QUASICLASSICAL APPROACH TO SYNERGIC SYNCHROTRON–CHERENKOV RADIATION IN POLARIZED VACUUM

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Keywords: synchrotron radiation, Cherenkov radiation, vacuum polarization, muons, quantum electrodynamics

Abstract
Photon emission by an ultrarelativistic charged particle in extremely strong magnetic field is analyzed, with vacuum polarization and photon recoil taken into account. Vacuum polarization is treated phenomenologically via refractive index. The photon emission occurs in the synergic (cooperative) synchrotron–Cherenkov process [J Schwinger et al 1976 Annals of Physics 96 303] which is similar to the synchrotron emission rather than to the Cherenkov one. For electrons, the effect of vacuum polarization on the emission spectrum is not evident at least below the probable onset of non-perturbative quantum electrodynamics (QED). However, the effect of vacuum polarization on the emission spectrum can be observable for muons already at \( \gamma B / B_S \approx 30 \), with \( \gamma \) the muon Lorentz factor, \( B \) the magnetic field strength and \( B_S \) the critical QED field. Nevertheless, vacuum polarization leads to only 10% enhancement of the maximum of the radiation spectrum.

1. Introduction
Quantum electrodynamics (QED) predicts nonlinear dielectric properties of the vacuum in strong magnetic field caused by virtual electron–positron pairs. The Kramers–Kronig relations connect vacuum refractive index with pair photoproduction probability, and the latter have been studied in strong crystalline fields [1] and in laser field [2, 3]. Still direct experimental evidence of vacuum refractive index is absent, and many set-ups have been proposed to detect and measure it, e.g. x-ray diffraction on a double-slit formed by two counterpropagating intense laser pulses [4–6], or changes in polarization of x or gamma photons due to vacuum birefrigence in strong laser field [7–10]. The idea behind these proposals is not only to measure vacuum refractive index but to test QED in a not-yet-investigated region of extreme laser fields. Moreover, investigation of vacuum polarization becomes important in the light of Ritus–Narozhny conjecture of perturbative QED breakdown at certain conditions [11–15].

The fields of intensity \( 10^{23}–10^{24} \) W cm\(^{-2} \) is expected in near future thanks to facilities such as ELI-NP [16], ELI-beamlines [17], Apollon [18], Vulcan 2020 [19], XCELS [20] and others. Therefore, the field of the order of \( 10^{-3} \times B_S \) will be available which results the vacuum refractive index \( n \) such that \( \delta n = n - 1 \approx 10^{-10} \) for photons with energy \( \lesssim 1 \) GeV [21], with \( B_S = m^2 c^3 / e \hbar \) the Sauter–Schwinger critical QED field [22], \( m \) and \( e > 0 \) the electron mass and charge magnitude, respectively, \( c \) the speed of light and \( \hbar \) the reduced Plank constant. Despite such small value of \( \delta n \), the Lorentz factor \( \gamma \sim 10^5 \), available for electrons nowadays, is enough to reach the speed of a charged particle greater than the phase speed of the photons, hence the Cherenkov emission may occur. Such estimates drives the recent interest to Cherenkov emission in the polarized vacuum [23–25]. However, the results of these papers should be reconsidered because of simplified approach used there. A charged particle in the strong field inevitably moves along a curved trajectory that prevents plain Cherenkov radiation. The trajectory curvature determines the radiation formation length and is crucial for the emission process. Furthermore, there are unified emission process [26], and it is not possible to distinguish ‘Cherenkov’ and ‘Compton’ emission mechanisms in the considered situation, as [23, 24] do. In [25] the authors, although consider Cherenkov emission and nonlinear Compton scattering as a single process, use expression for the emitted energy and
for the formation length [see (16) and (17) therein] as if the particle moves along the straight line and emits photons in a plain Cherenkov process. At the same time, earlier works on the considered topic contain not only qualitative estimates, but expressions for the spectrum and for the photon emission probability.

In 1966 Erber was the first who pointed out the possibility of Cherenkov radiation in polarized vacuum [27]. He used expression for pair photoproduction in a constant magnetic field and dispersion relation to compute the real part of the refractive index, following work of Toll in 1952, see reference [28] and references therein. In 1969 Ritus considered possibility of Cherenkov radiation in a constant crossed electromagnetic field [29], using photon Green’s function obtained year before by Narozhny. Thus, there is no need in two laser pulses which create regions with pure magnetic field, and it is enough to use single laser pulse to induce vacuum polarization. Then, in 1976 Schwinger, Tsai and Erber with QED mass operator method obtained the general expression for the spectrum of the photon emission by a charged particle which moves both in a constant magnetic field and in a medium with \( n \neq 1 \) [26]. They pointed out that ‘there is actually only a single emission act, synergic synchrotron–Cherenkov radiation, for which a correspondence with either Cherenkov emission or synchrotron radiation can be established only in the respective limits of vanishing field or matter density’, and that ‘the practical import of this synergism is that the radiation depends sensitively on both positive and negative values of \( n − 1 \)’. They demonstrated [26, 30] that depending on parameters, both amplification and suppression (quenching) of the photon emission may occur. Finally, synergic synchrotron–Cherenkov radiation in gases was observed in the experiment [31], which results agree well with the analysis which treats Cherenkov and synchrotron radiation as limiting manifestations of a unified process.

Another interesting result of Erber et al is that the Cherenkov condition for electrons \( v > c/n \) (with \( v \) the electron velocity) is not enough for spectrum of the synchrotron–Cherenkov radiation in polarized vacuum to be different from purely synchrotron spectrum. The sufficient condition for this occurs extremely strict when the photon emission is considered as a synergic synchrotron–Cherenkov process (see (8.8e) and (8.11) in [30]):

\[
\chi = \frac{B}{B_S} \gamma \gtrsim 10^3, \tag{1}
\]

which for \( B/B_S \sim 10^{-3} \) yields enormous energy \( mc^2 \gamma \sim 100 \text{TeV} \). Here \( B \) is the magnetic field strength, \( \gamma \) the electron Lorentz factor, and the electron velocity \( v \) is assumed to be perpendicular to the magnetic field. Erber then suppose that the condition (1) indicates that higher order QED corrections besides vacuum polarization should be also taken into account, i.e. QED is no longer a perturbative theory. Indeed the threshold \( \chi \) value (1) is even far beyond the conjectured value of the perturbative QED breakdown [11, 29] \( \chi \sim 1/\alpha^{3/2} \approx 1.6 \times 10^3 \).

The aim of the current paper is manifold. First, the physical picture of photon emission by ultrarelativistic particles is recalled and applied to the synchrotron emission in a medium with \( \delta n \ll 1 \), within the classical theory (section appendix A). A special attention is paid to the synchronism between the emitting particle and the emitted wave. Second, the general quantum formulas for spectral and angular distribution of the emitted photons in synergic synchrotron–Cherenkov process are obtained (section 2), for that quasiclassical theory of Baier et al [32] is used. This allows to take into account photon recoil neglected in references [26, 30]. Third, in section 3 the onset of Cherenkov corrections to the synchrotron spectrum is found. Following the proposal of Erber [30], synchrotron–Cherenkov emission by particles heavier than electrons is considered in details in section 3.2. It is shown that the onset of Cherenkov corrections to the synchrotron spectrum for muons occurs at much lower value of \( \chi \) than that for the electrons, due to enlarged formation length and weakened photon recoil. Section 4 is the conclusion.

2. Synchrotron–Cherenkov radiation in quantum electrodynamics

2.1. Quasiclassical theory of the synchrotron–Cherenkov radiation

In any realistic set-up the vacuum refractive index \( n = 1 + \delta n \) is extremely close to unity. For example, to formally fulfill the Cherenkov condition in the polarized vacuum for low-energy photons emitted by electrons, \( \chi = 70 \) is enough. Thus, one can choose \( \gamma = 7 \times 10^4 \) and \( B/B_S = 1 \times 10^{-3} \). As seen from section 3.1, this value of the magnetic field lead to the refractive index such that \( \delta n \approx 2 \times 10^{-10} \).

