Quadrature nonreciprocity in bosonic networks without breaking time-reversal symmetry

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Nonreciprocity means that the transmission of a signal depends on its direction of propagation. Despite vastly different platforms and underlying working principles, the realizations of nonreciprocal transport in linear, time-independent systems rely on Aharonov–Bohm interference among several pathways and require breaking time-reversal symmetry. Here we extend the notion of nonreciprocity to unidirectional bosonic transport in systems with a time-reversal symmetric Hamiltonian by exploiting interference between beamsplitter (excitation-preserving) and two-mode-squeezing (excitation non-preserving) interactions. In contrast to standard nonreciprocity, this unidirectional transport manifests when the mode quadratures are resolved with respect to an external reference phase. Accordingly, we dub this phenomenon ‘quadrature nonreciprocity’. We experimentally demonstrate it in the minimal system of two coupled nanomechanical modes orchestrated by optomechanical interactions. Next, we develop a theoretical framework to characterize the class of networks exhibiting quadrature nonreciprocity based on features of their particle–hole graphs. In addition to unidirectionality, these networks can exhibit an even–odd pairing between collective quadratures, which we confirm experimentally in a four-mode system, and an exponential end-to-end gain in the case of arrays of cavities.

In nonreciprocal systems, such as isolators, the transmission varies when interchanging the input and output, in an ideal case from unity to zero. Nonreciprocity is a resource for many applications, including sensing, constructing bosonic networks with general routing capabilities and realizing topological phases. Complemented with gain, nonreciprocity amplifies weak signals while protecting the source against noise, making it ideal for quantum information-processing applications. Magnetic-free isolators and directional amplifiers have been proposed based on parametric modulation, interfering parametric processes and reservoir engineering and have been experimentally demonstrated in different platforms, such as superconducting circuits and optomechanical systems.

In linear systems, achieving a nonreciprocal response relies on breaking time-reversal symmetry (TRS) in the Hamiltonian by employing real or synthetic magnetic fields and dissipation. The directionality in this standard kind of nonreciprocity (sNR) does not depend on the phase of the input signal. In this Article we extend the notion of nonreciprocity in linear bosonic systems by identifying a...
class of systems that show a kind of unidirectional signal transmission, positioned between reciprocal and standard nonreciprocal transmission, whose (un)directionality depends on the phase of the input signal. We dub the defining property of this class quadrature nonreciprocity (qNR), as it can be revealed when resolving the signal into its quadrature components. In contrast to sNR, a qNR Hamiltonian does not break TRS, but achieves unidirectional transport by interfering beamsplitter (excitation-preserving) and two-mode squeezing (excitation non-preserving) interactions. It does not require strong Kerr nonlinearity \([1,2,18]\) or spin-polarized emitters \([19,20]\). We report experimental realizations using an optomechanical network and construct a comprehensive theoretical framework that exposes an entire class of qNR systems with exciting properties, including an even–odd pairing between collective quadratures and exponential end-to-end gain in resonator chains. Our work introduces systematic tools to treat unidirectional phase-sensitive transport in bosonic lattices, which have recently sparked interest in connection with bosonic analogues of the Kitaev chain \([15,20]\). It further opens the door to studying exotic phenomena in these models, such as multimode entanglement \(^6\) and non-Hermitian topology \(^3\). From the point of view of applications, the concept of qNR opens up new avenues for signal routing and amplification.

**Defining qNR**

We consider a network of \(N\) driven dissipative bosonic modes. Their steady-state response to a coherent probe follows from the Heisenberg–Langevin equation of motion \(\dot{x}_j = \lambda a_j - \sqrt{\gamma} q_m\) for the field quadratures \(x_j \equiv (a_j + a_j^\dagger)/\sqrt{2}\) and \(p_j \equiv -i(a_j - a_j^\dagger)/\sqrt{2}\), where \(a_j\) denotes the annihilation operator for the bosonic mode \(j \in \{1, \ldots, N\}\). \(\lambda\) is the dynamical matrix and \(\gamma\) is the damping rate. We use the vector notation \(\mathbf{q}_1 = (x_1, p_1, \ldots, x_N, p_N)^T\) for the system’s quadratures and similarly \(\mathbf{q}_m\) for the quadrature inputs. We assume that each mode dissipates only via coupling to the input–output port, although our analysis extends to general temporal coupled-mode theory \([19,21]\). The scattering matrix \(S(\omega)\) connects input \(\mathbf{q}_m\) and output quadratures \(\mathbf{q}_{\text{out}}\) with frequency \(\omega\), satisfying the input–output boundary conditions \(\mathbf{q}_{\text{out}} = \mathbf{q}_m + \sqrt{\mathbf{q}} \) (refs. 34, 35).

\[
S(\omega) \equiv \mathbb{1} \pm \gamma(\mathbb{1} + \lambda M )^{-1} \mathbb{1} \pm \gamma \chi(\omega) \tag{1}
\]

with the susceptibility matrix \(\chi(\omega) \equiv (\mathbb{1} + \lambda M)^{-1}\). We will, from now on, consider resonant driving \(\omega = 0\) in a frame rotating at the mode frequencies and write \(\chi \equiv \chi(0)\) and \(S \equiv S(0)\) for brevity.

Let us consider a rotation of each quadrature pair \(\{x_j, p_j\}\) with respect to some external phase reference (Fig. 1a, inset), namely,

\[
\begin{pmatrix}
    x_j(\phi) \\
    p_j(\phi)
\end{pmatrix} =
\begin{pmatrix}
    \cos \phi & -\sin \phi \\
    \sin \phi & \cos \phi
\end{pmatrix}
\begin{pmatrix}
    x_j(0) \\
    p_j(0)
\end{pmatrix} \equiv R(\phi) \begin{pmatrix}
    x_j(0) \\
    p_j(0)
\end{pmatrix} \tag{2}
\]

Equation (2) can be understood as a \(U(1)\) gauge transformation in the mode basis \(\{a_j, a_j^\dagger\}\) (Supplementary Section IIA) \([15,20]\). The susceptibility matrix \(\chi\) transforms as

\[
\chi(\phi) \equiv U(\phi) \chi(0) U(\phi)^\dagger \tag{3}
\]

where we introduce \(U(\phi) \equiv \otimes_{j=1}^N R(\phi_j)\).

