Integrand reduction of one-loop scattering amplitudes through Laurent series expansion

Pierpaolo Mastrolia
Max-Planck-Institut für Physik, Föhringer Ring 6, D-80805 München, Germany;
Dipartimento di Fisica e Astronomia, Università di Padova, and INFN Sezione di Padova, via Marzolo 8, 35131 Padova, Italy
E-mail: pierpaolo.mastrolia@cern.ch

Edoardo Mirabella
Max-Planck-Institut für Physik, Föhringer Ring 6, D-80805 München, Germany
E-mail: mirabell@mppmu.mpg.de

Tiziano Peraro
Max-Planck-Institut für Physik, Föhringer Ring 6, D-80805 München, Germany
E-mail: peraro@mppmu.mpg.de

ABSTRACT: We present a semi-analytic method for the integrand reduction of one-loop amplitudes, based on the systematic application of the Laurent expansions to the integrand-decomposition. In the asymptotic limit, the coefficients of the master integrals are the solutions of a diagonal system of equations, properly corrected by counterterms whose parametric form is known a priori. The Laurent expansion of the integrand is implemented through polynomial division. The extension of the integrand-reduction to the case of numerators with rank larger than the number of propagators is discussed as well.
1. Introduction

The recent development of novel methods for computing one-loop scattering amplitudes has been highly stimulated by a deeper understanding of their multi-channel factorization properties under special kinematics enforced by on-shellness [1, 2] and generalized unitarity [3, 4]. Analyticity and unitarity of scattering amplitudes have then been strengthened by the complementary classification of the mathematical structures present in the residues at the singular points. They naturally arise after uncovering a relation between numerator and denominators of one-loop Feynman integrals, yielding the multipole decomposition of Feynman integrands originally proposed in a four-dimensional framework by Ossola, Papadopoulos and Pittau (OPP) [5, 6], and later extended to dimensionally regulated amplitudes by Ellis, Giele, Kunszt and Melnikov (EGKM) [7–9].

The use of unitarity-cuts and complex momenta for on-shell internal particles turned unitarity-based methods into very efficient tools for computing scattering amplitudes. These methods, recently reviewed in [10–12], exploit two general properties of scattering amplitudes, such as analyticity and unitarity: the former granting that amplitudes can be reconstructed from the knowledge of their (generalized) singularity-structure; the latter granting that the residues at the singular points factorize into products of simpler amplitudes. Unitarity-based methods are founded on the underlying representation of scattering amplitudes as a linear combination of master integrals (MI’s) [13, 14], and their principle
is the extraction of the coefficients entering in such a linear combination by matching the cuts of the amplitudes onto the cuts of each MI.

The multi-particle pole decomposition for the integrands of arbitrary scattering amplitudes emerges from the combination of analyticity and unitarity with the idea of a reduction under the integral sign.

The principle of an integrand-reduction method is the underlying multi-particle pole expansion for the integrand of any scattering amplitude, or, equivalently, a representation where the numerator of each Feynman integral is expressed as a combination of products of the corresponding denominators, with polynomial coefficients. Each residue is a (multivariate) polynomial in the irreducible scalar products (ISP’s) formed by the loop momenta and either external momenta or polarization vectors constructed out of them. The integrand reduction method has been recently shown to be applicable to scattering amplitudes beyond one-loop as well [15, 16].

The polynomial structure of the multi-particle residues is a qualitative information that turns into a quantitative algorithm for decomposing arbitrary amplitudes in terms of MI’s at the integrand level. In the context of an integrand-reduction, any explicit integration procedure and/or any matching procedure between cuts of amplitudes and cuts of MI’s is replaced by polynomial fitting, which is a simpler operation. Within this algorithm the (dimensionally regulated) integrand of a given scattering amplitude is the only input needed for sampling the integrand on the solutions of generalized on-shell conditions.

The original algorithm [5,7] is based on the solution of a triangular system of equations to be solved top-down, from the determination of the 5-point coefficients to the 1-point coefficients. At any step of the reduction, the Gauss-substitutions requires the subtraction of the of the coefficients determined in the previous steps. Accordingly, the algorithm proceeds by subtractions at the integrand level, requiring the knowledge of the already reconstructed residues. In other words the determination of the coefficients of the n-point residues (1 ≤ n ≤ 5), requires the subtraction of the residues of m-point functions with n < m ≤ 5. The integrand reduction method has been implemented in the publicly available libraries CutTools [17] and samurai [18], as well as in several multi-purpose codes [19–30].

In this paper, we exploit the asymptotic behaviour of the integrand decomposition, and propose a simpler technique for the integrand-reduction of one-loop scattering amplitudes in dimensional regularization. The use of Laurent series within the unitarity-based methods has been mainly developed in the context of analytic calculations [31–35].

Elaborating on the the techniques proposed by Forde [33] and Badger [35] we apply systematically the series expansion to the integrand decomposition formula of OPP/EGKM. We show how the advantages of the analytic techniques can be incorporated in a refined semi-analytic algorithm which determines the coefficients using Laurent expansions rather than polynomial fitting. The main features of this algorithm are the following.

- The coefficients of 5-point functions are never needed and do not have to be computed.
- The spurious coefficients of the 4-point functions do not enter the reduction and are not computed. The rational terms coming from higher-dimension 4-point functions
can be computed analytically from quadruple-cuts only.

- The computation of 3-, 2-, and 1-point coefficients is independent of the residues of the 4-point functions. In particular the 3-point coefficients are computed from triple cuts, without any subtraction.
- The subtraction at the integrand level is replaced by the subtraction at the \textit{coefficient level}. Indeed in the original reduction method the subtractions guarantee the polynomiality of each residue allowing its determination through polynomial fitting. The Laurent expansion makes each function entering the subtraction separately polynomial. Therefore the subtraction of higher point residues can be omitted during the reduction. Its effect is accounted for correcting the reconstructed coefficients.
- The correction terms of 2-, and 1-point functions are parametrized by universal functions in terms of the higher points coefficients.
- The application of the Laurent expansion for the determination of 3-, 2-, and 1-point coefficients is implemented via \textit{polynomial division}. The Laurent series is obtained as the \textit{quotient} of the division between the numerator and the product of the uncut denominators, \textit{neglecting the remainder}.

This algorithm has been implemented in \texttt{C++} and in \textsc{Mathematica} using the S\texttt{S@M} package [36]. The semi-analytic implementation in \texttt{C++} has been designed as a reduction library to be linked to codes like \textsc{GoSam} and \textsc{FormCalc} which provide analytic expression of the integrands, or to any package which can provide the tensor-structure of the integrand as described in [37, 38] and in the more recent Open-Loop technique [39]. The version in \textsc{Mathematica} has been used to obtain closed formulas for the coefficients, depending on the vector-basis associated to each cut and on the generic tensors appearing in the integrand. In this latter fashion, the reduction procedure is replaced by a simple pattern-matching.

