In this paper we discuss decomposition in the context of three-dimensional Chern-Simons theories. Specifically, we argue that a Chern-Simons theory with a gauged noneffectively-acting one-form symmetry is equivalent to a disjoint union of Chern-Simons theories, with discrete theta angles coupling to the image under a Bockstein homomorphism of a canonical degree-two characteristic class. On three-manifolds with boundary, we show that the bulk discrete theta angles (coupling to bundle characteristic classes) are mapped to choices of discrete torsion in boundary orbifolds. We use this to verify that the bulk three-dimensional Chern-Simons decomposition reduces on the boundary to known decompositions of two-dimensional (WZW) orbifolds, providing a strong consistency test of our proposal.
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1 Introduction

Decomposition is the observation that some local quantum field theories are equivalent to disjoint unions of other local quantum field theories, essentially a counterexample to old lore linking locality and cluster decomposition. It was first observed in [1] in two-dimensional gauge theories and orbifolds with trivially-acting subgroups (nonminimally-charged matter) [2–4], and since then has been developed in many other references, see e.g. [5–30] and e.g. [31–35] for reviews.

Decomposition is not limited to two dimensions, and indeed four-dimensional versions of decomposition have been described in [16, 18]. The common thread linking these different examples involves what is now called a higher-form symmetry: a quantum field theory in $d$ spacetime dimensions decomposes if it has a global $(d-1)$-form symmetry (possibly realized noninvertibly) [16,18].

In this paper, following up [30], we turn to decomposition in three-dimensional Chern-Simons theories with gauged noneffectively-acting one-form symmetries. Briefly, we find that

$$\text{[Chern-Simons}(H)/BA]\ = \coprod_{\theta \in \hat{K}} \text{Chern-Simons}(G)_{\theta},$$

where $G = H/(A/K)$, $K \subset A$ defines the trivially-acting subgroup, and $\theta$ indicates a discrete theta angle coupling to an appropriate characteristic class of $G$ bundles. On the boundary, this reduces to decomposition in noneffectively-acting orbifolds of two-dimensional WZW models. A key role is played by the fact that the bulk discrete theta angles (coupling to bundle characteristic classes) become discrete torsion on the boundary, a result we explain in detail. The fact that the bulk decomposition correctly implies a known decomposition of the two-dimensional boundary theory provides a strong consistency check on our proposal.

In two dimensions, decomposition has had a variety of applications, for example in giving nonperturbative constructions of geometries in phases of some gauged linear sigma models (GLSMs) [5,36–48], in Gromov-Witten theory [6,11], in computing elliptic genera to check claims about IR limits of pure supersymmetric gauge theories [17], and recently in understanding Wang-Wen-Witten anomaly resolution [27,29,49].

Chern-Simons theories are the starting point for many physics questions, and so we anticipate that the results of this paper should have a variety of applications. For example, as is well-known, three-dimensional AdS gravity can be understood as a Chern-Simons theory [50], making Chern-Simons theories a natural playground for addressing questions in three-dimensional gravity, an approach used in e.g. [51] to address Marolf-Maxfield factorization.

\footnote{For purposes of historical language translation, before the term ‘one-form symmetry’ was coined, theories with one-form symmetries were sometimes called ‘gerby’ theories, in reference to the fact that a gerbe is a fiber bundle whose fibers are higher groups.}
questions \[52\]. We anticipate that this work may have analogous uses.

Similarly, one of the original applications of two-dimensional decomposition was to understand phases of certain gauged linear sigma models, where decomposition was used locally (ala Born-Oppenheimer) to understand IR limits of certain theories as nonperturbatively-realized branched covers of spaces \[5\]. We expect that similar ideas could be used to understand the IR limits of certain Chern-Simons-matter theories.

We begin in section \[2\] with a review of decomposition in two-dimensional WZW orbifolds, which not only serves as a review of decomposition, but also describes the decomposition pertinent to boundaries in the three-dimensional Chern-Simons theories we discuss.

In section \[3\] we describe the primary proposal of this paper, namely decomposition in Chern-Simons theories with gauged one-form symmetry groups, which takes the form \(\text{(1.1)}\). All Chern-Simons theories are assumed to have levels such that the theories exist on the three manifolds over which they are defined. We describe how this bulk decomposition maps to boundary WZW models, and reproduces standard results on decomposition in two-dimensional noneffective orbifolds, which serves as a strong consistency test of our claims. We also observe that in all these examples, the boundary discrete theta angles (choices of discrete torsion in boundary WZW models) are trivial, which is often reflected in the bulk discrete theta angles.

In section \[4\] we discuss the spectra of these theories. We begin with an explanation and review of monopole operators, local operators (analogues of twist fields in two-dimensional orbifolds) which can be used to construct projection operators. We then discuss line operators. When gauging ordinary one-form symmetries, the standard technology of anyon condensation can be used to describe the line operators. However, to describe noneffectively-acting one-form symmetries (in which a subgroup acts trivially), as relevant for this paper, requires a minor extension, which we propose and utilize.

In section \[5\] we walk through the details of bulk and boundary decomposition, spectrum computations, and consistency tests such as level-rank duality in a variety of concrete examples.

Finally in section \[6\] we briefly discuss the related case of boundary \(G/G\) models. These two-dimensional theories decompose, and we briefly discuss their corresponding bulk theories.

In appendix \[A\] we summarize some results on line operators that are used in the main text. In appendix \[B\] we give a brief overview of crossed modules, to make this paper self-contained, as they are used in the description of three-dimensional decomposition. In appendix \[C\] we describe gauging effectively-acting one-form symmetries without appealing to line operators.
2 Warm-up: Decomposition in WZW orbifolds

As a warm-up exercise, let us briefly review decomposition in two dimensions, and apply it towards orbifolds of WZW models.

Consider an orbifold \([X/\Gamma]\) where a central subgroup \(K \subset \Gamma\) acts trivially on \(X\). As has been discussed previously (see e.g. [1]), for an ordinary (orientation-preserving) orbifold,

\[
\text{QFT}([X/\Gamma]) = \prod_{\theta \in \hat{K}} \text{QFT}([X/G]_{\theta(\omega)}),
\]

where \(\theta(\omega)\) is a choice of discrete torsion, given as the image of the extension class \([\omega] \in H^2(G,K)\) corresponding to

\[
1 \rightarrow K \rightarrow \Gamma \rightarrow G \rightarrow 1
\]

under the map \(\theta : K \rightarrow U(1)\), yielding \(\theta(\omega) \in H^2(K,U(1))\).

Consider a \(\Gamma\) orbifold of a WZW model for a group \(H\), with \(K \subset \Gamma\) acting trivially, and \(G = \Gamma/K\) a subset of the center of \(H\), acting freely on \(H\). Then, as a special case of the decomposition above, we have that

\[
[WZW(H)/\Gamma] = \prod_{\theta \in \hat{K}} WZW(H/G)_{\theta(\omega)},
\]

with both sides at the same level. That said, (ordinary) discrete torsion vanishes for cyclic subgroups, so the only occasion on which \(\theta(\omega)\) can be nontrivial will be if \(H = \text{Spin}(4n)\) and \(\Gamma/K = \mathbb{Z}_2 \times \mathbb{Z}_2\). (We will discuss that case in section 5.7.)

For example, consider a \(\mathbb{Z}_4\) orbifold of an \(SU(2)\) WZW model, where a \(\mathbb{Z}_2 \subset \mathbb{Z}_4\) acts trivially, and the \(\mathbb{Z}_2\) coset is the freely-acting center of \(SU(2)\). For an ordinary (orientation-preserving) orbifold, since there is no discrete torsion in a \(\mathbb{Z}_2\) orbifold, we have that

\[
[WZW(SU(2))/\mathbb{Z}_4] = \prod_{2} WZW(SO(3))
\]

(with all WZW models at the same level).

Although we will not utilize orientifolds in this paper, in principle one can also consider orientation-reversing orbifolds (orientifolds) of WZW models, see e.g. [53-58]. See [59] and references therein for discussions of discrete torsion in orientifolds.

So far we have discussed discrete torsion weighting different universes. In principle, WZW models can also be weighted by analogues of discrete theta angles. Although these are better
known in the case of gauge theories\footnote{Discrete theta angles in gauge theories in unrelated contexts have a long history, see e.g.\cite{60, 61, section 6}, \cite{36, section 4} for two-dimensional examples and \cite{62, 64} for four-dimensional examples.}, the point is that if a group manifold $G$ has a torsion characteristic class, some $w \in H^2(G, F)$ for some coefficient module $F$, then there exists a discrete theta angle $\theta \in \hat{F}$ that weights maps into $G$, via a term in the action of the form

$$\int_{\Sigma} \langle \theta, \phi^* w \rangle,$$

where $\Sigma$ is the worldsheets and $\phi : \Sigma \to G$ any map in the path integral. If $G = \tilde{G}/Z$ for some finite group $Z$, these discrete theta angles can also, for appropriate $w$, correspond to choices of discrete torsion in a $Z$ orbifold of a WZW model on $\tilde{G}$.

In section 3.3 we shall see that the choices of discrete theta angle above that arise in the WZW orbifolds appearing on boundaries of decompositions of one-form-gauged Chern-Simons theories, are the same as choices of discrete torsion.

3 Decomposition in noneffective one-form symmetry gaugings

In general terms, one expects a decomposition in a $d$-dimensional quantum field theory whenever it has a global $(d - 1)$-form symmetry \cite{16, 18}.

A typical example of a decomposition in two dimensions involves gauging a non-effective group action: a group action in which a subgroup acts trivially on the theory being gauged, in the sense that its generator commutes with the operators of that theory: $[J, O] = 0$. Gauging a trivially-acting group results in a global one-form symmetry, which is responsible for a decomposition.

In principle, an analogous phenomenon exists in three dimensions, involving the gauging of ‘trivially-acting’ one-form symmetries. Here, for a one-form action to be trivial means that it commutes with the line operators in the theory, as we shall elaborate below.

In this section, after a short overview of the notion of non-effective one-form symmetries, we make a precise prediction for decomposition.

3.1 Non-effective one-form symmetry group actions

We define a ‘trivially-acting’ one-form symmetry in terms of the fusion algebra of the corresponding lines, and a ‘non-effective’ one-form symmetry is one in which a subset of the lines acts trivially.
First, let us recall some basics of gauging one-form symmetries, which in three dimensions we will describe by the fusion algebra of line operators (see e.g. [65] section 3.1, [66, 67], [68] section II and references therein for a detailed discussion), with gauging as in e.g. [69]. Anomalies in such a gauging are discussed in e.g. [51, section 2.3], [70, section 2.1], [71–73].

In order to be gaugeable, its 't Hooft anomaly must vanish, which requires that the lines be mutually transparent, meaning that they have trivial mutual braiding. In particular, a one-form symmetry necessarily has abelian lines, for which the braiding is completely characterized by their spins (see e.g. [51, equ'n (2.28)], [70, section 2]), schematically

$$B(a, b) = \exp \left( 2\pi i \left( h(a \times b) - h(a) - h(b) \right) \right),$$

where $$a, b$$ denote lines, and $$h(a) \mod 1$$ is the spin of the line $$a$$. Note that if the spins are integers, then $$B = 1$$ and there is no obstruction. Conversely, if $$B = 1$$, then spins are integers or half integers.

We take a ‘trivially-acting BK’ to be described by a set of lines $$\{g\}$$ such that all other lines $$b$$ both

1. have trivial monodromy under $$g$$, meaning $$B(g, b) = 1$$, and also are
2. invariant under fusion with $$g$$, $$g \times b = b$$,

for all $$g$$. (In effect, there are two conditions in three dimensions, whereas invariance in two dimensions really boils down to a single constraint of the form $$[J, O] = 0$$.)

To be clear, this notion can be somewhat counterintuitive. Consider for example $$SU(2)$$ Chern-Simons theory. This theory has a $$BZ_2$$ one-form symmetry defined by the center of $$SU(2)$$. However, although the classical action is invariant under the center, the Wilson lines are not invariant, as the $$BZ_2$$ action multiplies Wilson lines by phases (corresponding to the n-ality of the corresponding representation with respect to the center). In particular, the $$BZ_2$$ action on $$SU(2)$$ Chern-Simons theory defined by the center of $$SU(2)$$ is not trivial.

---

3 We are using “B” to mean several different things in this section. We use $$BK$$ to denote a one-form symmetry, a standard notation in mathematics, going back decades. (In physics, the notation $$K^{[1]}$$ is sometimes used instead.) Later we will use $$BG$$ to denote a classifying space. In this section, we also use $$B(a, b)$$ to denote line monodromies.
3.2 Basic decomposition prediction

In [30], it was argued that in a quotient by a 2-group $\Gamma$ of the form

$$1 \to BK \to \Gamma \to G \to 1,$$

where the $BK$ acts trivially, the path integral sums over both $K$ gerbes and a subset of $G$ bundles, specifically $G$ bundles satisfying a constraint.

In general, if one has a group $H$ and an abelian group $A$ with a map $d : A \to H$ whose image is in the center of $H$, then the crossed module $\Gamma = \{ A \to H \}$ defines a 2-group we shall label $\Gamma$. So long as we are interested in flat bundles, we can apply the same analysis as [30], and argue that $\Gamma$ bundles on a three-manifold $M$ map to $G = H/\text{im} A$ bundles satisfying a condition. This 2-group fits into an exact sequence

$$1 \to K \to A \xrightarrow{d} H \to G \to 1,$$

where $K = \text{Ker} d$. (Physically, $d$ just encodes the $A$ action, by projecting it to a subgroup of the center of $H$.) This exact sequence defines an element

$$\omega \in H^3_{\text{group}}(G, K) = H^3_{\text{sing}}(BG, K),$$

which we will give explicitly in (3.5), and the condition that $G$ bundles must satisfy to be in the image of $\Gamma$ bundles is that

$$\phi^* \omega = 0,$$

for $\phi : M \to B\Gamma$ the map defining the $\Gamma$ bundle on $M$, for the same reasons discussed in [30].

Next, we describe the element $\omega$ corresponding to the extension (3.4), appearing in the constraint (3.6) above. Let $Z = \text{im} d \subset Z(H)$, the center of $H$, and $w_G$ the $Z$-valued degree-two characteristic class for $G$ corresponding to a generator of $H^2_{\text{sing}}(BG, Z)$. (For example, for $G = SO(n)$, $w_G$ is the second Stiefel-Whitney class $w_2$.) Let $\alpha \in H^2_{\text{group}}(Z, K)$ be the class of the extension

$$1 \to K \to A \to Z \to 1,$$

and let

$$\beta_{\alpha} : H^2_{\text{sing}}(BG, Z) \to H^3_{\text{sing}}(BG, K)$$

be the Bockstein homomorphism in the long exact sequence associated to the extension (3.7).

