On the Parameterization of Stabilizing Controllers using Closed-loop Responses

Yang Zheng\textsuperscript{1}, Luca Furieri\textsuperscript{2}, Maryam Kamgarpour\textsuperscript{2}, and Na Li\textsuperscript{1}

Abstract—In this paper, we study the problem of parameterizing all internally stabilizing controllers for strictly proper linear time-invariant (LTI) systems using closed-loop responses. It is known that the set of internally stabilizing controllers $\mathcal{C}_{\text{stab}}$ is non-convex, but it admits a convex representation using certain closed-loop maps. A classical result is the Youla parameterization, and two recent notions are the system-level parameterization (SLP) and input-output parameterization (IOP). This paper further examines all possible parameterizations of $\mathcal{C}_{\text{stab}}$ using certain closed-loop responses. Our main idea is to revisit the external transfer matrix characterization of internal stability, which uncovers that only four groups of stable closed-loop transfer matrices are equivalent to internal stability: one of them is used in SLP, another one is a classical result and is used in IOP, and the other two are new, leading to two new parameterizations for $\mathcal{C}_{\text{stab}}$. All these four parameterizations are convex in term of the respectively introduced parameters, allowing us to use convex optimization for controller synthesis. These results contribute to a more complete picture of the notion of closed-loop convexity for parameterizing $\mathcal{C}_{\text{stab}}$.

I. INTRODUCTION

Feedback systems should be stable in some appropriate sense for practical deployment, and thus one fundamental problem in control theory is to design a feedback controller that stabilizes a given dynamical system [1]. Indeed, many control synthesis problems include stability as a constraint while optimizing some performance [2]. However, it is well-known that the set of stabilizing controllers is non-convex, and hence, hard to search directly over. One standard approach is to parameterize all stabilizing controllers and the corresponding closed-loop responses in a convex way, and then to optimize the performance over the new parameter(s) using convex optimization [3].

A classical method for parameterizing the set of all internally stabilizing controllers is based on the celebrated Youla parameterization [4] which relies on a doubly coprime factorization of the system. It is shown in [3] that many performance specifications on the closed-loop system can be expressed in the Youla parameterization framework via convex optimization. Moreover, the foundational results of robust and optimal control are built on the Youla parameterization [1], [5]. More recently, a system-level parameterization (SLP) [6] and an input-output parameterization (IOP) [7] were proposed to characterize the set of internally stabilizing controllers, without relying on the doubly-coprime factorization technique. In principle, Youla, SLP and IOP all directly treat certain closed-loop responses as design parameters, and thus shift the controller synthesis from the design of a controller to the design of the closed loop responses. An explicit equivalence among Youla, SLP and IOP has been shown in [8]. We note that the idea of synthesizing closed-loop responses has been extensively discussed as closed-loop convexity in [3, Chapter 6].

In this paper, our motivation is to investigate the existence of other parameterizations using closed-loop responses beyond Youla, SLP and IOP. We present a positive answer to this question. In particular, we examine all possible parameterizations for the set of internally stabilizing controller using closed-loop responses from the disturbances on state, output, control signals $(\delta_x, \delta_y, \delta_u)$ to state, output, control signals $(x, y, u)$; see Figure 1 for illustration. Our main strategy is to examine the cases where the stability of external transfer matrices is equivalent to internal stability. It turns out that only four groups of stable disturbance-to-signal maps can guarantee internal stability (see Theorem 1): one of them is used in SLP [6], another one is a classical result and is used in IOP [7], and the other two have not been discussed before and thus can be used to derive two new parameterizations for internally stabilizing controllers (Propositions 3 & 4). Note that our results are exclusive in the sense that there are no other parameterizations for internally stabilizing controllers using closed-loop responses from $(\delta_x, \delta_y, \delta_u)$ to $(x, y, u)$.

The rest of this paper is organized as follows. We state the problem in Section II. The relationship between the stability of external transfer matrices and internal stability is discussed in Section III. Four parameterizations of stabilizing controllers using closed-loop responses, including SLP and IOP, are presented in Section IV. A numerical application is discussed in Section V. We conclude the paper in Section VI.

Notation: We use lower and upper case letters (e.g. $x$ and $A$) to denote vectors and matrices, respectively. Lower and upper case boldface letters (e.g. $x$ and $G$) are used to denote signals and transfer matrices, respectively. We denote the set of real-rational proper stable transfer matrices as $\mathcal{RH}_{\infty}$.

II. PROBLEM STATEMENT

A. System model

We consider a discrete-time linear time-invariant (LTI) plant of the form

$$x[t + 1] = Ax[t] + Bu[t] + \delta_x[t],$$
$$y[t] = Cx[t] + Du[t] + \delta_y[t],$$

(1)
Assumption 2. The set of all internally stabilizing controllers is defined as

\[ C_{\text{stab}} := \{ K \mid K \text{ internally stabilizes } G \}. \]  

where \( x[t], u[t], y[t] \) are the state vector, control action, and measurement vector at time \( t \), respectively; \( \delta_x[t] \) and \( \delta_y[t] \) are external disturbances on the state and measurement vectors, respectively. The transfer function from \( u \) to \( y \) is

\[ G = C(zI - A)^{-1}B + D. \]

Consider a dynamical controller

\[ u = Ky + \delta_u, \]

where \( \delta_u \) is the external disturbance on the control action. A state-space realization of (2) is

\[ \begin{align*}
\xi[t+1] &= A\xi[t] + B_ky[t], \\
u[t] &= C\xi[t] + D_ky[t] + \delta_u[t],
\end{align*} \]

where \( \xi[t] \) is the internal state of the controller at time \( t \).

