Central Limit Theorem And Moderate Deviation Principle For Inviscid Stochastic Burgers Equation

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Abstract: We establish a central limit theorem and prove a moderate deviation principle for inviscid stochastic Burgers equation. Due to the lack of viscous term, this is done in the framework of kinetic solution. The weak convergence method and doubling variables method play a key role.

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1 Introduction

This paper concerns the asymptotic behaviour of inviscid stochastic Burgers equation with multiplicative noise. More precisely, fix any $T > 0$ and let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t \in [0,T]}, \{\beta_k(t)\}_{t \in [0,T], k \in \mathbb{N}})$ be a stochastic basis. Without loss of generality, here the filtration $\{\mathcal{F}_t\}_{t \in [0,T]}$ is assumed to be complete and $\{\beta_k(t)\}_{t \in [0,T], k \in \mathbb{N}}$, are independent (one-dimensional) $\mathcal{F}_t$-Wiener processes. We use $E$ to denote the expectation with respect to $P$. Let $\mathbb{T}^1 \subset \mathbb{R}$ denote the one-dimensional torus (suppose the periodic length is 1). We are concerned with the following inviscid Burgers equation with stochastic forcing

\[
\begin{aligned}
du + \partial_x \left( \frac{u^2}{2} \right) dt &= \Phi(u) dW(t), \quad \text{in } [0, T] \times \mathbb{T}^1, \\
u(t, 0) &= 1 \quad \text{on } \mathbb{T}^1.
\end{aligned}
\] (1.1)
Here, $u : (\omega, x, t) \in \Omega \times T^1 \times [0, T] \mapsto u(\omega, x, t) := u(x, t) \in \mathbb{R}$, that is, the equation is periodic in the space variable $x \in T^1$, the coefficient $\Phi : \mathbb{R} \to \mathbb{R}$ is measurable and fulfill certain conditions specified later, and $W$ is a cylindrical Wiener process defined on $L^2(T^1)$ with the form $W(t) = \sum_{k \geq 1} \beta_k(t)e_k, t \in [0, T]$, where $(e_k)_{k \geq 1}$ is a complete orthonormal base in $L^2(T^1)$.

The deterministic Burgers equation was introduced in [9] to describe the turbulence phenomena in fluids, which can be solved by Cole-Hopf transform. The randomly forced Burgers equation is a prototype for range of problems in non-equilibrium statistical physics where strong effects are included, see [2, 3, 5, 10, 21, 24, 29, 32], etc. On the other hand, from the perspective of stochastic conservation laws on $d$–dimensional torus $T^d$

$$
\begin{aligned}
\left\{ 
\begin{array}{ll}
du + \text{div}(A(u))dt = \Phi(u)dW(t), & \text{in } [0, T] \times T^d, \\
u(\cdot, 0) = 1 & \text{on } T^d,
\end{array}
\right.
\end{aligned}
$$

the Burgers equation (1.1) is a special example with $d = 1$ and the flux function $A(\xi) = \frac{\xi^2}{2}$. Regarding to the conservation laws, both the deterministic ($\Phi = 0$) and stochastic cases have been studied extensively by many people due to its fundamental role in understanding the space-time evolution laws of interesting physical quantities. For more background on the conservation laws, we refer the readers to the monograph [11], the work of Ammar, Wittbold and Carrillo [1] and references therein. As we know, the Cauchy problem for the deterministic conservation laws does not admit any (global) smooth solutions, but there exist infinitely many weak solutions to the deterministic Cauchy problem. To solve the problem of non-uniqueness, an additional entropy condition was added to identify a physical weak solution. Under this condition, Kružkov [27, 28] introduced the notion of entropy solutions for the deterministic first-order scalar conservation laws. The kinetic formulation of weak entropy solution of the Cauchy problem for a general multi-dimensional scalar conservation laws (also called the kinetic system), was derived by Lions, Perthame and Tadmor in [30]. In recent years, the stochastic conservation laws has been developed rapidly. We refer the reader to [26, 33, 22, 16], etc. We particularly mention the paper [13] in which the authors proved the existence and uniqueness of kinetic solution to the Cauchy problem for (1.2) in any dimension. In addition, there are some works on the long time behavior/ergodicity of stochastic scalar conservation laws. In the space dimension one, E et al. [19] proved the existence and uniqueness of invariant measures for the periodic stochastic inviscid Burgers equation with additive forcing. Later, Debussche and Vovelle [14] studied scalar conservation laws with additive stochastic forcing on torus of any dimension and proved the existence and uniqueness of an invariant measure for sub-cubic fluxes and sub-quadratic fluxes, respectively. Recently, for the small noise asymptotic behaviour, Dong et al. [16] established Freidlin-Wentzell’s type large deviation principles (LDP) for the kinetic solution to the scalar stochastic conservation laws.
The purpose of this paper is to investigate the central limit theorem (CLT) and moderate deviation principle (MDP) for (1.1) driven by small multiplicative noise. Concretely, we consider

\[
\begin{aligned}
&\frac{du^\varepsilon + \partial_x \left(\frac{u^\varepsilon}{2}\right)}{\varepsilon}dt = \sqrt{\varepsilon} \Phi(u^\varepsilon)dW(t), \quad \text{in } [0, T] \times \mathbb{T}^1, \\
&u^\varepsilon(\cdot, 0) = 1 \quad \text{on } \mathbb{T}^1.
\end{aligned}
\]  

(1.3)

We aim to explore the deviations of \( u^\varepsilon \) from the deterministic solution \( \bar{u} \), as \( \varepsilon \to 0 \), that is, the asymptotic behavior of the trajectories,

\[ X^\varepsilon(t) = \frac{1}{\sqrt{\varepsilon \lambda(\varepsilon)}} (u^\varepsilon(t) - \bar{u}(t)), \quad t \in [0, T], \]

where \( \lambda(\varepsilon) \) is some deviation scale influencing the asymptotic behavior of \( X^\varepsilon \). Concretely, three cases are involved:

1. The case \( \lambda(\varepsilon) = \frac{1}{\sqrt{\varepsilon}} \) provides LDP, which has been proved by [16].

2. The case \( \lambda(\varepsilon) = 1 \) provides the central limit theorem (CLT). We will show that \( X^\varepsilon \) converges to a solution of a stochastic transport equation, as \( \varepsilon \) decrease to 0 in Section 3.

3. To fill in the gap between the CLT scale \( (\lambda(\varepsilon) = 1) \) and the large deviations scale \( (\lambda(\varepsilon) = \frac{1}{\sqrt{\varepsilon}}) \), we will study the so-called moderate deviation principle (MDP) in Section 4. Here, the deviations scale satisfies

\[ \lambda(\varepsilon) \to +\infty, \quad \sqrt{\varepsilon \lambda(\varepsilon)} \to 0 \quad \text{as } \varepsilon \to 0. \]  

(1.4)

Similar to LDP, MDP arises in the theory of statistical inference naturally, which can provide us with the rate of convergence and a useful method for constructing asymptotic confidence intervals (see, e.g. [18, 25, 26] and references therein). The proof of moderate deviations is mainly based on the weak convergence approach, which is developed by Dupuis and Ellis in [17]. The key idea is to prove some variational representation formula about the Laplace transform of bounded continuous functionals, which will lead to proving an equivalence between the Laplace principle and LDP. In particular, for Brownian functionals, an elegant variational representation formula has been established by Boué, Dupuis [8] and Budhiraja, Dupuis [6]. Recently, a sufficient condition to verify the large deviation criteria of Budhiraja, Dupuis and Maroulas [7] for functionals of Brownian motions is proposed by Matoussi, Sabbagh and Zhang in [31], which turns out to be more suitable for SPDEs arising from fluid mechanics. Thus, in the present paper, we adopt this new sufficient condition.
Up to now, there are plenty of results on the moderate deviations for fluid mechanics and other processes. For example, Wang et al. [34] established the CLT and MDP for 2D Navier-Stokes equations driven by multiplicative Gaussian noise in $C([0, T]; H) \cap L^2([0, T]; V)$. Further, Dong et al. [15] considered the MDP for 2D Navier-Stokes equations driven by multiplicative Lévy noises in $D([0, T]; H) \cap L^2([0, T]; V)$. In view of the characterization of the super-Brownian motion (SBM) and the Fleming-Viot process (FVP), Fatheddin and Xiong [20] obtained MDP for those processes. Recently, the CLT and MDP for stochastic Burgers equation with viscosity term have been studied by several authors, see [4, 35]. As stated above, the aim of this paper is to show two kinds of asymptotic behaviors of $X_\varepsilon$: the CLT and MDP in $L^1([0, T]; L^1(\mathbb{T}^1))$, which provide the exponential decay of small probabilities associated with the corresponding stochastic dynamical systems with small noise. We divide the proof into two parts. For the CLT, we show that $X_\varepsilon$ converges to a solution of a stochastic transport equation in $L^1([0, T]; L^1(\mathbb{T}^1))$, as $\varepsilon$ decrease to 0. For the MDP, it can be changed to prove that $X_\varepsilon$ satisfies a large deviation principle in $L^1([0, T]; L^1(\mathbb{T}^1))$ with $\lambda(\varepsilon)$ satisfying (1.4). Due to the lack of viscous term, the kinetic solutions of (1.1) are living in a rather irregular space, it is indeed a challenge to establish CLT and MDP for (1.1) with general noise force. During the proof process, the vanishing viscosity method and the doubling variables method play an essential role. Finally, we point out that we can only establish the results when the initial value is constant, and we cannot deal with the general initial value at this stage.

This paper is organized as follows. The mathematical framework of stochastic Burgers equation is in Section 2. We present the proof of the central limit theorem in Section 3. The moderate deviation principle is established in Section 4.

2 Framework

We will follow closely the framework of [13]. Let $\| \cdot \|_{L^p}$ denote the norm of usual Lebesgue space $L^p(\mathbb{T}^1)$ for $p \in [1, \infty]$. In particular, set $H = L^2(\mathbb{T}^1)$ with the corresponding norm $\| \cdot \|_H$. $C_b$ represents the space of bounded, continuous functions and $C^1_b$ stands for the space of bounded, continuously differentiable functions having bounded first order derivative. Define the function $f(x, t, \xi) := I_{u(x, t) > \xi}$, which is the characteristic function of the subgraph of $u$. We write $f := I_{u > \xi}$ for short. Moreover, denote by the brackets $\langle \cdot, \cdot \rangle$ the duality between $C^\infty_c(\mathbb{T}^1 \times \mathbb{R})$ and the space of distributions over $\mathbb{T}^1 \times \mathbb{R}$. In what follows, with a slight abuse of the notation $\langle \cdot, \cdot \rangle$, we denote the following integral by

$$
\langle F, G \rangle := \int_{\mathbb{T}^1} \int_{\mathbb{R}} F(x, \xi) G(x, \xi) dx d\xi, \quad F \in L^p(\mathbb{T}^1 \times \mathbb{R}), G \in L^q(\mathbb{T}^1 \times \mathbb{R}),
$$
where $1 \leq p \leq +\infty$, $q := \frac{p}{p-1}$ is the conjugate exponent of $p$. In particular, when $p = 1$, we set $q = \infty$ by convention. For a measure $m$ on the Borel measurable space $T^1 \times [0, T] \times \mathbb{R}$, the shorthand $m(\phi)$ is defined by

$$m(\phi) := \langle m, \phi \rangle([0, T]) := \int_{T^1 \times [0, T] \times \mathbb{R}} \phi(x, t, \xi) dm(x, t, \xi), \quad \phi \in C_b(T^1 \times [0, T] \times \mathbb{R}).$$

In the sequel, the notation $a \lesssim b$ for $a, b \in \mathbb{R}$ means that $a \leq D b$ for some constant $D > 0$ independent of any parameters.

### 2.1 Hypotheses

Let $A(\xi) := \frac{\xi^2}{2}$ be the flux function, then the derivative $a(\xi) := A'(\xi) = \xi$. For the coefficient $\Phi$, we assume that

**Hypothesis H** The map $\Phi(u) : H \rightarrow H$ is defined by $\Phi(u)e_k := g_k(\cdot, u)$, where $(e_k)_{k \geq 1}$ is a complete orthonormal base in the Hilbert space $H$ and each $g_k(\cdot, u)$ is a regular function on $T^1$. More precisely, we assume that $g_k \in C(T^1 \times \mathbb{R})$ satisfying the following bounds

$$G^2(x, u) := \sum_{k \geq 1} |g_k(x, u)|^2 \leq D_0 (1 + |u|^2), \quad (2.1)$$

$$\sum_{k \geq 1} |g_k(x, u) - g_k(y, v)|^2 \leq D_1 (|x - y|^2 + |u - v|^2), \quad (2.2)$$

for $x, y \in T^1, u, v \in \mathbb{R}$.

Based on the above notations, equation (1.1) can be rewritten as

$$\begin{cases}
    du(x, t) + \partial_x(A(u(x, t)))dt = \sum_{k \geq 1} g_k(x, u(x, t))d\beta_k(t) & \text{in } T^1 \times (0, T), \\
    u(\cdot, 0) = 1 & \text{on } T^1.
\end{cases} \quad (2.3)$$

From now on and in the sequel, we always assume Hypothesis H is in force.

### 2.2 Kinetic solution

Keeping in mind that we are working on the stochastic basis $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t \in [0, T]}, (\beta_k(t))_{k \in \mathbb{N}})$.

**Definition 2.1.** (Kinetic measure) A map $m$ from $\Omega$ to the set of non-negative, finite measures over $T^1 \times [0, T] \times \mathbb{R}$ is said to be a kinetic measure, if

1. $m$ is measurable, that is, for each $\phi \in C_b(T^1 \times [0, T] \times \mathbb{R}), \langle m, \phi \rangle : \Omega \rightarrow \mathbb{R}$ is measurable,
2. $m$ vanishes for large $\xi$, i.e.,
\[
\lim_{R \to +\infty} E[m(T^1 \times [0, T] \times B'_R)] = 0,
\] (2.4)
where $B'_R := \{\xi \in \mathbb{R}, |\xi| \geq R\}$

3. for every $\phi \in C_b(T^1 \times \mathbb{R})$, the process
\[
(\omega, t) \in \Omega \times [0, T] \mapsto \int_{T^1 \times [0, T] \times \mathbb{R}} \phi(x, \xi) dm(x, s, \xi) \in \mathbb{R}
\]
is predictable.

Let $M_0(\Omega \times T^1 \times \mathbb{R})$ be the space of all bounded, nonnegative random measures $m$ satisfying (2.4).

**Definition 2.2. (Kinetic solution)** A measurable function $u : T^1 \times [0, T] \times \Omega \to \mathbb{R}$ is called a kinetic solution to (2.3), if

1. $(u(t))_{t \in [0, T]}$ is predictable,
2. for any $p \geq 1$, there exists $C_p \geq 0$ such that
\[
E\left(\esssup_{0 \leq t \leq T} \|u(t)\|_{L_p(T^1)}^p \right) \leq C_p,
\]
3. there exists a kinetic measure $m$ such that $f := I_{u>\xi}$ satisfies the following
\[
\begin{align*}
&\int_0^T \langle f(t), \partial_t \phi(t) \rangle dt + \langle f_0, \phi(0) \rangle + \int_0^T \langle f(t), a(\xi) \cdot \nabla \phi(t) \rangle dt \\
&= -\sum_{k \geq 1} \int_0^T \int_{T^1} g_k(x, u(x,t)) \phi(x,t,u(x,t)) dx d\beta_k(t) \\
&\quad - \frac{1}{2} \int_0^T \int_{T^1} \partial_t \phi(x,t,u(x,t)) G^2(x,u(x,t)) dx dt + m(\partial_x \phi), \text{ a.s.,}
\end{align*}
\] (2.5)
for all $\phi \in C^1_b(T^1 \times [0,T] \times \mathbb{R})$, where $u(t) = u(\cdot, t, \cdot)$, $f_0 = I_{u>\xi}$ and $a(\xi) := A'(\xi)$.

Let $(X, \lambda)$ be a finite measure space. For some measurable function $u : X \to \mathbb{R}$, define $f : X \times \mathbb{R} \to [0, 1]$ by $f(z, \xi) = I_{u(z)>\xi}$ a.e. we use $\tilde{f} := 1 - f$ to denote its conjugate function. Define $\Lambda_f(z, \xi) := f(z, \xi) - I_{0>\xi}$, which can be viewed as a correction to $f$. Note that $\Lambda_f$ is integrable on $X \times \mathbb{R}$ if $u$ is.

It is shown in [13] that almost surely, for each kinetic solution $u$, the function $f = I_{u(\cdot, t)>\xi}$ admits left and right weak limits at any point $t \in [0, T]$, and the weak form (2.5) satisfied by a kinetic solution can be strengthened to be weak only respect to $x$ and $\xi$. More precisely, the following results are obtained.
Proposition 2.1. ([13], Left and right weak limits) Let \( u \) be a kinetic solution to (2.3). Then \( f = I_{u(x,t) \geq \xi} \) admits, almost surely, left and right limits respectively at every point \( t \in [0,T] \). More precisely, for any \( t \in [0,T] \), there exist functions \( f^{\pm} \) on \( \Omega \times \mathbb{T}^1 \times \mathbb{R} \) such that \( P \)-a.s.

\[
\langle f(t - r), \varphi \rangle \to \langle f^-, \varphi \rangle
\]

and

\[
\langle f(t + r), \varphi \rangle \to \langle f^+, \varphi \rangle
\]
as \( r \to 0 \) for all \( \varphi \in C_c^1(\mathbb{T}^1 \times \mathbb{R}) \). Moreover, almost surely,

\[
\langle f^+ - f^-, \varphi \rangle = - \int_{\mathbb{T}^1 \times [0,T] \times \mathbb{R}} \partial_\xi \varphi(x, \xi) I_{[t]}(s) dm(x, s, \xi).
\]

In particular, almost surely, the set of \( t \in [0,T] \) fulfilling \( f^+ \neq f^- \) is countable.

