ON THE NON-UNIQUENESS OF THE INSTANTANEOUS FREQUENCY

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ABSTRACT. In this article, we investigate the debated Instantaneous Frequency (IF) topic. Here, we show that IF is non-unique inherently. We explain how this non-uniqueness can be quantified and explained from a mathematical perspective. The non-uniqueness of the IF can also be observed if different methods of adaptive signal processing are used. We will also show that even if we know the physical origin of an oscillatory signal, e.g. linear second order ordinary differential equation, the non-uniqueness is still present. All in all, we will end up with the conclusion that, without any a priori assumption about the relationship of the envelope and phase function of an oscillatory signal, there is not any preferred neither best representation of the IF of such oscillatory signal.

1. INTRODUCTION

1.1. Problem Specification. Knowing that a signal \( s(t) = a(t) \sin(\theta(t)) \) is oscillatory having simple zeros, with unique extrema in between, for non-zero \( a(t) \) and strictly positive \( \frac{d\theta(t)}{dt} \), one can easily define \( \omega(t) = \frac{d\theta(t)}{dt} \) as the IF of \( s(t) \). On the other hand, having an oscillatory signal \( s(t) \) having simple zeros, with unique extrema in between, it is not straight forward to find an Intrinsic Mode Function (IMF) form \( a(t) \sin(\theta(t)) \) representing the signal. The problem that we are going to address in this paper is that even knowing \( s(t) = a(t) \sin(\theta(t)) \), there is no guarantee that this IMF representation is unique at all. As a reminder, in this paper, we only consider mono-component oscillatory functions having simple zeros and one extremum between two consequent zeros.
The organization of the paper is as follows: For the rest of this section, we will review a history of the IF. In Section 2, we will mathematically prove the non-uniqueness of presentations rigorously. Section 3 will include numerical examples regarding the non-uniqueness of the IF of an IMF. In Section 4, we will further show that the IF non-uniqueness shows up in differential equations IMF solutions. Finally, we will conclude the paper in Section 5.

1.2. History of IF. The concept of Intrinsic Frequency (IF) was first proposed by Van der Pol in 1930 [34]. He used the differential equation of a frequency modulated transmitter to express the current as a function of time. The frequency analysis of this function was essentially the first notion of IF. Later, Carson and Fry [6] generalized the definition of the IF in the context of a Frequency Modulated (FM) signal such as

\[ s(t) = e^{i(\omega_0 t + \lambda \int_0^t m(\tau) d\tau)}. \]

Van der Pol elaborated more on this type of definition and, at the same time, warned that the concept of IF is arbitrary but useful [35]. In all these approaches, IF was defined as the derivative of the phase function of an FM signal. Later presence of the IF notion can be found in a work by Gupta [12]. In that work, Gupta mentions two different forms of IF definitions: One that is based on a causal approach based on causality principle (e.g. Running Fourier Frequency), and the other from a non-casual one (e.g. Fourier Frequency). All the definitions, in that work, are based on a type of Fourier transform, average zero crossing mean frequency or second order linear differential equation.

Vakman and Vainshtein have defined IF and Instantaneous Amplitude (IA) from a physical point of view based on an analytic signal (AS) construction [33]. An analytic signal \( z(t) \) (see works by Gabor [11] and Ville [37]) is derived from the Hilbert Transform (HT) of a signal \( s(t) \) according to

\[ z(t) = s(t) + \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{s(\tau)}{t-\tau} d\tau = a(t) \exp(i \varphi(t)). \]

We should emphasize that AS is only meaningful for a mono-component oscillatory signal. AS approach to IF has also been mentioned by many other authors [20, 23, 27, 4]. Among
them, Boashash [4] has explicitly mentioned the inconsistencies and issues of the definitions of the IF, including AS, in depth. Boashash defines a “non-stationary” signal as one whose spectral characteristics, e.g. peaks, vary with time. In the stationary case, he mentions, the definition of the IF is clear: It can be related to the Fourier Transform. He believes that the IF is ambiguous, paradoxical, controversial, application-related, and empirically assessed. Citing Shekel [28], he explains that the analytic signal approach cannot have a unique physical representation, although the complex representation could be unique. Boashash mentions that the problem of IF definition comes from the ambiguity in IF, envelope (amplitude), and oscillation intuition. Despite all these, Boashash finally adopts the analytic signal approach to construct a theory for IF. Based on AS approach, Boashash further introduces the numerical algorithms for the extraction of the IF in [5]. Apparently, AS is the most commonly used approach to IF and IA.

There have been attempts to define IF based on Spectrum Time-Frequency (STF) methods [8, 29, 32]. For example, we can name methods like Short-Time Fourier Transform (STFT), Wavelet Transform (WV) [3] and Wigner Ville Distribution (WVD) [8]. These methods have inherent shortcomings in data analysis and signal processing. Consequently, the IF definitions that are based on these methods are not reliable. As mentioned in the work by Daubechies et al. [9], for linear time-frequency methods such as WT and STFT the signal is analyzed by its inner product with an a priori dictionary of basis functions. Hence, the main problem with these methods, specifically the STFT, would be the Heisenberg uncertainty principle.

Daubechies et al. [9] mention that for an IMF, having the form of $a(t) \cos \theta(t)$, the changes in time of the envelope $a(t)$ and frequency $\theta'(t) = \frac{d\theta(t)}{dt}$ should be much slower than the change of $\theta(t)$ itself. This means that for $[t - \delta, t + \delta]$, where $\delta \approx \frac{2\pi}{\theta'(t)}$, the IMF is essentially a harmonic signal with amplitude $a(t)$ and frequency $\theta'(t)$. This approach, however, would encounter difficulties in extraction of strongly frequency modulated signals [30]. In order to define the IF and an IMF in practice, Daubechies et al. use Synchrosqueezing Wavelet

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1Cohen, in [8], also explains that the exact mathematical description of the IF is not a settled question.
Transforms (SSWT) \[9\]. An in depth analysis of the method can be found in \[32\]. Using the notations of this method, the wavelet transform \(W_s\) of the signal \(s\) is defined by

\[
W_s(a,b) = \int s(t) \frac{1}{\sqrt{a}} \psi \left( \frac{t-b}{a} \right) dt,
\]

where \(\psi\) is a chosen wavelet. Later, in the same paper, a candidate for the IF is defined as

\[
\omega_s(a,b) = \frac{\partial}{\partial b} \frac{\partial}{\partial W_s(a,b)} + i W_s(a,b).
\]

This method, as can be seen from its definitions, is a STF method. They show that the IF candidate, Equation (1.4), would be a good approximation of the true frequency \(\phi'(t)\), if the following strict conditions are satisfied for an imaginary IMF \(s(t) = A(t) e^{i\phi(t)}\):

\[
A \in C^1(\mathbb{R}) \cap L_\infty, \phi \in C^2(\mathbb{R}),
\]

\[
\inf_{t \in \mathbb{R}} \phi'(t) > 0, \sup_{t \in \mathbb{R}} \phi'(t) < \infty,
\]

\[
|A'(t)|, |\phi''(t)| \leq \epsilon |\phi'(t)|, \forall t \in \mathbb{R},
\]

\[
M'' := \sup_{t \in \mathbb{R}} |\phi''(t)| < \infty.
\]

In fact, some of these conditions can be violated for strongly FM signal \[30\].

