THE MIXED CUBIC-QUARTIC FUNCTIONAL EQUATION

BY

A. BODAGHI, D. KANG and J.M. RASSIAS

Abstract. In this paper, we obtain the general solution of the following generalized mixed cubic and quartic functional equation

\[ f(x + kx) + f(x - ky) = k^2 \{ f(x + y) + f(x - y) \} - 2(k^2 - 1)f(x) - 2k^2(k^2 - 1)f(y) + \frac{4}{9}k^2(k^2 - 1)f(2y), \]

for fixed integers \( k \) with \( k \neq 0, \pm 1 \). The Hyers-Ulam stability problem for the mentioned functional equation is also proved.

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1. Introduction and preliminaries

In 1940, Ulam [15] proposed the following stability problem:

“When is it true that by slightly changing the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?”

In 1941, Hyers [7] solved this stability problem for additive mappings subject to the Hyers condition \( \|f(x+y) - f(x) - f(y)\| \leq \delta \) on approximately additive mappings \( f : \mathcal{X} \rightarrow \mathcal{Y} \) for a fixed \( \delta \geq 0 \) and all \( x, y \in \mathcal{X} \), where \( \mathcal{X} \) is a real normed space and \( \mathcal{Y} \) a real Banach space. In 1950, Aoki [1] generalized the Hyers theorem for additive mappings. In 1978, Rassias [14] provided a generalized version of the Hyers theorem which permitted the Cauchy difference to become unbounded.

The cubic function \( f(x) = ax^3 \) satisfies the functional equation

(1.1) \[ f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x). \]
So the equation (1.1) is called a cubic functional equation and every solution of equation (1.1) is said to be a cubic function. The stability result of equation (1.1) was obtained by Jun and Kim [8] for the first time. After that, they [9] introduced the following cubic functional equation
\[ f(x + 2y) + f(x - 2y) + 6f(x) = 4f(x + y) + 4f(x - y) \] and they established the general solution and the Hyers-Ulam stability problem for it. Recently, in [5], Bodaghi et al. introduced the following new form of cubic functional equations
\[ f(x + my) + f(x - my) = 2(2\cos\left(\frac{m\pi}{2}\right) + m^2 - 1)f(x) \]
(1.2)
\[ -\frac{1}{2}\left(\cos\left(\frac{m\pi}{2}\right) + m^2 - 1\right)f(2x) + m^2\{f(x + y) + f(x - y)\}, \]
where \( m \) is an integer with \( m \geq 2 \). They studied the Hyers-Ulam stability of (1.2).

The quartic functional equation introduced by the third author in [13], and then was employed by other authors. Rassias [13] also investigated stability properties of the following quartic functional equation
\[ f(x + 2y) + f(x - 2y) + 6f(x) = 4f(x + y) + 4f(x - y) + 24f(y). \]
(1.3)

It is easy to show that the function \( f(x) = bx^4 \) is a solution of (1.3). Every solution of the quartic functional equation is said to be a quartic mapping. For other forms of a quartic functional equation (see [11] and [12]). The second author [10] generalized (1.3) to the following equation
\[ f(mx + ny) + f(mx - ny) - 2m^2(m^2 - n^2)f(x) = (mn)^2[f(x + y) + f(x - y) - 2n^2(m^2 - n^2)f(y)], \]
for fixed integers \( m \) and \( n \) such that \( m \neq 0, n \neq 0, \) \( m+n \neq 0 \) (for the correction of some details in [10] see [2]). The Hyers-Ulam stability and the superstability for the functional equation (1.1) and quartic functional equations via a fixed point approach under certain conditions on Banach algebras are studied in [3] and [4].

In [6], Eshaghi et al. introduced the following mixed type cubic and quartic functional equation
\[ f(x + 2y) + f(x - 2y) = 9\{f(x + y) + f(x - y)\} \]
(1.4)
\[ -6f(x) - 24f(y) + 3f(2y). \]

In this paper we consider the following functional equation which is a
generalization of (1.4):

\[ f(x + ky) + f(x - ky) = k^2 \{ f(x + y) + f(x - y) \} - 2(k^2 - 1)f(x) - 2k^2(k^2 - 1)f(y) + \frac{1}{4}k^2(k^2 - 1)f(2y), \]  

(1.5)

where \( k \) is an integer with \( k \neq 0, \pm 1 \). Note that when \( k = 2 \), we have the equation (1.4). It is easily verified that the function \( f(x) = ax^3 + bx^4 \) is a solution of the functional equations (1.5).

The main purpose of the present paper is to solve and to prove the generalized Hyers-Ulam stability problem the functional equation (1.5).

2. Solution of equation (1.5)

We first solve the equation of (1.5) as follows:

**Theorem 2.1.** Let \( X \) and \( Y \) be real vector spaces. Then a function \( f : X \to Y \) satisfies the functional equation (1.4) if and only if it satisfies the functional equation (1.5).

