Structure of the low-lying positive parity states in the proton-neutron symplectic model

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Abstract. The proton-neutron symplectic model with $Sp(12,R)$ dynamical symmetry is applied for the simultaneous description of the microscopic structure of the low-lying states of the ground state, $\gamma$ and $\beta$ bands in $^{166}$Er. For this purpose, the model Hamiltonian is diagonalized in the space of stretched states by exploiting the $SU_p(3) \otimes SU_n(3)$ symmetry-adapted basis. The theoretical predictions are compared with experiment and some other microscopic collective models, like the one-component $Sp(6,R)$ symplectic and pseudo-$SU(3)$ models. A good description of the energy levels of the three bands under consideration, as well as the enhanced intraband $B(E2)$ transition strengths between the states of the ground and $\gamma$ bands is obtained without the use of effective charges. The results show the presence of a good $SU(3)$ dynamical symmetry. It is also shown that, in contrast to the $Sp(6,R)$ case, the lowest excited bands, e.g., the $\beta$ and $\gamma$ bands, naturally appear together with the ground state band within a single $Sp(12,R)$ irreducible representation.

1. Introduction

Experimental spectra in heavy nuclei show the emergence of simple collective patterns represented primarily by the nuclear collective rotation. The microscopic shell-model structure of these low-lying rotational states is still a challenge for the microscopic many-particle nuclear theory. This is particularly so because the model space dimensionalities rule out the use of standard shell-model theory. As a consequence, different algebraic models which capitalize on symmetries, exact or approximate, have been developed to reduce the model space in manageable size.

In the present work, the recently proposed [1, 2] proton-neutron symplectic model (PNSM) with $Sp(12,R)$ dynamical symmetry is applied for the simultaneous description of the microscopic structure of the low-lying states of the ground state, $\gamma$ and $\beta$ bands in $^{166}$Er. For this purpose, the model Hamiltonian is diagonalized in the space of stretched states by exploiting the $SU_p(3) \otimes SU_n(3)$ symmetry-adapted basis. The theoretical predictions are compared with experiment and some other microscopic collective models, like the one-component $Sp(6,R)$ symplectic [3] and pseudo-$SU(3)$ [4, 5, 6] models. A good description of the energy levels of the three bands under consideration, as well as the enhanced intraband $B(E2)$ transition strengths between the states of the ground and $\gamma$ bands is obtained without the use of effective charges. The results reveal almost the pure $SU(3)$ structure, traditionally assumed for well-deformed heavy mass nuclei in the macroscopic theory of nuclear rotation, like, e.g., the Interacting Boson Model [7].
2. The Proton-Neutron Symplectic Model

Collective observables of the proton-neutron symplectic model, which span the $Sp(12, R)$ algebra, are given by the following one-body operators [1]:

\[ Q_{ij}(\alpha, \beta) = \sum_{s=1}^{m} x_{is}(\alpha) x_{js}(\beta), \]
\[ S_{ij}(\alpha, \beta) = \sum_{s=1}^{m} \left( x_{is}(\alpha) p_{js}(\beta) + p_{is}(\alpha) x_{js}(\beta) \right), \]
\[ L_{ij}(\alpha, \beta) = \sum_{s=1}^{m} \left( x_{is}(\alpha) p_{js}(\beta) - x_{js}(\beta) p_{is}(\alpha) \right), \]
\[ T_{ij}(\alpha, \beta) = \sum_{s=1}^{m} p_{is}(\alpha) p_{js}(\beta), \]

where $i, j = 1, 2, 3$; $\alpha, \beta = p, n$ and $s = 1, \ldots, m = A - 1$. In Eqs. (1)-(4), $x_{is}(\alpha)$ and $p_{is}(\alpha)$ denote the coordinates and corresponding momenta of the translationally-invariant Jacobi vectors of the $m$-quasiparticle two-component nuclear system and $A$ is the number of protons and neutrons.

