COHOMOLOGY OF RESTRICTED LIE-RINEHART ALGEBRAS
AND THE BRAUER GROUP

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Abstract. We give an interpretation of the Brauer group of a purely inseparable extension of exponent 1, in terms of restricted Lie-Rinehart cohomology. In particular, we define and study the category $p$-LR$(A)$ of restricted Lie-Rinehart algebras over a commutative algebra $A$. We define cotriple cohomology groups $H_{p-LR}(L, M)$ for $L \in p$-LR$(A)$ and $M$ a Beck $L$-module. We classify restricted Lie-Rinehart extensions. Thus, we obtain a classification theorem for regular extensions considered by Hochschild.

Introduction

In classical theory of simple algebras it is known that if $E/F$ is a Galois extension of fields, then the Brauer group $B_{E/F}$ of this extension is isomorphic with the group of equivalence classes of group extensions of the Galois group $Gal(E/F)$ by the multiplicative group $F^*$ of $F$. Moreover, we have an isomorphism of groups relating the Galois cohomology and the Brauer group:

\[(0.1)\quad B_{E/F} \cong H^2(Gal(E/F), F^*)\]

In the context of purely inseparable extensions of exponent 1 there is a Galois theory due to Jacobson (see [18]). The role of Galois group is now played by the group of derivations. Purely inseparable extensions occur naturally in algebraic geometry. In particular, such extensions appear in the theory of elliptic curves in prime characteristic. If $k$ is a field such that $char k = p$ and $V$ an algebraic variety over $k$ of dimension greater than 0, then the function field $k(V)$ is a purely inseparable extension over the subfield $k(V)^p$ of $p$th powers.

Hochschild in [14] proves that the Brauer group $B^R_k$ of purely inseparable extension $K/k$ of exponent 1, is isomorphic with the set of equivalence classes of regular extensions of restricted Lie algebras of $Der_k(k)$ by $K$. We remark that the second Hochschild cohomology group $H^2_{Hoch}(L, M)$ where $L$ is a restricted Lie algebra and $M$ a restricted Lie -module classifies abelian extensions such that $M^{[p]} = 0$. This implies that, there is not an isomorphism between the Brauer group $B^R_k$ and a subgroup of the cohomology group $H^2_{Hoch}(Der_k(K), K)$. This remark has been the motivation to undertake this research. In order to classify regular extensions and obtain the analogue to (0.1) cohomological interpretation of the Brauer group, we are led to define and study the category of restricted Lie-Rinehart algebras. The concept of Lie-Rinehart algebra is the algebraic counterpart of the notion of Lie algebroid (see [23]). It seems that the notion of Lie-Rinehart algebra appears first under the name pseudo-algèbre de Lie in the paper [12] of Herz. Also, the notion

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has been examined by Palais under the name \textit{d-ring}. The first thorough study of the notion has been done by Rinehart in \cite{Rinehart}. Rinehart is the first who defined cohomology groups for the category of Lie-Rinehart algebras with\textcolor{black}{ coefficients} in a Lie-Rinehart module and further developments has been done by Huebschmann in \cite{Hu}. Moreover, cotriple cohomology for the category of Lie-Rinehart algebras has been defined in \cite{Casas} by Casas- Lada- Pirashvili. Besides, the notion of Lie-Rinehart algebra is closely related to the notion of Poisson algebra. Loday-Vallette in \cite{Loday} remarked that a Lie-Rinehart algebra is a Poisson algebra. Thus, all the constructions and properties of Poisson algebras apply to Lie-Rinehart algebras.

In Section 1 we define the category $p$-LR$(A)$ of restricted Lie-Rinehart algebras over a commutative algebra $A$. We give examples which occur naturally. In Section 2 we introduce the notion of restricted enveloping algebra and restricted Lie-Rinehart module. We prove a Poincaré-Birkhoff-Witt type theorem for the category of restricted Lie-Rinehart algebras. There is a notion of free Lie-Rinehart algebra made explicit by Casas-Pirashvili (see \cite{Casas}) and Kapranov (see \cite{Kapranov}). We extend this notion for the category $p$-LR$(A)$ and we construct free functor left adjoint to the forgetful functor. In Section 3 following the general scheme of Quillen-Barr-Beck cohomology theory for universal algebras, we determine the Beck modules and Beck derivations. In Section 4 we define cohomology groups $H^*_{p-LR(A)}(L, M)$ for $L \in p$-LR$(A)$ and $M$ a Beck $L$-module. We prove that Quillen-Barr-Beck cohomology for $p$-LR$(A)$ classifies extensions of restricted Lie-Rinehart algebras. Regular extensions considered by Hochschild are restricted Lie-Rinehart extensions. As a consequence in Section 5, we prove that if $K/k$ is purely inseparable extension of exponent 1, there is an isomorphism of groups
\[ B_k^1 \simeq H^1_{p-LR}(Der_k(K), K) \]

1. Restricted Lie-Rinehart algebras

In many cases when we study Lie algebras over a field of prime characteristic we are led to consider a richer structure than an ordinary Lie algebra. Indeed the notion of a Lie algebra has to be replaced by the notion of restricted Lie algebra introduced by N. Jacobson in \cite{Jacobson}. Let us recall the definition.

\textbf{Definition 1.1.} A restricted Lie algebra $(L, (-)^{[pL]})$ over a field $k$ of characteristic $p \neq 0$ is a Lie algebra $L$ over $k$ together with a map $(-)^{[pL]} : L \to L$ called the $p$-map such that the following relations hold:

\begin{align}
(\alpha x)^{[pL]} &= \alpha^p x^{[pL]} \\
[x, y]^{[pL]} &= \underbrace{[x, y, y, \ldots, y]}_{p} \\
(x + y)^{[pL]} &= x^{[pL]} + y^{[pL]} + \sum_{i=1}^{p-1} s_i(x, y)
\end{align}

where $s_i(x, y)$ is the coefficient of $\lambda^{i-1}$ in $ad_{x+y}^{p-1}\lambda(x)$, where $ad_x : L \to L$ denotes the adjoint representation given by $ad_x(y) := [y, x]$ and $x, y \in L$, $\alpha \in k$. We denote by $RLie$ the category of restricted Lie algebras over $k$. 
A Lie-module $A$ over a restricted Lie algebra $(L, (-)^[p])$ is called restricted if $x^[p]m = (x\cdots(x(xm)))$.

**Example 1.2.** Let $R$ be any associative algebra over a field $k$ with characteristic $p \neq 0$. We denote by $R_{Lie}$ the induced Lie algebra with the bracket given by $[x, y] := xy - yx$, for all $x, y \in R$. Then $(R, (-)^p)$ is a restricted Lie algebra where $(-)^p$ is the Frobenius map given by $x \mapsto x^p$. Thus, there is a functor $(-)_{RLie} : A_{Lie} \to R_{Lie}$ from the category of associative algebras to the category of restricted Lie algebras.

**Proposition 1.3.** Let $L$ be a restricted Lie algebra and $A$ a restricted $L$-module. If we denote by $A \oplus L$ the direct sum of the underlying vectors spaces of $A$ and $L$ then $A \oplus L$ is endowed with the structure of restricted Lie-algebra.

**Proof.** It is well known that $A \oplus L$ is endowed with the structure of a Lie algebra with bracket given by:

$$[a + X, b + Y] = (X(b) - Y(a)) + [X, Y]$$

for any $a, b \in A$ and $X, Y \in L$.

Moreover, we have:

$$[[a + X, Y], \cdots, Y] = -(Y(Y(\cdots(Y(a)))) + [X, Y], \cdots, Y]$$

$$= -(Y(Y(\cdots(Y(a)))) + [X, Y^[p]]$$

$$= [a + X, Y^[p]]$$

Besides,

$$[a + X, b], \cdots, b] = 0$$

Therefore, from Jacobson’s theorem there exists a unique $p$-map $(-)^{[p]} : A \oplus L \to A \oplus L$

which extents the $p$-map on $L$ and such that $a^{[p]} = 0$ for all $a \in A$. In particular we see that the $p$-map on $A \oplus L$ is given by:

$$(a + X)^{[p]} = (X(X(\cdots(X(a)))) + X^[p]$$

We denote this restricted Lie algebra structure on $A \oplus L$ by $A \ltimes L$.

