Generally covariant quantum mechanics on noncommutative configuration spaces

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Abstract

We generalize the previously given algebraic version of “Feynman’s proof of Maxwell’s equations” to noncommutative configuration spaces. By doing so, we also obtain an axiomatic formulation of nonrelativistic quantum mechanics over such spaces, which, in contrast to most examples discussed in the literature, does not rely on a distinguished set of coordinates.

We give a detailed account of several examples, e.g., $C^\infty(Q) \otimes M_n(C)$ which leads to nonabelian Yang-Mills theories, and of noncommutative tori $T_d^\theta$.

Moreover we examine models over the Moyal-deformed plane $\mathbb{R}^2_\theta$. Assuming the conservation of electrical charges, we show that in this case the canonical uncertainty relation $[x_k, \dot{x}_l] = ig_{kl}$ with metric $g_{kl}$ is only consistent if $g_{kl}$ is constant.

1 Introduction

In 1990, F. J. Dyson published a proof [1] of R. P. Feynman showing that the only interactions compatible with the canonical commutation relations of quantum mechanics are the electromagnetic ones, with the electric and magnetic fields satisfying the homogeneous Maxwell equations. This proof was generalized to the setting of gauge

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fields by C. R. Lee [3] and an attempt towards a relativistic theory was made by S. Tanimura [4]. Further considerations appear also in a review by J.F Carinena, L.A. Ibort, G.Marmo and A.Stern [2].

The idea of Feynman’s proof of Maxwell theory allowed in [5] a global and completely operational formulation of scalar quantum mechanics in close correspondence with the axioms of spectral geometry as given in [8] in which the configuration space $Q$ is described by the algebra of smooth coordinate functions $C_0^\infty (Q)$ vanishing at infinity and represented on the Hilbert space of the quantum particle.

This paper extends these results to noncommutative geometry by providing in Section 3 the needed framework and working out as a standard example the Moyal-deformed plane $\mathbb{R}^2_\theta$. We shall then present a number of almost commutative examples in Section 4: The case of $C_0^\infty (Q) \otimes M_n(\mathbb{C})$ in general leading to Quantum mechanics in interaction with a gauge field (for suitable $n$ interpretable as a spin connection). Then, in particular and in more detail the two-point model followed by a new example of a noncommutative deformation of the torus $T^2$. The usual noncommutative torus $T^2_\theta$ is investigated in Section 5.

As will become clear from these examples, our axioms may be used to define quantum mechanical systems over rather generic noncommutative configuration spaces, even though it is for most cases very hard to compute the most general Hamiltonian compatible with our axioms.

In any case, the noncommutative uncertainty relation turns out to be very restrictive as it essentially states that the Hamiltonian $H$ is “a differential operator of second order” in the sense of Connes’ Noncommutative Geometry. The usage of this property of $H$ as guiding principle might, at first sight, appear rather ad hoc. It turns out, however, that it should be viewed as a “nonrelativistic approximation”. It is, in fact, quite remarkable that the noncommutative case appears more naturally in the context of relativistic quantum field theory [6]. The difficulties with a relativistic one-particle quantum mechanics on noncommutative spaces will be given in [12].

However, to our point of view the main achievement of this paper is to clarify which $pre-C^*$-algebras, faithfully represented on a Hilbert space, may be good candidates to serve as noncommutative configuration spaces for quantum mechanics: Those that are small enough to allow enough room for functionally independent velocities and at the same time large enough to essentially generate together with these velocities the full algebra of observables. To illustrate the difficulties we shall give examples of algebras that are chosen inappropriately in Section 3.

2 Scalar quantum mechanics

Scalar quantum mechanics as given in [5] is a global formulation of quantum mechanics on a configuration manifold $Q$ which is in an algebraic setting fully captured by its algebra $\mathcal{A} = C_0^\infty (Q)$ of smooth functions vanishing at infinity. It consists of a family $(\mathcal{A}_t, t \in \mathbb{R})$ of unitary representations of the algebra $\mathcal{A}$ satisfying the following rather general postulates:
1. **Localizability**: The representations $\mathcal{A}_t$ contain in their representation space $\mathcal{H}$ a dense finite projective module $\mathcal{H}_\infty$ (It is isomorphic to representations of $C^\infty_0(Q)$ on the space $\mathcal{H} = L^2(Q, E)$ of square integrable sections of a complex line bundle $E$ over $Q$ vanishing at infinity).

2. **Scalarity**: At every time $t$, the commutant $\mathcal{A}_t'$ of $\mathcal{A}_t$ (the set of all operators commuting with $\mathcal{A}_t$) is just $\mathcal{A}_t$, the closure of $\mathcal{A}_t$ in the weak topology:

   \[ \mathcal{A}_t' = \bar{\mathcal{A}}_t \]

3. **Uncertainty relation**: The time evolution, i.e., the dependence of the representation $a_t$ of any $a \in \mathcal{A}$ is smooth in the parameter $t$ with respect to the strong topology and satisfies:

   \[ i[\mathcal{A}_t, \dot{\mathcal{A}}_t] \subset \mathcal{A}_t \quad \text{for all } t \in \mathbb{R} \]  

4. **Positivity and nontriviality**: For all $a_t \in \mathcal{A}_t$

   \[ -i[a_t, \dot{a}_t] \geq 0, \]

   and equality

   \[ [a_t, \dot{a}_t] = 0 \]  

   for some $a_t \in \mathcal{A}_t$ implies

   \[ \dot{a}_t = 0 \]

This set-up allows for easy algebraic expressions of geometric concepts. Linear connections on $E$ along a vector field $X$ are fully characterized as operators $\nabla_X : \mathcal{H}_\infty \rightarrow \mathcal{H}_\infty$ satisfying

\[ \nabla_X^* = \nabla_X, \]

\[ \nabla_X(a_t \psi) = iX(a_t)\psi + a_t \nabla_X \psi, \]

\[ \nabla_{a_tX + b_tY} = a_t \nabla_X + b_t \nabla_Y, \]

for any vector fields $X$, $Y$ and for any $a_t, b_t \in \mathcal{A}_t$, $\psi \in \mathcal{H}_\infty$. The freedom in the choice of a connection $\nabla$ is given by a form $A$ on $Q$ that relates it to an arbitrarily chosen reference connection $\partial$.

\[ \nabla_X = \partial_X + \langle A, X \rangle. \]

The following facts are derived in [5]
1. For each time $t$ and for any $a_t, b_t \in \mathcal{A}$, the expression
\[
ig_t(da_t, db_t) = [a_t, b_t]
\]
defines an inverse Riemannian metric $g_t$.