**Proof.** Replacing \( x \) by \( x + y \) and \( x - y \) in (1.4), respectively, and adding the results we have \( f(x + 3y) + f(x - 3y) = 9\{ f(x + y) + f(x - y) \} - 16f(x) - 144f(y) + 18f(2y) \). Similar to the above, we get \( f(x + 4y) + f(x - 4y) = 16\{ f(x + y) + f(x - y) \} - 30f(x) - 480f(y) + 60f(2y) \). Using the above method, we can deduce that \( f(x + ky) + f(x - ky) = k^2 \{ f(x + y) + f(x - y) \} - a_k f(x) - b_k f(y) + c_k f(2y) \) in which

\[
\begin{align*}
a_k &= -a_{k-2} + 4(k - 1)^2, \quad a_2 = 6, a_3 = 16 \\
b_k &= -b_{k-2} + 2b_{k-1} + 24(k - 1)^2, \quad b_2 = 24, b_3 = 144 \\
c_k &= -c_{k-2} + 2c_{k-1} + 3(k - 1)^2, \quad c_2 = 3, c_3 = 18.
\end{align*}
\]

Solving the above recurrence equations, we obtain \( a_k = 2(k^2 - 1), b_k = 2k^2(k^2 - 1) \) and \( c_k = \frac{1}{3}k^2(k^2 - 1) \), for all \( x, y \in X \) and each positive integer \( k \geq 3 \). The result for the negative integers is clear.

Conversely, suppose that \( f : X \to Y \) satisfies the functional equation (1.5) for any positive integer \( k \geq 3 \). We wish to show its correctness for the case \( k - 1 \). The mapping \( f \) satisfies (1.5) for each \( m \geq k \), in particular for
\[ m = k(k - 1). \] Hence for each \( x, y \in \mathcal{X} \), we have

\[
\begin{align*}
f(x + k(k - 1)y) + f(x - k(k - 1)y) \\
= k^2\{f(x + (k - 1)y) + f(x - (k - 1)y)} - 2(k^2 - 1)f(x) \\
- 2k^2(k^2 - 1)f((k - 1)y) + \frac{1}{4}k^2(k^2 - 1)f(2(k - 1)y).
\end{align*}
\] (2.1)

On the other hand,

\[
\begin{align*}
f(x + (k^2 - k)y) + f(x - (k^2 - k)y) \\
= (k^2 - k)^2\{f(x + y) + f(x - y)} - 2((k^2 - k)^2 - 1)f(x) \\
- 2(k^2 - k)^2((k^2 - k)^2 - 1)f(y) + \frac{1}{4}(k^2 - k)^2((k^2 - k)^2 - 1)f(2y),
\end{align*}
\] (2.2)

for all \( x, y \in \mathcal{X} \). Since \( n = k(k - 1) \geq 3 \), we have

\[
\begin{align*}
f(x + (k + 1)(k - 1)y) + f(x - (k + 1)(k - 1)y) \\
= (k + 1)^2\{f(x + (k - 1)y) + f(x - (k - 1)y)} \\
- 2((k + 1)^2 - 1)f(x) - 2(k + 1)^2((k + 1)^2 - 1)f((k - 1)y) \\
+ \frac{1}{4}(k + 1)^2((k + 1)^2 - 1)f(2(k - 1)y),
\end{align*}
\] (2.3)

for all \( x, y \in \mathcal{X} \). Also

\[
\begin{align*}
f(x + (k^2 - 1)y) + f(x - (k^2 - 1)y) \\
= (k^2 - 1)^2\{f(x + y) + f(x - y)} - 2((k^2 - 1)^2 - 1)f(x) \\
- 2(k^2 - 1)^2((k^2 - 1)^2 - 1)f(y) + \frac{1}{4}(k^2 - 1)^2((k^2 - 1)^2 - 1)f(2y),
\end{align*}
\] (2.4)

for all \( x, y \in \mathcal{X} \). It follows from (2.1) and (2.2) that

\[
\begin{align*}
f(x + (k - 1)y) + f(x - (k - 1)y) \\
= (k - 1)^2\{f(x + y) + f(x - y)} - 2(k - 2)f(x) \\
- 2(k - 1)^2((k^2 - k)^2 - 1)f(y) + \frac{1}{4}(k - 1)^2((k^2 - k)^2 - 1)f(2y) \\
+ 2(k^2 - 1)f((k - 1)y) - \frac{1}{8}f(2(k - 1)y),
\end{align*}
\] (2.5)
for all \(x, y \in X\). Plugging (2.3) into (2.4), we get

\[
2k(k + 2)\{f((k - 1)y) - \frac{1}{8}f(2(k - 1)y)\} = f(x + (k - 1)y) + f(x - (k - 1)y)
\]

\[
- (k - 1)^2\{f(x + y) + f(x - y)\} + 2(k^2 + 2k)f(x)
\]

\[
+ 2k^3(k - 1)^2(k - 2)f(y) - \frac{1}{4}k^3(k - 1)^2(k - 2)f(2y),
\]

for all \(x, y \in X\). Multiplying both sides of (2.5) by \(k(k + 2)\) and using (2.6), we have

\[
f(x + (k - 1)y) + f(x - (k - 1)y) = (k - 1)^2\{f(x + y) + f(x - y)\} - 2((k - 1)^2 - 1)f(x) - 2(k - 1)^2((k - 1)^2 - 1)f(y)
\]

\[
+ \frac{1}{4}(k - 1)^2((k - 1)^2 - 1)f(2y).
\]

This completes the proof. \(\square\)

Lemma 2.2. Let \(X\) and \(Y\) be real vector spaces.