**Remark 1.4.** Let $A$ be a restricted $L$-module. We consider the invariants for the Lie action $A^L = \{a \in A : la = 0, \text{ for all } l \in L, a \in A\}$. If $f : A \to A^L$ is a $p$-semi linear map, then it is easily seen that $A \oplus L$ is a restricted Lie algebra with $p$-map given by $(a + X)^{[p]} := (X(X(\cdots(X(a)))) + X^[p] + f(a)$. We denote this restricted Lie algebra by $A \ltimes f L$.

Let $A$ be a commutative algebra over a field $k$. A $k$-linear map $D : A \to A$ is called a $k$-derivation if

$$D(ab) = aD(b) + D(a)b$$
Let $\text{Der}_k(A)$ be the set of $k$-derivations of $A$. It is well known that if $D, D' \in \text{Der}(A)$, then $[D, D'] := DD' - D'D$ is a derivation. Thus, $(\text{Der}(A), [-, -])$ is a Lie algebra. Moreover, if $D \in \text{Der}_k(A)$ and $a, x \in A$ then $aD : A \to A$, given by: $(aD)(x) := aD(x)$ is a derivation. Therefore, $\text{Der}_k(A)$ has the structure of an $A$-module. Besides the following relation holds:

$$[D, aD'] = a[D, D'] + D(a)D'$$

The structure on $\text{Der}_k(A)$ described above is the prototype example of the notion of Lie-Rinehart algebra. Let us recall the definition.

**Definition 1.5.** A Lie-Rinehart algebra over $A$, or $(k - A)$-Lie algebra, is a pair $(A, L)$ where, $A$ is a commutative algebra over $k$, $L$ is a Lie algebra over $k$ equipped with the structure of an $A$-module together with a map called anchor $\alpha : L \to \text{Der}_k(A)$ which is an $A$-module and a Lie algebra morphism such that:

$$[X, aY] = a[X, Y] + \alpha(X)(a)Y$$

for all $a \in A$ and $X, Y \in L$.

In order to simplify the notation we denote $\alpha(X)(a)$ by $X(a)$. Moreover, we denote by LR$(A)$ the category of Lie-Rinehart algebras over $A$.

**Example 1.6.** We easily see that $(A, \text{Der}_k(A))$ is a Lie-Rinehart algebra with anchor map $id : \text{Der}_k(A) \to \text{Der}_k(A)$.

Suppose now that the ground field $k$ is a field of characteristic $p \neq 0$. Let $D \in \text{Der}_k(A)$, from Leibniz rule for all $a, b \in A$ we have

$$D^p(ab) = \sum_{i=0}^{i=p} \binom{p}{i} D^i(a)D^{p-i}(b)$$

Since $\text{char } k = p$ we get:

$$D^p(ab) = aD^p(b) + D^p(a)b$$

Therefore, $D^p$ is a derivation. In other words $\text{Der}_k(A)$ is equipped with the structure of restricted Lie algebra. Moreover, by Hochschild’s Lemma 1 in [13] we get the relation:

$$D^p = a^pD^p + (aD)^{p-1}(a)D$$

Therefore, we see in prime characteristic that the set of derivations $\text{Der}_k(A)$ has a richer structure than just a Lie-Rinehart algebra structure. We are naturally led to the following definition of restricted Lie-Rinehart algebra.

From now on we fix a field $k$ of characteristic $p \neq 0$.

**Definition 1.7.** A restricted Lie-Rinehart algebra $(A, L, (-)^{[p]})$ over a commutative $k$-algebra $A$, is a Lie-Rinehart algebra over $A$ such that: $(L, (-)^{[p]})$ is a restricted Lie algebra over $k$, the anchor map is a restricted Lie homomorphism, and the following relation holds:

$$(-)^{[p]}(aX) = a^pX^{[p]} + (aX)^{p-1}(a)X$$

for all $a \in A$ and $X \in L$. 
Let \((A, L, (-)^{[p]})\) and \((A', L', (-)^{[p]})\) be restricted Lie-Rinehart algebras. Then a Lie-Rinehart morphism \((\xi, f) : (A, L, (-)^{[p]}) \to (A', L', (-)^{[p]})\) is called restricted Lie-Rinehart morphism if \(f\) is a restricted Lie morphism, namely: \(f(x^{[p]}) = f(x)^{[p]}\) for all \(x \in L\). We will denote by \(p - \mathrm{LR}\) the category of restricted Lie-Rinehart algebras.

Example 1.8. As we have seen if \(A\) is a commutative algebra over \(k\), then \(\text{Der}_k(A)\) is a restricted Lie-Rinehart algebra.

Example 1.9. Any restricted Lie algebra over \(k\) is a restricted Lie-Rinehart algebra \((A, L, (-)^{[p]}), \text{where } A = k\).

The structure of restricted Lie Rinehart algebra appears in Jacobson-Galois theory of purely inseparable extensions of exponent 1.

Example 1.10. Let \(K/k\) be a purely inseparable field extension of exponent 1. Then, there is a one-to-one correspondence between intermediate fields and restricted Lie Rinehart sub-algebras of the restricted Lie Rinehart algebra \(\text{Der}_k(K)\) over \(K\) (see [15]).

The Jacobson-Galois correspondence for purely inseparable extensions of exponent 1 has been used by several authors in order to study isogenies of algebraic groups especially of abelian varieties. The notion of restricted Lie-Rinehart appears in this study.

Example 1.11. Let \(G\) be an algebraic group over \(k\), and \(K\) be the field of rationales functions of \(G\). Then, the \(K\)-space of derivations \(D_k(K)\) is a restricted Lie-Rinehart algebra over \(K\) (see [31]).

The structure of restricted Lie Rinehart algebras emerges also in connection with theory of characteristic classes.

Example 1.12. Let \(k \subset K\) be fields of characteristic \(p \neq 0\). Then, Maakestad considers the category \(\text{Lie}_{K/k}\) (see [22]) whose objects are restricted Lie Rinehart sub-algebras of the restricted Lie Rinehart algebra \(\text{Der}_k(K)\) over \(K\). Moreover, Maakestad construct a contravariant functor which associates \(g \in \text{Lie}_{K/k}\) to the Grothendiek ring \(K_0(g)\) of the category of \(g\)-connections. (for details see Theorem 3.2 in [22]).

Example 1.13. Let \(P\) be a Poisson algebra over \(k\). If we denote by
\[
C := \{c \in P \mid [c, -] = 0\}
\]
then we see that \(C\) is closed under the commutative and Lie bracket. Thus, \(C\) is a Poisson subalgebra of \(P\). We call a Poisson derivation a \(k\)-linear map \(D : P \to P\) which is at the same time a derivation with respect to commutative and Lie product. We can easily see that, \(\text{Der}_k(P)\) has the structure of Lie \(k\)-algebra. Moreover, by Leibniz rule we can see that \(D^p\) is a derivation with respect to commutative and the Lie product. Thus, \(\text{Der}_k(P)\) has the structure of restricted Lie \(k\)-algebra. Besides, if \(c \in C\) and \(D \in \text{Der}_k(P)\) then \(cD \in \text{Der}_k(P)\). Moreover, by Hochchild’s Lemma 1 (see [14]) we have the relation:
\[
(cD)^p = c^pD^p + (cD)^{p-1}(c)D
\]
for all \(c \in C\) Therefore, \(\text{Der}_k(P)\) is a restricted Lie-Rinehart algebra over \(C\).
2. Restricted enveloping algebras and restricted modules

Let \((A, L)\) be a Lie-Rinehart algebra. There is a notion of univeral enveloping algebra \(U(A, L)\) of \((A, L)\) defined by Rinehart in [28]. The universal enveloping algebra \(U(A, L)\) is an associative \(A\)-algebra which verifies the appropriate universal property (see [15]). We recall the definition of the enveloping associative algebra \(U(A, L)\).

