On the roots of $\sigma$-polynomials

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Abstract

Given a graph $G$ of order $n$, the $\sigma$-polynomial of $G$ is the generating function $\sigma(G, x) = \sum a_i x^i$ where $a_i$ is the number of partitions of the vertex set of $G$ into $i$ nonempty independent sets. Such polynomials arise in a natural way from chromatic polynomials. Brenti [1] proved that $\sigma$-polynomials of graphs with chromatic number at least $n - 2$ had all real roots, and conjectured the same held for chromatic number $n - 3$. We affirm this conjecture.

Keywords: $\sigma$-polynomial, real roots, chromatic number, chromatic polynomial, compatible polynomials

1 Introduction

Let $G$ be a graph of order $n$ with chromatic number $\chi(G)$. The $\sigma$-polynomial of $G$ (see [1]) is defined as the polynomial

$$\sigma(G, x) = \sum_{i=\chi(G)}^{n} a_i x^i$$

where $a_i$ denotes the number of partitions of the vertex set of $G$ into $i$ nonempty independent sets. The coefficients $a_i$ are also known as the graphical Stirling numbers [9, 10]. If a graph has no edges then $a_i$ is simply equal to the Stirling number of the second kind $S(n, i)$.

These polynomials first arose in the study of chromatic polynomials, since the chromatic polynomial of $G$ is $\sum a_i (x)_{\downarrow i}$, where $(x)_{\downarrow i} = x(x-1)\cdots(x-i+1)$ is the falling factorial of $x$ (the sequence $(a_i)$ has been called the chromatic vector of $G$ [11]). The $\sigma$-polynomial was first introduced by Korfhage [13] in a slightly different form (he refers to the polynomial

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\[(\sum_{i=\chi(G)}^n a_i x^i)/x^{\chi(G)} \text{ as the } \sigma\text{-polynomial}, \text{ and } \sigma\text{-polynomials have attracted considerable attention in the literature. Brenti [1] studied the } \sigma\text{-polynomials extensively and investigated both log-concavity and the nature of the roots. Chvátal [7] gave a necessary condition for a subsequence of the chromatic vector to be nondecreasing. Brenti, Royle and Wagner [2] proved that a variety of conditions are sufficient for a } \sigma\text{-polynomial to have only real roots. The } \sigma\text{-polynomial and its coefficients have connections to other graph polynomials and combinatorial structures as well. The partition polynomial studied by Wagner [18] reduces to a } \sigma\text{-polynomial and the } \sigma\text{-polynomial of the complement of a triangle free graph is just the well known matching polynomial [12]. The authors in [11] investigate the rook and chromatic polynomials, and prove that every rook vector is a chromatic vector. In [9] the authors explore relations among the } \sigma\text{-polynomial, chromatic polynomial, and the Tutte polynomial, and implications of these connections. A result on the ordinary Stirling numbers was generalized in [10] by considering the } \sigma\text{-polynomials of some graph families. Moreover, studying } \sigma\text{-polynomials is useful to find chromatically equivalent or chromatically unique graph families [15, 20]. Recently, in [3] the authors obtained upper bounds for the real parts of the roots of chromatic polynomials for graphs with large chromatic number by investigating the } \sigma\text{-polynomials of such graphs.}

It is known that } \sigma\text{-polynomials of several graph families such as chordal graphs and incomparability graphs have only real roots [18]. However, } \sigma\text{-polynomials do not always have only real roots. In [2], the authors exhibit all graphs of orders 8 and 9 whose } \sigma\text{-polynomials have nonreal roots (on the other hand, the } \sigma\text{-polynomial of every graph of order at most 7 has all real roots). Brenti [1] proved that } \sigma\text{-polynomials of all graphs of order } n \text{ with chromatic number at least } n-3 \text{ have all real roots, and proposed the following:}

**Conjecture 1.1.** [1] If } G \text{ is a graph of order } n \text{ and } \chi(G) \geq n-3, \text{ then } \sigma(G, x) \text{ has only real roots.}

In this paper we will prove Brenti’s conjecture.

