Quantum groups and nonabelian braiding in quantum Hall systems

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Abstract

Wave functions describing quasiholes and electrons in nonabelian quantum Hall states are well known to correspond to conformal blocks of certain coset conformal field theories. In this paper we explicitly analyse the algebraic structure underlying the braiding properties of these conformal blocks. We treat the electrons and the quasihole excitations as localised particles carrying charges related to a quantum group that is determined explicitly for the cases of interest. The quantum group description naturally allows one to analyse the braid group representations carried by the multi-particle wave functions. As an application, we construct the nonabelian braid group representations which govern the exchange of quasiholes in the fractional quantum Hall effect states that have been proposed by N. Read and E. Rezayi 1, recovering the results of C. Nayak and F. Wilczek 2 for the Pfaffian state as a special case.

1 Introduction

In a 2+1 dimensional setting, quantum mechanics leaves room for particles with exchange properties other than those of bosons and fermions and the exchanges of such n such particles are governed by a representation of the braid group $B_n$. These representations may be abelian, or, more excitingly, nonabelian. Quasihole excitations of fractional quantum Hall plateaus have already provided us with examples of the former possibility and may possibly reveal the latter as well. Several (series of) candidate nonabelian states have been proposed in the literature. Examples are the HR-state 3, the Pfaffian state 4, the spin singlet states of Ardonne and Schoutens 5 and the parafermionic generalisations of the Pfaffian proposed by Read and Rezayi 6. It is the last series of states that we will focus on in this paper, although the methods we use will also be applicable to the other cases. It has been suggested that the Read-Rezayi states should give a good description of quantum Hall plateaux which occur at several filling fractions 3, 4, 6. In particular, the Pfaffian is thought to describe the plateau observed 3 at filling fraction $\nu = \frac{5}{2}$. Numerical support for these claims has been provided in 3, 4, 6, where it was shown that some of the RR-states (among which the Pfaffian state) have large overlaps with the exact ground states for electrons with Coulomb interactions at the same filling fractions. Many aspects of the Read-Rezayi states have already been well-studied. For example, one may show

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(see [1, 4, 11]) that they are exact ground states of certain ultra-local Hamiltonians with $k+1$-body interactions, which gives hope that they will indeed represent new universality classes of two dimensional physical systems. Also, their zero modes have been counted and a suitable basis for the description of the quasiholes has been obtained [11, 12]. Finally, there is recent work which explains how the RR-states may be obtained as projections of abelian theories [13]. Still, the braiding of the quasiholes has been described explicitly only for the Pfaffian state [2].

One of the general aims of this paper is to analyse some of the physical properties of Hall systems, not by studying the explicit form of the wave functions but rather by exploiting the underlying algebraic structure, which in turn derives from the associated conformal field theories. This allows us for example to give an explicit description of the braid group representations that govern the exchange properties of the quasiholes for all of the RR-states. In order to do this, we first describe the electrons and quasiholes of the RR-states as particles that carry a representation of a certain quantum group. That such a description is possible is a logical consequence of the well known relation between quantum groups and conformal field theories and in fact, we expect that a similar description is possible for all the nonabelian quantum Hall states that have been proposed. We believe that the quantum group description of quantum Hall states will prove a useful complement to the existing conformal field theory and wave function methods, both technically, because it makes braiding calculations much easier, and conceptually. The reason that braiding calculations are so much simplified, is that the quantum group picture allows one to deal with quasiholes and electrons without dealing with their exact spatial coordinates. Exchanging two particles becomes a purely algebraic operation, simple enough to be carried out explicitly for large numbers of particles.

One of the useful features of the present paper is that it does bring together a number of sophisticated physical and mathematical ingredients, everybody knows to exist in principle, and integrates them into a rather self-contained toolbox to analyse the problems at hand. The outline of the paper is as follows. In section 2, we review the usual description of the Read-Rezayi states in terms of conformal blocks of parafermionic conformal field theories. We also count the number of independent states with a fixed number of quasiholes in fixed positions. In section 3, we give the motivation for the use of quantum groups in the description of particles with nonabelian braiding and provide the necessary background. In particular we describe the braid group representations that describe the exchanges in a system of localised particles with a hidden quantum group symmetry. In section 4, we recall the connection between quantum groups and conformal field theories and in particular, we obtain the quantum groups which can be used to describe the braiding of the parafermion CFTs which are important for the Read-Rezayi states. In section 5 we describe the RR-states as systems of point particles with a hidden quantum group symmetry and give the explicit form of the associated braid group representations. We also check that the results of Nayak and Wilczek for the case of the Pfaffian are recovered. A discussion of the results, including questions for future research can be found in section 6.

2 The CFT description of the Read-Rezayi states

In this section, we will review the conformal field theoretic description of the Read-Rezayi states. The definition of these states can be found in subsection 2.2, after the
required preliminaries on parafermionic conformal field theories in subsection 2.1. In subsection 2.3, we describe the fusion of the Read-Rezayi quasiholes in terms of paths on a Bratteli diagram and in the following subsection, we use this description to calculate the dimensions of the braid group representations which govern the exchanges of these quasiholes. The final subsection is devoted to a description of the results of Nayak and Wilczek on the braiding of the quasiholes of Pfaffian or Moore-Read state, which is the simplest state in the Read-Rezayi series.

2.1 The Parafermion CFT

Like many other (candidate) quantum Hall states, the RR-states can be described using conformal blocks of a rational conformal field theory. Here, the theory in question is the $\mathbb{Z}_k$ parafermionic theory of Zamolodchikov and Fateev. [14, 15] Before we write down any explicit expression for the RR-states, we recall some well known facts about this CFT. The $\mathbb{Z}_k$ parafermionic CFT may be described completely in terms of a chiral algebra generated by the modes of $k$ parafermionic currents (see [14, 15] and also [16] for some more recent work in this vein), but it also has two different coset descriptions. The cosets involved are $\hat{sl}(2)_k/\hat{U}(1)_k$ and $\hat{sl}(k)_1 \times \hat{sl}(k)_2/\hat{sl}(k)_2$. The first of these descriptions was used extensively already in [14, 15], to determine fusion rules, characters and partition functions for the parafermions. The treatment of the parafermions in most of the literature on the RR-states has been influenced by this description. The second coset was introduced by Bais et al. in [17, 18] and used in [19] to construct a Coulomb gas representation of the theory which led to alternative character formulae [20]. This coset description has recently also been used in the work of Cappelli, Georgiev and Todorov on the RR-states [13]. In the rest of this section, we will give a quick description of both pictures and indicate how they are connected.

2.1.1 The coset $\hat{sl}(2)_k/\hat{U}(1)_k$.

We start with the coset $\hat{sl}(2)_k/\hat{U}(1)_k$. For this coset, we have more information about the coset primaries than usual. In particular, one can decompose certain fields of the parent $\hat{sl}(2)$ WZW-theory as a product of a coset primary field and a $U(1)$ primary field (see formula (1) below). In order to describe this decomposition, it is convenient to start with a short description of the fields and fusion rules of the $\hat{sl}(2)$ theory before moving on to the parafermions (for much more detail on WZW-theories, see for example [21]). Note that when we speak of primary fields in the sequel, we will always mean chiral primary fields.

Recall that the spectrum generating algebra of the $\hat{sl}(2)_k$ model is the affine Lie algebra $\hat{sl}(2)$ at level $k$. The Virasoro algebra is embedded in the enveloping algebra of the affine algebra through the Sugawara construction. When discussing primary fields of the $\hat{sl}(2)_k$ model, we need to distinguish between primary fields of the affine algebra (affine primaries) and primary fields of the Virasoro algebra (Virasoro primaries). Each affine primary field is necessarily also a Virasoro primary, but not vice versa. In fact, one can always find infinitely many Virasoro primaries among the affine descendants of an affine primary.

Let us be more explicit. If $\theta$ is the highest root of a simple Lie algebra $g$, then the affine primaries of the $\hat{g}_k$ model are labelled by the dominant integral weights $\Lambda$ of $g$ for
which \((\Lambda, \theta) \leq k\). For \(g = sl(2)\) this just means \(0 \leq \Lambda \leq k\). Let us call these fields \(G^\Lambda\). The conformal dimension \(h_\Lambda\) of \(G^\Lambda\) is given by

\[
h_\Lambda = \frac{\Lambda(\Lambda + 2)}{4(k + 2)} \tag{1}
\]

The fusion rules of the \(G^\Lambda\) are

\[
G^\Lambda \times G^{\Lambda'} = \bigoplus_{\Lambda'' = |\Lambda - \Lambda'|}^{\min\{\Lambda + \Lambda', 2k - \Lambda - \Lambda'\}} G^{\Lambda''} \tag{2}
\]

There is an affine descendant field of \(G^\Lambda\) for each of the states in the \(\widehat{sl}(2)\) module with highest weight \(\Lambda\). Among these descendants, there are infinitely many Virasoro primaries, which we may name \(G^\Lambda_\Lambda\). The field \(G^\Lambda_\Lambda\) is by definition the field of lowest conformal dimension among the affine descendants of \(G^\Lambda\) which carry \(sl(2)\)-weight \(\lambda\). Naturally, we have \(G^\Lambda = G^\Lambda_\Lambda\). Also, we have to demand that \((\Lambda - \lambda) = 0 \mod(2)\), otherwise the weight \(\lambda\) will not appear in the representation with highest weight \(\Lambda\). One may check (see for instance [21]) that all the \(G^\Lambda_\Lambda\), defined this way are indeed Virasoro primary. Their conformal weights are given by

\[
h^\Lambda_\Lambda = \frac{\Lambda(\Lambda + 2)}{4(k + 2)} + n^\Lambda_\Lambda \tag{3}
\]

where \(n_{\Lambda, \Lambda}\) is the lowest grade at which the weight \(\lambda\) appears in the affine Lie algebra representation of highest weight \(\Lambda\). If \(\lambda\) is a weight in the (ordinary) Lie algebra representation of highest weight \(\Lambda\), then \(n_{\Lambda, \Lambda}\) will be zero and we will have \(h_\Lambda = h^\Lambda_\Lambda\). The fusion rules of the \(G^\Lambda_\Lambda\) are easily obtained from (2) and the sum rule for weights in operator products. They are

\[
G^\Lambda_\Lambda \times G^{\Lambda'_\Lambda} = \bigoplus_{\Lambda'' = |\Lambda - \Lambda'|}^{\min\{\Lambda + \Lambda', 2k - \Lambda - \Lambda'\}} G^{\Lambda''}_{\Lambda + \Lambda'}. \tag{4}
\]

Now let us turn to the \(\mathbb{Z}_k\) parafermion theory, as described by the coset \(\widehat{sl}(2)_{k}/\widehat{U}(1)_{k}\). As usual for cosets, the Virasoro primary fields of the parafermion CFT may be labelled by a highest weight \(\Lambda\) of the horizontal algebra of the parent theory \((sl(2))\) and by a similar weight \(P\lambda\) of the embedded theory \((U(1))\), which is obtained by a projection matrix \(P\) from a weight \(\lambda\) of the parent theory. These weights moreover have to satisfy a branching condition, which ensures that the representation \(P\lambda\) of the embedded algebra can occur as a summand in the decomposition of the representation \(\Lambda\) of the parent algebra into representations of the embedded algebra. If we denote by \(M\) the root lattice of the horizontal algebra of the parent algebra, then this branching condition is

\[
P\lambda - P\lambda \in P M. \tag{5}
\]

In the case of \(\widehat{sl}(2)_{k}/\widehat{U}(1)_{k}\), the projection matrix is trivial and the branching rule just says that the difference of the weights \(\Lambda\) and \(\lambda\) has to be an element of the root lattice of \(sl(2)\) i.e. the difference of \(\Lambda\) and \(\lambda\) has to be an even number. Thus, the parafermion theory has Virasoro primaries \(\Phi^\Lambda_\lambda\) labelled by a highest weight \(0 \leq \Lambda \leq k\) of \(sl(2)\) and a weight \(\lambda\) of \(sl(2)\) for which we have \(\Lambda - \lambda = 0 \mod(2)\).
Since the parafermion fields $\Phi^\Lambda_\lambda$ are now labelled in the same way as the Virasoro primary fields $G^\Lambda_\lambda$ of the $\hat{sl}(2)_k$ theory, one might hope that there is a simple relation between these fields. In fact, it was pointed out already in [14] that each of the fields $G^\Lambda_\lambda$ may be written as the product of a field $\Phi^\Lambda_\lambda$ from the parafermion theory and a vertex operator of the $\hat{U}(1)_k$ theory, which is just the theory of a free boson on a circle of radius $\sqrt{2k}$. This was further clarified in [15], using the results of [22]. One has

$$G^\Lambda_\lambda = \Phi^\Lambda_\lambda e^{i\lambda \phi/\sqrt{2k}}$$  \hspace{1cm} (6)

From this relation, one immediately reads off that the field $\Phi^\Lambda_\lambda$ must have conformal weight $\left(h'\right)^\Lambda_\lambda$ given by

$$\left(h'\right)^\Lambda_\lambda = h^\Lambda_\lambda - \frac{\lambda^2}{4k} = \frac{\Lambda(\Lambda + 2)}{4(k + 2)} - \frac{\lambda^2}{4k} + n^\Lambda_\lambda$$  \hspace{1cm} (7)

As in other coset theories, the labelling of the fields $\Phi^\Lambda_\lambda$ as we introduced it above is redundant. First of all, the $\hat{U}(1)^{(1)}_k$ label $\lambda$ is usually taken to be defined modulo $2k$, since the (extended) $\hat{U}(1)^{(1)}_k$ characters $\chi^\lambda$ and $\chi^{\lambda + 2k}$, that correspond to the vertex operators $e^{i\lambda \phi/\sqrt{2k}}$ and $e^{i(\lambda + 2k) \phi/\sqrt{2k}}$, are equal (see for example [21]). Because of this and because of the fusion rules (9) below, the label $\lambda$ is called the $\mathbb{Z}_{2k}$ charge of the field $\Phi^\Lambda_\lambda$. Also, in order to get proper behaviour of the fields' characters under modular transformations, one has to identify fields whose labels are sent onto each other by an external automorphism of the parent algebra [23]. In the case at hand, this means that we have to identify $\Phi^\Lambda_\lambda$ with $\Phi^{k - \Lambda}_\lambda$. Collecting, we get the field identifications

$$\Phi^\Lambda_\lambda \equiv \Phi^{\Lambda + 2k}_\lambda$$
$$\Phi^\Lambda_\lambda \equiv \Phi^{k - \Lambda}_{\lambda - k}.$$  \hspace{1cm} (8)

Using these identifications, we can choose a labelling of the primaries such that $\lambda$ is a weight in the representation $\Lambda$ of $sl(2)$, i.e. $-\Lambda \leq \lambda \leq \Lambda$. In fact, we may require $-\Lambda < \lambda \leq \Lambda$ and if we do this then every set of labels corresponds uniquely to a Virasoro primary. Thus, the number of Virasoro primaries is $\frac{1}{2}k(k + 1)$ (note: there are only $k$ primaries of the full parafermion algebra: the fields $\Phi^\Lambda_\lambda$).

One may check that the conformal weights given in (7) are equal for identified fields. Also, note that the grade $n^\Lambda_\lambda$ in (7) is zero if the labels $(\Lambda, \lambda)$ are in the range chosen above. Using the factorisation (8) and the field identifications, we may now also write down the fusion rules for the parafermion fields. They are

$$\Phi^\Lambda_\lambda \times \Phi'^\Lambda_\lambda' = \bigoplus_{\Lambda'' = |\Lambda - \Lambda'|}^{\min\{\Lambda + \Lambda', 2k - \Lambda - \Lambda''\}} \Phi^{\Lambda'\prime}_{\Lambda + \Lambda'}.$$  \hspace{1cm} (9)

In other words, they are the same as the fusion rules for the $G^\Lambda_\lambda$, except that the labels on the right hand side have to be brought back into the set chosen above, using the field identifications (8).

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\(^1\)Note that in the original parafermion theory of [14], there was a $\mathbb{Z}_k \times \mathbb{Z}_k$ symmetry. The $\mathbb{Z}_k \times \mathbb{Z}_k$ charge $(l, \bar{l})$ of the field $\Phi^\Lambda(z)\Phi^\Lambda(\bar{z})$ was given by $l = \frac{1}{2}(\Lambda + \bar{\Lambda})$, $\bar{l} = \frac{1}{2}(\Lambda - \bar{\Lambda})$, so that clearly in this theory, one needed $\Lambda + \bar{\Lambda}$ to be even. Here, we will not require this and thus allow chiral fields like $\Phi^1_1$. 

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2.1.2 The coset \( \hat{sl}(k)_1 \times \hat{sl}(k)_1 / \hat{sl}(k)_2 \)

We now turn to the coset \( \hat{sl}(k)_1 \times \hat{sl}(k)_1 / \hat{sl}(k)_2 \). This coset is a special case of the general class considered in [7], [13]. Its current algebra is denoted as a \( W \)-algebra and much is known about them. In the quantum Hall application however, the parafermion analyses is more directly relevant and applicable as is described in some detail in [13]. Nevertheless one should keep in mind that the discussion of the braid group representations that feature in these models and which will be extensively analysed later on, readily extend to the \( W \) representation theory. The Virasoro primaries of the coset may be labelled by an \( \hat{sl}(k)_1 \times \hat{sl}(k)_1 \) weight (or, equivalently, two \( \hat{sl}(k)_1 \) weights) and an \( \hat{sl}(k)_2 \) weight. Let us call the \( \hat{sl}(k)_1 \) weights \( \mu_1 \) and \( \mu_2 \) and the \( \hat{sl}(k)_2 \) weight \( \mu \), then we can write \( \Phi_{\mu_1,\mu_2}^\mu \). The weights \( \mu_1, \mu_2 \) and \( \mu \) once again have to satisfy the branching condition (5). In this case, the projection \( \mathcal{P} \) maps \( (\mu_1, \mu_2) \) onto \( \mu_1 + \mu_2 \) and it maps the root lattice of \( \hat{sl}(k) \times \hat{sl}(k) \) onto the root lattice of \( \hat{sl}(k) \). Hence we have the following requirement

\[
\mu_1 + \mu_2 - \mu \in M_{\hat{sl}(k)}
\]  

(10)

where \( M_{\hat{sl}(k)} \) is the root lattice of \( \hat{sl}(k) \). In other words, the weights \( \mu_1 + \mu_2 \) and \( \mu \) should be in the same conjugacy class (for details on this concept see for example [24, 21]). In terms of the Dynkin labels of the weights, this means that one has

\[
\sum_{j=1}^{k-1} j(\mu_1^{(j)} + \mu_2^{(j)} - \mu^{(j)}) = 0 \mod k
\]  

(11)

Now denote by \( e_i \) the \( \hat{sl}(k) \) weight whose Dynkin labels \( e_i^{(j)} \) are given by \( e_i^{(j)} = \delta_{ij} \) (These correspond to the fundamental representations of \( \hat{sl}(k) \)). Then \( \mu_1 \) is either zero or equal to one of the \( e_i \), since it is a level one weight. The same goes for \( \mu_2 \). For the level two weight \( \mu \), there are three possibilities: it can be zero, equal to one of the \( e_i \) or equal to the sum of two of the \( e_i \) (which may be the same). If we define \( e_0 = 0 \), then we may simplify this description and say that \( \mu_1 \) and \( \mu_2 \) will equal one of the \( e_i \) and \( \mu \) will equal the sum of two of the \( e_i \) (where \( i \in \{0, \ldots, k-1\} \)). The branching rule above then states that only triples \( (\mu_1, \mu_2, \mu_3) \) of the form \( (e_l, e_{m+n-l} \mod k, e_m + e_n) \) are admissible. This leaves \( \frac{1}{2}k^2(k+1) \) admissible triples. However, there are also field identifications, induced by the external automorphisms of \( \hat{sl}(k)_1 \times \hat{sl}(k)_1 \). These identifications take the form

\[
\Phi_{e_l+e_m}^{e_i,e_j} \equiv \Phi_{e_{l+s}+e_{j+s}}^{e_{i+s}+e_{j+s}}
\]  

(12)

for \( s \in \{1, \ldots, k-1\} \). The sums in the indices on the right hand side have to be taken modulo \( k \). Using these identifications, we can choose to set either \( \mu_1 \) or \( \mu_2 \) to zero. Say we set \( \mu_1 \) to zero. Then we are left with the triples \( (0, e_{m+n} \mod k, e_m + e_n) \). Clearly, \( \mu_2 \) is now uniquely determined by \( \mu \) and we may choose to label the fields by the \( \hat{sl}(k)_2 \) weight only: \( \Phi_\mu \). Every \( \hat{sl}(k)_2 \) weight is admissible and we are left with as many Virasoro primary fields as there are \( \hat{sl}(k)_2 \) weights: \( \frac{1}{2}k(k+1) \). This is just a reduction of the number of fields before identification by a factor of \( k \), as should be expected. Also, we get the same number of fields that we got in the other coset description of the parafermions.

The fractional part of the conformal weight of the field \( \Phi_{\mu_1,\mu_2}^\mu \) can be calculated directly from the coset description; it is the same as the fractional part of the difference between the conformal weight of field with labels \( (\mu_1, \mu_2) \) in the parent theory and the conformal
weight of the field with label \( \mu \) in the embedded theory. One may show that this recipe always yields the same fractional part, independently of the labels \( \mu_1, \mu_2, \mu \) that are chosen to represent a certain field (i.e., labels that are identified through (12) yield the same fractional part). Let us look at the field \( \Phi_{e_m+e_n} \), with \( m \leq n \). A particularly convenient choice of labels for this field, made in [13], is \((e_{k-n}, e_m, e_{k+m-n})\). The conformal dimension of the WZW-field labelled by the weight \( e_m \) is given by

\[
\Delta_p(e_m) = \frac{(e_m, e_m + 2\rho)}{2(p + k)} = \frac{m(k - m)(k + 1)}{2k(p + k)},
\]

where \( \rho \) is the Weyl-vector of \( sl(k) \) and \( p \) is the level (here, we have \( p=1 \) or \( p=2 \)). From this, we find

\[
\Delta_1(e_{k-n}) + \Delta_1(e_m) - \Delta_2(e_{k+m-n}) = \frac{m(k - n)}{2} + \frac{(n - m)(k + m - n)}{2k(k + 2)}
\]

\[
= \frac{(k + m - n)(k + m - n + 2)}{4(k + 2)} - \frac{(m + n - k)^2}{4k}.
\]

The middle expression is the one given in [13] and from the last expression, we see that it is equal to the weight of the field \( \Phi^\Lambda_\lambda \) with \( \Lambda = k + m - n \) and \( \lambda = m + n - k \) (cf. formula (11)). Thus, we have the correspondence

\[
\Phi^{k+m-n}_{m+n-k} \equiv \Phi_{e_m+e_n} \iff \Phi^\Lambda_\lambda \equiv \Phi_{e_{\Lambda+\lambda}+e_{2k-\Lambda+\lambda}}
\]

which is further supported by the fact that these fields have the same fusion rules. In fact, the \( \Phi_p \) fusion rules are the same as the fusion rules for the corresponding \( sl(k)_2 \) representations and these are the same as the fusion rules of the \( sl(2)_k/\bar{U}(1)_k \) coset as a consequence of level-rank duality (see [21] and references therein). One may also find the equality of the fusion rules directly by looking at the fusion rules of the field \( \Phi^1_1 \equiv \Phi_{e_1} \) with an arbitrary field. These fusion rules are easily seen to be the same and since the field \( \Phi^1_1 \) generates all the fields in the theory by repeated fusion, it follows that the fusion rules of all the fields that are identified through (15) are the same in both cosets.

### 2.2 Definition of the RR-states

As we mentioned before, the wave functions for the RR-states can be described (or even defined) as conformal blocks in a certain Conformal field theory. The CFT in question is the tensor product of the parafermion theory with the theory of a chiral boson. From the parafermion theory, one needs the chiral vertex operators \( \sigma := \Phi^1_1 = \Phi_{e_1} \) and \( \psi := \Phi^0_0 = \Phi_{2e_1} \) defined above and from the bosonic theory one uses the usual vertex operators \( e^{i\alpha \zeta} \), of weight \( \frac{1}{2} \alpha^2 \). Here, \( \zeta \) is the bosonic field (which should not be confused with the bosonic field \( \phi \) in the factorisation formula (11)). Electrons and quasiholes are represented by products of these bosonic and parafermionic vertex operators. Specifically, the electron is represented by the product \( \psi e^{i\sqrt{\frac{\alpha M}{k}}} \zeta \), where \( M \in \mathbb{Z} \) and the quasihole is represented by the product \( \sigma e^{i\sqrt{\frac{\alpha M}{k}}} \zeta \). These combinations of bosonic and parafermionic fields are not arbitrary; if the parafermion factor is given, then the bosonic factor for the

\[\text{Note that we could also identify the field } \Phi_{e_m+e_n} \text{ with the field } \Phi^{k+m-n}_{m+n-k}, \text{ which is the conjugate of the field } \Phi^{k+m-n}_{k-m-n}. \text{ It is impossible to decide between these identifications on the level of conformal weights and fusion rules.} \]
electron can be fixed by requiring that electrons are mutually local (that is, the OPE of two electron operators does not have a branch cut). This requirement is needed to make sure that the wave functions defined below are single valued in the electrons’ coordinates. If the electron is to have half-integer spin, then one should also require that $M$ be odd.

