Merton’s portfolio problem with power utility under Volterra Heston model

Bingyan Han ∗ Hoi Ying Wong†

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Abstract

This paper investigates Merton’s portfolio problem in a rough stochastic environment described by Volterra Heston model. The model has a non-Markovian and non-semimartingale structure. By considering an auxiliary random process, we solve the portfolio optimization problem with the martingale optimality principle. The optimal strategy is derived in a semi-closed form that depends on the solution of a Riccati-Volterra equation. Numerical studies suggest that investment demand decreases with the roughness of the market.

Keywords: Finance, optimal portfolio, rough volatility, Volterra Heston model, Riccati-Volterra equations

Mathematics Subject Classification: 93E20, 60G22, 49N90, 60H10.

1 Introduction

Empirical studies suggest that the volatility process of major financial indices tend to have rougher sample paths than the ones modeled by the standard Brownian motion Gatheral et al. (2018). The discovery stimulates a rapidly growing development in rough volatility models recently. Classic rough volatility models include fractional Brownian motion (fBm), fractional Ornstein-Uhlenbeck (fOU) process, and rough Bergomi (rBergomi) model. The popularity of the Heston model in the financial market leads to the introduction of the fractional Heston model Guennoun et al. (2018) and the rough Heston model El Euch and Rosenbaum (2019). Both are rough versions of the celebrated Heston stochastic volatility model. Remarkable recent advances include the derivation of the characteristic function of the rough Heston model El Euch and Rosenbaum (2019) and the affine Volterra processes in Abi Jaber et al. (2017). Specifically, the Volterra Heston model serves as an important

∗Department of Statistics, The Chinese University of Hong Kong, Hong Kong, byhan@link.cuhk.edu.hk
†Department of Statistics, The Chinese University of Hong Kong, Hong Kong, hywong@cuhk.edu.hk
specific example in Abi Jaber et al. (2017). In addition, the rough Heston model becomes a special case of Volterra Heston model under the so-called fractional kernel, $K(t) = t^{H-1/2}/\Gamma(H+1/2)$. The structure of characteristic functions in El Euch and Rosenbaum (2019) can be extended to affine Volterra processes using Riccati-Volterra equations as shown in Abi Jaber et al. (2017). Therefore, this paper focuses on the financial market with the Volterra Heston model.

We are interested in a question: how does the roughness of the market volatility affect investment demands? We address this by investigating the optimal investment demand with the Merton problem as it is probably the most classic financial economic approach to do so. The literature tends to focus more on the option pricing problems and portfolio optimization under rough volatility models is still at an early stage. However, some recent works do exist (Fouque and Hu, 2018a,b; Bäuerle and Desmettre, 2018; Glasserman and He, 2019; Han and Wong, 2019). The studies on (Fouque and Hu, 2018a,b) consider the expected power utility portfolio maximization with slow or fast varying stochastic factors driven by the fOU processes whereas the fractional Heston model is used in Bäuerle and Desmettre (2018) with the same objective function. Due to some market insights, it is suggested in Glasserman and He (2019) to use roughness as a trading signal. To the best of our knowledge, portfolio selection with Volterra Heston model is firstly studied in Han and Wong (2019) in the context of mean-variance objective.

In this paper, we investigate the Merton portfolio problem with an unbounded risk premium which is in contrast to the studies in (Fouque and Hu, 2018a,b) of assuming an essentially bounded risk premium. Therefore, their results are not directly applicable to our problem. Compared with Bäuerle and Desmettre (2018), we allow the correlation between stock and volatility to be non-zero, reflecting the well-known market leverage effect. Although the power utility maximization with the classic Heston model has been studied in Kraft (2005), the non-Markovian and non-semimartingale characteristic in Volterra Heston model prevents the use of the Hamilton-Jacobi-Bellman (HJB) framework for the classical model in Kraft (2005).

To overcome the aforementioned difficulty, we apply the martingale optimality principle and construct the Ansatz, which is inspired by the martingale distortion transformation (Zariphopoulou, 2001; Fouque and Hu, 2018a) and the exponential-affine representations Abi Jaber et al. (2017). The key finding is the auxiliary process $M_t$ in (3.5) and the properties of it presented in Theorem 3.1 below. Although certain auxiliary processes are also introduced in Fouque and Hu (2018a) and Han and Wong (2019) to circumvent difficulties for their problems, ours is significantly different from theirs because the $M_t$ in (3.5) is unbounded, making the proof of Theorem 3.2 different from both Fouque and Hu (2018a) and Han and Wong (2019). Consequently, we offer an explicit solution to the optimal portfolio policy that depends on a Riccati-Volterra equation, which can be solved by well-known numerical methods.