2 Background on } \sigma\text{-polynomials

In this section we summarize a number of known results on } \sigma\text{-polynomials that we will make use of in the sequel. For graph theory terminology, we follow [19] in general.

Let } G \text{ and } H \text{ be two graphs. We denote the union of } G \text{ and } H \text{ by } G \cup H \text{ and the disjoint union of } G \text{ and } H \text{ by } G \bar{\cup} H \text{ (for positive integer } l, lG \text{ denotes the disjoint union of } l \text{ copies of } G\). The join of } G \text{ and } H, \text{ denoted by } G \vee H, \text{ is the graph whose vertex set is } V(G) \cup V(H) \text{ and edge set is } E(G) \cup E(H) \cup \{uv| u \in V(G) \text{ and } v \in V(H)\}. \text{ In the following theorem we present some useful properties of } \sigma\text{-polynomials under these graph operations.}

**Theorem 2.1.** [1, 2] Let } G \text{ and } H \text{ be two graphs. Then,}

(i) } \sigma(G \vee H, x) = \sigma(G, x)\sigma(H, x),

(ii) If } \sigma(G, x) \text{ and } \sigma(H, x) \text{ have only real roots, then } \sigma(G \bar{\cup} H, x) \text{ has also only real roots,
(iii) If $\sigma(G, x)$ and $\sigma(H, x)$ have only real roots and $G \cap H$ is a complete graph, then $\sigma(G \cup H, x)$ has only real roots.

For two graphs $H$ and $G$, we denote by $\eta_G(H)$ the number of subgraphs of $G$ which are isomorphic to $H$. For example, if $G = K_4$ then we have $\eta_G(K_2) = 6$, $\eta_G(2K_2) = 3$, $\eta_G(K_3) = 4$ and $\eta_G(K_3 \cup K_2) = 0$. Let $G$ be a graph whose $\sigma$-polynomial is

$$\sigma(G, x) = \sum_{i=\chi(G)}^{n} a_i x^i.$$ 

For every partition $\sum_{j=1}^{k} m_j = i$ of a positive integer $i$, we associate a disjoint union of complete graphs $\bigcup_{j=1}^{k} K_{m_j} + 1$, an $i^{th}$ generation forbidden subgraph [14] (they are “forbidden” as the complement of any graph with chromatic number $n-k$ cannot contain any $(n-k-1)^{th}$ generation forbidden graph as a subgraph). For a partition of the $n$ vertices of $G$ into $n-i$ nonempty colour classes ($i \geq 1$), by ignoring the singleton classes, we see that $a_{n-i}$ counts the number of subgraphs of the form $\bigcup_{j=1}^{k} K_{m_j} + 1$ in $\overline{G}$ where $\sum_{j=1}^{k} m_j = i$ and $m_j \in \mathbb{Z}^+$. This fact was also observed by several other authors (see, for example, [14, 16]) and we will use it frequently in the next section. From this observation, we find that

$$a_n = 1,$$
$$a_{n-1} = \eta_{\overline{G}}(K_2) = \binom{n}{2} - |E(G)|,$$
$$a_{n-2} = \eta_{\overline{G}}(K_3) + \eta_{\overline{G}}(2K_2),$$
$$a_{n-3} = \eta_{\overline{G}}(K_4) + \eta_{\overline{G}}(K_3 \cup K_2) + \eta_{\overline{G}}(3K_2),$$
$$a_{n-4} = \eta_{\overline{G}}(K_5) + \eta_{\overline{G}}(K_4 \cup K_2) + \eta_{\overline{G}}(2K_3) + \eta_{\overline{G}}(K_3 \cup 2K_2) + \eta_{\overline{G}}(4K_2).$$

The matching polynomial $m(G, x)$ of a graph $G$ is defined as

$$m(G, x) = \sum_{i \geq 0} \eta_G(iK_2) x^i,$$

where $\eta_G(0K_2) \equiv 1$ by convention, and this polynomial is well known to have only real roots [12]. An important consequence is that if $G$ is a triangle-free graph then $\sigma(\overline{G}, x) = x^n m(G, 1/x)$ and hence $\sigma(\overline{G}, x)$ has only real roots.

