Nonexistence of a few binary orthogonal arrays

Peter Boyvalenkov
Institute of Mathematics and Informatics,
Bulgarian Academy of Sciences,
8 G. Bonchev Street, 1113, Sofia, BULGARIA
and Faculty of Mathematics and Natural Sciences,
South-Western University, Blagoevgrad, Bulgaria.
email: peter@math.bas.bg

Tanya Marinova, Maya Stoyanova
Faculty of Mathematics and Informatics, Sofia University,
5 James Bourchier Blvd., 1164 Sofia, BULGARIA
email: tanya.marinova@fmi.uni-sofia.bg
email: stoyanova@fmi.uni-sofia.bg

Abstract

We develop and apply combinatorial algorithms for investigation of the feasible distance distributions of binary orthogonal arrays with respect to a point of the ambient binary Hamming space utilizing constraints imposed from the relations between the distance distributions of connected arrays. This turns out to be strong enough and we prove the nonexistence of binary orthogonal arrays of parameters \((9, 6^4 = 96, 4), (10, 6^3.5), (10, 7^2 = 112, 4), (11, 7^2.5), (11, 7^3.4)\) and \((12, 7^5.5)\), resolving the first cases where the existence was undecided so far. For the existing arrays our approach allows substantial reduction of the number of feasible distance distributions which could be helpful for classification results (uniqueness, for example).

Keywords. Binary Hamming space orthogonal arrays Krawtchouk polynomials distance distributions nonexistence

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1 Introduction

Orthogonal arrays have many connections to other combinatorial designs and have applications in coding theory, the statistical design of experiments, cryptography, various types of software testing and quality control. We refer to the book [4] as excellent exposition of the theory and practice of orthogonal arrays. In fact, there are enormous material about orthogonal arrays in internet.

An orthogonal array (OA) of strength $\tau$ and index $\lambda$ in $H(n, 2)$ (or binary orthogonal array, BOA), consists of the rows of an $M \times n$ matrix $C$ with the property that every $M \times \tau$ submatrix of $C$ contains all ordered $\tau$-tuples of $H(\tau, 2)$, each one exactly $\lambda = M/2^\tau$ times as rows.

Let $C \subset H(n, 2)$ be an $(n, M, \tau)$ BOA. The distance distribution of $C$ with respect to $c \in H(n, 2)$ if the $(n+1)$-tuple

$$w = w(c) = (w_0(c), w_1(c), \ldots, w_n(c)),$$

where $w_i(c) = |\{x \in C | d(x, c) = i\}|$, $i = 0, \ldots, n$. All feasible distance distributions of BOA of parameters $(n, M, \tau)$ can be computed effectively for relatively small $n$ and $\tau$ as shown in [1]. Indeed, every distance distribution of $C$ satisfies the system

$$\sum_{i=0}^{\tau} w_i(c) \left(1 - \frac{2i}{n}\right)^k = b_k |C|, \quad k = 0, 1, \ldots, \tau, \quad (1)$$

where $b_k = \frac{1}{2^n} \sum_{d=0}^{n} \binom{n}{d} \left(1 - \frac{2d}{n}\right)^k$ and, in particular, $b_k = 0$ for $k$ odd.

The number $b_k$ is in fact the first coefficient in the expansion of the polynomial $t^k$ in terms of (binary) Krawtchouk polynomials. The Krawtchouk polynomials are zonal spherical functions for $H(n, 2)$ (see [3, 6, 7]) and can be the defined by the three-term recurrence relation

$$(n-k)Q_{k+1}^{(n)}(t) = ntQ_k^{(n)}(t) - kQ_{k-1}^{(n)}(t) \quad \text{for } 1 \leq k \leq n-1,$$

with initial conditions $Q_0^{(n)}(t) = 1$ and $Q_1^{(n)}(t) = t$.

Let $n$, $M$ and $\tau \leq n$ be fixed. We denote by $P(n, M, \tau)$ the set of all possible distance distributions of a $(n, M, \tau)$ BOA with respect to internal point $c$ (in the beginning – all admissible solutions of the system (1) with $w_0(c) \geq 1$) and by $Q(n, M, \tau)$ the set of all possible distance distributions of a $(n, M, \tau)$ BOA with respect to external point (in the beginning – all admissible solutions of the system (1) with $w_0(c) = 0$). Denote also $W(n, M, \tau) = P(n, M, \tau) \cup Q(n, M, \tau)$.  