(i) If an odd function \(f : X \rightarrow Y\) satisfies the functional equation (1.5), then \(f\) is cubic.

(ii) If an even function \(f : X \rightarrow Y\) satisfies the functional equation (1.5), then \(f\) is quartic.

Proof. The result follows from Theorem 2.1 and [6, Lemma 2.1 and Lemma 2.2]. \(\square\)

Theorem 2.3. Let \(X\) and \(Y\) be real vector spaces. Then a function \(f : X \rightarrow Y\) satisfies the functional equation (1.5), for all \(x, y \in X\) if and only if there exists a unique function \(C : X \times X \times X \rightarrow Y\) and a unique symmetric multiadditive function \(Q : X \times X \times X \times X \rightarrow Y\) such that

\[
f(x) = C(x, x, x) + Q(x, x, x, x),
\]

for all \(x \in X\), and that \(C\) is symmetric for each fixed one variable and is additive for fixed two variables.

Proof. Using Theorem 2.1 and [6, Theorem 2.3], one can obtain the desired result. \(\square\)

3. Hyers-Ulam stability of (1.5) in real Banach spaces

In this section, we investigate the generalized Hyers-Ulam stability problem for the functional equation (1.5). Throughout this section, we assume that \(X\) is a normed real linear space with norm \(\| \cdot \|_X\) and \(Y\) is a real Banach space with norm \(\| \cdot \|_Y\).
Let \( k \) be an integer such that with \( k \neq 0, \pm 1 \). We use the abbreviation for the given mapping \( f : \mathcal{X} \rightarrow \mathcal{Y} \) as follows: \( D_{k}f(x, y) := f(x + ky) + f(x - ky) - k^2\{f(x + y) + f(x - y)\} + 2(k^2 - 1)f(x) + 2k^2(k^2 - 1)f(y) - \frac{1}{4}k^2(k^2 - 1)f(2y) \), for all \( x, y \in \mathcal{X} \).

**Theorem 3.1.** Let \( f : \mathcal{X} \rightarrow \mathcal{Y} \) be an even mapping with \( f(0) = 0 \) for which there exists a function \( \phi : \mathcal{X} \times \mathcal{X} \rightarrow [0, \infty) \) such that

\[
\sum_{k=0}^{\infty} \frac{1}{16^k} \phi_{k}(2^{k}x) < \infty, \quad \lim_{k \to \infty} \frac{1}{16^k} \phi(2^{k}x, 2^{k}y) = 0
\]

and

\[
\|D_{k}f(x, y)\|_{\mathcal{Y}} \leq \phi(x, y),
\]

for all \( x, y \in \mathcal{X} \), where \( k \) is an integer with \( k \neq 0, \pm 1 \). Then, there exists a unique quartic mapping \( Q : \mathcal{X} \rightarrow \mathcal{Y} \) such that

\[
\|f(2x) - 4f(x) - Q(x)\|_{\mathcal{Y}} \leq \frac{1}{16} \sum_{n=0}^{\infty} \Phi_{q}(2^{n}x),
\]

for all \( x \in \mathcal{X} \), where the mapping \( Q(x) \) and \( \Phi_{q}(2^{n}x) \) are defined by \( Q(x) = \lim_{n \to \infty} \frac{1}{16^n} \{f(2^{n+1}x) - 4f(2^n x)\} \) and

\[
\Phi_{q}(2^{n}x) = \frac{4}{k^2(k^2 - 1)} \left[2\phi(k2^n x, 2^n x) + 2k^2\phi(2^n x, 2^n x)
\right.

\[
+2(k^2 - 1)\phi(0, 2^n x) + \phi(0, 2^{n+1} x)\right],
\]

for all \( x \in \mathcal{X} \).

