THE GREEN RINGS OF THE 2-RANK TAFT ALGEBRA
AND ITS TWO RELATIVES TWISTED

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Abstract. In the paper, the representation rings (or the Green rings) for a family of Hopf algebras of tame type, the 2-rank Taft algebra (at $q = -1$) and its two relatives twisted by 2-cocycles are explicitly described via a representation theoretic analysis. It turns out that the Green rings can serve to detect effectively the twist-equivalent Hopf algebras here.

1. Introduction

1.1. It is well-known that Drinfeld twist is a key method to yielding new Hopf algebras in quantum groups theory (see [13, 19], etc.). Dually, 2-cocycle twist or Doi-Majid twist including Drinfeld double as a kind of such twist (see [14, 28]) is extensively employed in various current researches. For instance, Andruskiewitsch et al ([11]) considered the twists of Nichols algebras associated to racks and cocycles. Guillot-Kassel-Masuoka ([18]) got some examples by twisting comodule algebras by 2-cocycles. In generic case, Pei-Hu-Rosso ([30]), Hu-Pei ([22]) found explicit 2-cocycle deformation formulae between multi-(resp. two-)parameter quantum groups and one-parameter quantum groups $U_{q,q^{-1}}(g)$ and an equivalence between the weight module categories $O$ as braided tensor ones. Likewise, in root of unity case, when a 2-cocycle twist exists under some conditions on the parameters, Benkart et al ([4]) used a result of Majid-Oeckl ([29]) to give a category equivalence between Yetter-Drinfeld modules for a finite-dimensional pointed Hopf algebra $H$ and those for its cocycle twist $H^\sigma$, and further to derive an equivalence of the categories of modules for $u_{q,q^{-1}}(sl_n)$ and $u_{q,q^{-1}}(sl_n)$ as Drinfeld doubles. In contrast, for particular choices of the parameters, there is no such cocycle twist, and in that situation the representation theories of $u_{q,q^{-1}}(sl_n)$ and $u_{q,q^{-1}}(sl_n)$ can be quite different (see Example 5.6 of [4]). Recently, Bazlov-Berenstein considered cocycle twists and extensions of braided doubles in a broader setting including.

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twisting the rational Cherednik algebra of the symmetric group into the Spin Cherednik algebra (see [3]).

A natural question is to ask how to detect two twist-equivalent Hopf algebras in nature? The article seeks to address this question through investigating the representation rings for a family of Hopf algebras, the 2-rank Taft Hopf algebra (at \( q = -1 \)) of which we introduced in [24] before, and its two relatives twisted by 2-cocycles.

1.2. Given a Hopf algebra \( H \), in the investigation of its monoidal module category, the decomposition problem of tensor products of indecomposables is of most importance and has received enormous attentions. One main approach is to explore the ring structure of the corresponding representation ring or say the Green ring of \( H \). Originally, the concept of the Green ring \( r(H) \) stems from the modular representations of finite groups (see [20] etc.). Since then, there are plenty of works on the Green rings. For finite groups, one can refer to the papers of Green [21], Benson [5, 6], etc. We also mention that Witherspoon computed the Green ring of the Drinfeld double of a finite group in [37]. Chen-Oystaeyen-Zhang studied the Green rings of Taft algebras in [9]. On the other hand, Cibils defined a quiver quantum group \( K\mathbb{Z}_n(q)/I_d \) and considered the decomposition of tensor products in [10]. This quiver quantum group is isomorphic to the generalized Taft algebra \( H_{n,d} \) defined in [25]. Recently, in order to investigate the Green ring of \( H_{n,d} \), Li-Zhang ([27]) reformulated the decomposition formulas given by Cibils in [10]. They determined all nilpotent elements in \( r(H_{n,d}) \). Wakui ([36]) also described the representation ring structures for all eight dimensional nonsemisimple Hopf algebras of finite type except for the only one of tame type. Erdmann et al ([16]) determined a large part of the structure of the Green ring of \( D(\Lambda_{n,d}) \) modulo projectives where \( D(\Lambda_{n,d}) \) stands for the Drinfeld doubles of a family of duals of generalized Taft Hopf algebras \( \Lambda_{n,d} \) using the different method.

1.3. In [11], Cibils showed that one half of small quantum groups are all wild when the rank \( n \geq 2 \) and \( \text{ord}(q) \geq 5 \). By contrast, the (generalized) Taft algebras are of finite type. On the other hand, Feldvoss-Witherspoon ([17]) proved a conjecture due to Cibils using support variety stating that all small quantum groups of rank at least two are wild. For the rank one case, as we know, \( u_q(\mathfrak{sl}_2) \) is tame, the study on indecomposables and their tensor product decomposition is already perfect (cf. [7, 26, 33, 38], etc).

In [23, 24], the second author defined the quantum divided power algebras and the quantized enveloping algebras of abelian Lie algebras. Denote by \( \mathcal{A}_q(n) \) the latter. When \( q \) is a root of unity, \( \mathcal{A}_q(n) \) admits a finite-dimensional Hopf quotient. We call it the \( n \)-rank Taft algebra (or small abelian quantum group), denoted \( \mathcal{A}_q^*(n) \). When \( n = 1 \), it comes back to the famous Taft algebra (see [34]). In what follows, we will concern the objects of tame type
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among the $n$-rank Taft (Hopf) algebras. According to the discussion of Ringel in [32], the only one being tame is the 2-rank Taft algebra $\bar{A}_q(2)$ at $q = -1$. Briefly set $\bar{A} := \bar{A}_{-1}(2)$.

1.4. The paper is organized as follows. In Section 2, we begin by giving the definition of the 2-rank Taft Hopf algebra $\bar{A}$ and a complete set of orthogonal primitive idempotents with the Gabriel quiver, and then describe its indecomposables. We decompose the various tensor products of indecomposables in Section 3. This leads to the description of the Green ring of $\bar{A}$, its Jacobson radical and the projective class algebra of the principle block in Section 4. Section 5 continues to give two 2-cocycle twisted Hopf algebras $\bar{A}_\sigma: D(H_4)$ and $H(\bar{H}_4) := H_4 \otimes H_4$, where $H_4$ is the Sweedler Hopf algebra of dimension 4, and explicitly determined their Green rings via a similar representation-theoretic analysis, together with the Jacobson radicals, the projective class algebras, etc. It is interesting to notice that even the Hopf algebras $H$, $\bar{A}$ and $D(H_4)$ are twist-equivalent to each other and are of dimension 16, they own the different number of blocks with 1, 2 and 3, respectively, whose diverse information on the Green rings are listed in the end of the paper.

As another evidence, we mention the work of Caenepeel-Dascalescu-Raianu et al. (see [12]) classifying all pointed Hopf algebras of dimension 16. After finishing the paper, we happen to find (and by comparison with those) that the Hopf algebras in question are the exact 3 iso-classes of pointed Hopf algebras with the Klein group algebra as the coradicals among the five iso-classes in their classification list, however, the rest do not exist any twist-equivalence with others. In general, it is not clear how the 2-cocycle twist of the multiplication affects the representation ring. Hopefully, it will stimulate a further research.

Throughout, we work over an algebraically closed field $K$ of characteristic 0. Unless otherwise stated, all (Hopf) algebras and modules defined over $K$ are finite-dimensional. Given an algebra $A$, let $A$-$mod$ denote the category of finite-dimensional left $A$-modules.

2. The 2-rank Taft algebra $\bar{A}$ and its indecomposable modules

2.1. From [32, (1.1)(a),(c),(9)], we know that the algebras $K[x, y, z]/(x^2, y^2, z^2, xy, yz, xz)$, $K[x, y]/(x^a, y^b)$, $a, b \geq 2$ are of infinite representation type. Explicitly, it is tame if $a = b = 2$ and wild if $a, b \geq 2$ but not both equal to 2. It means that if we want to figure out the representation ring of $\bar{A}_q(n)$, only the case for $q = -1$ and $n = 2$ is reachable when $n \geq 2$.

Now we describe the 2-rank Taft Hopf algebra $\bar{A}$ in detail. $\bar{A}$ has generators $g$, $h$, $x$, $y$, subjecting to the following relations,

\begin{align}
gh &= hg, \quad g^2 = h^2 = 1, \\
gx &= -xg, \quad gy = -yg, \quad hx = -xh, \quad hy = -yh, \\
x^2 &= y^2 = 0, \quad xy = -yx.
\end{align}
The comultiplication $\Delta$ is defined by

$$\Delta(g) = g \otimes g, \quad \Delta(h) = h \otimes h, \quad \Delta(x) = x \otimes 1 + g \otimes x, \quad \Delta(y) = y \otimes 1 + h \otimes y.$$ 

$\mathcal{A}$ has four orthogonal primitive idempotents

$$e_1 = \frac{1}{4}(1 + g + h + gh), \quad e_2 = \frac{1}{4}(1 + g - h - gh), \quad e_3 = \frac{1}{4}(1 - g + h - gh), \quad e_4 = \frac{1}{4}(1 - g - h + gh),$$

and two central primitive idempotents

$$f_1 = e_1 + e_4 = \frac{1}{4}(1 + gh), \quad f_2 = e_2 + e_3 = \frac{1}{4}(1 - gh).$$

**Lemma 2.1.** (1) There exist two signs $s_1, s_2 \in \{+,-\}$ such that $ge_i = s_1 e_i$, $he_i = s_2 e_i$ with $(s_1, s_2) = (+,+), (+,-), (-,+), (-,-)$, for $i = 1, 2, 3, 4$, successively.

(2) $xe_1 = e_4x$, $xe_4 = e_1x$, $ye_1 = e_4y$, $ye_4 = e_1y$, $xe_2 = e_3x$, $xe_3 = e_2x$.

Proof. It is straightforward to check. \qed

2.2. Let $S(s_1, s_2)$ denote the one dimensional simple module $\mathbb{K}$ of $\mathcal{A}$, defined by $g \cdot 1 = s_1 1, h \cdot 1 = s_2 1, x \cdot 1 = y \cdot 1 = 0$. Let $P(s_1, s_2)$ be the projective cover of $S(s_1, s_2)$, which coincides with the principle indecomposable module $\mathcal{A} e_i$ for some $i \in \{1, 2, 3, 4\}$. As $P(s_1, s_2)$ are non-isomorphic to each other with respect to the signs $s_1, s_2$, $\mathcal{A}$ is a basic algebra over $\mathbb{K}$. On the other hand, the radical $J(\mathcal{A}) = (x, y)$, thus from the above lemma, we know that the Gabriel quiver $Q_{\mathcal{A}}$ of $\mathcal{A}$ looks like:

![Diagram of Gabriel quiver](image)

where for $i = 1, 2, 3, 4$, the arrows $\alpha_i, \beta_i$ correspond to $e_i, x, e_j, y$, respectively. The admissible ideal $I$ has the following relations:

$$\alpha_1 \alpha_4 = \alpha_4 \alpha_1 = 0, \quad \alpha_2 \alpha_3 = \alpha_3 \alpha_2 = 0, \quad \beta_1 \beta_4 = \beta_4 \beta_1 = 0, \quad \beta_2 \beta_3 = \beta_3 \beta_2 = 0,$$

$$\alpha_1 \beta_4 + \beta_1 \alpha_4 = \alpha_4 \beta_1 + \beta_4 \alpha_1 = \alpha_2 \beta_3 + \beta_2 \alpha_3 = \alpha_3 \beta_2 + \beta_3 \alpha_2 = 0.$$ 

Hence, $\mathcal{A}$ decomposes into two block $\mathcal{A} f_1, \mathcal{A} f_2$. Their representation categories can transfer to each other via the involution functor.