The exponent of the bosonic factor for the quasihole is fixed up to integer times $\sqrt{\frac{k}{kM+2}}\zeta$ by the requirement that the quasihole and the electron are mutually local.

The linear space of RR-states $\Psi_{N,n}^k$ which have $N$ electrons with coordinates $z_1, \ldots, z_N$ and $n$ quasiholes located at positions $w_1, \ldots, w_n$ is now generated by the conformal blocks of a correlator of $N$ electron fields and $n$ quasihole fields inserted at these positions and supplemented by a homogeneously spread positive background charge, which ensures overall charge neutrality \[.\] This correlator may be factorized into parafermion and boson correlators, the latter of which may be evaluated explicitly (paying due attention to the background charge \[]), after which one obtains

$$
\Psi_{N,n}^k(z_1, \ldots, z_N, w_1, \ldots, w_n) = \langle \sigma(w_1) \ldots \sigma(w_n) \psi(z_1) \ldots \psi(z_N) \rangle \times \prod_{i<j}(z_i - z_j)^{M+2/k} \prod_{i=1}^{N} \prod_{j=1}^{n} (z_i - w_j)^{1/k} \times \prod_{i<j}(w_i - w_j)^{1/k} F_g(z_1, \ldots, z_N, w_1, \ldots, w_n) \tag{16}
$$

Here, the $z_i$ and $w_i$ are complex coordinates which parametrize the sample. $F_g$ is a factor which depends on the geometry of the sample. If the sample is a disc then this factor just implements the usual Gaussian factors which confine the electrons and quasiholes to the disc. If the sample is a sphere of radius $R$, then the $z_i$ and $w_i$ are related to the usual coordinates on the sphere through stereographic projection (see \[ for details) and one has

$$
F_g(z_1, \ldots, z_N, w_1, \ldots, w_n) = \prod_i (1 + |z_i|^2/4R^2)^{-N_o/2-1} \prod_j (1 + |w_j|^2/4(kM+2)R^2)^{-N_o/2-1} \tag{17}
$$

where $N_o = N/\nu - M - 2 + n$ is the number of flux quanta that go through the surface of the sphere and $\nu = \frac{k}{kM+2}$ is the filling fraction.

The construction of the Pfaffian and RR-states through conformal field theory is analogous to the construction of the Laughlin state and the Pfaffian state through conformal field theory as given in \[. Indeed, the $k = 2$ case of the above wave function just describes the Pfaffian state of \[ with $N$ electrons and $n$ quasiholes.

### 2.3 Fusion of quasiholes and the Bratteli diagram

It is interesting to know the number of independent states which the formula \[ encodes, i.e. the number of independent states with $N$ electrons that have $n$ quasiholes at fixed positions $w_1, \ldots, w_n$. This interest is twofold. First of all, we want to know which combinations $(N, n)$ are allowed. Second, the number of independent states is also the dimension of the braid group representation that governs the exchanges of electrons and quasiholes. Hence a necessary condition for nonabelian braiding is that it be larger than
A basis for the space of states that we are looking for is given by the states we obtain if we replace the parafermion correlator in (16) by its respective conformal blocks. The number of such blocks is equal to the number of fusion channels that make the correlator in (16) non-vanishing. Hence, the number we are looking for is just the number of ways in which $N$ electron fields $\psi$ and $n$ quasihole $\sigma$ fields may fuse into the vacuum.

Now the fusion of the $\psi$ fields is very simple; it just corresponds to addition of the $\mathbb{Z}_{2k}$ charges. Hence the $N$ electron fields fuse into the $\Phi^0_{2N} = \Phi_{2eN}$ sector. The fusion rules of the sigma fields, as given in equation (9), are a bit more complicated, but they have a nice graphical description in terms of a Bratteli diagram (see figures 1,2):

![Figure 1: fusion diagram for the field $\sigma$. The diagram must be thought extended indefinitely in the $\lambda$-direction and up to $\Lambda = k$ in the $\Lambda$-direction (the case $k = 3$ is as drawn here). On each line, we have drawn the Young diagram of the $sl(2)$ representation that resides on that line.]

![Figure 2: The same diagram as in figure 1, but this time each site in the diagram is labelled by the Young diagram for the $sl(k = 3)_2$ weight of the field that resides there. The dot represents the empty diagram. Again, generalisation to arbitrary $k$ is straightforward. Note that in this picture, the weights label the fields unambiguously, whereas in figure 1, one still has to take the field identifications (8) into account]

These diagrams must be read as follows: Each starting point or end point of an arrow has coordinates $(\Lambda, \lambda)$ and represents the $\Phi^0_{\Lambda, \lambda}$ sector of the parafermion CFT. Note that this means that coordinates related by the identifications (8) represent the same sector. In figure 2, one may see this explicitly for $k = 3$. Here, we have at each node of the
The correspondence between the fields of the parafermionic theory and such weights or diagrams is one to one and we see that the same diagram appears in different places. The fusion rules of the sigma field are encoded in the arrows; we start in the lower left corner, i.e. the $\Phi^0_0$ sector which is the vacuum sector of the theory. Then we take the operator product expansion with the field $\sigma = \Phi^1_1$, which naturally, following the arrow, lands us in the $\Phi^1_1$ sector. Once more taking the OPE with $\sigma$, we end up, following the arrows, in the $\Phi^2_2$ or in the $\Phi^0_2$ sector. In this way, each path of length $n$ through the diagram represents a fusion channel for $n \sigma$-fields.

To make the correlator in the wave function non vanishing, all the quasiholes and electron fields need to fuse into the vacuum sector. Now since the electron fields

$$\psi(z_1), \ldots, \psi(z_N)$$

in the correlator above fuse to $\Phi^0_{2N}$, the quasihole fields $\sigma(w_1), \ldots, \sigma(z_n)$ have to fuse to the field $\Phi^0_{-2N} = \Phi^k_{k-2N}$. The number of ways to do this is just the number of paths of length $n$ through the diagram of figure 1 which end up at a point whose co-ordinates $(\Lambda, \lambda)$ satisfy either $(\Lambda, \lambda) = (0, -2N \mod 2k)$ or $(\Lambda, \lambda) = (k, k - 2N \mod 2k)$. Clearly, for fixed $N$, such paths occur only for values of $n$ which are a multiple of $k$ apart, so quasiholes can only be created in multiples of $k$ at a time (maybe with the exception of the first few quasiholes if $N$ is not a multiple of $k$). Note that, although the same fields (or sectors) occur in different heights in the diagram, the same field never occurs more than once at given $\lambda$ and hence different paths are never identified by the field identifications. Thus, the number of fusion channels for the parafermion CFTs is the same as that for the corresponding WZW-theories.

### 2.4 Counting the independent $n$-quasiho!e states

Let us denote the number of paths through the Bratteli diagram which end up at the point $(\Lambda, n)$ by $D(\Lambda, n)$. Also, let us define $D(\Lambda, n) = 0$ if there is no point with coordinates $(\Lambda, n)$. The number of independent $n$-quasiho!e states encoded by (14) is then $D(0, n)$ in case $2N + n = 0 \pmod{2k}$, $D(k, n)$ in case $2N + n = k \pmod{2k}$, and zero otherwise. It should be obvious from looking at the Bratteli diagram that the $D(\Lambda, n)$ satisfy the following recursion relation:

$$D(\Lambda, n) = D(\Lambda - 1, n - 1) + D(\Lambda + 1, n - 1)$$

(18)

Using this relation and the fact that $D(1, 1)$ equals one, $D(\Lambda, n)$ can be easily calculated in each particular case. At least for low $k$, the recursion relation can also be used to prove simple closed expressions for the $D(\Lambda, n)$. In particular, we find for $k = 2$, $k = 3$ and $k = 4$

\[
\begin{align*}
D_2(0, 2n) &= D_2(1, 2n - 1) = 2^{n-1} \\
D_3(0, 2n) &= D_3(1, 2n - 1) = \text{Fib}(2n - 2) \\
D_3(2, 2n) &= D_3(3, 2n + 1) = \text{Fib}(2n - 1) \\
D_4(0, 2n) &= D_4(1, 2n - 1) = \frac{3^{n-1} + 1}{2} \\
D_4(2, 2n) &= 3^{n-1} \\
D_4(3, 2n + 1) &= D_4(4, 2n + 2) = \frac{3^{n-1} - 1}{2}
\end{align*}
\]
In these equations, we have written $D_k$ in stead of $D$ for clarity and we have used the notation Fib$(n)$ to denote the $n^{th}$ Fibonacci number, defined by

\[
\begin{align*}
\text{Fib}(0) &= \text{Fib}(1) = 1 \\
\text{Fib}(n + 1) &= \text{Fib}(n) + \text{Fib}(n - 1)
\end{align*}
\] (20)

It is also not that difficult to find and prove a closed formula for infinite $k$. We have

\[
D_\infty(\Lambda, n) = \Lambda + 1 \left( \frac{n + 1}{n - \Lambda} \right) \quad (n + \Lambda = 0 \pmod{2})
\] (21)

Of course this formula is valid for all $k$ as long as $n + \Lambda \leq 2k$.

To get formulae for other values of $k$ it is more convenient to rewrite the recursion relation (18) in matrix form. We consider the $D(\Lambda, n)$ at a fixed $n$ together as a $k$-vector and write the step from $n$ to $n + 1$ as multiplication with a $(k + 1) \times (k + 1)$ matrix $M_k$. that is, we have

\[
\begin{pmatrix}
D(0, n + 1) \\
\vdots \\
D(k, n + 1)
\end{pmatrix}
= M_k
\begin{pmatrix}
D(0, n) \\
\vdots \\
D(k, n)
\end{pmatrix}
\] (22)

where $M_k$ is given by

\[
(M_k)_{ij} = \delta_{i,j+1} + \delta_{i+1,j}
\] (23)

The asymptotic behaviour of the $D(\Lambda, n)$ for large $n$ will be related to the largest eigenvalue of the matrix $M_k$. The eigenvalues of the $M_i$ are just the zeros of their characteristic polynomials $P_k$. For these, we can easily deduce a recursion relation and “initial conditions:”

\[
\begin{align*}
P_2(\lambda) &= \lambda^2 - 1 \\
P_3(\lambda) &= \lambda^3 - 2\lambda \\
P_{i+1}(\lambda) &= \lambda P_i(\lambda) - P_{i-1}(\lambda)
\end{align*}
\] (24)

but these are just the defining relations for the Chebyshev polynomials, whose zeros are given by (see for example [25])

\[
\lambda_{k,m} = 2 \cos \left( \frac{(m + 1)\pi}{k + 2} \right)
\] (25)

Since we know all the eigenvalues of $M_k$, we can now in principle solve for the eigenvectors and using the solution, give explicit formulae for the $D_k(\Lambda, n)$ for any $k$. We will however content ourselves with giving the asymptotic behaviour of the $D_k(\Lambda, n)$ at large $n$. The largest eigenvalues (in absolute value) of the matrix $M_k$ are clearly $\lambda_0$ and $\lambda_k = -\lambda_0$. Hence, the asymptotic behaviour of the $D_k(\Lambda, n)$ is given by

\[
\begin{align*}
D_k(\Lambda, n) &\sim \left( 2 \cos \left( \frac{\pi}{k + 2} \right) \right)^n \quad (\Lambda + n \text{ even}) \\
D_k(\Lambda, n) &= 0 \quad (\Lambda + n \text{ odd})
\end{align*}
\] (26)

This conforms with the closed formulae we gave for $k = 2, 3, 4$.  

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2.5 Braiding for $k = 2$

In the previous section, we have calculated the dimensions of the braid group representations that govern the exchanges of the electrons and the quasiholes of the RR-states. We have seen that these dimensions increase with the number of quasiholes, which is an indication for nonabelian braiding. However, this indication is not conclusive evidence. To be sure, one needs to calculate the actual matrices that describe the braiding of the $\sigma$-fields in the conformal block in formula (16) above. Nayak and Wilczek [2] have done this calculation for the case $k = 2$ (the Pfaffian). The method they used was basically to compute the conformal block for four quasihole fields explicitly and then to extend the resulting braid group representation to a braid group representation for any even number of quasiholes. For general $k$, it is quite difficult to calculate conformal blocks for four, let alone for arbitrary numbers of quasiholes in the case. Fortunately it turns out that we can circumvent this problem by using the known duality between conformal field theory and quantum groups and using this, we will give a nice description of the braiding for arbitrary $k$. However, we will first briefly recall the results of Nayak and Wilczek for $k = 2$, for later reference.

The braid group representation for $n = 2m$ quasiholes has dimension $2^{m-1}$ (cf. (19)). Nayak and Wilczek describe this space as a subspace of a tensor product of $m$ two-dimensional spaces. Each of the two dimensional spaces has basis vectors $\{|+\rangle, \{-\rangle\}$ and the physical subspace of the tensor product is the space generated by the vectors whose overall sign is positive (so for $m = 2$, $|--\rangle$ is physical, but $|+--\rangle$ is not). On the tensor product space, there is a spinor representation of $SO(2m) \times U(1)$. The $U(1)$ acts as multiplicative factor, while the generators $\sigma_{ij}$ of the $SO(2m)$ may be written in terms of the Pauli matrices $\sigma_i$. We have

$$\sigma_{ij} = \frac{1}{4}i[\gamma_i, \gamma_j]$$

with

$$\gamma_1 = \sigma_1 \otimes \sigma_3 \otimes \ldots \otimes \sigma_3$$
$$\gamma_2 = \sigma_2 \otimes \sigma_3 \otimes \ldots \otimes \sigma_3$$
$$\gamma_3 = 1 \otimes \sigma_1 \otimes \sigma_3 \otimes \ldots \otimes \sigma_3$$
$$\gamma_4 = 1 \otimes \sigma_2 \otimes \sigma_3 \otimes \ldots \otimes \sigma_3$$
$$\vdots$$
$$\gamma_{2m} = 1 \otimes \ldots \otimes 1 \otimes \sigma_2$$

Here the states $|+\rangle$ and $|--\rangle$ are the spin up and spin down states for the Pauli matrices.

Now recall that the braid group $B_n$ is generated by $n - 1$ elementary exchanges $\tau_1, \ldots, \tau_n$ subject to the relations

$$\tau_i \tau_j = \tau_j \tau_i \quad (|i - j| \geq 2)$$
$$\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}$$

Note that the four point blocks in the case $k = 2$ are just the four point blocks for the chiral Ising model, which have, within a different context, been known for a long time (see for instance [26] for explicit expressions). The same is true for the corresponding braid group representations. However, the embedding of the resulting braid group representation into a rotation group, as given by Nayak and Wilczek (see below) seems to be new.
Here, $\tau_i$ represents an exchange of quasihole $i$ and quasihole $i+1$. The action of the braid group on the $n$-quasihole space is embedded in the action of $SO(2n) \times U(1)$, as follows:

$$\tau_i \equiv e^{i\pi i} e^{i\pi \sigma_{i,i+1}}$$

(30)

The $SO(2m)$ generators $\sigma_{i,i+1}$ which appear in this equation are given by

$$\begin{align*}
\sigma_{1,2} &= \frac{1}{2} \sigma_3 \otimes 1 \otimes \ldots \otimes 1 \\
\sigma_{2,3} &= \frac{1}{2} \sigma_2 \otimes \sigma_1 \otimes 1 \otimes \ldots \otimes 1 \\
\sigma_{3,4} &= \frac{1}{2} \sigma_1 \otimes \sigma_3 \otimes 1 \otimes \ldots \otimes 1 \\
\sigma_{4,5} &= \frac{1}{2} \sigma_2 \otimes \sigma_2 \otimes 1 \otimes \ldots \otimes 1
\end{align*}$$

(31)

So we see that, for odd $i$, $\tau_i$ acts only on the $i$th tensor factor, whereas for even $i$, $\tau_i$ acts only on the $(i-1)$th and $i$th tensor factors. Moreover, the $2 \times 2$-matrix by which describes the action for even $i$ and the $4 \times 4$-matrix which describes it for odd $i$ do not vary with $i$. Explicitly, they are given by

$$\begin{align*}
\tau_{2i+1} &\equiv \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \\
\tau_{2i} &\equiv \frac{1}{2} \begin{pmatrix} 1+i & 0 & 0 & -1+i \\ 0 & 1+i & 1-i & 0 \\ 0 & 1-i & 1+i & 0 \\ -1+i & 0 & 0 & 1+i \end{pmatrix}
\end{align*}$$

(32)

3 The quantum group picture

In this extensive section, we give a description of the braiding for a system of $n$ particles with a hidden quantum group symmetry. We expect that the braiding properties of a quantum Hall state with $n$ quasiholes is conveniently described in terms of such a system.

In the first subsection, we motivate such a description and mention some general features. In the remaining subsections, we give a fairly detailed description to the quantum group $U_q(sl(2))$ and its representation theory for $q$ a root of unity, which culminates in an explicit description of the braid group representations that are associated to this quantum group. We are well aware of the fact that most of the material treated in this section is not new, except possibly for the identities (12) for the $6j$-symbols, however the input came from quite a variety of sources and putting it together in comprehensive way seemed a nontrivial and even useful thing to do. It should serve as a quick introduction to braiding in systems with hidden quantum group symmetry, which is why we have tried to keep the treatment as self-contained as possible.

3.1 Using a quantum group rather than the full CFT

One may always choose to describe a quantum system in terms of its explicit wave functions but we know that it can be extremely profitable to exploit its operator algebra, in particular its symmetries. These allow one to extract many of the physical features without reference to the explicit realisation in terms of wave functions. Quite similarly one could in the present context remark that there is an aspect of the description of the Read-Rezayi states that is less than satisfactory: one has to use the full machinery of a (conformal) field theory to calculate wave functions or even just braiding properties for a finite number of quasiholes and electrons. There are many questions one may want to answer for which this seems like overkill: for example one would hope to be able to
describe the braiding of finitely many particles by means of a theory with only finitely many degrees of freedom. Indeed, there is such an alternative description and we pursue it here. It is well known that conformal field theories possess a hidden quantum group symmetry (see section 4 for details and references). What we propose is to describe the electrons and quasiholes of a quantum Hall state that would usually be described by a certain CFT as localised particles that carry representations of the quantum group that is associated with this CFT. Such a description has several advantages.

- It avoids the introduction of a field theory to describe a system with only a finite number of particle degrees of freedom.

- It provides a conceptual understanding of a phenomenon which emerges in the usual CFT description. This is the fact that, while a state with a low number of indistinguishable quasiholes can be described with a one component wave function, a system with a higher number of these quasiholes may need a several component wave function. Clearly, it should only be possible to distinguish between the components of the wave function by making a measurement that involves several holes (otherwise the holes would not be indistinguishable). Hence, there should be operators in the many hole Hilbert space that distinguish states that cannot be distinguished by operators that act only on the state of one of the particles. The quantum group picture provides these in a natural way. They are the operators that correspond to the global (i.e. invariant) quantum group charges of groups of quasiholes. Even though all individual quasiholes have the same quantum group charge i.e. belong to the same representation, a group of \( n \) such holes can occur in different representations leading to distinguishable \( n \)-hole states. As a simple example, suppose that the quasiholes carried the two dimensional representation of \( SU(2) \) (or equivalently, of the quantum group \( U(sl(2)) \)). In that case a two quasihole state could be either in the singlet or in the triplet representation and the singlet states could be distinguished from the triplet states by measuring the global charge.

- The quantum group picture allows for a very elegant description of the braiding properties of the \( n \)-quasihole states; all braiding properties are encoded into a single algebraic object: the quantum group’s \( R \)-matrix (cf. section 3.2.6). Starting from the \( R \)-matrix, braiding calculations can be done in a purely algebraic way and often a detailed picture of the braid group representation that governs the exchanges of particles can be constructed. In a CFT description, the information contained in the \( R \)-matrix of the quantum group would be much less manifest. In fact, to extract it from this description of the system, one would have to calculate the braiding and fusion matrices starting from the conformal blocks of the CFT, which is usually quite hard.

Of course the description we propose also has its disadvantages when compared to the CFT description. For instance, it seems much harder to describe dynamical aspects of the quantum Hall states in this picture. Still, we like to emphasise that the quantum group picture we propose is a useful complementary way of thinking about nonabelian quantum Hall states.
3.2 $U_q(sl(2))$ for pedestrians

In this section we have collected some basic information about quantum groups in general and the quantum group $U_q(sl(2))$ in particular. For those well versed in math there are excellent text books, for example [27], covering the full story.

3.2.1 The algebra and its unitary representations

Quantum groups are actually not groups, but algebras. To be a quantum group, an algebra has to have several structures associated with it. As an algebra, it is already a vector space with a multiplication, but next to this, there should also be a comultiplication, counit, antipode and *-structure, (which makes the algebra a Hopf-*-algebra), an $R$-matrix (which makes it a quasitriangular Hopf-*-algebra) and optionally a coassociator (which would make it a quasitriangular quasi-Hopf-*-algebra or something with a longer name). We briefly describe these structures and their use in our context in the following sections, always using the quantum group $U_q(sl(2))$ as an example. In this section, we will describe the algebra $U_q(sl(2))$ and its irreducible representations.

$U_q(sl(2))$ can be defined as the algebra generated by a unit $1$ and the three elements $H, L^+, L^-$. These satisfy the relations

$$[H, L^\pm] = \pm 2L^\pm$$
$$[L^+, L^-] = \frac{q^{H/2} - q^{-H/2}}{q^{1/2} - q^{-1/2}}$$

(33)

where $q$ may be set to any non-zero complex value. One may check that these relations reduce to those of the Lie algebra $sl(2)$ when $q$ goes to one. Hence, $U_1(sl(2))$ is just the universal enveloping algebra $U(sl(2))$ of $sl(2)$ and we say that $U_q(sl(2))$ is a $q$-deformation of $U(sl(2))$.

If $q$ is taken as a formal variable and also for all $q \in \mathbb{C}$, $q$ not a root of unity, the representation theory of $U_q(sl(2))$ is very similar to that of $U(sl(2))$. For each non-negative $j \in \frac{1}{2}\mathbb{Z}$ there is an irreducible highest weight representation of dimension $2j + 1$. We will denote this representation by $\pi^\Lambda$, where $\Lambda = d - 1 = 2j$ is the highest weight. The modules $V^\Lambda$ of these representations have an orthonormal basis that consists of kets $|j, m\rangle$, with $m = -j, -j + 1, \ldots, j$ and the generating elements $H, L^+$ and $L^-$ act on this basis as follows

$$H|j, m\rangle = 2m|j, m\rangle$$
$$L^\pm|j, m\rangle = \sqrt{|j \pm m\rangle_q [j \pm m + 1\rangle_q} |j, m \pm 1\rangle$$

(34)

Here, the $q$-number $|m\rangle_q$ is defined as

$$|m\rangle_q = \frac{q^{m/2} - q^{-m/2}}{q^{1/2} - q^{-1/2}}$$

(35)

These $q$-numbers enter the formulae for the representations through the commutation relation of $L^+$ and $L^-$, the right hand side of which can be written $[H]_q$. The $q$-number $|m\rangle_q$ approaches $m$ when $q$ goes to one and hence we see that the representations given
above reduce to the usual $U(sl(2))$ representations for $q = 1$. Furthermore, when $q \in \mathbb{R}$ or $|q| = 1$, the $q$-number $[x]_q$ is a real number when $x$ is a real number.

We will call a representation of $U_q(sl(2))$ unitary when $(L^+)^\dagger = L^-$, $H^\dagger = H$. More formally, we may define an antilinear algebra antihomomorphism $*$ on $U_q(sl(2))$ by

$$ *(L^\pm) = L^\mp, \quad *(H) = H $$

and a representation $\rho$ of $U_q(sl(2))$ is then called unitary, or a $*$-representation, exactly if

$$ \forall x \in U_q(sl(2)) : \rho(*(x)) = \rho(x)^\dagger. $$

In the representations defined above, we always have $\pi^\Lambda(H)^\dagger = \pi^\Lambda(H)$ and $\pi^\Lambda(L^\pm)^\dagger = \pi^\Lambda(L^\mp)$, where the bar denotes complex conjugation of the elements. Hence, these representations are unitary when the matrix elements of $L^\pm$ are real, i.e. if the square root in $\Lambda$ is real for all admissible values of $m$. This will certainly be the case if $q$ is real and positive and also if $q = e^{i\phi}$ with $\phi \in \mathbb{R}$, $|\phi| \leq \frac{2\pi}{2j+1}$. Thus we see that, for real $q$, all the representations above are $*$-representations, while for $q = e^{i\phi}$, the representations $\pi^{2j}$ with $|\phi| \leq \frac{2\pi}{2j+1}$ are $*$-representations.

