Quasi-normal modes for Dirac fields in Kerr-Newman-de Sitter black holes

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Abstract

We provide the full asymptotic description of the quasi-normal modes (resonances) in any strip of fixed width for Dirac fields in slowly rotating Kerr-Newman-de Sitter black holes. The resonances split in a way similar to the Zeeman effect. The method is based on the extension to Dirac operators of techniques applied by Dyatlov in [17], [18] to the (uncharged) Kerr-de Sitter black holes. We show that the mass of the Dirac field does not have effect on the two leading terms in the expansions of resonances.

Keywords: Resonances, quasi-normal modes, Dirac equation, Kerr-Newman-de Sitter black holes.

1 Introduction.

1.1 Background.

Kerr–Newman-de Sitter (KN-dS) black hole is an exact solution of the Einstein-Maxwell equations which describe electrically charged rotating black hole with positive cosmological constant.

We refer to [14] for detailed study in this background, including complete time-dependent scattering theory.

In Boyer-Lindquist coordinates, the exterior of a KN-dS black hole is described by the four-dimensional manifold

\[ \mathbb{R}_t \times M, \quad M = \{ r_-, r_+ \} \times S_\vartheta^2, \]

equipped with the Lorentzian metric (see Eq. (1.2) in [14])

\[ g = \frac{\Delta_r}{\rho^2} \left[ dt - \frac{a \sin^2 \vartheta}{E} d\varphi \right]^2 - \frac{\rho^2}{\Delta_r} dr^2 - \frac{\rho^2}{\Delta_\vartheta} d\vartheta^2 - \frac{\Delta_\vartheta \sin^2 \vartheta}{\rho^2} \left[ adt - \frac{r^2 + a^2}{E} \right]^2, \quad (1.1) \]

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where
\[ \rho^2 = r^2 + a^2 \cos^2 \vartheta, \quad E = 1 + \frac{a^2 \Lambda}{3}, \]
\[ \Delta_r = (r^2 + a^2) \left(1 - \frac{\Lambda r^2}{3}\right) - 2Mr + Q^2, \quad \Delta_\varphi = 1 + \frac{a^2 \Lambda \cos^2 \vartheta}{3}. \]

Here the parameters \( M > 0, Q \in \mathbb{R} \) and \( a \in \mathbb{R} \) are interpreted as the mass, the electric charge and the angular momentum per unit mass of the black hole, and \( \Lambda > 0 \) is the cosmological constant of the universe. In the case of KN-dS black hole the function \( \Delta_r \) has three simple positive roots \( 0 < r_c < r_- < r_+ \) and one negative root \( r_n = -(r_c + r_- + r_+) < 0 \), under the condition that (see [14])
\[ \frac{a^2 \Lambda}{3} \leq 7 - 4\sqrt{3} \approx 0.072, \quad M_{\text{crit}}^- < M < M_{\text{crit}}^+, \quad (1.2) \]
where
\[ M_{\text{crit}}^\pm = \frac{1}{\sqrt{18\Lambda}} \left(E_- \pm \sqrt{E_-^2 - F}\right) \left(2E_-^2 \mp \sqrt{E_-^2 - F}\right), \]
\[ E_- = 1 - \frac{a^2 \Lambda}{3}, \quad F = 4\Lambda(a^2 + Q^2). \]

In this paper we always assume conditions (1.2) to be fulfilled. By introduction of the Regge-Wheeler coordinate \( x \) via
\[ \frac{dx}{dr} = \frac{r^2 + a^2}{\Delta_r}, \quad (1.3) \]
the event and cosmological horizons are pushed away to \( \{x = -\infty\} \) and \( \{x = +\infty\} \) respectively. In this article we consider the exterior region of the black hole
\[ r_- < r < r_+ \Leftrightarrow -\infty < x < \infty. \quad (1.4) \]

We consider propagation of the Dirac fields with charge \( q \) and mass \( m \) in the exterior region of the black hole.

Let
\[ a(x) = \frac{\sqrt{\Delta_r}}{r^2 + a^2}, \quad c(x, D_\varphi) = \frac{aE}{r^2 + a^2} D_\varphi + \frac{qQr}{r^2 + a^2}, \quad b(x) = m \frac{r \sqrt{\Delta_r}}{r^2 + a^2}. \quad (1.5) \]

Note that in the non-rotating case \( a = 0 \) (de Sitter-Reissner-Nordström black hole, the massless and chargeless case was studied in [26]), \( \Delta_r = r^2 F(r) \), where \( F(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3} r^2 \) and \( a^2(x) = F(r)/r^2 = \Delta_r/r^4 \).

Let \( \Gamma^j, j = 0, 1, 2, 3 \), be any 4 by 4 Dirac matrices satisfying \( \Gamma^i \Gamma^j + \Gamma^j \Gamma^i = 2\delta_{ij} I_4 \), and
\[ \Gamma^5 = I^0 \Gamma^1 \Gamma^2 \Gamma^3. \]

Then \( \Gamma^5 \) anti-commutes with \( \Gamma^j, j = 0, 1, 2, 3 \). Let
\[ J = I_4 + \alpha(x, \vartheta) \Gamma^3, \quad \alpha(x, \vartheta) = \frac{\sqrt{\Delta_r}}{\sqrt{\Delta_\varphi}} \frac{a \sin \vartheta}{r^2 + a^2} = a(x) b(\vartheta), \quad b(\vartheta) = \frac{a \sin \vartheta}{\sqrt{\Delta_\varphi}}. \quad (1.6) \]
be the matrix-valued multiplication operator. Here sup|α(x, ϑ)| is exponentially decreasing at both horizons x → ±∞. By (4.16) in [2] we know that supx,ϑ α(x, ϑ) < 1. As ∥Γ3∥∞ = 1 we get that operator J is invertible and

\[ J^{-1} = (1 - α^2)(I_4 - αΓ^3) = \left(1 - \frac{Δr}{Δ_ϑ} \frac{a^2 \sin^2 ϑ}{r^2 + a^2} \right)^{-1} \left(I_4 - \frac{\sqrt{Δr}}{\sqrt{Δ_ϑ}} \frac{a \sin ϑ}{r^2 + a^2} Γ^3\right). \]  

We consider the charged Dirac fields with mass m represented by 4-spinors belonging to

\[ L^2 \left(\mathbb{R} \times S^2, \frac{\sin θ}{\sqrt{Δ_ϑ}} dxdθdϕ; \mathbb{C}^4\right) \]

and satisfying the evolution equation \( i∂_t ϕ = Hϕ, \) \( H = J^{-1}H_0, \) where

\[ H_0 = Γ^1D_x + b(x)Γ^0 + a(x) \left[\sqrt{Δ_ϑ} \left(Γ^2 \left(D_ϑ - \frac{\cot ϑ}{2}\right) + Γ^3 \frac{E}{Δ_ϑ \sin ϑ} D_ϕ\right) - am \cos ϑΓ^5\right] \]

+ c(x, D_ϕ).

Renormalizing spinors \( ψ = \left(\frac{\sin θ}{\sqrt{Δ_ϑ}}\right)^{\frac{1}{2}} ϕ, \) the new spinor \( ψ \) belongs to the Hilbert space independent of parameters of the black hole \( \mathcal{H} = L^2 (\mathbb{R} \times S^2, dxdθdϕ; \mathbb{C}^4) \) and satisfies the evolution equation

\[ i∂_t ψ = Dψ, \quad D = J^{-1}D_0, \quad D_0 = Γ^1D_x + b(x)Γ^0 + c(x, D_ϕ) + a(x)D_{S^2}. \]

Here \( D_{S^2} \) is an angular Dirac operator on 2-sphere \( S^2, \)

\[ D_{S^2} = \sqrt{Δ_ϑ} \left[Γ^2 \left(D_ϑ + \frac{iΛa^2 \sin(2ϑ)}{12Δ_ϑ} \right) + Γ^3 \frac{E}{Δ_ϑ \sin ϑ} D_ϕ\right] - am \cos ϑΓ^5. \]  

Note that \( D_0 \) is self-adjoint on \( \mathcal{H}, \) while \( D \) is self-adjoint on slightly modified Hilbert space \( \mathcal{G} \) given by the same space as \( \mathcal{H} \) but equipped with scalar product \( \langle..\rangle_\mathcal{G} = (.., J.)_\mathcal{H}. \) Note that in the expression of the hamiltonian \( D \) the mass of the Dirac field \( m \) appears in the coefficient \( b(x) \) and in the last term in (1.8). As \( b(x) \) decays exponentially as \( x → ±∞ \) (see (12.2)) it follows that the spectral properties of \( D \) are independent of the mass \( m, \) namely the spectrum of \( D \) is purely absolutely continuous and is given by \( \mathbb{R} \) (which is proved as in the massless case using Mourre theory, see [14]).

We show below that the cut-off resolvent

\[ R_χ(λ) = χ(D - λ)^{-1}χ, \quad χ ∈ C^∞_0(\mathbb{R}; \mathbb{C}^4) \]

has meromorphic continuation from the upper half-plane to \( \mathbb{C} \) with isolated poles of finite rank. The poles of this meromorphic continuation are called \textit{resonances}. 

3
1.2 Quasi-normal modes.

Resonances for the black holes are complex characteristic frequencies of the proper solutions of the perturbation equations which satisfy the boundary conditions appropriate for purely ingoing waves at the event horizon and purely outgoing waves at infinity (see [10], [9]).

These solutions are called quasi-normal modes (QNM). QNMs determine the late-time evolution of fields in the black hole exterior and eventually dominate the black hole response to any kind of perturbation, therefore containing informations about proper parameters of the black hole: the mass, the electric charge, the angular momentum per unit mass and De Sitter constant $\Lambda$. For the physics review we refer to [33] and more recent [9].

The subject has become very popular for the last few decades including the development of stringent mathematical theory of QNMs (see [3], [4], [12], [17], [18], [21]).

The paper [4] provides with mathematical justification for localization of QNMs for the wave equation on the de Sitter-Schwarzschild metric. In Regge-Wheeler coordinates the problem is reduced to the scattering problem for the Schrödinger equation on the line with exponentially decreasing potential. In the Schwarzschild case (zero cosmological constant, which corresponds to asymptotically flat Universe) the Regge-Wheeler potential is only polynomially decreasing and the method does not work due to the possible accumulation of resonances at the origin. A non-zero cosmological constant is needed in order to define an analytic continuation of the resolvent in a proper space of distributions. Later these works were complemented by [6], where the authors considered the local energy decay for the wave equation on the de Sitter-Schwarzschild metric and proved expansion of the solution in terms of resonances.

In [17] and [18] Dyatlov studied the slowly rotating Kerr-de Sitter black holes. Due to cylindrical instead of spherical symmetry the problem can no longer be simply reduced to a scattering problem on the line. The quasi-normal modes split in a way similar to the Zeeman effect. Dyatlov also extended [6] to the rotating black holes and showed the exponential decay of local energy of linear waves orthogonal to the zero quasi-normal mode. Note also the paper [14], where the authors extended their inverse scattering results from [15] to the scattering for massless Dirac fields by the (rotating) Kerr-Newman-de Sitter black holes. In the present paper we extend some techniques from [17], [18] to the framework of Dirac operators for the Kerr-Newman-de Sitter black holes.

Note that some results in [17] and [18] were extended in [35] to the case of perturbations of Kerr-de Sitter metrics when the separation of variables is not possible. For example resonance free regions and polynomials resolvent bounds were established by more general and flexible techniques. Recently, there appeared several new papers on applications and extensions of Vasy’s method (see [19], [20] and references therein).

As in the present paper we focus on the precise spectral asymptotics where separation of variables is crucial, we do not consider these methods here.

In [26], [27] we consider scattering of massless uncharged Dirac fields propagating in the outer region of de Sitter-Reissner-Nordström (dS-RN) black hole, which is spherically symmetric charged exact solution of the Einstein-Maxwell equations and is special case of non-rotating ($a = 0$) KN-dS black hole.
The resonances are approximated by the lattice associated to the trapped set which is a sphere of partially hyperbolic orbits - photon sphere. Due to radial symmetry, after separation of variables and a Regge-Wheeler transformation the problem is reduced to a family of one-dimensional Schrödinger operators on a line with potentials exponentially decaying at infinity and having unique non-degenerate maxima.

In [26], using a special super-symmetric form of the radial Dirac equation, we show that resonances for dS-RN black holes can be obtained as solutions of one-dimensional Schrödinger equations and using the method of semiclassical Birkhoff normal form (as in [29], [30]) we obtain complete asymptotic expansions. Moreover, we get similar results for the wave equation in de Sitter-Schwarzschild metric thus improving the results in [4]. Our results extend to the Dirac operators on spherically symmetric asymptotically hyperbolic manifolds (see [13]).

Note that the physicist treated the Dirac resonances exactly as solutions of the Schrödinger equation (see also [9], [8]). In [26] Theorem 1 shows a different point of view and gives exact relation between Schrödinger and Dirac resonances. Indeed, due to the symmetry of the equation, the set of non-zero Schrödinger resonances consists of two sets interposed: the set of Dirac resonances and its mirror image with respect to the imaginary axis.

In [27] we give an expansion of the massless Dirac fields in the outer region of dS-RN black hole in terms of resonances and describe the decay of local energy for compactly supported data. The methods extend to the Dirac operators on spherically symmetric asymptotically hyperbolic manifolds.

In the case of rotating KN-dS black hole as considered in the present paper the problem in no longer spherically symmetric, but still has cylindrical symmetry. Then following [14], [17] we show that it is still possible to decompose operator into the angular and radial parts. But contrary to the de Sitter-Reissner-Nordström black hole the radial Dirac operator has no longer super-symmetric form and only leading terms in asymptotic expansion of the resonances can be computed explicitly.

It is believed that, due to intense gravitation near the event and cosmological horizons of the black hole, even if the Dirac fields are massive, they propagate asymptotically as in the massless case. Below we support this claim by showing that the two leading terms in the expansion of the black hole resonances are in fact independent of the mass of the Dirac field.

In [21] and [11] the quasi-normal modes in rather different geometry of Anti-de-Sitter (AdS) black holes are discussed. Such black holes arise in superstring theory via AdS conformal field theory correspondence, that string theory in AdS space is equivalent to conformal field theory in one less dimension (see [5], [11]). The quasi-normal frequencies correspond to the thermalization time scale, which is very hard to compute directly. Gannot in [21] uses a black-box approach to define the quasi-normal modes after separation of variables and furthermore finds a sequence of quasi-normal frequencies approaching the real axis exponentially rapidly. Warnick in [11] uses a different approach which applies to asymptotically Anti-de-Sitter black holes and does not require any separability of the equations under consideration, nor any real analyticity of the metric. Moreover, the method can be extended to asymptotically de-Sitter black holes, where it is closely related to approach by Vasy in [35], [35],
and allows consideration of perturbations which do not vanish on the horizons.

2 Main results.

Theorem 2.1. For \(M, Q, \Lambda\) fixed satisfying (1.2), there exists a constant \(a_0 > 0\) such that for \(|a| < a_0\) and each \(\nu_0\), there exist constants \(C_\lambda, C_m\) such that the set of quasi-normal modes \(\lambda\) satisfying

\[
\text{Re} \lambda > C_\lambda, \quad \text{Im} \lambda > -\nu_0
\]

coincides modulo \(O(|\lambda|^{-\infty})\) with the set of pseudopoles

\[
\lambda = F(m, l, k), \quad m \in \mathbb{Z}, \quad l, k \in \mathbb{Z} + \frac{1}{2}, \quad 0 \leq m \leq C_m, \quad |k| \leq l.
\]

Here \(F\) is a complex valued classical symbol of order 1 in the \((l, k)\) variable, defined and smooth in the cone \(\{(m, l, k); m \in [0, C_m], |k| \leq l\} \subset \mathbb{R}^3\). The principal symbol \(F_0\) of \(F\) is real-valued and independent of index \(m\). The two leading terms \(F_0, F_1\) in the expansion of resonances are independent of mass \(m\) of the Dirac field. Moreover, for \(a = 0\)

\[
F = z_0(l + 1/2) - i \left( \frac{\alpha}{z_0} \right) \left( m + \frac{1}{2} \right) + O((l + 1/2)^{-1}), \tag{2.1}
\]

where

\[
z_0 = \left( \frac{M}{r_0^3} - \frac{Q^2}{r_0^4} - \frac{\Lambda}{3} \right)^{\frac{1}{2}}, \quad \alpha = \left( \frac{3M}{r_0} - \frac{4Q^2}{r_0^2} \right)^{\frac{1}{2}} z_0^2(x_0); \tag{2.2}
\]

for \(|a| < a_0\),

\[
\partial_\kappa F_0(m, \pm k, k) = \frac{a}{r_0^3} \left[ \frac{4F(r_0) r_0^2}{8Q^2 - 6Mr_0} \left( 1 - r_0 \frac{1}{2} F^{-1}(r_0) F'(r_0) \right) - 1 \right] - az_0 r_0^{-1} F^{\frac{1}{2}}(r_0), \tag{2.3}
\]

where \(F(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3} r^2\).

Remark 2.1. Formula (2.3) shows the Zeeman-type splitting of the resonances due to rotation.

Remark 2.2. The fact that the two leading terms in the expansion of resonances are independent of the mass of the Dirac field supports the well-known claim that the massive Dirac fields propagate asymptotically as in the massless case which is caused by intense gravitation near the event and cosmological horizons of the black hole.

Remark 2.3. Putting \(Q = 0, r_0 = 3M\), we recover the formulas (0.3), (0.4) by Dyatlov in the Kerr-de Sitter case [18]. Note that in that case \(1 - r_0 \frac{1}{2} F^{-1}(r_0) F'(r_0) = 0\).

Remark 2.4. Note that in the case of chargeless and massless Dirac fields propagating in the exterior of the static black holes \((a = 0)\) formula (2.1) was obtained in [20] using a...
special (super-symmetric) form of the Dirac operator. Moreover, using the method of Birkhoff normal forms, the next order term was obtained

\[ \mathcal{F} = z_0(l + 1/2) - i \left( \frac{\alpha}{z_0} \right) \left( m + \frac{1}{2} \right) \]

\[ - \left( \frac{\alpha}{z_0} \right) \left( \frac{m + 1/2}{l + 1/2} \right) \left[ - \frac{\alpha}{4z_0^2} (2m + 1) + \frac{1}{2} b_{0,2}(m + 1) + ib_{1,2} \right] + \mathcal{O}((l + 1/2)^{-2}). \]

Both massless and massive cases can be treated by similar methods, but as the massless Dirac fields can be represented by 2-spinors, the massless case is slightly less technical and the methods are more transparent. Therefore, the strategy of the proof of Theorem 2.1 will be the following. We explain all the techniques and give the proof of Theorem 2.1 in full details only in the massless case and in Section 11 we indicate how to extend the proofs to the massive Dirac fields.

In the present paper we do not consider the local energy decay or resonance expansion as studied in the rotationless case in [28] and other related properties, leaving these tasks to our forthcoming publications.

The paper is organized as follows. We start by the massless Dirac fields. In Section 3 we explain the decomposition of the Hilbert space using the cylindrical symmetry of the problem and consider the action of the Dirac operator on the associated subspaces.

In Section 4 we formulate the main results on meromorphic continuation of the resolvent. We show that the Dirac operator can be written as a tensor product of its angular and radial parts. The resolvent is then represented as a certain contour integral of a tensor product of angular and radial resolvents.

Properties of angular and radial resolvents are stated in Section 5. Subsection 5.2 contains resolvent identities connecting the Dirac (angular and radial) operators to the diagonal matrix Schrödinger-type operators, allowing to transfer some results from [17], [18] to the Dirac operators. The proof for the angular resolvent is given in subsection 5.3.

The proof for the radial resolvent is much longer and is given in Section 6. In Section 7 we reformulate Theorem 2.1 in semi-classical terms.

In Sections 8, 9 we deduce angular, respectively radial quantization conditions.

In Section 10 we combine both conditions, deduce formulas (2.1) and (2.3) and conclude the proof of Theorem 2.1 in the massless case.

Now, we pass to the general massive case. In Sections 11 – 14 we indicate how to extend the techniques to the massive Dirac fields and show that the results from previous sections extend to the massive case.
Massless Dirac fields.

3 Preliminaries.

3.1 Evolution equation.

We introduce the matrix-valued multiplication operator

\[ J = I_2 - \alpha(x, \vartheta)\sigma_2, \quad \alpha(x, \vartheta) = \frac{\sqrt{\Delta_r}}{\sqrt{\Delta_\vartheta}} \frac{a \sin \vartheta}{r^2 + a^2} = a(x)b(\vartheta), \quad b(\vartheta) = \frac{a \sin \vartheta}{\sqrt{\Delta_\vartheta}}, \] (3.1)

with \( \sup_\vartheta |\alpha(x, \vartheta)| \) is exponentially decreasing at both horizons \( x \to \pm \infty \). Here \( \sigma_j, i = 1, 2, 3, \) are the Pauli matrices

\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \] (3.2)

Note that in [14] the authors used notation \( \Gamma^1 = \sigma_3, \Gamma^2 = \sigma_1, \Gamma^3 = -\sigma_2 \).

Operator \( J \) is invertible and

\[ J^{-1} = (1 - \alpha^2)(I_2 + \alpha \sigma_2) = \left(1 - \frac{\Delta_r}{\Delta_\vartheta} \frac{a^2 \sin^2 \vartheta}{r^2 + a^2} \right)^{-1} \left(I_2 + \sqrt{\Delta_r} \frac{a \sin \vartheta}{\sqrt{\Delta_\vartheta}} \sigma_2 \right). \] (3.3)

The massless charged Dirac fields are represented by 2-spinors belonging to \( L^2(\mathbb{R} \times S^2, \frac{\sin \vartheta}{\sqrt{\Delta_\vartheta}} dx d\vartheta d\varphi; \mathbb{C}^2) \)

and satisfying the evolution equation \( i \partial_t \phi = H \phi, \quad H = J^{-1}H_0, \)

\[ H_0 = \sigma_3 D_x + a(x) \left[ \sqrt{\Delta_\vartheta} \left( \sigma_1 \left(D_\vartheta - i \cot \vartheta \frac{\vartheta}{2} \right) - \sigma_2 \frac{E}{\Delta_\vartheta \sin \vartheta} D_\varphi \right) \right] + c(x, D_\varphi). \]

Renormalizing spinors \( \psi = \left(\frac{\sin \vartheta}{\sqrt{\Delta_\vartheta}}\right)^{\frac{1}{2}} \phi \), the new spinor \( \psi \) belongs to the Hilbert space independent of parameters of the black hole \( \mathcal{H} = L^2(\mathbb{R} \times S^2, dx d\vartheta d\varphi; \mathbb{C}^2) \) and satisfies the evolution equation

\[ i \partial_t \psi = D \psi, \quad D = J^{-1}D_0, \quad D_0 = \sigma_3 D_x + c(x, D_\varphi) + a(x)D_{S^2}. \]

Here \( D_{S^2} \) is an angular Dirac operator on 2-sphere \( S^2 \),

\[ D_{S^2} = \sqrt{\Delta_\vartheta} \left[ \sigma_1 \left(D_\vartheta + \frac{i \Lambda a^2 \sin(2\vartheta)}{12 \Delta_\vartheta} \right) - \sigma_2 \left(\frac{E}{\Delta_\vartheta \sin \vartheta} \right) D_\varphi \right]. \] (3.4)

Note that \( D_0 \) is self-adjoint on \( \mathcal{H} \), while \( D \) is self-adjoint on slightly modified Hilbert space \( \mathcal{G} \) given by the same space as \( \mathcal{H} \) but equipped with scalar product \( \langle \cdot, \cdot \rangle_{\mathcal{G}} = \langle \cdot, J \cdot \rangle_{\mathcal{H}}. \)
### 3.2 Separation of variables for real $\lambda$

In this section we consider the decomposition of the Hilbert space using the cylindrical symmetry of the operator

$$D = J^{-1}D_0, \quad D_0 = \sigma_3 D_x + c(x, D_\varphi) + a(x)D_{S^2},$$

where $D_{S^2}$ is the angular Dirac operator on 2-sphere $S^2$ given in (3.4).

We start with the following identities

$$(D - \lambda)\psi = \phi \iff D(\lambda)\psi = J\phi, \quad (D - \lambda)^{-1} = [D(\lambda)]^{-1}J,$$  \hspace{1cm} (3.5)

where

$$D(\lambda) = (\sigma_3 D_x + c(x, D_\varphi) - \lambda) + a(x)(D_{S^2} + \lambda b(\vartheta) \sigma_2).$$  \hspace{1cm} (3.6)

Then the stationary Dirac equation $D\psi = \lambda\psi$ can be re-written as $D(\lambda)\psi = 0$.

Let

$$A_{S^2}(\lambda) = D_{S^2} + \lambda b(\vartheta) \sigma_2, \quad H_{S^2} = L^2(S^2, d\vartheta d\varphi; \mathbb{C}^2).$$  \hspace{1cm} (3.7)

In this section, following [14], we describe the decomposition of the Hilbert space in the case $\lambda \in \mathbb{R}$. The construction indicates the way of approaching the case $\lambda \in \mathbb{C}$.

We decompose $H_{S^2}$ onto the angular modes $\{e^{ik\varphi}\}_{k \in \frac{1}{2} + \mathbb{Z}}$ that are eigenfunctions for $D_\varphi$ with anti-periodic boundary conditions (see [2]). Then

$$H_{S^2} = \bigoplus_{k \in \frac{1}{2} + \mathbb{Z}} H^k_{S^2}, \quad H^k_{S^2} = L^2((0, \pi), d\varphi; \mathbb{C}^2).$$  \hspace{1cm} (3.8)

The reduced subspaces $H^k_{S^2}$ remain invariant under the action of $A_{S^2}(\lambda)$ and we denote $A_k(\lambda) = A_{S^2}(\lambda)|_{H^k_{S^2}}$. We have explicitly

$$A_k(\lambda) = \sqrt{\Delta_\varphi} \left[ \sigma_1 \left(D_\varphi + \frac{i\Delta a^2 \sin(2\vartheta)}{12\Delta_\varphi}\right) - \sigma_2 \left(\frac{kE}{\Delta_\varphi \sin \vartheta} - \frac{a \sin \vartheta}{\Delta_\varphi}\right) \right].$$  \hspace{1cm} (3.9)

For each $k \in \frac{1}{2} + \mathbb{Z}$, operator $A_k(\lambda)$ is self-adjoint and has discrete simple spectrum $\sigma(A_k(\lambda)) = \{\mu_{k,l}(\lambda)\}_{l \in \mathbb{Z}^*}$ with associated set of eigenfunctions $\{u_{k,l}^\lambda\}_{l \in \mathbb{Z}^*}$,

$$A_k(\lambda)u_{k,l}^\lambda(\vartheta) = \mu_{k,l}(\lambda)u_{k,l}^\lambda(\vartheta).$$

Here $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$. Since $\sigma(A_k(\lambda))$ is discrete, it has no accumulation point and thus

$$\forall k \in \frac{1}{2} + \mathbb{Z}, \quad |\mu_{k,l}(\lambda)| \to \infty \quad \text{as} \quad l \to \pm \infty.$$

Eigenvalues $\mu_{k,l}(\lambda)$ of $A_k(\lambda)$ are also the eigenvalues of $A_{S^2}(\lambda)$ with eigenfunctions $Y_{k,l}(\lambda, \vartheta, \varphi) = u_{k,l}^\lambda(\vartheta)e^{ik\varphi}$.

By Theorem 3.3 in [14] we have the following result
Lemma 1. Operator $A_{S^2}(\lambda), \lambda \in \mathbb{R}$, is self-adjoint on $\mathcal{H}_{S^2}$ and has pure point spectrum given by a sequence of eigenvalues $\{\mu_{kl}(\lambda)\} \in \mathbb{R}$ with associated eigenfunctions $Y_{kl}(\lambda) \in \mathcal{H}_{S^2}$. Moreover,

(i) $\mathcal{H}_{S^2} = \bigoplus_{(k,l) \in K} \text{Span} (Y_{kl}(\lambda)), \quad K = \left(\frac{1}{2} + \mathbb{Z}\right) \times \mathbb{Z}^*$,

(ii) $A_{S^2}(\lambda)Y_{kl}(\lambda) = \mu_{kl}(\lambda)Y_{kl}(\lambda),$

(iii) $D_{\lambda}Y_{kl}(\lambda) = kY_{kl}(\lambda)$.