From the quiver $Q_{\mathcal{A}}$, we know that $\mathcal{A}$ is a special biserial algebra. All indecomposable modules of such kind of algebras can be completely described. For the whole theory of special biserial algebras, we refer to [15, II]. Now we only focus on our target $\mathcal{A}$. 

Note that the simple modules $S(s_1, s_2), s_1, s_2 \in \{+, -\}$ exhaust all simple modules of $\mathcal{A}$, thus the projective modules $P(s_1, s_2), s_1, s_2 \in \{+, -\}$ are all indecomposable projective modules of $\mathcal{A}$. Moreover, $P(s_1, s_2) \cong P(+, +) \otimes S(s_1, s_2)$. Here we highlight $S(-, -), P(+, +), P(-, -)$ and write them as $S, P, P^-$, respectively for short. The module structure of $P$ can be presented by the following diagram:

$$
\begin{array}{c}
\xymatrix{
  x & e & y \\
  xe & & ye \\
  y & x & ye
}\end{array}
$$

where the arrow $\rightarrow$ (resp. $\leftarrow$) represents the action of $x$ (resp. $y$) on the module. We abbreviate them to $\xymatrix{\rightarrow}$ and $\xymatrix{\leftarrow}$. Given $M \in \mathcal{A}$-mod, for any $a \in \mathbb{K}, u, v \in M$, we use $u\xymatrix{\rightarrow} v$ (resp. $u\xymatrix{\leftarrow} v$) to represent $x \cdot u = av$ (resp. $y \cdot u = av$). Moreover, we omit the decoration of the arrow if the weight $a = 1$.

Note that any finite dimensional indecomposable module of $\mathcal{A}$ has the Loewy length at most 3. $P(s_1, s_2), s_1, s_2 = \pm$ exhaust all indecomposable modules whose Loewy length are 3. From the theory of special biserial algebras, those with the Loewy length 2 can be divided into string modules and band modules as follows.

There exist five groups of indecomposable modules: $\{M(r)\}_{r \in \mathbb{Z}^*}, \{W(r)\}_{r \in \mathbb{Z}^*}, \{N(r)\}_{r \in \mathbb{Z}^*}, \{N'(r)\}_{r \in \mathbb{Z}^*}, \{C(r, \eta)\}_{r \in \mathbb{Z}^*, \eta \in \mathbb{K}^\times}$, where the first four groups are all string modules, and the last one are band modules. $M(r) = (\oplus_{i=1}^r \mathbb{K} u_i) \oplus (\oplus_{i=1}^{r+1} \mathbb{K} v_i)$ is defined by the following diagram:

$$
\begin{array}{c}
\xymatrix{
  u_1 & u_2 & \cdots & u_r \\
  v_1 & v_2 & \cdots & v_r \\
  v_{r+1}
}\end{array}
$$

which is equivalent to the string module $M((\alpha_1 \beta_1^{-1})^r)$.

$W(r) = (\oplus_{i=1}^r \mathbb{K} u_i) \oplus (\oplus_{i=1}^{r+1} \mathbb{K} v_i)$ is defined by the following diagram:

$$
\begin{array}{c}
\xymatrix{
  u_1 & u_2 & \cdots & u_r \\
  v_1 & v_2 & \cdots & v_r \\
  v_{r+1}
}\end{array}
$$

which is equivalent to the string module $M((\alpha_1^{-1} \beta_1^r)^r)$.

$N(r) = (\oplus_{i=1}^r \mathbb{K} u_i) \oplus (\oplus_{i=1}^{r+1} \mathbb{K} v_i)$ is defined by the following diagram:

$$
\begin{array}{c}
\xymatrix{
  u_1 & u_2 & \cdots & u_r \\
  v_1 & v_2 & \cdots & v_r \\
  v_{r+1}
}\end{array}
$$

which is equivalent to the string module $M((\alpha_1^{-1} \beta_1^r)^{-1} \alpha_1^{-1})$. 

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$N'(r) = (\oplus_{i=1}^{r} \mathbb{K} u_i) \oplus (\oplus_{i=1}^{r} \mathbb{K} v_i)$ is defined by the following diagram:

\[ \begin{array}{ccccccc}
  & & & & u_r & & \\
  & & & & \downarrow & & \\
  & & & & v_r & & \\
 \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  & & & & u_2 & & \\
  & & & & \downarrow & & \\
  & & & & v_2 & & \\
 \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  & & & & u_1 & & \\
  & & & & \downarrow & & \\
  & & & & v_1 & & \\
  & & & & \cdot \cdot \cdot & & \\
 \end{array} \]

which is equivalent to the string module $M((\beta_1^{-1} \alpha_1)^{-1} \beta_1^{-1})$.

$C(r, \eta) = (\oplus_{i=1}^{r} \mathbb{K} u_i) \oplus (\oplus_{i=1}^{r} \mathbb{K} v_i)$ is defined by the following diagram:

\[ \begin{array}{ccccccc}
  & & & & u_r & & \\
  & & & & \downarrow & & \\
  & & & & v_r & & \\
 \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  & & & & u_2 & & \\
  & & & & \downarrow & & \\
  & & & & v_2 & & \\
 \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  & & & & u_1 & & \\
  & & & & \downarrow & & \\
  & & & & v_1 & & \\
  & & & & \cdot \cdot \cdot & & \\
 \end{array} \]

which is equivalent to the band module $M(\beta_1 \alpha_1^{-1}, r, \eta)$. Here $y \cdot u_i = \eta v_i + v_{i-1}$, $i = 1, \ldots, r$ and we set $v_0 = 0$, for convenience.

Meanwhile, for any $M \in \text{ind}(\mathcal{A})$ in the five groups above, we fix the trivial actions of $g, h$ on $u_1$ and extend it to the whole modules naturally. To change the diagonal actions of $g, h$ on $M$, one only needs to consider the tensor product of $M$ with some $S(s_1, s_2)$. In particular, we also use the notation $M(r)^{-}, W(r)^{-}, C(r, \eta)^{-}, N(r)^{-}, N'(r)^{-}$, when tensoring with $S(\cdot, \cdot)$. Referring to [15, II], we get

**Theorem 2.2.** The 5 groups of modules $\{M(r)\}_{r \in \mathbb{Z}^+}, \{W(r)\}_{r \in \mathbb{Z}^+}, \{C(r, \eta)\}_{r \in \mathbb{Z}^+, \eta \in \mathbb{K}^*}, \{N(r)\}_{r \in \mathbb{Z}^+}, \{N'(r)\}_{r \in \mathbb{Z}^+}$, together with $\{S, P\}$ provide a complete list of isomorphism classes of finite dimensional indecomposable modules of $\mathcal{A}$ when tensoring with all the simple modules $S(s_1, s_2)$.

### 3. Monoidal category $\mathcal{A}\text{-mod}$

#### 3.1. In this section, we decompose all the tensor products of indecomposable modules listed in Theorem 2.2. In [36], the author considered the quasitriangularity and the Green rings of all 8-dimensional nonsemisimple Hopf algebras over $\mathbb{K}$, except the unique tame case, $A_{C_2}$. Note that $A_{C_2 \times C_2}, A_{C_2}$ in Table 1 of [36] nicely serve as the Hopf quotients $\mathcal{A}/(g \cdot \cdot \cdot h)$, both of which are self-dual and quasi-triangular.

#### 3.2. Though $\mathcal{A}$ is not quasi-triangular, $M \otimes N \cong N \otimes M$ still holds for any $M, N \in \mathcal{A}\text{-mod}$ here. Hence, we only provide a one-sided version of decompositions of tensor products of indecomposable modules.

**Theorem 3.1.** For any $r, s \in \mathbb{Z}^+, \eta, \gamma \in \mathbb{K}^*$, we have

1. $P \otimes P \cong P^{\oplus 2} \oplus P^{-\oplus 2}$.
2. $M(r) \otimes P \cong P^{\oplus r} \oplus P^{-\oplus r+1}$, $W(r) \otimes P \cong P^{\oplus r+1} \oplus P^{-\oplus r}$. 
(3) \( C(r, \eta) \otimes P \cong N(r) \otimes P \cong N'(r) \otimes P \cong P^{\oplus r} \oplus P^{-\oplus r} \).
(4) \( M(r) \otimes M(s) \cong P^{\oplus rs} \oplus M(r + s)^- \).
(5) \( W(r) \otimes W(s) \cong P^{\oplus rs} \oplus W(r + s) \).
(6) \( M(r) \otimes W(s) \cong \begin{cases} P^{\oplus (s+1)} \oplus W(s-r)^-, & r < s, \\ P^{\oplus (r+1)} \oplus S^-, & r = s, \\ P^{\oplus (r+1)s} \oplus M(r-s), & r > s. \end{cases} \)
(7) \( M(r) \otimes C(s, \eta) \cong P^{\oplus rs} \oplus C(s, \eta)^- \).
(8) \( W(r) \otimes C(s, \eta) \cong P^{\oplus rs} \oplus C(s, \eta) \).
(9) \( M(r) \otimes N(s) \cong P^{\oplus rs} \oplus N(s)^- \), \( M(r) \otimes N'(s) \cong P^{\oplus rs} \oplus N'(s)^- \).
(10) \( W(r) \otimes N(s) \cong P^{\oplus rs} \oplus N(s) \), \( W(r) \otimes N'(s) \cong P^{\oplus rs} \oplus N'(s) \).
(11) \( C(r, \eta) \otimes C(s, \gamma) \cong \begin{cases} P^{\oplus rs}, & \eta \neq \gamma, \\ P^{\oplus (rs-\min(r, s))} \oplus C(\min(r, s), \eta) \oplus C(\min(r, s), \eta)^-, & \eta = \gamma. \end{cases} \)
(12) \( N(r) \otimes N(s) \cong P^{\oplus (rs-\min(r, s))} \oplus N(\min(r, s)) \oplus N(\min(r, s))^- \).
(13) \( N'(r) \otimes N'(s) \cong P^{\oplus (rs-\min(r, s))} \oplus N'(\min(r, s)) \oplus N'(\min(r, s))^- \).
(14) \( N(r) \otimes N'(s) \cong C(r, \eta) \otimes N(s) \cong C(r, \eta) \otimes N'(s) \cong P^{\oplus rs} \).

**Proof.** In order to simplify the proof, we only provide the diagrams for all the module structures. The elements in the second position of a tensor product will be added a superscript ’, thus we can distinguish the notations.

\[
\begin{array}{c}
\text{xe} \otimes \text{ye}'' + \text{xe} \otimes \text{xe}' \\
\text{ye} \otimes \text{xe}' + \text{ye} \otimes \text{ye}' \\
\text{xe} \otimes \text{ye}'' - \text{xe} \otimes \text{ye}'
\end{array}
\]

Since the collection of vectors in all the vertices forms a basis of \( P \otimes P \), we get the desired decomposition \( P^{\oplus 2} \oplus P^{-\oplus 2} \). All the decompositions below can be read similarly.
(2) In the case of $M(r) \otimes P$, we have

\[
\begin{align*}
&v_i \otimes e + u_i \otimes xe \\
&v_{i-1} \otimes e + u_i \otimes ye \\
&-v_i \otimes ye + v_{i+1} \otimes xe - u_i \otimes yxe
\end{align*}
\]

for $i = 1, \ldots, r, j = 1, \ldots, r + 1$, while in the case of $W(r) \otimes P$,

\[
\begin{align*}
&v_i \otimes e + u_i \otimes xe \\
&v_{i-1} \otimes e + u_i \otimes ye \\
&-v_i \otimes ye + v_{i+1} \otimes xe - u_i \otimes yxe
\end{align*}
\]

for $i = 1, \ldots, r + 1, j = 1, \ldots, r$, where we take $v_0 = 0$ for simplicity.

