HOLOMORPHIC HARMONIC MORPHISMS FROM COSYMPLECTIC ALMOST HERMITIAN MANIFOLDS

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Abstract. We study 4-dimensional orientable Riemannian manifolds equipped with a minimal and conformal foliation $\mathcal{F}$ of codimension 2. We prove that the two adapted almost Hermitian structures $J_1$ and $J_2$ are both cosymplectic if and only if $\mathcal{F}$ is Riemannian and its horizontal distribution $\mathcal{H}$ is integrable.

1. Introduction

The notion of a minimal submanifold of a given ambient space is of great importance in differential geometry. Harmonic morphisms $\phi : (M, g) \to (N, h)$ between Riemannian manifolds are useful tools for the construction of such objects. They are solutions to over-determined non-linear systems of partial differential equations determined by the geometric data of the manifolds involved. For this reason harmonic morphisms are difficult to find and have no general existence theory, not even locally.

For the existence of harmonic morphisms $\phi : (M, g) \to (N, h)$ it is an advantage that the target manifold $N$ is a surface i.e. of dimension 2. In this case the problem is invariant under conformal changes of the metric on $N^2$. Therefore, at least for local studies, the codomain can be taken to be the complex plane with its standard flat metric.

In this paper we are interested in 4-dimensional orientable Riemannian manifolds $(M^4, g)$ equipped with a minimal and conformal foliation $\mathcal{F}$ of codimension 2. These are important since they produce local complex-valued harmonic morphisms on $M$, see Section 2. Our following main result, gives a new relationship between the geometry of the foliation $\mathcal{F}$ and the cosymplecticity of both its adapted almost Hermitian structures.

Theorem 1.1. Let $(M^4, g)$ be a 4-dimensional orientable Riemannian manifold equipped with a minimal and conformal foliation $\mathcal{F}$ of codimension 2. Then the corresponding adapted almost Hermitian structures $J_1$ and $J_2$ are both cosymplectic if and only if $\mathcal{F}$ is Riemannian and its horizontal distribution $\mathcal{H}$ is integrable.
For the general theory of harmonic morphisms between Riemannian manifolds we refer to the excellent book [2] and the regularly updated on-line bibliography [4].

2. Harmonic morphisms and minimal conformal foliations

Let $M$ and $N$ be two manifolds of dimensions $m$ and $n$, respectively. A Riemannian metric $g$ on $M$ gives rise to the notion of a Laplacian on $(M, g)$ and real-valued harmonic functions $f : (M, g) \to \mathbb{R}$. This can be generalized to the concept of harmonic maps $\phi : (M, g) \to (N, h)$ between Riemannian manifolds, which are solutions to a semi-linear system of partial differential equations, see [2].

**Definition 2.1.** A map $\phi : (M, g) \to (N, h)$ between Riemannian manifolds is called a harmonic morphism if, for any harmonic function $f : U \to \mathbb{R}$ defined on an open subset $U$ of $N$ with $\phi^{-1}(U)$ non-empty, $f \circ \phi : \phi^{-1}(U) \to \mathbb{R}$ is a harmonic function.

The following characterization of harmonic morphisms between Riemannian manifolds is due to Fuglede and T. Ishihara. For the definition of horizontal (weak) conformality we refer to [2].

**Theorem 2.2.** [3, 7] A map $\phi : (M, g) \to (N, h)$ between Riemannian manifolds is a harmonic morphism if and only if it is a horizontally (weakly) conformal harmonic map.

Let $(M, g)$ be a Riemannian manifold, $\mathcal{V}$ be an involutive distribution on $M$ and denote by $\mathcal{H}$ its orthogonal complement distribution on $M$. As customary, we also use $\mathcal{V}$ and $\mathcal{H}$ to denote the orthogonal projections onto the corresponding subbundles of $TM$ and denote by $\mathcal{F}$ the foliation tangent to $\mathcal{V}$. The second fundamental form for $\mathcal{V}$ is given by

$$B^\mathcal{V}(U, V) = \frac{1}{2} \mathcal{H}(\nabla_U V + \nabla_V U) \quad (U, V \in \mathcal{V}),$$

while the second fundamental form for $\mathcal{H}$ satisfies

$$B^\mathcal{H}(X, Y) = \frac{1}{2} \mathcal{V}(\nabla_X Y + \nabla_Y X) \quad (X, Y \in \mathcal{H}).$$

The foliation $\mathcal{F}$ tangent to $\mathcal{V}$ is said to be conformal if there is a vector field $V \in \mathcal{V}$ such that

$$B^\mathcal{H} = g \otimes V,$$

and $\mathcal{F}$ is said to be Riemannian if $V = 0$. Furthermore, $\mathcal{F}$ is said to be minimal if trace $B^\mathcal{V} = 0$ and totally geodesic if $B^\mathcal{V} = 0$. This is equivalent to the leaves of $\mathcal{F}$ being minimal and totally geodesic submanifolds of $M$, respectively.

