Assigning Weights to Minimize the Covering Radius in the Plane

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Abstract

Given a set \( P \) of \( n \) points in the plane and a multiset \( W \) of \( k \) weights with \( k \leq n \), we assign each weight in \( W \) to a distinct point in \( P \) to minimize the maximum weighted distance from the weighted center of \( P \) to any point in \( P \). In this paper, we give two algorithms which take \( O(k^2n^2 \log^3 n) \) time and \( O(k^5n \log^3 k + kn \log^3 n) \) time, respectively. For a constant \( k \), the second algorithm takes only \( O(n \log^3 n) \) time, which is near linear.

1 Introduction

Consider a set of robots lying at different locations in the plane. Each robot is equipped with a locomotion module so that it moves to a nearby facility to recharge its battery and returns to its original location. We want to place a recharging facility for the robots such that the maximum travel time of them to reach the recharging facility is minimized. The Euclidean center of the robot locations may not be a good location for the recharging facility if the robots have huge differences in their speeds. For instance, consider three robots, each lying on a different corner of an equilateral triangle. If one of them has a much smaller speed compared to the speeds of the other two robots, the best location is very close to the corner where the low-speed robot lies. In this case, the recharging facility must be located at a weighted center of the robots by considering their speeds as weights of their placements.

In the weighted center problem, each input point is associated with a positive weight. The weighted distance between an input point and a point of the plane is defined to be their distance divided by the associated weight of the input point. Then the point of the plane that minimizes the maximum weighted distance to input points is the center of the weighted input points, which we call the weighted center. Dyer [8] gave a linear-time algorithm to compute the weighted center of a set of weighted points in the plane. Clearly, the weighted center coincides with the (unweighted) center if the associate weight is the same for every input point.

Imagine now that we are allowed to reassign the locomotion modules of the robots. Or, if the mobile robots are identical, except their speeds, we are allowed to relocate the robots. A relocation of robots (or a reassignment of locomotion modules) may change the weighted center and the maximum travel time for mobile robots to reach the weighted center. In other words, a clever assignment of robots (or their locomotion modules) to given locations may decrease the objective function value.

In this paper, we formally define this relocation problem and present algorithms for it. The weight assignment problem is defined as follows: given an input consisting of a set \( P \) of \( n \) points

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in the plane and a multiset \( W = \{ w_1, w_2, \ldots, w_k \} \) of \( k \) weights of positive real values with \( k \leq n \), find an assignment of the weights in \( W \) to input points such that the maximum weighted distance from the weighted center to input points is minimized. We assume that every input point of \( P \) is assigned the default weight 1. We assign the \( k \) weights to \( k \) points of \( P \) such that every weight of \( W \) is assigned to one of the \( k \) points, each of the \( k \) points gets one weight of \( W \), and the remaining \( n - k \) points of \( P \) have the default weight 1. We call this an assignment of weights of \( W \) to \( P \).

We regard an assignment of weights as a function. To be specific, for an assignment \( f \) of weights, let \( f(p) \) denote the weight assigned to a point \( p \in P \). We use \( c(f) \) to denote the weighted center of \( P \) by the assignment \( f \), and call the maximum weighted distance from \( c(f) \) to input points the covering radius of the assignment \( f \) and denote it by \( r(f) \).

Obviously, there are \( \binom{n}{k} \) different combinations of selecting \( k \) points from \( P \) and \( k! \) different ways of assigning the \( k \) weights to a combination, and therefore there are \( \Theta(n^k k!) \) different assignments of weights. Our goal is to find an assignment \( f \) of the weights of \( W \) to the points of \( P \) that minimizes the covering radius \( r(f) \) over all possible assignments of weights.

**Related Work.** As mentioned earlier, Dyer [8] studied the weighted center problem for a set of weighted points in the plane. He reformulated the problem as an optimization problem with linear inequalities and one quadratic inequality. Then he gave a linear-time algorithm to compute the weighted center using the technique by Megiddo [11]. Later, Megiddo [12] gave another linear-time algorithm for the same problem using a different technique.

In contrast, to our best knowledge, no algorithm is known for the weight assignment problem while there are works on several related problems. In the inverse 1-center problem on graphs, we are given a graph and a target vertex, and we are to increase or decrease the lengths of edges of the graph so that the target vertex becomes a center of the modified graph. A center of a graph is a vertex of the graph which minimizes the maximum distance from other vertices in the graph. Notice that a modification in the lengths of the edges is equivalent to assigning additive weights to the edges. The goal is to minimize the modification of the lengths. Cai et al. [6] showed that this problem is NP-hard on a general directed graph. Recently, Alizadeh and Burkard [3] gave an \( O(n^2 r) \)-time algorithm for this problem on a tree, where \( r \) is the compressed depth of the tree. A variant of this problem is the reverse 1-center problem, in which we are to decrease the lengths of edges of the graph under a given budget so that the maximum distance from a predetermined target vertex to any vertex of the graph is minimized. This problem is also known to be NP-hard even on a bipartite graph [3], and there is an \( O(n^2 \log n) \)-time algorithm on a tree by Zhang et al. [14].

Our weight assignment problem is closely related to the weight balancing problem which was studied by Barba et al. [14]. The input consists of a simple polygon, a target point inside the polygon, and a set of weights. The goal is to put the weights at points on the boundary of the polygon so that the barycenter (center of mass) of the weights coincides with the target point. They showed the existence of such a placement of weights under the condition that no input weight exceeds the sum of the other input weights. They also gave an algorithm to find such a placement in \( O(k + n \log n) \) time, where \( k \) is the number of the weights and \( n \) is the number of the vertices of \( P \). Our problem can be considered as a discrete version of this problem, but with a different criteria (minimizing the covering radius), in the sense that we place the weights on predetermined positions.

**Our Result.** In this paper, we present two algorithms that compute an assignment \( f \) of weights minimizing \( r(f) \). The first algorithm returns an optimal assignment of weights in \( O(k^2 n^2 \log^2 n) \) time using \( O(kn) \) space. We first observe that the number of all possible weighted
centers is $\Theta(k^3n^3)$ because a weighted center is determined by at most three weighted points. To achieve the $O(k^2n^2\log^3 n)$-time algorithm, we use an algorithm deciding, for a given real value $r > 0$, if there is an assignment $f$ of weights with $r(f) \leq r$. This algorithm makes use of the fact that there is a point $c$ satisfying $d(p,c)/f(p) \leq r$ for all points $p \in P$ if and only if $r(f) \leq r$, where $d(p,c)$ denotes the Euclidean distance between $p$ and $c$. Moreover, there is a point $p \in P$ and a weight $w \in W \cup \{1\}$ such that the circle $C$ centered at $p$ with radius $wr$ contains such a point $c$. Thus, the decision algorithm checks if there is such a point $c$ by considering each of the $O(kn)$ intervals on $C$ induced by the concentric circles centered at each point in $P \setminus \{p\}$ with radius $w'r$ for each distinct weight $w'$.