2. Integrand decomposition

In this section we collect the relevant formulae of the integrand reduction methods introduced in [5–9] following the notation of [18].

The reduction method is based on the general decomposition for the integrand of a generic one-loop amplitude. Any one-loop $n$-point amplitude can be written as

\[
\mathcal{A}_n = \int d^d \bar{q} \ A(\bar{q}, \epsilon) ,
\]

\[
A(\bar{q}, \epsilon) = \frac{\mathcal{N}(\bar{q}, \epsilon)}{D_0 D_1 \cdots D_{n-1}} ,
\]

\[
D_i = (\bar{q} + p_i)^2 - m_i^2 = (q + p_i)^2 - m_i^2 - \mu^2 , \quad (p_0 \neq 0) . \tag{2.1}
\]

We use a bar to denote objects living in $d = 4 - 2\epsilon$ dimensions, following the prescription

\[
\bar{q} = q + \mu , \quad \text{with} \quad q^2 = q^2 - \mu^2 . \tag{2.2}
\]
The most general numerator of one-loop amplitudes \( \mathcal{N}(q, \epsilon) \) can be thought as composed of three terms,
\[
\mathcal{N}(q, \epsilon) = N_0(q, \mu^2) + \epsilon N_1(q, \mu^2) + \epsilon^2 N_2(q, \mu^2).
\] (2.3)
The coefficients of this \( \epsilon \)-expansion, \( N_0, N_1 \) and \( N_2 \), are functions of \( q^{\nu} \) and \( \mu^2 \). In the following discussion we will denote by \( N \) any element of the set \( \{N_0, N_1, N_2\} \).

The numerator \( N(q, \mu^2) \) can be expressed in terms of denominators \( D_i \), as follows
\[
N(q, \mu^2) = \sum_{i<<m}^{n-1} \Delta_{ijk\ell m}(q, \mu^2) \prod_{h\neq i,j,k,\ell,m}^{} D_h + \sum_{i<<\ell}^{n-1} \Delta_{ijk\ell}(q, \mu^2) \prod_{h\neq i,j,k,\ell}^{} D_h + \sum_{i<<k}^{n-1} \Delta_{ijk}(q, \mu^2) \prod_{h\neq i,j,k}^{} D_h + \sum_{i<j}^{n-1} \Delta_i(q, \mu^2) \prod_{h\neq i}^{} D_h, \tag{2.4}
\]
where \( i<<m \) is the lexicographic ordering \( i<j<k<\ell<m \). The functions \( \Delta(q, \mu^2) \) are polynomials in the components of \( q \) and in \( \mu^2 \). The decomposition (2.3) expose the multi-pole nature of the integrand
\[
A(q, \mu^2) = \sum_{i<<m}^{n-1} \frac{\Delta_{ijk\ell m}(q, \mu^2)}{D_i D_j D_k D_\ell D_m} + \sum_{i<<\ell}^{n-1} \frac{\Delta_{ijk\ell}(q, \mu^2)}{D_i D_j D_k D_\ell} + \sum_{i<<k}^{n-1} \frac{\Delta_{ijk}(q, \mu^2)}{D_i D_j D_k} + \sum_{i<j}^{n-1} \frac{\Delta_i(q, \mu^2)}{D_i}. \tag{2.5}
\]

For each cut \( (ijk\cdots) \), obtained setting \( D_i = D_j = D_k = \cdots = 0 \), we introduce a basis of four massless vectors
\[
\mathcal{E}^{(ijk\cdots)} = \left\{ e_1^{(ijk\cdots)}, e_2^{(ijk\cdots)}, e_3^{(ijk\cdots)}, e_4^{(ijk\cdots)} \right\}, \tag{2.6}
\]
such that
\[
e_1^{(ijk\cdots)} e_1^{(ijk\cdots)} = 0, \quad e_1^{(ijk\cdots)} e_2^{(ijk\cdots)} = e_2^{(ijk\cdots)} e_3^{(ijk\cdots)} = e_2^{(ijk\cdots)} e_4^{(ijk\cdots)} = e_3^{(ijk\cdots)} e_4^{(ijk\cdots)} = 0, \quad e_2^{(ijk\cdots)} e_3^{(ijk\cdots)} = e_2^{(ijk\cdots)} e_4^{(ijk\cdots)} = 0, \quad e_1^{(ijk\cdots)} e_2^{(ijk\cdots)} = -e_3^{(ijk\cdots)} e_2^{(ijk\cdots)} = 1.
\]
The massless vectors \( e_1^{(ijk\cdots)} \) and \( e_2^{(ijk\cdots)} \) can be written as a linear combination of the two external legs at the edges of the propagator carrying momentum \( q + p_i \), say \( K_1 \) and \( K_2 \), along the lines of [18]. In the case of double-cut, \( K_1 \) is the momentum flowing through the corresponding 2-point diagram, and \( K_2 \) is an arbitrary massless vector. In the case of single-cut both \( K_1 \) and \( K_2 \) are chosen as arbitrary vectors. In the case of quadruple-cut \( (ijk\ell) \) we define
\[
v_\perp^{(ijk\ell)} = \left( K_3 \cdot e_4^{(ijk\ell)} \right) e_3^{(ijk\ell)} - \left( K_3 \cdot e_3^{(ijk\ell)} \right) e_4^{(ijk\ell)}, \quad
v^{(ijk\ell)} = \left( K_3 \cdot e_4^{(ijk\ell)} \right) e_3^{(ijk\ell)} + \left( K_3 \cdot e_3^{(ijk\ell)} \right) e_4^{(ijk\ell)}. \tag{2.7}
\]
The momentum \( K_3 \) is the third leg of the 4-point function associated to the considered quadruple-cut. To simplify our notation we will omit the indices of the cut \( (ijk\cdots) \) whenever possible.
The functions \( \Delta(q, \mu^2) \) are parametrized in terms of the basis (2.6) and of the vectors (2.7):

\[
\Delta_{ijk\ell m}(q, \mu^2) = c_{5,0}^{(ijk\ell m)} \mu^2 ,
\]
(2.8)

\[
\Delta_{ijk\ell}(q, \mu^2) = \Delta_{ijk\ell}^R(q, \mu^2) + c_{4,0}^{(ijk\ell)} + c_{4,2}^{(ijk\ell)} \mu^2 + c_{4,4}^{(ijk\ell)} \mu^4 ,
\]
(2.9)

\[
\Delta_{ijk}(q, \mu^2) = \Delta_{ijk}^R(q, \mu^2) + c_{3,0}^{(ijk)} + c_{3,7}^{(ijk)} \mu^2 ,
\]
(2.10)

\[
\Delta_{ij}(q, \mu^2) = \Delta_{ij}^R(q, \mu^2) + c_{2,0}^{(ij)} + c_{2,9}^{(ij)} \mu^2 ,
\]
(2.11)

\[
\Delta_i(q, \mu^2) = c_{1,0}^{(i)} + c_{1,1}^{(i)}(q + p_i) \cdot e_1 + c_{1,2}^{(i)}((q + p_i) \cdot e_2)
+ c_{1,3}^{(i)}((q + p_i) \cdot e_3) + c_{1,4}^{(i)}((q + p_i) \cdot e_4) .
\]
(2.12)

