ON THE SCATTERING LENGTH OF THE $K^-d$ SYSTEM

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Multiple-scattering approximations to Faddeev calculations of the $K^-d$ scattering length are reviewed and compared with published $\bar{K}N N - \pi YN$ fully reactive Faddeev calculations. A new multiple-scattering approximation which goes beyond the ‘fixed-center’ assumption for the nucleons is proposed, aiming at accuracies of $5 - 10\%$. A precise value of the $K^-d$ scattering length from the measurement of the $K^-d 1s$ atomic level shift and width, planned by the DEAR/SIDDHARTA collaboration, plus a precise value for the $K^-p$ scattering length by improving the $K^-p$ atom measurements, are essential for extracting the $K^-n$ scattering length, for resolving persistent puzzles in low-energy $\bar{K}N$ phenomenology and for extrapolating into $\bar{K}$-nuclear systems.

Keywords: $K^-d$ atom; $K^-d$ Faddeev calculations; multiple-scattering approximations.

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1. Introduction

The $\bar{K}N$ interaction near threshold is known, since the pioneering works by Dalitz and collaborators in the 1960s, to be strongly attractive as well as strongly absorptive. The experimental data which traditionally have been used to constrain it consist of elastic, charge-exchange and $K^-p \rightarrow \pi Y$ reaction cross sections at low energies, as low as $p_{lab} = 100$ MeV/c, and of the $\Lambda(1405)$ resonance shape extracted from $\pi \Sigma$ final-state interaction below threshold. In addition, one has also three accurately determined branching ratios for $K^-$ absorption from rest. A crucial experimental datum near threshold, the (complex) $K^-p$ scattering length, is progressively becoming accurately determined in recent years from measurements of the energy shift and width of the $K^-p$ atomic 1s state. The scattering length $a_{K^-p}$ may be expressed in terms of the $\bar{K}N I = 0, 1$ scattering lengths $a_0$ and $a_1$, respectively, as follows:

$$a_{K^-p} = \frac{\frac{1}{2}(a_0 + a_1) + k_0a_0a_1}{1 + k_0\frac{1}{2}(a_0 + a_1)},$$

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where $k_0$ is the $\bar{K}^0 n$ cm momentum with respect to the $K^- p$ threshold. Using the NLO corrected Deser formula proposed recently by Meißner et al. \cite{5}

$$\epsilon_{1s} - i\frac{\Gamma_{1s}}{2} = -2\alpha^3 \mu_{K^- p}^2 a_{K^- p}(1 - 2\alpha\mu_{K^- p}(\ln \alpha - 1)a_{K^- p}), \tag{2}$$

the value of $a_{K^- p}$ derived from the DEAR measurement \cite{3}

$$a_{K^- p} = (-0.45 \pm 0.090) + i(0.27 \pm 0.12) \text{ fm} \tag{3}$$

appears inconsistent with comprehensive fits to low-energy $K^- p$ scattering and reaction data, as discussed recently in Refs. \cite{6,7} and as shown pictorially in Fig. 1 taken from Ref. \cite{8}. Furthermore, such fits leave the $K^- n$ scattering length $a_{K^- n}$ poorly determined.

A precise measurement of the energy shift and width of the $K^- d$ atomic $1s$ state is called for, in order to determine $a_{K^- d}$, and this indeed is one of the prime aims of the DEAR/SIDDHARTA experimental collaboration (see e.g. Johann Zmeskal’s talk in these Proceedings). A theoretically meaningful extraction of $a_{K^- n}$ from $a_{K^- p}$ and $a_{K^- d}$ would require a relatively handy and accurate multiple-scattering (MS) approximation to the more involved coupled-channel Faddeev calculation of

![Fig. 1. Excluded and allowed regions for $a_0, a_1$, from Meißner et al. Ref. 8, using the DEAR value for $a_{K^- p}$ in Eq. 1. The values shown for $a_0$ from several calculations indicate a problem.](image-url)
the $K^-d$ system at low energy. Several Faddeev calculations of this sort have been reported. However, although the subject of MS approximations to the $K^-d$ scattering length has been discussed extensively in recent years, only one of these works produced a correct expression within the commonly used ‘fixed-center’ assumption for the nucleons, with a follow up recently in Ref. 8.

