Comparison of approaches to characteristic classes of foliations

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Abstract. It is shown that the characteristic classes of foliations that were defined by Losik and that take values in the de Rham cohomology of the space of infinite order frames over the leaf space may be mapped to the characteristic classes with values in the Čech-de Rham cohomology of the leaf space studied in details by Crainic and Moerdijk. This map is in general non-injective. All constructions are done using Losik’s approach to Gelfand formal geometry.

A similar result is obtained for the exotic characteristic classes as well as for the group actions of the diffeomorphisms. As illustrating examples, foliations of codimension one are discussed.

Keywords: foliation; leaf space of foliation; characteristic classes of foliations; Čech-de Rham cohomology; Gelfand-Fuchs cohomology

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Introduction

The theory of characteristic classes of foliations starts with the discovery of the Godbillon-Vey class [16] for a foliation of codimension one. Berstein and Rozenfeld [3], and Bott and Haefliger [5] defined characteristic classes for foliations of arbitrary codimension $n$. These classes are given by the generators of the relative Gelfand-Fuchs cohomology $H^*(W_n, O(n))$ of the Lie algebra of formal vector fields on $\mathbb{R}^n$ and take values in the cohomology $H^*(M)$ of the underlying manifold. The map $H^*(W_n, O(n)) \to H^*(M)$ may be factored through the cohomology $H^*(BG)$ of the classifying space $BG$ of the Haefliger groupoid $\Gamma_n$.

Losik [20, 21, 22] introduced the notion of a generalized atlas on the leaf space $M/F$ of a foliation $F$ on a smooth manifold $M$ showing that there is a reach smooth structure on $M/F$. Spaces with such structures are called $D_n$-spaces. In particular, this approach allowed him to define the characteristic classes as elements of the cohomology $H^*(S(M/F)/O(n))$, where $S(M/F)$ is the bundles of frames of infinite order over $M/F$. These classes may be projected to the usual ones in $H^*(M)$.

Following Haefliger, one may construct the characteristic map $H^*(W_n, O(n)) \to H^*(BG)$ and obtain characteristic classes with values in the cohomology of the classifying space of the holonomy groupoid of the foliation [4, 7, 17]. Again, these classes may be projected to the usual ones in $H^*(M)$. In the more recent work [10], Crainic and Moerdijk studied in details the characteristic classes of foliations as elements of the Čech-de Rham cohomology $H^*(M/F)$ of the leaf space $M/F$. The leaf space $M/F$ was considered in terms of the holonomy groupoid. There is an isomorphism $H^*(M/F) \cong H^*(BG)$ [10].

The aim of the present paper is to compare the approaches from the last two paragraphs. Using Losik’s theory, we recover the construction of Crainic and Moerdijk, and we show that Losik’s characteristic classes may be mapped to these of Crainic and Moerdijk. Let us sketch the idea. To a $D_n$-space $X$ is associated an étale groupoid $G_X$ such that there exists an isomorphism $H^*(X) \cong H^*(BG_X)$ [22]. The category of $D_n$-spaces possesses the terminal object, which is a point $pt$ with an additional structure. The associated groupoid to that space is exactly the Haefliger groupoid $\Gamma_n$. For a $D_n$-space, the unique map $X \to pt$ induces the homomorphism

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\( H^*(BT_n) \to \hat{H}^*(X) \), which in the case \( X = M/\mathcal{F} \) was defined by Crainic and Moerdijk. Next, there is a map from the de Rham cohomology \( H^*(S(X)/O(n)) \) to \( H^*(S(X)/O(n)) \); it holds \( H^*(S(pt)/O(n)) \cong H^*(W_n, O(n)) \); and we prove that \( \hat{H}^*(S(X)) \cong \hat{H}^*(X) \). This gives us the sequence of the maps

\[
H^*(W_n, O(n)) \cong H^*(S(pt)/O(n)) \to H^*(S(X)/O(n)) \to \hat{H}^*(S(X)/O(n)) \cong \hat{H}^*(X),
\]

which provides the characteristic homomorphism of Crainic and Moerdijk for the case \( X = M/\mathcal{F} \).

In particular, we see that the Losik classes living in \( H^*(S(X)/O(n)) \) are mapped to the classes contained in \( \hat{H}^*(M/\mathcal{F}) \). If we take \( X = \text{pt} \), then we obtain the new construction of the Bott map \( H^*(W_n, O(n)) \to H^*(BT_n) \).

Let \( n = 1 \). The Godbillon-Vey class is defined by a generator of \( H^3(W_1, O(1)) \cong \mathbb{R} \). Let \( \mathcal{F} \) be the Reeb foliation on the three dimensional sphere. It is known that the usual Godbillon-Vey class of \( \mathcal{F} \) is trivial in \( H^3(S^3) \) [24]. The triviality of the Godbillon-Vey class in \( H^3(S(S^3)/\mathcal{F})/O(1) \cong H^3(BG) \) may be shown by means of the noncommutative geometry: if the Godbillon-Vey class of a foliation is non-trivial in \( H^3(BG) \), then the corresponding von Neumann algebra has a non-trivial type III component [14], [8] p. 245, p. 261]. On the other hand, the von Neumann algebra of the Reeb foliation is of type I, [8] p. 54. In contrast, in [1], we show that the Godbillon-Vey class for some Reeb foliations \( \mathcal{F} \) is non-trivial in \( H^3(S(S^3)/\mathcal{F})/O(1) \). We conclude that the map \( H^*(S(M/\mathcal{F})/O(n)) \to \hat{H}^*(M/\mathcal{F}) \) that we constructed here is in general non-injective.

The above construction gives also the maps \( H^*(W_n, O(n)) \to H^*(S(M/G)/O(n)) \to H^*(G; O(M)) \) for a group \( G \) acting by diffeomorphisms on an \( n \)-dimensional manifold \( M \). For \( M = S^1 \) this allows to reconstruct the Bott-Thurston cocycle, which is an element of \( H^*(G; \mathbb{R}) \). Using results form [1], an example of \( G \) acting on \( S^1 \) is given with the Godbillon-Vey class trivial in \( H^*(G; \mathbb{R}) \) and non-trivial in \( H^*(S(S^3)/\mathcal{F})/O(1) \).

For foliations \( \mathcal{F} \) with the trivializable bundle of frames of the normal bundle one may consider characteristic classes are given by a map \( H^*(W_n) \to H^*(M) \). Losik considered these classes with values in \( H^*(S(M/\mathcal{F})) \). In a way as above, for a \( D_n \)-space \( X \) with the trivializable frame bundle we construct homomorphisms

\[
H^*(W_n) \to H^*(S(X)) \to \hat{H}^*(X), \quad H^*(W_n) \to H^*(BT_n) \to \hat{H}^*(X)
\]

defining the so-called exotic characteristic classes with values in the Čech-de Rham cohomology. Here \( T_n \) is the topological space with the space of object the frame of bundles over \( \mathbb{R}^n \) and morphisms the germs of the extensions of the local diffeomorphisms of \( \mathbb{R}^n \). In particular, we get a new construction of the Bott map \( H^*(W_n) \to H^*(BT_n) \).

Losik considered also the Chern classes of foliations that are defined by a map \( H^*(W_n, GL(n, \mathbb{R})) \to H^*(S(M/\mathcal{F})) \). Compared with the elements from \( H^*(W_n, O(n)) \), the elements \( H^*(W_n, GL(n, \mathbb{R})) \) do not give any new classes in \( H^*(M) \) and \( H^*(M/\mathcal{F}) \). On the other hand, elements of \( H^*(W_n, GL(n, \mathbb{R})) \) define classes in \( H^2(\mathcal{F}) \) similar to the Vey class [9] [13] [15]. Let \( n = 1 \). Denote by \( c_1 \) a generator of \( H^2(W_1, GL(1, \mathbb{R})) \cong \mathbb{R} \). The class defined by this generator is always trivial in \( H^2(M) \) and \( H^2(M/\mathcal{F}) \). This generator defines the Vey class in \( H^2(\mathcal{F}) \). Consider again the Reeb foliation. The Vey class for that foliation is trivial. In contrast, Losik [22] proved (see also [2]) that \( c_1 \) defines non-trivial class in \( H^2(S(S^3)/\mathcal{F}); GL(1, \mathbb{R}) \). We thus may conclude that Losik’s classes are most informative. In particular, the first Chern class indicates that the Reeb foliation contains a compact leaf with a non-trivial holonomy [2] (the Godbillon-Vey class is zero whenever only compact leaves have non-trivial holonomy [23]). The triviality of the Godbillon-Vey class with values in \( H^2(S(S^3)/\mathcal{F})/O(1) \) depends on the choice of the functions defining the Reeb foliation [1] and it may provide more delicate information about growth rate of the leaves near the compact leaf.