In the classical electrodynamics the key feature of the emission process is the synchronism between the emitting current and the emitted wave, as demonstrated in appendix A. The synchronism is the only thing which can be influenced by the vacuum polarization with such small \( \delta n \). Thus, to take into account the refractive index in the classical electrodynamics, one should use exact relation between the photon frequency and the wave vector \( k = n\omega/c \) only in the phase of (A.11).
In the quantum theory, similarly to the classical one, the refractive index should be taken into account first in the phase of the corresponding quasiclassical formulas (see equation (1.2) in [33]). However, an extra phase term \( \propto \omega^2 - c^2 k^2 \) appears in the quasiclassical formula [32, 34] which though can be neglected in the case of refractive index of polarized vacuum (see appendix C; this term probably should be taken into account for electrons with \( \chi \gtrsim 10^3 \)). Thus, in the polarized vacuum the energy emitted per unit frequency interval and per unit solid angle is determined by (C.9), where the refractive index is present only in the exponential function

\[
\exp[i\omega'(t - n\rho)],
\]  

with \( n = ck/\omega, |n| = n \) the refractive index, \( k \) and \( \omega \) the wave vector and the cyclic frequency of the emitted photon, \( \omega' = \omega - \epsilon' \epsilon' \epsilon \) the electron energy, \( \epsilon' = \epsilon - n\omega \) and \( \rho = r/c \) the normalized electron position.

Equation (C.9) differs from the classical one (A.11) by two quantum features. First, an additional 'spin' term appears in (C.9) (the last term, which originates from the spin flips [32, 35, 36]). Second, the radiation recoil arises, which is reflected, first of all, in the fact that \( \omega \) is substituted with a higher frequency \( \omega' \) in the exponential. Hence, if the photon energy is about the electron energy, the synchronism is strongly affected by the recoil. Particularly, the recoil effect squeezes the photon spectrum such that it is limited by the energy of the critical frequency, and \( \omega' = \omega_{\perp} = \omega_{\parallel} + 1 \).

All the terms in (C.9), including the spin term, contain the same exponential function. In local constant field approximation the phase of the exponential function can be expanded using the T aylor series as in appendix B, \( \exp[i\omega'(t - n\rho)] \approx \exp[i\phi(t)] \) with

\[
\phi(t) = 2\pi \left[ \frac{\omega_{\perp}}{\omega'}/\gamma^2 - 1/2\delta n \right]
\]

(note that if the local constant field approximation is not valid, the refractive index should be used not only for the emitted photons, but also for the laser field [37]). Therefore, as for the classical synchrotron emission, the spectrum of the synchrotron–Cherenkov emission at the given frequency \( \omega \) is governed by the only two timescales

\[
\tau_{\perp} = \left( \frac{12 \pi \gamma^2}{\omega_\perp^2 \omega B^2} \right)^{1/3},
\]

\[
\tau_1 = \frac{4\pi}{\omega' \theta^2 + 1/\gamma^2 - 2\delta n},
\]

with \( \theta \) the angle between the particle velocity at \( t = 0 \) and the wave vector \( k \), \( \omega_B = eB/mc \) the cyclotron frequency and \( B \) the magnetic field strength. Therefore the timescales differ from the timescales of the classical synchrotron spectrum given by (B.7) and (B.8).

The refractive index brings a novel effect: the sign of the linear term can be changed, i.e. if Cherenkov condition is fulfilled, \( \delta n > 1 \), then at least for \( \theta \approx 0 \) one has \( \zeta = -1 \). In the subsequent sections 2.2 and 3 it is demonstrated that in the case of \( \zeta < 0 \) the emission spectrum can differ dramatically from the synchrotron one. In the remaining part of this section the effect of the radiation recoil is considered in detail, because of its importance both for the case \( \zeta = +1 \) and \( \zeta = -1 \).

Similarly to the classical synchrotron emission, the critical frequency \( \omega_c \) can be introduced

\[
\omega_c = \frac{\epsilon \omega'_{\perp}}{\epsilon + \hbar \omega_{\perp}},
\]

\[
\omega'_{\perp} = \frac{3\omega_{\perp}^2 - 2\delta n \gamma^2}{|1 - 2\delta n \gamma^2|^{1/2}},
\]

such that for it \( \tau_{\perp} \) and \( \tau_1 \) are of the same order for \( \theta = 0 \), namely for \( \omega = \omega_c \) one has \( \tau_{\perp}/\tau_1 = 3/(4\pi)^{1/3} \approx 0.56 \).

If the quantum parameter is small, \( \chi = \gamma B / B_S \ll 1 \), quantum formulas tend to classical ones, i.e. radiation recoil is negligible, \( \omega_c \approx \omega'_{\perp} \approx 3\omega_{\perp}^2 \gamma^2 \), and the spin term is negligible. In this case (if additionally \( \delta n = 0 \)), the maximum of \( d^2I/d\omega d\Omega \) is in the point \( \theta = 0 \) and \( \omega \approx 0.42\omega_c \) that reveals the physical meaning of \( \omega_c \) in this case.

In the quantum limit, \( \chi \gg 1 \) (and for \( \delta n = 0 \)), (8) yields \( \hbar \omega'_{\perp} \gg \epsilon \), that looks non-physical if one neglects the effect of radiation recoil and sets \( \omega_c = \omega'_{\perp} \). Actually, the radiation recoil changes significantly the critical frequency, and \( \omega_c \) differs significantly from \( \omega'_{\perp} = \hbar \omega_c \approx \epsilon[1 - 1/(3\chi)] \). Thus \( \omega_c \) is very close to
the upper spectrum bound $\epsilon/\hbar$. Therefore, in the quantum limit almost for all frequencies one has $\tau_\perp \ll \tau_\parallel$, and the term $t/\tau_\parallel$ can be neglected in the phase of the exponential.

For the refractive index of the polarized vacuum the Cherenkov condition is fulfilled only in the quantum case, i.e. $n/\beta > 1$ can be reached only if $\chi \gg 1$ (this is discussed in section 1 and especially in section 3). As seen from (5) and (6), the refractive index affects $\tau_\parallel$ only, hence only the linear term in the phase. However, in the quantum case this term is negligible for almost all frequencies in the synchrotron spectrum. Therefore, in order to modify the synchrotron spectrum, the refractive index should be large enough not only to change $\tau_\parallel$, but make it much lower than in the case of $\delta n = 0$. That is why the Cherenkov condition is far from being enough to change the synchrotron spectrum.

2.2. Radiation formation length

The radiation formation length is the length of the electron path which contributes most to the integrals in (A.11) and (C.9). Obviously, the radiation formation length depends on the frequency of the emitted wave, although for synchrotron emission often $\omega \sim \omega_c$ is assumed.

The radiation formation time (i.e. the radiation formation length divided by $c$) can be estimated by consideration of the following integral:

$$I(t_a, t_b) = \int_{t_a}^{t_b} f(t) \sin[\phi(t)] dt,$$

with $f(t)$ and $\phi(t)$ slowly varying functions and $\sin[\phi(t)]$ contains many oscillation periods on the interval $[t_a, t_b]$. The contribution of a single oscillation period can be estimated as follows:

$$I(t_0, t_2) = \int_{t_0}^{t_2} f(t) \sin[\phi(t)] dt \approx 2f(\tau_\perp) T|_{(t_0+\tau_\perp)/2} - 2fT|_{(t_1+\tau_\perp)/2} \approx -\int_{t_0}^{t_2} \left( \frac{d}{dt} fT \right) dt.$$  \hspace{1cm} (10)

Here $t_0, t_1, t_2$ are the time instants which correspond to $\phi = 0, \pi, 2\pi$, respectively. The continuous function $T(t)$ is approximately equal to the local period of function $\sin[\phi(t)]$. Obviously, the estimate (10) for $I(t_a, t_b)$ can be extended to an arbitrary integer number of periods between $t_a$ and $t_b$. In this case the integral can be estimated as the difference between the integral of the first ‘bump’ (the first half-period) and the last one, whereas the intermediate ‘bumps’ do not contribute. Finally, the integral which determines the convergence speed of $I(-\infty, \infty)$ can be estimated as follows:

$$I(t, \infty) \sim f(t) T(t),$$

where we assume that $\lim_{t \to \infty} fT = 0$.

The local oscillation period for the phase (3) far from the saddle points is

$$T(t) \approx 2\pi \left( \frac{d\phi}{dt} \right)^{-1} = \left( \frac{\zeta}{\tau_\parallel} + \frac{3\tau_\parallel^2}{\tau_\perp^2} \right)^{-1}. \hspace{1cm} (12)$$

The integrals in (C.9) for $\beta_e$, from (B.3) and (B.4) contain $f(t) = 1$ and $f(t) = t$, both lead to the same radiation formation time, thus for simplicity we set $f(t) = t$ from here on.

