Spin-correlation functions in ultracold paired atomic-fermion systems: sum rules, self-consistent approximations, and mean fields

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The spin response functions measured in multi-component fermion gases by means of rf transitions between hyperfine states are strongly constrained by the symmetry of the interatomic interactions. Such constraints are reflected in the spin f-sum rule that the response functions must obey. In particular, only if the effective interactions are not fully invariant in SU(2) spin space, are the response functions sensitive to mean field and pairing effects. We demonstrate, via a self-consistent calculation of the spin-spin correlation function within the framework of Hartree-Fock-BCS theory, how one can derive a correlation function explicitly obeying the f-sum rule. By contrast, simple one-loop approximations to the spin response functions do not satisfy the sum rule, except in special cases. As we show, the emergence of a second peak at higher frequency in the rf spectrum, as observed in a recent experiment in trapped $^6$Li, can be understood as the contribution from the paired fermions, with a shift of the peak from the normal particle response proportional to the square of the BCS pairing gap.

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I. INTRODUCTION

Exploring pairing and superfluidity in ultracold trapped multicomponent-fermion systems poses considerable experimental and theoretical challenges [1, 2, 3, 4, 5]. Recently, Chin et al. [6] have found evidence, by rf excitation, of a pairing gap in a two-component trapped $^6$Li gas over a range of coupling strengths. The experiment, concentrating on the lowest lying hyperfine states, states $|\sigma\rangle = |1\rangle$, $|2\rangle$ and $|3\rangle$, with $m_F = 1/2$, $-1/2$, and $-3/2$ respectively, measures the long wavelength spin-spin correlation function, and is analogous to NMR experiments in superfluid $^3$He [7, 8]. While at high temperatures the rf field absorption spectrum shows a single peak from unpaired atoms, at sufficiently low temperature a second higher frequency peak emerges, attributed to the contribution from BCS paired atoms. Theoretical calculations at the “one-loop” level of the spin response [9, 10, 11] support this interpretation.

In this paper we carry out a fully self-consistent calculation of the spin-spin correlation function relevant to the rf experiment, at the Hartree-Fock-BCS level, in order to understand the dependence of the response on mean field shifts and the pairing gap. The calculation requires going beyond the one-loop level, and summing bubbles to all orders, and is valid in the weakly interacting BCS regime, away from the BEC-BCS crossover – the unitarity limit. An important constraint on the mean field shifts was brought out by Leggett [12] via a sum-rule argument: For a system with an interaction that is SU(2)-invariant in spin space, the spins in the long-wavelength limit simply precess as a whole at the Larmor frequency, without mean field effects; then the spin-spin correlation function is dominated by a single pole at the Larmor frequency. While the effective interactions between the three lowest hyperfine states of $^6$Li are not SU(2)-invariant the f-sum rule obeyed by the spin-spin correlation function still, as we shall show, implies strong constraints on the spin response, which are taken into account via a self-consistent calculation.

In order to bring out the physics of a self-consistent approach to the spin response, we consider a spatially uniform system, and work within the framework of simple BCS theory on the “BCS side” of the Feshbach resonance where the interactions between hyperfine states are attractive. We assume an effective Hamiltonian in terms of the three lowest hyperfine states explicitly involved in the experiments [6, 13] (we take $\hbar = 1$ throughout):

$$H = \int d\hat{r}\left\{ \sum_{\sigma = 1}^{3} \left( \frac{1}{2m} \nabla \psi_{\sigma}^\dagger(r) \cdot \nabla \psi_{\sigma}(r) + \epsilon_{\sigma}^2 \psi_{\sigma}^\dagger(r) \psi_{\sigma}(r) \right) + \frac{1}{2} \sum_{\sigma, \sigma' = 1}^{3} \tilde{g}_{\sigma\sigma'} \psi_{\sigma}^\dagger(r) \psi_{\sigma'}^\dagger(r) \psi_{\sigma'}(r) \psi_{\sigma}(r) \right\},$$