Then the overall algorithm finds an optimal assignment of weights and its covering radius by applying parametric search using the decision algorithm. In doing so, the algorithm computes the combinatorial structure of the arrangement of the circles $C_p$ for an optimal weight assignment $f^*$ without knowing $f^*$ and its covering radius $r^*$, where $C_p$ is a circle centered at $p$ with radius $f^*(p)r^*$ for every $p \in P$. This is done by computing, for each pair of a point $p \in P$ and a weight $w \in W \cup \{1\}$, the sorted list of intersection points on the circle centered at a point $p$ with radius $wr^*$ by the concentric circles centered at each point in $P \setminus \{p\}$ with radius $w'r^*$ for $w' \in W \cup \{1\}$ without knowing the optimal covering radius $r^*$. This takes $O(k^2n^2\log^3 n)$ time. Once the combinatorial structure is constructed, the algorithm computes an optimal assignment of weights and its covering radius in $O(k^2n^2\log^2 n)$ time using $O(kn)$ space. This is done by sorting $O(k^2n^2)$ candidate radii defined by three circles of the arrangement and applying binary search on the sorted list using the decision algorithm.

The second algorithm computes an optimal assignment of weights in $O(k^5n \log^3 k + kn \log^3 n)$ time assuming that every weight in $W$ is at most 1. The second algorithm is faster than the first algorithm when $k$ is sufficiently small ($k = o(n^{1/3})$). Moreover, it takes only $O(n \log^3 n)$ time when $k$ is a constant.

A merit of our algorithms is that they are based on useful geometric intuition and are easy to implement though they use parametric search, the optimization technique developed by Megiddo [10]. The parametric search technique is an important tool for solving many geometric optimization problems efficiently, but algorithms based on it are often not easy to be implemented [2]. A main difficulty lies in computing the roots of the polynomials exactly whose signs determine the outcome of the comparisons made by the algorithm. However, as we will see later, in our algorithms, such a root is a covering radius of at most three weighted points, so we can compute it easily instead of resorting to complicated methods.

2 Preliminaries

For any two points $p$ and $q$ in the plane, we use $d(p,q)$ to denote the Euclidean distance between $p$ and $q$. For a weighted point $p$ in the plane, let $w(p)$ denote the weight of $p$. For a weighted point $p$ and an unweighted point $q$ in the plane, their weighted distance is defined as $d(p,q)/w(p)$.

For a set of weighted points, the weighted center of the weighted points is a point in the plane that minimizes the maximum weighted distance to the weighted points.

Let $P$ be a set of $n$ points in the plane, and let $W = \{w_1, w_2, \ldots, w_k\}$ be a multiset of $k$ weights of positive real values. To ease the description, we assume that $k < n$ unless stated otherwise. In case that $k = n$, there is no difference in the proposed algorithm, except that every point of $P$ is assigned a weight of $W$. One way to deal with our problem is to find the weighted centers for all possible assignments of weights and choose the one with the minimum covering radius. Although there are $\Theta(n^k k!)$ different assignments of weights, there are only $O(k^3n^3)$ different weighted centers. This is because a weighted center is determined by at most three weighted points. That is, given an assignment of weights, there always exist at most three
weighted points of $P$ whose weighted center coincides with the weighted center of the whole weighted points. Figure 1 illustrates the cases that the weighted centers determined by three or two weighted points of $P$.

Thus the number of all possible weighted centers is at most the number of all possible 6-tuples $(p_1, p_2, p_3, w_1, w_2, w_3)$ such that $p_i \in P$ and $w_i \in W \cup \{1\}$ for $i = 1, 2, 3$, which is $O(k^3n^3)$. Moreover, this bound is asymptotically tight by the following lemma.

**Lemma 1.** There are $n$ points in the plane and $k$ weights for which the number of all possible weighted centers is $\Omega(k^3n^3)$.

**Proof.** Figure 2 illustrates an example in the plane with $n$ points and $k$ weights for which the number of all possible weighted centers is $\Omega(k^3n^3)$. Consider two concentric circles such that the smaller circle has a radius at most $1/3$ of the radius of the larger circle. We put $\lfloor n/4 \rfloor$ points along the small circle $C_2$ evenly and put the remaining points along the large circle $C_1$ such that they are clustered into three groups, each consisting of at most $\lceil n/4 \rceil$ points, as illustrated in the figure. Any two points in the same group are close to each other while any two points from two different groups are far from each other. Let $\varepsilon > 0$ be a sufficiently small real value and $k$ be a submultiple of $n$ satisfying $k \leq n/4$, and let $W = \{1/2 - \varepsilon, 1/2 - 2\varepsilon, \ldots, 1/2 - k\varepsilon\}$.

Every triplet $(p_1, p_2, p_3)$ of points, one point from each group on $C_1$, and every triplet $(w_1, w_2, w_3)$ of weights of $W \cup \{1\}$ define their weighted center if we assign weight 1 to all points on $C_1$ other than the three points. To see this, we first observe that the weighted center $c$ of the three points $p_i$ with weights $w_i$ for $i = 1, 2, 3$ lies inside $C_2$ by setting $\varepsilon$ to be sufficiently small.

We first show that $d(p_1, c) > d(q, c)$ for every point $q$ lying on $C_1$ other than $p_1, p_2$ and $p_3$. We have $d(p_1, c) > d(q, c)/2$ since $c$ lies inside $C_2$ and the radius of $C_2$ is at most $1/3$ of the radius of $C_1$. Therefore, the claim holds because $w_1$ is at most $1/2$.

Now we show that $d(p_1, c)/w_1 > d(q, c)/w$ for every point $q$ lying on $C_2$ and every weight $w$ in $W \cup \{1\}$. We have $d(p_1, c)/2 > d(q, c)$. By setting $\varepsilon > 0$ to be sufficiently small satisfying $1/(1 - 2k\varepsilon) < 2$, $\max_{w,w' \in W} w/w'$ is strictly less than 2. Thus, $d(p_1, c)/w_1 > d(q, c)/w$ for any point $q$ lying on $C_2$ and any weight $w \in W \cup \{1\}$, which proves the claim.

Moreover, we can choose $\varepsilon$ such that no two weighted centers lie on the same point. Therefore, the number of different weighted centers is $\Omega(k^3n^3)$. \hfill \Box

Figure 1: (a) The weighted center of three points $p_i$ with weight $w_i$ for $i = 1, 2, 3$ is the weighted center of whole weighted points. (b) The weighted center of two points $p_4$ with weight $w_4$ and $p_5$ with weight $w_5$ is the weighted center of whole weighted points.
if

As noted earlier, the weighted center and the covering radius of an assignment of weights are determined by at most three weighted points. But not every three weighted points define such a center and its covering radius. Here we show how to test this for three weighted points efficiently.

Let $W_1$ be the multiset consisting of the weights of $W$ and $n-k$ numbers of weight 1. Consider three points from $P$ and an assignment of three weights from $W_1$ to the points. We first test whether the three weighted points define a point in the plane at the same weighted distance from them. This can be done in constant time by checking if the three bisecting curves, each curve $\{x \in \mathbb{R}^2 \mid d(p,x)/w(p) = d(q,x)/w(q)\}$ defined by a pair $(p,q)$ of the three weighted points, have a common point. If such a point does not exist, there is no weighted center of the three weighted points. Otherwise, there is at most one such point as the weighted center of points in the plane is unique. We can decide in constant time if the common point of three bisecting curves is the weighted center of the three weighted points by checking if the point is contained in the convex hull of the three weighted points.

