A NEW PROOF OF KIRCHBERG’S $O_2$-STABLE CLASSIFICATION

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Abstract. I present a new proof of Kirchberg’s $O_2$-stable classification theorem: two separable, nuclear, stable/unital, $O_2$-stable $C^*$-algebras are isomorphic if and only if their ideal lattices are order isomorphic, or equivalently, their primitive ideal spaces are homeomorphic. Many intermediate results do not depend on pure infiniteness of any sort.

1. Introduction

After classifying all $A_T$-algebras of real rank zero [Ell93], Elliott initiated a highly ambitious programme of classifying separable, nuclear $C^*$-algebras by $K$-theoretic and tracial invariants. During the past three decades much effort has been put into verifying such classification results. In the special case of simple $C^*$-algebras the classification has been verified under the very natural assumptions that the $C^*$-algebras are separable, unital, simple, with finite nuclear dimension and in the UCT class of Rosenberg and Schochet [RSS7]. The purely infinite case is due to Kirchberg [Kir94] and Phillips [Phi00], and the stably finite case was recently solved by the work of many hands; in particular work of Elliott, Gong, Lin, and Niu [GLN15], [EGLN15], and by Tikuisis, White, and Winter [TWW15]. This makes this the right time to gain a deeper understanding of the classification of non-simple $C^*$-algebras which is the main topic of this paper.

The Cuntz algebra $O_2$ plays a special role in the classification programme as this has the properties of being separable, nuclear, unital, simple, purely infinite, and is $KK$-equivalent to zero. Hence if $A$ is any separable, nuclear $C^*$-algebra then $A \otimes O_2$ has no $K$-theoretic nor tracial data to determine potential classification, and one may ask if such $C^*$-algebras are classified by their primitive ideal space alone.

Predating the Kirchberg–Phillips theorem an important special case of this question was whether $A \otimes O_2 \cong O_2$ for any separable, nuclear, unital, simple $C^*$-algebra $A$. Some of the first major breakthroughs for verifying this were Elliott’s (unpublished) proof that $O_2 \otimes O_2 \cong O_2$, as well as Rørdam’s characterisation of when $A \otimes O_2 \cong O_2$, see [Rør94] or [Rør02, Theorem 7.2.2].

In Genève 1994 Kirchberg announced the $O_2$-embedding theorem; that any separable exact $C^*$-algebra embeds into the Cuntz algebra $O_2$. As an important consequence one gets that $A \otimes O_2 \cong O_2$ for any separable, nuclear, unital, simple $C^*$-algebra $A$. The $O_2$-embedding theorem also played an important role in the proof of the Kirchberg–Phillips theorem [Kir94], [Phi00], i.e. the classification of separable, nuclear, simple, purely infinite $C^*$-algebras. The first published proof of the $O_2$-embedding theorem appeared in [KP00].

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Kirchberg and Rørdam initiated the study of not necessarily simple (weakly/strongly) purely infinite $C^*$-algebras in [KR00] and [KR02]. Such $C^*$-algebras arise from many natural constructions. For instance, there are natural characterisations of when crossed products [KS17a], groupoids [BCS15], and Fell bundles [KS17b] are (strongly) purely infinite and it is an intriguing problem to classify them.

In [Kir00] Kirchberg outlined a far reaching generalisation of the Kirchberg–Phillips theorem by classifying all (not necessarily simple) separable, nuclear, strongly purely infinite $C^*$-algebras by ideal related $KK$-theory. A full proof of the result will appear in an upcoming book [Kir]. As an intermediate result Kirchberg obtains an ideal related version of the $O_2$-embedding theorem from which the following elegant result follows: any two separable, nuclear, stable/unital, $O_2$-stable $C^*$-algebras are isomorphic if and only if their primitive ideal spaces are homeomorphic.

The goal of this paper is to present an almost self-contained proof this result which is also shorter and more elementary than the proof contained in the widely distributed (far from finished) version of Kirchberg’s upcoming book [Kir]. At the moment a detailed proof of Kirchberg’s $O_2$-stable classification has never been published nor been publicly available, so this is the first published proof of the result.

The $O_2$-embedding theorem was the cornerstone of the Kirchberg–Phillips theorem. In the same way, the ideal related $O_2$-embedding theorem - the main intermediate result of this paper - plays a fundamental role in the proof of Kirchberg’s classification of strongly purely infinite $C^*$-algebras.

In general it is hard to determine when a strongly purely infinite $C^*$-algebra is $O_2$-stable. Dadarlat showed in [Dad09, Theorem 1.3] that if $X$ is a compact, metrisable space with finite covering dimension, and $A$ is a separable, unital $C(X)$-algebra for which each fibre of $A$ is isomorphic to $O_2$, then $A \cong C(X) \otimes O_2$. However, Dadarlat also gave examples [Dad09, Example 1.2] of separable, unital $C(X)$-algebras $A$ for which $X$ is the Hilbert cube such that each fibre is isomorphic to $O_2$, but for which $A \otimes O_2 \not\cong A$. Inspired by these results I showed in [Gab16], using Kirchberg’s non-simple classification, that a separable, nuclear, strongly purely infinite $C^*$-algebra $A$ is $O_2$-stable if and only if each two-sided, closed ideal in $A$ is $KK$-equivalent to zero.

1.1. The main results. In order to present the main results of the paper I introduce some notation. The ideal lattice of a $C^*$-algebra $A$ is denoted by $\mathcal{I}(A)$, and whenever $A$ is separable then $\mathcal{I}(A)$ is considered as an object in the category of abstract Cuntz semigroups. A $Cu$-morphism $\mathcal{I}(A) \to \mathcal{I}(B)$ is a map that preserves the Cuntz semigroup structure. It is shown in Lemma 2.12 that any $*$-homomorphism $\phi: A \to B$ induces a $Cu$-morphism $\mathcal{I}(\phi): \mathcal{I}(A) \to \mathcal{I}(B)$ via the assignment $I \mapsto B\phi(I)B$. More details will be given in Section 2.

The first main theorem, which is a special case of Corollary 6.11 gives a complete classification of $*$-homomorphisms between $A$ and $B \otimes O_2 \otimes \mathbb{K}$ whenever $A$ is separable, exact and $B$ is separable, nuclear.

**Theorem A.** Let $A$ and $B$ be separable $C^*$-algebras with $A$ exact and $B$ nuclear. Then the approximate unitary equivalence classes of $*$-homomorphisms $A \to B \otimes O_2 \otimes \mathbb{K}$ are in a natural one-to-one correspondence with $Cu$-morphisms $\mathcal{I}(A) \to \mathcal{I}(B)$ between the ideal lattices of $A$ and $B$. 
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As an almost immediate consequence the following classification result due to Kirchberg [Kir00] is obtained. A slightly more general result is provided in Theorem 6.13. Note that the classification is strong, i.e. that any isomorphism on the invariant is induced by an isomorphism of the $C^*$-algebras.

**Theorem B.** Let $A$ and $B$ be separable, nuclear $C^*$-algebras for which $A \cong A \otimes \mathcal{O}_2$ and $B \cong B \otimes \mathcal{O}_2$. Suppose that $A$ and $B$ are both stable or both unital, and that $f : \text{Prim} A \to \text{Prim} B$ is a homeomorphism. Then there exists an isomorphism $\phi : A \cong B$ such that $f(I) = \phi(I)$ for every $I \in \text{Prim} A$.

The proofs are based mainly on elementary or well-known results from the literature with the main exception being a deep structural result due to Kirchberg and Rørdam [KR05, Theorem 6.11], which shows that separable, nuclear, $\mathcal{O}_2$-stable $C^*$-algebras contain suitably well-behaved, commutative $C^*$-subalgebras. This will be used in the proof of Proposition 5.5.

The overall strategy of the proof is very classical: I provide an existence and a uniqueness result for $*$-homomorphisms into $\mathcal{O}_2$-stable $C^*$-algebras, using the ideal lattice of $C^*$-algebras as the classifying invariant. The ideal lattice is considered as a covariant functor with target category being the category $\text{Cu}$ of abstract Cuntz semigroups, as first introduced in [CEI08]. This is in contrast to Kirchberg’s approach which is to consider $C^*$-algebras with actions of topological spaces, an approach which is of a more contravariant nature. My approach allows for the use of compact containment of ideals, i.e. way-below in the Cuntz semigroup sense. Compact containment will play a crucial role, see Proposition 6.5. I will at points digress slightly from the main objective in order to present a more well-rounded theory applicable in a more general setting. For instance, certain results are proved for order zero maps, while the statements are only needed for $*$-homomorphisms.

**Outline of the paper.** The basic properties of ideal lattices are studied in Section 2. In Section 3, uniqueness results for nuclear maps are presented. The first uniqueness result Theorem 3.3 is the key ingredient. It shows that nuclear $*$-homomorphisms that define the same map between ideal lattices will approximately dominate each other. I then introduce an equivalence relation on $*$-homomorphisms weaker than approximate and asymptotic unitary equivalence, which I call approximate and asymptotic Murray–von Neumann equivalence. A key feature of this equivalence relation is that it is exactly the equivalence relation for which a certain $2 \times 2$-matrix trick of Connes [Con73] is applicable, see Proposition 3.10. Similar matrix tricks have recently appeared in the classification programme, notably in the work of Matui and Sato [MS14] and by Bosa et al. [BBS15]. This matrix trick is used to prove Theorem 3.23 which is a uniqueness result for so-called (strongly) $\mathcal{O}_2$-stable $*$-homomorphisms, showing that the approximate (resp. asymptotic) Murray–von Neumann equivalence class only depends on the induced maps between the ideal lattices.

In Section 4, a characterisation is given of when an approximate $*$-homomorphism (i.e. a $*$-homomorphism going into a sequence algebra) is approximately unitary equivalent to a $*$-homomorphism represented point-wise by constant sequences. This technique is a discrete version of [Phi00, Proposition 1.3.7]. Here no assumptions are put on our $C^*$-algebras.

With a uniqueness result in the utility belt I move on to study existence results for maps between ideal lattices (Section 5). This is done by using Michael’s selection theorem to

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1 At least when $A$ is separable, cf. Proposition 2.5.
produce well-behaved c.p. maps into commutative $C^*$-algebras, a method inspired by work of Blanchard [Bl96]. Similar results using the same method have previously been obtained by Harnisch and Kirchberg [HK05]. A lot of these results are very general and have (almost) no requirements on the $C^*$-algebras involved, e.g. pure infiniteness type criteria are never assumed.

Finally, in Section 6 I prove the main existence theorem which is an ideal related version of the $O_2$-embedding theorem. Kirchberg’s original proof uses (non-unital, ideal related versions of) $C^*$-systems as introduced in [Kir95]. The proof presented here only uses elementary $C^*$-algebraic techniques inspired by (though still quite different from) the proof of the $O_2$-embedding theorem presented by Kirchberg and Phillips in [KP00]. As a consequence I obtain Kirchberg’s classification of $O_2$-stable $C^*$-algebras.

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2. The basics

Throughout the entire paper every ideal is assumed to be two-sided and closed.

2.1. Compact containment. A very basic and somewhat overlooked property of ideal lattices of $C^*$-algebras is the compact containment relation which is exactly the way-below relation in complete lattices. This relation plays a crucial role in the study of ideal lattices. My motivation for considering this relation for ideals is inspired work on the Cuntz semigroup of $C^*$-algebras. See Proposition 2.5 for the connection to Cuntz semigroups.

Definition 2.1. Let $I$ and $J$ be ideals in a $C^*$-algebra $A$. Say that $I$ is compactly contained in $J$, written $I \ll J$, if whenever $(I_\lambda)_{\lambda \in \Lambda}$ is a family of ideals in $A$ such that $J \subseteq \sum_\lambda I_\lambda$, then there are finitely many $\lambda_1, \ldots, \lambda_n \in \Lambda$ such that $I \subseteq \sum_{k=1}^n I_{\lambda_k}$.

Note that in the definition above if the family $(I_\lambda)$ is upwards directed then it is always possible to find a single $\lambda$ such that $I \subseteq I_\lambda$.

Given a positive element $a \in A$ and an $\epsilon > 0$, $(a - \epsilon)_+ := f_\epsilon(a)$ denotes the element defined by functional calculus where $f_\epsilon(t) = \max\{0, t - \epsilon\}$.

Lemma 2.2. Let $A$ be a $C^*$-algebra and let $I, J$ be ideals in $A$. Then $I \subseteq J$ if and only if there is a positive element $a \in J$ and an $\epsilon > 0$ such that $I \subseteq \overline{A(a - \epsilon)_+A}$.

In particular, $\overline{A(a - \epsilon)_+A} \subseteq \overline{AaA}$ for any positive $a \in A$ and $\epsilon > 0$.

Proof. “In particular” clearly follows from the “if” statement. For the “if” part, suppose there are $a \in J_+$, and $\epsilon > 0$ such that $I \subseteq \overline{A(a - \epsilon)_+A}$, and let $(I_\lambda)$ be a family of ideals in $A$ such that $J \subseteq \sum_\lambda I_\lambda$. There are $\lambda_1, \ldots, \lambda_n$ and a positive element $b \in \sum_{k=1}^n I_{\lambda_k}$ such
that \( \|a - b\| < \epsilon/2 \). Note that \( a - \frac{\epsilon}{2} \cdot 1_A \leq b \). Let \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) be any continuous function which is 0 on \([0, \epsilon/2]\), and 1 on \([\epsilon, \infty)\). Then

\[
(a - \epsilon)_+ \leq f(a)(a - \frac{\epsilon}{2} \cdot 1_A) f(a) \leq f(a)bf(a).
\]

Thus \( I \subseteq A(a - \epsilon)_+A \subseteq AbA \subseteq \sum_{k=1}^n I_{\lambda_k} \).

For the “only if” part, suppose that \( I \in J \). The family of ideals \((A(a - \epsilon)_+A)_{a \in J, \epsilon > 0}\) is upwards directed. In fact, if \( a_1, a_2 \in J_+, \epsilon_1, \epsilon_2 > 0 \) let \( a = a_1 + a_2 \). By the “in particular”-part, \( A(a_i - \epsilon_i)_+A \subseteq AaA = \bigcup_{\epsilon > 0} A(a - \epsilon)_+A \) so there are \( \epsilon'_i > 0 \) such that \( A(a_i - \epsilon_i)_+A \subseteq A(a - \epsilon'_i)_+A \). Hence the family of ideals is upwards directed. So as \( J = \sum_{a \in J_+, \epsilon > 0} A(a - \epsilon)_+A \) it follows from compact containment that \( I \subseteq A(a - \epsilon)_+A \) for some \( a \in J_+ \) and \( \epsilon > 0 \).

The above lemma implies that whether or not an ideal is generated by a single element, i.e. contains a full element, can be determined from the ideal lattice.

**Corollary 2.3.** Let \( A \) be a \( C^* \)-algebra, and \( I \) be an ideal in \( A \). Then \( I \) has a full element if and only if there is a sequence \( I_1 \subseteq I_2 \subseteq \ldots \) of ideals such that \( I = \bigcup I_n \).

**Proof.** If \( a \in I \) is full then \( I_n := A(a^*a - 1/n)_{+}A \) gives the desired sequence. Conversely, suppose the sequence of ideals is given. By Lemma 2.2 we find positive contractions \( a_n \in A \) such that \( I_n \subseteq Aa_nA \subseteq I_{n+1} \). It is easy to see that \( \sum_{n=1}^\infty 2^{-n}a_n \) is full in \( I \).

2.2. The ideal lattice.

**Notation 2.4.** Let \( \mathcal{I}(A) \) denote the **ideal lattice** of a \( C^* \)-algebra \( A \).

Consider \( \mathcal{I}(A) \) as an ordered, abelian monoid (addition is addition of ideals). It has increasing suprema (closure of unions), and compact containment.

Let \( \text{Cu} \) denote the category of abstract Cuntz semigroups, i.e. the category of ordered, abelian monoids with countable increasing suprema and way-below such that these are suitably well-behaved. See [CE10, Page 170] or [APT11, Definition 4.1] for the precise definition.

One particular part of the definition which will be used below, is that for any element \( x \) in an abstract Cuntz semigroup there is a sequence \( x_1 \ll x_2 \ll \ldots \) such that \( \sup x_n = x \).

**Proposition 2.5.** Let \( A \) be a \( C^* \)-algebra. Then \( \mathcal{I}(A) \) is an object in \( \text{Cu} \) if and only if any ideal in \( A \) contains a full element. In particular, \( \mathcal{I}(A) \) is in \( \text{Cu} \) whenever \( A \) is separable.

**Proof.** If \( \mathcal{I}(A) \) is an object of \( \text{Cu} \), then (by definition) for any \( I \in \mathcal{I}(A) \) there is a sequence \( I_1 \subseteq I_2 \subseteq \ldots \) such that \( I = \bigcup I_n \). By Corollary 2.3 \( I \) contains a full element.

Conversely, suppose any ideal in \( A \) has a full element. As \( O_\infty \otimes K \) is simple and nuclear there is a canonical order isomorphism \( \mathcal{I}(A) \cong \mathcal{I}(A \otimes O_\infty \otimes K) \) given by \( I \mapsto I \otimes O_\infty \otimes K \). Note that any ideal in \( A \otimes O_\infty \otimes K \) also contains a full element.

As \( A \otimes O_\infty \otimes K \) is purely infinite by [KR00, Theorem 5.11], for any two positive elements \( a, b \) in \( A \otimes O_\infty \otimes K \) we have \( [a] \leq [b] \) in \( Cu(A \otimes O_\infty \otimes K) \) if and only if \( a \) is contained in the ideal generated by \( b \). Hence the canonical map \( Cu(A \otimes O_\infty \otimes K) \to \mathcal{I}(A \otimes O_\infty \otimes K) \) is an order isomorphism onto its image. Moreover, as every ideal in \( A \otimes O_\infty \otimes K \) has a full element this map is onto so \( Cu(A \otimes O_\infty \otimes K) \cong \mathcal{I}(A \otimes O_\infty \otimes K) \).
It is easy to see (by pure infiniteness) that addition in $Cu(A \otimes O_\infty \otimes K)$ is uniquely determined by the order. As sup and $\ll$ are also uniquely determined by the order it follows that the isomorphism $Cu(A \otimes O_\infty \otimes K) \cong I(A \otimes O_\infty \otimes K)$ preserves $(0, +, \sup, \ll)$. As $Cu(A \otimes O_\infty \otimes K)$ is an object of $Cu$, so is $I(A) \cong I(A \otimes O_\infty \otimes K)$.

One could alternatively also prove the “if”-statement above simply by checking that $I(A)$ satisfies the defining criteria for objects in $Cu$ instead of using that $Cu(A \otimes O_\infty \otimes K)$ is an object in $Cu$. This is straightforward but tedious.

The above proposition motivates the following definition.

**Definition 2.6.** A map $\Phi: I(A) \rightarrow I(B)$ is a generalised $Cu$-morphism if it is an ordered monoid homomorphism which preserves increasing suprema. Say that $\Phi$ is a $Cu$-morphism if in addition it preserves compact containment.

**Remark 2.7.** In the category $Cu$ a generalised $Cu$-morphisms is an ordered monoid homomorphism which preserves countable increasing suprema. A $Cu$-morphism is a generalised $Cu$-morphism which preserves compact containment.

The criterion that any $x$ in a $Cu$-semigroup is the supremum of a sequence $x_1 \ll x_2 \ll \ldots$ implies that whenever a (not necessarily countable) increasing net has a supremum then any (generalised) $Cu$-morphism preserves this supremum. So (generalised) $Cu$-morphisms in $Cu$ preserve (not necessarily countable) increasing suprema whenever these exist.

Hence whenever $I(A)$ and $I(B)$ are objects in $Cu$, see Proposition 2.5, a map $\Phi: I(A) \rightarrow I(B)$ is a (generalised) $Cu$-morphism in the sense of Definition 2.6 if and only if it is in the sense of $Cu$.

**Remark 2.8.** A map $\Phi: I(A) \rightarrow I(B)$ is a generalised $Cu$-morphism if and only if it preserves suprema (possibly empty, and not necessarily increasing).

In fact, as $\sup \emptyset = 0$ and as the supremum of finitely many ideals is the sum, it follows that any supremum preserving map is an ordered monoid homomorphism and thus a generalised $Cu$-morphism. Conversely, suppose $\Phi$ is a generalised $Cu$-morphism. Then $\Phi(\sup \emptyset) = \Phi(0) = 0 = \sup \emptyset$ so $\Phi$ preserves the supremum of $\emptyset$. Moreover, if $S \subseteq I(A)$ is non-empty, let $\sum S$ denote the set $\{\sum_{I \in S'} I : S' \subseteq S \text{ is finite}\}$. Clearly $\sup \sum S = \sup S$ and as $\sum S$ is upwards directed it follows that $\Phi(\sup S) = \Phi(\sup \sum S) = \sup \Phi(\sum S) = \sup \Phi(S)$ so $\Phi$ preserves the supremum of $S$.

**Notation 2.9.** If $\phi: A \rightarrow B$ is a completely positive (c.p.) map then $I(\phi): I(A) \rightarrow I(B)$ is the map $I(\phi)(I) = B\phi(I)B$ for $I \in I(A)$.

**Remark 2.10.** Given a $*$-homomorphism $\phi: A \rightarrow B$ it is common to consider the induced map on ideal lattices $\phi^*: I(B) \rightarrow I(A)$ given by $\phi^*(I) = \phi^{-1}(I)$. In this way the ideal lattice is a contravariant functor. I emphasise that this is not the same approach as used in this paper where the ideal lattice is considered a covariant functor.

