SHIFT OPERATORS AND STABILITY IN DELAYED DYNAMIC EQUATIONS

MURAT ADIVAR AND YOUSSEF N. RAFFOUL

Abstract. In this paper, we use what we call the shift operator so that general delay dynamic equations of the form

\[ x^\Delta(t) = a(t)x(t) + b(t)x(\delta^+(h, t)), \quad t \in [t_0, \infty) \]

can be analyzed with respect to stability and existence of solutions. By means of the shift operators we define a general delay function opening an avenue for the construction of Lyapunov functional on time scales. Thus, we use the Lyapunov’s direct method to obtain inequalities that lead to stability and instability. Therefore, we extend and unify stability analysis of delay differential, delay difference, delay \(h\)-difference, and delay \(q\)-difference equations which are the most important particular cases of our delay dynamic equation.

1. Introduction

Lyapunov functionals are widely used in stability analysis of differential and difference equations. However, the extension of utilization of Lyapunov functionals in dynamical systems on time scales has been lacking behind due to the constrained presented by the particular time scale. For example, in delay differential equations, a suitable Lyapunov functional will involve a term with double integrals, in which one of the integral’s lower limit is of the form \(t + s\). Such a requirement will restrict the time scale that can be considered.

For a few references on the study of stability in differential equations, using Lyapunov functionals, we refer the interested reader to [3], [5], [13]-[24]. The reader may consult Yoshizawa [24, pp. 183-213] (or any book on functional differential equations and Lyapunov’s direct method) for definitions of stability and for properties of Lyapunov functionals. For the stability analysis of the delay differential equation

\[ x'(t) = a(t)x(t) + b(t)x(t - h), \quad h > 0 \]  

we refer to [12]-[18], and [22]. In [6], the authors improved the results of [22] by considering the delay differential equation of the form

\[ x'(t) = a(t)x(t) + b(t)x(t - h(t)), \quad 0 < h(t) \leq r_0. \]  

On the other hand, stability analysis of delay difference equations of the form

\[ x(t + 1) = a(t)x(t) + b(t)x(t - \tau), \quad \tau \in \mathbb{Z}_+ \]

2000 Mathematics Subject Classification. Primary 34N05, 34K20; Secondary 39A12, 39A13.

Key words and phrases. Delay dynamic equation, instability, shift operators, stability, time scales.
is treated in [8], [20], and [21].

A time scale, denoted \( T \), is a nonempty closed subset of real numbers. The set \( \mathbb{T}^\kappa \) is derived from the time scale \( T \) as follows: if \( T \) has a left-scattered maximum \( M \), then \( \mathbb{T}^\kappa = T - \{ M \} \), otherwise \( \mathbb{T}^\kappa = T \). The delta derivative \( f^\Delta \) of a function \( f : T \to \mathbb{R} \), defined at a point \( t \in \mathbb{T}^\kappa \) by

\[
f^\Delta(t) := \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}, \quad \text{where } s \to t, \ s \in T \setminus \{ \sigma(t) \},
\]

was first introduced by Hilger [19] to unify discrete and continuous analyses. In (1.4), \( \sigma : T \to T \) is the forward jump operator defined by \( \sigma(t) := \inf \{ s \in T : s > t \} \). Hereafter, we denote by \( \mu(t) \) the step function \( \mu : T \to \mathbb{R} \) defined by \( \mu(t) := \sigma(t) - t \). A point \( t \in T \) is said to be right dense (right scattered) if \( \mu(t) = 0 \) \( (\mu(t) > 0) \). A point is said to be left dense if \( \sup \{ s \in T : s < t \} = t \). A function \( f : T \to \mathbb{R} \) is called rd-continuous if it is continuous at right dense points and its left sided limits exists (finite) at left dense points. Every rd-continuous function \( f : T \to \mathbb{R} \) has an anti-derivative \( F \) denoted by

\[
F(t) := \int_{t_0}^t f(t) \Delta t.
\]

To indicate the time scale interval \([a, b] \cap T \) we use the notation \([a, b]_T \). The intervals \([a, b]_T \), \((a, b]_T \), and \((a, b)_T \) are defined similarly. For brevity, we assume the reader is familiar with the basic calculus of time scales. A comprehensive review on dynamic equations on time scales can be found in [10] and [11].

In [3] and [7], the authors handle the stability analysis of the dynamic equation

\[
x^\Delta(t) = a(t)x(t) + b(t)x(\delta(t))\delta^\Delta(t),
\]

where the delay function \( \delta : [t_0, \infty)_T \to [\delta(t_0), \infty)_T \) is surjective, strictly increasing and is supposed to have the following properties

\[
\delta(t) < t, \quad \delta^\Delta(t) < \infty, \quad \delta \circ \sigma = \sigma \circ \delta.
\]

Afterwards, we point out in [3] that the assumption \( \delta \circ \sigma = \sigma \circ \delta \) is redundant whenever the delay function \( \delta : [t_0, \infty)_T \to [\delta(t_0), \infty)_T \) is surjective and strictly increasing.

Note that the delta derivative in (1.4) turns into the ordinary derivative \( f'(t) \) and the forward difference \( \Delta f(t) := f(t + 1) - f(t) \) when \( T = \mathbb{R} \) and \( T = \mathbb{Z} \), respectively. Hence, (1.5) is a general equation including the particular cases (1.1) - (1.3). However, this paper improves the results of [3].

In this paper, we define the general shift operator and make use of them in the construction of the Lyapunov functional to improve previous results on delay dynamic equations regarding stability and boundedness of solutions. In particular, we improve the results of Eq. (1.1), (1.3), and (1.5). The main task of this paper can be outlined as follows:

- To create a suitable Lyapunov function that leads to exponential stability of the zero solution.
- To give criteria for instability.
- To compare the results of this paper with the ones in the existing literature.
in [22], the author used the following
\[
V(t) = \left[ x(t) + \int_{t-h}^{t} b(s+h)x(s)ds \right]^2 \\
+ \lambda \int_{-h}^{0} \int_{t+s}^{t} b^2(z+h)x^2(z)dzds.
\]
(1.6)
to study the exponential stability of the zero solution of (1.1).

We do not adopt this type of Lyapunov functional since it requires the time scale to be additive. An additive time scale is a time scale which is closed under addition. There are many time scales that are not additive. To be more specific, the time scales to be additive. An additive time scale is a time scale which is closed under addition. There are many time scales that are not additive. However, the time scales considered in [1] is restricted to the ones having an initial point \( t_0 \in \mathbb{T} \) so that there exist the shift operators defined on \([t_0, \infty) \cap \mathbb{T}\). Afterwards, in [2] the definition of shift operators was extended so that they are defined on the whole time scale \( \mathbb{T} \). In this paper, our new and generalized shift operators include positive and negative values.

We end this section by giving some basic definitions and theorems that will be used in further sections.

**Definition 1.** A function \( h : \mathbb{T} \to \mathbb{R} \) is said to be regressive provided \( 1 + \mu(t)h(t) \neq 0 \) for all \( t \in \mathbb{T}^\alpha \), where \( \mu(t) = \sigma(t) - t \). The set of all regressive rd-continuous functions \( \varphi : \mathbb{T} \to \mathbb{R} \) is denoted by \( \mathcal{R} \) while the set \( \mathcal{R}^+ \) is given by \( \mathcal{R}^+ = \{ h \in \mathcal{R} : 1 + \mu(t)\varphi(t) > 0 \text{ for all } t \in \mathbb{T} \} \).

Let \( \varphi \in \mathcal{R} \) and \( \mu(t) > 0 \) for all \( t \in \mathbb{T} \). The exponential function on \( \mathbb{T} \) is defined by
\[
eq_{\varphi}(t,s) = \exp \left( \int_{s}^{t} \zeta_{\mu(r)}(\varphi(r))dr \right)
\]
(1.7)
where \( \zeta_{\mu(s)} \) is the cylinder transformation given by
\[
\zeta_{\mu(r)}(\varphi(r)) := \begin{cases} \\
\frac{1}{\mu(r)} \log(1 + \mu(r)\varphi(r)) & \text{if } \mu(r) > 0 \\
\varphi(r) & \text{if } \mu(r) = 0
\end{cases}
\]
(1.8)
It is well known that if \( p \in \mathcal{R}^+ \), then \( e_p(t,s) > 0 \) for all \( t \in \mathbb{T} \). Also, the exponential function \( y(t) = e_p(t,s) \) is the solution to the initial value problem \( y^\Delta = p(t)y \), \( y(s) = 1 \). Other properties of the exponential function are given in the following lemma:

**Lemma 1.** [10] Theorem 2.36 Let \( p, q \in \mathcal{R} \). Then
\begin{enumerate}
\item \( e_0(t,s) \equiv 1 \) and \( e_p(t,t) \equiv 1 \);
\item \( e_p(\sigma(t),s) = (1 + \mu(t)p(t))e_p(t,s) \);
\item \( \frac{1}{e_p(t,s)} = e_{\varnothing p}(t,s) \) where, \( \varnothing p(t) = -\frac{p(t)}{1 + \mu(t)p(t)} \);
\item \( e_p(t,s) = \frac{1}{e_{\varnothing p}(t,s)} = e_{\varnothing p}(s,t) \);
\end{enumerate}
v. \( e_p(t,s)e_p(s,r) = e_p(t,r); \)

vi. \( \left( \frac{1}{e_p(t,s)} \right)^{\Delta} = -\frac{p(t)}{e_p(t,s)}. \)

**Theorem 1.** ([10] Theorem 1.117) Let \( a \in \mathbb{T}^\kappa, b \in \mathbb{T} \) and assume that 
\( k : \mathbb{T} \times \mathbb{T}^\kappa \rightarrow \mathbb{R} \) is continuous at \( (t,t) \), where \( t \in \mathbb{T}^\kappa \) with \( t > a \). Also assume that \( k^\Delta(t,) \) is rd-continuous on \([a, \sigma(t)]\). Suppose that for each \( \varepsilon > 0 \) there exists a neighborhood \( U \) of \( t \), independent of \( \tau \in [t_0, \sigma(t)] \), such that
\[
|k(\sigma(t), \tau) - k(s, r) - k^\Delta(t, \tau)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|
\]
for all \( s \in U \), where \( k^\Delta \) denotes the derivative of \( k \) with respect to the first variable. Then
\[
g(t) := \int_a^t k(t, \tau) \Delta \tau \text{ implies } g^\Delta(t) = \int_a^t k^\Delta(t, \tau) \Delta \tau + k(\sigma(t), t)
\]
\[
h(t) := \int_t^b k(t, \tau) \Delta \tau \text{ implies } g^\Delta(t) = \int_t^b k^\Delta(t, \tau) \Delta \tau - k(\sigma(t), t).
\]

2. Shift operators

Next, we state the generalized shift operators. A limited version of it can be found in [1].

