PRESERVING THE MEASURE OF COMPATIBILITY BETWEEN QUANTUM STATES

LAJOS MOLNÁR
Institute of Mathematics and Informatics
University of Debrecen
4010 Debrecen, P.O.Box 12
Hungary
e-mail: molnarl@math.klte.hu

and

WERNER TIMMERMANN
Institut für Analysis
Technische Universität Dresden
D-01062 Dresden
Germany
e-mail: timmerma@math.tu-dresden.de

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Abstract
In this paper after defining the abstract concept of compatibility-like functions on quantum states, we prove that every bijective transformation on the set of all states which preserves such a function is implemented by an either unitary or antiunitary operator.
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In the last couple of years several communications have appeared in connection with the following problem raised by R. Peierls [8]: when different density matrices can characterize the knowledge available to different people about one and the same physical system? The first answer given by Peierls in [8, 9] was that the density matrices under consideration must commute and their product must be nonzero. However, C. Fuchs [4] (also see [5]) gave an example which made Peierls’ first condition questionable. After that several attempts have been made to find the proper solution of the problem (e.g., [1, 5]). All those attempts operate with the concept of compatibility of density matrices. According to them, we say that a collection of density matrices is compatible if the supports of the matrices under consideration (i.e., the orthogonal complements of their null spaces) have nontrivial intersection. So, it is just an easy task to determine whether a pair of density matrices is compatible or not. Having this in mind, it is now a natural problem to give sense to the following question: if a pair of density matrices is compatible, then ”how much” compatible they are. In other words, we arrive at the problem of measuring the compatibility. One possibility to define such a measure was described in [10]. Namely, in some analogy with the fidelity, C. Poulin and R. Blume-Kohout defined a compatibility function [10, Definition 1] which fulfils certain natural physical requirements and they proved some important properties.

In our recent paper [7] we have determined the structure of the bijective transformations on the set of all density operators which preserve the fidelity. This result is in close relation with Wigner’s theorem on symmetry transformations. In fact, it can be considered as a Wigner-type result for the set of all mixed states (recall that Wigner’s original
result concerns the pure states). In [7] we proved that the transformations in question are all implemented by unitary or antiunitary operators on the underlying Hilbert space. In view of this result and the analogy between the fidelity and the measure of compatibility defined by Poulin and Blume-Kohout, it is a natural problem to determine the structure of the bijective transformations of the set of all density matrices which preserve the compatibility function. We shall see below that the solution of this problem is the same as the one concerning fidelity. This is the content of the present paper.

Let us begin with the notation. Let $H$ be a (complex, not necessarily finite dimensional) Hilbert space. If not stated otherwise, all operators on $H$ are meant to be bounded and linear. The expression $\text{rng } A$ denotes the range of the operator $A$. If $A$ is positive, $A^{1/2}$ stands for its unique positive square root.

Denote by $S(H)$ the set of all states (or, in another terminology, density operators) on $H$ i.e., the positive trace-class operators on $H$ with trace 1. The set $S(H)$ is a convex subset of the space of all self-adjoint operators on $H$ and its extreme points (which are exactly the rank-one projections) are called pure states.

Now, instead of using the concept due to Poulin and Blume-Kohout, we define the abstract concept of compatibility-like functions which extends [10, Definition 1] to obtain a result of higher generality.

**Definition 1.** Let $C : S(H) \times S(H) \to [0, 1]$ be a function such that for any pair $A, B \in S(H)$ of states we have

(i) $C(A, B) = 0$ if and only if $\text{rng } A^{1/2} \cap \text{rng } B^{1/2} = \{0\},$

(ii) $C(A, B) = C(B, A),$

...
(iii) if $P$ is a pure state, then

$$C(A, P)^2 = \sup\{\lambda \in [0, 1] : \lambda P \leq A\}.$$  

We say that $C$ is a compatibility-like function on the set of all states on the Hilbert space $H$.

