Meta-modelling the climate of dry tide locked rocky planets

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\section*{ABSTRACT}

\textbf{Context.} Rocky planets hosted by close-in extrasolar systems are likely to be tidally locked in 1:1 spin-orbit resonance, a configuration where they exhibit permanent dayside and nightside. Because of the resulting day-night temperature gradient, the climate and large-scale circulation of these planets are strongly determined by their atmospheric stability against collapse, which designates the runaway condensation of greenhouse gases on the nightside.

\textbf{Aims.} To better constrain the surface conditions and climatic regime of rocky extrasolar planets located in the habitable zone of their host star, it is therefore crucial to elucidate the mechanisms that govern the day-night heat redistribution.

\textbf{Methods.} As a first attempt to bridge the gap between multiple modelling approaches ranging from simplified analytical greenhouse models to sophisticated 3-D General Circulation Models (GCM), we developed a General Circulation Meta-Model (GCMM) able to reproduce both the closed-form solutions obtained in earlier studies, the numerical solutions obtained from GCM simulations, and solutions provided by intermediate models, assuming the slow rotator approximation. We used this approach to characterise the atmospheric stability of Earth-sized rocky planets with dry atmospheres containing CO\textsubscript{2}, and we benchmarked it against 3-D GCM simulations using THOR GCM.

\textbf{Results.} We observe that the collapse pressure below which collapse occurs can vary by \(\sim 40\%\) around the value predicted by analytical scaling laws depending on the mechanisms taken into account among radiative transfer, atmospheric dynamics, and turbulent diffusion. Particularly, we find (i) that the turbulent diffusion taking place in the dayside planetary boundary layer (PBL) globally tends to warm up the nightside surface hemisphere except in the transition zone between optically thin and optically thick regimes, (ii) that the PBL also significantly affects the day-night advection timescale, and (iii) that the slow rotator approximation holds from the moment that the normalised equatorial Rossby deformation radius is greater than 2.

\textbf{Key words.} planets and satellites: atmospheres – planets and satellites: terrestrial planets – methods: numerical.

1. Introduction

Launched recently from Kourou’s spaceport in French Guiana, the \textit{James-Webb} Space Telescope (JWST; Deming et al. 2009) is on the point of unravelling the features of exoplanetary atmospheres at resolutions never reached before. With the current or upcoming transit searches of the TESS (Barclay et al. 2018) and PLATO (Ragazzoni et al. 2016) observatories, this telescope will accelerate the dynamics initiated by previous space missions by populating the continuum of extrasolar planets and constraining the properties of the detected objects. Many of these planets are rocky planets in close-in star-planet systems, especially planets orbiting brown dwarves and very-low-mass stars (e.g. Payne & Lodato 2007; Raymond et al. 2007; Kopparapu et al. 2017) such as the seven Earth-sized planets hosted by the TRAPPIST-1 ultra-cool dwarf star (Gillon et al. 2017; Grimm et al. 2018). Therefore it is crucial to better understand the mechanisms governing their climate, atmospheric circulation, and surface conditions.

Tidal locking in 1:1 spin-orbit resonance is the most probable final spin state of planets in close-in star-planet systems (Goldreich 1966). This evolution results from the action of the gravitational tides raised by the perturbing tidal potential of the star. Because of dissipative mechanisms, the tidal response of the planet is delayed with respect to the perturber. As a consequence, the resulting tidal torque tends to drive the spin towards the configuration where the star is motionless in the frame of reference rotating with the planet. This spin state corresponds to spin-orbit synchronisation, and is reached when the spin angular frequency of the planet \(\Omega\) equals its orbital frequency \(n_s\).

Additionally, gravitational tides act to decrease both the obliquity and the eccentricity of the planet, which is thereby driven towards the equilibrium configurations of coplanarity and circularity (Hut 1980, 1981) unless it spirals towards the star until being engulfed by it if the system is very close (Hut 1981; Levrard et al. 2009). Asynchronous final spin states may also exist. For instance, eccentric orbits maintained by orbital resonances in a multiple-planet system lead to spin-orbit resonances of higher degrees where the planet can be trapped (Correia et al. 2014; Auclair-Desrotour et al. 2019a). Similarly, it has been shown that significant thermal tides generated by stellar irradiation are able to prevent Venus-like planets to reach spin-orbit synchronisation by inducing a torque opposed to the solid tidal torque (Gold & Soter 1969; Ingersoll & Dobrovolskis 1978; Leconte et al. 2015; Auclair-Desrotour et al. 2019b).

The probability for a planet to be tidally locked in 1:1 spin-orbit resonance with temperate surface conditions is determined by the interplay between two radii: the tidal lock radius \(r_T\) and the radius of the habitable zone \(n_{HZ}\). While the tidal lock radius
indicates the size of the region where planets are likely to be tide locked in spin-orbit synchronisation, the radius of the habitable zone corresponds to the typical star-planet distance at which a planet can sustain liquid water at its surface (Kopparapu et al. 2013). By assuming that the planet behaves as a black body, and by writing the stellar luminosity as a function of the stellar mass with the empirical formula given by Barnes et al. (2008), it can be shown that \( r_{\text{HZ}} \propto M_\text{e}^{0.32} \) for \( M_\text{e} \lesssim 1 \) (Auclair-Desrotour & Heng 2020), whereas the tidal lock radius scales as \( r_T \propto M_\text{e}^{0.3} \) (Peale 1977; Kasting et al. 1993; Dobrovolskis 2009; Edson et al. 2011). Thus, the size of the habitable zone radius decays faster than the tidal lock radius with decreasing the stellar mass, which makes planets located in the habitable zone have more chance to be tide locked if they orbit low-mass stars than if they orbit Sun-like stars (Kasting et al. 1993).

For planets orbiting the low-mass M stars, tide-locking times are actually very short, and even extremely short in the case of lava planets (e.g. 55 Cancri e, Kepler 10b), with maximum values hardly reaching a few million years. For instance, the time required for the cool planet LHS1140 b to become tide locked is about 14 million years (Pierrehumbert & Hammond 2019), which is small compared with the typical ages of planetary systems hardly reaching a few million years. For instance, the time scale to be tide locked if they orbit low-mass stars than if they orbit Sun-like stars (Kasting et al. 1993).

Ding & Pierrehumbert 2020; Sergeev et al. 2020; Turbet et al. 2018), including intermediate semi-analytical or numerical approaches (e.g. Yang & Abbot 2014; Koll & Abbot 2016; Song et al. 2021) that cannot be listed here in an exhaustive way. Although based on robust methodologies, most of these approaches cannot be related self-consistently to each other due to major differences in modelling choices. These discrepancies raise two questions. How to disentangle the possible causes of different predictions between two models? How to assess the epistemic value of a given model? This can be reformulated in a more concrete way as: how to consistently characterise the climate of tide locked planets from multiple modelling approaches? This major concern was formulated explicitly by Held (2005), who argued for the need of model hierarchies on which to base one’s understanding in climate modelling. Such hierarchies appear as the only way to close the gap between idealised modelling and high-end simulations, as they allow for capturing the essence of each particular source of complexity.

The aim of the present work is to tackle these questions from the angle of atmospheric stability against collapse. In the continuity of a former study on the atmospheric stability of tide locked rocky planets (Auclair-Desrotour & Heng 2020), we developed a multi-dimensional model hierarchy that we call a General Circulation Meta-Model (GCMM) in order to bridge the gap between the analytic theory of planetary climates and simulations performed with 3-D GCMs. This model hierarchy is based on a systematic bottom-up approach in the spirit of Held (2005).

Let us specify the sense given here to meta-modelling. By meta-model, we mean that the model ought to be able to reproduce exactly the setups of both simplified greenhouse models and GCMs – as well as the configurations in-between – with the same intrinsic theoretical background. In that sense, such models are possible instances of the meta-model, which can generate any of them. Hence the so-defined GCMM allows for disentangling the effects of mechanisms that are either strongly coupled in standard GCMs or ignored in simplified analytic models. These effects are added or subtracted as a function of the number of degrees of freedom of the model. Increasing the number of degrees of freedom amounts to adding key sources of complexity.

Typically, radiative 0-D models are essentially based on radiative exchanges between the planet’s surface and the atmosphere (e.g. Wordsworth 2015; Auclair-Desrotour & Heng 2020). At the next level of complexity, 1-D models take the coupling between radiative transfer and the atmospheric structure into account (e.g. Robinson & Catling 2012). Two-column – or 1.5-D – models are the minimum setup to couple self-consistently the large-scale day-night overturning circulation with radiative transfer and the atmospheric structure (e.g. Yang & Abbot 2014; Koll & Abbot 2016). This coupling is refined at the level of 2-D GCMs, which allow for calculating self-consistently the interaction between physical mechanisms (clouds, turbulent diffusion in the planetary boundary layer, or PBL, convection) and mean flows in the slow rotation regime (e.g. Song et al. 2021). Finally, 3-D GCMs complete the picture by introducing Coriolis effects and non-axisymmetric flows where super-rotation can develop (e.g. Leconte et al. 2013; Carone et al. 2014; Haqq-Misra et al. 2018; Ding & Pierrehumbert 2020; Sergeev et al. 2020; Turbet et al. 2021).

Thus, the essence of a GCMM is to model all these levels of complexity at the same time so that the roles played by the different mechanisms involved in the planet’s climate can be clearly separated. As such a model has not been developed yet to our knowledge, the present work is a first attempt to design a GCMM dedicated to the study of tide locked rocky planets. For simplicity, we confine ourselves to the slow rotation regime and
ignore Coriolis effects in the dynamics. This allows us to avoid the mathematical complications related to the three-dimensional geometry and to speed up calculations. Our GCMM is thus designed to generate models ranging from 0-D configurations to 2-D configurations. Besides, we opt for a finite-volume method to solve the hydrostatitical primitive equations (HPES), following the approaches detailed by Yao & Stone (1987) and implemented in standard finite-volume GCMs such as the LMDZ (Hourdin et al. 2006) or THOR (Mendonça et al. 2016) GCMs. As the finite-volume method divides the atmosphere into cells, it is well appropriate to describe the radiative energy balance models on which the analytic theory is built. For instance, the one-cell configuration (1 × 1 grid) corresponds to the single-layer isothermal atmosphere of Wordsworth’s model (Wordsworth 2015). Similarly, increasing numbers of horizontal and vertical grid intervals generate the 1-D (1 × 50 grid), 1.5-D (2 × 50 grid), and 2-D (32 × 50 grid) model setups, respectively.

In order to minimise the size of the parameter space, we confine ourselves to the dry case in this study. The effects of moisture (clouds, latent heat transport, sedimentation, surface condensation or evaporation) are ignored. Radiative transfer is described in the double-gray approximation, meaning that the fluxes are divided into two bands – shortwave and longwave – characterised by effective absorption parameters (e.g. Heng 2017, Sect. 4.1). We also make the two-stream approximation, and consider that radiative fluxes only travel upwards and downwards (e.g. Heng 2017, Sect. 3.1). In addition with radiative transfer, the vertical turbulent diffusivity induced by the interactions between mean flows and the planet surface in the PBL is taken into account by making use of a model based upon the mixing length theory (Holtslag & Boville 1993). Finally, the thermal diffusion in the soil is included in the diffusion scheme of the GCMM with a 1-D finite-difference model following the method described by Wang et al. (2016). As shown by earlier studies (Wordsworth 2015; Koll & Abbot 2016; Auclair-Desrotour & Heng 2020), the three above physical ingredients (circulation, radiative transfer, turbulent diffusion) predominantly determine the nightside surface temperature, and thereby the atmospheric stability against collapse. Table 1 summarises the physics described by the studied instances of the meta-model and THOR 3-D GCM, which is used to benchmark the former. Physical mechanisms are gradually captured by the grid formats characterising the models, as the number of degrees of freedom increases.

Table 1. Physics described by the four studied instances of the meta-model and THOR 3-D GCM.

| Grid and Physics | 0-D | 1-D | 1.5-D | 2-D | THOR |
|------------------|-----|-----|-------|-----|------|
| Radiative transfer | X   | X   | X     | X   | X    |
| Thermal structure | X   | X   | X     | X   |      |
| Day-night circulation | X   | X   | X     | X   |      |
| Planetary boundary layer | X   | X   |       |     |      |
| Soil heat transfer | X   | X   |       |     |      |
| Coriolis effects |      |     |       |     | X    |

* All instances of the meta-model use the same parameters and theoretical background. Physical mechanisms are gradually captured by grid formats, which are defined by numbers of horizontal × vertical grid intervals. In ascending orders of grid resolution, the main physical features described by the meta-model’s instances and by THOR GCM are (i) the radiative exchanges between the planet’s surface and the atmosphere, (ii) the vertical thermal structure of the atmosphere (i.e. temperature-pressure profile) in radiative equilibrium, (iii) the day-night large-scale circulation, (iv) the convective turbulent diffusion due to friction in the PBL, (v) the soil heat diffusion, and (vi) the vortical components of mean flows due to Coriolis effects.

In Sect. 2, we introduce some control parameters and analytical scaling laws characterising the climate and circulation regime of tide locked planets. In Sect. 3, we detail the main features of the GCMM and the physical setup of the studied Earth-like and pure CO₂ atmospheres. Section 4 introduces the four instances of the meta-model used in this work: 0-D, 1-D, 1.5-D, and 2-D. In Sect. 5, we run grid simulations for these instances to characterise the atmospheric stability of Earth-sized synchronous planets against collapse as a function of the stellar flux and surface pressure. Particularly, this vertical inter-comparison highlights the influence of the planetary boundary layer on climate, day-night advection, and surface conditions. In Sect. 6, we investigate the limitations of the zero-spin rate approximation assumed in this approach by running simulations with THOR 3-D GCM. We show that the approximation holds from the moment that the dimensionless equatorial Rossby deformation length is greater than 2. Finally, in Sect. 7, we summarise the conclusions of the study.

2. Preliminary scalings

The circulation regime and thermal state of equilibrium of tide locked planets is controlled by a few parameters and scaling laws derived either from the shallow water approximation (e.g. Vallis 2006; Pierrehumbert & Hammond 2019) or from the weak temperature gradient – or WTG – approximation (e.g. Pierrehumbert 1995; Sobel et al. 2001). One ought to recall these scalings before introducing the physical setup of the numerical approach.

2.1. Circulation regimes of synchronous planets

If we assume that the planet surface is isotropic, the whole physics and dynamics governing the atmospheric circulation are symmetric about the star-planet axis except Coriolis terms. As a consequence, the circulation regime is essentially characterised by one control parameter depending upon the planet’s spin angular velocity, which determines whether mean flows are bi-dimensional and symmetric about the star-planet axis, or if they are sufficiently deviated by the planet’s rotation to become three-dimensional.

Such a parameter appears naturally in analyses making use of the Buckingham-Pi theorem (Buckingham 1914) in the primitive equations of fluid dynamics and thermodynamics (e.g. Koll & Abbot 2015). In the present study, following Leconte et al.
(2013) and Auclair-Desrotour & Heng (2020), we define the dimensionless equatorial Rossby deformation length \( \tilde{L}_{Ro} \) from the equatorial Rossby deformation radius \( L_{Ro} \) as (e.g. Menou et al. 2003)

\[
\tilde{L}_{Ro} = \frac{L_{Ro}}{R_{p}} = \sqrt{\frac{c_{\text{wave}}}{2\Omega R_{p}}} \tag{1}
\]

where \( R_{p} \) designates the planet radius, and \( c_{\text{wave}} \) the speed of horizontally propagating gravity waves. The dimensionless equatorial Rossby deformation length is essentially the square root of the distance – in radius unit – that fast gravity waves can travel before they feel the Coriolis effects and geostrophically adjust (Vallis 2006).

If \( \tilde{L}_{Ro} > 1 \), the Coriolis effects are small and the two-way force balance between advection and pressure-gradient accelerations leads to a day-night overturning circulation symmetric about the star-planet axis (Leconte et al. 2013; Pierrehumbert & Hammond 2019; Hammond & Lewis 2021). In this regime, the dynamics of mean flows is the same in all planes containing the star-planet axis, with high-altitude winds blowing from the dayside to nightside and near-surface winds blowing from the nightside to dayside. This essentially corresponds to the steady state expected in the Weak Temperature Gradient theory, where small Coriolis forces, friction, and nonlinearities make the heat advection unable to annihilate completely the day-night temperature gradient (Sobel et al. 2001; Hammond & Pierrehumbert 2017; Pierrehumbert & Hammond 2019).

Conversely, for \( \tilde{L}_{Ro} \lesssim 1 \), Showman & Polvani (2011) demonstrated that the formation of standing planetary-scale equatorial Rossby and Kelvin waves (i.e. waves restored by the Coriolis acceleration; see e.g. Lee & Saio 1997) favours the emergence of super-rotation by pumping angular momentum from the mid-latitudes towards the equator. In this regime, the equatorial Rossby deformation radius \( L_{Ro} \) essentially corresponds to the latitudinal width of the produced eastward equatorial jet, and mean flows take the form of the Matsuno-Gill standing wave pattern (Matsumo 1966b; Gill 1980; Showman & Polvani 2011; Tsai et al. 2014).

The dimensionless equatorial deformation length introduced in Eq. (1) can be related to the atmospheric parameters by considering the properties of gravity waves. Gravity waves are restored by the Archimedean force associated with the fluid buoyancy and their typical speed is given by \( c_{\text{wave}} = HN \), where \( H \) designates the characteristic vertical scale length of the atmosphere and \( N \) the Brunt-Vaisala frequency, which scales the strength of the atmospheric stratification against convection (Gerkema & Zimmerman 2008). In a dry stably stratified atmosphere, this frequency is expressed as (Gerkema & Zimmerman 2008)

\[
N^2 = \frac{g}{T} \left( \frac{g}{C_{p}} + \frac{\partial T}{\partial z} \right), \tag{2}
\]

where we have introduced the gravity \( g \), the heat capacity per unit mass of the gas \( C_{p} \), the temperature \( T \), and the partial derivative operator over the altitude \( \frac{\partial}{\partial z} \). In the idealised case of the vertically isothermal atmosphere (constant temperature), \( N = g/\sqrt{C_{p}T} \), and the vertical scale is the pressure height (Vallis 2006, Sect. 1.4),

\[
H = \frac{R_{d}T}{g}, \tag{3}
\]

where \( R_{d} = R_{GP}/M_{a} \) designates the specific gas constant for dry air, \( R_{GP} \) being the ideal gas constant and \( M_{a} \) the mean molecular weight of the atmosphere. Thus, in this configuration (e.g. Leconte et al. 2013),

\[
\tilde{L}_{Ro} = \frac{R_{d}T^{1/2}}{2\Omega R_{p}C_{p}^{1/2}}, \tag{4}
\]

which highlights the fact that the circulation regime depends on the the planet’s spin rotation, thermal state, and atmospheric composition.

The dimensionless equatorial Rossby deformation length conveys exactly the same information as the WTG parameter \( \Lambda \) introduced in the Weak Temperature Gradient theory (see e.g. Pierrehumbert & Hammond 2019), which is defined as the ratio of the Rossby radius of deformation – distinct from the equatorial deformation radius – normalised by the planet radius (Vallis 2006, Sect. 3.8.2). The two parameters are linked together by the relationship \( \tilde{L}_{Ro} = \sqrt{\Lambda / 2} \) (Pierrehumbert & Hammond 2019), meaning that either the former or the latter can be chosen to characterise the circulation regime. In the present study, we confine ourselves to the configuration of the WTG theory \( (\tilde{L}_{Ro} > 1) \) and consider thereby that mean flows are symmetric about the star-planet axis.

In addition with the slow and fast rotators regimes, there exists a third dynamical state that is proper to intermediate stellar cases in the range of 3000-3300 K and that is described as the Rhines rotation regime (Haqq-Misra et al. 2018; Sergeev et al. 2020). This regime is related to the Rhines length, which determines the maximum extent of zonally elongated turbulent structures (Rhines 1975). It occurs when the non-dimensional Rossby deformation radius is greater than one but the non-dimensional Rhines length less than one (Haqq-Misra et al. 2018). The slow rotation and fast rotation regimes occur when both the non-dimensional Rhines length and Rossby deformation radius are greater or less than one, respectively. The Rhines rotation regime is not considered here, meaning that we focus on the configuration where both the non-dimensional Rhines length and Rossby deformation radius are greater than one.