In what follows, all cohomology are considered with real coefficients.

1. Classical approaches

Let \( W_n \) denote the Lie algebra of formal vector fields on \( \mathbb{R}^n \). First we recall the description of the Gelfand-Fuchs cohomology of \( W_n \) [18]. Consider the differential graded algebra \( W_n \), with
the generators
\[ y_1, \ldots, y_n, c_1, \ldots, c_n \]
of degrees \( \deg y_i = 2i - 1 \), \( \deg c_i = 2i \), the relations
\[
y_{i}y_{j} = -y_{j}y_{i}, \quad y_{i}c_{j} = c_{j}y_{i}, \quad c_{i}c_{j} = c_{j}c_{i},
\]
\[
c_{i_1} \cdots c_{i_n} = 0 \quad \text{if} \quad i_1 + \cdots + i_n > n
\]
and the differential satisfying \( dy_i = c_i, \ dc_i = 0 \). The cohomology \( H^*(W_n) \) are isomorphic to \( H^*(\mathbb{W}_n) \). The relative cohomology \( H^*(W_n, O(n)) \) are isomorphic to the cohomology \( H^*(\mathbb{W}_n, O(n)) \) of the subcomplex generated by the elements \( y_1, y_3, y_5, \ldots, c_1, \ldots, c_n \); the cohomology \( H^*(W_n, \text{GL}(n, \mathbb{R})) \) are isomorphic to the cohomology \( H^*(\mathbb{W}_n, \text{GL}(n, \mathbb{R})) \) of the subcomplex generated by \( c_1, \ldots, c_n \).

Now we recall shortly the classical definitions of the characteristic classes of foliations \cite{3, 4}. Foliations \( \mathcal{F} \) of codimension \( n \) on smooth manifolds are classified by the homotopy classes of maps
\[
f_{\mathcal{F}} : M \to B\Gamma_n,
\]
where \( B\Gamma_n \) is the classifying space of the Haefliger groupoid \( \Gamma_n \). In this way one obtains the characteristic map
\[
f^*_\mathcal{F} : H^*(B\Gamma_n) \to H^*(M).
\]
Using the Chern-Weil theory, one may construct the map
\[
k_{\mathcal{F}} : H^*(\mathbb{W}_n) \to H^*(M).
\]
Using the maps \( k_{\mathcal{F}} \), Bott \cite{4} constructed the map
\[
k : H^*(\mathbb{W}_n) \to H^*(B\Gamma_n).
\]
It holds \( \lambda_{\mathcal{F}} = f^*_\mathcal{F} \circ k \). The images of the generators of \( H^*(\mathbb{W}_n) \) under the map \( \lambda_{\mathcal{F}} \) define characteristic classes of the foliation \( \mathcal{F} \).

Let \( P(\mathcal{F}) \) denote the frame bundle of the normal bundle of \( \mathcal{F} \). Suppose that the bundle \( P(\mathcal{F}) \) is trivial, i.e., there exists a section \( s : M \to P(\mathcal{F}) \). Then there is a classifying map
\[
\tilde{f}_{\mathcal{F}} : M \to B\Gamma_n,
\]
where \( \Gamma_n \) is the groupoid whose objects form the frame bundle \( P(\mathbb{R}^n) \) over \( \mathbb{R}^n \), and the morphisms are the germs of the extensions to \( P(\mathbb{R}^n) \) of the local diffeomorphisms of \( \mathbb{R}^n \) (the classifying space \( B\Gamma_n \) is the homotopy-theoretic fiber \( F\Gamma_n \) of the map \( B\Gamma_n \to B\text{GL}(n, \mathbb{R}) \)). There is a map
\[
\tilde{k} : H^*(\mathbb{W}_n) \to H^*(B\Gamma_n)
\]
constructed in \cite{4}. The images of the generators of \( H^*(\mathbb{W}_n) \) under the map
\[
\tilde{f}^*_\mathcal{F} \circ \tilde{k} : H^*(\mathbb{W}_n) \to H^*(M)
\]
define the so-called exotic characteristic classes of the foliation \( \mathcal{F} \).

Alternatively one may proceed in the following way \cite{3, 19}. Let \( S(\mathcal{F}) \) be the space of jets of infinite order at zero of submersions from \( M \) to \( \mathbb{R}^n \) that are constant on the leaves of \( \mathcal{F} \). Let \( S'(\mathcal{F}) = S(\mathcal{F})/\text{GL}(n, \mathbb{R}) \) and \( S''(\mathcal{F}) = S(\mathcal{F})/O(n) \). On each of the spaces \( S(\mathcal{F}), S'(\mathcal{F}), S''(\mathcal{F}) \), there is a canonical 1-from (the Gelfand-Kazhdan form) with values in \( W_n \) that delivers the homomorphisms
\[
H^*(W_n) \to H^*(S(\mathcal{F})) = H^*(P(\mathcal{F})),
\]
\[
H^*(W_n, \text{GL}(n, \mathbb{R})) \to H^*(S'(\mathcal{F})) = H^*(M),
\]
\[
H^*(W_n, O(n)) \to H^*(S''(\mathcal{F})) = H^*(M).
\]
If the bundle \( P(\mathcal{F}) \) is trivializable, then one gets the map
\[
H^*(W_n) \to H^*(P(\mathcal{F})) \to H^*(M).
\]
This gives another construction of the characteristic classes. The map
\[
H^*(W_n, \text{GL}(n, \mathbb{R})) \to H^*(M)
\]
does not give any new classes, since there is the following commutative diagram:

\[
\begin{array}{cccc}
H^*(W_n, GL(n, \mathbb{R})) & \longrightarrow & H^*(S'(\mathcal{F})) \\
\downarrow & & \downarrow \\
H^*(W_n, O(n)) & \longrightarrow & H^*(S''(\mathcal{F}))
\end{array}
\]

Below we will see that the generators of \(H^*(W_n, GL(n, \mathbb{R}))\) define certain interesting classes with values in the cohomology of a bundle over the leaf space of the foliation.

Finally note that the Godbillon-Vey class of a codimension one foliation is defined by the class \([y_1c_1] \in H^3(WO_1) = H^3(W_1, O(1))\).

2. Characteristic classes with values in \(\tilde{H}^*(M/\mathcal{F})\)

Let us briefly recall the construction of the Čech-de Rham cohomology following Crainic and Moerdijk [10]. Consider a foliation \(\mathcal{F}\) of codimension \(n\) on a smooth manifold \(M\). Let \(\mathcal{U}\) be a family of transversal sections of \(\mathcal{F}\). Such a family is called a transversal basis if for each transversal section \(V\) of \(\mathcal{F}\) and each point \(y \in V\), there exists a section \(U \in \mathcal{U}\) and a holonomy embedding \(h : U \rightarrow V\) such that \(y \in h(U)\). Consider the double complex

\[
(4) \quad C^{p,q} = \prod_{U_0 \sim U \sim \ldots \sim U_p} \Omega^q(U_0),
\]

where the product ranges over all \(p\)-tuples of holonomy embeddings between transversal sections from a fixed transversal basis \(\mathcal{U}\). The vertical differential is defined as

\[
(-1)^p d : C^{p,q} \rightarrow C^{p,q+1},
\]

where \(d\) is the usual de Rham differential. The horizontal differential

\[
\delta : C^{p,q} \rightarrow C^{p+1,q}
\]

is given by

\[
(5) \quad (\delta \omega)(h_1, \ldots, h_{p+1}) = h_1^* \omega(h_2, \ldots, h_{p+1}) + \sum_{i=1}^{p} (-1)^i \omega(h_1, \ldots, h_{i+1}h_i, \ldots, h_{p+1}) + (-1)^{p+1} \omega(h_1, \ldots, h_p).
\]

