On Double Vector Bundles

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Abstract

In this paper, we construct a category of short exact sequences of vector bundles and prove that it is equivalent to the category of double vector bundles. Moreover, operations on double vector bundles can be transferred to operations on the corresponding short exact sequences. In particular, we study the duality theory of double vector bundles in term of the corresponding short exact sequences. Examples including the jet bundle and the Atiyah algebroid are discussed.

1 Introduction

The notion of a double vector bundle was introduced by Pradines in [16] and further studied by Mackenzie in [12, 13], Konieczna and Urbański in [8], Grabowski in [9] and Gracia-Saz and Mehta in [7] (this list is not complete). As explained in [13], a double vector bundle is essentially a vector bundle object in the category of vector bundles. The most fundamental example of double vector bundles is the tangent double vector bundle \( TE \) of a vector bundle \( E \), which is also known as the tangent double and had been used in the connection theory since the late 1950s. It turns out that the framework of double vector bundles is very convenient for many important constructions like linear 1-forms, linear Poisson structures, special symplectic manifolds, linear connections and so on (see [2, 12, 17, 20, 8] for more details).

The theory of duality of double vector bundles is decisively different from that of ordinary vector bundles. A double vector bundle has two duals which are themselves in duality. In fact, for a double vector bundle \( (D; A; B; M)_C \) (see Diagram (1)), one can take the dual over \( A \) and obtain the double vector bundle \( (D^*A; A; C^*; M)_B \) as shown in Diagram (15). In the mean time, one may also take the dual over \( B \) and obtain the double vector bundle \( (D^*B; C^*; B; M)_A \) as shown as

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Diagram (18). The observation that there is a natural duality between $D^*A$ and $D^*B$ over $C^*$ is illustrated by \[14\] Theorem 9.2.2.

The present paper is concerned with the algebraic feature of double vector bundles. The main object introduced in this paper is a double vector bundle sequence (DVB sequence), namely a short exact sequence of vector bundles:

$$0 \to C \xrightarrow{\xi} \Omega \xrightarrow{p} A \otimes B \to 0.$$  

The main result of this paper is that such a DVB sequence encapsulates the structure of a double vector bundle. In fact, from a DVB sequence, one can construct a double vector bundle $(D; A, B; M)_C$ (this process is called the double realization). Conversely, from a double vector bundle $(D; A, B; M)_C$, we can also construct a DVB sequence $0 \to C \to C(D) \xrightarrow{p} A \otimes B \to 0$, which is called the associated DVB sequence. Moreover, we show that the category of double vector bundles (which will be denoted by $DVB$) and the category of DVB sequences (which will be denoted by $DVBS$) are equivalent.

The dual form of a DVB sequence is called a DVB $^*$ sequence. In an analogous way to the theory of duality of double vector bundles, we introduce a method of taking duals to DVB $^*$ sequences. The dual sequence obtained here (Sequence (29)) was also given by Gracia-Saz and Mehta in \[7\]. It plays an important role when the authors studied Lie algebroid structures on double vector bundles. Furthermore, it turns out that a DVB $^*$ sequence has two duals which are themselves in duality. Note that the pairing of such dual DVB $^*$ sequences takes values in a vector bundle. Actually, this idea already appeared in our work \[13, 14\], to study certain geometric structures.

This paper is organized as follows. In Section 2 we give a review of double vector bundles. In Section 3 we study DVB sequences of vector bundles. In Subsection 3.1 we prove that one can construct a double vector bundle associated to any DVB sequence (Theorem 3.3). In Subsection 3.2 we prove that one can also construct a DVB sequence associated to a double vector bundle. Moreover, the category of double vector bundles and the category of DVB sequences are equivalent (Theorem 3.9). In Subsection 3.3 we construct another short exact sequence of vector bundles associated to a double vector bundle and prove that it is isomorphic to the dual of the associated DVB sequence. In Section 4 we study the duality theory. In Subsection 4.1 we review the duality theory of double vector bundles. In Subsection 4.2 we introduce the notion of DVB $^*$ sequences and study their duals (taking values in a vector bundle). We prove that after taking three steps of duals, one can go back to the starting point (Theorem 4.6). This is totally different from the usual duality theory. In Subsection 4.3 we apply this method to the associated DVB $^*$ sequence of a double vector bundle and prove that this is the right process corresponding to taking the usual dual of a double vector bundle (Theorem 4.10). In Section 5 we give several examples including the Atiyah algebroid and the jet bundle.

## 2 Double vector bundles

Throughout the paper, we follow the conventions of \[14\].

**Definition 2.1.** A double vector bundle $(D; A, B; M)$ is a system of four vector bundle structures

\[
\begin{array}{ccc}
D & \xrightarrow{q_D^B} & B \\
\downarrow{q_D^A} & & \downarrow{q_B} \\
A & \xrightarrow{q_A} & M
\end{array}
\]
in which $D$ has two vector bundles structures, on bases $A$ and $B$, which are themselves vector bundles on $M$, such that each of the four structure maps of each vector bundle structure on $D$ (namely the bundle projection, zero section, addition and scalar multiplication) is a morphism of vector bundles with respect to the other structures.

In the above figure, we refer to $A$ and $B$ as the side bundles of $D$, and to $M$ as the double base. In the two side bundles, the addition, scalar multiplication and subtraction are denoted by usual symbols $+$, juxtaposition, and $\cdot$. We distinguish the two zero-sections, writing $\mathbf{0}^A : M \to A$, $m \mapsto \mathbf{0}^A_m$, and $\mathbf{0}^B : M \to B$, $m \mapsto \mathbf{0}^B_m$. We denote an element $d \in D$ by $(d; a, b; m)$ to indicate that $q^D_A (d) = a$, $q^D_B (d) = b$, $m = q_B(b) = q_A(a)$. We call $D_m = (q^D_A)^{-1}(A_m) = (q^D_B)^{-1}(B_m)$ the slice of $D$ at $m$.

In the vertical bundle structure on $D$ with base $A$, the vector bundle operations are denoted by $+_A$, $-_A$, $\cdot _A$, with $\mathbf{0}^A : A \to D$, $a \mapsto \mathbf{0}^A_a$, for the zero-section. Similarly, in the horizontal bundle structure on $D$ with base $B$ we write $+_B$, $-_B$, $\cdot _B$, with $\mathbf{0}^B : B \to D$, $b \mapsto \mathbf{0}^B_b$.

The two structures on $D$, namely $(D,q^D_B,B)$ and $(D,q^D_A,A)$ will also be denoted, respectively, by $\overline{D}_B$ and $\overline{D}_A$, and called the horizontal bundle structure and the vertical bundle structure.

The condition that each operation in $D$ is a morphism with respect to the other is equivalent to the following equalities, known as the interchange laws.

$$(d_1 +_B d_2) +_A (d_3 +_B d_4) = (d_1 +_A d_3) +_B (d_2 +_A d_4),$$

$$(t \cdot _A t_2) +_A (t_1 + d_2) = t \cdot _A (d_1 +_A d_2),$$

$$(t \cdot _B t_2) +_A (t_1 + d_2) = t \cdot _B (d_1 +_A t_2),$$

$$(t \cdot _A (s \cdot _B d)) = s \cdot _B (t \cdot _A d),$$

$$\mathbf{0}^A_{a_1 + a_2} = \mathbf{0}^A_{a_1} +_B \mathbf{0}^A_{a_2},$$

$$\mathbf{0}^A_{t_a} = t \cdot _B \mathbf{0}^A_a,$$

$$\mathbf{0}^B_{b_1 + b_2} = \mathbf{0}^B_{b_1} +_A \mathbf{0}^B_{b_2},$$

$$\mathbf{0}^B_{t_b} = t \cdot _A \mathbf{0}^B_b.$$

We denote by $C$ the intersection of the two kernels:

$$C = \{ c \in D \mid \exists m \in M \text{ such that } q^D_B(c) = \mathbf{0}^B_m, \ q^D_A(c) = \mathbf{0}^A_m \},$$

which is called the core, and together with the map $q_C : c \mapsto m$, $(C,q_C,M)$ is also a vector bundle over $M$. Although $C$ is affiliated with $D$, in this paper we prefer to use Diagram (1) below to emphasis the core of the relevant double vector bundle.

$$D \xrightarrow{q^D_B} B \xrightarrow{q^D_A} A \xrightarrow{q_C} M \xrightarrow{q_M} C.$$  \hspace{1cm} (1)

A double vector bundle as above will also be written as $(D; A,B; M)_C$.

