Kähler geometry of bounded pseudoconvex Hartogs domains

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Abstract

Let $\Omega$ be a bounded pseudoconvex Hartogs domain. There exists a natural complete Kähler metric $g^\Omega$ in terms of its defining function. In this paper, we study two problems. The first one is determining when $g^\Omega$ is Einstein or extremal. The second one is the existence of holomorphic isometric immersions of $(\Omega, g^\Omega)$ into finite or infinite dimensional complex space forms.

Key words: Kähler-Einstein metric, extremal metric, Kähler immersion, pseudoconvex Hartogs domain

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1 Introduction

In [8], Cheng and Yau proved that there exist complete Kähler-Einstein metrics with negative Ricci curvature on strictly pseudoconvex domains with $C^k$ boundary, $k \geq 5$. This metric is unique if the metric is normalized by multiplication by a constant to have the eigenvalues of the Ricci tensor identically $-1$. In [17], Mok and Yau extended this result to bounded pseudoconvex domains. The proof in [8] involves solution by the continuity method of a Monge-Ampère equation. The solution actually is a special Kähler potential function of the Kähler-Einstein metric. On homogeneous domains, the Bergman metric is the Kähler-Einstein metric with negative Ricci curvature $-1$. Up to a constant, the Bergman kernel is the solution of the Monge-Ampère equation introduced by Cheng and Yau. In general, it is very difficult to obtain the solution. In [22], Yin, Zhang, Roos and the second author of this paper obtained the explicit solutions for some nonhomogeneous Cartan-Hartogs domains by reducing the higher order nonlinear partial differential equation to an ordinary differential equation (see the definition of Cartan-Hartogs domain in Example 2.6). In [23], we generalized this result for some generalized Cartan-Hartogs domains over bounded symmetric domains. Notice that the Kähler-Einstein metric we obtain is just the natural complete Kähler metric given by a defining function of the domain. This Kähler metric was constructed by Cheng and Yau on the strictly pseudoconvex domain $\Omega$ with $C^k$, $k \geq 5$ boundary in $\mathbb{C}^n$ firstly. To be precise, let $F$ be a defining function for $\Omega$. Define $g^{\Omega}$ whose Kähler potential is $-\log(-F)$. Then they proved that $(\Omega, g^{\Omega})$ is a complete Kähler manifold whose Ricci tensor is “asymptotically Einstein”. Moreover, one can get a complete Kähler-Einstein metric by perturbing the metric $g^{\Omega}$.

On some special domains, we can describe the distinction between this natural Kähler metric and the Einstein-Kähler metric. In our case, the boundary condition is no more necessary. Recall the following definition firstly. Let $D \subset \mathbb{C}^d$ be a domain and $\varphi$ be a continuous positive function on $D$. The domain

$$
\Omega = \left\{ (z_0, z) \in \mathbb{C}^{d_0} \times D : ||z_0||^2 < \varphi(z) \right\}
$$

is called a Hartogs domain over $D$ with $d_0$-dimensional fibers. Hartogs domains have been investigated by many mathematicians for studying many problems in several complex variables. It is easy to see that $\Omega$ is pseudoconvex if and only if $D$ is pseudoconvex and $-\log \varphi$ is plurisubharmonic. Conversely, Hartogs domains can also be used to characterize pseudoconvex domains. Let $D$ be a domain, then $D$ is pseudoconvex if and only if the Hartogs-like domain $\{(z_0, z) \in \mathbb{C}^{d_0} \times D : z + \lambda z_0 \in D, \lambda \in \mathbb{C}, |\lambda| \leq 1\}$ over $D$ with balanced fibers is pseudoconvex [18].

From [16], we know Engliš had considered the bounded simply connected pseudoconvex Hartogs domain $(\Omega, g^{\Omega})$, where $g^{\Omega}$ is the natural Kähler metric whose Kähler potential is $-\log(\varphi(z) - ||z_0||^2)$. Let $g^B$ be the Bergman metric on $\Omega$. He proved that if $g^{\Omega} = \lambda g^B$ for some $\lambda \in \mathbb{R}^+$, then $g^{\Omega}$ is Kähler-Einstein. By the results in [23], we know that there also exist many examples whose natural Kähler metrics $g^{\Omega}$ are Einstein, while $g^{\Omega} \neq \lambda g^B$ for any $\lambda \in \mathbb{R}^+$. Notice that many important properties of Hartogs domain can be characterized by the base such as convex, smoothly (or real-analytically) bounded, Bergman kernel [13], the Bergman completeness of non hyperconvex domains [11], $D^*$-extension property [19], $k$-hyperbolic [20], $K$-exhaustive [12] and so on. So we prefer to characterize the Kähler-Einstein metric of $(\Omega, g^{\Omega})$ in terms of its base.
Our first result is Theorem 1.1 which characterizes two canonical metrics of the bounded pseudoconvex Hartogs domain in terms of the base. The first one is the Kähler-Einstein metric. The second one is the extremal Kähler metric which is one of the generalizations of Kähler-Einstein metric. It was introduced by Calabi [3, 4] for finding the canonical representant of a given Kähler classes $[\omega]$ of a complex compact Kähler manifold $(M, J)$. In the noncompact case, the problem of finding extremal metrics is quite natural but difficult [24]. In [7], Chang proved the existence of extremal metrics of a complete noncompact smooth surface. In [14], Loi and Zudda proved that the only extremal metric is the hyperbolic metric for a strongly pseudoconvex Hartogs domain. In [26], Zedda considered the Cartan-Hartogs domain endowed with a natural Kähler metric. He proved that this metric is extremal if and only if it is Einstein. Our theorem extends Zedda’s result for any bounded pseudoconvex Hartogs domain.

Theorem 1.1. Let
\[ \Omega = \left\{ (z_0, z) \in \mathbb{C}^{d_0} \times D : ||z_0||^2 < \varphi(z) \right\} \]  
(1.2)
be a bounded pseudoconvex Hartogs domain over $D \subset \mathbb{C}^d$, where $-\log \varphi$ is a $C^\infty$ strictly plurisubharmonic exhaustion function on $D$. Let $g^D$ and $g^\Omega$ be the Kähler metrics whose Kähler potentials are $-\log \varphi(z)$ and $-\log(\varphi(z) - ||z_0||^2)$ respectively. If $g^D$ is a Kähler-Einstein metric, then the following condition are equivalent:

(i) $g^\Omega$ is a Kähler-Einstein metric with Ricci curvature $-(d_0 + d + 1)$;

(ii) $g^\Omega$ is an extremal metric;

(iii) The scalar curvature of $g^\Omega$ is a constant;

(iv) The Ricci curvature of $g^D$ equals to $-(d + 1)$.

Actually, we study this problem in a more general case. The sufficient conditions for $g^\Omega$ is Einstein or extremal can be described by the curvatures respectively. Thus we also provide some extremal Kähler metrics. But they are not Einstein.

The study of holomorphic isometric immersions (called Kähler immersions in the sequel) between Kähler manifolds was started by Calabi [5]. He solved the problem of deciding about the existence of Kähler immersions between Kähler manifolds and complex space forms. If Kähler immersions exist, the Kähler manifolds are also called Kähler submanifolds of complex space forms. Afterwards, there appear many important studies about the characterization and classification of Kähler submanifolds of complex space forms. For example, Di Scala and Loi gave a complete description of the Kähler immersions of Hermitian symmetric spaces into complex space forms in [9]. Di Scala, Ishi and Loi also studied the Kähler immersions of homogeneous Kähler manifolds into complex space forms in [10]. In order to study the existence of nonhomogeneous Kähler-Einstein submanifolds of infinite dimensional complex projective forms, Loi and Zeddy studied the Kähler immersions of Cartan-Hartogs domains in [15]. Inspired by their work, we study the Kähler immersions of $(\Omega, g^\Omega)$ into complex space forms.

Theorem 1.2. Let $\Omega$ be as in (1.2). Suppose that $\Omega$ is a simply connected circular domain with center zero and the function $-\log \varphi$ is the special Kähler potential determined by the diastatic function of $g^D$. Then the existence of Kähler immersions of $(\Omega, g^\Omega)$ into complex space forms are completely determined by the existence of Kähler immersions of $(D, g^D)$ into complex space forms (see Table 2 of Section 3.5 for details).
The paper is organized as follows. In Section 2, we construct a natural Kähler metric on a generalized bounded pseudoconvex Hartogs domain. The Ricci curvature and scalar curvature will be calculated directly by the standard formulas. Then we study the problem when this metric is Einstein or extremal. By our results in this section, Theorem 1.1 can be obtained immediately. In Section 3, after recalling some definitions and criterions for Kähler immersions, we discuss the existence of Kähler immersions of \((\Omega, g)\) into three types of complex space forms respectively.