For synchrotron emission of low frequencies ($\tau_\perp \lesssim \tau_\parallel, \zeta = +1$) the leading ‘bump’ of the integrand has width about $\tau_\perp$ (see figure 1(a)), and for $t \gg \tau_\perp$ one gets $I(t, \infty) \propto 1/t$. Hence the radiation formation time is $t_{rf} \sim \tau_\perp$ in this case. For high frequencies $[\tau_\perp \gg \tau_\parallel, \zeta = +1$, see figure 1(b)] the integrand contribution decays, $I(t, \infty) \propto 1/t$, only if $t \gg t_i$, with

$$t_i = \sqrt{\frac{\tau_\parallel^2}{3\tau_\perp}} \hspace{1cm} (13)$$

a point where the linear and the cubic terms in the phase yield the same oscillation periods. Thus, here the radiation formation time is $t_{rf} = t_i \gtrsim \tau_\perp \gtrsim \tau_\parallel$.

A special consideration needed for the Cherenkov branch of the synchrotron–Cherenkov emission ($\zeta = -1$). If $\tau_\perp \lesssim \tau_\parallel$, the sign of the linear term is unimportant and $t_{rf} \sim \tau_\perp$. However, in the case $\tau_\perp \gg \tau_\parallel$ the integrand changes significantly [see figures 1(c) and (d)]: most contribution to the synchrotron–Cherenkov integrals comes from the regions around two saddle points $t = \pm t_i$. If $\tau_\parallel \ll \tau_\perp$,
the phase near a saddle point (say, \( t = +t_i \)) can be approximated with a parabolic dependency:

\[
\phi = 2\pi \left( -\frac{t}{\tau_\parallel} + \frac{t_i}{\tau_\perp} \right)^2 \approx \frac{6\pi t_i}{\tau_\perp} (t - t_i)^2 + \text{const},
\]

where the term \( 2\pi (t - t_i)^3/\tau_\perp^3 \) is neglected at the right-hand-side. Then the width of the bump at \( t_i \) can be found,

\[
T(t_i) \sim \left( \frac{t_i^3}{\tau_\perp} \right)^{1/2} \sim \tau_\perp^{3/4} \tau_\parallel^{1/4}.
\]

Hence, the cubic term is actually small: \( (t - t_i)^3/\tau_\perp^3 \sim T^3(t_i)/\tau_\perp^3 \sim (\tau_\parallel/\tau_\perp)^3/4 \ll 1 \). The parabolic phase dependency leads to quite fast convergence of the integrals, e.g. \( fT \sim 1/(t - t_i) \) for \( T(t_i) \ll |t - t_i| \ll t_i \).

An important consequence of the estimations above is that \( \tau_\parallel \ll T(t_i) \ll \tau_\perp \ll t_i \) in the case \( \varsigma = -1 \) and \( \tau_\parallel \ll \tau_\perp \). This means that the saddle points \( t = \pm t_i \) are far away from each other, and there is almost random phase shift between the integrals around these points. Thus, instead of coherent sum of the integrals one should sum resulting probabilities which are computed separately for \( t = +t_i \) and \( t = -t_i \) points. This also means that the radiation formation time in this case should be estimated not as the distance between these points, but as the width of the leading bumps, \( t_{rf} \sim T(t_i) \sim \tau_\perp^{3/4} \tau_\parallel^{1/4} \ll \tau_\perp \).

Summing up the above, the radiation formation length for synchrotron–Cherenkov radiation is

\[
t_{rf}/\tau_\perp \sim \begin{cases} 
1 & \text{for } \tau_\perp \lesssim \tau_\parallel, \\
(\tau_\parallel/\tau_\perp)^{1/2} & \text{for } \tau_\parallel \gtrsim \tau_\perp, \varsigma = 1, \\
(\tau_\parallel/\tau_\perp)^{1/4} & \text{for } \tau_\parallel \gtrsim \tau_\perp, \varsigma = -1. 
\end{cases}
\]

Besides the radiation formation length, the synchrotron–Cherenkov integrals depend on the ratio \( \tau_\perp/\tau_\parallel \) itself. For instance, one can note that the integrals decay exponentially with the increase of \( \tau_\parallel/\tau_\perp \), if \( \tau_\perp \gg \tau_\parallel \) and \( \varsigma = +1 \) [see figure 1(b)]. The emission probability becomes negligible in this case. Taking this into account, one notes from (16) that in all the noticeable cases the radiation formation length is less or about \( \tau_\perp \), which do not depend on the refractive index. Note also that in the regime of seemingly dominance of Cherenkov radiation, \( \delta n \gg 1/v^2 \), one has \( \varsigma = -1 \) and \( \tau_\parallel \ll \tau_\perp \) [see figure 1(d)], and the radiation formation time is small, \( t_{rf} \ll \tau_\perp \). This differs significantly from the case of plain Cherenkov radiation, where the radiation formation length can be extremely large.

The presence of the refractive index with \( \delta n \neq 0 \) can however significantly influence the emission probability. The effect of the refractive index in the case \( \varsigma = +1 \) is shown in figure 2, where \( I_0, I_\pm \) and \( I_\parallel \) are the emitted energy for \( n = 1, n = 1 + 0.1/v^2 \) and \( n = 1 - 0.1/v^2 \), respectively. Figures 2(a)–(c) show frequency and angular distribution, whereas figure 2(d) depicts the energy emitted per unit frequency interval. Distributions \( d^2I/d\omega d\theta \) for figures 2(a)–(c) are computed numerically as described in section 2.3, for \( \gamma = 1 \times 10^5 \) and \( B/B_0 = 3 \times 10^{-3} \) (hence \( \chi = 3 \)). Lines in figure 2(d) are computed by summing up these distributions along the \( \theta \) axis, except the black dotted line which corresponds to the analytical expression for the pure synchrotron emission \( \langle n \rangle = 1 \) found in the framework of BKS theory \([32, 36]\). Note that for such artificial \( \delta n \) the phase term \( \propto \omega^2 - \hat{c}^2 k^2 \) (not included in the computations) is not very small (see appendix G, \( \hat{r} \approx 0.2 \) for \( \hbar \omega \approx \varepsilon \)), thus figure 2 is rather qualitative.
Similarly to the transition from (A.10) and (A.11), one can find from $d^2\mathcal{I}/d\omega d\Omega$, for $n = 1$ ($\delta n = 0$), (b) $d^2\mathcal{I}/d\omega d\theta$, for $\delta n > 0$, and (c) $d^2\mathcal{I}/d\omega d\theta$, for $\delta n < 0$. (d) The spectrum of the synchrotron emission ($\delta n = 0$) computed (dotted line) analytically and (solid line) numerically, as well as the spectra of the synchrotron–Cherenkov radiation for (dashed line) $\delta n > 0$ and (dash-dotted line) $\delta n < 0$, computed numerically. In all the cases $\zeta = +1$ for the whole frequency and angular range. See text for details.

In the high-frequency region, $\omega \gtrsim \omega_c$, the timescales relate as $\tau_\perp \gtrsim \tau_\parallel$ in the absence of $\delta n$. Thus, as the presence of the refractive index with $\delta n > 0$ leads to the increase of $\tau_\parallel$, it also leads to the substantial increase of the emission probability [compare figures 1(b) and (a)]. In the opposite case of refractive index with $\delta n < 0$, the timescale $\tau_\parallel$ decreases that quenches the emission probability. The described picture is confirmed well by figure 2, where the critical frequency is $\hbar \omega_c = 0.9 \times \varepsilon$. For the photons of $\omega \sim \omega_c$, the radiation formation length can be estimated with (16) and (6) as $\tau_\text{eff} \lesssim \tau_\parallel \sim c/\omega_B \sim 10^{-2} \mu\text{m}$. Therefore, as expected, the curvature radius $R \sim \gamma_c/\omega_B$ and the scale of the laser wavelengths $\sim 1 \mu\text{m}$ are much larger than the radiation formation length.

In the low-frequency region ($\omega \ll \omega_c$, hence $\tau_\perp \ll \tau_\parallel$) the synchrotron integrals do not depend on $\tau_\parallel$ which only depend on the refractive index. Therefore, the presence of the refractive index with $\delta n \neq 0$ does not change the low-frequency spectrum, as seen in figure 2. If one increases the parameter $\chi$, the critical frequency tends to $\varepsilon/\hbar$, and the region where the effect of the refractive index is noticeable becomes narrow and pinned to $\varepsilon/\hbar$.