(1)

where $\psi_{\sigma}$ is the annihilation operator for state $|\sigma\rangle$, $\tilde{g}_{\sigma\sigma'}$ is the bare coupling constant between states $\sigma$ and $\sigma'$, which we assume to be constant up to a cutoff $\Lambda$ in momentum space. Consistent with the underlying symmetry we assume $\Lambda$ to be the same for all channels, and take $\Lambda \to \infty$ at the end of calculating physical observables. The renormalized coupling constants $g_{\sigma\sigma'}$ are related to those of the bare theory by

$$g_{\sigma\sigma'}^{-1} = \tilde{g}_{\sigma\sigma'}^{-1} + m\Lambda/2\pi^2,$$

(2)

where, in terms of the s-wave scattering length $a_{\sigma\sigma'}$, $g_{\sigma\sigma'} = 4\pi a_{\sigma\sigma'}/m$. In evaluating frequency shifts in normal states, we implicitly resum particle-particle ladders involving the bare couplings and generate the renormalized couplings. However, to treat pairing correlations requires working directly in terms of the bare $\tilde{g}_{\sigma\sigma'}$ [14].

It is useful to regard the three states $|\sigma\rangle$ as belonging to a pseudospin (denoted by $Y$) multiplet with the eigen-
values $m_{\sigma}$ of $Y_z$ equal to 1,0,-1 for $\sigma = 1,2,3$. In terms of $m_{\sigma}$ the Zeeman splitting of the three levels is

$$\epsilon_z^\sigma = \epsilon_z^1 - (\epsilon_z^2 - \epsilon_z^1)m_{\sigma}/2 + (\epsilon_z^2 + \epsilon_z^1 - 2\epsilon_z^0)m_{\sigma}^2/2. \quad (3)$$

The final term in $^6$Li is of order 4% of the middle term on the BCS side. The interatomic interactions in the full Hamiltonian for the six $F = 1/2$ and $F = 3/2$ hyperfine states are invariant under the SU(2) group of spin rotations generated by the total spin angular momentum $\mathbf{S}$. The effective Hamiltonian can be derived from the full Hamiltonian by integrating out the upper three levels. However, because the effective interactions between the lower three levels depend on the non-SU(2) invariant coupling of the upper states to the magnetic field, the interactions in the effective Hamiltonian [11] are no longer SU(2) invariant [12,16].

In the Chin et al. experiment equal numbers of atoms were loaded into states $|1\rangle$ and $|2\rangle$ leaving state $|3\rangle$ initially empty; transitions of atoms from $|2\rangle$ to $|3\rangle$ were subsequently induced by an rf field. Finally the residue atoms in $|2\rangle$ were imaged, thus determining the number of atoms transferred to $|3\rangle$. The experiment (for an rf field applied along the x-direction) basically measures the frequency dependence of the imaginary part of the correlation function $(−i)\int d^3 r \langle \hat{T} (\hat{\psi}_1^\dagger (r,t)\hat{\psi}_3 (r,t)\hat{\psi}_3^\dagger (0,0)\hat{\psi}_2 (0,0)) \rangle$ (although in principle atoms can make transitions from $|2\rangle$ to $|4\rangle$; such a transition, at higher frequency, is beyond the range studied in the experiment, and is not of interest presently). Here $T$ denotes time ordering. This correlation function can be written in terms of long-wavelength pseudospin-pseudospin correlation function, the Fourier transform of

$$\chi_{xx} (t) = −i\langle T (Y_x (t)Y_x (0)) \rangle; \quad (4)$$

where $Y_x = \int d^3 r y_x (r)$ is the x component of the total pseudospin of the system,

$$y_x (r) = \frac{1}{\sqrt{2}} \left[ \psi_1^\dagger (r)\psi_2 (r) + \psi_2^\dagger (r)\psi_1 (r) \right]
\left[ + \psi_2^\dagger (r)\psi_3 (r) + \psi_3^\dagger (r)\psi_2 (r) \right] \quad (5)$$

is the local pseudospin density along the x-axis. Since the experiment is done in a many-body state with $N_1 = N_3$, the contribution from transitions between $|1\rangle$ and $|2\rangle$ is zero [17]. The Fourier transform of $\chi_{xx} (t)$ has the spectral representation,

$$\chi_{xx} (\Omega) = \int_{−\infty}^{\infty} \frac{d \omega}{\pi} \frac{\chi''_{xx} (\omega)}{\Omega − \omega}, \quad (6)$$

where $\chi''_{xx} (\omega) = \text{Im} \chi_{xx} (\omega − i0^+)$. In the next section we discuss the f-sum rule in general, review Leggett’s argument, and illustrate how the sum rule works in simple cases. Then in Section III we carry out a systematic calculation, within Hartree-Fock-BCS theory, of the spin-spin correlation functions, generating them from the single particle Green’s functions.