In this paper, we make the following standard assumption.

**Assumption 1.** Both the plant and controller realizations are stabilizable and detectable, i.e., \((A, B)\) and \((A_k, B_k)\) are stabilizable, and \((A, C)\) and \((A_k, C_k)\) are detectable.

Also, we consider strictly proper plant dynamics exclusively in this paper, and make the following assumption.

**Assumption 2.** In (1), we assume \( D = 0 \), i.e., the plant dynamics are strictly proper.

Applying the controller (2) to the plant (1) leads to a closed-loop system shown in Fig. 1. Under Assumption 2, the closed-loop system is always well-posed [1, Lemma 5.1].

### B. Internal stability

The definition of internal stability is [1, Chapter 5.3]

**Definition 1.** The system in Fig. 1 is internally stable if it is well-posed, and the states \((x[t], \xi[t])\) converge to zero as \( t \to \infty \) for all initial states \((x[0], \xi[0])\) when \( \delta_x[t] = 0, \delta_y[t] = 0, \delta_u[t] = 0, \forall t \).

We say the controller \( K \) internally stabilizes the plant \( G \) if the closed-loop system in Fig. 1 is internally stable. The set of all internally stabilizing controllers is defined as

\[ C_{\text{stab}} := \{ K \mid K \text{ internally stabilizes } G \}. \]  

Note that when an infinite time-horizon is considered, a feedback system should at least be stable, and any controller synthesis will implicitly or explicitly involve a constraint \( K \in C_{\text{stab}} \). Therefore, it is fundamentally important to characterize \( C_{\text{stab}} \). Indeed, it is well-known that \( C_{\text{stab}} \) is non-convex and it is not difficult to find explicit examples where \( K_1, K_2 \in C_{\text{stab}} \) and \( \frac{1}{2}(K_1 + K_2) \notin C_{\text{stab}} \). Accordingly, it is not easy to directly search over \( K \in C_{\text{stab}} \) for control synthesis, and a suitable change of variables is used in many control synthesis procedures [1–8].

A standard state-space characterization of internal stabilization is as follows [1, Lemma 5.2].

**Lemma 1.** Under Assumptions 7 and 2, \( K \) internally stabilizes \( G \) if and only if

\[ A_{\text{cl}} = \begin{bmatrix}
A + BD_kC & BC_k \\
B_kC & A_k
\end{bmatrix} \]

is stable.

Note that the result in Lemma 1 is a simplified version of [1, Lemma 5.2] because we only focus on strictly proper plants, i.e., \( D = 0 \). Lemma 1 leads to an explicit state-space characterization of the set \( C_{\text{stab}} \) as follows:

\[ C_{\text{stab}} = \left\{ K \mid A_{\text{cl}} = \begin{bmatrix}
A + BD_kC & BC_k \\
B_kC & A_k
\end{bmatrix} \text{ is stable} \right\}, \]

where \( K = C_k(zI - A_k)^{-1}B_k + D_k \). Unfortunately the stability condition on \( A_{\text{cl}} \) in (6) is still non-convex in terms of the parameters \( A_k, B_k, C_k, D_k \).

Unlike the state-space parameterization (6), there are several frequency-domain characterizations for \( C_{\text{stab}} \) where only convex constraints are involved in the new parameter space. A classical approach is the celebrated Youla parameterization [4], where a doubly coprime factorization of the system is used. Two recent approaches are the system-level parameterization (SLP) [6] and input-output parameterization (IOP) [7], where no doubly coprime factorization is required. The explicit equivalence of Youla, SLP, and IOP has been recently shown in [8]. We refer the interested reader to [1], [4], [5] for details on Youla parameterization.

Both SLP and IOP use certain closed-loop responses for parameterizing \( C_{\text{stab}} \). Inspired by SLP and IOP, this paper aims to investigate all possible parameterizations for \( C_{\text{stab}} \) using the closed-loop responses from \((\delta_x, \delta_y, \delta_u)\) to \((x, y, u)\).

### III. External transfer matrix characterization of internal stability

In this section, we revisit the external transfer matrix characterization of internal stability, which will be applied to characterize \( C_{\text{stab}} \) in the next section.

Combining (1) with (2), we can write the closed-loop responses from \((\delta_x, \delta_y, \delta_u)\) to \((x, y, u)\) as

\[ \begin{bmatrix}
x \\
y \\
u
\end{bmatrix} = \begin{bmatrix}
\Phi_{xx} & \Phi_{xy} & \Phi_{xu} \\
\Phi_{yx} & \Phi_{yy} & \Phi_{yu} \\
\Phi_{ux} & \Phi_{uy} & \Phi_{uu}
\end{bmatrix} \begin{bmatrix}
\delta_x \\
\delta_y \\
\delta_u
\end{bmatrix}, \]

where we have \( \Phi_{xx} = (zI - A - BK) \) and

\[ \begin{align*}
\Phi_{xy} &= \Phi_{xx}BK, & \Phi_{xu} &= \Phi_{xx}B, \\
\Phi_{yx} &= C\Phi_{xx}, & \Phi_{yu} &= C\Phi_{xx}BK + I, \\
\Phi_{uy} &= C\Phi_{xx}B, & \Phi_{uu} &= KC\Phi_{xx}, \\
\Phi_{ux} &= K(C\Phi_{xx}BK + I), & \Phi_{uU} &= K(C\Phi_{xx}B + I).
\end{align*} \]
We define the closed-loop response transfer matrix as

$$\Phi = \begin{bmatrix} \Phi_{xx} & \Phi_{xy} & \Phi_{yu} \\ \Phi_{yx} & \Phi_{yy} & \Phi_{yu} \\ \Phi_{ux} & \Phi_{uy} & \Phi_{uu} \end{bmatrix}.$$  \hfill (9)

A notion of external transfer matrix stability is defined as

Definition 2 \((11, \text{Chapter 5})\). The closed-loop system is disturbance-to-signal stable if the closed-loop responses from \((\delta_x, \delta_y, \delta_u)\) to \((x, y, u)\) are all stable, i.e., \(\Phi \in \mathcal{RH}_\infty\).

