Poisson source localization on the plane: the smooth case

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Abstract
We consider the problem of localization of a Poisson source using observations of inhomogeneous Poisson processes. We assume that \( k \) detectors are distributed on the plane and each detector generates observations of the Poisson processes, whose intensity functions depend on the position of the source. We study asymptotic properties of the maximum likelihood and Bayesian estimators of the source position on the plane assuming that the amplitude of the intensity functions are large. We show that under regularity conditions these estimators are consistent, asymptotically normal and asymptotically efficient in the minimax mean-square sense. Then we propose some simple consistent estimators and these estimators are further used to construct asymptotically efficient One-step MLE-process.

Keywords Inhomogeneous Poisson process · Source localization · GPS-localization · Sensors · Maximum likelihood estimator · Bayes estimators · One-step MLE

1 Introduction

We consider the problem of estimation of the position \( \theta = (x_0, y_0) \) of a source emitting Poisson signals which are received by \( k \) sensors (Knoll 2010), distributed on the plane. We assume that the source starts emission at time \( t = 0 \) and the \( j \)-th sensor receives the data, which can be described as inhomogeneous Poisson process \( X_j = (X_j(t), 0 \leq t \leq T) \), whose intensity function \( \lambda_j(\theta_0, t) = \lambda_j(t - \tau_j) + \lambda_0, 0 \leq t \leq T \) increases, starting upon arrival of the signal at \( t = \tau_j \). Here \( \lambda_j(t - \tau_j) \geq 0 \) is the signal from the source received by the detector \( D_j \), \( \lambda_0 > 0 \) is the intensity of the Poisson noise and \( \tau_j \) is the time needed for the signal to reach the \( j \)-th detector.
For the $j$-th detector ($D_j$), positioned at the point $\vartheta_j = (x_j, y_j)$, we have $\tau_j (\vartheta_0) = \frac{1}{\nu} \| \vartheta_j - \vartheta_0 \|$, where $\nu > 0$ is the known rate of the signal propagation and $\| \cdot \|$ is the Euclidean norm on the plane. We assume that $\lambda_j (t) = 0$ for $t \leq 0$. Therefore we have $k$ independent inhomogeneous Poisson processes $X = (X_1, \ldots, X_k)$ with intensities depending on $\tau_j (\vartheta_0)$. We assume that the position of the source $\vartheta_0 \in \Theta$ is unknown and we have to estimate $\vartheta_0$ using the observations $X = (X_1, \ldots, X_k)$. Here $\Theta \subset \mathbb{R}^2$ is a convex bounded set (Fig. 1).

Note that this mathematical model is similar to the problem of GPS-localization on the plane (Luo 2013). In this case $k$ emitters have known positions and the receiver has to estimate its own position, using the arriving signals. More precisely, we observe $k$ inhomogeneous Poisson processes with intensity functions depending on the position of the receiver and we are interested in estimating its coordinates.

Due to the importance of such models in applications, much literature is concerned with design of various algorithms of localization [see the introduction in the work (Farinetto et al. 2018) and references therein]; however, rigorous mathematical analysis of the above class of models seems to be missing. The statistical models of inhomogeneous Poisson processes with intensity functions, discontinuous along parametric curves were considered in Kutoyants (1998, Sections 5.2 and 5.3.). Comprehensive accounts on the statistical inference of point processes can be found in Karr (1991), Snyder and Miller (1991) and Streit (2010).

Here we are interested in the observation models, which allow estimation with small errors, $\mathbb{E}_{\vartheta_0} \| \tilde{\vartheta} - \vartheta_0 \|^2 = o (1)$. As usual in such situations “small error” is interpreted as an asymptotic statement. Small estimation errors emerge, for example, when the intensity of the signal is large or in the case of Poisson processes with periodic intensity function of a known period. Another possibility is to have many sensors. We consider the model with large intensity functions $\lambda_j (\vartheta_0, t) = \lambda_{j,n} (\vartheta_0, t)$, which can be written as

$$\lambda_{j,n} (\vartheta_0, t) = n\lambda_j (t - \tau_j) + n\lambda_0, \quad 0 \leq t \leq T$$

or equivalently,

$$\lambda_{j,n} (\vartheta_0, t) = n\lambda_j (t - \tau_j) 1_{\{t \geq \tau_j (\vartheta_0)\}} + n\lambda_0, \quad 0 \leq t \leq T.$$
Here \( n \) is the “large parameter” and we study the estimators in the limiting regime \( n \to \infty \). For example, such model is relevant if there are \( k \) clusters and each consists of \( n \) detectors.

The likelihood ratio function \( L (\vartheta, X^n) \) is (see, e.g., Kutoyants 1998)

\[
\ln L \bigg( \vartheta, X^n \bigg) = \sum_{j=1}^{k} \int_{\tau_j}^{T} \ln \left( 1 + \frac{\lambda_j \left( t - \tau_j \right)}{\lambda_0} \right) dX_j(t) - n \sum_{j=1}^{k} \int_{\tau_j}^{T} \lambda_j \left( t - \tau_j \right) dt.
\]

Here \( \tau_j = \tau_j (\vartheta) \) and \( X = (X_j(t), 0 \leq t \leq T, j = 1, \ldots, k) \) are the counting processes from \( k \) detectors. Having this likelihood ratio formula, we define the maximum likelihood estimator (MLE) \( \hat{\vartheta}_n \) and the Bayesian estimator (BE) \( \tilde{\vartheta}_n \) by the “usual” relations

\[
L \bigg( \hat{\vartheta}_n, X^n \bigg) = \sup_{\vartheta \in \Theta} L \bigg( \vartheta, X^n \bigg)
\]

and

\[
\tilde{\vartheta}_n = \frac{\int_{\Theta} \vartheta \ p (\vartheta) \ L (\vartheta, X^n) \ d\vartheta}{\int_{\Theta} p (\vartheta) \ L (\vartheta, X^n) \ d\vartheta}.
\]

Here \( p (\vartheta) \), \( \vartheta \in \Theta \) is the prior density. We assume that it is a positive, continuous function on \( \Theta \). If Equation (1) has more than one solution then any of these solutions can be taken as the MLE. In Sect. 3 we consider another consistent estimator.

There are several types of statistical problems depending on the regularity of the function \( \lambda_j (\cdot) \). In particular, the rate of convergence of the mean squared error of the estimators \( \hat{\vartheta}_n = \hat{\vartheta}_n \) and \( \tilde{\vartheta}_n = \tilde{\vartheta}_n \) is

\[
E_{\vartheta_0} \left\| \tilde{\vartheta}_n - \vartheta_0 \right\|^2 = C \frac{1}{n^\gamma} (1 + o (1)),
\]

where the parameter \( \gamma > 0 \) depends on the regularity of the function \( \lambda (\cdot) \).

Let us recall some of such results using the following intensity functions

\[
\lambda_{j,n} (\vartheta_0, t) = an \left| t - \tau_j (\vartheta_0) \right|^\kappa \mathbb{1}_{\{t \geq \tau_j (\vartheta_0)\}} + n\lambda_0, \quad 0 \leq t \leq T.
\]

We assume that \( a > 0, \lambda_0 > 0 \) are known, the set \( \Theta \) is such that for all \( \vartheta \in \Theta \) the instants \( \tau_j (\vartheta) \in (0, T) \) (Fig. 2).

(a) Smooth case 1 Suppose that \( \kappa > \frac{1}{2} \), then the problem of parameter estimation is regular, the estimators are asymptotically normal and

\[
E_{\vartheta_0} \left\| \hat{\vartheta}_n - \vartheta_0 \right\|^2 = C \frac{1}{n} (1 + o (1)), \quad \gamma = 1.
\]
Fig. 2 Examples of signals with intensity functions (3): a $\kappa = \frac{5}{8}$, b $\kappa = \frac{1}{2}$, c $\kappa = \frac{1}{8}$, d $\kappa = 0$, e $\kappa = -\frac{3}{8}$

(b) Smooth case 2 If $\kappa = \frac{1}{2}$, then

$$E_{\tilde{\vartheta}_0} \left\| \tilde{\vartheta}_n - \vartheta_0 \right\|^2 = \frac{C}{n \ln n} \left( 1 + o \left( 1 \right) \right).$$

(c) Cusp-type case This case is intermediate between the smooth ($\kappa \geq \frac{1}{2}$) and change-point ($\kappa = 0$) cases. Suppose that $\kappa \in \left( 0, \frac{1}{2} \right)$. Then

$$E_{\tilde{\vartheta}_0} \left\| \tilde{\vartheta}_n - \vartheta_0 \right\|^2 = \frac{C}{n^{\frac{2}{2\kappa+1}}} \left( 1 + o \left( 1 \right) \right), \quad \gamma = \frac{2}{2\kappa + 1}.$$

(d) Change-point case Suppose that $\kappa = 0$. Then

$$E_{\tilde{\vartheta}_0} \left\| \tilde{\vartheta}_n - \vartheta_0 \right\|^2 = \frac{C}{n^2} \left( 1 + o \left( 1 \right) \right), \quad \gamma = 2.$$

(e) Explosion case Suppose that $\kappa \in (-1, 0)$. Then

$$E_{\tilde{\vartheta}_0} \left\| \tilde{\vartheta}_n - \vartheta_0 \right\|^2 = \frac{C}{n^{\frac{2}{2\kappa+1}}} \left( 1 + o \left( 1 \right) \right), \quad \gamma = \frac{2}{\kappa + 1}.$$
The smooth case (a) is studied in this paper. See also the work (Baidoo-Williams et al. 2015), where a similar model was considered. The case (b) is discussed below in Sect. 4. For cusp-type case (c) see Dachian (2003), Chernoyarov et al. (2018). The change-point case (d) is studied in Farinetto et al. (2018). Explosion case with slightly different intensity function was addressed in Dachian (2011) and the same technique also applies to the particular function in (e).