The cohomology of this complex is called the Čech-de Rham cohomology of the leaf space \(M/\mathcal{F}\) with respect to the cover \(\mathcal{U}\) and is denoted by

\[
\tilde{H}^*_\mathcal{U}(M/\mathcal{F}).
\]

A complete transversal basis \(\mathcal{U}\) may be obtained from a foliation atlas \(\mathring{\mathcal{U}}\) of \(M\). This defines a map of the complexes \(C^{p,q}(\mathcal{U}) \rightarrow C^{p,q}(\mathcal{U})\), which induces the map

\[
(6) \quad \tilde{H}^*_\mathcal{U}(M/\mathcal{F}) \rightarrow \tilde{H}^*_G(M) \cong H^*(M),
\]

where \(H^*(M)\) is the de Rham cohomology of \(M\). Next, there are natural isomorphisms

\[
(7) \quad \tilde{H}^*_\mathcal{U}(M/\mathcal{F}) \cong H^*(B\text{Hol}(M, \mathcal{F})) \cong H^*(B\text{Hol}_T(M, \mathcal{F})),
\]

where \(T\) is a complete transversal, \(\mathcal{U}\) is a basis of \(T\), \(B\text{Hol}(M, \mathcal{F})\) is the classifying space of the holonomy groupoid, and \(B\text{Hol}_T(M, \mathcal{F})\) is the classifying space of the holonomy groupoid restricted to \(T\).

Using the Chern-Weil theory, Crainic and Moerdijk constructed the characteristic map

\[
(8) \quad \tilde{k}_\mathcal{F} : H^*(WO_n) \rightarrow \tilde{H}^*_\mathcal{U}(M/\mathcal{F}).
\]

Even more, for a given étale groupoid \(G\), they defined the Čech-de Rham cohomology \(\tilde{H}\mathcal{U}(G)\) and constructed the map

\[
(9) \quad \tilde{k}_G : H^*(WO_n) \rightarrow \tilde{H}^*_\mathcal{U}(G) \cong H^*(BG).
\]
For the case $G = \Gamma_n$ this gives the new construction of the map (2). It is shown that the classifying map (1) induces the characteristic map

$$H^*(BT_n) \rightarrow H^*_\mathcal{U}(M/F).$$

Thus there is the following commutative diagram:

$$
\begin{array}{ccc}
H^*(\Omega^*_\mathcal{U}_n) & \xrightarrow{k} & H^*(BT_n) \\
\downarrow{k_F} & & \downarrow{\mathcal{U}} \\
H^*_\mathcal{U}(M/F) & & \\
\end{array}
$$

The images of the generators of $H^*(\Omega^*_\mathcal{U}_n)$ under the map $k_F$ are characteristic classes of the foliation $\mathcal{F}$ living in $H^*_\mathcal{U}(M/F)$. The map (3) sends these classes to the characteristic classes considered in Section 1.

3. Losik’s approach

The main idea of Losik’s approach to the leaf spaces of foliations is to introduce a notion of a generalized smooth atlas on such spaces in order to apply to them the technics from the theory of smooth manifolds [20, 21, 22]. In this approach, the main notion is a $D_n$-space, where $D_n$ is the category whose objects are open subsets of $\mathbb{R}^n$, and morphisms are étale (i.e., regular) maps.

The dimension $n$ may be infinite; in this case we use the definitions from [23] of manifolds with the model space $\mathbb{R}^\infty$.

Let us recall the definition of a $D_n$-space. Let $X$ be a set. A $D_n$-chart on $X$ is a pair $(U, k)$, where $U \subset \mathbb{R}^n$ is an open subset, and $k : U \rightarrow X$ is an arbitrary map. For two charts $k_i : U_i \rightarrow X$, a morphism of charts is an étale map $m : U_1 \rightarrow U_2$ such that $k_2 \circ m = k_1$. Let $\Phi$ be a set of charts and let $\mathcal{C}_\Phi$ be a category whose objects are elements of $\Phi$ and morphisms are some morphisms of the charts. The set $\Phi$ is called a $D_n$-atlas on $X$ if $X = \lim J$, where $J : \mathcal{C}_\Phi \rightarrow \text{Sets}$ is the obvious functor. A $D_n$-space is a set $X$ with a maximal $D_n$-atlas. An atlas $\Phi$ is called full if for each chart $(V, l)$ from the corresponding maximal atlas $\Phi$ and each point $y \in V$, there exists a chart $(U, k) \in \Phi$ and a morphism $m : (U, k) \rightarrow (V, l)$ such that $y \in m(U)$.

A map $f : X \rightarrow Y$ of $D_n$-spaces is called a morphism of $D_n$-spaces if, for any $D_n$-chart $k$ from the atlas on $X$, $f \circ k$ is a $D_n$-chart from the maximal atlas on $Y$.

If $\mathcal{F}$ is a foliation of codimension $n$ on a smooth manifold $M$, then the leaf space $M/F$ is a $D_n$-space. The maximal $D_n$-atlas on $M/F$ consists of the projections $U \rightarrow M/F$, where $U$ is a transversal which is the embedded $M$ open subset of $\mathbb{R}^n$. These transversals may be obtained from a foliation atlas on $M$. A full atlas may be obtained from a complete transversal as in the previous section.

A full atlas $\Phi$ of a $D_n$-space $X$ gives rise to a smooth groupoid $G_X$. The set of objects of $G_X$ is the union of the domains of the charts form $\Phi$, and the morphisms are germs of the morphisms of charts. The groupoid $G_X$ may be reduced essentially, if there is a surjection $(G_X)_0 \rightarrow M$ to a smooth manifold $M$. For the reduced groupoid $G_X$, $(G_X)_0 = M$, and the elements of $(G_X)_1$ are the germs of local diffeomorphisms of $M$ that can be lifted to the morphisms form $\mathcal{C}_\Phi$. If $X = M/F$, then the reduced groupoid $G_X$ coincides with the holonomy groupoid.

Generally $D_n$-spaces are orbit spaces of pseudogroups of local diffeomorphisms of smooth manifolds. Considering the space $\mathbb{R}^n$ and the pseudogroup of all local diffeomorphisms of open subsets of $\mathbb{R}^n$, we see that the point pt is a $D_n$-space. The atlas of pt consist of all pairs $(U, k)$, where $U \subset \mathbb{R}^n$ is an open subset and $k : U \rightarrow pt$ is the unique map. It is important to note that pt is the terminal objects in the category of $D_n$-spaces. The reduced groupoid corresponding to the $D_n$-space pt is exactly the Haefliger groupoid $\Gamma_n$.

Each (co)functor from the category $D_n$ to the category of sets may be extended to a (co)functor form the category of $D_n$-spaces. In this way one obtains, e.g., the de Rham complex $\Omega^*(X)$ of a $D_n$-space $X$, which defines the de Rham cohomology $H^*(X)$ of $X$. E.g., if $X = M/F$, then $\Omega^*(X)$ coincides with the complex of basic forms.
One may consider also the category \( \mathcal{M}_n \) of \( n \)-dimensional manifolds with the étale maps as morphisms. Then one gets the notion of an \( \mathcal{M}_n \)-space. The categories of \( \mathcal{D}_n \)-spaces and \( \mathcal{M}_n \)-spaces are equivalent. We do not restrict the attention to \( \mathcal{D}_n \)-spaces, since now we will consider \( \mathcal{M}_\infty \)-atlases on the spaces of frame of infinite order.