The direct sum \(A \oplus L\) of the underlying vector spaces has the structure of \(k\)-Lie algebra given in the Proposition 1.3. Let \((U(A \oplus L), \iota)\) be the enveloping algebra where \(\iota : A \oplus L \to U(A \oplus L)\) is the canonical embedding. We consider the subalgebra \(U^+(A \oplus L)\) generated by \(A \oplus L\). Moreover, \(A \oplus L\) has the structure of an \(A\)-module via

\[
a(a' + X) := aa' + aX
\]

for all \(a, a' \in A\) and \(X \in L\). Then, the enveloping algebra \(U(A, L)\) is defined as the quotient:

\[
U(A, L) := U^+(A \oplus L)/ < \iota(a)\iota(a' + X) - \iota(a(a' + X)) >
\]

The canonical map \(\iota_A\) is an \(A\)-algebra homomorphism. The canonical representation \(A \to End_k(A)\) given by the multiplication is faithful. Thus, by the universal property of \(U(A, L)\) we obtain that the \(A\)-algebra homomorphism \(\iota_A : A \to U(A, L)\) is injective. The canonical map \(\iota_L\) is a Lie algebra homomorphism. Moreover, in \(U(A, L)\) the following relations hold:

\[
\iota_A(a)\iota_L(X) = \iota_L(aX), \quad \text{and} \quad [\iota_L(X), \iota_A(a)] = \iota_A(X(a))
\]

for all \(a \in A\) and \(X \in L\).

The enveloping algebra \(U(A, L)\) has a canonical filtration:

\[
A = U_0(A, L) \subset U_1(A, L) \subset U_2(A, L) \cdots
\]

where \(U_n(A, L)\) is spanned by \(A\) and the powers \(\iota_L(L)^n\). Therefore, we can construct the associated graded algebra given by

\[
gr(U(A, L)) = \bigoplus_{n=0}^{\infty} U_n(A, L)/U_{n-1}(A, L)
\]

where we set \(U_{-1}(A, L) = 0\). We note that \(gr(U(A, L))\) is a commutative \(A\)-algebra. There is a theorem of Poincare-Birkhoff- Witt type due to Rinehart. In particular, it is proved (see [28], Theorem 3.1) that if \(L\) is a projective \(A\)-module and \(S_A(L)\) denotes the symmetric \(A\)-algebra on \(L\) then the canonical epimorphism \(\theta : S_A(L) \to gr(U(A, L))\) is an isomorphism of \(A\)-algebras. Moreover, we obtain that \(\iota_L : L \to U(A, L)\) is injective.

Let \((A, L)\) be a restricted Lie-Rinehart algebra over \(A\) and suppose that \(L\) is free as an \(A\)-module. Let \(\{u_i, i \in I\}\) be an ordered \(A\)-basis of \(L\). Let \(C(U(A, L))\) denote the center of \(U(A, L)\). Since \(L\) is a restricted \(A\)-algebra we obtain: for all \(u_i\) there is a \(z_i \in C(U(A, L))\) such that \(u_i^p = u_i^{|i|} = z_i\).

**Theorem 2.1.** Let \((A, L)\) be a restricted Lie-Rinehart algebra such that \(L\) is free as an \(A\)-module. Then the set,

\[
B := \{z_{i_1}^{h_1}z_{i_2}^{h_2} \cdots z_{i_r}^{h_r}, u_{i_1}^{k_1}u_{i_2}^{k_2} \cdots u_{i_r}^{k_r}\}
\]

where \(i_1 < i_2 < \cdots < i_r, \ h_i \geq 0\) and \(0 \leq k_i < p\) is an \(A\)-basis of \(U(A, L)\).
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Proof. It is proved in Theorem 3.1 in [28] that the standard monomials of the form $u_{i_1}^{s_1}u_{i_2}^{s_2}\cdots u_{i_r}^{s_r}$ where $i_1 < i_2 < \cdots < i_r$ and $s_i \geq 0$ form an $A$-basis of $U(A, L)$. Let $u_{i_1}^{s_1}u_{i_2}^{s_2}\cdots u_{i_r}^{s_r}$ be a standard monomial and $s = s_1 + \cdots + s_r$. By induction on $s$ we prove that the set $B$ generates $U(A, L)$. If all $s_i$ are such that $s_i < p$ then it is clear. Suppose that there is $s_{i_j} > p$ then we have

$$u_{i_1}^{s_1}u_{i_2}^{s_2}\cdots u_{i_r}^{s_r} = z_{i_j}u_{i_1}^{s_1}u_{i_2}^{s_2}\cdots u_{i_j}^{s_{i_j} - p}\cdots u_{i_r}^{s_r} + u_{i_1}^{s_1}u_{i_2}^{s_2}\cdots u_{i_j-1}^{s_{i_j-1}}u_{i_j+1}^{s_{i_j+1}}u_{i_r}^{s_r}.$$  

We notice that the terms $u_{i_1}^{s_1}u_{i_2}^{s_2}\cdots u_{i_j}^{s_{i_j} - p}\cdots u_{i_r}^{s_r}$ and $u_{i_1}^{s_1}u_{i_2}^{s_2}\cdots u_{i_j-1}^{s_{i_j-1}}u_{i_j+1}^{s_{i_j+1}}u_{i_r}^{s_r}$ belong in $U_{s-1}(A, L)$, thus by induction can be written as linear combination of elements of $B$. Next we prove that the elements of $B$ are linearly independent. Let $z_1^{h_1}z_2^{h_2}\cdots z_r^{h_r}u_{i_1}^{k_1}u_{i_2}^{k_2}\cdots u_{i_r}^{k_r}$ be an element of $B$. Since $z_i = u_i^p - u_i^p$, we get

$$(2.1) \hspace{1cm} z_1^{h_1}z_2^{h_2}\cdots z_r^{h_r}u_{i_1}^{k_1}u_{i_2}^{k_2}\cdots u_{i_r}^{k_r} = (u_1^p - u_1^p)^{h_1}(u_2^p - u_2^p)^{h_2}\cdots (u_r^p - u_r^p)^{h_r}u_{i_1}^{k_1}u_{i_2}^{k_2}\cdots u_{i_r}^{k_r}$$

$$= u_{i_1}^{h_1+p+k_1}u_{i_2}^{h_2+p+k_2}\cdots u_{i_r}^{h_r+p+k_r} \mod U_{s-1}(A, L)$$

where $s = \sum_{i=1}^{r} (h_i+p+k_i)$ and $1 < i_2 < \cdots < i_r$.

Let $(h_{i_1}, h_{i_2}, \ldots, h_{i_r})$, $(k_{i_1}, k_{i_2}, \ldots, k_{i_r})$ and $(h'_{i_1}, h'_{i_2}, \ldots, h'_{i_r})$, $(k'_{i_1}, k'_{i_2}, \ldots, k'_{i_r})$, be sequences such that $h_i+p+k_i = h'_i+p+k'_i$ for all $i = 1, 2, \cdots, r$. Since $0 < k_i < p$ and $0 < k'_i < p$ it is obliged to have $h_i = h'_i$ and $k_i = k'_i$ for all $i = 1, 2, \cdots, r$. Moreover, by Poincare-Birkhoff-Witt theorem we have an isomorphism of $A$-algebras $S_A(L) \simeq gr(U(A, L))$. Thus, by the relation (2.1) follows that the elements of $B$ are linearly independent. 

Proposition 2.2. Let $(A, L)$ be a Lie-Rinehart algebra such that $L$ is free as an $A$-module. Let $\{u_i, I\}$ be an ordered $A$-basis of $L$. If there is a map $u_i \to u_i^p$ such that $ad_{u_i}^{p^a} = ad_{u_i}^{[p]}$ for all $i \in I$ then $(A, L)$ can be equipped with the structure of restricted Lie-Rinehart algebra with a $p$-map which extends the map $(-)^{[p]}$.

Proof. Let $J$ be the ideal of $U(A, L)$ generated by the $z_i := u_i^p - u_i^p$. We consider the associative algebra $U_p(A, L) := U(A, L)/J$. By the previous theorem we get that the elements of the form

$$u_{i_1}^{k_1}u_{i_2}^{k_2}\cdots u_{i_r}^{k_r}$$

where $i_1 < i_2 < \cdots < i_r$, and $0 \leq k_i < p$ constitute an $A$-basis for $U_p(A, L)$.

