A group of processes related to the Poisson process

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A group of stochastic processes akin to the Poisson process is defined in terms of rules of interactions between two types of interacting entities and in terms of a parameter corresponding to the initial relative numbers of the two types of interacting entities. One limiting value of this parameter corresponds to the Poisson process and the exponential distribution function, and the other limiting value of the parameter corresponds to a special case among the group of stochastic processes defined and a statistical distribution function used previously for incomes and fracture toughness. The related processes in between, which correspond to the intermediate values of the parameter, correspond to an intermediate statistical distribution function. The transition between the limiting cases is smooth as evinced by the change of the mean and median with change in the parameter. The scale-invariant behaviour of the fields of stress and strain at the tips of cracks is used to support the introduction of a shape parameter into the special-case function. All the distribution functions considered are found to be stable extreme-value functions, either in the sense of multiplying probabilities or in the sense of summing the variable or in a mixture of both senses.

Keywords: probability; Poisson; stochastic processes; distribution functions; fracture

1. Introduction

In a previous paper (Neville 1987), a statistical distribution function for fracture toughness was derived on the basis of the nature of the stress and strain fields near the tips of sharp cracks, with a shape parameter being included. The function thus arrived at is that used by Fisk (1961) for describing distributions of incomes and is also known as the log-logistic function. Briefly, the basis of the derivation of the function for fracture toughness is that, in a piece under increasing load, the fields of stress and strain near the tips of sharp cracks are such that they expand away from the tips of cracks so that, in a scale-invariant fashion, more material is subjected to the same stresses and strains, whereas, remote from the tips of cracks, the same material is subjected to increasing stress and strain.

The statistical distribution function derived for fracture toughness has been found to fit many fracture data well, not only fracture toughness from pieces with single macroscopic cracks (Neville 1987), but also fracture stress from pieces with many microscopic cracks (Neville & Kennedy 1989; Neville 1990) and data equivalent to fracture toughness (Balankin et al. 1999).

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Balankin et al. (1999, p. 2573) expressed surprise that their fracture data were fitted best by the statistical distribution function derived for fracture toughness (the Fisk or log-logistic function). Furthermore, the derivation of and the inclusion of the shape parameter in the distribution function for fracture toughness (Neville 1987) seem unsatisfactorily arbitrary, even if intuitively reasonable. In this paper, therefore, support will be provided for the use of the distribution function for fracture toughness (the Fisk or log-logistic function) for failure strength by expanding the previous derivation in terms of Poisson-like stochastic processes and using the scale-invariant nature of the fields of stress and strain at the tips of cracks to justify the scale parameter.

The statistical distribution function derived for fracture toughness will be regarded as an extreme-value function and its relationship to the exponential distribution function and similar extreme-value functions, as it emerges from the consideration of stochastic processes, will also be investigated.

2. The group of processes and the corresponding distribution functions

The expanding fields of stress and strain near the tips of cracks will be regarded as entities of one type which interact with material in the particular sense that when the fields encounter the first sufficiently weak region of a material which is sufficiently near the tip of a crack, failure of the cracked piece occurs, whereby these weak regions will be regarded as entities of another type. Clearly, when failure occurs the fields cease to exist and the piece is eliminated from carrying load. In this view, the different entities are equally numerous: there is just one first sufficiently weak region for each crack-tip field of stress and strain. The first sufficiently weak region in each of a number of cracked pieces is taken to be randomly positioned in relation to the tip of the crack, so that a distribution of fracture toughness will be observed. Furthermore, the encounters of the two types of entity will be regarded as one of the stochastic processes defined below.

(a) Definition of the group of processes

A large number of entities of a first type interact with a larger number of entities of a second type according to the following rules:

(i) entities of the first type do not interact with one another,
(ii) entities of the second type do not interact with one another,
(iii) entities of the first and second types interact with one another by eliminating, annihilating or cancelling each other, and
(iv) the entities encounter one another at random and interact at each encounter according to rules (i)–(iii).

It is assumed that each encounter involves only two entities.

