Bikei, Involutory Biracks and unoriented link invariants

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Abstract

We identify a subcategory of biracks which define counting invariants of unoriented links, which we call \textit{involutory biracks}. In particular, involutory biracks of birack rank \( N = 1 \) are biquandles, which we call \textit{bikei} or 双圭. We define counting invariants of unoriented classical and virtual links using finite involutory biracks, and we give an example of a non-involutory birack whose counting invariant detects the non-invertibility of a virtual knot.

Keywords: Biquandles, Yang-Baxter equation, unoriented link invariants, enhancements of counting invariants

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1 Introduction

Much attention has recently been focused on the study of invariants of oriented knots and links defined using algebraic objects known as \textit{biquandles}, solutions to the set-theoretic Yang-Baxter equation satisfying certain invertibility conditions \([6, 8, 11, 15]\). Every oriented classical or virtual knot has a \textit{fundamental biquandle} whose isomorphism class is a strong invariant – indeed, a complete invariant of classical knots when considered up to ambient homeomorphism.

Comparing isomorphism classes of fundamental biquandles directly is generally impractical, so for more practical biquandle-derived invariants we can either look to functorial invariants like the Alexander and quaternionic biquandle polynomials studied in \([2, 3, 4, 5]\) which generalize the classical Alexander polynomial, or to representational invariants such as the counting invariant \( \Phi^X_Z(L) = |\text{Hom}(FB(L), X)| \) where \( X \) is a finite biquandle \([10, 15]\).

\textit{Quandles} are a special case of biquandles. Of particular interest are the CJKLS quandle 2-cocycle invariants of oriented classical and virtual knots and links defined in terms of homomorphisms from the fundamental quandle of a knot to a finite quandle, enhanced by a \textit{Boltzmann weight} defined from an element of the second cohomology of the finite quandle in question \([7]\). The study of quandle homology has recently turned to \textit{involutory quandles}, also known as \textit{kei} or 双圭; these are the type of quandles suitable for defining invariants of unoriented knots and links \([16]\). Kei were considered as far back as 1945 \([19]\).

\textit{Racks} are the objects analogous to quandles which are appropriate for defining representational invariants of blackboard-framed oriented knots and links \([12]\). Racks are a special case of \textit{biracks}, recently studied in papers such as \([3]\) and \([17]\).

In this paper we generalize the kei idea to the setting of biracks, defining counting invariants for unoriented framed and unframed knots and links. In section 2 we identify the necessary and sufficient conditions for a birack to be involutory, and we give examples of involutory biracks, involutory racks, and involutory biquandles (also known as bikei or 双圭) as well as a schematic map of the various types of biracks associated to categories of knots and links. In section 3 we define counting invariants associated to involutory biracks and give some computations and examples, including a biquandle whose counting invariant distinguishes a virtual knot from its inverse, answering a question posed to the second author by Xiao-Song Lin. In section 4 we list some open questions for future research.

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2 Bikei and Involutory Biracks

We begin with a definition. (See \[11, 17\]).

**Definition 1** A birack \((X, B)\) is a set \(X\) with a map \(B : X \times X \to X \times X\) which satisfies

- \(B\) is invertible, i.e there exists a map \(B^{-1} : X \times X \to X \times X\) satisfying \(B \circ B^{-1} = \text{Id}_{X \times X} = B^{-1} \circ B\),
- \(B\) is *sideways invertible*, i.e there exists a unique invertible map \(S : X \times X \to X \times X\) satisfying
  \[
  S(B_1(x, y), x) = (B_2(x, y), y),
  \]
  for all \(x, y \in X\),
- The sideways maps \(S\) and \(S^{-1}\) are *diagonally bijective*, i.e. the compositions \(S_1^{\pm 1} \circ \Delta\) and \(S_2^{\pm 1} \circ \Delta\) of the components of \(S^{\pm 1}\) with the diagonal map \(\Delta(x) = (x, x)\) are bijections, and
- \(B\) is a solution to the *set-theoretic Yang-Baxter equation*:
  \[
  (B \times \text{Id}) \circ (\text{Id} \times B) \circ (B \times \text{Id}) = (\text{Id} \times B) \circ (B \times \text{Id}) \circ (\text{Id} \times B)
  \]

The components of \(B\) and \(B^{-1}\) are sometimes written with the alternate notation \(B(x, y) = (y^x, x_y)\) and \(B^{-1}(x, y) = (x_y, y^x)\).

