SOME NEW INEQUALITIES OF HERMITE-HADAMARD TYPE
FOR h-CONVEX FUNCTIONS ON THE CO-ORDINATES VIA
FRACTIONAL INTEGRALS.

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Abstract. By making use of the identity obtained by Sarıkaya, some new
Hermite-Hadamard type inequalities for h-convex functions on the co-ordinates
via fractional integrals are established. Our results have some relationships
with the results of Sarıkaya[16].

1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function defined on the interval $I$ of real numbers
and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}.$$ 

holds, the double inequality is well known in the literature as Hermite-Hadamard
inequality [6].

Both inequalities hold in the reversed direction if $f$ is concave. For recent results,
generalizations and new inequalities related to the Hermite-Hadamard inequality
see ( [4], [12], [14], [18]).

The classical Hermite-Hadamard inequality provides estimates of the mean value
of a continuous convex function $f : [a, b] \rightarrow \mathbb{R}$

Let us now consider a bidimensional interval $\Delta := [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$
and $c < d$. A mapping $f : \Delta \rightarrow \mathbb{R}$ is said to be convex on if the following inequality:

$$f(tx + (1-t)z, ty + (1-t)w) \leq tf(x, y) + (1-t)f(z, w)$$

holds, for all $(x, y), (z, w) \in \Delta$ and $t \in [0, 1]$. If the inequality reversed then $f$ is
said to be concave on $\Delta$.

A function $f : \Delta \rightarrow \mathbb{R}$ is said to be on the co-ordinates on $\Delta$ if the partial
mappings $f_y : [a, b] \rightarrow \mathbb{R}, f_y(u) = f(u, y)$ and $f_x : [c, d] \rightarrow \mathbb{R}, f_x(v) = f(x, v)$ are
convex where defined for all $x \in [a, b]$ and $y \in [c, d]$. (see [5])

A formal definition for co-ordinated convex function may be stated as follows:

Definition 1. A function $f : \Delta \rightarrow \mathbb{R}$ will be called co-ordinated convex on $\Delta$, for
all $t, k \in [0, 1]$ and $(x, u), (y, w) \in \Delta$, if the following inequality holds:

$$f(tx + (1-t)y, ku + (1-k)w) \leq t(kf(x, u) + k(1-t)f(y, u) + t(1-k)f(x, w) + (1-t)(1-k)f(y, w))$$
Clearly, every convex mapping is convex on the co-ordinates, but the converse is not generally true ([9]). Some interesting and important inequalities for convex functions on the co-ordinates can be found in ([9], [10], [17]).

In [1], Alomari and Darus established the following definition of \( s \)-convex function in the second sense on co-ordinates.

**Definition 2.** Consider the bidimensional interval \( \Delta := [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d \). The mapping \( f : \Delta \to \mathbb{R} \) is \( s \)-convex on \( \Delta \) if

\[
f(\lambda x + (1 - \lambda) z, \lambda y + (1 - \lambda) w) \leq \lambda^s f(x, y) + (1 - \lambda)^s f(z, w)
\]

holds for all \( (x, y), (z, w) \in \Delta \) with \( \lambda \in [0, 1] \), and for some fixed \( s \in (0, 1] \).

A function \( f : \Delta \to \mathbb{R} \) is \( s \)-convex on \( \Delta \) is called \( \alpha \)-coordinated \( s \)-convex on \( \Delta \) if for all \( x \in [a, b] \) and \( y \in [c, d] \) with some fixed \( s \in (0, 1] \),

\[
\text{the co-ordinates on } \Delta \text{ is contained in the class of non-negative convex (concave) functions on the co-ordinates on } \Delta \text{ and if } h \text{ are } h \text{-convex on } \Delta \text{ belong to the class } \text{SX}(h, \Delta). \]

In [11], Latif and Alamori give the notion of \( h \)-convexity of a function \( f \) on a rectangle from the plane \( \mathbb{R}^2 \) and \( h \)-convexity on the co-ordinates on a rectangle from the plane \( \mathbb{R}^2 \) as follows

**Definition 3.** Let us consider a bidimensional interval \( \Delta := [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d \). Let \( h : J \subseteq \mathbb{R} \to \mathbb{R} \) be a positive function. A mapping \( f : \Delta \to \mathbb{R} \) is \( h \)-convex on \( \Delta \) if \( f \) is non-negative and if the following inequality:

\[
f(\lambda x + (1 - \lambda) z, \lambda y + (1 - \lambda) w) \leq h(\lambda) f(x, y) + h(1 - \lambda) f(z, w)
\]

holds, for all \( (x, y), (z, w) \in \Delta \) with \( \lambda \in (0, 1) \). Let us denote this class of functions by \( \text{SV}(h, \Delta) \). The function \( f \) is said to be \( h \)-concave if the inequality reversed. We denote this class of functions by \( \text{SV}(h, \Delta) \).

