Bernard Derrida · Martin Retaux

Finite size corrections to the large deviation function of the density in the one dimensional symmetric simple exclusion process

Received: date / Accepted: date

Abstract The symmetric simple exclusion process is one of the simplest out-of-equilibrium systems for which the steady state is known. Its large deviation functional of the density has been computed in the past both by microscopic and macroscopic approaches. Here we obtain the leading finite size correction to this large deviation functional. The result is compared to the similar corrections for equilibrium systems.

PACS 02.50.-r, 05.40.-a, 05.70 Ln, 82.20-w

February 7, 2014

1 Introduction

Over recent years there has been a growing interest in understanding the fluctuations and the large deviations of the density of systems in a non equilibrium steady state [1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21]. In such steady states, the generic situation is that the correlation range of density fluctuations extends through the whole system [22,23,24,25,26,27,28,29] and the large deviation functional of the density is non local [2,3,4,5,6,8,9,10,11,12,13]. This contrasts with systems (with short range interactions and far from a critical point) at equilibrium, where the range of correlations is microscopic and the large deviation functional is local.

B. Derrida · M. Retaux
Laboratoire de Physique Statistique,
École Normale Supérieure, Université Pierre et Marie Curie, Université Denis Diderot, CNRS
24, rue Lhomond, 75231 Paris Cedex 05 - France
E-mail: derrida@lps.ens.fr · E-mail: martin.retaux@ens.fr
Two main approaches have been followed recently to study these large deviations: for some microscopic models such as exclusion processes the steady state measure is known exactly \[ \text{[8,9,10,11,12]} \] and finding the large deviation functional is a matter of computing large scale properties (very much like when one tries to calculate the free energy in equilibrium systems starting from the Gibbs measure). Obviously this microscopic approach is limited to cases where the steady state is exactly known. The other approach is the macroscopic fluctuation theory \[ \text{[2,3,4,5,6,7]} \] for diffusive systems which calculates the large deviation functional by identifying the optimal path followed by the system to generate a given deviation. In systems at equilibrium, time reversal symmetry gives a simple relation between this path and the relaxation path starting from the same deviation, and so identifying this optimal path is easy. On the contrary in non equilibrium steady states this time reversal symmetry does not hold and the approach is limited to cases where the equations giving this optimal path can be solved.

One motivation to study the large deviation functional of the density is that it generalizes the notion of free energy to non equilibrium states \[ \text{[7,13]} \]. As analytic expressions of these large deviation functionals are usually hard to obtain, they are known so far for a rather limited number of models. The one dimensional symmetric simple exclusion process (SSEP) was one of the first models \[ \text{[3,6,8,9,13]} \] for which an explicit expression could be derived which showed the non local character of this large deviation functional. The goal of the present work is to obtain the leading finite size corrections to this large deviation functional and to compare it with the corrections one typically finds in equilibrium systems.

The SSEP describes a lattice of \( L \) sites in which each site \( i \) is either occupied by a single particle or empty \[ \text{[33,34,35,36,37,38]} \]. Each particle independently attempts to jump to its right neighboring site, and to its left neighboring site with rate one. It succeeds if the target site is empty; otherwise nothing happens. At the boundary sites, 1 and \( L \), particles are added or removed: a particle is added to site 1, when the site is empty, at rate \( \alpha \), and removed, when the site is occupied, at rate \( \gamma \); similarly particles are added to site \( L \) at rate \( \delta \) and removed at rate \( \beta \). These injection and removal rates at the boundaries correspond to the left and right boundaries being in contact with reservoirs at densities

\[
\rho_{a} = \frac{\alpha}{\alpha + \gamma} ; \quad \rho_{b} = \frac{\delta}{\beta + \delta}
\]  

(to justify \( 1 \) it is easy to check using detailed balance that, when one forbids the exchanges of particles at site \( L \) by setting \( \beta = \delta = 0 \), the steady state measure is a Bernoulli measure where all the sites are occupied with probability \( \rho_{a} \). Similarly one can check that when the contacts between site 1 and the left reservoir are broken, the system equilibrates at density \( \rho_{b} \).) The main advantage of the SSEP is that its steady state measure is known \[ \text{[23,30,8,9]} \] for arbitrary \( \alpha, \beta, \gamma, \delta \) and \( L \).

Here we try to determine the generating function of the density, which is simply the Legendre transform of the large deviation functional. Let \( P(n_1, \cdots, n_L) \) be the steady state measure of a one dimensional lattice gas on a lattice of \( L \) sites, where \( n_i \geq 0 \) is the number of particles on site \( i \) (for the SSEP the only possible values are \( n_i = 0 \) or 1 but in the more general case discussed in section 4 the occupation numbers \( n_i \) will be arbitrary). We want to calculate the following generating function \( Z_L(h_1, \cdots, h_L) \) (which, in the equilibrium case, is nothing but a partition
function in a varying field)

\[ Z_L(h_1, \cdots, h_L) = \sum \{ n_1, \cdots, n_L \} \exp \left( \sum_{i=1}^{L} h_i n_i \right) P(n_1, \cdots, n_L) \] (2)

where \( h_i \) depends on the site \( i \). Let us define \( G_L(h_1, \cdots, h_L) \) as

\[ G_L(h_1, \cdots, h_L) = \log Z_L(h_1, \cdots, h_L) \] (3)

We would like to obtain an expression of \( G_L(h_1, \cdots, h_L) \) for a slowly varying field, that is when \( h_i \) is of the form

\[ h_i = H \left( \frac{i}{\lambda} - \frac{1}{2\lambda} \right) \] (4)

where \( \lambda \) is large (the reason for the shift of \(-1/2\lambda \) in (4) is simply to make sites 1 and \( L \) play symmetric roles). This choice for the \( i \) dependence of \( h_i \) allows one to test density deviations which vary on a large length scale \( \lambda \), which might be different from the system size \( L \).

In the following we will consider the case where the two lengths \( \lambda \) and \( L \) are large (compared with the lattice spacing) but comparable

\[ L = \lambda y \] (5)

with \( y \) of order 1. For the symmetric exclusion process in contact at site \( i = 1 \) and at site \( i = L \) with two reservoirs at densities \( \rho_a \) and \( \rho_b \) it is known [8,9,3,4] (see for example eq. (80,81) of [13]) that, in the steady state,

\[ G_L(h_1, \cdots, h_L) \sim \lambda \min \left\{ F(x) \right\} \int_0^y dx \left( \log(1 + F(x)(e^{H(x)} - 1)) - \log \left( \frac{F'(x)}{\rho_b - \rho_a} \right) \right) \] (6)

where the minimum is over all the monotone functions \( F(x) \) which satisfy \( F(0) = \rho_a \) and \( F(1) = \rho_b \). From (6) it is easy to see that the equation satisfied by the optimal \( F \) is

\[ F''(x) = \frac{F'(x)^2 (e^{H(x)} - 1)}{1 - F(x) + F(x) e^{H(x)}} \] (7)

Note that the non local character of the functional (6) comes from the fact that the optimal \( F(x) \) depends on all values of \( H(z) \) for the whole range \( 0 < z < y = \frac{L}{\lambda} \).

On the other hand for a system at equilibrium (with short range interactions) one expects [7,13] that

\[ G_L(h_1, \cdots, h_L) \sim \lambda \int_0^y dx \ g(H(x)) \] (8)

where \( g(h) = \lim_{L \to \infty} G_L(h, \cdots, h) / L \) is the extensive part of the free energy in a constant \( h \). This is obviously a local functional of \( H(x) \).
The main result presented in the present work is that the leading correction to (6) is
\[ G_L(h_1, \cdots, h_L) \simeq \lambda \int_0^y dx \left( \log(1 + F(x)(e^{H(x)} - 1)) - \log \left( \frac{y E'(x)}{\rho_b - \rho_a} \right) \right) - a \log \left( \frac{y E'(0)}{\rho_b - \rho_a} \right) - b \log \left( \frac{y E'(y)}{\rho_b - \rho_a} \right) - \frac{1}{2} \log(\phi(0)) \] (9)
where \( F \) is the solution of (7), the parameters \( a \) and \( b \) are defined as in [27]
\[ a = \frac{1}{\alpha + \gamma}; \quad b = \frac{1}{\beta + \delta} \] (10)
and \( \phi(x) \) is the solution of the linear differential equation
\[ \phi''(x) = - \left( \frac{E''(x)}{E'(x)} \right) \phi(x) \] (11)
which satisfies the boundary conditions \( \phi(y) = 0 \) and \( \phi'(y) = -1/y \).