It is easy to see that the fibres of a horizontally conformal map (resp. Riemannian submersion) give rise to a conformal foliation (resp. Riemannian
foliation). Conversely, the leaves of any conformal foliation (resp. Riemannian foliation) are locally the fibres of a horizontally conformal map (resp. Riemannian submersion), see [2].

The next result of Baird and Eells gives the theory of harmonic morphisms, with values in a surface, a strong geometric flavour.

**Theorem 2.3.** [1] Let \( \phi : (M^m, g) \to (N^2, h) \) be a horizontally conformal submersion from a Riemannian manifold to a surface. Then \( \phi \) is harmonic if and only if \( \phi \) has minimal fibres.

### 3. Cosymplectic Almost Hermitian Structures

An almost Hermitian manifold \((M, g, J)\) is said to be cosymplectic if its almost complex structure \(J\) is divergence-free i.e.

\[
\delta J_k = \text{div} J = \sum_{k=1}^m (\nabla X_k) J(X_k) = 0,
\]

where \(\{X_1, \ldots, X_m\}\) is any local orthonormal frame for the tangent bundle \(TM\) of \(M\). As an application of a well-known result from [8] of A. Lichnerowicz, we have the following useful result.

**Proposition 3.1.** [6] Let \( \phi : (M, g, J) \to N \) be a holomorphic map from an almost Hermitian manifold to a Riemann surface. Then \( \phi \) is a harmonic morphism if and only if \( d\phi(\delta J) = 0 \).

In the light of the above discussion, the result of Proposition 3.1 has an equivalent formulation in terms of foliations.

**Proposition 3.2.** Let \((M, g, J)\) be an almost Hermitian manifold and \(F\) be a holomorphic minimal conformal foliation on \(M\) of codimension 2. Then \(F\) produces harmonic morphisms on \(M\) if and only if the divergence \(\delta J\) of the almost Hermitian structure \(J\) is vertical i.e. \(H\delta J = 0\).

We will now assume that \((M^4, g)\) is a 4-dimensional orientable Riemannian manifold equipped with a minimal and conformal foliation \(F\) of codimension 2. Then there exist, up to sign, exactly two almost Hermitian structures \(J_1\) and \(J_2\) on \(M\) which are adapted to the orthogonal decomposition \(TM = V \oplus H\) of the tangent bundle of \(M\). They are determined by

\[
J_1 X = Y, \quad J_1 Y = -X, \quad J_1 Z = W, \quad J_1 W = -Z, \\
J_2 X = Y, \quad J_2 Y = -X, \quad J_2 Z = -W, \quad J_2 W = Z,
\]

where \(\{X, Y, Z, W\}\) is any local orthonormal frame for the tangent bundle \(TM\) of \(M\) such that \(X, Y \in H\) and \(Z, W \in V\), respectively.

We are now ready to prove our main result stated in Theorem 1.1.

**Proof.** Let us assume that the almost complex structures \(J_1\) and \(J_2\) are both cosymplectic i.e. for \(k = 1, 2\) we have

\[
0 = \delta J_k
\]
\[\begin{align*}
\n&= (\nabla_X J)k(X) + (\nabla_Y Jk)(Y) + (\nabla_Z Jk)(Z) + (\nabla_W Jk)(W) \\
&= [X, Y] + (-1)^k[W, Z] - Jk(\nabla_X X + \nabla_Y Y + \nabla_Z Z + \nabla_W W).
\end{align*}\]

It now follows from Proposition 3.2 that
\[0 = \delta J_1 + \delta J_2 = V \delta J_1 + V \delta J_2 = 2V[X, Y].\]