Then

$$\omega = \beta_{\alpha}(w_G) \in H^3_{\text{sing}}(BG, K).$$

---

4See appendix B for an introduction to crossed modules, or alternatively [74 appendix A], [75 section 2].
When discussing boundary WZW models, it will be useful to describe $\omega$ differently. To that end, we use the fact that

$$H_n^{\text{sing}}(BG, Z) = \text{Map}(BG, K(Z, n)),$$

(3.10)

to write $w_G$ and $\alpha$ as maps

$$w_G : BG \to K(Z, 2), \quad \alpha : BZ(= K(Z, 1)) \to K(K, 2).$$

(3.11)

Since Eilenberg-Maclane spaces are in the stable category, where $B$ exists as a functor, we can define

$$B\alpha : K(Z, 2) \to K(K, 3),$$

(3.12)

hence

$$B\alpha \circ w_G : BG \to K(K, 3),$$

(3.13)

and so defines an element of $H_3^{\text{sing}}(BG, K)$. Furthermore, $B\alpha$ is just the Bockstein homomorphism $\beta_\alpha$, hence

$$\omega = B\alpha \circ w_G = \beta_\alpha(w_G),$$

(3.14)

and so we recover the description of $\omega$ above.

So far, we have argued that on general principles, our $\Gamma$ gauge theory should be described by a $G$ gauge theory such that the $G$ bundles satisfy the constraint (3.6). Just as in [1, 30], such a restriction on instantons can be implemented by a sum over universes. The constraint (3.6), namely $\phi^*\omega = 0$, is implemented by summing over $G$ Chern-Simons theories with discrete theta angles coupling to $\omega$, formally

$$[\text{Chern-Simons}(H)/BA] = \prod_{\theta \in K} \text{Chern-Simons}(G)_\theta,$$

(3.15)

where $\theta$ is the three-dimensional discrete theta angle coupling to $\phi^*\omega$, for levels and underlying three-manifolds for which these theories are defined. This is our prediction for decomposition in three-dimensional Chern-Simons theories.

\footnote{As has been noted in e.g. [69, appendix C], [77, appendix A], [78–80], not every Chern-Simons theory with every level is well-defined on every three-manifold. The basic issue is that Chern-Simons actions are not precisely gauge-invariant, but under gauge transformations shift by an amount proportional to $2\pi$. Depending upon the gauge group and the three-manifold, the proportionality factor may or may not be integral. If $k$ times that proportionality factor is integral, then the exponential of the action is gauge-invariant, and the theory is well-defined; if that product is not integral, then the path integral is not gauge-invariant and so not defined. Even if it is defined, it may depend upon subtle choices. For example, [77, appendix A] argues that the (ordinary, bosonic) $U(1)_1$ Chern-Simons theory is well-defined only on spin three-manifolds, and furthermore that the choices of values of the action, the Chern-Simons invariants in the sense of [81, 82], are in one-to-one correspondence with the spin structures. More generally, gauging one-form symmetries can create issues of this form, precisely because one twists gauge fields by gerbes, which results in ‘twisted’ bundles and connections not present in the original theory, of fractional instanton numbers.}
The $G$ Chern-Simons theory is defined to be the $B(\text{im } A)$ gauging of the $H$ Chern-Simons theory, at the same level as the $H$ Chern-Simons theory. This is important to distinguish because sometimes gauging one-form symmetries can shift levels. For example, [76, section C.1] argues that, schematically, $U(1)_{4m}/B\mathbb{Z}_2 = U(1)_m$, and not $U(1)_{4m}$, despite the fact that as groups, $U(1)/\mathbb{Z}_2 = U(1)$.

The reader should note that the decomposition statement above correctly reproduces ordinary one-form gaugings. Consider the case that $K = 1$, so that the map $d : A \rightarrow H$ is one-to-one into the center of $H$. Then, decomposition (3.15) correctly predicts that

$$[\text{Chern-Simons}(H)/BA] = \text{Chern-Simons}(G),$$

which is a standard result (see e.g. [69]). Decomposition becomes interesting in cases in which $K \neq 1$.

In section 5 we will check this statement in several examples, outlining how it both reproduces known results as well as explains new cases.

### 3.3 Boundary WZW models

Let us now turn to Chern-Simons theories on manifolds with boundary, and the corresponding theories on the boundaries. We will see that the bulk Chern-Simons decomposition of the previous section correctly predicts a decomposition of boundary WZW models, which matches existing results on decomposition in two-dimensional orbifolds. This matching involves a rather interesting relation between characteristic classes of bundles on three-manifolds and choices of discrete torsion in two-dimensional orbifolds. In particular, the fact that the three-dimensional decomposition correctly reproduces two-dimensional decomposition on the boundary is an important consistency test of our proposal.

Briefly, as has been discussed elsewhere (see e.g. [83–87], [88, section 4.2], [89, section 5.2], and in related contexts [90,91]), on a three-manifold with boundary, a bulk Chern-Simons theory for gauge group $G$ naturally couples to a (chiral) WZW model for the group $G$ on the boundary. If the Chern-Simons theory has level $k$, then (see e.g. [88 section 4.2]) the boundary WZW model has level $\tau(k)$, where

$$\tau : H^n_{\text{sing}}(BG, F) \rightarrow H^{n-1}_{\text{sing}}(G, F)$$

(3.17)
is the loop space map\(^6\) for any abelian group \(F\), and we take Chern-Simons level\(^7\) \(k \in H^4_{\text{sing}}(BG, \mathbb{Z})\), and WZW levels \(\tau(k) \in H^3_{\text{sing}}(G, \mathbb{Z})\). Similarly, if the Chern-Simons theory has a discrete theta angle coupling to some characteristic class defined by an element of \(\omega \in H^3(BG, F)\), then the boundary WZW model couples\(^8\) to a discrete theta angle defined by \(\tau(\omega) \in H^2(G, F)\). Such discrete theta angles in two-dimensional WZW models are reviewed in section \(2\).

Given that standard bulk Chern-Simons / boundary WZW model relationship reviewed above, the three-dimensional decomposition prediction \(3.15\) implies that in the associated boundary RCFT, an \(A\) orbifold of a WZW model for \(H\) is equivalent to a disjoint union of WZW models for \(G\),

\[
[WZW(H)/A] = \coprod_{\theta \in K} WZW(G)_{\theta},
\]

with levels and discrete theta angles related to those of the bulk theory by the map \(\tau\). We will see later in this section that although the WZW discrete theta angles \(\theta\) are derived from characteristic classes in the Chern-Simons theory, they nevertheless correspond to choices of discrete torsion in the boundary orbifolds.

As a consistency check, let us show that \(\tau\) commutes with gauging \(B A\), so that the levels on the left and right-hand sides of \(3.19\) match, just as they did\(^9\) in the bulk prediction \(3.15\).

First, for \(G\) any topological group, there is a natural homotopy equivalence between the loop space \(\Omega(BG)\) and \(G\) (meaning that \(BG\) is a delooping of \(G\)). Also, for any abelian group \(F\), the Eilenberg-Maclane space \(K(F,n-1)\) is homotopy equivalent to loop space \(\Omega(K(F,n))\). Since

\[
H^n_{\text{sing}}(BG, F) = \text{Map}(BG, K(F,n))
\]

and since \(\Omega\) is a functor, for any continuous homomorphism \(f : G_1 \to G_2\) between topological groups \(G_1, G_2\), there is a continuous map \(Bf : BG_1 \to BG_2\) and natural maps

\[
\text{Map}(BG_2, K(F,n)) \longrightarrow \text{Map}(BG_1, K(F,n)),
\]

\[
a \mapsto B(f \circ a)
\]

\(^6\)This is the natural map

\[
H^n_{\text{sing}}(BG, F) = \text{Map}(BG, K(F,n))
\]

\[
\longrightarrow \text{Map}(\Omega(BG), \Omega(K(F,n))) = \text{Map}(G, K(F,n-1)) = H^{n-1}_{\text{sing}}(G, F).
\]

which sends any \(f \in \text{Map}(BG, K(F,n))\) to \(\Omega(f)\). For later use, to construct explicit maps, one needs concrete choices of e.g. \(X \mapsto \Omega BX\), for which we refer the reader to e.g. \(92, 93\). As such choices do not alter cohomology classes, we will not discuss them explicitly in this paper.

\(^7\)As before, levels are assumed to be such that the theory exists.

\(^8\)We would like to thank Y. Tachikawa for a discussion of discrete theta angles in this context.

\(^9\)Modulo subtleties discussed there in special cases, such as those arising from the fact that \(U(1)/\mathbb{Z}_k = U(1)\) as a group, but the corresponding Chern-Simons theories have different levels.
and
\[
\text{Map}(G_2, K(F, n-1)) \to \text{Map}(G_1, K(F, n-1)),
\]
(3.23)
\[
b \mapsto f \circ b.
\]
(3.24)
Combining these maps, one finds that for any Lie group \(G\) with \(K\) a subgroup of the center, the following diagram commutes:
\[
\begin{array}{ccc}
H^3_{\text{sing}}(B(G/K), F) & \to & H^2_{\text{sing}}(G/K, F) \\
\downarrow & & \downarrow \\
H^3_{\text{sing}}(BG, F) & \to & H^2_{\text{sing}}(G, F).
\end{array}
\]
(3.25)
This tells us that the levels appearing on either side of the boundary WZW relation (3.19) match, as expected, consistent with the prediction (3.15) of the bulk Chern-Simons theory.

Now, we will argue that the WZW model discrete theta angles, arising as \(\tau\) of characteristic classes in the Chern-Simons theory, are the same as choices of discrete torsion in the boundary theory. This will be important in understanding how the three-dimensional Chern-Simons decomposition compares to two-dimensional decompositions as reviewed in section 2. For simplicity, we will assume that \(H\) is the universal covering, so that \(Z = \pi_1(G)\). (Similar results exist in more general cases.)

To that end, since \(\tau\) is the loop space functor, we can write
\[
\tau(\beta_\alpha(w_G)) = \tau(B\alpha \circ w_G) = \Omega(B\alpha \circ w_G) = \Omega(B\alpha) \circ \Omega(w_G).
\]
(3.26)
Now, \(\Omega(B\alpha) = \alpha\), and
\[
\Omega(w_G) \in \text{Map}(\Omega(BG), \Omega(K(Z, 2))) = \text{Map}(G, K(Z, 1)),
\]
(3.27)
so \(\Omega(w_G)\) is a map \(G \to BZ\). Now, we claim that \(\Omega(w_G)\) is also the cell attachment map \(p\) of the Postnikov tower, \(p : G \to B\pi_1(G) = BZ\), where \(Z = \pi_1(G)\).

To make this clear, recall that the Postnikov tower map is the classifying map for the universal cover. In other words, if \(\hat{G}\) is the universal covering group of \(G\), then \(p^*EZ = \hat{G}\). Now, on the other hand, \(BG \to B\hat{G}\) is a principal \(K(Z, 1)\) bundle on \(BG\), which corresponds to a map \(BG \to B(K(Z, 1)) = K(Z, 2)\), which is \(w_G\). Applying the loop space functor gives the map \(\Omega(w_G) : G \to K(Z, 1) = BZ\), which is then more or less tautologically \(p\).

In particular, we see that
\[
\tau(B\alpha \circ w_G) = \alpha \circ \Omega(w_G) = \alpha \circ p.
\]
(3.28)
The expression above relates the Chern-Simons discrete theta angles (coupling to bundle characteristic classes) to discrete torsion on the boundary. We can see this as follows. If
\( \phi : \Sigma \rightarrow G \) is any map from the worldsheet \( \Sigma \) into the target \( G \), then \( p \circ \phi : \Sigma \rightarrow BZ \) defines a \( Z \)-twisted sector over \( \Sigma \). In particular, the discrete theta angle phase

\[
\langle \theta, \phi^*(\alpha \circ p) \rangle,
\]

(3.29)

for for \( \theta : K \rightarrow U(1) \) any character of \( K \), corresponds to discrete torsion in the \( Z \)-twisted sector defined by \( p \circ \phi \), specifically discrete torsion given by \( \theta(\alpha) \in H^2_{\text{group}}(Z, U(1)) \), for \( \alpha \in H^2_{\text{group}}(Z, K) \). Thus, we see that \( \tau \) relates discrete theta angles coupling to bundle characteristic classes on three-manifolds, to discrete torsion in two-dimensional orbifolds on boundaries.

In passing, this phenomenon that three-dimensional bulk discrete theta angles become discrete torsion in boundary two-dimensional orbifolds is also visible in the case that the bulk theory is a finite 2-group orbifold, see [30, section 3.2].

Now, let us compare the decomposition (3.19) in boundary WZW models, implied by bulk Chern-Simons decomposition, to standard results [1] on decomposition in two-dimensional orbifolds, as reviewed earlier in section 2.

Certainly the form of the boundary decomposition (3.19) is identical to that arising in two-dimensional orbifolds with trivially-acting central subgroups, possibly modulo the form of the discrete theta angles. We have just argued that the discrete theta angles arising on the boundary correspond to choices of discrete torsion, and in fact, the discrete torsion phases arising in the boundary case match those in the ordinary two-dimensional case.

We can relate these two pictures of boundary discrete theta angles as follows. Recall \( \alpha \in H^2_{\text{group}}(Z, K) \) is the class of the extension

\[
1 \longrightarrow K \longrightarrow A \longrightarrow Z \longrightarrow 1.
\]

(3.30)

In two-dimensional decomposition in \( A \) orbifolds with trivially-acting central subgroups \( K \), the discrete torsion phase factors on a universe associated with \( \theta \in \hat{K} \) are precisely the image of \( \alpha \) under \( \theta \):

\[
\begin{align*}
H^2_{\text{group}}(Z, K) &\longrightarrow H^2_{\text{group}}(Z, U(1)), \\
\alpha &\mapsto \theta \circ \alpha.
\end{align*}
\]

(3.31)

(3.32)

These are the same as the discrete torsion phases arising in the boundary WZW decomposition (3.19), as we have just discussed, and we will confirm explicitly in examples in section 5 that the decomposition above in the boundary theory precisely coincides with the decomposition of WZW orbifolds given in (2.3). This matching is an important consistency test of our proposal.
3.4 Nontriviality of discrete theta angles

In the boundary WZW models appearing in these decompositions, the discrete torsion on each universe appearing in a decomposition is trivial. For most single group factors, this is because the center is usually a cyclic group, and cyclic group orbifolds have no discrete torsion. The exceptions are the groups Spin(4n), which have center $\mathbb{Z}_2 \times \mathbb{Z}_2$. That finite group admits discrete torsion; however, to generate the discrete torsion in a decomposition of a string orbifold, the orbifold group must be nonabelian, and so cannot arise as the boundary of a three-dimensional theory, as we will discuss in greater detail in section 5.7.