This can be compared to the case of a system at equilibrium where the form of the leading orders of \( G_L \) is
\[ G_L(h_1, \cdots, h_L) \simeq \lambda \int_0^y g(H(x)) \, dx + A^{\text{left}}(H(0)) + A^{\text{right}}(H(y)) \]
\[ + \frac{1}{\lambda} \left[ H'(0) B^{\text{left}}(H(0)) + H'(y) B^{\text{right}}(H(y)) \right] \]
\[ + \int_0^y C(H(x)) \, H'(x)^2 \, dx + O(1) \] (12)
where \( A^{\text{left}}(h), A^{\text{right}}(h), B^{\text{left}}(h), B^{\text{right}}(h) \) and \( C(h) \) are defined in [46,52,53]. We see that the leading correction (i.e. the term of order 0 in \( \lambda \)) is also non local [9] in the out of equilibrium SSEP whereas it corresponds to boundary contributions \( A^{\text{left}} \) and \( A^{\text{right}} \) in the equilibrium case [12]. At the next order (the order \( 1/\lambda \)), which we did not study in the non-equilibrium case, one can notice in [12] an integral containing the gradient term \( H'(x)^2 \) characteristic of the Ginzburg-Landau theory.

The paper is organized as follows. In section 2, we do a direct perturbative calculation when the \( h_i \)'s are small and we check that the expansion agrees with the prediction [9] for the SSEP. In section 3, we present the derivation of [9] for arbitrary \( h_i \)'s. In section 4, we discuss how [12] can be derived in the equilibrium case.

2 Perturbations for small \( h_i \)

In this section we present the straightforward calculation of \( G_L \) from the knowledge of the correlation functions of the density in the steady state. From the definition [5], one can relate the expansion of \( G_L \) in powers of the \( h_i \)'s to the steady state correlations. For example to second order in the \( h_i \)'s one has
\[ G_L(h_1, \cdots, h_L) = \sum_i h_i \langle n_i \rangle + \sum_i \frac{h_i^2}{2} \langle n_i^2 \rangle - \langle n_i \rangle^2 + \sum_{i<j} h_i h_j \langle n_i n_j \rangle_c + O(h^3) \] (13)
For the SSEP, it is known that in the steady state \([23, 27]\)

\[
\langle n_i \rangle = \langle n_i^2 \rangle = \frac{\rho_a (L + b - i) + \rho_b (i + a - 1)}{L + a + b - 1}, \quad (14)
\]

\[(n_i n_j)_{\varepsilon} = -\frac{(\rho_a - \rho_b)^2 (i + a - 1)(L + b - j)}{(L + a + b - 1)^2 (L + a + b - 2)} \quad \text{for } i < j \]

For large \(L\) and \(\lambda\) (keeping their ratio constant as in \([5]\)), when the \(h_i\) have the form \([3]\), the various sums in \([13]\) can be computed by the Euler MacLaurin formulae:

\[
\varepsilon \sum_{i=1}^{L} f \left( i \varepsilon - \frac{\varepsilon}{2} \right) = \int_{0}^{L} f(x)dx - \varepsilon \frac{f'(L\varepsilon) - f'(0)}{24} + O(\varepsilon^4)
\]

\[
\varepsilon^2 \sum_{i < j < L} f \left( \frac{i \varepsilon - \varepsilon}{2} \right) g \left( j \varepsilon - \frac{\varepsilon}{2} \right) = \int_{0}^{L} f(x)dx \int_{x}^{L} g(y)dy - \frac{\varepsilon}{2} \int_{0}^{L} f(x)g(x)dx + O(\varepsilon^2)
\]

and one gets

\[
\sum h_i(n_i) = \lambda \int_{0}^{y} \overline{\rho}(x)H(x)dx + (\rho_a - \rho_b) \int_{0}^{y} \frac{y - 2ay - 2x + 2ax + 2bx}{2y^2} H(x)dx
\]

\[
\sum h_i^2(n_i) = \lambda \int_{0}^{y} \overline{\rho}(x)H^2(x)dx + (\rho_a - \rho_b) \int_{0}^{y} \frac{y - 2ay - 2x + 2ax + 2bx}{2y^2} H^2(x)dx
\]

\[
\sum h_i^2(n_i)^2 = \lambda \int_{0}^{y} \overline{\rho}(x)H^2(x)dx + (\rho_a - \rho_b) \int_{0}^{y} \frac{y - 2ay - 2x + 2ax + 2bx}{y^2} \overline{\rho}(x)H^2(x)dx
\]

where \(\overline{\rho}(x)\) is the steady state profile

\[
\overline{\rho}(x) = \frac{\rho_a}{y}(y - x) + \frac{\rho_b}{y} x
\]

and

\[
\sum h_i h_j(n_i n_j)_{\varepsilon} = -\lambda (\rho_a - \rho_b)^2 \int_{0}^{y} dx \int_{x}^{y} \frac{x(y - z)}{y^3} H(x)H(z) + \frac{(\rho_a - \rho_b)^2}{2y^4} \left[ y \int_{0}^{y} x(y - x)H(x)^2 dx + \int_{0}^{y} dx \int_{x}^{y} H(x)H(z) \right] \times \left[ (1 - 2a)(y - x)(y - z) + (4a + 4b - 6)x(y - z) + (1 - 2b)xz \right]
\]

We have checked that these expressions coincide with \([5]\) at second order in \(H(x)\). For example at first order in \(H(x)\) the solutions of \([7]\) and \([11]\) are

\[
F(x) = \frac{\rho_a (y - x)}{y} + \frac{\rho_b x}{y} + (\rho_a - \rho_b) \left[ \int_{x}^{y} \frac{x(z - y)}{y^3} H(z)dz + \int_{0}^{H(x)} \frac{z(y - x)}{y^3} H(z)dz \right]
\]

\[
\varphi(x) = \frac{(y - x)}{y} + \frac{(\rho_a - \rho_b)}{y^2} \int_{x}^{y} (2z - y - x)H(z)dz
\]

and inserting these expressions into \([9]\) one gets \([15]\).
3 Derivation of the main result (9)

Our approach to obtain (9) consists in choosing \( h_i \) piecewise constant: \( h_i \) takes \( n \) possible values \( H_1, \cdots, H_n \) in \( n \) consecutive boxes. As in each of these boxes, \( h_i \) is constant we will use the expression (16,17) for a single box with a constant \( h \) which is much easier to obtain. Then we will use an additivity formula (71) to go from the expression for one box to the expression for \( n \) boxes. Finally we will take the limit \( n \to \infty \) to establish (9).

3.1 A single box

Using the matrix ansatz (see the Appendix), one gets, for large \( L \), the following expression for \( Z_L(h, h, \cdots, h) \) by dividing (69) by (62)

\[
Z_L(h, \cdots, h) \simeq \frac{(\rho_a - \rho_b)^{L+a+b} \mu_0^{-L-a-b}}{(1 + \rho_a(e^h - 1))^a (1 + \rho_b(e^h - 1))^b}
\]

(16)

where

\[
\mu_0 = \frac{1}{e^h - 1} \log \frac{1 + \rho_a(e^h - 1)}{1 + \rho_b(e^h - 1)}
\]

(17)

Remark: Let us check that these expressions agree with the claim (9) in the introduction: one has by solving (7) and (11) for a constant \( h \)

\[
F(x) = \frac{1}{e^h - 1} \left[ (1 + \rho_a(e^h - 1))^{1-x/y} (1 + \rho_b(e^h - 1))^{x/y} - 1 \right]
\]

and

\[
\varphi(x) = \frac{y-x}{y}
\]

from which it follows that

\[
\log[1 + (e^h - 1)F(x)] - \log \left( \frac{yF'(x)}{\rho_b - \rho_a} \right) = \log \left( \frac{\rho_a - \rho_b}{\mu_0} \right)
\]

\[
F'(0) = \frac{1 + \rho_a(e^h - 1)}{y(e^h - 1)} \log \left( \frac{1 + \rho_b(e^h - 1)}{1 + \rho_a(e^h - 1)} \right)
\]

\[
F'(y) = \frac{1 + \rho_b(e^h - 1)}{y(e^h - 1)} \log \left( \frac{1 + \rho_b(e^h - 1)}{1 + \rho_a(e^h - 1)} \right)
\]

and by replacing into (9) one finds an expression equivalent to (16) obtained by the direct calculation. This shows that (9) is valid in the case of a constant \( h_i \).
3.2 Several boxes: the prediction (9)

Let us now come to the case of several large boxes with a constant $h_i$ in each box. We will first write down the expressions predicted by the claim (9). Then we will see in the next subsection that these expressions coincide with those obtained by a direct microscopic calculation.