Nonreciprocity is typically defined by asymmetric transmission amplitudes \([1,3]\). We adopt this notion in a generalized sense, with signals split into their \(x, p\) quadratures, meaning \(\mathbb{1}^T \neq \mathbb{1}\) for equation (1) or, equivalently, \(|\gamma|^T \neq |\gamma| (\mathbb{1})^\dagger\) denotes taking the element-wise modulus. We say a system exhibits qNR if there exist at least two different sets of local gauges \(\phi^{(1,2)}\) corresponding to a nonreciprocal and reciprocal susceptibility matrix, respectively. Mathematically, this implies a pair of rotations \(U_{1,2} \equiv \otimes_{j=1}^N R(\phi^{(1,2)}_j)\) such that

\[
|U_1 \chi U_1^T| \neq |U_2 \chi U_2^T| \quad \text{and} \quad |U_2 \chi U_2^T| = |U_2 \chi U_2^T| \tag{4}
\]

**The qNR dimer, the simplest qNR system**

We now introduce the minimal system displaying qNR: the parametrically driven dimer shown in Fig. 1a. This comprises two modes \(a_{1,2}\) coupled via a beamsplitter (BS) coupling of strength \(J\) and two-mode-squeezing (TMS) coupling of strength \(\lambda\). In a frame rotating at the mode frequencies, the Hamiltonian reads

\[
H = J a_1^\dagger a_2 + J a_1 a_2^\dagger + h.c. \tag{5}
\]

corresponding to the most general bilinear coupling. As we explain below, we implement these couplings between mechanical resonators through a time-modulated optical spring. For \(J = \lambda\), we recover the well-known position–position coupling, which has been extensively studied, for example for implementing quantum non-demolition measurements \([18,19]\) or generating squeezing and entanglement \([20,21]\). Moreover, this Hamiltonian was also introduced as bidirectional phase-sensitive amplifier in ref. 39, which was implemented in a superconducting circuit and used for qubit readout \([22]\), and considered as a starting point for nonreciprocity with broken TRS in ref. 13.

As we now show, this coupling has an important impact in the context of nonreciprocal signal transduction. Equation (5) leads to the following equations of motion for the quadratures:

\[
\begin{align}
    x_1 &= -\frac{\gamma}{2} x_1 + (J - \lambda) p_2 - \sqrt{\gamma} x_{1,\text{in}} \\
    p_1 &= -\frac{\gamma}{2} p_2 + (J + \lambda) x_2 - \sqrt{\gamma} p_{1,\text{in}} \\
    x_2 &= -\frac{\gamma}{2} x_2 + (J - \lambda) p_1 - \sqrt{\gamma} p_{2,\text{in}} \\
    p_2 &= -\frac{\gamma}{2} p_2 - (J + \lambda) x_1 - \sqrt{\gamma} p_{2,\text{in}}. \\
\end{align} \tag{6}
\]

from which we can see that the quadratures can decouple, due to the fact that BS and TMS couplings enter with opposite sign. In particular, setting \(J = \lambda\) in equation (6) leads to perfect decoupling between \(x_1(x_2)\) and \(p_1(p_2)\), while \(p_2(p_1)\) still couples to \(x_1(x_2)\), in a way that is formally equivalent to a cascaded quantum system \([22,42]\). This is also reflected in the susceptibility matrix:

\[
\chi(\phi) =
\begin{pmatrix}
    \frac{2}{\gamma} & 0 & 0 & 0 \\
    0 & -\frac{2}{\gamma} & \frac{\gamma}{2} & 0 \\
    0 & 0 & -\frac{2}{\gamma} & 0 \\
    \frac{\gamma}{2} & \frac{\gamma}{2} & \frac{\gamma}{2} & \frac{\gamma}{2}
\end{pmatrix} \tag{7}
\]

We interpret this property by saying that a signal encoded in quadrature \(x_1\) can propagate from mode 1 to 2, emerging as \(p_2\), while the reverse transduction, that is, \(p_1 \rightarrow x_1\), does not take place. When \(J = \lambda > \gamma/8\), we further have \(S_{x_2 \rightarrow p_1} = \gamma\), \(S_{x_2 \rightarrow p_1} > 1\), which signifies phase-sensitive amplification.

To demonstrate this unidirectional transport between quadratures, we implement Hamiltonian (5) in a sliced photonic-crystal nanobeam—a nano-optomechanical network where multiple non-degenerate, megahertz-frequency flexural mechanical modes are coupled to a single broadband, telecom-frequency optical mode via radiation pressure. Two mechanical modes with equal linewidth \(\lambda\) serve as resonators \(a_q\) with effective interactions enabled via the cavity by temporally modulated optical driving of a detuned laser. Specifically, BS (TMS) coupling is stimulated by modulating the intensity of the...
drive laser at the mechanical frequency difference (sum)\(^3\), allowing control over both the strength and phase of the interaction through the depth and phase of the modulation tone, respectively (Methods). This particular protocol for parametric coupling between mechanical modes operates in the bad-cavity limit \(\kappa \gg \gamma\) and at large optical detuning \(\Delta = -\kappa/(2\sqrt{3})\), where \(\kappa\) is the cavity linewidth. Under these conditions, mechanical displacement modifies the intracavity photon number instantaneously on mechanical timescales. The aforementioned drive-laser modulations then create additional sidebands in the optical force that are mutually resonant between selected resonators (Methods). In effect, this protocol can implement arbitrary quadratic bosonic Hamiltonians for mechanical resonators\(^4\).

In addition, the optomechanical interaction allows optical read-out of the nanobeam displacement, as imprinted on the intensity of a spectrally resolved probe laser reflected from the cavity. The detected displacement signal \(h(t) = k_1 z_1(t) + k_2 z_2(t) + \ldots\) contains a superposition of the resonator coordinates \(z_j(t)\), transduced with strengths \(k_j\), which can be resolved in frequency as the resonators’ frequency separation greatly exceeds their linewidths.

In our measurements, we define (electronic) local oscillators (LOs) at the resonator frequencies \(\omega_j\) that demodulate—in parallel—the displacement signal, to obtain the amplitude envelope \(|a_j(t)|\) and relative phase \(\phi_j(t)\) of each resonator’s harmonic motion \(z_j(t) = |a_j(t)| \cos(\omega_j t + \phi_j(t))\). In effect, the LOs define a rotating frame of reference in which the resonator dynamics can be tracked by a complex amplitude \(|a_j\rangle = |a_j\rangle e^{i\eta_j}\), or equivalently by the quadrature amplitudes \(|x_j\rangle = \sqrt{2} \text{Re}(a_j)\) and \(|p_j\rangle = \sqrt{2} \text{Im}(a_j)\). Finally, signals to drive the resonator quadratures coherently through radiation pressure are derived from the same LOs, turning our experiment into a lock-in measurement. A sketch of the set-up is shown in Extended Data Fig. 2.

To connect the unidirectional response of equation (7) to the definition of qNR given in equation (4), we need to study how the dimer transforms under a change of gauge. We thus measure the dimer’s quadrature-resolved response when performing the gauge transformation in equation (3) (Fig. 1a, inset). This is experimentally achieved by referring both the interaction tones and LOs to a common time origin and subsequently adding a phase offset \(\phi\) to the LOs, rotating the quadratures they define. Note that even though the frequencies of LOs \((\omega_{1,2})\) and interaction tones \((\omega_{1,2} \pm \omega)\) are all distinct, the fact that the latter signals can be derived from the former through mixing leads to a well-defined relation between the LO and interaction phases (Methods).