In terms of the harmonic oscillator creation and annihilation operators

\[ b_{i\alpha,s}^{\dagger} = \sqrt{\frac{m_{\omega}\omega}{2\hbar}} (x_{i\alpha}(\alpha) - \frac{i}{m_{\omega}\omega} p_{i\alpha}(\alpha)), \quad b_{i\alpha,s} = \sqrt{\frac{m_{\omega}\omega}{2\hbar}} (x_{i\alpha}(\alpha) + \frac{i}{m_{\omega}\omega} p_{i\alpha}(\alpha)), \]

the many-particle realization of the $Sp(12, R)$ Lie algebra is given by [2]

\[ F_{ij}(\alpha, \beta) = \sum_{s=1}^{m} b_{i\alpha,s}^{\dagger} b_{j\beta,s}, \quad G_{ij}(\alpha, \beta) = \sum_{s=1}^{m} b_{i\alpha,s} b_{j\beta,s}, \quad A_{ij}(\alpha, \beta) = \frac{1}{2} \sum_{s=1}^{m} (b_{i\alpha,s}^{\dagger} b_{j\beta,s} + b_{j\beta,s}^{\dagger} b_{i\alpha,s}). \]

An $Sp(12, R)$ unitary irreducible representation is characterized by the $U(6)$ quantum numbers $\sigma = [\sigma_1, \ldots, \sigma_6]$ of its lowest-weight state $|\sigma\rangle$, i.e. $|\sigma\rangle$ satisfies

\[ G_{ab}|\sigma\rangle = 0; \quad A_{ab}|\sigma\rangle = 0, \quad a < b; \quad A_{aa}|\sigma\rangle = (\sigma_a + \frac{m}{2})|\sigma\rangle \]

for the indices $a \equiv i\alpha$ and $b \equiv j\beta$ taking the values $1, \ldots, 6$. If we introduce the $U(6)$ tensor product operators $P^{(n)}(F) = [F \times \ldots \times F]^{(n)}$, where $n = [n_1, \ldots, n_6]$ is a partition with even integer parts, then by an $U(6)$ coupling of these tensor products to the lowest-weight $U(6)$ state $|\sigma\rangle$, one constructs the whole basis of states for an $Sp(12, R)$ irrep

\[ |\Psi(\sigma n\rho E)\rangle = |P^{(n)}(F) \times |\sigma\rangle\rangle^{\rho E}, \]

where $E = [E_1, \ldots, E_6]$ indicates the $U(6)$ quantum numbers of the coupled state, $\eta$ labels a basis of states for the coupled $U(6)$ irrep $E$ and $\rho$ is a multiplicity index. In this way we obtain a basis of $Sp(12, R)$ states that reduces the subgroup chain $Sp(12, R) \supset U(6)$. To fix the basis $\eta$ one has to consider further the reduction of the $U(6)$ to the 3-dimensional rotational group $SO(3)$. Thus, in order to completely classify the basis states, we use the following reduction chain [2]:

\[ Sp(12, R) \supset U(6) \supset SU_p(3) \otimes SU_n(3) \supset SU(3) \supset SO(3) \supset SO(2), \]

\[ \sigma \quad n \rho \quad E \quad \gamma \quad (\lambda_p, \mu_p) \quad (\lambda_n, \mu_n) \quad g \quad (\lambda, \mu) \quad K \quad L \quad M \]
which defines a shell-model coupling scheme. The chain (9) corresponds to the following choice of the index $\eta = \gamma(\lambda_p, \mu_p)(\lambda_n, \mu_n)\varphi(\lambda, \mu)KLM$, labeling the basis states (8) of an $Sp(12,R)$ irrep. Each $Sp(12,R)$ irreducible representation is determined by a symplectic bandhead or an intrinsic $U(6)$ space, which in turn is fixed by the underlying proton-neutron shell-model structure. So, the theory becomes completely compatible with the Pauli principle.

3. Application

The first application of the one-component $Sp(6,R)$ symplectic model, using a phenomenological Hamiltonian consisting of harmonic oscillator and collective potential which is expressed as polynomial up to fourth order in the mass quadrupole moment, to $^{20}Ne$ was given in [8]. The results showed an excessive collectivity compared to the experimental data. A more realistic Hamiltonian with paring symplectic symmetry-breaking interaction was used subsequently for the same nucleus with obtaining a better agreement [9].

The contracted version of the symplectic model has been applied for the description of the ground state band ($^{20}Ne$) and ground state and $\gamma$ bands ($^{22}Ne$, $^{24}Mg$) within a single generic, triaxial, symplectic irreducible representation [10]. The contracted version was also extended and applied to some heavy mass even-even nuclei from the rare-earth and actinide regions for the description of the ground state and $\gamma$ bands [11, 12] by using the underlying pseudo-$SU(3)$ scheme.

