Dynamical localization of interacting bosons in the few-body limit

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The quantum kicked rotor is well-known to display dynamical localization in the non-interacting limit. In the interacting case, while the mean-field (Gross-Pitaevskii) approximation displays a destruction of dynamical localization, its fate remains debated beyond mean-field. Here we study the kicked Lieb-Liniger model in the few-body limit. We show that for any interaction strength, two kicked interacting bosons always dynamically localize, in the sense that the energy of the system saturates at long time. However, contrary to the non-interacting limit, the momentum distribution \( \Pi(k) \) of the bosons is not exponentially localized, but decays as \( C/k^4 \), as expected for interacting quantum particles, with Tan’s contact \( C \) which remains finite at long time. We discuss how our results will impact the experimental study of kicked interacting bosons.

I. INTRODUCTION

The Quantum Kicked Rotor (QKR) is a paradigmatic model of quantum chaos. It is most famous for displaying dynamical localization, which is the analog of Anderson localization in momentum space \([1]\). Experimental realizations of the atomic QKR and its variants have allowed for detailed studies of Anderson localization and two dimensions \([2]\), the Anderson transition in three dimensions \([3]\), as well as the study of the effects symmetries on weak localization \([4]\) and classical-to-quantum transitions \([3]\). We study two interacting bosons in a ring of circum-

II. THE INTERACTING QUANTUM KICKED ROTOR

We study two interacting bosons in a ring of circumference \( L = 2\pi \), with Hamiltonian \( \hat{H} = \hat{H}_{LL} + \hat{H}_K \). Here \( \hat{H}_{LL} \) describes the dynamics of the interacting bosons between the kicks, and is given by the Lieb-Liniger Hamiltonian \([23]\)

\begin{equation}
\hat{H}_{LL} = \frac{\hat{p}_1^2}{2} + \frac{\hat{p}_2^2}{2} + g \delta(\hat{x}_1 - \hat{x}_2),
\end{equation}

and the kick Hamiltonian reads

\begin{equation}
\hat{H}_K = K (\cos(\hat{x}_1) + \cos(\hat{x}_2)) \sum_n \delta(t - n).
\end{equation}

We use the standard units of the (non-interacting) kicked rotor: time is in units of the kick period \( T \), positions are in units of \( L/2\pi \) (which is also the inverse wavevector...
of the kicking potential), and momenta are in units of $ML/T$, with $M$ the mass of the bosons. The canonical commutation relations are then given by $[\hat{x}_i, \hat{p}_j] = \delta_{ij} \hbar$, with $\hbar = \frac{4\pi^2\kappa T}{ML}$ the effective Planck constant [24]. The dimensionless interaction strength $g$ is related to the one-dimensional scattering length $a$ by $g = -\frac{L}{a} \frac{\kappa^2}{4\pi}$ [25].

To study the dynamics of the system, it is convenient to use the eigenbasis of the Lieb-Liniger Hamiltonian. Following Lieb and Liniger, it is easily found using a Bethe ansatz, and the eigenfunctions of $\hat{H}_{LL}$ read

$$\Phi_m^n(x_1, x_2) = \frac{e^{i \frac{\kappa}{2}(x_1+x_2)}}{\sqrt{2\pi}} \sin \left( k_m^2 \left| x_1 - x_2 \right| - \frac{\theta_m}{2} \right) \sqrt{\frac{\pi}{\sin(\theta_m^2)}}. \tag{3}$$

