FINDING LIPSCHITZ DOMAINS IN THE COMPLEMENTS OF
BILATERALLY FLAT SETS

JARED KRANDEL

ABSTRACT. We study classes of sufficiently flat “d-dimensional” sets in \( \mathbb{R}^{d+1} \) whose complements are covered by star-shaped Lipschitz domains with bounded overlap and controlled total boundary measure. This study is motivated by the following result proved by Peter Jones as a piece of his proof of the Analyst’s Traveling Salesman Theorem in the complex plane: Any simply connected domain \( \Omega \subseteq \mathbb{C} \) with finite boundary length \( \ell(\partial \Omega) \) can be decomposed into Lipschitz domains with total boundary length bounded above by \( M\ell(\partial \Omega) \) for some \( M \) independent of \( \Omega \). In considering higher-dimensional sets, we use concepts from the theory of uniformly rectifiable sets such as the bilateral weak geometric lemma (BWGL) and prove a version of David and Toro’s result on Reifenberg parameterizations for sets with holes assuming global angle control. Our main result shows that a \( d \)-dimensional set \( E \) satisfying a BWGL which has good control on the turning of its best approximating planes has a complement which is covered by a family of star-shaped Lipschitz domains with bounded overlap and controlled total boundary measure. We also show how this result applies to uniformly rectifiable sets and Reifenberg flat sets in particular.

1. Introduction

1.1. Background. In the course of proving the Analyst’s Traveling Salesman Theorem in the complex plane, Peter Jones obtained a striking result concerning decompositions of simply connected domains with rectifiable boundaries:

**Theorem 1.1 ([Jon90] Theorem 2).** If \( \Omega \subseteq \mathbb{C} \) is a simply-connected domain and \( \ell(\partial \Omega) < \infty \), there is a rectifiable curve \( \Gamma \) such that \( \Omega \setminus \Gamma = \bigcup_j \Omega_j \) is a decomposition of \( \Omega \) into disjoint \( M \) Lipschitz domains, and

\[
\sum_j \ell(\partial \Omega_j) \leq C_0 \ell(\partial \Omega)
\]

where an \( M \) Lipschitz-domain of unit size centered at the origin has boundary defined by a function \( \theta \mapsto r(\theta)e^{i\theta} \) with \( \frac{1}{M+1} \leq r(\theta) \leq 1 \) and \( |r(\theta_1) - r(\theta_2)| \leq M|e^{i\theta_1} - e^{i\theta_2}| \).

The proof uses complex analysis. Specifically, Jones takes a conformal map \( g : \mathbb{D} \to \Omega \) and partitions \( \mathbb{D} \) into Carleson boxes on which he runs a stopping time argument to connect boxes into a collection of nice subdomains. The derivative of \( g \) varies only slightly on each domain by Koebe distortion estimates, so, after some pre-processing, one can ensure they are mapped into a collection of Lipschitz domains with total boundary controlled by integrals of \( g' \) over the disk.

*Date: October 2022.*

Jared Krandel was partially supported by the National Science Foundation under Grants No. DMS-1763973.
which, in turn, are bounded by $\ell(\Omega)$. (See [GM08] Section X Chapter 3 for a nice presentation of the proof.)

If one considers the set $\partial \Omega \cup \Gamma$ as an extension of $\partial \Omega$, then this result has the following geometric corollary: Any simple closed curve with finite length in the plane can be extended to a quasi-convex curve with at most $M$ times the length. In [AST12], the authors prove the following generalization of this corollary:

**Theorem 1.2.** Let $d \geq 2$. There exist constants $C_1, C_2 > 1$, $C_1$ depending on $d$, such that for any subset $K \subseteq \mathbb{R}^d$, there exists a connected set $\tilde{\Gamma} \subseteq \mathbb{R}^d$ such that

(i) $\tilde{\Gamma} \supseteq K$,

(ii) $\mathcal{H}^d(\tilde{\Gamma}) \leq C_1 \mathcal{H}^d(\Gamma)$ for any connected $\Gamma \supseteq K$,

(iii) For any $x, y \in \tilde{\Gamma}$, there is a path connecting $x$ and $y$, $\gamma_{x,y} \subseteq \tilde{\Gamma}$, with

$$\ell(\gamma_{x,y}) \leq C_2 |x - y|.$$ 

The proof uses a stopping time argument on trees of balls of decreasing radii centered in $\Gamma$, a connected extension of $K$. This result sprouts out of Theorem 1.1 in the direction of one-dimensional sets. It is natural to ask whether one can move forward in the direction of codimension one sets as well. In this paper, we prove the following result in the flavor of Theorem 1.1:

**Theorem 1.3.** Let $E \subseteq \mathbb{R}^{d+1}$ be a $d$-uniformly rectifiable set with $d$-Ahlfors-David regularity constant $C_0$ such that $0 < \mathcal{H}^d(E) < \infty$. Let $y \in E$ and $0 < r \leq \text{diam}(E)$. There exists a countable collection $\Lambda = \{\Omega_\alpha\}_{\alpha \in I}$ of $M(d)$-Lipschitz-star domains such that

(i) $\Omega_\alpha \subseteq \mathbb{R}^{d+1} \setminus E$,

(ii) $\chi_B(E, \text{diam}(E)) \setminus E \leq \sum_\alpha \chi_{\Omega_\alpha} \lesssim_d 1$,

and, for any $y \in E$ and $0 < r < \text{diam}(E)$,

(iii) $\sum_\alpha |\partial \Omega_\alpha \cap B(y, r)| \lesssim_{d, C_0, E} r^d$

where in (iii) the dependence of the constant on $E$ refers to the uniform rectifiability constants. This looks like Theorem 1.1 except we have replaced a general rectifiable set with a uniformly rectifiable set and the produced domains are not disjoint, but have bounded overlap and uniform local control on their boundary measure. See Definition 5.1 for the definition of Lipschitz-star. More concretely, the domains constructed in Theorem 1.3 (and the following theorems) fall into one of two collections: a disjoint collection $\Lambda_D$ or a “buffer zone” collection $\Lambda_B$ where $\Lambda = \Lambda_D \cup \Lambda_B$. The family $\Lambda_D$ is a collection of domains with disjoint interiors which occupy most of the space sufficiently close to $E$. The domains in $\Lambda_B$ form buffers around the domains in $\Lambda_D$, filling in the rest of the space not occupied by $\Lambda_D$.

We mention here that approximation of uniformly rectifiable sets by nice domains has played a vital role in the theory of harmonic measure in $\mathbb{R}^n$ for $n > 2$. [DJ90] gives a construction of Lipschitz domains which intersect a large part of the boundary of a class of nice domains. They use the result of [Dah77] on absolute continuity of harmonic measure with respect to surface measure in Lipschitz domains to translate absolute continuity from the approximating Lipschitz domain to the approximated domain. For improvements of their construction, see
The results of [ABHM19] give a characterization of rectifiability in terms of the existence of a countable collection of Lipschitz domains whose boundary covers almost all of $E$—this is essentially a qualitative version of Theorem 1.3. A similar result is given in [ABHM17] which characterizes the rectifiability of 1-sided NTA domains via countable coverings of boundaries of nice subdomains. A similar quantitative result is given in [BH17]. See also [GMT18] for sufficient conditions for uniform rectifiability in terms of properties of harmonic functions and in terms of the existence of a corona decomposition which interacts nicely with harmonic measure.

Harmonic measure interests many because of its connections to the solvability of the homogeneous Dirichlet problem in rough domains. In [AHM+20], the authors give a geometric characterization of open sets $\Omega \subseteq \mathbb{R}^{d+1}$ such that there exists $p < \infty$ such that the $L^p(\partial \Omega)$-Dirichlet problem is solvable given the background hypotheses that $\partial \Omega$ is $d$-Ahlfors-David regular and $\Omega$ satisfies the interior corkscrew condition. They prove that solvability is equivalent to $\partial \Omega$ being $d$-uniformly rectifiable and $\Omega$ satisfying a quantitative connectivity condition called the weak local John condition. A related line of research studies $L^p$ solvability of inhomogeneous problems on rough domains. In the course of preparing this work, the author was notified of [MPT22] in which the authors study equivalences of solutions to boundary value problems in rough domains and show that the regularity problem for so-called DKP operators is $L^p$-solvable on certain geometrically nice domains. In the course of their study, the authors derive a very similar result to Theorem 1.3 with the added assumption that $E$ bounds a domain $\Omega$ which satisfies the interior corkscrew condition and the added conclusion that the nice approximating domains are adapted to a DKP operator (see Section 4.3 of [MPT22] and see also [MT22] for an earlier version of their construction).

Moving away from uniform rectifiability, the second main result concerns Reifenberg flat sets:

**Theorem 1.4.** There exists a constant $\epsilon(d) < 1$ such that for any $\epsilon < \epsilon(d)$, if $\Sigma \subseteq \mathbb{R}^{d+1}$ is $(\epsilon,d)$-Reifenberg flat with $y \in \Sigma, R > 0$, there exists a countable collection $\Lambda = \{\Omega_\alpha\}_{\alpha \in I}$ of $M(d)$-Lipschitz-star domains such that

1. $\Omega_\alpha \subseteq \mathbb{R}^{d+1} \setminus \Sigma$,
2. $\chi_{B(y,R) \cap \Sigma} \leq \sum_\alpha \chi_{\Pi_\alpha} \lesssim_d 1$,
3. $\sum_\alpha |\partial \Omega_\alpha| \lesssim_{d,\epsilon} H^d(\Sigma \cap B(y, R))$.

The proof of this uses the results of Azzam and Schul in [AST18], which proves a $d$-dimensional traveling salesman theorem for lower content $d$-regular sets, which have Hausdorff $d$-content of any ball $B$ of radius $r$ bounded from below by $c r^d$ for some constant $c > 0$. One should also see [Hyd22] which gives a similar $d$-dimensional traveling salesman theorem for general sets in $\mathbb{R}^n$ and [Hyd21] which proves a version of Azzam and Schul’s theorem for Hilbert Space.

Theorems 1.3 and 1.4 differ in the control on the surface measure of their constructed domains’ boundaries. In each theorem, the domains near a certain location and scale in the set have surface measure bounded by a class of “bad” Christ-David dyadic cubes nearby in location and scale. The difference in these theorems is explained by the difference in each type of set’s control on “bad” cubes: A uniformly rectifiable set has uniform local control on its “non-flat” cubes and the angles between its best approximating planes for those cubes while Reifenberg flat sets (and...
general lower content \( d \)-regular sets) only have global control on angle turning (and no “non-flat” cubes).

Both of these results are corollaries of the more technical Theorem 3.15 below. The bulk of the paper is devoted to the proof of this result which utilizes a sharpened version of David-Toro’s results on parameterization of Reifenberg flat sets with holes, Theorem A.2, proven in Appendix A. In the outline below, we give a high-level overview of the pieces of the proof given in each section, as well as the applications of this result to proving Theorems 1.3 and 1.4.

1.2. Outline of the paper. In Section 3, we introduce relevant background on David and Toro’s Reifenberg parameterization results, Christ-David cubes, David and Semmes’s theory of uniform rectifiability, and Azzam and Schul’s results on lower content regular sets. We then introduce notation and definitions necessary for the following statement of Theorem 3.15.

The theorem takes in a set \( E \) with a collection of dyadic cubes \( \mathcal{D} \) and approximating \( d \)-planes \( \{L_Q\}_{Q \in \mathcal{D}} \) with \( \mathcal{D} \) divided into a “good” set \( \mathcal{G} \) and a “bad” set \( \mathcal{B} \) which contains all of the cubes in which \( E \) is not sufficiently close to a \( d \)-plane in Hausdorff distance. The good cubes are partitioned into stopping time regions \( \mathcal{F} = \{S\}_{S \in \mathcal{F}} \) where we impose two stopping conditions

1. We stop at a cube if one of its children \( R \) is in \( \mathcal{B} \) or satisfies \( \text{Angle}(L_R, L_Q(S)) > \delta \) where \( Q(S) \) is the top cube of the region.

In Section 4, these stopping time regions are used in conjunction with the main technical tool of the result: Theorem A.2, a sharpening of David and Toro’s result on parameterizations of Reifenberg flat sets with holes developed in [DT12] (See also [Ghi20], which gives parameterization results in the domain of \( C^{1,\alpha} \) rectifiability). A stopping time region \( S \subseteq \mathcal{D} \) gives net points \( \{x_Q\}_{Q \in S} \) and bilaterally approximating planes \( \{L_Q\}_{Q \in S} \) for \( E \) from which we can construct Coherent Collections of Balls and Planes (CCBPs) approximating \( E \) on the scales and locations covered in \( S \).

We describe how to construct a nice domain \( \Omega_p \) (which can be carved into domains of the desired type) that is roughly centered around any given point \( p \in \mathbb{R}^{d+1} \setminus E \). Given the point \( c \), we look at a set \( \mathcal{D}_p \subseteq \mathcal{D} \) of “top” scale cubes which intersect a large ball \( B_p = B(c, A' \text{dist}(p, E)) \) around \( c \). If any of these cubes are bad, we call the point \( c \) a “non-flat” point and add a collection of Whitney cubes near \( c \) to our overall collection of domains. Otherwise, \( \mathcal{D}_p \) contains only good cubes, and for each \( Q \in \mathcal{D}_p \), we construct a new stopping time region beginning at \( Q \) where we stop at a descendant of \( Q \) if either (1) one of its children \( R \) is a bad cube, or (2) a certain square function \( J_p(R) \) measuring an accumulation of angle changes between \( Q \) and \( R \) is too large. This gives a new good region \( \mathcal{D}_p \) of cubes near \( c \) from which we take the centers \( \{x_Q\}_{Q \in \mathcal{D}_p} \) and planes \( \{L_Q\}_{Q \in \mathcal{D}_p} \) as net points and planes for a CCBP \( \mathcal{Z}_p \) with initial surface \( P_0 \), a best-approximating plane for \( E \) inside \( B_p \).

Taking \( \mathcal{Z}_p \) as input, Theorem A.2 outputs a map \( g = g_p : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^{d+1} \) with

\[
|Dg_p(z) - I| \leq C\delta
\]

for appropriate \( z \) such that \( g_p(P_0 \cap B_p) \) is a \( C\epsilon \) Reifenberg flat surface approximating \( E \) well on the scales and locations determined by \( \mathcal{D}_p \). We will construct the domain \( \Omega_p \) as the image of a stopping time region \( T_p \) of Whitney cubes in the domain of \( g_p \). Indeed, by identifying the initial
surface $P_0$ with $\mathbb{R}^d \times \{0\}$ and the side of $P_0$ containing $c$ with $\mathbb{H}^{d+1}$, we take a decomposition of $\mathbb{H}^{d+1}$ into Whitney cubes $\mathcal{W}_p$ with respect to $\mathbb{R}^d$ whose distance to $\mathbb{R}^d$ is equal to their side length and set a “top” cube $W_p$ equal to the cube containing $c$. Then, we construct the region $D_p$ as the union of cubes in a collection $T_p$. This collection is formed out of cubes “below” (and including) $W_p$ with respect to $\mathbb{R}^d$ by adding a cube $W$ to $T$ if the image under $g_p$ of any cube which is “$A$-close” to $W$ in distance and diameter is surrounded only by cubes in the good region $D^p$. We finally show in Lemmas 4.7 and Lemma 4.9 that the constructed domain $g_p(D_p) = \Omega_p$ has surface measure bounded by nicely controlled sums over nearby bad cubes.

In Section 5, we describe how to carve a given domain $\tilde{\Omega}_p$ into Lipschitz-star domains without increasing the boundary measure too much. This follows from Lemma 5.2 which proves that $D_p$ is nicely carvable along with Proposition 5.5 which proves that the image of a Lipschitz-star domain under a map satisfying (1.1) is still Lipschitz-star. This concludes the construction of a nice collection of domains $\Lambda_p$ centered around a given point $p \in \mathbb{R}^{d+1} \setminus E$. We then describe how to choose a well-distributed collection $\mathcal{C}$ of points in $\mathbb{R}^{d+1} \setminus E$ around which to form $\Lambda_p$. We then prove in Proposition 5.8 that the “interior” domains $\{\Omega_p\}_{p \in \mathcal{C}}$ are pairwise disjoint, while the extensions $\{\tilde{\Omega}_p\}_{p \in \mathcal{C}}$ have bounded overlap and cover all of $B(y, \text{diam}(E)) \setminus E$ for any $y \in E$. Section 6 concludes the proof of the theorem by deducing the desired total surface measure bounds from these facts and the individual bounds on $|\partial \Omega_p|$ for any given $c$ achieved in Section 3.5.

In Section 7 we apply Theorem 3.15 to prove Theorems 1.3 and 1.4. In both cases, control on the square function $J_p(Q)$ comes from proving that the Hausdorff distance between well-approximating planes is controlled by appropriate $L^1$ beta numbers. The corresponding sums over these numbers are then controlled by the hypotheses of uniform rectifiability or Reifenberg flatness. Similarly, the hypotheses of uniform rectifiability and Reifenberg flatness imply control on total angle tilting and non-flatness. In the former case, these are standard results of David and Semmes while in the latter case, the tilt control follows from the work of Azzam and Schul on general lower content $d$-regular sets in [AS18].

Section 8 contains a number of further questions concerning possible directions for extending the results of this paper and applications of its results.

Finally, Appendix A contains the statement and proof of the sharpened parameterization result, Theorem A.2, as well as more general estimates on the variation of the derivative of the constructed parameterization function.

2. Acknowledgements

The author would like to thank Raanan Schul for many helpful discussions and suggestions. This work would not have been possible without his original inquiry into the problem and his general guidance on attacking it.

3. Preliminaries

3.1. Conventions and definitions. Whenever we write $A \lesssim B$, we mean that there exists some constant $C$ independent of $A$ and $B$ such that $A \leq CB$. If we write $A \lesssim_{a,b,c} B$ for
some constants $a, b, c$, then we mean that the implicit constant $C$ mentioned above is allowed to depend on $a, b, c$. We will sometimes write $A \simeq B$ to mean that both $A \lesssim B$ and $B \lesssim A$ hold.

In many computations, we use a constant $C$ to denote a catch-all, general constant which is allowed to vary significantly from one line to the next.

For $F, E \subseteq \mathbb{R}^n$, $a \in \mathbb{R}^n$, we let
\[
dist(E, F) = \inf\{|x - y| : x \in F, y \in E\},
\]
\[
dist(a, E) = \inf\{|x - y| : x \in F, y \in E\},
\]
and define
\[
diam(F) = \sup\{|x - y| : x, y \in F\}.
\]

For any $r > 0$, we let
\[
B(E, r) = \{x \in \mathbb{R}^{d+1} : \text{dist}(x, E) < r\}.
\]

For any subset $F \subseteq \mathbb{R}^{d+1}$, an integer $m \geq 0$, and constant $0 < \delta \leq \infty$, we define
\[
\mathcal{H}^m(F) = \inf\{\sum \text{diam}(E)^d : F \subseteq \bigcup E, \text{diam}(E) < \delta\}.
\]
The Hausdorff $m$-measure of $F$ is defined as
\[
\mathcal{H}^m(F) = \lim_{\delta \to 0} \mathcal{H}_\delta^m(F),
\]
We will only use this in the case $m = d$, and we often use the notation $|F| = \mathcal{H}^d(F)$. We refer to the function $\mathcal{H}_\infty^m$ as the $m$-dimensional Hausdorff content.

3.2. David-Toro maps and Reifenberg flatness. In this section, we record the details needed from [DT12] in order to construct stopping time regions using David-Toro maps.

3.2.1. Coherent Collections of Balls and Planes (CCBP). Set $r_k = 10^{-k}$ and let $x_{j,k} \in \mathbb{R}^{d+1}, j \in J_k$ satisfy
\[
|x_{j,k} - x_{i,k}| \geq r_k.
\]
Put $B_{j,k} = B(x_{j,k}, r_k)$ and for $\lambda > 0$ define $V^\lambda_k = \bigcup_{j \in J_k} \lambda B_{j,k} = \bigcup_{j \in J_k} B(x_{j,k}, \lambda r_k)$ where $\lambda B$ is always the ball with the same center as $B$ and radius dilated by a factor of $\lambda$. We also assume
\[
x_{j,k} \in V^2_{k-1}.
\]
We will always use a $d$-plane as the initial surface $\Sigma_0$. We require
\[
\text{dist}(x_{j,0}, \Sigma_0) \leq \epsilon \text{ for } j \in J_0.
\]
Finally, the coherent collection of planes is a collection of planes (in general of any dimension $m < d+1$, although here we only take $d$-planes) $P_{j,k}$ associated to $x_{j,k}$ such that the compatibility conditions
\[
d_{x_{j,k},100r_k}(P_{i,k}, P_{j,k}) \leq \epsilon \text{ for } k \geq 0 \text{ and } i, j \in J_k \text{ such that } |x_{i,k} - x_{j,k}| \leq 100r_k
\]
\[
d_{x_0,100}(P_{i,0}, P_x) \leq \epsilon \text{ for } i \in J_0 \text{ and } x \in \Sigma_0 \text{ such that } |x_{i,0} - x| \leq 2,
\]
\[
d_{x_{i,k},20r_k}(P_{i,k}, P_{j,k+1}) \leq \epsilon \text{ for } i \in J_k \text{ and } j \in J_{k+1} \text{ such that } |x_{i,k} - x_{j,k+1}| \leq 2r_k.
\]
With these conditions, we can define a CCBP

**Definition 3.1.** A CCBP is a triple \((\Sigma_0, \{B_{j,k}\}, \{P_{j,k}\})\) such that conditions \((3.1), (3.2), (3.3), (3.4), (3.5), (3.6)\) are satisfied with \(\epsilon\) sufficiently small in terms of \(d\).

CCBPs allow the construction of parametrizing maps we will denote by the letter \(g\) and call David-Toro maps. They give the following theorem.

**Theorem 3.2** ([DT12] Theorems 2.15, 2.23). Let \((\Sigma_0, \{B_{j,k}\}, \{P_{j,k}\})\) be a CCBP with \(\epsilon\) sufficiently small. Then there exists a bijection \(g : \mathbb{R}^n \to \mathbb{R}^n\) such that

\[
\begin{align*}
g(z) &= z \quad \text{when} \quad \text{dist}(z, \Sigma_0) \geq 2, \\
|g(z) - z| &\leq C \epsilon \quad \text{for} \quad z \in \mathbb{R}^n, \\
\frac{1}{4} |z' - z|^{1+C \epsilon} &\leq |g(z') - g(z)| \leq 3 |z' - z|^{1-C \epsilon}
\end{align*}
\]
for \(z, z' \in \mathbb{R}^n\) such that \(|z' - z| \leq 1\), and \(\Sigma = g(\Sigma_0)\) is a \(C\epsilon\)-Reifenberg flat set that contains the accumulation set

\[
E_\infty = \{x \in \mathbb{R}^n ; \ x \text{ can be written as } x = \lim_{m \to +\infty} x_{j(m), k(m)}, \ \text{with } k(m) \in \mathbb{N} \text{ and } j(m) \in J_{k(m)} \text{ for } m \geq 0, \ \text{and } \lim_{m \to +\infty} k(m) = +\infty \}.
\]

If in addition there exists \(M > 0\) such that

\[
\sum_{k \geq 0} \epsilon_k^2(f_k(z))^2 \leq M \quad \text{for all } z \in \Sigma_0,
\]
then \(g\) is bi-Lipschitz: there is a constant \(C(n, d, M) \geq 1\) such that

\[
C(n, d, M)^{-1}|z - z'| \leq |g(z) - g(z')| \leq C(n, d, M)|z - z'|.
\]

This theorem will be our main tool; we will apply it on CCBPs constructed in many scales and locations on \(E\) to get bi-Lipschitz parametrizations of little pieces of \(E\) inside \(E_\infty\) at a time which behave well on the space above \(E\) at the scales allowed by the net points inside the CCBP.

### 3.2.2. The definition of \(g\)

In order to prove the estimates we need on \(g\), we will need to know how it is constructed. Following Chapter 3 of [DT12], we take \(\psi_k\) to be a smooth function vanishing outside \(V^s_k\) and \(\theta_{j,k}\) to be a collection of smooth compactly supported functions in \(10B_{j,k}\) such that \(|\nabla^m \theta_{j,k}(y)| \leq C_m r^{-m}_k\) and \(\psi_k(y) + \sum_{j \in J_k} \theta_{j,k}(y) = 1\). We then define a sequence of maps \(f_k\) by

\[
f_0(y) = y, \quad f_{k+1} = \sigma_k \circ f_k
\]
where

\[
\sigma_k(y) = y + \sum_{j \in J_k} \theta_{j,k}(y) [\pi_{j,k}(y) - y] = \psi_k(y) y + \sum_{j \in J_k} \theta_{j,k}(y) \pi_{j,k}(y),
\]
where $\pi_{j,k}$ is orthogonal projection onto $P_{j,k}$. In our application, we only care about points inside $V_k$, so $\psi_k(y) = 0$ and the formula simplifies to

$$\sigma_k(y) = \sum_{j \in J_k} \theta_{j,k}(y) \pi_{j,k}(y).$$

The map $\sigma_k$ also satisfies

$$|\sigma_k(y) - y| \leq C \varepsilon r_k \tag{3.10}$$

for $k \geq 0$ and $y \in \Sigma_k$.