Let $c$ denote the weighted center and $r$ denote the covering radius of the three weighted points. Let $\langle p_1, \ldots, p_{n-3} \rangle$ be the sequence of the points in $P$, except the three (weighted) points, in increasing order with respect to the Euclidean distance from $c$. Let $\langle w_1, \ldots, w_{n-3} \rangle$ be the sequence of the weights in $W_1$, except the three weights, in increasing order. The following lemma directly gives us an $O(n)$-time algorithm to decide whether the three weighted points determine the weighted center and the covering radius of an assignment of weights assuming that we have the two sequences.

**Lemma 2.** There exists an assignment $f$ of weights such that $c(f) = c$ and $r(f) = r$ if and only if $d(p_i, c)/w_i \leq r$ for $1 \leq i \leq n-3$.

**Proof.** One direction is straightforward. If $d(p_i, c)/w_i \leq r$ for every index $1 \leq i \leq n-3$, we assign weight $w_i$ to point $p_i$ so that $f(p_i) = w_i$ for each $i$. Then this assignment $f$, together with the weight assignment for the three points, satisfies that $c(f) = c$ and $r(f) = r$.

Consider the other direction. Suppose that there exists an assignment $f$ such that $c(f)$ coincides with $c$ and $r(f) = r$, but $d(p_i, c)/w_i > r$ for some index $1 \leq i \leq n-3$. Let $j$ be the smallest index such that $d(p_j, c)/w_j > r$. Then $f(p_j) = w_\ell$ for some index $\ell > j$ because $d(p_j, c)/w_\ell > r$ for every $\ell' \leq \ell$. Moreover, there is a point $p_{\ell'}$ for $j < j' \leq n-3$ such that $f(p_{\ell'}) \leq w_j$. This is because the number of weights strictly larger than $w_j$ is less than $n-3-j$. Since $d(p_i, c)/f(p_i) \leq r$ for all $1 \leq i \leq n-3$, we have $d(p_{\ell'}, c)/f(p_{\ell'}) \leq r$. But this implies
weights that minimizes $r_f$. Thus it can be used for problems with optimization criteria other than the minimization of the weighted centers of all possible weight assignments is $O(W)$. While it suffices to sort the weights in $W$ just once. Thus, the total running time for finding the weighted centers of all possible weight assignments is $O(k^3n^4 \log n)$.

Note that this algorithm returns the weighted centers of all possible weight assignments. Thus it can be used for problems with optimization criteria other than the minimization of the covering radius. For example, we can find the assignment $f$ of weights such that the center $c(f)$ is the closest to a given point in the same time.

3 Algorithm for Computing an Optimal Assignment of Weights

In this section, we present an $O(k^2n^2 \log^3 n)$-time algorithm for finding an assignment $f$ of weights that minimizes $r_f$. This algorithm does not consider all the weighted centers of all possible weight assignments. Instead, it uses parametric search due to Megiddo [10]. To apply this technique, we need to devise a decision algorithm which is used as a subprocedure of the main algorithm.

3.1 Decision Algorithm

Let $r$ be an input of the decision algorithm. The decision algorithm decides whether there is an assignment $f$ of weights with $r_f \leq r$. In other words, it decides whether there are a point $c$ and an assignment $f$ of weights such that $d(p, c)/f(p) \leq r$ for all points $p \in P$. If this is the case, we call such a point $c$ an $r$-center with respect to $f$.

**Lemma 3.** For an assignment $f$ of weights with $r_f \leq r$, there is an $r$-center $c$ with respect to $f$ satisfying $d(q, c)/f(q) = r$ for some point $q \in P$.

**Proof.** Assume that an assignment $f$ of weights satisfies $r_f = \max_{p \in P} d(p, c(f))/f(p) \leq r$. Let $c$ be an $r$-center with respect to $f$. Imagine that we move $c$ along the horizontal line through $c$. Since the distance function $d$ is continuous, there always exists a point $c'$ on the line such that $\max_{p \in P} d(p, c')/f(p) = r$. Clearly, $c'$ is an $r$-center with respect to $f$ satisfying $d(q, c')/f(q) = r$ for some point $q \in P$. \[\square\]

By the above lemma, there is a point $p \in P$ and a weight $w \in W \cup \{1\}$ such that the circle centered at $p$ with radius $wr$ contains an $r$-center $c$ with respect to an assignment $f$ of weights if $r_f \leq r$. To find such an $r$-center, we do as follows. For each pair $(p, w)$ of a point $p \in P$ and a weight $w \in W \cup \{1\}$, we consider the circle $C$ centered at $p$ with radius $wr$. To check whether there exists an $r$-center on $C$, we subdivide $C$ into $O(kn)$ pieces such that for each piece $\mu$ on $C$, every point on $\mu$ is an $r$-center if and only if a point on $\mu$ is an $r$-center. Then we check for each piece if it contains an $r$-center.
3.1.1 Computing the Intervals on the Circles

Remind that $W_1$ denotes the multiset consisting of the weights of $W$ and $n - k$ numbers of weight 1. Let $W_1^w$ denote $W_1 \setminus \{w\}$. For each point $q \in P \setminus \{p\}$, we compute concentric circles centered at $q$ with radius $w'r$ for each distinct weight $w'$ of $W_1^w$. There are at most $k$ concentric circles centered at each point $q \in P \setminus \{p\}$. We compute the intersections of these circles with $C$ and sort them along $C$ in $O(kn \log(n)) = O(kn \log n)$ time. See Figure 3 for an illustration. In a degenerate case, there can be more than one circle passing through the same intersection point. In this case, we treat the intersection points as distinct points lying in the same position and handle them separately in any order. In the following, we assume that exactly one circle passes through one intersection point.

Now, we have $O(kn)$ intersection points sorted along $C$. These intersection points subdivide $C$ into $O(kn)$ pieces, which we call intervals on $C$. We say an assignment $f$ of weights is feasible for an interval on $C$ if for a point $c$ in the interval, $d(q,c)/f(q)$ is at most $r$ for all points $q \in P \setminus \{p\}$. We compute the intersections of these circles with $C$ and sort them along $C$ in $O(kn \log(n)) = O(kn \log n)$ time. See Figure 3 for an illustration. In a degenerate case, there can be more than one circle passing through the same intersection point. In this case, we treat the intersection points as distinct points lying in the same position and handle them separately in any order. In the following, we assume that exactly one circle passes through one intersection point.

Now, we have $O(kn)$ intersection points sorted along $C$. These intersection points subdivide $C$ into $O(kn)$ pieces, which we call intervals on $C$. We say an assignment $f$ of weights is feasible for an interval on $C$ if for a point $c$ in the interval, $d(q,c)/f(q)$ is at most $r$ for all points $q \in P \setminus \{p\}$. Note that $f$ is feasible for any point in the interval if $f$ is feasible for a point in the interval. Therefore, the set of all feasible assignments of weights is the same for any point $c$ lying in the same interval. We can test in $O(n \log n)$ time whether there exists a feasible assignment for an interval on $C$ by Lemma 2. Specifically, we choose an arbitrary point $c$ in the interval, sort all points in $P \setminus \{p\}$ with respect to their distances from $c$, and assign each weight in $W_1^w$ to a point in the sorted list of the points such that a smaller weight is assigned to a point closer to $c$. Then we check if $d(q,c)/f(q) \leq r$ for every point $q$ of $P \setminus \{p\}$ under the assignment $f$ of weights we just found.

