The asymptotic expansion of a function due to
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Abstract
We consider the asymptotic expansion for \( x \to \pm \infty \) of the entire function

\[
F_{n,\sigma}(x; \mu) = \sum_{k=0}^{\infty} \frac{\sin(n\gamma_k)}{\sin \gamma_k} \frac{x^k}{k!\Gamma(\mu - \sigma k)}
\]

\( \gamma_k = \frac{(k+1)\pi}{2n} \)

for \( \mu > 0 \), \( 0 < \sigma < 1 \) and \( n = 1, 2, \ldots \). When \( \sigma = \frac{\alpha}{2n} \), with \( 0 < \alpha < 1 \), this function was recently introduced by L.L. Karasheva [J. Math. Sciences, 250 (2020) 753–759] as a solution of a fractional-order partial differential equation.

By expressing \( F_{n,\sigma}(x; \mu) \) as a finite sum of Wright functions, we employ the standard asymptotics of integral functions of hypergeometric type to determine its asymptotic expansion. This is found to depend critically on the parameter \( \sigma \) (and to a lesser extent on the integer \( n \)). Numerical results are presented to illustrate the accuracy of the different expansions obtained.

Mathematics subject classification (2010): 33C15, 33C70, 34E05, 41A30, 41A60

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1. Introduction

In a recent paper, L.L. Karasheva [1] introduced the entire function

\[
\Theta_{n,\alpha}(x; \mu) := \sum_{k=0}^{\infty} \frac{\sin(n\gamma_k)}{\sin \gamma_k} \frac{x^k}{k!\Gamma(\mu - \sigma k)}
\]

\( \gamma_k := \frac{(k+1)\pi}{2n} \),

where \( \mu > 0 \), \( 0 < \alpha < 1 \) and \( n = 1, 2, \ldots \) and throughout \( x \) is a real variable. This function is of interest as it is involved in the fundamental solution of the differential equation

\[
\frac{\partial^\alpha u}{\partial t^\alpha} + (-)^n \frac{\partial^{2n} u}{\partial x^{2n}} = f(x, t)
\]

for positive integer \( n \), where the derivative with respect to \( t \) is the fractional derivative of order \( \alpha \). In the simplest case \( n = 1 \), the above function is the Wright function

\[
\Theta_{1,\alpha}(x; \mu) = \phi(-\sigma; \mu; x) = \sum_{k=0}^{\infty} \frac{x^k}{k!\Gamma(\mu - \sigma k)}, \quad \sigma := \frac{\alpha}{2n},
\]

which finds application as a fundamental solution of the diffusion-wave equation [2]. Under the above assumptions on \( n \) and \( \alpha \) it follows that the parameter \( \sigma < \frac{1}{2} \).
In this study, however, we shall allow the parameter $\sigma$ to satisfy $0 < \sigma < 1$ and consider the function

$$F_{n, \sigma}(x; \mu) := \sum_{k=0}^{\infty} \frac{\sin(n\gamma_k)}{\sin \gamma_k} \frac{x^k}{k! \Gamma(\mu - \sigma k)} \quad (0 < \sigma < 1),$$

(1.2)

which coincides with $\Theta_{n, \alpha}(x; \mu)$ when $\sigma = \alpha/(2n)$. From the well-known expansion

$$\sin(n\gamma_k) = \sum_{r=0}^{n-1} e^{i\gamma_k(2r-n+1)} = \sum_{r=0}^{n-1} e^{-i(k+1)\omega_r},$$

where

$$\omega_r := \frac{(n - 2r - 1)\pi}{2n} \quad (0 \leq r \leq n - 1),$$

(1.3)

it follows that (1.2) can be expressed as a finite sum of Wright functions with rotated arguments (compare [1, Eq. (4)])

$$F_{n, \sigma}(x; \mu) = \sum_{r=0}^{n-1} e^{-i\omega_r} \phi(-\sigma, \mu; xe^{-i\omega_r}).$$

(1.4)

We note that the extreme values of $\omega_r$ satisfy $\omega_0 = -\omega_{n-1} = (n - 1)\pi/(2n)$, whence $|\omega_r| < \frac{1}{2}\pi$ for $0 \leq r \leq n - 1$.

