Finite basis property for some classes of irreducible representations of the symmetric groups

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Abstract

In this paper we study the possibility to define irreducible representations of the symmetric groups with the help of finitely many relations. The existence of finite bases is established for the classes of representations corresponding to two-part partitions and to partitions from the fundamental alcove.

1. Introduction

In this paper we continue the research begun in [3]. Let us recall the basic notation used there. Let \( X_n = \{ x_t^r : t \in \mathbb{N}, s = 1, \ldots, n \} \) be a set of commuting variables, \( K \) be a field and \( F_n = K[X_n] \) be the free commutative algebra with 1. Denote by \( F_n^r \) the subspace of \( F_n \) spanned by monomials of the form \( x_{i_1}^{r_1} \cdots x_{i_r}^{r_r} \), where \( \{ i_1, \ldots, i_r \} = \{ 1, \ldots, r \} \). The symmetric group \( G(r) \) of degree \( r \) acts on \( F_n^r \) on the left by the formula

\[
\pi x_{i_1}^{r_1} \cdots x_{i_r}^{r_r} = x_{\pi i_1}^{r_1} \cdots x_{\pi i_r}^{r_r}.
\]

Put \( [x_{i_1}, \ldots, x_{i_k}] = \sum_{\sigma \in S(k)} \text{sgn}(\sigma) x_{i_1}^{\sigma(1)} \cdots x_{i_k}^{\sigma(k)} \), where \( k = 1, \ldots, n \). Let \( \lambda \) be a partition of \( r \) into no more than \( n \) parts. Denote by \( S^{\lambda} \) the subspace of \( F_n^r \) spanned by polynomials of the form \( \prod_{i=1}^{\lambda_1} [x_{i_1,i_1}, \ldots, x_{i_1,i_{\lambda_1}}] \), where \( \mu \) is the partition conjugate to \( \lambda \) and \( \{ t_{ij} : 1 \leq i \leq \lambda_1, 1 \leq j \leq \mu_1 \} = \{ 1, \ldots, r \} \). For any positive integer \( i \), denote by \( \lambda + (i^n) \) the partition \( (\lambda_1 + i, \ldots, \lambda_n + i) \) (see Section 3).

Let \( U \) be a \( KG(r) \)-submodule of \( F_n^r \). Denote by \( U^\uparrow \), or more precisely by \( U^\uparrow_n \) if we want to underline the role of \( n \), the \( KG(r+n) \)-submodule of \( F_n^{r+n} \) generated by the subspace \( U[x_{r+1}, \ldots, x_{r+n}] \). Note that \( U \subset S^{\lambda} \) implies \( U^\uparrow \subset S^{\lambda+(1^n)} \).

Conversely, let \( V \) be a \( KG(r+n) \)-submodule of \( F_n^{r+n} \). Denote by \( V \downarrow \) (or by \( V \downarrow_n \)) the set of all polynomials \( f \in F_n^r \) such that \( f[x_{r+1}, \ldots, x_{r+n}] \in V \). Clearly, \( V \downarrow \) is a \( KG(r) \)-module. Applying [1, Corollary 17.18] one can show that \( V \subset S^{\lambda+(1^n)} \) implies \( V \downarrow \subset S^{\lambda} \).

Let \( U^\uparrow_t \) or \( U^\downarrow_t \) denote the result of the \( t \)-fold application of \( \uparrow \) or \( \downarrow \) respectively to \( U \), of course, if this is possible (for \( U \downarrow \)). Let us remark the following simple properties: \( U^\uparrow \subset U < U^\uparrow \) and \( U^\uparrow \downarrow = U \downarrow \) for any submodule \( U \) of \( F_n^r \), where \( r \geq n \).

In [3] we formulated the following problem:

**Problem 2.** Let \( V_i \subset S^{\lambda+(q_i^n)}, i \in \mathbb{N} \), where \( 0 \leq q_1 < q_2 < \cdots \), be a sequence of submodules such that \( V_i \uparrow^{q_i+1} \subset V_{i+1} \) for every \( i \in \mathbb{N} \). Does there exist any \( N \) such that, for every \( i > N \), we have \( V_N \uparrow^{q_i+N} = V_i \)?

We considered this problem as some special, though important, case of the following problem:

**Problem 1.** Suppose \( \text{char } K = p > 0 \). Is every sequence of multilinear polynomials from \( A \) finitely based?

Here \( A \) is the free commutative \( K \)-algebra with free generators \( f_i(x_{t_1}, \ldots, x_{t_{n(i)}}) \), where \( i = 1, \ldots, k \) and \( t_1, \ldots, t_{n(i)} \in \mathbb{N} \), a polynomial of \( A \) is considered to be multilinear if it is multilinear with respect to the multiplicities of the variables \( x_j \) (for example,
Problem 1 can be regarded as the analog of Specht’s problem for forms. In that case form relations play the role of identities. An example of these relations is the well known $\lambda$ where

$$f(x_1, x_2)f(x_2, x_3)$$

is not a multilinear polynomial and implication is considered with respect to $K$-linear actions, multiplication by elements of $A$ and renaming the variables $x_j$.

Theorem 6] solves the partial case $n = 2$ of Problem 1 without any restriction on the submodules $V_i$. Note that we consider the case $n = 2$ only for illustration, since [8, Theorem 6] solves the partial case $n = 2$ of Problem 4 without any restriction on the submodules $V_i$.