It is known that the internal stability in Definition 1 and the disturbance-to-signal stability in Definition 2 are equivalent \([1, \text{Chapter 5}], \text{i.e.}, we have

$$C_{\text{stab}} = \{ K \mid \Phi \in \mathcal{RH}_\infty, \text{ where } \Phi \text{ is defined in } (9) \}.$$

In fact, it is sufficient to enforce a subset of elements in \(\Phi\) to be stable, as shown in \([1, \text{Lemma 5.3}]\).

Lemma 2. Under Assumption \(2\), \(K\) internally stabilizes \(G\) if and only if the closed-loop responses from \((\delta_u, \delta_u)\) to \((y, u)\) are stable, i.e.,

$$\begin{bmatrix} \Phi_{yy} & \Phi_{yu} \\ \Phi_{uy} & \Phi_{uu} \end{bmatrix} \in \mathcal{RH}_\infty.$$

For national simplicity, we denote

$$\begin{bmatrix} \delta_y \\ \delta_u \end{bmatrix} \rightarrow \begin{bmatrix} y \\ u \end{bmatrix} := \begin{bmatrix} \Phi_{yy} & \Phi_{yu} \\ \Phi_{uy} & \Phi_{uu} \end{bmatrix}.$$

The result in Lemma 2 motivates the question of whether we can select other elements in \(\Phi\) for internal stability. For example, if the closed-loop responses from \((\delta_x, \delta_y)\) to \((x, y)\) are stable, i.e.,

$$\begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \end{bmatrix} \in \mathcal{RH}_\infty,$$

can we guarantee the closed-loop system is internally stable? The answer is negative, as proved in Theorem 1 below.

In particular, we consider all possible combinations of 4 closed-loop responses that may guarantee internal stability. When choosing two disturbances and two outputs from \([7]\), we have in total \(\binom{4}{2} \times \binom{4}{2} = 9\) choices, i.e.,

$$\begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix} \rightarrow \begin{bmatrix} y \\ u \end{bmatrix}, \begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix} \rightarrow \begin{bmatrix} x \\ u \end{bmatrix}, \begin{bmatrix} \delta_y \\ \delta_u \end{bmatrix} \rightarrow \begin{bmatrix} y \\ u \end{bmatrix}, \begin{bmatrix} \delta_y \\ \delta_u \end{bmatrix} \rightarrow \begin{bmatrix} x \\ u \end{bmatrix}.$$

Note that it is in general not sufficient to select only one close-loop response since there are two dynamical parts in system \((1)\) and controller \((2)\). Our main result of this section shows that stability of any of the 4 closed-loop responses in the top-right corner of \((10)\), highlighted in black, is equivalent to internal stability.

Theorem 1. Under Assumption 1, the following statements are equivalent:

1) \(K\) internally stabilizes \(G\);

2) \(\begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix} \rightarrow \begin{bmatrix} y \\ u \end{bmatrix} \in \mathcal{RH}_\infty\),

3) \(\begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix} \rightarrow \begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{RH}_\infty\),

4) \(\begin{bmatrix} \delta_y \\ \delta_u \end{bmatrix} \rightarrow \begin{bmatrix} y \\ u \end{bmatrix} \in \mathcal{RH}_\infty\),

5) \(\begin{bmatrix} \delta_y \\ \delta_u \end{bmatrix} \rightarrow \begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{RH}_\infty\).

Moreover, stability of any other group of 4 closed-loop responses in \((10)\) is not sufficient for internal stability.

As shown in Theorem 1 to guarantee internal stability, it always requires to select \(\delta_y\) as an input and \(u\) as an output, leading to four possible groups of closed-loop responses. The other groups of closed-loop responses in \((10)\), highlighted in red, do not have either \(\delta_y\) or \(u\), and thus fail to guarantee internal stability. The proof of Theorem 1 is inspired by that in \([1, \text{Lemma 5.3}]\). We provide the proof in Appendix VI-A for completeness. Note that Theorem 1 is exclusive in the sense that there exists no other stable closed-loop responses that are equivalent to internal stability, and Lemma 2 is included as the equivalence between 1) and 4) in Theorem 1.

IV. PARAMETERIZATIONS OF STABILIZING CONTROLLERS

The results in Theorem 1 can be used to parameterize the set of internally stabilizing controllers \(C_{\text{stab}}\), leading to four equivalent parameterizations. One of them corresponds to the SLP \([6]\), another one is the IOP \([7]\), and the remaining two parameterizations are new and, to the best of the authors’ knowledge, have not been characterized before.

A. Four equivalent parameterizations

The closed-loop responses from \((\delta_x, \delta_y)\) to \((x, u)\) have been utilized in the SLP \([6]\). Specifically, consider

$$\begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} \Phi_{xx} & \Phi_{xy} \\ \Phi_{yx} & \Phi_{yu} \end{bmatrix} \begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix}.$$

We have the following result \([6]\).