The $q$-numbers satisfy many identities which are useful in representation theoretic calculations. Two examples of such identities, which hold for all $q \in \mathbb{C}$ are

$$ q^{n/2}[m]_q + q^{-m/2}[n]_q = [m+n]_q $$
$$ [n+m]_q[n-m]_q = [n]_q^2 - [m]_q^2. $$ (38)

### 3.2.2 Representations at $q = e^{2i\pi/(k+2)}$

When $q$ is set equal to a root of unity, the properties of most of the representations defined by (34) change quite drastically. Specifically, at $q = e^{2i\pi/(k+2)}$, the representations $\pi^{2j}$ with $j > \frac{k+1}{2}$ will no longer be irreducible. This can be traced back to the fact that $q = e^{2\pi i/(k+2)}$, the $q$-numbers satisfy the extra identity

$$ [m+k+2]_q = -[m]_q $$

and more specifically

$$ [k+2]_q = 0 $$

Because of this, one has $(L^+)^{k+2} = (L^-)^{k+2} = 0$ in all the representations defined by (34). Of course, in the representations with $j < \frac{k+2}{2}$, this was already the case and for these representations, nothing essential changes. In particular, they are still irreducible. However, in the representations with $j \geq \frac{k+2}{2}$, there will now be extra highest and lowest weight states, which are annihilated by $L^+$ resp. $L^-$. For example, the state $(L^+)^{k+1}|j,-j\rangle$ in the module $V^{2j}$ of the representation $\pi^{2j}$, $|j \leq \frac{k+2}{2}\rangle$ will now be an extra highest weight state, since $(L^+)^{k+2} = 0$ in this representation. The descendants of this highest weight state (that is, the states which can be obtained from it by applying powers of $L^-$) now span an invariant subspace $W$ of $V^{2j}$, so $\pi^{2j}$ is no longer irreducible. Figure 3.2.2 illustrates this situation in a simple case.

Although the module $\pi^{2j}$ is now reducible, it can not be written as a direct sum of irreducibles. One says that it is indecomposable. This indecomposability is directly related to the fact that $\pi^{2j}$ is not a $*$-representation. For a $*$-representation, the orthogonal
Figure 3: Diagram of an indecomposable representation as defined by (34). The dots represent the basis states $|j,m\rangle$, in particular, we have written $|h\rangle$ for the highest weight state and $|l\rangle$ for the lowest weight state. The arrows $\rightarrow$ and $\leftarrow$ indicate the action of $L^+$ and $L^-$ resp.

complement of an invariant submodule of the representation module is itself invariant and this guarantees that any finite dimensional representation has an orthogonal decomposition into irreducibles. The fact that $\pi^{2j}$ does not have a decomposition into irreducibles shows that it is not just non unitary, but even non unitarisable. That is, it is impossible to choose an inner product such that $\pi^{2j}$ is unitary with respect to it.

Summarising, for $q = e^{2\pi i/(k+2)}$, we are left with only $k + 2$ irreducibles out of the infinitude that we would usually get from (34). These are the unitary representations $\pi^{2j}$ with $j < \frac{k+2}{2}$. The other representations defined by (34) are no longer irreducible. They have become indecomposable, and therefore they are non unitarisable.

3.2.3 Tensor products of representations; the coproduct

The Hilbert space of a system of $N$ particles that carry quantum group representations $\pi^{\Lambda_1}, \ldots, \pi^{\Lambda_N}$ is just the tensor product $V^{\Lambda_1}, \ldots, V^{\Lambda_N}$ of the modules of these representations. Thus, we want to have representations of the quantum group on such tensor product spaces. The decomposition of these tensor product representations into irreducibles will then give the fusion rules for the particles in the theory. Clearly, it is desirable that the tensor product decomposition should be orthogonal, since otherwise the quantum group charges of fusion products would not be well-defined. Orthogonality of the decomposition can certainly be realised if the tensor product representation is unitarisable. Therefore, we will spend some time determining which tensor product representations of $U_q(sl(2))$ are unitarisable.

For any quantum group $A$, tensor product representations are formed with the help of a so called coproduct. This coproduct is an algebra homomorphism $\Delta$ from $A$ to $A \otimes A$. The coproduct of $U_q(sl(2))$ is given on the generators and the unit 1 by

$$
\begin{align*}
\Delta(H) &= 1 \otimes H + H \otimes 1 \\
\Delta(L^\pm) &= L^\pm \otimes q^{H/4} + q^{-H/4} \otimes L^\pm \\
\Delta(1) &= 1 \otimes 1.
\end{align*}
$$

(41)

Given two representations $\pi^\Lambda, \pi^{\Lambda'}$ of the quantum group, we may form the tensor product representation by the formula

$$
(\pi^\Lambda \otimes \pi^{\Lambda'})(x) := (\pi^\Lambda \otimes \pi^{\Lambda'})(\Delta(x)).
$$

(42)

Note that these irreps are by no means all the irreps at $q = e^{i2\pi/(k+2)}$: In fact, there are more irreps of dimensions $1, \ldots, k + 1$ (how many more depends on the precise definition of $U_q(sl(2))$, see e.g. [27, 28]) and there is a family of inequivalent representations of dimension $k + 2$, parametrised by a complex number $z$. However, these representations will not concern us here.

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Here, \( x \) is an arbitrary element of \( \mathcal{A} \) and one may check that, for \( \mathcal{A} = U_q(sl(2)) \) and \( q = 1 \), we recover the ordinary tensor product of \( U(sl(2)) \) representations. The fact that \( \Delta \) is an algebra homomorphism ensures that the tensor product we have just defined is indeed a representation of the algebra. Moreover, this tensor product of representations is associative, because \( \Delta \) is coassociative, that is, \( \Delta \) satisfies

\[
(1 \otimes \Delta) \Delta = (\Delta \otimes 1) \Delta
\]

A tensor product of \(*\)-representations will be itself a \(*\)-representation with respect to the standard inner product on the tensor product space if star and coproduct commute, that is, if one has

\[
(* \otimes *) \Delta(x) = \Delta(*(x)).
\]

If this does not hold, it may still be possible to choose an inner product other than the standard inner product with respect to which the tensor product representation is a \(*\)-representation. For \( U_q(sl(2)) \), star and coproduct commute only when \( q \) is real and positive. Thus, for \( q \in \mathbb{R}_+ \), the tensor product of two unitary representations is unitary and decomposes orthogonally with respect to the standard inner product. When \( q \) is not real, star and coproduct do not commute and hence, the tensor product of two unitary representations is not necessarily unitary with respect to the standard inner product. Hence, tensor product decompositions should be expected not to be orthogonal with respect to this inner product, but they may still be orthogonal with respect to a different inner product (see further on). \[\]

Tensor product decompositions and even Clebsch-Gordan coefficients for tensor products of unitary representations of \( U_q(sl(2)) \) may be calculated similarly as for \( U(sl(2)) \). The highest weight state \( |jj\rangle \) of each the irreducible representations in the tensor product may be found by solving the equations \( L^+|jj\rangle = 0 \) and \( (L^-)^2|jj\rangle = 0 \). The other states are produced by the action of \( L^- \) on the highest weight states. In the calculations, the following formula for the coproduct of \( (L^-)^n \) is a great help:

\[
\Delta((L^-)^n) = (\Delta(L^-))^n = \sum_{m=0}^{n} \left[ \begin{array}{c} n \\ m \end{array} \right]_q (L^{-1})^m q^{-m/4} \otimes (L^-)^{n-m} q^{m/4}
\]

The \( q \)-binomial \( \left[ \begin{array}{c} n \\ m \end{array} \right]_q \) in this formula is defined by

\[
\left[ \begin{array}{c} n \\ m \end{array} \right]_q := \frac{[n]_q!}{[m]_q![n-m]_q!}
\]

\[
[n]_q! := \prod_{m=1}^{n} [m]_q
\]

When \( q \) is not a root of unity, the tensor product representation \( \pi^A \otimes \pi^{A'} \) has the same decomposition into irreps as for \( q = 1 \), i.e.

\[\]

\footnote{Note that we do have \( \sigma \circ (* \otimes *) \Delta(x) = \Delta(*(x)) \) for \( |q| = 1 \). Here, \( \sigma \) is the operator which flips the factors of the tensor product. Thus, one might be tempted to take \( \sigma \circ (* \otimes *) \) as the \(*\)-structure on \( \mathcal{A} \otimes \mathcal{A} \) when \( |q| = 1 \). This is done for example in [29]. In our context, this is not the right course to pursue, since tensor product representations will still not be \(*\)-representations with the new \(* \) on \( \mathcal{A} \otimes \mathcal{A} \).}
\[ \pi^{\Lambda} \otimes \pi^{\Lambda'} = \bigoplus_{\Lambda''=|\Lambda-\Lambda'|} \pi^{\Lambda''}, \]  

where \( \Lambda'' \) increases in steps of 2.

Explicit Clebsch-Gordan coefficients may be calculated for any tensor product of irreducibles, using for example (45). One writes

\[ |j, m\rangle = \sum_{m_1, m_2} \left[ \begin{array}{ccc} j_1 & j_2 & j \\ m_1 & m_2 & m \end{array} \right]_q |j_1, m_1\rangle |j_2, m_2\rangle \]  

for the vector with \( H \)-eigenvalue \( 2m \) in the irrep \( \pi^{2j} \) in the decomposition of the tensor product \( \pi^{2j_1} \otimes \pi^{2j_2} \). The above formula is only meant to introduce the notation for the \( q \)-Clebsch-Gordan coefficients. Several general formulae for these coefficients are proved in [30] and [31] and collected in [32]. We will not give these explicit (complicated) formulae here, but we do give the coefficients for the case \( j_2 = \frac{1}{2} \), as an illustration and because this case is of interest to us later. For \( j > 0 \), one has

\[ |j + \frac{1}{2}, j + \frac{1}{2} - p\rangle = q^{p/4} \sqrt{\frac{[2j + 1 - p]_q}{[2j + 1]_q}} \ |j, j - p\rangle |\frac{1}{2}, \frac{1}{2}\rangle + \]

\[ q^{(p-2j-1)/4} \sqrt{\frac{[p]_q}{[2j + 1]_q}} \ |j, j - p + 1\rangle |\frac{1}{2}, -\frac{1}{2}\rangle \]

\[ |j - \frac{1}{2}, j - \frac{1}{2} - p\rangle = q^{(p-2j)/4} \sqrt{\frac{[p + 1]_q}{[2j + 1]_q}} \ |j, j - p - 1\rangle |\frac{1}{2}, \frac{1}{2}\rangle - \]

\[ q^{(p+1)/4} \sqrt{\frac{[2j - p]_q}{[2j + 1]_q}} \ |j, j - p\rangle |\frac{1}{2}, -\frac{1}{2}\rangle \]  

The decomposition for \( j = 0 \) should be obvious (it is the same as for \( U(sl(2)) \)). In making a decomposition such as the one above, one has the freedom to multiply all the states in each summand irrep by a constant phase factor. Here, the phases are chosen in such a way that, when \( q \) goes to one, the coefficients reduce to the usual Clebsch-Gordan coefficients for \( U(sl(2)) \). One may check (for example using (38)) that, when \( q \) is a real number, the tensor product vectors on the right hand side are orthonormal, as they should be, because the tensor product representation is unitary in this case. However, as expected, if \( q \) is not real, then the vectors above are no longer normalised or orthogonal with respect to the standard inner product on the tensor product space. This can be remedied by choosing the inner product on the tensor product space which is defined by the fact that it makes the states above orthonormal. With respect to this \( q \)-deformed inner product, the tensor product representation is a unitary representation of \( U_q(sl(2)) \) whenever all the irreps that appear in its decomposition are unitary. This happens when \( q \in \mathbb{R} \) (in which case the new inner product coincides with the old one) and also when \( |q| = e^{i\phi} \) with \( \phi \in \mathbb{R}, |\phi| \leq \frac{2\pi}{2j+2} \). For general tensor products of two irreps, we have a similar situation. Here, too, we may define an inner product by requiring that the different states given by (49) are orthonormal and this inner product will make the tensor product representation into a \( \ast \)-representation when the irreps into which it decomposes are \( \ast \)-representations.
From this point, one may go on and define inner products on \(N\)-fold tensor products of irreducibles by requiring that a complete set of states obtained by iterative use of (14) is orthonormal. The set of states that is declared orthonormal now depends on the order in which one takes the tensor product of the irreducibles, so \((V^{A_1} \otimes V^{A_2}) \otimes V^{A_3}\) and \(V^{A_1} \otimes (V^{A_2} \otimes V^{A_3})\) may in principle end up with a different inner product. We will say more about this issue in sections 3.2.7 and 3.2.8.

The orthogonality of the tensor product decomposition which holds when \(q\) is real is reflected in the following identity for the \(q\)-Clebsch-Gordan coefficients:

\[
\sum_{m_1,m_2} \left[ \begin{array}{ccc} j_1 & j_2 & j \\ m_1 & m_2 & m \end{array} \right]_q \left[ \begin{array}{ccc} j_1 & j_2 & j' \\ m_1 & m_2 & m' \end{array} \right]_q = \delta_{j,j'} \delta_{m,m'} \]

(51)

Note that, although tensor product decomposition is orthogonal with respect to the standard inner product only when \(q\) is real and positive, this equation holds by analytic continuation for all \(q\) where the summands are not singular.

Another useful identity (taken from (32)), which relates the coefficients for the tensor product \(\pi^{2j_1} \otimes \pi^{2j_2}\) with those for the opposite tensor product is:

\[
\left[ \begin{array}{ccc} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{array} \right]_q = (-1)^{j_1+j_2-j} \left[ \begin{array}{ccc} j_2 & j_1 & j_3 \\ m_2 & m_1 & m_3 \end{array} \right]_q^{-1} \]

(52)

In particular, this allows one to write down the Clebsch-Gordan coefficients for \(\pi^1 \otimes \pi^{2j}\) using (50).

3.2.4 The root of unity case: truncated tensor products

In the previous section, we described the tensor product of representations of \(U_q(sl(2))\) for the case that \(q\) is not a root of unity. In that case, many of the usual properties of tensor products at \(q = 1\) could be recovered. For example, the tensor product of two irreps could be decomposed into a direct sum of irreps (see (18)). Also, this decomposition is orthogonal with respect to a suitable deformation of the standard inner product.

When \(q\) is a root of unity, say \(q = e^{i2\pi/(k+2)}\), the situation is quite different. In this case, tensor products of two irreps will not split into a direct sum of irreps, but will contain indecomposable summands. This is not in itself surprising, because the representations \(\pi^\Lambda\) with \(\Lambda > k+1\) that would occur in the usual decomposition (18) become indecomposable for \(q = e^{i2\pi/(k+2)}\). However, what really happens is a bit more complicated. As an example, let us look at the decomposition of the tensor product of the spin \(\frac{1}{2}\) and the spin \(\frac{k+1}{2}\) module. As usual, the tensor product space may be decomposed into eigenspaces of the operator \(H\). These eigenspaces will be one dimensional for the extremal eigenvalues \(H = k+1\) and \(H = -(k+1)\) and two dimensional for the other eigenvalues. If \(q\) were not a root of unity, then we would have two highest weight states in the tensor product module. The \(H = k+1\) state \(|\frac{k+1}{2}, \frac{k+1}{2}\rangle\) and the \(H = k-1\) state \(|\frac{k-1}{2}, \frac{k-1}{2}\rangle\) given in (50). At \(q = e^{i2\pi/(k+2)}\), the coefficients of this second state diverge, but if we multiply the state by \(|k+2\rangle_q\), then this no longer happens and we still have two good highest weight states. However, we have a third candidate highest weight state, which is the \(H = k-1\) state one gets when one lets \((L^+)^{k+1}\) act on the lowest weight state \(|\frac{k+1}{2}, -\frac{k+1}{2}\rangle\) (remember \((L^+)^{k+2}\) gives zero for this value of \(q\)). This new highest weight state is just proportional to the state \(|\frac{k+1}{2}, \frac{k-1}{2}\rangle\) given in (50). Comparing this state with the other highest weight state at \(H = k - 1\), we see that although they would be linearly independent for any
arbitrary \( q \), they are actually proportional to each other for \( q = e^{2\pi i/(k+2)} \). It follows that the irreducible spin \( \frac{k-1}{2} \)-module has become a submodule of the module generated by the highest weight state at \( H = k + 1 \). Also, since we have only one highest weight state in the \( H = k - 1 \) eigenspace and since this space is two-dimensional, there must also be a non-highest weight state in this module. The two dimensional \( H \)-eigenspaces of the tensor product module will then be spanned by a descendant of the highest weight state at \( H = k - 1 \) and a descendant of the non-highest weight state at \( H = k - 1 \). We see thus that, at \( q = e^{2\pi i/(k+2)} \), the modules \( \pi^{k+2} \) and \( \pi^k \) have disappeared from the decomposition of \( \pi^1 \otimes \pi^{k+1} \) and in stead there is one indecomposable module, which has the module \( \pi^k \) as an irreducible submodule.

This general picture extends to all tensor products of irreps; in general, all the modules \( \pi^\Lambda \) with \( \Lambda > k \) and all the corresponding modules \( \pi^{2k-\Lambda} \) will disappear from the decomposition [18] and in stead, there will be indecomposable modules with the modules \( \pi^{2k-\Lambda} \) as irreducible submodule. The structure of these indecomposable modules is analogous to the structure of the module we described above and is illustrated in figure 3.2.4. For more detail on tensor product decomposition when \( q \) is a root of unity, one can consult for example [33, 28, 27].

![Diagram of an indecomposable representation](image)

**Figure 4:** Diagram of an indecomposable representation as it would occur in the tensor product of two \( U_q(sl(2)) \)-irreps at \( q = e^{2\pi i/(k+2)} \). The dots represent the basis states in the module, the arrows \( \rightarrow \) and \( \leftarrow \) indicate the action of \( L^+ \) and \( L^- \) resp. The split arrows are meant to indicate that the descendants of the state \(| \psi \rangle \) are mapped onto linear combinations of descendants of \(| \psi \rangle \) and \((L^+)_{k+1}|l\rangle\).

Clearly, the indecomposable representations which occur in the tensor products are non unitarisable; this follows from the indecomposability, but one can also see easily that any “inner product” that would make these representations unitary would give the states in the irreducible submodule zero norm. In fact, one can see this happen explicitly for the limit of \( q \to e^{2\pi i/(k+2)} \) of the \( q \)-deformed inner product we defined at the end of the previous section.

In the present context, the states in the indecomposable representations are considered non physical. Thus, we need to define a new “tensor product” \( \hat{\otimes} \) which is the the old tensor product with the indecomposable modules projected out. However, the tensor product defined this way would not be associative, since we would have for example \((\pi^1 \hat{\otimes} \pi^k) \hat{\otimes} \pi^{k+1} = 2\pi^{k+1} \) and \( \pi^1 \hat{\otimes} (\pi^k \hat{\otimes} \pi^{k+1}) = 0 \) for odd \( k \). (For even \( k \), we get similar problems). Also, the fusion rule \( \pi^\Lambda \otimes \pi^{k+1} = 0 \) (\( \Lambda \) even) is clearly unphysical; after adding a particle in the representation \( \pi^{k+1} \) we would be left without a Hilbert space! These problems can be solved both at once by projecting out not just the indecomposable modules, but also any modules of type \( \pi^{k+1} \) that may occur. The resulting tensor product is called the *truncated tensor product*. Similarly as in section (3.2.3), one may define an inner product on any truncated tensor product of irreducible \( U_q(sl(2)) \)-modules, which makes the tensor product decomposition orthogonal. Moreover, since the irreps into which a truncated tensor product factors are all \(*\)-representations, this inner product makes the truncated tensor product representation into a \(*\)-representation. The truncated tensor
product decomposition at \( q = e^{i2\pi/(k+2)} \) is given by the following formula, which is identical to the formula (2) for the fusion rules of \( sl(2)_k \) chiral primaries:

\[
\pi^\Lambda \otimes \pi^{\Lambda'} = \bigoplus_{\Lambda''=|\Lambda-\Lambda'|} \pi^{\Lambda''}.
\] (53)

From this formula, one may check easily that the truncated tensor product is indeed associative, that is, the tensor product modules \((\pi^{\Lambda_1} \otimes \pi^{\Lambda_2}) \otimes \pi^{\Lambda_3}\) and \(\pi^{\Lambda_1} \otimes (\pi^{\Lambda_2} \otimes \pi^{\Lambda_3})\) are isomorphic. Note however that these two modules are different subspaces of the ordinary tensor product, so we might say that the truncated tensor product is associative at the level of \( U_q(sl(2)) \)-modules, but not associative at the level of states. This might seem like a problem at first sight, because we want to have a unique three-particle Hilbert space, but this problem disappears if we can find a canonical \( U_q(sl(2)) \)-isomorphism between the two three-particle spaces which preserves the inner product. We will say more about this in sections (3.2.7) and (3.2.8).

As an illustration, let us take a closer look at the truncated tensor product of the 2-dimensional irrep \( \pi^1 \) with the unitary irreps \( \pi^0, \pi^1, \ldots, \pi^k \). For this case, the truncated tensor product decomposition is given by

\[
\begin{align*}
\pi^0 \otimes \pi^1 &= \pi^1 \\
\pi^\Lambda \otimes \pi^1 &= \pi^{\Lambda+1} \oplus \pi^{\Lambda-1} \quad (\Lambda \in \{1, \ldots, k-1\}) \\
\pi^k \otimes \pi^1 &= \pi^{k-1}
\end{align*}
\] (54)

As one can see, the only difference with the ordinary tensor product occurs in the last line. The decomposition on the level of states can be read off from (54). Using this formula, we can also give an example of the non-associativity at the level of states that we were talking about: At \( k = 1 \) (or \( q = e^{2\pi i/3} \)), the truncated tensor products \( V_1 = (\pi^1 \otimes \pi^1) \otimes \pi^1 \) and \( V_2 = \pi^1 \otimes (\pi^1 \otimes \pi^1) \) are both isomorphic to \( \pi^1 \) as \( U_q(sl(2)) \)-modules, but any state in \( V_1 \) may be written as

\[
\frac{1}{\sqrt{|2_q|}} \left( q^{-1/4} |\frac{1}{2}, -\frac{1}{2}\rangle |\frac{1}{2}, \frac{1}{2}\rangle - q^{1/4} |\frac{1}{2}, \frac{1}{2}\rangle |\frac{1}{2}, -\frac{1}{2}\rangle \right) \left( \alpha_1 |\frac{1}{2}, \frac{1}{2}\rangle + \alpha_2 |\frac{1}{2}, -\frac{1}{2}\rangle \right),
\] (55)

while any state in \( V_2 \) may be written as

\[
\frac{1}{\sqrt{|2_q|}} \left( \beta_1 |\frac{1}{2}, \frac{1}{2}\rangle + \beta_2 |\frac{1}{2}, -\frac{1}{2}\rangle \right) \left( q^{-1/4} |\frac{1}{2}, -\frac{1}{2}\rangle |\frac{1}{2}, \frac{1}{2}\rangle - q^{1/4} |\frac{1}{2}, \frac{1}{2}\rangle |\frac{1}{2}, -\frac{1}{2}\rangle \right)
\] (56)

From this, we see that a vector in \( V_1 \) can only equal a vector in \( V_2 \) if it is zero. Hence, \( V_1 \) and \( V_2 \) are different subspaces of \( \pi^1 \otimes \pi^1 \otimes \pi^1 \).

Before ending this section, let us write down two useful identities for truncated tensor decomposition which are related to the external automorphism of \( \bar{sl(2)}_k \) that we discussed in relation to the field identifications (8). If we define

\[
\hat{\Lambda} := k - \Lambda,
\] (57)

then we have
Here, we have written $\Lambda$ in stead of $\pi^\Lambda$ to avoid notational overload. These identities tell us that the truncated fusion rules of $U_q(sl(2))$ do not allow us to make a distinction between a particle that carries the representation $\Lambda$ and a particle that carries the representation $\hat{\Lambda}$.

### 3.2.5 Counit, antipode and quantum trace

Every quantum group $\mathcal{A}$ is required to have a one dimensional representation $\epsilon$, called the counit, which represents the vacuum (or $\mathcal{A}$-neutral) sector of the theory. The counit has to satisfy

$$\epsilon(\epsilon \otimes \text{id}) \Delta = (\text{id} \otimes \epsilon)(\Delta) = \text{id}$$

where $\text{id}$ is the identity map on $\mathcal{A}$. It follows that $\epsilon \otimes \pi = \pi \otimes \epsilon = \pi$ for any representation $\pi$ of the quantum group, so the vacuum (or an $\mathcal{A}$-neutral particle) has the fusion properties that one would expect. For $U_q(sl(2))$, we have already seen the counit; it is just the representation $\pi^0$. Explicitly, we have

$$\epsilon(1) = 1, \quad \epsilon(H) = 0, \quad \epsilon(L^\pm) = 0.$$  \hfill (60)

The counit of a quantum group is an analogue of the trivial representation of a group and hence, we will say that a state $|s\rangle$ in a representation $\pi$ of the quantum group $\mathcal{A}$ transforms trivially if $\pi(a)|s\rangle = \epsilon(a)|s\rangle$ for all $a \in \mathcal{A}$.

