EXISTENCE AND UNIQUENESS OF POSITIVE SOLUTIONS OF BOUNDARY-VALUE PROBLEMS FOR FRACTIONAL DIFFERENTIAL EQUATIONS WITH p-LAPLACIAN OPERATOR

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Abstract. In this article, we consider the following boundary-value problem of nonlinear fractional differential equation with p-Laplacian operator

\[ D_0^\beta \left( \phi_p(D_0^\alpha u(t)) \right) + a(t)f(u) = 0, \quad 0 < t < 1, \]
\[ u(0) = \gamma u(h) + \lambda, \quad u'(0) = \mu, \]
\[ \phi_p(D_0^\alpha u(0)) = (\phi_p(D_0^\alpha u(1)))' = (\phi_p(D_0^\alpha u(0)))'' = (\phi_p(D_0^\alpha u(0)))''' = 0, \]

where \( 1 < \alpha \leq 2, \ 3 < \beta \leq 4 \) are real numbers, \( D_0^\alpha, D_0^\beta \) are the standard Caputo fractional derivatives, \( \phi_p(s) = |s|^{p-2}s, \ p > 1, \ \phi_p^{-1} = \phi_{q}, \ 1/p+1/q = 1, \)
\( 0 \leq \gamma < 1, \ 0 \leq h \leq 1, \ \lambda, \mu > 0 \) are parameters, \( a : (0,1) \to [0,\infty) \) and \( f : [0,\infty) \to [0,\infty) \) are continuous. By the properties of Green function and Schauder fixed point theorem, several existence and noneexistence results for positive solutions, in terms of the parameters \( \lambda \) and \( \mu \) are obtained. The uniqueness of positive solution on the parameters \( \lambda \) and \( \mu \) is also studied. In the final section of this paper, we derive not only new but also interesting identities related special polynomials by which Caputo fractional derivative.

1. Introduction

In 1695, L’Hôpital asked Leibniz: What if the order of the derivative is \( \frac{1}{2} \)? To which Leibniz considered in an useful means, ”thus it follows that will be equal to \( x\sqrt{dx : x} \), an obvious paradox. In recent years, fractional calculus has been studied by many mathematicians from Leibniz’s time to the present.

Also, fractional differential equations arise in many engineering and scientific disciplines as the mathematical modelling of systems and processes in the fields of physics, fluid flows, electrical networks, viscoelasticity, aerodynamics, and many other branches of science. For details, see [7, 8, 9, 19, 25, 26, 27, 28, 32].

In the last few decades, fractional-order models are found to be more adequate than integer-order models for some real world problems. Recently, there have been some papers dealing with the existence and multiplicity of solutions (or positive solutions) of non linear initial fractional differential equations by the use of techniques of nonlinear analysis [10, 11, 12, 16, 29, 31, 33, 34, 36, 37, 38, 40], upper and lower solutions method [21, 23, 30], fixed point index [15, 35], coincidence theory [13], Banach contraction mapping principle [22], etc.

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Chai [11] investigated the existence and multiplicity of positive solutions for a class of boundary-value problem of fractional differential equation with $p$-Laplacian operator

$$D^3_{0+}(\phi_p(D^\alpha_{0+}u(t))) + f(t, u(t), D^\alpha_{0+}u(t)) = 0, \quad 0 < t < 1,$$

$$u(0) = 0, u(1) + \sigma D^\alpha_{0+}u(1) = 0, \quad D^\alpha_{0+}u(0) = 0,$$

where $1 < \alpha \leq 2$, $0 < \gamma \leq 1$, $0 \leq \alpha - \gamma - 1$, $\sigma$ is a positive constant number, $D^\alpha_{0+}, D^\beta_{0+}, D^\gamma_{0+}$ are the standard Riemann-Liouville derivatives. By means of the fixed point theorem on cones, some existence and multiplicity results of positive solutions are obtained.

Although the fractional differential equation boundary-value problems have been studied by several authors, very little is known in the literature on the existence and nonexistence of positive solutions of fractional differential equation boundary-value problems with $p$-Laplacian operator when a parameter $\lambda$ is involved in the boundary conditions. We also mention that, there is very little known about the uniqueness of the solution of fractional differential equation boundary-value problems with $p$-Laplacian operator on the parameter $\lambda$. Han et al [33] studied the existence and uniqueness of positive solutions for the fractional differential equation with $p$-Laplacian operator

$$D^\beta_{0+}(\phi_p(D^\alpha_{0+}u(t))) + a(t)f(u) = 0, \quad 0 < t < 1,$$

$$u(0) = \gamma u(\xi) + \lambda, \quad \phi_p(D^\alpha_{0+}u(0)) = (\phi_p(D^\alpha_{0+}u(1)))' = (\phi_p(D^\alpha_{0+}u(0)))'' = 0.$$

where $0 < \alpha \leq 1$, $2 < \beta \leq 3$ are real numbers; $D^\alpha_{0+}, D^\beta_{0+}$ are the standard Caputo fractional derivatives, $\phi_p(s) = |s|^{p-2}s$, $p > 1$. Therefore, to enrich the theoretical knowledge of the above, in this paper, we investigate the following $p$-Laplacian fractional differential equation boundary-value problem

$$D^\beta_{0+}(\phi_p(D^\alpha_{0+}u(t))) + a(t)f(u) = 0, \quad 0 < t < 1,$$

$$u(0) = \gamma u(h) + \lambda, \quad u'(0) = \mu,$$

$$\phi_p(D^\alpha_{0+}u(0)) = (\phi_p(D^\alpha_{0+}u(1)))' = (\phi_p(D^\alpha_{0+}u(0)))'' = (\phi_p(D^\alpha_{0+}u(0)))''' = 0,$$

where $1 < \alpha \leq 2$, $3 < \beta \leq 4$ are real numbers, $D^\alpha_{0+}, D^\beta_{0+}$ are the standard Caputo fractional derivatives, $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\phi_p^{-1} = \phi_q$, $1/p + 1/q = 1$, $0 \leq \gamma < 1$, $0 \leq h \leq 1$, $\lambda, \mu > 0$ are parameters, $a : (0, 1) \to [0, +\infty)$ and $f : [0, +\infty) \to [0, +\infty)$ are continuous. By the properties of Green function and Schauder fixed point theorem, several existence and nonexistence results for positive solutions, in terms of the parameters $\lambda$ and $\mu$ are obtained. The uniqueness of positive solution on the parameters $\lambda$ and $\mu$ is also studied.