The birack axioms are motivated by the oriented Reidemeister moves, where we interpret the map \(B\) as a map of semiarc labels going through a crossing:

As shown in \[17\], sideways invertibility defines bijections \(\alpha : X \to X\) and \(\pi : X \to X\) defined by \(\alpha = (S_2^{-1} \circ \Delta)^{-1}\) and \(\pi = S_1^{-1} \circ \Delta \circ \alpha\) which give the labels of semiarcs in a blackboard-framed type I move:

The bijection \(\pi(x)\) is known as the *kink map*, and its exponent \(N\), i.e. the smallest integer \(N\) such that \(\pi^N(x) = x\) for all \(x \in X\), is known as the *birack rank* or *birack characteristic* of \(X\).

We would like to modify the birack axioms to remove the orientation requirement with the goal of obtaining invariants of unoriented links. We will use the convention that if a crossing is positioned with the
understand on the right, then the upward map will be our $B$ map:

![Diagram](image)

We first observe that after rotating the crossing by $180^\circ$, we have $B(x, y) = (u, v)$ implies $B(v, u) = (y, x)$:

![Diagram](image)

For any set $X$, let $\tau : X \times X \to X \times X$ be the map $\tau(x, y) = (y, x)$. Then $B(v, u) = (y, x)$ implies $\tau \circ B \circ \tau(u, v) = (x, y)$, and we have

$$B^{-1} = \tau \circ B \circ \tau$$

or equivalently $(\tau \circ B)^2 = \operatorname{Id}$.

The Reidemeister II move then requires that the upward map at a crossing with the understrand on the left is $B^{-1} = \tau \circ B \circ \tau$, as well as that the sideways map $S$ is $B^{-1}$:

![Diagram](image)

With these observations, we can now define what it means for a birack to be involutory.

**Definition 2** An *involutory birack* $(X, B)$ is a set $X$ with a map $B : X \times X \to X \times X$ which satisfies

- $(\tau \circ B)^2 = \operatorname{Id}$ where $\tau(x, y) = (y, x)$,
- The compositions $B_1^{\pm 1} \circ \Delta$ and $B_2^{\pm 1} \circ \Delta$ of the diagonal map $\Delta(x) = (x, x)$ with the components of $B^{\pm 1}$ are bijections, and
- $B$ is a solution to the *set-theoretic Yang-Baxter equation*:

$$(B \times \operatorname{Id}) \circ (\operatorname{Id} \times B) \circ (B \times \operatorname{Id}) = (\operatorname{Id} \times B) \circ (B \times \operatorname{Id}) \circ (\operatorname{Id} \times B).$$

An involutory birack with birack rank $N = 1$ is an *involutory biquandle* or *bikei* (双圭).
Remark 1 The term “involutory” refers to the fact that the maps \( u_x, l_x : X \to X \) defined by \( u_x(y) = B_1(x, y) \) and \( l_x(y) = B_2(y, x) \) are involutions, i.e. \( u_x^2 = \text{Id} = l_x^2 \) for all \( x \in X \).

Example 1 Recall from [17] that any set \( X \) has the structure of a birack defined by \( B(x, y) = (\sigma(y), \rho(x)) \) where \( \sigma, \rho : X \to X \) are commuting bijections; these are known as constant action biracks. Such a birack is involutory iff \( \rho^2 = \sigma^2 = \text{Id} \), since
\[
\tau \circ B \circ \tau \circ B(x, y) = (\rho^2(x), \sigma^2(y)).
\]

Example 2 A birack in which \( B_2(x, y) = x \) is a rack. A rack is then involutory iff \( B_1(x, B_1(x, y)) = y \) for all \( x, y \in X \). Alternate notations for the rack operation \( B_1(x, y) \) include \( y \triangleright x \) and \( y^x \); using these conventions, a rack is involutory iff \( (y \triangleright x) \triangleright x = y \) or \( (y^x)^x = y \) for all \( x, y \in X \). A rack of birack rank \( N = 1 \) is known as a quandle; involutory quandles are also known as kei.

We summarize the relationships between these objects with the following Venn diagram. Note that the sizes of the circles are not meant to reflect proportions.