For the function \( f = I_{u(x,t) > \xi} \) in Proposition 2.1, define \( f^\pm \) by \( f^\pm(t) = f^{\pm t}, t \in [0,T] \). Since we are dealing with the filtration associated to Brownian motion, both \( f^+ \) and \( f^- \) are clearly predictable as well. Also \( f = f^+ = f^- \) almost everywhere in time and we can take any of them in an integral with respect to the Lebesgue measure or in a stochastic integral. However, if the integral is with respect to a measure, typically a kinetic measure in this article, the integral is not well-defined for \( f \) and may differ if one chooses \( f^+ \) or \( f^- \).

With the aid of Proposition 2.1, the following result was proved in [13].

Lemma 2.1. The weak form (2.5) satisfied by \( f = I_{u > \xi} \) can be strengthened to be weak only respect to \( x \) and \( \xi \). Concretely, for all \( t \in [0,T] \) and \( \varphi \in C_c^1(\mathbb{T}^1 \times \mathbb{R}) \), \( f = I_{u > \xi} \) satisfies

\[
\langle f^+(t), \varphi \rangle = \langle f_0, \varphi \rangle + \int_0^t \langle f(s), a(\xi) \cdot \nabla \varphi \rangle ds
+ \sum_{k \geq 1} \int_0^t \int_{\mathbb{T}^1} g_k(x, \xi) \varphi(x, \xi) d\nu_{x,k}(\xi) dx d\beta_k(s)
+ \frac{1}{2} \int_0^t \int_{\mathbb{T}^1} \partial_\xi \varphi(x, \xi) G^2(x, \xi) d\nu_{x,s}(\xi) dx ds - \langle \langle m, \partial_\xi \varphi \rangle([0,t]), a.s. \rangle, \tag{2.6}
\]

where \( \nu_{x,s}(\xi) = -\partial_\xi f(x, s, \xi) = \delta_{u(x,s) = \xi} \) and we set \( f^+(T) = f(T) \).

Remark 1. By making modification of the proof of Lemma 2.1, we have for all \( t \in (0,T] \) and \( \varphi \in C_c^1(\mathbb{T}^1 \times \mathbb{R}) \), \( f = I_{u > \xi} \) satisfies

\[
\langle f^-(t), \varphi \rangle = \langle f_0, \varphi \rangle + \int_0^t \langle f(s), a(\xi) \cdot \nabla \varphi \rangle ds
+ \sum_{k \geq 1} \int_0^t \int_{\mathbb{T}^1} g_k(x, \xi) \varphi(x, \xi) d\nu_{x,k}(\xi) dx d\beta_k(s)
+ \frac{1}{2} \int_0^t \int_{\mathbb{T}^1} \partial_\xi \varphi(x, \xi) G^2(x, \xi) d\nu_{x,s}(\xi) dx ds - \langle \langle m, \partial_\xi \varphi \rangle([0,t]), a.s. \rangle, \tag{2.7}
\]
and we set $f^-(0) = f_0$.

### 2.3 Global well-posedness of (2.3)

The following results were shown in [13].

**Theorem 2.2.** (Existence, Uniqueness) Assume Hypothesis $H$ holds. Then there is a unique kinetic solution $u$ to equation (2.3), and there exist $u^+$ and $u^-$, representatives of $u$ such that for all $t \in [0, T]$, $f^\pm(x, t, \xi) = I_{u^\pm(x,t) > \xi}$ a.s. for a.e. $(x, t, \xi)$.

**Corollary 2.3.** (Continuity in time) Assume Hypothesis $H$ is in force, then for every $p \in [1, +\infty)$, the kinetic solution $u$ to equation (2.3) has a representative in $L^p(\Omega; L^\infty(0, T; L^p(T^1)))$ with almost sure continuous trajectories in $L^p(T^1)$.

### 3 Central limit theorem

In this part, we devote to proving a central limit theorem for stochastic Burgers equation (1.1).

For any $\varepsilon > 0$, let $u^\varepsilon$ be the unique kinetic solution to (1.3). As the parameter $\varepsilon$ approaches zero, the solution $u^\varepsilon$ will tend to the solution of the following deterministic Burgers equation

\[
\begin{aligned}
\frac{d\bar{u}}{dt} + \frac{\partial_x}{2}\bar{u}^2 &= 0 \quad \text{in } T^1 \times (0, T], \\
\bar{u}(x, 0) &= 1 \quad \text{on } T^1.
\end{aligned}
\]  

(3.1)

For Eq.(3.1), we claim that it admits a unique solution $\bar{u} \equiv 1$. Indeed, for any $\eta > 0$, let us consider the following equation

\[
\begin{aligned}
\frac{d\bar{u}^\eta}{dt} + \frac{\partial_x}{2}\bar{u}^\eta &= \eta \Delta \bar{u}^\eta \quad \text{in } T^1 \times (0, T], \\
\bar{u}^\eta(x, 0) &= 1 \quad \text{on } T^1.
\end{aligned}
\]  

(3.2)

Clearly, Eq.(3.2) has a unique strong solution $\bar{u}^\eta(t, x) \equiv 1$, for any $(t, x) \in [0, T] \times T^1$. As a result, by vanishing viscosity method, Eq.(3.1) has a unique kinetic solution $\bar{u}(t, x) \equiv 1$, for any $(t, x) \in [0, T] \times T^1$.

In the following, we will investigate the fluctuation behaviour of $\frac{1}{\sqrt{\varepsilon}}(u^\varepsilon - \bar{u})$. Let $\bar{u}^1$ be the solution of the following SPDE

\[
\begin{aligned}
\frac{d\bar{u}^1}{dt} + \frac{\partial_x}{2}\bar{u}^1 &= \Phi(\bar{u})dW(t), \quad \text{in } T^1 \times (0, T], \\
\bar{u}^1(x, 0) &= 0, \quad \text{on } T^1.
\end{aligned}
\]  

(3.3)
Taking into account \( \bar{u} \equiv 1 \), equation (3.3) turns out to be a transport equation with an additive noise, and it can be solved directly as follows:

\[
\bar{u}^1(x, t) = \int_0^t \sum_{k \geq 1} g_k(x - \bar{u}(t - s), \bar{u})d\beta_k(s). \tag{3.4}
\]

It is readily to see that the law of \( \bar{u}^1 \) is Gaussian.

We will show that \( \frac{1}{\sqrt{\varepsilon}}(u^\varepsilon - \bar{u}) \) converges to \( \bar{u}^1 \) in the space \( L^1([0, T]; L^1(\mathbb{T}^1)) \). To achieve it, we will introduce some auxiliary approximation processes.

For \( u^\varepsilon \), consider the following viscosity approximating process \( u^{\varepsilon, \eta} \)

\[
\begin{cases}
  du^{\varepsilon, \eta} + \partial_x (\frac{(u^{\varepsilon, \eta})^2}{2})dt = \eta \Delta u^{\varepsilon, \eta}dt + \sqrt{\varepsilon} \Phi(u^{\varepsilon, \eta})dW(t), & \text{in } [0, T] \times \mathbb{T}^1, \\
  u^{\varepsilon, \eta}(x, 0) = 1 & \text{on } \mathbb{T}^1.
\end{cases} \tag{3.5}
\]

Referring to [13], (3.5) has a unique kinetic solution \( u^{\varepsilon, \eta} \) satisfying that

\[
\lim_{\varepsilon \to 0} \lim_{\eta \to 0} E\|u^{\varepsilon, \eta} - u^\varepsilon\|_{L^1([0, T]; L^1(\mathbb{T}^1))} = 0.
\]

We also consider the viscosity approximating process of \( \bar{u}^1 \) as follows

\[
\begin{cases}
  d\bar{u}^{1, \eta} + \partial_x (\bar{u}^{1, \eta})dt = \eta \Delta \bar{u}^{1, \eta}dt + \Phi(\bar{u})dW(t), & \text{in } \mathbb{T}^1 \times (0, T], \\
  \bar{u}^{1, \eta}(x, 0) = 1, & \text{on } \mathbb{T}^1.
\end{cases} \tag{3.6}
\]

Since \( \bar{u}^1 \equiv 1 \), it is easy to see that there is a unique strong solution \( \bar{u}^{1, \eta} \) to (3.6) satisfying

\[
\lim_{\varepsilon \to 0} E\|\bar{u}^{1, \eta} - \bar{u}^1\|_{L^1([0, T]; L^1(\mathbb{T}^1))} = 0.
\]

With the above approximation processes \( u^{\varepsilon, \eta}, \bar{u}^\eta \) and \( \bar{u}^{1, \eta} \), it gives

\[
\left\| \frac{u^\varepsilon - \bar{u}}{\sqrt{\varepsilon}} - \bar{u}^1 \right\|_{L^1([0, T]; L^1(\mathbb{T}^1))} \leq \left\| \frac{u^\varepsilon - \bar{u}}{\sqrt{\varepsilon}} - \frac{u^{\varepsilon, \eta} - \bar{u}^\eta}{\sqrt{\varepsilon}} \right\|_{L^1([0, T]; L^1(\mathbb{T}^1))} + \left\| \frac{u^{\varepsilon, \eta} - \bar{u}^\eta}{\sqrt{\varepsilon}} - \bar{u}^{1, \eta} \right\|_{L^1([0, T]; L^1(\mathbb{T}^1))} + \left\| \bar{u}^{1, \eta} - \bar{u}^1 \right\|_{L^1([0, T]; L^1(\mathbb{T}^1))}, \tag{3.7}
\]

From now on, we will estimate these three terms, separately. Firstly, we apply the doubling variables method to make estimation of \( \left\| \frac{u^\varepsilon - \bar{u}}{\sqrt{\varepsilon}} - \frac{u^{\varepsilon, \eta} - \bar{u}^\eta}{\sqrt{\varepsilon}} \right\|_{L^1([0, T]; L^1(\mathbb{T}^1))} \).

**Proposition 3.1.** We have

\[
\lim_{\varepsilon \to 0} \sup_{\varepsilon(0, 1)} E\left\| \frac{u^\varepsilon - \bar{u}}{\sqrt{\varepsilon}} - \frac{u^{\varepsilon, \eta} - \bar{u}^\eta}{\sqrt{\varepsilon}} \right\|_{L^1([0, T]; L^1(\mathbb{T}^1))} = 0.
\]
Proof. Denote by \( v^\varepsilon := \frac{\partial f}{\sqrt{\varepsilon}} \) and \( v^{\varepsilon,\eta} := \frac{\partial f^{\varepsilon,\eta}}{\sqrt{\varepsilon}} \). Due to the fact that \( \bar{u} = \bar{u}^\eta = 1 \), we derive from (1.3) and (3.1) that \( v^\varepsilon \) satisfies

\[
dv^\varepsilon + \partial_x \left( \frac{\sqrt{\varepsilon}(v^\varepsilon)^2}{2} + v^\varepsilon \right) = \Phi(\sqrt{\varepsilon}v^\varepsilon + 1) dW(t),
\]

with initial value \( v^\varepsilon(0) = 0 \). Clearly, it holds that

\[
\sup_{\varepsilon \in (0,1]} \mathbb{E} \|v^\varepsilon\|_{L^1([0,T] \times \mathbb{T}^1)} < \infty.
\]

Moreover, it follows from (3.2) and (3.5) that \( v^{\varepsilon,\eta} \) fulfills

\[
dv^{\varepsilon,\eta} + \partial_x \left( \frac{\sqrt{\varepsilon}(v^{\varepsilon,\eta})^2}{2} + v^{\varepsilon,\eta} \right) dt = \eta \Delta v^{\varepsilon,\eta} dt + \Phi(\sqrt{\varepsilon}v^{\varepsilon,\eta} + 1) dW(t),
\]

with initial value \( v^{\varepsilon,\eta}(0) = 0 \). For \( v^{\varepsilon,\eta} \), we have

\[
\sup_{\varepsilon,\eta \in (0,1]} \mathbb{E} \|v^{\varepsilon,\eta}\|_{L^1([0,T] \times \mathbb{T}^1)} < \infty. \tag{3.8}
\]

Denote by \( f_1(x, t, \xi) := I_{\varepsilon(x,t) = \xi} \) and \( f_2(y, t, \zeta) := I_{\varepsilon(y,t) > \zeta} \) with the corresponding kinetic measures \( m_1^\varepsilon, m_2^{\varepsilon,\eta} \) and initial values \( f_{1,0} = I_{0 < x_0}, f_{2,0} = I_{0 > \zeta_0} \), respectively. Similar to (2.6), for any \( \varphi_1 \in C^1_c(\mathbb{T}_x^1 \times \mathbb{R}_x^1) \) and \( t \in [0, T) \), it gives that

\[
\langle f_1^+(t), \varphi_1 \rangle = \langle f_{1,0}, \varphi_1 \rangle + \int_0^t \langle f_1(s), (\sqrt{\varepsilon} x + 1) \cdot \nabla \varphi_1(s) \rangle ds
\]

\[
+ \frac{1}{2} \int_0^t \int_{\mathbb{T}^1} \int_{\mathbb{R}} \partial_x \varphi_1(x, \xi) G^2(x, \sqrt{\varepsilon} \xi + 1) dv_{x,s}^{1,\varepsilon}(\xi) dx ds - \langle m^\varepsilon_1, \partial_x \varphi_1 \rangle([0, t])
\]

\[
+ \sum_{k \geq 1} \int_0^t \int_{\mathbb{T}^1} \int_{\mathbb{R}} g_k(x, \sqrt{\varepsilon} \xi + 1) \varphi_1(x, \xi) dv_{x,s}^{1,\varepsilon}(\xi) dx d\beta_k(s), \quad a.s.,
\]

where \( v_{x,s}^{1,\varepsilon}(\xi) = -\partial_x f_1(x, s, \xi) = \delta_{\varepsilon(x,s) = \xi} \). Moreover, for any \( \varphi_2 \in C^1_c(\mathbb{T}_y^1 \times \mathbb{R}_y^1) \) and \( t \in [0, T) \), it follows that

\[
\langle f_2^+(t), \varphi_2 \rangle = \langle f_{2,0}, \varphi_2 \rangle + \int_0^t \langle f_2(s), (\sqrt{\varepsilon} x + 1) \cdot \nabla \varphi_2(s) + \eta \Delta \varphi_2(s) \rangle ds
\]

\[
+ \frac{1}{2} \int_0^t \int_{\mathbb{T}^1} \int_{\mathbb{R}} \partial_y \varphi_2(y, \zeta) G^2(y, \sqrt{\varepsilon} \zeta + 1) dv_{y,s}^{2,\varepsilon,\eta}(\zeta) dy ds + \langle m_2^{\varepsilon,\eta}, \partial_y \varphi_2 \rangle([0, t])
\]

\[
+ \sum_{k \geq 1} \int_0^t \int_{\mathbb{T}^1} \int_{\mathbb{R}} g_k(y, \sqrt{\varepsilon} \zeta + 1) \varphi_2(x, \zeta) dv_{y,s}^{2,\varepsilon,\eta}(\zeta) dy d\beta_k(s) \quad a.s.,
\]

where \( v_{y,s}^{2,\varepsilon,\eta}(\zeta) = \partial_y f_2(y, s, \zeta) = \delta_{\varepsilon(y,s) = \zeta} \).
Denote the duality distribution over \( \mathbb{T}_x \times \mathbb{R}_x \times \mathbb{T}_y \times \mathbb{R}_y \) by \( \langle \cdot, \cdot \rangle \). Setting \( \alpha(x, \xi, y, \zeta) = \varphi_1(x, \xi)\varphi_2(y, \zeta) \). By the same method as Proposition 13 in [13], using (2.7) for \( f_1 \) and \( f_2 \), we obtain

\[
\langle f_1^+(t), \varphi_1 \rangle \langle \bar{f}_2^+(t), \varphi_2 \rangle = \langle \langle f_1^+(t) \bar{f}_2^+(t), \alpha \rangle \rangle
\]

satisfies

\[
E(\langle f_1^+(t) \bar{f}_2^+(t), \alpha \rangle) = \langle \langle f_{1,0,0} \rangle, \alpha \rangle + E \int_0^t \int_{\mathbb{T}_x^2} \int_{\mathbb{R}_x^2} f_1 f_2 ((\sqrt{\epsilon_\xi + 1})\partial_x + (\sqrt{\epsilon_\sigma + 1})\partial_y) \alpha d\xi d\zeta dx dy ds
\]

\[
-\frac{1}{2} E \int_0^t \int_{\mathbb{T}_x^2} \int_{\mathbb{R}_x^2} f_1(s, x, \xi) \partial_x \alpha G^2(y, \sqrt{\epsilon_\xi + 1}) d\xi d\nu_{y, s}^2(\zeta) dx dy ds
\]

\[
+\frac{1}{2} E \int_0^t \int_{\mathbb{T}_x^2} \int_{\mathbb{R}_x^2} \bar{f}_2(s, y, \zeta) \partial_y \alpha G^2(x, \sqrt{\epsilon_\xi + 1}) d\zeta d\nu_{x, s}^1(\xi) dx dy ds
\]

\[
-E \int_0^t \int_{\mathbb{T}_x^2} \int_{\mathbb{R}_x^2} G_{1,2}(x, y, \sqrt{\epsilon_\xi + 1}, \sqrt{\epsilon_\sigma + 1}) \alpha d\nu_{x, s}^{1,2} \otimes d\nu_{y, s}^{2,2}(\xi, \zeta) dx dy ds
\]

\[
+E \int_{(0,t]} \int_{\mathbb{T}_x^2} \int_{\mathbb{R}_x^2} f_1(s, x, \xi) \partial_x \alpha d\nu_{x, s}^{2,1}(y, \zeta, s) d\xi dx
\]

\[
-E \int_{(0,t]} \int_{\mathbb{T}_x^2} \int_{\mathbb{R}_x^2} \bar{f}_2(s, y, \zeta) \partial_y \alpha d\nu_{y, s}^{1,1}(x, \xi, s) d\zeta dy
\]