(3) For the case $C(r, \eta) \otimes P$, we have

\[
\begin{align*}
&v_i \otimes e + u_i \otimes xe \\
&\quad \quad \quad \quad (\eta v_i + v_{i-1}) \otimes e + u_i \otimes ye \\
&\quad \quad \quad \quad -v_i \otimes ye + (\eta v_i + v_{i-1}) \otimes xe \\
&\quad \quad \quad \quad -u_i \otimes yxe \\
\end{align*}
\]

where $i = 1, \ldots, r$.

For $N(r) \otimes P$,

\[
\begin{align*}
&v_i \otimes e + u_i \otimes xe \\
&v_{i-1} \otimes e + u_i \otimes ye \\
&-v_i \otimes ye + v_{i+1} \otimes xe - u_i \otimes yxe
\end{align*}
\]

where $i = 1, \ldots, r$ and we take $v_0 = 0$ for simplicity. It is similar for $N'(r) \otimes P$.

(4) We first prove the base case $M(1) \otimes M(s)$ as follows.
for \( i = 1, \ldots, s \). Meanwhile,

Now we prove the general case \( M(r) \otimes M(s) \) by induction on \( r \). From (2) and the base case, we see that

\[
(M(1) \otimes M(r)) \otimes M(s) \cong (P^{\otimes r} \oplus M(r+1)^{-}) \otimes M(s) \cong P^{\otimes r} \oplus P^{-\otimes r(s+1)} \oplus (M(r+1)^{-} \otimes M(s)).
\]

On the other hand, using (2) and the induction hypothesis, we have

\[
M(1) \otimes (M(r) \otimes M(s)) \cong M(1) \otimes (P^{\otimes r} \oplus M(r+s)^{-}) \cong (P^{\otimes r} \oplus P^{-\otimes 2r}) \oplus (P^{-\otimes r+s} \oplus M(r+s+1)).
\]

By the Krull-Schmidt theorem, they combine to give \( M(r+1) \otimes M(s) \cong P^{\otimes (r+1)} \oplus M(r+s+1)^{-} \).

(5) The base case \( W(1) \otimes W(s) \) is as follows.
(6) Here we only deal with the case \( r \leq s \). When \( r > s \), the proof is similar. First, the base case \( M(1) \otimes W(1) \) is as follows.

\[
\begin{array}{c}
\text{\( u_1 \otimes u'_1 \)} \\
v_1 \otimes u'_1 + u_1 \otimes u'_1 \\
v_2 \otimes u'_2 \\
v_1 \otimes u'_2 \\
v_2 \otimes u'_2 + u_1 \otimes u'_1 \\
-1
\end{array}
\]

Meanwhile, \( v_1 \otimes u'_1 + u_1 \otimes v'_1 + v_2 \otimes u'_2 \) spans the simple module \( S^- \).

Next, we consider \( M(1) \otimes W(s) \), \( s > 1 \) as follows. By the base case and (2), we have

\[
(M(1) \otimes W(1)) \otimes W(s-1) \equiv (P^{\otimes 2} \oplus S^-) \otimes W(s-1) \equiv P^{\otimes 2s} \oplus P^{-\otimes 2(s-1)} \oplus W(s-1)^-.
\]

On the other hand, from the base case, (2) and (5), we get

\[
M(1) \otimes (W(1) \otimes W(s-1)) \equiv M(1) \otimes (P^{\otimes s-1} \oplus W(s)) \equiv (P^{\otimes s-1} \oplus P^{-\otimes 2(s-1)}) \oplus (M(1) \otimes W(s)).
\]

They combine to give \( M(1) \otimes W(s) \equiv P^{\otimes s+1} \oplus W(s-1)^- \).

Now we decompose \( M(r) \otimes W(s) \), \( r \leq s \) by induction on \( r \). First, we have

\[
(M(1) \otimes M(r-1)) \otimes W(s) \equiv (P^{\otimes r-1} \oplus M(r^-)) \otimes W(s) \equiv P^{\otimes (r-1)(s+1)} \oplus P^{-\otimes (r-1)s} \oplus (M(r^-) \otimes W(s)).
\]

On the other hand, by induction, we have

\[
M(1) \otimes (M(r-1) \otimes W(s)) \equiv M(1) \otimes (P^{\otimes (r-1)(s+1)} \oplus W(s-r+1)^-)
\]

\[
\equiv P^{\otimes (r-1)(s+1)} \oplus P^{-\otimes 2(r-1)(s+1)} \oplus (M(1) \otimes W(s-r+1)^-).
\]

They combine to give

\[
M(r) \otimes W(s) \equiv P^{\otimes (r-1)(s+2)} \oplus (M(1) \otimes W(s-r+1)) \equiv \begin{cases} P^{\otimes (s+1)} \oplus W(s-r)^-, & r < s, \\ P^{\otimes (r+1)} \oplus S^-, & r = s. \end{cases}
\]

(7) First, the base case \( M(1) \otimes C(s, \eta) \) is as follows.

\[
\begin{array}{c}
\text{\( u_1 \otimes u'_1 \)} \\
u_1 \otimes u'_1 + v_1 \otimes u'_1 \\
u_1 \otimes (v'_1 + v'_{i-1}) + v_2 \otimes u'_1 \\
u_1 \otimes v'_1 + v_1 \otimes u'_1 \\
u_1 \otimes (v'_1 + v'_{i-1}) + v_2 \otimes u'_1 \\
-1
\end{array}
\]

for \( i = 1, \ldots, s \). Meanwhile,
Now we decompose \( M(r) \otimes C(s, \eta) \) by induction on \( r \). First, from (3) and (4), we have
\[
(M(1) \otimes M(r)) \otimes C(s, \eta) \equiv (P^{br} \oplus M(r+1)^{-}) \otimes C(s, \eta) \equiv (P^{brs} \oplus P^{\ominus brs}) \oplus (M(r+1)^{-} \otimes C(s, \eta)).
\]
On the other hand, by induction and (2), we have
\[
M(1) \otimes (M(r) \otimes C(s, \eta)) \equiv M(1) \otimes (P^{brs} \oplus C(s, \eta)^{-}) \equiv (P^{brs} \oplus P^{\ominus 2brs}) \oplus (P^{\ominus} \oplus C(s, \eta)).
\]
They combine to give \( M(r+1) \otimes C(s, \eta) \equiv F^{(r+1)x} \oplus C(s, \eta)^{-} \).

(8) First, the base case \( W(1) \otimes C(s, \eta) \) is as follows.

\[
\begin{array}{c}
\text{for } i = 1, \ldots, s. \text{ Meanwhile,}
\end{array}
\]

\[
U_i = u_2 \otimes (\eta u'_i + u'_{i-1}) + u_1 \otimes u'_i, V_i = u_2 \otimes (\eta v'_i + v'_{i-1}) + v_1 \otimes u'_i + u_1 \otimes v'_i, i = 1, \ldots, s.
\]

Note that \( y \cdot U_i = \eta V_i + V_{i-1} \).

Now we decompose \( W(r) \otimes C(s, \eta) \) by induction on \( r \). First, from (3) and (5), we have
\[
(W(1) \otimes W(r)) \otimes C(s, \eta) \equiv (P^{br} \oplus W(r+1)) \otimes C(s, \eta) \equiv (P^{brs} \oplus P^{\ominus brs}) \oplus (W(r+1) \otimes C(s, \eta)).
\]
On the other hand, by induction and (2), we get
\[
W(1) \otimes (W(r) \otimes C(s, \eta)) \equiv W(1) \otimes (P^{brs} \oplus C(s, \eta)) \equiv (P^{brs} \oplus P^{\ominus 2brs}) \oplus (P^{\ominus} \oplus C(s, \eta)).
\]
They combine to give \( W(r+1) \otimes C(s, \eta) \equiv F^{(r+1)x} \oplus C(s, \eta) \).

(9) First, the base case \( M(1) \otimes N(s) \) is as follows.

\[
\begin{array}{c}
\text{for } i = 1, \ldots, s, \text{ where we take } v'_0 = 0 \text{ for simplicity. Meanwhile,}
\end{array}
\]
Now we decompose $M(r) \otimes N(s)$ by induction on $r$. First, from (3) and (4), we have

$$(M(1) \otimes M(r)) \otimes N(s) \equiv (P^\oplus r \otimes M(r+1)^-) \otimes N(s) \equiv (P^{\oplus r} \otimes P^{-\oplus r}) \oplus (M(r+1)^- \otimes N(s)).$$

On the other hand, by induction and (2), we get

$$M(1) \otimes (M(r) \otimes N(s)) \equiv M(1) \otimes (P^{\oplus r} \otimes N(s)^-) \equiv (P^{\oplus r} \otimes P^{-\oplus r}) \oplus (P^{-\oplus r} \otimes N(s)).$$

They combine to give $M(r+1) \otimes N(s) \equiv P^{\oplus (r+1)} \otimes N(s)^-$. The case of $M(r) \otimes N'(s)$ is quite similar.

(10) First the base case $W(1) \otimes N(s)$ is as follows.

\[
\begin{array}{c}
\begin{array}{c}
 u_1 \otimes u_i' \\
 u_1 \otimes u_i' + v_1 \otimes u_i' \\
 u_1 \otimes u_i' + u_2 \otimes u_i' \\
 \vdots \\
 u_1 \otimes u_i' + v_1 \otimes u_i' + u_2 \otimes u_i' \\
 \end{array}
\end{array}
\begin{array}{c}
 u_1 \otimes u_i' \\
 u_1 \otimes u_i' + v_1 \otimes u_i' \\
 u_1 \otimes u_i' + u_2 \otimes u_i' \\
 \vdots \\
 u_1 \otimes u_i' + v_1 \otimes u_i' + u_2 \otimes u_i' \\
 \end{array}
\begin{array}{c}
 untuk j = 1, \ldots, s.\end{array}
\]

Now we decompose $W(r) \otimes N(s)$ by induction on $r$. First, from (3) and (5), we have

$$(W(1) \otimes W(r)) \otimes N(s) \equiv (P^\oplus r \otimes W(r+1)) \otimes N(s) \equiv (P^{\oplus r} \otimes P^{-\oplus r}) \oplus (W(r+1) \otimes N(s)).$$

On the other hand, by induction and (2), we obtain

$$W(1) \otimes (W(r) \otimes N(s)) \equiv W(1) \otimes (P^{\oplus r} \otimes N(s)) \equiv (P^{\oplus r} \otimes P^{-\oplus r}) \oplus (P^{\oplus r} \otimes N(s)).$$

They combine to give $W(r+1) \otimes N(s) \equiv P^{\oplus (r+1)} \otimes N(s)$. The case of $W(r) \otimes N'(s)$ is quite similar.