Proposition 1 \([6]\). Consider the system \((1)\) and the controller \((2)\). The following statements are true:

1) For any \(K \in C_{\text{stab}}\), the resulting closed-loop responses \((11)\) are in the following affine subspace

$$\begin{bmatrix} zI - A & -B \\ \Phi_{xx} & \Phi_{xy} \end{bmatrix} \begin{bmatrix} zI - A \\ \Phi_{ux} \end{bmatrix} = \begin{bmatrix} I & 0 \\ \Phi_{ux} & \Phi_{uy} \end{bmatrix} \begin{bmatrix} zI - A \\ -C \end{bmatrix} = \begin{bmatrix} I & 0 \\ \Phi_{ux} & \Phi_{uy} \end{bmatrix} \begin{bmatrix} zI - A \\ -C \end{bmatrix} \in \mathcal{RH}_\infty.$$

2) For any transfer matrices \(\Phi_{xx}, \Phi_{yx}, \Phi_{xy}, \Phi_{yu}\) satisfying \((12)\), \(K = \Phi_{uy} - \Phi_{ux} \Phi_{xx}^{-1} \Phi_{xy} \in C_{\text{stab}}\).

Also, the closed-loop responses from \((\delta_y, \delta_u)\) to \((y, u)\) have been used in the IOP \([7]\). Specifically, consider

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} \Phi_{yy} & \Phi_{yu} \\ \Phi_{uy} & \Phi_{uu} \end{bmatrix} \begin{bmatrix} \delta_y \\ \delta_u \end{bmatrix}.$$
We have the following result [7].

**Proposition 2 (17).** Consider the system (1) and the controller (2). The following statements are true:

1) For any $K \in C_{stab}$, the resulting closed-loop responses (13) are in the following affine subspace

$$[I - G] \begin{bmatrix} \Phi_{yy} & \Phi_{uy} \\ \Phi_{uy} & \Phi_{uu} \end{bmatrix} = [I 0],$$

$$\begin{bmatrix} \Phi_{yy} & \Phi_{uy} \\ \Phi_{uy} & \Phi_{uu} \end{bmatrix} \begin{bmatrix} -G \\ I \end{bmatrix} = [0 I],$$

$$\Phi_{yy}, \Phi_{uy}, \Phi_{uu} \in R\mathcal{H}_{\infty}.$$ (14)

2) For any transfer matrices $\Phi_{yy}, \Phi_{uy}, \Phi_{uu}$ satisfying (14), $K = \Phi_{uy}^{-1} \Phi_{yy} \in C_{stab}$.

Next, we consider the following closed-loop responses

$$\begin{bmatrix} y' \\ u \end{bmatrix} = \begin{bmatrix} \Phi_{yx} & \Phi_{yy} \\ \Phi_{uy} & \Phi_{uu} \end{bmatrix} \begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix}.$$ (15)

We have the following result about a new parametrization of $C_{stab}$. Its proof is presented in Appendix VI-B

**Proposition 3.** Consider the system (1) and the controller (2). The following statements are true:

1) For any $K \in C_{stab}$, the resulting closed-loop responses (15) are in the following affine subspace

$$[I - G] \begin{bmatrix} \Phi_{yx} & \Phi_{yy} \\ \Phi_{uy} & \Phi_{uu} \end{bmatrix} = [C(zI - A)^{-1} I],$$

$$\begin{bmatrix} \Phi_{yx} & \Phi_{yy} \\ \Phi_{uy} & \Phi_{uu} \end{bmatrix} \begin{bmatrix} zI - A \\ -C \end{bmatrix} = 0,$$

$$\Phi_{yx}, \Phi_{uy}, \Phi_{yy}, \Phi_{uu} \in R\mathcal{H}_{\infty}.$$ (16)

2) For any transfer matrices $\Phi_{yx}, \Phi_{uy}, \Phi_{yy}, \Phi_{uu}$ satisfying (16), $K = \Phi_{uy}^{-1} \Phi_{yy} \in C_{stab}$.

Finally, we consider the case

$$\begin{bmatrix} x' \\ u \end{bmatrix} = \begin{bmatrix} \Phi_{xy} & \Phi_{yu} \\ \Phi_{uy} & \Phi_{uu} \end{bmatrix} \begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix}.$$ (17)

The following result mirrors Proposition 3 for an additional new parametrization of $C_{stab}$. Its proof is presented in Appendix VI-C

**Proposition 4.** Consider the system (1) and the controller (2). The following statements are true:

1) For any $K \in C_{stab}$, the resulting closed-loop responses (17) are in the following affine subspace

$$\begin{bmatrix} zI - A & -B \end{bmatrix} \begin{bmatrix} \Phi_{xy} & \Phi_{yu} \\ \Phi_{uy} & \Phi_{uu} \end{bmatrix} = 0$$

$$\begin{bmatrix} \Phi_{xy} & \Phi_{yu} \\ \Phi_{uy} & \Phi_{uu} \end{bmatrix} \begin{bmatrix} -G \\ I \end{bmatrix} = \begin{bmatrix} (zI - A)^{-1} B \\ I \end{bmatrix},$$

$$\Phi_{xy}, \Phi_{yu}, \Phi_{uy}, \Phi_{uu} \in R\mathcal{H}_{\infty}.$$ (18)

2) For any transfer matrices $\Phi_{xy}, \Phi_{yu}, \Phi_{uy}, \Phi_{uu}$ satisfying (18), $K = \Phi_{uu}^{-1} \Phi_{yu} \in C_{stab}$.