\[
+\eta E \int_0^t \int_{\mathbb{T}_x^2} \int_{\mathbb{R}_x^2} f_1 \bar{f}_2 \Delta_y \alpha d\xi d\zeta dx dy ds
\]

\[
= \langle \langle f_{1,0,0} \rangle, \alpha \rangle + \sum_{i=1}^7 I_i(t),
\]

where \( G^2(x, \xi) = \sum_{k \geq 1} |g_k(x, \xi)|^2 \) and \( G_{1,2}(x, \xi, y, \zeta) = \sum_{k \geq 1} g_k(x, \xi)g_k(y, \zeta) \).
Similarly, we have

\[
E\langle\langle \tilde{f}_1^+(t)f_2^+(t), \alpha \rangle\rangle
= \langle\langle \tilde{f}_{1,0}f_{2,0}, \alpha \rangle\rangle + E \int_0^\prime \int_{\mathbb{T}^1} \int_{\mathbb{R}^2} \tilde{f}_1f_2\left(\sqrt{\mathbb{E}}\xi + 1\right)\partial_x + \left(\sqrt{\mathbb{E}}\xi + 1\right)\partial_y d\xi d\zeta dx dy ds
\]

\[+ \frac{1}{2} E \int_0^\prime \int_{\mathbb{T}^1} \int_{\mathbb{R}^2} \tilde{f}_1(s, x, \xi)\partial_\xi \alpha G^2(y, \sqrt{\mathbb{E}}\xi + 1)d\xi d\nu_{t,y,x}^2,\eta(\xi)d\xi ds dy ds
\]

\[\frac{1}{2} E \int_0^\prime \int_{\mathbb{T}^1} \int_{\mathbb{R}^2} f_2(s, y, \zeta)\partial_\zeta \alpha G^2(x, \sqrt{\mathbb{E}}\xi + 1)d\zeta d\nu_{t,y,x}^1,\eta(\xi)d\xi ds dy ds
\]

\[E \int_0^\prime \int_{\mathbb{T}^1} \int_{\mathbb{R}^2} G_{1,2}(x,y, \sqrt{\mathbb{E}}\xi + 1, \sqrt{\mathbb{E}}\xi + 1)\alpha d\nu_{t,y,x}^1,\eta \otimes d\nu_{t,y,x}^2,\eta(\xi, \zeta)d\xi ds dy ds
\]

\[-E \int_0^\prime \int_{\mathbb{T}^1} \int_{\mathbb{R}^2} \tilde{f}_1^+(s, x, \xi)\partial_\xi \alpha d\nu_{t,y,x}^2,\eta(y, \zeta, s)d\xi ds
\]

\[E \int_0^\prime \int_{\mathbb{T}^1} \int_{\mathbb{R}^2} f_2^+(s, y, \zeta)\partial_\zeta \alpha d\nu_{t,y,x}^1,\eta(x, \xi, s)d\zeta dy ds
\]

\[+ \eta E \int_0^\prime \int_{\mathbb{T}^1} \int_{\mathbb{R}^2} \tilde{f}_1f_2\Delta_\alpha d\xi d\zeta d\xi ds dy ds
\]

\[= \langle\langle \tilde{f}_{1,0}f_{2,0}, \alpha \rangle\rangle + \sum_{i=1}^7 \tilde{I}_i(t).
\]

Following the idea developed by [13], we can relax the conditions imposed on \(\alpha\). Specifically, we can take \(\alpha \in C^\infty_0(\mathbb{T}^1 \times \mathbb{R}_x \times \mathbb{T}^1 \times \mathbb{R}_x)\), which is compactly supported in a neighbourhood of the diagonal

\[\{(x, \xi, x, \xi); x \in \mathbb{T}^1, \xi \in \mathbb{R}\}.
\]

Taking \(\alpha = \rho(x - y)\psi(\xi - \zeta)\), then we have the following remarkable identities

\[(\nabla_x + \nabla_y)\alpha = 0, \quad (\partial_\xi + \partial_\zeta)\alpha = 0. \tag{3.9}
\]

Referring to Proposition 13 in [13], we know that \(I_5 + I_6 \leq 0\) and \(\tilde{I}_5 + \tilde{I}_6 \leq 0\). In view of (3.9), we deduce that

\[I_1 = \sqrt{\mathbb{E}}E \int_0^\prime \int_{\mathbb{T}^1} \int_{\mathbb{R}^2} f_1f_2(\xi - \zeta)\partial_x \alpha d\xi d\zeta d\xi ds dy ds.
\]

and

\[\tilde{I}_1 = \sqrt{\mathbb{E}}E \int_0^\prime \int_{\mathbb{T}^1} \int_{\mathbb{R}^2} \tilde{f}_1f_2(\xi - \zeta)\partial_x \alpha d\xi d\zeta d\xi ds dy ds.
\]
Moreover, by the same method as Proposition 13 in [13], it gives that

\[
\sum_{i=2}^{4} I_i = \sum_{i=2}^{4} \tilde{I}_i \\
= \frac{1}{2} E \int_{(\mathbb{T}^1)^2} \int_{\mathbb{R}^2} \alpha \sum_{k \geq 1} |g_k(x, \sqrt{\varepsilon} \xi + 1) - g_k(y, \sqrt{\varepsilon} \zeta + 1)|^2 d\nu_{x,s}^{\varepsilon,\eta}(\xi, \zeta) dxdy.
\]

Combining all the previous estimates, it follows that

\[
E \int_{(\mathbb{T}^1)^2} \int_{\mathbb{R}^2} \rho(x-y)\psi(\xi - \zeta)(f_1^+(x, t, \xi) \bar{f}_2^+(y, t, \zeta) + \bar{f}_1^+(x, t, \xi) f_2^+(y, t, \zeta))d\xi d\zeta dxdy \\
\leq \int_{(\mathbb{T}^1)^2} \int_{\mathbb{R}^2} \rho(x-y)\psi(\xi - \zeta)(f_{1,0}(x, \xi) \bar{f}_{2,0}(y, \zeta) + \bar{f}_{1,0}(x, \xi) f_{2,0}(y, \zeta))d\xi d\zeta dxdy \\
+ \sqrt{\varepsilon} E \int_{(\mathbb{T}^1)^2} \int_{\mathbb{R}^2} \int_{(\mathbb{T}^1)^2} \int_{\mathbb{R}^2} \alpha \sum_{k \geq 1} |g_k(x, \sqrt{\varepsilon} \xi + 1) - g_k(y, \sqrt{\varepsilon} \zeta + 1)|^2 d\nu_{x,s}^{\varepsilon,\eta}(\xi, \zeta) dxdy \\
+ \eta E \int_{(\mathbb{T}^1)^2} \int_{\mathbb{R}^2} \int_{(\mathbb{T}^1)^2} \int_{\mathbb{R}^2} (f_1 \bar{f}_2 + \bar{f}_1 f_2)(\xi - \zeta) \partial_s \alpha d\xi d\zeta dxdyds \\
=: \int_{(\mathbb{T}^1)^2} \int_{\mathbb{R}^2} \rho(x-y)\psi(\xi - \zeta)(f_{1,0}(x, \xi) \bar{f}_{2,0}(y, \zeta) + \bar{f}_{1,0}(x, \xi) f_{2,0}(y, \zeta))d\xi d\zeta dxdy \\
+ K_1 + K_2 + K_3.
\]  

(3.10)

Now, taking a sequence \( t_n \uparrow t \), since (3.10) holds for \( f^+(t_n) \). Letting \( n \to \infty \), we deduce that (3.10) holds for \( f^{-}_i(t) \). Thus, we have

\[
E \int_{(\mathbb{T}^1)^2} \int_{\mathbb{R}^2} \rho(x-y)\psi(\xi - \zeta)(f_1^+(x, t, \xi) \bar{f}_2^+(y, t, \zeta) + \bar{f}_1^+(x, t, \xi) f_2^+(y, t, \zeta))d\xi d\zeta dxdy \\
\leq \int_{(\mathbb{T}^1)^2} \int_{\mathbb{R}^2} \rho(x-y)\psi(\xi - \zeta)(f_{1,0}(x, \xi) \bar{f}_{2,0}(y, \zeta) + \bar{f}_{1,0}(x, \xi) f_{2,0}(y, \zeta))d\xi d\zeta dxdy \\
+ K_1 + K_2 + K_3.
\]

(3.11)

Let \( \rho_\gamma, \psi_\delta \) be approximations to the identity on \( \mathbb{T}^1 \) and \( \mathbb{R} \), respectively. That is, let \( \rho \in C^\infty(\mathbb{T}^1) \), \( \psi \in C^\infty_c(\mathbb{R}) \) be symmetric non-negative functions such as \( \int_{\mathbb{T}^1} \rho = 1 \), \( \int_{\mathbb{R}} \psi = 1 \) and \( \text{supp} \psi \subset (-1, 1) \). We define

\[
\rho_\gamma(x) = \frac{1}{\gamma} \rho\left(\frac{x}{\gamma}\right), \quad \psi_\delta(\xi) = \frac{1}{\delta} \psi\left(\frac{\xi}{\delta}\right).
\]
Letting $\rho := \rho_\gamma(x - y)$ and $\psi := \psi_\delta(\xi - \zeta)$ in (3.11), we deduce that

$$E \int_{(T^1)^2} \int_{\mathbb{R}^2} \rho_\gamma(x - y)\psi_\delta(\xi - \zeta)(f_1^+(x, t, \xi)\tilde{r}_1^+(y, t, \zeta) + f_1^+(x, t, \xi)f_2^+(y, t, \zeta))d\xi d\zeta dxdy$$

$$\leq \int_{(T^1)^2} \int_{\mathbb{R}^2} \rho_\gamma(x - y)\psi_\delta(\xi - \zeta)(f_{1,0}(x, \xi)\tilde{r}_{2,0}(y, \zeta) + f_{1,0}(x, \xi)f_{2,0}(y, \zeta))d\xi d\zeta dxdy$$

$$+ \tilde{K}_1(t) + \tilde{K}_2(t) + \tilde{K}_3(t),$$

where $\tilde{K}_1, \tilde{K}_2, \tilde{K}_3$ are the corresponding $K_1, K_2, K_3$ in (3.11) with $\rho, \psi$ replaced by $\rho_\gamma, \psi_\delta$, respectively.

For any $t \in [0, T]$, define the error term

$$\mathcal{E}_\epsilon(\gamma, \delta) := E \int_{(T^1)^2} \int_{\mathbb{R}^2} (f_1^+(x, t, \xi)\tilde{r}_2^+(x, t, \xi) + f_1^+(x, t, \xi)f_2^+(x, t, \xi))d\xi dx$$

$$- E \int_{(T^1)^2} \int_{\mathbb{R}^2} \rho_\gamma(x - y)(f_1^+(x, t, \xi)\tilde{r}_2^+(y, t, \xi) + \tilde{r}_1^+(x, t, \xi)f_2^+(y, t, \zeta))d\xi dxdy$$

$$+ E \int_{(T^1)^2} \int_{\mathbb{R}^2} \rho_\gamma(x - y)(f_{1,0}(x, \xi)\tilde{r}_{2,0}(y, \zeta) + \tilde{r}_{1,0}(x, \xi)f_{2,0}(y, \zeta))d\xi dxdy$$

$$- E \int_{(T^1)^2} \int_{\mathbb{R}^2} \rho_\gamma(x - y)(f_1^+(x, t, \xi)\tilde{r}_2^+(y, t, \zeta) + \tilde{r}_1^+(x, t, \xi)f_2^+(y, t, \zeta))d\xi dxdy$$

$$=: H_1 + H_2. \quad (3.12)$$

By utilizing $\int_{\mathbb{R}} \psi_\delta(\xi - \zeta)d\zeta = 1$, $\int_{\xi-\delta}^{\xi} \psi_\delta(\xi - \zeta)d\zeta = \frac{1}{2}$ and $\int_{(T^1)^2} \rho_\gamma(x - y)dxdy = 1$, we deduce
that for any $t \in [0, T]$,

$$
E \left| \int_{(\mathbb{T}^1)^2} \int_{\mathbb{R}} \rho_y(x-y) f_1^+(x, t, \xi) \tilde{f}_2^+(y, t, \xi) \, d\xi \, dx \right|
- \int_{(\mathbb{T}^1)^2} \int_{\mathbb{R}^2} f_1^+(x, t, \xi) \tilde{f}_2^+(y, t, \xi) \rho_y(x-y) \psi_\delta(\xi - \zeta) \, dx \, dy \, d\xi \, d\zeta
= E \left| \int_{(\mathbb{T}^1)^2} \int_{\mathbb{R}} \rho_y(x-y) \int_{\mathbb{R}} \psi_\delta(\xi - \zeta)(I_{\rho_y,\alpha}(y) \leq \xi) - I_{\rho_y,\alpha}(y) \leq \zeta) \, d\xi \, d\zeta \, dx \, dy \, d\xi \, d\zeta \right|
\leq E \int_{(\mathbb{T}^1)^2} \int_{\mathbb{R}} \rho_y(x-y) \int_{\mathbb{R}} \psi_\delta(\xi - \zeta)(I_{\rho_y,\alpha}(y) \leq \xi) \, d\xi \, d\zeta \, dx \, dy
+ E \int_{(\mathbb{T}^1)^2} \int_{\mathbb{R}} \rho_y(x-y) \int_{\mathbb{R}} \psi_\delta(\xi - \zeta)(I_{\rho_y,\alpha}(y) \leq \zeta) \, d\xi \, d\zeta \, dx \, dy
\leq \frac{1}{2} E \int_{(\mathbb{T}^1)^2} \int_{\mathbb{R}} \rho_y(x-y) \int_{\mathbb{R}} \psi_\delta(\xi - \zeta) \min\{\rho_y,\alpha\}(y, \xi) \, d\xi \, d\zeta \, dx \, dy
+ \frac{1}{2} E \int_{(\mathbb{T}^1)^2} \int_{\mathbb{R}} \rho_y(x-y) \int_{\mathbb{R}} \psi_\delta(\xi - \zeta) \min\{\rho_y,\alpha\}(y, \xi) \, d\xi \, d\zeta \, dx \, dy
\leq \delta. \tag{3.13}
$$

Similarly,

$$
E \left| \int_{(\mathbb{T}^1)^2} \int_{\mathbb{R}} \rho_y(x-y) f_1^+(x, t, \xi) \tilde{f}_2^+(y, t, \xi) \, d\xi \, dx \right|
- \int_{(\mathbb{T}^1)^2} \int_{\mathbb{R}^2} \tilde{f}_1^+(x, t, \xi) f_2^+(y, t, \xi) \rho_y(x-y) \psi_\delta(\xi - \zeta) \, dx \, dy \, d\xi \, d\zeta \leq \delta. \tag{3.14}
$$

(3.13) together with (3.14) imply that $H_1 \leq 2\delta$.

Moreover, when $\gamma$ is small enough, it follows that

$$
E \left| \int_{(\mathbb{T}^1)^2} \int_{\mathbb{R}} \rho_y(x-y) f_1^+(x, t, \xi) \tilde{f}_2^+(y, t, \xi) \, d\xi \, dx \right|
- \int_{(\mathbb{T}^1)^2} \int_{\mathbb{R}} \tilde{f}_1^+(x, t, \xi) f_2^+(y, t, \xi) \rho_y(x-y) \psi_\delta(\xi - \zeta) \, dx \, dy \, d\xi \, d\zeta
\leq \sup_{|\xi| < \gamma} E \int_{(\mathbb{T}^1)^2} \int_{\mathbb{R}} f_1^+(x, t, \xi) |\tilde{f}_2^+(y - z, t, \xi) - \tilde{f}_2^+(y, t, \xi)| d\xi \, dx
\leq \sup_{|\xi| < \gamma} E \int_{(\mathbb{T}^1)^2} \int_{\mathbb{R}} |f_2^-(x, t, \xi) - f_2^+(x, t, \xi)| d\xi \, dx,
$$

where $\Lambda f_2(\cdot, \cdot, \xi) := f_2(\cdot, \cdot, \xi) - I_{0>\xi}$. From (3.8), we know that $\Lambda f_2$ is integrable in $L^1(\Omega \times \mathbb{T}^1 \times \mathbb{R})$.
uniformly with respect to $\epsilon$ and $\eta$, hence
\[
\lim_{\gamma \to 0} \sup_{\epsilon, \eta \in [0,1]} E \left| \int_{(T^1)^2} \int_{\mathbb{R}} \rho_{\gamma}(x-y) f_{1\gamma}(x, t, \xi) \bar{f}_{2\gamma}(y, t, \xi) d\xi dy - \int_{(T^1)^2} \int_{\mathbb{R}} f_{1\gamma}(x, t, \xi) \bar{f}_{2\gamma}(x, t, \xi) d\xi dx \right| = 0.
\]
Similarly, it holds that
\[
\lim_{\gamma \to 0} \sup_{\epsilon, \eta \in [0,1]} E \left| \int_{(T^1)^2} \int_{\mathbb{R}} \rho_{\gamma}(x-y) f_{1\gamma}(x, t, \xi) \bar{f}_{2\gamma}(y, t, \xi) d\xi dx - \int_{(T^1)^2} \int_{\mathbb{R}} f_{1\gamma}(x, t, \xi) \bar{f}_{2\gamma}(x, t, \xi) d\xi dx \right| = 0.
\]
Therefore, we conclude that for any $t \in [0, T]$,
\[
\lim_{\gamma, \delta \to 0} \sup_{\epsilon, \eta \in [0,1]} |E_\gamma(\gamma, \delta)| = 0.
\]
In particular,
\[
\lim_{\gamma, \delta \to 0} \sup_{\epsilon, \eta \in [0,1]} |E_0(\gamma, \delta)| = 0. \tag{3.15}
\]
By the dominate convergence theorem, we have
\[
\lim_{\gamma, \delta \to 0} \sup_{\epsilon, \eta \in [0,1]} \int_0^T |E_\gamma(\gamma, \delta)|dt = 0. \tag{3.16}
\]
In the following, we aim to make estimates of $\bar{K}_1(t)$, $\bar{K}_2(t)$ and $\bar{K}_3(t)$. We begin with the estimation of $\bar{K}_1(t)$. By the same method as the proof of (31) in [13], it gives
\[
\bar{K}_1(t) \leq \sqrt{et} \delta^{-1}.
\]
Moreover, proceeding as Proposition 3.2 in [13], we have
\[
\bar{K}_2(t) \leq tD_1 \gamma^2 \delta^{-1} + \frac{\epsilon}{2} TD_1 \delta^2.
\]
For the remainder $\bar{K}_3$, it can be estimated as follows
\[
\bar{K}_3 \leq \eta E \int_0^T \int_{(T^1)^2} \int_{\mathbb{R}^2} (f_1 \bar{f}_2 + \bar{f}_1 f_2) \Delta_x \rho_{\gamma}(x-y) \psi_{\eta}(\xi - \zeta) d\xi d\zeta dx dy ds
\]
\[
= \eta E \int_0^T \int_{(T^1)^2} \Delta_x \rho_{\gamma}(x-y) \left[ \int_{\mathbb{R}^2} (f_1 \bar{f}_2 + \bar{f}_1 f_2) \psi_{\eta}(\xi - \zeta) d\xi d\zeta \right] dx dy ds
\]
\[
= 2\eta E \int_0^T \int_{(T^1)^2} \Delta_x \rho_{\gamma}(x-y) \left[ \int_{\mathbb{R}^2} l(\xi, \zeta) dv_{x,\gamma}^{1,\xi} \otimes dv_{y,\eta}^{2,\eta} \psi_{\eta}(\xi, \zeta) \right] dx dy ds,
\]
where \( \nu_{1,s}^{1,\epsilon}(\xi) = \partial_\xi \tilde{f}_1(x, s, \xi), \nu_{1,s}^{2,\epsilon}(\eta) = \partial_\zeta \tilde{f}_2(y, s, \zeta), \) and

\[
I(\xi, \zeta) = \int_{\xi}^{\infty} \int_{-\infty}^{\zeta} \psi_0(\xi' - \zeta') d\xi' d\zeta'.
\]