The map $g$ is constructed by, roughly speaking, interpolating between adjacent maps in the sequence $f_k$ at distance $r_k$ from the surface $\Sigma_k = f_k(\Sigma_0)$. In order to construct this, David and Toro define a collection of linear isometries $R_k$ on $\mathbb{R}^n$. The following proposition summarizes the properties of $R_k$ that we need

**Proposition 3.3 ([DT12] Proposition 9.29).** Let $R$ denote the set of linear isometries of $\mathbb{R}^n$. Also set

$$T_k(x) = T\Sigma_k(f_k(x)) \text{ for } x \in \Sigma_0 \text{ and } k \geq 0.$$  

There exist $C^1$ mappings $R_k: \Sigma_0 \to R$, with the following properties:

$$R_0(x) = I \text{ for } x \in \Sigma_0,$$

$$R_k(x)(T_0(x)) = T_k(x) \text{ for } x \in \Sigma_0 \text{ and } k \geq 0,$$

$$|R_{k+1}(x) - R_k(x)| \leq C \varepsilon \text{ for } x \in \Sigma_0 \text{ and } k \geq 0. \tag{3.11}$$

In addition, we record the bounds the distance between generations of tangent planes and between planes at different locations

**Lemma 3.4 ([DT12] Lemma 9.2).** We have that for $k \geq 0$ and $x, x' \in \Sigma_0$ such that $|x' - x| \leq 10$,

$$D(T\Sigma_{k+1}(f_{k+1}(x)), T\Sigma_k(f_k(x))) \leq C_1 \varepsilon,$$

$$D(T\Sigma_k(f_k(x')), T\Sigma_k(f_k(x))) \leq C_2 \varepsilon r_k^{-1}|f_k(x') - f_k(x)|. \tag{3.12}$$

Now, following Chapter 10 in [DT12], we define a collection $\rho_k$ of positive, smooth, radial functions such that $\sum_{k \geq 0} \rho_k(y) = 1$ for $y \in \mathbb{R}^n \setminus \{0\}$ and $\rho_k(y) = 0$ unless $r_k < |y| < 20r_k$. Because $[r_k, 20r_k] \cap [20r_k, 20r_k-2] = [r_k, 20r_k] \cap [100r_k, 200000r_k] = \emptyset$, we always have at most two values of $k$ such that $\rho_k(y) \neq 0$ for any fixed $y$. In order to single out specific values of $k$, we define functions $l, g: \mathbb{R}^+ \to \mathbb{N}$ by

$$l(y) = \min\{k \in \mathbb{N} : \rho_k(y) > 0\}, \tag{3.12}$$

$$n(y) = \max\{k \in \mathbb{N} : \rho_k(y) > 0\} = l(y) + 1. \tag{3.13}$$

More concretely, we have

$$n(y) = n \iff 20r_{n+1} = 2r_n < y \leq 20r_n \tag{3.14}$$
because then \( \rho_{n+1}(y) = 0 \) while \( \rho_n(y) > 0 \). We then define

\[
g(z) = \sum_{k \geq 0} \rho_k(y) \{ f_k(x) + R_k(x) \cdot y \}
\]

for \( z = x + y \in V \setminus \Sigma_0 \)

where \( x = \pi \Sigma_0(z) \) and \( y = z - x \) where \( \pi \) is the orthogonal projection onto the plane \( \Sigma_0 \). (Again, [DT12] Lemma 6.12 allow more general \( \Sigma_0 \), in which case the definition is slightly more complicated.) We will commonly use the notation \( z = x + y, \ z' = z' + y' \), etc., where we always take \( x = \pi \Sigma_0(z), \ x' = \pi \Sigma_0(z') \) and \( y = z - x, \ y' = z' - z' \) respectively where the specific underlying plane \( \Sigma_0 \) is understood.

3.2.3. Derivative estimates and regularity. In this section, we record more technical estimates necessary for specific estimates we make on the change in \( Dg \) for \( g \) given in Theorem 3.2. First, the surface \( \Sigma_k \) has a nice local Lipschitz representation:

Lemma 3.5 ([DT12] Lemma 6.12). For \( k \geq 0 \) and \( y \in \Sigma_k \), there is an affine \( d \)-plane \( P \) through \( y \) and a \( C \varepsilon \)-Lipschitz and \( C^2 \) function \( A : P \to P_1 \) such that \( |A(x)| \leq C \varepsilon r_k \) for all \( x \in B(y, 19r_k) \) and

\[
\Sigma_k \cap B(y, 19r_k) = \Gamma \cap B(y, 19r_k).
\]

where \( \Gamma \) denotes the graph of \( A \) over \( P \).

Now, we record distortion estimates for \( D\sigma_k \) as in [DT12] chapter 7. Importantly, \( D\sigma_k \) is very close to the identity in the following sense:

Lemma 3.6 ([DT12] Lemma 7.1). For \( k \geq 0 \), \( \sigma_k \) is a \( C^2 \)-diffeomorphism from \( \Sigma_k \) to \( \Sigma_{k+1} \) and, for \( y \in \Sigma_k \),

\[
D\sigma_k(y) : T\Sigma_k(y) \to T\Sigma_{k+1}(\sigma_k(y))
\]

is bijective and \((1 + C\varepsilon)\)-bi-Lipschitz.

In addition,

\[
|D\sigma_k(y) \cdot v - v| \leq C\varepsilon|v| \quad \text{for } y \in \Sigma_k \text{ and } v \in T\Sigma_k(y)
\]

\[
|\sigma_k(y) - \sigma_k(y') - y + y'| \leq C\varepsilon|y - y'| \quad \text{for } y, y' \in \Sigma_k.
\]

More precise estimates can be obtained when restricting \( D\sigma_k \) to its action on vectors tangent to \( \Sigma_k \). The best way to capture this is to define quantities which take into account exactly how close the nearby planes of appropriate scale in the CCBP are. These are the \( \epsilon'_k \) numbers, defined by

\[
\epsilon'_k(y) = \sup \{ d_{x_{i,l}, 100r_l}(P_{j,k}, P_{i,l}) ; j \in J_k, l \in \{k - 1, k\}, i \in J_l, \text{ and } y \in 10B_{j,k} \cap 11B_{i,l} \}
\]

The following lemma gives estimates in terms of these numbers

Lemma 3.7 ([DT12] Lemma 7.32). For \( k \geq 1 \) and \( y \in \Sigma_k \cap V^8_k \), choose \( i \in J_k \) such that

\[
|y - x_{i,k}| \leq 10r_k.
\]

Then

\[
|D\pi_{i,k} \circ D\sigma_k(y) \circ D\pi_{i,k} - D\pi_{i,k}| \leq C\epsilon'_k(y)^2,
\]

(3.15)
Lemma 3.5. So, assume $|f_k(x) - f_k(x')| > 18r_k$ and suppose first that there exists an integer $0 \leq m \leq k$ such that $|f_m(x) - f_m(x')| < 5r_m$. We calculate

$$\frac{Mr_m}{5^{k-m}} < 5r_m \iff \log_5 M - 1 < k - m$$

for every unit vector $v \in T \Sigma_k(y)$. 

Similarly, these numbers also control the distance between tangent planes to the surface and nearby $P_{j,k}$. For any $k \geq 0$ and $y \in \Sigma_k \cap V_k^8$ and $i \in J_k$ such that $|y - x_{i,k}| \leq 10r_k$, we have (\cite{DT12} (7.22))

$$\text{Angle}(T \Sigma_k(y), P_{i,k}) \leq C \epsilon_k(y).$$

Finally, we also use an estimate on $D^2 \sigma_k$ obtained in by Ghinassi in \cite{Ghi20} in work on constructing $C^{1,\alpha}$ parametrizations.

**Lemma 3.8** (\cite{Ghi20} Lemma 3.16). For $k \geq 0$, $y \in \Sigma_k \cap V_k^8$,

$$|D^2 \sigma_k(y)| \leq C \frac{\epsilon_k(y)}{r_k} \leq C \frac{\epsilon}{r_k}$$

where we interpret the norm on the tensor $D^2 \sigma_k$ as the Euclidean norm on $\mathbb{R}^n$. We also provide the following lemma and proof adapted from a proof of \cite{DT12} to fit our needs.

**Lemma 3.9** (cf. \cite{DT12} (11.22)). Suppose $\Sigma_0$ is such that for any $x, x' \in \Sigma_0$, there exists a curve $\gamma_0$ connecting $x$ and $x'$ with $\ell(\gamma_0) \leq (1 + C \epsilon)|x - x'|$. Let $1 < M \ll \epsilon^{-1}$, $k \geq 0$ be such that $|f_k(x) - f_k(x')| < Mr_k$. Then there is a curve $\gamma : I \to \Sigma_k$ such that $\ell(\gamma) \leq 2|f_k(x) - f_k(x')|$. 

**Proof.** We first prove the following claim:

**Claim:** For any $0 \leq p \leq k$,

$$|f_{k-p}(x) - f_{k-p}(x')| \leq \frac{Mr_{k-p}}{5^p}.$$ 

**Proof:** We prove this by induction. Indeed, observe that

$$|f_{k-p-1}(x) - f_{k-p-1}(x')| = |\sigma_{k-p-1}^{-1}(f_{k-p}(x)) - \sigma_{k-p-1}^{-1}(f_{k-p}(x'))| \leq (1 + C \epsilon)|f_{k-p}(x) - f_{k-p}(x')|.$$ 

by (3.10). Applying this for $p = 1$ gives

$$|f_{k-1}(x) - f_{k-1}(x')| < (1 + C \epsilon)(Mr_k) \leq \frac{Mr_{k-1}}{5}$$

This proves the base case. Assuming the claim holds for some $p$, we get

$$|f_{k-p}(x) - f_{k-p}(x')| \leq (1 + C \epsilon)\frac{Mr_{k-p}}{5^p} < \frac{Mr_{k-p-1}}{5^{p+1}}.$$
so that by the above claim we can assume \( k - m < \log_5 M < \log M \). Applying the Lipschitz graph lemma for \( B(f_m(x), 19r_m) \), we see that there exists a path \( \gamma_m \subseteq \Sigma_m \) such that
\[
\ell(\gamma_m) \leq (1 + C\epsilon)|f_m(x) - f_m(x')| \leq (1 + C\epsilon)(C\epsilon r_m + |f_k(x) - f_k(x')|)
\]
\[
\leq (1 + C\epsilon)|f_k(x) - f_k(x')| + C\epsilon r_k \log M.
\]

On the other hand, since \( |f_m(x) - f_m(x')| < 5r_m, \) we get \( \ell(\gamma_m) \leq 10r_m \) and so we can choose a chain of \( N \leq \frac{10r_m}{19r_m} = 10^{-k-m} \leq M^2 \) points contained in \( \gamma_m \) with consecutive points separated by a distance of at least \( 11r_k \) beginning at \( f_m(x) \) and ending at \( f_m(x') \). Call this collection of points \( \{f_m(x_i)\}_{i=1}^N \) for \( x_i \in \Sigma_0 \). This implies the total length of the string of points \( \{f_k(x_i)\} \) is
\[
L' = \sum_{i=1}^N |f_k(x_i) - f_k(x_{i+1})| \leq \sum_{i=1}^N [C\epsilon r_m + |f_m(x_i) - f_m(x_{i+1})|] \leq C\epsilon r_k M^2 \log M + \ell(\gamma_m)
\]
\[
\leq C\epsilon r_k M^2 \log M + (1 + C\epsilon)|f_k(x) - f_k(x')|.
\]

In addition, for any admissible \( l \) we can calculate
\[(3.19) \quad |f_k(x_i) - f_k(x_{i+1})| \leq C\epsilon r_m + |f_m(x_i) - f_m(x_{i+1})| \leq C\epsilon r_k \log M + 11r_k < 12r_k.\]

Using \((3.19)\) and Lemma 3.5 once again, we connect each pair \((f_k(x_i), f_k(x_{i+1}))\) by a curve \( \gamma_l \) of length \( \ell(\gamma_l) \leq (1 + C\epsilon)|f_k(x_i) - f_k(x_{i+1})| \) to get a curve \( \gamma \)
\[
\ell(\gamma) \leq (1 + C\epsilon)L' \leq (1 + C\epsilon)|f_k(x) - f_k(x')| + C\epsilon r_k M^2 \log M \leq 2|f_k(x) - f_k(x')|
\]
using the fact that \( |f_k(x) - f_k(x')| > 18r_k \) and \( M \ll \epsilon^{-1}, \) i.e. \( \epsilon \) is sufficiently small compared to \( M \). This completes the proof if there exists such an \( m \) where \( f_k(x) \) and \( f_k(x') \) pull back to a Lipschitz neighborhood in \( \Sigma_m \). If there does not exist such an \( m \) (i.e., \( k \) is too small), then we instead use the assumed curve \( \gamma_0 \subseteq \Sigma_0 \) in place of \( \gamma_m \) and argue as in the previous case. 

**Lemma 3.10.** (Reverse Triangle Inequality) Let \( u, v \in \mathbb{R}^{d+1} \) with \( \langle u, v \rangle \geq -\frac{1}{2}||u||v| \). Then
\[(3.20) \quad |u| + |v| \leq 2|u + v|.
\]

**Proof.** First, observe that \( 0 \leq (|u| - |v|)^2 = |u|^2 + |v|^2 - 2|u||v| \iff \frac{|u|^2 + |v|^2}{2} \geq |u||v| \). By first using the hypothesis on \( \langle u, v \rangle \), then this inequality twice, we get
\[
|u + v|^2 = |u|^2 + |v|^2 + 2\langle u, v \rangle \geq |u|^2 + |v|^2 - |u||v| \geq \frac{1}{2}(|u|^2 + |v|^2) = \frac{1}{4}(|u|^2 + |v|^2) + \frac{1}{4}(|u|^2 + |v|^2)
\]
\[
\geq \frac{1}{4}(|u|^2 + |v|^2) + \frac{|u||v|}{2} \geq \frac{1}{4}(|u|^2 + |v|^2 + 2|u||v|) = \frac{1}{4}(|u| + |v|)^2.
\]

3.3. Whitney cubes and Christ-David cubes.
3.3.1. Whitney cubes. Given a $d$-plane $P \subseteq \mathbb{R}^{d+1}$, we will often define a family of Whitney Cubes with respect to $P$. If we identify $P$ with $\mathbb{R}^d \times \{0\}$, then we define

$$\mathcal{W} = \{[k_12^{-n}, (k_1 + 1)2^{-n}] \times \cdots \times [k_d2^{-n}, (k_d + 1)2^{-n}] \times [2^{-n}, 2^{-n+1}] : k_1, \ldots, k_d, n \in \mathbb{Z}\}.$$ 

$\mathcal{W}$ consists of exactly the dyadic cubes in $\mathbb{R}^{d+1}$ which satisfy $\ell(W) = \text{dist}(W, P)$ where $\ell(W)$ denotes the side length of $W$. We also define the height of $W$ to be $h(W) = \text{dist}(W, P) = \ell(W)$.

We denote the orthogonal projection onto $P$ by $\pi_P : \mathbb{R}^{d+1} \to P$. Fix $W, R \in \mathcal{W}$. We call $W$ a child of $R$ if $h(W) = \frac{1}{2}h(R)$ and $\pi_P(W) \subseteq \pi_P(R)$. We use the notation $R = W^{(1)}$ and also call $R$ the parent of $W$. The child-parent relationship between any cube with the cubes sitting “below” it induces a partial order on the entire family $\mathcal{W}$ where $W < R$ if and only if $h(W) < h(R)$ and $\pi_P(W) \subseteq \pi_P(R)$. We will call the set $\{W \in \mathcal{W} : W \leq R\}$ the descendants of $R$ and the members of the set $\{W \in \mathcal{W} : W \geq R\}$ the ancestors of $R$.

3.3.2. Christ-David cubes. We will also need families of partitions of $E \subseteq \mathbb{R}^{d+1}$ which function as dyadic cubes do for $E = \mathbb{R}^{d+1}$. These were originally devised by Christ in [Chr90], but the formulation given here is due to Hytönen and Martikainen from [HMI12].

**Theorem 3.11.** Let $X$ be a doubling metric space. Let $X_k$ be a nested sequence of maximal $\rho^k$-nets for $X$ where $\rho < 1/1000$ and let $c_0 = 1/500$. For each $k \in \mathbb{Z}$ there is a collection $\mathcal{D}_k$ of “cubes,” which are Borel subsets of $X$ such that the following hold.

1. $X = \bigcup_{Q \in \mathcal{D}_k} Q$.
2. If $Q, Q' \in \mathcal{D} = \bigcup \mathcal{D}_k$ and $Q \cap Q' \neq \emptyset$, then $Q \subseteq Q'$ or $Q' \subseteq Q$.
3. For $Q \in \mathcal{D}_k$, let $k(Q)$ be the unique integer so that $Q \in \mathcal{D}_{k}$ and set $\ell(Q) = 5\rho^{k(Q)}$. Then there is $\zeta_Q \in X_k$ so that

$$B(x_Q, c_0 \ell(Q)) \subseteq Q \subseteq B(x_Q, \ell(Q))$$

and

$$X_k = \{x_Q : Q \in \mathcal{D}_k\}.$$

If in addition we assume $X \subseteq \mathbb{R}^{d+1}$ and $X$ is $d$–Ahlfors-David regular, then these cubes also satisfy

4. $|Q| \asymp_d (\text{diam } Q)^d \asymp_d \ell(Q)^d$.

We will refer to any family of Christ-David Cubes for the set $E$ by $\mathcal{D}$ and define

$$B_Q = B(x_Q, \ell(Q)).$$

3.4. Beta numbers, uniform rectifiability, and lower content regularity. The bounds on the boundary measure of our collection of subsets $\{\Omega_n\}$ in Theorem 3.15 come from translating the existence of stopped Whitney cubes above $E$ into collections of dyadic cubes on $E$ which satisfy packing estimates. Here, we record definitions necessary to state these estimates precisely. We define uniform rectifiability and lower content regularity, and give results concerning each used to derive bounds on bad cubes for uniformly rectifiable sets and Reifenberg flat sets (note that Reifenberg flat sets are a subclass of lower content regular sets).
We begin by giving the definitions of relevant beta numbers. Given two closed sets \( E, F \subseteq \mathbb{R}^{d+1} \), and a third set \( B \subseteq \mathbb{R}^{d+1} \) we define the Hausdorff distance between \( E \) and \( F \) inside \( B \) as
\[
d_B(E, F) = \frac{2}{\text{diam } B} \max \left\{ \sup_{y \in E \cap B} \text{dist}(y, F), \sup_{y \in F \cap B} \text{dist}(y, E) \right\}.
\]
Then, we define the bilateral beta number for \( E \) relative to a \( d \)-plane \( P \) inside \( B \) as
\[
b_{\beta} E(P) = d_B(E, P)
\]
and the bilateral beta number inside \( B \) by
\[
b_{\beta} E = \inf \{ b_{\beta} E(P) : P \text{ is a } d \text{-dimensional plane in } \mathbb{R}^{d+1} \}.
\]
We will often suppress the subscript \( E \) and simply write \( b_{\beta}(P) \).

Given a family of Christ-David cubes \( D \) for \( E \) and constants \( M, \epsilon > 0 \), we define the set
\[
\text{BWGL}(M, \epsilon) = \{ Q \in D : b_{\beta}(MB_Q) \geq \epsilon \}
\]
and let \( P_Q \) be a \( d \)-plane such that \( b_{\beta}(MB_Q, P_Q) = b_{\beta}(MB_Q) \). Define
\[
\text{BWGL}(Q, M, \epsilon) = \sum_{R \subseteq Q, R \in \text{BWGL}(M, \epsilon)} \ell(R)^d.
\]
We say that \( E \) satisfies the bilateral weak geometric lemma (BWGL) if for any \( Q \in D \),
\[
\text{BWGL}(Q, M, \epsilon) \lesssim_{d, M, \epsilon} \ell(Q)^d.
\]

Let \( B = B(x, r) \subseteq \mathbb{R}^{d+1} \) be a Euclidean ball and let \( P \) be a \( d \)-plane. To measure a notion of \( L^p \)-closeness rather than \( L^\infty \) closeness, we define
\[
beta_{E,p}^d(B, P) = \left( \frac{1}{r^d} \int_{B \cap E} \left( \frac{\text{dist}(y, P)}{r} \right)^p d \mathcal{H}^d(y) \right)^{1/p}
\]
and
\[
beta_{E,p}^d(B) = \inf \{ \beta_{E,p}^d(B, P) : P \text{ is a } d \text{-dimensional plane in } \mathbb{R}^{d+1} \}.
\]
b \( \beta \) and \( \beta_{E,p}^d \) give different equivalent definitions of uniform rectifiability. We note that a set \( E \) is called \( d \)-Ahlfors-David regular if there exists a constant \( C_0 > 0 \) such that for any \( x \in E \) and \( 0 < r < \text{diam}(E) \), we have
\[
C_0^{-1} r^d \leq \mathcal{H}^d(E \cap B(x, r)) \leq C_0 r^d.
\]

**Proposition 3.12** (cf. [DS93] Part I, Theorem 1.57 and Theorem 2.4; Part IV Proposition 2.1). Let \( E \subseteq \mathbb{R}^{d+1} \) be \( d \)-Ahlfors-David regular for \( d \geq 1 \), and let \( D \) be a system of dyadic cubes on \( E \). The following are equivalent:

(i) \( E \) is \( d \)-uniformly rectifiable,

(ii) For any \( Q \in D, M > 1, \) and \( 1 \leq p < \frac{2d}{d-2} \),
\[
\sum_{R \subseteq Q} \beta_{E,p}^d(MB_R)^2 |R| \lesssim_{M, d} |Q|
\]
(iii) $E$ satisfies the BWGL. That is, for all $M > 1$, $\epsilon > 0$, and any cube $Q \in \mathcal{D}$, we have

$$\text{BWGL}(Q, M, \epsilon) \lesssim M, \epsilon |Q|$$

(iv) For each $\epsilon, \delta > 0$, there exists a lattice $\mathcal{D} = \mathcal{G} \cup \mathcal{B}$ for $E$ with a collection of $d$-planes

$$\{P_Q\}_{Q \in \mathcal{D}}$$

such that $(\mathcal{G}, \mathcal{B}, \mathcal{F})$ is a coronization and

$$b\beta(MB_Q, P_Q) \leq \epsilon$$

for $Q \in \mathcal{G}$,

$$\text{Angle}(P_Q, P_{Q(S)}) \leq \delta$$

for $Q \in S \in \mathcal{F}$.

As used in item (iv) of Proposition 3.12, a coronization $(\mathcal{G}, \mathcal{B}, \mathcal{F})$ is a partition of $\mathcal{D}$ into disjoint sets $\mathcal{G}$ and $\mathcal{B}$ with a further partition of $\mathcal{G}$ into a collection $\mathcal{F}$ of stopping time regions which satisfy a number of nice properties. The main property we will need is packing estimates for cubes in $\mathcal{B}$ and the so-called top cubes $Q(S)$ of stopping time regions $S$. That is, for any $Q \in \mathcal{D}$

$$\sum_{Q(S) \subseteq Q} \ell(Q(S))^d \lesssim \ell(Q)^d$$

$$\sum_{R \in \mathcal{B}, R \subseteq Q} \ell(R)^d \lesssim \ell(Q)^d.$$ 

See [DS93] page 55 for the full definition of a coronization.

A set $E \subseteq \mathbb{R}^{d+1}$ is said to be lower content $d$-regular in a ball $B(x, r)$ if there exists a constant $c > 0$ and $r_B > 0$ such that

$$\mathcal{H}^d(\mathcal{B}) \geq cr^d$$

for all $x \in E \cap B$ and $r \in (0, r_B)$.

A set $E$ is lower content $d$-regular if there exists a constant $c$ such that $E$ is lower content regular with constant $c$ in every ball centered on $E$. In [AS18], Azzam and Schul proved a traveling salesman theorem for lower content $d$-regular sets involving a beta number analogue of $\beta_{E, 2}^d$ defined above which is well-defined on lower content regular sets ($\mathcal{H}^d$ is not a measure as $\mathcal{H}^d$ is, so one must be careful in providing a definition). This result was later extended by Hyde to lower content regular sets in Hilbert space in [Hyd21] and, with another different beta number, to general sets in $\mathbb{R}^n$ in [Hyd22].

In order to state the theorem for lower content regular sets in a Hilbert space $H$, we first define the appropriate beta number. For $B = B(x, r_B) \subseteq H$ and $P$ a $d$-dimensional plane in $H$, we write

$$\beta_{E, p}^d(B, P) = \left( \frac{1}{r_B^d} \int_0^{\infty} \mathcal{H}^d \{x \in B \cap E : \text{dist}(x, P) > tr_B \} t^{p-1} dt \right)^{1/p},$$

and we define

$$\beta_{E, p}^d(B) = \inf \{ \beta_{E, p}^d(B, P) : P \text{ is a } d \text{-dimensional plane in } H \}.$$ 

Hyde’s version of the traveling salesman theorem is contained in the following two theorems:
Theorem 3.13 ([Hyd21] Theorem 1.6). Let \( 1 \leq d < \dim(H) \), \( 1 \leq p < p(d) \), \( C_0 > 1 \), and \( A > 10^5 \). Let \( E \subseteq H \) be a lower content \( d \)-regular set with regularity constant \( c \) and Christ-David cubes \( \mathcal{D} \). There exists \( \epsilon > 0 \) small enough so that the following holds. Let \( Q_0 \in \mathcal{D} \) and 
\[
\beta_{E,C_0,d,p}(Q_0) = \ell(Q_0)^d + \sum_{Q \subseteq Q_0} \beta_{E,d}(C_0B_Q)^2 \ell(Q)^d.
\]
Then
\[
(3.22) \quad \beta_{E,C_0,d,p}(Q_0) \lesssim_{A,d,c,p,C_0,\epsilon} \mathcal{H}^d(Q_0) + \text{BWGL}(Q_0, A, \epsilon).
\]

Theorem 3.14 ([Hyd21] Theorem 1.7). Let \( 1 \leq d < \dim(H) \), \( 1 \leq p < \infty \), \( A > 1 \), \( \epsilon > 0 \), and \( C_0 > 2\rho^{-1} \) where \( \rho \) is as in the construction of the Christ-David lattice \( \mathcal{D} \). Let \( E \subseteq H \) be lower content \( d \)-regular with regularity constant \( c \) and Christ-David cubes \( \mathcal{D} \). For \( Q_0 \in \mathcal{D} \), we have 
\[
\mathcal{H}^d(Q_0) + \text{BWGL}(Q_0, A, \epsilon) \lesssim_{A,d,c,C_0,\epsilon} \beta_{E,C_0,d,p}(Q_0).
\]

3.5. The technical result. Before stating the general result, we develop some definitions and notation.

\( \text{BWGL}(M, \epsilon) \) identifies the cubes near which \( E \) is not sufficiently close to a \( d \)-plane, but it will also be necessary to identify the maximal descendants of a cube \( Q \in \mathcal{D} \setminus \text{BWGL}(M, \epsilon) \) which have well-approximating planes which make too large of an angle with \( P_Q \). Suppose we have a partition 
\[
\mathcal{D} = \mathcal{B} \cup \mathcal{G}
\]
where \( \mathcal{B} \cap \mathcal{G} = \emptyset \) and \( \text{BWGL}(M, \epsilon) \subseteq \mathcal{B} \). Given any \( Q \in \mathcal{G} \), we can form a stopping time region \( S_Q \) by inductively adding cubes \( R \) to \( S_Q \) which satisfy

1. \( R^{(1)} \in S_Q \),
2. Both \( U \in \mathcal{G} \) and \( \text{Angle}(P_U, P_Q) < \delta \), for every sibling \( U \) of \( R \) (including \( R \) itself).

We define
\[
\text{Stop}(S_Q) = \{ R \subseteq Q : R \text{ maximal, } R \text{ has a sibling } U \text{ with } U \notin \mathcal{B} \text{ or } \text{Angle}(P_U, P_Q) \geq \delta \},
\]
\[
m(S_Q) = \{ R \in S_Q : R \text{ has no children in } S_Q \}.
\]
Observe that \( \text{Stop}(S_Q) \) consists entirely of children of cubes in \( m(S_Q) \). We also call \( Q \), the top cube of \( S_Q \) and given some region \( S \) let \( Q(S) \) denote its top cube.