A function \( f : \Delta \to \mathbb{R} \) is said to be \( h \)-convex on the co-ordinates on \( \Delta \) if the partial mappings \( f_y : [a, b] \to \mathbb{R} \) and \( f_x : [c, d] \to \mathbb{R} \) are \( h \)-convex where defined for all \( x \in [a, b] \) and \( y \in [c, d] \). A formal definition of \( h \)-convex functions may also be stated as follows:

**Definition 4.** [11] A function \( f : \Delta \to \mathbb{R} \) is said to be \( h \)-convex on the co-ordinates on \( \Delta \), if the following inequality:

\[
(1.1)
\]

\[
f(tx + (1 - t) y, ku + (1 - k) w) \\
\leq h(t) h(k) f(x, u) + h(k) h(1 - t) f(y, u) \\
+ h(t) h(1 - k) f(x, w) + h(1 - t) h(1 - k) f(y, w)
\]

holds for all \( t, k \in [0, 1] \) and \( (x, u), (x, w), (y, u), (y, w) \in \Delta \).

Obviously, if \( h(\alpha) = \alpha \), then all the non-negative convex (concave) functions on \( \Delta \) belong to the class \( \text{SX}(h, \Delta) \) (\( \text{SV}(h, \Delta) \)) and if \( h(\alpha) = \alpha^s \), where \( s \in (0, 1) \), then the class of \( s \)-convex on \( \Delta \) belong to the class \( \text{SX}(h, \Delta) \). Similarly we can say that if \( h(\alpha) = \alpha \), then the class of non-negative convex (concave) functions on the co-ordinates on \( \Delta \) is contained in the class of \( h \)-convex (concave) functions on the co-ordinates on \( \Delta \) and if \( h(\alpha) = \alpha^s \), where \( s \in (0, 1) \), then the class of \( s \)-convex functions on the co-ordinates on \( \Delta \) is contained in the class of \( h \)-convex functions on the co-ordinates on \( \Delta \).
In the following we will give some necessary definitions which are used further in this paper. More details, one can consult [7], [8], [13].

**Definition 5.** Let \( f \in L^1([a, b]) \). The Riemann-Liouville integrals \( J_{a^+}^\alpha f \) and \( J_{b^-}^\alpha f \) of order \( \alpha > 0 \) and \( a \geq 0 \) are defined by

\[
J_{a^+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^a (x - t)^{\alpha - 1} f(t) \, dt, \quad x > a
\]

and

\[
J_{b^-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t - x)^{\alpha - 1} f(t) \, dt, \quad x < b
\]

respectively. Here \( \Gamma(\alpha) \) is the Gamma function, \( J_{a^+}^0 f = J_{b^-}^0 f = f \).

**Definition 6.** Let \( f \in L^1([a, b] \times [c, d]) \). The Riemann-Liouville integrals \( J_{a^+, c^+}^{\alpha, \beta} f \), \( J_{a^+, d^-}^{\alpha, \beta} f \), \( J_{b^-, c^+}^{\alpha, \beta} f \) and \( J_{b^-, d^-}^{\alpha, \beta} f \) of order \( \alpha, \beta > 0 \) with \( a, c \geq 0 \) are defined by

\[
J_{a^+, c^+}^{\alpha, \beta} f(x, y) = \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_a^x \int_c^y (x - t)^{\alpha - 1} (y - s)^{\beta - 1} f(t, s) \, ds dt, \quad x > a, y > c
\]

\[
J_{a^+, d^-}^{\alpha, \beta} f(x, y) = \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_a^x \int_c^d (x - t)^{\alpha - 1} (s - y)^{\beta - 1} f(t, s) \, ds dt, \quad x > a, y < d
\]

\[
J_{b^-, c^+}^{\alpha, \beta} f(x, y) = \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_x^b \int_c^y (t - x)^{\alpha - 1} (y - s)^{\beta - 1} f(t, s) \, ds dt, \quad x < b, y > c
\]

\[
J_{b^-, d^-}^{\alpha, \beta} f(x, y) = \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_x^b \int_c^d (t - x)^{\alpha - 1} (s - y)^{\beta - 1} f(t, s) \, ds dt, \quad x < b, y < d
\]

respectively. Here, \( \Gamma \) is the Gamma function, \( J_{a^+, c^+}^{0, 0} f = J_{a^+, d^-}^{0, 0} f = J_{b^-, c^+}^{0, 0} f = J_{b^-, d^-}^{0, 0} f = f \).

For some recent results connected with fractional integral inequalities see [2], [3], [15], [19].