2. Preliminaries and related lemmas

**Definition 1** ([11]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y : (0, +\infty) \to \mathbb{R}$ is given by

$$I^\alpha_{0+}y(t) = \frac{1}{\Gamma(\alpha)} \int^t_0 (t-s)^{\alpha-1}y(s)ds$$

provided the right side is pointwise defined on $(0, +\infty)$. 
Definition 2 ([19]). The Caputo fractional derivative of order \( \alpha > 0 \) of a continuous function \( y: (0, +\infty) \to \mathbb{R} \) is given by
\[
D_0^\alpha y(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t \frac{y^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds,
\]
where \( n \) is the smallest integer greater than or equal to \( \alpha \), provided that the right side is pointwise defined on \( (0, +\infty) \).

Remark 1 ([28]). By Definition 2, under natural conditions on the function \( f(t) \), for \( \alpha \to n \) the Caputo derivative becomes a conventional \( n \)-th derivative of the function \( f(t) \).

Remark 2 ([19]). As a basic example,
\[
D_0^\alpha t^\mu = \mu(\mu - 1) \ldots (\mu - n + 1) \frac{\Gamma(1 + \mu - n)}{\Gamma(1 + \mu - \alpha)} t^{\mu - \alpha}, \quad \text{for } t \in (0, \infty).
\]
In particular \( D_0^\alpha t^\mu = 0, \mu = 0, 1, \ldots, n - 1 \), where \( D_0^\alpha \) is the Caputo fractional derivative, \( n \) is the smallest integer greater than or equal to \( \alpha \).

From the definition of the Caputo derivative and Remark 2 we can obtain the following statement.

Lemma 1 ([19]). Let \( \alpha > 0 \). Then the fractional differential equation
\[
D_0^\alpha u(t) = 0
\]
has a unique solution
\[
u(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \quad c_i \in \mathbb{R}, i = 0, 1, 2, \ldots, n - 1,
\]
where \( n \) is the smallest integer greater than or equal to \( \alpha \).

Lemma 2 ([19]). Let \( \alpha > 0 \). Assume that \( u \in C^n[0, 1] \). Then
\[
I_0^\alpha D_0^\alpha u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},
\]
for some \( c_i \in \mathbb{R}, i = 0, 1, 2, \ldots, n - 1 \), where \( n \) is the smallest integer greater than or equal to \( \alpha \).

Lemma 3. Let \( y \in C[0, 1] \) and \( 1 < \alpha \leq 2 \). Then fractional differential equation boundary-value problem
\[
D_0^\alpha u(t) = y(t), \quad 0 < t < 1
\]
\[
u(0) = \gamma u(h) + \lambda, \quad u'(0) = \mu
\]
has a unique solution
\[
u(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds + \frac{\gamma}{1-\gamma} \int_0^h \frac{(h-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds + \frac{\lambda + \gamma \mu h}{1-\gamma}.
\]

Proof. We apply Lemma 2 to reduce (3) to an equivalent integral equation,
\[
u(t) = I_0^\alpha y(t) + c_0 + c_1 t, \quad c_0, c_1 \in \mathbb{R}
\]
Consequently, the general solution of (3) is
\[
u(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha)} y(s) ds + c_0 + c_1 t, \quad c_0, c_1 \in \mathbb{R}.
\]
By (4), we have
\[ c_0 = \frac{\gamma}{1 - \gamma} \int_0^h \frac{(h-s)^{\alpha-2}}{\Gamma(\alpha)} y(s) ds + \frac{\gamma c_1 h}{1 - \gamma} + \frac{\lambda}{1 - \gamma}. \]
and since \( u'(t) = c_1 \), we have by (4)
\[ c_1 = \mu. \]
Therefore, the unique solution of problem (3) and (4) is
\[ u(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha)} y(s) ds + \frac{\gamma}{1 - \gamma} \int_0^h \frac{(h-s)^{\alpha-2}}{\Gamma(\alpha)} y(s) ds + \frac{\lambda + \gamma \mu h}{1 - \gamma}. \]

\[ \square \]

**Lemma 4.** Let \( y \in C [0,1] \) and \( 1 < \alpha \leq 2, \ 3 < \beta \leq 4 \). Then fractional differential equation boundary-value problem
\[ D_0^\alpha u(\phi_p(D_0^\alpha u(t))) + y(t) = 0, \ 0 < t < 1, \]
\[ \begin{align*}
    &u(0) = \gamma u(h) + \lambda, \ u'(0) = \mu, \\
    &\phi_p(D_0^\alpha u(0)) = (\phi_p(D_0^\alpha u(1)))' = (\phi_p(D_0^\alpha u(0)))'' = (\phi_p(D_0^\alpha u(0)))''' = 0,
\end{align*} \]
has a unique solution
\[ u(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q \left( \int_0^1 H(s, \tau) y(\tau) d\tau \right) ds + \frac{\gamma}{1 - \gamma} \int_0^h \frac{(h-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q \left( \int_0^1 H(s, \tau) y(\tau) d\tau \right) ds + \frac{\lambda + \gamma \mu h}{1 - \gamma}, \]
where
\[ H(t, s) = \begin{cases}
    \frac{t(\beta-1)(1-s)^{\beta-2}-(t-s)^{\beta-1}}{\Gamma(\beta)}, & 0 \leq s \leq t \leq 1, \\
    \frac{t(\beta-1)(1-s)^{\beta-2}}{\Gamma(\beta)}, & 0 \leq t \leq s \leq 1.
\end{cases} \]