The first few excited states of the ground state rotational bands up to $L = 6$ in $^{126}Ba, ^{154}Sm, ^{164}Yb$ and $^{166}Er$ have been described in the stretched approximation of the symplectic model in Refs.[13, 14]. The latter is defined as the set of $SU(3)$ states $(\lambda_0 + 2n, \mu_0)$ [15], where $(\lambda_0, \mu_0)$ is the leading irreducible representation for the combined proton-neutron nuclear system and $n = 0, 1, 2, 3, \ldots$. The calculations within the framework of stretched approximation is often called $Sp(2,R)$ submodel of the $Sp(6,R)$ because the set of these basis states can be generated only by the raising symplectic generators adding oscillator quanta only along the $z$-axis which are generators of the subgroup $Sp(2, R) \subset Sp(6, R)$. The energy levels and $E2$ transitions between the states of the ground state band in $^{166}Er$ up to $L = 16$ were determined using the full $Sp(6,R)$ model with a Davidson interaction [16].

All calculations mentioned above proved the earlier observation of Arickx and collaborators for the case of light nuclei [17, 18, 19] that the dominant contributions to the wave functions are presented by the stretched $SU(3)$ states. Moreover, a comparison of the $Sp(2,R)$ and $Sp(6,R)$ models by their application to light and heavy nuclei up to spins $L = 6$ has been performed in Ref.[20]. It was confirmed there that not only the stretched $SU(3)$ states give dominant contribution but also that there is no practical difference from the full $Sp(6,R)$ calculations. The stretched $SU(3)$ states within the framework of the one-component $Sp(6,R)$ symplectic model and its contracted version usually contribute between 80% and 90% to the ground state band wave functions. For example, 90% of the $^{20}Ne$ ground state comes from the $(8, 0)$, $(10, 0)$ and $(12, 0)$ stretched states [8]. Similarly, the stretched states give rise up to 93.7% to the ground state in $^{238}U$ using the contracted symplectec model (CSM) [11]. The same picture was obtained in the recent applications of the symplectic $Sp(6,R)$ scheme with algebraic and schematic many-particle interactions to light and intermediate-mass nuclei [21, 22, 23]. Hence, the restriction of the full symplectic basis to the subset of stretched states turns out to be a valuable approximation for the symplectic model calculations in the heavy nuclei and is the first step toward the comprehensive microscopic description of the low-lying quadrupole collective dynamics. In the present work, we restrict ourselves to this approximation only.

In our application, we use the following model Hamiltonian

$$H = Nh\omega - \frac{1}{2} \chi [Q_p \cdot Q_n - (Q_p \cdot Q_n)_{TE}] - \xi C_2[SU(3)] + aL^2 + bK^2,$$  \hspace{1cm} (10)
Figure 1. (Color online) Comparison of the theoretical and experimental energy levels for the ground, $\gamma (K^+ = 2^+)$ and $\beta (K^+ = 0^+)$ bands in $^{166}$Er. The theoretical results of the $Sp(6,R)$ symplectic model [16] and the (pseudo)contracted symplectic model [12] for the ground band and $\gamma$ band, respectively, are also shown.

where $N = N_p + N_n$ and $Q_\alpha = Q(\alpha, \alpha)$ with $\alpha = p, n$ are given by Eq.(1). A similar Hamiltonian has been used in the pseudo-$SU(3)$ scheme calculations within the framework of the contracted symplectic model [11, 12]. The trace-equivalent part $(Q_p \cdot Q_n)_TE$ [24, 25, 26] is subtracted from the collective potential in order to preserve the mean-field shell structure under the action of the proton-neutron quadrupole-quadrupole interaction [11, 12, 10]. The $SU(3)$ second-order Casimir operator $C_2[SU(3)]$ splits energetically different $SU(3)$ multiplets and in this way determines the bandhead energies of excited bands with respect to the ground state band. Finally, the last two terms in (10), which represent a residual rotor part, allows the experimentally observed moment of inertia and band splitting features to be reproduced without altering the wave functions. The Hamiltonian (10) preserves the symplectic symmetry, thus having $Sp(12,R)$ as its dynamical symmetry. The full dynamics for it therefore occurs within a single irreducible representation of $Sp(12,R)$.