In this paper we describe an algorithm which works on the sets $P(n, M, \tau)$, $Q(n, M, \tau)$ and $W(n, M, \tau)$ utilizing connections between related BOAs. During the implementation of our algorithm these sets are changed by ruling out some distance distributions.

In Section 2 we prove several assertions which connect the distance distributions of arrays under consideration and their relatives. This imposes significant constraints on the targeted BOAs and therefore allows us to collect rules for removing distance distributions from the sets $P(n, M, \tau)$, $Q(n, M, \tau)$ and $W(n, M, \tau)$. The logic of our algorithm is described in Section 3. The new nonexistence results are described in Section 4.

Algorithms for dealing with distance distributions were proposed earlier in [1] and [2] but in these papers the set $P(n, M, \tau)$ was only examined. Moreover, two seemingly crucial observations (Theorem 1 together with Corollary 2 and Theorem 13 together with Corollary 13) are new. Also, all complete versions (for the set $W(n, M, \tau)$) of the remaining assertions from the next section are new.

2 Relations between distance distributions of $(n, M, \tau)$ BOA and its derived BOAs

We start with a simple observation.

**Theorem 1.** If the distance distribution $w = (w_0, w_1, \ldots, w_n)$ belongs to the set $W(n, M, \tau)$, then the distance distribution $\overline{w} = (w_n, w_{n-1}, \ldots, w_0)$ also belongs to $W(n, M, \tau)$.

**Proof.** Let $C \subset H(n, 2)$ be a BOA of parameters $(n, M, \tau)$ and $\overline{C}$ is the array which is obtained from $C$ by the permutation $(0 \rightarrow 1, 1 \rightarrow 0)$ in the whole $C$. Since the distances inside $C$ are preserved by this transformation, $\overline{C}$ is again $(n, M, \tau)$ BOA. On the other hand, distance $i$ from external for $C$ point to a point of $C$ correspond to distance $n - i$ to the transformed point of $\overline{C}$. This means that if $w = (w_0, w_1, \ldots, w_n)$ is the distance distribution of $C$ with respect to some point $c \in H(n, 2)$ (internal or external for $C$), then the distance distribution of $\overline{C}$ with respect to the same point (which can become either internal or external for $\overline{C}$, depending on whether $w_n > 0$ or $w_n = 0$) is $\overline{w} = (w_n, w_{n-1}, \ldots, w_0)$. $\square$

**Corollary 2.** The distance distribution $w = (w_0, w_1, \ldots, w_n) \in W(n, M, \tau)$ is ruled out if $\overline{w} = (w_n, w_{n-1}, \ldots, w_0) \notin W(n, M, \tau)$.

\(^1\)However, we prefer to keep the initial notation in order to avoid tedious notation.
Corollary 2 is important in all stages of our algorithm since it requires
the non-symmetric distance distributions to be paired off and infeasibility of
one element of the pair immediately implies the infeasibility for the other.

We proceed with analyzing relations between the BOA $C$ and BOAs
$C'$ of parameters $(n-1,M,\tau)$ which are obtained from $C$ by deletion of
one of its columns. Of course, the set $W(n-1,M,\tau)$ of possible distance
distributions of $C'$ is sieved by Corollary 2 as well.

It is convenient to fix the removing of the first column of $C$. Let the
distance distribution of $C$ with respect to $c=0=(0,0, \ldots, 0) \in H(n,2)$
be $w=(w_0,w_1,\ldots,w_n) \in W(n,M,\tau)$ and the distance distribution of $C$
with respect to $c'=(0,0, \ldots, 0) \in H(n-1,2)$ be $w'=(w'_0,w'_1,\ldots,w'_{n-1}) \in W(n-1,M,\tau)$.

For every $i \in \{0,1,\ldots,n\}$ the matrix which consists of the rows of $C$
of weight $i$ is called $i$-block. It follows from the above notations that the
cardinality of the $i$-block is $w_i$. Next we denote by $x_i$ ($y_i$, respectively)
the number of the ones (zeros, respectively) in the intersection of the first
column of $C$ and the rows of the $i$-block.