This shows that the horizontal distribution \(\mathcal{H}\) is integrable. Then employing the fact that \(J_1\) is cosymplectic, we see that
\[J_1 V(\nabla_X X + \nabla_Y Y) = V J_1(\nabla_X X + \nabla_Y Y) = -V[W, Z] - V J_1(\nabla_Z Z + \nabla_W W) = -(\nabla_W Z, W)W + (\nabla_Z W, Z)Z - J_1 V(\nabla_Z Z, W + (\nabla_W W, Z)Z) = 0.\]

Further it follows from \(V[X, Y] = 0\) and \(V(\nabla_X X + \nabla_Y Y) = 0\) that \(V \delta J_2 = 0\) is equivalent to
\[V[W, Z] - V J_2(\nabla_Z Z + \nabla_W W) = 0.\]

The fact that \(\mathcal{F}\) is conformal implies that for each \(X \in \mathcal{H}\)
\[2B^\mathcal{H}(X, X) = B^\mathcal{H}(X, X) + B^\mathcal{H}(Y, Y) = V(\nabla_X X + \nabla_Y Y) = 0.\]

Since the second fundamental form \(B^\mathcal{H}\) of the horizontal distribution \(\mathcal{H}\) is symmetric the polar identity tells us that \(B^\mathcal{H} \equiv 0\), so \(\mathcal{F}\) is Riemannian.

It is easily seen from the above calculations that the other part of the statement is also valid. \(\square\)

4. **Examples**

Let \(G\) be a 4-dimensional Lie group equipped with a left-invariant Riemannian metric. Let \(\mathfrak{g}\) be the Lie algebra of \(G\) and \(\{X, Y, Z, W\}\) be an orthonormal basis for \(\mathfrak{g}\). Let \(Z, W \in \mathfrak{g}\) generate a 2-dimensional left-invariant and integrable distribution \(\mathcal{V}\) on \(G\) which is conformal and with minimal leaves. We denote by \(\mathcal{H}\) the horizontal distribution, orthogonal to \(\mathcal{V}\), generated by \(X, Y \in \mathfrak{g}\). Then it is easily seen that the basis \(\{X, Y, Z, W\}\) can be chosen so that the Lie bracket relations for \(\mathfrak{g}\) are of the form
\[\begin{align*}
[W, Z] &= \lambda W, \\
[Z, X] &= \alpha X + \beta Y + z_1 Z + w_1 W, \\
[Z, Y] &= -\beta X + \alpha Y + z_2 Z + w_2 W,
\end{align*}\]
\[ [W, X] = aX + bY + z_3Z - z_1W, \]
\[ [W, Y] = -bX + aY + z_4Z - z_2W, \]
\[ [Y, X] = rX + \theta_1Z + \theta_2W \]

with real coefficients. It should be noted that these constants must be chosen in such a way that the Lie brackets for \( \mathfrak{g} \) satisfy the Jacobi identity. The solutions to that problem were recently classified in [5]. The following easy result describes the geometry of the situation.

**Proposition 4.1.** Let \( G \) be a 4-dimensional Lie group and \( \{X, Y, Z, W\} \) be an orthonormal basis for its Lie algebra as above. Then

(i) \( \mathcal{F} \) is totally geodesic if and only if \( z_1 = z_2 = z_3 + w_1 = z_4 + w_2 = 0 \),

(ii) \( \mathcal{F} \) is Riemannian if and only if \( \alpha = a = 0 \), and

(iii) \( \mathcal{H} \) is integrable if and only if \( \theta_1 = \theta_2 = 0 \).

The following lemma turns out to be useful later on.

**Lemma 4.2.** For the above situation we have the following:

i. The almost Hermitian structure \( J_1 \) is cosymplectic if and only if \( \theta_1 - 2a = 0 \) and \( \theta_2 + 2\alpha = 0 \).

ii. The almost Hermitian structure \( J_2 \) is cosymplectic if and only if \( \theta_1 + 2a = 0 \) and \( \theta_2 - 2\alpha = 0 \).