**Definition 2.** Let \( \mathbb{T}^* \) be a non-empty subset of the time scale \( \mathbb{T} \) and \( t_0 \in \mathbb{T}^* \) a fixed number such that there exist operators \( \delta_\pm : [t_0, \infty) \mathbb{T} \times \mathbb{T}^* \rightarrow \mathbb{T}^* \) satisfying the following properties:

P.1 The functions \( \delta_\pm \) are strictly increasing with respect to their second arguments, i.e., if
\[
(T_0, t), (T_0, u) \in D_\pm := \{(s, t) \in [t_0, \infty) \mathbb{T} \times \mathbb{T}^* : \delta_\pm(s, t) \in \mathbb{T}^* \},
\]
then
\[
T_0 \leq t < u \text{ implies } \delta_\pm(T_0, t) < \delta_\pm(T_0, u),
\]

P.2 If \( (T_1, u), (T_2, u) \in D_- \) with \( T_1 < T_2 \), then
\[
\delta_-(T_1, u) > \delta_-(T_2, u),
\]
and if \( (T_1, u), (T_2, u) \in D_+ \) with \( T_1 < T_2 \), then
\[
\delta_+(T_1, u) < \delta_+(T_2, u),
\]

P.3 If \( t \in [t_0, \infty) \mathbb{T} \), then \( (t, t_0) \in D_+ \) and \( \delta_+(t, t_0) = t \). Moreover, if \( t \in \mathbb{T}^* \), then \( (t_0, t) \in D_+ \) and \( \delta_+(t_0, t) = t \) holds,

P.4 If \( (s, t) \in D_+ \), then \( (s, \delta_+(s, t)) \in D_\pm \) and \( \delta_+(s, \delta_+(s, t)) = t \),

P.5 If \( (s, t) \in D_\pm \) and \( (u, \delta_\pm(s, t)) \in D_\mp \), then \( (s, \delta_\pm(u, t)) \in D_\pm \) and
\[
\delta_\mp(u, \delta_\pm(s, t)) = \delta_\pm(s, \delta_\mp(u, t)).
\]

Then the operators \( \delta_- \) and \( \delta_+ \) associated with \( t_0 \in \mathbb{T}^* \) (called the initial point) are said to be backward and forward shift operators on the set \( \mathbb{T}^* \), respectively. The variable \( s \in [t_0, \infty) \mathbb{T} \) in \( \delta_\pm(s, t) \) is called the shift size. The values \( \delta_\pm(s, t) \) and \( \delta_- (s, t) \) in \( \mathbb{T}^* \) indicate \( s \) units translation of the term \( t \in \mathbb{T}^* \) to the right and left, respectively. The sets \( D_\pm \) are the domains of the shift operators \( \delta_\pm \), respectively.
**Definition 3.** Let $\mathbb{T}$ be a time scale having an initial point such that there exist operators $\delta_{\pm} : [t_0, \infty)_{\mathbb{T}} \times \mathbb{T} \to \mathbb{T}$ satisfying P.3-P.5. A point $t^* (\neq t_0) \in \mathbb{T}$ is said to be a sticky point of $\mathbb{T}$ if

$$
\delta_{\pm}(s,t^*) = t^* \text{ for all } s \in [t_0, \infty)_{\mathbb{T}} \text{ with } (s,t^*) \in D_{\pm}.
$$

Hereafter, let $t^*$ and $\mathbb{T}^*$ denote the sticky point and the largest subset of $\mathbb{T}$ without sticky point, respectively.

**Corollary 1.** A sticky point $t^*$ cannot be included in the interval $[t_0, \infty)_{\mathbb{T}}$.

**Proof.** First, by P.3-P.5 we have $\delta_-(u,u) = \delta_-(u,\delta_+(u,t_0)) = t_0$, and hence, $(u,u) \in D_-$ for all $u \in [t_0, \infty)_{\mathbb{T}}$. If $t^* \in [t_0, \infty)_{\mathbb{T}}$ is a sticky point, then P.3-P.5 imply

$$
t^* = \delta_-(t^*, t^*) = t_0 \in \mathbb{T}^* = \mathbb{T} - \{t^*\}.
$$

This leads to a contradiction. $\square$

**Example 1.** Let $\mathbb{T} = \mathbb{R}$ and $t_0 = 1$. The operators

$$
\delta_-(s,t) = \begin{cases} 
t/s & \text{if } t \geq 0, \\
t & \text{if } t < 0
\end{cases}, \quad \text{for } s \in [1, \infty)
$$

(2.1)

and

$$
\delta_+(s,t) = \begin{cases} 
st & \text{if } t \geq 0, \\
t/s & \text{if } t < 0
\end{cases}, \quad \text{for } s \in [1, \infty)
$$

(2.2)

are backward and forward shift operators associated with the initial point $t_0 = 1$. Also, $t^* = 0$ is a sticky point (i.e. $\mathbb{T}^* = \mathbb{R} - \{0\}$) since

$$
\delta_\pm(s,0) = 0 \text{ for all } s \in [1, \infty).
$$

In the table below, we state different time scales with their corresponding shift operators.

| $\mathbb{T}$ | $t_0$ | $t^*$ | $\mathbb{T}^*$ | $\delta_-(s,t)$ | $\delta_+(s,t)$ |
|-------------|------|------|---------------|----------------|----------------|
| $\mathbb{R}$ | 0    | $\mathbb{R}$ | $\mathbb{R}$ | $t - s$         | $t + s$         |
| $\mathbb{Z}$ | 0    | $\mathbb{Z}$ | $\mathbb{Z}$ | $t - s$         | $t + s$         |
| $\mathbb{Q}^2 \cup \{0\}$ | 1    | 0    | $\mathbb{Q}^2$ | $t - s$         | $t + s$         |
| $\mathbb{N}^{1/2}$ | 0    | $\mathbb{N}^{1/2}$ | $\mathbb{N}^{1/2}$ | $\sqrt{t^2 - s^2}$ | $\sqrt{t^2 + s^2}$ |

The proof of the next lemma is a direct consequence of Definition 2.

**Lemma 2.** Let $\delta_-$ and $\delta_+$ be the shift operators associated with the initial point $t_0$. We have

i. $\delta_-(t,t) = t_0$ for all $t \in [t_0, \infty)_{\mathbb{T}}$.

ii. $\delta_-(t_0,t) = t$ for all $t \in \mathbb{T}^*$.

iii. If $(s,t) \in D_+$, then $\delta_+(s,t) = u$ implies $\delta_-(s,u) = t$. Conversely, if $(s,u) \in D_-$, then $\delta_-(s,u) = t$ implies $\delta_+(s,t) = u$.

iv. $\delta_+(t,\delta_-(s,t_0)) = \delta_-(s,t)$ for all $(s,t) \in D(\delta_+) \text{ with } t \geq t_0$.

v. $\delta_+(u,t) = \delta_+(t,u)$ for all $(u,t) \in ([t_0, \infty)_{\mathbb{T}} \times [t_0, \infty)_{\mathbb{T}}) \cap D_+$.

vi. $\delta_+(s,t) \in [t_0, \infty)_{\mathbb{T}}$ for all $(s,t) \in D_+$ with $t \geq t_0$.

vii. $\delta_-(s,t) \in [t_0, \infty)_{\mathbb{T}}$ for all $(s,t) \in ([t_0, \infty)_{\mathbb{T}} \times [s, \infty)_{\mathbb{T}}) \cap D_-$.

viii. If $\delta_+(s,\cdot)$ is $\Delta$-differentiable in its second variable, then $\delta_+(s,\cdot) > 0$.

ix. $\delta_+(\delta_-(u,s),\delta_-(s,v)) = \delta_-(u,v)$ for all $(s,v) \in ([t_0, \infty)_{\mathbb{T}} \times [s, \infty)_{\mathbb{T}}) \cap D_-$ and $(u,s) \in ([t_0, \infty)_{\mathbb{T}} \times [u, \infty)_{\mathbb{T}}) \cap D_-$.
x. If \((s, t) \in D_- \) and \(\delta_-(s, t) = t_0\), then \(s = t\).

Proof. (i) is obtained from P.3-5 since
\[\delta_-(t, t) = \delta_-(t, \delta_+(t, t_0)) = t_0\text{ for all } t \in T^*\.

(ii) is obtained from P.3-P.4 since
\[\delta_-(t_0, t) = \delta_-(t_0, \delta_+(t_0, t)) = t.\]

Let \(u := \delta_+(s, t)\). By P.4 we have \((s, u) \in D_-\) for all \((s, t) \in D_+\), and hence,
\[\delta_-(s, u) = \delta_-(s, \delta_+(s, t)) = t.\]

The latter part of (iii) can be done similarly. We have (iv) since P.3 and P.5 yield
\[\delta_+(t, \delta_-(s, t_0)) = \delta_-(s, \delta_+(t, t_0)) = \delta_-(s, t).\]

P.3 and P.5 guarantee that
\[t = \delta_+(t, t_0) = \delta_+(t, \delta_-(u, u)) = \delta_-(u, \delta_+(t, u))\]
for all \((u, t) \in ([t_0, \infty)_T \times [t_0, \infty)_T) \cap D_+\). Using (iii) we have
\[\delta_+(u, t) = \delta_+(u, \delta_-(u, \delta_+(t, u))) = \delta_+(t, u).\]

This proves (v). To prove (vi) and (vii) we use P.1-2 to get
\[\delta_+(s, t) \geq \delta_+(t_0, t) = t \geq t_0\]
for all \((s, t) \in ([t_0, \infty) \times [t_0, \infty)_T) \cap D_+\) and
\[\delta_-(s, t) \geq \delta_-(s, s) = t_0\]
for all \((s, t) \in ([t_0, \infty)_T \times [s, \infty)_T) \cap D_-\). Since \(\delta_+(s, t)\) is strictly increasing in its second variable we have (viii) by [11, Corollary 1.16]. (ix) is proven as follows: from P.5 and (v) we have
\[\delta_+(\delta_-(u, s), \delta_-(s, v)) = \delta_-(s, \delta_+(v, \delta_-(u, s)))\]
\[= \delta_-(s, \delta_-(u, \delta_+(v, s)))\]
\[= \delta_-(s, \delta_+(s, \delta_-(u, v)))\]
\[= \delta_-(u, v)\]
for all \((s, v) \in ([t_0, \infty)_T \times [s, \infty)_T) \cap D_-\) and \((u, s) \in ([t_0, \infty)_T \times [u, \infty)_T) \cap D_-\). Suppose \((s, t) \in D_- = \{(s, t) \in [t_0, \infty)_T \times T^* : \delta_-(s, t) \in T^*\}\) and \(\delta_-(s, t) = t_0\). Then by P.4 we have
\[t = \delta_+(s, \delta_-(s, t)) \in \delta_+(s, t_0) = s.\]