The effect of the refractive index becomes more dramatic if the Cherenkov condition is fulfilled, $\delta n > 1/(2\gamma^2)$. Such case for $\delta n = 1/(2\gamma^2)$ is shown in figure 3 for $\gamma = 1 \times 10^5$ and $B/B_0 = 3 \times 10^{-5}$ (hence $\chi = 3$ and $\hbar \omega_c = 0.9 \times \varepsilon$). Note that for such artificial $\delta n$ the phase term $\propto \omega^2 - c^2 k^2$ which is not included in the computations cannot be neglected (see appendix C, $\tau \approx 2$ for $\hbar \omega \approx \varepsilon$), thus figure 3 demonstrates the spectrum qualitatively. For the chosen refractive index and $\theta = 0$, $\tau_\parallel$ remains the same as for the case $\delta n = 0$, but the sign of the linear term in the phase (3) changes, $\zeta = -1$. As before, the low-frequency part of the spectrum remains the same in the both cases, $\delta n = 0$ [lower half of figure 3(a) and solid red curve in figure 3(c)] and $\delta n > 0$ [upper half of figure 3(a) and dashed blue curve in figure 3(c)]. However, in the case $\delta n > 0$ the high-frequency part of the spectrum is extremely enhanced, such that the overall emitted energy is more than an order of magnitude greater than in the case $\delta n = 0$.

Points A, B, C and D in figures 3(a) and (b) exactly correspond to $a \equiv \zeta \tau_\perp/\tau_\parallel$ used in figures 1(a)–(d), respectively. The fine structure in the radiation distribution at high frequencies is seen in figure 3(b), which shows in details the rectangular region of figure 3(a) marked with black solid line (note the different color scale in these figures). This fine structure emerges due to the interference of the contributions yielded by the two bumps seen in figure 1(d).

Figure 3 looks quite encouraging, however, the refractive index of the polarized vacuum depend on the photon frequency, and have both $\delta n > 0$ and $\delta n < 0$ parts. The latter corresponds to high photon energies. This, together with the fact that the high-frequency region (where the refractive index influences the radiation spectrum) becomes extremely narrow in the case $\chi \gg 1$, makes almost impossible to reveal the effect of vacuum polarization on the synchrotron radiation, at least for electrons (or positrons). The radiation spectrum for electrons is discussed in details in section 3, whereas the next section, section 2.3, is devoted to details of numerical computation of the spectrum.

### 2.3. Numerical implementation

Similarly to the transition from (A.10) and (A.11), one can find from $d^2\mathcal{I}/d\omega d\Omega$ [see (C.9)] the photon emission probability summed up for both polarizations, $W = \sum |C_n|^2$, which is more convenient for
numerical simulations:

\[
W = \frac{e^2 c^2 \pi \omega^2}{4 \hbar \omega^3 V} \left\{ \left( \frac{e^2}{2 \varepsilon^2} \right) \sum_n \left| \int dt \beta e \exp[i\omega(t-n\rho)] \right|^2 + \frac{1}{2} \left( \frac{\hbar mc^2}{\varepsilon^2} \right)^2 \int dt \exp[i\omega(t-n\rho)] \right\}.
\]

The integrals in (17) can be found analytically in some cases if Taylor series for the phase (3) and for the other factors are used. Also, analytical results of [26] can be considered. However, for the sake of simplicity, expression (17) is used here in the numerical computations. Equation (17) is implemented in the open-source code \textit{jE} [38] where a circle particle trajectory is used. As for \textit{jE} code of version 1.0.0, for the integrals in (17) the trapezoidal rule of integration with a fixed time step is implemented. The time step is computed as one-half of the minimal oscillation period reached on the integration interval \([t_b, t_f]\). Thus the resulting number of nodes (time steps) yielding proper accuracy becomes \(\approx 500\) for the artificial attenuation \(g\) is added:

\[
\int_{-\infty}^{\infty} f(t) \sin[\phi(t)] dt \approx \int_{-t_b}^{t_b} g(t)f(t) \sin[\phi(t)] dt.
\]

Here \(t_b\) should be just several times larger than \(t_{\text{tr}}\) (in the code \(t_b \approx 3t_{\text{tr}}\) is chosen), and the function \(g\) should fade smoothly near the boundaries of the integration interval from 1 to 0. Equation (18) can be easily proven by estimating the difference of its left-hand-side and its right-hand-side with formula (11): \((1-g)/T \approx 0\) at the point where \(g(t)\) just starts to fade as well as at \(t_b\). In the code the following attenuation function is chosen:

\[
g(t) = \frac{1}{4} \left\{ 1 - \tanh[8(t/t_b - 0.7)] \right\} \left\{ 1 + \tanh[8(t/t_b + 0.7)] \right\},
\]

which together with the given time step and the integration interval yields error less than 3% in comparison with the integrals without \(g\) computed on an extremely wide integration interval by a different numerical method, at least for \(\gamma \tau_\|/\tau_\perp \in [-48, 0.8]\).

In the code the integration method described above is used in the function which computes photon emission probability with (17). A number of tests is implemented for this function. For instance, in the classical limit (\(\gamma \ll 1\)) energy radiated per unit frequency interval per unit solid angle, \(d^2I/d\omega d\Omega\), computed numerically, is compared with analytical results, namely with (14.83) from [39]. This test shows accuracy of the code better than 0.5% for \(|\theta| \leq 1/\gamma\) and \(\omega \leq 1.6 \times \omega_c\). At this point the value of \(d^2I/d\omega d\Omega\) is already more than 500 times lower than the maximal value of \(d^2I/d\omega d\Omega\), thus although the error becomes greater with the increase of \(\omega\) and \(\theta\), the whole value of \(d^2I/d\omega d\Omega\) can be neglected there. Furthermore, a well-known asymptotic behavior of the full radiated energy is also tested (namely

**Figure 3.** The effect of the refractive index on the spectrum of the synchrotron radiation in the case of artificial \(dn > 1/(2\gamma^2)\). The energy emitted by an electron per unit photon frequency and per unit angle \(\theta\) for (a), lower half (b) synchrotron--Cherenkov emission with \(dn = 0\), and for (a), upper half (b) synchrotron--Cherenkov emission with \(dn = 1/\gamma^2\). (c) Radiation spectrum for (solid line) \(dn = 0\) and (dashed line) \(dn = 1/\gamma^2\) cases. The Cherenkov angle \(\theta_C = (2n - 1/\gamma^2)^{1/2} = 0.01\) mrad is shown by dotted white line in (b). Points A, B, C and D in (a) and (b) correspond to phase dependency shown in figures 1(a)–(d), respectively. See text for further details.
3. Possible experimental evidence of vacuum polarization

3.1. Synchrotron–Cherenkov radiation of electrons

One can ask for the conditions necessary to modify the well-known synchrotron spectrum because of vacuum polarization. To answer, one first needs to discuss an expression for the vacuum index of refraction in a strong field. The refractive index depends on the photon polarization. For example, in a constant magnetic field $\delta n$ for low-energy photons is about twice greater for the polarization perpendicular to the magnetic field, in comparison with the polarization parallel to the magnetic field. However, the most of the synchrotron photons are polarized perpendicularly to the magnetic field, and the following expression for the real part of the vacuum refractive index can be used [see [21, 27, 29] and references therein]:

$$n(\kappa) = 1 + \frac{\alpha}{4\pi} \left( \frac{B}{B_0} \right) N(\kappa),$$

with $N(\kappa)$ is presented in figure 4(a), $\kappa = (\hbar \omega/ m^2 c^2) (B/B_0)$ is the photon analogue of the $\chi$ parameter, and $B$ the (effective) magnetic field. The asymptotics of $N(\kappa)$ are given by:

$$N(\kappa) = \begin{cases} 
14/45 & \text{for } \kappa \ll 1 \\
-0.278 \times \kappa^{-4/3} & \text{for } \kappa \gg 1
\end{cases}$$

where $\kappa = (\hbar \omega/ m^2 c^2) (B/B_0)$. As seen from figure 4, $\delta n = n - 1$ is positive for $\kappa \lesssim 15$ and negative for $\kappa \gtrsim 15$.