Recall from [17] that any module over the ring \( \hat{\Lambda} = \mathbb{Z}[t^\pm 1, s, r^\pm 1] / (s^2 - (1 - tr)s) \) has the structure of a birack defined by \( B(x, y) = (sx + ty, rx) \), known as a \((t, s, r)\)-birack. The kink map of such a birack is \( \pi(x) = (tr + s)x \), so the birack rank of a \((t, s, r)\)-birack is the smallest integer \( N \) such that \( (tr + s)^N = 1 \). A \((t, s, r)\)-birack in which \( r = 1 \) is rack known as a \((t, s)\)-rack [12] [17] [9]; a \((t, s, r)\)-birack with rank \( N = 1 \) is an Alexander biquandle, and if we have both \( r = 1 \) and \( N = 1 \) we have an Alexander quandle.
**Proposition 1** A \((t, s, r)\)-birack is involutory iff we have \(t^2 = r^2 = 1\) and \((t + r)s = (1 - r)s = 0\).

**Proof.**

\[
\tau \circ B \circ \tau \circ B(x, y) = \tau \circ B \circ \tau \circ B(x, y) \\
= \tau \circ B \circ \tau (sx + ty, rx) \\
= \tau \circ B(rx, sx + ty) \\
= (r^2x, sx + ty + (t + r)sx) \\
\]

so setting \((\tau \circ B)^2 = \text{Id}\) we obtain \(t^2 = r^2 = 1\) and \((t + r)s = 0\).

Finally, note that in a \((t, s, r)\)-birack we have \(S(u, v) = (rv, t^{-1}u - t^{-1}sv)\). Then

\[
S \circ B(x, y) = S(sx + ty, rx) \\
= (r^2x, t^{-1}(sx + ty) - t^{-1}srx) \\
= (r^2x, y + (t^{-1}s - t^{-1}sr)x) \\
\]

so \(S \circ B = \text{Id}\) implies \(r^2 = 1\) and \(t^{-1}s - t^{-1}sr = 0\), and multiplication by \(t\) reduces the latter to \((1 - r)s = 0\).

\[\square\]

**Corollary 2** A \((t, s)\)-rack is involutory if and only if \(t^2 = 1\) and \((t + 1)s = 0\).

**Corollary 3** An Alexander biquandle is involutory if and only if \(t^2 = r^2 = 1\) and \((1 - t)(1 - r) = 0\).

**Proof.** In an Alexander biquandle, we have \(tr + s = 1\) which implies \(s = 1 - tr\). Then

\[
(t + r)s = (t + r)(1 - tr) = t - t^2r + r - tr^2 \\
\]

which is zero provided \(t^2 = r^2 = 1\). The condition that \((1 - r)s = 0\) is then

\[
(1 - r)(1 - tr) = 1 - r - tr + tr^2 = 1 - r - tr + t = 1 - r + t(1 - r) = (1 - t)(1 - r) = 0, \\
\]

as required.

These together imply the well-known result:

**Corollary 4** An Alexander quandle is involutory if and only if \(t^2 = 1\).

Let \(X = \{x_1, \ldots, x_n\}\) be a finite set. We can specify an involutory birack structure on \(X\) with a pair of \(n \times n\) matrices specifying the operation tables of \(y^x = B_1(x, y)\) and \(x^y = B_2(x, y)\), i.e. \(M_{X,B} = [U|L]\) where \(U(i, j) = k\) and \(L(i, j) = h\) where \(x_k = B_1(x_j, x_i)\) and \(x_h = B_2(x_i, x_j)\). This allows us to do computations with biracks for which we lack convenient formulas.

**Example 3** Let \(X = \mathbb{Z}_n = \{1, 2, 3, 4\}\). We can give \(X\) the structure of an involutory \((t, s, r)\)-birack on \(X\) by choosing invertible elements \(t, r \in \mathbb{Z}_n^*\) and an element \(s \in \mathbb{Z}_n\) satisfying the conditions \(s^2 = (1 - tr)s, \ t^2 = s^2 = 1\) and \((1 - r)s = (t + r)s = 0\). For example, \(X = \mathbb{Z}_4\) becomes an involutory \((t, s, r)\)-birack by setting \(s = 2, t = 1\) and \(r = 3\). Then we have \(s^2 = 4 = 0 = (1 - 1(3))(2), \ t^2 = 1, \ r^2 = 9 = 1, \ (t + r)s = (1 + 3)(2) = 0\) and \((1 - r)s = (1 - 3)2 = 0\). The birack matrix is given by

\[
M_{X,B} = \begin{bmatrix}
3 & 1 & 3 & 1 & 3 & 3 & 3 & 3 \\
4 & 2 & 4 & 2 & 2 & 2 & 2 & 2 \\
1 & 3 & 1 & 3 & 1 & 1 & 1 & 1 \\
2 & 4 & 2 & 4 & 4 & 4 & 4 & 4 \\
\end{bmatrix}
\]
Remark 2 The columns of the birack matrix are the images of the maps $u_x$ and $l_x$ mentioned in remark [1]. Hence, a birack whose matrix contains any column not representing an involution is not involutory.

