Structures of the Pareto set and their reduction in bicriteria discrete problems

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Abstract. We investigate discrete bicriteria problems and apply to them the axiomatic approach of the Pareto set reduction proposed by V. Noghin. Values of “quantum of information” parameter are identified, that guarantee the reduction of the Pareto set with special structures. We analyze the families of the bicriteria set covering instances in the scope of the Pareto set reduction.

1. Introduction

A discrete multicriteria problem (DMP) includes the following components: a finite set of feasible solutions \( X \) and vector criterion (vector-valued function) \( f = (f_1, f_2, \ldots, f_m) \) defined on set \( X \).

Optimal solution to the problem is usually supposed to be the Pareto set, defining via dominance relation. We say that one solution \( x^* \) dominates another solution \( x \) if the inequality \( f(x^*) \geq f(x) \) holds. The relation \( f(x^*) \geq f(x) \) means that \( f(x^*) \neq f(x) \) and \( f_i(x^*) \geq f_i(x) \) for \( i = 1, \ldots, m \).

This relation \( \geq \) is also called the Pareto relation. A set of all non-dominated solutions is called the set of pareto-optimal solutions \( P_f(X) = \{ x \in X \mid \nexists x^* \in X : f(x^*) \geq f(x) \} \). If we denote \( Y = f(X) \), then the Pareto set is defined as \( P(Y) = \{ y \in Y \mid \nexists y^* \in Y : y^* \geq y \} \).

The Pareto set is rather wide in real-world problems and rises difficulties in choosing a final solution. For that reason numerous state-of-the-art methods are developed [1]: multiattribute utility theory, outranking approaches, verbal decision analysis, various iterative procedures with man-machine interface, etc. In this paper, we investigate the axiomatic approach of the Pareto set reduction proposed in [2] and which has an alternative idea. The author introduced an additional information about the decision maker (DM) preferences in terms of so-called “quantum of information”, reflecting the compromise between criteria. The method shows how to construct a new bound of the optimal choice, which is narrower than the Pareto set. Practical applications of the approach could be found in [3, 4].

As far as we know, the axiomatic approach of the Pareto set reduction does not widely investigated in the case of discrete optimization problems. Computational experiment on instances of bicriteria asymmetric travelling salesman problem estimating the degree of the Pareto set reduction and how it depends on the parameters of the information about DM’s preferences could be found in [5, 6]. In this paper we identify values of “quantum of information” parameter that guarantee the reduction of the Pareto set of special structures (“cascade” and
“stairs” types) in bicriteria discrete problems. The families of instances are constructed for the bicriteria set covering problem, and the Pareto set reduction is analyzed.

2. Main approach
According to [2] we consider the bicriteria choice problem $<X, f, \succ>$:

- a set of feasible solutions $X$;
- a vector criterion $f=(f_1,f_2)$ defined on set $X$;
- an asymmetric binary preference relation of the DM $\succ$ defined on set of feasible outcomes $Y = f(X)$.

The notation $f(x') \succ f(x'')$ means that the DM prefers the solution $x'$ to $x''$.

Binary relation $\succ$ satisfies some axioms of the so-called “reasonable” choice, according which it is irreflexive, transitive, invariant with respect to a linear positive transformation and compatible with criteria $f_1$, $f_2$. The compatibility means that the DM is interested in increasing value of one criterion when value of another criterion is constant. Also, if for some feasible solutions $x'$, $x'' \in X$ the relation $f(x') \succ f(x'')$ holds, then solution $x''$ does not belong to the optimal choice within the whole set $X$.

In [2], the author established the Edgeworth–Pareto principle: under axioms of “reasonable” choice any set of selected outcomes $C(Y)$ belongs to the Pareto set $P(Y)$. Here the set of selected outcomes is interpreted as some abstract set corresponded to the set of solutions, that satisfy all hypothetic preferences of the DM. So, the optimal choice should be done within the Pareto set only if preference relation $\succ$ fulfills the axioms of “reasonable” choice.

In real-life multicriteria problems, particularly bicriteria, the Pareto set is rather wide. For this reason V. Noghin proposed a specific information on the DM’s preference relation $\succ$ to reduce the Pareto set staying within the set of selected outcomes [2]:

**Definition 1.** We say that there exists a “quantum of information” about the DM’s preference relation $\succ$ if vector $y' \in \mathbb{R}^2$ such that $y'_i = w_i > 0$ and $y'_j = -w_j < 0$ satisfies the expression $y' \succ 0_2$, where $0_2 = (0, 0)$. In such case we will say, that the component of criteria $i$ is more important than the component $j$ with given positive parameters $w_i$, $w_j$. Here $i,j \in \{1,2\}$, $i \neq j$.