In (145), Sariýaka establish the following inequalities of Hadamard’s type for co-ordinated convex mapping on a rectangle from the plane \( \mathbb{R}^2 \):
Theorem 1. Let \( f : \Delta \subset \mathbb{R}^2 \to \mathbb{R} \) be co-ordinated convex on \( \Delta := [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( 0 \leq a < b, 0 \leq c < d \) and \( f \in L_1(\Delta) \). Then one has the inequalities:

\[
(1.2) \quad f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \leq \frac{\Gamma (\alpha + 1) \Gamma (\beta + 1)}{4 (b - a)^\alpha (d - c)^\beta} \left[ J^\alpha_{a^+,c^+} f(b, d) + J^\beta_{a^-,d^-} f(b, c) + J^\alpha_{b^-,c^+} f(a, d) + J^\beta_{b^+,d^-} f(a, c) \right] \leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}.
\]

Theorem 2. Let \( f : \Delta \subset \mathbb{R}^2 \to \mathbb{R} \) be a partial differentiable mapping on \( \Delta := [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( 0 \leq a < b, 0 \leq c < d \). If \( \left| \frac{\partial^2 f}{\partial t \partial k} \right| \) is a convex function on the co-ordinates on \( \Delta \), then one has the inequalities:

\[
(1.3) \quad \left| \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} \right| + \frac{\Gamma (\alpha + 1) \Gamma (\beta + 1)}{4 (b - a)^\alpha (d - c)^\beta} \times \left[ J^\alpha_{a^+,c^+} f(b, d) + J^\beta_{a^-,d^-} f(b, c) + J^\alpha_{b^-,c^+} f(a, d) + J^\beta_{b^+,d^-} f(a, c) \right] = A \leq \frac{(b - a)(d - c)}{4 (\alpha + 1)(\beta + 1)} \times \left( \left| \frac{\partial^2 f}{\partial k \partial t} (a, c) \right| + \left| \frac{\partial^2 f}{\partial k \partial t} (a, d) \right| + \left| \frac{\partial^2 f}{\partial k \partial t} (b, c) \right| + \left| \frac{\partial^2 f}{\partial k \partial t} (b, d) \right| \right)
\]

where

\[
A = \frac{\Gamma (\beta + 1)}{4 (d - c)^\beta} \left[ J^\beta_{c^+} f(a, d) + J^\beta_{c^-} f(b, d) + J^\beta_{d^-} f(a, c) + J^\beta_{d^+} f(b, c) \right] + \frac{\Gamma (\alpha + 1)}{4 (b - a)^\alpha} \left[ J^\alpha_{a^+} f(b, c) + J^\alpha_{a^-} f(b, d) + J^\alpha_{b^-} f(a, c) + J^\alpha_{b^+} f(a, d) \right].
\]

Theorem 3. Let \( f : \Delta \subset \mathbb{R}^2 \to \mathbb{R} \) be a partial differentiable mapping on \( \Delta := [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( 0 \leq a < b, 0 \leq c < d \). If \( \left| \frac{\partial^2 f}{\partial t \partial k} \right|^q, q > 1, \) is a convex function
on the co-ordinates on $\Delta$, then one has the inequalities:

\[
\begin{align*}
&\left| f(a, c) + f(a, d) + f(b, c) + f(b, d) \right|/4 \\
&+ \left\{ \frac{\Gamma(\alpha + 1) \Gamma(\beta + 1)}{4(b-a)^{\alpha}(d-c)^{\beta}} \times \left[ J_{a^+,c^+}^{\alpha,\beta} f(b,d) + J_{a^+,d^-}^{\alpha,\beta} f(b,c) + J_{b^-,c^+}^{\alpha,\beta} f(a,d) + J_{b^-,d^-}^{\alpha,\beta} f(a,c) \right] \right\} - A \leq (b-a)(d-c)\left(\frac{1}{4}\right)^{\frac{1}{p}} \frac{1}{(\alpha p + 1)(\beta p + 1)}^{\frac{1}{q}} \\
&\times \left( \left| \frac{\partial^2 f}{\partial k \partial t} (a,c) \right|^q + \left| \frac{\partial^2 f}{\partial k \partial t} (a,d) \right|^q + \left| \frac{\partial^2 f}{\partial k \partial t} (b,c) \right|^q + \left| \frac{\partial^2 f}{\partial k \partial t} (b,d) \right|^q \right)^{\frac{1}{q}} \\
\end{align*}
\]

where

\[
A = \frac{\Gamma(\beta + 1)}{4(d-c)^{\beta}} \left[ J_{c^+}^{\beta} f(a,d) + J_{c^+}^{\beta} f(b,d) + J_{d^-}^{\beta} f(a,c) + J_{d^-}^{\beta} f(b,c) \right] \\
+ \frac{\Gamma(\alpha + 1)}{4(b-a)} \left[ J_{a^+}^{\alpha} f(b,c) + J_{a^+}^{\alpha} f(b,d) + J_{b^-}^{\alpha} f(a,c) + J_{b^-}^{\alpha} f(a,d) \right]
\]

and $\frac{1}{p} + \frac{1}{q} = 1$.

In order to prove our main results we need the following lemma (see [16]).