For later convenience, we define the reduced polynomials \( \Delta^R \) as,

\[
\Delta_{ijk\ell}(q, \mu^2) = \left( c_{4,1}^{(ijk\ell)} + c_{4,3}^{(ijk\ell)} \mu^2 \right)(q + p_i) \cdot v_{\perp} ,
\]
(2.13)

\[
\Delta_{ijk}(q, \mu^2) = \left( c_{3,1}^{(ijk)} + c_{3,8}^{(ijk)} \mu^2 \right)(q + p_i) \cdot e_3 + \left( c_{3,4}^{(ijk)} + c_{3,9}^{(ijk)} \mu^2 \right)(q + p_i) \cdot e_4
+ c_{3,2}^{(ijk)} ((q + p_i) \cdot e_3)^2 + c_{3,5}^{(ijk)} ((q + p_i) \cdot e_4)^2
+ c_{3,3}^{(ijk)} ((q + p_i) \cdot e_3)^3 + c_{3,6}^{(ijk)} ((q + p_i) \cdot e_4)^3 ,
\]
(2.14)

\[
\Delta_i(q, \mu^2) = c_{2,1}^{(i)}(q + p_i) \cdot e_2 + c_{2,2}^{(i)}((q + p_i) \cdot e_2)^2
+ c_{2,3}^{(i)}(q + p_i) \cdot e_3 + c_{2,4}^{(i)}((q + p_i) \cdot e_3)^2
+ c_{2,5}^{(i)}(q + p_i) \cdot e_4 + c_{2,6}^{(i)}((q + p_i) \cdot e_4)^2
+ c_{2,7}^{(i)}((q + p_i) \cdot e_2)((q + p_i) \cdot e_3) + c_{2,8}^{(i)}((q + p_i) \cdot e_2)((q + p_i) \cdot e_4) .
\]
(2.15)

Neglecting terms of \( \mathcal{O}(\epsilon) \), the one loop amplitude can be written in terms of master integrals and of the coefficients of \( \Delta_{ijk\ell m}, \Delta_{ijk\ell}, \Delta_{ijk}, \Delta_i \), and \( \Delta \),

\[
A_n = \sum_{i<j<k<\ell}^{n-1} \left\{ c_{4,0}^{(ijk\ell)} I_{ijk\ell} + c_{4,4}^{(ijk\ell)} I_{ijk\ell}[\mu^4] \right\}
+ \sum_{i<j<k}^{n-1} \left\{ c_{3,0}^{(ijk)} I_{ijk} + c_{3,7}^{(ijk)} I_{ijk}[\mu^2] \right\}
+ \sum_{i<j}^{n-1} \left\{ c_{2,0}^{(ij)} I_{ij} + c_{2,1}^{(ij)} I_{ij}[(q + p_i) \cdot e_2] + c_{2,2}^{(ij)} I_{ij}[(q + p_i) \cdot e_2]^2 + c_{2,9}^{(ij)} I_{ij}[\mu^2] \right\}
+ \sum_{i}^{n-1} c_{1,0}^{(i)} I_i ,
\]
(2.16)

where

\[
I_{i_1\cdots i_k}[\alpha] \equiv \int d^dq \frac{\alpha}{D_{i_1} \cdots D_{i_k}} , \quad I_{i_1\cdots i_k} \equiv I_{i_1\cdots i_k}[1].
\] 
(2.17)
As already noted in [5, 8, 12], some of the terms appearing in the integrand decomposition (2.5) vanish upon integration. They are called spurious and do not contribute to the amplitude (2.16). Beside the scalar boxes, triangles, bubbles and tadpoles, the other master integrals are the linear and quadratic two-points functions [40, 41] and the integrals containing powers of $\mu^2$ in the numerator. The latter can be traded with higher dimensional integrals [40, 41]

$$I_{i_1 \ldots i_k}[(\mu^2)^r f(q, \mu^2)] = \frac{1}{\pi^r} \prod_{\kappa=1}^r \left( \kappa - 3 + \frac{d}{2} \right) \int d^{d+2r} q \frac{f(q, \mu^2)}{D_{i_1} \cdots D_{i_k}}. \quad (2.18)$$

As already noticed in [42], eq. (2.16) is free of scalar pentagons.

3. Reduction algorithm

The one loop amplitude is completely known, provided the coefficients $c$ appearing the r.h.s. of eq. (2.16) are known. In this subsection we show how to get the coefficients of each polynomial $\Delta$ performing suitable series expansions.

Quintuple Cut

The coefficient $c_{5,0}^{(ijk\ell m)}$, eq. (2.8), can be computed using a quintuple cut. However its actual value it is not relevant in our reduction algorithm. Therefore its computation is omitted.

Quadruple cut

The solutions of the quadruple cut $D_i = \ldots = D_{\ell} = 0$, can be expressed as

$$q_{\pm}^{(ijk\ell)} = -p_i + x_1 e_1 + x_2 e_2 + x_\nu v \pm u v_\perp \quad u = \sqrt{\alpha_\perp + \frac{\mu^2}{v_\perp^2}} \quad (3.1)$$

where the coefficients $x_1, x_2, x_\nu$ and $\alpha_\perp$ are fully determined by the cut-conditions. The only two coefficients that are needed are obtained from [4, 35].

$$\frac{1}{2} \left[ \frac{N(q_+^{(ijk\ell)}, 0)}{\prod_{h \neq i,j,k,\ell} D_h(q_+^{(ijk\ell)}, 0)} + \frac{N(q_-^{(ijk\ell)}, 0)}{\prod_{h \neq i,j,k,\ell} D_h(q_-^{(ijk\ell)}, 0)} \right] = c_{4,0}^{(ijk\ell)}, \quad (3.2)$$

$$\left. \frac{N_+}{\prod_{h \neq i,j,k,\ell} D_{h,\pm}} \right|_{\mu^2 \to \infty} = c_{4,4}^{(ijk\ell)} \mu^4 + O(\mu^3). \quad (3.3)$$

