NEW INTEGRAL REPRESENTATIONS FOR RANKIN–SELBERG L-FUNCTIONS

ANDREW R. BOOKER AND M. KRISHNAMURTHY

Abstract. We derive integral representations for the Rankin–Selberg L-functions on \( \text{GL}(3) \times \text{GL}(1) \) and \( \text{GL}(3) \times \text{GL}(2) \) by a process of unipotent averaging at archimedean places. A key feature of our result is that it allows one to fix the choice of test vector at finite places, even when the underlying representations have common ramification.

1. Introduction

Let \( f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z} \) be a holomorphic newform of weight \( k \) and level \( \Gamma_0(N) \). Let \( \chi \) be a primitive Dirichlet character of conductor \( q \) and set \( f_\chi(z) = \sum_{n=1}^{\infty} a_n \chi(n)e^{2\pi i nz} \). Then, by \cite[Lemma 4.3.10]{16}, we have

\[
f_\chi = \frac{\tau(\chi)}{q} \sum_{a=1}^{q} \chi(-a)f\left(1 \frac{a}{q}\right),
\]

where \( \tau(\chi) \) is the associated Gauss sum. Taking the Mellin transform of both sides, we obtain

\[
\Lambda(s, f \times \chi) = \frac{\tau(\chi)}{q} \sum_{a=1}^{q} \chi(-a)\Lambda(s, f, a/q),
\]

where

\[
\Lambda(s, f, a/q) = \Gamma_C(s) \sum_{n=1}^{\infty} a_n e^{2\pi i a n/q} n^{-s} \quad \text{and} \quad \Lambda(s, f \times \chi) = \Gamma_C(s) \sum_{n=1}^{\infty} a_n \chi(n) n^{-s}.
\]

The above notion of unipotent averaging is a key ingredient in Weil’s proof of the converse theorem \cite[Theorem 4.3.15]{16} as well as in our prior work \cite{1, 3} on \( \text{GL}(2) \) converse theorems. The goal of the present paper is to prove an analogue of (1.1) for higher-rank Rankin–Selberg L-functions using a similar process of unipotent averaging; see Theorem 3.5. Our result can be interpreted as an integral representation for the Rankin–Selberg L-function in terms of the associated (Whittaker) essential functions, and we speculate that this may have applications to the study of algebraicity properties and nonvanishing of special values. We outline some necessary notation and explain our result in greater detail below.

Let \( F \) be a number field with adèle ring \( \mathbb{A}_F \), and let \( \pi \cong \bigotimes_v \pi_v \) (resp. \( \tau \cong \bigotimes_v \tau_v \)) be an irreducible generic (with respect to some fixed additive character \( \psi = \bigotimes_v \psi_v \) of \( F\setminus\mathbb{A}_F \)) automorphic representation of \( \text{GL}_m(\mathbb{A}_F) \) (resp. \( \text{GL}_n(\mathbb{A}_F) \)), with \( m < n \). For each place \( v \) of \( F \), let \( L(s, \pi_v \boxtimes \tau_v) \) be the local Rankin–Selberg factor \cite{9, 10} and set

\[
\Lambda(s, \pi \boxtimes \tau) = \prod_v L(s, \pi_v \boxtimes \tau_v).
\]

The completed L-function \( \Lambda(s, \pi \boxtimes \tau) \), defined initially for \( \Re(s) \gg 1 \), continues to a meromorphic function of \( s \in \mathbb{C} \) and satisfies a functional equation relating \( \Lambda(s, \pi \boxtimes \tau) \) to \( \Lambda(1-s, \pi \boxtimes \tau) \) \cite{9}. The

A. R. B. was partially supported by EPSRC Grant EP/K034383/1. No data were created in the course of this study.
collection of such functions for a fixed π and suitably varying τ plays a central role in converse theorems, whose purpose is to characterize automorphic representations π in terms of the analytic behavior of Λ(s, π ⊗ τ).

For each non-archimedean place v of F, let ξ_v^0 (resp. φ_v^0) denote the “essential vector” in the space of π_v (resp. τ_v) [8, 11, 15], and let W_v^0 ∈ W(π_v, ψ_v) (resp. W_φ_v^0 ∈ W(τ_v, ψ_v^{-1})) be the associated essential Whittaker functions (as in loc. cit.) formed relative to a ψ_v whose conductor is o_v. Suppose τ_v is unramified, it follows from [15, Corollary 3.3] that

\begin{equation}
L(s, π_v ⊗ τ_v) = \int_{U_m(F_v) \backslash GL_m(F_v)} W_v^0(h_v) W_φ_v^0(h_v) \det h_v||_v^{s-\frac{m}{2}} \, dh_v
\end{equation}

for a suitable normalization of the measure on U_m(F_v) \backslash GL_m(F_v). For n = m − 1, the above equality is a part of the characterization of the essential vector in [8, 11], which we review in the next section; the fact that it is true for any m ≤ n − 1 is a result of a concrete realization of essential functions in [15]. It follows from this that, if τ is such that τ_v is unramified at every finite v, then the Rankin–Selberg L-function Λ(s, π ⊗ τ) can be represented by a global integral of the corresponding automorphic forms, which unfolds to a product of local integrals of the above shape.

However, when τ is ramified at a place v, the local integral can vanish unless ξ_v^0 is adjusted in a suitable way. In fact in general one has that there is a tensor t_v^0 ∈ W(π_v, ψ_v) ⊗ W(σ_v, ψ_v^{-1}) whose local zeta integral is precisely the L-function L(s, π_v ⊗ τ_v). Now, let S be the finite set of finite places where τ_v is ramified. For v ∈ S, let t_v^0 be as described above, and for a finite place v not in S set t_v^0 = W_v^0 ⊗ W_φ_v^0. For any choice of a pair of archimedean Whittaker functions (W_∞, W'_∞), writing each t_v^0 for v ∈ S as a sum of pure tensors, we get a finite sum of pure (global) tensors such that

\[ \sum_j W_j ⊗ W'_j = (W_∞ ⊗ W'_∞) \prod_{v < \infty} t_v^0. \]

Fixing such a decomposition, let (φ_j, ϕ_j) be the automorphic forms associated with (W_j, W'_j). Then

\begin{equation}
P_∞(s)Λ(s, π ⊗ τ) = \sum_j \int_{GL_m(F) \backslash GL_m(ÅF)} \mathbb{P}_m^n(φ_j) \left( \begin{pmatrix} h & \\ 1_{n-m} & \end{pmatrix} \right) ϕ_j(h) ||_v^{s-\frac{m}{2}} \, dh,
\end{equation}

where P_∞(s) is an entire function depending only on the pair (W_∞, W'_∞), and \mathbb{P}_m is the projection operator defined in (2.3) below.

In this paper, we study integral representations for L(s, π ⊗ τ) in the case n = 3, m ∈ {1, 2}. We show that (in the above notation) one can take t_v^0 = W_v^0 ⊗ W_φ_v^0 at all finite places v, regardless of the ramification of τ, after going through a process of unipotent averaging on the right-hand side of (1.3). This is similar in spirit to the global Birch lemma of [13, 14], which plays a central role in the study of p-adic interpolation of special values of L(s, (π ⊗ χ) ⊗ τ) for a cuspidal pair (π, τ) and a Hecke character χ of finite order and non-trivial p-power conductor. While in loc. cit. the unipotent averaging is done at the finite place corresponding to p, here we do it over the archimedean places as in [1] which allows us to retain the essential vector at all finite places. In addition, it should be emphasized that χ can be taken to be trivial in our setting.

2. PRELIMINARIES

Let o_F denote the ring of integers of F. For each place v of F, let F_v denote the completion of F at v. For v < ∞, let o_v denote the ring of integers in F_v, p_v the unique maximal ideal in o_v and q_v the cardinality of o_v/p_v. Also, for every finite v, fix a generator π_v of p_v with absolute value ||π_v|| = q_v^{-1}. Let F_∞ = ∐_{v < ∞} F_v and let ÅF = F_∞ × ÅF,f denote the ring of adèles of F. Throughout the paper, we fix an unramified additive character ψ = ⊗ ψ_v of F ÅF whose local
conductor at every finite place \( v \) is \( \mathfrak{o}_v \), i.e., \( \psi_v \) is trivial on \( \mathfrak{o}_v \) but non-trivial on \( \mathfrak{p}_v^{-1} \). We write \( \psi_\infty \) to denote the character \( \prod_{v|\infty} \psi_v \) of \( F_\infty \).

For any \( n > 1 \), let \( B_n = T_n U_n \) be the Borel subgroup of \( \text{GL}_n \) consisting of upper triangular matrices; let \( P'_n \supset B_n \) be the standard parabolic subgroup of type \((n-1,1)\) with Levi decomposition \( P_n = M_n N_n \). Then \( M_n \cong \text{GL}_{n-1} \times \text{GL}_1 \) and

\[
N_n = \left\{ \left. \begin{pmatrix} I_{n-1} & * \\ 0 & 1 \end{pmatrix} \right| \begin{pmatrix} I_{n-1} & * \\ 0 & 1 \end{pmatrix} \right\}.
\]

If \( R \) is any \( F \)-algebra and \( H \) is any algebraic \( F \)-group, we write \( H(R) \) to denote the corresponding group of \( R \)-points. Let \( P_n(R) \subset P'_n(R) \) denote the mirabolic subgroup consisting of matrices whose last row is of the form \((0, \ldots, 0, 1)\), i.e.,

\[
P_n(R) = \left\{ \begin{pmatrix} h & y \\ 0 & 1 \end{pmatrix} : h \in \text{GL}_{n-1}(R), y \in R^{n-1} \right\} \cong \text{GL}_{n-1}(R) \rtimes N_n(R).
\]

Let \( w_n \) denote the long Weyl element in \( \text{GL}_n(R) \), and put \( \alpha_n = (w_{n-1}) \).

The character

\[
u \mapsto \psi\left( \sum_{i,i+1} u_{i,i+1} \right)
\]

defines a generic character of \( U_n(F) \setminus U_n(A_F) \), and by abuse of notation we continue to denote this character by \( \psi \). Similarly, for each \( v \), we also obtain a generic character \( \psi_v \) of \( U_n(F_v) \). Further, for any algebraic subgroup \( V \subset U_n \), \( \psi \) and \( \psi_v \) define characters of \( V(A_F) \) and \( V(F_v) \), respectively, via restriction. In particular, we may consider the character \( \psi|_{N_{n}(A_F)} \); its stabilizer in \( M_n(A_F) \) is then \( P_{n-1}(A_F) \), where we regard \( P_{n-1} \) as a subgroup of \( M_n \) via \( h \mapsto \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \).

For each \( v < \infty \), we will consider certain compact open subgroups of \( \text{GL}_n(F_v) \); namely, let \( K_v = \text{GL}_n(\mathfrak{o}_v) \), and for any integer \( m \geq 0 \), set

\[
K_{1,v}(\mathfrak{p}_v^m) = \left\{ g \in \text{GL}_n(\mathfrak{o}_v) : g \equiv \begin{pmatrix} * \\ \vdots \\ * \\ 0 & \cdots & 1 \\ \end{pmatrix} \pmod{\mathfrak{p}_v^m} \right\},
\]

\[
K_{0,v}(\mathfrak{p}_v^m) = \left\{ g \in \text{GL}_n(\mathfrak{o}_v) : g \equiv \begin{pmatrix} * \\ \vdots \\ * \\ 0 & \cdots & 0 * \\ \end{pmatrix} \pmod{\mathfrak{p}_v^m} \right\},
\]

so that \( K_{1,v}(\mathfrak{p}_v^m) \) is a normal subgroup of \( K_{0,v}(\mathfrak{p}_v^m) \), with quotient \( K_{0,v}(\mathfrak{p}_v^m)/K_{1,v}(\mathfrak{p}_v^m) \cong (\mathfrak{o}_v/\mathfrak{p}_v^m) \times \). Let \( K_f = \prod_{v < \infty} K_v \), and for an integral ideal \( \mathfrak{a} \) of \( F \), set

\[
K_i(\mathfrak{a}) = \prod_{v < \infty} K_{i,v}(\mathfrak{p}_v^m) \quad \text{for } i = 0, 1,
\]

where \( m_v \) are the unique non-negative integers such that \( \mathfrak{a} = \prod_v (\mathfrak{p}_v \cap \mathfrak{o}_F)^{m_v} \). Then \( K_1(\mathfrak{a}) \subseteq K_0(\mathfrak{a}) \subseteq K_f \) are compact open subgroups of \( \text{GL}_n(A_{F,f}) \). We may also form the corresponding principal congruence subgroups of \( \text{GL}_n(F_\infty) \), embedded diagonally, namely,

\[
\Gamma_i(\mathfrak{a}) = \{ \gamma \in \text{GL}_n(F) : \gamma_f \in K_i(\mathfrak{a}) \} \subset \text{GL}_n(F_\infty) \quad \text{for } i = 0, 1,
\]

where \( \gamma_f \) denotes the image of \( \gamma \) in \( \text{GL}_n(A_{F,f}) \).

From strong approximation for \( \text{GL}_n \), it follows that the set \( \text{GL}_n(F) \setminus \text{GL}_n(A_F) / \text{GL}_n(F_\infty) K_1(\mathfrak{a}) \) of double cosets is finite of cardinality \( h \), where \( h \) is the class number of \( F \). Let us write

\[
(2.1) \quad \text{GL}_n(A_F) = \prod_{j=1}^h \text{GL}_n(F) g_j \text{GL}_n(F_\infty) K_1(\mathfrak{a}),
\]
where each \( g_j \in \text{GL}_n(\mathbb{A}_{F,j}) \). Then

\[
(2.2) \quad \text{GL}_n(F) \backslash \text{GL}_n(\mathbb{A}_F)/K_1(\mathfrak{a}) \cong \prod_{j=1}^{h} \Gamma_{1,j}(\mathfrak{a}) \backslash \text{GL}_n(F_{\infty}),
\]

where \( \Gamma_{1,j}(\mathfrak{a}) = \{ \gamma \in \text{GL}_n(F) : \gamma \in g_j K_1(\mathfrak{a})g_j^{-1} \} \subset \text{GL}_n(F_{\infty}) \), embedded diagonally. Replacing \( K_1(\mathfrak{a}) \) by \( K_0(\mathfrak{a}) \) in this definition, we get the corresponding groups \( \Gamma_{0,j}(\mathfrak{a}) \).