In this way, we can test the feasibility for every interval on $C$ in $O(kn^2 \log n)$ time in total. In the following, we show how to do this in $O(kn \log n)$ time in total for all intervals on $C$.

3.1.2 Test of the Feasibility for Every Interval on $C$

Consider the intervals one by one in counterclockwise order along $C$. Note that for any two consecutive intervals, there is only one disk centered at a point in $P \setminus \{p\}$ with radius $rw'$ for some $w' \in W_1^w$ that contains exactly one of the two intervals.

We first show how to check the existence of a feasible assignment of weights for an interval $\mu$ on $C$ in $O(n \log n)$ time. Then we show how we do this for all intervals on $C$ efficiently. We
sort the weights of \( W \) and \( n - k \) numbers of weight 1 in the increasing order. The sorted list is denoted by \( \langle w_1, \ldots, w_n \rangle \). For each point \( q \) in \( P \), let \( \pi(q) \) be the smallest index such that 
\[
d(q, c)/w_{\pi(q)} \leq r,
\]
where \( c \) is a point in \( \mu \). Thus, \( \pi(p) \) is the smallest index of weight \( w \) in the sorted list. The indices for all points of \( P \) can be computed in \( O(n \log n) \) time in total. Then we sort the points in \( P \) in the increasing order with respect to \( \pi(p) \) in \( O(n \log n) \) time and denote the sorted list by \( \langle q_1, \ldots, q_n \rangle \). Then there exists a feasible assignment of weights for \( \mu \) if and only if 
\[
\pi(q_\ell) \leq \ell \text{ for all indices } 1 \leq \ell \leq n.
\]
This is because for every point \( q \in P \), we have 
\[
d(q, c)/w_{\pi(q)} \leq r
\]
for all indices \( j \geq \pi(q) \). Since we can check if \( \pi(q_\ell) \leq \ell \) for all indices \( 1 \leq \ell \leq n \) in \( O(n) \) time, we can decide whether there is a feasible assignment of weights for \( \mu \) in \( O(n \log n) \) time.

To check the existence of a feasible assignment of weights for the interval \( \mu' \) next to \( \mu \) in counterclockwise order along \( C \), we do not need to compute all such indices and compare them again. Recall that there is only one disk \( D \) centered at a point \( p' \in P \setminus \{p\} \) with radius \( rw' \) for some \( w' \in W^w_1 \) that contains exactly one of \( \mu \) and \( \mu' \). We observe that \( \pi(q) \) for every \( q \in P \setminus \{p'\} \) remains the same for \( \mu \) and \( \mu' \), because for any point \( c' \in \mu' \) we have 
\[
d(q, c')/w_{\pi(q)} \leq r
\]
and no smaller weight \( \bar{w} \in W^w_1 \) achieves \( d(q, c')/\bar{w} \leq r \). Now consider \( \pi(p') \) for \( \mu \). Let 
\[
w_{\mu'} := w_{\pi(p')} \text{ for } \mu.
\]
If \( D \) contains \( \mu \) but does not contain \( \mu' \), \( D \) has radius \( rw_{\mu} = rw' \) and \( d(p', c') > rw_{\mu} \) for any point \( c' \in \mu' \). Thus, for \( \mu' \), \( \pi(p') \) increases to the smallest index such that 
\[
d(p', c)/w_{\pi(p')} \leq r
\]
more specific, \( w_{\pi(p')} \) \( \mu' \) is the smallest weight larger than \( w_{\mu} \) among weights in \( W^w_1 \). If \( D \) contains \( \mu' \) but does not contain \( \mu \), \( \pi(p') \) decreases to the smallest index such that 
\[
d(p', c)/w_{\pi(p')} \leq r
\]
for \( \mu' \). Similarly, \( w_{\pi(p')} \) is the largest weight smaller than \( w_{\mu} \) among the weights in \( W^w_1 \).

To use this property, we maintain 2\( t \) pointers \( U_1, \ldots, U_t \) and \( L_1, \ldots, L_t \), where \( t \) with \( t \leq k \) is the number of distinct weights in \( W^w_1 \). For an index \( \ell \) with \( 1 \leq \ell \leq t \), the pointer \( U_\ell \) points to \( q_\ell \) with the largest index \( \ell \) such that \( \pi(q_\ell) \) is equal to the \( \ell \)th largest weight in \( W^w_1 \). Similarly, for an index \( \ell \) with \( 1 \leq \ell \leq t \), the pointer \( L_\ell \) points to \( q_\ell \) with the smallest index \( \ell \) such that \( \pi(q_\ell) \) is equal to the \( \ell \)th largest weight in \( W^w_1 \).

Note that we already know whether \( \pi(p') \) increases or decreases when we move from \( \mu \) to \( \mu' \). Here, we have to update not only \( \pi(p') \) but also the pointers. Moreover, we have to reorder \( \langle q_1, \ldots, q_n \rangle \) in the increasing order with respect to \( \pi(p') \).

We show how to do this for the case that \( \pi(p') \) increases. The other case can be handled analogously. We first find the point \( p'_u \) that the point \( U_{\pi(p')} \) points to. Then we swap the positions for \( p' \) and \( p'_u \) on the sequence \( \langle q_1, \ldots, q_n \rangle \) and let \( U_{\pi(p')} \) point to \( p'_u \). This does not violate the property that \( \pi(q_\ell) \leq \pi(q_{\ell+1}) \) for all indices \( 1 \leq \ell < n \) since \( \pi(p') = \pi(p'_u) \). Then we increase \( \pi(p') \) and update the pointers accordingly. To check the existence of a feasible assignment of weights for \( \mu' \), it suffices to check \( \pi(q_\ell) \leq \ell \) for \( q_\ell = p', p'_u \) only because there is no change to \( \pi(q) \) for the other points \( q \). This can be done in constant time, and thus the update for the next interval can be done in constant time. Since there are \( O(kn) \) intervals on \( C \), the update and existence test can be done in \( O(kn) \) time in total for the intervals. Therefore, the running time of the procedure is dominated by the time for sorting the intersection points along \( C \) and computing the intervals, which is \( O(kn \log n) \).

**Lemma 4.** We can decide whether or not there is a feasible assignment of weights for every interval on \( C \) in \( O(kn \log n) \) time in total.

Since there are \( O(kn) \) point-weight pairs, we have the following lemma.

**Lemma 5.** Given a real value \( r > 0 \), we can decide in \( O(k^2n^2 \log n) \) time using \( O(kn) \) space whether or not there exists an assignment \( f \) of weights with \( r(f) \leq r \).