With the optimal investment demand, we partially address the effect of market rough-
ness on investment decisions. Under rough Heston model, which is a Volterra Heston model with the fractional kernel, the rougher the market volatility, the lower the investment demand. It suggests investing less if the stock is rougher. Note that it does not directly contradict to the “buy rough, sell smooth” strategy in Glasserman and He (2019). Glasserman and He (2019) considers multiple stocks and benefits from the cross-sectional relationship between roughness and stock returns. A stock can be sold if it is not rough like others.

The rest of this paper is organized as follows. We present the problem formulation in Section 2 and solve the problem by the martingale optimality principle in Section 3. Section 4 offers numerical illustration for the investment demand under the rough Heston model. Section 5 concludes.

2 Problem formulation

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space, with a filtration \(\mathcal{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}\) satisfying the usual conditions, supporting a two-dimensional Brownian motion \(W = (W_1, W_2)\). The filtration \(\mathcal{F}\) is not necessarily the augmented filtration generated by \(W\).

Denote a kernel \(K(\cdot) \in L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{R})\) where \(\mathbb{R}_+ = \{t \in \mathbb{R} | t \geq 0\}\). Suppose the standing Assumption 2.1 holds throughout the paper, in line with (Abi Jaber et al., 2017; Keller-Ressel et al., 2018; Han and Wong, 2019). Recall that a function \(f\) is completely monotone if it is infinitely differentiable and \((-1)^k f^{(k)}(t) \geq 0\) for all \(t > 0, k = 0, 1, \ldots\). Assumption 2.1 is satisfied by positive constant kernels, fractional kernels and exponential kernels, with proper parameters, see Abi Jaber et al. (2017).

**Assumption 2.1.** The kernel \(K\) is strictly positive and completely monotone. There is \(\delta \in (0, 2]\) such that \(\int_0^h K(t)^2 dt = O(h^\delta)\) and \(\int_0^T (K(t + h) - K(t))^2 dt = O(h^\delta)\) for every \(T < \infty\).

The convolution \(K \ast L\) for a measurable kernel \(K\) on \(\mathbb{R}_+\) and a measure \(L\) on \(\mathbb{R}_+\) of locally bounded variation is defined by

\[
(K \ast L)(t) = \int_0^t K(t - s)L(ds)
\]

for \(t > 0\) under proper conditions. The integral is extended to \(t = 0\) by right-continuity if possible. If \(F\) is a function on \(\mathbb{R}_+\), let

\[
(K \ast F)(t) = \int_0^t K(t - s)F(s)ds.
\]

For a 1-dimensional continuous local martingale \(W\), the convolution between \(K\) and \(W\) is defined as

\[
(K \ast dW)_t = \int_0^t K(t - s)dW_s.
\]
A measure $L$ on $\mathbb{R}_+$ is called \textit{resolvent of the first kind} to $K$, if
\begin{equation}
K \ast L = L \ast K \equiv \text{id}.
\end{equation}

Kernel $R$ is called the \textit{resolvent}, or \textit{resolvent of the second kind}, to $K$ if
\begin{equation}
K \ast R = R \ast K = K - R.
\end{equation}

Further properties of these definitions can be found in (Gripenberg et al., 1990; Abi Jaber et al., 2017). Examples of kernels are available at Abi Jaber et al. (2017, Table 1).

The variance process of the Volterra Heston model is defined as
\begin{equation}
V_t = V_0 + \kappa \int_0^t K(t-s)(\phi - V_s) \, ds + \int_0^t K(t-s)\sigma \sqrt{V_s} dB_s,
\end{equation}
where $dB_s = \rho dW_{1s} + \sqrt{1 - \rho^2} dW_{2s}$ and $V_0, \kappa, \phi, \sigma$ are positive constants. The correlation $\rho$ between stock price and variance is also assumed constant. The process in (2.6) is non-Markovian and non-semimartingale in general. Rough Heston model in (El Euch and Rosenbaum, 2019, 2018) is a special case of (2.6) with $K(t) = \frac{t^{H-1/2}}{\Gamma(H+1/2)}$, $H \in (0,1/2]$.

An alternative definition for Heston model with rough paths in Guennoun et al. (2018) is known as fractional Heston model. The power utility maximization with the fractional Heston model has been investigated in Bäuerle and Desmettre (2018) for the case of zero correlation. Our consideration is much general because we incorporate the market leverage effect with a non-zero $\rho$ in the Volterra Heston model.

