Pion Physics at Low Energy and High Accuracy

H. Leutwyler

Institute for Theoretical Physics, University of Bern, Sidlerstr. 5, CH-3012 Switzerland

Abstract. The role of the quark condensate for the low energy structure of QCD is discussed in some detail. In particular, the dependence of $M_\pi$ on $m_u$ and $m_d$ and the low energy theorems for the $\pi\pi$ scattering amplitude are reviewed. The new data on $K_{e4}$ decay beautifully confirm the standard picture, according to which the quark condensate is the leading order parameter of the spontaneously broken chiral symmetry.

1. STANDARD MODEL FOR E $\leq M_W, M_Z, M_H = O(100\text{GeV})$

At energies that are small compared to $\leq M_W, M_Z, M_H = O(100\text{GeV})$, the weak interaction freezes out, because these energies do not suffice to excite the relevant degrees of freedom. As a consequence, the gauge group of the Standard Model, SU(3) $\times$ SU(2) $\times$ U(1), breaks down to the subgroup SU(3) $\times$ U(1) – only the photons, the gluons, the quarks and the charged leptons are active at low energies. Since the neutrini neither carry colour nor charge, they decouple.

The effective Lagrangian relevant at low energies is the one of QCD + QED. The strength of the interaction is characterized by the two coupling constants $g$ and $e$. In contrast to the Standard Model, the SU(3) $\times$ U(1) Lagrangian does contain mass terms: the quark and lepton mass matrices $m_q, m_l$. The field basis may be chosen such that $m_q$ and $m_l$ are diagonal and positive.

The two gauge fields behave in a qualitatively different manner: while the photons do not carry electric charge, the gluons do carry colour. This difference is responsible for the fact that the strong interaction becomes strong at low energies, while the electromagnetic interaction becomes weak there, in fact remarkably weak: the photons and leptons essentially decouple from the quarks and gluons. For the QCD part, on the other hand, perturbation theory is useful only at high energies. In the low energy domain, the strong interaction is so strong that it confines the quarks and gluons. For the same reason, a term in the Lagrangian of the form $\theta G_{\mu\nu} \tilde{G}^{\mu\nu}$ (where $G_{\mu\nu}$ is the gluon field strength) cannot a priori be dismissed, despite the fact that it represents a total derivative: it generates an electric dipole moment in the neutron, for instance. Conversely, the experimental fact that the dipole moment is smaller than $10^{-25}$ ecm implies that (in the basis where the quark mass matrix is diagonal, real and positive) the vacuum angle $\theta$ must be tiny, so that the Lagrangian is invariant under the discrete symmetries $P, C$ and $T$, to a very high degree of accuracy.

The resulting effective low energy theory is mathematically even more satisfactory than the Standard Model as such – it does not involve scalar degrees of freedom and has fewer free parameters. Remarkably, this simple theory must describe the structure
of cold matter to a very high degree of precision, once the parameters in the Lagrangian are known. It in particular explains the size of the atoms in terms of the scale

\[ a_0 = \frac{4\pi}{e^2 m_e} \]

which only contains the two parameters \( e \) and \( m_e \) – these are indeed known to an incredible precision. Unfortunately, our ability to solve the QCD part of the theory is rather limited – in particular, we are still far from being able to demonstrate on the basis of the QCD Lagrangian that the strong interaction actually confines colour. Likewise, our knowledge of the magnitude of the light quark masses leaves to be desired – we need to know these more accurately in order to test ideas that might lead to an understanding of the mass pattern, such as the relations with the lepton masses that emerge from attempts at unifying the electroweak and strong forces.