## 2. Two canonical metrics of bounded pseudoconvex Hartogs domains

In order to obtain more interesting properties, we prefer to study the bounded pseudoconvex Hartogs domain as follows:

\[
\Omega = \left\{ (z_0, z) \in \mathbb{C}^{d_0} \times D : ||z_0||^2 < \varphi(z) \right\}, \tag{2.1}
\]

where \(D = D_1 \times D_2 \times \cdots \times D_m \subset \mathbb{C}^{d_1} \times \mathbb{C}^{d_2} \times \cdots \times \mathbb{C}^{d_m}, m \in \mathbb{Z}^+,\) is the product of finite bounded pseudoconvex domains, and \(\varphi = \prod_{i=1}^{m} \varphi_i.\) Here \(\varphi_i\) is a function on \(D_i\) such that \(\varphi_i < \infty\) strictly plurisubharmonic exhaustion function on \(D_i.\) Since \(D_i\) is a bounded pseudoconvex domain, such \(\varphi_i\) always exists. Obviously, \(-\log \varphi = -\sum_{i=1}^{m} \log \varphi_i\) is a \(C^\infty\) strictly plurisubharmonic exhaustion function on \(D.\) Hartogs domain defined by (1.1) can be obtained from (2.1) by taking \(m = 1.\)

Let \(F(z_0, z) = ||z_0||^2 - \varphi(z).\) The boundary \(\partial \Omega\) of \(\Omega\) is consist of two parts, i.e.

\[
\partial \Omega = (\{0\} \times \partial D) \cup \partial_0 \Omega,
\]

where \(\{0\} \times \partial D = \{(0, z) : z \in \partial D\}\) and \(\partial_0 \Omega = \{(z_0, z) \in \mathbb{C}^{d_0} \times D : F(z_0, z) = 0, z_0 \neq 0\}.\) Now we claim that \(F\) is a local \(C^\infty\) defining function of \(\Omega\) at any fix boundary point \(\tilde{p} = (z_0, \bar{z}) \in \partial_0 \Omega.\) In fact, let \(V(\bar{z}) \subset D\) be a neighborhood of \(\bar{z},\) \(B(\bar{z}_0, r)\) be a ball with radius \(r < ||\bar{z}_0||,\) Then the neighborhood \(U(\tilde{p}) = B(\bar{z}_0, r) \times V(\bar{z})\) of \(\tilde{p}\) satisfies

\[
U(\tilde{p}) \cap \Omega = \{(z_0, z) \in U(\tilde{p}) : F(z_0, z) < 0\},
\]

and \(dF(z_0, z) \neq 0\) for \((z_0, z) \in \partial_0 \Omega.\) So the claim is true. Thus we know the boundary \(\partial_0 \Omega\) is always smooth.

### 2.1 Kähler-Einstein metric

Let \(\Omega\) be as in (2.1). Since \(-\log \varphi_i\) is a \(C^\infty\) strictly plurisubharmonic exhaustion function \(\varphi_i\) on \(D_i,\) it gives a global Kähler metric on \(D_i,\) denoted by \(g^{D_i}.\) The Kähler form \(\omega^{D_i} = \nabla_{\tilde{p}}(\log \varphi_i).\) Hence, \((D_i, g^{D_i})\) is a Kähler manifold. Let

\[
(D, g^D) = (D_1 \times \cdots \times D_m, g^{D_1} \times \cdots \times g^{D_m}). \tag{2.2}
\]

If \(g^{D_i}, i = 1, 2, \cdots, m,\) are Einstein metrics with the same Ricci curvature, then \(g^D\) is Einstein.
Notice that the function $- \log(-F)$ is a $C^\infty$ strictly plurisubharmonic function on $\Omega$, and $- \log(-F) \to \infty$ as $(z_0, z) \to \partial \Omega$. It gives a global Kähler metric $g^\Omega$ of $\Omega$, i.e.

$$\omega^\Omega = \frac{\sqrt{-1}}{2} \partial \bar{\partial} (- \log(-F)).$$

(2.3)

Let $(z_0, z) = (z_0, z_1, z_2, \cdots, z_m) \in \mathbb{C}^{d_0} \times \mathbb{C}^{d_1} \times \cdots \times \mathbb{C}^{d_m}$ and $d = \sum_{i=1}^{m} d_i$, where $z_i = (z_{i1}, z_{i2}, \cdots, z_{idi}) \in \mathbb{C}^{d_i}$. In this coordinate, the matrix of $g^\Omega$ in (2.3), also denoted by the same notation, can be written as follows:

$$g^\Omega = \begin{pmatrix} g^\Omega_{j\alpha, k\beta} \end{pmatrix},$$

(2.4)

where the elements

$$g^\Omega_{j\alpha, k\beta} = - \frac{\partial^2 \log (\varphi(z) - ||z_0||^2)}{\partial z_{j\alpha} \partial \bar{z}_{k\beta}}$$

for $0 \leq j, k \leq m$ and $1 \leq \alpha \leq d_j, 1 \leq \beta \leq d_k$.

**Lemma 2.1.** Let $\Omega$ be as in (2.1). If $g^{D_i}, i = 1, \cdots, m$, are Kähler-Einstein metrics with Ricci curvatures $c_i$ respectively, then there exist some real-valued functions $f_i$ on $D_i$ respectively, such that

$$\det(g^\Omega) = (\varphi - ||z_0||^2)^{-(d+d_0+1)} \prod_{i=1}^{m} \varphi_i^{d_i+1+c_i} e^{c_i f_i}.$$  

(2.5)

**Proof.** For convenient, we define

$$\varphi_j := \frac{\partial \varphi(z)}{\partial z_j}, \quad \varphi_{j\alpha, k\beta} := \frac{\partial^2 \varphi(z)}{\partial z_{j\alpha} \partial \bar{z}_{k\beta}}$$

for $1 \leq j, k \leq m$.

By a straightforward computation, the metric

$$g^\Omega = \frac{1}{(\varphi - ||z_0||^2)^2} \begin{pmatrix} \delta_{st} + \overline{z}_{0s} \overline{z}_{0t} & -\overline{z}_{0s} \varphi_{k\beta} \\ -\varphi_{j\alpha} \overline{z}_{0t} & \varphi_{j\alpha} \varphi_{k\beta} - \varphi_{j\alpha, k\beta}(\varphi - ||z_0||^2) \end{pmatrix},$$

(2.6)

where the upper left block is $(\varphi - ||z_0||^2)I^{(d_0)} + \overline{z}_{0s} \overline{z}_{0t}$ and the downer right block is a $d \times d$ submatrix. Now we should make some elementary determinant calculations. Indeed, fix $1 \leq s \leq m$, we know that

$$\sum_{t=1}^{m} \left( (\varphi - ||z_0||^2) \delta_{st} + \overline{z}_{0s} \overline{z}_{0t} \right) \frac{\overline{z}_{0t} \varphi_{k\beta}}{\varphi} = \overline{z}_{0s} \varphi_{k\beta} \sum_{t=1}^{m} -\varphi_{j\alpha} \overline{z}_{0t} \frac{\overline{z}_{0t} \varphi_{k\beta}}{\varphi} = - \frac{\varphi_{j\alpha} \varphi_{k\beta} ||z_0||^2}{\varphi}.$$  

(2.7)
Under the elementary transformations above, the matrix can be transformed into

\[
\begin{pmatrix}
\frac{1}{(\varphi - ||z_0||^2)^2} & (\varphi - ||z_0||^2)\delta_{st} + \overline{z}_0s \overline{z}_0t \\
-\varphi \overline{z}_0t & (\varphi - ||z_0||^2)\varphi \frac{\partial^2\varphi_j\overline{\varphi}_k}{\varphi^2}
\end{pmatrix}.
\]  

(2.8)

Let \( g^{D_i}_{\alpha \beta} \) be the \((\alpha, \beta)\)-entry of the Kähler-Einstein metric \( g^{D_i}_r \), i.e.

\[
g^{D_i}_{\alpha \beta} = \frac{\partial^2(- \log \varphi_j)}{\partial z_{i\alpha} \partial \overline{z}_{i\beta}} (z_i, \overline{z}_i).
\]  

(2.9)

Since

\[
\frac{\varphi_j \varphi_k - \varphi_j \varphi_k \varphi^2}{\varphi^2} = \begin{cases} 
g^{D_j}_{\alpha \beta} & j = k; \\
0 & j \neq k.
\end{cases}
\]  

(2.10)

This implies that the downer right block of (2.8) is a block diagonal matrix. If \( g^{D_i}_r \) is Kähler-Einstein, then there exists a real-valued pluriharmonic function \( f_i \) on \( D_i \) such that

\[
det g^{D_i}_r = \varphi^c_i e^{c_i f_i}.
\]  

(2.11)

In order to obtain the determinant of \( g^\Omega \), we recall a well known formulation:

\[
det(I^{(p)} + AB') = det(I^{(q)} + B'A),
\]  

where \( A \) is a \( p \times q \) matrix, \( B \) is a \( q \times p \) matrix. So we can get

\[
det((\varphi - ||z_0||^2)I^{(d_0)} + \overline{z}_0z_0) = (\varphi - ||z_0||^2)^{d_0} (1 + \frac{||z_0||^2}{\varphi - ||z_0||^2}) = (\varphi - ||z_0||^2)^{d_0-1} \varphi. \]  

(2.12)

Form (2.10) and (2.12), it follows that

\[
det(g^\Omega) = \frac{\varphi^{d+1}}{(\varphi - ||z_0||^2)^{d+d_0+1}} \prod_{i=1}^{m} det(g^{D_i}_{\alpha \beta}).
\]

Then, by (2.11) we obtain

\[
det(g^\Omega) = \frac{1}{(\varphi - ||z_0||^2)^{d+d_0+1}} \prod_{i=1}^{m} \varphi_i^{d+1+c_i} e^{c_i f_i}.
\]  

(2.13)

We complete the proof.

By using the standard formula of Ricci tensor, i.e.

\[
\text{Ric}_{ja,k\beta} = -\frac{\partial^2 \log \det g^\Omega}{\partial z_{ja} \partial \overline{z}_{k\beta}},
\]  

we can obtain the following lemma directly.
Lemma 2.2. Let $\text{Ric}_g$ be the Ricci tensor of $g^\Omega$. Then

$$\text{Ric}_g = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & \lambda_1 g^{D_1} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_m g^{D_m}
\end{pmatrix} - (d + d_0 + 1)g^\Omega, \quad (2.14)
$$

where $\lambda_i = d + 1 + c_i$. Here $c_i$ denotes the Ricci curvature of $g^{D_i}$.

Theorem 2.3. Let $\Omega$ be as in (2.1). If $g^{D_i}$, $1 \leq i \leq m$, are Kähler-Einstein metrics with Ricci curvatures $c_i$ respectively, then $g^\Omega$ is Einstein if and only if $c_i = -(d + 1)$ for all $1 \leq i \leq m$.

### 2.2 Extremal metric

In the following, we will try to obtain the criterion that $g^\Omega$ is extremal. Let $(M, g)$ be a $n$-dimensional Kähler manifold, and $(z_1, \ldots, z_n)$ be the local coordinate in a neighbourhood of $p \in M$. Let $s_g$ the scalar curvature of $g$, from Calabi’s result in [3], the extremal condition can be given by the following equations:

$$\frac{\partial}{\partial z_\alpha} \sum_{\beta=1}^{n} g^{\beta \gamma} \frac{\partial s_g}{\partial z_\beta} = 0, \quad (2.15)$$

for all $\alpha, \eta = 1, \cdots, n$.