Here, $n \in \mathbb{Z}$ is the momentum of the center-of-mass (in units of $\hbar$). The relative momentum $k_m = \frac{m+\theta_m}{2}$ (in units of $\hbar$) is parametrized by a positive integer $m$, and the phase-shift induced by the interaction $\theta_m$. The periodic boundary conditions and the delta-interaction give the constraints that $m + n$ must be odd, and

$$\theta_m = -2 \arctan \left( \frac{2k_m^2 m_n}{g} \right). \tag{4}$$

The energy of the state $|\Phi_m^n\rangle$ is $E_m^n = \frac{k_m^2}{\pi} (n^2 + 4k_m^2)$. The phase-shift $\theta_m$ is shown in Fig. 1 for different values of the interaction strength, and $k = 1$. It interpolates between 0 for small $m$, where the wave function effectively fermionizes, and $\theta_m \to -\pi$ as $m \to \infty$, where the bosons are almost free, as the (relative) kinetic energy dominates over the interaction. In the Tonks limit, $g \to \infty$, $\theta_m = 0$ and we recover the Tonks-Girardeau (TG) wave functions [26, 27].

The evolution operator over one period is given by

$$\hat{U} = e^{-i \frac{\kappa}{2} n \hat{p}} e^{-i \frac{\kappa}{2} m \hat{p}}, \tag{5}$$

and its matrix elements read

$$U_{mp}^{m \mp} \equiv \langle \Phi_m^n | \hat{U} | \Phi_p^q \rangle = e^{-i \frac{\kappa}{2} m \hat{p}} \langle \Phi_m^n | e^{-i \frac{\kappa}{2} m \hat{p}} | \Phi_p^q \rangle. \tag{6}$$

The matrix elements of kick operator must be computed numerically for finite $g$, and are given explicitly by

$$\langle \Phi_m^n | e^{-i \frac{\kappa}{2} m \hat{p}} | \Phi_p^q \rangle = \int_0^{2\pi} F_{q,n}(x) \psi_p(x) \psi_m(x), \tag{7}$$

with

$$\psi_m(x) = \frac{\sin(k_m x - \frac{\theta_m}{2})}{\sqrt{\pi - \sin(\theta_m^2)}} \frac{\sin(k_m x - \frac{\theta_m}{2})}{\sqrt{\pi - \sin(\theta_m^2)}}$$

and

$$F_n(x) = (-i)^n J_n \left( \frac{2k_m}{\kappa} \cos \left( \frac{\theta_m}{2} \right) \right), \quad J_n(z)$$

is the $n$-th Bessel function of the first kind. The asymptotic behavior of these matrix elements has been analyzed in Ref. [22]. There, it has been shown that for fixed $m$ and $p$, $|U_{mp}^{m \mp}|$ decays as $(|n - q|!)^{-1}$, much faster than an exponential, while at fixed $n, q, p$, it decays as $m^{-4}$ [28]. This power law decay has been interpreted by the authors of Ref. [22] to be the cause of the breakdown of dynamical localization in this model, see however the discussion of this argument in Sec. V.

To compute the time evolution of the system, we expand its wave function in the Lieb-Liniger basis, $|\Psi(t)\rangle = \sum_{n,m} c_{n,m}^q(t) |\Phi_n^m\rangle$, where the coefficients $c_{n,m}^q(t)$ obey the stroboscopic evolution $c_{n,m}^q(t+1) = \sum_{q,p} U_{mq}^{mp} c_{p,m}^q(t)$. Here and in the following, we always assume that the sum is performed over the allowed values of $m$ and $n$ ($m \in \mathbb{N}^+$, $n \in \mathbb{Z}$ and $n + m$ odd). To perform the time-evolution numerically, it is necessary to truncate the basis, and we only keep states with $|n| \leq n_{max}$ and $m \leq m_{max}$, with typical values of $n_{max} = 160$ and $m_{max} = 160$. We have checked that these values used in our numerics are such that our results are converged, in the sense that physical observables do not change when $n_{max}$ and $m_{max}$ are increased, and that the normalization of the wave function stays very close to one at all times (such that the states $|\Phi_n^m\rangle$ with $n > n_{max}$ and $m > m_{max}$ would not be significantly populated if they were included). Here and in the following, we will always assume that the system starts in the groundstate of the Lieb-Liniger Hamiltonian, $|\Psi_{t=0}\rangle = |\Phi_0^0\rangle$. We use $K = 3$ and $\kappa = 1$ in the numerics, which allows us to use a not too large basis.