The main purpose of this work is to review the existing MS approximations and to propose a new MS approximation for $a_{K^-d}$. The proposed version goes beyond the ‘fixed-center’ assumption. This should enable one to make a reasonable prediction for the $K^-d$ atom 1s level-shift and width on the basis of given values of $a_{K^-p}$ and $a_{K^-n}$ without having to perform new time-consuming Faddeev calculations or, vice-versa, to derive fairly reliably $a_{K^-n}$ from the joint measurement of the 1s level-shift and width in $K^-p$ and $K^-d$ atoms.

2. Multiple-Scattering Approximations

It was shown in the first fully reactive $KNN-\pi YN$ ($Y = \Lambda, \Sigma$) Faddeev calculation that $a_{K^-d}$ may be approximated to a few percent by neglecting the hyperonic channels altogether, in accordance with the conjecture by Schick and Gibson, provided the input to the $KNN$ Faddeev equations consists of $K\bar{N}$ matrices which implicitly account for the $K\bar{N}-\pi Y$ coupling. The Faddeev equations for the $K^-pn$ 3-body system, with initial $K^-d$ configuration, assume the form

$$T_1 = t_1 + t_1 G_0 (T_2 + T_3), \quad T_2 = t_2 + t_2 G_0 (T_3 + T_1), \quad T_3 = t_3 G_0 (T_1 + T_2), \quad (4)$$

where $G_0$ is the free Green’s function and the full $T$ matrix is given by its breakdown into partial $T$ matrices: $T = T_1 + T_2 + T_3$. The partial $T_1$ and $T_2$, upon a suitable choice of channels, stand for all the MS terms that start with a $K^-p$ collision (2-body $t_1$ matrix) and with a $K^-n$ collision (2-body $t_2$ matrix), respectively, whereas $T_3$ consists of the terms that start with $pn$ collisions (2-body $t_3$ matrix). The choice of $K^-d$ initial state dictates that the last of Eqs. 4 does not have an inhomogeneous term $t_3$. No antisymmetrization is explicitly applied yet for the $NN$ subsystem. The $T$ matrix which reduces in the limit of zero kinetic energy to the $K^-d$ scattering length, up to a kinematical factor, is $T_{K^-d} = T_1 + T_2$. We now outline several MS approximations, progressively by order of complexity, beginning with the Brueckner formula which was motivated by $\pi d$ scattering for static nucleons. Charge-exchange degrees of freedom are then introduced and, wherever charge independence holds, the Brueckner formula is generalized in terms of isoscalar and isovector meson-nuclear scattering lengths.

2.1. Brueckner formula

The Brueckner formula is obtained by neglecting $t_3$ and hence also $T_3$ in Eqs. 4,

$$T_1 = t_1 + t_1 G_0 T_2, \quad T_2 = t_2 + t_2 G_0 T_1, \quad (5)$$
and treating the nucleons as fixed centers. Furthermore, mostly for the sake of illustration, the charge-exchange two-body channel $K^- p \rightarrow \bar{K}^0 n$ is switched off, leading to the following expression:

$$T_{K^-d} = T_1 + T_2 = \frac{1}{1 - t_1 G_0 t_2 G_0} (t_1 + t_1 G_0 t_2) + \frac{1}{1 - t_2 G_0 t_1 G_0} (t_2 + t_2 G_0 t_1) . \quad (6)$$