Huang et. al., in \[19\], have also introduced newer definitions of IF including Direct Quadrature (DQ), Normalized Hilbert Transform (NHT), Generalized Zero-Crossing (GZC), and Teager Energy Operator (TEO). Some of these definitions are side products of the adaptive signal processing method that is being used in analyzing the data. For example, NHT is motivated by the Empirical Mode Decomposition (EMD) method \[18\].

Some authors have defined the IF from a polar coordinate perspective. They assume that an IMF \(s(t)\) has a polar coordinate if observed in a normalized \((s, \frac{ds}{dt})\) plane \[22\]. However,\footnote{In the context of the EMD method, the definition of the frequency of the IF was originally based on the AS approach \[18\].}
this normalization process is not unique, and as a result, the derived IF from this method is not unique as well.

Hou and Shi use different approaches to define an IF, and consequently an IMF \[15\,14\,13\]. Their initial approach was to define an IMF \(a(t)\cos(\theta(t))\) that has a smooth envelope \(a(t)\) \[14\]. The smoothness of the envelope was defined based on the Total Variation (TV) of the envelope. Later, they changed their approach, see \[15\], to define an IMF having \(a(t) \in V(\theta(t))\), with \(a(t) > 0\) and \(\theta'(t) > 0\), where

\[
V(\theta) = \text{span}\left\{1, \cos\left(\frac{\theta}{\lambda}\right), \sin\left(\frac{\theta}{\lambda}\right) | \lambda \geq 2\right\}.
\]

This approach is among few rigorous definitions of the IF and IMF.

Although the IF definition, as perceived from literature, is not well settled yet and most of the time is a side product of a specific adoptive signal processing method, there has been attempts to built a more general theory around this concept. For example, Wu has suggested that the definition of the IF should be extended to shape functions \[38\]. He proposes to have \(a(t)S(\theta(t))\), where \(S\) is a shape function and \(\frac{d\theta}{dt} > 0\). \(S(t)\) itself could be a summation of many IMFs such as \(A(t)\cos(\psi(t))\). In that work, the periodicity burden is put on the shape function. Essentially, the phase function \(\theta(t)\) is just a one-to-one map from the \(t\)-coordinate to the \(\theta\)-coordinate. However, if the physical properties of the signal are embedded in one or more of \(\psi(t)\)s, taking the whole signal as \(a(t)S(\theta(t))\) would dilute or mix the physical interpretation.

Other physicists have also tried to set forth definitions for IF from physical perspectives. For example, having a Hamiltonian \(\mathcal{H}(p, q)\), the IF of the system has been defined as \(\omega(I) = \frac{\partial\mathcal{H}(I)}{\partial I}\) \[21\]. For a conservative system, we have \(\mathcal{H} = E\) as the energy of the system, and \(I = \oint pdq/2\pi\) for

\[
\frac{\partial\mathcal{H}}{\partial q_i} = -\dot{p}_i,
\]

\[
\frac{\partial\mathcal{H}}{\partial p_i} = \dot{q}_i.
\]
Clearly, there is no consensus on a fixed definition of the IF. The only consensus is the positivity of the IF to ensure a meaningful physical interpretation of an IMF.

1.3. **IF vs IMF.** In all above mentioned literature, in order to extract an IF, an oscillatory signal (IMF) is needed to be defined either implicitly or explicitly. In some of the STF methods, e.g. STFT and WV, the IF is derived from an implicit notion of an IMF. On the other hand, methods like EMD define an IMF in clear explicit format [18]. In fact, since IF arises from time varying oscillations, it is logical to define the oscillation in the first place and then its time characteristics like IF and IA. This justification elaborates why methods like EMD are more successful in physical explanation of the non-stationary signals than others such as the STFT method [16, 17].

2. **Theoretical Aspects of the Non-uniqueness of the IF**

As mentioned earlier, in order to show the behavior of the IF of a mono-component oscillatory signal, we need to define the IMF in the first place. In our definition, with out loss of generality and for simplicity, we assume that an IMF starts from zero at time \( t_0 \) and ends at zero at some later instance \( t_{n-1} \).

In the following definitions, lemmas, and theorems, \( C^m_\xi \) is the space of all functions \( f(\xi) \) with continuous derivatives up to order \( m \in \{0,1,2,\ldots\} \). If \( \xi \) is not mentioned, it means \( \xi = t \).

**Definition 2.1.** For \( m \in \mathbb{N}, n \in \mathbb{N}, \) and \( n \geq 3 \), an \( n \)-zero-\( C^m \) IMF is a real function \( f(t) \) in \( C^m \), having \( n \) first-order (simple) zeros in \([t_0, t_{n-1}]\) and \((n - 1)\) extrema. There is only one extremum located between two consequent zeros. \( t_0 \) and \( t_{n-1} \) are two of the zeros pf \( f(t) \).

This definition is in fact more general than the modern definitions of an IMF found in literature; see [9, 15, 14, 13, 18, 19, 30, 32, 38]. Having this definition, one can construct phase functions with different regularities. During this section we provide phase functions \( \theta(t) \) with different regularities to establish the idea that the IF is non-unique.
2.1. $C^0$ Phase functions. Between any two zeros of an $n$-zero-$C^m$ IMF $f(t)$, one can define a piece-wise linear phase function $\theta(t)$. This construction would introduce a piece-wise constant IF. This linear phase function will result in a representation of the form $a(t) \sin \theta(t)$. However, as the phase function has a discontinuous first derivative on zeros of the IMF, the derivative of the envelope function defined as $a(t) = \frac{f(t)}{\sin \theta(t)}$ could be undetermined on all $t_i$, $i = 0, \dotsc, n - 1$. This definition of an IF is similar to GZC definition of the IF; see [19].

2.2. $C^1$ Phase functions. In order to alleviate the discontinuity problem in the previous part, we propose the next lemma, by which we can construct a smoother representation, for the envelope $a(t)$, of an $n$-zero-$C^m$ IMF. To do so, we first construct a $C^1$ map $\theta(t) : [t_0 = 0, t_{n-1} = 1] \to [0, (n-1)\pi]$, and then construct the envelope $a(t)$.

**Lemma 2.2.** There exists a $C^1$ map $\theta(t) : [t_0 = 0, t_1 = 1] \to [0, \pi]$ such that, $\theta(0) = 0$, $\theta(1) = \pi$, $\frac{d\theta}{dt} > 0$ and $\frac{d\theta}{dt}|_{t_0=0} = M > 0$.