Before proceeding further, let us introduce our choice of measures. For each place \( v \) of \( F \) and \( n \geq 1 \), we normalize the Haar measure on \( \text{GL}_n(F_v) \) and \( K_v \) so that \( \text{vol}(K_v) = 1 \). We fix the Haar measure on \( U_n(F_v) \) for which \( \text{vol}(U_n(F_v) \cap K_v) = 1 \). From these measures, we obtain a right-invariant measure on \( U_n(F_v) \backslash \text{GL}_n(F_v) \). Globally, on \( \text{GL}_n(\mathbb{A}_F) \) and \( U_n(\mathbb{A}_F) \), respectively, we take the corresponding product measure. On the compact quotient \( U(F) \backslash U_n(\mathbb{A}_F) \), we choose the right-invariant measure of unit volume. Since \( U_2(\mathbb{A}_F) \cong \mathbb{A}_F \), and \( \mathbb{A}_F = F + F_{\infty} + \prod_{v<\infty} \mathfrak{o}_v \), it follows from our normalization that \( \text{vol}(\mathfrak{o}_F \backslash F_{\infty}) = 1 \).

Suppose \((\pi, V_{\pi})\) is an irreducible admissible representation of \( \text{GL}_n(\mathbb{A}_F) \) (not necessarily automorphic) whose central character \( \omega_\pi \) is an idèle class character. We write \( \pi \cong \bigotimes \pi_v \) as a restricted tensor product with respect to a distinguished set of spherical vectors, where for each finite \( v \), \( \pi_v \) is an irreducible admissible representation of \( \text{GL}_n(F_v) \) on a complex vector space \( V_{\pi_v} \), and for each \( v \mid \infty \), \( \pi_v \) is an irreducible admissible Harish-Chandra module. However, for \( v \mid \infty \), we can pass to the canonical completion of \( \pi_v \) in the sense of Casselman and Wallach (see [10]) to obtain a smooth representation of moderate growth of \( \text{GL}_n(F_v) \). By abuse of notation we continue to write \((\pi_v, V_{\pi_v})\) to denote the canonical completion at an archimedean place \( v \) and let \((\pi_\infty, V_{\pi_\infty})\) denote the topological tensor product of these representations. Thus we may (and we will) take \( V_\pi \) to be the restricted tensor product \( V_{\pi_\infty} \otimes \bigotimes_{v<\infty} V_{\pi_v} \) and \( \pi \) the corresponding representation of \( \text{GL}_n(\mathbb{A}_F) \).

Now with \( \pi \cong \bigotimes \pi_v \) as in the previous paragraph, let us further assume that each \( \pi_v \) is generic, meaning there is a non-zero linear form (continuous when \( v \mid \infty \) \( \lambda_v : V_{\pi_v} \to \mathbb{C} \) satisfying \( \lambda_v(\pi_v(n)w) = \psi_v(n)\lambda_v(w) \) for \( w \in V_{\pi_v}, n \in N_n(F_v) \)). Let \( W(\pi_v, \psi_v) \) denote the Whittaker model of \( \pi_v \), viz. the space of functions on \( \text{GL}_n(F_v) \) defined by \( g \mapsto \lambda_v(\pi_v(g)w) \) for fixed \( w \in V_{\pi_v} \). By taking the tensor product of \( \lambda_{\pi_v} \) for \( v \mid \infty \), we can also form the space \( W(\pi_\infty, \psi_\infty) \). We note that for \( v < \infty \) where \( \pi_v \) is unramified, there is a unique vector \( W_{\pi_v}^0 \in W(\pi_v, \psi_v) \) that is fixed under \( K_v \) and takes the value 1 at the identity. The global Whittaker model \( W(\pi, \psi) \) of \( \pi \) is the space spanned by the functions \( W_{\infty} \prod_{v<\infty} W_v \) with \( W_{\infty} \in W(\pi_\infty, \psi_\infty) \), \( W_v \in W(\pi_v, \psi_v) \) and \( W_v = W_{\pi_v}^0 \) for almost all \( v \). For every \( \eta \in V_\pi \) we denote by \( W_\eta \) the corresponding element of \( W(\pi, \psi) \).

Following [6, 12], for each \( \eta \in V_\pi \) we set

\[
U_\eta(g) = \sum_{\gamma \in U_n(F) \backslash P_n(F)} W_\eta(\gamma g),
\]

which can also be written as

\[
U_\eta(g) = \sum_{\gamma \in U_{n-1}(F) \backslash \text{GL}_{n-1}(F)} W_\eta \left( \left( \begin{array}{c} \gamma \\ 1 \end{array} \right) g \right).
\]

It is shown in [12, Section 12] that this sum converges absolutely and uniformly on compact subsets to a continuous function on \( \text{GL}_n(\mathbb{A}_F) \), and that it is cuspidal along the unipotent radical of any standard maximal parabolic subgroup of \( \text{GL}_n \). Further, \( U_\eta \) is left-invariant under both \( P_n(F) \) and the center \( Z_n(F) \).

For \( m < n \), let \( Y = Y_{n,m} \) be the unipotent radical of the standard parabolic subgroup of \( \text{GL}_n \) of type \( (m + 1, 1, \cdots, 1) \). (In this notation, \( N_n = Y_{n,n-2} \).) For a function \( f \) on \( \text{GL}_n(\mathbb{A}_F) \) that is
where the local integrals are given by

\[ (2.3) \quad \mathbb{P}_m^n(f)(g) = \int_{Y(F) \backslash Y(A_F)} f(yg) \overline{\psi(y)} \, dy, \]

where \( dy \) is the Haar measure on \( Y(A_F) \) for which the quotient \( Y(F) \backslash Y(A_F) \) has unit volume. In particular, we can consider \( \mathbb{P}_m^n(U_\eta)(g) \), with the following Fourier series expansion:

\[ (2.4) \quad \mathbb{P}_m^n(U_\eta)(g) = \sum_{\gamma \in U_{m+1}(F) \backslash P_{m+1}(F)} W_\eta \left( \begin{pmatrix} \gamma & 0 \\ 0 & 1_{n-m-1} \end{pmatrix} g \right) \]

Viewed as a function on \( P_{m+1}(A_F) \), \( \mathbb{P}_m^n(U_\eta)(g) \) is left-invariant under \( P_{m+1}(F) \) and is cuspidal in the sense that all the relevant unipotent integrals are zero. Observe that \( \mathbb{P}_{n-1}^n \) is the identity.

Suppose \( \tau \) is an automorphic subrepresentation of \( GL_m(A_F) \), meaning \( V_\tau \) is a subspace of the space of automorphic forms on \( GL_m(A_F) \). Let \( \phi \) be an automorphic form in \( V_\tau \) and set

\[ (2.5) \quad I(s; U_\eta, \phi) = \int_{GL_m(F) \backslash GL_m(A_F)} \mathbb{P}_m^n(U_\eta) \left( \begin{pmatrix} h & 0 \\ 0 & 1_{n-m} \end{pmatrix} \right) \phi(h) \| \det h \|^{s-\frac{n-m}{2}} \, dh. \]

This integral is absolutely convergent for all \( s \), as we are integrating a cusp form against an automorphic form. For \( \Re(s) \gg 1 \) we can replace \( \mathbb{P}_m^n(U_\eta) \) by its Fourier expansion and unfold, to get

\[
I(s; U_\eta, \phi) = \int_{U_m(A_F) \backslash GL_m(A_F)} W_\eta \left( \begin{pmatrix} h & 0 \\ 0 & 1_{n-m} \end{pmatrix} \right) W_\phi(h) \| \det h \|^{s-\frac{n-m}{2}} \, dh,
\]

where \( W_\phi \) is the Whittaker on \( GL_m(A_F) \) associated to \( \phi \) with respect to \( \psi^{-1} \), i.e.

\[
W_\phi(h) = \int_{U_m(F) \backslash U_m(A_F)} \phi(uh) \psi(u) \, du.
\]

Now, if \( \eta \) (resp. \( \phi \)) corresponds to a pure tensor under \( \pi \cong \otimes_v \pi_v \) (resp. \( \tau \cong \otimes_v \tau_v \)), then by the uniqueness of the Whittaker model we have

\[
W_\eta(g) = \prod_v W_{\eta_v}(g_v), \quad W_\phi(g) = \prod_v W_{\phi_v}(g_v),
\]

and the integral now factors as

\[
I(s; U_\eta, \phi) = \prod_v \Psi_v(s; W_{\eta_v}, W_{\phi_v}),
\]

where the local integrals are given by

\[
\Psi_v(s; W_{\eta_v}, W_{\phi_v}) = \int_{U_m(F_v) \backslash GL_m(F_v)} W_{\eta_v} \left( \begin{pmatrix} h_v & 0 \\ 0 & 1_{n-m} \end{pmatrix} \right) W_{\phi_v}(h_v) \| \det h_v \|^{s-\frac{n-m}{2}} \, dh_v.
\]

It follows from the Rankin–Selberg theory of local factors that

\[
E_v(s) := \frac{\Psi_v(s; W_{\eta_v}, W_{\phi_v})}{L(s, \pi_v \boxtimes \tau_v)}
\]

is entire for all \( v \) \([9, 10]\). If \( v \) is non-archimedean then \( E_v(s) \in \mathbb{C}[q_v^s, q_v^{-s}] \), and \( E_v(s) = 1 \) for almost all finite \( v \). Setting \( E(s) = \prod_v E_v(s) \), for pure tensors \( \eta \) and \( \phi \) as above one has

\[
I(s; U_\eta, \phi) = E(s) \prod_v L(s, \pi_v \boxtimes \tau_v).
\]
For $\pi \cong \bigotimes \pi_v$ as above, let us now recall the notion of the essential vector of $\pi_v$. For each finite $v$, according to [8] (see also [11]), there is a unique positive integer $m(\pi_v)$ such that the space of $K_1(p^{m(\pi_v)})$-fixed vectors is of dimension 1. Further, as mentioned in the Introduction, by loc. cit., there is a unique vector $\xi_0^v$ in this space, called the essential vector, with the associated essential function $W_{\xi_0^v}^v$ in $W(\pi_v, \psi_v)$ satisfying the condition $W_{\xi_0^v}^v(gh_1) = W_{\xi_0^v}^v(g_1)$ for all $h \in GL_{n-1}(O_v)$ and $g \in GL_{n-1}(F_v)$. For $v < \infty$ and any unramified representation $\tau_v$ of $GL_{m}(F_v)$, let $W_{\tau_v}^{0, \psi_v} \in W(\tau_v, \psi_v)$ denote the normalized spherical function (cf. [11, pp.2]). If $m(\pi_v) = 0$ by uniqueness of essential functions one has the equality $W_{\xi_0}^v = W_{\tau_v}^{0, \psi_v}$ in $W(\pi_v, \psi_v)$. The integral ideal $n = \prod_{v < \infty} (p_v \cap o_F)^{m(\pi_v)} \subseteq o_F$ is called the conductor of $\pi$.

A crucial property of the conductor is that $K_0(n)$ acts on the space of $K_1(n)$-fixed vectors via the central character $\omega = \omega_{\pi}$ (cf. [6, Section 8]). Precisely, for $g_v = (g_{i,j}) \in K_0(n)(p_v^{m(\pi_v)})$, define

$$\chi_{\pi_v}(g_v) = \begin{cases} 1 & \text{if } m(\pi_v) = 0, \\ \omega_v(g_{n,n}) & \text{if } m(\pi_v) > 0, \end{cases}$$

and put $\chi_{\pi} = \bigotimes_{v < \infty} \chi_{\pi_v}$. Let $\xi_0^v = \bigotimes_{v < \infty} \xi_0^v$, which forms a basis for the space of $K_1(n)$-fixed vectors in $V_\pi$. Then it is shown in loc. cit. that $\chi_{\pi}$ is a character of $K_0(n)$ trivial on $K_1(n)$, and that for any finite $v$,

$$\pi_v(g)\xi_0^v = \chi_{\pi_v}(g)\xi_0^v \text{ for all } g \in K_0(n)(p_v^{m(\pi_v)}).$$

For each $j$, $\chi_{\pi}$ also determines a character of $\Gamma_{0,j}(n)$ that is trivial on $\Gamma_{1,j}(n)$, which we continue to denote by $\chi_{\pi}$.

We will also need a certain auxiliary function related to the local Rankin–Selberg $L$-factor $L(s, \pi_v \boxtimes \tau_v)$. In order to introduce this we pass to a more general setting and take $\pi_v$ (resp. $\tau_v$) to be an irreducible admissible representation of $GL_n(F_v)$ (resp. $GL_m(F_v)$), not necessarily the local component of a global representation. It is known that the local $L$-function $L(s, \tau_v)$ is of the form $P_{\pi_v}^{-s-1}$, where $P_{\pi_v} \in \mathbb{C}[X]$ has degree at most $n$ and satisfies $P_{\pi_v}(0) = 1$. We may then find $n$ complex numbers $\{\alpha_{v,i}\}$ (allowing some of them to be zero) such that

$$L(s, \pi_v) = \prod_{i=1}^{n} (1 - \alpha_{v,i}q_v^{-s})^{-1}.$$ 

We call the set $\{\alpha_{v,i}\}$ the Langlands parameters of $\pi_v$; if $\pi_v$ is spherical, they are the usual Satake parameters. Let $\{\beta_{v,j}\}$ be the Langlands parameters of $\tau_v$, and set

$$L(s, \pi_v \times \tau_v) = \prod_{i,j} (1 - \alpha_{v,i}\beta_{v,j}q_v^{-s})^{-1}.$$ 

Of course, $L(s, \pi_v \times \tau_v) = L(s, \pi_v \boxtimes \tau_v)$ if both $\pi_v$ and $\tau_v$ are spherical.

Next, we explore the connection between $L(s, \pi_v \times \tau_v)$ and $L(s, \pi_v \boxtimes \tau_v)$ in more detail, but let us first recall the full classification of irreducible admissible representations of $GL_n(F_v)$. Let $A_n$ denote the set of equivalence classes of such representations and put $A = \bigcup A_n$. The essentially square-integrable representations of $GL_n(F_v)$ have been classified by Bernstein and Zelevinsky; they are given as follows: If $\sigma_v$ is an essentially square-integrable representation of $GL_n(F_v)$, then there is divisor $a \mid n$ and a supercuspidal representation $\eta_v$ of $GL_a(F_v)$ such that if $b = \frac{n}{a}$ and $Q$ is the standard (upper) parabolic subgroup of $GL_n(F_v)$ of type $(a, \ldots, a)$, then $\sigma_v$ can be realized as the unique quotient of the (normalized) induced representation

$$\text{Ind}_{Q}^{GL_n(F_v)}(\eta_v \cdot \eta_v, \eta_v, \eta_v \| \cdot \|_{v}^{-1});$$
the integer $\alpha$ and the class of $\eta_0$, are uniquely determined by $\sigma_v$. In short, $\sigma_v$ is parametrized by $b$ and $\eta_v$, and we denote this by $\sigma_v = \sigma_b(\eta_v)$; further, $\sigma_v$ is square-integrable (also called “discrete series”) if and only if the representation $\eta_0 \parallel \parallel_v^b = \frac{1}{2}$ of $GL_\alpha(F_v)$ is unitary.