In addition to fulfilling the f-sum rule, our results are consistent with the emergence of the second absorption peak observed in the rf spectrum at low temperature from pairing of fermions.

## II. SUM RULES

The f-sum rule obeyed by the pseudospin-pseudospin correlation function arises from the identity,

$$\int_{−\infty}^{\infty} \frac{d \omega}{\pi} \chi''_{xx} (\omega) = \langle [Y_x, H], Y_x \rangle. \quad (7)$$

The need for self-consistency is driven by the fact that the commutator on the right side eventually depends on the single particle Green’s function, whereas the left side involves the correlation function. The static pseudospin susceptibility, $−\chi_{xx} (0)$, is related to $\chi''_{xx} (\omega)$ by

$$\chi_{xx} (0) = −\int_{−\infty}^{\infty} \frac{d \omega}{\pi} \frac{\chi''_{xx} (\omega)}{\omega}. \quad (8)$$

Leggett’s argument that an SU(2) invariant system gives an rf signal only at the Larmor frequency is the following: Let us assume that the $g_{\sigma\sigma'}$ are all equal, so that the interaction in Eq. (1) is SU(2) invariant in pseudospin space; in addition, let us assume, for the sake of the argument, that the Zeeman energy is $−\gamma m_{\sigma} B_z$ ($\gamma$ is the gyromagnetic ratio of the pseudospin). Then the right side of Eq. (7) becomes $\gamma B_z Y_x$, while the static susceptibility, $−\chi_{xx} (0)$, equals $\gamma B_z$, the Larmor frequency. The sum rule implies that neither mean field shifts nor pairing effects can enter the long wavelength rf spectrum of an SU(2) invariant system.

It is instructive to see how the sum rule [7] functions in relatively simple cases. We write the space and time dependent spin density correlation function as

$$D_{xx} (10) \equiv −i\langle T (y_x (1)y_x (0)) \rangle = \frac{1}{2} [D_{12} (1) + D_{21} (1) + D_{23} (1) + D_{31} (1)], \quad (9)$$

where

$$D_{\beta\alpha} (1) \equiv −i\langle T \left( \psi_{\beta}^\dagger (1)\psi_{\beta}^\dagger (0)\psi_\alpha (0) \right) \rangle, \quad (10)$$

and $\alpha, \beta = 1, 2, 3$. Here $\psi (1)$, with 1 standing for $\{r_1, t_1\}$, is in the Heisenberg representation, with Hamiltonian $H = H + \sum_{\sigma} \mu_{\sigma} N_{\sigma}$. Equation (9) implies that $\chi''_{xx}$ is a sum of $\chi''_{\beta\alpha}$, where

$$\chi_{\beta\alpha} (\Omega) = \frac{V}{2} D_{\beta\alpha} (\mathbf{q} = 0, \Omega + \mu_\alpha - \mu_\beta), \quad (11)$$

and $V$ is the system volume.
As a first example we consider free particles (denoted by superscript 0). For \( \alpha \neq \beta \),
\[
D_{\beta\alpha}^0(1) = -iG^0_\alpha(-1)G^0_\beta(1),
\]
where \( G^0_\alpha(1) \), the free single particle Green’s function, has Fourier transform, \( G^0_\alpha(k, z) = 1/(z - \epsilon^0_\alpha(k)) \), with \( z \) the Matsubara frequency and \( \epsilon^0_\alpha(k) = k^2/2m + \epsilon^g_Z - \mu_\alpha \). Then,
\[
\chi^{0\beta\alpha}(\Omega) = \frac{1}{2} \frac{N_\alpha - N_\beta}{\Omega + \epsilon^g_Z - \epsilon^g_Z},
\]
(13)
from which we see that \( \chi^{0\beta\alpha}(\omega) \) has a delta function peak at \( \epsilon^g_Z - \epsilon^g_Z \), as expected for free particles. This result is manifestly consistent with Eq. (7).