Now, we use the cylindrical symmetry and decompose the Hilbert space $\mathcal{H}$ onto the angular modes $\{e^{ik\varphi}\}_{k \in \frac{1}{2} + \mathbb{Z}}$,

$$\mathcal{H} = \bigoplus_{k \in \frac{1}{2} + \mathbb{Z}} \mathcal{H}_k, \quad \mathcal{H}_k = L^2(\mathbb{R} \times (0, \pi), dx d\vartheta; \mathbb{C}^2) = L^2(\mathbb{R}; \mathbb{C}^2) \otimes L^2((0, \pi), d\vartheta; \mathbb{C}^2). \quad (3.10)$$

We choose half-integers $k$ as we want the anti-periodic conditions in variable $\varphi$ : the spinors change the sign after a complete rotation (see [2]). Note that

$$\mu_{k,-l}(\lambda) = -\mu_{kl}(\lambda), \quad Y_{k,-l}(\lambda) = \sigma_3 Y_{kl}(\lambda).$$

Using these results we have the decomposition (see [14])

$$\mathcal{H} = \bigoplus_{(k,l) \in I} \mathcal{H}_{kl}(\lambda), \quad I = \left(\frac{1}{2} + \mathbb{Z}\right) \times \mathbb{N}^*, \quad \mathcal{H}_{kl}(\lambda) = L^2(\mathbb{R}; \mathbb{C}^2) \otimes Y_{kl}(\lambda).$$

We choose $I$ instead of $K$ in order to have subspaces $\mathcal{H}_{kl}$ remain invariant under the action of $\mathcal{D}(\lambda)$ (see Section 3.2 in [14] for details).

Let $\mathcal{D}_k(\lambda) := \mathcal{D}(\lambda)|_{\mathcal{H}_k} = \sigma_3 D_x + c(x, k) - \lambda + a(x) A_k(\lambda)$ be restriction of $\mathcal{D}(\lambda)$ to $\mathcal{H}_k$.

Radial operator $\sigma_3 D_x + c(x, k) - \lambda$ lets invariant $\mathcal{H}_{kl}$ and its action on $\psi = \psi_{kl} \otimes Y_{kl}(\lambda) \in \mathcal{H}_{kl}$ is given by

$$[\sigma_3 D_x + c(x, k) - \lambda] \psi = ([\sigma_3 D_x + c(x, k) - \lambda] \psi_{kl}) \otimes Y_{kl}(\lambda).$$

Angular operator $A_{S^2}(\lambda)$ lets invariant $\mathcal{H}_{kl}$ and its action on $\psi = \psi_{kl} \otimes Y_{kl}(\lambda) \in \mathcal{H}_{kl}$ is given by

$$A_{S^2}(\lambda) \psi = (\mu_{kl}(\lambda) \sigma_1 \psi_{kl}) \otimes Y_{kl}(\lambda). \quad (3.11)$$

Then the restriction of $\mathcal{D}(\lambda)$ to $\mathcal{H}_{kl}(\lambda)$ is given by

$$\mathcal{D}_{kl}(\lambda) = \sigma_3 D_x + c(x, k) - \lambda + \mu_{kl} A(x) \sigma_1, \quad (3.12)$$

with

$$c(x, k) = \frac{aEk + qQr}{r^2 + a^2}, \quad a(x) = \frac{\sqrt{\Delta r}}{r^2 + a^2}$$

satisfying

$$a(x) = a_{\pm} e^{\kappa x} + O(e^{3\kappa x}), \quad x \to \pm \infty, \quad (3.13)$$
\( c(x, k) = \Omega_\pm(k) + c_\pm e^{2\kappa_\pm x} + O\left(e^{4\kappa_\pm x}\right), \quad x \to \pm\infty, \)  
(3.14)

\[
\Omega_\pm = \frac{a E k + q Q r_\pm}{r_\pm^2 + a^2}.
\]

Here \( \kappa_+ < 0, \kappa_- > 0 \) are fixed constants (surface gravities at cosmological, event horizons respectively) depending on the parameters of the black hole.

### 3.3 Properties of eigenvalues \( \mu_{kl} \) of angular operator for real \( \lambda \).

In this section we collect some properties of the angular operator

\[ A_{S^2}(\lambda) = D_{S^2} + \lambda \frac{a \sin \vartheta}{\sqrt{\Delta \vartheta}} \sigma_2, \]

where \( D_{S^2} \) is given in (3.4).

For \( \lambda \in \mathbb{R} \) operator \( A_{S^2}(\lambda) \) is self-adjoint on \( \mathcal{H}_{S^2} = L^2(S^2; \mathbb{C}^2) \) and has positive discrete spectrum \( \sigma(A_{S^2}(\lambda)) = \{ \mu_{kl}(\lambda) \}_{(k,l) \in I} \), ordered in such a way that for each \( k \in \frac{1}{2} + \mathbb{Z} \) and \( l \in \mathbb{N}^* \) it follows \( 0 < \mu_{kl}(\lambda) < \mu_{k(l+1)}(\lambda) \). Let \( A_\lambda = A_{S^2}(\lambda)|_{\mathcal{H}_{S^2}} \). Below we recall some facts from [14] (see proof of Proposition A.1).

Let \( \zeta = \frac{a^2 \Lambda}{3}, \xi = a \lambda, \) and consider operator \( A_\lambda(\varnothing, \xi) \) as operator-valued function \( A_\lambda(\zeta, \xi) \) of complex parameters \( \zeta, \xi \). Put

\[ A_\lambda(\zeta, \xi) = A(\zeta) \mathbb{D}_{S^2}^k + B(\zeta, \xi), \]

(3.15)

with

\[ A(\zeta) = \sqrt{1 + \zeta \cos^2 \vartheta}, \quad B = i \sigma_1 \frac{\zeta \sin(2\vartheta)}{4 \sqrt{1 + \zeta \cos^2 \vartheta}} - \sigma_2 \frac{(\zeta k - \xi) \sin \vartheta}{\sqrt{1 + \zeta \cos^2 \vartheta}}. \]

Operator

\[ A_\lambda(0, 0) \equiv \mathbb{D}_{S^2}^k = \sigma_1 D_\vartheta - \sigma_2 \frac{\tau}{\sin \vartheta} \]

is the restriction of the standard Dirac operator on \( S^2 \) onto the angular mode \( \{ e^{ik\varphi}\} \), \( k \in 1/2 + \mathbb{Z} \). The domain of \( A_\lambda(0, 0) \) is given by

\[ \mathfrak{D} = \{ u \in \mathcal{H}_{S^2}^k, \ u \text{ is absolutely continuous, } \mathbb{D}_{S^2}^k u \in \mathcal{H}_{S^2}^k, \ u(\pi) = -u(0) \}. \]

The spectrum of \( A_\lambda(0, 0) \) is simple discrete given by

\[ \mu_{kl}(0, 0) = \text{sgn}(l) \left( |k| - \frac{1}{2} + |l| \right), \quad l \in \mathbb{Z}^* \]

(3.16)

and \( A_\lambda(0, 0) \) has compact resolvent.

According to (1.2) we have \( \zeta \in [0.7 - 4\sqrt{3}] \subset [0, \frac{1}{13.8}] \) and \( \xi \in \mathbb{R} \) respectively. Now, we allow parameters \( \zeta, \xi \) to be complex \( (\zeta, \xi) \in B(0, \frac{1}{13}) \times S, \) where \( B(0, r) = \{ z \in \mathbb{C}; |z| < r \} \) and \( S \) is a narrow strip containing the real axis.
The operators \( A(\zeta), B(\zeta, \xi) \) are bounded matrix-valued multiplications operators analytic in the variables \((\zeta, \xi) \in B(0, \frac{1}{13}) \times S\). Since the operator \( A(\zeta) \) is also invertible, the operators domain of \( A_k(\zeta, \xi) \) is independent on \((\zeta, \xi) \in B(0, \frac{1}{13}) \times S\).

Moreover, since for all \( u \in \mathcal{D} \), \( A_k(\zeta, \xi)u \) is a vector-valued analytic function in \((\zeta, \xi)\), and since for \((\zeta, \xi) \in [0, \frac{1}{13}] \times \mathbb{R}\) is self-adjoint on \( \mathcal{H}^k_{\xi} = L^2((0, \pi), d\vartheta; \mathbb{C}^2) \), then \( A_k(\zeta, \xi) \) forms a self-adjoint holomorphic family of type (A) in variable \((\zeta, \xi) \in B(0, \frac{1}{13}) \times S\) according to Kato’s classification.

Using the analytic perturbation theory by Kato [32] it was shown in [14] that \( A_k(\zeta, \xi) \) has compact resolvent for all \((\zeta, \xi) \in B(0, \frac{1}{13}) \times S\). Moreover, for a fixed \( k \in \frac{1}{2} + \mathbb{Z} \), the eigenvalues
\[
\mu_{kl}(\zeta, \xi), \quad k \in \frac{1}{2} + \mathbb{Z}, \quad l \in \mathbb{Z}^*,
\]
of \( A_k(\zeta, \xi) \) are simple and depend holomorphically on \((\zeta, \xi)\) in a complex neighborhood of \([0, \frac{1}{13}] \times \mathbb{R}\).

It was also shown in [14], Proposition A.1, that for all \( \lambda \in \mathbb{R} \), for all \( k \in \frac{1}{2} + \mathbb{Z} \) and for all \( l \in \mathbb{N}^* \), there exist constants \( C_1 \) and \( C_2 \) independent of \( k, l \) such that
\[
(2 - e^{\frac{1}{2\lambda}}) \left( |k| - \frac{1}{2} + l \right) - C_1|k| - C_2 - |a| |\lambda| \leq \mu_{kl}(\lambda) \leq e^{\frac{1}{2\lambda}} \left( |k| - \frac{1}{2} + l \right) + C_1|k| + C_2 + |a| |\lambda|.
\]
(3.17)

4 Resolvent.

We use the cylindrical symmetry of the operators \( \mathcal{D}, \mathcal{D}(\lambda) \) and decompose the Hilbert space \( \mathcal{H} \) onto the angular modes \( \{e^{ik\varphi}\}_{k \in \frac{1}{2} + \mathbb{Z}} \) as in (3.10):
\[
\mathcal{H} = \bigoplus_{k \in \frac{1}{2} + \mathbb{Z}} \mathcal{H}_k, \quad \mathcal{H}_k = L^2(\mathbb{R} \times (0, \pi), dxd\vartheta; \mathbb{C}^2) = L^2(\mathbb{R}; \mathbb{C}^2) \otimes L^2((0, \pi), d\vartheta; \mathbb{C}^2).
\]
Note that \( \mathcal{H}_k = \mathcal{H} \cap D'_k \), where
\[
D'_k = \{ u \in D'; (D_\varphi - k)u = 0, \quad k \in \frac{1}{2} + \mathbb{Z} \}.
\]
(4.1)
This space can be considered as a subspace of \( D'(\mathbb{R} \times \mathbb{S}^2, dxd\vartheta d\varphi; \mathbb{C}^2) \) or of \( D'(\mathbb{S}^2, d\vartheta d\varphi; \mathbb{C}^2) \) along.

In this section we consider the meromorphic continuation of the resolvent \( (\mathcal{D} - \lambda)^{-1} \) from the upper half-plane \( \mathbb{C}_+ \) to the whole complex plane \( \mathbb{C} \). Using (3.5), (3.6), (3.10) this task is reduced to defining an inverse \( [\mathcal{D}(\lambda)]^{-1} \) in some subspace of \( \mathcal{H}_k \).

We will prove the following results.

**Theorem 4.1.** 1) Let \( \mathcal{D}_k(\lambda) := \mathcal{D}(\lambda)|_{\mathcal{H}_k} = \sigma_3 D_+ + c(x, k) - \lambda + a(x) A_k(\lambda) \) be restriction of \( \mathcal{D}(\lambda) \) to \( \mathcal{H}_k \). Here \( A_k(\lambda) \) is given in (3.9). The operator \( R(\lambda, k) = [\mathcal{D}(\lambda)]^{-1} \) admits meromorphic continuation in \( \lambda \) from \( \mathbb{C}_+ \) into \( \mathbb{C} \) as a meromorphic family of operators
\[
R(\lambda, k) : L^2_{\text{comp}} \cap D'_k \rightarrow L^2_{\text{loc}} \cap D'_k
\]
with poles of finite rank.

2) Fix $\delta > 0$. Put $K_{\delta, r} = (r_- + \delta, r_+ - \delta)$ and $M_{\delta, r} = K_{\delta, r} \times \mathbb{S}^2$. Denote by the same letters also their images under the Regge-Wheeler change of variables $r \mapsto x$ given by (1.3). Let $1_{M_{\delta, r}}$ be the operator of multiplication by the characteristic function of $M_{\delta, r}$. Then there exists $a_0 > 0$ such that if the rotation speed of the black hole satisfies $|a| < a_0$, we have the following:

(i) Every fixed compact set can only contain $k$-resonances for a finite number of values of $k$. Therefore, quasi-normal modes form a discrete subset of $\mathbb{C}$.

(ii) The operators $1_{M_{\delta, r}} R(\lambda, k) 1_{M_{\delta, r}}$ define a family of operators $\mathcal{R}(\lambda) = [\mathcal{D}(\lambda)]^{-1}$

$$\mathcal{R}(\lambda) : L^2(M_{\delta, r}) \mapsto L^2(M_{\delta, r})$$

meromorphic in $\lambda \in \mathbb{C}$ with poles of finite rank.

In order to prove the theorem we follow [17] and represent $R(\lambda, k)$ as a certain contour integral of a tensor product of two operators acting in different spaces: the angular and the radial resolvents. This procedure replaces the separation of variables which can be performed in the non-rotating case of Reissner-Nordström black holes as in [26]. The properties of the angular and the radial resolvents are stated later in Propositions 5.1, 5.2 in Section 5 with the proofs given in Sections 5.3 and 6.

We start by representing the Hamiltonian as tensor product $H_k(\lambda) \otimes I_2 + I_2 \otimes A_k(\lambda)$ acting in $L^2(\mathbb{R}, dx; \mathbb{C}^2) \otimes \mathcal{H}_{S^2}^k$, where $\mathcal{H}_{S^2}^k = L^2((0, \pi), d\vartheta; \mathbb{C}^2)$. Here $H_k, A_k$ are operators specified below.

In the region (1.4) $a \neq 0$ and we introduce operator

$$\tilde{\mathcal{D}}(\lambda) = [a(x)]^{-1} \mathcal{D}(\lambda) = [a(x)]^{-1}(\sigma_3 D_x + c(x, D_\vartheta) - \lambda + (D_{S^2} + \lambda b(\vartheta))\sigma_2).$$

Denote $\tilde{\mathcal{D}}_k(\lambda) = \tilde{\mathcal{D}}(\lambda)|_{\mathcal{H}_k}$ its restriction to $\mathcal{H}_k$. For real $\lambda$, its radial $\tilde{\mathcal{D}}^r_k(\lambda)$ and angular $\tilde{\mathcal{D}}^\vartheta_k(\lambda)$ parts let invariant $\mathcal{H}_{kl}$ and the action $\tilde{\mathcal{D}}_k(\lambda) = \tilde{\mathcal{D}}^r_k(\lambda) + \tilde{\mathcal{D}}^\vartheta_k(\lambda)$ on $\psi = \psi_{kl} \otimes Y_{kl}(\lambda) \in \mathcal{H}_{kl}$ is given by

$$\tilde{\mathcal{D}}_k(\lambda)\psi = (\tilde{\mathcal{D}}^r_k(\lambda)\psi_{kl}) \otimes Y_{kl}(\lambda) + (\sigma_1 \psi_{kl}) \otimes (A_{S^2}(\lambda) Y_{kl}(\lambda)).$$

Let $\phi_{kl} = \sigma_1 \psi_{kl} \in L^2(\mathbb{R}, dx; \mathbb{C}^2)$ and

$$H_k(\lambda) := [a(x)]^{-1}(\sigma_3 D_x + c(x, k) - \lambda)\sigma_1, \quad A_k(\lambda) = A_{S^2}(\lambda)|_{\mathcal{H}_{S^2}^k}$$

acting in $L^2(\mathbb{R}, dx; \mathbb{C}^2)$, $\mathcal{H}_{S^2}^k = \text{Span}_{\mathbb{C}^2} (Y_{kl}(\lambda))$ respectively. Then for any $\lambda \in \mathbb{C}$,

$$\tilde{\mathcal{D}}_k(\lambda) = H_k(\lambda) \otimes I_2 + I_2 \otimes A_k(\lambda)$$

acts in $L^2(\mathbb{R}, dx; \mathbb{C}^2) \otimes \mathcal{H}_{S^2}^k$. Let $R_\vartheta(\lambda, \omega, k) = (H_k(\lambda) + \omega)^{-1}$ and $R_{\vartheta}(\lambda, \omega, k) = (A_k(\lambda) - \omega)^{-1}$.

We apply the method from [17], Section 3. Take $k \in 1/2 + \mathbb{Z}$ and an arbitrary $\delta > 0$.

Later on in Section 5 we summarize the properties of angular $R_\vartheta$ and radial $R_r$ resolvents. Here we need some of these results. From Propositions 5.1, 5.2 in Section 5 it follows that the resolvents

$$R_r(\lambda, \omega, k) = (H_k(\lambda) + \omega)^{-1} : L^2(\mathbb{R}, dx; \mathbb{C}^2) \mapsto H^1_{\text{loc}}(\mathbb{R}; \mathbb{C}^2),$$
are meromorphic families of operators in the sense of Definition 2.1 in \cite{17}. In particular, for a fixed values of \( \lambda \), these families are meromorphic in \( \omega \) with poles of finite rank (see Definition 2.2 in \cite{17}).

A point \( \lambda \) is called regular if the sets \( Z_r(\lambda) = \{ \omega \in \mathbb{C}; (\lambda, \omega) \in Z_r \} \), \( Z_\theta(\lambda) = \{ \omega \in \mathbb{C}; (\lambda, \omega) \in Z_\theta \} \) do not intersect. Here \( Z_r, Z_\theta \) are divisors of \( R_r, R_\theta \) respectively (see Definition 2.1 in \cite{17}). From part 3), Proposition 5.2 it follows that the set of all regular points is non-empty. Indeed, every \( \lambda \in \mathbb{R} \) is regular.

We fix an angle \( \psi \in (0, \pi) \) and a regular point \( \lambda \). Let \( \gamma \) be an admissible contour on \( \mathbb{C} \) at \( \lambda \), which mean that (see Definition 2.3 in \cite{17})

\( i) \gamma \) is smooth simple contour given by the rays \( \arg \omega = \pm \psi \) outside of some compact subset of \( \mathbb{C} \),

\( ii) \gamma \) separates \( \mathbb{C} \) into two regions, \( \Gamma_r \) and \( \Gamma_\theta \), such that sufficiently large positive real numbers lie in \( \Gamma_\theta \) and \( Z_\alpha(\lambda) \subset \Gamma_\alpha, \alpha \in \{ r, \theta \} \).

Now, the angle \( \psi \) of admissible contour \( \gamma \) at infinity is chosen as in part 2), Proposition 5.2.

From Propositions 5.1 and 5.2 it follows the following property which assures that admissible contour exists at every regular point.

For any compact \( K_\lambda \subset \Omega \subset \mathbb{C} \) there exist constants \( C \) and \( R \) such that for \( \lambda \in K_\lambda \) and \( |\omega| > R \),

\( i) \) for \( |\arg \omega| \leq \psi \) and \( |\pi - \arg \omega| \leq \psi \) we have \( (\lambda, \omega) \notin Z_r \) and \( \|R_r(\lambda, \omega, k)\| \leq 1/|\omega| \);

\( ii) \) for \( \psi \leq |\arg \omega| \leq \pi - \psi \), we have \( (\lambda, \omega) \notin Z_\theta \) and \( \|R_\theta(\lambda, \omega, k)\| \leq 1/|\omega| \).

Then we can construct restriction of the resolvent

\[ R(\lambda) = (D - \lambda)^{-1} = [D(\lambda)]^{-1}J = [\tilde{D}(\lambda)]^{-1}[a(x)]^{-1}J \]

to \( H_k \) at any regular point \( \lambda \) as a contour integral

\[ R(\lambda, k) = \tilde{R}(\lambda, k)[a(x)]^{-1}, \quad \tilde{R}(\lambda, k) = \frac{1}{2\pi i} \int_\gamma R_r(\lambda, \omega, k) \otimes R_\theta(\lambda, \omega, k) d\omega \quad (4.2) \]

for some admissible \( \gamma \). The orientation of \( \gamma \) is chosen so that \( \Gamma_r \) always stays on the left.

Integral (12.1) converges and is independent of the choice of an admissible contour \( \gamma \). The set of regular points is open and \( \tilde{R}(\cdot, k) \) is holomorphic on this set.

From Proposition 2.3 of Dyatlov in \cite{17} it follows that the set of all non-regular points is discrete and the operator \( \tilde{R}(\cdot, k) \) given by integral in (12.1) is defined as an operator on \( L^2_{\text{comp}} \cap D'_k \) with poles of finite rank. As this can be done for any \( \delta > 0 \) (\( \delta \) is the constant from the definition of an admissible contour in part 2), Proposition 5.2 then \( R(\lambda, k) \) is defined as an operator \( L^2_{\text{comp}} \cap D'_k \rightarrow L^2_{\text{loc}} \cap D'_k \) and is meromorphic in \( \lambda \) with poles of finite rank.

Now, we show that \( \tilde{D}_k(\lambda) \tilde{R}(\lambda, k) f = f \) in the sense of distributions for each \( f \in L^2_{\text{comp}}(\mathbb{R}, dh; \mathbb{C}^2) \otimes L^2((0, \pi), d\vartheta; \mathbb{C}^2) \). Let \( \lambda \) be a regular point. Then \( \tilde{R}(\lambda, k) \) is well-defined. By analyticity, we can assume that \( \lambda \) is real which allows us to use the results from Section 3. Then \( L^2((0, \pi), dh; \mathbb{C}^2) \) has an orthonormal basis of eigenfunctions \( \{ u_{kl}^k \} \) of \( A_k(\lambda) = A_{22}(\lambda)|_{H^k_{\text{loc}}} \), \( A_k(\lambda) u_{kl}^k(\vartheta) = \mu_{k,l}(\lambda) u_{kl}^k(\vartheta) \).
Let $Y_{kl}(\lambda) = u_{kl}^\lambda(\vartheta)e^{ik\varphi}$. Put $\omega_0 = \mu_{k,l}(\lambda)$. Then

$$R_{\vartheta}(\lambda, \omega, k) Y_{kl}(\lambda) = \frac{Y_{kl}(\lambda)}{\omega_0 - \omega}. \quad (4.3)$$

Let $f, h \in C^\infty(R, dx; \mathbb{C}^2)$, $\phi = Y_{kl}(\lambda)$ as above and $\chi \in C^\infty(S^2, d\vartheta d\varphi; \mathbb{C}^2)$. We need to prove

$$I := \langle \bar{R}(\lambda, k)(f(x) \otimes \phi(\vartheta, \varphi), \mathcal{D}_k(\lambda)(h(x) \otimes \chi(\vartheta, \varphi)) = \langle f, h \rangle \cdot \langle \phi, \chi \rangle. \quad (4)$$

If $\gamma$ is admissible contour, then

$$I = \frac{1}{2\pi i} \int_\gamma (R_r(\lambda, \omega, k)f, H_k(\lambda)h) \cdot (R_{\vartheta}(\lambda, \omega, k)\phi, \chi) + (R_r(\lambda, \omega, k)f, h) \cdot (R_{\vartheta}(\lambda, \omega, k)\phi, A_{\mathbb{S}^2}(\lambda)\chi)d\omega. \quad (5)$$

Using (4.3) we replace $\gamma$ by a closed bounded contour $\gamma'$ which contains $\omega_0$, but no poles of $R_r$. Then

$$I = \frac{1}{2\pi i} \int_{\gamma'}((1 - \omega R_r(\lambda, \omega, k))f, h) \cdot (R_{\vartheta}(\lambda, \omega, k)\phi, \chi) + (R_r(\lambda, \omega, k)f, h) \cdot ((1 + \omega R_{\vartheta}(\lambda, \omega, k))\phi, \chi)d\omega$$

$$= \frac{1}{2\pi i} \int_{\gamma'} \langle f, h \rangle \cdot (R_{\vartheta}(\lambda, \omega, k)\phi, \chi) + (R_r(\lambda, \omega, k)f, h) \cdot (\phi, \chi)d\omega$$

$$= \frac{1}{2\pi i} \int_{\gamma'} \frac{\langle f, h \rangle \cdot \langle \phi, \chi \rangle}{\omega_0 - \omega}d\omega = \langle f, h \rangle \cdot \langle \phi, \chi \rangle.$$
which shows $\tilde{D}_k(\lambda)\tilde{R}(\lambda, k)f = f$ and finishes the proof of the first part of Theorem 4.1.

Part 2) of Theorem 4.1 follows from an analogue of Proposition 3.3 in [17].

**Proposition 4.1.** Let $\chi \in C_0^\infty(\mathbb{R}; \mathbb{C}^2)$. Then there exists $a_0 > 0$ (depending on the support of $\chi$) and a constant $C_k$ such that for $|a| < a_0$ and $|k| \geq C_k(1 + |\lambda|)$, $\lambda$ is not a pole of $R(\cdot, k)$ and we have

$$\|\chi R(\lambda, k)\chi\|_{L^2(\mathbb{R} \cdot \mathbb{C}^2)} \leq \frac{C_k}{|k|}. \tag{4.4}$$

**Proof.** This fact follows from Propositions 5.1, 5.2. We choose the constants $\psi, C_r$ from part 2) of Proposition 5.2 and the constants $C_\vartheta, C_\psi$ from Proposition 5.1. Put $\omega_0 = k/3$. If $C_k$ is large enough, then

$$|k| > 1 + C_\vartheta |a\lambda|, \quad \omega_0 > C_\psi |a\lambda| + C_r (1 + |\lambda|).$$

We choose the contour $\gamma$ consisting of the rays $\{\arg \omega = \pm \psi, \pi - \arg \omega = \pm \psi, |\omega| \geq \omega_0\}$ and the arc $\{|\omega| = \omega_0, |\arg\omega| \geq \psi\}$.

By (5.2) and (5.4), all poles of $R_\vartheta$ lie inside $\gamma$ (namely, in the region $\{|\omega| \geq \omega_0, \psi \leq |\arg\omega| \leq \pi - \psi\}$) and

$$\|R_\vartheta(\lambda, \omega)\|_{L^2(\mathbb{S}^2) \cap L^2(\mathbb{S}^2)} \leq \frac{C}{|\omega|}. \tag{4.5}$$

for each $\omega$ on $\gamma$. Now, suppose that $|a| < a_0 = (3C_r)^{-1}$, then (5.5) is satisfied inside $\gamma$. Then (4.4) follows from (12.1): $R(\lambda, k) = \frac{1}{2\pi i} \int_\gamma R_r(\lambda, \omega, k) \otimes R_\vartheta(\lambda, \omega, k)d\omega$, (5.6) and (4.5), as

$$\left\|\frac{1}{2\pi i} \int_\gamma R_r(\lambda, \omega, k) \otimes R_\vartheta(\lambda, \omega, k)d\omega\right\| \leq \frac{C}{|\omega_0|}.$$ 

\[\Box\]

**Remark 4.1.** It follows that a number $\lambda \in \mathbb{C}$ is a pole of $[D(\lambda)]^{-1}$ if and only if there exist $k \in 1/2 + \mathbb{Z}, \omega \in \mathbb{C}$ such that $(\lambda, \omega, k)$ is a pole for both $R_r(\lambda, \omega, k)$ and $R_\vartheta(\lambda, \omega, k)$.

The proof of Theorem 4.1 is now finished.

The following important result comes from the fact that the only trapping in our problem is normally hyperbolic.

**Theorem 4.2** (Resonance free strip). Fix $\delta > 0$ and $s > 0$. Then there exist $a_0 > 0, \nu_0 > 0$, and $C$ such that for $|a| < a_0$,

$$\|R(\lambda)\|_{L^2(M_\delta,r) \rightarrow L^2(M_\delta,r)} \leq C|\lambda|^s, \quad |\text{Re} \lambda| \geq C, \quad |\text{Im} \lambda| \leq \nu_0.$$ 

The proof of this theorem follows the lines of the proof of Theorem 5 in [17] and is explained in Section 9.2.
5 Properties of angular and radial resolvents.

In this section we state the main technical results used in the proof of Theorem 4.1: Propositions 5.1 and 5.2. They are analogue of Propositions 3.1 and 3.2 by Dyatlov in [17]. Proposition 5.1 is proved in Subsection 5.3. Proof of Proposition 5.2 is much longer and will be given later in Section 6.