**Lemma 1.** Let $f : \Delta \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ be a partial differentiable mapping on $\Delta := [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $0 \leq a < b$, $0 \leq c < d$. If $\frac{\partial^2 f}{\partial k \partial t} \in L(\Delta)$, then the following equality holds:

\[
\begin{align*}
&\left| f(a, c) + f(a, d) + f(b, c) + f(b, d) \right|/4 \\
&+ \left\{ \frac{\Gamma(\alpha + 1) \Gamma(\beta + 1)}{4(b-a)^{\alpha}(d-c)^{\beta}} \times \left[ J_{a^+,c^+}^{\alpha,\beta} f(b,d) + J_{a^+,d^-}^{\alpha,\beta} f(b,c) + J_{b^-,c^+}^{\alpha,\beta} f(a,d) + J_{b^-,d^-}^{\alpha,\beta} f(a,c) \right] \right\} \\
&- \frac{\Gamma(\beta + 1)}{4(d-c)^{\beta}} \left[ J_{c^+}^{\beta} f(a,d) + J_{c^+}^{\beta} f(b,d) + J_{d^-}^{\beta} f(a,c) + J_{d^-}^{\beta} f(b,c) \right] \\
&- \frac{\Gamma(\alpha + 1)}{4(b-a)} \left[ J_{a^+}^{\alpha} f(b,c) + J_{a^+}^{\alpha} f(b,d) + J_{b^-}^{\alpha} f(a,c) + J_{b^-}^{\alpha} f(a,d) \right]
\end{align*}
\]
\[ (b - a) (d - c) \frac{1}{4} \left\{ \int_0^1 \int_0^1 t^\alpha k^\beta \frac{\partial^2 f}{\partial t \partial k} (ta + (1 - t) b, kc + (1 - k) d) \, dk \, dt \right. \\
- \int_0^1 \int_0^1 (1 - t)^\alpha k^\beta \frac{\partial^2 f}{\partial t \partial k} (ta + (1 - t) b, kc + (1 - k) d) \, dk \, dt \\
- \int_0^1 \int_0^1 t^\alpha (1 - k)^\beta \frac{\partial^2 f}{\partial t \partial k} (ta + (1 - t) b, kc + (1 - k) d) \, dk \, dt \\
- \int_0^1 \int_0^1 (1 - t)^\alpha (1 - k)^\beta \frac{\partial^2 f}{\partial t \partial k} (ta + (1 - t) b, kc + (1 - k) d) \, dk \, dt \right\}. \]

2. MAIN RESULTS

**Theorem 4.** Let \( f : \Delta \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) be \( h \)-convex function on the co-ordinates on \( \Delta := [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) and \( f \in L_2(\Delta) \). The one has the inequalities:

(2.1)

\[ f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \leq \left[ h \left( \frac{1}{2} \right) \right]^2 \Gamma (\alpha + 1) \Gamma (\beta + 1) \frac{(b - a)\alpha (d - c)^\beta}{(b - a)^\alpha (d - c)^\beta} \times \left[ J_{a+,c+}^{\alpha,\beta} f (b, d) + J_{a+,d-}^{\alpha,\beta} f (b, c) + J_{b-,c+}^{\alpha,\beta} f (a, d) + J_{b-,d-}^{\alpha,\beta} f (a, c) \right] \]

\[ \leq \left[ h \left( \frac{1}{2} \right) \right]^2 \alpha \beta \left[ f (a, c) + f (a, d) + f (b, c) + f (b, d) \right] \times \left[ \int_0^1 \int_0^1 t^{\alpha - 1} k^{\beta - 1} \, dt \, dk \right]. \]

**Proof.** According to (1.1) with \( x = t_1 a + (1 - t_1) b, y = (1 - t_3) a + t_1 b, u = k_1 c + (1 - k_1) d, w = (1 - k_1) c + k_1 d \) and \( t = k = \frac{1}{2} \), we find that

(2.2)

\[ f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \leq \left[ h \left( \frac{1}{2} \right) \right]^2 \times \left[ f (t_1 a + (1 - t_1) b, k_1 c + (1 - k_1) d) + f (t_1 a + (1 - t_1) b, (1 - k_1) c + k_1 d) \\
+ f ((1 - t_1) a + t_1 b, k_1 c + (1 - k_1) d) + f ((1 - t_1) a + t_1 b, (1 - k_1) c + k_1 d) \right]. \]
Thus, multiplying both sides of (2.2) by $t_1^{\alpha - 1}k_1^{\beta - 1}$, then by integrating with respect to $(t_1, k_1)$ on $[0, 1] \times [0, 1]$, we obtain

\[
\frac{1}{\alpha \beta}\int \left( \frac{a + b}{2} , \frac{c + d}{2} \right)
\leq \left[ h \left( \frac{1}{2} \right) \right]^2 \times \left[ \int_0^1 \int_0^1 t_1^{\alpha - 1}k_1^{\beta - 1} \right.
\times \left[ f(t_1 a + (1 - t_1) b, k_1 c + (1 - k_1) d) + f(t_1 a + (1 - t_1) b, (1 - k_1) c + k_1 d) + f((1 - t_1) a + t_1 b, k_1 c + (1 - k_1) d) \right]
\left. dk_1 ds_1. \right]
\]

