On the Exponent of Several Classes of Oscillatory Matrices

Yoram Zarai and Michael Margaliot

Abstract

Oscillatory matrices were introduced in the seminal work of Gantmacher and Krein. An \( n \times n \) matrix \( A \) is called oscillatory if all its minors are nonnegative and there exists a positive integer \( k \) such that all minors of \( A^k \) are positive. The smallest \( k \) for which this holds is called the exponent of the oscillatory matrix \( A \). Gantmacher and Krein showed that the exponent is always smaller than or equal to \( n - 1 \). An important and nontrivial problem is to determine the exact value of the exponent. Here we use the successive elementary bidiagonal factorization of oscillatory matrices, and its graph-theoretic representation, to derive an explicit expression for the exponent of several classes of oscillatory matrices, and an upper-bound on the exponent for several other classes.

Index Terms

Totally nonnegative matrices, totally positive matrices, oscillatory matrices, sequential elementary factorization, planar network, exponent of an oscillatory matrix.

I. INTRODUCTION

A matrix is called totally positive (TP) [totally nonnegative (TN)] if all its minors are positive [non-negative]. Such matrices arise in various branches of mathematics and in many applications, including oscillations in mechanical systems [1], stochastic processes and approximation theory [2], planar resistor networks [3], optimal allocation problems [4], and many more [5], [6], [7].

One reason for the importance of TP matrices is their variation diminishing property: if \( A \in \mathbb{R}^{n \times m} \) is TP, with \( n \geq m \), then for any \( x \in \mathbb{R}^m \setminus \{0\} \) the number of sign variations in \( Ax \) is smaller than or equal to the number of sign variations in \( x \). Recently, it was shown that this property has important implications in the asymptotic analysis of time-varying linear and nonlinear dynamical systems [8], [9], [10], [11], [12], [13]. In these dynamical systems the number of sign variations in the vector of derivatives is an integer-valued Lyapunov function.

Oscillatory matrices were introduced in the seminal work of Gantmacher and Krein [1] who studied small vibrations of mechanical systems. A matrix \( A \in \mathbb{R}^{n \times n} \) is called oscillatory if \( A \) is TN and there exists an integer \( k > 0 \) such that \( A^k \) is TP. Thus, oscillatory matrices are intermediary between TN and TP matrices. Oscillatory matrices enjoy many special properties [1]. For example, the eigenvalues of oscillatory matrices are real, positive and distinct, and the corresponding eigenvectors satisfy a special sign pattern.

An oscillatory matrix \( A \) must be non-singular, as \( \det(A^k) > 0 \). Two useful characterizations of oscillatory matrices are the following.

Proposition 1. [1] p. 139] Let \( A \in \mathbb{R}^{n \times n} \) be a TN matrix. Then \( A \) is oscillatory if and only if it is non-singular, and \( a_{i,i+1} > 0, a_{i+1,i} > 0 \), for all \( i = 1, \ldots, n-1 \).

Proposition 2. [5] Ch. 2] A matrix \( A \in \mathbb{R}^{n \times n} \) is oscillatory if and only if \( A \) is TN, non-singular and irreducible, and in this case \( A^{n-1} \) is TP.

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The exponent of an oscillatory matrix $A \in \mathbb{R}^{n \times n}$, denoted by $r = r(A)$, is the least positive integer such that $A^r$ is TP. Obviously, $r(A) \leq n - 1$. Yet, deriving a closed-form expression for $r$ is a nontrivial problem.

Oscillatory matrices have found many applications [1], [14], [5], [15]. Recently, Ref. [16] introduced the notion of oscillatory discrete-time systems and used it in the analysis of certain discrete-time, time-varying nonlinear systems. It was shown that if the mapping defining the nonlinear dynamics is $T$-periodic then any trajectory of the system either leaves any compact set or converges to a subharmonic trajectory, i.e. a trajectory that is periodic with a period of $mT$. The value $m > 0$ here is bounded by the exponent of an oscillatory matrix.

Ref. [17] identified classes of oscillatory matrices with $r(A) = n - 1$. Motivated by the work in [17], [18], we determine $r(A)$ explicitly for several classes of oscillatory matrices (see Props. 9 and 10 below), and provide nontrivial upper bounds on $r(A)$ for other classes (see Corollary 3).

The remainder of this paper is organized as follows. The next section reviews known tools and results that will be used later on. Let $A \in \mathbb{R}^{n \times m}$, $a_{ij}$ or $(i,j)$ denotes the entry of $A$ in row $i$ and column $j$, and $A^T$ is the transpose of $A$. The square identity matrix is denoted by $I$, with dimension that should be clear from the context.

II. Preliminaries

We begin by reviewing notations and results that will be used later on. Let $A \in \mathbb{R}^{n \times m}$. Pick $k_1 \in \{1, \ldots, n\}$ and $k_2 \in \{1, \ldots, m\}$, and let $\alpha [\beta]$ denote a set of $k_1 [k_2]$ integers $1 \leq i_1 < \cdots < i_{k_1} \leq n$ $[1 \leq j_1 < \cdots < j_{k_2} \leq m]$. Then $A[\alpha] [\beta]$ denotes the $k_1 \times k_2$ sub-matrix of $A$ containing the rows indexed by $\alpha$ and the columns indexed by $\beta$. When $k_1 = k_2$, we let $A(\alpha | \beta) := \det(A[\alpha | \beta])$, that is, the minor of $A$ corresponding to the rows indexed by $\alpha$ and columns indexed by $\beta$. A minor corresponding to the same set of row and column indexes (i.e. $A(\alpha | \alpha)$) is called a principal minor.

Pick $A \in \mathbb{R}^{n \times m}$, $B \in \mathbb{R}^{m \times p}$, and let $C := AB$. The Cauchy-Binet formula [5, Ch. 1] asserts that for any two sets $\alpha \subseteq \{1, \ldots, n\}$, $\beta \subseteq \{1, \ldots, p\}$, with the same cardinality $k \subseteq \{1, \ldots, \min\{n, m, p\}\}$, we have

$$C(\alpha | \beta) = \sum_{|\gamma|=k} A(\alpha | \gamma)B(\gamma | \beta),$$

(1)

where the sum is over all $\gamma = \{i_1, \ldots, i_k\}$, with $1 \leq i_1 < \cdots < i_k \leq m$. Thus, every minor of $AB$ is the sum of products of minors of $A$ and $B$. Note that (1) implies that if $A$ and $B$ are both TP [TN] then $AB$ is TP [TN], and that the product of an invertible TN and TP is TP. In particular, this implies that if $A \in \mathbb{R}^{n \times n}$ is oscillatory, then $A^k$ is TP for all $k \geq r(A)$.

Given $A \in \mathbb{R}^{n \times n}$ and $p \in \{1, \ldots, n\}$, the $p$th multiplicative compound (MC) $A^{(p)}$ is the $\binom{n}{p} \times \binom{n}{p}$ matrix that includes all the $p \times p$ minors of $A$ ordered lexicographically. For example, for $A \in \mathbb{R}^{3 \times 3}$,

$$A^{(2)} = \begin{pmatrix}
a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} & a_{12}a_{23} - a_{13}a_{22} \\
a_{13}a_{23} - a_{13}a_{23} & a_{11}a_{33} - a_{13}a_{31} & a_{12}a_{33} - a_{13}a_{32} \\
a_{23}a_{32} - a_{23}a_{32} & a_{21}a_{33} - a_{23}a_{31} & a_{22}a_{33} - a_{23}a_{32}
\end{pmatrix},$$

where the row [column] marks are the indexes in $\alpha [\beta]$.