(11) When $\eta \neq \gamma$, the decomposition of $C(r, \eta) \otimes C(s, \gamma)$ is given by:

\[
\begin{array}{c}
\begin{array}{c}
 u_1 \otimes u_i' \\
 v_1 \otimes u_i' + u_i \otimes v_i' \\
 (\eta v_i + v_{i-1}) \otimes u_i' \\
 + u_i \otimes (\gamma v_i' + v_{i-1}) \\
 \end{array}
\end{array}
\begin{array}{c}
 untuk i = 1, \ldots, r, j = 1, \ldots, s.\end{array}
\]
When \( \eta = \gamma \), assume \( r \leq s \) without loss of generality, then \( C(r, \eta) \otimes C(s, \gamma) \) decomposes as follows:

\[
\begin{array}{cccc}
U_1 & U_2 & \cdots & U_{r-1} & U_r \\
\eta & \eta & \cdots & \eta & \eta
\end{array}
\]

where \( U_k = \sum_{i=1}^{k} u_i \otimes u_{k+1-i}' \), \( V_k = \sum_{i=1}^{k} v_i \otimes v_{k+1-i}' + u_i \otimes v_{k+1-i}' \), \( k = 1, \ldots, r \). And \( \gamma \cdot U_i = \eta V_i + V_{i-1} \). Meanwhile,

\[
\begin{array}{cccc}
\eta & \cdots & \eta & \eta \\
\eta & \cdots & \eta & \eta
\end{array}
\]

Moreover, we take

\[
\begin{array}{cccc}
u_1 \otimes v'_j & u_2 \otimes v_s & \cdots & u_{r-1} \otimes v'_s & u_r \otimes v'_s \\
\eta & \cdots & \eta & \cdots & \eta
\end{array}
\]

for \( i = 1, \ldots, r, \ j = 2, \ldots, s \). Then \( C(r, \eta) \otimes C(s, \gamma) \equiv P^{(r(s-1))} \oplus C(r, \gamma) - C(r, \eta) \).

(12) Without loss of generality, we assume that \( r \leq s \). It gives

\[
\begin{array}{cccc}
u_1 \otimes v'_j & u_2 \otimes v_s & \cdots & u_{r-1} \otimes v'_s & u_r \otimes v'_s \\
\eta & \cdots & \eta & \cdots & \eta
\end{array}
\]

where \( U_k = \sum_{i=1}^{k} u_i \otimes u_{k+1-i}' \), \( V_k = \sum_{i=1}^{k} v_i \otimes v_{k+1-i}' + u_i \otimes v_{k+1-i}' \), \( k = 1, \ldots, r \). Meanwhile,

\[
\begin{array}{cccc}
u_1 \otimes v'_j & u_2 \otimes v_s & \cdots & u_{r-1} \otimes v'_s & u_r \otimes v'_s \\
\eta & \cdots & \eta & \cdots & \eta
\end{array}
\]

for \( i = 1, \ldots, r, \ j = 2, \ldots, s \), where we take \( v_0 = 0 \) for simplicity.

(13) It is similar to (12).
(14) For $N(r) \otimes N'(s)$, we have
\[
\begin{align*}
&\forall i = 1, \ldots, r, j = 1, \ldots, s, \\
&\text{where we take } v_0 = v'_0 = 0 \text{ for simplicity.}
\end{align*}
\]

For $C(r, \eta) \otimes N(s)$, we have
\[
\begin{align*}
&\forall i = 1, \ldots, r, j = 1, \ldots, s, \\
&\text{where we take } v'_0 = 0 \text{ for simplicity.}
\end{align*}
\]

4. The Green ring of $\mathcal{A}$ and its Jacobson radical

4.1. Let $H$ be a Hopf algebra. Recall that the Green (or representation) ring $r(H)$ and the Green algebra $R(H)$ of $H$ can be defined as follows. $r(H)$ is the abelian group generated by the isomorphism classes $[M]$ of $M \in H\text{-mod}$ modulo the relations $[M \oplus N] = [M] + [N]$. The multiplication of $r(H)$ is given by the tensor product of $H$-modules, that is, $[M][N] = [M \otimes N]$. Then $r(H)$ is an associative ring with the identity $[K_\varepsilon]$, where $K_\varepsilon$ is the trivial $H$-module. $R(H)$ is an associative $K$-algebra defined by $K \otimes_Z r(H)$. Note that $r(H)$ is a free abelian group with a $\mathbb{Z}$-basis $\{[M] \mid M \in \text{ind}(H)\}$, where $\text{ind}(H)$ denotes the category of finite-dimensional indecomposable $H$-modules. If $H$ is a quasitriangular Hopf algebra, then $M \otimes N \cong N \otimes M$ as $H$-modules. In this case, $r(H)$ is a commutative ring.

From Theorem 3.1, we derive the Green ring of $\mathcal{A}$. 
Corollary 4.1. The Green ring \( r(\mathcal{A}) \) of \( \mathcal{A} \) is a commutative ring generated by
\[
\left\{ [S(-,-)], [S(+,-)], [P], [M(1)], [W(1)] \right\} \cup \left\{ [C(r,\eta)], [N(r)], [N'(r)] \right\}_{r\in\mathbb{Z}^+,\eta\in\mathbb{K}^*},
\]
subjecting to the following relations
\[
S^2 = S_2^2 = 1,
\]
\[
P^2 = 2(1+S)P, \quad MP = (1+2S)P, \quad WP = (2+S)P, \quad C_{r_1}P = N_1P = N'_1P = r(1+S)P,
\]
\[
MW = 2P + S, \quad MC_{r_2} = rP + S r_1, \quad MN_1 = rP + S N_1, \quad MN'_1 = rP + S N'_1,
\]
\[
WC_{r_2} = rP + C_{r_2}, \quad WN_1 = rP + N_1, \quad WN'_1 = rP + N'_1,
\]
\[
C_{r_2} C_{s, \gamma} = \begin{cases} \frac{r s P}{\gamma}, & \gamma \neq \gamma, \\ (r s - \min\{r, s\}) P + (1+S) C_{\min\{r, s\}, \gamma}, & \gamma = \gamma, \end{cases}
\]
\[
N_1 N_1 = (r s - \min\{r, s\}) P + (1+S) N_{\min\{r, s\}}, \quad N'_1 N'_1 = (r s - \min\{r, s\}) P + (1+S) N'_{\min\{r, s\}},
\]
\[
C_{r_2} N_1 = C_{r_2} N'_1 = N_1 N'_1 = r s P,
\]
where we abbreviate \([S(-,-)], [S(+,-)], [P(+,+)], [M(1)], [W(1)], [C(r,\eta)], [N(r)], [N'(r)]\) to \(S, S_-, P, M, W, C_{r_2}, N_1, N'_1\), successively.

4.2. Let us focus on the principle block \( \mathcal{A}_1 = \mathcal{A}_{f_1} \). To recover the whole information, one can apply the functor \( \cdot \otimes \mathbb{K}[\mathbb{Z}_2] \), as \( \mathbb{K}[S_-]/(S_2^2 - 1) \cong \mathbb{K}[\mathbb{Z}_2] \). Recall that the projective class ring \( p(\mathcal{A}_1) \) of \( \mathcal{A}_1 \) is the subring of \( R(\mathcal{A}_1) \) generated by the projective and simple modules. We consider the projective class algebra \( p(\mathcal{A}_1) = \mathbb{K}[S, P]/(S^2 - 1, P^2 - 2(1+S)P) \). The Jacobson radical \( J(p(\mathcal{A}_1)) = (S-1)P \), thus
\[
p(\mathcal{A}_1)/J(p(\mathcal{A}_1)) = \mathbb{K}[S, P]/(S^2 - 1, P^2 - 4P) \cong \mathbb{K}[\mathbb{Z}_2 \times \mathbb{Z}_2].
\]

Note that all projective modules in \( R(\mathcal{A}_1) \) generate an ideal \( \mathcal{P} \), and one can define the stable Green ring of \( \mathcal{A}_1 \) as \( \text{St}(\mathcal{A}_1) = R(\mathcal{A}_1)/\mathcal{P} \). For its complexified counterpart, we have

Theorem 4.2. The Jacobson radicals of \( \text{St}(\mathcal{A}_1) \) and \( \mathcal{R}(\mathcal{A}_1) \) are respectively equal to
\[
J(\text{St}(\mathcal{A}_1)) = \left\{ (S-1) N_1, \ (S-1) N'_1, \ (S-1) C_{r_2}, \ r \in \mathbb{Z}^+, \eta \in \mathbb{K}^* \right\},
\]
\[
J(\mathcal{R}(\mathcal{A}_1)) = \left\{ (S-1) P, \ (S-1) N_1, \ (S-1) N'_1, \ (S-1) C_{r_2}, \ r \in \mathbb{Z}^+, \eta \in \mathbb{K}^* \right\}.
\]

Proof. We first deal with \( \text{St}(\mathcal{A}_1) \). Take \( J = \left\{ (S-1) N_1, \ (S-1) N'_1, \ (S-1) C_{r_2}, \ r \in \mathbb{Z}^+, \eta \in \mathbb{K}^* \right\} \) and \( A = \text{St}(\mathcal{A}_1)/J \), clearly, \( J \) is nilpotent, we only need to show that \( J \) is Jacobson semisimple. Note that for any \( a \in \mathbb{K}^*, a \neq 1 \), \( m_{a, \pm} = (S \pm 1, W - a) \) is a maximal ideal of \( A \). In fact, \( (W - a) N_1 = (1 - a) N_1 \) makes \( N_1 \in m_{a, \pm} \). Similarly, \( N'_1, C_{r_2} \in m_{a, \pm} \). Meanwhile, \( M(W - a) = S - a M \). Now combining all of them, we have \( m_{a, \pm} = \left\{ S \pm 1, W - a, M \pm a^{-1}, N_1, N'_1, C_{r_2}, \ r \in \mathbb{Z}^+, \eta \in \mathbb{K}^* \right\} \), which is maximal. Next consider the ideals \( m = (N_1 - 2), m' = (N'_1 - 2), m_\eta = (C_{1, \eta} - 2), \eta \in \mathbb{K}^* \). Since \( S(N_1 - 2) = N_1 - 2 S, \)
\[ W(N_1 - 2) = N_1 - 2W, \quad M(N_1 - 2) = N_1 - 2M, \quad N_r(N_1 - 2) = 2(N_1 - N_r), \quad N_r' (N_1 - 2) = -2N_r', \quad C_{r, q}(N_1 - 2) = -2C_{r, q}, \]

we know that \( S - 1, \quad W - 1, \quad M - 1, \quad N_r - 2, \quad N_r', \quad C_{r, q} \in \mathfrak{m} \), for any \( r \in \mathbb{Z}^+, \eta \in \mathbb{K}^x \), thus \( \mathfrak{m} \) is maximal. Similarly, we have \( S - 1, \quad W - 1, \quad M - 1, \quad N_r, \quad N_r', \quad C_{r, q} - 2, \quad C_{r, \gamma} \in \mathfrak{m}_\eta \), for any \( r \in \mathbb{Z}^+, \gamma \in \mathbb{K}^x, \gamma \neq \eta \), so \( \mathfrak{m}', \mathfrak{m}_\eta \) are maximal.