For piecewise constant $H(x)$, with

$$H(x) = H_m \quad \text{for} \quad x_{m-1} < x < x_m$$

with

$$x_0 = 0 \quad ; \quad x_m = x_{m-1} + y_m \quad ; \quad x_n = y$$

the solution of (7) in the interval $x_{m-1} < x < x_m$ is

$$F(x) = \frac{1}{e^{H_n} - 1} \left[ (1 + (e^{H_n} - 1)F_{m-1})^{\frac{x_n - x_{m-1}}{y_m}} - (1 + (e^{H_n} - 1)F_m) \right]$$

where $F_m = F(x_m)$. Writing that $F'(x)$ is continuous (i.e. $F'(x_m^-) = F'(x_m^+)$) at all the $x_m$’s leads to the $n - 1$ equations that these $F_m$’s should satisfy

$$\frac{1 + (e^{H_n} - 1)F_m}{y_m(e^{H_n} - 1)} \log \left( \frac{1 + (e^{H_n} - 1)F_m}{1 + (e^{H_n} - 1)F_{m-1}} \right) = \frac{1 + (e^{H_{n+1}} - 1)F_{m+1}}{y_{m+1}(e^{H_{n+1}} - 1)} \log \left( \frac{1 + (e^{H_{n+1}} - 1)F_{m+1}}{1 + (e^{H_{n+1}} - 1)F_m} \right)$$

Equations (20,21) fully determine the solution of (7) for a piecewise constant $H(x)$.

To solve the equation (11) for $\varphi(x)$, one can first notice that the discontinuity of the $\varphi'(x)$ at $x = x_m$ is

$$\varphi'(x_m^+) - \varphi'(x_m^-) = \frac{F''(x_m^-) - F''(x_m^+)}{F'(x_m)} \varphi(x_m)$$

Everywhere else the function $\varphi(x)$ is piecewise linear. These jumps of $\varphi'(x)$ and the fact that $\varphi(y) = 0$ and $\varphi'(y) = -1/y$ determine the function $\varphi(x)$ everywhere: in the interval $x_m < x < x_{m+1}$ one gets

$$\varphi(x) = \frac{1}{y} [y - x + \sum_{m_1 > m} (y - x_{m_1}) (x_{m_1} - x) W_{m_1}
+ \sum_{m_1 > m_2 > m} (y - x_{m_1}) (x_{m_1} - x_{m_2}) (x_{m_2} - x) W_{m_1} W_{m_2}
+ \sum_{m_1 > m_2 > m_3 > m} (y - x_{m_1}) (x_{m_1} - x_{m_2}) (x_{m_2} - x_{m_3}) (x_{m_3} - x) W_{m_1} W_{m_2} W_{m_3} + \ldots]$$

with $W_m$ defined by

$$W_m = \frac{F''(x_m^-) - F''(x_m^+)}{F'(x_m)}.$$

Using (20) and (21), one can show that

$$W_m = \frac{1}{y_m} \log \left( \frac{1 + (e^{H_n} - 1)F_m}{1 + (e^{H_n} - 1)F_{m-1}} \right) - \frac{1}{y_{m+1}} \log \left( \frac{1 + (e^{H_{n+1}} - 1)F_{m+1}}{1 + (e^{H_{n+1}} - 1)F_m} \right)$$
In summary in the case of several large boxes the claim (9) leads to

\[ Z_L(h_1, \ldots, h_L) = e^{G_L(h_1, \ldots, h_L)} \simeq \mathcal{B} \exp[\mathcal{C}(\rho_a, \rho_b)] \]  

(25)

with

\[ \mathcal{C} = -\sum_m y_m \log \left[ \frac{y}{y_m(\rho_b - \rho_a)(e^{H_m} - 1)} \log \left( \frac{1 + (e^{H_m} - 1)F_m}{1 + (e^{H_m} - 1)F_{m-1}} \right) \right] \]  

(26)

and

\[ \mathcal{B} = \left( \frac{\rho_b - \rho_a}{yF'(0)} \right)^a \left( \frac{\rho_b - \rho_a}{yF'(y)} \right)^b \frac{1}{\varphi(0)^{1/2}} \]  

(27)

with \( F(x) \) and \( \varphi(x) \) given by (20,22) and the \( F_m \)'s solutions of (21).

3.3 Several boxes: the microscopic approach

Let us now see how the microscopic calculation for the single box can be generalized to the case of several boxes and leads to expressions equivalent to (26,27).

We consider the case of several large boxes with a constant \( h_i \) in each box

\[ h_i = H_m \quad \text{for} \quad L_1 + \cdots + L_{m-1} < i \leq L_1 + \cdots + L_m \]  

(28)

uses the additivity formula (71) and the saddle point method.

Let us define

\[ z_i(\rho_a, \rho) = \frac{\langle \rho_a, a | (e^{H_1}D + E)_{L_1} | \rho, b \rangle}{\langle \rho_a, a | (D + E)_{L_1} | \rho, b \rangle} \]  

and for \( i \geq 2 \)

\[ z_i(\rho, \rho') = \frac{\langle \rho, 1-b | (e^{H_i}D + E)_{L_i} | \rho', b \rangle}{\langle \rho, 1-b | (D + E)_{L_i} | \rho', b \rangle} \]  

which are the generating functions for each box (see Appendix A). \( z_1 \) is special simply because in \( z_2, \ldots, z_n \) the parameter \( a \) has been replaced by \( 1-b \). Then using the additivity formula (71) derived in Appendix A one gets for \( Z_L(h_1, \ldots, h_L) \) when

\[ L_1 + L_2 + \cdots + L_n = L \]  

and the \( h_i \) are of the form (28)

\[ Z_L(h_1, \ldots, h_L) = \frac{\Gamma(L_1 + a+b)\Gamma(L_2 + 1)\cdots\Gamma(L_n + 1)}{\Gamma(L + a+b)} \times \]  

(29)

\[ \int \frac{d\rho_1}{2i\pi} \cdots \int \frac{d\rho_{n-1}}{2i\pi} \frac{(\rho_a - \rho_b)^{L-a+b} z_1(\rho_a, \rho_1) z_2(\rho_1, \rho_2) \cdots z_n(\rho_{n-1}, \rho_b)}{(\rho_a - \rho_1)^{L-a+b} \cdots (\rho_{n-1} - \rho_b)^{L-a+b}} \]  

where the integral contours verify \( \rho_b < |\rho_{n-1}| < \cdots < |\rho_1| < \rho_a \). So far (29) is exact for arbitrary \( H_m \)'s and \( L_m \)'s. The virtue of (71) is that it relates the properties of the whole system of those of the \( n \) subsystems.
When the lengths $L_m$ of the boxes become large, if we define the $y_m$'s by

$$L_m = \lambda y_m$$

one knows (16) from the single box calculation that

$$z_m(\rho, \rho') \sim B_m(\rho, \rho') e^{A_m(\rho, \rho') + \Delta(\rho, \rho')}$$

with

$$A_m(\rho, \rho') = -\log \left[ \frac{1}{e^{H_\rho} - 1} \log \frac{1 + \rho(e^{H_\rho} - 1)}{1 + \rho'(e^{H_\rho} - 1)} \right]$$

(30)

$$B_1(\rho, \rho') = \frac{(\rho - \rho') e^{A_1(\rho, \rho')}}{(1 + \rho(e^{H_\rho} - 1))^a (1 + \rho'(e^{H_\rho} - 1))^b}$$

(31)

and for $i \geq 2$

$$B_i(\rho, \rho') = \frac{(\rho - \rho') e^{A_i(\rho, \rho')}}{(1 + \rho(e^{H_\rho} - 1))^a (1 + \rho'(e^{H_\rho} - 1))^b}$$

(32)

Then using the saddle point method in (29) one finds that

$$Z_L(h_1, \ldots, h_L) \simeq e^{\lambda(\mathcal{R}^* + y \log(\rho_a - \rho_b))}$$

(33)

where

$$\mathcal{R}^* = \min_{\{r_m\}} \left[ y_1 A_1(\rho_a, r_1) + y_2 A_2(\rho_a, r_2) + \ldots + y_n A_n(\rho_a, r_n) - y \log y + \sum_{i=1}^n y_i \log y_i \right],$$

(34)

$$\mathcal{R}^* = \frac{(\rho_a - \rho_b)^a + b}{y^{a+b} - b} \frac{\partial^{a+b} \rho_a}{\rho_a - r_1} \frac{\partial B_2(\rho_a, r_2)}{r_1 - r_2} \ldots \frac{\partial B_n(\rho_a, r_n)}{r_{n-1} - r_n} (\det[\Delta])^{-1/2}$$