As described in the previous sections, the refractive index influences only the timescale $\tau_\parallel$ in which the electron becomes out of phase with the wave due to the difference of its velocity along the wave vector $k$ and the wave phase velocity. Hence the vacuum polarization influences only the linear term in the phase (3). At the same time the linear term of the phase influences the integrals in (C.9) only if the timescale $\tau_\parallel$ is less or about $\tau_\perp$. In the timescale $\tau_\perp$ the electron becomes out of the phase with the wave due to the trajectory curvature (because the curvature affects the electron velocity along the wave vector). Summing up, and taking into account (5)–(7), the necessary condition of the spectrum change at a given photon frequency $\omega$ is the following:

$$\begin{align*}
|\delta n(\omega)| &\gtrsim 1/\gamma^2, \\
\omega' &\gtrsim \omega'_c(\omega).
\end{align*}$$

The radiation spectrum calculated numerically and analytically for $n = 1$ and $\chi = 3$ is shown in figure 2(d), where the result of jE code is shown with solid red line and the analytical result with dotted black line. A number of tests also is written in order to demonstrate that the mass of the emitting particle, the spin term and the refractive index are treated correctly [38].
If condition (22) holds, then $\tau_{\perp}$ changes noticeably, and if (23) is holds too (with $\omega_{\epsilon}$ computed either with or without vacuum polarization taken into account), then the spectrum changes. Here $\omega_{\epsilon} = \omega / (\epsilon - h\omega)$, and $\omega_{\epsilon}$ is determined by (8) for a given frequency $\omega$ (note that $\delta n$ depends on $\omega$).

It should be noted that the conditions (22) and (23) are very weak: if $\delta n$ is, say, 10% of $1/\gamma^2$ it nevertheless can lead to sizable changes in the spectrum. Also, if $\omega_{\epsilon}$ is 10% of $\omega_{\epsilon}(\omega)$, the spectrum changes noticeably. For instance, for $\chi = 3$ and $\delta n = 0.1/\gamma^2$, one has $\omega_{\epsilon} \approx 0.1 \times \omega_{\epsilon}(\omega)$ for $h\omega \approx 0.5\epsilon$, however, the changes in the spectrum are evident for this frequency, as seen in figure 2(d).

For low-energy photons ($\varkappa \ll 1$) equation (22) can be rewritten using the electron $\chi$ parameter only:

$$\chi \gtrsim \left( \frac{90\pi}{7\alpha} \right)^{1/2} \approx 70. \quad (24)$$

However, condition (23) in this case much harder to fulfill: it also can be written in terms of $\chi$ and yields for $\varkappa = 1$

$$\alpha \chi^{2/3} \gtrsim \frac{3^{2/3} 45\pi}{7} \approx 42, \quad (25)$$

hence $\chi \gtrsim 4 \times 10^5$. One can note that the result (25) is far beyond the conjectured threshold of the perturbative QED breakdown [11], $\alpha \chi^{2/3} \gtrsim 1$. Therefore, the BKS approach used here and the expression for the refractive index (21) are hardly valid if $\alpha \chi^{2/3} \gtrsim 1$ and even more so for $\alpha \chi^{2/3} \gtrsim 2$. Thus the considered theory predicts no change in the synchrotron spectrum in the region of the perturbative QED.

For high-energy photons ($\varkappa \gg 1$) equation (22) yields $\chi \gtrsim 80 \varkappa^{2/3}$, and (23) yields $\alpha \chi^{2/3} \gtrsim 20 \varkappa (\chi - \varkappa) / \chi$.

To fit the latter to the region of perturbative QED applicability, one can try $\chi - \varkappa \ll \chi$, however, the former condition in this case yields $\chi^{1/3} \gtrsim 80$ hence $\chi \gtrsim 5 \times 10^5$ which is again beyond the region of perturbative QED. Therefore, the evidence of vacuum polarization in synchrotron spectrum for high-energy photons is also unreachable. The estimates above are in agreement with the results of reference [30] [see (8.8e) and (8.11) therein], which, however, do not take radiation recoil into account and do not discuss the region of the perturbative QED applicability.

### 3.2. Synchrotron–Cherenkov radiation of muons

The effect of the vacuum polarization on the synchrotron spectrum can be enhanced if heavy charged particles are used instead of electrons. For definiteness, and because of the recent progress in their acceleration technique [40], muons are considered here. The advantage of using muons is a two-fold. First, their big mass, $m_\mu \approx 207m$, yields much greater curvature radius hence much greater timescale $\tau_{\perp}$ than that for the electrons. For a given photon frequency this makes synchrotron spectrum much more sensible to the longitudinal synchronism between the particle and the emitted wave, i.e. to $\tau_{\parallel}$, which, opposite to $\tau_{\perp}$, depends on the refractive index. Second, high mass makes the critical frequency significantly lower, hence a more sizable part of the spectrum lies in the low-frequency region $\varkappa \lesssim 1$, in which $\delta n$ for the vacuum refractive index is maximal.

The classical critical frequency for muons is

$$\omega_{\epsilon,\mu} = 3\omega \gamma^2 m / m_\mu, \quad (26)$$

that gives the ratio of the photon energy to the muon energy: $h \omega_{\epsilon,\mu} / \epsilon_\mu = 3\chi (m / m_\mu)^2$ (with $\chi = \gamma B / B_\circ$ the same as for electrons). Therefore, this ratio is small, $h \omega_{\epsilon,\mu} / \epsilon_\mu \ll 1$, up to $\chi \sim 10^4$, and it is reasonable to neglect the radiation recoil for muons. In this case the conditions sufficient for the synchrotron spectrum to be noticeably modified due to vacuum polarization are the following:

$$|\delta n(\omega)| \gtrsim \frac{1}{\gamma^2}, \quad (27)$$

$$\omega \gtrsim \frac{\omega_{\epsilon,\mu}}{|1 - 2\delta n(\omega) \gamma^2|^{3/2}}. \quad (28)$$

Obviously, these conditions can be derived similarly to (22) and (23).

As a starting point, one can consider $\omega \sim \omega_{\epsilon}$ that ensures fulfillment of condition (28). By virtue of (5), the difference between $\tau_{\parallel}$ with $\delta n \neq 0$ taken into account, and $\tau_{\parallel}$ with $\delta n = 0$ ($\tau_{\parallel,0}$), is determined by $2\gamma^2 \delta n$, if this quantity is small:

$$\frac{\tau_{\parallel} - \tau_{\parallel,0}}{\tau_{\parallel}} \approx 2\gamma^2 \delta n, \quad (29)$$

where $\theta = 0$ is assumed. One can note that this quantity depends on $\chi$ and $\varkappa$ only [see (20)]. For $\omega = \omega_{\epsilon}/3$ (which corresponds to the maximum of the spectrum better than $\omega_{\epsilon}$ itself) the parameters $\chi$ and $\varkappa$ become related, $\chi^2 = \varkappa m_\mu / m$. Thus $2\gamma^2 \delta n$ can be expressed in terms of $\varkappa$ only, and $2\gamma^2 \delta n$ as function of $\varkappa$ is
Figures 4(b) and (c) demonstrate the radiation spectrum for synchrotron spectrum caused by vacuum polarization, that agrees well with the numerical results. With \( jE \) maximum. At the same time, for photon energies corresponding to \( \chi \) higher on it). At \( \chi \) changes in the spectrum occurs at frequencies lower than the frequency of the spectrum \( \approx 30 \) that ensures the fulfillment of the first of them. Then, similarly to the previous section, the low-energy part of the spectrum \( (\sim 1) \) can be considered. In this case condition (28) yields

\[
\alpha \chi^{2/3} \gtrsim 42 \times \left( \frac{m}{m_\mu} \right)^{2/3} \approx 1
\]  
(30)

[compare this with (25)]. Therefore, the change in the low-energy part of the synchrotron spectrum can be pronounced only near the conjectured breakdown threshold of the perturbative QED.

Figure 5 demonstrates the low-energy part of the radiation spectrum of muons for \( B/B_s = 0.01 \) (note that the subplots almost do not depend on this value). In figure 5(a) the ratio

\[
J = \left[ \frac{d^2 I}{d\omega d\theta} \right]_{\theta=0} / \left[ \frac{d^2 I}{d\omega d\theta} \right]_{\theta=0}
\]

computed for \( \theta = 0 \) and value of \( \omega \) providing \( \sim 2 \) is shown with green solid line. According with the estimate (30), value of \( J \) differs noticeable from unity for \( \chi \sim 10^5 \). The dashed brown line shows the ratio \( a = \xi_{\perp} / \tau_\parallel \) for the given frequency which corresponds to \( \sim 2 \). It is seen from the expression for the classical critical frequency [see (26)] that

\[
\frac{\hbar \omega_{c,\mu}}{m c^2} = \frac{3 \chi^2 m}{m_\mu} \approx \frac{3 \chi^2}{m^2} \approx 10^4
\]  
(32)

Hence, for \( \sim 10^6 \) the frequency which provides \( \sim 2 \) is lower than the critical frequency. Thus, one expects \( \tau_\perp \lesssim \tau_\parallel \) here, for \( n = 1 \). However, at \( \chi \sim 10^5 \) the timescales \( \tau_\perp \) and \( \tau_\parallel \) becomes of the same order thanks to vacuum polarization. At even higher values of \( \chi \) the timescale \( \tau_\parallel \) that together with \( \zeta = -1 \) leads to the interference patterns similar to that shown in figure 3(b). In figure 5(a) this is seen as the oscillations of \( J \) at high values of \( \chi \).