Moreover, let \( \xi'' = \xi' - \zeta' \), it follows that

\[
l(\xi, \zeta) \leq \int_{\xi}^{\infty} \left( \int_{\{\xi'' < \xi' \}} \psi_0(\xi'') d\xi'' \right) d\zeta'' \leq C\delta \int_{\xi}^{\infty} \|\psi_0\|_{L^\infty} d\zeta'' \leq C(|\xi| + |\zeta| + \delta).
\]

Then, using the property that the measures \( \nu_{1,s}^{1,\epsilon} \) and \( \nu_{1,s}^{2,\epsilon,\eta} \) vanish at the infinity, it yields

\[
K_3 \leq C(1 + \delta)\eta T \gamma^{-2}.
\]

Based on the above estimates, we get

\[
E \int_{\mathbb{T}^1} \int_{\mathbb{R}} (f_1^+ (x, t, \xi) \tilde{f}_1^+ (x, t, \xi) + \tilde{f}_2^+ (x, t, \xi) f_2^+ (x, t, \xi)) d\xi dx
\]

\[
= E \int_{\mathbb{T}^1} \int_{\mathbb{R}^2} \rho_\gamma (x-y)\psi_0(\xi-\zeta)(f_1^+ (x, t, \xi) \tilde{f}_1^+ (y, t, \zeta) + \tilde{f}_2^+ (x, t, \xi) f_2^+ (y, t, \zeta)) d\xi d\zeta dxdy + \mathcal{E}_0(\gamma, \delta)
\]

\[
\leq \int_{\mathbb{T}^1} \int_{\mathbb{R}} (f_{1,0} (x, \xi) \tilde{f}_2 (x, \xi) + \tilde{f}_{1,0} (x, \xi) f_2 (x, \xi)) d\xi dx + |\mathcal{E}_0(\gamma, \delta)| + \mathcal{E}_0(\gamma, \delta)
\]

\[
+ \sqrt{\epsilon t\delta} \gamma^{-1} + tD_1 \gamma^2 \delta^{-1} + \frac{\epsilon}{2} T D_1 \delta^2 + C(1 + \delta)\eta T \gamma^{-2}.
\]

Then

\[
E \int_{0}^{T} \int_{\mathbb{T}^1} \int_{\mathbb{R}} (f_1^+ (x, t, \xi) \tilde{f}_1^+ (x, t, \xi) + \tilde{f}_2^+ (x, t, \xi) f_2^+ (x, t, \xi)) d\xi dx dt
\]

\[
\leq T \int_{\mathbb{T}^1} \int_{\mathbb{R}} (f_{1,0} (x, \xi) \tilde{f}_2 (x, \xi) + \tilde{f}_{1,0} (x, \xi) f_2 (x, \xi)) d\xi dx + T |\mathcal{E}_0(\gamma, \delta)| + \int_{0}^{T} \mathcal{E}_0(\gamma, \delta) dt
\]

\[
+ \sqrt{\epsilon \delta} \gamma^{-1} T^2 + D_1 \gamma^2 \delta^{-1} T^2 + \frac{\epsilon}{2} D_1 \delta^2 T^2 + C(1 + \delta)\eta T \gamma^{-2} T^2.
\]

Utilizing the following identities

\[
\int_{\mathbb{R}} I_{\nu} \mathcal{I}_{\nu} d\xi = (\nu^+ - \nu^{-\eta})^+, \quad \int_{\mathbb{R}} \mathcal{I}_{\nu} I_{\nu} d\xi = (\nu^+ - \nu^{-\eta})^-,
\]

we get

\[
E \|\nu^+ - \nu^{-\eta}\|_{L^1([0,T];L^1(T^1))}
\]

\[
\leq T |\mathcal{E}_0(\gamma, \delta)| + \int_{0}^{T} \mathcal{E}_0(\gamma, \delta) dt + \sqrt{\epsilon \delta} \gamma^{-1} T^2 + D_1 \gamma^2 \delta^{-1} T^2 + \frac{\epsilon}{2} D_1 \delta^2 T^2 + C(1 + \delta)\eta T \gamma^{-2} T^2.
\]

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Lemma 3.1. We have
\[
\begin{align*}
\sup_{\varepsilon \in (0,1]} E \left\| \begin{array}{c}
u^\varepsilon - \bar{u} \\ \sqrt{\varepsilon}
\end{array} - \begin{array}{c}
u^{\varepsilon,\eta} - \bar{u}^\eta \\ \sqrt{\varepsilon}
\end{array} \right\|_{L^1([0,T];L^1(\mathbb{T}^1))} \\
\leq T (\mathcal{E}_0(\gamma, \delta) + \int_0^T \mathcal{E}_t(\gamma, \delta) dt + T^2 \eta^{\frac{1}{2}} + T^2 D_1 \eta^{\frac{5}{2}} + T^2 D_1 \eta^{\frac{3}{2}} + CT^2 (1 + \eta^{\frac{1}{2}}) \eta^{\frac{1}{2}}).
\end{align*}
\]
Thus, by (3.15) and (3.16), we have
\[
\lim_{\eta \to 0} \sup_{\varepsilon \in (0,1]} E \left\| \begin{array}{c}
u^\varepsilon - \bar{u} \\ \sqrt{\varepsilon}
\end{array} - \begin{array}{c}
u^{\varepsilon,\eta} - \bar{u}^\eta \\ \sqrt{\varepsilon}
\end{array} \right\|_{L^1([0,T];L^1(\mathbb{T}^1))} = 0,
\]
which is the desired result. \(\square\)

Now, we aim to make estimates of the second term of the right hand side of (3.7). That is,

Proposition 3.2. For any \(\eta > 0\), we have
\[
\lim_{\varepsilon \to 0} \sup_{t \in [0,T]} E \left\| \begin{array}{c}
u^{\varepsilon,\eta}(t) - \bar{u}^\eta(t) \\ \sqrt{\varepsilon}
\end{array} \right\|^2_{L^2(\mathbb{T}^1)} = 0.
\]

To achieve the above result, we need a priori estimate.

Lemma 3.1. We have
\[
\sup_{\varepsilon \in (0,1]} \left[ \sup_{t \in [0,T]} E \|u^{\varepsilon,\eta}(t)\|^2_{L^2(\mathbb{T}^1)} + 2\eta E \int_0^T \|\partial_x u^{\varepsilon,\eta}(s)\|^2_{L^2(\mathbb{T}^1)} ds \right] \leq C(T). \quad (3.18)
\]

Proof. Taking \(L^2\) inner product, using Itô formula and by Hypothesis H, we get
\[
\sup_{t \in [0,T]} E \|u^{\varepsilon,\eta}(t)\|^2_{L^2(\mathbb{T}^1)} + 2\eta E \int_0^T \|\partial_x u^{\varepsilon,\eta}(s)\|^2_{L^2(\mathbb{T}^1)} ds \leq 1 + \frac{\varepsilon}{2} E \int_0^T \int_{\mathbb{T}^1} \sum_{k \geq 1} \|g_k(x, u^{\varepsilon,\eta})\|^2 dx ds
\]
\[
\leq 1 + \frac{\varepsilon}{2} E \int_0^T \|u^{\varepsilon,\eta}(s)\|^2_{L^2(\mathbb{T}^1)} ds.
\]
Then Gronwall’s inequality yields the desired result. \(\square\)

Moreover, we need the following results.

Lemma 3.2. For any \(\eta > 0\), there exists constant \(C = C(T, D_0)\) such that
\[
\sup_{t \in [0,T]} E \|u^{\varepsilon,\eta}(t) - \bar{u}^\eta(t)\|^2_{L^2(\mathbb{T}^1)} + \eta E \int_0^T \|\partial_x (u^{\varepsilon,\eta}(s) - \bar{u}^\eta)\|^2_{L^2(\mathbb{T}^1)} ds \leq C \varepsilon, \quad (3.19)
\]
\[
E \int_0^T \|u^{\varepsilon,\eta}(s) - \bar{u}^\eta\|^4_{L^4(\mathbb{T}^1)} ds \leq C \varepsilon^2. \quad (3.20)
\]
Proof. Let \( w_n^{E,\eta} := u^{E,\eta} - \bar{u}^\eta \), by using \( \bar{u}^\eta = 1 \) and \( \partial_x \bar{u}^\eta = \partial_x^2 \bar{u}^\eta = 0 \), it follows from (3.2) and (3.5) that \( w_n^{E,\eta} \) satisfies
\[
dw_n^{E,\eta} + w_n^{E,\eta} \partial_x w_n^{E,\eta} dt + \partial_x w_n^{E,\eta} dt = \eta \Delta w_n^{E,\eta} dt + \sqrt{\varepsilon} \Phi(u^{E,\eta}) dW(t).
\]

Taking \( L^2 \) inner product, by Itô formula and Hypothesis H, we have
\[
\sup_{t \in [0,T]} E\|w_n^{E,\eta}(t)\|_{L^2(\mathbb{T}^1)}^2 + 2\eta E \int_0^T \|\partial_x w_n^{E,\eta}(s)\|_{L^2(\mathbb{T}^1)}^2 ds \leq \frac{\varepsilon D_0}{2} E \int_0^T (1 + \|u^{E,\eta}(s)\|_{L^2(\mathbb{T}^1)}) ds.
\]

With the aid of (3.18), we have
\[
\sup_{t \in [0,T]} E\|w_n^{E,\eta}(t)\|_{L^2(\mathbb{T}^1)}^2 \leq C(T, D_0) \varepsilon.
\]

Taking \( f_1(\xi) := \xi^2 \) and \( f_2(\xi) := \xi^3 \), applying Itô formula to \( f_1(||w_0^{E,\eta}(t)||_{L^2(\mathbb{T}^1)}) \) and \( f_2(||w_0^{E,\eta}(t)||_{L^2(\mathbb{T}^1)}) \), we get
\[
\sup_{t \in [0,T]} E\|w_0^{E,\eta}(t)\|_{L^2(\mathbb{T}^1)}^4 \leq 2E \int_0^T ||w_0^{E,\eta}(s)||_{L^2(\mathbb{T}^1)}^4 ds + \varepsilon E \int_0^T ||w_0^{E,\eta}(s)\|_{L^2(\mathbb{T}^1)}^2 \Phi(u^{E,\eta})_{L^2(\mathbb{T}^1)} ds \
\leq \varepsilon E \int_0^T ||w_0^{E,\eta}(s)||_{L^2(\mathbb{T}^1)}^4 ds + \varepsilon D_0 E \int_0^T ||w_0^{E,\eta}(s)||_{L^2(\mathbb{T}^1)}^2 ds + \varepsilon E \int_0^T ||w_0^{E,\eta}(s)||_{L^2(\mathbb{T}^1)}^2 ds \
\leq D_0 \varepsilon E \int_0^T ||w_0^{E,\eta}(s)||_{L^2(\mathbb{T}^1)}^4 ds + \varepsilon E \int_0^T ||w_0^{E,\eta}(s)||_{L^2(\mathbb{T}^1)}^2 ds + \varepsilon^2,
\]

where \( L^2(\mathbb{T}^1) \) denotes the space of Hilbert-Schmidt operator from \( L^2(\mathbb{T}^1) \) to \( L^2(\mathbb{T}^1) \). Similarly, we have
\[
\sup_{t \in [0,T]} E\|w_0^{E,\eta}(t)||_{L^6(\mathbb{T}^1)}^6 \leq D_0 \varepsilon E \int_0^T ||w_0^{E,\eta}(s)||_{L^6(\mathbb{T}^1)}^6 ds + \varepsilon E \int_0^T ||w_0^{E,\eta}(s)||_{L^2(\mathbb{T}^1)}^4 ds + \varepsilon^3.
\]

Applying Gronwall’s inequality, we get
\[
\sup_{t \in [0,T]} E\|w_0^{E,\eta}(t)||_{L^2(\mathbb{T}^1)}^4 \leq C(T, D_0) \varepsilon^2.
\]

Based on the above estimates, by Garliardo-Nirenberg’s inequality and Hölder’s inequality, we have
\[
\int_0^T E\|u^{E,\eta}(s) - \bar{u}^\eta\|_{L^4(\mathbb{T}^1)}^4 ds \leq C \left( \int_0^T E\|w_0^{E,\eta}(s)||_{H^1(\mathbb{T}^1)}^2 ds \right)^{\frac{1}{2}} \left( \sup_{t \in [0,T]} E\|w_0^{E,\eta}(t)||_{L^2(\mathbb{T}^1)}^6 \right)^{\frac{1}{2}} \
\leq C(T, D_0) \varepsilon^2.
\]

□
Now, we are ready to present the proof of Proposition 3.2.

**Proof of Proposition 3.2.** Fix \( \eta > 0 \). Denote by \( w_1^{\epsilon, \eta} := \frac{\epsilon}{\sqrt{\epsilon}} - \bar{u}^{1, \eta} \), from (3.2), (3.5) and (3.6), we get

\[
dw_1^{\epsilon, \eta} + \partial_x \left( \frac{w_1^{\epsilon, \eta} - \bar{u}^{\eta}}{\sqrt{\epsilon}} \right) dt = \eta \Delta w_1^{\epsilon, \eta} dt + (\Phi(u^{\epsilon, \eta}) - \Phi(\bar{u}))dW(t),
\]

with \( w_1^{\epsilon, \eta}(0) = 0 \).

Taking \( L^2 \) inner product, and by Itô formula and (2.2), we obtain

\[
\sup_{t \in [0, T]} E\|w_1^{\epsilon, \eta}(t)\|_{L^2(T^1)}^2 + 2\eta E \int_0^T \|\partial_x w_1^{\epsilon, \eta}(s)\|_{L^2(T^1)}^2 ds \\
\leq E \int_0^T | < \partial_x w_1^{\epsilon, \eta}(s), \frac{w_0^{\epsilon, \eta}(s)}{\sqrt{\epsilon}} + \bar{u}^{\eta}w_1^{\epsilon, \eta}(s) >_{L^2(T^1)} | ds + E \int_0^T \|w_1^{\epsilon, \eta}(s) - \bar{u}^{\eta}\|_{L^2(T^1)}^2 ds.
\]

By Young’s inequality, (3.19) and (3.20), we have

\[
\sup_{t \in [0, T]} E\|w_1^{\epsilon, \eta}(t)\|_{L^2(T^1)}^2 + \eta E \int_0^T \|\partial_x w_1^{\epsilon, \eta}(s)\|_{L^2(T^1)}^2 ds \\
\leq_{\eta} E \int_0^T \frac{\|w_0^{\epsilon, \eta}(s)\|_{L^2(T^1)}^4}{4\epsilon} ds + E \int_0^T \|w_1^{\epsilon, \eta}(s)\|_{L^2(T^1)}^2 ds + C(T)\epsilon \\
\leq_{\eta} C(T, D_0)\epsilon + E \int_0^T \|w_1^{\epsilon, \eta}(s)\|_{L^2(T^1)}^2 ds.
\]

Applying Gronwall’s inequality, we achieve the result.

Finally, we focus on the estimation of the third term of the right hand side of (3.7).

