ON SUBGRAPHS OF RANDOM CAYLEY SUM GRAPHS

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Abstract.

We prove that asymptotically almost surely, the random Cayley sum graph over a finite abelian group $G$ has edge density close to the expected one on every induced subgraph of size at least $\log^c |G|$, for any fixed $c > 1$ and $|G|$ large enough.

1 Introduction

Let $A$ be a subset of an additively written group $G$. We denote by $\text{Cay}(A, G)$ the Cayley sum graph induced by $A$ on $G$, which is the directed graph on the vertex set $G$ in which $(x, y) \in G \times G$ is an edge if and only if $x + y \in A$ ($x = y$ is allowed). Such graphs are classical combinatorial objects, see, e.g. [2]. B. Green [3] initiated to study the random Cayley sum graph, considering finite groups $G$ and selecting $A$ at random by choosing each $x \in G$ to lie in $A$ independently and at random with probability $1/2$. General random graphs are considered in [1]. Results about random Cayley sum graphs can be found, for example, in [3], [4], [6], [7]. R. Mrazović [6] proved the following theorem.

Theorem 1 Let $G$ be a finite group and $w : \mathbb{N} \to \mathbb{R}$ be a growing function that tends to infinity. Let $A \subset G$ be a random subset obtained by putting every element of $G$ into $A$ independently with probability $\frac{1}{2}$. Then with probability $1 - o(1)$, for all sets $X, Y \subset G$ with

$$|X| \geq w(|G|) \log |G| \quad \text{and} \quad |Y| \geq w(|G|) \log^2 |G|$$

one has

$$\sum_{x \in X} \sum_{y \in Y} A(x + y) = \frac{1}{2} |X||Y| + o(|X||Y|), \quad (|G| \to \infty), \quad (1)$$

where the rate of convergence implied by the $o$–notation depends only on $w$.

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In our paper for a set $A$ we use the same letter to denote its characteristic function $A : G \to \{0, 1\}$.

In the same paper Mrazović showed that there is no $C$ such that the assumption of Theorem 1 can be relaxed to $\min\{|X|, |Y|\} \geq C \log |G| \log \log |G|$.

Theorem 1 shows that with high probability, the edge density of the random Cayley sum graph on all induced subgraphs of size at least $\log^{2+\varepsilon} |G|$ is close to $1/2$.

Using some tools from Additive Combinatorics, we show that Theorem 1 can be improved.

**Theorem 2** Let $G$ be a finite abelian group of size $N$ and $w : \mathbb{N} \to \mathbb{R}$ be a growing function that tends to infinity. Let $A \subset G$ be a random subset obtained by putting every element of $G$ into $A$ independently with probability $1/2$. Then with probability $1 - o(1)$, for all sets $X, Y \subset G$ such that

$$|X| \geq w(|G|) \log |G| (\log \log |G|)^2, \quad |Y| \geq w(|G|) \log |G| (\log \log |G|)^{10},$$

one has

$$\sum_{x \in X} \sum_{y \in Y} A(x + y) = \frac{1}{2} |X||Y| + o(|X||Y|) \quad (|G| \to \infty),$$

where the rate of convergence implied by the $o$–notation depends only on $w$.

Thus lower and upper bounds for size of sets $X, Y$ differ by some powers of double logarithms.

Let us say a few words about the proof.

It was showed in [6] that if for some $X, Y$ the sum of the left-hand side of (1) deviates significantly from $\frac{1}{2} |X||Y|$, then the common energy (see the definition in the next section) of $X$ and $Y$ must be close to the trivial upper bound $|X||Y| \min\{|X|, |Y|\}$. Mrazović used a random choice to avoid such a situation (see details in [6]) and using structural results from [9], [11] we add one more twist to his arguments, hence proving that large portions of $X, Y$ must be very structured in this case. It follows that the number of such sets is much smaller than the number of all possible pairs of arbitrary sets $X, Y$. This allows us to relax the conditions on sizes of $|X|, |Y|$ and to obtain our bound $\log^c |G|$ with $c > 1$.

First we consider the case of elementary abelian 2–groups and prove Theorem 2 in this situation (with $c > 3/2$) using some arguments from [6]. For
such groups the proof is simpler and more transparent. For general case see sections 4, 5.

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2 Definitions and preliminary results

Let \( G \) be an abelian group. The \textit{additive energy} \( E(A, B) \) between two sets \( A \) and \( B \) from \( G \) is (see [13])

\[
E(A, B) = |\{(a_1, a_2, b_1, b_2) \in A \times A \times B \times B : a_1 + b_1 = a_2 + b_2\}|.
\]

The \textit{sumset} of \( A \) and \( B \) is

\[A + B := \{a + b : a \in A, b \in B\}.\]

By \( A \bigcup B \) denote the union of two disjoint sets \( A, B \).

Recall a simple lemma, see, e.g., [10, Lemma 12].

\textbf{Lemma 3} For any finite sets \( X, Y, Z \subset G \) one has

\[
E(X \cup Y, Z)^{1/2} \leq E(X, Z)^{1/2} + E(Y, Z)^{1/2},
\]

and for disjoint union of \( X \) and \( Y \) the following holds

\[
E(X \cup Y, Z) \geq E(X, Z) + E(Y, Z).
\]

Now let us recall the notion of the (additive) \textit{dimension} of a set. A finite set \( \Lambda \subset G \) is called \textit{dissociated} if any equality of the form

\[
\sum_{\lambda \in \Lambda} \varepsilon_\lambda \lambda = 0
\]

for \( \varepsilon_\lambda \in \{-1, 0, 1\} \) implies \( \varepsilon_\lambda = 0 \) for all \( \lambda \in \Lambda \). The notion of dissociativity appears naturally in analysis, see [8]. The size of a largest dissociated subset
of \( A \) is called the (additive) \textit{dimension} of the set \( A \) and is denoted by \( \dim(A) \).

