Instabilities in the Bogoliubov Spectrum of a condensate in a 1-D periodic potential

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We study the stability of standing wave solutions to a one-dimensional Gross-Pitaevskv equation with a periodic potential. We use some simple complex analysis and the Hamiltonian structure of the problem to give a simple rigorous criterion which guarantees the existence of non-real spectrum, which corresponds to exponential instability of the standing wave solution. This criterion can be stated simply in terms of the spectrum of one of these self-adjoint operators. When the standing wave has small amplitude this criterion simplifies further, and agrees with arguments based on the effective mass in the periodic potential.

The possibility of creating cold atom condensates in optical lattices has made it possible to observe both linear phenomena, such as Bloch oscillations [1] as well as genuinely nonlinear phenomena such as soliton generation [2] and an insulating transition [3]. While a great deal of effort has gone into the construction of exact or approximate solutions the question of stability is foremost, since it dictates what states can actually be observed, as well as governing the evolution of small disturbances to these states. Recent papers of Wu and Niu [4,5] and Burger et.al. [6] have helped elucidate the role of the dynamic and Landau instabilities. These dynamic instabilities have also been realized experimentally [7]. Most of the current theoretical stability work is either numerical or relies on either special properties of some exact solution [8–10], or some approximation, most notably a two-level approximation [5], the tight binding approximation [5,11,12], weak nonlinearity [13], or long wavelength [14]. In this paper we give a simple, rigorous, easily checked sufficient condition for the existence of exponential instabilities. In the case of a weak nonlinearity this condition reduces to a physically natural condition in terms of the standing wave profile φ is real and either periodic \( \phi(x + 1) = \phi(x) \) or antiperiodic \( \phi(x + 1) = -\phi(x) \). The linearized equation governing small disturbances is

\[ iu_t = \left( -\frac{1}{2} \partial_{xx} + V(x) + |\phi|^2(x) - \omega \right) v = L_+ v \]

\[ iv_t = \left( -\frac{1}{2} \partial_{xx} + V(x) + 3|\phi|^2(x) - \omega \right) u = L_- u \]

where \( u, v \) are the real and imaginary parts of the complex disturbance. Upon taking a Fourier transform in time this takes the form of the well-known Bogoliubov equation

\[ \mu u = L_+ v \]

\[ \mu v = L_- u. \]

Because of the Hamiltonian structure of the problem, the above eigenvalue problem can be written in vector form

\[ \vec{\psi}_t = JH(x, \mu) \vec{\psi} \]

with \( \vec{\psi} = (u, v, u_x, v_x)^t \), the matrix \( H = H^\dagger \) is Hermitian, and the matrix \( J \) is the canonical Hamiltonian form \( J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \). The standard results of Floquet theory guarantee that \( \mu \) is in the spectrum of the operator above if and only if the period map \( \mathbf{M}(\mu) \) has eigenvalues on the unit circle, where the period map \( \mathbf{M}(\mu) \) is defined by

\[ U_x = J\mathbf{H}(x, \mu) U \quad U(0, \mu) = J \]

\[ \mathbf{M}(\mu) = U(1, \mu) \]

In the second order case, which arises in the classical Floquet-Bloch theory, the location of the eigenvalues of the period map are determined entirely by the trace of the \( (2 \times 2) \) matrix \( \mathbf{M} \), there being two eigenvalues on the unit circle if \( \text{Tr}(\mathbf{M}) \in (-2, 2) \) and two eigenvalues off of the unit circle if \( \text{Tr}(\mathbf{M}) \notin [-2, 2] \). The case of the Bogoliubov operator is somewhat more complicated, since it consists of coupled second order problems, but can be treated in much the same way due to the Hamiltonian structure [15]. First we note that \( U \) and thus \( \mathbf{M} \) satisfy the relation

\[ U'(x, \mu)JU(x, \mu) = J, \]

a result known as the Poincare-Liouville theorem. From this it follows that \( \mathbf{M} \) is similar to its inverse transpose. This implies that the eigenvalues are invariant under inversion with respect to the unit circle, which in turn implies that the characteristic polynomial of \( \mathbf{M} \) has the following form:

\[ \det(\mathbf{M} - \lambda \mathbf{I}) = 1 + a\lambda + b\lambda^2 + a\lambda^3 + \lambda^4 \]

\[ a = -\text{Tr}(\mathbf{M}) \]

\[ b = \frac{1}{2} \left( \text{Tr}(\mathbf{M})^2 - \text{Tr}(\mathbf{M}^2) \right). \]
Because of the special form the above, the quartic admits an explicit factorization into quadratics,
\[
\det(M - \lambda I) = (1 - K_+ (\mu) \lambda + \lambda^2)(1 - K_-(\mu) \lambda + \lambda^2),
\]
where the quantities \(K_{\pm}\) are given by
\[
K_{\pm}(\mu) = \frac{\text{Tr}(M) \pm \sqrt{2 \text{Tr} M^2(\mu) - (\text{Tr} M(\mu))^2 + 8}}{2}.
\]
It then follows that the period map for the Bogoliubov operator has two eigenvalues on the unit circle if \(K_+\) is real and \(K_+ \in (-2, 2)\), and another two on the unit circle if \(K_-\) is real and \(K_- \in (-2, 2)\). It can be shown using complex analysis that the spectrum of the Bogoliubov operator consists of a union of piecewise smooth arcs in the complex plane, where the endpoints of the arcs are either band edges \(K_{\pm}(\mu) = \pm 2\), critical points \(K_{\pm}'(\mu) = 0\), or branch points \(K_{\pm}(\mu) = K_-(\mu)\). It is worth noting that a generic quartic does not admit a factorization into quadratics of the above form. It is only because of the orthogonal, which implies that \(\text{Tr} (\cdot) = 0\). Since \(\phi\) is either periodic or antiperiodic this implies that zero energy is a band edge for \(L_-\).