Next to the counit, $\mathcal{A}$ also required to have a linear algebra antihomomorphism $S : \mathcal{A} \to \mathcal{A}$ which satisfies

$$\mu(S \otimes \text{id}) \Delta(a) = \mu(\text{id} \otimes S)(\Delta)(a) = \epsilon(a) \text{id}$$

where, $\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is just the multiplication of $\mathcal{A}$. $S$ is called the antipode and $S(a)$ should be thought of as a quantum group analogue of the inverse of $a$. If we are given a representation $\pi$ of $\mathcal{A}$, then the antipode makes it possible to define the representation $\bar{\pi}$ conjugate to $\pi$ by the formula

$$\bar{\pi}(a) = (\pi(S(a)))^t$$

The fact that $S$ is an antihomomorphism then makes sure that $\bar{\pi}$ is a representation of $\mathcal{A}$, while the properties (61) ensure that the tensor product representations $\pi \otimes \bar{\pi}$ and $\bar{\pi} \otimes \pi$ will contain the trivial representation $\epsilon$ in their decomposition. Thus, a particle which carries the representation $\pi$ and its antiparticle, which carries the representation $\bar{\pi}$ may indeed annihilate.

The antipode also allows us to define the action of $\mathcal{A}$ on the space of linear operators on an $\mathcal{A}$-module $V$ by

$$\hat{\Lambda} \otimes \Lambda' = \bigoplus_{\Lambda''} \Lambda'' \quad \text{min}(\Lambda+\Lambda', 2k-\Lambda-\Lambda')$$

$$\hat{\Lambda} \hat{\Lambda}' = \bigoplus_{\Lambda''} \Lambda'' \quad \text{min}(\Lambda+\Lambda', 2k-\Lambda-\Lambda')$$

\hfill (58)
Here, $|v\rangle$ is any state in the module $V$ and the $a^k_1$ and $a^k_2$ are the left and right components of the coproduct of $a$, i.e.

$$\Delta(a) = \sum_k a^k_1 \otimes a^k_2 \tag{64}$$

Again using the fact that $S$ is an antihomomorphsim, one can see that (63) defines a representation of $\mathcal{A}$, while using the property (64), one can see that $\mathcal{A}$ acts trivially on operators that commute with the action of $\mathcal{A}$ on $V$.

For $U_q(sl(2))$, the antipode is given by

$$S(1) = 1, \quad S(H) = -H, \quad S(L^±) = -q^{±1}L^±. \tag{65}$$

The representations of $U_q(sl(2))$ are clearly all isomorphic to their conjugates (one may see this for example from the decomposition (18) without even using the explicit form of the antipode) and the action of $U_q(sl(2))$ on an operator $\hat{O}$ is given by

$$H \cdot \hat{O} = [H, \hat{O}], \quad L^± \cdot \hat{O} = L^±\hat{O}q^{-H/4} - q^{-(H±1)/4}\hat{O}L^±, \tag{66}$$

which reduces to the usual commutator for $q \to 1$.

One can define a kind of trace on operators, which has the property that it transforms trivially under $U_q(sl(2))$ when the operator is transformed. For $q = 1$, the ordinary trace has this property, since $\text{Tr}([a, \hat{O}]) = 0 = \epsilon(a)\text{Tr}(\hat{O})$ for all $a \in sl(2)$ and for arbitrary $\hat{O}$. However, for $q \neq 1$, we have to use a modified trace to get this property. This trace is usually called the quantum trace and we will denote it $\text{Tr}_q$. Of course, the quantum trace is supposed to preserve some nice properties of the ordinary trace. Most importantly, the trace of a tensor product of operators should be the product of the traces of the tensor factors. A quantum trace with this property can be defined for a large class of quantum groups (see cf. [27]). For $U_q(sl(2))$, it is given by

$$\text{Tr}_q(\hat{O}) = \text{Tr}(q^{H/2}\hat{O}) \tag{67}$$

One may verify readily that $\text{Tr}_q(a \cdot \hat{O}) = \epsilon(a)\text{Tr}_q(\hat{O})$. The fact that $\text{Tr}_q(\hat{O}_1 \otimes \hat{O}_2) = \text{Tr}_q(\hat{O}_1)\text{Tr}_q(\hat{O}_2)$ follows from the comultiplication $\Delta(q^{H/2}) = q^{H/2} \otimes q^{H/2}$.

Using the quantum trace, one may now define the quantum dimension $\dim_q(\pi)$ of a representation of the quantum group as the quantum trace of the unit operator in this representation. For the representations $\pi^{\Lambda}$ of $U_q(sl(2))$, this yields $\dim_q(\pi^{\Lambda}) = [\Lambda + 1]_q = [\dim(\pi^{\Lambda})]_q$. In particular, the quantum dimension of $\pi^{k+1}$ is zero. The quantum dimensions of all the indecomposable modules of dimension $2k + 4$ that appeared in the (untruncated) tensor products of the $\pi^{\Lambda}$ are also zero, since these modules were a (non-direct) sum of two modules of dimensions $k + 2 - d$ and $k + 2 + d$ and we have $[k+2-d]_q + [k+2+d]_q = [d]_q + [-d]_q = 0$. Since the quantum dimensions of the modules $\pi^1, \ldots, \pi^k$ are non zero, we see that we might also have defined the truncated tensor product of two modules in this set as the ordinary tensor product with the modules of quantum dimension zero projected out. With this definition, the truncated tensor product is automatically associative and the module $\pi^{k+1}$ does not need separate treatment.
Quantum traces may also be used to construct knot invariants (see for example [34],[27] and references therein). For $U_q(sl(2))$, one of the knot invariants which can be constructed this way is the famous Jones polynomial.

### 3.2.6 Braiding: the $R$-matrix

Suppose that we have two particles that carry the $U_q(sl(2))$ representation $\pi^\Lambda$. The total internal state of the system can then be represented by a state $|s\rangle$ in the (possibly truncated) tensor product $\pi^\Lambda \hat{\otimes} \pi^\Lambda$. Now if we exchange the particles, how does the state of the system change? In the $q = 1$ case, we can describe the exchange simply by exchanging the tensor factors in the state $|s\rangle$. Let us call this exchange of the factors in the tensor product $\sigma$. An exchange of two adjacent particles in a system of $N$ identical particles may then be described by the action of $\sigma$ on the corresponding factors of the $N$-fold tensor product that describes the system. For example, in a 4-particle system, the exchange of the second and third particles is affected by the operation $1 \otimes \sigma \otimes 1$. Such exchanges generate a representation of the permutation group $S_N$. Also, they commute with the action of the algebra at $q = 1$. This can be seen from the formulae for the coproduct at $q = 1$; the coproduct of any element of the algebra is invariant under the exchange of the tensor factors. We say that the coproduct is cocommutative. Because the algebra action and the permutation group action commute, the $N$-particle state space of the system can be decomposed into representations of $U(sl(2)) \times S_N$.

When $q \neq 1$, the situation is different. The coproduct is now no longer cocommutative and hence, exchanging the order in the tensor product no longer commutes with the $U_q(sl(2))$ action. However, this can be remedied with the use of a universal $R$-matrix. This is an invertible element of $U_q(sl(2) \otimes U_q(sl(2))$ which acts before $\sigma$ in the appropriate representation. For example, in a system of three particles, all of which carry the representation $\pi$ of $U_q(sl(2))$, the exchange of the first and second particles will now be affected by $(\sigma \circ \pi \otimes \pi(R)) \otimes 1$. Note that, in any tensor product of two representations, the universal $R$-matrix does indeed act as a matrix, but the matrix in question depends on the representations. The $R$-matrix is required to have the following properties:

\[
\Delta^{\text{op}} R = R \Delta \\
(\Delta \otimes 1) R = R_{13} R_{23} \\
(1 \otimes \Delta) R = R_{13} R_{12}
\]  

(68)

Here, $\Delta^{\text{op}}$ is the comultiplication, followed by an exchange of the tensor factors in $\mathcal{A} \otimes \mathcal{A}$ and $R_{ij}$ is an abbreviation for the action of $R$ on the factors $i$ and $j$ of $\mathcal{A}^\otimes 3$, so for example $R_{12} = R \otimes 1$. The first of the properties above ensures that the exchanges affected by $\sigma R$ in the tensor product representations commute with the action of the quantum group, as we wanted. The second and third property make sure that braiding of two particles around a third one and then fusing them together gives the same result as fusing the two particles first and then braiding the result around the third one. Using the above properties of the universal $R$-matrix, one may also prove the following equality, which is called the quantum Yang-Baxter equation.

\[
R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}
\]

(69)

This equation implies that in any representation, we have
It follows that, for a system of $N$ identical particles that carry a representation of the quantum group, the exchanges of adjacent particles, as performed using $\sigma R$, satisfy the relations (29). Hence, since $R$ is invertible, they generate a representation of the braid group on $N$ strands, $B_N$. Since the exchanges commute with the action of the quantum group $\mathcal{A}$, the system thus carries a representation of $\mathcal{A} \times B_N$.

When the particles do not all carry the same quantum group representation (and are hence not identical), then the $R$-matrix no longer gives us a representation of the braid group on the Hilbert space of the system, simply because the exchanges now act between different vector spaces; the flip operator $\sigma$ sends $V^{\Lambda_1} \otimes V^{\Lambda_2}$ into $V^{\Lambda_2} \otimes V^{\Lambda_1}$. This is not a problem, because exchanges of non-identical particles are not symmetries of the system. What we do still get from the $R$-matrix is a representation of a so called coloured braid group, which consists of the braids for which the final position of any particle is the original position of a particle of the same kind (or “colour”). These coloured braidings will still commute with the quantum group action. One should note that the colouring restriction leaves plenty of room for non-trivial and even nonabelian monodromies between distinguishable particles. All the braiding transformations in coloured braid groups may still be generated by elementary exchanges of adjacent particles, although some of these will no longer have any physical meaning and should be called half-monodromies rather than braidings. We will be a bit sloppy about this in the rest of this article, but we assume that this will not cause confusion.

Before going on, let us note that one could multiply the $R$-matrix by a constant phase factor. The matrices $(\sigma R)_{12}, (\sigma R)_{23}, \ldots, (\sigma R)_{n-1,n}$, which model the exchange of adjacent particles would then still generate a representation of the braid group and commute with the quantum group. The second and third condition in (68), which express compatibility of fusion and braiding, may also be satisfied if we give the coproduct $\Delta$ the same phase factor as the $R$-matrix. The tensor product “representation” defined in (42) will then in general no longer be a representation of the quantum group, but it will still be a projective representation, which is really what we should expect in a quantum mechanical setting. Thus, we may say that the universal $R$-matrix determines the braiding of the particles only up to a constant phase factor in each (clockwise) exchange (obviously, the counterclockwise exchanges get the inverse phase factor).

The universal $R$-matrix which fulfills the requirements (68) for $U_q(sl(2))$ is given by

$$\begin{equation}
R = q^{-\frac{\mu \otimes \mu}{4}} \sum_{n=0}^{\infty} \frac{(1 - q^{-1})^n}{[n]_q!} q^{n(1-n)/4} (q^{nH/4}(L^+)^n) \otimes (q^{-nH/4}(L^-)^n),
\end{equation}$$

(71)

We see that, when $q$ approaches one, only the $n = 0$ term in (71) contributes and we get $R = 1 \otimes 1$, as expected.

The action of the universal $R$-matrix on the module $V^{\Lambda_1} \otimes V^{\Lambda}$ of the tensor product representation $\pi^{\Lambda_1} \otimes \pi^{\Lambda}$ is given by

$$\begin{equation}
R|j_1, m_1\rangle |j_2, m_2\rangle = \sum_{n \geq 0} \sqrt{\frac{j_1 - m_1}{n}} \frac{j_2 + m_2}{n} \frac{[j_1 + m_1 + n]_q [j_2 - m_2 + n]_q}{[j_1 + m_1]_q [j_2 - m_2]_q} q^{n(1-n)/4} \times q^{\frac{1}{2}(m_2 n - m_1 n + 2m_1 m_2)} (1 - q^{-1})^n |j_1, m_1 + n\rangle |j_2, m_2 - n\rangle
\end{equation}$$

(72)
where the sum extends over all \( n \) for which the kets on the right hand side are well defined. Using this formula, one may easily find the exchange matrix \( \sigma R \) in any tensor product module. For example, in the tensor product \( V^1 \otimes V^1 \) of two two-dimensional modules, we have

\[
\sigma R^{1,1} := q^{-1/4} \begin{pmatrix}
q^{1/2} & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & q^{1/2} - q^{-1/2} & 0 \\
0 & 0 & 0 & q^{1/2}
\end{pmatrix}.
\]  

(73)

Note that, if \( q \neq 1 \), this is not a unitary matrix, which is not good if it is supposed to represent a symmetry transformation on a physical system. Still, we could hope to make \( \sigma R \) unitary by choosing a suitable inner product on the module \( V^1 \otimes V^1 \). This will certainly not succeed unless \(|q| = 1\). To see this, note that the eigenvalues of \( R^{1,1} \) are \( q^{1/4} \) (with multiplicity 3) and \(-q^{-3/4}\) and these will only have norm 1 (as required for the eigenvalues of a unitary transformation) if \( q \) does. However, if \(|q| = 1\), then \( R^{1,1} \) is indeed unitary with respect to the \( q \)-deformed inner product which we defined below formula (50) and which makes the tensor product decomposition orthogonal. Of course, the reason that this works is the fact that the action of \( R \) commutes with the quantum group action, which means that the eigenspaces of \( R \) are submodules of the tensor product module. We will say more about inner products and unitarity in section 3.2.8.

In the above, we have always implicitly assumed that \( q \) was not a root of unity. If it is, then the whole story is more complicated, because of the truncation of the tensor product which has to be taken into account for these values of \( q \). The \( R \)-matrix (71) still describes the braiding of two particles \( 1 \), but if we go to three or more particles, then we can get problems. For example, three particles in the representation \( \pi^\Lambda \) may be described by a state in the truncated tensor product space \((V^\Lambda \hat{\otimes} V^\Lambda) \hat{\otimes} V^\Lambda\) and we can exchange the two leftmost particles by means of \( \sigma(\pi^\Lambda \hat{\otimes} \pi^\Lambda)(R) \otimes 1 \), which gives us a state in \((V^\Lambda \hat{\otimes} V^\Lambda) \hat{\otimes} V^\Lambda\), as it should. However, if we want to exchange the two rightmost particles, then we can leave the space \((V^\Lambda \hat{\otimes} V^\Lambda) \hat{\otimes} V^\Lambda\) if we just apply \( 1 \otimes \sigma(\pi^\Lambda \hat{\otimes} \pi^\Lambda)(R) \). One may see this explicitly in the example we gave in formula (53); exchanging the last two particles in this state, we get a state which can clearly not be written in the same form and hence does not belong to \((V^1 \hat{\otimes} V^1) \hat{\otimes} V^1\). If we use the other bracketing of the truncated tensor product (i.e. \( V^\Lambda \hat{\otimes} (V^\Lambda \hat{\otimes} V^\Lambda) \)), then we can exchange the last two particles in the expected way, but then the problem occurs in the exchange of the first two. In this way, we can always expect problems when we try to exchange two particles over a bracket. Thus, we will not get a representation of the braid group on the truncated tensor product, unless we modify the way in which we exchange particles. We will explain the modification that is needed in some detail in section 3.2.8.

### 3.2.7 \( q \)-6j-symbols and their properties

If we take a tensor product of three \( U_q(sl(2)) \) modules \( \pi^\Lambda_1, \pi^\Lambda_2 \) and \( \pi^\Lambda_3 \), then there are two different ways to decompose this tensor product into irreducibles. We may either first decompose the product \( \pi^\Lambda_1 \otimes \pi^\Lambda_2 \) and then the resulting modules \( \pi^\Lambda \otimes \pi^\Lambda_3 \), or we may first

\footnote{Note that this \( R \)-matrix is not well defined if \( q = e^{i2\pi/(k+2)} \), since the \( q \)-factorial \(|n\rangle_q! \) which appears in the \((L^+)^n) \otimes (L^-)^n) \) term becomes zero for \( n \geq k + 2 \). This problem is usually resolved by adding the relations \((L^+)^{k+2} = (L^-)^{k+2} = 0 \) to the algebra for this value of \( q \). To us, this subtlety is not very important, since these relations already hold in the unitary representations we are interested in.}
decompose the product $\pi^A_2 \otimes \pi^A_3$ and then the resulting modules $\pi^A_1 \otimes \pi^A''$. These two procedures yield two different natural bases for the vector space $V^A_1 \otimes V^A_2 \otimes V^A_3$. In each case, the basis vectors are labelled by their $H$-eigenvalue, the label of their overall fusion channel and the label of their intermediate fusion channel (which is the representation into which $\pi^A_1$ and $\pi^A_2$ fuse in the first case and the representation into which $\pi^A_2$ and $\pi^A_3$ fuse in the second case). Let us call the vectors in the first basis $e^A_{j_1,j_2,j_3}$ and the vectors in the second basis $f^A_{j_1,j_2,j_3}$. Here, $j_1, j_2$ and $j_3$ correspond to $\Lambda_1, \Lambda_2$ and $\Lambda_3$, $m$ and $m'$ give the $H$-eigenvalues, $j$ and $j'$ give the overall fusion channels and $j_{12}$ and $j_{23}$ represent the intermediate fusion channels. The vectors $e^A_{j_1,j_2,j_3}$ and $f^A_{j_1,j_2,j_3}$, may be written in terms of the standard (product) basis for the tensor product by means of the Clebsch-Gordan coefficients. We have

$$e^A_{j_1,j_2,j_3} = \sum_{m_1,m_2,m_3} \left[ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ m_1 & m_2 & m_{12} \end{array} \right]_q \left[ \begin{array}{ccc} j_{12} & j_3 & j \\ m_{12} & m_3 & m \end{array} \right]_q |j_1,m_1 \rangle |j_2,m_2 \rangle |j_3,m_3 \rangle$$

$$f^A_{j_1,j_2,j_3} = \sum_{m_1,m_2,m_3} \left[ \begin{array}{ccc} j_2 & j_3 & j_{23} \\ m_2 & m_3 & m_{23} \end{array} \right]_q \left[ \begin{array}{ccc} j_1 & j_{23} & j \\ m_1 & m_{23} & m \end{array} \right]_q |j_1,m_1 \rangle |j_2,m_2 \rangle |j_3,m_3 \rangle$$

(74)

where $m_{12} = m_1 + m_2$ and $m_{23} = m_2 + m_3$. The vectors in the $e$-basis may also be expressed in terms of the $f$-basis vectors and this expression takes the following form:

$$e^A_{j_1,j_2,j_3} = \sum_{j_{23},j',m'} \delta_{j'j} \delta_{mm'} \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array} \right\} f^A_{j_{12},j',m'}$$

(75)

The coefficients represented by the curly brackets are now called the $6j$-symbols of $U_q(sl(2))$. By definition, these $q$-$6j$-symbols must be equal to the $6j$-symbols for $SU(2)$ when $q$ equals one. The $6j$-symbol in the formula above will clearly be zero unless the representation $j_{12}$ occurs in the tensor product of the representations $j_1$ and $j_2$, the representation $j$ occurs in the tensor product of the representations $j_{12}$ and $j_3$, etcetera. It follows that the $6j$-symbol will be zero unless its arguments satisfy the following requirements:

$$|j_1 - j_2| \leq j_{12} \leq j_1 + j_2, \quad j_1 + j_2 + j_{12} \in \mathbb{Z}$$

$$|j_2 - j_3| \leq j_{23} \leq j_2 + j_3, \quad j_2 + j_3 + j_{23} \in \mathbb{Z}$$

$$|j_{12} - j_3| \leq j \leq j_{12} + j_3, \quad j_{12} + j_3 + j \in \mathbb{Z}$$

$$|j_1 - j_{23}| \leq j \leq j_1 + j_{23}, \quad j_1 + j_{23} + j \in \mathbb{Z}$$

(76)

If these requirements are met, then the $6j$-symbol may be written in terms of Clebsch-Gordan coefficients; using (74) and the relations (51), one easily finds that

$$\left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array} \right\} = \sum_{m_1,m_2,m_3} \left[ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ m_1 & m_2 & m_{12} \end{array} \right]_q \left[ \begin{array}{ccc} j_{12} & j_3 & j \\ m_{12} & m_3 & m \end{array} \right]_q \left[ \begin{array}{ccc} j_2 & j_3 & j_{23} \\ m_1 & m_2 & m_{23} \end{array} \right]_q$$

(77)

From this formula, one may obtain explicit formulae for the $6j$-symbols. We will not do the (long) computations here, but just give one of the possible explicit answers, as given in [32] (see also [33]).
\[
\begin{align*}
\left\{ \begin{array}{ccc}
  j_1 & j_2 & j_{12} \\
  j_3 & j & j_{23}
\end{array} \right\} &= \\
\sqrt{[2j_{12} + 1]_q [2j_{23} + 1]_q} \Delta(j_1, j_2, j_{12}) \Delta(j_{12}, j_3, j) \Delta(j_2, j_3, j_{23}) \Delta(j_1, j_{23}, j) \\
\times \sum_{z} \left\{ \frac{(-1)^{\left|j - j_1 - j_2 - j_{12}\right|_q}}{[z - j_1 - j_2 - j_{12}]_q [z - j_1 - j_3 - j_{23}]_q [z - j_2 - j_3 - j_{23}]_q [z - j_1 - j_2 - j_{23}]_q} \right\} 
\end{align*}
\] (78)

where

\[
\Delta(a, b, c) := \sqrt{\frac{[-a + b + c]_q ![a - b + c]_q ![a + b - c]_q ![a + b + c]_q}{[a + b + c + 1]_q}}
\] (79)

The sum in (78) is taken over all \(z\) for which all the \(q\)-factorials in the summands are well-defined.

The \(q\)-6\(j\)-symbols are invariant under many symmetries (described in \([32, 33]\)) which are analogues of the symmetries of the 6\(j\)-symbols of \(SU(2)\) (see \([30]\)). For us, the most important of these are the so called classical symmetries. These symmetries can be treated slightly more elegantly if one works with the \(q\)-Racah coefficients in stead of the \(q\)-6\(j\)-symbols. The Racah coefficients are just the 6\(j\)-symbols with a different normalisation; they are given by the formula for the 6\(j\)-symbols above with the first square root root factor left out. Invariance under the classical symmetries means that the Racah coefficients remain unchanged under permutations of the columns and under exchanging the upper and lower entry in two columns simultaneously. In effect, this means that we have the following identities for the 6\(j\)-symbols

\[
\begin{align*}
\left\{ \begin{array}{ccc}
  j_1 & j_2 & j_{12} \\
  j_3 & j & j_{23}
\end{array} \right\} &= \left\{ \begin{array}{ccc}
  j_2 & j_1 & j_{12} \\
  j & j_3 & j_{23}
\end{array} \right\} = \sqrt{\frac{[2j_{12} + 1]_q [2j_{23} + 1]_q}{[2j_{23} + 1]_q [2j_{12} + 1]_q}} \left\{ \begin{array}{ccc}
  j_1 & j_{12} & j_2 \\
  j_3 & j_{23} & j
\end{array} \right\} \\
\left\{ \begin{array}{ccc}
  j_1 & j_2 & j_{12} \\
  j_3 & j & j_{23}
\end{array} \right\} &= \left\{ \begin{array}{ccc}
  j_1 & j & j_{23} \\
  j_3 & j_2 & j_{12}
\end{array} \right\}
\end{align*}
\] (80)

and all the identities generated by these. The other symmetries of the 6\(j\)-symbols are analogues of the Regge and reflection symmetries.

When \(q \in \mathbb{R}_+\), the bases for the three-fold tensor product given in (74) are orthonormal and hence the basis transformation between these bases is unitary. As a consequence, the 6\(j\)-symbols satisfy the following orthogonality relation (see cf. \([32]\))

\[
\sum_{j_{12}} \left\{ \begin{array}{ccc}
  j_1 & j_2 & j_{12} \\
  j_3 & j_4 & j_{23}
\end{array} \right\} \left\{ \begin{array}{ccc}
  j_1 & j_2 & j_{12} \\
  j_3 & j_4 & j_{23}
\end{array} \right\} = \delta_{j_{23}j_{23}'}
\] (81)

Here, we have used the fact that the 6\(j\)-symbols are real for \(q \in \mathbb{R}_+\). The equation above indeed tells us that the matrix for the coordinate transformation between the \(e\) and \(f\) bases for the charge \(j\) subspace of \(V^{j_1} \otimes V^{j_2} \otimes V^{j_3}\) is real-orthogonal. When \(q\) is not a positive real number, the above relation for the 6\(j\)-symbols remains valid by analytic continuation, as long as the summands are not singular, but it does not tell us that the matrix for the coordinate transformation we mentioned is orthogonal unless all the 6\(j\)-symbols that appear are real. For \(|q| = 1\), these 6\(j\)-symbols will certainly be real as long
as \( |\arg(q)| \) is small enough to make sure that all the \( q \)-numbers that appear in these 6\( j \)-symbols are positive. This will be the case (cf. formula (78)) when

\[
|\arg(q)| < \min\{ \frac{2\pi}{j_1 + j_2 + j_{12} + 1}, \frac{2\pi}{j_2 + j_3 + j_{23} + 1}, \frac{2\pi}{j_{12} + j_3 + j_4 + 1}, \frac{2\pi}{j_1 + j_2 + j_3 + j_4 + 1} \},
\]

(82)

where the minimum is over all \( j_{12} \) that appear in (81). Hence we see that also for \(|q| = 1, |\arg(q)| \) small enough, the matrix of the coordinate transformation from the \( e \) to the \( f \) basis of the charge \( j \) subspace of the space \( V^{2j_1} \otimes V^{2j_2} \otimes V^{2j_3} \) is real-orthogonal.