For zero-range interactions, and considering the necessary kinematical factors to transform from $t$ matrices to scattering lengths $a$, it is straightforward to obtain from Eq. (6) the Brueckner formula:

$$a_{K^-d} = \left( 1 + \frac{m_K}{m_d} \right)^{-1} \int a_{K^-d}(r) |\psi_d(r)|^2 dr , \quad (7)$$

$$a_{K^-d}(r) = \frac{\tilde{a}_p + \bar{a}_n + 2\tilde{a}_p \bar{a}_n/r}{1 - \tilde{a}_p \bar{a}_n/r^2} , \quad (8)$$

where $\tilde{a} = (1 + m_K/m_N)a$, with $a$ standing generically for $a_{K^-p} = \tilde{a}_p$ and for $a_{K^-n} = \bar{a}_n$ in the $K^-N$ cm system. The numerator in the Brueckner formula consists of single- and double-scattering terms, whereas the denominator provides for the renormalization of these terms to include all the higher-order scattering terms. The relevant expansion parameter is $\tilde{a} < 1/r >_d$, where $<1/r>_d \approx 0.5 \text{ fm}^{-1}$ for the deuteron wavefunction. Hence, this series faces divergence once the scattering length is of order $a \sim 1 \text{ fm}$ or more, as we encounter for the $I = 0 \bar{K}N$ channel. In contrast, for low-energy pions $a \sim 0.1 \text{ fm}$ and the single- and double-scattering terms (augmented by charge-exchange scattering as done below) provide an excellent approximation to the Brueckner formula for the $\pi^- d$ scattering length. The generalization of the Brueckner formula to include the charge exchange reaction on the proton (on-shell for $\pi^-$, off-shell for $K^-$) is given below.

### 2.2. Including charge exchange in the Brueckner formula

The inclusion of charge-exchange degrees of freedom for $K^- d$ scattering at threshold is due to Ref. [15]

$$a_{K^-d}(r) = \frac{\tilde{a}_p + \bar{a}_n + (2\tilde{a}_p \bar{a}_n - b_x^2)/r - 2b_x^2 \bar{a}_n/r^2}{1 - \tilde{a}_p \bar{a}_n/r^2 + b_x^2 \bar{a}_n/r^3} , \quad (9)$$

where $b_x^2 = \tilde{a}_x^2/(1 + \tilde{a}_u/r)$, with $a_x$ and $a_u$ the threshold scattering amplitudes for $K^- p \rightarrow \bar{K}^0 n$ and $\bar{K}^0 n \rightarrow K^0 n$ respectively. The derivation of Eq. (9) shows that it holds also for $a_{\pi^-d}$, provided that the replacement $(K^-, \bar{K}^0) \rightarrow (\pi^-, \pi^0)$ is made in the $\bar{K}N$ amplitudes. The charge-exchange scattering contribution to $a_{\pi^-d}$ as well as to $a_{K^-d}$ is substantial. When charge independence is assumed, all four $\bar{K}N$ amplitudes change everywhere $m_K \rightarrow m_x$.\(^\text{a}\)

\(^{a}\)for pions change everywhere $m_K \rightarrow m_x$.\(^\text{a}\)
amplitudes in Eq. (3) are given in terms of the isoscalar and isovector threshold amplitudes $b_0, b_1$ which are related to the isospin scattering lengths $a_0, a_1$ by

$$b_0 + b_1\tau_K \cdot \tau_N : \quad b_0 = \frac{1}{4}(3a_1 + a_0), \quad b_1 = \frac{1}{4}(a_1 - a_0). \quad (10)$$

Eq. (9) simplifies then to

$$a_{K-d}(r) = \frac{2\tilde{b}_0 - 2(b_0 + \tilde{b}_1)(\tilde{b}_1 - \tilde{b}_0)}{2 + (\tilde{b}_0 + b_1)(\tilde{b}_1 - \tilde{b}_0)} = \frac{2(3\tilde{a}_1 + \tilde{a}_0) + 2\tilde{a}_0\tilde{a}_1}{1 - \frac{1}{2}(\tilde{a}_1 - \tilde{a}_0)^2 - \frac{2}{3}\tilde{a}_0\tilde{a}_1/\tau}, \quad (11)$$