Here and in the following we use the abbreviation

$$f_\pm \equiv f(q_\pm^{(ijk\ell)}, \mu^2), \quad (3.4)$$

omitting whenever possible the indices of the cuts $\{i, j, \ldots\}$ as well as the $\mu^2$ dependence.
Triple cut

The solutions of the triple-cut, \( D_i = D_j = D_k = 0 \) can be parametrized as,

\[
q_+^{(ijk)} = -p_i + x_1e_1 + x_2e_2 + te_3 + \frac{\alpha_0 + \mu^2}{t} e_4 ,
\]

\[
q_-^{(ijk)} = -p_i + x_1e_1 + x_2e_2 + \frac{\alpha_0 + \mu^2}{t} e_3 + te_4 ,
\]

where \( x_1, x_2 \) and \( \alpha_0 \) are frozen by the triple-cut conditions. The coefficients can be obtained in the large-\( t \) limit according to

\[
\frac{N_+}{\prod_{h \neq i,j,k} D_{h,+}} \bigg|_{t\to\infty} = -\left( c_{3,4}^{(ijk)} + c_{3,9}^{(ijk)} \mu^2 \right) t + c_{3,5}^{(ijk)} t^2 - c_{3,6}^{(ijk)} t^3 + \mathcal{O}(1) ,
\]

\[
\frac{N_-}{\prod_{h \neq i,j,k} D_{h,-}} \bigg|_{t\to\infty} = -\left( c_{3,1}^{(ijk)} + c_{3,8}^{(ijk)} \mu^2 \right) t + c_{3,2}^{(ijk)} t^2 - c_{3,3}^{(ijk)} t^3 + \mathcal{O}(1) ,
\]

\[
\frac{1}{2} \left[ \frac{N_+}{\prod_{h \neq i,j,k} D_{h,+}} + \frac{N_-}{\prod_{h \neq i,j,k} D_{h,-}} \right] \bigg|_{t\to\infty} = \left( c_{3,0}^{(ijk)} + c_{3,7}^{(ijk)} \mu^2 \right) + \mathcal{O} \left( \frac{1}{t} \right) + \Omega(t).
\]

The Bachmann-Landau symbol \( \Omega(t) \) denotes terms which are non-negligible with respect to \( t \) as \( t \to \infty \). Eq. (3.3) has been introduced in [33,35]. As explained in [12], the average over the two solutions cancel the contributions of the spurious coefficient of the boxes. Eqs. (3.7) and (3.8) determine the spurious coefficients of the triangles. They are independent of the spurious coefficients of the boxes which are \( \mathcal{O}(1) \) in the \( t \to \infty \) expansion.

Double cut

The solutions of the double cut \( D_i = D_j = 0 \) are parametrized as follows

\[
q_+^{(ij)} = -p_i + x_1e_1 + (\alpha_0 + x_1\alpha_1)e_2 + te_3 + \frac{\mu^2 + \beta_0 + \beta_1x_1 + \beta_2x_1^2}{2t} e_4 ,
\]

\[
q_-^{(ij)} = -p_i + x_1e_1 + (\alpha_0 + x_1\alpha_1)e_2 + \frac{\mu^2 + \beta_0 + \beta_1x_1 + \beta_2x_1^2}{2t} e_3 + te_4 ,
\]

where \( \alpha_i \) and \( \beta_i \) are kinematical factors determined by the cut conditions. The coefficients can be extracted from the large-\( t \) expansion,

\[
\frac{N_+}{\prod_{h \neq i,j} D_{h,+}} - \sum_{k \neq i,j} \frac{\Delta R_{ijk,+}}{D_{k,+}} \bigg|_{t\to\infty} = c_{2,0}^{(ij)} + c_{2,9}^{(ij)} \mu^2 + c_{2,1}^{(ij)} x_1 + c_{2,2}^{(ij)} x_1^2 + c_{2,5}^{(ij)} t + c_{2,6}^{(ij)} t^2 - c_{2,8}^{(ij)} x_1t + \mathcal{O} \left( \frac{1}{t} \right) ,
\]

\[
\frac{N_-}{\prod_{h \neq i,j} D_{h,-}} - \sum_{k \neq i,j} \frac{\Delta R_{ijk,-}}{D_{k,-}} \bigg|_{t\to\infty} = c_{2,0}^{(ij)} + c_{2,9}^{(ij)} \mu^2 + c_{2,1}^{(ij)} x_1 + c_{2,2}^{(ij)} x_1^2 + c_{2,3}^{(ij)} t + c_{2,4}^{(ij)} t^2 - c_{2,7}^{(ij)} x_1t + \mathcal{O} \left( \frac{1}{t} \right) .
\]
Eqs. (3.11) and (3.12) hold only if the uncut denominators are linear in $t$, namely
\[
D_{h,\pm} \xrightarrow{t \to \infty} 2e_{3,4} \cdot (p_h - p_i) \cdot t + O(1) \quad \forall h \neq i, j.
\] (3.13)
Therefore the momentum $K_2$, entering in the definition of $e_{3,4}$, has to be chosen so that
\[
(p_h - p_i) \cdot e_{3,4} \neq 0
\] (3.14)
for all $h \neq i, j$.

The terms involving the reduced residues $\Delta_{ijk}^R$ remove the contributions of the spurious three-points coefficients. The treatment of the subtraction terms is thus different from the one proposed in [33–35], where the spurious three-points contamination is removed by subtracting all possible triple cuts constructed from the double cut $(ij)$.