The canonical $A$-algebra homomorphism $i_A : A \to U(A, L)$ induce an $A$-algebra homomorphism $i_A : A \to U_p(A, L)$. Moreover, the Lie homomorphism $i_L : L \to U(A, L)_{lie}$ induce a Lie algebra homomorphism $i_L : L \to U_p(A, L)_{lie}$ which is injective. Moreover, we have

$$(i_L(u_i))^p = u_i^p + J = u_i^p + J$$

and $(i_L(u_i))^p \in i_L(L)$. Besides,

$$(i_L(au_i))^p = (au_i)^p + J$$
By Hochschild’s relation Lemma 1 in [14] we get in $U(A, L)$ the relation

$$(au_i)^p = a^p u_i^p + \underbrace{[au_i, [au_i, \ldots [au_i, a]], \ldots]}_{p-1} u_i$$

$$= a^p u_i^p + (au_i)^{p-1}(a)u_i$$

Therefore,

$$(i_L(au_i))^p = a^p u_i^p + (au_i)^{p-1}(a)u_i + J$$

$$= a^p u_i^p + (au_i)^{p-1}(a)u_i + J$$

and $(i_L(au_i))^p \in i_L(L)$. Therefore, by relation (1.3) we get that for all $x \in L$ we have $x^p \in i_L(L)$ and $L$ is a restricted Lie $k$-subalgebra of $U_p(A, L)_{RLie}$. Obviously the relation (1.5) of the definition holds and $(A, L)$ is equipped with the structure of a restricted Lie-Rinehart algebra. □

**Remark 2.3.** Let $(A, L)$ be a Lie-Rinehart algebra such that $L$ is free as an $A$-module. Let $\{u_i, I\}$ be an ordered $A$-basis of $L$. We easily see using the relations 1.3 and 1.4, that the ideal of $U(A, L)$ generated by the elements $\{X^p - X^{[p]}, X \in L\}$ is equal to $J$.

Next, we introduce the notion of restricted enveloping algebra of a restricted Lie-Rinehart algebra. Let $(A, L, (-)^{[p]})$ be a restricted Lie-Rinehart algebra. We define as restricted enveloping algebra $U_p(A, L)$ of a restricted Lie-Rinehart algebra $(A, L, (-)^{[p]})$, the quotient:

$$U_p(A, L) := U(A, L)/\langle X^{[p]} - X^p \rangle$$

where, $a \in A$ and $X \in L$.

**Remark 2.4.** We note that in Proposition 2.2 is proved that the map $i_L : L \rightarrow U_p(A, L)_{RLie}$ is a restricted Lie monomorphism when $L$ is a free as an $A$-module.

The following proposition gives us the universal property of the restricted enveloping algebra.

**Proposition 2.5.** Let $B$ be an $A$-algebra such that there is an $A$-algebra homomorphism $\phi_A : A \rightarrow B$ and $\phi_L : L \rightarrow B_{RLie}$ a restricted Lie $k$-homomorphism. If we have:

$$\phi_A(a)\phi_L(X) = \phi_L(aX), \text{ and } [\phi_L(X), \phi_A(a)] = \phi_A(X(a))$$

for all $a \in A$ and $X \in L$ then there exists a unique homomorphism of associative algebras $\Phi_p : U_p(A, L) \rightarrow B$ such that $\Phi_p i_A = \phi_A$ and $\Phi_p i_L = \phi_L$.

**Proof.** We easily see that the map $f : A \oplus L \rightarrow B_{Lie}$ given by

$$f(a + X) = \phi_A(a) + \phi_L(X),$$
for all \( a \in A \) and \( X \in L \) is a Lie morphism. Therefore, there is a an algebra morphism 
\[ f' : U^+(A \oplus L) \to B. \]
Moreover,
\[
\begin{align*}
  f'(\iota(a(a' + X))) &= f(aa' + aX) \\
  &= \phi_A(aa') + \phi_L(aX) \\
  &= \phi_A(a)\phi_A(a') + \phi_A(a)\phi_L(X) \\
  &= f(a)f(a' + X) \\
  &= f'(\iota(a))f'(\iota(a' + X)) \\
  &= f'(\iota(a))(\iota(a' + X))
\end{align*}
\]
Thus, \( f' \) induces an algebra morphism \( \Phi : U(A, L) \to B. \) Since, \( \phi_L \) is a restricted Lie homomorphism we have \( \Phi(X[p]) = \Phi(X^p). \) Therefore, \( \Phi \) induces an algebra morphism \( \Phi_p : U_p(A, L) \to B. \)

**Remark 2.6.** The canonical representation \( A \to \text{End}_k(A) \) given by the multiplication, is faithful. By the universal property of \( U_p(A, L) \) we get that the \( A \)-algebra homomorphism \( i_A : A \to U_p(A, L) \) is injective.

**Definition 2.7.** Let \((A, L, (\cdot)[p])\) be a restricted Lie-Rinehart algebra. A restricted Lie-Rinehart module is a Lie-Rinehart \((A, L)\)-module \( M \) which additionally is a restricted Lie \( L \)-module. In other words, a restricted Lie-Rinehart \((A, L)\)-module is a \( k \)-module \( M \) equipped with the structures of an \( A \)-module and a Lie \( L \)-module such that:
\[
\begin{align*}
  (aX)m &= a(Xm) \\
  X(am) &= aX(m) + X(a)m \\
  X^{[p]}m &= (X(X(\ldots(Xm))))
\end{align*}
\]
for all \( a \in A, X \in L \) and \( m \in M \).

Let \((A, L, (\cdot)[p])\) be a restricted Lie-Rinehart algebra. The category of restricted \((A, L)\)-modules is equivalent to the category of \( U_p(A, L) \)-modules.

**Example 2.8.** The notion of a restricted Lie-Rinehart module recovers in a particular case the notion of regular module defined in \[6\]. Namely, if \( K/k \) is a purely inseparable extension of exponent 1 then a regular module is just a restricted Lie-Rinehart module over the restricted Lie Rinehart algebra \( \text{Der}_k(K) \).

**Example 2.9.** Let \( g \in \text{Lie}_{K/k} \) be a restricted Lie-Rinehart algebra (see 1.12 and [22]) then a \( p \)-flat connection is a restricted Lie-Rinehart \( g \)-module.

An important example of Lie algebroid is the transformation Lie algebroid for a differential manifold. There is an algebraic generalization of this notion called the transformation Lie-Rinehart algebra.

**Proposition 2.10.** Let \( g \in \text{RLie} \) be a restricted Lie \( k \)-algebra and \( A \) a commutative \( k \)-algebra. If there is a restricted Lie homomorphism \( \delta : g \to \text{Der}(A) \), then the transformation Lie-Rinehart algebra \((A, A \otimes_k g)\) can be endowed with the structure of a restricted Lie-Rinehart algebra.
The universal property of the free restricted Lie algebra $L$ is a left adjoint functor $\phi$ which is a restricted Lie homomorphism:

$$f \circ \psi = \phi$$

for all $a, a' \in A$ and $g, g' \in g$, and with anchor map $\alpha : A \otimes_k g \to \text{Der}(A)$ given by $\alpha(a \otimes g)(a') = a\delta(g)(a')$. Let $\{g_i, i \in I\}$ be a $k$-basis of $g$, then the elements $\{1 \otimes g_i, i \in I\}$ form an $A$-basis of $A \otimes g$. Let $\tau_{1 \otimes g_i}, \rho_{1 \otimes g_i} : A \otimes_k g \to A \otimes g$ be the $k$-linear maps given by $\tau_{1 \otimes g_i}(a \otimes g) := a \otimes [g, g_i]$ and $\rho_{1 \otimes g_i}(a \otimes g) := \delta(g_i)(a) \otimes g$ respectively. We note that:

$$\tau_{1 \otimes g_i}, \rho_{1 \otimes g_i} = \rho_{1 \otimes g_i}, \tau_{1 \otimes g_i}.$$ 

Since $\text{char } k = p$ we have:

$$\text{ad}_{1 \otimes g_i}^p = (\tau_{1 \otimes g_i} - \rho_{1 \otimes g_i})^p = \sum_{j=0}^{p} \binom{p}{j} \tau_{1 \otimes g_i}^j \rho_{1 \otimes g_i}^{p-j} = \tau_{1 \otimes g_i}^p - \rho_{1 \otimes g_i}^p = \text{ad}_{1 \otimes g_i}^p.$$ 

Therefore, by Proposition 2.2 above we get that $A \otimes_k g$ can be equipped with the structure of a restricted Lie-Rinehart algebra and the $p$-map is given by:

$$(a \otimes g)^{[p]} = a^p \otimes g^p - (a \delta(g))^{p-1}(a) \otimes g$$

By equation (1.4), we see that the anchor map $\alpha$ is actually a restricted Lie homomorphism.