Several remarks should be made concerning the above definition. First, we emphasize that our definition is formulated for both finite and infinite dimensional Hilbert spaces (in [10, Definition 1] only finite dimensional spaces were considered). Concerning the correctness of the definition we note the following. The quantity on the right hand side of the equality in (iii) also appears in relation with effects. A self-adjoint operator $T$ on $H$ with the property $0 \leq T \leq I$ ($I$ is the identity operator) is called an effect. The effects are well-known to play important role in the quantum theory of measurement (e.g., [3]). Now, it is clear that every state on $H$ as a linear operator can also be viewed as an effect. If $T$ is an effect, $\varphi$ is a unit vector in $H$ and $P_\varphi$ denotes the rank-one projection onto the subspace generated by $\varphi$, then the quantity

$$\lambda(T, P_\varphi) = \sup\{\lambda \in [0, 1] : \lambda P_\varphi \leq T\}$$

is called the strength of $T$ along the ray represented by $\varphi$. This concept was introduced by Busch and Gudder in [2]. It was proved in [2, Theorem 3] that $\lambda(T, P_\varphi) = 0$ if and only is $\varphi \notin \text{rng } T^{1/2}$ which is equivalent to $\text{rng } T^{1/2} \cap \text{rng } P^{1/2}_\varphi = \{0\}$. This means that there is no contradiction between the conditions (i) and (iii).

Observe that if $H$ is finite dimensional, then the ranges of a positive operator and its square root are the same and they are automatically
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Therefore, in the finite dimensional case we have

$$\text{rng } A^{1/2} = \text{rng } A = \overline{\text{rng } A} = (\ker A)$$

and hence (i) says that $C(A, B) > 0$ if and only if $A, B$ are compatible in the sense mentioned in the introduction. The meaning of (ii) is clear. Now, what about (iii)? One might think that this condition is quite restrictive and probably has no physical meaning for states. But it can be shown that the compatibility function defined by Poulin and Blume-Kohout satisfies (iii) (see either [10, Definition 1] itself or [10, Theorem 3]) as well as (i) and (ii). So, to sum up, our definition is a generalization of the one given by Poulin and Blume-Kohout and hence it certainly has sense at least from the mathematical point of view.

The reason why we assume (iii) is that there is a nice formula to compute $C(A, P)$. Namely, by [2, Theorem 4] for every unit vector $\varphi \in H$ we have

\[
C(A, P_\varphi)^2 = \lambda(A, P_\varphi) = \begin{cases} 
\|A^{-1/2}\varphi\|^{-2}, & \text{if } \varphi \in \text{rng}(A^{1/2}); \\
0, & \text{else.}
\end{cases}
\]

(Here $A^{-1/2}$ denotes the inverse of $A^{1/2}$ on $\text{rng } A^{1/2}$.) The proof of our result is based on this correspondence.

We further note that it would be another natural assumption to suppose that $C$ is invariant under unitary-antiunitary transformations. But, as we do not need it in our proof, we do not assume it.

To conclude our remarks, we show a natural example for a compatibility-like function which might also justify our definition. So,
for any pair $A, B \in S(H)$ define

$$C(A, B) = \sup\left\{ \sum_n \sqrt{\lambda_n \mu_n} : \lambda_n, \mu_n \in [0, 1], \sum_n \lambda_n = \sum_n \mu_n = 1 \text{ and } \exists \text{ pure states } Q_n \text{ with } \sum_n \lambda_n Q_n = A, \sum_n \mu_n Q_n = B \right\}.$$  

It is easy to verify that this function has the properties (i)-(iii). In fact, in accordance with the discussions in [1] and [5], we believe that this compatibility-like function represents the most natural way of defining a measure of compatibility between quantum states.

Now, our result reads as follows.