(35)

and $\Delta$ is a tridiagonal matrix

$$\Delta = \begin{pmatrix}
U_1 & V_1 & 0 & 0 & 0 \\
V_1 & U_2 & V_2 & 0 & 0 \\
0 & \ldots & \ldots & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & V_{n-2} \ U_{n-1}
\end{pmatrix}$$

(36)

with

$$U_m = \frac{\partial^2 [y_m A_m(\rho_{m-1}, \rho) + y_{m+1} A_{m+1}(\rho, \rho_{m+1})]}{\partial \rho^2} \bigg|_{\rho = \rho_m}$$

(37)

$$V_m = \frac{\partial^2 [y_{m+1} A_{m+1}(\rho, \rho')]}{\partial \rho \partial \rho'} \bigg|_{\rho = \rho_m, \rho' = \rho_{m+1}}$$

(38)

(Note that in (34) one takes the minimum and not the maximum because the integration contours are perpendicular to the real axis, and a maximum over $r_m$ along a contour becomes a minimum when $r_m$ varies along the real axis).
The saddle point values $r_1, \cdots, r_{n-1}$ (i.e. those which achieve the minimum in (34)) satisfy the $n-1$ equations

$$\frac{\partial}{\partial \rho} \left[ y_m A_m(r_{m-1}, \rho) + y_{m+1} A_{m+1}(\rho, r_m + 1) \right]_{\rho = r_m} = 0$$

These saddle point equations turn out to be the same equations as those satisfied (21) satisfied by the $F_m$'s. Therefore one has

$$r_m = F_m$$

(39)

This already allows one to verify, using (30,34), that the term proportional to $\lambda$ in (25) and (33) is the same.

One can also show by a direct computation that the $U_m$'s and the $V_m$'s defined in (37,38) can be expressed in terms of the function $F(x)$ given in (20).

$$U_m = \left( \frac{1}{y_m} + \frac{1}{y_{m+1}} \right) \frac{1}{F'(x_m)^2} + \frac{F''(x_m)(-) - F''(x_m)(+)}{F'(x_m)^3}$$

(40)

$$V_m = -\frac{1}{y_{m+1} F'(x_m) F'(x_{m+1})}.$$  

(41)

It is then easy to see that one can rewrite $U_m$ as

$$U_m = \left( \frac{1}{y_m} + \frac{1}{y_{m+1}} \right) \frac{1}{F'(x_m)^2} + \frac{W_m}{F'(x_m)^2}$$

with $W_m$ given in (23). Then the determinant of the matrix $\Delta$ defined in (36) can be computed

$$\det[\Delta] = \frac{1}{y_1 \cdots y_n |F'(x_1) \cdots F'(x_{n-1})|^2} \times$$

$$\left[ y + \sum_m (y - x_m) x_m W_m + \sum_{m_1 > m_2} (y - x_{m_1})(x_{m_1} - x_{m_2}) x_{m_2} W_{m_1} W_{m_2} \right.$$

$$+ \sum_{m_1 > m_2 > m_3} (y - x_{m_1})(x_{m_1} - x_{m_2})(x_{m_2} - x_{m_3}) x_{m_3} W_{m_1} W_{m_2} W_{m_3} + \ldots \right]$$

and using the fact (see (30,20)) that

$$\exp[A_m(F_{m-1}, F_m)] = -\frac{1 + (e^{H_m} - 1)F_m}{y_m F'(x_m)} = -\frac{1 + (e^{H_m} - 1)F_{m-1}}{y_m F'(x_{m-1})}$$

(43)

one gets for $B^+$

$$B^+ = \left( \frac{\rho_b - \rho_a}{y F'(0)} \right)^a \left( \frac{\rho_b - \rho_a}{y F'(y)} \right)^b \times$$

$$\left( 1 + \sum_m \frac{x_m(y - x_m)}{y} W_m + \sum_{m_1 > m_2} \frac{(y - x_{m_1})(x_{m_1} - x_{m_2}) x_{m_2}}{y} W_{m_1} W_{m_2} + \ldots \right)^{-1/2}$$

This expression coincides with the expected expression (27). Therefore this subsection has established the validity of (27) in the case of several boxes.
3.4 A large number of boxes

Let us now try to take the large $n$ limit of the above result. We consider that we have $n$ boxes, of equal length $L/n = \lambda y/n$, and that the field $H_m$ in the $m$-th box is given by

$$H_m = H \left( \frac{my}{n} - \frac{y}{2n} \right)$$

where $H(x)$ is a smoothly varying function. For simplicity we choose the boxes of equal lengths. Therefore one has

$$x_m = \frac{my}{n}.$$ 

One then need to solve the equations (21) satisfied by the $F_m$'s. For large $n$ one can show by a direct calculation that the solution of these equations is given by

$$F_m = F \left( \frac{my}{n} \right) + O \left( \frac{1}{n^2} \right)$$

where $F(x)$ is the solution of (7). (Note that from now on, $F(x)$ is the solution of (7) when $H(x)$ is a smoothly varying function. This solution $F(x)$ is not identical to (20) which was obtained for a piecewise $H(x)$. The difference is at the origin of the correction of order $O(1/n^2)$ in (44). This difference will lead to negligible terms anyway.) One has from (30)

$$A_m(F_{m-1}, F_m) = - \log \frac{y}{n} + \frac{F'(x_m)}{F(x_m)} \left( \frac{1 + F(x_m)(e^{H(x_m)} - 1)}{-F'(x_m)} \right) - \frac{y}{2n} \left( \log \frac{1 + F(x)(e^{H(x)} - 1)}{-F'(x)} \right) \bigg|_{x=x_m} + O \left( \frac{1}{n^2} \right)$$

which can be rewritten using the fact that $F(x)$ is solution of (7)

$$A_m(F_{m-1}, F_m) = - \log \frac{y}{n} + \frac{F'(x_m)}{F(x_m)} \left( \frac{1 + F(x_m)(e^{H(x_m)} - 1)}{-F'(x_m)} \right) - \frac{y}{2n} \left( \log \frac{1 + F(x)(e^{H(x)} - 1)}{-F'(x)} \right) \bigg|_{x=x_m} + O \left( \frac{1}{n^2} \right)$$

Using then the Euler McLaurin formula to perform the sum (34), one finds that

$$A^{\prime}(\rho_a, \rho_b) = -n \log \frac{y}{n} + \int_0^{\rho} \log \left( \frac{1 + F(x)(e^{H(x)} - 1)}{-F'(x)} \right) dx + O \left( \frac{1}{n} \right)$$

and this leads (33,34) to the term proportionnal to $\lambda$ in (9).

One can also obtain the large $n$ estimate of $W_m$

$$W_m = - \frac{y}{n} \left( \frac{F''(x)}{F'(x)} \right) \bigg|_{x=x_m}$$
Then by defining $W(x)$ by

$$W(x) = -\left(\frac{F''(x)}{F'(x)}\right)'$$

one can see that

$$B = (\rho_b - \rho_a) a + b (\rho_a - \rho_b) F'(0) - \frac{1}{2}$$

Then if one $\varphi(x)$ is solution of

$$\varphi''(x) = W(x) \varphi(x)$$

with $\varphi(y) = 0$ and $\varphi'(y) = -1/y$ one has

$$\varphi(x) = 1 - \frac{x}{y} + \int_y^x \frac{(y-z)(z-x)}{y} W(z) dz + \int_x^y \int_z^y \frac{(y'-z)(z-x)}{y} W(z) W(z') dz' + ...$$

and one finds

$$B = \left(\frac{\rho_b - \rho_a}{F'(0)}\right)^a \left(\frac{\rho_a - \rho_b}{F'(y)}\right)^b (\varphi(0))^{-1/2}$$

as claimed in (9).

4 Equilibrium case

Let us consider a one dimensional lattice gas on a lattice of $L$ sites, where each site $i$ is occupied by an integer $n_i \geq 0$ number of particles. We assume that the interactions are short range and that at equilibrium the system is homogeneous in the bulk with correlation functions decaying exponentially fast with the distance. We would like to obtain an expression of $G_L(h_1, \cdots, h_L)$ defined in (23) for a slowly varying field of the form (4) when both $\lambda$ and $L$ are much larger than the range $\xi$ of the correlations between the occupation numbers $n_i$.