An element $c \in C$ will be distinguished from its image in $D$, which will be denoted by $\overline{c}$. The linear structures of $C$ are inherited from those of $D$, due to the following facts:

$$\overline{c + c'} = \overline{c} +_A \overline{c'} = \overline{c} + _B \overline{c'}, \quad \overline{\mathbf{0} = r \cdot A \overline{c} = r \cdot B \overline{c}}.$$  \hspace{1cm} (2)

Let $(D; A,B; M)_C$ be a double vector bundle given as above. The flip of $(D; A,B; M)_C$ is the
Double vector bundle

\[
\begin{array}{ccc}
D & \xrightarrow{q_D} & A \\
\downarrow q_B & & \downarrow q_A \\
B & \xrightarrow{q_B} & M \xrightarrow{q_C} C
\end{array}
\]  

(3)

obtained by simply reversing the two side bundles. In this paper, we treat a double vector bundle and its flip as the same object.

**Definition 2.2.** A morphism of double vector bundles (see Diagram (4))

\[(\varphi; f_A, f_B; f_M) : (D; A, B; M) \rightarrow (D'; A', B'; M')\]

consists of maps \(\varphi : D \rightarrow D'\), \(f_A : A \rightarrow A'\), \(f_B : B \rightarrow B'\), \(f_M : M \rightarrow M'\), such that each of \((\varphi, f_B), (\varphi, f_A), (f_A, f_M)\) and \((f_B, f_M)\) is a morphism of the relevant vector bundles.

\[
\begin{array}{ccc}
D & \xrightarrow{f_B} & B \\
\varphi & \downarrow & \downarrow f_B \\
D' & \xrightarrow{f_B} & B' \\
A & \xrightarrow{f_A} & M \\
\downarrow f_A & & \downarrow f_A \\
A' & \xrightarrow{f_M} & M' \\
\end{array}
\]  

(4)

In this definition, \((f_C, f_M)\), where \(f_C = \varphi|_C : C \rightarrow C'\), is also a morphism of the associated cores.

We denote the category of double vector bundles by \(\text{DVB}\).

**Example 2.3.** Consider \(D = A \times_M B \times_M C\), the trivial double vector bundle over \(A\) and \(B\) with core \(C\) (see [14, Example 9.1.4]). The bundle projections \(q_D^A\) and \(q_D^B\) are exactly the projections to \(A\) and \(B\) respectively. The linear structures are given by

\[
\begin{align*}
\rho \cdot A (a, b_1, c_1) = & (a, rb_1 + b_2, rc_1 + c_2); \\
\rho \cdot B (a_1, b, c_1) = & (ra_1 + a_2, b, rc_1 + c_2).
\end{align*}
\]

In [6] and [7], it is proved that any double vector bundle is isomorphic to a trivial double vector bundle.

3 DVB sequences

In this section, \(q_O : \Omega \rightarrow M\), \(q_A : A \rightarrow M\), \(q_B : B \rightarrow M\) and \(q_C : C \rightarrow M\) are vector bundles over \(M\).

**Definition 3.1.** A DVB sequence\(^1\) on \(M\) is an exact sequence of vector bundles:

\[
0 \rightarrow C \xrightarrow{\epsilon} \Omega \xrightarrow{p} A \otimes B \rightarrow 0
\]  

(5)

\(^1\)DVB is the abbreviation of “double vector bundle”.

\[\text{over the identity map } \text{Id}_M.\]
Such a DVB sequence will be denoted by \( (\Omega \xrightarrow{p} A \otimes B; M)^{C} \). We refer to the vector bundles \( A, B \) as side bundles, and \( C \) the core.

**Definition 3.2.** A morphism of DVB sequences

\[
(\varpi; f_{A}, f_{B}; f_{M}): (\Omega \xrightarrow{p} A \otimes B; M)^{C} \rightarrow (\Omega' \xrightarrow{p'} A' \otimes B'; M')^{C'},
\]

consists of maps \( \varpi : \Omega' \rightarrow \Omega, f_{A} : A \rightarrow A', f_{B} : B \rightarrow B' \) and \( f_{M} : M \rightarrow M' \), each of \((\varpi, f_{M}), (f_{A}, f_{M}), (f_{B}, f_{M})\) is a morphism of the relevant vector bundles, such that the diagram

![Diagram](image)

commutes.

In this definition, for \( f_{C} = \varpi|_{C} : C \rightarrow C' \), \((f_{C}, f_{M})\) is also a morphism of vector bundles.

It is obvious that DVB sequences together with the above morphisms form a category, which will be denoted by \( \text{DVBS} \).

### 3.1 From DVB sequences to double vector bundles

We show how to construct a double vector bundle out of a DVB sequence \( (\Omega \xrightarrow{p} A \otimes B; M)^{C} \).

**Theorem 3.3.** Given a DVB sequence \( (\Omega \xrightarrow{p} A \otimes B; M)^{C} \), there is an associated double vector bundle

\[
\mathcal{D}(\Omega) \xrightarrow{q_{B}} B \xrightarrow{q_{A}} A \xrightarrow{q_{M}} M.
\]

Here \( \mathcal{D}(\Omega) \) is given by

\[
\mathcal{D}(\Omega) \triangleq \{(\omega, a, b) \in \Omega \times M \times M B|p(\omega) = a \otimes b\}.
\]

The vertical bundle projection \( q_{A}^{D} : D \rightarrow A \) is given by

\[
q_{A}^{D} : (\omega, a, b) \mapsto a, \quad \forall (\omega, a, b) \in \mathcal{D}(\Omega).
\]

The horizontal bundle projection \( q_{B}^{D} : D \rightarrow B \) is given by

\[
q_{B}^{D} : (\omega, a, b) \mapsto b, \quad \forall (\omega, a, b) \in \mathcal{D}(\Omega).
\]

**Proof.** Since \( p \) is surjective, both \( q_{A}^{D} \) and \( q_{B}^{D} \) are surjections. For the vector bundle structure over \( A \), we define (for any fixed \( a \in A \))

\[
r \cdot_{A} (\omega_{1}, a, b_{1}) +_{A} (\omega_{2}, a, b_{2}) \triangleq (r\omega_{1} + \omega_{2}, a, rb_{1} + b_{2}).
\]
Similarly, the vector bundle structure over $B$ are defined by (for any fixed $b \in B$):

$$r \cdot_B (\omega_1, a_1, b) +_B (\omega_2, a_2, b) \triangleq (r\omega_1 + \omega_2, ra_1 + a_2, b).$$

For $D(\Omega)$, the zero element above $a \in A$ is $\tilde{0}_a = (0, a, 0)$ and the zero element above $b \in B$ is $\tilde{0}_b = (0, 0, b)$. The core of $(D(\Omega); A, B; M)$ can be identified with the core $C = Ker(p)$ of the DVB sequence $(\Omega \xrightarrow{p} A \otimes B; M)_C$. In fact, any $c \in C$ is embedded into $D(\Omega)$ by $\pi = (\varphi(c), 0, 0)$. It is routine to check that the interchange laws are satisfied.

**Definition 3.4.** We call the double vector bundle $(D(\Omega); A, B; M)_C$ shown as in Diagram (7) the double realization of the DVB sequence $(\Omega \xrightarrow{\varphi} A \otimes B; M)_C$.

For any morphism of DVB sequences

$$(\varphi; f_A, f_B; f_M) : (\Omega \xrightarrow{\varphi} A \otimes B; M)_C \longrightarrow (\Omega' \xrightarrow{\varphi'} A' \otimes B'; M')_C,$$

define $D(\varphi) : D(\Omega) \longrightarrow D(\Omega')$ by

$$D(\varphi)(\omega, a, b) = (\varphi(\omega), f_A(a), f_B(b)), \quad \forall (\omega, a, b) \in D(\Omega).$$

It is easily seen that

$$(D(\varphi), f_A, f_B; f_M) : (D(\Omega); A, B; M)_C \longrightarrow (D(\Omega'); A', B'; M')_C$$

is a morphism of double vector bundles.