**Proof.** Take the function,

$$
\theta(t) = \begin{cases}
\pi t + k t e^{\frac{1}{q-1}} & t \in [0, 1) \\
\pi & t = 1
\end{cases}
$$

for $k = e (M - \pi), 0 < q < 1$. Taking the first derivative of $\theta(t)$ gives

$$
\frac{d\theta}{dt}(t) = \begin{cases}
\pi + k e^{\frac{1}{q-1}} - \frac{k q e^{\frac{1}{q-1}}}{(q-1)^2} & t \in [0, 1), \\
\pi & t = 1.
\end{cases}
$$

Now, we need to make sure that $q$ guarantees a positive IF $\frac{d\theta}{dt}(t)$. Taking the second derivative of the phase function and equating that to zero to find the location of the minimum of the first derivative would tell us that the relevant minimum would occur at

$$
t = \left(-\frac{(q + 2 - \sqrt{q(5q + 4)})}{2(q - 1)}\right)^{1/q}.
$$
The value of the first derivative of \( \theta (t) \) at this point is \( \pi + f(q)k \), where \( f(q) \) is a monotone decreasing function for \( 0 < q < 1 \). The bounds of this function are 0 and \(-\frac{5}{\pi}\). Hence, for any value of \( k \), one can always find an appropriate value of \( q \in (0, 1) \) such that \( \frac{d\theta}{dt} > 0 \). □

**Lemma 2.3.** There exists a \( C^1 \) map \( \theta (t) : [t_0 = 0, t_{n-1} = 1] \to [0, (n-1)\pi] \) such that, \( \theta (t_i) = i\pi \), for \( i = 0, \ldots, n-1 \), \( \frac{d\theta}{dt} > 0 \).

**Proof.** For \( t \in [t_0 = 0, t_1] \), set \( \theta (t) = \pi \frac{t}{t_1} \). For \( t \in [t_1, t_2] \), use Lemma 2.2 to construct \( \theta (t) \) having \( \dot{\theta} (t_1) = \frac{\pi}{t_1} \). In other words, in \( t \in [t_1, t_2] \) take

\[
(2.4) \quad \theta (t) = \begin{cases} 
\pi + \pi \left( \frac{t-t_1}{t_2-t_1} \right) + e\pi \left( 1 - \frac{t-t_1}{t_2-t_1} \right) e^{-i \frac{t-t_1}{t_2-t_1}} & t \in [t_1, t_2), \\
2\pi & t = t_2.
\end{cases}
\]

In general, for the \( i^{th} \) piece, \( i = 2, \ldots, n-1 \), take

\[
(2.5) \quad \theta (t) = \begin{cases} 
(i - 1) \pi + \pi \left( \frac{t-t_{i-1}}{t_i-t_{i-1}} \right) + e\pi \left( 1 - \frac{t-t_{i-1}}{t_i-t_{i-1}} \right) e^{-i \frac{t-t_{i-1}}{t_i-t_{i-1}}} & t \in [t_{i-1}, t_i), \\
i\pi & t = t_i.
\end{cases}
\]

The way that we have defined the map satisfies \( \theta (t) \in C^1 \), where \( \frac{d\theta}{dt} > 0 \). □

**Lemma 2.4.** If there is a map \( \theta (t) \in C^1 \), where \( \frac{d\theta}{dt} > 0 \), having fixed points \((t_0 = 0, 0), (t_1, \pi), (t_2, 2\pi), \ldots, (t_{n-1} = 1, (n-1)\pi)\), then an \( n \)-zero-\( C^1 \) IMF \( f(t) \) on \([t_0 = 0, t_{n-1} = 1]\) has a representation of \( a(t) \sin \theta(t) \) form, for \( a(t) \neq 0 \) and \( a(t) \in C \).

**Proof.** Without loss of generality, assume that between the first two zeros of \( f(t) \), the function \( f(t) \) is positive. Take \( a(t) = \frac{f(t)}{\sin \theta(t)} \) for all \( t \in [0, 1] \). For the nodes at \( t_i \) for \( i = 0, \ldots, n-1 \) take \( a(t_i) = \frac{f(t_i)}{\theta'(t_i)} \), then \( a(t) \in C \). □

**Note:** Other forms of mappings, with different regularities, can be designed using monotone shape preserving spline interpolations (see [10][25]).

These lemmas show that an \( n \)-zero-\( C^1 \) IMF has a representation of form \( a(t) \sin \theta(t) \).

At the same time, free parameters in Lemma 2.2 and 2.3 show that, in construction of the
phase function \( \theta(t) \), the non-uniqueness is inherent. Further more, we can show that the non-uniqueness behavior is not confined to the free parameters of Lemmas 2.2 and 2.3.

**Lemma 2.5.** If there exists a strictly increasing map \( \theta(t) \) on \( t \in [0, 1] \), having fixed points at \( 0, \pi, 2\pi, \ldots, n\pi \), then there are infinitely many strictly increasing maps having the same fixed points.

**Proof.** Since the map \( t \to \theta(t) \) is strictly increasing, it is invertible on the same domain. Take the map \( \chi(\theta) = \theta + c \sin \theta \). This map has the same fixed points as \( \theta \). If \( |c| < 1 \), then \( \chi \) would be strictly increasing in \( \theta \) and consequently in \( t \). As \( t \to \theta(t) \) is invertible, and \( \theta \to \chi(\theta) \) is also invertible, \( t \to \chi(t) \) is consequently invertible. Since there are infinitely many values satisfying \( |c| < 1 \), the proof is complete. \( \square \)

**Note:** From this lemma, one can construct other possibilities like \( \chi(\theta) = \theta + \sum c_j \sin (j\theta) \) as candidates for the IF non-uniqueness, as long as the sequence \( \{c_j\} \) keeps the monotonicity of \( \chi(\theta) \).

Finally, we can conclude that:

**Theorem 2.6.** An \( n \)-zero-C\(^1\) IMF has infinitely many \( a(t) \sin \psi(t) \) representations.

**Proof.** Using Lemmas 2.4 and 2.3, one can construct one \( a(t) \sin \theta(t) \) for the \( n \)-zero-C\(^1\) IMF. Using Lemma 2.5 on \( \theta(t) \in C^1 \), where \( \frac{d\theta}{dt} > 0 \), one can find \( \psi(t) \in C^1 \), where \( \frac{d\psi}{dt} > 0 \). Again using Lemma 2.4, one can find another representation for the same IMF, \( A(t) \sin \psi(t) \). \( \square \)

One can do better than a \( C^1 \) phase function. In the next part, we show how to construct \( C^\infty \) phase functions.

2.3. \( C^\infty \) Phase functions. Here, we present very smooth phase functions, which help to produce an \( a(t) \sin \theta(t) \) representation for an \( n \)-zero-C\(^m\) IMF having a \( C^m \) envelope \( a(t) \).

The following is our strategy for such constructions:

1. We first construct piecewise linear phase functions,
2. we let these piece-wise linear functions have some gaps between them at \( t_1, \ldots, t_{n-2} \),
(3) we connect the gaps, using mollifiers, in a way that the resulting phase function is a $C^\infty$ smooth function.

This approach is related closely to the analytic monotone interpolation \[36\], although, our derivation is fundamentally different.

**Definition 2.7.** A compact support mollifier is a $C^\infty$ smooth function that has a compact support in $\mathbb{R}$.