The Cauchy-Binet formula yields

$$A^{(p)} = A^{(p)}B^{(p)},$$

(2)

justifying the term multiplicative compound. In particular, $(A^{k})^{(p)} = (A^{(p)})^{k}$ for all $k \geq 1$. 
An upper-right [lower-left] corner minor of a matrix $A \in \mathbb{R}^{n \times m}$ is a minor $A(\alpha | \beta)$, where $\alpha$ consists of the first [last] $k$ indexes and $\beta$ consists of the last [first] $k$ indexes, for some $k \in \{1, \ldots, \min\{n, m\}\}$. A corner minor is one that is either a lower left or an upper right corner minor. Table 1 lists all the corner minors of a matrix $A \in \mathbb{R}^{n \times n}$, except for $\det(A)$. If $A \in \mathbb{R}^{n \times n}$ then the corner minors of $A$ are the entries in the first row and last column, and the last row and first column of every $A^{(k)}$, $k = 1, \ldots, n$.

| Upper-right | Lower-left |
|-------------|------------|
| $A(\{1\}|\{n\})$ | $A(\{n\}|\{1\})$ |
| $A(\{1, 2\}|\{n - 1, n\})$ | $A(\{n - 1, n\}|\{1, 2\})$ |
| $\vdots$ | $\vdots$ |
| $A(\{1, 2, \ldots, n - 1\}|\{2, 3, \ldots, n\})$ | $A(\{2, 3, \ldots, n\}|\{1, 2, \ldots, n - 1\})$ |

TABLE I: All upper-right and lower-left corner minors of $A \in \mathbb{R}^{n \times n}$ (except for $\det(A)$).

A matrix $A \in \mathbb{R}^{n \times n}$ is TP if all the $\sum_{p=1}^{n} (\binom{n}{p})^2$ minors of $A$ are positive. If $A$ is known to be TN then verifying that a small subset of the minors are positive implies that $A$ is TP. This is stated in the following result (see, e.g., [5, Thm. 3.1.10]).

**Proposition 3.** Suppose that $A \in \mathbb{R}^{n \times n}$ is a TN matrix. Then $A$ is TP if and only if all corner minors of $A$ are positive.

Our analysis of oscillatory matrices is motivated by the work of Fallat and Liu [17], and is based on the powerful successive elementary bidiagonal factorization of invertible TN matrices [19], [20] and their associated planar networks.

### A. Successive Elementary Bidiagonal Factorization

Let $E_{i,j} \in \mathbb{R}^{n \times n}^{+}$ denote the standard basis matrix whose only nonzero entry is a 1 that occurs in row $i$ and column $j$. For $q \in \mathbb{R}$ and $i \in \{2, \ldots, n\}$, let $L_i(q) := I + qE_{i,i-1}$ and $U_i(q) := (L_i(q))^T$. For example, for $n = 3$, $L_2(4) = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The matrices $L_i(q)$ and $U_i(q)$ are called elementary bidiagonal (EB) matrices. Several useful relations of EB matrices are:

$$
L_i(0) = U_i(0) = I, \quad i = 2, \ldots, n,
$$

$$
L_i(-x) = L_i(-x), \quad i = 2, \ldots, n,
$$

$$
U_i(-x) = U_i(-x), \quad i = 2, \ldots, n,
$$

$$
L_i(y)L_j(y) = L_j(y)L_i(x), \quad |i - j| > 1,
$$

$$
U_i(y)U_j(y) = U_j(y)U_i(x), \quad |i - j| > 1,
$$

$$
L_i(y)U_j(y) = U_j(y)L_i(x), \quad i \neq j.
$$

(3)

Let $k := (n-1)n/2$ (note that $k \geq n-1$ for all $n \geq 2$). Then any invertible TN (I-TN) matrix $A \in \mathbb{R}^{n \times n}$ can be factorized as [5]:

$$
A = [L_n(\ell_1) \cdots L_2(\ell_{n-1})][L_n(\ell_n) \cdots L_3(\ell_{2n-3})] \cdots [L_n(\ell_k)]D
$$

$$
[U_n(u_k)][U_{n-1}(u_{k-1})U_n(u_{k-2})] \cdots [U_2(u_{n-1}) \cdots U_n(u_1)],
$$

(4)

where, $\ell_i, u_i \geq 0$, $i = 1, \ldots, k$, and $D \in \mathbb{R}^{n \times n}$ is a diagonal matrix with positive diagonal entries. For example, for $n = 4$ we have $k = 6$, so all I-TN matrices $A \in \mathbb{R}^{4 \times 4}$ can be factorized as:

$$
A = [L_4(\ell_1)L_3(\ell_2)L_2(\ell_3)] [L_4(\ell_4)L_3(\ell_5)][L_4(\ell_6)] D[U_4(u_6)][U_3(u_5)U_4(u_4)][U_2(u_3)U_3(u_2)U_4(u_1)].
$$
The representation (4) is called the *successive elementary bidiagonal (SEB) factorization* of the I-TN matrix $A$, and this factorization is unique [5, p. 53]. The $\ell_j$’s and $u_k$’s are called the *multipliers* in the factorization.

The derivation of (4) is based on the well-known Neville elimination process [21]. The following example demonstrates this.

**Example 1.** Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ 2 & 10 & 29 \end{bmatrix}.$$  \hspace{1cm} (5)

It is straightforward to verify that $A$ is I-TN. The first step in the Neville elimination process is to use the second row to null entry $(3, 1)$ in $A$. This is done by multiplying $A$ from the left by $L_3(-1)$ yielding

$$L_3(-1)A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ 0 & 6 & 21 \end{bmatrix}.$$  

We next use the first row to null entry $(2, 1)$ in $L_3(-1)A$ using $L_2(-2)$:

$$L_2(-2)L_3(-1)A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 6 \\ 0 & 6 & 21 \end{bmatrix}.$$

Next, we use the second row to null entry $(3, 2)$ in $L_2(-2)L_3(-1)A$ using $L_3(-3)$:

$$L_3(-3)L_2(-2)L_3(-1)A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix} := P.$$

Applying similar row operations to $P^T$ yields $L_3(-2)L_2(-1)L_3(-1)P^T = D$, where $D := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

Thus,

$$L_3(-2)L_2(-1)L_3(-1)A^T L_3(-1)^T L_2(-2)^T L_3(-3)^T = D,$$

and using (3) yields

$$A = [L_3(1) L_2(2)][L_3(3)] D[U_3(2)][U_2(1) U_3(1)],$$  \hspace{1cm} (6)

which is the SEB factorization of $A$. \hspace{1cm} $\square$

TP and oscillatory matrices can be characterized in terms of their SEB factorization.

**Proposition 4.** [5] A matrix $A \in \mathbb{R}^{n \times n}$ is TP if and only if in the SEB factorization (4) $\ell_i > 0$, $u_i > 0$, $i = 1, \ldots, k$.

**Proposition 5.** [5] A matrix $A \in \mathbb{R}^{n \times n}$ is oscillatory if and only if in the SEB factorization (4) at least one of the multipliers from each of $L_n, L_{n-1}, \ldots, L_2$, and from each of $U_n, U_{n-1}, \ldots, U_2$ is positive.