**Proof.** From Lemma 2, the boundary-value problem (5) and (6) is equivalent to the integral equation
\[ \phi_p(D_0^\alpha u(t)) = -t_0^\alpha y(t) + c_0 + c_1 t + c_2 t^2 + c_3 t^3, \]
for some \( c_0, c_1, c_2, c_3 \in \mathbb{R} \); that is,
\[ \phi_p(D_0^\alpha u(t)) = - \int_0^t \frac{(t-\tau)^{\beta-1}}{\Gamma(\beta)} y(\tau) d\tau + c_0 + c_1 t + c_2 t^2 + c_3 t^3. \]
By the boundary conditions \( \phi_p(D_0^\alpha u(0)) = (\phi_p(D_0^\alpha u(1)))' = (\phi_p(D_0^\alpha u(0)))'' = (\phi_p(D_0^\alpha u(0)))''' = 0 \), we have
\[ c_0 = c_2 = c_3 = 0, \quad c_1 = \int_0^1 \frac{(\beta-1)(1-\tau)^{\beta-2}}{\Gamma(\beta)} y(\tau) d\tau. \]
Therefore, the solution \( u(t) \) of fractional differential equation boundary-value problem (5) and (6) satisfies
\[ \phi_p(D_0^\alpha u(t)) = - \int_0^t \frac{(t-\tau)^{\beta-1}}{\Gamma(\beta)} y(\tau) d\tau + t \int_0^1 \frac{(\beta-1)(1-\tau)^{\beta-2}}{\Gamma(\beta)} y(\tau) d\tau = \int_0^1 H(t, \tau) y(\tau) d\tau. \]
Consequently, $D_0^\alpha u(t) = \phi_q \left( \int_0^1 H(t, \tau)y(\tau)d\tau \right)$. Thus, fractional differential equation boundary-value problem (5) and (6) is equivalent to the problem

$$D_0^\alpha u(t) = \phi_q \left( \int_0^1 H(t, \tau)y(\tau)d\tau \right), \quad 0 < t < 1,$$

$$u(0) = \gamma u(h) + \lambda, \quad u'(0) = \mu.$$

Lemma 3 implies that fractional differential equation boundary-value problem (5) and (6) has a unique solution,

$$u(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha)} \phi_q \left( \int_0^1 H(s, \tau)y(\tau)d\tau \right)ds + \frac{\gamma}{1-\gamma} \int_0^h \frac{(h-s)^{\alpha-2}}{\Gamma(\alpha)} \phi_q \left( \int_0^1 H(s, \tau)y(\tau)d\tau \right)ds + \frac{\lambda + \gamma \mu h}{1-\gamma}.$$

The proof is complete. □

Lemma 5 (31). Let $1 < \alpha \leq 2, 3 < \beta \leq 4$. The function $H(t, s)$ is continuous on $[0, 1] \times [0, 1]$ and satisfies

1. $H(t, s) \geq 0, H(t, s) \leq H(1, s)$, for $t, s \in [0, 1]$;
2. $H(t, s) \geq t^{\beta-1}H(1, s)$, for $t, s \in (0, 1)$.

Lemma 6 (Schauder fixed point theorem [13]). Let $(E, d)$ be a complete metric space, $U$ be a closed convex subset of $E$, and $A : U \rightarrow U$ be a mapping such that the set $\{Au : u \in U\}$ is relatively compact in $E$. Then $A$ has at least one fixed point.

To prove our main results, we use the following assumptions.

(H1) $0 < \int_0^1 H(1, \tau)a(\tau)d\tau < +\infty$;

(H2) there exist $0 < \sigma < 1$ and $c > 0$ such that

$$f(x) \leq \sigma L \phi_p(x), \quad 0 \leq x \leq c,$$

where $L$ satisfies

$$0 < L < \phi_p \left[ \frac{1 + \gamma (h^\alpha - 1)}{\Gamma(\alpha + 1)(1 - \gamma)} \right] \int_0^1 H(1, \tau)a(\tau)d\tau^{-1};$$

(H3) there exist $d > 0$ such that

$$f(x) \leq M \phi_p(x), \quad d < x < +\infty,$$

where $M$ satisfies

$$0 < M < \phi_p \left[ \frac{1 + \gamma (h^\alpha - 1)}{\Gamma(\alpha + 1)(1 - \gamma)} 2^{\beta-1} \right] \int_0^1 H(1, \tau)a(\tau)d\tau^{-1};$$

(H4) there exist $0 < \delta < 1$ and $e > 0$ such that

$$f(x) \geq N \phi_p(x), \quad e < x < +\infty,$$

where $N$ satisfies

$$N > \phi_p \left[ c_3 \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)} \phi_q(s^{\beta-1})ds \right] \int_0^1 H(1, \tau)a(\tau)d\tau^{-1};$$

with

$$c_3 = \int_0^\delta (1-s)^{\alpha-2} \phi_q(s^{\beta-1})ds \in (0, 1);$$

(H5) $f(x)$ is nondecreasing in $x$;
Remark 3. Let
\[
f_0 = \lim_{x \to 0^+} \frac{f(x)}{\phi_p(x)}, \quad f_\infty = \lim_{x \to +\infty} \frac{f(x)}{\phi_p(x)}.
\]
Then, (H2) holds if \( f_0 = 0 \), (H3) holds if \( f_\infty = 0 \), and (H4) holds if \( f_\infty = +\infty \).