The main result of this paper is the following:

**Theorem 1** Suppose that \(\phi\) is a standing wave solution to the GP equation for which the \(L_+\) operator is in the interior of a band. Then this solution to the GP equation is exponentially unstable.

**Proof:** When \(\mu = 0\) the equations for \(u\) and \(v\) decouple, and the period map for the GP equation is block diagonal, with one \(2 \times 2\) block (denoted by \(m_{L_+}\) corresponding to \(L_+\), and the other block of \(m_{L_-}\) corresponding to \(L_-\). If \(\mu = 0\) is not a band edge for \(L_+\) it follows that \((K_+(0) - K_-(0))^2 - \text{Tr} (m_{L_+}) - \text{Tr} (m_{L_-})) \neq 0\), so the Floquet discriminants are analytic in a neighborhood of \(\mu = 0\). Since \(K_{\pm}\) are even functions it follows that both are real on some interval of the imaginary axis. If \(\mu = 0\) is in the interior of a band for \(L_+\) it follows that \(K_+(0) \in (-2, 2)\), and so there is some interval along the imaginary axis on which \(K_+\) is real and \((-2, 2)\). Thus the GP equation has a band of spectrum along the imaginary axis.

An important special case of this is given by the following:

**Corollary 1** Small amplitude solutions of the focusing GP equation corresponding to lower band edges are modulationally unstable, small amplitude solutions of the defocusing GP equation corresponding to upper band edges are modulationally unstable.

**Proof:** The proof is perturbative, considering \(L_-\) as a perturbation of \(L_+\). Since \(L_+ = L_- \pm 2 |\phi|^2\) the perturbation is strictly positive (resp. negative) and it follows that the band edges of the \(L_+\) and \(L_-\) operators satisfy \(\mu_+(L_+) \geq \mu_-(L_-)\) (resp. \(\mu_+(L_-) \leq \mu_-(L_-)\)). Since the unperturbed operator \(L_-\) has a band edge at \(\mu = 0\) it follows that, for \(||\phi||_\infty \ll 1\), \(\mu = 0\) will be in the interior of a band of \(L_+\) for the focusing case and a lower band edge, or defocusing case and an upper band edge. From the above result this guarantees the existence of an instability.

This result provides a rigorous justification of the intuition provided by the effective mass theories, such as those proposed by Konotop and Salerno [13] and Taylor and Zaremba [14] where the effective mass is given by the curvature of the dispersion relation. Since this curvature has the same sign as the bare mass at lower band edges and the opposite sign at upper band edges this kind of argument leads to the same conclusion. Another way to look at this result is that in either of the
above cases (lower band edge with a focusing nonlinearity or upper band edge with a defocusing nonlinearity) the periodic solution is unstable to formation of a gap soliton. The obvious advantage of the more general criterion is that it applies for strong nonlinearity, when the band-gap structure of L_µ can differ considerably from the linear problem, and effective mass arguments based on the linear problem can no longer be expected to hold.

We have conducted some numerical experiments to illustrate these results. For our basic model we take the one dimensional Gross-Pitaevsky equation with a Jacobi elliptic function potential:

\[ i\psi_t = -\frac{1}{2}\psi_{xx} \pm |\psi|^2\psi + V_0\text{sn}^2(x, k)\psi. \]

As is shown in previous work [8] this equation has a family of elliptic function solutions. In the first experiment we take the focusing sign of the nonlinearity. In this case we have an exact solution

\[ \psi(x, t) = \sqrt{-(V_0 + k^2)\text{sn}(x, k)} \exp(-i\omega t) \quad V_0 \leq -k^2 \]

which represents a solution bifurcating from the third band edge - the lower band edge of the second band. The corollary implies that for |V_0 + k^2| < 1 there exists a band of spectrum along the imaginary axis. Figure 1 shows a plot of the Floquet discriminants K_±(µ) along the real µ axis for |V_0 + k^2| = 5, V_0 = -0.6. Note that at K_-(0) = -2, via Noether’s theorem and the fact that the third band edge represents an antiperiodic eigenfunction. The other Floquet discriminant satisfies K_+(0) ≈ -1.4, and is thus in the interior of a band. From the first theorem it follows that there is a band of spectrum along the imaginary µ axis. Both Floquet discriminants are real and ε (-2, 2) near the origin, implying that the Bogoliubov eigenvalue problem has a band of spectrum along the imaginary axis. This corresponds to an exponential instability in the evolution of small disturbances to this standing wave solution.

In summary, by using the Hamiltonian structure and symmetries of the Gross-Pitaevsky equation with a periodic potential, we have presented a sufficient condition for the exponential instabilities of standing wave solutions.

It is our hope, that this criterion will be a complement to the numerous already known solutions in studying the Gross-Pitaevsky equation, and condensates in optical lattices.

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