In at least some examples, not only are the boundary discrete theta angles (discrete torsions) trivial, but the bulk discrete theta angles are also trivial. For example\(^\text{10}\) in bulk theories, for cases in which $K = \mathbb{Z}_2$, $Z = \mathbb{Z}_2$, and $A = \mathbb{Z}_4$, so that the extension $\alpha$ is

$$1 \rightarrow \mathbb{Z}_2 \rightarrow \mathbb{Z}_4 \rightarrow \mathbb{Z}_2 \rightarrow 1,$$

(3.33)

the bulk discrete theta angle couples to the Bockstein $\beta_\alpha$ of a distinguished element $w_G \in H^2(M_3, \mathbb{Z}_2)$. Now, for this $\alpha$,

$$\beta_\alpha(w_G) = \text{Sq}^1(w_G),$$

(3.34)

and as we will argue in section 5.3,

$$\text{Sq}^1(w_G) = w_1(TM_3) \cup w_G,$$

(3.35)

hence it can only be nonzero on nonorientable spaces. However, we only define Chern-Simons theories on oriented three-manifolds, so for all cases we consider, these bulk discrete theta angles vanish.

Similarly, if the three-manifold is $T^3$, the pertinent Bockstein homomorphism will vanish, and one cannot get a nonzero bulk discrete theta angle. Briefly, for any short exact sequence

$$1 \rightarrow K \rightarrow A \rightarrow Z \rightarrow 1,$$

(3.36)

for $K, A, Z$ abelian, the induced map

$$H^2(T^3, A) \rightarrow H^2(T^3, Z)$$

(3.37)

is surjective (since each of those cohomology groups is just $\text{Hom}$ from a free abelian group into the coefficients), which implies that in the long exact sequence

$$H^2(T^3, K) \rightarrow H^2(T^3, A) \rightarrow H^2(T^3, Z) \xrightarrow{\beta} H^3(T^3, K),$$

(3.38)

the Bockstein $\beta = 0$, and so the bulk discrete theta angles are trivial in corresponding cases.

\(^{10}\)We would like to thank Y. Tachikawa for making this observation.
For another example, consider Lens spaces. From example 3E.2, for the Bockstein associated to the short exact sequence

\[ 1 \to \mathbb{Z}_m \to \mathbb{Z}_{m^2} \to \mathbb{Z}_m \to 1, \]  

(3.39)

the associated Bockstein maps generators of \( H^1(L, \mathbb{Z}_m) \) to generators of \( H^2(L, \mathbb{Z}_m) \), for \( L \) a Lens space, but \( \beta^2 = 0 \), hence the associated Bockstein map

\[ \beta : H^2(L, \mathbb{Z}_m) \to H^3(L, \mathbb{Z}_m) \]  

(3.40)

necessarily vanishes, and so the bulk discrete theta angles are trivial in corresponding cases.

More generally, whether the bulk discrete theta angles are always trivial is a reflection of the map \( \tau : H^3_{\text{sing}}(BG, K) \to H^2_{\text{sing}}(BG, K) \). For example, if \( \tau \) is injective, then triviality of the boundary discrete theta angles implies triviality of the bulk discrete theta angles. We leave general questions about the injectivity of \( \tau \) for future work.

In passing, note that in the bulk, orientability plays a key role. At least abstractly, it is tempting to speculate about more general cases involving e.g. orientifolds of boundary WZW models, as might arise if the three-manifold descends to a solid Klein bottle (a three-manifold whose boundary is the two-dimensional Klein bottle). On such a nonorientable space, at least sometimes the discrete theta angles would be nontrivial. Furthermore, in orientifolds, discrete torsion is counted by \( H^2_{\text{group}}(\mathbb{Z}, U(1)) \) with a nontrivial action on the coefficients (see e.g. [53, 54, 58, 59]), so that for example \( H^2_{\text{group}}(\mathbb{Z}_2, U(1)) \) can be nonzero, which again would result in boundary WZW models with nonzero discrete theta angle contributions.

## 4 Spectra

In this section we briefly describe the spectra of monopole operators and line operators in a theory with a gauged trivially-acting one-form symmetry, and argue that the results are consistent with decomposition (3.15).

### 4.1 Monopole operators

In two dimensional theories, when one gauges a non-effectively-acting group, one gets twist fields and Gukov-Witten operators corresponding to conjugacy classes in the trivially-acting subgroup. In three dimensional theories, instead of twist fields, one has monopole operators (see e.g. [97, 98]), which play the same role. In this section we will outline their properties.

In two dimensions, twist fields generate branch cuts, which in the language of topological defect lines are real codimension-one walls that implement the gauging of the zero-form
symmetry. In three dimensions, when gauging a one-form symmetry, from thinking about topological defect ones one sees the theory has codimension-two lines, which end in monopole operators, in the same way that in two dimensions, the orbifold branch cuts terminate in twist fields.

We can think of the monopole operators in three dimensions as local disorder operators: on a sphere surrounding the monopole operator associated to a $BG$ symmetry, one has a nontrivial $G$ gerbe, corresponding to an element of $H^2(S^2, G)$ (for $G$ assumed finite), just as on a circle surrounding a twist field in two dimensions one has a nontrivial bundle.

In two dimensions, the twist fields associated to trivially-acting gauged zero-form symmetries are local dimension-zero operators, which can be used to form projectors onto the universes of decomposition. In three dimensions, the monopole operators associated to trivially-acting gauged one-form symmetries are closely analogous, and can again be used to form projection operators, in exactly the same fashion. In \cite[section 4.1.4]{30}, projection operators are explicitly constructed from monopole operators in three-dimensional theories, and we encourage the reader to consult that reference for further details.

### 4.2 Line operator spectrum

Given a ‘gaugable’ one-form symmetry, described by a subset of the lines in the theory, there is a standard procedure for computing the spectrum of lines in the gauged theory, given as follows (see e.g. \cite[section 2]{69}, \cite[section 2.5]{99}, \cite{51}). For $B\mathbb{Z}_n$, let $g$ denote a line generating the others, and then:

- Exclude from the spectrum all lines $a$ which have monodromy $B(g, a) \neq 1$, under a generator $g$, where the $g$ action on a line $b$ is determined by the process

$$b g = B(g, b) b$$

- Identify any two lines $b, g \times b$ that differ by fusion with $g$, and finally

- Lines $b$ that are invariant under fusion (meaning $a = g \times a$) become $n$ lines in the spectrum of the gauged theory.

This is closely analogous to two-dimensional orbifolds, in which one omits non-invariant operators, and fixed points lead to twist fields.

\footnote{\textsuperscript{11}In terms of the S-matrix, $B(a, b) = S_{ab}/S_{0b}$, see e.g. \cite[equ’n (40)]{68}.}
This is a special case of a more general procedure, sometimes known as anyon condensation, which is also applicable to noninvertible symmetries, unlike the basic algorithm above. See e.g. [51, section 4.1], [25, 99–102] for further details.

Now, in our case, the (noneffectively-acting) one-form symmetry we wish to gauge is not described by a set of lines within the original theory. Sometimes, in some special cases, we can describe it by adding lines to the theory, such as in the case that the entire gauged one-form symmetry acts trivially. In general, however, that procedure is not well-defined. Consider for example the case of $SU(2)_4$ Chern-Simons theory, whose spectrum of lines $\{(0), (1), (2), (3), (4)\}$ is as described in appendix [A]. Let us consider gauging a $B\mathbb{Z}_4$. Now, $SU(2)_4$ has a $B\mathbb{Z}_2$ symmetry, corresponding to the lines $(0), (1)$, so one could imagine extending it to $B\mathbb{Z}_4$ by replacing $\{(0), (1)\}$ with $\{(0), \ell_1, \ell_2, \ell_3\}$ which obey

$$\ell_i \times \ell_j = \ell_{i+j \mod 4}, \quad (4.2)$$

with $\ell_0 = (0)$. In order for the image of $\mathbb{Z}_2 \hookrightarrow \mathbb{Z}_4$ to act trivially, we require

$$\ell_2 \times (2, 3, 4) = (2, 3, 4), \quad (4.3)$$

and for this to descend to the ordinary $SU(2)_4$, we also require

$$\ell_1 \times (2, 3, 4) = \ell_3 \times (2, 3, 4), \quad (4.4)$$

which must match $(1) \times (2, 3, 4)$ in the $SU(2)_4$ fusion algebra given in appendix [A]

$$\ell_{1,3} \times (2) = (3), \quad \ell_{1,3} \times (3) = (2), \quad \ell_{1,3} \times (4) = (4). \quad (4.5)$$

The new lines $\ell_i$ are then defined to have trivial monodromy with all other lines

$$B(\ell_i, x) = 1 \quad (4.6)$$

for all lines $x$. This much is uniquely specified by the statement of the extension.

Now, we have not completely specified the extension of $SU(2)_4$; for example, the product $(2) \times (3)$ in $SU(2)_4$ contains a $(1)$, so we would still need to decide whether to replace $(1)$ with $\ell_1$ or $\ell_3$, for example. However, in making such choices, we find an internal contradiction with the structure we have already described, namely a failure of associativity. For example, in $SU(2)_4$, as described in appendix [A]

$$(2) \times (3) = (1) + (4), \quad (3) \times (3) = (0) + (4). \quad (4.7)$$

We could replace the $(1)$ above with either $\ell_1$ or $\ell_3$. Suppose we take

$$(2) \times (3) = \ell_1 + (4). \quad (4.8)$$

Then,

$$\ell_1 \times ((2) \times (3)) = \ell_1 \times (\ell_1 + (4)) = \ell_2 + (4), \quad (4.9)$$
(ℓ_1 × (2)) × (3) = (3) × (3) = (0) + (4). \hfill (4.10)

However, ℓ_2 ≠ (0), so we see that
\[ ℓ_1 × ((2) × (3)) ≠ (ℓ_1 × (2)) × (3), \]
and so associativity is broken. We encounter a similar problem if we choose
\[ (2) × (3) = ℓ_3 + (4) \]
instead and consider fusion with ℓ_3. Put simply, we cannot enlarge the BZ_2 inside SU(2)_4 to a noneffective BZ_4, without breaking associativity.

With this in mind, we outline here a minor extension of the prescription of [69, section 2], [99, section 2.5], [51] for counting line operators in three-dimensional theories with gauged one-form symmetries. (As we will not be using noninvertible symmetries, we will not attempt to describe the analogous construction for condensation algebra objects here.)

Our approach is motivated by the action of a group G on a set M: we distinguish G and M, G is not a subset of M in general, though we can still define an action of G on M that enables us to make sense of the quotient M/G. Let G be a finite abelian group, so that BG is a group of one-form symmetries, and associate lines to elements of G. Consider a set of simple lines C (objects in a braided tensor category, which we will gauge).

An action of BG on C is then described by giving, for each \( g ∈ G \) and line \( L ∈ C \),

- a monodromy \( B(g, L) \), such that
  \[ B(g_1, L) B(g_2, L) = B(g_1 g_2, L), \]
  \hfill (4.13)

  and

- a fusion \( g × L ∈ \mathbb{Z}|C| \), meaning \( g × L = \sum_c N_{gL}^c c \) for \( c ∈ C \) and \( N_{gL}^c ∈ \mathbb{Z} \), with the property that
  \[ g_1 × (g_2 × L) = (g_1 g_2) × L. \]
  \hfill (4.14)

We say that a line in BG corresponding to \( g ∈ G \) acts trivially if, for all \( L ∈ C \),
\[ B(g, L) = 1 \quad \text{and} \quad g × L = L, \]
and then to say BG acts noneffectively, as in section 3.1, means that some for some \( g ≠ 1 \) in G, the line corresponding to G acts trivially.

Now, given an action of BG on C, we propose to construct the lines of the quotient C/BG as follows, by close analogy with [69, section 2], [99, section 2.5], [51].

\[ \text{12Although we are only interested in isomorphism classes of objects, presumably the full categorical description is in terms of module categories, or as a minor variation on the group actions described in [68, section III.B]. As we are only interested here in counting (isomorphism classes of) objects, and this is merely a minor variation on existing methods, we will be very schematic.} \]
• Exclude any $L$ such that for some $g \in G$, $B(g, L) \neq 1$,
• Identify $L \sim g \times L$ for each $g \in G$,
• For each $g \in G$ such that $g \times L = L$, we get a copy of $L$ in $\mathcal{C}/BG$.

It is straightforward to check that in the special case the lines in $BG$ are a subset of those in $\mathcal{C}$, this reduces to the prescription reviewed earlier and in [69, section 2], [99, section 2.5], [51].

As another special case, note that if all of $BG$ acts trivially, then in the quotient $\mathcal{C}/BG$,

• No lines in $\mathcal{C}$ are excluded, since $B(g, L) = 1$ for all $L$,
• No lines are identified, since $g \times L = L$ for all $L$, so fusion does not relate different lines,
• Since $g \times L = L$ for each $g \in G$ and each $L \in \mathcal{C}$, we get $|G|$ copies of the lines in $\mathcal{C}$.

This is consistent with the expectations of decomposition in this case: if we gauge a $BG$ that acts completely trivially on a theory, in the sense above, one expects to get $|G|$ copies of the theory.

We will apply this computation in specific examples in Chern-Simons theories later in this paper, but for the moment, we give two toy examples, to illustrate the idea.

First, consider $B\mathbb{Z}_2/B\mathbb{Z}_2$. Let the lines of $\mathcal{C} = B\mathbb{Z}_2$ be generated over $\mathbb{Z}$ by $\{(0), (1)\}$, where

$$(0) \times (0) = (0), \quad (0) \times (1) = (1), \quad (1) \times (1) = (0),$$

and $B\mathbb{Z}_2$ acts as

$g \times (0) = (1), \quad g \times (1) = (0),$

and we take all monodromies $B = 1$. Then, applying the procedure above, to get the lines of $\mathcal{C}/B\mathbb{Z}_2$,

• Since $B(g, L) = 1$ for all $L \in \mathcal{C}$, no lines are excluded,
• Since $g \times (0) = (1), (0) \sim (1),$
• No lines are invariant.

Hence the quotient is generated by one single line, as one would expect.
Next, consider $B\mathbb{Z}_2/B\mathbb{Z}_4$, where the $B\mathbb{Z}_4$ acts noneffectively. Let the lines of $\mathcal{C} = B\mathbb{Z}_2$ be as above, and the generator $g$ of $B\mathbb{Z}_4$ acts as
\[ g \times (0) = (1), \quad g \times (1) = (0). \] (4.18)

As before, we take all monodromies $B = 1$. Applying the procedure above,

- Since $B(g, L) = 1$ for all $L \in \mathcal{C}$, no lines are excluded.
- Since $g \times (0) = (1), \ (0) \sim (1)$,
- Since $g^2 \times (0) = (0)$ and $g^2 \times (1) = (1), \ (0) \sim (1)$ appears twice in the quotient.