For a subset \( S = \{s_1, \ldots, s_l\} \subset G \) one can define

\[
\text{Span}(S) := \left\{ \sum_{j=1}^{l} \varepsilon_j s_j : \varepsilon_j \in \{0, -1, 1\} \right\}.
\]

It is easily seen that if \( S \) is a dissociated subset of \( A \) of size \( |S| = \dim(A) \), then \( A \subset \text{Span}(S) \).

Notice that if \( G \) is a finite group of exponent 2 (hence, a linear space over \( \mathbb{F}_2 \)), then \( \text{Span}(S) \) is the linear span of \( S \), and \( \dim S \) is its dimension.

We need the main result from [11], also see [9, Theorem 19].

**Theorem 4** Let \( A, B \) be finite non-empty subsets of an abelian group, \( |A| \geq |B| \). If \( E(A, B) \geq \frac{|A||B|^2}{K} \), then there exist a non-empty set \( B_* \subset B \) such that

\[
\dim(B_*) \ll K \log |A|, \quad (2)
\]

and

\[
E(A, B_*) \geq 2^{-5}E(A, B). \quad (3)
\]

Theorem 4 shows that if \( E(A, B) \) is large, then \( B \) contains a large, well-structured subset \( B_* \).

Let us derive a simple consequence of the theorem above.

**Corollary 5** Let \( A, B \) be finite subsets of an abelian group with \( |A| \geq |B| \geq 2 \). Suppose that \( E(A, B) = \frac{|A||B|^2}{K} \), and \( M \geq K \) be a parameter. Then there is a partition \( B = B' \sqcup B'' \) such that

\[
\dim(B') \ll M \log |A| \cdot \log(|B| M/K), \quad E(A, B') \gg E(A, B), \quad (4)
\]

and

\[
E(A, B'') \leq \frac{|A||B''|^2}{M}. \quad (5)
\]
Proof. Our arguments is a sort of an algorithm similar to that found in [5, 12]. We construct an increasing sequence of sets $\emptyset = B_1' \subset B_2' \subset \cdots \subset B_k'$ and a decreasing sequence of sets $B = B_1'' \supset B_2'' \supset \cdots \supset B_k''$ such that for any $j = 1, 2, \ldots, k$ the sets $B_j'$ and $B_j''$ are disjoint and moreover $B = B_j' \cup B_j''$. If at some step $j$ we have either $E(A, B_j'') < |A||B_j''|^2/M$ or $B_j'' = \emptyset$ (notice that due to the definition of $K$ and the supposition $M \geq K$ this can happen only for $j > 1$) then we stop our algorithm putting $B_j'' = B_j''$, $B_j' = B_j'$, and $k = j$. In the opposite situation where $E(A, B_j'') \geq |A||B_j''|^2/M$ we apply Theorem 4 to the set $B_j''$, finding a non-empty subset $G_j$ of $B_j''$ such that
\[
\dim(G_j) \ll M \log |A|, \tag{6}
\]
and
\[
E(A, G_j) \geq 2^{-5}E(A, B_j''). \tag{7}
\]
After that we put $B_{j+1}'' = B_j'' \setminus G_j$, $B_{j+1}' = B_j' \cup G_j$ and repeat the procedure. Clearly, $B_k' = \bigcup_{j=1}^k G_j$. In view of Lemma 3 and (7), we get
\[
E(A, B_j'') \geq E(A, G_j) + E(A, B_j''_{j+1}) \geq 2^{-5}E(A, B_j'') + E(A, B_j'')
\]
whence $E(A, B_{j+1}'') \leq \frac{31}{32}E(A, B_j'')$. It follows that our algorithm stops after at most $k \ll \log(|B|M/K)$ steps. Because $G_1 \subset B_j'$, $j \geq 2$, we have in view of (7) that for $j \geq 2$ one has
\[
E(A, B_j') \geq E(A, G_1) \geq 2^{-5}E(A, B_1'') = 2^{-5}E(A, B)
\]
and thus inequality $E(A, B') \gg E(A, B)$ holds. Finally, from estimate (6), we obtain
\[
\dim(B') \leq \sum_{j=1}^{k-1} \dim(G_j) \ll kM \log |A| \ll M \log |A| \cdot \log(|B|M/K).
\]
This completes the proof of the corollary. \qed

We finish this section with a result on the number of sets with small dimension.

**Lemma 6** Let $G$ be a finite abelian group, and write $N = |G|$. Let $n, d \in \mathbb{N}$ with $n \geq 2 \log N$. Then the number of sets $X \subset G$ with $0 < |X| \leq n$ and $\dim X \leq d$ is at most $e^{2nd}$. 