Now we calculate the scalar curvature of $g^\Omega$ firstly. Let

$$g^\Omega = \begin{pmatrix} g_{j\alpha, k\beta} \end{pmatrix} \quad \text{(2.16)}$$

be the inverse matrix of $g^\Omega$, where $0 \leq j, k \leq m$ and $1 \leq \alpha \leq d_j$, $1 \leq \beta \leq d_k$. Let $g_\Omega^{D_i} = (g_{D_i}^{\alpha \beta})$ be the inverse matrix of $g^{D_i} = (g^{D_i}_{\alpha \beta})$ in (2.9). By the same method in Lemma 2.1 a direct computation implies

$$g_{j\alpha, k\beta}^\Omega = \begin{cases} \frac{\varphi - ||z_0||^2}{\varphi} g_{D_j}^{\alpha \beta} & j = k; \\
0 & j \neq k,
\end{cases} \quad (2.17)$$

where $j \neq 0, k \neq 0$. This shows that the downer right block of (2.16) is a $d \times d$ block diagonal matrix.

Lemma 2.4. Let $s_{g^\Omega}$ be the scalar curvature of $g^\Omega$, then

$$s_{g^\Omega} = \frac{\tau (\varphi - ||z_0||^2)}{\varphi} - (n + 1)n,$$

where $\tau = (d + 1)d + \sum_{i=1}^{m} c_i d_i$. Furthermore, $s_{g^\Omega}$ is constant if and only if $\tau = 0$. 


Proof. According to the formula of the scalar curvature, it follows that

$$s_{g^\Omega} = \sum_{i=1}^{m} \sum_{i \alpha, i \beta = 1}^{d_i} \lambda_i \varphi \frac{||z_0||^2}{\varphi} g_{\partial \Omega} \left( g_{\partial \Omega}^{d_i} - (d_d + d_0 + 1)(d_d + d_0) \right)$$

$$= \sum_{i=1}^{m} \lambda_i d_i \varphi \frac{||z_0||^2}{\varphi} - (d_d + d_0 + 1)(d_d + d_0).$$

where $\lambda_i = (d_d + 1 + c_i)$, and $c_i$ is the Ricci curvature of $g^{D_i}$. Let $\tau = \sum_{i=1}^{m} \lambda_i d_i$, then $\tau = (d_d + 1)d + \sum_{i=1}^{m} c_i d_i$. Hence, we can complete the proof.

**Theorem 2.5.** Let $\Omega$ be as in (2.1) If $g^{D_i}$ is a Kähler-Einstein metric with Ricci curvature $c_i$, then $g^\Omega$ in (2.3) is extremal if and only if its scalar curvature $s_{g^\Omega}$ is a constant.

Proof. By Calabi’s result, it suffices to show that $g^\Omega$ is not an extremal metric if $\tau \neq 0$. By (2.6), for any fixed $0 \leq t \leq d_d$ and $1 \leq (k \beta) \leq d_k$, $k \neq 0$, we know

$$z_{0t} = \left( \varphi - ||z_0||^2 \right)^2 \sum_{s=1}^{m} g_{0s, \partial \Omega} z_{0s} \quad \text{and} \quad ||z_0||^2 \varphi_{k \beta} = -(\varphi - ||z_0||^2)^2 \sum_{s=1}^{m} g_{0s, k \beta} z_{0s}, k \neq 0.$$

By Lemma 2.4 and the equations above, we can obtain

$$\frac{\partial s_{g}}{\partial z_{0t}} = -\tau z_{0t} \frac{\varphi}{\varphi} = -\tau \left( \varphi - ||z_0||^2 \right)^2 \sum_{s=1}^{m} g_{0s, \partial \Omega} z_{0s},$$

$$\frac{\partial s_{g}}{\partial z_{k \beta}} = \tau ||z_0||^2 \varphi_{k \beta} \frac{\varphi}{\varphi^2} = -\tau \left( \varphi - ||z_0||^2 \right)^2 \sum_{s=1}^{m} g_{0s, k \beta} z_{0s}, k \neq 0.$$

Hence, we have

$$\sum_{t=1}^{d_d} g_{\partial \Omega}^{0t, iT} \frac{\partial s_{g}}{\partial z_{0t}} + \sum_{k=1}^{d_k} \sum_{\beta=1}^{d_k} g_{k \beta, \partial \Omega} \frac{\partial s_{g}}{\partial z_{k \beta}}$$

$$= -\tau \left( \varphi - ||z_0||^2 \right)^2 \sum_{t=1}^{d_d} g_{\partial \Omega}^{0t, iT} \sum_{s=1}^{m} g_{0s, \partial \Omega} z_{0s} + \sum_{k=1}^{d_k} \sum_{\beta=1}^{d_k} g_{k \beta, \partial \Omega} \sum_{s=1}^{m} g_{0s, k \beta} z_{0s}$$

$$= -\tau z_{01} \left( \varphi - ||z_0||^2 \right)^2.$$

By comparing it with Calabi’s extremal condition (2.15), we know that $g^\Omega$ is not extremal if $\tau \neq 0$. Consequently, we complete the proof.

Immediately, Theorem 1.1 will be a direct consequence by taking $m = 1$.

**The proof of Theorem 1.1.** By Theorem 2.3 (i)$\Leftrightarrow$(iv). By Lemma 2.2 Lemma 2.4 (iv)$\Leftrightarrow$(iii). By Theorem 2.5 (iii)$\Leftrightarrow$(ii).
2.3 Some classical examples

Now we will show some classical examples of bounded pseudoconvex Hartogs domains as an application of our results in previous section. Recall some basic notions firstly. Let Aut(D) be the group of automorphisms of a bounded domain $D \subset \mathbb{C}^d$. $D$ is called homogeneous if there exists a map $\Phi \in \text{Aut}(\Omega)$ such that $\Phi(a) = b$ for two arbitrary points $a, b \in D$. Moreover, $D$ is called symmetric if for every point $a \in D$, there exists an automorphism $\Phi \in \text{Aut}(\Omega)$ such that $\Phi(a) = a$, $\Phi \circ \Phi = id$, and $a$ is an isolated point of the set $\{z \in D : \Phi(z) = z\}$. The irreducible bounded symmetric domains have been completely classified up to a biholomorphic isomorphism due to E. Cartan \cite{6}. Each bounded symmetric domain is biholomorphic to a Cartesian product of domains belongs to the following six Cartan types.

Let $M_{m,n}$ be the space of $m \times n$-complex matrices, $I$ be the unit $p \times p$-matrix, $\bar{z}$ be the conjugate matrix of $z$, $z^t$ be the transposed matrix of $z$. If a square matrix $A$ is positive definite, then we denote it by $A > 0$. The list of irreducible bounded symmetric domains and the corresponding generic norms is as follows.

**Type I** ($1 \leq m \leq n$): $D_I = \{z \in M_{m,n}(\mathbb{C}) : I - zz^t > 0\}$, $N(z, w) = det(I - z\bar{w})$.

**Type II** ($m = n \geq 5$): $D_{II} = \{z \in D_I : z = -z^t\}$, $N(z, w) = det(I + z\bar{w})$.

**Type III** ($m = n \geq 2$): $D_{III} = \{z \in D_I : z = z^t\}$, $N(z, w) = det(I - z\bar{w})$.

**Type IV** ($m \geq 5$): $D_{IV} = \{z \in \mathbb{C}^m : 1 - 2q(z, \bar{z}) + |q(z, z)|^2 > 0, |q(z, \bar{z})| < 1\}$,

$$N(z, w) = 1 - q(z, w) + q(z, z)q(w, w), \text{ where } q(z, w) = \sum_{j=1}^{m} z_j w_j.$$ 

**Type V:** $D_V = \{z \in M_{2,1}(O_C) : 1 - (z|z) + (z^t|z^t) > 0, 2 - (z|z) > 0\}$,

$$N(z, w) = 1 - (z|w) + (z^t|w^t).$$

**Type VI:** $D_{VI} = \{z \in M_{3,3}(O_C) : 1 - (z|z) + (z^t|z^t) - |\text{det } z|^2 > 0, 3 - 2(z|z) + (z^t|z^t) > 0, 3 - (z|z) > 0\}$,

$$N(z, w) = 1 - (z|w) + (z^t|w^t) - \text{det } z \text{det } w.$$ 

where $O_C = \mathbb{C} \otimes \mathbb{O}$ is complex 8 dimensional Cayley algebra. $M_{3,3}(O_C)$ is the space of $3 \times 3$ matrices with entries in the space $O_C$ of octonions over $\mathbb{C}$, which are Hermitian with respect to the Cayley conjugation. $z^t$ is the adjoint matrix in $M_{3,3}(O_C)$ and $(z|w)$ is the standard Hermitian product in $M_{3,3}(O_C)$. $M_{2,1}(O_C)$ is a subspace of $M_{3,3}(O_C)$.

The domains of types I – IV are classical, $D_V$ and $D_{VI}$ are the exceptional 16 and 27 dimensional domains. These domains are also called Cartan domains. Let $g_B^D$, $K(z, z)$, $r$, $\gamma$, $V(D)$ be the Bergman metric, Bergman kernel, rank, genus, volume of Cartan domain $D$ respectively, then

$$\det g_B^D = \gamma^d V(D) K(z, z). \quad (2.19)$$

Obviously, the Ricci curvature is $-1$. This result is also true for any bounded homogeneous domain. For Cartan domain, the connection between the generic norm $N(z, z)$ and the Bergman kernel $K(z, z)$ is

$$V(D) K(z, z) = N(z, z)^{-\gamma}. \quad (2.20)$$
The Wallach set can be described as follow.

\[ W(D) = \left\{ 0, \frac{a}{2}, \cdots, (r-1) \frac{a}{2} \right\} \cup \left( (r-1) \frac{a}{2}, \infty \right), \quad (2.21) \]

where \(a, b\) are invariant numbers. For more details we refer the reader to [2].