### 2. SYMMETRIES OF MASSLESS QCD

In the following, I focus on the QCD part and switch the electromagnetic interaction off. It so happens that the interactions of \( u_d s \) with the Higgs fields are weak, so that the masses \( m_u, m_d, m_s \) are small. Let me first set these parameters equal to zero and, moreover, send the masses of the heavy quarks, \( m_c, m_b, m_t \), to infinity. In this limit, the theory becomes a theoreticians paradise: the Lagrangian contains a single parameter, \( g \). In fact, since the value of \( g \) depends on the running scale used, the theory does not contain any dimensionless parameter that would need to be adjusted to observation. In principle, this theory fully specifies all dimensionless observables as pure numbers, while dimensionful quantities like masses or cross sections can unambiguously be predicted in terms of the scale \( \Lambda_{QCD} \) or the mass of the proton. The resulting theory – QCD with three massless flavours – is among the most beautiful quantum field theories we have. I find it breathtaking that, at low energies, nature reduces to this beauty, as soon as the dressing with the electromagnetic interaction is removed and the Higgs condensate is replaced by one that does not hinder the light quarks, but is impenetrable for \( W \) and \( Z \) waves as well as for heavy quarks.

The Lagrangian of the massless theory, \( L_{QCD}^0 \), has a high degree of symmetry that originates in the fact that the interaction among the quarks and gluons is flavour-independent and conserves helicity: \( L_{QCD}^0 \) is invariant under independent flavour rotations of the three right- and left-handed quark fields. These form the group \( G = SU(3)_R \times SU(3)_L \). The corresponding 16 currents \( V_i^\mu = q \gamma^\mu \gamma_5 \lambda_i q \) and \( A_i^\mu = q \gamma^\mu \gamma_5 \lambda_i q \) are conserved, so that their charges commute with the Hamiltonian:

\[ [Q_i^\mu, H_{QCD}^0] = [Q_i^\mu, H_{QCD}^0] = 0 ; \quad i = 1; \ldots; 8 \]

Vafa and Witten [1] have shown that the state of lowest energy is necessarily invariant under the vector charges: \( Q_i^\mu J^\perp = 0 \). For the axial charges, however, there are the two possibilities characterized in table 1.

The observed spectrum does not contain parity doublets. In the case of the lightest meson, the pion, for instance, the lowest state with the same spin and flavour quantum
TABLE 1. Alternative realizations of the symmetry group $G = SU(3)_R \times SU(3)_L$.

| $Q_i^\dagger \bar{Q} \parallel = 0$                | $Q_i^\dagger \bar{Q} \parallel \notin 0$                |
|-------------------------------------------------|-------------------------------------------------|
| Wigner-Weyl realization of $G$                   | Nambu-Goldstone realization of $G$                |
| ground state is symmetric                        | ground state is asymmetric                        |
| $\not\!{\bar q}_R q_L \parallel i = 0$         | $\not\!{\bar q}_R q_L \parallel \notin 0$         |
| ordinary symmetry                                | spontaneously broken symmetry                     |
| spectrum contains parity partners                | spectrum contains Goldstone bosons                |
| degenerate multiplets of $G$                     | degenerate multiplets of SU(3)$_R \times SU(3)_L$ |

numbers, but opposite parity is the $a_0 (980)$. So, experiment rules out the first possibility: for dynamical reasons that yet remain to be understood, the state of lowest energy is an asymmetric state. Since the axial charges commute with the Hamiltonian, there must be eigenstates with the same energy as the ground state:

$$H_{QCD}^0 Q_i^\dagger \bar{Q} \parallel = Q_i^\dagger H_{QCD}^0 \bar{Q} \parallel = 0 :$$

The spectrum must contain 8 states $Q_i^\dagger \bar{Q} \parallel i$; $\cdots$; $Q_8^\dagger \bar{Q} \parallel i$ with $E = P = 0$, describing massless particles, the Goldstone bosons of the spontaneously broken symmetry. Moreover, these must carry spin 0, negative parity and form an octet of SU(3).

3. QUARK MASSES AS SYMMETRY BREAKING PARAMETERS

Indeed, the 8 lightest hadrons, $\pi^+; \pi^0; \pi^-; K^+; K^0; \bar{K}^0; \bar{K}^-; \eta$, do have these quantum numbers, but massless they are not. This has to do with the deplorable fact that we are not living in paradise: the masses $m_u; m_d; m_s$ are different from zero and thus allow the left-handed quarks to communicate with the right-handed ones. The full Hamiltonian is of the form

$$H_{QCD} = H_{QCD}^0 + H_{QCD}^1 ; \quad H_{QCD}^1 = \int d^3x \bar{q}_R m q_L + \bar{q}_L m^\dagger q_R ; \quad m = m_u m_d m_s :$$