Example 4 Let $K$ be a blackboard-framed classical or virtual knot or link diagram. The fundamental involutory birack of $K$, denoted $IB(K)$, is the set of equivalence classes of involutory birack words modulo the equivalence relation generated by the involutory birack axioms and the crossing relations in $K$. More precisely, let:

- $G$, the set of generators, correspond bijectively with the set of semiarcs in $k$,
- $W(G)$, the set of involutory birack words in $G$, be defined inductively by the rules that (1) $G \subseteq W(G)$ and (2) $B(g,h) \in W(G)$ for all $g,h \in W(G)$, and
- $\sim$ be the smallest equivalence relation on $W(G)$ containing each of the crossing relations together with $(\tau \circ B)^2(g,h) \sim (g,h)$ and $(B \times \text{Id})(\text{Id} \times B)(B \times \text{Id})(g,h,k) \sim (\text{Id} \times B)(B \times \text{Id})(\text{Id} \times B)(g,h,k)$ for all $g,h,k \in W(G)$.

We will specify the fundamental involutory birack with a list of generators $G$ and crossing relations $R$, $IB(K) = \langle G | R \rangle$ with the birack axiom relations understood. For example, the trefoil knot below has the listed fundamental involutory birack presentation:

$$IB = \langle a, b, c, d, e, f \mid B(a,b) = (c,d), B(c,d) = (e,f), B(e,f) = (a,b) \rangle.$$ 

For virtual knots and links, we ignore virtual crossings, with semiarcs going from one classical over or undercrossing point to the next:

$$IB = \langle a, b, c, d, e, f \mid B(a,b) = (c,d), B(c,d) = (e,f), B(e,f) = (b,a) \rangle.$$ 

Remark 3 The fundamental involutory birack is analogous to the fundamental birack $BR(L)$ of a knot or link (see [11] or [17]). Indeed, there is a functor $I : Br \rightarrow IBr$ from the category of finitely generated biracks to the category of finitely generated involutory biracks defined by setting $B^{-1} = \tau B \tau = S$ in presentations of the objects of $Br$ in addition to the inclusion functor $IBr \rightarrow Br$.

As with other algebraic structures, we have the following standard definitions:

Definition 3 A map $f : X \rightarrow Y$ between involutory biracks is a homomorphism if for all $x,y \in X$ we have

$$B(f(x), f(y)) = (f(B_1(x,y)), f(B_2(x,y))).$$

Definition 4 A subset $Y \subseteq X$ of an involutory birack $X$ is a subbirack if $B(Y \times Y) \subseteq Y \times Y$. 

6
3 Invariants of Unoriented Links

We begin this section by recalling the counting invariant of oriented classical and virtual links associated to a finite involutory birack defined in [17]. Let $X$ be a finite set and $B : X \times X \to X \times X$ a birack structure on $X$. Let $L$ be an oriented link diagram (classical or virtual) with $c$ components. A framing of $L$ is given by an element $w \in \mathbb{Z}^c$, where the $k$th entry of $w$ gives the writhe of the $k$th component of $L$, i.e. the sum of the crossing signs of the crossings where both strands are from component $k$ using the convention below.

A birack labeling of $L$ by $X$ is a homomorphism $f : BR(L) \to X$ from the fundamental birack of $L$ to $X$. In particular, every homomorphism $f : BR(L) \to X$ assigns an element of $X$ to each generator of $BR(L)$ and hence to each semiarc in $L$, and such an assignment determines a homomorphism if and only if the crossing relations are satisfied in $X$ by the assignment. Thus, we can visualize birack homomorphisms $f : BR(L) \to X$ as labelings of the semiarcs in a diagram of $L$ by their images in $X$.

By construction, changing a diagram by the blackboard-framed Reidemeister moves

induces a bijection on the sets of birack homomorphisms. In particular, the number of birack labelings of a link diagram by a finite birack $X$ is an integer-valued invariant of blackboard-framed isotopy.