in close analogy to Deloff’s expression\(22\) for $\pi^{-}d$ scattering at threshold:

$$a_{\pi-d}(r) = \frac{2\tilde{b}_0 - 2(b_0 + \tilde{b}_1)(\tilde{b}_1 - \tilde{b}_0)}{1 - b_1^2 + (b_0 + b_1)(\tilde{b}_1 - \tilde{b}_0)} = \frac{2(2\tilde{a}_1 + \tilde{a}_0) + 2\tilde{a}_0\tilde{a}_1}{1 - 3\tilde{a}_1 - \tilde{a}_0/\tau}, \quad (12)$$

where the isoscalar and isovector $\pi N$ threshold amplitudes are now given in terms of the isospin scattering lengths $a_{1/2}, a_{3/2}$ by

$$b_0 + b_1\tau_{\pi} \cdot \tau_N : \quad b_0 = \frac{1}{3}(2a_{3/2} + a_{1/2}), \quad b_1 = \frac{1}{3}(a_{3/2} - a_{1/2}). \quad (13)$$

Eqs. (11, 12) may be obtained directly from Eq. (9) treating it as an operator relationship in isospace.

| Ref.  | basis   | $a_{K-p}$ | $a_{K-n}$ | Fad $a_{K-d}$ | MS $a_{K-d}$ |
|-------|---------|-----------|-----------|---------------|-------------|
| TGE\(9\) | isospin | -0.655 + i 0.705 | 0.350 + i 0.660 | -1.67 + i 1.00 | -1.46 + i 1.09 |
| TDD\(10\) | isospin | -0.645 + i 0.725 | 0.320 + i 0.700 | -1.34 + i 1.04 | -1.42 + i 1.09 |
| BFM\(11\) | particle | -0.888 + i 0.867 | 0.544 + i 0.644 | -1.80 + i 1.55 | -1.73 + i 1.06 |

In Table 1 we show the quality of the ‘fixed center’ MS approximation. Eqs. (11, 12) were used to compare with values for $a_{K-d}$ reported in Faddeev calculations done in the isospin basis, with the further approximation of replacing $r^{-n}$ by the TPE matrix elements $<r^{-n}>$ for $n = 1, 2$ from Table 1 in Ref. 25. The values of $a_p$ and $a_n$ listed in the table are sufficient to determine the input scattering lengths $a_0$ and $a_1$ for Eq. (11). This version of MS approximation reproduces the real part of $a_{K-d}$ to better than about 15% and the imaginary part to within 5 – 10%. For the particle-basis Faddeev calculation of Ref. 12 Eqs. (11, 12) were used. The extra scattering lengths, beyond $a_p$ and $a_n$, are not listed here. The renormalized TPE value for $<r^{-3}>$ was used\(22\). The resulting value of $a_{K-d}$ is sensitive to the value

\(b\) the isospin basis may still be used when charge independence is only broken by different $KN$ thresholds\(3\).
used for this matrix element which diverges unless renormalized; scaling it down from \( <r^{-2}> \) by the ratio of \( <r^{-2}> / <r^{-1}> \), we get \( a_{K-d} = -1.96 + i1.37 \) fm, in better agreement with the Faddeev value \( -1.80 + i1.55 \) fm \cite{12}.

In Table 2 we list two MS evaluations of \( a_{K-d} \) using a deuteron wavefunction based on the Paris potential and \( \bar{K}N \) input which is similar to that used in the TDD\cite{10} Faddeev calculation. The accuracy of these MS evaluations is typically 25%. Also listed is our own MS approximation (from Table 1) which, perhaps fortuitously, works to about 5%.

| Faddeev \cite{10} | MS \cite{15} | MS \cite{8} | present MS |
|-------------------|-------------|-------------|------------|
| \(-1.34 + i1.04\) | \(-1.54 + i1.29\) | \(-1.66 + i1.28\) | \(-1.42 + i1.09\) |