### 2. Preliminaries

Let us fix an algebraically closed field $K$ of positive characteristic $p$ and an integer $n \geq 2$. Let $E$ be the linear $K$-space with basis $v_1, \ldots, v_n$ and $GL_n(K)$ be the group of all $K$-linear automorphisms of $E$. The group $GL_n(K)$ is naturally endowed with the structure of an algebraic $K$-group and we may consider polynomial (rational) $GL_n(K)$-modules with respect to this structure. Denote by $\Lambda$ the set of all sequences of whole numbers of length $n$. We write $\lambda \triangleright \mu$, where $\lambda, \mu \in \Lambda$, if we have $\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$ for any $i = 1, \ldots, n$. For a polynomial $GL_n(K)$-module $V$ and a sequence of whole numbers $\lambda = (\lambda_1, \ldots, \lambda_n)$, we denote by $V^\lambda$ the set of all $v \in V$ such that $\text{diag}(t_1, \ldots, t_n)v = t_1^{\lambda_1} \cdots t_n^{\lambda_n}v$ for any $t_1, \ldots, t_n \in K^*$. Clearly, $V^\lambda$ is a $K$-linear space. This space is called the $\lambda$-weight space of $V$. If $V^\lambda \neq 0$ then we call $\lambda$ a weight of $V$. Denote by $\Lambda^+$ the subset of $\Lambda$ consisting of sequences $(\lambda_1, \ldots, \lambda_n)$ such that $\lambda_1 \geq \cdots \geq \lambda_n$.

In the papers [3] and [7] the theory of the so-called tilting modules is developed. Here we assume that any tilting module is a finite dimensional polynomial $GL_n(K)$-module. We will not give definitions of these objects. Instead, let us enumerate their basic properties necessary for the sequel.

1) $E$ is a tilting module (by the definitions of [3] and [7]).

2) The tensor product of tilting modules is also a tilting module (the first full proof in the general case was obtained by O. Mathieu [11]).

3) For any $\lambda \in \Lambda^+$ there exists an indecomposable module $P(\lambda)$ such that $\dim P(\lambda)^\lambda = 1$ and if $\dim P(\lambda)^\mu > 0$ for some $\mu \in \Lambda$, then $\mu \leq \lambda$ (see [3, Theorem (1.1)]).

4) Any tilting module is a direct sum of tilting modules isomorphic to $P(\lambda)$, $\lambda \in \Lambda^+$ (see [3, Theorem (1.1)]).
For any positive integer \( r \), we denote by \( G(r) \) the symmetric group of degree \( r \), i.e. the set of bijections \( \sigma \) of the set of positive integers such that \( \sigma(i) = i \) for any \( i > r \). This realization of the symmetric groups implies the inclusions \( G(r) \subset G(r') \) for \( r \leq r' \). Let \( \text{sgn}(\sigma) \) denote the sign of a permutation \( \sigma \). Let \( \Lambda^+ \) denote the subset of \( \Lambda \) consisting of the sequences of nonnegative integers with sum \( r \). For any set \( S \) we denote by \( i_s \) the identity map from \( S \) to itself.

Consider an arbitrary decomposition \( E^{\otimes r} = \bigoplus h_{ij} M_i(r) \) into a direct sum of indecomposable \( GL_n(K) \)-modules. By properties \([1]\) and \([2]\) the module \( E^{\otimes r} \) is tilting. Therefore by property \([4]\) and the Krull-Schmidt theorem the modules \( M_i(r) \) are tilting indecomposable. Put \( A(r) = \text{End}_{GL_n(K)} E^{\otimes r} \). We have the natural projections \( \pi_i^r : E^{\otimes r} \to M_i(r) \) and embeddings \( \varepsilon^r_i : M_i(r) \to E^{\otimes r} \). Define \( e^r_i = \varepsilon^r_i \pi_i^r \). The endomorphisms \( e^r_i \) are primitive idempotents and we have \( e^r_i e^r_j = 0 \) for \( i \neq j \).

One can see directly from the definition that any weight of \( E^{\otimes r} \) has no negative entries with sum \( r \). Therefore we have the partition \( \{1, \ldots, h_r\} = \bigcup_{\lambda \in \Lambda^+} I_\lambda \), where \( I_\lambda \) are possibly empty sets and for each \( \lambda \in \Lambda^+ \)

\[ A(r) = \bigoplus_{\lambda \in \Lambda^+, \lambda \neq 0} S(r) e_\lambda \oplus \bigoplus_{\lambda, \mu \in \Lambda^+} e_\lambda J(r) e_\mu. \]

In addition \( S(r) \) is a subalgebra of \( A(r) \) and \( S(r) e_\lambda = e_\lambda S(r) = e_\lambda S(r) e_\lambda \) for any \( \lambda \in \Lambda^+ \).

Given \( \pi \in G(r) \) is a subalgebra of \( A(r) \) and \( S(r) e_\lambda = e_\lambda S(r) = e_\lambda S(r) e_\lambda \) for any \( \lambda \in \Lambda^+ \).

Lemma 1 (De Concini, Procesi \([2]\)). The map \( \sigma_r \) is an epimorphism. If \( r \leq n \) then \( \sigma_r \) is an isomorphism and if \( r \geq n + 1 \) then \( \text{Ker} \sigma_r \) is generated as a two-sided ideal of \( K(G(r)) \) by \( \sum_{\sigma \in \text{Ker}(G(n+1))} \text{sgn}(\sigma) \).