5.1 Statement of results.

Proposition 5.1 (Angular resolvent estimates). There exists a two-sided inverse
\[ R_\vartheta(\lambda, \omega) = (A_{S^2}(\lambda) - \omega)^{-1} : L^2(S^2) \mapsto H^1(S^2), \quad (\lambda, \omega) \in \mathbb{C}^2, \]
with the following properties:
1) \( R_\vartheta(\lambda, \omega) \) is meromorphic with poles of finite rank and it has the following meromorphic
decomposition near \( \operatorname{Im} \lambda = 0, \omega = \mu_{k,l}(\lambda) \):
\[ R_\vartheta(\lambda, \omega) = S_\vartheta(\lambda, \omega) / (\omega - \mu_{k,l}(\lambda)), \quad (5.1) \]
where \( S_\vartheta(\lambda, \omega) \) and \( \mu_{k,l}(\lambda) \) are holomorphic in some \( a \)-independent neighborhoods of \( \operatorname{Im} \lambda = 0, \omega = \mu_{k,l}(\lambda) \). Moreover, if \( \operatorname{Im} \lambda = 0 \), then \( \mu_{k,l}(\lambda) \) satisfies (3.17).
2) There exists a constant \( C_\vartheta \) such that
\[ \| R_\vartheta(\lambda, \omega) \|_{L^2(S^2) \cap D'_k \mapsto L^2(S^2)} \leq C_\vartheta |k| \quad \text{for} \quad |\omega| \leq 0.4|k|, |k| \geq C_\vartheta |a\lambda| ; \]
(5.2)
and
\[ \| R_\vartheta(\lambda, \omega) \|_{L^2(S^2) \cap D'_k \mapsto L^2(S^2)} \leq \frac{1}{|\operatorname{Im} \omega|} \quad \text{for} \quad |\operatorname{Im} \omega| > 2a|\operatorname{Im} \lambda|. \]
(5.3)
3) For every \( \psi > 0 \), there exists a constant \( C_\psi \) such that
\[ \| R_\vartheta(\lambda, \omega) \|_{L^2(S^2) \cap D'_k \mapsto L^2(S^2)} \leq C_\psi |\omega| \quad \text{for} \quad |\psi| \leq |\arg \omega| \leq \pi - \psi, |\omega| \geq C_\psi |a\lambda|. \]
(5.4)

Proposition 5.2 (Radial resolvent estimates). There exists a family of operators
\[ R_r(\lambda, \omega, k) = (H_k(\lambda) + \omega)^{-1} : L^2_{\text{comp}}(\mathbb{R}, dx; \mathbb{C}^2) \mapsto H^1_{\text{loc}}(\mathbb{R}; \mathbb{C}^2), \quad (\lambda, \omega) \in \mathbb{C}^2 \]
with the following properties:
1) For each \( k \in 1/2 + \mathbb{Z} \), \( R_r(\lambda, \omega, k) \) is meromorphic with poles of finite rank.
2) Take \( \delta > 0 \). There exists \( \psi > 0 \) and \( C_\varphi \) such that
\[ |\omega| \geq C_\varphi, \quad |\arg \omega| \leq \psi \quad \text{or} \quad |\pi - \arg \omega| \leq \psi, \quad |ak| \leq |\omega|/C_\varphi, \quad |\lambda| \leq |\omega|/C_\varphi, \]
(5.5)
\( (\lambda, \omega, k) \) is not a pole of \( R_r \) and we have
\[ \| 1_{K_{\delta, r}} R_r(\lambda, \omega, k) 1_{K_{\delta, r}} \|_{L^2 \mapsto L^2} \leq \frac{C_r}{|\omega|} . \]
(5.6)
Here $1_{K_{\delta,r}}$ is the operator of multiplication by the characteristic function of the Regge-Wheeler image of $(r_- + \delta, r_+ - \delta)$ (see Theorem \ref{thm:regge-wheeler}.

3) Resolvent $R_r(\lambda,\omega,k)$ does not have any poles for real $\omega$ and real $\lambda$.

4) Assume that $R_r(\lambda,\omega,k)$ has pole at $(\lambda,\omega,k)$. Then there exists a nonzero solution $f \in C^\infty(r_-, r_+)$ to the equation

$$(H_k(\lambda) + \omega)f = [a(x)]^{-1} ((\sigma_3 D_x + c(x,k) - \lambda)\sigma_1 + \omega) f = 0$$

such that the functions

$$|r - r_\pm|^\frac{(\lambda - \Omega_{\pm}(k))}{2\pi_\pm} f(r)$$

are real analytic at $r_\pm$ respectively.

### 5.2 Reduction to the Schrödinger-type operators.

In this section we derive useful resolvent identities connecting the angular and radial Dirac operators to the diagonal matrix Schrödinger-type operators similar to those studied in \cite{17,18}. This allows us in many cases to apply the results obtained in these papers. The results of this section will be used throughout the whole paper and we will formulate them for the semi-classically scaled version of Dirac operators.

**Angular operator.** We consider an $h$-dependent version of the operator $A_S(\lambda)$ from Section 3.3

$$A_{S,h}(\lambda) = \Delta_{\theta}^{\frac{1}{2}} (\sigma_1 [h D_\theta + h q_1(\vartheta)] - \sigma_2 [q_2(\vartheta)h D_\varphi - \lambda q_3(\vartheta)]) = \left( \begin{array}{cc} 0 & D_+ \\ D_- & 0 \end{array} \right)$$

$$= \Delta_{\theta}^{\frac{1}{2}} \left( \begin{array}{cc} 0 & h D_\theta + h q_1(\vartheta) + i (q_2(\vartheta)h D_\varphi - \lambda q_3(\vartheta)) \\ h D_\theta + h q_1(\vartheta) - i (q_2(\vartheta)h D_\varphi - \lambda q_3(\vartheta)) & 0 \end{array} \right),$$

where

$$q_1(\vartheta) = \frac{i \Lambda a^2 \sin(2 \vartheta)}{12 \Delta_\theta}, \quad q_2(\vartheta) = \frac{E}{\Delta_\theta \sin \vartheta}, \quad q_3(\vartheta) = \frac{a \sin \vartheta}{\Delta_\theta}.$$ 

Then

$$[A_{S,h}(\lambda)]^2 = \left( \begin{array}{cc} \mathcal{P}_+ & 0 \\ 0 & \mathcal{P}_- \end{array} \right),$$

where

$$\mathcal{P}_+ = D_+ D_- = \Delta_{\theta} \left\{ [h D_\theta + h q_1(\vartheta)]^2 + [q_2(\vartheta)h D_\varphi - \lambda q_3(\vartheta)]^2 - h q'_2(\vartheta) h D_\varphi + \lambda h q'_3(\vartheta) \right\},$$

$$\mathcal{P}_- = D_- D_+ = \Delta_{\theta} \left\{ [h D_\theta + h q_1(\vartheta)]^2 + [q_2(\vartheta)h D_\varphi - \lambda q_3(\vartheta)]^2 + h q'_2(\vartheta) h D_\varphi - \lambda h q'_3(\vartheta) \right\}.$$ 

It is well-known (see \cite{22,23}) that for any $\lambda \in \mathbb{C}$ the non-zero spectrum of operators $\mathcal{P}_\pm(\lambda)$ coincides. Moreover,

$$\sigma(A_{S,h}(\lambda)) \setminus \{0\} = \{\omega \in \mathbb{C}; \; \omega^2 \in \sigma(\mathcal{P}_+(\lambda))\} \setminus \{0\} = \{\omega \in \mathbb{C}; \; \omega^2 \in \sigma(\mathcal{P}_-(\lambda))\} \setminus \{0\}. \quad (5.8)$$
Note that

\[
[q_2(\vartheta)hD_\varphi - \lambda q_3(\vartheta)]^2 = \left(1 + \frac{\Lambda a^2}{3} \sin \vartheta \Delta_{\vartheta} \right)^2 \left(\lambda a \sin \vartheta \Delta_{\vartheta} \right)^2
\]

\[
= \left(1 + \frac{\Lambda a^2}{3} \right)^2 \left(hD_\varphi - \lambda a \sin^2 \vartheta \frac{1}{1 + \frac{\Lambda a^2}{3}} \right)^2.
\]

The leading term of \( \mathcal{P}_\pm \) is given by

\[
\mathcal{P}_{+,0}(\lambda) = \mathcal{P}_{-,0}(\lambda) = \Delta_\vartheta h^2 D_\vartheta^2 + [q_2(\vartheta)hD_\varphi - \lambda q_3(\vartheta)]^2
\]

\[
= \Delta_\vartheta h^2 D_\vartheta^2 + \left(1 + \frac{\Lambda a^2}{3} \right)^2 \left(hD_\varphi - \lambda a \sin^2 \vartheta \frac{1}{1 + \frac{\Lambda a^2}{3}} \right)^2.
\]

We compare it with the leading term of the angular operator in Dyatlov [18], Formula (1.11) (after introducing parameter \( h \)):

\[
P_{\vartheta,0}(\lambda) = \Delta_\vartheta h^2 D_\vartheta^2 + \left(1 + \frac{\Lambda a^2}{3} \right)^2 \left(hD_\varphi - \lambda a \sin^2 \vartheta \right)^2,
\]

which is the same as ours (with \( \tilde{\lambda} = \lambda/(1 + \frac{\Lambda a^2}{3}) = \lambda/E \) instead of \( \lambda \)). Let \( \omega \in \rho(A_{S^2,h}(\lambda)) \).

Then we have the following resolvent identity

\[
(A_{S^2,h}(\lambda) - \omega)^{-1} = (A_{S^2} + \omega) \begin{pmatrix}
(P_+ - \omega^2)^{-1} & 0 \\
0 & (P_- - \omega^2)^{-1}
\end{pmatrix}
\]

or

\[
(A_{S^2,h}(\lambda) - \omega)^{-1} = \begin{pmatrix}
\omega(P_+ - \omega^2)^{-1} & D_+(P_- - \omega^2)^{-1} \\
D_-(P_+ - \omega^2)^{-1} & \omega(P_- - \omega^2)^{-1}
\end{pmatrix},
\]

where \( D_\pm = \Delta_{\vartheta}^{\frac{1}{2}} \left[ hD_{\vartheta} + hq_1(\vartheta) \pm i (q_2(\vartheta)hD_\varphi - \lambda q_3(\vartheta)) \right] \).

**Radial operator.** In \( \mathcal{H} = L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \) we consider two Dirac operators

\[
\mathcal{D}_{h,\pm}(\lambda) := -\mathcal{D}_{h,0} \pm (c(x) - \lambda)I_2, \quad \mathcal{D}_{h,0} := -ih\sigma_3 \partial_x + q(x)\sigma_1 = \begin{pmatrix}
-ih\partial_x & q(x) \\
q(x) & i\partial_x
\end{pmatrix}.
\]

Here we only assume the natural conditions on the real-valued functions \( q, c \) so that \( \mathcal{D}_{h,\pm}(\lambda) \) and all operators below are well-defined and for \( \lambda \in \mathbb{R} \) self-adjoint in \( \mathcal{H} \).

The product of the operators is given by

\[
\mathcal{D}_{h,+}\mathcal{D}_{h,-} = \mathcal{D}_{h,0}^2 - (c - \lambda)^2 I_2 - [\mathcal{D}_{h,0}, (c - \lambda)I_2].
\]

Here

\[
\mathcal{D}_{h,0}^2 = -I_2 h^2 \partial_x^2 + \begin{pmatrix}
q^2 & -ihq'(x) \\
-ihq'(x) & q^2
\end{pmatrix},
\]

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is the matrix Schrödinger operator. The commutator is given by

\[ [\mathcal{D}_{h,0}, (c - \lambda)I_2] = -i\hbar \sigma_3 c'(x). \]

The operator \( \mathcal{D}_{h,0}^2 \) is self-adjoint in \( \mathcal{H} = L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \) and unitary equivalent to

\[ U \mathcal{D}_{h,0}^2 U^{-1} = \begin{pmatrix} \mathcal{P}_{h,0}^- & 0 \\ 0 & \mathcal{P}_{h,0}^+ \end{pmatrix}, \quad \mathcal{P}_{h,0}^\pm = -h^2 \partial_x^2 + q^2 \pm hq'. \quad (5.12) \]

Here,

\[ U = \frac{i}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}, \quad U^{-1} = -\frac{i}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}. \]

We get also

\[ U (\mathcal{D}_{h,0}^2 - (c - \lambda)^2 I_2) U^{-1} = \begin{pmatrix} \mathcal{P}_{h}^- (\lambda) & 0 \\ 0 & \mathcal{P}_{h}^+ (\lambda) \end{pmatrix}, \quad \mathcal{P}_{h}^\pm (\lambda) = -h^2 \partial_x^2 + q^2 - (c - \lambda)^2 \pm hq'. \quad (5.13) \]

Now, using (5.11) we get

\[ \mathcal{D}_{h,+} \mathcal{D}_{h,-} = U^{-1} \begin{pmatrix} \mathcal{P}_{h}^- (\lambda) & 0 \\ 0 & \mathcal{P}_{h}^+ (\lambda) \end{pmatrix} U + i\hbar c'(x)\sigma_3. \quad (5.14) \]

If \( \mathcal{P}_{h}^\pm (\lambda) \) are invertible, denote

\[ \mathcal{R}(\lambda) = U^{-1} \begin{pmatrix} [\mathcal{P}_{h}^- (\lambda)]^{-1} & 0 \\ 0 & [\mathcal{P}_{h}^+ (\lambda)]^{-1} \end{pmatrix} U \]

and write

\[ \mathcal{D}_{h,+} \mathcal{D}_{h,-} \mathcal{R}(\lambda) = I + i\hbar c'(x)\sigma_3 \mathcal{R}(\lambda). \]

Then we get

\[ [\mathcal{D}_{h,+}(\lambda)]^{-1} \left(I_2 + i\hbar \sigma_3 c'(x) (\mathcal{D}_{h,0}^2 - (c - \lambda)^2 I_2)^{-1}\right) = \mathcal{D}_{h,-}(\lambda) (\mathcal{D}_{h,0}^2 - (c - \lambda)^2 I_2)^{-1} \]

which leads to

\[ [\mathcal{D}_{h,+}(\lambda)]^{-1} = \mathcal{D}_{h,-}(\lambda) U^{-1} \begin{pmatrix} [\mathcal{P}_{h}^- (\lambda)]^{-1} & 0 \\ 0 & [\mathcal{P}_{h}^+ (\lambda)]^{-1} \end{pmatrix} U \]

\[ \cdot \left(I_2 + i\hbar c'(x) U^{-1} \begin{pmatrix} [\mathcal{P}_{h}^- (\lambda)]^{-1} & 0 \\ 0 & [\mathcal{P}_{h}^+ (\lambda)]^{-1} \end{pmatrix} U \right)^{-1}. \quad (5.15) \]

As \( U \) is constant matrix, identity (13.5) can be extended to \( q = \omega a \) complex with \( \omega \in \mathbb{C} \). This is used in Section 6.3.
5.3 Angular resolvent estimates, proof of Proposition 5.1

Here, we prove Proposition 5.1.

1) In Section 3.3 it was shown that for $a^2 \lambda$, $|a \text{Im} \lambda|$ small enough $A_k(\lambda) = A_{S^2}(\lambda)_{|\mathcal{H}^k_{S^2}}$ has compact resolvent. Then for each $\omega$, operator $A_{S^2}(\lambda) - \omega : H^1(S^2) \mapsto L^2(S^2)$ is Fredholm and by Proposition 2.2 in [17], $R_\theta(\lambda, \omega) = (A_{S^2}(\lambda) - \omega)^{-1}$ is meromorphic families of operators $L^2(S^2) \mapsto H^1(S^2)$. Decomposition (5.1) and the meromorphic property of the resolvent for $|a \text{Im} \lambda|$ small follow from Section 3.3. General case follows from representation (5.9) in Section 5.2 (putting $h = 1$ there):

$$R_\theta(\lambda, \omega) = (A_{S^2}(\lambda) - \omega)^{-1} = (A_{S^2}(\lambda) + \omega) \begin{pmatrix} (\mathcal{P}_+ (\lambda) - \omega^2)^{-1} & 0 \\ 0 & (\mathcal{P}_- (\lambda) - \omega^2)^{-1} \end{pmatrix},$$

where

$$\mathcal{P}_\pm (\lambda) = \Delta_\theta \left\{ [D_\theta + q_1(\vartheta)]^2 + [q_2(\vartheta)D_\vartheta - \lambda q_3(\vartheta)]^2 \mp (q_2'(\vartheta)D_\vartheta + \lambda q_3'(\vartheta)) \right\},$$

where

$$q_1(\vartheta) = \frac{i \lambda a^2 \sin(2\vartheta)}{12 \Delta_\theta}, \quad q_2(\vartheta) = \frac{E}{\Delta_\theta \sin \vartheta}, \quad q_3(\vartheta) = \frac{a \sin \vartheta}{\Delta_\theta}.$$

We observe that operators $\mathcal{P}_\pm (\lambda)$ are holomorphic families of elliptic second order differential operators on the sphere of the same type as operator $P_\theta(\lambda)$ in the proof of Proposition 3.1 in [17]. Therefore, for each $\omega$, we know that the operators $\mathcal{P}_\pm (\lambda) - \omega^2 : H^2(S^2) \mapsto L^2(S^2)$ are Fredholm and by Proposition 2.2 in [17], $R_\theta(\lambda, \omega)$ is a meromorphic family of operators $L^2(S^2) \mapsto H^1(S^2)$.

In order to prove (5.2) and (5.3) we need some preliminary inequalities following from Section 3.3. Recall that for each $\lambda \in \mathbb{R}$ and $k \in 1/2 + \mathbb{Z}$, $\mu_{kl}$ are the eigenvalues of the self-adjoint operator $A_k(\lambda)$ on $\mathcal{H}_{S^2}^k = L^2((0, \pi), d\vartheta; \mathbb{C}^2)$.

Let $\zeta = \frac{a^2 \lambda}{3}$, $\xi = a \lambda$, and consider operator $A_k(\lambda) = A_{S^2}(\lambda)_{|\mathcal{H}^k_{S^2}}$ as operator-valued function $A_k(\zeta, \xi)$ of complex parameters $\zeta, \xi$ as in (3.15): $A_k(\zeta, \xi) = A(\zeta)D^k_{S^2} + B(\zeta, \xi)$ with $A(\zeta) = \sqrt{1 + \zeta \cos^2 \vartheta}$,

$$B(\zeta, \xi) = \frac{1}{4\sqrt{1 + \zeta \cos^2 \vartheta}} \begin{pmatrix} 0 & i(\zeta \sin(2\vartheta) + 4(\zeta k - \xi) \sin \vartheta) \\ i(\zeta \sin(2\vartheta) - 4(\zeta k - \xi) \sin \vartheta) & 0 \end{pmatrix}.$$

Operator $A_k(0, 0) \equiv D^k_{S^2} = \sigma_1 D_\theta - \sigma_2 \frac{k}{\sin \vartheta}$ is restriction of the standard Dirac operator on $S^2$ onto the angular mode $\{e^{ik\varphi}\}, k \in 1/2 + \mathbb{Z}$, with domain

$$\mathcal{D} = \{u \in \mathcal{H}_{S^2}^k, u \text{ is absolutely continuous } D_{S^2}^k u \in \mathcal{H}_{S^2}^k, u(\pi) = -u(0)\},$$

and simple discrete spectrum $\mu_{k,l}(0, 0) = \text{sgn}(k) \left(|k| - \frac{1}{2} + |l|\right), l \in \mathbb{Z^*}$. 

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Now, using \[32\] (Chap. VII, Sect. 3, Theorem 3.6) as in \[14\] we get for all $\zeta \in [0, \frac{1}{13}]$, $k \in \frac{1}{2} + \mathbb{Z}$ and $l \in \mathbb{N}^*$,

$$|\mu_{kl}(\zeta, 0) - (k - \frac{1}{2} + l)| \leq (e^{\frac{1}{26}} - 1) (|k| - \frac{1}{2} + l) + 2 \left( e^{\frac{1}{26}} - 1 \right) \left( 1 + \frac{1}{26} \right) \left( |k| + \frac{1}{4} \right),$$

$$\mu_{kl}(\zeta, 0) \geq \left( 2 - e^{\frac{1}{26}} \right) (|k| - \frac{1}{2} + l) - 2 \left( e^{\frac{1}{26}} - 1 \right) \left( 1 + \frac{1}{26} \right) \left( |k| + \frac{1}{4} \right) \geq \left( 4 + \frac{1}{13} - \frac{3 + 1}{13} e^{\frac{1}{26}} \right) \left( |k| + \frac{1}{4} \right) > 0.8 \left( |k| + \frac{1}{4} \right).$$

For $l < 0$ we use $\mu_{k,-l} = -\mu_{kl}$ and get $\mu_{kl}(\zeta, 0) < -0.8 \left( |k| + \frac{1}{4} \right)$. Therefore, if $u \in H^1(S^2) \cap D'_k$, then

$$\|u\|_{L^2} \leq \frac{\|(A(\zeta)D_{S^2}^k + B(\zeta, 0) - \omega)u\|_{L^2}}{d(\omega, \mathbb{R} \setminus (0.8 \left( |k| + \frac{1}{4} \right))(-1, 1))}.$$ 

Now,

$$B(\zeta, \xi) - B(\zeta, 0) = \frac{1}{\sqrt{1 + \xi^2}} \left( \begin{array}{cc} 0 & -i\xi \sin \vartheta \\ i\xi \sin \vartheta & 0 \end{array} \right),$$

$$(B(\zeta, \xi) - B(\zeta, 0))^2 = \frac{\xi^2}{1 + \xi^2} \left( \begin{array}{cc} \vartheta^2 & 0 \\ 0 & \vartheta^2 \end{array} \right) I_2$$

and we get

$$\|B(\zeta, \xi) - B(\zeta, 0)\|_{L^2(S^2) \cap \mathcal{D}'_k} \leq |\xi| = a|\lambda|.$$ 

Therefore,

$$\|u\|_{L^2} \leq \frac{\|A_k(\zeta, \xi) - \omega)u\|_{L^2}}{d(\omega, \mathbb{R} \setminus (0.8 \left( |k| + \frac{1}{4} \right))(-1, 1)) - |a\lambda|}, \quad (5.16)$$

provided that the denominator is positive.

2) Using \[5.16\] we prove \[5.2\]. Let $|\omega| \leq 0.8 \left( |k| + \frac{1}{4} \right)/2$. Then $d(\omega, \mathbb{R} \setminus (0.8 \left( |k| + \frac{1}{4} \right))(-1, 1)) \geq 0.8 \left( |k| + \frac{1}{4} \right)/2$ and

$$d(\omega, \mathbb{R} \setminus (0.8 \left( |k| + \frac{1}{4} \right))(-1, 1)) - |a\lambda| \geq 0.4 \left( |k| + \frac{1}{4} \right) - |a\lambda| \geq 0.2|k|$$

if $|k| \geq |a\lambda|/0.2$.

In order to prove \[5.3\] we calculate $\text{Im} A_k(\zeta, \xi) = \frac{1}{2} (A_k(\zeta, \xi) - A_k^*(\zeta, \xi))$ and get

$$\text{Im} A_k(\zeta, \xi) = \text{Im} (B(\zeta, \xi) - B(\zeta, 0)) = \frac{1}{4\sqrt{1 + \xi^2}} \left( \begin{array}{cc} \xi^2 & 0 \\ 0 & \xi^2 \end{array} \right)$$

$$= \frac{1}{4\sqrt{1 + \xi^2}} \left( \begin{array}{cc} 0 & -i4\xi \sin \vartheta \\ i4\xi \sin \vartheta & 0 \end{array} \right).$$
As
\[
(\text{Im } A_k(\zeta, \xi))^2 = \frac{(\text{Im } \xi)^2 \sin^2 \vartheta}{1 + \zeta \cos^2 \vartheta} I_2
\]
we get
\[
\|\text{Im } A_k(\zeta, \xi)\|_{L^2(S^2) \cap D'_k} \leq |\text{Im } \xi| = a|\text{Im } \lambda|.
\]
However, for \( u \in H^1(S^2) \cap D'_k \),
\[
\|(A_k(\zeta, \xi) - \omega)u\| \cdot \|u\| \geq |\text{Im}(A_k(\zeta, \xi) - \omega)u| \geq |\text{Im } \omega| \cdot \|u\|^2 - |(\text{Im } A_k(\zeta, \xi), u, u)|
\]
\[
\geq (|\text{Im } \omega| - a|\text{Im } \lambda|)\|u\|.
\]
If \( |\text{Im } \omega| > 2a|\text{Im } \lambda| \) we get \( \|(A_k(\zeta, \xi) - \omega)u\| \geq a|\text{Im } \lambda| \) and therefore (5.3).

3) We use (5.16). If \( \psi \leq |\arg \omega| \leq \pi - \psi \) (equivalently \( \psi \leq \arg \omega \leq \pi - \psi \) or \( -\pi + \psi \leq \arg \omega \leq -\psi \)) then \( d(\omega, \mathbb{R} \setminus (0.8(k + \frac{1}{4}))(-1, 1)) \geq (|k| + |\omega|)/C_2 \) with some constant \( C_2 \) depending on \( \psi \). We get
\[
d(\omega, \mathbb{R} \setminus (0.8(k + \frac{1}{4}))(-1, 1)) - |a\lambda| \geq (|k| + |\omega|)/C_2 - C_1|a\lambda| \geq |\omega|/C_2 - C_1|a\lambda|,
\]
which implies (5.4).

### 6 Radial resolvent estimates, proof of Proposition 5.2

Here we prove Proposition 5.2

#### 6.1 Preliminaries.

In this section we construct the outgoing (Jost) solutions to the radial Dirac equation. We consider the radial resolvent \( R_r(\lambda, \omega, k) = (H_k(\lambda) + \omega)^{-1} \), where
\[
H_k(\lambda) := [a(x)]^{-1}(\sigma_3 D_x + c(x, k) - \lambda)\sigma_1.
\]

Note that \( R_r(\lambda, \omega, k) \) satisfies
\[
([a(x)]^{-1}(\sigma_3 D_x + c(x, k) - \lambda)\sigma_1 + \omega) R_r(\lambda, \omega, k) f = f, \quad f \in C_0^\infty(\mathbb{R}, dx : \mathbb{C}^2),
\]
\[
\Leftrightarrow (\sigma_3 D_x + c(x, k) - \lambda + \omega a(x)\sigma_1) \sigma_1 R_r(\lambda, \omega, k) f = a(x) f
\]
\[
\Leftrightarrow R_r(\lambda, \omega, k) = \sigma_1 [\sigma_3 D_x + c(x, k) - \lambda + \omega a(x)\sigma_1]^{-1} a(x).
\]

Let \( R(\lambda, \omega, k) = [D_k^r(\lambda, \omega)]^{-1} \) be resolvent of \( D_k^r(\lambda, \omega) = \sigma_3 D_x + c(x, k) - \lambda + \omega a(x)\sigma_1 \).
Here
\[
c(x, k) = \frac{aEk + qQr}{r^2 + a^2}, \quad a(x) = \frac{\sqrt{\Delta r}}{r^2 + a^2}
\]
satisfying (see (3.13), (3.14))

\[ a(x) = a_\pm e^{\kappa_\pm x} + O\left(e^{3\kappa_\pm x}\right), \quad x \to \pm \infty, \]  
(6.1)

\[ c(x, k) = \Omega_\pm(k) + c_\pm e^{2\kappa_\pm x} + O\left(e^{4\kappa_\pm x}\right), \quad x \to \pm \infty, \]  
(6.2)

\[ \Omega_\pm = \frac{AEk + aQr_\pm}{r_\pm^2 + a^2}, \quad \Omega_- > \Omega_. \]

Write

\[ V(x; \lambda, \omega, k) = c(x, k) - \lambda + \omega a(x)\sigma_1, \quad \mathcal{D}_k^x(\lambda, \omega) := \sigma_3 D_x + V(x; \lambda, \omega, k). \]

We consider the Dirac equation (6.3)

\[ \mathcal{D}_k^x(\lambda, \omega) f = (\sigma_3 D_x + V(x; \lambda, \omega, k)) f = 0 \]  
(6.3)

for a vector valued function \( f(x) = f_1(x)e_+ + f_2(x)e_- \), where \( f_1, f_2 \) are the functions of \( x \in \mathbb{R} \),

\[ f(x) = f_1(x)e_+ + f_2(x)e_-, \quad e_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

\[ \begin{cases} -if'_1 + c(x, k)f_1 + \omega a(x)f_2 = \lambda f_1 \\ if'_2 + c(x, k)f_2 + \omega a(x)f_1 = \lambda f_2 \end{cases} \quad \lambda, \omega \in \mathbb{C}^2. \]  
(6.4)

Note that in [31] we studied the properties of similar equation with \( c(x, k) = 0 \) and \( a(x) \) with compact support. Note also the following property which will be used in the proof of part 3) of Proposition 5.2.