**Lemma 3.5.** With the above notations,

$$D : (\Omega \xrightarrow{\varphi} A \otimes B; M)_C \longrightarrow (D(\Omega); A, B; M)_C,$$

$$D : (\varphi; f_A, f_B; f_M) \longrightarrow (D(\varphi), f_A, f_B; f_M)$$

is a covariant functor (called the doubling functor) from the category of DVB sequences DVBS to the category of double vector bundles DVB.

The proof is straightforward and thus omitted.

### 3.2 From double vector bundles to DVB sequences

In this part we show that a double vector bundle also yields a DVB sequence. Suppose that we are given a double vector bundle $(\tilde{D}; A, B; M)_C$ as in Diagram (7). Fix $m \in M$ and let $\tilde{D}_m$ denote the free vector space generated by all the elements in the slice $D_m$. In other words, an element of $\tilde{D}_m$ is a formal linear combination of some elements of $D_m$ with coefficients in $\mathbb{R}$, i.e. it has an expression of the form

$$\Sigma_{i=1}^l r_i \cdot d_i, \quad r_i \in \mathbb{R}, d_i \in D_m.$$

Let $\overline{D}_m$ be the subspace of $\tilde{D}_m$ generated by elements of the following types:

$$\begin{align*}
    d_1 + d_2 - (d_1 +_A d_2), \\
    d_3 + d_4 - (d_3 +_B d_4), \\
    r \cdot d - (r \cdot_1 d), \\
    r \cdot d - (r \cdot_B d).
\end{align*}$$

(8)
Let $\mathcal{C}(D)_m = \mathcal{D}_m / \mathcal{T}_m$. For each $d \in D_m$, let $[d]$ denote its image in $\mathcal{C}(D)_m$. Then, $\mathcal{C}(D)_m$ is generated by elements of the form $[d]$, and by definition, one has

$$[d_1 +_A d_2] = [d_1] + [d_2], \quad [d_3 +_B d_4] = [d_3] + [d_4], \quad [r \cdot_A d] = [r \cdot_B d] = r[d].$$

In particular, we know that

$$[\tilde{0}_A^a] = [\tilde{0}_B^b] = 0, \quad \forall a \in A_m, \ b \in B_m.$$

**Proposition 3.6.** The vector space $\mathcal{C}(D)_m$ has dimension $\dim A \times \dim B + \dim C$.

**Proof.** Since any double vector bundle can be trivialized, it suffices to assume $D_m = A_m \times B_m \times C_m$. In this case, the subspace $\mathcal{T}_m$ of $\mathcal{D}_m$ is generated by elements of the form (according to (3))

$$(a, b_1, c_1) + (a, b_2, c_2) - (a, b_1 + b_2, c_1 + c_2),$$

$$(a_3, b, c_3) + (a_4, b, c_4) - (a_3 + a_4, b, c_3 + c_4),$$

$$r \cdot (a, b, c) = (a, rb, rc),$$

$$r \cdot (a, b, c) = (ra, b, rc).$$

Hence the quotient space $\mathcal{C}(D)_m = \mathcal{D}_m / \mathcal{T}_m$ is isomorphic to $A_m \otimes B_m \otimes C_m$ (c.f. [1] Chp.2) by canonically identifying $[(a, b, c)]$ with $a \otimes b + c$. This completes the proof. \qed

It is also clear that the map $[d] \mapsto a \otimes b$, for any $(d; a, b, m) \in D_m$, extends to a linear map

$$p_m : \mathcal{C}(D)_m \to A_m \otimes B_m,$$

which is surjective.

In summary, we have

**Theorem 3.7.** The set $\mathcal{C}(D) = \bigcup_{m \in M} \mathcal{C}(D)_m$ is a vector bundle over $M$ and fit into the following DVB sequence:

$$0 \to C \xrightarrow{\varepsilon} \mathcal{C}(D) \xrightarrow{p} A \otimes B \to 0. \tag{10}$$

where $p$ is induced by $[d] \mapsto q_A^D(d) \otimes q_B^D(d)$ and $\varepsilon$ is given by $c \mapsto \tilde{c}$.

We call (10) the **associated DVB sequence** of the double vector bundle $(D; A, B; M)_C$.

Let

$$(\varphi; f_A, f_B; f_M) : (D; A, B; M)_C \to (D'; A', B'; M')_C,$$

be a morphism of double vector bundles, define $\mathcal{C}(\varphi) : \mathcal{C}(D) \to \mathcal{C}(D')$ by setting

$$\mathcal{C}(\varphi)([d]) = [\varphi(d)].$$

Obviously, it is a morphism of vector bundles and satisfies

$$p' \circ \mathcal{C}(\varphi) = (f_A \otimes f_B) \circ p.$$

Hence $(\mathcal{C}(\varphi); f_A, f_B; f_M)$ is a morphism of DVB sequences.

**Lemma 3.8.** With the above notations,

$$\mathcal{C} : (D; A, B; M)_C \to (\mathcal{C}(D) \xrightarrow{p} A \otimes B; M)_C,$$

$$\mathcal{C} : (\varphi; f_A, f_B; f_M) \mapsto (\mathcal{C}(\varphi); f_A, f_B; f_M)$$

is a covariant functor from the category of double vector bundles DVB to the category of DVB sequences DVBS.
Again we omit the proof of this lemma. The main result of this paper is the following theorem.

**Theorem 3.9.** The category of double vector bundles $DVB$ and the category of $DVB$ sequences $DVBS$ are equivalent.

**Proof.** For the preceding $\mathcal{D}$-functor and the $\mathcal{C}$-functor, respectively, given by Lemma 3.5 and 3.8, we establish a natural transformation $t$ from the identity functor $I_{DVB}$ to $\mathcal{DC}$. For any double vector bundle $(D; A, B; M)$ and $(d, a, b; M) \in D$, as a canonical element $[d] \in \mathcal{E}(D)$, $(d, a, b) \in \mathcal{E}(D)$, define $t_D : D \to \mathcal{DC}(D)$ by setting

$$t_D(d) = ([d], a, b).$$

Since it preserves the side bundles $A$, $B$ and the core $C$,

$$(t_D; \text{Id}_A, \text{Id}_B; \text{Id}_M) : (D; A, B; M)_C \to (\mathcal{DC}(D); A, B; M)_C$$

must be an isomorphism. Further, $t$ is natural in the sense that the following diagram

$$
\begin{array}{ccc}
(D; A, B; M)_C & \xrightarrow{\pi p} & (\mathcal{E}(D); A, B; M)_C \\
\downarrow{(\mathcal{DC}(\phi)f_A, f_B; f_M)} & & \downarrow{(\mathcal{DC}(\phi)f_A, f_B; f_M)} \\
(D'; A', B'; M')_C & \xrightarrow{\pi' p'} & (\mathcal{E}(D'); A', B'; M')_C
\end{array}
$$

is commutative, for every morphism of double vector bundles $(\phi; f_A, f_B; f_M)$.

On the other hand, there is a natural transformation $\pi$ from $\mathcal{E}D$ to the identity functor $I_{DVBS}$. In fact, for any $DVB$ sequence $(\Omega \longrightarrow A \otimes B; M)_C$, we can establish a natural isomorphism

$$(\pi_\Omega; \text{Id}_A, \text{Id}_B; \text{Id}_M) : (\mathcal{E}(\Omega \longrightarrow A \otimes B; M)_C \to (\Omega \longrightarrow A \otimes B; M)_C.$$  

This is done as follows. Write $D = \mathcal{E}(\Omega)$ and recall that an element in $D_m$ is of the form $d = (\omega, a, b)$, satisfying $\pi(\omega) = a \otimes b$, where $\omega \in \Omega_m, a \in A_m, b \in B_m$.

So we define a map $\tilde{\pi} : D_m \to \Omega$ (recall that $D_m$ is freely generated by $D_m$) by setting $(\omega, a, b) \mapsto \omega$ and then it linearly extends to $\tilde{D}_m$. It is easy to check that $\tilde{\pi}$ sends all the four types of elements $\mathcal{S}$ to zero. Therefore, $\tilde{\pi}$ induces a well-defined linear map $\pi_\Omega$ from $\mathcal{E}(D)_m = D_m/\tilde{D}_m$ to $\Omega$, i.e.

$$\pi_\Omega([(\omega, a, b)]) = \omega.$$  

Obviously, $\pi_\Omega : \mathcal{E}(\Omega) \to \Omega$ is a morphism of vector bundles.