We remark that in general neither
\[
\frac{N_+}{\prod_{h \neq i,j} D_{h,\pm}} \quad \text{nor} \quad \frac{\Delta_{ijk,\pm}^R}{D_{k,\pm}}
\]
are polynomial in $t$ and $1/t$, but only their difference so it is. Instead their Laurent expansion has the same polynomial structure of the r.h.s. of eqs. (3.11) and (3.12). For the “+” case we have
\[
\bigg| \frac{N_+}{\prod_{h \neq i,j} D_{h,\pm}} \bigg|_{t \to \infty} = a_{2,0}^{(ij)} + a_{2,9}^{(ij)} \mu^2 + a_{2,1}^{(ij)} x_1 + a_{2,2}^{(ij)} x_1^2 +
\]
\[
- a_{2,5}^{(ij)} t + a_{2,6}^{(ij)} t^2 - a_{2,8}^{(ij)} x_1 t + O\left( \frac{1}{t} \right),
\] (3.15)
\[
\bigg| \frac{\Delta_{ijk,\pm}^R}{D_{k,\pm}} \bigg|_{t \to \infty} = b_{2,0}^{(ij|k)} + b_{2,9}^{(ij|k)} \mu^2 + b_{2,1}^{(ij|k)} x_1 + b_{2,2}^{(ij|k)} x_1^2 +
\]
\[
- b_{2,5}^{(ij|k)} t + b_{2,6}^{(ij|k)} t^2 - b_{2,8}^{(ij|k)} x_1 t + O\left( \frac{1}{t} \right).
\] (3.16)
The “−” case is obtained by replacing $(5, 6, 8) \to (3, 4, 7)$. Therefore in our algorithm the coefficients $a^{(ij)}$ and $b^{(ij|k)}$ can be computed separately, obtaining the coefficient $c^{(ij)}$ by their difference,
\[
c^{(ij)}_{2,m} = a^{(ij)}_{2,m} - \sum_{k \neq i,j}^{n-1} b^{(ij|k)}_{2,m}.
\] (3.17)
In other words the subtraction is implemented at the coefficient-level rather than at the integrand-level. Moreover, given the known structure of $\Delta_{ijk}^R$, the analytic expression of the coefficients $b^{(ij|k)}$ can be computed once and for all, in terms of the 3-point spurious coefficients, the corresponding basis, and the basis of the cut $(ij)$. The actual semi-numerical procedure has to be applied only to the term involving the numerator, in order to determine the coefficients $a^{(ij)}$. 


Single cut

We consider the following solution of the single cut $D_i = 0$,

$$q_+^{(i)} = -p_i + x_1 e_1 + \alpha_0 + \frac{\mu^2}{2x_1} e_2$$

(3.18)

with $\alpha_0$ fixed by the cut conditions. The coefficient $c_{1,0}^{(i)}$ is extracted from the large-$x_1$ limit,

$$\left[ \frac{N_+}{\prod_{h \neq i} D_{h,+}} - \sum_{j < k \neq i} \frac{\Delta^R_{ijk,+}}{D_{j,+}D_{k,+}} - \sum_{j \neq i} \frac{\Delta^R_{ij,+}}{D_{j,+}} \right] \xrightarrow{x_1 \to \infty} c_{1,0}^{(i)} + \mathcal{O} \left( \frac{1}{x_1} \right) + \Omega (x_1) \quad (3.19)$$

The symbol $\Omega(x_1)$ denotes terms which are not negligible with respect to $x_1$ as $x_1 \to \infty$. Eq. (3.19) holds only if the uncut denominators are linear in $x_1$,

$$D_{h,+} \xrightarrow{x_1 \to \infty} 2e_1 \cdot (p_h - p_i) x_1 + \mathcal{O}(1) , \quad \text{for} \ h \neq i \quad \text{.} \quad (3.20)$$

Therefore $K_1$ and $K_2$, entering the definition of the basis $e_{1,2}$, have to be chosen accordingly.

The contributions from the spurious two- and three-points coefficients are discarded subtracting the reduced residues $\Delta^R_{ij}$ and $\Delta^R_{ijk}$. The subtraction procedure differs from the one presented in [34], where the spurious 2- and 3-point contributions are removed subtracting the double and triple cuts constructed from the single cut $(i)$.

Also in this case we remark that

$$\left[ \frac{N_+}{\prod_{h \neq i} D_{h,+}} \right] \xrightarrow{x_1 \to \infty} a_{1,0}^{(i)} + \mathcal{O} \left( \frac{1}{x_1} \right) + \Omega (x_1) \quad (3.21)$$

and

$$\left[ \frac{\Delta^R_{ijk,+}}{D_{j,+}D_{k,+}} \right] \xrightarrow{x_1 \to \infty} b_{1,0}^{(ijk)} + \mathcal{O} \left( \frac{1}{x_1} \right) + \Omega (x_1) \quad (3.22)$$

The coefficients $a^{(i)}$, $b^{(ijk)}$, and $b^{(ij)}$ can be computed separately, and finally the 1-point coefficient read,

$$c_{1,0}^{(i)} = a_{1,0}^{(i)} - \sum_{j < k \neq i} b_{1,0}^{(ijk)} - \sum_{j \neq i} b_{1,0}^{(ij)} \quad \text{.} \quad (3.24)$$

The semi-numerical procedure has to be used to determine the coefficient $a_{1,0}^{(i)}$ only. Indeed, given the known structure of $\Delta^R_{ijk}$ and $\Delta^R_{ij}$, the parametric form of the coefficients $b_{1,0}^{(ijk)}$ and $b_{1,0}^{(ij)}$ is universal and can be computed once and for all.
4. Implementation

As shown in the previous section, our method requires, from quadruple- to single-cut, one-dimensional asymptotic expansions. In this section we present their semi-numerical implementation.

**Quadruple cut**

The coefficient $c_{4,4}^{(ijk\ell)}$ is computed performing a large $\mu^2$ expansion of

$$ \frac{N_+}{\prod_{h \neq i,j,k,\ell}^{n-1} D_h} , $$

 equation (3.3). Both $N$ and $D_h$ are polynomial in $u$

$$ N_+ = \sum_{i=1}^{r} f_i u^i = N(u v_{\perp}, v_{\perp}^2 u^2) + \mathcal{O}(u^{r-1}) , \quad D_{h,+} = d_{h,0} + d_{h,1} u . $$

By power counting, the coefficient $c_{4,4}^{(ijk\ell)}$ is non-vanishing only if $r = n$ and it is proportional to $f_r$,

$$ c_{4,4}^{(ijk\ell)} = \frac{f_r}{(v_{\perp}^2)^r \prod_{h \neq i,j,k,\ell}^{n-1} d_{h,1}} = \frac{f_r}{(v_{\perp}^2)^r \prod_{h \neq i,j,k,\ell}^{n-1} (2 p_h \cdot v_{\perp})} . $$

The coefficient $f_r$ can be obtained from the analytic expression of $N(u v_{\perp}, v_{\perp}^2 u^2)$. For instance this procedure can be easily implemented in FORMCALC and GoSAM which use the symbolic manipulations programs MATHEMATICA and/or FORM [43].

**Triple, double, and single cuts**

Along the reduction procedure, the integrand of the $n$-ple cut is a multivariate function of $(5 - n)$ variables, corresponding to the parameters of the loop momentum not fixed by the on-shell conditions. Each expansion is performed with respect to one variable only, say $\tau$.