Now, let $P$ be an upper parabolic subgroup of $GL_n(F_v)$ of type $(n_1, \ldots, n_r)$. For each $i = 1, \ldots, r$, let $\tau_{i,v}$ be a discrete series representation of $GL_{n_i}(F_v)$. Let $(s_1, \ldots, s_r)$ be a sequence of real numbers satisfying $s_1 \geq \cdots \geq s_r$, and put $\tau_{i,v} = \tau_{i,v}^0 \parallel \parallel_v^b$ (an essentially square-integrable representation), for $i = 1, \ldots, r$. Then the induced representation

$$\xi_v = \text{Ind}^{GL_n(F_v)}_{P}(\tau_{1,v} \otimes \cdots \otimes \tau_{r,v})$$

is said to be a representation of $GL_n(F_v)$ of Langlands type. If $\tau_v \in A_n$, then it is well known that it is uniquely representable as the quotient of an induced representation of Langlands type. We write $\tau_v = \tau_{1,v} \otimes \cdots \otimes \tau_{r,v}$ to denote this realization of $\tau_v$. Thus one obtains a sum operation on the set $A$ [9, (9.5)]. It follows easily from the definition that $L(s, \pi_v \times \tau_v)$ is bi-additive:

$$L(s, \pi_v \times (\tau_v \otimes \tau'_v)) = L(s, \pi_v \times \tau_v)L(s, \pi_v \times \tau'_v)$$
$$L(s, (\pi_v \otimes \tau'_v) \times \tau_v) = L(s, \pi_v \times \tau_v)L(s, \pi_v \times \tau'_v)$$

for all $\pi_v, \pi'_v, \tau_v, \tau'_v \in A$. The local factor $L(s, \pi_v \otimes \tau_v)$ is also bi-additive in the above sense, according to [9, (9.5) Theorem].

**Lemma 2.1.** Given $v < \infty$, suppose $\pi_v \in A_n$ and $\tau_v \in A_m$. Then

$$L(s, \pi_v \times \tau_v) = P(q_v^{-s})L(s, \pi_v \otimes \tau_v)$$

for some polynomial $P \in \mathbb{C}[X]$.

**Proof.** Since $\pi_v$ is a sum of essentially square-integrable representations and both $L(s, \pi_v \times \tau_v)$ and $L(s, \pi_v \otimes \tau_v)$ are additive with respect to this sum, it suffices to prove the lemma for essentially square-integrable representations $\pi_v$. So for the remainder of the proof we assume $\pi_v$ is an essentially square-integrable representation of the form $\pi_v = \sigma_v(\eta_v)$. We proceed by induction on $m$. If $m = 1$, then $\tau_v = \chi_v$ is a quasi-character of $F_v^\times$ and $L(s, \pi_v \otimes \chi_v) = L(s, \pi_v \otimes \chi_v)$, where $\pi_v \otimes \chi_v$ is the representation of $GL_n(F_v)$ defined by $g \mapsto \pi_v(g)\chi_v(\det g)$. If $\chi_v$ is unramified, then

$$L(s, \pi_v \otimes \chi_v) = L(s, \pi_v \times \chi_v)$$

and consequently $P = 1$; on the other hand if $\chi_v$ is ramified then $L(s, \pi_v \times \chi_v) = 1$, and the assertion follows since $L(s, \pi_v \otimes \chi_v)^{-1}$ is a polynomial in $q_v^{-s}$.

We now assume $m > 1$. Suppose $\tau_v$ is an essentially square-integrable representation of $GL_m(F_v)$, say $\tau_v = \sigma_v(\eta_v')$, $\eta_v' \in A_{m'}$ is supercuspidal, and $a'b' = m$. Then the standard $L$-factor $L(s, \tau_v)$ is given by $L(s, \tau_v) = L(s + b' - 1, \eta_v')$ (see [9]). Consequently, $L(s, \tau_v) = 1$ unless $a' = 1$ and $\eta_v' = \chi_v$ is an unramified quasi-character of $F_v^\times$. However, if $L(s, \tau_v) = 1$ then $L(s, \pi_v \times \tau_v) = 1$, and the assertion of the lemma follows. Hence we may assume $\tau_v = \sigma_m(\chi_v)$ for an unramified quasi-character $\chi_v$ of $F_v^\times$, in which case

$$L(s, \pi_v \times \tau_v) = L(s, \pi_v \times \chi_v) \parallel \parallel_v^{m-1}$$

$$= L(s, \sigma_m(\pi_v \parallel \parallel_v^{b-1}) \parallel \parallel_v^{m-1})$$

$$= L(s + m - 1 + b - 1, \eta_v \otimes \chi_v).$$

On the other hand, it also follows from [9, (8.2) Theorem] that

$$L(s, \pi_v \otimes \tau_v) = \begin{cases} 
\prod_{j=0}^{m-1} L(s + j + b - 1, \eta_v \otimes \chi_v) & \text{if } m \leq n, \\
\prod_{i=0}^{b-1} L(s + m - 1 + i, \eta_v \otimes \chi_v) & \text{if } m > n.
\end{cases}$$
Now, from (2.6) and (2.7), one sees that the ratio \( \frac{L(s, \pi \times \tau)}{L(s, \pi \otimes \tau)} \) is a polynomial in \( q_v^{-s} \), thus proving the lemma when \( \tau_v \) is an essentially square-integrable representation.

If \( \tau_v \) is not essentially square-integrable, then as discussed above there is a partition \( \sum_{i=1}^{k} m_i = m \), with each \( m_i < m \), and essentially square-integrable representations \( \tau_{i,v} \) of \( GL_{m_i}(F_v) \), \( i = 1, \ldots, k \), so that \( \tau_v = \bigoplus_i \tau_{i,v} \). We may then use additivity and apply the induction hypothesis to get the desired conclusion. \( \square \)

**Corollary 2.2.** If \( L(s, \pi_v \boxtimes \tau_v) = 1 \), then either \( L(s, \pi_v) = 1 \) or \( L(s, \tau) = 1 \).

**Proof.** If \( L(s, \pi_v \boxtimes \tau_v) = 1 \), then Lemma 2.1 implies that \( L(s, \pi_v \times \tau_v) \) is a polynomial in \( q_v^{-s} \) and hence must be 1. This in turn implies the conclusion. \( \square \)

3. \( GL_3 \times GL_2 \)

We keep the notation of the previous section and take \( n = 3 \), \( m = 2 \). Thus \( \pi \) is an irreducible admissible generic representation of \( GL_3(A_F) \) with conductor \( \mathfrak{n} \) and central character \( \omega_\pi \), and \( \tau \) is an automorphic subrepresentation of \( GL_2(A_F) \) with central character \( \omega_\tau \). We fix an arbitrary integral ideal \( q \) and consider only those \( \tau \) whose conductor is \( q \). Recall that we also have the associated characters \( \chi_\pi \) and \( \chi_\tau \), respectively.

Now, choose ideals \( a_j \subseteq \mathfrak{o}_F \) representing the ideal classes of \( F \) so that \( a = \prod_j a_j \) is co-prime to \( q \). Let \( t_j \) be a finite idèle such that \( a_j = (t_j) \), and take \( t_1 = 1 \). For each \( j \), put \( h_j = \left( \begin{smallmatrix} t_j & 0 \\ 0 & 1 \end{smallmatrix} \right) \) and \( g_j = \left( \begin{smallmatrix} h_j & 1 \\ 0 & 1 \end{smallmatrix} \right) \in GL_3(A_{F,f}) \). Then \( \{g_j\} \) is a valid set of representatives in (2.1) (with \( n = 3 \)). In what follows, we will also consider strong approximation for \( GL_2(A_F) \), i.e.

\[
GL_2(A_F) = \prod_{j=1}^{h} GL_2(F)h_j GL_2(F_\infty)K,
\]

where the compact open subgroup \( K \subseteq GL_2(A_{F,f}) \) is either \( K_f = GL_2(\mathfrak{o}_F), K_1(q) \) or \( K_0(q) \). We then have the group \( G_j = \Gamma_{1,j}(\mathfrak{o}_F) = \Gamma_{0,j}(\mathfrak{o}_F) \) along with the congruence subgroups \( \Gamma_{0,j}(q), \Gamma_{1,j}(q) \) and \( \Gamma_j(q) \). Explicitly,

\[
G_j = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, d \in \mathfrak{o}_F, c \in a_j^{-1}, b \in \mathfrak{a}_j, ad - bc \in \mathfrak{a}_F^* \},
\]

which is precisely the stabilizer (acting on the right) of the lattice \( \mathfrak{o}_F \oplus \mathfrak{a}_j \subset F \oplus F \), and

\[
\Gamma_{0,j}(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_j : c \in qa_j^{-1} \right\},
\]

\[
\Gamma_{1,j}(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{0,j}(q) : d - 1 \in q \right\},
\]

\[
\Gamma_j(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{1,j}(q) : a - 1 \in q, b \in qa_j \right\}.
\]

Note that \( \Gamma_j(q) \subseteq \Gamma_{1,j}(q) \subseteq \Gamma_{0,j}(q) \subseteq G_j \), and that \( \Gamma_j(q) \) is normal in \( G_j \).

For any \( \alpha \in F \hookrightarrow F_\infty \) embedded diagonally, we form the adele \( (\alpha, 0) \in F_\infty \times A_{F,f} \) which we continue to denote by \( \alpha \). In this way, we view \( \beta = (\beta_1, \beta_2) \in F^2 \) as an element of \( \mathbb{A}_F^2 \). For \( \xi_\infty \in V_{\pi_\infty} \), note that

\[
h \mapsto U_{\xi_\infty \otimes \xi_\infty^j} \left( \begin{array}{c} h \\ 1 \end{array} \right), \quad \text{for } h \in GL_2(A_F),
\]

is a rapidly decreasing automorphic function on \( GL_2(A_F) \) (cf. [7, Theorem 1.1]) which allowed us to consider integrals of the type (2.5) while defining \( \Lambda(s, \pi \boxtimes \tau) \). We now follow [2] in order to define
additive twists (see loc. cit.) in the current context. To that end, we modify the above function by inserting a unipotent element. To be precise, with \( \beta = (\beta_1, \beta_2) \) as above, consider the function

\[
h \mapsto U_{\xi \otimes \xi_j} \left( \begin{array}{c} h \\ \beta \\ 1 \end{array} \right), \quad \text{for } h \in \text{GL}_2(\mathbb{A}_F);
\]

it is rapidly decreasing (by the gauge estimates in loc. cit.), but (as one can easily check) is not \( \text{GL}_2(F) \)-invariant, i.e., not an automorphic function on \( \text{GL}_2(\mathbb{A}_F) \), unless of course \((\beta_1, \beta_2) = (0, 0)\).

### 3.1. The function \( \Phi_{\xi \otimes \beta} \) and its Fourier expansion

For each \( j = 1, \ldots, h \), put

\[
\Phi_{\xi \otimes \beta}(g) = U_{\xi \otimes \xi_j} \left( \frac{g}{g_j} \right), \quad \text{for } g \in \text{GL}_n(F_{\infty}),
\]

and for \( \beta_1 \in \mathfrak{a}q^{-1} \) and \( \beta_2 \in q^{-1} \), consider the function

\[
\Phi_{\xi \otimes \beta}(g) = U_{\xi \otimes \xi_j} \left( \frac{\beta}{\beta_j} \right), \quad \text{for } h \in \text{GL}_2(F_{\infty}).
\]

Our goal in this section is to derive its Fourier series expansion, which will also reveal it to be (classical) automorphic form on \( \text{GL}_2(F_{\infty}) \). For \( k \in \{1, \ldots, h\} \), choose distinct prime ideals \( p_k \) and \( p_k' \) such that \( p_k \sim q p_k' \sim a_k \) and \( p_k \) is coprime to \( q a_1 \cdots a_h \), and let \( \alpha_k \) be a generator of the principal fractional ideal \( p_k^{-1} q p_k' \). Then for each fixed \( j \), \( \alpha_k a_j + \sigma_F = p_k^{-1} \) runs through a set of representatives of the class group of \( F \). Let \( p_k \) be a finite idèle such that \( (p_k) = p_k \).

**Lemma 3.1.** Let \( \mathfrak{o}_q^\times \) denote the image of the natural map \( \mathfrak{o}_F^\times \to (\mathfrak{o}_F/q)^\times \). Then \( \Gamma_j(\mathfrak{q}) \backslash \Gamma_1(\mathfrak{q}) \) is represented by elements of the form \( (s, t) \), where \( s \in \mathfrak{o}_F^\times \) runs through representatives for \( \mathfrak{o}_q^\times \) and \( x \in \mathfrak{a}_j \) runs through representatives for \( \mathfrak{a}_j/\mathfrak{a}_j q \). In particular, \( [\Gamma_1(\mathfrak{q}) : \Gamma_j(\mathfrak{q})] = N(\mathfrak{q})|\mathfrak{o}_q^\times| \).

**Proof.** Note that \( \mathfrak{o}_q^\times \) acts on \( \mathfrak{a}_j/\mathfrak{a}_j q \) via \( x \mapsto ex \). The map \( \Gamma_1(\mathfrak{q}) \to (\mathfrak{a}_j/\mathfrak{a}_j q) \times \mathfrak{o}_q^\times \) given by \( (a b \chi) \mapsto (b \mod q, ad - bc \mod q) \) is an epimorphism with kernel \( \Gamma_j(\mathfrak{q}) \). \( \square \)

**Lemma 3.2.** Let \( R_1, R_2 \) be sets of representatives for \( F^\times/\mathfrak{o}_F^\times \). Then for any fixed \( j \),

\[
\left\{ \begin{vmatrix} \gamma_1 & \gamma_2 \\ \gamma_1 & 1 \end{vmatrix} : \gamma_1 \in R_1, \gamma_2 \in R_2, 1 \leq k \leq h \right\}
\]

is a set of representatives for \( U_2(F) \backslash \text{GL}_2(F)/G_j \).