**Proposition 3.3.** We have

\[
\lim_{\eta \to 0} \int_0^T E\|\bar{u}^{1, \eta}(t) - \bar{u}^1(t)\|_{L^2(T^1)}^2 dt = 0.
\]

**Proof.** Recall that \( \bar{u}^1 \) satisfies (3.4) and it can be written as

\[
\bar{u}^1(x, t) = \int_0^\infty \sum_{k \geq 1} g_k(x - \bar{u}(t - s), \bar{u})d\beta_k(s).
\]

(3.21)

For the solution \( \bar{u}^{1, \eta} \) of (3.6), it has the following representation

\[
\bar{u}^{1, \eta}(x, t) = \int_0^\infty \sum_{k \geq 1} (e^{\eta(t-s)}g_k(\cdot - \bar{u}(t - s), \bar{u}))(x)d\beta_k(s).
\]

(3.22)

Recall that \( \{e_k\}_{k \geq 1} \) is an orthogonal normal basis for \( L^2(T^1) \) satisfying

\[
\Delta e_k = -\lambda_k e_k,
\]

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with \( 0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \to +\infty \). From (3.21) and (3.22), by Hypothesis H, we deduce that
\[
E\|\bar{u}^{1,\eta}(t) - \bar{u}^{1}(t)\|_{L^{2}(\mathbb{T}^{1})}^2 \leq \int_{0}^{T} \sum_{k \geq 1} \|(e^{i\lambda(t-s) - i\delta})g_{k}(\cdot - (t-s), 1)\|_{L^{2}(\mathbb{T}^{1})}^2 ds \\
\leq \int_{0}^{T} \sum_{k \geq 1} \sum_{j \geq 1} |e^{-\eta\lambda(t-s)} - 1|^2 < g_{k}(\cdot - (t-s), 1), e_{j} \geq \frac{1}{L^{2}(\mathbb{T}^{1})} ds \\
\leq \int_{0}^{T} \sum_{k \geq 1} \|g_{k}(\cdot - (t-s), 1)\|_{L^{2}(\mathbb{T}^{1})}^2 ds \\
\leq C(T, D_{0}).
\]

With the help of dominated convergence theorem, we get
\[
\lim_{\eta \to 0} \int_{0}^{T} E\|\bar{u}^{1,\eta}(t) - \bar{u}^{1}(t)\|_{L^{2}(\mathbb{T}^{1})}^2 dt \\
\leq \int_{0}^{T} \int_{0}^{T} \sum_{k \geq 1} \sum_{j \geq 1} \lim_{\eta \to 0} |e^{-\eta\lambda(t-s)} - 1|^2 < g_{k}(\cdot - (t-s), 1), e_{j} \geq \frac{1}{L^{2}(\mathbb{T}^{1})} dsdt \\
= 0.
\]

Based on all the previous estimates, we are able to proceed with the proof of a central limit theorem. It reads as follows.

**Theorem 3.4. (Central Limit Theorem)** Assume Hypothesis H is in force, then
\[
\lim_{\varepsilon \to 0} E\left\| \frac{u^{\varepsilon} - \bar{u}}{\sqrt{\varepsilon}} - \bar{u}^{1} \right\|_{L^{1}([0,T];L^{1}(\mathbb{T}^{1}))} = 0.
\]

**Proof.** Recall that for any \( \eta > 0 \), we have
\[
\left\| \frac{u^{\varepsilon} - \bar{u}}{\sqrt{\varepsilon}} - \bar{u}^{1} \right\|_{L^{1}([0,T];L^{1}(\mathbb{T}^{1}))} \\
\leq \left\| \frac{u^{\varepsilon} - \bar{u}}{\sqrt{\varepsilon}} - \frac{u^{\varepsilon,\eta} - \bar{u}^{\eta}}{\sqrt{\varepsilon}} \right\|_{L^{1}([0,T];L^{1}(\mathbb{T}^{1}))} + \left\| \frac{u^{\varepsilon,\eta} - \bar{u}^{\eta}}{\sqrt{\varepsilon}} - \bar{u}^{1,\eta} \right\|_{L^{1}([0,T];L^{1}(\mathbb{T}^{1}))} \\
+ \left\| \bar{u}^{1,\eta} - \bar{u}^{1} \right\|_{L^{1}([0,T];L^{1}(\mathbb{T}^{1}))}.
\]

From Proposition 3.1 and Proposition 3.3, we know that for any \( \delta > 0 \), we can choose \( \eta_{0} > 0 \) small enough, such that for all \( \varepsilon > 0 \),
\[
\left\| \frac{u^{\varepsilon} - \bar{u}}{\sqrt{\varepsilon}} - \frac{u^{\varepsilon,\eta_{0}} - \bar{u}^{\eta_{0}}}{\sqrt{\varepsilon}} \right\|_{L^{1}([0,T];L^{1}(\mathbb{T}^{1}))} < \frac{\delta}{3}, \quad \left\| \bar{u}^{1,\eta_{0}} - \bar{u}^{1} \right\|_{L^{1}([0,T];L^{1}(\mathbb{T}^{1}))} < \frac{\delta}{3}.
\]

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Letting $\eta = \eta_0$, we deduce from (3.23) that
\[
\left\| \frac{u^\varepsilon - \bar{u}}{\sqrt{\varepsilon}} - \bar{u} \right\|_{L^1([0,T];L^1(\mathbb{T}^1))} \leq \frac{2}{3} \delta + \left\| \frac{u^{\varepsilon,\eta_0} - \bar{u}^{\eta_0}}{\sqrt{\varepsilon}} - \bar{u}^{1,\eta_0} \right\|_{L^1([0,T];L^1(\mathbb{T}^1))}.
\] (3.24)

By Proposition 3.2, for the above $\delta > 0$, there exists small enough $\varepsilon_0 > 0$ such that for any $\varepsilon \leq \varepsilon_0$,
\[
\left\| \frac{u^{\varepsilon,\eta_0} - \bar{u}^{\eta_0}}{\sqrt{\varepsilon}} - \bar{u}^{1,\eta_0} \right\|_{L^1([0,T];L^1(\mathbb{T}^1))} \leq \frac{\delta}{3}.
\]
Thus, for any $\varepsilon \leq \varepsilon_0$, we deduce from (3.24) that
\[
\left\| \frac{u^\varepsilon - \bar{u}}{\sqrt{\varepsilon}} - \bar{u} \right\|_{L^1([0,T];L^1(\mathbb{T}^1))} < \delta.
\]

By the arbitrary of $\delta$, we conclude the result.

\[\square\]

### 4 Moderate deviation principle

In this section, we will prove that $\frac{u^\varepsilon - \bar{u}}{\sqrt{\varepsilon}}$ satisfies an LDP on $L^1([0, T]; L^1(\mathbb{T}^1))$ with $\lambda(\varepsilon)$ satisfying (1.4), which is called moderate deviation principle.

#### 4.1 Large deviation principle

We first introduce some notations and recall a general criteria for large deviation principle given by [31]. Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $\{\mathcal{F}_t\}_{t \geq 0}$ is a filtration satisfying the usual condition. Denote by $\mathcal{E}$ a Polish space with metric $d$, and $\mathcal{B}(\mathcal{E})$ is the Borel $\sigma$-algebra produced by the metric $d$.

**Definition 4.1.** (Rate function) A function $I : \mathcal{E} \to [0, \infty]$ is called a rate function if $I$ is lower semicontinuous. A rate function $I$ is called a good rate function if the level set $\{x \in \mathcal{E} : I(x) \leq M\}$ is compact for each $M < \infty$.

**Definition 4.2.** (Large deviation principle) Let $I$ be a rate function on $\mathcal{E}$. A family $\{X^\varepsilon\}$ of $\mathcal{E}$-valued random elements is said to satisfy the large deviation principle on $\mathcal{E}$ with rate function $I$, if the following two conditions hold.

(i) (Upper bound) For each closed subset $F$ of $\mathcal{E}$,
\[
\limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P}(X^\varepsilon \in F) \leq -\inf_{x \in F} I(x).
\]
(ii) (Lower bound) For each open subset $G$ of $\mathcal{E}$,

$$
\liminf_{\varepsilon \to 0} \varepsilon \log P(X^\varepsilon \in G) \geq -\inf_{x \in G} I(x).
$$

Assume that $W$ is a cylindrical Wiener process on $L^2(\mathbb{T}^1)$ defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, P)$ (that is, the paths of $W$ take values in $C([0,T]; \mathcal{U})$, where $\mathcal{U}$ is another Hilbert space such that the embedding $L^2(\mathbb{T}^1) \subset \mathcal{U}$ is Hilbert-Schmidt). The Cameron-Martin space of the Wiener process $\{W(t), t \in [0,T]\}$ is given by

$$
\mathcal{H}_0 := \{h : [0,T] \to L^2(\mathbb{T}^1); h \text{ is absolutely continuous and } \int_0^T \|\dot{h}(s)\|^2_{L^2(\mathbb{T}^1)} ds < \infty\}.
$$

The space $\mathcal{H}_0$ is a Hilbert space with inner product

$$
<h_1, h_2>_{\mathcal{H}_0} := \int_0^T \langle \dot{h}_1(s), \dot{h}_2(s) \rangle_{L^2(\mathbb{T}^1)} ds.
$$

Denote by $\mathcal{A}$ the class of $\{\mathcal{F}_t\}$-predictable processes $\phi$ belonging to $\mathcal{H}_0$, $P$-a.s. Let $S_N := \{h \in \mathcal{H}_0; \int_0^T \|\dot{h}(s)\|^2_{L^2(\mathbb{T}^1)} ds \leq N\}$. Here and in the sequel of this paper, we will always refer to the weak topology on the set $S_N$. Then, the set $S_N$ is a Polish space. Define $\mathcal{A}_N := \{\phi \in \mathcal{A}; \phi(\omega) \in S_N, P$-a.s.$\}$.

Recently, a new sufficient condition (condition (a) and condition (b) in the following Theorem 4.1) to verify the large deviation principle is proposed by Matoussi, Sabagh and Zhang in [31]. It turns out this new sufficient condition is suitable for establishing the large deviation principle for inviscid stochastic Burgers equation. Combining Budhiraja et al. [6] and [31], it follows that

**Theorem 4.1.** For $\varepsilon > 0$, let $\Gamma^\varepsilon$ be a measurable mapping from $C([0,T]; \mathcal{U})$ into $\mathcal{E}$. Let $X^\varepsilon := \Gamma^\varepsilon(W(\cdot))$. Suppose that $\{\Gamma^\varepsilon\}_{\varepsilon > 0}$ satisfies the following assumptions: there exists a measurable map $\Gamma^0 : C([0,T]; \mathcal{U}) \to \mathcal{E}$ such that

(a) for every $N < \infty$ and any family $\{h^\varepsilon; \varepsilon > 0\} \subset \mathcal{A}_N$, and for any $\delta > 0$,

$$
\lim_{\varepsilon \to 0} P(d(Y^\varepsilon, Z^\varepsilon) > \delta) = 0,
$$

where $Y^\varepsilon := \Gamma^\varepsilon(W(\cdot) + \frac{1}{\sqrt{\varepsilon}} \int_0^\cdot h^\varepsilon(s) ds)$, $Z^\varepsilon := \Gamma^0(\int_0^\cdot \dot{h}^\varepsilon(s) ds)$.

(b) for every $N < \infty$, the family $\{h_m\}_{m \geq 1} \subset S_N$ that converges to some element $h$ as $m \to \infty$, $\Gamma^0(\int_0^\cdot \dot{h}_m(s) ds)$ converges to $\Gamma^0(\int_0^\cdot \dot{h}(s) ds)$ in the space $\mathcal{E}$.

Then the family $\{X^\varepsilon\}_{\varepsilon > 0}$ satisfies a large deviation principle in $\mathcal{E}$ with the rate function $I$ given by

$$
I(g) := \inf_{h \in \mathcal{H}_0; g = \Gamma^0(\int_0^\cdot \dot{h}(s) ds)} \left\{ \frac{1}{2} \int_0^T \|\dot{h}(s)\|^2_{L^2(\mathbb{T}^1)} ds \right\}, \quad g \in \mathcal{E},
$$

with the convention $\inf\{0\} = \infty$. 

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4.2 Skeleton equations

We begin by introducing the map $\Gamma_0$ that will be used to define the rate function and also used to verify conditions (a) and (b) in Theorem 4.1.

For any $h \in \mathcal{H}_0$, consider the following deterministic integral equation

$$\begin{cases} dX_h(t) + \bar{u} \partial_x X_h(t)dt = \Phi(\bar{u})h(t)dt, \\ X_h(x, 0) = 0, \end{cases} \tag{4.2}$$

where $\bar{u} \equiv 1$. In fact, the equation (4.2) is called the skeleton equation for MDP of stochastic Burgers equation (2.3) and is derived briefly as follows. Let $X^\varepsilon := \frac{\delta - \partial_x}{\sqrt{\varepsilon}(\cdot)}$, it satisfies

$$\begin{cases} dX^\varepsilon + \sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon \partial_x X^\varepsilon dt + \bar{u} \partial_x X^\varepsilon dt = \lambda(\varepsilon)^{-1}\Phi(\bar{u} + \sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon) dW(t), \\ X^\varepsilon(0) = 0, \end{cases} \tag{4.3}$$

Using the solution $X^\varepsilon$ to (4.3), we can define the map $\Gamma^\varepsilon$ as $\Gamma^\varepsilon(W(\cdot)) := X^\varepsilon(\cdot)$.

For any $h \in \mathcal{H}_0$, consider the following SPDE

$$\begin{cases} dX_h^\varepsilon + \sqrt{\varepsilon}\lambda(\varepsilon)X_h^\varepsilon \partial_x X_h^\varepsilon dt + \bar{u} \partial_x X_h^\varepsilon dt = \lambda(\varepsilon)^{-1}\Phi(\bar{u} + \sqrt{\varepsilon}\lambda(\varepsilon)X_h^\varepsilon) (dW(t) + \lambda(\varepsilon)\dot{h}(t)dt), \\ X_h^\varepsilon(0) = 0. \end{cases}$$

Letting $\varepsilon \to 0$, it follows that $X_h^\varepsilon$ converges to $X_h$ in some suitable space with $X_h$ satisfying (4.2).

Regarding to (4.2), we introduce the definition of kinetic solution.

**Definition 4.3.** (Kinetic solution) A measurable function $X_h : \mathbb{T}^1 \times [0, T] \to \mathbb{R}$ is said to be a kinetic solution to (4.2), if for any $p \geq 1$, there exists $C_p \geq 0$ such that

$$\text{ess sup}_{0 \leq t \leq T} ||X_h(t)||_{L^p(\mathbb{T}^1)} \leq C_p,$$

and if there exists a measure $m_h \in \mathcal{M}_0^*(\mathbb{T}^1 \times [0, T] \times \mathbb{R})$ such that $f_h := I_{X_h > \bar{\xi}}$ satisfies that for all $\varphi \in C^1(\mathbb{T}^1 \times [0, T] \times \mathbb{R}),$

$$\int_0^T (f_h(t), \partial_t \varphi(t))dt + \langle f_0, \varphi(0) \rangle + \int_0^T \langle f_h(t), \partial_x \varphi(t) \rangle dt = -\sum_{k \geq 1} \int_0^T \int_{\mathbb{T}^1} g_k(x, \bar{u}) \varphi(x, t, X_h(x, t)) \dot{h}^k(t) dx dt + m_h(\partial_x \varphi),$$

where $f_0(x, \xi) = I_{X_0 > \bar{\xi}} = I_{0 > \bar{\xi}}$.

Referring to [16], we have the following well-posedness result for (4.2).
Theorem 4.2. (Well-posedness) Under Hypothesis H, for any $T > 0$, (4.2) has a unique kinetic solution $X_h$ on $[0, T]$.

In view of Theorem 4.2, we can define $\Gamma^0 : C([0, T]; U) \to L^1([0, T]; L^1(T^1))$ by

$$\Gamma^0(\tilde{h}) := \begin{cases} 
X_h, & \text{if } \tilde{h} = \int_0^T \dot{h}(s) ds, \text{ for some } h \in H_0, \\
0, & \text{otherwise},
\end{cases}$$

where $X_h$ is the solution of equation (4.2).

4.3 The continuity of the skeleton equation

In this part, we aim to prove the continuity of the mapping $\Gamma^0$. Namely, let $X_{ha}$ denote the kinetic solution of (4.2) with $h$ replaced by $h_m$ and we will show that $X_{ha}$ converges to the kinetic solution $X_h$ of the skeleton equation (4.2) in $L^1([0, T]; L^1(T^1))$, if $h_m \to h$ weakly in $L^2([0, T]; L^2(T^1))$. For technical reasons, we will introduce an auxiliary approximation process.

For any family $\{ h_m; m \geq 1 \} \subset SN$ and $\eta > 0$, let us consider the following parabolic approximation of the skeleton equation

$$dX^\eta_h(t) + \bar{u} \partial_x X^\eta_h(t) dt = \eta \Delta X^\eta_h(t) dt + \Phi(\bar{u}) \dot{h}(t) dt,$$  (4.4)

with $X^\eta_h(0) = 0$. Referring to [16], the equation (4.4) admits a unique strong solution $X^\eta_h$ satisfying that for any $N > 0$,

$$\lim_{\eta \to 0} \sup_{h \in SH_N} \int_0^T \|X^\eta_h(t) - X_h(t)\|^2_{L^1(T^1)} dt = 0.$$  (4.5)

With the above process (4.4), for any $\eta > 0$, we have

$$\|X_{ha} - X_h\|_{L^1([0, T] \times T^1)}$$

$$\leq \|X_{ha} - X^\eta_{ha}\|_{L^1([0, T] \times T^1)} + \|X^\eta_h - X^\eta_h\|_{L^1([0, T] \times T^1)} + \|X^\eta_h - X_h\|_{L^1([0, T] \times T^1)}.$$

With the aid of (4.5), for any $\delta > 0$, we can choose small enough $\eta_0 > 0$ such that

$$\sup_{m \geq 1} \|X_{ha} - X^\eta_{ha}\|_{L^1([0, T] \times T^1)} < \frac{\delta}{3}, \quad \|X^\eta_h - X_h\|_{L^1([0, T] \times T^1)} < \frac{\delta}{3}.$$

Thus, in order to prove the continuity of the skeleton equation, it suffices to show that for fixed $\eta > 0$, $\lim_{m \to \infty} \|X^\eta_{ha} - X^\eta_h\|_{L^1([0, T] \times T^1)} = 0$.