**Theorem 3.** The numbers $x_i$ and $y_i$, $i = 0,1,\ldots,n$, satisfy the following
system of linear equations

\[
\begin{align*}
x_i + y_i &= w_i, \quad i = 1,2,\ldots,n-1 \\
x_{i+1} + y_i &= w'_i, \quad i = 0,1,\ldots,n-1 \\
y_0 &= w_0 \\
x_n &= w_n \\
x_i, y_i \in \mathbb{Z}, \quad x_i \geq 0, \quad y_i \geq 0, \quad i = 0,1,\ldots,n
\end{align*}
\]

**Proof.** The equalities $x_i + y_i = w_i$, $i = 1,\ldots,n-1$, $x_n = w_n$ and $y_0 = w_0$
follow directly from the definition of the numbers $x_i$ and $y_i$. The relations
$x_{i+1} + y_i = w'_i$, $i = 0,1,\ldots,n-1$, connecting $w$ and $w'$, follow from the
fact that the rows of $C'$, which are at distance $i$ from $c'$, are obtained in
two ways: from the $y_i$ rows of $C$ at distance $i$ from $c$ and first coordinate 0,
and from the $x_{i+1}$ rows of $C$ at distance $i+1$ from $c$ and first coordinate 1.
\[\square\]

**Corollary 4.** The distance distribution $w=(w_0,w_1,\ldots,w_n) \in W(n,M,\tau)$
is ruled out if no system (2) obtained when $w'$ runs $W(n-1,M,\tau)$ has a
solution.

**Remark 5.** Theorem 3 was firstly proved and used in 2013 by Boyvalenkov-
Kulina \[\square\] for $w \in P(n,M,\tau)$.  

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Corollary 4 rules out some distance distributions \( w \) but it mainly serves to produce feasible pairs \( (w, w') \) which will be investigated further.

Our next step is based on the following property of BOAs: if we take the rows of \( C \) with first coordinate 0 (1, respectively) and remove that first coordinate then we obtain a BOA \( C_0 \) (\( C_1 \), respectively) of parameters \((n - 1, M/2, \tau - 1)\) (see Figure 1). At this stage the BOAs \( C_0 \) and \( C_1 \) have the same sets of admissible distance distributions – all those which have passed the sieves of Corollaries 2 and 4 for the set \( W(n - 1, M/2, \tau - 1) \).

Figure 1.
\[
\begin{array}{c|c}
0 & y = (y_0, y_1, \ldots, y_{n-1}) \\
\vdots & C_0 = (n - 1, M/2, \tau - 1) \\
0 & \\
1 & x = (x_1, x_2, \ldots, x_n) \\
\vdots & C_1 = (n - 1, M/2, \tau - 1) \\
1 & \\
\end{array}
\]

We continue with relations between the BOAs \( C, C', C_0 \) and \( C_1 \) using the numbers \( x_i \) and \( y_i \), \( i = 0, 1, \ldots, n \).

**Theorem 6.** The distance distribution of the \((n - 1, M/2, \tau - 1)\) BOA \( C_0 \) with respect to \( c' \) is \( y = (y_0, y_1, \ldots, y_{n-1}) \), i.e. \( y \in W(n - 1, M/2, \tau - 1) \).

**Proof.** The number \( y_i \) is equal to the number of the points of \( C_0 \) at distance \( i \) from the point \( c' \). \( \square \)

**Remark 7.** We have two possibilities in Theorem 6 – if \( y_0 \geq 1 \), then \( c' \in C_0 \) and therefore \( y \in P(n - 1, M/2, \tau - 1) \) (this is Theorem 1a) in [2], or \( y_0 = 0 \) when \( c' \not\in C_0 \) and therefore \( y \in Q(n - 1, M/2, \tau - 1) \).

**Corollary 8.** The pair \((w, w')\) is ruled out if \( y \not\in W(n - 1, M/2, \tau - 1) \) or if \( \overline{y} = (y_{n-1}, y_{n-2}, \ldots, y_0) \not\in W(n - 1, M/2, \tau - 1) \).

**Proof.** Follows from Theorem 6 and Corollary 2 for \( C_0 \). \( \square \)
Theorem 9. The distance distribution of the $(n-1, M/2, \tau - 1)$ BOA $C_1$ with respect to $c'$ is $x = (x_1, x_2, \ldots, x_n)$, i.e. $x \in W(n-1, M/2, \tau - 1)$.

Proof. The number $x_i$ is equal to the number of the points of $C_1$ at distance $i - 1$ from the point $c'$. □

Remark 10. Similarly to above, we have two possibilities in Theorem 9 - if $x_1 \geq 1$, then $c' \in C_1$ and therefore $x \in P(n-1, M/2, \tau - 1)$ (this is Theorem 2a) in [2], or $x_1 = 0$ when $c' \notin C_1$ and therefore $x \in Q(n-1, M/2, \tau - 1)$.