**Example 2.6.** Let \( D \subset \mathbb{C}^d \) be a Cartan domain. Let \( g^D \) be the metric generated by the function \(-\mu \log N(z, z), \mu > 0\). Thus \( g^D = \frac{\mu}{2} g^D_B \) is a Kähler-Einstein metric with Ricci curvature \(-\frac{\mu}{2}\). Now, we take \( \varphi(z) = N(z, z)^\mu \), then we get the so-called Cartan-Hartogs domain introduced by Yin and Roos [22]:

\[ \Omega_{CH} = \left\{ (z_0, z) \in \mathbb{C}^{d_0} \times D : ||z_0||^2 < N(z, z)^\mu, \mu > 0 \right\}. \quad (2.22) \]

It is homogeneous if and only if \( \Omega_{CH} \) is a ball, i.e. \( D = \mathbb{B}^d \) and \( \mu = 1 \) (see [1]). Let \( g^{\Omega_{CH}} \) be the metric generated by \(-\log(N(z, z)^\mu - ||z_0||^2)\). By Theorem 1.1, \( g^{\Omega_{CH}} \) is extremal if and only if it is Kähler-Einstein, i.e. \( \mu = \frac{2}{d+1} \). This is coincide with Loi and Zedda’s results in [20]. Now we consider a Hartogs domain over Cartan-Hartogs domain.

\[ \tilde{\Omega}_{CH} = \left\{ (z_0, z, z) \in \mathbb{C}^{d_0} \times \mathbb{C}^d \times \Omega_{CH} : ||z_0||^2 < \varphi(z) \right\}. \quad (2.23) \]

where \(-\log \varphi(z) = -\log(N(z, z)^\mu - ||z_0||^2)\) is a strictly plurisubharmonic exhaustion function on \( \Omega_{CH} \). Hence, \(-\log(N(z, z)^\mu - ||z_0||^2 - ||z_0||^2)\) generates a complete Kähler-Einstein metric with Ricci curvature \(-\frac{\mu}{2} \cdot 2\). In fact, \( \tilde{\Omega}_{CH} \) is also a Cartan-Hartogs domain with \((d_0 + d_0)\)-dimensional fibers, i.e.

\[ \tilde{\Omega}_{CH} = \left\{ (z_0, z, z) \in \mathbb{C}^{d_0} \times \mathbb{C}^d \times D : ||z_0||^2 + ||z_0||^2 < N(z, z)^\mu \right\}. \quad (2.24) \]

**Example 2.7.** Let \( D \subset \mathbb{C}^d \) be a bounded homogeneous domain, \( K(z, z) \) be the Bergman kernel, \( V(D) \) be the volume of \( D \). Let \( g^D \) be the metric generated by \( \log K(z, z)^\nu \) with \( \nu > 0 \). Thus \( g^D = \nu g^D_B \) is a Kähler-Einstein metric with the Ricci curvature \(-\frac{\nu}{6}\). We take \( \varphi(z) = K(z, z)^{-\nu} \), then we get the so-called Bergman-Hartogs domain:

\[ \Omega_{BH} = \left\{ (z_0, z) \in \mathbb{C}^{d_0} \times D : ||z_0||^2 < K(z, z)^{-\nu}, \nu > 0 \right\}. \quad (2.25) \]

Let \( g^{\Omega_{BH}} \) be the metric generated by \(-\log(K(z, z)^{-\nu} - ||z_0||^2)\). Then \( g^{\Omega_{BH}} \) is extremal if and only if it is Kähler-Einstein, i.e. \( \nu = \frac{1}{d+1} \).

**Example 2.8.** Fock-Bargmann-Hartogs domain:

\[ \Omega_{FBH} = \left\{ (z_0, z) \in \mathbb{C}^{d_0} \times \mathbb{C}^d : ||z_0||^2 < e^{-\mu ||z||^2}, \mu > 0 \right\}. \quad (2.26) \]

Although this domain is unbounded, our result is also valid. The metric of \( \mathbb{C}^d \) given by \(-\log \varphi = \mu ||z||^2\) is flat, thus the Kähler metric whose Kähler potential is \(-\log(e^{-\mu ||z||^2} - ||z_0||^2)\) is not Einstein.
3 Kähler immersions in complex space forms

In this section, we will study the existence of Kähler immersions of bounded pseudoconvex Hartogs domains into complex space forms. The complex space forms are Kähler manifolds of constant holomorphic sectional curvatures. Assume that they are complete and simply connected. According to the sign of the constant holomorphic sectional curvature, there are three types:

1. Complex Euclidean space \((\mathbb{C}^N, g_0)\), \(N \leq +\infty\), where \(g_0\) denotes the flat metric. Here \(\mathbb{C}^\infty\) is the complex Hilbert space \(\ell^2(\mathbb{C})\) consisting of sequences \(z_j \in \mathbb{C}\), \(j = 1, \cdots\), such that \(\sum_{j=1}^{\infty} |z_j|^2 < +\infty\).

2. Complex hyperbolic space \(\mathbb{CH}^N, N \leq +\infty\), namely the unit ball in \(\mathbb{C}^N\), \(\sum_{j=1}^{N} |z_j|^2 < +\infty\) endowed with the hyperbolic metric \(g_{hyp}\) of holomorphic sectional curvature being \(-4\), whose associated Kähler form is

\[
\omega_{hyp} = -\frac{\sqrt{-1}}{2} \partial \overline{\partial} \log \sum_{j=1}^{N} (1 - |z_j|^2).
\]

3. Complex projective space \(\mathbb{CP}^N, N \leq +\infty\), with the Fubini-Study metric \(g_{FS}\) of holomorphic sectional curvature being \(4\). If \(\omega\) denotes the Kähler form associated to \(g_{FS}\). Let \([Z_0, \cdots, Z_N]\) be the homogeneous coordinates in \(\mathbb{CP}^N\). In the affine chart \(U_0 = \{Z_0 \neq 0\}\) endowed with coordinates \((z_1, \cdots, z_N)\), \(z_j = \frac{Z_j}{Z_0}\),

\[
\omega_{FS} = -\frac{\sqrt{-1}}{2} \partial \overline{\partial} \log \sum_{j=1}^{N} (1 + |z_j|^2).
\]

3.1 Some criterions

A fundamental criterion is Calabi’s criterion which is based on the diastatic function. Let us introduce the definition firstly. Let \((M, g)\) be a \(n\)-dimensional Kähler manifold. In a local complex coordinate \((U, z)\), the Kähler form \(\omega = \frac{\sqrt{-1}}{2} \partial \overline{\partial} \Phi\), where \(\Phi\) is a local Kähler potential. If \((M, g)\) admits a Kähler immersion into a complex space from, Calabi proved that \(g\) must be real analytic (see Theorem 8 in [5]). In this case, the Kähler potential \(\Phi(z)\) can be expressed as a power series in the \(2n\) real variables in \(U\). Substitute \(\Phi(z, \overline{z})\) for \(\Phi(z)\). Let \(p\) and \(q\) denote two arbitrary points in \(U\), \(z(p)\) and \(z(q)\) denote the local complex coordinate of \(p\) and \(q\) respectively, then \(\Phi(z(p), \overline{z(q)})\) is a real analytic function in \(U \times \overline{U}\). Then Calabi introduced the diastatic function:

\[
D(p, q) = \Phi(z(p), \overline{z(p)}) + \Phi(z(q), \overline{z(q)}) - \Phi(z(p), \overline{z(q)}) - \Phi(z(q), \overline{z(p)}).
\]  

One of the elementary properties of the diastatic function is that it is uniquely determined by the Kähler metric and independent of the local Kähler potential function (see [5]). Obviously, \(D(p, q) = D(q, p)\). Suppose that the origin \(o \in U\), then \(D(o, q)\) is a special Kähler
potential that determined by the diastatic function. In a neighbourhood of the origin \( o \), the power series of \( D(o, q) \) is

\[
D_o(q) = D(o, q) = \sum_{\alpha, \beta \geq 0} a_{\alpha \beta} z^\alpha \overline{z}^\beta, \quad (3.2)
\]

where the multi-indexes \( \alpha = (\alpha_1, \cdots, \alpha_n), \beta = (\beta_1, \cdots, \beta_n) \), \( z^\alpha = \prod_{j=1}^{n} (z_j)^{\alpha_j}, \overline{z}^\beta = \prod_{j=1}^{n} (\overline{z}_j)^{\beta_j} \).

Now define an ordering for the set of multi-indexes. Consider two arbitrary different multi-indexes \( \alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n), \tilde{\alpha} = (\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) \).

(1) If \( |\alpha| \neq |\tilde{\alpha}| \), then \( \alpha > \tilde{\alpha} \iff |\alpha| > |\tilde{\alpha}| \);

(2) If \( |\alpha| = |\tilde{\alpha}| \), then \( \alpha > \tilde{\alpha} \iff \exists 1 \leq j \leq n \text{ such that } \alpha_j < \tilde{\alpha}_j \), and \( \alpha_i = \tilde{\alpha}_i \) for any \( 1 \leq i \leq j - 1 \).

Under the ordering, the multi-indexes can be denoted by \( m_0 = (0, 0 \cdots, 0), m_1 = (1, 0 \cdots, 0), \cdots, m_n = (0, \cdots, 0, 1), \cdots \). The diastatic function can be described in terms of Bochner coordinate and

\[
D_o(q) = D(o, q) = \sum_{j,k=0}^{\infty} a_{j,k}(z)^{m_j}(\overline{z})^{m_k},
\]

where the multi-index \( m_j = (m_{j,1}, m_{j,2}, \cdots, m_{j,n}), m_k = (m_{k,1}, m_{k,2}, \cdots, m_{k,n}) \), \( |m_j| = |m_{j,1} + m_{j,2} + \cdots + m_{j,n}|, |m_k| = |m_{k,1} + m_{k,2} + \cdots + m_{k,n}| \).