Firstly, we prove the compactness of $\{X^\eta_{ha}\}$ with fixed $\eta > 0$. For simplicity, we denote by $X^\eta_{m} := X^\eta_{ha}$.
As in [23], we introduce the following space. Let $K$ be a separable Banach space with the norm $\| \cdot \|_K$, for $p > 1, \alpha \in (0, 1)$, let $W^{\alpha,p}([0, T]; K)$ be the Sobolev space of all functions $f \in L^p([0, T]; K)$ such that

$$\int_0^T \int_0^T \frac{\|f(t) - f(s)\|_K^p}{|t-s|^{1+\alpha p}} dt ds < \infty,$$

which endowed with the norm

$$\|f\|_{W^{\alpha,p}([0,T];K)}^p = \int_0^T \|f(t)\|_K^p dt + \int_0^T \int_0^T \frac{\|f(t) - f(s)\|_K^p}{|t-s|^{1+\alpha p}} dt ds.$$

We will use the following compactness criterion, which can be found in [23].

**Lemma 4.1.** Let $B_0 \subset B \subset B_1$ be three Banach spaces, assume that both $B_0$ and $B_1$ are reflexive, and $B_0$ is compactly embedded in $B$. Let $p \in (1, \infty)$ and $\alpha \in (0, 1)$ be given. Let $\Lambda$ be the space

$$\Lambda := L^p([0, T]; B_0) \cap W^{\alpha,p}([0, T]; B_1)$$

endowed with the natural norm. Then the embedding of $\Lambda$ in $L^p([0, T]; B)$ is compact.

With the help of the above lemma, we have

**Proposition 4.3.** For any $\eta > 0$, $\{X^\eta_m\}_{m \geq 1}$ is compact in $L^2([0, T]; L^2(\mathbb{T}^1))$.

**Proof.** Firstly, using Proposition 5.1 in [16], we have

$$\sup_{m \geq 1} \left\{ \sup_{t \in [0, T]} \|X^\eta_m(t)\|_{L^2(\mathbb{T}^1)}^2 + \int_0^T \|\nabla X^\eta_m(s)\|_{L^2(\mathbb{T}^1)} ds \right\} \leq C(N).$$

From (4.4), $X^\eta_m$ can be written as

$$X^\eta_m(t) = \eta \int_0^t \Delta X^\eta_m(s) ds - \int_0^t \partial_t X^\eta_m(s) ds + \int_0^T \Phi(\bar{u}) \hat{h}_m(s) ds$$

$$=: I^\eta_1 + I^\eta_2 + I^\eta_3.$$

Since $\Delta : H^1 \rightarrow H^{-1}$ is an isometry operator, and by Minkowski inequality and Hölder inequality, we have

$$\|I^\eta_1(t) - I^\eta_1(s)\|_{H^{-1}} = \eta \left\| \int_s^t \Delta X^\eta_m(s) ds \right\|_{H^{-1}}$$

$$\leq C\eta |t-s| \int_s^t \|\Delta X^\eta_m(s)\|_{H^{-1}}^2 ds$$

$$\leq C\eta |t-s| \int_s^t \|\nabla X^\eta_m(s)\|_{L^2(\mathbb{T}^1)}^2 ds.$$
With the aid of (4.6), for \( \alpha \in (0, \frac{1}{2}) \), we have

\[
\sup_{m \geq 1} \| I^m_1 \|_{W^{2,1}(0,T;H^{-1}(T^1))}^2 \leq \int_0^T \| I^m_1(s) \|_{H^{-1}}^2 ds + \int_0^T \int_0^T \frac{\| I^m_1(t) - I^m_1(s) \|_{H^{-1}}^2}{|t-s|^{1+2\alpha}} ds dt \\
\leq C_1(\alpha).
\]

For \( I^m_2 \), by integration by part formula, we have

\[
\| I^m_2(t) - I^m_2(s) \|_{H^{-1}}^2 = \| \int_s^t \partial_s X^n_m(r) dr \|_{H^{-1}}^2 \\
\leq C(t-s) \int_s^t \| \partial_s X^n_m(r) \|_{H^{-1}}^2 dr \\
\leq C(t-s) \int_s^t \sup_{\| v \|_{H^1} \leq 1} | \partial_s X^n_m \cdot v \|_{L^2(T^1)}^2 dr \\
\leq C(t-s) \int_s^t \| X^n_m(r) \|_{L^2(T^1)}^2 dr \\
\leq C(t-s)^2 \sup_{m \geq 1, r \in [0,T]} \| X^n_m(r) \|_{L^2(T^1)}^2.
\]

Hence, we deduce that for \( \alpha \in (0, \frac{1}{2}) \),

\[
\sup_{m \geq 1} \| I^m_2 \|_{W^{2,1}(0,T;H^{-1}(T^1))}^2 \leq C_2(\alpha).
\]

At last, we focus on \( I^m_3 \),

\[
\| I^m_3(t) - I^m_3(s) \|_{L^2(T^1)}^2 = \| \int_s^t \Phi(\tilde{u}) \hat{h}_m(r) dr \|_{L^2(T^1)}^2 \\
\leq C(t-s) \int_s^t \int_{T^1} \left( \sum_{k \geq 1} g_k(x, \tilde{u}) h^k_m(r) \right)^2 dx dr \\
\leq C(t-s) \int_s^t \int_{T^1} \left( \sum_{k \geq 1} g_k(x, \tilde{u}) h^k_m(r) \right)^2 dx dr \\
\leq CND_0(t-s).
\]

So that for \( \alpha \in (0, \frac{1}{2}) \), we have

\[
\sup_{m \geq 1} \| I^m_3 \|_{W^{2,1}(0,T;L^2(T^1))}^2 \leq C_3(\alpha).
\]

With the help of (4.6) and lemma 4.1, we obtain the compactness result. \( \square \)
Finally, we are able to proceed with the proof of the continuity of the skeleton equation.

**Theorem 4.4.** Fix \( N > 0 \). Assume \( \{h_m\}_{m \geq 1} \subset S_N \) and \( h_m \rightharpoonup h \) weakly in \( L^2([0, T]; L^2(\mathbb{T}^1)) \), then \( X_m \) converges to \( X_h \) in \( L^1([0, T] \times \mathbb{T}^1) \), where \( X_m \) is the kinetic solution to the skeleton equation (4.2) with \( h \) replaced by \( h_m \).

**Proof.** Recall that for any \( \eta > 0 \), we have

\[
\|X_m - X_h\|_{L^1([0,T] \times \mathbb{T}^1)} \\
\leq \|X_m - X_h^0\|_{L^1([0,T] \times \mathbb{T}^1)} + \|X_h^0 - X_h\|_{L^1([0,T] \times \mathbb{T}^1)} + \|X_h^0 - X_h\|_{L^1([0,T] \times \mathbb{T}^1)},
\]

(4.7)

Referring to (4.5), for any \( \delta > 0 \), we can choose small enough \( \eta_0 > 0 \) such that

\[
\sup_{m \geq 1} \|X_m - X_h^0\|_{L^1([0,T] \times \mathbb{T}^1)} < \frac{\delta}{3}, \quad \|X_h - X_h^0\|_{L^1([0,T] \times \mathbb{T}^1)} < \frac{\delta}{3}.
\]

Letting \( \eta = \eta_0 \) in (4.7), it follows that

\[
\|X_m - X_h\|_{L^1([0,T] \times \mathbb{T}^1)} \leq \frac{2}{3} \delta + \|X_h^0 - X_h\|_{L^1([0,T] \times \mathbb{T}^1)}.
\]

As a result, in order to show \( \|X_m - X_h\|_{L^1([0,T] \times \mathbb{T}^1)} \rightarrow 0 \), it suffices to prove that for any fixed \( \eta_0 > 0 \), we have \( \lim_{m \rightarrow \infty} \|X_h^0 - X_h\|_{L^1([0,T] \times \mathbb{T}^1)} = 0 \).

From (4.2), by the chain rule and due to the boundary condition, we have

\[
\|X_h^0(t) - X_h^0(t)\|_{L^2(\mathbb{T}^1)} + 2\eta_0 \int_0^T \|
abla(X_m^0 - X_h^0)\|^2_{L^2(\mathbb{T}^1)} ds \\
\leq \int_0^T < \partial_s(X_m^0 - X_h^0), X_m^0 - X_h^0 >_{L^2(\mathbb{T}^1)} ds + 2 \int_0^T < \Phi(\bar{u})(h_m(s) - h(s)), X_m^0 - X_h^0 >_{L^2(\mathbb{T}^1)} ds \\
= \int_0^T < \Phi(\bar{u})(h_m(s) - h(s)), X_m^0 - X_h^0 >_{L^2(\mathbb{T}^1)} ds.
\]

To show \( \lim_{m \rightarrow \infty} \sup_{t \in [0,T]} \|X_m^0(t) - X_h^0(t)\|_{L^2(\mathbb{T}^1)} = 0 \), it is sufficient to prove

\[
\lim_{m \rightarrow \infty} \sup_{t \in [0,T]} \left| \int_0^T < \Phi(\bar{u})(h_m(s) - h(s)), X_m^0 - X_h^0 >_{L^2(\mathbb{T}^1)} ds \right| = 0.
\]

This will be achieved if we show that for any sequence \( m_k \rightarrow \infty \), one can find a subsequence \( m_{k_i} \rightarrow \infty \) such that

\[
\lim_{l \rightarrow \infty} \sup_{t \in [0,T]} \left| \int_0^T < \Phi(\bar{u})(h_{m_{k_i}}(s) - h(s)), X_{m_{k_i}}^0 - X_h^0 >_{L^2(\mathbb{T}^1)} ds \right| = 0.
\]

(4.8)
By the compactness of \( \{X_{m\geq 1}^0\} \) in \( L^2([0, T] \times \mathbb{T}^1) \), we know that for any sequence \( m_k \to \infty \), there exist a subsequence \( m_{k_l} \to \infty \) and \( \tilde{X} \in L^2([0, T] \times \mathbb{T}^1) \) such that \( X_{m_{k_l}}^0 \to \tilde{X} \) in \( L^2([0, T] \times \mathbb{T}^1) \).

With the help of \( \tilde{X} \), we have

\[
\left| \int_0^t < \Phi(\tilde{u})(h_{m_{k_l}} - h), X_{m_{k_l}}^0 - X_h^0 >_{L^2(\mathbb{T}^1)} ds \right| \\
\leq \left| \int_0^t < \Phi(\tilde{u})(h_{m_{k_l}} - h), X_{m_{k_l}}^0 - \tilde{X} >_{L^2(\mathbb{T}^1)} ds \right| + \left| \int_0^t < \Phi(\tilde{u})(h_{m_{k_l}} - h), \tilde{X} - X_h^0 >_{L^2(\mathbb{T}^1)} ds \right|.
\]

By Hölder’s inequality and Hypothesis H, we deduce that

\[
\left| \int_0^t < \Phi(\tilde{u})(h_{m_{k_l}} - h), X_{m_{k_l}}^0 - \tilde{X} >_{L^2(\mathbb{T}^1)} ds \right| \leq D_0 N \|X_{m_{k_l}}^0 - \tilde{X}\|_{L^2([0, T] \times \mathbb{T}^1)}^2 
\to 0, \quad \text{as } l \to \infty.
\]

Since \( h_{m_{k_l}} \to h \) weakly in \( L^2([0, T]; L^2(\mathbb{T}^1)) \), for any \( t > 0 \), we have

\[
\left| \int_0^t < \Phi(\tilde{u})(h_{m_{k_l}} - h), \tilde{X} - X_h^0 >_{L^2(\mathbb{T}^1)} ds \right| \to 0, \quad (4.9)
\]
as \( l \to \infty \).

On the other hand, by the assumption on \( h \), for \( 0 < t_1 < t_2 \leq T \), we have

\[
\left| \int_{t_1}^{t_2} < \Phi(\tilde{u})(h_{m_{k_l}} - h), \tilde{X} - X_h^0 >_{L^2(\mathbb{T}^1)} ds \right| \\
\leq \sqrt{D_0} \int_{t_1}^{t_2} \|\tilde{X} - X_h^0\|_{L^2(\mathbb{T}^1)} \|h_{m_{k_l}} - h\|_{L^2(\mathbb{T}^1)} ds \\
\leq \sqrt{D_0} (2N)^{1/2} \left( \int_{t_1}^{t_2} \|\tilde{X} - X_h^0\|_{L^2(\mathbb{T}^1)}^2 ds \right)^{1/2}. \quad (4.10)
\]

Combining (4.9) and (4.10), we deduce that

\[
\lim_{l \to \infty} \sup_{0 \leq t \leq T} \left| \int_0^t < \Phi(\tilde{u})(h_{m_{k_l}} - h), \tilde{X} - X_h^0 >_{L^2(\mathbb{T}^1)} ds \right| = 0.
\]

Based on all the above results, we obtain (4.8) holds. We complete the proof. \( \square \)

### 4.4 Moderate deviation principle

In this section, we focus on the proof of the main result. It reads as follows.

**Theorem 4.5.** For the kinetic solution of stochastic Burgers equations \( u^\varepsilon \), \( \frac{u^\varepsilon - \bar{u}}{(\sqrt{\varepsilon})} \) satisfies LDP on \( L^1([0, T]; L^1(\mathbb{T}^1)) \) with speed \( \lambda^2(\varepsilon) \) and with rate function \( I \) defined by (4.1), that is
(I) for any closed subset $F$ of $L^1([0, T]; L^1(\mathbb{T}^1))$,
\[
\limsup_{\varepsilon \to 0} \frac{1}{\lambda(\varepsilon)^2} \log P\left(\frac{u^\varepsilon - \bar{u}}{\sqrt{\varepsilon \lambda(\varepsilon)}} \in F\right) \leq -\inf_{x \in F} I(x);
\]

(II) for each open subset $G$ of $L^1([0, T]; L^1(\mathbb{T}^1))$,
\[
\liminf_{\varepsilon \to 0} \frac{1}{\lambda(\varepsilon)^2} \log P\left(\frac{u^\varepsilon - \bar{u}}{\sqrt{\varepsilon \lambda(\varepsilon)}} \in G\right) \geq -\inf_{x \in G} I(x).
\]

According to Theorem 4.1, we only need to verify sufficient conditions (a) and (b) to establish Theorem 4.5. In Section 4.3, the condition (b) has been proved by Theorem 4.4, hence, it remains to prove condition (a).

For any $\{h^\varepsilon\}_{\varepsilon > 0} \subset \mathcal{A}_N$, we consider
\[
\begin{cases}
\begin{aligned}
d\hat{X}^\varepsilon + \sqrt{\varepsilon \lambda(\varepsilon)} \hat{X}^\varepsilon \partial_x \hat{X}^\varepsilon dt + \partial_x \hat{X}^\varepsilon dt = \lambda^{-1}(\varepsilon) \Phi(1 + \sqrt{\varepsilon \lambda(\varepsilon)} \hat{X}^\varepsilon) dW(t) + \Phi(1 + \sqrt{\varepsilon \lambda(\varepsilon)} \hat{X}^\varepsilon) h^\varepsilon(t) dt,
\end{aligned}
\end{cases}
\]
\[
\hat{X}^\varepsilon(0) = 0.
\]

Denote by $A_\varepsilon(\xi) := \sqrt{\varepsilon \lambda(\varepsilon)} \frac{\xi}{2} + \xi$, then $a_\varepsilon(\xi) := A'_\varepsilon(\xi) = \sqrt{\varepsilon \lambda(\varepsilon)} \xi + 1$. Combining techniques from Theorem 2.2 and Theorem 4.2, we conclude that there exists a unique kinetic solution $\hat{X}^\varepsilon$ satisfying that for any $p \geq 1$
\[
E\left(\esssup_{0 \leq t \leq T} ||\hat{X}^\varepsilon(t)||^p_{L^p(\mathbb{T}^1)}\right) \leq C_p,
\]
and there exists a kinetic measure $\bar{m}^\varepsilon \in \mathcal{M}^+_0(\mathbb{T}^1 \times [0, T] \times \mathbb{R})$ such that $f^\varepsilon := I_{\hat{X}^\varepsilon > \xi}$ satisfies for any $\varphi \in C^1_c(\mathbb{T}^1 \times [0, T] \times \mathbb{R})$,
\[
\int_0^T < f^\varepsilon(t), \partial_t \varphi(t) > dt + < f_0, \varphi(0) > + \int_0^T < f^\varepsilon(t), a_\varepsilon(\xi) \partial_x \varphi(t) > dt
\]
\[
= - \lambda^{-1}(\varepsilon) \sum_{k \geq 1} \int_0^T \int_{\mathbb{T}^1} g_k(x, 1 + \sqrt{\varepsilon \lambda(\varepsilon)} \hat{X}^\varepsilon) \varphi(x, t, \hat{X}^\varepsilon) dx \beta_k(t)
\]
\[
- \frac{1}{2 \lambda^2(\varepsilon)} \int_0^T \int_{\mathbb{T}^1} \partial_x \varphi(x, t, \hat{X}^\varepsilon) G^2(x, 1 + \sqrt{\varepsilon \lambda(\varepsilon)} \hat{X}^\varepsilon) dx dt
\]
\[
- \sum_{k \geq 1} \int_0^T \int_{\mathbb{T}^1} \varphi(x, t, \hat{X}^\varepsilon) g_k(x, 1 + \sqrt{\varepsilon \lambda(\varepsilon)} \hat{X}^\varepsilon) \dot{h}^{\varepsilon,k}(t) dx dt + \bar{m}^\varepsilon(\partial_x \varphi), \ a.s.
\]
where $\{\dot{h}^{\varepsilon,k}\}_{k \geq 1}$ are the Fourier coefficients of $\dot{h}^\varepsilon$, that is $\dot{h}^\varepsilon(t) = \sum_{k \geq 1} \dot{h}^{\varepsilon,k}(t) e_k$. Recall that the map $\Gamma^\varepsilon$ is defined by $\Gamma^\varepsilon(W(\cdot)) = X^\varepsilon(\cdot)$, where $X^\varepsilon$ is the solution to (4.3). By the definition of $\Gamma^\varepsilon$, we have $\Gamma^\varepsilon(W(\cdot) + \lambda(\varepsilon) \int_0^t \dot{h}^\varepsilon(s) ds) = \hat{X}^\varepsilon(\cdot)$.