\[\text{Theorem 7. Assume that (H1), (H2) hold. Then the fractional differential equation boundary-value problem (1.1) and (1.2) has at least one positive solution for } 0 < \lambda + \gamma \mu \leq (1 - \gamma) (1 - \phi_q(\sigma)) c.\]

\[\text{Proof. Let } c > 0 \text{ be given in (H2). Define } K_1 = \{ u \in C[0,1]: 0 \leq u(t) \leq c \text{ on } [0,1]\} \text{ and an operator } T_\lambda : K_1 \to C[0,1] \text{ by }\]
\[
T_\lambda u(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha)} \phi_q \left( \int_0^1 H(s,\tau)a(\tau)f(u(\tau))d\tau \right) ds + \frac{\lambda + \gamma \mu h}{1-\gamma} \left( \int_0^1 H(s,\tau)a(\tau)f(u(\tau))d\tau \right) ds.
\]
\[
\text{Then, } K_1 \text{ is a closed convex set. From Lemma 4, } u \text{ is a solution of fractional differential equation boundary-value problem (1) and (2) if and only if } u \text{ is a fixed point of } T_\lambda. \text{ Moreover, a standard argument can be used to show that } T_\lambda \text{ is compact.}
\]

\[\text{For any } u \in K_1, \text{ from (3) and (4), we obtain }\]
\[
f(u(t)) \leq \sigma L \phi_p(u(t)) \leq \sigma L \phi_p(c), \quad \text{on } [0,1],
\]
\[
\frac{1 + \gamma (h^{\alpha - 1})}{\Gamma(\alpha + 1) (1 - \gamma)} \phi_q(L) \phi_q \left( \int_0^1 H(1,\tau)a(\tau)d\tau \right) \leq 1.
\]

\[\text{Let } 0 < \lambda + \gamma \mu \leq (1 - \gamma) (1 - \phi_q(\sigma)) c. \text{ Then, from Lemma } 5 \text{ and (15), it follows that }\]
\[
0 \leq T_\lambda u(t) \leq \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha)} \phi_q \left( \int_0^1 H(1,\tau)a(\tau)f(u(\tau))d\tau \right) ds + \frac{\lambda + \gamma \mu h}{1-\gamma} \left( \int_0^1 H(1,\tau)a(\tau)f(u(\tau))d\tau \right) ds \leq \frac{1 + \gamma h^{\alpha}}{\Gamma(\alpha + 1) (1 - \gamma)} \phi_q(L) \phi_q \left( \int_0^1 H(1,\tau)a(\tau)f(u(\tau))d\tau \right) + (1 - \phi_q(\sigma)) c
\]
\[
= \frac{1 + \gamma h^{\alpha - 1}}{\Gamma(\alpha + 1) (1 - \gamma)} \phi_q \left( \int_0^1 H(1,\tau)a(\tau)f(u(\tau))d\tau \right) + (1 - \phi_q(\sigma)) c
\]
\[
\leq \phi_q(\sigma)c + (1 - \phi_q(\sigma))c = c, \quad t \in [0,1].
\]
Thus, $T_{\lambda}(K_{1}) \subseteq K_{1}$. By Schauder fixed point theorem, $T_{\lambda}$ has a fixed point $u \in K_{1}$; that is, the fractional differential equation boundary-value problem (1) and (2) has at least one positive solution. The proof is complete.

**Corollary 8.** Assume that (H1) holds and $f_{0} = 0$. Then the fractional differential equation boundary-value problem (1) and (2) has at least one positive solution for sufficiently small $\lambda > 0$.

**Theorem 9.** Assume that (H1), (H3) hold. Then the fractional differential equation boundary-value problem (1) and (2) has at least one positive solution for all $\lambda > 0$.

**Proof.** Let $\lambda > 0$ be fixed and $d > 0$ be given in (H3). Define $D = \max_{0 \leq x \leq d} f(x)$. Then

$$f(x) \leq D, \quad \text{for } 0 \leq x \leq d. \quad \text{(16)}$$

From (10), we have

$$1 + \gamma(h^{\alpha} - 1) \frac{2^{\alpha - 1}}{\Gamma(\alpha + 1)(1 - \gamma)} \phi_{q}(M) \phi_{q}\left( \int_{0}^{1} H(1, \tau)a(\tau)d\tau \right) < 1.$$

Thus, there exists $d^{*} > d$ large enough so that

$$1 + \gamma(h^{\alpha} - 1) \frac{2^{\alpha - 1}}{\Gamma(\alpha + 1)(1 - \gamma)} \phi_{q}(D) + \phi_{q}(M)d^{*} \phi_{q}\left( \int_{0}^{1} H(1, \tau)a(\tau)d\tau \right) + \frac{\lambda + \gamma \mu h}{1 - \gamma} \leq d^{*}. \quad \text{(17)}$$

Let

$$K_{2} = \{u \in C[0, 1] : 0 \leq u(t) \leq d^{*} \text{ on } [0, 1]\}.$$

For $u \in K_{2}$, define

$$I_{1}^{u} = \{t \in [0, 1] : 0 \leq u(t) \leq d\},$$

$$I_{2}^{u} = \{t \in [0, 1] : d < u(t) \leq d^{*}\}.$$

Then, $I_{1}^{u} \cup I_{2}^{u} = [0, 1], I_{1}^{u} \cap I_{2}^{u} = \emptyset$, and in view of (19), we have

$$f(u(t)) \leq M\phi_{p}(u(t)) \leq M\phi_{p}(d^{*}), \quad \text{for } t \in I_{2}^{u}. \quad \text{(18)}$$