It follows by iterative use of this fact that, for \(|q| = 1\), the \( q \)-deformed inner product on a given \( N \)-fold tensor product of irreducible \( U_q(sl(2)) \)-modules that we defined in section 3.2.3 can be made independent of the choice of the order in which the tensoring is performed by taking \(|\arg(q)| \) small enough. For the case \( N = 3 \), the different sets of vectors that are declared orthonormal are just the \( e \) and the \( f \)-vectors and the fact that the matrix which relates these is unitary shows that declaring one set to be orthonormal is equivalent to declaring the other set to be orthonormal. On the other hand, given any fixed value of \(|\arg(q)|\), it will not be difficult to construct tensor product representations in which the inner product does depend on the order of the tensoring. In fact, we can expect this to happen as soon as the decomposition of the tensor product module contains non-unitary irreps.

### 3.2.8 Truncated 6\( j \)-symbols; the coassociator

When \( q \) is a root of unity \((q = e^{\pi i/(k+2)})\), we can define truncated 6\( j \)-symbols, related to the truncated tensor product. For these to be non-zero, the conditions (77) have to be changed in such a way that they require that \( j_{12} \) be not just in the tensor product, but even in the truncated tensor product of \( j_1 \) and \( j_2 \), etcetera. This means that the upper bounds \( j_1 + j_2, \ldots, j_1 + j_{23} \) in (76) are sharpened to \( \min\{ j_1 + j_2, k - j_1 - j_2, \ldots, j_1 + j_{23}, k - j_1 - j_{23} \} \). When the arguments satisfy these sharpened conditions, the truncated 6\( j \)-symbols are still given by the formula (78). The truncated 6\( j \)-symbols defined in this way give a canonical isomorphism between the truncated tensor product modules \((V^{2j_1} \otimes V^{2j_2}) \otimes V^{2j_3}\) and \(V^{2j_1} \otimes (V^{2j_2} \otimes V^{2j_3})\). Moreover, this isomorphism is preserves the \( q \)-deformed inner products on the truncated tensor product spaces. To see this, note first of all that the truncated 6\( j \)-symbols are real (this follows easily from (78)). Also, it is known that the truncated 6\( j \)-symbols satisfy an analogue of the orthogonality relations (81). We have

\[
\sum_{j_{12}} \left\{ \begin{array}{ll} j_1 & j_2 \\ j_3 & j_4 \\ j_{12} & j_{23} \end{array} \right\} \left\{ \begin{array}{ll} j_1 & j_2 \\ j_3 & j_4 \\ j_{12} & j_{23} \end{array} \right\} = \delta_{j_{23}j_{23}'} \tag{83}
\]

where the sum is now restricted to the \( j_{12} \) that are allowed by the truncated tensor product. It follows that the matrix of the mapping between \((V^{2j_1} \otimes V^{2j_2}) \otimes V^{2j_3}\) and \(V^{2j_1} \otimes (V^{2j_2} \otimes V^{2j_3})\) is real-orthogonal and hence that the mapping preserves the inner product. The proof of the relations (83) uses the usual orthogonality relations and the formula (rfrsyms) in section 3.2.3.

Using the isomorphism given by the truncated 6\( j \)-symbols, we can identify the spaces \((V^{2j_1} \otimes V^{2j_2}) \otimes V^{2j_3}\) and \(V^{2j_1} \otimes (V^{2j_2} \otimes V^{2j_3})\), so that we have a well-defined three-particle Hilbert space. The isomorphism may also be used to define braiding transformations...
on truncated tensor products. Recall from the end of section 3.2.6 that we could use the $R$-matrix to define braiding of two particles, but that there were difficulties if we wanted to braid particles “over a single bracket” in a multi-particle Hilbert space. These difficulties can now be resolved using the mappings given by the truncated 6$j$-symbols. For example, if we want to exchange the two rightmost particles in the representation $(\pi^{2j_1} \otimes \pi^{2j_2}) \otimes \pi^{2j_3} \otimes \pi^{2j_4}$, then we can first use the 6$j$-symbols to map the representation space onto that for $\pi^{2j_1} \otimes (\pi^{2j_2} \otimes \pi^{2j_3} \otimes \pi^{2j_4})$, then use the $R$-matrix to exchange the particles and finally use the inverse of the mapping given by the 6$j$-symbols to get back to the representation space of $(\pi^{2j_1} \otimes \pi^{2j_2}) \otimes \pi^{2j_3} \otimes \pi^{2j_4}$. Similarly, any braiding in a multiple truncated tensor product may now be achieved by using the 6$j$-symbols to move the brackets around before and after the actual braiding.

The whole procedure of truncating the tensor product so that it is no longer associative and then defining braiding by identification mappings may be elegantly formalised and brought to the level of the algebra, at the cost of making the connection with non-truncated $U_q(sl(2))$ somewhat less apparent. This has been done in [29] and the resulting structure is called a weak quasitriangular quasi-Hopf algebra, or a weak quasi quantum group. Let us call this $\mathcal{A}$. Some important features of the resulting picture are the following. The coproduct is modified in such a way that it has the truncation built in. As a result, one no longer has $\Delta(1) = 1 \otimes 1$ and one also loses coassociativity. A so-called coassociator is introduced to compensate for this loss. This coassociator is an element $\phi = \sum_k \phi_k^1 \otimes \phi_k^2 \otimes \phi_k^3$ of $\mathcal{A}^\otimes 3$ which is not invertible in $\mathcal{A}$, but which has a quasi-inverse called $\phi^{-1}$ which is the inverse in all the representations that one is interested in and which has the following important property for all $a \in \mathcal{A}$:

$$\phi(\Delta \otimes 1)\Delta(a) = (1 \otimes \Delta)\Delta(a)\phi \quad (84)$$

This ensures that the representations $(\pi^{2j_1} \otimes \pi^{2j_2}) \otimes \pi^{2j_3} \otimes \pi^{2j_4}$ and $\pi^{2j_1} \otimes (\pi^{2j_2} \otimes \pi^{2j_3} \otimes \pi^{2j_4})$ are isomorphic, with the isomorphism given by $\pi^{2j_1} \otimes \pi^{2j_2} \otimes \pi^{2j_3} \otimes \pi^{2j_4}(\phi)$. Of course, this isomorphism is just the one given by the truncated 6$j$-symbols. Clearly, one would like to be able to go from one bracketing of a multiple tensor product to another, using $\phi$, in such a way that it does not matter which individual steps are taken on the way. This will be the case if the diagram in figure 3.2.8 commutes.

![Diagram](attachment:image.jpg)

Figure 5: This diagram shows two ways of going from one bracketing of a fourfold tensor product to another. The arrows denote the canonical isomorphisms given by the coassociator (or the truncated 6$j$-symbols). The diagram will commute if the condition (85) on the coassociator is satisfied.

To make this diagram commute, we need to impose the following condition on the
coassociator [37]:

\[(1 \otimes 1 \otimes \Delta)(\phi)(\Delta \otimes 1 \otimes 1)(\phi) = (1 \otimes \phi)(1 \otimes \Delta \otimes 1)(\phi)(\phi \otimes 1)\] (85)

In terms of 6j-symbols, this condition becomes

\[\{ j_{12} \quad j_3 \quad j_{123} \} \{ j_1 \quad j_2 \quad j_{12} \} = \sum_{j_{23}} \left\{ j_1 \quad j_2 \quad j_{12} \right\} \left\{ j_1 \quad j_{23} \quad j_{123} \right\} \] (86)

This condition will clearly be satisfied for non-truncated 6j-symbols, since the sides of the equation just correspond to two ways of doing the same coordinate transformation in that case. For non-truncated 6j-symbols the coordinate transformations change to mappings that really do something, but the equation above still holds.

When there is a non-trivial coassociator, the last two conditions in (88), which guaranteed the compatibility of fusion and braiding, change to

\[(\Delta \otimes 1)(R) = \phi_{132} R_{13} \phi_{132}^{-1} R_{23} \phi_{123}\]
\[(1 \otimes \Delta)(R) = \phi_{231}^{-1} R_{13} \phi_{231} R_{12} \phi_{123}^{-1}\] (87)

and these in turn imply the following quasi-Yang-Baxter equation [37], which is the analogue of (70):

\[R_{12} \phi_{132} R_{13} \phi_{132}^{-1} R_{23} \phi_{123} = \phi_{321} R_{23} \phi_{231}^{-1} R_{13} \phi_{213} R_{12}\] (88)

This relation ensures that the recipe that we gave for performing braidings does indeed give a representation of the braid group.

### 3.2.9 Symmetries of the truncated 6j-symbols

The truncated 6j-symbols still satisfy the classical, Regge and reflection symmetries of the 6j-symbols, but as it turns out, we also have two sets of extra symmetries, which do not have an analogue at \(q = 1\).

The first of these two sets was mentioned already in [32] and it is important in the proof of the truncated orthogonality relations (83). If we define \(\overline{j} := k + 1 - j\), then the symmetries in this set can all be generated from the untruncated symmetries (such as the classical symmetries (80)) and the identity

\[\{ \overline{j}_1 \quad \overline{j}_2 \quad \overline{j}_{12} \} = (-1)^{j_3 + j_{23} - j - j_{12} + 2j_{1} + 1} \{ j_1 \quad j_2 \quad j_{12} \}\] (89)

In particular, we get from this that

\[\{ j_1 \quad j_2 \quad j_{12} \} = i(-1)^{j_3 + j_{12} - j - j_{1} + 2j_{12} + 1} \{ \overline{j}_1 \quad \overline{j}_2 \quad \overline{j}_{12} \}\] (90)

To prove the truncated orthogonality relations (83), we may now start from the the untruncated orthogonality relations (81). We may split the sum in (81) into three parts as in
We have that are associated with (truncated) tensor products of \( U \). In this section, we will give a systematic description of the braid group representations.

3.2.10 Braiding and \( 6j \)-symbols

Now if \( \min\{j_1 + j_2 + j_3 + j_4, k - j_1 - j_2, k - j_3 - j_4\} \) equals \( j_1 + j_2 \) or \( j_3 + j_4 \), then all the \( 6j \)-symbols in the last two summations are zero, because their arguments don’t satisfy the conditions (76). If \( \min\{j_1 + j_2 + j_3 + j_4, k - j_1 - j_2, k - j_3 - j_4\} \) equals \( k - j_1 - j_2 \), then the second summation on the right hand side is empty and the third is zero because the \( j_{12} \) and \( j_{12} \) terms cancel each other using (90) (if there is a middle term in the summation then this also vanishes using (90)). Finally, if \( \min\{j_1 + j_2, j_3 + j_4, k - j_1 - j_2, k - j_3 - j_4\} \) equals \( k - j_3 - j_4 \), then one can use the explicit formula (78) for the \( 6j \)-symbols to show that all the terms of the middle summation vanish, while (90) still makes sure that the last summation vanishes because of pairwise cancellation of terms. In any case, the summation on the left, which is the summation in (51), equals the first summation on the right, which is the summation in (83) and this shows the validity of the truncated orthogonality relations.

We will now present the second set of symmetries of the truncated \( 6j \)-symbols. These symmetries are related to the identities (53) for the truncated fusion rules and the seem to have gone unnoticed until now. They are generated by the following identities

\[
\left\{ \begin{array}{ccc}
  j_1 & j_2 & j_{12} \\
  j_3 & j & j_{23}
\end{array} \right\} = (-1)^{k+j_1+j_3+j_{12}+j_{23}} \left\{ \begin{array}{ccc}
  \hat{j}_1 & \hat{j}_2 & \hat{j}_{12} \\
  \hat{j}_3 & \hat{j} & \hat{j}_{23}
\end{array} \right\} = (-1)^{k+j_{12}+j_3+j} \left\{ \begin{array}{ccc}
  \hat{j}_1 & \hat{j}_2 & \hat{j}_{12} \\
  \hat{j}_3 & \hat{j} & \hat{j}_{23}
\end{array} \right\}.
\]

(92)

Here, we have defined \( \hat{j} := \frac{k}{2} - j \), in accordance with the definition of (54) of \( \hat{\Lambda} \). Of course all the identities related to these by the classical, Regge and reflection symmetries are also symmetries. The above identities may be proved in the following way. First notice that the replacements of spins are made in a way that is consistent with the truncated tensor product decomposition. Hence, the arguments of the \( 6j \)-symbol on the left satisfy the truncated version of the conditions (76) exactly if the arguments in the other two \( 6j \)-symbols do. This means we can fill in formula (78) in all three cases. To show that the results are equal, one needs an identity which holds for \( q \)-factorials at \( q = e^{2\pi i/(k+2)} \).

We have

\[
|k + 1 - a|_q! = \frac{(k + 1)_q!}{(a)_q!}.
\]

(93)

Using this identity, it is easy to show that the \( \Delta \)-factors are equal for all three \( 6j \)-symbols in (72). For the middle \( 6j \)-symbol in (92), we can now see that the sum over \( z \) in (78) is equal to that for the untransformed \( 6j \)-symbol by making the substitution \( z \rightarrow z + k - (j_1 + j_3 + j_{12} + j_{23}) \) and using the \( q \)-factorial identity above twice. The proof for the rightmost \( 6j \)-symbol in (92) is similar, but uses the substitution \( z \rightarrow -z + k + j_3 + j + j_{12} \).

3.2.10 Braiding and \( 6j \)-symbols

In this section, we will give a systematic description of the braid group representations that are associated with (truncated) tensor products of \( U_q(sl(2)) \) representations. First of all, let us look at the braiding in a tensor product of two irreps \( \pi^{\Lambda_1} \) and \( \pi^{\Lambda_2} \). We
can decompose this tensor product into irreps as in equation (48) or (53). From these formulae, we see that any irrep can occur at most once in this decomposition; we say that the tensor product decomposition is multiplicity-free. It follows from this, using Schur’s lemma, that any map from the tensor product module \( V^{\Lambda_1} \otimes V^{\Lambda_2} \) to the tensor product module \( V^{\Lambda_2} \otimes V^{\Lambda_1} \) that commutes with the quantum group action on these modules, is a constant on each of the irreducible summands of \( V^{\Lambda_1} \otimes V^{\Lambda_2} \). The exchange matrix \( \sigma \equiv \sigma(\pi^{\Lambda_1} \otimes \pi^{\Lambda_2})(R) \) is such a map. Hence, we can choose bases for \( V^{\Lambda_1} \otimes V^{\Lambda_2} \) and \( V^{\Lambda_2} \otimes V^{\Lambda_1} \) such that the action of \( \sigma \) is described by a diagonal matrix with respect to these bases. Of course, the basis vectors in each case are just the basis vectors of each irreducible summand \( \pi^\Lambda \) and the action of \( \sigma \) on these will depend on \( \Lambda_1, \Lambda_2 \) and \( \Lambda \) and not on \( m \). Explicitly, one has

\[
\sigma(\pi^{\Lambda_1} \otimes \pi^{\Lambda_2})(R)|_{\pi^\Lambda} = (-1)^{\Lambda_1 + \Lambda_2} q^{\frac{1}{2} \left(c_{\Lambda_1} - c_{\Lambda_1} - c_{\Lambda_2}\right)}
\]

where \( c_{\Lambda_i} = \frac{1}{2} (\frac{\Lambda_i}{2} + 1) \) is the value of the undeformed Casimir for the representation \( \pi^{\Lambda_i} \). This can be derived from the formula (72) for the elements of the \( R \)-matrix, using the formulae for the Clebsch-Gordan coefficients given in [32]. For the case \( \Lambda_2 = 2 \), one may also check it from (50), using (52). Note that the action of \( \sigma \) given above is effected by a unitary matrix exactly if \( q \) is a root of unity. Therefore, if we use the \( q \)-deformed inner product on the tensor product space that makes the basis described above orthonormal, then \( \sigma \) is a unitary operation, as we already remarked in a special case below formula (73).

Now let us look at a tensor product of \( n \) quantum group irreps \( \pi^{\Lambda_1}, \ldots, \pi^{\Lambda_n} \). In such a tensor product, there are a number of natural bases which reflect the structure of the tensor product. More precisely, there is on such basis for each way in which the tensor product can be built up by adding subsequent factors. We have already described the situation for three tensor factors in the section 3.2.7. In this case, there were two of these natural bases and the transformation that related these was described by the \( 6j \)-symbols. In the case of \( n \) factors, we will choose to work with the natural basis one gets by adding subsequent tensor factors on the right, i.e. the basis induced by the following “bracketing” of the tensor product:

\[
\pi^{\Lambda_1} \otimes \pi^{\Lambda_2} \otimes \ldots \otimes \pi^{\Lambda_n} = (\ldots (\pi^{\Lambda_1} \otimes \pi^{\Lambda_2}) \otimes \pi^{\Lambda_3}) \ldots \otimes \pi^{\Lambda_{n-1}}) \otimes \pi^{\Lambda_n})
\]

The elements of this basis can be labelled by their overall \( H \) eigenvalue \( m \), their overall fusion channel \( j_n \) and and \( n - 1 \) intermediate fusion channels \( j_1, \ldots, j_{n-1} \). We may thus write these basis elements as \( e_{j_1, \ldots, j_n, m}^{\sigma_1, \ldots, \sigma_n} \), where we have defined \( \sigma_i = \frac{\sigma - 1}{2} \). Here, we have \( j_1 = J_1 \) and \( j_i \) is one of the summands in \( j_{i-1} \otimes J_i \) for \( i > 1 \). If there is no cause for confusion, we will suppress the upper indices and write \( e_{j_1, \ldots, j_n, m} \). It is easy to show that the set of \( e_{J_1, \ldots, j_n, m} \) for which all the \( j \)'s are equal forms a basis for an irrep of \( U_q(sl(2)) \) of type \( \pi^{2j_n} \), i.e. the action on this set corresponds to the action given in formula (34). Hence, it follows that the tensor product representation becomes a \( * \)-representation if we take the inner product which makes the \( e_{j_1, \ldots, j_n, m} \) orthonormal and if each of the possible \( \pi^{2j_n} \) is itself a \( * \)-representation. Note that if we are working with a truncated tensor product, then there will be a different truncated tensor product space for each bracketing, because of the non-associativity of this tensor product. The bases we have described here then provide canonical bases for the different subspaces of the ordinary tensor product that one gets from the different bracketings.
The basis of $e_{j_1,...,j_n,m}$ is very suited to a description of the braiding. Suppose we want to exchange particles $i$ and $i+1$, i.e. we want to calculate the action of the exchange $\tau_i$ on $e_{j_1,...,j_n,m}$. We can do this in three steps:

1. Move particle $i$ completely to the left, using right-over-left exchanges. Since the representations $\pi^{\Lambda_1},...,\pi^{\Lambda_{i-1}}$ fuse together to the representation $\pi^{2j_i-1}$ and since the fusion of this $\pi^{2j_i-1}$ with $\pi^{\Lambda_i}$ gives $\pi^{2j_i}$, this operation gives us just a constant factor. We have

$$e_{J_1,...,J_n}^{J_{i_1},...,J_{i_m},...} \rightarrow (-1)^{j_i-J_i-j_{i-1}} q^\frac{1}{2}(c_{j_i-1}+c_{j_i}) f_{J_1,...,J_n}^{J_{i_1},...,J_{i_m},...}$$

Here we have defined $c_j = j(j+1)$, in accordance with the definition of $c_\Lambda$ above. Also, the vector $f_{J_1,...,J_n}^{J_{i_1},...,J_{i_m},...}$ is an element of the natural basis for the tensor product that one gets by first tensoring together $\pi^{\Lambda_1},...,\pi^{\Lambda_{i-1}}$, adding successive factors on the right, then tensoring on $\pi^{\Lambda_i}$ from the left and finally tensoring on the remaining factors from the right. To get the result (96), one uses (94) and, repeatedly, (68) or, for truncated tensor products, (57).

2. Now change the bracketing, using the $6j$-symbols, so that we end up in a basis in which the representations $\pi^{\Lambda_i}$ and $\pi^{\Lambda_1},...,\pi^{\Lambda_{i-1}}$ no longer fuse to a fixed representation, but the representations $\pi^{\Lambda_1},...,\pi^{\Lambda_{i-1}}$ and $\pi^{\Lambda_{i+1}}$ do. The new basis is the natural basis for the tensor product which one gets by first tensoring together $\pi^{\Lambda_1},...,\pi^{\Lambda_{i-1}}$, adding successive factors on the right, then tensoring on $\pi^{\Lambda_{i+1}}$ from the right, then tensoring on $\pi^{\Lambda_i}$ from the left and finally tensoring on the remaining factors from the right. In the new basis, the label $j_i$ (which gave the overall quantum group charge of particles 1 to $i$) is replaced by a new label $j'$, which gives the overall quantum group charge of particles 1, 2, ..., $i-1$, $i+1$. All the other labels are as before. If we denote the elements of the new basis by $g_{J_1,...,J_n}^{J_{i_1},...,J_{i_m},...}$, then the $f$-basis can be written in terms of the $g$'s as

$$f_{J_1,...,J_n}^{J_{i_1},...,J_{i_m},...} = \sum_{j'} \left\{ \begin{array}{ccc} \Lambda & \Lambda & \Lambda \\ J_i & j_{i-1} & j_i \\ J_{i+1} & j_{i+1} & j' \end{array} \right\} g_{J_1,...,J_n}^{J_{i_1},...,J_{i_m},...}$$

where we have used the fact that the representations carried by the particles 1, ..., $i-1$ fuse to $\pi^{2j_i}$ and that these particles can thus be treated as one particle that carries the representation $\pi^{2j_i+1}$.

3. Now we move particle $i$ to the right, using left-over-right exchanges, until it has reached the position to the right of particle $i-1$. At the end of this process, we have effectively only produced a left-over-right exchange of the particles $i$ and $i+1$, as we wanted. In the $g$ basis, the process of exchanging particle $i$ past particles 1, ..., $i-1$ and $i+1$ is described once again by a simple phase factor (compare the first step of the calculation), since the representations on particles 1, ..., $i-1$, $i+1$ fuse to $\pi^{2j'}$ and this fuses with $\pi^{2J_i}$ into the fixed fusion channel $\pi^{2j_{i+1}}$. We get

$$g_{J_1,...,J_{i},...,J_n}^{J_{i},...,J_{i+1},...,J_n} \rightarrow (-1)^{j'+J_i-j_{i+1}} q^\frac{1}{2}(c_{j_{i+1}}-c_{j'}) e_{J_1,...,J_{i+1},...,J_n}^{J_{i},...,J_{i+1},...,J_n}$$

where $e_{J_1,...,J_{i+1},...,J_n}^{J_{i},...,J_{i+1},...,J_n}$ is an element of the basis we started with.
We may now write down the action of the elementary exchange $\tau_i$ on the $e$-basis as the cumulative effect of these three steps. We have

$$\tau_i e_{J_1, \ldots, J_i, \ldots, J_n, j_1, \ldots, j_i, \ldots, j_n, m} = \sum_{j'} (-1)^{J_i-j_{i-1}+J_{i+1}-j_i} q^{\frac{1}{2}(c_{j_i-1}-c_{j_i}+c_{j_{i+1}}-c_{j_i})} \left\{ \begin{array}{ccc} J_i & j_i-1 & j_i \\ J_{i+1} & j_{i+1} & j' \end{array} \right\} e_{J_1, \ldots, J_i, \ldots, J_{i+1}, \ldots, J_n, j_1, \ldots, j', \ldots, j_n, m} \quad (99)$$

Using equation (83), one may check easily that the matrix that describes this transformation is unitary if $q$ is a root of unity, which is the case we are interested in. Hence, if we take the inner product which makes the $e_{j_1, \ldots, j_n, m}$ orthonormal, then the braid group representation which governs the exchanges of particles with $U_q(sl(2))$-charges is unitary, as it should be. If either $J_i = J_{i+1}$ or $j_{i-1} = j_{i+1}$, then it follows from the classical symmetries (80) that the matrix for $\tau_i$ is also symmetric.