2.2. Thermal states predicted by radiative box models

Over the past decade, analytic solutions and scalings characterising the thermal state of equilibrium of tide locked planets have been obtained both for hot Jupiters (Komacek & Showman 2016; Zhang & Showman 2017; Koll & Komacek 2018), lava planets (Hammond & Pierrehumbert 2017), and cooler rocky planets orbiting in the habitable zone of their host star (Showman et al. 2013; Wordsworth 2015; Koll & Abbot 2016; Pierrehumbert & Ding 2016; Koll 2022; Auclair-Desrotour & Heng 2020). Most of them were derived in the framework of the WTG theory and involve simplified atmospheric physics and structure. The present study builds on these works, and particularly those based on box model approaches, where the atmosphere and surface are reduced to large scale energy reservoirs exchanging heat with each other (Wordsworth 2015; Koll & Abbot 2016; Auclair-Desrotour & Heng 2020). Although they are based on strong simplifications (isothermal atmosphere, large-scale averages, no self-consistent coupling between the dynamics and the thermodynamics), these models provide scalings that capture the behaviour of the thermal state of tide locked rocky planets with a minimum set of physical parameters. Particularly, they lead to closed-form solutions for the nightside surface temperature \( T_{n} \).
which determines the whole atmospheric stability against collapse.

By considering a globally isothermal atmosphere interacting with dayside and nightside surface hemispheres, Wordsworth (2015) shows that the pure radiative equilibrium of the surface-atmosphere system corresponds, in the optically thin layer approximation (i.e. transparent in the visible and optically thin in the infrared), to the nightside temperature scaling (Wordsworth 2015, Eq. (29))

$$T_{n;low} = T_{eq} \left[ \frac{1}{2} (1 - A_s) \tau_{s;L} \right]^{\frac{1}{4}}, \quad (5)$$

which can be generalised to optically thick atmospheres with scattering (Auclair-Desrotour & Heng 2020). In the above expression, $A_s$ designates the surface albedo in the shortwave, $\tau_{s;L}$ the longwave optical depth at planet’s surface, and $T_{eq}$ the black body equilibrium temperature, which is defined as

$$T_{eq} = \left( \frac{F_s}{4\sigma_{SB}} \right)^{\frac{1}{4}}, \quad (6)$$

with $F_s$, the incident stellar flux and $\sigma_{SB} = 5.670367 \times 10^{-8}$ W m$^{-2}$ K$^{-4}$ the Stefan-Boltzmann constant (Mohr et al. 2016).

Since it ignores all types of energy exchanges except radiative transfer, the estimate given by Eq. (5) can be interpreted as a lower bound for the nightside surface temperature of a rocky tide locked planet. In reality, the strong convection generated by the thermal forcing of the atmosphere in the dayside planetary boundary layer increases surface-atmosphere heat fluxes, which significantly affects the nightside temperature (Sergeev et al. 2020). The friction of the flow against the surface generates sensible heat exchanges in dry thermodynamics. Additionally, in moist atmospheres, surface evaporation generates latent heat exchanges, which results from the energy taken from or released in the fluid during the changes of phases of the component (Pierrehumbert 2010).

We consider here that the atmosphere is dry and thereby ignore the latter contribution. Sensible heat exchanges can be introduced in the radiative box model by including, in the energy balance equations, the hemisphere-averaged sensible heat flux given by (e.g. Pierrehumbert 2010, Eq. (6.11), p. 396)

$$\overline{F}_{sen} = C_D p_s v_{sen} \left( T_d - T_n \right), \quad (7)$$

where $p_s$ is the atmospheric density at planet’s surface, $C_D$ the bulk drag coefficient characterising the strength of friction in the surface layer, and $v_{sen}$ the typical horizontal wind speed of the flow. Among these parameters, $v_{sen}$ accounts for the circulation, meaning that it cannot be self-consistently related to the thermal state of the system in this simplified approach. Nevertheless, it can be scaled from a dimensional analysis of the thermodynamic equation (e.g. Wordsworth 2015), or by making use of the heat engine theory (e.g. Koll & Abbot 2016; Koll & Komacek 2018; Auclair-Desrotour & Heng 2020). For instance, by modelling the overturning circulation as an ideal heat engine and using Carnot’s theorem (Bruhat 1968), Koll & Abbot (2016) found an upper bound of the dayside average surface wind speed (Koll & Abbot 2016, Eq. (12)),

$$v_{sen} = \left\{ \left( T_d - (1 - A_s) \right) \frac{1}{2} T_{eq} (1 - e^{-\tau_{s;L}}) \left( 1 - A_s \right) \frac{R_d F_s}{2C_D p_s} \right\}^{\frac{1}{2}}, \quad (8)$$

which agrees well with the values obtained numerically from 3-D GCM simulations (Koll & Abbot 2016; Koll & Komacek 2018).

For an isentropic cycle (i.e. an idealised Carnot’s heat engine), the weight of dayside sensible heating1 relatively to radiative heating is controlled by the dimensionless parameter (Auclair-Desrotour & Heng 2020, Eq. (63))

$$L_{sen} = \frac{2C_D p_s}{\tau_{s;L} R_d F_s} \left( \frac{F_s}{2\sigma_{SB}} \right)^{\frac{1}{4}} \left( 1 - A_s \right), \quad (9)$$

which is written here for an atmosphere optically transparent in the shortwave and thin in the longwave. The notation $Q_{sen} \propto v_{sen}^{\frac{3}{2}}$ (e.g. Koll & Abbot 2016) designates the amount of power per unit area available to drive atmospheric motion. Looking at the zero-convection limit ($L_{sen} = 0$) we recover the purely radiative regime, while the opposite asymptotic regime ($L_{sen} = +\infty$) implies that $T_d = T_s$ and provides an upper bound for the nightside surface temperature of a tide-locked rocky planet (e.g. Auclair-Desrotour & Heng 2020, Eq. (85)),

$$T_{n;up} = T_s \left[ 2 (1 - A_s) \tau_{s;L} \right]^{\frac{1}{4}} = \sqrt{2T_{n;low}}, \quad (10)$$

which is valid in the optically thin layer approximation as well as Eq. (5). Therefore, for a globally isothermal and optically thin atmosphere, the nightside surface temperature falls into the interval

$$T_{n;low} \leq T_n \leq \sqrt{2T_{n;low}}. \quad (11)$$

However, we shall remark that $T_d = T_s$ actually corresponds to an extreme regime that is never reached in standard configurations, and we shall thus consider $T_{n;up}$ as a theoretical upper limit.

Similarly as the dayside convective planetary layer, the nightside atmospheric structure alters the nightside equilibrium temperature. Its effect can be quantified by relaxing the isothermal approximation and by dividing the atmosphere into dayside and nightside air columns, which is the essence of two-column models (Yang & Abbot 2014; Koll & Abbot 2016; Auclair-Desrotour & Heng 2020). As shown by Koll & Abbot (2016), the nightside subsidence induced by the day-night overturning circulation generates a temperature inversion in the lowest layers of the atmosphere if the subsidence timescale is slightly greater than the radiative cooling timescale. The resulting atmospheric structure leads to large day-night differences.

3. A General Circulation Meta-model (GCMM)

We introduce in this section the main features of the meta-model and the used physical setup.

3.1. Primitive equations

At a given time $t$, the dynamical core of the GCMM solves the hydrostatical primitive equations over the Cartesian rectangular domain defined by the colatitude $\theta \in \left[0^\circ, 180^\circ\right]$ of the

1 Sensible heating designates surface-atmosphere heat exchanges due to the vertical turbulent diffusion generated by friction of mean flows against the planet’s surface within the surface layer in dry thermodynamics. This mechanism is distinct from latent heat exchanges, which designates the energy exchanges resulting from the change of phase of a condensable substance (e.g. Pierrehumbert 2010, Sect. 6.3).
tially locked coordinates (or TLC; see Koll & Abbot 2015, Appendix B) and the mass-based vertical coordinate defined, in the absence of the topography, as (e.g. Yao & Stone 1987; Carone et al. 2014)
\[ \sigma = \frac{p - p_t}{p - p_t} \in [0, 1], \] (12)
where we have introduced the pressure \( p \), the surface pressure \( p_s \), and the pressure at the top of the atmosphere \( p_t \). In these coordinates, \( \sigma = 0 \) and \( \sigma = 180^\circ \) correspond to the substellar and anti-stellar points, respectively, while \( \sigma = 0 \) and \( \sigma = 1 \) correspond to the top and the bottom of the atmosphere, respectively. While \( p_s \) and \( p \) vary over time and spatial coordinates, \( p_t \) is set to \( p_t = 0 \) in the model, which corresponds to the usual sigma-coordinate \( \sigma = p/p_t \). The vertical coordinate given by Eq. (12) is well suited to the study of the tide locked planets since it follows the distortion of the atmosphere induced by the differential day-night thermal forcing: the pressure of an altitude level may differ by orders of magnitude between the dayside and nightside, which would possibly generate numerical issues with the altitude coordinate.

The relationship between the altitude \( z \) and the generalised vertical coordinate \( \sigma \) is contained in the so-called pseudo-density (e.g. Kasahara 1974),
\[ \rho = -\frac{\hat{\rho}}{\hat{\sigma}} \frac{\partial z}{\partial \sigma}, \] (13)
where \( \rho \) denotes the density. The pseudo-density is proportional to the mass contained in a generalised volume where the vertical dimension is not a length but an interval of the generalised coordinate \( \sigma \). With the chosen mass-based coordinate (Eq. (12)) and the assumed hydrostatic balance, it is simply expressed as (e.g. Yao & Stone 1987)
\[ p = p_s - p_t. \] (14)
We note that \( p \) would be the density to a constant factor if the vertical coordinate were the altitude. Using the pseudo-density instead of the density allows for writing the HPEs given further in the same form for any chosen vertical coordinate.

The HPEs are the mass continuity equation (e.g. Kasahara 1974),
\[ \frac{\partial p}{\partial t} + \nabla \cdot (p\mathbf{v}) + \frac{\partial}{\partial \sigma} (p\dot{\sigma}) = 0, \] (15)
the horizontal momentum equation,
\[ \frac{\partial}{\partial t} (p v_\sigma \sin \theta) + \frac{1}{R_p} \frac{\partial}{\partial \theta} (pv_\theta \sin \theta) + \frac{\partial}{\partial \sigma} \left( p v_\sigma \sin \theta \right) + \frac{1}{R_p} \frac{\partial}{\partial \theta} (pv_\theta \sin \theta) + \frac{\partial}{\partial \sigma} \left( p v_\sigma \sin \theta \right) = \frac{\partial \phi}{\partial \sigma} + \frac{\partial E}{\partial \theta} = p \sin \theta F_\theta, \] (16)
the potential temperature equation,
\[ \frac{\partial \Theta}{\partial t} + \nabla \cdot (p \Theta \mathbf{v}) + \frac{\partial}{\partial \sigma} (p \Theta \dot{\sigma}) = \frac{Q}{E}, \] (17)
and the hydrostatic equation combined with the ideal gas law,
\[ \frac{\partial \rho}{\partial t} + \frac{\partial E}{\partial \sigma} = 0, \] (18)
where we have introduced the horizontal velocity vector \( \mathbf{v}_\sigma = v_\sigma \mathbf{e}_\sigma \), the sigma-velocity \( \dot{\sigma} = \frac{d \sigma}{dt} \) (with \( \frac{d}{dt} \) the material time derivative), the geopotential \( \phi = gz \), the potential temperature \( \Theta \), the Exner function \( E \) (e.g. Vallis 2006, Sect. 3.9), and the horizontal divergence operator at constant \( \sigma \),
\[ \nabla \cdot \frac{1}{R_p} \frac{\partial}{\partial \theta} \frac{\partial}{\partial \theta}. \] (19)
The potential temperature and Exner function are defined as
\[ \Theta = T \left( \frac{p}{p_{ref}} \right)^{-\kappa}, \quad E = C_p \frac{T}{\Theta} = C_p \left( \frac{p}{p_{ref}} \right)^{\kappa}, \] (20)
where \( T \) is the temperature, \( p_{ref} \) a constant reference pressure set to \( p_{ref} = 1 \) bar, and \( \kappa = R_d/C_p \). We note that the Exner function is a proxy for pressure. It is used for convenience, as it facilitates the integration of the hydrostatic equation. In right-hand members of Eqs. (15–18), the source-sink terms are the force per unit mass \( F_\theta \), and the heat power per unit mass \( Q \). We note that the primitive equations are written in their conservative forms, which involve mass flows and mass-integrated quantities rather than the original quantities themselves. Besides, these equations are given here for any vertical coordinate varying monotonically with altitude for the sake of generality.

The nondimensional HPEs derived from Eqs. (15–18) (see Appendix A) are solved for \( \{p, v_\sigma, \Theta, \phi\} \) on a staggered Arakawa C grid (Arakawa & Lamb 1977) with uniformly spaced horizontal intervals, and \( \sigma \)-dependent vertical intervals refined near the model bottom and top (see Appendix B). Following the method implemented in the LMDZ GCM (Hourdin et al. 2006), the integration is done using a leapfrog scheme with a periodic predictor/corrector timestep. The source-sink terms associated with the physics, \( \{F_\theta, Q\} \), as well as other physical variables, are updated periodically using implicit schemes except radiative transfer equations, which are solved directly from the current thermodynamical state. In the 2-D GCM-like configurations, integrating the HPEs on a discrete domain generates numerical instabilities that develop at grid scale and may strongly disrupt calculations. In GCMs, this concern is usually handled by introducing horizontal hyper-diffusion, which is ideally designed to damp efficiently the numerical instabilities at grid scale while leaving the mean flows unchanged (e.g. Thrastarson & Cho 2011). Therefore we include in the model a fourth-order hyper-diffusion (or bi-harmonic diffusion) using an anisotropic super-diffusivity that vanishes at the substellar and anti-stellar points (see Appendix C.1) in order to avoid the stability concerns associated with isotropic diffusion near the poles (Lauritzen et al. 2011, Sect. 13.3). The corresponding hyperdiffusion terms for the momentum and temperature equations are given by
\[ F_{\text{diff},x} = -K_4 \sin^{2\alpha} (\theta) \nabla^2_x v_\theta, \] (21)
\[ F_{\text{diff},T} = -K_4 \sin^{2\alpha} (\theta) \nabla^2_T T, \] (22)
where \( \nabla^2_x = \nabla^2_x \nabla^2_x \) designates the second order horizontal hyper-Laplacian operator, \( \alpha \) the anisotropy exponent (\( \alpha = 1 \) in the model), and \( K_4 \) the super-diffusivity defined by Eq. (C.5). Validation test simulations were run to verify that the mean flow and temperature distribution are insensitive to the hyper-diffusion scheme (see Fig. C.1).

In very hot cases, exponentially growing gravity waves propagating upwards and reflected by the upper boundary of the model can lead to extreme fluctuations of the dynamical quantities in the upper atmosphere. To address these instabilities, it can be necessary to use a sponge layer in addition with horizontal hyper-diffusion. In the present work, we introduce a linear Rayleigh friction sponge (e.g. Lauritzen et al. 2011,
3.2. Radiative transfer

In the model, radiative transfer is described through the double-gray approximation, which consists in (i) decoupling the stellar radiation (shortwave flux) and the planet radiation (longwave flux) – each band being characterised by an effective optical depth \( \tau \), and in (ii) assuming that radiative fluxes only propagate upwards and downwards, which is the essence of the two-stream approximation (e.g., Heng 2017, Sects. 3.1 and 4.1). This allows for calculating fluxes in the shortwave and in the longwave separately (see Appendix E). We denote upward and downward fluxes by \( F_+ \) and \( F_- \), respectively. The equations governing the propagation of the wavelength-integrated total flux \( F_+ = F_+ + F_- \) and net flux \( F_s = F_+ - F_- \) have the same formulation in both bands. They are written as (Heng 2017)

\[
\frac{dF_+}{d\tau} = \frac{1}{\beta_0} F_-, \quad \frac{dF_-}{d\tau} = \beta_0 (F_+ - 2B),
\]

where \( B = \alpha_{SB} T^4 \) designates the black body radiation of the gas, which is zero in the shortwave since the atmosphere is assumed to radiate in the infrared only.

In these equations, \( \tau \) designates the optical depth of the associated band, and \( \beta_0 \) the scattering parameter (\( \beta_0 = 1 \) for pure absorption; \( 0 < \beta_0 < 1 \) in the presence of scattering). The fluxes equations given by Eqs. (23) and (24) are solved numerically in the code by computing the transmission functions of each atmospheric layer as a first step, and by solving the boundary condition problem as a second step. We note that the solution obtained numerically for the single-layer atmosphere this way exactly corresponds to that derived in radiative box models based on the isothermal atmosphere approximation (see e.g., Wordsworth 2015; Auclair-Desrotour & Heng 2020). The optical depths in the shortwave \( \tau_s \) and longwave \( \tau_l \) are both assumed to scale linearly with pressure, and are defined as

\[
\tau_s = \frac{\kappa_s p}{g}, \quad \tau_l = \frac{\kappa_l p}{g},
\]

where we have introduced the effective absorption coefficients of the gas in the short- and longwave, \( \kappa_s \) and \( \kappa_l \), respectively.

We remark that the optical depths defined by Eq. (25) depict horizontally averaged vertical profiles rather than local profiles varying as functions of the propagation angles of radiative fluxes, \( \alpha_s \) and \( \alpha_l \). Therefore, the effective absorption coefficients \( \kappa_s \) and \( \kappa_l \) introduced in Eq. (25) are related both to the absorption properties of the gas and to the mean cosine of the propagation angles in the visible and in the infrared, \( \cos \alpha_s \) and \( \cos \alpha_l \). Typically, these coefficients are related to Wordsworth’s absorption coefficients (Wordsworth 2015, Eq. (12)) – denoted by \( \kappa_s^W \) and \( \kappa_l^W \) – by the equations \( \kappa_s = \kappa_s^W / \cos \alpha_s \) and \( \kappa_l = \kappa_l^W / \cos \alpha_l \). Therefore, changing the value of the mean cosine of the propagation angle in this definition amounts to changing the value of the absorption coefficient in Eq. (25). One shall also bear in mind that the absorption coefficients defined in the double-gray approach are not fundamental parameters of the gas but parameters that mimic the averaged effect of highly frequency-dependent atmospheric opacities, as shown by Wordsworth et al. (2010a) for CO\(_2\)-dominated atmospheres.

Although it does not fully account for the complex physics of radiative transfer, the adopted double-gray approximation with average optical depths captures the dependence of optical depths upon pressure, and it allows for fast numerical computation of radiative fluxes. The radiative transfer scheme may be refined in future works by using more sophisticated approaches such as the correlated-k distribution method (e.g., Lacis & Oinas 1991), but this goes beyond the scope of the present study.