Corollary 11. The pair $(w, w')$ is ruled out if $x \notin W(n-1, M/2, \tau - 1)$ or if $\Xi = (x_n, x_{n-1}, \ldots, x_1) \notin W(n-1, M/2, \tau - 1)$.

Proof. Follows from Theorem 9 and Corollary 2 for $C_1$. □

In our next step we consider the effect of the permutation $(0 \to 1, 1 \to 0)$ in the first column of $C$. This transformation does not change the distances from $C$ and thus we obtain a BOA $C^{1,0}$ of parameters $(n, M, \tau)$ again.

Theorem 12. If the distance distribution of $C$ with respect to $c = 0 \in H(n, 2)$ is $w = (w_0, w_1, \ldots, w_{n-1}, w_n) = (y_0, x_1 + y_1, \ldots, x_{n-1} + y_{n-1}, x_n)$, then the distance distribution of $C^{1,0}$ with respect to $c$ is $\hat{w} = (x_1, x_2 + y_0, \ldots, x_n + y_{n-2}, y_{n-1})$, i.e. $\hat{w} \in W(n, M, \tau)$.

Proof. There are $x_i$ points in $C^{1,0}$ (coming from $C_1$) at distance $i - 1$ from $c$. Analogously, there are $y_i$ points in $C^{1,0}$ (coming from $C_0$) at distance $i + 1$ from $c$. This means that the number of the points of $C^{1,0}$ at distance 0 from $c$ is $x_1$, the number of the points of $C^{1,0}$ at distance 1, 1 \leq i \leq n - 1, from $c$ is $y_{i-1} + x_{i+1}$, and, finally, the number of the points of $C^{1,0}$ at distance $n$ from $c$ is $y_{n-1}$. Therefore the distance distribution of $C^{1,0}$ with respect to $c$ is $\hat{w} = (x_1, x_2 + y_0, \ldots, x_n + y_{n-2}, y_{n-1})$. □

Corollary 13. The pair $(w, w')$ is ruled out if $\hat{w} \notin W(n, M, \tau)$ or if $\Xi \notin W(n, M, \tau)$.

Corollary 14. The distance distribution $w$ is ruled out if all possible pairs $(w, w')$, $w' \in W(n-1, M, \tau)$, are ruled out.

Otherwise, we proceed with the remaining pairs as follows. Let

$$(x_0^{(j)} = 0, x_1^{(j)}, \ldots, x_n^{(j)}, y_0^{(j)}, y_1^{(j)}, \ldots, y_{n-1}^{(j)}, y_n^{(j)} = 0), \quad j = 1, \ldots, s, \quad (3)$$

are all solutions of Theorem 8 when $w'$ runs $W(n-1, M, \tau)$ which have passed the sieves of Corollaries 8, 11 and 13. We now free the cutting and
thus consider all possible \( n \) cuts of columns of \( C \). These cuts produce pairs \((w, w')\) (where \( w \) is fixed) and corresponding solutions (3). Let the solutions (3) appear with multiplicities \( k_1, k_2, \ldots, k_s \), respectively.

**Theorem 15.** The nonnegative integers \( k_1, k_2, \ldots, k_s \) satisfy the equations

\[
\begin{align*}
  k_1 + k_2 + \cdots + k_s &= n \\
  k_1 x_1^{(1)} + k_2 x_1^{(2)} + \cdots + k_s x_s^{(s)} &= w_1 \\
  k_1 x_2^{(1)} + k_2 x_2^{(2)} + \cdots + k_s x_s^{(s)} &= 2w_2 \\
  &\vdots \\
  k_1 x_n^{(1)} + k_2 x_n^{(2)} + \cdots + k_s x_s^{(s)} &= nw_n
\end{align*}
\]

(4)

**Proof.** This follows for counting in two ways the number of the ones in the \( i \)-block of \( C \). For fixed \( i \in \{1, 2, \ldots, n\} \), this number is obviously \( iw_i \), and, on the other hand, it is equal to the sum \( k_1 x_i^{(1)} + k_2 x_i^{(2)} + \cdots + k_s x_i^{(s)} \). \( \square \)

**Corollary 16.** The distance distribution \( w \) is ruled out if the system (4) does not have solutions.

**Corollary 17.** Let \( j \in \{1, 2, \ldots, s\} \) be such that all solutions of the system (4) have \( k_j = 0 \). Then the pair \((w, w')\), which corresponds to \( j \), is ruled out.