In the next subsection we extend for the category $p - \text{LR}(A)$ the notion of free Lie-Rinehart algebra defined by Casas-Pirashvili in [9] and Kapranov in [20].

2.1. Free restricted Lie-Rinehart algebra. Let $A$ be a commutative $k$-algebra. We denote by $\text{Vect/\text{Der}(A)}$ the category whose objects are $k$-linear morphisms $\psi : V \to \text{Der}(A)$ where $V \in \text{Vect}$ and morphisms $f : \psi \to \psi'$ are $k$-linear morphisms $f : V \to V'$ such that $\psi'f = \psi$. We denote by $\mathcal{V} : p - \text{LR}(A) \to \text{Vect/\text{Der}(A)}$ the forgetful functor from the category of restricted Lie-Rinehart $A$-algebras to the category $\text{Vect/\text{Der}(A)}$ which assigns a restricted Lie-Rinehart algebra $L$ over $A$ to $\alpha : L \to \text{Der}(A)$ the anchor map of $L$.

**Proposition 2.11.** There is a left adjoint functor $F : \text{Vect/\text{Der}(A)} \to p - \text{LR}(A)$ to the functor $\mathcal{V} : p - \text{LR}(A) \to \text{Vect/\text{Der}(A)}$

$$\text{Hom}_{p - \text{LR}(A)}(F(\psi), L) \simeq \text{Hom}_{\text{Vect/\text{Der}(A)}}(\psi, \mathcal{V}(L))$$

**Proof.** Let $\psi : V \to \text{Der}(A)$. Then, by the universal property of the free restricted Lie algebra $L_p(V)$ generated by $V$ there is a restricted Lie homomorphism $\Phi : L_p(V) \to \text{Der}(A)$ such that $\Phi|_V = \psi$, where $i_V : V \to L_p(V)$ denotes the canonical map. By Proposition 2.2, we get that $A \otimes L_p(V)$ is equipped with the structure of a restricted Lie-Rinehart algebra. Therefore, we construct a functor $F : \text{Vect/\text{Der}(A)} \to p - \text{LR}$ which assigns $\psi \in \text{Vect/\text{Der}(A)}$ to $A \otimes_k L_p(V)$.

Let $f \in \text{Hom}_{\text{Vect/\text{Der}(A)}}(\psi, \mathcal{V}(L))$. Then, we have that $\alpha f = \psi$. Moreover, by the universal property of the free restricted Lie algebra $L_p(V)$ generated by $V$, there is a restricted Lie homomorphism $\phi : L_p(V) \to L$ such that $\phi|_V = f$ and $\Phi = \alpha \phi$. 
Let $f_p : A \otimes_k L_p(V) \to L$ be the homomorphism of restricted Lie-Rinehart algebras given by $f_p(a \otimes x) := a \phi(x)$, where $a \in A$ and $x \in L_p(V)$. Thus, we construct a map $f \mapsto f_p$. Conversely, for $f_p \in Hom_{p-LR(A)}(F(\psi), L)$ we consider the $k$-linear map $f : V \to L$ given by $f := f_p\overline{g}$ where $\overline{g} : V \to A \otimes_k L_p(V)$ given by $v \mapsto (1 \otimes v)$, and $v \in V$. We easily see that the maps $f \mapsto f_p$ and $f_p \mapsto f$ are inverse to each other. \hfill $\square$

3. Beck modules and Beck derivations

Beck in his desertion (see [5]) gave an answer of what should be the right notion of coefficient module for cohomology. The notion of Beck-module encompasses for various categories, the usual known notions of coefficient module for cohomology (see [3]). In this section we determine the category of Beck modules and the group of Beck derivations (see [3, 4]) for the category of restricted Lie-Rinehart algebras $p$-LR(A) over a commutative algebra $A$.

**Definition 3.1.** Let $L$ be an object in a category $p$-LR(A). We denote by $(p$-LR(A)/$L)_{ab}$ the category of abelian group objects of the comma category $p$-LR(A)/$L$ and by $I_L : (p$-LR(A)/$L)_{ab} \to p$-LR(A)/$L$ the forgetful functor. An object $M \in (p$-LR(A)/$L)_{ab}$ is called a Beck $L$-module. Let $g \in p$-LR(A)/$L$ and $M$ a Beck-$L$-module. The group $Hom_{p$-LR(A)/$L}(g, I_L(M))$ is called the group of Beck derivations of $g$ by $M$.

**Notation 3.2.** Let $M \xrightarrow{\mu} L$ be a Beck $L$-module. We denote by $\bar{M} := \ker \mu$ and by $p_M$ the restriction of the $p$-map $p_M$ of $M$ to $\bar{M}$.

**Theorem 3.3.** Let $L \in p$-LR(A) and $M \xrightarrow{\mu} L$ be Beck $L$-module. Then, $\bar{M} \rtimes_{p_M} L$ is defined and endowed with the structure of a restricted Lie-Rinehart algebra. Moreover, there is an isomorphism of restricted Lie-Rinehart algebras:

$$\bar{M} \rtimes_{p_M} L \simeq M$$

**Proof.** Since $M \xrightarrow{\mu} L$ is an abelian group object in $(p$-LR(A)/$L)_{ab}$ a fortiori is an abelian group object in $(Lie/L)_{ab}$, where $Lie/L$ denotes the slice category of Lie algebras over $L$. It is well known that the category of abelian group objects $(Lie/L)_{ab}$ is equivalent to the category of Lie $L$-modules. In particular if $z : L \to M$ denotes the zero map for the structure of group object we have a split extension in the category of Lie algebras:

$$0 \to \bar{M} \to M \xrightarrow{\varepsilon} L \to 0$$

Moreover, there is an isomorphism of Lie algebras $\psi : \bar{M} \oplus L \simeq M$ given by $\psi(\bar{m} + X) := \bar{m} + z(X)$, for all $\bar{m} \in \bar{M}$ and $X \in L$. Since $z$ is a restricted Lie homomorphism we have:

$$X^{[\mu]} \bar{m} = [z(X^{[\mu]}), \bar{m}]$$

$$= [z(X)^{[\mu]}, \bar{m}]$$

$$= [z(X), \cdots, [z(X), [z(X), \bar{m}]] \cdots]$$
Therefore, $\tilde{M}$ is endowed with the structure of restricted $L$-module. Besides $\tilde{M}$ is abelian thus,

$$[z(X), \bar{m}]_p^{[pM]} = [z(X), \bar{m} \ldots, \bar{m}]_p = 0$$

Therefore, $p_{\tilde{M}} : \tilde{M} \to \tilde{M}^L$ and $\tilde{M} \rtimes_{p_{\tilde{M}}} L$ is defined. Since $\tilde{M}$ is abelian

$$(\psi(X + \bar{m}))^{pM} = z(X)^{pM} + \bar{m}^{pM} + \sum_{i=1}^{i=p-1} s_i(z(X), \bar{m})$$

$$= z(X)^{pM} + \bar{m}^{pM} + ad_{z(X)}^{p-1}(\bar{m})$$

Therefore, the isomorphism $\psi$ is an isomorphism of a restricted Lie algebras

$$\psi : \tilde{M} \rtimes_{p_{\tilde{M}}} L \simeq M$$

The $k$-module $\tilde{M} \rtimes_{p_{\tilde{M}}} L$ has the structure of an $A$-module given by the formula

$$a(\bar{m} + X) := a\bar{m} + aX, \ a \in A$$

and $\psi$ is an $A$-module homomorphism. We easily see that, $\tilde{M} \rtimes_{p_{\tilde{M}}} L$ is endowed with the structure of a Lie-Rinehart algebra with anchor map $\alpha : \tilde{M} \rtimes_{p_{\tilde{M}}} L \to Der_k(A)$ given by

$$\alpha(\bar{m} + X)(a) := X(a) = z(X)(a)$$

In this way, $\psi$ becomes a Lie-Rinehart isomorphism. Besides, we have

$$\psi((a(X + \bar{m}))^{[p]} = (\psi(a(X + \bar{m})))^{[p]}$$

$$= (a(\psi(X + \bar{m})))^{[p]}$$

$$= a^p(\psi(X + \bar{m}))^{[p]} + (a\psi(X + \bar{m}))^{p-1}(a)\psi(X + \bar{m})$$

$$= a^p(\psi(X + \bar{m}))^{[p]} + \psi((a(z(X) + \bar{m}))^{p-1}(a)(X + \bar{m}))$$

$$= \psi((a^p(X + \bar{m}))^{[p]} + \psi((a(X + \bar{m}))^{p-1}(a)(X + \bar{m}))$$

Therefore, $M \rtimes_{p_{\tilde{M}}} L$ is a restricted Lie-Rinehart algebra and $\psi : \tilde{M} \rtimes_{p_{\tilde{M}}} L \simeq M$ is a restricted Lie-Rinehart isomorphism.