Thus, “quantum of information” shows that the DM is ready to compromise by decreasing the criterion $f_j$ by amount $w_i$ for increasing the criterion $f_i$ by amount $w_i$. The quantity of relative loss is set by the so-called coefficient of relative importance $\theta = w_j/(w_i + w_j)$, therefore $\theta \in (0,1)$.

Due to invariant of relation $\succ$ with respect to a linear positive transformation Definition 1 is equivalent to the existence of such vector $y'' \in \mathbb{R}^2$ with components $y''_i = 1-\theta$ and $y''_j = -\theta$ that the relation $y'' \succ 0_2$ holds, $i,j \in \{1,2\}$, $i \neq j$. Further, we consider “quantum of information” exactly in terms of coefficient $\theta$.

In [2], the author established the rule of taking into account “quantum of information”. This rule consists in constructing a “new” vector criterion using the components of the “old” one and coefficient $\theta$. Then one should find the Pareto set of “new” multicriteria problem with the same set of feasible solutions and “new” vector criterion. The obtained set will belong to the Pareto set of the initial problem and give a narrower upper bound on the optimal choice, as a result the Pareto set will be reduced.

**Theorem 1 [2].** Given a “quantum of information” criterion $i$ is more important than criterion $j$ with coefficient $\theta$, the inclusions $C(Y) \subseteq \hat{P}(Y) \subseteq P(Y)$ are valid for any set of selected outcomes $C(Y)$. Here $\hat{P}(Y) = f(P_f(X))$, and $P_f(X)$ is the set of pareto-optimal solutions with respect to vector criterion $\hat{f} = (\hat{f}_1, \hat{f}_2)$, where $\hat{f}_j = \theta f_i + (1-\theta)f_j$, $\hat{f}_i = f_i$. Here $i,j \in \{1,2\}$, $i \neq j$. 


Thus, “new” vector criterion \( f \) differs from the “old” one only by less important component \( j \). In [3, 4] one can find results on applying particular collections of “quanta of information” and scheme to arbitrary collection [2].

Following [2] dominance of some vector \( \hat{y} \in \mathbb{R}^2 \) over another vector \( y \in \mathbb{R}^2 \) by the Pareto relation \( \geq \), i.e. \( y \geq \hat{y} \), means that their difference \( \hat{y} - y \) belongs to convex cone \( \mathbb{R}^2_+ = \text{cone}\{e^1, e^2\} \setminus \{0_2\} \) (the non-negative orthant), where \( e^1 \) and \( e^2 \) are unit vectors of space \( \mathbb{R}^2 \).

Given “quantum of information” extends the non-negative orthant by vector \( y' \) to convex cone \( M = \text{cone}\{e^1, e^2, y'\} \setminus \{0_2\} \), and reduced Pareto set \( \hat{P}(Y) \) in Theorem 1 is the set of non-dominated vectors with respect to cone \( M \). The preference relation \( \succ \) is called cone relation and assigned to cone \( M \) [2].

3. Reduction of the Pareto set with special structures

We investigate the degree of the Pareto set reduction with respect to values of coefficient of relative importance \( \theta \). In any multicriteria discrete problem there exists such increasing sequence of coefficients of relative importance \( 0 < \theta_1 < \theta_2 < \ldots < \theta_k < 1 \) that on each fixed interval \( I \) from the set of intervals \( \mathcal{I} = \{(0, \theta_1), \ldots, (\theta_i, \theta_{i+1}), \ldots, (\theta_k, 1)\} \) for any \( \theta \in I \) the reduction of the Pareto set \( \hat{P}(Y) \) will be the same. Thus, the goal is to find bordered values of coefficient of relative importance and reduced Pareto set on each interval.

In the paper we identify the following classes of the bicriteria discrete problem upon its Pareto set structure: cascade and stairs.

3.1. Cascade structure

We say that the Pareto set has cascade structure if its elements lay on \( p \) parallel lines \( l_1, \ldots, l_p \).