2 Main result

Suppose that there exists a source at some point \( \vartheta_0 = (x_0, y_0) \in \Theta \subset R^2 \) and \( k \geq 3 \) sensors (detectors) are located at the points \( \vartheta_j = (x_j, y_j), \ j = 1, \ldots, k \) on the same plane. The source is activated at the (known) instant \( t = 0 \) and the signals from the source (inhomogeneous Poisson processes) are registered by all \( k \) detectors. The signal arrives at the \( j \)-th detector at the instant \( \tau_j = \tau_j(\vartheta_0) \), which is the time necessary for the signal to arrive at the \( j \)-th detector:

\[
\tau_j(\vartheta_0) = v^{-1} \| \vartheta_j - \vartheta_0 \|,
\]

where \( v > 0 \) is the known speed of propagation of the signal and \( \| \cdot \| \) is the Euclidean norm (distance) in \( R^2 \).

The intensity function of the Poisson process \( X^n_j = \{ X_j(t), 0 \leq t \leq T \} \) registered by the \( j \)-th detector is

\[
\lambda_{j,n}(\vartheta, t) = n\lambda_j(t - \tau_j) + n\lambda_0, \quad 0 \leq t \leq T.
\]

Here \( n\lambda_j(t - \tau_j) \) is the intensity function of the signal and \( n\lambda_0 > 0 \) is the intensity of the noise. For simplicity of the presentation we assume that the noise level in all detectors is the same.

We introduce the following notations

\[
\alpha_j = \inf_{\vartheta \in \Theta} \tau_j(\vartheta), \quad \beta_j = \sup_{\vartheta \in \Theta} \tau_j(\vartheta), \quad j = 1, \ldots, k,
\]

\[
J_j(\vartheta) = \frac{1}{v^2 \| \vartheta_j - \vartheta \|^2} \int_{\tau_j(\vartheta)}^T \frac{\lambda_j'(t - \tau_j(\vartheta))^2}{\lambda_j(t - \tau_j(\vartheta)) + \lambda_0} \, dt,
\]

\[
\langle a, b \rangle_\vartheta = \sum_{j=1}^k a_j b_j J_j(\vartheta), \quad \| a \|_\vartheta^2 = \sum_{j=1}^k a_j^2 J_j(\vartheta).
\]

Recall that \( \lambda_j'(t - \tau_j) = 0 \) for \( 0 \leq t \leq \tau_j \) and note that \( \langle a, b \rangle_\vartheta \) and \( \| a \|_\vartheta \) are formally the scalar product and the norm in \( R^k \) of the vectors \( a = (a_1, \ldots, a_k)^T \), \( b = (b_1, \ldots, b_k)^T \) with weights \( \rho_j = J_j(\vartheta) \), but both depend on \( \vartheta \) in a very special way. The Fisher information matrix \( I_n(\vartheta) = nI(\vartheta) \), where \( \vartheta = (x, y) \) and

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\[ \mathbb{I}(\vartheta) = \begin{pmatrix} \|x - x_0\|_\vartheta^2, & \langle (x - x_0), (y - y_0) \rangle_\vartheta, \\ \langle (x - x_0), (y - y_0) \rangle_\vartheta, & \|y - y_0\|_\vartheta^2 \end{pmatrix}. \]

Here \( x = (x_1, \ldots, x_k), y = (y_1, \ldots, y_k) \) and \( x_0 = (x_0, \ldots, x_0) \) etc.

Further, we suppose that \( \beta_j < T \) and that the functions \( \lambda_j(t), j = 1, \ldots, k \) are defined on the sets \( T_j = [-\beta_j, T - \alpha_j] \).

Regularity conditions \( \mathcal{R} \).

\( \mathcal{R}_1. \) For all \( j = 1, \ldots, k \) the functions satisfy
\[
\lambda_j(t) = 0, \quad t \in [-\beta_j, 0], \quad \text{and} \quad \lambda_j(t) > 0, \quad t \in (0, T - \alpha_j].
\]

\( \mathcal{R}_2. \) The functions \( \lambda_j(t), t \in T_j, j = 1, \ldots, k \) have two continuous derivatives \( \lambda_j'(\cdot) \) and \( \lambda_j''(\cdot) \).

\( \mathcal{R}_3. \) The Fisher information matrix is uniformly non degenerate
\[
\kappa_1 = \inf_{\vartheta \in \Theta} \inf_{|e|=1} e^T \mathbb{I}(\vartheta) e > 0.
\]

\( \mathcal{R}_4. \) There are at least three detectors which are not on the same line.

Note that if all detectors are located on the same line, then the consistent identification is impossible by the following reason. Consider two points located symmetrically with respect to this line. Then the time needed for the signals to reach detectors from these two different sources coincide and therefore identification of the position of source is impossible.

According to Lemma 1 below the family of measures \( \left( P^{(n)}_{\vartheta}, \vartheta \in \Theta \right) \) induced by the Poisson processes \( X^n = (X^n_1, \ldots, X^n_k) \) on the space of their realizations is locally asymptotically normal and therefore we have the following minimax Hajek-Le Cam’s lower bound for the mean square error of any estimator \( \vartheta_n \): for any \( \vartheta_0 \in \Theta \)
\[
\lim_{\delta \to 0} \lim_{n \to \infty} \sup_{\|\vartheta - \vartheta_0\| \leq \delta} n E_{\vartheta} \| \vartheta_n - \vartheta \|^2 \geq E_{\vartheta_0} \| \xi \|^2, \quad \xi \sim \mathcal{N}(0, \mathbb{I}(\vartheta_0)^{-1}).
\]

We call the estimator \( \vartheta_n \) asymptotically efficient, if for all \( \vartheta_0 \in \Theta \) we have the equality
\[
\lim_{\delta \to 0} \lim_{n \to \infty} \sup_{\|\vartheta - \vartheta_0\| \leq \delta} n E_{\vartheta} \| \vartheta_n - \vartheta \|^2 = E_{\vartheta_0} \| \xi \|^2.
\]

For the proof of this bound see, e.g., Ibragimov and Khasminskii (1981, Theorem 2.12.1).

**Theorem 1** Let the conditions \( \mathcal{R} \) be fulfilled, then the MLE \( \hat{\vartheta}_n \) and BE \( \tilde{\vartheta}_n \) are uniformly consistent, asymptotically normal
\[
\sqrt{n} \left( \hat{\vartheta}_n - \vartheta_0 \right) \to \mathcal{N} \left( 0, \mathbb{I}(\vartheta_0)^{-1} \right), \quad \sqrt{n} \left( \tilde{\vartheta}_n - \vartheta_0 \right) \to \mathcal{N} \left( 0, \mathbb{I}(\vartheta_0)^{-1} \right),
\]
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and for any $p > 0$

$$\lim_{n \to \infty} n^{p} E_{\vartheta_0} \left\| \hat{\vartheta}_n - \vartheta_0 \right\|^p = E_{\vartheta_0} \left\| \zeta \right\|^p, \quad \lim_{n \to \infty} n^{p} E_{\vartheta_0} \left\| \tilde{\vartheta}_n - \vartheta_0 \right\|^p = E_{\vartheta_0} \left\| \zeta \right\|^p,$$

where $\zeta \sim \mathcal{N} \left(0, \mathbb{I} (\vartheta_0)^{-1}\right)$ and both estimators are asymptotically efficient.

**Proof** The proof of this theorem is based on two general results by Ibragimov and Khasminskii (1981), given in the Theorems 1.10.1 and 1.10.2. The conditions of these theorems are formulated in terms of normalized likelihood ratio

$$Z_n (u) = \frac{L \left( \vartheta_0 + \frac{u}{\sqrt{n}}, X^n \right)}{L \left( \vartheta_0, X^n \right)}, \quad u \in \mathbb{U}_n = \left\{ u : \vartheta_0 + \frac{u}{\sqrt{n}} \in \Theta \right\}.$$

Introduce the limit likelihood ratio

$$Z (u) = \exp \left\{ \langle u, \Delta (\vartheta) \rangle - \frac{1}{2} u^T \mathbb{I} (\vartheta_0) u \right\}, \quad u \in \mathcal{R}^2.$$

Here $\Delta (\vartheta_0) \sim \mathcal{N} \left(0, \mathbb{I} (\vartheta_0)\right)$.

Suppose that we already proved the weak convergence

$$Z_n (\cdot) \Longrightarrow Z (\cdot).$$

Then the limit distributions of the above estimators are obtained by a change of variable $\tilde{\vartheta} = \vartheta_0 + \frac{u}{\sqrt{n}}$, see Ibragimov and Khasminskii (1981). Let $\mathcal{B} \subset \mathcal{R}^2$ be a bounded Borel set. Then for the MLE we have

$$P_{\vartheta_0} \left( \sqrt{n} (\hat{\vartheta}_n - \vartheta_0) \in \mathcal{B} \right)$$

$$= P_{\vartheta_0} \left\{ \sup_{\sqrt{n}(\vartheta-\vartheta_0) \in \mathcal{B}} L \left( \vartheta, X^T \right) > \sup_{\sqrt{n}(\vartheta-\vartheta_0) \in \mathcal{B}^c} L \left( \vartheta, X^T \right) \right\}$$

$$= P_{\vartheta_0} \left\{ \sup_{u \in \mathcal{B}, u \in \mathbb{U}_n} Z_n (u) > \sup_{u \in \mathcal{B}^c, u \in \mathbb{U}_n} Z_n (u) \right\}$$

$$\longrightarrow P_{\vartheta_0} \left\{ \sup_{u \in \mathcal{B}} Z (u) > \sup_{u \in \mathcal{B}^c} Z (u) \right\} = P_{\vartheta_0} (\zeta \in \mathcal{B}).$$

It is easy to see that $\zeta = \arg \max_u Z (u) \sim \mathcal{N} \left(0, \mathbb{I} (\vartheta_0)^{-1}\right)$.

