The Pound- Rebka experiment and torsion in the Schwarzschild spacetime

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Abstract

We develop some ideas discussed by E. Schucking [arXiv:0803.4128] concerning the geometry of the gravitational field. First, we address the concept according to which the gravitational acceleration is a manifestation of the spacetime torsion, not of the curvature tensor. It is possible to show that there are situations in which the geodesic acceleration of a particle may acquire arbitrary values, whereas the curvature tensor approaches zero. We conclude that the spacetime curvature does not affect the geodesic acceleration. Then we consider the Pound-Rebka experiment, which relates the time interval $\Delta \tau_1$ of two light signals emitted at a position $r_1$, to the time interval $\Delta \tau_2$ of the signals received at a position $r_2$, in a Schwarzschild type gravitational field. The experiment is determined by four spacetime events. The infinitesimal vectors formed by these events do not form a parallelogram in the $(t,r)$ plane. The failure in the closure of the parallelogram implies that the spacetime has torsion. We find the explicit form of the torsion tensor that explains the nonclosure of the parallelogram.

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1 Introduction and notation

The spacetime geometry is determined by the metric tensor $g_{\mu\nu}$, and the dynamics of the metric tensor is determined by Einstein's equations. For a given metric tensor there exists an infinity of tetrad fields $e^a_{\mu}$ that are compatible with the spacetime geometry. Tetrad fields may be interpreted as reference frames adapted to a class of observers in spacetime. Einstein's equations may be written in the traditional form in terms of the metric tensor, in which case the curvature tensor plays a prominent role, or in terms of the tetrad field. In the latter case the field equations are constructed out of the torsion tensor. Therefore the dynamics of the gravitational field admits a description either in terms of the curvature tensor (of the Levi-Civita connection), or of the torsion tensor (of the Weitzenböck connection) [1].

However, there is a point of view according to which the gravitational force that acts on a particle or on a frame, in a given gravitational field, is due to the torsion tensor only, not to the spacetime curvature. Of course the curvature tensor is responsible for the tidal forces, but the force on a particle that moves along a particular geodesic worldline $x^{\mu}(s)$, with tangent vector $u^\mu = dx^\mu/ds$, is due to the torsion tensor. This is one of the issues discussed by Schucking [2], and we will address it in this paper, in some detail, in terms of the acceleration tensor. This tensor is a coordinate invariant quantity that describes the accelerations that are necessary to maintain a reference frame in spacetime in a given inertial state (for instance, to maintain the frame in stationary state). The reference frame is fixed by identifying the timelike components of the inverse tetrad field with the velocity field of the class of observers, i.e., $e^{(0)}_{\mu} = u^\mu$.

A second issue to be considered here is the interpretation of the Pound-Rebka experiment as a manifestation of the spacetime torsion, as suggested by Schucking [2]. Suppose that at the top of a tower, at a distance $r_1$ from the center of the Earth a light signal is emitted radially downwards at the instant $t_1$, and received at a position $r_2$ at the instant $t_2$. After a proper time interval $\Delta \tau_1$ a second light signal is emitted downwards, and is received at the position $r_2$ at the instant $t_2 + \Delta \tau_2$. It is known that timelike and null vectors formed by the events $(t_1, r_1)$, $(t_1 + \Delta \tau_1, r_1)$, $(t_2, r_2)$ and $(t_2 + \Delta \tau_2, r_2)$ do not form a parallelogram in the $(t,r)$ plane. The nonclosure of the parallelogram may be interpreted as a manifestation of the torsion of the spacetime. By establishing the frame of stationary observers in the Schwarzschild spacetime we arrive at...
the torsion tensor that precisely explains the breaking of the parallelogram. This issue will be investigated in detail in the present analysis.