**Remark 2.11.** $I(\phi)$ is defined for any c.p. map $\phi$. However, $I(-)$ is not functorial on the category of $C^*$-algebras with c.p. maps as morphisms. For instance, if $\phi: C \rightarrow M_2(\mathbb{C})$ is the embedding into the $(1, 1)$-corner, and $\psi: M_2(\mathbb{C}) \rightarrow C$ is the compression to the $(2, 2)$-corner, then $\psi \circ \phi = 0$, but $I(\psi) \circ I(\phi) = id_{I(\mathbb{C})} \neq I(0) = I(\psi \circ \phi)$.

However, $I(-)$ is functorial on the category of $C^*$-algebras with c.p. order zero maps as morphisms, cf. Proposition 2.15 below.
Lemma 2.12. Let \( \phi: A \to B \) be a c.p. map. The following hold.

(i) \( \mathcal{I}(\phi)(I) = B\phi(I+)B \) for all \( I \in \mathcal{I}(A) \).
(ii) \( \mathcal{I}(\phi) \) is a generalised \( \text{Cu} \)-morphism.
(iii) \( \mathcal{I}(\phi) \) is a \( \text{Cu} \)-morphism whenever \( \phi \) is a \( * \)-homomorphism.

Proof. (i): As \( \phi(a)^*\phi(a) \leq \phi(a^*a) \) it easily follows that \( \mathcal{I}(\phi)(I) = B\phi(I+)B \).

(ii): Clearly \( \mathcal{I}(\phi) \) preserves zero and order. As \( (I + J)_+ = I_+ + J_+ \) for ideals \( I, J \) in \( A \), it follows from (i) that

\[
\mathcal{I}(\phi)(I + J) = B\phi(I_+ + J_+)B = B(\phi(I_+) \cup \phi(J_+))B = \mathcal{I}(\phi)(I) + \mathcal{I}(\phi)(J).
\]

So \( \mathcal{I}(\phi) \) is an ordered monoid homomorphism. Let \( (I_\lambda) \) be an increasing net of ideals in \( A \), and let \( I = \bigcup I_\lambda \). Clearly \( \bigcup \mathcal{I}(\phi)(I_\lambda) \subseteq \mathcal{I}(\phi)(I) \) as \( \mathcal{I}(\phi)(I_\lambda) \subseteq \mathcal{I}(\phi)(I) \) for each \( \lambda \). To show the other inclusion, let \( x \in I \) and pick a sequence \( (x_n) \) in \( \bigcup I_\lambda \) so that \( \|x_n - x\| \to 0 \). Then \( \|\phi(x_n) - \phi(x)\| \to 0 \), and as \( \phi(x_n) \in \bigcup \mathcal{I}(\phi)(I_\lambda) \) it follows that \( \phi(x) \in \bigcup \mathcal{I}(\phi)(I_\lambda) \). Hence \( \mathcal{I}(\phi) \) preserves increasing suprema and is thus a generalised \( \text{Cu} \)-morphism.

(iii): If \( \phi \) is a \( * \)-homomorphism we want to see that \( \mathcal{I}(\phi) \) preserves compact containment, so suppose that \( I \subseteq J \). Note that \( \mathcal{I}(\phi)(\overline{AaA}) = B\phi(b)B \) for any \( b \in A_+ \) as \( \phi \) is multiplicative. By Lemma \ref{lemma2.2} there are \( a \in J_+ \) and \( \epsilon > 0 \) such that \( I \subseteq \overline{A(a - \epsilon)_+A} \). As

\[
\mathcal{I}(\phi)(I) \subseteq \mathcal{I}(\phi)(\overline{A(a - \epsilon)_+A}) = B(\phi(a) - \epsilon)_+B \subseteq B\phi(a)B = \mathcal{I}(\phi)(\overline{AaA}) \subseteq \mathcal{I}(\phi)(J)
\]

it follows that \( \mathcal{I}(\phi) \) preserves compact containment. \( \square \)

2.3. On functoriality of \( \mathcal{I}(-) \). It is shown that \( \mathcal{I}(-) \) is functorial on the category of \( C^* \)-algebras with c.p. order zero maps as morphisms. Although the generality of order zero maps is not actually needed in this paper, I believe that this level of generality could be applicable in other contexts. A few additional lemmas on functoriality of \( \mathcal{I}(-) \) is also provided which will be used in Section 3.

Recall that a c.p. map \( \phi: A \to B \) is called order zero if \( \phi(a)\phi(b) = 0 \) whenever \( a, b \in A_+ \) are such that \( ab = 0 \).

Remark 2.13 (Basics of order zero maps). Suppose \( \phi: A \to B \) is a c.p. order zero map. It follows from \cite{WZ09} Theorem 3.3 that if one lets \( C = C^*(\phi(A)) \), there is a positive element \( h \in \mathcal{M}(C) \cap \phi(A)' \) and a \( * \)-homomorphism \( \pi: A \to \mathcal{M}(C) \cap \{h\}' \) such that \( \phi = h\pi(-) \). It is easy to see (using \cite{WZ09} Proposition 3.2 for the case when \( A \) is non-unital) that the pair \((\pi, h)\) with this property is unique.

This is used to do functional calculus of order zero maps c.f. \cite{WZ09} Corollary 4.2. In fact, for \( f: [0, \infty) \to [0, \infty) \) a continuous map for which \( f(0) = 0 \), one defines \( f(\phi)(-) := f(h)\pi(-): A \to B \) which is also a c.p. order zero map.

This clearly implies that c.p. order zero maps have the following bi-module type property: for any \( a, b, c \in A \)

\[
(\phi(abc) = \lim_{k \to \infty} \phi^{1/k}(a)\phi(b)\phi^{1/k}(c)).
\]

Lemma 2.14. Let \( \phi: A \to B \) be a c.p. order zero map. For any \( a, b, c \in A \), \( \phi(abc) \in B\phi(b)B \).

Proof. This follows from equation (2.1). \( \square \)
Proposition 2.15. Let $A$, $B$ and $C$ be $C^*$-algebras, and let $\phi: A \to C$ and $\psi: C \to B$ be c.p. maps. If $\psi$ is order zero, then $\mathcal{I}(\psi \circ \phi) = \mathcal{I}(\psi) \circ \mathcal{I}(\phi)$.

Proof. Let $I \in \mathcal{I}(A)$. Then

\[ \mathcal{I}(\psi \circ \phi)(I) = \overline{B\psi(\phi(I))B} \subseteq \overline{B\psi(\mathcal{I}(\phi)(I))B} = \mathcal{I}(\psi) \circ \mathcal{I}(\phi)(I). \]

To show the other inclusion, it suffices by Lemma 2.12(i) to show that $\psi((\mathcal{I}(\phi)(I))_+) \subseteq \overline{B\psi(\phi(I))B}$. Fix a positive element $x \in \mathcal{I}(\phi)(I)$. As $\mathcal{I}(\phi)(I)$ is generated by $\phi(I_+)$, we may for any $\epsilon > 0$ find $c_1, \ldots, c_n \in C$ and $y_1, \ldots, y_n \in I_+$ such that $(x - \epsilon)_+ = \sum_{i=1}^n c_i^* \phi(y_i) c_i$. Then

\[ \psi((x - \epsilon)_+) = \sum_{i=1}^n \psi(c_i^* \phi(y_i) c_i) \leq \sum_{i=1}^n B\psi(\phi(y_i))B \subseteq \overline{B\psi(\phi(I))B}. \]

Hence $\psi(x) \in \overline{B\psi(\phi(I))B}$. \qed

The following two lemmas about when $\mathcal{I}(-)$ preserves composition will be used in Section 5.

Lemma 2.16. Let $A$, $B$ and $C$ be $C^*$-algebras with $C$ commutative, and let $\phi: A \to C$ and $\psi: C \to B$ be c.p. maps. Then $\mathcal{I}(\psi) \circ \mathcal{I}(\phi) = \mathcal{I}(\psi \circ \phi)$.

Proof. Let $I \in \mathcal{I}(A)$. By (2.2), $\mathcal{I}(\psi \circ \phi)(I) \subseteq \mathcal{I}(\psi) \circ \mathcal{I}(\phi)(I)$ so it remains to prove the other implication. For this it is enough to prove that $\psi((\mathcal{I}(\phi)(I))_+) \subseteq \mathcal{I}(\psi \circ \phi)(I)$. To see this it suffices to show that $\psi((f - \epsilon)_+) \in \mathcal{I}(\psi \circ \phi)(I)$ for every positive $f \in \mathcal{I}(\phi)(I)$ and every $\epsilon > 0$.

As $\mathcal{I}(\phi)(I) = \mathcal{C}\phi(I_+)$, and as $C$ is commutative, we may pick $a_1, \ldots, a_n \in I_+$ and $g_1, \ldots, g_n \in C_+$ such that $\|f - \sum_{k=1}^n g_k \phi(a_k)\| < \epsilon$. As $C$ is commutative, this implies that

\[ (f - \epsilon)_+ \leq \sum_{k=1}^n g_k \phi(a_k) \leq \sum_{k=1}^n \|g_k\| \phi(a_k). \]

Thus

\[ \psi((f - \epsilon)_+) \leq \sum_{k=1}^n g_k \psi(\phi(a_k)) \leq \sum_{k=1}^n B\psi(\phi(a_k))B \subseteq \mathcal{I}(\psi \circ \phi)(I). \] \qed

Let $\otimes$ denote the minimal tensor product. If $B$ and $D$ are $C^*$-algebras, and $\eta$ is a state on $D$, $\lambda_\eta: B \otimes D \to B$ denotes the induced (left) slice map given on elementary tensors by $\lambda_\eta(b \otimes d) = b\eta(d)$.

Lemma 2.17. Let $D$ be an exact $C^*$-algebra, and suppose that $\eta$ is a faithful state on $D$. For any c.p. map $\phi: A \to B \otimes D$ we have $\mathcal{I}(\lambda_\eta \circ \phi) = \mathcal{I}(\lambda_\eta) \circ \mathcal{I}(\phi)$.

Proof. First note that $\lambda_\eta$ is faithful: if $x \in B \otimes D$ is positive and non-zero, then there are positive linear functionals $\mu_1$ on $B$ and $\mu_2$ on $D$ such that $\mu_2(\rho_{\mu_1}(x)) = (\mu_1 \otimes \mu_2)(x) > 0$\(^2\) where $\rho_{\mu_1}: B \otimes D \to D$ is the induced right slice map. Hence $(\rho_{\mu_1})(x) \in D$ is positive and non-zero so $\rho_1(\lambda_\eta(x)) = \eta(\rho_{\mu_1}(x)) > 0$, and thus $\lambda_\eta(x) \neq 0$.

\(^2\)If $\pi_1: B \to \mathcal{B}(\mathcal{H}_1), \pi_2: D \to \mathcal{B}(\mathcal{H}_2)$ are faithful representations, then $\pi_1 \times \pi_2: B \otimes D \to \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is faithful (and well-defined, by the definition of minimal tensor products). Hence there are $\xi_i \in \mathcal{H}_i$ such that $(\pi_1 \otimes \pi_2)(x)(\xi_1 \otimes \xi_2) \neq 0$. Letting $\mu_1 = \langle \pi_i(-)\xi_i, \xi_i \rangle$ does the trick.
Let $x \in B \otimes D$ be positive and $J \in \mathcal{I}(B)$. We claim that $x \in J \otimes D$ if and only if $\lambda_\eta(x) \in J$. “Only if” is trivial, so for “if” we assume that $\lambda_\eta(x) \in J$. As $D$ is exact, $(B \otimes D)/(J \otimes D) = (B/J) \otimes D$, so $\lambda_\eta$ descends to a faithful slice map $\overline{\lambda}_\eta: (B/J) \otimes D \to B/J$. Thus $\overline{\lambda}_\eta(x + J \otimes D) = \lambda_\eta(x) + J = 0$. So $x + J \otimes D = 0$, and hence $x \in J \otimes D$.

So for any subset $S \subseteq B \otimes D$ of positive elements, $\lambda_\eta(S)$ generates the ideal $J$ if and only if $J \otimes D = \bigcup_{J \in \mathcal{I}(B)} (J_0 \otimes D)$. Let $I \in \mathcal{I}(A)$, $S := \phi(I_+)$ and $J = B\lambda_\eta(S)\overline{B}$. Then $\mathcal{I}(\phi)(I) \subseteq J \otimes D$ so

$$\mathcal{I}(\lambda_\eta) \circ \mathcal{I}(\phi)(I) \subseteq \mathcal{I}(\lambda_\eta)(J \otimes D) = J = \mathcal{I}(\lambda_\eta \circ \phi)(I).$$

The implication $\mathcal{I}(\lambda_\eta \circ \phi)(I) \subseteq (\mathcal{I}(\lambda_\eta) \circ \mathcal{I}(\phi))(I)$ is easy, see [22].

\end{proof}

3. Uniqueness of nuclear maps via ideals

3.1. Approximate domination. In this section several uniqueness results for nuclear maps are presented. The main result of this subsection, Theorem 3.3, is the key ingredient in all of these results, and it characterises when nuclear order zero maps approximately dominate each other in terms of their behaviour on ideals.

Although the results in this paper only need the results in the generality of $*$-homomorphisms, and not order zero maps, I believe that the generality of order zero maps could potentially be applicable in other contexts.

Definition 3.1. Let $\phi, \psi: A \to B$ be c.p. maps and suppose that $\phi$ is order zero. Say that $\phi$ approximately dominates $\psi$ if for any finite subset $F \subseteq A$ and any $\epsilon > 0$, there are an $n \in \mathbb{N}$ and $b_1, \ldots, b_n \in B$ such that

$$\|\psi(a) - \sum_{k=1}^{n} b_k^* \phi(a) b_k\| < \epsilon, \quad a \in F.$$

Recall that a map is called nuclear if it is a point-norm limit of maps factoring by c.p. maps through matrix algebras. Note that nuclear maps are c.p. by definition.

Clearly any c.p. map which is approximately dominated by a nuclear map is itself nuclear.

Notation 3.2. For generalised $Cu$-morphisms $\Phi, \Psi: \mathcal{I}(A) \to \mathcal{I}(B)$, write $\Phi \leq \Psi$ if $\Phi(I) \subseteq \Psi(I)$ for all $I \in \mathcal{I}(A)$.

The following is a slight modification of similar results appearing in work of Kirchberg and Rørdam [KR02 Lemma 7.18], [KR05 Proposition 4.2], and the proof presented here is virtually identical to theirs. A similar result also appeared in [Gab16 Theorem 2.5]. The true strength of the result is that condition (i) is a uniform on compact sets type condition whereas condition (iii) is a point-wise condition.

Theorem 3.3. Let $A$ and $B$ be $C^*$-algebras with $A$ exact, let $\phi, \rho: A \to B$ be nuclear maps and suppose that $\phi$ is order zero. The following are equivalent

(i) $\phi$ approximately dominates $\rho$,
(ii) $\mathcal{I}(\rho) \leq \mathcal{I}(\phi)$,
(iii) $\rho(a) \in B\phi(a)\overline{B}$ for all $a \in A_+$.

Proof. (i) $\Rightarrow$ (ii): Obvious.
(ii) ⇒ (iii): Assume $\mathcal{I}(\rho) \leq \mathcal{I}(\phi)$ and let $a \in A_+$ be positive. As $\rho(a) \in \mathcal{I}(\rho)(AaA) \subseteq \mathcal{I}(\phi)(AaA)$, it suffices to show that $\mathcal{I}(\phi)(AaA) \subseteq B\phi(a)B$. However, $\phi(xay) \in B\phi(a)B$ for all $x, y \in A$ by Lemma 2.14, so this is obvious.

(iii) ⇒ (i): Assume $\rho(a) \in B\phi(a)B$ for all $a \in A_+$. Let $\mathcal{C}$ be the convex cone of all c.p. maps $A \to B$ which are approximately dominated by $\phi$. Clearly every map in $\mathcal{C}$ is nuclear. As $\mathcal{C}$ is point-norm closed, it follows by a Hahn–Banach separation argument that it is point-weakly closed. Hence, to show that $\rho \in \mathcal{C}$ it suffices to show that given $a_1, \ldots, a_n \in A$, $\epsilon > 0$ and $f_1, \ldots, f_n \in B^*$ then there is a $\psi \in \mathcal{C}$ such that

$$|f_i(\rho(a_i)) - f_i(\psi(a_i))| < \epsilon, \quad \text{for } i = 1, \ldots, n.$$ 

By the Radon–Nikodym theorem for $C^*$-algebras, see e.g. [KR02, Lemma 7.17 (i)], we may find a cyclic representation $\pi: B \to \mathbb{B}(\mathcal{H})$ with cyclic vector $\xi \in \mathcal{H}$, and elements $c_1, \ldots, c_n \in \pi(B)' \cap \mathbb{B}(\mathcal{H})$ such that $f_i(b) = \langle \pi(b)c_i, \xi \rangle$ for $i = 1, \ldots, n$. Let $C = C^*(c_1, \ldots, c_n)$ and $\iota: C \to \mathbb{B}(\mathcal{H})$ be the inclusion. For any c.p. map $\eta: A \to B$ there is an induced positive linear functional on $A \max \otimes C$ given by the composition

$$A \max \otimes C \xrightarrow{\eta \otimes \text{id}_C} B \max \otimes C \xrightarrow{\pi \times \iota} \mathbb{B}(\mathcal{H}) \xrightarrow{\omega_\xi} \mathbb{C},$$

where $\omega_\xi$ is the vector functional induced by $\xi$, i.e. $\omega_\xi(T) = \langle T\xi, \xi \rangle$. If $\eta$ is nuclear, then $\eta \otimes \text{id}_C$ above factors through the spatial tensor product $A \otimes C$, see e.g. [BO08, Lemma 3.6.10], so if $\eta$ is nuclear it induces a positive linear functional $\gamma_\eta$ on $A \otimes C$. Thus, as $\rho$ and any $\psi \in \mathcal{C}$ are nuclear, we get induced positive functionals $\gamma_\rho$ and $\gamma_\psi$ on $A \otimes C$.

Let $\mathcal{K}$ be the weak$^\ast$ closure of $\{\gamma_\psi : \psi \in \mathcal{C}\} \subseteq (A \otimes C)^\ast$. It suffices to show that $\gamma_\rho \in \mathcal{K}$. In fact, if $|\gamma_\rho(a_i \otimes c_i) - \gamma_\psi(a_i \otimes c_i)| < \epsilon$ for some $\psi \in \mathcal{C}$, then

$$f_i(\rho(a_i)) = \langle \pi(\rho(a_i))c_i, \xi \rangle = \gamma_\rho(a_i \otimes c_i) \approx \epsilon \gamma_\psi(a_i \otimes c_i) = \langle \pi(\psi(a_i))c_i, \xi \rangle = f_i(\psi(a_i)),$$

for $i = 1, \ldots, n$, which is what we want to prove. It is easily verified (e.g. by checking on elementary tensors $a \otimes c$) that $\gamma_\psi_1 + \gamma_\psi_2 = \gamma_{\psi_1 + \psi_2}$, and that $t\gamma_\psi = \gamma_{t\psi}$ for $t \in \mathbb{R}_+$. Hence $\mathcal{K}$ is a weak$^\ast$ closed convex cone of positive linear functionals.

We want to show that if $\gamma \in \mathcal{K}$ and $d \in A \otimes C$ then $d^\ast \gamma d := \gamma(d^\ast(-)d) \in \mathcal{K}$. Since $\mathcal{K}$ is weak$^\ast$ closed it suffices to show this for $\gamma = \gamma_\psi$ where $\psi = \sum e_i^* \phi(-)e_i$ for $e_1, \ldots, e_m \in B$, and $d = \sum_{j=1}^k x_j \otimes y_j$ where $x_1, \ldots, x_k \in A$ and $y_1, \ldots, y_k \in C$. Let $\mathcal{F} \subset A \otimes C$ be a finite set of elementary tensors and let $\delta > 0$ so that we wish to find $\psi_0 \in \mathcal{C}$ such that $d^\ast \gamma_\psi d(a \otimes c) \approx_\delta \gamma_{\psi_0}(a \otimes c)$ for $a \otimes c \in \mathcal{F}$. Since $\xi$ is cyclic for $\pi$ we may find $b_1, \ldots, b_k \in B$
such that $\|\pi(b_j)\xi - y_j\xi\|$ is as small, and $N$ sufficiently large, such that

$$
\gamma_\psi(d^*(a \otimes c)d) = \sum_{j,l=1}^k \gamma_\psi((x_j^*ax_l) \otimes (y_j^*cy_l))
$$

$$
= \sum_{j,l=1}^k \langle \pi(x_j^*ax_l)cyl, y_j\xi \rangle
$$

$$
\approx_{\delta/2} \sum_{j,l=1}^k \langle \pi(x_j^*ax_l)c\pi(b_l)\xi, \pi(b_j)\xi \rangle
$$

$$
= \langle \pi(\sum_{j,l=1}^k b_j^*\psi(x_j^*ax_l)b_l)c\xi, \xi \rangle
$$

$$
= \langle \pi(\sum_{j,l=1}^k \sum_{i=1}^m b_j^*e_i^*\phi(x_j^*ax_l)e_ib_l)c\xi, \xi \rangle
$$

$$
\approx_{\delta/2} \langle \pi(\sum_{j,l=1}^k \sum_{i=1}^m b_j^*e_i^*\phi^{1/N}(x_j^*)\phi(x_l)e_ib_l)c\xi, \xi \rangle
$$

$$
= \gamma_\psi_0(a \otimes c)
$$

for all $a \otimes c \in \mathcal{F}$, where

$$
\psi_0(-) = \sum_{i=1}^m \sum_{j,l=1}^k b_j^*e_i^*\phi^{1/N}(x_j^*)\phi(-)\phi^{1/N}(x_l)e_ib_l = \sum_{i=1}^m f_i^* \phi(-)f_i \in \mathcal{C}
$$

for $f_i := \sum_{j=1}^k \phi^{1/N}(x_j)e_ib_j$. Thus $d^*\gamma d \in \mathcal{K}$ for any $\gamma \in \mathcal{K}$ and $d \in A \otimes C$. Let $J$ be the subset of $A \otimes C$ consisting of elements $d$ such that $\gamma(d^*d) = 0$ for all $\gamma \in \mathcal{K}$. By [KR02, Lemma 7.17 (ii)] it follows that $J$ is a closed two-sided ideal in $A \otimes C$, and that $\gamma_\rho \in \mathcal{K}$ if $\gamma_\rho(d^*d) = 0$ for all $d \in J$. In other words, it suffices to show that $J$ is contained in the left kernel $L := L_{\gamma_\rho}$ of $\gamma_\rho$.