To summarize, Propositions 1-4 establish four equivalent methods to parameterize the set of internally stabilizing controllers using closed-loop responses.

$$C_{stab} = \{ K = \Phi_{uy} - \Phi_{uu} \Phi_{xy}, \Phi_{xx}, \Phi_{ux}, \Phi_{xy}, \Phi_{uy} \in \text{affine subspace (12)} \}.$$ (19)

$$C_{stab} = \{ K = \Phi_{uy} \Phi_{yy}^{-1} \Phi_{yy}, \Phi_{yy}, \Phi_{uy}, \Phi_{uu} \in \text{affine subspace (14)} \}.$$ (20)

$$C_{stab} = \{ K = \Phi_{xy} \Phi_{yy}^{-1} \Phi_{yy}, \Phi_{yy}, \Phi_{xy}, \Phi_{yu}, \Phi_{uu} \in \text{affine subspace (16)} \}.$$ (21)

Unlike the state-space characterization (6), the constraints (12), (14), (16), and (18) are all affine in the new parameters. Based on (12), (14), (16), and (18), convex optimization problems can be derived for optimal controller synthesis; see [6], [7] for details. We will present an example in the next section.

**Remark 1.** Explicit equivalence relationship between Propositions 1 & 2 and Youla parameterization has been derived in [8]. It is possible to derive explicit relationship among Propositions 1 & 2 and Youla parameterization using the approach of [7]. Besides, while there are four parameters in (12), (14), (16), or (18), there is only one freedom due to the affine constraints. This is consistent with the Youla parameterization, where only one parameter is involved with no explicit affine constraints; see [8] for details. Finally, note that while being convex, the decision variables in (12), (14), (16), and (18) are infinite-dimensional, and thus finite approximations are needed for numerical computations. The interested reader is referred to Remark 2 in [8].

**V. Case Studies**

**A. Application to optimal controller synthesis**

Here, we consider the following optimal controller synthesis problem

$$\min_K \lim_{T \to \infty} \mathbb{E} \left\{ \frac{1}{T} \sum_{t=0}^{T} (y[t]Qy[t] + u[t]Ru[t]) \right\}$$

subject to $x[t + 1] = Ax[t] + Bu[t],$

$$y[t] = Cx[t] + \delta_y[t],$$

$$u = Ky,$$

where disturbance $\delta_y[t] \sim \mathcal{N}(0, I)$, $Q \succ 0$ and $R \succ 0$ are performance-weight matrices with appropriate dimensions, and the decision variable $K$ is a dynamical controller. Note that (19) can be equivalently reformulated as an optimization problem in the frequency domain as follows

$$\min_K \left\| \left[ \frac{Q^{\frac{1}{2}}}{R^{\frac{1}{2}}} \right] \left[ (I - GK)^{-1} \right] \right\|_{\mathcal{H}_2}^2$$

subject to $K \in C_{stab},$
where $G = C(zI - A)^{-1}B$ and $C_{\text{stab}}$ denotes the set of internally stabilizing controllers.

It is easy to see that the optimal synthesis problem (20) is non-convex in terms of $K$ since both the cost function and constraints are non-convex. Using a change of variables as suggested in Propositions 1-4, it is straightforward to replace the constraint $K \in C_{\text{stab}}$ with the affine constraints (12), (14), (16), or (18). Now, it remains to reformulate the cost function in terms of these new variables. Simple rote calculations show that

$$ \| \begin{bmatrix} Q^\frac{1}{2} & R^\frac{1}{2} \\ R^\frac{1}{2} & \Phi \end{bmatrix} \left( I - GK \right)^{-1} \|_{H_2}^2 = \| Q^\frac{1}{2} R^\frac{1}{2} \Phi_{uy} \|_{H_2}^2 = \| Q^\frac{1}{2} R^\frac{1}{2} \left( C\Phi_{xy} + I \right) \|_{H_2}^2. $$

Therefore, (20) is equivalent to any of the following convex optimization problems (21)-(24) corresponding to Propositions 1-4, respectively.

\begin{align*}
\min_{\Phi_{xx}, \Phi_{uu}, \Phi_{yy}, \Phi_{xy}} & \| Q^\frac{1}{2} R^\frac{1}{2} \left( C\Phi_{xy} + I \right) \|_{H_2}^2 \\
\text{subject to} & \quad (12).
\end{align*}

\begin{align*}
\min_{\Phi_{yy}, \Phi_{xy}, \Phi_{yy}, \Phi_{yy}} & \| Q^\frac{1}{2} R^\frac{1}{2} \Phi_{yy} \|_{H_2}^2 \\
\text{subject to} & \quad (14).
\end{align*}

\begin{align*}
\min_{\Phi_{xx}, \Phi_{uu}, \Phi_{yy}, \Phi_{uy}} & \| Q^\frac{1}{2} R^\frac{1}{2} \Phi_{uu} \|_{H_2}^2 \\
\text{subject to} & \quad (16).
\end{align*}

\begin{align*}
\min_{\Phi_{xy}, \Phi_{uy}, \Phi_{uu}, \Phi_{uu}} & \| Q^\frac{1}{2} R^\frac{1}{2} \Phi_{uy} \|_{H_2}^2 \\
\text{subject to} & \quad (18).
\end{align*}

Since the decision variables in problems (21)-(24) are infinite-dimensional, we further impose a finite impulse response (FIR) constraint with horizon length $N$ [6], [7], i.e., we optimize over finitely many real matrices $T_k$ such that

$$ T = \sum_{k=0}^{N} \frac{1}{2^k} T_k, $$

where $T$ is any corresponding transfer function matrix in (21)-(24).