| Partition of 4 | Associated 4th generation forbidden subgraph |
|---------------|---------------------------------------------|
| 4             | $K_5$                                       |
| 3 + 1         | $K_4 \cup K_2$                             |
| 2 + 2         | $2K_3$                                     |
| 2 + 1 + 1     | $K_3 \cup 2K_2$                           |
| 1 + 1 + 1 + 1 | $4K_2$                                     |

Table 1: Fourth generation forbidden subgraphs
If $\mathcal{F}$ is a finite set system (that is a collection of finite sets, called blocks) then its partition polynomial $\rho(\mathcal{F}, x)$ is defined as

$$\rho(\mathcal{F}, x) = \sum_{i \geq 1} a_i(\mathcal{F}) x^i$$

where $a_i(\mathcal{F})$ is the number of ways to partition the vertex set of $\mathcal{F}$ (that is $\cup_{A \in \mathcal{F}} A$) into $i$ nonempty blocks [18]. The independence complex of a graph $G$ is the simplicial complex (that is a collection of sets, called faces, closed under containment – see [5], for example) on the vertex set of $G$ whose faces correspond to independent sets of the graph. Thus the partition polynomial of the independence complex of a graph is equal to the $\sigma$-polynomial of the graph. A graph is called chordal if it does not contain a cycle of order 4 or more as an induced subgraph. The comparability graph of a partially ordered set $(V, \preceq)$ has vertex set $V$ and has an edge $uv$ whenever $u \preceq v$ or $v \preceq u$; a graph is called a comparability graph if it is the comparability graph of some partial order. In [18] is was shown that the partition polynomial of the independence complex of a chordal graph or the complement of a comparability graph has only real roots. Hence, the same is true for the $\sigma$-polynomials of such graphs.

Computer aided computations show that $\sigma$-polynomials of all graphs of order at most 7 have only real roots [2]. Also, Brenti [1] showed that all graphs $G$ with $\chi(G) \geq n - 2$ has only real roots. In the following theorem we summarize all these results.

**Theorem 2.2.** [1, 2, 12, 18] If graph $G$ has any of the following properties then $\sigma(G, x)$ has only real roots:

(i) $G$ has order at most 7.

(ii) $G$ has order $n$ with $\chi(G) \geq n - 2$.

(iii) $G$ is chordal.

(iv) $\overline{G}$ is triangle-free.

(v) $\overline{G}$ is a comparability graph.

For an edge $e = uv$ of a graph $G$, the graph $G - e$ denotes the subgraph of $G$ obtained by deleting the edge $e$, and $G - \{u, v\}$ denotes the subgraph induced by the vertex set $V(G) - \{u, v\}$. The following result gives a recursive formula to calculate the $\sigma$-polynomial of the complement of a graph, but it can be applied only to edges in the original graph which satisfy a particular condition.

**Lemma 2.3.** [15] Let $G$ be a graph and $e = uv$ be an edge of $G$ such that $e$ is not contained in any triangle of $G$. Then,

$$\sigma(\overline{G}, x) = \sigma(\overline{G - e}, x) + x\sigma(\overline{G - \{u, v\}}, x)$$
3 Main Results

In this section, we will prove that if $G$ is a graph of order $n$ with $\chi(G) = n - 3$, then $\sigma(G,x)$ has only real roots. We will use a characterization of the complements of such graphs, obtained in \[14\]; specifically, $\chi(G) = n - 3$ if and only if $G \cong H \vee K_{n-r}$ where $|V(H)| = r \leq n$, and either $H$ is a proper 3-star graph (whose definition will follow shortly), or $H$ is one of the graphs of the families described in Figures 1, 4 and 5.