One can directly verify that the diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & C \\
\downarrow{\text{Id}_C} & & \downarrow{\text{Id}} \\
0 & \longrightarrow & \mathcal{E}(\Omega) \\
\downarrow{\pi_\Omega} & & \downarrow{\text{Id}} \\
0 & \longrightarrow & \Omega \\
\downarrow{\pi} & & \downarrow{\text{Id}} \\
0 & \longrightarrow & A \otimes B \\
\downarrow{\pi} & & \downarrow{\text{Id}} \\
0 & \longrightarrow & 0
\end{array}
$$

commutes and it implies that $\pi_\Omega$ is an isomorphism of vector bundles. Finally, $\pi$ is natural because for any morphism $(\omega; f_A, f_B; f_M)$ of $DVB$ sequences, we have the following commutative diagram:

$$
\begin{array}{ccc}
(\mathcal{E}(\Omega) \longrightarrow A \otimes B; M)_C & \xrightarrow{\pi_\Omega} & (\Omega \longrightarrow A \otimes B; M)_C \\
\downarrow{(\mathcal{E}(\omega)f_A, f_B; f_M)} & & \downarrow{(\omega)f_A, f_B; f_M) \\
(\mathcal{E}(\Omega') \longrightarrow A' \otimes B'; M')_C & \xrightarrow{\pi'_\Omega} & (\Omega' \longrightarrow A' \otimes B'; M')_C.
\end{array}
$$

This finishes the proof of the equivalence of the categories $DVB$ and $DVBS$. \[\square\]
3.3 The dual of the associated DVB sequence

Consider the dual of the associated DVB sequence (10):

\[
\begin{align*}
0 & \longrightarrow A^* \otimes B^* \overset{\varepsilon}{\longrightarrow} \mathcal{E}(D)^* \overset{i^*}{\longrightarrow} C^* \longrightarrow 0 \, .
\end{align*}
\] (11)

In this part of the paper, we give an explanation of this sequence.

**Definition 3.10.** For the double vector bundle \((D; A, B; M)\) shown as in Diagram (1), a function \(\sigma\) on the slice \(D_m\) is said to be **double-linear**, if it satisfies the following equalities

\[
\begin{align*}
\sigma(d_1 +_A d_2) &= \sigma(d_1) + \sigma(d_2), \quad \text{if} \quad q^D_A(d_1) = q^D_A(d_2); \\
\sigma(r \cdot_A d) &= r \sigma(d); \\
\sigma(d_3 +_B d_4) &= \sigma(d_3) + \sigma(d_4), \quad \text{if} \quad q^D_B(d_3) = q^D_B(d_4); \\
\sigma(r \cdot_B d) &= r \sigma(d);
\end{align*}
\]

where \(d, d_1, d_2, d_3, d_4 \in D_m, r \in \mathbb{R}\).

For any \(\theta \in A^*_m \otimes B^*_m\), define a function \(i(\theta)\) on \(D_m\) by

\[
i(\theta)(d) \triangleq \langle \theta, q^D_A \otimes q^D_B(d) \rangle = \langle \theta, a \otimes b \rangle, \quad \forall \, (d; a, b; m) \in D_m,
\]

which is obviously double-linear.

Let \((\mathcal{X}(D))_m\) be the collection of double linear functions on \(D_m\). With respect to the usual addition and multiplication of functions, \((\mathcal{X}(D))_m\) is a vector space. It is easy to see that

\[
\mathcal{X}(D) = \bigcup_{m \in M} (\mathcal{X}(D))_m
\]
is a vector bundle over \(M\) and \(i : A^* \otimes B^* \rightarrow \mathcal{X}(D)\) given by (12) is a vector bundle morphism.

**Proposition 3.11.** Any double-linear function \(\sigma : D_m \rightarrow \mathbb{R}\) uniquely determines some \(\chi \in C^*_m\), such that

\[
\sigma(\overline{c}) = \langle \chi, c \rangle, \quad \forall \, c \in C_m.
\]

Write \(\chi = j(\sigma)\), then \(j\) is a vector bundle morphism and

\[
0 \longrightarrow A^* \otimes B^* \overset{i}{\longrightarrow} \mathcal{X}(D) \overset{j}{\longrightarrow} C^* \longrightarrow 0 \quad (13)
\]
is an exact sequence.

**Proof.** By relation (2), \(\sigma|_{C_m}\) is a linear function on \(C_m\) and this clearly defines a linear map \(j : \sigma \rightarrow \sigma|_{C_m} \in C^*_m\). To see that \(j\) is surjective it is sufficient to work locally. In fact, if we assume that \(D_m = A_m \times B_m \times C_m\), then a double linear function on \(D_m\) is indeed an element in \(A^* \otimes B^* \otimes C^*\). In this case, \(j\) is the projection to \(C^*\). It is also trivial that \(i\) is an injection. \(\blacksquare\)

Below we establish a canonical way to identify the sequence (13) with the dual of the associated DVB sequence (10). In fact, there is a standard pairing between \(\mathcal{E}(D)\) and \(\mathcal{X}(D)\):

\[
\langle \cdot, \cdot \rangle : \quad \mathcal{E}(D) \times_M \mathcal{X}(D) \rightarrow \mathbb{R},
\]
induced by

\[
\langle [d], \sigma \rangle \triangleq \sigma(d),
\]
for all \((d; a, b; m) \in D_m\) and \(\sigma \in (\mathfrak{X}(D))_m\). It is not hard to see that this well defines a pairing between \(\mathfrak{E}(D)\) and \(\mathfrak{X}(D)\). Moreover, it satisfies

\[
\begin{cases}
\langle \omega, i(\theta) \rangle = \langle p(\omega), \theta \rangle, & \forall \omega \in \mathfrak{E}(D)_m, \theta \in A^*_m \otimes B^*_m; \\
\langle c(e), \sigma \rangle = \langle c, j(\sigma) \rangle, & \forall c \in C_m, \sigma \in (\mathfrak{X}(D))_m.
\end{cases}
\]  

(14)

Fix \(\sigma \in \mathfrak{X}(D)_m\), the condition "\(|d|, \sigma\rangle = \sigma(d) = 0\, \forall d \in D_m\)" clearly implies \(\sigma = 0\). Thus the pairing must be nondegenerate.

\section{Duality theory}

\subsection{Duality of double vector bundles}

Dualizing the vertical structure on \(D\) leads again to a double vector bundle \((D^*A; A, C^*; M)_{B^*}\), called the vertical dual or dual over \(A\) of \([1]\), which is denoted by

\[
\begin{array}{ccc}
D^*A & \xrightarrow{q^A} & C^* \\
\downarrow{q^*A} & & \downarrow{q^{c*}} \\
A & \xrightarrow{q_A} & M \\
\downarrow{q_A} & & \downarrow{q_B} \\
B^*.
\end{array}
\]  

(15)

The vertical structure in \([15]\) is the usual dual of the bundle structure on \(\hat{D}_A\), and \(q^{c*}\) is the usual dual of \(q_C : C \to M\). The additions and scalar multiplications in the side bundles will be denoted by \(+_A\), \(-_A\), \(+_{C^*}\), \(-_{C^*}\), \(\cdot_{C^*}\) and \(-_{C^*}\). The zero of \(D^*A\) above \(a \in A\) is denoted by \(\tilde{0}^A_a\).

The vector bundle projection \(q^{c*} : D^*A \to C^*\) is defined by

\[\langle q^{c*}(\Phi), c \rangle = \langle \Phi, \tilde{0}^A +_B \psi \rangle, \forall a \in A_m, c \in C_m, \Phi \in (q^A)^{-1}(a).\]  

(16)

The addition \(+_{C^*}\) in \(D^*A \to C^*\) is defined by

\[r \cdot_{C^*} \Phi, d = (\Phi, d) + (\Phi', d').\]  

(17)

Here \(\Phi \in (q^A)^{-1}(a), \Phi' \in (q^A)^{-1}(a')\), and \(d \in (q^D)^{-1}(a), d' \in (q^D)^{-1}(a')\). Note the important hypothesis that \(q^{c*}(\Phi) = q^{c*}(\Phi')\), which ensures that \(+_{C^*}\) is well-defined.