The case where exactly one of the multipliers above is positive defines a *basic oscillatory matrix*.

**Definition 1.** [5] A matrix is called *basic oscillatory* if and only if in the SEB factorization (4) exactly one of the multipliers from each of $L_n, \ldots, L_2$, and exactly one from each of $U_n, \ldots, U_2$ is positive.
For example, the factorization of

\[ A = \begin{bmatrix} 1 & 6 & 0 & 0 \\ 2 & 13 & 4 & 20 \\ 2 & 13 & 5 & 25 \\ 0 & 0 & 3 & 16 \end{bmatrix}, \]

is \( A = L_3(1)L_2(2)L_4(3)IU_3(4)U_4(5)U_2(6) \), so \( A \) is a basic oscillatory matrix.

Basic oscillatory matrices may be viewed as minimal oscillatory matrices in the sense of the number of SEB factors involved. The following result shows that any oscillatory matrix is a product of a basic oscillatory and I-TN matrices.

**Proposition 6.** [5 Thm. 2.6.9] Any oscillatory matrix \( A \in \mathbb{R}^{n \times n} \) can be written in the form \( A = A_1BA_2 \), where \( B \in \mathbb{R}^{n \times n} \) is basic oscillatory and both \( A_1, A_2 \in \mathbb{R}^{n \times n} \) are I-TN.

Let \( A \in \mathbb{R}^{n \times n} \) be oscillatory. Ref. [17] derived a necessary and sufficient condition for its exponent to be \( n - 1 \) using the SEB factorization. For example, let

\[ A = L_2(\ell_1)L_3(\ell_2) \cdots L_n(\ell_{n-1})DU_n(u_{n-1}) \cdots U_3(u_2)U_2(u_1), \]

with \( \ell_i, u_i > 0 \). Then \( A \) is a tridiagonal basic oscillatory matrix (also referred to as a Jacobi matrix), and it is not difficult to see that the entries \((n, 1)\) and \((1, n)\) in \( A^{n-2} \) are zero. Thus, \( r(A) > n - 2 \) and since \( r(A) \leq n - 1 \), \( r(A) = n - 1 \). In addition, the cases where

\[ A = L_n(\ell_1)L_{n-1}(\ell_2) \cdots L_2(\ell_{n-1})DU_2(u_{n-1}) \cdots U_1(u_2)U_1(u_1), \]

with \( \ell_i, u_i > 0 \), also yield \( r(A) = n - 1 \) [17].

More generally, Ref. [17] established conditions on \( \ell_i, u_i, \ i = 1, \ldots, k \), in [4] that yield \( r(A) = n - 1 \), and conditions that yield \( r(A) \leq n - 2 \). For example, for \( n = 4 \), pick one form of \( L \) from:

\[ L = L_4(\ell_1)L_2(\ell_2)L_3(\ell_3), \]
\[ L = L_3(\ell_1)L_2(\ell_2)L_4(\ell_3), \]

and one form of \( U \) from

\[ U = U_4(u_1)U_2(u_2)U_3(u_3), \]
\[ U = U_3(u_1)U_2(u_2)U_4(u_3), \]

with all \( \ell_i, u_i > 0 \). Then \( A = LDU \) implies that \( A^2 \) is TP for any diagonal \( D \) with positive diagonal entries, so \( r(A) = 2 \) (note that \( A \) is not TP). On the other hand, \( A = L_2(\ell_1)L_3(\ell_2)L_4(\ell_3)DU \) or \( A = L_4(\ell_1)L_3(\ell_2)L_2(\ell_3)DU \), with \( \ell_i > 0 \) and any \( U \), implies that \( A^2 \) is not TP, so \( r(A) > 2 \) and thus \( r(A) = 3 \).

### B. Planar Networks

An *elementary weighted diagram* is a graph consisting of \( n \) source vertices (on the left of the graph) and \( n \) sink vertices (on the right). The sources and sinks are numbered consecutively from bottom to top. All edges in the diagram are directed from a source to a sink.

The elementary weighted diagrams of the EB matrices \( L_i(\ell), U_i(u) \), and a diagonal matrix \( D \) are depicted in Fig. 1. All edges in the figure are directed from left to right, and unmarked edges have weight one. The source [sink] vertices on the left [right] side represent the matrix rows [columns] indexes, and an edge of weight \( q \) from row index \( i \) to column index \( j \) implies that the entry \((i, j)\) in the associated matrix is equal to \( q \).

Given an elementary diagram, pick \( k \in \{1, \ldots, n\} \), \( 1 \leq i_1 < \cdots < i_k \leq n \) and \( 1 \leq j_1 < \cdots < j_k \leq n \). We define a family of paths that connect the sources \( \{i_1, \ldots, i_k\} \) to the sinks \( \{j_1, \ldots, j_k\} \) as follows. Each family contains \( k \) vertex-disjoint paths (i.e. paths that are non-intersecting and non-touching) joining the vertices \( \{i_1, \ldots, i_k\} \) on the left side of the diagram with the vertices \( \{j_1, \ldots, j_k\} \) on the right side. The
weight of a path is defined to be the product of the weights of its edges. Note that in an elementary diagram a path contains a single edge, but in the more general diagrams defined below a path consists of several edges. The weight of the family is defined to be the product of the weights of its \( k \) paths.

**Example 2.** Consider the diagram on the left-hand side of Fig. 1 corresponding to \( Q = L_i(\ell) \), and consider the sources \( \{i-1, i\} \) and the sinks \( \{i-1, i\} \). There is one corresponding family, namely, \( \{i-1 \to i-1, i \to i\} \) (the path \( i \to i-1 \) cannot be included, as the paths must be vertex-disjoint). The weight of each of the two paths is one, and so the weight of the family is also one.

As another example, suppose that \( Q = L_i(\ell) \), with \( i > 2 \), and consider the sources \( \{1, i\} \) and the sinks \( \{1, i-1\} \). There is one corresponding family, namely, \( \{1 \to 1, i \to i-1\} \). The weight of the first [second] path is \( 1[\ell] \), and so the weight of the family is \( \ell \).

An important observation on the associated elementary diagrams is the following \cite{5}. Let \( Q \in \mathbb{R}^{n \times n} \) be a matrix represented by any one of the diagrams in Fig. 1. For any \( k \in \{1, \ldots, n\} \) and any \( 1 \leq i_1 < \cdots < i_k \leq n \) and \( 1 \leq j_1 < \cdots < j_k \leq n \) consider all the families of \( k \) vertex-disjoint paths joining the vertices \( \{i_1, \ldots, i_k\} \) on the left side of the diagram with the vertices \( \{j_1, \ldots, j_k\} \) on the right side. This set of families is unique. We define the weight of the set of families as the sum of the weights of each family in the set. Then the minor

\[
Q(\{i_1, \ldots, i_k\}|\{j_1, \ldots, j_k\})
\]

is equal to weight of the set of families (and is zero if and only if there is no such family). For example, consider again the diagram on the left-hand side of Fig. 1 corresponding to \( Q = L_i(\ell) \). Then

\[
Q(\{i-1, i\}|\{i-1, i\}) = \det\begin{bmatrix} 1 & 0 \\ \ell & 1 \end{bmatrix} = 1, \quad (9)
\]

and

\[
Q(\{1, i\}|\{1, i-1\}) = \det\begin{bmatrix} 1 & 0 \\ 0 & \ell \end{bmatrix} = \ell, \quad (10)
\]

and this agrees with the results in Example 2.