Thus, the quotient $\mathcal{C}/B\mathbb{Z}_4$ is generated over $\mathbb{Z}$ by two lines, as expected since a $\mathbb{Z}_2 \subset \mathbb{Z}_4$ describes trivially-acting lines.

We should also briefly observe that the theories we are describing, which decompose, have the property that they violate the axiom of remote detectability in a topological order, see e.g. [103–105]. This axiom says that there are no invisible lines in the bulk theory (technically, that the category of lines has trivial center). Violation of remote detectability signals multiple vacua and therefore a decomposition, much as cluster decomposition in other contexts [1].

### 4.3 Bulk-boundary map

Let us now consider the bulk-boundary map between lines in the three-dimensional bulk and on the two-dimensional boundary. Let $\mathcal{C}$ be the category of lines which act trivially in the bulk. Suppose we have a line in $\mathcal{C}$ which ends on the boundary, defining an object in the two-dimensional vertex operator algebra $V$. We can describe this bulk-boundary relation by a functor
\[ F : \mathcal{C} \rightarrow \text{Rep}(V), \] (4.19)
(for $\text{Rep}(V)$ the category of representations of $V$) which takes a line to the vector space of ways the line can end on the boundary, giving point operators.

As observed in [106 section 3.5], a one-form symmetry that acts trivially in the bulk might act nontrivially on the boundary, and the theory can still decompose, much as with Chan-Paton factors and D-branes in two-dimensional theories. Broadly speaking, the different line operators in the three-dimensional bulk end on the various two-dimensional sectors of the boundary theory.

A three-dimensional theory may have surface operators which are not totally determined by the line operators. In the case where the three-dimensional theory has only a local
vacuum, all the surfaces can build as condensations, i.e. networks of lines. However, when there are multiple vacua, as in the cases we are interested in, then this fails to be true. The surfaces which are not built as a network of lines will end on a line on the boundary. These lines define the ‘action’ of a trivially acting zero-form symmetry. In two dimensions, if one gauges a trivially-acting zero-form symmetry, then one obtains an emergent global one-form symmetry (and hence a decomposition).

From the decomposition conjecture (3.15), the different universes and hence the different ground states are labeled by elements of the Pontryagin dual of the one-form symmetry group. On the other hand, the surfaces in the bulk which enact a 2-form symmetry, come from gauging a trivially acting one-form symmetry. So while the lines that the surface ends on has trivial action on the boundary, the surface itself is not necessarily trivial in the bulk. This is summarized in the following diagram, where $F$ is the functor that makes objects to the boundary:

\[
\begin{array}{ccc}
\text{Bulk symmetry} & \rightarrow & \text{Boundary symmetry} \\
\text{trivially-acting one-form} & \xrightarrow{F} & \text{trivially-acting zero-form} \\
gauge & \downarrow & \text{gauge} \\
\text{global two-form} & \xrightarrow{F} & \text{global one-form}
\end{array}
\]

5 Examples

In the next several subsections we will walk through examples of the decomposition proposed in section 3. Where possible, we will apply level-rank duality to perform self-consistency tests. In all cases, we will compare to the decomposition of the boundary WZW model. In particular, as reviewed in section 2, decomposition is reasonably well-understood in two-dimensional theories, and so we get solid consistency tests by checking that the boundary WZW decomposition implied by the bulk Chern-Simons decomposition matches existing two-dimensional results.

In each case, we will assume that levels are chosen so that the theories are well-defined, but will not list those conditions explicitly.

5.1 Chern-Simons($SU(2)/BZ_2$, $K = 1$)

In this section, we will reproduce a well-known result as a special case of the decomposition prediction (3.15).
Specifically, we consider gauging the $BZ_2$ central one-form symmetry in $SU(2)$ Chern-Simons theory.

Here, this $BZ_2$ is not trivially-acting, and so no decomposition is expected. In particular, this gauging is known (see e.g. [69]) to be equivalent to the $SO(3)$ Chern-Simons theory at the same level. At the level of the path integral for the gauge theory, this is discussed in appendix C.

We can understand this as a special case of the decomposition prediction (3.15). In the language of that statement, we identify $A = \mathbb{Z}_2$, $H = SU(2)$, and $d : A \to H$ is the inclusion map of the center, $\mathbb{Z}_2 \hookrightarrow SU(2)$. Then, the kernel of $d$ vanishes, so $K = 1$, and $G = H/A = SO(3)$. This corresponds to the exact sequence

$$1 \longrightarrow 1 \longrightarrow \mathbb{Z}_2 \overset{d}{\longrightarrow} SU(2) \longrightarrow SO(3) \longrightarrow 1. \quad (5.1)$$

Furthermore, in the case, since $K = 1$, the extension class $[\omega] \in H^3(G, K)$ is trivial, $\omega = 1$, so $\phi^*\omega = 1$ and there is no discrete theta angle.

Putting this together, we see that the decomposition prediction (3.15) in this case is

$$[\text{Chern-Simons}(SU(2))/BZ_2] = \text{Chern-Simons}(SO(3)), \quad (5.2)$$

which reproduces known results.

Let us also compute the line operator spectrum in this example. This is a standard computation, but we will quickly outline it using the tools of section 4.2, with an eye towards later, more obscure, versions. There are five line operators in $SU(2)_4$ Chern-Simons, as listed in appendix A, which we denote

$$(0), (1), (2), (3), (4). \quad (5.3)$$

We gauge a $BZ_2$, with lines $\{\ell_0, \ell_1\}$, where

$$\ell_i \times \ell_j = \ell_{i+j \mod 2}, \quad (5.4)$$

and which act on the $SU(2)_4$ lines as

$$\ell_0 \times L = L, \quad \ell_1 \times L = (1) \times L, \quad (5.5)$$

and with

$$B(\ell_0, L) = +1, \quad B(\ell_1, 0) = B(\ell_1, 1) = B(\ell_1, 4) = +1, \quad B(\ell_1, 2) = B(\ell_1, 3) = -1. \quad (5.6)$$

(Clearly, we can identify the action of this $BZ_2$ with the action of the lines (0), (1) in $SU(2)_4$.) It is straightforward to check that this gives a well-defined action in the sense of section 4.2. Applying the procedure there, to get the lines of $SU(2)_4/BZ_2$,
• the lines (2), (3) are not invariant under monodromies and so should be excluded,
• from \((1) \times (1) = (0)\), the lines (0) and (1) should be identified in the quotient, and
• from \((1) \times (4) = (4)\), the line (4) is duplicated,

so that the \(SU(2)_4/B\mathbb{Z}_2\) spectrum consists of the vacuum line and two copies of (4), which is the standard result for \(SO(3)_4\).

Now, let us turn to the boundary theory. On the boundary, this reduces to the statement

\[
[WZW(SU(2))/\mathbb{Z}_2] = WZW(SO(3)),
\]

(5.7)

which is standard.

### 5.2 Chern-Simons\((SU(2)) \times [\text{point}/B\mathbb{Z}_2], K = \mathbb{Z}_2\)

Now, let us apply the decomposition prediction (3.15) to a different case, namely one in which we gauge a trivially-acting \(B\mathbb{Z}_2\) ‘acting’ on an \(SU(2)\) Chern-Simons theory, uncoupled from the center one-form symmetry of the \(SU(2)\) theory.

This is perhaps the cleanest example of a \(B\mathbb{Z}_2\) gauging that acts trivially: we gauge a \(B\mathbb{Z}_2\) in bulk that does nothing at all to the \(SU(2)\).

Let us apply the decomposition prediction (3.15) to this case. Here, in the notation of (3.15), we take \(H = SU(2)\) and \(A = \mathbb{Z}_2\); however, the map \(d : A \to H\) maps all of \(\mathbb{Z}_2\) to 1. In this case, the kernel of \(d\), \(K\), is all of \(\mathbb{Z}_2\), and \(G = H = SU(2)\). The decomposition prediction for this case is that

\[
[\text{Chern-Simons}(SU(2))/B\mathbb{Z}_2] = \bigoplus_{\theta \in \hat{K}} \text{Chern-Simons}(SU(2)),
\]

(5.8)

two copies of the \(SU(2)\) Chern-Simons theory. Furthermore, in this case there are no non-trivial discrete theta angles, hence the decomposition prediction can be written more simply as

\[
[\text{Chern-Simons}(SU(2))/B\mathbb{Z}_2] = \bigoplus_{2} \text{Chern-Simons}(SU(2)).
\]

(5.9)

Let us briefly consider the spectrum of line operators, following the procedure discussed in section 4.2. We describe the trivially-acting \(B\mathbb{Z}_2\) in terms of two lines \(\{\ell_0, \ell_1\}\), where

\[
\ell_i \times \ell_j = \ell_{i+j} \mod 2,
\]

(5.10)
and with an action on the lines of $SU(2)_4$ given by

$$B(\ell, L) = +1, \quad \ell \times L = L.$$  \hspace{1cm} (5.11)

It is straightforward to check that this gives a well-defined action in the sense of section 4.2.

Next, we compute the spectrum of $SU(2)_4/B\mathbb{Z}_2$, for this trivially-acting $B\mathbb{Z}_2$. From the rules in section 4.2,

- None of the original lines of the $SU(2)$ Chern-Simons theory are omitted, as they all have trivial monodromy under the generator $(a)$,
- Since $(a) \times (a) = (0)$, we see that in the gauged theory, $(a)$ and $(0)$ are identified with one another,
- Since all of the original lines are invariant under fusion $((a) \times (x) = (x))$, they are all duplicated.

As a result, the line operator spectrum of the gauged theory is two copies of the line operator spectrum of the original $SU(2)$ Chern-Simons theory, consistent with decomposition. This result could also be obtained by adding one new line $a$ to the lines of $SU(2)_4$, which interacts trivially with all other lines, and then condensing $\{(0), a\}$ in the ordinary fashion, though as we discussed in section 4.2 it will not always be possible to do that.

Next, we turn to the boundary theory. In the boundary WZW model, bulk decomposition becomes the statement that

$$[\text{WZW}(SU(2))/\mathbb{Z}_2] = \coprod_2 \text{WZW}(SU(2)).$$  \hspace{1cm} (5.12)

In the $\mathbb{Z}_2$ orbifold on the left, the $\mathbb{Z}_2$ acts trivially on the $SU(2)$ WZW model, for which case ordinary two-dimensional decomposition predicts exactly the statement above, that the completely-trivially-acting $\mathbb{Z}_2$ orbifold of a WZW model is just two copies of the same WZW model. Thus, the boundary theory matches results from two-dimensional decomposition, as expected.

### 5.3 Chern-Simons($SU(2))/B\mathbb{Z}_4$, $K = \mathbb{Z}_2$

Consider a $SU(2)$ Chern-Simons theory in three dimensions, and gauge a $B\mathbb{Z}_4$ that acts via projecting to a $B\mathbb{Z}_2$ which acts as the center symmetry. In this case, there is a trivially-acting $B\mathbb{Z}_2$, so in broad brushstrokes one expects two copies of a $B\mathbb{Z}_2$-gauged $SU(2)$ Chern-Simons theory.
Let us walk through the prediction of the decomposition prediction (3.15) in this case. Here, we have \( H = SU(2) \) and \( A = \mathbb{Z}_2 \), with the map \( d : A \to SU(2) \) mapping the \( \mathbb{Z}_4 \) onto the center \( \mathbb{Z}_2 \) of \( SU(2) \). Thus, the map \( d \) is surjective, but not injective: its kernel \( K = \mathbb{Z}_2 \). Similarly,

\[
G = H/\text{im} \ d = SU(2)/\mathbb{Z}_2 = SO(3).
\]

Putting this together, we see in this case that the decomposition prediction (3.15) is

\[
[\text{Chern-Simons}(SU(2))/B\mathbb{Z}_4] = \text{Chern-Simons}(SO(3)) + \bigoplus \text{Chern-Simons}(SO(3)) - ,
\]

where the \( \pm \) denote the two values of the discrete theta angle coupling to the characteristic class defined by \( \beta_\alpha(w_G = w_{SO(3)}) \), for \( \alpha \) the class of the extension

\[
1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2 \longrightarrow 1,
\]

and where here, \( w_{SO(3)} = w_2 \), the second Stiefel-Whitney class.

Next, we will argue\(^\textsuperscript{13}\) that the characteristic class \( \beta_\alpha(w_2) \) is the third Stiefel-Whitney class \( w_3 \). From the Wu formula \([107, \text{prob. 8-A}]\) for Steenrod squares, which map \( \text{Sq}^k : H^\bullet(X, \mathbb{Z}_2) \to H^{\bullet+k}(X, \mathbb{Z}_2), k \geq 0: \)

\[
\text{Sq}^k(w_m(\xi)) = \sum_{t=0}^{k} \binom{k - m}{t} w_{k-t}(\xi) \cup w_{m+t}(\xi) = \sum_{t=0}^{k} \binom{m - k + t - 1}{t} w_{k-t}(\xi) \cup w_{m+t}(\xi),
\]

where each \( w_j = w_j(\xi) \) for a real vector bundle \( \xi \), and in the equality, we have used the fact that

\[
\left( \begin{array}{c} k - m \\ t \end{array} \right) = \frac{(k-m)(k-m-1)\cdots(k-m-t+1)}{t!},
\]

\[
= (\pm) \frac{(m-k)(m-k+1)\cdots(m-k+t-1)}{t!} \equiv \left( \begin{array}{c} m - k + t - 1 \\ t \end{array} \right) \mod 2.
\]

(See e.g. [108] for this and related observations.) As a result, for any real vector bundle,

\[
\text{Sq}^1(w_2) = w_1 \cup w_2 + w_0 \cup w_3 = w_1 \cup w_2 + w_3,
\]

so if \( w_1 = 0 \), as is the case for \( SO(3) \) bundles, then \( w_3 = \text{Sq}^1(w_2) \). (In principle, this is one explanation of why all \( SO(3) \) bundles can be constructed by twisting \( SU(2) \) bundles by \( \mathbb{Z}_2 \) gerbes: the gerbe characteristic class determines not only the second Stiefel-Whitney class \( w_2 \) of the \( SO(3) \) bundles, but also \( w_3 \) via \( \text{Sq}^1 \), as above.)

Furthermore, the action of \( \text{Sq}^1 \) is the Bockstein homomorphism \( \beta \) associated to the extension

\[
1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2 \longrightarrow 1,
\]

\(^{13}\)E.S. would like to thank Y. Tachikawa for observing the pertinent properties of \( w_3 \).
(see e.g. [96, section 4.1],) meaning
\[ Sq^1(x) = \beta(x) \]  
(5.20)
for any \( x \). The extension (5.19) above coincides with \( \alpha \) in the present case, so we see that in this example, the discrete theta angle couples to
\[ \beta_\alpha(w_2) = Sq^1(w_2), \]  
(5.21)
using (3.14). We also see that in this example, this class can be described even more simply as \( w_3 \), the third Stiefel-Whitney class, as \( w_3 = Sq^1(w_2) \).