Note that any two of the ideals \( \mathfrak{m}_{a, x}, \mathfrak{m}, \mathfrak{m}', \mathfrak{m}_\eta (a, \eta \in \mathbb{K}^x) \) are coprime, i.e., the sum of them is the whole ring. If two ideals \( a, b \) are coprime, then \( a \cap b = ab. \) Hence, we have

\[
\begin{align*}
\mathfrak{m} \cap \mathfrak{m}' &\cap \bigcap_{\eta \in \mathbb{K}^x} \mathfrak{m}_\eta = (S - 1, \quad W - 1, \quad M - 1), \\
\bigcap_{x \in \mathbb{K}^x} \mathfrak{m}_a &\cap \bigcap_{x \in \mathbb{K}^x} \mathfrak{m}_b = \left( N_r, \quad N_r', \quad C_{r, q} \mid r \in \mathbb{Z}^+, \quad \eta \in \mathbb{K}^x \right).
\end{align*}
\]

We denote these two ideals by \( a, b \) respectively and show that \( a \cap b = 0, \) thus \( A \) is Jacobson semisimple. Suppose \( x \in a \cap b \setminus \{0\}, \) then there exist \( y_r, y'_r, y_{r, \eta} \in A, \quad r \in \mathbb{Z}^+, \quad \eta \in \mathbb{K}^x \) such that \( x = \sum_{r \in \mathbb{Z}^+, \eta \in \mathbb{K}^x} (y_rN_r + y'_rN'_r + y_{r, \eta}C_{r, \eta}). \) Note that \( N_r/2, N'_r/2, C_{r, \eta}/2, \; r \in \mathbb{Z}^+, \; \eta \in \mathbb{K}^x \) are orthogonal idempotents. Without loss of generality, we assume that there exists a largest \( r_0 \) such that \( y_{r_0}N_{r_0} \neq 0. \) Then from the multiplication of \( A, \) we see that \( N_{r_0} \neq 0. \) But this is impossible, as \( N_{r_0}(S - 1) = N_{r_0}(W - 1) = N_{r_0}(M - 1) = 0 \) in \( A. \) Hence, \( A \) is Jacobson semisimple, and \( J = J(\text{St}(\tilde{\mathcal{A}}_1)). \)

Next we deal with the case of \( R(\tilde{\mathcal{A}}_1). \) From the short exact sequence \( 0 \to \mathcal{P} \to R(\tilde{\mathcal{A}}_1) \to \text{St}(\tilde{\mathcal{A}}_1) \to 0, \) we see that

\[
J(R(\tilde{\mathcal{A}}_1)) \subset \left( P, \quad SP, \quad (S - 1)N_r, \quad (S - 1)N'_r, \quad (S - 1)C_{r, \eta} \mid r \in \mathbb{Z}^+, \quad \eta \in \mathbb{K}^x \right).
\]

Meanwhile, since \( (S - 1)P, (S - 1)N_r, (S - 1)N'_r, (S - 1)C_{r, \eta} \quad (r \in \mathbb{Z}^+, \; \eta \in \mathbb{K}^x) \) are all nilpotent in \( R(\tilde{\mathcal{A}}_1), \) they all lie in \( J(R(\tilde{\mathcal{A}}_1)). \) In order to see that they generate \( J(R(\tilde{\mathcal{A}}_1)), \) one only needs to check that \( (S + 1)P/8 \) is an idempotent, thus \( (S + 1)P \notin J(R(\tilde{\mathcal{A}}_1)). \)

\[ \square \]

5. The Green rings of two 2-cocycle twists of \( \mathcal{A} \)

In this section, we will discuss two 2-cocycle twists of \( \mathcal{A}, \) both of which involve the 4-dimensional Sweedler algebra \( H_4. \) \( H_4 = \mathbb{K}<a, b>/<(a^2 - 1, b^2, ab + ba)> \) is the simplest example of noncommutative and noncocommutative Hopf algebra with \( S^2 \neq 1. \)

5.1 Two 2-cocycle twists of \( \mathcal{A}, \) \( H_4 \otimes H_4 \) and \( D(H_4). \) First of all, let us briefly recall the definition of 2-cocycle twist of a Hopf algebra (see [14]). Associated with a 2-cocycle \( \sigma \) as a bilinear form defined on a bialgebra algebra \( H, \) which is invertible under the convolution product and satisfies

\[
\begin{align*}
\sigma(a, 1) &= \sigma(1, a) = \varepsilon(a), \quad a \in H, \\
\sigma(a_1, b_1) \sigma(a_2, b_2, c) &= \sigma(b_1, c_1) \sigma(a, b_2)c_2, \quad a, b, c \in H,
\end{align*}
\]

In this article, we consider the following 2-cocycle twist of the Hopf algebra \( H_4. \) \( \sigma(a, b) = \varepsilon(a), a \in H_4. \) \( \sigma(a, b) \neq 0 \) for any \( a, b \in H_4. \) Then we will see that \( \mathcal{A}_1 \) is a 2-cocycle twist of \( \mathcal{A}. \)
one can construct a new bialgebra \((H', \cdot, \Delta, \varepsilon)\) with
\[
a \cdot_{\sigma} b = \sigma(a(1), b(1)) a(2) b(2) \sigma^{-1}(a(3), b(3)), \quad a, b \in H.
\]
Moreover, if \(H\) is a Hopf algebra with the antipode \(S\), then the antipode \(S'\) of \(H'\) is given by \(S'(a) = \langle \mathcal{U}, a(1) \rangle S(a(2)) \langle \mathcal{U}^{-1}, a(3) \rangle\), for \(a \in H'\), where \(\mathcal{U} = \sigma \circ (\text{id} \otimes S) \circ \Delta \in H^*\) with the inverse \(\mathcal{U}^{-1} = \sigma^{-1} \circ (S \otimes \text{id}) \circ \Delta\).

The first twist involved is just the tensor product \(H_4 \otimes H_4\) of the Sweedler Hopf algebra \(H_4\), with the corresponding 2-cocycle \(\sigma_1\) on \(\mathcal{A}\) given by:
\[
\sigma_1(u, v) = \begin{cases} (-1)^{a_1b_2}, & u = g^{a_1} h^{b_2}, \quad v = g^{b_1} h^{a_2}, \\ 0, & \text{if } u, v \notin \mathbb{K}(g, h), \end{cases}
\]
where \(a_1, a_2, b_1, b_2 \in \mathbb{Z}\).

The other twisted we concern is the Drinfel’d double \(D(H_4)\), whose Green ring has been computed in [8]. Here we point out that it is also 2-cocycle twist-equivalent to \(H_4 \otimes H_4\), thus to \(\mathcal{A}\). Now we recall the presentation of \(D(H_4)\) given in [8]. It has generators \(g, h, x, y\), which subject to the following relations:
\[
\begin{align*}
g^2 &= h^2 = 1, \quad x^2 = y^2 = 0, \quad gx = -xg, \quad gy = -yg, \\
hx &= -xh, \quad hy = -yh, \quad xy + yx = 1 - gh.
\end{align*}
\]
Meanwhile, we define the comultiplication of \(D(H_4)\) to be the same as \(\mathcal{A}\), opposite to the one used in [8]. Due to [14], for two given Hopf algebras \(A, B\), if there exists a skew pairing \(\langle , \rangle : B \otimes A \rightarrow \mathbb{K}\), satisfying
\[
\langle bb', a \rangle = \langle b \otimes b', \Delta(a) \rangle, \quad \langle ba', a' \rangle = \langle \Delta^0(b), a \otimes a' \rangle, \\
\langle 1, a \rangle = \varepsilon(a), \quad \langle b, 1 \rangle = \varepsilon(b),
\]
then one can define a 2-cocycle \(\sigma\) on \(A \otimes B\) by
\[
\sigma(a \otimes b, a' \otimes b') = \varepsilon(a) \langle b, b' \rangle \varepsilon(b), \quad a, a' \in A, \quad b, b' \in B,
\]
such that the twist \((A \otimes B)^{\sigma}\) is again a Hopf algebra. Now as \(\{b^i a^l\}_{0 \leq i, j, k, l \leq 1}\) forms a basis of \(H_4\), we can define a skew pairing \(\langle , \rangle : H_4 \otimes H_4 \rightarrow \mathbb{K}\) by
\[
\langle a', a \rangle = (-1)^{ij}, \quad \langle b, b \rangle = 1, \quad \langle a', b \rangle = \langle b, a' \rangle = 0
\]
with the corresponding 2-cocycle \(\sigma_2\).

**Proposition 5.1.** There exists a Hopf algebra isomorphism \(\phi\) between \((H_4 \otimes H_4)^{\sigma_2}\) and \(D(H_4)\) defined by \(\phi(b^i a^l \otimes b^k a^l) = x^i g^j h^l\), for \(0 \leq i, j, k, l \leq 1\).
Proof. One only needs to show that $\phi$ is an algebra homomorphism.

$$\begin{align*}
(1 \otimes b) \cdot_{\mathcal{C}_2} (b \otimes 1) &= \sigma_2(1 \otimes b, b \otimes 1)\sigma_2^{-1}(1 \otimes 1, 1 \otimes 1)1 \otimes 1 \\
&+ \sigma_2(1 \otimes a, a \otimes 1)\sigma_2^{-1}(1 \otimes b, b \otimes 1)a \otimes a \\
&+ \sigma_2(1 \otimes a, a \otimes 1)\sigma_2^{-1}(1 \otimes 1, 1 \otimes 1)b \otimes b = 1 \otimes 1 - a \otimes a - b \otimes b, \\
(1 \otimes b) \cdot_{\mathcal{C}_2} (a \otimes 1) &= \sigma_2(1 \otimes b, b \otimes 1)\sigma_2^{-1}(1 \otimes 1, a \otimes 1)a \otimes 1 \\
&+ \sigma_2(1 \otimes a, a \otimes 1)\sigma_2^{-1}(1 \otimes b, a \otimes 1)a \otimes a \\
&+ \sigma_2(1 \otimes a, a \otimes 1)\sigma_2^{-1}(1 \otimes 1, a \otimes 1)a \otimes b = -a \otimes b, \\
(1 \otimes a) \cdot_{\mathcal{C}_2} (b \otimes 1) &= \sigma_2(1 \otimes a, b \otimes 1)\sigma_2^{-1}(1 \otimes a, 1 \otimes 1)1 \otimes a \\
&+ \sigma_2(1 \otimes a, a \otimes 1)\sigma_2^{-1}(1 \otimes a, 1 \otimes 1)b \otimes a \\
&+ \sigma_2(1 \otimes a, a \otimes 1)\sigma_2^{-1}(1 \otimes a, b \otimes 1)a \otimes a = -b \otimes a.
\end{align*}$$

Hence,

$$\begin{align*}
\phi((1 \otimes b) \cdot_{\mathcal{C}_2} (b \otimes 1)) &= 1 - gh - xy = \phi((1 \otimes b)\phi(b \otimes 1)), \\
\phi((1 \otimes b) \cdot_{\mathcal{C}_2} (a \otimes 1)) &= -gy = \phi((1 \otimes b)\phi(a \otimes 1)), \\
\phi((1 \otimes a) \cdot_{\mathcal{C}_2} (b \otimes 1)) &= -xh = \phi((1 \otimes a)\phi(b \otimes 1)).
\end{align*}$$

\[\square\]

5.2. The Green ring of $D(H_4)$. Motivated by the construction of a complete set of primitive orthogonal idempotents of $\mathcal{U}_r(sl_2)$ in [35], we figure out those of $D(H_4)$.