This is (x). The proof is complete. \(\Box\)

Notice that the shift operators \(\delta_\pm\) are defined once the initial point \(t_0 \in T^*\) is known. For instance, we choose the initial point \(t_0 = 0\) to define shift operators \(\delta_\pm(s, t) = t \pm s\) on \(T = \mathbb{R}\). However, if we choose \(\lambda \in (0, \infty)\) as the initial point, then the new shift operators associated with \(\lambda\) are defined by \(\tilde{\delta}_\pm(s, t) = t \mp \lambda \pm s\). In terms of \(\tilde{\delta}_\pm\), the operators \(\delta_\pm\) can be given as
\[\tilde{\delta}_\pm(s, t) = \delta_\mp(\lambda, \delta_\pm(s, t)).\]
Example 2. In the following, we give some particular time scales to show
the change in the formula of shift operators as the initial point changes.

|                  | $T = \mathbb{N}^{1/2}$ | $T = h\mathbb{Z}$ | $T = 2\mathbb{N}$ |
|------------------|-------------------------|-------------------|------------------|
| $t_0$            | $0$                     | $0$               | $1$              |
| $\delta_-(s,t)$ | $\sqrt{T^2 - s^2}$     | $t - s$           | $\frac{t}{s}$   |
| $\delta_+(s,t)$ | $\sqrt{T^2 + s^2}$     | $t + s$           | $2^{\lambda}ts^{-1}$ |

where $\lambda \in \mathbb{Z}_+$, $\mathbb{N}^{1/2} := \{\sqrt{n} : n \in \mathbb{N}\}$, $2\mathbb{N} := \{2^n : n \in \mathbb{N}\}$, and $h\mathbb{Z} := \{hn : n \in \mathbb{Z}\}$.

3. Delay function

In this section we introduce the delay function on time scales that will be
used for the construction of the Lyapunov functional.

Definition 4. Let $T$ be a time scale that is unbounded above and $t_0 \in T^*$
an element such that there exist the shift operators $\delta_\pm : [t_0, \infty) \times T^* \to T^*$
associated with $t_0$. Suppose that $h \in (t_0, \infty)_T$ is a constant such that $(h,t) \in
D_\pm$ for all $t \in [t_0, \infty)_T$, the function $\delta_-(h,t)$ is differentiable with an rd-
continuous derivative, and $\delta_-(h,t)$ maps $[t_0, \infty)_T$ onto $[\delta_-(h,t_0), \infty)_T$. Then
the function $\delta_-(h,t)$ is called the delay function generated by the shift $\delta_-$ on
the time scale $T$.

It is obvious from P.2 and (iii) of Lemma 2 that

$$\delta_-(h,t) < \delta_-(t,t_0) = t \text{ for all } t \in [t_0, \infty)_T. \quad (3.1)$$

Notice that $\delta_-(.,.)$ is strictly increasing and it is invertible. Hence, by P.4-5
$\delta_-(h,t) = \delta_+(h,t)$.

Hereafter, we shall suppose that $T$ is a time scale with the delay function
$\delta_-(h,.) : [t_0, \infty)_T \to [\delta_-(h,t_0), \infty)_T$, where $t_0 \in T$ is fixed. Denote by $T_1$
and $T_2$ the sets

$$T_1 = [t_0, \infty)_T \text{ and } T_2 = \delta_-(h,T_1). \quad (3.2)$$

Evidently, $T_1$ is closed in $\mathbb{R}$. By definition we have $T_2 = [\delta_-(h,t_0), \infty)_T$. Hence, $T_1$ and $T_2$ are both time scales. Let $\sigma_1$ and $\sigma_2$ denote the forward
jump operators on the time scales $T_1$ and $T_2$, respectively. By (3.1) and (3.2)

$$T_1 \subset T_2 \subset T.$$ 

Thus,

$$\sigma(t) = \sigma_2(t) \text{ for all } t \in T_2$$

and

$$\sigma(t) = \sigma_1(t) = \sigma_2(t) \text{ for all } t \in T_1.$$ 

That is, $\sigma_1$ and $\sigma_2$ are the restrictions of the forward jump operator $\sigma : T \to T$ to the time scales $T_1$ and $T_2$, respectively, i.e.,

$$\sigma_1 = \sigma|_{T_1} \text{ and } \sigma_2 = \sigma|_{T_2}.$$ 

Recall that the Hilger derivatives $\Delta$, $\Delta_1$, and $\Delta_2$ on the time scales $T$, $T_1$, and $T_2$ are defined in terms of the forward jumps $\sigma$, $\sigma_1$, and $\sigma_2$, respectively. Hence, if $f$ is a differentiable function at $t \in T_2$, then we have

$$f^{\Delta_2}(t) = f^{\Delta_1}(t) = f^{\Delta}(t), \quad \text{for all } t \in T_1.$$
Similarly, if \(a, b \in \mathbb{T}_2\) are two points with \(a < b\) and if \(f\) is a rd-continuous function on the interval \((a, b)\), then
\[
\int_a^b f(s) \Delta_2 s = \int_a^b f(s) \Delta s.
\]

The next result is essential for future calculations.

**Lemma 3.** The delay function \(\delta_-(h,t)\) preserves the structure of the points in \(\mathbb{T}_1\). That is,
\[
\sigma_1(\hat{t}) = \hat{t} \text{ implies } \sigma_2(\delta_-(h,\hat{t})) = \delta_-(h,\hat{t}).
\]
\[
\sigma_1(\hat{t}) > \hat{t} \text{ implies } \sigma_2(\delta_-(h,\hat{t})) > \delta_-(h,\hat{t}).
\]

**Proof.** By definition \(\sigma_1(t) \geq t\) for all \(t \in \mathbb{T}_1\). Thus,
\[
\delta_-(h,\sigma_1(t)) \geq \delta_-(h,t).
\]

Since \(\sigma_2(\delta_-(h,t))\) is the smallest element satisfying
\[
\sigma_2(\delta_-(h,t)) \geq \delta_-(h,t),
\]
we get
\[
\delta_-(h,\sigma_1(t)) \geq \sigma_2(\delta_-(h,t)) \text{ for all } t \in \mathbb{T}_1.
\]
(3.3)

If \(\sigma_1(\hat{t}) = \hat{t}\), then we have
\[
\delta_-(h,\hat{t}) = \delta_-(h,\sigma_1(\hat{t})) \geq \sigma_2(\delta_-(h,\hat{t})).
\]
That is,
\[
\delta_-(h,\hat{t}) = \sigma_2(\delta_-(h,\hat{t})).
\]

If \(\sigma_1(\hat{t}) > \hat{t}\), then
\[
(\hat{t},\sigma_1(\hat{t}))_{\mathbb{T}_1} = (\hat{t},\sigma_1(\hat{t}))_{\mathbb{T}} = \emptyset
\]
and
\[
\delta_-(h,\sigma_1(\hat{t})) > \delta_-(h,\hat{t}).
\]

Suppose the contrary. That is \(\delta_-(h,\hat{t})\) is right dense; namely \(\sigma_2(\delta_-(h,\hat{t})) = \delta_-(h,\hat{t})\). This along with (3.3) implies
\[
(\delta_-(h,\hat{t}),\delta_-(h,\sigma_1(\hat{t}))_{\mathbb{T}_2} \neq \emptyset.
\]

Pick one element \(s \in (\delta_-(h,\hat{t}),\delta_-(h,\sigma_1(\hat{t}))_{\mathbb{T}_2}\). Since \(\delta_-(h,t)\) is strictly increasing in \(t\) and invertible, there should be an element \(t \in (\hat{t},\sigma_1(\hat{t}))_{\mathbb{T}_1}\) such that \(\delta_-(h,t) = s\). This leads to a contradiction. Hence, \(\delta_-(h,\hat{t})\) must be right scattered. \(\square\)

Using the preceding lemma and applying the fact that \(\sigma_2(u) = \sigma(u)\) for all \(u \in \mathbb{T}_2\) we arrive at the following result.

**Corollary 2.** We have
\[
\delta_-(h,\sigma_1(t)) = \sigma_2(\delta_-(h,t)) \text{ for all } t \in \mathbb{T}_1.
\]
Thus,
\[
\delta_-(h,\sigma(t)) = \sigma(\delta_-(h,t)) \text{ for all } t \in \mathbb{T}_1.
\]
(3.4)
By (3.4) we have
\[ \delta_-(h, \sigma(s)) = \sigma(\delta_-(h, s)) \text{ for all } s \in [t_0, \infty)_\tau. \]
Substituting \( s = \delta_+(h, t) \) we obtain
\[ \delta_-(h, \sigma(\delta_+(h, t))) = \sigma(\delta_-(h, \delta_+(h, t))) = \sigma(t). \]
This and (iv) of Lemma 2 imply
\[ \sigma(\delta_+(h, t)) = \delta_+(h, \sigma(t)) \text{ for all } t \in [\delta_-(h, t_0), \infty)_\tau. \]

**Example 3.** In the following, we give some time scales with their shift operators:

| \( \mathbb{T} \) | \( h \) | \( \delta_-(h, t) \) | \( \delta_+(h, t) \) |
|----------------|-------|----------------|----------------|
| \( \mathbb{R} \) | \( \in \mathbb{R}_+ \) | \( t-h \) | \( t+h \) |
| \( \mathbb{Z} \) | \( \in \mathbb{Z}_+ \) | \( t-h \) | \( t+h \) |
| \( \mathbb{Q} \cup \{0\} \) | \( \in \mathbb{Q}_+ \) | \( \frac{1}{h} \) | \( ht \) |
| \( \mathbb{N}\cup\{0\} \) | \( \in \mathbb{Z}_+ \) | \( \sqrt{t^2-h^2} \) | \( \sqrt{t^2+h^2} \) |

**Example 4.** There is no delay function \( \delta_-(h, : [0, \infty)_\tau \to [\delta_-(h, 0), \infty)_\tau \) on the time scale \( \mathbb{T} = (-\infty, 0) \cup [1, \infty) \).