Let us call a sequence of integers \( \lambda = (\lambda_1, \ldots, \lambda_s) \) such that \( \lambda_1 \geq \cdots \geq \lambda_s \geq 0 \) and \( \lambda_1 + \cdots + \lambda_s = r \) a partition of \( r \). We call this sequence \( p \)-regular if there are no \( p \) numbers \( \lambda_i \) equal to a positive integer. Otherwise this sequence is called \( p \)-singular. Any \( p \)-regular partition \( \lambda \) of \( r \) labels in the standard way (see \([1]\) some irreducible \( K(G(r)) \)-module denoted by \( D^\lambda \). In addition, any irreducible \( K(G(r)) \)-module is isomorphic to some \( D^\lambda \). Using the epimorphism \( \sigma_r \) defined above, one can make any irreducible \( K(G(r)) \)-module \( N \) isomorphic to some \( D^\lambda \) for a \( p \)-regular \( \lambda \in \Lambda^+ \) into an \( A(r) \)-module by the formula \( am = x nm \), where \( m \in N, a \in A(r), x \in K(G(r)) \) and \( \sigma_r(x) = a \). This definition is correct, since by Lemma \([4]\) and the direct construction of \( D^\lambda \) we have \( (\text{Ker} \sigma_r) D^\lambda = 0 \). By Lemma \([4]\) one can easily check that any irreducible \( A(r) \)-module is isomorphic to \( D^\lambda \) for some \( p \)-regular \( \lambda \in \Lambda^+ \).

We say an irreducible representation \( D^\mu \) corresponds to an idempotent \( e_\lambda \) if \( e_\lambda D^\mu \neq 0 \). It is obvious that \( e_\lambda m = m \) for any \( m \in D^\mu \) and \( e_\nu D^\mu = 0 \) for \( \nu \neq \lambda \). From the general
theory it follows that for any nonzero primitive idempotent there is one up to isomorphism irreducible representation corresponding to this idempotent.

**Lemma 2.** The representation $D^\lambda$, where $\lambda$ is a $p$-regular partition of $\Lambda^+_p$, corresponds to $e^\lambda$. If $\lambda$ is a $p$-singular partition, then $e^\lambda = 0$.

**Proof.** For brevity, we put $G = G(r)$, $A = A(r)$, $M_i = M_i(r)$ and $e_i = e_i^r$. Denote by $X$ the set consisting of $\lambda \in \Lambda^+_p$ such that $e_\lambda \neq 0$. For $\lambda \in X$ we denote by $(t(\lambda))$ the $p$-regular partition of $\Lambda^+_p$ such that $e_\lambda D^{t(\lambda)} \neq 0$. Clearly, $t$ is a bijection from $X$ to the subset of $\Lambda^+_p$ consisting of $p$-regular partitions.

For any $\lambda \in \Lambda^+_p$ we put

$$v(\lambda) = c_1 \cdots c_{\lambda_1} \underbrace{v_1 \otimes \cdots \otimes v_1}_\text{\lambda_1 times} \cdots \underbrace{v_n \otimes \cdots \otimes v_n}_\text{\lambda_n times}$$

where $c_i$ is the sum of the elements $\text{sgn}(\sigma)\sigma$ taken over $\sigma \in G$ such that $\sigma(i) = i$ for any $i$ not belonging to $\{i + \sum_{j=1}^s \lambda_j : 0 \leq s \leq n-1, i \leq \lambda_{s+1}\}$. It is obvious that $Av(\lambda) = KGV(\lambda)$ is isomorphic to a $KG$-module to the Specht module $S^\lambda$ defined in $\Pi$. Indeed, let us identify an element $v_1 \otimes \cdots \otimes v_r$ of $(E_{\otimes r})^\lambda$ with the tabloid that has each $s$, $s = 1, \ldots, r$ in row $i_s$. This correspondence extended linearly to $(E_{\otimes r})^\lambda$ gives an isomorphism of $KG$-modules $(E_{\otimes r})^\lambda$ and $M^\lambda$. Tabloids and $M^\lambda$ are understood in the sense of $\Pi$ Definitions 3.9 and 4.1] with the only difference that in our case the symmetric group acts on the left. Clearly, the restriction of this isomorphism to $Av(\lambda)$ gives the required correspondence.

Let $\lambda$ be an arbitrary partition of $X$. Denote by $P$ the maximal $A$-submodule of $Av(t(\lambda))$. The modules $Av(t(\lambda))/P$ and $D^{t(\lambda)}$ are isomorphic as $KG$-modules and therefore are isomorphic as $A$-modules. It follows from this fact and from $e_\lambda D^{t(\lambda)} \neq 0$ that $e_\lambda Av(t(\lambda)) \neq 0$ and $e_i Av(t(\lambda)) \neq 0$ for some $i \in I_\lambda$. All elements of the nonzero space $e_i Av(t(\lambda))$ have weight $t(\lambda)$. On the other hand $e_i Av(t(\lambda)) \subset M_i \simeq P(\lambda)$. Hence $t(\lambda)$ is a weight of $P(\lambda)$ and $t(\lambda) \leq \lambda$ by property $[3]$

Let us prove the reverse inequality. Without loss of generality we may assume $I_\lambda = \{1, \ldots, l\}$. Here $l \geq 1$ since $e_\lambda \neq 0$. As is well known, $KGL_n(K)v(\lambda)$ is isomorphic to the so-called Weyl module with highest weight $\lambda$. But this same module is embeddable in $P(\lambda)$. Thus there is an embedding of $GL_n(K)$-modules $\tau : KGL_n(K)v(\lambda) \rightarrow M_1$. Let $v = \sigma(v(\lambda))$.