### 3.2 Overall Algorithm

In this section, we present an algorithm for computing an optimal assignment of weights in \( O(k^2n^2 \log^3 n) \) time using \( O(kn) \) space. To obtain an optimal solution, we apply parametric
Let \( r^* \) be the minimum of \( r(f) \) over all possible assignments \( f \) of weights. We use \( A(r) \) to denote the arrangement of the circles \( C_{p,w}(r) \) for all point-weight pairs \((p, w)\), where \( C_{p,w}(r) \) is the circle centered at \( p \) with radius \( rw \). Let \( C(r) \) be the set of the circles \( C_{p,w}(r) \) for all point-weight pairs \((p, w)\). Here, \( r > 0 \) is a variable. Notice that the combinatorial complexity of \( A(r) \) is \( \Theta(k^2n^2) \) in the worst case. Thus we cannot maintain the whole structure of \( A(r) \) explicitly using \( O(kn) \) space.

### 3.2.1 Combinatorial Structure of the Arrangement

Imagine that \( r \) increases from 0. Then the combinatorial structure of \( A(r) \) changes at certain \( r \) values. To be specific, we observe that three circles of \( C(r) \) meet at a point or two circles of \( C(r) \) become tangent to each other when the combinatorial structure of \( A(r) \) changes. We call a value of \( r \) where the combinatorial structure of \( A(r) \) changes an \( r \)-value. For any three point-weight pairs, there are at most two \( r \)-values for which the circles \( C_{p,w}(r) \) of the three pairs \((p, w)\) have a common intersection because the trajectory of the intersections between two increasing circles forms a hyperbolic curve and two hyperbolic curves cross at most twice. Also, for any two point-weight pairs, there are at most two \( r \)-values for which the circles \( C_{p,w}(r) \) of the pairs \((p, w)\) are tangent to each other. Thus, the combinatorial structure of \( A(r) \) changes \( O(k^3n^3) \) times, and there are \( O(k^3n^3) \) \( r \)-values.

Given a real value \( r > 0 \), we first introduce a simple way to compute the arrangement \( A(r) \). For a point \( p \in P \) and a weight \( w \in W \cup \{1\} \), we compute the intersections of \( C_{p,w}(r) \) and \( C_{p',w'}(r) \) for all points \( p' \in P \setminus \{p\} \) and all distinct weights \( w' \) of \( W \setminus \{w\} \). Each circle \( C_{p',w'}(r) \) intersects \( C_{p,w}(r) \) at most twice for any fixed \( r > 0 \). Let \( I_{p,w}(r) \) be the set of all such \( O(kn) \) intersection points. We sort the points in \( I_{p,w}(r) \) along \( C_{p,w}(r) \) in counterclockwise order. Once this is done for all points \( p \in P \) and all distinct weights \( w \) of \( W \cup \{1\} \), we can construct \( A(r) \).

As \( r \) increases, the combinatorial structure of \( A(r) \) changes only when the sorted list of the points in \( I_{p,w}(r) \) along \( C_{p,w}(r) \) for a point-weight pair \((p, w)\) changes. Clearly, the sorted list changes if two points in \( I_{p,w}(r) \) cross each other, a point appears on \( I_{p,w}(r) \), or a point disappears from \( I_{p,w}(r) \). Note that an intersection point \( x \) appears on \( I_{p,w}(r) \) or disappears from \( I_{p,w}(r) \) if \( C_{p',w'}(r) \) becomes tangent to \( C_{p,w}(r) \) for a point-weight pair \((p', w')\).

These \( r \)-values partition the real value space \( \mathbb{R} \) into intervals such that for any value \( r \) in the same interval the combinatorial structure of \( A(r) \) remains the same. We search for the interval where \( r^* \) lies using the decision algorithm in Section 3.1. There are \( \Theta(k^3n^3) \) such \( r \)-values, but we do not consider all of them. Instead, as we will see later, we do the search in \( O(\log n) \) iterations. In each iteration, we consider \( O(k^2n^2) \) \( r \)-values and reduce the search space (intervals). In each iteration, we apply the decision algorithm \( O(\log n) \) times, which leads to the running time of \( O(k^2n^2 \log^2 n) \).

### 3.2.2 Computing the Combinatorial Structure of \( A(r^*) \)

In this subsection, we compute the combinatorial structure of \( A(r^*) \) without computing \( r^* \) explicitly. We first compute the intersection points in \( I_{p,w}(r^*) \) for each point-weight pair \((p, w)\), and then sort them along \( C_{p,w}(r^*) \). For both procedures, we need the following technical lemma.

**Lemma 6.** Let \( \mathcal{L}_i \) be a set of \( M \) real numbers for an index \( i \) with \( 1 \leq i \leq N \). We can find the interval containing \( r^* \) among the intervals on \( \mathbb{R} \) induced by the numbers of \( \bigcup_{i=1}^{N} \mathcal{L}_i \) in \( O((k^2n^2 \log n + NM) \log(N + M)) \) time using \( O(kn + M) \) space assuming that we can obtain the numbers in \( \mathcal{L}_i \) in \( O(M) \) time.

**Proof.** If we are allowed to use \( O(kn + NM) \) space, we can compute the intervals on \( \mathbb{R} \) induced by the numbers in \( \bigcup_{i=1}^{N} \mathcal{L}_i \) explicitly, and then apply binary search on the intervals using the
decision algorithm in Section 3.1. This takes \( O((k^2n^2 \log n + NM) \log(N + M)) \) time.

We can improve the space complexity to \( O(kn + M) \) as follows. Basically, we apply binary search on the intervals on \( \mathbb{R} \) induced by the numbers in each set \( L_i \). Initially, \( \mathbb{R} \) itself is the search space for every set \( L_i \). For each \( i \) with \( 1 \leq i \leq N \), the search space for \( L_i \) gets smaller in over iterations. After \( O(\log(NM)) \) iterations, the search space for \( L_i \) is an interval of \( \mathbb{R} \) induced by the real numbers in \( L_i \). Then the intersection of all search spaces is the interval we want to find.

In each iteration, we compute the median \( a_i \) of the real numbers in \( L_i \) in the current search space for each set \( L_i \) in \( O(NM) \) time in total. We also compute the number \( b_i \) of the real numbers of \( L_i \) in the current search space in the same time. We assign weight \( b_i/b \) to \( a_i \) for each set \( L_i \), where \( b \) is the sum of \( b_i \) for all indices \( i \) with \( 1 \leq i \leq N \). Then we compute the weighted median of all medians \( a_i \)'s of \( L_i \)'s. In specific, we compute \( a_i \) with smallest index \( i \) such that the sum of \( a_jb_j \) for all sets \( L_j \) with \( a_i \geq a_j \) is at most \( 1/2 \). After computing \( a_i \) and \( b_i \) for every set \( L_i \), we can find the weighted median \( a \) of them in \( O(N) \) time. Let \( a' \) be the smallest median larger than \( a \) among the medians for all \( L_i \). We apply the decision algorithm to \( a \) (and \( a' \)) to determine if \( r^* \leq a \), \( a < r^* \leq a' \), or \( a' < r^* \). If \( r^* \leq a \), we halve the search space for \( L_i \) such that \( a_i \geq a \). If \( a < r^* \leq a' \), we halve the search space for \( L_i \) such that \( a_i = a \). Otherwise, we halve the search space for \( L_i \) such that \( a_i \leq a \). Each iteration takes \( O(k^2n^2 \log n + NM) \) time using \( O(kn + M) \) space, and reduces the search space for some sets.