Its symmetry follows from the commutativity of $\mathcal{A}$ and its consistency with the Leibniz rule follows from the commutator $[\cdot, \cdot]$ satisfying the Leibniz rule and from the symmetry of $g$.

2. There exists (up to a topological obstruction) a Hamiltonian operator $H$ that generates the time evolution. Thus, surprisingly, the uncertainty relation (2) enforces the time evolution to be unitary, i.e. $\|a_t\|$ is independent of $t$.

3. By (2) it is bound to be a differential operator of order two. Together with (1), Newton’s law (the expression of second time derivatives in terms of first and zeroth ones) can be shown to hold.

The Hamiltonian has the form
\[
H = \frac{1}{2} \Delta(g_t, A_t) + \phi_t
\]
where $\Delta(g_t, A_t)$ is the covariant Laplacian associated with $g_t, A_t$ and $\phi_t \in \bar{\mathbb{A}}_t$.

This shows that the postulates of scalar quantum mechanics, although general in their appearance, lead to a tightly specified dynamics: A Hamiltonian of second order in spatial derivatives of the particular form (11) and a Newton’s law expressing second time derivatives in terms of lower ones.

3 Noncommutative configuration spaces

Being formulated in terms of (commutative) operator-algebras of observables it is tempting to try to generalize the axioms of scalar quantum mechanics [5] to generic noncommutative algebras for at least two reasons:

First of all, there might be additional quantum numbers like isospin, which render the "configuration space" noncommutative.

Second, one might ask whether spacetime itself is a noncommutative space, like the Moyal-deformed plane
\[
[x, y] = i\theta
\]
that has been quite fashionable over the last years. In that case, one should of course try to find a sound basis to set up quantum mechanical models on such a space.

To our point of view, this in particular means that the physical content of such a model should be independent of the coordinates we have chosen to describe it. In other words, the model should not depend on the choice of generators of the algebra of functions on the configuration space. This is, however, not the case for many examples discussed in the literature. Yet, if spacetime is really noncommutative, then there should be a dynamical mechanism leading to this noncommutativity and we believe this mechanism to be related to the quantized gravitational field. As the (classical) dynamics of the
gravitational field are governed by the demand of general covariance, i.e., the independence of the chosen coordinate system, we do therefore not believe that models which heavily violate general covariance could be of physical relevance. One might of course argue that, at very small scales there might appear an anomaly to diffeomorphism invariance and that violations of the latter, i.e., the appearance of preferred coordinates, may therefore be visible at such a scale. But even then, it seems to us, such effects should not be visible in nonrelativistic one-particle systems. We do not exclude that in many particle systems such effects might accumulate and thus be rendered observable at macroscopic scales. Yet, we do not consider such systems here, but leave their investigation open as a future project.

It is usually assumed in the literature that the noncommutativity parameter $\theta$ in $[x, y] = i\theta$ is a physical parameter that can be determined experimentally just as $\hbar$ can be measured. Yet, to us this seems to be a wrong analogy (to quantum mechanics). The point is that due to Darboux' theorem there do exist canonical, i.e. uniquely given momenta $p_i$ to (almost) any reasonable choice of coordinates $q_i$, such that $[q_i, p_j] = i\delta_{ij}\hbar$. Thus $\hbar$ does not depend on the coordinates we have chosen on the configuration space, but only makes reference to the symplectic structure, which in turn is uniquely given by the dynamics. In contrast to that, if the coordinate $x_1$ is given, then there is no canonical choice of the remaining coordinates $x_k$. Thus, we can always achieve by an appropriate choice of coordinates that $\theta = 1$, say. In fact, the algebras generated by $[x, y] = i\theta$ are isomorphic for all nonvanishing values of $\theta$ and our models will be equivalent for isomorphic algebras, and hence $\theta$ will not be observable in the sense of general relativity. But there is a crucial difference if $\theta$ is vanishing, as we shall see.

3.1 Axioms for noncommutative scalar quantum mechanics

It is well known, that the space of sections of vector bundles can equivalently be described as a finitely generated projective module over $\mathcal{A}$. This notion, however, is also well-defined for noncommutative algebras, so we shall replace the corresponding axiom, by requiring

NC-"Localizability". For all $t \in \mathbb{R}$ the representations are finitely generated projective modules over $\mathcal{A}$.

As the methods used in [3] to infer the existence of a Hamiltonian are not available in the noncommutative case, we shall for simplicity also require the

Unitarity of Time-evolution, i.e., we shall require the existence of a (possibly time-dependent) selfadjoint Hamiltonian $H$ generating the time evolution $\mathcal{A}_t$.

If the algebra $\mathcal{A}$ is noncommutative, it is in general no longer possible to require that the commutator of "functions", i.e., of algebra elements $a_t \in \mathcal{A}_t$ and "velocities" $b_t \in \mathcal{A}_t$ is an algebra element: Suppose $a_t, b_t \in \mathcal{A}_t$ and

$$[a_t, b_t] \in \mathcal{A}_t.$$
Then one has

$$[a_t, (b_t \dot{c}_t)] = [a_t, b_t \dot{c}_t] + b_t[a_t, c_t] + [a_t, b_t \dot{c}_t] + \dot{b}_t[a_t, c_t]$$

which will not be in the algebra for all $c_t \in \mathcal{A}_t$, unless $\dot{b}_t$ is either an algebra element itself, or the factor $[a_t, c_t]$ vanishes. It is thus still possible to demand that the commutant of elements of the center, that we shall denote $Z_t := Z(\mathcal{A}_t)$, with time derivatives of algebra elements are in the algebra:

**weak NC-Uncertainty** We assume that

$$[z_t, b_t] \in \mathcal{A}_t$$

for all $z_t \in Z_t$ and all $b_t \in \mathcal{A}_t$.

**Proposition 3.1.** Suppose the NC-Uncertainty holds. Then

$$[\mathcal{A}_t, \dot{Z}_t] \subset \mathcal{A}_t$$

$$[Z_t, \dot{Z}_t] \subset Z_t$$

**proof:**

Assume $z_t \in Z_t$ and $a_t \in \mathcal{A}_t$. Then:

$$[z_t, a_t] = 0 \Rightarrow [z_t, \dot{a}_t] = [a_t, \dot{z}_t]$$

(by differentiation with respect to $t$). This proves the statement $[\mathcal{A}_t, \dot{Z}_t] \subset \mathcal{A}_t$.

Consider now additional elements $w_t \in Z_t$. The statement $[Z_t, \dot{Z}_t] \subset Z_t$ then follows from the Jacobi identity:

$$0 = [a_t, [z_t, w_t]] + [z_t, [w_t, a_t]] + [w_t, [a_t, z_t]]$$

As we shall see in the next section, the NC-Uncertainty is in some cases very restrictive. However, in the generic case, the center of $\mathcal{A}$ might be too small, or even trivial, so that we shall need additional assumptions for such cases.