Suppose the contrary that there exists such a delay function on \( \mathbb{T} \). Then since 0 is right scattered in \( \mathbb{T}_1 := [0, \infty)_\tau \) the point \( \delta_-(h, 0) \) must be right scattered in \( \mathbb{T}_2 = [\delta_-(h, 0), \infty)_\tau \), i.e., \( \sigma_2(\delta_-(h, 0)) > \delta_-(h, 0) \). Since \( \sigma_2(t) = \sigma(t) \) for all \( t \in [\delta_-(h, 0), 0)_\tau \), we have
\[ \sigma(\delta_-(h, 0)) = \sigma_2(\delta_-(h, 0)) > \delta_-(h, 0). \]
That is, \( \delta_-(h, 0) \) must be right scattered in \( \mathbb{T} \). However, in \( \mathbb{T} \), we have \( \delta_-(h, 0) < 0 \), that is, \( \delta_-(h, 0) \) is right dense. This leads to a contradiction.

**Theorem 2.** (Substitution) Assume \( \nu : \mathbb{T} \to \mathbb{R} \) is strictly increasing and \( \mathbb{T} := \nu(\mathbb{T}) \) is a time scale. If \( f : \mathbb{T} \to \mathbb{R} \) is an rd-continuous function and \( \nu \) is differentiable with rd-continuous derivative, then for \( a, b \in \mathbb{T} \),
\[ \int_a^b g(t, s) \nu^\Delta(s) \Delta s = \int_{\nu(a)}^{\nu(b)} g(t, \nu^{-1}(s)) \Delta s, \quad (3.5) \]
First, since the operator \( \delta : [t_0, \infty)_\tau \to [\delta(t_0), \infty)_\tau \) is strictly increasing, it is bijection. If we substitute \( \nu(t) = \delta_-(h, t) \)
\[ f(t, s) = g(t, \delta_-(h, s)) = g(t, \delta_+(h, s)) \]
into (3.5), we obtain
\[ \int_a^b f(t, \delta_-(h, s)) \delta_+^\Delta(h, s) \Delta s = \int_{\delta_-(h, a)}^{\delta_-(h, b)} f(t, s) \Delta s, \quad (3.6) \]
for $a, b \in T_1$. For any $t \in T_1$, we have $[\delta_-(h,t_0), t]_{T_1} \subset T_2$. This and (3.6) yield
\[
\int_{\delta_-(h,t)}^{t} f(t,s) \Delta s = \int_{\delta_-(h,t)}^{t} f(t,s) \Delta_2 s = \int_{\delta_-(h,t_0)}^{t} f(t,s) \Delta_2 s + \int_{\delta_-(h,t_0)}^{t} f(t,s) \Delta_2 s = \int_{t_0}^{t} f(t,\delta_-(h,s)) \Delta_1 (h,s) \Delta_2 s + \int_{\delta_-(h,t_0)}^{t} f(t,s) \Delta_2 s.
\]

(3.7)

The the formula
\[
\left[ \int_{\delta_-(h,t)}^{t} f(t,s) \Delta s \right]^{\Delta} = f(\sigma(t),t) - f(\sigma(t),\delta_-(h,t)) \Delta \Delta_2 (h,t)
\]
\[+ \int_{\delta_-(h,t)}^{t} f^\Delta (t,s) \Delta s
\]

(3.8)

follows from (3.7) and Theorem 1

**Theorem 3.** Let $k$ be an rd-continuous function. Then
\[
\int_{\delta_-(h,t)}^{t} \Delta s \int_{s}^{t} k(u) \Delta u = \int_{\delta_-(h,t)}^{t} \Delta u \int_{\delta_-(h,t)}^{u} k(u) \Delta s.
\]

(3.9)

**Proof.** Substituting
\[f(s) = s - \delta_-(h,t), \quad g(s) = \int_{s}^{t} k(u) \Delta u\]

into the formula
\[\int_{a}^{z} f(\sigma(x)) g(x) \Delta x = [f(x)g(x)]_{a}^{z} - \int_{a}^{z} f^\Delta (x) g(x) \Delta x\]

(see [10] Theorem 1.77) and using Lemma 2 we get
\[
\int_{\delta_-(h,t)}^{t} \Delta s \int_{s}^{t} k(u) \Delta u = \int_{\delta_-(h,t)}^{t} [\sigma(s) - \delta_-(h,t)] k(s) \Delta s
\]
\[= \int_{\delta_-(h,t)}^{t} \Delta u \int_{\delta_-(h,t)}^{\sigma(u)} k(u) \Delta s.
\]

(3.10)

4. **Stability analysis using Lyapunov’s method**

Let $T$ be a time scale having a delay function $\delta_-(h,t)$ where $h \geq t_0$ and $t_0 \in T$ is nonnegative and fixed. In this section we consider the equation
\[x^\Delta(t) = a(t)x(t) + b(t)x(\delta_-(h,t)) \delta_-(h,t), \quad t \in [t_0, \infty)_T
\]

(4.1)

and assume that
\[|\delta_-(h,t)| \leq M < \infty \text{ for all } t \in [t_0, \infty)_T.
\]

(4.2)
Lemma 4. Let

\[ A(t) := x(t) + \int_{\delta_-(h,t)}^{t} b(\delta_+(h,s))x(s)\Delta s \]  

(4.4)

and

\[ \beta(t) := t - \delta_-(h,t). \]  

(4.5)

Assume that there exists a \( \lambda > 0 \) such that

\[ -\frac{\lambda \delta_-(h,t)}{\beta(t) + \lambda [\beta(t) + \mu(t)]} \leq Q(t) \leq -\lambda [\beta(t) + \mu(t)] b(\delta_+(h,t))^2 - \mu(t)Q^2(t) \]  

(4.6)

for all \( t \in [t_0, \infty)_\tau \). If

\[ V(t) = A(t)^2 + \lambda \int_{\delta_-(h,t)}^{t} \Delta s \int_{\delta_-(h,t)}^{s} b(\delta_+(h,u))^2 x(u)^2 \Delta u \]  

(4.7)

then, along the solutions of Eq. (4.1) we have

\[ V^\Delta(t) \leq Q(t)V(t) \]  

for all \( t \in [t_0, \infty)_\tau \).  

(4.8)

Proof. It is obvious from (4.3) and (4.4) that

\[ A^\Delta(t) = Q(t)x(t). \]

Then by (3.8) and the formula \( A(\sigma(t)) = A(t) + \mu(t)A(t) \) we have

\[ V^\Delta(t) = \left[ A(t) + A(\sigma(t)) \right] A^\Delta(t) + \lambda \int_{\delta_-(h,t)}^{\sigma(t)} b(\delta_+(h,u))^2 x(u)^2 \Delta u \]

\[ - \lambda \delta_-(h,t) \int_{\delta_-(h,t)}^{\sigma(t)} b(\delta_+(h,u))^2 x(u)^2 \Delta u + \lambda (t - \delta_-(h,t)) b(\delta_+(h,t))^2 x(t)^2 \]

\[ = [2A(t) + \mu(t)Q(t)x(t)] Q(t)x(t) - \lambda \delta_-(h,t) \int_{\delta_-(h,t)}^{\sigma(t)} b(\delta_+(h,u))^2 x(u)^2 \Delta u \]

\[ + \lambda [\beta(t) + \mu(t)] b(\delta_+(h,t))^2 x(t)^2. \]

Using the identity

\[ 2A(t)x(t) = x^2(t) + A^2(t) - \left( \int_{\delta_-(h,t)}^{t} b(\delta_+(h,s))x(s)\Delta s \right)^2 \]

(4.9)
and condition (4.10) we have

\[ V^\Delta(t) = Q(t)V(t) + R(t) + x^2(t) \left[ \lambda (\beta(t) + \mu(t)) b(\delta_+(h,t))^2 + Q(t) + \mu(t)Q^2(t) \right] \leq Q(t)V(t) + R(t), \]  

(4.10)

where

\[
R(t) = -Q(t) \left( \int_{\delta_-(h,t)}^t b(\delta_+(h,s))x(s)\Delta s \right)^2 \\
- \lambda \delta_+^2(h,t) \int_{\delta_-(h,t)}^{\sigma(t)} b(\delta_+(h,u))^2x(u)^2\Delta u \\
- \lambda Q(t) \int_{\delta_-(h,t)}^t \Delta s \int_{\delta_-(h,t)}^s b(\delta_+(h,u))^2x(u)^2\Delta u. 
\]

(4.11)

Hereafter, we will show that (4.5) implies \( R(t) \leq 0 \). This and (4.11) will enable us to derive the desired inequality (4.8). First we have

\[
\int_{\delta_-(h,t)}^{\sigma(t)} b(\delta_+(h,u))^2x(u)^2\Delta u \geq \int_{\delta_-(h,t)}^t b(\delta_+(h,s))^2x(s)^2\Delta s. 
\]

(4.12)

From Hölder’s inequality [10, Theorem 6.13] we get

\[
\left( \int_{\delta_-(h,t)}^t b(\delta_+(h,s))x(s)\Delta s \right)^2 \leq \beta(t) \int_{\delta_-(h,t)}^t b(\delta_+(h,s))^2x(s)^2\Delta s. \]

(4.13)

On the other hand, (3.9) yields

\[
\int_{\delta_-(h,t)}^t \Delta s \int_{\delta_-(h,t)}^s b(\delta_+(h,u))^2x(u)^2\Delta u = \int_{\delta_-(h,t)}^t \Delta u \int_{\sigma(t)}^{\delta_-(h,t)} b(\delta_+(h,u))^2x(u)^2\Delta s \\
= \int_{\delta_-(h,t)}^t [\sigma(u) - \delta_-(h,t)] b(\delta_+(h,u))^2x(u)^2\Delta u \\
\leq [\beta(t) + \mu(t)] \int_{\delta_-(h,t)}^t b(\delta_+(h,u))^2x(u)^2\Delta u. 
\]

(4.14)

Substituting (4.13) and (4.14) into (4.11) and using (4.12) together with \( \delta_+^2(h,t) > 0 \) we deduce

\[
R(t) \leq - \left\{ (\beta(t) + [\lambda \beta(t) + \mu(t)]) Q(t) + \lambda \delta_+^2(h,t) \right\} \int_{\delta_-(h,t)}^t b(\delta_+(h,s))^2x(s)^2\Delta s. 
\]
Hence, using the left-hand side of (1.6) we arrive at the inequality \( R(t) \leq 0 \). The proof is complete. \( \square \)

In preparation for the proof of the next theorem we state the following lemma.

**Lemma 5.** If \( \varphi \in \mathcal{R}^+ \), then

\[
0 < e_{\varphi}(t, s) \leq \exp \left( \int_s^t \varphi(r) \Delta r \right) \tag{4.15}
\]

for all \( t \in [s, \infty)_T \).