The solution of the triple, of the double and of the single cut reads as follows

$$ q^\mu_{\text{cut}} = \frac{1}{\tau} \eta_\perp^\mu + \eta_0^\mu + \tau \eta_1^\mu , $$

in terms of the (cut-dependent) momenta $\eta_{\perp}, \eta_0,$ and $\eta_1$. For each cut the generic term to be expanded is a ratio of the type,

$$ \frac{F(\tau)}{\prod_{h=0}^{k-1} D_h(\tau)} , $$

where $D_h$ is an uncut propagators,

$$ D_h(\tau) = \frac{D_h(\tau)}{\tau} , \quad D_h(\tau) = \sum_{i=0}^{2} d_{h,i} \tau^i . $$

The function $F$ can either be the original numerator $N$ or any of the reduced residues $\Delta^R$. 


If $F$ is a reduced residue $\Delta^R$ the large $\tau$ expansion of eq. (4.5) is universal and can be performed analytically, cfr. section 3.

If $F$ is the numerator $N$ we have

$$ F(\tau) = \frac{\mathcal{F}(\tau)}{\tau^r}, \quad \mathcal{F}(\tau) = \tau^r N (q_{\text{cut}}, \mu^2) \equiv \sum_{i=0}^{2r} f_i \tau^i, \quad (4.7) $$

where $r$ is the rank of the numerator. The coefficients $f_i$ depend implicitly on the cut through the momenta $\eta_-, \eta_0$, and $\eta_1$,

$$ f_i = f_i (\eta_-, \eta_0, \eta_1). \quad (4.8) $$

Their parametric expression can be obtained from either the tensor structure of the integrand or the analytic form of the numerators, as provided by codes like FORMCALC and GoSam. In the $F = N$ case the large $\tau$ expansion is numerically implemented through polynomial division, as described below. In the following we assume $r \geq k$, otherwise the ratio (4.5) vanishes in the $\tau \to \infty$ limit.

Step 1. We start by dividing $\mathcal{F}$ by $D_0$ obtaining

$$ \frac{\mathcal{F}(\tau)}{D_0(\tau)} = Q_0(\tau) + \frac{R_0(\tau)}{D_0}. \quad (4.9) $$

The quotient $Q_0$ is a polynomial of degree $2r - 2$, while the remainder $R_0$ is a polynomial of degree one. In the large-$\tau$ limit, the contribution of the latter can be neglected, since

$$ \frac{R_0(\tau)}{\prod_{h=0}^{k-1} D_h(\tau)} \xrightarrow{\tau \to \infty} O \left( \frac{1}{\tau^{2k-1}} \right), \quad (4.10) $$

therefore

$$ \frac{\mathcal{F}(\tau)}{\prod_{h=0}^{k-1} D_h(\tau)} \xrightarrow{\tau \to \infty} \frac{Q_0(\tau)}{\prod_{h=1}^{k-1} D_h(\tau)}. \quad (4.11) $$

Step 2. We perform the division by the successive denominator $D_1$

$$ \frac{Q_0(\tau)}{D_1(\tau)} = Q_1(\tau) + \frac{R_1(\tau)}{D_1}. \quad (4.12) $$

The quotient $Q_1$ is a polynomial of degree $2r - 4$ while the remainder $R_1$ has degree one. In the large-$\tau$ limit, the contribution of $R_1$ drops out as well, hence

$$ \frac{\mathcal{F}(\tau)}{\prod_{h=0}^{k-1} D_h(\tau)} \xrightarrow{\tau \to \infty} \frac{Q_1(\tau)}{\prod_{h=2}^{k-1} D_h(\tau)}. \quad (4.13) $$

Last step. After reiterating this procedure over the remaining denominators, $D_2, \ldots, D_{k-1}$, we get

$$ \frac{\mathcal{F}(\tau)}{\prod_{h=0}^{k-1} D_h(\tau)} \xrightarrow{\tau \to \infty} Q_{k-1}(\tau) \equiv \sum_{i=0}^{2(r-k)} s_i \tau^i, \quad (4.14) $$
Finally, the Laurent expansion of eq. (4.5) is given by

\[
\frac{F(\tau)}{\prod_{h=0}^{k-1} D_h(\tau)} = \frac{\mathcal{F}(\tau)}{\tau^{r-k} \prod_{h=0}^{k-1} D_h(\tau)} = \sum_{i=0}^{r-k} s_{i+r-k} \tau^i + \mathcal{O}\left(\frac{1}{\tau}\right).
\]  

(4.15)

It is worth to notice that the large \( \tau \) expansion can be achieved by using a smaller polynomial

\[
\mathcal{F}^R(\tau) = \sum_{i=r+k}^{2r} f_i \tau^i,
\]

(4.16)

instead of the polynomial \( \mathcal{F} \) defined in eq. (4.7). Indeed

\[
\frac{F(\tau)}{\prod_{h=0}^{k-1} D_h(\tau)} = \frac{\mathcal{F}^R(\tau)}{\tau^{r-k} \prod_{h=0}^{k-1} D_h(\tau)} = \sum_{i=0}^{r-k} s_{i+r-k} \tau^i + \mathcal{O}\left(\frac{1}{\tau}\right).
\]  

(4.17)

We have implemented this algorithm in C++ and in Mathematica and verified its correctness reconstructing the integrands of up to sixth rank 6-point functions. Two numerical examples are described in the Appendix A. A complete implementation in GoSam and FormCalc is planned.

5. Example: reducing a second rank 3-point integrand

In this section we apply the reduction procedure described above considering a rank-two three-point integrand of the type

\[
\frac{N(q)}{D_0 D_1 D_2} = \frac{4(q \cdot v)(q \cdot w)}{D_0 D_1 D_2},
\]

(5.1)

where

\[
D_0 = q^2 - m^2, \quad D_1 = (q - k_1)^2 - m^2, \quad D_2 = (q + k_2)^2 - m^2.
\]

(5.2)

the “external” momenta \( k_1 \) and \( k_2 \) taken as massless. For simplicity we consider only the four-dimensional part of the reduction. The extension to \( d \)-dimensions is straightforward. For illustration purposes, we use polynomial division.