4.1 The constant field case

Let us discuss first the case of a constant field ($h_i = h$). If $g(h)$ is the extensive part of the free energy $G_L$

$$g(h) = \lim_{L \to \infty} \frac{G_L(h, \cdots, h)}{L}$$
one expects that in the large $L$ limit
\begin{equation}
G_L(h) = Lg(h) + A^{\text{left}}(h) + A^{\text{right}}(h) + O\left(\exp\left[-L/\xi(h)\right]\right) \tag{46}
\end{equation}
where $A^{\text{left}}(h)$ and $A^{\text{right}}(h)$ represent the contributions of the left and right boundaries respectively and $\xi(h)$ is the correlation length (in presence of the constant field $h$). The form (46) can be easily understood by the transfer matrix method, in particular $\exp(-1/\xi(h))$ is the ratio of the two largest eigenvalues of the transfer matrix. These two contributions $A^{\text{left}}$ and $A^{\text{right}}$ are not necessarily equal as they may differ if one imposes different boundary conditions at the two ends.

In a constant field, one can also define the average density $\langle n_i \rangle$ at site $i$ by
\begin{equation}
\langle n_i \rangle = \frac{\partial \log Z_L}{\partial h_i}(h,\ldots h)
\end{equation}

and the pair correlation function
\begin{equation}
\langle n_i n_j \rangle_c = \frac{\partial^2 \log Z_L}{\partial h_i \partial h_j}(h,\ldots h)
\end{equation}

In the large $L$ limit, far from the boundaries (i.e. for $i \gg \xi(h)$ and $L - i \gg \xi(h)$), the average density $\langle n_i \rangle$ has a limit independent of $i$
\begin{equation}
\langle n_i \rangle \to g'(h) \tag{47}
\end{equation}

the pair correlation function $\langle n_i n_j \rangle_c$ depends only on the distance $j - i$
\begin{equation}
\langle n_i n_j \rangle_c \to c_{j-i}(h) \tag{48}
\end{equation}

and one has
\begin{equation}
g''(h) = \sum_{k=-\infty}^{m} c_k(h) \tag{49}
\end{equation}

On the other hand, close to the left or to the right boundary, i.e. as long as $i \sim \xi(h)$ or $L - i \sim \xi(h)$ these quantities keep in general a dependence on $i$ even in the large $L$ limit. For example
\begin{equation}
\langle n_i \rangle - g'(h) \to a^{\text{left}}_i(h) ; \quad \langle n_{L-i} \rangle - g'(h) \to a^{\text{right}}_i(h) \tag{50}
\end{equation}

4.2 The slowly varying field case

Now for a slowly varying field of the form \cite{4}, when $\lambda \gg \xi(h)$ (more precisely $\lambda \gg \max_i \xi(h_i)$) one expects \cite{7,13} that to leading order
\begin{equation}
G_L(h_1, \ldots h_L) \simeq \lambda \int_0^y g(H(x)) \, dx
\end{equation}

where $y$ is defined in \cite{5}. This can be easily understood by cutting the system of length $L$ into many subsystems of size $\lambda dx$ much larger than $\xi$ but much smaller than $\lambda$. In each of these subsystems the field $h_i$ is essentially constant, and the free energies of these subsystems can simply be added.
As explained in Appendix B, the leading corrections to this formula when \( L \) and \( \lambda \) are much larger than the correlation length \( \xi(h) \) is

\[
G_L(h_1, \cdots, h_L) \simeq \lambda \int_0^y g(H(x)) \, dx + A^{\text{left}}(H(0)) + A^{\text{right}}(H(y)) + \frac{1}{\lambda} \left[ H'(0) B^{\text{left}}(H(0)) + H'(y) B^{\text{right}}(H(y)) + \int_0^y C(H(x)) H'(x)^2 \, dx \right] + \mathcal{O} \left( \frac{1}{\lambda^2} \right)
\]

which is the result announced in (12) where

\[
B^{\text{left}}(h) = \frac{g'(h)}{24} + \sum_{i=1}^\infty \left( i - \frac{1}{2} \right) a_i^{\text{left}}(h) \quad ; \quad B^{\text{right}}(h) = -\frac{g'(h)}{24} - \sum_{i=0}^\infty \left( i + \frac{1}{2} \right) a_i^{\text{right}}(h)
\]

and

\[
C(h) = -\frac{1}{2} \sum_{k \geq 1} k^2 c_k(h)
\]

Remark: for the same system on ring with periodic boundary conditions, implying in particular that \( H(y + x) = H(x) \), the boundary terms disappear and one gets

\[
G_L(h_1, \cdots, h_L) \simeq \lambda \int_0^y g(H(x)) \, dx + \frac{1}{\lambda} \int_0^y C(H(x)) H'(x)^2 \, dx + \cdots
\]

Note that at order \( \frac{1}{\lambda^2} \) the integral of \( H'(x)^2 \) in (51) and in (54) is nothing but the square of the gradient of the Ginsburg Landau theory.

5 Conclusion

In this paper we have obtained the first correction (9) to the large deviation functional of the density for the non equilibrium steady state of the SSEP and compared it with the corresponding term for equilibrium systems (12). Like in the equilibrium case, this first correction does not depend on the system size. On the other hand in the non-equilibrium case (9) the correction has a non-local character, very much like the leading term. Our derivation is based on the knowledge of the steady state measure, as given by the matrix ansatz.

An interesting question would be to try to recover our result by the macroscopic approach: in the macroscopic fluctuation theory [2,3], the large deviation functional of the density is given by the contribution of the optimal trajectory in the space of all the time dependent density profiles which produces a given deviation starting from the steady state profile. A natural question would be to try to calculate the correction by integrating over all the profiles in the neighborhood of this optimal profile. Such an approach was successful in understanding the first corrections to the large deviation function of the current [39,40], and it would be of course interesting to see whether it works as well for the deviations of the
density. If this is the case, one could try to determine similar corrections for other models such as generalizations of the SSEP [12,15,41].

Recently, it has been noticed that the large deviation functional could exhibit phase transitions [42,43,44]. Whether the corrections calculated here would become singular at such phase transitions is another question one could try to investigate.

It has also been shown that the SSEP, in a non equilibrium steady state, could be mapped by a non local change of variables onto a system at equilibrium [45]. It would be interesting to know whether this transformation could be sued to confirm our prediction [29] and establish a connection with [12].

Acknowledgements

We would like to thank Vincent Hakim for helpful discussions.

A Additivity formulae

It is known [30,31,13] and has been used in several previous works [8,9] that the steady state measure of the SSEP with injection and removal rates $\alpha$, $\beta$, $\gamma$, $\delta$, as defined in the introduction, can be calculated by the matrix ansatz. [30,31]. The probability of any microscopic configuration $\{n_1,\cdots, n_L\}$ (with $n_i = 0$ or 1) is given by

$$P(\{n_1,\cdots, n_L\}) = \frac{\langle W | X_1 X_2 \cdots X_L | V \rangle}{\langle W | (D+E)^{L} | V \rangle}$$

(55)

where each matrix $X_i$ depends on the occupation $n_i$ of site $i$

$$X_i = n_i D + (1-n_i) E$$

(56)

and the matrices $D, E$ and the vectors $|V\rangle, \langle W|$ satisfy the following algebraic rules

$$DE - ED = D + E$$

$$\langle W | (\alpha E - \gamma D) = |W\rangle$$

$$\langle \beta D - \delta E | V \rangle = |V\rangle.$$  

(57)

Given these algebraic rules, one can define a family of left and right eigenvectors $|\rho_a, a\rangle$ and $|\rho_b, b\rangle$ by

$$\langle \rho_a, a \rangle \left( \rho_a E - (1 - \rho_a) D \right) = a \langle \rho_a, a \rangle$$

$$\left( 1 - \rho_b \right) D - \rho_b E \langle \rho_b, b \rangle = b \langle \rho_b, b \rangle.$$  

(58)

(59)

The vectors $\langle W \rangle$ and $|V\rangle$ which appear in (55) are examples of such eigenvectors

$$\langle W \rangle = |\rho_a, a\rangle,$$

$$|V\rangle = |\rho_b, b\rangle.$$  

(60)

when $\rho_a = \alpha/(\alpha + \gamma)$, $\rho_b = \delta/(\delta + \beta)$ and $a = 1/(\alpha + \gamma)$, $b = 1/(\delta + \beta)$ as in [10].