Using the change of the varible, we get

\[
f \left( \frac{a + b}{2} , \frac{c + d}{2} \right)
\leq \left[ h \left( \frac{1}{2} \right) \right]^2 \frac{\alpha \beta}{(b - a)^\alpha (d - c)^\beta} \left\{ \int_a^b \int_c^d (b - x)^\alpha (d - y)^\beta f (x, y) dy dx + \int_a^b \int_c^d (b - x)^\alpha (y - c)^\beta f (x, y) dy dx + \int_a^b \int_c^d (x - a)^\alpha (d - y)^\beta f (x, y) dy dx + \int_a^b \int_c^d (x - a)^\alpha (y - c)^\beta f (x, y) dy dx \right\}
\]

which the first inequality is proved.

For the proof of second inequality (2.1), we first note that if $f$ is a $h$-convex function on $\Delta$, then, by using (1.1) with $x = a, y = b, u = c, w = d$, it yields

\[
f (ta + (1 - t) b, k_1 c + (1 - k_1) d)
\leq h (t) h (k) f (a, c) + h (t) h (1 - k) f (a, d) + h (1 - t) h (k) f (b, c) + h (1 - t) h (1 - k) f (b, d)
\]

\[
f ((1 - t) a + tb, k_1 c + (1 - k_1) d)
\leq h (1 - t) h (k) f (a, c) + h (1 - t) h (1 - k) f (a, d) + h (t) h (k) f (b, c) + h (t) h (1 - k) f (b, d)
\]

\[
f (ta + (1 - t) b, (1 - k) c + kd)
\leq h (t) h (1 - k) f (a, c) + h (t) h (k) f (a, d) + h (1 - t) h (1 - k) f (b, c) + h (1 - t) h (k) f (b, d)
\]
Here, using the change of the variable we have

\[
f ((1 - t) a + t b, (1 - k) c + k d) \\
\leq h (1 - t) h (1 - k) f (a, c) + h (1 - t) h (k) f (a, d) \\
+h (t) h (1 - k) f (b, c) + h (t) h (k) f (b, d) .
\]

By adding these inequalities we have

\[
(2.3)
\]

\[
f (t a + (1 - t) b, k c + (1 - k) d) + f ((1 - t) a + t b, k c + (1 - k) d)
\]

\[
+f (t a + (1 - t) b, (1 - k) c + k d) + f ((1 - t) a + t b, (1 - k) c + k d)
\]

\[
\leq [f (a, c) + f (b, c) + f (a, d) + f (b, d)]
\]

\[
\times \left[ h (t) h (k) + h (t) h (1 - k) + h (1 - t) h (k) + h (1 - t) h (1 - k) \right].
\]

Then, multiplying both sides of (2.3) by \( t^{\alpha - 1} k^{\beta - 1} \) and integrating with respect to \((t, k)\) over \([0, 1] \times [0, 1]\) , we get

\[
\int_0^1 \int_0^1 t^{\alpha - 1} k^{\beta - 1}
\]

\[
\times [f (t a + (1 - t) b, k c + (1 - k) d) + f ((1 - t) a + t b, k c + (1 - k) d)
\]

\[
+f (t a + (1 - t) b, (1 - k) c + k d) + f ((1 - t) a + t b, (1 - k) c + k d)] d k d t
\]

\[
\leq [f (a, c) + f (b, c) + f (a, d) + f (b, d)]
\]

\[
\times \left\{ \int_0^1 \int_0^1 t^{\alpha - 1} k^{\beta - 1}
\right\}
\]

\[
\times [h (t) h (k) + h (t) h (1 - k) + h (1 - t) h (k) + h (1 - t) h (1 - k)] d k d t \}.
\]

Here, using the change of the variable we have

\[
\left[ h \left( \frac{1}{2} \right) \right]^2 \Gamma (\alpha + 1) \Gamma (\beta + 1) \frac{(b - a)\alpha (d - e)^\beta}{(b - a)\alpha (d - e)^\beta}
\]

\[
\times \left[ J_{a+,c+}^{\alpha,\beta} f (b, d) + J_{a+,c-}^{\alpha,\beta} f (b, c) + J_{b-,d-}^{\alpha,\beta} f (a, d) + J_{b-,d+}^{\alpha,\beta} f (a, c) \right]
\]

\[
\leq \left[ h \left( \frac{1}{2} \right) \right]^2 \alpha \beta [f (a, c) + f (a, d) + f (b, c) + f (b, d)]
\]

\[
\times \left\{ \int_0^1 \int_0^1 t^{\alpha - 1} k^{\beta - 1}
\right\}
\]

\[
\times [h (t) h (k) + h (t) h (1 - k) + h (1 - t) h (k) + h (1 - t) h (1 - k)] d k d t \}.
\]

The proof is completed. \( \Box \)
Remark 1. If we take $h(\alpha) = \alpha$ in Theorem [3], then the inequality (2.7) becomes the inequality (1.3) of Theorem [4].