However, we have not yet generalized the axiom of “scalarity”. A noncommutative algebra can, of course, not equal its commutant. However, we might require it to be antiisomorphic to it. Note that this follows immediately if there exists a cyclic and separating vector for the algebra $\mathcal{A}_t$, as is the case for the vacuum representation of the algebra of local observables of a quantum field theory. In that case the operator $J$ is related to the PCT-operator. Thus, in a sense the above requirement would mean that we should not be able to distinguish a relativistic scalar particle and its PCT transformation on such a noncommutative spacetime. As has been pointed out in [6], this has, in fact to be expected. We therefore consider our requirement as naturally motivated from QFT to which a nonrelativistic QM should be an approximation.
NC-Scalarity We shall assume, that there exists for each $t$ an antiunitary operator $J_t$, such that

$$J_t A_t J_t^{-1} = A'_t. \quad (14)$$

In other words, we shall assume that the algebras $A_t \pi_t(A)$ are anti-isomorphic to a dense subse of their commutant. Equivalently that then means that besides the family of representations $\pi_t$ of the algebra $A$ of “functions” over the noncommutative configuration space there is also a family $\pi^o_t = J_t \pi_t^* J_t^{-1}$ of “opposite” representations given, i.e., one has:

$$\pi^o_t(ab) = \pi^o_t(b)\pi^o_t(a) \quad \forall t \in \mathbb{R}, \quad \forall a_t, b_t \in A_t.$$  

For brevity, we shall use the notation $\pi^o_t(a) = a^o_t$, thus we have

$$[a_t, b^o_t] = 0 \quad \forall t, \quad \forall a, b \in A_t.$$  

In the commutative case one can choose $J_t$ to be given for all times $t$ by the complex conjugation of functions, i.e. $J_t a_t J_t^{-1} = \overline{a_t}$. In view of this example it therefore appears natural to assume the reality-condition:

$$\frac{d}{dt} (J_t a_t J_t^{-1}) = J_t \left( \frac{d}{dt} a_t \right) J_t^{-1} \quad \forall a_t \in A_t$$

which is obviously fulfilled in the classical case. Note that this condition is already slightly weaker than requiring $J$ to commute with $H$. However, this condition still can not always be met in the noncommutative case, as we shall see in the examples. In any case, the property NC-scalarity now enables us to require a stronger version of the uncertainty relation:

**strong NC-Uncertainty:** If the above property of NC-Scalarity holds, then it is assumed that

$$[\dot{a}_t, b^o_t] \in A^o_t \otimes A_t \quad \forall t, \quad \forall a_t, b_t \in A_t.$$  

The notation $A^o_t \otimes A_t$ should be understood as the set of all operators that can be written in the form $\sum_i a_{t,(i)} b^o_{t,(i)}$. 

Hence, the commutators with time derivatives $\dot{a}_t$ are required to be derivations on the algebras $A^o_t$. Note that this condition then also implies that $J_t \dot{a}_t J_t^{-1}$ is a derivation on the algebra $A$. If the above reality property holds, then one also has $\langle a_t^o \rangle = J_t \dot{a}_t J_t^{-1}$ and thus the strong NC-Uncertainty is completely symmetric in $A_t$ and $A^o_t$.

**Proposition 3.2.** Strong NC-Uncertainty implies weak NC-Uncertainty, i.e

$$[z_t, \dot{a}_t] \in A_t \quad \forall z_t \in Z_t, \quad a_t \in A_t.$$  

**Proof:** Since the commutant of $A^o_t = J_t A_t J_t^{-1}$ equals $A_t$ it is sufficient to prove that $[z_t, \dot{a}_t]$ commutes for all $z_t \in Z_t$ and $a_t \in A_t$ with all $b^o_t \in A^o_t$. Considering the
Jacobi-Identity again:
\[
0 = \left[[z_t, \dot{a}_t], b_t\right] + \left[[b_t, z_t], \dot{a}_t\right] = 0
\]

\[
\left[[\dot{a}_t, b_t], z_t\right] = \left[[\dot{a}_t, z_t], b_t\right] = 0
\]

**Proposition 3.3.** If \(\mathcal{A} \otimes \mathcal{A}^\circ = \mathcal{Z}(\mathcal{A})'\) then weak NC-Uncertainty implies strong NC-uncertainty.

**proof:**
To show strong NC-Uncertainty, it is by the assumptions sufficient to show that \([\dot{a}_t, b_t]\) is in the commutant of \(\mathcal{Z}_t(\mathcal{A}_t)\), i.e.,
\[
[[\dot{a}_t, b_t], z_t] = 0
\]
for all \(a_t, b_t \in \mathcal{A}_t, z_t \in \mathcal{Z}_t(\mathcal{A}_t)\) (15)

But this holds due to the Jacobi identity
\[
[[\dot{a}_t, b_t p], z_t] = \left[[\dot{a}_t, z_t], b_t p\right]
\]
and since \([\dot{a}_t, z_t]\) is in \(\mathcal{A}_t\) by weak NC-Uncertainty and thus commutes with \(b_t p\). □

However, for infinite dimensional algebras \(\mathcal{A} \otimes \mathcal{A}^\circ\) will in general only be a (dense) subset of the commutant of \(\mathcal{Z}(\mathcal{A}_t)\) and weak and strong uncertainty are equivalent:

Let \(\mathcal{A} = \bigoplus_{k=1}^{K} M_{n_k}(\mathbb{C})\) be any finite dimensional \(C^*\)-algebra. Define \(P_k = 1_{n_k}\), i.e.,
\[
P_k P_i = \delta_{ik} P_i, \quad P_k a \in M_{n_k}(\mathbb{C}) \quad \text{for all } a \in \mathcal{A}, k = 1, \ldots, K
\]
(16) (17)

The centre of \(\mathcal{A}\) is then given as
\[
\mathcal{Z}(\mathcal{A}) = \{ \sum_k c_k P_k \mid c_k \in \mathbb{C} \}.
\]

It is well known (see, e.g., [9]) that the only representation \(\pi\) of \(\mathcal{A}\) such that there exists an antunitary automorphism \(J : \mathcal{H} \to \mathcal{H}\) with \(J AJ^{-1} = A'\) is given by taking \(\mathcal{H} = \mathcal{A}\) (as a vector space) and representing \(a\) by left multiplication on matrices:
\[
\pi(a) \psi = a \psi
\]
(18)

\(J\) is (up to unitary equivalence) given as
\[
J \psi = \psi^*,
\]
(19)

where \(\psi\) is considered as a block-diagonal matrix. It follows that \(J^{-1} = J\).