Now, on a three-manifold \( M \), we can write \( Sq^1(x) \) for any \( x \) in terms of the Wu class \( \nu_1 \in H^1(M, \mathbb{Z}_2) \) as [107, chapter 11]
\[ Sq^1(x) = \nu_1 \cup x. \]  
(5.22)
Furthermore, [107, theorem 11.14]
\[ \nu_1 = w_1(TM), \]  
(5.23)
so assembling these pieces, we have that
\[ w_3(\xi) = Sq^1(w_2(\xi)) = w_1(M) \cup w_2(\xi). \]  
(5.24)
As a result, the third Stiefel-Whitney class \( w_3 \) will only be nontrivial on a nonorientable three-manifold \( M \). However, Chern-Simons theories are not defined on nonorientable spaces.

In section 5.9, we will use level-rank duality to perform a self-consistency check of decomposition in this case.

Now, let us check this prediction by computing the line spectrum in this gauged Chern-Simons theory. First, following section 4.2 we define a \( B\mathbb{Z}_4 \) by lines \( \{\ell_0, \ell_1, \ell_2, \ell_3\} \) such that
\[ \ell_i \times \ell_j = \ell_{i+j \mod 4}, \]  
(5.25)
and which act on the lines of \( SU(2)_4 \) (described in appendix A) as follows:
\[ B(\ell_{0,2}, L) = +1, \ B(\ell_{1,3}, 0) = B(\ell_{1,3}, 1) = B(\ell_{1,3}, 4) = +1, \ B(\ell_{1,3}, 2) = B(\ell_{1,3}, 3) = -1, \]  
(5.26)
\[ \ell_0 \times L = \ell_2 \times L = L, \ \ell_1 \times L = \ell_3 \times L = (1) \times L. \]  
(5.27)
It is straightforward to check that this action of \( B\mathbb{Z}_4 \) on the lines of \( SU(2)_4 \) is well-defined in the sense of section 4.2. As \( \ell_2 \) acts trivially, this is also a non-effective action, in the sense of section 3.1.

Next, we follow the procedure outlined in section 4.2 to get the lines of \( SU(2)_4/B\mathbb{Z}_4 \):
• Lines (2), (3) have $B(\ell_{1,3}, L) \neq +1$, and so are omitted.

• Since $\ell_{1,3} \times (1) = (0)$, we identify the lines $(0) \sim (1)$.

• Since $\ell_i \times (4) = (4)$ for all $i$, we get four copies of (4) in the spectrum of $SU(2)_4/B\mathbb{Z}_4$, and since $\ell_{0,2} \times (1) = (1)$, $\ell_{0,2} \times (0) = (0)$, we get two copies of $(0) \sim (1)$.

Thus, we see that we get two copies of the lines of $SO(3)_4$, consistent with expectations from decomposition.

Before going on, let us compute the lines in one more example, specifically $SU(2)_4/B\mathbb{Z}_{2p}$, where the $\mathbb{Z}_{2p}$ projects to the $\mathbb{Z}_2$ center of $SU(2)_4$, with kernel $\mathbb{Z}_4$. The lines of $B\mathbb{Z}_{2p}$ are $\{\ell_0, \cdots, \ell_{2p-1}\}$, where

\[ \ell_i \times \ell_j = \ell_{i+j} \mod 2p, \]

and their action on $SU(2)_4$ is given by

\[ B(\ell_{\text{even}}, L) = +1, \quad B(\ell_{\text{odd}}, 0) = +1 = B(\ell_{\text{odd}}, 1) = B(\ell_{\text{odd}}, 4), \]

\[ B(\ell_{\text{odd}}, 2) = -1 = B(\ell_{\text{odd}}, 3), \]

\[ \ell_{\text{even}} \times L = L, \quad \ell_{\text{odd}} \times L = (1) \times L. \]

As before, it is straightforward to check that this action of $B\mathbb{Z}_{2p}$ is well-defined in the sense of section 4.2, and since $\{\ell_{\text{even}}\}$ act trivially, it is a non-effective action, in the sense of section 3.1.

Next, we follow the procedure outlined in section 4.2 to get the lines of $SU(2)_4/B\mathbb{Z}_{2p}$:

• Lines (2), (3) have $B(\ell_{\text{odd}}, L) \neq +1$, and so are omitted.

• Since $\ell_{\text{odd}} \times (1) = (0)$, we identify the lines $(0) \sim (1)$.

• Since $\ell_i \times (4) = (4)$ for all $i$, we get $2p$ copies of (4), and since $\ell_{\text{even}} \times (1) = (1)$, $\ell_{\text{even}} \times (0) = (0)$, we get $p$ copies of $(0) \sim (1)$.

Altogether, we find $p$ copies of the lines of $SO(3)_4$, consistent with expectations from decomposition, since $B\mathbb{Z}_p$ acts trivially.

Before going on, let us briefly discuss the boundary theory. The Chern-Simons decomposition (5.14) becomes a decomposition of WZW models, formally

\[ [\text{WZW}(SU(2))/\mathbb{Z}_4] = \text{WZW}(SO(3))_+ \coprod \text{WZW}(SO(3))_- \]

Here the $\mathbb{Z}_2$ discrete theta angle couples to the image of the element of $H^3(BSO(3), \mathbb{Z}_2)$ (corresponding to third Stiefel-Whitney classes) in $H^2(SO(3), \mathbb{Z}_2) = \mathbb{Z}_2$. However, the generator of this group is $\text{Sq}^1(a)$, where $a$ generates $H^1(SO(3), \mathbb{Z}_2)$ and for reasons discussed
previously, \( Sq^1(a) = w_1(TM) \cup a \), hence is nonzero only if the two-dimensional space is nonorientable.

We will consider various generalizations of this example, returning to this example for special levels to utilize level-rank duality consistency checks in section 5.9.

### 5.4 Chern-Simons \((SU(n))/BZ_{np}, K = \mathbb{Z}_p\)

Next, we will consider gauging the action of \( B\mathbb{Z}_{np} \) on \( SU(n) \) Chern-Simons, where the \( \mathbb{Z}_{np} \) acts by projecting to the center \( \mathbb{Z}_n \) of \( SU(n) \), and study the discrete theta angles for special values of \( n \) and \( p \) beyond those discussed already.

In terms of the decomposition prediction (3.15), we take \( A = \mathbb{Z}_{np}, H = SU(n), \) and \( d: A \rightarrow H \) acts by projecting to \( Z = \mathbb{Z}_n \subseteq Z(H) \). Then, the kernel \( K = \mathbb{Z}_p, G = SU(n)/\mathbb{Z}_n \), and we have the long exact sequence

\[
1 \rightarrow \mathbb{Z}_\ell \rightarrow \mathbb{Z}_{n\ell} \rightarrow SU(n) \rightarrow SU(n)/\mathbb{Z}_n \rightarrow 1.
\]

(5.33)

In general terms, decomposition (3.15) then predicts that

\[
[\text{Chern-Simons}(SU(n))/BA] = \coprod_{\theta \in K} \text{Chern-Simons}(SU(n)/\mathbb{Z}_n)_{\theta(\omega)},
\]

(5.34)

where the \( \theta(\omega) \) are discrete theta angles coupling to the characteristic class defined by \( \beta_{\alpha}(w_{SU(n)/\mathbb{Z}_n}) \), where \( w_{SU(n)/\mathbb{Z}_n} \in H^2_{\text{sing}}(BSU(n)/\mathbb{Z}_n, \mathbb{Z}_n) \) is a generalization of the second Stiefel-Whitney class to \( n \geq 2 \), and \( \beta_{\alpha} \) is the Bockstein map in the long exact sequence associated to the extension

\[
1 \rightarrow K(= \mathbb{Z}_p) \rightarrow A(=\mathbb{Z}_{np}) \rightarrow Z(= \mathbb{Z}_n) \rightarrow 1.
\]

(5.35)

with extension class \( \alpha \in H^2_{\text{group}}(Z, K) \).

We will evaluate this expression for some special cases in which we will simplify the expression for discrete theta angles. We will use [109], which provides the cohomology of \( SU(n)/\mathbb{Z}_n \), which (modulo a degree shift) is essentially the same. (See also [110–115].)

First, consider the case that \( p \) is a prime number that does not divide \( n \). Then, from [109, section 7],

\[
H^\bullet_{\text{sing}}(BSU(n)/\mathbb{Z}_n, \mathbb{Z}_p) = H^\bullet_{\text{sing}}(BSU(n), \mathbb{Z}_p),
\]

(5.36)

and so there is no \( \mathbb{Z}_p \)-valued characteristic class in degree three, hence no discrete theta angle. In this case, the decomposition above can be written more simply as

\[
[\text{Chern-Simons}(SU(n))/BA] = \coprod_p \text{Chern-Simons}(SU(n)/\mathbb{Z}_n).
\]

(5.37)
Next, suppose that $p = 2$, and $n = 2m$ for $m$ odd. From [109] cor. 4.2, the group $H^3_{\text{sing}}(BSU(n)/\mathbb{Z}_n,\mathbb{Z}_2) \neq 0$, and so for $w_{SU(n)/\mathbb{Z}_n} \in H^2_{\text{sing}}(BSU(n)/\mathbb{Z}_n,\mathbb{Z}_2)$, we get a discrete theta angle coupling to $\beta_\alpha(w_{SU(n)/\mathbb{Z}_n})$, the image of $w_{SU(n)/\mathbb{Z}_n}$ under the Bockstein map associated to the extension

$$1 \to \mathbb{Z}_p \to \mathbb{Z}_{2m} \to \mathbb{Z}_n \to 1,$$

(5.38)

with extension class $\alpha \in H^2_{\text{group}}(\mathbb{Z}_n,\mathbb{Z}_p)$. Since $p = 2$, we can write $\beta_\alpha(w_{SU(n)/\mathbb{Z}_n}) = \text{Sq}^1(w_{SU(n)/\mathbb{Z}_n})$, as before, and also just as before, it is only nonzero on nonoriented spaces, as we saw for the case of $SU(2)$ and $SO(3)$ theories in section 5.3.

Now, let us consider the corresponding boundary WZW model. The bulk decomposition above predicts

$$[\text{WZW}(SU(n))/\mathbb{Z}_{np}] = \bigsqcup_{\theta \in K} \text{WZW}(SU(n)/\mathbb{Z}_n)_{\theta(\omega)}.$$

(5.39)

Now, from ordinary two-dimensional decomposition, since there is no discrete torsion in $\mathbb{Z}_n$,

$$[\text{WZW}(SU(n))/\mathbb{Z}_{np}] = \bigsqcup_{\ell} \text{WZW}(SU(n)/\mathbb{Z}_n).$$

(5.40)

This is certainly consistent with the special cases computed above, in which the bulk discrete theta angle vanishes.

### 5.5 Chern-Simons(Spin($n$))/B$\mathbb{Z}_{2p}$, $K = \mathbb{Z}_p$

Next, we consider a simple generalization of the example above, in which we gauge a $B\mathbb{Z}_{2p}$ action on $\text{Spin}(n)$ Chern-Simons, in which the $B\mathbb{Z}_{2p}$ acts by first projecting to $B\mathbb{Z}_2$ which acts through (a subgroup of) the center. We begin by discussing the case that the $\mathbb{Z}_2$ is such that $\text{Spin}(n)/\mathbb{Z}_2 = \text{SO}(n)$. In the case that $n$ is divisible by four, there is a second choice of $\mathbb{Z}_2$ subgroup, for which the quotient $\text{Spin}(n)/\mathbb{Z}_2 \neq \text{SO}(n)$. We will discuss the second case at the end of this section.

In terms of the decomposition prediction (3.15), we take $A = \mathbb{Z}_{2p}$, $H = \text{Spin}(n)$, and $d : A \to H$ is the map that projects $\mathbb{Z}_{2p}$ onto the $\mathbb{Z}_2$ in the center of $\text{Spin}(n)$ such that $\text{Spin}(n)/\mathbb{Z}_2 = \text{SO}(n)$. Then, the kernel of $d$ is $K = \mathbb{Z}_p$, $G = H/A = \text{SO}(n)$, and we have the exact sequence

$$1 \to \mathbb{Z}_p \to \mathbb{Z}_{2p} \to \text{Spin}(n) \to \text{SO}(n) \to 1.$$

(5.41)

This extension is nontrivial, and defines a discrete theta angle coupling to $\beta_\alpha(w_{\text{SO}(n)})$, with $w_{\text{SO}(n)} = w_2$, the second Stiefel-Whitney class, as before, and the Bockstein homomorphism $\beta_\alpha$ is associated to the extension

$$1 \to \mathbb{Z}_p \to \mathbb{Z}_{2p} \to \mathbb{Z}_2 \to 1$$

(5.42)
of extension class $\alpha \in H^2_{\text{group}}(\mathbb{Z}_2, \mathbb{Z}_p)$.

Decomposition then predicts (5.15)

$$[\text{Chern-Simons(Spin}(n)/B\mathbb{Z}_{2p}] = \coprod_{\theta \in \hat{\mathbb{Z}}_p} \text{Chern-Simons}(SO(n))_\theta, \quad (5.43)$$

where the $\theta$ denotes the discrete theta angle coupling.

In the case that $p = 2$, for the same reasons as discussed in section 5.3, we can identify $\beta_\alpha(w_2)$ with $w_3$, the third Stiefel-Whitney class. However, by the same reasoning as described in subsection 5.3, the third Stiefel-Whitney class will only be nontrivial on nonorientable three-manifolds. Therefore, on orientable three-manifolds, for $p = 2$, the statement of decomposition reduces to

$$[\text{Chern-Simons(Spin}(n)/B\mathbb{Z}_{4}] = \coprod_2 \text{Chern-Simons}(SO(n)). \quad (5.44)$$

Next, let us briefly compare to the boundary WZW model. On the boundary, from the decomposition (5.43), we have

$$[\text{WZW(Spin}(n))/\mathbb{Z}_{2p}] = \coprod_{\theta \in \hat{\mathbb{Z}}_p} \text{WZW}(SO(n))_\theta, \quad (5.45)$$

For the case $p = 2$, for the same reasons as noted in section 5.3, for oriented spaces, the discrete theta angles are trivial, as the characteristic class they couple to vanish. As a result, on oriented spaces, for $p = 2$ we can equivalently write

$$[\text{WZW(Spin}(n))/\mathbb{Z}_{4}] = \coprod_2 \text{WZW}(SO(n)). \quad (5.46)$$

This is consistent with the prediction of decomposition in two dimensions in this case. As reviewed in section 2, essentially because there is no discrete torsion in a $\mathbb{Z}_2$ orbifold, in a two-dimensional WZW orbifold by $\mathbb{Z}_{2p}$ with trivially-acting $\mathbb{Z}_p$, we have

$$[\text{WZW(Spin}(n))/\mathbb{Z}_{4}] = \coprod_p \text{WZW}(SO(n)). \quad (5.47)$$

For $p = 2$ this is certainly consistent with the bulk description.