### 3.2.11 Hidden quantum group symmetry

We will say that a quantum mechanical system has a hidden quantum group symmetry if there is an action of a quantum group $\mathcal{A}$ on the Hilbert space of the theory which has the property that it commutes with all the observables of the theory. For a system of particles which carry $U_q(sl(2))$-representations, this means in particular that the $H$-eigenvalues associated to the particles will not be observable, while on the other hand, one can allow observables which make it possible to determine the $U_q(sl(2))$-representation associated to each of the particles. In other words, the total “quantum spin” of each particle would be measurable, but the components of this quantum spin would not be measurable. The above definition of hidden quantum group symmetry is just what we have distilled from various sources in the literature that mention hidden quantum group symmetries (see section 4 for references). Note however that there does not seem to be a completely standard definition of the concept of this concept. Let us say more about what the above definition means within our context. Suppose we have a system of $n$ particles that carry representations $\pi^{\Lambda_1}, \ldots, \pi^{\Lambda_n}$ of a quantum group $\mathcal{A}$. In that case the whole system will be in a state in the tensor product space $V^{\Lambda_1} \otimes \ldots \otimes V^{\Lambda_n}$. If this tensor product may be decomposed into irreducibles then the decomposition will take the form

$$V^{\Lambda_1} \otimes \ldots \otimes V^{\Lambda_n} = \bigoplus_{\Lambda} U^{\Lambda}_{\Lambda_1, \ldots, \Lambda_n} \otimes V^{\Lambda} \quad (100)$$

Here, $U^{\Lambda}_{\Lambda_1, \ldots, \Lambda_n}$ is a vector space whose dimension equals the multiplicity of the irrep $V^{\Lambda}$ of $\mathcal{A}$ in the tensor product. When no confusion seems possible, we will just write $U^{\Lambda}$. If the $\mathcal{A}$-symmetry of this system is a hidden symmetry, then it follows that all the observable operators act only on the spaces $U^{\Lambda}$ without mixing these. That is, every observable $\hat{O}$ should take the form

$$\hat{O} = \sum_{\Lambda} \hat{O}^{\Lambda} \otimes I_{V^{\Lambda}} \quad (101)$$

where each $\hat{O}^{\Lambda}$ is an operator acting on $U^{\Lambda}$ and $I_{V^{\Lambda}}$ is the identity operator on $V^{\Lambda}$. Since all the observables have this structure, the state of the system can be uniquely characterised by a list of vectors, one for each of the spaces $U^{\Lambda}$. Usually, the overall quantum group charge(s) of the system will be well-defined. In other words, the state of
the system will be described by a vector in one of the summands in the decomposition \( [100] \). In fact, there may be superselection rules which prevent superposition of states from different summands in \( [100] \). If the system as a whole is in the quantum group representation \( \pi^\Lambda \), then the state of the system may be described by a vector in the space \( U^\Lambda \). Now note that the braid group representation on the tensor product \( V^{\Lambda_1} \otimes \ldots \otimes V^{\Lambda_n} \) which comes from the action of the \( R \)-matrix of \( \mathcal{A} \), induces an action of the braid group on each of the \( U^\Lambda \). This follows from the fact that the action of the braid group elements, like the action of any observable, commutes with the action of \( \mathcal{A} \). Any operator that represents a braid group element will thus be of the general form given above for observables. Thus, if one wants to describe only the monodromy or braid group representation that governs the statistics of the system at a fixed number of particles with given overall quantum group charge \( \Lambda \), one can restrict oneself to the space \( U^\Lambda \). It should be clear from the previous section what form such a representation would take for a system of \( n \) particles with a hidden \( U_q(sl(2)) \) symmetry. In this case we have the canonical basis of the \( e_{j_1,\ldots,j_n,m}^{\mathcal{J}} \) for the tensor product of the \( n \) representations of \( U_q(sl(2)) \). Of these, we need only retain the ones whose overall charge \( j_n \) is equal to the fixed total charge of the system, say \( j_n = j \). The remaining vectors may then be written as tensor product vectors:

\[
e^{\mathcal{J}_1,\ldots,\mathcal{J}_n}_{j_1,\ldots,j,m} = e^{\mathcal{J}_1,\ldots,\mathcal{J}_n}_{j_1,\ldots,j} \otimes | j, m \rangle
\]

where \( e^{\mathcal{J}_1,\ldots,\mathcal{J}_n}_{j_1,\ldots,j} \) now denotes a vector in the space \( U^j \). The braid group representation on \( U^j \) may now be read off immediately from the formula (99) which gave the braiding for the full tensor product of \( U_q(sl(2)) \)-representations. The matrix elements between the \( e^{\mathcal{J}_1,\ldots,\mathcal{J}_n}_{j_1,\ldots,j,m} \) which are given in this formula can be used in unchanged form for the vectors \( e^{\mathcal{J}_1,\ldots,\mathcal{J}_n}_{j_1,\ldots,j} \), since they already did not depend on \( m \) and did not mix different \( j_n \).

A similar treatment of braid group representations for systems with hidden quantum group symmetry is possible in any situation in which \( 6j \)-symbols may be defined for the quantum group representations involved. This is the case if the tensor products of these representations have a multiplicity free decomposition into irreducibles.

### 3.2.12 Braiding of identical particles and fusion diagrams

In the previous subsections, we have described the braiding for a system of \( n \) particles with a hidden \( U_q(sl(2)) \)-symmetry and we have indicated that a similar description should hold for many systems that involve other quantum groups. Let us now look at the special case in which the particles are identical. This case is of interest for the description of the braiding of identical quasiholes in the RR-states. When the particles are identical, they all carry the same quantum group representation \( \pi^{2\mathcal{J}} \) and hence the upper indices on the elements \( e^{\mathcal{J}_1,\ldots,\mathcal{J}_n}_{j_1,\ldots,j} \) of the canonical basis for the space \( U^j \) are all equal to \( \mathcal{J} \). Fixing \( \mathcal{J} \), we may thus forget about the upper indices and write just \( e_{j_1,\ldots,j_n} \). As in the previous section, we also fix \( j_n = j \). Now the \( n \)-tuple \((j_1,\ldots,j_n)\) may be seen as a path of length \( n \) through the space of representation labels of \( U_q(sl(2)) \), which starts at the trivial representation \( j_0 = 0 \) and ends at \( j \). Of course, not all paths through the space of representation labels will correspond to an element of the canonical basis. A path will represent a basis vector precisely if the representation at position \( m \) of the path may always be found in the tensor product of the representation at position \( m - 1 \) with the representation \( \pi^{2\mathcal{J}} \). This in turn means precisely that the path lies on the fusion (or Bratteli) diagram for the representation \( \pi^{2\mathcal{J}} \). Thus, the paths of length \( n \) on the Fusion diagram of the representation \( \pi^{2\mathcal{J}} \) may be
taken as a basis for the braid group representation that describes exchanges in a system of \( n \) particles that carry the quantum group representation \( \pi^{27} \). From equation (99), one may now easily read off that the braid group generator \( \tau_m \) will only mix paths that are identical everywhere except at position \( m \).

As an example let us look at the case of \( n \) particles in the 2-dimensional representation of \( U_q(sl(2)) \). The fusion diagram for this representation is just the diagram drawn in figure 4. Let \( p = (p^{(1)}, \ldots, p^{(n)}) = ((\Lambda_p^{(1)}, 1), \ldots, (\Lambda_p^{(n)}, n)) \) be a path on this diagram which starts at the point \( p^{(1)} = (\Lambda_p^{(1)}, 1) = (0, 0) \) and ends at the point \( p^{(n)} = (\Lambda_p^{(n)}, n) = (\Lambda, n) \). Then there is either no path which differs from \( p \) only at its \( m^{th} \) vertex or there is exactly one such path. If there is such a path, we will call it \( \sigma_m(p) \). Let us write down the action of the exchange \( \tau_m \) on a path \( p \). We start with the cases in which \( p \) does not have a partner path. Using equation (99), we see that such paths just get a phase factor. There are four cases:

\[
\begin{align*}
\Lambda_p^{(m-1)} &= 2j < \Lambda_p^{(m)} < \Lambda_p^{(m+1)} \quad \Rightarrow \quad \tau_m p = q^{1/4} p \\
\Lambda_p^{(m)} &= 2j > \Lambda_p^{(m+1)} \quad \Rightarrow \quad \tau_m p = q^{1/4} p \\
\Lambda_p^{(m-1)} &= \Lambda_p^{(m+1)} = 0, \quad \Lambda_p^{(m)} = 1 \quad \Rightarrow \quad \tau_m p = -q^{-3/4} p \\
\Lambda_p^{(m-1)} &= \Lambda_p^{(m+1)} = k, \quad \Lambda_p^{(m)} = k - 1 \quad \Rightarrow \quad \tau_m p = -q^{-3/4} p
\end{align*}
\tag{103}
\]

These equations may be summarised by saying that the path \( p \) gets a factor of \( q^{1/4} \) if it does not change direction at its \( m^{th} \) vertex, whereas it gets a factor \(-q^{-3/4}\) if it does change direction (which can only happen at the boundary of the diagram). In obtaining the equations, we used the following values for the 6\( j \)-symbols involved:

\[
\left\{ \begin{array}{ccc}
\frac{1}{2} & j & j + \frac{1}{2} \\
\frac{1}{2} & j + 1 & j + \frac{1}{2}
\end{array} \right\} = \left\{ \begin{array}{ccc}
\frac{1}{2} & j & j - \frac{1}{2} \\
\frac{1}{2} & j - 1 & j - \frac{1}{2}
\end{array} \right\} = \left\{ \begin{array}{ccc}
\frac{1}{2} & 0 & 0 \\
\frac{1}{2} & 0 & \frac{1}{2}
\end{array} \right\} = 1
\]

\[
\left\{ \begin{array}{ccc}
\frac{1}{2} & \frac{3}{2} & \frac{k+1}{2} \\
\frac{1}{2} & \frac{k}{2} & \frac{k+1}{2}
\end{array} \right\} = -1
\tag{104}
\]

We are now left with the case in which \( p \) does have a partner path \( \sigma(p) \). In this case, we will certainly have \( \Lambda_p^{(m-1)} = \Lambda_p^{(m+1)} = \Lambda_{\sigma(p)}^{(m-1)} = \Lambda_{\sigma(p)}^{(m+1)} = 2j \) and, exchanging \( p \) with \( \sigma(p) \) if needed, we can also make sure that \( \Lambda_p^{(m)} > \Lambda_{\sigma(p)}^{(m)} \), so that \( \Lambda_p^{(m)} = 2j + 1, \quad \Lambda_{\sigma(p)}^{(m)} = 2j - 1 \). The relevant 6\( j \)-symbols for this case are given by

\[
\left\{ \begin{array}{ccc}
\frac{1}{2} & j & j + \frac{1}{2} \\
\frac{1}{2} & j & j + \frac{1}{2}
\end{array} \right\} = \frac{1}{[2j+1]_q}
\]

\[
\left\{ \begin{array}{ccc}
\frac{1}{2} & j & j - \frac{1}{2} \\
\frac{1}{2} & j & j - \frac{1}{2}
\end{array} \right\} = -\frac{1}{[2j+1]_q}
\]

\[
\left\{ \begin{array}{ccc}
\frac{1}{2} & j & j + \frac{1}{2} \\
\frac{1}{2} & j & j + \frac{1}{2}
\end{array} \right\} = \left\{ \begin{array}{ccc}
\frac{1}{2} & j & j - \frac{1}{2} \\
\frac{1}{2} & j & j + \frac{1}{2}
\end{array} \right\} = -\sqrt{\frac{[2j+2]_q [2j]_q}{[2j+1]_q}}
\tag{105}
\]

and combining this with the phase factors in (99), we see that, in the linear space with basis \( \{p, \sigma(p)\} \), the exchange \( \tau_m \) is represented by the matrix

\[
\tau_m = \frac{q^{-1/4}}{[d]_q} \left( \begin{array}{cc}
-q^{-d/2} & -\sqrt{[d+1]_q [d-1]_q} q^{d/2} \\
-\sqrt{[d+1]_q [d-1]_q} & q^{d/2}
\end{array} \right)
\tag{106}
\]
where we have defined \( d := 2j + 1 \). This matrix for \( \tau_m \) is obviously symmetric. It is also unitary, as can be easily seen, using the fact that \( |d + 1|_q |d - 1|_q \) equals \( |d|_q^2 - 1 \).

We will denote the braid group representation on the paths which start from \((0,0)\) and end at \((\Lambda, n)\) by \( \rho_n^\Lambda \) and the corresponding modules by \( U_n^\Lambda \). An induction argument taken from [38] shows that the \( \rho_n^\Lambda \) are all irreducible and that they are non-isomorphic for different \( \Lambda \): The representation \( \rho_0^1 \) of the trivial group \( B_0 \) is irreducible because it is one dimensional and \( \Lambda = 1 \) is the only possibility at \( n = 1 \). Now suppose that, for all \( \Lambda \) and all \( n < m \), all \( \rho_n^\Lambda \) are irreducible and non-isomorphic for different \( \Lambda \). Then the representations \( \rho_m^\Lambda \) are irreducible and mutually non-isomorphic for all \( \Lambda \). For let \( S \) be a non-zero \( B_{m-1} \)-invariant submodule of \( U_m^\Lambda \) and suppose for convenience that \( \Lambda \) does not equal 0 or \( k \). Then \( S \) has a unique decomposition into \( B_m \)-invariant submodules which is clearly given by \( U_m^\Lambda = U_{m-1}^{\Lambda-1} \oplus U_{m-1}^{\Lambda+1} \) (just forget the last step in the paths). Because the \( \rho_m^{\Lambda-1} \) are non-isomorphic for different values of \( \Lambda \) (by the induction hypothesis), it follows immediately that the \( \rho_m^{\Lambda} \) are also non-isomorphic for different values of \( \Lambda \); their modules have different decompositions into irreducible \( B_{m-2} \)-modules. Moreover, since \( \rho_m^{\Lambda-1} \) and \( \rho_m^{\Lambda+1} \) are irreducible and non-isomorphic, it follows that the only possible proper \( B_{m-1} \)-invariant submodules of \( U_m^\Lambda \) are \( U_{m-1}^{\Lambda-1} \) and \( U_{m-1}^{\Lambda+1} \). However, these will clearly be mixed by the exchange \( \tau_{m-1} \), so that \( U_m^\Lambda \) has no proper \( B_{m-1} \)-invariant subspaces. Hence \( S = U_m^\Lambda \) and \( \rho_m^\Lambda \) is irreducible. Of course if \( \Lambda \) equals 0 or \( k \) then the argument becomes even simpler and we need not repeat it.

4 Conformal field theory and quantum groups

In this section, we review the correspondence between conformal field theory and quantum groups. In section 4.1, we give a short general description of this correspondence, illustrated with the example of \( U_q(\mathfrak{sl}(2)) \) versus the \( \widehat{\mathfrak{sl}}(2)_k \) WZW-theory. In the next section, we go on to describe the quantum group \( U_q(\mathfrak{sl}(m)) \) and its relation to the \( \widehat{\mathfrak{sl}}(m)_k \) WZW-theory. In section 4.3, we describe representations of the braid group \( B_n \) which factor over the Hecke algebra \( H_{n,q} \). These are important in the description of the braiding of a system of \( n \) particles with hidden \( U_q(\mathfrak{sl}(m)) \)-symmetry. In section 4.4, we describe a quantum group for the chiral boson. Finally, in section 4.5, we indicate quantum groups which correspond to the parafermion theory that is used in the description of the Read-Rezayi states.

4.1 The CFT-QG relation

The relation between quantum groups and conformal field theories has been much studied over the years and it is believed that every conformal field theory has associated to it some quantum group (or generalisation thereof) with the following properties:

- Each chiral primary field of the CFT (or equivalently: each irreducible representation of the chiral algebra) corresponds to an irreducible representation of the quantum group.

- The fusion algebra of the CFT is identical to the representation ring of the quantum group, i.e. fusion of chiral primaries corresponds to taking the tensor product of quantum group irreps.
The braiding of the chiral primary fields in conformal blocks corresponds to the braiding in the tensor product of quantum group representations, as described by means of an $R$-matrix and, if needed, a coassociator.

The points above can be illustrated by the case of the $\widehat{sl}(2)_k$ WZW-theory, whose associated quantum group is $U_q(sl(2))$ at $q = e^{\frac{2\pi i}{k+\hat{g}}}$ for this value of $q$, the unitary irreducible representations $\pi^\Lambda$ of $U_q(sl(2))$ that have positive quantum dimension are indeed in one to one correspondence with the affine primary fields $G^\Lambda$ of the WZW-theory. Moreover, comparing equations (2) and (53), we see that the fusion rules for the WZW-fields are the same as the decomposition rules for tensor products of $U_q(sl(2))$-representations. We described the braid group representations associated to the fundamental representation of $U_q(sl(2))$ in section 3.2.12. The braiding of the corresponding conformal blocks of the WZW-theory was calculated by Tsuchiya and Kanie [39, 40] and this braiding is indeed the same as that described in section 3.2.12, up to a renormalisation of the blocks. In connection with this, the $q$-6$j$-symbols may be identified with the fusion matrix of the $\widehat{sl}(2)_k$ conformal field theory as defined by Moore and Seiberg [41, 42]. The pentagon equation for this fusion matrix then corresponds to the equation (86) (see also figure 3.2.8) and the hexagon equation is just the quasi-Yang-Baxter equation (88), written in terms of 6$j$-symbols by means of (99).

Note that it is essential in the above, that the truncated tensor product of $U_q(sl(2))$-representations is used, rather than the ordinary one. In other words, we may say that it is essential that one uses a weak quasi-quantum group rather than an ordinary quantum group. This is not a very special situation; the fusion rules of many CFTs cannot be reproduced by those of ordinary quantum groups (or quantum groups with an ordinary tensor product). On the other hand, there is mathematical work [43, 44] in which it is shown that, given a CFT, one may always find weak quasi quantum groups that will reproduce its fusion and braiding properties. This does not mean that it is known for all conformal field theories how the quantum group generators can be represented in terms of operators in the conformal field theory. In fact, no general construction for these operators seems to be known, although several proposals have been made for CFTs that have a Coulomb gas description [45, 46, 47]. Through this work, much is known about the quantum groups for the WZW-models. In particular, it is well known that for any semisimple Lie algebra $g$, the $\hat{g}_k$ WZW-model and the quantum group $U_q(g)$ at $q = e^{2\pi i/(k+\hat{g})}$ are related in the way we have described above (here $\hat{g}$ is the dual Coxeter number of $g$). In the following, we shall be especially interested in the case $g = sl(m)$, because of the close relation between the parafermion theory that describes the RR-states and the $\widehat{sl(2)}_k$ and $\widehat{sl(k)}_2$ WZW-theories.

Before we go on, let us cite a few general references on the relation between CFTs and quantum groups. Books that include information on this are for example [34, 48] and a review article is [26]. An early description of the correspondence between $U_q(sl(2))$ and the $sl(2)_k$ WZW-theory can be found in [49].

### 4.2 $U_q(sl(m))$ and the $\widehat{sl(m)}_k$ WZW-theory

In this subsection, we recall some facts about the quantum group $U_q(sl(m))$ that is associated with the $\widehat{sl(m)}_k$ WZW-theory. $U_q(sl(m))$ is a $q$-deformation of the universal enveloping algebra $U(sl(m))$ of $sl(m)$. Such a $q$-deformation can be constructed for any
simple Lie algebra $g$. If we denote the simple roots of $g$ by $\alpha_i$, then we can associate to each of these three generators $H_i, L_i^+, L_i^-$ and these will generate $U_q(g)$ as an algebra, subject to the relations

\[
[H_i, H_j] = 0 \\
[H_i, L_j^\pm] = \pm 2 \frac{\alpha_i \alpha_j}{(\alpha_i, \alpha_j)} L_j^\pm \\
[L_i^+, L_j^-] = \delta_{ij} [H_i]_q \\
[L_i^+, L_j^+] = \begin{cases} 0 & \text{if } A_{ij} = 0 \text{ or } i = j \\
1 - A_{ij} & \sum_{s=0}^{1-A_{ij}} (-1)^s q^{(\alpha_i, \alpha_i) s(s+A_{ij}-1)/2} \left[ 1 - \frac{A_{ij}}{s} \right] q^{(L_i^\pm)^{1-A_{ij}-s}} L_j^s (L_i^\pm)^s & \text{otherwise}
\end{cases}
\]

(107)

Here, $A$ is the Cartan matrix of $g$. When $q = 1$, these relations reduce to the relations for the Chevalley-Serre basis of $U(g)$ and when $g = sl(2)$, they reduce to the relations we gave in section 3.2. If $q$ is not a root of unity, the irreducible representation of $U_q(g)$ are labelled by dominant integral weights of $g$ and one may give formulae for the action of the generators which are similar to those given in [34]. When $q$ is a root of unity, one finds again that all these representations remain well-defined, but many are no longer irreducible and in particular there are indecomposable representations.

The coproduct $\Delta$, counit $\epsilon$ and antipode $S$ are given by

\[
\Delta(H_i) = 1 \otimes H_i + H_i \otimes 1 \\
\Delta(L_i^\pm) = L_i^\pm \otimes q^{H_i/4} + q^{-H_i/4} \otimes L_i^\pm \\
\epsilon(1) = 1, \quad \epsilon(L_i^\pm) = \epsilon(H_i) = 0 \\
S(H_i) = -H_i, \quad S(L_i^\pm) = -q^{\rho/2} L_i^\pm q^{-\rho/2}
\]

(108)

Here, $\rho$ is the Weyl-vector of $g$, which is equal to half the sum of the positive roots, or equivalently, to the sum of the fundamental weights. One may check easily that this comultiplication, counit and antipode satisfy the conditions given in section 3.2. As usual, one can define the tensor product of representations through the formula (42) and as in the case of $U_q(sl(2))$, this tensor product will usually not be fully decomposable if $q$ is a root of unity. However, it is once more possible to define a truncated tensor product which involves only a finite set of unitary irreducible representations which is fully decomposable. If $q = e^{2\pi i/(k+\hat{g})}$, where $\hat{g}$ is the dual Coxeter number of $g$, then the irreducible representations involved are each labelled by a dominant integral weight $\Lambda$ such that $(\Lambda, \theta) \leq k$, where $\theta$ is the highest root of $g$. Hence, they are in one to one correspondence with the affine primary fields of the $\hat{g}k$ WZW-theory. Moreover, as in the case of $sl(2)$, the decomposition rules of the truncated tensor product are identical to the fusion rules of the WZW-primaries. One may also define a quantum trace and a corresponding quantum dimension and one may then go from the ordinary to the truncated tensor product by projecting out modules of zero quantum dimension.

An $R$-matrix that satisfies the requirements [38] with the given comultiplication is also known (see for example chapter eight of [27] for details and references), but it is in general not so easy to obtain the exchange matrices in any given tensor product of representations from this $R$-matrix. The reason for this is that, to calculate the action of
the \( R \)-matrix on a tensor product of representations, one needs formulae for the action of the elements of \( U_q(g) \) associated to the roots of \( g \) on both representations in the tensor product. Although it is quite easy to obtain formulae similar to (34) for the action of the raising and lowering operators \( L^+_{\alpha_i} \) associated with the simple roots \( \alpha_i \), the same does not go for the action of the raising and lowering operators that correspond to non-simple roots. Nevertheless, the exchange matrices have been calculated in special cases, one of which is important to us. This is the case of the tensor product of the fundamental \( m \)-dimensional representation of \( U_q(sl(m)) \) with itself (see [27] for a detailed calculation).

Let us denote this fundamental representation by \( \pi_{e_1} \), where \( e_1 \) denotes the highest weight of the representation as in section 2.1.2. We may then write

\[
\sigma(\pi_{e_1} \otimes \pi_{e_1})(R) = q^{\frac{1}{2m}} \left( q^{1/2} \sum_{i=1}^{m} E_{ii} \otimes E_{ii} + \sum_{i \neq j} E_{ij} \otimes E_{ji} + (q^{1/2} - q^{-1/2}) \sum_{i<j} E_{jj} \otimes E_{ii} \right)
\]

where \( E_{ij} \) denotes the matrix whose \((i,j)\) entry is one and whose other entries are zero. One may check easily that this formula gives back the matrix (73) in the case of \( sl(2) \).

In the following, we want to describe the braid group representation that is associated with an \( n \)-fold truncated tensor product \( \pi_{e_1}^{\otimes n} \). Since tensor products that involve multiplicity free, we can apply the methods described in sections 3.2.10 and onwards. That is, we may define \( 6j \)-symbols for the tensor products involved and describe the braiding by formula (93) and finally graphically, in terms of paths on the fusion diagram of the representation \( \pi_{e_1} \). In fact, we can already say quite a lot about the braiding just from the fusion diagram, without a detailed knowledge of the \( 6j \)-symbols. So let us describe this fusion diagram. For \( q = e^{2\pi i/(k+\theta)} \), each vertex of the fusion diagram may be labelled the number of fundamental representations that have been tensored up to that point, together with a dominant integral weight of \( sl(m) \) which satisfies the requirement \((\Lambda, \theta) \leq k \). We may equivalently represent this weight by its Young diagram and if we do this, then the requirement that \((\Lambda, \theta) \leq k \) translates to the restriction that the diagrams should not have more than \( k \) columns. Fusion diagrams of this kind have already been drawn in figures 1 and 2. Instead of using the particle number and the Young diagram for the overall \( U_q(sl(m)) \)-charge, one may also use just a Young diagram to represent each vertex. This Young diagram is then the diagram which reduces to the Young diagram for \( U_q(sl(m)) \)-charge if columns of \( m \) boxes are removed and whose number of boxes is equal to the number of representations tensored up to that vertex. As an example, we show a diagram for \( U_q(sl(3)) \) in figure 13.

The connections between the different vertices are of course determined by the fusion rules for the fundamental representation. These can be elegantly described in terms of Young diagrams. The truncated tensor product of the fundamental representation with the representation that has Young diagram \( Y \) decomposes into the sum of the representations whose Young diagrams have at most \( m \) columns and may be formed by adding one box to \( Y \) and removing any columns of length \( m \) that result. If one keeps the columns of length \( m \) then one obtains the Young diagrams which label the vertices of the Bratteli diagram. The restriction on the number of columns then applies only to the number of columns of length less than \( m \).