Suppose there is a risk-free asset with deterministic bounded risk-free rate $r_t > 0$. Following (Abi Jaber et al., 2017; Kraft, 2005), we assume the risky asset (stock or index) price $S_t$ follows
\begin{equation}
dS_t = S_t(r_t + \theta V_t) \, dt + S_t \sqrt{V_t} dW_{1t}, \quad S_0 > 0,
\end{equation}
with constant $\theta \neq 0$. Then the market price of risk (risk premium) is given by $\theta \sqrt{V_t}$.

We need the following existence and uniqueness result.

**Theorem 2.2.** (Theorem 7.1 in Abi Jaber et al. (2017)) Under Assumption 2.1, the stochastic Volterra equation (2.6)-(2.7) has a unique in law $\mathbb{R}_+ \times \mathbb{R}_+$-valued continuous weak solution for any initial condition $(S_0, V_0) \in \mathbb{R}_+ \times \mathbb{R}_+$.

Pathwise uniqueness for (2.6)-(2.7) is still an open problem. For weak solutions, Brownian motion is also a part of the solution. However, expected power utility (2.9) only depends on the expectation of the wealth process. In the sequel, we fix a version of the solution $(S, V, W_1, W_2)$ to (2.6)-(2.7) as other solutions have the same law.

Let $\alpha_t \triangleq \sqrt{V_t} \pi_t$ be the investment strategy, where $\pi_t$ is the proportion of wealth invested in the stock. Then, the wealth process $X_t$ reads
\begin{equation}
dX_t = (r_t + \theta \sqrt{V_t} \alpha_t) X_t dt + \alpha_t X_t dW_{1t}, \quad X_0 = x_0 > 0.
\end{equation}
Definition 2.3. An investment strategy \( \alpha(\cdot) \) is said to be admissible if

1. \( \alpha(\cdot) \) is \( \mathbb{F} \)-adapted and \( \int_0^T \alpha_s^2 ds < \infty, \mathbb{P}\text{-a.s.} \);
2. the wealth process (2.8) has a unique solution in the sense of Yong and Zhou (1999, Chapter 1, Definition 6.15), with \( \mathbb{P}\text{-a.s.} \) continuous paths;
3. \( X_t \geq 0, \mathbb{P}\text{-a.s., a.e. } t \in [0, T] \);
4. \( \mathbb{E}\left[ \frac{1}{\gamma} X_T^\gamma \right] < \infty, \ 0 < \gamma < 1 \).

The set of all admissible investment strategies is denoted as \( A \).

We are interested in the Merton problem with a power utility optimization:

\[
\sup_{\alpha(\cdot) \in A} \mathbb{E}\left[ \frac{1}{\gamma} X_T^\gamma \right], \ 0 < \gamma < 1. \quad (2.9)
\]

To ease notation burden, we simply write \( X, \) instead of \( X_{x_0, \alpha}, \) as the wealth process (2.8) under \( \alpha \in A \) with initial condition \( X_0 = x_0 > 0. \)

3 Optimal strategy

The classical martingale optimality principle, see, e.g., Pham (2009, Section 6.6.1) or Jeanblanc et al. (2012), states that the Problem (2.9) can be solved by constructing a family of processes \( \{J^\alpha_t\}_{t \in [0, T]}, \alpha \in A, \) satisfying conditions:

1. \( J^\alpha_T = \frac{1}{\gamma} X_T^\gamma \) for all \( \alpha \in A; \)
2. \( J^\alpha_0 \) is a constant, independent of \( \alpha \in A; \)
3. \( J^\alpha_t \) is a supermartingale for all \( \alpha \in A, \) and there exists \( \alpha^* \in A \) such that \( J^{\alpha^*} \) is a martingale.

Indeed, if we can find \( J^\alpha_t, \) then for all \( \alpha \in A, \)

\[
\mathbb{E}\left[ \frac{1}{\gamma} X_T^\gamma \right] = \mathbb{E}[J^\alpha_T] \leq J^\alpha_0 = J^\alpha^* = \mathbb{E}[J^{\alpha^*}_T] = \mathbb{E}\left[ \frac{1}{\gamma} (X^*_T)^\gamma \right],
\]

where \( X^* \) is the wealth process under \( \alpha^*. \)

To construct \( J^\alpha_t, \) we introduce a new probability measure \( \hat{\mathbb{P}} \) together with \( \hat{W}_t \triangleq W_t - \frac{\theta}{1-\gamma} \int_0^t \sqrt{\nu_s} ds \) as the new Brownian motion by Abi Jaber et al. (2017, Lemma 7.1). Under \( \hat{\mathbb{P}}, \)

\[
V_t = V_0 + \int_0^t K(t-s) (\kappa \phi - \lambda V_s) ds + \int_0^t K(t-s) \sigma \sqrt{\nu_s} d\hat{B}_s, \quad (3.1)
\]
with $\lambda = \kappa - \frac{\gamma}{1-\gamma} \rho \theta \sigma$ and $d\tilde{B}_s = \rho d\tilde{W}_1 + \sqrt{1-\rho^2} dW_2$.