Given the factorization \cite{4}, the corresponding planar network associated with \( A \) is obtained by concatenating (in order) left to right the elementary diagrams associated with the EB matrices and \( D \) in \cite{4}. For example, Fig. 2 depicts the planar network associated with the matrix \( A \) in \cite{6}.
Fig. 2: The planar network associated with the I-TN matrix $A$ in (6). All edges are directed from left to right.

We note in passing that any planar network can be associated with a TN matrix \cite{22}. This association is unique in case of I-TN matrices.

**Remark 1.** Since the planar network of any I-TN matrix is associated with an SEB factorization, it admits a special structure. First, all horizontal edges have positive weights, where horizontal edges corresponding to the matrix $D$ (in the center of the network) have weight $d_{ii}$, $i = 1, \ldots, n$, and all other horizontal edges have weight one. In addition, the left-hand side [right-hand side] of the network, corresponding to the product of all $L$ [$U$] matrices, consists of only downward [upward] pointing diagonal edges. This structure holds for all I-TN matrices. Only the weights of the diagonal edges and the horizontal edges corresponding to the matrix $D$ differ among different I-TN matrices.

The following important result associates the minors of $A$ to its planar network. Its proof follows from the Cauchy-Binet formula.

**Proposition 7.** \cite{5} Let $A := A_1 \cdots A_p$, where each $A_i$ is either an EB matrix ($L$ or $U$) or a diagonal matrix with positive diagonal entries. Recall that the network associated with $A$ is obtained by concatenating left to right the diagrams associated with $A_1, \ldots, A_p$, respectively. Then $A(\{i_1, \ldots, i_k\}\{j_1, \ldots, j_k\})$ is equal to the weight of the set of vertex-disjoint families in the network connecting the sources $\{i_1, \ldots, i_k\}$ to the sinks $\{j_1, \ldots, j_k\}$. In particular, $a_{ij} = A(\{i\}\{j\})$ is equal to the sum of all the weights of paths that join source $i$ to sink $j$.

The next example demonstrates Prop. 7.

**Example 3.** Let

$$A = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0.1 \\ 0.1 & 1 & 5 & 3 \\ 0 & 0 & 2 & 7 \end{bmatrix}. \quad (11)$$

It is straightforward to verify that $A$ is I-TN. The associated network of $A$ is depicted in Fig. 3 (all numerical values in this paper are to four-digit accuracy).

There is only one directed path joining the source 2 with the sink 4, and the weight of this path is $3.6667 \cdot 0.2727 \cdot 0.1 \approx 0.1$. Indeed, $a_{2,4} = 0.1$. 
Fig. 3: The planar network associated with the matrix $A$ in (11).

There are two paths that join source 2 with sink 2: the horizontal path with weight $0.3333 \cdot 3 \cdot 0.3333 \approx 0.3333$. The sum of these two path weights is 4, and this is equal to $a_{22}$.

As another example, consider $A(\{1,3\}|\{2,3\})$. In this case, there are three families of vertex-disjoint paths joining the sources $\{1,3\}$ with the sinks $\{2,3\}$:

1) The path joining source 1 with sink 2 with weight $3 \cdot 0.3333 \approx 1$, and the path joining source 3 with sink 3 with weight $0.1636 \cdot 0.3333 \approx 0.1$. This family weight is then $1 \cdot 0.1 = 0.1$.

2) The path joining source 1 with sink 2 with weight $3 \cdot 0.3333 \approx 1$, and the path joining source 3 with sink 3 with weight $0.1 \cdot 3.6667 \cdot 0.2727 \approx 0.1636$. This family weight is then $1 \cdot 0.1636 = 0.1636$.

3) The path joining source 1 with sink 2 with weight $3 \cdot 0.3333 \approx 1$, and the path joining source 3 with sink 3 with weight $0.1 \cdot 3.6667 \cdot 0.2727 \approx 0.1636$. This family weight is then $1 \cdot 0.1636 = 0.1636$.

The sum over these three families is $4.7364 + 0.1 + 0.1636 = 5$, and $A(\{1,3\}|\{2,3\}) = \det\left(\begin{bmatrix} 1 & 0 \\ 1 & 5 \end{bmatrix}\right) = 5$. □

Prop. 7 links the minors of I-TN matrices to the topology of the associated planar network and this has many theoretical and practical implications. For example, the following known proposition follows immediately from Prop. 7 as horizontal edges with positive weights always exist in any planar network of an I-TN matrix (see Remark 1).

Proposition 8. [6] Let $A \in \mathbb{R}^{n \times n}$ be an I-TN matrix. Then every principal minor of $A$ is positive.

Prop. 7 is one of the main tools used to derive the results below.

III. MAIN RESULTS

We start by introducing a notation that simplifies the representation of the factorization in (4). For $i \in \{2, \ldots, n\}$, define: $x_i := 1 + (i - 2)n - \sum_{j=1}^{i-2} j = (i - 2)n - \frac{(i-3)i}{2}$.

\[ \ell^i := \left[ \ell_{x_i} \ldots \ell_{x_i+n-i} \right]^T, \text{ and } u^i := \left[ u_{x_i} \ldots u_{x_i+n-i} \right]^T. \] (12)
Thus,
\[
\ell^2 = [\ell_1 \ldots \ell_{n-1}]^T \in \mathbb{R}^{n-1}_+,
\]
\[
\ell^3 = [\ell_n \ldots \ell_{2n-3}]^T \in \mathbb{R}^{n-2}_+,
\]
\[
\vdots
\]
\[
\ell^n = \ell_k \in \mathbb{R}_+,
\]
and similarly for \(u^j\). Let \(\ell^j_{ij} [u^j]_{ij}\) denote the \(j\)'th entry in \(\ell^j [u^j]\), and let
\[
W_i(\ell^i) := \prod_{j=n}^i L_j(\ell^i_{n-j+1}),
\]
\[
Q_i(u^i) := \prod_{j=i}^n U_j(u^i_{j-i+1}).
\]
Then the SEB factorization (4) can be written more succinctly as
\[
A = W_2(\ell^2) \cdots W_n(\ell^n) D Q_n(u^n) \cdots Q_2(u^2).
\]

A. Explicit Expressions for \(r(A)\)

We now characterize explicitly the exponent of certain classes of oscillatory matrices. For \(y \in \mathbb{R}\), let \([y]\) denote the smallest integer that is larger than or equal to \(y\).

**Proposition 9.** Assume that an I-TN matrix \(A \in \mathbb{R}^{n\times n}\) is factored as
\[
A = W_2(\ell^2) \cdots W_s(\ell^s) D Q_s(u^s) \cdots Q_2(u^2),
\]
for some \(s_\ell \in \{2, \ldots, n\}\) and \(s_u \in \{2, \ldots, n\}\), where all the multipliers are positive. Let \(s := \min\{s_\ell, s_u\}\). Then
\[
r(A) = \left\lfloor \frac{n-1}{s-1} \right\rfloor.
\]

**Example 4.** Assume that \(s_\ell = s_u = n\). In this case, \(\ell_i > 0, u_i > 0, i = 1, \ldots, k,\) and by Prop. 4, \(A\) is TP. Indeed (16) gives \(r = \left\lfloor \frac{n}{s-1} \right\rfloor = 1\). Now assume that \(s_\ell = s_u = 2\). Then \(A\) is as in (8) and (16) gives \(r(A) = \left\lfloor \frac{n-1}{2-1} \right\rfloor = n-1\). This recovers one of the results in [17].