**Remark 6.1.** Note that if \( f = (f_1(\lambda, \omega), f_2(\lambda, \omega))^T \) is solution of (6.3) with \( \lambda, \omega \in \mathbb{C}^2 \), then \( \tilde{f} := (\overline{f_2(\lambda, \omega)}, \overline{f_1(\lambda, \omega)})^T \) is also the solution of (6.3) with the same \( \lambda, \omega \).

Due to (6.1) and (6.2) we have

\[ V(x; \lambda, \omega, k) = \Omega_\pm(k) - \lambda + \omega a_\pm\sigma_1 e^{\kappa_\pm x} + c_\pm e^{2\kappa_\pm x} + O\left(e^{3\kappa_\pm x}\right), \quad x \to \pm \infty \]  
(6.5)

and the system (6.4) in the limit \( x \to \pm \infty \) is given by

\[ \begin{cases} -if'_1 + (\Omega_\pm(k) + O\left(e^{2\kappa_\pm x}\right)) f_1 + \omega (a_\pm e^{\kappa_\pm x} + O\left(e^{3\kappa_\pm x}\right)) f_2 = \lambda f_1 \\ if'_2 + (\Omega_\pm(k) + O\left(e^{2\kappa_\pm x}\right)) f_2 + \omega (a_\pm e^{\kappa_\pm x} + O\left(e^{3\kappa_\pm x}\right)) f_1 = \lambda f_2 \end{cases} \quad \lambda, \omega \in \mathbb{C}^2. \]

The Regge-Wheeler coordinate \( x \) is given by

\[ x = \int_{r_0}^r \frac{s^2 + a^2}{\Delta_r(s)} ds, \]

where \( r_0 \in (r_-, r_+) \) is a fixed number or explicitly

\[ x = \frac{1}{2\kappa_-} \ln(r - r_-) + \frac{1}{2\kappa_+} \ln(r_+ - r) + \frac{1}{2\kappa_c} \ln(r - r_c) + \frac{1}{2\kappa_n} \ln(r - r_n) + C \in \mathbb{R} \]  
24
with an integration constant $C$. Here
\[
\kappa_\sigma = \frac{\Delta'_r(r_\sigma)}{2(r_\sigma^2 + a^2)}, \quad \sigma = -, +, c, n,
\]
and $\kappa_- > 0$, $\kappa_+ < 0$.

We recall the following result by Dyatlov [18], Proposition 4.1

**Proposition 6.1** (Dyatlov). There exists a constant $X_0$ such that for $\pm x > X_0$, it follows $r = r_\pm \mp F_\pm(e^{\kappa x})$, where $F_\pm(w)$ are real analytic on $[0, e^{\pm\kappa X_0})$ and holomorphic in the discs $\{ |w| < e^{\pm\kappa X_0} \} \subset \mathbb{C}$.

**Proof.** For $x(r)$ near $r = r_+$ we have $2\kappa_+ x(r) = \ln(r_+ - r) + G(r)$, where $G$ is holomorphic near $r = r_+$. Then
\[
w_2 := e^{2\kappa_+ x} = (r_+ - r)e^{G(r)}.
\]

We apply the inverse function theorem to solve for $r$ as a function of $w$ near zero.

Together with the similar analysis near $r = r_-$ it implies that there exists a constant $X_0 > 0$ such that for $\pm x > X_0$, we have $r = r_\pm \mp F_\pm(e^{\kappa x})$, where $F_\pm(w)$ are real analytic on $[0, e^{\pm\kappa X_0})$ and holomorphic in the discs $\{ |w| < e^{\pm\kappa X_0} \} \subset \mathbb{C}$.

We get by this proposition that
\[
V(x, \lambda, \omega, k) = V_\pm(e^{\kappa x}), \quad \pm x > X_0,
\]
where $V_\pm(w)$ are functions holomorphic in the discs $\{ |w| < e^{\pm\kappa X_0} \}$, and $V_\pm(e^{\kappa x}) \to \Omega_\pm(k) - \lambda$ as $x \to \pm\infty$.

Using (6.5) we can consider Hamiltonian $\sigma_3 D_x + V(x, \lambda, \omega, k)$ as asymptotic perturbation for $x \to \pm\infty$ of the “free” operators $\sigma_3 D_x + \Omega_\pm(k) - \lambda$. Then the “free” Jost solutions $f^{0,\pm}$ such that $[\sigma_3 D_x + \Omega_\pm(k) - \lambda] f^{0,\pm} = 0$ are given by (compare with [31])
\[
f^{0-,}(x; \lambda, k) = \begin{pmatrix} 0 \\ e^{-i(\lambda - \Omega_-(k))x} \end{pmatrix}, \quad f^{0,+}(x; \lambda, k) = \begin{pmatrix} e^{i(\lambda - \Omega_+(k))x} \\ 0 \end{pmatrix}.
\]

**Definition 6.1** (Outgoing solution). Let $\lambda$ and $(k, l) \in I$ be fixed constants. A function $f(x) := f(x; \lambda, \omega, k)$ is called outgoing at $-\infty$ if and only if
\[
f(x) = e^{-i(\lambda - \Omega_-(k))x} \begin{pmatrix} v^-_1(e^{\kappa_- x}; \lambda, \omega, k) \\ v^-_2(e^{\kappa_- x}; \lambda, \omega, k) \end{pmatrix}
\]
and $v^-_1(\bullet; \lambda, \omega, k)$, $v^-_2(\bullet; \lambda, \omega, k)$ are meromorphic in a neighborhood of zero.

A function $f(x) := f(x; \lambda, \omega, k)$ is called outgoing at $+\infty$ if and only if
\[
f(x) = e^{i(\lambda - \Omega_+(k))x} \begin{pmatrix} v^+_1(e^{\kappa_+ x}; \lambda, \omega, k) \\ v^+_2(e^{\kappa_+ x}; \lambda, \omega, k) \end{pmatrix}
\]
and $v^+_1(\bullet; \lambda, \omega, k)$, $v^+_2(\bullet; \lambda, \omega, k)$ are holomorphic in a neighborhood of zero.

If $f$ is outgoing at both infinities, we call it outgoing.
In variable $w = e^{\kappa x}$ we get $e^{i(\lambda - \Omega_+)(k)x} = \frac{w^{i(\lambda - \Omega_+)(k)}}{w_{1+}}$. Moreover, as $w^2 = (r_+ - r)e^{G(r)}$, with function $G$ holomorphic near $r_+$, we get

$$e^{i(\lambda - \Omega_+)(k)x} = (r_+ - r)^{i(\lambda - \Omega_+)(k)} \frac{w^{i(\lambda - \Omega_+)(k)}}{w_{1+}} e^{G(r)} \frac{w^{i(\lambda - \Omega_+)(k)}}{w_{1+}}.$$ 

Therefore, for the outgoing at $+\infty$ function we have

$$f(x) = (r_+ - r)^{i(\lambda - \Omega_+)(k)} \frac{w^{i(\lambda - \Omega_+)(k)}}{w_{1+}} u_+(r; \lambda, \omega, k),$$

for the outgoing at $-\infty$ function we have

$$f(x) = (r - r_-)^{-i(\lambda - \Omega_-)(k)} \frac{w^{i(\lambda - \Omega_-)(k)}}{w_{1-}} u_-(r; \lambda, \omega, k),$$

with $\mathbb{C}^2$-valued functions $u_\pm(r; \lambda, \omega, k)$ real analytic at $r = r_\pm$ respectively.

We get

**Proposition 6.2.** Let $D^r_k(\lambda, \omega) = \sigma_3 D_x + V(x; \lambda, \omega, k)$, $V(x; \lambda, \omega, k) = c(x, k) - \lambda + \omega a(x) \sigma_1$.

1) There exist outgoing at $\pm\infty$ solutions $f^\pm$ to the equation $D^r_k(\lambda, \omega)f^\pm = 0$ of the form

$$f^\pm(x) = e^{\pm i(\lambda - \Omega_\pm)(k)x} \begin{pmatrix} v_1^\pm(e^{\kappa_\pm x}; \lambda, \omega, k) \\ v_2^\pm(e^{\kappa_\pm x}; \lambda, \omega, k) \end{pmatrix},$$

where $v_\pm(w; \lambda, \omega, k)$ are holomorphic in $\{|w| < W_\pm(e^{\pm\kappa_\pm x_0})\}$ and

$$v_1^- (0; \lambda, \omega, k) = 0, \quad v_2^- (0; \lambda, \omega, k) = \frac{1}{\Gamma\left(1 + 2\frac{\lambda - \Omega_-}{\kappa_-}\right)}, \quad (6.7)$$

$$v_1^+ (0; \lambda, \omega, k) = \frac{1}{\Gamma\left(1 - 2\frac{\lambda - \Omega_+}{\kappa_+}\right)}, \quad v_2^+ (0; \lambda, \omega, k) = 0. \quad (6.8)$$

These solutions are holomorphic in $(\lambda, \omega)$ and are unique unless

$$-2\frac{\lambda - \Omega_-}{\kappa_-} \quad \text{or respectively} \quad 2\frac{\lambda - \Omega_+}{\kappa_+} \quad \text{is positive integer.} \quad (6.9)$$

Equivalently

$$\lambda \in \Omega_+ + i\frac{1}{2}\kappa_+ Z_+ \in \mathbb{C}_- \quad \text{(for } v^+) \quad \lambda \in \Omega_- - i\frac{1}{2}\kappa_- Z_- \in \mathbb{C}_- \quad \text{(for } v^-). \quad (6.10)$$

2) For $\lambda$ as in (6.9) the solutions $f^\pm$ can be identically zero. However, assume that one of the solutions $f_\pm$ is identically zero. Then every solution $f$ to the equation $D^r_k(\lambda, \omega)f = 0$ is outgoing at the corresponding infinity.
Proof.
1) We consider only function \( f^+ \), as construction of \( f^- \) is similar. Let

\[
f^+(x) = e^{i(\lambda-\Omega_+)x} \left( \begin{array}{c} v_1^+(e^{\kappa_+}; \lambda, \omega, k) \\ v_2^+(e^{\kappa_+}; \lambda, \omega, k) \end{array} \right).
\]

Recall that \( V(x; \lambda, \omega, k) = c(x, k) - \lambda + \omega a(x) \sigma_1 \). We write \( v^+(e^{\kappa_+}) := v^+(e^{\kappa_+}; \lambda, \omega, k) \), \( v^+ = (v_1^+, v_2^+)^T \). Now,

\[
(D_x + c(x, k) - \lambda) e^{i(\lambda-\Omega_+)x} v_1^+(e^{\kappa_+}) + \omega a(x) v_2^+(e^{\kappa_+})
= e^{i(\lambda-\Omega_+)x} (\lambda - \Omega_+ + c(x, k) - \lambda + D_x) v_1^+(e^{\kappa_+}) + \omega a(x) v_2^+(e^{\kappa_+}) e^{i(\lambda-\Omega_+)x},
\]

and we get the first equation with \( w = e^{\kappa_+} \)

\[
(w\kappa_+ D_w + c(x, k) - \Omega_+) v_1^+(w) + \omega a(x) v_2^+(w) = 0.
\]

Now, from

\[
(-D_x + c(x, k) - \lambda) e^{i(\lambda-\Omega_+)x} v_1^+(e^{\kappa_+}) + \omega a(x) e^{i(\lambda-\Omega_+)x} v_2^+(e^{\kappa_+})
= e^{i(\lambda-\Omega_+)x} (-\lambda + \Omega_+ + c(x, k) - \lambda - D_x) v_1^+(e^{\kappa_+}) + \omega a(x) v_2^+(e^{\kappa_+}) e^{i(\lambda-\Omega_+)x},
\]

we get \((-w\kappa_+ D_w + \Omega_+ + c(x, k) - 2\lambda) v_2^+(w) + \omega a(x) v_1^+(w) = 0.\)

Then equation \( D_x^2(\lambda, \omega) f^+ = 0 \) is equivalent to the system

\[
\begin{cases}
(w\kappa_+ D_w + c(x, k) - \Omega_+) v_1^+(w) + \omega a(x) v_2^+(w) = 0 \\
(-w\kappa_+ D_w + \Omega_+ + c(x, k) - 2\lambda) v_2^+(w) + \omega a(x) v_1^+(w) = 0.
\end{cases}
\]

Let \( x > X_0 \). Then there are holomorphic functions \( c^+(w), a^+(w) \) such that

\[
c(x, k) = c^+(e^{\kappa_+}), \quad a(x) = a^+(e^{\kappa_+}), \quad x > X_0.
\] (6.11)

Now, we consider

\[
\begin{cases}
(w\kappa_+ D_w + c^+(w) - \Omega_+) v_1^+(w) + \omega a^+(w) v_2^+(w) = 0 \\
(-w\kappa_+ D_w + \Omega_+ + c^+(w) - 2\lambda) v_2^+(w) + \omega a^+(w) v_1^+(w) = 0.
\end{cases}
\]

We construct the Taylor series of \( v^+ = (v_1^+, v_2^+)^T :\)

\[
v_i^+(w) = \sum_{j \geq 0} v_{i,j} w^j, \quad i = 1, 2.
\]

Now, we omit + in all indexes. Let

\[
c = \sum_{j \geq 0} c_j w^j, \quad c_0 = \Omega, \quad a = \sum_{j \geq 1} a_j w^j.
\]
At order $w^0$, $j = 0$, the system

$$
\begin{align*}
  c_0 - \Omega &= 0 \\
  c_0 v_{2,0} + (\Omega - 2\lambda)v_{2,0} &= 0.
\end{align*}
$$

(6.12)

is satisfied as $c_0 - \Omega = 0$ and if $(\Omega_+ + c_{0,+} - 2\lambda)v_{2,0}^+ = 0$ which force $v_{2,0}^+ = 0$ unless $\lambda = \Omega_+$. We get the system of equations of order $w^j$, $j \geq 1$:

$$
\begin{align*}
  -i\kappa j v_{1,j} + \sum_{1 \leq l \leq j} c_l v_{1,j-l} + \omega \sum_{1 \leq l \leq j} a_l v_{2,j-l} &= 0 \\
  i\kappa v_{2,j} + \sum_{0 \leq l \leq j} c_l v_{2,j-l} + (\Omega - 2\lambda)v_{2,j} + \omega \sum_{1 \leq l \leq j} a_l v_{1,j-l} &= 0.
\end{align*}
$$

(6.13)

We solve system (6.13) by induction. For $j = 1$ we get

$$
\begin{align*}
  -i\kappa v_{1,1} + c_1 v_{1,0} &= 0 \quad \Rightarrow \quad v_{1,1} = \frac{c_1}{i\kappa} v_{1,0} \\
  i\kappa v_{2,1} + c_0 v_{2,1} + (\Omega - 2\lambda)v_{2,1} + \omega a_1 v_{1,0} &= 0 \quad \Rightarrow \quad v_{2,1} = -\frac{\omega a_1}{i\kappa + 2(\Omega - \lambda)} v_{1,0}.
\end{align*}
$$

as $c_0 = \Omega$. We can choose $v_{1,0} = 1$ if we want a Jost solution $\psi^+$. But then solution will not be holomorphic! We will chose Gamma function with simple zeros precisely at poles of the solution, see below. In order to get $v_{2,1}$ we need that the denominator is non-zero:

$$
i\kappa + 2(\Omega - \lambda) \neq 0 \quad \Leftrightarrow \quad \lambda \neq \Omega_+ + i\frac{1}{2}\kappa_+ \in \mathbb{C}_-
$$

as $\kappa = \kappa_+ < 0$. If $\lambda = \Omega_+ + i\frac{1}{2}\kappa_+$ and $\omega \neq 0$ then there is no solution satisfying $v_{1,0} = 1$ as necessarily $v_{1,0} = 0$ and $v_{1,1} = 0$, $v_{2,1}$ is arbitrary.

Now, suppose we know $\{v_{1,l}, v_{2,l}\}_{l \leq j}$. Then the system of equations of order $w^j$

$$
\begin{align*}
  -i(j + 1)\kappa v_{1,j+1} + \sum_{1 \leq l \leq j+1} c_l v_{1,j+1-l} + \omega \sum_{1 \leq l \leq j+1} a_l v_{2,j+1-l} &= 0 \\
  i\kappa(j + 1) v_{2,j+1} + \sum_{0 \leq l \leq j+1} c_l v_{2,j+1-l} + (\Omega - 2\lambda)v_{2,j+1} + \omega \sum_{1 \leq l \leq j+1} a_l v_{1,j+1-l} &= 0.
\end{align*}
$$

(6.14)

has solutions

$$
\begin{align*}
  v_{1,j+1} &= \frac{1}{i(j + 1)\kappa} \left( \sum_{1 \leq l \leq j+1} c_l v_{1,j+1-l} + \omega \sum_{1 \leq l \leq j+1} a_l v_{2,j+1-l} \right), \\
  v_{2,j+1} &= -\frac{1}{i\kappa(j + 1) + 2(\Omega - \lambda)} \left( \sum_{1 \leq l \leq j+1} c_l v_{2,j+1-l} + \omega \sum_{1 \leq l \leq j+1} a_l v_{1,j+1-l} \right).
\end{align*}
$$

The last denominator is non-zero if $\lambda \neq \Omega_+ + i\frac{1}{2}\kappa_+(j + 1) \in \mathbb{C}_-$, i.e. if $\frac{2\lambda - \Omega}{\kappa_+}$ is not positive integer.
Now, if \( \lambda = \Omega + i_2 \kappa (j + 1) \) for some \( j = j_0 \), then solution \( v_{2,j+1} \) exists and is arbitrary if and only if
\[
\sum_{1 \leq l < j + 1} c_l v_{2,j+1-l} + \omega \sum_{1 \leq l < j + 1} a_l v_{1,j+1-l} = 0
\]
- which is only possible for discrete values of \( \omega \) - or if solution is identically zero in these discrete values. Thus if we want a holomorphic solution we chose
\[
v_{1,0} = \frac{1}{\Gamma \left( 1 - 2 \frac{\lambda - \Omega_+}{i \kappa_+} \right)}.
\]

The convergence of the series is proved by induction.

2) The same argument as in the proof of Proposition 4.3 in [17] applies. Assume that \( f_+^+(x; \lambda_0, \omega_0, k_0) \equiv 0 \) (similar for \( f_-^- \)). Then \( \nu = 2(\lambda_0 - \Omega_+)/i \kappa_+ \) has to be a positive integer. As in part 1), we can construct a nonzero solution \( f_1 \) to the equation \( D_k^r(\lambda, \omega)g = 0 \) with
\[
f_1(x) = e^{-i(\lambda_0 - \Omega_+)x} \left( \hat{v}_1(e^{\kappa_+ x}) \right)
\]
and \( \hat{v} \) is holomorphic at zero. We can see that
\[
f_1(x) = e^{i(\lambda_0 - \Omega_+)x} e^{-2i(\lambda_0 - \Omega_+)x} \hat{v}'(e^{\kappa_+ x}) = e^{i(\lambda_0 - \Omega_+)x} \frac{e^{-2i(\lambda_0 - \Omega_+)x}}{2\kappa_+ x} \hat{v}'(e^{\kappa_+ x})
\]
\[
= e^{i(\lambda_0 - \Omega_+)x} w^\nu \hat{v}(e^{\kappa_+ x}) = e^{i(\lambda_0 - \Omega_+)x} v(e^{\kappa_+ x}),
\]
where \( v(w) = w^\nu \hat{v}(w) \) is holomorphic at zero. Therefore, \( f_1 \) is outgoing at \( +\infty \). Note that \( f_1(x) = o(e^{i(\lambda_0 - \Omega_+)x}) \) as \( x \to +\infty \).

Now, since \( f_+^+(x; \lambda_0, \omega_0, k_0) \equiv 0 \), we can define
\[
f_2(x) = \lim_{\lambda \to \lambda_0} \Gamma \left( 1 - 2 \frac{\lambda - \Omega_+}{i \kappa_+} \right) f_+^+(x; \lambda, \omega_0, k_0),
\]
which is an outgoing solution to \( D_k^r(\lambda, \omega)f_2 = 0 \) and satisfies
\[
f_2(x) = e^{i(\lambda_0 - \Omega_+)x} \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} + o(1) \right),
\]
where \( \nu_\pm(w; \lambda, \omega, k) \) are holomorphic in \( \{|w| < W_\pm(e^{\pm \kappa_0 X_0})\} \) and
\[
v_2^- (0; \lambda, \omega, k) = \frac{1}{\Gamma \left( 1 + 2 \frac{\lambda - \Omega_+}{i \kappa_-} \right)}, \quad v_1^+ (0; \lambda, \omega, k) = \frac{1}{\Gamma \left( 1 - 2 \frac{\lambda - \Omega_+}{i \kappa_+} \right)}.
\]
6.2 Proof of parts 1), 3) and 4) in Proposition 5.2.

Proof of 1). The integral kernel of the resolvent

\[ R(\lambda, \omega, k) := [\mathcal{D}_k^T(\lambda, \omega)]^{-1}, \quad \mathcal{D}_k(\lambda, \omega) = \sigma_2 D_x + c(x, k) - \lambda + \omega a(x) \sigma_1, \]

is given by

\[ R(x, y; \lambda, \omega, k) = \begin{cases} \frac{1}{\det(\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial y})} f^+(x, \lambda) f^-(y, \lambda) \big|_{\lambda = \lambda_0} & \text{if } y < x, \\ \frac{1}{\det(\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial y})} f^+(x, \lambda) f^-(y, \lambda) \big|_{\lambda = \lambda_0} & \text{if } x < y, \end{cases} \]

where \( f^+ = \psi^+ \), \( f^- = \varphi^- \) are the outgoing solutions.

The Wronskian \( W(\lambda, \omega, k) := \det(f^+, f^-) \) is independent of \( x \). We define the integral operator \( S \) by its kernel \( S(x, y; \lambda, \omega, k) = W(\lambda, \omega) \cdot R(x, y; \lambda, \omega, k). \)

Using Definition 6.1 of the outgoing solutions we have

\[ \det(f^+, f^-) = \begin{vmatrix} e^{i(\lambda-\Omega_+)(k)x} v^+_1(e^{k_+x}; \lambda, \omega, k) & e^{-i(\lambda-\Omega_+)(k)x} v^-_1(e^{k_-x}; \lambda, \omega, k) \\ e^{i(\lambda-\Omega_-)(k)x} v^+_2(e^{k_+x}; \lambda, \omega, k) & e^{-i(\lambda-\Omega_-)(k)x} v^-_2(e^{k_-x}; \lambda, \omega, k) \end{vmatrix} = e^{i(\Omega_-(k)-\Omega_+(k))x} (v^+_1(e^{k_+x}; \lambda, \omega, k) v^-_2(e^{k_-x}; \lambda, \omega, k) - v^-_1(e^{k_-x}; \lambda, \omega, k) v^+_2(e^{k_+x}; \lambda, \omega, k)). \]

Then \( \mathcal{D}_k^T(\lambda, \omega) R(\lambda, \omega, k) f = f. \)

Now, let \( R_r(\lambda, \omega, k) = \sigma_1 R(\lambda, \omega, k) a(x). \) Then \( (H_k(\lambda) + \omega) R_r(\lambda, \omega, k) = I. \) Resolvent \( R_r(\lambda, \omega, k) \) is a meromorphic family of operators \( L^2_{\text{comp}} \Rightarrow H^1_{\text{loc}}. \) The proof of the fact that \( R_r \) has poles of finite rank is done by induction as in [17], Section 4. It follows by differentiating \( l \) times the identity \( (H_k(\lambda) + \omega) R_r(\lambda, \omega, k) = I_2 \) in \( \lambda. \)

Proof of 3) Resolvent \( R(\lambda, \omega, k) \) does not have any poles for real \( \omega \) and real \( \lambda. \)

Assume that \( \lambda \) and \( \omega \) are both real and \( R \) has a pole at \( (\lambda, \omega, k) \). Then by Proposition 6.2 there exists the corresponding resonant state \( f, \) i.e. a nonzero solution \( f \in C^\infty(\mathbb{R}, \mathbb{C}^2) \) to the equation \( \mathcal{D}_k^T(\lambda, \omega) f = 0 \) such that \( f \) is outgoing in the sense of Definition 6.1. We know that it has the asymptotics

\[ f^+(x) = e^{i(\lambda-\Omega_+)(k)x} v^+, \quad v^+ = \begin{pmatrix} C_+ \\ 0 \end{pmatrix} + \mathcal{O}(e^{k_+x}), \quad x \to +\infty, \]

\[ f^-(x) = e^{-i(\lambda-\Omega_-)(k)x} v^-, \quad v^- = \begin{pmatrix} 0 \\ C_- \end{pmatrix} + \mathcal{O}(e^{k_-x}), \quad x \to -\infty, \]

for some nonzero constants \( C_\pm. \) Since entries of \( V \) are real-valued and due to Remark 6.1 both \( f(\lambda, \omega, k) \) and \( \tilde{f}(\lambda, \omega, k) \) solve Dirac equation (6.3). Then the Wronskian \( W = \det(f, \tilde{f}) \) must be constant. However,

\[ \det(f, \tilde{f}) \to \pm |C_\pm|^2 \quad \text{as } x \to \pm \infty \]

which leads to the contradiction.

Proof of 4) follows the lines of the proof of 4) of Proposition 3.2 in [17]. If neither \( f^\pm \) is identically zero, then the resolvent \( R \) has a pole if and only if the functions \( f^\pm \) are linearly dependent, or, equivalently, if there exists a nonzero outgoing (at both ends) solution \( f \) to the equation \( \mathcal{D}_k^T(\lambda, \omega) f = 0. \) Now, if one of \( f^\pm, \) say \( f^+, \) is identically zero, then by part 2) of Proposition 6.2 \( u^- \) will be an outgoing solution at both infinities.
6.3 Proof of 2) in Proposition 5.2.

Now, we are able to prove part 2) from Proposition 5.2. Take $h > 0$ such that $|\text{Re } \omega| = h^{-1}$ and let $s = \text{sign } \text{Re } \omega$. Put
\[
\tilde{\lambda} = h\lambda, \quad \tilde{k} = hk, \quad \tilde{\Omega}_\pm = h\Omega_\pm, \quad \tilde{\mu} = h \text{Im } \omega.
\]
Then (5.5) implies
\[
|\tilde{\mu}| \leq \epsilon_r, \quad |ak\tilde{k}| \leq \epsilon_r, \quad |\tilde{\lambda}| \leq \epsilon_r,
\]
where $\epsilon_r > 0$ and $h$ can be made arbitrary small by choice of $C_r$ and $\psi$. Let
\[
\tilde{D}_{h,+}(\lambda) := hD_x^\pm(\lambda, \omega) = \sigma_3 hD_x + \tilde{V}_h, \quad \tilde{V}_h(x, \tilde{\lambda}, \tilde{\mu}, \tilde{k}) = (s + i\tilde{\mu})a(x)\sigma_1 + \tilde{c}_h(x, \tilde{k}) - \tilde{\lambda},
\]
\[
\tilde{c}_h(x, \tilde{k}) = \frac{aE\tilde{k}}{r^2 + a^2} + h\frac{qQr}{r^2 + a^2}.
\]
Now, we apply the resolvent identity (13.5)
\[
[D_{h,+}(\lambda)]^{-1} = D_{h,-}(\lambda)U^{-1} \begin{pmatrix} [P_\lambda^+(\lambda)]^{-1} & 0 \\ 0 & [P_\lambda^-(\lambda)]^{-1} \end{pmatrix} U^{-1} \cdot \left( I_2 + ih\sigma_3c'(x)U^{-1} \begin{pmatrix} [P_\lambda^+(\lambda)]^{-1} & 0 \\ 0 & [P_\lambda^-(\lambda)]^{-1} \end{pmatrix} U \right)^{-1}.
\]
Here $P_\lambda^\pm(\lambda) = (hD_x)^2 + \tilde{W}_h^\pm(x)$, where
\[
\tilde{W}_h^\pm(x) = \tilde{W}_h^\pm(x, \tilde{\lambda}, \tilde{\mu}, \tilde{k}) = (s + i\tilde{\mu})a^2(x) - (\tilde{c}_h(x, \tilde{k}) - \tilde{\lambda})^2 \pm h(s + i\tilde{\mu})a'(x).
\]
Note that similarly to (4.4) in [17] and (6.6) we get
\[
\tilde{W}_h^\pm(x) = W_\pm^\pm(e^{\pm x}), \quad \pm x > X_0,
\]
where $W_\pm^\pm(w)$ are functions holomorphic in the discs $|w| < e^{\pm X_0}$, and $W_\pm^\pm(e^{\pm x}) \rightarrow -(\Omega_\pm(k) - \lambda)^2$ as $x \rightarrow \pm \infty$.