The rate of occurrence of interactions that eliminate one of each type of entity (rule (iii)) is proportional to the remaining proportion of entities of type one and to the remaining proportion of entities of the second type such that

\[ \delta F(t) = \delta t \cdot k \cdot R_1(t) \cdot R_2(t), \]  

(2.1)
where \( F(t) \) is the proportion of entities of type one that have been eliminated at time \( t \); \( k \) is a constant of proportionality; \( R_1(t) \) is the proportion of entities of type one remaining at time \( t \); and \( R_2(t) \) is the proportion of entities of type two remaining at time \( t \). Constant \( k \) is proportional to the initial number of entities of type one and proportional to the initial number of entities of type two and inversely proportional to the size of the space within which encounters occur.

The initial number of entities of the second type is \( 4 \) times as large as the initial number of entities of the first type, whereby \( 4R_1 \).

The definition set out above is a simple version of the well-known collision theory of chemistry for the case of a one-way interaction, the emphasis here being more on the proportion of interacting entities eliminated than on the rates of elimination.

(b) Derivation of the distribution functions for \( \phi \neq 1 \)

\( F(t) \) is a cumulative probability of elimination. \( R_1(t) \) is equal to \( 1 - F(t) \). \( R_2(t) \) is given by

\[
R_2 = ((\phi - 1)/\phi) + R_1/\phi,
\]

so that

\[
\delta F/(R_1 \cdot (((\phi - 1)/\phi) + R_1/\phi)) = k \cdot \delta t.
\]

By partial fractions, equation (2.3) becomes

\[
((\Phi/R_1) - \Phi/(\phi - 1 + R_1)) \cdot \delta F = k \cdot \delta t,
\]

where \( \Phi = \phi/(\phi - 1) \) and \( \phi \neq 1 \), that is \( \phi > 1 \), or

\[
((\Phi/(1 - F)) - \Phi/(\phi - F)) \cdot \delta F = k \cdot \delta t.
\]

Integrating with the initial condition that \( t = F = 0 \) and rearranging gives

\[
F(t) = (1 - e^{-kt/\Phi})/(1 - e^{-kt/\phi}).
\]

When \( \phi = \infty \), \( \Phi = 1 \), the process reduces to the Poisson process and equation (2.6) reduces to the exponential distribution function

\[
F(t) = 1 - e^{-kt}.
\]

(c) Derivation of a distribution function for the special case \( \phi = 1 \)

If \( \phi = 1 \), \( R_2 = R_1 \) so that

\[
\delta F(t)/R_1^2 = \delta t \cdot k.
\]

Integrating with the initial condition that \( t = F = 0 \) and rearranging gives

\[
F(t) = kt/(1 + kt).
\]

This derivation of the special-case distribution function results in the distribution function for fracture toughness, which is the Fisk or log-logistic function, with its shape parameter equal to 1 and its scale parameter, the median, equal to \( 1/k \).
3. The relationships among the processes and distribution functions

The processes and the corresponding distribution functions of §2 form three groups: \( \varphi = \infty \), the Poisson process and the exponential function; \( 1 < \varphi < \infty \), the intermediate processes and functions; and \( \varphi = 1 \), the special-case process and function. In order to compare these different groups, the well-established derivation of the Poisson-exponential case is reviewed and then parallels are drawn for the special case and the intermediate cases. The behaviour of the means and medians of the various cases with a change in the parameter \( \varphi \) will be investigated.

(a) The Poisson process

The Poisson process has to do with the chance occurrence of independent events. In a homogeneous Poisson process, the probability of occurrence of an event in any of a series of small non-overlapping intervals of time is defined as constant and as so small that the probability of more than one concurrent or simultaneous event occurring in any such interval is negligibly small.

The probability that no event occurs in a small interval, \( \delta t \), of time, for instance, when the average frequency of events (i.e. the number of events expected per unit time) is \( k \), where \( 1/k \) is the mean time before an event and is given by

\[
(1 - k\delta t),
\]

and the probability that no event occurs in a unit of time \( n\delta t \) is given by

\[
(1 - k\delta t)^n,
\]

which, if \( k\delta t \) tends to being small in relation to 1 and \( n \) tends towards \( \infty \), tends towards

\[
e^{-k}.
\]

The probabilities of one, two, \( m \), etc., events in unit time are

\[
e^{-k}/k, e^{-k}/k^2, \ldots, e^{-k}/k^m m!, \text{ etc.,}
\]

provided that \( m \) is small in relation to \( n \).