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Proof. Take \( X \subset X' = \text{Span}(\Lambda) \) where \(|\Lambda| = \dim X \leq d\). The number of sets \( \Lambda \) is at most \( N^d \). For a fixed \( \Lambda \), we have \(|X'| \leq 3^d\), and the number of sets \( X \subset X' \) (with fixed \( \Lambda \)) is at most \(|X'|^n \leq 3^{nd}\). Therefore, the total number of sets \( X \) is at most
\[
N^d 3^{nd} < e^{(\log N + 1.1n)d} \leq e^{2nd},
\]
as required. \( \square \)

3 A model case

The main result of this section is the following

**Theorem 7** Let \( G \) be a finite group of exponent 2 and \( w : \mathbb{N} \to \mathbb{R} \) be a growing function that tends to infinity. Let \( A \subset G \) be a random subset obtained by putting every element of \( G \) into \( A \) independently with probability \( \frac{1}{2} \). Then with probability \( 1 - o(1) \), for all sets \( X, Y \subset G \) such that
\[
|X|, |Y| \geq w(|G|)(\log \log |G|) \cdot \log^{3/2} |G|,
\]
one has
\[
\sum_{x \in X} \sum_{y \in Y} A(x + y) = \frac{1}{2} |X||Y| + o(|X||Y|) \quad (|G| \to \infty),
\]
where the rate of convergence implied by the \( o \)-notation depends only on \( w \).

We notice that the groups considered in Theorem 7 are abelian and they can be treated as vector spaces over the field \( \mathbb{F}_2 \).

For finite, non-empty subsets \( X, Y \), and \( A \) of \( G \), let
\[
\sigma_A(X, Y) := \frac{1}{|X||Y|} \sum_{x \in X} \sum_{y \in Y} A(x + y) - \frac{1}{2} = \frac{1}{|X||Y|} \sum_{x \in X} \sum_{y \in Y} \left( A(x + y) - \frac{1}{2} \right).
\]

The following technical result is the heart of [6] (see section 4 of that paper).
Proposition 8 Let $G$ be a finite abelian group, and write $N = |G|$. If $A$ is a random subset of $G$ obtained by putting every element of $G$ into $A$ independently with probability $\frac{1}{2}$, then for any $r, K \geq 1$ and $\varepsilon \in (0, 1]$, the probability that there exist $X, Y \subset G$ satisfying

$$|Y| \geq |X| \geq 2000\varepsilon^{-4} \log N, |Y| \geq r, E(X, Y) \leq \frac{|X|^2|Y|}{K},$$

and

$$|\sigma_A(X, Y)| \geq \varepsilon$$

is at most

$$C \exp\left(\frac{2000 \log^2 N}{\varepsilon^4} - \frac{\varepsilon^2 r K}{40}\right)$$

with an absolute constant $C$.

Proposition 8 was not stated in [6] explicitly, and for completeness, we prove it in Appendix.

Let us show quickly how Proposition 8 implies Theorem 1 for abelian groups $G$. Choosing $K = 1$ and $r = C\varepsilon^{-6} \log^2 N$ with $C$ large enough, we obtain that the probability of existence $X, Y$, $|Y| \geq |X| \geq 2000\varepsilon^{-4} \log |G|$, $|Y| \geq r$ such that $|\sigma_A(X, Y)| \geq \varepsilon$ is less than

$$C_1 \exp(-C_2 \varepsilon^{-4} \log^2 |G|) = o(1) \quad \text{as} \quad |G| \to +\infty,$$

where $C_1, C_2 > 0$ are some absolute constants.

The next corollary immediately follows from Proposition 8 (as applied with $K = M$ and $r = (\varepsilon/4)w(N)(\log \log N) \cdot \log^{3/2} N$).

Corollary 9 Let $G$ be a finite abelian group, and write $N = |G|$. If $A$ is a random subset of $G$ obtained by putting every element of $G$ into $A$ independently with probability $\frac{1}{2}$, then for $M = (\log \log N)^{-1}(\log N)^{1/2}$, any $\varepsilon \in (0, 1]$ such that

$$\varepsilon^7 \geq \frac{2^{25}}{w(N)\log \log N \sqrt{\log N}},$$

and any growing function $w: \mathbb{N} \to \mathbb{R}$ tending to infinity, the probability that there exist $X, Y \subset G$ satisfying

$$|Y| \geq |X| \geq (\varepsilon/4)w(N)(\log \log N) \cdot \log^{3/2} N, E(X, Y) \leq \frac{|X|^2|Y|}{M},$$
and
\[ |\sigma_A(X, Y)| \geq \varepsilon/2, \]
tends to 0 as \( N \to \infty \).

**Proof of Theorem 7.** Take a random set \( A \) and suppose that for some \( X, Y \) one has \( |\sigma_A(X, Y)| \geq \varepsilon \). Without loss of generality, suppose that \( |Y| \geq |X| \geq (\varepsilon/4)w(N)(\log \log N) \cdot \log^{3/2} N \). In view of Theorem 1, we can assume that \( |X|, |Y| \ll \log^{5/2} N \), say. Otherwise the probability of the event \( |\sigma_A(X, Y)| \geq \varepsilon \) is \( o(1) \).

Denote \( K = \frac{|X|^2|Y|}{E(X, Y)} \), \( M = (\log \log N)^{-1}(\log N)^{1/2} \).

If \( M \leq K \) then we can apply Corollary 9 and conclude that the probability of this event is \( o(1) \). Thus we can assume that \( M \geq K \). Applying Corollary 5 to the sets \( X, Y \), we find \( X', X'' \subset X \) such that \( X = X' \bigcup X'' \), \( \dim(X') \ll M(\log \log N)^2 \), \( E(X', Y) \gg E(X, Y) \) and \( E(X'', Y) \leq |Y||X''|^2/M \).