Next we take interactions into account within the Hartree-Fock approximation (denoted by \( H \)) for the single particle Green’s function, with an implicit resummation of ladders to change bare into renormalized coupling constants. It is tempting to factorize \( D \) as in the free particle case as \( 9, 13, 11, 18, 19 \),
\[
D_{\beta\alpha}^H(1) = -iG^H_\alpha(-1)G^H_\beta(1),
\]
where \( G^H_\alpha(k, z) \), with
\[
\xi^g_\alpha = \frac{k^2}{2m} + \epsilon^g_Z + \sum_{(\beta \neq \alpha)} g_{\alpha\beta} n_\beta - \mu_\alpha
\]
(15)
and \( n_\beta \) the density of particles in hyperfine level \( \beta \). Then
\[
D_{\beta\alpha}^H(0, \Omega) = \frac{n_\alpha - n_\beta}{\Omega + \epsilon^g_Z - \epsilon^g_Z},
\]
(16)
and
\[
\chi_{\beta\alpha}(\Omega) = \frac{N_\alpha - N_\beta}{2 \Omega + \epsilon^g_Z + \sum_{(\beta \neq \alpha)} g_{\alpha\beta} n_\beta - \epsilon^g_Z - \sum_{(\beta \neq \alpha)} g_{\beta\sigma} n_{\sigma}.
\]
(17)
Consequently
\[
\chi''_{\beta\alpha}(\omega) = \frac{-\pi}{2} (N_\beta - N_\alpha) \delta(\omega - \Delta E_{\beta\alpha}),
\]
(18)
where
\[
\Delta E_{\beta\alpha} = \epsilon^g_\beta + \sum_{(\beta \neq \alpha)} g_{\alpha\beta} n_\alpha - \epsilon^g_Z - \sum_{(\beta \neq \alpha)} g_{\beta\sigma} n_\sigma
\]
(19)
is the energy difference of the single particle levels \( |\alpha\rangle \) and \( |\beta\rangle \). The response function \( \chi''_{\beta\alpha}(\omega) \) is non-zero only at \( \omega = \Delta E_{\beta\alpha} \).

On the other hand, \( \chi''_{\beta\alpha}(\omega) \) obeys the sum rule
\[
\int_{-\infty}^{+\infty} \frac{d\omega}{\pi} \chi''_{\beta\alpha}(\omega) = V \int d^3 r \langle [\psi^0_\alpha(r) \psi_\beta(0), H], \psi^0_\beta(r) \psi_\alpha(0) | \rangle
\]
\[
= \frac{1}{2} (N_\alpha - N_\beta) \left( \Delta E_{\beta\alpha} - g_{\alpha\beta}(n_\beta - n_\alpha) \right),
\]
(20)
where the final line holds for the Hartree-Fock approximation. The sum rule (21) is violated in this case unless \( g_{\alpha\beta} = 0 \).

The self-consistent approximation for the correlation function (detailed in the following section) that maintains the sum rule and corresponds to the Hartree approximation for the single particle Green’s function includes a sum over bubbles in terms of the renormalized \( g \):
\[
D_{\beta\alpha}^H(q, \Omega) = \frac{D_{\beta\alpha}^0(q, \Omega)}{1 + g_{\beta\alpha}D_{\beta\alpha}^H(q, \Omega)}. \tag{22}
\]
Then with (22),
\[
\chi_{\beta\alpha}''(\omega) = \frac{\pi}{2} (N_\alpha - N_\beta) \delta(\omega + (\epsilon^g_Z - \epsilon^g_Z)
\]
\[
+ \sum_{\sigma(\neq \alpha)} g_{\alpha\sigma} n_\sigma - \sum_{(\beta \neq \alpha)} g_{\beta\sigma} n_\rho + g_{\alpha\beta}(n_\beta - n_\alpha))\). \tag{23}
\]
Note that \( \chi_{32}''(\omega) \) peaks at \( \omega_H = \epsilon^g_Z - \epsilon^g_Z + (g_{13} - g_{12})n_1 \), indicating that the mean field shift is \( (g_{13} - g_{12})n_1 \).