**Theorem 4.** Let \( a \in \mathcal{R}^+ \) and \( Q \in \mathcal{R} \). Suppose the hypothesis of Lemma 4.

If there exists an \( \alpha \in (t_0, h)_T \) such that

\[
(a, t) \in D_{\pm} \text{ for all } t \in [t_0, \infty)_T \tag{4.16}
\]

and

\[
\delta_{\pm}(h, t) \leq \frac{\delta_{\pm}(\alpha, t) + \delta_{\pm}(h, \delta_{\pm}(\alpha, t))}{2} \text{ for all } t \in [\alpha, \infty)_T, \tag{4.17}
\]

then any solution \( x(t) = x(t, t_0, \varphi) \) of (4.1) satisfies the exponential inequalities

\[
|x(t)| \leq \frac{2}{\lambda \Delta(t)} V(t_0) e^{\frac{\lambda}{2} \int_{t_0}^{\delta_{\pm}(\alpha, t)} Q(s) \Delta s} \tag{4.18}
\]

for all \( t \in [\alpha, \infty)_T \) and

\[
|x(t)| \leq \lVert \psi \rVert e^{\int_{t_0}^{t} a(s) \Delta s} \left[ 1 + M \int_{t_0}^{t} b(s) \Delta s \right] \left[ 1 + \int_{t_0}^{t} \frac{b(s)}{1 + \mu(s) a(s)} \Delta s \right] e^{-\int_{t_0}^{\delta_{\pm}(\alpha, t)} a(u) \Delta u} \tag{4.19}
\]

for all \( t \in [t_0, \alpha)_T \), where \( M \) is as defined by (4.2),

\[
\xi(t) := 1 + \frac{\lambda \Lambda(t)}{\beta(t)} > 1,
\]

and \( \Lambda(t) := \delta_{\pm}(h, t) - \delta_{\pm}(h, \delta_{\pm}(\alpha, t)) \).

**Proof.** Since \( t_0 < \alpha < h \) the condition (4.17) implies

\[
\delta_{\pm}(h, t) < \delta_{\pm}(\alpha, t) \text{ for all } t \in [\alpha, \infty)_T \tag{4.20}
\]

and

\[
0 < \Lambda(t) \leq \delta_{\pm}(\alpha, t) - \delta_{\pm}(h, t) \text{ for all } t \in [\alpha, \infty)_T. \tag{4.21}
\]

Let \( V(t) \) be defined by (4.17). First we get by (3.10), (4.7) and (4.20) \( \Leftrightarrow (4.21) \) that

\[
V(t) \geq \lambda \int_{\delta_{\pm}(h, t)}^{t} \int_{s}^{t} b(\delta_{\pm}(h, u))^2 x(u)^2 \Delta u \Delta s,
\]

\[
= \lambda \int_{\delta_{\pm}(h, t)}^{t} \int_{s}^{t} [\sigma(u) - \delta_{\pm}(h, t)] b(\delta_{\pm}(h, u))^2 x(u)^2 \Delta u \Delta s,
\]

\[
\geq \lambda \int_{\delta_{\pm}(h, t)}^{t} \int_{s}^{t} [\sigma(u) - \delta_{\pm}(h, t)] b(\delta_{\pm}(h, u))^2 x(u)^2 \Delta u \Delta s,
\]

\[
\geq \lambda [\delta_{\pm}(a, t) - \delta_{\pm}(h, t)] \int_{\delta_{\pm}(a, t)}^{t} b(\delta_{\pm}(h, u))^2 x(u)^2 \Delta u.
\]
This along with (4.21) yields

\[ V(t) \geq \lambda \Lambda(t) \int_{\delta_-(\alpha,t)}^{\alpha,t} b(\delta_+(h,u))^2 x(u)^2 \Delta u \]  

(4.22)

for all \( t \in [\alpha, \infty)_T \). Similarly, we get

\[ V(\delta_-(\alpha,t)) \geq \lambda \int_{\delta_-(\alpha,t)}^{\delta_-(\alpha,t)} \left[ \sigma(u) - \delta_-(h,\delta_-(\alpha,t)) \right] b(\delta_+(h,u))^2 x(u)^2 \Delta u \]

\[ \geq \lambda \int_{\delta_-(h,t)}^{\delta_-(\alpha,t)} \left[ \sigma(u) - \delta_-(h,\delta_-(\alpha,t)) \right] b(\delta_+(h,u))^2 x(u)^2 \Delta u \]

\[ \geq \lambda \Lambda(t) \int_{\delta_-(h,t)}^{\delta_-(\alpha,t)} b(\delta_+(h,u))^2 x(u)^2 \Delta u \]  

(4.23)

for all \( t \in [\alpha, \infty)_T \) since \( \delta_-(\alpha,t) \leq \delta_-(t_0,t) = t \). Utilizing (4.22), (4.23), and (4.27) we obtain

\[ V(t) + V(\delta_-(\alpha,t)) \geq A(t)^2 + \lambda \Lambda(t) \int_{\delta_-(\alpha,t)}^{\alpha,t} b(\delta_+(h,u))^2 x(u)^2 \Delta u \]

\[ + \lambda \Lambda(t) \int_{\delta_-(h,t)}^{\delta_-(\alpha,t)} b(\delta_+(h,u))^2 x(u)^2 \Delta u \]

\[ \geq A(t)^2 + \lambda \Lambda(t) \int_{\delta_-(h,t)}^{\delta_-(\alpha,t)} b(\delta_+(h,u))^2 x(u)^2 \Delta u \]  

(4.24)

for all \( t \in [\alpha, \infty)_T \). Substituting (4.13) and (4.4) into (4.24) we find

\[ V(t) + V(\delta_-(\alpha,t)) \geq \left( 1 - \frac{1}{\xi(t)} \right) x^2(t) \]

\[ + \left[ \frac{1}{V\xi(t)} x(t) + \sqrt{\xi(t) \left( \int_{\delta_-(h,t)}^{\alpha,t} b(\delta_+(h,u)) x(u) \Delta u \right)} \right]^2 \]

\[ \geq \left( 1 - \frac{1}{\xi(t)} \right) x^2(t) \]  

(4.25)

for all \( t \in [\alpha, \infty)_T \). Since \( V^\Delta(t) \leq 0 \), we get by (4.25) that

\[ \left( 1 - \frac{1}{\xi(t)} \right) x^2(t) \leq V(t) + V(\delta_-(\alpha,t)) \leq 2V(\delta_-(\alpha,t)) \]

(4.26)

for all \( t \in [\alpha, \infty)_T \). Multiplying (4.8) by \( e_{\in\mathbb{Q}}(\sigma(s),t_0) \) and integrating the resulting inequality from \( t_0 \) to \( t \) we derive

\[ 0 \geq \int_{t_0}^{t} \left[ V^\Delta(s) - Q(s) V(s) \right] e_{\in\mathbb{Q}}(\sigma(s),t_0) \Delta s \]

\[ = \int_{t_0}^{t} [V(s)e_{\in\mathbb{Q}}(s,t_0)]^\Delta \Delta s \]

\[ = V(t)e_{\in\mathbb{Q}}(t,t_0) - V(t_0). \]  

(4.27)

That is,

\[ V(t) \leq V(t_0)e_{\in\mathbb{Q}}(t,t_0) \text{ for all } t \in [t_0, \infty)_T. \]  

(4.28)
Combining (4.26) and (4.28) we arrive at
\[ x^2(t) \leq \frac{2}{1 - \frac{1}{\xi(t)}} V(t_0)e_Q(\delta_-(\alpha, t), t_0) \]
for all \( t \in [\alpha, \infty)_T \). The hypothesis \( Q \in \mathcal{R} \) and the condition (4.6) guarantee that \( Q(t) \in \mathcal{R}^+ \). Thus, (4.15) implies
\[ |x(t)| \leq \sqrt{\frac{2}{1 - \frac{1}{\xi(t)}} V(t_0)e^{\int_{t_0}^t \delta_-(\alpha,t) Q(s)\Delta s}} \]
for all \( t \in [\alpha, \infty)_T \).

Multiplying (4.11) by \( e_{\sigma(t)}(\sigma(t), t_0) \) and integrating the resulting equation from \( t_0 \) to \( t \) we have
\[ x(t) = x(t_0)e_{\sigma(t)}(t_0, t_0) + \int_{t_0}^t \frac{b(s)}{1 + \mu(s)a(s)} e_{\sigma(t)}(t_0, s)x(\delta_-(h, s))\delta_+^\alpha(h, s)\Delta s. \] (4.29)

Since \( \delta_-(h, t) < \delta_-(\alpha, t) \leq \delta_-(\alpha, \alpha) = t_0 \) for all \( t \in [t_0, \alpha)_T \), (4.15) along with Eq. (4.29) yields
\[
|x(t)| = e_{\sigma(t)}(t_0, t_0) \left[ \psi(t_0) + \int_{t_0}^t \frac{b(s)}{1 + \mu(s)a(s)} e_{\sigma(t)}(t_0, s)\psi(\delta_-(h, s))\delta_+^\alpha(h, s)\Delta s \right]
\leq \| \psi \| \left[ e^{\int_{t_0}^t a(s)\Delta s} + M \int_{t_0}^t \left| \frac{b(s)}{1 + \mu(s)a(s)} \right| e^{\int_{t_0}^s a(u)\Delta u} \Delta s \right]
\leq \| \psi \| e^{\int_{t_0}^t a(s)\Delta s} \left[ 1 + M \int_{t_0}^t \left| \frac{b(s)}{1 + \mu(s)a(s)} \right| e^{\int_{t_0}^s a(u)\Delta u} \Delta s \right].
\]

The proof is complete. \( \square \)

Notice that Theorem 4 does not work for the time scales in which \((t_0, h)_T = \emptyset\).

For instance, let \( T = \mathbb{Z} \), \( t_0 = 0 \), \( \delta_-(h, t) = t - h \) and \( h = 1 \). It is obvious that \((t_0, h)_\mathbb{Z} = (0, 1)_\mathbb{Z} = \emptyset\). That is, there is no \( \alpha \) so that (4.16) and (4.17) hold. In preparation for the proof of the next theorem we give the following lemma.