**Proof.** For \( M = (a b \chi) \in \text{GL}_2(F) \), let \( I_j(M) \) denote the fractional ideal \( c a_j + d \sigma_F \). It is easy to see that \( I_j(M) \) depends only on the coset of \( M \) in \( U_2(F) \backslash \text{GL}_2(F)/G_j \), i.e., \( I_j(u M g) = I_j(M) \) for all \( u \in U_2(F) \), \( g \in G_j \). For a given \( M \) there is a unique choice of \( k \in \{1, \ldots, h\} \) and \( \gamma_1 \in R_1 \) such that \( I_j(M) = \gamma_1 p_k^{-1} \). Thus, \( I_j(\gamma_1^{-1} M) = p_k^{-1} = \alpha_k a_j + \sigma_F \).

Suppose that \( \gamma_1^{-1} M \) has bottom row \( (c d) \). Then \( c a_j + d \sigma_F = \alpha_k a_j + \sigma_F \), which in turn implies \( c a_F + d a_j^{-1} = \alpha_k \sigma_F + a_j^{-1} \). Thus, we have

\[
(\alpha_k, 1) = (c d) \begin{pmatrix} A & B \\ C & D \end{pmatrix}
\]

for some \( A \in \mathfrak{o}_F, B \in \mathfrak{a}_j, C \in \mathfrak{a}_j^{-1}, D \in \mathfrak{o}_F \). It follows that the determinant \( AD - BC \) is an element of \( \mathfrak{o}_F \), but it need not be a unit. However, the choice of \( (A B) \) in (3.2) is not unique. For any particular solution \( (A B) \), it is straightforward to see that \( (A' B') \) is another solution if and only if \( (A' B') = (A B) + s(\chi) + (B D) \) for some \( s \in p_k \) and \( t \in \mathfrak{a}_j p_k \). Thus,

\[
\det \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} = \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = s \det \begin{pmatrix} d & B \\ -c & D \end{pmatrix} + t \det \begin{pmatrix} A & -d \\ C & c \end{pmatrix} = s + t \alpha_k,
\]
so by appropriate choice of $s$ and $t$ we may adjust the determinant by any element of $p_k + \alpha_k a_j p_k = \mathfrak{a}_F$.

Let $\epsilon \in \mathfrak{a}_F^\times$ be the unique unit such that $\epsilon \gamma_1^{-2} \det M \in R_2$. We choose $g = \left( \begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$ in the above with $\det g = \epsilon$. Then $g \in G_j$ and $\gamma_1^{-1} Mg$ has bottom row $(\alpha_k \ 1)$. Multiplying on the left by a suitable $u \in U_2(F)$, we can make it lower-triangular, i.e. $u \gamma_1^{-1} Mg = (\gamma_2 \ 1)$ for some $\gamma_2 \in F^\times$. In fact, evaluating the determinant, we have $\gamma_2 = \epsilon \gamma_1^{-2} \det M \in R_2$.

Thus, $u Mg = (\gamma_2 \gamma_1)(\gamma_2 \ 1)$ is of the required form. Moreover, although the pair $(u, g)$ is not unique in the above construction, it is clear that $k, \gamma_1$ and $\gamma_2$ are. This completes the proof. \ \[ \square \]

**Lemma 3.3.** We have

$$W_{\xi,0}^0 \left( \begin{pmatrix} \gamma_1 \gamma_2 t_j \\ \gamma_1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ t_j \alpha_k \\ 1 \end{pmatrix} \right) = \psi_{v_k}(\alpha_k^{-1} \gamma_2) W_{\xi,0}^0 \left( \begin{pmatrix} \gamma_1 \gamma_2 t_j p_k \\ \gamma_1 p_k^{-1} \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ t_j \alpha_k \\ 1 \end{pmatrix} \right),$$

and both sides vanish unless $\gamma_1 \in p_k, \gamma_2 \in \mathfrak{a}_j^{-1} p_k^{-2}$.

**Proof.** Suppose $v \neq v_k$ is a finite place of $F$. Then the corresponding local factors agree on both sides since $\xi_0^0$ is fixed by matrices of the form $(\begin{smallmatrix} g & \ 0 \\ 0 & 1 \end{smallmatrix})$ for $g \in \mathrm{GL}_2(\mathfrak{a}_v)$. Hence, it is enough to prove the local equality for $v = v_k$. To that end, let us write the $v$-component of $t_j \alpha_k$ as $u/\omega_v$ for $u \in \mathfrak{a}_v$. We have

$$\begin{pmatrix} 1 \\ u/\omega_v \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} \omega_v \\ \omega_v^{-1} \end{pmatrix} \begin{pmatrix} 0 & -u^{-1} \\ u & \omega_v \end{pmatrix},$$

and consequently

$$W_{\xi,0}^0 \left( \begin{pmatrix} \gamma_1 \gamma_2 t_j \\ \gamma_1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ u/\omega_v \\ 1 \end{pmatrix} \right) = \psi_v(\gamma_2 t_j u^{-1} \omega_v) W_{\xi,0}^0 \left( \begin{pmatrix} \gamma_1 \gamma_2 t_j \omega_v \\ \gamma_1 \omega_v^{-1} \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right).$$

This concludes the proof of the first assertion. Next, for any finite place $v$ of $F$ and $g = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \mathrm{GL}_2(F_v)$, one has

$$W_{\xi,0}^0 \begin{pmatrix} g \\ 0 \end{pmatrix} \neq 0 \implies c, d \in \mathfrak{a}_v.$$

It follows from this that $\gamma_1 \in p_k$. On the other hand, for $x \in \mathfrak{a}_v$, note that

$$W_{\xi,0}^0 \left( \begin{pmatrix} \gamma_1 \gamma_2 t_j \omega_v \\ \gamma_1 \omega_v^{-1} \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ x \\ 1 \end{pmatrix} \right) = \psi_v(\gamma_2 t_j \omega_v^x) W_{\xi,0}^0 \left( \begin{pmatrix} \gamma_1 \gamma_2 t_j \omega_v \\ \gamma_1 \omega_v^{-1} \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right).$$

Thus, if $W_{\xi,0}^0 \left( \begin{pmatrix} \gamma_1 \gamma_2 t_j \omega_v \\ \gamma_1 \omega_v^{-1} \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right) \neq 0$ then $\gamma_2 t_j \omega_v^2 \in \mathfrak{a}_v$, or in other words (globally) $\gamma_2 \in \mathfrak{a}_j^{-1} p_k^{-2}$. \ \[ \square \]
In what follows, we use the notation $G^x$ to denote $x^{-1}Gx$. Now

$$
\Phi_{\xi,\beta}(h_1, \beta) = \sum_{\gamma \in U_2(F) \setminus GL_2(F)} W_{\xi,\beta}(\gamma \xi_\beta)(h_1, \beta) W_{\xi,\beta}(\gamma \xi_\beta) \cdot \psi_\infty((\gamma h_2) \xi_\beta) W_{\xi,\beta}(\gamma h_2)
$$

which by Lemmas 3.2 and 3.3 can be written as

$$
\sum_k \frac{1}{n_{jk}(q)} \sum_{\gamma_1 \in R_1, \gamma_2 \in R_2} \psi_{\nu_k}(\alpha_k^{-1} \gamma_2) W_{\xi,\beta}(\gamma_1 \gamma_2 \xi_\beta)(h_1, \beta) W_{\xi,\beta}(\gamma_1 \xi_\beta) \cdot \psi_\infty((\gamma_1 \eta h_2) \xi_\beta) W_{\xi,\beta}(\gamma_1 \eta h_2)
$$

(3.3)

where $n_{jk}(q) = [U_2(F)^{\ell_k} \cap \Gamma_j(q) : U_2(F)^{\ell_k} \cap G_j]$. One can check that

$$
U_2(F)^{\ell_k} \cap G_j = \{ \ell_k^{-1}(1 \ell) \ell_k : x \in a_j p_k^2 \}.
$$

From this and Lemma 3.3, it follows that the final summand in (3.3), viewed as a function of $\eta$, is constant on left cosets of $U_2(F)^{\ell_k} \cap G_j$ in $G_j$, so (3.3) is well defined. Similarly, we have

$$
U_2(F)^{\ell_k} \cap \Gamma_j(q) = \{ \ell_k^{-1}(1 \ell) \ell_k : x \in a_j q p_k^2 \}.
$$

In particular, $n_{jk}(q) = N(q)$.

Since $\beta_1 \in a_j q^{-1}$ and $\beta_2 \in q^{-1}$, the function $\eta \mapsto \psi_\infty((\gamma_1 \eta \beta_2) \xi_\beta)$ is constant on left cosets of $\Gamma_j(q)$ in $G_j$. Thus (3.3) may be rewritten as

$$
\sum_k \frac{1}{N(q)} \sum_{\gamma_1 \in R_1, \gamma_2 \in R_2} \psi_{\nu_k}(\alpha_k^{-1} \gamma_2) W_{\xi,\beta}(\gamma_1 \gamma_2 \xi_\beta)(h_1, \beta) W_{\xi,\beta}(\gamma_1 \xi_\beta) \cdot \psi_\infty((\gamma_1 \eta h_2) \xi_\beta) W_{\xi,\beta}(\gamma_1 \eta h_2)
$$

(4.4)

In particular, it follows that (3.1) is left-invariant under $G_j$.

### 3.2. Additive twists.

We continue with the above set-up. Recall that at every finite place $v$ we have the essential vector $\xi_v^0$ in the space of $\pi_v$, with the associated essential function $W_{\xi_v}^0$ which satisfies

$$
W_{\xi_v}^0 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1.
$$

Let $\varphi \in V_\tau$ be any decomposable vector which is locally the new vector $\varphi_v^0$ at any finite place $v$ of $F$ and put $\varphi_j(h) = \varphi(h, h_j), h \in GL_2(F_\infty)$, so that $\varphi_j \in A(\Gamma_0, \Gamma_1(q), GL_2(F_\infty), \omega_{\tau,\pi}, \chi^{-1})$. Here again the essential function $W_{\varphi_v}^0$ satisfies

$$
W_{\varphi_v}^0 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1.
$$
Suppose $\tau_v$ is unramified, then (1.2) with $n = 3$ reads as

\[(3.5) \quad \int_{U_2(F_v) \setminus GL_2(F_v)} W_{\xi_0}(g_v) \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \Phi_{\tau} \left( \begin{array}{cc} h & \beta \\ 1 & 1 \end{array} \right) \det g_v^v \, dg_v = L(s, \pi_v \boxtimes \tau_v). \]

For each $j = 1, \ldots, h$, put

\[\Lambda_j(s, \pi, \tau, \xi_\infty, \varphi_\infty, \beta) = N(a_j)^{\frac{1}{2}} \int_{\Gamma_j(q) \setminus GL_2(F_\infty)} \Phi_{\xi_\infty, j} \left( \begin{array}{cc} h & t^j \beta \\ 1 & 1 \end{array} \right) \varphi_j(h) \, \det h^{|\frac{s-1}{2}} \, dh, \]

\[(3.6) \quad = \frac{N(a_j)^{\frac{1}{2}}}{[\Gamma_j(q) : \Gamma_j(q)]} \int_{\Gamma_j(q) \setminus GL_2(F_\infty)} \Phi_{\xi_\infty, j} \left( \begin{array}{cc} h & t^j \beta \\ 1 & 1 \end{array} \right) \varphi_j(h) \, \det h^{|\frac{s-1}{2}} \, dh. \]

In view of (3.4) this integral is clearly well defined and we call it a (generalized) additive twist.

**Lemma 3.4.** For $x \in a_j$, $j = 1, \ldots, h$, we have

\[\Lambda_j(s, \pi, \tau, \xi_\infty, \varphi_\infty, (\beta_1, \beta_2)) = \Lambda_j(s, \pi, \tau, \xi_\infty, \varphi_\infty, (\beta_1 + x, \beta_2)). \]

**Proof.** Since $U_{\xi_\infty \otimes \xi_j}$ is invariant under $Z_3(F)P_3(F)$ and $\xi_j$ is $K_1(n)$-invariant, it follows that each $\Phi_{\xi_\infty, j}$ is left-invariant by

\[Z_3(F)P_3(F) \cap GL_3(F_\infty)g_jK_1(n)g_j^{-1}. \]

From this we see that

\[\Lambda_j(s, \pi, \tau, \xi_\infty, \varphi_\infty, (\beta_1, \beta_2)) = \Lambda_j(s, \pi, \tau, \xi_\infty, \varphi_\infty), \]

for all $x \in a_j$. \qed

If $(\beta_1, \beta_2) = (0, 0)$ then strong approximation implies that

\[\sum_j \Lambda_j(s, \pi, \tau, \xi_\infty, \varphi_\infty, \beta) = \Lambda(s, \pi, \tau, \xi_\infty, \varphi_\infty), \]

where

\[\Lambda(s, \pi, \tau, \xi_\infty, \varphi_\infty) := \frac{1}{\text{vol}(K_1(q))} \int_{GL_2(F) \setminus GL_2(A_F)} U_{\xi_\infty \otimes \xi_j} \left( \begin{array}{cc} h & 1 \\ 1 & 1 \end{array} \right) \varphi(h) \, \det h^{|\frac{s-1}{2}} \, dh. \]

If, in addition, $\tau$ is unramified (so that $q = a_F$), then it follows from (3.5) that

\[\Lambda(s, \pi, \tau, \xi_\infty, \varphi_\infty) = \Psi_\infty(s, \xi_\infty, \varphi_\infty) L(s, \pi \boxtimes \tau) \]

\[(3.8) \quad = \Psi_\infty(s, \xi_\infty, \varphi_\infty) L(s, \pi \times \tau), \]

where

\[L(s, \pi \boxtimes \tau) = \prod_{v < \infty} L(s, \pi_v \boxtimes \tau_v) \quad \text{and} \quad L(s, \pi \times \tau) = \prod_{v < \infty} L(s, \pi_v \times \tau_v). \]

We now state our main result, which is convenient to formulate in terms of $L(s, \pi \times \tau)$ rather than $L(s, \pi \boxtimes \tau)$.