Similarly, we define

\[r \cdot_{C^*} \Phi, d = r \langle \Phi, \frac{1}{r} \cdot_B d \rangle,\]  

for \(r \neq 0\) and \(d \in D\) with \(q^D(d) = r \cdot q^A(\Phi)\).

The zero above \(\chi \in C^*_m\) is denoted by \(\tilde{0}^A_{\chi, C^*}\) and is defined by

\[\langle \tilde{0}^A_{\chi, C^*} +_A \psi \rangle = \langle \chi, c \rangle, \forall b \in B_m, c \in C_m.\]

The core element \(\psi\) corresponding to \(\psi \in B^*_m\) is

\[\langle \psi, \tilde{0}^B_{\chi} +_A \psi \rangle = \langle \psi, b \rangle,\]  

i.e. \(\langle \tilde{0}^A_{\chi} +_{C^*} \psi, d \rangle = \langle \psi, q^B_{d}(d) \rangle, \forall d \in D.\)

There is of course also a horizontal dual \((D^*B; C^*, B; M)_{A^*}\):

\[
\begin{array}{ccc}
D^*B & \xrightarrow{q^B} & B \\
\downarrow{q^A} & & \downarrow{q_B} \\
C^* & \xrightarrow{q^{c*}} & M \\
\downarrow{q_A} & & \downarrow{q_{A^*}} \\
A^*,
\end{array}
\]  

(18)

defined in an analogous way.
4.2 Duality of DVB\(^\ast\) sequences

In this part of the paper, \(U, V, K\) and \(\Pi\) are all vector bundles over the base space \(M\). The dual form of DVB sequences will be called DVB\(^\ast\) sequences.

**Definition 4.1.** An exact sequence of vector bundles over \(M\) as follows

\[
0 \longrightarrow U \otimes V \xrightarrow{i} \Pi \xrightarrow{j} K \longrightarrow 0,
\]

is called a DVB\(^\ast\) sequence. Again, we refer to \(U\) and \(V\) as the side bundles.

Obviously, the short exact sequence (13) is a DVB\(^\ast\) sequence, which we call the associated DVB\(^\ast\) sequence of the double vector bundle \((D; A, B; M)\)\(_C\).

**Definition 4.2.** Let

\[
0 \longrightarrow U \otimes K^\ast \xrightarrow{e} \Delta \xrightarrow{p} V^* \longrightarrow 0
\]

be a DVB\(^\ast\) sequence. The DVB\(^\ast\) sequences (19) and (20) are said to be in duality with respect to \(U\), or \(U\)-duality, if there is a bilinear pairing taking values in \(U\):

\[
\{\cdot, \cdot\}_U : \Delta \times_M \Pi \to U,
\]

satisfying the following two equalities

\[
\begin{align*}
\{\epsilon, i(\theta)\}_U &= p(\epsilon)_\ast \theta, \quad \forall \epsilon \in \Delta, \theta \in U \otimes V, \\
\{\epsilon(\kappa), \sigma\}_U &= j(\sigma)_\ast \kappa, \quad \forall \sigma \in \Pi, \kappa \in U \otimes K^\ast.
\end{align*}
\]

**Remark 4.3.** It is easy to check the following facts:

\[
\{\cdot, \sigma\}_U = 0 \iff \sigma = 0;
\]

\[
\{\epsilon, \cdot\}_U = 0 \iff \epsilon = 0.
\]

Thus the \(U\)-valued pairing \(\{\cdot, \cdot\}_U\) in this definition must be nondegenerate.

Below we use a constructive approach to show the existence of such duals of a given DVB\(^\ast\) sequence. Again consider the DVB\(^\ast\) sequence (19) and its dual form:

\[
0 \longrightarrow K^* \xrightarrow{j^*} \Pi^* \xrightarrow{i^*} U^* \otimes V^* \longrightarrow 0.
\]

Applying the “\(U \otimes\)” operation to this sequence, we get another exact sequence

\[
0 \longrightarrow U \otimes K^* \xrightarrow{\text{Id}_U \otimes j^*} U \otimes \Pi^* \xrightarrow{\text{Id}_U \otimes i^*} U \otimes U^* \otimes V^* \longrightarrow 0.
\]

Using the standard decomposition \(U \otimes U^* = \text{gl}(U) \equiv \text{sl}(U) \oplus \mathbb{R}\text{Id}_U\), the right side of the above sequence becomes \(\text{sl}(U) \otimes V^* \oplus V^*\). So we are able to obtain a sub-vector bundle of \(U \otimes \Pi^*\), which is the pull back of \(V^*\) and will be denoted by \(\Pi_U^\ast\). Moreover, we have the following exact sequence:

\[
0 \longrightarrow U \otimes K^* \xrightarrow{e} \Pi_U^\ast \xrightarrow{p} V^* \longrightarrow 0,
\]

where \(e = \text{Id}_U \otimes j^*, \quad p = (\text{Id}_U \otimes i^*)|_{\Pi_U^\ast}\).

Since \(\Pi_U^\ast\) is the pull back of \(V^*\), it can be directly described as follows:

\[
(\Pi_U^\ast)_m = \{\epsilon \in \text{Hom}(\Pi_m, U_m) \mid \exists v^\ast \in V^*_m, \text{ s.t., } \epsilon \circ i(\theta) = v^\ast_\ast \theta, \forall \theta \in U \otimes V\}.
\]

It is a straightforward verification to prove the following fact:
Proposition 4.4. The DVB\(^\ast\) sequences (19) and (22) are in U-duality.

In a similar manner, one is able to get another DVB\(^\ast\) sequence

\[ 0 \to K^\ast \otimes V \xrightarrow{\epsilon} \Pi^\ast_V \xrightarrow{p} U^\ast \to 0, \]

which is a V-dual of (19).

Remark 4.5. In the special case that \(U = M \times \mathbb{R}\), the DVB\(^\ast\) sequence (19) is of the form

\[ 0 \to V \xrightarrow{i} \Pi \xrightarrow{j} K \to 0. \]

It is easy to see that its \((M \times \mathbb{R})\)-dual \(\Pi^\ast_{M \times \mathbb{R}}\) is exactly \(\Pi^\ast\). So we get the usual dual of an exact sequence.

Let us introduce another operation of DVB\(^\ast\) sequences. In short, transpositions exchange the side bundles and add a minus sign. In specific, the transposition of the DVB\(^\ast\) sequence (19) is the following DVB\(^\ast\) sequence:

\[ 0 \to V \otimes U \xrightarrow{i^t} \Pi \xrightarrow{j} K \to 0, \tag{23} \]

where \(i^t\) is given by

\[ v \otimes u \mapsto -i(u \otimes v), \quad \forall u \in U, v \in V. \]

A DVB\(^\ast\) sequence remains unchanged if we take twice transpositions. An interesting phenomenon is the next theorem, which claims that after taking three steps of duals with respect to different side bundles, one gets the transposition.

Theorem 4.6. Let

\[ 0 \to V \otimes K^\ast \xrightarrow{r} (\Pi^\ast_U)_K \xrightarrow{q} U^\ast \to 0 \tag{24} \]

be the \(K^\ast\)-dual of (22). Then the DVB\(^\ast\) sequences (24) and (23) are in V-duality, or, in terms of a diagram:

\[ \begin{array}{c}
\text{(19)} & \text{transposition} & \text{(23)} \\
\text{U-dual} & \searrow & \text{V-dual} \\
\text{(22)} & \text{K}^\ast\text{-dual} & \text{(23)}
\end{array} \tag{25} \]

Proof. Let us denote \(\Delta = \Pi^\ast_U\), \(\Xi = (\Pi^\ast_U)_K^\ast\). By the assumption, we have a \(U\)-valued pairing \(\{\cdot, \cdot\}_U\) of \(\Delta\) and \(\Pi\) satisfying the two equalities in (21). We also have a \(K^\ast\)-valued pairing \(\{\cdot, \cdot\}_K^\ast\) of \(\Xi\) and \(\Delta\) satisfying

\[
\{\eta, e(\kappa)\}_K^\ast = q(\eta) \ast \kappa, \quad \forall \eta \in \Xi, \kappa \in U \otimes K^\ast; \\
\{r(\zeta), e\}_K^\ast = p(e) \ast \zeta, \quad \forall e \in \Delta, \zeta \in V \otimes K^\ast.
\]