As a concrete example, consider
\[
A = \begin{bmatrix}
1 & 1 & 2 & 2 \\
2 & 3 & 7 & 9 \\
6 & 9 & 22 & 30 \\
6 & 9 & 22 & 31
\end{bmatrix}
\]
which is I-TN, but not TP (it has zeros minors, for example, \(A(\{2, 3\}\{1, 2\}) = 0\)). The SEB factorization of \(A\) is
\[
A = L_4(1)L_3(3)L_2(2)IU_3(1)U_4(2)U_2(1)U_3(2)U_4(1)
\]
\[
= W_2(\ell^2)Q_n(u^n)Q_2(u^2),
\]
with \(\ell^2 = [1 3 2]^T\), \(u^3 = [1 2]^T\), and \(u^2 = [1 2 1]^T\), thus \(A\) is oscillatory. This is as in (15) with
\( s_1 = 2 \) and \( s_u = 3 \) (and \( D = 1 \)), and by Prop. 9 \( r(A) = 3/(\min\{2, 3\} - 1) = 3 \). Now,
\[
A^2 = \begin{bmatrix}
27 & 40 & 97 & 133 \\
104 & 155 & 377 & 520 \\
336 & 501 & 1219 & 1683 \\
342 & 510 & 1241 & 1714
\end{bmatrix},
\]
and so \((A^2)(\{2, 3, 4\}, \{1, 2, 3\}) = 0\), thus \( A^2 \) is not TP, implying that indeed \( r(A) = 3 \).

\[\square\]

**Proof of Prop. 9** For \( v \in \{1, \ldots, \left\lceil \frac{n-1}{s} \right\rceil \} \), consider a corner minor \( A^v(\alpha|\beta) \). Recall that \( A^v(\alpha|\beta) > 0 \) if and only if there is a family of vertex-disjoint paths connecting the sources in \( \alpha \) to the sinks in \( \beta \) in the planar network associated with \( A^v \).

We start by analyzing all lower-left corner minors of \( A^v \). Suppose that the I-TN matrix \( A \in \mathbb{R}^{n \times n} \) is factored as in (15) for some \( s_1, s_u \in \{2, n\} \), where all the multipliers are positive. The left-hand side of the associated network represents the product \( \prod_{i=1}^n W_{\ell_i} \). This includes horizontal lines, and \( s_1 - 1 \) diagonal lines with an ‘upper-left to lower-right orientation’: the first goes from 1 to 3, the second from 3 to 5, and so on. Fig. 4 depicts the left-hand side of this network for \( n = 6 \) and \( s_1 = 2, 3, \ldots, 6 \).

Consider all the lower-left corner minor indexes (see Table 1). It is clear that only the diagonal lines with an “upper-left to lower-right orientation” in the left-hand side of the associated planar network (and all the horizontal lines) contribute to the family of vertex-disjoint paths. Thus, we will only use the left-hand side of the associated network to analyze all lower-left corner minors.

The minimal value \( v \) for which every lower-left corner minor of \( A^v \) is positive is determined by \( A^v(\alpha|\beta) \) with \( \alpha := \{2, 3, \ldots, n\} \) and \( \beta := \{1, 2, \ldots, n-1\} \). This is due to the specific structure of the left-hand side of the planar network associated with \( A \), and the number of vertex-disjoint paths corresponding to this minor as compared to all others.

Let \( z_\ell := \left\lceil \frac{n-1}{s_1-1} \right\rceil \), \( \alpha := \{2, 3, \ldots, n\} \), and \( \beta := \{1, 2, \ldots, n-1\} \). Pick \( v_\ell \in \{1, \ldots, z_\ell\} \). Since each copy of \( A \) contains \( s_1 - 1 \) diagonal lines corresponding to \( \prod_{i=1}^n W_{\ell_i} \), the \( n - 1 \) paths in \( A^{v_\ell}(\alpha|\beta) \) require at least \( z_\ell \) copies of \( A \) in order to be vertex-disjoint paths. Thus,

\[
r(A) \geq z_\ell.
\]

(17)

We now show that \( A^{z_\ell}(\alpha|\beta) > 0 \). We use the following notation. An arc denoted \( \searrow \) represents a diagonal arc from one of \( \prod_{i=1}^n W_{\ell_i} \) in some copy of \( A \), and is accompanied by an explanation of which \( L_j \) is used. The paths in the family are as follows. The first path is
\[
p_2: 2 \rightarrow 2 \rightarrow \cdots \rightarrow 2 \searrow 1 \rightarrow 1 \rightarrow \cdots \rightarrow 1,
\]
where \( \searrow \) is from \( L_2 \) in the first copy of \( W_2 \). The second path is
\[
p_3: 3 \rightarrow 3 \rightarrow \cdots \rightarrow 3 \searrow 2 \rightarrow 2 \rightarrow \cdots \rightarrow 2,
\]
where \( \searrow \) is from \( L_3 \) in the first copy of \( W_3 \) if \( s_1 > 2 \) or from \( L_3 \) in second copy of \( W_2 \) if \( s_1 = 2 \). Note that this implies that \( p_2 \) and \( p_3 \) are vertex-disjoint. The last path is
\[
p_\ell: n \rightarrow n \rightarrow \cdots \rightarrow n \searrow n-1 \rightarrow n-1 \rightarrow \cdots \rightarrow n-1,
\]
where \( \searrow \) is from \( L_n \) in the first possible copy of some \( W_i \) such that \( p_\ell \) and \( p_{\ell-1} \) are vertex-disjoint. Note that the paths are vertex-disjoint by construction. We now calculate the number of copies of \( A \) that are needed. The paths \( p_2, \ldots, p_{s_1} \) use the first copy of \( \prod_{i=1}^n W_{\ell_i} \). The paths \( p_{s_1+1}, \ldots, p_{2s_1-1} \) use the second copy of \( \prod_{i=1}^n W_{\ell_i} \), and so on. This implies that at least \( z_\ell \) copies are enough to realize the family of \( n - 1 \) vertex-disjoint paths. Summarizing, \( A^{z_\ell}(\alpha|\beta) > 0 \), implying that all lower-left corner minors (except for the determinant) of \( A^{z_\ell} \) are positive, and that there is at least one lower-left corner minor of \( A^p, p \in \{1, \ldots, z_\ell - 1\} \) that is zero.
Fig. 4: The left-hand side of the planar networks corresponding to all possible values of $s_\ell$ in (15) for the case $n = 6$. Upper-left figure corresponds to $s_\ell = 2$, upper-middle to $s_\ell = 3$, upper-right to $s_\ell = 4$, lower-left to $s_\ell = 5$, and lower-right to $s_\ell = 6$.

The analysis of all upper-right corner minors of $A^v$ is similar. In this case, we use the right-hand side of the associated planar network, and the upper-right corner minor $A^v(\{1, \ldots, n-1\}|\{2, \ldots, n\})$ to show that $z_u := \lceil \frac{n-1}{s_u-1} \rceil$ copies of $A$ are needed to guarantee that all upper-right corner minors (except for the determinant) of $A^z$ are positive.