It is also possible to derive a continuous distribution function for the time passing before a first event occurs. The cumulative probability \( F(t) \) is equal to 1 minus the probability of survival \( R(t) \) up to some time \( t \) equal to \( r\delta t \) and is given by

\[
F(t) = 1 - (1 - k\delta t)^r = 1 - e^{-kt},
\]

and the corresponding probability density function is given by

\[
f(t) = ke^{-kt}.
\]
The mean of this exponential distribution is $1/k$, and the instantaneous rate of occurrence of events or hazard rate $h(t)$, which equals $f(t)$ divided by $R(t)$, is constant and given by

$$h(t) = k. \quad (3.7)$$

(b) The special case

In the development of a statistical function for fracture toughness (Neville 1987), a somewhat similar process was considered. In this process, the effect of the passage of time $g(t)$ on the remaining population $R(t)$ is to cause events to occur among the remaining population at a frequency $1/B$. Accordingly, in a small interval of time $\delta t$ the proportion of trials in which a first event has occurred will increase by the increment $\delta F(t)$ such that

$$\frac{\delta F(t)}{R(t)} = \frac{\delta g(t) \cdot R(t)}{(\delta t/B) \cdot R(t)}. \quad (3.8)$$

Rearranging, putting $R(t)$ equal to $1 - F(t)$ and integrating with the condition that $t = g(t) = F(t) = 0$ leads straightforwardly to

$$F(t) = g(t)/(1 + g(t)) = (t/B)/(1 + (t/B)). \quad (3.9)$$

Whereas from equation (3.7), the hazard rate and the rate of effect $dg/dt$ are equal to the expected frequency $k$ and are constant for the Poisson process, for this special-case process the hazard rate is equal to $R(t)/B$ and thus is dependent on the number of events having occurred, although the frequency is consistent and equal to the reciprocal of the median value of $t$, $B$. This is taken to mean that the special-case process is not memoryless, though in a manner that is nonetheless consistent.

For the special-case process, instead of equation (3.1), we have

$$R(\delta t) = (1 + \delta t/B)^{-1} = (1 - \delta t/B) \text{ for small } \delta t/B. \quad (3.10)$$

In other words, both processes are similar at the start, when $t$ is small and $R(t)$ is close to 1. Instead of equations (3.2) and (3.3), we have, from equation (3.9),

$$R(t) = (1 + t/B)^{-1} = (1 + g(t))^{-1}, \quad (3.11)$$

which means that time has been summed, or its effect $g(t)$ scaled with time, in order to proceed from equations (3.10) to (3.11), in contrast to the multiplicative development from equations (3.1) to (3.2) and (3.3). The Poisson process depends on the independence of time intervals, whereas such independence is not necessary, at least in the same sense, for the special-case process.

(c) The intermediate case

For the intermediate processes we have, from equation (2.6),

$$R(t) = e^{-kt/\Phi}/(\Phi \cdot (1 - e^{-kt/\Phi}/\varphi)), \quad (3.12)$$

and instead of equation (3.1) we have, for small $kt/\Phi$,

$$R_1(t) = (1 - kt/\Phi) \cdot (1 + kt/\varphi)^{-1}. \quad (3.13)$$
Since $B$ in equations (3.8)–(3.11) is the same as $1/k$ in equations (2.8) and (2.9), both being the median, equation (3.13) combines the character of equation (3.1) in the first bracketed term, which stems from the numerator of equation (3.12), and the character of equation (3.10) in the second bracketed term, which stems from the denominator of equation (3.12). The parameter $\varphi$ determines which character dominates: if $\varphi$ is large ($\Phi$ little more than 1), then the intermediate process is predominantly Poisson-like; but if $\varphi$ is little more than 1 ($\Phi$ large), then the intermediate process is predominantly like the special case, even if $kt$ is large although small compared with $\Phi$. Since $1/B = 1/k + 1/\Phi$ equals 1, intermediate processes have, in some sense, according to equation (3.13) one $\varphi$th the character of the special-case process and one $\Phi$th that of the Poisson process.