**Theorem 3.5.** Let $\pi, \tau, q$ and $a$ be as in the beginning of §3. Let $c$ be the conductor of $\chi_\tau$, and define $m = \prod_{(p; \text{ord}_p(q) > \max(1, \text{ord}_p(c)))} p^{\text{ord}_p(q)}$. Then there exist $\beta_2 \in q^{-1}$ and $c \in \mathbb{C}^\times$ such that

\[\Psi_\infty(s, \xi_\infty, \varphi_\infty) L(s, \pi \times \tau) = c \sum_{\beta_1 \in m^{-1}/a} \sum_{j=1}^{h} \Lambda_j(s, \pi, \tau, \xi_\infty, \varphi_\infty, (\beta_1, \beta_2)), \]

where

\[\Psi_\infty(s, \xi_\infty, \varphi_\infty) = \int_{U_2(F_\infty) \setminus GL_2(F_\infty)} W_{\xi_\infty} \left( \begin{array}{cc} h \\ 1 \end{array} \right) W_{\varphi_\infty} (h) \, \det h^{|\frac{s-1}{2}} \, dh. \]
In particular, if \( q \) is squarefree or \( \chi_\tau \) is primitive then

\[
\Psi_\infty(s, \xi_\infty, \varphi_\infty)L(s, \pi \times \tau) = c \sum_{j=1}^{h} \Lambda_j(s, \pi, \tau, \xi_\infty, \varphi_\infty, (0, \beta_2)).
\]

### 3.3. Proof of Theorem 3.5.

The proof will be broken into a series of lemmas. First, substituting (3.4) into (3.6) and changing \( h \mapsto \eta^{-1}h \), we get

\[
\Lambda_j(s, \pi, \tau, \xi_\infty, \varphi_\infty, \beta)
\]

\[
= N(a_j)^{\frac{1}{2} - s} N(q)^{-1} \left[ \Gamma_1 : \Gamma_j(q) \right] \sum_k \sum_{\gamma_1 \in R_1, \gamma_2 \in R_2} \psi_{v_k}(\alpha_k^{-1} \gamma_2) W_{\xi_j}^{(1)}(\gamma_1 \gamma_2 \ell_j \ell_k \eta h, 1) \gamma_1 \gamma_2 \ell_j \ell_k \eta h, 1)
\]

\[
\psi_{\infty}(\gamma_1(\ell_k \eta \gamma_2)) \int_{U_2(F) \cap \Gamma_j(q) \cap \Gamma_j(q)} W_{\xi_j}(\gamma_1 \gamma_2 \ell_j \ell_k \eta h, 1) \varphi_j(\eta^{-1}h) \| \det h \|_\infty^{-\frac{1}{2}} dh.
\]

Since \( \varphi_j \) is left-invariant under \( \Gamma_j(q) \), we may collapse the integral and the sum over \( \epsilon \), and make the change of variable \( h \mapsto \ell_k^{-1}h \) to get

\[
\Lambda_j(s, \pi, \tau, \xi_\infty, \varphi_\infty, \beta)
\]

\[
= N(a_j)^{\frac{1}{2} - s} N(q)^{-1} \left[ \Gamma_1 : \Gamma_j(q) \right] \sum_k \sum_{\gamma_1 \in R_1, \gamma_2 \in R_2} \psi_{v_k}(\alpha_k^{-1} \gamma_2) W_{\xi_j}^{(1)}(\gamma_1 \gamma_2 \ell_j \ell_k \eta h, 1) \gamma_1 \gamma_2 \ell_j \ell_k \eta h, 1)
\]

\[
\psi_{\infty}(\gamma_1(\ell_k \eta \gamma_2)) \int_{U_2(F) \cap \Gamma_j(q) \cap \Gamma_j(q)} W_{\xi_j}(\gamma_1 \gamma_2 \ell_j \ell_k \eta h, 1) \varphi_j(\eta^{-1}h) \| \det h \|_\infty^{-\frac{1}{2}} dh.
\]

Let us now consider the archimedean integral in the above equation. We change \( h \mapsto \gamma_1^{-1}h \) to get

\[
\omega_{\tau_\infty}(\gamma_1) \| \gamma_1 \|_{\infty}^{1-2s} \int_{U_2(F) \cap \Gamma_j(q) \cap \Gamma_j(q) \cap \Gamma_j(q) \cap \Gamma_j(q)} W_{\xi_j}(\gamma_1 \gamma_2 \ell_j \ell_k \eta h, 1) \varphi_j(\eta^{-1}h) \| \det h \|_{\infty}^{-\frac{1}{2}} dh.
\]

Since \( \varphi(\eta^{-1} \ell_k^{-1}h, h_j) = \varphi(h, \ell_k \eta h_j) \), this becomes

\[
\omega_{\tau_\infty}(\gamma_1) \| \gamma_1 \|_{\infty}^{1-2s} \int_{U_2(F_\infty) \cap \Gamma_j(q) \cap \Gamma_j(q) \cap \Gamma_j(q) \cap \Gamma_j(q)} W_{\xi_j}(\gamma_1 \gamma_2 \ell_j \ell_k \eta h, 1) \varphi_j(\eta^{-1}h) \| \det h \|_{\infty}^{-\frac{1}{2}} dh,
\]

where

\[
W_\varphi(h) = \int_{U_2(F) \cap \Gamma_j(q) \cap \Gamma_j(q) \cap \Gamma_j(q) \cap \Gamma_j(q)} \varphi(u(x)h, \ell_k \eta h_j) \psi_{\infty}(\gamma_2 x) du(x).
\]

Here \( u(x) \) denotes the upper unipotent matrix \( \left( \begin{array}{cc} 1 & x \\ 0 & 1 \end{array} \right) \). Then it is clear that the function \( h \mapsto W_\varphi(h) \left( \begin{array}{cc} \gamma_2^{-1} \gamma_1 \end{array} \right) \) belongs to the Whittaker model of \( \tau_\infty \) with respect to the character \( \psi_{\infty}^{-1} \). Hence,
by the local multiplicity one theorem, there exists a constant $a_{jk}^\varphi(\gamma_2, \eta)$ so that

$$W'_\varphi(h) = a_{jk}^\varphi(\gamma_2, \eta) W_{\varphi_\infty}\left(\begin{pmatrix} \gamma_2 & 1 \\ 1 & 1 \end{pmatrix} h\right), \text{ for all } h \in \text{GL}_2(F_\infty),$$

where $W_{\varphi_\infty}$ is the Whittaker function (with respect to $\psi^{-1}_\infty$) associated to $\varphi_\infty$. Consequently, (3.11) can now be written as

$$\omega_{\varphi_\infty}^{-1}(\gamma_1) a_{jk}^\varphi(\gamma_2, \eta) \parallel \gamma_2 \parallel^\frac{1}{2} \cdot s \int_{\text{U}_2(F_\infty) \setminus \text{GL}_2(F_\infty)} W_{\xi_\infty}\left(\begin{pmatrix} \gamma_2 & 1 \\ 1 & 1 \end{pmatrix} h\right) W_{\varphi_\infty}\left(\begin{pmatrix} \gamma_2 & 1 \\ 1 & 1 \end{pmatrix} h\right) \parallel \det h\parallel^\frac{s}{2} \, dh,$$

which after the change $h \mapsto \left(\gamma_2^{-1} \right) h$ becomes

$$\omega_{\varphi_\infty}^{-1}(\gamma_1) a_{jk}^\varphi(\gamma_2, \eta) \parallel \gamma_2 \parallel^\frac{1}{2} \cdot s \int_{\text{U}_2(F_\infty) \setminus \text{GL}_2(F_\infty)} W_{\xi_\infty}\left(h_1\right) W_{\varphi_\infty}(h) \parallel \det h\parallel^\frac{s}{2} \, dh.$$  

We note that this is precisely the local Rankin–Selberg integral for $\pi \times \tau$ at $\infty$, viz. $\prod_{\nu|\infty} \Psi_\nu(s; W_{\xi_\nu}, W_{\varphi_\nu})$, in the notation of Section 2.

**Lemma 3.6.** We have

$$a_{jk}^\varphi(\gamma_2, \eta) = N(a_j p_k^2 q) W_{\varphi_f}\left(\begin{pmatrix} \gamma_2 & 1 \\ 1 & 1 \end{pmatrix} \ell_k \eta h_j\right).$$

**Proof.** We have $\varphi(g) = \sum_{\gamma \in F^\times} W_{\varphi}(\gamma_1 g)$ for all $g \in \text{GL}_2(A_F)$, and thus

$$\varphi(h, \ell_k \eta h_j) = \sum_{\gamma \in F^\times} W_{\varphi_f}\left(\begin{pmatrix} \gamma & 1 \\ 1 & 1 \end{pmatrix} \ell_k \eta h_j\right) W_{\varphi_\infty}\left(\begin{pmatrix} \gamma & 1 \\ 1 & 1 \end{pmatrix} h\right).$$

Further, a calculation similar to that in the proof of Lemma 3.3 shows that if $W_{\varphi_f}(\gamma_1 \ell_k \eta h_j)$ is non-zero then $\gamma \in p_k^{-2} m^{-1} a_j^{-1}$. Plugging this into (3.12), we get

$$W'_\varphi(h) = \int_{\text{U}_2(F) \cap \Gamma_j(q)_{k}^{-1} \setminus \text{U}_2(F_\infty)} \sum_{\gamma \in F^\times} W_{\varphi_f}\left(\begin{pmatrix} \gamma & 1 \\ 1 & 1 \end{pmatrix} \ell_k \eta h_j\right) W_{\varphi_\infty}\left(\begin{pmatrix} \gamma & 1 \\ 1 & 1 \end{pmatrix} h\right) \psi_{\infty}(\gamma_2 x) \, du(x)$$

$$= \sum_{\gamma \in F^\times} W_{\varphi_f}\left(\begin{pmatrix} \gamma & 1 \\ 1 & 1 \end{pmatrix} \ell_k \eta h_j\right) \int_{\text{U}_2(F) \cap \Gamma_j(q)_{k}^{-1} \setminus \text{U}_2(F_\infty)} W_{\varphi_\infty}\left(\begin{pmatrix} \gamma & 1 \\ 1 & 1 \end{pmatrix} u(x) h\right) \psi_{\infty}(\gamma_2 x) \, du(x)$$

$$= \sum_{\gamma \in F^\times} W_{\varphi_f}\left(\begin{pmatrix} \gamma & 1 \\ 1 & 1 \end{pmatrix} \ell_k \eta h_j\right) W_{\varphi_\infty}\left(\begin{pmatrix} \gamma & 1 \\ 1 & 1 \end{pmatrix} h\right) \int_{\text{U}_2(F) \cap \Gamma_j(q)_{k}^{-1} \setminus \text{U}_2(F_\infty)} \psi_{\infty}(\gamma_2 - \gamma x) \, du(x)$$

$$= N(a_j p_k^2 q) W_{\varphi_f}\left(\begin{pmatrix} \gamma_2 & 1 \\ 1 & 1 \end{pmatrix} \ell_k \eta h_j\right) W_{\varphi_\infty}\left(\begin{pmatrix} \gamma_2 & 1 \\ 1 & 1 \end{pmatrix} h\right).$$

Here we have used the fact that the integral vanishes unless $\gamma = \gamma_2$, and that $\text{vol}(U_2(F) \cap \Gamma_j(q)_{k}^{-1} \setminus U_2(F_\infty)) = N(a_j p_k^2 q)$ since the additive measure is normalized so that $\text{vol}(F_\infty/\mathfrak{o}_F) = 1$. Our assertion now follows from (3.13).
By (3.14) and Lemma 3.6, we now have
\begin{equation}
\Lambda_j(s, \pi, \tau, \xi, \varphi, \beta) = N(a_j p_k^2)N(a_j)^{\frac{1}{2-s}} \prod_{\nu \in \infty} \Psi_{\nu}(s; W_{\xi_{\nu}}, W_{\varphi_{\nu}})
\end{equation}
\begin{equation}
\sum_k \sum_{\gamma_1, \gamma_2 \in F \times / \mathcal{F}} \|\gamma_1^2 \gamma_2 \|_{2, \infty} \|\gamma_2\|_{\infty} \psi_{\nu_k}(\alpha_k \gamma_2) W_{\nu_k}(\gamma_1 \gamma_2 \gamma_k p_k^{-1})
\end{equation}
Next we study the inner sum
\begin{equation}
\frac{1}{[\Gamma_{1,j} : \Gamma_j(q)]} \sum_{\gamma \in \Gamma_j(q) \setminus \Gamma_j} \psi_{\nu}(\gamma_1(\ell_k \eta \beta)_2) W_{\varphi_f}(\gamma_1 \gamma_2 \gamma_{1j} \ell_k \eta h_j).
\end{equation}
To that end, we need the following basic group-theoretic result:

**Lemma 3.7.** Suppose \( K \subseteq H \) are subgroups of a group \( G \), with \( K \) normal of finite index in \( G \). If \( \{g_i\} \) is a set of representatives for \( G/H \) and \( \{h_j\} \) is a set of representatives for \( K/H \), then \( \{g_i h_j\} \) is a set of representatives for \( K/G \).

**Proof.** We have
\begin{equation}
G = \bigcup_i g_i H = \bigcup_i g_i \bigcup_j K h_j = \bigcup_i g_i K h_j = \bigcup_i K g_i h_j,
\end{equation}
where in the last equality we have used that \( g_i K g_i^{-1} = K \). The fact that \( \{g_i h_j\} \) is a minimal set of representatives follows from the index formula \([G : K] = [G : H][H : K]\).

Now, since \( \Gamma_j(q) \) is normal in \( G_j \) and is contained in \( \Gamma_{0,j}(q) \), we may apply Lemma 3.7 with \( K = \Gamma_j(q), H = \Gamma_{0,j}(q) \) and \( G = G_j \) to (3.16) and obtain
\begin{equation}
\frac{1}{[\Gamma_{1,j} : \Gamma_j(q)]} \sum_g \sum_{\eta} \psi_{\nu}(\gamma_1(\ell_k \eta \beta)_2) W_{\varphi_f}(\gamma_1 \gamma_2 \gamma_{1j} \ell_k \eta h_j),
\end{equation}
where \( g \) runs through a set of representatives for \( G_j/\Gamma_{0,j}(q) \) and \( \eta \) runs through a set of representatives for \( \Gamma_j(q) \setminus \Gamma_{0,j}(q) \). Since \( h_{\eta}^{-1} \eta h_j \in K_0(q), \eta \in \Gamma_{0,j}(q) \subseteq \Gamma_{0,j}(m), \) and \( \varphi_f \) transforms via the central character \( \chi_{\tau} \) under \( K_0(q) \), we see that (3.17) in turn becomes
\begin{equation}
\sum_g W_{\varphi_f}(\gamma_1 \gamma_2 \gamma_{1j} \ell_k g h_j) \frac{1}{[\Gamma_{1,j}(q) : \Gamma_j(q)]} \sum_{\eta} \psi_{\nu}(\gamma_1(\ell_k \eta \beta)_2) \chi_{\tau}(\eta).
\end{equation}
Applying Lemma 3.7 again with \( K = \Gamma_j(q), H = \Gamma_{1,j}(q) \) and \( G = \Gamma_{0,j}(q) \), we have
\begin{equation}
\frac{1}{[\Gamma_{1,j}(q) : \Gamma_j(q)]} \sum_{\eta} \psi_{\nu}(\gamma_1(\ell_k \eta \beta)_2) \chi_{\tau}(\eta) = \sum_{\mu} \chi_{\tau}(\mu) \left( \frac{1}{[\Gamma_{1,j}(q) : \Gamma_j(q)]} \sum_{\nu} \psi_{\nu}(\gamma_1(\ell_k \nu \beta)_2) \right),
\end{equation}
where \( \mu \) runs through a set of representatives for \( \Gamma_{0,j}(q)/\Gamma_{1,j}(q) \) and \( \nu \) runs through a set of representatives for \( \Gamma_j(q)/\Gamma_{1,j}(q) \). By Lemma 3.1, we may choose \( \nu \) of the form \( \nu = (\epsilon, \xi) \) for \( \epsilon \in \mathfrak{o}_F, x \in a_j \).