**Lemma 6.** Let \( T \) be a time scale and \( t_0 \) a fixed point. Suppose that the shift operators \( \delta_\pm(\alpha, t) \) associated with the initial point \( t_0 \) are defined on \( T \). Suppose also that there is a delay function \( \delta_-(h, t) \) defined on \( T \). If \((t_0, h)_T = \emptyset\), then the time scale \( T \) is isolated (i.e., \( T \) consists only of right scattered points). Moreover,
\[ \sigma(t) = \delta_+(h, t) \] (4.30)
for all \( t \in [\delta_-(h, t_0), \infty)_T \) or equivalently
\[ \sigma(\delta_-(h, t)) = t \] (4.31)
for all \( t \in [t_0, \infty)_T \).
Proof. Suppose that \((t_0, h)_\mathcal{T} = \emptyset\). Define \(\delta_+^0(h, t_0) = t_0\) and \(\delta_+^k(h, t_0) = \delta_+(h, \delta_+^{k-1}(h, t_0))\) for \(k \in \mathbb{Z}_+\). Since \(\delta_+(h, t)\) is surjective and strictly increasing we have \[
(\delta_+^{k-1}(h, t_0), \delta_+^k(h, t_0))_\mathcal{T} = \emptyset \quad \text{for all} \quad t_0 \leq t.
\]

Thus, one can show by induction that \[
(\delta_+^{k-1}(h, t_0), \delta_+^k(h, t_0))_\mathcal{T} = \emptyset \quad \text{for all} \quad k \in \mathbb{Z}_+.
\]

That is, \(\sigma(\delta_+^{k-1}(h, t_0)) = \delta_+^k(h, t_0)\) for \(k \in \mathbb{Z}_+\). On the other hand, we can write \[
[t_0, \infty)_\mathcal{T} = \bigcup_{k=1}^{\infty} \sigma(\delta_+^{k-1}(h, t_0), \delta_+^k(h, t_0))_\mathcal{T}.
\]

Hence, for any \(t \in [t_0, \infty)_\mathcal{T}\) there is a \(k_0 \in \mathbb{Z}_+\) so that \(t \in [\delta_+^{k_0-1}(h, t_0), \delta_+^{k_0}(h, t_0))_\mathcal{T}\). By \(4.32\) we have \(t = \delta_+^{k_0-1}(h, t_0)\). This shows that \[
\sigma(t) = \sigma(\delta_+^{k_0-1}(h, t_0)) = \delta_+^{k_0}(h, t_0) = \delta_+(h, \delta_+^{k_0-1}(h, t_0)) = \delta_+(h, t)
\]

for all \(t \in [\delta_-(h, t_0), \infty)_\mathcal{T}\). This along with \(\sigma(\phi(h, t)) = \delta_-(h, \sigma(t))\) yields \(4.31\). The proof is complete. 

\[\square\]

**Theorem 5.** Let \(a \in \mathcal{R}^+, Q \in \mathcal{R}\). Assume the hypothesis of Lemma 4. If \((t_0, h)_\mathcal{T} = \emptyset\), then any solution \(x(t) = x(t, t_0, \varphi)\) of \((4.1)\) satisfies the exponential inequality \[
|x(t)| \leq \sqrt{\left(1 + \frac{1}{\lambda}\right) V(t_0) e^{\frac{1}{2} \int_{t_0}^{t} Q(s) \Delta s}}
\]

for all \(t \in [t_0, \infty)_\mathcal{T}\).

**Proof.** Let \(H\) be defined by \[
H(t) = \int_{\delta_-(h, t)}^{t} \Delta s \int_{s}^{t} b(\delta_+(h, u))^2 x(u)^2 \Delta u.
\]

From \(3.10, 4.31,\) and \(4.33\) we get \[
H(t) = \int_{\delta_-(h, t)}^{t} [\sigma(u) - \delta_-(h, t)] b(\delta_+(h, u))^2 x(u)^2 \Delta u
\]

\[
\geq [\sigma(\delta_-(h, t)) - \delta_-(h, t)] \int_{\delta_-(h, t)}^{t} b(\delta_+(h, u))^2 x(u)^2 \Delta u
\]

\[
= \beta(t) \int_{\delta_-(h, t)}^{t} b(\delta_+(h, u))^2 x(u)^2 \Delta u
\]

\[
\geq \left( \int_{\delta_-(h, t)}^{t} b(\delta_+(h, u)) x(u) \Delta u \right)^2.
\]
Hence, by (4.7) we have
\[
V(t) = A^2(t) + \lambda H(t)
\]
\[
\geq \left( x(t) + \int_{\delta_- (h,t)}^t b(\delta_+ (h, s)) x(s) \Delta s \right)^2 \\
+ \lambda \left( \int_{\delta_- (h,t)}^t b(\delta_+ (h, u)) x(u) \Delta u \right)^2 \\
= \left( 1 - \frac{1}{1 + \lambda} \right) x^2(t) \\
+ \frac{1}{\sqrt{1 + \lambda}} x(t) + \sqrt{1 + \lambda} \left( \int_{\delta_- (h,t)}^t b(\delta_+ (h, u)) x(u) \Delta u \right)^2 \\
\geq \left( 1 - \frac{1}{1 + \lambda} \right) x^2(t).
\]
This along with (4.28) yields
\[
|x(t)| \leq \sqrt{\left( 1 + \frac{1}{\lambda} \right)} V(t_0) e^{\frac{1}{2} \int_{t_0}^t Q(s) \Delta s}.
\]
The proof is complete. \(\Box\)

In the next corollary, we summarize the results obtained in Theorem 4 and Theorem 5.

**Corollary 3.** Assume the hypothesis of Lemma 4. Let \( a \in \mathbb{R}^+ \) and \( Q \in \mathbb{R} \). Suppose that there exists a \( \lambda > 0 \) such that (4.6) holds for all \( t \in [t_0, \infty) \).

1. If there exists an \( \alpha \in (t_0, h) T \) such that (4.16) and (4.17) hold, then any solution \( x(t) = x(t, t_0, \phi) \) of (4.1) satisfies
\[
|x(t)| \leq \sqrt{\left( 1 + \frac{1}{\lambda} \right)} V(t_0) e^{\frac{1}{2} \int_{t_0}^t [\lambda (\beta(s) + \mu(s)) b(\delta_+ (h, s))^2 + \mu(s) Q^2(s)] \Delta s}.
\]

Thus, if
\[
\lim_{t \to \infty} \int_{t_0}^{\delta_-(\alpha, t)} \left[ \lambda (\beta(s) + \mu(s)) b(\delta_+ (h, s))^2 + \mu(s) Q^2(s) \right] \Delta s = \infty,
\]
then the zero solution of Eq. (4.7) is exponentially stable.

2. If \((t_0, h) T = \emptyset \), then any solution \( x(t) = x(t, t_0, \phi) \) of (4.7) satisfies
\[
|x(t)| \leq \sqrt{\left( 1 + \frac{1}{\lambda} \right)} V(t_0) e^{\frac{1}{2} \int_{t_0}^t [\lambda (\beta(s) + \mu(s)) b(\sigma(s))^2 + \mu(s) Q^2(s)] \Delta s}.
\]

Thus, if
\[
\lim_{t \to \infty} \int_{t_0}^t [\lambda (\beta(s) + \mu(s)) b(\sigma(s))^2 + \mu(s) Q^2(s)] \Delta s = \infty,
\]
then the zero solution of Eq. (4.1) is exponentially stable.

Let $q > 1$, $\mathbb{T} = \mathbb{q}^\mathbb{T} = \{0\} \cup \{q^n : n \in \mathbb{Z}\}$, $\delta_-(h, t) = q^{-h}t$, and $h \in \mathbb{Z}_+$. Then, Eq. (4.1) turns into the $q$-difference equation

$$D_q x(t) = a(t)x(t) + b(t)x(q^{-h}t)q^{-h}, \quad t \in \{1, q, q^2, \ldots\},$$

(4.34)

where $D_q x(t) = \frac{x(q^h) - x(t)}{(q-1)^t}$. Next, we use Corollary 3 to derive a stability criteria for the $q$-difference equation (4.34).

**Example 5.** Suppose that $1 + \mu(t)a(t) > 0$, $1 + \mu(t)Q(t) \neq 0$, and

$$-\frac{\lambda q^{-h}}{\omega(t) + \lambda(\omega(t) + \mu(t))} \leq Q(t) \leq -\lambda(\omega(t) + \mu(t)) |b(\delta_+(h, t))|^2 - \mu(t)Q^2(t)$$

for all $t \in \{1, q, q^2, \ldots\}$, where $\omega(t) := t(1 - q^{-h})$ and $\mu(t) = t(q - 1)$.

1. If $(1, q^h)_{q^\mathbb{T}} \neq \emptyset$, then condition (4.17) holds. By Corollary 3, we conclude that any solution $x(t) = x(t, t_0, \varphi)$ of the $q$-difference equation (4.34) satisfies the exponential inequalities

$$|x(t)| \leq \sqrt{2 \left(1 - \frac{1}{\lambda(t)}\right)V(t_0) \exp \left(\frac{1}{2} \sum_{s \in [1, q^{-h}t]_{q^\mathbb{T}}} \mu(s)Q(s)\right)}$$

for all $t \in [q^n, \infty)_{q^\mathbb{T}}$ and

$$|x(t)| \leq \|\psi\| \exp \left(\sum_{s \in [1, t]_{q^\mathbb{T}}} \mu(s)a(s)\right)$$

$$\times \left[1 + \sum_{s \in [1, t]_{q^\mathbb{T}}} G(s) \exp \left(-\sum_{u \in [1, s]_{q^\mathbb{T}}} \mu(u)a(u)\right)\right]$$

for all $t \in [1, q^n)_{q^\mathbb{T}}$, where

$$G(s) := q^{-h}\mu(s) \left|\frac{b(s)}{1 + \mu(s)a(s)}\right|.$$  

Hence, if

$$\lim_{t \to \infty} \sum_{s \in [1, q^{-h}t]_{q^\mathbb{T}}} s^2 \left[\lambda(q - q^{-h})b(q^hs)^2 + (q - 1)Q^2(s)\Delta s\right] = \infty,$$

then the zero solution of Eq. (4.34) is exponentially stable.

2. If $(1, q^h)_{q^\mathbb{T}} = \emptyset$, then $h = 1$ and

$$|x(t)| \leq \sqrt{\left(1 + \frac{1}{\lambda}\right)V(t_0) \exp \left(\frac{1}{2} \sum_{s \in [1, t]_{q^\mathbb{T}}} \mu(s)Q(s)\right)}$$

Hence, if

$$\lim_{t \to \infty} \sum_{s \in [1, t]_{q^\mathbb{T}}} s^2 \left[\lambda(q - q^{-1})b(gs)^2 + (q - 1)Q^2(s)\Delta s\right] = \infty,$$
then the zero solution of Eq. (4.34) is exponentially stable.

