can a posteriori justify this by noting that as we shall see, there are no indications in the free energy of further instabilities in the Fermi-surface distorted phase, unlike in the classical spin glass case which does demand replica symmetry breaking. Presumably this has to do with the much simpler free-energy landscape in the Pomeranchuk case, indicating lack of glassiness in our system. Replica symmetry means we can drop the replica indices from now on. The saddle point equations $\delta S_{\text{sp}}/\delta Q_{n\mathbf{k}} = 0$ and $\delta S_{\text{sp}}/\delta \Lambda_{n\mathbf{k}} = 0$ give

$$Q_{n\mathbf{k}} = \frac{1}{i\omega_n - \epsilon_k - i\Lambda_{n\mathbf{k}}},$$

$$i\Lambda_{n\mathbf{k}} = \frac{1}{2\pi N_0\tau} \sum_{\mathbf{p}} Q_{n\mathbf{p}} + \sum_{\mathbf{p}} V_0(k\mathbf{p}) T \sum_{m} Q_{m\mathbf{p}}. \tag{10}$$

First, let’s check that the ansatz eq. (7) recovers known results. For free electrons with quenched disorder, setting $V_0(k,\mathbf{p}) = 0$ leads to the standard Born approximation result; at the saddle point, $Q_{n\mathbf{k}}$ is just the electron propagator with a disorder-induced lifetime $\tau$: $Q_{n\mathbf{k}} \approx g_0 = [i\omega_n - \epsilon_k + \frac{q}{2\pi} \text{sgn}(\omega_n)]^{-1}$. For clean electrons with the quadrupolar interaction, setting $\tau \to \infty$, and assuming a spatially homogeneous order parameter (i.e., only the $\mathbf{q} = 0$ component of $V_q(k,\mathbf{p})$ is involved), we define the Pomeranchuk order parameter

$$\phi_k \Delta_0 = T \sum_{\mathbf{p}} V_0(k,\mathbf{p}) \left( \psi_{\mathbf{p}}^{\dagger} \psi_{\mathbf{p}} \right) = 2\phi_k T \sum_{\mathbf{p}} g_{\mathbf{p}\mathbf{p}} Q_{\mathbf{p}\mathbf{p}}. \tag{11}$$

We then recover the clean case mean-field order parameter equation $\Delta_0 = 2g \sum_k \phi_k f_T(\epsilon_k + \phi_k \Delta_0)$, where $f_T(x) = [\exp x/T + 1]^{-1}$ is the usual Fermi distribution.

**Results.** – Now we consider the case of weakly disordered electrons with an interaction favoring a Pomeranchuk instability. Equations (11), (9), (10) lead to the order parameter equation

$$\Delta_0 = 2gT \sum_k \phi_k \frac{1}{i\omega_n - \epsilon_k - i\Lambda_{n\mathbf{k}}} \tag{12}.$$

This is in fact the non-s-wave, non-magnetic analogue of the Stoner instability of a ferromagnet. The extra angular dependence in the momentum sum means that this order parameter equation does not reduce to the clean case. By contrast, the s-wave Stoner Pomeranchuk instability is unaffected, to leading order, by impurities [19]). Thus, just as for non-s-wave superconductors, there is no Anderson’s theorem for $l \neq 0$ Pomeranchuk instabilities, because of its angular dependence in momentum space. Equation (12) simplifies to

$$\Delta_0 = g \frac{2}{\pi} \sum_k \phi_k \text{Im} \psi \left( \frac{1}{2} + \frac{1}{4\pi T} \epsilon_k + \phi_k \Delta_0 \right). \tag{13}$$

As expected, disorder smears out the Fermi distribution to give the digamma function $\psi(x)$; crudely speaking, disorder raises the effective temperature.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig1.png}
\caption{The effect of disorder on the magnitude of the order parameter $\Delta$ (all energies and scattering rates expressed in units of 2$\tau$). (a) At $T = 0$ as disorder decreases the lifetime, the critical coupling, $g/2\pi^2$ increases and a first-order transition becomes second order. (b) At fixed coupling, the order parameter and $T_c$ are rapidly suppressed by weak disorder.}
\end{figure}

In contrast to weak-coupling superconductivity where there is always a second-order transition for arbitrary weak interactions, the Pomeranchuk instability requires a critical coupling and the transition can either be first or second order. To check the order of the transition, we need to evaluate the free energy. Substituting the mean-field eqs. (9), (10) into the saddle point action eq. (8), together with the approximation (just as for the disordered free electrons case) $\sum_{\mathbf{p}} Q_{\mathbf{p}} \approx -i\pi N_0 \text{sgn}(\omega_n)$ and the definition of the order parameter equation (11), the saddle point free energy becomes

$$F_{\text{sp}} = -\text{Tr} \ln \left[ -i(\omega_n + \frac{\text{sgn}(\omega_n)}{2\tau}) + \phi_k - \mu \right] - \frac{\Delta_0^2}{2g} + \text{const},$$

where the renormalized dispersion $\tilde{\epsilon}_k = \epsilon_k + \mu + \Delta_0 \phi_k$. Defining the renormalized density of state of state $N_\Delta (\epsilon) = \sum_k \delta(\epsilon - \tilde{\epsilon}_k)$, the order parameter equation (12) becomes

$$\frac{\Delta_0}{g} = 4T \int_{-4t}^{4t} \frac{d\epsilon}{\partial N_\Delta (\epsilon)} \text{Re} \ln \Gamma \left( \frac{1}{2} + \frac{1}{4\pi T} + i \frac{\epsilon}{2\pi T} \right). \tag{14}$$