**Remark 2** (Computation via quadratic programs). Note that the $H_2$ norm for FIR transfer functions admits the following expression [3]

$$ \| T \|_{H_2}^2 = \sum_{k=0}^{N} \text{trace}(T_k^T T_k). $$

Thus, after imposing FIR constraints, problems (21)-(24) can all be solved via quadratic programs (QPs).

Fig. 2. Car-following control.

| TABLE I |
| --- |
| $H_2$ NORM FOR DIFFERENT FIR LENGTHS WHEN SOLVING THE CAR-FOLLOWING PROBLEM. |
| **FIR** $N$ | 10 | 20 | 30 | 40 | 50 | 75 | 100 |
| $H_2$ norm | 11.88 | 3.06 | 1.97 | 1.69 | 1.59 | 1.49 | 1.45 |

†: The $H_2$ norms from [21]-[24] have no difference up to four significant figures. ‡: The true $H_2$ norm from $h_2$syn in MATLAB is 1.42.

**B. Numerical experiments**

Here, a car-following control example [9] is used to demonstrate the numerical performance of the parameterizations in Propositions 1-4. As shown in Fig. 2, the acceleration, velocity and position of the leading vehicle and following vehicle are denoted as $(p_0(t), v_0(t), a_0(t))$ and $(p_1(t), v_1(t), a_1(t))$, respectively. It is assumed that the leading vehicle is running at a constant velocity, i.e., $a_0(t) = 0$. The dynamics from the control input $u$ (throttle opening/braking pressure) to the vehicle’s acceleration response can be approximated as [9], [12] $\tau a(t) + a_1(t) = u(t)$.

Defining the state as $x = [p_1 - p_0 + d, v_1 - v_0, a_1 - a_0]^T$, where $d$ is the desired spacing between two vehicles, the car-following dynamics are expressed as

$$ \dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{1}{\tau} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{\tau} \end{bmatrix} u. $$

We assume that only the relative spacing and relative velocity can be measured. The measurements are affected by i.i.d. Gaussian noise $\delta_y \sim N(0, I)$, i.e., $y = Cx + \delta_y$ with

$$ C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. $$

Using a forward Euler-discretization of (25) with a sampling time of $\Delta_T = 0.1s$, we formulate the corresponding optimal controller synthesis problem (19) in discrete-time with $\tau = 0.5$s, $Q = I$ and $R = I$. This can be solved via any of the convex problems (21)-(24). The results are listed in Table I. As expected, when increasing the FIR length, the optimal cost from (21)-(24) converges to the true value returned by the standard synthesis $h_2$syn in MATLAB.

**VI. CONCLUSIONS**

In this paper, we have characterized all possible parameterizations for the set of stabilizing controllers using closed-loop maps. We have revealed two other parallel choices beyond the recent notion of SLP [6] and IOP [7]. In fact, our analysis...
allows to treat SLP [6] and IOP [7] in the same viewpoint. All these four parameterizations are convex in the new parameter space, which allows using convex optimization for control synthesis, as demonstrated in the car-following example. It should be noted that after imposing the FIR approximation, the ability of the four parameterizations for $C_{	ext{stab}}$ becomes different. Future work will make this point more precise. Also, it would be interesting to address decentralized control (e.g., the notion of quadratic invariance (QI) [13] and sparsity invariance (SI) [14]) using different parameterizations. Finally, similar to SLP [15], [16], it is extremely interesting to investigate the features of different parameterizations in robust analysis and synthesis for uncertain systems.

APPENDIX

In this appendix, we provide the proofs for Theorems 1 and Propositions 3 & 4. The proof strategy for Theorem 1 is classical and is based on the following result [1]:

**Lemma 3.** Consider $G(s) = C(sI - A)^{-1}B + D$. We have

- If $(A, B, C)$ is detectable and stabilizable, then $G(s) \in RH_{\infty}$ if and only if $A$ is stable;
- If $(A, B, C)$ is not detectable or stabilizable, then the stability of $A$ is only sufficient but not necessary for $G(s) \in RH_{\infty}$.

Similar to [7], [8], the proofs of Propositions 3 & 4 are purely algebraic. The following push-through identify [17] is used multiple times: $G(I + HG)^{-1} = (I + GH)^{-1}G$, for any compatible transfer matrices $G, H$ such that $(I + GH)$ is invertible.

**A. Proof of Theorem 1**

Combining (1) with (3), leads to

$$
\begin{bmatrix}
\dot{x}[t + 1] \\
\xi[t + 1]
\end{bmatrix} = A_{d3} \begin{bmatrix}
\dot{x}[t] \\
\xi[t]
\end{bmatrix} + \begin{bmatrix}
B & BD_k \\
0 & B
\end{bmatrix} \begin{bmatrix}
\delta_x[t] \\
\delta_y[t]
\end{bmatrix} + \begin{bmatrix}
\delta_x[t] \\
\delta_y[t]
\end{bmatrix},
$$

where $A_{d3}$ is defined in (5). Now, we prove the equivalences in Theorem 1: (1) $\Leftrightarrow$ (2), (1) $\Leftrightarrow$ (3), (1) $\Leftrightarrow$ (4), and (1) $\Leftrightarrow$ (5).

Note that (1) $\Rightarrow$ (2), (1) $\Rightarrow$ (3), (1) $\Rightarrow$ (4), and (1) $\Rightarrow$ (5) is true by the definition of internal stability.