**Proof.** A standard calculation involving the Koszul formula

\[
2\langle \nabla_X Y, Z \rangle = \langle [Z, X], Y \rangle + \langle [Z, Y], X \rangle + \langle Z, [X, Y] \rangle
\]

shows that for the Levi-Civita connection of \((G, g)\) we have

\[
\nabla_X X = rY + \alpha Z + aW, \quad \nabla_Y Y = \alpha Z + aW, \\
\nabla_Z Z = -z_1X - z_2Y, \quad \nabla_W W = z_1X + z_2Y - \lambda Z.
\]

Then the divergence of the almost complex structure \( J_1 \) is given by

\[
\delta J_1 = [X, Y] - [W, Z] - J_1(\nabla_X X + \nabla_Y Y + \nabla_Z Z + \nabla_W W) \\
= -(\theta_1 - 2a)Z - (\theta_2 + 2\alpha)W.
\]

This proves i. and ii. is obtained in exactly the same way. \(\square\)

In the case when \( \lambda = 0, r \neq 0 \) and \( (a\beta - ab) \neq 0 \) the solutions are given by the following 5-dimensional family \( \mathfrak{g}_5(\alpha, a, \beta, b, r) \), see Case (C) of [5].

**Example 4.3** \((\mathfrak{g}_5(\alpha, a, \beta, b, r))\).

\[
[Z, X] = \alpha X + \beta Y + \frac{r(\beta b - \alpha a)}{2(a\beta - ab)}Z + \frac{r(\alpha^2 - \beta^2)}{2(a\beta - ab)}W, \\
[Z, Y] = -\beta X + \alpha Y + \frac{r(ab + \beta a)}{2(a\beta - ab)}Z - \frac{r\alpha\beta}{(a\beta - ab)}W, \\
[W, X] = aX + bY + \frac{r(b^2 - a^2)}{2(a\beta - ab)}Z + \frac{r(\alpha a - \beta b)}{2(a\beta - ab)}W,
\]
\[
\begin{align*}
[W,Y] &= -bX + aY + \frac{r_{ab}}{(a\beta - ab)}Z - \frac{r(a\beta + \beta a)}{2(a\beta - ab)}W, \\
[Y,X] &= rX - \frac{ar^2}{2(a\beta - ab)}Z + \frac{\alpha r^2}{2(a\beta - ab)}W.
\end{align*}
\]

Since \( r \neq 0 \) and \( \alpha^2 + a^2 \neq 0 \) each of the induced foliations \( \mathcal{F} \) is neither Riemannian nor does it have an integrable horizontal distribution. This tells us that at most one of the almost Hermitian structures is cosymplectic.

In the case when \( \lambda = 0, r = 0 \) and \( ab - a\beta = 0 \) we have several interesting families of solutions, see Case (F) of [5]. Two of those are presented below.

**Example 4.4.** If we assume \( \alpha = a = 0 \) and \( \beta \neq 0 \neq b \) then we obtain the family \( g_{18}(\beta, b, z_3, z_4, \theta_1, \theta_2) \) of the following form

\[
\begin{align*}
[Z,X] &= \beta Y + \frac{\beta z_3}{b}Z - \frac{\beta^2 z_3}{b^2}W, \\
[Z,Y] &= -\beta X + \frac{\beta z_4}{b}Z - \frac{\beta^2 z_4}{b^2}W, \\
[W,X] &= bY + z_3Z - \frac{\beta z_3}{b}W, \\
[W,Y] &= -bX + z_4Z - \frac{\beta z_4}{b}W, \\
[Y,X] &= \theta_1Z + \theta_2W.
\end{align*}
\]

The corresponding foliations \( \mathcal{F} \) are all Riemannian so the almost Hermitian structures \( J_1 \) and \( J_2 \) are both cosymplectic if and only if the horizontal distribution \( \mathcal{H} \) is integrable i.e. \( \theta_1^2 = \theta_2^2 = 0 \).

**Example 4.5.** If we assume \( \alpha \neq 0 \neq a \) and \( \beta \neq 0 \neq b \) then we get the family \( g_{20}(\alpha, a, \beta, w_1, w_2) \) of the following form

\[
\begin{align*}
[Z,X] &= \alpha X + \beta Y - \frac{aw_1}{\alpha}Z + w_1W, \\
[Z,Y] &= -\beta X + \alpha Y - \frac{aw_2}{\alpha}Z + w_2W, \\
[W,X] &= aX + \frac{\beta a}{\alpha}Y - \frac{a^2 w_1}{\alpha^2}Z + \frac{a}{\alpha}w_1W, \\
[W,Y] &= -\frac{\beta a}{\alpha}X + aY - \frac{a^2 w_2}{\alpha^2}Z + \frac{a}{\alpha}w_2W.
\end{align*}
\]

The almost Hermitian structure \( J_1 \) is cosymplectic if and only if \( 2a^2 + aw_2 = 0 \). The same applies to \( J_2 \) if and only if \( 2a^2 - aw_2 = 0 \). It is clear that in none of the cases is the foliation \( \mathcal{F} \) Riemannian. The horizontal distribution \( \mathcal{H} \) is integrable in all the cases.