So far we have discussed the case that the $\mathbb{Z}_{2p}$ maps to $\mathbb{Z}_2 \subset \text{Spin}(n)$ such that

$$\text{Spin}(n)/\mathbb{Z}_2 = SO(n). \quad (5.48)$$
In the case that $n$ is divisible by four, there is another choice of $\mathbb{Z}_2$ subgroup of the center of $\text{Spin}(n)$, which leads to a quotient
\[
\text{Spin}(n)/\mathbb{Z}_2 \neq SO(n),
\] (5.49)
which for example projects out the vector representation. (See e.g. [116] for a discussion in a different context.) This second quotient group is sometimes denoted Semi-spin$(n)$, abbreviated $Ss(n)$ (see e.g. [115 section 11]). Relevant material on the cohomology of $Ss(n)$ can be found in e.g. [109 section 9].

### 5.6 Chern-Simons $(\text{Spin}(4n + 2))/B\mathbb{Z}_{4p}$, $K = \mathbb{Z}_p$

Let us consider the case of a Chern-Simons theory with gauge group $\text{Spin}(4n + 2)$ and a gauged $B\mathbb{Z}_{4p}$, where the $\mathbb{Z}_4$ maps to the center ($\mathbb{Z}_2$) of $\text{Spin}(4n + 2)$, with kernel $K = \mathbb{Z}_p$.

In terms of the decomposition prediction (3.15), we take $A = \mathbb{Z}_{4p}$, $H = \text{Spin}(4n + 2)$, and $d : A \rightarrow H$ projects $\mathbb{Z}_{4p}$ onto the central $\mathbb{Z}_4 \subset \text{Spin}(4n + 2)$. The kernel of $d$ is $K = \mathbb{Z}_p$, $G = H/A = \text{SO}(4n + 2)/\mathbb{Z}_2$, and we have the exact sequence
\[
1 \rightarrow \mathbb{Z}_p \rightarrow \mathbb{Z}_{4p} \rightarrow \text{Spin}(4n + 2) \rightarrow \text{SO}(4n + 2)/\mathbb{Z}_2 \rightarrow 1. \tag{5.50}
\]
Decomposition then predicts (3.15)
\[
[\text{Chern-Simons}(\text{Spin}(4n + 2))/B\mathbb{Z}_{4p}] = \bigoplus_{\theta \in \mathbb{Z}_p} \text{Chern-Simons}(\text{SO}(4n + 2)/\mathbb{Z}_2)_{\theta(\omega)}, \tag{5.51}
\]
where the discrete theta angle couples to a characteristic class $\beta_\alpha(w_{\text{Spin}(4n + 2)/\mathbb{Z}_2})$ for $\beta_\alpha$ the Bockstein map associated to the short exact sequence
\[
1 \rightarrow \mathbb{Z}_p \rightarrow \mathbb{Z}_{4p} \rightarrow \mathbb{Z}_4 \rightarrow 1 \tag{5.52}
\]
of extension class $\alpha \in H^2_{\text{group}}(\mathbb{Z}_4, \mathbb{Z}_p)$.

Consider for example the case $p = 2$. From [109 lemma 8.1], $\text{SO}(4n + 2)/\mathbb{Z}_2$ has one characteristic class in $H^3(B\text{SO}(4n + 2)/\mathbb{Z}_2, \mathbb{Z}_2)$, related to $w_3$ of a covering $\text{SO}(4n + 2)$ bundle.

In the boundary WZW model, the decomposition (5.51) predicts
\[
[\text{WZW}(\text{Spin}(4n + 2))/\mathbb{Z}_{4p}] = \bigoplus_{\theta \in \mathbb{Z}_p} \text{WZW}(\text{SO}(4n + 2)/\mathbb{Z}_2)_{\theta}. \tag{5.53}
\]
Ordinary two-dimensional decomposition predicts in this case that
\[
[\text{WZW}(\text{Spin}(4n + 2))/\mathbb{Z}_{4p}] = \bigoplus_p \text{WZW}(\text{SO}(4n + 2)/\mathbb{Z}_2), \tag{5.54}
\]
essentially because there is no discrete torsion in a $\mathbb{Z}_4$ orbifold.
5.7 Chern-Simons(Spin(4n))/B(\mathbb{Z}_2 × \mathbb{Z}_{2p}), K = \mathbb{Z}_p

Next, we consider the case of a $B(\mathbb{Z}_2 × \mathbb{Z}_{2p})$ action on a Spin(4n) Chern-Simons theory. Here, Spin(4n) has center $\mathbb{Z}_2 × \mathbb{Z}_2$, and the $\mathbb{Z}_2 × \mathbb{Z}_{2p}$ acts by first mapping to the center.

In terms of the decomposition prediction (3.15), we take $A = \mathbb{Z}_2 × \mathbb{Z}_{2p}$, $H = \text{Spin}(4n)$, $d : A → H$ maps $A$ onto the center, $K = \text{Ker} d = \mathbb{Z}_p$, hence we predict

$$[\text{Chern-Simons(Spin}(4n)))/B(\mathbb{Z}_2 × \mathbb{Z}_{2p})] = \prod_{θ ∈ \hat{\mathbb{Z}}_p} \text{Chern-Simons(SO}(4n)/\mathbb{Z}_2)_θ,$$

(5.55)

where the discrete theta angle couples to $β_α(w_{\text{Spin}(4n)/\mathbb{Z}_2 × \mathbb{Z}_2})$, for $β_α$ the Bockstein map associated to the short exact sequence

$$1 → \mathbb{Z}_p → \mathbb{Z}_2 × \mathbb{Z}_{2p} → \mathbb{Z}_2 × \mathbb{Z}_2 → 1$$

(5.56)

of extension class $α ∈ H^2_{\text{group}}(\mathbb{Z}_2 × \mathbb{Z}_2, \mathbb{Z}_p)$.

Consider for example $p = 2$. From [109, lemma 8.1], $SO(4n)/\mathbb{Z}_2$ has one characteristic class in $H^3(\text{BSO}(4n)/\mathbb{Z}_2, \mathbb{Z}_2)$, related to $w_3$ of a covering $SO(4n)$ bundle.

Now, let us consider this in the boundary WZW model. The bulk decomposition (5.55) predicts that

$$[\text{WZW(Spin}(4n))/\mathbb{Z}_2 × \mathbb{Z}_{2p})] = \prod_{θ ∈ \hat{\mathbb{Z}}_p} \text{WZW(SO}(4n)/\mathbb{Z}_2)_θ,$$

(5.57)

where, as discussed in section 3.3, the boundary discrete theta angles $θ$ correspond to choices of discrete torsion, here in a $G = \mathbb{Z}_2 × \mathbb{Z}_2$ orbifold.

We can understand those boundary discrete theta angles more precisely by comparing to the predictions of two-dimensional decomposition. We have a $Γ = \mathbb{Z}_2 × \mathbb{Z}_{2p}$ orbifold, with trivially-acting $K = \mathbb{Z}_p$, and $G = Γ/K = \mathbb{Z}_2 × \mathbb{Z}_2$. In principle, $G$ can contain discrete torsion, since $H^2(\mathbb{Z}_2 × \mathbb{Z}_2, U(1)) = \mathbb{Z}_2$, so we should compute to see if we get nontrivial discrete torsion in any factors. Any such discrete torsion is the image of the extension class in $H^2(G, K)$ corresponding to

$$1 → K → Γ → G → 1$$

(5.58)

under the map $K → U(1)$ defined by the representation of $K$ corresponding to that universe, and the extension class is nontrivial; nevertheless, as discussed in [26, section 6.1], its image in $H^2(G, U(1))$ is trivial for both irreducible representations of $K$. As a result, two-dimensional decomposition predicts

$$[\text{WZW(Spin}(4n))/\mathbb{Z}_2 × \mathbb{Z}_{2p})] = \prod_p \text{WZW(SO}(4n)/\mathbb{Z}_2).$$

(5.59)
In particular, the boundary discrete theta angles vanish.

In passing, we should observe that this is a nontrivial constraint. The two choices of discrete torsion in the WZW model for Spin(4n)/Z_2 × Z_2 correspond to two distinct quantum theories, each of which can be described as the WZW model for SO(4n), see e.g. [117–121]. Furthermore, in two dimensions, certainly there exist examples in which both choices of discrete torsion appear. For example, only slightly generalizing results in [1],

\[ [\text{WZW}(\text{Spin}(4n))/D_4] = \text{WZW}(SO(4n)/Z_2)_+ \coprod \text{WZW}(SO(4n)/Z_2)_, \quad (5.60) \]
\[ [\text{WZW}(\text{Spin}(4n))/\mathbb{H}] = \text{WZW}(SO(4n)/Z_2)_+ \coprod \text{WZW}(SO(4n)/Z_2)_, \quad (5.61) \]

where in both D_4 and \( \mathbb{H} \) the Z_2 center is taken to act trivially, and the ± indicate the two choices of discrete torsion.

However, because both the dihedral group D_4 and the group of unit quaternions \( \mathbb{H} \) are nonabelian, there is no Chern-Simons version of the decompositions above. That is fortuitous, as of the two \( SO(4n)/Z_2 \) WZW models, the one with nonzero discrete torsion also does not have a Chern-Simons dual [121][122].

More generally, in order to get a two-dimensional decomposition of \( [\text{WZW}(\text{Spin}(4n))/\Gamma] \) to copies of \( \text{WZW}(SO(4n)/Z_2) \) with nontrivial discrete torsion, it is straightforward to check that \( \Gamma \) must be nonabelian, and so does not admit a Chern-Simons description.

5.8 Chern-Simons(\( Sp(n) \))/\( B\mathbb{Z}_{2p} \), \( K = \mathbb{Z}_p \)

Next, consider the case of a Chern-Simons theory with gauge group \( Sp(n) \) and a gauged \( B\mathbb{Z}_{2p} \), where the \( \mathbb{Z}_{2p} \) maps to the center (\( Z_2 \)) of \( Sp(n) \).

In terms of the decomposition prediction (3.15), we take \( A = \mathbb{Z}_{2p} \), \( H = Sp(n) \), and \( d : A \to H \) projects \( \mathbb{Z}_{2p} \) onto the central \( Z_2 \subset Sp(n) \), with \( K = \text{Ker} \ d = \mathbb{Z}_p \). Decomposition then predicts (3.15)

\[ [\text{Chern-Simons}(Sp(n))/B\mathbb{Z}_{2p}] = \coprod_{\theta \in \mathbb{Z}_p} \text{Chern-Simons}(Sp(n)/Z_2)_{\theta}, \quad (5.62) \]

where the discrete theta angle couples to a characteristic class \( \beta_{\alpha}(w_{Sp(n)/Z_2}) \) for \( \beta_\alpha \) the Bockstein map associated to the short exact sequence

\[ 1 \to \mathbb{Z}_p \to \mathbb{Z}_{2p} \to \mathbb{Z}_2 \to 1 \quad (5.63) \]

of extension class \( \alpha \in H^2_{\text{group}}(Z_2, \mathbb{Z}_p) \). See e.g. [109] section 8] for results on pertinent characteristic classes.
In the boundary WZW model, the bulk decomposition \(5.62\) predicts
\[
[WZW(Sp(n))/\mathbb{Z}_2] = \coprod_{\theta \in \hat{\mathbb{Z}}_p} WZW(Sp(n)/\mathbb{Z}_2)_\theta.
\] (5.64)

Because there is no discrete torsion in a \(\mathbb{Z}_2\) orbifold, two-dimensional decomposition predicts in this case that
\[
[WZW(Sp(n))/\mathbb{Z}_2] = \coprod_p WZW(Sp(n)/\mathbb{Z}_2).
\] (5.65)

### 5.9 Chern-Simons(\(U(1)\))\(_k\)/\(B\mathbb{Z}_{\ell p}\), \(K = \mathbb{Z}_p\)

Consider a \(U(1)_k\) Chern-Simons theory in three dimensions. This theory has a global \(B\mathbb{Z}_k\) symmetry which can be gauged (see e.g. \([123,124], [70, \text{appendix C}]\)). It has slightly different properties depending upon whether \(k\) is even or odd (see e.g. \([51, \text{section 2.2}]\)):

- When \(k\) is even, this theory has \(k\) line operators, labelled by elements of \(\mathbb{Z}_k\). If \(k\) is 0 mod 8, then the \(B\mathbb{Z}_k\) one-form symmetry generator has integer spin. If \(k\) is 2 mod 8, then the one-form generator has spin 1/4 and if \(k\) is 4 mod 8, then the one-form symmetry generator is spin 1/2.

- When \(k\) is odd, the theory has \(2k\) lines labelled by elements of \(\mathbb{Z}_{2k}\) and is moreover a spin TQFT. The line with the label \(k\) is the transparent fermion.

Now, consider gauging a \(B\mathbb{Z}_{\ell p}\), where \(\ell\) divides \(n\), where the \(\mathbb{Z}_{\ell p}\) projects to \(\mathbb{Z}_\ell \subset \mathbb{Z}_k\), for that \(B\mathbb{Z}_k\) above, with kernel \(B\mathbb{Z}_p\). Let us apply the decomposition prediction \((3.15)\) to this case.

In the language of \((3.15)\), \(A = \mathbb{Z}_{\ell p}\) and \(H = U(1)\). Here, the map \(d : A \to H\) is given by projecting \(A = \mathbb{Z}_{\ell p}\) to a \(\mathbb{Z}_\ell \subset \mathbb{Z}_k \subset U(1)\), and so it has kernel \(K = \mathbb{Z}_p\). Furthermore,
\[
G = H/\im\ d = U(1)/\mathbb{Z}_\ell = U(1).
\] (5.66)

In this case, \(BU(1) = \mathbb{C}P^\infty\) has no odd degree cohomology, so there cannot be any discrete theta angles. Thus, the decomposition prediction \((3.15)\) for this case is that
\[
[\text{Chern-Simons}(U(1)_k)/B\mathbb{Z}_{\ell p}] = \coprod_p [\text{Chern-Simons}(U(1)_k)/B\mathbb{Z}_\ell],
\] (5.67)
a sum of \(p\) theories (consistent with a trivially-acting \(B\mathbb{Z}_p\)) with no discrete theta angles.
In particular, note that the right-hand side is a sum of $U(1)_k/B\mathbb{Z}_k$ Chern-Simons theories, which is not necessarily the same as a union of $U(1)_k$ Chern-Simons theories. Although as groups $U(1)/\mathbb{Z}_k = U(1)$, gauging a Chern-Simons theory by a one-form symmetry is a bit different. For example, $U(1)_{4m}/B\mathbb{Z}_2 = U(1)_{m}$, from [76, section C.1]. (On the boundary, one has a $U(1)$ WZW model, meaning a sigma model on $S^1$, with radius determined by the level. Gauging the $B\mathbb{Z}_k$ in bulk becomes gauging a $\mathbb{Z}_k$ rotation in the boundary theory, which changes the radius and hence the level.)