3.3. Planetary boundary layer

In the planetary boundary layer, the shear instability generates turbulence, which acts to mix the flow. This turbulent mixing induces a vertical diffusion of momentum, heat, and potentially moisture in moist cases, near the planet’s surface. The associated eddy diffusivities are controlled by the gradient Richardson number \( Ri \) defined as (Vallis 2006)

\[
Ri = \frac{g}{\Theta} \left( \frac{\partial \Theta}{\partial z} \right)^{-1},
\]

which characterises fluid stratification. The Richardson number is essentially the ratio of the production of turbulent energy due to the shear instability over the restoring force induced by buoyancy. For a given quantity \( X = v_B \Theta \) (or \( q \) in moist cases, \( q \) being the specific humidity), the vertical diffusion equation is written, in the gradient-flux theory (e.g., Garratt 1994), as

\[
\frac{dX}{dt} = -\frac{1}{\rho} \frac{\partial}{\partial z} \left( \rho K_X \frac{\partial X}{\partial z} \right),
\]

the upward diffusive flux being given by

\[
F_{\text{diff},X} = -\rho K_X \frac{\partial X}{\partial z}.
\]

In these equations, \( K_X \) is the eddy diffusivity associated with turbulent mixing for \( X \). This parameter is a function of the mean fields. The lower boundary condition is a continuity condition determined by the exchanges with the planet’s surface, while the upper condition is a zero-flux condition. In a dry atmosphere, the
tendencies for the momentum and potential temperature equations in the dry case are expressed from the diffusive terms given by Eq. (27) as

$$F_m = \frac{d\nu_m}{dt}, \quad Q = \left( \frac{p}{\rho \kappa} \right)^{\frac{1}{2}} \frac{d}{dt} (C_p \theta),$$

(29)

To calculate the eddy diffusivities, we make use of the formulation given by Holtslag & Boville (1993). In that model, the eddy diffusivities of Eq. (27) are expressed as functions of a mixing length scale $\ell$, the local shear $\frac{\partial \nu_m}{\partial z}$, and the gradient Richardson number $Ri$ defined by Eq. (26). They read (e.g. Holtslag & Boville 1993, Eq. (3.2))

$$K_X = \ell^2 \left( \frac{\partial \nu_m}{\partial z} \right)_F (Ri),$$

(30)

where $F_X (Ri)$ describes the functional dependence of $K_X$ on the gradient Richardson number. The form of $F_X$ is determined by the turbulent regime, which can be either stable ($Ri \geq 0$) or unstable ($Ri < 0$). The mixing length is expressed as (Blackadar 1962)

$$\ell = \frac{\ell_0 K_z}{K_z + \ell_0},$$

(31)

the parameter $K \approx 0.4$ being the von Karman constant (e.g. Garratt 1994), and $\ell_0$ the asymptotic length scale ($\ell \approx \ell_0$ for $K \ell_0 \gg 1$), which varies as a function of $z$ (see Appendix F.1).

The turbulent friction of mean flows against the soil in the surface layer leads to sensible momentum and heat exchanges that are described in the form of parametrised surface fluxes. Denoting by $M$ and $H$ the subscripts for the momentum and heat exchanges due to turbulent flows in the case of stable stratification, the functions $F_m$ and $F_H$ are two functions of the bulk Richardson number,

$$F_M = -C_M \nu_{|r_{SL}|} v_{|SL|}, \quad F_H = -C_H \nu_{|r_{SL}|} (\Theta_{|SL|} - \Theta_r),$$

(32)

(33)

where the subscripts $s$ and $SL$ refer to values at the surface and at the top of the surface layer, respectively. The surface-layer exchange coefficients $C_M$ and $C_H$ are defined as (Holtslag & Boville 1993)

$$C_M = C_N f_M (Ri_0), \quad C_H = C_N f_H (Ri_0).$$

(34)

(35)

Here, $C_N$ designates the neutral exchange coefficient (e.g. Holtslag & Boville 1993),

$$C_N = \left[ \frac{K}{\ln (1 + z_{SL}/z_C)} \right]^2,$$

(36)

where $z_C$ denotes the roughness height, while $f_M$ and $f_H$ are two functions of the bulk Richardson number,

$$Ri_0 = \frac{g z_{SL} (\Theta_{|SL|} - \Theta_r)}{\Theta_{|SL|} |v_{r,SL}|},$$

(37)

which controls the stability of the surface layer. We note that the bulk Richardson number $Ri_0$ corresponds to the local gradient Richardson number $Ri$ (Eq. (26)) characterising the surface layer. The functions $f_M$ and $f_H$, as well as the functions $F_X$ introduced in Eq. (30), are detailed in Appendix F.1. The physical tendencies resulting from turbulent diffusion are evaluated every physical time step using an implicit scheme (see Appendix F.2).

As it accounts for the dependence of eddy diffusivities on the gradient Richardson number, the above turbulent diffusion scheme describes both the regime of strong convection occurring on dayside, and the regime of stable stratification associated with the nightside temperature inversion (see Figs. 1 and 2 in the following). It thus captures the evolution of turbulent diffusivities in the PBL between the dayside – where they are high –, and the nightside – where they are low. However, we note that the used standard formulation of turbulent diffusion is not sophisticated enough to account properly for the heat and momentum exchanges due to turbulent flows in the case of strong stratification ($Ri > 1$). In this regime, the vertical momentum mixing continues even at relatively high Ri due to the momentum transport of vertically propagating internal gravity waves, which may increase the surface-atmosphere sensible heat exchanges and warm up the nightside surface (e.g. Sukoriansky et al. 2005; Joshi et al. 2020). Considering this effect, the used turbulent scheme might tend to underestimate the atmospheric stability against collapse. Nevertheless, we adopt it as a convenient compromise between simplicity and realism, which can be refined in the future by using more advanced turbulent diffusion models (e.g. Sukoriansky et al. 2005).

3.4. Soil heat transfer

The soil heat transfer is solved by using a classical 1-D soil heat conduction approach (e.g. Hourdin et al. 1993; Wang et al. 2016). The evolution of the ground temperature $T$ due to vertical diffusion is governed by the heat equation

$$C_g \frac{\partial T}{\partial t} = -\frac{\partial F_c}{\partial u},$$

(38)

where $u = -z$ is the depth from surface ($u \geq 0$), $C_g$ the heat capacity of the ground per unit volume, and $F_c$ the conductive flux propagating downwards. The conductive flux is expressed as

$$F_c = -\lambda_f \frac{\partial T}{\partial \bar{u}},$$

(39)

the parameter $\lambda_f$ being the thermal conductivity of the material. Both $C_g$ and $\lambda_f$ are assumed to be constants in the model.

We introduce the normalised pseudo-depth

$$\bar{u} = u \sqrt{\frac{C_g}{\lambda_f}},$$

(40)

which has dimensions of $s^{1/2}$. Expressed in terms of $\bar{u}$, the vertical diffusion is controlled by one parameter solely, the thermal inertia

$$I_{gr} = \sqrt{\lambda_f C_g}.$$ 

(41)

The downward conductive flux given by Eq. (39) then becomes

$$F_c = -I_{gr} \frac{\partial T}{\partial \bar{u}},$$

(42)

and the vertical diffusion equation simply reads

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial \bar{u}^2}.$$ 

(43)
Table 2. Values of parameters used in the two reference cases of the present work. The acronyms SW and LW are used in place of ‘shortwave’ and ‘longwave’, respectively.

| SYMBOL | DESCRIPTION | UNITS | EARTH-LIKE CASE | PURE CO₂ CASE |
|--------|-------------|-------|----------------|--------------|
| \( R_p \) | Planet radius\(^a\) | \( R_\oplus \) | 1.0 | 1.0 |
| \( g \) | Surface gravity\(^a\) | m \( \cdot \) s\(^{-2} \) | 9.8 | 9.8 |

### Atmospheric properties

| \( R_d \) | Gas constant for dry air\(^a\) | J kg\(^{-1} \) K\(^{-1} \) | 287 | 188.9 |
| \( C_p \) | Heat capacity per unit mass\(^a\) | J kg\(^{-1} \) K\(^{-1} \) | 1005 | 909.3 |
| \( \kappa_S \) | SW absorption coefficient\(^b\) | m\(^2\) kg\(^{-1} \) | 10\(^{-6} \) | 10\(^{-6} \) |
| \( \kappa_L \) | LW absorption coefficient\(^b\) | m\(^2\) kg\(^{-1} \) | 10\(^{-4} \) | 2.5 \( \times \) 10\(^{-4} \) |
| \( \beta_{SW} \) | SW scattering parameter | – | 1.0 | 1.0 |
| \( \beta_{LO} \) | LW scattering parameter | – | 1.0 | 1.0 |

### Surface properties

| \( \varepsilon_s \) | Surface emissivity | – | 1.0 | 1.0 |
| \( A_s \) | Surface albedo\(^d\) | – | 0.2 | 0.2 |
| \( I_H \) | Thermal inertia\(^c\) | J m\(^{-2} \) s\(^{-1/2} \) K\(^{-1} \) | 2000 | 2000 |
| \( z_o \) | Roughness height\(^c\) | m | \( 3.21 \times 10^{-5} \) | \( 3.21 \times 10^{-5} \) |

\(^a\) Values for the Earth’s atmosphere given by Deitrick et al. (2020), Table 2, in the Earth-like case. In the pure CO₂ case, \( R_d \) is calculated from Meija et al. (2016a) and \( C_p \) is evaluated for \( T = 350 \) K from Yaws (1996), Appendix E.

\(^b\) Defined so that the optical depths in the longwave and in the shortwave at \( p = 1 \) bar are \( \tau_L = 1 \) and \( \tau_S = 10^{-2} \tau_L \), respectively, in the Earth-like case. In the pure CO₂ case, the optical depth in the longwave is set to an effective value for which the collapse pressure computed using the 2-D instance of the meta-model in Sect. 5 corresponds to that obtained by Wordsworth (2015) from numerical simulations performed with a 3-D GCM with correlated-\( k \) radiative transfer (Lacis & Oinas 1991).

\(^c\) Typical values for Venus-like radiative transfer (e.g. Lebonnois et al. 2010).

\(^d\) Defined so that \( C_N = 10^{-3} \) for a 10 m-thick surface layer, following Frierson et al. (2006).

The above equation is solved by means of the finite difference method between the surface (\( \ddot{u} = 0 \)) and an inner boundary (\( \ddot{u} = \dot{b}_{\text{bot}} \)) using an implicit time scheme. At the surface, a continuity condition is applied: the incoming heat fluxes equalise the outgoing fluxes. At the inner boundary, a zero-flux condition is applied, which enforces the assumption that the planet interior is in thermal equilibrium. These two conditions are respectively formulated as

\[
-F_c + F_L - F_{1S} - F_H - \varepsilon_s \sigma_{SB} T_s^4 = 0 \quad \text{at} \quad \ddot{u} = 0 ,
\]

\[
F_c = 0 \quad \text{at} \quad \ddot{u} = \dot{b}_{\text{bot}} ,
\]

where \( F_L \) designates the downward radiative flux (i.e. the sum of the shortwave and longwave contributions), \( F_{1S} \) the reflected shortwave flux, \( T_s \) the surface temperature, and \( \varepsilon_s \) the surface emissivity in the longwave. The scheme used to solve the soil heat transfer is detailed in Appendix G. Similarly as the radiative transfer and turbulent diffusion equations, the soil heat transfer equation is discretised and integrated as a boundary condition problem by means of the tridiagonal matrix algorithm (see Appendix H).

Several simulations were run in order to assess the sensitivity of mean fields to the heat transfer scheme. The obtained results are discussed in Appendix G. They show that mean flows and temperatures do not depend much on the ground thermal inertia that parametrises the soil heat transfer scheme (Eq. (41)). Particularly, the nightside surface temperature is almost insensitive to the value chosen for \( I_g \). Nevertheless, we recall that horizontal diffusion is ignored in the scheme since we focus on dry rocky planets, while it might play an important role for an ocean planet due to heat advection by oceanic flows, as discussed by Wordsworth (2015).

### 3.5. Physical setup

In the following, we perform simulations for the two cases defined in Table 2: (i) and Earth-like atmosphere, and (ii) a pure CO₂ atmosphere. For the Earth-like case, we use the values given by Deitrick et al. (2020) in the synchronous Earth case (see Deitrick et al. 2020, Table 2). These values correspond to a tide-locked Earth-sized planet with an atmosphere having the thermodynamical properties of the Earth’s atmosphere. Following Wordsworth (2015), we assume that \( \tau_L = 1 \) at \( p = 1 \) bar. Similarly, we fix \( \tau_S = 0.01 \) at \( p = 1 \) bar to enforce the assumption that the atmosphere is optically thin in the visible. The effective absorption coefficients introduced in Eq. (25) are set accordingly. Besides, we consider the case of pure absorption (no scattering). The scattering parameter is therefore set to \( \beta_L = 1 \) both for the shortwave and longwave. Assuming that the planet’s surface is made of bare rocks, we use the typical values commonly assumed for Venus-like soils (e.g. Lebonnois et al. 2010) to set the planet’s surface properties. Following Frierson et al. (2006), the roughness height is set to \( z_o = 3.21 \times 10^{-5} \) m so that \( C_N = 10^{-3} \) for \( z_{\text{SL}} = 10 \) m.

The pure CO₂ case is similar to the Earth-like case except for the specific gas constant, \( R_d = 188.9 \) J kg\(^{-1} \) K\(^{-1} \) (calculated from Meija et al. 2016b), the heat capacity per unit mass, \( C_p = 909.3 \) J kg\(^{-1} \) K\(^{-1} \) (evaluated for \( T = 350 \) K from Yaws 1996, Appendix E), and the absorption coefficient in the longwave, \( \kappa_L = 2.5 \times 10^{-4} \) m\(^2\) kg\(^{-1} \). The latter is an effective value of \( \kappa_L \) for which the 2-D instance of the GCM presented in Sect. 4 approximately reproduces the stability diagram obtained by Wordsworth (2015) from 3-D GCM simulations with correlated-\( k \) radiative transfer (Wordsworth 2015, Fig. 12), as shown in Sect. 5. We note that this value is less than that obtained by Wordsworth (2015) by adjusting the analytic solution given by Eq. (5) to GCM simulations (\( \kappa_L = 3.2 \times 10^{-4} \) m\(^2\) kg\(^{-1} \) by
taking into account the factor 2 difference between the definition of $k_L$ given by Eq. (12) of the article and ours, given by Eq. (25)). As highlighted by Sect. 5, this discrepancy is due to the fact that the planetary boundary layer acts to warm up the nightside surface in the general case, which consequently requires a weaker greenhouse effect to reach the same thermal state.

4. Climate regime: from 0-D to 2-D models

The GCMM described in the preceding section can be used at the same time for various grid configurations, each of them being a possible instance of the meta-model. This allows for comparing the results obtained from a bench of models of various complexities albeit sharing the same intrinsic theoretical background and physical setup. Four instances of the meta-model are examined in the present study (Table 1). There are introduced below in as-
cending order of complexity, while the resulting pressure, temperature, and vertical wind speed snapshots are plotted in Fig. 1.

We note that, for legibility, the arrows representing wind speeds are uniformly spaced both horizontally and vertically in the figure, rather than being centred at the points where winds speeds are really evaluated. For instance, in the 1.5-D model, the arrows are determined from a linear interpolation and located at $\theta = 45^\circ$, $135^\circ$, while horizontal wind speeds are evaluated at $\theta = 90^\circ$. In the range $180^\circ \leq \theta \leq 360^\circ$, the distributions are plotted by applying a symmetric transformation to the distributions calculated in the range $0^\circ \leq \theta \leq 180^\circ$. The additional inner ring in temperature snapshots (middle column of Fig. 1) corresponds to the soil temperature.

The 0-D model ($1 \times 1$ grid) is the simplest configuration. Here the atmosphere is vertically and horizontally isothermal, and it exchanges heat with dayside and nightside isothermal surface hemispheres through radiative transfer only. This configuration corresponds exactly to that of the two-layer grey radiative model described in Sect. 3 of Auclair-Desrotour & Heng (2020), which provides closed-form solutions. In the optically thin limit, it reproduces the analytic model introduced by Wordsworth (2015).

The next level of complexity is the 1-D model ($1 \times 50$ grid). In this configuration, the vertical structure of the atmosphere is allowed to adjust with radiative transfer although it is still horizontally isothermal. Similarly as in the 0-D configuration, the planet’s surface is divided into the dayside and nightside hemispheres, and the surface-atmosphere heat exchanges are purely radiative. The 1-D model thus provides a more realistic vertical temperature profile than that assumed in the 0-D model. This profile can be interpreted as an approximation of the globally averaged atmospheric structure. This instance of the GCMM appears as a simplified version of more sophisticated one-dimensional models (e.g. Robinson & Catling 2012).

Then we introduce the 1.5-D model ($2 \times 50$ grid). This level corresponds to the two-column approach, where the atmosphere is modelled by dayside and nightside hemispherical air columns, similarly as the planet’s surface (e.g. Yang & Abbot 2014; Koll & Abbot 2016). Therefore the vertical structure ceases to be horizontally uniform and differences appear between the dayside and the nightside. However these differences are mitigated by the fact that the dayside and nightside planetary boundary layers are both controlled by the horizontal wind speed at the planet’s terminator ($\theta = 90^\circ$) here, whereas horizontal wind speeds strongly differ between the dayside and nightside in reality. Besides, this configuration allows for taking into account the coupling between the thermodynamics and the day-night overturning circulation, and particularly the contribution of heat advection to the planet’s thermal state of equilibrium.

Finally, the 2-D model ($32 \times 50$ grid) is the more complex instance of the meta-model. In this configuration, the latter behaves similarly as a 2-D GCM (e.g. Song et al. 2021), or as a 3-D GCM in the slow rotation regime (e.g. Leconte et al. 2013; Haqq-Misra et al. 2018; Pierre Humbert & Hammond 2019; Turbet et al. 2021). Both the atmospheric structure and mean flows are fully resolved, which describes the global heat engine circulation. The planetary boundary layer strongly differs between the dayside, where it is unstable ($Ri < 0$) due to convection (see e.g. Koll & Abbot 2016), and the nightside, where it is stable ($Ri > 0$). Also, the circulation is highly asymmetric with respect to the terminator: as shown by Fig. 1 (top panels), it exhibits strong upwelling flows in a small region around the substellar point, and weak subsidence over a large area that spreads from $\theta \approx 60^\circ$ to $\theta = 180^\circ$ (anti-stellar point).

We recover the day-night evolution of the atmospheric structure in the potential temperature distribution shown by Fig. 2, which was obtained using the 2-D model. In this figure, the latitude is measured with respect to the terminator, located at $0^\circ$. The sub- and anti-stellar point thus correspond to the latitudes $90^\circ$ and $-90^\circ$, respectively. The nightside stably stratified region is characterised by a positive vertical potential temperature gradient. On dayside, the convective mixing induced by the thermal forcing makes the temperature gradient converge towards the adiabat. As a consequence, convective regions are indicated by vertically uniform profiles of potential temperature. We note that the thickness of the planetary boundary layer grows as the latitude increases, and that it is maximal at the sub-stellar point.

Owing to spatial bi-dimensionality, the solver is relatively fast even for the 2-D instance, where one day of simulation with a two-minute dynamical timestep is equivalent to 0.61 seconds of CPU time on one CPU. This allows for running grid simulations, which is the purpose of the next section.

5. Stability diagrams

In this section, we examine both the role played by several physical features in the atmospheric stability, and the sensitivity of model predictions to the simplifications made in the different approaches. To do so, we proceed to a vertical inter-comparison between the four models introduced in Sect. 4 for the two Earth-sized synchronous planets defined in Table 2. For each instance of the meta-model, simulations were performed on a $15 \times 13$ grid in the space of stellar flux and initial surface pressure, with $0.2 F_{\odot} \leq F_* \leq 3 F_{\odot}$ and $0.01 \text{ bar} \leq p_s \leq 10 \text{ bar}$, $F_{\odot} = 1366 \text{ W m}^{-2}$ being the Earth’s incident stellar flux. Starting from isothermal and zero-velocity initial conditions, simulations were run for a period $t_{\text{sim}} = \min \{ \max \{ t_{\text{rad}}, t_{\text{min}} \}, t_{\text{max}} \}$ ranging between $t_{\text{min}} = 300$ and $t_{\text{max}} = 30,000$ Earth days, $t_{\text{rad}}$ being an empirical estimate of the timescale necessary to reach radiative equilibrium that includes the dependence of the radiative cooling timescale on surface pressure and stellar flux (e.g.,
At the end of simulations, two-day averaged distributions of mean fields were computed, as well as the resulting minimum (or nightside) surface temperature ($T_n$).