The paper is organized as follows. In section 2 we review the construction of the acceleration tensor. The values of this tensor characterize the inertial state of the frame, i.e., it provides the nongravitational accelerations (translational and angular velocity of the local spatial frame with respect to a nonrotating Fermi-Walker transported frame) that are exerted on the frame. In section 3 we discuss the possibility of having a situation in which the curvature tensor approaches zero, whereas the geodesic (gravitational) acceleration of a particle may acquire arbitrary values. The geodesic acceleration is related to some components of the acceleration tensor (constructed out of the torsion tensor) for a stationary frame in spacetime. In section 4 we consider the Pound-Rebka experiment and explain the breaking of the parallelogram in terms of the spacetime torsion. We conclude that the torsion tensor is an important entity in the description of the spacetime geometry and of the gravitational field.

Notation: space-time indices $\mu, \nu, \ldots$ and SO(3,1) indices $a, b, \ldots$ run from 0 to 3. Time and space indices are indicated according to $\mu = 0, i$, $a = (0), (i)$. The tetrad field is denoted $e^a_{\mu}$, and the torsion tensor reads $T_{a\mu\nu} = \partial_{\mu}e_{av} - \partial_{\nu}e_{am}$. The flat, Minkowski space-time metric tensor raises and lowers tetrad indices and is fixed by $\eta_{ab} = e_{a\mu}e_{b\nu}g^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. The determinant of the tetrad field is represented by $e = \det(e^a_{\mu})$.

The torsion tensor defined above is often related to the object of anholonomity $\Omega^\lambda_{\mu\nu}$ via $\Omega^\lambda_{\mu\nu} = e^a_{\mu}T^a_{\mu\nu}$. However, we assume that the spacetime geometry is defined by the tetrad field only, and in this case the only possible nontrivial definition for the torsion tensor is given by $T^a_{\mu\nu}$. This torsion tensor is related to the antisymmetric part of the Weitzenböck connection $\Gamma^\lambda_{\mu\nu} = e^a_{\lambda}\partial_{\mu}e_{av}$, which is frame dependent and establishes the Weitzenböck spacetime. The curvature of the Weitzenböck connection vanishes. However, the tetrad field also yields the metric tensor, which establishes the Riemannian geometry. Therefore in the framework of a geometrical theory based only on the tetrad field one may use the concepts of both Riemannian and Weitzenböck geometries.
The tetrad field as reference frame and the acceleration tensor

We recall the discussion presented in refs. [3, 4] regarding the characterization of tetrad fields as reference frames in spacetime. A frame may be characterized in a coordinate invariant way by its inertial accelerations, represented by the acceleration tensor.

We denote by $x^\mu(s)$ the worldline $C$ of an observer in spacetime, where $s$ is the proper time of the observer. The velocity of the observer on $C$ reads $u^\mu = dx^\mu/ds$. We identify the observer’s velocity with the $a = (0)$ component of $e_a^\mu$: $u^\mu(s) = e_{(0)}^\mu$. The acceleration $a^\mu$ of the observer is given by the absolute derivative of $u^\mu$ along $C$ [5],

$$a^\mu = \frac{Du^\mu}{ds} = \frac{De_{(0)}^\mu}{ds} = u^\alpha \nabla_\alpha e_{(0)}^\mu; \quad (1)$$

where the covariant derivative is constructed out of the Christoffel symbols. Thus $e_a^\mu$ and its derivatives determine the velocity and acceleration along the worldline of an observer. The set of tetrad fields for which $e_{(0)}^\mu$ describe a congruence of timelike curves is adapted to a class of observers characterized by the velocity field $u^\mu = e_{(0)}^\mu$ and by the acceleration $a^\mu$.

We may consider not only the acceleration of observers along trajectories whose tangent vectors are given by $e_{(0)}^\mu$, but the acceleration of the whole frame along $C$. The acceleration of the frame is determined by the absolute derivative of $e_a^\mu$ along the path $x^\mu(s)$. Thus, assuming that the observer carries an orthonormal tetrad frame $e_a^\mu$, the acceleration of the latter along the path is given by [6]