We conclude that $z := \max\{z_\ell, z_u\} = \left\lceil \frac{n-1}{\min\{s_\ell, s_u\} - 1} \right\rceil = \left\lceil \frac{n-1}{s-1} \right\rceil$ copies of $A$ are needed to guarantee that all corner minors (except for the determinant) of $A^z$ are positive. Also, $\det(A^z) > 0$, as $A$ is invertible. By Prop. 3, $A^z$ is TP and $A^1, \ldots, A^{z-1}$ are not TP, so $r(A) = z$. This completes the proof. □

The next result provides an explicit expression for the exponent of another class of oscillatory matrices.

**Proposition 10.** Assume that $A \in \mathbb{R}^{n \times n}$ is I-TN and that there exist $q_\ell, q_u \in \{2, \ldots, n\}$ such that

\[
A = L_2(\ell_1) \cdots L_{q_\ell-1}(\ell_{q_\ell-2}) W_{q_\ell}(\ell_{q_\ell}) \cdots W_n(\ell_n) \cdot DQ_n(u^n) \cdots Q_{q_u}(u^{q_u}) U_{q_u-1}(u_{q_u-2}) \cdots U_2(u_1),
\]

where all the multipliers are positive. Let $q := \max\{q_\ell, q_u\}$. Then

\[
r(A) = q - 1.
\]

**Example 5.** The case $q_\ell = q_u = 2$ (i.e. $q = 2$) corresponds to the case $s_\ell = s_u = n$ (i.e. $s = n$) in Prop. 9 yielding $r(A) = \left\lceil \frac{n-1}{n-1} \right\rceil = 1 = q - 1$. The case $q_\ell = q_u = n$ yields $A$ as in (7) (note that $W_n(\ell_n) = L_n(\ell_n)$ and $Q_n(u^n) = U_n(u_n)$), and in this case $r(A) = n - 1$ \[7\].

As a concrete example, consider

\[
A = \begin{bmatrix}
1 & 2 & 0 & 0 & 0 \\
2 & 5 & 3 & 0 & 0 \\
0 & 2 & 7 & 2 & 6 \\
0 & 8 & 29 & 11 & 34 \\
0 & 24 & 89 & 41 & 131
\end{bmatrix}.
\]
which is I-TN, but clearly not TP. The SEB factorization of $A$ is

$$A = L_2(2)L_5(3)L_4(4)L_3(2)L_5(2)L_4(1)L_5(2)IU_5(1)U_4(2)U_5(3)U_3(3)U_2(2)$$

$$= L_2(2)W_3(\ell^0)W_4(\ell^0)W_5(\ell^0)Q_5(u^5)Q_4(u^4)U_3(3)U_2(2),$$

with $\ell^0 = [3 \ 4 \ 2]^T$, $\ell^4 = [2 \ 1]^T$, $\ell^5 = 2$, $u^5 = 1$, and $u^4 = [2 \ 3]^T$, thus $A$ is oscillatory. This is as in (18) with $q_\ell = 3$ and $q_u = 4$ (and $D = 1$), and by Prop. 10 $r(A) = \max\{3, 4\} - 1 = 3$. Indeed it can be easily verified that $A^2$ is not TP (e.g. its $(1, 4)$ entry is zero), and that $A^3$ is TP. □

Proof of Prop. 10. Suppose that the I-TN matrix $A \in \mathbb{R}^{n \times n}$ is factored as in (18) for some $q_\ell, q_u \in \{2, \ldots, n\}$, where all the multipliers are positive. The left-hand side of the associated network represents the product

$$L_2(\ell_1) \cdots L_{q_\ell-1}(\ell_{q_\ell-2})W_{q_\ell}(\ell_{q_\ell}) \cdots W_n(\ell_n).$$

This includes horizontal lines, and $n - 1$ diagonal lines with an ‘upper-left to lower-right orientation’: the first goes from $2 \{n\}$ to $1$ if $q_\ell > 2 \{q_\ell = 2\}$, the second from $3 \{n\}$ to $2$ if $q_\ell > 3 \{q_\ell \leq 3\}$, and so on. Fig. 5 depicts the left-hand side of this network for $n = 6$ and $q_\ell = 2, 3, \ldots, 6$.

We first analyze the lower-left corner minors of $A^v$ based on the left-hand side of the associated network. Observe that in this case, the minimal $v$ such that any lower-left corner minor of $A^v$ is positive is determined by the requirement that $A^v(\alpha|\beta) > 0$ for $\alpha := \{n\}$ and $\beta := \{1\}$. To see this, consider for example the case $n = 6$ depicted in Fig. 5 and, without loss of generality, $q_\ell = 6$. We again use the notation $\searrow$ to represent a diagonal arc from some $L_j$ in some $W_i$. Clearly five copies of $A$ are required to obtain a path from $6$ to $1$: $6 \searrow 5 \searrow 4 \searrow 3 \searrow 2 \searrow 1$. However, only four copies of $A$ are required to obtain a family of two vertex-disjoint paths connecting the source indexes $\{5, 6\}$ to the sink indexes $\{1, 2\}$ (corresponding to the lower-left corner minor $A^v(\{5, 6\}|\{1, 2\})$. The first path is: $5 \searrow 4 \searrow 3 \searrow 2 \searrow 1$, and the second path is: $6 \searrow 5 \searrow 4 \searrow 3 \searrow 2$. Note that these are vertex-disjoint paths. Continuing this way, we see that indeed at least five copies of $A$ are required to guarantee that all lower-left corner minors (except for the determinant) are positive.

Let $z_\ell := q_\ell - 1$. In each copy of $A$, the longest diagonal with an ‘upper-left to lower-right orientation’ contains $n - q_\ell + 1$ consecutive edges, and is the $(q_\ell - 1)$’th left most diagonal. All diagonals left to it (if they exist) include a single edge. This implies that at least $n - (n - q_\ell + 1) = z_\ell$ copies of $A$ are needed to realize a path from the source $n$ to the sink $1$. Given this number of copies, the path is:

$$n \searrow q_\ell - 1 \searrow q_\ell - 2 \searrow \cdots \searrow 1.$$ 

Summarizing, $A^v(\{n\}|\{1\}) > 0$, implying that all lower-left corner minors (except for the determinant) of $A^v$ are positive, and for any $p < z_\ell$ there is at least one lower-left corner minor of $A^p$ that is zero.

The analysis of all upper-right corner minors of $A^v$ is similar. In this case, we use the right-hand side of the associated planar network, and the upper-right corner minor $A^v(\{1\}|\{n\})$ to show that at least $z_u := q_u - 1$ copies of $A$ are needed to guarantee that all upper-left corner minors (except for the determinant) of $A^{z_u}$ are positive.

We conclude that at least

$$z := \max\{z_\ell, z_u\} = \max\{q_\ell, q_u\} - 1 = q - 1$$

copies of $A$ are needed to guarantee that all corner minors (except for the determinant) of $A^z$ are positive. Also, $\det(A^z) > 0$, as $A$ is invertible. By Prop. 3 $r(A) = z$. □

B. Generalizations

The following remark implies that the results in Props. 9 and 10 may be applicable more generally.
Fig. 5: The left-hand side of the planar networks corresponding to all possible values of $q_\ell$ in (18) for the case $n = 6$. Upper-left figure corresponds to $q_\ell = 6$, upper-middle to $q_\ell = 5$, upper-right to $q_\ell = 4$, lower-left to $q_\ell = 3$, and lower-right to $q_\ell = 2$.