**Lemma 3.4.** Let $L$ be a Lie-Rinehart algebra over $A$. Then, the following relation holds

$$(aX)^{p-1}(ab) = a^pX^{p-1}(b) + (aX)^{p-1}(a)b$$

for all $X \in p\text{-}LR(A)$ and $a, b \in A$.

**Proof.** We consider the restricted enveloping algebra $U(A, L)$ of the Lie-Rinehart algebra $L$ over $A$. By Hochschild’s Lemma 1, we get in $U(A, L)$ that:

$$(a(b + i_L(X)))^p = a^p(b + i_L(X))^p + (aX)^{p-1}(b + i_L(X))$$

$$= a^p(b + i_L(X))^p + X^{p-1}(b) + (aX)^{p-1}(a)b + (aX)^{p-1}(a)i_L(X)$$
Besides,
\[(a(b + i_L(X)))^p = (ab + ai_L(X))^p = ((ab)^p + ai_L(X))^p + (aX)^{p-1}(ab) = a^p b^p + a^p i_L(X)^p + (aX)^{p-1}(a)i_L(X) + (aX)^{p-1}(ab)\]
Therefore, we get:
\[(aX)^{p-1}(ab) = a^p X^{p-1}(b) + (aX)^{p-1}(a)b\]
for all \(X \in L\) and \(a, b \in A\). 

Let \(L\) be a Lie-Rinehart algebra over \(A\). Let \(M\) be a Lie-Rinehart \((A - L)\)-module. We consider the commutative algebra semi-direct product \(A \oplus M\) with product given by:
\[(a + m)(a' + m') := aa' + (am' + a'm)\]
There is an anchor map \(\alpha : L \to \text{Der}(A \oplus M)\) given by
\[\alpha(X)(a + m) := X(a) \oplus X(m)\]
for all \(X \in L, a \in A\) and \(m \in M\). Moreover, \(L\) becomes an \(A \oplus M\)-module via the action \((a + m)L := aX\). Then, we easily see that, \(L\) becomes a Lie-Rinehart algebra over \(A \oplus M\). Therefore, by Lemma 3.4 above we get:
\[(3.1) \quad (aX)^{p-1}(am) = a^p X^{p-1}(m) + (aX)^{p-1}(a)m\]
where \(X \in L\) and \(a \in A, m \in M\).

**Proposition 3.5.** Let \(L\) be a restricted Lie-Rinehart algebra over \(A\) and \(M\) a restricted Lie-Rinehart module. Then, \(M \times L\) is endowed with the structure of restricted Lie-Rinehart algebra.

**Proof.** We observe that \(\tilde{M} \times L\) is an \(A\)-module via the action \(a(\tilde{m} + X) := a\tilde{m} + aX\) where \(a \in A, \tilde{m} \in M\) and \(X \in L\). Moreover, there is anchor map \(\alpha : M \times L \to \text{Der}(A)\) given by \(\alpha((\tilde{m} + X))(a) := X(a)\). We easily see that \(\tilde{M} \times L\) is endowed with the structure of Lie-Rinehart algebra over \(A\). Besides, by the relation (3.1) above we get
\[(a(\tilde{m} + X))^{[p]} = (aX)^{[p]} + (aX)^{p-1}(a\tilde{m}) = a^p X^{[p]} + (aX)^{p-1}(a) X + a^p X^{p-1}(\tilde{m}) + (aX)^{p-1}(a)\tilde{m} = a^p (\tilde{m} + X)^{[p]} + (a(\tilde{m} + X))^{p-1}(a)(\tilde{m} + X)\]
Thus, \(\tilde{M} \times L\) is a restricted Lie-Rinehart algebra. 

Let \(A[P]\) be the polynomial ring given by
\[A[P] := \left\{ \sum_{i=0}^{n} a_i P_i : Pa = a^p P \text{ for all } a_i, a \in A \right\}\]
We consider the ring \(W(A, L)\) which as an \(A\)-module is given by
\[W(A, L) := A[P] \otimes_A U_p(A, L)\]
and such that $A[P] \to W(A, L)$ and $U_p(A, L) \to W(A, L)$ are $A$-algebra homomorphisms and the multiplication is such that

\begin{align}
(3.2) & \quad (P \otimes 1)(1 \otimes i_L(X)) = P \otimes i_L(X) \\
(3.3) & \quad (1 \otimes i_L(X))(P \otimes 1) = 0
\end{align}

**Proposition 3.6.** Let $L$ be a restricted Lie-Rinehart algebra over $A$. Then, the category of Beck $L$-modules is equivalent to the category of $W(A, L)$-modules.

**Proof.** Let $M$ be a $W(A, L)$-module. Using the homomorphism $U_p(A, L) \to W(A, L)$ we can see $M$ as a $U_p(A, L)$-module which we denote by $\tilde{M}$. Besides, the $A[P]$ action endows $\tilde{M}$ with a $p$-semi-linear map $p_{\tilde{M}} : \tilde{M} \to \tilde{M}$ given by

$$p_{\tilde{M}}(\tilde{m}) := P\tilde{m}, \quad \tilde{m} \in \tilde{M}$$

From Proposition 3.5 and Remark 1.4 we see that $\tilde{M} \rtimes_{p_{\tilde{M}}} L$ is a restricted Lie-Rinehart algebra. Thus, by Theorem 3.3, any $W(A, L)$-module $M$ is associated to the Beck $L$-module $M \rtimes_{p_{\tilde{M}}} L$. Conversely, let $M$ be a Beck $L$-module, then by Theorem 3.3 we have that $M \simeq M \rtimes_{p_{\tilde{M}}} L$. Moreover, we observe that $\tilde{M}$ is a $W(A, L)$-module. Thus, we have an equivalence of categories. \qed

### 3.1. Beck derivations.

Let $g$ be a restricted Lie-Rinehart algebra and $M = \tilde{M} \rtimes_{p_{\tilde{M}}} L$ be a Beck $g$-module. Then, $\tilde{M}$ is a Lie-Rinehart $g$-module. We denote by $\text{Der}(g, \tilde{M})$ the group of Lie-Rinehart derivations. We recall that a $k$-Lie algebra derivation $d : g \to \tilde{M}$ is called Lie Rinehart if $d$ is $A$-linear. The group of Beck derivations is defined as follows:

$$\text{Der}_p(g, M) := \{d \in \text{Der}_A(g, \tilde{M}) : d(X^{[p]}) = X^{[p]} + (d(X))^{|p_{\tilde{M}}|}, \quad X \in g\}$$

We note that $\text{Der}_p(g, M)$ is a group under the addition since $p_{\tilde{M}}$ is a $p$-semi-linear map.