Since $A$ is exact it follows that $J = \text{span}\{a \otimes c : a \in A, c \in C, a \otimes c \in J\}$, see e.g. [BO08, Corollary 9.4.6]. As the left kernel $L$ of $\gamma_\rho$ is a closed linear subspace of $A \otimes C$, it suffices to show that $a \otimes c \in L$ for all elementary tensors $a \otimes c \in J$, so fix such $a \in A$ and $c \in C$.

By assumption $\rho(a^*a) \in B\phi(a^*a)B$. Thus for any $\delta > 0$ we may choose $b_1, \ldots, b_m \in B$ such that

$$
\|\rho(a^*a) - \sum_{j=1}^m b_j^*\phi(a^*a)b_j\| < \delta.
$$

Define $\psi = \sum_{j=1}^m b_j^*\phi(-)b_j$ which is in $\mathcal{C}$, and note that $\|\rho(a^*a) - \psi(a^*a)\| < \delta$. Since $\gamma_\psi(a^*a \otimes c^*c) = 0$ we get that

$$
|\gamma_\rho(a^*a \otimes c^*c)| = |\gamma_\rho(a^*a \otimes c^*c) - \gamma_\psi(a^*a \otimes c^*c)|
$$

$$
= |\langle \pi(\rho(a^*a) - \psi(a^*a))c\xi, c\xi \rangle|
$$

$$
< \delta\|c\xi\|^2.
$$
Since $\delta$ was arbitrary we get that $\gamma_{\rho}(a^*a \otimes c^*c) = 0$ so $a \otimes c \in L$, which finishes the proof. □

3.2. **Murray–von Neumann equivalence of $*$-homomorphisms.** It is customary to consider approximate or asymptotic unitary equivalence of $*$-homomorphisms as the correct equivalence relation for $*$-homomorphisms (at least when it comes to classification). However, this does not always seem to be the right framework to consider for not necessarily unital maps in general, in the same sense as unitary equivalence of projections has its downsides compared to Murray–von Neumann equivalence.

For instance, if $B$ is a unital, infinite $C^*$-algebra with a non-unitary isometry $v$ and $\phi: A \to B$ is a unital $*$-homomorphism, then $\phi$ and $v\phi(-)v^*$ are clearly not approximately unitary equivalent as one map is unital and the other is not. However, these two maps will agree on all usual invariants used for classification (e.g. ideal related $KK$-theory or the Cuntz semigroup), see also Corollary 3.11.

Thus, if one’s hope is to classify (not necessarily unital) $*$-homomorphisms, approximate (or asymptotic) unitary equivalence is often too strong of an equivalence relation. This motivates the weaker version which I call approximate/asymptotic Murray–von Neumann equivalence.

In many cases approximate/asymptotic Murray–von Neumann equivalence is the same as approximate/asymptotic unitary equivalence, cf. Proposition 3.13.

**Definition 3.4.** Let $A$ and $B$ be $C^*$-algebras, and let $\phi, \psi: A \to B$ be two $*$-homomorphisms. We say that $\phi$ and $\psi$ are **approximately Murray–von Neumann equivalent** if for any finite set $F \subset A$, and any $\epsilon > 0$, there is $u \in B$ such that

$$
\|u^*\phi(a)u - \psi(a)\| < \epsilon, \quad \|u\psi(a)u^* - \phi(a)\| < \epsilon, \quad a \in F.
$$

When $A$ is separable we say that $\phi$ and $\psi$ are **asymptotically Murray–von Neumann equivalent** if there is a contractive$^4$ continuous path $(u_t)_{t \in \mathbb{R}_+}$ in $B$ (where $\mathbb{R}_+ := [0, \infty)$) such that

$$
\lim_{t \to \infty} u_t^*\phi(a)u_t = \psi(a), \quad \lim_{t \to \infty} u_t\psi(a)u_t^* = \phi(a), \quad a \in A.
$$

A standard argument shows that two $*$-homomorphisms $\phi, \psi: \mathbb{C} \to B$ are approximately Murray–von Neumann equivalent if and only if $\phi(1)$ and $\psi(1)$ are Murray–von Neumann equivalent.

More generally, if $A$ is unital then $u$ in the definition of approximate Murray–von Neumann equivalence can be taken to be a partial isometry such that $u^*u = \psi(1_A)$ and $uu^* = \phi(1_A)$.

Recall that maps are **approximately/asymptotically unitary equivalent** if they satisfy the condition in Definition 3.4 with $u$ (resp. each $u_t$) a unitary in $\mathcal{M}(B)$.

**Lemma 3.5.** In the definition of approximate Murray–von Neumann equivalence one may always pick $u$ to be contractive.

---

$^3$I would be surprised if this definition has never appeared before but I know of no reference to such a definition.

$^4$A proof similar to that of Lemma 3.5 implies that we do not need to assume that $(u_t)$ is contractive or even bounded. Contractivity will be assumed for convenience.
Clearly, B ∈ D((resp. how this characterisation will be applied by replacing D below with either B∞ or Bas).

**Remark 3.6.** For a C*-algebra B let $B_\infty := \prod \mathbb{N} B / \bigoplus \mathbb{N} B$ be the sequence algebra, and let $B_{as} := C_b(\mathbb{R}_+, B)/C_0(\mathbb{R}_+, B)$ be the path algebra (or asymptotic corona) where $\mathbb{R}_+ = [0, \infty)$. Clearly B embeds into both $B_\infty$ and $B_{as}$ as constant sequences/paths, so we consider B as a C*-subalgebra of both $B_\infty$ and $B_{as}$.

Due to Lemma 3.5 in the approximate case, the following observation is immediate.

**Observation 3.7.** If A is separable then $\phi, \psi: A \to B$ are approximately (resp. asymptotically) Murray–von Neumann equivalent if and only if there is a contraction $v \in B_\infty$ (resp. $v \in B_{as}$) such that $v^*\phi(-)v = \psi$ and $v\psi(-)v^* = \phi$.

The above characterisation of approximate/asymptotic Murray–von Neumann equivalence will be used throughout the paper without reference. The following lemma indicates how this characterisation will be applied by replacing D below with either $B_\infty$ or $B_{as}$.

**Lemma 3.8.** Let $\phi, \psi: A \to D$ be *-homomorphisms and suppose that there is a contraction $v \in D$ such that $v^*\phi(a)v = \psi(a)$ for all $a \in A$. Then

(i) $vv^* \in D \cap \phi(A)'$,
(ii) $v^*\psi(a) = \psi(a)$ for all $a \in A$,
(iii) $\phi(a)v = v\psi(a)$ for all $a \in A$.

**Proof.** (i) is a standard trick: using that the map $v^*\phi(-)v$ is multiplicative one easily checks $((1-vv^*)^{1/2}\phi(a)vv^*)^*((1-vv^*)^{1/2}\phi(a)vv^*) = 0$ for any $a \in A$. Thus $(1-vv^*)^{1/2}\phi(a)vv^* = 0$, so $\phi(a)vv^* = vv^*\phi(a)vv^*$ for any $a \in A$. By symmetry $\phi(a)vv^* = vv^*\phi(a)$.

(ii): Given $a \in A$, pick $b, c \in A$ such that $a = bc$. Then

$\psi(a) = \psi(b)\psi(c) = v^*\phi(b)vv^*\phi(c)v \overset{(i)}{=} v^*vv^*\phi(bc)v = v^*\psi(a)$.

(iii): By (i), $\phi(a)vv^* = vv^*\phi(a)v = v\psi(a)$. Also, we have

$(1-v^*v)v^*\phi(a^*)\phi(a)v(1-v^*v) = (1-v^*v)\psi(a^*)\phi(a)v(1-v^*v) \overset{(ii)}{=} 0$,

so $\phi(a)v(1-v^*v) = 0$, and thus $\phi(a)v = \phi(a)vv^*v = v\psi(a)$. □

**Remark 3.9.** If $\phi: A \to B$ is a *-homomorphism let

$$\text{Ann}(\phi(A)) := \{b \in B_\infty : b\phi(A) + \phi(A)b = \{0\}\}$$
be the annihilator of \( \phi(A) \) in \( B_\infty \). Then \( \text{Ann}(\phi(A)) \) is an ideal in the relative commutant \( B_\infty \cap \phi(A)' \). If \( A \) is \( \sigma \)-unital and \( (a_n)_{n \in \mathbb{N}} \) is an approximate unit for \( A \), then the image of \( (\phi(a_n))_{n \in \mathbb{N}} \) in \( B_\infty \) induces a unit in \( B_\infty \cap \phi(A)'/\text{Ann}(\phi(A)) \), so this \( C^* \)-algebra is unital.

The analogous results hold for \( B_{as} \) (taking a continuous path \( \mathbb{R}_+ \ni t \mapsto a_t \in A \) which is an approximate unit), and we again use the notation \( \text{Ann}(\phi(A)) \) for the annihilator of \( \phi(A) \) in \( B_{as} \).

Considering relative commutants of the above form is inspired by work of Kirchberg in \cite{Kir06}.

The equivalence of \((i)\) and \((ii)\) in the proposition below is a generalisation of the fact that two projections \( p, q \) are Murray–von Neumann equivalent if and only if \( p \oplus 0 \) and \( q \oplus 0 \) are unitary equivalent. Condition \((iii)\) is a version of a 2 \( \times \) 2-matrix trick of Connes \cite{Con73}, and \((i)\) \( \Leftrightarrow \) \((iii)\) below shows that approximate/asymptotic Murray–von Neumann equivalence is exactly the equivalence relation on \( * \)-homomorphisms for which this “trick” is applicable. It will be used in the main result of this section Theorem 3.23.

My inspiration for considering such matrix tricks comes from recent work of Matui and Sato \cite{MS14} and by Bosa, Brown, Sato, Tikuisis, White and Winter \cite{BBS+15}.

It is not hard to see that one may replace \( M_2(B)_\infty \) below with \( M_2(B)_{\omega} \) for some/any free filter \( \omega \) on \( \mathbb{N} \). However, in this paper it suffices to work with \( M_2(B)_{\infty} \).

I would like to thank the referee for suggesting improving item \((ii)\) below to have the unitaries in \( M_2(B)^{\sim} \), which is the optimal version of approximate/asymptotic unitary equivalence.

**Proposition 3.10.** Let \( A \) and \( B \) be \( C^* \)-algebras with \( A \) separable, and let \( \phi, \psi: A \to B \) be \( * \)-homomorphisms. The following are equivalent.

\[
(i) \quad \phi \text{ and } \psi \text{ are approximately Murray–von Neumann equivalent,}
\]

\[
(ii) \quad \phi \oplus 0, \psi \oplus 0: A \to M_2(B) \text{ are approximately unitary equivalent (with unitaries in the minimal unitisation } M_2(B)^{\sim}),
\]

\[
(iii) \quad \text{The two projections}
\]

\[
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix} \in \frac{M_2(B)_\infty \cap (\phi \oplus \psi)(A)'}{\text{Ann}(\phi \oplus \psi)(A)}
\]

are Murray–von Neumann equivalent.

The above is also true if one replaces “approximately” with “asymptotically”, and “\( M_2(B)_\infty \)” with “\( M_2(B)_{as} \)”.

Note that the projections in \((iii)\) can be represented by \((\phi(a_n) \oplus 0)_{n \in \mathbb{N}} \) and \((0 \oplus \psi(a_n))_{n \in \mathbb{N}} \) respectively where \((a_n)_{n \in \mathbb{N}} \) is an approximate identity in \( A \), and similarly in the asymptotic case. Also, it is crucial that the Murray–von Neumann equivalence in \((iii)\) happens in the specified \( C^* \)-algebra.

**Proof of Proposition 3.10.** The asymptotic version of the proof is identical to the approximate version, so we only prove the approximate one.

\((i) \Rightarrow (iii)\): By Lemma 3.8 we find a \( u \in B_\infty \) such that \( u^* \phi(-)u = \psi \) and \( u \psi(-)u^* = \phi \). Using Lemma 3.8(i) it follows that

\[
(\phi(a) \oplus \psi(a))(u \otimes e_{12}) = \phi(a)u \otimes e_{12} = u \psi(a) \otimes e_{12} = (u \otimes e_{12})(\phi(a) \oplus \psi(a)),
\]
so $u \otimes e_{12} \in M_2(B)_\infty \cap (\phi \oplus \psi)(A)'$. Moreover, by Lemma 3.8(ii) we get
\[(u \otimes e_{12})(u \otimes e_{12})^*(\phi(a) \oplus \psi(a)) = uu^*(\phi(a) \oplus 0 = \phi(a) \oplus 0 = (1 \oplus 0)(\phi(a) \oplus \psi(a)) \text{,}
\]and similarly $(u \otimes e_{12})^*(u \otimes e_{12})(\phi(a) \oplus \psi(a)) = (0 \oplus 1)(\phi(a) \oplus \psi(a))$. Hence
\[(3.1) \quad V := u \otimes e_{12} + \text{Ann}((\phi \oplus \psi)(A)) \subseteq \frac{M_2(B)_\infty \cap (\phi \oplus \psi)(A)'}{\text{Ann}((\phi \oplus \psi)(A))}
\]is a partial isometry for which $VV^* = 1 \oplus 0$ and $V^*V = 0 \oplus 1$.

(iii) $\Rightarrow$ (ii): Note that the inclusion $M_2(B) \hookrightarrow M_2(B)^\sim$ induces an isomorphism
\[
\frac{M_2(B)_\infty \cap (\phi \oplus \psi)(A)'}{\text{Ann}((\phi \oplus \psi)(A))} \cong \frac{(M_2(B)^\sim)_\infty \cap (\phi \oplus \psi)(A)'}{\text{Ann}((\phi \oplus \psi)(A))}
\]as the unit of the left hand side is represented by an element in $M_2(B)_\infty$. Let $V$ be a partial isometry with $VV^* = 1 \oplus 0$ and $V^*V = 0 \oplus 1$. Then $U = V + V^*$ is a unitary such that $U^*(1 \oplus 0)U = 0 \oplus 1$.

Since $U$ is a self-adjoint unitary we may lift $U$ to a unitary $u \in (M_2(B)^\sim)_\infty \cap (\phi \oplus \psi)(A)'$. We get
\[
u^*(\phi(a) \oplus 0)u = u^*(1 \oplus 0)(\phi(a) \oplus \psi(a))u = u^*(1 \oplus 0)u(\phi(a) \oplus \psi(a)) = (0 \oplus 1)(\phi(a) \oplus \psi(a)) = 0 \oplus (\phi(a) \oplus \psi(a))
\]

for all $a \in A$. As the inclusion $M_2(B)^\sim \rightarrow M_2(B)$ induces an isomorphism
\[
\frac{(M_2(B)^\sim)_\infty \cap (\psi \oplus \psi)(A)'}{\text{Ann}((\psi \oplus \psi)(A))} \cong \frac{M_2(B)_\infty \cap (\psi \oplus \psi)(A)'}{\text{Ann}((\psi \oplus \psi)(A))}
\]
we may lift the self-adjoint unitary $W = e_{12} + e_{21} \in \frac{M_2(B)_\infty \cap (\phi \oplus \psi)(A)'}{\text{Ann}((\phi \oplus \psi)(A))}$ to a unitary $w \in (M_2(B)^\sim)_\infty \cap (\psi \oplus \psi)(A)'$. One checks exactly as above that
\[
w^*(0 \oplus \psi(a))w = \psi(a) \oplus 0
\]
for all $a \in A$. Hence $uw \in (M_2(B)^\sim)_\infty$ is a unitary such that $w^*u^*(\phi(a) \oplus 0)uw = \psi(a) \oplus 0$.

(ii) $\Rightarrow$ (i): Let $(a_n)_{n \in \mathbb{N}}$ be an approximate identity in $A$. If $u \in M_2(E)_\infty$ is a unitary such that $u^*(\phi(a) \oplus 0)u = \psi(a) \oplus 0$, then
\[
v := [(\phi(a_n) \oplus 0)_{n \in \mathbb{N}}]u[(\psi(a_n) \oplus 0)_{n \in \mathbb{N}}] \in (B \oplus 0)_\infty = B_\infty
\]
induces an approximate Murray–von Neumann equivalence.

The following essentially states that any functor used for classification which does not take the class of the unit in $K_0$ or similar into consideration, will not be able to distinguish $*$-homomorphisms which are approximately/asymptotically Murray–von Neumann equivalent. Hence, if one wants uniqueness results of (not necessarily unital) $*$-homomorphisms,

\[\text{At this point it would be tempting to use a rotation unitary to conclude that } \phi \oplus 0 \text{ and } \psi \oplus 0 \text{ are approximately unitary equivalent. However, then the resulting sequence of unitaries would be in } M_2(B) \text{ and not in } M_2(B)^\sim.\n\]
\[\text{Note that we use } \psi \oplus \psi \text{ here and not } \phi \oplus \psi.\]
approximate/asymptotic Murray–von Neumann equivalence is a more appropriate equivalence relation than approximate/asymptotic unitary equivalence.

**Corollary 3.11.** Let $F$ be any functor for which the domain is the category of separable $C^*$-algebras. Suppose that $F$ is invariant under approximate/asymptotic unitary equivalence and is $M_2$-stable, i.e. $F(id_B \oplus 0) : F(B) \xrightarrow{\cong} F(M_2(B))$ is an isomorphism for any $B$. Then $F$ is invariant under approximate/asymptotic Murray–von Neumann equivalence.

**Proof.** Suppose $\phi, \psi : A \to B$ are approximately/asymptotically Murray–von Neumann equivalent. By Proposition 3.10, $\phi \oplus 0$ and $\psi \oplus 0$ are approximately/asymptotically unitary equivalent, so

$$F(\phi) = F(id_B \oplus 0)^{-1} \circ F(\phi \oplus 0) = F(id_B \oplus 0)^{-1} \circ F(\psi \oplus 0) = F(\psi).$$

□

**Lemma 3.12.** Let $\mathcal{H}$ be an infinite dimensional, separable Hilbert space with orthonormal basis $(\xi_n)_{n \in \mathbb{N}}$, and let $T_1 \in \mathbb{B}(\mathcal{H})$ be given by $T_1 \xi_n = \xi_{2n-1}$ for $n \in \mathbb{N}$. Then the two $\ast$-homomorphisms $id_{\mathbb{K}^2(\mathcal{H})}, T_1(\cdot)T_1^* : \mathbb{K}(\mathcal{H}) \to \mathbb{K}(\mathcal{H})$ are asymptotically unitary equivalent.

**Proof.** We construct a norm-continuous path $(U_t)_{t \in [1, \infty)}$ of unitaries in $\mathbb{B}(\mathcal{H})$ implementing the asymptotic unitary equivalence as follows: Let $U_1 = 1$ and suppose that we have constructed $U_t$ for $t \in [1, k-1]$ for some integer $k \geq 2$. We may fix a continuous path $(V_k)_{t \in [k-1,k]}$ of unitaries such that $V_{k,k-1} = 1$, $V_{k,t} \xi_n = \xi_n$ for $n \neq k, 2k-1$ and all $t \in [k-1,k]$, and such that $V_{k,k} \xi_k = \xi_{2k-1}$ and $V_{k,k} \xi_{2k-1} = \xi_k$. Let $V_k := V_k \xi_k$ and define $U_t := V_{k,t}U_{k-1} = V_{k,t}U_{k-1} \ldots V_1$ for $t \in [k-1,k]$. This gives a norm-continuous path $(U_t)_{t \in [1, \infty)}$ of unitaries in $\mathbb{B}(\mathcal{H})$.