Consider the functor \( S \) assigning to each open subset \( U \subset \mathbb{R}^n \) the space of frames of infinite order, i.e., the space of jets at \( 0 \in \mathbb{R}^n \) of regular maps from \( \mathbb{R}^n \) to \( U \). Then for each \( \mathcal{D}_n \)-space \( X \), we obtain the space of frames of infinite order \( S(X) \). This space is a \( \mathcal{M}_\infty \)-space. Each chart \( U \to X \) defines the char \( S(U) \to S(X) \), and each morphism of charts \( h : U \to V \) defines the morphism \( S(h) : S(U) \to S(V) \). In this way we get an \( \mathcal{M}_\infty \)-atlas on \( S(X) \). Similarly, let \( S'(U) = S(U)/GL(n, \mathbb{R}) \) and \( S''(U) = S(U)/O(n) \). We obtain the \( \mathcal{M}_\infty \)-spaces \( S'(X) \) and \( S''(X) \) with the atlases similar to the above one.

Let us consider the point \( pt \) as a \( \mathcal{D}_n \)-space. Each of the spaces \( S(pt) \), \( S'(pt) \), and \( S''(pt) \) consists of a single point, on the other hand, the complexes \( \Omega^*(W_n) \), \( \Omega^*(S(pt)) \), and \( \Omega^*(S''(pt)) \), are naturally isomorphic to the complexes \( C^*(W_n) \), \( C^*(W_n, GL(n, \mathbb{R})) \), and \( C^*(W_n, O(n)) \), respectively.

Let now \( X \) be a \( \mathcal{D}_n \)-space. The unique morphism
\[
p_X : X \to pt
\]
of \( \mathcal{D}_n \)-spaces induces the morphisms
\[
S(X) \to S(pt), \quad S'(X) \to S'(pt), \quad S''(X) \to S''(pt)
\]
and the characteristic morphisms
\[
\chi : H^*(W_n) \cong H^*(S(pt)) \to H^*(S(X)),
\]
\[
\chi' : H^*(W_n, GL(n, \mathbb{R})) \cong H^*(S'(pt)) \to H^*(S'(X)),
\]
\[
\chi'' : H^*(W_n, O(n)) \cong H^*(S''(pt)) \to H^*(S''(X)).
\]
The images of the generators of the cohomology under these maps give Losik’s characteristic classes. In the case \( X = M/\mathcal{F} \), these classes may be projected to the characteristic classes from Section 1 as follows. The projection \( p : M \to M/\mathcal{F} \) defines the map \( S(p) : S(\mathcal{F}) \to S(M/\mathcal{F}) \) and two similar maps \( S'(p) \) and \( S''(p) \), and one obtains the following chains of homomorphisms:
\[
\begin{align*}
H^*(W_n) & \xrightarrow{\chi} H^*(S(M/\mathcal{F})) \xrightarrow{S(p)^*} H^*(S(\mathcal{F})) = H^*(P(\mathcal{F})), \\
H^*(W_n, GL(n, \mathbb{R})) & \xrightarrow{\chi'} H^*(S'(M/\mathcal{F})) \xrightarrow{S'(p)^*} H^*(S'(\mathcal{F})) = H^*(M), \\
H^*(W_n, O(n)) & \xrightarrow{\chi''} H^*(S''(M/\mathcal{F})) \xrightarrow{S''(p)^*} H^*(S''(\mathcal{F})) = H^*(M).
\end{align*}
\]

4. Comparison of the approaches

Generalizing the construction from [10], Losik [22] defined the Čech-de Rham cohomology for an \( \mathcal{M}_n \)-space \( X \). Let \( \Phi \) be a full \( \mathcal{M}_n \)-atlas on \( X \). The disjoint union of the domains of the charts from \( \Phi \) is an \( n \)-dimensional manifold. Let \( \mathcal{U} \) be a base of the topology on this manifold. We will refer to \( \mathcal{U} \) as to a complete cover of \( X \). The word „complete” stresses that the cover is obtained from a full atlas. Consider the double complex
\[
C^{p,q}_\mathcal{U}(X) = \prod_{U_0 \in \mathcal{U}_0} \Omega^q(U_0),
\]
where product is taken over the strings of composable arrows from \( \mathcal{C}_\Phi \). The differentials of this double complex are defined in the same way as for the double complex [4] above. The obtained cohomology are called the Čech-de Rham cohomology of the \( \mathcal{M}_n \)-space defined by the cover \( \mathcal{U} \) and are denoted by
\[
\tilde{H}^*_\mathcal{U}(X).
\]
If \( X = M/\mathcal{F} \) is a leaf space and \( \mathcal{U} \) is as in Section 1 then both complexes coincide.
Losik proved\(^1\) the existence of the natural isomorphism
\[
\hat{H}^*_\mathcal{U}(X) \cong H^*(BG_X),
\]
where \(BG_X\) is the classifying space of the groupoid \(G_X\) defined above. This implies that the cohomology \(\hat{H}^*_\mathcal{U}(X)\) does not depend on the choice neither of a full \(D\)-atlas \(\Phi\), nor on the base \(\mathcal{U}\).

In particular, for the \(D\)-space \(pt\) we get
\[
\hat{H}^*_\mathcal{V}(pt) \cong H^*(BG_n),
\]
where \(\mathcal{V}\) is a base of the topology on \(\mathbb{R}^n\).

Now we are going to describe the relation between the characteristic classes defined by Crainic and Moerdijk with the classes of Losik.

Let \(X\) be a \(D\)-space with a full atlas \(\Phi\). A cover \(\mathcal{U}\) for \(X\) defines the cover \(\mathcal{U}_S\) for \(S(X)\) consisting of the elements \(S(U)\). Each morphism of charts \(h : U \to V\) defines the morphism \(S(h) : S(U) \to S(V)\). We consider in the definition of the Čech-de Rham cohomology for \(S(X)\) only such morphisms. We use similar notations for the spaces \(S'(X), S''(X)\) and \(P(X)\), where \(P(X)\) is the frame bundle of \(X\).

**Proposition 1.** The projection \(\pi : S''(X) \to X\) induces the isomorphism
\[
\pi^* : \hat{H}^*_\mathcal{U}(X) \to \hat{H}^*_\mathcal{U}_S(S''(X)).
\]

**Proof.** The map of the complexes
\[
\pi^* : C^p_q(\mathcal{U}_S) \to C^p_q(\mathcal{U}_S(S''(X)))
\]
is defined by the equality
\[
(\pi^*\omega)(S''(h_1), \ldots, S''(h_p)) = \pi^*(\omega(h_1, \ldots, h_p)),
\]
where \(\pi^*\) on the right hand side is the map acting on the differential forms, and the chain of arrows
\[
U_0 \stackrel{h_1}{\to} \cdots \stackrel{h_p}{\to} U_p
\]
is the one that uniquely defines the chain
\[
S''(U_0) \xrightarrow{S''(h_1)} \cdots \xrightarrow{S''(h_p)} S''(U_p).
\]
It is clear that the map \(\pi^*\) induces the isomorphisms
\[
\pi^* : H^q(C^p_q(\mathcal{U}_S(X)) \to H^q(C^p_q(\mathcal{U}_S(S''(X))).
\]
This and Theorem 1.1 from [11] imply the proof of the proposition. \(\square\)

**Remark.** It is known that for a manifold \(M\), the spaces \(S''(M)\) and \(M\) are weakly homotopy equivalent and consequently have isomorphic de Rham cohomology. This is not the case for a \(D\)-space \(X\), consider, e.g., \(X = pt\).