Let the compact operator $T_{\lambda}$ be defined by (15). Then from Lemma 5, 9 and 10, we have

$$0 \leq T_{\lambda}u(t)$$

$$\leq \int_{0}^{1} \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha)} \phi_{q}\left( \int_{0}^{1} H(1, \tau)a(\tau)f(u(\tau))d\tau \right)ds$$

$$+ \frac{\gamma}{1 - \gamma} \int_{0}^{h} \frac{(h-s)^{\alpha-2}}{\Gamma(\alpha)} \phi_{q}\left( \int_{0}^{1} H(1, \tau)a(\tau)f(u(\tau))d\tau \right)ds + \frac{\lambda + \gamma \mu h}{1 - \gamma}$$

$$\leq \frac{1}{\Gamma(\alpha + 1)} \phi_{q}\left( \int_{0}^{1} H(1, \tau)a(\tau)f(u(\tau))d\tau \right)$$

$$+ \frac{\gamma h^{\alpha}}{\Gamma(\alpha + 1)(1 - \gamma)} \phi_{q}\left( \int_{0}^{1} H(1, \tau)a(\tau)f(u(\tau))d\tau \right) + \frac{\lambda + \gamma \mu h}{1 - \gamma}$$

$$= \frac{1 + \gamma(h^{\alpha} - 1)}{\Gamma(\alpha + 1)(1 - \gamma)} \phi_{q}\left( \int_{I_{1}^{u}} H(1, \tau)a(\tau)f(u(\tau))d\tau + \int_{I_{2}^{u}} H(1, \tau)a(\tau)f(u(\tau))d\tau \right) + \frac{\lambda + \gamma \mu h}{1 - \gamma}.$$
From (17) and the inequality (18), we obtain

\[
0 \leq T \lambda u(t) \leq \frac{1 + \gamma(h^\alpha - 1)}{\Gamma(\alpha + 1)(1 - \gamma)} \phi_q \left( D \int_{I_1^\alpha} H(1, \tau) a(\tau) d\tau + M \phi_p(d^*) \right) + \frac{\lambda + \gamma \mu h}{1 - \gamma}.
\]

From (17) and the inequality \((a + b)^r \leq 2^r(a^r + b^r)\) for any \(a, b, r > 0\) (see, for example, (18)), we obtain

\[
0 \leq T \lambda u(t) \leq \frac{1 + \gamma(h^\alpha - 1)}{\Gamma(\alpha + 1)(1 - \gamma)} 2^{q-1} (\phi_q(D) + \phi_q(M)d^*) \phi_q \left( \int_0^1 H(1, \tau) a(\tau) d\tau \right) + \frac{\lambda + \gamma \mu h}{1 - \gamma} \leq d^*.
\]

Thus, \(T \lambda : K_2 \rightarrow K_2\). Consequently, by Schauder fixed point theorem, \(T \lambda\) has a fixed point \(u \in K_2\), that is, the fractional differential equation boundary-value problem (1) and (2) has at least one positive solution. The proof is complete. \(\square\)

**Corollary 10.** Assume that (H1) holds and \(f_\infty = 0\). Then the fractional differential equation boundary-value problem (1) and (2) has at least one positive solution for all \(\lambda > 0\).

4. **Uniqueness**

**Definition 3** ([17]). A cone \(P\) in a real Banach space \(X\) is called solid if its interior \(P^0\) is not empty.

**Definition 4** ([17]). Let \(P\) be a solid cone in a real Banach space \(X\), \(T : P^0 \rightarrow P^0\) be an operator, and \(0 \leq \theta < 1\). Then \(T\) is called a \(\theta\)-concave operator if

\[
T(\theta u) \geq \theta^\theta Tu \quad \text{for any } 0 < \theta < 1 \text{ and } u \in P^0.
\]

**Lemma 11** ([17] Theorem 2.2.6]). Assume that \(P\) is a normal solid cone in a real Banach space \(X\), \(0 \leq \theta < 1\), and \(T : P^0 \rightarrow P^0\) is a \(\theta\)-concave increasing operator. Then \(T\) has only one fixed point in \(P^0\).

**Theorem 12.** Assume that (H1), (H5), (H6) hold. Then the fractional differential equation boundary-value problem (1) and (2) has a unique positive solution for any \(\lambda > 0\).

**Proof.** Define \(P = \{u \in C[0, 1] : u(t) \geq 0 \text{ on } [0, 1]\}\). Then \(P\) is a normal solid cone in \(C[0, 1]\) with

\[
P^0 = \{u \in C[0, 1] : u(t) > 0 \text{ on } [0, 1]\}.
\]

For any fixed \(\lambda > 0\), let \(T \lambda : P \rightarrow C[0, 1]\) be defined by (15). Define \(T : P \rightarrow C[0, 1]\) by

\[
Tu(t) = \int_0^t \frac{(t - s)^{\alpha-2}}{\Gamma(\alpha)} \phi_q \left( \int_0^1 H(s, \tau) a(\tau) f(u(\tau)) d\tau \right) ds
\]

\[
+ \frac{\gamma}{1 - \gamma} \int_0^h \frac{(h - s)^{\alpha-2}}{\Gamma(\alpha)} \phi_q \left( \int_0^1 H(s, \tau) a(\tau) f(u(\tau)) d\tau \right) ds
\]