Then
\[
a^{op} \psi = J a^* J \psi = \psi a
\]
(20)
Thus $\mathcal{A}^{\text{op}}$ is left multiplication with $\mathcal{A}$. Note now that, with $H_k = P_k \mathcal{H}$ one has that

$$\pi(\mathcal{A}) : H_k \to H_k \quad \text{for all } k \quad (21)$$

$$\pi^{\text{op}}(\mathcal{A}) = J \pi(\mathcal{A}) J : H_k \to H_k \quad \text{for all } k, \quad (22)$$

as they both commute with $P_k$.

One then easily proves that

$$P_k(\pi(\mathcal{A}) \otimes \pi^{\text{op}}(\mathcal{A})) P_k \cong \text{End}(H_k) = P_k \text{End}(\mathcal{H}) P_k. \quad (23)$$

Since $\sum_k P_k = 1_\mathcal{H}$, we have

$$\sum_k P_k(\pi(\mathcal{A}) \otimes \pi^{\text{op}}(\mathcal{A})) P_k = \pi(\mathcal{A}) \otimes \pi^{\text{op}}(\mathcal{A}), \quad (24)$$

and thus

$$\pi(\mathcal{Z}(\mathcal{A}))' = \{ M : H \to \mathcal{H} \mid MP_k = P_k M \} = \sum_k P_k (\text{End } \mathcal{H}) P_k = \pi(\mathcal{A}) \otimes \pi^{\text{op}}(\mathcal{A}). \quad (25)$$

Thus, for finite dimensional algebras, weak NC-Uncertainty and strong NC-Uncertainty are equivalent. As we shall see in the example of the noncommutative torus, which has a trivial center, the strong uncertainty is in fact very restrictive for infinite dimensional algebras, however.

### 3.2 Heuristic construction for the Moyal-plane

Let us illustrate the difficulties on the example of the Moyal-deformed plane $[x_1, x_2] = i\theta$. One might then require the canonical uncertainty relations

$$[x_k, \dot{x}_l] \in \mathcal{A} \quad (30)$$

for the generators $x_k$. As explained above, it would then not be true that $[a, \dot{b}] \in \mathcal{A}$ for generic algebra elements $a, b \in \mathcal{A}$. In particular, if one would choose another set of generators of the algebra, then these new generators would not fulfill the canonical uncertainty relation. Thus, models which are constructed in this way depend on the choice of generators!

Nevertheless, it is quite instructive to proceed with the construction of models based on (30). To this end we shall use the standard (star-product) representation of the Moyal deformed plane on the space $\mathcal{A} = L^2(\mathbb{R}^2)$ (which is just the representation of the
algebra on itself). Note that there do exist the standard derivations

\[ \delta_i(x_j) = \delta_{ij} \]
on the algebra and hence they are also represented (as hermitean operators upon multiplying with \( i \)) on the Hilbert space \( \mathcal{H} \). Then, as usual one has \( [a, \delta_k] = i \delta_k(a) \) for all \( a \in \mathcal{A} \). The representations of the \( \delta_i \) are not in the algebra. Recall that for any \( b \in \mathcal{A} \) that is not in the center, \( [a, b\delta_k] \) will not be an algebra element (for all \( a \)). Thus \( b\delta_k \) does not define a derivation on \( \mathcal{A} \).

For simplicity we shall only consider derivations of the algebra of the form

\[ P(a) = \sum_i \zeta_i \delta_i(a) + [A, a] \]
with \( \zeta_i \in \mathbb{C} \) and \( A \in \mathcal{A} \). (Most probable all derivations of \( \mathcal{A} \) are actually of this form. However we are not aware of any proof of this statement.)

Let us define the non-vanishing operators

\[ p_k = \delta_k - \varepsilon_{kl} \theta x_l \]

Obviously these operators commute with all the algebra elements. In fact, one can prove that the commutant \( \mathcal{A}' \) of \( \mathcal{A} \) is generated by \( p_1, p_2 \).

Let us now return to (30). It then immediately follows (from the Leibniz rule for commutators), that for every algebra element (power series in the generators) \( a \in \mathcal{A} \) one has

\[ [a, \dot{x}_k] \in \mathcal{A}. \]

Thus commuting with \( \dot{x}_k \) is a derivation on the algebra \( \mathcal{A} \). If we then allow for \( H \) only derivations on \( \mathcal{A} \) of the form given above, there do thus exist \( \zeta_{kl}, A_k \) such that

\[ \dot{x}_k = i \sum_l \zeta_{kl} \delta_l + A_k + \omega_k, \]
where \( \omega_k \in \mathcal{A}' \). We suppress \( \omega_k \) in the remainder. Therefore

\[ [x_l, \dot{x}_k] = i \zeta_{kl} + [x_l, A_k]. \]

Note also that, upon differentiating \( [x_k, x_l]i \varepsilon_{kl} \theta \) with respect to \( t \) we get the consistency relation:

\[ i \zeta_{kl} + [x_l, A_k] = i \zeta_{lk} + [x_k, A_l]. \]

Recall now the classical case: There we found \( \dot{x}_k i \sum_l g_{kl} \delta_l + A_k \), but \( g_{kl} \), i.e. the components of the metric being in \( \mathcal{A} \). So here it seems, that the metric has to be constant. Well, not really, since now the vector potential \( A_k \) renders the uncertainty nontrivial. As we shall point out in the section on the noncommutative Torus (where we obtain a similar result) the term \( [x_l, A_k] \) does in fact correspond to the metric on this space.

In conclusion, one can consistently construct quantum mechanical models over the Moyal deformed plane. However, these models turn out to be fairly restricted as compared to the commutative case.
4 Almost commutative Spaces

We now turn our attention to simple examples for algebras of the form \( \mathcal{A} = C_0^\infty(\mathbb{Q}) \otimes \mathcal{A}_f \), where \( \mathcal{A}_f \) is a (semi-simple) finite-dimensional algebra. Such algebras do have a very large center. As we shall see, our NC-Uncertainty condition is therefore already a very strong (sometimes even too strong) requirement.

4.1 \( C_0^\infty(\mathbb{Q}) \otimes M_n(\mathbb{C}) \) and nonabelian gauge theories

We first consider the algebra \( \mathcal{A} = C_0^\infty(\mathbb{Q}) \otimes M_n(\mathbb{C}) \).