One difficulty in the study of the dynamics of this problem is that the various observables typically display large fluctuations during time-evolution. This also happens in the context of the QKR, and in that case, one usually averages over the quasi-momentum $\beta$, which is a dynamically conserved quantity. Changing the quasi-momentum here corresponds to a change of the disorder realization of the corresponding Anderson model [24]. In order to simplify the analysis of our numerics, we introduce an artificial “quasi-momentum” in the energy of the Lieb-Liniger model, i.e. we replace $E_m^n$ by $E_m^{n+2\beta}$, equivalent to add a magnetic flux in the system. This way of introducing the quasi-momentum is consistent with what is done in the non-interacting limit. In practice, we average typically over 100 and 500 values of $\beta$ sampled uniformly in $[0, 1/2]$, and write the average of an observable $O$ by an overline, $\bar{O}$.
III. DYNAMICAL LOCALIZATION OF INTERACTING BOSONS

The top panel of Fig. 2 shows the time-evolution of the energy of the system $E_{\text{tot}}(t) = \langle \Psi_t | \hat{H}_{\text{LL}} | \Psi_t \rangle$ for various values of $g$, up to 2500 kicks. We observe a behavior similar to that of the dynamical localization of the non-interacting QKR: at very short times, the energy increases linearly, with a rate independent of $g$ (dashed line) – which hints that the classical diffusion constant might be rather insensitive to interactions. This initial behavior is followed by a decrease of diffusion and ultimately by a saturation of the energy. We conclude that, even in presence finite interactions, the system does not heat to infinite energy, which is a hallmark of localization for interacting systems. In this sense, the system dynamically localizes.

To check that the system does truly localize asymptotically (i.e., that delocalization of the energy does not happen at longer time scales), we have computed the energy after $2^N$ kicks, with $N$ up to 28, by computing $(U)^{2N}$. The bottom panel of Fig. 2 shows that the total energy of the system indeed saturates to a finite value and no sub-diffusive behavior seems to occur even at very large kick numbers. For some finite values of $g$, the localization time (i.e. the time needed for the full saturation of the energy) is significantly longer than in the non-interacting case. Finally, we have checked that the wave-function coefficients $c_n(t)$ do converge at long times to a finite steady-state value.

We now proceed to analyze the dependence of various observables as a function of the interaction strength $g$. Fig. 3 shows the total energy at long-time $E_{\text{tot}}(t) = \lim_{t \to \infty} E_{\text{tot}}(t)$ as a function of $g$. We observe a non-monotonous dependence of the energy as a function of the interaction. This is not too surprising, since in both limits $g = 0$ and $g = \infty$, the energy is given by that of non-interacting quasi-particles. In the non-interacting limit, the two bosons start in the zero-momentum state and localize with the same wave function described by the non-interacting QKR. In the opposite limit $g = \infty$, the Tonks limit, the system can be described in terms of non-interacting fermions [26, 27]. In particular, the energy of the Tonks gas is given by the kinetic energy of those free fermions. The fermions start in the state $\pm \frac{1}{2}$ and localize with wave functions described by the same localization length $p_{\text{loc}}$ (and hence the same final kinetic energy) as the free bosons. Moreover, because the interaction energy also vanishes in the Tonks limit due to the fermionization of the bosons, we therefore expect this two limits to have roughly the same total energy in the long-time limit. Fig. 4 shows the ratio between the interaction energy, $E_{\text{pot}} = \lim_{t \to \infty} \langle \Psi_t | g \delta(\hat{x}_1 - \hat{x}_2) | \Psi_t \rangle$, and the total energy in the localized regime. The interaction energy corresponds to a very small contribution, at most 1.5% for $g \approx 10$, to the total energy, which is therefore dominated by the kinetic energy. The interaction energy
vanishes both in the non-interacting limit \( g \to 0 \) and in the Tonks regime \( g \to \infty \), due to the fermionization of the bosons.