1) Suppose the first criterion \( f_1 \) is more important than the second one \( f_2 \) with coefficient of relative importance \( \theta \). If \( \theta \in (0, k/(k + 1)) \) then the reduced Pareto set \( \hat{P}(Y) \) coincides with the Pareto set \( P(Y) \), in the case of \( \theta \in [k/(k + 1), 1) \) the set \( \hat{P}(Y) \) includes at most \( p \) elements.

2) Suppose the second criterion \( f_2 \) is more important than the first one \( f_1 \) with coefficient of relative importance \( \theta \). If \( \theta \in (0, 1/(k + 1)) \) then the reduced Pareto set \( \hat{P}(Y) \) includes at least \( \hat{n} \) elements. In the case of \( \theta \in [1/(k + 1), 1) \) the reduced Pareto set \( \hat{P}(Y) \) consists of one element.

1) Firstly we consider case 1), secondly case 2).

Case 1). According to Definition 1 there exists such vector \( y' \in \mathbb{R}^2 \) with components \( y'_1 = 1 - \theta \) and \( y'_2 = -\theta \) that the expression \( y' > 0_2 \) holds. Thus, the preference relation \( \succ \) is assigned to a convex cone \( M = \text{cone}\{e^1, e^2\} \setminus \{0_2\} \). Later on, if we obtain \( \bar{y} - \hat{y} \in M \) for some vectors \( \hat{y}, \bar{y} \in Y \) it means that vector \( \hat{y} \) dominates vector \( \bar{y} \) with respect to cone \( M \). In other words, vector \( \hat{y} \) does not belong to reduced Pareto set \( \hat{P}(Y) \).

Let us show that \( \forall \bar{y}, \hat{y} \in l_i, \bar{y}_1 > \hat{y}_1 \), the inclusion \( \bar{y} - \hat{y} \in M \) is valid for all \( i = 1, \ldots, p \) if \( \theta \geq k/(k + 1) \).

We choose an arbitrary \( i \in \{1, \ldots, p\} \) and consider \( \bar{y}, \hat{y} \in l_i \) with \( \bar{y}_1 > \hat{y}_1 \) such that \( \bar{y} - \hat{y} \in M \). According to definition of cone \( M \) we get the following vector equation \( y' = \lambda_1 e^1 + \lambda_2 e^2 + \lambda_3 y' \), where parameters \( (\lambda_1, \lambda_2, \lambda_3) \geq 0_3 \). Since \( \bar{y}, \hat{y} \in l_i \) each component of vector equation gives

\[ \Delta y_1 = \lambda_1 + \lambda_3(1 - \theta), \]

\[ -k\Delta y_1 = \lambda_2 - \lambda_3\theta, \]

where \( \Delta y_1 = \bar{y}_1 - \hat{y}_1 \). Due to \( \lambda_1, \lambda_2 \geq 0 \) using simple transformation we obtain inequalities \( k\Delta y_1/\theta \leq \lambda_3 \leq \Delta y_1/(1 - \theta) \), that lead us to expression \( \theta \geq k/(k + 1) \).
If we consider inequality \( \theta \geq k/(k + 1) \) and arbitrary vectors \( \hat{y}, \tilde{y} \in l_i \) with \( \hat{y}_i > \tilde{y}_i \) for any fixed \( i \in \{1, \ldots, p\} \), then there exists such nonnegative \( \lambda_3 \in \mathbb{R} \) that expression \( k\Delta y_1/\theta \leq \lambda_3 \leq \Delta y_1/(1 - \theta) \) is valid, where \( \Delta y_1 = \hat{y}_i - \tilde{y}_i \). Making the aforementioned transformations inversely we obtain inclusion \( \hat{y} - \tilde{y} \in M \).

Now, let us show that \( \forall i, j \in \{1, \ldots, p\}, i > j, \forall \hat{y} \in l_i, \forall y \in l_j : \hat{y} - \tilde{y} \in M \) implies \( \theta \geq (k\Delta y_1 - a)/((k + 1)\Delta y_1 - a) \), where \( \Delta y_1 = \hat{y}_i - \tilde{y}_i, a = a^{(i)} - a^{(j)} \). Similarly using previous arguments we obtain the following equations for some \( (\lambda_1, \lambda_2, \lambda_3) \geq 0 \)

\[
\Delta y_1 = \lambda_1 + \lambda_3(1 - \theta),
\]

\[
a - k\Delta y_1 = \lambda_2 - \lambda_3\theta.
\]

From here we have inequalities \( (k\Delta y_1 - a)/\theta \leq \lambda_3 \leq \Delta y_1/(1 - \theta) \) due to \( \lambda_1, \lambda_2 \geq 0 \), and get \( \theta \geq (k\Delta y_1 - a)/((k + 1)\Delta y_1 - a) \). Thus we obtain \( \theta \geq (k\Delta y_1 - a)/((k + 1)\Delta y_1 - a) > k/(k + 1) \), as \( \Delta y_1 > 0, a < 0 \). The inverse transformations are also valid for corresponding \( i, j, \hat{y}, \tilde{y} \).