For different phases \(\phi\), we independently reconstruct the susceptibility matrix (3) for \(f = \lambda\) in Fig. 1a. For the full expression of equation (3) we refer to Supplementary Section I.A; here we focus on the matrix elements shown in the bottom row of Fig. 1, namely

\[
\chi_{x_1 \rightarrow p_2} = \frac{S_j \cos^2(\phi)}{\gamma^2}, \\
\chi_{p_1 \rightarrow x_2} = \frac{S_j \sin^2(\phi)}{\gamma^2}.
\]
It is clear that nonreciprocity only reveals itself in particular rotated quadratures. Although maximal nonreciprocity is obtained for $\phi = 0$, as in equation (7), a gauge transformation reduces the ‘contrast’ of the nonreciprocity until, at $\phi = \pi/4$, the transport is completely reciprocal, that is, $|x_{i\to j}| = |x_{j\to i}| = 4/\gamma^2$, and in fact $|\chi| = |\chi_0|$. This confirms that the dimer is indeed a qNR system, as per our definition (4).

Further increasing $\phi$ swaps the direction of nonreciprocity, $a_j \to a_d$, with complete reversal at $\phi = \pi$, when $x_{i\to j} = 0$, $|x_{j\to i}| = 8/\gamma^2$.

In the qNR dimer, the cancellation at $\phi = \pi/4$ and the reversal of directionality fundamentally stems from TRS. We identify a system’s TRS from its Hamiltonian, that is, its dynamics in the absence of local dissipation or gain. This is motivated by the fact that dissipation or gain by themselves, while breaking TRS in a ‘trivial’ way, cannot induce nonreciprocal behaviour. In Supplementary Sections II.B–D we show that TRS implies the constraints $|x_{i\to j}| = |x_{j\to i}|$ for any gauges $\phi$, and reciprocity for at least one set of gauges. This is strikingly different from a system that breaks TRS, such as the isolator of ref. 13. In Fig. 1b we show the measured susceptibility matrix for the ‘SNR isolator’, which is implemented in our optomechanical system from two equal-linewidth mechanical modes $a_1$ and $a_2$. The experiment is carried out with the drive symmetric in time, systems with drives symmetric in time ($t \to -t$) do not exhibit nonreciprocity (Fig. 1c), and ‘holes’, respectively, that are coupled in the equations of motions are annihilating ‘particles’ and ‘holes’, respectively, that are coupled in the equations of motions are connected through a line. a, if the graph forms a double loop as is the case for the qNR dimer, TRS is always preserved. b, if the resulting graph decomposes into two loops as is the case for the trivial BS dimer, TRS is always preserved. The spectral density of the thermal fluctuations in mode $a_0$ serves as an experimental signature of TRS (right column). In a gauge where all other interactions are real, we vary the phase $\phi_0$ of the BS coupling between $a_1$ and $a_2$. For a fixed $\phi_0$, this does not affect the system’s eigenfrequencies, given by the real part of the dynamical matrix eigenvalues (white lines), whereas for it signals TRS breaking. The experimental parameters are identical to those used in Fig. 1.

**TRS and qNR**

The results above point to the fact that preserving TRS in the Hamiltonian is a key element that sets qNR apart from sNR. A TRS Hamiltonian imposes certain symmetries on the dynamical matrix that underlie the unique transformation properties of a qNR susceptibility matrix. This motivates us to provide a general characterization of TRS Hamiltonians in bosonic networks. TRS means that there exists a $U(1)$ transformation

$$a_j \to e^{i\phi_0} a_j$$

(10)

that renders the coefficients in the Hamiltonian real. In concrete experiments that realize the Hamiltonian terms through parametric driving, finding the phases $\phi_0$ in equation (10) that lead to a real Hamiltonian is equivalent to making the time-dependent parametric drives symmetric in time ($t \to -t$). As we see here, even though there is one gauge in which the drive is symmetric in time, systems with phase-dependent transmission can feature unidirectional transmission. Although TRS is often associated with reciprocity, TRS only requires $|\chi|_n$ to be symmetric for one set of phases $\phi_0^{(n)}$ in equation (10) (Supplementary Section II.B). Indeed, in qNR systems, reciprocity only occurs when $\phi_0 = \phi_0^{(0)}$ (the qNR dimer), $\phi_0^{(0)} = \phi_0^{(3)} = \pi/4$, $3\pi/4$ and nonreciprocity occurs for all other $\phi_0 \neq \phi_0^{(0)}$. This feature, which is the best of our knowledge has not been recognized before, positions qNR precisely between reciprocity and gauge-independent sNR.

We develop a criterion to identify TRS for arbitrary quadratic bosonic Hamiltonians, based on the graph representation of the Hamiltonian matrix in the field basis, inspired by Bogoliubov–de Gennes theory (Supplementary Section II.B). We associate the Hamiltonian matrix with a graph in which the ladder operators $a_j$ (annihilating ‘particle’ excitations) and $a_j^\dagger$ (annihilating ‘hole’ excitations) are represented as vertices and the interactions as edges, that is, connecting $a_j$ to $a_1$ (and $a_j^\dagger$ to $a_1^\dagger$) for BS between sites $j$ and $\ell$. For the BS coupling between $a_j$ and $a_j^\dagger$, the graph representation of the three systems of Fig. 1 display manifestly different structures (Fig. 2). The graph of the qNR dimer in Fig. 2a connects all vertices in a double loop visiting each site twice. Such double loops guarantee TRS for arbitrary, complex, coupling constants. In contrast, the graph of the sNR isolator (Fig. 2b) decomposes into two disjoint loops. This structure allows TRS to be broken through a non-vanishing relative phase between the coupling constants. The BS dimer (Fig. 2c) displays no loops and trivially preserves TRS.

The different graph structures, which embody the behaviour under time reversal, are catalogued in general by a $Z_2$ invariant, which we call loop product $\mathcal{P}$. To define such a quantity, we consider the particle–hole graph for a general loop with BS and TMS interactions. Then, by multiplying by $(-1)$ for a line crossing (TMS coupling) and $(+1)$ for an uncrossed pair of lines (BS couplings), the loop product $\mathcal{P}$ distinguishes between an even (disjoint loops) and odd number (double loop) of line crossings:

$$\mathcal{P} \equiv (-1)^{n_{BS}}(+1)^{n_{TMS}} = \begin{cases} -1 : & \text{double loop (TRS)} \\ +1 : & \text{disjoint loops} \end{cases}$$

(11)

with $n_{BS}$ being the number of BS couplings and $n_{TMS}$ the number of TMS couplings. Equation (11) indicates that the only requirement for TRS in a general loop is an odd number of TMS couplings. In other words, their position in the loop is irrelevant.