For a \(D\)-space \(X\), one defines the natural map (cf. [10] Sec. 6.2)]
\[
j : \Omega^*(X) \to C^0_{\mathcal{U}}(X)
\]
as follows:
\[
j(\omega)(U_0) = \omega|_{U_0},
\]
here \(U_0\) corresponds in [14] to the strings of length zero. This defines the homomorphism of cohomology
\[
j : H^*(X) \to \hat{H}^*_\mathcal{U}(X).
\]
Taking, \(X = pt\), we immediately obtain the new construction of the map (2):
\[
k : H^*(\bigcup_{n=1}^\infty) \cong H^*(W_n, O(n)) \cong H^*(S''(pt)) \xrightarrow{j} \hat{H}^*_\mathcal{V}_S(S''(pt)) \cong \hat{H}^*_\mathcal{V}(pt) \cong H^*(BG_n).
\]
\(^1\)In the proof it is implicitly assumed that the cover consists of contactable sets. To avoid this assumption one may use the proof of the isomorphism (7) from [10]. For that it is enough to apply Theorem 3 from [10] to the étale groupoid \(G_X\).
Let \( X \) be a \( \mathcal{D}_n \)-space with an atlas \( \Phi \). It is clear that the domain of each chart from \( \Phi \) is a domain of the chart on \( \text{pt} \), and any morphism of charts from \( \Phi \) is a morphism of the corresponding charts on \( \text{pt} \). This shows that the unique map 

\[ p_X : X \to \text{pt} \]

induces a homomorphism of cohomology

\[ \hat{p}_X : \hat{H}_V^*(\text{pt}) \to \hat{H}_V^*(X). \]

Together with the isomorphism (16) this gives the map

\[ H^\ast(B\Gamma_n) \cong \hat{H}_V^*(\text{pt}) \to \hat{H}_V^*(X). \]

Using the just constructed map \( k \), we instantly obtain the characteristic homomorphism

\[ (20) \] 

\[ \hat{k}_X : H^\ast(W\Omega_n) \to \hat{H}_V^*(X). \]

**Theorem.** The homomorphism (18) coincides with the homomorphism (2). If \( X = M/F \) is the leaf space of a foliation, then the homomorphism (19) coincides with the homomorphism (10); consequently, the homomorphism (20) coincides with the homomorphism (8).

**Proof.** Suppose that \( X = M/F \) is the leaf space of a foliation. A morphism of \( \mathcal{D}_n \)-spaces \( f : X \to Y \) induces the morphism of the corresponding groupoids \( \hat{f} : G_X \to G_Y \) and the morphism \( \hat{f} : BG_X \to BG_Y \) of the classifying spaces [22]. By the naturality of the isomorphism (15), the map \( p_X : X \to \text{pt} \) delivers the commutative diagram

\[ \begin{array}{ccc}
\hat{H}_V^*(\text{pt}) & \cong & H^\ast(B\Gamma_n) \\
\downarrow \hat{p}_X & & \downarrow \hat{p}_X \\
\hat{H}_V^*(X) & \cong & H^\ast(BG_X)
\end{array} \]

Since \( X = M/F \), \( G_X \) is the holonomy groupoid, and the map \( \hat{p}_X : BG_X \to B\Gamma_n \) is the one defined by (1) in the standard way. According to (10), the composition of the map \( \hat{p}_X \) with the isomorphism (7) gives the map (10). This, the above construction and the diagram imply that (19) coincides with (10).

Let us denote the map (18) by \( \tilde{k} \). We must prove that \( \tilde{k} \) coincides with \( k \) given by (2).

Let us first prove the equality

\[ f^\ast \circ \tilde{k} = f^\ast \circ k. \]

For the moment let us assume that \( X \) is an arbitrary \( D_n \)-space. Note that the domain of each chart from the \( M_\infty \)-atlas on \( S''(X) \) is a domain of the corresponding chart on \( S''(\text{pt}) \), and a morphism of the charts on \( S''(X) \) is a morphism of the corresponding charts on \( S''(\text{pt}) \). This gives the homomorphism

\[ (S''(p_X))^* : \hat{H}_{gr}^*(S''(\text{pt})) \to \hat{H}_{gr}^*(S''(X)). \]

From the definitions of the maps \( j \) and \( (S''(p_X))^* \) it follows that we have the commutative diagram

\[ \begin{array}{ccc}
H^\ast(S''(\text{pt})) & \cong & \hat{H}_{gr}^*(S''(\text{pt})) \\
\downarrow j & & \downarrow (S''(p_X))^* \\
H^\ast(S''(X)) & \cong & \hat{H}_{gr}^*(S''(X))
\end{array} \]

It is clear, that the isomorphism from Proposition 1 gives the following commutative diagram:

\[ \begin{array}{ccc}
\hat{H}_V^*(S''(\text{pt})) & \cong & \hat{H}_V^*(\text{pt}) \\
\downarrow (S''(p_X))^* & & \downarrow (p_X)^* \\
\hat{H}_{gr}^*(S''(X)) & \cong & \hat{H}_{gr}^*(S''(X))
\end{array} \]
Let now again $X = M/\mathcal{F}$. Let $\mathcal{U}_M$ be a cover of $M$ obtained from a maximal foliated atlas on $M$. Let $\mathcal{U}$ be the cover of $X$ obtained from the full $\mathcal{D}_n$-atlas of $X$ defined by the foliated atlas of $M$. The cover $\mathcal{U}_M$ gives the cover $\mathcal{U}_M''(\mathcal{F})$ of the manifold $S''(\mathcal{F})$. The projection $p : M \to X$ provides the commutative diagrams

$$
\begin{array}{c}
\xymatrix{
S''(X) & X \\
S''(\mathcal{F}) & M \\
\uparrow{S''(p)} & \uparrow{p} \\
\end{array}
$$

and

$$
\begin{array}{c}
\xymatrix{
\check{H}^*_{\mathcal{U}_M''}(S''(X)) & \check{H}^*_M(X) \\
\check{H}^*_{\mathcal{U}_M''}(S''(\mathcal{F})) & \check{H}^*_M(M) \\
(\mathcal{S}''(p)) & p^* \\
\downarrow{\check{H}^*_{\mathcal{U}_M''}(S''(\mathcal{F}))} & \downarrow{\check{H}^*_M(M)} \\
\end{array}
$$

Since both $M$ and $S''(\mathcal{F})$ are smooth manifolds, the maps $j$ become isomorphisms and we get (recall that $S''(\mathcal{F})$ and $M$ are homotopy equivalent)

$$
\begin{array}{c}
\xymatrix{
\check{H}^*_M(S''(\mathcal{F})) & \check{H}^*_M(M) \\
H^*(S''(\mathcal{F})) & H^*(M) \\
\downarrow{j} & \downarrow{\sim} \\
\end{array}
$$

Summarizing the above diagrams and using the isomorphisms $H^*(W\mathcal{O}_n) \cong H^*(S''(pt))$ and (10), we obtain the following diagram:

$$
\begin{array}{c}
\xymatrix{
H^*(W\mathcal{O}_n) & \check{H}^*_V(S''(pt)) & H^*(B\mathcal{F}_n) \\
H^*(S''(X)) & \check{H}^*_V(S''(X)) & \check{H}^*_M(X) \\
\downarrow{\check{H}^*_{\mathcal{U}_M''}(S''(\mathcal{F}))} & \downarrow{\check{H}^*_M(M)} & \downarrow{\check{H}^*_M(M)} \\
H^*(S''(\mathcal{F})) & H^*(\mathcal{F}) & H^*(M) \\
\end{array}
$$

A part of the sequence of the maps of the diagram coincides with (13) and it gives the map $f^*_{\mathcal{F}} \circ k$. The top row of the maps give the map $\tilde{k}$. Since the maps (13) and (2) coincide, according to (10), the right column of the maps coincides with $f^*_{\mathcal{F}}$. Thus, $f^*_{\mathcal{F}} \circ \tilde{k} = f^*_{\mathcal{F}} \circ k = k_{\mathcal{F}}$.

By Bott’s construction [4], the map $k$ is uniquely defined by the property $f^*_{\mathcal{F}} \circ k = k_{\mathcal{F}}$ for all foliations $\mathcal{F}$. This implies the equality $\tilde{k} = k$. The theorem is proved. \hfill \square

**Theorem 2.** Let $X$ be a $\mathcal{D}_n$-space with a full atlas $\Phi$ and the corresponding cover $\mathcal{U}$. Then there exists the following commutative diagram

$$
\begin{array}{c}
\xymatrix{
H^*(W_n, O(n)) & \check{H}^*_M(X) \\
\uparrow{k} & \uparrow{\check{k}_X} \\
H^*(S''(X)) & \check{H}^*_M(X) \\
\downarrow{\alpha} & \downarrow{\check{H}^*_M(X)} \\
\end{array}
$$

**Proof.** The proof of the theorem deliver the diagrams (21) and (22), and the isomorphisms $H^*(W_n, O(n)) \cong H^*(S''(pt))$ and (10). \hfill \square
Corollary 1. Let $\mathcal{F}$ be a foliation of codimension $n$ on a smooth manifold $M$. Let $T$ be a complete transversal for $T$ and $U$ a basis of $T$. Then there exists the following commutative diagram:

$$
\begin{array}{ccc}
H^*(WO_n) & \xrightarrow{k} & H^*(B\Gamma_n) \\
\chi'' \downarrow & & \downarrow \chi_x \\
H^*(S''(M/\mathcal{F})) & \xrightarrow{\alpha} & H^*_U(M/\mathcal{F}) \\
& & \downarrow \\
& & H^*(M)
\end{array}
$$

The images of the generators of $H^*(WO_n)$ under the map $\chi''$ are characteristic classes defined by Losik. The images of the generators of $H^*(WO_n)$ under the map $\chi_x \circ k$ are characteristic classes defined by Crainic and Moerdijk. These classes are projected to the usual ones living in $H^*(M)$.