This result agrees with the rf experiment done in a two level \(^6\)Li system away from the resonance region \( 20 \). This experiment finds that no matter whether the atoms in states \( |1\rangle \) and \( |2\rangle \) are coherent or incoherent, the rf signal of the transition between \( |1\rangle \) and \( |2\rangle \) never shows a mean field shift. As explained in \( 21 \), in a coherent sample, the internal degrees of freedom of all the fermions are the same, and thus there is no interaction between them. In the incoherent case, the above calculation gives \( \chi_{32}''(\omega) = (\pi/2)(N_2 - N_1) \delta(\omega + \epsilon^g_Z - \epsilon^g_Z) \), always peaking at the difference of the Zeeman energy, and therefore without a mean field contribution. \( 21 \)

In an rf experiment using all three lowest hyperfine states, the mean field shifts appear in \( \chi_{32}'' \) as \( (g_{13} - g_{12})n_1 \). Since \( g_{a1\sigma} = 4\pi a_{a\sigma}(k^2/m) \), our result \( (g_{13} - g_{12})n_1 \) agrees with Eq. (1) of Ref. \( 13 \). However, from \( B = 660 \) to 900 G (essentially the region between the magnetic fields at which \( a_{13} \) and \( a_{12} \) diverge) no obvious deviation of the rf signal from the difference of the Zeeman energies is observed in the unpaired state \( 13, 21 \). The frequency shifts estimated from the result \( 23 \) taken literally in this region do not agree with experiment; one should not, however, trust the Hartree-Fock mean field approximation around the unitarity limit. The disappearance of the mean field shifts in the unitary regime has been attributed to the \( s \)-wave scattering process between any two different species of atoms becoming unitary-limited \( 6, 9 \); however, the situation is complicated by the fact that the two two-particle channels do not become unitarity limited simultaneously.

### III. SELF-CONSISTENT APPROXIMATIONS

References \( 22 \) and \( 24 \) laid out a general method to generate correlation functions self-consistently from the single particle Green’s functions. To generate the correlation function \( \chi_{xx}(t) \), defined in Eq. (11), we couple the
and calculate the right side of Eq. (7) directly, we decompose the Hamiltonian as \( H = H_{\text{nuc}} + H_{\text{var}} \), where \( H_{\text{nuc}} \) is invariant under \( SU(2) \) and the remainder

\[
H_{\text{var}} = \epsilon_2^Z + (\epsilon_3^Z + \epsilon_1^Z - 2\epsilon_2^Z) Y_2^Z / 2 - \left( \epsilon_1^Z - \epsilon_2^Z \right) Y_2^Z / 2
\]

\[
+ \left( \bar{g}_{12} - \bar{g}_{13} \right) \int \psi_1^{\dagger} \psi_1 \psi_2 \psi_3 + \left( \bar{g}_{23} - \bar{g}_{12} \right) \int \psi_3^{\dagger} \psi_2 \psi_3 \psi_4.
\]

(25)

is not invariant. We evaluate the right side of Eq. (7), \( \langle |v_{\text{var}}|, x \rangle \rangle \), term by term in the case that the states have particle number \( N_1 = N_2 = N \) and \( N_3 = 0 \). The Zeeman energy in \( H_{\text{var}} \) gives \( (\epsilon_2^Z - \epsilon_2^Z)N \), and the second term gives \( \left( \bar{g}_{12} - \bar{g}_{13} \right) \int \langle \psi_1^{\dagger} \psi_1 \psi_2 \psi_3 \rangle \).