When $kt/\Phi$ becomes large, $R(t)$ tends towards $e^{-kt/\Phi}$ and the intermediate cases tend to be Poisson-like.

\[(d) \quad \text{The means and medians as they vary with } \varphi\]

The mean of the intermediate functions is given by $(\varphi/k) \ln \Phi$, which changes smoothly from $\infty$, the mean of the special-case function, to $1/k$, the mean of the exponential function, as $\varphi$ goes from 1 to $\infty$.

Similarly, the median of the intermediate functions is given by $(\Phi/k) \ln(1 + 1/\Phi)$, which changes smoothly from $1/k$, the median of the special-case function, to $(1/k) \ln 2$, the median of the exponential function, as $\varphi$ goes from 1 to $\infty$.

\section{4. The effect of the passage of time}

Since for the special case ($\varphi = 1$), as well as for cases for which $\varphi > 1$, all the entities of type one are eventually exhausted and the same rules apply, the processes might in all cases be viewed at any moment as the effect of time, $\delta g(t)$, being to cause events among the remaining populations of the entities of both types and being $\delta t \cdot k$, that is, from equation (2.1),

$$\delta F(t) = k \cdot \delta t \cdot R_2(t) \cdot R_1(t) = \delta g(t) \cdot R_1(t) \cdot R_2(t). \quad (4.1)$$

For the Poisson process ($\varphi = \infty$), $R_2(t) = 1$ and we have, following §3b,

$$F(t) = 1 - e^{-g(t)}, \quad (4.2)$$

instead of equation (2.7) or (3.5), and for the intermediate cases ($1 < \varphi < \infty$), we have

$$F(t) = (1 - e^{-g(t)/\varphi})/(1 - e^{-g(t)/\Phi/\varphi}), \quad (4.3)$$

instead of equation (2.6). For the special-case, equation (3.9) applies, with $1/B$ being equal to $k$.

Since according to the special-case process, the effect of time is scaled (see equations (3.10) and (3.11)), scale-invariant functions for $g(t)$ will not disturb the consistency of the special-case process. Furthermore, the scale-invariant
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expansion of fields of stress and strain near to the tips of cracks suggests that \( g(t) \) might be expected to be a scale-invariant function. For \( g(t) \) to be scale invariant, it must fulfil the condition

\[
g(t) = \eta^{-D} \cdot g(\eta t),
\]

where \( \eta \) is a scalar and \( D \) is an exponent. The monomial or power law

\[
g(t) = (t/B)^D
\]

fulfils this condition.

Putting equation (4.5) into equation (3.9) yields

\[
F(t) = (t/B)^D/(1 + (t/B)^D).
\]

If the same reasoning is applied to the view of the other processes presented above, \( g(t) \) as \( kt \) being similarly replaced by \((kt)^D\), equation (4.2) gives

\[
F(t) = 1 - e^{-(kt)^D},
\]

which is the extreme-value function with a lower limit presented by Fisher & Tippett (1928) with the lower limit set to zero.

For the intermediate cases, equation (4.3) becomes

\[
F(t) = (1 - e^{-(kt)^p/\phi})/(1 - e^{-(kt)^p/\phi}/\phi). (4.8)
\]

(a) The means and medians as they vary with \( \phi \) and \( D \)