The quark masses may be viewed as symmetry breaking parameters: the QCD-Hamiltonian is only approximately symmetric under independent rotations of the right- and left-handed quark fields, to the extent that these parameters are small. Chiral symmetry is thus broken in two ways:

- spontaneously $\not\!{\bar q}_R q_L \parallel \notin 0$
- explicitly $m_u m_d m_s \notin 0$

The consequences of the fact that the explicit symmetry breaking is small may be worked out by means of an effective field theory, “chiral perturbation theory” [2, 3, 4]. In this context, the heavy quarks do not play an important role – as the corresponding fields are singlets under SU(3)$_R \times SU(3)_L$, we may include their contributions in the symmetric part of the Hamiltonian, irrespective of the size of their mass.
Since the masses of the two lightest quarks are particularly small, the Hamiltonian of QCD is almost exactly invariant under the subgroup \(SU(2)_R \times SU(2)_L\). The ground state spontaneously breaks that symmetry to the subgroup \(SU(2)_V\) — the good old isospin symmetry discovered in the thirties of the last century [5]. The pions represent the corresponding Goldstone bosons [6], while the kaons and the \(\eta\) remain massive if the limit \(m_u, m_d \to 0\) is taken at fixed \(m_s\). In the following, I consider this framework and, moreover, ignore isospin breaking, setting \(m_u = m_d = \hat{m}\).

If \(SU(2)_R \times SU(2)_L\) was an exact symmetry, the pions would be strictly massless. According to Gell-Mann, Oakes and Renner [7], the square of the pion mass is proportional to the product of the quark masses and the quark condensate:

\[
M_\pi^2 \approx \frac{1}{F_\pi^2} \langle m_u + m_d \rangle \not\!\not\!\not\!u \not\!\not\!\not\!d = \frac{1}{F_\pi^2} \langle m_u + m_d \rangle \not\!\not\!\not\!u \not\!\not\!\not\!d \not\!\not\!\not\!i \not\!\not\!\not\!j \not\!\not\!\not\!k \not\!\not\!\not\!l \not\!\not\!\not\!m \not\!\not\!\not\!n \not\!\not\!\not\!o \not\!\not\!\not\!p \not\!\not\!\not\!q \not\!\not\!\not\!r \not\!\not\!\not\!s \not\!\not\!\not\!t \not\!\not\!\not\!u \not\!\not\!\not\!v \not\!\not\!\not\!w \not\!\not\!\not\!x \not\!\not\!\not\!y \not\!\not\!\not\!z \] \tag{1}
\]

The factor of proportionality is given by the pion decay constant \(F_\pi\). The term \(m_u + m_d\) measures the explicit breaking of chiral symmetry, while the quark condensate,

\[
\not\!\not\!\not\!u \not\!\not\!\not\!i = \not\!\not\!\not\!u \not\!\not\!\not\!i + \text{c.c.} = \not\!\not\!\not\!d \not\!\not\!\not\!i ;
\]

is a measure of the spontaneous symmetry breaking: it may be viewed as an order parameter and plays a role analogous to the spontaneous magnetization of a magnet.

### 4. ROLE OF THE QUARK CONDENSATE

The approximate validity of the relation (1) was put to question by Stern and collaborators [8], who pointed out that there is no experimental evidence for the quark condensate to be different from zero. Indeed, the dynamics of the ground state of QCD is not understood — it could resemble the one of an antiferromagnet, where, for dynamical reasons, the most natural candidate for an order parameter, the magnetization, happens to vanish. There are a number of theoretical reasons indicating that this scenario is unlikely:

(i) The fact that the pseudoscalar meson octet satisfies the Gell-Mann-Okubo formula remarkably well would then be accidental.

(ii) The value obtained for the quark condensate on the basis of QCD sum rules, in particular for the baryonic correlation functions [9], confirms the standard picture.

(iii) The lattice values [10] for the ratio \(m_v = \hat{m}\) agree very well with the result of the standard chiral perturbation theory analysis [11], corroborating this picture further.