Use of the reflection formula for the gamma function shows that

$$\phi(-\sigma, \mu; x) = \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{x^k}{k!} \Gamma(1 - \mu + \sigma k) \sin \pi(\mu - \sigma k)$$

$$= \frac{1}{2\pi} \left\{ e^{\pi i \vartheta} \Psi(x e^{-\pi i \sigma}) + e^{-\pi i \vartheta} \Psi(x e^{-\pi i \sigma}) \right\}, \quad \vartheta := \frac{1}{2} - \mu,$$

where the associated function $\Psi(z)$ is defined by

$$\Psi(z) := \sum_{k=0}^{\infty} \frac{z^k}{k!} \Gamma(\sigma k + \delta) \quad (0 < \sigma < 1, \ \delta = 1 - \mu)$$

(1.5)

valid for $|z| < \infty$. Hence we obtain the representation

$$F_{n, \sigma}(x; \mu) = \frac{1}{2\pi} \sum_{r=0}^{n-1} e^{-i\omega_r} \left\{ e^{\pi i \vartheta} \Psi(x e^{-\pi i \sigma - i\omega_r}) + e^{-\pi i \vartheta} \Psi(x e^{-\pi i \sigma - i\omega_r}) \right\},$$

which on account of the symmetry of the $\omega_r$ in (1.3) (with $x$ a real variable) can be expressed in the alternative form

$$F_{n, \sigma}(x; \mu) = \frac{1}{\pi} \sum_{r=0}^{N-1} e^{-i\omega_r} \left( e^{\pi i \vartheta} \Psi(x e^{-\pi i \sigma - i\omega_r}) + e^{-\pi i \vartheta} \Psi(x e^{-\pi i \sigma - i\omega_r}) \right) + \Delta_n e^{\pi i \vartheta} \Psi(x e^{-\pi i \sigma}).$$

(1.6)

Here we have defined the quantities

$$N = \lfloor n/2 \rfloor, \quad \Delta_n = \begin{cases} 0 & (n \text{ even}) \\ 1 & (n \text{ odd}) \end{cases}$$

The values of $\omega_r$ appearing in the above expression satisfy

$$\{\omega_0, \omega_1, \ldots, \omega_{N-1}\} = \left\{ \frac{(n-1)\pi}{2n}, \frac{(n-3)\pi}{2n}, \ldots, \frac{\pi}{2n}, \epsilon_n \right\}, \quad \epsilon_n = \begin{cases} 1 & (n \text{ even}) \\ 2 & (n \text{ odd}) \end{cases}$$

(1.7)
We shall use the representation in (1.6), with the above values of $\omega_r$, to determine the asymptotic expansion of $F_{n,\sigma}(x; \mu)$ for $x \to \pm \infty$ by application of the asymptotic theory of the integral function $\Psi(z)$. A summary of this expansion is given in Section 2, which is based on the presentation discussed in [5, 6].

2. The asymptotic expansion of $\Psi(z)$ for $|z| \to \infty$

We first present the large-$|z|$ asymptotics of the function $\Psi(z)$ in (1.6) from the presentation described in [5, Section 4]; see also [6, Section 4.2], [7, §2.3]. We introduce the following parameters:

$$\kappa = 1 - \sigma, \quad h = \sigma^\sigma, \quad \vartheta = \delta - \frac{1}{2}, \quad \delta = 1 - \mu,$$

(2.1)

together with the associated (formal) exponential and algebraic expansions

$$E(z) := Z^\vartheta e^Z \sum_{j=0}^\infty A_j(\sigma) Z^{-j}, \quad H(z) := \frac{1}{\sigma} \sum_{k=0}^\infty \left( \frac{-k}{k!} \right) \Gamma \left( \frac{k+\delta}{\sigma} \right) z^{-(k+\delta)/\sigma},$$

(2.2)

where

$$Z := \kappa(hz)^{1/\kappa}, \quad A_0(\sigma) = (2\pi/\kappa)^{1/2} (\sigma/\kappa)^{\delta}.$$

(2.3)

Then, since $0 < \kappa < 1$, we obtain from [7, p. 57] the large-$z$ expansion

$$\Psi(z) \sim \begin{cases} E(z) + H(ze^{\mp \pi i}) & (|\arg z| < \frac{1}{2}\pi \kappa) \\ H(ze^{\mp \pi i}) & (\frac{1}{2}\pi \kappa < |\arg z| < \pi), \end{cases}$$

(2.4)

where the upper or lower signs are chosen according as $\arg z > 0$ or $\arg z < 0$, respectively.