We can partition \( \mathcal{G} \) into a collection of such stopping time regions as follows. Set \( Q_0 = E \) and let \( \mathcal{Q}_0 \) be a collection of maximal disjoint cubes in \( \mathcal{G} \). Put \( \mathcal{F}_0 = \{ S_Q \}_{Q \in \mathcal{Q}_0} \). Given the collection \( \mathcal{F}_j \) for some \( j \geq 0 \), define
\[
\mathcal{Q}_{j+1} = \{ Q \in \mathcal{G} : Q \text{ maximal, } Q \subseteq R \in \text{Stop}(S), S \in \mathcal{F}_j \},
\]
\[
\mathcal{F}_{j+1} = \{ S_Q : Q \in \mathcal{Q}_{j+1} \},
\]
\[
\mathcal{F} = \bigcup_{j=0}^{\infty} \mathcal{F}_j.
\]
Then $\mathcal{G} = \bigcup_{S \in \mathcal{F}} S$. In analogy to the BWGL quantities defined above, we define

$$\text{Tilt}(\mathcal{G}, \delta) = \bigcup_{S \in \mathcal{F}} \{Q \in \text{Stop}(S) \setminus \mathcal{B} : \text{Angle}(P_Q, P_{Q(S)}) \geq \delta\},$$

$$\text{Tilt}(Q, \mathcal{G}, \delta) = \sum_{R \subseteq Q \cap \text{Tilt}(\delta)} \ell(R)^d.$$

Using this, we define new good sets and bad sets for $\mathcal{D}$.

$$\text{Good}(\mathcal{D}) = \mathcal{D} \setminus (\mathcal{B} \cup \text{Tilt}(\mathcal{G}, \delta)) = \mathcal{D} \setminus \text{Tilt}(\mathcal{G}, \delta)$$

$$\text{Bad}(\mathcal{D}) = \mathcal{D} \setminus \text{Good}(\mathcal{D}).$$

Given a set $E \subseteq \mathbb{R}^{d+1}$ with diam$(E) < \infty$, a Christ-David lattice $\mathcal{D}$ for $E = Q_0$, and a collection of planes $\{L_Q\}_{Q \in \mathcal{D}}$, we define epsilon numbers adapted to the lattice $\mathcal{D}$ and the collection $\{P_Q\}_{Q \in \mathcal{D}}$. Given a large constant $M > 0$, we let $\frac{M}{\rho} < K < 10^{-1} \rho^2 M$ and define

$$e^M(Q) = \epsilon(Q) = \sup \left\{ d_{KB_R}(P_Q, P_R) : k(R) \in \{k(Q), k(Q) - 1\}, k(U) = k(Q), x_Q \in \frac{K}{10} B_Q \cap \frac{K}{10} B_R \right\}.$$  

This is essentially a version of David and Toro's $\epsilon_k$ numbers which is adapted to a cube structure rather than a general collection of nets. We can now state the result.

**Theorem 3.15.** Let $E \subseteq \mathbb{R}^{d+1}$ satisfy diam$(E) < \infty$ and $0 < \mathcal{H}^d(E) < \infty$. There exists a constant $M(d) > 0$ such that for any $M > M(d)$, there exist constants $\delta = \delta(d,M)$, $\epsilon = \epsilon(d,M,\delta) \ll \delta$ such that the following holds: If there exists a Christ-David lattice $\mathcal{D} = \mathcal{B} \cup \mathcal{G}$ for $E$ with BWGL$(M, \epsilon) \subseteq \mathcal{B}$ and a corresponding family of $d$-planes $\{L_Q\}_{Q \in \mathcal{D}}$ with $x_Q \in L_Q$ such that, for a constant $0 < \Gamma < \infty$,

\begin{equation}
\label{eq:3.23}
b\beta(KB_Q, L_Q) \lesssim \epsilon \quad \text{for } Q \in \mathcal{G}, \text{ and}
\end{equation}

\begin{equation}
\label{eq:3.24}
\text{Tilt}(E, \mathcal{G}, \delta) + \sum_{Q \in \mathcal{B}} \ell(Q)^d + \sum_{Q \in \mathcal{D}} \epsilon(Q)^2 \ell(Q)^d \leq \Gamma.
\end{equation}

Then, there exists a countable collection $\Lambda = \{\Omega_\alpha\}_{\alpha \in I}$ of $M$-Lipschitz-star domains such that

\begin{enumerate}[(i)]
\item $\Omega_\alpha \subseteq \mathbb{R}^{d+1} \setminus E,$
\item $\chi_{\mathcal{H}(E, \text{diam}(E))}(E) \leq \sum_\alpha \chi_{\Omega_\alpha \setminus \Gamma \setminus \{E\}} \lesssim_d 1,$
\item $\sum_{\alpha \in I} |\partial \Omega_\alpha| \lesssim_d \Gamma + |E|.$
\end{enumerate}

Suppose further that for any $Q \in \mathcal{D}$

\begin{equation}
\label{eq:3.25}
\text{Tilt}(Q, \mathcal{G}, \delta) + \sum_{Q \subseteq R \subseteq \mathcal{B}} \ell(R)^d + \sum_{Q \subseteq R \subseteq \mathcal{D}} \epsilon(R)^2 \ell(R)^d \leq C\ell(Q)^d.
\end{equation}

Then for any $y \in E$ and $r < \text{diam}(E)$, we can improve (iii) above to

$$\sum_{\alpha \in I} |\partial \Omega_\alpha \cap B(y, r)| \lesssim_d r^d + |E \cap B(y, A'r)|,$$

for some constant $1 \leq A' \lesssim_d 1$. 


4. Stopping time constructions

In this section, we develop definitions and notation related to stopping time arguments used to prove Theorem 3.15. Throughout this section, we assume that the set $E$ satisfies the hypotheses of Theorem 3.15.

4.1. The CCBPs adapted to $\mathcal{F}$. Given a point $p \in \mathbb{R}^{d+1} \setminus E$ with $\text{dist}(p, E) \leq \text{diam}(E)$, we want to use the collection of stopping time regions $\mathcal{F}$ to form a CCBP centered around $p$ satisfying the hypotheses of Theorem A.2. For any $k \in \mathbb{Z}$, let $s(k)$ be an integer such that

$$50 \rho^{s(k)} \leq r_k < 50 \rho^{s(k)-1}$$

and define

$$\text{Good}(k) = \text{Good}(\mathcal{F}) \cap \mathcal{D}_{s(k)}.$$ 

Let $k_0$ be the least integer such that $r_{k_0} \leq \frac{1}{100} \text{dist}(p, E)$. The integer $s(k_0)$ determines the top scale of cubes we will consider in forming the CCBP. Define

$$Q_p = \{ Q \in \mathcal{D}_{s(k_0)} : x_Q \in B(p, A' \text{dist}(p, E)) \}.$$ 

This is the family of top scale cubes with centers in $B(p, A' \text{dist}(p, E)) \cap E$. For the rest of this section assume that

$$Q_p \subseteq \text{Good}(k_0).$$

We call such $p$ admissible. Define $B_p = B(p, A' \text{dist}(p, E))$ and let $Q \in \mathcal{Q}_p$. Observe that $Q \in \text{Good}(k_0)$ means $Q \in \mathcal{D} \setminus \text{BWGL}(M, \epsilon)$ so that $b\beta(B_p) \lesssim_M b\beta(MB_p) < \epsilon$ as long as $M$ is sufficiently large with respect to $A'$. Let $P_p$ be a $d$-plane such that $b\beta(B_p, P_p) = b\beta(B_p)$.

Define a localized lattice

$$\mathcal{D}' = \{ Q \in \mathcal{D} : Q \subseteq R \text{ for some } R \in \mathcal{Q}_p \}.$$ 

For any cube $Q \in \mathcal{D}'$, we let $Q_p$ denote the unique cube such that $Q \subseteq Q_p \in \mathcal{Q}_p$ and define $\text{Chain}(Q) = \{ U \in \mathcal{D}' : Q \subseteq U \subseteq Q_p \}$. Define the square function $J_p(Q)$ by

$$J_p(Q) = \sum_{Q \subseteq U \subseteq Q_p} \epsilon(U)^2 = \sum_{U \in \text{Chain}(Q)} \epsilon(U)^2.$$ 

We now use this to define a union of stopping time regions $\mathcal{D}^p$ adapted to $p$. Let $\mathcal{D}^p$ be the maximal collection of cubes $Q \in \mathcal{D}'$ such that

1. $Q^{(1)} \in \mathcal{D}^p$,
2. $U \in \text{Good}(\mathcal{D})$ and $J_p(U) < \epsilon$ for every sibling $U$ of $Q$ (including $Q$ itself).

With this, we define

$$m(\mathcal{D}^p) = \{ Q \in \mathcal{D}^p : Q \text{ has no children in } \mathcal{D}^p \},$$

$$\text{Stop}(\mathcal{D}^p) = \{ Q \in \mathcal{D} : Q^{(1)} \in m(\mathcal{D}^p) \},$$

$$\mathcal{D}^p_k = \mathcal{D}^p \cap \mathcal{D}_{s(k)}.$$ 

This gives us an appropriate coherent collection of cubes near $p$ from which we now construct a CCBP.
Therefore, using the fact that \( K > 10^4 \rho^{-1} \) and \( \ell(Q) = \ell(Q_{i,k}) \). 

**Lemma 4.1.** Fix \( k \geq 0 \) and \( Q \in \mathcal{D}_{s(k)} \). For any \( z \in \mathbb{R}^{d+1} \) such that \( |z - x_Q| < 200 \rho^{-1} \ell(Q) \),

\[ \epsilon_i^k(z) \leq K \epsilon(Q). \]

**Proof.** We first show that the supremum in the definition of \( \epsilon(Q) \) is over a larger collection of pairs of planes than that in the definition of \( \epsilon_i^k(z) \). We note that (4.1) implies

\[ 10 \ell(Q) < r_k \leq 10 \rho^{-1} \ell(Q). \]

Let \( i \in J_k \) be such that \( z \in 10B_{i,k} \). Then

\[ |x_Q - x_{i,k}| < |x_Q - z| + |z - x_{i,k}| < 200 \rho^{-1} \ell(Q) + 10r_k < 300 \rho^{-1} \ell(Q_{i,k}) < \frac{K}{10} \ell(Q) \]

because \( K > 10^4 \rho^{-1} \) and \( \ell(Q) = \ell(Q_{i,k}) \). Therefore, \( x_Q \in \frac{K}{10} B_{Q_{i,k}} \). If instead \( z \in 11B_{i,k-1} \) for some \( i \in J_{k-1} \), then

\[ |x_Q - x_{i,k-1}| < |x_Q - z| + |z - x_{i,k-1}| < 200 \rho^{-1} \ell(Q) + 11r_{k-1} < 310 \rho^{-1} \ell(Q_{i,k-1}) < \frac{K}{10} \ell(Q_{i,k-1}). \]

Therefore, \( x_Q \in \frac{K}{10} B_{Q_{i,k-1}} \). In addition, for any admissible \( x_{i,l} \) in the definition of \( \epsilon_i^k(z) \) we can write

\[ 100r_l \leq 1000 \rho^{-1} \ell(Q_{i,l}) < K \ell(Q_{i,l}) \]

so that \( 100B_{i,l} \subseteq KB_{Q_{i,l}} \). Let \( P_{i,k} \) and \( P_{m,l} \) be planes which achieve the supremum in the definition of \( \epsilon_i^k(z) \). Then

\[ d_{x_{i,l},100B_{m,l}}(P_{i,k}, P_{m,l}) \leq \frac{K \ell(B_{Q_{m,l}})}{100r_l} d_{KB_{Q_{m,l}}(P_{Q_{i,k}}, P_{Q_{m,l}})} \leq K d_{KB_{Q_{m,l}}(P_{Q_{i,k}}, P_{Q_{m,l}})} \]

using the fact that \( \ell(Q_{m,l}) < r_l \).

Applying this result for \( z = x_{j,k} \) shows that \( \epsilon_i^k(x_{j,k}) \leq \epsilon(Q_{j,k}) \).

**Lemma 4.2.** For any admissible point \( p \), the triple \( \mathcal{Z}_p = (P_0, \{B_{j,k}\}_{x_{j,k} \in x_k}, \{P_{j,k}\}_{x_{j,k} \in x_k}) \) is a CCBP satisfying the hypotheses of Theorem 4.2. That is,

\[ (i) \sum_{k=1}^\infty \epsilon_i^k(f_k(x)))^2 \leq \epsilon \text{ for all } x \in P_0, \]

\[ (ii) \text{ Angle}(P_{j,k}, P_0) \leq \delta \text{ for all } k \geq k_0, \ j \in J_k. \]

Therefore, there exists a map \( g_p : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1} \) which is \((1+2 \delta)\)-bi-Lipschitz with \( |Dg(z) - I| \leq 2 \delta \) on any quasiconvex domain \( \Omega \subseteq I(\mathcal{Z}_p) \).
Proof. Without loss of generality, it suffices to assume $k_0 = 0$ so that $r_{k_0} = 1$ and hence $100 \leq \text{dist}(p, E) < 1000$. We first check that this triple is a CCBP using Lemma A.1. First, we will show that for any $j \in J_0$, $\text{dist}(x_{j,0}, P_0) \lesssim \epsilon$. Indeed, $x_{j,0} = x_Q$ for some $Q \in \mathscr{G}_x^{0}$. Hence, $x_Q \in B_p$ so that $b\beta(B_p, P_p) < \epsilon$ implies
\[
\text{dist}(x_Q, P_0) \leq b\beta(B_p)A' \text{dist}(p, E) < 1000A'\epsilon.
\]
Assuming $\epsilon$ is sufficiently small in terms of $A'$, we get the result. Now, we fix $k > 0$ and $j \in J_k$ and prove the following claim:

**Claim:** There exists $i \in J_{k-1}$ such that $x_{j,k} \in B_{i,k-1}$

**Proof:** Indeed, let $x_{j,k} = x_{Q,j,k}$. If $s(k) = s(k - 1)$, then $Y_{k-1}^p = Y_k^p$ so that $x_{Q,j,k} \in Y_k^p$. The claim follows since $X_{k-1}^p$ is an $r_{k-1}$-net for $Y_k^p$. If instead $s(k) > s(k - 1)$, then $x_{Q,j,k} \in Y_{k-1}^p$ so that there exists $i \in J_{k-1}$ such that $x_{Q_{j,k}} \in B_{i,k-1}$. We have
\[
\ell(Q_{j,k}^{(1)}) = 5\rho^{s(k)-1} \leq 5\rho^{s(k)-1} \leq r_{k-1}
\]
so that
\[
\text{dist}(x_{Q,j,k}, x_{i,k-1}) \leq \text{dist}(x_{Q,j,k}, x_{Q_{j,k}}^{(1)}) + \text{dist}(x_{Q_{j,k}}^{(1)}, x_{i,k-1}) \leq \ell(Q_{j,k}^{(1)}) + r_{k-1} \leq 2r_{k-1}
\]
which proves $x_{Q,j,k} \in 2B_{i,k-1}$.

By Lemma 4.1 it suffices to show that $\epsilon(Q_{j,k}) \lesssim \epsilon$. We will use the following lemma

**Lemma 4.3 (AT15 Lemma 6.4).** Suppose $P_1$ and $P_2$ are $n$-planes in $\mathbb{R}^d$ and $X = \{x_0, \ldots, x_n\}$ are points so that

(a) $\eta = \eta(X) = \min\{\text{dist}(x_i, \text{Span } X \setminus \{x_i\})/\text{diam } X \in (0, 1) \text{ and}
(b) \text{dist}(x_i, P_j) < \nu \text{ diam } X \text{ for } i = 0, \ldots, n \text{ and } j = 1, 2, \text{ where } \nu < \eta d^{-1}/2.$

Then
\[
\text{dist}(y, P_i) \leq \nu \left\{ \frac{2d}{\eta} \text{dist}(y, X) + \text{diam } X \right\} \text{ for all } y \in P_2.
\]

We use the inequalities $b\beta(KB_U, L_U), b\beta(KB_R, L_R) \lesssim \epsilon$ from (3.23) and the fact that $\frac{K}{10}B_U \cap \frac{K}{10}B_R \neq \emptyset$ to find a collection of well-distributed points $X = \{x_0, \ldots, x_d\} \subseteq KB_U \cap KB_R \cap E$ such that $\eta(X) \approx 1$ and both $\text{dist}(x_i, L_U) \lesssim \epsilon K\ell(U)$, and $\text{dist}(x_i, L_R) \lesssim \epsilon K\ell(R)$. We then apply Lemma 4.3 to conclude
\[
\text{dist}(y, L_U) \lesssim \epsilon K\ell(R) \text{ for all } y \in L_R \cap KB_R,
\]
and, interchanging $L_R$ and $L_U$,
\[
\text{dist}(z, L_R) \lesssim \epsilon K\ell(R) \text{ for all } y \in L_U \cap KB_R.
\]
We conclude that $d_{KB_R}(L_U, L_R) \lesssim \epsilon$, proving that $\epsilon(Q_{j,k}) \lesssim \epsilon$. This suffices to prove that the triple is a CCBP. We now prove that this CCBP satisfies properties (i) and (ii).

**Claim:** $\sum_{k=1}^{\infty} \epsilon_k'(f_k(x)))^2 \lesssim \epsilon$ for all $x \in P_0$. 

Proof: Suppose first that there exists a greatest integer $N$ such that $\epsilon^N(f_N(x)) > 0$. Let $Q_{s(N)} \in D_{s(N)}$ be such that $x_{Q_{s(N)}} \in X_N^p$ and $|f_N(x) - x_{Q_{s(N)}}| < 10r_N$. $Q_{s(N)}$ exists because $\epsilon^N(f_N(x)) > 0$ implies $f_N(x) \in V_N^{10}$. For any $k \leq N$, let $Q_{s(k)}$ be the unique cube such that $Q_{s(N)} \subseteq Q_{s(k)}$. We claim that $\epsilon_k^k(f_k(x)) \leq \epsilon(Q_{s(k)})$. Indeed, by Lemma 4.1 we only need to show that $|f_k(x) - x_{Q_{s(k)}}| < 200\rho^{-1}\ell(Q_{s(k)})$. But we have

$$|f_k(x) - x_{Q_{s(k)}}| \leq |f_k(x) - f_N(x)| + |f_N(x) - x_{Q_{s(N)}}| + |x_{Q_{s(N)}} - x_{Q_{s(k)}}|$$

$$\leq C\epsilon r_N + 10r_N + \ell(Q_{s(k)}) \leq (C\epsilon + 100)\rho^{-1}\ell(Q_{s(N)}) + \ell(Q_{s(K)}) \leq 102\rho^{-1}\ell(Q_{s(k)})$$

as desired. Because the set $\{n : s(k) = s(n)\}$ has a uniformly bounded number of elements in terms of $\rho$, it follows that

$$\sum_{k=0}^{\infty} \epsilon_k^k(f_k(x))^2 = \sum_{k=0}^{N} \epsilon_k^k(f_k(x))^2 \leq \sum_{k=0}^{N} \epsilon(Q_{s(k)})^2 \leq \sum_{Q_{s(N)} \subseteq Q \subseteq Q_{s(0)}} \epsilon(Q)^2 = J_p(Q_{s(N)}) < \epsilon.$$

If instead $\epsilon_k^k(f_k(x)) > 0$ for all $k \geq 0$, then $f(x) \in E$ so that there is an infinite chain of cubes $Q_{s(0)} \supseteq Q_{s(1)} \supseteq \cdots$ containing $f(x)$. The above proof shows that $\epsilon_k^k(f_k(x)) \leq \epsilon(Q_{s(k)})$ for all $k \geq 0$ and the result follows from the same computation as above.

Claim: Angle($P_{j,k}, P_0$) $\lesssim \delta$ for all $k \geq k_0$, $j \in J_k$.

Proof: Let $Q_{s(0)} \in D_{s(0)}^p$ be such that $Q_{j,k} \subseteq Q_{s(0)}$ and let $Q_{s(0)} \in S \in \mathcal{F}$. The fact that $Q_{j,k}, Q_{s(0)} \in D^p$ implies

$$\text{Angle}(L_{Q_{j,k}}, L_{Q_{s(0)}}) \leq \text{Angle}(L_{Q_{j,k}}, L_{Q(S)}) + \text{Angle}(L_{Q(S)}, L_{Q_{s(0)})} < 2\delta.$$

Hence, it suffices to show that Angle($L_{Q_{s(0)}}, P_0$) $\lesssim \delta$. Indeed, by a nearly identical argument to the proof of $\epsilon(Q_{j,k}) \leq \epsilon$ above, we use that $B_{Q_{s(0)}} \subseteq B_p$ and $b\beta(B_{Q_{s(0)}}, L_{Q_{s(0)}})$, $b\beta(B_p, P_0) \lesssim_{\epsilon M} \epsilon$ to find a collection of points $X = \{x_0, \ldots, x_d\} \subseteq B_{Q_{s(0)}} \cap E$ such that $\eta(X) \simeq 1$ and $\nu \lesssim_{\epsilon M} \epsilon$ in the notation of Lemma 4.3. Applying the lemma shows that $d_{\ell_{Q_{s(0)}}}(L_{Q_{s(0)}}, P_0) \lesssim \epsilon$ so that Angle($L_{Q_{s(0)}}, P_0$) $\lesssim_{\epsilon M} \epsilon < \delta$.

We have proven items (i) and (ii), so Theorem A.2 applies and the desired map $g_p$ exists. $

Lemma 4.2$ gives us a nice map $g_p$ from which we want to build a domain $\mathcal{D}_p$ in the domain of $g_p$ which we can map forward to a nice bi-Lipschitz image $\Omega_p = g_p(\mathcal{D}_p)$ which approaches $E$ at scales and locations determined by the CCBP. We now turn to using the map $g_p$ to build such a domain $\mathcal{D}_p$.

4.2. The stopping time construction outside of $E$. Recall that we’ve assumed $p$ is admissible so that $b\beta(B_p, P_p) = b\beta(B_p) < \epsilon$. Identify the component of $\mathbb{R}^{d+1} \setminus P_p$ containing $p$ with the upper half plane $\mathbb{H}^{d+1}$. Decompose $\mathbb{H}^{d+1}$ into the standard collection of dyadic Whitney cubes $W = W_p$ with sides parallel to the coordinate axes such that $W \in W$ implies $h(W) = \text{dist}(W, P_p) = \ell(W)$. We choose $W$ such that there exists a cube $W_p$ such that $p \in W_p$ and $\ell(W_p) \leq \text{dist}(p, E) \leq 2\ell(W_p)$. We call two subsets (most often Whitney cubes) $W, R \subseteq \mathbb{R}^{d+1}$
$A$-close (as in [DS93] pg. 59) if the following hold:

\begin{align*}
&1 \leq \frac{1}{A} \text{diam } W \leq \text{diam } R \leq A \text{diam } W, \\
&\text{dist}(W, R) \leq A(\text{diam } W + \text{diam } R).
\end{align*}

We will also write $W \simeq_A R$.

For Whitney cubes $W, R \in \mathcal{W}_p$, if $W \simeq_A R$ then $W^{(1)} \simeq_A R^{(1)}$. Indeed, assuming $h(W) \geq h(R)$, we can estimate

\begin{align*}
\text{diam } R^{(1)} &= 2 \text{diam } R \leq A(2 \text{diam } W) = A \text{diam } W^{(1)}, \\
\text{diam } R^{(1)} &= 2 \text{diam } R \geq \frac{1}{A} (2 \text{diam } W) = \frac{1}{A} \text{diam } W^{(1)}, \\
\text{dist}(R^{(1)}, W^{(1)}) &\leq \text{dist}(R, W) + \text{diam}(W) \leq A(\text{diam } R + \text{diam } W) + \text{diam } W \\
&< A(\text{diam } R^{(1)} + \text{diam } W^{(1)}).
\end{align*}

If one of the sets involved is a Chist-David cube $Q$, then we replace $\text{diam } Q$ with $\ell(Q)$ in the above definition. We now fix a constant $A = \max(1000, 120\sqrt{d})$.

Given $z = x + y \in \mathbb{R}^{d+1}$, and $W \in \mathcal{W}_p$ let

\begin{align*}
\mathcal{Q}^z &= \{ Q \in \mathcal{D}_{s(n(y))} : f_{n(y)}(x) \in 4\rho^{-1}B_Q \}, \\
\mathcal{Q}^W &= \bigcup_{z \in W} \mathcal{Q}^z.
\end{align*}

We define the good region of the domain of $g_p$ as

$G(\mathcal{Q}^p) = \{ z = x + y \in \mathbb{R}^{d+1} : \mathcal{Q}^z \cap \mathcal{D}^p \subseteq \mathcal{D}^p \}$.

We define the stopping time region $T_p \subseteq \mathcal{W}_p$ as the maximal collection of Whitney cubes $W \in \mathcal{W}_p$ such that $W \in T_p$ implies

1. $W \leq W_p$, 
2. $W^{(1)} \in T_p$, 
3. For every sibling $R$ of $W$ and cube $V \in \mathcal{W}_p$ such that $V \simeq_A R$, 

$V \subseteq G(\mathcal{Q}^p)$.

One can equivalently phrase condition (3) as

$\mathcal{Q}^V \subseteq \mathcal{D}^p$

which roughly says that all cubes in $Q \in \mathcal{D}$ near $V$ in location and scale are contained in the good region $\mathcal{D}^p$ of flat cubes with small $J_p$ and making small angle with the top cube of their region $S(Q)$. $T_p$ is defined essentially as a maximal region of Whitney cubes living below $W_p$ such that the nice estimates on the map $g_p$ constructed via Lemma 4.2 from the CCBP $\mathcal{Q}^p$ apply in an $A$-neighborhood around each cube in $T_p$. 
We now divide points $p \in \mathbb{R}^{d+1} \setminus E$ into two classes. We call $p$ flat if $p$ is admissible and $W_p \in T_p$. That is, if $W_p$ satisfies condition (3) above. Otherwise, we call $p$ non-flat.

In the following definitions, assume first that $p$ is flat. We will often suppress the subscript $p$ and write $T = T_p$. We refer to $W_p$ as $W(T)$ and call it the top cube of the region $T$. The mapping $g_p = g_T = g$ is that given by Lemma 4.2. For any such region $T$, we define

$$D_p = D_T = \bigcup_{W \in T} W.$$  

We abuse nomenclature and often refer to this set as a stopping time region as well. Each stopping time region comes with an associated stopping time domain defined by

$$\Omega_p = \Omega_T = g_T(D_T).$$

We will also need to define expanded versions of these objects. Given a stopping time region $T$, we define the extended stopping time region as

$$\hat{T} = \hat{T}_p = \{ W \in \mathcal{W}_p : \exists W' \in T, W \text{ is } A\text{-close to } W' \},$$

$$\hat{D}_p = \hat{D}_T = \bigcup_{W \in \hat{T}} W.$$  

the extended region naturally comes with an extended stopping time domain

$$\hat{\Omega}_p = \hat{\Omega}_T = g_T(\hat{D}_T).$$

The result of this section so far is a mapping $p \mapsto \hat{T}_p$ of a flat point $p \in \mathbb{R}^{d+1} \setminus E$ to an associated extended stopping time region. We finish the definition of this mapping by handling the case when $p$ is non-flat.