### IV. MOMENTUM DISTRIBUTION OF THE DYNAMICALLY LOCALIZED LIEB-LINIGER GAS

We shall now address the momentum distribution \( \Pi_t(k) \) of the interacting system, which is a relevant quantity for experiments, and point out key differences with respect to the non-interacting case. The momentum distribution \( \Pi_t(k) \) of the system is the Fourier transform of the one-body reduced density matrix (OBRDM) \( \rho_t(x,y) \),

\[
\Pi_t(k) = \frac{1}{2\pi} \int_0^{2\pi} dx \int_0^{2\pi} dy \ e^{ik(x-y)} \rho_t(x,y),
\]

with the momentum (in unit of \( k \) \( k \in \mathbb{Z} \) due to the periodic boundary conditions, and where the OBRDM is defined as:

\[
\rho_t(x,y) = 2 \int_0^{2\pi} dz \ \Psi_t^*(x,z) \Psi_t(y,z).
\]

It is normalized such that \( \int_0^{2\pi} dx \rho(x,x) = 2 \) is the number of particles of the system. For a given state \( |\Psi_t\rangle \), the momentum distribution \( \Pi_t(k) \) is such that \( \sum_k \Pi_t(k) = 2 \) and \( \sum_k \frac{k^2e^{ikx}}{2} \Pi_t(k) = E_{\text{kin}}(t) = E_{\text{tot}}(t) - E_{\text{int}}(t) \).

Leaving the details of the calculation to App. A, the momentum distribution in the localized regime reads

\[
\Pi(k) = \lim_{t \to \infty} \sum_{n,m,p} \left( c_{m}(t) \right)^* c_{p}(t) \Pi_{m,p}^{n,n}(k),
\]

with

\[
\Pi_{m,p}^{n,n}(k) = \delta_{n,q} \pi \frac{A_m A_p}{(2k-n)^2 - 4k_m^2} \frac{(2k-n)^2 - 4k_p^2}{A_p},
\]

and

\[
A_m = \frac{8k_m \cos \left( \frac{2\pi}{2k_m} \right)}{\sqrt{\pi - \sin(\theta_m)}}. \tag{12}
\]

Noting that \( \Pi_{m,p}^{n,n}(k) \) decays at large momenta as \( k^{-4} \) for all \( n,m,p \), we find that

\[
\lim_{k \to \infty} k^4 \Pi(k) = \mathcal{C}, \tag{13}
\]

where

\[
\mathcal{C} = \lim_{t \to \infty} \sum_{n,m,p} \left( c_{m}(t) \right)^* c_{p}(t) \frac{A_m A_p}{16\pi}, \tag{14}
\]

is the effective Tan’s contact in the dynamically localized regime.

The momentum distribution in the dynamically localized regime is shown in Fig. 5. The distributions \( \Pi(k) \) displays an exponential decay at small enough momenta, with a characteristic localization length which depends on the interaction strength. However, when \( g \neq 0 \) the \( k^{-4} \) tail dominates at large momenta. This tail is a universal feature of interacting quantum systems \([25, 29]\), and is in sharp contrast with the non-interacting limit of the kicked rotor, where the momentum distribution decays exponentially.

This feature is also dependent of the value of interactions, and is captured in the evolution of Tan’s contact
Our results are in stark contrast with the conclusions of Qin et al. [22], who found for the same model and parameter range that interactions lead to delocalization. This affirmation was based on two results: i) By computing the variance of the momentum up to 5000 kicks, they observed a somewhat increasing trend, which they interpreted as delocalization; ii) Their major argument was that the coefficient $c_m^n(t)$ behaved as $m^{-4}$ at long time (contrary to the exponential decay in the non-interacting limit), which they also interpreted as a sign of delocalization.