**Proposition 3.7.** Let $L$ be a restricted Lie-Rinehart algebra over $A$ and $g \in p$-LR($A$)/$L$. If $M$ is a Beck $L$-module, then we have the following isomorphism

$$\text{Hom}_{p-\text{LR}(A)/L}(g, M \rtimes_{p_{\tilde{M}}} L) \simeq \text{Der}_p(g, M)$$

**Proof.** Let $f : g \to \tilde{M} \rtimes_{p_{\tilde{M}}} L$ and $\pi : M \rtimes_{p_{\tilde{M}}} L \to \tilde{M}$ be the canonical projection. We easily see that, $d_f := \pi f$ is a Beck derivation and therefore, is defined a map $\Phi : f \mapsto d_f$. Moreover, let $d : g \to \tilde{M}$ be a Beck derivation we consider the map $f_d : g \to \tilde{M} \rtimes_{p_{\tilde{M}}} L$ given by $f_d(X) := d(X) + \gamma(X)$, where $X \in g$ and $\gamma : g \to L$ is the structural map. The maps $\Psi : d \mapsto f_d$ and $\Phi : f \mapsto d_f$ are inverse to each other. \qed

### 4. Quillen-Barr-Beck cohomology for restricted Lie-Rinehart algebras

There is a general theory of cohomology for universal algebras due to Quillen-Barr-Beck (see [25], [9], [1]). Moreover, Quillen in [26] proves that the cohomology theory for universal algebras defined in [25] is a special case of the general definition of sheaf cohomology due to Grothendieck. Following the general scheme of Quillen-Barr-Beck cohomology theory we define cohomology groups for the category of restricted Lie-Rinehart algebras. By Proposition 2.8 there is a functor

$$F : \text{Vect}/\text{Der}(A) \to p-\text{LR}(A)$$
left adjoint to the functor
\[ \mathcal{V} : p-LR(A) \to \text{Vect}/\text{Der}(A) \]

The adjoint pair \((F, \mathcal{V})\) induce a cotriple \(G = (G_*, \epsilon, \delta)\) such that \(G_* : \mathcal{V} : p-LR(A) \to \text{Vect}/\text{Der}(A)\) known as cotriple resolution or Godement resolution (see [10], [4]).

**Definition 4.1.** Let \(L\) be a restricted Lie-Rinehart algebra and \(M\) a Beck \(L\)-module. Then, for \(n \in \mathbb{N}^*\), Quillen-Barr-Beck cohomology groups are defined by the following formula
\[ H^n_{p-LR}(L, M) := H^n(\text{Hom}_{p-LR(A)/L}(G_*(L, M))) \]
where in the right hand side of the formula \(H\) denotes the cohomology of a cosimplicial object.

**Remark 4.2.** We observe that in degree 0 we have \(H^0_{p-LR}(L, M) = \text{Der}_p(L, M)\).

**4.1. Cohomology in degree 1 and extensions.** A principal bundle gives rise to Atiyah sequence introduced by Atiyah in [2] (see for details [16]). The algebraic analogue of ”Atiyah sequence” is an extension of Lie-Rinehart algebras. In this subsection we consider extensions of restricted Lie-Rinehart algebras. We prove that the Quillen-Barr-Beck cohomology in degree one classifies restricted Lie-Rinehart extensions.

**Definition 4.3.** An extension \((e)\) of restricted Lie-Rinehart algebras (of \(L\) by \(M\)) is a short exact sequence of restricted Lie-Rinehart algebras
\[ 0 \to M \to E \to L \to 0 \quad (e) \]
such that \([M, M] = 0\).

**Remark 4.4.** Let \(L\) be a restricted Lie-Rinehart algebra over \(A\) and \((e)\) an extension of \(L\) by \(M\). Since \(M\) is abelian a section \(s : L \to E\) defines a restricted Lie \(L\)-action on \(M\). Moreover, we easily see that \(M\) becomes a \(U_p(A, L)\)-module. Besides, there is an \(A[P]\) action on \(M\) such that \(Pm := m^{[Pm]}\), \(m \in M\). Thus, we obtain that \(M\) is endowed with the structure of a \(W(A, L)\)-module.

Two restricted Lie-Rinehart extensions \((e), (e')\) of \(L\) by \(M\) are called equivalent if there is a restricted Lie-Rinehart isomorphism \(f : E \to E'\) such that the following diagram commutes
\[
\begin{array}{ccc}
0 & \longrightarrow & M & \longrightarrow & E & \longrightarrow & L & \longrightarrow & 0 \\
\| & & \| & & f & & \| & & \\
0 & \longrightarrow & M & \longrightarrow & E' & \longrightarrow & L & \longrightarrow & 0 \\
\end{array}
\]

We denote the set of equivalent classes by \(\text{Ext}_p(L, M)\).

**Baer sum of restricted Lie-Rinehart algebras.** Let \((e), (e')\) be two Lie algebra extensions of \(L\) by \(M\)
\[ 0 \to M \to E \xrightarrow{f} L \to 0 \]
and
\[ 0 \to M \to E' \xrightarrow{f'} L \to 0 \]
Let \( E \times_L E' = \{(e, e'): f(e) = f'(e')\} \) be the pullback in \( \text{Lie} \). We denote by \( I \) the ideal

\[
I := \langle \{(m, 0) - (0, m'), \ m, m' \in M\} \rangle
\]

and we consider the Lie algebra

\[
Y := E \times_L E'/I
\]

The Baer sum of \((e)\) and \((e')\) is the extension of Lie algebras of \( L \) by \( M \)

\[
0 \to M \xrightarrow{\iota} Y \xrightarrow{\psi} L \to 0
\]

where \( \iota(m) := (m, 0) \) and the last one \( \psi((e, e')) := f(e) = f(e') \). The set \( \text{Ext}_{\text{Lie}}^p(L, M) \) of equivalence classes of extensions of Lie algebras of \( L \) by \( M \) endowed with the operation of Baer sum is a group. Moreover, let \((e), (e')\) be restricted Lie-Rinehart extensions of \( L \) by \( M \). There is an action of \( A \) on \( E \times_L E' \) given by \( a \in A \) and \((X, Y) \in E \times_L E'\). Then, there is an action of \( A \) on \( E \times_L E' \) given by \( a(X, Y) := (aX, aY) \). Besides, there is an action \( E \times_L E' \to \text{Der}(A) \) given by \((X, Y)(a) := X(a) = Y(a)\). The above actions endow \( E \times_L E' \) with the structure of a restricted Lie-Rinehart algebra. Moreover, the ideal \( I \) is a restricted Lie-Rinehart ideal. Thus, \( Y \) is a restricted Lie-Rinehart algebra.

Theorem 4.5. Let \( L \) be a restricted Lie-Rinehart algebra over \( A \) and \( M \) a Beck \( L \)-module. There is a bijection

\[
H^1_{p-LR}(L, M) \simeq \text{Ext}_p(L, M)
\]

Proof. Duskin in [7] develops the theory of torsors and gives an interpretation of cotriple cohomology. There is a bijection between the first cohomology group \( H^1_{\text{LR}}(L, M) \) and the set \( \text{Tors}_{p-LR}(L, M) \) of the isomorphism classes of objects \( E \in p-LR(A)/L \) which are torsors for the abelian group object \( M \). An object \( E \in p-LR(A)/L \) is \( M \) torsor, if \( E \to L \) is an epimorphism and there is a restricted Lie-Rinehart morphism

\[
\omega : (\bar{M} \rtimes_{p_{\text{ad}}} L) \times_L E \to E
\]

such that the map

\[
(\omega, \pi) : (\bar{M} \rtimes_{p_{\text{ad}}} L) \times_L E \to E \times_L E
\]

where \( \pi \) denotes projection, is a restricted Lie-Rinehart isomorphism.

Let \( E \in \text{Ext}_p(L, M) \), then we have a short exact sequence in \( p-LR(A) \)

\[
0 \to M \to E \xrightarrow{f} L \to 0
\]

such that \([M, M] = 0\) and the induced \( W(A, L) \)-action on \( M \), recovers the given \( W(A, L) \)-action on \( M \). We define a map

\[
\omega : (\bar{M} \rtimes_{p_{\text{ad}}} L) \times_L E \to E
\]
Therefore, there is a bijection of sets $\tilde{M} \times_{p_{\tilde{M}}} L \times_L E \rightarrow E \times_L E$ is a restricted Lie-Rinehart isomorphism. Therefore, $E \rightarrow L$ is torsor for $\tilde{M}$.