Indeed, suppose $p \in \mathbb{R}^{d+1} \setminus E$ is non-flat. We fix an auxiliary Whitney decomposition $\mathcal{W}_0$ of $\mathbb{R}^{d+1} \setminus E$ such that for any $W \in \mathcal{W}_0$, we have

$$\text{diam } W \leq \text{dist}(W, E) \leq 4 \text{ diam } W. \tag{4.6}$$

Then, there exists $W_p$ such that $p \in W_p \in \mathcal{W}_0$, and we define

$$T_p = \{ W_p \},$$

$$\hat{T}_p = \{ W' \in \mathcal{W}_0 : W' \text{ is } A\text{-close to } W_p \},$$

$$g_p = I,$$

$$\Omega_p = D_p = W_p,$$

$$\hat{\Omega}_p = \hat{D}_p = \bigcup_{W \in \hat{S}} W.$$  

This completes the definition of the mapping $p \mapsto \hat{T}_p$ for any $p \in \mathbb{R}^{d+1} \setminus E$. 
4.3. Properties of the stopping time regions. For any $T$ associated to a flat point $p$, we
define the set of minimal cubes in $T$ as

$$m(T) = \{ W \in T : W \text{ has no children in } T \}.$$ 

Lemma 4.4. For any flat point $p$ and associated stopping time region $T = T_p$,

$$\hat{D}_T \subseteq I(\mathcal{Z}_p).$$

Proof. Let $z \in \hat{D}_T$. Then there exists a cube $V \in \mathcal{W}_p$ such that $z \in V \simeq_A R$ for some $R \in T$.
Write $z = x + y$ and put $n = n(y)$. We need to show that $f_n(x) \in V^n$. We will prove the
following claim by induction (assuming $r_{k_0} = 1$):

Claim: For any $0 \leq k \leq n$, $f_k(x) \in V_k^{3/2}$.

Proof: We begin with the base case $k = 0$. Indeed, $x \in P_0 \cap B(p, 2A' \text{dist}(p, E))$ so that $b\beta(B_p, P_0) \lesssim \epsilon$ implies that there exists $y \in E$ such that $|x - y| \lesssim \epsilon$. There exists $Q_0 \in \mathcal{D}_p$ such that $y \in Q_0$ so that $\text{dist}(x, Q_0) \lesssim \epsilon$ and

$$|x - x_{Q_0}| \leq |x - y| + |y - x_{Q_0}| < C\epsilon + \ell(Q_0) < \frac{3}{2}\ell(Q_0) < \frac{1}{5}r_0.$$ 

because, by definition, $r_{k_0} = r_0 = 1 < 10p^{-1}\ell(Q_0)$. Because $p$ is flat, $Q_0 \in \text{Good}(0)$ and therefore $x_{Q_0} \in Y_0^p$. Recall that the underlying scale $k$ net $X_k^p$ of the CCBP $\mathcal{Z}_p$ is an $r_k$-net of $Y_k^p$. If $x_{Q_0} \in X_0^p$, then $x \in V_0^{1/5}$. Otherwise, there must exist $Q \in \text{Good}(0)$ such that $x_Q \in X_0^p$ and $|x_Q - x_{Q_0}| < r_0$. Then we have

$$|x - x_Q| \leq |x - x_{Q_0}| + |x_{Q_0} - x_Q| \leq \frac{1}{5}r_0 + r_0 < \frac{6}{5}r_0$$ 

so that $x \in V_0^{6/5} \subseteq V_0^{3/2}$.

Now, we handle the inductive step. Suppose that $f_k(x) \in V_k^{3/2}$ and choose $x_{j,k} \in X_k^p$ such that $|f_k(x) - x_{j,k}| < \frac{3}{2} r_k$. By definition, $Q_{j,k} \in \text{Good}(k)$ so that $b\beta(MB_{Q_{j,k}}, L_{Q_{j,k}}) \lesssim \epsilon$ and Lemma 3.5 and (3.17) imply

$$(4.7) \quad d_{f_k(x), 19r_k}(\Sigma_k, E) \leq d_{f_k(x), 19r_k}(\Sigma_k, L_{Q_{j,k}}) + d_{f_k(x), 19r_k}(L_{Q_{j,k}}, E) \lesssim \epsilon.$$ 

Therefore, there exists $y \in E$ such that $|f_k(x) - y| \lesssim \epsilon r_k$. Let $Q \in \mathcal{Z}_{s(k+1)}$ such that $y \in Q$.

Then

$$|f_{k+1}(x) - x_Q| \leq |f_{k+1}(x) - f_k(x)| + |f_k(x) - y| + |y - x_Q| \leq C\epsilon r_k + C\epsilon r_k + \ell(Q) < (1 + C\epsilon)\ell(Q).$$ 

We now want to verify the following claim, which states that $x_Q$ is a candidate net point:

Claim: $x_Q \in Y_{k+1}^p$

Proof: By the definition of $Y_{k+1}^p$, this is equivalent to showing $Q \in \mathcal{Z}_{s(k+1)}$. We must prove that every sibling $U$ of $Q$ satisfies $U \in \text{Good}(\mathcal{D})$ and $J_p(U) < \epsilon$.

Let $Q = Q_{s(k+1)} \subseteq Q_{s(k)} \subseteq Q_{s(k-1)} \subseteq \ldots \subseteq Q_{s(0)}$ be the chain of ancestors of $U$ in $\mathcal{D}^p$. By the geometry of $\hat{D}_T$, for any point $z_m = x + r_m e_{d+1}$ for $0 \leq m \leq k + 1$, there exists a Whitney cube
Corollary 4.5. For any flat \( p \in \mathbb{R}^{d+1} \), the map \( g_T : \hat{D}_T \to \hat{\Omega}_T \) is \((1+C\delta)\)-bi-Lipschitz and, in fact, satisfies
\[
|Dg_T(z) - I| \leq C\delta. 
\]

We can also conclude that \( g_T \) preserves distance to \( E \) inside of \( \hat{D}_T \):

**Corollary 4.6.** For any \( z = x + y \in \hat{D}_T \)
\[
(1 - C\epsilon|y|) = (1 - C\epsilon) \text{dist}(z, P_0) \leq \text{dist}(g_T(z), E) \leq (1 + C\epsilon) \text{dist}(z, P_0) = (1 + C\epsilon)|y|.
\]

**Proof.** Put \( n = n(y) \) and write
\[
g_T(z) - f_n(x) = \sum_k \rho_k(y) \{ f_k(x) - f_n(x) + R_k(x) \cdot y \}
\]
where \( |f_k(x) - f_n(x)| \leq c\rho_n \) and \( R_k(x) \cdot y \) is a vector of norm \(|y|\) which is orthogonal to the tangent space \( T_k(x) \) to \( \Sigma_p \) at \( f_n(x) \). We can apply the same reasoning as in equation (4.7) to conclude
\[
d_{f_n(x),19\rho_n}(T_n(x) + f_n(x), E) < C\epsilon, \text{ which implies } (1 - C\epsilon)|y| \leq \text{dist}(g_T(z), E) \leq (1 + C\epsilon)|y|. \]

In order to describe the interaction between \( \mathcal{D} \) and \( \mathcal{W}_T \), we use the notation
\[
\mathcal{D}_T = \mathcal{D}_{T_p} = \mathcal{D}^p, \\
\text{Stop}(\mathcal{D}) = \text{Stop}(\mathcal{D}_{T_p}) = \text{Stop}(\mathcal{D}^p), \\
m(\mathcal{D}_T) = m(\mathcal{D}_{T_p}) = m(\mathcal{D}^p).
\]
That is, the subscripts \( p \) and \( T \) are interchangeable where \( T = T_p \) is the stopping time region built around \( p \). We further partition \( m(\mathcal{D}_T) \) based on what kind of stopped cubes sit below a
given minimal cube. Define
\[ m_\mathcal{D}(\mathcal{D}_T) = \{ Q \in m(\mathcal{D}_T) : Q \text{ has a child } R \in \text{Bad}(\mathcal{D}) \}, \]
\[ m_t(\mathcal{D}_T) = m(\mathcal{D}_T) \setminus m_\mathcal{D}(\mathcal{D}_T). \]

**Lemma 4.7.** There exists a mapping \( \Psi : m(T) \to \text{Bad}(\mathcal{D}) \) such that

(i) If \( p \) is flat, then \( \Psi(m(T_p)) \subseteq m(\mathcal{D}_T) \),

(ii) \( \Psi(W) \simeq_A W \),

(iii) \( \Psi \) is at most \( C \) to one.

**Proof.** First, assume that \( T = T_p \) where \( p \) is flat. Then \( W(T) \in T \) so that \( m(T) \neq \emptyset \). Let \( W \in m(T) \). Then, by the definition of \( T, W \) has a child \( R \) which has a Whitney cube \( V \simeq_A R \) of maximal height \( h(V) \) such that there exists a point \( z = x + y \in V \setminus G(\mathcal{Z}_p) \). Let \( n = n(y) \) so that \( 2r_n < |y| \leq 20r_n \). Let \( V' \) be the least ancestor of \( V \) such that \( h(V') > 20r_n \), implying \( n(h(V')) = n - 1 \). Because \( h(V) \leq |y| \leq 2h(V) \), we know \( r_n < h(V) \) so that

\[ (4.10) \quad h(V') < 32h(V). \]

We claim that there exists an ancestor \( R' \in T \) of \( R \) such that \( V' \simeq_A R' \). Indeed, we know from (4.10) that \( V' = V^{(k)} \) for some \( 0 \leq k \leq 5 \). If \( R^{(k)} \in T \), then the claim follows from the implication

\[ V \simeq_A R \implies V^{(k)} \simeq_A R^{(k)} \]
for any \( k \geq 0 \) by taking \( R' = R^{(k)} \). If \( R^{(k)} \notin T \), then \( R^{(m)} = W(T) \) for some \( 0 \leq m < k \). Then \( V^{(m)} \simeq_A W(T) \) while \( V \notin A W(T) \) so that, because \( A > 1000 \), we also have \( V^{(k)} \simeq_A W(T) \), so we take \( R' = W(T) \).

Because \( R' \in T \) and \( V' \simeq_A R' \), we know \( V' \subseteq G(\mathcal{Z}_p) \) and \( z' = x + h(V')e_{d+1} \in V' \) with \( n(h(V')) = n - 1 \) by the definition of \( V' \). By Lemma 4.4, \( f_{n-1}(x) \in V^{8}_{n-1} \) so that the argument of (4.7) applies and

\[ d_{f_{n-1}(x), 19r_{n-1}}(\Sigma_{n-1}, E) < C \epsilon. \]

We conclude that there exists a cube \( Q \in \mathcal{D}_{s(n)} \) such that \( \text{dist}(Q, f_{n-1}(x)) \lesssim \epsilon r_n \) so that \( f_n(x) \in 4\rho^{-1}B_Q \), implying \( \mathcal{Z}_{V} \neq \emptyset \). Now, \( z \in V \setminus G(\mathcal{Z}_p) \) implies there exists \( Q_W \in \mathcal{D}_{s(n)} \) such that \( f_n(x) \in 4\rho^{-1}B_{Q_W} \) but \( Q_W \notin \mathcal{D}_T \). By assuming \( A' \) is sufficiently large, we can ensure that \( Q_W \in \mathcal{D}_T \), the localized lattice, so that actually \( Q_W \in \text{Bad}(\mathcal{D}) \) and \( Q_W \not\subseteq Q \) for some \( Q \in m(\mathcal{D}_T) \). In fact, we claim that we can take \( Q = Q^{(1)}_W \), implying \( Q^{(1)}_W \in m(\mathcal{D}_T) \).

Put \( U = Q^{(1)}_W \). We want to show that \( U \in \mathcal{D}_T \). First, notice that because \( p \) is flat, \( W(T) \in T \) and there exists a cube \( Q_p \in \mathcal{D}_T \) such that \( Q_W \subseteq Q_p \). Because \( Q_p \in \mathcal{D}_{s(0)} \), this means that there exists \( 0 \leq m < n \) such that \( U \in \mathcal{D}_{s(m)} \) with \( 10\ell(U) < r_m \leq 10\rho^{-1}\ell(U) \). Therefore, by a similar argument to that proving the existence of \( V'' \) and \( R' \), there exist ancestors \( V'' \supseteq V \) and \( R'' \supseteq R \) such that \( R'' \in T \) and \( V'' \simeq_A R'' \) and \( n(h(V'')) = m \). Therefore, \( \mathcal{Z}_{V''} \subseteq \mathcal{D}_T \) and it suffices to show that \( U \in \mathcal{Z}_{V''} \).
Because \( x + h(V^n)c_{d+1} \in V^n \), it suffices to show that \( f_m(x) \in 4\rho^{-1}B_U \). We compute

\[
|f_m(x) - x_U| \leq |f_m(x) - f_n(x)| + |f_n(x) - x_Q| + |x_Q - x_U| \\
\leq C\epsilon_m + 4\rho^{-1}\ell(Q) + \ell(U) \leq 6\ell(U) < 4\rho^{-1}\ell(U).
\]

Therefore, \( U \in \mathcal{D}_T \) and \( U = Q^{(1)}_W \in m(\mathcal{D}_T) \). We define \( \Psi(W) = U \). This concludes the definition in the case when \( T \) is associated to a flat point \( p \) and verifies [1i] We now construct the definition when \( p \) is non-flat.

Recall that \( p \) is non-flat if either \( p \) is not admissible, i.e. there exists a cube \( Q \in \mathcal{D}_p \cap \text{Bad}(\mathcal{D}) \), or \( p \) is admissible and there exists a Whitney cube \( V \approx_A W(T) \) with \( 2^V \cap \text{Bad}(\mathcal{D}) \neq \emptyset \). In the first case, we choose \( \Psi(W_p) = Q \) and in the second case we take \( U \in 2^V \cap \text{Bad}(\mathcal{D}) \) and define \( \Psi(W_p) = U \).

We now verify [ii] with \( U = \Psi(W) \). If \( p \) is flat, then we compute

\[
\text{dist}(U, W) \leq 2\ell(U) + \text{dist}(U, f_m(x)) + \text{dist}(f_m(x), V'') + \text{diam } V'' \lesssim_{A, \rho} \ell(U) + \text{diam } R'' \\
\lesssim A'(\ell(U) + \text{diam } W), \\
\ell(U) \approx \rho \, r_m \approx_A \text{diam } R'' \approx \rho \, \text{diam } W.
\]

So, \( U \approx_{A'} W \) as long as \( A' \) is sufficiently large. If instead \( p \) is non-flat, then in the first case,

\[
\text{dist}(Q, W_p) \leq \frac{A'}{10} \text{dist}(p, E) \leq \frac{A'}{10} (\text{diam } W_p + \text{dist}(W_p, E)) \leq A' \text{diam } W_p, \\
\ell(Q) \approx \rho \, r_0 \approx \text{dist}(p, E) \approx \text{diam } W_p.
\]

In the second case, we argue just as in the case when \( p \) is flat, but with \( m = 0 \), concluding the proof of [ii] Item [iii] follows from the fact that for any fixed \( Q \in m(\mathcal{D}_T) \), number of Whitney cubes \( W \in \mathcal{W}_p \) such that \( W \approx_{A'} Q \) is uniformly bounded.

Given the mapping \( \Psi \), we can now partition the collection of minimal Whitney cubes \( m(T_p) \) for flat \( p \) by which class of \( m(\mathcal{D}_T) \) the cube \( \Psi(W) \) lies in. Define

\[
m_{\mathcal{F}}(T) = \{ W \in m(T) : \Psi(W) \in m_{\mathcal{F}}(S), \ S \in \mathcal{F} \}, \\
m_{\mathcal{T}}(T) = m(T) \setminus m_{\mathcal{F}}(T).
\]

We want to use the mapping \( \Phi \) to control \( |\partial \Omega_T| \) in terms of the size of cubes in \( m(\mathcal{D}_T) \). Before doing this, we prove the following necessary lemma which gives a packing estimate for disjoint sub-collections of cubes living inside the nice region \( \mathcal{D}_T \).

**Lemma 4.8.** Let \( Q \in \mathcal{D}_T \). For any disjoint collection of cubes \( \mathcal{D} \subseteq \mathcal{D}_T \) with \( U \subseteq Q \) for all \( U \in \mathcal{D} \),

\[
\sum_{U \in \mathcal{D}} \ell(U)^d \lesssim_d \ell(Q)^d.
\]

**Proof.** Let \( U \in \mathcal{D} \) and suppose \( U \in \mathcal{D}_{s(k)} \) for some \( k \). Then \( U \in \mathcal{D}_T \) implies \( x_U \in Y^p_k \) so that there exists \( x_R \in X^p_k \) such that \( |x_U - x_R| < r_k \). Equation (4.7) implies

\[
d_{x_U, 10r_k}(\Sigma_k, E) < C\epsilon.
\]
Therefore, we get
\[ \left| \frac{1}{2} c_0 B_U \cap \Sigma_k \right| \simeq_d \ell(U)^d \]
and the fact that \( g \) (and the associated surface map \( f \)) is \((1 + C\delta)\)-bi-Lipschitz means that
\[ f \left( f_k^{-1} \left( \frac{1}{2} c_0 B_U \cap \Sigma_k \right) \right) \subseteq c_0 B_U \cap \Sigma \]
so that
\[ \left| c_0 B_U \cap \Sigma \right| \simeq_d \ell(U)^d. \]
Because \( c_0 B_U \cap E \subseteq U \) and \( b\beta(MB_U) < \epsilon \), the collection \( \{c_0 B_U\}_{U \in \mathcal{Q}} \) is disjoint. Hence, we conclude
\[ \sum_{U \in \mathcal{Q}} \ell(U)^d \lesssim_d \sum_{U \in \mathcal{Q}} \left| c_0 B_U \cap \Sigma \right| = \left| \bigcup_{U \in \mathcal{Q}} c_0 B_U \cap \Sigma \right| \leq |\Sigma \cap B_Q| \lesssim \ell(Q)^d. \]

Now, we are able to bound \( |\partial \Omega_T| \).

**Lemma 4.9.** For any \( T \in \mathcal{T} \),
\[ |\partial \Omega_T| \lesssim_{\epsilon,d} |\partial \Omega_T \cap E| + \sum_{Q \in m_\mathcal{F}(\mathcal{T})} \ell(Q)^d + \sum_{U \in \mathcal{Q}} \epsilon(U)^2 \ell(U)^d. \]

**Proof.** The geometry of \( \mathcal{D}_T \) guarantees that for any cube \( W \in T \),
\[ |\partial W| \lesssim_d |\text{Bot}(W)| = \sum_{R \in m(T)} |\text{Bot}(R)| + |\partial \mathcal{D}_T \cap \pi_T(W)|. \]
We also have
\[ |\partial \mathcal{D}_T| \lesssim_d |\text{Bot}(W(T))| = |\partial \mathcal{D}_T \cap P_T| + \sum_{W \in m(T)} |\text{Bot}(W)| \]
due to the equality in (4.12) and the fact that \( \partial \mathcal{D}_T \subseteq \partial W(T) \cup \partial \tilde{W}(T) \cup m(W) \partial W \cup \partial \tilde{W} \)
where \( \tilde{W} = W - h(W) e_{d+1} \). Using Corollary 4.5 and (4.13) we write
\[ |\partial \Omega_T| \lesssim_d |\partial \mathcal{D}_T| \lesssim_d |\partial \mathcal{D}_T \cap P_T| + \sum_{W \in m(T)} |\text{Bot}(W)| \]
\[ \lesssim |\partial \Omega_T \cap E| + \sum_{W \in m(T)} |\text{Bot}(W)| + \sum_{W \in m(T)} |\text{Bot}(W)|. \]
We have
\[ \sum_{W \in m(T)} |\text{Bot}(W)| \lesssim_d \sum_{W \in m(T)} \ell(\Psi(W))^d \lesssim \sum_{Q \in m_\mathcal{F}(\mathcal{T})} \ell(Q)^d \]
where we used Lemma 4.7 items (ii) and (iii) consecutively. To handle the second sum, we similarly compute

\[
\sum_{W \in m_r(T)} |\text{Bot}(W)| \lesssim_{d, \epsilon} \sum_{W \in m_r(T)} \ell(\Psi(W))^d \cdot J_p(\Psi(W)) \lesssim \sum_{Q \in m_s(\mathcal{Q}_T)} \ell(Q)^d \cdot J_p(Q)
\]

\[
= \sum_{Q \in m_s(\mathcal{Q}_T)} \ell(Q)^d \sum_{U \in \text{Chain}(Q)} \epsilon(U)^2 \leq \sum_{U \in \mathcal{Q}_T} \epsilon(U)^2 \sum_{Q \in m_s(\mathcal{Q}_T)} \ell(Q)^d
\]

where we used Lemma 4.7 items (ii) and (iii) consecutively in the first line and Lemma 4.8 in the final line. The result follows from (4.14) and (4.15).

5. Lipschitz-star domains and the definition of \( \Lambda \)

We will form our final collection \( \Lambda \) in the statement of Theorem 3.15 as a union

\[
\Lambda = \bigcup_{p \in \mathcal{C}} \Lambda_{T_p}
\]

where \( \Lambda_{T_p} \) is a collection of domains whose union is \( \hat{\Omega}_{T_p} \) and \( \mathcal{C} \) is a well-chosen collection of points in \( \mathbb{R}^{d+1} \setminus E \). We begin by fixing some stopping time region \( T \) and building the definition of \( \Lambda_T \). The primary work in this definition is describing a way of breaking \( \hat{\Omega}_T \) into a collection of star-shaped Lipschitz domains. If we fix a stopping time region \( T \) and take for granted the existence of a nice \( d \)-dimensional set \( \Sigma_T \) such that \( D_T \setminus \Sigma_T = \bigcup_{j \in J_T} D^j_T \), a disjoint union where \( D^j_T \) is Lipschitz-star, then we define \( \Omega^j_T = g(D^j_T) \) and set

\[
\Lambda_T = \{ g(W) : W \in \hat{T} \setminus T \} \cup \{ \Omega^j_T : j \in J_T \}
\]

That is, for every Whitney cube in \( W \in \hat{T} \setminus T \), we take the image \( g(W) \) as a domain in its own right and add in the collection of domains \( \{ \Omega^j_T \} \) which are images of the domains \( D^j_T \) carved out of \( D_T \) by the set \( \Sigma_T \). In the remainder of this section, we give the definition of the set \( \Sigma_T \), show that each of the domains defined above is Lipschitz-star, and define the set \( \mathcal{C} \).

5.1. Defining \( \Sigma_T \). First, we give the definition of \( M \)-Lipschitz-star:

**Definition 5.1** (Lipschitz-star domains). We say that an open, connected set \( \Omega \subseteq \mathbb{R}^{d+1} \) is \( M \)-Lipschitz-star if after applying a translation and dilation to \( \Omega \), there exists a function \( r : \mathbb{S}^d \to \mathbb{R}^+ \) such that

\[
\partial \Omega = \{ r(\theta) \theta : \theta \in \mathbb{S}^d \}
\]
and, for any $\theta, \psi \in S^d$

$$|r(\theta) - r(\psi)| \leq M|\theta - \psi|,$$

$$\frac{1}{1 + M} \leq r(\theta) \leq 1.$$  

We want to prove the following proposition:

**Proposition 5.2.** There exists a constant $M(d) > 0$ such that for any $p \in \mathbb{R}^{d+1} \setminus E$ with associated stopping time region $T = T_p$, there exists a $d$-Ahlfors-David upper regular set $\Sigma_T$ which is a union of subsets of $d$-planes such that

$$D_T \setminus \Sigma_T = \bigcup_{j \in J_T} D_T^j$$

where

$$\sum_{j \in J_T} |\partial D_T^j| \lesssim_d |D_T|$$

and $D_T^j$ is $M$-Lipschitz-star.

$\Sigma_T$ will be defined as a union of more local sets $\Sigma_W$ for $W \in m(T)$. The basic idea is to use a “cover” emanating from the bottom face of every minimal cube $W$ downwards at a $\pi/4$ angle with the vertical in order to turn the jagged right angles made by stopped cubes into smoother $\pi/4$ angles which look Lipschitz to a point sitting above them higher up in the domain. This is essentially a modification of Peter Jones’s algorithm for turning chord arc domains composed of Whitney boxes in the disk into lipschitz domains in his classic proof of the Analyst’s Traveling Salesman Theorem in the complex plane (see pg. 8 of [Jon90]). We now construct $\Sigma_W$.

Fix $T$ and $W \in m(T)$. By translating and dilating, we can without loss of generality assume $W = [-1,1]^d \times [2,4]$ with $\mathbb{R}^d \times \{0\}$ the plane associated to $\mathcal{W}_T$ for ease of notation. For any function $f : \mathbb{R}^d \rightarrow \mathbb{R}$, we let $\text{Graph}(f)$ denote the graph of $f$ in $\mathbb{R}$ over $\mathbb{R}^d \times \{0\}$. We begin by defining, for $1 \leq j \leq d$,

$$H_0(x) = 2,$$

$$H_{2j-1}(x) = 3 + x_j,$$

$$H_{2j}(x) = 3 - x_j.$$  

The graphs of these functions (except $H_0$) over $\mathbb{R}^d$ are planes which make an angle of $\pi/4$ with $\mathbb{R}^d$ and contain the edges of $\text{Bot}(W)$ with $x_j = -1$ and $x_j = 1$ respectively. We define

$$H_W(x) = \min_{0 \leq i \leq 2d} H_i(x),$$

$$\text{Cover}(W) = \text{Graph}(H_W) \cap \mathbb{H}^{d+1}.$$  

$\text{Cover}(W)$ is the lower envelope of the collection of planes given by the graphs of the $H_i$. In $\mathbb{R}^3$, $\text{Cover}(W)$ forms the sides of a square pyramid minus its tip with base $[-3,3]^2 \times \{0\}$. In general, $\text{Cover}(W)$ divides $\mathbb{H}^{d+1}$ into two components: a bounded component which we call $C_W$
with boundary \( \text{Cover}(W) \cup [-3, 3]^d \times \{0\} \) and the unbounded complimentary component. It also follows that

\[
(5.3) \quad |\text{Cover}(W)| \lesssim_d |\text{Bot}(W)|.
\]

\( \text{Cover}(W) \) is one of two parts of \( \Sigma_W \). The second part will be called \( \text{Divider}(W) \) because its purpose will be to ensure that all future domains look similar to the top domain by separating future domains from one another with vertical plane extensions of the sides of cubes sliced by \( \text{Cover}(W) \).