Quite irrespective, however, of whether or not the scenario advocated by Stern et al. is theoretically appealing, the issue can be subject to experimental test. In fact, significant progress has recently been achieved in this direction [12, 13]. The remainder of the talk concerns this matter.

The Gell-Mann-Oakes-Renner formula is not exact. The expansion of \(M_\pi^2\) in powers of \(m_u, m_d\) contains an infinite sequence of contributions. The expansion starts with a term linear in the quark masses:

\[
M_\pi^2 = M^2 \frac{\hat{m}^2}{32\pi^2 F^2} + O \left( M^6 \right) ; \quad \hat{m}^2 \langle m_u + m_d \rangle B \] \tag{2}
The coefficient $B$ of the linear term is given by the value of $\int_0^\infty \mu^j (\int_0^\mu dF_\pi^2)$ in the limit $m_u, m_d \to 0$, and $F$ is the value of $F_\pi$ in that limit. The Gell-Mann-Oakes-Renner formula is obtained by dropping the higher order contributions. These are dominated by the term of order $M^2$, which involves one of the coupling constants occurring in the effective Lagrangian at order $\rho^4$. More precisely, the formula involves the value of the running coupling constant $\gamma_3$ at scale $\mu = M$, which logarithmically depends on $M$. Expressed in terms of the corresponding intrinsic scale $\Lambda_3$, we have

$$\gamma_3 = \ln \frac{\Lambda_3^2}{M^2}.$$  \hspace{1cm} (3)

The symmetry does not determine the numerical value of this scale. The crude estimates underlying the standard version of chiral perturbation theory [3] yield numbers in the range

$$0.2 \text{ GeV} < \Lambda_3 < 2 \text{ GeV} :$$  \hspace{1cm} (4)

The term of order $M^4$ is then very small compared to the one of order $M^2$, so that the Gell-Mann-Oakes-Renner formula is obeyed very well. Stern and collaborators investigate the more general framework, referred to as “generalized chiral perturbation theory”, where arbitrarily large values of $\gamma_3$ are considered. The quartic term in eq. (2) can then take values comparable to the “leading”, quadratic one. If so, the dependence of $M_\pi^2$ on the quark masses would fail to be approximately linear, even for values of $m_u$ and $m_d$ that are small compared to the intrinsic scale of QCD. A different bookkeeping for the terms occurring in the chiral perturbation series is then needed [8] – the standard chiral power counting is adequate only if $\gamma_3$ is not too large.

5. QUARK MASS DEPENDENCE OF $M_\pi$ AND $F_\pi$

The behaviour of the ratio $M_\pi^2$ as a function of $\hat{m}$ is indicated in fig. 1, taken from ref. [14]. The fact that the information about the value of $\Lambda_3$ is very meagre shows up through very large uncertainties. In particular, with $\Lambda_3 \sim 0.5 \text{ GeV}$, the ratio $M_\pi^2 = M^2$ would remain close to 1, on the entire interval shown. Note that outside the range (4), the dependence of $M_\pi^2$ on the quark masses would necessarily exhibit strong curvature.

The figure illustrates the fact that brute force is not the only way the very small values of $m_u$ and $m_d$ observed in nature can be reached through numerical simulations on a lattice. It suffices to equip the strange quark with the physical value of $m_s$ and to measure the dependence of the pion mass on $m_u, m_d$ in the region where $M_\pi$ is comparable to $M_K$. A fit to the data based on eq.(2) should provide an extrapolation to the physical quark masses that is under good control\textsuperscript{1}. Moreover, the fit would allow a determination of the scale $\Lambda_3$ on the lattice. This is of considerable interest, because that scale also shows up in other contexts, in the $\pi\pi$ scattering lengths, for example. For recent work in this direction, I refer to [17, 18].