Showman & Guillot (2002),

$$t_{	ext{cad}} = 900 \text{ days} \times \left( \frac{p_s}{1 \text{ bar}} \right) \left( \frac{F_*}{1366 \text{ W m}^{-2}} \right)^{-3/4}. \quad (45)$$

As highlighted by Wang & Wordsworth (2020) in the case of sub-Neptunes, the convergence time of 3-D numerical simulations can be extremely long ($t_{\text{sim}} \sim 250,000$ Earth days, typically) for massive atmospheres ($p_s \gtrsim 80$ bar) owing to the long radiative timescale of deep atmospheric layers. In the present study, the maximum surface pressure (10 bar) is smaller than that usually assumed for sub-Neptunes, the circulation described by our 2-D model is simpler than that described by 3-
D GCMs – where complex structures and super-rotating zonal jets can emerge, and the major part of the incident stellar flux reaches the planet’s surface, which tends to facilitate vertical energy transport. Therefore the resulting convergence times are expected to be smaller than those reached in the case of sub-Neptunes by one order of magnitude approximately. However, we verified a posteriori that both the energy balance between the outgoing longwave radiation (OLR) and the absorbed stellar radiation (ASR) on the one hand, and the circulation on the other hand, had reached a steady state at the end of several test simulations. Besides, by running grid simulations with larger values of $f_{\text{run}}$, we noticed that increasing this parameter did not affect the considered mean fields.

Following earlier studies (Wordsworth 2015; Koll & Abbot 2016; Auclair-Desrotour & Heng 2020), we assume that the greenhouse effect is mainly due to the presence of CO$_2$ in the atmosphere. The condensation temperature of CO$_2$ is given, in K, by (Fanale et al. 1982; Wordsworth et al. 2010b; Wordsworth 2015)

$$T_{\text{cond,CO}_2}(p) = \begin{cases} 
3167.8 & \text{if } p < p_{\text{tr}} , \\
23.23 - \ln (0.01 p) & \text{if } p \geq p_{\text{tr}}, 
\end{cases}$$

where the partial pressure of the gas $p$ is given in Pa, and $p_{\text{tr}} = 5.18 \times 10^5$ Pa designates the triple point pressure. The stability diagrams are therefore obtained by comparing the minimum surface temperature calculated from simulations with the condensation temperature of CO$_2$ at the planet’s surface, $T_{\text{cond,CO}_2}(\chi_{\text{ps}})$, where $\chi$ designates the volume mixing ratio of CO$_2$. In the Earth-like case, the volume mixing ratio of CO$_2$ is set to the value of Earth at the beginning of the twenty-first century, namely $\chi = 370$ ppm (e.g. Etheridge et al. 1996), while $\chi = 1.0$ in the pure CO$_2$ case. The atmosphere is considered to be stable if $T_n(F_*, p_s) > T_{\text{cond,CO}_2}(\chi_{\text{ps}})$ and unstable else. We remark that the collapse itself, when it occurs, is not described by the model since the changes of phases of CO$_2$ are not taken into account.

Figure 3 shows the simulation results. The minimum surface temperature is plotted in both cases as a function of the normalised stellar flux $F_*/F_\oplus$ and the surface pressure $p_s$ in logarithmic scale. The stability diagrams are plotted too, with large red dots indicating stability and small blue dots collapse. The collapse pressures associated with the lower and upper bounds of the nightside temperature given by Eqs. (5) and (10), are obtained by solving for $p_s$ the equations

$$T_{n,\text{low}}(F_*, p_s) = T_{\text{cond,CO}_2}(\chi_{p_s}),$$

$$T_{n,\text{up}}(F_*, p_s) = T_{\text{cond,CO}_2}(\chi_{p_s}),$$

and are denoted by $p_{\text{C,low}}$ (orange dashed line) and $p_{\text{C,up}}$ (pink dotted line), respectively.

With the 0-D model, we recover the behaviour of the nightside temperature predicted by the closed-form solutions of Auclair-Desrotour & Heng (2020) in the purely radiative regime. This is due to the fact that the two models are actually the same in this configuration, the temperatures being computed numerically here instead of analytically. Compared with the other models, the 0-D model tends to underestimate the nightside temperature in the high surface pressure regime. As the surface pressure increases, $T_n$ reaches a plateau corresponding to the planet’s equilibrium temperature $T_{\text{eq}}(F_*)$ given by Eq. (6), and does not evolve with $p_s$ any more. This unrealistic behaviour is a consequence of the isothermal approximation, which does not account for the strong vertical temperature gradient characterising thick atmospheres, especially their convective regions. In the low stellar flux regime, the 0-D model captures the stability decrease observed for pure CO$_2$ atmospheres in the 3-D GCM simulations performed by Wordsworth (2015). However, this feature is due to the isothermal temperature profile too since it vanishes from the moment that the atmospheric structure is allowed to adjust with radiative transfer, in models of higher dimensions. This effect is a caveat of the limitations of idealised models in explaining predictions of much more sophisticated 3-D GCMs.

The 1-D model exhibits the same behaviour as the 0-D model for surface pressures less than 1 bar, which corresponds to the regime where the vertically isothermal approximation holds. Beyond $p_s \approx 1$ bar, the nightside surface temperature increases as a function of both the stellar flux and surface pressure, which makes the collapse pressure associated with $T_{n,\text{low}}$ capture the threshold of the stability region for the whole stellar flux interval. The two-column configuration (1.5-D model) relaxes the horizontally isothermal atmospheric approximation. As a consequence, the nightside temperature becomes dependent upon the efficiency of the interhemispheric heat redistribution, and can thereby be less than the lower bound obtained in the horizontally isothermal atmospheric approximation, $T_{n,\text{low}}$. The coarse spatial resolution of the 1.5-D model for the horizontal direction does not account for the strong convection generated in the substellar region. The wind speed is therefore underestimated, and so the strength of the overturning circulation and the heat advected from dayside to nightside are. This leads to the observed stability decrease: the collapse pressure is approximately increased by $\sim 25\%$ with respect to $p_{\text{C,low}}$ in the Earth-like case, and by $\sim 80\%$ in the pure CO$_2$ case.

Conversely, the 2-D model predicts a wider stability region. The collapse pressure is lowered by $\sim 10\%$–$40\%$ with respect to $p_{\text{C,low}}$ for $F_* \gtrsim 0.7F_\oplus$, which is significant albeit less than the 75% maximum decrease predicted by the analytic theory (see Auclair-Desrotour & Heng 2020, Eq. (86)) in the case of intense sensible heating ($L_{\text{def}} \rightarrow +\infty$, see Eq. (9)). This stability increase results from the effect of the planetary boundary layer. The vertical turbulent diffusion generated by the friction of mean flows against the planet’s surface in the planetary layer acts both (i) to increase the thermal forcing of the atmosphere by intensifying sensible exchanges, and (ii) to enhance the day-night heat advection by strengthening the overturning circulation. As a consequence, the nightside surface is warmer by $\sim 4$–$14$ K in the vicinity of the threshold between the stability and collapse regions.

The nightside temperature increase induced by the planetary boundary layer was quantified by running simulations in the 2-D configuration without turbulent diffusion both for Earth-like and pure CO$_2$ atmospheres. In these simulations, the surface-atmosphere heat exchanges are induced by radiative transfer only, and there is no friction of mean flows against the surface. Figure 4 shows the resulting stability diagrams, as well as the corresponding nightside surface temperature difference between the cases with and without turbulent diffusion, denoted by the superscripts TD (Turbulent Diffusion) and NTD (No Turbulent Diffusion), respectively (top panels). In addition with the nightside surface temperature, we consider the day-night advection timescale $t_{\text{adv}}$, which is defined here as the mean period necessary for a fluid parcel to accomplish one full cycle of the day-night overturning circulation (see Appendix 1),

$$t_{\text{adv}} = \frac{4R_{\oplus}}{\int_0^{90} |\nu_0| \, \text{d}\theta}.$$
where the subscript 90° indicates that the integral of the mass flow rate is performed over the terminator annulus (θ = 90°). The day-night advection timescale also corresponds to the mean renewal time of the air contained in one atmospheric hemisphere (dayside or nightside). This timescale can be compared to the advection timescales introduced in earlier studies, such as the analytic expression of the lower bound obtained by Koll & Abbot (2016) from the heat engine theory (Koll & Abbot 2016, Eq. (12)),

\[ t_{\text{adv}}^{\text{KA}} = \frac{R_p}{v_{\text{sen}}} \]  

(50)

where the typical speed \( v_{\text{sen}} \), given by Eq. (8), is a function of the stellar flux, the surface pressure, the surface albedo, the dayside surface temperature, the optical depth in the longwave at surface, the specific gas constant, and the bulk drag coefficient of the surface layer. The timescale \( t_{\text{adv}}^{\text{KA}} \) is estimated by setting the bulk drag coefficient to the typical value \( C_D = 10^{-3} \) for convenience and
by taking the maximum surface temperature for the dayside temperature. Thus, in addition with the nightside temperature difference mentioned above, we plot in Fig. 4 the day-night advection timescale in the case with turbulent diffusion (bottom panels), the ratio of this timescale over \( t_{adv}^{KA} \), and the ratio between advection timescales in the cases with and without turbulent diffusion (middle panels).

We first consider the stability diagrams obtained in the absence of turbulent diffusion (Fig. 4, top and middle panels). Similarly as in the 1-D configuration, the threshold of the stability region coincides with the lower bound of \( T_n \) associated with the purely radiative regime in the radiative box model, namely \( p_C,low \). This indicates that the bulk atmosphere is horizontally isothermal for \( p_s \geq 0.1 \) bar, which corresponds to an efficient interhemispheric heat redistribution. As highlighted by temperature differences (Fig. 4, top panels), turbulent diffusion tends to warm up the nightside surface in the general case with, for instance a temperature increase of \( 6 - 28 \) K in the Earth-like case. However this temperature increase does not vary monotonically with the stellar flux and surface pressure, but instead it exhibits a bi-modal behaviour with maxima and minima depending on surface pressure.

Particularly, turbulent diffusion in the PBL somehow counterintuitively acts to decrease the nightside surface temperature instead of increasing it in a region centred on \( p_s \sim 1 \) bar for pure CO\(_2\) atmospheres, with a minimum of \(-8\) K for \( p_s = 1 \) bar and \( F_s = 1.2 F_\odot \). A similar – although slightly smaller – negative difference (\(-5\) K) was obtained by running gray gas simulations with the fully global 3-D GCM THOR (Mendonça et al. 2016; Deitrick et al. 2020) in this configuration using the values given by Table 2 and assuming a zero-spin angular velocity, which tends to corroborate the prediction of the 2-D model. This effect of the planetary boundary layer can be predicted through the interplay between the day-night advection timescale given by Eq. (49) and the dayside radiative timescale, \( t_{rad} \), which is the typical timescale needed for a warm fluid parcel located in the upper atmosphere to cool down radiatively. Both parameters are altered by the turbulent diffusion taking place within the planetary boundary layer.

Notwithstanding the high pressure – and optically thick – regime, where strong convection develops, the effect of turbulent diffusion reaches a maximum around \( p_s \sim 0.1 \) bar for Earth-like atmospheres and around \( p_s \sim 0.03 \) bar for pure CO\(_2\) atmospheres (Fig. 4, top panels). This maximum is consistent with the fact that the additional thermal forcing due to turbulent diffusion is all the more significant as the radiative absorption is weak, which tends to maximise the impact of turbulent diffusion for small optical thicknesses. However, the radiative timescale of the atmosphere scales as \( t_{rad} \propto p_s/T^4 \) (e.g. Showman & Guillot 2002, Eq. (10)) in the optically thin regime, meaning that the heat surplus provided by sensible exchanges is radiated towards space over timescales that become extremely short as the surface pressure tends to zero. Thus, in spite of the decay of \( t_{adv} \) induced by turbulent diffusion, a fluid parcel is radiatively cooled before being advected to nightside by mean flows in the optically thin limit \( t_{rad} \ll t_{adv} \), which mitigates the impact of the PBL in this regime and makes the temperature difference decay as \( p_s \rightarrow 0 \).

Similarly, as the surface pressure increases, the atmosphere switches from the optically thin regime to the optically thick regime, with a transition occurring at lower pressures in the pure CO\(_2\) case than in the Earth-like case due to the difference between optical depths in the infrared. In this transition regime, the PBL strongly affects the advection timescale, which is increased up to three times (Fig. 4, middle panels) and becomes thereby greater than the radiative timescale. As a consequence, less heat is advected towards nightside than in the absence of PBL although the latter generates a heat surplus on dayside, and the nightside temperature difference falls to the observed minimum valley where it reaches negative values in the pure CO\(_2\) case.

We note that this behaviour could be significantly altered by the presence of gas, dust, and aerosols inducing the so-called anti-greenhouse effect by increasing the shortwave scattering and absorption, as observed on Titan (McKay et al. 1991). The anti-greenhouse effect refers to the cooling of the planet surface resulting from the fact that a substantial part of the incident stellar flux is absorbed and re-radiated towards space in the infrared by the upper layers of the atmosphere, and that only a fraction of it reaches the surface (e.g. Pierrehumbert 2010). This effect is likely to play a major role on rocky planets with thick atmospheres similar to Venus, where only \~0.1 – 1% of the incident solar flux reaches the surface (Lacis 1975).

We now consider the evolution of the advection timescale itself, which is plotted in the case including turbulent diffusion in the PBL (Fig. 4, bottom panels). Similarly as the nightside surface temperature (Fig. 3), \( t_{adv} \) exhibits a relatively smooth dependence on the incident stellar flux and surface pressure, with values spanning over two orders of magnitude from \~8\) days in the low pressure-high stellar flux regime to \~500\) days in the high pressure-low stellar flux regime. These values are larger than those given by the analytic scaling law of Koll & Abbot (2016) by one order of magnitude owing to the difference in the used definitions: since it is computed from the horizontal wind speed in the substellar region, the analytic advection timescale \( t_{adv}^{KA} \) is necessarily smaller than the advection timescale defined by Eq. (49), which is computed from mass flows going through the terminator annulus. Notwithstanding this scaling factor, \( t_{adv}^{TD} \) matches \( t_{adv}^{KA} \) relatively well for both Earth-like and pure CO\(_2\) atmospheres in the optically thin regime, where the ratio varies by a factor of two. The two quantities diverge from each other as the stellar flux decreases and the surface pressure increases, the model of the present study predicting larger advection timescales in this regime. However, we remark that this discrepancy is less significant for pure CO\(_2\) atmospheres than for Earth-like atmospheres, which suggests that the thermodynamic and absorption properties of the gas affect the large scale overturning circulation in a non-negligible way when the atmosphere is optically thick.

The strength of the overturning circulation can be characterised by examining the behaviour of the Eulerian mean streamfunction, which is defined in tidally locked coordinates as (e.g. Pauluis et al. 2008)

\[
\Psi = \frac{2\pi R_p}{g} \int_0^{\mu_p} v_b \sin \theta \, dp. \tag{51}
\]

The Eulerian mean streamfunction measures here the strength of longitude-averaged cells in the TLC. It accounts for the vorticity of the flow in a plane containing the planet-star axis. In the slow rotation regime, the large-scale cell of the predominant overturning circulation corresponds to a large region centred on a maximum of \( \Psi \) in absolute value, the latter taking negative values owing to the flow direction (see e.g. Fig. 6 in next section). The maximum value of \( \Psi \) over the atmospheric domain, defined as

\[
\Psi_{max} = \max (-\Psi), \tag{52}
\]

can be scaled analytically, as shown by Innes & Pierrehumbert (2021) who established for sub-Neptunes the scaling law
In the case of rocky planets, we need to account for the presence of the planet’s surface, which sizes the thickness of the atmosphere. The mass flow thus depends on the atmospheric mass in addition with the stellar flux. As the atmospheric mass is directly proportional to the surface pressure, we introduce for rocky planets a scaling law of the form

$$\Psi_{\text{max}} \propto F_r^{3/4}. \tag{52}$$

where $F_r$ is the planetary flux and $F_\oplus$ the Earth flux.

We remark that $(\alpha, \beta) = (3/4, 0)$ corresponds to the scaling law obtained by Innes & Pierrehumbert (2021) for super-Neptunes. However, for the Earth-sized rocky planets of the present study, we adopt the empirical values $(\alpha, \beta) = (1/2, 1)$, which seem to be more appropriate as discussed further.

Figure 5 shows the evolution of $\Psi_{\text{max}}$ and of the ratio $\Psi_{\text{max}}/\Psi_{\text{SL}}$ as functions of the planet’s surface pressure, where $\Psi_{\text{max}}$ increases faster with $F_r$. Conversely, the slow evolution of $\Psi_{\text{max}}/\Psi_{\text{SL}}$ with $p_s$ at low surface pressures suggests that the scaling law $\Psi_{\text{max}} \propto p_s$ captures well the dependence of $\Psi_{\text{max}}$ on the surface pressure in the regime of optically thin atmospheres, while this dependence is weaker in the regime of thick atmospheres.

6. From slow to fast rotation

Since the GCMM is designed to study the asymptotic regime of slowly rotatating planets, where mean flows are predominantly driven by the day-night temperature gradient, we have ignored the effects of rotation on the general circulation until now. As discussed in Sect. 2, the impact of these effects is mainly controlled by the nondimensional equatorial Rossby deformation radius $L_{\text{eo}}$ introduced in Eq. (1). The zero-spin rate limit treated by the model corresponds to $L_{\text{eo}} = +\infty$. In reality Coriolis acceleration tends to deviate the divergent winds blowing between the substellar and the anti-stellar points. This alters the large-scale circulation regime, which switches from the bi-dimensional Hadley cell described by the 2-D model to the three-dimensional structure characterising fast rotators, where superrotation develops, as $L_{\text{eo}}$ decays. In this section, we investigate the limitations of the bi-dimensional approach by benchmarking the results obtained with the 2-D instance of the GCMM against the simulations performed with THOR GCM (Mendonça et al. 2016; Deitrick et al. 2020), which solves the three-dimensional nonhydrostatic Euler equations on an icosahedral grid. Particularly, THOR fully accounts for the horizontal vertical component of mean flows that is ignored in the 2-D model.

Four simulations were performed with THOR for the pure CO$_2$ atmospheres defined by Table 2. They each correspond to a given spin period $P$ taken in a range spanning from fast to slow rotators all things being equal ($P = 10^{-3}, 10^{-2}, 10^{-1}, 10^0$ $P_0$
with $P_0 = 365$ days). For comparison, one additional simulation was run with the 3-D GCM in the slow rotator case ($P = P_0$) by assuming no turbulent diffusion or sensible surface-atmosphere heat exchanges. In these simulations, the values of the surface pressure ($p_s = 0.18$ bar) and stellar flux ($F_s = F_{\text{GB}}$) were chosen so that the studied cases are in the vicinity of the threshold between the stability and collapse regions predicted by the 2-D instance of the meta-model (see Fig. 3, top right panel). The horizontal resolution of the icosahedral grid used in the model was set to $\sim 4$ degrees, and the atmosphere was divided into 40 vertical intervals logarithmically refined in the vicinity of the surface with a top altitude of 37000 m and a 2 m-thick lowest layer. For the sake of consistency, the simulations were run using the double-gray approximation for radiative transfer and a physical setup as close as possible that in the GCMM. After the circulation and radiative transfer have reached a state of equilibrium, the physical quantities were averaged over the longitude of the tidally locked coordinates to transform the three-dimensional fields into two-dimensional fields similar to those displayed in Fig. 1. The bulk dimensionless equatorial Rossby deformation length ($L_{\text{RD}}$) was computed from the resulting mass-averaged temperature using the expression given by Eq. (4). In parallel of the simulations performed with Thor, a simulation corresponding to the zero-spin limit was run with the 2-D instance of the meta-model.