Set \( \beta' = \ell(\beta_1, \beta_2) := \mu \beta \). Then since \( \beta_1 \in a_j q^{-1} \) and \( \beta_2 \in q^{-1} \), we have \( \beta_1' \in a_j q^{-1} \) and \( \beta_2' \in q^{-1} \). (In fact, if \( \beta_2 = 0 \) then \( \beta_2' \in \mathfrak{o}_F \).) With \( \nu \) as above we have \( \nu \beta' = \ell(\epsilon \beta_1 + x \beta_2, \beta_2') \). Now, writing
\[ g = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \in G_j, \text{ we see that the matrix } \ell_k g \text{ has bottom row } (\alpha_k A + C \quad \alpha_k B + D), \text{ so that} \\
\gamma_1(\ell_k g \nu \beta_2) = \gamma_1(\alpha_k A + C)(\epsilon_1' + x_2') + \gamma_1(\alpha_k B + D)\beta_2' \equiv \gamma_1 C(\epsilon_1' + x_2') + \gamma_1 D \beta_2' \pmod{a_j}, \]

since \( \gamma_1 \in p_k, \alpha_k \in p_{q_k}^{-1}, \beta_1' \in a_j q^{-1} \text{ and } \beta_2' \in q^{-1}. \) Therefore the average over \( \nu \) in (3.18) equals

\[ \psi_{\infty}(\gamma_1 D \beta_2') \cdot \frac{1}{N(q)} \sum_{x \in a_j / a_j q} \psi_{\infty}(\gamma_1 C x \beta_2') \cdot \frac{1}{|a_j|} \sum_{e \in a_j} \psi_{\infty}(\gamma_1 C \epsilon_1'). \]

Next, we have

\[ \frac{1}{N(q)} \sum_{x \in a_j / a_j q} \psi_{\infty}(\gamma_1 C x \beta_2') = \begin{cases} 1 & \text{if } \gamma_1 C \beta_2' \in a_j^{-1}, \\ 0 & \text{otherwise}, \end{cases} \]

and thus (3.18) becomes

(3.19) \[ \sum_{\mu \gamma_1 C(\mu / a_j) \in a_j^{-1}} \chi_{\tau}(\mu) \psi_{\infty}(\gamma_1 D(\mu / a_j) \beta_2) \cdot \frac{1}{|a_j|} \sum_{e \in a_j} \psi_{\infty}(\epsilon_1 C(\mu / a_j)). \]

Let \( \mu = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_{0,j}(q) \). Then \( \beta_1' = a \beta_1 + b \beta_2 \) and \( \beta_2' = c \beta_1 + d \beta_2 \equiv d \beta_2 \pmod{a_F} \). Since \( d \) is invertible modulo \( q \), the condition \( \gamma_1 C(\mu / a_j) \in a_j^{-1} \) holds if and only if \( \gamma_1 C \beta_2 \in a_j^{-1} \). In this case, we also have \( \gamma_1 C(\mu / a_j) = \gamma_1 C(a \beta_1 + b \beta_2) \equiv c \gamma_1 C a / \beta_1 \pmod{a_F} \). Note that \( a / \text{det} \mu \equiv d \pmod{q} \), where \( d \in a_F \) is a multiplicative inverse of \( d \pmod{q} \). Hence, (3.19) equals

(3.20) \[ \left\{ \begin{array}{ll} \frac{1}{|a_j|} \sum_{e \in a_j} \sum_{d \in (a_F / q) / a_F} \chi_{\tau}(d) \psi_{\infty}(\gamma_1 D d \beta_2 + \epsilon_1 C d \beta_1) & \text{if } \gamma_1 C \beta_2 \in a_j^{-1}, \\ 0 & \text{otherwise}. \end{array} \right\} \]

Now, choose \( \nu \in a_F \setminus \cup_{p | q} p \) and \( \beta_2 \in \nu q^{-1} \setminus \cup_{p | q} p q^{-1} \), where \( q_1 = \prod_{p | q} p_{\max(1, \text{ord}_p(c))} \). Then \( \beta_2^{-1} \in q_1^{-1} \) and \( \gamma_1 D \nu \in a_F \). Further note that

\[ \frac{1}{N(m)} \sum_{\beta \in a_m^{-1} / a} \psi_{\infty}(\epsilon_1 C d \beta_1) = \begin{cases} 1 & \text{if } \gamma_1 C \in ma_j^{-1}, \\ 0 & \text{otherwise}, \end{cases} \]

so if we average (3.20) over \( \beta_1 \in a_1 \cdots a_h m^{-1} / a_1 \cdots a_h \), since \( q_1 \cap m = q \), we obtain

\[ \left\{ \begin{array}{ll} \sum_{d \in (a_F / q)} \chi_{\tau}(d) \psi_{\infty}(\gamma_1 D d \beta_2) & \text{if } \gamma_1 C \in qa_j^{-1}, \\ 0 & \text{otherwise}. \end{array} \right\} \]

In summary, we have shown that

(3.21) \[ \frac{1}{N(m)} \sum_{\beta_1 \in a_m^{-1} / a} \frac{1}{[\Gamma_{1,j} : \Gamma_j(q)]} \sum_g \sum_\eta \psi_{\infty}(\gamma_1 (\ell_k g \eta \beta_2)) W_{\varphi_f} \left( \left( \begin{array}{c} \gamma_2 \\ \gamma_1 \end{array} \right) \ell_k g \eta h_j \right) \\
= \sum_g W_{\varphi_f} \left( \left( \begin{array}{c} \gamma_2 \\ \gamma_1 \end{array} \right) \ell_k g h_j \right) \left\{ \begin{array}{ll} \sum_{d \in (a_F / q)} \chi_{\tau}(d) \psi_{\infty}(\gamma_1 D d \beta_2) & \text{if } \gamma_1 C \in qa_j^{-1}, \\ 0 & \text{otherwise}, \end{array} \right\} \]

where \( g = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \) runs through a set of representatives for \( G_j / \Gamma_{0,j}(q) \).

Next, note that for a fixed \( \gamma_1 \) satisfying \( \gamma_1 C \in qa_j^{-1} \), the ideal \( \gamma_1 D v \sigma_F + q \) is independent of the choice of representative for \( \gamma_1 \). We will show that the sum over those \( g \) for which \( \gamma_1 D v \sigma_F + q \neq \sigma_F \) vanishes. To that end, fix a prime ideal \( p \mid q \) and split the coset representatives \( g = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \) as \( g_1 g_2 \), where \( g_1 = \left( \begin{array}{cc} A_1 & B_1 \\ C_1 & D_1 \end{array} \right) \) runs through representatives for \( G_j / \Gamma_{0,j}(p^{-1} q) \) and \( g_2 = \left( \begin{array}{cc} A_2 & B_2 \\ C_2 & D_2 \end{array} \right) \) runs through representatives for \( \Gamma_{0,j}(p^{-1} q) / \Gamma_{0,j}(q) \). For fixed \( \gamma_1 \in \sigma_F \), \( \text{ord}_p(\gamma_1 D v) > 0 \) holds
if and only if $\text{ord}_p(\gamma_1 D_1 v) > 0$. Fix a $g_1$ for which this holds. Then, for every choice of $g_2$, $\psi_\infty(\gamma_1 D_2) = \psi_\infty(\gamma_1 D_1 D_2 \beta_2)$ and $\gamma_1 C \in qa_j^{-1} \iff \gamma_1 C_1 \in qa_j^{-1}$. Define

$$f_{\gamma_1, g_1}(g_2) = \begin{cases} \sum_{d \in (\mathcal{O}_F/q)^\times} \chi_r(d) \psi_\infty(\gamma_1 D_1 D_2 \beta_2) & \text{if } \gamma_1 C_1 \in qa_j^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Then it is plain that $f_{\gamma_1, g_1}(k g_2 k') = f_{\gamma_1, g_1}(g_2) x_r(k')$ for any $k \in \Gamma_{1,j}(p^{-1} q)$, $k' \in \Gamma_{0,j}(q)$. As the following lemma shows, these terms therefore contribute nothing to (3.21).

**Lemma 3.8.** Suppose $p$ is prime ideal dividing $q$. Let $f : \Gamma_{0,j}(p^{-1} q) \to \mathbb{C}$ be left-invariant under $\Gamma_{1,j}(p^{-1} q)$ and transform by $\chi_r$ on the right under $\Gamma_{0,j}(q)$. Then for any $x \in \text{GL}_2(\mathcal{A}_F, f)$,

$$\sum_{g \in \Gamma_{0,j}(p^{-1} q)/\Gamma_{0,j}(q)} W_{\varphi_j}(xgh_j) f(g) = 0.$$

**Proof.** Let $v$ be the place corresponding to $p$, and put $m = \text{ord}_v(q)$. For any subgroup $G \subseteq \text{GL}_2$, let $G^j$ denote $G \cap \text{SL}_2$. The natural map

$$\Gamma_{1,j}(p^{-1} q) \to K_{1,v}^j(p^{m-1})/K_v^1(p^m)$$

is surjective by strong approximation for $\text{SL}_2(\mathcal{A}_F)$. Fix $k_1 \in K_{1,v}^j(p^{m-1})$ and choose $k \in \Gamma_{1,j}(p^{-1} q)$ that maps to the coset $k_1 K_v^1(p^m)$ above. We have

$$\sum_{g \in \Gamma_{0,j}(p^{-1} q)/\Gamma_{0,j}(q)} W_{\varphi_j}(xgh_j) f(g) = \sum_{g \in \Gamma_{0,j}(p^{-1} q)/\Gamma_{0,j}(q)} W_{\varphi_j}(xkg_j) f(g).$$

For any finite place $w \neq v$, we have

$$W_{\varphi_w}(xkg_j) = W_{\varphi_w}(xgh_j),$$

since $(gh_j)^{-1} k_w (gh_j) \in K_{1,w}(p^{\text{ord}_w(q)})$. At $v$ we have $k_v = k_1 k'$ for some $k' \in K_v^1(p^m)$, so that

$$W_{\varphi_v}(xkg_j) = W_{\varphi_v}(xk_1 k' g) = W_{\varphi_v}(xk_1 g g^{-1} k' g) = W_{\varphi_v}(xk_1 g) = W_{\varphi_v}(xg g^{-1} k_1 g),$$

since $g^{-1} k' g \in g^{-1} K_v^1(p^m) g = K_v^1(p^m)$.

Now, if $k_1$ runs through a set of representatives for $K_{1,v}^j(p^{m-1})/K_v^1(p^m)$, so does $g^{-1} k_1 g$, since $g$ normalizes both $K_{1,v}^j(p^{m-1})$ and $K_v^1(p^m)$. Thus, averaging over $k_1$, our sum becomes

$$\sum_{g \in \Gamma_{0,j}(p^{-1} q)/\Gamma_{0,j}(q)} f(g) \prod_{w \neq v} W_{\varphi_w}(xgh_j) \cdot \frac{1}{[K_{1,v}^j(p^{m-1}): K_v^1(p^m)]} \sum_{k_1} W_{\varphi_v}(xkgk_1).$$

Consider the inner sum $\sum_{k_1} W_{\varphi_v}(tk_1)$ as a function of $t$. Fixing $k \in K_{1,v}(p^{m-1})$, we have $k \in (y_1) K_{1,v}(p^{m-1})$ for some $y \in p_0^\times$. Hence,

$$\sum_{k_1} W_{\varphi_v}(tk_1) = \sum_{k_1} W_{\varphi_v}(t(y_1) k_1) = \sum_{k_1} W_{\varphi_v}(tk_1(y_1)) = \sum_{k_1} W_{\varphi_v}(tk_1),$$

since $(y_1) \in K_{1,v}(p^m)$. Therefore $\sum_{k_1} W_{\varphi_v}(tk_1) = 0$. \hfill \Box

Hence, we can assume that $(\gamma_1 Dv, q) = 1$. For that case, we have the following:

**Lemma 3.9.** If $\gamma_1 C \in qa_j^{-1}$ and $(\gamma_1 Dv, q) = 1$ then $g \in \Gamma_{0,j}(q)$ and

$$\sum_{d \in (\mathcal{O}_F/q)^\times} \chi_r(d) \psi_\infty(\gamma_1 Dd \beta_2) = \chi_r(\gamma_1 Dv) \tau_q(\chi_r, \beta_2 v^{-1}),$$

17
where
\[
\tau_q(\chi, \beta_2 v^{-1}) := \sum_{d \in (\mathbb{Q}/q)^*} \chi(d)\psi_\infty(d\beta_2 v^{-1})
\]
is non-zero.