Corollary 1. If we take $h(\alpha) = \alpha^s$ in Theorem [4], we have the following inequality:

$$f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \leq \left( \frac{1}{2} \right)^{2s} \frac{\Gamma (\alpha + 1) \Gamma (\beta + 1)}{(b - a)^\alpha (d - c)^\beta} \times \left[ J_{a^+,c^+}^{\alpha,\beta} f (b, d) + J_{a^-,d^-}^{\alpha,\beta} f (b, c) + J_{b^-,c^+}^{\alpha,\beta} f (a, d) + J_{b^+,d^-}^{\alpha,\beta} f (a, c) \right]$$

$$\leq \left( \frac{1}{2} \right)^{2s} \alpha \beta \left[ f (a, c) + f (a, d) + f (b, c) + f (b, d) \right] \times \left( \frac{1}{\alpha + s} + B (\alpha, s + 1) \right) \left( \frac{1}{\beta + s} + B (\beta, s + 1) \right)$$

where $B$ is the Beta function,

$$B (x, y) = \int_0^1 t^{x-1} (1 - t)^{y-1} dt.$$  

Theorem 5. Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on $\Delta := [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $0 < a < b$, $0 < c < d$. If $\left| \frac{\partial^2 f}{\partial t \partial k} \right|$ is a h-convex function on the co-ordinates on $\Delta$, then one has the inequalities:

$$\left| \frac{f (a, c) + f (a, d) + f (b, c) + f (b, d)}{4} \right|$$

$$+ \left\{ \frac{\Gamma (\alpha + 1) \Gamma (\beta + 1)}{(b - a)^\alpha (d - c)^\beta} \times \left[ J_{a^+,c^+}^{\alpha,\beta} f (b, d) + J_{a^-,d^-}^{\alpha,\beta} f (b, c) + J_{b^-,c^+}^{\alpha,\beta} f (a, d) + J_{b^+,d^-}^{\alpha,\beta} f (a, c) \right] \right\} - A$$

$$\leq \frac{(b - a) (d - c)}{4} \times \left\{ \left| \frac{\partial^2 f}{\partial t \partial k} (a, c) \right| \int_0^1 \int_0^1 (t^\alpha + (1 - t)^\alpha) \left( k^\beta + (1 - k)^\beta \right) h (t) h (k) dk dt \right.$$

$$+ \left| \frac{\partial^2 f}{\partial t \partial k} (b, c) \right| \int_0^1 \int_0^1 (t^\alpha + (1 - t)^\alpha) \left( k^\beta + (1 - k)^\beta \right) h (1 - t) h (k) dk dt \right.$$

$$+ \left| \frac{\partial^2 f}{\partial t \partial k} (a, d) \right| \int_0^1 \int_0^1 (t^\alpha + (1 - t)^\alpha) \left( k^\beta + (1 - k)^\beta \right) h (t) h (1 - k) dk dt \right.$$

$$+ \left| \frac{\partial^2 f}{\partial t \partial k} (b, d) \right| \int_0^1 \int_0^1 (t^\alpha + (1 - t)^\alpha) \left( k^\beta + (1 - k)^\beta \right) h (1 - t) h (1 - k) dk dt \right\}.$$
where

\[
A = \frac{\Gamma (\beta + 1)}{4(d-c)^\beta} \left[ J^\beta_{c^+} f (a, d) + J^\beta_{c^-} f (b, d) + J^\beta_{d^+} f (a, c) + J^\beta_{d^-} f (b, c) \right] + \frac{\Gamma (\alpha + 1)}{4(b-a)^\alpha} \left[ J^\alpha_{a^+} f (b, c) + J^\alpha_{b^-} f (b, d) + J^\alpha_{b^-} f (a, c) + J^\alpha_{d^-} f (a, d) \right].
\]

Proof. From Lemma 1, we have

\[
\left| f (a, c) + f (a, d) + f (b, c) + f (b, d) \right| + \frac{\Gamma (\alpha + 1) \Gamma (\beta + 1)}{4(b-a)^\alpha (d-c)^\beta} \times \left[ J^\alpha_{a^+} f (b, d) + J^\alpha_{b^-} f (b, c) + J^\alpha_{b^-} f (a, d) + J^\alpha_{d^-} f (a, c) \right] - A
\]

\[
\leq \frac{(b-a) (d-c)}{4} \left\{ \int_0^1 \int_0^1 t^\alpha k^\beta \left| \frac{\partial^2 f}{\partial t \partial k} (ta + (1-t) b, kc + (1-k) d) \right| dk dt + \int_0^1 \int_0^1 (1-t)^\alpha k^\beta \left| \frac{\partial^2 f}{\partial t \partial k} (ta + (1-t) b, kc + (1-k) d) \right| dk dt + \int_0^1 \int_0^1 t^\alpha (1-k)^\beta \left| \frac{\partial^2 f}{\partial t \partial k} (ta + (1-t) b, kc + (1-k) d) \right| dk dt - \int_0^1 \int_0^1 (1-t)^\alpha (1-k)^\beta \left| \frac{\partial^2 f}{\partial t \partial k} (ta + (1-t) b, kc + (1-k) d) \right| dk dt \right\}.
\]