We shall represent it on the Hilbert space \( \mathcal{H}L^2(\mathbb{Q}, E) \otimes \mathbb{C}^n \), which corresponds to a "line bundle" over \( \mathcal{A} \), i.e., it is given (up to closure...) as \( \mathcal{H} = p\mathcal{A} \) where \( p = p^2 \in \mathcal{A} \) has \( \text{tr}p = 1 \). It is well known, that in this case one has:

\[ \mathcal{A}' = \mathcal{Z}, \quad \mathcal{Z}' = \mathcal{A}. \]

Choosing generators \( T^a \) of \( \text{su}(n) \) in the fundamental representation (they then together with the unit matrix do form a basis in \( M_n(\mathbb{C}) \)), we thus have (using local coordinates):

\[ [x^k, x^l] = [x^k, T^a] = 0 \quad [T^a, T^b] = i f^{abc} T^c. \]

If we then demand the NC-Uncertainty

\[ [x^k, \dot{a}] \in \mathcal{A} \quad \forall a \in \mathcal{A}, \]

then we immediately get from the above lemma:

\[ [x^k, \dot{x}^l] \in \mathcal{Z} \quad [\dot{x}^k, T^a] \in \mathcal{A}. \]

Analogously to the classical case, we can thus infer:

\[ \dot{x}^k = g^{kl}(-i \partial_l - A_l) \]

where \( g^{kl} \in \mathcal{Z} = C_0^\infty(\mathbb{Q}) \) and \( A_l = a_l^* \in \mathcal{A} \). Note that \( A_l \) thus defines a \( \mathfrak{u}(n) = \mathfrak{u}(1) \oplus \mathfrak{su}(n) \) gauge connection. But then we have

\[ [x^k, \dot{T}^a] = [T^a, \dot{x}^k] = g^{kl}[T^a, A_l] \]

from which we immediately get

**Lemma 4.1.** There exist hermitean elements \( A_0^a \in \overline{\mathcal{A}} = \mathcal{Z}' \) such that

\[ \dot{T}^a = -\dot{x}^k[A_k, T^a] + A_0^a \]

Constructing the most general Hamiltonian, that would lead to these time derivatives, we then obtain:
Theorem 4.2. There exists a metric $g^{kl}$ and a $u(n)$-gauge connection $A_{\mu}$ ($\mu = 0, 1, \ldots, d$) on $Q$ (respectively on $\mathcal{M}Q \times \mathbb{R}$) such that

$$H = \frac{1}{2} g^{kl} (-i \partial_k - A_k) (-i \partial_l - A_l) + A_0.$$ 

Hence we obtain that our system describes a scalar particle that is (minimally) coupled to gravity and $u(n)$ Yang-Mills Theory.

It is quite surprising that our axioms are that restrictive in this case. In particular, we obtain the nonabelian gauge invariance just from the NC-uncertainty relation (that is defined on the center only!).

Remark 4.3. Let us restrict to $n = 2$ in which case $T^k = \sigma^k$, $k = 1, 2, 3$ and $Q = \mathbb{R}^3$. Then one would also expect to obtain the Pauli-Term $\frac{e}{4m^2r} \nabla \sigma L$ and the fine structure term

$$\frac{e}{4m^2r} \frac{\partial \varphi}{\partial r} \sigma L$$

(for radially symmetric scalar potential $\varphi$) to appear in $H$.

They are, in fact, present: Choose $A_0 = \frac{e}{m} \vec{B} \vec{\sigma}$ to obtain the Pauli term and $\vec{A} = -\frac{ie}{8m^2} \vec{\sigma} \times \vec{\nabla} \phi (r)$ to obtain the fine structure. The physical difference of spinors and $SU(2) - doublets$, i.e. the correct spin-orbit coupling is, of course, only seen in the relativistic theory.

Remark 4.4. Since $\mathcal{H} = L^2(Q, E) \otimes C^n$, we only obtain the $su(n)$ gauge theory on a trivial bundle, while the $u(1)$ part might live on an arbitrary line bundle. If one would like to get gauge theories on arbitrary bundles of rank $n$, then one would simply work with the algebra

$$\mathcal{A} = pM_N(C_0^\infty(Q))p$$

where $p \in M_N(C_0^\infty(Q))$, $p^2 = p$ and $tr p = n$.

4.2 The two point model

We now come to another classic in NCG, the two-point-space, i.e.

$$\mathcal{A} = C_0^\infty(Q) \otimes (\mathbb{C} \oplus \mathbb{C}).$$

Algebra elements $a \in \mathcal{A}$ can then be viewed as diagonal two-by-two matrices

$$a = \left( \begin{array}{cc} a_1 & 0 \\ 0 & a_2 \end{array} \right)$$

with entries $a_1, a_2$ from $C_0^\infty(Q)$. We shall represent $\mathcal{A}$ on the space $\mathcal{H} = L^2(Q, E) \otimes \mathbb{C}^2$. In this case, the commutant of the algebra equals the (weak closure of the) algebra itself

$$\mathcal{A}' = \mathcal{A}.$$
Since $A = Z = A^o$ is commutative, our NC-Uncertainty agrees with the commutative one. Once again, it will tell us that
\[
\dot{b}_t = \begin{pmatrix} X_{b_1} & 0 \\ 0 & X_{b_2} \end{pmatrix} + A(b)
\]
where $A(b)$ lies in the commutant of $A$, i.e. in $A$ itself. Moreover, the map $b \rightarrow A(b)$ is linear and fulfills the Leibniz rule. Thus there will exist one forms $A_i, \alpha_i$ such that
\[
A(b) = \begin{pmatrix} g_1(A_1, db_1) + g_1(\alpha_1, db_2) & 0 \\ 0 & g_2(A_2, db_2) + g_2(\alpha_2, db_1) \end{pmatrix}
\]
where $g_1, g_2$ denote the metrics on the different world sheets. The terms $g_1(\alpha_1, db_2)$ (and resp. ...) seem to carry particles from one world sheet to the other. However, there are not present, i.e., $\alpha_k = 0$! In fact, if one decomposes the Hamiltonian according to
\[
H = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}.
\]
Then one easily computes
\[
[H, a] = \begin{pmatrix} [h_{11}, a_1] & h_{12}a_2 - a_1h_{12} \\ h_{21}a_1 - a_2h_{21} & [h_{22}, a_2] \end{pmatrix}.
\]
Thus, a term like the one above can never be generated by a Hamiltonian.