Then from (H5), we have \(T\) is increasing in \(u \in P^0\) and

\[
T \lambda u(t) = Tu(t) + \frac{\lambda + \gamma \mu h}{1 - \gamma}.
\]
Clearly, \( T_\lambda : P^o \to P^o \). Next, we prove that \( T_\lambda \) is a \( \theta \)-concave increasing operator. In fact, for \( u_1, u_2 \in P \) with \( u_1(t) \geq u_2(t) \) on \([0, 1] \), we obtain

\[
T_\lambda u_1(t) \geq T u_2(t) + \frac{\lambda + \gamma \mu h}{1 - \gamma} = T_\lambda u_2(t);
\]

i.e., \( T_\lambda \) is increasing. Moreover, \( (H6) \) implies

\[
T_\lambda (ku)(t) \geq k^\theta \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha)} \phi_q \left( \int_0^1 H(s, \tau)a(\tau)f(u(\tau)) d\tau \right) ds
\]

\[
+ k^\theta \frac{\gamma}{1 - \gamma} \int_0^h \frac{(h-s)^{\alpha-2}}{\Gamma(\alpha)} \phi_q \left( \int_0^1 H(s, \tau)a(\tau)f(u(\tau)) d\tau \right) ds + \frac{\lambda + \gamma \mu h}{1 - \gamma}
\]

\[
= k^\theta T u(t) + \frac{\lambda + \gamma \mu h}{1 - \gamma}
\]

\[
\geq k^\theta (T u(t) + \frac{\lambda + \gamma \mu h}{1 - \gamma}) = k^\theta T_\lambda u(t);
\]

i.e., \( T_\lambda \) is \( \theta \)-concave. By Lemma 11, \( T_\lambda \) has a unique fixed point \( u_\lambda \) in \( P^o \), that is, the fractional differential equation boundary-value problem (1) and (2) has a unique positive solution. The proof is complete. □

5. Nonexistence

In this section, we let the Banach space \( C[0, 1] \) be endowed with the norm \( \| u \| = \max_{0 \leq t \leq 1} | u(t) | \).

**Lemma 13.** Assume \( (H1) \) holds and let \( 0 < \delta < 1 \) be given in \( (H4) \). Then the unique solution \( u(t) \) of fractional differential equation boundary-value problem (5) and (6) satisfies

\[
u(t) \geq c_\delta \| u \| \quad \text{for} \quad \delta \leq t \leq 1,
\]

where \( c_\delta \) is defined by \( (13) \).

**Proof.** In view of Lemma 5 and Eq. (6), we have

\[
u(t) \leq \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha)} \phi_q \left( \int_0^1 H(1, \tau)y(\tau) d\tau \right) ds
\]

\[
+ \frac{\gamma}{1 - \gamma} \int_0^h \frac{(h-s)^{\alpha-2}}{\Gamma(\alpha)} \phi_q \left( \int_0^1 H(s, \tau)y(\tau) d\tau \right) ds + \frac{\lambda + \gamma \mu h}{1 - \gamma}
\]

for \( t \in [0, 1] \), and

\[
u(t) \geq \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha)} \phi_q \left( \int_0^1 s^{\beta-1} H(1, \tau)y(\tau) d\tau \right) ds
\]

\[
+ \frac{\gamma}{1 - \gamma} \int_0^h \frac{(h-s)^{\alpha-2}}{\Gamma(\alpha)} \phi_q \left( \int_0^1 H(s, \tau)y(\tau) d\tau \right) ds + \frac{\lambda + \gamma \mu h}{1 - \gamma}
\]

\[
= \int_0^t \alpha(t-s)^{\alpha-2} \phi_q (s^{\beta-1}) ds \frac{1}{\Gamma(\alpha + 1)} \phi_q \left( \int_0^1 H(1, \tau)y(\tau) d\tau \right)
\]
This contradiction completes the proof $\square$

Corollary 15. Assume that (H1) holds and $f_\infty = +\infty$. Then the fractional differential equation boundary-value problem (1) and (2) has no positive solution for sufficiently large $\lambda > 0$. 

Theorem 14. Assume that (H1), (H4) hold. Then the fractional differential equation boundary-value problem (1) and (2) has no positive solution for $\lambda + \gamma \mu h > (1 - \gamma) e$.

Proof. Assume, to the contrary, the fractional differential equation boundary-value problem (1) and (2) has a positive solution $u(t)$ for $\lambda + \gamma \mu h > (1 - \gamma) e$. Then by Lemma 4, we have

$$u(t) = \int_0^1 \frac{(t - s)^{\alpha - 2}}{\Gamma(\alpha)} \phi_q \left( \int_0^1 H(s, \tau) a(\tau) f(u(\tau)) d\tau \right) ds$$

$$+ \frac{\gamma}{1 - \gamma} \int_0^1 \frac{(h - s)^{\alpha - 2}}{\Gamma(\alpha)} \phi_q \left( \int_0^1 H(s, \tau) a(\tau) f(u(\tau)) d\tau \right) ds + \frac{\lambda + \gamma \mu h}{1 - \gamma}$$

Therefore, $u(t) > e$ on $[0,1]$. In view of (11) and (12), we obtain

$$f(u(t)) \geq N \phi_q (u(t)) \quad \text{on } [0,1],$$

$$c_5 \phi_q (N) \phi_q \left( \int_0^1 H(1, \tau) d\tau \right) \int_0^1 \frac{(1 - s)^{\alpha - 2}}{\Gamma(\alpha)} \phi_q (s^{\beta - 1}) ds > 1.$$ 