**Proof.** Replacing \((x, y)\) by \((0, x)\) in (3.2) and using the evenness of \( f \), we get

\[
\left\|2f(kx) + 2k^2(k^2 - 2)f(x) - \frac{1}{4}k^2(k^2 - 1)f(2x)\right\|_{\mathcal{Y}} \leq \phi(0, x),
\]

for all \( x \in \mathcal{X} \). Interchanging \((x, y)\) by \((kx, x)\) in (3.2), we deduce that

\[
\left\|f(2kx) - k^2[f((k + 1)x) + f((k - 1)x)] + 2(k^2 - 1)f(kx)
\right.

\[
+2k^2(k^2 - 1)f(x) - \frac{1}{4}k^2(k^2 - 1)f(2x)\right\|_{\mathcal{Y}} \leq \phi(kx, x),
\]
for all $x \in \mathcal{X}$. Putting $x = y$ in (3.2), we obtain
\[
\left\| \left[ f((k + 1)x) + f((k - 1)x) \right] - k^2 f(2x) + 2(k^2 - 1)f(x) \\
+ 2k^2(k^2 - 1)f(x) - \frac{1}{4}k^2(k^2 - 1)f(2x) \right\|_Y \leq \phi(x, x),
\]
for all $x \in \mathcal{X}$. Thus we have
\[
\left\| \left[ f((k + 1)x) + f((k - 1)x) \right] + 2(k^4 - 1)f(x) - \frac{1}{4}k^2(k^2 + 3)f(2x) \right\|_Y \leq \phi(x, x),
\]
for all $x \in \mathcal{X}$. The above inequality implies that
\[
(3.7) \quad \left\| k^2 \left[ f((k + 1)x) + f((k - 1)x) \right] + 2k^2(k^4 - 1)f(x) \\
- \frac{1}{4}k^4(k^2 + 3)f(2x) \right\|_Y \leq k^2\phi(x, x),
\]
for all $x \in \mathcal{X}$. It follows from (3.6), (3.7) and triangular inequality that
\[
(3.8) \quad \left\| f(2kx) + 2(k^2 - 1)f(kx) + 2k^2(k^2 - 1)(k^2 + 2)f(x) \\
- \frac{1}{4}k^2(k^4 + 4k^2 - 1)f(2x) \right\|_Y \leq \phi(kx, x) + k^2\phi(x, x),
\]
for all $x \in \mathcal{X}$. Multiplying both sides of (3.5) by $k^2 - 1$, we get
\[
(3.9) \quad \left\| 2(k^2 - 1)f(kx) + 2k^2(k^2 - 1)(k^2 - 2)f(x) \\
- \frac{1}{4}k^2(k^2 - 1)^2f(2x) \right\|_Y \leq (k^2 - 1)\phi(0, x),
\]
for all $x \in \mathcal{X}$. Plugging (3.8) into (3.9), we have
\[
(3.10) \quad \left\| f(2kx) + 8k^2(k^2 - 1)f(x) - \frac{1}{2}k^2(3k^2 - 1)f(2x) \right\|_Y \\
\leq \phi(kx, x) + k^2\phi(x, x) + (k^2 - 1)\phi(0, x),
\]
for all $x \in \mathcal{X}$. It also follows from (3.5) that
\[
(3.11) \quad \left\| 2f(2kx) + 2k^2(k^2 - 2)f(2x) - \frac{1}{4}k^2(k^2 - 1)f(4x) \right\|_Y \leq \phi(0, 2x),
\]
for all \( x \in \mathcal{X} \). Multiplying both sides of (3.10) by 2 and then adding the result to (3.11), we obtain
\[
\| 5k^2(k^2 - 1)f(2x) - 16k^2(k^2 - 1)f(x) - \frac{1}{4}k^2(k^2 - 1)f(4x) \|_Y
\leq 2\phi(kx, x) + 2k^2\phi(x, x) + 2(k^2 - 1)\phi(0, x) + \phi(0, 2x),
\]
for all \( x \in \mathcal{X} \). Thus
\[
\| 20f(2x) - 64f(x) - f(4x) \|_Y
\leq \frac{4}{k^2(k^2 - 1)} \left[ 2\phi(kx, x) + 2k^2\phi(x, x) + 2(k^2 - 1)\phi(0, x) + \phi(0, 2x) \right],
\]
for all \( x \in \mathcal{X} \). The above relation implies that
\[
(3.12) \quad \| g(2x) - 16g(x) \|_Y \leq \Phi_q(x),
\]
for all \( x \in \mathcal{X} \) in which \( g(x) = f(2x) - 4f(x) \) and
\[
\Phi_q(x) = \frac{4}{k^2(k^2 - 1)} \left[ 2\phi(kx, x) + 2k^2\phi(x, x) + 2(k^2 - 1)\phi(0, x) + \phi(0, 2x) \right],
\]
for all \( x \in \mathcal{X} \). The equality (3.12) shows that
\[
(3.13) \quad \left\| \frac{1}{16}g(2x) - g(x) \right\|_Y \leq \frac{1}{16}\Phi_q(x),
\]
for all \( x \in \mathcal{X} \). Now replacing \( x \) by \( 2x \) and dividing by 16 in (3.13), we obtain
\[
(3.14) \quad \left\| \frac{1}{16^2}g(4x) - \frac{1}{16}g(2x) \right\|_Y \leq \frac{1}{16^2}\Phi_q(2x),
\]
for all \( x \in \mathcal{X} \). From (3.13) and (3.14), we arrive at
\[
(3.15) \quad \left\| \frac{1}{16^2}g(4x) - g(x) \right\|_Y \leq \frac{1}{16} \left( \Phi_q(x) + \frac{1}{16}\Phi_q(2x) \right),
\]
for all \( x \in \mathcal{X} \). In general for any positive integer \( n \), we get
\[
(3.16) \quad \left\| \frac{1}{16^n}g(2^n x) - g(x) \right\|_Y \leq \frac{1}{16} \sum_{j=0}^{n-1} \frac{\Phi_q(2^j x)}{16^j},
\]
for all $x \in \mathcal{X}$. In order to prove the convergence of the sequence $\left\{ \frac{g(2^n x)}{16^n} \right\}$, replace $x$ by $2^n x$ and divide by $16^n$ in (3.16). For any $m, n > 0$, we have