Denote $\tilde{E}[]$ and $\tilde{E}_t[.] = \tilde{E}[][\mathcal{F}_t]$ as the $\tilde{P}$-expectation and conditional $\tilde{P}$-expectation, respectively. The forward variance under $\tilde{P}$ is the conditional $\tilde{P}$-expected variance, that is, $\tilde{E}_t[V_s] \equiv \xi_t(s)$. It is shown in Keller-Ressel et al. (2018, Proposition 3.2) and Abi Jaber et al. (2017, Lemma 4.2) that

$$\xi_t(s) = \tilde{E}[V_s|\mathcal{F}_t] = \xi_0(s) + \int_0^t \frac{1}{\lambda} R_\lambda(s-u) \sigma \sqrt{V_u} d\tilde{B}_u,$$

where

$$\xi_0(s) = \left(1 - \int_0^s R_\lambda(u) du \right) V_0 + \frac{\kappa \phi}{\lambda} \int_0^s R_\lambda(u) du,$$

and $R_\lambda$ is the resolvent of $\lambda K$ such that

$$\lambda K * R_\lambda = R_\lambda * (\lambda K) = \lambda K - R_\lambda.$$

Consider the stochastic process,

$$M_t = \exp \left[ \int_0^T \left( \gamma r_s + \frac{\gamma^2 \theta \xi_t(s)}{2(1-\gamma)} + \frac{c \sigma^2}{2} \psi^2(T-s) \xi_t(s) \right) ds \right],$$

where $c = \frac{1-\gamma}{1-\gamma+\gamma \rho^2}$ and $\psi(\cdot)$ satisfies the Riccati-Volterra equation

$$\psi(t) = \int_0^t K(t-s) \left[ \frac{\sigma^2}{2} \psi^2(s) - \lambda \psi(s) + \frac{\gamma \theta^2}{2c(1-\gamma)} \right] ds, \quad \psi(0) = 0.$$

Existence and uniqueness of the solution to (3.6) are established in Han and Wong (2019, Lemma A.2 and A.3) based on the results of (Gatheral and Keller-Ressel, 2018; El Euch and Rosenbaum, 2018). Indeed, if $\lambda > 0$ and $\lambda^2 - \frac{\gamma \theta^2 \sigma^2}{c(1-\gamma)} > 0$, then (3.6) has a unique non-negative global solution. These assumptions are also in line with Kraft (2005, Proposition 5.2). Furthermore, there is a tighter result for (3.6) with the fractional kernel in El Euch and Rosenbaum (2018, Theorem 3.2).

By considering $M_t$, we overcome the non-Markovian and non-semimartingale difficulty in the variance process (2.6). Main properties of $M_t$ are summarized in Theorem 3.1. We highlight that the $M_t$ in (3.5) is unbounded so that it is very different from the one considered in (Fouque and Hu, 2018a; Han and Wong, 2019).

**Theorem 3.1.** Assume

$$\kappa^2 - 6 \frac{\gamma^2}{(1-\gamma)^2} \theta^2 \sigma^2 > 0, \quad \lambda > 0, \quad \lambda^2 - 2p \frac{\gamma}{1-\gamma} \theta^2 \sigma^2 > 0,$$

for some $p > 1/(2c)$. Then $M$ has following properties:
(1) $M_t \geq l > 0$ for some positive constant $l$. And $\mathbb{E}[\sup_{t \in [0,T]} |M_t|^p] < \infty$;

(2) Apply Itô’s lemma to $M$ on $t$, then

$$dM_t = -\left[\gamma r_t + \frac{\gamma}{2(1-\gamma)} \theta^2 V_t\right] M_t dt - \frac{\gamma}{2(1-\gamma)} \left[2\theta \sqrt{V_t} U_{1t} + \frac{U_{2t}^2}{M_t}\right] dt$$

$$+ U_{1t} dW_{1t} + U_{2t} dW_{2t},$$

where

$$U_{1t} = \rho c \sigma M_t \sqrt{V_t} \psi(T-t),$$

$$U_{2t} = \sqrt{1-\rho^2 c \sigma M_t \sqrt{V_t} \psi(T-t)}.$$  (3.8)

(3) $\mathbb{E}\left[\left( \int_0^T U_{2t}^2 dt \right)^{p/4}\right] < \infty$ for $i = 1, 2$.