Then by Lemmas 5 and 13 we obtain

$$||u|| = u(1) > \int_0^1 \frac{(1 - s)^{\alpha - 2}}{\Gamma(\alpha)} \phi_q \left( \int_0^1 H(s, \tau) a(\tau) f(u(\tau)) d\tau \right) ds$$

$$\geq \int_0^1 \frac{(1 - s)^{\alpha - 2}}{\Gamma(\alpha)} \phi_q (s^{\beta - 1}) ds \phi_q \left( \int_0^1 H(1, \tau) a(\tau) f(u(\tau)) d\tau \right)$$

$$\geq \int_0^1 \frac{(1 - s)^{\alpha - 2}}{\Gamma(\alpha)} \phi_q (s^{\beta - 1}) ds \phi_q (N) \phi_q \left( \int_0^1 H(1, \tau) a(\tau) f(u(\tau)) d\tau \right)$$

$$\geq ||u|| c_5 \int_0^1 \frac{(1 - s)^{\alpha - 2}}{\Gamma(\alpha)} \phi_q (s^{\beta - 1}) ds \phi_q (N) \phi_q \left( \int_0^1 H(1, \tau) a(\tau) d\tau \right)$$

$$> ||u||.$$ 

This contradiction completes the proof $\square$
6. Conclusion: Identities on the special polynomials whereby Caputo Fractional derivative

In this final part, we will focus on the new interesting identities related special polynomials by means of Caputo fractional derivative.

As well known, the Bernoulli polynomials may be defined to be:

$$F(t, z) = \frac{z}{e^z - 1} e^{\beta z} = e^{\beta z} = \sum_{n=0}^{\infty} B_n(t) \frac{z^n}{n!},$$  \hspace{1cm} (19)

which usual convention about replacing $B^n$ by $B_n$ in, is used. Also, we note that the Bernoulli polynomials is analytic on the region $D = \{ z \in \mathbb{C} \mid |z| < 2\pi \}$ (see [1]).

Let $\frac{d}{dt}$ be familiar normal derivative, says us the following identity:

$$\frac{d}{dt} t^n = nt^{n-1}.$$  \hspace{1cm} (20)

Differentiating in both sides of (19), we have

$$\frac{d}{dt} B_n(t) = nB_{n-1}(t) \hspace{1cm} (see \ [1]).$$  \hspace{1cm} (21)

When $t = 0$ in (19), we have $B_n(0) := B_n$ are called Bernoulli numbers, which can be generated by

$$F(z) = \frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}.$$  \hspace{1cm} (22)

By (19) and (22), we have the following functional equation:

$$F(t, z) = e^{tz} F(z)$$

and this equation yields to

$$B_m(t) = \sum_{k=0}^{m} \binom{m}{k} t^{m-k} B_k = \sum_{k=0}^{m} \binom{m}{k} t^k B_{m-k} \hspace{1cm} (see \ [1]).$$

Let us now take $y(t) = B_m(t)$ in Definition 2, leads to

$$D_{0+}^\alpha B_m(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{B_m(t) \big|_{s=s}}{(t-s)^{n-\alpha+1}} ds$$

$$= m (m-1) \cdots (m-n+1) \sum_{k=0}^{m-n} \binom{m-n}{k} B_{m-n-k} \left[ \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{s^k}{(t-s)^{\alpha-n+1}} ds \right]$$

$$= \frac{\Gamma(m+n+1)}{\Gamma(m-n+1)} \sum_{k=0}^{m-n} \frac{k! (m-n)_k B_{m-n-k} t^{k-\alpha+n}}{\Gamma(n+k-\alpha+1)}.$$

Therefore, we procure the following theorem.

**Theorem 16.** The following identity holds true:

$$D_{0+}^\alpha B_m(t) = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} \sum_{k=0}^{m-n} \frac{k! (m-n)_k B_{m-n-k} t^{k-\alpha+n}}{\Gamma(n+k-\alpha+1)}.$$
Theorem 17. The following identity holds true:

\[ \frac{z}{e^z - 1} \frac{z}{e^z - 1} \cdots \frac{z}{e^z - 1} e^{tz} = \sum_{m=0}^{\infty} B_m^{(l)}(t) \frac{z^n}{m!}, \quad (23) \]

we note that \( B_m^{(l)}(t) \) is analytic on \( D \). It follows from (23), we have

\[ \frac{d}{dt} B_m^{(l)}(t) = m B_m^{(l)}(t) \quad \text{and} \quad \frac{d^n}{dt^n} B_m^{(l)}(t) = \frac{\Gamma (m + 1)}{\Gamma (m - n + 1)} B_m^{(l)}(t) \quad \text{see [4]} \quad (24) \]

Substituting \( x = 0 \) into (23), \( B_m^{(l)}(0) := B_m^{(l)} \) are called Bernoulli polynomials of higher order.

Owing to (23) and (24), we readily see that

\[ D_{0+}^\alpha B_m^{(l)}(t) = \frac{1}{\Gamma (n - \alpha)} \int_0^t \frac{d^n}{dt^n} B_m^{(l)}(t) |_{t=s} (t-s)^{n-\alpha+1} ds \]

\[ = m (m - 1) \cdots (m - n + 1) \sum_{k=0}^{m-n} \binom{m-n}{k} B_{m-n-k}^{(l)} \left[ \frac{1}{\Gamma (n - \alpha)} \int_0^t (t-s)^{n-\alpha+1} ds \right] \]

\[ = \frac{\Gamma (m + 1)}{\Gamma (m - n + 1)} \sum_{k=0}^{m-n} k! (\binom{m-n}{k}) B(m-n-k-k^{\alpha+n}) \]

\[ = \frac{\Gamma (m + 1)}{\Gamma (m - n + 1)} \sum_{k=0}^{m-n} \frac{k! (\binom{m-n}{k})}{(n+k-\alpha+1)} \]

\[ \times \left( \sum_{s_1+s_2+\cdots+s_l=m-n-k} \binom{m-n-k}{s_1, s_2, \cdots, s_l} \left( \prod_{j=1}^l B_{s_j} \right) \right). \]

Therefore, we can state the following theorem.