**Proof.** For any prime \( p \mid q \), we have \( \text{ord}_p(v) = 0 \) by definition, and by hypothesis we have \( \text{ord}_p(\gamma_1 D v) = 0 \). Further \( \text{ord}_p(\gamma_1 C) \geq \text{ord}_p(q) - \text{ord}_p(a_j) \). Hence,
\[
\text{ord}_p(C) \geq \text{ord}_p(q) - \text{ord}_p(\gamma_1 a_j) = \text{ord}_p(D a_j^{-1} q) \geq \text{ord}_p(a_j^{-1} q).
\]

Therefore, \( C \in a_j^{-1} q \), so that \( g \in \Gamma_{0,j}(q) \). \( \square \)

Taking \( g = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \), (3.21) can now be written as
\[
\tau_q(\chi, \beta_2 v^{-1})\omega_{\infty}(v)W_{\varphi_f} \left( \left( \begin{array}{cc} \gamma_2 & 1 \\ 1 & 1 \end{array} \right) \ell_k h_j \right) \omega^{-1}_{\infty}(\gamma_1 v)\chi(\gamma_1 v).
\]

Consequently the finite part of \( \frac{1}{N(m)} \sum_{\beta_1 \in a^{-1} / a} L_j(s, \tau, \xi, \varphi, \beta) \) takes the form
\[
N(a_j)^{1/2 - s} \sum_k N(a_j p_k^2) \|\gamma_2\|_\infty
\]
\[
\cdot \sum_{\gamma_1 \in \phi_{p_k} \cap F^*} \sum_{\gamma_2 \in \phi_{a_j^{-1} p_k^{-2} \cap F^*}} \|\gamma_1^2 \gamma_2\|_{\infty}^{1/2 - s} W_{\varphi_f} \left( \left( \begin{array}{cc} \gamma_1^2 & 1 \\ 1 & 1 \end{array} \right) \ell_k h_j \right) \omega^{-1}_{\infty}(\gamma_1 v)\chi(\gamma_1 v).
\]

For \( \gamma_1, \gamma_2 \) as above, let \( m_1 = (\gamma_1)p_k^{-1} \) and \( m_2 = (\gamma_2)a_j p_k^2 \). Then \( \|\gamma_1^2 \gamma_2\|_\infty = N(m_1^2 m_2 a_j^{-1}) \) and \( (\gamma_1 v) = v p_k m_1 \).

**Lemma 3.10.** We have
\[
\psi_{v_k}(\alpha_k^{-1} \gamma_2) W_{\varphi_f} \left( \left( \begin{array}{cc} \gamma_2 p_k & 1 \\ 0 & 1 \end{array} \right) \ell_k h_j \right) = W_{\varphi_f} \left( \left( \begin{array}{cc} \gamma_2 p_k^2 & 1 \\ 0 & 1 \end{array} \right) h_j \right).
\]

**Proof.** We check this locally at every place \( v \) by the same computation as in the proof of Lemma 3.3. \( \square \)

We define
\[
\lambda_{\pi}(m_1, m_2) = N(m_1 m_2)W_{\varphi_f} \left( \left( \begin{array}{cc} m_1 m_2 & 1 \\ 0 & 1 \end{array} \right) \right)
\]
and
\[
\lambda_{\pi}(m_2) = \sqrt{N(m_2)} W_{\varphi_f} \left( \left( \begin{array}{cc} m_2 & 1 \\ 0 & 1 \end{array} \right) \right),
\]
where \( m_1 \) and \( m_2 \) are finite idèles such that \( m_1 = (m_1) \) and \( m_2 = (m_2) \). Then
\[
\sum_{\beta_1 \in a^{-1} / a} L_j(s, \tau, \xi, \varphi, \beta) = N(m)\tau_q(\chi, \beta_2 v^{-1}) \prod_{v \mid \infty} \Psi_v(s; W_{\xi_v}, W_{\varphi_v})
\]
\[
\cdot \sum_k \sum_{m_1 \sim p_k^{-1} m_2 \sim a_j p_k^2} \lambda_{\pi}(m_1, m_2) \lambda_{\pi}(m_2) \chi_{\omega_r}(m_1) N(m_1^2 m_2)^{-s}.
\]
We can now sum over \( j \) to get
\[
\sum_{\beta_1} \sum_j \Lambda_j(s, \pi, \tau, \xi_\infty, \varphi_\infty, \beta) = N(m)\tau_q(\chi_\tau, \beta_2v^{-1}) \prod_{v|\infty} \Psi_v(s; W_{\xi_v}, W_{\varphi_v}) \\
\cdot \sum_{m_1, m_2} \lambda_\tau(m_1, m_2)\lambda_\tau(m_2)\chi_{\omega_\tau}(m_1)N(m_1^2m_2)^{-s}.
\]
It remains to identify the Dirichlet series in the above expression.

**Lemma 3.11.** We have
\[
\sum_{m_1, m_2} \lambda_\pi(m_1, m_2)\lambda_\tau(m_2)\chi_{\omega_\tau}(m_1)N(m_1^2m_2)^{-s} = L(s, \pi \times \tau).
\]

**Proof.** For a fixed choice of \( \tau \) and a non-zero integral ideal \( a \), define
\[
c_{\pi, \tau}(a) = \sum_{m_1, m_2} \lambda_\pi(m_1, m_2)\lambda_\tau(m_2)\chi_{\omega_\tau}(m_1).
\]
Then, for any unramified idèle class character \( \omega \), we have
\[
\lambda_\tau \otimes \omega(m_2) = \lambda_\tau(m_2)\chi_\omega(m_2) \quad \text{and} \quad \chi_{\omega_\tau \otimes \omega}(m_1) = \chi_{\omega_\tau}(m_1)\chi_\omega(m_1)^2,
\]
so that
\[
c_{\pi, \tau \otimes \omega}(a) = c_{\pi, \tau}(a)\chi_\omega(a).
\]
Next define \( \lambda_{\pi \times \tau} \) to be the Dirichlet coefficients of \( L(s, \pi \times \tau) \), so that, for any unramified \( \omega \), we have
\[
L(s, \pi \times (\tau \otimes \omega)) = \sum_a \lambda_{\pi \times \tau}(a)\chi_\omega(a)N(a)^{-s}.
\]
Note that both \( c_{\pi, \tau} \) and \( \lambda_{\pi \times \tau} \) are multiplicative, so it suffices to show that they agree at prime powers.

Given integers
\[
\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0,
\]
the Schur polynomial \( s_{\lambda_1, \lambda_2, \lambda_3}(x_1, x_2, x_3) \) is the ratio
\[
\det \begin{pmatrix} x_1^{\lambda_1+2} & x_2^{\lambda_2+2} & x_3^{\lambda_3+2} \\
 x_1^{\lambda_1+1} & x_2^{\lambda_2+1} & x_3^{\lambda_3+1} \\
 x_1^{\lambda_1} & x_2^{\lambda_2} & x_3^{\lambda_3} \end{pmatrix}.
\]

By the Cauchy identity [5, Theorem 43.3], for sufficiently small \( \alpha_1, \alpha_2, \alpha_3, \gamma_1, \gamma_2, \gamma_3 \in \mathbb{C} \), we have
\[
\prod_{i=1}^3 \prod_{j=1}^3 \frac{1}{1 - \alpha_i \gamma_j} = \sum_\lambda s_\lambda(\alpha_1, \alpha_2, \alpha_3)J_{\lambda}(\gamma_1, \gamma_2, \gamma_3),
\]
where the sum runs over all \( \lambda = (\lambda_1, \lambda_2, \lambda_3) \) satisfying (3.26). Note that if \( \gamma_3 = 0 \) then only terms with \( \lambda_3 = 0 \) contribute to the above sum. Writing \( \lambda_2 = k_1, \lambda_1 = k_1 + k_2 \) and replacing \( (\gamma_1, \gamma_2, \gamma_3) \) by \( (x_{\gamma_1}, x_{\gamma_2}, 0) \) for a small \( x \in \mathbb{C} \), we obtain
\[
\prod_{i=1}^3 \prod_{j=1}^2 \frac{1}{1 - \alpha_i \gamma_j x} = \sum_{k_1=0}^\infty \sum_{k_2=0}^\infty s_{k_1+k_2,k_1,0}(\alpha_1, \alpha_2, \alpha_3)J_{k_1+k_2,k_1,0}(\gamma_1, \gamma_2, 0)x^{2k_1+k_2}.
\]
Now fix a prime ideal \( \mathfrak{p} \), and let \( \nu \) be the corresponding place of \( F \); then \( q_\nu = N(\mathfrak{p}) \). Let \( \{ \alpha_1, \alpha_3, \alpha_3 \} \) (resp. \( \{ \gamma_1, \gamma_2 \} \)) denote the Langlands parameters of \( \pi_\nu \) (resp. \( \tau_\nu \)), so that

\[
L(s, \pi_\nu) = \prod_{i=1}^{3} \frac{1}{1 - \alpha_i N(\mathfrak{p})^{-s}} \quad \text{and} \quad L(s, \tau_\nu) = \prod_{j=1}^{2} \frac{1}{1 - \gamma_j N(\mathfrak{p})^{-s}}.
\]

Then by the above we have

\[
L(s, \pi_\nu \times \tau_\nu) = \prod_{i=1}^{3} \prod_{j=1}^{2} \frac{1}{1 - \alpha_i \gamma_j N(\mathfrak{p})^{-s}}
\]

\[
= \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} s_{k_1+k_2,k_1,0}(\alpha_1, \alpha_2, \alpha_3) s_{k_1+k_2,k_1,0}(\gamma_1, \gamma_2, 0) N(\mathfrak{p})^{-(2k_1+k_2)s}.
\]

On the other hand, we have

\[
L(s, \pi_\nu \times \tau_\nu) = \sum_{k=0}^{\infty} \lambda_{\pi \times \tau}(p^k) N(\mathfrak{p})^{-ks},
\]

whence

\[
\lambda_{\pi \times \tau}(p^k) = \sum_{2k_1+k_2=k} s_{k_1+k_2,k_1,0}(\alpha_1, \alpha_2, \alpha_3) s_{k_1+k_2,k_1,0}(\gamma_1, \gamma_2, 0).
\]

Further, by (3.28) and the identity \( L(s, \tau_\nu) = \sum_{k=0}^{\infty} \lambda_\tau(p^k) N(\mathfrak{p})^{-ks} \), we have

\[
\lambda_\tau(p^{k_2}) = \sum_{j=0}^{k_2} \gamma_j \gamma_{2-j} = \frac{\gamma_{k_2+1} - \gamma_{k_2+1}}{\gamma_1 - \gamma_2}.
\]

Moreover, \( \chi_{\omega_\nu}(p^{k_1}) = (\gamma_1 \gamma_2)^{k_1} \), so that

\[
\lambda_\tau(p^{k_2}) \chi_{\omega_\nu}(p^{k_1}) = (\gamma_1 \gamma_2)^{k_1} \gamma_{k_2+1} - \gamma_{k_2+1} \gamma_1 - \gamma_2 = s_{k_1+k_2,k_1,0}(\gamma_1, \gamma_2, 0),
\]

by (3.27). Thus,

\[
c_{\pi, \tau}(p^k) = \sum_{2k_1+k_2=k} \lambda_\tau(p^{k_1}, p^{k_2}) s_{k_1+k_2,k_1,0}(\gamma_1, \gamma_2, 0).
\]

Suppose now that \( \tau \) is unramified. Then we have \( m = \sigma_F \), so we may take \( \beta_1 = \beta_2 = 0 \). In that case, we have (cf. (3.8))

\[
\sum_j \Lambda_j(s, \pi, \tau, \xi_\infty, \varphi_\infty, \beta) = \Psi_\infty(s, \xi_\infty, \varphi_\infty) L(s, \pi \times \tau).
\]

Choosing \( \xi_\infty \) and \( \varphi_\infty \) such that \( \Psi_\infty(s, \xi_\infty, \varphi_\infty) \neq 0 \), we conclude that (3.23) holds. Replacing \( \tau \) by an unramified twist \( \tau \otimes \omega \), by (3.24) and (3.25), we have

\[
\sum_{a} c_{\pi, \tau}(a) \chi_{\omega}(a) N(a)^{-s} = \sum_{a} \lambda_{\pi \times \tau}(a) \chi_{\omega}(a) N(a)^{-s}
\]

for any unramified \( \omega \). By [4, Lemma 4.2] it follows that \( c_{\pi, \tau}(a) = \lambda_{\pi \times \tau}(a) \) for all \( a \). In particular, taking \( a = p^k \), by (3.29) and (3.31) we find that

\[
(3.32) \sum_{2k_1+k_2=k} s_{k_1+k_2,k_1,0}(\alpha_1, \alpha_2, \alpha_3) s_{k_1+k_2,k_1,0}(\gamma_1, \gamma_2, 0) = \sum_{2k_1+k_2=k} \lambda_\tau(p^{k_1}, p^{k_2}) s_{k_1+k_2,k_1,0}(\gamma_1, \gamma_2, 0),
\]
whenever \( \tau \) is unramified. Applying this with \( \tau = \| i t \| \) for arbitrary \( t_1, t_2 \in \mathbb{R} \), the Satake parameters \((\gamma_1, \gamma_2) = (N(p)^{-it_1}, N(p)^{-it_2})\) are Zariski-dense in \( \mathbb{C}^2 \), so (3.32) holds for arbitrary \( \gamma_1, \gamma_2 \in \mathbb{C} \). Further, from (3.30) it is easy to see that the polynomials \( s_{k_1+k_2, k_1, 0}(x_1, x_2, 0) \), for \( k_1, k_2 \) ranging over all non-negative integers, are linearly independent. Therefore, from (3.32) we conclude that \( \lambda_\pi(p^{k_1}, p^{k_2}) = s_{k_1+k_2, k_1, 0}(\alpha_1, \alpha_2, \alpha_3) \).

Finally, applying this together with (3.29) and (3.31) for an arbitrary \( \tau \) (not necessarily unramified), we conclude that \( c_{\pi, \tau}(p^{k^2}) = \lambda_\pi(x, \beta v^{-1}) \).

This concludes the proof of Theorem 3.5 with \( c = N(m) \tau_\pi(\chi_r, \beta v^{-1}) \).

4. \( GL_3 \times GL_4 \)

In this section, \( \tau = \omega \) is an idèle class character of \( F \) of conductor \( q \), and \( \pi \) is an irreducible admissible generic representation of \( GL_3(\mathbb{A}_F) \). For each finite place \( v \) of \( F \), the essential vector \( \xi_v^0 \) and the corresponding \( W_{\xi^0} \) is as discussed in the previous section. For \( \beta \in F^\times \), we embed \( \beta \) in \( \mathbb{A}_F \) via \( \beta \mapsto (\beta, 0) \in F_\infty \times \mathbb{A}_{F,F} \) as before, and for \( \xi_\infty \in V_{\pi_\infty}, j = 1, \ldots, h \), we define the function \( \Phi_{\xi_\infty,j} \) on \( \mathbb{A}^\times_F \) via

\[
\Phi_{\xi_\infty,j}(y) = \| \xi^0 \| (U_\xi) \begin{pmatrix} \gamma(0, t_j) & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ for } \xi = \xi_\infty \otimes \xi^0_j, y \in \mathbb{A}^\times_F.
\]

This function is not invariant under \( F^\times \) due to the presence of \( (\beta, 0) \), but as before it will follow from its Fourier expansion (see below) that it is invariant under a suitable congruence subgroup when viewed as a function on \( F^\times_\infty \). We will from here onwards take \( \beta \in a q^{-1}, \) where \( a \) as before is the product \( \prod_j a_j \).