Figure 2. (Color online) Calculated and experimental intraband $B(E2)$ values between the states of the ground and $\gamma (K^+ = 2^+)$ bands in $^{166}$Er. For comparison, the theoretical results of the $Sp(6,R)$ symplectic model [16] and the pseudo-$SU(3)$ model [6] for the ground band and $\gamma$ band, respectively, are also shown. In the pseudo-$SU(3)$ model calculations [6], the used value for the effective charge is $e_{eff} = 1.5e$.
The first point in the practical application of the theory for description of the low-lying collective states in strongly deformed nuclei is the determination of the relevant irreducible representation of $Sp(12, R)$. Different approaches exist to determine the symplectic irrep by fixing the shell-model structure of the ground state using isotropic or anisotropic harmonic oscillator with or without spin-orbit interaction. It is well known that, for heavy mass nuclei from the rare-earth and actinide regions, the latter is strong and destroys the oscillator structure. Due to this, we use the pseudo-$SU(3)$ scheme to determine the relevant irreducible representation of $Sp(12, R)$. The shell-model considerations based on the pseudo-$SU(3)$ thus give the symplectic irrep $\langle \sigma \rangle = (82 + 165 / 4, 52 + 165 / 4, 44 + 165 / 4, 44 + 165 / 4, 44 + 165 / 4, 44 + 165 / 4)$ for $^{166}Er$, which is fixed by the direct product proton-neutron shell-model structure of the ground state $(10, 4) \otimes (20, 4)$. The symplectic band structure so obtained contains a plethora of $SU(3)$ multiplets, namely:

$$(10, 4) \otimes (20, 4) \rightarrow (30, 8), (31, 6), (32, 4), \ldots$$
$$(28, 9), (29, 7), (30, 5), \ldots$$
$$(26, 10), (27, 8), (28, 6), \ldots$$

which are appropriate for the simultaneous shell-model description of the lowest collective bands, observed in the spectrum of $^{168}Er$.

After obtaining the appropriate symplectic irrep, the model Hamiltonian (10) is further used to determine the microscopic structure of the low-lying collective states in $^{166}Er$. For this purpose, we perform its diagonalization in the space of stretched states $(\lambda_0 + 2n, \mu_0)$ using the $SU_p(3) \otimes SU_n(3)$ symmetry-adapted basis. The results for the low-lying energy levels of the ground state, $\gamma$ $(K^+ = 2^+)$ and $\beta$ $(K^+ = 0^+)$ bands, compared with experiment [27], are shown in Fig.1. The major shell separation energy $\hbar \omega$ is determined by the standard formula $41 A^{-1/3}$ MeV. The adopted values for the model parameters (in MeV) are as follows: $\chi = 0.0029, \xi = 0.0259, a = 0.010$ and $b = 0.187$. For comparison, the (two-parameter) $Sp(6, R)$ [16] and (four-parameter) (pseudo)contracted symplectic model [12] calculations for the ground state and $\gamma$ bands, respectively, are also shown. The results of the CSM for the $\gamma$ band are for $^{168}Er$, but the pseudo-$SU(3)$ scheme gives the same leading irreducible representation $(30, 8)$ of the combined proton-neutron system for both $^{166}Er$ and $^{168}Er$ nuclei. So, the theory predicts similar collective properties and the usage of the $\gamma$ band energies for $^{168}Er$ is a reasonable choice. From Fig.1 we see almost the same results for the ground state band between the PNSM and the full $Sp(6, R)$ calculations from one side, and for the $\gamma$ band between the PNSM and the full CSM calculations from the other one. We recall that within the framework of the contracted symplectic model, the $\gamma$ band together with the ground band belong to the single triaxial $Sp(6, R)$ irrep $(30, 8)$ and no low-lying $\beta$ band exists. A low-lying $\beta$ band can be obtained within the framework of one-component symplectic model by including symplectic symmetry-breaking terms, like, e.g., the spin-orbit or pairing interactions (see, e.g., Ref.[15]). The symplectic irrep of the $Sp(6, R)$ model calculations used in Ref.[16] is the axially symmetric one $(78, 0)$, so only the ground state band appears. From Fig.1 one also sees that, in contrast to the CSM, no staggering for the $\gamma$ band levels is obtained in the PNSM, as observed in experiment.