Complementing our graph-based theoretical criterion above, the experimental response of a system to incoherent excitation serves as...
obtain a nonreciprocal N with χ between collective quadratures of even and odd sites in the susceptibility matrix.

non-conserving processes. This condition brings us to the necessary

destructive interference between particle-conserving and particle condition discussed along with equation (6) as constructive and destructive interference between particle-conserving and particle non-conserving processes. This condition brings us to the necessary

contrast compared to b. For ϕ = π/4, χ is reciprocal, showing the qNR of transmission of the ring. d. Susceptibility matrices in the basis of even and odd collective quadratures show their pairing for ϕ = 0, when χ is block-diagonal. Similarly, reciprocal susceptibility is obtained for ϕ = π/4. e,f. The values of selected susceptibility matrix elements (e, resonator basis, dashed boxes in c,f. collective basis, dashed boxes in d) show continuous tuning as function of quadrature angle ϕ, for theory (solid lines) and experiment (circles). Even in theory, the isolation for ϕ = 0, n/2, n in the resonator basis (e) is not perfect, whereas in the collective basis (f) it is. Error bars are obtained by repeating the measurement sweep ten times and represent the statistical ± 2σ spread around the average value.

a signature of a TRS Hamiltonian\(^{44}\) as non-trivial fluxes manifest in the eigenfrequencies (Fig. 2, right column). We choose a gauge in which all interactions are real, except for the BS coupling between modes \(a_1\) and \(a_2\) present in all three systems studied so far, whose phase \(ψ\) we vary. Thermal fluctuations stochastically drive all mechanical quadratures homogeneously and lead to a power spectrum insensitive to \(ψ\) if there is a gauge transformation (equation (10)) that removes this BS phase. In our experiment this indicates the TRS of the qNR (Fig. 2a) and BS Hamiltonians (Fig. 2c). Conversely, if the eigenfrequencies tune with \(ψ\), such a gauge transformation cannot exist, marking the broken TRS of the sNR isolator (Fig. 2b).

Finally, we stress another consequence of TRS, already visible in the dimer (equation (9)), that is, that quadratures travel in pairs in opposite directions. In fact, the transduction \(x_j → p_j\) is accompanied by \(x_j → p_j\) with equal transmission in the opposite direction, for example, as in equation (7). \(|x_j → p_j\| = |x_j → p_j|\) (see Supplementary Sections IIC,D for a proof). This requirement of counter-propagating pairs of quadratures is reminiscent of other TRS systems, such as quantum spin Hall systems\(^{5,6}\).

Construcuting qNR ring networks

We now take inspiration from the particle–hole graphs and look for qNR in TRS-preserving N-mode ring networks (\(P = -1\), dubbed ‘N-rings’) from now on. To obtain a ring with qNR transmission, a TRS Hamiltonian is only necessary, not sufficient. We also have to guarantee that BS and TMS couplings can interfere. The particle–hole representation allows a reinterpretation of the quadrature decoupling condition discussed along with equation (6) as constructive and destructive interference between particle-conserving and particle non-conserving processes. This condition brings us to the necessary and sufficient condition for qNR. Besides the odd number of TMS couplings (equation (11)), achieving qNR requires that the ring consists of an even number of modes. To understand why, we take a closer look at the four-mode ring with a single TMS interaction (Fig. 3a). The corresponding equations of motion for equal couplings (\(J = 1\)) read

\[ x_1 = -\frac{\sqrt{3}}{2}(p_2 - p_4) - \frac{\sqrt{3}}{2}x_1 - \sqrt{3}x_{1,\text{lin}}, \]

\[ p_2 - p_4 = \frac{\sqrt{3}}{2}(p_2 - p_4) - \sqrt{3}(p_{2,\text{lin}} + p_{4,\text{lin}}). \tag{12} \]

The coupling to \(x_1\) vanishes in the second equation for the collective quadrature \((p_2 - p_4)\) due to the interference of BS and TMS couplings. Equations (12) have a similar structure as those for the qNR dimer (equation (6)), the main difference being that \(x_1\) couples nonreciprocally to the collective quadrature \((p_2 - p_4)\) instead of a local quadrature. Analogously, \(x_2\) couples to \((p_1 - p_3)\) nonreciprocally. As a consequence, the dominant elements of the susceptibility matrix are coupling \((p_1 - p_3)\) to \(-x_2\) and \(x_2\) as well as \((p_2 - p_4)\) to \(-x_1\) and \(x_1\). This peculiar non-local pairing between even and odd quadratures is shown in Fig. 3b for a four-ring and carries over to larger N-rings with even \(N\), independent of system size. The pairing is only possible if \(N\) is even, as it requires an equal number of quadratures on even and odd sites, respectively. We rigorously derive this condition using the particle–hole graphs and a reduction technique detailed in Supplementary Section IV.

We implemented the four-ring shown in Fig. 3a experimentally. The measured susceptibility matrix in the resonator quadrature basis (Fig. 3c) illustrates its nonreciprocal response for \(ϕ = 0\), while the non-reciprocity vanishes completely for \(ϕ = π/4\), establishing qNR in this system. For maximum nonreciprocity \((ϕ = 0\), the largest-magnitude
Towards qNR lattices

Going one step further, we use qNR rings as building blocks for constructing qNR lattices. It follows from Supplementary Sections IIIC and IV that, if a lattice overall preserves TRS and contains at least one qNR ring, it is itself qNR. Of particular interest are translational-invariant chains of qNR N-rings, which can combine qNR transmission with non-Hermitian topology. One such example is given by the bosonic Kitaev chain of ref. 29, which corresponds to a two-ring chain with a specific choice of inter-ring coupling phases, and is indeed a qNR system. Thanks to our particle–hole graph framework, we can now consider more complex scenarios. As an example, we take a chain of qNR four-rings connected through BS interactions. In Figure 4a,b, we show the simulated steady-state field quadratures for this qNR lattice for \( \phi = 0 \), given a quadrature-resolved input at different sites. Nonreciprocal transmission is clearly visible, and changing the gauge to \( \phi = \pi/4 \) (not shown) leads to complete reciprocity; that is, we recover the characteristic feature of qNR, equation (1). The pairing we characterized in the previous sections also manifests: an input at the 1,1 quadrature (in which the indices stand for row and column in the array) (Fig. 4a) procures the largest steady-state response at two sites at the opposite end of the chain. The pairing is evidenced in larger qNR ring sizes, in which the steady-state response of an eight-ring chain is most prominent in every second site of the last plaquette shaping a zigzag line (Fig. 4c).