Notwithstanding the vertical coordinates used in dynamical cores – altitude-based in Thor, and mass-based in the GCMM – the main differences between the two models essentially lie in the description of the surface thermal response, planetary boundary layer, and numerical energy dissipation. In Thor, the surface temperature evolution is integrated using a 0-D thermodynamic equation parametrised by an effective surface heat capacity ($C_s = 10^7$ J K$^{-1}$ m$^{-2}$), while a 1-D soil heat transfer scheme is used in the GCMM (see Appendix G). In the treatment of the planetary boundary layer, the two models closely follow the method proposed by Holtslag & Boville (1993), except for some parameters, which are set to different values. For instance, the asymptotic length scale characterising the evolution of the mixing length with altitude (Eq. (31)) for the heat equation is set to three times the function used for the momentum equation (Eq. (F.6)) in Thor, while the same function is used in both cases in the GCMM. As regards numerical energy dissipation, all THOR simulations utilized 4th order horizontal hyperdiffusion and 3-D divergence damping with a nondimensional coefficient $D_{\text{hyp}} = D_{\text{div}} = 0.002$ (see Mendoza et al. 2016; Deitrick et al. 2020, for descriptions of the hyperdiffusion scheme). We addi-

![Fig. 6. Dayside and nightside surface temperatures vs. normalised equatorial Rossby deformation radius ($\tilde{R}_D$) in the gray 3-D GCM simulations performed with THOR. Top left: maximum (red solid line) and hemisphere-averaged (purple solid line) dayside surface temperatures. Top middle: minimum (blue solid line) and hemisphere-averaged (green solid line) nightside surface temperature. Top right: Day-night averaged (grey solid line) and extremal (black solid line) surface temperature differences. Dashed lines, dotted lines, star symbols, and square points indicate (i) the asymptotic surface temperatures computed using the 2-D model, (ii) the hemisphere-averaged surface temperatures predicted by Wordsworth’s greenhouse model (W15, Wordsworth 2015), (iii) the surface temperatures obtained in the 3-D simulation without turbulent diffusion in the slow rotator case, and (iv) those obtained in the simulation performed with the same asymptotic scale lengths in the PBL as in the GCMM, respectively. Bottom, from left to right: Averaged snapshots of the Eulerian mean streamfunction (e.g. Pauluis et al. 2008) obtained from the 2-D simulation (zero-spin rate limit), the 3-D simulation for $P = 365$ days (slow rotator), and the 3-D simulation for $P = 0.365$ days (fast rotator).]
lectronically used 6th order vertical hyperdiffusion with a nondimensional coefficient of $D_{\text{vert}} = 5 \times 10^{-4}$. Finally, a sponge layer was applied to the upper 25% of the model domain to damp spuriously reflected waves off the top boundary, with a damping time scale of 5000 s (see Mendonça et al. 2018; Deitrick et al. 2020, for the sponge layer description).

Figure 6 shows the simulation results. The dayside and nightside surface temperatures, as well as the day-night surface temperature difference, are plotted as a function of the bulk equatorial Rossby deformation radius (top panels). Solid lines designate the temperatures obtained in THOR 3-D simulations, and the horizontal dashed lines the temperatures obtained in the 2-D simulation performed with the meta-model. The dotted lines indicate the corresponding averaged dayside and nightside surface temperatures predicted by Wordsworth’s purely radiative greenhouse model (Wordsworth 2015). The temperatures obtained from the 3-D simulation performed in the absence of turbulent diffusion are designated by the star symbol *. In addition, the Eulerian mean streamfunction defined in Eq. (51) is plotted for the 2-D model and the two extrema of the 3-D model ($P = 365$ days and $P = 0.365$ days) as functions of pressure (or sigma-coordinate) and the latitude of the TLC, where the North and South poles correspond to the substellar and antistellar points, respectively (bottom panels).

We first consider the evolution of surface temperatures with $L_{Ro}$ (Fig. 6, top panels). Whereas the dayside surface temperature predicted by 3-D GCM simulations is hardly affected by the planet’s rotation, the nightside surface temperature increases monotonically with the equatorial Rossby deformation radius until it converges towards the asymptotic limit of slow rotators described by the 2-D model. The effect of rotation is particularly significant for the minimum surface temperature, which varies by ~35 K between the two extremal cases. This difference results from the decay of day-night advection provoked by the breaking of the day-night overturning circulation in the fast rotator regime ($L_{Ro} \sim 1$). The super-rotating equatorial jets induced by Coriolis effects do not compensate the overturning circulation in terms of mass flux crossing the terminator annulus, which makes the nightside surface temperature decrease. However we remark that the nightside temperature stays close from the asymptotic limit of slow rotators from the moment that $L_{Ro} > 2$, the difference of the cases $P = 36.5$ days and $P = 365$ days to this limit being less than 3 K. This suggests that the 2-D model is relevant to describe the climate and large-scale circulation of the planet for equatorial Rossby deformation radii exceeding the critical value $L_{Ro} \sim 2$. Below this value, the circulation and heat redistribution are strongly affected by the planet’s rotation.

Besides we remark that the 2-D model and THOR GCM are in agreement with each other for the nightside surface temperature, the latter being greater by ~ 2 K only in 3-D simulations. The discrepancy between the two models is more significant for the dayside average temperature, where the effects of the PBL in surface-atmosphere heat exchanges are predominant. As observed in the 2-D model, the frictional interaction of mean flows with the planet’s surface taking place in the PBL tends to increase the atmospheric stability against collapse by warming up the nightside surface hemisphere. To understand whether the differences in PBL parameters between the 2-D and 3-D models could be the source of the discrepant dayside temperatures, one additional THOR simulation was run at the slowest rotation rate. In this simulation, the asymptotic scale lengths (for heat and momentum mixing in the boundary layer) were set equal to those used in the 2-D model. The resulting values are shown as the square points in top panels of Fig. 6. Interestingly, these results are nearly indistinguishable from the simulation with different scale lengths used for the heat and momentum equations, indicating that the scenario is insensitive to the asymptotic scale lengths used in Equation 31.

We now consider the snapshots of the Eulerian mean streamfunction (Fig. 6, bottom panels). We remark that the vertical coordinates used for the plots in the 2-D and 3-D cases differ from each other, which leads to slight distortions of the distributions. The sigma-coordinate is such that $\sigma = 1$ everywhere at planet’s surface, while the surface pressure is less than the globally averaged surface pressure on the dayside, and greater on the nightside due to the day-night thermal forcing gradient. As a consequence, isobars in sigma-coordinate are slightly shifted downwards on dayside and upwards on nightside. The 3-D simulation performed with THOR GCM in the slow rotation regime is in good agreement with the 2-D simulation performed with the GCM in the zero-spin rate limit (Fig. 6, bottom left and middle panels). The two snapshots both exhibit large day-night cells of comparable strengths albeit slightly weaker in the 2-D model. The snapshot corresponding to the fast rotation regime (bottom right panel) exhibits more complex features due to Coriolis effects although a weak day-night cell still remains visible.

7. Concluding remarks

In this work we have developed a General Circulation Meta-Model (GCMM) to bridge the gap between the analytic solutions provided by simplified greenhouse models for synchronous planets and the numerical simulations obtained from 3-D GCMs in the asymptotic regime of slow rotators. This model hierarchy is based on a systematic bottom-up approach in the spirit of Held (2005), wherein the number of degrees of freedom determines the key properties of complexity that are added or subtracted. The solver of the GCMM integrates the hydrostatical primitive equations using the finite-volume method for arbitrary numbers of horizontal and vertical intervals, each configuration being an instance of the meta-model. Consistently with a previous analytical study (Auclair-Desrotour & Heng 2020), the physics implemented in the meta-model includes double-gray radiative transfer, turbulent diffusion in the planetary boundary layer, and soil heat conduction. Particularly, the meta-model was designed so that the solutions obtained with the 0-D instance exactly correspond to the analytic solutions of the purely radiative box model detailed in Sect. 3.3 of Auclair-Desrotour & Heng (2020).

As a first step, we proceeded to a vertical model inter-comparison by running grid simulations for four instances of the meta-model (0-D, 1-D, 1.5-D, and 2-D) in the cases of dry Earth-like and pure CO$_2$ atmospheres. In each case, we computed from simulations the nightside surface temperature of the planet as a function of the stellar flux and surface pressure, and the resulting stability diagrams of the atmosphere against collapse. These diagrams are compared to the scaling laws predicted by the analytic theory. With the 0-D and 1-D instances of the meta-model, we recovered the stability diagrams predicted by simplified radiative models in the optically thin regime, which shows that the globally isothermal approximation used in these models is relevant to this regime. The 1.5-D instance tends to underestimate the atmospheric stability, by predicting a collapse pressure 25% to 80% larger than that given by radiative box models. Conversely, the collapse pressure computed using the 2-D instance of the meta-model is 10% to 40% smaller than the analytic estimate owing to the warming effect of the PBL, which is less than
the theoretical 75% maximum decrease predicted by radiative models albeit still significant.

As a second step, we investigated the role played by the planetary boundary layer in the thermal state of equilibrium and atmospheric circulation of the planet by examining with the 2-D instance of the GCM how the turbulent diffusion taking place in the PBL alters the nightside temperature, the day-night advection timescale, and the collapse pressure. We compared the advection timescale obtained from simulations with the analytic scaling law proposed by Koll & Abbot (2016). We observed that the turbulent diffusion taking place in the PBL increases the nightside surface temperature by 4–14 K around the threshold of the stability region. However, we found that the PBL can also contribute to cool down the nightside surface of the planet by acting on the day-night advection timescale in the transition zone between optically thin and optically thick atmospheres. This result was corroborated a posteriori by 3-D GCM simulations. The effect of the PBL on the large-scale circulation is complex and depends upon the interplay between the advection and radiative timescales. The day-night advection timescale estimated with the 2-D model varies over two orders of magnitude in the studied domain of stellar fluxes and surface pressures, with values ranging between 8 and 500 days. In the optically thin regime, its evolution matches relatively well the scaling law derived by Koll & Abbot (2016) but it diverges from it at high pressures and low stellar fluxes. We noticed that this behaviour also depends on the thermodynamic and absorption properties of the atmosphere in the optically thick regime. In addition we empirically obtained, for the circulation of slowly rotating rocky planets, a scaling law analogous to that established analytically by Innes & Pierrehumbert (2021) for sub-Neptunes.

As a third and final step, we characterised the limitations of the slow rotator approximation that forms the foundations of the meta-model by performing simulations with THOR 3-D GCM, the latter fully accounting for the effects of rotation on mean flows. We computed from these simulations the evolution of the planet’s dayside and nightside surface temperatures as functions of the dimensionless equatorial Rossby deformation radius, which controls the large-scale circulation regime of the planet. The results obtained with the 3-D GCM were compared with the outcomes of the 2-D instance of the meta-model. The 3-D GCM simulations highlight the transition between the slow and fast rotation regimes. We found that the 2-D model properly accounts for the climate and large-scale atmospheric circulation from the moment that the normalised equatorial Rossby deformation radius is greater than the critical value \( L_{\text{Ro}} \sim 2 \), which corresponds to a broad region of the parameter space. In the slow rotation regime, the circulation and surface temperature predicted by THOR 3-D GCM and the 2-D instance of the meta-model are similar.

This study is a first attempt to fill the continuum between analytic greenhouse models and 3-D GCMs in a self-consistent way. The obtained results show that the meta-modelling approach is efficient to disentangle the mechanisms determining the climatic state of the planet, which are narrowly coupled together in GCMs. This approach allows for running in parallel series of models of various complexities albeit sharing the very same physical setup, so that each simulation can be interpreted using the others. As a consequence, the final outcome of the meta-model conveys information not only on the climate itself but also on the separated contributions of key physical ingredients such as radiative transfer, atmospheric structure, dynamics, and turbulent friction in the PBL. In addition with these diagnostic aspects, the meta-modelling approach appears as a robust method to refine the analytic theory of planetary climates given that it allows for assessing the relevance of solutions obtained at low spatial dimensionality with consistently generated GCM numerical solutions. Finally, we note that the meta-modelling approach can be extended to moist atmospheres and rapidly rotating planets, but we leave that to the content of a future study.

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The model solves the HPEs given by Eqs. (15-18) in their nondimensional form. Although it is not used in the present study, the moisture conservation equation is included in the solver and we shall give it here for the sake of generality. In its conservative form, this equation reads (e.g. Yao & Stone 1987)

\[
\frac{\partial (\rho q)}{\partial t} + \nabla \cdot (\rho q \mathbf{v}) + \frac{\partial}{\partial \sigma} (\rho q \sigma) = q, 
\]

where \( q \) designates the specific humidity of any tracer, and \( \sigma \) the corresponding net evaporation or condensation rate per unit mass. Since the elemental unit of time of the discretised equations is the dynamical time step \( \Delta t \) used in the time-differencing scheme, it is convenient to normalise the time by \( \Delta t \) so that the time lapse between two dynamical time steps is always unity.

We then introduce the reference pressure \( p_0 \), temperature \( T_0 \), and specific humidity \( q_0 \), from which we can define the reference velocity \( v_0 \), energy per unit mass \( e_0 \), density \( \rho_0 \), force per unit mass \( F_0 \), heat power per unit mass \( Q_0 \), evaporation per unit mass \( q_0 \), Exner function \( E_0 \), and potential temperature \( \Theta_0 \).

\[
\begin{align*}
V_0 &= \sqrt{\frac{C_p T_0}{\Delta t}}, & e_0 &= C_p T_0, & \rho_0 &= \frac{p_0}{C_p T_0}, \\
F_0 &= \sqrt{\frac{C_p T_0}{\Delta t}}, & Q_0 &= \frac{C_p T_0}{\Delta t}, & q_0 &= \frac{q_0}{\Delta t}, \\
E_0 &= C_p \left( \frac{p_0}{\rho_0} \right)^{\gamma}, & \Theta_0 &= \frac{C_p}{E_0}.
\end{align*}
\]

Besides, we make the horizontal coordinate vary within the range \([0, 1]\) like the vertical coordinate by renormalising the co-latitude (\( \theta \)). The normalised variables and cotangent are therefore defined as

\[
\begin{align*}
\tilde{p} &= \frac{p_0}{p_0}, & \tilde{p} &= p, & \tilde{v} &= \frac{v}{v_0}, & \tilde{\Theta} &= \frac{\Theta}{\Theta_0}, \\
\tilde{\Sigma} &= \frac{\Theta}{\Theta_0}, & \tilde{T} &= \frac{T}{T_0}, & \tilde{\Theta} &= \frac{\Theta}{\Theta_0}, & \tilde{\rho} &= \frac{\rho}{\rho_0}, \\
\tilde{q} &= \frac{q}{q_0}, & \tilde{F} &= \frac{F_0}{F_0}, & \tilde{Q} &= \frac{Q}{Q_0}, & \tilde{\varphi} &= \frac{\varphi}{\varphi_0}, \\
\tilde{\vartheta} &= \frac{\vartheta}{\vartheta_0}, & \tilde{\sigma} &= \frac{\sigma}{\sigma_0}, & \tilde{\varphi} &= \varphi.
\end{align*}
\]

As a second step, the spatial coordinates \((\tilde{\theta}, \tilde{\sigma})\) are converted to grid coordinates\(^4\), \((Y, Z)\). In this transformation, the

\[4 \] The notation \( Y \) is chosen in place of \( X \) for the horizontal coordinate in order to be consistent with notations conventionally used in GCMs (\( X \) and \( Y \) for the longitudinal and latitudinal directions, and \( Z \) for the vertical direction).
normalised colatitude $\tilde{\theta} = \tilde{\theta}(Y)$ and vertical coordinate $\sigma = \sigma(Z)$ are assumed to be monotonic functions of $Y$ and $Z$, respectively. These functions are defined further so that the interval between the two boundaries of a finite volume is equal to 1 both in the horizontal and in the vertical directions, as showed by Fig. A.1. The transformation $(\tilde{\theta}, \sigma) \rightarrow (Y, Z)$ is defined at any point by the horizontal and vertical metric coefficients, $c_Y$ and $c_Z$, defined as

\begin{equation}
\frac{c_Y}{c_Y} = \frac{\partial \tilde{\theta}}{\partial Y}, \quad \frac{c_Z}{c_Y} = \frac{\partial \sigma}{\partial Z} \tag{A.4}
\end{equation}

which yields the change relations of partial derivatives

\begin{equation}
\frac{\partial}{\partial \tilde{\theta}} = \frac{1}{c_Y} \frac{\partial}{\partial Y}, \quad \frac{\partial}{\partial \sigma} = \frac{1}{c_Z} \frac{\partial}{\partial Z} \tag{A.5}
\end{equation}

\begin{equation}
\frac{\partial}{\partial Y} = \frac{c_Y}{c_Y} \frac{\partial}{\partial \tilde{\theta}}, \quad \frac{\partial}{\partial Z} = \frac{c_Z}{c_Y} \frac{\partial}{\partial \sigma} \tag{A.6}
\end{equation}

We remark that the change of coordinate is just a dilation if the adopted horizontal and vertical spacings are uniform since $c_Y$ and $c_Z$ are constants in this case. We also notice that $c_Z < 0$ with the chosen sigma coordinate given that $\sigma$ decreases monotonically as the altitude increases.

To include the metric in the primitive equations, we introduce the respective covariant and contravariant horizontal velocities,

\begin{equation}
\tilde{v} = c_Y \tilde{v}_0, \quad \tilde{v} = \frac{\tilde{v}_0}{c_Y}, \tag{A.7}
\end{equation}

the normalised area density (area per unit length),

\begin{equation}
\tilde{A} = c_Y \sin(\theta) \tag{A.8}
\end{equation}

mass density\(^5\),

\begin{equation}
\tilde{m} = -\tilde{v} \tilde{A} c_Z \tag{A.9}
\end{equation}

\(^5\) The minus sign in the expression of $\tilde{m}$ is here to ensure $\tilde{m} > 0$, which is the convention used in the model.

and horizontal and vertical mass flux densities,

\begin{equation}
V = \tilde{m} \tilde{v}, \quad W = \tilde{m} \tilde{v} \frac{\tilde{c}}{c_Z} \tag{A.10}
\end{equation}

After the above manipulations, the system of HPEs given by Eqs. (15-18) and Eq. (A.1) become the normalised primitive equations

\begin{equation}
\frac{\partial \tilde{m}}{\partial t} + b \frac{\partial V}{\partial Y} + \frac{\partial W}{\partial Z} = 0, \tag{A.11}
\end{equation}

\begin{equation}
\frac{\partial}{\partial t}(\tilde{m} \tilde{v}_0) + b \frac{\partial}{\partial Y}(\tilde{V} \tilde{v}_0) + \frac{\partial}{\partial Z}(\tilde{W} \tilde{v}_0) = 0, \tag{A.12}
\end{equation}

\begin{equation}
\frac{\partial}{\partial t}(\tilde{m} \tilde{v}) + b \frac{\partial}{\partial Y}(\tilde{V} \tilde{v}) + \frac{\partial}{\partial Z}(\tilde{W} \tilde{v}) = \frac{\tilde{m} \tilde{Q}}{\tilde{E}}, \tag{A.13}
\end{equation}

\begin{equation}
\frac{\partial}{\partial t}(\tilde{m} \tilde{q}) + b \frac{\partial}{\partial Y}(\tilde{V} \tilde{q}) + \frac{\partial}{\partial Z}(\tilde{W} \tilde{q}) = \tilde{m} \tilde{q}, \tag{A.14}
\end{equation}

where the normalised Exner function, derived from Eq. (20), is expressed as a function of the normalised pressure $\tilde{p}$,

\begin{equation}
\tilde{E} = \frac{E}{E_0} = \tilde{p}^k. \tag{A.16}
\end{equation}

As may be noticed, with the normalisation adopted in Eq. (A.3), the dynamics of the nondimensional HPEs is controlled by one unique dimensionless parameter,

\begin{equation}
b = \frac{v_\Delta}{\pi R_p} = \frac{\sqrt{C_\rho T_0 \Delta t}}{\pi R_p}, \tag{A.17}
\end{equation}

which is a global Courant number weighting the contribution of horizontal advection\(^6\). The density is not a variable of the primitive equations but can be evaluated using the perfect gas equation,

\begin{equation}
\tilde{\rho} = \kappa \tilde{p} \tilde{T}, \tag{A.18}
\end{equation}

while the vertical velocity can be calculated from the conversion formula

\begin{equation}
v_z = \frac{v_0}{\tilde{m}} \left[ \tilde{m} \tilde{c} \tilde{v} + b \tilde{v} \tilde{c} + W \tilde{c} \tilde{v} \right], \tag{A.19}
\end{equation}

where we have introduced the characteristic vertical velocity

\begin{equation}
v_{z0} = \frac{c_0}{g \Delta t}. \tag{A.20}
\end{equation}

**Appendix B: Dynamical core**

The system of prognostic equations given by Eqs. (A.11-A.15) is integrated numerically using a standard finite-volume method. This appendix details the main features of the dynamical core of our model.

