Chapter 12
Optimal Time Evolution for Hermitian and Non-Hermitian Hamiltonians

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The shortest path between two truths in the real domain passes through the complex domain.
Jacques Hadamard, The Mathematical Intelligencer 13 (1991)

12.1 Introduction

Interest in optimal time evolution dates back to the end of the seventeenth century, when the famous brachistochrone problem was solved almost simultaneously by Newton, Leibniz, l’Hôpital, and Jacob and Johann Bernoulli. The word brachistochrone is derived from Greek and means shortest time (of flight). The classical brachistochrone problem is stated as follows: A bead slides down a frictionless wire from point A to point B in a homogeneous gravitational field. What is the shape of the wire that minimizes the time of flight of the bead? The solution to this problem is that the optimal (fastest) time evolution is achieved when the wire takes the shape of a cycloid, which is the curve that is traced out by a point on a wheel that is rolling on flat ground.

In the past few years there has been much interest in the quantum brachistochrone problem, which is formulated in a somewhat similar fashion: Consider two fixed quantum states, an initial state $|\psi_I\rangle$ and a final state $|\psi_F\rangle$ in a Hilbert space. We then consider the set of all Hamiltonians satisfying the energy constraint that the difference between the largest and smallest eigenvalues is a fixed energy $\omega$: $E_{\text{max}} - E_{\text{min}} = \omega$. Some of the Hamiltonians in this set allow the initial state $|\psi_I\rangle$ to evolve into the final state $|\psi_F\rangle$ in time $t$:

$$|\psi_F\rangle = e^{-iHt/\hbar}|\psi_I\rangle .$$  
(12.1)

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The quantum brachistochrone problem is to find the \textit{optimal} Hamiltonian, that is, the Hamiltonian that accomplishes this evolution in the shortest possible time, which we denote by $\tau$.

In this chapter we show that for Hermitian Hamiltonians the shortest evolution time $\tau$ is a nonzero quantity whose size depends on the Hilbert-space distance between the fixed initial and final state vectors. However, for complex non-Hermitian Hamiltonians, the value of $\tau$ can be made arbitrarily small. Thus, non-Hermitian Hamiltonians permit arbitrarily fast time evolution.

Of course, a non-Hermitian Hamiltonian may be physically unrealistic because it may possess complex eigenvalues and/or it may generate nonunitary time evolution, that is, time evolution in which probability is not conserved. However, there is a special class of non-Hermitian Hamiltonians that are $PT$ symmetric, that is, Hamiltonians that are invariant under combined space and time reflection. Although such Hamiltonians are not Hermitian in the Dirac sense, they do have entirely real spectra and give rise to unitary time evolution. Thus, such Hamiltonians define consistent and acceptable theories of quantum mechanics. We show in this chapter that if we use Hamiltonians of this type to solve the quantum brachistochrone problem, we can achieve arbitrarily fast time evolution without violating any principles of quantum mechanics. Thus, if it were possible to implement faster-than-Hermitian time evolution, then non-Hermitian Hamiltonians might have important applications in quantum computing.

This chapter is organized as follows: In Sect. 12.2 we introduce and describe $PT$ quantum mechanics and explain how a Hamiltonian that is not Dirac Hermitian can still define a consistent theory of quantum mechanics. Then in Sect. 12.3 we explain why complex classical mechanics allows for faster-than-conventional time evolution. In Sect. 12.4 we discuss the quantum brachistochrone for Hermitian Hamiltonians. Then, in Sect. 12.5 we extend the discussion in Sect. 12.4 to Hamiltonians that are not Dirac Hermitian. In Sect. 12.6 we explain how it might be possible for a complex Hamiltonian to achieve faster-than-Hermitian time evolution.

12.2 $PT$ Quantum Mechanics

Based on the training that one receives in a traditional quantum mechanics course, one would expect a theory defined by a non-Hermitian Hamiltonian to be physically unacceptable for a closed system\footnote{N. of E.: For \textit{open} systems non-Hermitian Hamiltonians and non unitary evolution may be perfectly physical, see e.g., Chap. 6 by G. Hegerfeldt or Chap. 4 by A. Ruschhaupt et al., this volume.} because the energy levels would most likely be complex and the time evolution would most likely be nonunitary (not probability conserving). However, theories defined by a special class of non-Hermitian Hamiltonians called $PT$-symmetric Hamiltonians can have positive real energy levels and can exhibit unitary time evolution. Such theories are consistent quantum theories. It