We note, that if we take arbitrary \( i, j \in \{1, \ldots, p\}, i > j \), arbitrary \( \hat{y} \in l_i \), and arbitrary \( y \in l_j \), then \( \hat{y} - y \notin M \) for any \( \theta \) due to definition of cone \( M \).

Thus, if the condition \( \theta < k/(k + 1) \) holds, then there is no any point from the Pareto set that is dominated with respect to cone \( M \) by another point from the Pareto set. So, we do not get a reduction. The condition \( \theta \geq k/(k + 1) \) says that on each line all points, except for the lower line \( l_p \), are dominated with respect to cone \( M \). It means that reduced Pareto set \( \bar{P}(Y) \) consists of at most \( p \) points. The reduction up to more than \( p \) points depends on relative position of lines and points on them.

Case 2). Following Definition 1 the expression \( y' > 0 \) is valid, where vector \( y' \in \mathbb{R}^2 \) has components \( \theta' = -\theta \) and \( \lambda_2 = 1 - \theta \). Also the preference relation \( \succeq \) is assigned to a convex cone \( M = \text{cone}\{e^1, e^2, y'\} \setminus \{0\} \). Similarly to case 1) we prove that \( \forall \hat{y}, \tilde{y} \in l_i, \bar{y} = \hat{y} \), the inclusion \( \hat{y} - \tilde{y} \in M \) is valid for all \( i \in \{1, \ldots, p\} \).

Inclusion \( \hat{y} - \tilde{y} \in M \) implies the following equations

\[
\Delta y_1 = \lambda_1 - \lambda_3\theta,
\]

\[
-k\Delta y_1 = \lambda_2 + \lambda_3(1 - \theta),
\]

where \( \Delta y_1 = \hat{y}_1 - \tilde{y}_1 \), and parameters \( (\lambda_1, \lambda_2, \lambda_3) \geq 0 \). One could obtain inequalities

\[
-\Delta y_1/\theta \leq \lambda_3 \leq -\Delta y_1k/(1 - \theta),
\]

that lead to \( \theta \geq 1/(k + 1) \). Sufficiency is proved also analogous to case 1).

Now, we consider arbitrary \( i, j \in \{1, \ldots, p\}, i < j \), and any \( \hat{y} \in l_i, \bar{y} \in l_j \) such that \( \bar{y} = \tilde{y} \in M \). There exist such \( (\lambda_1, \lambda_2, \lambda_3) \geq 0 \) that the following equations

\[
\Delta y_1 = \lambda_1 - \lambda_3\theta,
\]

\[
a - k\Delta y_1 = \lambda_2 + \lambda_3(1 - \theta),
\]

hold, where \( a = a^{(i)} - a^{(j)} > 0 \) due to \( i < j \). Thus we get inequality \( \theta \geq \Delta y_1/\Delta y_1(k + 1) - a \).

For any \( i, j \in \{1, \ldots, p\}, i < j \), and \( \hat{y} \in l_i, \bar{y} \in l_j \) such that \( \theta \geq \Delta y_1/\Delta y_1(k + 1) - a \) the same arguments in the opposite direction are valid.

We note, that consideration arbitrary \( i, j \in \{1, \ldots, p\}, i > j \), arbitrary \( \hat{y} \in l_i \), and arbitrary \( \hat{y} \in l_j \) implies \( \hat{y} - \tilde{y} \notin M \) for any \( \theta \) due to definition of cone \( M \).