Let $U$ be the subgroup of $GL_n(K)$ generated by endomorphisms $\varphi$ such that $\varphi(v_k) = v_k$ for $k \neq i$ and $\varphi(v_i) = v_i + \alpha v_j$, where $\alpha \in K$ and $j < i$. Corollary 17.18 of $\Pi$ actually means that an element $m \in (E_{\otimes r})^\lambda$ belongs to $Av(\lambda)$ if and only if $um = m$ for any $u \in U$. Using this fact and that $\tau$ is an embedding of $GL_n(K)$-modules, we get $v \in Av(\lambda)$.

If $e_\lambda$ had annihilated each composition factor of $Av$, we would have got $e_\lambda Av = (e_\lambda)^* Av = 0$. However $v \in e_\lambda Av$. Hence $e_\lambda D^\mu \neq 0$, where $D^\mu$ is a composition factor of $Av$ and therefore a composition factor of $Av(\lambda)$. Since $Av(\lambda) \simeq S^\lambda$, we have $\mu \geq \lambda$. By definition $\mu = t(\lambda)$ and therefore $\mu \leq \lambda$ by what was proved above. Hence we obtain $t(\lambda) = \lambda$ and $Av = Av(\lambda)$.

During the proof we additionally obtained

**Corollary 1.** Let $v$ be a nonzero element of $M_i \simeq P(\lambda)$ of weight $\lambda$. Then the $GL_n(K)$-module generated by $v$ is isomorphic to the Weyl module with highest weight $\lambda$ and the $G(r)$-module generated by $v$ is isomorphic to the Specht module $S^\lambda$.

Lemma $\Pi$ is very important for our theory. It gives a description via the primitive idempotents of $A$ of the same irreducible modules that were described in $\Pi$ and that were
studied in $\mathcal{C}$. Without this lemma the approach taken here would be ineffective. The author does not know for sure how to cite this assertion. Therefore its proof is given in this article.

3. Embedding operators

In the previous section we chose an arbitrary decomposition of $E^\otimes r$ into a direct sum of irreducible tilting modules. However, it is convenient that these decompositions for different $r$ be consistent. Thus we will build them inductively as follows.

First we put $M_1(1) = E$. Suppose that we have a decomposition $E^\otimes r = \bigoplus_{i=1}^{h_r} M_i(r)$. By properties [1] and [2] each module $M_i(r) \otimes E$ is tilting. Therefore by property [4] it is decomposable into a direct sum of indecomposable tilting modules. Collecting all these summands, we obtain a decomposition of $E^\otimes r+1$.

Let $r < r'$ be two positive integers. Let us define an embedding operator $f_{r',r} : A(r) \to A(r')$ by the formula $f_{r',r}(\varphi) = \varphi \otimes i_{E^\otimes r' - r}$. Then we have $f_{r',r} \circ \sigma_r = \sigma_{r'}|_{KG(r)}$ and $f_{r'',r'} \circ f_{r',r} = f_{r'',r}$. It is obvious that $M_i(r) \otimes E^\otimes r = \bigoplus_{j \in X} M_j(r')$ implies $f_{r',r}(e_i^r) = \sum_{j \in X} e_j^{r'}$.

Let us recall the terminology and the definitions used in $\mathcal{C}$. A partition $\lambda \in \Lambda^+_+$ is called degenerate ($n$-degenerate) if $\lambda_n = 0$. For two partitions $\lambda, \mu \in \Lambda^+_+$ and two integers $\alpha, \beta$, we denote by $\alpha \lambda + \beta \mu$ the componentwise linear combination of these partitions. Let $(i^n)$ denote the sequence of length $n$, whose every entry is $i$. Denote by $d_r$ the sum of the idempotents $e_\lambda$ over all degenerate $\lambda \in \Lambda^+_+$.

Now suppose that $\lambda$ is a partition of $\Lambda^+_+$ satisfying

**Condition 1.** There exists a such that for any degenerate $\mu \in \Lambda^+_+$ and $k \geq a$ the module $P(\mu) \otimes E^\otimes(k-a)n$ does not have a direct summand isomorphic to $P(\lambda + (k^n))$.

The absence of the direct summand mentioned above for any degenerate $\mu$ is equivalent to $e_{\lambda+(k^n)} f_{r+kn,r+an}(d_{r+an}) = 0$. We will see in Section [4] that this assumption is not meaningless.

**Theorem 1.** Suppose that $\lambda + (1^n)$ is a $p$-regular partition. Let $k \geq \frac{a^2}{a} + (2r+1)a + a^2n$ and $\mu$ be a degenerate $p$-regular partition of $\Lambda^+_+$. Then any $KG(r+an)$-module $M$ of length 2 with factors $D^{\lambda+(k^n)}$ and $D^\mu$ is decomposable.

**Proof.** Considering, if necessary, the dual modules instead of the initial ones, we may suppose without loss of generality that there exists a submodule $N \subset M$ such that $N \cong D^\mu$ and $M/N \cong D^{\lambda+(k^n)}$.