The operators $P_\lambda^\pm(\lambda)$ are of the same type as considered by Dyatlov (his operator $P_x$ given by (4.2) in [17] in unscaled version $h = 1$). Note that the notion of the outgoing vector-function from Definition 6.1 is consistent with Definition 4.1 of outgoing function in [17] in the sense that each component $f_i, \ i = 1, 2$, of the outgoing at $+\infty (-\infty)$ vector-function $f = (f_1, f_2)^T$ is outgoing at the same $\infty$ in the sense of Definition 4.1 in [17].

We apply Proposition 4.4 from [17] to $P = (hD_x)^2 + W(x)$, where $W(x)$ is either $\tilde{W}_h^+(x)$ or $\tilde{W}_h^-(x)$, given in (6.17).

We get that, if $u \in H^2_{\text{loc}}(\mathbb{R}; \mathbb{C})$ is any outgoing function in the sense of Definition 4.1 in [17] and if $v = P u$ is supported in $K_r$, then
1) $u$ can be extended holomorphically to the two half-planes $\{ \pm \text{Re } z > X_0 \}$ and $P_x u = 0$ there. Here $P_x = (hD_x)^2 + W(z)$, and $W(z)$ is well defined by (6.18).
Moreover, if $\gamma$ is a contour in $\mathbb{C}$ given by $\text{Im} z = F(\text{Re} z)$, $x_- \leq \text{Re} z \leq x_+$ with $F(x) = 0$ for $|x| \leq X_0$, then there is a restriction to $\gamma$ of the holomorphic extension of $u$ by

$$u_\gamma(x) = u(x + iF(x)), \quad \mathcal{P}_\gamma u_\gamma = v, \quad \mathcal{P}_\gamma = \left( \frac{1}{1 + iF'(x)hD_x} \right)^2 + W(x + iF(x)).$$

(6.19)

3) If $\gamma$ is as above, $x_\pm = \pm \infty$, and $F'(x) = C = \text{const}$ for large $|x|$. Then

$$u_\gamma(x) = \mathcal{O} \left( e^{\pm \text{Im}((1+iC)(\lambda - \Omega_\pm)x)} \right), \quad \text{as } x \to \pm \infty.$$ 

Moreover, if $\text{Im}(1 + iC)(\lambda - \Omega_\pm) > 0$, then $u_\gamma \in H^2(\mathbb{R}; \mathbb{C}^2)$. Let $W := \tilde{W}_h^\pm(x) = \tilde{W}_h^\pm(x, \tilde{\lambda}, \tilde{\mu}, \tilde{k}) = (s + i\tilde{\mu})^2 a^2(x) - (\tilde{c}_h(x, \tilde{k}) - \tilde{\lambda})^2 \pm h(s + i\tilde{\mu})a'(x)$. Let $u$ be an outgoing function in the sense of Definition 4.1 in [17] and define $u_\gamma(x)$, $x_- \leq x \leq x_+$, by (6.19).

Then $\tilde{P}_{h,\gamma} u_\gamma = v$, where $v$ is supported in $K_r$, Regge-Wheeler image of $(r_- + \delta_r, r_+ - \delta_r)$. Then as in [17], Section 6, it follows that

$$\|u_\gamma\|_{L^2} \leq C\|v\|_{L^2}.$$ 

Note that though Dyatlov proved it for $\tilde{P}_0$, his proof also extends to $\tilde{P}_{h,\gamma}$. Then we get

$$\|1_{K_r}[\tilde{P}_h^\pm(\lambda)]^{-1}1_{K_r}\|_{L^2 \to L^2} \leq C.$$ 

(6.20)

In order to transfer bound (6.20) over to the Dirac operator we use the resolvent identity (13.5) and two lemmas which we formulate below. We omit the proofs as they repeat the arguments from the proofs of Lemmas 1, 2 in [28]. Let

$$H^s = \{u \in L^2(\mathbb{R}, \mathbb{C}), \|u\|_{H^s} < \infty\}, \|u\|_{H^s}^2 := \sum_{k=0}^s \int_{\mathbb{R}} |(h\partial_x)^k u(x)|^2 dx$$

(6.21)

be the standard semi-classical Sobolev spaces, and we define $\mathcal{H}^s = H^s \oplus \bar{H}^s$.

**Lemma 2.** Suppose (6.16). Let $\mathcal{P}_h$ be either $\tilde{P}_h^+(\lambda)$ or $\tilde{P}_h^-(\lambda)$ and $\chi \in C^\infty_0(\mathbb{R}; \mathbb{C})$. Then for $j = 1, 2$

$$\|\chi \mathcal{P}_h^{-1}\chi\|_{\mathcal{L}(H^0, H^s)} \lesssim \|\chi \mathcal{P}_h^{-1}\chi\|_{\mathcal{L}(H^0, \bar{H}^s)}.$$ 

(6.22)

From the radial resolvent identity (13.5) it follows
Lemma 3. Suppose \((6.16)\). Let \(\tilde{D}_{h,+}(\lambda), \tilde{P}_h^\pm\) be as before. Let \(\chi \in C_0^\infty(\mathbb{R}; \mathbb{C}^2)\) and \(\chi_i \in C_0^\infty(\mathbb{R}; \mathbb{C}), i = 1, 2\). Then

\[
\|\chi[\tilde{D}_{h,+}(\lambda)]^{-1}\chi\|_{L(\mathcal{H},\mathcal{H})} \lesssim \left(\|\chi_1[\tilde{P}_h^-]^{-1}\chi_1\|_{L(\mathcal{H},\mathcal{H})} + \|\chi_2[\tilde{P}_h^+]^{-1}\chi_2\|_{L(\mathcal{H},\mathcal{H})}\right).
\] (6.23)

These two lemmas imply

\[
\|1_{K_r}[\tilde{D}_{h,+}(\lambda)]^{-1}1_{K_r}\|_{L^2 \to L^2} \leq C
\]

or \((5.6)\)

\[
\|1_{K_r} R_r(\lambda, \omega, k)1_{K_r}\|_{L^2 \to L^2} \leq \frac{C_r}{|\omega|}.
\]
7 Semi-classical reformulation of Theorem 2.1

Now, we pass to the proof of Theorem 2.1. We need to reformulate the statement in the semi-classical terms.

We start by recalling Definition 1.1 from [18].

**Definition 7.1.** Let $h > 0$ and $R(\lambda; h) : \mathcal{H}_1 \mapsto \mathcal{H}_2$ be a meromorphic family of operators for $\lambda \in \mathcal{U}(h) \subset \mathbb{C}$, with $\mathcal{H}_j$ Hilbert spaces. Assume $\Omega(h) \subset \mathcal{U}(h)$ be open and $\mathcal{Z}(h) \subset \mathbb{C}$ be a finite subset (the elements of $\mathcal{Z}(h)$ may have multiplicities). The poles of $R$ in $\Omega(h)$ are simple with polynomial resolvent estimate, given modulo $O(h^\infty)$ by $\mathcal{Z}(h)$, if for sufficiently small $h$ there exist maps $Q$ and $\Pi$ from $\mathcal{Z}(h)$ to $\mathbb{C}$ and the algebra of bounded operators $\mathcal{H}_1 \mapsto \mathcal{H}_2$, such that:

(i) for each $\hat{\lambda}' \in \mathcal{Z}(h)$, $\hat{\lambda} = Q(\hat{\lambda}')$ is a pole of $R$, $|\hat{\lambda} - \hat{\lambda}'| = O(h^\infty)$, and $\Pi(\hat{\lambda}')$ is a rank one operator;

(ii) there is a constant $N$ such that $\|\Pi(\hat{\lambda}')\|_{\mathcal{H}_1 \mapsto \mathcal{H}_2} = O(h^{-N})$ for each $\hat{\lambda}' \in \mathcal{Z}(h)$ and

$$R(\lambda; h) = \sum_{\hat{\lambda}' \in \mathcal{Z}(h)} \frac{\Pi(\hat{\lambda}')}{\lambda - Q(\hat{\lambda}')} + O(h^{-N}), \quad \lambda \in \Omega(h).$$

So every pole of $R$ in $\Omega(h)$ lie in the image of $Q$.

Theorem 2.1 results from the following $h$–dependent version (similarly to Proposition 1.2 by Dyatlov [18]).

**Proposition 7.1.** Let $\nu_0 > 0$ and $h > 0$. Then, for sufficiently small $h$ (independently of $\nu_0$), the poles of $R(\lambda)$ in the region

$$|\text{Im } \lambda| < \nu_0, \quad h^{-1} < |\text{Re } \lambda| < 2h^{-1},$$

are simple with a polynomial resolvent estimate, given modulo $O(h^\infty)$ by

$$\lambda = h^{-1} F^\lambda(m, hl, hk; h), \quad m \in \mathbb{Z}, \quad l, k \in \mathbb{Z} + \frac{1}{2}, \quad 0 \leq m \leq C_m, \quad C_l^{-1} \leq hl \leq C_l, \quad |k| \leq l. \quad (7.2)$$

Here $C_m$ and $C_l$ are some constants and

$$F^\lambda(m, \bar{l}, \bar{k}; h) \sim \sum_{j \geq 0} h^j F_j^\lambda(m, \bar{l}, \bar{k})$$

is a classical symbol. The principal symbol $F_0$ of $F$ is real-valued and independent of $m$. Moreover, for $a = 0$

$$F^\lambda = z_0(\bar{l} + h/2) - ih \left(\frac{\alpha}{z_0}\right) \left(m + \frac{1}{2}\right) + O(h^2), \quad (7.3)$$

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where $z_0$, $\alpha$ are given in \cite{2.2}; for $|a| < a_0$,

$$\partial_k \mathcal{F}_0(m, \pm \tilde{k}, \tilde{k}) = \frac{a}{r_0^2} \left[ \frac{4F(r_0) r_0^2}{8Q^2 - 6Mr_0} \left( 1 - r_0 \frac{1}{2} F^{-1}(r_0) F'(r_0) \right) - 1 \right] - az_0 r_0^{-1} F^{\frac{1}{2}}(r_0), \quad (7.4)$$

where $F(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \Lambda^2 r^2$.

Proposition \ref{prop:7.1} is consequence of the following results.

Proposition 7.2. Take $\delta > 0$ and let $M_{\delta,r}$ be the image of $(r_- + \delta, r_+ - \delta) \times \mathbb{S}^2$ under the Regge-Wheeler transformation $r \mapsto x$ (see (1.3)). Let $1_{M_{\delta,r}}$ be the operator of multiplication by the characteristic function of $M_{\delta,r}$. Then, for a small enough and fixed $\nu_0$, the poles of the cut-off resolvent $R_M(\lambda) = 1_{M_{\delta,r}} R(\lambda) 1_{M_{\delta,r}} : L^2(M_{\delta,r}) \mapsto L^2(M_{\delta,r})$ in the region (7.1) are simple with polynomial resolvent estimate $L^2 \mapsto L^2$, given modulo $O(h^\infty)$ by (7.2).

Proposition 7.3. Let $R_M(\lambda)$ be as in Proposition \ref{prop:7.2} and denote its restriction $R_M(\lambda, k) = R_M(\lambda)|_{H_k}$, where $H_k = H \cap D_k$ is the subspace of angular momentum $k \in \frac{1}{2} + \mathbb{Z}$ as in (4.1).

There exists a constant $C_k$ such that for each $k \in \frac{1}{2} + \mathbb{Z}$,

1) if $h|k| > C_k$, then $R_M(\lambda, k)$ has no poles in the region (7.1) and its $L^2 \mapsto L^2$ norm is $O(|k|^{-1})$.

2) if $h|k| \leq C_k$, then the poles of $R_M(\lambda, k)$ in the region (7.1) are simple with a polynomial resolvent estimate $L^2 \mapsto L^2$, given modulo $O(h^\infty)$ by (7.3).

Propositions \ref{prop:7.2}, \ref{prop:7.3} are analogue for the Dirac case of Propositions 1.3, 1.4 in \cite{18}.

The construction of the cut-off resolvent in Propositions \ref{prop:7.2} \ref{prop:7.3} follows from Part 2) of Theorem 4.1.

Part 1) of Proposition \ref{prop:7.3} is a reformulation of Proposition 4.1.

In Sections 8, 9, 10 we prove the second part of Proposition \ref{prop:7.3}, namely expansion (7.2).

8 Angular quantization condition.

8.1 Main result.

For small $h > 0$ we put

$$\tilde{\lambda} = h \text{Re} \lambda, \quad \tilde{\nu} = \text{Im} \lambda, \quad \tilde{k} = hk, \quad \tilde{\Omega}_\pm = h \Omega_\pm, \quad \tilde{\omega} = h \text{Re} \omega, \quad \tilde{\mu} = \text{Im} \omega.$$

The following results are analogue for the Dirac case of Proposition 1.6 in \cite{18}.

Proposition 8.1. (Angular) The poles $\tilde{\omega} + ih\tilde{\mu}$ of $R_\phi(\lambda, \omega, k) = (A_k(\lambda) - \omega)^{-1}$ as a function of $\omega$ in the region

$$1 < \tilde{\lambda} < 2, \quad |\tilde{\nu}| < \nu_0, \quad |\tilde{k}| < C_k, \quad C_{\phi}^{-1} < \tilde{\omega} < C_\phi, \quad |\tilde{\mu}| < C_\mu \quad (8.1)$$
are simple with polynomial resolvent estimate $L^2 \mapsto L^2$, given modulo $O(h^\infty)$ by
\[
(\hat{\omega} + ih\hat{\mu})^2 = \mathcal{F}^\varphi(hl, \hat{\lambda}, \hat{\nu}, \hat{k}; h), \quad l + 1/2 \in \mathbb{Z}, \quad 0 \leq m \leq C_m, \quad \max(\vert \hat{k} \vert, C_i^{-1}) \leq h\vert l \vert \leq C_l,
\]
(in a sense of Definition 7.1) for some constant $C_l$. The principal part $\mathcal{F}^\varphi_0$ of the classical symbol $\mathcal{F}^\varphi$ is real-valued, independent of $\hat{\nu}$, and (in the non-rotating case)
\[
\mathcal{F}^\varphi_0 = \tilde{l}(\tilde{l} + h) + O(h^\infty) \quad \text{for } a = 0.
\]
Moreover,
\[
\mathcal{F}^\varphi_0(\pm \hat{k}, \hat{\lambda}, \hat{k}) = (E\hat{k} - a\hat{\lambda})^2, \quad \partial_{\hat{k}}\mathcal{F}^\varphi_0(\pm \hat{k}, \hat{\lambda}, \hat{k}) = \pm 2\hat{k} + O(a^2), \quad E = 1 + \frac{\Lambda a^2}{3}, \quad (8.3)
\]
and consequently, $\partial_{\hat{k}}\mathcal{F}^\varphi_0(\pm \hat{k}, \hat{\lambda}, \hat{k}) = -2a\hat{\lambda} + O(a^2)$.

In the following sections 8.2–8.3 we prove Proposition 8.1.

### 8.2 Joint spectrum and diagonalization.

We start by recalling Definition A.1 from [18]. We say that $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{C}^n$ belongs to the joint spectrum of (matrix-valued) pseudodifferential operators $P_1, \ldots, P_n \in \Psi^{k_j}, k_j \geq 0$, on a compact manifold $M$, if the joint eigenspace
\[
\{u \in C^\infty(M); P_j u = \omega_j u, j = 1, \ldots n\}
\]
is nontrivial. The classes $\Psi^k$ are the natural generalization to the matrix-valued case of the usual classes of pseudodifferential operators defined in [18], Section 2.

Put
\[
P_1(\hat{\lambda}, \hat{\nu}; h) := hA_{g^2}(\lambda) = A_{g^2, h}(\tilde{\lambda} + ih\tilde{\nu})
\]
\[
= \sqrt{\Delta_\varphi} \left[ \sigma_1 \left( hD_\varphi + \frac{ih\Lambda a^2 \sin(2\varphi)}{12\Delta_\varphi} \right) - \sigma_2 \left( \frac{E}{\Delta_\varphi \sin \varphi} \right) hD_\varphi \right] + (\tilde{\lambda} + ih\tilde{\nu}) \frac{a \sin \varphi}{\sqrt{\Delta_\varphi}} \sigma_2,
\]
\[
P_2(h) := I_2 hD_\varphi.
\]
Then, $(h\lambda, h\omega, hh)$ is a pole of $R^\varphi_0(\omega) = hA_k(\lambda) - h\omega)^{-1}$ if and only if $(\hat{\omega} + ih\hat{\mu}, \hat{k})$ lies in the joint spectrum of the operators $(P_1, P_2)$. If $h\text{ Im } \lambda = \tilde{\nu} = 0$ then from Section 3.3 we know that the joint spectrum is given by
\[
(h\mu_{kl}(\lambda), hh), \quad (k, l) \in (1/2 + \mathbb{Z}) \times \mathbb{Z}^*.
\]

In the rotationless case $a = 0$
\[
P_1(\hat{\lambda}, \hat{\nu}; h)|_{a=0} = \sqrt{\Delta_\varphi} \left[ \sigma_1 hD_\varphi - \sigma_2 \frac{1}{\sin \varphi} hD_\varphi \right] = h\sqrt{\Delta_\varphi} D_{g^2},
\]
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where $D_{S^2} = \sigma_1 D_\theta - \sigma_2 \frac{1}{\sin \theta} D_\varphi$ is the usual Dirac operator on $S^2$ (with our choice of weight on the spinor) and the joint spectrum of $(P_1, P_2)$ is given by the spherical harmonics

$$(\pm (\tilde{l} + h/2), \tilde{k}), \quad \tilde{k} \in h(1/2 + \mathbb{Z}), \quad \tilde{l} \in h(1/2 + \mathbb{N}_0), |\tilde{k}| \leq |\tilde{l}|.$$  

Now, we use the results obtained in Section 5.2 Formula (5.7):

$$[P_1(\tilde{\lambda}, \tilde{\nu}; h)]^2 = \left[ A_{S^2, h}(\tilde{\lambda} + ih\tilde{\nu}) \right]^2 = \left( \begin{array}{cc} \mathcal{P}_+(\tilde{\lambda} + ih\tilde{\nu}) & 0 \\ 0 & \mathcal{P}_-(\tilde{\lambda} + ih\tilde{\nu}) \end{array} \right),$$

where

$$\mathcal{P}_\pm(\lambda) = \Delta_\theta \left\{ [hD_\theta + hq_1(\vartheta)]^2 + [q_2(\vartheta)hD_\varphi - \lambda q_3(\vartheta)]^2 \mp h(q_2'(\vartheta)hD_\varphi - \lambda q_3'(\vartheta)) \right\}$$  

$$= \Delta_\theta [hD_\theta + hq_1(\vartheta)]^2 + \frac{E^2}{\sin^2 \vartheta \Delta_\theta} \left( hD_\varphi - \lambda \sin^2 \vartheta \frac{1}{E} \right)^2 \mp h (\Delta_\theta q_2'(\vartheta)hD_\varphi - \lambda \Delta_\theta q_3'(\vartheta)), $$

where

$$q_1(\vartheta) = \frac{i \Lambda \sin^2(2\vartheta)}{12 \Delta_\theta}, \quad q_2(\vartheta) = \frac{E}{\Delta_\theta \sin^2 \vartheta}, \quad q_3(\vartheta) = \frac{a \sin \vartheta}{\Delta_\theta}. $$

Note that for any $\lambda \in \mathbb{C}$ (see 5.8)

$$\sigma(A_{S^2, h}(\lambda)) \setminus \{0\} = \{ \omega \in \mathbb{C}; \omega^2 \in \sigma(\mathcal{P}_+(\lambda)) \setminus \{0\} = \{ \omega \in \mathbb{C}; \omega^2 \in \sigma(\mathcal{P}_-(\lambda)) \setminus \{0\}. $$

The principal symbol of both $\mathcal{P}_\pm(\tilde{\lambda} + ih\tilde{\nu})$ is

$$p = \Delta_\theta \frac{E^2}{\sin^2 \vartheta \Delta_\theta} \left( \xi_\varphi - \tilde{\lambda} \sin^2 \vartheta \frac{1}{E} \right)^2, \quad E = 1 + \frac{\Lambda a^2}{3}. $$

Note that in [18], page 1130, Dyatlov considered symbol

$$p_{10}(\vartheta, \xi_\vartheta, \xi_\varphi; \tilde{\lambda}) = \Delta_\vartheta \frac{E^2}{\sin^2 \vartheta \Delta_\vartheta} \left( \xi_\varphi - \tilde{\lambda} \sin^2 \vartheta \frac{1}{E} \right)^2 $$

which is analogue to our $p$ and coincides with ours after rescaling $\tilde{\lambda} = E\tilde{\lambda}$. Then we follow [18]. Let

$$\begin{align*}
\mu_{10\pm}(\vartheta, \xi_\vartheta, \xi_\varphi; \tilde{\lambda}) &= \pm \Delta_\vartheta \frac{1}{2} \sqrt{p_{10}} = \pm \Delta_\vartheta \frac{1}{2} \left( \Delta_\vartheta \xi_\vartheta^2 + \frac{E^2}{\sin^2 \vartheta \Delta_\vartheta} \left( \frac{1}{\sqrt{\Delta_\vartheta}} \xi_\varphi - \tilde{\lambda} \sin^2 \vartheta \frac{1}{E} \right)^2 \right)^{\frac{1}{2}} \\
p_{20} = \xi_\varphi
\end{align*}$$

and $p(\vartheta, \xi_\vartheta, \xi_\varphi) := (\mu_{10\pm}, p_{20})$. In the non-rotating case $a = 0$ we have

$$\begin{align*}
\mu_{10\pm}(\vartheta, \xi_\vartheta, \xi_\varphi; \tilde{\lambda}) &= \pm \left( \xi_\vartheta^2 + \frac{\xi_\varphi^2}{\sin^2 \vartheta} \right)^{\frac{1}{2}} \\
p_{20} = \xi_\varphi.
\end{align*}$$
So in \(a = 0\) case we get \(|\mu| \geq \xi\). Then the set \(p^{-1}(\tilde{\omega}, \tilde{k})\) is non-empty if \(|\tilde{\omega}| \geq \tilde{k}\). If \(a \neq 0\) small, we get \(|\mu| \geq \xi\) and the set \(p^{-1}(\tilde{\omega}, \tilde{k})\) is non-empty if \(|\tilde{\omega}| \geq E(k - \lambda a)\).

Fix \(\xi \geq \epsilon_k\). Then \(\mu_{10 \pm}(\theta, \xi; \lambda)\) as a function of \((\theta, \xi; \lambda)\) for small \(a\) has unique critical point at \((\theta, \xi) = (\pi/2, 0)\) with critical value \(\pm E(\xi - \lambda a)\) with signature of the hessian \((\pm 1, \pm 1)\). It is enough to verify for \(a = 0\):

\[
\begin{align*}
\mu_{10 \pm}(\theta, \xi) &= \begin{cases}
(\xi \theta + \frac{\xi}{\sin^2 \theta})^{-\frac{1}{2}} \sin^{-3}(\theta) \cos \theta = 0 & \implies \theta = \pi/2 \\
(\xi \theta + \frac{\xi}{\sin^2 \theta})^{-\frac{1}{2}} \xi \theta = 0 & \implies \xi \theta = 0.
\end{cases}
\end{align*}
\]

Moreover, by the same Proposition 2.10 of Dyatlov from [18] to the function \(\mu_{10 \pm}(\cdot, \xi; \lambda)\) and obtain a function \(F_+ (\tilde{\omega}; \tilde{k})\) on \(K_\epsilon\) such that \(F_+|_{K_\epsilon} = 0\) and \(\partial_\theta F_+ > 0\). If \(a = 0\) then \(F_+ = \tilde{\omega} \mp \tilde{k}\).

We have the following version of Proposition 3.1 by Dyatlov from [18] applied to either \(\mathcal{P}_+(\lambda + ih\tilde{\nu})\) or \(\mathcal{P}_-(\lambda + ih\tilde{\nu})\) defined in [8,4]:

**Proposition 8.2.** Let \(K = \{(\tilde{\omega}, \tilde{k})\} \subset R^2\), \(C_0 \leq \tilde{\omega} \leq C_0, \tilde{\omega}^2 \geq 2^2 \leq C_0\}\) and \(K_\pm = \{(\tilde{\omega}, \tilde{k} \in K; \tilde{\omega} = \pm E(k - aE^{-1}/\lambda)\}\).

There are functions \(G_\pm(\tilde{\omega}, \tilde{k}; h)\) such that:

1) \(G_\pm\) is a complex-valued classical symbol in \(h\), smooth in a fixed neighborhood of \(K\). For \((\tilde{\omega}, \tilde{k})\) near \(K\) and \(|\mu| \leq C_0\), symbol \(G_\pm(\tilde{\omega} + ih\tilde{\mu}, \tilde{k})\) can be defined as an asymptotic analytic Taylor series for \(G_\pm\) at \(\tilde{\omega}, \tilde{k}\).

2) For \(a = 0\), \(G_\pm(\tilde{\omega}, \tilde{k}; h) = -h/2 + \sqrt{\tilde{\omega} + h^2/4} \mp \tilde{k}\).

3) \(G_-(\tilde{\omega}, \tilde{k}; h) - G_+(\tilde{\omega}, \tilde{k}; h) = 2\).

4) The principal symbol \(F_\pm\) of \(G_\pm\) is real-valued, \(\partial_\omega F_\pm > 0\) on \(\tilde{\omega} > 0\) on \(K\) and \(F_\pm|_{K_\pm} = 0\).

5) For sufficiently small \(h\), the set of elements \(((\tilde{\omega} + ih\tilde{\mu})^2, \tilde{k})\) of the joint spectrum of \(((P_1)^2, P_2)\) lies within \(\mathcal{O}(h)\) of \(K\) and coincides modulo \(\mathcal{O}(h^\infty)\) with the set of solutions to the quantization condition

\[
\tilde{k} \in h(\mathbb{Z} + \frac{1}{2}), \quad G_\pm(\tilde{\omega} + ih\tilde{\mu}, \tilde{k}) \in \mathbb{hN}^*.
\]

It is required \(G_\pm \geq 0\), the condition \(G_+ \in \mathbb{Z}\) and \(G_- \in \mathbb{Z}\) are equivalent, the corresponding joint eigenspaces are one dimensional.
The function $\mathcal{F}^\vartheta(\tilde{l}, \tilde{\lambda}, \kappa; \ell; h)$ in Proposition 8.1 can be defined as the solution $\tilde{\omega} + i\hbar \tilde{\mu}$ to the equation

$$G_+(\tilde{\omega} + i\hbar \tilde{\mu}, \tilde{k}, \tilde{\lambda} + i\hbar \tilde{\nu}; h) = \tilde{l} - \tilde{k}. \quad (8.5)$$

Function $\mathcal{F}^\vartheta$ is uniquely defined as $\mathcal{P}_+(\tilde{\lambda} + i\hbar \tilde{\nu})$ and $\mathcal{P}_-(\tilde{\lambda} + i\hbar \tilde{\nu})$ have the same non-zero spectrum.

### 8.3 The bottom of the well asymptotics for the angular eigenvalues.

In this section we show (8.3).