Now, we are ready to verify the condition (a) in Theorem 4.1.

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**Theorem 4.6.** For every $N < \infty$, let $\{h^\epsilon\}_{\epsilon > 0} \subset \mathcal{A}_N$. Then
\[
\left\| \Gamma^\epsilon \left( W(\cdot) + \lambda(\epsilon) \int_0^\cdot h^\epsilon(s)ds \right) - \Gamma^0 \left( \int_0^\cdot h^\epsilon(s)ds \right) \right\|_{L^1([0,T],L^1)} \to 0,
\]
in probability, as $\epsilon \to 0$.

**Proof.** Recall that $\bar{X}^\epsilon = \Gamma^\epsilon \left( W(\cdot) + \lambda(\epsilon) \int_0^\cdot h^\epsilon(s)ds \right)$ is the kinetic solution to (4.11) with the corresponding kinetic measure $\bar{m}_1^\epsilon$. Moreover, $Y^\epsilon(\cdot) := \Gamma^0 \left( \int_0^\cdot h^\epsilon(s)ds \right)$ is the kinetic solution to the skeleton equation (4.2) with $h$ replaced by $h^\epsilon$ and the corresponding kinetic measure is denoted by $m_2^\epsilon$.

Let $f_1(x, t, \xi) := I_{\bar{X}^\epsilon(\cdot,t) > \xi}$, and $f_2(y, t, \zeta) := I_{Y^\epsilon(\cdot,t) > \zeta}$. Applying the same procedure as Lemma 2.1, for any $\varphi_1 \in C^1(\mathbb{T}_x \times \mathbb{R}_\xi)$, we have
\[
< f_1^\epsilon(t), \varphi_1 > - < f_{1,0}, \varphi_1 > + \int_0^t < f_1(s), (\sqrt{\lambda(\epsilon)} \xi + 1) \partial_s \varphi_1(x, \xi) > ds \\
+ \lambda(\epsilon)^{-1} \sum_{k \geq 1} \int_0^t \int_{\mathbb{T}_x} g_k(x, 1 + \sqrt{\lambda(\epsilon)} \xi) \varphi_1(x, \xi) dv_{x,\xi}^1(\xi) dx d\beta_k(s) \\
+ \frac{\lambda(\epsilon)^{-2}}{2} \int_0^t \int_{\mathbb{T}_x} \varphi_1(\xi) G^2(x, 1 + \sqrt{\lambda(\epsilon)} \xi) dv_{x,\xi}^1(\xi) dx ds \\
+ \sum_{k \geq 1} \int_0^t \int_{\mathbb{T}_x} \varphi_1(x, \xi) g_k(x, \xi) h^\epsilon_k(s) dv_{x,\xi}^1(\xi) dx ds < \bar{m}_1^\epsilon, \partial_\xi \varphi_1 > ([0, t]), \ a.s.
\]
where $f_{1,0}(\xi) := I_{\bar{X}^\epsilon(\cdot,t) > \xi} = I_{0 > \xi}$ and $v_{x,\xi}^1(\xi) = -\partial_\xi f_1(s, x, \xi) = \partial_\xi f_1(s, x, \xi) = \delta_{\bar{X}^\epsilon(\cdot,t) = \xi}$. Similarly, for $\varphi_2 \in C^1(\mathbb{T}_y \times \mathbb{R}_\zeta)$, we have
\[
< f_2^\epsilon(t), \varphi_2 > - < f_{2,0}, \varphi_2 > + \int_0^t < f_2(s), \partial_s \varphi_2(y, \zeta) > ds \\
- \sum_{k \geq 1} \int_0^t \int_{\mathbb{T}_y} g_k(y, 1) \varphi_2(y, \zeta) h^\epsilon_k(s) dv_{y,\zeta}^2(\zeta) dy ds < m_2^\epsilon, \partial_\zeta \varphi_2 > ([0, t]), \ a.s.
\]
where $f_{2,0}(\zeta) := I_{Y^\epsilon(\cdot,t) > \zeta} = I_{0 > \zeta}$ and $v_{y,\zeta}^2(\zeta) = \partial_\zeta f_2(s, y, \zeta) = -\partial_\xi f_2(s, y, \zeta) = \delta_{Y^\epsilon(\cdot,t) = \zeta}$.
Taking \( \alpha(x, \xi, y, \zeta) = \varphi_1(x, \xi) \varphi_2(y, \zeta) \), using Itô formula, we have

\[
\langle\langle f'_1(t) \bar{f}_2^+, \alpha \rangle\rangle
= \langle\langle f'_{1,0}(t) \bar{f}_2^+, \alpha \rangle\rangle + \sqrt{\lambda(\varepsilon)} \int_0^t \int_{\mathbb{R}^2} f_1 \bar{f}_2 \xi \partial_x \alpha d\xi d\zeta dx dy ds
+ \lambda(\varepsilon)^{-1} \int_0^t \int_{\mathbb{R}^2} \xi \partial_x \alpha f_2(s, y, \zeta) G^2(x, 1 + \sqrt{\lambda(\varepsilon)\xi}) \partial_x^2 \dot{\zeta}^2 d\xi dx dy ds
+ \sum_{k \geq 1} \int_0^t \int_{(T^1)^2} \int_{\mathbb{R}^2} \tilde{f}_2(s, x, \xi) \alpha g_k(x, 1 + \sqrt{\lambda(\varepsilon)\xi}) \dot{x}_k(s) \partial_x^2 \dot{\zeta}^2 d\xi dx dy ds
- \sum_{k \geq 1} \int_0^t \int_{(T^1)^2} \int_{\mathbb{R}^2} f_1(s, x, \xi) \alpha g_k(y, 1) \dot{x}_k(s) \partial_x \dot{\zeta} d\xi dx
+ \int_0^t \int_{(T^1)^2} \int_{\mathbb{R}^2} f_1(s, x, \xi) \partial_x \alpha m_1^2(x, \xi, s) d\xi dx
\]

\[=: \langle\langle f_{1,0} \bar{f}_2^+, \alpha \rangle\rangle + J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7 + J_8, \quad a.s..
\]
Similarly, we get
\[
\langle \langle \tilde{f}_1^+(t) f_2^+(t), \alpha \rangle \rangle \\
= \langle \langle \tilde{f}_{1,0} f_{2,0}, \alpha \rangle \rangle + \sqrt{\varepsilon} \lambda(\varepsilon) \int_0^T \int_{\mathbb{T}^1} \int_{\mathbb{R}^2} \tilde{f}_1 f_2 \xi \partial_x \alpha d\xi d\zeta dxdyds \\
+ \int_0^T \int_{\mathbb{T}^1} \int_{\mathbb{R}^2} \tilde{f}_1 f_2 (\partial_x \alpha + \partial_\zeta \alpha) d\xi d\zeta dxdyds \\
- \frac{\lambda(\varepsilon)^{-1}}{2} \int_0^T \int_{\mathbb{T}^1} \int_{\mathbb{R}^2} \partial_x \alpha f_2(s, y, \zeta) G^2(x, 1 + \sqrt{\varepsilon} \lambda(\varepsilon) \xi) d\xi d\zeta dxdyds \\
- \sum_{k \geq 1} \int_0^T \int_{\mathbb{T}^1} \int_{\mathbb{R}^2} f_2(s, y, \zeta) \alpha g_k(x, 1) h^{\varepsilon,k}(s) d\xi d\zeta d\nu_{x,s}(\xi) dxdyds \\
+ \sum_{k \geq 1} \int_0^T \int_{\mathbb{T}^1} \int_{\mathbb{R}^2} \tilde{f}_1(s, x, \xi) \alpha g_k(y, 1) h^{\varepsilon,k}(s) d\xi d\zeta d\nu_{x,s}(\xi) dxdyds \\
+ \int_0^T \int_{\mathbb{T}^1} \int_{\mathbb{R}^2} f_2^-(s, y, \zeta) \partial_x \alpha d \tilde{m}_1(x, \xi, \zeta) d\xi dy \\
- \int_0^T \int_{\mathbb{T}^1} \int_{\mathbb{R}^2} \tilde{f}_1^-(s, x, \xi) \partial_\zeta \alpha d \tilde{m}_2(x, \xi, \zeta) d\xi dx \\
= \langle \langle \tilde{f}_{1,0} f_{2,0}, \alpha \rangle \rangle + \tilde{J}_1 + \tilde{J}_2 + \tilde{J}_3 + \tilde{J}_4 + \tilde{J}_5 + \tilde{J}_6 + \tilde{J}_7 + \tilde{J}_8, \ a.s.
\]

Taking \( \alpha(x, y, \xi, \zeta) = \rho_\gamma(x - y) \psi_\delta(\xi - \zeta) \), where \( \rho_\gamma \) and \( \psi_\delta \) are approximations to the identity on \( \mathbb{T}^1 \) and \( \mathbb{R} \), respectively. Clearly, we have
\[
\partial_x \alpha + \partial_\zeta \alpha = 0, \quad \partial_\xi \alpha + \partial_\zeta \alpha = 0. \tag{4.13}
\]

Then, it follows that \( J_2 = \tilde{J}_2 = 0 \). Utilizing the same method as the proof of Theorem 15 in [13], it follows that
\[
\sup_{t \in [0, T]} (J_6(t) + J_7(t)) \leq 0, \quad a.s., \quad \sup_{t \in [0, T]} (\tilde{J}_6(t) + \tilde{J}_7(t)) \leq 0, \quad a.s.
\]

Hence, we get
\[
\int_{\mathbb{T}^1} \int_{\mathbb{R}^2} \rho_\gamma(x - y) \psi_\delta(\xi - \zeta) (f_1^+(x, t, \xi) \tilde{f}_2(y, t, \zeta) + \tilde{f}_1^+(x, t, \xi) f_2^+(y, t, \zeta)) d\xi d\zeta dxdy \\
\leq \int_{\mathbb{T}^1} \int_{\mathbb{R}^2} \rho_\gamma(x - y) \psi_\delta(\xi - \zeta) (f_{1,0}(x, \xi) \tilde{f}_2(y, \zeta) + \tilde{f}_{1,0}(x, \xi) f_{2,0}(y, \zeta)) d\xi d\zeta dxdy \\
+ J_1(t) + \tilde{J}_1(t) + J_3(t) + \tilde{J}_3(t) + J_4(t) + \tilde{J}_4(t) + J_5(t) + \tilde{J}_5(t) + J_6(t) + \tilde{J}_6(t), \ a.s.
\]

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Applying the same method as the proof of Theorem 15 in [13], it gives
\[ E \sup_{t \in [0, T]} |J_1(t)| \leq TC_p \sqrt{E} \Lambda(\varepsilon) \gamma^{-1}, \quad E \sup_{t \in [0, T]} |\tilde{J}_1(t)| \leq TC_p \sqrt{E} \Lambda(\varepsilon) \gamma^{-1}. \]

By (4.13) and using (2.1), we have
\[
\tilde{J}_3 = J_3 \\
= \frac{\lambda(\varepsilon)^{-2}}{2} \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{R}^2} \alpha G^2(x, 1 + \sqrt{E} \Lambda(\varepsilon) \xi) \nu_{x,s}^{1,\varepsilon} \otimes \nu_{y,s}^{2,\varepsilon}(\xi, \zeta) dxdyds \\
\leq \frac{\lambda(\varepsilon)^{-2}}{2} D_0 \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{R}^2} \alpha (1 + \varepsilon \Lambda(\varepsilon)^2 |\xi|^2) \nu_{x,s}^{1,\varepsilon} \otimes \nu_{y,s}^{2,\varepsilon}(\xi, \zeta) dxdyds \\
\leq \frac{\lambda(\varepsilon)^{-2}}{2} D_0 \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{R}^2} \alpha |\xi|^2 \nu_{x,s}^{1,\varepsilon} \otimes \nu_{y,s}^{2,\varepsilon}(\xi, \zeta) dxdyds \\
+ \frac{\varepsilon}{2} D_0 \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{R}^2} \alpha |\xi|^2 \nu_{x,s}^{1,\varepsilon} \otimes \nu_{y,s}^{2,\varepsilon}(\xi, \zeta) dxdyds.
\]

Clearly, it holds that
\[
E \int_{\mathbb{T}^d} \int_{\mathbb{R}^2} \alpha |\xi|^2 \nu_{x,s}^{1,\varepsilon} \otimes \nu_{y,s}^{2,\varepsilon}(\xi, \zeta) dxdy \\
\leq E \|\psi_\delta\|_{L^\infty} \int_{\mathbb{T}^d} \int_{\mathbb{R}^2} \rho_\gamma(x - y) \nu_{x,s}^{1,\varepsilon} \otimes \nu_{y,s}^{2,\varepsilon}(\xi, \zeta) dxdy \\
\leq \|\psi_\delta\|_{L^\infty} \int_{\mathbb{T}^d} \rho_\gamma(x - y) dxdy \\
\leq \delta^{-1}. \tag{4.14}
\]

Moreover, by utilizing the property that measures \(\nu_{x,s}^{1,\varepsilon}\) and \(\nu_{y,s}^{2,\varepsilon}\) vanish at infinity, it follows that
\[
E \int_{\mathbb{T}^d} \int_{\mathbb{R}^2} \alpha |\xi|^2 \nu_{x,s}^{1,\varepsilon} \otimes \nu_{y,s}^{2,\varepsilon}(\xi, \zeta) dxdy \\
\leq E \int_{\mathbb{T}^d} \rho_\gamma(x - y) \int_{\mathbb{R}^2} \psi_\delta(\xi - \zeta) |\xi|^2 \nu_{x,s}^{1,\varepsilon} \otimes \nu_{y,s}^{2,\varepsilon}(\xi, \zeta) dxdy \\
\leq E \|\psi_\delta\|_{L^\infty} \int_{\mathbb{T}^d} \rho_\gamma(x - y) \int_{\mathbb{R}^2} |\xi|^2 \nu_{x,s}^{1,\varepsilon} \otimes \nu_{y,s}^{2,\varepsilon}(\xi, \zeta) dxdy \\
\leq C \delta^{-1} \int_{\mathbb{T}^d} \rho_\gamma(x - y) dxdy \\
\leq C \delta^{-1}. \tag{4.15}
\]

Hence, combining (4.14) and (4.15), we deduce that
\[
E \sup_{t \in [0, T]} J_3(t) = E \sup_{t \in [0, T]} \tilde{J}_3(t) \leq \frac{\lambda(\varepsilon)^{-2}}{2} D_0 T \delta^{-1} + \frac{\varepsilon}{2} C D_0 T \delta^{-1}.
\]
Define

\[ \Gamma(\zeta, \xi) := \int_{\zeta}^{\xi} \psi_\delta(\xi - \zeta^*) d\zeta^* = \int_{-\infty}^{\xi} \psi_\delta(\xi^* - \zeta) d\xi^*. \tag{4.16} \]

Using similar arguments as in the proof of Theorem 5.1 in [16], we have

\[
\begin{align*}
\tilde{J}_4(t) + \tilde{J}_5(t) & = J_4(t) + J_5(t) \\
& = \sum_{k \geq 1} \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \Gamma(\zeta, \xi) \rho_y(x-y) \left( g_k(x, 1 + \sqrt{e^\lambda(t)} \xi) - g_k(y, 1) \right) h^{e,k}(s) dV_{x,\delta}^1 \otimes V_{y,\delta}^2(\xi, \xi) dx dy ds \\
& \leq \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \Gamma(\zeta, \xi) \rho_y(x-y) \left( \sum_{k \geq 1} |g_k(x, 1 + \sqrt{e^\lambda(t)} \xi) - g_k(y, 1)|^2 \right)^{1/2} \left( \sum_{k \geq 1} |h^{e,k}(s)|^2 \right)^{1/2} dV_{x,\delta}^1 \otimes V_{y,\delta}^2(\xi, \xi) dx dy ds \\
& \leq \sqrt{D_1} \int_0^t |h^e(s)|_{L^2(\mathbb{T}^1)} \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \Gamma(\zeta, \xi) \rho_y(x-y) |x-y| dV_{x,\delta}^1 \otimes V_{y,\delta}^2(\xi, \xi) dx dy ds \\
& \quad + \sqrt{e^\lambda(t)} \sqrt{D_1} \int_0^t |h^e(s)|_{L^2(\mathbb{T}^1)} \int_{\mathbb{T}^2} \gamma_2(\zeta, \xi) dV_{x,\delta}^1 \otimes V_{y,\delta}^2(\xi, \xi) dx dy ds \\
& =: J_{4,1}(t) + J_{5,1}(t).
\end{align*}
\]

Due to

\[
\int_{\mathbb{T}^2} \rho_y(x-y) |x-y| dx dy \leq \gamma,
\]

\[
E \int_{\mathbb{T}^2} \Gamma(\zeta, \xi) dV_{x,\delta}^1 \otimes V_{y,\delta}^2(\xi, \xi) \leq 1,
\]

it follows that

\[
E \sup_{t \in [0, T]} J_{4,1}(t) \leq \sqrt{D_1} \gamma T^{1/2} N^{1/2}.
\]