$$\frac{De_a^\mu}{ds} = \phi_a^\ b e_b^\mu; \quad (2)$$

where $\phi_{ab}$ is the antisymmetric acceleration tensor. According to ref. [6], in analogy with the Faraday tensor we can identify $\phi_{ab} \rightarrow (a, \Omega)$, where $a$ is the translational acceleration ($\phi_{(0)(i)} = a_{(i)}$) and $\Omega$ is the angular velocity of the local spatial frame with respect to a nonrotating (Fermi-Walker transported) frame. It follows that

$$\phi_a^\ b = e_b^\mu \frac{De_a^\mu}{ds} = e_b^\mu u^\lambda \nabla_\lambda e_a^\mu. \quad (3)$$
Therefore given any set of tetrad fields for an arbitrary gravitational field configuration, its geometrical interpretation may be obtained by suitably interpreting the velocity field $u^\mu = e_{(0)}^\mu$ and the acceleration tensor $\phi_{ab}$. The acceleration vector $a^\mu$ defined by Eq. (1) may be projected on a frame in order to yield

$$a^b = e^b_\mu a^\mu = e^b_\mu u^\alpha \nabla_\alpha e_{(0)}^\mu = \phi_{(0)}^b. \quad (4)$$

Thus $a^\mu$ and $\phi_{(0)(i)}$ are not different accelerations of the frame.

The acceleration $a^\mu$ given by Eq. (1) may be rewritten as

$$a^\mu = u^\alpha \nabla_\alpha e_{(0)}^\mu = u^\alpha \nabla_\alpha u^\mu = \frac{dx^\alpha}{ds} \left( \frac{\partial u^\mu}{\partial x^\alpha} + 0\Gamma^\mu_{\alpha\beta} u^\beta \right)$$

$$= \frac{d^2 x^\mu}{ds^2} + 0\Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds}, \quad (5)$$

where $0\Gamma^\mu_{\alpha\beta}$ are the Christoffel symbols. Thus if $u^\mu = e_{(0)}^\mu$ represents a geodesic trajectory, then the frame is in free fall and $a^\mu = 0 = \phi_{(0)(i)}$. Therefore we conclude that nonvanishing values of $\phi_{(0)(i)}$ represent inertial accelerations of the frame.

Following ref. [3], we take into account the orthogonality of the tetrads and write Eq. (3) as $\phi_a^b = -u^\lambda e_a^\mu \nabla_\lambda e^b_\mu$, where $\nabla_\lambda e^b_\mu = \partial_\lambda e^b_\mu - 0\Gamma^b_{\sigma\lambda} e^\sigma_\mu$. Next we consider the identity $\partial_\lambda e^b_\mu - 0\Gamma^b_{\sigma\lambda} e^\sigma_\mu + 0\omega^b_\lambda e^c_\mu = 0$, where $0\omega^b_\lambda e^c_\mu$ is the metric compatible Levi-Civita connection, and express $\phi_a^b$ according to

$$\phi_a^b = e_{(0)}^\mu (0\omega^b_\mu a^\lambda). \quad (6)$$

Finally we take into account the identity $0\omega^a_\mu b = -K^a_\mu b$, where $-K^a_\mu b$ are the Ricci rotation coefficients defined by

$$K_{ab\mu} = \frac{1}{2} e_a^\lambda e_b^\nu (T_{\lambda\mu\nu} + T_{\nu\lambda\mu} + T_{\mu\lambda\nu}), \quad (7)$$

and $T_{\lambda\mu\nu} = e^a_\lambda T_{a\mu\nu}$. After simple manipulations we arrive at

$$\phi_{ab} = \frac{1}{2} [T_{(0)ab} + T_{a(0)b} - T_{b(0)a}], \quad (8)$$

The expression above is not invariant under local SO(3,1) transformations, and for this reason the values of $\phi_{ab}$ may characterize the frame. How-
ever, eq. (8) is invariant under coordinate transformations. We interpret $\phi_{ab}$ as the inertial accelerations of the frame along the trajectory $C$.