Nevertheless one might like to construct a model which "mixes the two manifolds". In order to see what has been wrong with our assumptions, let us consider such a toy Hamiltonian
\[
H = (\Delta(g, A) + V) \mathbb{1} + \begin{pmatrix} 0 & \phi \\ \bar{\phi} & 0 \end{pmatrix}
\]
where $\phi$ is a complex scalar field. Such a term could be reminiscent of spontaneous breaking via a Higgs field of $U(1) \times U(1)$ (which of course, does not happen here – it can only happen in a quantum field theory...we consider our theory to arise as a limit...). Then, after a short computation one gets
\[
[a, \dot{b}] = ig(da, db) \mathbb{1} + i \begin{pmatrix} 0 & \phi(a_1 - a_2)(b_1 - b_2) \\ -\bar{\phi}(a_2 - a_1)(b_2 - b_1) & 0 \end{pmatrix}.
\]
So, such a term would violate the canonical uncertainty relation, which might therefore be a too strong requirement.

### 4.3 The noncommutative double-torus

Let us now come to a much more noncommutative, yet still very simple example. The algebra we shall consider can be viewed as the bi-cross product of the two-torus $C(T^2)$ with the diffeomorphism that is not generated by a vector field, i.e. the one which interchanges the two circles in $T^2 = S^1 \times S^1$. 

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Thus $\mathcal{A}$ is generated by two mutually commuting unitaries $U, V$ and one additional generator $\sigma$, subject to the relation
\[
\sigma U = V \sigma, \quad \sigma^* = \sigma, \quad \sigma^2 = 1.
\]
Note that every algebra element $a \in \mathcal{A}$ can be written in the form $a = f_- + f_+ \sigma$, where $f_\pm$ are power series in $U$ and $V$, i.e. they can be identified with functions over the torus $T^2$. The product in the algebra $\mathcal{A}$ is then given as
\[
(f_- + f_+ \sigma)(g_- + g_+ \sigma) = (f_- \cdot g_- + f_+ \cdot \tilde{g}_+) + (f_- \cdot g_+ + f_+ \cdot \tilde{g}_-) \sigma.
\]
Note that every algebra element $a \in \mathcal{A}$ can be written in the form $a = f_- + f_+ \sigma$, where $f_\pm$ are power series in $U$ and $V$, i.e. they can be identified with functions over the torus $T^2$. The product in the algebra $\mathcal{A}$ is then given as
\[
(f_- + f_+ \sigma)(g_- + g_+ \sigma) = (f_- \cdot g_- + f_+ \cdot \tilde{g}_+) + (f_- \cdot g_+ + f_+ \cdot \tilde{g}_-) \sigma.
\]
where the dot “.” denotes the usual point-by-point product of functions over $T^2$ and $\tilde{g}_\pm = \sigma g_\pm \sigma$. As is easily seen the center $\mathcal{Z}$ of the algebra $\mathcal{A}$ is given by elements of the form $f + \sigma f \sigma$ with $f \in C(T^2)$. Thus, not expected, the center consists of functions on the torus which are invariant under $\sigma$.

This algebra can then be represented on the Hilbert space $L^2(T^2) \oplus L^2(T^2)$, with the standard orthonormal basis
\[
|n, m, \pm\rangle, \quad n, m \in \mathbb{Z}
\]
as follows
\[
U |n, m, \pm\rangle = |n + 1, m, \pm\rangle \quad V |n, m, \pm\rangle = |n, m + 1, \pm\rangle \quad \sigma |n, m, \pm\rangle = |m, n, \mp\rangle.
\]
Note that the vector $|0, 0, -\rangle$ cyclic and separating. We can thus apply the Tomita-Takesaki Theorem to obtain
\[
J |n, m, +\rangle = |- n, -m, +\rangle \quad J |n, m, -\rangle = |- m, -n, -\rangle,
\]
and the corresponding “opposite algebra”:
\[
U^o |n, m, +\rangle = |n + 1, m, +\rangle \quad U^o |n, m, -\rangle = |n, m + 1, -\rangle \quad V^o |n, m, +\rangle = |n, m + 1, +\rangle \quad V^o |n, m, -\rangle = |n + 1, m, -\rangle \quad \sigma^o |n, m, \pm\rangle = |n, m, \mp\rangle.
\]
We can now start to investigate the consequences of the strong NC-Uncertainty relation $[\hat{a}_t, b_t^*] \in \mathcal{A} \otimes \mathcal{A}^*$ for families of representations of $\mathcal{A}$ which are all unitarily equivalent to the one described above, the unitaries being generated by a Hamiltonian $H$. For simplicity we shall assume that $H$ is time independent and commutes with $J$,
\[
[H, J] = 0 \quad \Rightarrow \quad H |n, m, \pm\rangle = h_\pm |n, m, \pm\rangle.
\]
with two operators \( h_\pm \) obeying \( h_+ = \bar{h}_+ \) respectively \( \bar{h}_- = \sigma h_- \).

A lengthy but straightforward calculation of the commutators \( [\hat{U}, U^o], [\hat{V}, V^o] \) \([\hat{V}, V^o]\) then shows that they are in \( \mathcal{A} \otimes \mathcal{A}^o \) if and only if \( h_+ = h_- =: h \) and such that \([h, a] \) is a derivation on \( \mathcal{A} \) for all \( a \) that are power series in \( U \) and \( V \), i.e.

\[
[[h, a], b] \in \mathcal{A} \quad \forall b \in \mathcal{A}.
\]

Note that it then immediately follows that \( h = \sigma h\sigma \) and hence that \( \dot{\sigma} = [H, \sigma] = 0 \).

In order to obtain the most general Hamilton-Operator it is convenient to introduce the two standard vector fields on the torus which generate the \(U(1) \times U(1)\) -Symmetry, \(\delta_1|n, m, \pm\rangle = n|n, m, \pm\rangle\) \(\delta_2|n, m, \pm\rangle = m|n, m, \pm\rangle\) \((31)\) \((32)\).

Note that they are not derivations on the algebra, since they are not compatible with the relation \(U \sigma = \sigma V\). However, the combination \(P = z(\delta_1 + \delta_2)\) is a derivation on \(\mathcal{A}\) whenever \(z \in \mathcal{A}\), i.e. whenever \(z\) is a function on the torus that is invariant under \(\sigma\). One then easily infers that the most general Hamilton-Operator is of the form

\[
H = z_1(\delta_1^2 + \delta_2^2) + z_2\delta_1\delta_2 + z_3(\delta_1 + \delta_2) + z_4\sigma \quad z_i \in \mathbb{Z} \quad i \in \{1, 2, 3, 4\}.
\]

The interesting piece here is the term \(z_4\sigma\) which, since the diffeomorphism \(\sigma\) is not generated by a vector field is highly nonlocal on \(T^2\).