By how each $\xi_n$ is constructed it holds that $V_t \xi_{2n-1} = \xi_{2n-1}$ for $l = 1, \ldots, n-1$. Hence for $t \geq n$ we get

$$U_tT_1 \xi_n = V_{[t],t}V_{[t]} \ldots V_{n}V_{[t]-1} = V_{[t],t}V_{[t]} \ldots V_{n}V_{[t]-1} = V_{[t],t}V_{[t]} \ldots V_{n+1} \xi_n = \xi_n.$$

Hence if $\theta_{\xi,\xi'}$ for $\xi, \xi' \in \mathcal{H}$ is the rank 1 operator $\theta_{\xi,\xi'} \eta = \xi (\eta, \xi')$, we get

$$U_tT_1 \xi_n \xi_m \xi_m = \theta_{U_tT_1 \xi_n, U_tT_1 \xi_m} = \theta_{\xi_n, \xi_m}, \quad n, m \in \mathbb{N}, t \geq \max\{n, m\}.$$

As span$_{n,m \in \mathbb{N}} \theta_{\xi_n, \xi_m}$ is dense in $\mathcal{K}(\mathcal{H})$ it follows that $id_{\mathbb{K}^2(\mathcal{H})}$ and $T_1(\cdot)T_1^*$ are asymptotically unitary equivalent.

□

**Proposition 3.13.** Let $A$ and $B$ be $C^*$-algebras with $A$ separable, and let $\phi, \psi : A \to B$ be $\ast$-homomorphisms. Consider the following conditions.

(a) $A, B, \phi$ and $\psi$ are all unital,

(b) $B$ is stable,

(c) $B$ has stable rank 1.

If $\phi$ and $\psi$ are approximately Murray–von Neumann equivalent and (a), (b) or (c) holds, then $\phi$ and $\psi$ are approximately unitary equivalent.

If $\phi$ and $\psi$ are asymptotically Murray–von Neumann equivalent and (a) or (b) holds, then $\phi$ and $\psi$ are asymptotically unitary equivalent.

**Proof.** Case (a) is obvious for both approximate and asymptotic equivalences.

Case (c) follows since if $v \in B_\infty$ implements the approximate Murray–von Neumann equivalence, we may by stable rank 1 decompose $v = u|v|$ with $u \in (B)_\infty$ a unitary. Hence

$$w^*w$$

for $w$ given by Lemma 3.8:

$$u(v^*v)^{1/2} \psi(a)(v^*v)^{1/2}u^* = v \psi(a)v^* = \phi(a)$$
for all \( a \in A \), so \( \phi \) and \( \psi \) are approximately unitary equivalent.

For case (b) we only prove the asymptotic version, as the approximate version is essentially identical. We may replace \( B \) with \( B \otimes K(H) \) where \( H \) is an infinite dimensional, separable Hilbert space. Let \((\xi_n)_{n \in \mathbb{N}}\) be an orthonormal basis of \( H \), and let \( T_1, T_2 \in \mathbb{B}(H) \) be the isometries \( T_1 \xi_n = \xi_{2n-1} \) and \( T_2 \xi_n = \xi_{2n} \). Let \( s_i := 1_{\mathcal{M}(B)} \otimes T_i \in \mathcal{M}(B \otimes K(H)) \). By Lemma 3.12 it follows that \( \phi = id * \phi \) is asymptotically unitary equivalent to \( s_1 \phi(-) s_1^* \), and that \( \psi \) is asymptotically unitary equivalent to \( s_1 \psi(-) s_1^* \). As \( s_1, s_2 \) are isometries with \( s_1 s_1^* + s_2 s_2^* = 1 \), there is an isomorphism \( M_2(B \otimes K(H)) \cong B \otimes K(H) \) given by \( b \otimes e_{ij} \mapsto s_i b s_j^* \). Hence, if \( \phi \) and \( \psi \) are approximately asymptotically Murray–von Neumann equivalent, then \( s_1 \phi(-) s_1^* \) and \( s_1 \psi(-) s_1^* \) are approximately asymptotically unitary equivalent by Proposition 3.10. This proves the result in case (b).

The following corollary shows that approximate Murray–von Neumann equivalence is still a strong enough equivalence relation to get classification up to stable isomorphism.

**Corollary 3.14.** Let \( A \) and \( B \) be separable \( C^* \)-algebras, and let \( \phi: A \to B \) and \( \psi: B \to A \) be \(*\)-homomorphisms. If \( \psi \circ \phi \) and \( id_A \) are approximately Murray–von Neumann equivalent, and \( \phi \circ \psi \) and \( id_B \) are approximately Murray–von Neumann equivalent, then \( A \otimes K \cong B \otimes K \).

**Proof.** By Proposition 3.13 the maps \( (\psi \circ \phi) \otimes id_K = (\psi \otimes id_K) \circ (\phi \otimes id_K) \) and \( id_A \otimes id_K = id_A \otimes id_K \) are approximately unitary equivalent, and \( (\phi \otimes id_K) \circ (\psi \otimes id_K) \) and \( id_B \otimes id_K \) are approximately unitary equivalent. Hence \( A \otimes K \cong B \otimes K \) by an intertwining argument a la Elliott, see e.g. [Rør02, Corollary 2.3.4].

**Remark 3.15.** Let \( \phi: \mathcal{O}_2 \to \mathcal{O}_2 \otimes K \) and \( \psi: \mathcal{O}_2 \otimes K \to \mathcal{O}_2 \) be injective \(*\)-homomorphisms. By Theorem 3.28 below, \( \psi \circ \phi \) and \( id_{\mathcal{O}_2} \), and \( \phi \circ \psi \) and \( id_{\mathcal{O}_2} \otimes id_K \) are approximately Murray–von Neumann equivalent. This shows that \( A \not\cong B \) in general in Corollary 3.14.

### 3.3. \( \mathcal{O}_\infty \)-stable and \( \mathcal{O}_2 \)-stable \(*\)-homomorphisms.

**Definition 3.16.** Let \( \mathcal{D} \) be either \( \mathcal{O}_2 \) or \( \mathcal{O}_\infty \). Let \( A \) and \( B \) be \( C^* \)-algebras with \( A \) separable, and let \( \phi: A \to B \) be a \(*\)-homomorphism. We say that \( \phi \) is

- \( \mathcal{D} \)-stable if \( \mathcal{D} \) embeds unitally in \( \mathcal{D}_\infty \cap \phi(A)' / \text{Ann}(\phi(A)) \).
- strongly \( \mathcal{D} \)-stable if \( \mathcal{D} \) embeds unitally in \( \mathcal{D}_{\text{as}} \cap \phi(A)' / \text{Ann}(\phi(A)) \).

**Remark 3.17.** In Corollary 4.5 it is shown that \( \mathcal{O}_2 \)- and \( \mathcal{O}_\infty \)-stable \(*\)-homomorphisms have a McDuff type property, which really is the motivation for why the maps have been given this name.

There is an obvious generalisation of \( \mathcal{D} \)-stable maps for any strongly self-absorbing \( C^* \)-algebra \( \mathcal{D} \). However, with this definition it seems unlikely that they satisfy the McDuff type property and therefore I do not believe that this is the correct generalisation of \( \mathcal{D} \)-stable \(*\)-homomorphisms for more general strongly self-absorbing \( C^* \)-algebras.

In [Rør02, Definition 8.2.4] Rørdam introduces \( \mathcal{O}_2 \)-absorbing and \( \mathcal{O}_\infty \)-absorbing \(*\)-homomorphisms. These are unital \(*\)-homomorphisms \( \phi: A \to B \) such that \( \mathcal{O}_2 \) (resp. \( \mathcal{O}_\infty \)) embeds unitally in the commutant \( B \cap \phi(A)' \). I emphasise that although these notions are closely related to Definition 3.16, they are not the same. For instance, the identity map \( id_{\mathcal{O}_2} \) is \( \mathcal{O}_2 \)-stable (in the above sense), but not \( \mathcal{O}_2 \)-absorbing in the sense of Rørdam.
Proposition 3.18. Let $A$ and $B$ be $C^*$-algebras with $A$ separable, and let $\phi: A \to B$ be a $*$-homomorphism. Let $D$ be either $O_2$ or $O_\infty$. If either $A$ or $B$ is $D$-stable then $\phi$ is strongly $D$-stable.

Proof. Let $\iota_n: D \to D \otimes D \otimes \cdots =: D^{\otimes \infty}$ be the embedding into the $n$'th tensor. By [DW09] Theorem 2.2, $\iota_n$ and $\iota_{n+1}$ are asymptotically unitary equivalent with unitaries in $D_{n,n+1} := 1 \otimes \cdots \otimes 1 \otimes D \otimes D \otimes 1 \otimes \cdots$, where the two $D$'s are the $n$'th and $(n+1)$'st tensors. Hence there are unital homotopies $\sigma_i: D \to C([n,n+1], D^{\otimes \infty})$ from $\iota_n$ to $\iota_{n+1}$, such that $\sigma_i(d) \in D_{n,n+1}$. We get an induced unital embedding of $\sigma_i: D \to C_b(\mathbb{R}_+, D)$, as $D \cong D^{\otimes \infty}$. By construction, $||x, \sigma_i(y)|| \to 0$ for all $x, y \in D$.

Let $(a_t)$ be a continuous approximate unit of $A$. If $\alpha: A \otimes D \xrightarrow{\cong} A$ is an isomorphism, then $\eta: D \to C_0(\mathbb{R}_+, B)$ given by $\eta(d)(t) = \phi(a_t \otimes \sigma_t(d))$ induces strong $D$-stability.

Suppose $\beta: B \otimes D \xrightarrow{\cong} B$ is an isomorphism. Then $\eta: D \to C_0(\mathbb{R}_+, B)$ given by $\eta(d)(t) = \phi(a_t) \mathcal{M}(\beta)(1, \mathcal{M}(B) \otimes \sigma_t(d))$ induces strong $D$-stability. \hfill $\Box$

Proposition 3.19. Let $D$ be either $O_2$ or $O_\infty$. A separable $C^*$-algebra $A$ is $D$-stable if and only if $id_A$ is $D$-stable.

Proof. If $A$ is $D$-stable then $id_A$ is $D$-stable by Proposition 3.18. If $id_A$ is $D$-stable then $A$ is $D$-stable by [Kir06] Proposition 4.4(4,5)]. \hfill $\Box$

Lemma 3.20. Let $A, B$ and $C$ be $C^*$-algebras with $A$ separable, and let $\phi: A \to B$, $\psi: B \to C$ be $*$-homomorphisms. Let $D$ be either $O_2$ or $O_\infty$.

(i) If $\phi$ is (strongly) $D$-stable then $\psi \circ \phi$ is (strongly) $D$-stable.

(ii) If $B$ is separable and $\psi$ is (strongly) $D$-stable then $\psi \circ \phi$ is (strongly) $D$-stable.

Proof. We only do the $D$-stable cases, as the strongly $D$-stable cases are virtually identical.

(i): It is straightforward to check that $\psi_\infty: B_\infty \to C_\infty$ induces a $*$-homomorphism

$$\psi_\infty: \frac{B_\infty \cap \phi(A)'}{\text{Ann}(\phi(A))} \to \frac{C_\infty \cap \psi \circ \phi(A)'}{\text{Ann}(\psi \circ \phi(A))}.$$ 

Let $(a_n)_{n \in \mathbb{N}}$ be an approximate identity for $A$. Then the unit of the left (resp. right) hand side above is represented by $(\phi(a_n))_{n \in \mathbb{N}}$ (resp. $(\psi \circ \phi(a_n))_{n \in \mathbb{N}} = \psi_\infty((\phi(a_n))_{n \in \mathbb{N}}$). Hence the map above is unital. Thus, if $D$ embeds unitally into the left hand side above then it embeds unitally into the right hand side.

(ii): There is an obvious map

$$\frac{C_\infty \cap \psi(B)'}{\text{Ann}(\psi(B))} \to \frac{C_\infty \cap \psi \circ \phi(A)'}{\text{Ann}(\psi \circ \phi(A))}.$$ 

As in (i), it suffices to show that this map is unital. The unit of the left hand side is represented by $(\psi(b_n))_{n \in \mathbb{N}}$, where $(b_n)$ is an approximate identity in $B$. For any $a \in A$, we have that

$$(1 - \psi(b_n))\psi(\phi(a) - b_n\phi(a)) \to 0, \quad n \to \infty,$$

so $(\psi(b_n))_{n \in \mathbb{N}}$ also induces the unit in the right hand side above, so the map above is unital. \hfill $\Box$

---

\footnote{Alternatively, let $\sigma: D \to (A_\infty \cap A')/\text{Ann}(A)$ be a unital $*$-homomorphism, and $\Psi: D \to A_\infty \cap A'$ be any map that lifts $\sigma$. We obtain a $*$-homomorphism $A \otimes D \to A_\infty$ given on elementary tensors by $a \otimes d \mapsto a\Psi(d)$. In particular, $a \otimes 1_D \mapsto a$, so [TW07] Theorem 2.3 implies that $A \otimes D \cong A$.}
Remark 3.21. By Proposition 3.18, examples where this is not the case. A nice example comes from the study of KK-theory and Ext-theory. Given any *-homomorphism \( \phi: A \to M(B) \) one can form the infinite repeat by \( \phi \otimes 1_M \): \( A \to M(B \otimes \mathbb{K}) \). Since such an infinite repeat factors through \( M(B) \otimes \mathcal{O} \subseteq M(B \otimes \mathbb{K}) \), such infinite repeats are always \( \mathcal{O}_2 \)-stable by Lemma 3.20. Thus, all types of Weyl–von Neumann–Voiculescu theorems such as those due to Kasparov [Kas80], Kirchberg [Kir94] and Elliott–Kucerovsky [EK01] are essentially about characterising when (full, weakly nuclear) *-homomorphisms \( A \to M(B \otimes \mathbb{K}) \) are \( \mathcal{O}_2 \)-stable.

The following is a Stinespring type theorem for (strongly) \( \mathcal{O}_\infty \)-stable maps. Recall that we consider \( B \) as a \( C^* \)-subalgebra of \( B_{\infty} \) and of \( B_{as} \).

Theorem 3.22. Let \( A \) and \( B \) be \( C^* \)-algebras with \( A \) separable and exact, let \( \phi: A \to B \) be a nuclear *-homomorphism, and let \( \rho: A \to B \) be a nuclear c.p. map such that \( \mathcal{I}(\rho) \leq \mathcal{I}(\phi) \).

(i) If \( \phi \) is \( \mathcal{O}_\infty \)-stable then there is an element \( v \in B_\infty \) of norm \( \|\rho\|^{1/2} \) such that \( v^*\phi(a)v = \rho(a) \) for all \( a \in A \).

(ii) If \( \phi \) is strongly \( \mathcal{O}_\infty \)-stable then there is an element \( v \in B_{as} \) of norm \( \|\rho\|^{1/2} \) such that \( v^*\phi(a)v = \rho(a) \) for all \( a \in A \).

Proof. (i): By Theorem 3.3 \( \phi \) approximately dominates \( \rho \). We first show that we may take \( n = 1 \) in the definition of approximate domination, Definition 3.1. Given \( F \subset A \) finite and \( \epsilon > 0 \), find \( b_1, \ldots, b_n \) such that

\[
\|\rho(a) - \sum_{k=1}^n b_k^*\phi(a) b_k \| < \epsilon, \quad a \in F.
\]

Passing to \( B_\infty \), fix elements \( s_1, s_2, \ldots \) in \( B_\infty \cap \phi(A)' \) such that \( s_1 + \text{Ann}(\phi(A)), s_2 + \text{Ann}(\phi(A)), \ldots \) are isometries with orthogonal range projections. Then

\[
\sum_{k=1}^n b_k^*\phi(a) b_k = \sum_{k=1}^n b_k^*s_k\phi(a) s_k b_k = c^*\phi(a)c
\]

where \( c = \sum_{k=1}^n s_k b_k \). Let \( (s_k^{(m)})_{m=1}^\infty \) be a lift of \( s_k \) for each \( k \), and let \( c_m = \sum_{k=1}^n s_k^{(m)} b_k \). For large \( m \) we have

\[
\|\rho(a) - c_m^*\phi(a)c_m\| < \epsilon, \quad a \in F.
\]

Hence we obtain approximate domination with \( n = 1 \). By separability of \( A \), there is a sequence \( (d_n) \) in \( B \) such that \( d_n^*\phi(a)d_n \to \rho(a) \) for all \( a \in A \). Let \( (a_n)_{n \in \mathbb{N}} \) be an approximate identity for \( A \). By passing to a subsequence of \( (d_n) \) we may assume that \( \|d_n^*\phi(a)^2d_n - \rho(a_2^2)\| \to 0 \). Let \( v_n = \phi(a_n)d_n \). As \( \|\rho(a_n^2)\| \to \|\rho\| \) it follows that \( \limsup_{n \to \infty} \|v_n\| = \|\rho\|^{1/2} \), so \( (v_n) \) induces an element \( v \in B_\infty \) of norm at most \( \|\rho\|^{1/2} \) satisfying \( v^*\phi(a)v = \rho(a) \) for all \( a \in A \).

(ii): By part (i) we find a sequence \( (w_n) \) in \( B \) of norm \( \|\rho\|^{1/2} \) such that \( w_n^*\phi(a)w_n \to \rho(a) \) for all \( a \in A \). As \( \mathcal{O}_\infty \) embeds unitarily in \( (B_\infty \cap \phi(A)')/\text{Ann}(\phi(A)) \) we may find a sequence of contractions \( s_1, s_2, \ldots \in C_0(\mathbb{R}_+, B) \), which induces such a unital copy of \( \mathcal{O}_\infty \) (so that the \( s_i \) induce isometries, with orthogonal range projections).

Let \( (a_k)_{k \in \mathbb{N}} \) be a dense sequence in \( A \). We will find a continuous function \( r: \mathbb{R}_+ \to \mathbb{R}_+ \) such that \( \|s_j(r(t))^\epsilon\phi(a_k)s_j(r(t)) - \phi(a_k)\| < 1/n \) and \( \|s_i(r(t))^\epsilon\phi(a_k)s_j(r(t))\| < 1/n \) for
Let \( r(0) = 0 \) and suppose that we have constructed \( r(t) \) on \([0, n-1]\). Pick \( r(n-1) \leq R_n \in \mathbb{R}_+ \) such that \(|s_i(x)^*\phi(a_k)s_j(x) - \phi(a_k)| < 1/n\) and \(|s_i(x)^*\phi(a_k)s_j(x)| < 1/n\) for \( i, j, k = 1, \ldots, n+1, i \neq j \) and all \( x \geq R_n \). Let \( r(t) = r(n-1)(n-t) + (t-n+1)R_n \) for \( t \in [n-1, n] \). This gives a continuous function \( r : \mathbb{R}_+ \to \mathbb{R}_+ \) with the desired property.

Now, define \( v_t := (n+1-t)^{1/2}s_n(r(t))w_n + (t-n)^{1/2}s_{n+1}(r(t))w_{n+1} \) for \( t \in [n, n+1] \), \( n \in \mathbb{N} \). Clearly \( \lim_{t \to \infty} \|v_t\| = \|\rho\|^{1/2} \). For every \( k \in \mathbb{N} \), every \( n \geq k \) and every \( t \in [n, n+1] \), we have

\[
\|v_t^{*}\phi(a_k)v_t - \rho(a_k)\| \\
\leq (n+1-t)^{1/2}\left(\|w_n^{*}\phi(a_k)w_n - \rho(a_k)\| + \|\rho\|\|s_n^{*}(r(t))\phi(a_k)s_n(r(t)) - \phi(a_k)\|\right) \\
+ (t-n)^{1/2}\left(\|w_{n+1}^{*}\phi(a_k)w_{n+1} - \rho(a_k)\| + \|\rho\|\|s_{n+1}^{*}(r(t))\phi(a_k)s_{n+1}(r(t)) - \phi(a_k)\|\right) \\
+ (n+1-t)^{1/2}(t-n)^{1/2}\|\rho\|\|s_n^{*}(r(t))\phi(a_k)s_n(r(t))\| \\
+ (n+1-t)^{1/2}(t-n)^{1/2}\|\rho\|\|s_{n+1}^{*}(r(t))\phi(a_k)s_{n+1}(r(t))\| \\
\leq \|w_n^{*}\phi(a_k)w_n - \rho(a_k)\| + \|w_{n+1}^{*}\phi(a_k)w_{n+1} - \rho(a_k)\| + 4\|\rho\|/n
\]

From this it easily follows that \( v_t^{*}\phi(a)v_t \to \rho(a) \) for any \( a \in A \). \( \square \)

**Theorem 3.23.** Let \( A \) and \( B \) be C*-algebras with \( A \) separable and exact, and let \( \phi, \psi : A \to B \) be nuclear *-homomorphisms.

(i) If \( \phi \) and \( \psi \) are \( \mathcal{O}_2 \)-stable, then \( \phi \) and \( \psi \) are approximately Murray–von Neumann equivalent if and only if \( \mathcal{I}(\phi) = \mathcal{I}(\psi) \).

(ii) If \( \phi \) and \( \psi \) are strongly \( \mathcal{O}_2 \)-stable, then \( \phi \) and \( \psi \) are asymptotically Murray–von Neumann equivalent if and only if \( \mathcal{I}(\phi) = \mathcal{I}(\psi) \).

Moreover, if either \( A, B, \phi \) and \( \psi \) are all unital, or if \( B \) is stable, then we may replace “approximately/asymptotically Murray–von Neumann equivalent” with “approximately/asymptotically unitary equivalent” above.

**Proof.** The proof of the two statements are almost identical simply by interchanging \( B_\infty \) and \( B_{\text{sa}} \). So we only prove (i).