However, first we need a theorem that determines whether a real polynomial (that is, a polynomial with real coefficients) has all real roots. The Sturm sequence of a real polynomial $f(t)$ of positive degree is a sequence of polynomials $f_0, f_1, f_2, \ldots$, where $f_0 = f$, $f_1 = f'$, and, for $i \geq 2$, $f_i = -\text{rem}(f_{i-1}, f_{i-2})$, where $\text{rem}(h, g)$ is the remainder upon dividing $h$ by $g$. The sequence is terminated at the last nonzero $f_i$. The Sturm sequence of $f$ has gaps in degree if there exist integers $j \leq k$ such that $\deg f_j < \deg f_{j-1} - 1$. Sturm’s well known theorem (see, for example, \[4\]) is the following:

**Theorem 3.1** (Sturm’s Theorem). Let $f(t)$ be a real polynomial whose degree and leading coefficient are positive. Then $f(t)$ has all real roots if and only if its Sturm sequence has no gaps in degree and no negative leading coefficients.

For the next result, we consider the family of graphs $F$ depicted in Figure 1. In each of the eight subfamilies, the vertex $v$ is joined to each vertex in an independent set of size $m$.

**Lemma 3.2.** Let $G$ be a graph whose complement $\overline{G}$ is in the $F$ family (see Figure 1). Then $\sigma(G, x)$ has only real roots.
Proof. It is clear that if $\overline{G}$ is equal to $F(0,m)$, $F(1,m)$, $F(2,m)$ or $F(3,m)$ then it is triangle-free, hence $\sigma(G,x)$ has only real roots by Theorem 2.2 [X].

If $\overline{G} = F(2,m)$ then we find that $\sigma(G,x)/x^{m+3} = x^3 + (m+7)x^2 + (5m+12)x + (5m+4)$. Calculations show that the leading coefficients of this polynomial’s Sturm sequence are

$$1, \frac{2}{9}(m^2 - m + 13) \quad \text{and} \quad \frac{9(5m^4 - 16m^3 + 88m^2 - 92m + 272)}{4(m^2 - m + 13)^2},$$

all of which are strictly positive. Hence, we get the result by Theorem 3.1. Also, if $\overline{G} = F(3,m)$ then we find that $\sigma(G,x)/x^{m+3} = x^3 + (m+8)x^2 + (5m+16)x + (5m+7)$. The leading coefficients of this polynomial’s Sturm sequence turn out to be

$$1, \frac{2}{9}(m^2 + m + 16) \quad \text{and} \quad \frac{9(5m^4 + 2m^3 + 99m^2 + 46m + 469)}{4(m^2 + m + 16)^2},$$

all of which are obviously strictly positive for $m \geq 0$, and we conclude as above.

Now let $\overline{G} = F(4,m)$ and $v$ be the vertex of $\overline{G}$ which is adjacent to $m$ leaves in $\overline{G}$ and $u$ be the vertex which is not adjacent to $v$ in $\overline{G}$. Let $H$ be the edge induced by $u$ and $v$ in $G$. Now, $G = (C_5 \lor K_m) \cup H$, and the intersection of $C_5 \lor K_m$ and $H$ is equal to $\{u\}$ in $G$. Note that $\sigma(C_5,x)$ has only real roots by Theorem 2.2 [X]. Also, $\sigma(C_5 \lor K_m,x) = x^m\sigma(C_5,x)$ holds by Theorem 2.1 [I], so the polynomial $\sigma(C_5 \lor K_m,x)$ has only real roots. Hence, the result follows from Theorem 2.1 [III].

Lastly, suppose that $\overline{G} = F(5,m)$, then $G = (C_5 \lor K_m) \cup K_1$. Now, we obtain the result from Theorem 2.1 [III], since both $\sigma(C_5 \lor K_m,x)$ and $\sigma(K_1,x) = x$ have only real roots.

The proof of the realness of the roots of the $\sigma$-polynomials of the other classes of graphs will require a more subtle argument than just Sturm sequences, and we rely on an approach taken by Chudnovsky and Seymour [X] in their proof for the realness of the roots of independence polynomials of claw-free graphs. Following [X], we say that polynomials $f_1, \ldots, f_k$ in $\mathbb{R}[x]$ are compatible if for all $c_1, \ldots, c_k \geq 0$, all the roots of the linear combination $\sum_{i=1}^{k} c_i f_i(x)$ are real, and the polynomials are called pairwise compatible if for all $i,j$ in $\{1, \ldots, k\}$, the polynomials $f_i(x)$ and $f_j(x)$ are compatible. The following observation will be utilized later.