\textsuperscript{1} The logarithmic singularities occurring at next-to-next-to-leading order are also known [15] – for a detailed discussion, I refer to [16].
For the pion decay constant, the expansion analogous to eq. (2) reads

$$F_\pi = F_1 + \frac{3\Lambda_4^2}{16\pi^2 F^2} + O(M^4) ; \quad \bar{\gamma}_4 = \ln \frac{\Lambda_4^2}{M^2} :$$

(5)

In this case, the relevant effective coupling constant is known rather well: chiral symmetry implies that it also determines the slope of the scalar form factor of the pion,

$$F_s(\gamma) = i\pi(p^0)\bar{q}iu + \bar{d}d \pi(p) = F_s(0) + \frac{1}{6} i\pi^2 \pi^2 + O(\gamma^2) :$$

As shown in ref. [3], the expansion of $i\gamma^2 i_s$ in powers of $m_u; m_d$ starts with

$$i\gamma^2 i_s = \frac{6}{(4\pi F)^2} \bar{\gamma}_4 \left( \frac{13}{12} + O(M^2) \right) :$$

(6)

Analyticity relates the scalar form factor to the $I = 0$ $S$–wave phase shift of $\pi\pi$ scattering [19]. Evaluating the relevant dispersion relation with the remarkably accurate information about the phase shift that follows from the Roy equations [16], one finds $i\gamma^2 i_s = 0.61 \pm 0.04 \text{ fm}^2$. Expressed in terms of the scale $\Lambda_4$, this amounts to

$$\Lambda_4 = 1.26 \pm 0.14 \text{ GeV} :$$

(7)

Fig. 1 shows that this information determines the quark mass dependence of the decay constant to within rather narrow limits. The change in $F_\pi$ occurring if $\hat{m}$ is increased from the physical value to $\frac{1}{2} m_s$ is of the expected size, comparable to the difference between $F_K$ and $F_\pi$. The curvature makes it evident that a linear extrapolation from values of order $\hat{m} = \frac{1}{2} m_s$ down to the physical region is meaningless.

6. $\pi\pi$ SCATTERING

The experimental test of the hypothesis that the quark condensate represents the leading order parameter relies on the fact that $i0 \bar{q}q \not{j}$ not only manifests itself in the depen-
dence of the pion mass on \( m_u \) and \( m_d \), but also in the low energy properties of the \( \pi \pi \) scattering amplitude.

At low energies, the scattering amplitude is dominated by the contributions from the \( S- \) and \( P- \) waves, because the angular momentum barrier suppresses the higher partial waves. Bose statistics implies that configurations with two pions and \( I = 0 \) are symmetric in flavour space and thus carry either isospin \( I = 0 \) or \( I = 2 \), so that there are two distinct \( S- \) waves. For \( I = 1 \), on the other hand, the configuration must be antisymmetric in flavour space, so that there is a single \( P- \) wave, \( I = 1 \). If the relative momentum tends to zero, only the \( S- \) waves contribute, through the corresponding scattering lengths \( a_0^0 \) and \( a_0^2 \) (the lower index refers to angular momentum, the upper one to isospin).

As shown by Roy [20], analyticity, unitarity and crossing symmetry subject the partial waves to a set of coupled integral equations. These equations involve two subtraction constants, which may be identified with the two \( S- \) wave scattering lengths \( a_0^0, a_0^2 \). If these two constants are given, the Roy equations allow us to calculate the scattering amplitude in terms of the imaginary parts above 800 MeV and the available experimental information suffices to evaluate the relevant dispersion integrals, to within small uncertainties [21]. In this sense, \( a_0^0, a_0^2 \) represent the essential parameters in low energy \( \pi \pi \) scattering.

As a general consequence of the hidden symmetry, Goldstone bosons of zero momentum cannot interact with one another. Hence the scattering lengths \( a_0^0 \) and \( a_0^2 \) must vanish in the symmetry limit, \( m_u; m_d \to 0 \). These quantities thus also measure the explicit symmetry breaking generated by the quark masses, like \( M_\pi^2 \). In fact, Weinberg’s low energy theorem [22] states that, to leading order of the expansion in powers of \( m_u \) and \( m_d \), the scattering lengths are proportional to \( M_\pi^2 \), the factor of proportionality being fixed by the pion decay constant:\[^2\]

\[
\begin{align*}
   a_0^0 &= \frac{7 M_\pi^2}{32 \pi F_\pi^2} + O(\hat{m}^2) \quad ; \quad a_0^2 = \frac{M_\pi^2}{16 \pi F_\pi^2} + O(\hat{m}^2)
\end{align*}
\]

Chiral symmetry thus provides the missing element: in view of the Roy equations, Weinberg’s low energy theorem fully determines the low energy behaviour of the \( \pi \pi \) scattering amplitude. The prediction (8) corresponds to the dot on the left of fig. 2.