We begin by defining \( t_n = 1 + \sum_{j=0}^{n-1} 2^{-j} \) and

\[
\mathcal{Q}_n = \left\{ Q \in \Delta^d([-3, 3]^d \times \{0\}) : \ell(Q) = t_{n+1} - t_n = 2^{-n}, \right. \\
\left. \exists j, \ 1 \leq j \leq d, \ a_j = \pm t_n, \ Q = \prod_{j=1}^{d} [a_j, b_j] \right\}.
\]

Intuitively, we think of \( t_n \) as the radii of growing balls in the \( \ell_\infty \) metric centered at 0, and the cubes inside \( \mathcal{Q}_n \) as the natural collection of dyadic cubes tiling the set difference between successive balls with side length exactly equal to the gap between the two square rings forming the boundaries of the \( \ell_\infty \) balls (See Figure 2). Set \( \mathcal{Q} = \bigcup_{n=1}^{\infty} \mathcal{Q}_n \) and define

\[
\text{Divider}(W) = C_W \cap \bigcup \{ F_j \times [0, 2\ell(Q)] : F_j \in \text{Faces}(Q), \ Q \in \mathcal{Q} \}.
\]
Figure 2. A representation of $[-3, 3]^2 \times \{0\}$ split into $Q_1$ in yellow, $Q_2$ in red, and $\bigcup_{n=3}^{\infty} Q_n$ left uncolored at the edge of $Q_2$ (The white square in the middle sits below the cube $W \in m(T)$, hence nothing above it lies in $D_T$). The set $\text{Divider}(W)$ shoots out of the page as a union of extensions of the sides of the squares up to the points at which they hit the slanting top of $\text{Cover}(W)$.

Because $\sum_{j=1}^{2d} |F_j \times [0, 2\ell(Q)]| \lesssim_d |Q|$ and $[-3, 3]^d \times \{0\} = \bigcup_{Q \in \mathcal{Q}} Q$ is a disjoint union, it follows immediately that

$$|\text{Divider}(W)| \lesssim_d |\text{Bot}(W)|.$$  

Now, we define

$$\Sigma_W = \text{Cover}(W) \cup \text{Divider}(W),$$

$$\Sigma_T = \bigcup_{W \in m(T)} \Sigma_W \cap D_T.$$  

We first prove the upper regularity claim of Proposition 5.2.

Lemma 5.3. $\Sigma_T$ is upper $d$-Ahlfors-David upper regular with constant $C \lesssim_d 1$.

Proof. Fix $R > 0$ and $x \in \Sigma_W \subseteq \Sigma_T$ for some $W \in m(T)$. We write

$$|\Sigma_T \cap B(x, R)| = \sum_{W \in m(T)} |\Sigma_W \cap B(x, R)| + \sum_{W \in m(T)} |\Sigma_W \cap B(x, R)|.$$
We note that $\pi_T(W)$ and $\pi_T(W')$ have disjoint interiors for any $W, W' \in m(T)$ with $W \neq W'$, so that
\[
\sum_{W \in m(T)} |\Sigma_W \cap B(x, R)| \lesssim_d \sum_{W \in m(T)} |\text{Bot}(W)| \leq (20R)^d.
\]
On the other hand, there are a uniformly bounded number of stopped cubes $N(d)$ with $h(W) \geq 10R$ such that $B(x, R) \cap \Sigma_W \neq \emptyset$ so that
\[
\sum_{W \in m(T)} |\Sigma_W \cap B(x, R)| \leq N(d) \cdot c(d)R^d \lesssim_d R^d
\]
because $|\Sigma_W \cap B(x, R)| \leq c(d)R^d$ for any particular $W$ by construction. Therefore, $\Sigma_T$ is upper regular. \hfill \blacksquare

We now finish the proof of Proposition 5.2.

**Proof of Proposition 5.2.** It follows from (5.3) and (5.4) that
\[
|\Sigma_T| \leq \sum_{W \in m(T)} |\Sigma_W| \lesssim_d \sum_{W \in m(T)} |\text{Bot}(W)| \leq |\text{Bot}(W(T))| \leq |\partial D_T|
\]
which proves (5.2). We now need to show that the resulting domains $D_T^j$ are Lipschitz-star. If $D_T^j$ is the domain containing $W(T)$, then the claim follows with the choice of central point $c_{W(T)}$. Indeed, the cube $W(T)$ is clearly Lipschitz-star with respect to $c_{W(T)}$, and any boundary point of $D_T^j$ not $\partial W(T)$ is either in a vertical plane containing one of the vertical faces of $W(T)$, or is part of the Lipschitz graph consisting of the horizontally planar faces Bot($W$) for $W \in m(T)$ and the planes of Cover($W$) making $\pi_4$ angles with the bottom faces.

Now, suppose $D_T^j \cap W(T) = \emptyset$. We have set up the construction such that this will not differ too much from the top cube case. Let $W \in m(T)$ be a cube of minimal height such that $D_T^j \subseteq C_W$ and $|\partial D_T^j \cap \text{Cover}(W)| > 0$. Such $W$ exists because its minimality implies that for any $W' \in m(T)$ of smaller side length than $W$, Cover($W'$) can only be part of the “lower” boundary of $D_T^j$ while the only non-vertical planar pieces in $D_T^j$ are Lipschitz-star. If $\partial W(T)$ is the domain containing $W(T)$, then the claim follows. Otherwise, $D_T^j$ has nonempty intersection with some children of $R$. By the geometry described above, $D_T^j$ contains the convex hull of $c_T$ and Bot($T$), so we can justifiably claim that $D_T^j$ is Lipschitz-star with respect to $\frac{1}{2}(c_T + c_{\text{Bot}(T)})$. Indeed, Lipschitz-starliness follows for points in $T \cap D_T^j$, and follows for the rest of $D_T^j$ because the remaining boundary consists of vertical planes containing one of the vertical faces of $R$ or is part of a Lipschitz graph consisting of horizontally planar faces Bot($W'$) for $R \geq W' \in m(T)$ and the planes of Cover($W'$) making $\pi_4$ angles with the bottom faces. \hfill \blacksquare
5.2. A domains are Lipschitz-star. We must now show that the Lipschitz-star property is preserved under the image of a \((1 + C\delta)\)-bi-Lipschitz map satisfying (4.8).

We begin with a lemma which relates being Lipschitz-star to properties of cones at boundary points. In preparation, define \(\ell_z\) for any \(z \in \mathbb{R}^{d+1}\) to be the line passing through 0 and \(z\) and let \(P_z = \ell_z^+ + z\). Define the radial cone at \(x\) of aperture \(\alpha\) and radius \(R\) as

\[
C_x(\alpha, R) = \left\{ y \in B(x, R) : \frac{\text{dist}(y, \ell_x)}{\text{dist}(y, P_x)} < \tan(\alpha) \right\} \setminus \{x\}.
\]

**Lemma 5.4.** Let \(\Omega\) be a star-shaped domain with respect to 0, and let \(r : \mathbb{S}^d \to \mathbb{R}^+\) be such that \(\partial \Omega = \{r(\theta) : \theta \in \mathbb{S}^d\}\). If \(\Omega\) is \(M\)-Lipschitz-star, then for every \(x \in \partial \Omega\), \(\delta > 0\), there exists \(R > 0\) such that \(C_x(\arctan((M/|x| + \delta)^{-1}), R) \cap \partial \Omega = \emptyset\). Conversely, if the latter condition holds, then there exists a constant \(M' \lesssim_{M,\delta} 1\) such that \(\Omega\) is \(M'\)-Lipschitz-star.

**Proof.** First, suppose \(\Omega\) is \(M\)-Lipschitz-star. Assume without loss of generality that \(\frac{1}{M+1} \leq |x| \leq 1\). Let \(y \in \partial \Omega \cap B(x, r)\) for \(r\) sufficiently small and let \(0 < \theta < \frac{\pi}{4}\) be the angle between the lines \(\ell_x\) and \(\ell_y\). We have \(\text{dist}(y, \ell_x) = |y| \sin \theta\) while \(\text{dist}(y, P_x) = |x| - |y| \cos \theta\). Because \(\Omega\) is \(M\)-Lipschitz-star, we know

\[
| |x| - |y| | = r \left( \frac{x}{|x|} \right) - r \left( \frac{y}{|y|} \right) \leq M \left| \frac{x}{|x|} - \frac{y}{|y|} \right| \leq M \theta.
\]

With this, we calculate

\[
\frac{\text{dist}(y, \ell_x)}{\text{dist}(y, P_x)} = \frac{|y| \sin \theta}{|x| - |y| \cos \theta} \leq \left| \frac{|x| \sin \theta + M \theta \sin \theta}{|x| - |x| \cos \theta - M \theta \cos \theta} \right| \xrightarrow{\theta \to 0} \frac{|x|}{M}.
\]

For the second statement, we note that the cone hypothesis implies that \(r\) is locally \(\frac{2M}{|x|} \leq 2M(1+M)\) Lipschitz on \(\mathbb{S}^d\) from which the result follows. □

**Proposition 5.5.** Suppose \(\Omega \subseteq \mathbb{R}^{d+1}\) is an \(M\)-Lipschitz-star domain, and \(\varphi : \overline{\Omega} \to \varphi(\overline{\Omega})\) is a \(C^1\) map satisfying

\[
(D\varphi(z) - I) \leq C\delta
\]

for all \(z \in \overline{\Omega}\) where \(C\delta \ll \frac{1}{M}\). Then there exists \(M' \lesssim_{M,\delta} 1\) such that \(\varphi(\Omega)\) is a \(M'\)-Lipschitz-star domain.

**Proof.** By translating, we can assume that the central point of \(\Omega\) is 0 and \(\varphi(0) = 0\). We first verify that the radial defining function \(r : \mathbb{S}^d \to \mathbb{R}^+\) is well-defined, i.e., the domain is star-shaped with respect to 0. Let \(\varphi(x) \in \partial \Omega\) and let \(\gamma(t) = t \varphi(x)\). We want to show \(\gamma \cap \partial \varphi(\Omega) = \{\varphi(x)\}\). Set \(\tilde{\gamma}(t) = \varphi^{-1}(\gamma(t))\). We would like to prove

\[
|\tilde{\gamma}'(t) - x| \leq C\delta |x|
\]

for all \(t \in [0, 1]\). First note that

\[
|D\varphi(z)^{-1} - I| = |D\varphi(z)^{-1} \cdot [I - D\varphi(z)^{-1}]| \leq C\delta |D\varphi(z)^{-1}| \leq C\delta
\]
using the bound $|D\varphi(z)^{-1}| \leq \frac{1}{\sigma_{\min}(D\varphi(z))} \leq (1 + C\delta)$ where $\sigma_{\min}(D\varphi(z))$ is the smallest singular value of $D\varphi(z)$. This means

$$|\tilde{\gamma}'(t) - x| = |D\varphi^{-1}(\gamma(t)) \cdot \gamma'(t) - x| = |[D\varphi(\tilde{\gamma}(t))^{-1} - I] \cdot \gamma'(t) + \gamma'(t) - x| \leq C\delta|\varphi(x)| + |\varphi(x) - x| \leq C\delta|x|$$

where the final line follows from the fact that $\varphi(x) = \int_0^1 D\varphi(tx) \cdot x \ dt = x + \int_0^1 (D\varphi(tx) - I) \cdot x \ dt$ so that $|\varphi(x) - x| \leq C\delta|x|$. Now, it follows that $\tilde{\gamma} \subseteq C_x(\partial \Omega, \{x\})$. Indeed, suppose there existed a time $s \in [0,1]$ for which $\tilde{\gamma}(s)$ were outside this cone. Since $\tilde{\gamma}(1) = x$, $\tilde{\gamma}$ must have a large perpendicular component to $x$ on a set of positive measure in $[s,1]$, contradicting (5.6). It follows from Lemma 5.4 that $C_x(C\delta, \{x\}) \cap \partial \Omega = \emptyset$. Therefore, $\gamma \cap \partial \Omega = \{x\}$ so that $\gamma \cap \partial \varphi(\Omega) = \{\varphi(x)\}$ as desired.

By Lemma 5.4 to prove that $\Omega$ is $M'$-Lipschitz-star it suffices to show that there exists $K \lesssim_{M,d} 1$ satisfying the following claim: For every $x \in \partial \Omega$, there exists $R > 0$ such that $C_{\varphi(x)}(\frac{|\varphi(x)|}{R}, R) \cap \partial \varphi(\Omega) = \emptyset$. To prove this claim, choose $R > 0$ such that $C_x(\frac{|x|}{2M}, R) \cap \partial \Omega = \emptyset$ and let $y \in \partial \Omega \cap B(x, R)$. Put $z = (1 - |x - y|)x$, let $\alpha$ be the angle $\angle zxy$, and let $\alpha'$ be $\angle(\phi(z)\phi(x)\phi(y))$. By the law of cosines,

$$\cos \alpha = \frac{|z - x|^2 + |x - y|^2 - |z - y|^2}{2|z - x||x - y|} = 1 - \frac{|z - y|^2}{2|z - x|^2},$$

$$\cos \alpha' = \frac{|\varphi(z) - \varphi(x)|^2 + |\varphi(x) - \varphi(y)|^2 - |\varphi(z) - \varphi(y)|^2}{2|\varphi(x) - \varphi(z)||\varphi(x) - \varphi(y)|} \leq \frac{2(1 + C\delta)^2|z - x|^2 - (1 - C\delta)^2|z - y|^2}{2(1 - C\delta)^2|z - x|^2} \leq 1 - \frac{|z - y|^2}{2|z - x|^2} + C\delta = \cos \alpha + C\delta.$$

Assume first that $\alpha < \pi/2$. Since $C_x(\frac{|x|}{2M}, R) \cap \partial \Omega = \emptyset$ and $y \in \partial \Omega \cap B(x, R)$, we know that

$$\tan \alpha \geq \frac{|x|}{2M'}.$$ 

Hence, $\alpha' \geq \frac{\alpha}{2}$ with $C\delta \ll \frac{\alpha}{2}$. Now, we only need to measure the angle between the line segment $[\varphi(x), \varphi(z)]$ and the line $\ell_{\varphi(x)}$. By the proof of star-shapedness this must be less than $C\delta$ since the curve $\gamma(t) = \varphi(tx)$ has derivative close to $x$ everywhere by (5.6). Therefore, if we let $\theta$ be the angle between $[\varphi(x), \varphi(y)]$ and $\ell_{\varphi(x)}$, then we have $\theta \geq \alpha' - C\delta$. Choosing $K$ large enough in terms of $\alpha'$ (dependent only on $M$) such that $\tan \theta > \frac{|\varphi(y)|}{K}$ for any chosen $y \in \partial \Omega \cap B(x, R)$, we see that $\frac{\text{dist}(\varphi(y), \ell_{\varphi(x)})}{\text{dist}(\varphi(y), \phi(x))} > \tan \theta > \frac{|\varphi(x)|}{K'}$ so that $\varphi(y) \notin C_{\varphi(x)}(\frac{|\varphi(x)|}{K'}, \frac{R}{2})$, proving that $C_{\varphi(x)}(\frac{|\varphi(x)|}{K}, \frac{R}{2}) = \emptyset$ because $\varphi$ is $(1 + C\delta)$-bi-Lipschitz.

If instead $\alpha \geq \frac{\pi}{2}$, we can argue nearly identically to the above where we use the fact that $\alpha$ is bounded away from $\pi$ in terms of $M$ in place of $\alpha$ being bounded away from 0. 

We now collect what we’ve proven so far about $\Lambda$ into the following proposition:
Corollary 5.6. There exists a constant $M > 0$ dependent on the dimension $d$ such that for every $\Omega_\alpha \in \Lambda_T$, $\Omega_\alpha$ is $M$-Lipschitz-star.

Proof. If $\Omega_\alpha = g_T(\mathcal{D}_T^j)$ for some $j \in J_T$ then by Proposition 5.2, $\mathcal{D}_T^j$ is $c(d)$-Lipschitz-star so that Proposition 5.5 implies that $g(\mathcal{D}_T^j)$ is $M(d)$-Lipschitz-star. If $\Omega = g(W)$ for some $W \in \hat{\mathcal{D}}_T \setminus \mathcal{D}_T$, then the result again follows from Proposition 5.5 because the cube $W$ is $c(d)$-Lipschitz-star. □

The next step in the setup is to give an algorithm for choosing the collection $\mathcal{C}$.

5.3. Defining $\mathcal{C}$. We define a new series of scales for $n \geq 0$ by

$$s_n = \frac{3}{2}2^{-n+n_0}$$

where $n_0 \in \mathbb{Z}$ is the least integer such that $2^{n_0} > 10\diam(E)$. Let $C_n = \{p_{i,n}\}$ be an $s_n$-net for the set $\{z \in B(E, 10\diam(E)) : \dist(z, E) = s_n\}$.

Set $\mathcal{C}_0 = \bigcup_n C_n$. Put an ordering on the points of $\mathcal{C}_0$ by choosing some ordering on each finite set $C_n$ and then imposing $p_n < p_m$ for any $p_n \in C_n$, $p_m \in C_m$ with $n < m$. $\mathcal{C}_0$ has a least element which we call $c_0$ and we define the auxiliary collection $\mathcal{P}_0 = \{p_0\}$. Given the definitions of $\mathcal{C}_0$ and $\mathcal{P}_0$, we define $\mathcal{C}_{n+1}$ and $\mathcal{P}_{n+1}$ inductively for any $n \geq 0$ by

$$\mathcal{C}_{n+1} = \mathcal{C}_n \setminus \left\{ p \in \mathcal{C}_n : \dist\left(p, \bigcup_{p' \in \mathcal{P}_n} \Omega_{p'}\right) < \frac{A}{30} \dist(p, E) \right\}$$

(5.7)

$$\mathcal{P}_{n+1} = \mathcal{P}_n \cup \{p_{n+1}\}$$

(5.8)

where $p_{n+1}$ is the least element of $\mathcal{C}_{n+1}$ with respect to the ordering inherited from $\mathcal{C}_0$. Finally, put

$$\mathcal{C} = \bigcup_{n=0}^{\infty} \mathcal{P}_n.$$

We also collect the regions $T = T_p$ into $\mathcal{J} = \{T_p\}_{p \in \mathcal{C}}$.

5.4. Properties of $\{\Omega_T\}_{T \in \mathcal{J}}$. This subsection is devoted to proving properties (1) and (2) in the statement of Theorem 3.15. First, we show that the “buffer” region $\hat{\Omega}_T \setminus \Omega_T$ contains a cone around $\Omega_T$ with respect to the distance to $E$:

Lemma 5.7. For any $T \in \mathcal{J}$, $\hat{\Omega}_T$ contains a $A/10$-cone around $\Omega_T$ with respect to distance from $E$. That is,

$$F = \left\{ w \in \mathbb{R}^{d+1} \setminus E : \dist(w, \Omega_T) < \frac{A}{10} \min \{\dist(w, E), \dist(g(W(T)), E)\} \right\} \subseteq \hat{\Omega}_T$$

(5.9)

Proof. Let $z \in F$ and suppose first that $p$ is flat. Since $\hat{\Omega}_T = g(\hat{\mathcal{D}}_T)$ where $g$ is $(1 + C\delta)$-bi-Lipschitz by Corollary 4.5 and translates distance in the domain to $P_T$ to distance to $E$ in the image by (4.6), it suffices to show

$$\left\{ z \in \mathbb{R}^{d+1} \setminus E : \dist(z, \mathcal{D}_T) < \frac{A}{4} \min \{\dist(z, P_T), \dist(W(T), P_T)\} \right\} \subseteq \hat{\mathcal{D}}_T$$

(5.10)
because the desired containment then follows by mapping \[5.10\] forward. Now, there exists $W \in T$ such that $\text{dist}(z, W) = \text{dist}(z, \mathcal{D}_T)$ and there exists a cube $W_z \in \mathcal{W}_\mathcal{F}$ such that $z \in W_z$.

By the definition of $\mathcal{D}_T$, it suffices to show that $W$ and $W_z$ are $\mathcal{A}$-close. We can estimate

\[
\text{dist}(W, W_z) \leq \text{dist}(z, \mathcal{D}_T) \leq \frac{A}{4} \min\{\text{dist}(z, P_T), \text{dist}(W(T), P_T)\} \leq \frac{A}{2} \min\{h(W_z), h(W_T)\} = \frac{A}{2} \min\{\ell(W_z), \ell(W_T)\}.
\]

Using this we get

\[
\ell(W) = h(W) \leq \text{dist}(W, W_z) + \text{diam} W_z + h(W_z) \leq \left(\frac{A}{2} + \sqrt{d+1} + 1\right) \ell(W_z) \leq A \ell(W_z)
\]

given that $A \geq 4\sqrt{d}$. Now, if $W < W_p$, then $\ell(W_z) = h(W_z) \leq h(W) = \ell(W)$ because otherwise we would have $\text{dist}(W_z, W^{(1)}) < \text{dist}(W_z, W)$, contradicting the definition of $W$. If instead $W = W(T)$, then we can calculate similarly to the above:

\[
\ell(W_z) = h(W_z) \leq \text{dist}(W_z, W(T)) + \text{diam} W_p + h(W(T)) \leq \left(\frac{A}{2} + \sqrt{d+1} + 1\right) \ell(W(T)) \leq A \ell(W(T)).
\]

This completes the proof in the case when $p$ is flat. If $p$ is non-flat, then $\Omega_T = W(T)$ where $p \in W_T \in \mathcal{W}_0$. Let $z \in W_z \in \mathcal{W}_0$. By the definition of $\mathcal{D}_T$ for non-flat $p$, it suffices to show that $W(T)$ and $W_z$ are $\mathcal{A}$-close. We have

\[
\text{dist}(W(T), W_z) \leq \text{dist}(z, W(T)) \leq \frac{A}{10} \text{dist}(W(T), E) \leq \frac{A}{2} \text{diam} W(T),
\]

\[
\text{dist}(W(T), W_z) \leq \text{dist}(z, W(T)) \leq \frac{A}{10} \text{dist}(z, E) \leq \frac{A}{10} (\text{diam} W_z + \text{dist}(W_z, E)) \leq \frac{A}{10} (5 \text{diam} W_z) \leq \frac{A}{2} \text{diam} W_z,
\]

where we used \[4.6\] and each part of the minimum in \[5.9\]. Then

\[
\text{diam} W_z \leq \text{dist}(W_z, E) \leq \text{dist}(W_z, W(T)) + \text{diam} W(T) + \text{dist}(W(T), E) \leq \left(\frac{A}{2} + 5\right) \text{diam} W(T) \leq A \text{diam} W(T),
\]

\[
\text{diam} W(T) \leq \text{dist}(W(T), E) \leq \text{dist}(W(T), W_z) + \text{diam} W_z + \text{dist}(W_z, E) \leq \left(\frac{A}{2} + 5\right) \text{diam} W_z \leq A \text{diam} W_z.
\]

This show that $W_z$ and $W(T)$ are $\mathcal{A}$-close, completing the proof.

Now that we have regularity of $g$ on $\mathcal{D}_T$, we conclude this section with a proof of some key properties of the collection $\{\Omega_T\}_{T \in \mathcal{F}}$.\]
Proposition 5.8. The following hold:

(i) $\Omega_T \cap \Omega_{T'} = \emptyset$ for $T, T' \in \mathcal{T}$, $T \neq T'$,
(ii) $B(E, \text{diam}(E)) \setminus E \subseteq \bigcup_{T \in \mathcal{D}} \Omega_T$,
(iii) $\sum_{T \in \mathcal{D}} \sum_{W \in \mathcal{T}} \chi_g(W) \lesssim_d 1$,
(iv) $\hat{\Omega}_T \cap E = \emptyset$.

Proof. We begin with proving (i). Using the partial order on $\mathcal{C}$, assume without loss of generality that $p < p'$. By the definition of $\mathcal{C}$, we have

$$\text{dist}(p', \Omega_p) \geq A \frac{1}{30} \text{dist}(p', E).$$

Hence, $\Omega_{p'} \subseteq B(p', 3\sqrt{d} \text{dist}(p', E)) \subseteq B(c', \frac{4}{30} \text{dist}(p', E))$ because $A > 120\sqrt{d}$.

We now prove (ii). Let $z \in B(E, \text{diam}(E)) \setminus E$ and let $k \geq 0$ be such that $s_{k+1} \leq \text{dist}(z, E) \leq s_k$. By the definition of $C_k$, there exists $p_k \in C_k$ such that

$$|z - p_k| \leq 3s_k = 6s_{k+1} \leq 6 \text{dist}(z, E).$$

Now, if $p_k \in \mathcal{C}$, then by Lemma 5.7, $z \in \hat{\Omega}_{p_k}$. Otherwise, $p_k \notin \mathcal{C}$ so that by (5.7) there exists $p \in \mathcal{C}$ such that $p < p_k$ and $\text{dist}(p_k, \Omega_p) < \frac{4}{30} \text{dist}(p_k, E) = \frac{4}{30}s_k = \frac{4}{15}s_{k+1}$. But then

$$\text{dist}(z, \Omega_p) \leq |z - p_k| + \text{dist}(p_k, \Omega_p) \leq 6s_{k+1} + \frac{4}{15}s_{k+1} \leq A \frac{1}{10} \text{dist}(z, E)$$

so that $z \in \hat{\Omega}_p$ by Lemma 5.7 as long as $\text{dist}(z, \Omega_p) \leq A \frac{4}{10} \text{dist}(g(W_p), E)$ which follows from the fact that $p < p_k$ ($p$ is a net point of larger scale).

We now prove (iii). Let $z \in B(E, \text{diam}(E)) \setminus E$ and define $\mathcal{C}' = \{ p \in \mathcal{C} : z \in \hat{\Omega}_p \}$. Because the individual elements of $\mathcal{S}_p$ have disjoint interiors, it suffices to prove that $\mathcal{C}'$ is finite with size independent of $z$. By definition, for any $p \in \mathcal{C}'$, there exists $W \in \mathcal{D}_p$ such that $z \in g(W)$. The definition of $\hat{\Omega}_p$ then implies that there exists $R_p \in \mathcal{D}_p$ such that $R_p$ and $W$ are $A$-close, which gives a mapping $p \mapsto R_p$ where $R_p \in \mathcal{W}_p$ with $g(R_p) \supset B(c_{R_p}, r)$ where $r \geq \text{diam } R_p \gtrsim_A \text{diam } W \gtrsim \text{dist}(z, E)$ and similarly $\text{dist}(g(R_p), z) \lesssim \text{diam } R_p + \text{dist}(R_p, W) + \text{diam } W \lesssim_A \text{dist}(z, E)$. Since $\Omega_p \cap \Omega_{p'} = \emptyset$ for $p \neq p'$, it follows that the set $\{ g(c_{R_p}) : p \in \mathcal{C}' \}$ is a collection of points contained in a ball of radius comparable to $\text{dist}(z, E)$ with mutual distances comparable to $\text{dist}(z, E)$ where comparability constants are in terms of $A$ and $d$. Since there can only be a uniformly bounded number of such points with bound in terms of $A$ and $d$, the result follows.