In ref. [3] we applied definition (8) to the analysis of two simple configurations of tetrad fields in the flat Minkowski spacetime. We considered the frame adapted to linearly accelerated observers, and to a stationary frame whose four-velocity is $e_{(0)} \mu = (1, 0, 0, 0)$ and which rotates around the $\alpha$ axis. The components $\phi_{(0)(i)}$ and $\phi_{(i)(j)}$ yield the known values of the translational acceleration and of the angular velocity of the frame, respectively. As we will see in section 3, in suitable situations the values of $\phi_{(0)(i)}$ which are necessary to maintain a frame in stationary state exactly cancel the geodesic acceleration exerted on a particle or observer in spacetime.

3 Stationary frame and geodesic acceleration in the Schwarzschild spacetime

In order to obtain the radial geodesic acceleration of a particle in the Schwarzschild spacetime, as discussed in ref. [2], we will address a more general situation, namely, we will obtain the inertial accelerations that are necessary to impart to a frame such that it remains stationary in spacetime. The Schwarzschild spacetime is described by the line element

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 (\sin \theta)^2 d\phi^2.$$  \hspace{1cm} (9)

A field of stationary observers in spacetime is characterized by a vector field $u^\mu$ such that $u^\mu = (u^0, 0, 0, 0)$, i.e., the spatial components of $u^\mu$ vanish. Thus in the construction of the tetrad field we require

$$e_{(0)}^i = u^i = 0.$$  \hspace{1cm} (10)

In view of the orthogonality of the tetrad components this condition implies $e^{(k)}_0 = 0$. A simple form of $e_{a\mu}$ in $(t, r, \theta, \phi)$ coordinates that satisfies this property and yields (9) is given by

$$e_{a\mu} = \begin{pmatrix} -\beta & 0 & 0 & 0 \\ 0 & \alpha \sin \theta \cos \phi & r \cos \theta \cos \phi & -r \sin \theta \sin \phi \\ 0 & \alpha \sin \theta \sin \phi & r \cos \theta \sin \phi & r \sin \theta \cos \phi \\ 0 & \alpha \cos \theta & -r \sin \theta & 0 \end{pmatrix},$$  \hspace{1cm} (11)
where

\[ \alpha = \left(1 - \frac{2m}{r}\right)^{-1/2} \]

\[ \beta = \left(1 - \frac{2m}{r}\right)^{1/2}. \]  

In (11) \( a \) and \( \mu \) label lines and rows, respectively. It is possible to show that in the asymptotic limit \( r \to \infty \) the inverse tetrad components in \((t, x, y, z)\) coordinates satisfy

\[ e^{(1)}_{\mu}(t, x, y, z) \cong (0, 1, 0, 0), \]

\[ e^{(2)}_{\mu}(t, x, y, z) \cong (0, 0, 1, 0), \]

\[ e^{(3)}_{\mu}(t, x, y, z) \cong (0, 0, 0, 1). \]  

(13)

We proceed now to determine the acceleration tensor \( \phi_{ab} \). After a number of manipulations we find that (11) represents a nonrotating frame, i.e.,

\[ \phi_{(i)(j)} = 0. \]  

(14)

Altogether, conditions (13) and (14) fix the orientation of the frame in spacetime.

The translational acceleration, however, is nonvanishing. From definition (8) we find

\[ \phi_{(0)(i)} = T_{(0)(0)(i)} = e^{(0)\mu} e^{(i)\nu} T_{(0)\mu\nu}. \]  

(15)

For \( a = (0) \) the only nonvanishing component of \( T_{a\mu\nu} \) is \( T_{(0)01} = \partial_1 \beta \). The equation above yields

\[ \phi_{(0)(i)} = g^{00} g^{11} e_{(0)0} e_{(i)1} T_{(0)01}, \]  

(16)

from what follows

\[ \phi_{(0)(1)} = \frac{m}{r^2} \left(1 - \frac{2m}{r}\right)^{-1/2} \sin \theta \cos \phi, \]

\[ \phi_{(0)(2)} = \frac{m}{r^2} \left(1 - \frac{2m}{r}\right)^{-1/2} \sin \theta \sin \phi, \]

\[ \phi_{(0)(3)} = \frac{m}{r^2} \left(1 - \frac{2m}{r}\right)^{-1/2} \cos \theta. \]  