The expansion $E(z)$ is exponentially large as $|z| \to \infty$ in the sector $|\arg z| < \frac{1}{2}\pi \kappa$, being oscillatory (multiplied by the algebraic factor $Z^{\vartheta/\kappa}$) on the anti-Stokes lines $\arg z = \pm \frac{1}{2}\pi \kappa$. In the adjacent sectors $\frac{1}{2}\pi \kappa < |\arg z| < \pi \kappa$, the expansion $E(z)$ continues to be present, but is exponentially small reaching maximal subdominance relative to the algebraic expansion on the Stokes lines $\arg z = \pm \pi \kappa$. In our treatment of $F_{n,\sigma}(x; \mu)$ we shall not be concerned with exponentially small contributions, except in one special case when $x \to -\infty$ where the expansion of $F_{n,\sigma}(x; \mu)$ is exponentially small.

The first few normalised coefficients $c_j = A_j(\sigma)/A_0(\sigma)$ are [5, 6]:

$$c_0 = 1, \quad c_1 = \frac{1}{24\sigma} \{ 2 + 7\sigma + 2\sigma^2 - 12\delta(1 + \sigma) + 12\delta^2 \}, \quad c_2 = \frac{1}{1152\sigma^2} \{ 4 + 172\sigma^2 + 417\sigma^3 + 172\sigma^3 + 4\sigma^4 - 24\delta(6 + 41\sigma + 41\sigma^2 + 6\sigma^3) + 120\delta^2(4 + 11\sigma + 4\sigma^2) - 480\delta^3(1 + \sigma) + 144\delta^4 \}, \quad c_3 = \frac{1}{414720\sigma^3} \{ (-1112 + 9636\sigma + 163734\sigma^2 + 336347\sigma^3 + 163734\sigma^4 + 9636\sigma^5 - 1112\sigma^6) - \delta(3600 + 220320\sigma + 929700\sigma^2 + 929700\sigma^3 + 220320\sigma^4 + 3600\sigma^5) + \delta^2(65520 + 715680\sigma + 1440180\sigma^2 + 715680\sigma^3 + 65520\sigma^4) - \delta^3(161280 + 816480\sigma + 816480\sigma^2 + 161280\sigma^3) + \delta^4(151200 + 378000\sigma + 151200\sigma^2) - 60480\delta^5(1 + \sigma) + 8640\delta^6 \}.$$

(2.5)

In addition to the Stokes lines $\arg z = \pm \pi \kappa$, where $E(z)$ is maximally subdominant relative to the algebraic expansion, the positive real axis is also a Stokes line. Here the algebraic expansion

\footnote{The dependence of the coefficients $A_j(\sigma)$ on the parameter $\delta$ is not indicated.}

\footnote{On these rays $E(z)$ undergoes a Stokes phenomenon where it switches off in a smooth manner (see [3, p. 67]).}
is maximally subdominant relative to $E(z)$. As the positive real axis is crossed from the upper
to the lower half plane the factor $e^{-\pi i}$ appearing in $H(ze^{-\pi i})$ changes to $e^{\pi i}$, and vice versa.
The details of this transition will not be considered here; see [4, Eq. (3.17)] for the case of the
confluent hypergeometric function $\, _1F_1(a; b; z)$.

3. The asymptotic expansion of $F_{n,\sigma}(x; \mu)$ for $x \to +\infty$

3.1 Asymptotic character as a function of $\sigma$

Let us denote the arguments of the $\Psi$ functions appearing in (1.6) by

\[ z^\pm_r = x \exp[i\phi^\pm_r], \quad \phi^\pm_r = \pm \pi \sigma - \omega_r. \]

The representation of the asymptotic structure of the functions $\Psi(z^\pm_r)$ is illustrated in Fig. 1 for
different values of $\sigma$. The figures show the rays $\arg z = \pm \pi \sigma$ and the anti-Stokes lines (dashed lines) $\arg z = \pm \frac{1}{2} \pi \kappa$. In the case $\sigma = \frac{2}{3}$, the exponentially large sector is $|\arg z| < \frac{1}{3} \pi$ and it is seen from Fig. 1(a) that the arguments $z^\pm_r$ for $0 \leq r \leq N - 1$ and $xe^{\pm \pi i a}$ all lie in the domain where $\Psi(z)$ has an algebraic expansion; this conclusion applies \textit{a fortiori} when $\frac{2}{3} < \sigma < 1$. When $\sigma = \frac{1}{3}$, the exponentially large sector is $|\arg z| < \frac{1}{3} \pi$; when $n = 2$ we have $\omega_0 = \frac{1}{4} \pi$ so that $z^+_0$ is situated on the boundary of the exponentially large sector. Other values of $n \geq 3$ will have some $z^+_r$ inside this sector, whereas the $z^-_r$ are in the algebraic sector for $n \geq 2$. Similarly, the case $\sigma = \frac{1}{2}$, where the rays $\arg z = \pm \pi \sigma$ and $\arg z = \pm \frac{1}{2} \pi \kappa$ coincide, has all the $z^+_r$ situated in the
exponentially large sector, with the $z^-_r$ situated in the algebraic domain. Finally, when $\sigma = \frac{1}{6}$
the exponentially large sector $|\arg z| < \frac{5}{12} \pi$ encloses the rays $\arg z = \pm \pi \sigma$, with the result
that all the $z^+_r$ lie in the exponentially large sector, whereas the $z^-_r$ lie in the algebraic domain
(except when $n = 2$ when $z^-_0$ lies on the lower boundary of the exponentially large sector).