The representation of \( B_n \) that describes exchanges for a system of \( n \) particles with \( U_q(sl(m)) \) symmetry may now be described in terms of these Bratteli diagrams. In fact, given the overall \( U_q(sl(m)) \)-charge of the system, we may find the Young diagram \( Y \) with
Figure 6: The Bratteli diagram for the fundamental representation of $U_q(sl(3))$ at $q = e^{2i\pi/5}$. This is in fact the same diagram as that shown in figure 2, but this time each site in the diagram is uniquely labelled by a Young diagram only. The diagrams in figure 2 may be recovered by removing columns of 3 boxes.

$n$ boxes which gives this overall charge and then the Braid group representation space is just the space of paths on the diagram which start at the empty diagram and end at $Y$. Moreover, each of the exchanges $\tau_i$ will only mix paths that are the same everywhere except possibly at the $i^{th}$ vertex. Note that such paths occur at most in pairs, since there are no more than two orders in which one can place the two boxes that are added in going from vertex $i - 1$ to vertex $i + 1$ (if the boxes are added in the same row, for example, then there is only one admissible order and thus only one path). The paths which are mixed transform into each other by means of unitary matrices and since the diagrams all become periodic after a while, one needs only to find a finite number of such matrices (see also section 5.2 for this). To find these matrices exactly, one should calculate the $6j$-symbols of $U_q(sl(m))$. However, we will not do this here, but in stead take a short cut by using the fact that all the Braid group representations we need are related to representations of the Hecke algebra $H_{n,q}$, whose representation theory has been well studied.

4.3 The Hecke algebra $H_{n,q}$

In this paragraph, we give a short description of an algebra which plays an important role in our understanding of the braiding of $U_q(sl(m))$-representations: the Hecke algebra $H_{n,q}$. We will also describe the irreps of this algebra that are relevant to us.

$H_{n,q}$ may be defined as the complex algebra with generators $1, g_1, g_2, \ldots, g_{n-1}$, subject to the relations

$$
\begin{align*}
g_i g_j &= g_j g_i \quad (|i - j| \geq 2) \\
g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1} \\
g_i^2 &= (q - 1) g_i^1 + q
\end{align*}
$$

(110)

From these relations, we see that $H_{n,q}$ is a $q$-deformation of the group algebra $\mathbb{C}S_n$ of the symmetric group, to which it reduces at $q = 1$. The reason that the Hecke algebra comes into play in the braiding of $U_q(sl(2))$ representations is that the exchange matrix for the fundamental representation of $U_q(sl(m))$, given in (109), satisfies the following extra relation next to the braiding relation given in (70):

$$
(\sigma R^{e_1,e_1})^2 = (q^{\frac{m-1}{2m}} - q^{-\frac{m-1}{2m}})\sigma R^{e_1,e_1} + q^{-1/m}(1 \otimes 1)
$$

(111)
As a consequence of this relation, the braid group representation that can be constructed from the $R$-matrix also gives a representation of the Hecke algebra $H_{n,q}$. This representation is given by the prescription
\[
g_i \mapsto q^{\frac{m+1}{2m}} (R_{e_1,e_1}^{e_1})_{i,i+1}
\] (112)
and one may easily verify that (68) and (111) guarantee that the relations (110) for the $g_i$ are satisfied.

The representations of the Hecke algebra which are induced by $U_q(sl(m))$ in this way all factor over a quotient of the Hecke algebra, the so called $m$-row quotient. This is because the exchange matrix (109) satisfies even further relations apart from the ones already given. For example, the 2-row quotient of the Hecke algebra (which is also called the Temperley-Lieb-Jones algebra) can be defined by adding the following relations to those given in (110):
\[
1 + g_i + g_{i+1} + g_i g_{i+1} + g_{i+1} g_i + g_i g_{i+1} g_i = 0
\] (113)
and one may check that the matrix $q^{3/4} R_{e_1,e_1}$ for $U_q(sl(2))$ satisfies the corresponding equation. Note that the situation we are describing already occurs in the $q = 1$ case. In that case, we are describing representations of $\mathbb{C}S_n$ in which the $m+1$-row antisymmetriser vanishes (as it should do for exchanges in a tensor product of $m$-dimensional spaces). The Young tableaux for these representations thus have at most $m$ rows. The equation above indeed just says that the 3-row antisymmetriser vanishes, if one identifies the generators $\sigma_i$ of the permutation group with the elements $-g_i$ of the Hecke algebra at $q = 1$. We have kept this minus sign for better compatibility with the literature on Hecke algebra representations (for example [38]).

The representation theory of the Hecke algebra is analogous to that of the group algebra of the symmetric group as long as $q$ is not a root of unity, but when $q$ is a root of unity ($q \neq 1$), there are complications, similar to those that arise in the representation theory of $U_q(sl(m))$. In particular, the representation ring of $H_{n,q}$ is no longer semisimple for these values of $q$. The same is true for the representation ring of the $m$-row quotients. However, one may restrict oneself to representations that factor over a certain subquotient of the $m$-row quotient and these representations do form a semisimple ring. We will now give a quick description of the irreducible representations of $H_{n,q}$ at $q = e^{2\pi i/(k+2)}$, that factor over this semisimple quotient. These representations (among many others) have been constructed by Wenzl [38] and, independently, by Ocneanu [50]. Another relevant early reference is [51]. They are $q$-deformations of Young’s orthogonal representations of the symmetric group (see for example [52]). Each one of the representations we are interested in is characterised by a Young diagram $Y$ that has at most $m$ rows and at most $k$ columns of length less than $m$. The module of the representation characterised by $Y$ is the module generated by all paths on the Bratteli diagram of the fundamental representation of $U_q(sl(m))$ that start at the empty diagram and end at $Y$. Thus, we do indeed get the same representation modules that we described in section 4.2. Also, the action of the elementary exchanges is of the kind we described in section 4.2; $g_i$ does not mix paths that differ in any place other than at their $i$th vertex. Using the work of Wenzl and Ocneanu, we may now write down the matrices through which $g_i$ acts on the spaces...
of paths that do differ only at this vertex. Let us denote the Young diagram at the $i^{th}$ vertex of the path $p$ as $Y^{(i)}_p$, so that $p = (Y^{(1)}_p, Y^{(2)}_p, \ldots, Y^{(n)}_p) = Y$. Then two paths $p$ and $p'$ can only be mapped into each other by $g_i$ if one has $Y^{(n)}_p = Y^{(n)}_{p'}$ for all $n \neq i$. As we have already remarked, the spaces of paths that are mapped into each other by $g_i$ are at most two dimensional, since the last two boxes that are added in going from the Young diagram at vertex $i-1$ to that at vertex $i+1$ can be added in at most two different orders. Suppose that there are indeed two admissible orders. These two orders then correspond to two paths $p$ and $p'$ that form a basis for the space that we are interested in. One may define the distance between the two boxes involved as the number of hops from box to box that one has to make if one walks from the first to the second box along the right hand side of the Young diagram $Y^{(i)}_p$. One may also define a signed version of this distance, which we will denote $d_{p,i}$. Suppose that the first box is added in row $r_1$ and column $c_1$ of the Young diagram $Y^{(i)}_p$ and that the second is added in row $r_2$ and column $c_2$ of the Young diagram $Y^{(i+1)}_p$, then this signed distance is given by

$$d_{p,i} = r_2 - r_1 + c_1 - c_2$$

Clearly, this is equal to the ordinary distance if the first box is located higher up and more to the right than the second. In the opposite case the formula above gives minus the distance. Using this signed distance, we may now write the action of the exchange $g_i$ on the path $p$ as

$$g_i p = -q^{(1-d_{p,i})/2} \frac{[d_{p,i}]_q}{[d_{p,i}]_q} p + \text{sign}(d_{p,i}) \sqrt{[d_{p,i} + 1]_q [d_{p,i} - 1]_q} \frac{[d_{p,i}]_q}{[d_{p,i}]_q} p'$$

Hence if $d_{p,i} > 0$ (which may be achieved by exchanging $p$ and $p'$ if necessary), then the matrix for the action of the exchange $g_i$ on the basis $\{p, p'\}$ is given by

$$g_i \equiv -q^{1/2} \frac{[d_{p,i}]_q}{[d_{p,i}]_q} \left( \sqrt{[d_{p,i} + 1]_q [d_{p,i} - 1]_q} \sqrt{[d_{p,i} + 1]_q [d_{p,i} - 1]_q} \right)$$

and one may check easily that this matrix is symmetric and unitary.

The action of the exchange $g_i$ on a path $p$ that does not have a partner path, i.e. for which there is no other path $p' \neq p$ such that $Y^{(n)}_p = Y^{(n)}_{p'}$ for all $n \neq i$, is just multiplication by a phase factor. This phase factor equals $q$ if $d_{p,i} = -1$, which means that the two boxes were added in the same row, and it equals $-1$ if $d_{p,i} = 1$, which means the boxes were added to the same column, or if $d_{p,i} = k+1$, which happens at the “edges” of the fusion diagrams.

Clearly, the representations we have just described are the right ones for the description of the braiding of particles with a hidden $U_q(sl(m))$-symmetry. For a system of $n$ particles with overall quantum group charge $\Lambda$, we need the representation labelled by the Young diagram that consists of $n$ boxes and which reduces to the Young diagram of the charge $\Lambda$ on removal of all columns with $m$ boxes. With this correspondence, one may indeed check that the formulae given in this section are the same as the ones we gave for $U_q(sl(2))$ in section 3.2.12, up to the phase factor in (112). Using the explicit form of the exchange matrices, it is not difficult to prove some nice mathematical properties of the representations above. For example, Wenzl has proved [38] that they are irreducible and that representations labelled by different Young diagrams are non-isomorphic. The argument used in the proof is essentially the same as the one we described at the end of section 3.2.12 for the case of $U_q(sl(2))$. 

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4.4 A quantum group for the chiral boson

In this section, we will indicate how a quantum group may reproduce the fusion and braiding of the CFT that describes a chiral boson on a circle of radius $\sqrt{m}$, where $m \in \mathbb{Z}$. This CFT has $m$ chiral primary fields, which are the vertex operators $\nu_l = e^{il\phi/\sqrt{m}}$, for $m \in \{0, \ldots, k - 1\}$ The vertex operator $\nu_l$ has conformal weight $l^2/2m$ and the fusion is very simple, one has

$$\nu_{l_1} \times \nu_{l_2} = \nu_{l_1 + l_2 \mod m} \quad (117)$$

All conformal blocks of primaries may be calculated explicitly, giving

$$\langle \nu_{l_1}(z_1), \ldots, \nu_{l_n}(z_n) \rangle = \prod_{i<j} (z_i - z_j)^{l_i l_j/m} \quad (118)$$

It follows that the braiding of these blocks is abelian; the half-monodromy which takes a $\nu_{l_1}$ around a $\nu_{l_2}$ gives the block a factor of $e^{i\pi/m}$.

We would like to reproduce the above data through a quantum group, that is we would like to find a quantum group which has $m$ irreducible representations whose fusion rules and braiding are identical to those of the boson vertex operators. It turns out that this can not be achieved by a quantum group whose coassociator is trivial. This can be most easily seen in the case $m = 2$. In this case, we would need a quantum group with two representations $\pi_0$ and $\pi_1$ satisfying

$$\pi_0 \otimes \pi_0 = \pi_0$$
$$\pi_0 \otimes \pi_1 = \pi_1$$
$$\pi_1 \otimes \pi_1 = \pi_0 \quad (119)$$

Now if we look at the threefold tensor product $\pi_1 \otimes \pi_1 \otimes \pi_1$, then we know on the one hand that braiding the left $\pi_1$ over the other two must give two factors of $e^{i\pi/2}$ yielding a total factor of $e^{i\pi} = -1$. On the other hand, the braiding factor $bf$ is also given by the following formula (which is only valid if the coassociator is trivial)

$$bf = \pi_1 \otimes \pi_1 \otimes \pi_1 ((1 \otimes \Delta)(R)) = \pi_1 \otimes \pi_0 (R) \quad (120)$$

Here we have used the information that the two rightmost representations must fuse to $\pi_0$. We are thus really just exchanging the left $\pi_1$ over a $\pi_0$ and this should give a factor of $e^{i\pi/m} = 1$, which yields a contradiction. To describe the chiral boson, we should thus either use a quantum group with a non-trivial coassociator or relax the demands on the correspondence between quantum group and CFT. A good candidate for a non-coassociative quantum group would be $U_q(sl(m))$ at $q = e^{2i\pi/3}$ (or $k = 1$). This weak quasi-quantum group does have $m$ representations with the right fusion rules and the braiding for the fundamental representation does reproduce that for the vertex operator $\nu_1$, up to a trivial scalar factor in every exchange. However, checking the correctness of the braiding for all the other representations of $U_q(sl(m))$ seems to be rather complicated and therefore we will choose for a different approach.

If we relax the demands on our quantum group such that more than one quantum group representation may correspond to the same charge sector in the CFT, then we can reproduce the braiding of the chiral boson CFT by a very simple finite dimensional coassociative quantum group. As a Hopf algebra this quantum group is the group algebra of
the cyclic group $\mathbb{Z}_{2m}$. A convenient basis for this algebra is given by the primitive idempotents $e_0, e_1, \ldots, e_{2m-1}$ which project on the isotypical components of the representations of $\mathbb{Z}_{2m}$ in the group algebra. In formulae: the $e_i$ are elements that satisfy

$$e_i e_j = \delta_{ij} e_i$$

and the full set $\pi_0, \ldots, \pi_{2m-1}$ of irreps of $\mathbb{Z}_{2m}$ is given by

$$\pi_i(e_j) = \delta_{ij}$$

The coproduct reads

$$\Delta(e_j) = \sum_i e_i \otimes e_{j-i \mod 2m},$$

and one may easily check that this leads to the fusion rules

$$\pi_i \otimes \pi_j = \pi_{i+j \mod 2m}$$

Counit and antipode are given by

$$\epsilon(e_i) = \pi_0(e_i) = \delta_{0i}, \quad S(e_i) = e_{2m-i}$$

So far, we have just described the group algebra of $\mathbb{Z}_{2m}$ in terms of the $e_i$. Now let us introduce an $R$-matrix. One may easily check that the most general $R$-matrix which satisfies the requirements (68) is of the form

$$R = \sum_{i,j} q^{ij} e_i \otimes e_j$$

where $q$ is an $m$th root of unity. If we take $q = 1$, then this is just the identity on $\mathbb{C} \mathbb{Z}_{2m} \otimes \mathbb{C} \mathbb{Z}_{2m}$ and braiding is trivial. On the other hand, if $q \neq 1$ then we have non-trivial braiding and the braiding factor we get when taking a $\pi_i$ around a $\pi_j$ will be $q^{ij}$.

Let us call the quantum group we have just described $\mathbb{C} \mathbb{Z}_{2m,q}$. The correspondence between this simple quantum group and the chiral boson CFT can now be made as follows: The chiral sector corresponding to $\nu_l$ is represented by the two quantum group representations $\pi_l$ and $\pi_{l+m}$. Of these, the $\pi_l$ with $0 \leq l < m$ represent the primary fields and their even conformal descendants, while the $\pi_l$ with $m \leq l < 2m$ represent all the odd descendants. The fusion rules of the quantum group are then consistent with those of the CFT. In particular, it is impossible to distinguish particles represented by the representations $\pi^l$ and $\pi^{l+m}$ by means of the tensor product decomposition rules for these representations, just as it is impossible to distinguish the primary field $\nu_l$ from one of its descendants by means of the CFTs fusion rules. The braiding is also correct, if we choose $q = e^{i\pi/m}$. The braiding factors we get for two different representatives of the same CFT-sector may now differ by a minus sign, but this is in fact just what we want. To clarify this, let us look once more at the example we gave for the case $m = 2$. For this case, the vacuum sector will now be represented by the representation $\pi_0$ and also by the representation $\pi_2$, but if we exchange a $\pi_1$ with a $\pi_0$ then we get a factor of 1, while if we exchange a $\pi_1$ with a $\pi_2$, we get factor of $-1$. We already know that if we have three $\nu_1$ fields and we exchange the first over the last two, we will get a factor of $-1$. This is due to the fact that the correlator (118) will have a single zero at any point where two
\( \nu_1 \)-fields are brought together. If one would just place a \( \nu_0 \) at this point there would be no such zero and hence also no braiding factor. Hence we can think of \( \pi_0 \) as representing the vacuum sector, while we can think of \( \pi_2 \) as representing a charge-neutral bound state of two \( \nu_1 \) fields.

### 4.5 Quantum groups for the parafermions

In this section, we shall describe quantum groups for the \( \mathbb{Z}_k \)-parafermion conformal field theory that is used in the description of the RR-states. Since there are two different coset descriptions of this CFT (cf sections 2.1.2 and 2.1.1), one can also expect to get two different quantum group theoretic descriptions.

#### 4.5.1 The quantum group for \( \widehat{sl(2)}_k/U(1)_k \)

Let us start with the coset \( \widehat{sl(2)}_k/U(1)_k \). For this coset, we have the factorisation formula (6), which describes a Virasoro primary field of the \( \widehat{sl(2)}_k \) theory as a product of a parafermion field and a vertex operator for a chiral boson that lives on a circle of radius \( \sqrt{2k} \). We already used this formula to explain the conformal weights and fusion rules of the parafermions and clearly, it can also be used to calculate braiding. To see this, look at the following equality of correlators which follows from (9):

\[
\langle G^{\Lambda_1}(z_1), \ldots, G^{\Lambda_n}(z_n) \rangle = \langle \Phi^{\Lambda_1}(z_1), \ldots, \Phi^{\Lambda_n}(z_n) \rangle \langle e^{i\lambda_1 \phi}(z_1), \ldots, e^{i\lambda_n \phi}(z_n) \rangle \tag{127}
\]

The braiding on the left hand side of this equation is just a braiding of \( \widehat{sl(2)}_k \)-fields and the matrices which describe this are known to be the same matrices that describe the braiding of \( U_q(sl(2)) \) representations at \( q = e^{2i\pi/(k+2)} \) (the labels \( \lambda_i \) do not play a role in the braiding). The braiding on the right hand side will be described by matrices which are products of a matrix for braiding the parafermions and a known scalar factor for braiding the boson’s vertex operators. Thus, we may obtain the braiding matrices for the parafermion fields by just bringing the scalar factor obtained from the bosonic correlator to the left.

The braiding matrices which are obtained from this recipe are the same braiding matrices that one gets for the quantum group \( \mathcal{A}_{q_1,q_2} := U_{q_1}(sl(2)) \otimes \mathbb{C}Z_{4k,q_2} \), where \( q_1 = e^{2i\pi/(k+2)} \) and \( q_2 = e^{-i\pi/2k} \). The irreducible representations of this quantum group are tensor products of \( U_{q_1}(sl(2)) \)-irreps and \( \mathbb{C}Z_{4k,q_2} \)-irreps and hence they are labelled by an \( sl(2) \)-weight \( 0 \leq \Lambda \leq k \) and an integer \( 0 \leq \lambda < 4k \). We will write these representations as \( \pi^{\Lambda}_\lambda \). The representation \( \pi^{\Lambda}_\lambda \) will represent the \( \Phi^{\Lambda} \mod 2k \)-sector of the parafermion CFT. As in the case of the chiral boson we thus have more than one quantum group representation that corresponds to the same sector of the CFT. In fact, we now have four quantum group representations for every sector of the CFT, since not only have we doubled the period of the label \( \lambda \) (as we did for the boson), but we have also not taken the second of the field identifications (8) into account. Looking at this identification, we see that the labels \( (\Lambda, \lambda) \) and \( (k-\Lambda, \lambda-k) \), that should be identified, usually stand for quantum group representations of different dimensions \( (\Lambda+1) \) and \( k-\Lambda+1 \) respectively), although their quantum dimensions are equal \( ([\Lambda+1]_q = [k+2-(\Lambda+1)]_q = [k-\Lambda+1]_q) \). Nevertheless, we believe that the quantum group \( \mathcal{A}_{q_1,q_2} \) will give a good description of
the CFTs braiding properties. To motivate this statement, let us first look at the tensor product decomposition of $A_{q_1,q_2}$. This is given by

$$
\pi^A_{\lambda} \otimes \pi^N_{\chi} = \bigoplus_{\Lambda' = [\Lambda - \Lambda']} \pi^N_{\Lambda + \chi'},
$$

which is the same as the fusion rules \([1]\) for the parafermions, except that the identifications \([8]\) are not incorporated. Nevertheless, using the formulae \([53]\) for the truncated tensor product of the $U_{q_1}(sl(2))$-representations, on can see that it is impossible to distinguish particles in representations that correspond to the same parafermion sector by means of these fusion rules alone. In other words, it is consistent with these fusion rules to declare that particles in the representations $\pi^A_{\lambda}, \pi^{\Lambda+}_{\lambda}$ and $\pi^{\Lambda-}_{\lambda,k}$ are indistinguishable, just as it is consistent with the fusion rules of a CFT to declare all descendants of a field indistinguishable.

Now let us look at the braiding of $A_{q_1,q_2}$-representations. To describe this braiding, we can use the bases that we introduced for $U_{q_1}(sl(2))$ in section \([3.2.10]\), because the representations of $\mathbb{C}Z_{4k,q_2}$ are one-dimensional. The matrices that describe the braiding w.r.t. these bases will be the product of the matrices for $U_{q_2}(sl(2))$ that we gave in \([94]\) with the powers of $q_2$ that we get from the $R$-matrix \([120]\) for $\mathbb{C}Z_{4k,q_2}$. Using the symmetries \([72]\) of the truncated $6j$-symbols, one may then check that, when one changes the representations which represent CFT-sectors in a way which is consistent with the quantum group’s fusion rules, the elements of the braiding matrices will at most get minus signs. Again, this situation is similar to the situation for the chiral boson that we discussed in section \(4.4\).

We are now left with the difficulty of choosing the quantum group representations which should represent the fields $\Phi^0_2$ and $\Phi^1_4$, which are important for the description of electrons and quasiholes in the $RR$-states. We will use the representations $\pi^0_2$ and $\pi^1_2$ for this (rather than for example $\pi^{k-2}_2$ and $\pi^{k+1}_2$). This choice keeps comparison to the CFT-picture easy and it gives good results. Also, it gives results which are consistent with those of the quantum group for the coset $\widehat{sl(k)}_1 \times \widehat{sl(k)}_1/\widehat{sl(k)}_2$, for which there is a one-to-one correspondence between quantum group representations and CFT-sectors.