(17)
We define the acceleration

\[ \mathbf{a} = \phi_{(0)(1)} \hat{\mathbf{x}} + \phi_{(0)(2)} \hat{\mathbf{y}} + \phi_{(0)(3)} \hat{\mathbf{z}}, \]  

which may be written as

\[ \mathbf{a} = \frac{m}{r^2} \left( 1 - \frac{2m}{r} \right)^{-1/2} \hat{\mathbf{r}}. \]  

Equation (19) represents the inertial acceleration necessary to maintain the frame in stationary state in spacetime. Therefore it exactly cancels the geodesic acceleration that is exerted on the frame. In fact,

\[ a = \frac{m}{r^2} \left( 1 - \frac{2m}{r} \right)^{-1/2} \]  

is precisely the geodesic acceleration obtained in ref. [2] by means of Cartan’s structural equations or, for instance, in ref. [7] by taking the absolute derivative (according to eq. (1)) of the velocity of a body in free fall in the Schwarzschild spacetime. We note, however, that eq. (15) (and consequently (19)) is invariant under coordinate transformations.

Now we analyse a very interesting consequence of eq. (20), considering that the acceleration \( a \) is kept constant. Equation (20) is a quadratic equation for the mass \( m \), which can be written as

\[ m^2 + 2a^2 r^3 m - a^2 r^4 = 0. \]  

Solving this equation for \( m \) we find

\[ m = -a^2 r^3 + ar^2 (1 + a^2 r^2)^{1/2}, \]  

which, after simple manipulations, leads to

\[ \frac{m}{r} = a^2 r^2 \left[ \sqrt{1 + \frac{1}{(ar)^2}} - 1 \right]. \]  

Keeping in mind that \( a \) is assumed to be nonvanishing and constant, we define the variable \( x \) according to

\[ \frac{1}{ar} = x, \]  

and therefore

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When \( r \to \infty \), \( x \to 0 \) and consequently

\[
\frac{1}{x^2} \left[ \sqrt{1 + x^2} - 1 \right] \cong \frac{1}{2} - \frac{x^2}{8} \equiv \frac{1}{2} - \epsilon ,
\]

where \( \epsilon << 1 \), in the limit \( r \to \infty \). Thus in this limit we have

\[
\frac{m}{r} \cong \frac{1}{2} - \epsilon ,
\]

which implies \( r \cong 2m(1 + 2\epsilon) \). However the component of the curvature tensor (of the Levi-Civita connection) in the \((t, r)\) plane is given by

\[
R_{0101} = \frac{2m}{r^3}.
\]

Therefore, in view of eq. (27), in the limit when both \( r \to \infty \) and \( m \to \infty \) the curvature tensor vanishes,

\[
R_{0101} = \frac{m}{r^2} \frac{2}{r^2} \cong \frac{1}{r^2} \cong 0.
\]

Thus we see that if (i) \( r \to \infty \) and (ii) \( m/r \cong 1/2 - \epsilon \), then the curvature tensor approaches zero whereas the geodesic acceleration \( a \) may acquire arbitrary values. This is not a realistic physical situation, but it proves that the curvature tensor is not responsible for the geodesic acceleration given by (20). This is the argument presented by Schucking [2]: there may exist gravitational field configurations such that the curvature tensor approaches zero, whereas the geodesic acceleration may acquire arbitrary values. The action of gravity on a particle that undergoes geodesic acceleration is not affected by the vanishing value of the curvature tensor. As we have seen, the geodesic acceleration may be obtained from the acceleration tensor given by eq. (8), and the latter is constructed out of the torsion tensor, which is ultimately responsible for the geodesic acceleration. We note that it is impossible to write eq. (8) in terms of the curvature tensor.