To summarise, we have the following asymptotic character of $F_{n,\sigma}(x; \mu)$ when $x \to +\infty$ as a function of the parameter $\sigma$:

\[
\begin{align*}
0 < \sigma < \frac{1}{2} & \quad \text{EL + A (for } n \geq 2) \\
\frac{1}{2} \leq \sigma < \frac{2}{3} & \quad \text{EL (dependent on } n) + \text{A} \\
\frac{2}{3} \leq \sigma < 1 & \quad \text{A (for } n \geq 2),
\end{align*}
\]

(3.1)

where EL and A denote ‘exponentially large’ and ‘algebraic’ behaviour, respectively.

3.2 Asymptotic expansion

From (1.6) and (2.2), we have the algebraic expansion associated with $F_{n,\sigma}(x; \mu)$ given by

\[
H(x) = \frac{1}{\sigma} \sum_{k=0}^{\infty} \frac{x^{-K}}{k! \Gamma(1 - K)} \theta_{n,k}, \quad K := \frac{k + \delta}{\sigma},
\]

(3.2)

where, with appropriate choices of the factors $e^{\pm \pi i}$ in $H(z)$,

\[
\begin{align*}
\theta_{n,k} &= \frac{(-)^k}{\sin \pi K} \Re \left\{ \sum_{r=0}^{N-1} e^{\pi i \theta - i \omega_r} (e^{\pi i \sigma - i \omega_r} - e^{-\pi i})^{-K} + e^{-\pi i \theta - i \omega_r} (e^{-\pi i \sigma - i \omega_r} + e^{\pi i})^{-K} \right. \\
& \quad \left. + \Delta_n e^{\pi i \theta} (e^{\pi i \sigma} + e^{-\pi i})^{-K} \right\} \\
& = \frac{(-)^k}{\sin \pi K} \Re \left\{ \sum_{r=0}^{N-1} e^{(K-1)i \omega_r} (e^{\pi i(\theta + \kappa K)} + e^{-\pi i(\theta + \kappa K)}) + \Delta_n e^{\pi i(\theta + \kappa K)} \right\} \\
& = \Re \left\{ \sum_{r=0}^{N-1} e^{(K-1)i \omega_r} + \Delta_n \right\}.
\end{align*}
\]

(3.3)
Figure 1: Diagrams representing the rays \( \arg z = \pm \pi \sigma \) and the boundaries of the exponentially large sector (shown by dashed rays) \( |\arg z| < \frac{1}{2} \pi \kappa \), \( \kappa = 1 - \sigma \) for (a) \( \sigma = 2/3 \), (b) \( \sigma = 1/2 \), (c) \( \sigma = 1/3 \) and (d) \( \sigma = 1/6 \). Outside the exponentially large sector the expansion of \( \Psi(z) \) is algebraic in character. The circular quadrants represent the range of the arguments \( \arg z = \pm \pi \sigma - \omega r \) for \( 0 \leq r \leq \lfloor n/2 \rfloor - 1 \), with \( n \geq 2 \) and the arrow-head corresponding to \( n = \infty \). When \( \sigma = 1/3 \) the rays \( \arg z = \pm \pi \sigma \) and \( \arg z = \pm \frac{1}{2} \pi \kappa \) coincide.

since \( \cos \pi(\theta + \kappa K) = \cos \pi(K - k - \frac{1}{2}) = (-)^k \sin \pi K \).