The search space for \( L_i \) becomes an interval of \( \mathbb{R} \) induced by the real numbers in the set in \( O(\log(N + M)) \) iterations. This is simply because the sum of \( b_i \)'s reduces by a constant factor for each iteration. Therefore, we can compute the interval containing \( r^* \) among the intervals on \( \mathbb{R} \) induced by the real numbers of \( \bigcup_{i=1}^{N} L_i \) in \( O((k^2n^2 \log n + NM) \log(N + M)) \) time using \( O(kn + M) \) space.

\( \square \)

### Computing the Intersection Points in \( \mathcal{I}_{p,w}(r^*) \).

To compute the intersection points in \( \mathcal{I}_{p,w}(r^*) \), we compute every \( r \)-value for which a point appears on or disappears from \( \mathcal{I}_{p,w}(r^*) \). Recall that a circle \( C_{p',w'}(r) \) is tangent to \( C_{p,w}(r) \) at such a \( r \)-value. There are \( O(k^2n^2) \) such \( r \)-values for all point-weight pairs \((p, w)\), and we can compute them in \( O(k^2n^2) \) time. By applying binary search on such \( r \)-values together with the decision algorithm in Section 3.1, we find the interval containing \( r^* \) among the intervals on \( \mathbb{R} \) induced by such \( r \)-values. This takes \( O(k^2n^2 \log^2 n) \) time with \( O(k^2n^2) \) space.

But we can improve the space complexity to \( O(kn) \) without increasing the time complexity using Lemma 6. Here, we have a set of \( r \)-values for each point-weight pair \((p, w)\). We can compute the \( r \)-values in each set in \( O(kn) \) time. We want to compute an interval on \( \mathbb{R} \) induced by all \( r \)-values for all point-weight pairs. We apply Lemma 6 to the sets of \( r \)-values. Then we can compute the interval containing \( r^* \) among the intervals on \( \mathbb{R} \) induced by such \( r \)-values in \( O(k^2n^2 \log^2 n) \) time with \( O(kn) \) space.

Since for any point-weight pair, no point appears on or disappears from \( \mathcal{I}_{p,w}(r) \) in this interval, we can obtain the intersection points in \( \mathcal{I}_{p,w}(r^*) \) by computing the intersection points in \( \mathcal{I}_{p,w}(r) \) for any value \( r \) in the interval.

### Sorting the Intersection Points in \( \mathcal{I}_{p,w}(r^*) \) along \( C_{p,w}(r^*) \) in Clockwise Order.

Since \( C_{p,w}(r) \) is a circle for any value \( r > 0 \), we use an arbitrary point in \( \mathcal{I}_{p,w}(r^*) \), denoted by \( p_0(r) \), as the reference point of the sorted list. That is, the points in \( \mathcal{I}_{p,w}(r) \) are sorted along \( C_{p,w}(r) \), say in counterclockwise order, from \( p_0(r) \). To make the description clear, we first handle the \( r \)-values for which \( u(r) \) crosses \( p_0(r) \) for every intersection point \( u(r) \) in \( \mathcal{I}_{p,w}(r^*) \). We compute the \( r \)-values for each point-weight pair in \( O(kn) \) time, and then apply binary search on the \( r \)-values for all point-weight pairs as we did before in \( O(k^2n^2 \log^2 n) \) time with \( O(kn) \) space. We
obtain an interval on \( \mathbb{R} \) such that no intersection point \( u(r) \) crosses \( p_0(r) \) in this interval for any point-weight pair.

Suppose that we want to compute the relative positions of two points \( u_1(r^*) \) and \( u_2(r^*) \) in the sorted list of the points of \( \mathcal{I}_{p,w}(r^*) \) along \( C_{p,w}(r^*) \). As \( r \) increases, both \( u_1(r) \) and \( u_2(r) \) move along \( C_{p,w}(r) \) and cross each other at most twice. Moreover, they cross each other when \( C_{p,w}(r) \) and the two circles defining \( u_1(r) \) and \( u_2(r) \) meet at a point. This occurs at most twice. Let \( r_1 \) and \( r_2 \) be the two \( r \)-values with \( r_1 \leq r \leq r_2 \) at which \( u_1(r) \) and \( u_2(r) \) cross each other. Then we decide whether \( r^* \leq r_1 \), \( r_1 \leq r^* \leq r_2 \), or \( r_2 \leq r^* \) using the decision algorithm in Section 3.1. With this, we can decide the relative position of \( u_1(r^*) \) and \( u_2(r^*) \) with respect to \( p_0(r^*) \) in the sorted list of the points in \( \mathcal{I}_{p,w}(r^*) \) without computing \( r^* \) explicitly. Thus, we can compare two points in \( \mathcal{I}_{p,w}(r^*) \) in \( O(k^2n^2\log n) \) time. By applying this comparison \( O(kn\log n) \) times, we can sort the points in \( \mathcal{I}_{p,w}(r^*) \) along \( C_{p,w}(r^*) \) without knowing \( r^* \) in \( O(k^3n^3\log^2 n) \) time. Since there are \( O(kn) \) point-weight pairs, the total running time is \( O(k^4n^4\log^2 n) \).

Here, we apply the decision algorithm in Section 3.1 twice for each comparison, once with \( r = r_1 \) and once with \( r = r_2 \). We reduce the running time of the overall algorithm by reducing the number of executions of the decision algorithm. Suppose that we want to do \( m \) comparisons which are independent to each other. As we did in the previous procedure, we compute at most two \( r \)-values from each comparison where two points cross each other. Then we have at most \( 2m \) \( r \)-values. We sort them and apply binary search to compute the smallest interval containing \( r^* \). After applying the decision algorithm \( O(\log m) \) times, we can complete the \( m \) comparisons.

In our problem, we use Cole’s parallel algorithm to sort \( m \) elements in \( O(\log m) \) time using \( O(m) \) processors [7]. Note that comparisons performed in different processors are independent to each other. In each iteration, we compare \( O(kn) \) elements for each point-weight pair. To do this, we consider a set of \( O(kn) \) \( r \)-values for each point-weight pair, and apply Lemma 6 to the sets for all point-weight pairs. Then we can obtain an interval containing \( r^* \) among the intervals on \( \mathbb{R} \) induced by the \( r \)-values for each point-weight pair in \( O(k^2n^2\log^2 n) \). In other words, we can complete the \( O(kn) \) comparisons for every point-weight pair. After \( O(\log n) \) iterations, we can obtain an interval containing \( r^* \) among the intervals on \( \mathbb{R} \) induced by all \( r \)-values. This takes \( O(k^2n^2\log^3 n) \) time using \( O(kn) \) space. The combinatorial structure of \( A(r) \) remains the same in this interval. Thus we obtain the combinatorial structure of \( A(r^*) \) by computing the combinatorial structure of \( A(r) \).

Lemma 7. The combinatorial structure of \( A(r^*) \) can be computed in \( O(k^2n^2\log^3 n) \) time using \( O(kn) \) space.

3.2.3 Finding an Optimal Assignment of Weights

We have the combinatorial structure of \( A(r^*) \) while \( r^* \) is not known yet. Using the combinatorial structure, we show how to compute an optimal assignment of weights and its covering radius \( r^* \) in \( O(k^2n^2\log^2 n) \) time in this subsection.