Item (iv) follows from (4.6).

With this, we now have a collection of domains which satisfy (i) and (ii) of Theorem 3.15

Corollary 5.9. $\Lambda = \{ \Omega_{\alpha} \}_{\alpha \in I}$ is a collection of Lipschitz-star domains such that

(i) $\Omega_{\alpha} \subseteq \mathbb{R}^{d+1} \setminus E$,
(ii) $\chi_{B(E, \text{diam}(E)) \setminus E} \leq \sum_{\alpha} \chi_{\overline{\Omega}_{\alpha}} \lesssim_d 1$.

Proof. $\Omega_{\alpha}$ is Lipschitz-star by Corollary 5.6. Item (i) above follows from Proposition 5.8 item (iv) while item (ii) above follows from items (ii) and (iii) of the same proposition. ■
6. Boundary measure estimates

This section finishes the proof of Theorem 3.15 by proving the requisite estimates (3.24) and (3.25) on the $\mathcal{H}^d$ measure of the boundaries of domains in $\Lambda$. Both of these estimates will follow from the general framework given below centered around a point $y \in E$ and $0 < r < \text{diam}(E)$ as in (3.25).

We begin with some definitions. In order to pick out the pieces of the domains which actually intersect $B(y, r)$, for any $T \in \mathcal{T}$ we define

$$T'_{y, r} = \{T \in \mathcal{T} : \hat{\Omega}_T \cap B(y, r) \neq \emptyset\}.$$

Break up $T'_{y, r}$ into regions with large and small top cubes:

$$T_L = \{T \in T'_{y, r} : h(W(T)) > 10r\},$$

$$T_{y, r} = T'_{y, r} \setminus T_L.$$

The desired estimates will follow from the following proposition:

**Proposition 6.1.**

(6.1)

$$\sum_{\alpha \in I_T} |\partial \Omega_{\alpha} \cap B(y, r)| \lesssim r^d + |E \cap B(y, A'r)| + \sum_{T \in T'_{y, r}} \sum_{Q \in m_T(D_T)} \ell(Q)^d + \sum_{T \in T'_{y, r}} \sum_{U \in \mathcal{D}_T} \epsilon(U)^2 \ell(U)^d$$

We will prove this proposition in two lemmas. The first gives a bound for the domains in $T_L$ while the second gives a bound for those in $T_{y, r}$:

**Lemma 6.2.**

$$\sum_{T \in T_L} \sum_{\alpha \in I_T} |\partial \Omega_{\alpha}^T \cap B(y, r)| \lesssim_d r^d.$$  

**Proof.** We will show that $\#(T_L)$ is bounded in terms of $d$. For any $T \in T_L$ we claim that there exists some $W_T \in T$ such that $h(W_T) \simeq_d r$ and $\text{dist}(g(W_T), y) \simeq_d r$. Indeed, by definition there exists $R_T \in \hat{T}$ such that $g(R_T) \cap B(y, r) \neq \emptyset$ so that there is a cube $V_T \in T$ such that $V_T \simeq A R_T$. There then exists an cube $W_T \in T$ with $W_T \geq V_T$ with the desired properties because $g$ is $(1 + C\delta)$-bi-Lipschitz and $h(W(T)) > 10r$. But, since the collection $\{g(W_T)\}_{T \in \mathcal{T}_L}$ is disjoint by Proposition 5.8 item (i), $N = \#(T_L) = \#(\{g(W_T)\}_{T \in \mathcal{T}_L}) \lesssim_d 1$. Therefore,

$$\sum_{T \in T_L} \sum_{\alpha \in I_T} |\partial \Omega_{\alpha}^T \cap B(y, r)| \lesssim N \sum_{\alpha \in I_T} |\partial \Omega_{\alpha}^T \cap B(y, r)| \lesssim_d r^d$$

where the final inequality follows because $\bigcup_{\alpha \in I_T} \partial \Omega_{\alpha}^T$ is $d$-Ahlfors-David upper regular for any $T \in \mathcal{T}$ by Lemma 5.3, the fact that $\partial D_T$ is regular, and the fact that $\sum_{R \simeq A W} |\partial R| \lesssim_{A,d} |\partial W|$ for any $W \in \mathcal{W}_T$.  

We now handle the regions with small top cubes:
Lemma 6.3.
\[
\sum_{T \in \mathcal{F}_{y,r}} \sum_{\alpha \in I_T} |\partial \Omega_T^\alpha \cap B(y,r)| \lesssim_{\epsilon,d} |E \cap B(y,A'r)| + \sum_{T \in \mathcal{F}_{y,r}} \sum_{Q \in m_{\mathcal{F}}(\mathcal{P}_T)} \ell(Q)^d + \sum_{T \in \mathcal{F}_{y,r}, U \in \mathcal{P}_T} \epsilon(U)^2 \ell(U)^d
\]

Proof. Begin by fixing \(T \in \mathcal{F}_{y,r}\). The definition of \(A\)-closeness implies that each cube \(R \simeq_A W\) for some “outer” \(W \in T\) such that \(|\partial W \cap \partial D_T| \gtrsim_d |\partial W|\). Therefore, we have
\[
\sum_{\alpha \in I_T} |\partial \Omega_T^\alpha \cap B(y,r)| \leq \sum_{R \in \hat{T}, T \setminus T} |g(R)| + |\partial \Omega_T| + |\Sigma_T|
\]
\[
\lesssim_A |\partial \Omega_T| + |\Sigma_T \cap \Omega_T| \lesssim |\partial \Omega_T|
\]
\[
\lesssim_{\epsilon,d} |\partial \Omega_T \cap E| + \sum_{Q \in m_{\mathcal{F}}(\mathcal{P}_T)} \ell(Q)^d + \sum_{U \in \mathcal{P}_T} \epsilon(U)^2 \ell(U)^d
\]
where we used the fact that \(\Sigma_T\) is upper regular and \(\partial \Omega_T\) is lower regular up to scale \(\text{diam}(\Omega_T)\) in the second line and Lemma 4.9 in the final line.

Summing this inequality over \(T_{y,r}\), we get
\[
\sum_{T \in \mathcal{F}_{y,r}} \sum_{\alpha \in I_T} |\partial \Omega_T^\alpha \cap B(y,r)| \lesssim_{\epsilon,d} \sum_{T \in \mathcal{F}_{y,r}} |\partial \Omega_T \cap E| + \sum_{T \in \mathcal{F}_{y,r}} \sum_{Q \in m_{\mathcal{F}}(\mathcal{P}_T)} \ell(Q)^d + \sum_{T \in \mathcal{F}_{y,r}, U \in \mathcal{P}_T} \epsilon(U)^2 \ell(U)^d
\]

(6.2)
\[
\leq |E \cap B(y,A'r)| + \sum_{T \in \mathcal{F}_{y,r}} \sum_{Q \in m_{\mathcal{F}}(\mathcal{P}_T)} \ell(Q)^d + \sum_{T \in \mathcal{F}_{y,r}, U \in \mathcal{P}_T} \epsilon(U)^2 \ell(U)^d
\]

where the final inequality comes from the fact that \(\{g_T(\partial D_T \cap P_T)\}_{T \in \mathcal{F}_{y,r}}\) is a pairwise disjoint collection contained in \(E \cap B(y,A'r)\) as long as \(A'\) is sufficiently large with respect to \(A\).  

Corollary 6.4. Let \(E\) and \(\mathcal{D}\) be as in Theorem 3.15. If (3.24) holds, then
\[
\sum_{\alpha \in I} |\partial \Omega_\alpha| \lesssim_{\epsilon,d} |\Gamma| + |E|.
\]
If (3.25) holds, then for any \(y \in E\) and \(0 < r < \text{diam}(E)\),
\[
\sum_{\alpha \in I} |\partial \Omega_\alpha \cap B(y,r)| \lesssim_{\epsilon,d} r^d.
\]

Proof. For any \(Q \in \mathcal{D}\), define
\[
\mathcal{K}_Q = \{T \in \mathcal{F} : \exists W_T \in T, \ Q \simeq_{A'} g(W_T)\}.
\]
We claim that \(#(\mathcal{K}_Q) \lesssim_{1} d^1\). Indeed, Proposition 5.8 item (i) says \(\Omega_T \cap \Omega_T' = \emptyset\) for any unequal \(T,T' \in \mathcal{F}\) so that the collection \(\{g(W_T)\}_{T \in \mathcal{K}_Q}\) is a collection of disjoint subsets containing balls of radius \(\simeq_d \text{diam} \ Q\) at distance \(\simeq_d \text{diam} \ Q\) from \(Q\), thus has a bound on its size in terms of \(A'\) and independent of \(Q\). Similarly, there are a uniformly bounded number of cubes within any particular region \(T \in \mathcal{F}_Q\) so that \(#(\{W \in T : T \in \mathcal{F}_Q, \ Q \simeq_{A'} g(W)\}) \lesssim_{A'} 1\).
Suppose first that (3.24) holds. Let \( y \in E \) and \( r = 100 \operatorname{diam}(E) \) so that \( \mathcal{T}_L = \emptyset \) and \( \mathcal{T} = \mathcal{T}_{y,r} \). Applying Lemma 6.3, we get

\[
\sum_{\alpha \in I} |\partial \Omega_\alpha| = \sum_{T \in \mathcal{T}_{y,r}} \sum_{\alpha \in \mathcal{I}_T} |\partial \Omega^\alpha_T \cap B(y, r)| 
\leq \varepsilon, d \left| E \cap B(y, A'r) \right| + \sum_{T \in \mathcal{T}_{y,r}} \sum_{Q \in \mathcal{Q}(\mathcal{P}_T)} \ell(Q)^d + \sum_{T \in \mathcal{T}_{y,r}} \sum_{U \in \mathcal{P}_T} \epsilon(U)^2 \ell(U)^d 
= |E| + \sum_{T \in \mathcal{T}} \sum_{Q \in \mathcal{Q}(\mathcal{P}_T)} \ell(Q)^d + \sum_{T \in \mathcal{T}} \sum_{U \in \mathcal{P}_T} \epsilon(U)^2 \ell(U)^d 
\leq |E| + \sum_{Q \in \mathcal{Q}_{\text{Bad}}(\mathcal{P})} \ell(Q)^d + \sum_{Q \in \mathcal{P}} \epsilon(U)^2 \ell(U)^d 
\leq |E| + \Gamma
\]

where the last line follows because \( \mathcal{Q}_{\text{Bad}}(\mathcal{P}) = \text{Tilt}(E, \mathcal{G}, \delta) \cup \mathcal{B} \).

Now, suppose that (3.25) holds and let \( y \in E \) and \( 0 < r < \operatorname{diam}(E) \). Any \( Q \in \mathcal{Q}_T \) for some \( T \in \mathcal{T}_{y,r} \) is \( A' \)-close to a Whitney cube near \( y \) of diameter \( \lesssim r \) so that it is contained in a uniformly bounded number of top scale cubes \( Q_0, \ldots, Q_N \) of diameter \( \lesssim A'r \) and distance \( \lesssim A'r \) from \( y \). Therefore, Proposition 6.1 implies

\[
\sum_{\alpha \in I} |\partial \Omega_\alpha \cap B(y, r)| \lesssim_{d, \varepsilon} r^d + |E \cap B(y, A'r)| + \sum_{T \in \mathcal{T}_{y,r}} \sum_{Q \in \mathcal{Q}(\mathcal{P}_T)} \ell(Q)^d + \sum_{T \in \mathcal{T}_{y,r}} \sum_{U \in \mathcal{P}_T} \epsilon(U)^2 \ell(U)^d 
\lesssim r^d + |E \cap B(y, A'r)| + \sum_{j=0}^{N} \sum_{Q_{\subseteq Q_j} \in \mathcal{Q}_{\text{Bad}}(\mathcal{P})} \ell(Q_j)^d 
\lesssim C r^d + |E \cap B(y, A'r)| + \sum_{j=0}^{N} \ell(Q_j)^d 
\lesssim_A' r^d + |E \cap B(y, A'r)|.
\]

This corollary finishes the proof of Theorem 3.15. We collect the pieces of the proof below

**Proof of Theorem 3.15.** Form the family \( \Lambda \) as defined in (5.1). By Corollary 5.6, these domains satisfy items [i] and [ii] of the theorem’s conclusions. The final conclusions on the boundary measure are proven in Corollary 6.4.  

### 7. Application to uniformly rectifiable and Reifenberg flat sets

The goal of this section is to provide the proofs of Theorems 1.3 and 1.4 regarding uniformly rectifiable and general Reifenberg flat sets respectively. We begin by introducing preliminary results needed for the proofs.
7.1. Preliminary lemmas and definitions. In order to apply Theorem 3.15, we must choose a well-approximating collection of planes \( \{L_Q\}_{Q \in \mathcal{D}} \) for which \( \sum_{R \subseteq Q} \epsilon(R) \geq (R)^d \) is controlled. For both uniformly rectifiable and Reifenberg flat sets, we will choose \( L_Q \) as a plane which minimizes appropriate beta numbers. The following two lemmas give tools for controlling the Hausdorff distance, and therefore epsilon numbers, for such \( L_Q \).

7.1.1. Preliminary lemmas.

Lemma 7.1 ([AS18] Lemma 2.16). Suppose \( E \subseteq \mathbb{R}^n \) and \( B \) is a ball centered on \( E \) such that for all balls \( B' \subseteq B \), \( \mathcal{H}_\infty^d(B') \geq cr_{B'}^d \). Let \( P \) and \( P' \) be two \( d \)-planes. Then

\[
d_B(P, P') \lesssim_{d, c} \left( \frac{r_B}{r_{B'}} \right)^{d+1} \beta_{E}^d(B, P) + \beta_{E}^d(B', P').
\]

This lemma is stated with respect to the lower content beta number, but in the case that \( E \) is \( d \)-Ahlfors-regular with constant \( C \), we have

\[
\beta_{E, p}^d(B, L) \approx_{C, d} \frac{1}{r_B^d} \int_B \left( \frac{\text{dist}(y, L)}{r_B} \right)^p d\mathcal{H}_d(y)
\]

\[
= \frac{1}{r_B^d} \int_0^\infty \mathcal{H}_d(\{x \in B \cap E : \text{dist}(x, L) > tr_B\}) t^{p-1} dt
\]

\[
\approx_{C, d} \frac{1}{r_B^d} \int_0^\infty \mathcal{H}_\infty^d(\{x \in B \cap E : \text{dist}(x, L) > tr_B\}) t^{p-1} dt = \beta_{E}^d(B, L)
\]

so that it applies to \( \beta_{E, 1}^d \) when \( E \) is \( d \)-Ahlfors regular. The next lemma applies this to show that \( \epsilon(Q) \) is bounded by an appropriate beta number.

Lemma 7.2. Let \( \mathcal{D} \) be a Christ-David lattice for a lower content \( d \)-regular set \( E \) and \( K, M > 0 \) be constants such that \( 2000 \rho^d < K < 10^{-1} \rho^2 M \). If \( \{L_Q\}_{Q \in \mathcal{D}} \) is a family of planes satisfying \( \beta_{E, 1}^d(2\rho^{-1} KB_Q, L_Q) \leq C \beta_{E, 1}^d(4\rho^{-1} KB_Q) \), for some constant \( C > 0 \), then

\[
\epsilon(Q) \lesssim_{C, \rho, M, d} \beta_{E, 1}^d(MB_Q).
\]

If \( E \) is \( d \)-uniformly rectifiable, then \( \beta_{E, 1}^d(2\rho^{-1} KB_Q, L_Q) \lesssim_{C_0} \beta_{E, 1}^d(4\rho^{-1} KB_Q) \) and

\[
\epsilon(Q) \lesssim_{C, \rho, M, d} \beta_{E, 1}^d(MB_Q).
\]

Proof. Let \( U, R \in \mathcal{D} \) be cubes which achieve the supremum in the definition of \( \epsilon(Q) \). Then

\[
\epsilon(Q) = d_{KB_R}(L_U, L_R).
\]

We want to apply Lemma 7.1 with \( B = B' = KB_R \). First, we prove some ball inclusions. We claim

\[
KB_R \subseteq 2\rho^{-1} KB_U.
\]
Indeed, we let \( y \in KB_R \) and we compute
\[
|y - x_U| \leq |y - x_R| + |x_R - x_Q| + |x_Q - x_U|
\leq K\ell(R) + \frac{K}{10}\ell(R) + \frac{K}{10}\ell(U) \leq 2K\ell(R) \leq 2\rho^{-1}K\ell(U).
\]

Second, we claim
\[
(7.3) \quad 4\rho^{-1}KB_U \subseteq MB_Q \text{ and } 4\rho^{-1}KB_R \subseteq MB_Q.
\]
Because \( \ell(R) \geq \ell(U) \), it suffices to prove \( 4\rho^{-1}KB_R \subseteq MB_Q \). We let \( y \in 4\rho^{-1}KB_R \) and compute
\[
|y - x_Q| \leq |y - x_R| + |x_R - x_Q| \leq 4\rho^{-1}K\ell(R) + \frac{K}{10}\ell(R) \leq 10\rho^{-2}\ell(Q) < M\ell(Q).
\]
Now, we apply Lemma 7.1 with \( B = B' = KB_R \), then
\[
d_{KB_R}(L_U, L_R) \lesssim \beta^{d,1}_E(KB_R, L_R) + \beta^{d,1}_E(KB_R, L_U)
\lesssim \rho \beta^{d,1}_E(2\rho^{-1}KB_R, L_R) + \beta^{d,1}_E(2\rho^{-1}KB_U, L_U)
\lesssim C \beta^{d,1}_E(4\rho^{-1}KB_R) + \beta^{d,1}_E(4\rho^{-1}KB_U)
\lesssim M \beta^{d,1}_E(MB_Q)
\]
where the second line follows from (7.2), the third line follows from the hypothesis on \( L_Q \), and the final line from (7.3). If \( E \) is uniformly rectifiable, the further claim follows immediately from (7.1) because \( E \) is \( d \)-Ahlfors regular. \( \blacksquare \)

Using the previous two lemmas will require that we choose the planes \( \{L_Q\} \) to be planes which approximately minimize the \( L^1 \) (or \( L^p \)) beta numbers. But, in order to apply Reifenberg parameterization results, we must have these planes pass through \( x_Q \). The following lemma shows that one can assume this for lower content \( d \)-regular sets.

**Lemma 7.3.** Let \( E \) be lower content \( d \)-regular with regularity constant \( c \) and Christ-David cubes \( \mathcal{D} \). For any \( Q \in \mathcal{D} \setminus BWGL(M, \epsilon) \), one can choose a cube center \( x_Q \) such that there exists a \( d \)-plane \( L_Q \) such that \( x_Q \in L_Q \) and
\[
\beta^{d,1}_E(2K\rho^{-1}B_Q, L_Q) \lesssim_{K, \rho, c} \beta^{d,1}_E(4K\rho^{-1}B_Q)
\]

**Proof.** Suppose \( Q \in \mathcal{D} \) and let \( L_Q' \) be a \( d \)-plane such that
\[
\beta^{d,1}_E(3\rho^{-1}KB_Q, L_Q') = \beta^{d,1}_E(3\rho^{-1}KB_Q).
\]
Fix a constant \( \eta > 0 \) to be taken sufficiently large later. The plan of the proof is to find a point \( y_Q \in E \) whose distance from \( x_Q \) is at most some multiple of \( \eta \beta^{d,1}_E(3\rho^{-1}KB_Q)\ell(Q) \). We will then re-define the center of \( Q \) to be \( y_Q \) and the corresponding plane to be the translate of \( L_Q' \) passing through \( y_Q \). First, we prove the existence of such a nice point \( y_Q \):

**Claim:** There exists \( y_Q \in \frac{\eta}{2}B_Q \cap E \) such that
\[
\text{dist}(y_Q, L_Q') \leq \eta \beta^{d,1}_E(3\rho^{-1}KB_Q)\ell(Q).
\]
Proof: By way of contradiction, suppose that
\[
\left\{ x \in \frac{c_0}{2} B_Q \cap E : \text{dist}(x, L'_Q) < \eta \beta_E^{d,1}(3\rho^{-1} KB_Q) \ell(Q) \right\} = \emptyset.
\]
It follows that for any \( t < \eta \beta_E^{d,1}(3\rho^{-1} KB_Q) \), we have
\begin{equation}
\mathcal{H}_\infty^d \left( \{ x \in 3 \rho^{-1} KB_Q \cap E : \text{dist}(x, L'_Q) > t \ell(Q) \} \right) \geq \mathcal{H}_\infty^d \left( \frac{c_0}{2} B_Q \cap E \right) \gtrsim t \ell(Q)^d.
\end{equation}
Then, we compute
\[
\beta_E^{d,1}(3\rho^{-1} KB_Q) \simeq_{K, \rho} \frac{1}{\ell(Q)^d} \int_0^\infty \mathcal{H}_\infty^d \left( \{ x \in 3 \rho^{-1} KB_Q \cap E : \text{dist}(x, L'_Q) > t \ell(Q) \} \right) dt
\]
\[
\geq \frac{1}{\ell(Q)^d} \int_0^{\eta \beta_E^{d,1}(3\rho^{-1} K \ell(Q))} \mathcal{H}_\infty^d \left( \{ x \in 3 \rho^{-1} KB_Q \cap E : \text{dist}(x, L'_Q) > t \ell(Q) \} \right) dt
\]
\[
\gtrsim_{c, \eta} \eta \beta_E^{d,1}(3\rho^{-1} K \ell(Q))
\]
where we used (7.4) in the final inequality. By choosing \( \eta > 0 \) sufficiently large in terms of \( c, c_0, K, \rho \), we get the desired contradiction.

Let \( y_Q \) be as in the statement of the claim and define \( L_Q \) to be the translate of \( L'_Q \) passing through \( y_Q \). Set \( B_{y_Q} = B(y_Q, 2 \rho^{-1} K \ell(Q)) \subseteq 3 \rho^{-1} KB_Q \). We now prove a required \( \beta_E^{d,1} \) estimate for \( L_Q \):

Claim: \( \beta_E^{d,1}(y_Q, L_Q) \simeq_{K, \rho} \beta_E^{d,1}(y_Q, 2B_{y_Q}) \).

Proof: First, define \( D = \text{dist}(L_Q, L'_Q) = \text{dist}(y_Q, L'_Q) \leq \eta \beta_E^{d,1}(3\rho^{-1} KB_Q) \ell(Q) \) and notice that the triangle inequality implies \( \text{dist}(x, L_Q) \leq \text{dist}(x, L'_Q) + D \) for any \( x \in \mathbb{R}^{d+1} \). If \( t \ell(Q) > 2D \), then
\begin{equation}
\{ x \in B_{y_Q} \cap E : \text{dist}(x, L_Q) > t \ell(Q) \} \subseteq \{ x \in B_{y_Q} \cap E : \text{dist}(x, L'_Q) > t \text{dist}(x, L_Q)/2 \}.
\end{equation}
Therefore, we can compute
\[
\beta_E^{d,1}(B_{y_Q}, L_Q) \simeq_{K, \rho} \frac{1}{\ell(Q)^d} \int_0^\infty \mathcal{H}_\infty^d \left( \{ x \in B_{y_Q} \cap E : \text{dist}(x, L_Q) > t \ell(Q) \} \right) dt
\]
\[
\leq \int_0^{2D/\ell(Q)} \mathcal{H}_\infty^d \left( \{ x \in B_{y_Q} \cap E : \text{dist}(x, L_Q) > t \ell(Q) \} \right) dt + \int_{2D/\ell(Q)}^\infty \mathcal{H}_\infty^d \left( \{ x \in B_{y_Q} \cap E : \text{dist}(x, L'_Q) > t \ell(Q)/2 \} \right) dt
\]
\[
\lesssim_{K, \rho} \eta \beta_E^{d,1}(3\rho^{-1} KB_Q) + \int_0^\infty \mathcal{H}_\infty^d \left( \{ x \in 3 \rho^{-1} KB_Q \cap E : \text{dist}(x, L'_Q) > t \ell(Q) \} \right) dt
\]
\[
\lesssim_{c, 0, \rho} \beta_E^{d,1}(3\rho^{-1} KB_Q) \lesssim_{K, \rho} \beta_E^{d,1}(y_Q, 2B_{y_Q}).
\]
If we re-define the center of $Q$ to be $y_Q$, then because $y_Q \in \frac{c_0}{2} B(x_Q, \ell(Q))$, $y_Q$ will satisfy all of the requirements of the central point if we reduce $c_0$ by a factor of two and possibly increase $\rho$ by a small factor to ensure $Q \subseteq B(y_Q, \ell(Q))$. With this definition, the computations above prove the desired result. ■

This final lemma gives a packing estimate needed for packing estimates on $\text{Tilt}(\mathcal{G}, \delta)$ in the applications

**Lemma 7.4.** Suppose $E$ is lower content $d$-regular with regularity constant $c > 0$ and $Q \in \mathcal{D}$. Let $\mathcal{Q}$ be a collection of disjoint cubes contained in $Q$. Then

$$\ell(Q)^d \lesssim_{c,d} \sum_{R \in \mathcal{Q}} \ell(R)^d + \mathcal{H}^d \left( Q \setminus \bigcup_{R \in \mathcal{Q}} R \right).$$

**Proof.** If $\{V_i\}_{i \in I}$ is any covering of $Q \setminus \bigcup_{R \in \mathcal{Q}} R$, then the collection

$$\{B_R : R \in \mathcal{Q}\} \cup \{V_i\}_{i \in I}$$

is a covering of $Q$ so that, by the definition of $\mathcal{H}^d_\infty$ and the lower content $d$-regularity of $E$,

$$\ell(Q)^d \lesssim_{c,d} \mathcal{H}^d_\infty(Q) \lesssim \sum_{R \in \mathcal{Q}} \ell(R)^d + \sum_{i \in I} \text{diam}(V_i)^d.$$  

By taking the infimum over coverings $\{V_i\}$, we conclude

$$\ell(Q)^d \lesssim_{c,d} \sum_{R \in \mathcal{Q}} \ell(R)^d + \mathcal{H}^d \left( Q \setminus \bigcup_{R \in \mathcal{Q}} R \right).$$

7.1.2. **Stopping time constructions of Azzam and Schul for Reifenberg flat sets.** We will require small technical tweaks of the stopping time machinery of Azzam and Schul on Reifenberg flat sets in order to prove that $\text{Tilt}(\Sigma, \mathcal{G}, \delta) \lesssim \mathcal{H}^d(\Sigma)$. We review the necessary definitions here, but refer to [AS18] sections 5-8 for a full treatment of the construction.