We can assume that with high probability the following holds:
\[ |\sigma_A(X', Y)| \geq \varepsilon/2, \quad |X'| \geq (\varepsilon/2)|X|. \tag{9} \]

Indeed, if one of these two inequalities does not hold, then
\[
\left| \sum_{x \in X', y \in Y} \left( A(x + y) - \frac{1}{2} \right) \right| \leq \varepsilon|X||Y|/2
\]
and we have
\[
\varepsilon|X||Y| \leq |\sigma_A(X, Y)||X||Y| = \left| \sum_{x \in X, y \in Y} \left( A(x + y) - \frac{1}{2} \right) \right| \leq \left| \sum_{x \in X', y \in Y} \left( A(x + y) - \frac{1}{2} \right) \right| + \left| \sum_{x \in X'', y \in Y} \left( A(x + y) - \frac{1}{2} \right) \right| \leq \varepsilon|X||Y|/2 + |X''||Y||\sigma_A(X'', Y)|. \tag{10}
\]
Whence
\[
\frac{\varepsilon|X||Y|}{2} \leq |X''||Y||\sigma_A(X'', Y)|.
\]
The last bound implies that $|X''| \geq \varepsilon |X|/2$ and $|\sigma_A(X'', Y)| \geq \varepsilon / 2$. Using Corollary 9 with the sets $X''$, $Y$, we see that the probability of the last inequality is $o(1)$. Thus, we will assume that (9) holds.

Split the set $Y$ onto sets $\hat{Y}_j$ (using lexicographical ordering, say) such that $|X'| / 2 \leq |\hat{Y}_j| \leq |X'|$. Then arguing as in (10), we see that for some $\hat{Y}_j$ one has $\sigma_A(X', \hat{Y}_j) \geq \varepsilon / 2$. Indeed

$$2^{-1} \varepsilon \cdot |X'| \sum_j |\hat{Y}_j| = 2^{-1} \varepsilon \cdot |X'||Y| \leq |\sigma_A(X', Y)||X'||Y| \leq$$

$$\leq \sum_j \left| \sum_{x \in X', y \in \hat{Y}_j} \left( A(x + y) - \frac{1}{2} \right) \right| = |X'| \sum_j \sigma_A(X', \hat{Y}_j) |\hat{Y}_j|$$

and thus there is $j$ with $\sigma_A(X', \hat{Y}_j) \geq \varepsilon / 2$. Put $\hat{Y} = \hat{Y}_j$. After that taking into account (9) and applying Corollary 5 to the sets $X'$, $\hat{Y}$, we find $Y'$, $Y'' \subset \hat{Y}$ such that $\hat{Y} = Y' \cup Y''$, $\dim(Y') \ll M(\log \log N)^2$, $E(X', Y') \gg E(X', \hat{Y})$ and $E(X', Y'') \leq |X'||Y''|^2 / M$ (if $E(X', \hat{Y})$ := $|X'||\hat{Y}|^2 / K < |X'||\hat{Y}|^2 / M$, then it is nothing to prove, just put $Y'' = \hat{Y}, Y' = \emptyset$, otherwise $M \geq K$).

Again, we can assume with probability $1 - o(1)$

$$|\sigma_A(X', Y')| \geq \varepsilon / 4, \quad |Y'| \geq (\varepsilon / 4)|\hat{Y}|.$$  

For fixed $\varepsilon$ and large $N$ in view of the assumption $|X| \geq w(N)(\log \log N) \cdot \log^{3/2} N$, we have

$$\min\{\varepsilon |\hat{Y}|, |X'|\} \gg \varepsilon^{-4} \log N.$$  

Up to this point we did not use a specific structure of the group $G$. Notice that in this group for any subset $A$ the additive dimension $\dim(A)$ is just the ordinary dimension of its linear span. Consider the set $L := \text{Span}(Y' \cup X')$ of dimension

$$\dim(L) \leq \dim(X') + \dim(Y') := d \ll M(\log \log N)^2.$$  

Recall now that $G$ is a linear space over $\mathbb{F}_2$. Therefore, $L$ is also an abelian group, and we can apply to $L$ Proposition 8 implying that the probability of the inequality $\sigma_A(X', Y') \geq \varepsilon / 4$ is less than (also, see the calculations after the Proposition)

$$C' \exp \left( \frac{2^{19} d^2}{\varepsilon^4} - \frac{\varepsilon^2 \max\{|X'|, |Y'|\}^2}{160} \right),$$  

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where $C' > 0$ is some absolute constant. Because the number of sets $L$ is roughly bounded by $N^d$, and the number of sets $\hat{Y}_j$ is bounded by $O(|Y|/|X'|)$, we obtain that the total probability tends to zero, if

$$
\varepsilon^2 \max\{|X|, |Y|\} \gg \frac{d^2}{\varepsilon^4} + d \log N + \log |Y| \gg \frac{d^2}{\varepsilon^4} + d \log N .
$$

Since due to (9)

$$d \ll (\log \log N)(\log N)^{1/2}, \quad |X'| \geq (\varepsilon/2)w(N)(\log \log N) \cdot \log^{3/2} N ,
$$

we see that (11) holds for large $N$. This completes the proof.