**Remark 3.3.** Suppose that $f(x), g(x) \in \mathbb{R}[x]$ are two polynomials with positive leading coefficients and all roots real. Then, $f$ and $g$ are compatible if and only if for all $c > 0$, the polynomial $cf(x) + g(x)$ has all real roots.

We need a few more definitions. Let $a_1 \geq \cdots \geq a_m$ and $b_1 \geq \cdots \geq b_n$ be two sequences of real numbers. We say that the first interleave the second if $n \leq m \leq n + 1$ and $a_1 \geq b_1 \geq a_2 \geq b_2 \geq \cdots$. If $f$ is a polynomial of degree $d$ with only real roots, let $r_1 \geq \cdots \geq r_d$ be the roots of $f$. Then the sequence $(r_1, \ldots, r_d)$ is called the root sequence of $f$. Let $f_1, \ldots, f_k$ be polynomials with positive leading coefficients and all roots real. A
common interleaver for \(f_1, \ldots, f_k\) is a sequence that interleaves the root sequence of each \(f_i\).

The key analytic result we need from [6] is the following:

**Theorem 3.4.** [6] Let \(f_1, \ldots, f_k\) be polynomials with positive leading coefficients and all roots real. Then the following statements are equivalent:

(i) \(f_1, \ldots, f_k\) are pairwise compatible,
(ii) for all \(s, t\) such that \(1 \leq s < t \leq k\), the polynomials \(f_s\) and \(f_t\) have a common interleaver,
(iii) \(f_1, \ldots, f_k\) have a common interleaver,
(iv) \(f_1, \ldots, f_k\) are compatible.

We now return to proving the realness of the roots of the \(\sigma\)-polynomials for the remaining classes of graphs with \(\chi(G) = n - 3\). We say that a subset of vertices \(S\) of a graph \(G\) is a vertex cover of \(G\) if every edge of \(G\) contains at least one vertex of \(S\). The vertex cover number, \(\alpha_0(G)\), is the cardinality of a minimum vertex cover. Note that \(S\) is a vertex cover of \(G\) if and only if \(V(G) - S\) induces an independent set, and that if \(\alpha_0(G) = k\) then \(G\) contains a complete subgraph of order \(n - k\), and hence \(\chi(G) \geq n - k\). A graph \(G\) is called a proper \(k\)-star [14] if \(\alpha_0(G) = k\) and \(G\) contains at least one \(k\)th generation forbidden subgraph. In the following proof, \(n_G\) and \(n_H\) denotes the number of vertices of the graph \(G\) and subgraph \(H\), respectively.

**Theorem 3.5.** Let \(G\) be a graph such that \(\alpha_0(G) \leq 3\). Then \(\sigma(G, x)\) has only real roots.

**Proof.** We may assume that \(\alpha_0(G) = 3\) and \(\chi(G) = n_G - 3\), since otherwise \(\chi(G) \geq n_G - 2\) and the result holds by Theorem 2.2 (ii). Also, we may assume that \(G\) has no isolated vertices by Theorem 2.1 (i). Let \(S = \{u_1, u_2, u_3\}\) be a vertex cover of \(G\), so that \(G - S\) is an independent set. We set \(V = V(G) = V(G)\). There are four cases we need to consider: \(S\) induces either (i) an independent set, (ii) \(K_3\), (iii) \(P_3\), or (iv) \(K_2 \cup K_1\) in \(G\).

For case (i), if \(S\) induces an independent set in \(G\), then \(G\) is a triangle-free graph and we are done by Theorem 2.2 (iv).

In case (ii), the subgraph of \(G\) induced by \(S\) is isomorphic to \(K_3\). Here \(G\) can be partitioned into a clique and independent set, so one can check that \(G\) is, in fact, chordal and hence the result follows from Theorem 2.2 (iii).