Example 1. Let $X$ be a $\mathcal{D}_3$-space. The Godbillon-Vey class with values in $\hat{H}^3(X)$ is obtained form the generator $[y_1 c_1] \in H^3(W_1, O(1))$, it is defined by the form of type $2, 1$ given by the formula

$$
(23) \quad gv(h_1, h_2) = \ln |h'_1| \, d \ln |h'_2 \circ h_1|.
$$

For foliations of codimension one this formula is given in [10].

The Godbillon-Vey class with values in $H^3(S''(X))$ may be described in the following way. Let $U \subset \mathbb{R}$ be the domain of a chart on $X$. A map $h$ from a neighborhood of $0$ in $\mathbb{R}$ to $U$ defines the coordinates $z_i = h^i(0)$, $z = 0, 1, 2, \ldots$, on $S(U)$. Consider the following coordinates on $S''(U)$: $x_0 = z_0, x_1 = \ln|z_1|, x_k = z_k/z_k^1, k = 2, 3, \ldots$. The Godbillon-Vey class is given by the form

$$
dx_0 \wedge dx_1 \wedge dx_2
$$

with respect to each such coordinate system.

The formula (23) defines a form of type $2, 1$ on $S''(X)$ if the arrows are changed to their extensions to $S''(X)$. It is easy to check that

$$
dx_0 \wedge dx_1 \wedge dx_2 = gv + D(\omega^{(0,2)} + \omega^{(1,1)} + \omega^{(2,0)}),
$$

where $D$ is the total differential and

$$
\omega^{(0,2)} = x_1 dx_2 \wedge dx_1, \quad \omega^{(1,1)}(h) = x_2 \ln|h'(h_0)| dx_0, \quad \omega^{(2,0)}(h_1, h_2) = \ln|h'_1| \ln|h'_2 \circ h_1|.
$$

This confirms the fact that the homomorphism $H^*(S''(X)) \to \hat{H}^*(X)$ respects the corresponding Godbillon-Vey classes.

Let now $\mathcal{F}$ be a Reeb foliation on the three dimensional sphere. It is known that the usual Godbillon-Vey class of $\mathcal{F}$ is trivial in $\hat{H}^3(S^3)$. As it is explained in the Introduction, this class is also trivial in $\hat{H}^3(S^3/\mathcal{F})$. In [1] we show that for some specific choice of $\mathcal{F}$, the Godbillon-Vey class is non-trivial in $\hat{H}^3(S(S^3/\mathcal{F})/O(1))$. Hence, it allows to detect non-diffeomorphic foliations.

5. Characteristic classes of groups diffeomorphisms

Let $G$ be an (abstract) group acting by diffeomorphisms on an $n$-dimensional manifold $M$. Let $H^*(G; \Omega(M))$ be the cohomology of the group $G$ with the coefficients in the forms on $M$ (which is defined in the same way as the Čech-de Rham cohomology with all arrows being the global diffeomorphisms of $M$ from $G$). The orbit space $M/G$ is a $\mathcal{D}_n$-space. It is obvious that there is a homomorphism

$$
\hat{H}^*(M/G) \to H^*(G; \Omega(M)).
$$

From the above we obtain the sequence of maps

$$
H^*(W_n, O(n)) \to H^*(S''(M/G)) \to \hat{H}^*(M/G) \to H^*(G; \Omega(M)).
$$

This gives the character map

$$
H^*(W_n, O(n)) \to H^*(G; \Omega(M)).
Example 2. Let $G$ be a group of orientation preserving diffeomorphisms of the circle $S^1$. Then the formula \((23)\), for $h_1, h_2 \in G$, gives the Godbillon-Vey class with values in $H^3(G; \Omega(S^1))$. The integration of $g(h_1, h_2)$ over $S^1$ gives the Bott-Thurston formula for the Godbillon-Vey class in $H^2(G; \mathbb{R})$. An important Theorem by Duminy and Sergiescu \cite{12} states that if $G$ does not contain crossed elements, then this class is trivial.

On the other hand, let $F$ be a Reeb foliation with non-trivial Godbillon-Vey class in $H^3(S^3/F)$ constructed in \cite{11}. Let $\xi$ be the product of two generators of the holonomy group of the compact leaf such that $\xi(x) < x$. Then $\xi$ is a diffeomorphism on its image of a neighborhood of $0 \in \mathbb{R}$ and $0$ is the only fixed point of $\xi$. It is clear that $\xi$ may be extended to a diffeomorphism of $\mathbb{R}$ satisfying $\xi(x+1) = \xi(x) + 1$ and with the set $\mathbb{Z}$ of fixed points, i.e., $\xi$ defines a diffeomorphism of the circle with exactly one fixed point. This diffeomorphism generates a group $G$. It is clear that $G$ does not contain crossed elements. From the results of \cite{11} it follows that the Godbillon-Vey class of the obtained action is non-trivial in $H^3(S''(S^1/G))$. This shows that the Godbillon-Vey class in $H^3(S''(S^1/G))$ may be used to detect non-conjugate group actions.

6. Chern classes

Till now, we considered only the cohomology $H^*(W_n, O(n))$. It is clear that in the same way we may consider the cohomology $H^*(W_n, GL(n, \mathbb{R}))$. Then the Proposition \cite{11} Theorem \cite{2} and Corollary \cite{1} hold true with $S''(X)$ replaced by $S'(X)$. Moreover, there are the obvious maps $H^*(W_n, GL(n, \mathbb{R})) \rightarrow H^*(W_n, O(n))$ and $S''(X) \rightarrow S'(X)$. For any $D_n$-space we obtain the diagram

$$
\begin{align*}
H^*(W_n, GL(n, \mathbb{R})) & \xrightarrow{\chi'} H^*(W_n, O(n)) \xrightarrow{k} H^*(B\Gamma_n) \\
H^*(S'(X)) & \xrightarrow{\chi''} H^*(S''(X)) \xrightarrow{\alpha} H^*_G(X)
\end{align*}
$$

Corollary 2. Let $X$ be a $D_n$-space. Then the characteristic classes defined by the elements of the kernel of the homomorphism $H^*(W_n, GL(n, \mathbb{R})) \rightarrow H^*(W_n, O(n))$ are zero in $H^*_G(X)$.

Example 3. Let $X$ be a $D_1$-space. The class $[c_1] \in H^2(W_1, GL(1, \mathbb{R}))$ defines the first Chern class $C_1 \in H^2(S'(X))$. If $x_0, x_1, x_2, \ldots$ are the standard coordinates on $S(U)$, then $y_0 = x_0, y_2 = \frac{x_2}{x_0}, y_3 = \frac{x_3}{x_0}, \ldots$ are coordinates on $S'(U)$. With respect to these coordinates, $C_1$ is given by the form $dy_2 \wedge dy_0$.

Let $X = M/F$. The image of the generator $[c_1]$ of $H^2(W_1, GL(1, \mathbb{R}))$ under the map $H^2(W_1, GL(1, \mathbb{R})) \rightarrow H^2(W_1, O(1))$ is trivial, consequently the first Chern class is always trivial in the Čech-de Rham cohomology $H^2(M/F)$ and in $H^2(M)$. This is also directly proved in \cite{10}. On the other hand, Losik \cite{22} (see also \cite{2}) proved that the first Chern class of the Reeb foliation on the three-dimensional sphere is non-trivial in the cohomology $H^2(S'(M/F))$.