Similarly, for the BE, let $\theta_u = \vartheta_0 + \frac{u}{\sqrt{n}}$, we obtain:
\[ \tilde{\vartheta}_n = \frac{\int_{\Theta} \theta p(\theta) L(\theta, X^T) d\theta}{\int_{\Theta} p(\theta) L(\theta, X^T) d\theta} = \vartheta_0 + \frac{1}{\sqrt{n}} \frac{\int_{\mathbb{U}_n} u p(\theta_u) L(\theta_u, X^T) du}{\int_{\mathbb{U}_n} p(\theta_u) L(\theta_u, X^T) du} \]

Hence
\[ \sqrt{n} \left( \tilde{\vartheta}_n - \vartheta_0 \right) = \frac{\int_{\mathbb{U}_n} u p(\theta_u) Z_n(u) du}{\int_{\mathbb{U}_n} p(\theta_u) Z_n(u) du} \Rightarrow \frac{\int_{\mathcal{R}^2} u Z(u) du}{\int_{\mathcal{R}^2} Z(u) du} = \zeta. \]

Recall that \( p(\theta_u) \to p(\vartheta_0) > 0 \) and note that
\[ \int_{\mathcal{R}^2} u Z(u) du = \zeta \int_{\mathcal{R}^2} Z(u) du. \]

Recall that the equality of limit distributions of the MLE and BE for a wide class of symmetric loss functions in regular statistical experiments follows from the general results [see details in the proofs of the Theorems 3.1.1 and 3.2.1 in Ibragimov and Khasminskii (1981)].

The properties of \( Z_n(u) \) required in Theorems 1.10.1 and 1.10.2 Ibragimov and Khasminskii (1981) are checked in the three lemmas below. This approach to the study of asymptotic properties of estimators was taken in Kutoyants (1979, 1998). Here we use some inequalities, obtained therein.

Introduce the vector of partial derivatives
\[ \Delta_n(\vartheta_0, X^n) = \frac{1}{\sqrt{n}} \left( \frac{\partial \ln L(\vartheta_0, X^n)}{\partial x_0}, \frac{\partial \ln L(\vartheta_0, X^n)}{\partial y_0} \right)^T. \]

The convergence of finite-dimensional distributions of the random field \( Z_n(u), u \in \mathbb{U}_n \) to the finite-dimensional distributions of the limit random field \( Z(u), u \in \mathcal{R}^2 \) follows from Lemma 1 below.

**Lemma 1** Let the conditions \( R_1 - R_3 \) be fulfilled, then the family of measures \( \{P_\vartheta, \vartheta \in \Theta\} \) is locally asymptotically normal (LAN), i.e., the random process \( Z_n(u), u \in \mathbb{U}_n \) for any \( \vartheta_0 \in \Theta \) admits the representation
\[ Z_n(u) = \exp \left\{ (u, \Delta_n(\vartheta_0, X^n)) - \frac{1}{2} u^T \Pi(\vartheta_0) u + r_n \right\}, \quad u \in \mathbb{U}_n, \quad (4) \]

where the vector
\[ \Delta_n(\vartheta_0, X^n) \Rightarrow \Delta(\vartheta_0) \sim \mathcal{N}(0, \Pi(\vartheta_0)) \quad (5) \]

and \( r_n \to 0 \) in probability.
Proof. Let us denote \( \lambda_j(t, u) = \lambda_j(t - \tau_j(\theta_u)) \) and \( d\pi_{j, n}(t) = dX_j(t) - n [\lambda_j(t - \tau_j(\theta_0)) + \lambda_0] \, dt \). Then we can write

\[
\ln Z_n(u) = \sum_{j=1}^{k} \int_0^T \ln \left( \frac{\lambda_j(t, u) + \lambda_0}{\lambda_j(t, 0) + \lambda_0} \right) \, d\pi_{j, n}(t) \\
- n \sum_{j=1}^{k} \int_0^T \left[ \frac{\lambda_j(t, u) + \lambda_0}{\lambda_j(t, 0) + \lambda_0} - 1 - \ln \left( \frac{\lambda_j(t, u) + \lambda_0}{\lambda_j(t, 0) + \lambda_0} \right) \right] \lambda_j(t, 0) + \lambda_0 \, dt.
\]

Using the Taylor formula we obtain the relations

\[
\tau_j \left( \theta_0 + \frac{u}{\sqrt{n}} \right) = \tau_j(\theta_0) - \frac{1}{\sqrt{n}} \langle m_j, u \rangle + O \left( \frac{1}{n} \right),
\]

\[
m_j = \left( \frac{x_j - x_0}{\rho_j}, \frac{y_j - y_0}{\rho_j} \right), \quad \|m_j\| = 1,
\]

\[
\lambda_j(t - \tau_j(\theta_0 + n^{-1/2}u)) - \lambda_j(t - \tau_j(\theta_0)) = -n^{-1/2} \lambda_j'(t - \tau_j(\theta_0)) \left( u, \frac{\partial \tau(\theta_0)}{\partial \vartheta} \right) + n^{-1} O \left( \|u\|^2 \right)
\]

\[
= -n^{-1/2} \sqrt{n} \lambda_j'(t - \tau_j(\theta_0)) \|m_j, u\| + n^{-1} O \left( \|u\|^2 \right),
\]

\[
\ln \left( \frac{\lambda_j(t, u) + \lambda_0}{\lambda_j(t, 0) + \lambda_0} \right) = \frac{\lambda_j'(t - \tau_j(\theta_0))}{\sqrt{n} [\lambda_j(t - \tau_j(\theta_0)) + \lambda_0]} \left( u, \frac{\partial \tau(\theta_0)}{\partial \vartheta} \right) + O \left( \frac{1}{n} \right)
\]

\[
= \frac{\lambda_j'(t - \tau_j(\theta_0))}{\sqrt{n} [\lambda_j(t - \tau_j(\theta_0)) + \lambda_0]} \langle m_j, u \rangle + O \left( \frac{1}{n} \right),
\]

\[
\frac{\lambda_j(t, u) + \lambda_0}{\lambda_j(t, 0) + \lambda_0} - 1 - \ln \left( \frac{\lambda_j(t, u) + \lambda_0}{\lambda_j(t, 0) + \lambda_0} \right)
\]

\[
= \frac{1}{2n} \lambda_j'(t - \tau_j(\theta_0))^2 \left( u, \frac{\partial \tau(\theta_0)}{\partial \vartheta} \right)^2 + O \left( \frac{1}{n^{3/2}} \right)
\]

\[
= \frac{1}{2n} \lambda_j'(t - \tau_j(\theta_0))^2 \langle m_j, u \rangle + O \left( \frac{1}{n^{3/2}} \right).
\]

Note that

\[
\frac{\partial \tau_j(\theta_0)}{\partial x_0} = - \frac{x_j - x_0}{v \| \theta_j - \theta_0 \|}, \quad \frac{\partial \tau_j(\theta_0)}{\partial y_0} = - \frac{y_j - y_0}{v \| \theta_j - \theta_0 \|}.
\]

Therefore we can write

\[
\frac{\partial \ln L(\theta_0, X^n)}{\partial x_0} = \sum_{j=1}^{k} \frac{1}{v \| \theta_j - \theta_0 \|} \int_{\tau_j(\theta_0)}^{T} \frac{\lambda_j'(t - \tau_j(\theta_0))}{\lambda_j(t - \tau_j(\theta_0)) + \lambda_0} \, d\pi_{j, n}(t).
\]
Hence

\[
E_{\vartheta_0} \left[ \frac{\partial \ln L(\vartheta_0, X^n)}{\partial x_0} \right]^2 = n \sum_{j=1}^{k} \frac{(x_j - x_0)^2}{\nu^2 \| \vartheta_j - \vartheta_0 \|} \int_{\tau_j(\vartheta_0)}^{T} \frac{\lambda_j'(t - \tau_j(\vartheta_0))^2}{\lambda_j(t - \tau_j(\vartheta_0)) + \lambda_0} \, dt
\]

and

\[
E_{\vartheta_0} \left[ \frac{\partial \ln L(\vartheta_0, X^n)}{\partial x_0} \frac{\partial \ln L(\vartheta_0, X^n)}{\partial y_0} \right] = n \sum_{j=1}^{k} (x_j - x_0) (y_j - y_0) J_j(\vartheta_0).
\]

These equalities justify the form of the Fisher information matrix \( I(\vartheta_0) \), introduced above.