Concerning the first point, we note that their numerical simulation were not averaged, which makes it difficult to interpret the absence of localization (as can happen in the non-interacting QKR for some specific values of the parameters if not averaged over the quasimomentum). Concerning the second point, we do agree with the $m^{-4}$ behavior of $c_m^n(t)$. However, this power law decay does not imply delocalization. Indeed, as we have shown above, the total energy (which has a term proportional to $\sum_{n,m} m^2 |c_m^n(t)|^2$) does saturate at long times. Furthermore, the coefficients converge to finite steady-state values. Finally, and more importantly, it is known that in some disordered model with power law (but short-range) hopping, corresponding here to $|U_{mq}| \sim m^{-\mu}$ for large $m$ and fixed $n,q,p$, the states are localized as long as $\mu > 3/2$ [30]. Since the matrix element of the present problem decay with $\mu = 4$, dynamical localization is therefore expected. To support this, we analyze in App. C a modified QKR with matrix elements decaying as $m^{-4}$, and we show that indeed it dynamically localizes.
VI. CONCLUSIONS

We studied the outcome of dynamical localization with the kicked rotor model of two interacting bosons, and demonstrated its survival for arbitrary interaction strengths. The localization energy is found equal in the non-interacting (free bosons) and TG limits, and displays a non-monotonous behavior. Moreover, new features are predicted for the shape of the momentum distribution, namely the subsistence of an exponentially-localized ‘core’, at low momenta, and the existence of a power law decay at large momenta – a key characteristic of interacting quantum particles. Both features depend, yet in different manners, on the strength of the interaction.

An interesting question is the outcome of dynamical localization in the many-body limit. For interacting bosons the TG limit, our localization argument still holds: the energy is rigorously equal to that of \( N \) free fermions, and thus saturates at long times to a finite value, with the same localization time scale. This has already been predicted in [21]. However, the nature of this localized state is still to be determined – for instance the shape of the momentum distribution, relevant for experimental observations, is expected to display the long tails characteristic of Tan’s contact, demonstrated here. A comprehensive study of these features is under way [31].

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### Appendix A: Calculation of the momentum distribution

The momentum distribution is obtained from the OBRDM,

\[ \Pi_{\ell}(k) = \frac{1}{2\pi} \int_0^{2\pi} dx \int_0^{2\pi} dy e^{ik(x-y)} \rho_{\ell}(x,y), \tag{A1} \]

with

\[ \rho_{\ell}(x,y) = \int_0^{2\pi} dz \Psi_{\ell}^*(x,z) \Psi_{\ell}(y,z). \tag{A2} \]

The OBRDM can be expressed as

\[ \rho_{\ell}(x,y) = \sum_{n,m,q,p} \left( c_{m,n}^{*,q} \right)^* c_{p,n}^{q} \rho^{n,q}_{m,p}(x,y), \tag{A3} \]

\[ \rho^{n,q}_{m,p}(x,y) = 2 \int_0^{2\pi} dz \left( \Phi_{m,n}^*(x,z) \right)^* \Phi_{p,q}^y(y,z). \tag{A4} \]

To get the momentum distribution, we need to compute \( \Pi^{n,q}_{m,p}(k) \), the Fourier transform of \( \rho^{n,q}_{m,p}(x,y) \). Noting that the invariance per translation of \( H_{LL} \) implies that \( \Pi^{n,q}_{m,p}(k) \) vanishes if \( n \neq q \), we obtain after a straightforward though rather tedious calculation

\[ \Pi^{n,q}_{m,p}(k) = \delta_{n,q} \frac{A_m A_p}{(2k-n)^2 - 4k^2}, \tag{A5} \]

where

\[ A_m = \frac{8k_m \cos \left( \frac{\theta_m}{2} \right)}{\sqrt{\pi - \sin(2\theta_m)}}. \tag{A6} \]

For a given state \( |\Phi_m^n\rangle \), one can check that its momentum distribution \( \Pi^m_n(k) \) obeys

\[ \sum_k \Pi^m_n(k) = 2, \]

\[ \sum_k k \Pi^m_n(k) = nk, \tag{A7} \]

\[ \sum_k \frac{k^2 k^2}{2} \Pi^m_n(k) = E_{\text{kin}} = E^n_m - E^n_{m,\text{int}}, \]

where \( E^n_{m,\text{int}} = (\Phi_m^n | g \delta(\hat{x}_1 - \hat{x}_2) | \Phi_m^n) \) is the interaction energy.