**Theorem 2.** Let $H$ be a Hilbert space and let $C$ be a compatibility-like function on $S(H)$. Let $\phi : S(H) \rightarrow S(H)$ be a bijective function which preserves $C$, that is, assume that

$$C(\phi(A), \phi(B)) = C(A, B) \quad (A, B \in S(H)).$$

Then there exists an either unitary or antiunitary operator $U$ on $H$ such that $\phi$ is of the form

$$\phi(A) = UAU^* \quad (A \in S(H)).$$

**Proof.** Clearly, we can assume that $\dim H \geq 2$. For temporary use, we say that the states $D, A$ are compatible (resp. incompatible) if $C(D, A) > 0$ (resp. $C(D, A) = 0$). It is useful to introduce the following notation. If $\mathcal{M}$ is a subset of $S(H)$, then denote

$$\mathcal{M}^{\text{ic}} = \{ D \in S(H) : C(D, A) = 0 \text{ for all } A \in \mathcal{M} \}.$$  

By condition (i) in Definition[1], $C(D, A) = 0$ means that the subspaces $\text{rng } D^{1/2}, \text{ rng } A^{1/2}$ of $H$ have trivial intersection. It can be easily verified
that the operator \( A \in S(H) \) has rank one (which means that \( A \) is a pure state) if and only if

\[
(\{A\}^ic)^ic = \{A\}.
\]

Since \( \phi \) preserves the compatibility in both directions, it follows from this characterization that \( \phi \) preserves the pure states in both directions. This means that \( A \in S(H) \) is a pure state if and only if so is \( \phi(A) \).

Next we assert that \( \phi \) maps independent pure states to independent ones. Here, a set of \( n \) pure states (rank-one projections) is called independent if their ranges generate an \( n \)-dimensional subspace of \( H \). To prove the assertion we use induction. The statement is obvious if the set has only one element. Let \( \{P_1, \ldots, P_n, P_{n+1}\} \) be a set of \( n + 1 \) pure states such that the subset \( \{P_1, \ldots, P_n\} \) is independent. It is easy to see that \( \{P_1, \ldots, P_n, P_{n+1}\} \) is dependent if and only if for any \( A \in S(H) \) with \( C(A, P_1) > 0, \ldots, C(A, P_n) > 0 \) we have that \( C(A, P_{n+1}) > 0 \) holds too. Indeed, this follows from the fact that for any pure state \( P \) we have \( C(A, P) > 0 \) if and only if the range of \( P \) is included in the range of \( A^{1/2} \) (see the remarks after the Definition 1). Using the above description of dependence, it is now clear that assuming \( \phi \) maps independent sets of \( n \) pure states to sets of the same kind, we have the same property of \( \phi \) for \( n + 1 \) in the place of \( n \). Since \( \phi^{-1} \) has the same properties as \( \phi \), we deduce that \( \phi \) preserves the independence of the sets of pure states in both directions.

It is easy to see that an operator \( A \in S(H) \) has rank \( n \) if and only if there exists an independent set of \( n \) pure states such that \( C(A, P) > 0 \) for every element of that set, but there does not exist a set of \( n + 1 \) elements having the same property. This gives us that \( \phi \) preserves the rank.
Now we prove that \( \phi \) preserves the transition probability between pure states. Recall that for any pair \( P, Q \) of pure states, the transition probability between them is \( \text{tr} \, PQ \), where \( \text{tr} \) is the usual trace-functional. To verify the mentioned preserver property of \( \phi \), first let \( P, Q \) be rank-one projections with orthogonal ranges. Define

\[
A = \lambda P + \mu Q,
\]

where \( \lambda, \mu \) are fixed and satisfy \( 0 < \lambda < \mu < 1, \lambda + \mu = 1 \). Clearly, \( A \) acts on the 2-dimensional subspace \( H_A \) of \( H \) generated by the ranges of \( P, Q \). (Here, the phrase that \( A \) acts on \( H_A \) means that \( (\ker A)^\perp = \overline{\text{rng} \, A} = \text{rng} A = H_A \).) We assert that \( \phi(A) \) acts on the subspace generated by the ranges of the independent pure states \( \phi(P), \phi(Q) \). Indeed, \( \phi(A) \) has rank 2 and taking into account that

\[
C(\phi(A), \phi(P)) = C(A, P) > 0, \quad C(\phi(A), \phi(Q)) = C(A, Q) > 0,
\]