Proof. First of all, we point out that there exists a unique continuous solution to (3.6) over $[0,T]$ under Assumption (3.7). Then, we claim

$$M_t^{1/c} = \mathbb{E}_t \left[ \exp \left( \int_t^T \left( \frac{\gamma r_s}{c} + \frac{\gamma}{2c(1-\gamma)} \theta^2 V_s ds \right) \right) \right].$$  (3.11)

Indeed, by Abi Jaber et al. (2017, Theorem 4.3),

$$\exp \left[ \int_t^T \left( \frac{\gamma \theta^2}{2c(1-\gamma)} \xi_t(s) + \frac{\sigma^2}{2} \psi^2(T-s) \xi_t(s) ds \right) \right] = \tilde{\mathbb{E}}_t \left[ \exp \left( \int_t^T \gamma \theta^2 ds \right) \right].$$

The martingale assumption in Abi Jaber et al. (2017, Theorem 4.3) is guaranteed by Abi Jaber et al. (2017, Lemma 7.3) for Volterra Heston model.

As $V_t$ is non-negative, $r_t > 0$ is deterministic, and $1-\gamma \leq c \leq 1$, we have $M_t \geq l > 0$ in view of (3.11).

Let $L \triangleq \exp \left( \int_0^T \frac{\gamma \theta^2}{2c(1-\gamma)} V_t ds \right)$ and the Radon-Nikodym derivative at $\mathcal{F}_T$ as

$$R = \exp \left( -\frac{\gamma^2 \theta^2}{(1-\gamma)^2} \int_0^T V_t dt + \gamma \theta \int_0^T \sqrt{V_t} dW_{1t} \right).$$

Then

$$\mathbb{E}\left[ \sup_{t \in [0,T]} |M_t|^p \right] \leq C \mathbb{E}\left[ \sup_{t \in [0,T]} \tilde{\mathbb{E}}_t \left[ \exp \left( \int_t^T \frac{\gamma \theta^2}{2c(1-\gamma)} V_s ds \right) \right]^{pc} \right]$$

$$\leq C \mathbb{E}\left[ \sup_{t \in [0,T]} \tilde{\mathbb{E}}_t \left[ \exp \left( \int_0^T \frac{\gamma \theta^2}{2c(1-\gamma)} V_s ds \right) \right]^{pc} \right]$$

$$\leq C \mathbb{E}\left[ \sup_{t \in [0,T]} \tilde{\mathbb{E}}_t [L]^{2pc} \right]^{1/2} \mathbb{E} [R^{-1}]^{1/2}. $$
By Han and Wong (2019, Theorem 2.5 and Lemma A.2), we have \( \tilde{E}[L] < \infty \) if \( \lambda > 0 \) and \( \lambda^2 - \frac{2\gamma^2}{1-\gamma}\theta^2\sigma^2 > 0 \). Therefore, \( \tilde{E}_t[L] \) is a martingale under \( \tilde{P} \). Note \( 2p \sigma > 1 \), by Doob’s maximal inequality,

\[
\tilde{E}\left[ \sup_{t \in [0,T]} \tilde{E}_t[L]^{2p} \right] \leq C \tilde{E}\left[ \exp \left( \frac{p\gamma\theta^2}{1-\gamma} \int_0^T V_s ds \right) \right] < \infty.
\]

The last inequality holds under the assumption that \( \lambda > 0 \) and \( \lambda^2 - 2p\gamma\theta^2\sigma^2 > 0 \). The argument is the same for \( \tilde{E}[L] < \infty \). Moreover, by Hölder’s inequality,

\[
\mathbb{E}[R^{-1}] \leq \mathbb{E}\left[ \exp \left( \frac{3\gamma^2\theta^2}{(1-\gamma)^2} \int_0^T V_t dt \right) \right]^{1/2} \times \mathbb{E}\left[ \exp \left( -\frac{2\gamma^2\theta^2}{(1-\gamma)^2} \int_0^T V_t dt - \frac{2\gamma\theta}{1-\gamma} \int_0^T \sqrt{V_t} dW_t \right) \right]^{1/2} < \infty.
\]

\( \mathbb{E}[e^{\frac{3\gamma^2\theta^2}{(1-\gamma)^2} \int_0^T V_t dt}] \) is finite since \( \kappa^2 - 6\frac{\gamma^2}{(1-\gamma)^2}\theta^2\sigma^2 > 0 \). Therefore, \( \mathbb{E}\left[ \sup_{t \in [0,T]} |M_t|^p \right] < \infty \) holds.