In the next result, we will display a Lyapunov functional that involves $|x|^{\Delta}$. Thus, in preparation we have the following. Using the product rule $(fg)^\Delta = f^{\Delta}g^{\sigma} + fg^{\Delta}$ and differentiating both sides of $x^2(t) = |x(t)|^2$ we obtain the derivative $|x(t)|^{\Delta}$ as follows

$$|x|^{\Delta} = \frac{x + x^{\sigma}}{|x| + |x^{\sigma}|} x^{\Delta}$$  \hspace{.5cm} \text{for} \ x \neq 0. \hspace{.5cm} (4.35)

So $|x|^{\Delta}$ depends on $\frac{x(t)}{|x(t)|}$ and $\frac{x(t)}{|x(t)|}$ (i.e., signs of $x$ and $x^{\sigma}$, respectively). Given $x : T \to \mathbb{R}$, let the sets $T_x^+$ and $T_x^-$ be defined by

$$T_x^+ = \{ t \in T : x(t)x^{\sigma}(t) \geq 0 \},$$

$$T_x^- = \{ t \in T : x(t)x^{\sigma}(t) < 0 \},$$

respectively. The set $T_x^-$ consists only of right scattered points of $T$. Since the time scale $T = \mathbb{R}$ has no any right scattered points, we have $T_x^- = \emptyset$. Thus for all differentiable functions $x : \mathbb{R} \to \mathbb{R}$, the formula (4.35) turns into $|x|^\Delta = \frac{1}{|x|}x^\Delta$. However, for an arbitrary time scale (e.g. $T = \mathbb{Z}$) the set $T_x^-$ may not be empty. For simplicity, we need to have a formula for $|x|^\Delta$ which does not include $x^\sigma$. The next result provides a relationship between $|x|^\Delta$ and $\frac{1}{|x|}x^\Delta$. Its proof can be found in [5].

**Lemma 7.** [Lemma 5] Let $x \neq 0$ be $\Delta$-differentiable. Then

$$|x(t)|^\Delta = \begin{cases} \frac{x(t)}{|x(t)|} x^\Delta(t) & \text{if} \ t \in T_x^+ \\ -\frac{2}{\mu(t)}|x(t)| - \frac{x(t)}{|x(t)|} x^\Delta(t) & \text{if} \ t \in T_x^- \end{cases}.$$  \hspace{.5cm} (4.36)

**Theorem 6.** Define a continuous function $\eta(t) \geq 0$ by

$$\eta(t) := \frac{e_a(t, t_0)}{1 + \lambda \int_{\delta_-(h, t_0)}^{t} e_a(\delta_+(h, s), t_0) \Delta s}.$$  \hspace{.5cm} (4.37)

Suppose that $a \in \mathcal{R}^+$ and that

$$|b(t)| - \lambda \eta^\sigma(t) \delta^\Delta(h,t) \leq 0$$  \hspace{.5cm} (4.38)

holds for all $t \in [t_0, \infty)_{\mathbb{T}}$. Then any solution of Eq. (4.1) satisfies the inequality

$$|x(t)| \leq V(t_0, x_{t_0}) e_\gamma(t, t_0)$$  \hspace{.5cm} \text{for all} \ t \in [t_0, \infty)_{\mathbb{T}},$$  \hspace{.5cm} (4.39)

where

$$V(t_0, x_{t_0}) := |x(t_0)| + \lambda \eta(t_0) \int_{\delta_-(h,t_0)}^{t_0} |x(s)| \Delta s,$$

$$\gamma(t) := a(t) + \lambda \tilde{M}\eta^\sigma(t), \hspace{.5cm} \tilde{M} = \max \{1, M\}, \text{and} \ M \hspace{.5cm} \text{is as in (4.2)}.$$  \hspace{.5cm}

**Proof.** For convenience define

$$\zeta(t) := 1 + \lambda \int_{\delta_-(h,t)}^{t} e_a(\delta_+(h, s), t_0) \Delta s.$$  \hspace{.5cm}
This and a differentiation of (4.37) yield

\[ \eta^\Delta(t) = \frac{e_a(t,t_0)}{\zeta(t)} \left( \frac{a \zeta(t) - \zeta^\Delta(t)}{\zeta^\sigma(t)} \right) \]

\[ = \eta(t) \left( \frac{a \zeta(t) + a \mu(t) \zeta^\Delta(t) - a \mu(t) \zeta(t) - \zeta^\Delta(t)}{\zeta(t) + \mu(t) \zeta^\Delta(t)} \right) \]

\[ = a(t) \eta(t) - \left[ (1 + \mu(t) a(t)) \eta(t) \frac{\zeta(t)}{\zeta^\sigma(t)} \right] \frac{\zeta^\Delta(t)}{\zeta(t)} \]

\[ = a(t) \eta(t) - \eta^\sigma(t) \frac{\zeta^\Delta(t)}{\zeta(t)} \]

\[ = a(t) \eta(t) + \eta^\sigma(t) \eta(t) \zeta^\Delta(h,t) - \lambda \eta^\sigma(t) e_a(\delta_+(h,t),t) \]

\[ \leq \eta(t) \left[ a(t) + \lambda \bar{M} \eta^\sigma(t) \right], \quad (4.41) \]

where we also used \( \zeta^\sigma(t) = \zeta(t) + \mu(t) \zeta^\Delta(t) \) and

\[ (1 + \mu(t) a(t)) \eta(t) \frac{\zeta(t)}{\zeta^\sigma(t)} = \eta^\sigma(t). \]

Define

\[ V(t,x_t) := |x(t)| + \lambda \eta(t) \int_{\delta_-(h,t)}^t |x(s)| \Delta s. \quad (4.42) \]

Let \( t \in T^+_x \cap [t_0, \infty) \). Then by (4.36) we have \( |x(t)|^\Delta = \frac{x(t)}{|x(t)|} x^\Delta(t) \). Differentiating (4.42) and utilizing (4.38) and (4.41) we arrive at

\[ \begin{align*}
V^\Delta(t,x_t) &= |x(t)|^\Delta + \lambda \eta^\Delta(t) \int_{\delta_-(h,t)}^t |x(s)| \Delta s \\
&\quad + \lambda \eta^\sigma(t) \left[ |x(t)| - |x(\delta_-(h,t))| \delta^\Delta(h,t) \right] \\
&\leq \frac{x(t)}{|x(t)|} x^\Delta(t) + \lambda \eta(t) \left[ a(t) + \lambda \bar{M} \eta^\sigma(t) \right] \int_{\delta_-(h,t)}^t |x(s)| \Delta s \\
&\quad + \lambda \eta^\sigma(t) \left[ |x(t)| - |x(\delta_-(h,t))| \delta^\Delta(h,t) \right] \\
&= \left( a(t) + \lambda \bar{M} \eta^\sigma(t) \right) |x(t)| + \left( |b(t)| - \lambda \delta^\Delta(h,t) \eta^\sigma(t) \right) |x(\delta_-(h,t))| \\
&\quad + \lambda \eta(t) \left[ a(t) + \lambda \bar{M} \eta^\sigma(t) \right] \int_{\delta_-(h,t)}^t |x(s)| \Delta s \\
&\leq \gamma(t) V(t,x_t). 
\end{align*} \]
Similarly, if \( t \in T_x \cap [t_0, \infty)_T \), then \( |x(t)|^\Delta = -\frac{2}{\mu(t)} |x(t)| - \frac{x(t)}{|x(t)|} x^\Delta(t) \) by (4.36). Hence,
\[
V^\Delta(t, x_t) \leq |x(t)|^\Delta + \eta(t) \left[ a(t) + \lambda \tilde{M} \eta^\sigma(t) \right] \int_{\delta_-(h,t)}^t |x(s)| \Delta s + \lambda \eta^\sigma(t) \left[ |x(t)| - |x(\delta_-(h, t))| \delta^\Delta(h, t) \right]
\]
\[
\leq \left( -\frac{2}{\mu(t)} - a(t) + \lambda \tilde{M} \eta^\sigma(t) \right) |x(t)| + \lambda \eta(t) \left[ a(t) + \lambda \tilde{M} \eta^\sigma(t) \right] \int_{\delta_-(h,t)}^t |x(s)| \Delta s
\]
\[
\leq \left( a(t) + \lambda \tilde{M} \eta^\sigma(t) \right) |x(t)| + \lambda \eta(t) \left[ a(t) + \lambda \tilde{M} \eta^\sigma(t) \right] \int_{\delta_-(h,t)}^t |x(s)| \Delta s
\]
\[
= \gamma(t)V(t, x_t).
\]

since \( 1 + \mu(t)a(t) > 0 \) implies
\[
-\frac{2}{\mu(t)} - a(t) < a(t).
\]

Thus,
\[
V^\Delta(t, x_t) \leq \gamma(t)V(t, x_t) \text{ for all } t \in [t_0, \infty)_T. \quad (4.43)
\]

An integration of (4.43) and applying the fact that \( V(t, x_t) \geq |x(t)| \) we arrive at the desired result. \[\square\]

In the next section we give a criteria for instability.

5. A Criteria for Instability

**Theorem 7.** Suppose there exists positive constant \( D \) such that
\[
\beta(t) < D \leq \frac{Q(t)}{b(\delta_+(h, t))^2}
\]
for all \( t \in [t_0, \infty)_T \), where \( \beta(t) \) is as defined in (4.34). Let the function \( A \) be defined by (4.4). If
\[
V(t) = A(t)^2 - D \int_{\delta_-(h,t)}^t b(\delta_+(h, s))^2 x(s)^2 \Delta s
\]
then along the solutions of Eq. (4.1) we have
\[
V^\Delta(t) \geq Q(t)V(t) \text{ for all } t \in [t_0, \infty)_T. \quad (5.3)
\]
Proof. Let $V$ be defined by (5.2). Using (4.9) and (4.13) we obtain

$$V^\Delta(t) = [A(t) + A(\sigma(t))] A^\Delta(t) - Db(\delta_+(h, t))^2 x(t)^2$$

$$+ Db(t)^2 x(\delta_- (h, t))^2 \delta^\Delta(h, t)$$

$$\geq [2A(t) + \mu(t) Q(t) x(t)] Q(t) x(t) - Db(\delta_+(h, t))^2 x(t)^2$$

$$\geq 2Q(t) A(t) x(t) - Db(\delta_+(h, t))^2 x(t)^2$$

$$= Q(t) \left[ x^2(t) + A^2(t) - \left( \int_{\delta_- (h, t)}^t b(\delta_+(h, s)) x(s) \Delta s \right)^2 \right]$$

$$- Db(\delta_+(h, t))^2 x(t)^2$$

$$\geq Q(t) V(t) + [Q(t) - Db(\delta_+(h, t))^2] x(t)^2.$$ 

This along with (5.1) implies (5.3). $\square$

To prove the next theorem we will need to use the following lemma:

**Lemma 8.** [9, Remarks 2] If $\varphi$ is rd-continuous and nonnegative, then

$$1 + \int_s^t \varphi(u) \Delta u \leq e^{\varphi(t, s)} \leq \exp \left\{ \int_s^t \varphi(u) \Delta u \right\} \text{ for all } t \geq s. \quad (5.4)$$