The prediction is of limited accuracy, because it only holds to leading order of the expansion in powers of the quark masses. In the meantime, the chiral perturbation series of the scattering amplitude has been worked out to two loops [23]. At first nonleading order of the expansion in powers of momenta and quark masses, the scattering amplitude can be expressed in terms of \( F_\pi, M_\pi \) and the coupling constants \( \gamma_1; \ldots; \gamma_4 \) that occur in the derivative expansion of the effective Lagrangian at order \( p^4 \). The terms \( \gamma_1 \) and \( \gamma_2 \) manifest themselves in the energy dependence of the scattering amplitude and can thus be determined phenomenologically. As discussed in section 5, the coupling constant \( \gamma_4 \) is known rather accurately from the dispersive analysis of the scalar form factor. The crucial term is \( \gamma_3 \) – the range considered for this coupling constant makes the difference between standard and generalized chiral perturbation theory. In the standard framework,

[^2]: The standard definition of the scattering length corresponds to \( a_0^0=M_\pi \). It is not suitable in the present context, because it differs from the invariant scattering amplitude at threshold by a kinematic factor that diverges in the chiral limit.
FIGURE 2. $S$–wave scattering lengths. The Roy equations only admit solutions in the “universal band”, spanned by the two tilted lines. The dot indicates Weinberg’s leading order result, while the small ellipse includes the higher order corrections, evaluated in the framework of standard chiral perturbation theory. In the generalized scenario, there is no prediction for $a_0^0$, but there is a correlation between $a_0^0$ and $a_2^0$, shown as a narrow strip. The triangle with error bars indicates the phenomenological range permitted by the old data, $a_0^0 = 0.26 \pm 0.05$, $a_2^0 = 0.028 \pm 0.012$ [27].

where the relevant scale is in the range (4), one finds that the leading order result is shifted into the small ellipse shown in fig. 2, which corresponds to [24, 25]:

$$a_0^0 = 0.220 \pm 0.005; \quad a_2^0 = 0.0444 \pm 0.0010.$$ (9)

The numerical value quoted includes the higher order corrections (in the standard framework, the contributions from the corresponding coupling constants are tiny).

The corrections from the higher order terms in the Gell-Mann-Oakes-Renner relation can only be large if the estimate (4) for $\Lambda_3$ is totally wrong. As pointed out long ago [26], there is a low energy theorem that holds to first nonleading order and relates the $S$–wave scattering lengths to the scalar radius:

$$2a_0^0 = 5a_2^0 - \frac{3M_\pi^2}{4\pi F_\pi^2} \left( 1 + \frac{1}{3} M_\pi^2 r_\pi^2 \right) + \frac{41M_\pi^2}{192\pi^2 F_\pi^2} + O(\hat{m}^3) :$$ (10)

In this particular combination of scattering lengths, the term $\gamma_3$ drops out. The theorem thus correlates the two scattering lengths, independently of the numerical value of $\Lambda_3$. The correlation holds both in standard and generalized chiral perturbation theory. The corrections occurring in eq. (10) at order $\hat{m}^3$ have also been worked out. These are responsible for the fact that the narrow strip, which represents the correlation in fig. 2, is slightly curved.

7. IMPACT OF THE NEW K DECAY DATA

The final state interaction theorem implies that the phases of the form factors relevant for the decay $K \rightarrow \pi\pi e\nu$ are determined by those of the $I = 0$ $S$–wave and of the $P$–wave of
elastic $\pi\pi$ scattering, respectively. Conversely, the analysis of the final state distribution observed in this decay yields a measurement of the phase difference $\delta_0(s)$, $\delta_1(s)$, in the region $4M_K^2 < s < M_K^2$. As discussed above, the Roy equations determine the behaviour of the phase shifts in terms of the two $S$–wave scattering lengths. Moreover, in view of the correlation between the two scattering lengths, $a_0^2$ is determined by $a_0^0$, so that the phase difference $\delta(s)$ can be calculated as a function of $a_0^0$ and $q$, where $q$ is the c.m. momentum in units of $M_\pi$. In the region of interest ($q < 1$, $0.18 < a_0^0 < 0.26$), the prediction reads