We consider

$$P_1(\tilde{\lambda}, \tilde{\nu}; h)_{\mathcal{H}^{k_2}_x} = hA_{\mathbb{S}^2}(\lambda)|_{\mathcal{H}^{k_2}_x} = U\Delta^2_{\vartheta} \tilde{P}_1(\tilde{\lambda}, \tilde{\nu}; h)U^*|_{\mathcal{H}^{k_2}_x},$$

where

$$\Delta^2_{\vartheta} \tilde{P}_1 = \Delta^2_{\vartheta} \begin{pmatrix} \mu_+(\vartheta, hD_\vartheta, hD_\varphi; \tilde{\lambda}) & 0 \\ 0 & \mu_-(\vartheta, hD_\vartheta, hD_\varphi; \tilde{\lambda}) \end{pmatrix} + \Delta^2_{\vartheta} W(h).$$

Here $W(h)$ is an admissible pseudodifferential operator of order 0 (see [23]) and

$$\Delta^2_{\vartheta} \mu_{10,\pm}(\vartheta, \xi_\vartheta, \xi_\varphi; \tilde{\lambda}) = \pm \sqrt{P_{10}} = \pm \left( \Delta_{\vartheta} \xi_\vartheta^2 + \frac{E^2}{\sin^2 \vartheta \Delta_{\vartheta}} \left( \xi_\varphi - \tilde{\lambda} a \sin^2 \vartheta \right)^2 \right)^{1/2}, \quad \tilde{\lambda} = E^{-1} \tilde{\lambda}.$$

So in the leading order the quantization condition for $P_1(\tilde{\lambda}, \tilde{\nu}; h)_{\mathcal{H}^{k_2}_x}$ is the same as for

$$\begin{pmatrix} +P_{10}^2(\vartheta, hD_\vartheta, \kappa; \tilde{\lambda}) & 0 \\ 0 & -P_{10}^2(\vartheta, hD_\vartheta, \kappa; \tilde{\lambda}) \end{pmatrix}.$$

Following [18], Section B.3, we introduce new variable $y = \cos \vartheta$ and consider (for $h = 1$)

$$\begin{pmatrix} +P_y^2 & 0 \\ 0 & -P_y^2 \end{pmatrix}, \quad P_y = D_y(1 - y^2)(1 + (E - 1)y^2)D_y + \frac{E^2(a\tilde{\lambda}(1 - y^2) - k)^2}{(1 - y^2)(1 + (E - 1)y^2)}. \quad (8.6)$$

We study the bottom of the well asymptotics for the eigenvalues of $P_y^2$. The critical point for the principal symbol of $P_y$ is $(0, 0)$. In order to pass from bottom of the well to the barrier-top we rescale the operator, introducing the parameter $y' = e^{i\pi/4} y$. Eigenvalue $-i\omega^2$ of the rescaled operator $-iP_y$ is given by

$$-i\omega^2 = -iE^2(k - a \Re \tilde{\lambda})^2 - i(2m + 1)\sqrt{U_0(0)} + \ldots,$$

$$U_0(0) = -V''(0)/2 = k^2 - (a \Re \tilde{\lambda}^2 - (E - 1)(a \Re \tilde{\lambda} - k)^2 = 2a \Re \tilde{\lambda} + \mathcal{O}(a^2) > 0.$$

Now, multiplying by $i$ and rescaling with $h$ we get

$$(\tilde{\omega} + i\hbar \tilde{\mu})^2 = E^2(\tilde{k} - a \Re \tilde{\lambda})^2 + (2m + 1)h\sqrt{U_0(0)} + \ldots,$$

which shows (8.3).

Note that (8.6) gives the leading part of $P_1$. In [18] the full angular Hamiltonian is given by $P_y$ with complex parameter $\lambda$ instead of real $\tilde{\lambda}$. In order to get the quantization condition to any order, to calculate function $G_+$ in (8.5) and function $\mathcal{F}^\vartheta$ in (8.2), we apply the above method to the original operators $\mathcal{P}_\pm(\lambda)$ defined in (8.4).
9 Radial quantization condition.

9.1 Formulation of the radial quantization condition.

As in previous section, for small \( h > 0 \) we put

\[
\tilde{\lambda} = h \text{Re} \lambda, \quad \tilde{\nu} = \text{Im} \lambda, \quad \tilde{k} = hk, \quad \tilde{\Omega}_\pm = h \Omega_\pm, \quad \tilde{\omega} = h \text{Re} \omega, \quad \tilde{\mu} = \text{Im} \omega.
\]  

(9.1)

**Proposition 9.1 (Radial).** Let \( C_\omega \) be a fixed constant and put \( K_r = (r_+ + \delta, r_+ - \delta) \). Recall that \( R_r(\lambda, \omega, k) = (H_k(\lambda) + \omega)^{-1} \), where \( H_k(\lambda) := [a(x)]^{-1}(\sigma_3 D_x + c(x, k) - \lambda)\sigma_1 \). Then, the poles of \( 1_{K_r} R_r(\lambda, \omega, k) 1_{K_r} \) as a function of \( \omega \) in the region

\[
1 < \tilde{\lambda} < 2, \quad |\tilde{\nu}| < \nu_0, \quad |\tilde{k}| < C_k, \quad |\tilde{\omega}|, |\tilde{\mu}| < C_\omega,
\]

(9.2)

are simple with polynomial resolvent estimate \( L^2 \mapsto L^2 \), given modulo \( O(h^\infty) \) by

\[
\tilde{\omega} + ih\tilde{\mu} = F^{r,+}_0(m, \tilde{\lambda}, \tilde{\nu}, \tilde{k}; h) \quad \text{or} \quad \tilde{\omega} + ih\tilde{\mu} = F^{r,-}_0(m, \tilde{\lambda}, \tilde{\nu}, \tilde{k}; h), \quad m \in \mathbb{Z}, \quad 0 \leq m \leq C_m,
\]

(9.3)

for some constant \( C_m \). The principal part \( F^{r,\pm}_0 \) of the classical symbol \( F^{r,\pm} \) is real-valued, independent of \( m \) and \( \tilde{\nu} \). Moreover,

\[
F^{r,+}_0(\tilde{\lambda}, \tilde{k}) = \tilde{\lambda}_0 F^{-\frac{1}{2}}(r_0) + \frac{a}{r_0} \left[ H \left( 1 - r_0 \frac{1}{2} F^{-1}(r_0) F'(r_0) \right) - F^{-\frac{1}{2}}(r_0) \tilde{k} \right]
\]

and

\[
F^{r,-}_0(\tilde{\lambda}, \tilde{k}) = -F^{r,+}_0(\tilde{\lambda}, \tilde{k}), \quad H = \frac{\tilde{k} 4 F^{\frac{1}{2}}_0(r_0) r_0^2}{8Q^2 - 6Mr_0}, \quad F(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3} r^2.
\]

For \( \lambda, k \) satisfying (9.2), every pole \( \omega \) satisfies \( |\tilde{\omega}| > \epsilon \) for some \( \epsilon > 0 \).

Note that this proposition is an analogue for the Dirac case of Proposition 1.5, [18], for the Kerr-de Sitter black holes. In that case instead of two symbols \( F^{r,\pm} \) there is one symbol \( F^{r} \) and

\[
F^{r} = ih(m + 1/2) + \frac{3\sqrt{3} M}{\sqrt{1 - 9\Lambda M^2}} (\tilde{\lambda} + ih\tilde{\nu}) + O(h) \quad \text{for} \ a = 0,
\]

\[
F^{r}_0(\tilde{\lambda}, \tilde{k}) = \frac{3\sqrt{3} M}{\sqrt{1 - 9\Lambda M^2}} \tilde{\lambda} - \frac{\tilde{k}}{M\sqrt{3\sqrt{1 - 9M^2}\Lambda}} a + O(a^2).
\]

In the next sections we prove Proposition 9.1.

9.2 Resonance free strip

As in [17], the definition of an outgoing solution implies the following proposition.
Proposition 9.2. Let $\delta_r > 0$ and $K_r$ be the image of the set $(r_- + \delta_r, r_+ - \delta_r)$ via the Regge-Wheeler change of variables $r \mapsto x$. Suppose that $X_0$ is sufficiently large so that Proposition 6.1 holds and $K_r \subset (-X_0, X_0)$. Let $f \in H^1_{\text{loc}}(\mathbb{R}; \mathbb{C}^2)$ be any outgoing function in the sense of Definition 6.1 and suppose that $g = D^r_k(\lambda, \omega)f$ is supported in $K_r$. Then

1) $f$ has an analytic extension to the two half-planes $\{ \pm \text{Re } z > X_0 \}$ and satisfies the equation $D^r_k(z, \lambda, \omega)f = 0$ there. Here $D^r_k(z, \lambda, \omega) = \sigma_3 D_x + V(z; \lambda, \omega, k)$, $V(z; \lambda, \omega, k) = c(z, k) - \lambda + \omega a(z) \sigma_3$, an $V(z; \lambda, \omega, k)$ is well defined by (6.6).

2) Suppose $\gamma$ be a contour in $\mathbb{C}$ given by $\text{Im } z = F(\text{Re } z)$, $x_- \leq \text{Re } z \leq x_+$ and $F(x) = 0$ for $|x| \leq X_0$. Then there is a restriction to $\gamma$ of the holomorphic extension of $f$ by

$$f_\gamma(x) = f(x + iF(x))$$

and $f_\gamma$ satisfies the equation $D_\gamma f_\gamma = g$, where

$$D_\gamma = \frac{1}{1 + iF'(x)} \sigma_3 D_x + V(x + iF(x); \lambda, \omega, k).$$

3) If $\gamma$ is as above, with $x_\pm = \pm \infty$, and $F'(x) = c = \text{const}$ for large $|x|$. Then

$$f_\gamma(x) = O \left(e^{\pm \text{Im}(1+ic)(\lambda - \Omega_\pm)x}\right), \quad \text{as } x \to \pm \infty.$$

Therefore, if $\text{Im}(1 + ic)(\lambda - \Omega_\pm) > 0$, then $f_\gamma \in H^1(\mathbb{R}; \mathbb{C}^2)$.

Put $h = |\text{Re } \lambda|^{-1}$, $\sigma = \text{sign } \text{Re } \lambda$ and

$$\tilde{\nu} = \text{Im } \lambda, \quad \tilde{k} = h k, \quad \tilde{\Omega}_\pm = h \Omega_\pm, \quad \tilde{\omega} = h \text{Re } \omega, \quad \tilde{\mu} = \text{Im } \omega.$$

Consider the rescaled operator

$$\tilde{D}^r_k(\lambda, \omega) := h D^r_k(\lambda, \omega) := \sigma_3 h D_x + \tilde{V}_h(x, \lambda, \tilde{\mu}, \tilde{k}),$$

$$\tilde{V}_h(x, \lambda, \tilde{\mu}, \tilde{k}) = (\tilde{\omega} + i \tilde{\mu}) a(x) \sigma_1 + \tilde{c}_h(x, \lambda, \tilde{k}) - (\sigma + i \tilde{\nu}), \quad \tilde{c}_h(x, \lambda, \tilde{k}) = \frac{a \tilde{E}}{r^2 + a^2} + h \frac{q \tilde{Q} \tilde{r}}{r^2 + a^2}.$$

As in [17] one can show that in order to prove Theorem 4.2 it suffices to prove the following:

Let $h$ be small enough and suppose the conditions

$$|\tilde{\omega}| \leq C', \quad |\tilde{k}| \leq C', \quad |\tilde{\mu}| \leq 1/C'', \quad |\tilde{\nu}| \leq 1/C''. \quad (9.4)$$

For each $g(x) \in L^2 \cap \mathcal{E}'(K_r)$ let $f(x)$ be solution to the equation $\tilde{D}^r_k(\lambda, \omega)f = g$ which is outgoing in the sense of Definition 6.1. Then

$$\|f\|_{L^2(K_r)} \leq Ch^{-1-\epsilon}\|g\|_{L^2} \quad (9.5)$$

for some $\epsilon > 0$.  

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Bound (9.5) will follow from (9.7) which itself is proved by the construction of an escape function and conjugation by exponential weight as in [17]. We will explain below why the method also works for the Dirac operator.

We consider the leading part of \( \mathcal{D}_k^r \)

\[
\mathcal{D}_h - 1 = \begin{pmatrix}
 hD_x + c_0(x, \tilde{k}) - s & \tilde{w}a(x) \\
 \tilde{w}a(x) & -hD_x + c_0(x, \tilde{k}) - s
\end{pmatrix}
\]

with the principal symbol (which is also the principal symbol of \( \mathcal{D}_k^r \))

\[
p(x, \xi) - I_2 = \begin{pmatrix}
 \xi + c_0(x, \tilde{k}) - s & \tilde{w}a(x) \\
 \tilde{w}a(x) & -\xi + c_0(x, \tilde{k}) - s
\end{pmatrix}.
\]

The eigenvalues \( \mu_\pm \) of \( p(x, \xi) \) are zeros of the determinant

\[
\det(p(x, \xi) - (s + \mu)I_2) = -\xi^2 + (c_0(x, \tilde{k}) - (s + \mu))^2 - \tilde{w}^2a^2(x).
\]

We get

\[
\mu_\pm = c_0(x, \tilde{k}) - s \pm \sqrt{\xi^2 + \tilde{w}^2a^2(x)}.
\]

We apply the method of complex scaling (see for example [17]). Consider the contour \( \gamma \) in the complex plane given by \( \text{Im} x = F(\text{Re} x) \), with \( F \) defined by

\[
F(x) = \begin{cases}
0, & |x| \leq R; \\
F_0(x - R), & x \geq R; \\
-F_0(-x - R), & x \leq -R.
\end{cases}
\] (9.6)

Here \( R > X_0 \) is large and \( F_0 \in C_0^\infty(0, \infty) \) is fixed function such that \( F'_0 \geq 0 \) and \( F''_0 \geq 0 \) for all \( x \) and \( F'_0(x) = 1 \) for \( x \geq 1 \).

Now, let \( f(x) \) be an outgoing in the sense of Definition 6.1 solution to the equation \( \mathcal{D}_k^r(\lambda, \omega) f = g \in L^2 \cap E'(K_r) \). By Proposition 9.2 we can define the restriction \( f_\gamma \) of \( f \) to \( \gamma \) and \( \mathcal{D}_\gamma(\lambda, \omega) f_\gamma = g \), where

\[
\mathcal{D}_\gamma = \frac{1}{1 + iF'(x)} \sigma_3D_x + V(x + iF(x); \lambda, \omega, k).
\]

For \( \alpha \) and \( h \) small \( f_\gamma \in H^1(\mathbb{R}) \), and in order to prove (9.5), it is enough to show that for each \( f_\gamma \in H^1(\mathbb{R}) \), we have

\[
\|f_\gamma\|_{L^2(K_r)} \leq Ch^{-1-\epsilon}\|\mathcal{D}_\gamma(\lambda, \omega) f_\gamma\|_{L^2}. \tag{9.7}
\]

The eigenvalues of the semi-classical principal symbols of \( \mathcal{D}_k^r(\lambda, \omega) \) and \( \mathcal{D}_\gamma \) are given by \( \mu_\pm \), \( \mu_{\gamma, \pm} \), respectively. Here

\[
\mu_\pm = c_0(x, \tilde{k}) - s \pm \sqrt{\xi^2 + \tilde{w}^2a^2(x)},
\]

\[
\mu_{\gamma, \pm} = c_0(x + iF(x), \tilde{k}) - s \pm \sqrt{\frac{\xi^2}{(1 + iF'(x))^2} + \tilde{w}^2a^2(x + iF(x), \tilde{k})}.
\]
Now, recall (6.2):
\[
c_0(x, \tilde{k}) = \Omega_{0, \pm}(\tilde{k}) + c_{0, \pm}e^{2\kappa_\pm x} + O(e^{4\kappa_\pm x}), \quad x \to \pm \infty, \quad \Omega_{0, \pm} = \frac{aE\tilde{k}}{r_\pm^2 + a^2},
\]
and (6.11)
\[
c_0(x, k) = c_0^\pm(e^{\kappa \pm x}), \quad a(x) = a^\pm(e^{\kappa \pm x}), \quad \pm x > X_0, \quad c_0^\pm(0) = \Omega_{0, \pm}, \quad a^\pm(0) = 0
\]
for some holomorphic functions $c_0^\pm(w)$, $a^\pm(w)$.

Bound (9.7) follows by the construction of an escape function and conjugation by exponential weight as in [17] using an analogue of Proposition 7.2 in that paper:

Suppose $x > X_0 > 0$. There exists a constant $C$ such that for $R$ large enough and $\delta > 0$ small enough,

\[
\begin{align*}
\text{if } x \geq R + 1, & \text{ then } |\mu_{\gamma, \pm}| \geq 1/C > 0, \quad (9.8) \\
\text{if } |\mu_{\gamma, \pm}(x, \xi)| \leq \delta, & \text{ then } |\mu_{\pm}(x, \xi)| \leq C\delta, \quad |\nabla(\Re \mu_{\pm}(x, \xi))|^2 \leq C\delta. \quad (9.9) \\
\text{if } |\mu_{\gamma, \pm}(x, \xi)| \leq e^{2R + 1}, & \text{ then } \Im \mu_{\pm}(x, \xi) \leq 0. \quad (9.10)
\end{align*}
\]

Similar facts hold if $x < -X_0 < 0$ with $\kappa_-$ instead of $\kappa_+$.

It is enough to prove this statement for $s = 1$.

Formula (9.8). Suppose $x \geq R + 1$. Then
\[
\mu_{\gamma, \pm} = c_0(x + iF(x), \tilde{k}) - 1 \pm \sqrt{-i\xi^2/2 + \tilde{\omega}^2a^2(x + iF(x), \tilde{k})} = \Omega_{0, \pm} - 1 \pm \sqrt{-i\xi^2/2 + O(e^{-R}) = \Omega_{0, \pm} - 1 \pm (1 - i)|\xi|/2 + O(e^{-R})}
\]
and
\[
|\mu_{\gamma, \pm}|^2 = (\Omega_{0, \pm} - 1 \pm |\xi|/2)^2 + |\xi|^2/4 + O(e^{-R}).
\]

Minimizing the function $f_\pm(x) = (\Omega_{0, \pm} - 1 \pm x)^2 + x^2$ we get $f_+(x) \geq f(\pm \frac{2}{9}(\Omega_{0, \pm} - 1)) = C_\pm > 0$, where $C_+ = \frac{5}{9}(\Omega_{0, \pm} - 1)^2$, $C_- = \frac{20}{9}(\Omega_{0, \pm} - 1)^2$.

Taking $R$ large enough we get (9.8).

For the rest of the proof we assume that $R \leq x \leq R + 1$.

Formulas (9.9), (9.10). Suppose that $|\mu_{\gamma, \pm}(x, \xi)| \leq \delta$, then as in [17] we get
\[
\frac{\xi^2}{(1 + iF'(x))^2} = (\Omega_{0, \pm} - 1)^2 + O(\delta + e^{-RF(x)})
\]
which implies $F'(x) \leq c\delta$ and leads to (9.8) and (9.10) (by repeating the proof in [17]).

Then we can apply the method of [17] involving construction of escape function and conjugation by exponential weights.

This achieves the proof of Theorem 4.2 and shows a resonance free strip. The only trapping in our situation is normally hyperbolic and generate the radial poles which are studied in the next section.
9.3 The trapping point asymptotics for the radial resonances.

In this section we study the radial poles in the region (9.2) appearing in the trapping case. In the non-trapping case there is an arbitrary large strip free of radial poles.

We rescale operator $\tilde{\mathcal{D}}_k^r(\lambda, \omega)$ and use notation (9.1). Let

$$\tilde{\mathcal{D}}_k^r(\lambda, \omega) := h\mathcal{D}_k^r(\lambda, \omega) := \sigma_3 h D_x + \tilde{V}_h(x, \tilde{\lambda}, \tilde{\mu}, \tilde{k}),$$

and that it does not depend on the sign of $\mu$.

We consider the leading part of $\tilde{\mathcal{D}}_k^r(\lambda, \omega)$

$$\mathcal{D}_h(\tilde{\lambda}, \tilde{\omega}, \tilde{k}) = \mathcal{D}_h(\tilde{\omega}, \tilde{k}) - \tilde{\lambda} = \left( \begin{array}{cc} hD_x + c_0(x, \tilde{k}) - \tilde{\lambda} & \tilde{\omega}a(x) \\ \tilde{\omega}a(x) & -hD_x + c_0(x, \tilde{k}) - \tilde{\lambda} \end{array} \right), \quad (9.11)$$

The principal symbol of $\mathcal{D}_h(\tilde{\lambda}, \tilde{\omega}, \tilde{k})$ is given by

$$p(x, \xi; \tilde{\lambda}, \tilde{\omega}, \tilde{k}) = p(x, \xi; \tilde{\omega}, \tilde{k}) - \tilde{\lambda} I_2 = \left( \begin{array}{cc} \xi + c_0(x, \tilde{k}) - \tilde{\lambda} & \tilde{\omega}a(x) \\ \tilde{\omega}a(x) & -\xi + c_0(x, \tilde{k}) - \tilde{\lambda} \end{array} \right), \quad (9.12)$$

with the eigenvalues

$$\mu_{\pm} = \mu_{\pm}(x, \xi; \tilde{\lambda}, \tilde{\omega}, \tilde{k}) = c_0(x, \tilde{k}) - \tilde{\lambda} \pm \sqrt{\xi^2 + \tilde{\omega}^2 a^2(x)}.$$

Note the oddness property of the principal symbol: $\mu_{-}(x, \xi; \tilde{\lambda}, \tilde{\omega}, \tilde{k}) = -\mu_{+}(x, \xi; -\tilde{\lambda}, \tilde{\omega}, -\tilde{k})$ and that it does not depend on the sign of $\tilde{\omega}$.

We study the trapping properties of the eigenvalues of $\mu_{\pm}$. Bicharacteristics are the solutions of the Hamilton system

$$\begin{cases} \\ x_{\pm} = \partial_\xi \mu_{\pm}(x_{\pm}(t), \xi_{\pm}(t)), \\ \xi_{\pm} = -\partial_x \mu_{\pm}(x_{\pm}(t), \xi_{\pm}(t)) \end{cases} \Leftrightarrow \begin{cases} \\ \dot{x}_{\pm} = \pm \frac{1}{2} \frac{1}{\sqrt{\xi^2 + \tilde{\omega}^2 a^2(x)}} 2\xi, \\ \dot{\xi}_{\pm} = -c_0'(x) \mp \frac{1}{2} \frac{1}{\sqrt{\xi^2 + \tilde{\omega}^2 a^2(x)}} \tilde{\omega}^2(a^2(x))' \end{cases}$$

with initial condition $(x_{\pm}(0), \xi_{\pm}(0)) = (x_0, \xi_0^0, \xi_0^1)$.

The only critical point of the Hamiltonian $\mu_{\pm}$ is given by $(x, \xi) = (x_{\pm}, 0)$, where $x_{\pm}$ is solution to $[c_0 \pm |\tilde{\omega}| a']'(x) = 0$.

Note that by symmetry of the symbol we get that passing from $x_\pm$ to $x_\mp$ is equivalent to the change of sign $\tilde{k} \mapsto -\tilde{k}$.

At $(x_{\pm}, 0)$ we get

$$\begin{aligned} 
(\mu_{\pm})''_{xx} &= (c \pm |q|)'(x_{\pm}), \quad (\mu_{\pm})''_{xx} = \pm |q(x_{\pm})|^{-1}, 
(\mu_{\pm})''_{xx} &= 0. \\
(\mu_{\pm})''_{xx} &= (c_0 \pm |\tilde{\omega}| a')(x_{\pm}), \quad (\mu_{\pm})''_{xx} = \pm (|\tilde{\omega}| a(x_{\pm}))^{-1}, 
(\mu_{\pm})''_{xx} &= 0.
\end{aligned}$$

Note that for $a$ small enough $(\mu_{+})''_{xx}(x_{+}, 0) < 0$, $(\mu_{-})''_{xx}(x_{-}, 0) > 0$, and the critical points are of hyperbolic type.
Now, using that there is an arbitrary large strip free of radial poles in the nontrapping cases we know that the only radial poles in the region (9.2) appear in the trapping case. Then we may assume that $|\tilde{\omega}^2 - \tilde{\omega}_0^2(\tilde{\lambda}, \tilde{k})| < \varepsilon_r$, where $\tilde{\omega}_0^2$ is the value of the function

$$F_V(r; \tilde{\lambda}, \tilde{k}) = \frac{\alpha^2}{\lambda - c_0(x)} = \frac{\Delta_r}{[\tilde{\lambda}(r^2 + a^2) - aE\tilde{k}]^2} \tag{9.13}$$

at its only maximum point. Under assumption (9.2), $1/C \leq \tilde{\omega}_0^2 \leq C$ for some constant $C$.

We see that the only critical points of $\mu_{\pm}$ in the set $\{\mu_{\pm} \geq -\varepsilon_r^3\}$ are non-degenerate hyperbolic critical points at $(x_{\pm}, 0)$.

Now, we establish a microlocal form for $\mathcal{D}_k^\nu(\lambda, \omega)$ near $(x, \xi) = (x_{\pm}, 0)$. Firstly, we will diagonalize the 2 by 2 matrix $\mathcal{D}_k^\nu(\lambda, \omega)$ modulo $h^\infty$, using the method of M. Taylor explained in [25] and [7].

The principal symbol $p(x, \xi; \tilde{\lambda}, \tilde{\omega}, \tilde{k})$ (9.12) can be diagonalized exactly via unitary Foldy-Wouthuysen transform $U_0$ (see [34])

$$U_0(x, \xi)^* p(x, \xi; \tilde{\lambda}, \tilde{\omega}, \tilde{k}) U_0(x, \xi) = \begin{pmatrix} \mu_+(x, \xi) & 0 \\ 0 & \mu_-(x, \xi) \end{pmatrix}, \tag{9.14}$$

with $U_0 = U_0(x, \xi) = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{\Delta^2 + (\xi + c_0(x))\Delta}} \begin{pmatrix} \xi + c_0(x) + \Delta & -(\tilde{\omega}a(x)) \\ \tilde{\omega}a(x) & \xi + c_0(x) + \Delta \end{pmatrix}$, $\Delta = \sqrt{(\xi + c_0(x))^2 + (\tilde{\omega}a(x))^2}$.

By Weyl quantization of the symbol identity (9.14) we diagonalize the principle part of the operator $\mathcal{D}_h = \mathcal{D}_h(\tilde{\lambda}, \tilde{\omega}, \tilde{k})$ in (9.11)

$$\mathcal{D}_h(\lambda) = U(x, hD_x)^* \mathcal{D}_h U(x, hD_x) = \begin{pmatrix} \mu_+(hD_x) & 0 \\ 0 & \mu_- (hD_x) \end{pmatrix} + hR(h),$$

where pseudodifferential operator $R(h)$ has classical symbol and $\|R(h)\| = \mathcal{O}(1)$.

Now, by iteration as in Section 3.1 of [25] and Section 4.1 of [7] we can diagonalise operator $\mathcal{D}_k$ to the infinite order in $h$.

**Proposition 9.3** (Decoupling). *There exist unitary $U = U(x, hD_x, h)$ such that*

$$U^* \mathcal{D}_k^\nu(\lambda, \omega) U = \begin{pmatrix} \mu_+(x, hD, h) & 0 \\ 0 & \mu_- (x, hD, h) \end{pmatrix} + \mathcal{O}(h^\infty) \tag{9.15}$$

in $\mathcal{L}(L^2, L^2)$. Here $\mu_{\pm}(x, hD, h)$ is pseudodifferential operator with classical symbol $\mu_{\pm}(x, \xi, h) = \sum_{k=0}^\infty h^k \mu_{k,\pm}(x, \xi)$ and $\mu_{0,\pm}(x, \xi) = \mu_{\pm}(x, \xi)$.

We pass now to reduction of $\mu_{\pm}(x, hD, h)$ to the microlocal normal form. It is enough to consider $\mu_+(x, hD, h)$ as the other case is similar.

The proof of Proposition 4.3 in [18] can also be applied here using that $\mu_{\pm}(x, \xi)$ is of the same type as symbol $p_0(x, \xi)$ in Section 4.3 in [18]. The idea is to apply an analogue of
Theorem 12 in [16] to the symbol \( \mu_+(x, \xi) - \mu_+(x_+, 0) \). Here \( \mu_+(x, \xi) \) is the principal symbol of \( \mu_+(x, hD, h) \). Then operator
\[
\mu_+(x, hD_x) - \mu_+(x_+, 0)
\]
can be transformed microlocally near \((x_+, 0)\) into \( f(hD_x) \) by constructing a symplectomorphism \( \Phi \) from a neighborhood of \( K_0 = \{|x - x_+| \leq \epsilon_0, |\xi| \leq \epsilon_0\} \subset T^*\mathbb{R} \)
onto a neighborhood of the origin in \( T^*\mathbb{R}, \Phi(x_+, 0) = (0, 0) \), and operators \( B_1, B_2 \) quantizing \( \Phi \) near \( K_0 \times \Phi(K_0) \).

Let \( f_0 \) be the principal part of \( f \). Then, \( \mu_+(x, \xi) \circ \Phi^{-1} = f_0(x\xi) \). The level set \( \{\mu_+(x, \xi) = \mu_+(x_+, 0)\} \) at the trapped energy contains in particular the outgoing trajectory
\[
\{x > x_+, \xi = \sqrt{(c_0(x_+) - c_0(x) + \tilde{\omega}a(x_+))^2 - (\tilde{\omega}a(x))^2}\}.
\]
Then we can choose \( \Phi \) mapping this trajectory into \( \{x > x_+, \xi = 0\} \).