An example of a mollifier is

\[
\chi(x) = \begin{cases} 
  e^{-\frac{1}{x}} & x > 0, \\
  0 & x \leq 0.
\end{cases}
\]

This function is a $C^\infty$ smooth function. One can easily convert this function into a compact support mollifier by introducing the function

\[
\mu(x) = \begin{cases} 
  \chi(x) \chi(1 - x) & 0 < x < 1, \\
  0 & \text{otherwise}.
\end{cases}
\]

This function is symmetric (see Figure 1) and it can be used to construct a connector mollifier:

\[
\eta(x) = \frac{\int_0^x \mu(\xi) \, d\xi}{\int_0^1 \mu(\xi) \, d\xi}.
\]

**Definition 2.8.** A connector mollifier (see Figure 2) is defined as follows:
Lemma 2.9. For a connector mollifier defined in Definition 2.8, we have

\[ \int_0^1 \eta(x) \, dx = \frac{1}{2}. \]  

Proof. We have

\[
\int_0^1 \eta(x) \, dx = \int_0^1 \frac{\int_0^x \mu(\xi) \, d\xi}{\int_0^x \mu(\xi) \, d\xi} \, dx \\
= \frac{1}{\int_0^x \mu(\xi) \, d\xi} \int_0^x \int_0^{\frac{1}{e^{\frac{1}{y}}}} \, dy \, dx \\
= \frac{1}{\int_0^x \mu(\xi) \, d\xi} \int_0^1 \int_y^{\frac{1}{e^{\frac{1}{y}}}} \, dy \, dx \\
= \frac{1}{\int_0^x \mu(\xi) \, d\xi} \int_0^1 (1 - y) \, e^{\frac{1}{y(1-y)}} \, dy \\
= \frac{\int_0^1 (1-y) \, e^{\frac{1}{y(1-y)}} \, dy}{\int_0^1 e^{\frac{1}{y(1-y)}} \, dy} \\
= 1 - \frac{\int_0^1 y \, e^{\frac{1}{y(1-y)}} \, dy}{\int_0^1 e^{\frac{1}{y(1-y)}} \, dy} = 1 - \frac{1}{2} = \frac{1}{2}.
\]

Remember that \( \int_0^1 y \, e^{\frac{1}{y(1-y)}} \, dy = \frac{1}{2} \) comes from the symmetric behavior of \( e^{\frac{1}{y(1-y)}} \) around \( y = 0.5 \). \qed

The connector mollifier defined in Definition 2.8 can connect two discrete values \( a, b \) on an interval \((\varepsilon_1, \varepsilon_2)\) by

\[ \eta_{\varepsilon_1}^\varepsilon (x; a, b) = (b - a) \eta \left( \frac{x - \varepsilon_1}{\varepsilon_2 - \varepsilon_1} \right) + a. \]

later, we will use this to construct a \( C^\infty \) phase function.

Lemma 2.10. There exists \( 0 < \varepsilon_1 < \varepsilon_2 < 1 \) and \( f_{\varepsilon_1} > 0 \), such that

\[ \int_0^{\varepsilon_1} \eta_{\varepsilon_1}^{\varepsilon_1} (x; a, f_{\varepsilon_1}) \, dx + \int_{\varepsilon_1}^{\varepsilon_2} \eta_{\varepsilon_1}^{\varepsilon_1} (x; f_{\varepsilon_1}, b) \, dx = b\varepsilon_2, \]
for any \(a > 0, b > 0\) and \(a \neq b\).

**Proof.** Case 1 \((b > a)\):

For sufficiently small \(\varepsilon_1 (0 < \varepsilon_1 \ll 1)\) one can always find \(f_{\varepsilon_1} > 0\) satisfying the inequality

\[
(2.11) \quad b + (b - a) \varepsilon_1 < f_{\varepsilon_1} < b + (b - a).
\]

Now set \(\varepsilon_2 = \left(\frac{b-a}{f_{\varepsilon_1} - b}\right) \varepsilon_1\). This condition, based on inequality \((2.11)\), will satisfy the condition \(0 < \varepsilon_1 < \varepsilon_2 < 1\):

i) \(f_{\varepsilon_1} < b + (b - a) \Rightarrow f_{\varepsilon_1} - b < (b - a) \Rightarrow 1 < \frac{(b-a)}{f_{\varepsilon_1} - b} \Rightarrow \varepsilon_1 < \frac{(b-a)}{f_{\varepsilon_1} - b} \varepsilon_1 \Rightarrow \varepsilon_1 < \varepsilon_2\).

ii) \(b + (b - a) \varepsilon_1 < f_{\varepsilon_1} \Rightarrow (b - a) \varepsilon_1 < f_{\varepsilon_1} - b \Rightarrow \frac{(b-a)}{f_{\varepsilon_1} - b} \varepsilon_1 < 1 \Rightarrow \varepsilon_2 < 1\).

Using this value of \(\varepsilon_2\) we will prove that Equation \((2.10)\) is satisfied:

\[
\int_0^{\varepsilon_1} \eta_0^{\varepsilon_1} (x; a, f_{\varepsilon_1}) \, dx + \int_{\varepsilon_1}^{\varepsilon_2} \eta_1^{\varepsilon_2} (x; f_{\varepsilon_1}, b) \, dx = \int_0^{\varepsilon_1} \left((f_{\varepsilon_1} - a) \eta \left(\frac{x}{\varepsilon_1}\right) + a\right) \, dx \\
+ \int_{\varepsilon_1}^{\varepsilon_2} \left((b - f_{\varepsilon_1}) \eta \left(\frac{x - \varepsilon_1}{\varepsilon_2 - \varepsilon_1}\right) + f_{\varepsilon_1}\right) \, dx \\
= \varepsilon_1 \int_0^1 ((f_{\varepsilon_1} - a) \eta (x) + a) \, dx \\
+ (\varepsilon_2 - \varepsilon_1) \int_0^1 (b - f_{\varepsilon_1}) \eta (x) \, dx \\
+ (\varepsilon_2 - \varepsilon_1) \int_0^1 f_{\varepsilon_1} \, dx.
\]

Now, using the fact that \(\int_0^1 \eta (x) \, dx = \frac{1}{2}\), we have

\[
\int_0^{\varepsilon_1} \eta_0^{\varepsilon_1} (x; a, f_{\varepsilon_1}) \, dx + \int_{\varepsilon_1}^{\varepsilon_2} \eta_1^{\varepsilon_2} (x; f_{\varepsilon_1}, b) \, dx = \varepsilon_1 \frac{a-b}{2} + \varepsilon_2 \frac{b + f_{\varepsilon_1}}{2} \\
= \varepsilon_2 \left(\frac{b-f_{\varepsilon_1}}{a-b} \frac{a-b}{2} + \frac{b+f_{\varepsilon_1}}{2}\right) \\
= b \varepsilon_2.
\]

Case 2 \((b < a)\):

For sufficiently small \(\varepsilon_1 (0 < \varepsilon_1 \ll 1)\) one can always find \(f_{\varepsilon_1} > 0\) satisfying the inequality

\[
(2.12) \quad b - (a - b) < f_{\varepsilon_1} < b - (a - b) \varepsilon_1.
\]

Now set \(\varepsilon_2 = \left(\frac{b-a}{f_{\varepsilon_1} - b}\right) \varepsilon_1\). This condition, based on inequality \((2.12)\), will satisfy the condition \(0 < \varepsilon_1 < \varepsilon_2 < 1\):

i) \(f_{\varepsilon_1} < b - (a - b) \varepsilon_1 \Rightarrow (a - b) \varepsilon_1 < b - f_{\varepsilon_1} \Rightarrow \frac{(a-b)}{b-f_{\varepsilon_1}} \varepsilon_1 < 1 \Rightarrow \varepsilon_2 < 1\).

ii) \(b - (a - b) < f_{\varepsilon_1} \Rightarrow b - f_{\varepsilon_1} < a - b \Rightarrow \frac{(a-b)}{b-f_{\varepsilon_1}} \varepsilon_1 > \varepsilon_1 \Rightarrow \varepsilon_2 > \varepsilon_1\).
Again, using this value of $\varepsilon_2$ we will prove that Equation (2.10) is satisfied.  