We can use level-rank duality to perform a consistency test. Begin with the decomposition described in section 5.3 at level 1, namely,

$$[\text{Chern-Simons}(SU(2)_1)/B\mathbb{Z}_4] = \text{Chern-Simons}(SO(3)_1) \bigoplus \bigoplus \text{Chern-Simons}(SO(3)_1).$$

(5.68)

Here we have kept track of the discrete theta angle; we only consider Chern-Simons theories on orientable manifolds, so no discrete theta angle is visible, so the prediction of section 5.3 in this case is more simply

$$[\text{Chern-Simons}(SU(2)_1)/B\mathbb{Z}_4] = \bigoplus \text{Chern-Simons}(SO(3)_1).$$

(5.69)

From level-rank duality, we know [125, sections 3.1, 3.2]

$$U(1)_2 = U(1)_{-2} \leftrightarrow SU(2)_1,$$

(5.70)

so we have that

$$[\text{Chern-Simons}(U(1)_2)/B\mathbb{Z}_2] = [SU(2)_1/B\mathbb{Z}_2] = \text{Chern-Simons}(SO(3)_1).$$

(5.71)

Thus, we see from level-rank duality that our decomposition in section 5.3 implies

$$[\text{Chern-Simons}(U(1)_2)/B\mathbb{Z}_4] = \bigoplus \text{Chern-Simons}(U(1)_2)/B\mathbb{Z}_2],$$

(5.72)

which is a special case of the result (5.67), confirming in this case that the decomposition prediction (3.15) is giving results compatible with this example of level-rank duality.

Next, we compute the spectrum of line operators in $U(1)_8/B\mathbb{Z}_{2p}$, using the methods of section 4.2. Here in the gauging, the $\mathbb{Z}_{2p}$ projects to $\mathbb{Z}_2$ with trivially acting $\mathbb{Z}_p$. We describe the $\mathbb{Z}_{2p}$ by a set of lines $\{\ell_i\}, i \in \{0, \cdots, 2p-1\}$, where

$$\ell_i \times \ell_j = \ell_{i+j} \mod 8.$$

(5.73)

$U(1)_8$ has eight lines, labelled

$$(0), (1), (2), (3), (4), (5), (6), (7)$$

(5.74)
whose properties are listed in appendix A and for which \{0, 4\} encode a \(BZ_2\). The action of \(BZ_{2p}\) on the lines of \(U(1)_8\) is given as follows:

\[
B(\ell_{\text{even}}, L) = +1, \quad B(\ell_{\text{odd}}, L) = B(4, L),
\]

\[
\ell_{\text{even}} \times L = L, \quad \ell_{\text{odd}} \times L = (4) \times L,
\]

using the monodromies and fusion algebra described in appendix A. It is straightforward that this gives a well-defined action in the sense of section 4.2.

Next, we compute the spectrum of lines in \(U(1)_8/BZ_8\), following the procedure of section 4.2.

- The lines (1), (3), (5), (7) have \(B(\ell_{\text{odd}}, L) = -1 \neq +1\), and so are excluded.
- \(\ell_1 \times (0) = (4), \ell_1 \times (2) = (6)\), so we identify \(0 \sim (4), (2) \sim (6)\).
- \(\ell_{\text{even}} \times L = L\), so we get \(p\) copies of \((0) \sim (4)\) and \((2) \sim (6)\).

Thus, the resulting spectrum is \(p\) copies of \{0 \sim (4), (2) \sim (6)\}, which is the same as \(p\) copies of the line operator spectrum of \(U(1)_8/BZ_2\), as expected from decomposition, since there is a trivially-acting \(Z_p\).

Next, let us compare to boundary WZW models. A (boundary) WZW model for the group \(U(1)\) is the same as a \(c = 1\) free scalar, of radius determined by the level. (See e.g. [76, appendix C.1] for discussions of the RCFTs arising at particular values of the level.) Gauging the bulk one-form symmetry corresponds to orbifolding the boundary \(c = 1\) theory, which just changes the radius of the target-space circle in that boundary \(c = 1\) theory.

In a two-dimensional sigma model with target \(S^1\), if we orbifold by a \(Z_{kp}\) where \(Z_p \subset Z_{kp}\) acts trivially, then from two-dimensional decomposition, the resulting theory is equivalent to \(p\) copies of the effectively-acting \(Z_k\) orbifold, precisely matching (5.67), as expected.

### 5.10 Exceptional groups

So far we have discussed quotients of Chern-Simons theories for the gauge groups \(SU(n)\), \(\text{Spin}(n)\), and \(\text{Sp}(n)\). We can also consider cases with exceptional gauge groups. Although \(G_2, F_4,\) and \(E_8\) have no center, the group \(E_6\) has center \(Z_3\), and \(E_7\) has center \(Z_2\) (see e.g. [126 appendix A]).

For example, applying decomposition (3.15), for a \(Z_{3p}\) that acts on \(E_6\) by projecting to the \(Z_3\) center with kernel \(Z_p\),

\[
\text{Chern-Simons}(E_6)/BZ_{3p} = \prod_{\theta \in Z_p} \text{Chern-Simons}(E_6/Z_3\theta),
\]

(5.77)
where the discrete theta angle couples to $\beta_\alpha(w_{E_6/Z_3})$, for $\beta_\alpha$ the Bockstein map associated to the short exact sequence

$$1 \longrightarrow \mathbb{Z}_p \longrightarrow \mathbb{Z}_{3p} \longrightarrow \mathbb{Z}_3 \longrightarrow 1$$

(5.78)
of extension class $\alpha \in H^2_{\text{group}}(\mathbb{Z}_3, \mathbb{Z}_p)$.

Similarly, from decomposition (3.15), for a $\mathbb{Z}_{2p}$ that acts on $E_7$ by projecting to the $\mathbb{Z}_2$ center with kernel $\mathbb{Z}_p$,

$$[\text{Chern-Simons}(E_7)/B\mathbb{Z}_{2p}] = \coprod_{\theta \in \hat{\mathbb{Z}}_p} \text{Chern-Simons}(E_7/\mathbb{Z}_2)_\theta,$$

(5.79)
where the discrete theta angle couples to $\beta_\alpha(w_{E_7/\mathbb{Z}_2})$, for $\beta_\alpha$ the Bockstein map associated to the short exact sequence

$$1 \longrightarrow \mathbb{Z}_p \longrightarrow \mathbb{Z}_{2p} \longrightarrow \mathbb{Z}_3 \longrightarrow 1$$

(5.80)
of extension class $\alpha \in H^2_{\text{group}}(\mathbb{Z}_2, \mathbb{Z}_p)$.

In both cases, in the boundary WZW model, this reduces to two-dimensional decomposition of a WZW orbifold, with the discrete theta angles becoming choices of discrete torsion. In both cases, as the orbifolds involve cyclic groups, discrete torsion is trivial, so the boundary decomposition yields just a disjoint union of copies of the same WZW orbifold.

### 5.11 Chern-Simons($H_1 \times H_2$)/$BA$

For completeness, let us also briefly discuss decomposition in gauged Chern-Simons theories whose gauge groups are a product of Lie groups. Specifically, consider the gauge of a gauged $BA$ action, for $A$ finite and abelian, on a Chern-Simons theory for $H_1 \times H_2$ (at various levels, such that the gauge theory is well-defined on the given three-manifold). Bulk decomposition takes the same form as (3.15):

$$[\text{Chern–Simons}(H_1 \times H_2)/BA] = \coprod_{\theta \in \hat{K}} \text{Chern–Simons}(G)_\theta,$$

(5.81)
where

$$1 \longrightarrow K \longrightarrow A \xrightarrow{d} H_1 \times H_2 \longrightarrow G \longrightarrow 1,$$

(5.82)
and the discrete theta angle couples to $\beta_\alpha(w_G)$, for $\beta_\alpha$ the Bockstein homomorphism associated to

$$1 \longrightarrow K \longrightarrow A \longrightarrow \mathbb{Z} \longrightarrow 1,$$

(5.83)
classified by $\alpha \in H^2_{\text{group}}(\mathbb{Z}, K)$, where $\mathbb{Z}$ is a subgroup of the product of the centers of $H_{1,2}$, given by the image of $d$.  

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On the boundary, as before, this reduces to decomposition in the two-dimensional theory, here
\[ [WZW(H_1 \times H_2)/A] = \prod_{\theta \in K} WZW(G_\theta), \quad (5.84) \]
where the discrete theta angles \( \theta \) now correspond to choices of discrete torsion in a
\[ [WZW(H_1 \times H_2)/Z] \quad (5.85) \]
orbifold. Essentially because \( A \) is abelian, for ultimately the same reasons as in section 5.7, the discrete torsion is trivial on each universe.

### 5.12 Finite 2-group orbifolds

So far we have focused on Chern-Simons theories in three dimensions, but the same ideas apply to the finite 2-group orbifolds discussed in [30]. There, orbifolds by 2-groups \( \Gamma \) were described, where \( \Gamma \) is an extension
\[ 1 \rightarrow BK \rightarrow \Gamma \rightarrow G \rightarrow 1, \quad (5.86) \]
where \( G, K \) are both finite and \( K \) is abelian, determined by \([\omega] \in H^3_{\text{group}}(G, K)\). Now, \( \Gamma \) can also be described by a crossed module \( \{d : A \rightarrow H\} \), corresponding to a four-term exact sequence of ordinary groups
\[ 1 \rightarrow K \rightarrow A \xrightarrow{d} H \rightarrow G \rightarrow 1, \quad (5.87) \]
also determined (up to equivalences) by \([\omega] \in H^3_{\text{group}}(G, K)\) (see e.g. [127] section IV.9 for related observations).

In this language, we can write the 2-group orbifold \([X/\Gamma]\) in terms of the crossed module as
\[ [X/\Gamma] = [(X/H)/BA], \quad (5.88) \]
at least for a presentation in which \( A \) is abelian.

For this slightly different physical realization in terms of finite groups, the statement of decomposition (3.15) is modified, but only slightly:
\[ [X/\Gamma] = [(X/H)/BA] = \bigoplus_{\theta \in K} [X/G]_{\omega(\theta)}, \quad (5.89) \]
where the discrete torsion (formerly discrete theta angle) \( \omega(\theta) \) is defined by \( \phi^*\omega \). In this sense, the decomposition described in this paper is simply a variation on the 2-group orbifold decomposition described in [30]. The fact that bulk discrete theta angles (here, \( C \)-field analogues of discrete torsion) become (ordinary) discrete torsion in the boundary theory was also observed in [30 section 3.2].
In passing, we should also observe that results in finite 2-group orbifolds have a qualitatively different form. For example, [30, section 4.4] described an orbifold by a 2-group extension
\[ 1 \to B\mathbb{Z}_2 \to \Gamma \to (\mathbb{Z}_2)^3 \to 1. \] (5.90)
In this case, \([X/\Gamma]\) is equivalent to a pair of copies of \([X/(\mathbb{Z}_2)^3]\) orbifolds, each with a different field discrete torsion in \(H^3_{\text{group}}((\mathbb{Z}_2)^3, U(1))\), which is nontrivial even on \(T^3\). One could imagine an analogous theory here, such as a quotient of \(SU(2)^3\) Chern-Simons by \(B\mathbb{A}\) (for \(\mathbb{A}\) a finite abelian group, with \(K = \mathbb{Z}_2\) kernel, say) that leads to a disjoint union of \(SO(3)^3\) Chern-Simons theories. Here, however, in the case of Chern-Simons theories, no analogue of \(C\) field discrete torsion is present for \(T^3\), partly because (as noted in section 3.4) the pertinent Bockstein homomorphism vanishes. Part of the difference between these two theories is that in the Chern-Simons case, the pertinent exact sequence of finite groups has the form
\[ 1 \to \mathbb{Z}_2 \to A \to (\mathbb{Z}_2)^3 \to 1, \] (5.91)
whereas by contrast the analogous sequence in [30], namely (5.90), can be alternately encoded as a four-term sequence
\[ 1 \to \mathbb{Z}_2 \to P' \to Q' \to (\mathbb{Z}_2)^3 \to 1, \] (5.92)
which realizes an element of \(H^3_{\text{group}}((\mathbb{Z}_2)^3, \mathbb{Z}_2)\). By contrast, the short exact sequence (5.91) realizes an element of \(H^2_{\text{group}}((\mathbb{Z}_2)^3, \mathbb{Z}_2)\), cohomology of different degree; the crossed module construction realizes a 2-group, but involves different groups.

6 Boundary \(G/G\) models

For completeness, in this section we include a different example of a decomposition.

Consider gauged WZW models \(G/H\) at level \(k\), on the boundary of a three-dimensional theory. Because the \(H\) action being gauged is an adjoint action [128], if the center \(Z(H)\) of \(H\) is nonzero, it acts trivially, and in two dimensions, the resulting gauged WZW model decomposes into universes indexed by irreducible representations of \(Z(H)\).

Now, let us compare to the bulk theory. From [69, section 3], for the gauged WZW model \(G/H\) at level \(k\), the bulk three-dimensional theory is a \((G \times H)/Z\) gauge theory, with \(Z\) the commmon center of \(G\) and \(H\), with action
\[ k\ell S_{CS}(G) - kS_{CS}(H), \] (6.1)
where \(\ell\) is the index of the embedding \(H \hookrightarrow G\).

Consider the special case of the two-dimensional \(G/G\) model, on the boundary of a three-dimensional theory. The \(G/G\) model decomposes into universes indexed by the integrable
representations. (In principle this is because it is a unitary topological field theory \[129,130\]; the specific relation to decomposition is via noninvertible symmetries, as discussed in \[20,21\].) From the discussion above, the bulk dual to the boundary $G/G$ model appears to have an identically-zero action (6.1). Since the boundary theory is a topological field theory, this would be trivially consistent.

For more general boundary $G/H$ gauged WZW models, the bulk action (6.1) does not vanish identically. Decomposition of the boundary suggests that the bulk may also decompose, in which case the bulk theory should admit a global two-form symmetry. We leave elucidating that symmetry for future work.