**Theorem 17.** The following identity holds true:

\[ D_{0+}^\alpha B_m^{(l)}(t) = \frac{\Gamma (m + 1)}{\Gamma (m - n + 1)} \sum_{k=0}^{m-n} \frac{k! (\binom{m-n}{k})}{(n+k-\alpha+1)} \]

\[ \times \left( \sum_{s_1+s_2+\cdots+s_l=m-n-k} \binom{m-n-k}{s_1, s_2, \cdots, s_l} \left( \prod_{j=1}^l B_{s_j} \right) \right) \]

in which \( B_{s_j} \) and \( \binom{m-n-k}{s_1, s_2, \cdots, s_l} \) are Bernoulli numbers and multi-binomial coefficients.

In the region \( T = \{ z \in \mathbb{C} \mid |z| < \pi \} \), the Euler polynomials and the Euler polynomials of higher order are given, respectively, with the help of the following generating functions:

\[ \frac{2}{e^z + 1} e^{tz} = \sum_{m=0}^{\infty} E_m(t) \frac{z^m}{m!}, \quad (25) \]

\[ \frac{2}{e^z + 1} \frac{2}{e^z + 1} \cdots \frac{2}{e^z + 1} e^{tz} = \sum_{m=0}^{\infty} \left( \sum_{s_1+s_2+\cdots+s_l=m} \binom{m}{s_1, s_2, \cdots, s_l} \left( \prod_{j=1}^l E_{s_j} \right) t^u \right) \frac{z^m}{m!} \]
Theorem 18. The following identity

\[ \sum_{m=0}^{\infty} E_m^{(l)} (t) \frac{z^n}{n!}, \]

which \( E_n \) are Euler numbers in \( \text{see [1], [2], [3] and [4]} \). From the last equation, we discover the followings:

\[ \frac{dt}{dt} E_m (t) = mE_{m-1} (t) \quad \text{and} \quad \frac{dt}{dt} E_m^{(l)} (t) = mE_m^{(l)} (t) \quad \text{(see [2])}. \]

(26)

Obviously that

\[ E_m^{(l)} (t) := E_m (t). \]

Taking \( y (t) = E_m^{(l)} (t) \) in Definition 2, by \( \text{[1]} \) and \( \text{[20]} \), we compute

\[
D_{0+}^{\alpha} E_m^{(l)} (t) = \frac{1}{\Gamma (n-\alpha)} \int_{0}^{t} \frac{d^m E_m^{(l)} (t) |_{t=s}}{(t-s)^{\alpha+n+1}} ds
\]

\[ = m (m-1) \cdots (m-n+1) \sum_{k=0}^{m-n} \binom{m-n}{k} E_{m-n-k} \left[ \frac{1}{\Gamma (n-\alpha)} \int_{0}^{t} s^{k-\alpha+n} ds \right] \]

\[ = \frac{\Gamma (m+1)}{\Gamma (m-n+1)} \sum_{k=0}^{m-n} k! \binom{m-n}{k} E_{m-n-k} \frac{s^{k-\alpha+n}}{\Gamma (n+k-\alpha+1)} \]

\[ \times \left( \sum_{s_1+s_2+\cdots+s_l=m-n-k} \binom{m-n-k}{s_1, s_2, \cdots, s_l} \left( \prod_{j=1}^{l} E(s_j) \right) \right). \]

Therefore, we obtain the following theorem.

Theorem 18. The following identity

\[ D_{0+}^{\alpha} E_m^{(l)} (t) = \frac{\Gamma (m+1)}{\Gamma (m-n+1)} \sum_{k=0}^{m-n} k! \binom{m-n}{k} E_{m-n-k} \frac{s^{k-\alpha+n}}{\Gamma (n+k-\alpha+1)} \left( \prod_{j=1}^{l} E(s_j) \right) \]

is true. Obviously that

\[ D_{0+}^{\alpha} E_m (t) = \frac{\Gamma (m+1)}{\Gamma (m-n+1)} \sum_{k=0}^{m-n} k! \binom{m-n}{k} E_{m-n-k} \frac{s^{k-\alpha+n}}{\Gamma (n+k-\alpha+1)}. \]

In the region \( T = \{ z \in \mathbb{C} \mid |z| < \pi \} \), Genocchi polynomials, \( G_m (x) \), and Genocchi polynomials of higher order, \( G_m^{(l)} (x) \), are defined as an extension of Genocchi numbers \( G_m \) defined in \( \text{[1], [5], [6]} \), respectively:

\[ \frac{2z}{e^z+1} e^{tz} = \sum_{m=0}^{\infty} G_m (t) \frac{z^m}{m!}, \]

(27)

\[ \frac{2z}{e^z+1} \frac{2z}{e^z+1} \cdots \frac{2z}{e^z+1} e^{tz} = \sum_{m=0}^{\infty} G_m^{(l)} (t) \frac{z^n}{n!}. \]

By the similar method, in this final section, we arrive at the following theorem.
Theorem 19. The following identity

\[ D_{0+}^\alpha G_m^{(l)}(t) = \frac{\Gamma(m + 1)}{\Gamma(m - n + 1)} \sum_{k=0}^{m-n} \frac{k! (m-n)}{\Gamma(n + k - \alpha + 1)} \left( \sum_{s_1 + s_2 + \cdots + s_l = m-n-k, s_j \geq 0} k! (m-n-k) G_{s_1, s_2, \cdots, s_l} \left( \prod_{j=1}^l G_s^j \right) \right) t^{k-\alpha+n}, \]

is true. Obviously that,

\[ D_{0+}^\alpha G_m(t) = \frac{\Gamma(m + 1)}{\Gamma(m - n + 1)} \sum_{k=0}^{m-n} \frac{k! (m-n)}{\Gamma(n + k - \alpha + 1)} G_{m-n-k} t^{k-\alpha+n}. \]

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