Conversely, let $E \in p\text{-LR}(A)/L$ be an object torsor for $\tilde{M} \times_{p_{\tilde{M}}} L \rightarrow L$. Then, the structural map $f : E \rightarrow L$, is a restricted Lie-Rinehart epimorphism. If $K := \ker f$, then there is an injection $i : K \hookrightarrow E \times_L E$ with $i(k) := (k, 0)$. Besides, there is an injection $j : \tilde{M} \hookrightarrow (\tilde{M} \times_{p_{\tilde{M}}} L) \times_L E$ given by $j(\tilde{m}) := ((\tilde{m}, 0), 0)$. The restriction of the restricted Lie-Rinehart isomorphism $(\omega, \pi)$ on $\tilde{M}$ implies an isomorphism of restricted Lie-Rinehart algebras $(A, \tilde{M}, (-)_{p_{\tilde{M}}}) \simeq (A, K, (-)_{p_{\bar{K}}})$. Moreover, let $X \in L$, $m \in \tilde{M}$ and $e \in E$ such that $f(e) = X$. Then, we have

$$\omega(Xm, 0) = \omega(((X, e), (m, 0))) = [\omega(X, e), \omega(m, 0)]$$

Besides, we have

$$(\omega, \pi)((X, e), e) = (\omega(X, e), e)$$

it follows that $f(\omega(X, e)) = X$. Therefore, $\tilde{M}$ and $K$ are isomorphic as restricted Lie $L$-modules. Since $\omega$ is a restricted Lie-Rinehart homomorphism, it follows that, $\tilde{M}$ and $K$ are isomorphic as $U_p(A, L)$-modules. Thus, we get an extension of restricted Lie-Rinehart algebras

$$0 \rightarrow M \rightarrow E \rightarrow L \rightarrow 0$$

Therefore, there is a bijection of sets $H^1_{p-LR}(L, M) \simeq Ext_p(L, M)$. \hfill \Box

Remark 4.6. We note that, in terms of Quillen-Barr-Beck cohomology, there is a shift in the dimension, in compare to the classical notation, concerning classification theorems of extensions.

If $L$ is a Lie algebra and $M$ a $\mathcal{U}(L)$-module, then cotriple cohomology groups $H^1_{Lie}(L, M) := H^1(\text{Hom}_{Lie}(G_{Lie}(L), M))$ are defined (see [3]). The cotriple which is used is given by $G_{Lie} := LF$ where $L : \text{Vect} \rightarrow \text{Lie}$ is the free Lie algebra functor left adjoint to the forgetful functor $\mathcal{F} : \text{Lie} \rightarrow \text{Vect}$.

Corollary 4.7. There is an isomorphism of groups

$$H^1_{p-LR}(L, M) \simeq Ext_p(L, M)$$
Proof. Duskin in [7],[8] and Glen in [11], develop the theory of torsors. In particular, is proved (see Section 4 in [8] and [11]) that the set of torsors can be endowed with the structure of a group. It follows from the general theory (see Section 5 in [8] and [11]) that, there is a group isomorphism
\[ H^1_{p-LR}(L, M) \simeq \text{Tors}_{p-LR}(L, M) \]
and respectively for the category Lie algebras
\[ H^1_{Lie}(L, M) \simeq \text{Tors}_{Lie}(L, M) \]

Besides, the category of Lie algebras is a category of interest in sense of Orzech (see [24]). Therefore by a general result of Vale in [32], we obtain a group isomorphism
\[ \text{Ext}_{Lie}(L, M) \simeq \text{Tors}_{Lie}(L, M) \]
If \( L \in p-LR(A) \) and \( M \) a \( W(L) \)-module, then there is a natural embedding \( H^1_{p-LR}(L, M) \hookrightarrow H^1_{Lie}(L, M) \). Thus, we have the following commutative diagram

\[
\begin{array}{ccc}
H^1_{p-LR}(L, M) & \xrightarrow{\sim} & \text{Tors}_{p-LR}(L, M) \xrightarrow{\sim} \text{Ext}_p(L, M) \\
\downarrow & & \downarrow & \uparrow \\
H^1_{Lie}(L, M) & \xrightarrow{\sim} & \text{Tors}_{Lie}(L, M) \xrightarrow{\sim} \text{Ext}_{Lie}(L, M)
\end{array}
\]

Therefore, the bijection (4.1)
\[ H^1_{p-LR}(L, M) \simeq \text{Ext}_p(L, M) \]
is an isomorphism of groups. \( \square \)

5. Brauer group and cohomology

The problem of classification of finite-dimensional central simple algebras is related to the notion of Brauer group. The theory of Brauer groups has strong ties with number theory, algebraic geometry (see [30], Local fields) and algebraic \( k \)-theory [19]. In connection with Galois theory we have that if \( E/F \) is a Galois extension then the relative Brauer group \( B^E_F \) is isomorphic with the group of equivalence classes of group extensions of the Galois group \( \text{Gal}(E/F) \) by the multiplicative group \( F^* \) of \( F \). Moreover, we have an isomorphism of groups relating the Galois cohomology and the Brauer group:
\[ B^E_F \simeq H^2(\text{Gal}(E/F), F^*) \]
In the context of purely inseparable extension of exponent 1 there is a Galois theory due to Jacobson (see [18]). The role of Galois group is now played by the group of derivations. In particular, let \( K/k \) be a finite purely inseparable extension of exponent 1 and \( A \) a finite dimensional algebra with center \( k \) which contains \( K \) as a maximal commutative subring. Let \( S := \{ s \in A, | D_s(K) \subset K \} \), where \( D_s \) denotes the derivation \( D_s(a) := sa - as \). Then, Hochschild in [14] considers particular class of restricted Lie algebra extensions of \( \text{Der}_k(K) \) by \( K \)
\[ 0 \rightarrow K \rightarrow S \rightarrow \text{Der}_k(K) \rightarrow 0 \]
called regular extensions. A regular extension is nothing more than a restricted Lie-Rinehart extension of the restricted Lie-Rinehart algebra \( \text{Der}_k(K) \) by \( K \). We note that \( K \) is an abelian restricted Lie algebra over \( k \) with its natural \( p \)-map (see [14] for details). Moreover, Hochschild in [14] proves that there is an isomorphism between the Brauer group \( B^k_F \) and the set of equivalence classes of regular restricted Lie algebra extensions of \( \text{Der}_k(K) \) by \( K \).
As motivation to carry out this research was to establish an isomorphism in terms of cohomology of restricted Lie-Rinehart algebras analogue to the isomorphism (5.1) in terms of Galois cohomology. Hochschild in [13] defined cohomology groups $H^*_\text{Hoch} := H^*(L, M)$ where $L \in R\text{Lie}$ and $M$ a restricted Lie module. Since the second cohomology group $H^2_{\text{Hoch}}(L, M)$ classifies extensions where $M$ is strongly abelian, i.e such that $M^{[p]} = 0$ we do not have an isomorphism of the Brauer group $B_K^k$ with a sub group of $H^2_{\text{Hoch}}(\text{Der}_k(K), K)$. Nevertheless, using the Quillen-Barr-Beck cohomology for $p$-LR(A) we can establish the analogue isomorphism to (5.1).

**Theorem 5.1.** Let $K/k$ be a finite purely inseparable extension of exponent 1. Then, we have the following isomorphism of groups

$$B^K_K \simeq H^1_{p-LR}(\text{Der}_k(K), K)$$

**Proof.** The proof follows from Corollary (4.7) and the isomorphism proved by Hochschild in [13] between the Brauer group $B^K_K$ and the group of equivalence classes of regular extensions of $\text{Der}_k(K)$ by $K$. □

**Remark 5.2.** Given a commutative ring $C$ and a commutative $C$-algebra $A$, Amitsur in [1] defined cohomology groups $H^*_\text{Am}(A)$. Let $K$ be a purely inseparable extension field of $k$ of exponent 1. Then, Rosenberg and Zelinsky in Section 4 in [29] exhibit an isomorphism between $H^3_{\text{Am}}(K)$ and the Brauer group $B^K_K$. Therefore, by the above theorem we get an isomorphism

$$H^3_{\text{Am}}(K) \simeq H^1_{p-LR}(\text{Der}_k(K), K)$$

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