If \( \phi \) and \( \psi \) are approximately Murray–von Neumann equivalent, then clearly \( \mathcal{I}(\phi) = \mathcal{I}(\psi) \). Conversely, suppose \( \mathcal{I}(\phi) = \mathcal{I}(\psi) \). Let

\[
D := \frac{M_2(B_\infty) \cap (\phi \oplus \psi)(A)^\prime}{\text{Ann}(\phi \oplus \psi)(A)}.
\]

By Theorem 3.22 there is a \( v \in B_\infty \) such that \( v^*\phi(-)v = \psi \). By Lemma 3.8 it follows that \( V := v \otimes e_{12} + \text{Ann}(\phi \oplus \psi)(A) \in D \) is well-defined (as \( \phi(-)v = \psi(-) \)), that \( VV^* \leq 1 \oplus 0 \), and \( V^*V = 0 \oplus 1 \) (as \( v^*v(-) = \psi \)). Similarly, \( 1 \oplus 0 \) is subequivalent to \( 0 \oplus 1 \), so the projections \( 1 \oplus 0 \) and \( 0 \oplus 1 \) generate the same ideal in \( D \). As their sum is \( 1_D \) it follows that \( 1 \oplus 0 \) and \( 0 \oplus 1 \) are both full projections in \( D \). As

\[
(1 \oplus 0)D(1 \oplus 0) = \frac{B_\infty \cap (\phi(A)^\prime)}{\text{Ann}(\phi(A))} \oplus 0,
\]

it follows from \( \mathcal{O}_2 \)-stability of \( \phi \), that \( 1 \oplus 0 \) is properly infinite and \( [1 \oplus 0]_0 = 0 \) in \( K_0(D) \). The same holds for \( 0 \oplus 1 \). Thus, by a result of Cuntz [Cun81], \( 1 \oplus 0 \) and \( 0 \oplus 1 \) are Murray–von
Neumann equivalent. Proposition 3.10 implies that \( \phi \) and \( \psi \) are approximately Murray–von Neumann equivalent.

The “moreover” part follows from Proposition 3.13. \( \square \)

**Remark 3.24.** Using the Kirchberg–Phillips theorem and a result of Lin, one can deduce that \( \mathcal{O}_2 \)-stable \(*\)-homomorphisms are not always strongly \( \mathcal{O}_2 \)-stable. Let \( A, B \) be unital Kirchberg algebras in the UCT class and suppose that \([1_A]_0 = 0 \in K_0(A)\) and \([1_B]_0 = 0 \in K_0(B)\). Any unital \(*\)-homomorphism \( \theta : A \to B \) that factors through \( \mathcal{O}_2 \) is strongly \( \mathcal{O}_2 \)-stable and \( KK(\theta) = 0 \). By Theorem 3.23 any strongly \( \mathcal{O}_2 \)-stable, unital \(*\)-homomorphism \( \phi : A \to B \) is asymptotically unitary equivalent to \( \theta \), so in particular \( KK(\phi) = 0 \) (the converse is also true by the uniqueness part in the Kirchberg–Phillips theorem).

Now, if \( \text{Ext}(K_n(A), K_{1-s}(B)) \neq 0 \), apply the UMCT \([\text{DL}96]\) and the Kirchberg–Phillips theorem \([\text{Kir}94] , \text{Phi}00\) to find a unital \(*\)-homomorphism \( \phi : A \to B \) such that \( KK(\phi) \neq 0 \) but such that \( KL(\phi) = 0^8 \). Then \( \phi \) is not strongly \( \mathcal{O}_2 \)-stable, but as \( KL(\phi) = KL(\theta) \) it follows from \([\text{Lin}02]\) Theorem 4.10 that \( \phi \) and \( \theta \) are approximately unitary equivalent, so \( \phi \) is \( \mathcal{O}_2 \)-stable.

**Question 3.25.** Are \( \mathcal{O}_\infty \)-stable \(*\)-homomorphisms always strongly \( \mathcal{O}_\infty \)-stable?

## 4. From Approximate Morphisms to \(*\)-Homomorphisms

Recall that \( B_\infty := \prod B / \bigoplus B \) for a \( C^* \)-algebra \( B \). In this section a characterisation is given of when a \(*\)-homomorphism \( A \to B_\infty \) is unitary equivalent to a constant \(*\)-homomorphism, i.e. a \(*\)-homomorphism factoring through \( B \).

**Lemma 4.1.** Let \( A \) and \( B \) be \( C^* \)-algebras with \( A \) separable and \( B \) unital, and let \( \phi, \psi : A \to B_\infty \) be continuous maps. If \( \phi \) and \( \psi \) are approximately unitary equivalent, then they are unitary equivalent.

**Proof.** The proof is a standard “diagonal” argument. Let \( a_1, a_2, \ldots \) be a dense sequence in \( A \). For each \( n \in \mathbb{N} \) find a sequence \( u_n \in B_\infty \) of unitaries such that \( u_n^* \phi(a_i) u_n \approx_1 \psi(a_i) \) for \( i = 1, \ldots, n \). Let \( (\phi_j), (\psi_j) : A \to \prod B \) be (set-theoretical) lifts of \( \phi \) and \( \psi \) respectively, and let \( (u_n^{(j)})_{j \in \mathbb{N}} \) be a unitary lift of \( u_n \) for each \( n \in \mathbb{N} \). Let \( k_n \in \mathbb{N} \) be such that

\[
\| u_n^{(k_n)} \phi_k(a_1) u_n^{(k_n)} - \psi_k(a_1) \| \leq 1, \quad k \geq k_1.
\]

Having found \( k_{n-1} \) we let \( k_n > k_{n-1} \) be such that

\[
\| u_n^{(k_n)} \phi_k(a_i) u_n^{(k_n)} - \psi_k(a_i) \| \leq 1/n, \quad i = 1, \ldots, n, \text{ and } k \geq k_n.
\]

Let \( v_k = 1_B \) for \( k < k_1 \), and \( v_k = u_n^{(k)} \) if \( k_n \leq k < k_{n+1} \). We let \( v \) be the induced unitary in \( B_\infty \). Then \( v^* \phi(a_i) v = \psi(a_i) \) for all \( i \in \mathbb{N} \) so by continuity of \( \phi \) and \( \psi \), \( v^* \phi(a) v = \psi(a) \) for all \( a \in A \). \( \square \)

Note that whenever \( \eta : \mathbb{N} \to \mathbb{N} \) is a map for which \( \lim_{k \to \infty} \eta(k) = \infty \), then there is an induced \(*\)-endomorphism \( \eta^* : B_\infty \to B_\infty \) given by

\[
\eta^*([(b_1, b_2, \ldots)]) = [(b_{\eta(1)}, b_{\eta(2)}, \ldots)].
\]

---

8The \( KL \)-groups were originally defined by Rørdam in \([\text{Ror}95]\) in the presence of a universal coefficient theorem, and were later treated by Dadarlat in \([\text{Dad}05]\) in the general case.
Lemma 4.2. Let $A$ and $B$ be $C^*$-algebras with $A$ separable and $B$ unital. Suppose $\phi: A \to B$ is a $*$-homomorphism with the following property: for any map $\eta: \mathbb{N} \to \mathbb{N}$ for which $\lim_{k \to \infty} \eta(k) = \infty$, the maps $\phi$ and $\eta^* \circ \phi$ are approximately unitary equivalent as maps into $B_\infty$.

Let $(\phi_n): A \to \prod_\mathbb{N} B$ be any (not necessarily linear) lift of $\phi$. For every finite $F \subset A$, every $\epsilon > 0$, and every $m \in \mathbb{N}$, there is an integer $k \geq m$ such that for every integer $n \geq k$ there is a unitary $u \in B$ for which

$$
\|u^* \phi_n(a) u - \phi_k(a)\| < \epsilon, \quad a \in F.
$$

Proof. Suppose for contradiction that the lemma is false. Then there is a finite set $F \subset A$, an $\epsilon > 0$ and an $m \in \mathbb{N}$, such that for any integer $k \geq m$ there exists an integer $n_k \geq k$ for which

$$
\max_{a \in F} \|u_k^* \phi_n_k(a) u_k - \phi_k(a)\| \geq \epsilon,
$$

for every unitary $u_k \in B$. Let $\eta: \mathbb{N} \to \mathbb{N}$ be the map $\eta(k) = n_k$ whenever $k \geq m$ and $\eta(k) = 1$ for $k < m$. As $n_k \geq k$ for $k \geq m$, it follows that $\lim_{k \to \infty} \eta(k) = \infty$. As $\phi$ and $\eta^* \circ \phi$ are approximately unitary equivalent in $B_\infty$, there is a unitary $u \in B_\infty$ for which

$$
\|u^* (\eta^* \circ \phi(a)) u - \phi(a)\| < \epsilon, \quad a \in F.
$$

Let $(u_k)_{k \in \mathbb{N}} \in \prod_\mathbb{N} B$ be a unitary lift of $u$. It follows that

$$
\limsup_{k \to \infty} \|u_k^* \phi_n_k(a) u_k - \phi_k(a)\| = \|u^* (\eta^* \circ \phi(a)) u - \phi(a)\| < \epsilon
$$

for all $a \in F$. However, this contradicts (4.1) so the lemma is true. \qed

The following is essentially a discrete version of [Phi00, Proposition 1.3.7] and the proofs are very similar.

Theorem 4.3. Let $A$ and $B$ be $C^*$-algebras with $A$ separable and $B$ unital, and let $B_\infty = \prod_\mathbb{N} B/\bigoplus_\mathbb{N} B$. Suppose $\phi: A \to B_\infty$ is a $*$-homomorphism. Then $\phi$ is unitary equivalent to a $*$-homomorphism $\psi: A \to B \subset B_\infty$ if and only if for any map $\eta: \mathbb{N} \to \mathbb{N}$ with $\lim_{k \to \infty} \eta(k) = \infty$, the maps $\phi$ and $\eta^* \circ \phi$ are approximately unitary equivalent.

Proof. “Only if”: Suppose $u \in B_\infty$ is a unitary such that $\operatorname{Ad} u \circ \psi = \phi$, where $\psi$ factors through $B$. As $\eta^* \circ \psi = \psi$ for all $\eta: \mathbb{N} \to \mathbb{N}$ with $\lim_{n \to \infty} \eta(n) = \infty$, we get

$$
\eta^* \circ \phi = \eta^* \circ \operatorname{Ad} u \circ \psi = \operatorname{Ad} \eta^*(u) \circ \psi = \operatorname{Ad} (\eta^*(u) u^*) \circ \phi.
$$

“If”: let $F_1 \subseteq F_2 \subseteq \cdots \subseteq A$ be finite sets such that $\bigcup F_n$ is dense in $A$. Let $(\phi_k): A \to \prod_\mathbb{N} B$ be any function which is a lift of $\phi$. Pick $k_0 := 1 < k_1 < k_2 < \ldots$ recursively by applying Lemma 4.2 to $F = F_n$, $\epsilon = 1/2^n$ and $m = k_{n-1} + 1$ and then obtain $k = k_n$. For every $n \in \mathbb{N}$ we may find a unitary $u_n \in B$ for which

$$
\|u_n^* \phi_k(a) u_n - \phi_k(a)\| < 1/2^n, \quad a \in F_n.
$$

Let $v_n = u_n u_{n-1} \cdots u_1$. We claim that $\psi(a) := \lim_{n \to \infty} v_n^* \phi_k(a) v_n$ is a well-defined $*$-homomorphism which is approximately unitary equivalent to $\phi$ as maps into $B_\infty$.

To see that the map is well-defined it suffices to check that $(v_n^* \phi_k(a) v_n)_{n \in \mathbb{N}}$ is a Cauchy sequence for every $a \in A$. Given $\epsilon > 0$, pick an $n \in \mathbb{N}$ and $b \in F_n$ such that $\|a - b\| < \epsilon/3$. For $m > l \geq n$ we have

$$
v_m^* \phi_k(b) v_m \approx v_m^* \phi_{k_{m-1}}(b) v_m \approx v_{m-1}^* \phi_{k_{m-1}}(b) v_{m-1} \approx \cdots \approx v_1^* \phi_k(b) v_1,
$$
so \( \| v_m^* \phi_k_m(b) v_m - v_l^* \phi_k_l(b) v_l \| < \sum_{k=l+1}^\infty 2^{-k} < \sum_{k=l+1}^\infty 2^{-k} \). As \( \limsup_{k \to \infty} \| \phi_k(a) - \phi_k(b) \| < \epsilon/3 \), and since \( k_1 \to \infty \), we may pick \( N \geq n \) such that \( \| \phi_k(a) - \phi_k(b) \| < \epsilon/3 \) and \( \sum_{k=l+1}^\infty 2^{-k} < \epsilon/3 \) for \( l \geq N \). It easily follows that

\[
\| v_m^* \phi_k_m(a) v_m - v_l^* \phi_k_l(a) v_l \| < \epsilon
\]

for \( m, l \geq N \), so \( (v_m^* \phi_k_m(a) v_m)_{m \in \mathbb{N}} \) is a Cauchy sequence for every \( a \in A \). Hence \( \psi : A \to B \) is a well-defined map.

To see that \( \psi \) is approximately unitary equivalent to \( \phi \) in \( B_\infty \), let \( v \in B_\infty \) be the unitary induced by \( (v_1, v_2, \ldots) \) and \( \eta : \mathbb{N} \to \mathbb{N} \) be the map \( \eta(n) = k_n \). Then as maps into \( B_\infty \)

\[
\psi = v^* (\eta^* \circ \phi(-)) v,
\]

and thus \( \psi \) and \( \phi \) are approximately unitary equivalent in \( B_\infty \) by assumption. By Lemma \[11\] they are also unitary equivalent. As \( \psi \) composed with the embedding into \( B_\infty \) is a *-homomorphism, so is \( \psi \).

The above theorem can be applied to give a nice McDuff type characterisation of \( O_2 \)- and \( O_\infty \)-stable *-homomorphisms.

Recall from \[18\] that \( K_0(O_2) \cong K_1(O_2) \cong K_1(O_\infty) \cong \mathbb{Z} \), and that \( K_0(O_\infty) \cong \mathbb{Z} \) for which \([1_{O_\infty}]_0\) is a generator. This is essentially computed by realising each \( O_n \otimes \mathbb{K} \) as a crossed product by an extendible endomorphism on an AF algebra, an argument that also implies that \( O_n \) satisfies the universal coefficient theorem (UCT) of Rosenberg and Schochet in \[87\]. In particular, it follows from the UCT that \( O_2 \) is KK-equivalent to zero, and that the unital inclusion \( \mathbb{C} \to O_\infty \) is a KK-equivalence.

The following well-known consequence of Kirchberg’s version of the Kirchberg–Phillips theorem is needed to give the McDuff type characterisation.

**Proposition 4.4.** Let \( D \) be either \( O_2 \) or \( O_\infty \). Any two unital embeddings of \( D \) are asymptotically unitary equivalent.

**Proof.** Let \( \phi, \psi : D \to C \) be unital embeddings. By replacing \( C \) with \( C^*(\phi(D), \psi(D)) \) we may assume that \( C \) is separable. It holds that \( \phi \) and \( \psi \) define the same element in KK-theory. In fact, the case \( D = O_2 \) follows since \( O_2 \) is KK-equivalent to zero. In the case \( D = O_\infty \), one uses that the unital inclusion is \( \iota : \mathbb{C} \to O_\infty \) is a KK-equivalence to note that \( KK(\phi) = KK(\psi) \) if and only if \( KK(\phi \circ \iota) = KK(\psi \circ \iota) \). The latter is obviously true since \( \phi \circ \iota = \psi \circ \iota : \mathbb{C} \to C \), so \( KK(\phi) = KK(\psi) \).

Hence by \[7\] Proposition 2.8(i) and Theorem 2.9(ii) (relying on Kirchberg’s version of the Kirchberg–Phillips theorem \[94\]), \( \phi \) and \( \psi \) are asymptotically unitary equivalent.

In the case \( D = O_2 \), Theorem 5.23 provides an alternative proof of the above result.

As a corollary, the following McDuff type characterisation of \( O_2 \)- and \( O_\infty \)-stable *-homomorphisms is obtained.

**Corollary 4.5.** Let \( A \) and \( B \) be \( C^* \)-algebras with \( A \) separable and for which \( M(B) \) is properly infinite, let \( \phi : A \to B \) be a *-homomorphism, and let \( D \) be either \( O_2 \) or \( O_\infty \). The following are equivalent.

---

9) emphasise that these results will not be needed when proving the classification of \( O_2 \)-stable \( C^* \)-algebras, Theorem 6.13. So overall, the proof of this main theorem is still pretty self-contained with the main exception of a result of Kirchberg and Rørdam \[65\] Theorem 6.11 used in Proposition 5.5.
(i) $\phi$ is $D$-stable,

(ii) there exists a $*$-homomorphism $\psi: A \otimes D \to B$ such that $\phi$ and $\psi \circ (id_A \otimes 1_D)$ are approximately Murray–von Neumann equivalent.

Proof. (ii) $\Rightarrow$ (i): Let $v \in B_\infty$ be such that $v^*\phi(a)v = \psi(a \otimes 1)$ and $v\psi(a \otimes 1)v^* = \phi(a)$. As $v\psi(a \otimes 1) = \phi(a)v$ for all $a \in A$ by Lemma [3.8] it easily follows that the c.p. map $v^*(-)v: B_\infty \to B_\infty$ restricts to $B_\infty \cap \phi(A)' \to B_\infty \cap \psi(A \otimes 1)'$, which in turn descends to an isomorphism

$$v^*(-)v: \frac{B_\infty \cap \phi(A)'}{\text{Ann}(\phi(A))} \cong \frac{B_\infty \cap \psi(A \otimes 1)'}{\text{Ann}(\psi(A \otimes 1))}.$$ 

The map $\psi \circ (1_A \otimes id_D)$ induces a unital embedding of $D$ in $B_\infty \cap \psi(A \otimes 1)'/\text{Ann}(\psi(A \otimes 1))$, so it follows that $\phi$ is $D$-stable.

(i) $\Rightarrow$ (ii): Fix isometries $s_1, s_2 \in \mathcal{M}(B)$ with orthogonal range projections and $p = s_1s_1^* + s_2s_2^*$. Then $s_1, s_2$ implement an isomorphism $p\mathcal{B}_p \cong M_2(B)$ such that $s_1\phi(-)s_1^*: A \to p\mathcal{B}_p$ corresponds to the map $\phi \circ 0$ via this identification. As $\phi$ and $s_1\phi(-)s_1^*$ are approximately Murray–von Neumann equivalent, it suffices to construct $\psi: A \otimes D \to M_2(B)$ such that $\phi \oplus 0$ and $\psi \circ (id_A \otimes 1)$ are approximately Murray–von Neumann equivalent.

Let $\theta: \mathcal{D} \to \frac{B_\infty \cap \phi(A)'}{\text{Ann}(\phi(A))}$ be a unital embedding, and let $\eta: \mathbb{N} \to \mathbb{N}$ be a map such that $\lim_{n \to \infty} \eta(n) = \infty$. Then $\eta^*: B_\infty \to B_\infty$ induces a $*$-homomorphism

$$\eta^*: \frac{B_\infty \cap \phi(A)'}{\text{Ann}(\phi(A))} \to \frac{B_\infty \cap \phi(A)'}{\text{Ann}(\phi(A))}.$$ 

By Proposition [4.4] $\theta$ and $\eta^* \circ \theta$ are asymptotically unitary equivalent, so let $u_n \in \frac{B_\infty \cap \phi(A)'}{\text{Ann}(\phi(A))}$ be a sequence of unitaries such that $u_n^*\theta(d)u_n \to \eta^* \circ \theta(d)$ for all $d \in \mathcal{D}$. Let $\overline{\theta}: A \to B_\infty \cap \phi(A)'$ be any map that lifts $\theta$, and $v_n \in B_\infty \cap \phi(A)'$ be a lift of $u_n$ for each $n$.

Let $\phi \times \theta: A \otimes \mathcal{D} \to B_\infty$ be the induced $*$-homomorphism given on elementary tensors by $\phi \times \theta(a \otimes x) = \phi(a)\overline{\theta}(x)$. Then

$$v_n^* (\phi \times \theta)(a \otimes d)v_n = \phi(a)v_n^*\overline{\theta}(d)v_n \to \phi(a)\eta^*(\overline{\theta}(d)) = \eta^* \circ (\phi \times \theta)(a \otimes d)$$

for all $a \in A$ and $d \in \mathcal{D}$. Thus $\phi \times \theta$ and $\eta^* \circ (\phi \times \theta)$ are approximately Murray–von Neumann equivalent. By Proposition [3.10] $(\phi \times \theta) \oplus 0$ and

$$(\eta^* \circ (\phi \times \theta)) \oplus 0 = \eta^* \circ ((\phi \times \theta) \oplus 0)$$

are approximately unitary equivalent with unitaries in $(M_2(B)_{\infty})^\sim \subseteq (M_2(B)^\sim)_{\infty}$. By Theorem [4.3] there is a $*$-homomorphism $\psi: A \otimes \mathcal{D} \to M_2(B)$ which is approximately unitary equivalent to $(\phi \times \theta) \oplus 0$. Thus the result follows.

If $\phi: A \to B$ is $O_{\infty}$-stable it is not hard to see that for any $a \in A_+$ the element $\phi(a)$ is properly infinite in $B$ in the sense of Kirchberg and Rørdam [KR00]. The following is a special case of [KR00 Question 3.4]. An affirmative answer would imply that the requirement that $\mathcal{M}(B)$ is properly infinite is not needed in the above corollary.