For the term \( J_{5,1}(t) \), we have

\[
\begin{align*}
J_{5,1}(t) & \leq \sqrt{\sqrt{e^\lambda(t)} \int_0^t |h^e(s)|_{L^2(\mathbb{T}^1)} \int_{\mathbb{T}^2} \rho_y(x-y) |\tilde{X}^e(x, s)| dx dy ds \\
& \leq \sqrt{\sqrt{e^\lambda(t)} \int_0^t |h^e(s)|_{L^2(\mathbb{T}^1)} ||\tilde{X}^e(s)||_{L^1(\mathbb{T}^1)} ds \\
& \leq \sqrt{\sqrt{e^\lambda(t)} \int_0^T T^{1/2} N^{1/2} \sup_{0 \leq t \leq T} ||\tilde{X}^e(t)||_{L^1(\mathbb{T}^1)}.}
\end{align*}
\]

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For the error term, we use the same symbol as (3.12). Then, it gives

\[
\begin{align*}
\int_{T^1} \int_{\mathbb{R}} & \left( f_1^+(x, t, \xi) \tilde{f}_2^+(x, t, \xi) + f_1^+(x, t, \xi) f_2^+(x, t, \xi) \right) d\xi dx \\
\leq & \int_{T^1} \int_{\mathbb{R}} \left( f_{1,0}(x, \xi) \tilde{f}_{2,0}(x, \xi) + f_{1,0}(x, \xi) f_{2,0}(x, \xi) \right) dxd\xi + |E_0(\gamma, \delta)| + |E_1(\gamma, \delta)| + |J_1(t)| + |\bar{J}_1(t)| \\
& + J_3(t) + \bar{J}_3(t) + J_{4,1}(t) + \sqrt{\varepsilon \lambda(\varepsilon)} \sqrt{D_1 T^1 N_1 T^{1/2} \text{ess sup}} \| \tilde{X}_\varepsilon(t) \|_{L^1(T^1)} + J_8(t) + \bar{J}_8(t) \\
=: & \int_{T^1} \int_{\mathbb{R}} \left( f_{1,0}(x, \xi) \tilde{f}_{2,0}(x, \xi) + f_{1,0}(x, \xi) f_{2,0}(x, \xi) \right) dxd\xi \\
& + J_8(t) + \bar{J}_8(t) + r(\varepsilon, \gamma, \delta, t), \quad \text{a.s.,} \quad (4.17)
\end{align*}
\]

where the remainder is given by

\[
r(\varepsilon, \gamma, \delta, t) = |J_1(t)| + |\bar{J}_1(t)| + J_3(t) + \bar{J}_3(t) + J_{4,1}(t) \\
& + \sqrt{\varepsilon \lambda(\varepsilon)} \sqrt{D_1 T^1 N_1 T^{1/2} \text{ess sup}} \| \tilde{X}_\varepsilon(t) \|_{L^1(T^1)} + |E_0(\gamma, \delta)| + |E_1(\gamma, \delta)|.
\]

Applying Burkholder-Davis-Gundy inequality, utilizing (4.16) and (2.1), we deduce that

\[
\begin{align*}
E \sup_{t \in [0, T]} |J_8(t)| \\
\leq & \sqrt{\varepsilon \lambda(\varepsilon)} E \sup_{t \in [0, T]} \left| \sum_{k \geq 1} \int_{0}^{t} \int_{(T^1)^2} \int_{\mathbb{R}^2} \tilde{f}_2(s, y, \zeta) \rho_\varepsilon (x-y) g_k(x, 1 + \sqrt{\varepsilon \lambda(\varepsilon)} c_\varepsilon d\nu_{x,s}^{1,\varepsilon}(\xi) dxdy \beta_k(s) \right| \\
= & \sqrt{\varepsilon \lambda(\varepsilon)} E \sup_{t \in [0, T]} \left| \sum_{k \geq 1} \int_{0}^{t} \int_{(T^1)^2} \int_{\mathbb{R}^2} \tilde{f}_2(s, y, \zeta) \partial_\xi T(\xi, \zeta) \rho_\varepsilon (x-y) g_k(x, 1 + \sqrt{\varepsilon \lambda(\varepsilon)} d\zeta d\nu_{x,s}^{1,\varepsilon}(\xi) dxdy \beta_k(s) \right| \\
= & \sqrt{\varepsilon \lambda(\varepsilon)} E \sup_{t \in [0, T]} \left| \sum_{k \geq 1} \int_{0}^{t} \int_{(T^1)^2} \int_{\mathbb{R}^2} T(\xi, \zeta) \rho_\varepsilon (x-y) g_k(x, 1 + \sqrt{\varepsilon \lambda(\varepsilon)} d\nu_{x,s}^{1,\varepsilon}(\xi, \zeta) dxdy \lambda \beta_k(s) \right| \\
\leq & \sqrt{\varepsilon \lambda(\varepsilon)} \sqrt{D_0} E \left[ \int_{0}^{T} \int_{(T^1)^2} \int_{\mathbb{R}^2} T^2(\xi, \zeta) \rho_\varepsilon^2 (x-y) \left( \sum_{k \geq 1} g_k^2(x, 1 + \sqrt{\varepsilon \lambda(\varepsilon)} c_\varepsilon d\nu_{x,s}^{1,\varepsilon}(\xi, \zeta) dxdy \right)^2 \right]^{1/2} \\
\leq & \sqrt{\varepsilon \lambda(\varepsilon)} \sqrt{D_0} E \left[ \int_{0}^{T} \int_{(T^1)^2} \int_{\mathbb{R}^2} T^2(\xi, \zeta) \rho_\varepsilon^2 (x-y) \left( 1 + \varepsilon \lambda(\varepsilon) |\tilde{\xi}|^2 \right) d\nu_{x,s}^{1,\varepsilon}(\xi, \zeta) dxdy \right]^{1/2} \\
\leq & \sqrt{\varepsilon \lambda(\varepsilon)} \sqrt{D_0} \left[ E \int_{0}^{T} \int_{(T^1)^2} \int_{\mathbb{R}^2} T^2(\xi, \zeta) \rho_\varepsilon^2 (x-y) \left( 1 + \varepsilon \lambda(\varepsilon) |\tilde{\xi}|^2 \right) d\nu_{x,s}^{1,\varepsilon}(\xi, \zeta) dxdy \right]^{1/2} \\
\leq & \sqrt{\varepsilon \lambda(\varepsilon)} \sqrt{D_0} \gamma^{-1} \left[ E \int_{0}^{T} \int_{(T^1)^2} \int_{\mathbb{R}^2} T^2(\xi, \zeta) (1 + \varepsilon \lambda(\varepsilon) |\tilde{\xi}|^2) d\nu_{x,s}^{1,\varepsilon}(\xi, \zeta) dxdy \right]^{1/2}.
\end{align*}
\]
Taking into account the following fact

\[
E \int_0^T \int_{(T')^2} \int_{R^2} f^2(\xi, \zeta)(1 + \varepsilon \lambda^2(\varepsilon)|\xi|^2)dv_{x,s}^{1,\varepsilon} \otimes v_{y,s}^{2,\varepsilon}(\xi, \zeta)dxdyds
\]

\[
\leq E \int_0^T \int_{(T')^2} \int_{R^2} (1 + \varepsilon \lambda^2(\varepsilon)|\xi|^2)dv_{x,s}^{1,\varepsilon} \otimes v_{y,s}^{2,\varepsilon}(\xi, \zeta)dxdyds
\]

\[
\leq T + \varepsilon \lambda^2(\varepsilon) Te \text{ess sup } \|\bar{X}(t)\|^2_{L^2(T')}.
\]

we further deduce that

\[
E \sup_{t \in [0, T]} |J_8(t)| \leq \sqrt{\varepsilon \lambda(\varepsilon)} \sqrt{D_0} \gamma^{-1} \left[ T + \varepsilon \lambda^2(\varepsilon) Te \text{ess sup } \|\bar{X}(t)\|^2_{L^2(T')} \right]^\frac{1}{2}.
\]

By the same method as above, it gives

\[
E \sup_{t \in [0, T]} |\tilde{J}_8(t)| \leq \sqrt{\varepsilon \lambda(\varepsilon)} \sqrt{D_0} \gamma^{-1} \left[ T + \varepsilon \lambda^2(\varepsilon) Te \text{ess sup } \|\bar{X}(t)\|^2_{L^2(T')} \right]^\frac{1}{2}.
\]

Moreover, for the remainder \(r(\varepsilon, \gamma, \delta, t)\), we get

\[
\text{ess sup } r(\varepsilon, \gamma, \delta, t) \leq TC \rho \sqrt{\varepsilon \lambda(\varepsilon)} \gamma^{-1} + \frac{A(\varepsilon)^2}{2}D_0 T \delta^{-1} + \frac{\varepsilon}{2}CD_0 T \delta^{-1} + \sqrt{D_1 T^2 N^2} \text{ess sup } \|\bar{X}(t)\|_{L^1(T')} + \text{ess sup } \|E_r(\gamma, \delta)\|_{0 \leq t \leq T}.
\]

In the following, we aim to make estimates of the error term \(E \text{ess sup } |E_r(\gamma, \delta)|\) by utilizing a similar method as the proof of Proposition 6.1 and Theorem 6.2 in [12].

For any \(t \in [0, T]\), we have

\[
E_r(\gamma, \delta) = \int_{T'} \int_{R^1} (f^1_1(x, t, \xi) \tilde{f}^1_2(x, t, \xi) + \tilde{f}^1_1(x, t, \xi) f^1_2(x, t, \xi))d\xi dx
\]

\[
- \int_{(T')^2} \int_{R^2} (f^1_1(x, t, \xi) \tilde{f}^1_2(y, t, \xi) + \tilde{f}^1_1(x, t, \xi) f^1_2(y, t, \xi))\rho_{y}(x - y)\psi_{\delta}(\xi - \zeta)dxdydx\xi\zeta
\]

\[
= \left[ \int_{T'} \int_{R^1} (f^1_1(x, t, \xi) \tilde{f}^1_2(x, t, \xi) + \tilde{f}^1_1(x, t, \xi) f^1_2(x, t, \xi))d\xi dx
\]

\[
- \int_{(T')^2} \int_{R^2} \rho_{y}(x - y)(f^1_1(x, t, \xi) \tilde{f}^1_2(y, t, \xi) + \tilde{f}^1_1(x, t, \xi) f^1_2(y, t, \xi))d\xi dy dx dy \right]
\]

\[
+ \left[ \int_{(T')^2} \int_{R^2} \rho_{y}(x - y)(f^1_1(x, t, \xi) \tilde{f}^1_2(y, t, \xi) + \tilde{f}^1_1(x, t, \xi) f^1_2(y, t, \xi))d\xi dy dy \right]
\]

\[
- \int_{(T')^2} \int_{R^2} (f^1_1(x, t, \xi) \tilde{f}^1_2(y, t, \xi) + \tilde{f}^1_1(x, t, \xi) f^1_2(y, t, \xi))\rho_{y}(x - y)\psi_{\delta}(\xi - \zeta)dxdydx\xi\zeta
\]

\[
= H_1 + H_2.
\]
Applying the same method as (3.13) and (3.14), it follows that
\[
|H_2(t)| \leq 2\delta, \quad a.s..
\]
Moreover, it is easy to deduce that
\[
|H_1(t)| \leq \left| \int_{(T)} \rho_\gamma(x-y) \left( I_{\{x>(x,t)>\xi\}}(I_{Y^{x,\xi}(x,t)\leq\xi} - I_{Y^{x,\xi}(x,t)\leq\xi}) \right) d\xi dy \right|
\]
\[
+ \left| \int_{(T)} \rho_\gamma(x-y) \left( I_{\{x>(x,t)\leq\xi\}}(I_{Y^{x,\xi}(x,t)>\xi} - I_{Y^{x,\xi}(x,t)>\xi}) \right) d\xi dy \right|
\]
\[
\leq 2 \int_{(T)} \rho_\gamma(x-y) |Y^{x,\xi}(x,t) - Y^{x,\xi}(y,t)| d\xi dy.
\]
By (3.17), we get
\[
E_{\text{ess sup}} \int_{0 \leq t \leq T} \rho_\gamma(x-y) |Y^{x,\xi}(x,t) - Y^{x,\xi}(y,t)| d\xi dy
\]
\[
= E_{\text{ess sup}} \int_{0 \leq t \leq T} \int_{\mathbb{R}^2} \rho_\gamma(x-y) \left( f_1^2(x,t,\xi)f_2^2(y,t,\xi) + f_2^2(x,t,\xi)f_2^2(y,t,\xi) \right) d\xi dy
\]
\[
\leq E_{\text{ess sup}} \int_{0 \leq t \leq T} \int_{\mathbb{R}^2} \rho_\gamma(x-y) \psi_\delta(\xi - \xi)(f_1^2(x,t,\xi)f_2^2(y,t,\xi) + f_2^2(x,t,\xi)f_2^2(y,t,\xi)) d\xi d\xi dy + 2\delta
\]
\[
\leq |E_{0}(\gamma, \delta)| + D_0^{\frac{1}{2}}\gamma T^2 N^\frac{1}{2} + 2\delta,
\]
where \(|E_{0}(\gamma, \delta)| \to 0\), when \(\gamma, \delta \to 0\). Hence, we deduce that
\[
E_{\text{ess sup}} |E_{\xi}(\gamma, \delta)| \leq |E_{0}(\gamma, \delta)| + D_0^{\frac{1}{2}}\gamma T^2 N^\frac{1}{2} + 4\delta. \quad (4.19)
\]
With the aid of (4.19), we deduce from (4.18) that
\[
E_{\text{ess sup}} r(\varepsilon, \gamma, \delta, t)
\]
\[
\leq TC_p \sqrt{E_{\varepsilon}(\gamma)^{-1}} + \frac{\lambda(\varepsilon)^{-2}}{2}D_0 T\delta^{-1} + \frac{\varepsilon}{2}CD_0 T\delta^{-1} + \sqrt{D_1 T^\frac{1}{2} N^\frac{1}{2}} E_{\text{ess sup}} \|\tilde{X}^\xi(t)\|_{L^1(\mathbb{T}^1)} + |E_{0}(\gamma, \delta)| + D_0^{\frac{1}{2}}\gamma T^2 N^\frac{1}{2} + 4\delta. \quad (4.20)
\]
Notice that \(f_1 = I_{\tilde{X}^{x,\xi}}\) and \(f_2 = I_{Y^{x,\xi}}\) with initial data \(f_{1,0} = I_{\tilde{X}^{0,\xi}} = I_{0>\xi}\) and \(f_{2,0} = I_{Y^{0,\xi}} = I_{0>\xi}\), respectively. With the help of identity (3.17), we deduce from (4.17) and (4.20) that
\[
E_{\text{ess sup}} \|\tilde{X}^\xi(t) - Y^{x,\xi}(t)\|_{L^1(\mathbb{T}^1)}
\]
\[
\leq 2 \sqrt{E_{\varepsilon}(\gamma) T} + E_{\varepsilon}(\gamma) T E_{\text{ess sup}} \|\tilde{X}^\xi(t)\|_{L^1(\mathbb{T}^1)}^2 \right)^{\frac{1}{2}} + TC_p \sqrt{E_{\varepsilon}(\gamma)^{-1}}
\]
\[
+ \frac{\lambda(\varepsilon)^{-2}}{2}D_0 T\delta^{-1} + \frac{\varepsilon}{2}CD_0 T\delta^{-1} + \sqrt{D_1 T^\frac{1}{2} N^\frac{1}{2}} E_{\text{ess sup}} \|\tilde{X}^\xi(t)\|_{L^1(\mathbb{T}^1)} + |E_{0}(\gamma, \delta)| + D_0^{\frac{1}{2}}\gamma T^2 N^\frac{1}{2} + 4\delta.
\]
For any \( \iota > 0 \), one can choose \( \gamma, \delta > 0 \) small enough such that
\[
\sqrt{D_1} \gamma T^{1/2} N^{1/2} + |E_0(\gamma, \delta)| + D_0^{1/2} \gamma T^{1/2} N^{1/2} + 4\delta < \frac{\iota}{2}.
\]
Since \( \sqrt{\varepsilon \lambda(\varepsilon)} \to 0 \), by (4.12), for fixed \( \gamma, \delta > 0 \), we can find \( \varepsilon > 0 \) small enough such that
\[
2 \sqrt{\epsilon \lambda(\epsilon)} \sqrt{D_0} \gamma^{-1} \left[ T + \epsilon \lambda^2(\epsilon) T \operatorname{ess sup}_{0 \leq t \leq T} \left\| \bar{X}_\epsilon(t) \right\|_{L^2(\mathbb{T}^1)} \right]^2 + TC_\rho \sqrt{\epsilon \lambda(\epsilon)} \gamma^{-1}
\]
\[
+ \frac{C^2 D_0 T \delta^{-1}}{2} + \frac{\epsilon}{2} C D_0 T \delta^{-1} + \sqrt{\epsilon \lambda(\epsilon)} \sqrt{D_1} T^{1/2} N^{1/2} \operatorname{ess sup}_{0 \leq t \leq T} \left\| \bar{X}_\epsilon(t) \right\|_{L^1(\mathbb{T}^1)} < \frac{\iota}{2}.
\]
Thus, we reach
\[
\lim_{\varepsilon \to 0} \operatorname{ess sup}_{0 \leq t \leq T} \left\| \bar{X}_\epsilon(t) - Y_\epsilon(t) \right\|_{L^1(\mathbb{T}^1)} = 0.
\]
As a result, it gives
\[
E \left\| \bar{X}_\epsilon - Y_\epsilon \right\|_{L^1(\mathbb{T}^1)}^{L^1([0,T];L^1(\mathbb{T}^1))} \leq T \cdot \operatorname{ess sup}_{0 \leq t \leq T} \left\| \bar{X}_\epsilon(t) - Y_\epsilon(t) \right\|_{L^1(\mathbb{T}^1)} \to 0,
\]
which implies that \( \left\| \bar{X}_\epsilon - Y_\epsilon \right\|_{L^1([0,T];L^1(\mathbb{T}^1))} \to 0 \) in probability, as \( \epsilon \to 0 \).

\[ \square \]

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