2) $\Rightarrow$ 1): From (1) and (3), we have

$$
\begin{bmatrix}
y[t] \\
u[t]
\end{bmatrix} = \begin{bmatrix}
C \\
D_k C_c C_k
\end{bmatrix} \begin{bmatrix}
\dot{x}[t] \\
\xi[t]
\end{bmatrix} + \begin{bmatrix}
0 & I \\
0 & D_k
\end{bmatrix} \begin{bmatrix}
\delta_x[t] \\
\delta_y[t]
\end{bmatrix} + \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix} \begin{bmatrix}
\delta_x[t] \\
\delta_y[t]
\end{bmatrix}.
$$

Then, the closed-loop responses from $(\delta_x, \delta_y)$ to $(y, u)$ are

$$
\begin{bmatrix}
C & 0 \\
D_k C_c C_k
\end{bmatrix} (zI - A_{d3})^{-1} \begin{bmatrix}
I & BD_k \\
0 & B
\end{bmatrix} + \begin{bmatrix}
0 & I \\
0 & D_k
\end{bmatrix}.
$$

Suppose that the transfer matrix in (26) is stable. We now prove that $A_{d3}$ is stable. It is sufficient to show that

$$(A_{d3}, \begin{bmatrix}
I & BD_k \\
0 & B
\end{bmatrix}, \begin{bmatrix}
C \\
D_k C_c C_k
\end{bmatrix})$$

is stabilizable and detectable. Letting $\hat{F} = \begin{bmatrix}
F & 0 \\
C & \hat{F}_k
\end{bmatrix}$ leads to

$$
A_{d3} - \begin{bmatrix}
I & BD_k \\
0 & B
\end{bmatrix} \hat{F} = \begin{bmatrix}
A - F & BC_k - BD_k F_k \\
0 & A_k - B_k F_k
\end{bmatrix},
$$

which can be made stable by choosing appropriate $F$ and $F_k$ since $(A, I)$ and $(A_k, B_k)$ are stabilizable. Similarly, we let $\hat{L} = \begin{bmatrix}
L & B \\
0 & L_k
\end{bmatrix}$, leading to

$$
A_{d3} - \hat{L} \begin{bmatrix}
C & 0 \\
D_k C_c C_k
\end{bmatrix} = \begin{bmatrix}
A - LC & 0 \\
B_k C - L_k D_k C & A_k - L_k C_k
\end{bmatrix},
$$

which can be made stable by choosing appropriate $L$ and $L_k$ since $(A, C)$ and $(A_k, C_k)$ are detectable. Therefore,

$$
(A_{d3}, \begin{bmatrix}
I & BD_k \\
0 & B
\end{bmatrix}, \begin{bmatrix}
C \\
D_k C_c C_k
\end{bmatrix})
$$

is stabilizable and detectable. According to Lemma 3, the closed-loop system is internally stable.

The equivalence of 1) $\Leftrightarrow$ 3), 1) $\Leftrightarrow$ 4), and 1) $\Leftrightarrow$ 5) can be proved in a similar fashion. In particular,

- 1) $\Leftrightarrow$ 3): we can show that the closed-loop responses from $(\delta_x, \delta_y)$ to $(x, u)$ are

$$
\begin{bmatrix}
I & 0 \\
D_k C_c C_k
\end{bmatrix} (zI - A_{d3})^{-1} \begin{bmatrix}
I & BD_k \\
0 & B
\end{bmatrix} + \begin{bmatrix}
0 & I \\
0 & D_k
\end{bmatrix}.
$$

It is not difficult to check that

$$(A_{d3}, \begin{bmatrix}
I & BD_k \\
0 & B
\end{bmatrix}, \begin{bmatrix}
I \\
D_k C_c C_k
\end{bmatrix})
$$

is stabilizable and detectable.

- 1) $\Leftrightarrow$ 4): The closed-loop responses from $(\delta_y, \delta_u)$ to $(y, u)$ are

$$
\begin{bmatrix}
C & 0 \\
D_k C_c C_k
\end{bmatrix} (zI - A_{d3})^{-1} \begin{bmatrix}
BD_k & B \\
0 & D_k
\end{bmatrix} + \begin{bmatrix}
I & 0 \\
0 & I
\end{bmatrix}.
$$

We can check that

$$(A_{d3}, \begin{bmatrix}
BD_k & B \\
0 & D_k
\end{bmatrix}, \begin{bmatrix}
C \\
D_k C_c C_k
\end{bmatrix})
$$

is stabilizable and detectable.

- 1) $\Leftrightarrow$ 5): The closed-loop responses from $(\delta_y, \delta_u)$ to $(x, u)$ are

$$
\begin{bmatrix}
I & 0 \\
D_k C_c C_k
\end{bmatrix} (zI - A_{d3})^{-1} \begin{bmatrix}
BD_k & B \\
0 & D_k
\end{bmatrix} + \begin{bmatrix}
I & 0 \\
0 & I
\end{bmatrix}.
$$

We can check that

$$(A_{d3}, \begin{bmatrix}
BD_k & B \\
0 & D_k
\end{bmatrix}, \begin{bmatrix}
I \\
D_k C_c C_k
\end{bmatrix})
$$

is stabilizable and detectable.