4.1. The Fourier expansion of \( \Phi_{\xi_\infty,j} \). By definition, for \( y \in F^\times_\infty, \) we have

\[
\Phi_{\xi_\infty,j}(y) = \sum_{\gamma \in F^\times} W_{\xi_\infty \otimes \xi^0_j}(\gamma(0, t_j) \begin{pmatrix} \gamma(0, t_j) & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}).
\]

One checks that

\[
W_{\xi^0_j}(\begin{pmatrix} \gamma t_j & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}) \neq 0 \Rightarrow \gamma \in a_j^{-1} \cap F^\times,
\]

so that

\[
\Phi_{\xi_\infty,j}(y) = \sum_{\gamma \in a_j^{-1} \cap F^\times} W_{\xi^0_j}(\begin{pmatrix} \gamma t_j & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}) W_{\xi_\infty}(\begin{pmatrix} \gamma y & \gamma \beta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}).
\]

Let us put \( a_{\xi^0_j}(t_j, \gamma) = W_{\xi^0_j}(\begin{pmatrix} \gamma t_j & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}) \), and note that it is invariant under \( \gamma \mapsto \eta \gamma \) for \( \eta \in \mathfrak{a}^\times_F \). Thus the above becomes

\[
\Phi_{\xi_\infty,j}(y) = \sum_{\gamma \in \mathfrak{a}^\times_F} a_{\xi^0_j}(t_j, \gamma) \sum_{\eta \in \mathfrak{a}^\times_F} \psi_\infty(\eta \gamma \beta) W_{\xi_\infty}(\begin{pmatrix} \eta \gamma y & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}).
\]

Since \( \gamma \beta \in q^{-1} \), the map \( \eta \mapsto \psi_\infty(\eta \gamma \beta) \) factors through \( \Gamma_q = \{ \epsilon \in \mathfrak{a}^\times_F : \epsilon \equiv 1(\text{mod } q) \} \), and consequently

\[
\Phi_{\xi_\infty,j}(y) = \sum_{\gamma \in \mathfrak{a}^\times_F} a_{\xi^0_j}(t_j, \gamma) \sum_{\eta \in \Gamma_q} \psi_\infty(\eta \gamma \beta) \sum_{\epsilon \in \Gamma_q} W_{\xi_\infty}(\begin{pmatrix} \epsilon \eta \gamma y & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}).
\]
In particular, it follows that \( \Phi_{\infty, j}(y) \) is \( \Gamma_q \)-invariant.

4.2. **Additive twists.** For \( \beta \in a_q^{-1}, \xi_\infty \in V_{\pi, \infty} \), and with the rest of the notation as above, we define additive twist (the analogue of (3.6)) in this situation to be

\[
\Lambda_j(s, \pi, \omega, \xi_\infty, \beta) = \frac{N(a_j)^{1-s}}{[a_q^\infty : \Gamma_q]} \int_{\Gamma_q \backslash F_q^\infty} \Phi_{\xi_\infty, j}(y) \omega_\infty(y) \|y\|_\infty^{s-1} d^\times y,
\]

where \( \omega_\infty \) is the archimedean component of \( \omega \). We insert (4.1) into the above expression and collapse the integral and the sum over \( \epsilon \) to get

\[
\Lambda_j(s, \pi, \omega, \xi_\infty, \beta) = \frac{N(a_j)^{1-s}}{[a_q^\infty : \Gamma_q]} \sum_{\gamma \in a_q^\infty \backslash a_j^{-1} \cap F^\times} \psi_\infty(\eta \gamma \beta) \cdot \int_{F_\infty^\times} W_{\xi_\infty} \left( \begin{array}{ccc} \eta \gamma \beta y & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) \omega_\infty(y) \|y\|_\infty^{s-1} d^\times y.
\]

Changing \( y \mapsto (\gamma \eta \beta)^{-1} y \), we obtain

\[
\Lambda_j(s, \pi, \omega, \xi_\infty, \beta) = \frac{N(a_j)^{1-s}}{[a_q^\infty : \Gamma_q]} \sum_{\gamma \in a_q^\infty \backslash a_j^{-1} \cap F^\times} \psi_\infty(\eta \gamma \beta) \omega_\infty(\eta \gamma \beta)^{-1} \int_{F_\infty^\times} W_{\xi_\infty} \left( \begin{array}{ccc} y & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) \omega_\infty(y) \|y\|_\infty^{s-1} d^\times y.
\]

In the notation of [2], the average over \( \eta \) is \( e_q((\gamma \beta), \omega_\infty^{-1}) \). Putting all of this together we obtain

\[
\Lambda_j(s, \pi, \omega, \xi_\infty, \beta) = \sum_{\gamma \in a_q^\infty \backslash a_j^{-1} \cap F^\times} N((\gamma) a_j)^{1-s} a_{\xi_j}^\gamma(t_j, \gamma) e_q((\gamma \beta), \omega_\infty^{-1}) \prod_{\nu|\infty} \Psi_\nu(s; W_{\xi_\nu}, \omega_\nu).
\]

For \( \gamma \in a_q^\infty \backslash a_j^{-1} \cap F^\times \), let \( a = (\gamma) a_j \), then note that

\[
\lambda_\pi(a, a) = a_{\xi_j}^\gamma(t_j, \gamma) N(a).
\]

Consequently

\[
\Lambda_j(s, \pi, \omega, \xi_\infty, \beta) = \frac{\lambda_\pi(a, a) e_q(aa_j^{-1}(\beta), \omega_\infty^{-1})}{N(a)^s} \cdot \prod_{\nu|\infty} \Psi_\nu(s; W_{\xi_\nu}, \omega_\nu).
\]

**Lemma 4.1.** Let \( \lambda_\pi(a) \) denote the Dirichlet coefficients of \( L(s, \pi) \), so that

\[
L(s, \pi \otimes \omega) = \sum_{a \in a_F} \frac{\lambda_\pi(a) \chi_\omega(a)}{N(a)^s}
\]

for every unramified idèle class character \( \omega \). Then \( \lambda_\pi(a) = \lambda_\pi(a, a) \).

**Proof.** This essentially follows from Lemma 3.11, where we have identified the double Dirichlet coefficient \( \lambda_\pi(m_1, m_2) \) of \( \pi \). To be precise, by multiplicativity it suffices to verify the desired identity at prime powers. To that end, fix a prime \( p \) and let \( v \) be the corresponding place of \( F \). Let \( \alpha_1, \alpha_2, \alpha_3 \) be the Langlands parameters of \( \pi \), and let \( k \geq 0 \) be any integer. From the proof of Lemma 3.11 we have \( \lambda_\pi(a_F, p^k) = s_{k, 0, 0}(\alpha_1, \alpha_2, \alpha_3) \). On the other hand

\[
s_{k, 0, 0}(\alpha_1, \alpha_2, \alpha_3) = \sum_{m_1 + m_2 + m_3 = k} \alpha_1^{m_1} \alpha_2^{m_2} \alpha_3^{m_3},
\]
where \( m_1, m_2 \) and \( m_3 \) are non-negative integers. The right-hand side of this expression is precisely \( \lambda_\pi(p^k) \) thus proving our claim at prime powers. \( \square \)

Let \( L_j(s, \pi, \omega, \xi_\infty, \beta) \) denote the finite part of \( \Lambda_j(s, \pi, \omega, \xi_\infty, \beta) \). The following analogue of Theorem 3.5 for \( \text{GL}(3) \times \text{GL}(1) \) is a consequence of \([1, \text{Proposition 3.1}]\).

**Corollary 4.2.** For any idèle class character \( \omega \), there are numbers \( \beta_i \in F \) and \( c_{ij} \in \mathbb{C} \) (depending on \( \omega \)) such that
\[
L(s, \pi \times \omega) = \sum_{i,j} c_{ij} L_j(s, \pi, \omega, \xi_\infty, \beta_i).
\]

To end, we note the following corollary, which is another consequence of identifying the Dirichlet coefficients of \( L(s, \pi) \) in terms of the associated essential function. Although this directly follows from (1.2) by a local calculation (see \([15, \text{Corollary 3.3}]\)), the argument here is global in nature. Moreover, it can be extended to \( \text{GL}(n) \times \text{GL}(m) \) for arbitrary \( n > m \), i.e. we can identify the Dirichlet coefficients of \( L(s, \pi \times \tau) \) for any pair \((\pi, \tau)\) in terms of the associated essential functions. We will investigate this for \( n > 3 \) in a forthcoming paper.

**Corollary 4.3.** Let \( \xi \in V_\pi \) be a decomposable vector with \( \xi_v = \xi_v^0 \) for all finite \( v \). Then
\[
\int_{F^\times \backslash \mathbb{A}_F^\times} \mathbb{P}_1^3(U_\xi)(h) \prod_{v|\infty} \Psi_v(s; W_{\xi_v}),
\]
where \( \Psi_v(s; W_{\xi_v}) = \int_{F_v^*} W_{\xi_v}(a) ||a||_v^{-1} d^x a \).

**Proof.** We have the Fourier expansion \( \mathbb{P}_1^3(U_\xi)(g) = \sum_{\gamma \in F^\times} W_\xi(\gamma \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} g) \). Inserting this in the left-hand side above, we get
\[
\int_{F^\times \backslash \mathbb{A}_F^\times} \sum_{\gamma} W_\xi(\gamma \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} h) ||h||_v^{-1} d^x h.
\]
Using \( \mathbb{A}_F^\times = \prod_j F^\times F_\infty^\times j_1(\prod_{v<\infty} \mathfrak{m}_v^\times), \) we may write this as
\[
\sum_j \int_{\mathfrak{m}_v^\times \backslash F_\infty^\times} \sum_{\gamma \in \mathfrak{m}_v^{-1} \cap F^\times} W_{\xi_v}(\gamma h) ||h||_v^{-1} ||h||_\infty^{-1} d^x h_v,
\]
which in turn is the same as
\[
\sum_j \int_{\mathfrak{m}_v^\times \backslash F_\infty^\times} \sum_{\gamma \in \mathfrak{m}_v^{-1} \cap F^\times} a_{\xi_v}(t_j, \gamma) \sum_{\eta \in \mathfrak{m}_v^\times} W_{\xi_v}(\gamma \eta h) ||h||_\infty^{-1} d^x h_v.
\]
Thus the left-hand side becomes (by a similar calculation to the above)
\[
\sum_j \sum_{\gamma \in \mathfrak{m}_v^{-1} \cap F^\times} a_{\xi_v}(t_j, \gamma) ||t_j||_v^{-1} ||t_j||_\infty^{-1} \prod_{v|\infty} \Psi_v(s; W_{\xi_v}).
\]
Since \( a_{\xi_v}(t_j, \gamma) = N(a)^{-1} \lambda_\pi(\mathfrak{m}_F, a) \) for \((\gamma)a_j = a\), by Lemma 4.1, this is
\[
\sum_j \sum_{a \sim a_j} \frac{\lambda_\pi(a)}{N(a)^s} \prod_{v|\infty} \Psi_v(s; W_{\xi_v}),
\]
proving our assertion. \( \square \)
References

[1] Andrew R. Booker and M. Krishnamurthy, *A strengthening of the GL(2) converse theorem*, Compos. Math. **147** (2011), no. 3, 669–715.

[2] ———, *Further refinements of the GL(2) converse theorem*, Bull. Lond. Math. Soc. **45** (2013), no. 5, 987–1003.

[3] ———, *Weil’s converse theorem with poles*, Int. Math. Res. Not. IMRN (2014), no. 19, 5328–5339.

[4] ———, *A converse theorem for GL(n)*, Adv. Math. **296** (2016), 154–180. MR 3490766

[5] Daniel Bump, *Lie groups*, second ed., Graduate Texts in Mathematics, vol. 225, Springer, New York, 2013.

[6] J. W. Cogdell and I. I. Piatetski-Shapiro, *Converse theorems for GL_n*, Inst. Hautes Études Sci. Publ. Math. (1994), no. 79, 157–214. MR 1307299 (95m:22009)

[7] James W. Cogdell, *L-functions and converse theorems for GL_n*, Automorphic forms and applications, IAS/Park City Math. Ser., vol. 12, Amer. Math. Soc., Providence, RI, 2007, pp. 97–177.

[8] H. Jacquet, I. I. Piatetski-Shapiro, and J. Shalika, *Conducteur des représentations du groupe linéaire*, Math. Ann. **256** (1981), no. 2, 199–214. MR 620708 (83c:22025)

[9] H. Jacquet, I. I. Piatetskii-Shapiro, and J. A. Shalika, *Rankin-Selberg convolutions*, Amer. J. Math. **105** (1983), no. 2, 367–464. MR 701565 (85g:11044)

[10] ———, *Archimedean Rankin-Selberg integrals*, Automorphic forms and L-functions II. Local aspects, Contemp. Math., vol. 489, Amer. Math. Soc., Providence, RI, 2009, pp. 57–172. MR 2533003 (2011a:11103)

[11] ———, *A correction to conducteur des représentations du groupe linéaire [mr620708]*, Pacific J. Math. **260** (2012), no. 2, 515–525.

[12] ———, Ilja Iosifovitch Piatetski-Shapiro, and Joseph Shalika, *Automorphic forms on GL(3). II*, Ann. of Math. (2) **109** (1979), no. 2, 213–258. MR 528964 (80i:10034b)

[13] Fabian Januszewski, *Modular symbols for reductive groups and p-adic Rankin-Selberg convolutions over number fields*, J. Reine Angew. Math. **653** (2011), 1–45.

[14] D. Kazhdan, B. Mazur, and C.-G. Schmidt, *Relative modular symbols and Rankin-Selberg convolutions*, J. Reine Angew. Math. **519** (2000), 97–141.

[15] Nadir Matringe, *Essential Whittaker functions for GL(n)*, Doc. Math. **18** (2013), 1191–1214.

[16] Toshitsune Miyake, *Modular forms*, English ed., Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2006, Translated from the 1976 Japanese original by Yoshitaka Maeda.

School of Mathematics, University of Bristol, University Walk, Bristol, BS8 1TW, United Kingdom

E-mail address: andrew.booker@bristol.ac.uk

Department of Mathematics, University of Iowa, 14 MacLean Hall, Iowa City, IA 52242-1419, USA

E-mail address: muthu-krishnamurthy@uiowa.edu