Lemma 2.10 will help us to match together the piecewise continuous instantaneous frequencies and make a $C^\infty$ phase function. However, before doing so, we need more prerequisites.

**Lemma 2.11.** If the function $f(\theta) \in C^m_\theta$ and $\theta(t) \in C^m_t$, then $f(\theta(t)) \in C^m_t$, for $m \geq 0$.

**Proof.** For $m = 0$ the proof is trivial, and for $m > 0$ one can use the Chain Rule. □

**Lemma 2.12.** If the function $f(u) \in C^m_u$ and $\theta(u) \in C^m_u$ and $\frac{d\theta}{du} > 0$, then $f(\theta) \in C^m_\theta$, for $m \geq 1$.

**Proof.** We show the proof for one step, which can be extended by induction. Define $y = f(u)$ and $x = \theta(u)$. As a result, $y = f(\theta^{-1}(x))$. Now apply the Chain Rule:

$$\frac{dy}{dx}\bigg|_{x=p} = \frac{du}{du}\bigg|_{u=\theta^{-1}(p)} \frac{dy}{dx}\bigg|_{x=p} = f'(\theta^{-1}(p)) \frac{du}{dx}\bigg|_{x=p} = f'(\theta^{-1}(p)) \frac{d\theta}{d\theta^{-1}(p)}.$$

In short, $\frac{dy}{dx}(x) = \left( \frac{d\theta}{d\theta^{-1}} \right)(\theta^{-1}(x))$. This procedure can be continued for higher derivatives as well. As $\theta' > 0$, the conclusion follows that $y = f(\theta)$ is in $C^m_\theta$. □

**Lemma 2.13.** There exists a non-unique $C^\infty$ map $\theta(t) : [t_0 = 0, t_{n-1} = 1] \rightarrow [0, (n-1)\pi]$ such that, $\theta(t_i) = i\pi$, for $i = 0, \ldots, n-1$, and $\frac{d\theta}{dt} > 0$.

**Proof.** At the first step, fit a piecewise map between the nodes $\theta(t_i) = i\pi$. This piecewise map has constant frequencies in $(t_i, t_{i+1})$ namely $\omega_i$ for $i = 0, \ldots, n-2$. In order to convert this piecewise constant frequency map into a $C^\infty$ map, use Lemma 2.10. For $[t_0, t_1]$, $\omega_0 = \frac{\pi}{t_1-t_0}$. For $[t_1, t_1 + \varepsilon]$, where $t_1 + \varepsilon < t_2$, use Lemma 2.10 with $a = \omega_0$ and $b = \omega_1 = \frac{\pi}{t_2-t_1}$; hence, for $[t_1 + \varepsilon, t_2]$, we set $\omega_1 = \frac{\pi}{t_2-t_1}$. Doing this in $[t_0 = 0, t_{n-1} = 1]$ one can construct a $C^\infty$ instantaneous frequency $\omega(t)$. Integrating such a function would produce our desired map $\theta(t) = \int_0^t \omega(\xi) \, d\xi$. □

**Notes:** There are three subtle points that we need to address to finalize the proof:
1- When using Lemma 2.10, we implicitly use a linear map between \([t_i, t_{i+1}]\) and \([0, 1]\). This map preserves the \(C^\infty\) properties of \(\omega(t)\) due to the way we have defined \(\eta(x)\) in Lemma 2.8.

2- The non-uniqueness property of the map comes from the free parameters in Lemma 2.10.

3- The way we have defined \(\eta(x)\) and the essential integral in Definition 2.8 and Lemma 2.10 would guarantee \(\pi = \int_{t_i}^{t_{i+1}} \omega(\xi) \, d\xi\).

We can now summarize:

**Theorem 2.14.** Having a map \(\theta(t) \in C^\infty_t\), where \(\frac{d\theta}{dt} > 0\), with fixed points \((t_0 = 0, 0)\), \((t_1, \pi)\), \((t_2, 2\pi)\), \ldots, \((t_{n-1} = 1, (n-1)\pi)\), then an \(n\)-zero-\(C^m\) IMF \(f(t)\) on \([t_0 = 0, t_{n-1} = 1]\), for \(m \geq 1\), has a representation of \(a(t) \sin \theta(t)\) with \(a(t) \in C^m_t\).

**Proof.** Without loss of generality, we investigate the regularity of \(a(\theta)\) around \(\theta = 0\). Take

\[
\frac{d^q}{d\theta^q}(a(\theta)) = \begin{cases} 
\frac{d^q}{d\theta^q} \left( \frac{f(\theta)}{\sin \theta} \right) & \theta \neq 0, \\
\lim_{\theta \to 0} \frac{d^q}{d\theta^q} \left( \frac{f(\theta)}{\sin \theta} \right) & \theta = 0.
\end{cases}
\]

The first term, on the right hand side equation 2.13 is valid for \(q = 0, 1, \ldots, m\) following from the the fact that \(f(\theta) \in C^m_{\theta}\) and \(\sin \theta \in C^m_{\theta}\). However, the second term needs more scrutiny. We can use series expansion around \(\theta = 0\) and write

\[
\frac{f(\theta)}{\sin \theta} = \left( \sum_{j=1}^{m-1} \frac{1}{j!} \frac{d^j f(0)}{d\theta^j} \theta^j + \frac{1}{m!} \frac{d^m f(\xi)}{d\theta^m} \theta^m \right)
\]

\[
= \left( \sum_{j=1}^{m-1} \frac{1}{j!} \frac{d^j f(0)}{d\theta^j} \theta^j + \frac{1}{m!} \frac{d^m f(\xi)}{d\theta^m} \theta^m \right)
\]

\[
= \left( \sum_{j=1}^{m-1} \frac{1}{j!} \frac{d^j f(0)}{d\theta^j} \theta^j + \frac{1}{m!} \frac{d^m f(\xi)}{d\theta^m} \theta^m \right) \left( 1 + \frac{\theta^2}{6} + \frac{7\theta^4}{360} + \cdots \right)
\]

for \(|\theta| < \pi\). This expansion is convergent and the differentiation for \(q = 0, 1, \ldots, m\) exists. Hence, \(\lim_{\theta \to 0} \frac{d^q}{d\theta^q} \left( \frac{f(\theta)}{\sin \theta} \right)\) is well defined in 2.13. Finally, using Lemma (2.11) and (2.12), the proof is complete.

Finally, putting all pieces together, we can bring about the following:
Corollary 2.15. An $n$-zero-$C^m$ IMF, for $m \in \mathbb{N}$, has infinitely many $a(t)\sin\psi(t)$ representations for $\psi(t) \in C^\infty$ and $a(t) \in C^m$.