Proposition 5.2. A complete set of primitive orthogonal idempotents of $D(H_4)$ is given by $\{e_1, \ldots, e_6\}$, where

$$\begin{align*}
e_1 &= \frac{1}{6}(1 + g + h + gh), & e_2 &= \frac{1}{6}(1 - g - h + gh), \\
e_3 &= \frac{1}{4}xy(1 + g - h - gh), & e_4 &= \frac{1}{6}(2 - xy)(1 + g - h - gh), \\
e_5 &= \frac{1}{8}xy(1 - g + h - gh), & e_6 &= \frac{1}{8}(2 - xy)(1 - g + h - gh).
\end{align*}$$

Proof. It is straightforward to check. \[\square\]

Remark 5.3. In [2], Arike described a complete set of primitive orthogonal idempotents of $\mathcal{U}_q(sl_2)$ with $q$ a primitive $2p$-th root of unity, $p \geq 2$. Later, Kondo, Saito [26] decomposed the indecomposable decomposition of tensor products of modules over $\mathcal{U}_q(sl_2)$ based on Arike’s work. However, the truncation relations in their definition of $\mathcal{U}_q(sl_2)$ are distinct from $D(H_4)$ in [8]. From this point of view, the double $D(H_4)$ considered by Chen does not belong to those small quantum groups studied by Kondo and Saito in [26]. Hence, we prefer to use the version (close to ours) of restricted two-parameter quantum groups in [35].
Note that $D(H_4)$ is not basic, as $D(H_4)e_3 \cong D(H_4)e_4$, $D(H_4)e_5 \cong D(H_4)e_6$. Each of them is a two-dimensional simple projective module. We denote $P_+ = D(H_4)e_3$, $P_- = D(H_4)e_5$, whose diagrams are respectively:

$$
\begin{align*}
&\begin{array}{c}
\cdot e_1 \\
\vdots \\
\cdot e_5
\end{array} \\
&\begin{array}{c}
\cdot y \\
\vdots \\
\cdot y
\end{array}
\end{align*}
$$

Meanwhile, the radical $J(D(H_4)) = (x(1 + gh), \gamma(1 + gh))$. Hence, the Gabriel quiver of the basic subalgebra $D(H_4)e_1 \oplus D(H_4)e_2 \oplus D(H_4)e_3 \oplus D(H_4)e_5$ of $D(H_4)$ looks like:

$$
ge_1, e_2, e_3, e_4, e_5
$$

Note that $D(H_4)$-mod has three blocks, the modules from the block $D(H_4)e_1 \oplus D(H_4)e_2$-mod are the same as those from $\mathcal{A}f_1$-mod, on which the action of $1 - gh$ vanishes. So all the tensor product decompositions from $\mathcal{A}f_1$-mod in Theorem 3.1 work on $D(H_4)e_1 \oplus D(H_4)e_2$-mod as well.

Now due to Corollary 4.1, the Green ring of $D(H_4)$ can be obtained directly. One should compare our list with the result in [8], which is described in a different language, using syzygy, cosyzygy functors, etc.

**Corollary 5.4.** The Green ring $r(D(H_4))$ of $D(H_4)$ is a commutative ring generated by

$$
\{ [S(-, -)], [P], [P_+], [M(1)], [W(1)] \} \cup \{ [C(r, \eta)], [N(r)], [N'(r)] \}_{r \in \mathbb{Z}^+, \eta \in \mathbb{K}^*},
$$

subjecting to the following relations:

- $S^2 = 1$, $P_+P = 2(1+SP_+)$, $P_+^2 = SP$,
- $MP_+ = (1 + 2S)P_+$, $WP_+ = (2 + S)P_+$, $C_{r,\eta}P_+ = N_\eta P_+ = N'_\eta P_+ = r(1 + S)P_+$,
- $MW = 2P + S$, $MC_{r,\eta} = rP + S C_{r,\eta}$, $MN_\eta = rP + SN_\eta$, $MN'_\eta = rP + SN'_\eta$,
- $WC_{r,\eta} = rP + C_{r,\eta}$, $WN_\eta = rP + N_\eta$, $WN'_\eta = rP + N'_\eta$,
- $C_{r,\eta}C_{s,\gamma} = \begin{cases} rsP, & \eta \neq \gamma, \\ (rs - \min(r,s))P + (1 + S)C_{\min(r,s),\eta \gamma}, & \eta = \gamma, \end{cases}$
- $N_\eta N_\gamma = (rs - \min(r,s))P + (1 + S)N_{\min(r,s)}$, $N'_\eta N'_\gamma = (rs - \min(r,s))P + (1 + S)N'_{\min(r,s)}$,
- $C_{r,\eta}N_\gamma = C_{r,\eta}N'_\gamma = N_\eta N'_\gamma = rsP$,

where $S$, $P$, $M$, $W$, $C_{r,\eta}$, $N_\eta$, $N'_\eta$ are still for $[S(-, -)]$, $[P(+, +)]$, $[M(1)]$, $[W(1)]$, $[C(r, \eta)]$, $[N(r)]$, $[N'(r)]$, successively.
The Jacobson radical $J(p(D(H_4))) = \mathbb{K}[S, P_+]\langle S^2 - 1, P_+^3 - 2(1+S)P_+ \rangle$. The Jacobson radical $J(p(D(H_4))) = ((1-S)P_+)$, thus
\[
p(D(H_4))/J(p(D(H_4))) = \mathbb{K}[S, P_+]\langle S^2 - 1, P_+^3 - 4P_+ \rangle \cong \mathbb{K}[^2]\mathbb{K}[^2] \cong \mathbb{K}[^2].
\]
Meanwhile, similar to Theorem 4.2, we know that the Jacobson radical of $p(D(H_4))$ is $\mathbb{K}[S, P_+]\langle S^2 - 1, P_+^3 - 4P_+ \rangle$.

5.3. The Green ring of $\tilde{\mathcal{H}}$. Write $\mathcal{H} := H_4 \otimes H_4$ for short. The rest of the paper will be devoted to computing the Green ring of $\mathcal{H}$. We use the following presentation of $\mathcal{H}$. It has generators $g, h, x, y$ as well, but subjecting to the following different relations,
\[
\begin{align*}
g x &= xg, \quad g^2 = h^2 = 1, \\
g y &= yg, \quad hx = xh, \quad hy = -yh, \\
x^2 &= y^2 = 0, \quad xy = yx.
\end{align*}
\]
The comultiplication $\Delta$ is defined by
\[
\Delta(g) = g \otimes g, \quad \Delta(h) = h \otimes h, \quad \Delta(x) = x \otimes 1 + g \otimes x, \quad \Delta(y) = y \otimes 1 + h \otimes y.
\]
$\tilde{\mathcal{H}}$ also has 4 orthogonal primitive idempotents
\[
e_1 = \frac{1}{4}(1 + g + h + gh), \quad e_2 = \frac{1}{4}(1 + g - h - gh), \quad e_3 = \frac{1}{4}(1 - g + h - gh), \quad e_4 = \frac{1}{4}(1 - g - h + gh),
\]
but the unit 1 is the unique central idempotent of $\mathcal{H}$.

Lemma 5.5. (1) There exist two signs $s_1, s_2 \in \{+, -\}$ such that $ge_i = s_1e_i$, $he_i = s_2e_i$ with $(s_1, s_2) = (+, +)$, $(+, -)$, $(-, +)$, $(-, -)$ for $i = 1, 2, 3, 4$, successively.

(2) $xe_1 = e_3x$, $xe_3 = e_1x$, $ye_1 = e_2y$, $ye_2 = e_1y$, $xe_2 = e_4x$, $xe_4 = e_2x$.

$ye_4 = e_3y$, $ye_3 = e_4y$.

Similar to $\tilde{\mathcal{A}}$, we denote by $S(s_1, s_2)$ the one-dimensional simple module $\mathbb{K}$ with $g \cdot 1 = s_1 1, h \cdot 1 = s_2 1, x \cdot 1 = y \cdot 1 = 0$, and $P(s_1, s_2)$ as the projective cover of $S(s_1, s_2)$.

As $P(s_1, s_2)$ are non-isomorphic to each other with respect to the signs $(s_1, s_2)$, $\tilde{\mathcal{H}}$ is also a basic algebra over $\mathbb{K}$. On the other hand, the radical $J(\mathcal{H}) = (x, y)$, thus from the above lemma, we know that the Gabriel quiver $Q_{\tilde{\mathcal{H}}}$ of $\tilde{\mathcal{H}}$ looks like:
where for \( i = 1, 2, 3, 4 \), the arrows \( \alpha_i, \beta_i \) correspond to \( e_i, x, e_i, y \), respectively. The admissible ideal \( I \) has the following relations:

\[
\begin{align*}
\alpha_1 \alpha_3 &= \alpha_3 \alpha_1 = 0, \\
\alpha_2 \alpha_4 &= \alpha_4 \alpha_2 = 0, \\
\beta_1 \beta_2 &= \beta_2 \beta_1 = 0, \\
\beta_3 \beta_4 &= \beta_4 \beta_3 = 0, \\
\alpha_1 \beta_3 - \beta_1 \alpha_2 &= \alpha_2 \beta_4 - \beta_2 \alpha_1 = \alpha_3 \beta_1 - \beta_3 \alpha_4 = \alpha_4 \beta_2 - \beta_4 \alpha_3 = 0.
\end{align*}
\]

Hence, \( \mathcal{H} \) has a unique block.

From the quiver \( Q_{\mathcal{H}} \), we know that \( \mathcal{H} \) is also a special biserial algebra. We describe all its indecomposable modules of Loewy length 2 as follows. For any \( r, s \in \mathbb{Z}^+, \eta, \gamma \in \mathbb{K}^\times \), the string modules are

\[
M(r) = \begin{cases} 
M \left( (\alpha_1 \beta_1^{-1} \alpha_4 \beta_4^{-1})^{r/2} \right), & \text{if } r \text{ is even}, \\
M \left( (\alpha_1 \beta_1^{-1} \alpha_4 \beta_4^{-1})^{(r-1)/2} \right), & \text{if } r \text{ is odd}, 
\end{cases}
\]

\[
W(r) = \begin{cases} 
M \left( (\beta_1^{-1} \alpha_4 \beta_4^{-1} \alpha_1)^{r/2} \right), & \text{if } r \text{ is even}, \\
M \left( (\beta_1^{-1} \alpha_4 \beta_4^{-1} \alpha_1)^{(r-1)/2} \right), & \text{if } r \text{ is odd}, 
\end{cases}
\]

all of dimension \( 2r + 1 \).

\[
N(r) = \begin{cases} 
M \left( (\alpha_1 \beta_1^{-1} \alpha_4 \beta_4^{-1})^{(r-2)/2} \right), & \text{if } r \text{ is even}, \\
M \left( (\alpha_1 \beta_1^{-1} \alpha_4 \beta_4^{-1})^{(r-1)/2} \right), & \text{if } r \text{ is odd}, 
\end{cases}
\]

\[
N'(r) = \begin{cases} 
M \left( (\beta_1^{-1} \alpha_4 \beta_4^{-1} \alpha_1)^{(r-2)/2} \right), & \text{if } r \text{ is even}, \\
M \left( (\beta_1^{-1} \alpha_4 \beta_4^{-1} \alpha_1)^{(r-1)/2} \right), & \text{if } r \text{ is odd}, 
\end{cases}
\]

all of dimension \( 2r \).