Remark 2. Many SEB factorizations can be rewritten as in (15) or as in (18). We demonstrate this using an example. Consider the case $n = 5$ and the product

$$P := [L_5(\ell_1)L_4(\ell_2)L_3(\ell_3)L_2(\ell_4)]L_5(\ell_5)L_4(\ell_6)L_3(\ell_7)]L_5(\ell_8)L_4(\ell_9)]L_5(\ell_{10}),$$

with $\ell_3 = \ell_4 = \ell_6 = \ell_8 = 0$, and all the other multipliers positive, that is,

$$P = L_5(\ell_1)L_4(\ell_2)L_5(\ell_5)L_3(\ell_7)L_4(\ell_9)L_5(\ell_{10}).$$

This cannot be written as a product of $W_i$’s as in (15) nor as in (18). However, using (3) yields

$$P = L_5(\ell_1)L_4(\ell_2)L_3(\ell_7)L_5(\ell_5)L_4(\ell_9)L_5(\ell_{10}),$$

and this can be written as

$$P = W_3(\ell^3)W_4(\ell^4)W_5(\ell^5),$$

with

$$\ell^3 := [\ell_1 \ \ell_2 \ \ell_7]^T, \ \ell^4 := [\ell_5 \ \ell_9]^T, \ \ell^5 := \ell_{10}.$$

This implies that many factorizations are equivalent to the factorizations defined in Props. 9 and 10, and thus their exponent can be determined explicitly.

Recall that the proofs of Props. 9 and 10 use the fact that certain edges in the associated planar networks have positive weights, while ignoring their actual value. Thus, another generalization is to arbitrary products of oscillatory matrices.

Corollary 1. Consider a set of matrices $\{A_i\}_{i=1}^m$ with every $A_i \in \mathbb{R}^{n \times n}$ I-TN. Suppose that there exist $s_\ell$ and $s_u$ with $s_\ell, s_u \in \{2, \ldots, n\}$ such that every $A_i$ can be factorized as in (15) with the same $s_\ell$ and $s_u$, but possibly every $A_i$ has a different set of multipliers. Let $s := \min\{s_\ell, s_u\}$. Then a product of $c$ matrices from the set is TP if and only if $c \geq \lceil \frac{n-1}{s-1} \rceil$.

Corollary 2. Consider a set of matrices $\{A_i\}_{i=1}^m$ with every $A_i \in \mathbb{R}^{n \times n}$ I-TN. Suppose that there exist $q_\ell$ and $q_u$, with $q_\ell, q_u \in \{2, \ldots, n\}$, such that every $A_i$ can be factorized as in (18) with the same $q_\ell$ and $q_u$, but possibly every $A_i$ has a different set of multipliers. Let $q := \max\{q_\ell, q_u\}$. Then any product of $c$ matrices from the set is TP if and only if $c \geq q - 1$. 

Example 6. Let
\[
A_1 = \begin{bmatrix} 2 & 8 & 16 & 48 \\ 8 & 33 & 67 & 203 \\ 8 & 39.5 & 89.5 & 289.5 \\ 20 & 134.5 & 350.5 & 1219.5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 2 & 8 & 24 \\ 2 & 6 & 28 & 88 \\ 6 & 23 & 117 & 376 \\ 30 & 145 & 789 & 2580 \end{bmatrix}.
\]

The SEB factorizations of these are
\[
A_1 = L_4(2.5)L_3(1)L_2(4)L_4(5.5)L_3(6.5)L_4(1)D_1U_3(1)U_4(2)U_2(4)U_3(2)U_4(3),
\]
\[
A_2 = L_4(5)L_3(3)L_2(2)L_4(6)L_3(2.5)L_4(2)D_2U_3(2)U_4(1)U_2(2)U_3(4)U_4(3),
\]
where
\[
D_1 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}.
\]

This implies that \(A_1\) and \(A_2\) are oscillatory matrices, but not TP (see Props. 4 and 5). Note that both \(A_1\) and \(A_2\) are factorized as in (15) with \(s_\ell = 4\) and \(s_u = 3\), and also that each has a different set of parameters. Thus, \(r(A_1) = r(A_2) = 2\). It is straightforward to verify that \(A_1^2\), \(A_2^2\), \(A_1A_2\), and \(A_2A_1\) are all TP matrices. \(\Box\)

We note that in case where each \(A_i\) can be factorized as in (15) with the same \(s\), but not the same \(s_\ell\) and \(s_u\), a product of less than \(\left\lceil \frac{n-1}{s-1} \right\rceil\) matrices from the set \(\{A_i\}_{i=1}^m\) may be sufficient to obtain a TP matrix. This is demonstrated by the following example.

Example 7. Consider the matrices
\[
A_1 = \begin{bmatrix} 1 & 1 & 2 & 6 \\ 3 & 4 & 9 & 28 \\ 3 & 4 & 10 & 32 \\ 6 & 8 & 20 & 65 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 3 & 3 & 6 \\ 2 & 7 & 7 & 14 \\ 6 & 23 & 24 & 48 \\ 6 & 25 & 27 & 55 \end{bmatrix},
\]
and their factorizations
\[
A_1 = L_4(2)L_3(1)L_2(3)U_3(1)U_4(1)U_2(1)U_3(2)U_4(3),
\]
\[
A_2 = L_4(1)L_3(3)L_2(2)L_4(1)U_3(2)U_3(1)U_4(2).
\]

Thus, \(A_1\) \([A_2]\) is factorized as in (15) with \(s_\ell = 2\) and \(s_u = 3\) \([s_\ell = 3\) and \(s_u = 2\)]. So both are factorized with the same \(s = \min\{s_\ell, s_u\} = 2\), but they differ in their \(s_\ell\) and \(s_u\) values. Prop. 9 implies that \(r(A_1) = r(A_2) = \left\lceil \frac{4-1}{2-1} \right\rceil = 3\). However, it can be verified that both \(A_1A_2\) and \(A_2A_1\) are TP, i.e. we do not necessarily need a product of three matrices from the set \(\{A_1, A_2\}\) to get a TP matrix. \(\Box\)

C. Upper-Bounds for \(r(A)\)

For many classes of oscillatory matrices that cannot be written as in (15), nor as in (18) (even after applying the idea described in Remark 2), upper-bounds for their exponent value can be obtained. The next result follows immediately from the proofs of Props. 9 and 10.

Corollary 3. Let \(A \in \mathbb{R}^{n \times n}\) be factorized as in (15) \([18]\), plus some additional multipliers. Then \(r(A) \leq \left\lceil \frac{n-1}{s-1} \right\rceil\) \([r(A) \leq q-1\].

For example, if \(A \in \mathbb{R}^{5 \times 5}\) can be factored as
\[
A = L_3(\ell_3)L_2(\ell_4)L_3(\ell_7)W_4(\ell^4)W_5(\ell^5)DQ_5(u^5)Q_4(u^4)U_3(u_7)U_2(u_4),
\]
then $A$ has the form as in (18) with $n = 5$, $q_\ell = q_u = 4$, plus the additional term $L_3(\ell_3)$, and thus $r(A) \leq \max\{4, 4\} - 1 = 3$.

The mapping from the SEB factorization of $A$ to $r(A)$ is nontrivial. There are known cases where adding specific additional terms necessarily reduces the exponent [17]. The following example demonstrates that in general adding terms with positive multipliers to the factorization (18) does not necessarily reduce $r(A)$ (similar examples exist for the case of the factorization (15)), and thus in these cases the bounds in Corollary 3 may be tight.