It suffices to prove that the DVB\(^\ast\) sequences (24) and (23) are in V-duality and thus it amounts to establish a \(V\)-valued pairing:

\[ \{\cdot, \cdot\}_V : \Xi \times_M \Pi \to V, \]
such that
\[
\{\eta, i^*(\theta)\}_V = q(\eta) \cdot \theta, \quad \forall \eta \in \Xi, \theta \in V \otimes U, \quad (26)
\]
\[
\{r(\zeta), \sigma\}_V = j(\sigma) \cdot \zeta, \quad \forall \epsilon \in \Delta, \zeta \in V \otimes K^*.
\] (27)

In fact, the pairing \(\{\eta, \sigma\}_V, \) for \(\eta \in \Xi, \sigma \in \Pi,\) can be defined by
\[
\langle\{\eta, \sigma\}_V, v^*\rangle = \langle\{\eta, \epsilon\}_{K^*}, j(\sigma)\rangle - \langle\{\epsilon, \sigma\}_U, q(\eta)\rangle, \quad \forall \ v^* \in V^*, \quad (28)
\]
where an arbitrary \(\epsilon \in \Delta\) satisfying \(p(\epsilon) = v^*\) is chosen. This is well defined because if one chooses another \(\epsilon' = \epsilon + \epsilon(\kappa),\) for some \(\kappa \in U \otimes K^*\), then
\[
\langle\{\eta, \epsilon'\}_{K^*}, j(\sigma)\rangle - \langle\{\epsilon', \sigma\}_U, q(\eta)\rangle \]
\[
= \langle\{\eta, \epsilon\}_{K^*}, j(\sigma)\rangle - \langle\{\epsilon, \sigma\}_U, q(\eta)\rangle + \langle\{\eta, \epsilon(\kappa)\}_{K^*}, j(\sigma)\rangle - \langle\{\epsilon(\kappa), \sigma\}_U, q(\eta)\rangle \]
\[
= \langle\{\eta, \epsilon\}_{K^*}, j(\sigma)\rangle - \langle\{\epsilon, \sigma\}_U, q(\eta)\rangle + \langle\{\eta, \epsilon(\kappa)\}_{K^*}, j(\sigma)\rangle - \langle\{\epsilon(\kappa), \sigma\}_U, q(\eta)\rangle \]
\[
= \langle\{\eta, \epsilon\}_{K^*}, j(\sigma)\rangle - \langle\{\epsilon, \sigma\}_U, q(\eta)\rangle.
\]
We finally prove (26) and (27):
\[
\langle\{\eta, i^*(\theta)\}_V, v^*\rangle = \langle\{\eta, \epsilon\}_{K^*}, 0\rangle - \langle\{\epsilon, i^*(\theta)\}_U, q(\eta)\rangle 
\]
\[
= \langle v^* \cdot \theta, q(\eta)\rangle = \langle q(\eta) \cdot \theta, v^*\rangle.
\]
\[
\langle\{r(\zeta), \sigma\}_V, v^*\rangle = \langle\{r(\zeta), \epsilon\}_{K^*}, j(\sigma)\rangle - \langle\{\epsilon, \sigma\}_U, 0\rangle 
\]
\[
= \langle v^* \cdot \zeta, j(\sigma)\rangle = \langle j(\sigma) \cdot \zeta, v^*\rangle.
\]
This completes the proof. \(\blacksquare\)

4.3 Duality of associated DVB* sequences of double vector bundles

This part of the paper is devoted to show the relevance of duals of double vector bundles and that of the associated DVB* sequences. Given a double vector bundle \((D; A, B; M)_C\) as in Diagram (11), we have its dual over \(A^*\), which is again a double vector bundle, shown as in Diagram (15). In the meantime, the associated DVB* sequence (13) of \((D; A, B; M)_C\) has an \(A^*\)-dual, which is denoted by \(\mathcal{D}(D')\).

We will show that the associated DVB* sequence of the dual double vector bundle (15) is exactly the DVB* sequence (29). For this aim, we first need another description of the vector bundle \(\mathcal{D}(D)\).

**Proposition 4.7.** Let \((D^*; A, C^*; M)_B\) be the dual double vector bundle of \(D\) over \(A\) given by (13). Then,
\[
(\mathcal{D}(D))_m \cong \left\{ \text{linear map } \underline{\mathcal{g}}: A_m \to (q^*_C)^{-1}(\chi), \text{ for some } \chi \in C^*_m, \text{ s.t., } q^*_A \circ \underline{\mathcal{g}} = \text{Id}_{A_m} \right\}. \quad (30)
\]
Under this identification, the maps \(i\) and \(j\) in (16) are given by
\[
i : \theta \mapsto (a \mapsto \bar{0}_a^* + C^* \to a \cdot \theta, \quad \forall a \in A_m), \quad \forall \theta \in A^*_m \otimes B^*_m.
\]
\[
j : \underline{\mathcal{g}} \mapsto \chi,
\]
This sequence appeared in [2], where linear sections of a double vector bundle are considered.
\textbf{Proof.} Recall that an element $\sigma \in (\mathcal{X}(D))_m$ is a double-linear function on $D_m$. Notice that $\sigma$ determines a map $\underline{\sigma}$ from $A_m$ to $(D^*A)_m$ by sending each $a \in A_m$ to $\underline{\sigma}(a) \in (q_A^*)^{-1}(a)$, where

$$\langle \underline{\sigma}(a), d \rangle \triangleq \sigma(d), \quad \forall \ (d; a, b; m) \in D_m. \quad (31)$$

The fact that $\underline{\sigma}(a) \in D^*A$ is due $\sigma$ being double-linear. Moreover, we have

$$\langle \underline{\sigma}(a), \tilde{0}_a^A +_B \mathfrak{r} \rangle = \sigma(\tilde{0}_a^A +_B \mathfrak{r}) = \langle j(\sigma), c \rangle, \quad \forall \ c \in C_m.$$ 

This implies that $q_B^A(\underline{\sigma}(a)) = j(\sigma)$ (see Eq.\,(10)). Hence we obtain a map

$$\underline{\sigma} : A_m \to (q_B^A)^{-1}(j(\sigma)),$$

which is $\mathbb{R}$-linear and satisfies $q_B^A \circ \underline{\sigma} = \text{Id}_{A_m}.$

Conversely, the element $\underline{\sigma}$ in (30) determines a double linear function $\sigma$ via relation (31). In fact, for any $(d_1; a_1, b_1; m) \in D_m$, we have

$$\sigma(d +_A d_1) = \langle \underline{\sigma}(a), d +_A d_1 \rangle = \langle \underline{\sigma}(a), d \rangle + \langle \underline{\sigma}(a), d_1 \rangle = \sigma(d) + \sigma(d_1),$$

which implies that $\sigma$ is linear with respect to $+_A$. For every $(d_2; a_2, b_2; m)$, by (17), we have

$$\sigma(d +_B d_2) = \langle \underline{\sigma}(a +_C a_2), d +_B d_2 \rangle = \langle \underline{\sigma}(a), d \rangle + \langle \underline{\sigma}(a_2), d_2 \rangle = \sigma(d) + \sigma(d_2),$$

which implies that $\sigma$ is also linear with respect to $+_C$. Therefore, $\sigma$ is double linear. $\blacksquare$

\textbf{Corollary 4.8.} Let $(D^*B; C^*, B; M)_A$ be the dual double vector bundle of $D$ over $B$ shown as in Diagram (13), then we have

$$(\mathcal{X}(D))_m \cong \{ \text{linear map} \ \underline{\sigma} : B_m \longrightarrow (q_B^A)^{-1}(\chi), \text{for some} \ \chi \in C_m^*, \ s.t., \ q_B^A \circ \underline{\sigma} = \text{Id}_{B_m} \} . \quad (32)$$