4 The Pound-Rebka experiment

The relevance of the torsion tensor to the spacetime geometry is revealed by the Pound-Rebka experiment [8]. Let us consider the emission of two radial
light signals at the position $r + \Delta r$ to the position $r$, in the Schwarzschild spacetime. At the position $r + \Delta r$ the time elapsed between the first and second signals is the proper time $d\tau_1$, and at $r$ the second signal is received after a proper time $d\tau_2$. As in the previous section, we assume that the Schwarzschild spacetime is described the coordinates $(t, r, \theta, \phi)$. In this case we have

$$
\begin{align*}
    d\tau_1 &= \beta(r + \Delta r)dt , \\
    d\tau_2 &= \beta(r)dt ,
\end{align*}
$$

where $\beta(r) = (-g_{00})^{1/2}$. If $\Delta r/r << 1$, then

$$
    d\tau_1 \approx \left[ 1 + \frac{\Delta r}{r^2} \left( \frac{GM}{c^2} \right) \right] d\tau_2 ,
$$

where we have used $m = GM/c^2$. The experimental verification of eq. (31) is the result of the Pound-Rebka experiment \cite{9}, which may be described in Figure 1.

![Figure 1: The Pound-Rebka experiment](image)

In Figure 1 the vectors $v^\mu$ and $b^\mu$ are null vectors that represent the light signals. Null vectors satisfy the condition $v^\mu v^\nu g_{\mu\nu} = 0$. For radial null vectors we have $v^0_v^0 g_{00} + v^1 v^1 g_{11} = 0$, and therefore $v^1 = (-g_{00}/g_{11})^{1/2} v^0$. In the
Schwarzschild spacetime we have \( v^1 = (-g_{00})v^0 \). Thus a radial null vector in the Schwarzschild spacetime may be written as \( v^\mu = v^0(1, -g_{00}, 0, 0) \).

The vectors \( a^\mu, b^\mu, v^\mu \) and \( w^\mu \) in Figure 1 have dimension of length. Except for the factor \( c \) (the speed of light), the zero components of these vectors represent the time elapsed between two events. The time elapsed between \( (t_1 + d\tau_1, r + dr) \) and \( (t_1, r + dr) \) is \( d\tau_1 = \beta(r + dr)dt \), and between \( (t_2 + d\tau_2, r) \) and \( (t_2, r) \) is \( d\tau_2 = \beta(r)dt \). Thus,

\[
\begin{align*}
a^\mu &= \beta(r + dr)(cdt, 0, 0, 0), \\
w^\mu &= \beta(r)(cdt, 0, 0, 0).
\end{align*}
\]  
(32)

Let us denote \( dT \) the time elapsed between the events \( (t_2, r) \) and \( (t_1, r + dr) \), or between \( (t_2 + d\tau_2, r) \) and \( (t_1 + d\tau_1, r + dr) \). We write

\[
\begin{align*}
b^\mu &= (c dT, -g_{00} c dT, 0, 0), \\
v^\mu &= (c dT, -g_{00} c dT, 0, 0).
\end{align*}
\]  
(33)

However, ingoing radial null geodesics in the Schwarzschild spacetime satisfy (see, for instance, section 16.4 of [10])

\[
\frac{cdT}{dr} = -\frac{r}{r - 2m} = -\frac{1}{1 - 2m/r} = -g_{11} = \frac{1}{g_{00}}.
\]  
(34)

Thus, \( c dT = (1/g_{00})dr \), and finally we have

\[
\begin{align*}
b^\mu &= dr\left(\frac{1}{g_{00}}, -1, 0, 0\right), \\
v^\mu &= dr\left(\frac{1}{g_{00}}, -1, 0, 0\right).
\end{align*}
\]  
(35)

The breaking of the parallelogram in Figure 1 is verified by the following operation,

\[
(a^\mu + b^\mu) - (v^\mu + w^\mu) = \left(\beta(r + dr) - \beta(r)\right)(cdt, 0, 0, 0),
\]

\[
\cong \frac{m}{r^2} \frac{cdt}{d\tau} \frac{dr}{d\tau} \left(-g_{00}\right)^{1/2} (1, 0, 0, 0).
\]  
(36)
The nonclosure of the parallelogram can also be obtained by means of an alternative procedure. Let us consider two infinitesimal vectors, \( A^\mu = dx^\mu \) and \( B^\mu = \delta x^\mu \), as in Figure 2 below.