**Example 8.** Let

$$B = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & 10 \\ 0 & 2 & 13 \end{bmatrix}.$$  

This matrix can be written as $B = AU_3(4)$, where $A := L_2(1)L_3(2)IU_3(2)U_2(1)$. Note that $A$ can be written as in (18) with $q_\ell = q_u = 3$. By Corollary 3 $r(B) \leq r(A) \leq q - 1 = 2$, and indeed in this case this bound is tight. □

### IV. Conclusion

Oscillatory matrices are a class of matrices that are intermediary between TN and TP matrices. The exponent $r(A)$ of an oscillatory matrix $A \in \mathbb{R}^{n \times n}$ is the least positive integer such that $A^r$ is TP. It is known that $r(A) \leq n - 1$. Ref. [17] used the SEB factorization and its associated planar network to describe classes of oscillatory matrices satisfying $r(A) = n - 1$.

Here, we provide explicit expressions for $r(A)$ for several classes of oscillatory matrices. In addition, we provide upper-bounds on $r(A)$ for several other classes of oscillatory matrices. Our analysis is based on the planar network associated with the SEB factorization of oscillatory matrices. This allows to verify positivity of all matrix minors via the existence of families of vertex-disjoint paths in the planar network. As such, results obtained for a particular oscillatory matrix are immediately applicable to different combinations of oscillatory matrices, provided that all are from the same class. We believe that the SEB factorization and the associated planar network are powerful tools for analyzing I-TN matrices in general, and will find more applications in the analysis of oscillatory matrices.

Possible topics for further research include the following. First, a natural extension to this work is identifying additional classes of oscillatory matrices whose exponent can be determined explicitly. Also, Corollaries 1 and 2 considered the arbitrary products of oscillatory matrices that share the same $s_\ell$ and $s_u$ values, and the same $q_\ell$ and $q_u$ values, respectively. Can these results be extended to product of matrices that share only the values of $s$ and $q$, while the values of $s_\ell$ and $s_u$, and $q_\ell$ and $q_u$ differ, respectively? As suggested by Example 7, these cases seem to be different than the results in Corollaries 1 and 2.

Finally, let $\lambda_1 > \lambda_2 > \cdots > \lambda_n$ denote the eigenvalues of $A \in \mathbb{R}^{n \times n}$. This means that for large $k$ we have $\text{trace}(A^k) = \lambda_1^k + \cdots + \lambda_n^k \approx \lambda_1^k$, where $\text{trace}(B)$ denotes the sum of the diagonal entries of $B$. Assume that $A$ is I-TN, and thus it admits an SEB factorization. Note that $A^k$ also admits a SEB factorization (and its associated planar network in obtained by concatenating $k$ times the planar network associated with $A$). Using the SEB factorization of $A^k$ (and the associated planar network), what can we say about $\text{trace}(A^k)$?

### References

[1] F. R. Gantmacher and M. G. Krein, *Oscillation Matrices and Kernels and Small Vibrations of Mechanical Systems.* Providence, RI: American Mathematical Society, 2002, translation based on the 1941 Russian original.

[2] S. Karlin, *Total Positivity.* Stanford, CA: Stanford University Press, 1968, vol. 1.

[3] E. B. Curtis, D. Ingerman, and J. A. Morrow, “Circular planar graphs and resistor networks,” *Linear Algebra Appl.*, vol. 283, no. 1-3, pp. 115–150, 1998.

[4] J. Bartroff, L. Goldstein, Y. Rinott, and E. Samuel-Cahn, “On optimal allocation of a continuous resource using an iterative approach and total positivity,” *Advances in Applied Probability*, vol. 42, no. 3, pp. 795–815, 2010.

[5] S. M. Fallat and C. R. Johnson, *Totally Nonnegative Matrices.* Princeton, NJ: Princeton University Press, 2011.
[6] A. Pinkus, *Totally Positive Matrices*. Cambridge, UK: Cambridge University Press, 2010.

[7] S. Fomin and A. Zelevinsky, “Total positivity: tests and parametrizations,” *The Mathematical Intelligencer*, vol. 22, no. 1, pp. 23–33, 2000.

[8] M. Margaliot and E. D. Sontag, “Revisiting totally positive differential systems: A tutorial and new results,” *Automatica*, vol. 101, pp. 1–14, 2019.

[9] T. Ben-Avraham, G. Sharon, Y. Zarai, and M. Margaliot, “Dynamical systems with a cyclic sign variation diminishing property,” *IEEE Trans. Automat. Control*, 2019, to appear. [Online]. Available: [https://ieeexplore.ieee.org/document/8706539/](https://ieeexplore.ieee.org/document/8706539/)

[10] M. Margaliot and E. D. Sontag, “Analysis of nonlinear tridiagonal cooperative systems using totally positive linear differential systems,” in *Proc. 57th IEEE Conf. on Decision and Control*, 2018, pp. 3104–3109.

[11] R. Alseidi, M. Margaliot, and J. Garloff, “On the spectral properties of nonsingular matrices that are strictly sign-regular for some order with applications to totally positive discrete-time systems,” *J. Math. Anal. Appl.*, vol. 474, pp. 524–543, 2019.

[12] E. Weiss and M. Margaliot, “A generalization of linear positive systems with applications to nonlinear systems: Invariant sets and the Poincaré-Bendixon property,” 2019, submitted. [Online]. Available: [arXiv:1902.01630](http://arxiv.org/abs/1902.01630)

[13] R. Alseidi, M. Margaliot, and J. Garloff, “Discrete-time k-positive linear systems,” 2019, submitted. [Online]. Available: [http://arxiv.org/abs/1910.08125](http://arxiv.org/abs/1910.08125)

[14] H. S. Price, “Monotone and oscillatory matrices applied to finite difference approximations,” *Mathematics of Computation*, vol. 22, no. 103, pp. 489–516, 1968.

[15] M. Kardell, “Total positivity and oscillatory kernels: An overview, and applications to the spectral theory of the cubic string,” Master’s thesis, Linköping University, Sweden, 2010.

[16] R. Katz, M. Margaliot, and E. Fridman, “Entrainment to subharmonic solutions in oscillatory discrete-time systems,” *Automatica*, 2019, to appear. [Online]. Available: [https://arxiv.org/abs/1904.06547](https://arxiv.org/abs/1904.06547)

[17] S. Fallat and X. P. Liu, “A class of oscillatory matrices with exponent \( n - 1 \),” *Linear Algebra Appl.*, vol. 424, no. 2-3, pp. 466–479, 2007.

[18] S. M. Fallat, “A remark on oscillatory matrices,” *Linear Algebra Appl.*, vol. 393, pp. 139–147, 2004.

[19] A. M. Whitney, “A reduction theorem for totally positive matrices,” *J. d’Analyse Mathématique*, vol. 2, no. 1, pp. 88–92, 1952.

[20] C. W. Cryer, “Some properties of totally positive matrices,” *Linear Algebra Appl.*, vol. 15, pp. 1–25, 1976.

[21] M. Gasca and J. M. Pena, “Total positivity and Neville elimination,” *Linear Algebra Appl.*, vol. 165, pp. 25–44, 1992.

[22] B. Lindström, “On the vector representations of induced matroids,” *Bull. London Math. Soc.*, vol. 5, no. 1, pp. 85–90, 1973.