In this system, qNR is accompanied by amplification, with the end-to-end gain growing exponentially with the chain length (Fig. 4d). This generalizes the phase-dependent directional amplification predicted in a bosonic Kitaev chain. Directional amplification with exponential end-to-end gain has been identified as proxy of non-trivial topology. Based on previous results for two-ring chains, we conjecture a connection with a non-trivial non-Hermitian winding number, calculated from the determinant of the non-Hermitian matrix, the dynamical matrix in our case (Methods). In Fig. 4c, we plot the determinant of each of the diagonal blocks of the Bloch matrix (each of the blocks accounts for a set of collective quadratures; Fig. 4e) for the four-ring qNR chain of Fig. 4a under periodic boundary conditions. We find that the determinant of each of these blocks can be assigned a non-trivial winding number that equals in magnitude and differs in sign corresponding to the two directions of directional end-to-end gain. The opposite winding sense of these two matrix blocks is an expression of the TRS Hamiltonian.

**Outlook**

In conclusion, we have introduced the novel phenomenon of qNR. In contrast to standard nonreciprocity, qNR does not break TRS in the Hamiltonian and presents a characteristic gauge dependence. We have identified the set of bosonic networks that display qNR and have reported the first experimental realizations.

Our results point to a close connection of qNR with the existence of quantum nondemolition (QND) variables and with backaction evading measurements. Indeed, our characterization of qNR simultaneously provides a powerful recipe to design systems with collective QND variables. In this context, exploring the noise properties of the qNR networks we introduced will also be interesting. As they rely on interference between coherent interactions, without the necessity of dissipation, we can expect quantum-limited performance on suitable platforms. Future qNR devices may play a part in quantum applications such as efficient, noiseless sensors and quantum information routers and the generation and measurement of non-classical states, including entangled states. Indeed, while we have demonstrated the concept in the domain of nanomechanics, it could find application in classical electrical circuits, acoustics, superconducting circuits and spin ensembles, for example.

Going forward, we envision the construction of lattices from qNR rings that inherit the qNR properties similar to those of Fig. 4, demonstrating the generality of our framework and relevance for other domains such as non-Hermitian topology. In the optomechanical realm, real and synthetic dimensions offer scaling potential, for example by assembling multiple nanobeams or by addressing higher-order overtones. Moreover, it will be interesting to explore qNR Hamiltonians with other platforms that have already demonstrated larger lattice sizes, such as in ref. 55. Our research opens new avenues for exploring qNR lattices and networks in which new topological phenomena may emerge similar to the quantum spin Hall effect.

**Online content**

Any methods, additional references, Nature Portfolio reporting summaries, source data, extended data, supplementary information, acknowledgements, peer review information; details of author contributions.
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Methods

Linear response and interference of BS and squeezing interactions

A general quadratic, bosonic Hamiltonian can be written in the field basis of creation and annihilation modes, namely \( \{a_i, a_i^\dagger\} \), as

\[
H = \sum_{ij} \left( a_i^\dagger a_j a_i + \frac{1}{2}(a_i^\dagger a_j^\dagger + a_i a_j^\dagger a_i a_j) \right)
\]

(13)

where overall constant shifts have been removed. Here, the matrix elements \( A_i = f_i e^{-i\omega_i t} \), \( A_i^\dagger = A_i^\dagger e^{i\omega_i t} \) of the Hermitian hopping matrix \( A \) encode BS interactions that conserve the total number of excitations. Similarly, we define the symmetric squeezing matrix \( \Sigma \) that encodes the particle-non-conserving squeezing interactions in its elements \( \Sigma_{ij} = \lambda_{ij} e^{i\phi_{ij}} \).

In the quadrature basis \( \chi_{ij} = (a_i + a_i^\dagger)/\sqrt{2} \) and \( p_i = i(a_i^\dagger - a_i)/\sqrt{2} \), equation (13) reads

\[
H = \sum_{ij} \left( T_{ij}p_i p_j + V_{ij} x_i x_j + U_{ij} x_i p_j + U_{ij}^T p_i x_j \right)
\]

(14)

where we define the effective potential matrices \( U = \text{Im}(B - A) \), \( V = \text{Re}(A + B) \) and kinetic energy \( T = \text{Re}(A - B) \) (ref. 56). The corresponding Heisenberg equations of motion read as

\[
\frac{\dot{q}_i}{\sqrt{2}} = \sum_{j=1}^{N} \left( U_{ij} q_j + T_{ij} p_j \right) - \left( V_{ij} x_j + T_{ij} p_j \right).
\]

(15)

In our platform, quadratic Hamiltonians of the form of equation (13) are effectively conceived in a rotating frame of reference that oscillates at the natural frequencies of the resonators. As such, free energy terms \( a_i^\dagger a_i \), \( a_i a_i^\dagger \) are absent, and the matrix elements for \( U \), \( V \) and \( T \) read

\[
\chi_{ij} = (a_i + a_i^\dagger)/\sqrt{2} \quad \text{and} \quad p_i = i(a_i^\dagger - a_i)/\sqrt{2}.
\]

(16)

These expressions show that matching interaction amplitudes and complex interaction phases can lead to a cancellation of different contributions in equation (14), decoupling quadratures from different resonators in the dynamics governed by equation (15) (ref. 50). This is the case for \( A_{ij} = f_j \) and \( \theta_\delta = \phi_{ij} = m\pi/2, n \in \mathbb{Z} \), where \( T = V = 0 \). This can be interpreted as destructive interference in the particle–hole space (ref. 44 and main text).

From the coefficients in equation (15), we define the dynamical matrix \( M \) for the quadrature vector \( \mathbf{q} = (x_1, p_1, \ldots, x_N, p_N)^\top \), as the matrix that fulfills \( \dot{\mathbf{q}} = M \mathbf{q} - \mathbf{F} \). Here, \( \mathbf{F} \) is a column vector grouping \( q_i = \chi_\omega \) and \( p_i = (\partial \chi_\omega/\partial \omega_\delta) \), \( \omega_\delta \) being the frequency difference (sum) frequency \( \omega_i = \omega_j = \omega_k \). The dynamical matrix \( M \) is the susceptibility matrix \( \chi \). On resonance (in the rotating frame, this is equivalent to \( \omega = 0 \)), this matrix will present asymmetries if and only if \( M \) is a nonzero asymmetric matrix. A more exhaustive discussion is provided in Supplementary Section IIB.

Device

The device, a suspended, sliced photonic-crystal nanobeam hosting a telecom-frequency optical mode coupled simultaneously to multiple non-degenerate flexural mechanical modes, was used in a previous study (ref. 44), where details can be found on its design and fabrication. For completeness, a micrograph along with the thermo-mechanical spectrum of the device is shown in Extended Data Fig. 1.

The optical mode has a free-space wavelength \( \lambda_0 = 1.533.5 \text{ nm} \) and linewidth \( \kappa = 2\pi \times 320 \text{ GHz} \). The four mechanical modes (resonators) we use have frequencies \( \omega_j = 2\pi \times 3 - 17 \text{ MHz} \), dissipation rates \( \gamma_j = 2\pi \times 1 - 4 \text{ kHz} \) and photon–phonon coupling rates \( g_{ij} = 2\pi \times 3 - 6 \text{ MHz} \).