Now, let $N$ be the birack rank of $X$. For any birack labeling of $L$ by $X$, there is a unique corresponding birack labeling of any framed link diagram related to $L$ by blackboard framed Reidemeister moves together with the $N$ phone cord move:

Thus, the numbers of labelings $|\text{Hom}(RB(L), X)|$ are periodic in the writhe of each component with period $N$, and we obtain an invariant of unframed isotopy by summing these numbers of labelings over a complete period of writhes:

$$\Phi^\mathbb{Z}_{(X,B)}(L) = \sum_{w \in (\mathbb{Z}_N)^c} |\text{Hom}(BR(L, w), X)|$$

is the integral birack counting invariant, defined in [17].
This same definition applies unmodified in the involutory birack case with unoriented links: while crossing signs are undefined for intercomponent crossings in unoriented links, they are well defined for intracomponent crossings since both choices of orientation of a given component determine the same sign for each intracomponent crossing. Thus, we have:

**Definition 5** Let $L$ be an unoriented classical or virtual link with $c$ components and let $(X, B)$ be a finite involutory birack. The integral involutory birack counting invariant is

$$
\Phi_{(X, B)}^\mathbb{Z}(L) = \sum_{w \in (\mathbb{Z}_N)^c} |\text{Hom}(IB(L, w), X)|.
$$

**Example 5** The well-known Fox 3-coloring invariant is a special case of the involutory birack counting invariant. Specifically, let $X = \mathbb{Z}_3$ with $t = 2$, $s = 2$ and $r = 1$; then $s^2 = 1 = (1 - 2(1))2 = (1 - tr)s$, so we have a $(t, s, r)$-birack. Moreover, $t^2 = r^2 = 1$, $(t + r)s = 3(2) = 0$ and $(1 - r)s = (1 - 1)2 = 0$, so $X$ is involutory, and $t + s = 4 = 1$, so $X$ is a kei. As a labeling rule, we have

$$
\begin{array}{c}
\includegraphics{example_diagram.png}
\end{array}
$$

which amounts to “all three colors agree or all three are distinct”. The fact that $\Phi_{(X, B)}^\mathbb{Z}(3_1) = 9 \neq 3 = \Phi_{(X, B)}^\mathbb{Z}(\text{Unknot})$ is perhaps the easiest proof that the trefoil is nontrivially knotted.

An enhancement of the birack counting invariant assigns a blackboard-framed and $N$-phone cord invariant signature to each birack labeling or homomorphism in $\text{Hom}(BR(L), X)$; the multiset of such signatures over a complete set of writhe vectors is then an enhanced invariant which determines the counting invariant value but is generally stronger. All of the enhancements of $\Phi_{(X, B)}^\mathbb{Z}(L)$ defined in [17] are also defined for involutory biracks. These include:

- **The image-enhanced counting invariant.** Here the signature is the cardinality of the image subbirack:

$$
\Phi_{(X, B)}^{\text{Im}}(L) = \sum_{w \in (\mathbb{Z}_N)^c} \left( \sum_{f \in \text{Hom}(BR(L, w), X)} u^{\text{Im}(f)} \right);
$$

- **The writhe-enhanced counting invariant.** Here we keep track of which writhe vectors contribute which labelings:

$$
\Phi_{(X, B)}^{\text{Im}}(L) = \sum_{w \in (\mathbb{Z}_N)^c} \left( \sum_{f \in \text{Hom}(BR(L, w), X)} |\text{Hom}(BR(L, w), X)|q^w \right),
$$

where $q^{(w_1, \ldots, w_c)} = q_1^{w_1} \ldots q_c^{w_c}$, and

- **The birack polynomial enhanced counting invariant.** Here the signature is the subbirack polynomial $\rho_{\text{Im}(f) \subset X}$ of the image subbirack (see [17] for more):

$$
\Phi_{(X, B)}^{\text{Im}}(L) = \sum_{w \in (\mathbb{Z}_N)^c} \left( \sum_{f \in \text{Hom}(BR(L, w), X)} u^{\rho_{\text{Im}(f) \subset X}} \right),
$$

and
• **The column group enhanced counting invariant.** Here the signature is the subgroup \( CG(\text{Im}(f)) \) of \( S_{|X|} \) generated by the permutations \( u_x \) and \( l_x \) for \( x \in \text{Im}(f) \):

\[
\Phi_{(X,B)}^{CG}(L) = \sum_{w \in (\mathbb{Z}_N)^e} \left( \sum_{f \in \text{Hom}(BR(L,w),X)} u_{CG(\text{Im}(f))} w^{CG(\text{Im}(f))} \right).
\]

See [14] for more.