3 Our algorithm

We organize the results from the previous section to work together as follows.

All BOAs (in fact, their current sets of feasible distance distributions \( P, Q \) and \( W \)) of interest for the targeted BOA \( C = (n, M, \tau) \) are collected in a table starting with first row

\[(\tau, M, \tau) \ (\tau + 1, M, \tau) \ (\tau + 2, M, \tau) \ \ldots \ C = (n, M, \tau)\]

The next row consist of the derived BOAs

\[(\tau - 1, M/2, \tau - 1) \ (\tau, M/2, \tau - 1) \ (\tau + 1, M/2, \tau - 1) \ \ldots \ (n - 1, M/2, \tau - 1)\]

and so on until it makes sense. We apply Corollaries 4, 14 and 16 in every row separately from left to right to reduce the sets \( P, Q \) and \( W \). Of course, this process is fueled with information from the columns (starting from the bottom end) according to Corollaries 8, 11, 13 and 17. Every nonsymmetric distance distribution \( w \) which is ruled out, forces its mirror image \( \overline{w} \) to be ruled out according to Corollary 2.
The algorithm stops when no new rulings out are possible. An entry at the right end, showing that some of the sets \( P, Q \) and \( W \) is empty\(^2\), means nonexistence of the corresponding BOA. Otherwise, we collect the reduced sets for further analysis and classification results (in some cases, possibly, uniqueness).

Here is the pseudocode of the module of our algorithm which deals with the sets \( W(n, M, \tau), W(n-1, M, \tau) \) and \( W(n-1, M/2, \tau-1) \).

*Algorithm 1*

```
1: procedure NDDA(\( W(n, M, \tau), W(n-1, M, \tau), W(n-1, M/2, \tau-1) \))
2:   filteredW = empty set
3:   for \( w \in W(n, M, \tau) \) do
4:     allX = empty set
5:     for \( w' \in W(n-1, M, \tau) \) do
6:       \( x, y = \) solve system \(^2\) for integer nonnegative solutions
7:       if \( x, \bar{x} \in W(n-1, M/2, \tau-1) \) and \( y, \bar{y} \in W(n-1, M/2, \tau-1) \)
8:       and \( \hat{w}, \tilde{w} \in W(n, M, \tau) \) and \( \hat{w}, \tilde{w} \notin \text{filteredW} \) then
9:         add \( x \) to \( \text{allX} \)
10:        if \( \text{allX} \) is empty then
11:          add \( w \) to \( \text{filteredW} \)
12:        else
13:          if system \(^4\) has no integer nonnegative solutions then
14:            add \( w \) to \( \text{filteredW} \)
15:          if \( \text{filteredW} \) is nonempty then
16:            return NDDA(\( W(n, M, \tau) \setminus \text{filteredW}, W(n-1, M, \tau), W(n-1, M/2, \tau-1) \))
17:          else
18:            return \( W(n, M, \tau) \)
```

We believe that the above description is enough for smooth reproduction of our algorithm. Anyway, we are ready to supply the interested reader with all our programs and databases \[^9\].

\(^2\)In fact, in all cases where we arrived at an empty set, the other two also became empty at the same step.
4 New nonexistence results

4.1 Nonexistence of \((9,96,4)\) BOA and consequences

We apply the algorithm from the previous section on the table below targeting the \((9,96,4)\) BOA.

\[
(4,96,4) \quad (5,96,4) \quad \cdots \quad (9,96,4) \\
(3,48,3) \quad (4,48,3) \quad \cdots \quad (8,48,3) \\
(2,24,2) \quad (3,24,2) \quad \cdots \quad (7,24,2)
\]

The frame of the implementation is showed in the next table. In every entry we first show the number of distance distributions in the beginning and then (after the arrow) the number of the remaining distance distributions in the end of the implementation. The numbers in the brackets show how many distance distributions were left possible after [2].