Now, suppose that case (iii) holds, namely that \(S\) induces in \(G\) a \(P_3\). Without loss of generality, we may assume that \(u_2\) is adjacent to both \(u_1\) and \(u_3\) in \(G\). Let \(H_1\) (respectively \(H_2\)) be the subgraph induced by \(V - \{u_1, u_3\}\) (respectively \(V - \{u_2\}\)) in \(G\). Clearly, \(H_1 \cap H_2\) is a complete graph in \(G\) and \(H_1 \cup H_2 = G\). Also, \(\sigma(H_1, x)\) and \(\sigma(H_2, x)\) have only real roots by Theorem 2.2 (iii) because \(\chi(H_1) \geq n_{H_1} - 2\) and \(\chi(H_2) \geq n_{H_2} - 2\). Therefore, we obtain the result by Theorem 2.1 (iii).
Lastly, suppose that the subgraph induced by $S$ in $\overline{G}$ is isomorphic to $K_2 \cup K_1$ – this is the final case (iv). Without loss, let $u_1$ and $u_2$ be adjacent to each other in $\overline{G}$; we will partition the remaining vertices into sets by their neighbourhood in $S$ (see Figure 2). Let $P$ be the set of vertices which are adjacent to $u_1$ in $\overline{G}$, and $m_i = |M_i|$. Also, let $R$ be the set of all common neighbours of $u_1$ and $u_2$ in $\overline{G}$. Similarly, let $J$ (respectively, $K$) be the set of all common neighbours of $u_2$ and $u_3$ ($u_1$ and $u_3$, respectively) in $\overline{G}$. Let $r = |R|$, $j = |J|$ and $k = |K|$. If $j = 0$ or $k = 0$, then $\overline{G}$ is a comparability graph (see Figure 3 for $k = 0$) and we obtain the result from Theorem 2.2(iii). Hence, we may assume that $j, k \geq 1$. Now, let $H$ be the subgraph of $\overline{G}$ induced by $V - (M_3 \cup \{u_3\})$. Let also $H_j$ (respectively $H_K$) be a subgraph of $\overline{G}$ induced by $V - (M_3 \cup \{u_3, v_j\})$ (respectively $V - (M_3 \cup \{u_3, v_K\})$) where $v_j$ (respectively $v_K$) is a vertex of $J$ (respectively $K$). None of the edges incident to $u_3$ are contained in a triangle in $\overline{G}$. We now apply the recursive formula in Lemma 2.3 to all edges incident to $u_3$ successively. We set $G_i$ be an induced subgraph of $\overline{G}$ which is obtained from $\overline{G}$ by deleting $i$ vertices of $M_3$. Beginning with the edges between $M_3$ and $u_3$, we find from Lemma 2.3 (and the fact from Theorem 2.1(i)) that any isolated vertex in the complement of a graph adds a factor of $x$ to the $\sigma$-polynomial) that

$$
\sigma(G, x) = \sigma(G_0, x) = x\sigma(G_1, x) + x \cdot x^{m_3 - 1}\sigma(\overline{H}, x) = x\sigma(G_1, x) + x^{m_3}\sigma(\overline{H}, x) = x^2\sigma(G_2, x) + 2x^{m_3}\sigma(\overline{H}, x) = \ldots = x^{m_3}\sigma(G_{m_3}, x) + m_3x^{m_3}\sigma(\overline{H}, x).
$$

We then continue to successively remove the other edges incident to $u_3$ in $G_{m_3}$, and using a similar argument, we find that $\sigma(G_{m_3}, x) = jx\sigma(\overline{H}_j, x) + kx\sigma(\overline{H}_K, x) + x\sigma(\overline{H}, x)$, so that

$$
\sigma(G, x) = x^{m_3} \left( x\sigma(\overline{H}, x) + jx\sigma(\overline{H}_j, x) + kx\sigma(\overline{H}_K, x) + m_3\sigma(\overline{H}, x) \right)
$$

The chromatic number of each of the graphs $\overline{H}$, $\overline{H}_j$ and $\overline{H}_K$ is at least the order of the graph minus 2, as none of these graphs contain a third generation forbidden subgraph. Hence, their $\sigma$-polynomials have only real roots by Theorem 2.2(iii).