$$(3.17) \quad \left\| \frac{g(2^{n+m} x)}{16^{n+m}} - \frac{g(2^m x)}{16^m} \right\|_Y \leq \frac{1}{16} \sum_{j=0}^{n-1} \frac{\Phi g(2^{j+m} x)}{16^{j+m}},$$

for all $x \in \mathcal{X}$. Since the right hand side of the inequality (3.17) tends to 0 as $m$ tends to infinity, the sequence $\left\{ \frac{g(2^n x)}{16^n} \right\}$ is Cauchy. The completeness of $Y$ allows us to assume that there exists a map $Q : \mathcal{X} \rightarrow Y$ such that

$$(3.18) \quad Q(x) = \lim_{n \to \infty} \frac{g(2^n x)}{16^n} \quad (x \in \mathcal{X}).$$

Letting $n \to \infty$ in (3.16), we see that (3.3) holds for all $x \in \mathcal{X}$. To prove that $Q$ satisfies (1.5), replace $(x, y)$ by $(2^n x, 2^n y)$ and divide by $16^n$ in (3.2).

Then, we obtain

$$\frac{1}{16} \cdot \frac{1}{16^n} \| D_k g(2^n x, 2^n y) \|_Y \leq \frac{1}{16^{n+1}} \| D_k f(2^{n+1} x, 2^{n+1} y) - 4D_k f(2^n x, 2^n y) \|_Y \leq \frac{1}{16^{n+1}} \| D_k f(2^{n+1} x, 2^{n+1} y) \|_Y + \frac{4}{16^n} \| D_k f(2^n x, 2^n y) \|_Y \leq \frac{\phi(2^{n+1} x, 2^{n+1} y)}{16^{n+1}} + 4 \cdot \frac{\phi(2^n x, 2^n y)}{16^n},$$

for all $x, y \in \mathcal{X}$. Letting $n \to \infty$ in the above inequality and using (3.1), we observe that $D_k Q(x, y) = 0$, for all $x, y \in \mathcal{X}$. Therefore, by the part (ii) of Lemma 2.2, $Q$ is a quartic mapping. Now, let $Q' : \mathcal{X} \rightarrow Y$ be another quartic mapping satisfying (3.3). Then we have

$$\| Q(x) - Q'(x) \|_Y = \frac{1}{16^n} \| Q(2^n x) - Q'(2^n x) \|_Y \leq \frac{1}{16^n} \left( \| Q(2^n x) - g(2^n x) \|_Y + \| g(2^n x) - Q'(2^n x) \|_Y \right) \leq \frac{1}{16^n} \frac{1}{8} \sum_{j=0}^{\infty} \frac{\Phi g(2^j x)}{16^j},$$

for all $x \in \mathcal{X}$. Taking $n \to \infty$ in the preceding inequality, we immediately find the uniqueness of $Q$. This finishes the proof. \(\square\)

The following corollary is an immediate consequence of Theorem 3.1 concerning the stability of (1.5).
Corollary 3.2. Let $\alpha$ and $p, q$ be nonnegative real numbers. Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is an even mapping fulfilling

$$
\|D_k f(x, y)\| \leq \begin{cases}
\alpha & 0 \leq p + q < 4 \\
\alpha \|x\|_\mathcal{X}^p \|y\|_\mathcal{X}^q, & 0 \leq p < 4 \\
\alpha (\|x\|_\mathcal{X}^p + \|y\|_\mathcal{X}^q), & 0 \leq p < 2
\end{cases},
$$

for all $x, y \in \mathcal{X}$. Then there exists a unique quartic function $Q : \mathcal{X} \to \mathcal{Y}$ such that

$$
\|f(2x) - 4f(x) - Q(x)\|_\mathcal{Y} \leq \begin{cases}
\lambda_1 \alpha, & 0 \leq p + q < 4 \\
\frac{\alpha \|x\|_\mathcal{X}^{p+q}}{16 - 2p+q} \lambda_2, & 0 \leq p < 4 \\
\frac{\alpha \|x\|_\mathcal{X}^p}{16 - 2p} \lambda_3, & 0 \leq p < 2
\end{cases},
$$

where

$$
\lambda_1 = \frac{4(4k^2 + 1)}{15k^2(k^2 - 1)}, \quad \lambda_2 = \frac{8(k^p + k^2)}{k^2(k^2 - 1)},
\lambda_3 = \frac{8(k^p + 3k^2 + 2^{p-1})}{k^2(k^2 - 1)}, \quad \lambda_4 = \frac{8(k^p + 4k^2 + 2^{p-2} + 2^{2p-1})}{k^2(k^2 - 1)}.
$$