For the exponential component we introduce the quantities

\[
X = \kappa(hx)^{1/\kappa}, \quad \Phi_r^\pm = \pm \frac{\pi \vartheta}{\kappa} - \omega_r \left(1 + \frac{\vartheta}{\kappa}\right)
\]

and the formal asymptotic sum

\[
S(Xe^{i\Omega}) := \sum_{j=0}^{\infty} A_j(\sigma)(Xe^{i\Omega/\kappa})^{-j}.
\]

Then, from (3.4) and (3.5), we have the exponential expansion in the form

\[
E(x) = \frac{X^\vartheta}{\pi} \Re \left\{ \sum_{r=0}^{N-1} \left( \exp[Xe^{i\vartheta - i\Phi_r^+}] S(Xe^{i\vartheta}) + \exp[Xe^{i\vartheta - i\Phi_r^-}] S(Xe^{i\vartheta}) \right) + \Delta_n \exp[Xe^{i\vartheta + \pi \vartheta/\kappa}] S(Xe^{i\vartheta}) \right\}.
\]
It is important to stress that only the exponential terms with \(|\phi^+| \leq \frac{1}{2}\pi\kappa\), that is with

\[|\pm \pi\sigma - \omega r| \leq \frac{1}{2}\pi\kappa,\]

are to be retained in \(E(x)\) in (3.6). In addition, it is seen by inspection of Fig. 1 that the second term involving \(S(Xe^{i\phi+})\) does not contribute to \(E(x)\) when \(\frac{1}{4} \leq \sigma < 1\), since for this range of \(\sigma\) the ray \(\arg z = -\pi\sigma\) lies outside (or, when \(\sigma = \frac{1}{4}\), on the lower boundary of) the exponentially large sector \(|\arg Z| < \frac{1}{2}\pi\kappa\). Thus, when \(\frac{1}{4} \leq \sigma < \frac{2}{3}\), the exponential expansion is significant if \(\pi\sigma - \omega_0 \leq \frac{1}{2}\pi\kappa\); that is, if \(n \geq n_0 = 1/(2-3\sigma)\).

In summary, we have the following theorem:

**Theorem 1.** The following expansion holds for \(x \to +\infty\):

\[
F_{n,\sigma}(x; \mu) \sim \begin{cases} 
E(x) + H(x) & (0 < \sigma < \frac{1}{2}; \ n \geq 2) \\
E(x) + H(x) & (\frac{1}{2} \leq \sigma < \frac{2}{3}; \ n \geq n_0) \\
H(x) & (\frac{1}{2} \leq \sigma < \frac{2}{3}; \ n < n_0) \\
H(x) & (\frac{2}{3} \leq \sigma < 1; \ n \geq 2),
\end{cases}
\]

where \(n_0 = 1/(2-3\sigma)\) and the exponential and algebraic expansions \(E(x)\) and \(H(x)\) are defined in (3.2) and (3.6).

### 3.3 Karasheva’s estimate for \(|\Theta_{n,\alpha}(x; \mu)|\)

When \(\sigma = \alpha/(2n) < \frac{1}{4}\), we see from Theorem 1 that the dominant exponential expansion as \(x \to +\infty\) corresponds to \(r = 0\) yielding

\[
\Theta_{n,\alpha}(x; \mu) \sim \frac{A_0(\sigma)X^\theta}{\pi} \Re \left[X e^{i(\pi\sigma - \omega_0)/\kappa + i\Phi_0^+}\right]
= \frac{A_0(\sigma)X^\theta}{\pi} \exp \left[X \cos(\pi\sigma - \omega_0)/\kappa\right] \cos \left[X \sin(\pi\sigma - \omega_0)/\kappa\right] + \Phi_0^+,
\]

where \(\pi\sigma - \omega_0 = \frac{2n\pi\sigma - (n - 1)\pi}{2n - \alpha} = \frac{(\alpha + 1 - n)\pi}{2n - \alpha}\).

Thus we have the leading order estimate

\[
\Theta_{n,\alpha}(x; \mu) \sim \frac{A_0(\sigma)X^\theta}{\pi} \exp \left[X \cos \left(\frac{(n - 1 - \alpha)\pi}{2n - \alpha}\right)\right] \cos \left[X \sin \left(\frac{(n - 1 - \alpha)\pi}{2n - \alpha}\right) - \Phi_0^+\right] \quad (3.7)
\]

as \(x \to +\infty\). When expressed in our notation, Karasheva’s estimate for \(|\Theta_{n,\alpha}(x; \mu)|\) in [11] §8 agrees with (3.7) (when the second cosine term is replaced by 1), except that she did not give the value of the multiplicative constant \(A_0(\sigma)/\pi\). However, the presentation of her result as an upper bound is not evident due to the presence of possibly less dominant exponential expansions and also the subdominant algebraic expansion.