\(^6\) The global Courant number given by Eq. (A.17) should be multiplied by the number of horizontal cells to obtain the Courant number used in the CFL (Courant, Friedrichs, and Lewy) numerical stability condition (Courant et al. 1928).
Appendix B.1: Grid

The primitive equations are discretised and solved on a staggered Arakawa C grid (Arakawa & Lamb 1977). The atmosphere is divided into elemental cells. Mass fluxes and velocities are evaluated at cell interfaces, and volume quantities \((\dot{m}, \Theta, \dot{q}, \phi, \hat{E})\) at cells centers, except the pressure, which is evaluated at horizontal cells interfaces. A diagram representing the grid is shown by Fig. B.1.

As a first step, the spatial domain defined by \(\{\hat{t}, \sigma\} \in [0, 1]^2\) is divided into non-uniform intervals both along the horizontal and vertical directions. There are \(M\) intervals for the colatitude, and \(N\) intervals for the vertical coordinate, which represents \(M \times N\) cells. The vertical spacing can be specified arbitrarily. It determines the coordinates of cell boundaries. We note that the \(\sigma\) levels of cell centers can be explicitly computed from the \(\sigma\) levels of cell boundaries if \(p_i = 0\) (standard sigma-coordinate) as discussed in Sect. B.2.

As a second step, the change of coordinates \((\theta, \sigma) \to (Y, Z) \in [0, M] \times [0, N]\) is applied, and the corresponding metric coefficients \(c_Y\) and \(c_Z\) are calculated. Typically, for a uniform grid, \(\theta = Y/M\), and thus \(c_Y = 1/M\) everywhere. The case of non-uniform horizontal intervals is more complicated, since this implies \(\theta\)-dependent coefficients. To treat the general case, we adapt the method employed for the horizontal grid in the LMDZ GCM to the one-dimensional case: each surface area is divided into two subsurface areas, as shown by Fig. B.2, and the \(c_Y\) are evaluated at subareas centers (Sadourny 1975a,b). The \(c_Z\) are just the difference between two vertical levels.

We introduce the difference and average operators,\(\delta_x\psi(x) = \psi(x + \frac{1}{2}) - \psi(x - \frac{1}{2})\), \(\overline{\psi}(x) = \frac{1}{2}[\psi(x + \frac{1}{2}) + \psi(x - \frac{1}{2})]\),

\[
\delta_x \psi(X) = \psi \left( X + \frac{1}{2} \right) - \psi \left( X - \frac{1}{2} \right), \quad \overline{\psi}(X) = \frac{1}{2} \left[ \psi \left( X + \frac{1}{2} \right) + \psi \left( X - \frac{1}{2} \right) \right],
\]

where \(X\) is a placeholder for \(Y\) or \(Z\) and \(\psi\) any variable. In this formalism, the \(c_Z\) are simply expressed as \(c_Z = \delta_Z \sigma\).

In the present work, we use a vertical grid refined near the surface and the upper boundary. The sigma-coordinates of vertical levels are given by the function

\[
\sigma(x) = \frac{1}{2} \left[ 1 + \cos \left( \pi x^a \right) \right],
\]

where \(x = Z/N\) takes its values between 0 and 1. Here, the exponent \(a\) is a dilation coefficient that controls the growth or decay rate of vertical intervals in the vicinity of the lower and upper bounds. Introducing the vertical coordinate difference between the lowest model level and the surface, \(\Delta \sigma_{SL}\), this parameter is defined as

\[
a = \frac{\ln \left( \pi^{-1} \arccos \left( 1 - 2 \Delta \sigma_{SL} \right) \right)}{\ln \left( 1/N \right)}.
\]

For an isothermal temperature profile near planet’s surface, \(\Delta \sigma_{SL} \approx z_{SL}/H\), where \(H\) is the local pressure height. In practice, we set \(\Delta \sigma_{SL} = 3.0 \times 10^{-3}\).

Appendix B.2: Exner function

The calculation of the Exner function follows the method described by Arakawa & Lamb (1977) and Hourdin (1994), which is summarised here. Instead of computing \(\hat{E}\) at the middle of the layer by extrapolating the pressure at the middle of the layer from the level pressures at the interfaces, which is computationally expensive, one rather uses a supplementary relationship between the interface levels and the Exner function. This relationship is directly derived from considerations about energy conservation principles within the atmospheric air column. Particularly, in the hydrostatic approximation, the internal and potential energy are proportional, which implies (Arakawa & Lamb 1977; Hourdin 1994)

\[
\int_0^{m_{col}} \phi dm = \int_0^{m_{col}} R_d T dm, \quad (B.5)
\]

where \(m = \rho dz\) is the infinitesimal parcel of mass per unit surface, and \(m_{col}\) the mass of the air column per unit surface. The expression of \(\phi\) as a function of \(\Theta\) and \(E\) is simply obtained by making use of the hydrostatic balance equation given, in grid vertical coordinates, by

\[
\frac{\partial \phi}{\partial Z} + \Theta \frac{\partial E}{\partial Z} = 0, \quad (B.6)
\]

and reads

\[
\phi = \int_0^Z \frac{\partial \phi}{\partial Z'} dZ' = - \int_0^Z \Theta \frac{\partial E}{\partial Z'} dZ', \quad (B.7)
\]

which allows us, noticing that \(R_d T = \kappa E\), to rewrite Eq. (B.5) as

\[
\int_0^{m_{col}} \left[ \int_0^Z \Theta \frac{\partial E}{\partial Z'} dZ' + \kappa \Theta E \right] dm = 0. \quad (B.8)
\]

The discretised form of the first term is given by

\[
\int_0^{Z_{col}} \Theta \frac{\partial E}{\partial Z'} dZ' = \sum_{k=1}^N \left( \Theta_k + \Theta_{k-1} \right) \left( E_k - E_{k-1} \right) + \Theta_0 (E_0 - E_1), \quad (B.9)
\]

where \(\Theta_k, E_k\) are the potential temperature and Exner function evaluated at the middle of the layer, indexed by \(l = 0, \ldots, N - 1\) (for \(N\) vertical intervals), and \(E_k = C_p (p/p_{ref})^\kappa\) is the Exner function evaluated at the planet’s surface. By making use of the difference and average operators introduced in Eqs. (B.1) and (B.2), the preceding equation can be rewritten in the compact form

\[
\int_0^{Z_{col}} \Theta \frac{\partial E}{\partial Z'} dZ' = \sum_{l=0}^{N-1} \left[ \Theta \frac{\partial E}{\partial Z'} \right]_k, \quad (B.10)
\]

with

\[
\left[ \Theta \frac{\partial E}{\partial Z'} \right]_k = \left( \Theta_k + \Theta_{k-1} \right) \left( E_k - E_{k-1} \right), \quad k = 1, \ldots, N - 1,
\]

\[
\left[ \Theta \frac{\partial E}{\partial Z'} \right]_0 = \Theta_0 (E_0 - E_1). \quad (B.11)
\]

Then, introducing the mass \(m_l\) per unit area of layer \(l\) and interchanging the sums in the double integral, we obtain

\[
\sum_{l=0}^{N-1} m_l \int_0^{Z_l} \Theta \frac{\partial E}{\partial Z'} dZ' = \sum_{l=0}^{N-1} \frac{1}{m_l} \int_0^{Z_l} \Theta \frac{\partial E}{\partial Z'} dZ' = \sum_{k=0}^{N-1} \left[ \Theta \frac{\partial E}{\partial Z'} \right]_k \sum_{l=k}^{N-1} m_l. \quad (B.12)
\]
pressed in the framework of the hydrostatic approximation as Fig. B.2.

In the above equation, we recognise the atmospheric mass per unit surface above the \( k \) interfaces (\( k = 0, \ldots, N \)), which is expressed in the framework of the hydrostatic approximation as

\[
\sum_{l=k}^{N} m_l = \frac{p_k}{g}.
\]

(B.14)

It follows

\[
\sum_{l=0}^{N-1} m_l \int_0^{\Theta} \frac{\partial E}{\partial Z} \, dZ' = \frac{1}{g} \sum_{k=0}^{N-1} \left[ \Theta \frac{\partial E}{\partial Z} \right]_k p_k.
\]

(B.15)

Finally, expanding the coefficients of the sum, we remark that this later may be rewritten

\[
\sum_{k=0}^{N-1} \left[ \Theta \frac{\partial E}{\partial Z} \right]_k p_k = \sum_{k=0}^{N-1} \Theta_k \left[ p \delta Z E \right]_k,
\]

(B.16)

with the conventions

\[
\begin{align*}
[p \delta Z E]_0 &= \frac{1}{2} (E_1 - E_0) p_1 + (E_0 - E_s) p_0, \\
[p \delta Z E]_N &= \frac{1}{2} (E_N + E_{N-2}) p_{N-1} + (E_1 - E_{N-1}) p_1.
\end{align*}
\]

(B.17) (B.18)

Thus, in the general case, the Exner function can be evaluated at cell centers by applying, at all vertical levels, the relation

\[
-p \delta \sigma E = -\kappa E \delta Z p.
\]

(B.19)

The use of standard sigma coordinates appreciably reduces the complexity of the problem. Assuming \( p_1 = 0 \) and introducing the notation \( s = \sigma^q \), the preceding equation simplifies to

\[
\sigma \frac{\partial E}{\partial Z} = kq \delta \sigma, \quad \text{for} \ k = 1, \ldots, N - 2,
\]

which allows to pre-calculate the sigma coordinates of cell centers levels. Indexing these levels by \( k = 0, \ldots, N - 1 \) and the intermediate (or interface) \( \sigma \) levels by \( k = 0, \ldots, N \), we have

\[
\begin{align*}
\sigma \frac{\partial E}{\partial Z} \bigg|_k &= kq \delta \sigma \bigg|_k \quad \text{for} \ k = 1, \ldots, N - 2, \\
\sigma \frac{\partial E}{\partial Z} \bigg|_0 &= \frac{1}{2} (s_1 - s_0) \sigma_1 + (s_0 - s_1) \sigma_0 = kq \delta \sigma_0, \\
\sigma \frac{\partial E}{\partial Z} \bigg|_{N-1} &= \frac{1}{2} (s_{N-1} - s_{N-2}) \sigma_{N-1} = kq \delta \sigma_{N-1}.
\end{align*}
\]

(B.20) (B.21)

where \( s_1 = 1 \) and \( s_0 = 0 \) are the value of \( s \) at planet’s surface and atmospheric upper boundary, respectively.

In practice, the Exner function is evaluated at grid centers levels by solving an algebraic equation of the form \( AX = B \).
In the case of standard sigma coordinates \((\rho_z = 0)\), this calculation is performed only once, when the grid is constructed. The algebraic system to solve then writes

\[
\begin{bmatrix}
A_{00} & A_{01} & A_{01} & A_{11} & A_{11} & A_{11} & A_{11} & A_{11} \\
A_{01} & A_{01} & A_{01} & A_{01} & A_{01} & A_{01} & A_{01} & A_{01} \\
A_{k,k-1} & A_{k,k} & A_{k,k} & A_{k,k} & A_{k,k} & A_{k,k} & A_{k,k} & A_{k,k} \\
A_{N-2,N-3} & A_{N-2,N-2} & A_{N-2,N-2} & A_{N-2,N-2} & A_{N-2,N-2} & A_{N-2,N-2} & A_{N-2,N-2} & A_{N-2,N-2} \\
B_{s} & B_{s} & B_{s} & B_{s} & B_{s} & B_{s} & B_{s} & B_{s}
\end{bmatrix}
\begin{bmatrix}
s_0 \\
s_1 \\
s_k \\
s_{N-2} \\
s_{N-1} \\
s_{N-1} \\
s_{N-1} \\
s_{N-1}
\end{bmatrix}
= \begin{bmatrix}
B_0 \\
B_1 \\
B_k \\
B_{N-2} \\
B_{N-1} \\
B_{N-1} \\
B_{N-1} \\
B_{N-1}
\end{bmatrix}
\]  

(B.22)

where the \(s_k\) is the coordinate \(s = \sigma^s\) evaluated at the mid-level of the \(k\)-layer, \(A\) the tridiagonal matrix of coefficients

\[
A_{k,k} = \left( \frac{1}{2} + \kappa \right) (\sigma_k - \sigma_{k+1}) \quad \text{for} \quad k = 1, \ldots, N - 2;
\]

(B.23)

\[
A_{0,0} = (1 + \kappa) \sigma_0 - \left( \frac{1}{2} + \kappa \right) \sigma_1,
\]

(B.24)

\[
A_{N-1,N-1} = \left( \frac{1}{2} + \kappa \right) \sigma_{N-1} - (1 + \kappa) \sigma_N,
\]

(B.25)

\[
A_{k,k+1} = -A_{k+1,k} = \frac{1}{2} \sigma_{k+1},
\]

(B.26)

and \(B\) the vector of coefficients

\[
B_{0} = 0 \quad \text{for} \quad k = 1, \ldots, N - 2;
\]

(B.27)

\[
B_{0} = \sigma_0 s_n,
\]

(B.28)

\[
B_{N-1} = -\sigma_N s_1.
\]

(B.29)

In these equations, \(s_k = 1\) and \(s_0 = 0\) are the values of \(s\) at planet’s surface and atmospheric top, respectively. This procedure can be applied from the moment that \(N > 1\). If there is only one layer, then the mid-level coordinate of the unique layer is set to \(s_0 = (\sigma_0 + \sigma_1)/2\).

**Appendix B.3: Discretisation of the primitive equations**

The normalised prognostic equations given by Eqs. (A.11-A.15) are discretised by making use of the difference and average operators defined by Eqs. (B.1) and (B.2), and become

\[
\frac{\partial \tilde{m}}{\partial t} + b_\psi Y V + \delta_z W = 0,
\]

(B.30)

\[
\frac{\partial}{\partial t} \left( \tilde{m} \tilde{V}_\psi \right) + b_\psi \left( \tilde{V}^T \tilde{V}_\psi \right) + \delta_z \left( \tilde{W}^T \tilde{V}_\psi \right) = \tilde{m} \tilde{F}_\psi,
\]

(B.31)

\[
\frac{\partial}{\partial t} \left( \tilde{m} \tilde{\Theta} \right) + b_\psi \left( \tilde{V}^T \tilde{\Theta} \right) + \delta_z \left( \tilde{W}^T \tilde{\Theta} \right) = \tilde{m} \tilde{Q},
\]

(B.32)

\[
\frac{\partial}{\partial t} \left( \tilde{m} \tilde{g} \right) + b_\psi \left( \tilde{V}^T \tilde{g} \right) + \delta_z \left( \tilde{W}^T \tilde{g} \right) = \tilde{m} \tilde{q},
\]

(B.33)

\[
\frac{\partial}{\partial t} \left( \delta_z \phi \right) + \delta_z \left( \delta_z \tilde{E} \right) = 0.
\]

(B.34)

In the right-hand members of these equations, the horizontal force is specified at horizontal interface levels, where the horizontal mass fluxes and velocities are also evaluated, and \(\tilde{Q}\) and \(\tilde{g}\) at cell centers. The vertical velocity can be calculated afterwards, when it is necessary, using the formula

\[
v_z = \frac{v_z}{\tilde{m}} \left[ \frac{\partial \tilde{m}}{\partial t} + b_\psi \delta_z \tilde{V}_\psi + \delta_z \tilde{W} \tilde{V}_\psi \right].
\]

(B.35)

**Appendix B.4: Time-differencing scheme**

Following the method used in many general circulation models (Hansen et al. 1983; Yao & Stone 1987; Houben et al. 2006), the temporal integration of the primitive equations is based on the so-called leapfrog scheme, which is a centred explicit scheme (e.g. Press et al. 2007). Since the leapfrog scheme tends to generate a spurious growth of numerical instabilities over time (e.g. Lauritzen et al. 2011, Sect. 5.5), a Matsuno time step (Matsuno 1966a) is introduced every \(n_{MT}\) steps (with \(n_{MT} = 5\)) to stabilise the integration, as illustrated by Fig. B.3. Mathematically, denoting the time derivative operator by \(\dot{M}\), any dynamical variable by \(\psi\), and indexing time steps by \(n\), a leapfrog step is expressed as

\[
\psi_n = \psi_{n-2} + 2\Delta t M (\psi_{n-1}),
\]

(B.36)

while a Matsuno step (Matsuno 1966a) consists in the succession of two steps,

\[
\psi_n = \psi_{n-1} + \Delta t M (\psi_{n-1}),
\]

(B.37)

\[
\psi_n = \psi_{n-1} + \Delta t M (\psi_n^*),
\]

(B.38)

where the superscript * is used to designate the intermediate virtual time step. In Fig. B.3, the leapfrog time step is used to compute variables at all dates except \(t_0\) and \(t_5\), where it is replaced by a Matsuno step. Sink terms associated with numerical dissipation (hyper-diffusion, sponge layer, etc.) are evaluated periodically every \(n_{MT}\) dynamical time step, before Matsuno time steps. Physical tendencies are computed every \(n_p\) dynamical time step (with \(n_p = 10\)).

**Appendix C: Numerical dissipation**

**Appendix C.1: Horizontal hyper-diffusion**

To dissipate energy at grid scale, we introduce a bi-harmonic diffusion (Lauritzen et al. 2011), which is a fourth-order diffusion. This is obtained by applying twice the Laplacian operator to the temperature and horizontal velocity. For any scalar quantity \(\phi\), the horizontal Laplacian operator reads, in our two-dimensional system of coordinates,

\[
\nabla_z^2 \phi = R_p^{-2} \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \phi}{\partial \theta} \right).
\]

(C.1)

In grid coordinates, it is expressed as

\[
\nabla^2_z \phi = \left( \pi R_p \right)^{-2} (c_Y \sin \theta)^{-1} \frac{\partial}{\partial Y} \left( c_Y \frac{\partial \phi}{\partial Y} \right)
\]

(C.2)
where $\nabla_\sigma^2$ designates the normalised horizontal Laplacian operator,
\[
\nabla_\sigma^2 \psi = \left( c_y \sin \theta \right)^{-1} \frac{\partial}{\partial Y} \left( \frac{\sin \theta}{c_y} \frac{\partial \psi}{\partial Y} \right). 
\]  
(C.3)

We introduce the hyper-Laplacian operator, which is the Laplacian of order $q$, defined as $\nabla^2_q = \nabla^2 \cdots \nabla^2$. Basically, for the fourth-order hyper-diffusion, $q = 2$, and $\nabla^2 = \nabla_\sigma^2 \nabla_\sigma^2$. In the general case, the 2$^q$-order hyper-diffusion term of any variable $\psi$ is defined by
\[
F_{\text{diff}} = (-1)^q \frac{1}{2q} K_{2q} \nabla^2_q \psi, 
\]  
(C.4)

where $K_{2q}$ is the hyper-diffusivity. This parameter can be written as a function of the diffusion timescale $\tau_{\text{diff}}$ and mean horizontal grid spacing $\Delta \theta = \pi / M$ (Lauritzen et al. 2011, Sect. 13.3),
\[
K_{2q} = \frac{1}{2\tau_{\text{diff}}} \left( \frac{R_0 \Delta \theta}{2} \right)^{2q}. 
\]  
(C.5)

The normalised hyper-diffusivity $\tilde{K}_{2q}$ is thus given by
\[
\tilde{K}_{2q} = \frac{K_{2q}}{\pi R_{\text{p}}}^{2q} = \frac{1}{2\tau_{\text{diff}} (2M)^{2q}}. 
\]  
(C.6)

In order to adapt the hyper-diffusivity to the horizontal grid resolution, it is convenient to introduce the nondimensional diffusion parameter (Tomita & Satoh 2004; Mendonça et al. 2016)
\[
\gamma = \frac{\Delta t}{2^{2q+1} \tau_{\text{diff}}}, 
\]  
(C.7)

which allows us to rewrite diffusivity parameters as
\[
K_{2q} = \gamma \frac{\Delta t}{M^{2q}}, \quad \tilde{K}_{2q} = \frac{1}{\Delta t M^{2q}}. 
\]  
(C.8)

Bi-harmonic diffusion is used in the model, with a diffusion parameter set to $\gamma = 6.25 \times 10^{-4}$ in agreement with the order of magnitude of commonly used values (Lauritzen et al. 2011, Sect. 13.3).