5. **Integrable almost Hermitian structures**

We conclude this paper with Theorem 5.1 giving another relationship between the geometry of the horizontal conformal foliation \( \mathcal{F} \) and conditions
on the almost Hermitian structure \( J_1 \) and \( J_2 \). The result follows from Lemma 3.6 (iii) and Proposition 3.9 of [9], but here we give a more direct proof.

**Theorem 5.1.** Let \((M^4, g)\) be a 4-dimensional orientable Riemannian manifold equipped with a minimal and conformal foliation \( \mathcal{F} \) of codimension 2. Then the corresponding adapted almost Hermitian structures \( J_1 \) and \( J_2 \) are both integrable if and only if \( \mathcal{F} \) is totally geodesic.

**Proof.** Here we use the same notation as in Section 3. For \( k = 1, 2 \) the skew-symmetric Nijenhuis tensor \( N_k \) of the almost Hermitian structure \( J_k \) is given by

\[
N_k(E, F) = [E, F] + J_k[J_kE, F] + J_k[E, J_kF] - [J_kE, J_kF].
\]

It is easily seen that this satisfies

\[
N_k(X, Y) = N_k(Z, W) = 0
\]

and that the horizontal conformality of \( \mathcal{F} \) is equivalent to

\[
\mathcal{H}N_k(X, Z) = \mathcal{H}N_k(X, W) = \mathcal{H}N_k(Y, Z) = \mathcal{H}N_k(Y, W) = 0.
\]

This means that \( J_1 \) and \( J_2 \) are both integrable if and only if for \( k = 1, 2 \)

\[
\forall N_k(X, Z) = \forall N_k(X, W) = \forall N_k(Y, Z) = \forall N_k(Y, W) = 0.
\]

We define the 1-forms \( \alpha, \beta : C^\infty(\mathcal{H}) \to \mathbb{R} \) by

\[
\alpha(E) = 2(B^\mathcal{V}(Z, Z) - B^\mathcal{V}(W, W), E),
\]

\[
\beta(E) = 2(B^\mathcal{V}(Z, W) + B^\mathcal{V}(W, Z), E).
\]

Since we are assuming that the fibres are minimal i.e.

\[
B^\mathcal{V}(Z, Z) + B^\mathcal{V}(W, W) = 0
\]

we see that the foliation \( \mathcal{F} \) is totally geodesic if and only if \( \alpha \) and \( \beta \) vanish. Now a standard calculation shows that

\[
\langle (N_1 + N_2)(X, Z), Z \rangle = \alpha(X) = -\langle (N_1 + N_2)(X, W), W \rangle,
\]

\[
-\langle (N_1 - N_2)(X, Z), Z \rangle = \beta(Y) = \langle (N_1 - N_2)(X, W), W \rangle,
\]

\[
\langle (N_1 + N_2)(X, Z), W \rangle = \beta(X) = \langle (N_1 + N_2)(X, W), Z \rangle,
\]

\[
\langle (N_1 - N_2)(X, Z), W \rangle = \alpha(Y) = \langle (N_1 - N_2)(X, W), Z \rangle,
\]

\[
\langle (N_1 + N_2)(Y, Z), Z \rangle = \alpha(Y) = -\langle (N_1 + N_2)(Y, W), W \rangle,
\]

\[
\langle (N_1 - N_2)(Y, Z), Z \rangle = \beta(X) = -\langle (N_1 - N_2)(Y, W), W \rangle,
\]

\[
\langle (N_1 + N_2)(Y, Z), W \rangle = \beta(Y) = \langle (N_1 + N_2)(Y, W), Z \rangle,
\]

\[
\langle (N_1 - N_2)(Y, Z), W \rangle = -\alpha(X) = \langle (N_1 - N_2)(Y, W), Z \rangle,
\]

The statement of Theorem 5.1 is a direct consequence of these equations. \(\square\)
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