We fix constants $0 < \epsilon \ll \alpha < \delta$ with $\alpha$ to be chosen sufficiently small in terms of $\delta$. For any cube $Q \in \mathcal{D}$, we define a stopping time region $S_Q^\alpha$ by adding cubes $R \subseteq Q$ to $S_Q$ if

1. $R^{(1)} \in S_Q^\alpha$,
2. We have $\text{Angle}(P_U, P_Q) < \alpha$ for any sibling $U$ of $R$ (including $R$ itself).

For any collection of cubes $\mathcal{Q}$, define a distance function

$$d_{\mathcal{Q}}(x) = \inf \{\ell(Q) + \text{dist}(x, Q) : Q \in \mathcal{Q}\}.$$  

For any $Q \in \mathcal{D}$, define

$$d_{\mathcal{Q}}(Q) = \inf_{x \in Q} d_{\mathcal{Q}}(x) = \inf \{\ell(R) + \text{dist}(Q, R) : R \in \mathcal{Q}\}.$$
We let $m(S)$ be the set of minimal cubes of $S$, those which have no children contained in $S$ and define

$$z(S) = Q(S) \setminus \bigcup_{Q \in m(S)} Q.$$ 

Define

$$\text{Stop}(-1) = D_0$$

and fix a small constant $\tau \in (0, 1)$. Suppose we have defined $\text{Stop}(N - 1)$ for some integer $N \geq 0$ and define

$$\text{Layer}(N) = \bigcup \{ S^\alpha_Q : Q \in \text{Stop}(N - 1) \}.$$ 

We then set $\text{Up}(-1) = \emptyset$ and put

$$\text{Stop}(N) = \{ Q \in D : Q \text{ maximal such that } Q \text{ has a sibling } Q' \text{ with } \ell(Q') < \tau \ell(\text{Layer}(N))(Q') \},$$

$$\text{Up}(N) = \text{Up}(N - 1) \cup \{ Q \in D : Q \supset R \text{ for some } R \in \text{Stop}(N) \cup \text{Layer}(N) \}.$$

[ASIS] Lemma 5.5 says that, in fact

$$\text{Up}(N) = \{ Q \in D : Q \not\subset R \text{ for any } R \in \text{Stop}(N) \}.$$ 

Essentially, $\text{Layer}(N)$ is a layer of stopping time regions $S^\alpha_Q$ beginning at the stopped cubes of the previous generation and continuing until reaching a cube $R$ with a child $R'$ such that $\text{Angle}(P_Q, P_{R'}) > \delta$. $\text{Stop}(N)$ is formed by taking a “smoothing” of $\text{Layer}(N)$ that ensures that nearby minimal cubes in $\text{Stop}(N)$ are of similar size. One forms a CCBP from the centers and $b\beta$-minimizing planes of cubes in $\text{Up}(N)$ which gives a surface $\Sigma_N$ for any $N \geq 0$ which converges to $\Sigma$ as $N \to \infty$. Azzam and Schul give tools for proving bounds on the degree of stopping in this construction in the following lemma

**Lemma 7.5 [Hyd21] Lemma 4.4 (5)).** Let $\Sigma$ be Reifenberg flat and $D$ a Christ-David lattice for $\Sigma$. Let $N \geq 0$. For any $Q_0 \in D$,

$$\sum_{N \geq 0} \sum_{Q \in \text{Stop}(N)} \ell(Q)^d \lesssim_{d,\alpha,\epsilon} H^d(Q_0).$$

### 7.2. Proofs of the theorems.

We can now complete the proofs of Theorems 1.3 and 1.4.

**Proof of Theorem 1.3.** By Theorem 3.15, it suffices to exhibit a Christ-David lattice $D = \mathcal{G} \cup \mathcal{B}$ and an associated collection of $d$-planes $\{ L_Q \}_{Q \in \mathcal{D}}$ such that

(7.6) $b\beta(KB_Q, L_Q) \lesssim \epsilon$ for $Q \in \mathcal{G}$,

(7.7) $\text{Tilt}(Q, \mathcal{G}, \delta) + \sum_{R \subseteq Q} \ell(R)^d + \sum_{R \subseteq Q} \epsilon(R)^2 \ell(R)^d \leq C\ell(Q)^d$.

We choose

$$\mathcal{B} = \text{BWGL}(M, \epsilon),$$

$$\mathcal{G} = D \setminus \mathcal{B}.$$
We now define the plane $L_Q$ to be that given by Lemma 7.3. So, $x_Q \in L_Q$ and

$$\beta^{d}_{E,1}(2K\rho^{-1}B_Q, L_Q) \lesssim_{K,\rho,d} \beta^{d}_{E,1}(4K\rho^{-1}B_Q).$$

Because $Q \in \mathcal{G}$, $b\beta(MB_Q) < \epsilon$ so that (7.6) follows immediately. We now prove (7.7). We will prove a bound on each term in the inequality separately in the next three claims:

**Claim:** $\sum_{R \subseteq Q} \epsilon(R)^2\ell(R)^d \lesssim \ell(Q)^d$

**Proof:** Using Lemma 7.2 and the characterization of uniform rectifiability in Proposition 3.12 item (iii) we get

$$\sum_{R \subseteq Q} \epsilon(R)^2\ell(R)^d \lesssim_{\rho,M,d} \sum_{R \subseteq Q} \beta^{d}_{E,1}(MB_R)\ell(R)^d \lesssim_{M,d} \ell(Q)^d$$

**Claim:** $\sum_{R \in \mathcal{G}} \ell(R)^d \lesssim \ell(Q)^d$.

**Proof:** It follows from Proposition 3.12 item (iii) that

$$\sum_{R \subseteq Q} \ell(R)^d = BWGL(Q, M, \epsilon) \lesssim_{M,d,\epsilon} \ell(Q)^d.$$ 

**Claim:** $\text{Tilt}(Q, \mathcal{G}, \delta) \lesssim \ell(Q)^d$.

**Proof:** Fix $Q \in \mathcal{G}$ and let $R \in \text{Tilt}(\mathcal{G}, \delta)$ with $R \subseteq Q$. Set

$$\mathcal{D}_R = \{U \in \mathcal{D} : U \text{ maximal such that } U^{(1)} \in m(S^d_R)\}.$$ 

Suppose $U \in \mathcal{D}_R$ has a sibling in $\text{Tilt}(\mathcal{G}, \delta)$. Use Proposition 3.12 item (iv) to get a collection of stopping time regions $\mathcal{F}_{\delta'}$ and planes $\{P_Q\}_{Q \in \mathcal{G}}$ satisfying the stated conclusions of the proposition with $\epsilon$ there as above and $\delta$ there equal to $\epsilon \ll \delta' \lesssim \delta$ sufficiently small. Because of (7.6) and $\delta'$ sufficiently small in terms of $\delta$, for any cube $U \in \mathcal{D}_R$ such that $\text{Angle}(L_U, L_R) > \delta$ there exists some cube $U' \supseteq U$ such that $U = U(S)$ for some $S \in \mathcal{F}_{\delta'}$, that is, $U'$ is a top cube where in the construction of $S$, $U'$ was stopped at because $\text{Angle}(P_{Q(S)}, P_{U'}) > \delta'$. Define

$$\text{Stop}(R) = \{U \in \mathcal{D} : U \text{ maximal such that } U = U(S), S \in \mathcal{F}_{\delta'}\}.$$ 

By the above discussion, $\bigcup \mathcal{D}_R \subseteq \bigcup \text{Stop}(R)$. Therefore, by Lemma 7.4 we have

$$\ell(R)^d \lesssim \sum_{U \in m(S^d_R)} \ell(U)^d + \mathcal{H}^d(z(S^d_R))$$

$$\lesssim \sum_{U \subseteq R, \text{such that } U \notin \bigcup_{\text{Stop}(R)} U \in \text{BWGL}(M, \epsilon)} \ell(U)^d + \sum_{U \subseteq R, \text{such that } U \notin \text{Stop}(R)} \ell(U)^d + \mathcal{H}^d(z(S^d_R)).$$ 

We note that the maximality of cubes in $\text{Stop}(R)$ implies that for any distinct $R, R' \in \text{Tilt}(\mathcal{G}, \delta)$, $\text{Stop}(R) \cap \text{Stop}(R') = \emptyset$. Similarly, the $\text{BWGL}(M, \epsilon)$ cubes summed over are distinct for distinct $R$ so
that summing this inequality gives

\[ \sum_{R \subseteq Q} \ell(R)^d \lesssim_d \sum_{R \in \text{Tilt}(\mathcal{G}, \delta)} \left( \sum_{U \subseteq R, U \text{ maximal}} \ell(U) + \sum_{U \subseteq R} \ell(U)^d + \mathcal{H}^d(z(S_R^d)) \right) \]

\[ \lesssim \sum_{U \subseteq Q} \ell(Q)^d + \sum_{S \in \mathcal{F}_R} \ell(Q(S))^d + \mathcal{H}^d(Q) \]

\[ \lesssim_{M,d,\varepsilon} \ell(Q)^d. \]

where in the last line we used the fact that \( \text{BWGL}(Q, M, \varepsilon) \lesssim_{M,\varepsilon,d} \ell(Q)^d \), the packing property of coronizations, and the fact that \( z(S_R^d) \cap z(S_R'^d) = \emptyset \) for distinct \( R, R' \in \text{Tilt}(\mathcal{G}, \delta) \).

These three claims prove (7.7). This means the hypotheses of Theorem 3.15 are satisfied and the result follows.

**Proof of Theorem 1.4.** Define \( \Sigma' = \Sigma \cap B(y, 100r) \). By Theorem 3.15, it suffices to exhibit a Christ-David Lattice \( \mathcal{D} = \mathcal{G} \cup \mathcal{B} \) for \( \Sigma' \) with an associated collection of \( d \)-planes \( \{L_Q\}_{Q \in \mathcal{D}} \) such that

\[ b\beta(MB_Q, L_Q) \lesssim \varepsilon \text{ for } Q \in \mathcal{G}, \]

\[ \text{Tilt}(\Sigma', \mathcal{G}, \delta) + \sum_{R \subseteq \mathcal{D}} \ell(R)^d + \sum_{R \subseteq Q} \varepsilon(R)^2 \ell(R)^d \lesssim \mathcal{H}^d(\Sigma'). \]

We choose

\[ \mathcal{G} = \mathcal{D}, \]

\[ \mathcal{B} = \emptyset. \]

For any \( Q \in \mathcal{D} \), we let \( L_Q \) be the \( d \)-plane given by Lemma 7.3. Because \( Q \in \mathcal{G} \), \( b\beta(MB_Q) < \varepsilon \) and (7.8) follows immediately. The inequality (7.9) can be written

\[ \text{Tilt}(\Sigma', \mathcal{D}, \delta) + \sum_{R \subseteq Q} \varepsilon(R)^2 \ell(R)^d \lesssim \mathcal{H}^d(\Sigma'). \]

We will prove each of the terms on the left hand side is bounded by \( C\mathcal{H}^d(\Sigma') \) separately. We define

\[ \text{Stop}(-1, \delta) = \emptyset, \]

and, given \( \text{Stop}(N - 1, \delta) \) for some integer \( N \geq 0 \), we define

\[ \text{Stop}(N, \delta) = \{ R \in \mathcal{D} : R^{(1)} \in m(S_Q^\delta), Q \in \text{Stop}(N - 1, \delta) \} \]

where \( S_Q^\delta \) is a stopping time region formed via the same process as \( S_Q^{\alpha} \) with \( \alpha \) replaced by \( \delta \). With this, we have

\[ \text{Tilt}(\Sigma', \mathcal{G}, \delta) = \bigcup_{N \geq 0} \text{Stop}(N, \delta). \]
Claim: \( \text{Tilt}(\Sigma', \mathcal{G}, \delta) \lesssim \mathcal{H}^d(\Sigma') \).

Proof: Fix \( Q \in \text{Stop}(N, \delta) \) and let \( x \in Q \setminus z(S) \). Then there exists \( R \in m(S_Q^\delta) \) such that \( x \in R \) and, assuming \( \delta \gtrsim \alpha \), there exists a cube \( R' \in \text{Stop}(K) \) for some \( K \geq 0 \) such that \( R \subseteq R' \subseteq Q \). Set
\[
\text{Stop}(Q) = \left\{ R \in \mathcal{D} : R \text{ maximal such that } R \in \bigcup_{N \geq 0} \text{Stop}(N) \text{ and } R \subseteq Q \right\}.
\]
The above argument has shown that \( Q \setminus z(S_Q^\delta \cap R \subseteq \bigcup_{R \in \text{Stop}(Q)} R \). By applying Lemma 7.4, we see
\[
\ell(Q)^d \lesssim \sum_{R \in \text{Stop}(Q)} \ell(R)^d + \mathcal{H}^d(z(S_Q^\delta)).
\]

This means
\[
\text{Tilt}(\Sigma', \mathcal{G}, \delta) = \sum_{N \geq 0} \sum_{Q \in \text{Stop}(N, \delta)} \ell(Q)^d \lesssim \sum_{N \geq 0} \sum_{Q \in \text{Stop}(N, \delta)} \left( \sum_{R \in \text{Stop}(Q)} \ell(R)^d + \mathcal{H}^d(z(S_Q^\delta)) \right) \lesssim \mathcal{H}^d(\Sigma') + \sum_{K \geq 0} \sum_{R \in \text{Stop}(K)} \ell(R)^d \lesssim_{d, \delta, \epsilon} \mathcal{H}^d(\Sigma'),
\]
where the penultimate line follows from the fact that \( \text{Stop}(Q) \cap \text{Stop}(Q') = \emptyset \) for \( Q, Q' \in \text{Tilt}(\Sigma', \mathcal{G}, \delta) \), \( Q \neq Q' \), and the final line follows from Lemma 7.5.

Claim: \( \sum_{Q \in \mathcal{D}} \epsilon(Q)^2 \ell(Q)^d \lesssim \mathcal{H}^d(\Sigma') \).

Proof: Fix \( Q \in \mathcal{D} \). By the definition of \( L_Q \), Lemma 7.2 implies
\[
\epsilon(Q) \lesssim_M \beta_{\Sigma'}^{d,1}(MB_Q).
\]

Therefore, by Theorem 3.13,
\[
\sum_{Q \in \mathcal{D}} \epsilon(Q)^2 \ell(Q)^d \lesssim_{M,d,\rho} \sum_{Q \in \mathcal{D}} \beta_{\Sigma'}^{d,1}(MB_Q)^2 \ell(Q)^d \lesssim_{M,d,\epsilon} \mathcal{H}^d(\Sigma').
\]

This concludes the proof of (7.9). The corollary follows from an application of Theorem 3.15.

8. Further questions

Question 8.1. Can one find a disjoint decomposition rather than one of bounded overlap in Theorem 3.15?

In the construction given in this paper, the only overlap between domains occurs in intersections between \((1+C\delta)\)-bi-Lipschitz images of Whitney cubes of comparable side length. It seems
plausible that one could devise an algorithm to decompose the pieces of these intersections into domains of the desired type while maintaining the desired surface measure bounds.

**Question 8.2.** Does Theorem 1.3 have an application to the study of harmonic measure?

As stated in the introduction, the approximation of uniformly rectifiable sets by Lipschitz domains has played an important role in the study of harmonic measure. It is not clear to the author whether this result has any application in that domain. Most techniques involving Lipschitz domains are concerned with showing there exists a Lipschitz domain inside a ball whose boundary intersects the boundary of the approximated domain in a set of large measure. However, the existence of such domains is not possible in the case of a general uniformly rectifiable set. Is there some other way in which this result could be useful?

**Question 8.3.** Can Theorem 1.4 be extended to general lower-content $d$ regular sets?

It seems possible that the necessary tools to handle this extension to non-Reifenberg flat sets are present in the curvature estimates for general sets in [Hyd22]. The technical disconnect between this paper and that one caused by the “smoothing” procedure needed there in defining the stopping time regions makes this generalization non-obvious to the author. It seems that a proof would require new ideas.

**Appendix A. A codimension one Reifenberg parameterization theorem with tilt control**

The goal of this appendix is to prove a version of David-Toro’s Reifenberg parameterization result where we use an additional hypothesis bounding the total angle between any two planes in the CCBP. We first state a small modification of a lemma in [AS18] which gives criteria for a triple $(\Sigma_0, \{B_{j,k}\}, \{P_{j,k}\})$ to be a CCBP.

**Lemma A.1** (cf. [AS18] Theorem 2.5). For any $k \in \mathbb{N} \cup \{0\}$, let $r_k = 10^{-k}$. Let $\{x_{j,k}\}_{j \in J_k}$ be a collection of points such that for some $d$-plane $P_0$ we have

$$\text{dist}(x_{j,0}, P_0) < \epsilon,$$

$$|x_{j,k} - x_{i,k}| \geq r_k,$$

and, with $B_{j,k} = B(x_{j,k}, r_k)$,

$$x_{i,k} \in V_{k-1}^2,$$

where

$$V_k^\lambda = \bigcup_{j \in J_k} \lambda B_{j,k}.$$

Let $P_{j,k}$ be a $d$-plane such that $x_{j,k} \in P_{j,k}$. There is $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$, if

$$\epsilon_k(x_{j,k}) \lesssim \epsilon \text{ for all } k \geq 0 \text{ and } j \in J_k$$

then $(P_0, \{B_{j,k}\}, \{P_{j,k}\})$ is a CCBP.
In order to state the parameterization result, we define an “interior” of a CCBP $\mathcal{Z}$:

$$I(\mathcal{Z}) = \{ z = x + y \in \mathbb{R}^{d+1} : f_n(y)(x) \in V_8 \text{ or } |y| \geq 2 \}.$$ 

This consists of points $z$ whose distance to the initial plane $P_0$ is roughly greater than the smallest scale at which the trajectory of points of the CCBP. It is when $f_k(x)$ is near net points of scale $k$ that the computations involving $Dg$ are valid.

**Theorem A.2.** Let $0 < \epsilon < \delta \ll 1$ and let $\mathcal{Z} = (P_0, \{B_{j,k}\}, \{P_{j,k}\})$ be a CCBP in $\mathbb{R}^{d+1}$ such that $P_0$ and $P_{j,k}$ are $d$-planes. If both

(i) $\sum_{k=1}^{\infty} c_k(f_k(x))^2 \leq \epsilon$ for all $x \in P_0$, and

(ii) $\text{Angle}(f_{j,k}, P_0) \leq \delta$ for all $k \geq 0$, $j \in J_k$,

then there exists a map $g : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ such that $g$ obtains all of the conclusions of Theorem 3.2 and the following: For any quasiconvex domain $\Omega \subseteq I(\mathcal{Z})$ such that $\text{diam}(\Omega) < M \ll \epsilon^{-1}$ and there exists $z \in \Omega$ such that $Dg(z) = I$, $g|_{\Omega}$ is $(1 + C\delta)$-bi-Lipschitz, and, in fact,

$$|Dg(w) - I| \leq C\delta$$

for any $w \in \Omega$.

This result follows from a series of computations involving the derivative of the map $g$ produced by Theorem 3.2 for the given CCBP. The primary tool for proving $|Dg - I| \leq C\delta$ is the more general Proposition A.3 which, given a “central” point $z \in \mathbb{R}^{d+1}$ and an inflation factor $0 < M \ll \epsilon^{-1}$, gives a set $G_z^M$ such that $w \in G_z^M$ means $Dg(w)$ is very close to $Dg(z)$ in the sense of (A.1). In our application, we will choose the central point $z$ to lie in a region where $g$ is the identity so that $Dg(z) = I$. Proposition A.3 is more general than needed for the results of this paper in that it allows one to make similar derivative estimates around a point $z$ at which $Dg(z)$ is arbitrary. This complicates the proof, as $g$ in general behaves very differently in the “horizontal” $x$-coordinates than it does in the “vertical” $y$-coordinate.

Proposition A.3 follows from technical Lemmas A.4 and A.6 which give separate horizontal and vertical estimates respectively. That is, they show how to appropriately bound the individual pieces of the difference $Dg(x + y) - Dg(x' + y)$ and $Dg(x' + y) - Dg(x' + y')$ between points $z = x + y, z' = x' + y'$ respectively when acting on horizontal or vertical vectors $v \in \mathbb{R}^d \times \{0\} \cup \{0\}^d \times \mathbb{R} = T\Sigma_0 \cup T\Sigma_0^\perp$. Corollaries A.5 and A.7 put these pieces together to get the requisite $Dg$ estimates, from which we prove Proposition A.3.

**Proposition A.3.** Fix $1 < M \ll \epsilon^{-1}$ and $z \in \mathbb{R}^{d+1}$ with $z = x + y$ with respect to a CCBP $(\Sigma_0, \{B_{j,k}\}, \{P_{j,k}\})$ where $p = l(y)$ (see (3.12), (3.13), and the following discussion). Let $z' \in \mathbb{R}^{d+1}$ with $z' = x' + y'$ where $|y'| = |y|$, and let be $m = n(y')$. Assume $f_{p+1}(x) \in V_{p+1}^m$ and $f_m(x') \in V_m$. Define

$$G_z^M = \{ z' = x' + y' \in \mathbb{R}^{d+1} : |f_p(x) - f_p(x')| < Mr_p, \sum_{k=p}^m c_k(f_k(x'))^2 < \epsilon, \text{ Angle}(T_k(x'), T_p(x')) \leq C\delta \}.$$
Then for any \( w \in G^M_z \), we have
\[
(A.1) \quad |Dg(w) \cdot Dg(z)^{-1} - I| \leq C\delta.
\]
\[\text{Lemma A.4 (Horizontal Estimates). Let } z, z' \text{ be as in Proposition A.3 and let } v \in T\Sigma_0 \cup T\Sigma_0^\perp. \]
Let \( k \) be such that \( \rho_k(y) > 0 \). If \( |f_k(x) - f_k(x')| < M\rho_k \), for some \( 1 < M \ll \epsilon^{-1} \) then
\[
(A.2) \quad |Df_k(x) \cdot v - Df_k(x') \cdot v| \leq C\epsilon|Df_k(x) \cdot v|,
\]
\[
(A.3) \quad |(D_x(R_k(x) \cdot y)) \cdot v - (D_x(R_k(x') \cdot y)) \cdot v| \leq C\epsilon|Df_k(x) \cdot v|.
\]
In any case,
\[
(A.4) \quad \left| \frac{\partial g}{\partial y}(x + y) - \frac{\partial g}{\partial y}(x' + y) \right| \leq C\epsilon \left| \frac{\partial g}{\partial y}(x + y) \right|
\]
where the constant \( C \) depends on \( M \).
\[\text{Proof. We begin with proving (A.2). We have}
\]
\[
|Df_k(x) \cdot v - Df_k(x') \cdot v| = \left| \left[ D\sigma_{k-1}(f_k-1(x)) - D\sigma_{k-1}(f_k-1(x')) \right] Df_{k-1}(x) \cdot v \\
+ D\sigma_{k-1}(f_k-1(x')) \left[ Df_{k-1}(x) - Df_{k-1}(x') \right] \cdot v \right| \\
\leq |Df_{k-1}(x) \cdot v| |D\sigma_{k-1}(f_k-1(x)) - D\sigma_{k-1}(f_k-1(x'))| \\
+ |D\sigma_{k-1}(f_k-1(x'))| \left( |Df_{k-2}(x) \cdot v| |D\sigma_{k-2}(f_{k-2}(x)) - D\sigma_{k-2}(f_{k-2}(x'))| \\
+ |D\sigma_{k-2}(f_{k-2}(x'))| |Df_{k-2}(x) \cdot v - Df_{k-2}(x') \cdot v| \right)
\]
\[
(A.5) \quad \leq |Df_{k-1}(x) \cdot v| |D\sigma_{k-1}(f_k-1(x)) - D\sigma_{k-1}(f_k-1(x'))| \\
+ \sum_{p=1}^{k} \left( \prod_{m=1}^{p} |D\sigma_{k-m}(f_{k-m}(x'))| \right) |Df_{k-p-1}(x) \cdot v| \cdot |D\sigma_{k-p-1}(f_{k-p-1}(x)) - D\sigma_{k-p-1}(f_{k-p-1}(x'))|.
\]
Now, Lemma 3.6 implies
\[
(A.6) \quad \prod_{m=1}^{p} |D\sigma_{k-m}(f_{k-m}(x'))| \leq (1 + C\epsilon)^p,
\]
and
\[
(A.7) \quad |Df_{k-p-1}(x) \cdot v| = \left| \prod_{m=1}^{p+1} D\sigma_{k-m}^{-1}(f_{k-m+1}(x)) Df_k(x) \cdot v \right| \leq (1 + C\epsilon)^{p+1} |Df_k(x) \cdot v|.
\]
Using Lemma 3.9, we see that (3.18) implies \( |f_{k-p-1}(x) - f_{k-p}(x')| < \frac{Mr_{k-p-1}}{5p+1} \leq M r_{k-p-1} \) so that we get a rectifiable curve \( \gamma_{k-p-1} \) connecting \( f_{k-p-1}(x) \) and \( f_{k-p}(x') \) such that \( \ell(\gamma_{k-p-1}) \leq 2|f_{k-p-1}(x) - f_{k-p}(x')| \). Lemma 3.8 gives

\[
|D\sigma_{k-p-1}(f_{k-p-1}(x)) - D\sigma_{k-p-1}(f_{k-p}(x'))| = \left| \int_I D^2\sigma_{k-p-1}(\gamma_{k-p-1}(t)) \cdot \gamma'_{k-p-1}(t) \, dt \right| \\
\leq \int_I |D^2\sigma_{k-p-1}(\gamma_{k-p-1}(t))| |\gamma'_{k-p-1}(t)| \, dt \\
\leq C \frac{\epsilon}{r_{k-p-1}} \cdot 2|f_{k-p-1}(x) - f_{k-p}(x')| \\
\leq C \frac{\epsilon}{r_{k-p-1}} \frac{Mr_{k-p-1}}{5p+1} \\
\leq CM \frac{\epsilon}{5p+1}.
\]

(A.8)

Applying (A.6), (A.7), and (A.8) to (A.5) gives

\[
|Df_k(x) \cdot v - Df_k(x') \cdot v| \leq (1 + C\epsilon)C\frac{\epsilon}{5}|Df_k(x) \cdot v| + \sum_{p=1}^{k} (1 + C\epsilon)^p(1 + C\epsilon)^{p+1}|Df_k(x) \cdot v|CM \frac{\epsilon}{5p+1} \\
\leq C\epsilon|Df_k(x) \cdot v| + C\epsilon M|Df_k(x) \cdot v| \sum_{p=1}^{k} \frac{(1 + C\epsilon)^{2p}}{5p+1} \\
\leq C\epsilon|Df_k(x) \cdot v|.
\]