In analogy with Theorem 3.1, we have the following theorem for the stability of (1.5) when $f$ is an odd mapping.

**Theorem 3.3.** Let $f : \mathcal{X} \to \mathcal{Y}$ be an odd mapping with $f(0) = 0$ for which there exists a function $\phi : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ such that

$$
\sum_{j=0}^{\infty} \frac{1}{8^j} \phi(0, 2^j x) < \infty, \quad \lim_{j \to \infty} \frac{1}{8^j} \phi(2^j x, 2^j y) = 0
$$

and

$$
\|D_k f(x, y)\|_\mathcal{Y} \leq \phi(x, y),
$$
for all \( x, y \in X \), where \( k \) is an integer with \( k \neq 0, \pm 1 \). Then, there exists a unique cubic mapping \( C : X \rightarrow Y \) such that

\[
\| f(x) - C(x) \|_Y \leq \sum_{n=0}^{\infty} \frac{\psi_c(2^n x)}{8^n},
\]

for all \( x \in X \), where the mapping \( C(x) \) and \( \Phi(2^n x) \) are defined by

\[
C(x) = \lim_{n \to \infty} \frac{1}{8^n} f(2^n x)
\]

(3.21)

and

\[
\psi_c(2^n x) = \frac{1}{2k^2(k^2 - 1)} \phi(0, 2^n x),
\]

for all \( x \in X \).

**Proof.** Replace \((x, y)\) by \((0, x)\) in (3.20). By the oddness of \( f \) we have

\[
\left\| 2k^2(k^2 - 1)f(x) - \frac{1}{4} k^2(k^2 - 1)f(2x) \right\|_Y \leq \phi(0, x),
\]

for all \( x \in X \). Hence

\[
\| 8f(x) - f(2x) \|_Y \leq \frac{4}{k^2(k^2 - 1)} \phi(0, x),
\]

(3.23)

for all \( x \in X \). In other words

\[
\left\| f(x) - \frac{1}{8} f(2x) \right\|_Y \leq \psi_c(x),
\]

(3.24)

for all \( x \in X \) in which \( \psi_c(x) = \frac{1}{2k^2(k^2 - 1)} \phi(0, x) \). Interchanging \( x \) by \( 2x \) and then dividing both sides by 8 in the above inequality, we deduce that

\[
\left\| \frac{1}{8} f(2x) - \frac{1}{8^2} f(2^2 x) \right\|_Y \leq \frac{1}{8} \psi_c(2x),
\]

(3.25)

for all \( x \in X \). The inequalities (3.24) and (3.25) imply that

\[
\left\| f(x) - \frac{1}{8^2} f(2^2 x) \right\|_Y \leq \left( \psi_c(x) + \frac{1}{8} \psi_c(2x) \right),
\]

(3.26)

for all \( x \in X \). This method can be repeated to obtain

\[
\left\| f(x) - \frac{1}{8^n} f(2^n x) \right\|_Y \leq \sum_{j=0}^{n-1} \frac{\psi_c(2^j x)}{8^j},
\]

(3.27)
for all \( x \in \mathcal{X} \). Putting \( x \) by \( 2^m x \) and then dividing both sides by \( 8^m \) in (3.27), we get

\[
(3.28) \quad \left\| \frac{f(2^m x)}{8^m} - \frac{f(2^n x)}{8^n} \right\|_Y \leq \sum_{j=0}^{n-1} \frac{\psi_c(2^{j+m} x)}{8^{j+m}},
\]

for all \( x \in \mathcal{X} \) and all positive integers \( m, n \). Thus, we conclude from (3.19) and (3.28) that the sequence \( \{ \frac{f(2^m x)}{8^m} \} \) is Cauchy. Since the space \( Y \) is complete, this sequence converges in \( Y \) to the mapping \( C \). Indeed,

\[
(3.29) \quad C(x) = \lim_{n \to \infty} \frac{f(2^n x)}{8^n}, \quad (x \in \mathcal{X}).
\]