The cut (012)

In order to deal with relatively compact expressions we use the basis \( \{e_1, e_2, e_3, e_4\} \), where

\[
e_1^\mu = k_1^\mu, \quad e_2^\mu = k_2^\mu, \quad e_3^\mu = \frac{(1|\gamma^\mu|2)}{2}, \quad e_4^\mu = \frac{(2|\gamma^\mu|1)}{2}.
\]

(5.3)

The basis does not fulfill the normalization conditions \( e_1 \cdot e_2 = -e_3 \cdot e_4 = 1 \). Therefore the formulae (3.7–3.9) have to be modified performing the substitutions

\[
c_{3,i} \rightarrow (e_1 \cdot e_2)c_{3,i}, \quad \text{if } i = 1, 4, 8, 9; \\
c_{3,i} \rightarrow (e_1 \cdot e_2)^2 c_{3,i}, \quad \text{if } i = 2, 5; \\
c_{3,i} \rightarrow (e_1 \cdot e_2)^3 c_{3,i}, \quad \text{if } i = 3, 6.
\]
The solutions of the triple cut are
\[ q_+^{(012)} = t \, e_4 - \frac{m^2}{2(k_1 \cdot k_2)} \, e_4 \, , \quad q_-^{(012)} = t \, e_3 - \frac{m^2}{2(k_1 \cdot k_2)} \, e_3 \, . \] (5.4)

The functions appearing the l.h.s. of eqs. (3.7–3.9) are
\[ N_+ = -\frac{m^2}{2(k_1 \cdot k_2)} \left( \langle 1|v|2 \rangle \, \langle 2|w|1 \rangle + \langle 2|v|1 \rangle \, \langle 1|w|2 \rangle \right) + \langle 1|v|2 \rangle \, \langle 1|w|2 \rangle \, t^2 + O \left( \frac{1}{t^2} \right) \, , \] (5.5)
\[ N_- = -\frac{m^2}{2(k_1 \cdot k_2)} \left( \langle 1|v|2 \rangle \, \langle 2|w|1 \rangle + \langle 2|v|1 \rangle \, \langle 1|w|2 \rangle \right) + \langle 2|v|1 \rangle \, \langle 2|w|1 \rangle \, t^2 + O \left( \frac{1}{t^2} \right) \, , \] (5.6)
since all the propagators are cut. No polynomial division is needed and the coefficients can be immediately computed. The non vanishing ones are
\[ c_{3,0}^{(012)} = -\frac{m^2}{2(k_1 \cdot k_2)} \left( \langle 1|v|2 \rangle \, \langle 2|w|1 \rangle + \langle 2|v|1 \rangle \, \langle 1|w|2 \rangle \right) \, , \] (5.7)
\[ c_{3,2}^{(012)} = \frac{\langle 2|v|1 \rangle \, \langle 2|w|1 \rangle}{(k_1 \cdot k_2)^2} \, , \] (5.8)
\[ c_{3,5}^{(012)} = \frac{\langle 1|v|2 \rangle \, \langle 1|w|2 \rangle}{(k_1 \cdot k_2)^2} \, . \] (5.9)

The reduced residue reads as follows
\[ \Delta_{012}^R(q) = c_{3,5}^{(012)} \left( \frac{\langle 2|q|1 \rangle}{2} \right)^2 + c_{3,2}^{(012)} \left( \frac{\langle 1|q|2 \rangle}{2} \right)^2 \, . \] (5.10)

The cut (21)

The basis used for this cut is obtained from the momenta
\[ K_1^\mu = k_1^\mu + k_2^\mu \, , \quad K_2^\mu = -\frac{\langle 1|\gamma^\mu|2 \rangle}{2} - \frac{\langle 2|\gamma^\mu|1 \rangle}{2} \, , \] (5.11)
and its elements are
\[ e_1^\mu = \frac{1}{2} (K_1^\mu - K_2^\mu) \, , \quad e_2^\mu = \frac{1}{2} (K_1^\mu + K_2^\mu) \, , \quad e_3^\mu = \frac{\langle e_1|\gamma^\mu|e_2 \rangle}{2} \, , \quad e_4^\mu = \frac{\langle e_2|\gamma^\mu|e_1 \rangle}{2} \, . \] (5.12)

The element of the basis are not canonically normalized thus eqs. (3.11) and (3.12) have to be modified performing the substitutions
\[ c_{2,i} \rightarrow (e_1 \cdot e_2) c_{2,i} \, , \quad \text{if } i = 1, 3, 5 \, ; \]
\[ c_{2,i} \rightarrow (e_1 \cdot e_2)^2 c_{2,i} \, , \quad \text{if } i = 2, 4, 6, 7, 8 \, . \]

The solutions of the double cut (12) are
\[ q_+^{(21)} = -k_2 + x_1 e_1 + (1 - x_1) e_2 + t \, e_3 + \left( -\frac{m^2}{2(k_1 \cdot k_2)} + x_1 - x_1^2 \right) \frac{1}{t} e_4 \, , \] (5.13)
\[ q_-^{(21)} = -k_2 + x_1 e_1 + (1 - x_1) e_2 + t \, e_4 + \left( -\frac{m^2}{2(k_1 \cdot k_2)} + x_1 - x_1^2 \right) \frac{1}{t} e_3 \, . \] (5.14)

The coefficients \( c_{2,0}^{(21)}, c_{2,1}^{(21)} \) and \( c_{2,5}^{(21)} \) are obtained from eq. (3.11). The large \( t \) expansion is obtained performing the polynomial division with respect to \( t \), along the lines of section 4.
• **Contribution of the reduced residue** – The only reduced residue entering the subtractions is

\[
\frac{\Delta_{012}^R}{D_{0,+}} = b_{2,0}^{(21)(0)} + b_{2,1}^{(21)(0)} x_1 - b_{2,5}^{(21)(0)} t .
\] (5.15)

The coefficients \(b^{(21)(0)}\) are universal functions of the spurious coefficients and of kinematic invariants. In the case of a rank-2 three-point integrand with \(p_0 = 0\) they read as follows

\[
b_{2,0}^{(21)(0)} = \frac{-1}{4(e_3^{(21)} \cdot p_2)^2} \left\{ \left[ (p_2^2 + m_2^2 - m_0^2) - 2\alpha_0 \left( e_2^{(21)} \cdot p_2 \right) \right] \left( e_3^{(21)} \cdot e_3^{(012)} \right) e_3^{(012)} 
\right.
\]

\[
+ 2 \left[ c_{3,1}^{(012)} + 2\alpha_0 \left( e_2^{(21)} \cdot e_3^{(012)} \right) e_3^{(012)} \right] \left( e_3^{(21)} \cdot p_2 \right) \left( e_3^{(21)} \cdot e_3^{(012)} \right)
\]

\[
+ \left[ c_{3,1} \to c_{3,4}; c_{3,2} \to c_{3,5}; e_3^{(012)} \to e_4^{(012)} \right] \right\} = \frac{3 \langle 2[v] | 2[w] | 1 \rangle - \langle 1[v] | 2[w] | 2 \rangle}{4(k_1 \cdot k_2)} ,
\]

\[
b_{2,1}^{(21)(0)} = \frac{1}{2(e_3^{(21)} \cdot p_2)^2} \left\{ \left[ e_1^{(21)} \cdot p_2 + \alpha_1 e_2^{(21)} \cdot p_2 \right] \left( e_3^{(21)} \cdot e_3^{(012)} \right) 
\right.
\]