Optically mediated mechanical interactions

The non-degenerate flexural nanobeam modes do not interact mechanically. However, by suitable modulation of radiation pressure backaction, cavity light can mediate controllable effective mechanical interactions (44, 52).

Our cavity operates in the unresolved sideband regime or bad-cavity limit (\( \omega_\delta < \kappa \)), such that external drives of the cavity field modulate the photon population \( n_c \) instantaneously on mechanical timescales. Moreover, for the optimal detuning \( \Delta = -\kappa/(2\sqrt{\kappa}) \), mechanical displacements maximally modulate \( n_c \) and, in turn, the radiation pressure force that acts back on the displacement. For a single resonator, such instantaneous optomechanical backaction leads to the well-known optical spring effect (ref. 44), namely an effective shift \( \delta \omega = 2\omega_\delta^2\Delta/(\Delta^2 + \kappa^2) \) in the resonator’s frequency \( \omega_i \to \omega_i + \delta \omega \), where \( \delta \omega \) is the intrinsic mechanical frequency.

In multimode systems, the cavity \( n_c \) contains contributions for each resonator’s motion. For fixed drive-laser intensity, contributions from different resonators are mutually off-resonant, and only each resonator’s self-backaction (that is, optical spring) is relevant. However, a modulation of the drive-laser intensity can cause the motion of one resonator to influence that of another one, by dynamically modulating the ‘cross-resonator’ optical spring. In practice, harmonic modulation of the drive-laser intensity with frequency \( \omega_m \), depth \( c_m \), and phase \( \phi_m \) mixes with the displacement contributions in \( n_c \) to create additional sidebands in the optical force. By setting \( \omega_m \) equal to a mechanical difference (sum) frequency \( \omega_i = \omega_j = \omega_k \), these sidebands are mutually resonant between resonators \( j \) and \( k \). Consequently, by stimulating the resonant frequency conversion process selected by \( \omega_m \), these sidebands induce a light-mediated effective mechanical BS (squeezing) interaction after adiabatic elimination of the cavity field.

The effective BS and squeezing interaction rates \( J_{jk} \) and \( \lambda_{jk} \), respectively, are tuned by the depth \( c_m \) of the corresponding modulation tone and given by (44, 45)

\[
[J_{jk}, \lambda_{jk}] = c_m \frac{g_{jk} \Delta}{\Delta^2 + \kappa^2/4} = c_m \sqrt{\frac{\delta \omega \omega_n}{2}}
\]

(18)

where \( g_{jk} = \sqrt{n_c} \delta \omega_j \) is the optomechanical coupling rate enhanced by the laser-driven average cavity population \( n_c \). The last expression relates the interaction rates to the (easily measured) optical spring shifts \( \delta \omega_j \), avoiding the need to know \( n_c \) and the cavity in-coupling efficiency precisely. We note that \( \delta \omega_j \) and \( \delta \omega_j \) always have the same sign. Moreover, the modulation phase \( \phi_m \) is imprinted on the effective interactions in the appropriate rotating frame (section ‘Phase-coherent interaction tones’) and controls TRS. Finally, as all mechanical frequencies are distinct and incommensurate, modulation tones can be applied simultaneously to induce multiple interactions.

Experimental set-up and procedures

In this study we employed the same set-up as in ref. 44, including the same procedures to characterize the system, to calibrate the experiments and to attain phase coherence between the driven and detected signals. For completeness, here we recapitulate those methods as presented in ref. 44.

A schematic of the experimental set-up is displayed in Extended Data Fig. 2. The nanobeam sample was placed in a vacuum chamber at room temperature and a pressure of \( 2 \times 10^{-7} \text{ mbar} \) aligned at 45° relative to the vertical polarization light focused from free space at normal incidence. A detection laser (Toptica CTL 1550) far detuned from the cavity resonance \( (\omega_{det} - \omega_c = -2.5\kappa) \) at incident power \( P_{det} = 2 - 4 \text{ mW} \) served to detect the mechanical resonators’ displacement. The
detection laser light reflected from the cavity was filtered using a cross-polarized detection scheme, fibre-coupled, separated from the other laser light using a tunable bandpass filter (DiCon), and detected on a fast photodetector (New Focus 1811, d.c.-coupled). Intensity modulations of the detection laser encoding resonator displacements were analysed using a Zurich Instruments UHFLI lock-in amplifier (LIA).

The intrinsic resonator frequencies $\omega_j$ and linewidths $\gamma_j$ were obtained from thermomechanical spectra, and vacuum optomechanical coupling rates $g_{zz,j}/(2\pi) = (5.30 \pm 0.14, 5.86 \pm 0.17, 3.29 \pm 0.30, 3.12 \pm 0.89)$ MHz were estimated from nonlinear transduction for the four resonators, respectively. The cavity linewidth was estimated from the detuning dependence of the spring shift. A drive laser (Toptica CTL1500) was launched through the same fibre and collimator into the set-up after modulation by a Thorlabs LN91S-FC intensity modulator (IM) at an incident power of $P_{\text{drive}} = 1.0$ mW. A small part of the modulated drive-laser light was split off to be monitored by a fibre-coupled fast photodetector (New Focus 1811, d.c.-coupled).

Control signals, including sum- and difference-frequency tones to generate interactions as well as resonant excitation signals, were generated by the lock-in amplifier, amplified (Mini-Circuits ZHL-32A+ with 9-dB attenuation), and connected to the radiofrequency (RF) port of the IM to modulate the drive laser’s radiation pressure.

Measurement-based feedback. Additionally, driving signals to regulate mechanical damping rates through measurement-based feedback were generated by filtering the electronic displacement signal using a digital signal processor (DSP, RedPitaya STEMlab 125-14) that implemented a configurable electronic bandpass filter with tunable gain and phase shift (using the PyRPL suite). The DSP output was combined with the control signals before the RF amplifier. Choosing a proper phase shift allowed the effective feedback of a signal proportional to the mechanical velocity of a given resonator as a radiation pressure force to dampen the resonator, in an approach similar to that in refs. 60,61. The desired damping rates were realized by adjusting the electronic feedback gain coefficients for each individual resonator. Further details on the procedure to calibrate the required electronic phase shifts and gains are available in ref. 44.

Characterization and preparation of resonators. A standard procedure was performed immediately before every experiment to compensate for variations in in-coupling and out-coupling efficiency associated with drifts of the optical sample alignment. The spring-shifted resonator frequencies $\omega_p$, linewidths $\gamma_j$, and root-mean-squared (r.m.s.) displacement voltage levels $\epsilon_{\text{rms}}$ were obtained from thermomechanical spectra with modulations off. The displacement voltage corresponding to a single phonon amplitude was calculated using $\epsilon_p = \epsilon_{\text{rms}}/(\hbar \omega_p/\gamma_p)$. Here, $\hbar = k_B T / \omega_p$ is the thermal occupancy of the resonator’s bath at $T = 295$ K and $\hbar \omega_p/\gamma_p$ compensates for thermo-optically induced dynamical backaction.