It follows from (3.20) that \( \frac{1}{8^n} \| D_k f(2^n x, 2^n y) \|_Y \leq \frac{\phi(2^n x, 2^n y)}{8^n} \), for all \( x, y \in \mathcal{X} \). Letting \( n \to \infty \) in the above inequality and applying (3.19), (3.29), we get \( D_k C(x, y) = 0 \), for all \( x, y \in \mathcal{X} \). Hence, the part (i) of Lemma 2.2 shows that \( C \) is a cubic mapping. Also the relations (3.27) and (3.29) imply that (3.21) holds for all \( x \in \mathcal{X} \). For the uniqueness of \( C \), assume that \( C' : \mathcal{X} \to Y \) is another cubic mapping satisfying (3.21). Then, we have

\[
\|C(x) - C'(x)\|_Y = \frac{1}{8^n} \|C(2^n x) - C'(2^n x)\|_Y
\]

\[
\leq \frac{1}{8^n} \left( \|C(2^n x) - f(2^n x)\|_Y + \|f(2^n x) - C'(2^n x)\|_Y \right)
\]

\[
\leq \frac{2}{8^n} \sum_{j=0}^{\infty} \frac{\psi_c(2^n x)}{8^n},
\]

for all \( x \in \mathcal{X} \). Taking \( n \to \infty \) in the last inequality, we have \( C(x) = C'(x) \), for all \( x \in \mathcal{X} \).

**Corollary 3.4.** Let \( \alpha \) and \( p, q \) be nonnegative real numbers. Suppose that \( f : \mathcal{X} \to Y \) is an odd mapping fulfilling

\[
\|D_k f(x, y)\| \leq \begin{cases} 
\alpha & 0 \leq p < 3, \\
\alpha(\|x\|_\mathcal{X}^p + \|y\|_\mathcal{X}^p) & 0 \leq p \leq 3,
\end{cases}
\]

for all \( x, y \in \mathcal{X} \). Then there exists a unique cubic function \( C : \mathcal{X} \to Y \) such that

\[
\|f(x) - C(x)\|_Y \leq \begin{cases} 
\lambda_5 \alpha & 0 \leq p < 3, \\
\frac{\lambda_5 \alpha}{8 - 2^p} & 0 \leq p \leq 3,
\end{cases}
\]
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where $\lambda_5 = \frac{4}{k^2(k^2-1)}$, $\lambda_6 = \frac{4}{k^2(k^2-1)}$.

In particular, if $\|D_k f(x, y)\| \leq \alpha \|x\|_X^p \|y\|_X^q$ where $p + q \neq 3$, then the mapping $f$ is cubic.

**Theorem 3.5.** Let $\phi : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be a function such that

\begin{equation}
\sum_{n=0}^{\infty} \frac{1}{8^n} \phi(0, 2^n x) < \infty, \sum_{n=0}^{\infty} \frac{1}{16^n} \phi(0, 2^n x) < \infty
\end{equation}

and

\begin{equation}
\lim_{n \to \infty} \frac{1}{8^n} \phi(2^n x, 2^n y) = \lim_{n \to \infty} \frac{1}{16^n} \phi(2^n x, 2^n y) = 0.
\end{equation}

Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is an mapping with $f(0) = 0$ satisfies

\begin{equation}
\|D_k f(x, y)\|_Y \leq \phi(x, y),
\end{equation}

for all $x, y \in \mathcal{X}$, where $k$ is an integer with $k \neq 0, \pm 1$. Then there exists a unique cubic mapping $C : \mathcal{X} \to \mathcal{Y}$ and a unique quartic mapping $Q : \mathcal{X} \to \mathcal{Y}$ such that

\begin{equation}
\|f(2x) - 4f(x) - C(x) - Q(x)\|_Y \\
\leq \frac{1}{32} \sum_{n=0}^{\infty} \frac{\Phi_q(2^n x) + \Phi_q(-2^n x)}{16^n} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{\Phi_c(2^n x) + \Phi_c(-2^n x)}{8^n},
\end{equation}

for all $x \in \mathcal{X}$, where $\Phi_c(2^n x) = \frac{1}{2k^2(k^2-1)} [\phi(0, 2^{n+1} x) + 4\phi(0, 2^n x)]$ and $\Phi_q(2^n x)$ is given in (3.4).

**Proof.** We decompose $f$ into the even part and odd part by setting $f_e(x) = \frac{f(x) + f(-x)}{2}$, $f_o(x) = \frac{f(x) - f(-x)}{2}$, for all $x \in \mathcal{X}$. Obviously, $f(x) = f_e(x) + f_o(x)$, for all $x \in \mathcal{X}$. Then $\|D_k f_e(x, y)\|_Y = \frac{1}{2} \|D_k f(x, y) + D_k f(-x, -y)\|_Y \leq \frac{1}{2} (\|D_k f(x, y)\|_Y + \|D_k f(-x, -y)\|_Y) \leq \frac{1}{2} (\phi(x, y) + \phi(-x, -y))$ and