Then for $0 < b < 1$ and $\rho_a > \rho_b$, one can prove the following key additivity formula

$$\frac{\langle \rho_a, a | X_1 X_2 | \rho_b, b \rangle}{\langle \rho_a, a | \rho_b, b \rangle} = \int \frac{dp}{2\pi i} \left( \frac{\rho_a - \rho_b}{\rho_a - \rho} \right)^{a+b} - \left( \frac{\rho_a - \rho_b}{\rho_a - \rho} \right)^{a\rho_b} - \left( \frac{\rho - \rho_b}{\rho - \rho_b} \right)^{b\rho_a} - \left( \frac{\rho - \rho_b}{\rho - \rho_b} \right)^{b(1-b)} |\rho_a, a\rangle |\rho_b, b\rangle.$$  

(61)

where $X_1$ and $X_2$ are arbitrary polynomials of $D$'s and $E$'s and the contour is such that $\rho_b < |\rho | < \rho_a.$
Proof of (61): Let us first derive of the following identity [3]

\[
\frac{\langle \rho_a, a | (D + E)^{i+1} | \rho_b, b \rangle}{\langle \rho_a, a | \rho_b, b \rangle} = \frac{\Gamma(a + b + L)}{\Gamma(a + b)} (\rho_a - \rho_b)^{i+L}.
\]  
(62)

To do so one can notice that in the steady state of the SSEP, as defined in the introduction, the average occupations satisfy

\[\alpha - (\alpha + \gamma)(n_1) = (n_1 - n_2) = \cdots = (n_{i+1} - n_{i+1}) = (\beta + \delta)(n_t) - \delta \]

These \(L\) equations which express simply that in the steady state the current is conserved, can be solved. From the solution (53) one can see that

\[\langle n_t - n_{i+1} \rangle = \frac{(\rho_a - \rho_b)}{L + a + b - 1}.
\]

On the other hand using the matrix representation (55,57,60) one has

\[\langle n_t - n_{i+1} \rangle = \frac{(\rho_a, a | (D + E)^{i+1} | \rho_b, b)}{(\rho_a, a | \rho_b, b)} \]

These two identities give the recursion

\[
\frac{(\rho_a, a | (D + E)^{i+1} | \rho_b, b)}{(\rho_a, a | \rho_b, b)} = \frac{(\rho_a - \rho_b)}{L + a + b - 1}
\]

which establishes the veracity of (62).

Now to prove (61) (as in [13]) one can first notice that the discussion can be limited to \(X_1\) and \(X_2\) of the form

\[X_1 = \{\rho_aE - (1 - \rho_a)D\}^{\rho_1} (D + E)^{\rho_2}\]
\[X_2 = \{D + E\}^{\rho_2} \{1 - \rho_0\}D - \rho_0E\]  

as any polynomial in \(D\)'s and \(E\)'s can be written as a sum of such terms (this is because \(D\) and \(E\) are linear functions of the operators \(A\) and \(B\) defined by \(A = D + E\) and \(B = \rho_aE - (1 - \rho_a)D\) and that \(AB - BA = A\). Thus word made up of \(A\)'s and \(B\)'s can be ordered as a sum of terms of the form \(B^{\rho_1}A^{\rho_2}\) or \(A^{\rho_2}B^{\rho_1}\). Then the left hand side of (61) becomes

\[
\frac{(\rho_a, a | X_1X_2 | \rho_b, b)}{(\rho_a, a | \rho_b, b)} = \frac{(\rho_a, a | (D + E)^{\rho_1 + \rho_2} | \rho_b, b)}{(\rho_a, a | \rho_b, b)}
\]  
(63)

while the right hand side of (61) becomes

\[
a^{\rho_1} b^{\rho_2} \int \frac{d\rho}{2\pi i} \frac{(\rho_a - \rho)^{\rho_1 + \rho} (\rho - \rho_b)}{(\rho_a, a | \rho_b, b)} \frac{(\rho_a, a | (D + E)^{\rho_1} | \rho, b)}{(\rho_a, a | \rho, b)} \frac{(\rho, 1 - b | (D + E)^{\rho_2} | \rho_b, b)}{(\rho, 1 - b | \rho_b, b)}
\]  
(64)

and the equality of (63) and (64) follows from the expression (62) and the Cauchy theorem. This completes the derivation of (61).

First consequence of (61). It is possible to show directly from the algebra (55,57) that

\[
\frac{(\rho_a, a | D | \rho_b, b)}{(\rho_a, a | \rho_b, b)} = \frac{b\rho_a + a\rho_b}{\rho_a - \rho_b}; \quad \frac{(\rho_a, a | E | \rho_b, b)}{(\rho_a, a | \rho_b, b)} = \frac{b(1 - \rho_a) + a(1 - \rho_b)}{\rho_a - \rho_b}
\]
which becomes by replacing \( \rho_a \) by \( \rho \) and \( a \) by \( 1-b \)

\[
\frac{(\rho, 1-b)D(\rho_a, b)}{(\rho, 1-b)(\rho_b, b)} = b + \frac{\rho_b}{\rho - \rho_b} ; \quad \frac{(\rho, 1-b)E(\rho_a, b)}{(\rho, 1-b)(\rho_b, b)} = -b + \frac{1 - \rho_b}{\rho - \rho_b}
\]

Therefore (61) becomes after integration

\[
\frac{\langle \rho_{a \rho} a, X_0 \rangle D(\rho_a, b)}{(\rho_a, a)[p_b, b]} = b \left[ \frac{\langle \rho_{a \rho} a, X_0 \rangle}{(\rho_a, a)[p_b, b]} + \rho_b \frac{d}{dp} \left( \frac{\rho_a - \rho_b}{\rho_a - \rho} \right) \right]^{a+b} \left[ \frac{\langle \rho_{a \rho} a, X_0 \rangle}{(\rho_a, a)[p_b, b]} \right] \bigg|_{\rho = \rho_b} (65)
\]

\[
\frac{\langle \rho_{a \rho} a, X_0 \rangle E(\rho_a, b)}{(\rho_a, a)[p_b, b]} = -b \left[ \frac{\langle \rho_{a \rho} a, X_0 \rangle}{(\rho_a, a)[p_b, b]} + (1 - \rho_b) \frac{d}{dp} \left( \frac{\rho_a - \rho_b}{\rho_a - \rho} \right) \right]^{a+b} \left[ \frac{\langle \rho_{a \rho} a, X_0 \rangle}{(\rho_a, a)[p_b, b]} \right] \bigg|_{\rho = \rho_b} (66)
\]

These last two formulae are exact and valid for all values of \( \rho_a, \rho_b, a, b \). (They have been derived from (61) under the condition that \( \rho_a > \rho_b \) and \( 0 < b < 1 \), but as all expressions are rational functions of all their arguments, they remain valid everywhere.)

From (65) it is possible to show that \( \Phi(\mu, h) \) defined by

\[
\Phi(\mu, h) = \frac{\langle W| \exp[(e^h D+E)\mu]|V \rangle}{\langle W|V \rangle}
\]

satisfies the following equation

\[
\frac{d \Phi}{d \mu} = \frac{b(1 + \rho_a(e^h - 1)) + a(1 + \rho_b(e^h - 1))}{(1 + \rho_a(e^h - 1))} \Phi + \frac{1 + \rho_b(e^h - 1)}{1 + \rho_b(e^h - 1)} \frac{d \Phi}{d \rho_b}
\]

This equation can be solved by the method of characteristics, which tells us that the solution is of the form

\[
\Phi(\mu, h) = \frac{(\rho_a - \rho_b)e^h}{(1 + \rho_b(e^h - 1))} \mathcal{F} \left( (1 + \rho_b(e^h - 1)) \exp[\mu(e^h - 1)] \right)
\]

The fact that \( \Phi(0, h) = 1 \) determines the unknown function \( \mathcal{F} \) and one gets

\[
\Phi(\mu, h) = \left( \frac{\rho_a - \rho_b(e^h - 1)}{(1 + \rho_a(e^h - 1)) - \exp[\mu(e^h - 1)](1 + \rho_b(e^h - 1))} \right)^{a+b} \exp[\mu(e^h - 1)]
\]

(see eq (3.7-3.10) of [3]).