we see that the ranges of \( \phi(P), \phi(Q) \) are included in \( \text{rng} \phi(A)^{1/2} = \text{rng} \phi(A) \). This clearly implies our assertion. In what follows we restrict the considerations onto those 2-dimensional subspaces, that is, to the ranges of \( A \) and \( \phi(A) \), respectively. By property (iii) in the definition of compatibility-like functions, we see that

\[
\lambda \leq C(A, R)^2 \leq \mu
\]

holds for every rank one projection \( R \) on the range of \( A \). As \( \phi \) preserves \( C \), we have

\[
\lambda \leq C(\phi(A), \phi(R))^2 \leq \mu
\]

for every rank one projection \( \phi(R) \) on the range of \( \phi(A) \). Moreover, we have

\[
C(\phi(A), \phi(P))^2 = C(A, P)^2 = \lambda, \quad C(\phi(A), \phi(Q))^2 = C(A, Q)^2 = \mu.
\]
Now we refer to a result in [6]. Namely, Lemma 3 given there states that if $T$ is an effect and $0 < \epsilon < \delta \leq 1$ are scalars such that $\epsilon I \leq T \leq \delta I$ and we have unit vectors $\varphi, \psi \in H$ such that $\lambda(T, P_\varphi) = \epsilon$ and $\lambda(T, P_\psi) = \delta$, then $\varphi, \psi$ are eigenvectors of $T$ and the corresponding eigenvalues are $\epsilon, \delta$, respectively. Using this result and the correspondence between compatibility-like functions and the strength, we obtain that the range of $\phi(P)$ is the eigensubspace of $\phi(A)$ corresponding to the eigenvalue $\lambda$ and the range of $\phi(Q)$ is the eigensubspace of $\phi(A)$ corresponding to the eigenvalue $\mu$. Therefore, we have

\begin{equation}
\phi(A) = \lambda \phi(P) + \mu \phi(Q).
\end{equation}

Now let $P, R$ be arbitrary rank-one projections. Pick a rank-one projection $Q$ which is orthogonal to $P$ such that the subspace generated by the ranges of $P$ and $Q$ includes the range of $R$. Let $\lambda, \mu$ and $A$ be as above. It is easy to check that by the formula (1) we have

\begin{equation}
C^2(A, R) = \frac{1}{\frac{1}{\lambda} \text{tr} PR + \frac{1}{\mu} \text{tr} QR} = \frac{\lambda \mu}{\mu \text{tr} PR + \lambda \text{tr} QR} = \frac{\lambda \mu}{\mu \text{tr} PR + \lambda (1 - \text{tr} PR)} = \frac{\lambda \mu}{(\mu - \lambda) \text{tr} PR + \lambda}.
\end{equation}

As the spectral resolution of $\phi(A)$ is (3), we similarly have

\begin{equation}
C^2(\phi(A), \phi(R)) = \frac{\lambda \mu}{(\mu - \lambda) \text{tr} \phi(P)\phi(R) + \lambda}.
\end{equation}

Since $\phi$ preserves $C$, it follows from (3) and (4) that

$$\text{tr} \phi(P)\phi(R) = \text{tr} PR,$$

which means that $\phi$ preserves the transition probability between pure states. It follows from Wigner's theorem that $\phi$, when restricted onto
the set of all pure states, is of the form
\[ \phi(P) = UPU^* \]
for some unitary or antiunitary operator \( U \) on \( H \).

It remains to show that the above formula holds for every state as well. The proof goes as follows. Let \( A \in S(H) \). For every rank-one projection \( P \) we compute
\[
\lambda(UAU^*, P) = \lambda(A, U^*PU) = C(A, U^*PU)^2 = C(\phi(A), \phi(U^*PU))^2 = C(\phi(A), P)^2 = \lambda(\phi(A), P).
\]
Now we refer to [2, Corollary 1] which states that if the strengths of two effects are the same along every ray, then the effects in question are equal. This gives us that
\[
\phi(A) = UAU^* \quad (A \in S(H))
\]
and the proof is complete.

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