However, the other terms apart from \(z_4\sigma\), have a rather clear interpretation:
The above construction is, in fact a standard technique in noncommutative geometry. Given the algebra \(C(T^2)\) and the diffeomorphism \(\sigma\) one may construct the algebra \(C(T^2)/\sim\) of equivalence classes under the equivalence relation given by \(\sigma\) (i.e. \(f \sim g \iff f \sigma g\sigma\)). Obviously \(C(T^2)/\sim\) is isomorphic to \(\mathbb{Z}\) from above. But it is also Morita-equivalent to the full algebra \(\mathcal{A}\). Thus, in order to do some differential topology on \(C(T^2)/\sim\) one may also use \(\mathcal{A}\) to obtain the same invariants.

But the above Hamilton Operator is just the most general Laplace Operator on the spectrum of \(C(T^2)/\sim= \mathbb{Z}\) plus the highly nonlocal term \(z_4\sigma\) whose presence is due to the fact that the algebras are "only" Morita equivalent but not isomorphic.

### 5 Noncommutative Tori

Finally we examine the noncommutative torus, i.e., the algebra generated by two unitaries \(U_1, U_2\) subject to the relation

\[
U_1U_2 = \lambda U_2U_1, \quad \lambda = e^{i2\pi \theta}.
\]

More precisely algebra elements \(a\) are power series \(a \sum_{kl} a_{kl} U_2^k U_1^l\) with coefficients \(a_{kl}\) which vanish faster than any polynomial for \(k, l \to \infty\).
Without loss of generality, we assume \( \theta \) being irrational. It is well known that for \( \theta \) rational the algebra \( \mathcal{A} \) isomorphic to the algebra of endomorphisms of the space of sections of a vector bundle over the commutative torus, i.e. it is of the form \( pM_n(c_0^\infty(T^2))p \). Thus for rational \( \theta \) we would be back to the almost commutative case already described. We would like to treat the noncommutative Torus as a deformation of the commutative one which corresponds to \( \theta = 0 \).

Now, it is well-known [10], [11], that derivations \( P \) on the noncommutative Torus are of the form
\[
P = \sum ic_i\delta_i + a, \quad a \in \mathcal{A}, \quad c_i \in \mathbb{C}
\]
where \( \delta_i \) are the standard derivations
\[
\delta_iU_j = \delta_{ij}U_j.
\]

We have not yet said anything about the representation of \( \mathcal{A} \). We take \( \mathcal{H} = L^2(T^2) \), with its basis \( |n_1, n_2\rangle, n_k \in \mathbb{Z} \). The representation (that is also used for the construction of the spectral triple...) is then given by
\[
U_1|n_1, n_2\rangle = \lambda^{n_2}|n_1 + 1, n_2\rangle
\]
\[
U_2|n_1, n_2\rangle = |n_1, n_2 + 1\rangle.
\]

Note that this representation possesses a cyclic, separating vector, namely \( |0, 0\rangle \). According to the Tomita-Takesaki-Theorem there then exists an antiisomorphism \( J \) from \( \mathcal{A} \) to its commutant \( \mathcal{A}' \), which is therefore isomorphic to "opposite algebra" \( \mathcal{A}^o \), i.e. \( \mathcal{A} \) equipped with the reversed product \( ab^o(ba)^o \). Explicitly one finds that \( \mathcal{A}' = \mathcal{A}^o \) is generated by
\[
U_1^o|n_1, n_2\rangle = |n_1 + 1, n_2\rangle
\]
\[
U_2^o|n_1, n_2\rangle = \lambda^{n_1}|n_1, n_2 + 1\rangle.
\]

In the commutative case (\( \lambda = 1 \)) obviously \( \mathcal{A} = \mathcal{A}' \). However, for irrational \( \theta \) the situation is completely different. Then the center of \( \mathcal{A} \) is trivial, and thus
\[
\mathcal{A} \cap \mathcal{A} = \mathbb{C} \, \mathbb{1}.
\]

It is therefore natural to replace the algebra \( \mathcal{A} \) by the larger algebra \( \mathcal{A} \otimes \mathcal{A}' \). The resulting algebra has trivial center. We therefore need a reasonable candidate for the uncertainty relation. Before we come to that, it is helpful to make the following observation:

The standard derivations \( \delta_k \) are represented on \( \mathcal{H} \) as
\[
-i\delta_k|n_1, n_2\rangle = n_k|n_1, n_2\rangle.
\]

One then calculates:
\[
[-i\delta_k, U_1^o] = \delta_{kl}U_1^o
\]

The algebra \( \mathcal{A}^o \) does possess, of course, the same derivations as \( \mathcal{A} \), i.e. only inner derivatives and complex linear combinations of the standard derivations \( \delta_k \).
Proposition 5.1. Assume that $\frac{1}{|\lambda|} = O(n^k)$ for some $k$ (Note, that this is satisfied by generic $\lambda$, while those $\lambda$ not satisfying this condition are of measure zero). Then any derivation $\delta : A \rightarrow A \otimes A^o$ from $A$ into the $A$-bimodule $A \otimes A^o$ can be uniquely decomposed into

$$\delta := c^i \delta_i + \tilde{\delta},$$

(33)

where $c^i$ are in $A^o$, $\delta_i$ are the standard derivations on $A \cong A \otimes 1$ and $\tilde{\delta}$ is an inner derivation on $A \otimes A^o$.

proof: We extend and follow closely the proof given in [11].

Any derivation $\delta$ can be given by its action on the generators $U_i$:

$$\delta(U_1) = \sum_{n,m,p,q} c^1_{mnpq} U_1^{n+1} U_2^m U_1^{op} U_2^q$$

(34)

$$\delta(U_2) = \sum_{n,m,p,q} c^2_{mnpq} U_1^n U_2^{m+1} U_1^{op} U_2^q$$

(35)

with the coefficients $c^i_{mnpq}$ being in Schwartz space and satisfying consistency relations due to the commutation relations between $U_1$ and $U_2$:

$$c^1_{mnpq}(\lambda^{-n} - 1) + c^2_{mnpq}(\lambda^{-m} - 1) = 0$$

(36)

An inner derivation of $A \otimes A^o$ is formally given by the commutator with

$$B = \sum_{m,n,p,q} d_{mnpq} U_1^n U_2^m U_1^{op} U_2^q,$$

where the coefficients $d_{mnpq}$ have to satisfy the following to match (34):

$$d_{mnpq} \begin{cases} 
  c^1_{mnpq} & \text{if } m \neq 0 \\
  c^2_{mnpq} & \text{if } n \neq 0
\end{cases}$$

(37)

Thus, an inner derivation can provide for all coefficients except for $c^i_{00pq}$ which has in this case to be set to be zero,

$$c^i_{00pq} = 0.$$

However, these are exactly given by linear combinations of the standard derivations with coefficients $U_1^{op} U_2^q \in A^o$. Thus, any derivation can be uniquely decomposed into a combination of standard ones with coefficients in $A^o$ and into an inner part.