3. Which Representation Is the Best?

The results in previous section were all of theoretical taste. However, the implications in practice are real. For example, take the signal $s(t) = \sin\left(20t + \frac{1}{2} \sin(20t)\right)$. This FM signal is shown in Figure 3. Comparing to a simple sinusoid having the same period as $s(t)$, we can observe the shifted sharp peaks and troughs; see Figure 3. Signal $s(t)$ has many different envelopes depending on the way the phase function is defined. For example, if the phase function is picked to be $\theta(t) = 20t + \frac{1}{2} \sin(20t)$, the envelope would be a smooth constant curve; see the blue curve in Figure 4. On the other hand, if the phase function is picked to be $\theta(t) = 20t$, the envelope would be a smooth oscillatory curve; see the red curve in Figure 4. Both phase functions are in $C^\infty$. But, which one is better? Should the burden of oscillation be put on the phase of envelope? If it is put on the envelope, the IF of the signal is constant, i.e. $\omega = 20$. If it is on the phase function, the IF is $\omega = 20 + 10 \cos(20t)$. In fact, there is no preference from a mathematical point of view, as depicted in previous sections. Generally, the non-uniqueness of representation can also be observed when different adaptive signal processing methods, with different definitions of IF, are being used. In the following examples, we show the non-uniqueness of these representation for different adaptive signal processing methods.

Example 3.1. Take the oscillatory signal $f(t) = (1 + 0.1 \cos(2\pi t)) \sin(100\pi t)$. The signal is shown in Figure 5. The IFs of the signal using using different methods are shown in Figure 18. The IFs definition are based on HT (1.2) of an AS, $C^0$ Phase function frequency, Sparse Time-Frequency Representation (STFR) method (1.9) presented in [15], and NHT method [19], respectively. Both $C^0$ Phase functions frequency and NHT method show ripples in IF, see Figure 18. This is because these methods rely on the sampling rate of the recorded
Figure 3. Blue Curve: \( s(t) = \sin \left( 20t + \frac{1}{2} \sin(20t) \right) \). Black Curve: \( y(t) = \sin(20t) \).

Figure 4. Blue Curve: envelope for \( \theta(t) = 20t + \frac{1}{2} \sin(20t) \). Red Curve: envelope for \( \theta(t) = 20t \).

signal to find either the zeros or the extrema. The HT method shows instability issue at the boundary that is due to the global definition of the HT (1.2). As can be seen from this
simple example, all these different definitions of the IF show different results and consequently different representations of the same IMF.

**Example 3.2.** In this example, we take an FM signal

\[ f(t) = (1 + 0.1 \cos(2\pi t)) \sin(100\pi t) + \sin(50\pi t) \]

The signal is shown in Figure 7. The IFs of the signal using using different methods are shown in Figure 8. This time, the HT (1.2) definition of the IF is drastically different from STFR IF (1.9) presented in [15], and also NHT IF [19]. The STFR and NHT IFs overlap with very small differences. Again, all these different definitions of the IF possibly show different results and consequently different representations of the same IMF.
Figure 6. Comparison of Instantaneous Frequencies: HT frequency in red, $C^0$ Phase function frequency in dashed blue, STFR frequency in pink, and EMD NHT in black.

Figure 7. $f(t) = (1 + 0.1 \cos(2\pi t)) \sin(100\pi t + \sin(50\pi t))$

Based on these examples, it seems that the STFR IF (1.9) and NHT IF [19] are in general closer to each other in what they extract as an IF. These two methods apparently prefer to put the burden of oscillation on the IF rather than the envelope. However, we know that for
Intrawave signals this agreement would not hold \cite{30}. In fact, some adaptive methods like the work in \cite{30} allow oscillatory envelopes. As can be seen, different methods of adaptive signal processing would possibly adopt different representations of the same IMF. From a numerical and pure signal processing perspective, it is just a matter of preference to put the burden of oscillation on either the envelope or the IF. Can we answer the question of “a best representation” from another perspective? We will address this in the next section.

4. Physical Non-Uniqueness

In this section, we show that even from a physical perspective, we cannot have the best representation among all infinite representations of an IMF. We will depict our perspective by showing that the oscillatory signals arising from linear second order Ordinary Differential Equations (ODEs) would not accept a unique IMF representation. In this part we first confirm the assertion that many physical signals have IMF manifestations. In other words, we will first talk about the oscillatory solutions of linear second order ODEs, and then show that solutions are showing the behavior of non-unique IF representation.
4.1. **Linear Second Order ODEs.** Usually a second order homogeneous ODE is expressed as

\[
\ddot{u} + a(t) \dot{u} + b(t) u = 0. \tag{4.1}
\]

However, to extract the IMF solutions, equations like

\[
\frac{d}{dt} \left( p(t) \frac{dv}{dt} \right) + q(t) v = 0, \tag{4.2}
\]

or

\[
\ddot{v} + q(t) v = 0, \tag{4.3}
\]

are of more interest. To convert (4.1) into (4.2), we can substitute \( v = u, \ p(t) = e^{\int a(\xi) d\xi}, \) and \( q(t) = b(t) e^{\int a(\xi) d\xi}. \) In order to convert (4.1) into (4.3), we only need to take \( v = e^{\frac{1}{2} \int a(\xi) d\xi} u, \) and \( q(t) = b(t) - \frac{1}{2} a^2(t) - \frac{1}{2} \dot{a}(t). \) This transformation is interesting since \( e^{\pm \frac{1}{2} \int a(\xi) d\xi} \) is always positive and will not change the zeros of the solution. Hence, if \( u \) has an IMF representation, so does \( v, \) and vice versa.

As will be shown, the solutions of many of the second order ODEs are IMFs. Theory accompanied by examples show how linear and nonlinear homogeneous second order ODEs have solutions that are essentially IMFs.

4.2. **Prufer Transformation for Linear Second Order ODEs.** One classical way of extracting IMF-like solutions of second order ODEs is using the Prufer transform. The Prufer transformation is a mathematical transformation that proves that the solution of certain linear second order ODEs are IMFs.

Take the following differential equation in the canonical form

\[
\frac{d}{dt} \left( P(t) \frac{du}{dt} \right) + Q(t) u = 0,
\]

\[
\begin{align*}
  u(A) &= u_0, \\
  \dot{u}(A) &= \dot{u}_0,
\end{align*}
\]
where \( P > 0, P \in C^1 [A, B] \) and \( Q \in C [A, B] \). Remember that it is possible to have \( B = \infty \).

The solution of this differential equation can be represented in a new coordinate system using the Prufer transformation

\[
\begin{align*}
  u &= r \sin \theta, \\
  P \frac{du}{dt} &= r \cos \theta.
\end{align*}
\]

(4.5)

This transformation explicitly shows that the envelope \( r \) is strictly positive:

\[
r = \sqrt{u^2 + (P \dot{u})^2} > 0.
\]

The last inequality holds true since for any non-zero initial conditions, the solution should not get to zero for both \( u \) and \( \dot{u} \). Using (4.3), one can convert (4.4) into

\[
\begin{align*}
  \dot{\theta} &= Q \sin^2 \theta + \frac{1}{P} \cos^2 \theta, \\
  \dot{r} &= \left( \frac{1}{P} - Q \right) r \cos \theta \sin \theta, \\
  u_0 &= r_0 \sin \theta_0, \\
  P (A) \dot{u}_0 &= r_0 \cos \theta_0, \\
  \theta_0 &\in [0, 2\pi).
\end{align*}
\]

(4.6)

If \( Q > 0 \), the phase derivative is always positive and \( \theta \) is strictly increasing. In other words, considering \( r > 0 \), the solution is nothing but an IMF. This transformation shows that a large class of second order linear ODEs have solutions that are in the form of IMFs. Furthermore, this transformation shows that the oscillatory solutions of Legendre, Hermite, Laguerre and Chebychev equations are all of the IMF type in certain domains.