We have the representations

\[
\sum_{j=1}^{k} \int_{0}^{T} \ln \left( \frac{\lambda_j(t, u) + \lambda_0}{\lambda_j(t, 0) + \lambda_0} \right) d\pi_{j,n}(t)
\]

\[
= \frac{1}{\nu \sqrt{n}} \sum_{j=1}^{k} \langle m_j, u \rangle \int_{\tau_j(\vartheta_0)}^{T} \frac{\lambda_j'(t - \tau_j(\vartheta_0))}{\lambda_j(t - \tau_j(\vartheta_0)) + \lambda_0} \, d\pi_{j,n}(t) + o(1),
\]

\[
= \langle u, \Delta_n(\vartheta_0, X^n) \rangle + o(1),
\]

and

\[
\sum_{j=1}^{k} \int_{0}^{T} \left[ \frac{\lambda_j(t, u) + \lambda_0}{\lambda_j(t, 0) + \lambda_0} - 1 - \ln \left( \frac{\lambda_j(t, u) + \lambda_0}{\lambda_j(t, 0) + \lambda_0} \right) \right] \left[ \frac{\lambda_j(t, 0) + \lambda_0}{\lambda_j(t, 0) + \lambda_0} \right] dt
\]

\[
= \frac{1}{2\nu^2} \sum_{j=1}^{k} \langle m_j, u \rangle^2 \int_{\tau_j(\vartheta_0)}^{T} \frac{\lambda_j'(t - \tau_j(\vartheta_0))^2}{\lambda_j(t - \tau_j(\vartheta_0)) + \lambda_0} \, dt + o(1)
\]

\[
= \frac{1}{2} u^T I(\vartheta_0) u + o(1),
\]

which imply \((4)\). To verify the convergence \((5)\) we introduce the vector \( I_n = (I_{1,n}, I_{2,n}) \), where

\[
I_{1,n} = \frac{1}{\sqrt{n}} \sum_{j=1}^{k} a_j \int_{\tau_j(\vartheta_0)}^{T} \frac{\lambda_j'(t - \tau_j(\vartheta_0))}{\lambda_j(t - \tau_j(\vartheta_0)) + \lambda_0} \, d\pi_{j,n}(t),
\]

\[
I_{2,n} = \frac{1}{\sqrt{n}} \sum_{j=1}^{k} b_j \int_{\tau_j(\vartheta_0)}^{T} \frac{\lambda_j'(t - \tau_j(\vartheta_0))}{\lambda_j(t - \tau_j(\vartheta_0)) + \lambda_0} \, d\pi_{j,n}(t),
\]
with vectors $a, b \in \mathbb{R}^k$. Then the asymptotic normality of $I_n$ follows from the central limit theorem for stochastic integrals, see, e.g., Theorem 1.1 in Kutoyants (1998):

$$I_{1,n} \Longrightarrow \sum_{j=1}^k a_j \int_{\tau_j(\theta_0)}^T \frac{\lambda_j'(t - \tau_j(\theta_0))}{\lambda_j(t - \tau_j(\theta_0))} + \lambda_0 \, dW_j(\Lambda(\theta_0, t)),$$

where $W_j(\cdot), j = 1, \ldots, k$ are independent Wiener processes. The conditions of this theorem can be readily verified using the expression for $\Delta_1(n, X_n)$.

Lemma 2 Let the conditions $R_1 - R_3$ be fulfilled, then there exists a constant $C > 0$, which does not depend on $n$ such that for any $R > 0$

$$\sup_{\theta_0 \in \Theta} \sup_{\|u_1 - u_2\|} \left| u_1 - u_2 \right|^4 \mathbb{E}_{\theta_0} \left| Z_{j,n}^1(u_1) - Z_{j,n}^1(u_2) \right|^4 \leq C \left( 1 + R^2 \right). \quad (6)$$

Proof The proof of this lemma follows from the proof of Lemma 2.2 in Kutoyants (1998) if we put there $m = 2$.

Lemma 3 Let the conditions $R$ be fulfilled, then there exists a constant $\kappa > 0$, which does not depend on $n$ such that

$$\sup_{\theta_0 \in \Theta} \mathbb{E}_{\theta_0} Z_{j,n}^2(u) \leq e^{-\kappa \|u\|^2}. \quad (7)$$

Proof Let us denote $\theta_u = \theta_0 + \frac{u}{\sqrt{n}}$ and set

$$Z_{j,n}(u) = \exp \left\{ \int_0^T \ln \left( \frac{\lambda_{j,n}(\theta_u, t)}{\lambda_{j,n}(\theta_0, t)} \right) dX_j(t) - \int_0^T \left[ \lambda_{j,n}(\theta_u, t) - \lambda_{j,n}(\theta_0, t) \right] dt \right\}.$$

Recall that [see Lemma 2.2 in Kutoyants (1998)]

$$\mathbb{E}_{\theta_0} Z_{j,n}^2(u) = \exp \left\{ -\frac{1}{2} \int_0^T \left[ \sqrt{\lambda_{j,n}(\theta_u, t)} - \sqrt{\lambda_{j,n}(\theta_0, t)} \right]^2 dt \right\}. $$

Therefore we have the equality

$$\mathbb{E}_{\theta_0} Z_{n}^2(u) = \prod_{j=1}^k \mathbb{E}_{\theta_0} Z_{j,n}^2(u)$$

$$= \exp \left\{ -\frac{1}{2} \sum_{j=1}^k \int_0^T \left[ \sqrt{\lambda_{j,n}(\theta_u, t)} - \sqrt{\lambda_{j,n}(\theta_0, t)} \right]^2 dt \right\}. \quad (8)$$
By the Taylor formula, for $\|h\| \leq \delta$, we can write
\[
\sum_{j=1}^{k} \int_{0}^{T} \left[ \sqrt{\lambda_{j,n}(\vartheta_{0} + h, t)} - \sqrt{\lambda_{j,n}(\vartheta_{0}, t)} \right]^2 dt = \frac{n}{4} h^T \Pi(\vartheta_{0}) h \left( 1 + O(\delta) \right).
\]

Hence we can choose a small $\delta > 0$ such that for $\|u\|/\sqrt{n} \leq \delta$
\[
\sum_{j=1}^{k} \int_{0}^{T} \left[ \sqrt{\lambda_{j,n}(\vartheta_{0} + u/\sqrt{n}, t)} - \sqrt{\lambda_{j,n}(\vartheta_{0}, t)} \right]^2 dt \geq \frac{1}{8} u^T \Pi(\vartheta_{0}) u \geq \frac{\kappa_1}{8} \|u\|^2,
\]
where $\kappa_1 > 0$ is from condition $\mathcal{R}_3$.

Let us denote
\[
g(\delta) = \frac{1}{n} \inf_{\vartheta_0 \in \Theta} \inf_{\|\vartheta - \vartheta_0\| \geq \delta} \frac{1}{k} \sum_{j=1}^{k} \int_{0}^{T} \left[ \sqrt{\lambda_{j,n}(\vartheta, t)} - \sqrt{\lambda_{j,n}(\vartheta_{0}, t)} \right]^2 dt,
\]
and show that $g(\delta) > 0$. Note that $g(\delta)$ does not depend on $n$. Indeed,
\[
\frac{1}{n} \int_{0}^{T} \left[ \sqrt{\lambda_{j,n}(\vartheta, t)} - \sqrt{\lambda_{j,n}(\vartheta_{0}, t)} \right]^2 dt \\
= \frac{1}{n} \int_{0}^{T} \frac{\left[ \lambda_{j,n}(\vartheta, t) - \lambda_{j,n}(\vartheta_{0}, t) \right]^2}{\left[ \sqrt{\lambda_{j,n}(\vartheta, t)} + \sqrt{\lambda_{j,n}(\vartheta_{0}, t)} \right]^2} dt \\
\geq \frac{1}{4(\lambda_M + \lambda_{0})} \int_{0}^{T} \left[ \lambda_j(t - \tau_j(\vartheta)) - \lambda_j(t - \tau_j(\vartheta_0)) \right]^2 dt.
\]

Therefore if we assume that $g(\delta) = 0$, there exists at least one point $\vartheta^* \in \Theta$ such that $\|\vartheta^* - \vartheta_0\| \geq \delta$ and for all $j = 1, \ldots, k$ we have
\[
\int_{0}^{T} \left[ \lambda_j(t - \tau_j(\vartheta^*)) - \lambda_j(t - \tau_j(\vartheta_0)) \right]^2 dt = 0.
\]

Note that by condition $\mathcal{R}_1$ we get consistent estimation of all “delays” $\tau_j$. Indeed, the identifiability condition
\[
g_j(\delta) \equiv \inf_{|\tau - \tau_0| \geq \delta} \int_{0}^{T} \left[ \lambda_j(t - \tau) - \lambda_j(t - \tau_0) \right]^2 dt > 0
\]
is fulfilled for all $j = 1, \ldots, k$ and $\delta > 0$. If for some $j$ and $\delta > 0$ we have $g_j(\delta) = 0$, then this implies that there exists $\tau_* \neq \tau_0$ such that the equality $\lambda_j(t - \tau_*) =
\( \lambda_j(t - \tau_0) \) holds for all \( t \in [0, T] \). This equality is impossible by the following reason. Suppose that \( \tau_0 < \tau_\ast \), then on the interval \((\tau_0, \tau_\ast)\) the function \( \lambda_j(t - \tau_0) > 0 \) and the function \( \lambda_j(t - \tau_\ast) = 0 \).

Therefore condition (10) is always fulfilled, which verifies consistency of the MLE \( \hat{\tau}_{j,n} \) and BE \( \tilde{\tau}_{j,n} \)

\[
\hat{\tau}_{j,n} \rightarrow \tau_j(\vartheta_0), \quad \tilde{\tau}_{j,n} \rightarrow \tau_j(\vartheta_0), \quad j = 1, \ldots, k
\]
as well as their asymptotic normality. For the proofs see Kutoyants (1998, Theorem 2.8).

If \( g(\delta) = 0 \), then there exist two points \( \vartheta^\ast \) and \( \vartheta_0 \), such that the two sets \((\tau_j(\vartheta^\ast), j = 1, \ldots, k)\) and \((\tau_j(\vartheta_0), j = 1, \ldots, k)\) are equal, i.e., the distances \( \|\vartheta_j - \vartheta^\ast\| \) and \( \|\vartheta_j - \vartheta_0\| \) for all \( j = 1, \ldots, k \) coincide. This is impossible, however, due to the geometric properties of the set of points \( \vartheta_1, \ldots, \vartheta_k \), satisfying the condition \( \mathcal{A}_4 \).

Hence, \( g(\delta) > 0 \) and, for \( \frac{\|u\|}{\sqrt{n}} > \delta \), we can write

\[
\sum_{j=1}^{k} \int_{0}^{T} \left( \sqrt{\lambda_{j,n}(\vartheta_0 + \frac{u}{\sqrt{n}}, t)} - \sqrt{\lambda_{j,n}(\vartheta_0, t)} \right)^2 dt \geq n g(\delta) \geq g(\delta) \frac{\|u\|^2}{D^2},
\]

where \( D = \sup_{\vartheta_1, \vartheta_2 \in \Theta} \|\vartheta_1 - \vartheta_2\| \).