The function \( f(s; h) \) is not uniquely defined; however, its Taylor decomposition at \( s = 0, h = 0 \) is and we can compute
\[
\mu_+(x, \xi) = \mu_+(x_+, 0) + \frac{1}{2} |\tilde{\omega}a(x_+)|^{-1} \xi^2 - \frac{1}{2} |(c_0 + |\tilde{\omega}|a)''(x_+)| (x - x_+)^2 + \ldots
\]
\[
= \mu_+(x_+, 0) + \frac{1}{2} |\tilde{\omega}a(x_+)|^{-1} [\xi^2 - |\tilde{\omega}|a(x_+)] (c_0 + |\tilde{\omega}|a)''(x_+)| (x - x_+)^2] + \ldots,
\]
where we used that \( (\mu_+)'''(x_+, 0) = (c_0 + |\tilde{\omega}|a)''(x_+) < 0 \). Denote
\[
\sqrt{|\tilde{\omega}|a(x_+)| (c_0 + |\tilde{\omega}|a)''(x_+)|} = \sigma.
\]
Performing the first symplectic change of variables \( x \mapsto \tilde{x} := \sigma^{\frac{1}{2}}(x - x_+), \xi \mapsto \tilde{\xi} = \sigma^{-\frac{1}{2}}\xi \) we get
\[
\mu_+(x, \xi) = \mu_+(x_+, 0) + \frac{1}{2} |\tilde{\omega}a(x_+)|^{-1} \sigma \left[\tilde{\xi}^2 - \tilde{x}^2\right] + \ldots
\]
Then the second linear symplectic change of variables \( \tilde{\xi} - \tilde{x} = \sqrt{2}\eta, \tilde{\xi} + \tilde{x} = \sqrt{2}y \) leads to
\[
\mu_+(x, \xi) = \mu_+(x_+, 0) + |\tilde{\omega}a(x_+)|^{-1} \sigma [y\eta] + \ldots
\]
The composition of these two linear maps is the linear part of the symplectomorphism \( \Phi \). Recall \( \mu_+(x, \xi) \circ \Phi^{-1} = f_0(x\xi) \) and we get for \( f_0(s), s = y\eta \)
\[
f_0(s) = \mu_+(x_+, 0) + |\tilde{\omega}a(x_+)|^{-1} \sigma s + \mathcal{O}(s^2)
\]
\[
= c_0(x_+) - \tilde{\lambda} + \tilde{\omega}a(x_+) + |\tilde{\omega}a(x_+)|^{-\frac{1}{2}} \sqrt{|(c_0 + |\tilde{\omega}|a)''(x_+)|} |s + \mathcal{O}(s^2)|.
\]
Similarly, we can consider the case of critical point \((x, \xi) = (x_-, 0)\).

As result, by applying Proposition 4.3 in [18] we get an analogue for Dirac operators of Proposition 4.3 in [18].
Proposition 9.4. The microlocal normal form of $\mathcal{D}^{\nu}_{k}(\lambda, \omega)$ is given by
\[
\begin{pmatrix}
 hxD_{x} - \beta_{+} & 0 \\
 0 & -hxD_{x} + \beta_{-}
\end{pmatrix},
\]
where $\beta_{\pm} = \beta_{\pm}(\bar{\lambda}, \bar{\nu}, \bar{\omega}, \bar{\mu}, \bar{k}; h)$ is classical symbol. Moreover, the principal part $\beta_{0,\pm}$ of $\beta_{\pm}$ is real-valued, independent of $\bar{\nu}$, $\bar{\mu}$, and vanishes if and only if $\bar{\omega} = \bar{\omega}_{0}(\bar{\lambda}, \bar{k})$, where $\bar{\omega}_{0}^{-1}$ is defined by (9.13). Moreover,
\[
\beta_{0, +} = -\frac{(c_{0}(x_{+}) - \bar{\lambda} + \bar{\omega}a(x_{+}))\sqrt{|\bar{\omega}|a(x_{+})}}{\sqrt{|(c_{0} + |\bar{\omega}|a)^{\prime\prime}(x_{+})|}} + O\left(c_{0}(x_{+}) - \bar{\lambda} + \bar{\omega}a(x_{+})\right)^{2}
\]
and $\beta_{0, -}(\bar{\lambda}, \bar{\nu}, \bar{\omega}, \bar{k}) = \beta_{0, +}(-\bar{\lambda}, \bar{\nu}, \bar{\omega}, -\bar{k})$.

The radial quantization symbol $\mathcal{F}^{r,+}(m, \bar{\lambda}, \bar{\nu}, \bar{k}; h)$ can be obtained as the solution $\bar{\omega} + ih\bar{\mu}$ to the equation
\[
\beta_{+}(\bar{\lambda}, \bar{\nu}, \bar{\omega}, \bar{k}; h) = -ihm, \quad m \in \mathbb{Z}, \quad 0 \leq m \leq C_{m}.
\]
So we get in the leading order $h = 0$ near $x = x_{+}$
\[
-(c_{0}(x_{+}) - \bar{\lambda} + \bar{\omega}a(x_{+}))(|\bar{\omega}|a)^{\frac{1}{2}}(x_{+}) = -ihm\sqrt{|(c_{0} + |\bar{\omega}|a)^{\prime\prime}(x_{+})|}, \quad c_{0} = \frac{aE_{k}}{r^{2} + a^{2}} \quad (9.16)
\]
We need to solve it with respect to $\bar{\omega}$. Let $r_{+}$ be defined via Regge-Wheeler transform $x_{+} = x(r_{+})$ and let $x_{0} = x(r_{0})$, where $r_{0} = \frac{3M}{2} + \sqrt{\left(\frac{3M}{2}\right)^{2} - 2Q^{2}}$, is the critical point for $a = 0$, i.e. $F^{r}(r_{0})r_{0} - 2F(r_{0}) = 0$. Then we find
\[
r_{+} \sim r_{0} + \frac{a}{\omega}H + O(a^{2}), \quad H = \frac{\vec{k}^{4}F^{\frac{1}{2}}(r_{0})}{r_{0}(F^{r}(r) - 2F(r))^{\prime}(r_{0})} = \frac{\vec{k}^{4}F^{\frac{1}{2}}(r_{0})r_{0}^{2}}{8Q^{2} - 6Mr_{0}}
\]
and
\[
r_{+} := r(x_{+}) = r_{0} + \frac{\vec{k}^{4}F^{\frac{1}{2}}(r_{0})r_{0}^{2}}{\omega(8Q^{2} - 6Mr_{0})} + O(a^{2}(\vec{k}^{2} + |\bar{\omega}|^{2})).
\]
Considering order $h^{0}$ in (9.16) we get for small $a$
\[
\mathcal{F}^{r,+}_{0}(\bar{\lambda}, \bar{k}) = \frac{\bar{\lambda}}{a(x_{0})} + \left(\mathcal{F}^{r,+}_{0}(\bar{\lambda}, \bar{k})\right)^{\prime}_{a=a_{0}} + O(a^{2}).
\]
Then we expand
\[
r_{+}^{2} = r_{0}^{2} + 2r_{0}H\frac{a}{\omega} + O(a^{2}), \quad F^{\frac{1}{2}}(r_{+}) = F^{\frac{1}{2}}(r_{0}) + \frac{1}{2}F^{-\frac{1}{2}}(r_{0})F^{r}(r)H\frac{a}{\omega} + O(a^{2}).
\]
Now, consider the following equation in $\bar{\omega} : c_{0}(x_{+}) - \bar{\lambda} + \bar{\omega}a(x_{+}) = 0$. As $c_{0} = \frac{aE_{k}}{r^{2} + a^{2}} = \frac{\vec{k}}{r_{0}} + O(a^{2})$ and
\[
a(x_{+}) = \frac{F^{\frac{1}{2}}(r_{+})}{r_{+}} = \frac{F^{\frac{1}{2}}(r_{0})}{r_{0}} + \frac{a}{\omega}H\left(\frac{1}{2}F^{-\frac{1}{2}}(r_{0})F^{r}(r_{0}) - \frac{F^{\frac{1}{2}}(r_{0})}{r_{0}^{2}}\right) + O(a^{2}),
\]
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we get equation in $\tilde{\omega}$:

$$\frac{a\tilde{k}}{r_0^2} - \tilde{\lambda} + \tilde{\omega} \left[ \frac{F^\frac{1}{2}(r_0)}{r_0} + \frac{a}{\tilde{\omega}} H \left( \frac{1}{2} \frac{F^{-\frac{1}{2}}(r_0)F'(r_0)}{r_0^2} - \frac{F^\frac{1}{2}(r_0)}{r_0^2} \right) \right] = \mathcal{O}(a^2).$$

We get

$$\mathcal{F}_0^{r,+}(\tilde{\lambda}, \tilde{k}) = \tilde{\lambda}_0 F^{-\frac{1}{2}}(r_0) + \frac{a}{r_0} H \left( 1 - r_0 \frac{1}{2} F^{-1}(r_0)F'(r_0) \right) - F^{-\frac{1}{2}}(r_0)\tilde{k}$$

(9.17)

The radial quantization symbol $\mathcal{F}^{r,-}(m, \tilde{\lambda}, \tilde{\nu}, \tilde{\mu}; h)$ can be obtained as the solution $\tilde{\omega} + ih\tilde{\mu}$ to the equation

$$\beta_-(\tilde{\lambda}, \tilde{\nu}, \tilde{\omega}, \tilde{\mu}, \tilde{k}; h) = -ihm, \quad m \in \mathbb{Z}, \quad 0 \leq m \leq C_m.$$

In the leading order near $x = x_-$ we have

$$(c_0(x_-) - \tilde{\lambda} - \tilde{\omega}a(x_-)) \sqrt{\tilde{\omega}a(x_-)} = -ihm\sqrt{(c_0 - \tilde{\omega}|a|')(x_-), \quad c_0 = \frac{aE\tilde{k}}{r^2 + a^2}.}$$

As for the case $x_+$, we solve (9.18) with respect to $\tilde{\omega}$. Then we get

$$\mathcal{F}_0^{r,-}(\tilde{\lambda}, \tilde{k}) = -\tilde{\lambda}_0 F^{-\frac{1}{2}}(r_0) - \frac{a}{r_0} \left[ H \left( 1 - r_0 \frac{1}{2} F^{-1}(r_0)F'(r_0) \right) - F^{-\frac{1}{2}}(r_0)\tilde{k} \right]$$

(9.19)

and

$$\mathcal{F}_0^{r,-}(\tilde{\lambda}, \tilde{k}) = -\mathcal{F}_0^{r,+}(\tilde{\lambda}, \tilde{k}).$$

(9.20)

We compare with [18]. Put $Q = 0$. Then

$$H = -\frac{\tilde{k}4F^\frac{1}{2}(r_0)r_0}{6M}, \quad r_0 = 3M, \quad F(r) = 1 - \frac{2M}{r} - \frac{3/2}{3} r^2, \quad F'(r) = \frac{2M}{r^2} - \frac{2}{3} \Lambda r$$

and

$$F(r_0) = \frac{1}{3} - 3M^2 \Lambda, \quad F'(r_0) = \frac{2}{9} M - 2\Lambda M, \quad H = -\tilde{k} \frac{2}{\sqrt{3}} \sqrt{1 - 9M^2 \Lambda}.$$

We get the leading term at $a = 0$

$$\tilde{\lambda}_0 F^{-\frac{1}{2}}(r_0) = \frac{3M}{\sqrt{\frac{1}{3} - 9M^2 \Lambda}} \tilde{\lambda} = \frac{3\sqrt{3}M}{\sqrt{1 - 9M^2 \Lambda}},$$

Order $a^1$:

$$F^{-\frac{1}{2}}(r_0) \left[ H \left( F^\frac{1}{2}(r_0)r_0^{-1} - \frac{1}{2} F^{-\frac{1}{2}}(r_0)F'(r_0) \right) - \tilde{k}r_0^{-1} \right]$$

$$= H \left( r_0^{-1} - \frac{1}{2} F^{-1}(r_0)F'(r_0) \right) - \tilde{k}r_0^{-1} F^{-\frac{1}{2}}(r_0)$$

$$= -\frac{\tilde{k}}{2} \sqrt{\frac{1}{3} - 9M^2 \Lambda} \left( \frac{1}{3M} - \frac{\Lambda M}{\frac{7}{3} - 3M^2 \Lambda} \right) - \frac{\tilde{k}}{3M} \frac{1}{\sqrt{\frac{1}{3} - 3M^2 \Lambda}}$$

$$= -\frac{\tilde{k}}{M\sqrt{3\sqrt{1 - 9M^2 \Lambda}}}.$$
Then
\[ F_0^{r,\pm}(\tilde{\lambda}, \tilde{k})^2 = \frac{27M^2}{1 - 9M^2\Lambda} \tilde{\lambda}^2 - \frac{6a\lambda\tilde{k}}{1 - 9M^2\Lambda} + \mathcal{O}(a^2) \]
coinsides with expression obtained in [18], Proposition 1.5.

\section{Combination of both quantization conditions.}

In this section we calculate the poles of the resolvent \( R(\lambda, k) \) by combining the poles of the angular and radial resolvents (see Remark 4.1) Proposition 7.3. We define \( F_{\lambda,\pm}(m, \tilde{l}, k; h) \) to be solution \( \tilde{\lambda} + ih\tilde{\nu} \) to the equation
\[ \left( F_{\lambda,\pm}(m, \tilde{l}, \tilde{\nu}, \tilde{k}; h) \right)^2 = F_{\vartheta}(\tilde{l}, \tilde{\lambda}, \tilde{\nu}, \tilde{k}; h). \quad (10.1) \]

Note that such solutions are not unique.

The proof of (7.2) repeats the arguments from the proof of Proposition 1.2 in [18] (with few straightforward modifications) and we do not need to reproduce it here. Note that in our case we need to choose the contour \( \gamma = \gamma_+ \cup \gamma_- \) in Fig. 1 containing two parts: with \( \gamma_+ \) as in Figure 1 in [18] and \( \gamma_- \) the mirror image with respect to the imaginary axis of the contour \( \gamma_+ \). We prove (7.4).

As \( F_{\vartheta,-}(\lambda, \tilde{k}) = -F_{\vartheta,+}(\lambda, \tilde{k}) \) (see (9.20) it is enough to consider \( F_{\vartheta,+}(\lambda, \tilde{k}) \). We use (8.3) from Proposition 8.1:
\[ F_{\vartheta,0}(\pm \tilde{k}, \tilde{\lambda}, \tilde{k}) = E^2(\tilde{k} - \frac{a}{E}\tilde{\lambda})^2, \quad \partial_\tilde{\nu} F_{\vartheta,0}(\pm \tilde{k}, \tilde{\lambda}, \tilde{k}) = \pm \tilde{k} + \mathcal{O}(a^2), \]

implying \( \partial_\tilde{k} F_{\vartheta,0}(\pm \tilde{k}, \tilde{\lambda}, \tilde{k}) = -2a\tilde{\lambda} + \mathcal{O}(a^2) \). Then we get
\[ \partial_\tilde{k} \sqrt{F_{\vartheta,0}(\pm \tilde{k}, \tilde{\lambda}, \tilde{k})} = -\frac{a\tilde{\lambda}}{E\tilde{k}} + \mathcal{O}(a^2). \quad (10.2) \]

We use (9.17). Suppose \( \tilde{\lambda} \) is solution to the equation
\[ F_{\vartheta,0}(\tilde{\lambda}, \tilde{k}) = \pm \left( F_{\vartheta,0}(\tilde{l}, \tilde{\lambda}, \tilde{k}) \right)^{1/2}. \]

In \( F_{\vartheta,0}(\pm \tilde{k}, \tilde{\lambda}, \tilde{k}) = E^2(\tilde{k} - \frac{a}{E}\tilde{\lambda})^2 \), \( \tilde{\lambda} \) can be exchanged with
\[ \tilde{\lambda}_{|_{a=0, \tilde{l}=\pm \tilde{k}}} = \pm Ez_0\tilde{k}, \quad z_0 = \left( \frac{M}{r_0^3} - \frac{Q^2}{r_0^4} - \frac{\Lambda}{3} \right)^{1/2}. \]

Then \( \tilde{k} - \frac{a}{E}\tilde{\lambda} = \tilde{k}(1 \mp az_0) \).

We have
\[ \partial_\tilde{k} F_{\vartheta,0}(\tilde{\lambda}, \tilde{k}) = r_0F^{-\frac{1}{2}}(r_0)\partial_\tilde{k} \tilde{\lambda} + \frac{a}{r_0} \left[ \frac{4F^{\frac{1}{2}}(r_0)r_0^2}{8Q^2 - 6Mr_0} \left( 1 - r_0 \frac{1}{2}F^{-1}(r_0)F'(r_0) \right) - F^{-\frac{1}{2}}(r_0) \right]. \]
Comparing with (10.2) we get that
\[ r_0 F^{-\frac{1}{2}}(r_0) \partial_k \tilde{\lambda} + \frac{a}{r_0} \left[ \frac{4F^{\frac{1}{2}}(r_0)r_0^2}{8Q^2 - 6Mr_0} \left( 1 - r_0 \frac{1}{2} F^{-1}(r_0)F'(r_0) \right) - F^{-\frac{1}{2}}(r_0) \right] = \mp \frac{a\tilde{\lambda}|_{a=0}}{E_k}. \]

Here
\[ \tilde{\lambda}|_{a=0} = E_0 \tilde{k}, \quad z_0 = \left( \frac{M}{r_0^3} - \frac{Q^2}{r_0^4} - \frac{\Lambda}{3} \right)^{\frac{1}{2}}. \]

We get (7.4)
\[ \partial_k \tilde{\lambda} = -\frac{a}{r_0^2} \left[ \frac{4F(r_0)r_0^2}{8Q^2 - 6Mr_0} \left( 1 - r_0 \frac{1}{2} F^{-1}(r_0)F'(r_0) \right) - 1 \right] \mp a z_0 r_0^{-1} F^{\frac{1}{2}}(r_0). \]

If \( Q = 0 \), we get \( 1 - r_0 \frac{1}{2} F^{-1}(r_0)F'(r_0) = 0 \) and we get formula (0.4) in [18] if we choose sign \(-\).
11 Preliminaries.

11.1 Evolution equation and separation of variables for real \( \lambda \).

We consider the charged Dirac fields with mass \( m \) represented by 4-spinors \( \psi \) belonging to the Hilbert space \( \mathcal{H} = L^2(\mathbb{R} \times S^2, dx \, d\vartheta \, d\varphi; \mathbb{C}^4) \) and satisfying the evolution equation

\[
i\frac{\partial}{\partial t} \psi = \mathcal{D} \psi, \quad \mathcal{D} = J^{-1} \mathcal{D}_0, \quad \mathcal{D}_0 = \Gamma^1 D_x + b(x) \Gamma^0 + c(x, D_\varphi) + a(x) \mathcal{D}_{S^2}.
\]

Here \( \mathcal{D}_{S^2} \) is an angular Dirac operator on 2-sphere \( S^2 \) given in (1.8)

\[
\mathcal{D}_{S^2} = \sqrt{\Delta_\vartheta} \left[ \Gamma^2 \left( D_\vartheta + \frac{i\Lambda a^2 \sin(2\vartheta)}{12\Delta_\vartheta} \right) + \Gamma^3 \frac{E}{\Delta_\vartheta \sin \vartheta} D_\varphi \right] - am \cos \vartheta \Gamma^5.
\]

Now, note the following identities

\[
(D - \lambda) \psi = \phi \iff J^{-1} \left[ \Gamma^1 D_x + b(x) \Gamma^0 + c(x, D_\varphi) + a(x) \mathcal{D}_{S^2} - \lambda J \right] \psi = \phi
\]

\[
\iff \left[ \Gamma^1 D_x + b(x) \Gamma^0 + c(x, D_\varphi) + a(x) \mathcal{D}_{S^2} - \lambda \left( I_4 + a(x) b(\vartheta) \Gamma^3 \right) \right] \psi = J \phi
\]

\[
\iff \left[ \Gamma^1 D_x + b(x) \Gamma^0 + c(x, D_\varphi) - \lambda + a(x) \left\{ \mathcal{D}_{S^2} - \lambda b(\vartheta) \Gamma^3 \right\} \right] \psi = J \phi.
\]

Then the stationary Dirac equation \( \mathcal{D} \psi = \lambda \psi \) can be re-written as \( \mathcal{D}(\lambda) \psi = 0 \), where

\[
\mathcal{D}(\lambda) = \Gamma^1 D_x + b(x) \Gamma^0 + c(x, D_\varphi) - \lambda + a(x) \left\{ \mathcal{D}_{S^2} - \lambda b(\vartheta) \Gamma^3 \right\}, \quad (11.1)
\]

and \( (D - \lambda) \psi = f \) is equivalent to \( \mathcal{D}(\lambda) \psi = J f \) and \( \psi = (D - \lambda)^{-1} f = [\mathcal{D}(\lambda)]^{-1} J f \). Therefore, we have for the resolvent of \( \mathcal{D} \)

\[
(D - \lambda)^{-1} = [\mathcal{D}(\lambda)]^{-1} J.
\]

Let

\[
A_{S^2}(\lambda) = \mathcal{D}_{S^2} - \lambda b(\vartheta) \Gamma^3, \quad \mathcal{H}_{S^2} = L^2(S^2, d\vartheta d\varphi; \mathbb{C}^4).
\]

(11.2)

For real \( \lambda \), we decompose \( \mathcal{H}_{S^2} \) onto the angular modes \( \{e^{ik\varphi}\}_{k \in \mathbb{Z}} \) that are eigenfunctions for \( D_\varphi \) with anti-periodic boundary conditions (see [2]). Then

\[
\mathcal{H}_{S^2} = \bigoplus_{k \in \frac{1}{2} + \mathbb{Z}} \mathcal{H}^k_{S^2}, \quad \mathcal{H}^k_{S^2} = L^2([0, \pi), d\vartheta; \mathbb{C}^4). \quad (11.3)
\]

The reduced subspaces \( \mathcal{H}^k_{S^2} \) remain invariant under the action of \( A_{S^2}(\lambda) \) and we denote \( A_k(\lambda) = A_{S^2}(\lambda)_{|\mathcal{H}^k_{S^2}} \). We have explicitly

\[
A_k(\lambda) = \sqrt{\Delta_\vartheta} \left[ \Gamma^2 \left( D_\vartheta + \frac{i\Lambda a^2 \sin(2\vartheta)}{12\Delta_\vartheta} \right) + \Gamma^3 \left( \frac{kE}{\Delta_\vartheta \sin \vartheta} - \lambda \frac{a \sin \vartheta}{\Delta_\vartheta} \right) \right] - am \cos \vartheta \Gamma^5. \quad (11.4)
\]
For each $k \in 1/2 + \mathbb{Z}$, operator $A_k(\lambda)$ is self-adjoint and has discrete simple spectrum $\sigma(A_k(\lambda)) = \{ \mu_{kl}(\lambda) \}_{l \in \mathbb{Z}^*}$ with associated set of eigenfunctions $\{ u_{kl}^\lambda \}_{l \in \mathbb{Z}^*}$,

$$A_k(\lambda)u_{kl}^\lambda(\vartheta) = \mu_{k,l}(\lambda)u_{kl}^\lambda(\vartheta).$$

Here $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$. Since $\sigma(A_k(\lambda))$ is discrete, it has no accumulation point and thus

$$\forall k \in 1/2 + \mathbb{Z}, \quad |\mu_{k,l}(\lambda)| \to \infty \quad \text{as} \quad l \to \pm \infty.$$

Eigenvalues $\mu_{k,l}(\lambda)$ of $A_k(\lambda)$ are also the eigenvalues of $A_{S^2}(\lambda)$ with eigenfunctions $Y_{k,l}(\lambda) := Y_{k,l}(\lambda, \vartheta, \varphi) = u_{kl}^\lambda(\vartheta)e^{ik\varphi}$.

The analogue of Lemma 1 is still valid in the massive case. By using the cylindrical symmetry we decompose the Hilbert space $\mathcal{H}$ onto the angular modes $\{ e^{ik\varphi} \}_{k \in 1/2 + \mathbb{Z}},$

$$\mathcal{H} = \bigoplus_{k \in 1/2 + \mathbb{Z}} \mathcal{H}_k, \quad \mathcal{H}_k = L^2(\mathbb{R} \times (0, \pi), dx d\vartheta; \mathbb{C}^4) = L^2(\mathbb{R}; \mathbb{C}^4) \otimes L^2((0, \pi), d\vartheta; \mathbb{C}^4). \quad (11.5)$$

We choose half-integers $k$ as we want the anti-periodic conditions in variable $\varphi$ : the spinors change the sign after a complete rotation (see [2]). Note that

$$\mu_{k,-l}(\lambda) = -\mu_{kl}(\lambda), \quad Y_{k,-l}(\lambda) = \Gamma_1 Y_{kl}(\lambda).$$

Using these results we have the decomposition

$$\mathcal{H} = \bigoplus_{(k,l) \in I} \mathcal{H}_{kl}(\lambda), \quad I = \left( \frac{1}{2} + \mathbb{Z} \right) \times \mathbb{N}^*, \quad \mathcal{H}_{kl}(\lambda) = L^2(\mathbb{R}; \mathbb{C}^4) \otimes Y_{kl}(\lambda).$$

We choose $I$ instead of $K$ in order to have subspaces $\mathcal{H}_{kl}$ remain invariant under the action of $\mathcal{D}(\lambda)$ (see Section 3.2 in [14] for details).

Let $\mathcal{D}_k(\lambda) := \mathcal{D}(\lambda)|_{\mathcal{H}_k} = \Gamma_1 D_x + b(x)\Gamma^0 + c(x, D_\varphi) - \lambda + a(x)A_k(\lambda)$ be restriction of $\mathcal{D}(\lambda)$ to $\mathcal{H}_k$.

Radial operator $\Gamma_1 D_x + b(x)\Gamma^0 + c(x, k) - \lambda$ lets invariant $\mathcal{H}_{kl}$ and its action on $\psi := \psi_{kl} \otimes Y_{kl}(\lambda) \in \mathcal{H}_{kl}$ is given by

$$[\Gamma_1 D_x + b(x)\Gamma^0 + c(x, k) - \lambda] \psi = \left( [\Gamma_1 D_x + b(x)\Gamma^0 + c(x, k) - \lambda] \psi_{kl} \right) \otimes Y_{kl}(\lambda).$$

Angular operator $A_{S^2}(\lambda)$ lets invariant $\mathcal{H}_{kl}$ and its action on $\psi := \psi_{kl} \otimes Y_{kl}(\lambda) \in \mathcal{H}_{kl}$ is given by

$$A_{S^2}(\lambda)\psi = (\mu_{kl}(\lambda)\Gamma^2 \psi_{kl}) \otimes Y_{kl}(\lambda). \quad (11.6)$$

### 11.2 Properties of eigenvalues $\mu_{kl}$ of angular operator for real $\lambda$.

The angular operator

$$A_{S^2}(\lambda) = \mathcal{D}_{S^2} + \lambda \frac{a \sin \vartheta}{\sqrt{\Delta \vartheta}} \Gamma^3,$$
where $D_{S^2}$ is given in (13.8) has similar properties as its massless version considered in Section 3.3.

For $\lambda \in \mathbb{R}$ operator $A_{S^2}(\lambda)$ is self-adjoint on $H_{S^2} = L^2(S^2; \mathbb{C}^4)$ and has positive discrete spectrum $\sigma(A_{S^2}(\lambda)) = \{\mu_{kl}(\lambda)\}_{(k,l) \in I}$, ordered in such a way that for each $k \in \frac{1}{2} + \mathbb{Z}$ and $l \in \mathbb{N}^*$ it follows $0 < \mu_{kl}(\lambda) < \mu_{k(l+1)}(\lambda)$. Put $A_k(\lambda) = A_{S^2}(\lambda)|_{H_{S^2}}$.

Let $\zeta = \frac{a^2A}{3}$, $\xi = a\lambda$, $\nu = a\mathbf{m}$ and consider operator $A_k(\lambda)$ as operator-valued function $A_k(\zeta, \xi, \nu)$ of complex parameters $\zeta, \xi, \nu$. Put

$$A_k(\zeta, \xi, \nu) = A(\zeta)D_{S^2}^k + B(\zeta, \xi, \nu),$$

with

$$A(\zeta) = \sqrt{1 + \zeta \cos^2 \vartheta}, \quad B = i\Gamma^2 - \frac{\zeta \sin(2\vartheta)}{4\sqrt{1 + \zeta \cos^2 \vartheta}} + \Gamma^3 \frac{(\zeta k - \xi) \sin \vartheta}{\sqrt{1 + \zeta \cos^2 \vartheta}} - \nu \cos \vartheta \Gamma^5.$$

Operator

$$A_k(0,0,0) \equiv D_{S^2}^k = \Gamma^2 D_\vartheta + \Gamma^3 \frac{k}{\sin \vartheta}$$

is the restriction of the standard Dirac operator on $S^2$ onto the angular mode $\{e^{ik\vartheta}\}$, $k \in 1/2 + \mathbb{Z}$. The domain of $A_k(0,0,0)$ is given by

$$\mathcal{D} = \{u \in H_{S^2}^k, \ u \ \text{is absolutely continuous}, \ D_{S^2}^k u \in H_{S^2}^k, \ u(\pi) = -u(0)\}.$$

The spectrum of $A_k(0,0,0)$ is simple discrete given by

$$\mu_{k,l}(0,0,0) = sgn(l) \left(|k| - \frac{1}{2} + |l|\right), \quad l \in \mathbb{Z}^*$$

and $A_k(0,0,0)$ has compact resolvent.