This dynamical backaction was also actively controlled in the qNR dimer experiments to equalize the two resonators’ loss rates, as an alternative to the measurement-based feedback described above. Fine-tuning the laser intensity allowed matching the linewidths of modes 3 and 4, as those exhibited different scaling with mean photon number.

Calibration of control signals. To induce effective interactions with desired rates $\gamma_j$, the required modulation depths $c_{\text{mod}}$ were determined from equation (18). For every individual tone, the d.c.-coupled modulation monitor detector was used to calibrate the relation between $c_{\text{mod}}$ and the electronic control signal amplitude $V_{\text{in}}$, compensating the frequency-dependent transmission of the RF chain.

The control tones were synthesized digitally by the lock-in amplifier. This allowed us to reference their phase offsets to an effective internal time origin, which then defined a deterministic gauge in which the modulation phases were set, and the response was analysed.

Analysis of the displacement signal. As described in ref. 44, the electronic displacement signal was demodulated in parallel at each resonator’s frequency $\omega_j$ using electronic LOs internal to the lock-in amplifier that were referenced to the same clock as the control tones. For each resonator, the demodulated in-phase ($i$) and quadrature ($q$) components were filtered (third-order low-pass filter, 3-dB bandwidth 50 kHz) and combined into a complex amplitude $\zeta_j(t) = i(t) + i_q(t)$ that is formally equivalent to the resonator amplitude in the rotating frame.

The total signal delay through the set-up, from the LIA control outputs via the sample to the LIA input, was determined by driving each of the resonators and measuring the coherent response. The phase offset $\alpha_j$ between drive tone and coherent response of resonator $j$ was extracted and fitted linearly against the resonator frequencies $\omega_j$. The fitted delay (on the order of 100 ns) was used to relate the quadratures of the demodulated amplitudes $\zeta_j(t)$ to those defined by the control tones.

Phase-coherent interaction tones

Even though the frequencies of LOs ($\omega_f$) and interaction tones ($\omega_i \pm \omega_j$) are all distinct, a phase relation between them can nevertheless be established. To do so, we absorbed the explicit time dependence of each tone, $\cos(\omega_f t + \phi_j) = \cos \beta_j(t)$, into the instantaneous phase $\beta_j(t) = \omega_f t + \phi_j$. Phase coherence was then attained between an interaction tone $\ell \pm \ell$ (at $\omega_j \pm \omega_i$) and the combination of both LOs $\ell$, $\ell (\omega_j)$, because the phase difference

$$\Delta \phi_{\text{int}} = \beta_{\ell}(t) - \left( \beta_{\ell}(t) \pm \beta_j(t) \right)$$

is stable (time-independent). Physically, this phase difference may be evaluated by generating a tone with the combined instantaneous phase $\beta_{\ell}(t) \pm \beta_j(t)$ through mixing of the LOs and subsequent high-pass (low-pass) filtering, and comparing that to the interaction tone.

Experimental settings

For the experiments shown in Figs. 1 and 2, the mechanical resonances labelled 3 and 4 (frequencies $2\pi \times 12.8$ and $2\pi \times 17.6$ MHz) were used, with their decay rates matched thermo-optically to $\gamma = 2\pi \times 3.7$ kHz by appropriately adjusting the average drive-laser intensity. For the three-mode experiments shown in Figs. 1b and 2b, the mechanical resonance labelled 2 (frequency $2\pi \times 5.3$ MHz, thermo-optically tuned dissipation rate $\gamma' = 2\pi \times 1.3$ kHz) was utilized as the auxiliary mode.

The four-mode experiments shown in Fig. 3 were performed using the mechanical resonances labelled 1 (frequency $2\pi \times 3.7$ MHz) to 4 at maximum drive-laser power (thermo-optically tuned dissipation rates $2\pi \times (1.9, 0.9, 4.2, 3.5)$ kHz, respectively). In this case, simultaneous feedback cooling on all mechanical resonance was employed to increase and equalize their decay rates to $\gamma = 2\pi \times 10$ kHz. In addition to homogenizing the system, the feedback cooling reduces the amplitude of thermal fluctuations, thereby ensuring that the cavity photon population responds linearly to mechanical displacement, and rendering nonlinear transduction and nonlinear reduction of the optical spring shift unimportant.

Reconstructing the susceptibility matrices

The central relation used to analyse the results of our experiments is the steady-state response $q_{\text{ss}}$ of a system of $N$ resonators described by Hamiltonian (13) to resonant forces driving its quadratures with amplitudes $f^{(\ell)} = f^{(\ell_1)} \ldots f^{(\ell_N)}$. This response is measured in units of the zero-point fluctuations $x_{\text{ss},\ell}$, and given by

$$q_{\text{ss}} = x_{\text{ss},\ell} f^{(\ell)}$$

(20)
Harmonic driving forces were generated by modulation of the drive-laser intensity \( I(t) = I_0 (1 + c \cos (\omega t_d + \phi_d)) \), where \( I_0 \) is the average drive-laser intensity, \( c \) the depth of the modulation, \( \omega_t \) its frequency, matched to the mechanical resonance of interest, and \( \phi_d \) its phase. Due to the short photon lifetime \( \langle k \rangle \approx \omega_t / \omega_c \), the cavity population \( n_c(t) = \mu(t) - \kappa \tau \) responds instantaneously to the drive-laser intensity on the mechanical timescale, as do the resulting optical forces \( F_j(t) = h g_k \mu_j(t) / X_p \), on each resonator \( j \). Although all mechanical resonators are coupled to the same cavity and thus experience similar optical forces, their large frequency separation \( (\omega_c - \omega_l) \ll \gamma_j \) ensures that only a single mode is resonantly driven, allowing us to neglect the driving on the other resonators.