The \( m \)th moment about 0 of the intermediate functions, including the shape parameter, \( D \), is given by \((\phi/(k^m \cdot \Phi((D-m)/D))) \cdot (\Gamma((D + m)/D)) \cdot \sum_{n=1}^{\infty} (n^{\phi/D} \cdot \phi^n)^{-1}. The mean (\( m=1 \)) goes for \( D=1 \) to \( (\phi/k)\ln\Phi \) as in §3d, for \( D=\infty \) to \( 1/k \) and for \( \phi = \infty \) to the well-known result \((1/k) \cdot \Gamma((D + 1)/D)). The mean of the special-case function, which is given by \((\pi/kD) \cdot \csc(\pi/D) \) or \((1/k) \cdot \Gamma((D-1)/D)) \cdot (\Gamma((D + 1)/D)) \) is approached ever more slowly as \( \phi \) is reduced towards 1 because the ratio of adjacent terms in the sum series is \((n/(n+1))^{1/2}/\phi \) (approximately \( 1/\phi \) for large \( n \)) so that the terms decrease ever more slowly as \( \phi \) nears 1 (\( \Phi \) approaches \( \infty \)). This convergence might be so tardy because \( \phi \) may not be equal to 1 (see §2b). Some approximate sample values for \( D=2 \) are 0.886/\( k \), 0.960/\( k \), 1.01/\( k \), 1.25/\( k \), 1.45/\( k \) and 1.53/\( k \) for \( \Phi=1 \), 3/2, 2, 10, 100 and 1000 respectively, the mean of the special-case function being \( \pi/2k \) for \( D=2 \).

The median of the intermediate functions, including the shape parameter, \( D \), is given by \((1/k) \cdot (\Phi \cdot \ln(1+1/\Phi))^{1/D} \), which, for \( \Phi = \infty \), becomes \( 1/k \), the median of the special-case function, for \( \Phi=1 \) becomes the well-known result \((1/k) \cdot (\ln 2)^{1/D} \), for \( D=\infty \) becomes \( 1/k \) and for \( D=1 \) becomes \((\Phi/k) \cdot \ln(1+1/\Phi) \) as in §3d.

Thus, a smooth transition is again observed between the special case and Poisson-exponential case.

5. Space rather than time

The variable in the Poisson process may be some form of space rather than time. For instance, one may search through material for flaws (see Kennedy & Neville 1985). Similarly, the variable in the special-case process may be some form of
space rather than time, so that fields of stress and strain expand through material and encounter weak regions. If time $t$ is replaced by the variable ‘sampling’ $S$, which was defined as a space in which scale-invariant conditions of stress and strain pertain near the front of a crack (Neville 1987), then exactly the statistical function derived for fracture toughness is obtained

$$F(S) = (S/B)^D/(1 + (S/B)^D). \quad (5.1)$$

For the intermediate cases, taking sampling $S$ as space, equation (4.8) becomes

$$F(t) = (1 - e^{-(S/B)^D}) / (1 - e^{-(S/B)^D/\phi}). \quad (5.2)$$

### 6. Stability and extreme-value functions

Extreme-value functions obey a criterion of stability (Fisher & Tippett 1928), shifting and scaling according to

$$F_J(x) = F(a_Jx + b_J), \quad (6.1)$$

where $b_J$ is zero if the lower bound is zero. The exponential distribution function derived from the Poisson process obeys this criterion in the sense that the frequency parameter $k$ is scaled accordingly: if the population subject to the effect of the passage of time is doubled, for example, the frequency is also doubled and the mean time before an event is halved.

The special-case statistical distribution function (equations (2.9), (3.9), (4.6) and (5.1)) is an extreme-value function because it deals with values of time or sampling at which first events occur and is inherently stable in the sense that the variable (time or sampling) is additive as shown by the additive, rather than multiplicative, step from equations (3.10) to (3.11). This stability of the special-case function is borne out empirically by a correct prediction of the effect of size (Neville 1987) and maintained applicability after summing of the sampling (Neville 1990), even with effect $g(t)$ in its power-law form.

In view of equation (3.13), it would appear that the intermediate function is also its own extreme-value function, since, at least for small $t$ and therefore the first events, increasing the number of entities involved without changing $\phi$ will have the effect of scaling $k$ according to the increase in numbers, so that the times to the first events will correspondingly shorten, with equation (3.13) thus still applying and also exhibiting the required stability.