Since \( \frac{\partial^2 f}{\partial t \partial k} \) is h-convex function on the co-ordinates on \( \Delta \), then one has:

\[
\left| f (a, c) + f (a, d) + f (b, c) + f (b, d) \right| + \frac{\Gamma (\alpha + 1) \Gamma (\beta + 1)}{4(b-a)^\alpha (d-c)^\beta} \times \left[ J^\alpha_{a^+} f (b, d) + J^\alpha_{b^-} f (b, c) + J^\alpha_{b^-} f (a, d) + J^\alpha_{d^-} f (a, c) \right] - A
\]
Corollary 2. If we take \( h(\alpha) = \alpha \) in Theorem 3 then the inequality (1.3) of Theorem 2 becomes the inequality (1.3) of Theorem 3.

Theorem 6. Let \( f : \Delta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R} \) be a partial differentiable mapping on \( \Delta := [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( 0 \leq a < b, \ 0 \leq c < d \). If \( \left| \frac{\partial^2 f}{\partial t \partial k} \right|^q, q > 1 \), is a \( h \)-convex function.
function on the co-ordinates on \( \Delta \), then one has the inequalities:

\[
\left| \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} \right| + \left\{ \frac{\Gamma (\alpha + 1) \Gamma (\beta + 1)}{4 (b - a)^\alpha (d - c)^\beta} \right\} \times \left[ J_{a^+, c^+}^\alpha f(b, d) + J_{a^+, d^-}^\alpha f(b, c) + J_{b^-, c^+}^\alpha f(a, d) + J_{b^-, d^-}^\alpha f(a, c) \right] \}
\]

\[
\leq \frac{(b - a) (d - c)}{[(\alpha p + 1) (\beta q + 1)]^\frac{1}{p}} \left[ \frac{\partial^2 f}{\partial k \partial t} (a, c) \right]^q \int_0^1 \int_0^1 h(t) h(k) \, dk \, dt + \frac{\partial^2 f}{\partial k \partial t} (a, d) \right|^q \int_0^1 \int_0^1 h(t) h(1 - k) \, dk \, dt + \frac{\partial^2 f}{\partial k \partial t} (b, c) \right|^q \int_0^1 \int_0^1 h(1 - t) h(k) \, dk \, dt + \frac{\partial^2 f}{\partial k \partial t} (b, d) \right|^q \int_0^1 \int_0^1 h(1 - t) h(1 - k) \, dk \, dt \]

where

\[
A = \frac{\Gamma (\beta + 1)}{4 (d - c)^\beta} \left[ J_{c^+}^\beta f(a, d) + J_{d^-}^\beta f(b, d) + J_{d^-}^\beta f(a, c) + J_{d^-}^\beta f(b, c) \right] + \frac{\Gamma (\alpha + 1)}{4 (b - a)^\alpha} \left[ J_{a^+}^\alpha f(b, c) + J_{a^+}^\alpha f(b, d) + J_{a^+}^\alpha f(a, c) + J_{a^+}^\alpha f(a, d) \right]
\]

and \( \frac{1}{p} + \frac{1}{q} = 1 \).

**Proof.** From Lemma [1], we have

\[
\left| \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} \right| + \left\{ \frac{\Gamma (\alpha + 1) \Gamma (\beta + 1)}{4 (b - a)^\alpha (d - c)^\beta} \right\} \times \left[ J_{a^+, c^+}^\alpha f(b, d) + J_{a^+, d^-}^\alpha f(b, c) + J_{b^-, c^+}^\alpha f(a, d) + J_{b^-, d^-}^\alpha f(a, c) \right] \}
\]
\[
\begin{align*}
&\leq \frac{(b-a)(d-c)}{4} \left\{ \int_0^1 \int_0^1 t^\alpha \kappa \beta \left| \frac{\partial^2 f}{\partial t \partial k} (ta + (1-t)b, kc + (1-k)d) \right| \, dk \, dt \\
&+ \int_0^1 \int_0^1 (1-t)^\alpha \kappa \beta \left| \frac{\partial^2 f}{\partial t \partial k} (ta + (1-t)b, kc + (1-k)d) \right| \, dk \, dt \\
&+ \int_0^1 \int_0^1 t^\alpha (1-k)^\beta \left| \frac{\partial^2 f}{\partial t \partial k} (ta + (1-t)b, kc + (1-k)d) \right| \, dk \, dt \\
&- \int_0^1 \int_0^1 (1-t)^\alpha (1-k)^\beta \left| \frac{\partial^2 f}{\partial t \partial k} (ta + (1-t)b, kc + (1-k)d) \right| \, dk \, dt \right\}.
\end{align*}
\]