\textbf{Corollary 4.9.} Let

$$0 \longrightarrow A^* \otimes C \underset{i_1}{\longrightarrow} \mathcal{X}(D^*A) \underset{j_1}{\longrightarrow} B \longrightarrow 0. \quad (33)$$

be the associated DVB* sequence of $(D^*A; A, C^*; M)_B$. Then,

$$(\mathcal{X}(D^*A))_m \cong \{ \text{linear map} \ \underline{\xi} : A_m \longrightarrow (q_B^D)^{-1}(b), \text{for some} \ b \in B_m, \ s.t., \ q_B^D \circ \underline{\xi} = \text{Id}_{A_m} \} .$$

Under this identification, the maps $i_1$ and $j_1$ in Sequence (33) are given by

$$j_1 : \xi \mapsto b, \quad i_1 : \kappa \mapsto (a \mapsto \tilde{0}_a^A +_B \underline{\sigma}_a \kappa), \quad \forall \ a \in A_m, \ \forall \ k \in A_m^* \otimes C_m.$$ 

Now we are able to show that the associated DVB* sequence (33) of the double vector bundle $D^*A$ is isomorphic to (29), the $A^*$-dual of the associated DVB* sequence (13) of $(D; A, B; M)_C$, as the following theorem claims.
Theorem 4.10. The associated DVB* sequence \(i_{\mathfrak{D}}\) of the double vector bundle \((D; A, B; M)_C\) and the associated DVB* sequence \(i_{\mathfrak{D}}\) of the dual double vector bundle \((D^*; A, C^*; M)_{B^*}\) are in \(A^*-\text{duality}\).

**Proof.** We need to establish an \(A\)-pairing \(\{\cdot, \cdot\}_A\) between \(\mathfrak{X}(D)\) and \(\mathfrak{X}(D^*A)\) satisfying the following two equalities:

\[
\{i(\theta), \epsilon\}_A = j_1(\epsilon \cdot \theta), \quad \forall \epsilon \in \mathfrak{X}(D^*A), \quad \theta \in A^*_m \otimes B^*_m, \\
\{\sigma, i(\kappa)\}_A = j(\sigma \cdot \kappa), \quad \forall \sigma \in \mathfrak{X}(D), \quad \kappa \in A^*_m \otimes C_m.
\]

This can be done by defining \(\{\sigma, \epsilon\}_A\), for \(\sigma \in \mathfrak{X}(D)_m\) and \(\epsilon \in \mathfrak{X}(D^*A)_m\) to be

\[\{\sigma, \epsilon\}_A(a) = \sigma(\epsilon(a)) = \epsilon(\sigma(a)).\]

It is obviously bilinear and symmetric. Moreover, Proposition 4.7 and Corollary 4.9 imply that

\[
\{\sigma, \epsilon\}_A(a) = \sigma(\epsilon(a)) = \epsilon(\sigma(a)).
\]

Thus, by definition of \(i(\theta)\) in (12), we have

\[
\{i(\theta), \epsilon\}_A(a) = i(\theta)(\epsilon(a)) = (\theta \cdot a \otimes \epsilon) = (j_1(\epsilon \cdot \theta))(a).
\]

Similarly, the map \(i_1\) in (33) is given by

\[
i_1(\kappa)(\Phi) = \langle \kappa, q^*_A(\Phi) \otimes q^*_C(\Phi) \rangle, \quad \forall \Phi \in D^*A.
\]

So we obtain

\[
\{\sigma, i_1(\kappa)\}_A(a) = i_1(\kappa)(\sigma(a)) = \langle \kappa, a \otimes j(\sigma) \rangle = (j(\sigma \cdot \kappa))(a)
\]

This completes the proof. \(\blacksquare\)

Since the categories \(\text{DVB}\) and \(\text{DVBS}\) are equivalent (Theorem 3.9), a direct corollary is the following:

**Corollary 4.11.** Suppose that \((D; A, B; M)_C\) and \((E; A, C^*; M)_{B^*}\) are double vector bundles with a side bundle \(A\) in common and with cores \(C\) and \(B^*\) respectively. Suppose that their associated DVB* sequences

\[
0 \to A^* \otimes B^* \xrightarrow{i} \mathfrak{X}(D) \xrightarrow{j} C^* \to 0,
\]

\[
0 \to A^* \otimes C \xrightarrow{i} \mathfrak{X}(E) \xrightarrow{j} B \to 0.
\]

are in \(A^*-\text{duality}\) as DVB* sequences. Then there induces a pairing between \(D\) and \(E\) over \(A\) so that they are mutually duals as double vector bundles.

We also recover a well known fact [14, Theorem 9.2.2].

**Corollary 4.12.** The double vector bundles \(D^*A\) and \(D^*B\) are mutually duals over \(C^*\).

**Proof.** By Theorem 4.10 and 4.11, the associated DVB* sequences of \(D^*A\) and \(D^*B\) are in \(C^*-\text{duality}\). Then the conclusion follows directly from Corollary 4.11. \(\blacksquare\)
5 The Atiyah algebroid and the jet bundle.

Let $E \xrightarrow{q} M$ be a vector bundle and $E^* \xrightarrow{q^*} M$ the dual bundle of $E$. The tangent double vector bundle of $E$ is a double vector bundle $(TE; TM, E; M)_E$ which fits the following diagram:

$$
\begin{array}{ccc}
TE & \xrightarrow{p_E} & E \\
\downarrow{q} & & \downarrow{q} \\
TM & \xrightarrow{p} & M & \xrightarrow{q} & E.
\end{array}
$$ (34)

Here $TE \xrightarrow{Tq} TM$ is the tangent to the $E \xrightarrow{q} M$ and $TE \xrightarrow{p_E} E$ is the usual tangent bundle. We denote elements of $TE$ by $\xi, \eta, \zeta$ ... and we write $(\xi; x, e; m)$ to indicate that $e = p_E(\xi)$, $x = Tq(\xi)$, and $m = p \circ Tq(\xi) = q \circ p_E(\xi)$. With respect to the tangent bundle structure $(TE, p_E, E)$, we use usual notations. The zero element in $T_0 E$ is denoted by $\tilde{0}$. In the prolonged tangent bundle structure, $(TE, Tq, TM)$, we use $+_{TM}$ for addition, $-_{TM}$ for subtraction, and $\cdot_{TM}$ for scalar multiplication. The zero element of the fiber $(Tq)^{-1}(x)$ is denoted by $T(0)(x)$.

The embedding of the core $(\cdot): E \rightarrow TE$ is to identify the tangent space $T_0 E$ canonically with $E_m$. That is, the element $\overline{\tau}$ of $T_0_m(E_m)$ corresponding to $e \in E_m$ is given by

$$
\overline{\tau} = \frac{d}{dt}|_{t=0}(te).
$$

The vertical dual of (34) is actually the tangent double vector bundle of $E^*$:

$$
\begin{array}{ccc}
TE^* & \xrightarrow{p_{E^*}} & E^* \\
\downarrow{Tq^*} & & \downarrow{q^*} \\
TM & \xrightarrow{q} & M & \xrightarrow{q^*} & E^*.
\end{array}
$$ (35)

Dualizing the double vector bundle $TE$ in Diagram (34) over $E$ leads to a double vector bundle $(T^*E; E^*; M)_{T^*M}$:

$$
\begin{array}{ccc}
T^*E & \xrightarrow{r_E} & E^* \\
\downarrow{c_E} & & \downarrow{q^*} \\
E & \xrightarrow{q} & M & \xrightarrow{p^*} & T^*M,
\end{array}
$$ (36)

which will be referred to as the cotangent double vector bundle of $E$, or cotangent dual of $TE$. An element of $T^*E$ will be denoted by $(\epsilon, e, \varphi; m)$ to indicate that $e = c_E(\epsilon)$, $\varphi = r_E(\epsilon)$, $m = q(\epsilon) = q_*(\varphi)$. The map $r_E: T^*E \rightarrow E^*$ is defined as follows. Take $\epsilon \in T^*_0(\epsilon)$, where $\epsilon \in E_m$. Define

$$
\langle r_E(\epsilon), e' \rangle = \langle \epsilon, \tilde{0} \rangle +_{TM} \langle e', \epsilon \rangle, \quad \forall e' \in E_m.
$$

The embedding of $w \in T^*_m M$ into $T^*E$ is given by

$$
\langle \overline{\tau}, T(0)(x) + \overline{\epsilon} \rangle = \langle w, x \rangle, \quad \forall e \in E_m, x \in T_m M, \quad (\text{i.e. } \overline{\tau} = q^*(w)).
$$

There is of course an analogously

$$
\begin{array}{ccc}
T^*E^* & \xrightarrow{r_{E^*}} & E^* \\
\downarrow{c_{E^*}} & & \downarrow{q^*} \\
E^* & \xrightarrow{q^*} & M & \xrightarrow{p^*} & T^*M,
\end{array}
$$ (37)
which is the cotangent double vector bundle of \( E^* \).