Let $x$ and $y$ be elements of $KG(r+an)$ such that $\sigma_{r+an}(x) = 1 - d_{r+an}$ and $\sigma_{r+an}(y) = d_{r+an}$. In addition we have $\sigma_{r+kn}(x) = f_{r+kn,r+an}(1-d_{r+an})$ and $\sigma_{r+kn}(y) = f_{r+kn,r+an}(d_{r+an})$.

We have the following equalities:

$$\begin{align*}
\sigma_{r+kn}(y)e_{\lambda+(k^n)} &= 0, \\
\sigma_{r+kn}(x)e_\mu &= 0, \\
\sigma_{r+kn}(y)e_\mu &= e_\mu, \\
\sigma_{r+kn}(x)e_{\lambda+(k^n)} &= e_{\lambda+(k^n)}. 
\end{align*}$$

The first one follows from the assumption about $\lambda$ we have made. The second one follows from the fact that for any nondegenerate $\nu \in \Lambda^+_+$ the module $P(\nu) \otimes E^\otimes(k-a)n$ does not have a direct summand isomorphic to $P(\mu)$. The last two equalities follow from the two equalities already proved and from the formula $\sigma_{r+kn}(x) + \sigma_{r+kn}(y) = 1$.

Consider the translates of $x$ defined by the formula $x_i = \pi_i x \pi_i^{-1}$, $i = 1, \ldots, r+an + 1$, where $\pi_i = \prod_{j=1}^{i+an} (j + (i-1)(r + an), j)$. Note that $\text{supp}(x_i) = \{1 + (i-1)(r + an), \ldots,$
which follow from the first two equalities (1) and Lemma 2.

Choose an arbitrary \( m \in M \setminus N \) and prove that \( x_1 \cdots x_{r+an+1}m \notin N \). Assume the contrary. Then \( \sigma_{r+kn}(x_1) \cdots \sigma_{r+kn}(x_{r+an+1})m = 0 \), where \( m \) is the image of \( m \) under the projection \( M \to M/N \). Hence by the last formula of (1) and Lemma 2 we obtain

\[
\sigma_{r+kn}(x_i)m = \sigma_{r+kn}(\pi_i)\sigma_{r+kn}(x)\sigma_{r+kn}(\pi_i^{-1})m = \sigma_{r+kn}(\pi_i)\sigma_{r+kn}(x)e_{\lambda+(kn)}\sigma_{r+kn}(\pi_i^{-1})m = m.
\]

This implies \( m = 0 \), which leads to a contradiction.

Denote by \( P \) the \( KG(r+kn) \)-module generated by \( x_1 \cdots x_{r+an+1}m \). Let us calculate what \( yP \) is. Let \( \alpha \in G(r+kn) \). The cardinality of \( \text{supp}(\alpha^{-1}y\alpha) \) does not exceed \( r+an \). Thus there is an \( i = 1, \ldots, r+an+1 \) such that \( \alpha^{-1}y\alpha \) and \( x_i \) commute. We have

\[
y\alpha x_1 \cdots x_{r+an+1}m \in \alpha x_i(\alpha^{-1}y\alpha)M \subset \alpha x_iN = 0.
\]

Thus \( yP = 0 \). Since \( yN \neq 0 \), we have \( P \cap N = 0 \) and \( M = P \oplus N \). \( \square \)

By induction on the length one can easily prove that under the hypothesis of Theorem 1 any \( KG(r+kn) \)-module \( M \) of a finite length such that one of its composition factors is isomorphic to \( D^{\lambda+(kn)} \) and the remaining factors are degenerate, contains a submodule isomorphic to \( D^{\lambda+(kn)} \).

4. Examples

We proved Theorem 1 under the assumption that \( \lambda \) satisfies Condition 1. In this section we are going to give two examples of such partitions.

Example 1. Suppose \( n < p \). For any positive integer \( R \) we put \( c_0(R) = \{ \lambda \in \Lambda_R^+ : \lambda_1 - \lambda_n \leq p - n \} \). Lemma 12(2) of [6] asserts that in the case where \( \mu \in \Lambda_R^+ \setminus c_0(R) \), the module \( P(\mu) \otimes E \) does not have a direct summand isomorphic to \( P(\nu) \), where \( \nu \in c_0(R+1) \). It obviously follows from this fact that for any \( i \) the module \( P(\mu) \otimes E^i \) does not have a direct summand isomorphic to \( P(\nu) \), where \( \nu \in c_0(R+i) \).

Choose an arbitrary \( \lambda \) of \( c_0(r) \) and prove that this partition satisfies Condition 1.

Lemma 3. Let \( R \geq (n-1)(p-n) + 1 \) and \( \mu \) be a degenerate partition of \( \Lambda_R^+ \). Then \( \mu \notin c_0(R) \).

Proof. We have \( (n-1)\mu_1 \geq \mu_1 + \cdots + \mu_{n-1} = R > (n-1)(p-n) \). Hence \( \mu_1 - \mu_n = \mu_1 > p-n \) and \( \mu \notin c_0(R) \). \( \square \)

Choose any, for example minimal, \( a \) satisfying the inequality \( r+an \geq (n-1)(p-n)+1 \). Then by [6] Lemma 12(2)] and Lemma 3 the partition \( \lambda \) satisfies Condition 1 for \( a \) defined above. Thus Theorem 1 is applicable to \( \lambda \). However taking into account that we consider the special case \( n < p \), the estimate of this theorem can be sharpened.