For property (2), the proof is in the same spirit of Han and Wong (2019, Theorem 4.1 (2)). Let

\[
Z_t = \int_t^T \left[ \gamma r_s + \frac{\gamma\theta^2 \xi_t(s)}{2(1-\gamma)} + \frac{c\sigma^2}{2} \psi^2(T-s)\xi_t(s) \right] ds.
\]

Then \( M_t = e^{Z_t} \). Applying Itô’s lemma to \( \xi_t(s) \) on time \( t \) yields

\[
d\xi_t(s) = \frac{1}{\lambda} R_\lambda(s-t)\sigma \sqrt{V_t} d\hat{B}_t
\]

from (3.2). Then

\[
dZ_t = \left[ -\gamma r_t - \frac{\gamma\theta^2}{2(1-\gamma)} V_t - \frac{c\sigma^2}{2} \psi^2(T-t)V_t \right] dt
\]

\[
+ \frac{\gamma\theta^2}{2(1-\gamma)} \int_t^T \frac{1}{\lambda} R_\lambda(s-t)\sigma \sqrt{V_t} d\hat{B}_s ds + \frac{c\sigma^2}{2} \int_t^T \psi^2(T-s) \frac{1}{\lambda} R_\lambda(s-t)\sigma \sqrt{V_t} d\hat{B}_s ds
\]

\[
= \left[ -\gamma r_t - \frac{\gamma\theta^2}{2(1-\gamma)} V_t - \frac{c\sigma^2}{2} \psi^2(T-t)V_t \right] dt
\]

\[
+ \frac{\gamma\theta^2}{2(1-\gamma)} \int_t^T \sigma \frac{1}{\lambda} R_\lambda(s-t) ds \sqrt{V_t} d\hat{B}_t + \frac{c\sigma^2}{2} \int_t^T \sigma \psi^2(T-s) \frac{1}{\lambda} R_\lambda(s-t) ds \sqrt{V_t} d\hat{B}_t.
\]

The second equality relies on the stochastic Fubini theorem Veraar (2012).
Next, we show
\[
\int_t^T \left[ \frac{c \sigma^2}{2} \psi^2(T_s) + \frac{\gamma \theta^2}{2(1 - \gamma)} \right] \frac{1}{\lambda} R_\lambda(s-t) ds = c \psi(T-t). \tag{3.14}
\]

In fact,
\[
\int_t^T \left[ \frac{c \sigma^2}{2} \psi^2(T_s) + \frac{\gamma \theta^2}{2(1 - \gamma)} \right] \frac{1}{\lambda} R_\lambda(s-t) ds - c \psi(T-t)
\]
\begin{align*}
&= \left[ \frac{c \sigma^2}{2} \psi^2 + \frac{\gamma \theta^2}{2(1 - \gamma)} \right] \frac{1}{\lambda} R_\lambda(T-t) - cK \frac{\sigma^2}{2c(1 - \gamma)} \psi + \frac{\gamma \theta^2}{2c(1 - \gamma)} (T-t) \\
&= \left[ \frac{c \sigma^2}{2} \psi^2 + \frac{\gamma \theta^2}{2(1 - \gamma)} \right] \frac{1}{\lambda} R_\lambda - K(T-t) + c\lambda K \psi(T-t) \\
&= -R_\lambda * K \left[ \frac{c \sigma^2}{2} \psi^2 + \frac{\gamma \theta^2}{2(1 - \gamma)} \right] (T-t) + c\lambda K \psi(T-t) \\
&= c(\lambda K - R_\lambda - \lambda K * R_\lambda) \psi(T-t) = 0.
\end{align*}

We have used the equality
\[
R_\lambda * \psi = R_\lambda * K \left[ \frac{\sigma^2}{2} \psi^2 - \lambda \psi + \frac{\gamma \theta^2}{2c(1 - \gamma)} \right]. \tag{3.15}
\]