This expression becomes singular as \( \mu \to \mu_0 \) with

\[
\mu_0 = \frac{1}{e^h - 1} \log \left( \frac{1 + \rho_a(e^h - 1)}{1 + \rho_b(e^h - 1)} \right)
\]

and by analysing the power law singularity one can get the asymptotic expression valid for large \( L \)

\[
\frac{\langle W|\langle e^h D+E \rangle^L|V \rangle}{\langle W|V \rangle} \approx \frac{\Gamma(a + b + L)(\rho_a - \rho_b)^{a+b} \mu_0^{L-a-b}}{\Gamma(a+b)(1 + \rho_a(e^h - 1))^a(1 + \rho_b(e^h - 1))^b}
\]

Second consequence of (61):
Another important consequence which can be obtained by dividing (61) by (62) is the following additivity formula

\[
\langle \rho_a | X_1 X_2 | \rho_b, b \rangle = \Gamma(L + a + b) \Gamma(L' + 1) \int_{|\rho_b| < |\rho_a|} \frac{d\rho}{2\pi} \times \langle \rho_a | X_1 | \rho_b, b \rangle \langle \rho_a, a | (D + E)^L | \rho, b \rangle \langle \rho, 1 - b | X_2 | \rho_b, b \rangle \langle \rho, 1 - b | (D + E)^{L'} | \rho_b, b \rangle.
\]

which is the same as eq. (65) of [13] up to the prefactor which was wrong in [13] and which is corrected here. This formula allows one to compute the properties of a lattice of \(L + L'\) sites if one knows those of two systems of size \(L\) and \(L'\).

**Third consequence of (61).**

Using (65) and (62) one can write an exact recursion for \(Z_L\) defined in (2)

\[
Z_{L+1} = \left[ 1 + \rho_b e^{\psi_{L+1} + b} \frac{\rho_a - \rho_b}{L + a + b} e^{\psi_{L+1}} \right] Z_L + \frac{\rho_a - \rho_b}{L + a + b} \frac{dZ_L}{d\rho_b}.
\]

We won't use this recursion relation in this paper, but we believe that it could be an alternative starting point to recover the result (79) and possibly further corrections.
B Derivation of (51)-(53) in the equilibrium case

Let us consider a site dependent field \( h_i \) with small variations

\[ z_i = h_i - h \]

around a certain value \( h \). One can then expand \( G_L \) defined in (213) in powers of the \( z_i \)'s

\[ G_L(h_1, \cdots h_L) = G_L(h, \cdots h) + \sum_i z_i \langle n_i \rangle + \frac{1}{2} \sum_{i,j} z_i z_j \langle n_i n_j \rangle_c + O(z^3) \]

where \( \langle \rangle \) denotes an average in the constant field \( h \). Far from the boundaries i.e. when \( i \gg 1 \) and \( L-i \gg 1 \), the correlations become translational invariant (because the system is at equilibrium)

\[ \langle n_i \rangle = g'(h) ; \quad \langle n_i n_j \rangle_c = c_{j-i}(h) \]

(72)

and

\[ g''(h) = \sum_i c_k(h) \].

One can rewrite \( G_L(h_1, \cdots h_L) \) as

\[ G_L(h_1, \cdots h_L) = G_L(h, \cdots h) + g'(h) \sum_i z_i + \frac{c_0(h)}{2} \sum_i z_i^2 + \sum_{k \geq 1} c_k(h) \sum_{i=1}^{L-k} z_i z_{i+k} + \sum_i z_i \left( \langle n_i \rangle - g'(h) \right) + \frac{1}{2} \sum_{i,j} z_i z_j \left( \langle n_i n_j \rangle_c - c_{j-i}(h) \right) + O(z^3) \]

(73)

and using the fact (which follows from (73) by looking at the term proportional to \( L \) when all the \( h_i \)'s are equal) that

\[ g(h_i) = g(h) + z_i g'(h) + \frac{z_i^2}{2} \sum_{k=-\infty}^\infty c_k(h) + O(z^3) \]

one gets

\[ G_L(h_1, \cdots h_L) - \sum_i g(h_i) = G_L(h, \cdots h) - L g(h) \]

(74)

\[ + \sum_{k \geq 1} c_k(h) \sum_{i=1}^{L-k} \left( z_i z_{i+k} - \frac{z_i^2 + z_{i+k}^2}{2} \right) - \frac{1}{2} \sum_{k \geq 1} \sum_{i \in L-k+1} z_i^2 \]

\[ + \sum_i z_i \left( \langle n_i \rangle - g'(h) \right) + \frac{1}{2} \sum_{i,j} z_i z_j \left( \langle n_i n_j \rangle_c - c_{j-i}(h) \right) + O(z^3) \].

In the large \( L \) limit, the correlation functions, near the boundaries, have a limit which is not translational invariant

\[ \langle n_i \rangle - g'(h) \rightarrow d_i^\text{left}(h) ; \quad \langle n_{L-i} \rangle - g'(h) \rightarrow d_i^\text{right}(h) \]

whereas

\[ \langle n_i n_j \rangle_c - c_{j-i}(h) \rightarrow b_{i,j}^\text{left}(h) \quad \langle n_{L-i} n_{L-j} \rangle_c - c_{j-i}(h) \rightarrow b_{i,j}^\text{right}(h) \]

One then should have

\[ \frac{dA^\text{left}(h)}{dh} = \sum_{i=1}^\infty d_i^\text{left}(h) ; \quad \frac{dA^\text{right}(h)}{dh} = \sum_{i=0}^\infty d_i^\text{right}(h) \]

(75)
\[
\frac{d^2 A_1(h)}{dh^2} = \sum_{i,j \geq 1} b_{i,j}(h) - \sum_{k \geq 1} k c_k(h) \quad : \quad \frac{d^2 A_2(h)}{dh^2} = \sum_{i \geq 0, j \geq 0} b_{i,j}(h) - \sum_{k \geq 1} k c_k(h)
\]

so that using (45) and the fact that \( c_k(h) = c_{-k}(h) \)

\[
\frac{d^2 A_1(h)}{dh^2} = \sum_{i,j \geq 1} b_{i,j}(h) - \sum_{k \geq 1} k c_k(h) \quad : \quad \frac{d^2 A_2(h)}{dh^2} = \sum_{i \geq 0, j \geq 0} b_{i,j}(h) - \sum_{k \geq 1} k c_k(h)
\]

For large \( L \) this becomes

\[
G_L(h_1, \cdots, h_L) = A_1(h_1) + A_2(h_L)
\]

\[
+ \sum_{i,j \geq 1} c_i(h) \left( \sum_{h_{i+j} = 0} \left( \frac{h_{i+j}}{2} \right)^2 - \frac{1}{2} \sum_{i=1}^k \frac{h_{i+j}}{2} \sum_{j=1}^k \frac{h_{j+i}}{2} \right)
\]

\[
+ \sum_{i,j \geq 1} c_i(h) \left( n_i - g'(h) \right) + \frac{1}{2} \sum_{i,j \geq 1} c_i(h) \left( n_{i+j} - c_{i,j}(h) \right) + O(z^3)
\]

which can be rewritten, up to terms of third order in the \( z \)'s

\[
G_L(h_1, \cdots, h_L) - \sum_{i \geq 1} g(h_i) = D^{left} + D^{right} - \sum_{k \geq 1} c_k(h_1) \left( \sum_{i=1}^{k+1} \left( h_i - h_{i+1} \right)^2 \right)
\]

where

\[
D^{left} = A_1(h_1) - \sum_{k \geq 1} c_k(h_1) \left( \sum_{i=1}^{k} \left( h_i - h_{i+1} \right)^2 \right) + \sum_{i \geq 1} \left( h_i - h_1 \right) a^{left}_i(h_1)
\]

\[
+ \frac{1}{2} \sum_{i,j \geq 1} \left( h_i - h_1 \right) \left( h_j - h_1 \right) b^{left}_{i,j}(h_1) + O(z^3)
\]

and

\[
D^{right} = A_2(h_L) - \sum_{k \geq 1} c_k(h_L) \left( \sum_{i=1}^{k} \left( h_i - h_{i+1} \right)^2 \right) + \sum_{i \geq 1} \left( h_i - h_L \right) a^{right}_i(h_L)
\]

\[
+ \frac{1}{2} \sum_{i,j \geq 1} \left( h_i - h_L \right) \left( h_j - h_L \right) b^{right}_{i,j}(h_L) + O(z^3)
\]

All the differences \( h_i - h_j \) which appear in (77) are between nearby sites \( i, j \). Under this form, the differences \( h_i - h_j \) between remote sites do not need to be small. In what follows we will assume that (77) remains true as long as these differences \( h_i - h_j \) remain small for nearby sites (i.e. for \( |i - j| \ll \lambda \)) even if these differences could be large when \( |i - j| \sim \lambda \).