Theorem 2. Let \( \lambda \in c_0(r) \), \( \lambda_n \geq (n-1)(p-n) + 2 \) and \( \mu \) be a degenerate partition of \( \Lambda_R^+ \). Then any \( KG(r) \)-module \( M \) of length 2 with factors \( D^\lambda \) and \( D^\mu \) is decomposable.
Proof differs from that of Theorem 1 mainly by the choice of $x$ and $y$. Similarly to Theorem 1 we assume without loss of generality that there is a submodule $N \subset M$ such that $N \cong D^\mu$ and $M/N \cong D^\lambda$. For brevity, we put $R = (n - 1)(p - n) + 1$.

Let $x = \frac{1}{n!} \sum_{\sigma \in S(n)} \text{sgn}(\sigma) x_{\sigma}$ and $x_i = \pi_i g_i \pi_i^{-1}$, $i = 1, \ldots, R + 1$, where $\pi_i = \prod_{r=1}^n (j + (i - 1)n, j)$. Choose $y \in KG(R)$ so as to have $\sigma(y) = d_R$. Then $\sigma(y) = f_{r,R}(d_R)$. By Lemma 12(2) we obtain $\sigma(y)e_j = 0$. The first and the third of formulas (2) can be checked similarly to what was done in Theorem 1. The second formula follows from the fact that $\mu$ contains no more than $n - 1$ nonzero parts. From the direct construction of $D^\lambda$ as a quotient module of the Specht module $S^\lambda$ and from the inequality $\lambda_0 \geq R + 1$ it follows that there exists some $m \in M$ such that $x_1 \cdots x_{R+1}m \notin N$. Denote by $P$ the $KG(r)$-submodule of $M$ generated by $x_1 \cdots x_{R+1}m$ and similarly to Theorem 1 show that $yP = 0$. Hence $M = P \oplus N$.

Example 2. Suppose $n = 2$. Choose an arbitrary $\lambda \in \Lambda_r$. Let $\mu \in \Lambda_{r+1}$ be a partition such that $P(\mu)$ is a direct summand of $P(\lambda) \otimes E$. Suppose that $k$ is a positive integer such that $\lambda_1 - \lambda_2 \geq k^2 - 1$. Let us prove that in this case $\mu_1 - \mu_2 \geq k^2 - 1$. The proof proceeds by applying the formulas for the product of tilting modules form [1 Section 1]. These formulas deal with tilting modules over $SGL_2(K)$. However, it is clear that if a true formula of the form $T(s) \otimes T(1) = T(s')$ is replaced by $P((\frac{s+1}{2}, \frac{s-1}{2}) \otimes E = P((\frac{s+1}{2}, \frac{s+1}{2} - \frac{1}{2}))$, then we have a true formula again. In what follows we will take account of such reformulations while citing formulas of [1].

For brevity, we put $m = \lambda_1 - \lambda_2$ and $m' = \mu_1 - \mu_2$. We assume $m > 0$ and $k > 0$, since in the contrary case the assertion being proved is obvious. Let $m = \sum_{s=0}^{\infty} i_s p^s$, where $0 \leq i_s \leq p - 1$, be the $p$-adic expansion of $m$.

If $i_0 = p - 1$ then by [1 Lemma 1.5(1)] we have $m' = m + 1 \geq k^2$. If $0 \leq i_0 \leq p - 3$ (the case $p > 2$) then $m' \geq k^2$. By [1 Lemma 1.5(2),(3)] we have $m' = m + 1$ or $m' = m - 1$. In both cases we have $m' \geq k^2 - 1$.

Now suppose that $i_0 = p - 2$. Denote by $t$ the positive integer such that $i_s = p - 1$ for $s = 1, \ldots, t - 1$ and $i_t < p - 1$. We put $\sigma = \sum_{s=t+1}^{\infty} i_s p^{s-t-1}$. Then $m + 2 = (i_t + 1)p^t + \sigma p^{t+1}$.

Case $p > 2$. For the calculation we will use [1 Lemma 1.7.3]. To prove the inequality $m' \geq k^2 - 1$ it suffices to prove that $m + 1 - 2p^{t-1} \geq k^2 - 1$ and that $m + 1 - 2p^t \geq k^2 - 1$ for $i_t = 1$ and $\sigma > 0$ for $i_t = 2$.

Suppose that $t < k$. Since $m \geq k^2$, we have $i_{k'} > 0$ for some $k' \geq k$. Hence $m + 2 \geq (i_t + 1)p^t + p^k$ and $m + 1 - 2p^{t-1} \geq p^k - 1 + p^t - 2p^{t-1} > p^k - 1$. If $i_t \geq 1$ then $m + 1 - 2p^t \geq p^k - 1 + (i_t - 1)p^t \geq p^k - 1$.

Suppose that $t = k$. Since $m \geq p^k$, we have $i_k > 0$ and $\sigma = 0$ or $\sigma > 0$. In the first case $m + 2 \geq 2p^k$ and $m + 1 - 2p^{t-1} \geq 2p^k - 1 + 2p^{t-1} > 2p^k - 1 - 2p^k - 1 > 2p^k - 1$. In addition, if $i_k \geq 2$ then $m + 2 \geq 3p^k$ and $m + 1 - 2p^t \geq p^k - 1$. In the second case $m + 2 \geq 3p^k$ and $m + 1 - 2p^t \geq p^k - 1$.