**Question 4.6.** Let $A$ be separable and let $\phi: A \to B$ be an $O_{\infty}$-stable $*$-homomorphism. Is $\mathcal{M}(\phi(A)B\phi(A))$ properly infinite?
5. Existence results for generalised $Cu$-morphisms

In this section existence results are produced for lifting generalised $Cu$-morphisms on ideal lattices of $C^*$-algebras to c.p. maps. The main idea is to use Michael’s selection theorem to produce large enough c.p. maps $\phi$ into commutative $C^*$-algebras which preserve the ideal structure. This general method was first used by Blanchard [Bla96] and has later been used by Harnisch and Kirchberg [HK05] to produce similar results.

Basically everything in this section is contained in [Gab16] but in the language of actions on $C^*$-algebras instead of generalised $Cu$-morphisms. I include self-contained proofs here for the sake of completeness.

An important tool is a version of Michael’s selection theorems [Mic66, Theorem 1.2] which is a slight variation of his iconic selection theorem [Mic56, Theorem 3.2”]. I will recall the statement in the special case which will be needed, as well as the required terminology.

Let $Y$ and $Z$ be topological spaces. A carrier from $Y$ to $Z$ is a map $\Gamma: Y \to 2^Z$ where $2^Z$ denotes the set of non-empty subsets of $Z$. The purpose of Michael’s selection theorems is to find continuous selections of carriers $\Gamma$, i.e. continuous maps $\gamma: Y \to Z$ such that $\gamma(y) \in \Gamma(y)$ for all $y \in Y$.

A carrier $\Gamma$ is called lower semicontinuous if for every $U \subseteq Z$ open, the set 
\[ \{ y \in Y : \Gamma(y) \cap U \neq \emptyset \} \]

is an open subset of $Y$. The following is a special case of [Mic66, Theorem 1.2].

**Theorem 5.1** (Michael’s selection theorem). Let $Y$ be a compact Hausdorff space, let $A$ be a separable Banach space and equip the dual space $A^*$ with the weak$^*$ topology. Let $\Gamma: Y \to 2^{A^*}$ be a lower semicontinuous carrier such that $\Gamma(y)$ is a closed convex subset of the closed unit ball of $A^*$ for each $y \in Y$. Then there exists a continuous selection of $\Gamma$, i.e. there is a weak$^*$ continuous map $\gamma: Y \to A^*$ such that $\gamma(y) \in \Gamma(y)$ for all $y \in Y$.

The sets $P(A) \subseteq QS(A) \subseteq A^*$ are the sets of pure states, quasi-states and the dual space of $A$ respectively, all equipped with the weak$^*$ topology.

**Lemma 5.2.** Let $A$ be a $C^*$-algebra, let $Y$ be a compact Hausdorff space, and let $\Phi: I(A) \to I(C(Y))$ be a generalised $Cu$-morphism. For every $y \in Y$, define
\[ I_{\Phi,y} := \sum_{I \in I(A) : \Phi(I) \subseteq C_0(Y \{y\})} I \in I(A). \]

The carrier $\Gamma: Y \to 2^{A^*}$ given by
\[ \Gamma(y) = \{ \eta \in QS(A) : \eta(I_{\Phi,y}) = 0 \}, \quad y \in Y, \]

is lower semicontinuous.

**Proof.** Define the carrier $\Gamma': Y \to 2^{A^*}$ by
\[ \Gamma'(y) = \{ \eta \in P(A) \cup \{0\} : \eta(I_{\Phi,y}) = 0 \}, \quad y \in Y. \]

Note that $\Gamma(y)$ (resp. $\Gamma'(y)$) can be naturally identified with the set of states (resp. pure states) in the forced unitisation $(A/I_{\Phi,y})^\dagger$. Hence $\Gamma(y)$ is the weak$^*$ closure of the convex

---

10 This follows immediately since the closed unit ball of the dual of a separable Banach space is compact and metrisable in the weak$^*$ topology.

11 A quasi-state is a positive linear functional of norm at most 1.
hull of $\Gamma'(y)$ for each $y \in Y$. By [Mic56, Propositions 2.3 and 2.6] it thus follows that $\Gamma$ is lower semicontinuous if $\Gamma'$ is lower semicontinuous. Thus it suffices (in order to finish the proof) to show that $\Gamma'$ is lower semicontinuous.

Let $U \subseteq A^*$ be open. Suppose that $0 \in U$. As $0 \in \Gamma'(y)$ for every $y$, it follows that

$$\{y \in Y : \Gamma'(y) \cap U \neq \emptyset\} = Y$$

which is open. So suppose that $0 \notin U$. By [Ped79, Theorem 4.3.3], the continuous map $F : P(A) \to \text{Prim} A$, $F(\eta) = \ker \pi_\eta$ is open where $\pi_\eta$ is the GNS representation of $\eta$. So there is an induced map $\mathcal{O}(P(A)) \xrightarrow{F} \mathcal{O}(\text{Prim} A)$. Also, there is a canonical order isomorphism $\mathcal{O}(\text{Prim} A) \cong \mathcal{I}(A)$ given by $U \mapsto \bigcap_{p \in \text{Prim} A \setminus U} p$. The inverse map is given by $I \mapsto \{p \in \text{Prim} A : I \not\subset p\}$. Let $\Psi$ be the composition

$$\mathcal{O}(P(A)) \xrightarrow{F} \mathcal{O}(\text{Prim} A) \xrightarrow{\cong} \mathcal{I}(A) \xrightarrow{\Phi} \mathcal{I}(C(Y)) \cong \mathcal{O}(Y).$$

We claim that

$$\{y \in Y : \Gamma'(y) \cap U \neq \emptyset\} = \Psi(U \cap P(A)),$$

which will imply that $\Gamma'$ is lower semicontinuous as $\Psi(U \cap P(A)) \subseteq Y$ is open.

As generalised $Cu$-morphisms of ideal lattices preserve arbitrary suprema by Remark 2.8, it follows that $\Phi(I_{\Phi,y}) \subseteq C_0(Y \setminus \{y\})$, and that $I_{\Phi,y}$ is the largest ideal in $A$ with this property. Hence, for any $I \in \mathcal{I}(A)$ we have $I \subseteq I_{\Phi,y}$ if and only if $\Phi(I) \subseteq C_0(Y \setminus \{y\})$. Thus we get the following chain of reasoning for $y \in Y$:

$$y \notin \Psi(U \cap P(A)) \iff \Phi\left(\bigcap_{p \in \text{Prim} A \setminus F(U \cap P(A))} p\right) \subseteq C_0(Y \setminus \{y\})$$

$$\iff \bigcap_{p \in \text{Prim} A \setminus F(U \cap P(A))} p \subseteq I_{\Phi,y}$$

$$\iff F(U \cap P(A)) \subseteq \{p \in \text{Prim} A : I_{\Phi,y} \not\subset p\}$$

$$\iff \text{for every } \eta \in U \cap P(A) \text{ we have } I_{\Phi,y} \not\subset \ker \pi_\eta$$

$$\iff \text{for every } \eta \in U \cap P(A) \text{ we have } \eta(I_{\Phi,y}) \neq 0$$

$$\iff \Gamma'(y) \cap U \cap P(A) = \emptyset$$

$$\iff \Gamma'(y) \cap U = \emptyset.$$

This means that

$$\{y \in Y : \Gamma'(y) \cap U \neq \emptyset\} = \Psi(U \cap P(A)).$$

As $\Psi(U \cap P(A))$ is open, $\Gamma'$ is lower semicontinuous. This finishes the proof since implies that $\Gamma$ is lower semicontinuous (as seen early on in the proof).

Lemma 5.3. With the same setup as in Lemma 5.2, suppose that $\gamma : Y \to A^*$ is a continuous selection of $\Gamma$. Then the map $\phi : A \to C(Y)$ given by

$$\phi(a)(y) = \gamma(y)(a), \quad a \in A, y \in Y,$$

is a contractive c.p. map satisfying $\mathcal{I}(\phi) \leq \Phi$. 

\[\square\]
Proof. Recall that positive \( \ast \)-linear maps into \( C(Y) \) are completely positive. As \( \gamma \) is weak* continuous and each \( \gamma(y) \) has norm at most 1, it follows that \( y \mapsto \gamma(y)(a) \) is a continuous map of norm at most \( \|a\| \) for every \( a \in A \), and thus the map \( \phi: A \to C(Y) \) given by \( \phi(a)(y) = \gamma(y)(a) \) is a contractive c.p. map.

To see that \( \mathcal{I}(\phi) \leq \Phi \), fix \( I \in \mathcal{I}(A) \). We want to show that \( \phi(I) \subseteq \Phi(I) \). Let \( U_I \subseteq Y \) be the open subset such that \( \Phi(I) = C_0(U_I) \). As \( \Phi(I) = \bigcap_{y \notin U_I} C_0(Y \setminus \{y\}) \), it suffices to show that \( \phi(I) \subseteq C_0(Y \setminus \{y\}) \) for any \( y \in Y \setminus U_I \). Fix such a \( y \).

As \( \Phi(I) \subseteq C_0(Y \setminus \{y\}) \) it follows that \( I \subseteq I_{\Phi,y} \) by how \( I_{\Phi,y} \) was constructed. Thus, for any \( \gamma \in I_{\Phi,y} \) we have \( \phi(a)(y) = \gamma(y)(a) = 0 \). Hence \( \phi(I) \subseteq C_0(Y \setminus \{y\}) \), so \( \mathcal{I}(\phi) \leq \Phi \). \( \square \)

**Lemma 5.4.** Let \( A \) be a separable \( C^* \)-algebra, let \( C \) be a separable, commutative \( C^* \)-algebra and let \( \Phi: \mathcal{I}(A) \to \mathcal{I}(C) \) be a generalised \( \text{Cu} \)-morphism. Then there exists a c.p. map \( \phi: A \to C \) such that \( \mathcal{I}(\phi) = \Phi \).

Proof. If \( C \) is not unital, let \( \iota: C \to \tilde{C} \) be the inclusion. If we can find a c.p. map \( \tilde{\phi}: A \to \tilde{C} \) such that \( \mathcal{I}(\tilde{\phi}) = \mathcal{I}(\iota) \circ \Phi \), then \( \phi \) corestricts to a c.p. map \( \phi: A \to C \) such that \( \mathcal{I}(\phi) = \Phi \). Hence we may assume that \( C \) is unital. We may assume that \( C = C(Y) \) for a compact, metrisable space \( Y \).

We use the same notation as in Lemma 5.2. Since \( A \) is separable, it follows from [Ped79, Corollary 4.3.4] that \( \mathcal{I}(A) \) has a countable basis \( \{I_m \}_{m \in \mathbb{N}} \). Let \( U_m \) be the open subset of \( Y \) such that \( \Phi(I_m) = C_0(U_m) \). Note that \( I_m \nsubseteq I_{\Phi,y} \) whenever \( y \in U_m \). In fact, as \( \Phi \) preserves suprema, it follows from the definition of \( I_{\Phi,y} \) that \( \Phi(I_{\Phi,y}) \subseteq C_0(Y \setminus \{y\}) \). Thus, if \( I_m \subseteq I_{\Phi,y} \) then

\[
\Phi(I_m) \subseteq C_0(U_m) \cap C_0(Y \setminus \{y\}) = C_0(U_m \setminus \{y\})
\]

which is a contradiction if \( y \in U_m \).

Hence, for every pair \((y, m)\) where \( \Phi(I_m) \neq 0 \) and \( y \in U_m \), we may pick a positive contraction \( a_{y,m} \in I_m \) such that \( \|a_{y,m} + I_{\Phi,y}\| = 1 \). Moreover, we may fix a quasi-state \( \eta_{y,m} \in \Gamma(y) \) such that \( \eta_{y,m}(a_{y,m}) = 1 \).

For each such pair \((y, m)\) where \( \Phi(I_m) \neq 0 \) and \( y \in U_m \), we construct carriers \( \Gamma_{y,m}: Y \to 2^{A^*} \) by

\[
\Gamma_{y,m}(z) = \begin{cases} \{\eta_{y,m}\}, & \text{if } z = y \\ \Gamma(z), & \text{otherwise.} \end{cases}
\]

As \( \Gamma \) is lower semicontinuous by Lemma 5.2, it follows from [Mic56, Example 1.3*] that \( \Gamma_{y,m} \) is lower semicontinuous. By Michael’s selection theorem, Theorem 5.1, there are continuous maps \( \gamma_{y,m}: Y \to A^* \) for each \( y, m \in \mathbb{N} \) such that \( \gamma_{y,m}(z) \in \Gamma_{y,m}(z) \) for every \( z \in Y \). As each \( \gamma_{y,m} \) is also a selection for \( \Gamma \), it follows from Lemma 5.3 that we may construct contractive c.p. maps \( \phi_{y,m}: A \to C(Y) \) by

\[
\phi_{y,m}(a)(z) = \gamma_{y,m}(z)(a), \quad a \in A, z \in Y,
\]

which satisfy \( \mathcal{I}(\phi_{y,m}) \leq \Phi \). Note that \( \phi_{y,m}(a)(y) = \eta_{y,m}(a) \) for every \( a \in A \).

For each \((y, m)\) as above, the set \( V_{y,m} := \{x \in Y : \phi_{y,m}(a_{y,m})(x) > 0\} \) is an open neighbourhood of \( y \) by construction. If \( x \notin U_m \), then \( \Phi(I_m) = C_0(U_m) \subseteq C_0(Y \setminus \{x\}) \), so \( I_m \subseteq I_{\Phi,x} \) by definition of \( I_{\Phi,x} \). Hence \( a_{y,m} \in I_{\Phi,x} \), so \( \phi_{y,m}(a_{y,m})(x) = 0 \) whenever \( x \notin U_m \). Thus \( V_{y,m} \subseteq U_m \), so \( (V_{y,m})_{y \in U_m} \) is an open cover of \( U_m \).

---

12 This follows as an element in \( C(Y, M_n) \) is positive exactly when its evaluation in \( y \) is positive for every \( y \in Y \), and since positive linear functionals are completely positive, see [Ror08, Example 1.5.2].
As $U_m$ is $\sigma$-compact, we may find a countable sequence $(y_{(n,m)})_{n \in \mathbb{N}}$ in $U_m$ such that $(V_{y_{(n,m)},m})_{n \in \mathbb{N}}$ is an open cover of $U_m$.

Let

$$\phi = \sum_{n,m \in \mathbb{N}, \Phi(I_m) \neq 0} 2^{-n-m} \phi_{n,m} \quad \text{(point-wise convergence)}.$$ 

We claim that $\mathcal{I}(\phi) = \Phi$. As $\mathcal{I}(\phi_{n,m}) \leq \Phi$ it easily follows that $\mathcal{I}(\phi) \leq \Phi$, so it remains to prove the other inequality.

As $\Phi$ and $\mathcal{I}(\phi)$ are both generalised $Cu$-morphisms they both preserve suprema, so it suffices to show that $\Phi(I_m) = C_0(U_m) \subseteq \mathcal{I}(\phi)(I_m)$ for each $m \in \mathbb{N}$.

If $\Phi(I_m) = 0$ this is obvious, so we assume that $\Phi(I_m) \neq 0$. Let $W_m \subseteq Y$ be the open set such that $\mathcal{I}(\phi)(I_m) = C_0(W_m)$, and recall that

$$W_m = \{ y \in Y : f(y) \neq 0 \text{ for some } f \in \mathcal{I}(\phi)(I_m) \}.$$ 

For $y \in U_m$ we will find $f \in \mathcal{I}(\phi)(I_m)$ such that $f(y) \neq 0$ which implies $y \in W_m$. Pick an $n$ such that $y \in V_{y_{(n,m)},m}$. Pick an $n$ such that $y \in V_{y_{(n,m)},m}$. In particular, $I = a_{y_{(n,m)},m} \in I_m$ satisfies

$$\phi(a_{y_{(n,m)},m})(y) \geq 2^{-n-m} \phi_{n,m}(a_{y_{(n,m)},m})(y) > 0.$$

Hence $f = \phi(a_{y_{(n,m)},m}) \in \mathcal{I}(\phi)(I_m)$ satisfies $f(y) \neq 0$, so $y \in W_m$. This implies that $U_m \subseteq W_m$ and thus $\Phi(I_m) = C_0(U_m) \subseteq C_0(W_m) = \mathcal{I}(\phi)(I_m)$, which completes the proof.

\begin{proposition}
Let $A$ and $B$ be separable $C^*$-algebras with $B$ nuclear, and let $\Phi: \mathcal{I}(A) \to \mathcal{I}(B)$ be a generalised $Cu$-morphism. Then there exists a c.p. map $\phi: A \to B$ such that $\mathcal{I}(\phi) = \Phi$.
\end{proposition}

\begin{proof}
Say that $B$ has Property $(\lozenge)$ if the following holds: there exist a separable, commutative $C^*$-algebra $C$, a c.p. map $\psi: C \to B$, and a generalised $Cu$-morphism $\Psi: \mathcal{I}(B) \to \mathcal{I}(C)$, such that $\mathcal{I}(\psi) \circ \Psi = \text{id}_{\mathcal{I}(B)}$.

We will finish the proof assuming $B$ has Property $(\lozenge)$ and afterwards apply a result of Kirchberg and Rørdam [KR05 Theorem 6.11] to conclude that any separable, nuclear $C^*$-algebra $B$ has this property.

So suppose that $B$ has Property $(\lozenge)$ and let $C$, $\psi$ and $\Psi$ be given as above. As $\psi \circ \Phi: \mathcal{I}(A) \to \mathcal{I}(C)$ is a generalised $Cu$-morphism we may apply Lemma 5.3 to obtain a c.p. map $\phi_0: A \to C$ such that $\mathcal{I}(\phi_0) = \Psi \circ \Phi$. Let $\phi = \psi \circ \phi_0$. By Lemma 2.16 we get

$$\mathcal{I}(\phi) = \mathcal{I}(\psi) \circ \mathcal{I}(\phi_0) = \mathcal{I}(\psi) \circ \Psi \circ \Phi = \Phi.$$ 

It remains to show that any separable, nuclear $B$, has Property $(\lozenge)$. By [KR05 Theorem 6.11], $B \otimes \mathcal{O}_2$ contains a commutative $C^*$-subalgebra which separates the ideals of $B \otimes \mathcal{O}_2$ and such that $(I \cap C) + (J \cap C) = (I + J) \cap C$ for every $I, J \in \mathcal{I}(B \otimes \mathcal{O}_2)$. It follows that the map $\Psi': \mathcal{I}(B \otimes \mathcal{O}_2) \to \mathcal{I}(C)$ given by $\Psi'(J) = J \cap C$ is a generalised $Cu$-morphism.\footnote{For any $C^*$-subalgebra $C \subseteq B$, the map $J \mapsto J \cap C$ will preserve zero, order and increasing suprema, but it will in general not be additive.}

Let $\iota: C \hookrightarrow B \otimes \mathcal{O}_2$ be the inclusion. As $C$ separates the ideals it follows that $\mathcal{I}(\iota) \circ \Psi' = \text{id}_{\mathcal{I}(B \otimes \mathcal{O}_2)}$.

Let $\eta$ be a faithful state on $\mathcal{O}_2$ and let $\lambda_\eta: B \otimes \mathcal{O}_2 \to B$ be the induced slice map. Clearly $\mathcal{I}(\lambda_\eta)$ is the inverse of the isomorphism $\Theta: \mathcal{I}(B) \cong \mathcal{I}(B \otimes \mathcal{O}_2)$, $\Theta(I) = I \otimes \mathcal{O}_2$. Let
ψ := Ψ′ ◦ Θ and ψ := λ_η ◦ τ. Then
\[ \mathcal{I}(ψ) \circ Ψ \overset{\text{Lem. 2.17}}{=} \mathcal{I}(λ_η) \circ \mathcal{I}(τ) \circ Ψ′ \circ Θ = id_{\mathcal{I}(B)}. \]
Hence B has Property (\circ) thus finishing the proof. □

Remark 5.6. It would be desirable to have a more elementary proof of the above proposition without having to go through the deep structural result of Kirchberg and Rørdam. As emphasised in the proof, B does not need to be separable and nuclear for the proof to work. What is really needed is that B has Property (\circ) as defined in the proof above.

By the Dauns–Hofmann theorem any σ-unital B with Prim \( B \) second countable, Hausdorff has Property (\circ). This easily follows by letting C = \( C_0(\text{Prim } B) \), \( \psi = h_t(-)h \), and Ψ = \( \mathcal{I}(ψ)^{-1} \), where \( ν: C_0(\text{Prim } B) \to \mathcal{M}(B) \) is the canonical \( * \)-homomorphism coming from the Dauns-Hofmann theorem, and where \( h \in B \) is a strictly positive element.

Similarly, if Prim \( B \) is second countable and zero dimensional (i.e. has a basis of compact open sets), then it is not hard to apply the construction of [BEL78] to find an AF \( C^* \)-subalgebra \( D \subseteq B \otimes \mathcal{O}_2 \) such that the inclusion induces an isomorphism \( \mathcal{I}(D) \cong \mathcal{I}(B \otimes \mathcal{O}_2) \).

One easily sees that AF algebras have Property (\circ) so by slicing away \( \mathcal{O}_2 \) using Lemma 2.17 it follows that any such \( B \) has Property (\circ). Hence, in these cases one does not have to rely on the result of Kirchberg and Rørdam.