### 4. The expansion of \(F_{n,\sigma}(x; \mu)\) for \(x \to -\infty\)

To examine the case of negative \(x\) we replace \(x\) by \(e^{\mp \pi i}x\), with \(x > 0\), and use the fact that \(\Psi(xe^{\mp \pi i}) = \Psi(z)\) to find from (3.7) that

\[
F_{n,\sigma}(-x; \mu) = \frac{1}{\pi} \left\{ \sum_{r=0}^{N-1} e^{-i\omega} \left( e^{\pi i \theta} \Psi(xe^{-\pi i\kappa - i\omega}) + e^{-\pi i \theta} \Psi(xe^{\pi i\kappa - i\omega}) \right) + \Delta_n e^{\pi i \phi} \Psi(xe^{-\pi i\kappa}) \right\},
\]

(4.1)
The rays $\arg z = \pm \pi\kappa$ in Fig. 1 are now replaced by the Stokes lines $\arg z = \pm \pi\kappa$. The Stokes and anti-Stokes lines $\arg z = \pm \frac{1}{2}\pi\kappa$ are illustrated in Fig. 2 when $0 < \sigma < \frac{1}{2}$ and $\frac{1}{2} < \sigma < 1$. In the sectors $\frac{1}{2}\pi\kappa < |\arg z| < \pi\kappa$, we recall that the exponential expansion $E(z)$ is still present but is exponentially small as $|z| \to \infty$.

For the algebraic component of the expansion two cases arise when the argument of the second $\Psi$ function $\pi\kappa - \omega_r$ is either (i) positive or (ii) negative. In case (i) the algebraic expansion $H(z)$ does not encounter a Stokes phenomenon as its argument does not cross $\arg z = 0$, whereas in case (ii) a Stokes phenomenon arises for those values of $r$ that make $\pi\kappa - \omega_r < 0$. In case (i), the algebraic component contains the factor inside the sum over $r$

$$e^{\pi i \theta} (e^{-\pi i \kappa - i \omega_r} \cdot e^{\pi i})^{-K} + e^{-\pi i \theta} (e^{\pi i \kappa - i \omega_r} \cdot e^{-\pi i})^{-K}$$

$$= e^{i \omega_r K} (e^{i(\theta - \sigma K)} + e^{-i(\theta - \sigma K)}) = 2e^{i \omega_r K} \cos \pi (k + \frac{1}{2}) \equiv 0$$

upon recalling the definition of $K$ in (3.32) and noting that $\delta - \theta = \frac{1}{2}$. Similarly, the final term involves the factor $\Re e^{\pi i \theta} (e^{-\pi i \kappa} \cdot e^{\pi i})^{-K} = \cos \pi (\theta - \sigma K) = 0$. Thus the algebraic contribution to $F_{n,\sigma}(-x; \mu)$ vanishes in case (i).

For case (ii) to apply, we require that $\pi\kappa - \omega_0 < 0$; that is, $n > n^* = 1/(2\sigma - 1)$. Suppose that $\pi\kappa - \omega_r < 0$ for $0 \leq r \leq r_0$. Then the algebraic component resulting from the terms with $r \leq r_0$ becomes

$$\frac{1}{\pi \sigma} \Re \left\{ \sum_{k=0}^{\infty} \frac{(-)^k \Gamma(K)}{k!} x^{-K} \sum_{r=0}^{r_0} e^{(K-1)i \omega_r} \left( e^{\pi i \theta} (e^{-\pi i \kappa} \cdot e^{\pi i})^{-K} + e^{-\pi i \theta} (e^{\pi i \kappa} \cdot e^{-\pi i})^{-K} \right) \right\}$$

$$= \frac{2}{\pi \sigma} \Re \left\{ \sum_{k=0}^{\infty} \frac{(-)^k \Gamma(K)}{k!} x^{-K} \sum_{r=0}^{r_0} e^{(K-1)i \omega_r - \pi i K} \cos \pi (\theta - \sigma K + \pi K) \right\},$$

where in the second term in round braces we have taken account of the Stokes phenomenon (the first term and that multiplied by $\Delta_n$ are unaffected). Some routine algebra then produces the algebraic contribution

$$\hat{H}(x) := \frac{2}{\pi \sigma} \sum_{k=0}^{\infty} \frac{x^{-K}}{k! \Gamma(1 - K)} \hat{\theta}_{n,k}, \quad \hat{\theta}_{n,k} := \sum_{r=0}^{r_0} \cos \left\{ \pi K - (K - 1)\omega_r \right\}$$
when \( n > n^* \), and \( \hat{\mathcal{H}}(x) \equiv 0 \) when \( n < n^* \).