For the second part of Theorem 1 we first prove that stability of

$$
\begin{bmatrix}
\delta_x \\
\delta_y
\end{bmatrix} \rightarrow \begin{bmatrix}
x \\
y
\end{bmatrix}
$$

is not sufficient for internal stability. It is not difficult to derive that the closed-loop responses from $(\delta_x, \delta_y)$ to $(x, y)$ are

$$
\begin{bmatrix}
I & 0 \\
C & 0
\end{bmatrix} (zI - A_{d3})^{-1} \begin{bmatrix}
B & B \\
0 & 0
\end{bmatrix} + \begin{bmatrix}
I & 0 \\
0 & I
\end{bmatrix}.
$$

Since

$$(A_{d3}, \begin{bmatrix}
I & 0 \\
C & 0
\end{bmatrix})$$

is not detectable in general, the stability of

$$
\begin{bmatrix}
\delta_x \\
\delta_y
\end{bmatrix} \rightarrow \begin{bmatrix}
x \\
y
\end{bmatrix}
$$

cannot guarantee the stability of
Therefore, it is not sufficient for internal stability. The other claims can be proved similarly: the corresponding state-space realization of the closed-loop transfer matrix is not stabilizable and/or detectable.

B. Proof of Proposition 3

Proof of Statement 1: Given any $K \in C_{\text{stab}}$, we know that the closed-loop responses are $\Phi_{yx} = (I - GK)^{-1}C(zI - A)^{-1}$, $\Phi_{yy} = (I - GK)^{-1}$, $\Phi_{ux} = K(I - GK)^{-1}C(zI - A)^{-1}$, $\Phi_{uy} = K(I - GK)^{-1}$, which are all stable by definition. Then, it is not difficult to verify that

$$\Phi_{yx} = G\Phi_{ux},$$

$$\Phi_{yx} - G\Phi_{ux} = ((I - GK)^{-1} - G(I - GK)^{-1})C(zI - A)^{-1} \in C_{zI - A}^{-1},$$

and

$$\Phi_{yy} - G\Phi_{uy} = (I - GK)^{-1} - G(I - GK)^{-1} = I.$$ 

Also, we have $\Phi_{yx}(zI - A) - \Phi_{yy}C = 0, \Phi_{ux}(zI - A) - \Phi_{uy}C = 0$. Therefore, the closed-loop responses $\Phi_{yx}, \Phi_{yy}, \Phi_{ux}, \Phi_{uy}$ satisfy (16).

Proof of Statement 2: Consider any $\Phi_{yx}, \Phi_{yy}, \Phi_{ux}, \Phi_{uy}$ satisfying (16), and let $K = \Phi_{uy}^{-1}\Phi_{yx}$. We now verify the resulting closed loop responses in (15) are all stable. In particular, we have

$$\Phi_{yx} = (I - GK)^{-1}C(zI - A)^{-1} \in C_{zI - A},$$

and

$$(I - P_{22}K)^{-1} = \Phi_{yy} \in C_{zI - A}^{-1},$$

$$K(I - P_{22}K)^{-1}C(zI - A)^{-1} = \Phi_{ux} \in C_{zI - A},$$

$$K(I - P_{22}K)^{-1} = \Phi_{uy} \in C_{zI - A}.$$ 

Therefore, we have $\left[\begin{array}{c} \delta x \\ \delta y \end{array}\right] \rightarrow \left[\begin{array}{c} y \\ u \end{array}\right] \in C_{zI - A}$. By the equivalence between disturbance-to-signal stability and internal stability in Theorem 1, we have $K = \Phi_{uy}^{-1}\Phi_{yx} \in C_{\text{stab}}$.

C. Proof of Proposition 4

Proof of Statement 1: Given $K \in C_{\text{stab}}$, we have the following transfer matrices are all stable:

$$\Phi_{yx} = \Phi_{xx}BK, \quad \Phi_{yu} = \Phi_{xx}B,$$

$$\Phi_{uy} = K(\Phi_{xx}BK + I), \quad \Phi_{uu} = KC\Phi_{xx}B + I,$$

where $\Phi_{xx} = (zI - A - BK)^{-1} \in C_{zI - A}$.

We now verify that

$$\Phi_{yx} = \Phi_{xx}BK,$$

$$\Phi_{yu} = \Phi_{xx}B,$$

$$\Phi_{uy} = K(\Phi_{xx}BK + I),$$

and

$$\Phi_{uu} = KC\Phi_{xx}B + I.$$

Furthermore, it is not difficult to verify that

$$-\Phi_{yu}G + \Phi_{uu} = -\Phi_{xx}BK + \Phi_{xx}B = \Phi_{xx}(I - KG) = (zI - A - BK)^{-1}B(I - KG) = (zI - A)^{-1}B,$$

and

$$-\Phi_{yu}G + \Phi_{uu} = I.$$ 

Proof of Statement 2: Consider any $\Phi_{yx}, \Phi_{yy}, \Phi_{xx}, \Phi_{uu}$ satisfying (18) and let $K = \Phi_{yu}^{-1}\Phi_{yx}$. Using the relationship (18), we can verify algebraically that

$$(zI - A - BK)^{-1}BK = \Phi_{yx} \in C_{zI - A}^{-1},$$

$$(zI - A - BK)^{-1}B = \Phi_{uu} \in C_{zI - A}^{-1},$$

$$K(C_{xx}BK + I) = \Phi_{uy} \in C_{zI - A}^{-1}.$$

Thus, we derive

$$\left[\begin{array}{c} \delta x \\ \delta y \end{array}\right] \rightarrow \left[\begin{array}{c} y \\ u \end{array}\right] \in C_{zI - A}.$$ 

By Theorem 1, we conclude that $K = \Phi_{yu}^{-1}\Phi_{yx} \in C_{\text{stab}}$.

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