Figure 5(b) shows the radiated energy per unit frequency and per unit angle \( \theta \) for the vacuum refractive index taken into account (upper half) and \( \delta n = 0 \) (lower half), for \( \chi = 800 \) \( (B/B_s = 0.01 \) and \( \gamma = 8 \times 10^4 \). Note that even such high value of \( \chi \) the phase term \( \sim (\omega^2 - c^2 \ell^2) \) which is neglected here...
(see appendix C) is small, \( \tilde{r} \lesssim 2 \times 10^{-3} \). Figure 5(c) shows the energy spectra which correspond to the distributions in figure 5(b). Although the difference between dashed and solid curves in figure 5(c) is dramatic, it is hardly fit as possible experimental evidence of vacuum polarization. First, it rises only at high \( \chi \) values, where some other high-order terms of QED can give even bigger contribution. Second, the change in the spectrum occurs only for frequencies for which \( \chi \lesssim 10 \), which is much lower than for the critical frequency, hence this difference occurs for a small fraction of the photons. Therefore, photon emission by muons with \( \chi \approx 30 \) is still the most promising probe for vacuum polarization effect in the radiation spectrum.

One more interesting prospect of QED study should be noted regarding the photon emission by muons. Let muons and electrons are of the same Lorentz factor \( \gamma \). For the electrons the energy of the emitted photons in the regime of \( \chi \gg 1 \) is limited due to the recoil effect by \( mc^2\gamma \). Despite higher curvature radius, for the muons the photon energy can be much higher, because the recoil effect for them is negligible. Definitely, one can reach \( \alpha \chi^{2/3} \sim 1 \) for photons emitted by the muons already at \( \chi \sim 300 \), for which \( \alpha \chi^{2/3} \sim 0.3 \). This potentially opens perspectives to reach the non-perturbative QED [11–15] in future experiments.

4. Conclusion

The general formula which describes the photon emission by an ultrarelativistic electron in a strong magnetic field can be found in the framework of the quasiclassical theory of BKS [32]. BKS formula can be extended to the case of a constant non-unity refractive index \( n, |n-1| \ll 1 \) [see (C.9)]; strictly speaking, this formula can be used for vacuum refractive index for electrons with \( \chi \lesssim 10^3 \), for other cases the additional phase term should be taken into account as shown in appendix C]. From this, one can find photon emission probability that generalizes both the synchrotron and the Cherenkov emission, and takes into account photon recoil and spin flips. The obtained expression clearly shows that the emission probability is not the sum of the synchrotron emission probability and the Cherenkov emission probability. Hence, the photon emission occurs in the synergic (cooperative) synchrotron–Cherenkov radiation process.

The electron motion along its curved trajectory prevents the pure Cherenkov radiation. The trajectory curvature determines the radiation formation time for the synchrotron–Cherenkov radiation [see (16)], which is much shorter than that for the pure Cherenkov radiation (which can be extremely large in the case of the Cherenkov synchronism, \( n\beta = 1 \)). Furthermore, the radiation formation time for the synchrotron–Cherenkov radiation is less or about of that for the pure synchrotron emission.

The photon emission probability is determined not only by the radiation formation time, and the probability can be either greater or less for the synchrotron–Cherenkov radiation (\( n \neq 1 \)) than that for the synchrotron one (\( n = 1 \)). The radiation spectrum is sensible to \( \delta n = n - 1 \) in the both cases, \( \delta n > 0 \) and \( \delta n < 0 \), however, the changes in the spectrum occur first for frequencies which are higher than the critical frequency \( \omega_c \) [see (7)]. If the Cherenkov condition holds, \( v > c/n \), the overall emitted energy can be much higher than in the case \( n = 1 \). For numerical simulations of the synchrotron–Cherenkov spectrum the open source code \( I^E \) is implemented [38], and the numerical results are in a good agreement with the analytical predictions.

One can use formulas for the refractive index of vacuum polarized by a strong external magnetic field (see reference [28] and references therein), in order to find how the synchrotron spectrum is modified due to vacuum polarization. The estimates and numerical simulations demonstrate that in the framework of the considered model the changes in the spectrum of the photons emitted by electrons become noticeable far beyond the Cherenkov threshold \( v = c/n \) and even far beyond the conjectured breakdown of the perturbative QED \( \alpha \chi^{2/3} \sim 1 \). The considered model of synchrotron–Cherenkov radiation is based on the perturbative QED and is applicable if \( \alpha \chi^{2/3} < 1 \), hence no change in the synchrotron spectrum in the region of the perturbative QED is expected. The cause of this is that the vacuum refractive index depend on the photon frequency, and for the photon energies greater or about \( \hbar \omega_c \) (for which the spectrum modification is expected first) \( \delta n \) is negative and \( |\delta n| \) is very small. Moreover, the critical frequency for electrons with \( \chi \gg 1 \) is very close to the electron energy.

Muons have much larger curvature radius of the trajectory in a strong field than the electrons, if the muons and the electrons are of the same Lorentz factor. This makes the radiation spectrum of the muons much more sensible to the refractive index than that of the electrons. Opposite to the electrons with \( \chi \gg 1 \) for which the critical frequency always yields \( (\hbar \omega_c/mc^2)(B/B_0) \gg 1 \), for the muons this is not the case. E.g., for \( \chi = 30 \) the maximum of the spectrum corresponds to \( \kappa = (\hbar \omega_c/mc^2)(B/B_0) \approx 2 \) which is favourable for the vacuum refractive index. The radiation spectrum is enhanced up to 10% in this case thanks to the vacuum polarization [see figure 4(b)]. From the point of view of possible experiments, the
muons with $\chi \approx 30$ probable are the most promising tool to probe the influence of the vacuum polarization on the synchrotron spectrum.

Regarding possible experiments using laser pulses and muon accelerators, a simple head-on collision geometry with single laser pulse can be considered. The expression for the vacuum refractive index for the emitted photons in this case is quite close to that for a constant magnetic field [21, 29]. However, the results of this paper cannot be applied directly to the laser field. First, as the most of the emitted photons have $\gamma \gtrsim 1$, the pair photoproduction and the photon emission by the secondary electrons and positrons should be taken into account, as well as possible Sokolov–Ternov effect. Second, $\kappa \equiv m c^2 / m_\mu c^2 = 207$ (with $m_\mu$ and $c$ the magnetic field amplitude and the photon frequency of the laser pulse, respectively), e.g. for $a_0 = 800$ (for $h \omega_L = 1$ eV) and $\gamma = 2 \times 10^4$ (this value is in between the Lorentz factor reached for the electrons at SLAC, $\gamma \approx 10^3$, and that for the protons at LHC, $\gamma \approx 6 \times 10^3$). For such values of $a_0$ the local constant field approximation applied here should be used with caution [41]. This constraint becomes even more pronounced for protons, for which the dipole–Cherenkov radiation should be considered rather than the synchrotron–Cherenkov radiation. Third, in the linearly polarized laser the refractive index is not uniform (however, it probably can be considered uniform on a scale of the radiation formation length). Therefore, the realistic proposal for probing vacuum polarization with synchrotron emission of heavy charged particles in laser fields needs further investigations.

Acknowledgments

We thank A M Fedotov and A A Mironov for fruitful conversations and comprehensive references.

This research is supported in part by the Russian Science Foundation Grant No. 18-72-00121 (analysis of synchrotron-Cherenkov radiation properties and development of the code $jE$), by the Russian Foundation for Basic Research under Grants No. 20-52-12046 and 20-21-00150 [analysis of the general formulas for the photon emission]. IA would like to acknowledge the support from the Foundation for the Advancement of Theoretical Physics and Mathematics ‘BASIS’ under Grant No. l7-11-101.

Appendix A. Photon emission by ultrarelativistic particle in classical theory

Calculation of the radiation of ultrarelativistic charged particle for $n = 1$ can be found in many textbooks, e.g. in [39]. However, these calculations are often difficult to tailor to the case $n \neq 1$ because of the terms which can be singular for the Cherenkov radiation. Here the general formulas for angular and spectral distributions of the emitted energy are recalled and obtained in a handy form. Despite $n = 1$ is used in this section, the approach used here allows obvious generalization to the case $n \neq 1$, if $\delta n \ll 1$.