The diagrams of \( M(r), N(r) \) look like:

\[
\begin{array}{cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
u_1 & u_2 & \cdots & u_k & \cdots & u_1 \\
v_1 & v_2 & \cdots & v_k & \cdots & v_1 \\
\end{array}
\]

while the diagrams of \( W(r), N'(r) \) look like:

\[
\begin{array}{cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
u_1 & u_2 & \cdots & u_k & \cdots & u_1 \\
v_1 & v_2 & \cdots & v_k & \cdots & v_1 \\
\end{array}
\]

where \( u_1 \) always has the sign \((+, +)\), and we again use the arrow \( \longrightarrow \) (resp. \( \longleftarrow \)) to represent the action of \( x \) (resp. \( y \)) on the modules.

Meanwhile, the band modules are

\[
C(r, \eta) = M \left( \beta_4 \alpha_1^{-1} \beta_1 \alpha_4^{-1}, r, \eta \right)
\]
of dimension 4r, whose diagrams are as follows:

\[
\begin{array}{c}
\begin{array}{ccccccc}
u_1 & u_1 & u_2 & u_3 & u_4 & \cdots & u_{2r-2} & u_{2r} \\
v_1 & v_2 & & v_3 & & \cdots & v_{2r-2} & v_{2r} \\
\end{array}
\end{array}
\]

where \( y \cdot u_{2i-1} = \eta v_{2i} + v_{2i-2}, \ i = 1, \ldots, r \) and \( v_0 = 0 \). Moreover, \( u_{2i-1}, u_{2j} \) have the signs \((-,-), (+,+), \) respectively.

For \( M \in \{ S(+,+), P(+,+), M(r), W(r), N(r), N'(r), C(r, \eta) \} \), we define \( M_{s_1s_2} = M \otimes S(s_1, s_2) \) (omitting the subscript ++ for short), which complete the list of indecomposable modules of \( \mathcal{H}^r \).

Now for any \( t \in \mathbb{Q} \), we define \([r]\) (resp. \([t]\)) as the lower (resp. upper) bound of integers bigger (resp. smaller) than \( t \). For any \( r, s \in \mathbb{Z}^+ \), we write \( \min\{r, s\} \), \( \max\{r, s\} \) as \( \underline{r}, \underline{s}, \overline{r}, \overline{s} \), respectively for short. Moreover, we define the parity

\[
[r] = \begin{cases} 
- & \text{if } r \text{ is odd}, \\
+ & \text{if } r \text{ is even}.
\end{cases}
\]

**Theorem 5.6.** For any \( r, s \in \mathbb{Z}^+, \eta, \gamma \in \mathbb{K} \), we have

1. \( P \otimes P \cong P \oplus P_{--} \oplus P_{+--} \oplus P_{++,} \).
2. \( M(r) \otimes P \cong P_{\overline{r}/2} \oplus P_{\overline{r}/2} \oplus P_{\overline{r}/2} \oplus P_{\overline{r}/2} \).
3. \( W(r) \otimes P \cong P_{\overline{r}/2} \oplus P_{\overline{r}/2} \oplus P_{\overline{r}/2} \oplus P_{\overline{r}/2} \).
4. \( N(r) \otimes P \cong P_{\overline{r}/2} \oplus P_{\overline{r}/2} \oplus P_{\overline{r}/2} \oplus P_{\overline{r}/2} \).
5. \( M(r) \otimes M(s) \cong P_{\overline{r}/2} \oplus P_{\overline{r}/2} \oplus M(r+s)_{++} \).
6. \( M(r) \otimes W(s) \cong P_{\overline{r}/2} \oplus P_{\overline{r}/2} \oplus W(r+s) \).
7. \( M(r) \otimes C(s, \eta) \cong P_{\overline{r}/2} \oplus P_{\overline{r}/2} \oplus C(s, \eta)_{++} \).
8. \( W(r) \otimes C(s, \eta) \cong P_{\overline{r}/2} \oplus P_{\overline{r}/2} \oplus C(s, \eta)_{++} \).
9. \( M(r) \otimes N(s) \cong P_{\overline{r}/2} \oplus P_{\overline{r}/2} \oplus N(s)_{++} \).
10. \( W(r) \otimes N(s) \cong P_{\overline{r}/2} \oplus P_{\overline{r}/2} \oplus N(s)_{++} \).
Proof. (Sketch) For (1), (2), (3), note that the decomposition of the tensor product of $M$ with projective modules only depends on the composition factors of $M$. On the other hand, by the construction of band modules, $M(\beta_4 \alpha_1^{-1} \beta_1 \alpha_4^{-1}, r, \eta) \cong M(\beta_1 \alpha_4^{-1} \beta_1 \alpha_4^{-1}, r, \eta)$. That is, $C(r, \eta) \otimes S_{-\gamma} \cong C(r, \eta)$.

For (4), the base case $M(1) \otimes M(s) \cong P^{[s]2} \oplus P_{--}^{-}[s]2 \oplus M(s+1)_{-\gamma}$ should be as follows,

Now we prove the general case $M(r) \otimes M(s)$ by induction on $r$. From (2) and the base case, we see that

\[
(M(1) \otimes M(r)) \otimes M(s) \cong \left( P^{[r]2} \oplus P_{--}^{-}[r]2 \oplus M(r+1)_{-\gamma} \right) \otimes M(s) \\
\cong P^{[r,s]2} \oplus P^{[r]2}_{--} \oplus P^{[r,s+1]2}_{--} \oplus P^{[r,s+1]2}_{++} \oplus (M(r+1)_{-\gamma} \otimes M(s)).
\]

On the other hand, using (2) and the induction hypothesis, we have

\[
M(1) \otimes (M(r) \otimes M(s)) \cong M(1) \otimes \left( P^{[r,s]2} \oplus P^{[r]2}_{--} \oplus M(r+s)_{-\gamma} \right) \\
\cong \left( P^{[r,s]2} \oplus P^{[r,s]2}_{--} \oplus P^{[r,s+1]2}_{--} \oplus P^{[r,s+1]2}_{++} \right) \left( P^{[r,s+1]2}_{--} \oplus P^{[r,s+1]2}_{++} \oplus M(r+s+1) \right).
\]

By the Krull-Schmidt theorem, they combine to give $P^{[r,s]2}_{--} \oplus P^{[r,s+1]2}_{--} \oplus M(r+s+1)_{-\gamma}$. One can prove the decompositions (5) and (6) by induction similarly.
For (7), we first prove the base case $M(1) \otimes C(s, \eta)$. The decomposition is given by

$$u_1 \otimes u_2 \cdots u_{n-1}$$

for $i = 1, \ldots, s$, and

Now one can prove the general case by induction on $r$ via (1), (2) and (4).

For (8), it is complicated to describe the decomposition even for the base case $W(1) \otimes C(s, \eta)$. Instead, we use (6), (7) to deal with it.

$$W(1) \otimes (M(1) \otimes C(s, \eta)) \cong W(1) \otimes (P^{b^2s} \oplus P^{b^2s} \oplus C(s, \eta))$$

On the other hand, we have

$$(W(1) \otimes (M(1))) \otimes C(s, \eta) \cong (P \oplus P^{b^2s} \oplus S_\pm) \otimes C(s, \eta)$$

Hence, by the Krull-Schmidt theorem, they combine to give

$$W(1) \otimes C(s, \eta) \cong P^{b^2s} \oplus P^{b^2s} \oplus C(s, \eta).$$

Now one can prove the general case by induction on $r$ via (1), (2) and (5).

For (9), the base case $M(1) \otimes N(s)$ is as follows.

$$u_1 \otimes v_2 \cdots v_{s-1}$$

for $i = 1, \ldots, s$, where we take $v_{s+1} = 0$ for simplicity. Meanwhile,
Now we decompose $M(r) \otimes N(s)$ by induction on $r$. First, from (3) and (4), we have
\[
(M(1) \otimes M(r)) \otimes N(s) \cong (P^\oplus[r/2]_m \oplus P^\oplus[r/2]_n \oplus M(r+1)_-) \otimes N(s)
\cong \left( P^\oplus[r/2]_m \oplus P^\oplus[r/2]_n \oplus P^\oplus[r/2]_s \otimes P^\oplus[r/2]_t \right) \oplus (M(r+1)_- \otimes N(s)).
\]
On the other hand, by induction and (2), we get
\[
M(1) \otimes (M(r) \otimes N(s)) \cong M(1) \otimes \left( P^\oplus[r/2]_m \oplus P^\oplus[r/2]_n \oplus N(s)_- \right)
\cong \left( P^\oplus[r/2]_m \oplus P^\oplus[r/2]_n \oplus P^\oplus[r/2]_s \oplus P^\oplus[r/2]_t \right) \oplus \left( P^\oplus[r/2]_s \oplus P^\oplus[r/2]_t \oplus N(s)_- \right).
\]
They combine to give $M(r+1) \otimes N(s) \cong P^\oplus[r(r+1)/2]_m \oplus P^\oplus[(r+1)/2]_s \oplus N(s)_-$. The case of $M(r) \otimes N'(s)$ is quite similar. The decomposition (10) is also straightforward to check.

For (11), when $\eta \neq \gamma$, the decomposition can be given by
\[
\begin{align*}
-\eta_2 \otimes \eta'_{2,j-1} & \quad \eta_2 \otimes \eta'_{j-1} & \quad -\eta_2 \otimes \eta'_{2,j} & \quad -\eta_2 \otimes \eta'_{2,j-1} \\
\eta_2 \otimes \eta'_{2,j-1} & \quad -\eta_2 \otimes \eta'_{2,j-2} & \quad \eta_2 \otimes \eta'_{2,j} & \quad \eta_2 \otimes \eta'_{2,j-1} \\
\eta_2 \otimes \eta'_{j-1} & \quad \eta_2 \otimes \eta'_{j-2} & \quad \eta_2 \otimes \eta'_{j-1} & \quad \eta_2 \otimes \eta'_{j-1} \\
\end{align*}
\]
for $i = 1, \ldots, r$, $j = 1, \ldots, s$.