We now prove (A.3). For any \( t > 0 \), Proposition 3.3 implies that the quantity \( R_k(x + tv) \cdot e_d - R_k(x) \cdot e_d \) is the difference between the unit normal vectors to the linear subspaces \( TS_k(f_k(x + tv)) \) and \( TS_k(f_k(x)) \). But by Lemma 3.4 we have

\[
|R_k(x + tv) \cdot e_d - R_k(x) \cdot e_d| \leq D(TS_k(f_k(x + tv)), TS_k(f_k(x))) \leq C \frac{\epsilon}{r_k} |f_k(x + tv) - f_k(x)|.
\]

(A.9)

Hence, we can write

\[
|(D_x(R_k(x) \cdot y)) \cdot v| \leq |y| \lim_{t \to 0} \frac{|R_k(x + tv) \cdot e_d - R_k(x) \cdot e_d|}{|t|} \leq C\epsilon \frac{|y|}{r_k} \lim_{t \to 0} \frac{|f_k(x + tv) - f_k(x)|}{|t|} \\
\leq C\epsilon |Df_k(x) \cdot v|
\]

(A.10)

where \(|y| \lesssim r_k\) since \( \rho_k(y) > 0 \). We then have

\[
|(D_x(R_k(x) \cdot y)) \cdot v - (D_x(R_k(x') \cdot y)) \cdot v| \leq C\epsilon(|Df_k(x) \cdot v| + |Df_k(x') \cdot v|) \leq C\epsilon |Df_k(x) \cdot v|
\]

using (A.2).
Finally, we prove (A.4). First, we compute
\[
\frac{\partial g}{\partial y}(x + y) - \frac{\partial g}{\partial y}(x' + y) = \sum_{k \geq 0} \frac{\partial \rho_k}{\partial y}(y) \{ f_k(x) - f_k(x') + R_k(x) \cdot y - R_k(x') \cdot y \} + \sum_{k \geq 0} \rho_k(y) (R_k(x) \cdot e_{d+1} - R_k(x') \cdot e_{d+1})
\]
\[
= I + II
\]
Let \( p, p + 1 \) be the values of \( k \) such that \( \rho_k(y) > 0 \). Since \( \rho_p(y) + \rho_{p+1}(y) = 1 \), we have \( \frac{\partial \rho_p}{\partial y}(y) + \frac{\partial \rho_{p+1}}{\partial y}(y) = 0 \). This implies
\[
I = \frac{\partial \rho_p}{\partial y}(y) (f_p(x) - f_{p+1}(x) + R_p(x) \cdot y - R_{p+1}(x) \cdot y)
\]
\[
+ \frac{\partial \rho_p}{\partial y}(y) (f_p(x') - f_{p+1}(x') + R_p(x') \cdot y - R_{p+1}(x') \cdot y)
\]
But using (3.10) and (3.11), we have
\[
|I| \leq \frac{C}{r_p} (Cer_p + C\epsilon|y|) \leq C\epsilon.
\]
By (A.9) we have
\[
|II| \leq |\rho_p(y)| \frac{C\epsilon}{r_p} |f_p(x) - f_{p+1}(x')| + |\rho_{p+1}(y)| \frac{C\epsilon}{r_{p+1}} |f_{p+1}(x) - f_{p+1}(x')| 
\]
\[
\leq C M \epsilon (|\rho_p(y)| + |\rho_{p+1}(y)|) \leq C\epsilon.
\]
We’ve proven that \( \left| \frac{\partial g}{\partial y}(x + y) - \frac{\partial g}{\partial y}(x' + y) \right| \leq C\epsilon \). We will complete the proof of (A.4) by showing that \( \left| \frac{\partial g}{\partial y}(x + y) \right| \geq 1 \). Indeed,
\[
(A.11) \quad \frac{\partial g}{\partial y}(x + y) = \left| \left[ \frac{\partial \rho_p}{\partial y}(y) (f_p(x) - f_{p+1}(x) + R_p(x) \cdot y - R_{p+1}(x) \cdot y) \right] + \left[ \rho_p(y) R_p(x) \cdot e_{d+1} + \rho_{p+1}(y) R_{p+1}(x) \cdot e_{d+1} \right] \right|.
\]
But the previous computation shows that the first expression has norm \( \leq C\epsilon \), while the second expression is a convex combination of two nearly parallel unit vectors because \( R_p(x) \) and \( R_{p+1}(x) \) are orthogonal matrices which are \( C\epsilon \) close. Hence, we get
\[
(A.12) \quad \left| \frac{\partial g}{\partial y} \right| \geq 1.
\]

**Corollary A.5.** Let \( z, z' \) be as in Lemma A.4. Then for any vector \( v \in T\Sigma_0 \cup T\Sigma_0^+ \), we have
\[
|Dg(x + y) \cdot v - Dg(x' + y) \cdot v| \leq C\epsilon|Dg(x + y) \cdot v|
\]
Proof. First, suppose \( v = v_x \in T \Sigma_0 \). Since \( v_x \cdot e_{d+1} = 0 \), we have
\[
Dg(x + y) \cdot v_x = \sum_{k \geq 0} \rho_k(y) \left\{ Df_k(x) \cdot v_x + D(R_k(x) \cdot y) \cdot v_x \right\}.
\]

Therefore, we get
\[
(Dg(x + y) - Dg(x' + y)) \cdot v_x = \sum_{k \geq 0} \rho_k(y) \left\{ |Df_k(x) \cdot v_x - Df_k(x') \cdot v_x| + |D(R_k(x) \cdot y) \cdot v_x - D(R_k(x') \cdot y) \cdot v_x| \right\}
\leq CM\epsilon \sum_{k \geq 0} \rho_k(y) \left\{ |Df_k(x) \cdot v_x| \right\}.
\]

Using (A.16), we get
\[
|Dg(x + y) \cdot v_x| = |Dg(v_1 + v_2) \cdot v_x| = |Dg(v_1 + v_2) \cdot (s + tu)|.
\]

Putting \( v = v_1 + v_2 \), \( u = u_1 + u_2 \), we have \( Dg(x + y) \cdot v_x = sv + tu \). In this notation,
\[
|v_1 - u_1| \leq C\epsilon|v_1|,
\]
\[
|v_2, |u_2\| \leq C\epsilon|v_1|,
\]

by Lemma 3.6 and (A.10). We then want to prove the following claim:

Claim: \( s|v_1| + t|u_1| \leq |sv + tu| \).

Proof: Using (A.10), we get \( s|v_1| \leq s|v_1| + s|v_2| \leq s|v| \) and similarly \( t|u_1| \leq t|u| \). We now just need to show that \( |sv| + |tu| \leq |sv + tu| \). By Lemma 3.10, this follows if we can show \( (sv, tu) \geq -\frac{1}{2}|sv||tu| \). Indeed, we have
\[
(sv, tu) = st(\langle v_1, u_1 \rangle + \langle v_1, u_2 \rangle + \langle v_2, u_1 \rangle + \langle v_2, u_2 \rangle),
\]
\[
\geq st(|v_1|^2 - \langle v_1, u_1 - v_1 \rangle - C\epsilon|v_1|^2) \geq st(1 - C\epsilon)|v_1|^2 \geq 0.
\]

This completes the proof for \( v = v_x \). If instead \( v = v_y \in T \Sigma_0^+ \), then \( Dg(z) \cdot v_y = v_y \cdot \frac{\partial g}{\partial y}(z) \) and the result follows from (A.4) in Lemma A.4.  

Lemma A.6 (Vertical Estimates). Let \( z, z', v \) be as in Lemma A.4. If \( \sum_{k=p}^m \epsilon_k(f_k(x'))^2 \leq C\epsilon \) and Angle\((T\Sigma_p(f_p(x'))), T\Sigma_m(f_m(x'))\) \( \leq C\delta \), we have
\[
\sum_{k \geq 0} (\rho_k(y) - \rho_k(y')) Df_k(x') \cdot v \leq C\delta|Df_p(x') \cdot v|,
\]
Proof. We begin by proving (A.17). First, since $D\sigma_k$ is $(1+C\epsilon)$-bi-Lipschitz for any $k$, we have
$$|Df_p(x') \cdot v - Df_{p+1}(x') \cdot v| \leq C\epsilon |Df_p(x') \cdot v|.$$  
This implies
$$\sum_{k \geq 0} \rho_k(y) Df_k(x') \cdot v - Df_p(x') \cdot v \leq \sum_{k \geq 0} \rho_k(y) |Df_k(x') \cdot v - Df_p(x') \cdot v| \leq C\epsilon |Df_p(x')|$$
because $\rho_k(y) \neq 0$ only for $k = p, p + 1$. An identical argument gives (A.20) with $y$ replaced by $y'$ and $p$ replaced by $m$. We now want a similar bound for $|Df_m(x') \cdot v - Df_p(x') \cdot v|$. For ease of notation, define $u = Df_p(x') \cdot v$ and $w = \prod_{k=p}^{m-1} D\sigma_k(f_k(x')) \cdot u$. We can then write
$$|Df_m(x') \cdot v - Df_p(x') \cdot v| = \left| \prod_{k=p}^{m-1} D\sigma_k(f_k(x')) \right| \cdot u - u = |w - u|.$$  
The fact that $\sum_{k=p}^m \epsilon_k(f_k(x'))^2 \leq C\epsilon$ means
$$\prod_{k=p}^{m-1} D\sigma_k(f_k(x')) \leq \prod_{k=p}^{m-1} 1 + CM^2 \epsilon_k(f_k(x'))^2 \leq 1 + CM^2 \epsilon.$$  
Hence, $||w| - |u|| \leq CM^2 \epsilon |u|$. Since $w \in T\Sigma_m(f_m(x'))$, $u \in T\Sigma_p(f_p(x'))$, and we’ve assumed that $\operatorname{Angle}(T\Sigma_p(f_p(x')))$, $\operatorname{Angle}(T\Sigma_m(f_m(x')))$ $\leq C\delta$, we have $\operatorname{Angle}(w, u) \leq C\delta$ and it follows that
$$|w - u| \lesssim_M \delta |u|$$
as long as $\delta$ and $\epsilon$ are sufficiently small. Finally, using (A.20) and (A.22), we see
$$\sum_{k \geq 0} (\rho_k(y) - \rho_k(y')) Df_k(x') \cdot v$$
$$\leq \sum_{k \geq 0} \rho_k(y) Df_k(x') \cdot v - Df_p(x') \cdot v + \sum_{k \geq 0} \rho_k(y') Df_k(x') \cdot v - Df_m(x') \cdot v$$
$$+ |Df_p(x') \cdot v - Df_m(x') \cdot v|$$
$$\leq C\epsilon |Df_p(x') \cdot v| + C\epsilon |Df_m(x') \cdot v| + C\delta |Df_p(x') \cdot v|$$
$$\leq C\delta |Df_p(x') \cdot v|.$$
The proof of (A.18) follows from (A.10) and (A.17). Indeed,
\[
\left| \sum_{k \geq 0} \rho_k(y) D(R_k(x') \cdot y) \cdot v - \rho_k(y') D(R_k(x') \cdot y') \cdot v \right|
\leq C \epsilon |D f_p(x') \cdot v| + C \delta |D f_p(x') \cdot v| + C \epsilon |D f_m(x') \cdot v|
\leq C \delta |D f_p(x') \cdot v|.
\]

Finally, we prove (A.19). We have
\[
\left| \frac{\partial g}{\partial y}(x' + y) - \frac{\partial g}{\partial y}(x' + y') \right| \leq \sum_{k \geq 0} \left| \frac{\partial \rho_k(y)}{\partial y} \right| \left( |f_k(x') - f_{k+1}(x')| + |R_p(x') - R_{p+1}(x')| \cdot |y| \right) \leq \frac{C \epsilon}{r_p} (C \epsilon r_p + C \epsilon r_p) \leq C \epsilon
\]
by (3.10) and (3.11). A nearly identical calculation gives the same bound for \( \delta_2 \). We now handle \( \delta_3 \). First, notice that
\[
(A.23) \quad \left| \sum_{k \geq 0} \rho_k(y) R_k(x') \cdot e_{d+1} - R_p(x') \cdot e_{d+1} \right| \leq \sum_{k \geq 0} |\rho_k(y)| |R_k(x') \cdot e_{d+1} - R_p(x') \cdot e_{d+1}| \leq C \epsilon
\]
by (3.11). Because \( R_k(x') \) is an isometry such that \( R_k(x')(T \Sigma_0(x')) = T \Sigma_k(f_k(x)) \), \( R_k(x') \cdot e_{d+1} \) is the unit normal to \( T \Sigma_k(f_k(x')) \) so that
\[
(A.24) \quad |R_p(x') \cdot e_{d+1} - R_m(x') \cdot e_{d+1}| \leq C \angle(T \Sigma_p(f_p(x')), T \Sigma_m(f_m(x'))) \leq C \delta.
\]
Finally, (A.23) and (A.24) imply
\[
\delta_3 \leq \sum_{k \geq 0} \rho_k(y) R_k(x') \cdot e_{d+1} - R_p(x') \cdot e_{d+1} + \sum_{k \geq 0} \rho_k(y') R_k(x') \cdot e_{d+1} - R_m(x') \cdot e_{d+1}
+ |R_p(x') \cdot e_{d+1} - R_m(x') \cdot e_{d+1}|
\leq C \epsilon + C \epsilon + C \delta \leq C \delta \left| \frac{\partial g}{\partial y}(y') \right|.
\]
where the final inequality uses (A.12).

Corollary A.7. Let \( z, z' \) be as in Lemma A.6. Then for any vector \( v \in T \Sigma_0 \cup T \Sigma_0^\perp \), we have
\[
|D g(x' + y) \cdot v - D g(x' + y') \cdot v| \leq C \delta |D g(x' + y) \cdot v|.
\]
Proof. Suppose first that \( v = v_x \in T\Sigma_0 \). Then using (A.13), we compute

\[
|Dg(x' + y) \cdot v_x - Dg(x' + y') \cdot v_x|
\]

(A.25)

\[
= \left| \sum_{k \geq 0} (\rho_k(y) - \rho_k(y')) Df_k(x') \cdot v_x + \rho_k(y) D(R_k(x') \cdot y) \cdot v_x - \rho_k(y') D(R_k(x') \cdot y') \cdot v_x \right|
\]

\[
\leq C\delta|Df_p(x') \cdot v_x| \leq C\delta|Dg(x + y) \cdot v_x|
\]

(A.26)

\[
\leq C\delta(1 + C\delta)|Dg(x + y) \cdot v_x| \leq C\delta|Dg(x + y) \cdot v_x|
\]

(A.27)

using (A.17) and (A.18) in the first inequality, (A.10) in the second, and (A.14) in the third. If instead \( v = v_y \in T\Sigma_0^\perp \), then \( Dg(x' + y) = v_y \cdot \frac{\partial R}{\partial g}(x' + y) \) and the result follows from (A.19) and (A.4).

Using Corollaries A.5 and A.7 we can prove Proposition A.3.

Proof of Proposition A.3. Let \( z' = x' + y' \in G^M_0 \). We will show that for any vector \( v \in T\Sigma_0 \times T\Sigma_0^\perp \),

\[
|Dg(x + y) \cdot v - Dg(x' + y) \cdot v| \leq C\delta|Dg(x + y) \cdot v|.
\]

(A.28)

The set \( G^M_0 \) is designed exactly so that \( z' \in G^M_0 \) implies that the hypotheses of Lemmas A.4 and A.6 are satisfied. Hence, we can apply Corollaries A.5 and A.7 so that

\[
|Dg(x + y) \cdot v - Dg(x' + y') \cdot v| \leq |Dg(x + y) \cdot v - Dg(x' + y) \cdot v| + |Dg(x' + y) \cdot v - Dg(x' + y') \cdot v|
\]

\[
\leq C\delta|Dg(x + y) \cdot v| + C\delta|Dg(x' + y) \cdot v|
\]

\[
\leq C\delta|Dg(x + y) \cdot v|.
\]

By decomposing an arbitrary \( v' \in \mathbb{R}^{d+1} \) as \( v' = v_x + v_y \) where \( v_x \in T\Sigma_0 \) and \( v_y \in T\Sigma_0^\perp \), we write

\[
|Dg(x + y) \cdot v' - Dg(x' + y') \cdot v'| \leq |Dg(x + y) \cdot v_x - Dg(x' + y') \cdot v_x| + |Dg(x + y) \cdot v_y - Dg(x' + y') \cdot v_y|
\]

\[
\leq C\delta(|Dg(x + y) \cdot v_x| + |Dg(x + y) \cdot v_y|)
\]

(A.29)

\[
\leq C\delta|Dg(x + y) \cdot v'|.
\]

Where the final inequality follows from an application of the reverse triangle inequality in Lemma 3.10. We justify the application of the lemma by looking at the equations (A.13) and (A.11). These imply that the vector \( Dg(x + y) \cdot v_x \) is nearly parallel to \( T\Sigma_k(x) \) while the vector \( Dg(x+y) \cdot v_y \) is nearly perpendicular to \( T\Sigma_k(x) \) for some value of \( k \) where the deviations described
are on the order of $\epsilon$. This implies $|\langle Dg(x+y)\cdot v_x, Dg(x+y)\cdot v_y \rangle| \leq \frac{1}{2} |Dg(x+y)\cdot v_x| \cdot |Dg(x+y)\cdot v_y|$, so that the lemma applies. With this, we now compute,

$$|Dg(z') \cdot Dg(z)^{-1} \cdot v' - v'| = |[Dg(z') - Dg(z)] \cdot Dg(z)^{-1} \cdot v'| \leq C\delta |Dg(z) \cdot Dg(z)^{-1} \cdot v'| = C\delta |v'|.$$ 

This concludes the computations we need to bound the change in $Dg$. By integrating $Dg$ over paths in a quasiconvex domain $\Omega$, we get a companion result to Proposition [A.3] which roughly states that the map $g|_\Omega$ is a $(1 + C\delta)$-bi-Lipschitz perturbation of $Dg(z_0)$ for any $z_0 \in \Omega$. More precisely, for any $z \in \mathbb{R}^{d+1}$ define

(A.30) $L_{z_0}(z) = z_0 + Dg(z_0)(z - z_0).$

This is the affine transformation which approximates $g$ near $z_0$. Define

(A.31) $\varphi_{z_0} = g \circ L_{z_0}^{-1}$

**Proposition A.8.** Let $\Omega \subseteq \mathbb{R}^{d+1}$ be a quasiconvex domain such that $\Omega \subseteq G_{z_0}^M$ for some $z_0 \in \Omega$ and $M \ll \epsilon^{-1}$. Then the map $\varphi_{z_0} : L_{z_0}(\Omega) \to g(\Omega)$ is $(1 + C\delta)$-bi-Lipschitz and

(A.32) $|D\varphi_{z_0}(w) - I| \leq C\delta$

for all $w \in L_{z_0}(\Omega)$.

**Proof.** Because $w \in L_{z_0}(G_{z_0}^M)$ by assumption, we get

$$D\varphi_{z_0}(w) = Dg(L_{z_0}^{-1}(w)) \cdot DL_{z_0}^{-1}(w) = Dg(z) \cdot Dg(z_0)^{-1}$$

for $z' = L_{z_0}^{-1}(w) \in G_{z_0}^M$. Equation (A.32) follows from (A.3).

To prove that $\varphi_{z_0}$ is $(1 + C\delta)$-bi-Lipschitz, let $\gamma : [0, 1] \to \mathbb{R}^{d+1}$ be a path with $\gamma(0) = z_0, \gamma(1) = z$, and $\ell(\gamma) \lesssim |z_0 - z|$. Put $\tilde{\gamma}(t) = L_{z_0}(\gamma(t))$ and $w_0 = L_{z_0}(z) = z$. Observe that

$$L_{z_0}^{-1}(w) = z_0 + Dg(z_0)^{-1}(w - z_0).$$

We estimate

$$|\varphi_{z_0}(w) - \varphi_{z_0}(w_0)| = \left| \int_0^1 D(g \circ L_{z_0}^{-1})(\tilde{\gamma}(t)) \cdot \tilde{\gamma}'(t) dt \right|$$

$$= \left| \int_0^1 Dg(\gamma(t)) \cdot DL_{z_0}^{-1}(\tilde{\gamma}(t)) \cdot \tilde{\gamma}'(t) dt \right|$$

$$= \left| \int_0^1 Dg(\gamma(t)) \cdot Dg(z_0)^{-1} \cdot \tilde{\gamma}'(t) dt \right|$$

$$= \left| w - w_0 + \int_0^1 [Dg(\gamma(t)) \cdot Dg(z_0)^{-1} - I] \cdot \tilde{\gamma}'(t) dt \right|.$$
Using the fact that $\gamma(t) \in G^M_\omega$ for all $t$, Proposition A.3 implies, on one hand
\[
|\varphi_{z_0}(w) - \varphi_{z_0}(w_0)| \leq |w - w_0| + \int_0^1 \left| Dg(\gamma(t)) \cdot Dg(z_0)^{-1} - I \right| \cdot |\gamma'(t)| dt \\
\leq |w - w_0| + C\delta|Dg(z_0)| \cdot \ell(\gamma) \\
\leq (1 + C\delta)|w - w_0|.
\]
On the other,
\[
|\varphi_{z_0}(w) - \varphi_{z_0}(w_0)| \geq |w - w_0| - \int_0^1 \left| Dg(\gamma(t)) \cdot Dg(z_0)^{-1} - I \right| \cdot |\gamma'(t)| dt \\
\geq |w - w_0| - C\delta|Dg(z_0)| \cdot \ell(\gamma) \\
\geq (1 - C\delta)|w - w_0|,
\]
where the final inequality on both hands comes from the fact that $|w' - w| = |Dg(z) \cdot (z' - z)| \leq |Dg(z)| \cdot |z - z'|$ and our assumption that $\ell(\gamma) \lesssim |z - z'|$.

With these results, we can now prove Theorem A.2.

**Proof of Theorem A.2** Construct the map $g$ as in Theorem 3.2. We need to show that $g$ is $(1 + C\delta)$-bi-Lipschitz and has $|Dg(w) - I| \leq C\delta$ for all $w \in \Omega$. Because $Dg(z) = I$, by Proposition A.8, it suffices to show that $\Omega \subseteq G^M_z$ for some $M \ll C\delta^{-1}$ because $\varphi_{z_0} = g \circ L_z^{-1} = g$.

Let $z' = x' + y' \in \Omega$. Because $\text{diam}(\Omega) < \hat{M}$, we have $|f_0(x) - f_0(x')| = |x - x'| < \hat{M}$. We also need to verify that $\sum_{k=0}^n (x'_{k+1} - x'_k)^2 < C\epsilon$ and $\text{Angle}(T_k(x'), T_k(x')) \leq C\delta$, but these follow from hypotheses (i) and (ii) on the given CCBP, concluding the proof.

**References**

[ABHM17] M. Akman, M. Badger, S. Hofmann, and J. M. Martell. Rectifiability and elliptic measures on 1-sided NTA domains with Ahlfors-David regular boundaries. *Trans. Amer. Math. Soc.*, 369(8):5711–5745, 2017.

[ABHM19] M. Akman, S. Bortz, S. Hofmann, and J. M. Martell. Rectifiability, interior approximation and harmonic measure. *Ark. Mat.*, 57(1):1–22, 2019.

[AHM+20] J. Azzam, S. Hofmann, J. M. Martell, M. Mourgoglou, and X. Tolsa. Harmonic measure and quantitative connectivity: geometric characterization of the $L^p$-solvability of the Dirichlet problem. *Invent. Math.*, 222:881–993, 2020.

[AS12] J. Azzam and R. Schul. How to take shortcuts in Euclidean space: making a given set into a short quasi-convex set. *Proc. Lond. Math. Soc.*, 105(2):367–392, 2012.

[AS18] J. Azzam and R. Schul. An analyst’s traveling salesman theorem for sets of dimension larger than one. *Math. Ann.*, 370(3-4):1389–1476, 2018.

[AT15] J. Azzam and X. Tolsa. Characterization of $n$-rectifiability in terms of Jones’ square function: Part II. *Geom. Funct. Anal.*, 25:1371–1412, 2015.

[Azz18] J. Azzam. Tangents, rectifiability, and corkscrew domains. *Publ. Mat.*, 62(1):161–176, 2018.

[Bad10] M. Badger. Null sets of harmonic measure on NTA domains: Lipschitz approximation revisited. *Math. Z.*, 270:241–262, 2010.

[BH17] S. Bortz and S. Hofmann. Harmonic measure and approximation of uniformly rectifiable sets. *Rev. Mat. Iberoam.*, 33(1):351–373, 2017.
M. Christ. A T(b) theorem with remarks on analytic capacity and the Cauchy integral. *Colloq. Math.*, 60/61(2):601–628, 1990.

B. E. J. Dahlberg. Estimates of harmonic measure. *Arch. Rational Mech. Anal.*, 65(3):275–288, 1977.

G. David and D. Jerison. Lipschitz approximation to hypersurfaces, harmonic measure, and singular integrals. *Indiana Univ. Math. J.*, 39(3):831–845, 1990.

G. David and S. Semmes. *Analysis of and on uniformly rectifiable sets*, volume 38 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1993.

G. David and T. Toro. Reifenberg parameterizations for sets with holes. *Mem. Amer. Math. Soc.*, 215(1012):vi+102, 2012.

S. Ghinassi. Sufficient condition for $C^{1,\alpha}$ parametrization and rectifiability. *Ann. Aca. Sci. Fenn. Math.*, 45:1065–1094, 2020.

J. Garnett and D. E. Marshall. *Harmonic measure*, volume 2 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2008.

J. Garnett, M. Mourgoglou, and X. Tolsa. Uniform rectifiability from Carleson measure estimates and $\varepsilon$-approximability of bounded harmonic functions. *Duke Math. J.*, 167(8):1473–1524, 2018.

T. Hytönen and H. Martikainen. Non-homogeneous $Tb$ theorem and random dyadic cubes on metric measure spaces. *J. Geom. Anal.*, 22(4):1071–1107, 2012.

M. Hyde. A $d$-dimensional Analyst’s Traveling Salesman Theorem for subsets of Hilbert space. Preprint, 2021.

M. Hyde. The restricted content and the $d$-dimensional Analyst’s Travelling Salesman Theorem for general sets. *Adv. Math.*, 397, 2022. Paper No. 108189.

P. W. Jones. Rectifiable sets and the traveling salesman problem. *Invent. Math.*, 102(1):1–15, 1990.

M. Mourgoglou, B. Poggi, and X. Tolsa. $L^p$-solvability of the Poisson-Dirichlet problem and its applications to the regularity problem. Preprint, 2022.

M. Mourgoglou and X. Tolsa. The regularity problem for the Laplace equation in rough domains. Preprint, 2022.