In the finite-volume approach, hyper-diffusion can lead to numerical instabilities near the poles because of the singularity $\sin^{-1} (\theta)$ of the Laplacian operator. Typically, assuming that $\psi$ is an oscillatory function of the form $\psi (\theta) = \sin(n \pi \theta)$ yields
\[
\nabla_\sigma^2 \psi = \frac{(n \pi)^2 \sin \theta}{\sin \theta} \left[ (n \pi)^{-1} \cos (\theta) \cos (n \pi \theta) - \sin (\theta) \sin (n \pi \theta) \right], 
\]  
(C.9)

and thus $\nabla_\sigma^2 \psi \propto \sin^{-1} (\theta)$ in polar regions. To remediate to this problem, it is often chosen to compensate the singularity introduced by the Laplacian by annihilating the diffusivity parameter at the poles (Lauritzen et al. 2011, Sect. 13.3). Basically, the isotropic diffusivity introduced in Eq. (C.4) is multiplied by $\sin^{4\alpha}(\theta)$, where $\alpha$ is a coefficient of anisotropy ($\alpha = 0.5$–1 typically). This anisotropic diffusion solves the stability problem but can also induce unphysical boundary effects near the poles. Following Majewski et al. (2002), we apply a bi-harmonic diffusion ($q = 2$) to the temperature and horizontal velocity $v_\theta$. We use the same diffusion coefficient for both terms. Thus,
\[
F_{\text{diff}, v_\theta} = -\tilde{K}_2 \sin^2 (\theta) \nabla_\sigma^4 v_\theta, 
\]  
(C.10)
\[
F_{\text{diff}, T} = -\tilde{K}_2 \sin^2 (\theta) \nabla_\sigma^4 T. 
\]  
(C.11)

In addition, we apply to the top layer of the model an harmonic diffusion of the form
\[
F_{\text{diff}, v_\theta} = -\tilde{K}_2 \sin (\theta) \nabla_\sigma^2 v_\theta, 
\]  
(C.12)
\[
F_{\text{diff}, T} = -\tilde{K}_2 \sin (\theta) \nabla_\sigma^2 T. 
\]  
(C.13)

In order to verify that the mean flow and temperature distribution are insensitive to the hyper-diffusion scheme, simulations were run for various values of the nondimensional diffusion parameter $\gamma$ introduced in Eq. (C.7), namely $\gamma = 10^{-5}, 10^{-4}, 10^{-3}$. These validation tests were performed for the Earth-like case of Table 2 with the same surface pressure and stellar flux as in Fig. 1. Figure C.1 shows the two-day averaged temperature snapshots obtained for each value of $\gamma$, as well as the associated mean flows. We observe that varying the value of $\gamma$ over two orders of magnitude hardly alters the temperatures and wind speeds. The minimum nightside temperature after convergence is $T_n = 231.5$ K, $T_n = 231.7$ K, and $T_n = 231.8$ K, for $\gamma = 10^{-3}, 10^{-4}, 10^{-5}$, respectively. Similarly, the maximum wind speed varies between 59.55 m s$^{-1}$ and 62.08 m s$^{-1}$ (see Fig. C.1). These variations correspond to a 0.1% difference for
the minimum surface temperature, and to a 4.2% difference for the maximum wind speed. Winds are thus more sensitive to the hyper-diffusion scheme than the nightside surface temperature, although the observed dependence is relatively weak in both cases. Nevertheless, this dependence is expected to be more important for extreme values of \( \gamma \) because such values would lead either to underdissipated or to overdissipated flow fields, as shown by Thrastarson & Cho (2011). Typically, with a value of \( \gamma \) smaller than the adopted one by several orders of magnitude, the flow rapidly becomes numerically unstable at grid scale, which induces spurious fluctuations and may cause the run to abort.

### Appendix C.2: Sponge layer

In extreme cases (low surface pressure and high stellar flux), the strong thermal forcing of the atmosphere on the dayside generates instabilities that can lead to negative pressures near the top of the model. Particularly, it generates internal gravity waves that propagate upwards with amplitudes becoming very large as the atmospheric density tends to zero\(^7\). These waves are reflected downwards by the upper boundary, which acts as a wall (no vertical mass flow). Owing to the weak density, such extreme fluctuations have dramatic repercussions on the computation of mass flows and are thereby a source of unphysical values that make runs abort. Thus, in addition with the hyperdiffusion scheme, the use of a sponge layer is necessary in cases where the surface pressure is low and the stellar irradiation is strong.

The sponge layer is a numerical dissipation process that strongly damps wind flows diverging from a prescribed equilibrium profile in the upper regions of the atmosphere, with an efficiency increasing with the altitude. In the present model, we use a Rayleigh friction sponge, which is based on a linear relaxation term of generic form (e.g. Lauritzen et al. 2011, Sect. 13.4)

\[
\frac{\partial \tilde{v}_\theta}{\partial t} = -k_{\text{SL}} (v_\theta - v_{\theta,\text{SL}}),
\]

\( k_{\text{SL}} \) designates the Rayleigh damping coefficient of the sponge layer and \( v_{\theta,\text{SL}} \) the equilibrium velocity profile near the upper boundary. This profile is set to the standard value \( v_{\theta,\text{SL}} = 0 \). Following Polvani & Kushner (2002), we opt for a vertical profile of the Rayleigh coefficient of the form

\[
k_{\text{SL}} (\sigma) = \begin{cases} 
0 & \text{if } \sigma \geq \sigma_{\text{SL}}^\text{max}, \\
 k_{\text{SL}}^\text{max} \left(1 - \frac{\sigma}{\sigma_{\text{SL}}^\text{max}}\right)^2 & \text{if } \sigma < \sigma_{\text{SL}}^\text{max}.
\end{cases}
\]

(C.15)

In the above piecewise function, \( \sigma_{\text{SL}}^\text{max} \) corresponds to the critical normalised pressure below which the sponge layer is applied while \( k_{\text{SL}}^\text{max} \) is the maximum value of the Rayleigh friction coefficient. This parameter has dimensions of a frequency and is the inverse of the minimum damping timescale of the sponge layer \( \tau_{\text{SL}}^\text{min} \). The smaller \( \tau_{\text{SL}}^\text{min} \), the stronger the damping in the sponge layer. In the model, the maximum Rayleigh coefficient is set to \( k_{\text{SL}}^\text{max} = 0.5 \text{ day}^{-1} \), while the thickness of the sponge layer is defined as a function of the stellar flux and surface pressure.

### Appendix D: Convective adjustment scheme

The turbulent diffusion scheme implemented described by Sect. 3.3 is not sophisticated enough to prevent superadiabatic vertical temperature gradients,

\[
\frac{\partial \Theta}{\partial z} < 0.
\]

(D.1)

If such an unstable profile is produced by the model, it may generate numerical errors in the solution and lead the run to crash. In order to prevent this behaviour, a convective adjustment scheme can be activated in simulations. This scheme tends to regularise the potential temperature profile every physical timestep by correcting the tendencies in heat fluxes while conserving the entropy over the air column. The convective adjustment scheme used in the present work is similar to that implemented in the LMDZ and THOR GCMs (Hourdin et al. 1993; Mendonça & Buchhave 2020).

![Fig. D.1. Construction of the stable temperature profile in the convective adjustment scheme. An initially unstable region of the air column spreading from layer \( l_1 \) to layer \( l_2 \) is extended both downwards and upwards until the temperature profile is stable. Every step, the mass-averaged potential temperature of the region is adjusted to include the contribution of the current unstable layer.](image)

The principle of the scheme is contained in two steps. During the first step, the unstable intervals of the vertical temperature profiles extrapolated from tendencies are detected and a stable profile is constructed by averaging the potential temperature over unstable layers. Starting from the ground, the interval bounded by layers \( l_1 \) and \( l_2 \) with \( l_1 \leq l_2 \) is extended both downwards and upwards incrementally until \( l_1 \) and \( l_2 \) correspond to the bottom and top layers of the atmosphere, respectively. In an unstable region, the potential temperature is set to the mass-averaged adiabatic potential temperature,

\[
\Theta = \frac{\int_{\Theta_{\text{top}}}^{\Theta_{\text{bot}}} \Theta \ dm}{\int_{\Theta_{\text{bot}}}^{\Theta_{\text{top}}} dm},
\]

where \( dm = \rho dz \) is an infinitesimal parcel of mass of the air column, \( \Theta_{\text{bot}} \) the mass of the air column below the lower boundary of the mixed region, and \( \Theta_{\text{top}} \) the mass below the upper boundary of the mixed region. If \( \Theta_l > \Theta \) for a layer \( k \) underneath the mixed region, the mass-averaged adiabatic potential temperature is adjusted by including the contribution of the layer and \( \Theta_l \) is set to \( \Theta \). Figure D.1 illustrates how the adiabatic profile is adjusted to stabilise the layer \( l_1 \), where the vertical gradient of potential temperature is negative.

The second step of the scheme consists in evaluating the new tendencies resulting from the stable temperature profile. The tendency for the entropy equation is straightforwardly obtained from the adiabatic temperature profile constructed at the first step. For the momentum equation, an estimate of the instability of the atmosphere is computed from the relative enthalpy

---

\( \gamma \) (prob. 1.0)
exchange that is necessary to restore the adiabatic profile from the original profile,
\[ \alpha = \frac{\sum_{m=0}^{\text{max}} \Theta - \Theta | \; \text{dm} }{\sum_{m=0}^{\text{max}} \Theta | \; \text{dm}}. \tag{D.3} \]

This parameter corresponds to the fraction of the mesh on which the angular momentum is mixed (in practice, the condition \( \alpha < 1 \) is always verified in simulations; e.g. Hourdin et al. 2006). The extrapolated horizontal velocity used to compute the tendency for the horizontal momentum is corrected by a factor \( \alpha (v_0 - v_y) \), where \( v_0 \) is the mass-averaged velocity defined as
\[ v_0 = \frac{\sum_{m=0}^{\text{max}} v_y | \; \text{dm}}{\sum_{m=0}^{\text{max}} | \; \text{dm}}. \tag{D.4} \]

**Appendix E: Radiative transfer scheme**

The two radiative transfer equations given by Eqs. (23) and (24) are solved periodically every \( n_{\text{RT}} \) physical time step (with \( n_{\text{RT}} = 6 \) as illustrated by Fig. E.1. First we integrate them analytically between two intermediate vertical levels, indexed by \( k \) and \( k + 1 \), with \( k = 0, \ldots, N - 1 \). This leads to
\[ F_{tk} = \eta_k^{-1} \left\{ \lambda_k F_{tk+1} + \mu_k F_{tk} - \lambda_k B_k + [\eta_k - \mu_k] B_k - (\xi_+^2 - \xi_0^2) (1 - T_k) (\xi_+ + \xi_0 - T_k) \frac{\text{dB}}{\text{d}\tau_k} \right\}, \tag{E.1} \]

\[ F_{tk+1} = \eta_k^{-1} \left\{ \lambda_k F_{tk+1} + \mu_k F_{tk+1} - \lambda_k B_{k+1} + [\eta_k + \mu_k] B_{k+1} + (\xi_+^2 - \xi_0^2) (1 - T_k) (\xi_+ + \xi_0 - T_k) \frac{\text{dB}}{\text{d}\tau_k} \right\}, \tag{E.2} \]

where the \( B_k \) designate the values of the blackbody emission \( B \) introduced in Eq. (24) \((B = 0\) for the shortwave) interpolated at intermediate vertical levels, \( \frac{\text{dB}}{\text{d}\tau_k} \) the values of its derivatives evaluated at internal vertical levels, \( \xi_\pm \) the usual coupling coefficients,
\[ \xi_\pm = \frac{1}{2} (1 \pm \beta_0), \tag{E.3} \]

and where we have introduced the transmission functions
\[ T_k = e^{\tau_0 + \tau_k}, \tag{E.4} \]

and the coefficients
\[ \eta_k = \xi_+^2 - (\xi_0 - T_k)^2, \quad \lambda_k = (\xi_+ - \xi_0) T_k, \tag{E.5} \]

These relations can be put in the matrix form
\[ \begin{bmatrix} -\mu_k & \eta_k & 0 & -\lambda_k \\ -\lambda_k & 0 & \eta_k - \mu_k \end{bmatrix} \begin{bmatrix} F_{tk} \\ F_{tk+1} \end{bmatrix} = \begin{bmatrix} b_k^1 \\ b_k^2 \end{bmatrix}, \tag{E.6} \]

where \( b_k^1 \) and \( b_k^2 \) are given by
\[ b_k^1 = - \lambda_k B_{k+1} + (\eta_k - \mu_k) B_k \]
\[ - (\xi_+^2 - \xi_0^2) (1 - T_k) (\xi_+ + \xi_0 - T_k) \frac{\text{dB}}{\text{d}\tau_k} \], \tag{E.7} \]

\[ b_k^2 = - \lambda_k B_k + (\eta_k + \mu_k) B_{k+1} + (\xi_+^2 - \xi_0^2) (1 - T_k) (\xi_+ + \xi_0 - T_k) \frac{\text{dB}}{\text{d}\tau_k}. \tag{E.8} \]

They are completed by relations derived from the lower and upper boundary conditions, which, in the general case, are of the form
\[ a_{s0} F_{t0} + a_{s1} F_{t1} + a_{s2} F_{t1} + a_{s3} F_{t1} = b_s, \tag{E.9} \]
\[ a_{t0} F_{tN-1} + a_{t1} F_{tN-1} + a_{t2} F_{tN} + a_{t3} F_{tN} = b_t, \tag{E.10} \]

where the subscripts \( s \) and \( t \) denote the coefficients associated with the planet’s surface or with the top of the atmosphere, respectively. Therefore, introducing the vectors \( F_k = (F_{tk}, F_{tk+1})^T \), we can write the discretised equations as a linear algebraic system of the form
\[ \begin{bmatrix} B_0 \\ A_1 \end{bmatrix} = \begin{bmatrix} C_0 \\ A_k \end{bmatrix} \begin{bmatrix} F_0 \\ F_1 \end{bmatrix}, \tag{E.11} \]
\[ A_k = \begin{bmatrix} -\lambda_k & 0 \\ 0 & -\lambda_k \end{bmatrix}, \quad C_k = \begin{bmatrix} 0 & -\mu_k \\ -\mu_k & \eta_k \end{bmatrix}, \tag{E.12} \]
\[ A_{N-1} = \begin{bmatrix} -\lambda_{N-1} & 0 \\ 0 & -\lambda_{N-1} \end{bmatrix}, \quad B_N = \begin{bmatrix} \eta_{N-1} & 0 \\ 0 & -\lambda_{N-1} \end{bmatrix}, \tag{E.13} \]

and the corresponding vectors are expressed as
\[ d_k = \begin{bmatrix} b_k^1 \\ b_k^2 \end{bmatrix}, \quad k = 1, \ldots, N - 1, \tag{E.14} \]
\[ d_0 = \begin{bmatrix} b_0^1 \\ b_0^2 \end{bmatrix}, \quad d_N = \begin{bmatrix} b_N^1 \\ b_N^2 \end{bmatrix}. \tag{E.15} \]

As the matrix of the system is a block tridiagonal matrix, the system can be solved by making use of Thomas algorithm (see Appendix H). We note that the shortwave and longwave fluxes can be integrated in parallel since there are decoupled. In practice, the coefficients of boundaries conditions introduced in Eqs. (E.9) and (E.10) are, for the shortwave,
\[ a_{s0} = 1, \quad a_{s1} = -A_s, \quad a_{s2} = 0, \quad a_{s3} = 0, \quad b_s = 0, \tag{E.16} \]
and, for the longwave,
\[ a_{t0} = 1, \quad a_{t1} = 0, \quad a_{t2} = 0, \quad a_{t3} = 0, \quad b_t = 0, \tag{E.17} \]
where \( F_s \) designates the incident stellar flux, given by
\[ F_s (\theta) = \begin{cases} F_s \cos \theta, & \text{if } 0^\circ \leq \theta \leq 90^\circ, \\ 0, & \text{if } 90^\circ < \theta \leq 180^\circ. \end{cases} \tag{E.18} \]
Appendix F: Turbulent diffusion scheme

Appendix F.1: Stability and asymptotic length scale functions

The functions \( f_M \) and \( f_H \) introduced in Eqs. (34) and (35) are piecewise functions of the surface-layer bulk Richardson number given by Eq. (37). They are defined in the model following the formulation proposed by Holtslag & Boville (1993), which was established experimentally in the Earth case. In the unstable regime (\( \text{Ri}_b < 0 \)), they are given by

\[
f_M(\text{Ri}_b) = 1 - \frac{10 \text{Ri}_b}{1 + 75 \text{C}_N \sqrt{(1 + \text{SL}/z)} |\text{Ri}_b|}.
\]

\[
f_H(\text{Ri}_b) = 1 - \frac{15 \text{Ri}_b}{1 + 75 \text{C}_N \sqrt{(1 + \text{SL}/z)} |\text{Ri}_b|}.
\]

and, in the stable regime (\( \text{Ri}_b \geq 0 \)), by

\[
f_M(\text{Ri}_b) = f_H(\text{Ri}_b) = \frac{1}{1 + 10 \text{Ri}_b (1 + 8 \text{Ri}_b)}.
\]

The functions \( \mathcal{F}_X \) introduced in Eq. (30) to characterise the dependence of eddy diffusivities upon the gradient Richardson number are the same for both momentum and heat diffusion. This function is defined, in the unstable regime (\( \text{Ri}_b < 0 \)), as

\[
\mathcal{F}_X(\text{Ri}) = \sqrt{1 - 18 \text{Ri}_b},
\]

and, in the stable regime (\( \text{Ri}_b \geq 0 \)), as

\[
\mathcal{F}_X(\text{Ri}) = f_M(\text{Ri}) = \frac{1}{1 + 10 \text{Ri}_b (1 + 8 \text{Ri}_b)}.
\]

Finally, following Holtslag & Boville (1993), the asymptotic length scale is defined, for both momentum and heat diffusivities, as the piecewise function

\[
\ell(z) = \begin{cases} 
300 & \text{if } z \leq 1 \text{ km} \\
30 + 270 \exp(1 - z/1000) & \text{if } z > 1 \text{ km},
\end{cases}
\]

where \( z \) and \( \ell \) are expressed in meters. This function enforces a mixing length of 300 m from the surface to \( z = 1 \) km, and a smooth interpolation to the free atmospheric value, which is set to 30 m.