We factorize the correlation function within the Hartree-Fock-BCS theory for the contact pseudopotential in (1), implicitly resumming ladders to renormalize the coupling constant in the direct and exchange terms 27, to write,

\[
\left( \bar{g}_{12} - \bar{g}_{13} \right) \int \langle \psi_1^{\dagger} \psi_1 \psi_2 \psi_3 \rangle
\]

\[
= \left( g_{12} - g_{13} \right) \int \langle \psi_1^{\dagger} \psi_2 \rangle \langle \psi_1^{\dagger} \psi_1 \rangle
\]

\[
+ \left( g_{12} - g_{13} \right) \int \langle \psi_2^{\dagger} \psi_3 \rangle \langle \psi_1^{\dagger} \psi_2 \rangle.
\]

(26)

Using Eq. (2), we find a contribution from the second term, \( V(1g_{23} - 1g_{23}) \), assumed to be real and positive. The last term gives \( (g_{12} - g_{23}) \int \int \langle \psi_1^{\dagger} \psi_3 \psi_4 \psi_2 \psi_3 \psi_4 \rangle \psi_2 \psi_3 \psi_4 \rangle = 0 \); altogether,

\[
\int \frac{d\omega}{\pi} \omega \chi_{xx}(\omega)
\]

\[
= \left( \epsilon_2^Z - \epsilon_2^Z \right) N - V(g_{12} - g_{13}) \left( n_1 n_2 + \Delta^2 / g_{12} g_{13} \right). \)

(27)

The absence of \( g_{23} \) arises from \( N_3 = 0 \). We are not \( SU(2) \) invariant both mean field shifts and the pairing gap contribute to the sum rule, allowing the possibility of detecting pairing via the rf absorption spectrum.

To turn now to calculating the full Hartree-Fock-BCS pseudospin-pseudospin correlation function. For convenience we define the spinor operator

\[
\Psi = \left( \psi_1, \psi_2, \psi_3, \psi_1^{\dagger}, \psi_2^{\dagger}, \psi_3^{\dagger} \right),
\]

(28)

and calculate the single particle Green’s function

\[
G_{ab}(1, 1') \equiv (-i) \langle T \Psi_a(1) \Psi_b^{\dagger}(1') \rangle,
\]

(29)

where \( a \) denotes \( r_1, t_1 \), etc., and the subscripts \( a \) and \( b \) run from 1 to 6 (in the order from left to right in Eq. (28); the subscripts 4, 5, and 6 should not be confused with the label for the upper three hyperfine states), and \( \Psi_a(1) \) is in the Heisenberg representation with Hamiltonian \( H'' = H' + H_{\text{probe}}(t) \). For \( F(r, t) = 0 \) and with BCS pairing between \( |1\rangle \) and \( |2\rangle \),

\[
G = \begin{pmatrix}
G_{11} & 0 & 0 & 0 & G_{15} & 0 \\
0 & G_{22} & 0 & G_{24} & 0 & 0 \\
0 & 0 & G_{33} & 0 & 0 & 0 \\
0 & G_{42} & G_{44} & 0 & 0 & 0 \\
G_{51} & 0 & 0 & G_{55} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & G_{66}
\end{pmatrix}. \quad (30)
\]

To obtain a closed equation for \( G_{ab}(1, 1') \), we factorize the four-point correlation functions in the equation of motion for \( G \) as before, treating the Hartree-Fock (normal propagator) and BCS (abnormal propagator) parts differently. In the dynamical equation for \( G_{11}(1, 2) \), the term \( \bar{g}_{12}(\psi_3^2(1) (\psi_2(1) \psi_1(1) (1)^3)) \) is approximated by \( g_{12}(\psi_2^2(1) (\psi_2(1) (\psi_3(1))) \) for the normal part, but \( \bar{g}_{12}(\psi_2(1) \psi_1(1) (1)^3)) \) for the abnormal part 14. Since \( n_1 = n_2, \epsilon_2^Z + g_{12} n_1 + g_{13} n_3 - \mu_2 = \epsilon_2^Z + g_{12} n_1 + g_{13} n_3 - \mu_2 \equiv -\mu_0 \), where \( \mu_0 \) is the free particle Fermi energy; \( \mu_0 \) enters into the single particle Green’s function as usual via the dispersion relation \( E_k \equiv [(\kappa^2 / 2m - \mu_0)^2 + \Delta^2]^{1/2} \) for the paired states.