Since \( 1/(k + 1) > \Delta y_1(\Delta y_1(k + 1) - a)/a > 0 \) parameters \( i, j, \hat{y}, \bar{y} \) were chosen arbitrary, when coefficient \( \theta \in (0, 1/(k + 1)) \) the maximum number of points that could be dominated with respect to cone \( M \) equals the number of points on all lines except the upper line \( l_1 \), and set \( \bar{P}(Y) \) has at least \( n \) elements. If \( \theta \in [1/(k + 1), 1) \), only upper point on line \( l_1 \) is not dominated with respect to cone \( M \), and the reduction of the Pareto set consists of one element.
3.2. Stairs structure

We say that the Pareto set has stairs structure if its elements lay on $p$ parallel lines $l_1, \ldots, l_p$. 

$$P(Y) = \bigcup_{i=1}^{p} \{(y_1^{(i)}, y_2^{(i)}) : y_2^{(i)} = a^{(i)} - ky_1^{(i)}, y_1^{(i)} \in \tilde{Y}_1^{(i)}\},$$

where $\tilde{Y}_1^{(i)} = \{(a^{(i)} - y_2^{i1})/k, \ldots, (a^{(i)} - y_2^{iq})/k \mid y_2^1 > \ldots > y_2^q\}$ for all $i \in \{1, \ldots, p\}$, such that $(a^{(p)} - y_2^{jq})/k < (a^{(1)} - y_2^{1j+1})/k$ for all $j \in \{1, \ldots, q - 1\}$. Here, $q$ is the number of points on each line, $a^{(1)} < a^{(2)} < \ldots < a^{(p)}$, (see, e.g. Fig 1).

![Figure 1](image1)

Figure 1. Examples of cascade ($p = 3$) and stairs ($p = 4$, $q = 3$) structures of the Pareto set

**Theorem 3.** Let the Pareto set $P(Y)$ has stairs structure with $p$ lines $l_1, \ldots, l_p$, each of them having $q$ points.

1) Suppose the first criterion $f_1$ is more important than the second one $f_2$ with coefficient of relative importance $\theta$. Then if $\theta \in (0, k/(k + 1))$ the reduced Pareto set $\tilde{P}(Y)$ includes at least $q$ elements, in the case of $\theta \in [k/(k + 1), 1)$ the set $\tilde{P}(Y)$ consists of one element.

2) Suppose the second criterion $f_2$ is more important than the first one $f_1$ with coefficient of relative importance $\theta$. Then if $\theta \in (0, 1/(k + 1))$ the reduced Pareto set $\tilde{P}(Y)$ includes at least $p + q - 1$ elements. In the case of $\theta \in [1/(k + 1), 1)$ the set $\tilde{P}(Y)$ contains at most $p$ elements.

Firstly we consider case 1), secondly case 2).

Case 1). According to Definition 1 there exists such vector $y' \in \mathbb{R}^2$ with components $y'_1 = 1 - \theta$ and $y'_2 = -\theta$ that the expression $y' > 0_2$ holds.

Repeating the corresponding arguments from proof of Theorem 2, case 1) one could check that $\forall \tilde{y}, \tilde{y} \in l_1$, $\tilde{y} > \tilde{y}_1$, the inclusion $\tilde{y} - \tilde{y}_1 > 0$ implies $\theta \geq k/(k + 1)$.

Furthermore, similarly to proof of Theorem 2, case 1) we obtain that $\forall i, j \in \{1, \ldots, p\}$, $i > j$, $\forall \tilde{y} \in l_i$, $\forall \tilde{y} \in l_j$, $\Delta y_1 = \tilde{y}_1 - \tilde{y}_1 > 0 : \tilde{y} - \tilde{y}_1 \in M$ implies $\theta \geq (k\Delta y_1 - a)/((k + 1)\Delta y_1 - a)$, where $a = a^{(i)} - a^{(j)} > 0$. The inverse transformation is valid for corresponding values of $i, j, \tilde{y}, \tilde{y}$. Meanwhile inequalities $0 < (k\Delta y_1 - a)/((k + 1)\Delta y_1 - a) < k/(k + 1)$ hold due to $\Delta y_1 > 0$, $\Delta y_2 > 0$, and $a > 0$. Also the same inequality $\theta \geq (k\Delta y_1 - a)/((k + 1)\Delta y_1 - a)$ are obtained when $i < j$ and other conditions are invariant. But in such situation $0 < k/(k + 1) < (k\Delta y_1 - a)/((k + 1)\Delta y_1 - a) = \theta$.

Thus, if $\theta \in (0, k/(k + 1))$ the maximum number of points that could be dominated with respect to cone $M$ is the total number of points minus $q$ (except points on the upper line $l_p$). Later on, if $\theta \in [k/(k + 1), 1)$ only one point is not dominated on the upper line $l_p$. 