| BOA | 1st entry | 2nd entry | 3rd entry | 4th entry | 5th entry | 6th entry | 7th entry | 8th entry | 9th entry | 10th entry | 11th entry | 12th entry |
|-----|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| \(P(n,96,4)\) | 1 \rightarrow 1 | 6 \rightarrow 6 | 12 \rightarrow 12 | 20(10) \rightarrow 10 | 34(12) \rightarrow 9 | 37(10) \rightarrow 0 |
| \(Q(n,96,4)\) | 0 \rightarrow 0 | 1 \rightarrow 1 | 4 \rightarrow 4 | 12 \rightarrow 6 | 41 \rightarrow 11 | 97 \rightarrow 0 |
| \(W(n,96,4)\) | 1 \rightarrow 1 | 7 \rightarrow 7 | 16 \rightarrow 16 | 32 \rightarrow 16 | 75 \rightarrow 20 | 134 \rightarrow 0 |
| \(P(n,48,3)\) | 1 \rightarrow 1 | 6 \rightarrow 6 | 13 \rightarrow 13 | 31(25) \rightarrow 25 | 53(41) \rightarrow 38 | 96(65) \rightarrow 62 |
| \(Q(n,48,3)\) | 0 \rightarrow 0 | 1 \rightarrow 1 | 4 \rightarrow 4 | 13 \rightarrow 9 | 41 \rightarrow 30 | 110 \rightarrow 85 |
| \(W(n,48,3)\) | 1 \rightarrow 1 | 7 \rightarrow 7 | 17 \rightarrow 17 | 44 \rightarrow 34 | 94 \rightarrow 68 | 206 \rightarrow 147 |
| \(P(n,24,2)\) | 1 \rightarrow 1 | 6 \rightarrow 6 | 13 \rightarrow 13 | 30 \rightarrow 28 | 49 \rightarrow 47 | 74 \rightarrow 69 |
| \(Q(n,24,2)\) | 0 \rightarrow 0 | 1 \rightarrow 1 | 5 \rightarrow 5 | 19 \rightarrow 17 | 54 \rightarrow 52 | 130 \rightarrow 125 |
| \(W(n,24,2)\) | 1 \rightarrow 1 | 7 \rightarrow 7 | 18 \rightarrow 18 | 49 \rightarrow 45 | 103 \rightarrow 99 | 204 \rightarrow 194 |

**Theorem 18.** There exist no binary orthogonal arrays of parameters \((9,96,4)\).

**Proof.** The zero entries in the right upper cells of the last table imply that there exists no binary orthogonal array of parameters \((9,96,4)\). □ □

The implementation of the algorithm for Theorem 18 created database which is available at [9]. Note that intermediate results are also included.

In 1966 Seiden and Zemach [8] (see also [4, Theorem 2.24]) proved that BOAs of parameters \((n,N,\tau = 2k)\) and \((n + 1,2N,\tau + 1 = 2k + 1)\) coexist. Therefore we have the following nonexistence result as well.

**Corollary 19.** There exist no binary orthogonal arrays of parameters \((10,192,5)\).

The last Corollary follows also from the implementation of our algorithm with \((10,192,5)\) as target. This is illustrated in the next table.
The nonexistence results of Theorem 18 and Corollary 19 give improvements in two entries of Table 12.1 from [4]. We have $7 \leq L(n, \tau) \leq 8$ instead of $6 \leq L(n, \tau) \leq 8$ for the pairs $(n, \tau) = (9, 4)$ and $(10, 5)$.

4.2 Nonexistence of $(10, 112, 4)$ BOA and consequences

Here we work in the table with $C = (11, 112, 4)$ as target.

The results are shown below. Again, in every entry we first show the number of distance distributions in the beginning and then (after the arrow) the number of the remaining distance distributions in the end of the implementation. The numbers in the brackets show how many distance distributions were left possible after the implementation of the algorithm from [2].

The zero entries in the upper right corner imply the nonexistence of BOAs of parameters $(10, 112, 4)$.
**Theorem 20.** There exist no binary orthogonal arrays of parameters $(10, 112, 4)$ and $(11, 112, 4)$.

**Proof.** The zero entry in the right upper cell of the last table means that there exists no binary orthogonal array of parameters $(10, 112, 4)$. This immediately implies the nonexistence of BOAs of parameters $(11, 224, 4)$. 

The data from the implementation of the algorithm for Theorem 20 is available at [9] with intermediate results included.

As above, we use the coexistence of $(n, N, 2k)$ and $(n + 1, 2N, 2k + 1)$ BOAs to obtain further nonexistence results.