### 4 On large deviations

We use the notations of section 3. Recall, that for finite, non-empty subsets $X, Y$, and $A$ of $G$, we denote

$$
\sigma_A(X, Y) := \frac{1}{|X||Y|} \sum_{x \in X} \sum_{y \in Y} A(x + y) - \frac{1}{2} = \frac{1}{|X||Y|} \sum_{x \in X} \sum_{y \in Y} \left(A(x + y) - \frac{1}{2}\right).
$$

In this section we fix $X \subset G$ and estimate the probabilities that the deviations $|\sigma_A(X, Y)|$ are large where $Y \subset G$ (see precise statements below). If $Y = \{y\}$ we will write for simplicity $\sigma_A(X, y)$ rather than $\sigma_A(X, \{y\})$.

**Lemma 10** Let $G$ be a finite abelian group. Fix $X \subset G$ with $|X| = n$. Let $\varepsilon \in (0, 1/2]$ and $y_1, \ldots, y_k \in G$ satisfy the condition

$$
|X + y_i) \cap \left(\bigcup_{j=1}^{i-1}(X + y_j)\right) \leq \varepsilon n \quad (i = 2, \ldots, k).
$$

If $A$ is a random subset of $G$ obtained by putting every element of $G$ into $A$ independently with probability $\frac{1}{2}$, then the probability of the event

$$
|\sigma_A(X, y_j)| \geq \varepsilon \quad (j = 1, \ldots, k)
$$

is at most

$$
\exp\left(-\frac{\varepsilon^2 kn}{2}\right).
$$
Proof. Denote by $H_i$ ($i = 0, \ldots, k$) the event
\[ |\sigma_A(X, y_j)| \geq \varepsilon \quad (j = 1, \ldots, i). \]
We will prove by induction on $i$ that the probability $P_i$ of the event $H_i$ is at most
\[ \exp \left( -\frac{\varepsilon^2 i n}{2} \right). \]
The claim is obvious for $i = 0$. Now we prove that it is true for each $i = 1, \ldots, k$ whenever it holds for $i - 1$.

Let
\[ X'_i = \{ x \in X : \exists j \in \{1, \ldots, i - 1\} : x + y_i \in X + y_j \}, \quad X_i = X \setminus X'_i. \]
By (12) we have
\[ |X'_i| \leq \varepsilon n. \]
Therefore,
\[ \left| \sum_{x \in X'} A(x + y_i) - \frac{|X'_i|}{2} \right| \leq \frac{\varepsilon n}{2}. \]
Assuming that $H_i$ holds, we see that
\[ \left| \sum_{x \in X} A(x + y_i) - \frac{|X|}{2} \right| \geq \varepsilon n. \]
Hence,
\[ \left| \sum_{x \in X_i} A(x + y_i) - \frac{|X_i|}{2} \right| \geq \left| \sum_{x \in X} A(x + y_i) - \frac{|X|}{2} \right| \]
\[ - \left| \sum_{x \in X'_i} A(x + y_i) - \frac{|X'_i|}{2} \right| \geq \frac{\varepsilon n}{2}. \]
Thus, denoting by $H'_i$ the event
\[ \left| \sum_{x \in X_i} A(x + y_i) - \frac{|X_i|}{2} \right| \geq \frac{\varepsilon n}{2}, \]
we conclude that $H'_i$ holds if $H_i$ holds. Since for $x \in X_i$ the element $x + y_j$, $j < i$, the event $H'_i$ is independent of the events $H_1, \ldots, H_{i-1}$. Therefore, if $P'_i$ is the probability of the event $H'_i$, then
\[ P_i \leq P_{i-1} P'_i. \quad (14) \]

By Proposition 3 from [6] (Hoeffding’s theorem) we get
\[ P'_i \leq \exp \left( -\frac{1}{2} \left( \frac{\varepsilon n / 2}{\sqrt{|X_i|/2}} \right)^2 \right) \leq \exp \left( -\frac{\varepsilon^2 n}{2} \right). \]

Plugging in this estimate into (14) and using the induction hypothesis we complete the proof of the lemma.

\[ \square \]

**Corollary 11** Let $G$ be a finite abelian group, and write $N = |G|$. Fix $X \subset G$ with $|X| = n$. Let $\varepsilon \in (0, 1/2]$ and $k \in \mathbb{N}$. Then the probability that there exist $y_1, \ldots, y_k \in G$ satisfying (12) and (13) is at most
\[ \left( N \exp \left( -\frac{\varepsilon^2 n}{2} \right) \right)^k. \]

**Corollary 12** Let $G$ be a finite abelian group, and write $N = |G|$. Fix $X \subset G$ with $|X| = n$. Let $\varepsilon \in (0, 1/2]$ and $k \in \mathbb{N}$. If
\[ n \geq \frac{4 \log N}{\varepsilon^2}, \]
then the probability that there exist $y_1, \ldots, y_k \in G$ satisfying (12) and (13) is at most
\[ \exp \left( -\frac{\varepsilon^2 nk}{4} \right). \]

Now we will show that if $Y \subset G$ is a large set and $\sigma_A(X, Y)$ is also large then for an appropriate $\varepsilon$ there are many elements $y_1, \ldots, y_k$ satisfying (12) and (13). Observe that if we even do not assume that $y_1, \ldots, y_k$ are distinct, this would follow from (12).
Lemma 13 Let $G$ be a finite abelian group, $X, Y \subseteq G$ with $|X| = n$, and let $\varepsilon \in (0, 1/2]$. If

$$E(X, Y) = |X|^2|Y|/K,$$

then for some

$$k > \varepsilon^2|Y|K/n$$

there are $y_1, \ldots, y_k \in Y$ satisfying condition (12).