On this complex manifold \( M \), Calabi defined two new Kähler metrics whose Kähler potentials are \( e^{D_o} - 1, 1 - e^{-D_o} \) respectively. Their power series can be written as follows.

\[
e^{D_o} - 1 = \sum_{j,k \geq 0} b_{jk}(g) z^{m_j} \overline{z}^{m_k}, \quad 1 - e^{-D_o} = \sum_{j,k \geq 0} c_{jk}(g) z^{m_j} \overline{z}^{m_k}, \quad (3.3)
\]

Let us introduce two definitions before giving Calabi’s criterion.

**Definition 3.1 (Calabi [5]).** A Kähler immersion \( f \) of \( (M, g) \) into \( \mathbb{C}^N \) (resp. \( \mathbb{CP}^N \) or \( \mathbb{CP}_N^N \)) is said to be full if \( f(M) \) cannot be contained in any complex totally geodesic hypersurface of \( \mathbb{C}^N \) (resp. \( \mathbb{CP}^N \) or \( \mathbb{CP}_N^N \)).

**Definition 3.2 (Calabi [5]).** The Kähler metric \( g \) on a complex manifold \( M \) is resolvable (resp. 1-resolvable or \(-1\)-resolvable) of rank \( N \) at \( p \) if the \( \infty \times \infty \) matrix \( a_{jk}(g) \) (resp. \( b_{jk}(g) \) or \( c_{jk}(g) \)) given by formula (\ref{2}) (resp. (\ref{3})) is positive semidefinite and of rank \( N, N \leq \infty \).

In [9], Di Scala and Loi reorganized Calabi’s results as follows.

**Theorem 3.3 (Calabi’s criterion).** Let \( M \) be a complex manifold endowed with a real analytic Kähler metric \( g \).

(i) If \( g \) is resolvable (resp. 1-resolvable or \(-1\)-resolvable) of rank \( N \) at \( p \in M \) then it is resolvable (resp. 1-resolvable or \(-1\)-resolvable) of rank \( N \) at every point in \( M \).

(ii) A neighborhood of a point \( p \) admits a (full) Kähler immersion into \( \mathbb{C}^N \) (resp. \( \mathbb{CP}^N \) or \( \mathbb{CP}_N^N \)) if and only if \( g \) is resolvable (resp. 1-resolvable or \(-1\)-resolvable) of rank at most (exactly) \( N \) at \( p \).
(iii) Two full Kähler immersions into \( \mathbb{C}^N \) (resp. \( \mathbb{C}P^N \) or \( \mathbb{C}H^N \)) are congruent under the isometry group of \( \mathbb{C}^N \) (resp. \( \mathbb{C}P^N \) or \( \mathbb{C}H^N \)).

**Theorem 3.4** (Calabi’s criterion). Let \( (M, g) \) be a simply connected Kähler manifold. If a neighborhood of a point \( p \in M \) can be Kähler immersed into a complex space form \( (S, G) \) then the whole \( (M, g) \) admits a Kähler immersion into \( (S, G) \).

Besides Calabi’s criterion, we also need the follow theorems about the relations between the Kähler submanifolds of complex space forms.

**Lemma 3.5** (Umehara [21]). If a Kähler manifold \( (M, g) \) admits a Kähler immersion into \( \mathbb{C}^N \), \( N < \infty \), then it can not be Kähler immersed into any finite dimensional complex hyperbolic space or complex projective space. If it can be Kähler immersed into \( (\mathbb{C}H^N, g_{h_{\text{hyp}}}) \), \( N < \infty \), then it can not be into any finite dimensional complex projective space.

**Lemma 3.6** (Di Scala, Ishi, Loi [10]). If a Kähler manifold \( (M, g) \) admits a Kähler immersion into \( (\mathbb{C}H^N, g_{h_{\text{hyp}}}) \), \( N \leq \infty \), then it also can be Kähler immersed into \( (\mathbb{C}^\infty, g_0) \).

**Lemma 3.7** (Zedda [25]). A Kähler manifold \( (M, h g^M) \) admits a local Kähler immersion into \( (\mathbb{C}P^N, g_{FS}) \) for all \( h > 0 \) if and only if \( (M, g^M) \) admits a local Kähler immersion into \( (\mathbb{C}^\infty, g_0) \).

### 3.2 The diastatic function of \( (\Omega, g^\Omega) \)

Let \( \Omega \) be as in (1.2). Let \( g^D \) and \( g^\Omega \) be Kähler metrics whose Kähler potentials are \( -\log \varphi(z) \) and \( -\log(\varphi(z) - ||z_0||^2) \) respectively. The following lemma shows a relation between their diastatic functions.

**Lemma 3.8.** Let \( h \) be a positive number. If \( -\log \varphi \) is the special Kähler potential that determined by the diastatic function of \( g^D \), then \( -h \log(\varphi(z) - ||z_0||^2) \) is the special Kähler potential that determined by the diastatic function of \( h g^\Omega \).

**Proof.** Let \( \Phi \) be the extension of \( -\log \varphi \) on \( U \times \overline{U} \), i.e. \( \Phi(z, \overline{z}) = -\log \varphi(z) \), then

\[
D(z, w) = \Phi(z, \overline{w}) - \Phi(z, \overline{w}) - \Phi(w, \overline{z}).
\]

Since \( D(0, w) = -\log \varphi(w) \), we know \( \Phi(0, 0) - \Phi(0, \overline{w}) - \Phi(w, 0) = 0 \). Let \( \tilde{\Phi} \) be the extension of \( -h \log(\varphi(z) - ||z_0||^2) \) that satisfies

\[
\tilde{\Phi}((0, 0), (\overline{z}_0, \overline{z})) = -h \log(\varphi(z) - ||z_0||^2) = -h \log(e^{-\Phi(z, \overline{z})} - ||z_0||^2),
\]

then \( \tilde{\Phi}((0, 0), (\overline{w}_0, \overline{w})) = -h \log(e^{-\Phi(z, \overline{w})} - ||\overline{w}_0||^2) \). Hence,

\[
D((0, 0), (w_0, w)) = -h \log(\varphi(w) - ||w_0||^2).
\]

We complete the proof. \( \square \)
In the following sections, we will focus on the case that Ω contains the origin and Ω is circular with the origin. It implies that D is also circular with the origin and φ(e√−1θz) = φ(z). Conversely, if D is circular with the origin and φ(e√−1θz) = φ(z), then Ω is circular with the origin.

### 3.3 Immersion in complex Euclidean space

Let f : (M, g^M) → (C^N, g_0), N ≤ ∞, be a Kähler immersion, then √hf, h > 0, gives a Kähler immersion of (M, h^M) into (C^N, g_0). There are no difference to prove one of them.

**Theorem 3.9.** Let Ω be as in (1.2). Suppose that Ω is a simply connected circular domain with center zero and the function − log φ is the special Kähler potential determined by the diastatic function of g^Ω. Then (Ω, g^Ω) admits a full Kähler immersion into (C^∞, g_0) if and only if (D, g^D) admits a Kähler immersion into (C^N, g_0), N ≤ ∞.

**Proof.** Although Ω is a bounded pseudoconvex Hartogs domain with d_0-dimensional fibers, it suffices to prove the case d_0 = 1. In fact, suppose it is true for d_0 = 1, · · · , k. Define

Ω_k = \{(z_0, · · · , z_k, z) ∈ C^k × D : |z_0|^2 + · · · + |z_k|^2 < φ(z)\}. \tag{3.4}

The Kähler potential of g^Ω_k is − log(φ(z) − |z_0|^2 − · · · − |z_k|^2). Let (D, g^D) be a Kähler submanifold of (C^N, g_0). By the assumption, (Ω_k, g^Ω_k) is a full Kähler submanifold of (C^N, g_0). We now prove it is also true for d_0 = k + 1. Note that Ω_k+1 can be written by the following equivalent form.

Ω_k+1 = \{(z_0(k+1), z_0, · · · , z_k, z) ∈ C × Ω_k : |z_0(k+1)|^2 < φ(z) − |z_0|^2 − · · · − |z_k|^2\}.

Hence, Ω_k+1 is a bounded pseudoconvex domain over Ω_k with 1-dimensional fibers. By the assumption, (Ω_k+1, g^Ω_k+1) is a full Kähler submanifold of (C^∞, g_0). Conversely, if (Ω_k+1, g^Ω_k+1) is a full Kähler submanifold of (C^∞, g_0), then (Ω_k, g^Ω_k) is a full Kähler submanifold of (C^∞, g_0). Moreover, (D, g^D) is a Kähler submanifold of (C^N, g_0). Thus we prove that the result is also true for any finite dimensional fibers by induction.

Now we assume that d_0 = 1. Let o = (0, o_1) be the origin of C × C^d, η = (z_0, z) = (z_0, z_1, · · · , z_d) be the coordinate of the point q ∈ Ω. Consider the domain Ω with Kähler metric g^Ω whose globally defined Kähler potential around the origin o is

D(o, q) = − log(φ(z) − |z_0|^2). \tag{3.5}

Let p_1 ∈ D be the projection point of p, then z is the complex coordinate of p_1. Let D(o_1, q_1) = − log φ be the diastatic function for g^D around the origin. By Lemma 3.8 we know D(o, q) is the diastatic function for g^Ω around the origin. The power expansion

D(o, q) = \sum_{j,k=0}^∞ a_{j,k}(η)^m_j (\overline{η})^m_k. \tag{3.6}

The property of the matrix of coefficients can be described as follows: the elements of the row vector have the same multi-index m_j, and |m_k| increases with the increase of column
number; the elements of the column vector have the same multi-index $m_k$, and $|m_j|$ increases with the increase of row number; The matrix of coefficient can be given by the following block matrix.

$$
(a_{j,k}) = \begin{pmatrix}
A_{0,0} & A_{0,1} & A_{0,2} & \cdots \\
A_{1,0} & A_{1,1} & A_{1,2} & \cdots \\
A_{2,0} & A_{2,1} & A_{2,2} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix},
$$

(3.7)

where the element of matrix $A_{s,t}(s, t \in \mathbb{N})$ satisfies that $|m_j| = s$ and $|m_k| = t$.