Let us denote \( \kappa_2 = D^{-2} g(\delta) \) and define \( \kappa = \min\left( \frac{\kappa_1}{16}, \frac{\kappa_2}{2} \right) \), then the estimate (7) follows from (8), (9) and (11).

The properties of the likelihood ratio field \( Z_n(\cdot) \), established in the lemmas 1-3, form the sufficient conditions in Theorems 1.10.1 and 1.10.2 in Ibragimov and Khasminskii (1981). Therefore the MLE \( \hat{\vartheta}_n \) and the BE \( \tilde{\vartheta}_n \) satisfy all the properties claimed in Theorem 1.

\[ \square \]

### 3 Simple consistent estimator

Finding MLE by solving (1) can be computationally challenging. In the case of three detectors, \( k = 3 \), a simple estimator was proposed by Pu (2009).

For any \( k \geq 3 \), we construct can construct a practical estimator in two steps. First, we solve \( k \) one dimensional problems of estimation of arrival times \( \tau_j, j = 1, \ldots, k \) and then with \( k \) estimators \( \hat{\tau}_{1,n}, \ldots, \hat{\tau}_{k,n} \) at hand, we estimate the parameter \( \vartheta_0 \).

We have \( k \) inhomogeneous Poisson processes \( X^n = (X^n_1, \ldots, X^n_k) \), where \( X^n_j = (X_j(t), 0 \leq t \leq T) \) is a Poisson process with the intensity function

\[
\lambda_{j,n}(\tau_j, t) = n\lambda_j(t - \tau_j) + n\lambda_0, \quad 0 \leq t \leq T.
\]
Therefore we have \( k \) likelihood ratios
\[
L(\tau_j, X^n_j) = \exp \left\{ \int_{\tau_j}^T \ln \left( 1 + \frac{\lambda_j (t - \tau_j)}{\lambda_0} \right) dX_j(t) - n \int_{\tau_j}^T \lambda_j (t - \tau_j) dt \right\}
\]
and can introduce \( k \) MLEs \( \hat{\tau}_{j,n} \):
\[
L \left( \hat{\tau}_{j,n}, X^n_j \right) = \sup_{\tau_j \in \Theta_j} L \left( \tau_j, X^n_j \right), \quad j = 1, \ldots, k.
\]
The true value of \( \tau = (\tau_1, \ldots, \tau_k) \) is \( \tau_0 = (\tau_1(\vartheta_0), \ldots, \tau_k(\vartheta_0)) \). Let us denote the vector MLE \( \hat{\tau}_n = (\hat{\tau}_{1,n}, \ldots, \hat{\tau}_{k,n}) \) and introduce the corresponding Fisher information matrix \( \mathbb{I}_\tau (\vartheta_0) = (\mathbb{I}_\tau (\vartheta_0)_{j,i})_{j,i=1,\ldots,k} \)
\[
\mathbb{I}_\tau (\vartheta_0)_{j,i} = \int_{\tau_j}^T \frac{\lambda'(t - \tau_j)^2}{\lambda(t - \tau_j) + \lambda_0} dt \delta_{i,j}
\]
where \( \delta_{j,i} = 1_{\{j=i\}} \), i.e., this matrix is diagonal. Then we have the following result

**Theorem 2** Let the conditions \( \mathcal{R} \) be fulfilled, then the MLE \( \hat{\tau}_n \) is consistent, asymptotically normal
\[
\sqrt{n} \left( \hat{\tau}_n - \tau_0 \right) \implies \xi \sim N \left( 0, \mathbb{I}_\tau (\vartheta_0)^{-1} \right),
\]
we have the convergence of polynomial moments and this MLE is asymptotically efficient.

**Proof** The proof of this theorem is similar to that of Theorem 2.4 in Kutoyants (1998) and, in fact, the BE \( \hat{\tau}_n \) has the same properties.

Suppose that this vector MLE \( \hat{\tau}_n \) produces estimates close to the true values of the corresponding parameters. Our goal is to construct an estimator of the position \( \vartheta_0 = (\vartheta_0, \gamma_0) \). We propose the linear system which gives us an estimator of \( \vartheta_0 \) as follows. To this end, we have
\[
z_{j,n} := v^2 \hat{\tau}_{j,n}^2 = (x_j - \hat{x}_0)^2 + (y_j - \hat{y}_0)^2 = x_j^2 + y_j^2 + \hat{x}_0^2 + \hat{y}_0^2 - 2x_j \hat{x}_0 - 2y_j \hat{y}_0 = r_j^2 + \hat{\tau}_0^2 - 2x_j \hat{x}_0 - 2y_j \hat{y}_0 = r_j^2 - 2x_j \hat{\gamma}_1 - 2y_j \hat{\gamma}_2 + \hat{\gamma}_3,
\]
where we denoted \( \hat{\gamma}_1 := \hat{x}_0, \hat{\gamma}_2 := \hat{y}_0, \hat{\gamma}_3 := \|\hat{\vartheta}_0\|^2, r_j^2 := x_j^2 + y_j^2, \hat{\tau}_0^2 := \hat{x}_0^2 + \hat{y}_0^2 \).

Consider the problem of estimation of the vector \( \gamma_0 = (\gamma_{0,1}, \gamma_{0,2}, \gamma_{0,3}) \) from the observations
\[
z_{j,n} = r_j^2 - 2x_j \gamma_{0,1} - 2y_j \gamma_{0,2} + \gamma_{0,3} + \varepsilon_{j,n}, \quad j = 1, \ldots, k.
\]
Here $\varepsilon_{j,n}$ is the noise process and the proposed model is an approximation of our model. Using the method of least squares

$$\frac{\partial}{\partial \gamma_l} \sum_{j=1}^{k} \left[ z_{j,n} - r_j^2 + 2x_j \gamma_1 + 2y_j \gamma_2 - \gamma_3 \right]^2 = 0, \quad l = 1, 2, 3,$$

we obtain the system of equations

$$-2 \sum_{j=1}^{k} x_j \gamma_{1,n}^* - 2 \sum_{j=1}^{k} y_j \gamma_{2,n}^* + k \gamma_{3,n}^* = \sum_{j=1}^{k} \left( z_{j,n} - r_j^2 \right),$$

$$-2 \sum_{j=1}^{k} x_j^2 \gamma_{1,n}^* - 2 \sum_{j=1}^{k} x_j y_j \gamma_{2,n}^* + \sum_{j=1}^{k} x_j \gamma_{3,n}^* = \sum_{j=1}^{k} \left( z_{j,n} - r_j^2 \right),$$

$$-2 \sum_{j=1}^{k} y_j x_j \gamma_{1,n}^* - 2 \sum_{j=1}^{k} y_j^2 \gamma_{2,n}^* + \sum_{j=1}^{k} y_j \gamma_{3,n}^* = \sum_{j=1}^{k} \left( z_{j,n} - r_j^2 \right),$$

which reduces to

$$A_n \gamma_n^* = Z_n,$$

where

$$A = \begin{pmatrix}
-2 \sum_{j=1}^{k} x_j, & -2 \sum_{j=1}^{k} y_j, & k,
-2 \sum_{j=1}^{k} x_j^2, & -2 \sum_{j=1}^{k} x_j y_j, & \sum_{j=1}^{k} x_j,
-2 \sum_{j=1}^{k} x_j y_j, & -2 \sum_{j=1}^{k} y_j^2, & \sum_{j=1}^{k} y_j
\end{pmatrix},$$

and $Z_n = \left( \sum_{j=1}^{k} \left( z_{j,n} - r_j^2 \right), \sum_{j=1}^{k} x_j \left( z_{j,n} - r_j^2 \right), \sum_{j=1}^{k} y_j \left( z_{j,n} - r_j^2 \right) \right)^T$.

We consider now the three dimensional parameter $\gamma = (\gamma_1, \gamma_2, \gamma_3)^T$ with independent components. Note that $\gamma_3^2 = \gamma_1^2 + \gamma_2^2$ holds, but we do not use it here. We assume that the matrix $A$ is non-degenerate.

Let us introduce the $k \times 3$ matrix $C = (c_{j,r})$

$$c_{j,1} = 2\nu^2 \tau_{0,j} \sigma_j, \quad c_{j,2} = 2\nu^2 x_j \tau_{0,j} \sigma_j, \quad c_{j,3} = 2\nu^2 y_j \tau_{0,j} \sigma_j, \quad j = 1, \ldots, k,$$

where

$$\sigma_j^2 = \left( \int_{t_j(\vartheta_0)}^{T_j(\vartheta_0)} \frac{\lambda_j^\prime \left( t - \tau_j(\vartheta_0) \right)^2}{\lambda_j \left( t - \tau_j(\vartheta_0) \right) + \lambda_0} \, dt \right)^{-1},$$

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and define
\[
\mathbb{D} (\vartheta_0) = A^{-1} A^T C A^{-1}.
\]

The properties of the estimator \( \gamma_n^* = A^{-1} Z_n \) are given in the following proposition.

**Proposition 1** Let the conditions \( \mathcal{R} \) be fulfilled and the matrix \( A \) be non-degenerate, then the estimator \( \gamma_n^* \) is consistent, asymptotically normal
\[
\sqrt{n} (\gamma_n^* - \gamma_0) \xrightarrow{D} N (0, \mathbb{D} (\vartheta_0)),
\]
where the moments converge as well.