7 Conclusions

In this paper, we have discussed decomposition in three-dimensional Chern-Simons theories with gauged noneffectively-acting one-form symmetries. In the bulk decomposition, the different universes of the decomposition have discrete theta angles coupling to bundle characteristic classes, specifically, images under Bockstein maps of canonical degree-two characteristic classes. On the boundary, those map to choices of discrete torsion, and the bulk decomposition becomes a standard orbifold decomposition, involving WZW models, which serves as a strong consistency test.

There are many directions this work could be taken. One example would be to consider decomposition in gauged Chern-Simons theories in which the original theory has a discrete theta angle, analogous to decomposition in two-dimensional orbifolds with discrete torsion \[26\]. Another example would be to consider decomposition in Chern-Simons-matter theories, rather than pure Chern-Simons. Similarly, it would be interesting to consider decomposition in holomorphic Chern-Simons \[131\], or deformations of Chern-Simons theories, as arise when studying disk instanton corrections in string compactifications.

It would also be interesting to understand dimensional reduction of decomposition to two dimensions. The dimensional reduction of pure Chern-Simons is the two-dimensional $G/G$ model (which as a unitary TFT already admits a decomposition \[20,21,129,130\]), and the $BK$ symmetry in three dimensions should become a $K \times BK$ symmetry in the two-dimensional theory.

In condensed matter physics, there exists a realization of Chern-Simons theories known as the Levin-Wen model \[134\], and it would be interesting to consider this story in that setting.

In a different direction, Chern-Simons theories can also arise on boundaries of four-dimensional theories, and it would be interesting to study decomposition in that context, perhaps relating it to the decomposition arising after instanton restriction in \[16\].
the instanton restriction resulted in a disjoint union of four-dimensional Yang-Mills theories with theta angle terms of the form
\[ \frac{1}{8\pi^2} \frac{2\pi m}{k} \int \text{Tr} F \wedge F, \tag{7.1} \]
for \( m \in \{0, 1, \cdots, k-1\} \), which implements the restriction on instantons. On a boundary, that would become a disjoint union of theories, whose actions have Chern-Simons terms of the form
\[ \frac{1}{8\pi^2} \frac{2\pi m}{k} \int \omega_{\text{CS}}, \tag{7.2} \]
clearly related to the disjoint unions of Chern-Simons theories we discuss in this paper. We leave such considerations for future work.

8 Acknowledgements

We would like to thank C. Closset, S. Datta, J. Distler, D. Berwick Evans, T. Gomez, S. Gukov, L. Lin, D. Robbins, I. Runkel, U. Schreiber, J. Stasheff, Y. Tachikawa, T. Vandermeulen, and K. Waldorf for useful discussions. We would further like to thank M. Yu for initial collaboration and many discussions. T.P. was partially supported by NSF/BSF grant DMS-2200914, NSF grant DMS-1901876, and Simons Collaboration grant number 347070. E.S. was partially supported by NSF grant PHY-2014086.

A Line operators

In this appendix we briefly review some basics of line operators in Chern-Simons theories and their quantum numbers, to make this paper self-contained.

In general, the line operators in a Chern-Simons theory at level \( k \) correspond to integrable representations, which for a model at level \( k \), are the representations of highest weight \( \lambda \) satisfying the unitarity bound \[ \psi \cdot \lambda \leq k, \tag{A.1} \]
for \( \psi \) the highest weight of the adjoint representation. (For example, for \( SU(n) \) the integrable representations at any level are classified by Young diagrams of width bounded by the level.) Similarly, for a given WZW primary associated to an integrable representation of highest weight \( \lambda \), the \( L_0 \) eigenvalue is \[ h = \frac{(\lambda, \lambda + 2\rho)}{2(k + g)}, \tag{A.2} \]
where $g$ is the dual Coxeter number and $\rho$ the Weyl vector (half-sum of positive roots). In passing, a representation is integrable if and only if its dual is integrable, and it and its dual define WZW primaries of the same conformal weight, see e.g. [126, section 8.3]. Similarly, the quantum dimension is given by [133, equ'n (16.66)]

$$\prod_{\alpha > 0} \frac{\sin \left( \frac{\pi (\lambda + \rho, \alpha)}{k+g} \right)}{\sin \left( \frac{\pi (\rho, \alpha)}{k+g} \right)}.$$  

(A.3)

For use in examples in the text, the line operators of $SU(2)_4$ are[14]

| $SU(2)_4$ | Integrable rep. | $\tilde{\lambda}$ | $h$ | q-dim |
|-----------|----------------|------------------|----|-------|
| (0)       | $1$            | [0, 4]           | 0  | 1     |
| (1)       | $\boxed{1}$   | [4, 0]           | 1  | 1     |
| (2)       | $\boxed{1}$   | [1, 3]           | $\frac{1}{3}$ | $\sqrt{3}$ |
| (3)       | $\boxed{1}$   | [3, 1]           | $\frac{5}{8}$ | $\sqrt{3}$ |
| (4)       | $\boxed{1}$   | [2, 2]           | $\frac{1}{3}$ | 2     |

where $\tilde{\lambda}$ denotes the Dynkin label of each line, $h$ is the conformal weight of the corresponding boundary chiral primary as above, and q-dim denotes the quantum dimension.

The fusion algebra of $SU(2)_4$ lines can be computed with the program Kac [135], and that algebra is given below:

$$(0) \times (0) = (0), \quad (2) \times (2) = (0) + (4),$$
$$(0) \times (1) = (1), \quad (2) \times (3) = (1) + (4),$$
$$(0) \times (2) = (2), \quad (2) \times (4) = (2) + (3),$$
$$(0) \times (3) = (3), \quad (3) \times (3) = (0) + (4),$$
$$(0) \times (4) = (4), \quad (3) \times (4) = (2) + (3),$$
$$(1) \times (1) = (0), \quad (4) \times (4) = (0) + (1) + (4).$$
$$(1) \times (2) = (3),$$
$$(1) \times (3) = (2),$$
$$(1) \times (4) = (4),$$

We see that the lines (0), (1) are mutually transparent, and their fusion products have the structure of the group $\mathbb{Z}_2$.

From the table above, it is straightforward to compute the monodromies of the line (1) about other lines, using

$$B(a, b) = \exp (2\pi i \ (h(a \times b) - h(a) - h(b))),$$

(A.4)

[14] We would like to thank M. Yu for providing the results for line operators of $SU(2)_4$ and $U(1)_8$ listed in this appendix.
and one finds

\[ B(1,1) = +1, \]  
\[ B(1,2) = -1, \]  
\[ B(1,3) = -1, \]  
\[ B(1,4) = +1, \]

so that all monodromies are in \( \{ \pm 1 \} \), as expected for a \( B\mathbb{Z}_2 \), and also consistent with the fact that (2) and (3) correspond to Wilson lines for an odd number of copies of the \( k \) representation.

Similarly, it will be useful later to write down the fusion algebra for \( U(1)_8 \). Here, there are eight lines, labelled (0) through (7), with conformal weights and quantum dimensions

| \( U(1)_8 \) | h     | q-dim |
|-----------|-------|-------|
| (0)       | 0     | 1     |
| (1)       | 1/16  | 1     |
| (2)       | 1/4   | 1     |
| (3)       | 9/16  | 1     |
| (4)       | 1     | 1     |
| (5)       | 9/16  | 1     |
| (6)       | 1/4   | 1     |
| (7)       | 1/16  | 1     |

and the fusion algebra acts by addition, as

\[ (a) \times (b) = (a + b \mod 8). \]  

From the table of lines above, it is clear that there is a \( B\mathbb{Z}_2 \) corresponding to the lines \{ (0), (4) \}. For use in section 5.9 we list here pertinent monodromies:

\[ B((0),L) = +1, \quad B(4,0) = B(4,2) = B(4,4) = B(4,6) = +1, \]  
\[ B(4,1) = B(4,3) = B(4,5) = B(4,7) = -1. \]

B Overview of crossed modules

In this paper we have described 2-groups using crossed modules. As they play an important role in the decomposition statement in three-dimensional Chern-Simons theories, to make this paper self-contained we include a brief overview here.

Briefly, a crossed module consists of the following data:
• a pair of groups \( G_0, G_1 \),
• a group homomorphism \( d : G_1 \to G_0 \),
• a group homomorphism \( \alpha : G_0 \to \text{Aut}(G_1) \),

such that

1. the composition
\[
G_1 \xrightarrow{d} G_1 \xrightarrow{\alpha} \text{Aut}(G_1)
\] (B.1)
is the conjugation action of \( G_1 \) on itself, meaning
\[
\alpha(d(g_1))(h) = g_1 h g_1^{-1},
\] (B.2)
for \( g_1, h \in G_1 \), or equivalently that
\[
G_1 \times G_1 \xrightarrow{d \times \text{Id}} G_0 \times G_1 \xrightarrow{\alpha} G_1
\] (B.3)
commutes,

2. \( d \) is equivariant for the \( G_0 \) action on the source and target, meaning
\[
d(\alpha(g_0))(h) = g_0 d(h) g_0^{-1}
\] (B.4)
for \( g_0, h \in G_0 \), or equivalently that
\[
G_0 \times G_1 \xrightarrow{\alpha} G_1 \xrightarrow{d} G_1 \xrightarrow{\alpha} G_0
\] (B.5)
commutes.

In the description above, \( \text{Ad} : G \to \text{Aut}(G) \) denotes the adjoint action of \( G \) to itself, namely \( \text{Ad}(g)(x) = gxg^{-1} \).

Some examples of crossed modules include the following:

• For \( G_1 \) any group, let \( G_0 = \text{Aut}(G_1) \), with \( d : G_1 \to \text{Aut}(G_1) \) the natural inclusion (meaning \( d(g) = \text{Ad}(g) \)) and \( \alpha : \text{Aut}(G_1) \to \text{Aut}(G_1) \) the identity.
• Let $G_0$ be any group and $G_1$ a normal subgroup of $G_0$, with $d : G_1 \to G_0$ inclusion, and $\alpha : G_0 \to \text{Aut}(G_1)$ by conjugation.

A crossed module can be encoded in a four-term exact sequence:

$$1 \to \text{Ker} \ d \to G_1 \to G_0 \to \text{Coker} \ d \to 1.$$  \hspace{1cm} (B.6)

In the case that Ker $d$ is abelian, this is sometimes alternatively expressed as the extension

$$1 \to B(\text{Ker} \ d) \to \Gamma \to \text{Coker} \ d \to 1,$$  \hspace{1cm} (B.7)

for $\Gamma$ the 2-group corresponding to the crossed module.

Physically, in this paper, the map $d$ encodes the action of the noneffectively-acting $B A$, by mapping $A$ to a subset of the center of the Chern-Simons gauge group, which acts nontrivially.

For more information on crossed modules, see for example [136] for further mathematics background, or [74, appendix A], [75, section 2] in physics.

C Generalities on gauging effectively-acting one-form symmetries

For most of this paper, we have discussed gauging one-form symmetries in terms of line operators, but it is worth observing that this operation can also be understood in terms of local actions, which we will briefly review in this section.

Suppose in general terms we have a $G$ gauge theory, and we gauge the action of a one-form symmetry $B K$, where $B K$ acts nontrivially on the line operators of the theory. (For example, this is the case if $K$ is a subset of the center of $G$.)

In general terms, when gauging the $B K$ on a $G$ gauge theory,

• the path integral sums over $K$ gerbes, and
• for each $K$ gerbe, the path integral sums over gerbe-twisted $G$ bundles, defined by transition functions which close on triple overlaps only up to a cocycle representing the gerbe characteristic class.

Consider for example gauging an effectively-acting $B \mathbb{Z}_n$ in an $SU(n)$ gauge theory. The twisted $SU(n)$ gauge fields above are all the same as ordinary $SU(n)/\mathbb{Z}_n$ gauge fields, and the gerbe characteristic classes correspond to (some) characteristic classes of $SU(n)/\mathbb{Z}_n$ bundles. Let us look at this in more detail:
1. The transition functions $g_{ij}$ of a twisted bundle no longer close on triple overlaps, but rather obey

$$g_{ij}g_{jk}g_{ki} = h_{ijk}$$  \hspace{1cm} (C.1)

for a cocycle $h_{ijk}$ representing an element of $H^2(Y, \mathbb{Z}_n)$ corresponding to the gerbe characteristic class, and

2. Across overlaps, the gauge field $A$ obeys

$$A_i = g_{ij}A_jg^{-1}_{ij} + g^{-1}_{ij}dg_{ij} - IA_{ij},$$  \hspace{1cm} (C.2)

where $I$ is the identity and $\Lambda_{ij}$ is a locally-defined one-form field, with the property that if the gerbe were to admit a connection $B$, then on the same overlaps

$$B_i = B_j + d\Lambda_{ij}. \hspace{1cm} (C.3)$$

Now, this procedure should generate all $G/K$ bundles. One example of this involves the relation between $SU(2)$ and $SO(3)$ bundles in three-dimensional theories. As is well-known,

$$\text{Chern-Simons}(SU(2))/B\mathbb{Z}_2 = \text{Chern-Simons}(SO(3)), \hspace{1cm} (C.4)$$

for the $B\mathbb{Z}_2$ corresponding to the center one-form symmetry. Viewed as a $B\mathbb{Z}_2$ quotient of an $SU(2)$ gauge theory, the path integral

- sums over $\mathbb{Z}_2$ gerbes, whose characteristic class is $w \in H^2(M, \mathbb{Z}_2)$, and
- sums over $w$-twisted $SU(2)$ bundles, meaning that the $SU(2)$ transition functions close on triple overlaps only up to $w$, and that gauge transformations across patches only have to match up to a $\mathbb{Z}_2$ shift.

Interpreted in terms of $SO(3)$ bundles, the characteristic class $w \in H^2(M, \mathbb{Z}_2)$ is the second Stiefel-Whitney class of an $SO(3)$ bundle. (The other possibly nonzero characteristic class, the third Stiefel-Whitney class $w_3 \in H^3(M, \mathbb{Z}_2)$, is determined by $w_3 = Sq^1(w_2)$, see section [5.3]) The fact that gauge transformations only respect $SU(2)$ up to $\mathbb{Z}_2$ shifts, and that $SU(2)$ transition functions only close up to $w$, are indicative of general aspects of $SO(3)$ bundles.

Thus, we see that the $B\mathbb{Z}_2$-gauged $SU(2)$ theory really does recover all $SO(3)$ bundles, even those with nonzero $w_3$, as expected.

If we instead gauged a $BA$ action on a $G$ Chern-Simons theory with a trivially-acting subgroup $BK$, then, for reasons detailed in [30], we would recover $G/(A/K)$ gauge theory, with a restriction on $G/(A/K)$ bundles. One role of decomposition is to implement that restriction.
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