The drive forces the evolution \( \mathbf{a} = \mathcal{M}^{(d)} \mathbf{a} - \mathcal{F}^{(d)}(t) \) of the mode vector \( \mathbf{a} = (a_1, a_2, \ldots, a_N, a_N^\dagger) \) (expressed in the frame rotating along with the resonators) through the driving vector \( \mathcal{F} \), with dynamical matrix \( \mathcal{M}^{(d)} \) as defined in Supplementary Section IIIB. The odd-index elements \( f_{j+1}^{(d)}(t) = \text{e}^{i \phi_j / 2} f_{j}^{(d)}(t) \) encode the force \( F_j(t) \) driving operator \( a_j \), whereas the even-index elements \( f_{j}^{(d)}(t) = \left( f_{j}^{(d)}(t) \right)^\dagger \) are related by conjugation and drive the adjoint operator \( a_j^\dagger \). For a driving force \( F_j(t) = h g_k \cos (\omega t_d + \phi) \) resonant with resonator \( j \), with phase \( \phi \), the rotating factor \( \text{e}^{i \phi_j / 2} \) selects the positive frequency sideband of \( F_j(t) \). We neglect the non-resonant negative frequency sideband in a similar spirit to the rotating-wave approximation. This results in time-independent driving terms

\[
 f_{j}^{(d)}(t) = \frac{1}{4 \text{mod}_j X_p(t)} f_{j}^{(d)}(t). \tag{21}
\]

Before each experiment we performed a driven reference measurement for each resonator to quantify how the modulation depth \( c \) is transduced into the driving term \( f_{j}^{(d)}(t) \). We disabled all interaction modulations, such that the dynamical matrix was diagonal and only contained damping terms \( \mathcal{M}^{(d)} = -\gamma_j / 2 \). We then applied a modulation of depth in the range \( c = 1 - 5 \times 10^{-3} \) and swept its frequency \( \omega_d \) across the resonance frequency \( \omega_c \) of resonator \( j \). We normalized the response to the zero-point fluctuation amplitude, \( c \), and only contained damping terms \( \gamma_j \), reading

\[
 x^j_\text{ss} = \frac{-\gamma_j}{2} x^j_\text{ss} + \frac{\gamma_j}{2} x^j_\text{ss} - \sqrt{\gamma_j^2} \text{in} \nonumber
\]

\[
 \rho^j_\text{ss} = \frac{-\gamma_j}{2} \rho^j_\text{ss} + \gamma_j \rho^j_\text{ss} - \sqrt{\gamma_j^2} \text{in} \nonumber
\]

\[
 \rho^j_\text{ss} = \frac{-\gamma_j}{2} \rho^j_\text{ss} - \frac{\gamma_j}{2} \rho^j_\text{ss} - \sqrt{\gamma_j^2} \text{in} \nonumber
\]

\[
 \rho^j_\text{ss} = \frac{-\gamma_j}{2} \rho^j_\text{ss} - \frac{\gamma_j}{2} \rho^j_\text{ss} - \sqrt{\gamma_j^2} \text{in} \nonumber
\]

The fact that these two sets decouple is a consequence of the pairing we found for qNR rings. We now switch to reciprocal space, \(|f \rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \text{e}^{-i \mathbf{q} \cdot \mathbf{r}_i} |k \rangle \), in which we use Dirac notation to denote the basis vectors \(|j \rangle \) and \(|k \rangle \), and \( \mathbf{q} \) mimics a lattice spacing (we set \( a = 1 \)). We obtained two dynamical matrices for each of these sets

\[
 \mathcal{M}_1(k) = \begin{pmatrix}
 -\gamma & 0 & -\gamma & J \\
 0 & -\gamma & 0 & J \\
 -\gamma & 0 & -\gamma & 0 \\
 J & J & J & J
 \end{pmatrix}
\]

\[
 \mathcal{M}_2(k) = \begin{pmatrix}
 0 & -\gamma & 0 & J \\
 -\gamma & 0 & 0 & J \\
 J & J & J & J \\
 J & J & J & J
 \end{pmatrix}
\]

We calculated a topological invariant from each matrix, namely the winding number of their determinant

\[
 \Psi_{1,2} = \frac{1}{2\pi i} \int_0^{2\pi} dq \det \mathcal{M}_{1,2}(q) \tag{25}
\]

and find

\[
 \det \mathcal{M}_{1,2}(k) = -2^\delta \cos (k) + \frac{1}{16} (4^\delta + 16^\delta + 12^\delta \phi) \nonumber
\]

\[
 \pm i (2^\delta + 16^\phi \phi / 2) \sin (k) \nonumber
\]

in which we choose the \((-)\) sign for \( \det \mathcal{M}_1(k) \) and the \((+)\) sign for \( \det \mathcal{M}_2(k) \). These two curves wind in opposite directions in the complex plane, as \( k \) evolves from 0 to \( 2\pi \), inducing the opposite sign for \( \Psi_{1} \) and \( \Psi_{2} \) (Fig. 4e).

Above, we used the plane-wave basis to identify the non-trivial topology of each block \( \mathcal{M}_1(k), \mathcal{M}_2(k) \). The eigenvalues of \( \mathcal{M}_1(k) \) are also the eigenvalues of \( \mathcal{M}_2(k) \) albeit for a different \( k \). Together they form a degenerate sub-space. To obtain the physical (and real) eigenvectors,
that is, pairs of canonically conjugated, real-valued quadratures, we need to superpose eigenvectors from these sub-spaces.

Data availability
The data in this study are available from the Zenodo repository at https://doi.org/10.5281/zenodo.7927433. Source data are provided with this paper.

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Author contributions
C.C.W. developed the theoretical framework with J.d.P. and M.B. J.J.S. fabricated the sample, performed the experiments and analysed the data. E.V. and A.N. supervised the project. All authors contributed to the interpretation of the results and writing of the manuscript.

Competing interests
The authors declare no competing interests.

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Extended Data Fig. 1| Optomechanical device. (a) Scanning electron micrograph of a device as used in our experiments. Three suspended beams are defined in the top silicon device layer (thickness 220 nm). The flexural motion of each pair of adjacent beams is coupled to an optical resonance, hosted by a point defect in the sliced photonic crystal defined by the teeth. In the presented experiments, we address only one of the cavities from free-space, at normal incidence, to drive and read out the mechanical motion of a single beam pair. 

The inset shows a top view, revealing the narrow slit separating the beam pair. More details can be found in ref. 44. (b) Spectrum of thermal fluctuations imprinted on the intensity of a read-out laser reflected off the optical cavity. In this frequency range, four mechanical resonances can be identified, with the corresponding simulated displacement profiles shown above. These mechanical modes serve as the resonators in the presented experiments.
Extended Data Fig. 2 | Schematic of the experimental set-up. A sample holding the sliced nanobeam device is placed in a room-temperature vacuum chamber rotated by 45° relative to the vertical polarization of the incoming light. Spectrally resolved drive and detection lasers are combined in-fibre, launched into free-space, and focused onto the sample, coupling into the nanocavity from normal incidence. In a cross-polarized detection scheme, the horizontal component of the light radiated out from the cavity is transmitted by a polarizing beamsplitter (PBS). Subsequently, a fibre-based tunable bandpass filter (BPF) rejects light at the drive laser frequency and transmits detection laser light onto a fast photodetector (PD2). A high-frequency lock-in amplifier (LIA) serves to analyze (In) the intensity modulations of the drive laser (for calibration) and detection laser, while also driving (Out) the drive laser intensity modulator (IM) through an amplification stage (not shown). In addition, the electronic displacement signal is routed through a digital signal processor (DSP) that optionally generates a feedback signal to modify resonator damping rates. LP, linear polarizer.