Reference to Fig. 2 shows that there is no exponential contribution to \( F_{n,\sigma}(-x; \mu) \) from the terms \( \Psi(x e^{-\pi i \kappa}) \) and \( \Psi(x e^{-\pi i \kappa - i \omega_r}) \). From (2.2) and (4.1), we find the exponential expansion results from the terms \( \Psi(x e^{\pi i \kappa - i \omega_r}) \) and is given by

\[
\hat{E}(x) := X^\sigma\pi^\frac{\sigma}{\mu} \sum_{r=0}^{N-1} \exp[-X e^{-i \omega_r / \kappa} - i\Phi] S(-X e^{-i \omega_r / \kappa}),
\]

(4.3)
where \( X \) and the asymptotic sum \( S \) are defined in (3.4) and (4.5) with \( \Phi := \omega_r(1 + \theta / \kappa) \). For \( \sigma < \frac{1}{2} \) (when the algebraic expansion vanishes) the expansion of \( F_{n,\sigma}(-x; \mu) \) will be exponentially small as \( |x| \to \infty \) provided \( \pi \kappa - \omega_0 > \frac{1}{2} \pi \kappa \); that is, when \( n < 1 / \sigma \). If \( n = 1 / \sigma \), there is an exponentially oscillatory contribution and when \( n > 1 / \sigma \), the expansion is exponentially large.

To summarise we have the theorem:

**Theorem 2.** The following expansion holds for \( x \to -\infty \):

\[
F_{n,\sigma}(-x; \mu) \sim \begin{cases} 
\hat{E}(x) & (0 < \sigma \leq \frac{1}{2}) \\
\hat{E}(x) + \hat{H}(x) & (\frac{1}{2} < \sigma < 1),
\end{cases}
\]

(4.4)
where the exponential expansion \( \hat{E}(x) \) is defined in (4.3), the algebraic expansion \( \hat{H}(x) \) is given by

\[
\hat{H}(x) := \begin{cases} 
\frac{2}{\sigma} \sum_{k=0}^{\infty} \frac{x^{-K}}{k! \Gamma(1-K)} \hat{\theta}_{n,k} & (n > n^*) \\
0 & (n < n^*),
\end{cases}
\]

(5.1)

\( n^* = 1/(2\sigma - 1) \) and \( K, \hat{\theta}_{n,k} \) are specified in (3.2) and (4.2).

5. Numerical results

In this section we describe numerical calculations that support the expansions given in Theorems 1 and 2. The function \( F_{n,\sigma}(x; \mu) \) was evaluated using the expression in terms of Wright functions (valid for real \( x \))

\[
F_{n,\sigma}(x; \mu) = 2^R \sum_{r=0}^{N-1} e^{i \omega_r} \phi(-\sigma, \mu; x e^{i \omega_r}) + \Delta_n \phi(-\sigma, \mu; x), \quad N = \lfloor n/2 \rfloor,
\]

(5.1)
which follows from (1.4) and the symmetry of \( \omega_r \).

In Table 1 we present the results of numerical calculations for \( x \to +\infty \) compared with the expansions given in Theorem 1. We choose four representative values of \( \sigma \) that focus on the different cases of Theorem 1 and \( n = 2, 3 \) and 4. The exact value of \( F_{n,\sigma}(x; \mu) \) was obtained by high-precision evaluation of (5.1). The exponential expansion \( \mathcal{E}(x) \) was computed with truncation index \( j = 3 \) and the algebraic expansion \( \hat{H}(x) \) was optimally truncated (that is, at or near its smallest term). The first case \( \sigma = \frac{1}{4} \) has an exponentially large expansion with a subdominant algebraic contribution for all three values of \( n \). The second case \( \sigma = \frac{1}{2} \) corresponds to \( n_0 = 2 \); when \( n = 2 \), \( \mathcal{E}(x) \) is oscillatory and makes a similar contribution as \( \hat{H}(x) \), whereas when \( n = 3 \) and 4, \( \mathcal{E}(x) \) is exponentially large. The third case \( \sigma = \frac{3}{4} \) corresponds to \( n_0 = 3 \); when \( n = 2 \) there is no exponential contribution, whereas when \( n = 3 \), \( \mathcal{E}(x) \) is oscillatory and so makes a similar contribution as \( \hat{H}(x) \); when \( n = 4 \), \( \mathcal{E}(x) \) is exponentially large. Finally, when \( \sigma = \frac{2}{3} \) the expansion of \( F_{n,\sigma}(x; \mu) \) is purely algebraic in character.