The equation of the single particle Green’s function in matrix form is

\[
\int d\Omega \left( G_0^{-1} (\Omega) - F(1) \right) \sigma(1 - \Omega)
\]

\[-\Sigma(1) G(\Omega') = \delta(1 - \Omega'), \]

(31)

where the inverse of the free single-particle Green’s function is

\[
G_{\text{free}}^{-1}(1') = \left( \frac{\delta}{\Omega_1} + \frac{\sqrt{\Omega}\pm \mu_a}{2m} \right) \delta(1 - \Omega') \delta_{ab}, \quad (32)
\]

with the upper sign for \( a = 1, 2, 3 \), and the lower for
\[ a=4,5,6. \] The matrix \( \tau \) is

\[
\tau = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -1
\end{pmatrix}
\]

(33)

where here \( G_{ab} \) denotes \( G_{ab}(1^+)\delta(1 - 1^+) \) with \( 1^+ = \{ r_1, t_1 + 0^+ \} \).

We generate the correlation functions as

\[
D_{ab}(12) = -i\sqrt{2} \frac{\delta G_{ab}(1^+)}{\delta F(2)} \bigg|_{F=0};
\]

(35)

(where the factor \( \sqrt{2} \) cancels that from the coupling of \( F(r, t) \) to the atoms via \( y_x \) so that from Eq. (31),

\[
D(q, \Omega) = \frac{\sqrt{2}}{\beta V} \sum_{k, z} G(k, z) (\tau
+ \frac{\delta \Sigma}{\delta B_{ii}}(q, \Omega)) G(k - q, z - \Omega).
\]

(36)

Using Eq. (30) in (36), we derive

\[
D_{23} = \frac{D^0_{23}}{1 + g_{12} D^0_{23}},
\]

(37)

\[
D_{12} = \frac{D^0_{12}}{1 + g_{12} D^0_{12}},
\]

(38)

where

\[
D^0_{23} = \Pi_{2233} + \frac{\Pi_{2433} \Pi_{6511}}{1 - g_{13} \Pi_{6611}},
\]

(39)

and

\[
D^0_{12} = \Pi_{1122} - \Pi_{1542};
\]

(40)

the bubble \( \Pi_{abcd}(q, \Omega) \) is given by

\[
\Pi_{abcd}(q, \Omega) = \frac{1}{\beta V} \sum_{k, z} G_{ab}(k, z) G_{cd}(k - q, z - \Omega),
\]

(41)

and the summation on \( k \) is up to \( \Lambda \). When \( \Delta \to 0 \), Eqs. (37) and (38) reduce to (22), since \( \Pi_{2433} \) and \( \Pi_{6511} \) are both proportional to \( \Delta \). Furthermore, when the interaction is \( SU(2) \) invariant, \( \chi''_{12}(\omega) \) is proportional to \( \delta(\omega - (\epsilon^a_{12} - \epsilon^b_{12})) \). If only \( \bar{g}_{12} \) is non-zero, the response function \( D_{23} \) reduces to the single loop, \( \Pi_{2233} \) (as calculated in Ref. [11]), and in fact satisfies the f-sum rule (7).

To see that the result (37) for the correlation function obeys the sum rule (27), we expand Eq. (37) as a power series in \( 1/\Omega \) in the limit \( \Omega \to \infty \) and compare the coefficients of \( 1/\Omega^2 \) of both sides. In addition, with \( n_1 = n_2 \), we find \( \int (d\omega/2\pi) \omega \chi''_{12}(\omega) = 0 \).