For the final result of this section, recall (a possible construction of) the Kasparov–Stinespring dilation, cf. [Kas80] Theorem 3. Given a contractive c.p. map \( φ: A \to B \), where A is separable and B is σ-unital and stable, one constructs the (countably generated, right) Hilbert B-module \( E := A \otimes_φ B \) where \( φ: A \to \mathcal{M}(B) \) is the minimal unitisation. There is an induced \( * \)-homomorphism \( φ_0: A \to \mathcal{B}(E) \subseteq \mathcal{B}(E \oplus B) \) given by multiplication on the left tensor of \( E \). As B is stable, \( B \otimes \ell^2(\mathbb{N}) \cong B \) as Hilbert B-modules. Thus, by Kasparov’s stabilisation theorem [Kas80] Theorem 2, there is a unitary \( u \in \mathcal{B}(B, E \oplus B) \). Define \( φ_0: A \to \mathcal{B}(B) = \mathcal{M}(B) \) by \( φ_0 := u^*φ_0(-)u \) which is a \( * \)-homomorphism. By letting \( W \in \mathcal{B}(B, E \oplus B) \) be given by \( W(b) = (1 \otimes b, 0) \), and \( V := u^*W \in \mathcal{M}(B) \), one obtains \( V^*φ_0(-)V = φ \). I will refer to \( (φ_0, V) \) as a Kasparov–Stinespring dilation of \( φ \).

Corollary 5.7. Let A and B be separable \( C^* \)-algebras with B nuclear and stable, and let \( Φ: \mathcal{I}(A) \to \mathcal{I}(B) \) be a generalised Cu-morphism. Then there exists a \( * \)-homomorphism \( φ_0: A \to \mathcal{M}(B) \) such that \( Φ(I) = Bφ_0(I)B \) for all \( I \in \mathcal{I}(A) \).

Proof. As B is separable and nuclear, Proposition 5.5 implies the existence of a c.p. map \( φ: A \to B \) for which \( \mathcal{I}(φ) = Φ \). We may assume that \( φ \) is contractive. Let \( (φ_0, V) \) be a Kasparov–Stinespring dilation as described above. For \( I \in \mathcal{I}(A) \) we have
\[ Φ(I) = Bφ_0(I)B = BV^*φ_0(I)V = Bφ_0(I)B. \]

We wish to show that \( Bφ_0(I)B \subseteq Φ(I) \). As the canonical inner product on \( Bφ_0(I)B \) is given by \( \langle a, b \rangle = a^*b \), we have \( Bφ_0(I)B = \langle φ_0(I)B, φ_0(I)B \rangle \), so it suffices to show that \( \langle φ_0(I)B, φ_0(I)B \rangle \subseteq Φ(I) \). Since
\[ \langle φ_0(I)B, φ_0(I)B \rangle = Φ(I)E \oplus 0, \]
it suffices to check that \( \langle y \oplus 0, \phi'_0(x)y \oplus 0 \rangle \in \Phi(I) \) for any \( x \in I \), and any \( y \in E \) induced from the algebraic tensor product \( A \hat{\otimes}_C B \). So let \( a_1, \ldots, a_n \in A \) and \( b_1, \ldots, b_n \in B \) be such that \( y \) is induced by \( \sum a_i \otimes b_i \). Then
\[
\langle y \oplus 0, \phi'_0(x)y \oplus 0 \rangle = \sum_{i,j=1}^n b_j^* \phi(a_i x a_j)b_j \in \overline{B\phi(I)B} = \mathcal{I}(\phi)(I) = \Phi(I). \quad \square
\]

6. An ideal related \( O_2 \)-embedding theorem

The main goal of this section is to prove the following ideal related \( O_2 \)-embedding result of Kirchberg, which essentially should be considered as an existence result.

**Theorem 6.1.** Let \( A \) be a separable, exact \( C^* \)-algebra, let \( B \) be a separable, nuclear, \( O_\infty \)-stable \( C^* \)-algebra, and let \( \Phi: \mathcal{I}(A) \to \mathcal{I}(B) \) be a \( Cu \)-morphism. Then there exists a strongly \( O_2 \)-stable \( * \)-homomorphism \( \phi: A \to B \) such that \( \mathcal{I}(\phi) = \Phi \).

**Remark 6.2.** If in the above theorem one takes \( B = O_2 \) and \( \Phi \) to be the map \( \Phi(I) = O_2 \) for \( I \neq 0 \) and \( \Phi(0) = 0 \), then one obtains Kirchberg’s classical \( O_2 \)-embedding theorem: for any separable, exact \( C^* \)-algebra \( A \) there is an injective \( * \)-homomorphism \( A \hookrightarrow O_2 \).

**Remark 6.3.** Kirchberg and Rørdam have shown in [KR02] that a separable, nuclear \( C^* \)-algebra is \( O_{\infty} \)-stable if and only if it is strongly purely infinite. As strong pure infiniteness in general is weaker than \( O_{\infty} \)-stability, it would be more natural to replace \( O_{\infty} \)-stability in the above theorem with strong pure infiniteness. However, to keep the proof more self contained the results will be stated with \( O_{\infty} \)-stability instead.

**Remark 6.4.** Here is the main idea of how the proof of Theorem 6.1 goes:

Apply Corollary 5.7 to find a \( * \)-homomorphism \( \phi_0: A \to M(B) \) such that \( B\phi_0(I)B = \Phi(I) \) for all \( I \in \mathcal{I}(A) \). We pick a suitably well-behaved positive element in \( B_{\infty} \cap \phi_0(A)' \) with spectrum \([0,1]\). By considering \( M(B) \subseteq M(B)_{\infty} \) and \( B_{\infty} \subseteq M(B)_{\infty} \), there is an induced \( * \)-homomorphism \( C_0(0,1) \hat{\otimes} A \to M(B)_{\infty} \) which factors through \( B_{\infty} \). The uniqueness result Theorem 3.23 will be used to show that there is a unitary in \( M(B)_{\infty} \) implementing a certain automorphism \( \alpha \) on \( C_0(0,1) \hat{\otimes} A \), so there is an induced \( * \)-homomorphism \( (C_0(0,1) \hat{\otimes} A) \rtimes_\alpha \mathbb{Z} \to B_{\infty} \). The automorphism \( \alpha \) is chosen such that the crossed product is isomorphic to \( C(\mathbb{T}) \otimes \mathbb{K} \otimes A \). As \( A \) embeds into \( C(\mathbb{T}) \otimes \mathbb{K} \otimes A \), this will produce a \( * \)-homomorphism \( \psi: A \to B_{\infty} \). One now combines the uniqueness result Theorem 3.23 with Theorem 4.3 to produce a \( * \)-homomorphism \( \phi: A \to B \) which is unitary equivalent to \( \psi \) in \( B_{\infty} \). This \( \phi \) will satisfy \( \mathcal{I}(\phi) = \Phi \).

Obviously compact containment of ideals must play an important part in the proof of Theorem 6.1, as any generalised \( Cu \)-morphism which lifts to a \( * \)-homomorphism must necessarily preserve compact containment by Lemma 2.12(iii). The following proposition says that one can compute certain ideals in \( B_{\infty} \) by using compact containment whenever \( B \) is weakly purely infinite.

\[15\text{Technically one should also use } [\text{Kir06 } \text{Proposition 4.4(4,5)}] \text{ or } [\text{TW07 } \text{Corollary 3.2}] \text{ to reduce from the stable case to the general case.}
\[16\text{This is the same trick as in the proof of the } O_2 \text{-embedding theorem due to Kirchberg and Phillips in } [\text{KP00}].\]
Recall from [BK04]\(^\text{17}\) that a $C^*$-algebra $B$ is $n$-purely infinite for $n \in \mathbb{N}$ if $\ell^\infty(B)$ has no quotients of dimension $\leq n^2$, and if for any positive $a,b \in B$ such that $a \in BbB$, and any $\epsilon > 0$, there are $d_1, \ldots, d_n \in B$ such that $\|a - \sum_{k=1}^n d_k^* bd_k\| < \epsilon$. A $C^*$-algebra is weakly purely infinite if it is $n$-purely infinite for some $n$, and is purely infinite if it is 1-purely infinite.

For any $C^*$-algebra $B$, $B \otimes \mathcal{O}_2$ and $B \otimes \mathcal{O}_\infty$ are purely infinite by [KR00] Theorem 5.11. A similar statement as the one below (with virtually the same proof) holds for the sequence algebra $B_\omega$ for any free filter $\omega$ on $\mathbb{N}$.

**Proposition 6.5.** Let $B$ be a weakly purely infinite $C^*$-algebra and let $I$ be an ideal of $B$. Then

$$B_\infty IB_\infty = \bigcup_{J \in I} J_\infty$$

where the union above is taken over ideals $J$ in $B$ which are compactly contained in $I$.

**Proof.** “$\subseteq$”: To show $B_\infty IB_\infty \subseteq \bigcup_{J \in I} J_\infty$ it suffices to check that whenever $c \in I$ is positive and $\epsilon > 0$, then $(c - \epsilon)_+ \in \bigcup_{J \in I} J_\infty$. However, this is obvious as $B(c - \epsilon)_+ B \subseteq I$ by Lemma 2.2 so

$$(c - \epsilon)_+ \in (B(c - \epsilon)_+ B)_\infty \subseteq \bigcup_{J \in I} J_\infty.$$  

“$\supseteq$”: To show $B_\infty IB_\infty \supseteq \bigcup_{J \in I} J_\infty$, it suffices to check that $J_\infty \subseteq B_\infty IB_\infty$ for every ideal $J \in I$. Fix such a $J$. By Lemma 2.2 there is a positive $c \in I$ and $\epsilon > 0$, such that $J \subseteq B(c - \epsilon)_+ B$. Let $x \in J_\infty$ be a positive contraction and let $(x_n)_{n \in \mathbb{N}} \subseteq \prod_n J$ be a lift of $x$ for which each $x_n$ is a positive contraction. As $B$ is weakly purely infinite it is $m$-purely infinite for some $m \in \mathbb{N}$. Thus, as each $x_n \in B(c - \epsilon)_+ B$, we may find $d_n^{(1)}, \ldots, d_n^{(m)} \in B$ such that

$$\|x_n - \sum_{k=1}^m d_n^{(k)*}(c - \epsilon)_+ d_n^{(k)}\| < 1/n, \quad \text{for all } n \in \mathbb{N}.$$  

Let $y_n^{(k)} = (c - \epsilon)^{1/2} d_n^{(k)*}$ for $n \in \mathbb{N}, k = 1, \ldots, m$. These elements are clearly bounded, as $\|x_n\| \leq 1$ and $\sum_{k=1}^m y_n^{(k)*} y_n^{(k)}$ is $1/n$-close to $x_k$. Let $y^{(k)}$ be the image of $(y_n^{(k)})_{n \in \mathbb{N}}$ in $B_\infty$. Pick an element $c_0 \in C^*(c)$ such that $c_0(c - \epsilon)_+ = (c - \epsilon)_+$. Then

$$x = \sum_{k=1}^m y^{(k)*} y^{(k)} = \sum_{k=1}^m y^{(k)*} c_0 y^{(k)} \in B_\infty c B_\infty \subseteq B_\infty IB_\infty.$$  

As $x \in J_\infty$ was an arbitrary positive contraction it follows that $J_\infty \subseteq B_\infty IB_\infty$. \hfill $\Box$

**Lemma 6.6.** Let $A$ and $B$ be $C^*$-algebras with $B$ stable, let $\phi: A \to B_\infty$ be a contractive c.p. map, and let $\tilde{\phi}: \hat{A} \to \mathcal{M}(B)_\infty$ be the induced unital c.p. map. For $I \in \mathcal{I}(\hat{A})$ we have

$$\mathcal{I}(\tilde{\phi})(I) = \begin{cases}  
\mathcal{I}(\phi)(I), & \text{if } I \subseteq A \\
\mathcal{M}(B)_\infty, & \text{otherwise}. 
\end{cases}$$

\(^{17}\)This differs slightly from the definition of $n$-purely infinite in [KR02], but the definitions of weakly purely infinite are still the same by [BK04] Proposition 4.12.]
Proof. Clearly $\mathcal{I}(\tilde{\phi})(I) = \mathcal{I}(\phi)(I)$ whenever $I \subseteq A$. If $I \not\subseteq A$, then $I$ contains an element of the form $1 - a$ with $a \in A$. Lift $\phi(a)$ to a bounded sequence $(b_n)_{n \in \mathbb{N}} \in \prod_{n} B$. As $B$ is stable we may for each $n$ find an isometry $v_n \in \mathcal{M}(B)$ such that $\|v_n^* b_n v_n\| < 1/n$. Let $v$ be the isometry in $\mathcal{M}(B)_\infty$ induced by $(v_n)$. Then $v^* \phi(a)v = 0$, so $v^* \phi(1 - a)v = 1$, and thus $\tilde{\phi}(1 - a)$ is full. Hence $\mathcal{I}(\tilde{\phi})(I) = \mathcal{M}(B)_\infty$.

Lemma 6.7. Let $A$ and $B$ be $C^*$-algebras with $A$ separable, and suppose that $B$ is $\mathcal{O}_2$-stable. For any $*$-homomorphism $\phi: A \to B_\infty$, the unitisation $\tilde{\phi}: \hat{A} \to \mathcal{M}(B)_\infty$ is $\mathcal{O}_2$-stable.

Proof. Lift $\phi$ to a map $(\phi_n)_{n \in \mathbb{N}}: A \to \prod_{n} B$, and let $a_1, a_2, \cdots \in A$ be dense. As $B$ is $\mathcal{O}_2$-stable, we may find unital $*$-homomorphisms $\psi_n: \mathcal{O}_2 \to \mathcal{M}(B)$ such that $\|\psi_n(s_i), \phi_n(a_j)\| < 1/n$ for $i = 1, 2$ and $j = 1, \ldots, n$. The induced unital $*$-homomorphism $\psi: \mathcal{O}_2 \to \mathcal{M}(B)_\infty$ commutes with $\phi(A)$ and thus also with $\tilde{\phi}(A)$, so $\tilde{\phi}$ is $\mathcal{O}_2$-stable.

For convenience of the reader I include a few lemmas on nuclear maps which should be well-known to experts. These are not stated in their most general forms, but just in a form which is applicable in the proof of Theorem 6.1 and easy to prove using only well-known results.

Lemma 6.8. Let $A$ be a separable, exact $C^*$-algebra, and let $B$ be a separable, nuclear, stable $C^*$-algebra. Then any c.p. map $A \to \mathcal{M}(B)$ is nuclear.

Proof. Let $\eta: A \to \mathcal{M}(B)$ be a c.p. map. By normalising and unitising we may assume $A$ and $\eta$ are unital. Pick a unital embedding $A \to \mathcal{M}(\mathbb{K})$ such that $A \cap \mathbb{K} = \{0\}$. As $B$ is stable we may find unital embeddings $A \to \mathcal{M}(\mathbb{K})$ so that $\mathcal{M}(\mathbb{K}) / \mathcal{M}(B \otimes \mathbb{K}) \cong \mathcal{M}(B)$. As $A$ is exact it is nuclearly embeddable, see for instance [BO08 Theorem 3.9.1], so the composition of these maps is nuclear. By Kasparov’s Weyl–von Neumann–Voiculescu type theorem [Kas80 Theorem 5], this composition approximately 1-dominates $\eta$. Hence $\eta$ is nuclear.

Lemma 6.9. Let $A$, $B$ and $C$ be $C^*$-algebras with $C$ nuclear, and let $\phi: A \to B$ and $\psi: C \to B$ be c.p. maps with commuting images. If $\phi$ is nuclear then the induced c.p. map $\phi \otimes \psi: A \otimes C \to B$ is nuclear.

Proof. Let $D$ be a unital $C^*$-algebra. As in [BO08 Corollary 3.8.8] it suffices to show that $id_D \otimes _{\text{alg}} (\phi \otimes \psi): D \otimes _{\text{alg}} (A \otimes C) \to D \otimes _{\max} B$ is $\|\cdot\|_{\min}$-continuous, i.e. continuous with respect to the minimal tensor product norm. By nuclearity of $\phi$, the c.p. map $id_D \otimes \phi: D \otimes A \to D \otimes _{\max} B$ is well-defined, and as $1_D \otimes \psi: C \to D \otimes _{\max} B$ commutes with the image of $id_D \otimes \phi$ there is an induced c.p. map $\eta := (id_D \otimes \phi) \times (1_D \otimes \psi): (D \otimes A) \otimes C \to D \otimes _{\max} B$. Under the canonical identification $(D \otimes A) \otimes C \cong D \otimes (A \otimes C)$, $\eta$ is a c.p. extension of $id_D \otimes _{\text{alg}} (\phi \otimes \psi)$ so this map is $\|\cdot\|_{\min}$-continuous.

Lemma 6.10. Let $G$ be a countable, discrete, amenable group, let $A$ be a unital $G$-$C^*$-algebra, let $B$ be a $C^*$-algebra, and let $\eta: A \times G \to B$ be a $*$-homomorphism. If $\eta|A$ is nuclear, then $\eta$ is nuclear.

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18 An application of [21] implies that the result also holds for order zero maps.
Proof. We use [BO08, Lemma 4.2.3]. For any finite set \( F \subset G \) they construct c.p. maps \( \phi_F : A \times G \to A \otimes M_F(\mathbb{C}) \) and \( \psi_F : A \otimes M_F(\mathbb{C}) \to A \times G \) such that \( \psi_F \circ \phi_F = \text{id} \) point-norm for any Følner sequence \( (F_n) \) for \( G \). Let \( \alpha \) denote the \( G \)-action on \( A \) and \( \lambda_g \) be the canonical unitaries in \( A \times G \) for \( g \in G \). The map \( \psi_F \) is given by
\[
\psi_F(a \otimes e_{g,h}) = \frac{1}{|F|} \alpha_g(a)\lambda_{gh^{-1}} = \frac{1}{|F|} \lambda_g a \lambda_h^*, \quad a \in A, g, h \in F.
\]
Thus, under the canonical identification of \( A \otimes M_F(\mathbb{C}) \cong M_F(A) \), one has
\[
\psi_F = \frac{1}{|F|}(\lambda_g)_{g \in F}(-)(\lambda_g)^*_{g \in F} : M_F(A) \to A \times \Gamma
\]
where \((\lambda_g)_{g \in F}\) is considered a row vector. The amplification \( \lambda_{f,F(\mathbb{C})^{\ast}} \) is nuclear, so \( \eta \circ \psi_F = \frac{1}{|F|} \eta(\lambda_g)_{g \in F} = \eta(\lambda_g)(-)(\eta(\lambda_g))^*_{g \in F} \) is nuclear. For any Følner sequence \( (F_n) \), \( \eta \circ \psi_F \circ \phi_F \) is thus nuclear and converges point-norm to \( \eta \) which is therefore nuclear.

\[\Box\]

**Proof of Theorem 6.1.** By Proposition 3.18 any \(*\)-homomorphism into \( B \otimes \mathcal{O}_2 \otimes K \) is strongly \( \mathcal{O}_2 \)-stable. As the composition of a strongly \( \mathcal{O}_2 \)-stable \(*\)-homomorphism with any \(*\)-homomorphism is again strongly \( \mathcal{O}_2 \)-stable by Lemma 3.20 and as any inclusion \( \mathcal{O}_2 \otimes K \hookrightarrow \mathcal{O}_\infty \) induces an isomorphism of ideal lattices
\[
\mathcal{I}(B \otimes \mathcal{O}_2 \otimes K) \cong \mathcal{I}(B \otimes \mathcal{O}_\infty) \cong \mathcal{I}(B)
\]

we may assume that \( B \) is stable and \( \mathcal{O}_2 \)-stable. Write \( B = B_1 \otimes \mathcal{O}_2 \) with \( B_1 \cong B \). Let \( \Phi_1 : \mathcal{I}(A) \to \mathcal{I}(B_1) \) be the \( Cu \)-morphism induced by \( \Phi \) and the obvious identification \( \mathcal{I}(B_1) \cong \mathcal{I}(B) \), i.e. \( \Phi(I) = \Phi_1(I) \otimes \mathcal{O}_2 \) for all \( I \in \mathcal{I}(A) \). By Corollary 5.7 we pick a \(*\)-homomorphism \( \phi_1 : A \to \mathcal{M}(B_1) \) such that \( B_1 \phi_1(I) B_1 = \Phi_1(I) \) for all \( I \in \mathcal{I}(A) \). Let \( \phi_0 = \phi_1 \otimes 1_{\mathcal{O}_2} : A \to \mathcal{M}(B_1 \otimes \mathcal{O}_2) = \mathcal{M}(B) \). Clearly \( B \phi_0(I) B = \Phi(I) \) for all \( I \in \mathcal{I}(A) \). By Lemma 6.8 \( \phi_0 \) is nuclear. Let \( (b_n)_{n \in \mathbb{N}} \) be a countable approximate identity (of positive contractions) for \( B_1 \) which is quasi-central with respect to \( \phi_1(A) \), and let \( h \in \mathcal{O}_2 \) be a positive element with spectrum \([0,1] \). Let \( b = \pi_\infty((b_n)) \in (B_1)_{\text{\infty}} \). As \( b \otimes h \in (B_1)_{\text{\infty}} \otimes \mathcal{O}_2 \) is a positive contraction which commutes with the image of \( \phi_1 \otimes 1_{\mathcal{O}_2} = \phi_0 \), there is a \(*\)-homomorphism \( \Psi : C_0(0,1) \otimes A \to \mathcal{M}(B)_{\infty} \) given on elementary tensors by
\[
\Psi(f \otimes a) = f(b \otimes h)\phi_0(a), \quad f \in C_0(0,1), a \in A.
\]
The map \( \Psi \) clearly factors through \( B_\infty \) as each \( f(b \otimes h) \in B_\infty \), and \( \Psi \) is nuclear by Lemma 6.9. Let \( \iota : B \hookrightarrow B_\infty \) be the canonical inclusion and let \( \hat{\Phi} = \iota \circ \Phi \), i.e. \( \hat{\Phi}(I) = B_\infty \Phi(I) B_\infty \).