Example 4. Let $F$ be a foliation of codimension one on a three-dimensional manifold. Suppose that the foliation $F$ is defined by a non-vanishing 1-form $\omega$. Consider the complex

$$
A^m = \Omega^{m-1}(M) \wedge \omega
$$

with the differential being the usual exterior derivative. The cohomology of this complex are denoted by $H^*_F(M)$. Let $\eta$ be any 1-form such that $d\omega = \eta \wedge \omega$. Then the Vey class is the class of the form $d\eta$ in $H^3_F(M)$. A Riemannian metric $g$ on $M$ defines a map $\sigma : M \rightarrow S'(M/F)$ such that $\sigma^*(dy_2 \wedge dy_0) = d\eta$, where $\eta = -L_X\omega$ and $X$ is the vector field orthogonal with respect to $g$ to the distribution tangent to $F$ and such that $\omega(X) = 1$. Unfortunately, the map $\sigma^*$ does not induce a map from $H^*(S'(M/F))$ to $H^*_F(X)$. If $F$ is the Reeb foliation, then $d\eta = 0$ \cite{24}, i.e., the Vey class of the Reeb foliation is trivial.
7. Classes from the cohomology $H^*(W_n)$

Let $X$ be a $\mathcal{D}_n$-space with a full atlas $\Phi$, and $\mathcal{U}$ the corresponding cover. The proof of the following proposition is similar to the proof of Proposition 1.

**Proposition 2.** The projection $S(X) \to P(X)$ induces the isomorphism

$$\tilde{H}^*_{\mathcal{U}^Q}(S(X)) \cong \tilde{H}^*_{\mathcal{U}^P}(P(X)).$$

Let $X$ be a $\mathcal{D}_n$-space with a full atlas $\Phi$ that defines a complete cover $\mathcal{U}$. Let $Q(X)$ be one of the spaces $S(X)$, $S'(X)$, $S''(X)$, $P(X)$. Note that the cover $\mathcal{U}_Q$ is not complete. Denote by $\hat{U}_Q$ the cover obtaining as a base of the topologies on $Q(U)$ for all $U \in \mathcal{U}$. The cover $\hat{U}_Q$ is complete. We need a complete cover in order to use the isomorphism (15). Let $U, V \in \mathcal{U}$. Let $\hat{U} \subset Q(U)$ and $\hat{V} \subset Q(V)$ be open subsets. We will consider the morphisms $g : \hat{U} \to \hat{V}$ that are restrictions of the extensions $Q(h) : Q(U) \to Q(V)$ of the morphisms $h : U \to V$ from $C_\Phi$. This gives a full atlas of $Q(X)$ and the corresponding complete cover $\hat{U}_Q$. The proof of the following proposition will be given in Appendix.

**Proposition 3.** There is a natural isomorphism

$$\tilde{H}^{*\mathcal{U}}_{\mathcal{U}^Q}(Q(X)) \cong \tilde{H}^{*\mathcal{U}}_{\mathcal{U}^P}(Q(X)).$$

Consider the frame bundle $P(pt)$ for the space $pt$. It is obvious that the reduced groupoid $G_{P(pt)}$ coincides with the groupoid $\tilde{\Gamma}_n$. The isomorphism (15) applied to $P(pt)$ gives

$$\tilde{H}^*_\mathcal{U}(P(pt)) \cong H^*(B\tilde{\Gamma}_n).$$

Let $X$ be a $\mathcal{D}_n$-space. The projection $p_X : X \to pt$ induces the map

$$H^*(W_n) \xrightarrow{\sim} H^*(S(X)) \xrightarrow{\sim} \tilde{H}^*_{\mathcal{U}_Q}(S(X)) \cong \tilde{H}^*_{\mathcal{U}_P}(P(X)) \cong \tilde{H}^*_{\mathcal{U}^P}(P(X)).$$

Taking $X = pt$, we get the map

$$\tilde{k} : H^*(W_n) \to H^*(B\tilde{\Gamma}_n).$$

Applying to the projection $p_X : X \to pt$ the functor $P$, we get the maps

$$P(p_X) : P(X) \to P(pt)$$

and

$$\tilde{H}^*(B\tilde{\Gamma}_n) \cong \tilde{H}^*_{\mathcal{U}^P}(P(pt)) \to \tilde{H}^*_{\mathcal{U}^P}(P(X)).$$

We say that the frame bundle $P(X)$ is trivializable if there is a section $s : X \to P(X)$, i.e., for the domain $U$ of each chart form $\Phi$, there exists a section $s_U : U \to P(U)$ such that for each morphism of charts $m : U \to V$, the diagram

$$\begin{array}{ccc}
P(U) & \xrightarrow{P(m)} & P(V) \\
| & s_U & | \\
\downarrow & & \downarrow \ s_V \\
U & \xrightarrow{m} & V \\
\end{array}$$

is commutative. It is clear that this notion does not depend of the choice of a full atlas. It is obvious that if $X = M/\mathcal{F}$ is the leaf space of a foliation $\mathcal{F}$ on a manifold $M$, then the bundle $P(X) \to X$ is trivializable if and only if the bundle $P(\mathcal{F}) \to M$ is trivializable. A section $s : X \to P(X)$ induces the map

$$\tilde{H}^*_{\mathcal{U}^P}(P(X)) \cong \tilde{H}^*_{\mathcal{U}^P}(P(X)) \to \tilde{H}^*_{\mathcal{U}}(X).$$

Together with (25) and (27) this delivers the characteristic homomorphisms

$$H^*(W_n) \to \tilde{H}^*_\mathcal{U}(X)$$

and

$$\tilde{f}_\mathcal{U}^* : H^*(B\tilde{\Gamma}_n) \to \tilde{H}^*_\mathcal{U}(X).$$
By the construction, the map (28) coincides with the composition $\tilde{f}_X \circ \tilde{k}$. This defines the exotic characteristic classes of a $D_n$-space $X$ with values in the Čech-de Rham cohomology of $X$. We obtain

**Theorem 3.** Let $X$ be a $D_n$-space with a trivializable bundle of frames, then there exist a characteristic map $\tilde{f}_X$ defined by (29) and a characteristic map (28) that coincides with the composition $\tilde{f}_X \circ \tilde{k}$.

If $X = M/F$ is the leaf space of a foliation $\mathcal{F}$ on a smooth manifold $M$ with a trivializable bundle $P(\mathcal{F}) \to M$, then there is the following commutative diagram:

$$
\begin{array}{ccc}
H^*(W_n) & \longrightarrow & H^*(S(M/F)) \\
\downarrow & & \downarrow \\
H^*(M) & \longrightarrow & H^*_\mathcal{F}(M/F)
\end{array}
$$

Showing the relation of the Losik classes from $H^*(S(M/F))$, the secondary classes with values in the Čech-de Rham cohomology of the leaf space and the usual secondary characteristic classes of the foliation $\mathcal{F}$.

The proof of the following theorem is similar to the proof of Theorem 1.

**Theorem 4.** The homomorphism (26) coincides with the homomorphism (3).

Completing this section note that the projection $S(\text{pt}) \to S''(\text{pt})$ together with the maps $j$ give the commutative diagram

$$
\begin{array}{ccc}
H^*(S''(\text{pt})) & \longrightarrow & H^*(S(\text{pt})) \\
\downarrow & & \downarrow \\
\tilde{H}^*_\psi(S''(\text{pt})) & \longrightarrow & \tilde{H}^*_\psi(S(\text{pt}))
\end{array}
$$

Applying the above described isomorphisms, we get the well-known diagram

$$
\begin{array}{ccc}
H^*(WQ_n) & \longrightarrow & H^*(W_n) \\
\downarrow & & \downarrow \\
H^*(B\Gamma_n) & \longrightarrow & H^*(B\bar{\Gamma}_n)
\end{array}
$$

Note that the map $H^*(B\Gamma_n) \to H^*(B\bar{\Gamma}_n)$ is in fact induced by the projection $P(\text{pt}) \to \text{pt}$.