Figure 11 shows the paired fermion contribution to \( \chi''_{12}(\omega) \), calculated from Eqs. (37) and (11), as a function of \( \omega \), with \( g_{\sigma, \sigma'} = 4\pi\hbar^2 a_{\sigma, \sigma'}/m \). This graph corresponds to the \( ^6Li \) experiment done in a spatially uniform system. The origin is the response frequency of unpaired atoms, which is \( \omega_{\text{rr}} = \epsilon_1^a - \epsilon_2^a + (g_{13} - g_{12})n_1 \). We have not included the normal particle response in our calculation and do not show this part of the response in the figure. The parameters used are \( k_F a_{12} = -\pi/4 \) and \( a_{13} = a_{23} = 0.1 a_{12} \), for which, \( T_c = 0.084 a_{10} \). As the pairing gap, \( \Delta \), grows with decreasing temperature, the most probable frequency, \( \omega_{\text{pair}} \), in the response shifts to higher value. Within the framework of BCS theory, we can interpret the peak at higher frequency observed in the rf experiment as the contribution from the paired atoms.

We now ask how the most probably frequency \( \omega_{\text{pair}} \) is related to the pairing gap \( \Delta \). To do this we use the sum rule (27) on \( \chi''_{12}(\omega) \), written in terms of \( \chi''_{32}(\omega) \). Since \( \chi''_{xx}(\omega) = \chi''_{12}(\omega) + \chi''_{23}(\omega) \) and \( \chi''_{32}(-\omega) = -\chi''_{23}(\omega) \), we
have
\[ \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \chi''(\omega) = 2 \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \chi''(\omega). \]

Formally expanding Eq. (42) as a power series in $1/\Omega$ and comparing the coefficients of $1/\Omega$ on both sides, we find
\[ \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \chi''(\omega) = (Y_z)/2. \]  

Then assuming that the rf peak due to pairing is single and narrow (as found experimentally), we approximate $\chi''(\omega)$ as $\pi(Y_z)\delta(\omega - \omega_{\text{pair}})/2$. Using Eqs. (27), (42) and (43), we finally find
\[ \omega_{\text{pair}} - \omega_H = (g_{13} - g_{12}) \frac{\Delta^2}{n_0 g_{12} g_{13}}, \]

where $n_0 = n_1 = n_2$. Thus BCS pairing shifts the spectrum away from the normal particle peak by an amount proportional to $\Delta^2$. The shift of the peak vanishes if the interactions within the lowest three states is SU(2) invariant. Although the results given here are for the particular case of the lowest hyperfine states in $^4$Li, the present calculation can be readily extended to other multi-component fermion systems, as well as extended to include effects of the finite trap in realistic experiments.

**IV. CONCLUSION**

As we have seen, the experimental rf result on the BCS side can be understood by means of a self-consistent calculation of the pseudospin response within the framework of BCS theory in the manifold of the lowest three hyperfine states. The second peak observed at low temperature arises from pairing between fermions, with the displacement of the peak from the normal particle peak proportional to the square of the pairing gap $\Delta$. The shift of the peak vanishes if the interactions within the lowest three states is SU(2) invariant. Although the results given here are for the particular case of the lowest hyperfine states in $^4$Li, the present calculation can be readily extended to other multiple component fermion systems, as well as extended to include effects of the finite trap in realistic experiments.

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The coupling of the rf field to the atoms at the magnetic fields of interest in the Chin et al. experiment is primarily through the electron spin. The contribution to the electron spin-spin correlation function from transitions between the lower three hyperfine states is

\[
\langle T (S_x(t)S_y(0)) \rangle / V = |\langle 2|S_z|3 \rangle|^2 \int d^3r \langle \langle T\psi_1^*(\mathbf{r}, t)\psi_3(\mathbf{r}, t)\psi_2^*(0, 0)\psi_2(0, 0) \rangle_{\text{c.c.}} \rangle + c.c.
\]

\[
+ |\langle 1|S_z|2 \rangle|^2 \int d^3r \langle \langle T\psi_1^*(\mathbf{r}, t)\psi_2(\mathbf{r}, t)\psi_2^*(0, 0)\psi_1(0, 0) \rangle_{\text{c.c.}} \rangle,
\]

where \( V \) is the volume. Since the latter term vanishes in the many-body state with \( N_1 = N_2, N_3 = 0 \), we have simply,

\[
\langle T (S_x(t)S_y(0)) \rangle = \frac{|\langle 2|S_z|3 \rangle|^2}{|\langle 2|Y_2^*|3 \rangle|^2} \langle T (Y_2(t)Y_2(0)) \rangle.
\]

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