As an application, we are able to answer a question posed to the second author by Xiao-Sing Lin in 2005: can birack counting invariants be used to distinguish non-invertible knots from their inverses? We are happy to say that the answer is yes, as we demonstrate in the next example.

**Example 6** Consider the virtual knot numbered 3.3 in the knot atlas [1]; it is the closure of the virtual braid diagram below. If we orient the braid first downward and then upward, we have the listed fundamental birack presentations, \( B(3.3_↓) \) and \( B(3.3_↑) \) respectively.

\[
\begin{align*}
B(3.3_↓) &= \langle a, b, c, d, e, f \mid (b, a) = B(e, d), (c, e) = B(b, f), (d, f) = B(a, c) \rangle \\
B(3.3_↑) &= \langle a, b, c, d, e, f \mid B(a, b) = (d, e), B(c, e) = (f, b), B(f, d) = (c, a) \rangle
\end{align*}
\]

Note that these are related by \( B_↓ = \tau \circ B_↑ \circ \tau \), so both oriented versions of 3.3 have the same fundamental involutory birack.

To see that the two oriented versions are non-isotopic, we will need a non-involutory birack. Let \( X = \mathbb{Z}_{11} \) and set \( t = 6, s = 5 \) and \( r = 3 \). Then we have \( s^2 = 5^2 = 5(1-6(3)) = 5(1-tr) \) and we have a \( (t,s,r) \)-birack; moreover, \( t^2 = 6^2 = 3 \neq 1 \) and \( X \) is non-involutory. Since \( \mathbb{Z}_{11} \) is a field, \( s \) is invertible and \( X \) is biquandle, so \( N = 1 \) and we do not need to compute labelings for multiple writhes.

Then setting \( B(x,y) = (5x + 6y, 3x) \), the crossing relations for \( B_↓ \) give us a system of linear equations over \( \mathbb{Z}_{11} \) with homogeneous matrix

\[
\begin{bmatrix}
0 & 10 & 0 & 6 & 5 & 0 \\
10 & 0 & 0 & 0 & 3 & 0 \\
0 & 5 & 10 & 0 & 0 & 6 \\
0 & 3 & 0 & 0 & 10 & 0 \\
5 & 0 & 6 & 10 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 10
\end{bmatrix}
\]

which has full rank, and hence the only solution is the trivial labeling of all semiarcs by 0, and the counting invariant is \( \Phi_{(X,B)}(3.3_↓) = u^1 = u \). On the other hand, the crossing relations for \( B(3.3_↑) \) give us the system

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]
of linear equations over $\mathbb{Z}_{11}$ with homogeneous matrix

$$
\begin{pmatrix}
5 & 6 & 0 & 10 & 0 & 0 \\
3 & 0 & 0 & 0 & 10 & 0 \\
0 & 0 & 6 & 0 & 5 & 10 \\
0 & 10 & 0 & 0 & 3 & 0 \\
0 & 0 & 10 & 6 & 0 & 5 \\
10 & 0 & 0 & 0 & 10 & 0
\end{pmatrix}
$$

is row equivalent to

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 8 \\
0 & 1 & 0 & 0 & 0 & 6 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 10 \\
0 & 0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

which has rank 1, and thus the space of solutions is 1-dimensional, giving us a counting invariant value of $\Phi^2_{(X,B)}(3,3) = u^{11} \neq u$, and the counting invariant detects the non-invertibility of the virtual knot 3.3.

4 Questions

In this section we collect a few questions for future research.

What new enhancements of the counting invariant require $X$ to be involutory? Does the condition that $(X,B)$ is involutory determine anything about the homology groups, column groups or birack polynomials of $(X,B)$?

In remark 1 we observed that the component maps of an involutory birack must be involutions. Is the converse true? That is, do the conditions $(\tau \circ B)^2 = \text{Id}$ and $S \circ B = \text{Id}$ follow from the condition that $u_x$ and $l_x$ are involutions for all $x \in X$?

What conditions on two non-involutory finite biracks $B, B'$ imply that $I(B) \cong I(B')$? That is, when do two non-involutory birack involutize to the same involutory birack?

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