**8. The full picture**

Finally, considering the projections $S(X) \to S''(X) \to S'(X)$, we obtain the following theorem.

**Theorem 5.** Let $X$ be a $D_n$-space with a section $s : X \to P(X)$. Then there exists the following commutative diagram
Here the images of the maps $\chi$, $\chi'$ and $\chi''$ give the characteristic classes defined by Losik, and the images of the maps from $H^*(W_n, O(n))$ and $H^*(W_n)$ to $\tilde{H}^*_q(M)$ give the characteristic classes with values in the Čech-de Rham cohomology of $X$.

Note that the map $H^*(W_n, GL(n, \mathbb{R})) \to H^*(W_n)$ is zero. This implies

**Corollary 3.** Let $X$ be a $\mathcal{D}_n$-space with a section $s : X \to P(X)$. Then the Chern classes, i.e., the characteristic classes defined by the cohomology $H^*(W_n, GL(n, \mathbb{R}))$, are zero in $\tilde{H}^*_q(X)$.

We also get.

**Corollary 4.** Let $X$ be a $\mathcal{D}_n$-space with a section $s : X \to P(X)$. Then the characteristic classes defined by the elements of the kernel of the homomorphism $H^*(W_n, O(n)) \to H^*(W_n)$ are zero in $\tilde{H}^*_q(X)$.

Let $n = 1$. The homomorphism $H^*(W_1, O(1)) \to H^*(W_1)$ is an isomorphism. The last two corollaries do not give new information comparing to Corollary 2 and Example 3.

Let $n = 2$. The element $c_2$ is trivial in $H^*(W_2)$, and it is non-trivial in both $H^*(W_2, GL(n, \mathbb{R}))$ and $H^*(W_2, O(n))$. Consequently, if $X$ is a $\mathcal{D}_2$-space with trivial $P(X)$, then the image of $[c_2]$ is trivial in $\tilde{H}^*_q(X)$. It would be interesting to construct a foliation of codimension two with trivializable $P(M/\mathcal{F})$ and such that $[c_2]$ is non-trivial in $H^4(S^2(M/\mathcal{F}))$ (and consequently non-trivial in $H^4(S^2(M/\mathcal{F}))$).

**Appendix A. Proof of Proposition 3**

Recall that the cochains form $C_{\mathcal{U}^q}(Q(X))$ map the strings of morphisms

$$V_0 \xrightarrow{g_1} \cdots \xrightarrow{g_k} V_k$$

to $\Omega^q(V_0)$, here each $V_i$ is an open subset of some $Q(U_i)$, $U_i \in \mathcal{U}$, $g_i : V_{i-1} \to V_i$ is the restriction to $V_{i-1}$ of the extension $Q(h_i) : Q(U_{i-1}) \to Q(U_i)$ of a morphism of charts $h_i : U_{i-1} \to U_i$. It is clear that we may assume that $p(V_i) = U_i$, where $p : Q(U_i) \to U_i$ is the projection. The cochains form $\tilde{C}_{\mathcal{U}^q}(Q(X))$ map the strings of morphisms

$$Q(U_0) \xrightarrow{Q(h_1)} \cdots \xrightarrow{Q(h_k)} Q(U_k)$$

to $\Omega^q(Q(U_0))$. We define the morphisms of the complexes

$$\mu : C_{\mathcal{U}^q}(Q(X)) \to C_{\mathcal{U}^q}(Q(X)),$$

$$\lambda : \tilde{C}_{\mathcal{U}^q}(Q(X)) \to \tilde{C}_{\mathcal{U}^q}(Q(X))$$

by setting

$$\mu(\varphi) \left( V_0 \xrightarrow{g_1} \cdots \xrightarrow{g_k} V_k \right) = \varphi \left( Q(U_0) \xrightarrow{Q(h_1)} \cdots \xrightarrow{Q(h_k)} Q(U_k) \right) \bigg|_{V_0}, \quad \varphi \in \tilde{C}_{\mathcal{U}^q}(Q(X)),$$

$$\lambda(\epsilon) \left( Q(U_0) \xrightarrow{Q(h_1)} \cdots \xrightarrow{Q(h_k)} Q(U_k) \right) = \epsilon \left( Q(U_0) \xrightarrow{Q(h_1)} \cdots \xrightarrow{Q(h_k)} Q(U_k) \right), \quad \epsilon \in \tilde{C}_{\mathcal{U}^q}(Q(X)).$$
It holds
\[ \lambda \circ \mu = \text{id}, \]
i.e. \( \mu \) induces a monomorphism in cohomology. We are going to construct a chain homotopy between the maps \( \mu \circ \lambda \) and \( \text{id} \). We define the map
\[ F : C^k_{\mathcal{U}_Q}(Q(X)) \to C^{k-1}_{\mathcal{U}_Q}(Q(X)) \]
in the following way:
\[
F(\varphi) \left( V_0 \xrightarrow{g_1} \cdots \xrightarrow{g_{k-1}} V_{k-1} \right) = \sum_{s=0}^{k-1} (-1)^s \varphi \left( V_0 \xrightarrow{g_1} \cdots \xrightarrow{g_s} V_s \xrightarrow{i_s} Q(U_s) \xrightarrow{Q(h_{s+1})} \cdots \xrightarrow{Q(h_{k-1})} Q(U_{k-1}) \right),
\]
where \( i_s : V_s \to Q(U_s) \) is the inclusion. It is clear that \( F \) commutes with the differential \( d \). Next, for the total differential \( D \) and \( \varphi \in C^k_{\mathcal{U}_Q}(Q(X)) \) it holds
\[
D(F(\varphi)) + F(D(\varphi)) = (\delta + (-1)^{k-1}d)(F(\varphi)) + F((\delta + (-1)^k d)\varphi) = \delta(F(\varphi)) + F(\delta(\varphi)).
\]
Hence in order to show that \( F \) is a cochain homotopy it is sufficient to prove the equality
\[ (30) \]
\[ (\mu \circ \lambda - \text{id})(\varphi)(g_1, g_2) = \varphi(Q(h_1), Q(h_2))|_{V_0} - \varphi(g_1, g_2). \]

This equality may be checked directly. To illustrate this let us for simplicity suppose that \( \varphi \in C^2_{\mathcal{U}_Q}(Q(X)) \). It holds
\[
(\mu \circ \lambda - \text{id})(\varphi)(g_1, g_2) = \varphi(Q(h_1), Q(h_2))|_{V_0} - \varphi(g_1, g_2).
\]

Next,
\[
F(\delta \varphi)(g_1, g_2) = \left( \delta \varphi \right)(i_0, Q(h_1), Q(h_2)) - \left( \delta \varphi \right)(g_1, i_1, Q(h_2)) + \left( \delta \varphi \right)(g_1, g_2, i_2)
\]
\[
= \varphi(Q(h_1), Q(h_2))|_{V_0} - \varphi(Q(h_1))|_{i_0, Q(h_2)} + \varphi(i_0, Q(h_2))|_{Q(h_1)} - \varphi(i_0, Q(h_1))
\]
\[
- (g_1^* \varphi(i_1, Q(h_2)) - \varphi(i_1 \circ g_1, Q(h_2)) + \varphi(g_1, Q(h_2) \circ i_1) - \varphi(g_1, i_1))
\]
\[
+ g_1^* \varphi(g_2, i_2) - \varphi(g_2 \circ g_1, i_2) + \varphi(g_1, i_2 \circ g_2) - \varphi(g_1, g_2).
\]

\[
\delta(F(\varphi))(g_1, g_2) = g_1^* F(\varphi)(g_2) - F(\varphi)(g_2 \circ g_1) + F(\varphi)(g_1)
\]
\[
= g_1^* \varphi(i_1, Q(h_2)) - g_1^* \varphi(g_2, i_2) - \varphi(i_0, Q(h_2) \circ i_1) + \varphi(g_2 \circ g_1, i_2) + \varphi(i_0, Q(h_1)) - \varphi(g_1, i_1).
\]

Noting that \( Q(h_{s+1}) \circ i_s = i_{s+1} \circ g_{s+1} \), we see that \( (30) \) holds true for the \( \varphi \) under consideration. The proposition is proved. \( \square \)

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