**Proof** Since by Theorem 2, \( \hat{\tau}_{j,n} \) are asymptotically normal, we can write
\[
\hat{\tau}_{j,n} = \tau_{0,j} + n^{-1/2} \sigma_j \xi_{j,n} \quad \text{and} \quad \xi_{j,n} \xrightarrow{D} \xi_j.
\]
Recall that \( (\xi_1, \ldots, \xi_k) \) are i.i.d. \( N (0, 1) \). Therefore
\[
Z_{1,n} = \sum_{j=1}^{k} (z_{j,n} - r_j^2) = \sum_{j=1}^{k} \left( v^2 \hat{\tau}_{j,n}^2 - r_j^2 \right)
\]
\[
= \sum_{j=1}^{k} \left( v^2 \left( \tau_{0,j} + n^{-1/2} \sigma_j \xi_{j,n} \right)^2 - r_j^2 \right)
\]
\[
= \sum_{j=1}^{k} \left( v^2 \tau_{0,j}^2 - r_j^2 \right) + \frac{2v^2}{\sqrt{n}} \sum_{j=1}^{k} \tau_{0,j} \sigma_j \xi_{j,n} + O \left( \frac{1}{n} \right)
\]
\[
= Z_{1,0} + \frac{1}{\sqrt{n}} \sum_{j=1}^{k} c_{j,1} \xi_{j,n} + O \left( \frac{1}{n} \right),
\]
and
\[
Z_{2,n} = \sum_{j=1}^{k} x_j \left( v^2 \tau_{0,j}^2 - r_j^2 \right) + \frac{2v^2}{\sqrt{n}} \sum_{j=1}^{k} x_j \tau_{0,j} \sigma_j \xi_{j,n} + O \left( \frac{1}{n} \right)
\]
\[
= Z_{2,0} + \frac{1}{\sqrt{n}} \sum_{j=1}^{k} c_{j,2} \xi_{j,n} + O \left( \frac{1}{n} \right),
\]
\[
Z_{3,n} = \sum_{j=1}^{k} y_j \left( v^2 \tau_{0,j}^2 - r_j^2 \right) + \frac{2v^2}{\sqrt{n}} \sum_{j=1}^{k} y_j \tau_{0,j} \sigma_j \xi_{j,n} + O \left( \frac{1}{n} \right)
\]
\[
= Z_{3,0} + \frac{1}{\sqrt{n}} \sum_{j=1}^{k} c_{j,3} \xi_{j,n} + O \left( \frac{1}{n} \right).
\]
The limit covariance matrix is
\[
\mathbb{R}_{l,m} = \lim_{n \to \infty} n \mathbf{E}_{\vartheta_0} (Z_{l,n} - Z_{l,0}) (Z_{m,n} - Z_{m,0}) = \sum_{j=1}^{k} c_{j,l} c_{j,m}.
\]

We can write
\[
\sqrt{n} (\gamma_n^* - \gamma_0) = A^{-1} \sqrt{n} (Z_n - Z_0) \Rightarrow A^{-1} C^T \xi.
\]

The convergence of moments follow from that of the estimator \( \hat{\tau}_n \).

Let us denote \( \vartheta_n^* = (\gamma_{1,n}^*, \gamma_{2,n}^*) \), to be referred below as \textit{mean squared estimator} (MSE) of \( \vartheta \).

\( \blacksquare \)

Then Proposition 1 implies the following:

**Corollary 1** Let the conditions \( \mathcal{R} \) be fulfilled and the matrix \( A \) be non-degenerate, then the estimator \( \vartheta_n^* \) is consistent and asymptotically normal
\[
\sqrt{n} (\vartheta_n^* - \vartheta_0) \Rightarrow \mathcal{N} (0, \mathbb{M} (\vartheta_0)),
\]
where
\[
\mathbb{M} (\vartheta_0) = \begin{pmatrix}
\mathbb{D} (\vartheta_0)_{1,1} & \mathbb{D} (\vartheta_0)_{1,2} \\
\mathbb{D} (\vartheta_0)_{2,1} & \mathbb{D} (\vartheta_0)_{2,2}
\end{pmatrix}.
\]

To avoid large errors we can introduce the following condition
\[
S_n = \left| \gamma_{3,n}^2 - \gamma_{1,n}^2 - \gamma_{2,n}^2 \right| < n^{-1/4}. \tag{12}
\]

For the large values of \( n \) it has to be fulfilled because \( \gamma_1^2 + \gamma_2^2 - \gamma_3^2 = 0 \), we have \( \gamma_{3,n}^2 - \gamma_{1,n}^2 - \gamma_{2,n}^2 \to 0 \) and
\[
\sqrt{n} \left[ \gamma_{1,n}^2 + \gamma_{2,n}^2 - \gamma_3^2 - \gamma_{1,n}^2 + \gamma_{2,n}^2 + \gamma_{3,n}^2 \right] = \sqrt{n} \left[ \gamma_{1,n}^2 + \gamma_{2,n}^2 - \gamma_{3,n}^2 \right]
\]
is bounded in probability. Therefore (12) is equivalent to
\[
\left| \gamma_{1,n}^2 + \gamma_{2,n}^2 - \gamma_{3,n}^2 \right| \sqrt{n} < n^{1/4},
\]
where the left hand side is bounded in probability.
4 The case $\kappa = \frac{1}{2}$

Let us consider the case of intensity function (3) in the case $\kappa = \frac{1}{2}$, i.e.,

$$\lambda_{j,n}(\vartheta,t) = an\left| t - \tau_j \right|^{1/2} \mathbb{1}_{\{t \geq \tau_j\}} + n\lambda_0, \quad 0 \leq t \leq T,$$

where $\tau_j = \tau_j(\vartheta)$ is a smooth function of $\vartheta$. Recall that $\kappa \in (0, \frac{1}{2})$ correspond to the cusp case (Chernoyarov et al. 2018) and $\kappa > \frac{1}{2}$ fits the smooth case considered in this work. We start with the problem of estimating the parameter $\tau = (\tau_1, \ldots, \tau_k)$. Since the Poisson processes with such intensity functions are independent, it is sufficient to consider the estimation problem only for one $\tau_j$, which we denote as $\tau$ for brevity. Hence we assume that the intensity function of the observed Poisson process $X_n = (X(t), 0 \leq t \leq T)$ is

$$\lambda(t - \tau) = a\left| t - \tau \right|^{1/2} \mathbb{1}_{\{t \geq \tau\}} + \lambda_0, \quad 0 \leq t \leq T.$$

Note that the integral (the Fisher information)

$$\mathbb{I}_\tau = \int_0^T \left( \frac{\partial \lambda(t - \tau)}{\partial \tau} \right)^2 \lambda(t - \tau)^{-1} \, d\tau$$

$$= \frac{a^2}{4} \int_\tau^T \frac{d\tau}{\left| t - \tau \right| \left[ a\left| t - \tau \right|^{1/2} + \lambda_0 \right]} = \infty.$$ 

Introduce the normalizing function $\phi_n = (n \ln n)^{-1/2}$ and the corresponding log-likelihood ratio process (below $\tau_0$ is the true value and $u > 0$)

$$\ln Z_n(u) = \int_{\tau_0}^T \ln \frac{\lambda(t - \tau_0 - \varphi_n u)}{\lambda(t - \tau_0)} \, d\pi_n(t)$$

$$- n \int_{\tau_0}^T \left[ \frac{\lambda(t - \tau_0 - \varphi_n u)}{\lambda(t - \tau_0)} - 1 - \ln \frac{\lambda(t - \tau_0 - \varphi_n u)}{\lambda(t - \tau_0)} \right] \lambda(t - \tau_0) \, dt$$

$$= I_n(u) - J_n(u)$$

with the obvious notations.

The following asymptotics holds

$$\mathbb{E}_{\tau_0} I_n(u)^2 = n \int_{\tau_0}^T \left( \ln \frac{\lambda(t - \tau_0 - \varphi_n u)}{\lambda(t - \tau_0)} \right)^2 \lambda(t - \tau_0) \, dt$$

$$= n \int_{\tau_0}^{\tau_0 + \varphi_n u} \left( \ln \frac{\lambda_0}{a(t - \tau_0)^{1/2} + \lambda_0} \right)^2 \lambda(t - \tau_0) \, dt$$

$$= n \int_{\tau_0}^T \left( \ln \frac{a(t - \tau_0 - \varphi_n u)^{1/2} + \lambda_0}{a(t - \tau_0)^{1/2} + \lambda_0} \right)^2 \lambda(t - \tau_0) \, dt$$

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\[
\begin{align*}
&= o \left( \frac{u}{\ln n} \right) + n \int_{\varphi_n u}^{T-\tau_0} \left( \ln \left( 1 + \frac{a \sqrt{t - \varphi_n u} - a \sqrt{T}}{a \sqrt{t} + \lambda_0} \right) \right)^2 \left( a \sqrt{t} + \lambda_0 \right) dt \\
&= o \left( \frac{u}{\ln n} \right) + na^2 \int_{\varphi_n u}^{T-\tau_0} \frac{\left( \sqrt{t - \varphi_n u} - \sqrt{T} \right)^2}{a \sqrt{t} + \lambda_0} dt (1 + o(1)).
\end{align*}
\]

Below we let \( t = s \varphi_n \) and use the expansion (for large \( s \))
\[
\left( \sqrt{s - u} - \sqrt{s} \right)^2 = s \left( \sqrt{1 - \frac{u}{s}} - 1 \right)^2 = \frac{u^2}{4s} \left( 1 + O \left( \frac{1}{s} \right) \right).
\]