From the above results, the momentum distribution reads

\[ \Pi_{\ell}(k) = \sum_{n,m,p} \left( c_{m,n}^{*,q} \right)^* c_{p,n}^{q} \Pi^m_n(k). \tag{A8} \]

### Appendix B: Momentum distribution in the asymptotic regimes

#### 1. Non-interacting limit

In the limit \( g \to 0 \), the initial wave function is given by \( \Phi_0^m(x_1, x_2) = (2\pi)^{-1} + O(g) \), i.e. the two bosons start into the zero-momentum state. The dynamics is that of two independent bosons (up to \( O(g) \) corrections), and we can therefore assume that at long times, the two bosons are described by the same dynamically localized wave function of the non-interacting QKR \( \psi_0(x) \), i.e.

\[ \Psi(x_1, x_2) = \psi_0(x_1) \psi_0(x_2) + O(g). \tag{B1} \]

It is then straightforward to show that in the Lieb-Liniger basis, the coefficients \( c_{m,n}^p \) are given in the localized regime by

\[ c_1^n = \psi_0 \left( \frac{n}{2} \right) + O(g), \]

\[ c_{m>1}^n = \sqrt{2} \psi_0 \left( \frac{n+m+1}{2} \right) \psi_0 \left( \frac{n-m+1}{2} \right) + O(g). \tag{B2} \]
where \( \hat{f} \) is the Fourier transform of the function \( f \).

In the weak interaction limit, we find that the coefficients \( A_m \) that enter in the momentum distribution (see App. A) are such that

\[
\frac{A_{1}}{\sqrt{\pi} ((2k-n)^2 - 4k^2)} = \sqrt{2} \delta_{n,2k} + \mathcal{O}(g),
\]

\[
\frac{A_{m>1}}{\sqrt{\pi} ((2k-n)^2 - 4k^2_m)} = \delta_{n,2k+m-1} + \delta_{n,2k-m+1} + \mathcal{O}(g),
\]

which immediately gives

\[
\Pi(k) = 2|\hat{\psi}_0(k)|^2 + \mathcal{O}(g),
\]

as expected for free bosons.

However, for momenta very large compared to the localization length \( p_{loc} \) of the non-interacting QKR, \( |\hat{\psi}_0(k)|^2 \) is exponentially small compared to the \( \mathcal{O}(g) \) corrections, and the momentum distribution is dominated by the contact,

\[
\Pi(k) \sim \frac{\mathcal{C}}{k^4}.
\]

In this regime, we find

\[
\mathcal{C} = \frac{g^2}{\pi^2} \sum_{n,m,p} a_m a_p (c^*_m c^*_p) c^n p + \mathcal{O}(g^3),
\]

with \( a_1 = 1/\sqrt{2} \) and \( a_{m>1} = 1 \), where we can use Eq. (B2) to the same accuracy. We can now use the fact that the phases of the QKR wave functions are essentially random, such that when averaging over \( \beta \), only the diagonal terms \( p = m \) survive, i.e. \( (c^*_m c^*_p) c^n \approx \delta_{m,p} \frac{c^n}{|c^n|^2} \).

We then obtain

\[
\mathcal{C} = \frac{g^2}{\pi^2} \left( 1 - \frac{2}{\pi} \sum_q |\psi_0(q)|^4 \right) + \mathcal{O}(g^3).
\]

For localized state, we expect \( \frac{1}{2} \sum_q |\psi_0(q)|^4 \approx \frac{1}{8p_{loc}} \) to be small and the contact is thus

\[
\mathcal{C} \approx \frac{g^2}{\pi^2}.
\]

In summary, the momentum distribution is decays exponentially as \( 2|\hat{\psi}_0(k)|^2 \) for \( |k| \ll p_c \) and as a power law \( g^2/(\pi^2 k^4) \) for \( |k| \gg p_c \), where the cross-over scale is given by

\[
2|\hat{\psi}_0(p_c)|^2 \approx \frac{g^2}{\pi^2 p_c^2}.
\]

A similar calculation shows that the contact in the the ground state is \( g^2/2\pi^2 \).