Proof. Let $\{y_1, \ldots, y_k\}$ be the maximal subset of $Y$ satisfying (12). Denote

$$Z = \bigcup_{i=1}^{k}(X + y_i).$$

For any $z \in Z$ we denote by $f(z)$ the number of solutions of the equation

$$x + y = z, \quad x \in X, y \in Y.$$ 

By the choice of $k$, for any $y \in Y$ there are more than $\varepsilon n$ values $x \in X$ such that $x + y \in Z$. Hence,

$$\sum_{z \in Z} f(z) > \varepsilon n|Y|.$$

By the Cauchy–Schwarz inequality

$$\sum_{z \in Z} f(z)^2 \geq |Z|^{-1} \left( \sum_{z \in Z} f(z) \right)^2 > \frac{\varepsilon^2 n|Y|^2}{k}.$$ 

Since

$$E(X, Y) = \sum_{z \in G} f(z)^2,$$

we conclude that

$$n^2|Y|/K > \frac{\varepsilon^2 n|Y|^2}{k}$$

implying the required inequality for $k$.

\[\square\]

Lemma 14 Let $G$ be a finite abelian group, $X, Y \subseteq G$, and let $\varepsilon \in (0, 1/2]$. Also, let $A \subseteq G$ be a set. If

$$|\sigma_A(X,Y)| \geq \varepsilon,$$
then there is a set $Y' \subset Y$ such that $|Y'| \geq \varepsilon |Y|$ and any $y \in Y'$ satisfies the condition 
\[ |\sigma_A(X, y)| \geq \varepsilon / 2. \]

**Proof.** Denote 
\[ Y' = \{ y \in Y : |\sigma_A(X, y)| \geq \varepsilon / 2 \}, \quad Y'' = Y \setminus Y'. \]
We have 
\[
\left| \sum_{x \in X, y \in Y''} A(x + y) - \frac{|X||Y''|}{2} \right| \leq \sum_{y \in Y''} \left| \sum_{x \in X} A(x + y) - \frac{|X|}{2} \right| \leq \sum_{y \in Y''} \varepsilon |X| / 2 \leq \varepsilon |X||Y| / 2.
\]
Therefore, 
\[
\left| \sum_{x \in X, y \in Y'} A(x + y) - \frac{|X||Y'|}{2} \right| \geq \sum_{x \in X, y \in Y} A(x + y) - \frac{|X||Y|}{2} \geq \varepsilon |X||Y| / 2.
\]
On the other hand, 
\[
\left| \sum_{x \in X, y \in Y'} A(x + y) - \frac{|X||Y'|}{2} \right| \leq |X||Y'| / 2.
\]
Thus, $|Y'| \geq \varepsilon |Y|$ as required. 

Combining Lemmas 14 and 13, we get the following corollary.

**Corollary 15** Let $G$ be a finite abelian group, $X, Y \subset G$ with $|X| = n$, and let $\varepsilon \in (0, 1/2]$. Also, let $A \subset G$ be a set. If 
\[
E(X, Y) \leq |X|^2 |Y| / K, \quad |\sigma_A(X, Y)| \geq \varepsilon,
\]

then for some
\[ k > \varepsilon^4 |Y| K / (4n) \]
there are \( y_1, \ldots, y_k \in Y \) satisfying conditions
\[ |(X + y_i) \cap \left( \bigcup_{j=1}^{i-1} (X + y_j) \right) | \leq \varepsilon n / 2 \quad (i = 2, \ldots, k) \quad (15) \]
and the condition
\[ |\sigma_A(X, y_j)| \geq \varepsilon / 2 \quad (j = 1, \ldots, k). \]

**Proof.** We take a subset \( Y' \subset Y \), in accordance with Lemma 14. Let
\[ E(X, Y') = |X|^2 |Y'| / K'. \]
Since \( |Y'| \geq \varepsilon |Y| \), \( E(X, Y') \leq E(X, Y) \leq |X|^2 |Y| / K \), we have
\[ K' \geq \varepsilon K. \]
Applying Lemma 13 (with \( \varepsilon / 2 \) instead of \( \varepsilon \)) to the set \( Y' \) we get desired \( y_1, \ldots, y_k \in Y' \) with
\[ k > \varepsilon^2 |Y'| K' / (4n) \geq \varepsilon^4 |Y| K / (4n). \]

Corollaries 12 and 15 immediately imply the main result of this section.

**Proposition 16** Let \( G \) be a finite abelian group, and write \( N = |G| \). Fix \( X \subset G \) with \( |X| = n, \varepsilon \in (0, 1/2], K \geq 1 \), and \( m \in \mathbb{N} \). Let \( A \) be a random subset of \( G \) obtained by putting every element of \( G \) into \( A \) independently with probability \( 1 / 2 \). If
\[ n \geq \frac{4 \log N}{\varepsilon^2}, \]
then the probability that there exist \( Y \subset G \) satisfying
\[ |Y| \geq m, \quad |\sigma_A(X, Y)| \geq \varepsilon, \quad E(X, Y) \leq |X|^2 |Y| / K, \]
is at most
\[ \exp \left( -\varepsilon^6 m K / 64 \right). \]
5 The proof of the main result