Suppose $t > k$. Since $m + 2 \geq p^t$, we have $m + 1 - 2p^{t-1} \geq p^{t-1} - 1 \geq p^k - 1$. If $\sigma > 0$ or $i_t \geq 2$, then $m + 2 \geq 3p^k$ and $m + 1 - 2p^t \geq p^k - 1 - 1 > p^k - 1$.

Case $p = 2$. For the calculation we will use [1 Lemma 1.7.2]. In this case to prove the inequality $m' \geq 2^k - 1$ it suffices to prove that $m + 1 - 2^{t-1} \geq 2^{k} - 1$ and that $m + 1 - 2^t \geq 2^k - 1$ for $\sigma > 0$.

Suppose $t < k$. Similarly to the case $p > 2$ we obtain $m + 2 \geq 2^t + 2^k$. Hence $m + 1 - 2^t \geq 2^k - 1$.

Suppose $t = k$. Similarly to the case $p > 2$ we obtain $i_k > 0$ or $\sigma > 0$. In both cases we have $m + 2 \geq 2^{k+1}$ and $m + 1 - 2^t \geq 2^{k+1} - 1 - 2^k = 2^k - 1$.

Suppose $t > k$. Since $m + 2 \geq 2^t$, we have $m + 1 - 2^t \geq 2^{t-1} - 1 \geq 2^k - 1$. If $\sigma > 0$ then $m + 2 \geq 2^t + 1$ and $m + 1 - 2^t \geq 2^t - 1 > 2^k - 1$.  


Now we are ready to check Condition \( \square \) for any partition \( \lambda \in \Lambda^+_1 \). Choose any, for example minimal, \( k \) such that \( \lambda_1 - \lambda_2 < p^k - 1 \) and put \( a = p^k - 1 - r \). From the above calculations one can see that Condition \( \square \) is satisfied for \( a \) we have chosen.

5. Finite basis property

Let us study what consequences of the results of Section \( \sqref{4} \) one can get as far as the operation \( \uparrow \) introduced in \( \sqref{8} \) is concerned.

For any \( KG(r) \)-module \( V \) we denote by \( d(V) \) the dual module of \( V \). In addition, there exists the nondegenerate form \( \langle , \rangle : V \times d(V) \to K \) such that \( \langle sv, u \rangle = \langle v, \sigma^{-1}u \rangle \) for any \( v \in V, u \in d(V) \) and \( \sigma \in G(r) \). For a subset \( A \) of \( V \) or of \( d(V) \) we denote by \( V^\perp \) the set \( \{u \in d(V) : \langle A, u \rangle = 0\} \) or the set \( \{v \in V : \langle v, A \rangle = 0\} \) respectively.

**Lemma 4.** Let \( \lambda \) be a \( p \)-regular nondegenerate partition of \( \Lambda^+_1 \) and \( V \) be a proper submodule of the Specht module \( S^\lambda \). Consider the following conditions:

1) \( V\downarrow \uparrow \nsubseteq V \).

2) There is a \( KG(r) \)-module \( L \) and a nonisomorphic embedding \( \iota : d(S^\lambda/V) \to L \) such that \( \text{Im} \iota \) is an essential submodule of \( L \) and all composition factors of \( L/\text{Im} \iota \) are degenerate.

Condition \( \square \) always follows from Condition \( \square \). The reverse implication is true for \( n < p \).

**Proof.** \( \square \)\( \Rightarrow \square \) Define \( L = d(S^\lambda/V\downarrow \uparrow) \) and \( \iota \) to be the embedding induced by the projection \( S^\lambda/V\downarrow \uparrow \to S^\lambda/V \). From the obvious equality \( V\downarrow \uparrow \downarrow = V\downarrow \) (see Introduction) and \( \sqref{8} \) Theorem 5(a) it follows that \( V/V\downarrow \uparrow \). Therefore all composition factors of \( L/\text{Im} \iota \) are also degenerate. It remains to show that \( \text{Im} \iota \) is an essential submodule of \( L \). Let \( N \) be a submodule of \( L \) and \( N\cap \text{Im} \iota = 0 \). Then we have \( N^\perp + (V/V\downarrow \uparrow) = S^\lambda/V\downarrow \uparrow \). Since \( P^\lambda/V\downarrow \uparrow \) is a unique maximal submodule of \( S^\lambda/V\downarrow \uparrow \) and \( V \subset P^\lambda \), we have \( N^\perp = S^\lambda/V\downarrow \uparrow \). Hence \( N = 0 \).

\( \square \Rightarrow \square \) subject to \( n < p \). Without loss of generality we may assume that \( L/\text{Im} \iota \cong D^\mu \), where \( \mu \) is a degenerate \( p \)-regular partition.

In the remaining part of the proof we will follow the definitions of \( \sqref{5} \). Choose an arbitrary \( N \geq r \) and put \( G = G(r), S = S_K(N, r), \omega = (1^r, 0^{N-r}), e = \xi_\omega \). Let us fix the sequence \( u(r) = (1, \ldots, r) \). The map \( \xi_{u(r)} \pi_{u(r)} \to \pi \) defines an isomorphisms of the rings \( eSe \) and \( KG \). Thus we may consider any \( KG \)-module as an \( eSe \)-module and vice versa.

Let \( V_{\lambda,K} \) be the Weyl module with highest weight \( \lambda \). In the paper \( \sqref{5} \) the isomorphism (6.3d) was used to establish an isomorphism of \( V_{\lambda,K}^\omega \) and \( d(S^\chi) \) (\( \square_{T,K} \) in the notation of \( \sqref{5} \)).