![Figure 2: The breaking of the parallelogram](image)

The parallel transport of \( A^\mu \) along \( \delta x^\mu \), and of \( B^\mu \) along \( dx^\mu \) are given by, respectively,

\[
\delta A^\mu = -\Gamma^\mu_{\alpha \beta} A^\alpha \delta x^\beta, \\
\delta B^\mu = -\Gamma^\mu_{\alpha \beta} B^\alpha dx^\beta,
\]

where \( \Gamma^\mu_{\alpha \beta} \) is a spacetime connection with torsion. The nonclosure of the parallelogram is obtained as follows,

\[
\left[ A^\mu + (B^\mu + \delta B^\mu) \right] - \left[ B^\mu + (A^\mu + \delta A^\mu) \right] = (\Gamma^\mu_{\alpha \beta} - \Gamma^\mu_{\beta \alpha}) dx^\alpha \delta x^\beta = T^\mu_{\alpha \beta} dx^\alpha \delta x^\beta. 
\]

As in the previous section, the Schwarzschild spacetime is described by the set of tetrad fields given by (11), i.e., by stationary observers in spacetime. Without going into details of calculations we just assert that the frame determined by (11) yields only three components of the torsion tensor \( T^\mu_{\alpha \beta} = e^\mu_a T^a_{\alpha \beta} \) (note that there are six nonvanishing components of \( T_{\alpha \mu \nu} \)),

\[
T^0_{01} = -\frac{1}{\beta} \partial_1 \beta, \\
T^2_{12} = T^3_{13} = \frac{1}{r}(1 - \alpha).
\]
Now we identify

\[ dx^\alpha = a^\alpha = \beta(r + dr)(c\, dt, 0, 0, 0), \]
\[ \delta x^\beta = v^\beta = dr\left(\frac{1}{g_{00}}, -1, 0, 0\right). \]  

(40)

It is straightforward to verify that

\[ T^0_{\alpha\beta} dx^\alpha \delta x^\beta = T^0_{01} a^0 v^1 \]
\[ = \frac{m}{r^2(-g_{00})} \beta(r + dr) c\, dt \, dr \]
\[ \cong \frac{m}{r^2(-g_{00})^{1/2}}. \]  

(41)

As a consequence of eq. (39) no other breaking of parallelogram takes place in the \((t, r)\) plane of Figure 1. Taking into account eq. (30), we may also write

\[ T^0_{\alpha\beta} dx^\alpha \delta x^\beta = \frac{m}{r^2(-g_{00})^{1/2}}. \]  

(42)

In view of the agreement between eqs. (36) and (41) we conclude that the reference frame determined by (11) is indeed suitable to describe the emergence of torsion in the Schwarzschild spacetime.

5 Concluding remarks

We have investigated two manifestations of torsion in the Schwarzschild spacetime in the framework of a set of tetrad fields adapted to stationary observers. In order to maintain a frame in stationary state in spacetime it is necessary to impart to the frame a translational, inertial acceleration that exactly cancels the gravitational, geodesic acceleration. By means of the acceleration tensor defined by eq. (8), which is a coordinate invariant definition, we have obtained the inertial acceleration and consequently the geodesic acceleration on a radial trajectory. The investigation of the expression of the geodesic acceleration led to the conclusion that for certain values of \(m\) and \(r\) in the Schwarzschild spacetime the curvature tensor approaches
zero, whereas the geodesic acceleration may acquire arbitrary values. We concluded that it is the torsion tensor (of the Weitzenböck connection), and not the curvature tensor (of the Levi-Civita connection) that is responsible for the geodesic acceleration of a particle.

The same set of tetrad fields (eq. (11)) explains the Pound-Rebka experiment in terms of breaking of parallelogram, as in Figure 1. The set of tetrad fields given by (11) yields the torsion tensor components (39), which are crucial to the agreement between (36) (the nonclosure of the parallelogram directly from the Pound-Rebka experiment) and (41) (the breaking of the parallelogram obtained by parallel transport).