The convergences of series in the formal argument above can be checked following [11].

From the strong NC-uncertainty condition we know that the derivation $[\bullet, [b^o, H]] : A \rightarrow A \otimes A^o$ can be decomposed as above:

$$[\bullet, [b^o, H]] = d^i(b^o) \delta_i + [\bullet, A_0], \quad \text{with } A_0 \in A \otimes A^o, d^i(b^o) \in A^o.$$  

(38)
It is easily checked that \( d^i : \mathcal{A}^o \to \mathcal{A}^o \) is a derivation and can be decomposed accordingly:

\[
d^i = c^{ij} \delta_j + [\bullet, A^i] \quad \text{with} \quad A^i \in \mathcal{A} \oplus \mathcal{A}^o, \quad c^{ij} \in \mathbb{C}. \tag{39}
\]

Thus, the general Hamiltonian is given by

\[
H = \frac{1}{2} \sum_{kl} c_{kl} (\delta_k - A_k)(\delta_l - A_l) + A_0,
\]

where \( c_{kl} \in \mathbb{C}, \ A_k \in \mathcal{A} \oplus \mathcal{A}^o \) and \( A_0 \in \mathcal{A} \otimes \mathcal{A}^o \). We would first like to point out, that the above Hamiltonian coincides (if \( A_0 = 0 \)) with the most general Laplace operator, that one get from Connes’ notion of spectral triples: If one takes the square of the most general Dirac-Operator on the noncommutative torus [10] one obtains precisely the above. Then, however one would have have to assume that \( H \) commutes with \( J \).

More explicitly, the Dirac-Operator, acting on the space \( L(T^2) \otimes \mathbb{C}^2 \) is given as

\[
D = \sum_{i=1}^2 \sigma^i (\delta_i - a_i - a^o_i)
\]

where \( \sigma^i \) denote selfadjoint two by two matrices such that \( \sigma^i \sigma^j + \sigma^j \sigma^i = 2c_{ij} \) and the \( a_i \) are selfadjoint algebra elements. One then has obviously \( D^2 = H \) with \( A_k = a_k + a^o_k \), as required by commutation with \( J \).

At first sight, it may appear as if this Dirac-operator describes the flat metric \( c_{ij} \) on the noncommutative torus. Yet, in view of Connes distance formula for two states \( \chi, \varphi \) on \( \mathcal{A} \):

\[
d(\chi, \varphi) = \sup_{\|D[a]\| \leq 1} \{|\chi(a) - \varphi(a)|\}
\]

this is not necessarily true: As the algebra elements \( a_i \) do not commute with algebra elements, they do affect the set over which the supremum is to be taken, and thus the distance itself. Thus the metric is affected by the choice of the gauge potential \( A_k \).

This, however, is not unexpected. The gauge potential refers to inner automorphisms of the algebra, while the metric (or rather the Levi-Civita connection) gauges the outer automorphisms. But the noncommutative torus essentially has only inner automorphisms, and hence one would expect that curvature only arises from the gauge potential. Note however, that the freedom in the choice of \( A_k \) is very large – unlike the commutative case, where it would in fact be zero if it had to commute with \( J \).

6 Conclusions and Outlook

In this paper, we have presented an axiomatic framework for nonrelativistic quantum mechanics on noncommutative configuration spaces. One of the main virtues of our approach as compared to approaches which work with the “canonical momentum” \( p_k \) which is usually taken as translation operators, and thus in particular as derivations on the algebra \( \mathcal{A} \), is that it is not restricted to algebras which possess such derivations. Moreover, the velocities \( \dot{x}_k \), that we laid emphasis on, do have a clear physical meaning
and one may immediately propose experiments to measure them, in contrast to the momentum operators $p_k$ which are usually measured via the velocities.

More importantly, our approach does not depend on the choice of coordinates. As we have pointed out, this is of particular importance on noncommutative spaces.

As we have shown in several examples, our generalized uncertainty relations do, despite their rather general appearance, pose tight restrictions on the dynamics of such a theory. In particular, on the Moyal deformed plane we find only such Hamiltonians which would correspond to a flat metric. (The same would hold in three dimensions). Our uncertainty relation is the equivalent to the usual description via the canonical momenta $[p_k, x_l] = i\hbar \delta_{kl}$. They are, however, valid in any coordinate system. We should stress that modified uncertainty relations of the form

$$[x_k, \dot{x}_l] = i\hbar \delta_{kl} + \mathcal{O}(\frac{1}{p_k^2})$$

which are found for instance in string theory can also be discussed in our framework, as the modification of the usual uncertainty relation is given by a compact operator. However, we have not found a convincing route to the most general Hamiltonian for such a relation. They are thus left for future work. More importantly however, we shall then try to set up scattering theory in our framework, as that might lead to definite experimental predictions.

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References

[1] F.J. Dyson *Feynman’s proof of the Maxwell equations* Amer. J. Phys. 58 (1990) 209

[2] J.F. Carinena, L.A. Ibort, G. Marmo, A. Stern *The Feynman problem and the inverse problem for Poisson dynamics* Phys. Rep. 263 (1995) 153

[3] C.R. Lee *The Feynman-Dyson proof of the gauge field equation* Phys. Lett. 148 A (1990) 146

[4] S. Tanimura *Relativistic Generalization and Extension to Non-Abelian Gauge Theory of Feynman’s Proof of the Maxwell Equations* Ann. Phys. 220 (1992) 229

[5] M. Paschke *Time evolutions in Quantum mechanics and (Lorentzian) geometry*, Preprint, ArXiv: math-ph/0301040
[6] M. Paschke and R. Verch *Local covariant quantum field theory over spectral geometries* Class. Quant. Grav. 21, No. 23 (2004) 5299-5316

[7] T. Kopf, M. Paschke *A spectral quadruple for de Sitter space* J. Math. Phys. 43 (2002) 818

[8] A. Connes *Gravity coupled with matter and the foundation of noncommutative geometry* Comm. Math. Phys. 182 (1996) 155-176

[9] Mario Paschke, Andrzej Sitarz *Discrete Spectral Triples and Their Symmetries*, Journal of Mathematical Physics 39 (1998) 6191-6205.

[10] H. Figueroa, J. C. Gracia-Bondía, J. C. Várilly *Elements of noncommutative geometry*, Birkhäuser Boston (2001)

[11] O. Bratteli, G. A. Elliott, P. E. T. Jorgensen *Decomposition of unbounded derivations into invariant and approximately inner parts*. J. Reine Angew. Math. 346 (1984), 166-193

[12] T. Kopf, M. Paschke *in preparation*