In this section we obtain our main Theorem 2. Let $N = |G|$. Without loss of generality we assume that

$$w(N) \leq \log \log(N + 3). \quad (16)$$

Denote

$$w_1(N) = \sqrt{w(N)}, \quad \varepsilon = w(N)^{-1/13} \quad (17)$$

and

$$\tilde{n}_0 = \left[ w_1(N) \log N (\log \log N)^2 \right], \quad \tilde{n}_1 = 2\tilde{n}_0.$$

Assume that

$$|\tilde{X}| \geq w(N) \log N (\log \log N)^2, \quad |Y| \geq m := w(N) \log N (\log \log N)^{10},$$

$$|Y| \geq |\tilde{X}|, \quad |\sigma_A(\tilde{X}, Y)| \geq \varepsilon/2,$$

and $w(N)$ is large enough as well as $N$. We have to prove that the probability of existence of such sets $\tilde{X}, Y$ is small. In view of Theorem 1, we can assume that $|\tilde{X}|, |Y| \ll \log^{5/2} N$, say, because otherwise the probability of the event $|\sigma_A(\tilde{X}, Y)| \geq \varepsilon/2$ is $o(1)$. Since $|\tilde{X}| \geq \tilde{n}_1$, we can split $\tilde{X}$ into sets $X_i$ with $
_0 \leq |X_i| \leq \tilde{n}_1$ in an arbitrary way. For some $i$ we have $|\sigma_A(X_i, Y)| \geq |\sigma_A(\tilde{X}, Y)| \geq \varepsilon/2$. Take $X = X_i$. If these sets $X, Y$ exist, then the following event $H_0$ happens:

there exist $X, Y \subset G$ such that

$$|X| \leq 2w_1(N) \log N (\log \log N)^2, \quad |Y| \geq m,$$

$$\sum_{x \in X} \sum_{y \in Y} \left( A(x + y) - \frac{1}{2} \right) \geq \tilde{\varepsilon}w_1(N) (\log N)(\log \log N)^2|Y|,$$

where $\tilde{\varepsilon} = \varepsilon/4$. Our aim is to prove that the probability $P_0$ of the event $H_0$ tends to 0 as $N \to \infty$. We will consider the family of events $\{H_j\}, j \geq 0$. We say that $H_j$ happens if there exist $X, Y \subset G$ such that

$$|X| \leq 2w_1(N) \log N (\log \log N)^2, \quad (18)$$

$$|Y| \geq m = w(N) \log N (\log \log N)^{10}, \quad (19)$$

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\[
E(X, Y) \leq |X|^2 |Y| 10^{-j},
\]

\[
\left| \sum_{x \in X} \sum_{y \in Y} \left( A(x + y) - \frac{1}{2} \right) \right| \geq \left( 1 - \frac{j}{\log \log N} \right) \tilde{\varepsilon} w_1(N)(\log N)(\log \log N)^2 |Y|,
\]

where \( j \geq 0 \). We denote by \( P_j \) the probability of the event \( H_j \). Let \( j_0 = \lfloor (\log \log N)^2/2 \rfloor \). We observe that, due to (16) and (18),

\[
10^{j_0} > |X|.
\]

Hence, \( P_{j_0} = 0 \) (because (20) does not hold for \( j = j_0 \) due to \( E(X, Y) \geq |X||Y| \)), and we will consider that \( j < j_0 \).

We will estimate \( P_j \) in terms of \( P_{j+1} \). Let \( X, Y \subset G \) satisfy (18)–(21).

Denote

\[
K = \frac{|X|^2 |Y|}{E(X, Y)}, \quad M = 10^{j+1}.
\]

If \( M \leq K \) then the event \( H_{j+1} \) holds. Thus we can assume that \( M \geq K \). Applying Corollary 5 to the sets \( X, Y \), we find \( X', X'' \subset X \) such that \( X = X' \cup X'' \), \( \dim(X') \ll M(\log \log N)^2 \ll 10^{j+1}(\log \log N)^2 \), \( E(X', Y) \gg E(X, Y) \) and \( E(X'', Y) \leq |Y||X''|^2/M \).

Firstly, consider the case where the inequality

\[
\left| \sum_{x \in X'} \sum_{y \in Y} \left( A(x + y) - \frac{1}{2} \right) \right| \geq \tilde{\varepsilon} w_1(N)(\log N)(\log \log N)|Y|
\]

does not hold. Then

\[
\left| \sum_{x \in X''} \sum_{y \in Y} \left( A(x + y) - \frac{1}{2} \right) \right| \geq \left| \sum_{x \in X'} \sum_{y \in Y} \left( A(x + y) - \frac{1}{2} \right) \right| - \left| \sum_{x \in X'} \sum_{y \in Y} \left( A(x + y) - \frac{1}{2} \right) \right| \geq \left( 1 - \frac{j + 1}{\log \log N} \right) \tilde{\varepsilon} w_1(N)(\log N)(\log \log N)^2 |Y|,
\]
and (18)–(21) hold for $j + 1$ instead of $j$ and $X''$ instead of $X$. Again, $H_{j+1}$ holds.

Now consider the case where (22) holds. Then we have

$$|X'| \geq \bar{\varepsilon} w_1(N)(\log N)(\log \log N) := n_0.$$  

Let

$$n_\nu = 2^\nu n_0, \quad \nu \leq \nu_0,$$

where $\nu_0$ is defined by

$$n_{\nu_0} \leq 2w_1(N) \log N \log \log N < n_{\nu_0+1}.$$  

Clearly, $\nu_0 \ll \log \log \log N + \log(1/\varepsilon)$. By Lemma 6, the number of such sets $X'$ with $|X'| = n, n_{\nu} \leq n < n_{\nu+1}$, is at most

$$e^{Cn_{\nu} 10^{j}(\log \log N)^2} \leq e^{2C10^{j}w_1(N)\log \log N^4},$$  \hspace{1cm} (23)

where $C$ is an absolute constant.