Appendix F.2: Discretisation of diffusion equations

The contribution of turbulent diffusion to the physical tendencies is computed every physical time step. Between the surface and the top of the atmosphere (internal levels corresponding to \( k = 1, \ldots, N - 2 \)), the discretised equations derived from Eq. (27) are given by

\[
\frac{X^n_{k+1/2} - X^{n-1}_{k+1/2}}{\Delta t} = \frac{\mathcal{A}}{m_{k+1/2}} \left[ \rho_k^n \bar{K}_{X,k} \bar{d}_k X^n_{k+1} - \rho_k^n \bar{K}_X \frac{\delta X^n_{k+1}}{\delta z_k} \right],
\]

with \( \mathcal{A} \) the area of the surface parcels defined by horizontal intervals. We note that integer indices in subscripts indicate levels separating vertical intervals, and non-integer indices centers of vertical intervals (see Fig. F.1). For \( k = 0 \) (lower boundary condition),

\[
\frac{X^n_{1/2} - X^{n-1}_{1/2}}{\Delta t} = \frac{\mathcal{A}}{m_{1/2}} \left[ \rho_1^n \bar{K}_1 X^n_{1/2} \delta X^n_{1/2} - F_{\text{turb}} \right],
\]

and, for \( k = N - 1 \) (upper boundary condition),

\[
\frac{X^n_{N-1/2} - X^{n-1}_{N-1/2}}{\Delta t} = -\frac{\mathcal{A}}{m_{N-1/2}} \left[ \rho_0^n \bar{K}_0 X^n_{N-1/2} \frac{\delta X^n_{N-1/2}}{\delta z_{N-1}} \right].
\]

where \( F_{\text{turb}} \) is the downward turbulent flux at the surface-atmosphere interface. Similarly as the equations of vertical diffusion, these equations are put into the form

\[
c_{k+1/2} \left( X^n_{k+1/2} - X^{n-1}_{k+1/2} \right) = d_k \delta X^0_{k+1} - d_k \delta X^n_{k+1},
\]

\[
c_{1/2} \left( X^n_{1/2} - X^{n-1}_{1/2} \right) = d_1 \delta X^0_{1/2} - F_{\text{turb}},
\]

\[
c_{N-1/2} \left( X^n_{N-1/2} - X^{n-1}_{N-1/2} \right) = -d_{N-1} \delta X^n_{N-1/2},
\]

where we have introduced the coefficients

\[
c_{k+1/2} = \frac{m_{k+1/2}}{\mathcal{A} \Delta t}, \quad d_k = \frac{\rho_k^n \bar{K}_k}{\delta z_k}.
\]

This algebraic system is solved using the tridiagonal matrix algorithm (Appendix H). We introduce, for \( k = 1, \ldots, N - 1 \), the recursion relation

\[
X^n_{k+1/2} = \alpha_k X^n_{k-1/2} + \beta_k,
\]

where the coefficients \( \alpha_k \) and \( \beta_k \) are given by

\[
\alpha_k = \frac{d_k}{\Delta t}, \quad \beta_k = \frac{1}{\Delta t} \left( c_{k+1/2} X^{n-1}_{k+1/2} + d_{k+1} \beta_{k+1} \right).
\]

with \( \Delta t = c_{k+1/2} + d_k + (1 - \alpha_k \beta_k) d_{k+1} \). The \( \alpha_k \) and \( \beta_k \) are first computed downwards starting from the upper boundary, where

\[
\alpha_{N-1} = \frac{d_{N-1}}{c_{N-1/2} + d_{N-1}}, \quad \beta_{N-1} = \frac{c_{N-1/2} X^{n-1}_{N-1/2}}{c_{N-1/2} + d_{N-1}}.
\]
Fig. F.1. Discretisation of the atmosphere and ground into vertical levels.

Then, the $X^n_{k+1/2}$ are computed upwards, from $k = 0$ to $k = N - 1$, using the recursion relation given by Eq. (F.12) and starting from

$$X^n_{1/2} = \frac{d_t \beta_1 - F_{\text{urb}} + c_{1/2} X^{N-1}_{N/2}}{\alpha_1 + d_1 (1 - \alpha_1)}.$$  

(F.15)

This allows for calculating the source terms of the momentum, thermodynamic, and moisture conservation equations. Basically,

$$\frac{d\vartheta}{dr} = \left( \frac{p}{p_{\text{ref}}} \right)^{\frac{k}{\gamma}} \frac{d}{dr} (C_p \Theta), \quad \dot{\vartheta} = \frac{d\vartheta}{dr}.$$  

(F.16)

### Appendix G: Soil heat transfer scheme

The one-dimensional heat conduction equation given by Eq. (G.3) is solved by means of a finite difference method adapted from Appendix B.1 of Wang et al. (2016). The domain is discretised into $N_{gr}$ vertical intervals of $\tilde{u}$ following a geometric law of scale factor $\alpha$ (Fig. F.1). Denoting by $k = 0, \ldots, N_{gr}$ the vertical grid levels ($k = 0$ corresponding to the surface, and $k = N_{gr}$ to the inner boundary of the domain), we express all of the $\tilde{u}_k$ as functions of the thickness of the first layer $\tilde{u}_1$. Literally, the depths of the full and intermediate levels are respectively given by

$$\tilde{u}_k = \frac{\alpha^k - 1}{\alpha - 1} \tilde{u}_1, \quad k = 0, \ldots, N_{gr}.$$  

(G.1)

$$\tilde{u}_{k+1/2} = \frac{\alpha^{k+1/2} - 1}{\alpha - 1} \tilde{u}_1, \quad k = 0, \ldots, N_{gr} - 1.$$  

(G.2)

We take $N_{gr} = 6$, $\alpha = 2$ and $\tilde{u}_1 = 0.1$ s$^{1/2}$. To solve the heat equation, we use an implicit scheme. The set of discretised equations is written, for $1 \leq k \leq N_{gr} - 1$, as

$$\frac{T^n_{k+1/2} - T^{n-1}_{k+1/2}}{\Delta t} = \frac{1}{3} \left[ \frac{\delta \vartheta T^n_{k+1} + \beta_k T^n_{k+1/2} - \delta \vartheta T^n_{k+1/2}}{\delta \tilde{u}_{k+1} + \delta \tilde{u}_{k+1/2}} \right],$$  

(G.3)

with $\delta \vartheta T^n_{k+1} = T^n_{k+1/2} - T^n_{k+1}$, and the boundary conditions yield

$$\frac{T^n_{1/2} - T^{n-1}_{1/2}}{\Delta t} = \frac{1}{\delta \tilde{u}_{1/2}} \left[ \delta \vartheta T^n_1 + \frac{1}{l_{gr}} \left( \sum F_i (T_s) - \varepsilon \sigma_{SB} T^n_1^4 \right) \right],$$  

(G.4)

and

$$\frac{T^n_{N_{gr} - 1/2} - T^{n-1}_{N_{gr} - 1/2}}{\Delta t} = -\frac{1}{\delta \tilde{u}_{N_{gr} - 1/2}} \left[ \delta \vartheta T^n_{N_{gr} - 1/2} \right].$$  

(G.5)

Introducing the coefficients $c_{k+1/2}$ and $d_k$ defined as

$$c_{k+1/2} = \frac{\delta \tilde{u}_{k+1/2}}{\Delta t}, \quad d_k = \frac{1}{\delta \tilde{u}_k},$$  

(G.6)

the above equations (Eqs. (G.3-G.5)) are rewritten as

$$c_{k+1/2} \left( T^n_{k+1/2} - T^{n-1}_{k+1/2} \right) = d_k + \delta \vartheta n - d_k \delta \vartheta T^n_k,$$

(G.7)

$$c_{1/2} \left( T^n_{1/2} - T^{n-1}_{1/2} \right) = d_1 \delta \vartheta T^n_1 + \frac{1}{l_{gr}} \left( \sum F_i (T_s) - \varepsilon \sigma_{SB} T^n_1^4 \right),$$

$$c_{N_{gr} - 1/2} \left( T^n_{N_{gr} - 1/2} - T^{n-1}_{N_{gr} - 1/2} \right) = -d_{N_{gr} - 1} \delta \vartheta T^n_{N_{gr} - 1},$$

and the system is put into the standard algebraic form

$$\begin{bmatrix} B_0 & C_0 & \ldots & \ldots & C_{N_{gr} - 2} & C_{N_{gr} - 1} \\ A_1 & B_1 & \ldots & \ldots & C_{N_{gr} - 2} & C_{N_{gr} - 1} \\ A_2 & B_2 & \ldots & \ldots & C_{N_{gr} - 2} & C_{N_{gr} - 1} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ A_{N_{gr} - 2} & B_{N_{gr} - 2} & \ldots & \ldots & C_{N_{gr} - 2} & C_{N_{gr} - 1} \\ A_{N_{gr} - 1} & B_{N_{gr} - 1} & \ldots & \ldots & C_{N_{gr} - 2} & C_{N_{gr} - 1} \end{bmatrix} \begin{bmatrix} T^n_{1/2} \\ T^n_{3/2} \\ \vdots \\ T^n_{N_{gr} - 1/2} \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{N_{gr} - 1} \end{bmatrix},$$  

(G.8)

with the coefficients

$$A_k = -d_k,$$

(G.9)

$$B_k = d_k + d_{k+1} + c_{k+1/2},$$

for $k = 1, \ldots, N_{gr} - 1, 1,$

$$C_k = d_{k+1},$$

for $k = 0, \ldots, N_{gr} - 2, 1,$

$$B_{k+1/2} = c_{k+1/2} T^n_{k+1/2} + \frac{1}{l_{gr}} \left( \sum F_i (T_s) - \varepsilon \sigma_{SB} T^n_{k+1/2}^4 \right).$$

(G.10)

The algebraic system given by Eq. (G.8) is solved by making use of Thomas algorithm (Appendix H). As a first step, the temperatures of two consecutive levels are linked together by the recursion relation

$$T^n_{k+1/2} = \alpha_k T^n_{k+1/2} + \beta_k,$$

(G.11)

where the coefficients $\alpha_k$ and $\beta_k$ are defined, for $k = 1, \ldots, N_{gr} - 2$, as

$$\alpha_k = \frac{d_k}{\Delta k}, \quad \beta_k = \frac{1}{\Delta k} \left( c_{k+1/2} T^n_{k+1/2} + d_{k+1} \beta_{k+1} \right),$$

(G.12)

with $\Delta_k = c_{k+1/2} + d_k + (1 - \alpha_{k+1}) d_{k+1}$. At the inner boundary, the zero-flux condition leads to

$$\alpha_{N_{gr} - 1} = \frac{d_{N_{gr} - 1}}{\Delta k}, \quad \beta_{N_{gr} - 1} = \frac{c_{N_{gr} - 1} - 1}{\Delta k}.$$
with $\Delta N_{n-1} = d N_{n-1} + c N_{n-1}/2$. Thus, the coefficients $\alpha_k$ and $\beta_k$ are integrated upwards from the inner boundary.

As a second step, the equation of the surface boundary condition is put into the form

$$C_s \frac{T_s - T_{s-1}}{\Delta \tau} = F_s + \sum F_i - \varepsilon_s \sigma_{SB} T_s^4 - F_{1S}. \quad \text{(G.14)}$$

where $C_s^*$ and $F_s^*$ are expressed as

$$C_s^* = \int_{\alpha_s} \Delta \tau \left[ c_{1/2} + d_1 \left( 1 - \alpha_1 \right) \right],$$

$$F_s^* = \int_{\alpha_s} d_1 \left[ \beta_1 + (\alpha_1 - 1) \right] T_{s-1}^{4/2}. \quad \text{(G.16)}$$

In order to obtain an equation for the surface temperature, we proceed to a linear interpolation near the surface, which yields

$$T_s = (1 + \mu) T_{1/2} - \mu T_{3/2}, \quad \text{with} \quad \mu = \frac{\bar{u}_{1/2} - \bar{u}_{3/2}}{\bar{u}_{3/2} - \bar{u}_{1/2}}. \quad \text{(G.17)}$$

Combining the above equation with the recursion relation $T_{3/2} = \alpha_1 T_{1/2} + \beta_1$, we rearrange Eq. (G.14) into

$$C_s \frac{T_s - T_{s-1}}{\Delta \tau} = F_s + \sum F_i - F_{1S} - \varepsilon_s \sigma_{SB} \left( T_s^4 - 4 \varepsilon_s \sigma_{SB} \left( T_{s-1}^4 - T_s^4 \right) \right). \quad \text{(G.18)}$$

where the heat capacity per unit surface $C_s$ and upcoming flux $F_s$ are given by

$$C_s = \frac{C_s^*}{1 + \mu (1 - \alpha_1)},$$

$$F_s = F_s^* + \frac{C_s^*}{\Delta \tau} \left( T_{s-1}^4 - T_s^4 \right) \left( T_s - T_{s-1} \right) \left( T_{s-1}^4 - T_s^4 \right). \quad \text{(G.20)}$$

In practice, this equation is linearised and solved with an implicit scheme. The parameters $C_s$ and $F_s$ can be set to constants if one does not wish to solve the vertical diffusion within the ground. This yields the surface temperature of the current step $T_s^n$, which allows for calculating the temperatures at ground levels by using the recursion relation downwards.

Linearising and discretising the surface temperature evolution equation, given by Eq. (G.18), we obtain

$$C_s \frac{T_s^n - T_{s-1}^n}{\Delta \tau} = F_s + \sum F_i - F_{1S} - F_{\text{turb}} - \varepsilon_s \sigma_{SB} \left( T_{s-1}^4 - T_s^4 \right) + 4 \varepsilon_s \sigma_{SB} \left( T_{s-1}^4 - T_s^4 \right) \left( T_s^4 - T_{s-1}^4 \right). \quad \text{(G.21)}$$

The downward heat flux associated with turbulent diffusion is expressed in its general form as

$$F_{\text{turb}} = -\Delta T_s^n + B, \quad \text{(G.22)}$$

which allows for writing the surface temperature of the current step as

$$T_s^n = \frac{C_s^*}{\Delta \tau} T_{s-1}^{n-1} + 3 \varepsilon_s \sigma_{SB} \left( T_{s-1}^4 \right)^4 + F_s + \sum F_i - F_{1S} + B \frac{C_s^*}{\Delta \tau} + 4 \varepsilon_s \sigma_{SB} \left( T_{s-1}^4 \right)^3 + A.$$
The evolution of the surface content of any tracer in liquid or solid phase can be described with a similar equation. It induces a source-sink term in the moisture conservation equation. In the case of temperature the downward turbulent flux is given by

\[ F_{\text{turb}} = -F_H = C_{HPSL} C_p |\nabla_T| (\Theta_{SL} - \Theta_s) , \]

which is a function of the mean fields near the surface (see Eq. (33)). The coefficients \( A \) and \( B \) introduced in Eq. (G.22) are thus expressed as

\[ A = C_{HPSL} C_p |\nabla_T| \left( \frac{p_s}{p_{\text{ref}}} \right)^{-x} , \quad B = C_{HPSL} C_p |\nabla_T| \Theta_{SL} . \]

As shown by Fig. G.1, the soil heat transfer scheme is coupled with the turbulent diffusion scheme (Appendix F) through the equation of surface thermal evolution, which may lead to consistency issues. In order to preserve the consistency of the diffusion scheme from the lowest level of the atmosphere to the highest level of the ground conduction model, the two steps of the calculation are permuted in chronological order: temperatures are computed first, and the coefficients \( \alpha_k \) and \( \beta_k \) are computed then, and conserved for the next step.

Simulations were run for various values of the ground thermal inertia given by Eq. (41) \( (I_{\text{gr}} = 10^2, 10^3, 10^4 \text{ J} \text{ m}^{-2} \text{ s}^{-1/2} \text{ K}^{-1}) \) for the Earth-like planet of Fig. 1 in order to investigate how numerical solutions depend upon the soil thermal response. The two-day averaged temperature snapshots obtained from these simulations are shown by Fig. G.2. We observe that varying the ground thermal inertia over two orders of magnitude does not significantly alter the climate state of equilibrium. The change in \( I_{\text{gr}} \) essentially affects the maximum wind speed, which varies by \( \approx 9\% \). The minimum surface temperature on nightside \( T_n \) hardly varies, as it switches from 231.3 K for \( I_{\text{gr}} = 10^2 \text{ J} \text{ m}^{-2} \text{ s}^{-1/2} \text{ K}^{-1} \) to 231.8 K for \( I_{\text{gr}} = 10^4 \text{ J} \text{ m}^{-2} \text{ s}^{-1/2} \text{ K}^{-1} \). This insensitivity of mean fields to the soil vertical conduction is consistent with the fact that the circulation reaches a steady state where mean flows are essentially invariant in time. With variations, mean flows might be more substantially affected by vertical thermal diffusion in the soil.

**Appendix H: Thomas algorithm for block tridiagonal matrices**

The tridiagonal matrix algorithm (TDMA, or Thomas algorithm, Thomas 1949) can be used to solve a system of equations that involves a block tridiagonal matrix of the form

\[
\begin{bmatrix}
B_0 & C_0 & 0 & 0 \\
A_1 & B_1 & C_1 & 0 \\
A_2 & B_2 & C_2 & 0 \\
\vdots & \vdots & \vdots & \ddots \\
A_{N-2} & B_{N-2} & C_{N-2} & 0 \\
A_{N-1} & B_{N-1} & C_{N-1} & 0
\end{bmatrix}
\begin{bmatrix}
x_0 \\
x_1 \\
x_2 \\
\vdots \\
x_{N-2} \\
x_{N-1}
\end{bmatrix}
= \begin{bmatrix}
d_0 \\
d_1 \\
d_2 \\
\vdots \\
d_{N-2} \\
d_{N-1}
\end{bmatrix},
\]

where the \( A_k, B_k, C_k \) are sub-matrices indexed by \( k = 0, \ldots, N \), and the \( x_k \) and \( d_k \) vectors of appropriate dimensions. As a first step, the matrix is triangularised, meaning that the system is transformed into a system where the matrix is block triangular. The new system is written as

\[
\begin{bmatrix}
1 & \Gamma_0 & 0 & \ldots & 0 \\
1 & \Gamma_1 & \Gamma_1 & \ldots & 0 \\
1 & \Gamma_{N-2} & \Gamma_{N-2} & \ldots & 0 \\
& \vdots & \vdots & \ddots & \vdots \\
1 & \Gamma_1 & 0 & \ldots & \Gamma_1
\end{bmatrix}
\begin{bmatrix}
x_0 \\
x_1 \\
x_2 \\
\vdots \\
x_{N-2} \\
x_{N-1}
\end{bmatrix}
= \begin{bmatrix}
\beta_0 \\
\beta_1 \\
\beta_2 \\
\vdots \\
\beta_{N-2} \\
\beta_{N-1}
\end{bmatrix}.
\]

The matrices \( \Gamma_k \) and vectors \( \beta_k \) are computed forwards using the recursion relations

\[ \Gamma_0 = B_0^{-1} C_0 , \]

\[ \Gamma_k = (B_k - A_k \Gamma_{k-1})^{-1} C_k , \quad k = 1, \ldots, N - 2; \]

and

\[ \beta_0 = B_0^{-1} d_0 , \]

\[ \beta_k = (B_k - A_k \Gamma_{k-1})^{-1} (d_k - A_k \beta_{k-1}) , \quad k = 1, \ldots, N - 1 . \]

As a second step, the solution vectors \( x_k \) are computed backwards (backward sweep) using the recursion relation

\[ x_{N-1} = d_{N-1} , \]

\[ x_k = \beta_k - \Gamma_k x_{k+1} , \quad k = N - 2, \ldots, 0 . \]
Appendix I: Interhemispheric mass flow rate

At the terminator ($\theta = 90^\circ$ in tidally locked coordinates), the total mass flow rate (i.e. mass that passes through the terminator annulus per unit of time in one direction) is given by

$$F_{\text{mass}} = \pi R_p \int_0^{\pi} |v_\theta| \rho \, dz.$$  \hspace{1cm} (I.1)

The day-night advection timescale $t_{\text{adv}}$ corresponds to the mean timescale necessary for one particle to accomplish one full cycle of the day-night overturning circulation. It measures the renewal rate of the air and the strength of the cell. The smaller $t_{\text{adv}}$ and the faster air is advected from dayside to nightside. Introducing the total mass of the atmosphere $M_{\text{atm}} = (4\pi R_p^2 \rho) / g$, this timescale can be defined as

$$t_{\text{adv}} = \frac{M_{\text{atm}}}{F_{\text{mass}}}.$$  \hspace{1cm} (I.2)

In sigma-coordinate, the circulation timescale is expressed as

$$t_{\text{adv}} = \frac{4R_p}{\int_0^1 |v_\sigma| \, d\sigma},$$  \hspace{1cm} (I.3)

the integral being performed at the terminator ($\theta = 90^\circ$).

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