• The Atiyah algebroid \( \mathfrak{D}E \)

For a vector bundle \( E \xrightarrow{\pi} M \), we denote the gauge Lie algebroid of the frame bundle \( F(E) \) by \( \mathfrak{D}E \), which is also called the Atiyah algebroid, also known as the covariant differential operator bundle of \( E \) (see [14] Example 3.3.4 and [15]). Here we treat each element \( \mathfrak{d} \) of \( \mathfrak{D}E \) at \( m \in M \) as an \( \mathbb{R} \)-linear operator \( \Gamma(E) \to E_m \) with some \( x \in T_m M \) (which is uniquely determined by \( \mathfrak{d} \) and called the anchor of \( \mathfrak{d} \)) such that
\[
\mathfrak{d}(f) = f(m)\mathfrak{d}(u) + x(f)u(m), \quad \forall f \in C^\infty(M), u \in \Gamma(E).
\]
It is known that \( \mathfrak{D}E \) is a transitive Lie algebroid over \( M \) [11]. The anchor of \( \mathfrak{D}E \) is given by \( J_2(\mathfrak{d}) = x \) and the Lie bracket \( [\cdot, \cdot] \) of \( \Gamma(\mathfrak{D}E) \) is just the commutator. As a transitive Lie algebroid, \( \mathfrak{D}E \) has an associated exact sequence, called the Atiyah sequence:
\[
0 \to E^* \otimes E \xrightarrow{l_\pi} \mathfrak{D}E \xrightarrow{J_\mathfrak{d}} TM \to 0.
\]
(38)
We have an equivalent description of \( \mathfrak{D}E \):
\[
(\mathfrak{D}E)_m = \{ (\mathfrak{d}, x) \mid x \in T_m M, \mathfrak{d} \in \text{Hom}(E_{m_1}^*, (T_0(q))^{-1}(x)) \text{ and } p_{E^*} \circ \mathfrak{d} = \text{Id}_{E_m^*} \}.
\]
From this point of view, a section of \( \mathfrak{D}E \), namely a derivation \( (\mathfrak{d}, x) \), is a linear vector field \( \mathfrak{d} \) on \( E^* \) which projects to a vector field \( x \in \mathfrak{X}(M) \), such that for any \( u \in \Gamma(E) \), if it is treated as a fiber-wise linear function \( l_u \), then
\[
\mathfrak{d}(l_u) = l_{\mathfrak{d}(u)}.
\]
One may refer to [14] Definition 3.4.1 for more details.

By Corollary 3.8, we are able to draw the following conclusion.

Corollary 5.1. The exact sequence (38) associated with the Atiyah algebroid \( \mathfrak{D}E \) is the associated \( \text{DVB}^* \) sequence of the cotangent double vector bundle of \( E^* \) shown as in Diagram (37), i.e.
\[
\mathfrak{D}E = \mathfrak{X}(T^*E^*).
\]

• The Jet bundle \( \mathfrak{J}E \)

Given a vector bundle \( E \xrightarrow{\pi} M \) and a point \( m \in M \), for any \( e \in E_m \), the tangent space \( T_e(E_m) \), called the vertical subspace, is a linear subspace of the full tangent space of \( E \) at \( e \). The full tangent space can be decomposed into a direct sum of \( T_e(E_m) \) and a complementary horizontal subspace. We can define a fiber bundle \( \mathfrak{J}E \) over \( E \) whose fiber at \( e \) is the set of all possible horizontal subspaces:
\[
(\mathfrak{J}E)_e = \{ \text{linear map } \mu : T_m M \to T_e E, \text{ such that } Tq \circ \mu = \text{Id}_{T_m M} \}.
\]

There is a linear structure of \( \mathfrak{J}E \) as follows: for \( \mu_1 \in (\mathfrak{J}E)_{e_1}, \mu_2 \in (\mathfrak{J}E)_{e_2} \) with \( q(e_1) = q(e_2) = m \), \( r\mu_1 + \mu_2 \in (\mathfrak{J}E)_{r e_1 + e_2} \) (where \( r \in \mathbb{R} \)) is a linear map \( T_m M \to T_{re_1 + e_2} E \) defined by:
\[
(r\mu_1 + \mu_2)(x) = r \cdot T_{T_m M} \mu_1(x) + T_{T_m M} \mu_2(x), \quad \forall x \in T_m M.
\]
Equipped with this operation, \( \mathfrak{J}E \) is a vector bundle over \( M \), called the first order jet bundle over \( M \), or simply the jet bundle of \( E \) (see [13] for more details about jet bundles). In summary, the fiber of \( \mathfrak{J}E \) at \( m \) is the collection
\[
\mathfrak{J}E_m = \{ (\mu, e), \text{ where } e \in E_m, \mu \in \text{Hom}(T_m M, T_e E) \text{ such that } Tq \circ \mu = \text{Id}_{T_m M} \},
\]

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with the linear structure defined by
\[ r(\mu_1, e_1) + (\mu_2, e_2) = (r\mu_1 + \mu_2, re_1 + e_2), \]
for \((\mu_p, e_p) \in \mathcal{J}E_m \ (p = 1, 2), \) \(r \in \mathbb{R}.\) It is well known that there is an associated exact sequence:
\[ 0 \rightarrow T^*M \otimes E \xrightarrow{\sigma} \mathcal{J}E \xrightarrow{\mathcal{D}} E \rightarrow 0. \quad (39) \]

By Proposition 4.7, we have

**Corollary 5.2.** The exact sequence (39) associated with \(\mathcal{J}E\) is the associated DVB* sequence of the tangent double vector bundle of \(E^*\) shown as in Diagram (35), i.e.
\[ \mathcal{J}E = \mathfrak{T}(TE^*). \]

- **The relations between \(\mathcal{D}E, \mathcal{J}E, \mathcal{D}E^*\) and \(\mathcal{J}E^*\)**

Since \(TE\) and \(TE^*\) are double vector bundles in duality over \(TM,\) and \(\mathcal{J}E^* = \mathfrak{T}(TE),\) \(\mathcal{J}E = \mathfrak{T}(TE^*)\) (Corollary 5.2), we have an induced \(T^*M\)-pairing between \(\mathcal{J}E^*\) and \(\mathcal{J}E\) (Theorem 4.10).

Similarly, \(T^*E^*\) and \(TE^*\) are in duality over \(E^*\) as double vector bundles, and \(\mathcal{J}E = \mathfrak{T}(TE^*).\) \(\mathcal{D}E = \mathfrak{T}(T^*E^*)(\text{Corollary 5.1}),\) we come to the conclusion that \(\mathcal{J}E\) and \(\mathcal{D}E\) are mutually \(E\)-duals as DVB* sequences. The pairing between \(\mathcal{J}E\) and \(\mathcal{D}E\) is exactly the one defined in [4]. For the same reasons, \(\mathcal{J}E^*\) and \(\mathcal{D}E^*\) are mutually \(E^*\)-duals.

In [4, Lemma 2.3], it is shown that \(\mathcal{D}E\) is canonically isomorphic to \(\mathcal{D}E^*\). The isomorphism is just the transposition of a DVB* sequence. Thus, by Theorem 3.9 and Corollary 5.1 we recover a well known result:

**Corollary 5.3.** The double vector bundle associated to \(T^*E,\) is isomorphic to the double vector bundle associated to \(T^*E^*\).

Recall Diagram (25), we can again summarize the relations between \(\mathcal{D}E, \mathcal{J}E, \mathcal{D}E^*\) and \(\mathcal{J}E^*\) by the following diagram:

\[ \begin{array}{ccc}
\mathcal{D}E^* & \xrightarrow{\text{transposition}} & \mathcal{D}E \\
E^*\text{-dual} & & \mathcal{J}E^* \\
&T^*M\text{-dual} & \mathcal{J}E.
\end{array} \]

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