### 7. Discussion

It is noted that equations (3.10) and (3.11) represent a statement of the special-case process, whereby the associated distribution function (equation (3.9)) is immediately derivable from equation (3.11). Similarly, equations (3.1) and (3.2) represent a statement of the Poisson process, from which the exponential function results straightforwardly. While equation (3.13) might be regarded as a similar statement of the intermediate processes, the route to equation (2.6) is less directly obvious.
The special-case statistical distribution function is seen to be self-consistent in the sense that the same rule of summation of the variable is always applied and never leads to any discrepancy or inconsistency. Effect \( g(t) \) in the form \( kt \) may be replaced by a power-law form or another scale-invariant function without invalidating the definitions and derivations given above, and this leads to a process and corresponding function, which are in agreement with a large body of experimental evidence. Stress fields moving through a piece of material will, at some stage, encounter a first stationary flaw or weak patch, where failure will initiate, and when a failure occurs, both the stress field and the piece of material in which it has been moving are eliminated; so it is the special-case function that would appear appropriate. This position is in agreement with a wealth of experimental evidence (Neville 1987, 1990; Neville & Kennedy 1989; Balankin et al. 1999).

The proportion of entities of type two that have been eliminated at time \( t \) is readily derived. This does not represent a distribution function because the final value of this proportion is less than 1, except in the special case where \( \phi = 1 \).

An example of the special case has already been given in §§1, 2 and 5, the interacting entities being, first, fields of stress and strain spreading out from sharp cracks, and then, weak patches of material near to cracks. An example of the Poisson-exponential case is radioactive decay, whereby one can envisage, for instance, a target containing a limited number of particles of one type, for example Lithium-6, being bombarded by a stream containing an essentially unlimited \( (\phi = \infty) \) number of particles of another type, for example deuterons.

The following example of an intermediate case is, in a sense, half way between the special case and the Poisson-exponential case, with \( \phi = \Phi = 2 \). A number of persons are considered to move randomly in a large, entirely darkened room in which half as many stationary chairs are randomly placed. When encountering a chair, a person sits down and occupies it. The procedure continues until all chairs are occupied. The proportion of chairs occupied will be according to equation (2.6), or conceivably equation (4.8), with \( \phi = \Phi = 2 \). If there had been half as many persons as chairs, the proportion of persons seated would also have been according to equation (2.6) or (4.8) with \( \phi = \Phi = 2 \), with \( k \) being the same whether there is an excess of persons or chairs because equation (2.1) is symmetrical in \( R_1 \) and \( R_2 \). Had there been a large excess of persons \( (\phi \) large), something close to a Poisson process would have resulted, the mean time to a chair being occupied not being great, whereby, by contrast, had there been only an excess of one person \( (\phi \) little more than 1), then something approaching the special case would have resulted, the mean time to a chair being occupied being large because, as is easily imagined, the last two persons would clearly be expected to take some time in finding the last chair.

Similar processes involving the interaction of more than two different types of entities and the corresponding distribution functions might be obtained by means of rules and considerations analogous to those presented in §§2 and 4.

### 8. Conclusions

Starting from a very basic set of rules of interaction between two different types of entity, a group of stochastic processes has been developed and the corresponding probability distribution functions have been derived. In one
limiting form, the process is the Poisson process to which the exponential
distribution corresponds. In the other limiting form, the process is a special
case to which the statistical distribution function derived previously for fracture
toughness (Neville 1987), which is the Fisk or log-logistic function,
corresponds. The processes and functions in between the limiting forms represent a
transition between the Poisson process and the other limiting process and a
transition between the exponential distribution function and the distribution
function for fracture toughness. The mean and median of the intermediate
distribution function change in smooth transition between those of the expo-
nential function and those of the special-case function. In this way, a relationship
between the limiting processes and the limiting distribution functions has
been established.

The processes considered are akin to the Poisson process in that they coincide
when the variable is small and also in that each is associated with an extreme-
value distribution function. The essential difference lies between the additive
character of the special-case process, which entails summation of the variable,
and the multiplicative character of the Poisson process, which entails
multiplication of probabilities, whereby the intermediate processes show some
additive character and some multiplicative character.

Some justification for the inclusion of the shape parameter in cases in which
the distribution of strength when failure occurs near sharp cracks is considered
has been given based on the scale-invariant nature of the fields of stress and
strain near the tips of sharp cracks.

Some support has thus been lent to the use of the distribution function for
fracture toughness for failure strength and further reasons for this function
providing the best description of fracture data as has been found previously (see
Neville & Kennedy 1989; Neville 1990; Balankin et al. 1999) have been given.

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