2. Tonks-Girardeau regime

In the limit \( g \to \infty \), thanks to the Bose-Fermi mapping, we can write the wave function of the bosons in the localized regime as

\[
\Psi(x_1, x_2) = \frac{\text{sign}(x_1 - x_2)}{\sqrt{2}} \det \left( \begin{array}{cc} \psi_+(x_1) & \psi_-(x_1) \\ \psi_+(x_2) & \psi_-(x_2) \end{array} \right),
\]

(B10)

where \( \psi_\pm(x) \) are the wave functions of non-interacting fermions, evolving according the non-interacting QKR Hamiltonian, with anti-periodic boundary conditions. The initial condition is such that the two fermions start in the momentum state \( p_\pm = \pm \frac{1}{2} \). At long time, \( \hat{\psi}_\pm(q) \) are exponentially localized with localization length \( p_{loc} \) similar to that of free bosons. In particular, for large enough \( p_{loc} \), we expect \( |\hat{\psi}_\pm(q)|^2 \approx |\hat{\psi}_0(q)|^2 \) where \( \hat{\psi}_0(q) \) is the localized wave function of a boson starting at zero-momentum.

In the Lieb-Liniger basis, the coefficients \( c^*_m \) are then given by

\[
c^*_m = \sum_{\sigma = \pm 1} \sigma \hat{\psi}_+ \left( \frac{n + \sigma m}{2} \right) \hat{\psi}_- \left( \frac{n - \sigma m}{2} \right),
\]

(B11)

Therefore, the momentum distribution reads

\[
\Pi(k) = \frac{1}{\pi^2} \sum_{p_1, p_2, q_1, q_2} B_{p_1, p_2, q_1, q_2} (k) \hat{\psi}_+^*(p_1) \hat{\psi}_-^*(p_2) \hat{\psi}_+(q_1) \hat{\psi}_-(q_2),
\]

(B12)

where

\[
B_{p_1, p_2, q_1, q_2} (k) = \frac{(p_1 - p_2)(q_1 - q_2)}{(k - p_1)(k - p_2)(k - q_1)(k - q_2)},
\]

(B13)

and the sum \( \sum_{p_1, p_2, q_1, q_2} \) is over half-integers such that \( p_1 + p_2 = q_1 + q_2 \).

Upon averaging over \( \beta \), we expect

\[
\hat{\psi}_+^*(p_1) \hat{\psi}_-^*(p_2) \hat{\psi}_+(q_1) \hat{\psi}_-(q_2) \approx \delta_{p_1, q_1} \delta_{p_2, q_2} |\hat{\psi}_+(q_1)|^2 |\hat{\psi}_-(q_2)|^2,
\]

since the phases of two different localized state of the QKR have (almost) uncorrelated phases.

The averaged momentum distribution reads

\[
\bar{\Pi}(k) = \frac{1}{\pi^2} \sum_{q_1, q_2} \frac{(q_1 - q_2)^2}{(k - q_1)^2(k - q_2)^2} |\hat{\psi}_+(q_1)|^2 |\hat{\psi}_-(q_2)|^2.
\]

(B15)

We have observed numerically that for small enough momenta, \( \bar{\Pi}(k) \) is well described by

\[
\bar{\Pi}(k) \approx |\hat{\psi}_+(k)|^2 + |\hat{\psi}_-(k)|^2, \approx 2|\hat{\psi}_0(k)|^2,
\]