For quantitative results for effects of disorder, we evaluate the free energy and mean-field equations numerically for the 2D square lattice with bare dispersion $\epsilon_k = -2t[\cos k_x + \cos k_y] - \mu$, and the d-wave form factor: $\phi_k = \cos k_x - \cos k_y$, with the corresponding renormalized density of states (see refs. [11,12] for the actual form). In the following, energies and scattering rates, $\tau^{-1}$, are measured in units of $2t$.

First we consider how disorder changes the evolution of the order parameter. In fig. 1(a) we see for fixed chemical potential, $\mu$, how the Pomeranchuk order parameter is modified at $T = 0$ by scattering. Disorder both increases the critical coupling and turns at $T = 0$ from first order in the clean limit, to second order. In fig. 1(b) we see that rather weak disorder dramatically reduces $T_c$.

We next consider the disorder dependence of the Pomeranchuk transition temperature, $T_c$. In fig. 2(a) we choose a parameter region where the clean transition is second
order \((g/2\pi^2 = 0.051\) and \(\mu/2t = 0, 0.05\)). Since the order parameter goes smoothly to zero, we can simplify the order parameter equation (eq. (12)) to get the second-order transition order parameter equation

\[
-\frac{1}{g} = T \sum_{n,k} \phi_n^2 \left( \omega_n + \frac{1}{2\pi^2 g \mu n \omega_n} \right)^2 + \phi_k^2.
\]  

We then note that this order parameter equation is identical to the one determining the critical temperature for \(d\)-wave superconductor with non-magnetic disorder. Note that this is true only for the \(T_c\) equation for a second-order transition: there are first-order transitions at larger \(\mu\), and furthermore, the full order parameter equation for the disordered Pomeranchuk instability has a different form to the \(d\)-wave superconductor with disorder.

Thus, from eq. (15), we get the familiar Abrikosov-Gor’kov form [20] for the disordered gap equation

\[
\ln \left( \frac{T_{c0}}{T_c} \right) = \psi \left( \frac{1}{2} + \frac{1}{4\pi T_c \tau} \right) - \psi \left( \frac{1}{2} \right).
\]  

In figs. 2(a) and (b), the solid curve is this universal Abrikosov-Gor’kov form, while the small points are direct numerical evaluation of the general \(i.e.\) not just for second-order transition) order parameter equation (14). At \(\mu/2t = 0\), the direct evaluation coincide with the universal form, while for finite chemical potential \(\mu/2t = 0.05\) shown in fig. 2(b), some small deviation can be seen at larger disorder, due to the approximation in the \(k\)-integral that goes into deriving eq. (16).

We have also taken existing data of on \(Sr_3Ru_2O_7\) at 7.95\(T\) [2,21] where the resistivity minimum and the zero-field residual resistivity are taken to indicate \(T_c\) and \(1/\tau\), respectively, and plotted them as large dots in fig. 2(a). The reasonable fit shows that the putative transition in \(Sr_3Ru_2O_7\) (which experimentally, is found not to be of superconductivity type) does follow the universal Abrikosov-Gor’kov form for disordered Pomeranchuk transition, even if our actual model interaction of eq. (1) may be too simplistic.

Finally we consider the disorder dependence on the transition temperature where, at low temperatures, the transition can become first order [22] such as when the system is further away from half-filling, \textit{e.g.} with \(\mu = 0.10\) (fig. 3). Then, with larger disorder (larger \(1/\tau\)), the transition turns from second to first order (bold line). Surprisingly, the effect of increasing disorder is opposite to increasing \(T\). Higher \(T\) smears out the Fermi function and leads to a smaller order parameter and eventually a second-order transition results [12]. Of course, it is possible that including Gaussian fluctuations around the saddle point may turn first-order transitions into second-order ones in the presence of disorder. Future work is needed to resolve this issue. What is clear is that even at the mean-field level, there is a strong suppression of \(T_c\) for the Pomeranchuk instability with increasing disorder.

In summary, we have calculated the strong dependence of the \(d\)-wave Fermi surface distortion transition on dilute disorder and shown that the effect is reminiscent both qualitatively and quantitatively of the strong dependence on impurities of the order parameter and \(T_c\) in non-\(s\)-wave superconductors. Our results suggest that detailed disorder dependence of “hidden order” transitions could be used to indicate Pomeranchuk type order just as is the case for low \(T_c\) unconventional superconductors. The extreme sensitivity to disorder [2] of the low-temperature metal-to-metal transition in \(Sr_3Ru_2O_7\) is suggestive of this (see fig. 2(a)). Our theoretical results are intended to motivate a more comprehensive systematic experimental study to compare with our quantitative predictions for the putative ordered state in \(Sr_3Ru_2O_7\).
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