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Case 2). The expression \( y' > 0_2 \) is valid, where vector \( y' \in \mathbb{R}^2 \) has components \( y'_1 = -\theta \) and \( y'_2 = 1 - \theta \). Similarly to arguments in proof of Theorem 2, case 2) one could prove that \( \forall i, j \in \{1, \ldots, p\}, i > j, \) and any \( \bar{y} \in l_i, \tilde{y} \in l_j, \Delta y_1 = \bar{y}_1 - \tilde{y}_1 < 0 \) such that \( \bar{y} - \bar{y} \in M \), inequalities \( 1/(k + 1) > \theta > \Delta y_1/(\Delta y_1(k + 1) - \alpha) > 0 \) hold, and vice versa. In case \( i < j \) we have inequalities \( \theta > \Delta y_1/(\Delta y_1(k + 1) - \alpha) > 1/(k + 1) > 0 \).

Also \( \forall \bar{y}, \tilde{y} \in l_i, \bar{y}_1 < \tilde{y}_1, \) the inclusion \( \bar{y} - \tilde{y} \in M \) is valid for all \( i = 1, \ldots, p \) iff \( \theta \geq 1/(k + 1) \).

4. Bicriteria Set Covering Problem

Here we consider a modification of the classical bicriteria set covering problem (bi-SCP). Let \( R = (r_{ji}) \) be a binary matrix with \( N_r \) rows and \( N_c \) columns, and \( c = (c_i) \), \( u = (u_i) \) be \( N_c \) dimensional vectors, representing weights of columns. It is supposed that \( c_i > 0 \) and \( u_i > 0 \), \( i = 1, \ldots, N_c \). We say that a column \( c_i \) covers a row \( j \) when \( r_{ji} = 1 \). A subset \( S \) of columns is feasible, if each row \( j \) is covered by at least one column \( i \in S \). The total sums of weights \( f_1(S) = \sum_{i \in S} u_i \) and \( f_2(S) = \sum_{i \in S} (-c_i) \) are maximized over all feasible subsets \( S \) such that \( |S| \leq b \), where \( 1 \leq b \leq N_r \) is the given constant.

The application of bi-SCP is the supply management problem. Here the rows correspond to types of orders and the columns are interpreted as transportation companies that can supply some types of the orders. The value \( r_{ji} = 1 \) means that company \( i \) can provide order type \( j \), \( i = 1, \ldots, N_r, j = 1, \ldots, N_c \). The parameters \( c_i \) and \( u_i \) give the cost of service and the level of performance for company \( i = 1, \ldots, N_c \). The constant \( b \) represents the upper bound on the number of the selected companies. An analogous interpretation gives the production management problem, where a row represents a type of jobs and production companies specify columns.

The next application is the problem of centers allocation. The centers can be allocated in \( N_r \) points of some region, which consists of \( N_r \) zones (or which has \( N_r \) transportation routes). The matrix \( R = (r_{ji}) \) indicates zones (or routes), which may be served by the center, placed in a point. The allocation cost and the coefficient of efficiency for the center at point \( i \) are given by \( c_i \) and \( u_i \), respectively. The maximal number of opened centers is \( b \). Various interpretations of the set covering problem and its generalizations may be found in [7–11].

Now we present the families of bi-SCP instances, for which the Pareto set has stairs structure and/or cascade structure.

**Stairs and Cascade Structures.** Let \( N_r = 2, b = 2 \) and \( N_c = p + q \), where \( p, q \in \mathbb{N} \). The first \( p \) columns cover only the first row and the next \( q \) columns cover only the second row, i.e. \( r_{1,i} = 1, r_{2,i} = 0, r_{1,l+p} = 0, r_{2,l+p} = 1 \) for \( i = 1, \ldots, p, l = 1, \ldots, q \). Any feasible solution consists of one of the first \( p \) columns and one of the next \( q \) columns.