The following observations tell us that the coefficients which satisfy some conditions are zero. This can be proved as follows.

(1) Take the transformation $\Psi(z_{01}, z) = (e^{i\theta} z_{01}, z)$, by the expression (3.5) of $D(o, q)$, we know $D(o, q)$ is invariant, i.e.

$$
\sum_{j,k=0}^{\infty} a_{j,k}(\eta)^m_j (\eta)^m_k = \sum_{j,k=0}^{\infty} a_{j,k}(\eta)^{m_j,1},
$$

then the coefficient $a_{j,k} = 0$ if $m_{j,1} \neq m_{k,1}$.

(2) Take the transformation $\Psi(z_{01}, z) = (z_{01}, e^{i\theta} z)$, then $D(o, q) = D(0, \Psi(q))$, i.e.,

$$
\sum_{j,k=0}^{\infty} a_{j,k}(\eta)^m_j (\eta)^m_k = \sum_{j,k=0}^{\infty} a_{j,k}(\eta)^{m_j,1},
$$

This implies $a_{j,k} = 0$ if $|m_{j,2} + \cdots + m_{j,d+1}| \neq |m_{k,2} + \cdots + m_{k,d+1}|$.

One direct result is that the block matrix (3.7) is a block diagonal matrix. In fact, every element of $A_{s,t}(s \neq t)$ satisfies $|m_j| \neq |m_k|$. This means at list one of the following inequalities is true, $m_{j,1} \neq m_{k,1}$ or $|m_{j,2} + \cdots + m_{j,d+1}| \neq |m_{k,2} + \cdots + m_{k,d+1}|$. Thus $A_{s,t} = 0$ for $s \neq t$. Hence, the matrix (3.7) can be written as

$$
(a_{jk}) = \begin{pmatrix}
A_{0,0} & 0 & 0 & \cdots \\
0 & A_{1,1} & 0 & \cdots \\
0 & 0 & A_{2,2} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix},
$$

where $A_{0,0} = -\log \varphi(0) = D(o_1, o_1)$ and

$$
A_{i,i} = \begin{pmatrix}
A_{z_{01}(i)}(0) & 0 & \cdots & 0 \\
0 & A_{z_{01}(i-1)}(0) & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & A_{z_{01}(i)}(0)
\end{pmatrix},
$$

and $A_{z_{01}(\sigma)}(0), (\sigma = 0, 1, \cdots , i)$ contains derivatives $\partial(z_{01}z)^{m_j}, \partial(z_{01}z)^{m_k}$ of order $2i$ with $|m_j| = |m_k| = i$ such that $m_{j,1} = m_{k,1} = \sigma$. This also implies the positive definite of matrix $(b_{j,k})$ is determined by metrics $A_{z_{01}(\sigma)}(0), \sigma = 0, 1, \cdots , i$, where $i = 1, 2, \cdots , \infty$.  

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For convenience, define multi-index \( \alpha_j = (m_{j,2}, \cdots, m_{j,n}) \). Then \( m_j = (m_{j,1}, \alpha_j) \), \( m_k = (m_{k,1}, \alpha_k) \)

\[
\sum_{j,k=0}^{\infty} a_{j,k}(\eta)^{m_j}(\overline{\eta})^{m_k} = \sum_{j,k=0}^{\infty} a_{j,k}(z_{01})^{m_{j,1}}(\overline{z}_{01})^{m_{k,1}}(\overline{\tau})^{\alpha_k},
\]

where

\[
a_{j,k} = \frac{\partial^{m_j + |m_k|} D(o, q)}{\partial z_{01}^{m_{j,1}} \partial (z)^{m_{j,1}} \partial (\overline{z})^{\alpha_j} \partial (\overline{\tau})^{\alpha_k}} \bigg|_{(z_{01}, z) = 0}.
\]  

(3.8)

Now, we will study when \( A_{z_{01}(\sigma)}(0) \), \( \sigma = 0, 1, \cdots, i \) are positive semidefinite.

(i) \( A_{z_{01}(i)}(0) \) is always positive.

\[
a_{j,k} = \frac{\partial^{2i} D(o, q)}{\partial z_{01}^{2i} \partial (\overline{z})^{2i}} \bigg|_{\eta = 0} = \Gamma(i)(i + 1)(\varphi(z) - |z_{01}|^2)^{-i} \bigg|_{\eta = 0} > 0.
\]  

(3.9)

(ii) Consider the matrices \( A_{z_{01}(\sigma)}(0) \), \( \sigma = 1, 2, \cdots, i - 1 \). By direct computation,

\[
\frac{\partial^\sigma D(o, q)}{\partial z_{01}^\sigma} = \Gamma(\sigma)(\varphi(z) - |z_{01}|^2)^{-\sigma} z_{01}^{\sigma},
\]

\[
\frac{\partial^{2\sigma} D(o, q)}{\partial z_{01}^{\sigma} \partial (\overline{z})^{\sigma}} = \Gamma(\sigma) \sum_{k+j = \sigma} \binom{\sigma}{k} \frac{\partial^k (\varphi(z) - |z_{01}|^2)^{-\sigma}}{\partial z_{01}^k} \frac{\partial^j (\varphi(z) - |z_{01}|^2)^{-\sigma}}{\partial (\overline{z})^j}.
\]

Hence, we have

\[
a_{j,k} = \frac{\partial^{m_j + |m_k|} D(o, q)}{\partial (z)^{m_j} \partial (\overline{z})^{m_k}} \bigg|_{\eta = 0} = \Gamma(\sigma) \Gamma(\sigma + 1) \frac{\partial^{\alpha_j + |\alpha_k|} (\varphi(z) - |z_{01}|^2)^{-\sigma}}{\partial (z)^{\alpha_j} \partial (\overline{z})^{\alpha_k}} \bigg|_{\eta = 0}
\]

\[
= \Gamma(\sigma) \Gamma(\sigma + 1) \frac{\partial^{\alpha_j + |\alpha_k|} \varphi(z)^{-\sigma}}{\partial (z)^{\alpha_j} \partial (\overline{z})^{\alpha_k}} \bigg|_{z = 0}.
\]

Thus \( \varphi(z)^{-\sigma} - 1 = e^{|D(o_{1,p})} - 1 \). By Lemma 3.7, if \( (D, g_D) \) is a Kähler submanifold of \( (\mathbb{C}^N, g_0) \), then \( (D, \sigma g_D) \) is a Kähler submanifold of \( (\mathbb{C}^N, g_{FS}) \) for any \( \sigma > 0 \). By Calabi’s criterion, \( \sigma g_D \) is 1-resolvable at \( o_1 \in D \). This implies the metrics \( A_{z_{01}(\sigma)}(0) \), \( \sigma = 1, 2, \cdots, i - 1 \) are all positive semidefinite.

(iii) Consider the matrix \( A_{z_{01}(0)}(0) \). Because

\[
a_{j,k} = \frac{\partial^{m_j + |m_k|} D(o, q)}{\partial (z)^{m_j} \partial (\overline{z})^{m_k}} \bigg|_{\eta = 0} = \frac{\partial^{m_j + |m_k|} \log \varphi(z)}{\partial (z)^{m_j} \partial (\overline{z})^{m_k}} \bigg|_{z = 0}.
\]  

(3.10)

Notice that \( -\log \varphi(z) = D(o_1, p_1) \). By the same reason in (ii), we know the metric \( A_{z_{01}(0)}(0) \) is positive semidefinite if \( (D, g_D) \) admits a local Kähler immersion into \( (\mathbb{C}^N, g_0) \) in the neighbourhood of \( o_1 \).
In summary, if \((D, g^D)\) admits a Kähler immersion into \((\mathbb{C}^N, g_0)\), then \(g^\Omega\) is resolvable of rank \(\infty\) at \(o \in \Omega\). By Calabi’s criterion, \((\Omega, g^\Omega)\) admits a local full Kähler immersion into \((\mathbb{C}^\infty, g_0)\). Since the domain is simple connected, the immersion can be extended to a global one. Hence, \((\Omega, g^\Omega)\) is a full Kähler submanifold of \(\mathbb{C}^\infty\).

The converse of this theorem can be easily obtained by studying the leading minors in terms of (iii).

**Remark 3.10.** By this theorem and Lemma 3.7, \((\Omega, hg^\Omega)\) also admits a Kähler immersion of \((\mathbb{C}^\infty, g_{FS})\) for any \(h > 0\) if \((D, g^D)\) admits a Kähler immersion into \((\mathbb{C}^N, g_0)\).

### 3.4 Immersion in complex projective space

When the ambient space is \((\mathbb{C}^\infty, g_{FS})\), there is a special phenomenon: although we have know \((M, g^M)\) is a Kähler submanifold of \((\mathbb{C}^\infty, g_{FS})\), it is still hard to determine whether or not \((M, hg^M)\) is a Kähler submanifold of \((\mathbb{C}^\infty, g_{FS})\) for a positive number \(h\). In compact case, the condition implies that \((M, g^M)\) is also a Kähler submanifold of \((\mathbb{C}^N, g_{FS})\), \(N < \infty\). Thus \(h\) must be a positive integer. In noncompact case, Loi and Zedda shows that the irreducible bounded symmetric domain \(D\) equipped with Bergman metric admits an equivalent Kähler immersion into \((\mathbb{C}^\infty, g_{FS})\) if and only if \(h \gamma \in W(D) \setminus \{0\}\), where \(\gamma\) denotes the genus of \(D\). \(W(D)\) denotes the Wallach set (2.21). Our following result shows that the bounded circular pseudoconvex Hartogs domains preserve the similar property of the base.