Therefore,
\[
dM_t = M_t dZ_t + \frac{1}{2} M_t dZ_t dZ_t
\]
\begin{align*}
&= M_t \left[ -\gamma r_t - \frac{\gamma \theta^2}{2(1 - \gamma)} V_t - \frac{c \sigma^2}{2} \psi^2(T-t) V_t \right] dt + \frac{U_1^2 + U_2^2}{2M_t} dt + U_1 dt W_1 + U_2 dt W_2 \\
&= - \left[ \frac{\gamma r_t}{2(1 - \gamma)} \theta^2 V_t \right] M_t dt - \frac{\gamma \theta}{1 - \gamma} \theta \sqrt{V_t} U_1 dt - \frac{\gamma \theta}{2(1 - \gamma)} \frac{U_1^2}{M_t} dt \\
&+ U_1 dt W_1 + U_2 dt W_2.
\end{align*}

Finally, for property (3),
\begin{align*}
\mathbb{E}\left[ \left( \int_0^T U_1^2 dt \right)^{p/4} \right] &\leq C \mathbb{E}\left[ \sup_{t \in [0,T]} |M_t|^{p/2} \left( \int_0^T V_t dt \right)^{p/4} \right] \\
&\leq C \mathbb{E}\left[ \sup_{t \in [0,T]} |M_t|^p \right]^{1/2} \mathbb{E}\left[ e^{a \int_0^T V_t dt} \right]^{1/2} < \infty,
\end{align*}
where \( a > 0 \) is constant.
Now we are ready to give the Ansatz for $J_t^\alpha$. Consider

$$J_t^\alpha = \frac{X_t^\gamma}{\gamma} M_t. \tag{3.16}$$

Then we have the following verification result.

**Theorem 3.2.** Suppose the conditions (3.7) in Theorem 3.1 hold and $\kappa^2 - 2\eta \sigma^2 > 0$, where

$$\eta = \max \left\{ 2q|\theta| \sup_{t \in [0,T]} |A_t|, (8q^2 - 2q) \sup_{t \in [0,T]} |A_t|^2 \right\},$$

for some $q > 1$ and $A_t = \frac{1}{\gamma} [\theta + \rho c \sigma \psi (T - t)]$. Then $J_t^\alpha = \frac{X_t^\gamma}{\gamma} M_t$ satisfies the martingale optimality principle, and the optimal strategy is given by

$$\alpha_t^* = A_t \sqrt{V_t}. \tag{3.17}$$

Moreover,

$$\mathbb{E} \left[ \sup_{t \in [0,T]} |X_t^\alpha|^q \right] < \infty, \tag{3.18}$$

and $\alpha^*$ is admissible.

**Proof.** (1). Clearly, as $M_T = 1$, $J_T^\alpha = \frac{X_T^\gamma}{\gamma}$.

(2). As $M_0$ is a constant independent of $\alpha \in A$, $J_0^\alpha = \frac{X_0^\gamma}{\gamma} M_0$ is a constant independent of $\alpha \in A$.

(3). By Itô’s lemma,

$$dJ_t^\alpha = \left[ \theta \alpha_t \sqrt{V_t} + \frac{1}{\gamma} \alpha_t^2 \right] M_t X_t^\gamma dt + \alpha_t X_t^\gamma U_{1t} dt - \frac{1}{2(1 - \gamma)} \left[ \theta \sqrt{V_t} + U_{1t} \right]^2 M_t X_t^\gamma dt$$

$$+ \left[ \frac{X_t^\gamma U_{1t}}{\gamma} + M_t \alpha_t X_t^\gamma \right] dW_{1t} + \frac{X_t^\gamma U_{2t}}{\gamma} dW_{2t}$$

$$\triangleq J_t^\alpha F(\alpha, t) dt + J_t^\alpha \left[ U_{1t} + \gamma \alpha_t \right] dW_{1t} + J_t^\alpha \frac{U_{2t}}{M_t} dW_{2t},$$

where

$$F(\alpha, t) = \frac{\gamma(\gamma - 1)}{2} \alpha^2 + \gamma \left[ \theta \sqrt{V_t} + \frac{U_{1t}}{M_t} \right] \alpha - \frac{\gamma}{2(1 - \gamma)} \left[ \theta \sqrt{V_t} + \frac{U_{1t}}{M_t} \right]^2.$$

$\alpha^*$ in (3.17) is derived from $\frac{\partial F}{\partial \alpha} = 0$. Note $F(\alpha, t)$ is a quadratic function on $\alpha$ and $\gamma - 1 < 0$. Since $F(\alpha^*, t) = 0$, then $F(\alpha, t) \leq 0$.