Define the weights as \( u_i = c_i = i \) for columns \( i = 1, \ldots, p \), and \( u_{i+p} = T + (l-1)p, c_{l+p} = T + (l-1)(2p-1) \) for columns \( i + p, l = 1, \ldots, q \). Here and later, \( T \) is an arbitrary positive constant. Images of all feasible solutions compose the Pareto set

\[
P(Y) = \{(T + (l-1)p + i; -T - (l-1)(2p-1) - i), i = 1, \ldots, p, l = 1, \ldots, q\}.
\]

This Pareto set illustrates stairs and cascade structures simultaneously. In the case of stairs structure we have \( p \) parallel lines, containing \( q \) points of the form \( y^l_i = -\frac{2p-1}{p}i^l + \left(T\frac{p-1}{p} + i^l\frac{2-1}{p}\right), \)

\[
y^l_i \in Y^l_i = \{(T + (l-1)p + i, l = 1, \ldots, q), i = 1, \ldots, p\}
\]

For each line the distance between two consecutive points is \( p \) on the first criterion, and it is \( (2p-1) \) on the second criterion. In the case of cascade structure we have \( q \) parallel lines, having \( p \) points of the form \( y^l_i = -y^l_1 + (l-1)(1-p), y^l_i \in Y^l_i = \{(T + (l-1)p + i, i = 1, \ldots, p), l = 1, \ldots, q\} \) for each line we have the unit distance between two consecutive points on both criteria.
For the situation when the first criterion is more important than the second one with coefficient of relative importance \( \theta \) and \( p > 1 \) we conclude (from Theorems 2 and 3):

- if \( \theta \in (0, \frac{1}{2}) \) the reduction does not hold;
- if \( \theta \in [\frac{1}{2}, \frac{2p-1}{3p-1}] \) the reduced Pareto set consists of \( q \) elements;
- if \( \theta \in [\frac{2p-1}{3p-1}, 1) \) the reduced Pareto set consists of one element.

For the situation when the second criterion is more important than the first one with coefficient of relative importance \( \theta \) and \( p > 1 \) we obtain (from Theorems 2 and 3):

- if \( \theta \in (0, \frac{p}{3p-1}) \) the reduced Pareto set contains at least \( p + q - 1 \) elements;
- if \( \theta \in [\frac{p}{3p-1}, \frac{1}{2}) \) the reduced Pareto set consists of \( p \) elements;
- if \( \theta \in [\frac{1}{2}, 1) \) the reduced Pareto set consists of one element.

The second and the third intervals give the exact estimate on reduced Pareto set, specifying the number of its elements. At the same time the first interval in the second situation could not guarantee the reduction of the Pareto set, only indicating the lower bound.

**Stairs Structure.** Let \( N_r = 2, b = 2 \) and \( N_c = p + q \), where \( p, q \in \mathbb{N} \). The first \( p \) columns cover only the first row and the next \( q \) columns cover only the second row, i.e. \( r_{1,i} = 1, r_{2,i} = 0, r_{1,i+p} = 0, r_{2,i+p} = 1 \) for \( i = 1, \ldots, p, l = 1, \ldots, q \). So, any feasible solution consists of one of the first \( p \) columns and one of the next \( q \) columns.

Define the weights as \( u_i = i, c_i = \frac{i(i+1)}{2p} \) for columns \( i = 1, \ldots, p \), and \( u_{i+p} = T + (l-1)p, c_{i+p} = T + (l-1)(p+1) \) for columns \( l + p, l = 1, \ldots, q \). Images of all feasible solutions compose the Pareto set

\[
P(Y) = \left\{ \left( T + (l-1)p + i; -T - (l-1)(p+1) - \frac{i(i+1)}{2p} \right), i = 1, \ldots, p, l = 1, \ldots, q \right\}.
\]

This Pareto set illustrates stairs structure, but does not represent cascade structure. We have \( p \) parallel lines, containing \( q \) points of the form \( y_i^p = \frac{-p+1}{p} y_i^1 + \left( \frac{T}{p} + \frac{(2p+i+1)}{2p} \right) \), \( y_i^1 \in Y_i^1 = \{ T + (l-1)p + i; l = 1, \ldots, q \} \), \( i = 1, \ldots, p \). For each line the distance between two consecutive points is \( p \) on the first criterion, and it is \( (p+1) \) on the second criterion.

For the situation when the first criterion is more important than the second one with coefficient of relative importance \( \theta \) we conclude (from Theorem 3):

- if \( \theta \in (0, \frac{p+1}{2p+1}) \) the reduced Pareto set contains at least \( q \) elements;
- if \( \theta \in [\frac{p+1}{2p+1}, 1) \) the reduced Pareto set consists of one element.

For the situation when the second criterion is more important than the first one with coefficient of relative importance \( \theta \) we indicate (from Theorem 3):

- if \( \theta \in (0, \frac{p}{2p+1}) \) the reduced Pareto set contains at least \( p + q - 1 \) elements;
- if \( \theta \in [\frac{p}{2p+1}, 1) \) the reduced Pareto set contains at most \( p \) elements.