Therefore
\[
\begin{align*}
n \int_{\varphi_n u}^{T-\tau_0} \frac{\left( \sqrt{t - \varphi_n u} - \sqrt{T} \right)^2}{a \sqrt{t} + \lambda_0} dt &= n \varphi_n^2 \int_{\varphi_n u}^{T-\tau_0} \frac{\left( \sqrt{s - u} - \sqrt{s} \right)^2}{a \sqrt{s \varphi_n} + \lambda_0} ds \\
&= \frac{n \varphi_n^2 u^2}{4} \int_{\varphi_n u}^{T-\tau_0} \frac{1}{s \left( a \sqrt{s \varphi_n} + \lambda_0 \right)} ds (1 + o(1)) \\
&= \frac{n \varphi_n^2 u^2}{4} \int_{\varphi_n u}^{T-\tau_0} \frac{1}{s \left( a \sqrt{s \varphi_n} + \lambda_0 \right)} ds (1 + o(1)) \\
&\approx \frac{n \varphi_n^2 u^2}{4 \left( a \sqrt{T - \tau_0} + \lambda_0 \right)} \ln \left( \frac{T - \tau_0}{\varphi_n} \right) \\
&\approx \frac{n \varphi_n^2 u^2}{4 \left( a \sqrt{T - \tau_0} + \lambda_0 \right)} \left( \ln (n \ln n)^{1/2} + \ln (T - \tau_0) \right) \\
&\approx \frac{u^2}{8 \left( a \sqrt{T - \tau_0} + \lambda_0 \right)}.
\end{align*}
\]

The stochastic integral admits the representation
\[
I_n(u) = a \varphi_n \sqrt{n} \int_{\varphi_n u}^{T-\tau_0} \frac{\sqrt{s - u} - \sqrt{s}}{a \sqrt{s \varphi_n} + \lambda_0} dW_n(s) (1 + o(1)) \\
= \frac{a \varphi_n \sqrt{n} u}{2} \int_{\varphi_n u}^{T-\tau_0} \frac{1}{\sqrt{s \left( a \sqrt{s \varphi_n} + \lambda_0 \right)}} dW_n(s) (1 + o(1)).
\]

Here we denoted
\[
W_n(s) = \frac{1}{\sqrt{n \varphi_n}} \int_{\tau_0}^{\tau_0 + s \varphi_n} [dX(v) - n \lambda (v - \tau_0) dv] \\
= \frac{X(\tau_0 + s \varphi_n) - X(\tau_0) - n \int_{\tau_0}^{\tau_0 + s \varphi_n} \lambda (t - \tau_0) dt}{\sqrt{n \varphi_n}}.
\]
Using the characteristic function of the stochastic integral we can verify that as \(n \to \infty\)
\[
\frac{1}{\sqrt{\ln n}} \int_{\tau_0}^{T} \sqrt{n \ln n} \, dW_t(s) \sqrt{s} \left( a \sqrt{\varphi_n + \lambda_0} \right) \implies \mathcal{N} \left( 0, \frac{1}{a \sqrt{T - \tau_0 + \lambda_0}} \right).
\]
Similar calculations give the following asymptotics for the ordinary integral \(J_n(u)\):
\[
J_n(u) = \frac{u^2 a^2}{16 \left( a \sqrt{T - \tau_0 + \lambda_0} \right)} + o(1).
\]
Therefore for the likelihood ratio process we have
\[
Z_n(u) = \exp \left\{ \gamma \left[ u \Delta_n - \frac{\gamma^2 u^2}{2} + r_n \right] \right\} \tag{13}
\]
where \(r_n \to 0\),
\[
\gamma^2 = \frac{a^2}{8 \left( a \sqrt{T - \tau_0 + \lambda_0} \right)}, \quad \Delta_n \implies \mathcal{N}(0, 1). \tag{14}
\]
The estimates (6) and (7) can be obtained along the same lines. These estimates and the representation (13)–(14) allow to verify the convergences
\[
\sqrt{n \ln n} (\hat{\tau}_n - \tau_0) \implies \mathcal{N} \left( 0, \gamma^{-2} \right)
\]
and
\[
E_{\tau_0} \left( \hat{\tau}_n - \tau_0 \right)^2 = \frac{1}{\gamma^2 n \ln n} \left( 1 + o(1) \right).
\]
Therefore this is also the smooth or regular case with asymptotically normal MLE.

5 One-step MLE

The observation model remains the same, but our problem now is to construct an estimator-process \(\vartheta^* = (\vartheta^*_{t,n}, 0 \leq t \leq T)\), where the estimator \(\vartheta^*_{t,n}\) retains the asymptotic efficiency, on one hand, and can be easily computed, on the other. Obviously, finding the MLE at all \(t \in (0, T]\) is a formidable computational task.

A relevant construction in the case of ergodic diffusion processes was proposed in the work (Kutoyants 2017). In Khasminskii and Kutoyants (2018) a similar approach was applied to parameter estimation of the hidden telegraph process. The case of inhomogeneous Poisson processes was considered in Dabye et al. (2018). Here we apply
the techniques, developed therein. The main advantage of this method is that it is capable of producing computationally feasible estimators, which are also asymptotically efficient.

We need a consistent and asymptotically normal estimator \( \vartheta_n^* \)

\[
n^q \left( \vartheta_n^* - \vartheta_0 \right) \overset{\text{d}}{\longrightarrow} N(0, \mathbb{M}(\vartheta_0)),
\]

where \( q \in \left( \frac{1}{4}, \frac{1}{2} \right) \) and \( \mathbb{M}(\vartheta_0) \) is some non degenerate matrix. To construct such an estimator we follow the work (Khasminskii 2009, Section 3.3) and use the thinning of a Poisson process. Let \( X^T = (X(t), 0 \leq t \leq T) \) be the Poisson process with intensity function \( \lambda(t) \). Let \( t_i, i = 1, 2, \ldots \) be the events of the process \( X^T \), so that

\[
X(t) = \sum_i 1_{[t_i < t]}.
\]

Let \( \eta_1, \eta_2, \ldots \) be i.i.d. random variables such that \( P(\eta_i = 1) = p = 1 - P(\eta_i = 0) \).

Introduce the new process

\[
Y(t) = \sum_i \eta_i 1_{[t_i < t]}, \quad 0 \leq t \leq T.
\]

**Lemma 4** (Ross 2006, Proposition 5.2) The process \( Y(t), 0 \leq t \leq T \) is a Poisson process with the intensity function \( p\lambda(t) \). The process \( \tilde{X}(t) = X(t) - Y(t) \) is also Poisson with intensity function \( (1 - p)\lambda(t) \). The processes \( Y(t) \) and \( \tilde{X}(t) \) are independent.

Note that in Ross (2006) the proof is given for Poisson processes with constant intensity function, but it can be easily modified to cover inhomogeneous Poisson processes considered in our work.

The observed \( k \) Poisson processes \( X^n = (X_j(t), 0 \leq t \leq T, j = 1, \ldots, k) \) with intensity functions

\[
\lambda_{j,n}(\vartheta_0) = n\lambda_j(t - \tau_j(\vartheta_0)) + n\lambda_0, \quad 0 \leq t \leq T, \quad j = 1, \ldots, k
\]

can be represented, using the thinning procedure, as the sum of \( 2k \) independent Poisson processes

\[
Y^n = (Y_j(t), 0 \leq t \leq T, j = 1, \ldots, k), \quad \tilde{X}^n = (\tilde{X}_j(t), 0 \leq t \leq T, j = 1, \ldots, k),
\]

where \( Y_j(t) = X_j(t) - \tilde{X}_j(t) \) with the intensity functions

\[
\lambda_{j,n}^Y(\vartheta_0, t) = np_n\lambda_j(t - \tau_j(\vartheta_0)) + np_n\lambda_0, \quad 0 \leq t \leq T, \\
\lambda_{j,n}^\tilde{X}(\vartheta_0, t) = n(1 - p_n)\lambda_j(t - \tau_j(\vartheta_0)) + n(1 - p_n)\lambda_0, \quad 0 \leq t \leq T,
\]
The modification concerns the weights $J_j$ respectively. Define the probability $p_n = n^{-b}$, where $b \in (0, \frac{1}{2})$.

Let us denote by $\vartheta_n$ the MLE constructed using the observations $Y^n$ in Sect. 3 and note that it is asymptotically normal with $q = \frac{1-b}{2}$, see (15). Note that we need not to use all $k$ detectors and it is sufficient to construct the preliminary estimator using the Poisson processes, generated by any three detectors, not positioned on the same line. We introduce the One-step MLE-process $\vartheta_{\ast,n}$ as a slight modification of the counterpart from Dabye et al. (2018). In our case it is

$$
\vartheta_{\ast,n} = \vartheta_n + \mathbb{I}_t (\vartheta_n)^{-1} \sum_{j=1}^k \partial \tau_j (\vartheta_n) \int_{\tau_j(\vartheta_n)}^t \frac{\ell_j (s, \vartheta_n)}{n} \left[ d\tilde{X}_j (s) - \lambda_j (\vartheta_n, s) \right] ds.
$$

Here we use the convention $\frac{0}{0} = 0$ and assume that any three detectors are not on the same line. The function

$$
\ell_j (s, \vartheta_n) = \frac{\lambda'_{j} (s - \tau_j (\vartheta_n))}{\lambda_j (s - \tau_j (\vartheta_n)) + \lambda_0} \mathbb{I}_{s > \tau_j (\vartheta_n)}
$$

and the Fisher information matrix $\mathbb{I}_t (\vartheta)$ is a slight modification of the matrix $\mathbb{I} (\vartheta)$. The modification concerns the weights $J_{j,t} (\vartheta)$ in the definition of the norm $\|a\|_{\vartheta}$ and the scalar product $\langle a, b \rangle_{t, \vartheta}$. The modified weights are

$$
J_{j,t} (\vartheta) = \frac{1}{\nu^2 \| \vartheta_j - \vartheta \|} \int_{\tau_j(\vartheta)}^t \frac{\lambda'_{j} (s - \tau_j (\vartheta))}{\lambda_j (s - \tau_j (\vartheta)) + \lambda_0} \mathbb{I}_{s > \tau_j (\vartheta)} \, ds
$$

and we write $\langle a, b \rangle_{t, \vartheta}$, $\|a\|_{t, \vartheta}$ accordingly. Therefore

$$
\mathbb{I}_t (\vartheta) = \left(\begin{array}{cc}
\|x - x_0\|^2_{t, \vartheta} & \langle (x - x_0), (y - y_0) \rangle_{t, \vartheta} \\
\langle (x - x_0), (y - y_0) \rangle_{t, \vartheta} & \|y - y_0\|^2_{t, \vartheta}
\end{array}\right).
$$

Let us put the estimators $\tau_j (\vartheta_n)$ in the increasing order

$$
\tau_{(1)} (\vartheta_n) < \tau_{(2)} (\vartheta_n) < \cdots < \tau_{(k)} (\vartheta_n).
$$

It is evident that on the time interval $[0, \tau_{(1)} (\vartheta_n)]$ we have $\vartheta_{\ast,n} = \vartheta_n$. Moreover, for the values $\tau_{(1)} (\vartheta_n) \leq t \leq \tau_{(2)} (\vartheta_n)$ the Fisher information matrix $\mathbb{I}_t (\vartheta_n)$ is degenerate. In the case $\tau_{(2)} (\vartheta_n) \leq t \leq \tau_{(3)} (\vartheta_n)$ this matrix is non-degenerate and the estimator $\vartheta_{\ast,n}$ is asymptotically normal.