Moreover, $J_t^\alpha = \frac{M_0 \gamma}{\gamma} e^{\int_0^t F(\alpha, s) ds} G_t$, where

$$G_t = \exp \left[ - \frac{1}{2} \int_0^t \left( \frac{U_{1s}}{M_s} + \gamma \alpha_s \right)^2 + \frac{U_{2s}}{M_s^2} \right] ds + \int_0^t \frac{U_{1s}}{M_s} + \gamma \alpha_s \right] dW_{1s} + \int_0^t \frac{U_{2s}}{M_s} dW_{2s}. \tag{3.19}$$
Since \( \int_0^t \alpha_s^2 ds < \infty \), \( \mathbb{P} \)-a.s., \( G_t \) is a local martingale. Furthermore, \( J_t^\alpha \) is a supermartingale because \( e^{\int_0^t F(\alpha_s, s) ds} \) is non-increasing and \( J_t^\alpha \) is bounded below.

For \( \alpha_t = \alpha_t^* \), \( G_t \) is a martingale by Abi Jaber et al. (2017, Lemma 7.3). Subsequently, \( J_t^\alpha \) is a true martingale. We have verified all conditions required by martingale optimality principle, except for the admissibility of \( \alpha^* \).

By Doob’s maximal inequality,

\[
E\left[ \sup_{t \in [0,T]} |X_t^\alpha|^q \right] \leq C E\left[ \sup_{t \in [0,T]} e^{2q \int_0^t \theta A_s V_s ds} \right]^{1/2} \times \\
E\left[ \sup_{t \in [0,T]} \exp \left( -\int_0^t \frac{A_s^2}{2} V_s ds + \int_0^t A_s \sqrt{V_s} dW_1 \right) \right]^{2q - 1/2} \\
\leq C E\left[ e^{2q \int_0^T \theta A_s |V_s| ds} \right]^{1/2} \times \\
E\left[ \exp \left( -q \int_0^T A_s^2 V_s ds + 2q \int_0^T A_s \sqrt{V_s} dW_1 \right) \right]^{1/2}.
\]

The first term is finite. The second term is also finite. In fact, by Hölder’s inequality,

\[
E\left[ \exp \left( -\int_0^T q A_s^2 V_s ds + \int_0^T 2q A_s \sqrt{V_s} dW_1 \right) \right] \\
\leq E\left[ e^{(8q^2 - 2q) \int_0^T A_s^2 V_s ds} \right]^{1/2} \times E\left[ \exp \left( -8q^2 \int_0^T A_s^2 V_s ds + 4q \int_0^T A_s \sqrt{V_s} dW_1 \right) \right]^{1/2} \\
< \infty.
\]

\( E\left[ \sup_{t \in [0,T]} |X_t^\alpha|^q \right] < \infty \) is proved. It becomes straightforward to verify \( \alpha^* \) is admissible.

\( \square \)

4 Numerical illustration

In this section, we consider the fractional kernel \( K(t) = \frac{t^H}{\Gamma(H + 1/2)}, \) \( H \in (0, 1/2) \). The smaller the Hurst parameter \( H \), the rougher the volatility of the stock. We use the Adams method (El Euch and Rosenbaum, 2019; Han and Wong, 2019) to solve the Riccati-Volterra equation (3.6) numerically. Figure 1 shows the \( A_t \) in \( \alpha^* \) defined in (3.17), under different values of \( H \). Assumptions in Theorem 3.1 and 3.2 are satisfied by the parameter setting detailed in figure descriptions. \( H = \frac{1}{2} \) corresponds to the strategy in Kraft (2005) under the classical Heston model. Figure 1 exhibits that if the stock volatility is rougher, the investment demand (3.17) is smaller. It is partially due to the market leverage. In the equity market, the correlation \( \rho \) is usually negative. \( \psi(t) \) is positive for \( t > 0 \). Moreover, the value of \( \psi(t) \) becomes larger for a smaller \( H \).
Figure 1: $A_t$ under different $H$. We set risk aversion $\gamma = 0.5$, volatility of volatility $\sigma = 0.02$, mean-reversion speed $\kappa = 0.1$, risk premium parameter $\theta = 0.5$, correlation $\rho = -0.7$, and time horizon $T = 1$.

5 Concluding remarks

In this paper, we solve Merton’s portfolio optimization with the power utility under Volterra Heston model. Interestingly, the investment demand suggests buying less on a single asset when volatility gets rougher. The novelty of the solution approach stems on the proper use of the martingale optimality principle and the novel auxiliary stochastic process $M_t$ in the text. A future research may consider a general concave utility for the Merton’s problem under the Volterra Heston model.

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