**Theorem 8.** Suppose all hypotheses of Theorem 7 hold. Suppose also that $\beta(t)$ is bounded above by $\beta_0$ with $0 < \beta_0 < D$. Then the zero solution of Eq. (4.1) is unstable, provided that

$$\lim_{t \to \infty} \int_{t_0}^t b(\delta_+(h, s))^2 \Delta s = \infty.$$

Proof. As we did in (4.27), an integration of (5.3) from $t_0$ to $t$ gives

$$V(t) \geq V(t_0) e^{Q(t, t_0)} \text{ for all } t \in [t_0, \infty). \quad (5.5)$$

Let $V(t)$ be given by (5.2). Then

$$V(t) = x(t)^2 + 2x(t) \int_{\delta_- (h, t)}^t b(\delta_+(h, s)) x(s) \Delta s + \left( \int_{\delta_- (h, t)}^t b(\delta_+(h, s)) x(s) \Delta s \right)^2$$

$$- D \int_{\delta_- (h, t)}^t b(\delta_+(h, s))^2 x(s)^2 \Delta s. \quad (5.6)$$

Let $C := D - \beta_0$. Then from

$$\left( \sqrt{\frac{\beta_0}{C}} K - \sqrt{\frac{C}{\beta_0}} L \right)^2 \geq 0,$$

we have

$$2KL \leq \frac{\beta_0}{C} K^2 + \frac{C}{\beta_0} L^2.$$
With this in mind we arrive at
\[ 2 |x(t)| \int_{\delta_-(h,t)}^t |b(\delta_+(h,s))||x(s)| \Delta s \leq \frac{\beta_0}{C} x^2(t) + \frac{C}{\beta_0} \left( \int_{\delta_-(h,t)}^t b(\delta_+(h,s))x(s) \Delta s \right)^2. \]

A substitution of the above inequality into (5.6) yields
\[ V(t) \leq \left( 1 + \frac{\beta_0}{C} \right) x(t)^2 + \left( 1 + \frac{C}{\beta_0} \right) \left( \int_{\delta_-(h,t)}^t b(\delta_+(h,s))x(s) \Delta s \right)^2 \]
\[- D \int_{\delta_-(h,t)}^t b(\delta_+(h,s))^2 x(s)^2 \Delta s \]
\[= \frac{D}{C} x(t)^2 + \frac{D}{\beta_0} \left( \int_{\delta_-(h,t)}^t b(\delta_+(h,s))x(s) \Delta s \right)^2 \]
\[- D \int_{\delta_-(h,t)}^t b(\delta_+(h,s))^2 x(s)^2 \Delta s. \]

Using (4.13) we find
\[ V(t) \leq \frac{D}{C} x(t)^2. \]

By (5.1), (5.4), and (5.5) we get
\[ |x(t)| \geq \sqrt{\frac{C}{D} V(t_0) e^{Q(t,t_0)}} \]
\[ \geq \sqrt{\frac{C}{D} V(t_0) \left( 1 + \int_{t_0}^t Q(s) \Delta s \right)} \]
\[ \geq \sqrt{CV(t_0) \left( \int_{t_0}^t b(\delta_+(h,s))^2 \Delta s \right)}.
\]

This completes the proof. \(\square\)

We end this paper by comparing our results to the existing ones.

6. Some applications

In [3], by means of Lyapunov’s direct method the authors investigated the stability analysis of the delay dynamic equation
\[ x^A(t) = a(t)x(t) + b(t)x(\delta(t))\delta^A(t), \tag{6.1} \]
where \(a : \mathbb{T} \to \mathbb{R}\) and \(b : \mathbb{T} \to \mathbb{R}\) are functions and \(a \in \mathbb{R}^+\). Moreover, the delay function \(\delta : [t_0, \infty)_\mathbb{T} \to [\delta(t_0), \infty)_\mathbb{T}\) is surjective, strictly increasing and is supposed to have the following properties
\[ \delta(t) < t, \quad \delta^A(t) < \infty, \quad \delta \circ \sigma = \sigma \circ \delta. \]

It is concluded in [3, Theorem 6] that
\[ |b(t)| \leq N \text{ and } a(t) < -N \tag{6.2} \]
are the sufficient conditions guaranteeing stability of the zero solution of Eq. (6.1). Next, we furnish an example to show that Theorem 4 allows us to relax condition (6.2) that leads to exponential stability of zero solution Eq. (6.1).

**Example 6.** Let $\mathbb{T} = \mathbb{R}$, $a(t) = 1$, $b(t) = -\frac{3}{2}$, $\delta(t) = t - \frac{1}{3}$, and $N = 1$. It is obvious that the condition (6.3) does not hold. So, [3, Theorem 6] does not imply the stability of the zero solution of the delayed differential equation

$$x'(t) = x(t) - \frac{3}{2}x(t - \frac{1}{3}).$$

(6.3)

On the other hand, setting $\mathbb{T} = \mathbb{R}$, $\lambda = \frac{1}{3}$, and $\delta_-(h, t) = t - \frac{1}{3}$ Eq. (4.1) turns into (6.3) and the condition (4.6) becomes

$$-\frac{3}{4} \leq Q(t) \leq -\frac{1}{9}b(\delta_+(h, t))^2,$$

which holds for all $t \in [0, \infty)$ since $Q(t) = a(t) + b(\delta_+(h, t)) = -\frac{1}{2}$. One may easily verify that condition (4.17) is satisfied for $\delta_-(\alpha, t) = t - \frac{1}{6}$ and $\delta_-(h, t) = t - \frac{1}{3}$. Thus, we conclude the exponential stability of the zero solution of (6.3) by Corollary 3.

Now, let us consider the equation

$$x^{\Delta}(t) = b(t)x(\delta_-(h, t))\delta^{\Delta}(h, t), \quad t \in [t_0, \infty)_T.$$  

(6.4)

We observe the following by combining Corollary 3 and Theorem 5.

**Remark 1.** Let $b \in \mathcal{R}$. Suppose that there exists a $\lambda > 0$ such that

$$-\frac{\lambda \delta_-(h, t)}{\beta(t) + \lambda [\beta(t) + \mu(t)]} \leq b(\delta_+(h, t)) \leq -b(\delta_+(h, t))^2 [\lambda \beta(t) + (1 + \lambda)\mu(t)],$$

(6.5)

holds for all $t \in [t_0, \infty)_T$.

1. If there exists an $\alpha \in (t_0, h)_T$ such that (4.10) and (4.17) hold and if

$$\lim_{t \to \infty} \int_{t_0}^{\delta_-(\alpha, t)} [\lambda \beta(s) + (1 + \lambda)\mu(s)] b(\delta_+(h, s))^{\Delta} s = \infty,$$

(6.6)

then the zero solution of Eq. (6.4) is exponentially stable.

2. If $(t_0, h)_T = \emptyset$ and if

$$\lim_{t \to \infty} \int_{t_0}^{t} [\lambda \beta(s) + (1 + \lambda)\mu(s)] b(\sigma(s))^{\Delta} s = \infty,$$

then the zero solution of Eq. (6.4) is exponentially stable.

3. Suppose that $a \in \mathcal{R}^+$ and that

$$|b(t)| - \lambda \eta^\tau(t) \delta^{\Delta}(h, t) \leq 0$$

holds for all $t \in [t_0, \infty)_T$, where

$$\eta(t) := \frac{1}{1 + \lambda \beta(t)}.$$

Then any solution of Eq. (4.1) satisfies the inequality

$$|x(t)| \leq V(t_0, x_{t_0})e^{\frac{1}{2} \int_{t_0}^{t} \gamma(s)ds} e_\gamma(t, t_0) \text{ for all } t \in [t_0, \infty)_T,$$
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where

\[ V(t_0, x_{t_0}) := |x(t_0)| + \lambda \eta(t_0) \int_{\delta_-(h,t_0)}^{t_0} |x(s)| \Delta s, \]

\[ \gamma(t) := \lambda \bar{M} \eta^p(t), \quad \bar{M} = \max \{1, M\}, \text{and M is as in (4.2)}. \]

In [3, Theorem 7], the authors utilized fixed point theory and deduced that the conditions

\[ p(t) := b(\delta_+(h, t)) \neq 0 \quad \text{for all } t \in [t_0, \infty)_T, \quad (6.7) \]

and

\[ \lim_{t \to \infty} e_p(t, t_0) = 0, \]

\[ \int_{\delta_-(h, t)}^{t} |p(s)| \Delta s + \int_{t_0}^{t} |\oplus p(s)| e_p(t, s) \left( \int_{\delta_-(h, s)}^{s} |p(u)| \Delta u \right) \Delta s \leq N < 1 \]

lead to stability of solution \( x(t, t_0; \psi) \) of Eq. (6.4). Notice that [3] generalizes all the results of [20].

Moreover, Wang (see [22, Corollary 1]) proposed the inequality

\[ -\frac{1}{2h} \leq a(t) + b(t + h) \leq -hb^2(t + h) \]

as sufficient condition for uniform asymptotic stability of the zero solution of the delay differential equation

\[ x'(t) = a(t) + b(t)x(t - h), \quad h > 0. \]

It can be easily seen that the conditions (6.8-6.9) are not satisfied for the data given in the following example.

**Example 7.** Let \( a(t) = 0, \quad T = \mathbb{R}, \quad \delta_-(h, t) = t - h, \) and \( p < 0 \) be fixed. Then Eq. (4.1) becomes

\[ x'(t) = b(t)x(t - h). \]

We can simplify condition (6.5) as follows

\[ h |p| (2 - e^{pt}) \leq N < 1. \]

If \( h = \frac{2}{3}, \) and \( b(t) = -\frac{9}{10}, \) then (6.7) implies

\[ h |p| (2 - e^{pt}) = \frac{3}{5} \left( 2 - e^{-\frac{9}{10}t} \right) \geq 1 \]

for all \( t \geq -\frac{10}{9} \ln \left( \frac{1}{3} \right) \approx 1.22. \) Thus, the condition (6.10) does not hold. On the other hand, for \( h = \frac{2}{3}, \) and \( \lambda = \frac{3}{2}, \) condition (6.5) turns into

\[ -\frac{9}{10} \leq b(\delta_+(h, t)) \leq -b(\delta_+(h, t))^2. \]

The last inequality holds for \( b(t) = -\frac{9}{10}. \) In addition, setting \( \delta_-(\alpha, t) = t - \frac{4}{3}, \) one may easily verify that conditions (4.10), (4.17), and (6.7) are satisfied. Hence, the first part of Remark [7] yields exponential stability while [3, Theorem 7] and [22, Corollary 1] cannot.
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