Note also that in the stochastic integral, used in the definition of the One-step MLE-process, the random vector $\vartheta_n$ is independent of the “observations” $\tilde{X}_j (\cdot)$ because the Poisson processes $Y^n$ and $\tilde{X}^n$ are independent. Therefore the stochastic integral is well defined.

Below we argue, omitting strict proofs, why the estimator-process $\vartheta_{\ast,n}$ is asymptotically normal with the same parameters as the MLE $\hat{\vartheta}_{t,n}$. Here $\hat{\vartheta}_{t,n}$ is the MLE constructed by the first observations $X_{t,n} = (X_j (s), 0 \leq s \leq t, j = 1, \ldots, k)$. " Springer
Of course, this MLE is not even consistent for the values \( t \in (0, \tau_{(3)}) \) because up to \( \tau_{(1)} \) the observations \( X^{t,n} \) do not contain any information about \( \vartheta_0 \). Its consistency is possible for the values \( t > \tau_{(3)} \) only. Indeed for these values we have the consistent estimators of \( \tau_{(1)} (\vartheta_0), \tau_{(2)} (\vartheta_0), \tau_{(3)} (\vartheta_0) \) and if the corresponding detectors are not on the same line then \( \vartheta_0 \) is identifiable. Note that the conditions required by One-step \( \vartheta_{t,n}^* \), and MLE \( \hat{\vartheta}_t \) estimators are not the same because in the former case we assume that we have the observations \( Y^n \). The proof of this asymptotic equivalence is a slight modification of the proof given in Dabye et al. (2018).

We can write

\[
\sqrt{n} \left( \vartheta_{t,n}^* - \vartheta_0 \right) = \sqrt{n} \left( \vartheta_n^* - \vartheta_0 \right) + \frac{1}{n^{\frac{1}{2} - b}} + \sum_{j=1}^{k} \frac{\partial \tau_j (\vartheta_0)}{\partial \vartheta} \int_{\tau_j (\vartheta_0)}^{T} \frac{\ell_j (s, \vartheta_n^*)}{\sqrt{n}} d\bar{\pi}_j (s) \int_{\tau_j (\vartheta_n^*)}^{T} \frac{\ell_j (s, \vartheta_0)}{\sqrt{n}} \left[ \lambda_{j,n} (\vartheta_0, s) - \lambda_{j,n} (\vartheta_n^*, s) \right] ds + o (1)
\]

Here \( d\bar{\pi}_j (s) = d\hat{X}_j (s) - \lambda_{j,n} (\vartheta_0, s) ds \) and we used the consistency of the estimator \( \vartheta_n^* = \vartheta_0 + O \left( \frac{1}{\sqrt{n}^{1-b}} \right) \), Taylor formula and the equality

\[
\bar{\pi}_t (\vartheta_n^*) = \sum_{j=1}^{k} \frac{\partial \tau_j (\vartheta_0)}{\partial \vartheta} \int_{\tau_j (\vartheta_0)}^{T} \frac{\ell_j (s, \vartheta_n^*)}{\sqrt{n}} \left[ \lambda_{j,n} (\vartheta_0, s) - \lambda_{j,n} (\vartheta_n^*, s) \right] ds + O \left( \frac{1}{\sqrt{n}^{1-b}} \right).
\]

Hence (see (15))

\[
\sqrt{n} \left( \vartheta_{t,n}^* - \vartheta_0 \right) O \left( n^{\frac{1}{2} - b} \right) = n^q \left( \vartheta_n^* - \vartheta_0 \right) O \left( n^{\frac{1}{2} - q - \frac{1}{2}} \right) = n^q \left( \vartheta_n^* - \vartheta_0 \right) o (1).
\]

Now the asymptotic normality

\[
\sqrt{n} \left( \vartheta_{t,n}^* - \vartheta_0 \right) \Longrightarrow N \left( 0, \bar{\pi}_t (\vartheta_0)^{-1} \right)
\]
follows from the central limit theorem for (independent) stochastic integrals
\[
\frac{1}{\sqrt{n}} \int_0^t \ell_j (s, \vartheta) d\tilde{\pi}_j (s), \quad j = 1, \ldots, k.
\]
Therefore, the One-step MLE-process \( \hat{\vartheta}_{T,n}^* \) is asymptotically equivalent to the MLE \( \hat{\vartheta}_{T,n} \). It is also possible to verify the uniform convergence of moments and the asymptotic efficiency of this estimator.

Recall that the One-step MLE-process is uniformly consistent, i.e., for any \( \varepsilon > 0 \) and any \( \nu > 0 \)
\[
P_{\vartheta_0} \left( \sup_{\tau_3(\vartheta_0) + \varepsilon \leq t \leq T} \| \hat{\vartheta}_{T,n}^* - \vartheta_0 \| > \nu \right) \longrightarrow 0.
\]
Here we have to assume that any three detectors are not on the same line. Moreover, it is possible to verify the weak convergence of the random process \( u_{T,n}^* = \sqrt{n} \left( \hat{\vartheta}_{T,n}^* - \vartheta_0 \right) \), \( \tau_3(\vartheta_0) + \varepsilon \leq t \leq T \) to the limit Gaussian process (see Dabye et al. 2018).

We can also consider the One-step MLE \( \hat{\vartheta}_{T,n}^* = \hat{\vartheta}_{T,n} \)
\[
\vartheta_n^* = \vartheta_n^* + \|
\hat{\vartheta}_{T,n}^* \|^{-1} \sum_{j=1}^k \frac{\partial \tau_j}{\partial \vartheta} \left( \vartheta_n^* \right) \int_{\tau_j(\vartheta_n^*)}^T \frac{\ell_j (t, \vartheta_n^*)}{n} d\tilde{\pi}_j (t) - \lambda_{j,n} \left( \vartheta_n^*, t \right) dr.
\]
This estimator have the same asymptotic normality
\[
\sqrt{n} \left( \vartheta_n^* - \vartheta_0 \right) \Longrightarrow \mathcal{N} \left( 0, \mathbb{I}_T \left( \vartheta_0 \right)^{-1} \right).
\]
For proving the asymptotic efficiency of this estimator we can verify the uniform convergence of the moments (see Dabye et al. 2018).

Another possibility is to use the CUSUM type estimators for \( \tau_j \). Then we can initialize the One-step MLE of the source position with the estimators, corresponding to the there sensors, closest to the source, and proceed, using the observations of other sensors.

### 6 Discussion

In this work we assumed that the source starts its emission at the instant \( t = 0 \). It is interesting to consider the more general setup, when the emission starts at an unknown time point \( \tau_* \). Therefore the signal received by the \( j \)-th detector arrives at the moment \( \bar{\tau}_j = \tau_* + \tau_j \), where \( \tau_j = \nu^{-1} \| \vartheta_j - \vartheta_0 \| \). Let us denote \( \hat{\tau}_{j,n} \) the MLE of the arrival time at the \( j \)-th detector. Then we have
\[
\nu^2 \left( \bar{\tau}_j - \tau_* \right)^2 = \left( x_j - x_0 \right)^2 + \left( y_j - y_0 \right)^2, \quad j = 1, \ldots, k,
\]
and
\[
\nu^2 \bar{\tau}_j^2 = x_j^2 + y_j^2 + x_0^2 + y_0^2 - \nu^2 \tau_*^2 - 2x_j x_0 - 2y_j y_0 + 2\nu^2 \bar{\tau}_j \tau_*.\]
Let us denote by $\gamma_1 = x_0$, $\gamma_2 = y_0$, $\gamma_3 = \tau_\star$, $\gamma_4 = x_0^2 + y_0^2 - \nu^2 \tau_\star^2$ and $r_j^2 = x_j^2 + y_j^2$. Then the parameter $\gamma = (\gamma_1, \ldots, \gamma_4)$ can be estimated by $\gamma_n^\star = (\gamma_1^\star_n, \ldots, \gamma_4^\star_n)$, obtain by solving the system

$$-2x_j \gamma_{1,n}^\star - 2y_j \gamma_{2,n}^\star + 2\nu^2 \tau_{j,n}^\star \gamma_{3,n}^\star + \gamma_{4,n}^\star = \nu^2 \tau_{j,n}^2 - r_j^2, \quad j = 1, \ldots, k$$

and so on. The corresponding matrix $A$ is already random and the estimator needs a special study. Consistency of the estimator $\gamma_n^\star$ can be deduced from consistency of the estimators $\tau_{j,n}^\star$, $j = 1, \ldots, k$.

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