Let \( K_s \) denote the sign representation of \( G \). There is the standard embedding of \( V \otimes K_s \) into \( d(S^\chi) \) and therefore into \( V_{\lambda,K}^\omega \). Denote by \( V' \) the image of \( V \) under this embedding.

Put \( U = SV' \). This module is a submodule of \( V_{\lambda,K} \). The paper \( \sqref{5} \) introduces the left \( S \)-modules \( D_{\lambda,K} \) and \( A_K(N, r) \). The last module is also a right \( S \)-module. In addition \( D_{\lambda,K} \) is a submodule of \( XA_K(N, r) \) that is a right weight space of \( A_K(N, r) \).

According to \( \sqref{5} \) Section 5.1] the restriction of the natural form \( \langle , \rangle \) defined on \( E^\otimes r \) gives a nondegenerate contravariant form \( \langle , \rangle : V_{\lambda,K} \times D_{\lambda,K} \to K \). Put \( W = \{w \in D_{\lambda,K} : (U, w) = 0\} \). The module \( W \) is the contravariant dual to \( V_{\lambda,K}/U \) in the sense of \( \sqref{5} \).
we have that is identified with the Specht module $S$. Therefore $\psi$ is an embedding of $eW$ to $M$. In addition, $\text{Im}\varphi$ is an essential submodule of $M$ and $M/\text{Im}\varphi \cong D' \otimes K$.

Put $h(M) = Se \otimes eSe M$. Denote by $W'$ the $K$-subspace of $h(M)$ spanned by the elements of the form $se \otimes \varphi(ew)$, where $s \in S$ and $w \in W$. Clearly, $W'$ is an $S$-module. Consider the map $\psi : W' \rightarrow W$ defined by the formula $\psi(se \otimes \varphi(ew)) = sew$.

Since $W \subset \lambda A_K(N, r)$ and the latter module is injective, there exists an extension $\overline{\psi} : h(M) \rightarrow \lambda A_K(N, r)$ of $\psi$. Consider the isomorphism $\varepsilon : M \rightarrow eh(M)$ defined by the formula $\varepsilon(m) = e \otimes m$. Let $\tau = \overline{\psi}\varepsilon$ be their composition. It is obvious that $\tau \varphi = i_{eW}$. Therefore $\tau$ isomorphically takes $\text{Im}\varphi$ to $eW$. Since $\text{Im}\varphi$ is an essential submodule of $M$, we have that $\tau$ is an embedding of $M$ in $e\lambda A_K(N, r)$.

The module $e\lambda A_K(N, r)$ can be identified with $M'$ in the notation of [1]. Then $eD_{\lambda, K}$ is identified with the Specht module $S'$. Applying the results of [1] Section 17, we obtain that there is a filtration of $e\lambda A_K(N, r)/eD_{\lambda, K}$ with factors isomorphic to $S'$, where $\nu \triangleright \lambda'$. Therefore either $\tau(M) \subset eD_{\lambda, K}$ or some $S'$, where $\nu \triangleright \lambda'$, contains a submodule isomorphic to $D' \otimes K$.

Let us show that the latter case is impossible. Since $S' \subset M'$ (see [1]) and the modules $D' \otimes K$ and $M'$ are self-dual, there exists an epimorphism $\pi : M' \rightarrow D' \otimes K$. Put $g = \sum_{\sigma \in G(n)} \sigma$. Since $\nu \triangleright \lambda'$, we have $\nu_1 \geq n$. It follows from this fact and from $n < p$ that $M'$ has a cyclic generator of the form $gv$, where $v \in M'$. Applying $\pi$, we obtain $\pi(gv) = g\pi(v) \in g(D' \otimes K) = 0$. The last equality follows from the fact that $\mu$ is degenerate. This leads to a contradiction.

Thus we have obtained that $eW \subset \tau(M) \subset eD_{\lambda, K}$. Applying the form $(\cdot, \cdot)$ and multiplying by $K$, we obtain that there exists some submodule $V_0$ of $V$ such that $V/V_0 \cong D'$. By [3] Corollary 6 we have $V \downarrow \nsubseteq V$. □

**Note.** In the first part of the proof it is possible to consider an arbitrary field, while in the second part the field should be infinite. However, the second part is given only for an illustration of the relation between properties [1] and [2] and is never used in the present paper.

For any $p$-regular partition $\nu$, we denote by $P'_{\nu}$ the unique maximal submodule of the Specht module $S'_{\nu}$. One can easily check that if $\nu$ is nondegenerate, then $P'_{\nu} \downarrow = P'_{\nu-1}$. Now by the first part of Lemma[4] we have that under the hypothesis of Theorems[1] and [2] the equalities

$$P^{\lambda+(k-1)^n} \uparrow = P^{\lambda+(k^n)}, \quad P^{\lambda-(1^n)} \uparrow = P^\lambda$$

hold respectively. If $K$ is not algebraically close, then applying the natural isomorphism $S_K' \otimes \overline{\mathbb{R}} \cong S_K'$ which takes $P_k' \otimes \overline{\mathbb{R}}$ to $P_k'$ we obtain formulas [4] for any field of the same characteristic. It follows from these formulas that for any $\lambda$ satisfying Condition[1] there exists finitely many relations defining all irreducible modules $D^{\lambda+(k^n)} \cong S^{\lambda+(k^n)}/P^{\lambda+(k^n)}$, $k \in \mathbb{N}$.

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