\begin{align*}
\|D_k f_o(x, y)\|_Y &= \frac{1}{2} \|D_k f(x, y) - D_k f(-x, -y)\|_Y \\
&\leq \frac{1}{2} (\|D_k f(x, y)\|_Y + \|D_k f(-x, -y)\|_Y) \\
&\leq \frac{1}{2} (\phi(x, y) + \phi(-x, -y)),
\end{align*}
for all \(x \in \mathcal{X}\). By Theorems 3.1 and 3.3, there exists a unique quartic function \(Q_0 : \mathcal{X} \rightarrow \mathcal{Y}\) and a unique cubic function \(C_0 : \mathcal{X} \rightarrow \mathcal{Y}\) such that

\[
\|f_e(2x) - 4f_e(x) - Q_0(x)\|_\mathcal{Y} \leq \frac{1}{32}\sum_{n=0}^\infty \frac{\Phi_q(2^n x) + \Phi_q(-2^n x)}{16^n}
\]

and

\[
\|f_o(x) - C_0(x)\|_\mathcal{Y} \leq \frac{1}{2}\sum_{n=0}^\infty \frac{\Phi_q(2^n x) + \Phi_q(-2^n x)}{8^n}
\]

for all \(x \in \mathcal{X}\). Put \(Q(x) = Q_0(x)\) and \(C(x) = 4C_0(x)\). Since \(C_0(x)\) is odd and satisfies the equation (1.5), it is easy to check that \(C_0(2x) = 8C_0(x)\). Thus we have

\[
\|f(2x) - 4f(x) - Q(x) - C(x)\|_\mathcal{Y} = \|f(2x) - 4f(x) - Q_0(x) - 4C_0(x)\|_\mathcal{Y}
\]

\[
\leq \|f_e(2x) - 4f_e(x) - Q_0(x)\|_\mathcal{Y} + \|f_o(2x) - 8C_0(x)\| + 4\|f_o(x) - C_0(x)\|_\mathcal{Y}
\]

\[
\leq \|f_e(2x) - 4f_e(x) - Q_0(x)\|_\mathcal{Y} + \|f_o(2x) - 4C_0(x)\|_\mathcal{Y} + 4\|f_o(x) - C_0(x)\|_\mathcal{Y}
\]

\[
\leq \frac{1}{32}\sum_{n=0}^\infty \frac{\Phi_q(2^n x) + \Phi_q(-2^n x)}{16^n} + \frac{1}{2}\sum_{n=0}^\infty \frac{\Phi_q(2^n x) + \Phi_q(-2^n x)}{8^n}
\]

in which \(\Phi_q(2^n x) = \frac{1}{2^{n+1}}[\phi(0, 2^n x) + \phi(0, 2^n x)]\) and \(\Phi_q(2^n x)\) is given in (3.4).

The following corollary is a direct consequence of Theorem 3.5 concerning the stability of (1.5).

**Corollary 3.6.** Let \(\alpha\) and \(p, q\) be nonnegative real numbers. Suppose that \(f : \mathcal{X} \rightarrow \mathcal{Y}\) is an even mapping fulfilling

\[
\|\mathcal{D}_k f(x, y)\| \leq \begin{cases} 
\alpha, & 0 \leq p + q < 3 \\
\alpha\|x\|^p\|y\|^q_{\mathcal{X}}, & 0 \leq p < 3 \\
\alpha(\|x\|^p + \|y\|^p_{\mathcal{X}}), & 0 \leq p < 3 \\
\alpha(\|x\|^p\|y\|^q_{\mathcal{X}} + \|x\|^2p + \|y\|^2p_{\mathcal{X}}), & 0 \leq p < \frac{3}{2}
\end{cases}
\]
for all $x, y \in X$. Then there exists a unique cubic mapping $C : X \to Y$ and a unique quartic mapping $Q : X \to Y$ such that

\[
\|f(2x) - 4f(x) - C(x) - Q(x)\|_Y \\
\leq \begin{cases} 
(\lambda_1 + \lambda_5)\alpha, \\
\alpha\|x\|^{p+q}_{X^p} \lambda_2, \\
\frac{1}{16 - 2^{p+q}} \lambda_3 + \frac{4 + 2^p}{8 - 2^p \lambda_6} \alpha\|x\|^{p}_{X}, \\
\frac{1}{16 - 2^{p}} \lambda_4 + \frac{4 + 2^p}{8 - 2^p \lambda_6} \alpha\|x\|^{2p}_{X},
\end{cases} \\
0 \leq p + q < 3,
\]

where $\lambda_j (j = 1, 2, 3, 4, 5, 6)$ are given in Corollaries 3.2 and 3.4.

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Department of Mathematics,
Garmsar Branch,
Islamic Azad University,
Garmsar,
IRAN
abasalt.bodaghi@gmail.com

Department of Mathematical Education,
Dankook University,
Gyeonggi, South Korea 448-701,
KOREA
dskang@dankook.ac.kr

Section of Mathematics and Informatics,
Pedagogical Department,
National and Capodistrian University of Athens,
Aghia Paraskevi, 15342, Athens,
GREECE
jrassias@primedu.uoa.gr