When $\eta = \gamma$, we assume $r \leq s$, without loss of generality. First, there exist the following two submodules isomorphic to $C(r, \eta)_-$, $C(r, \eta)_+$, respectively.
where for \( k = 1, \ldots, r, \)

\[
U_{2k-1} = \sum_{i=1}^{k} u_{2i-1} \otimes v_{2k-2} + u_{2i} \otimes v_{2k-2} + \sum_{i=1}^{k-1} u_{2i} \otimes u'_{2(k-i)} + \eta u_{2i} \otimes u'_{2(k-i)+2},
\]

\[
U_{2k} = \sum_{i=1}^{k} u_{2i-1} \otimes u'_{2(k-i)+2} + u_{2i} \otimes u'_{2(k-i)+1},
\]

\[
V_{2k-1} = \sum_{i=1}^{k} v_{2i-1} \otimes u'_{2(k-i)+1} - v_{2i-1} \otimes v'_{2(k-i)+1} + \eta \left( v_{2i} \otimes u'_{2(k-i)+2} + u_{2i} \otimes v'_{2(k-i)+2} \right)
\]

\[
\quad + \sum_{i=1}^{k-1} v_{2i} \otimes u'_{2(k-i)} + u_{2i} \otimes v'_{2(k-i)},
\]

\[
V_{2k} = \sum_{i=1}^{k} v_{2i-1} \otimes u'_{2(k-i)+2} - v_{2i-1} \otimes v'_{2(k-i)+2} + v_{2i} \otimes u'_{2(k-i)+1} + u_{2i} \otimes v'_{2(k-i)+1}.
\]

One can check that \( y \cdot U_{2k-1} = \eta V_{2k} + V_{2k-2}. \) Meanwhile, we choose \( 4rs - 2r \) projective submodules listed above for \( i = 1, \ldots, r, \ j = 2, \ldots, s. \) One should examine that all vectors in the diagrams of the chosen submodules form a basis of \( C(r, \eta) \otimes C(s, \eta). \) Hence, They combine to give the decomposition

\[
C(r, \eta) \otimes C(s, \eta) \cong P^0_{1} \oplus P^0_{2} \oplus (C(r, \eta) \otimes (r, \eta) - \cdots - (r, \eta) +).\]

For (13), we assume that \( r \leq s \) without loss of generality. It gives

\[
\begin{array}{cccccc}
U_1 & U_2 & \cdots & U_{r-1} & U_r \\
V_1 & V_2 & \cdots & V_{r-1} & V_r \\
\end{array}
\]

where \( U_k = \sum_{i=1}^{k} u_i \otimes u'_{k+1-i}, \ V_k = \sum_{i=1}^{k} v_i \otimes u'_{k+1-i} \ (1)^{j-1} u_i \otimes v'_{k+1-i}, \ k = 1, \ldots, r. \)

Meanwhile,

\[
\begin{array}{cccccc}
u_1 \otimes v'_1 & u_2 \otimes v'_2 & \cdots & u_{r-1} \otimes v'_{r-1} & u_r \otimes v'_{r} \\
v_1 \otimes v'_1 & v_2 \otimes v'_2 & \cdots & v_{r-1} \otimes v'_{r-1} & v_r \otimes v'_{r} \\
\end{array}
\]

for \( i = 1, \ldots, r, \ j = 2, \ldots, s, \) where we take \( v_0 = 0 \) for simplicity.

The decompositions (12) and (14) are similar to prove. \( \square \)
Corollary 5.7. The Green ring $r(\mathcal{H})$ of $\mathcal{H}$ is a commutative ring generated by
\[
\left\{ [S(-,-)], [S(+,-)], [P(+,+)], [M(1)], [W(1)], [C(r,\eta)], [N(r)], [N'(r)] \right\}_{r \in \mathbb{Z}^+, \eta \in \mathbb{K}^*},
\]
subjecting to the following relations
\[
S^2 = S^2 = 1, \quad C_{r,\eta}S = C_{r,\eta},
P^2 = (1+S)(1+S\_\)P, \quad MP = (1+S+S\_\)P,
WP = (1+S+S\_)P, \quad C_{r,\eta}P = r(1+S+S\_\)P,
N_rP = \lfloor r/2 \rfloor (1+S\_)P + \lceil r/2 \rceil (S+S\_)P,
N'_rP = \lfloor r/2 \rfloor (1+S\_)P + \lceil r/2 \rceil (S+S\_)P,
MW = (1+S)P + S\_,
MN_r = \lfloor r/2 \rfloor P + \lceil r/2 \rceil S P + S\_N_r,
MN'_r = \lfloor r/2 \rfloor P + \lceil r/2 \rceil S P + S\_N'_r,
WN_r = \lfloor r/2 \rfloor P + \lceil r/2 \rceil S P + S N_r,
WN'_r = \lfloor r/2 \rfloor P + \lceil r/2 \rceil S P + N'_r,
MC_{r,\eta} = r(1+S)P + S\_C_{r,\eta}, \quad WC_{r,\eta} = r(1+S)P + S C_{r,\eta},
C_{r,\eta}C_{s,\gamma} = \begin{cases} 2rs(1+S)P, & \eta \neq \gamma, \\
(2rs - r_s)(1+S)P + (S+S\_)C_{r,\eta}, & \eta = \gamma. 
\end{cases}
N_rN_s = [(rs - r_s)/2]P + [(rs - r_s)/2]S P + (S\_\)N_rN_s,
N'_rN'_s = [(rs - r_s)/2]P + [(rs - r_s)/2]S P + (1+S\_\)N'_rN'_s,
N_rN'_s = [rs/2]P + [rs/2]S P, \quad C_{r,\eta}N_s = C_{r,\eta}N'_s = N_rN'_s = rs(1+S)P,
\]
where we abbreviate $[S(-,-)], [S(+,-)], [P(+,+)], [M(1)], [W(1)], [C(r,\eta)], [N(r)], [N'(r)]$ to $S, \ S\_, \ P, \ M, \ W, \ C_{r,\eta}, \ N_r, \ N'_r$, successively.

The projective class algebra $p(\mathcal{H}) = \mathbb{E}[S, S\_, P]/\left( S^2 - 1, S^2 - 1, P^2 - (1+S)(1+S\_\)P \right)$. The Jacobson radical $J(p(\mathcal{H})) = ((1-S)P, (1-S\_)P)$, thus
\[
p(\mathcal{H}) / J(p(\mathcal{H})) = \mathbb{E}[S, S\_, P]/(S^2 - 1, S^2 - 1, P^2 - 4P) \cong \mathbb{E}[\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2].
\]
Meanwhile, similar to Theorem 4.2, we know that the Jacobson radical $J(R(\mathcal{H}))$ equals
\[
\left\{ (1-S)P, (1-S\_)P, (S^{-1}S\_)N_r, (1-S^{-1}S\_)N'_r, (1-S\_)C_{r,\eta} \right\}_{r \in \mathbb{Z}^+, \eta \in \mathbb{K}^*}. \]

5.4. Diverse information list on the Green rings. Finally, in order to get a clear overview on the differences of the Green rings of $\mathcal{H}$, $\mathcal{A}$, and $D(H_4)$ from the point of ring structural view, we summarize all computation results as follows:
| Hopf algebra | (co)quasitri. | blocks | projective class algebra | modulo radical |
|--------------|-------------|--------|------------------------|----------------|
| $\mathcal{H}$ | QT, CQT     | 1      | $\mathbb{K}[s, s_i, p] / (s^2 - 1, p^2 - 1 + (1 + s_i) p)$ | $\mathbb{K}[Z_2 \times Z_2 \times Z_2]$ |
| $\mathcal{A}$ | CQT         | 2      | $\mathbb{K}[s, s_i, p] / (s^2 - 1, p^2 - 2 + 2(1 + s_i) p)$ | $\mathbb{K}[Z_2 \times Z_2 \times Z_2]$ |
| $D(H_3)$     | QT, CQT     | 3      | $\mathbb{K}[s, p] / (s^2 - 1, p^1 - 2 + 2(1 + s) p)$ | $\mathbb{K}[Z_2] \oplus \mathbb{K}[Z_2] \oplus \mathbb{K}[Z_2]$ |

| Hopf algebra | the Jacobson radical of the Green algebra |
|--------------|------------------------------------------|
| $\mathcal{H}$ | $((1 - s) p, (1 - s) p, (s^r - 1 - s^r) n_r, (1 - s^r - 1 - s^r) n_r', (1 - s) c_{r,q})_{r \in \mathbb{Z}^+, q \in \mathbb{K}^x}$ |
| $\mathcal{A}$ | $(1 - s) p, (1 - s) n_r, (1 - s) n_r', (1 - s) c_{r,q})_{r \in \mathbb{Z}^+, q \in \mathbb{K}^x}$ |
| $D(H_3)$     | $(1 - s) p, (1 - s) n_r, (1 - s) n_r', (1 - s) c_{r,q})_{r \in \mathbb{Z}^+, q \in \mathbb{K}^x}$ |

| Hopf algebra | the Green algebra modulo the Jacobson radical |
|--------------|---------------------------------------------|
| $\mathcal{H}$ | $\mathbb{K}[s, s_i, p, m, w, c_{r,q}, n_r, n_r', r, n_r'] / (s^2 - 1, s^2 - 1, (p - 4) p, (m - 3) p, (w - 3) p, (c_{r,q} - 4 r) p, (n_r - 2 r) p, (n_r' - 2 r) p, m w - 2 p - s, (1 - s) c_{r,q}, (m - s) n_r - r p, (m - s) n_r' - r p, (w - s) n_r - r p, (w - 1) n_r' - r p, (m - 1) c_{r,q} - 2 r p, (w - 1) c_{r,q} - 2 r p, c_{r,q} c_{r,q} - 4 r t p, \eta \neq \gamma, c_{r,q} c_{r,q} - 2(2 r t - r) p - 2 c_{r,q} t p, n_r n_r - (r - r) t p - (s^r - 1 + s^r) n_r t, n_r' n_r' - (r - r) t p - (1 + s^r - 1 - s^r) n_r' t, n_r n_r' - r t p, c_{r,q} n_r n_r - r t p, c_{r,q} n_r' n_r' - r t p) |
| $\mathcal{A}$ | $\mathbb{K}[s, s_i, p, m, w, c_{r,q}, n_r, n_r', r, n_r'] / (s^2 - 1, s^2 - 1, (p - 4) p, (m - 3) p, (w - 3) p, (c_{r,q} - 2 r) p, (n_r - 2 r) p, (n_r' - 2 r) p, m w - 2 p - s, (m - 1) n_r - r p, (m - 1) n_r' - r p, (w - 1) n_r - r p, (w - 1) n_r' - r p, (m - 1) c_{r,q} - r p, (w - 1) c_{r,q} - r p, c_{r,q} c_{r,q} - r t p, \eta \neq \gamma, c_{r,q} c_{r,q} - (r - r) t p - 2 c_{r,q} t p, n_r n_r - (r - r) t p - 2 n_r t, n_r' n_r' - (r - r) t p - 2 n_r' t, n_r n_r' - r t p, c_{r,q} n_r n_r - r t p, c_{r,q} n_r' n_r' - r t p) |
| $D(H_3)$     | $\mathbb{K}[s, p, m, w, c_{r,q}, n_r, n_r', r, n_r'] / (s^2 - 1, (p - 4) p, (m - 3) p, (w - 3) p, (c_{r,q} - 2 r) p, (n_r - 2 r) p, (n_r' - 2 r) p, m w - 2 p - s, (m - 1) n_r - r p, (m - 1) n_r' - r p, (w - 1) n_r - r p, (w - 1) n_r' - r p, (m - 1) c_{r,q} - r p, (w - 1) c_{r,q} - r p, c_{r,q} c_{r,q} - r t p, \eta \neq \gamma, c_{r,q} c_{r,q} - (r - r) t p - 2 c_{r,q} t p, n_r n_r - (r - r) t p - 2 n_r t, n_r' n_r' - (r - r) t p - 2 n_r' t, n_r n_r' - r t p, c_{r,q} n_r n_r - r t p, c_{r,q} n_r' n_r' - r t p) |

- In the tables, the generators of algebras or ideals stand for the respective modules.
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