(B16)

where we have assume that the width of the wave functions (given by \( p_{loc} \)) is much larger than one to go from
the first to the second line. For large momenta we have \( \tilde{\Pi}(k) \approx \tilde{C}/k^4 \) with the averaged contact

\[
\tilde{C} \approx \frac{1}{\pi^2} \sum_{q_1, q_2} (q_1 - q_2)^2 |\bar{\psi}_+(q_1)|^2 |\bar{\psi}_-(q_2)|^2,
\]

\[
\approx \frac{2E_{\text{tot}}}{\pi^2 k^4}, \tag{B17}
\]

where the averaged total energy is given by \( E_{\text{tot}} = \frac{\hbar^2}{2\pi} \sum_q q^2 \left( |\psi_+(q)|^2 + |\psi_-(q)|^2 \right) \). To go from the first to the second line, we have assumed that the wave functions are broad enough such that we can neglect \( \sum_q q |\psi_\pm(q)|^2 \).

A cross-over scale between the exponential and power law decay of the momentum distribution can be defined similarly as in the weak interaction regime.

### Appendix C: Dynamical localization of a modified QKR

We analyze a modified QKR model engineered such that the evolution operator decays as a power law similar to that of the kicked Lieb-Liniger gas, and we show that this power law behavior does not change the localization properties.

We introduce the toy model

\[
\hat{H}' = \frac{\hat{p}^2}{2} + KV'(\hat{x}) \sum_n \delta(t - n), \tag{C1}
\]

with \( [\hat{x}, \hat{p}] = i\hbar \), and the kick potential

\[
V'(x) = \frac{2x^4}{\pi^4} - \frac{4x^2}{\pi^2} + 1, \tag{C2}
\]

for \( x \in [-\pi, \pi] \), and \( V'(x) \) is of period 2\( \pi \). This potential and its first and second derivative are continuous, whereas its third derivative is piece-wise continuous, which implies that its Fourier coefficients \( \hat{V}_n \) decay as \( n^{-4} \). The corresponding evolution operator over one period is

\[
\hat{U}' = e^{-i\frac{\hat{p}^2}{2}} V'(\hat{x}) e^{-i\frac{\hat{p}^2}{2}}, \tag{C3}
\]

and by the same argument, one has

\[
\lim_{|p' - p| \to \infty} \langle p'|\hat{U}'|p \rangle \propto |p' - p|^{-4}. \tag{C4}
\]

This behavior is demonstrated in Fig. 8.

The numerical analysis of this model is much simpler than that of the kicked Lieb-Liniger model, and one convinces oneself rather quickly that for generic values of the parameters (choosing \( k \) not rational multiple of \( \pi \) to avoid quantum resonances), the kinetic energy of the system always saturates at long times, see Fig. 9. In the localized regime, we observe that similarly to the Lieb-Liniger case, the wave function take a steady-state shape, and decay as \( \langle |p|\psi| \rangle^4 \propto p^{-4} \) in momentum space for large momenta, see Fig. 9. However, this power law tail does not change the fact that the inverse partition ratio \( P = \sum_p \langle |p|\psi| \rangle^4 \) is always finite, which is a hallmark of localization. Because of the power law nature of the momentum coupling of \( \hat{U}' \), the momentum distribution features a long power law tail even after a single kick. We note that the large-momentum power law tails localizes over longer time scales than the system energy, but still ends up localizing to a constant value.

To push the analysis further, we can also analyze the shape of the wave function in the Lieb-Liniger basis \( |c_n| \). This is shown in Fig. 10. While we observe an exponential localization in the center of mass direction \( n \), the shape of the wave function coefficients \( |c_m| \) display the characteristic power law \( 1/m^6 \) tails along the relative momentum direction \( m \).
FIG. 10. Localized two-body wave function in the Lieb-Liniger basis (top), at $t = 2^{28}$ kicks ($K = 3$, $k = 1$, $g = 1$). The wave function is exponentially localized in the center of mass direction $n$ (bottom left), and displays a long-range $1/m^8$ tail (bottom right), characteristic of the power law coupling (similar to the $V'(x)$ potential of the modified single-particle QKR).