**Cascade Structure.** Let \( N_r = 3, b = 2 \) and \( N_c = p+q \), where \( p \) and \( q \) are even positive numbers. The first \( \frac{p}{2} \) columns cover only the first row, i.e. \( r_{1,i} = 1, r_{2,i} = 0, r_{3,i} = 0 \) for \( i = 1, \ldots, \frac{p}{2} \), the next \( \frac{p}{2} \) columns cover the first and the second rows, i.e. \( r_{1,i} = 1, r_{2,i} = 1, r_{3,i} = 0 \) for \( i = 1, \ldots, \frac{p}{2} \). The odd columns \( p + 2l - 1 \) cover the second and the third rows, i.e. \( r_{1,p+2l-1} = 0, r_{2,p+2l-1} = 1, r_{3,p+2l-1} = 1 \) for \( l = 1, \ldots, \frac{q}{2} \), and the even columns \( p + 2l \) cover only the third row, i.e. \( r_{1,p+2l} = 0, r_{2,p+2l} = 0, r_{3,p+2l} = 1 \) for \( l = 1, \ldots, \frac{q}{2} \). So, the set of
feasible solutions consists of columns pairs $i$ and $(p+2l-1)$, where $i=1,\ldots,p$ and $l=1,\ldots,\frac{q}{2}$, and columns pairs $i$ and $(p+2l)$, where $i=\frac{q}{2}+1,\ldots,p$ and $l=1,\ldots,\frac{q}{2}$.

Define the weights as $u_i = c_i = i$ for columns $i = 1, \ldots, p$, and $u_{p+2l-1} = T + (l-1)\frac{p}{2} + (l-1)p$, $u_{p+2l} = T + l\frac{p}{2} + (l-1)p$, $c_{p+2l-1} = T + (l-1)\frac{p}{2} + (l-1)p + (2l-2)(p-1)$, $c_{p+2l} = T + l\frac{p}{2} + (l-1)p + (2l-2)(p-1)$ for $l = 1, \ldots, \frac{q}{2}$.

Images of all feasible solutions compose the Pareto set

\[ P(Y) = \left\{ (T + (l-1)\frac{p}{2} + (l-1)p + i; -T - (l-1)\frac{p}{2} - (l-1)p - (2l-2)(p-1) - i), \right. \]
\[ \left. i = 1, \ldots, p, \ l = 1, \ldots, \frac{q}{2} \right\} \cup \]
\[ \left\{ (T + l\frac{p}{2} + (l-1)p + p + i; -T - l\frac{p}{2} - (l-1)p - (2l-1)(p-1) - \frac{p}{2} - i), \right. \]
\[ \left. i = 1, \ldots, p, \ l = 1, \ldots, \frac{q}{2} \right\} \]

This Pareto set illustrates cascade structure, but does not represent stairs structure. We have $q$ parallel lines, containing $p$ or $\frac{q}{2}$ points of the form $y_{2l-1} = -y_{2l-2} - (2l-2)(1-p)$, $y_{2l-1} \in Y_{2l-1} = \{ T + (l-1)\frac{p}{2} + (l-1)p + i, \ i = 1, \ldots, p \}$, $l = 1, \ldots, \frac{q}{2}$; $y_{2l} = -y_{2l-2} + (2l-1)(1-p)$, $y_{2l} \in Y_{2l} = \{ T + (l+1)\frac{p}{2} + (l-1)p + i, \ i = 1, \ldots, \frac{q}{2} \}$, $l = 1, \ldots, \frac{q}{2}$. For each line the distance between two consecutive points is 1 on both criteria.

For the situation when the first criterion is more important than the second one with coefficient of relative importance $\theta$ we state (from Theorem 2):

- if $\theta \in (0, \frac{1}{2})$ the reduction does not hold;
- if $\theta \in [\frac{1}{2}, 1)$ the reduced Pareto set contains at most $q$ elements.

For the situation when the second criterion is more important than the first one with coefficient of relative importance $\theta$ we conclude (from Theorem 2):

- if $\theta \in (0, \frac{1}{2})$ the reduced Pareto set contains at least $p$ elements;
- if $\theta \in [\frac{1}{2}, 1)$ the reduced Pareto set consists of one element.

All presented results allow to help the DM in the final choice and correction of coefficient $\theta$.

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