**Corollary 21.** There exist no binary orthogonal arrays of parameters $(11, 224, 5)$ and $(12, 224, 5)$.

The last Corollary follows also from the implementation of our algorithm with $(11, 224, 5)$ as target. This is illustrated in the next table, where the first two columns are missed.

| $P(n, 224, 5)$ | $Q(n, 224, 5)$ | $W(n, 224, 5)$ |
|----------------|----------------|----------------|
| $15(11) → 11$ | $4 → 2$        | $19 → 13$      |
| $32(19) → 4$  | $16 → 2$       | $48 → 6$       |
| $63(15) → 5$  | $47 → 4$       | $110 → 9$      |
| $74(11) → 2$  | $141 → 4$      | $215 → 6$      |
| $108(6) → 0$  | $337 → 4$      | $445 → 0$      |

The nonexistence results of Theorem 20 and Corollary 21 give improvements in four entries of Table 12.1 from [4]. We have $L(n, τ) = 8$ instead of $7 ≤ L(n, τ) ≤ 8$ for the pairs $(n, τ) = (10, 4), (11, 4), (11, 5)$ and $(12, 5)$.

4.3 Other nonexistence results

Our algorithm gives other nonexistence results which however are superseded by the result of Khalyavin [5] from 2010. We list these in the next assertion.

**Theorem 22.** ([5] and our algorithm) There exist no binary orthogonal arrays of parameters $(10, 7.2^6 = 448, 6)$, $(11, 7.2^7 = 896, 7)$, $(12, 10.2^8 = 2560, 8)$, $(13, 10.2^9 = 5120, 9)$, $(12, 11.2^8 = 2816, 8)$, $(13, 11.2^9 = 5632, 9)$ and $(15, 13.2^{10} = 13312, 10)$. 

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5 Updated table for $L(n, \tau)$

We present an updated version of the situation with the possible values of function $L(n, \tau)$ – the minimum possible index $\lambda$ of an $(n, M = \lambda 2^\tau, \tau)$ binary orthogonal array. Our table covers the range $4 \leq n \leq 16$ and $4 \leq \tau \leq 10$ (see Table 1).

All calculations in this paper were performed by programs in Maple. All results (in particular all possible distance distributions in the beginning) can be seen at [9]. All programs are available upon request.

References

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Table 1: (see Table 12.1 in [4]) Minimum possible index $\lambda$ of binary orthogonal array of length $n$, $4 \leq n \leq 16$, and strength $\tau$, $4 \leq \tau \leq 10$.

| $n \div \tau$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|----------------|---|---|---|---|---|---|----|
| 4              | 1 |   |   |   |   |   |    |
| 5              | 1 | 1 |   |   |   |   |    |
| 6              | 2 | 1 | 1 |   |   |   |    |
| 7              | $s^24$ | 2 | 1 | 1 |   |   |    |
| 8              | $4^c$ | $s^24$ | 2 | 1 | 1 |   |    |
| 9              | $bms^7-8$ | $4^c$ | 4 | 2 | 1 | 1 |    |
| 10             | $bms^8$ | $bms^7-8$ | $kh^8$ | 4 | 2 | 1 | 1 |
| 11             | $bms^8$ | $bms^8$ | $8^c$ | $kh^8$ | 4 | 2 | 1 |
| 12             | $bkm^8$ | $bms^8$ | $12-16$ | $8^c$ | $kh^8$ | 4 | 2 |
| 13             | $8$ | $bkms^8$ | $16$ | $12-16$ | $kh^{16}$ | $kh^8$ | 4 |
| 14             | $8$ | $8$ | $16$ | $16$ | $16^c$ | $kh^{16}$ | $kh^8$ |
| 15             | $8^{nr}$ | $8$ | $16^{rh}$ | $16$ | $26-32$ | $16^c$ | $kh^{16}$ |
| 16             | $10-16$ | $8^{nr}$ | $21-32$ | $16^{rh}$ | $39-64$ | $26-32$ | $kh^{32}$ |

Key:

- $bkms$: Boyvalenkov, Kulina, Marinova, Stoyanova in [2]
- $c$: Cyclic code
- $kh$: Khalyavin (2010) (and see also Section 6 and [2, Section 4.3] and )
- $nr$: Nordstrom-Robinson (1967) code
- $rh$: Rao-Hamming construction
- $sz$: Seiden and Zemach (1966) bound
- $bms$: Theorem 18 and Corollary 19 or Theorem 20 and Corollary 21