$$\begin{align*}
\delta_0^0 & = \frac{q}{1 + q^2} a_0^0 + q^2 b + q^4 c + q^6 d \\
b & = 0.2527 + 0.141 \Delta a_0^0 + 1.14 (\Delta a_0^0)^2 + 3.55 (\Delta a_0^0)^3; \\
c & = 0.0063 + 0.145 \Delta a_0^0; \\
d & = 0.0096;
\end{align*}$$

with $\Delta a_0^0 = a_0^0 - 0.22$. The uncertainty in this relation mainly stems from the experimental input used in the Roy equations and is not sensitive to $a_0^0$:

$$e = 0.0035 q^3 + 0.0015 q^5.$$  

The prediction (11) is illustrated in fig. 3, where the energy dependence of the phase difference is shown for $a_0^0 = 0.18$, $0.22$ and $0.26$. The width of the corresponding bands indicates the uncertainties, which according to (12) grow in proportion to $q^3$ – in the range shown, they amount to less than a third of a degree. The figure shows that the data of ref. [28] barely distinguish between the three values of $a_0^0$ shown. The results of the E865 experiment at Brookhaven [13] are significantly

![FIGURE 3.](image-url)
more precise, however. The best fit to these data is obtained for $a_0^0 = 0.218$, with $\chi^2 = 5.7$ for 5 degrees of freedom. This beautifully confirms the value in eq. (9), obtained on the basis of standard chiral perturbation theory. There is a marginal problem only with the bin of lowest energy: the corresponding scattering lengths are outside the region where the Roy equations admit solutions. In view of the experimental uncertainties attached to that point, this discrepancy is without significance: the difference between the central experimental value and the prediction amounts to 1.5 standard deviations. Note also that the old data are perfectly consistent with the new ones: the overall fit yields $a_0^0 = 0.221$ with $\chi^2 = 8.3$ for 10 degrees of freedom.

The relation (11) can be inverted, so that each one of the values found for the phase difference yields a measurement of the scattering length $a_0^0$. The result is shown in fig. 4. The experimental errors are remarkably small. It is not unproblematic, however, to treat the data collected in the different bins as statistically independent: in the presence of correlations, this procedure underestimates the actual uncertainties. Also, since the phase difference rapidly rises with the energy, the binning procedure may introduce further uncertainties. To account for this, the final result given in ref. [12],

$$a_0^0 = 0.221 \pm 0.026;$$  \hspace{1cm} (13)

corresponds to the 95% confidence limit – in effect, this amounts to stretching the statistical error bar by a factor of two.

We may translate the result into an estimate for the magnitude of the coupling constant $\tilde{\gamma}_3$: the range (13) corresponds to $|\tilde{\gamma}_3| < 16$. Although this is a coarse estimate, it implies that the Gell-Mann-Oakes-Renner relation does represent a decent approximation: more than 94% of the pion mass stems from the first term in the quark mass expansion (2), i.e. from the term that originates in the quark condensate. This demonstrates that there is no need for a reordering of the chiral perturbation series based on SU(2)$_R$ SU(2)$_L$. In that context, the generalized scenario has served its purpose and can now be dismissed.
A beautiful experiment is under way at CERN [29], which exploits the fact that $\pi^+\pi$ atoms decay into a pair of neutral pions, through the strong transition $\pi^+\pi \rightarrow \pi^0\pi^0$. Since the momentum transfer nearly vanishes, only the scattering lengths are relevant: at leading order in isospin breaking, the transition amplitude is proportional to $a_0^2$. The corrections at next-to-leading order are now also known [30]. Hence a measurement of the lifetime of a $\pi^+\pi$ atom amounts to a measurement of this combination of scattering lengths. At the planned accuracy of 10% for the lifetime, the experiment will yield a measurement of the scattering lengths to 5%, thereby subjecting chiral perturbation theory to a very sensitive test.

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