According to (12.2) we have $\zeta \in [0, 7 - 4\sqrt{3}] \subset [0, \frac{1}{13}]$ and $\xi, \nu \in \mathbb{R}$ respectively. Now, we allow parameters $\zeta, \xi, \nu$ to be complex $\langle \zeta, \xi, \nu \rangle \in B(0, \frac{1}{13}) \times S^2$, where $B(0, r) = \{z \in \mathbb{C}; \ |z| < r\}$ and $S$ is a narrow strip containing the real axis.

The operators $A(\zeta)$, $B(\zeta, \xi, \nu)$ are bounded matrix-valued multiplications operators analytic in the variables $\langle \zeta, \xi, \nu \rangle \in B(0, \frac{1}{13}) \times S^2$. Since the operator $A(\zeta)$ is also invertible, the operators domain of $A_k(\zeta, \xi, \nu)$ is independent on $\langle \zeta, \xi, \nu \rangle \in B(0, \frac{1}{13}) \times S^2$.

Moreover, since for all $u \in \mathcal{D}$, $A_k(\zeta, \xi, \nu)u$ is a vector-valued analytic function in $\langle \zeta, \xi, \nu \rangle$, and since for $\langle \zeta, \xi, \nu \rangle \in [0, \frac{1}{13}] \times \mathbb{R}^2$ is self-adjoint on $H_{S^2}^k = L^2((0, \pi), d\vartheta; \mathbb{C}^4)$, then $A_k(\zeta, \xi, \nu)$ forms a self-adjoint holomorphic family of type (A) in variable $\langle \zeta, \xi, \nu \rangle \in B(0, \frac{1}{13}) \times S^2$ according to Kato’s classification.

Then, using the analytic perturbation theory by Kato [32], $A_k(\zeta, \xi, \nu)$ has compact resolvent for all $\langle \zeta, \xi, \nu \rangle \in B(0, \frac{1}{13}) \times S^2$, and for a fixed $k \in 1/2 + \mathbb{Z}$, the eigenvalues

$$\mu_{kl}(\zeta, \xi, \nu), \quad k \in 1/2 + \mathbb{Z}, \ l \in \mathbb{Z}^*,$$
of \( A_k(\zeta, \xi, \nu) \) are simple and depend holomorphically on \((\zeta, \xi, \nu)\) in a complex neighborhood of \([0, \frac{1}{13}] \times \mathbb{R}^2\).

Moreover, for all \( \zeta \in [0, \frac{1}{13}], \ k \in \frac{1}{2} + \mathbb{Z} \) and \( l \in \mathbb{N}^* \),

\[
|\mu_{kl}(\zeta, 0) - (|k| - \frac{1}{2} + l)| \leq (e^{\frac{3\pi}{4}} - 1) (|k| - \frac{1}{2} + l) + 2 \left( e^{\frac{3\pi}{13}} - 1 \right) \left( 1 + \frac{1}{26} \right) (|k| + \frac{1}{4}) + |a|m,
\]

\( (11.9) \)

\[
\mu_{kl}(\zeta, 0) \geq \left( 2 - e^{\frac{3\pi}{13}} \right) (|k| - \frac{1}{2} + l) - 2 \left( e^{\frac{3\pi}{13}} - 1 \right) \left( 1 + \frac{1}{26} \right) (|k| + \frac{1}{4}) - |a|m
\]

\[
\geq \left( 4 + \frac{1}{13} - (3 + \frac{1}{13})e^{\frac{3\pi}{4}} \right) (|k| + \frac{1}{4}) - |a|m > 0.8 \left( |k| + \frac{1}{4} \right) - |a|m > 0.7 \left( |k| + \frac{1}{4} \right)
\]

for \(|a|m < 0.1 \left( |k| + \frac{1}{4} \right) \). For \( l < 0 \) we use \( \mu_{k,-l} = -\mu_{kl} \) and get \( \mu_{kl}(\zeta, 0) < -0.7 \left( |k| + \frac{1}{4} \right) \).

Note that

\[
B(\zeta, \xi, \nu) - B(\zeta, 0, \nu) = -\Gamma^3 \frac{\xi \sin \vartheta}{\sqrt{1 + \zeta \cos^2 \vartheta}}
\]

and the same argument works as in Section 5.2 which implies an analogue of \((5.16)\) in the massive case:

\[
\|u\|_{L^2} \leq \frac{\|A_k(\zeta, \xi) - \omega\| u\|_{L^2}}{d(\omega, \mathbb{R} \setminus (0.7 (|k| + \frac{1}{4}))(-1, 1)) - |a\lambda|}
\]

provided that the denominator is positive and \(|a|m\) small enough.

Note that from \((11.9)\) it follows and an analogue of \((3.17)\):

for all \( \lambda \in \mathbb{R} \), for all \( k \in \frac{1}{2} + \mathbb{Z} \) and for all \( l \in \mathbb{N}^* \), the eigenvalues \( \mu_{kl}(\lambda) \) for some constants \( C_1 \) and \( C_2 \) independent of \( k, l \), satisfy bound

\[
\left( 2 - e^{\frac{3\pi}{4}} \right) (|k| - \frac{1}{2} + l) - C_1 |k| - C_2 - |a|(|\lambda| + m) \leq \mu_{kl}(\lambda)
\]

\( (11.10) \)

\[
\leq e^{\frac{3\pi}{4}} (|k| - \frac{1}{2} + l) + C_1 |k| + C_2 + |a|(|\lambda| + m).
\]

### 12 Resolvent.

#### 12.1 Decomposition in radial and angular parts.

Now, we apply the method from [17], Section 3. Take \( k \in 1/2 + \mathbb{Z} \) and an arbitrary \( \delta > 0 \). Let the angle of admissible contours (see Section 4). Then it follows that the resolvents

\[
R_r(\lambda, \omega, k) = (H_k(\lambda) + \omega)^{-1} : L^2_{comp}(\mathbb{R}, dx; \mathbb{C}^4) \mapsto H^1_{loc}(\mathbb{R}; \mathbb{C}^4),
\]

\[
R_o(\lambda, \omega, k) = (A_k(\lambda) - \omega)^{-1} : L^2((0, \pi), d\vartheta; \mathbb{C}^4) \mapsto H^1((0, \pi), d\vartheta; \mathbb{C}^4)
\]

are meromorphic families of operators in the sense of Definition 2.1 in [17]. In particular, for a fixed values of \( \lambda \), these families are meromorphic in \( \omega \) with poles of finite rank (see Definition 2.2 in [17]).
We have the following property which assures that admissible contour exists at every regular point.

For any compact $K_\lambda \subset \Omega \subset \mathbb{C}$ there exist constants $C$ and $R$ such that for $\lambda \in K_\lambda$ and $|\omega| \geq R$,

(i) for $|\arg \omega| \leq \psi$ and $|\pi - \arg \omega| \leq \psi$ we have $(\lambda, \omega) \notin Z_r$ and $\|R_r(\lambda, \omega, k)\| \leq 1/|\omega|$;

(ii) for $\psi \leq |\arg \omega| \leq \pi - \psi$, we have $(\lambda, \omega) \notin Z_\theta$ and $\|R_\theta(\lambda, \omega, k)\| \leq 1/|\omega|$.

Then we can construct restriction of the resolvent

$$R(\lambda) = (\mathcal{D} - \lambda)^{-1} = [\mathcal{D}(\lambda)]^{-1}J = [\tilde{\mathcal{D}}(\lambda)]^{-1}[a(x)]^{-1}J$$

to $\mathcal{H}_k$ at any regular point $\lambda$ as a contour integral

$$R(\lambda, k) = \tilde{R}(\lambda, k)[a(x)]^{-1}, \quad \tilde{R}(\lambda, k) = \frac{1}{2\pi i} \int_{\gamma} \tilde{R}_r(\lambda, \omega, k) \otimes \tilde{R}_\theta(\lambda, \omega, k)d\omega \quad (12.1)$$

for some admissible $\gamma$. The orientation of $\gamma$ is chosen so that $\Gamma_r$ always stays on the left.

We represent the Hamiltonian as tensor product $H_k(\lambda) \otimes I_2 + I_2 \otimes A_k(\lambda)$ acting in $L^2(\mathbb{R}, dx; \mathbb{C}^4) \otimes \mathcal{H}_{\mathbb{S}^2}^k$, where $\mathcal{H}_{\mathbb{S}^2} = L^2((0, \pi), d\vartheta; \mathbb{C}^4)$. Here $H_k, A_k$ are the operators specified below.

In the exterior region $a \neq 0$ and we introduce operator

$$\tilde{\mathcal{D}}(\lambda) = [a(x)]^{-1}\mathcal{D}(\lambda) = [a(x)]^{-1}(\Gamma^1D_x + b(x)\Gamma^0 + c(x, k) - \lambda) + (\mathcal{D}_2 - \lambda b(\vartheta))\Gamma^3).$$

Denote $\tilde{\mathcal{D}}_k(\lambda) = \tilde{\mathcal{D}}(\lambda)|_{\mathcal{H}_k}$ its restriction to $\mathcal{H}_k$. For real $\lambda$, its radial $\tilde{\mathcal{D}}^r_k(\lambda)$ and angular $\tilde{\mathcal{D}}^\theta_k(\lambda)$ parts let invariant $\mathcal{H}_{kl}$ and the action $\tilde{\mathcal{D}}_k(\lambda) = \tilde{\mathcal{D}}^r_k(\lambda) + \tilde{\mathcal{D}}^\theta_k(\lambda)$ on $\psi = \psi_{kl} \otimes Y_{kl}(\lambda) \in \mathcal{H}_{kl}$ is given by

$$\tilde{\mathcal{D}}_k(\lambda)\psi = (\tilde{\mathcal{D}}^r_k(\lambda)\psi_{kl}) \otimes Y_{kl}(\lambda) + (\Gamma^2\psi_{kl}) \otimes (A_{\mathbb{S}^2}(\lambda)Y_{kl}(\lambda)).$$

Let $\phi_{kl} = \Gamma^2\phi_{kl} \in L^2(\mathbb{R}, dx; \mathbb{C}^4)$ and

$$H_k(\lambda) := [a(x)]^{-1}(\Gamma^1D_x + b(x)\Gamma^0 + c(x, k) - \lambda)\Gamma^2, \quad A_k(\lambda) = A_{\mathbb{S}^2}(\lambda)|_{\mathcal{H}_{\mathbb{S}^2}^k}$$

acting in $L^2(\mathbb{R}, dx; \mathbb{C}^4)$, $\mathcal{H}_{\mathbb{S}^2}^k = \text{Span}_{i \in \mathbb{Z}^3}(Y_{kl}(\lambda))$ respectively. Then for any $\lambda \in \mathbb{C}$,

$$\tilde{\mathcal{D}}_k(\lambda) = H_k(\lambda) \otimes I_2 + I_2 \otimes A_k(\lambda)$$

acts in $L^2(\mathbb{R}, dx; \mathbb{C}^4) \otimes \mathcal{H}_{\mathbb{S}^2}^k$.

Let $R_r(\lambda, \omega, k) = (H_k(\lambda) + \omega)^{-1}$, $R_\theta(\lambda, \omega, k) = (A_k(\lambda) - \omega)^{-1}$.

### 12.2 Square of the angular operator.

Here, we extend results of Section 5.2 for the angular operator to the massive case.

We consider an $h$-dependent version of the operator $A_{\mathbb{S}^2}(\lambda)$

$$A_{\mathbb{S}^2, h}(\lambda) = \sqrt{\Delta_\vartheta} \left[ \Gamma^2 (hD_\vartheta + hq_1(\vartheta)) + \Gamma^3 (q_2(\vartheta)hD_\varphi - \lambda q_3(\vartheta)) - hq_4(\vartheta)\Gamma^5 \right],$$

55
\[ q_1(\vartheta) = \frac{i \Lambda a^2 \sin(2\vartheta)}{12 \Delta \vartheta}, \quad q_2(\vartheta) = \frac{E}{\Delta \vartheta \sin \vartheta}, \quad q_3(\vartheta) = \frac{a \sin \vartheta}{\Delta \vartheta}, \quad q_4(\vartheta) = \frac{a \cos \vartheta}{\sqrt{\Delta \vartheta}}. \]

Let \( \sigma_j, \quad i = 1, 2, 3, \) be the Pauli matrices (3.2) and \( \sigma_0 = I_2. \) We choose the following representation of Dirac matrices satisfying \( \Gamma_i \Gamma_j + \Gamma_j \Gamma_i = 2 \delta_{ij} I_4, \quad j = 0, 1, 2, 3, 4, 5 : \)

\[ \Gamma^1 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}, \quad \Gamma^2 = \begin{pmatrix} 0 & -i \sigma_2 \\ i \sigma_2 & 0 \end{pmatrix}, \quad \Gamma^3 = \begin{pmatrix} 0 & -i \sigma_1 \\ i \sigma_1 & 0 \end{pmatrix}, \quad \Gamma^4 = \begin{pmatrix} 0 & -i \sigma_3 \\ i \sigma_3 & 0 \end{pmatrix}, \quad \Gamma^5 = \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3. \]

Using that \( \Gamma^2 \Gamma^3 = \begin{pmatrix} -i \sigma_3 & 0 \\ 0 & -i \sigma_3 \end{pmatrix}, \quad \Gamma^2 \Gamma^5 = \Gamma^0 \Gamma^1 \Gamma^3 = \begin{pmatrix} -i \sigma_2 & 0 \\ 0 & i \sigma_2 \end{pmatrix}, \)

we get

\[ [A_{S^2, h}(\lambda)]^2 = \begin{pmatrix} \mathcal{P}_+ & 0_2 \\ 0_2 & \mathcal{P}_- \end{pmatrix}, \]

\[ [\Delta \vartheta]^{-1} \mathcal{P}_\pm = [h D \vartheta + h q_1(\vartheta)]^2 + [q_2(\vartheta) h D \varphi - \lambda q_3(\vartheta)]^2 - h \sigma_3 [q'_2(\vartheta) h D \varphi - \lambda q'_3(\vartheta)] \pm h \sigma_2 q'_4(\vartheta) \]

\[ - h^2 q'_4(\vartheta). \]

The leading term of order \( h^0 \) of \( \mathcal{P}_\pm \) is given by

\[ \mathcal{P}_{\pm, 0}(\lambda) = \mathcal{P}_{-0}(\lambda) = I_2 \left( \Delta \vartheta h^2 D^2 \vartheta + [q_2(\vartheta) h D \varphi - \lambda q_3(\vartheta)]^2 \right) \]

\[ = I_2 \left( \Delta \vartheta h^2 D^2 \vartheta + \left( \frac{1 + \frac{\Lambda a^2}{3}}{\sin^2 \vartheta \Delta \vartheta} \right)^2 \left( h D \varphi - \lambda a \sin^2 \vartheta \frac{1}{1 + \Lambda a^2 / 3} \right)^2 \right), \]

where the operator on the diagonal is exactly the same as in the massless case.

### 12.3 Radial operator.

Here we extend the content of Section [6] to the massive case.

We consider the radial resolvent \( R_r(\lambda, \omega, k) = (H_k(\lambda) + \omega)^{-1}, \) where

\[ H_k(\lambda) := [a(x)]^{-1} (\Gamma^1 D_x + b(x) \Gamma^0 + c(x, k) - \lambda) \Gamma^2. \]
Note that $R_r(\lambda, \omega, k)$ satisfies

$$( [a(x)]^{-1} (\Gamma^1 D_x + b(x) \Gamma^0 + c(x, k) - \lambda) \Gamma^2 + \omega ) R_r(\lambda, \omega, k) f = f, \quad f \in C^\infty_0(\mathbb{R}, dx : \mathbb{C}^2),$$

$\Leftrightarrow \ (\Gamma^1 D_x + b(x) \Gamma^0 + c(x, k) - \lambda + \omega a(x) \Gamma^2 ) \Gamma^2 R_r(\lambda, \omega, k) f = a(x) f$

$\Leftrightarrow R_r(\lambda, \omega, k) = \Gamma^2 \left[ \Gamma^1 D_x + b(x) \Gamma^0 + c(x, k) - \lambda + \omega a(x) \Gamma^2 \right]^{-1} a(x).$

Let $R(\lambda, \omega, k) = \left[ \mathcal{D}_k(\lambda, \omega) \right]^{-1}$ be resolvent of

$$\mathcal{D}_k(\lambda, \omega) = \Gamma^1 D_x + b(x) \Gamma^0 + c(x, k) - \lambda + \omega a(x) \Gamma^2.$$

Here

$$c(x, k) = \frac{aEk + qQr}{r^2 + a^2}, \quad a(x) = \frac{\sqrt{\Delta_r}}{r^2 + a^2}, \quad b(x) = \frac{m r \sqrt{\Delta_r}}{r^2 + a^2},$$

satisfying

$$a(x) = a_\pm e^{\kappa x} + \mathcal{O} \left( e^{3\kappa x} \right), \quad b(x) = b_\pm e^{\kappa x} + \mathcal{O} \left( e^{3\kappa x} \right) \quad x \to \pm \infty, \quad (12.2)$$

$$c(x, k) = \Omega_\pm(k) + c_\pm e^{2\kappa x} + \mathcal{O} \left( e^{4\kappa x} \right), \quad x \to \pm \infty, \quad (12.3)$$

$$\Omega_\pm = \frac{aEk + qQr_\pm}{r^2_\pm + a^2}, \quad \Omega_- > \Omega_+.$$

Outgoing solutions $f^\pm$ can be defined similarly to the massless case, using that as $x \to \pm \infty$ they satisfy $[\Gamma^1 D_x + \Omega_\pm(k) - \lambda] f = 0$,

$$f(x) = \sum_{i=1}^4 f_i(x)e_i, \quad e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \ldots$$

or

$$\begin{cases} -if_1' + \Omega_\pm(k) f_1 = \lambda f_1 \\ -if_2' + \Omega_\pm(k) f_2 = \lambda f_2 \\ if_3' + \Omega_\pm(k) f_3 = \lambda f_3 \\ if_4' + \Omega_\pm(k) f_4 = \lambda f_4 \end{cases}, \quad \lambda \in \mathbb{C}.$$

The outgoing solutions $f^\pm$ are then defined by their asymptotics

$$f^- \sim f^{0,-}(x; \lambda, k) = \begin{pmatrix} 0_2 \\ e^{-i(\lambda - \Omega_-)(k)x} I_2 \end{pmatrix}, \quad x \to -\infty;$$

$$f^+ \sim f^{0,+}(x; \lambda, k) = \begin{pmatrix} e^{i(\lambda - \Omega_+)(k)x} I_2 \\ 0_2 \end{pmatrix}, \quad x \to +\infty.$$
13 Semiclassical leading part and square of the radial operator.

Here, we extend results of Section 5.2 for the radial operator to the massive case. We consider semiclassical version of the Dirac operator $\Gamma D_x + b(x)\Gamma^0 + c(x,k) - \lambda + \omega a(x)\Gamma^2$ of the form

$$\Gamma D_x + h b(x)\Gamma^0 + c(x,k) - (\tilde{\lambda} + ih\tilde{\nu}) + (\tilde{\omega} + ih\tilde{\mu})a(x)\Gamma^2$$

with the leading part $D_{h} = D_{h,+} = \Gamma h D_x + c_0(x,k) - \tilde{\lambda} + \tilde{\omega}a(x)\Gamma^2$. Note that the leading part is independent of the field mass $m$. Put $c = c_0$, $q = \tilde{\omega}a(x)$.

In $\mathcal{H} = L^2(\mathbb{R})^4$ we consider two Dirac operators

$$D_{h,\pm}(\lambda) := -D_{h,0} \pm (c(x) - \lambda)I_4, \quad D_{h,0} := -ih\Gamma^1\partial_x + q(x)\Gamma^2.$$  

The product of the operators is given by

$$D_{h,+} D_{h,-} = D_{h,0}^2 - (c - \lambda)^2 I_4 - [D_{h,0}, (c - \lambda)I_4]. \quad (13.1)$$

Here

$$D_{h,0}^2 = (-h^2 \partial_x^2 + q^2(x))I_4 - \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix} hq'(x)$$

is the matrix Schrödinger operator. The commutator is given by

$$[D_{h,0}, (c - \lambda)I_4] = -ih\Gamma^1c'(x).$$

The operator $D_{h,0}^2$ is self-adjoint in $\mathcal{H} = L^2(\mathbb{R})^4$ and unitary equivalent to

$$U D_{h,0}^2 U^{-1} = \begin{pmatrix} P_{h,0}^- & 0 \\ 0 & P_{h,0}^+ \end{pmatrix}, \quad P_{h,0}^\pm = (-h^2 \partial_x^2 + q^2 \pm hq')I_2. \quad (13.2)$$

Here $U$ is constant matrix.

We get also

$$U (D_{h,0}^2 - (c - \lambda)^2 I_4) U^{-1} = \begin{pmatrix} P_{h}^- & 0 \\ 0 & P_{h}^+ \end{pmatrix}, \quad (13.3)$$

$$P_{h,0}^\pm(\lambda) = (-h^2 \partial_x^2 + q^2 - (c - \lambda)^2 \pm hq')I_2$$

Now, using (13.1) we get

$$D_{h,+} D_{h,-} = U^{-1} \begin{pmatrix} P_{h}^- & 0 \\ 0 & P_{h}^+ \end{pmatrix} U + ihc'(x)\Gamma^1. \quad (13.4)$$

If $P_{h}^\pm(\lambda)$ are invertible, denote

$$\mathcal{R}(\lambda) = U^{-1} \begin{pmatrix} [P_{h}^-(\lambda)]^{-1} & 0 \\ 0 & [P_{h}^+(\lambda)]^{-1} \end{pmatrix} U.$$
and write
\[
\mathcal{D}_{h,+} \mathcal{D}_{h,-} \mathcal{R}(\lambda) = I_4 + i h \epsilon'(x) \Gamma_1 \mathcal{R}(\lambda)
\]
\[
\mathcal{D}_{h,-} \mathcal{R}(\lambda) = [\mathcal{D}_{h,+}]^{-1} + [\mathcal{D}_{h,+}]^{-1} i h \epsilon'(x) \Gamma_1 \mathcal{R}(\lambda) = [\mathcal{D}_{h,+}]^{-1} \left( I_4 + i h \epsilon'(x) \Gamma_1 \mathcal{R}(\lambda) \right)
\]
or
\[
[\mathcal{D}_{h,+}(\lambda)]^{-1} \left( I_4 + i h \Gamma^1 \epsilon'(x) \left( \mathcal{D}_{h,0}^2 - (c - \lambda)^2 I_4 \right)^{-1} \right) = \mathcal{D}_{h,-}(\lambda) \left( \mathcal{D}_{h,0}^2 - (c - \lambda)^2 I_4 \right)^{-1}
\]

which leads to

\[
[\mathcal{D}_{h,+}(\lambda)]^{-1} = \mathcal{D}_{h,-}(\lambda) U^{-1} \left( \begin{array}{cc} [\mathcal{P}_h^-(\lambda)]^{-1} & 0 \\ 0 & [\mathcal{P}_h^+(\lambda)]^{-1} \end{array} \right) U \cdot \left( I_4 + i h \Gamma^1 \epsilon'(x) \mathcal{D}_{h,0}^2 - (c - \lambda)^2 I_4 \right)^{-1}.
\]

As \( U \) is constant matrix, identity \([13.5]\) can be extended to \( q = \omega a \) complex with \( \omega \in \mathbb{C} \).

### 14 Normal form of the radial operator.

In this section we extend Proposition 9.4 to the massive case which together with the properties of the angular operator and quantization conditions as in the massless case concludes the proof of Theorem 2.1.

Let \( \mathcal{D}_k^r = \Gamma^1 h D_x + h b(x) \Gamma^0 + c(x, \tilde{k}) - (\tilde{\lambda} + i h \nu) + (\tilde{\omega} + i h \tilde{\mu}) a(x) \Gamma^2 \) with the leading part
\[
\mathcal{D}_{h,0} = \Gamma^1 h D_x + c_0(x, \tilde{k}) - \tilde{\lambda} + \tilde{\omega} a(x) \Gamma^2 =
\[
\begin{pmatrix}
h D_x + c_0(x, \tilde{k}) - \tilde{\lambda} & 0 & 0 & \tilde{\omega} a(x) \\
0 & h D_x + c_0(x, \tilde{k}) - \tilde{\lambda} & -\tilde{\omega} a(x) & 0 \\
0 & -\tilde{\omega} a(x) & -h D_x + c_0(x, \tilde{k}) - \tilde{\lambda} & 0 \\
\tilde{\omega} a(x) & 0 & 0 & -h D_x + c_0(x, \tilde{k}) - \tilde{\lambda}
\end{pmatrix}.
\]

The determinant of its principal symbol
\[
\det \begin{pmatrix}
\xi + c_0(x, \tilde{k}) - \tilde{\lambda} & 0 & 0 & \tilde{\omega} a(x) \\
0 & \xi + c_0(x, \tilde{k}) - \tilde{\lambda} & -\tilde{\omega} a(x) & 0 \\
0 & -\tilde{\omega} a(x) & -\xi + c_0(x, \tilde{k}) - \tilde{\lambda} & 0 \\
\tilde{\omega} a(x) & 0 & 0 & -\xi + c_0(x, \tilde{k}) - \tilde{\lambda}
\end{pmatrix}
\]
is equal to \((\xi^2 + (\tilde{\omega} a(x))^2) - (c_0 - \tilde{\lambda})^2)^2\). Thus eigenvalues of the principal symbol are of multiplicity two and equal to
\[
\mu_{\pm} = \mu_{\pm}(x, \xi; \tilde{\lambda}, \tilde{\omega}, \tilde{k}) = c_0(x, \tilde{k}) - \tilde{\lambda} \pm \sqrt{\xi^2 + \tilde{\omega}^2 a^2(x)},
\]
which are the same as in the massless case (where the multiplicity was one).

Now, by iteration as in Section 3.1 of [25] and Section 4.1 of [7] we can reduce operator \( \mathcal{D}_k^r \) to the block-diagonal form up to the infinite order of \( h \).
Proposition 14.1 (Decoupling). There exist unitary $U = U(x, hD_x, h)$ such that

$$U^\ast \tilde{D}_k^r(\lambda, \omega) U = \begin{pmatrix} \mu_+(x, hD_x, h) & 0 \\ 0 & \mu_-(x, hD_x, h) \end{pmatrix} + O(h^\infty) \quad (14.1)$$

in $L(L^2, L^2)$. Here $\mu_\pm(x, hD_x, h)$ is pseudodifferential operator with classical symbol $\mu_\pm(x, \xi, h) = \sum_{j=0}^\infty h^j \mu_{j,\pm}(x, \xi)$ and $\mu_{0,\pm}(x, \xi) = \mu_\pm(x, \xi) I_2$.

Proposition 14.2. The microlocal normal form of $\tilde{D}_k^r(\lambda, \omega)$ is given by

$$\begin{pmatrix} h x I_2 D_x - \beta_+ & 0 \\ 0 & -h x I_2 D_x + \beta_- \end{pmatrix},$$

where $\beta_\pm = \beta_\pm(\tilde{\lambda}, \tilde{\nu}, \tilde{\omega}, \tilde{\mu}, \tilde{k}; h)$ is classical symbol. Moreover, the principal part $\beta_{0,\pm}$ of $\beta_\pm$ is real-valued, independent of $\tilde{\nu}, \tilde{\mu}$ and mass $m$, and vanishes if and only if $\tilde{\omega} = \tilde{\omega}_0(\tilde{\lambda}, \tilde{k})$, where $\tilde{\omega}_0^{-2}$ is defined by (9.13). Moreover,

$$\beta_{0,+} = -\frac{(c_0(x_+) - \tilde{\lambda} + \tilde{\omega} a(x_+)) \sqrt{|\tilde{\omega}| a(x_+)} I_2}{\sqrt{|(c_0 + |\tilde{\omega}| a^\prime(x_+)|}}} + O \left( c_0(x_+) - \tilde{\lambda} + \tilde{\omega} a(x_+) \right)^2$$

and $\beta_{0,-}(\tilde{\lambda}, \tilde{\nu}, \tilde{\omega}, \tilde{k}) = \beta_{0,+}(\tilde{\lambda}, \tilde{\nu}, \tilde{\omega}, \tilde{k})$.

Now, using that the leading parts of the angular and radial operators are independent of mass $m$, the angular and radial quantization conditions are independent are independent of mass in the leading order and Theorem (2.1) follows.
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