We avoid here consideration of the algebraic contribution when \( \pi \kappa - \omega_r = 0 \), that is, on the Stokes line \( \arg z = 0 \).
is exponentially large for $n \gg 0$ whereas for $n \sigma = 2$ and exponentially large for $n = 3/4, 1/2, 1/4, \sigma$.

Table 2: The values of the exponential and algebraic expansions compared with $F_{\sigma}(x; \mu)$ for large $x > 0$ for different values of $\sigma$ and $n$ when $\mu = 3/4$ and $x = 8$.

| $\sigma$ | $n = 2$ | $n = 3$ | $n = 4$ |
|----------|---------|---------|---------|
| $1/3$    | $E(x)$  | $-1.8114881 \times 10^{2}$ | $-1.08294258 \times 10^{3}$ | $-3.08231679 \times 10^{3}$ |
|          | $H(x)$  | $+0.34241316$  | $+0.17280892$  | $+0.34947729$  |
|          | $E(x) + H(x)$ | $-1.8107648 \times 10^{2}$ | $-1.08276977 \times 10^{3}$ | $-3.08197181 \times 10^{3}$ |
|          | $F_{\sigma}(x; \mu)$ | $-1.80709370 \times 10^{2}$ | $-1.08284759 \times 10^{3}$ | $-3.08254767 \times 10^{3}$ |
| $1/2$    | $E(x)$  | $+0.06317153$  | $+1.15957937 \times 10^{3}$ | $-4.47945373 \times 10^{4}$ |
|          | $H(x)$  | $+0.74012019$  | $+1.09449277$  | $+1.45169481$  |
|          | $E(x) + H(x)$ | $+0.80329172$  | $+1.16067387 \times 10^{3}$ | $-4.47930856 \times 10^{4}$ |
|          | $F_{\sigma}(x; \mu)$ | $+0.80329527$  | $+1.16069221 \times 10^{3}$ | $-4.47921506 \times 10^{4}$ |
| $5/9$    | $E(x)$  | $—$       | $-0.14805870$  | $+2.77243091 \times 10^{2}$ |
|          | $H(x)$  | $+0.79825166$  | $+1.17615555$  | $+1.55857242$  |
|          | $E(x) + H(x)$ | $+0.79825166$  | $+1.02809685$  | $+2.78801663 \times 10^{2}$ |
|          | $F_{\sigma}(x; \mu)$ | $+0.79825119$  | $+1.02809649$  | $+2.78801134 \times 10^{2}$ |
| $2/3$    | $H(x)$  | $+0.84046066$  | $+1.23266920$  | $+1.63072031$  |
|          | $E(x) + H(x)$ | $+0.84046066$  | $+1.23266920$  | $+1.63072031$  |

In Table 2 we present illustrative examples of Theorem 2 when $x \to -\infty$. The first case $\sigma = 1/4$ ($\kappa = 3/4$), has an expansion that is exponential in character; for $n < 1/\sigma = 4$, $\hat{E}(x)$ is exponentially small, whereas for $n = 4$ the argument $\pi \kappa - \omega_0 = 3\pi$ lies on the upper boundary of the exponentially large sector $|\arg z| < 3\pi$ and so $\hat{E}(x)$ is oscillatory. For $n \geq 5$, $\hat{E}(x)$ becomes exponentially large as $x \to -\infty$. In the second case $\sigma = 3/8$ ($\kappa = 3/4$), $\hat{E}(x)$ is exponentially small for $n = 2$ and exponentially large for $n \geq 3$. In the third case $\sigma = 3/4$, $\hat{E}(x)$ is oscillatory for $n = 2$ and exponentially large for $n \geq 3$. Finally, when $\sigma = 3/4$ ($\kappa = 3/4$) the function $F_{\sigma}(x; \mu)$ is exponentially large for $n = 2, 3$, and $n \geq 5$. But for $n = 4$, the two values $\omega_0 = 3\pi, \omega_1 = 3\pi$ yield arguments $\pi \kappa - \omega_r$ ($r = 0, 1$) situated on both boundaries of the exponentially large sector $|\arg z| < 3\pi$. In this case $\hat{E}(x)$ is oscillatory and, since $\kappa = 2$, there is in addition an algebraic
contribution $\hat{H}(x)$.

It is seen from Tables 1 and 2 that the asymptotic values agree well with the numerically computed values of $F_{n,\sigma}(\pm x; \mu)$.

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