In addition, the solution can be an IMF in the complete classical sense. This can be observed as \( P \dot{u} = r \cos \theta \) shows that the derivative of the solution \( u \) goes to zero only once between two consecutive zeros of the solution \( u \) itself. In the following, a few examples support the ideas expressed so far.

\[^3\theta \text{ and } r \text{ are called the Prufer variables; namely the Prufer angle and Prufer radius, respectively.}\]
Example 4.1. (Linear IMF) The solution of

\[
\frac{d}{dt} \left( \frac{du}{dt} \right) + u = 0
\]

is an IMF of a constant envelope and constant IF: \( u = c_1 \cos(t) + c_2 \sin(t) \).

Example 4.2. (Chebychev IMF) The Chebyshev’s Differential Equation, for \( |t| < 1, \alpha > 0 \) is

\[
\frac{d}{dt} \left( \sqrt{1-t^2} \frac{du}{dt} \right) + \frac{\alpha^2}{\sqrt{1-t^2}} u = 0.
\]

The solution of this equation is

\[
u = c_1 \cos \left( -\alpha \cos^{-1} t \right) + c_2 \sin \left( -\alpha \cos^{-1} t \right)
\]

for some constant envelopes \( c_1, c_2 \). Taking \( \theta = -\alpha \cos^{-1} t \),

\[
\frac{d}{dt} \left( -\alpha \cos^{-1} t \right) = \frac{\alpha}{\sqrt{1-t^2}} > 0.
\]

In other words, the solution of the Chebychev differential equation, in this representation of the phase angle, is a constant envelope Frequency Modulated (FM) IMF.

Example 4.3. (Bessel IMFs) Take the zero order Bessel ODE on \((0, \infty)\)

\[
t^2 \frac{d^2 u}{dt^2} + t \frac{du}{dt} + t^2 u = 0.
\]

It is possible to convert this into

\[
\frac{d}{dt} \left( t \frac{du}{dt} \right) + tu = 0.
\]

Using the Prufer transformation, one gets

\[
\dot{\theta} = t \sin^2 \theta + \frac{\cos^2 \theta}{t} > 0, \forall t \in (0, \infty),
\]

which clearly shows the IMF behavior of the solution in \((0, \infty)\).
4.3. Oscillatory Solutions of ODEs in Literature. What we showed in previous part was showing the possibility of IMF solutions for some ODE equations. In this part, we will bring about more theory about IMF solutions of ODEs.

As mentioned in [26], it is not hard to show that if \( x(t) \) is an IMF, having a representation \( a(t) \cos \theta(t) \), then one possible governing differential equation is

\[
\ddot{x} + \left(-\frac{\ddot{\theta}}{\theta} - 2\frac{\dot{a}}{a}\right) \dot{x} + \left(\ddot{\theta}^2 + \frac{\dot{a} \ddot{\theta}}{a \theta} + 2 \left(\frac{\dot{a}}{a}\right)^2 - \frac{\ddot{a}}{a}\right) x = 0. \tag{4.10}
\]

Knowing the coefficients in front of \( \dot{x} \) and \( x \), one can solve for the unknown envelope \( a \) and the phase function \( \theta \). However, equation (4.10) is not always easy to solve for \( a \) and \( \theta \). Hence, a better method is needed to find possible IMF solutions. Furthermore, one needs methods to prove that equations of the form \( \ddot{u} + a(t) \dot{u} + b(t) u = 0 \) are sources of IMFs. Some of the theorems and transformations in Oscillation Theory that prove the oscillatory nature of linear equation solutions are viewed here. In particular, we present theorems from Oscillation Theory literature that show the possible oscillatory nature of linear second order ODEs.

The first theorem, mentioned in this part, expresses the interval increase in the phase function if the solution is oscillatory. The second explains how it is possible to understand whether an equation has oscillatory solutions and how one can find the minimum number of zeros. These theorems could act as companions to the Prüfer transformation.

**Theorem 4.4.** There is at most one value of \( t_n \in [A, B] \) such that \( \theta(t_n) = n\pi, n \in \mathbb{N} \). Also, \( \theta(t) \) will remain strictly above that in \((t_n, B)\), and strictly below that in \((A, t_n)\) for (4.4) using (4.5).

For the proof of this theorem, see [1] Theorem 8.4.3.

**Theorem 4.5.** Let \( p_1, p_2, g_1 \) and \( g_2 \) be piecewise continuous on \([A, B] \), satisfying \( 0 < p_2(t) \leq p_1(t) \), \( g_1(t) \leq g_2(t) \) \( \forall t \in [A, B] \). Let \( u_1 \) and \( u_2 \) be solutions to the equation

\[
\frac{d}{dt} \left(p_i(t) \frac{du_i}{dt}\right) + g_i(t) u_i = 0 \text{ for } i = 1, 2. \]

We know \( \theta_1 \) and \( \theta_2 \) are defined by \( \tan \theta_i = \frac{u_i}{p_i u_i} \),
using (4.5). Let \( \theta_2 (A) \geq \theta_1 (A) \). Then \( \theta_2 (t) \geq \theta_1 (t) \) \( \forall t \in [A, B] \). Moreover, if \( g_1 (t) < g_2 (t) \) \( \forall t \in [A, B] \), then \( \theta_2 (t) > \theta_1 (t) \) \( \forall t \in [A, B] \).

For the proof of this theorem, see [7] Theorem 8.1.2. In this theorem, if one chooses

\[
p_1 = \max_{t \in [A, B]} \{ p_2 (t) \},
g_1 = \min_{t \in [A, B]} \{ g_2 (t) \}
\]

and if \( \frac{g_1}{p_1} > 0 \), then the solution of

\[
\frac{d^2 u}{dt^2} + \frac{g_1}{p_1} u = 0
\]

is necessarily oscillatory. As a result, the solution of \( \frac{d}{dt} \left( p_2 (t) \frac{du}{dt} \right) + g_2 (t) u = 0 \) is indeed oscillatory and has at least the same number of zeros as (4.11). The following theorems express the conditions under which there will be oscillation in a semi-infinite domain [2 31].

**Theorem 4.6.** 1- In (4.4), if \( \int_A^\infty \frac{dt}{P(t)} < \infty \) and \( \int_A^\infty |Q(t)| \, dt < \infty \), then the solution is non-oscillatory on \([A, \infty)\).

2- In (4.4), if \( \int_A^\infty \frac{dt}{P(t)} = \infty \) and \( \int_A^\infty Q(t) \, dt = \infty \), then the solution is oscillatory on \([A, \infty)\).

**Theorem 4.7.** (Kneser’s Theorem) Having the differential equation

\[
- \ddot{u} + q(t) \, u = 0
\]

on \((0, \infty)\), then

1- \( \lim_{t \to \infty} \inf (t^2 q(t)) > -\frac{1}{4} \) implies that the solution of (4.12) is not oscillatory;

2- \( \lim_{t \to \infty} \inf (t^2 q(t)) < -\frac{1}{4} \) implies that the solution of (4.12) is oscillatory.

More literature on the oscillatory nature of these equations for periodic cases (i.e. Hill’s equation (4.12)) can be found in [24].