We say that three circles, each defined by a point-weight pair, determine an \( r \)-value \( r' \) if they intersect at one point for \( r = r' \). We already showed that there are at most three circles \( C_1^*, C_2^*, \) and \( C_3^* \) each defined by a point-weight pair for \( r \)-value \( r^* \), that determine \( r^* \). Thus, in the combinatorial structure of \( A(r^*) \), two intersection points on one of the three circles, say \( C_1^* \), made by the other two are consecutive along \( C_1^* \).

Let \( R \) be the set of all \( r \)-values determined by three circles in the combinatorial structure of \( A(r^*) \) such that two intersection points on one of the three circles, denoted by \( C \), made by the other two are consecutive along \( C \). Since there are \( O(k^2n^2) \) edges in the arrangement \( A(r^*) \), there are the same number of such \( r \)-values.
Imagine that we sort the \( r \)-values of \( R \), apply binary search on the sorted list using the decision algorithm in Section \[ \ref{sec:binary-search} \] and find the smallest \( r \)-value of \( R \) to which the decision algorithm returns “yes”. Then the smallest \( r \)-value is \( r^* \). This takes \( O(k^2n^2\log^2 n) \) time as the \( r \)-values of \( R \) can be sorted in \( O(k^2n^2\log n) \) time and the decision algorithm is called \( O(\log n) \) times.

We do this without computing the whole arrangement \( A(r^*) \). We consider a point-weight pair \((p, w)\) first. We compute the edges and the vertices lying on \( C_{p,w}(r^*) \). This takes \( O(kn\log n) \) time because we already have the interval \( \mu \) containing \( r^* \) such that the combinatorial structure of \( A(r) \) remains the same for any \( r \in \mu \). Then we apply the algorithm in Lemma \[ \ref{lem:algorithm} \] on the edges and vertices lying on \( C_{p,w}(r^*) \). The algorithm returns the minimum \( r \)-value in \( \mu \) to which the decision algorithm returns “yes”. We do this for all point-weight pairs \((p, w)\). Then we have \( O(kn) \) \( r \)-values one of which is exactly \( r^* \). We again apply binary search on these \( r \)-values to find the minimum value over such \( r \)-values to which the decision algorithm returns “yes”. Clearly, the minimum value is exactly \( r^* \). This algorithm takes \( O(k^2n^2\log^2 n) \) time using \( O(kn) \) space.

**Lemma 8.** Given the combinatorial structure of \( A(r^*) \), an optimal assignment of weights and its covering radius \( r^* \) can be computed in \( O(k^2n^2\log^2 n) \) time using \( O(kn) \) space.

**Theorem 9.** Given a set \( P \) of \( n \) points in the plane and a multiset \( W \) of \( k \) weights with \( k \leq n \), an assignment \( f \) of weights that minimizes \( r(f) \) can be found in \( O(k^2n^2\log^2 n) \) time using \( O(kn) \) space.

### 4 Faster Algorithm for a Small Set of Weights

We can improve the algorithm in Section \[ \ref{sec:algorithm} \] for the case that \( k \) is sufficiently small compared to \( n \) and the input weights are at most 1. Specifically, we can compute an optimal assignment of weights in \( O(k^5n\log^3 k + kn\log^3 n) \) time using \( O(kn) \) space. Let \( f^* \) be an optimal assignment of weights, that is, \( r(f^*) = r^* \). A point \( p \) in \( P \) is called a determinator of \( f^* \) if \( d(p, c(f^*))/f^*(p) = r^* \). We already observed that there are two or three determinators for an optimal assignment of weights. Moreover, if there exist exactly two determinators, then the two determinators and \( c(f^*) \) are collinear.

The algorithm in this section is based on the observation that if \( f^*(p) = 1 \) for a determinator \( p \), then \( d(p, c(f^*)) \geq d(q, c(f^*)) \) for any point \( q \) in \( P \). The following lemma provides a more general observation.

**Lemma 10.** Let \( f^* \) be an optimal assignment of weights with minimum number of determinators. If a determinator \( p \) is the \( i \)th closest point of \( P \) from \( c(f^*) \), it is assigned the \( i \)th weight value in the sorted list of the weights of \( W_1 \) in increasing order, where \( W_1 \) is the multiset consisting of the weights of \( W \) and \( n - k \) numbers of weight 1.

**Proof.** Consider a determinator \( p \) that is the \( i \)th closest point of \( P \) from \( c(f^*) \). For any point \( q \) in \( P \) with \( d(q, c^*) \geq d(p, c^*) \), we have \( f^*(q) \geq f^*(p) \). This is because \( d(p, c^*)/f^*(p) = r^* \geq d(q, c^*)/f^*(q) \).

Consider a point \( q \) in \( P \setminus \{p\} \) with \( d(q, c^*) < d(p, c^*) \). If \( f^*(q) > f^*(p) \), then we have \( d(p, c^*)/f^*(q) < r^* \) and \( d(q, c^*)/f^*(p) < r^* \). This implies that we can remove one determinator, \( p \), of \( f^* \) without increasing the covering radius by swapping the weights assigned to \( p \) and \( q \), which is a contradiction. Thus, we have \( f^*(q) \leq f^*(p) \).

Therefore, there are \( i - 1 \) points \( q \in P \setminus \{p\} \) satisfying \( f^*(q) \leq f^*(p) \) and \( n - i \) points \( q' \in P \setminus \{p\} \) satisfying \( f^*(q') \geq f^*(p) \). This implies that \( f^*(p) \) is the \( i \)th weight value in the sorted list of the weights of \( W_1 \) in increasing order. \qed
We will see that the number of candidates for a determinator with its weight we consider can be reduced by Lemma 10. Recall that the algorithm in Section 3 considers each of $O(kn)$ point-weight pairs as a determinator and its weight. Also, we use the following corollary, which is a paraphrase of Lemma 2.

**Corollary 11.** There exists an optimal assignment $f^*$ of weights that assigns the $i$th closest point of $P$ from $c(f^*)$ the $i$th weight value in the sorted list of the weights of $W_1$ in increasing order, where $W_1$ is the multiset consisting of the weights of $W$ and $n - k$ numbers of weight 1.

We consider two possible cases of optimal weight assignments with respect to the determinators and handle them in different ways. The first case is that every determinator is assigned a weight smaller than 1. The second case is that a determinator is assigned weight 1.

### 4.1 Every Determinator is Assigned a Weight Smaller than 1.

By Lemma 10, a determinator is one of the $k$ closest points from $c(f^*)$. This is related to the order-$k$ Voronoi diagram. The order-$k$ Voronoi diagram is a generalization of the standard Voronoi diagram. Given sites in the plane, it partitions the plane into regions such that every point in the same region has the same $k$ closest sites. The complexity of the order-$k$ Voronoi diagram of $n$ point sites is $O(kn)$ \[^9\]. There are a number of algorithms to compute the order-$k$ Voronoi diagram with different running times \[^11\][^13\]. We compute the order-$k$ Voronoi diagram of $P$ using the algorithm in \[^13\], which runs in $O(n \log n + nk2^\alpha \log^* k) \leq O(n \log n + kn \log k)$ time using $O(kn)$ space, where $\alpha$ is a constant. We also compute the farthest-point Voronoi diagram of $P$, also known as the order-$(n - 1)$ Voronoi diagram of $P$, in $O(n \log n)$ time.