We remark that the discussions and results of sections 3 and 4 are valid for a wider class of spacetimes, namely, to all spacetimes determined by the metric tensor

\[ ds^2 = -\beta^2 dt^2 + \alpha^2 dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 , \]

for which \( \beta(r) = 1/\alpha(r) \) is an arbitrary function of the radial coordinate \( r \). This class of spacetime metrics includes, for instance, the Schwarzschild-de Sitter and the Reissner-Nordstrom spacetimes. Let us briefly consider the de Sitter spacetime. We have \( \beta^2 = 1 - Kr^2 \), where \( K \) is related to the positive cosmological constant \( \Lambda \) by means of \( \Lambda = 3K \). By repeating the analysis of section 3 it follows from eqs. (15) and (16) that the inertial acceleration that is necessary to impart to the frame such that it remains stationary in spacetime (in the notation of eq. (18)) is given by

\[ a = (\partial_1 \beta) \hat{r} = -\frac{K r}{(1 - Kr^2)^{1/2}} \hat{r} . \]

We see that when \( r \) approaches the cosmological horizon \( R = 1/\sqrt{K} = \sqrt{3/\Lambda} \), \( a \) acquires arbitrarily large values, whereas the curvature tensor component \( R_{0101} \) remains finite and constant: \( R_{0101} = -K \). Once again we see that the values of the acceleration tensor have no direct relationship to the curvature tensor.

The question finally arises: what is the connection of the Schwarzschild spacetime? Is it simply the frame independent, metric connection,

\[ \Gamma^\lambda_{\mu\nu} = 0 \Gamma^\lambda_{\mu\nu} , \]

given by the Christoffel symbols, or a connection with torsion,
\[ \Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + T^\lambda_{\mu\nu}, \tag{45} \]

where \( T^\lambda_{\mu\nu} \) (obtained in the frame of stationary observers) is given by (39)? Since \( T^\lambda_{\mu\nu} \) is antisymmetric in the \( \mu\nu \) indices, the geodesic equations obtained from connections (44) and (45) are the same. \( T^\lambda_{\mu\nu} \) does not affect the standard geodesic motion of particles in spacetime, and not any of the experimental tests of general relativity. Moreover we note that in the limit \( m/r << 1 \) the three torsion components in eq. (39) fall off as \( -m/r^2 \). It is likely that \( T^\lambda_{\mu\nu} \) is relevant to small scale gravitational phenomena. However, (45) is not a metric compatible connection, as it leads to \( \nabla_\alpha g_{\mu\nu} \neq 0 \). The answer to the question above will probably require the investigation of further experimental consequences of the Schwarzschild geometry.

References

[1] F. W. Hehl, J. D. McCrea, E. W. Mielke and Y. Ne’eman, Phys. Rep. 258, 1 (1995).

[2] E. Schucking, Gravitation is torsion [arXiv:0803.4128].

[3] J. W. Maluf, F. F. Faria and S. C. Ulhoa, Class. Quantum Grav. 24 (2007) 2743 [arXiv:0704.0986].

[4] J. W. Maluf and F. F. Faria, Ann. Phys. (Berlin) 17 (2008) 326 [arXiv:0804.2502].

[5] F. H. Hehl, J. Lemke and E. W. Mielke, “Two Lectures on Fermions and Gravity”, in Geometry and Theoretical Physics, edited by J. Debrus and A. C. Hirshfeld (Springer, Berlin Heidelberg, 1991).

[6] B. Mashhoon and U. Muench, Ann. Phys. (Berlin) 11 (2002) 532 [gr-qc/0206082].

[7] J. B. Hartle, Gravity: an introduction to Einstein’s general relativity (Addison Wesley, San Francisco, 2003), section 20.4.

[8] R. V. Pound and G. A. Rebka, Phys. Rev. Lett. 4 (1960) 337.

[9] S. Weinberg, Gravitation and Cosmology (Wiley, New York, 1972).
[10] R. D’Inverno, *Introducing Einstein’s Relativity* (Clarendon Press, Oxford, 2002).