Now, if we sum up all the mentioned theorems in this part, we can conclude that if the solution of (4.4) is oscillatory, by using any of Theorems 4.5 4.6 or 4.7 then using (4.5) and Theorem 4.5 we can say that the phase function of the solution is always increasing; Which is to say that the solution is an IMF.
4.4. **Fundamental IMF Solutions of Linear Second Order ODEs.** Previously, the Prufer transformation was used to express an IMF solution for a general initial value problem. The Prufer transformation is only one way of expressing IMF solutions. Here, we investigate the fundamental IMF solutions of second order ODEs. In other words, we investigate a model that is more general than Prufer transformation. It will depict non-uniqueness in IF and IA of IMF solutions of the second order ODEs.

The solution of $\ddot{y} + q(t)y = 0$ in the form of $r \cos \theta$ is detailed in [24]. Here, we present a formalization of fundamental solutions of such equations. Take the following Second Order ODE

$$
\ddot{y} + q(t)y = 0,
$$

where $q(t) \in \mathbb{C}$. Having this, the solution exists for any initial value problem. Furthermore, all solutions of (4.13) could possibly show IMF behavior:

**Theorem 4.8.** $y_1 = r \cos \theta$ and $y_2 = r \sin \theta$ are two linearly independent solutions of (4.13) if and only if

$$
\begin{cases}
\dddot{r} - r \dot{\theta}^2 + rq = 0, \\
r^2 \dot{\theta} = 1.
\end{cases}
$$

**Proof.** First assume that $y_1 = r \cos \theta$ and $y_2 = r \sin \theta$ are linearly independent solutions. One can easily put them in (4.13) separately to find:

$$
- \left( \ddot{r} - r \dot{\theta}^2 + rq \right) \sin \theta + \left( -r \ddot{\theta} - 2 \dot{r} \dot{\theta} \right) \cos \theta = 0,
$$

$$
\left( \dddot{r} - r \dot{\theta}^2 + rq \right) \cos \theta + \left( -r \ddot{\theta} - 2 \dot{r} \dot{\theta} \right) \sin \theta = 0,
$$

which has the unique solution

$$
\begin{cases}
\dddot{r} - r \dot{\theta}^2 + rq = 0, \\
- r \ddot{\theta} - 2 \dot{r} \dot{\theta} = 0.
\end{cases}
$$
Hence, \( r \) cannot be zero since this would make \( y_1 \) and \( y_2 \) linearly dependent. Hence, \( r \ddot{\theta} + 2r \dot{\theta} = 0 \Rightarrow \frac{d(r^2 \dot{\theta})}{dt} = 0. \) \( \dot{\theta} \) cannot be zero either, since this would make \( y_1 \) and \( y_2 \) linearly dependent. Without loss of generality, one can take \( r^2 \dot{\theta} = 1. \)

On the other direction, put \( y_1 = r \cos \theta \) and \( y_2 = r \sin \theta \) in \( \ddot{y} + q(t)y = 0 \) and check that based on (4.14), the original equation (4.13) is satisfied. Finally, one can check the independence by constructing the Wronskian

\[
W(y_1, y_2) = y_1 \dot{y}_2 - \dot{y}_1 y_2 = r^2 \dot{\theta} = 1 \neq 0.
\]

This argument is saying that all solutions of (4.13) can be written as

\[
y = c_1 r \cos \theta + c_2 r \sin \theta
\]

for some real constants \( c_1, c_2, \) and \( r, \theta \) satisfying (4.14).

Remember that the fundamental solution (4.15), and the fundamental conditions (4.14) are not necessarily conditions for the existence of IMFs as solutions. They are proposing solutions represented in \( r \cos \theta \) and \( r \sin \theta \) format. If certain conditions are satisfied on \( q(t) \), e.g. Theorem 4.7, then the solutions are IMFs.

Equation (4.14) shows that the IF is strictly positive. Another consequence is that as the IF becomes small, the envelope \( r \) increases to compensate, and as the IF increases in time, the envelope is damped.

We can further analyze fundamental conditions (4.14). Assume that, using Theorem 4.5, 4.6, 4.7, or 4.5, the fundamental solutions are IMFs. Based on \( r^2 \dot{\theta} = 1 \), one can observe that the envelope (IA) \( r \) and the IF \( \dot{\theta} \) are not independent from each other; and they are not unique. They are dependent part of the IMF produced by the linear second order ODE.

It is important to note that when the solution of an initial value problem is needed, one should use (4.14) with caution. Assume that the solution of (4.13) is to be found under the initial conditions

\[
y(0) = A, \\
\dot{y}(0) = B.
\]
In (4.14), for simplicity and without loss of generality, one can set \( \theta (0) = 0 \) and \( r (0) = 1 \).

However, there is no way to find \( \dot{r} (0) \) using the initial conditions given. In fact, \( \dot{r} (0) \triangleq \dot{r}_0 \) remains as a free parameter. Using the initial conditions given, the solution (4.15) would become

\[
y = Ar \cos \theta + (B - A\dot{r}_0) r \sin \theta.
\]

This is nothing but the dependency of \( r \) and \( \theta \); and also the non-uniqueness of the representation of and IMF solution. The dependency of \( \theta \) and \( r \) in (4.14) would bring about the nonlinear second order ODE

\[
\ddot{r} - \dot{r}^3 + rq(t) = 0,
\]

\[
r (0) = 1,
\]

having \( \dot{r} (0) \) as a free parameter. This free parameter is the source of different representations (manifestations) of the same IMF. This can also be seen in

(4.16)

\[
\theta (t) = \int_{r_0}^{r(t)} \frac{d\xi}{\sqrt{r^2 (\xi)}}.
\]

However, having \( \dot{r} (0) \) as a free parameter will not deter the uniqueness of the IMF solution itself.

There is significant difference between the Prufer transformation (4.5), and the fundamental solution conditions (4.14). The fundamental solution conditions (4.14) provide an observation that the solution of all linear second order ODEs are necessarily IMF-like functions. Equation (4.14) does not say that the solution is necessarily an IMF. However, the Prufer transformation (4.5) is a good method to perceive whether or not the solution is essentially an IMF, when combined with appropriate theorems.

5. Discussion and Conclusion

In this paper we could show that the IF non-uniqueness in the representation of an IMF, arises at two different places: It shows up either at the regularity type of \( \theta (t) \), or it will show up as a infinitely many acceptable forms of \( \theta (t) \), for a fixed regularity. The vague nature of the IF occurs when its regularity, or relation with the envelope (IA) of an IMF.
is under-looked. The latter happens so often if one compares the different regularities that
different researchers set forth in defining the matter. We showed the latter by some numerical
examples. For instance, the regularity and definition of the IF in EMD method is not exactly
the same as HT [1,2]. Hence, one should not expect similar extraction of IF among different
adaptive signal processing methods.

The IMF representation non-uniqueness can also be observed and quantified for physical
systems following second order ODEs. Many of these systems have IMF solutions. However,
the IMFs are all non-unique in manifestation. Even referring to such simple systems, it is
impossible to define, or set, the best possible representation for the IF and IA.

In general, there is an uncertainty in defining the IF and the envelope (IA) of an IMF,
irrespective of the source of production. Not IF by its own, nor envelope by itself can explain
the whole dynamics of an IMF.

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