Now, by Theorem 3.4, it suffices to show that the polynomials

$$
\sigma(\overline{H}, x), \ x\sigma(\overline{H}, x), \ x\sigma(\overline{H}_j, x), \text{ and } x\sigma(\overline{H}_K, x)
$$

are pairwise compatible. Let $\alpha = m_2 + j + r$ and $\beta = m_1 + k + r$. Now the number of $K_2$’s, $K_3$’s and $2K_2$’s in $\overline{H}$ are, respectively, $\alpha + \beta + 1$, $r$ and $\alpha \beta - r$, and hence

$$
\sigma(\overline{H}, x) = x^{n_H - 2} \left( x^2 + (\alpha + \beta + 1)x + \alpha \beta \right).
$$
Figure 2: The graph $G$ with vertex cover $\{u_1, u_2, u_3\}$

Figure 3: A comparability graph of a subclass of graphs from Figure 2
Moreover, as \( \overline{H}_j \) and \( \overline{H}_k \) are graphs of the same form as \( H \) with \( j \) replaced by \( j-1 \) and \( k \) replaced by \( k-1 \) respectively (and hence \( \alpha \) and \( \beta \) decreased by 1, respectively), we see that
\[
xx(\overline{H}_j, x) = x^{n-2} \left( x^2 + (\alpha + \beta)x + (\alpha - 1)\beta \right), \quad \text{and} \quad
xx(\overline{H}_k, x) = x^{n-2} \left( x^2 + (\alpha + \beta)x + \alpha(\beta - 1) \right).
\]

Let \( 0 = r_1 \geq r_2 \geq r_3 \) be the roots of \( x^3 + (\alpha + \beta + 1)x^2 + \alpha \beta x \) and \( t_1 \geq t_2 \) be the roots of \( x^2 + (\alpha + \beta)x + (\alpha - 1)\beta \), so
\[
r_2 = -\frac{(\alpha + \beta + 1) + \sqrt{(\alpha + \beta + 1)^2 - 4\alpha \beta}}{2},
\]
\[
r_3 = -\frac{(\alpha + \beta + 1) - \sqrt{(\alpha + \beta + 1)^2 - 4\alpha \beta}}{2},
\]
\[
t_1 = -\frac{(\alpha + \beta) + \sqrt{(\alpha + \beta)^2 - 4(\alpha - 1)\beta}}{2}, \quad \text{and}
\]
\[
t_2 = -\frac{(\alpha + \beta) - \sqrt{(\alpha + \beta)^2 - 4(\alpha - 1)\beta}}{2}.
\]

It is not difficult to verify that \( 0 = r_1 > t_1 > r_2 > t_2 > r_3 \), which shows that \( \sigma(\overline{H}, x) \), \( xx(\overline{H}, x) \), and \( xx(\overline{H}_j, x) \) have a common interleaver. Since \( j \) and \( k \) play symmetric roles, it is also clear that the same argument works to prove that \( \sigma(\overline{H}, x) \), \( xx(\overline{H}, x) \), and \( xx(\overline{H}_K, x) \) also have a common interleaver.

Finally, we need to show that \( \sigma(\overline{H}_j, x) \) and \( \sigma(\overline{H}_K, x) \) are compatible. So, we shall prove that \( x^2 + (\alpha + \beta)x + (\alpha - 1)\beta \) and \( x^2 + (\alpha + \beta)x + \alpha(\beta - 1) \) are compatible. We use Remark 3.3 and show that \( c(x^2 + (\alpha + \beta)x + (\alpha - 1)\beta) + x^2 + (\alpha + \beta)x + \alpha(\beta - 1) \) has all real roots for all \( c > 0 \).