In terms of the order-$k$ Voronoi diagram, Lemma 10 can be interpreted as follows. The points of $P$ are the sites of the Voronoi diagram and the $k$ sites corresponding to the cell $V^*$ containing $c(f^*)$ are the closest $k$ sites from $c(f^*)$. Note that the weights of $W$ are the smallest $k$ weights among the weights of $W$ and $n - k$ numbers of weight 1. By Lemma 2, all the $k$ sites corresponding to $V^*$ must be assigned the weights of $W$ and all the other sites must be assigned weight 1. Since every determinator is assigned a weight smaller than 1, it must be one of the $k$ sites corresponding to $V^*$.

To use this observation, we consider each cell of the order-$k$ Voronoi diagram of $P$. For each cell, we assign the $k$ weights of $W$ to the $k$ sites corresponding to the cell, each weight to a distinct site by applying the algorithm in Section 3 to the $k$ sites. This takes $O(k^4 \log^3 k)$ time. Let $c$ and $r$ be the weighted center and the covering radius of this assignment of weights with respect to the $k$ sites. We assign weight 1 to all the other $n - k$ sites. Let $f$ be the resulting assignment of weights.

Then we check whether $c$ is $c(f)$ and $r$ is $r(f)$. By Lemma 2, it suffices to check if the farthest point from $c$, which is assigned weight 1, lies at distance at most $r$ from $c$. This can be done in $O(\log n)$ time by finding the farthest point of $P$ from $c$ using the farthest-point Voronoi diagram of $P$. In total, this takes $O(k^3 n \log^3 k + kn \log n)$ time.

### 4.2 A Determinator is Assigned Weight 1.

We first apply the procedure that deals with the first case in Section 4.1. Let $r_U$ be the minimum radius of the results of this procedure. To handle the case that a determinator is assigned weight 1, we present a decision algorithm that returns “yes” for an input $r$ with $0 < r < r_U$ if and only if there exist an assignment $f$ of weights and a point $c \in \mathbb{R}^2$ such that $d(p, c)/f(p) \leq r$ for all points $p \in P$ and $d(q, c) = r$ for some point in $q \in P$. In other words, the decision algorithm returns “yes” if and only if there is an assignment of weights with covering radius $r$ one of whose determinators is assigned weight 1. For a radius $r$, we call such a point $c$ a center with radius $r$. 


The following lemma enables us to apply parametric search.

**Lemma 12.** If a determinator of an optimal weight assignment is assigned weight 1, then the decision problem for any input $r$ with $r^* \leq r < r_U$ returns “yes”.

*Proof.* Assume to the contrary that the decision problem for an input $r$ with $r^* \leq r < r_U$ returns “no”. By Lemma 3, there is an assignment $f$ of weights with covering radius $r$. However, since the answer for the decision problem is “no”, every determinator of $f$ is assigned a weight smaller than 1. Then such an assignment should have been dealt by the procedure for the first case in Section 4.1, which contradicts the assumption that $r < r_U$. Therefore, the lemma holds. $\square$

**Decision Algorithm.** Given a covering radius $r$, the decision algorithm first computes the intersection $I$ of the disks $D(p,r)$ for all points $p \in P$, where $D(p,r)$ is the disk centered at $p$ with radius $r$. If the answer for the decision problem is “yes”, there is a center corresponding to covering radius $r$ that lies on the boundary of $I$ by the definition.

Thus the decision algorithm searches the boundary of $I$ and checks whether there exists a center on the boundary of $I$. The boundary of $I$ consists of circular arcs. We call an endpoint of a maximal circular arc on the boundary of $I$ a breakpoint. Here, we follow the framework of the algorithm in Section 3. That is, we consider $O(kn)$ circles $C_{p,w}(r)$ for all points $p \in P$ and all distinct weights $w$ of $W \setminus \{1\}$, where $C_{p,w}(r)$ is the circle centered at $p$ with radius $rw$. We compute $O(kn)$ intersection points of the circles with the boundary of $I$. We sort them together with the breakpoints of $I$ along the boundary of $I$ in $O(kn \log n)$ time. Then we apply the procedure in the proof of Lemma 4 which checks whether there exists a center corresponding to covering radius $r$ lying on the boundary of $I$ in $O(kn)$ time. The decision algorithm takes $O(kn \log n)$ time.

**Overall Algorithm.** As we did in the decision algorithm, we first compute the intersection $I(r^*)$ of the disks $D(p,r^*)$ for all points $p \in P$. Here, we are not given $r^*$. Instead of computing the intersection explicitly, we compute its combinatorial structure.

**Lemma 13.** The combinatorial structure of the intersection of the disks $D(p,r^*)$ for all points $p \in P$ can be computed in $O(n \log n + T(n) \log n)$ time, where $T(n)$ is the running time of the decision algorithm.

*Proof.* As $r$ increases from 0 to $r_U$, the combinatorial structure of the intersection of the disks $D(p,r)$ may change. We have two types of events where the combinatorial structure changes: an arc of a disk starts to appear in the structure or an existing arc of a disk disappears from the structure. At both types of events, such an arc becomes a point which is a degenerate arc. Moreover, this point is a vertex of the farthest-point Voronoi diagram of $P$.

To use this fact, we compute the farthest-point Voronoi diagram of $P$ in $O(n \log n)$ time. Then for each vertex $v$ of the diagram, we compute the Euclidean distance between $v$ and its farthest point in $P$. There are $O(n)$ distances, and we sort them in increasing order. Then we apply binary search on the distances using the decision algorithm to find the smallest interval containing $r^*$ whose endpoints are the distances we have. This can be done in $O(T(n) \log n)$ time.

Therefore, for any value of $r$ in the interval, the combinatorial structure of the intersection of the disks remains the same. $\square$

Now we have the combinatorial structure of the intersection $I(r^*)$. As we did in the algorithm of Section 3, we sort the intersections of $O(kn)$ circles $C_{p,w}(r^*)$ for all point $p \in P$ and all distinct weights $w$ of $W$ with the boundary of $I(r^*)$ without explicitly computing $r^*$. This can be done
in $O(kn \log n + T(n) \log^2 n)$ time, where $T(n) = O(kn \log n)$ is the running time of the decision algorithm, in a way similar to Lemma 7. Then we find an optimal solution in a way similar to Lemma 8 in $O(kn)$ time if it belongs to the second case. In total, the second case can be dealt in $O(kn \log^2 n)$ time using $O(kn)$ space.

Combining the three cases, we have the following theorem.

**Theorem 14.** Given a set $P$ of $n$ points in the plane and a multiset $W$ of $k$ weights smaller than or equal to 1 with $k \leq n$, we can compute an assignment $f$ of weights that minimizes $r(f)$ in $O(k^5 n \log^3 k + kn \log^3 n)$ time using $O(kn)$ space.

### 5 Concluding Remarks

We would like to mention that the approaches presented in this paper also work under any convex distance function, including the $L_p$ metric for $p \geq 1$. For the $L_1$ or the $L_\infty$ metric, there can be more than one optimal weighted center though. The weight assignment problem can also be considered in higher dimensions. A future work to consider is to devise efficient algorithms for the problem in higher dimensions under various distance functions.

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