We also computed the reduced intraband $E2$ electromagnetic transition strengths between the states of the ground and $\gamma$ bands

$$B(E2; L_i \rightarrow L_f) = \frac{2L_f + 1}{2L_i + 1} \left( \frac{5}{16\pi} \right) \left( \frac{eZ}{A - 1} \right)^2 |\langle f || Q(p, p) || i \rangle|^2,$$

(12)

Note that in the definition of the operator $Q(p, p)$ (c.f. Eq.(1)), the summation is over the $(A - 1)$ Jacobi quasiparticles. Thus, in order to obtain the charge quadrupole operator, the
Figure 3. (Color online) Calculated probability distributions for the wave functions of the $0^+$ state of the ground, $2^+$ state of the $\gamma$ and $0^+$ state of the $\beta$ bands as a function of $n$.

$Q(p, p)$ operator is multiplied by the factor $Z/(A-1)$. The results are shown in Fig.2 and compared with experiment [27] and the theoretical predictions of the $Sp(6, R)$ symplectic model [16] and the pseudo-$SU(3)$ model [6] for the ground state band and $\gamma$ band, respectively. Again, we see that the PNSM, the full $Sp(6, R)$ and the pseudo-$SU(3)$ calculations reproduce equally well the general trend. Within the symplectic schemes which take the core polarization effects by construction no effective charge is used, i.e. $e = 1$. In the pseudo-$SU(3)$ model calculations [6], the used value for the effective charge is $e_{eff} = 1.5e$.

Concerning the interband transitions, the calculations within the stretched approximation give zero for $\beta \to g$ and $\beta \to \gamma$, which is a reasonable approximation to the small experimentally observed transition strengths, e.g., $0^+_g \to 2^+_g$ (2.7 w.u.) and $0^+_g \to 2^+_\gamma$ (2.4 w.u.). For $\gamma \to g$, we have non-zero values since both the ground and $\gamma$ bands belong to the same set of stretched $SU(3)$ irreps ($30 + 2n, 8$). For instance, for $2^+_\gamma \to 0^+_g$ we obtain 18.45 w.u. compared with the experimental value 5.7 w.u. It is clear that the extension of the model space beyond the stretched approximation will produce a mixing of the two sets of $SU(3)$ irreps corresponding to the ground ($\gamma$) and $\beta$ bands which will result in non-zero interband transitions between them.

In Fig.3, the squares of the amplitudes (probabilities) of the $SU(3)$ components from each $n\hbar \omega$ shell are plotted for the $0^+$ state of the ground, $2^+$ state of the $\gamma$ and $0^+$ state of the $\beta$ bands as a function of $n$. From the latter the structure of the eigenstates obtained in the diagonalization of the model Hamiltonian becomes evident. In particular, we see that the structure of the ground and $\gamma$ bands is predominantly determined by a single $SU(3)$ irrep which exhausts up to $\approx 97.5\%$ of the structure. For the $\beta$ band we see that the probability of the $0^+$ state is fully exhausted ($\approx 100\%$) by the $SU(3)$ irrep $(32, 4)$. The results obtained for this particular nucleus, exploiting the stretched approximation in the $SU_p(3) \otimes SU_n(3)$ symmetry-adapted basis, thus clearly show the presence of a good $SU(3)$ dynamical symmetry.

4. Conclusions

In the present paper, the proton-neutron symplectic model with $Sp(12, R)$ dynamical symmetry is applied for the simultaneous description of the microscopic structure of the low-lying states of the ground state, $\gamma$ and $\beta$ bands in $^{166}$Er. For this purpose, the model Hamiltonian is diagonalized in the space of stretched states by exploiting the $SU_p(3) \otimes SU_n(3)$ symmetry-adapted basis. The model Hamiltonian, which exhibits a symplectic dynamical symmetry, is restricted to the spherical harmonic oscillator shell-model part and the full major shell-mixing proton-neutron quadrupole-quadrupole interaction, plus an $SU(3)$ scalar term which take into account the band-head energies and a residual rotational part which allows the inertial and band splitting features of the observed spectrum to be reproduced without altering the wave functions. The theoretical predictions are compared with experiment and some other microscopic collective models, like the
one-component $Sp(6, R)$ symplectic and pseudo-$SU(3)$ models. A good description of the energy levels of the three bands under consideration, as well as the enhanced intraband $B(E2)$ transition strengths between the states of the ground and $\gamma$ bands is obtained without the use of effective charges. The results show the presence of a good $SU(3)$ dynamical symmetry. It is also shown that, in contrast to the $Sp(6, R)$ case, the lowest excited bands, e.g., the $\beta$ and $\gamma$ bands, naturally appear together with the ground state band within a single $Sp(12, R)$ irreducible representation.

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