**Theorem 3.11.** Let \(\Omega\) be as in (1.2) and \(h\) be a positive number. Suppose that \(\Omega\) is a simply connected circular domain with center zero and the function \(− \log \varphi\) is the special Kähler potential determined by the diastatic function of \(g^D\). Then \((\Omega, hg^\Omega)\) admits a full Kähler immersion into \((\mathbb{C}^\infty, g_{FS})\) if and only if \((D, (h + \sigma)g^D)\) admits a Kähler immersion into \((\mathbb{C}^N, g_{FS})\), \(N \leq \infty\), for all \(\sigma \in \mathbb{N}\).

**Proof.** By the same reason in the proof of Theorem 3.9, it suffices to prove the case \(d_0 = 1\). Let \(o = (o_0, o_1)\) be the origin of \(\mathbb{C} \times \mathbb{C}^d\), \(\eta = (z_{01}, z) = (z_{01}, z_1, \ldots, z_d)\) be the coordinate of the point \(q \in \Omega\). Consider the domain \(\Omega\) with Kähler metric \(hg^\Omega\), then the globally defined Kähler potential function around the origin \(o\) is

\[
D(o, q) = −h \log (\varphi(z) − |z_{01}|^2).
\]

By Lemma 3.8 we know \(D(o, q)\) is the diastatic function for \(hg^\Omega\) around the origin. The power expansion

\[
e^{D(o, q)} − 1 = (\varphi(z) − |z_{01}|^2)^{-h} − 1 = \sum_{j, k=0}^{\infty} b_{j,k}(\eta)^m_j (\bar{\eta})^m_k.
\]

The property of the matrix of coefficients can be described as follows: the elements of the row vector have the same multi-index \(m_j\), and \(|m_k|\) increases with the increase of column number; the elements of the column vector have the same multi-index \(m_k\), and \(|m_j|\) increases with the increase of row number; The matrix of coefficient can be given by the following block.
where the element of matrix $B_{s,t}(s, t \in \mathbb{N})$ satisfies that $|m_j| = s$ and $|m_k| = t$.

The matrix $(b_{j,k})$ has the similar properties of the matrix $(a_{j,k})$ in (3.7). Hence, the matrix (3.13) can be written as follows.

\[
(b_{j,k}) = \begin{pmatrix}
B_{0,0} & 0 & 0 & \cdots \\
0 & B_{1,1} & 0 & \cdots \\
0 & 0 & B_{2,2} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]

(3.13)

where $B_{0,0} = \varphi(0)^{-h} - 1 = e^{hD(o_1,o_1)} - 1$ and

\[
B_{i,i} = \begin{pmatrix}
B_{201(i)}(0) & 0 & 0 & \cdots & 0 \\
0 & B_{201(i-1)}(0) & 0 & \cdots & 0 \\
0 & 0 & B_{201(0)}(0) & \cdots & 0 \\
0 & 0 & 0 & \cdots & B_{201(0)}(0)
\end{pmatrix},
\]

and $B_{201(\sigma)}(0), (\sigma = 0, 1, \cdots, i)$ contains derivatives $\partial(z_{01}z)^{m_j}, \partial(z_{01}\overline{z})^{m_k}$ of order $2i$ with $|m_j| = |m_k| = i$ such that $m_{j,1} = m_{k,1} = \sigma$. This also implies the positive definite of matrix $(b_{j,k})$ is determined by metrics $B_{201(\sigma)}(0), \sigma = 0, 1, \cdots, i$, where $i = 1, 2, \cdots, \infty$. For convenience, define $\alpha_j = (m_{j,2}, \cdots, m_{j,n})$. Then $m_j = (m_{j,1}, \alpha_j), m_k = (m_{k,1}, \alpha_k)$ and

\[
\sum_{j,k=0}^{\infty} b_{j,k}(\eta)^{m_j}(\overline{\eta})^{m_k} = \sum_{j,k=0}^{\infty} b_{j,k}(z_{01})^{m_j,1}(z)^{\alpha_j}(\overline{z}_{01})^{m_k,1}(\overline{z})^{\alpha_k}.
\]

where

\[
b_{j,k} = \frac{\partial^{m_j,1+m_k,1}(\varphi(z) - |z_{01}|^2)^{-h} - 1}{\partial(z_{01})^{m_j,1}\partial(z)^{\alpha_j}\partial(\overline{z}_{01})^{m_k,1}\partial(\overline{z})^{\alpha_k}}|_{z_{01},z=0}.
\]

(3.14)

We now study under what conditions, $B_{201(\sigma)}(0), \sigma = 0, 1, \cdots, i$ can be positive semidefinite.

(i) $B_{201(i)}(0)$ is always positive.

\[
b_{j,k} = \frac{\partial^{2i}(\varphi(z) - |z_{01}|^2)^{-h}}{\partial z_{01}^{2i}\partial z_{01}^i} |_{\eta=0} = \frac{\Gamma(h+i)\Gamma(i+1)}{\Gamma(h)}(\varphi(z) - |z_{01}|^2)^{-(h+i)} |_{\eta=0} > 0.
\]

We now study under what conditions, $B_{201(\sigma)}(0), \sigma = 0, 1, \cdots, i$ can be positive semidefinite.
(ii) Consider the matrices $B_{2\sigma_1}(\sigma)(0)$, $\sigma = 1, 2, \ldots, i - 1$. By direct computation, we have

$$b_{j,k} = \frac{\partial^{m_j+|m_k|}(\varphi(z) - |z_{01}|^2)^{-h}}{\partial z^{m_j} \partial \bar{z}^{m_k}}|_{z=0} \Gamma(h + \sigma) \Gamma(\sigma + 1) \frac{\partial^{(\alpha_j + |\alpha_k|)(\varphi(z) - |z_{01}|^2)^{-h + \sigma}}}{\partial (\bar{z})^{\alpha_j} \partial \bar{z}^{\alpha_k}}|_{z=0}. $$

Since $\varphi(z)^{-h - 1} = e^{(h + \sigma)D(\alpha_1, \alpha_2)} - 1$. If $(D, (h + \sigma)g^D)$ is a Kähler submanifold of $(\mathbb{CP}^\infty, g_{FS})$ for $\sigma \in \mathbb{N}$, by Calabi’s criterion, the coefficient matrix of power series of the right side is positive semidefinite. This implies the metrics $B_{2\sigma_1}(\sigma)(0)$, $\sigma = 1, 2, \ldots, i - 1$ are all positive semidefinite.

(iii) Consider the matrix $B_{2\sigma_0}(0)$. Because

$$b_{j,k} = \frac{\partial^{m_j+|m_k|}(\varphi(z) - |z_{01}|^2)^{-h}}{\partial z^{m_j} \partial \bar{z}^{m_k}}|_{z=0} = \frac{\partial^{m_j+|m_k|}(\varphi(z) - |z_{01}|^2)^{-h}}{\partial z^{m_j} \partial \bar{z}^{m_k}}|_{z=0}. \quad (3.15)$$

Notice that $\varphi(z)^{-h - 1} = e^{HD(\alpha_1, \alpha_2)} - 1$. By the same reason in (ii), we know that the metric $B_{2\sigma_0}(0)$ is positive semidefinite if $(D, h^D)$ is a Kähler submanifold of $(\mathbb{CP}^\infty, g_{FS})$.

Finally, if $(D, (h + \sigma)g^D)$ admits a Kähler immersion into $(\mathbb{CP}^\infty, g_{FS})$ for all $\sigma \in \mathbb{N}$, then $(\Omega, h^D)$ admits a local full Kähler immersion into $(\mathbb{CP}^\infty, g_{FS})$. Since $\Omega$ is simple connected, the immersion can be extended to a global one.

The converse of this theorem can be easily obtained by (ii) and (iii).

**Remark 3.12.** By Theorem 3.11 we know $(\Omega, h^\Omega)$ is a Kähler submanifold of $(\mathbb{CP}^\infty, g_{FS})$, then $(\Omega, (h + \sigma)g^\Omega)$, $\sigma \in \mathbb{N}$, are all Kähler submanifolds of $(\mathbb{CP}^\infty, g_{FS})$. Suppose that there exists a positive number $h_0$ such $h_0g^D$ is not a Kähler submanifold of $(\mathbb{CP}^\infty, g_{FS})$. By Theorem 3.11 $(\Omega, h_0g^\Omega)$ is not a Kähler submanifold of $(\mathbb{CP}^\infty, g_{FS})$. By Lemma 3.7 $(\Omega, h_0g^\Omega)$ is not a Kähler submanifold of $(\mathbb{CP}^\infty, g_0)$. By Lemma 3.6 $(\Omega, h^\Omega)$ is not a Kähler submanifold of $(\mathbb{CP}^\infty, g_{hyp})$ for any $h > 0$.

### 3.5 Immersion in complex hyperbolic space

In this section, we deal with Kähler immersions of $(\Omega, h^\Omega)$ into complex hyperbolic space $(\mathbb{CH}^N, g_{hyp})$, $N \leq \infty$.