For the sake of simplicity one can consider emission of electromagnetic waves by a current density $j$ inside a virtual superconductive rectangular box (resonator or cavity) of size $L_x \times L_y \times L_z$. The emitted field can be decomposed by complex resonator modes with well-known sine-cosine spatial and $\exp(-i \omega_s t)$ temporal structure:

$$E = \sum_s C_s E_s, \quad B = \sum_s C_s B_s,$$

where $s$ is the generalized mode number and $\omega_s$ is the mode cyclic frequency, $E$ and $B$ are the electric and magnetic field, respectively. The modes can be chosen orthogonal, with the following normalization:

$$\frac{1}{8\pi} \int_V \left( E_s^* E_s + B_s^* B_s \right) \, dV = h \omega_s \delta_{d,}$$(A.2)

where the symbol $^*$ means complex conjugate. Hence the energy of the emitted field is

$$I = \frac{1}{8\pi} \int_V \left( E E^* + B B^* \right) \, dV = \sum_s h \omega_s |C_s|^2,$$(A.3)

and $|C_s|^2$ can be interpreted as the emission probability of the photon of mode $s$.

To find $C_s$, one can start from Maxwell’s equations:

$$\nabla \times E = -\frac{1}{c} \frac{\partial B}{\partial t}, \quad \nabla \times B = \frac{1}{c} \frac{\partial E}{\partial t} + \frac{4\pi}{c} j,$$

$$\nabla E = 4\pi \rho, \quad \nabla B = 0,$$
with $\rho$ and $j$ the charge and the current density, respectively. Let the current $j$ emits during $t \in (t_1, t_2)$, and $j = 0$, $\rho = 0$ for $t < t_1$ and $t > t_2$. Thus, the decomposition (A.4) is valid for $t > t_2$. One can multiply equation (A.4) on $B_j'$, and subtract it from equation (A.5) multiplied on $E_j'$. Then the result can be integrated over the space and time that yields

$$\int_V \left( E_j' E + B_j' B \right) \, dV \bigg|_{t_1}^{t_2} + 4\pi \int_{t_1}^{t_2} \int_V j E_j' \, dV \, dt = e \int_{t_1}^{t_2} \oint_S \left( B \times E_j' - E \times B_j' \right) \, dS \, dt,$$

(A.7)

with $V$ a volume of the virtual box and $S$ its boundary.

The cavity can be chosen big enough such that $E = B = 0$ at the boundary, in this case the right-hand side of (A.7) is zero. Furthermore, $E = B = 0$ at $t = t_1$, hence from (A.7), taking into account (A.1) and (A.2), one gets

$$C_s = -\frac{1}{2\hbar \omega C_s} \int_j j E_s' \, dV \, dt.$$

(A.8)

Equation (A.8) has a clear physical meaning. Being multiplied by $\hbar \omega C_s^*$, it expresses the equality between the energy emitted into the mode $s$, and the work of the current $j$ over the one-half field of the emitted mode. This work peaks if there is a synchronism between the current and the field of the mode. Note also that formula (A.8) is similar to one for the amplitude of an oscillator driven by an external force.

For an ultrarelativistic electron, which emits mostly in the forward direction, the computation of $C_s$ can be further simplified. First, the current of the electron is

$$j = -e v \delta (r - r(t)),$$

(A.9)

with $r(t)$ the electron position. Second, each of the complex modes is formed by eight complex plane waves $\propto \exp(-i\omega t + ik r)$ (except a few modes with wave vector parallel to the box boundaries). This yields eight terms in the integral over $t$ in (A.8). It can be noted that one of the terms oscillates much slower than the others which hence can be dropped (e.g., if $k_s \approx \omega / c$ and $x(t) \approx ct$, then $\exp[i\omega t - i k_s x(t)]$ cannot be dropped, whereas $\exp[i\omega t + i k_s x(t)]$ can be). Let the remaining term corresponds to a wave with the polarization direction $e_i$ (with $e_i^2 = 1$). The amplitude of this remaining wave, $a_i$, can be found from the normalization (A.2): the wave energy is $\hbar \omega / 8$ hence $a_i = (2\pi \hbar \omega / V)^{1/2}$. Therefore,

$$C_s = -\frac{e^2}{2} \sqrt{\frac{\pi}{\hbar \omega}} \int_j v e_i \exp[i\omega t - i k e_i r(t)] \, dt.$$

(A.10)

An ultrarelativistic particle emits photons in a narrow cone around the direction of the particle velocity. Thus the energy radiated in a certain direction can be readily computed from the energy of the modes. The density of the modes which has a plane-wave component in some certain unit solid angle and unit frequency interval can be found from the boundary conditions for the virtual superconducting box. From this, the full emitted energy can be expressed using the energy radiated per unit frequency interval and per unit solid angle:

$$I = \sum_s \hbar \omega_s |C_s|^2 = \frac{\hbar V}{\pi^2 c^5} \int \omega^5 \sum_s |C_s|^2 \, d\omega \, d\Omega$$

$$= \frac{e^2}{4\pi^2 c^5} \int \omega^3 \sum_s \left| \int v(t) \exp[i\omega t - i k e_i r(t)] \, dt \right|^2 \, d\omega \, d\Omega,$$

(A.11)

with $e_i$ the polarization directions. Equation (A.11) is very useful in estimating the radiation timescales and the radiation formation length, that discussed for the synchrotron radiation in the next section, and for the synchrotron–Cherenkov radiation in section 2.2. Note also that (A.11) (contrary to (14.65) from [39]) do not contain terms proportional to $1/(1 - v k / \omega)$, thus it can be used even in the case of exact Cherenkov synchronism.

**Appendix B. Classical synchrotron emission and the timescales**

The key feature of the photon emission by an ultrarelativistic particle is a synchronism between the particle and the emitted wave, as seen from (A.8), (A.10) and (A.11). The phase of the exponential function in these equations in the case $n = 1$ (hence $k = \omega / c$), varies slowly in vicinity of the point where the angle between $v$ and $k$ is minimal. For the sake of simplicity we assume that there is only one such point, and it is in the origin of the local coordinates (figure B1), and the particle is in the origin at $t = 0$.
Figure B1. Local coordinates used in the computations. For a given wave number \( k \) the origin is the point on the electron trajectory (thick blue line) where \( k \) is perpendicular to the normal vector of the trajectory. Thus, the \( x \) axis is tangent to the trajectory, the \( y \) axis is parallel to the normal vector (hence the \( xy \) plane is the osculating plane), and the \( z \) axis is chosen by the right-hand rule. The polarization vector \( e_1 \) is chosen to be on the \( y \) axis, and \( e_2 \) to be perpendicular to \( e_1 \) and \( k \).

In the synchrotron approximation, or local-constant-field approximation, the particle trajectory is described locally like a circular orbit fully determined by the local curvature radius \( R \) and the Lorentz factor \( \gamma \):

\[
x \approx R \sin \left( \frac{vt}{R} \right) \approx vt - \frac{(vt)^3}{6R^2}, \tag{B.1}
\]

\[
y \approx R \left[ \cos \left( \frac{vt}{R} \right) - 1 \right] \approx -\frac{(vt)^2}{2R}. \tag{B.2}
\]

Then, the pre-exponential functions in the integrand of (A.11) can be written as follows:

\[
v e_1 \approx c^2 t / R, \tag{B.3}
\]

\[
v e_2 \approx c \sin \theta, \tag{B.4}
\]

\[
\exp[\ii \omega t - \ii k r(t)] \approx \exp[\ii \phi(t)], \tag{B.5}
\]

where the phase \( \phi \) is obtained similar to (14.76) and (14.77) of [39] and contains only linear and cubic terms:

\[
\phi(t) = 2\pi \left[ \frac{t}{\tau_\parallel} + \left( \frac{t}{\tau_\perp} \right)^3 \right]. \tag{B.6}
\]

Here \( \tau_\parallel \) and \( \tau_\perp \) are the timescales of dephasing between the electron and the emitted wave caused by the longitudinal (along the \( x \) axis) and transverse (along the \( y \) axis) electron motion, respectively:

\[
\tau_\parallel = \frac{4\pi}{\omega(\theta^2 + 1/\gamma^2)}, \tag{B.7}
\]

\[
\tau_\perp = \frac{12\pi \gamma^3}{\left( \omega \omega_B^2 \right)^{1/3}}. \tag{B.8}
\]

where the effective magnetic field strength \( B \) is introduced for convenience such that \( v = \omega_B R / \gamma \), and

\[
\omega_B = \frac{eB}{mc} \tag{B.9}
\]

is the cyclotron frequency in this field. Note that \( \tau_\parallel \) depends on \( v \) (hence on \( \gamma \)) and does not depend on \( R \), whereas \( \tau_\perp \) depends on \( R \propto \gamma / \omega_B \) and does not depend separately on \( \gamma \).