By using the well known Hölder’s inequality for double integrals, we get

\[
\begin{align*}
&\left| f(a,c) + f(a,d) + f(b,c) + f(b,d) \right| \\
&+ \left\{ \Gamma(\alpha + 1) \Gamma(\beta + 1) \left[ \frac{\partial^2 f}{\partial t \partial k} (ta + (1-t)b, kc + (1-k)d) \right] \right\} - A \\
&\leq \frac{(b-a)(d-c)}{4} \left\{ \left( \int_0^1 \int_0^1 t^{-\alpha} k^{\beta} \, dk \, dt \right)^{\frac{1}{\gamma}} + \left( \int_0^1 \int_0^1 (1-t)^{-\alpha} k^{\beta} \, dk \, dt \right)^{\frac{1}{\gamma}} \\
&+ \left( \int_0^1 \int_0^1 t^{\alpha} (1-k)^{\beta} \, dk \, dt \right)^{\frac{1}{\gamma}} + \left( \int_0^1 \int_0^1 (1-t)^{\alpha} (1-k)^{\beta} \, dk \, dt \right)^{\frac{1}{\gamma}} \right\} \\
&\left( \int_0^1 \int_0^1 \left| \frac{\partial^2 f}{\partial t \partial k} (ta + (1-t)b, kc + (1-k)d) \right|^q \, dk \, dt \right)^{\frac{1}{q}}.
\end{align*}
\]

Since \( \left| \frac{\partial^2 f}{\partial t \partial k} \right|^q \) is h-convex function on the co-ordinates on \( \Delta \), then one has:

\[
\begin{align*}
&\left| f(a,c) + f(a,d) + f(b,c) + f(b,d) \right| \\
&+ \left\{ \Gamma(\alpha + 1) \Gamma(\beta + 1) \left[ \frac{\partial^2 f}{\partial t \partial k} (ta + (1-t)b, kc + (1-k)d) \right] \right\} - A \\
&\leq \frac{(b-a)(d-c)}{4} \left\{ \left( \int_0^1 \int_0^1 t^{-\alpha} k^{\beta} \, dk \, dt \right)^{\frac{1}{\gamma}} + \left( \int_0^1 \int_0^1 (1-t)^{-\alpha} k^{\beta} \, dk \, dt \right)^{\frac{1}{\gamma}} \\
&+ \left( \int_0^1 \int_0^1 t^{\alpha} (1-k)^{\beta} \, dk \, dt \right)^{\frac{1}{\gamma}} + \left( \int_0^1 \int_0^1 (1-t)^{\alpha} (1-k)^{\beta} \, dk \, dt \right)^{\frac{1}{\gamma}} \right\} \\
&\left( \int_0^1 \int_0^1 \left| \frac{\partial^2 f}{\partial t \partial k} (ta + (1-t)b, kc + (1-k)d) \right|^q \, dk \, dt \right)^{\frac{1}{q}}.
\end{align*}
\]
Corollary 3. If we take $h(\alpha) = \alpha$ in Theorem 3, then the inequality \((\text{1.4})\) of Theorem 3 becomes the inequality \((\text{1.4})\) of Theorem 3.

Remark 3. If we take $h(\alpha) = \alpha^s$ in Theorem 5, we have

$$\left| f(a, c) + f(a, d) + f(b, c) + f(b, d) \right| \leq \left( \frac{(b-a)(d-c)}{[\alpha(\alpha + 1)\beta(\beta + 1)]^{\frac{1}{p}}} \right) \times \left( \frac{1}{(s+1)^2} \left\{ \left| \frac{\partial^2 f}{\partial k^2t} (a, c) \right|^q + \left| \frac{\partial^2 f}{\partial k^2t} (a, d) \right|^q + \left| \frac{\partial^2 f}{\partial k^2t} (b, c) \right|^q + \left| \frac{\partial^2 f}{\partial k^2t} (b, d) \right|^q \right\} \right) ^{\frac{1}{q}}$$

where

$$A = \frac{\Gamma(\beta+1)}{4(d-c)^{\frac{1}{2}}} \left[ J_{c+}^\beta f(a, d) + J_{c+}^\beta f(b, d) + J_{d-}^\beta f(a, c) + J_{d-}^\beta f(b, c) \right]$$

and

$$+ \frac{\Gamma(\alpha+1)}{4(b-a)^{\frac{1}{2}}} \left[ J_{c-}^\alpha f(c, d) + J_{c-}^\alpha f(a, d) \right].$$

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