### 4.5.2 The quantum group for $\widehat{sl(k)}_1 \times \widehat{sl(k)}_1/\widehat{sl(k)}_2$

For the coset $\widehat{sl(k)}_1 \times \widehat{sl(k)}_1/\widehat{sl(k)}_2$, we do not have a factorisation formula like \([1]\) and therefore we cannot find the braiding matrices for this coset by the method we used for $\widehat{sl(2)}_k/\widehat{U(1)}_k$ in the previous section (cf. formula \([127]\)). Still, the results of the previous section and also the fusion rules and modular properties \([1]\) of $\widehat{sl(k)}_1 \times \widehat{sl(k)}_1/\widehat{sl(k)}_2$ suggest a natural candidate for a quantum group related to this coset: the quantum group $U_{q_2}(sl(k)) \otimes U_{q_3}(sl(k)) \otimes U_{q_4}(sl(k))$ with $q_3 = e^{i\pi/(k+1)}$ and $q_4 = e^{-i\pi/k+2} = (q_1)^{-1}$. The irreducible representations of this quantum group are tensor products of those of the factors and hence they are labelled by two $\widehat{sl(k)}_1$-weights and an $\widehat{sl(k)}_2$-weight, just like the fields $\Phi^{\mu_1,\mu_2}$ of section \(2.1.2\). Let us thus denote the quantum group irreps as $\pi^{\mu_1,\mu_2}_\mu$. We are now in the same situation that we encountered in the case of the coset $\widehat{sl(2)}_k/\widehat{U(1)}_k$; we have several quantum group representations per CFT-sector, because

\footnote{For modular properties of cosets, one can consult for example \([24,22]\). The relationship between modular and braiding properties of CFTs and quantum groups is clarified in \([20]\).}
the quantum group does not take the identifications (12) into account. However, in this case, there is a subset of representations of the quantum group which closes under fusion and which contains exactly one representation for each CFT-sector. In fact, there are two such subsets: the set of $\pi_{\mu}^{\mu_1, \mu_2}$ with $\mu_1 = 0$ and the set of $\pi_{\mu}^{\mu_1, \mu_2}$ with $\mu_2 = 0$. If we restrict to one of these sets (clearly, it does not matter which of the two we use), then we are effectively forgetting about one of the $U_{q_1}(sl(k))$-factors of the quantum group and hence we may say that the quantum group for the coset $\hat{sl}(k)_{1} \times \hat{sl}(k)_{1}/\hat{sl}(k)_{2}$ is $B_{q_3,q_4} := U_{q_3}(sl(k)) \otimes U_{q_4}(sl(k))$. The irreducible representations of this quantum group are labelled by an $\hat{sl}(k)_{1}$-weight $\mu_1$ and an $\hat{sl}(k)_{2}$-weight $\mu_2$. For the irreps that are relevant to the description of the coset CFT, the $\hat{sl}(k)_{1}$-weight is uniquely determined by the $\hat{sl}(k)_{2}$-weight through the branching rule (111), so that we may choose to label the relevant irreps by just a single $\hat{sl}(k)_{2}$-weight $\mu_2$. We may write these representations $\pi_{\mu}$ and they are in one-to-one correspondence with the fields $\Phi_{\mu}$ we defined in section (2.1.2). Clearly, the fusion rules of the $\pi_{\mu}$ are the same as those of the $\Phi_{\mu}$; they are identical to the fusion rules for the corresponding $\hat{sl}(k)_{2}$-fields or $U_{q_2}(sl(k))$-representations. We have not checked if all the braiding representations we get from the quantum group $U_{q_1}(sl(k)) \otimes U_{q_2}(sl(k))$ are equivalent to those one gets from the quantum group $A_{q_1,q_2}$ of the previous section, but we do know this for the representations that are related to the quasiholes of the $RR$-states. In section 4.5.1, the quasihole was represented by the irrep $\pi_{1}^{1}$ of $A_{q_1,q_2}$, whereas here, it must clearly be represented by the irrep $\pi_{e_1}$ of $B_{q_3,q_4}$ (for the notation $e_1$, see section 2.1.2). Explicit calculation of braiding matrices, using formulae (110), (112) and (120) shows that the braid group representations related to these irreps are indeed equivalent. Rather than writing out all these calculations in detail here, we will make some remarks which make this result very plausible. First of all, the braid group representations we get from the tensor products $(\pi_{1}^{1})^{\otimes n}$ and $\pi_{e_1}^{\otimes n}$ have the same fusion diagram associated to them. This guarantees for example that the representations will have a similar structure (see the discussion at the end of section 4.3, and in particular that their dimensions are equal. Second, the eigenvalues of the matrices that represent the fundamental exchanges may be easily found if we note that the representation matrices of the canonical Hecke algebra generators always have eigenvalues $-1$ and $q$ (this follows directly from the last relation in (110)). Thus, if we denote the eigenvalues of the braiding for $\pi_{1}^{1}$ by $\alpha_1$ and $\alpha_2$, then we have, using (112) and (120):

$$\alpha_1 = (q_1)^{-3/4} q_1 q_2 = e^{\frac{-i\pi}{4(k+2)}}$$
$$\alpha_2 = (q_1)^{-3/4} (-1) q_2 = -e^{\frac{-i(2k+1)\pi}{4(k+2)}}$$

(129)

On the other hand, the eigenvalues $\beta_1$ and $\beta_2$ for the braiding associated with $\pi_{e_1}$ can be found using (112) and this yields

$$\beta_1 = (q_4)^{-\frac{k+1}{2k}} (-1)(q_2)^{-\frac{k+1}{2k}} q_2 = -e^{\frac{-i(2k+1)\pi}{4(k+2)}} = \alpha_2$$
$$\beta_2 = (q_4)^{-\frac{k+1}{2k}} (-1)(q_2)^{-\frac{k+1}{2k}} (-1) = e^{\frac{i\pi}{4(k+2)}} = \alpha_1$$

(130)

so that the eigenvalues of the braidings are equal, as they should be.
5 Quantum group picture and braiding for the Read-Rezayi states

In this section, we will describe the Read-Rezayi states as systems of point particles with a hidden quantum group symmetry. We also give an explicit description of braiding representations that are associated with these states. All of this will be done in subsection 5.1. In section 5.2, we will give an alternative description of the resulting braid group representations, which does not make any explicit use of quantum groups. In this description, it is also somewhat easier to change the number of quasiholes in the system. Finally, in section 5.3, we show how the results of Nayak and Wilczek [2] for the Pfaffian state arise as a special case.

5.1 RR-states and hidden quantum group symmetry

In section 2.2, we described the RR-states as conformal blocks in a CFT which was the tensor product of the parafermion CFT and a CFT for a chiral boson. In section 4.3, we derived a relation between the parafermion CFT and the quantum groups $A_{q_1,q_2} = U_{q_1}(sl(2)) \otimes \mathbb{C}Z_{4k,q_2}$ and $B_{q_3,q_4} = U_{q_3}(sl(k)) \otimes U_{q_4}(sl(k))$. In section 4.4, we gave a quantum group for the chiral boson. Clearly, we can thus make a quantum group which will describe fusion and braiding for the Read-Rezayi states by tensoring the parafermion and boson quantum groups. However, since the extra boson factor does not affect the fusion of the relevant representations and only adds some scalar factors to the braiding matrices, we will choose to work with $A_{q_1,q_2}$ and $B_{q_3,q_4}$ and to add the scalar factors by hand. Thus, we see the following picture of the RR-states emerge. The RR-system of electrons and quasiholes can be seen as a system of point particles with hidden quantum group symmetry (cf. section 3.2.11). The electrons, which were represented by the operator $\psi = \Phi_2 = \Phi_{e_1}$ in the CFT-picture, are now point particles which carry the representation $\pi_2^0$ of $A_{q_1,q_2}$ or the representation $\pi_{e_1}$ of $B_{q_1,q_4}$. Similarly, the quasiholes, which used to be represented by the field $\sigma = \Phi_1 = \Phi_{e_1}$, now carry the representation $\pi_1^0$ of $A_{q_1,q_2}$ or the representation $\pi_{e_1}$ of $B_{q_1,q_4}$. The state of an RR-system with $N$ electrons and $n$ quasiholes may then be described as a vector in the tensor product of $N$ $\pi_2^0$ (or $\pi_{e_1}$) modules and $n$ $\pi_1^0$ (or $\pi_{e_1}$) modules. However, not all of the vectors in this tensor product correspond to physical states. First of all, we have to restrict to the states in a truncated tensor product with a given bracketing, as explained in sections 3.2.4 and 3.2.8. Second, there is a restriction that comes from the fact that the conformal block in (10) vanishes unless all the fields that appear in it fuse into the vacuum sector. This now means that the system as a whole is in one of the $A_{q_1,q_2}$-representations $\pi_0^0$, $\pi_k^0$ or in the $B_{q_3,q_4}$-representation $\pi_0$. Thus, the physical states in the tensor product are those that lie in a truncated tensor product and are in a global quantum group representation that corresponds to the CFT’s vacuum sector. The second condition has the same consequence as the corresponding condition for the CFT; one has to have $N + n$ equal to zero modulo $k$, because otherwise there are no states that fulfill this condition. The condition can be interpreted as saying that the incorporation of more electrons in an RR-system and the creation of quasiholes in such a system are $A$-charge preserving processes. It follows as in the CFT-picture that quasiholes may only be created in multiples of $k$ at a time (if the number of electrons is kept fixed).

Now let us look at the braiding of electrons and quasiholes. For convenience, we will do this in terms of $A_{q_1,q_2}$, but the treatment in terms of $B_{q_3,q_4}$ will give equivalent results (see
the discussion at the end of section 4.5.2. Since the representation $\pi^0_2$ is one dimensional, the braiding between electrons is abelian. This means any exchange of electrons will just give a phase factor. To find this factor, one may use the formulae (71) and (34) for the universal $R$-matrix and for the representations of $U_q(sl(2))$, the analogous formulae (120) and (122) for the universal $R$-matrix of $\mathbb{CZ}_{4k,q_2}$, and the explicit factors from the boson vertex operators which appear in the expression (16) for the wave function. All this together just gives a factor of $-1$, as is appropriate for fermions. Similarly, one may show that there is no nonabelian braiding between electrons and quasiholes and that the braiding factor for electron-quasihole exchanges is equal to one. Hence, as far as the braiding is concerned, the electrons and quasiholes can be treated separately. Since only the quasiholes have nonabelian braiding, we will from now on focus on these.

The braiding associated to a system of identical particles with hidden quantum group symmetry can be elegantly described in terms of a basis that is labelled by the paths on the fusion diagram of the quantum group representation carried by the particles (we described this in detail in sections 3.2.10 to 3.2.12). The quasiholes of the RR-states carry the $A_{q_1,q_2}$-representation $\pi^1_1$ and the fusion diagram for this representation is the same as that for the parafermionic field $\Phi^1_1$, which is in turn the same as the fusion diagram for the spin-$\frac{1}{2}$-representation of $U_q(sl(2))$. The braiding representations associated to this $U_q(sl(2))$-representation were described in detail in section 3.2.12 (and they were also included in the material of section 1.3). The only difference between the braid representations described there and the braiding for the RR-quasiholes lies in a scalar factor for every exchange, which comes from the $\mathbb{CZ}_{4k,q_2}$ part of $A_{q_1,q_2}$ and from the explicit factors in the wave function (16). Thus, a basis for the space of states with $n$ quasiholes in fixed positions is labelled by the paths on the fusion diagram of figure 1 which start at the point $(0,0)$ and which end at the point $(n,\Lambda)$, where $\Lambda = -N \mod k$, so that the $A_{q_1,q_2}$-charge of the whole system corresponds to the vacuum sector of the CFT. We will call the braid group representation on this space $\rho^A_n$. In this representation, the braid group generator $\tau_m$ will only mix paths which differ from each other only at the $m$th node. Given any path $p$, there will be at most one path $p'$ which differs from $p$ only at the $m$th node and we will call this the partner path of $p$ at node $m$. If a path does not have a partner path at node $m$, then the action of $\tau_m$ on this path will be multiplication by a scalar factor. To give this factor, let us first take $q = e^{2\pi i/(k+2)}$, so that we have

$$q_1 = q, \quad q_2 = q^{-\frac{1}{2}}.$$  \hspace{1cm} (131)

The path will then get a factor of $-q^{-1 + \frac{1-M}{2(kM+2)}}$ if it changes direction at the $m$th node and a factor of $q^{\frac{1-M}{2(kM+2)}}$ otherwise. When $M$ takes its lowest physical value of 1, these factors reduce to $-q^{-1}$ and 1 respectively. If a path does have a partner path at its $m$th node, then the path and its partner path will have the same representations at nodes $m-1$ and $m+1$ and these representations will have the same dimension. Let us call this dimension $d$. The exchange $\tau_m$ will then act on the vector space generated by the path and its partner path through the matrix

$$\rho^A_n(\tau_m) \equiv -q^{-\frac{1}{2} + \frac{1-M}{2(kM+2)}} \begin{pmatrix} q^{-d/2} & \sqrt{[d+1]_q[d-1]_q} \\ \sqrt{[d+1]_q[d-1]_q} & -q^{d/2} \end{pmatrix}.$$  \hspace{1cm} (132)

Here we have ordered the basis so that the first of the basis vectors corresponds to the path with the highest representation at node $m$ of the diagram. The matrix above is
symmetric and unitary and which should be compared with (106). The braid group representations \( \rho_n^A \) are irreducible by the same arguments as those we used for the braid group representations associated with \( U_q(sl(2)) \). Information on their dimensions has been gathered in section 2.4.

5.2 Tensor product description of the braiding

We will now set up an alternative description for the braid group representations \( \rho_n^A \) of the previous section. In this description, the representation spaces are seen as subspaces of \( n \)-fold tensor product spaces. This makes it somewhat closer in spirit to the description Nayak and Wilczek have given of the braiding for the Pfaffian state [2], a fact we will utilise in section 5.3. We also feel that the description of this section is useful in itself, because it shows very clearly how braiding in systems with an arbitrary number of quasiholes can be performed by a recipe that depends very little on this number.

Let us start by defining \( V_{k,l} \) to be the \( k \)-dimensional vector space spanned by orthonormal vectors which represent the possible \( l \)th steps in a path on the fusion diagram of figure [1]. We write:

\[
V_{k,l} = \begin{cases} 
\text{Span}\{v_{0,1}, v_{2,3}, \ldots, v_{k-2,k-1}, v_{2,1}, v_{4,3}, \ldots, v_{k,k-1}\} & (k \text{ even}, l \text{ odd}) \\
\text{Span}\{v_{1,2}, v_{3,4}, \ldots, v_{k-1,k}, v_{1,0}, v_{3,2}, \ldots, v_{k-1,k-2}\} & (k \text{ even}, l \text{ even}) \\
\text{Span}\{v_{0,1}, v_{2,3}, \ldots, v_{k-1,k}, v_{2,1}, v_{4,3}, \ldots, v_{k-1,k-2}\} & (k \text{ odd}, l \text{ odd}) \\
\text{Span}\{v_{1,2}, v_{3,4}, \ldots, v_{k-2,k-1}, v_{1,0}, v_{3,2}, \ldots, v_{k,k-1}\} & (k \text{ odd}, l \text{ even}) 
\end{cases}
\]

(133)

Here the indices on each basis vector represent the weights at the starting points and end points of the piece of path represented by the vector. In order to simplify the description, we have also, for \( l \leq k \), included some vectors which do not actually correspond to bits of path in the fusion diagram of figure [1] (for example the vector \( v_{2,1} \) at \( l = 1 \)). Clearly, any continuous path of length \( n \) through the fusion diagram may be represented by a canonical basis vector of the “domino” form \( v_{A_1A_2} \otimes v_{A_2A_3} \otimes v_{A_3A_4} \otimes \ldots \otimes v_{A_{n-1}A_n} \) in the tensor product space \( V_{k,1} \otimes V_{k,2} \otimes \ldots \otimes V_{k,n} \). The paths in the representation space of \( \rho_n^A \) can be isolated by requiring \( \Lambda_0 = 0 \) and \( \Lambda_n = \Lambda \).

We can now define a matrix representation \( \Gamma_{k,n} \) of the relations (29) on the given tensor product space which has a very simple form and which reduces to the braid group representation \( \rho_n^A \) when one restricts to the subspace of the tensor product which corresponds to the paths in \( \rho_n^A \). The action of the \( \tau_i \) is defined as follows. \( \Gamma_{k,n}(\tau_i) \) is always of the form \( 1 \otimes \ldots \otimes 1 \otimes R_{k,i} \otimes 1 \otimes \ldots \otimes 1 \), where the matrix \( R_{k,i} \) acts only on \( V_{k,i} \otimes V_{k,i+1} \). On the basis vectors which are of domino form in the \( i \)th and \((i+1)\)th factor, we take

\[
\begin{align*}
R_{k,i}v_{A_1,A_{i+1}} & = \alpha v_{A_1,A_{i+1}} \\
R_{k,i}v_{A_1,A_{i-1}} & = \alpha q^{-\frac{A_i}{2}}v_{A_1,A_{i-1}} \\
R_{k,i}v_{A_1,A_{i+1}} & = -q^{\frac{A_i}{2}-1}v_{A_1,A_{i+1}} - \frac{\alpha q^{-\frac{A_i}{2}}}{[A_i+1]_q}v_{A_1,A_{i+1}}
\end{align*}
\]

(135)

We have defined the relations for \( 0 \leq i < k \), and to these we add the condition

\[
\alpha = \frac{q^{rac{A_i}{2}}}{[A_i+1]_q}
\]

(136)

for an arbitrary \( 1 \leq A_i \leq k - 1 \). The conditions (135) and (136) completely determine the action of \( \Gamma_{k,n} \) on the basis vectors.
\[ R_{k,i}v_{\Lambda_i-1} \otimes v_{\Lambda_i-1} = \frac{\alpha q^{\frac{\Lambda_i}{2}}}{[\Lambda_i + 1]_q} v_{\Lambda_i-1} \otimes v_{\Lambda_i-1} \]
\[ = \frac{\alpha q^{\frac{\Lambda_i}{2}}}{[\Lambda_i + 1]_q} v_{\Lambda_i-1} \otimes v_{\Lambda_i-1} \]
\[ R_{k,i}v_{\Lambda_i} \otimes v_{\Lambda_i} = -\alpha q^{-1} v_{\Lambda_i} \otimes v_{\Lambda_i} \]
\[ R_{k,i}v_{\Lambda_i-1} \otimes v_{\Lambda_i-1} = -\alpha q^{-1} v_{\Lambda_i-1} \otimes v_{\Lambda_i-1} \]

where we have defined
\[ \alpha = q^{\frac{1-M}{2}} \]

This factor of course reduces to 1 when \( M = 1 \). For basis vectors \( v \) which are not of the domino form in \( V_{k,i} \otimes V_{k,i+1} \), we define \( R_{k,i}v = 0 \). With this definition, the matrices in \( \Gamma_{k,n} \) satisfy the relations (137), but they are not invertible (of course they are invertible if we restrict to the space generated by vectors of the domino form). Alternatively, one may set \( R_{k,i}v = v \). In that case, the matrices in \( \Gamma_{k,n} \) are invertible, but they no longer satisfy the second relation in (137) (of course, they still do satisfy this relation on the “domino state space”). It is not difficult to see that, on the set of vectors which corresponds to the paths of \( \rho_n^\Lambda \), the representation defined here indeed reduces to \( \rho_n^\Lambda \). The matrices \( R_{k,i} \), for given \( k \), depend only on the parity of \( i \), which means that knowledge of \( R_{k,1} \) and \( R_{k,2} \) is enough to determine the representation \( \Gamma_{k,n} \) and hence also all the \( \rho_n^\Lambda \) completely. This is quite useful, because it gives us an easy way to go from a description of \( n \) particles to a description of \( n + 1 \) particles; we just add another tensor factor and use the same matrices \( R_{k,1} \) and \( R_{k,2} \) as before to implement particle exchanges. We hope that this explicit recipe can be a small first step towards a second quantised description of particles with nonabelian statistics.

### 5.3 Reproducing the results for the Pfaffian state

Now let us check that our results reproduce those of Nayak and Wilczek [2] for \( k = 2 \), \( n = 2m \) even, \( N \) even and \( M = 1 \). In this case, the relevant paths on the fusion diagram have to end at the coordinates \((0, 2m)\) in case \( m \) is even and at the coordinates \((2, 2m)\) in case \( m \) is odd. The fusion diagram for \( k = 2 \) is given in figure [4]

\[ \Lambda \uparrow \]
\[ 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \]

Figure 7: fusion diagram for the quasiholes at \( k = 2 \). The diagram must be thought extended indefinitely in the \( \lambda \)-direction

Each of these paths can be uniquely characterised by stating whether or not it changes direction at each of its odd numbered vertices. If \( m \) is even, then the paths have to change direction an even number of times in order to end up at the point \((0, 2m)\). If \( m \) is odd then the number of changes of direction also has to be odd in order for the path to end up...
at the point \((2, 2m)\). Thus, for \(m\) even, we may represent any path of length \(2m\) by a ket 
\(|s_1, s_2, \ldots, s_m\rangle\), where each of the \(s_i\) is a sign, a plus sign denoting a change of direction
and a minus sign no change. The physically relevant paths are then the paths for which
the product of all these signs is a plus sign. For \(m\) odd, we may do the same, but now
with a minus sign denoting a change of direction and a plus sign denoting no change.
The relevant states are then once more the ones whose overall sign is positive. Both for
\(m\) odd and for \(m\) even, we thus describe a \(2^{m-1}\) dimensional space whose basis vectors
are labelled in the same way as those of Nayak and Wilczek. Just as Nayak and Wilczek
have done, we will interpret this space as a subspace of an \(m\)-fold tensor product of two
dimensional spaces, each of which has basis \(|+\rangle, |−\rangle\).

Now let us check that the action of the braiding matrices on these states is also the same as in
[2].

For \(k = 2\), the tensor product \(V_{k,1} \otimes V_{k,2}\) contains four states with the domino property:
the states \(v_{0,1} \otimes v_{1,0}, v_{0,1} \otimes v_{1,2}, v_{2,1} \otimes v_{1,2}\) and \(v_{2,1} \otimes v_{1,0}\). Using these as an ordered basis
of relevant states, the matrix \(R_{k,1}\) can now be found by filling in (134). It is given by

\[
R_{k,1} = \begin{pmatrix}
i & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & i & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

From this, we may read off that the action of the braid group generator \(\tau_{2l+1}\) on the sign
states \(|s_1, \ldots, s_m\rangle\) is given by

\[
\tau_{2l+1}|s_1, \ldots, s_m\rangle = \begin{cases} 
|1\rangle s_1, \ldots, s_m\rangle & (s_{2l+1} = m + 1 \mod 2) \\
i|s_1, \ldots, s_m\rangle & (s_{2l+1} = m \mod 2)
\end{cases}
\]

In this equation, we let the value + of the symbol \(s_{2l+1}\) correspond to 0 mod 2 and we let
the value − correspond to 1 mod 2. We see that \(\tau_{2l+1}\) acts only on the \((2l + 1)\)th factor
of the tensor product of sign spaces and on this factor it is given by the following \(2 \times 2\) matrix:

\[
\tau_{2l+1} \equiv \begin{cases} 
\begin{pmatrix}1 & 0 \\
i & 0
\end{pmatrix} & (m = 1 \mod 2) \\
\begin{pmatrix}i & 0 \\
0 & 1
\end{pmatrix} & (m = 0 \mod 2)
\end{cases}
\]

In Nayak and Wilczek’s work, the action of \(\tau_{l+1}\) also corresponds to the action of a diagonal
matrix in the \((l + 1)\)th tensor product factor. In this case, the matrix does not depend on
\(m\) and it is given by (cf. (32))

\[
\tau_{NW}^{l+1} \equiv e^{\frac{i\pi}{4} e^{-\frac{i\pi}{4}}} = \begin{pmatrix}i & 0 \\
0 & 1
\end{pmatrix}
\]

Here, \(\sigma_3\) denotes the third Pauli matrix. We see that the Nayak-Wilczek matrix is the
same as ours, up to a change in the order of the basis when \(m\) is odd.

The tensor product \(V_{k,2} \otimes V_{k,3}\) contains only two states with the domino property: the
states \(v_{1,0} \otimes v_{0,1}\) and \(v_{0,1} \otimes v_{1,0}\). Using this as an ordered basis for the relevant states, the
matrix \(R_{k,2}\) can again be found from (134) and is given by

\[
R_{k,2} = \begin{pmatrix}
\frac{1+i}{2} & \frac{-1+i}{2} \\
\frac{1-i}{2} & \frac{1+i}{2}
\end{pmatrix}
\]

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From this, we may read off the action of the braid group generator $\tau_{2l}$ on the states $|s_1, \ldots, s_m\rangle$. This generator acts only on the $(2l)^{th}$ and $(2l + 1)^{th}$ tensor factors of the sign space and on those it is given by the matrix:

$$
\tau_{2l} \equiv \begin{pmatrix}
\frac{1+i}{2} & 0 & 0 & -\frac{1+i}{2} \\
0 & \frac{1+i}{2} & -\frac{1+i}{2} & 0 \\
0 & -\frac{1+i}{2} & \frac{1+i}{2} & 0 \\
-\frac{1+i}{2} & 0 & 0 & \frac{1+i}{2}
\end{pmatrix}
$$

(141)

where the basis on which this matrix acts, is $\{|++,|--\rangle, |+-\rangle, |-+\rangle, |--\rangle\}$. The matrix above is identical to Nayak and Wilczek’s matrix for $\tau_{2l}$, which we gave in (32). Hence, we have reproduced Nayak and Wilczek’s result. Note that the change of order in the basis which was needed for $\tau_{2l+1}$ when $m$ is odd has no effect on the matrix for $\tau_{2l}$, which is why the result does not depend on the parity of $m$ this time.

### 6 Discussion and outlook

In this paper, we have shown how quantum groups may be used to give an algebraic description of the braiding and fusion properties of the excitations of nonabelian quantum Hall systems. Due to the well-established relationship between conformal field theory and quantum groups, it is in principle be possible to find such a description for any quantum Hall state that has a CFT-description. As an application, we obtained the explicit braiding matrices for the quasihole excitations over the Read-Rezayi series of states. In a special case, these reduce to the matrices given by Nayak and Wilczek in [2], as they should.

The obvious question to ask is now whether one can somehow make predictions about physical quantities from the results we have derived. The answer to this depends very much on what quantities one considers as physical. For example, one may fairly easily calculate amplitudes for Aharonov-Bohm scattering of quasiholes from the braiding matrices we have given, but it seems unlikely that the control over quasiholes that one needs to test these will soon be reached in experiments. One would probably have better chances of making contact with experiment if one could find effects of the nonabelian braiding in some transport properties of the quantum Hall state. To be able to make predictions about such quantities, one would most probably need to have a better understanding of the relation between the overcomplete set of states with localised quasiholes which we deal with in this article and a basis of the Hilbert space of the quantum Hall state. Suitable bases of the Hilbert space for the Read-Rezayi-states are constructed in [11, 12] and a logical next step in the program of understanding the consequences of nonabelian braiding in quantum Hall states would thus be to express the states we deal with in this article in terms of these bases and vice versa. Another important question is whether there is some intuitive way of understanding which features of the underlying theory cause the quantum symmetry exhibited by the effective theories at the plateaus. If such an intuitive picture could be found it would probably be very helpful in extracting physics from the effective theories.

There are also some questions of a more mathematical nature which arise naturally from our work. For example, one would like to generalise the way we associated quantum groups to coset CFTs to more general cosets than the ones we considered. A first generalisation would be to look at the generalisations of the parafermions that were defined by Gepner [53]. We expect that most arguments we gave for the parafermions will go
through unchanged for these theories. In connection with this, there should be identities like (92) for the “$6j$-symbols” of quantum universal enveloping algebras more general than $U_q(sl(2))$: one identity for each external automorphism of the corresponding Affine Lie algebra. Another interesting mathematical issue is whether the groups generated by the braiding matrices we have found are finite and/or can be characterised in a nice way. For the case studied by Nayak and Wilczek, this is certainly so: the group generated by the fundamental exchange matrices is just a double cover of the permutation group in this case. However, for the general case we have not yet obtained interesting results.

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