Well-known equations (14.78) and (14.83) from the textbook [39] which describe angular and spectral distribution of the synchrotron photons can be easily got from (A.11) and (B.3)–(B.6). The key feature of the synchrotron spectrum is the critical frequency (see (14.85) in [39]; note that some other textbooks use different numerical coefficient here)

\[
\omega_c = \frac{3\gamma^3 c}{R} = 3\omega_B \gamma^2. \tag{B.10}
\]

The energy emitted per unit frequency interval per unit solid angle, \( d^2I / d\omega \ d\Omega \), has maximum at \( \theta = 0 \) and \( \omega \approx 0.42 \times \omega_c \).

If \( \omega \gg \omega_c \) or \( \theta \gg 1/\gamma \), the emitted energy sharply tends to zero, that can be explained with \( \tau_\parallel \) and \( \tau_\perp \). The critical frequency corresponds to \( \tau_\perp / \tau_\parallel = 3/(4\pi)^{2/3} \approx 0.56 \). If \( \omega \) increases beyond \( \omega_c \), or if \( \omega \sim \omega_c \) and \( \theta \) increases beyond 1/\( \gamma \), then \( \tau_\parallel \) becomes smaller than \( \tau_\perp \). Hence, the exponential function (B.5) and
(B.6) oscillates strongly hence the synchrotron integrals tend to zero. The presence of the refractive index, spin contribution and photon recoil change the basic equation for the photon emission probability. However, the emission probability is still governed by the synchronism between the emitted wave and the electron, hence, by \( \tau_\parallel \) and \( \tau_\perp \), though equations for them should be corrected.

**Appendix C. Quasiclassical theory of the photon emission in the polarized vacuum**

In order to take into account the refractive index \( n = 1 + \delta n \) (which is assumed close to unity, \(|\delta n| \ll 1\)) in the classical formula \((\text{A.11})\), one should not change anything in the reasoning of appendix A, except the relation between the photon frequency and the wave vector in the phase, \( k = n \omega / c \). The mode structure, the energy of the modes and their normalization can be taken unchanged in the case \(|\delta n| \ll 1\). This situation almost replicates in QED.

If one follows the BKS quasiclassical derivation of the spectral and angular distribution of the synchrotron photons \([32]\), he/she finds that the presence of the vacuum refractive index changes nothing in it except the phase in the exponential (note that the refractive index is taken into account up to \((2.26)\) in \([32]\); see also \([34]\)). However, in the BKS theory two assumptions are used which should be mentioned. First, it is assumed that the angle between the propagation direction of the emitted photon and the electron velocity is small, \( \theta \ll 1 \). As seen from \((5)\), the timescale \( \tau_\parallel \) decreases with the increase of \( \theta \) if \( \theta \gtrsim 1/\gamma \) (for \(1/\gamma^2 \gtrsim \delta n\)) or if \( \theta \gtrsim \sqrt{\delta n} \) (otherwise). The decrease of \( \tau_\parallel \) lead to the decrease of the emission power (see section 2.2) hence the photons are emitted to the cone \( \theta \lesssim \max(1/\gamma, \sqrt{\delta n}) \ll 1 \).

The second notable assumption of the BKS theory is \( \omega^2 - c^2 k^2 = 0 \), that is not the case in the presence of \( n \neq 1 \). As shown in \([32]\) [equation \((2.26)\)], an additional phase term appears if \( \omega \neq c k \). This additional term should be taken into account e.g. for the transition radiation \([34]\). The exponential function in \((2.27)\) from \([32]\) with this term is (here we omit terms with time instant \( t_2 \) for brevity)

\[
\exp \left\{ \frac{i \epsilon}{\hbar \omega} \left[ \omega t_1 - k \cdot r(t_1) - \frac{\hbar (\omega^2 - c^2 k^2) t_1}{2 \epsilon} \right] \right\}, \tag{C.1}
\]

with \( \epsilon \) the electron energy. Using Taylor series for the circular electron trajectory (see appendix B), one gets the linear term in the phase

\[
\frac{i \epsilon}{\hbar \omega} \left[ \omega t_1 - k \cdot r(t_1) - \frac{\hbar (\omega^2 - c^2 k^2) t_1}{2 \epsilon} \right]. \tag{C.2}
\]

Thus the ratio \( \hat{r} \) of the second term in the phase \((C.2)\) to the leading linear term is about \( \hat{r} \sim 2 \gamma^2 \delta n \hbar \omega / \epsilon \) which yields \( \hat{r} \sim (2\pi)^{-1} \alpha \chi \propto N(\chi) \lesssim 4 \times 10^{-4} \chi \) for electrons and \( \hat{r} \sim (2\pi)^{-1} \alpha \chi \propto N(\chi)m/m_{\mu} \lesssim 2 \times 10^{-6} \chi \) for muons [here \((20)\) and \((21)\) are used; note that the estimate given here is valid even for the range \( \hbar \omega / \epsilon \lesssim 1/\chi \ll 1 \) where the vacuum refraction index peaks]. Therefore, for the quantum parameter below the threshold of the perturbative QED breakdown \((\chi \ll 1600)\), the ratio is small, \( \hat{r} \ll 1 \). As seen from \((C.2)\), the term \( \propto \omega^2 - c^2 k^2 \) just corrects the linear phase term, i.e. it corrects the timescale \( \tau_\parallel \). Hence if \( \hat{r} \ll 1 \), the additional phase term \( \propto (\omega^2 - c^2 k^2) \) can be neglected and the timescale \( \tau_\parallel \) is determined by \((5)\).

Quasiclassical formula for the photon emission can be rewritten in form close to the classical one, as shown in \([42]\) for \( n = 1 \). However, to take into account \( \delta n \) in the formula, one should first isolate the phase, i.e. get rid of \( 1/(1 - v \cdot k / \omega) \) which originates from the phase. To do so, one can start from photon emission of unpolarized electron beams, equation \((34)\) of \([42]\) (see also, for example, \((6)\) in the supplementary material of reference \([43]\)):

\[
\frac{d^2 I}{d\omega \, d\Omega} = \frac{e^2}{4 \pi^2 c} \left\{ \frac{e^2 + e' e'}{2 \epsilon^2} \left[ \int dt \frac{n \times [(n - \beta) \times \dot{\beta}]}{(1 - n \beta)^2} \exp[i \omega'(t - n \rho)] \right]^2 + \frac{1}{2} \left( \frac{\hbar \omega m_e^2}{\epsilon^2} \right)^2 \left[ \int dt \frac{n \beta}{(1 - n \beta)^2} \exp[i \omega'(t - n \rho)] \right]^2 \right\} \tag{C.3}
\]

with \( \beta = v / c, \rho = r / c, n = c k / \omega \) (still \(|n| = 1\) here), and

\[
e' = \epsilon - \hbar \omega, \tag{C.4}\]

\[
\omega' = \omega / \epsilon', \tag{C.5}
\]
with $\varepsilon$ the electron energy before the emission of the photon. One can note that

$$\frac{d}{dr} \left( \frac{1}{1 - n\beta} \right) = \frac{n\beta}{(1 - n\beta)^2},$$

$$(C.6)$$

$$\frac{d}{dr} \left( \frac{n \times [n \times \beta]}{1 - n\beta} \right) = \frac{n \times [(n - \beta) \times \beta]}{(1 - n\beta)^2},$$

$$(C.7)$$

$$\frac{d}{dr} \exp[i\omega'(t - n\rho)] = i\omega'(1 - n\beta) \exp[i\omega'(t - n\rho)].$$

$$(C.8)$$

Hence, integrating by parts one gets

$$\frac{d^2 I}{d\omega d\Phi} = \frac{\varepsilon^2}{4\pi^2 c} \left\{ \frac{2}{\varepsilon^2} \sum \int dt \beta e \exp[i\omega'(t - n\rho)] + \frac{1}{\varepsilon^2} \int dt \exp[i\omega'(t - n\rho)] \right\},$$

where the product $n \times [n \times \beta]$ is rewritten using $\beta_1$ and $\beta_2$. Now, to take into account the refractive index $n$, one should set $|n| \equiv |\mathbf{k}/\omega| = n = 1 + \delta n$ in the exponential functions in (C.9).

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