**Theorem 3.13.** Let $\Omega$ be as in (1.2). Suppose that $\Omega$ is a simply connected circular domain with center zero and the function $-\log \varphi$ is the special Kähler potential determined by the diastatic function of $g^D$. Then $(\Omega, h^\Omega)$ is a full Kähler submanifold of $(\mathbb{CH}^\infty, g_{hyp})$ if and only if $(D, h^D)$ is a Kähler submanifold of $(\mathbb{CH}^\infty, g_{hyp})$, $N \leq \infty$, and $0 < h \leq 1$.
Proof. Let $d_0 = 1$ and $o = (o_0, o_1)$ be the origin of $\mathbb{C} \times \mathbb{C}^d$, $\eta = (z_{01}, z) = (z_{01}, z_1, \cdots, z_d)$ be the coordinate of the point $q \in \Omega$. Consider the domain $\Omega$ with Kähler metric $h g^\Omega$, then the globally defined Kähler potential function around the origin $o$ is

$$D(o, q) = -h \log \left( \varphi(z) - |z_{01}|^2 \right).$$  \hspace{1cm} (3.16)

By Lemma $3.8$ we know $D(o, q)$ is the diastic function for $h g^\Omega$ around the origin. In Bochner coordinate,

$$1 - e^{-D(o, q)} = 1 - (\varphi(z) - |z_{01}|^2)^h = \sum_{j,k=0}^\infty c_{j,k}(\eta)^{m_j}(\overline{\eta})^{m_k}.$$ \hspace{1cm} (3.17)

The property of the matrix of coefficients can be described as follows: the elements of the row vector have the same multi-index $m_j$, and $|m_k|$ increases with the increase of column number; the elements of the column vector have the same multi-index $m_k$, and $|m_j|$ increases with the increase of row number; The matrix of coefficient can be given by the following block matrix.

$$(c_{j,k}) = \begin{pmatrix} C_{0,0} & C_{0,1} & C_{0,2} & \cdots \\ C_{1,0} & C_{1,1} & C_{1,2} & \cdots \\ C_{2,0} & C_{2,1} & C_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$ \hspace{1cm} (3.18)

where the element of matrix $C_{s,t}(s, t \in \mathbb{N})$ satisfies that $|m_j| = s$ and $|m_k| = t$. The matrix $(c_{j,k})$ has the similar properties of the matrix $(a_{j,k})$ in (3.7). Hence, the matrix (3.18) can be written as follows.

$$(c_{j,k}) = \begin{pmatrix} C_{0,0} & 0 & 0 & \cdots \\ 0 & C_{1,1} & 0 & \cdots \\ 0 & 0 & C_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $c_{0,0} = 1 - \varphi(0)^h = 1 - e^{-D(o_1, o_1)}$ and

$$C_{i,i} = \begin{pmatrix} C_{01}(i)(0) & 0 & \cdots & 0 \\ 0 & C_{01}(i-1)(0) & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & C_{01}(0)(0) \end{pmatrix},$$

and $C_{01}(\sigma)(0)$, $(\sigma = 0, 1, \cdots, i)$ contains derivatives $\partial(z_{01} z)^{m_j}, \partial(z_{01} \overline{z})^{m_k}$ of order $2i$ with $|m_j| = |m_k| = i$ such that $m_{j,1} = m_{k,1} = \sigma$. This also implies the positive definite of matrix $(c_{j,k})$ is determined by matrices $C_{01}(\sigma)(0), \sigma = 0, 1, \cdots, i$, where $i = 1, 2, \cdots, \infty$. For convenience, define $\alpha_j = (m_{j,2}, \cdots, m_{j,n})$. Then $m_j = (m_{j,1}, \alpha_j)$, $m_k = (m_{k,1}, \alpha_k)$ and

$$\sum_{j,k=0}^\infty c_{j,k}(\eta)^{m_j}(\overline{\eta})^{m_k} = \sum_{j,k=0}^\infty c_{j,k}(z_{01})^{m_j,1}(z)^{\alpha_j}(z_{01})^{m_k,1}(\overline{z})^{\alpha_k},$$

where

$$c_{j,k} = \frac{\partial^{m_j + m_k}(1 - (\varphi(z) - |z_{01}|^2)^h)}{\partial(z_{01})^{m_j,1}\partial(z)^{\alpha_j}\partial(z_{01})^{m_k,1}\partial(\overline{z})^{\alpha_k}}(z_{01} = 0).$$ \hspace{1cm} (3.19)
In the following two cases, we will study sufficient and necessary conditions for $C_{z_{01}(\sigma)}(0)$ are positive semidefinite.

(1) **The case that $h$ is a integer.**

(i) Consider $C_{z_{01}(i)}(0)$.

\[
C_{j,k} = \begin{cases} 
\frac{(-1)^i+i\Gamma(h+1)\Gamma(i+1)}{\Gamma(h-i+1)}(\varphi(z) - |z_{01}|^2)^{h-i}|_{\eta=0} & \text{for } i \leq h; \\
0 & \text{for } i > h. 
\end{cases} 
\tag{3.20}
\]

So we can get $C_{z_{01}(i)}(0) \geq 0$ for all $1 \leq i < \infty$ if and only if $h = 1$.

In the following, we only need to consider the case that $h = 1$.

(ii) The matrices $C_{z_{01}(\sigma)}(0) \equiv 0$ for $\sigma = 1, 2, \ldots, i-1$ when $h = 1$.

(iii) Consider the matrix $C_{z_{01}(0)}(0)$ when $h = 1$. Because

\[
c_{j,k} = -\frac{\partial^{m_j+|m_k|}(\varphi(z) - |z_{01}|^2)}{\partial(z)^{\alpha_j} \partial(\overline{z})^{\alpha_k}}|_{z_{01}=0} = -\frac{\partial^{m_j+|m_k|}\varphi(z)}{\partial(z)^{\alpha_j} (\overline{z})^{\alpha_k}}|_{z=0} 
\]

By Calabi’s criterion and the discussion above, $(\Omega, g^{D})$ admits a local Kähler immersion into $(\mathbb{C}^N, g_{hyp})$ if and only if $(D, g^{D})$ admits a local Kähler immersion into $(\mathbb{C}^N, g_{hyp})$, $N \leq \infty$. By Calabi’s criterion, we know the Kähler immersion can be extended to a global one.

(2) **The case that $h$ is not a integer.**

(i) Consider $C_{z_{01}(i)}(0)$.

\[
c_{j,k} = \begin{cases} 
\frac{(-1)^{2i}i\Gamma(h+1)\Gamma(i+1)}{\Gamma(h-i+1)}(\varphi(z) - |z_{01}|^2)^{h-i}|_{\eta=0} & \text{for } i < h; \\
\frac{(-1)^{2i}i\Gamma(h+1)\Gamma(i+1)}{\Gamma(h-i+1)}(\varphi(z) - |z_{01}|^2)^{h-i}|_{\eta=0} & \text{for } i > h. 
\end{cases} 
\]

If $0 < h < 1$, then $h < 1 \leq i$. Thus $c_{j,k} > 0$ for any $i \in \mathbb{N}^+$. If $1 < h < 2$, then $c_{j,k} > 0$ for $i \geq 2$. If $h > 2$, then $c_{j,k} < 0$ for $i = 2$. Hence, $C_{z_{01}(i)}(0)$ can be always non negative if and only if $0 < h < 1$.

(ii) Consider the matrices $C_{z_{01}(\sigma)}(0), \sigma = 1, 2, \ldots, i-1$, when $0 < h < 1$.

\[
c_{j,k} = \frac{(-1)^{2\sigma}\Gamma(h+1)\Gamma(\sigma-h)\Gamma(\sigma+1)}{\Gamma(h-\sigma)\Gamma(h+\sigma)} \frac{\partial^{\sigma+|\sigma_k|}\varphi(z)^{\sigma-\sigma}}{\partial(z)^{\alpha_j} \partial(\overline{z})^{\alpha_k}}|_{z=0}.
\]

Notice that $\varphi(z)^{\sigma-\sigma} - 1 = e^{\sigma-hD(o_1,p_1)} - 1$. By Lemma 3.6 and Lemma 3.7, $(\sigma-h)g^D$ is always 1-resolvable if $g^D$ is a Kähler submanifold of $(\mathbb{C}^N, g_{hyp})$. 

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(iii) Consider the matrix $C_{z_{01}(0)}(0)$ when $0 < h < 1$. Because

$$c_{j,k} = -\frac{\partial^{|m_j|+|m_k|}(\varphi(z) - |z_{01}|^2)^h}{\partial (z)^{\alpha_j} \partial (\overline{z})^{\alpha_k}} |_{z_{01},z=0} = -\frac{\partial^{|m_j|+|m_k|}(\varphi(z)^h)}{\partial (z)^{\alpha_j} \partial (\overline{z})^{\alpha_k}} |_{z=0}.$$

By (i) and Calabi’s criterion, $(\Omega, h g^\Omega)$ admits a local Kähler immersion in $\mathbb{C}H\infty$ if and only $(D, h g^D)$ admits a local Kähler immersion in $\mathbb{C}H^N, N \leq \infty$ for $0 < h < 1$. By Calabi’s criterion again, we know the Kähler immersion can be extended to a global one.

The converse of this theorem can be easily obtained by the discussion above.

Remark 3.14. By Lemma 3.6 if $(D, g^\Omega)$ is a Kähler submanifolds of $(\mathbb{C}H^N, g_{h_{hyp}}), N \leq \infty$, then $(D, g^D)$ is a Kähler submanifolds of $(C^\infty, g_0)$. Thus we can reduce the problem of studying the existence of Kähler immersions from $(\Omega, g^\Omega)$ into complex space forms to the case in Theorem 3.9.

By (i) in Theorem 3.9, Theorem 3.11 and Theorem 3.13, it implies that there does not exist a Kähler immersion of $(\Omega, g^\Omega)$ into any finite dimensional complex space forms. For the reader’s convenience, we summarize the results obtained so far by the following table.

Table 1: Conclusion

|   | $\mathbb{C}^n$ | $\mathbb{C}\infty$ | $\mathbb{CP}^n$ | $\mathbb{CP}\infty$ | $\mathbb{CH}^n$ | $\mathbb{CH}\infty$ |
|---|---------|---------|---------|---------|---------|---------|
| A | $\times$ | $\sqrt{\ }$ | $\times$ | $\times$ | $\times$ | $\times$ |
| B | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ |
| C | $\times$ | $\sqrt{\ }$ | $\times$ | $\sqrt{\ }$ | $\times$ | $\sqrt{\ }$ |

The notations A, B, C denote the sufficient conditions in Theorem 3.9, Theorem 3.11 and Theorem 3.13 respectively. $\sqrt{\ }$ denote “There exists a Kähler immersion”, $\times$ denote “There does not exist a Kähler immersion”. $-$ denote “Not sure”. By Table 1, we can obtain Theorem 1.2 immediately.

Note that the property of circular plays an important role in our proof. It makes the Bochner coordinate of the diastasis very simple. If we remove this property, then the problem will be too complicated to deal with. A possible way is to consider the minimal domain which is a generalization of circular domain. We will explore it in the near future.

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