**Claim:** \( \Psi(f \otimes a) \) is full in \( \hat{\Phi}(\mathbb{A}a\mathbb{A}) \) for all positive \( a \in A \) and all positive, non-zero \( f \in C_0(0,1) \).

So fix such \( a, f \).

To see that \( \Psi(f \otimes a) \in \hat{\Phi}(\mathbb{A}a\mathbb{A}) \), it suffices to check that \( \Psi(f \otimes (a - \epsilon)_+) \in \hat{\Phi}(\mathbb{A}a\mathbb{A}) \) for any \( \epsilon > 0 \). Fix such an \( \epsilon \). As \( B \) is purely infinite, it follows from Proposition 6.5 that
\[
\hat{\Phi}(\mathbb{A}a\mathbb{A}) = \bigcup_{J \in \hat{\Phi}(\mathbb{A}a\mathbb{A})^\ast} J_\infty.
\]

\footnote{This implies that \( \mathcal{I}(\Psi) = \hat{\Phi} \circ \mathcal{I}(\rho_\mu) \), where \( \rho_\mu : C_0(0,1) \otimes A \to A \) is the right slice map with respect to a faithful state \( \mu \) on \( C_0(0,1) \).}
Recall that $B\phi_0(A(a-\epsilon)_+A)B = \Phi(A(a-\epsilon)_+A)$, and that $A(a-\epsilon)_+A \subseteq AaA$ by Lemma 2.2. As $\Phi$ preserves compact containment it follows that $B\phi_0(A(a-\epsilon)_+A)B \subseteq \Phi(AaA)$. Thus, as $f(b \otimes h) \in B_\infty$, we have

$$\hat{\Phi}(AaA) \subseteq B_\infty \Phi(f \otimes a)B_\infty,$$

which finishes the proof of the claim. As $B_1$ is separable, $\Phi_1(AaA)$ contains a full, positive element, say $c$. In particular, as $O_2$ is simple, $f$ is non-zero, and $h$ has spectrum $[0,1]$, $c \otimes f(h)$ is full in $\Phi_1(AaA) \otimes O_2 = \Phi(AaA)$ and thus also full in $\hat{\Phi}(AaA)$. So it suffices to show that $(c - \epsilon)_+ \otimes f(h)$ is in the ideal generated by $\Psi(f \otimes a)$ for every $\epsilon > 0$. Fix such an $\epsilon$.

Recall that $B = B_1 \otimes O_2$. Note that $b \otimes 1$ and $1 \otimes h$ are in the relative commutant $M(B)_\infty \cap (B_1 \otimes 1_{O_2})'$, and that $b \otimes 1 + \text{Ann}(B_1 \otimes 1_{O_2})$ is the unit of $(M(B)_\infty \cap (B_1 \otimes 1))'/\text{Ann}(B_1 \otimes 1)$. Thus

$$b \otimes h + \text{Ann}(B_1 \otimes 1_{O_2}) = 1 \otimes h + \text{Ann}(B_1 \otimes 1_{O_2}),$$

so in particular

$$f(b \otimes h) + \text{Ann}(B_1 \otimes 1_{O_2}) = 1 \otimes f(h) + \text{Ann}(B_1 \otimes 1_{O_2}).$$

Hence, for any $d \in B_1$ we have

$$(6.3) \quad f(b \otimes h)(d \otimes 1) = (d \otimes 1)f(b \otimes h) = (d \otimes 1)(1 \otimes f(h)).$$

As $c \in B_1\phi_1(a)B_1$ we may find $d_1, \ldots, d_n \in B$ such that $(c - \epsilon)_+ = \sum_{k=1}^n d_k^n \phi_1(a)$. It follows that

$$(c - \epsilon)_+ \otimes f(h) = \sum_{k=1}^n (d_k^n \otimes 1)(\phi_1(a) \otimes 1)(d_k \otimes 1)(1 \otimes f(h))$$

$$(6.3) = \sum_{k=1}^n (d_k^n \otimes 1)\phi_0(a)f(b \otimes h)(d_k \otimes 1)$$

$$= \sum_{k=1}^n (d_k^n \otimes 1)\Psi(f \otimes a)(d_k \otimes 1).$$

As $\epsilon > 0$ was arbitrary, it follows that $c \otimes f(h)$ is in the ideal generated by $\Psi(f \otimes a)$, so it follows that

$$(6.4) \quad \hat{\Phi}(AaA) \subseteq B_\infty \Psi(f \otimes a)B_\infty.$$

Combining this with equation (6.2) proves our claim above.

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20Separability of $B_1$ is actually not needed to conclude that $\Phi_1(AaA)$ has a full element. In fact, an easy consequence of Corollary 2.3 is that $Cu$-morphisms map ideals with full elements to ideals with full elements.
Let \( \alpha \) be an automorphism on \( C_0(0, 1) \) such that \( C_0(0, 1) \rtimes_\alpha \mathbb{Z} \cong C(\mathbb{T}) \otimes \mathbb{K} \).\(^{23}\) Let \( \beta = \alpha \otimes id_A \) be the induced automorphism on \( C_0(0, 1) \otimes A \), and let \( \tilde{\beta} \) be the unitisation which is an automorphism on \( (C_0(0, 1) \otimes A)^\sim \).

For any \( a \in A^+ \) and any non-zero \( f \in C_0(0, 1)_+ \) it follows from what we showed above that both \( \Psi(f \otimes a) \) and \( \Psi \circ \beta(f \otimes a) = \Psi(\alpha(f) \otimes a) \) are full in \( \hat{\Phi}(AaA) \). Hence \( I(\Psi) \) and \( I(\Psi \circ \beta) \) agree on all ideals generated by elements of the form \( f \otimes a \). As such ideals form a basis for \( I(C_0(0, 1) \otimes A) \), and as \( I(\Psi) \) and \( I(\Psi \circ \beta) \) preserve suprema by Lemma 2.12 and Remark 2.8, it follows that \( I(\Psi) = I(\Psi \circ \beta) \).

Consider the unital extension \( \tilde{\Psi} : (C_0(0, 1) \otimes A)^\sim \to \mathcal{M}(B)_\infty \) of \( \Psi \). As unitisations of nuclear maps are nuclear (cf. BO08 Proposition 2.2.4]), \( \tilde{\Psi} \) is nuclear. Moreover, as \( \Psi \) factors through \( B_\infty \), it follows from Lemma 6.7 that \( \tilde{\Psi} \) is \( O_2 \)-stable. Also, as \( \Psi \) takes values in \( B_\infty \), and as \( I(\Psi) = I(\Psi \circ \beta) \), it follows from Lemma 6.6 that \( I(\tilde{\Psi}) = I(\tilde{\Psi} \circ \tilde{\beta}) \).

As \( \tilde{\Psi} \) is unital, nuclear and \( O_2 \)-stable, so is \( \tilde{\Psi} \circ \tilde{\beta} \). Thus by Theorem 3.23 \( \tilde{\Psi} \circ \tilde{\beta} \) and \( \tilde{\Psi} \) are approximately unitary equivalent. By Lemma 4.1 there exists a unitary \( u \in \mathcal{M}(B)_\infty \) such that \( u\tilde{\Psi}(-)u^* = \tilde{\Psi} \circ \tilde{\beta} \). Thus there is an induced \(*\)-homomorphism
\[
\psi_1 : (C_0(0, 1) \otimes A)^\sim \rtimes_{\tilde{\beta}} \mathbb{Z} \to \mathcal{M}(B)_\infty
\]
given by \( \psi_1(xv^n) = \tilde{\Psi}(x)u^n \) for \( x \in (C_0(0, 1) \otimes A)^\sim \) and \( n \in \mathbb{Z} \), where \( v \) is the canonical unitary in the crossed product. As the restriction of \( \psi_1 \) to \( (C_0(0, 1) \otimes A)^\sim \) is \( \tilde{\Psi} \) which is nuclear, it follows from Lemma 6.10 that \( \psi_1 \) is nuclear. As \( \psi_1(xv^n) = \tilde{\Psi}(x)v^n \in B_\infty \) for any \( x \in C_0(0, 1) \otimes A \) and \( n \in \mathbb{Z} \), it follows that \( \psi_1 \) restricts to a \(*\)-homomorphism
\[
\psi_0 : (C_0(0, 1) \otimes A) \rtimes_{\tilde{\beta}} \mathbb{Z} \to B_\infty.
\]
As \( \psi_0 \) is nuclear when considered as a map into \( \mathcal{M}(B)_\infty \) (as this is just the restriction of \( \psi_1 \)), and as \( B_\infty \) is an ideal in \( \mathcal{M}(B)_\infty \), it follows that \( \psi_0 \) is nuclear.

Since \( \beta = \alpha \otimes id_A \) we have a natural isomorphism
\[
\theta : (C_0(0, 1) \rtimes_\alpha \mathbb{Z}) \otimes A \xrightarrow{\cong} (C_0(0, 1) \otimes A) \rtimes_{\tilde{\beta}} \mathbb{Z}.
\]
As \( C_0(0, 1) \rtimes_\alpha \mathbb{Z} \cong C(\mathbb{T}) \otimes \mathbb{K} \) we may fix a full projection \( p \in C_0(0, 1) \rtimes_\alpha \mathbb{Z} \). We get an induced \(*\)-homomorphism
\[
\psi : A \to B_\infty, \quad \psi(a) = \psi_0(\theta(p \otimes a))
\]
for \( a \in A \). As \( \psi_0 \) is nuclear, so is \( \psi \). We will show that \( \psi(a) \) is full in \( \hat{\Phi}(AaA) \) for every positive \( a \in A^+ \), so fix such an \( a \).

Let \( w \in \mathcal{M}(C_0(0, 1) \rtimes_\alpha \mathbb{Z}) \) be the canonical unitary. Then
\[
\psi_0(\theta(fw^n \otimes a)) = \Psi(f \otimes a)w^n \in \hat{\Phi}(AaA), \quad f \in C_0(0, 1), n \in \mathbb{Z}.
\]
It follows that
\[
\psi(a) = \psi_0(\theta(p \otimes a)) \in \hat{\Phi}(AaA).
\]
Let \( f \in C_0(0, 1) \) be positive and non-zero. As \( p \) is full in \( C_0(0, 1) \rtimes_\alpha \mathbb{Z} \), it follows that
\[
\hat{\Phi}(AaA) \subseteq B_\infty \Psi(f \otimes a)B_\infty = B_\infty \psi_1(\theta(f \otimes a))B_\infty \subseteq B_\infty \psi_1(\theta(p \otimes a))B_\infty = B_\infty \psi(a)B_\infty.
\]
Thus \( \psi(a) \) is full in \( \hat{\Phi}(AaA) \) for all positive \( a \in A \), so \( I(\psi) = \hat{\Phi} \).

\(^{23}\)For instance, the automorphism \( \sigma \) on \( C_0(\mathbb{R}) \) which shifts the variable by 1, satisfies \( C_0(\mathbb{R}) \rtimes_\sigma \mathbb{Z} \cong C(\mathbb{T}) \otimes \mathbb{K} \).
We wish to apply Theorem 4.3 so let \( \eta : N \to N \) be a map such that \( \lim_{n \to \infty} \eta(n) = \infty \), and let \( \eta^* : \mathcal{M}(B)_\infty \to \mathcal{M}(B)_\infty \) be the induced \(*\)-homomorphism. Let \( I \in \mathcal{I}(A) \), and let \( c \in B_+ \) be such that \( \hat{\Phi}(I) = B_\infty \top B_\infty \). We get
\[
\mathcal{I}(\eta^*)(\hat{\Phi}(I)) = \mathcal{M}(B)_\infty \eta^*(c) \mathcal{M}(B)_\infty = B_\infty \top B_\infty = \hat{\Phi}(I).
\]
It follows, as \( \mathcal{I}(\psi) = \hat{\Phi} \), that
\[
\mathcal{I}(\eta^*) \circ \mathcal{I}(\psi) = \mathcal{I}(\psi) \circ \mathcal{I}(\eta^*) = \hat{\Phi} = \hat{\Phi} = \mathcal{I}(\psi).
\]
Let \( \tilde{\psi} : \tilde{A} \to \mathcal{M}(B)_\infty \) be the unital extension of \( \psi \) which is nuclear. Then \( \eta^* \circ \tilde{\psi} \) is the unital extension of \( \eta^* \circ \psi \) which is also nuclear. It follows from Lemma 6.7 that both \( \psi \) and \( \eta^* \circ \tilde{\psi} \) are \( \mathcal{O}_2 \)-stable, and by Lemma 6.10 \( \mathcal{I}(\tilde{\psi}) = \mathcal{I}(\eta^* \circ \tilde{\psi}) \). So by our uniqueness result Theorem 3.23 \( \hat{\psi} \) and \( \eta^* \circ \tilde{\psi} \) are approximately unitary equivalent. By Theorem 4.3 it follows that there is a \(*\)-homomorphism \( \hat{\phi} : \tilde{A} \to \mathcal{M}(B) \) such that \( \hat{\phi} \) and \( \hat{\psi} \) are unitary equivalent as maps into \( \mathcal{M}(B)_\infty \).

It follows that
\[
\mathcal{I}(\iota) \circ \mathcal{I}(\phi) = \mathcal{I}(\iota \circ \phi) = \mathcal{I}(\psi) = \mathcal{I}(\iota) \circ \Phi.
\]
where \( \iota : B \hookrightarrow B_\infty \) is the constant inclusion. Obviously the map \( \mathcal{I}(\iota) \) is injective, so it follows that \( \mathcal{I}(\phi) = \Phi \), thus finishing the proof. \( \square \)

6.1. Applications. Let \( \text{Hom}_{\mathcal{O}_2}(A, B) \) and \( \text{Hom}_{s\mathcal{O}_2}(A, B) \) denote the sets of \( \mathcal{O}_2 \)-stable and strongly \( \mathcal{O}_2 \)-stable \(*\)-homomorphisms from \( A \) to \( B \) respectively (Definition 3.16), and let \( \sim_{aMvN} \) and \( \sim_{asMvN} \) denote approximate and asymptotic Murray–von Neumann equivalence respectively (Definition 3.3). Let \( \text{Cu}(\mathcal{I}(A), \mathcal{I}(B)) \) denote the semigroup of \( Cu \)-morphisms from \( \mathcal{I}(A) \to \mathcal{I}(B) \) (Definition 2.6).

Corollary 6.11. Let \( A \) be a separable, exact \( C^* \)-algebra and let \( B \) be a separable, nuclear, \( \mathcal{O}_\infty \)-stable \( C^* \)-algebra. Then the natural maps
\[
\text{Hom}_{s\mathcal{O}_2}(A, B)/\sim_{asMvN} \to \text{Hom}_{\mathcal{O}_2}(A, B)/\sim_{aMvN} \to \text{Cu}(\mathcal{I}(A), \mathcal{I}(B))
\]
are both bijective.

In particular, if either \( A \) or \( B \) is \( \mathcal{O}_2 \)-stable, then the natural maps
\[
\text{Hom}(A, B)/\sim_{asMvN} \to \text{Hom}(A, B)/\sim_{aMvN} \to \text{Cu}(\mathcal{I}(A), \mathcal{I}(B))
\]
are both bijective.

Moreover, if \( B \) is also stable, then \( \sim_{asMvN} \) and \( \sim_{aMvN} \) may be replaced with asymptotic and approximate unitary equivalence respectively (with unitaries in \( \mathcal{M}(B) \)).

Proof. Injectivity is Theorem 3.23 and surjectivity is Theorem 6.1. The “in particular” part follows from Proposition 3.18. \( \square \)

Similarly, suppose \( A \) and \( B \) are unital. Let \( \text{Hom}_{((u)\mathcal{O}_2)}(A, B)_1 \) denote the set of all ((strongly) \( \mathcal{O}_2 \)-stable) unital \(*\)-homomorphisms, let \( \sim_{asu} \) and \( \sim_{au} \) denote asymptotic and approximate unitary equivalence respectively, and let \( \text{Cu}(\mathcal{I}(A), \mathcal{I}(B))_1 \) denote the set of \( Cu \)-morphisms \( \Phi \in \text{Cu}(\mathcal{I}(A), \mathcal{I}(B)) \) such that \( \Phi(A) = B \).
Corollary 6.12. Let $A$ be a separable, exact, unital $C^*$-algebra and let $B$ be a separable, nuclear, unital, $O_{\infty}$-stable $C^*$-algebra such that $[1_B]_0 = 0 \in K_0(B)$. Then the natural maps
$$\text{Hom}_{O_2}(A,B)_1/\sim_{asu} \rightarrow \text{Hom}_{O_2}(A,B)_1/\sim_{au} \rightarrow \text{Cu}(I(A),I(B))_1$$
are both bijective.

In particular, if either $A$ or $B$ is $O_2$-stable, then the natural maps
$$\text{Hom}(A,B)_1/\sim_{asu} \rightarrow \text{Hom}(A,B)_1/\sim_{au} \rightarrow \text{Cu}(I(A),I(B))_1$$
are both bijective.

Proof. Injectivity again follows from Theorem 3.23. For surjectivity let $\Phi \in \text{Cu}(I(A),I(B))_1$. Use Theorem 6.1 to construct a strongly $O_2$-stable $*$-homomorphism $\phi_0: A \rightarrow B$ such that $I(\phi_0) = \Phi$. As $\Phi(A) = B$, $p := \phi_0(1_A)$ is a full projection in $B$. As $\phi_0$ is $O_2$-stable, $O_2$ embeds unitally in $(pBp)_0 \cap \phi_0(A)'$. By semiprojectivity of $O_2$, this embedding lifts to a unital $*$-homomorphism $O_2 \rightarrow (pBp)_0$, so $O_2$ embeds unitally in $pBp$. Hence $p \in B$ is a full, properly infinite projection with $[p]_0 = 0 \in K_0(B)$. Thus, a result of Cuntz \cite{Cun81} implies that $1_B$ and $\phi_0(1_A)$ are Murray–von Neumann equivalent. Let $v \in B$ be an isometry with $vv^* = \phi_0(1_A)$. Then $\phi := v\phi_0(-)v^*: A \rightarrow B$ is unital, strongly $O_2$-stable and satisfies $I(\phi) = \Phi$.

“In particular” is again Proposition 3.18. □

The main application is the following strong classification result which was originally proved by Kirchberg, see \cite{Kir00}.

Theorem 6.13. Let $A$ and $B$ be separable, nuclear, $O_2$-stable $C^*$-algebras which are either both stable or both unital.

(a) If $\Phi: I(A) \rightarrow I(B)$ is an order isomorphism, then there exists an isomorphism $\phi: A \rightarrow B$ such that $I(\phi) = \Phi$, i.e. such that $\phi(I) = \Phi(I)$ for all $I \in I(A)$.

(b) If $f: \text{Prim} A \rightarrow \text{Prim} B$ is a homeomorphism, then there exists an isomorphism $\phi: A \rightarrow B$ such that $\phi(I) = f(I)$ for all $I \in \text{Prim} A$.

Proof. (a): The stable (resp. unital) case follows from Corollary 6.11 (resp. Corollary 6.12) and an intertwining argument a la Elliott, see \cite{Rør02}, Corollary 2.3.4.

(b): By \cite{Ped79} Theorem 4.1.3 there is an induced order isomorphism $\Phi: I(A) \rightarrow I(B)$ such that $\Phi(I) = f(I)$ for all $I \in \text{Prim} A$. Hence the result follows from part (a). □

Corollary 6.14. Let $A$ and $B$ be separable, nuclear $C^*$-algebras which are either both stable or both unital. The following are equivalent.

(i) $A \otimes O_2$ and $B \otimes O_2$ are isomorphic,
(ii) $I(A)$ and $I(B)$ are order isomorphic,
(iii) $\text{Prim} A$ and $\text{Prim} B$ are homeomorphic.

Proof. (ii) $\Leftrightarrow$ (iii): For any separable $C^*$-algebra $C$, $\text{Prim} C$ is sober\footnote{A topological space $X$ is sober (or spectral, or point complete) if the prime open subsets $U$ are exactly the sets of the form $X \setminus \{x\}$ for a unique $x \in X$. An open set $U$ is prime if whenever $V, W$ are open and $V \cap W \subseteq U$, then $V \subseteq U$ or $W \subseteq U$.} by \cite{Ped79} Proposition 4.3.6. It is well-known (and easily verified) that two sober spaces are homeomorphic.
if and only if their lattices of open subsets are order isomorphic. Since $\mathcal{I}(C)$ is order isomorphic to the lattice of open subsets of $\text{Prim} C$ by [Ped79, Theorem 4.1.3], we get that $\mathcal{I}(A) \cong \mathcal{I}(B)$ if and only if $\text{Prim} A \cong \text{Prim} B$.

(i) $\Rightarrow$ (ii): This follows since $I \mapsto I \otimes \mathcal{O}_2$ is an isomorphism $\mathcal{I}(C) \cong \mathcal{I}(C \otimes \mathcal{O}_2)$.

(ii) $\Rightarrow$ (i): This follows from Theorem 6.13.

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