2.3. Beyond ‘fixed centers’

Under the ‘fixed center’ assumption we have suppressed the terms \( t_1 G_0 T_3 \) and \( t_2 G_0 T_3 \) in the coupled Faddeev equations for \( T_1 \) and \( T_2 \), respectively, in Eq. \cite{4}. These terms may be rewritten in the form

\[
t_1 G_0 T_3 = t_1 G_0 t_3 G_0 (T_1 + T_2) = t_1 (\Delta G_0) (T_1 + T_2),
\]

(14)

\[
t_2 G_0 T_3 = t_2 G_0 t_3 G_0 (T_1 + T_2) = t_2 (\Delta G_0) (T_1 + T_2),
\]

(15)

where \( \Delta G_0 = G_3 - G_0 = G_0 t_3 G_0 \), so that \( G_3 \) is a Green’s function which takes full account of the \( NN \) interaction but is still free with respect to the meson-nucleon interactions. This leads to the following coupled equations for \( T_1 \) and \( T_2 \):

\[
(1 - t_1 (\Delta G_0)) T_1 = t_1 + t_1 G_3 T_2, \quad (1 - t_2 (\Delta G_0)) T_2 = t_2 + t_2 G_3 T_1,
\]

(16)

which are in the form of the ‘fixed center’ Faddeev equations \cite{5} with \( G_0 \) replaced by \( G_3 \) and with renormalized \( t_1 \) and \( t_2 \):

\[
t_j' = (1 - t_j (\Delta G_0))^{-1} t_j = t_j + t_j (\Delta G_0) t_j + \cdots, \quad j = 1, 2.
\]

(17)

We thus obtain the following improvement over the Brueckner formula Eq. \cite{5}:

\[
T_{K-d} = T_1 + T_2 = \frac{1}{1 - t_1' G_3 t_2'} \left( t_1' + t_1' G_3 t_2' \right) + \frac{1}{1 - t_2' G_3 t_1'} \left( t_2' + t_2' G_3 t_1' \right).
\]

(18)

This expression for \( T_{K-d} \) provides as transparent and applicable MS expansion, with \( t_j \rightarrow t_j' \), \( j = 1, 2 \), and \( G_0 \rightarrow G_3 \), as the Brueckner formula MS expansion is. Charge exchange degrees of freedom can be introduced in a straightforward manner, as done in subsection \cite{22}. It is conceivable that one can reach in this way a level of accuracy of \( 5 - 10\% \) for approximating fully-reactive Faddeev calculations.
3. Discussion and conclusions

In this talk I have surveyed schematically the derivation of MS approximations to the few available fully-reactive Faddeev calculations for $a_{K-d}$ under the generally accepted practice of treating the nucleons as fixed centers. These ‘fixed center’ MS approximations, which provide an excellent approximation scheme already at the double-scattering order for $a_{\pi-d}$, require the full summation of the MS series for $a_{K-d}$, using Eq. (9) in the particle basis or Eq. (11) in the isospin basis. The accuracy thus provided is in the range of 10 – 25%. It should be stressed that Effective Field Theory approaches do not yet offer any alternative scheme at present. I have subsequently outlined a theoretical MS approach for going beyond the ‘fixed center’ assumption while keeping the formal appearance of the ‘fixed center’ MS formulae, at the expense of renormalizing the input $\bar{K}N$ scattering amplitude and the free Green’s function which for fixed centers gives rise to the $r^{-n}$ dependence of the MS series terms. It is conceivable that one can reach in this way a level of accuracy of 5 – 10% for approximating fully-reactive Faddeev calculations. The aim of this project is to free oneself of depending on available Faddeev calculations, which might become obsolete if the DEAR measurement of the $K^{-p}$ atomic 1s level shift and width is confirmed and the error bars further reduced, since all the published Faddeev calculations of $a_{K-d}$ use considerably larger values for the $a_{K-p}$ input.

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