Now for a slowly varying field of the form (46), with \( L = \gamma \lambda \) as in (45) one can evaluate the different terms using the Euler Mac Laurin formula

\[
\int \frac{g(x)}{H(x)} dx - \frac{H'(y)g(H(y)) - H'(0)g(H(0))}{24 \lambda}
\]

\[
D^{left} \approx A_1(h_0) + \frac{H'(0)}{\lambda} \sum_{i \geq 1} \left( i - \frac{1}{2} \right) a^{left}_i(h_0)
\]

\[
D^{right} \approx A_2(h_0) - \frac{H'(y)}{\lambda} \sum_{i \geq 1} \left( i + \frac{1}{2} \right) a^{right}_i(h_Y)
\]

\[
\sum_{k \geq 1} c_k(h_1) \left( \sum_{i=1}^{k} \left( h_i - h_{i+1} \right)^2 \right) \approx \frac{1}{2 \lambda} \int_0^h \sum_{k \geq 1} k^2 c_k(h(x)) H'(x)^2 dx
\]
References

1. C. Kipnis, S. Olla, S. R. S. Varadhan, Hydrodynamics and large deviation for simple exclusion processes Communications on Pure and Applied Mathematics 2, 115-137 (1989)
2. L. Bertini, A. De Sole, D. Gabrielli, G. Jona–Lasinio, C. Landim, Fluctuations in stationary non equilibrium states of irreversible processes, Phys. Rev. Lett. 87 040601 (2001)
3. L. Bertini, A. De Sole, D. Gabrielli, G. Jona–Lasinio, C. Landim, Macroscopic fluctuation theory for stationary non equilibrium states, J. Stat. Phys. 107, 635-675 (2002)
4. L. Bertini, A. De Sole, D. Gabrielli, G. Jona–Lasinio, C. Landim, Large deviations for the boundary driven symmetric simple exclusion process, Math. Phys. Analysis and Geometry 6, 231-267 (2003)
5. L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, C. Landim, Minimum dissipation principle in stationary non equilibrium states, J. Stat. Phys. 116 831-841 (2004)
6. L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, C. Landim, Stochastic interacting particle systems out of equilibrium J. Stat. Mech. P07014 (2007)
7. L. Bertini, D. Gabrielli, G. Jona-Lasinio, C. Landim, Thermodynamic transformations of nonequilibrium states J. Stat. Phys. 149, 773-802 (2012)
8. B. Derrida, J. L. Lebowitz, E. R. Speer, Free energy functional for nonequilibrium systems: an exactly solvable case, Phys. Rev. Lett. 87 150601 (2001)
9. B. Derrida, J.L. Lebowitz, E.R. Speer, Large deviation of the density profile in the steady state of the symmetric simple exclusion process J. Stat. Phys. 107, 599-634 (2002)
10. B. Derrida, J.L. Lebowitz, E.R. Speer, Exact free energy functional for a driven diffusive open stationary nonequilibrium system Phys. Rev. Lett. 89 030601 (2002)
11. B. Derrida, J.L. Lebowitz, E.R. Speer, Exact large deviation functional of a stationary open driven diffusive system: the asymmetric exclusion process J. Stat. Phys. 110, 775-810 (2003)
12. C. Enaud, B. Derrida, Large deviation functional of the weakly asymmetric exclusion process J. Stat. Phys. 114, 537-562 (2004)
13. B. Derrida, Non-equilibrium steady states: fluctuations and large deviations of the density and of the current J. Stat. Mech. P07023 (2007)
14. W. E, W. Ren and E. Vanden-Eijnden, Minimum action method for the study of rare events Commun. Pure Appl. Math. 57, 637-656 (2004)
15. L. Bertini, D. Gabrielli, J. Lebowitz, Large deviation for a stochastic model of heat flow, J. Stat. Phys. 121, 843-885 (2005)
16. F. van Wijland and Z.Racz, Large deviations in weakly interacting boundary driven lattice gases J. Stat. Phys. 118, 27-54 (2005)
17. C. Giardinà, J. Kurchan, and L. Peliti, Direct Evaluation of Large-Deviation Functions Phys. Rev. Lett. 96, 120603 (2006)
18. V. Lecomte and J. Tailleur, A numerical approach to large deviations in continuous time J. Stat. Mech. P03004 (2007)
19. C. Giardinà, J. Kurchan, V. Lecomte, J. Tailleur, Simulating rare events in dynamical processes J. Stat. Phys. 45, 787-811 (2011)
20. H. Touchette, The large deviation approach to statistical mechanics Physics Reports 478, 1-69 (2009)
21. G. Bunin, Y. Kafri, D. Podolsky, Large deviations in boundary-driven systems: Numerical evaluation and effective large-scale behavior Europhys. Lett. 99, 20002 (2012)
22. T. R. Kirkpatrick, E. G. D. Cohen, J. R. Dorfman, Fluctuations in a nonequilibrium steady state: Basic equations Phys. Rev. A 26, 950-971 (1982)
23. H. Spohn, Long range correlations for stochastic lattice gases in a non-equilibrium steady state. J. Phys. A 16, 4275-4291 (1983)
24. R. Schmitz, E.G.D. Cohen, Fluctuations in a ??uid under a stationary heat-??ux. 1. General theory. J. Stat. Phys. 39, 285-316 (1985)
25. J.R. Dorfman, T.R. Kirkpatrick, J.V. Sengers, Generic long-range correlations in molecular ??uids Annu. Rev. Phys. Chem. 45, 213-239 (1994)
26. J.M. Ortiz de Zerate, J.V. Sengers, On the physical origin of long-ranged ??uctuations in ??uids in thermal nonequilibrium states J. Stat. Phys. 115, 1341-1359 (2004)
27. B. Derrida, J.L. Lebowitz, E.R. Speer, Entropy of open lattice systems J. Stat. Phys. 126, 1083-1108 (2007)
28. T. Bodineau, B. Derrida, V. Lecomte, F. van Wijland, Long range correlations and phase transition in non-equilibrium diffusive systems J. Stat. Phys. 133, 1013-1031 (2008)
29. L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, C. Landim, On the long range correlations of thermodynamic systems out of equilibrium. cond-mat/0705.2996 (2007)
30. B. Derrida, M.R. Evans, V. Hakim, V. Pasquier, Exact solution of a 1d asymmetric exclusion model using a matrix formulation J. Phys. A 26, 1493-1517 (1993)
31. R.A. Blythe, M.R. Evans, Nonequilibrium steady states of matrix-product form: a solver’s guide J. Phys A 40, R333-R441 (2007)
32. L. Bertini, A. D. Gabrielli, G. Jona–Lasinio, C. Landim, Thermodynamic Transformations of Nonequilibrium States J. Stat. Phys. 149, 773-802 (2012)
33. F. Spitzer, Interaction of Markov Processes, Advances in Mathematics 5 246-290 (1970)
34. G. Eyink, J. L. Lebowitz, and H. Spohn, Hydrodynamics of stationary non-equilibrium states for some stochastic lattice gas models Comm. Math. Phys. 132, 253-283 (1990)
35. G. Eyink, J. L. Lebowitz, and H. Spohn, Lattice gas models in contact with stochastic reservoirs: local equilibrium and relaxation to the steady state Comm. Math. Phys. 140, 119 (1991)
36. H. Spohn, Large Scale Dynamics of Interacting Particles 1991 Springer, Berlin
37. T.M. Liggett T.M., Stochastic Interacting Systems: Contact, Voter and Exclusion Processes. Grundlehren der mathematischen Wissenschaften 324, Springer, 1999
38. C. Kipnis, C. Landim, Scaling limits of interacting particle systems, Springer Berlin, 1999
39. C. Appert-Rolland, B. Derrida, V. Lecomte, F. van Vlaid, Universal cumulants of the current in diffusive systems on a ring Phys. Rev. E 78, 021122 (2008)
40. A. Imparato, V. Lecomte, F. Van Vlaid, Equilibrium-like fluctuations in some boundary-driven open diffusive systems Phys. Rev. E 80 011131 (2009)
41. G. Carinci, C. Giardinà, C. Giberti, F. Redig, Duality for stochastic models of transport [arXiv:1212.3154]
42. L. Bertini, A. De Sole, D. Gabrielli, G. Jona–Lasinio, C. Landim, Lagrangian phase transitions in nonequilibrium thermodynamic systems J. Stat. Mech. L11001 (2010)
43. G. Bunin, Y. Kafri, D. Podolsky, Cusp singularities in boundary-driven diffusive systems [arXiv:1301.1708]
44. G. Bunin, Y. Kafri, D. Podolsky, Non differentiable large-deviation functionals in boundary-driven diffusive systems J. Stat. Mech. L10001 (20012)
45. J. Tailleur, J. Kurchan, V. Lecomte, Mapping out-of-equilibrium into equilibrium in one-dimensional transport models J. Phys. A and Theoretical 41 505001 (2008)