Let \( c > 0 \). Then \( (c + 1)(\alpha - \beta)^2 > -4(c \beta + \alpha) \) which is equivalent to \( (c + 1)(\alpha + \beta)^2 > 4(c + 1)\alpha \beta - 4c \beta - 4\alpha \) or \( (c + 1)^2(\alpha + \beta)^2 > 4(c + 1)(c(\alpha - 1)\beta + \alpha(\beta - 1)) \). This implies that the discriminant of the quadratic \( (c + 1)x^2 + (c + 1)(\alpha + \beta)x + c(\alpha - 1)\beta + \alpha(\beta - 1) \) is nonnegative, and hence \( x^2 + (\alpha + \beta)x + (\alpha - 1)\beta \) and \( x^2 + (\alpha + \beta)x + \alpha(\beta - 1) \) are compatible. This completes the proof.

We are ready to tie everything all together in a proof of Brenti’s conjecture.

**Theorem 3.6.** Let \( G \) be a graph on \( n \) vertices. If \( \chi(G) = n - 3 \), then \( \sigma(G, x) \) has only real roots.

**Proof.** In [14], it was shown that for a graph \( G \) with \( n \) vertices, \( \chi(G) = n - 3 \) if and only if \( G \) is isomorphic to \( H \vee K_{n-r} \) where \( |V(H)| = r \leq n \) and \( \overline{H} \) is a proper 3-star graph or \( \overline{H} \) is one of the graphs of the \( F \), \( S \) and \( L \) families. So, by Theorem 2.7[9], it suffices to show that \( \sigma(H, x) \) has only real roots. As we already noted earlier, the \( \sigma \)-polynomials of all graphs of order at most 7 have all real roots. Hence, the result is clear if \( \overline{H} \) is a graph in one of the \( S \) or \( L \) families (see Figures 4 and 3). Also, if \( \overline{H} \) is in the \( F \) family, we get the desired result by Lemma 3.2. Finally, if \( \overline{H} \) is a proper 3-star, then the result is established by Theorem 3.3. \(\)
Figure 4: The $S$ family

Figure 5: $L$ family
4 Concluding remarks

As the $\sigma$-polynomials of graphs of order $n$ with chromatic number at least $n - 3$ have all real roots, the question remains how far down can the chromatic number go before nonreal roots arise? For chromatic number $n - 5$ there are indeed such graphs. Figure 6 shows the two smallest examples (known as Royle graphs [17, pg. 265]), of order 8 (as mentioned earlier, any such graphs must have order at least 8); it is interesting to observe that the first is a subgraph of the second. Moreover, by taking the join of such a graph with a complete graph, we see that there are graphs of order $n \geq 8$ with chromatic number $n - 5$ whose $\sigma$-polynomials have a nonreal root. So the question remains – are there any graphs of order $n$ with chromatic number $n - 4$ whose $\sigma$-polynomials have nonreal roots? In [2] all graphs of order $n \leq 9$ whose $\sigma$-polynomials have a nonreal root are listed, and none of these have chromatic number $n - 4$. We have verified as well that all of the $\sigma$-polynomials of the 113, 272 6-chromatic graphs of order 10 have all real roots, so that if there is a graph with chromatic number $n - 4$ whose $\sigma$-polynomial has a nonreal root, then it has order at least 11.

![Figure 6: The graphs of order 8 whose $\sigma$-polynomials have nonreal roots.](image)

Finally, a well known result due to Newton (see [8, pp. 270–271]) states that if a real polynomial $\sum_{i=0}^{d} a_i x^i$ has only real roots then the sequence $a_0, a_1, \ldots, a_d$ is log-concave, that is, $a_i^2 \geq a_{i-1} a_{i+1}$ for $i = 1, \ldots, d - 1$ (if a log concave sequence has no internal zeros, then it is unimodal in absolute value). Brenti [1] posed the question of whether the coefficients of $\sigma$-polynomials of all graphs are log-concave. As a corollary of Theorem 3.6, we obtain that the coefficients of $\sigma$ polynomials of all graphs with $\chi(G) \geq n - 3$ are log-concave.

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