Next, for these sets $X'$ inequality (22) implies that

$$|\sigma_A(X', Y)| \geq \varepsilon' = \bar{\varepsilon} w_1(N)(\log N)(\log \log N)/n_{\nu+1} \geq$$

$$\geq \bar{\varepsilon} w_1(N)(\log N)(\log \log N)/(2n_{\nu}).$$

We have

$$n_\nu(\varepsilon')^2/(4 \log N) \geq n_\nu \bar{\varepsilon}^2 w_1(N)^2(\log N)^2(\log \log N)/ (16n_{\nu}^2 \log N)$$

$$= \varepsilon^2 w_1(N)^2(\log N)(\log \log N)^2/(16n_{\nu}) \geq \varepsilon^2 w_1(N)/32 > 1,$$

where we have used (17), and we are in position to use Proposition 16. We have

$$E(X', Y) \leq E(X, Y) \leq |X|^2|Y|/10^j$$

$$\leq (2w_1(N) \log N(\log \log N)^2)|Y|/10^j = n_{\nu}^2|Y|/K' \leq |X'|^2|Y|/K',$$

where

$$K' = 10^j \left( n_{\nu}/(2w_1(N) \log N(\log \log N)^2) \right)^2$$

$$= 10^j n_{\nu}^2/ \left( 4w_1(N)^2(\log N)^2(\log \log N)^4 \right).$$

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Thus, for any such $X'$ the probability $P_{j, \nu}(X')$ of the existence of a set $Y$ satisfying this inequality is at most

$$\exp\left(-\frac{6(m \max\{K', 1\})}{64}\right) \leq \exp\left(-\frac{(\varepsilon')^6 m K'}{256}\right).$$

We have

$$(\varepsilon')^6 K' = 10^j \varepsilon^6 w_1(N)^4 (n_\nu)^{-4} (\log N)^4 (\log \log N)^2 / 256 \geq 10^j \varepsilon^6 (\log \log N)^{-6} / 2^{18}. \quad (24)$$

Next, the probability $P_{j, \nu}$ of the existence of a set $X'$, $|X'| = n$, $n_\nu \leq |X'| < n_{\nu+1}$, that can be obtained by our construction, is bounded by (see (23), (24))

$$\exp \left(2C w_1(N) 10^j \log N (\log \log N)^4 - 10^j \varepsilon^6 (\log \log N)^{-6} m / 2^{26}\right).$$

Taking into account (17) and the definition of the parameter $m$, we get

$$P_{j, \nu} \leq \exp \left(-10^j w_1(N) \log N (\log \log N)^4\right).$$

Taking the sum over $\nu$, we find

$$P_j \leq P_{j+1} + \sum_{\nu} P_{j, \nu} \leq P_{j+1} + \exp \left(-10^j w_1(N) \log N (\log \log N)^4 / 2\right).$$

Finally,

$$P_0 \leq \sum_{j=0}^{j_0-1} \exp \left(-10^j w_1(N) \log N (\log \log N)^4 / 2\right) \leq \exp \left(-w_1(N) \log N (\log \log N)^4 / 3\right).$$

\[ \square \]

## 6 Appendix

In this section we prove Proposition 8.
By sections 3, 4 of [6] there are sets $S \subset X$, $T \subset Y$, $s = |S|$, $t = |T|$ such that
\[ E(S, T) \leq 2st + \frac{2s^2t^2}{|X|^2|Y|^2} \cdot E(X, Y), \tag{25} \]
and
\[ |\sigma_A(X, Y) - \sigma_A(S, T)| \leq 6 \sqrt{\frac{|Y|}{st}}. \tag{26} \]

Here $s, t$ are parameters and we choose $s = \frac{2000 \log N}{\varepsilon^4} \leq |X|$ and $t = \frac{K |Y| \varepsilon^2}{10 \log N}$. The left-hand side of (26) is less than $\varepsilon/2$. On the other hand, by the large deviations low (see Proposition 3 and calculations after this proposition from [6]) and (25) the probability $\mathbb{P}$ of $|\sigma_A(S, T)| \geq \varepsilon/2$ is bounded by
\[ \mathbb{P} \ll \exp \left( - \frac{\varepsilon^2 s^2 t^2}{2E(S, T)} \right) \ll \exp \left( - \min \left\{ \frac{\varepsilon^2 st}{8}, \frac{\varepsilon^2 |X|^2|Y|^2}{8E(X, Y)} \right\} \right) \ll \exp \left( - \frac{\varepsilon^2 |Y|}{8} \right). \]

Thus, the final probability (8) does not exceed
\[ N^{s+t} \exp \left( - \frac{\varepsilon^2 K |Y|}{8} \right) \ll \exp \left( \frac{2000 \log^2 N}{\varepsilon^4} + \frac{K |Y| \varepsilon^2}{10} - \frac{\varepsilon^2 K |Y|}{8} \right) \leq \exp \left( \frac{2000 \log^2 N}{\varepsilon^4} - \frac{\varepsilon^2 r K}{40} \right) \]
as required.

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