\[
- 2 \left( e_1^{(21)} \cdot e_3^{(012)} + \alpha_1 e_2^{(21)} \cdot e_3^{(012)} \right) \left( e_3^{(21)} \cdot p_2 \right) \left( e_3^{(21)} \cdot e_3^{(012)} \right) c_{3,2}^{(012)}
\]

\[
+ \left[ c_{3,2} \to c_{3,5}; e_3^{(012)} \to e_4^{(012)} \right] \right\} = \frac{\langle 1[v] | 2[w] | 2 \rangle - \langle 2[v] | 1[w] | 1 \rangle}{k_1 \cdot k_2} .
\]

\[
b_{2,5}^{(21)(0)} = \frac{c_{3,2}^{(012)} \left( e_3^{(21)} \cdot e_3^{(012)} \right)^2 + c_{3,5}^{(012)} \left( e_3^{(21)} \cdot e_4^{(012)} \right)^2}{2(e_3^{(21)} \cdot p_2)}
\]

\[
= \frac{\langle 1[v] | 2[w] | 2 \rangle + \langle 2[v] | 1[w] | 1 \rangle}{4(k_1 \cdot k_2)} .
\] (5.16)

• **Contribution of the numerator** – We perform the polynomial division to compute the large \(t\) expansion of the ratio

\[
\frac{N_+}{D_{0,+}} = \frac{\mathcal{F}(t)}{t D_0(t)} \quad t \to \infty \Rightarrow \frac{\mathcal{F}^R(t)}{t} = \frac{\mathcal{F}^R(t)}{t D_0(t)} .
\] (5.17)

The function \(\mathcal{F}\) defined in eq. (4.17) is given by \(\mathcal{F} = t^2 N_+\). The reduced polynomial \(\mathcal{F}^R\), eq. (4.16), is obtained from \(\mathcal{F}\) neglecting terms of \(O(t^2)\),

\[
\mathcal{F}^R(t) = f_4 t^4 + f_3 t^3 ,
\] (5.18)
where
\[ f_4 = \frac{1}{4} \left( \langle 1|v|1 \rangle - \langle 1|v|2 \rangle + \langle 2|v|1 \rangle - \langle 2|v|2 \rangle \right) \left( \langle 1|w|1 \rangle - \langle 1|w|2 \rangle + \langle 2|w|1 \rangle - \langle 2|w|2 \rangle \right) \]
\[ f_3 = \frac{1}{2} \left( \langle 1|v|1 \rangle - \langle 1|v|2 \rangle - \langle 2|v|1 \rangle - \langle 2|v|2 \rangle \right) \left( \langle 1|w|1 \rangle - \langle 1|w|2 \rangle - \langle 2|w|1 \rangle - \langle 2|w|2 \rangle \right) \]
\[ + \frac{1}{2} \left( \langle 1|v|1 \rangle - \langle 1|v|2 \rangle + \langle 2|v|1 \rangle - \langle 2|v|2 \rangle \right) \left( \langle 1|w|2 \rangle + \langle 2|w|1 \rangle \right) + (v \leftrightarrow w) \right] x_1. \]

The polynomial \( D_0 \) in the denominator is related to the propagator \( D_{0,+} \),
\[ D_0(t) = t \, D_{0,+} = d_{0,2} t^2 + d_{0,1} t + d_{0,0}, \quad (5.19) \]
with
\[ d_{0,2} = d_{0,1} = -(k_1 \cdot k_2), \quad d_{0,0} = (k_1 \cdot k_2) \left( \frac{m^2}{2k_1 \cdot k_2} - x_1 + x_1^2 \right). \quad (5.20) \]

In the notation of eq. (4.15), the result of the polynomial division reads,
\[ \frac{\mathcal{F}^R(t)}{D_0(t)} = s_2 t^2 + s_1 t + s_0 + \frac{R_0}{D_0} \implies \frac{N_+}{D_0} \xrightarrow{t \to \infty} \frac{\mathcal{F}^R(t)}{t D_0(t)} \xrightarrow{t \to \infty} s_2 t + s_1, \quad (5.21) \]
where
\[ s_2 = \frac{f_4}{d_{0,2}}, \quad \text{and} \quad s_1 = \frac{f_3 d_{0,2} - f_4 d_{0,1}}{d_{0,2}^2} = s_{1,0} + x_1 s_{1,1}. \quad (5.22) \]

By comparing eqs. (3.16) and (5.21) we get
\[ a_{2,0}^{(21)} = s_{1,0}, \quad a_{2,1}^{(21)} = \frac{s_{1,1}}{k_1 \cdot k_2}, \quad a_{2,5}^{(21)} = -\frac{s_2}{k_1 \cdot k_2} \quad (5.23) \]
while \( a_{2,2}^{(21)}, a_{2,6}^{(21)} \) and \( a_{2,8}^{(21)} \) vanish.

- **Computation of the coefficients** – The coefficients \( c_{21}^{(21)} \) are obtained subtracting the coefficients \( b_{21(0)}^{(21)} \) to the coefficients \( a_{21}^{(21)} \), according to eq. (3.17). The non-vanishing ones are:

\[ c_{2,0}^{(21)} = \frac{1}{4(k_1 \cdot k_2)} \left( \langle 1|v|2 \rangle \langle 1|w|1 \rangle + \langle 1|w|1 \rangle \langle 2|v|1 \rangle - \langle 1|w|2 \rangle \langle 2|v|1 \rangle \right) \]
\[ + \langle 1|w|1 \rangle \langle 2|v|2 \rangle - \langle 1|w|2 \rangle \langle 2|v|2 \rangle - \langle 2|v|2 \rangle \langle 2|w|1 \rangle \]
\[ - \frac{\langle 2|v|2 \rangle \langle 2|w|2 \rangle}{2} - \frac{\langle 1|v|1 \rangle \langle 1|w|1 \rangle}{2} \right) + (v \leftrightarrow w), \]
\[ c_{2,1}^{(21)} = \frac{1}{2(k_1 \cdot k_2)^2} \left( \langle 1|w|2 \rangle \langle 2|v|2 \rangle - \langle 1|v|2 \rangle \langle 1|w|1 \rangle - \langle 1|w|1 \rangle \langle 2|v|1 \rangle \right) \]
\[ + \langle 2|v|2 \rangle \langle 2|w|1 \rangle \right) + (v \leftrightarrow w), \]
\[ c_{2,5}^{(21)} = -\frac{1}{4(k_1 \cdot k_2)^2} \left( \langle 1|v|2 \rangle \langle 1|w|1 \rangle - \langle 1|w|1 \rangle \langle 2|v|1 \rangle + \langle 1|w|2 \rangle \langle 2|v|1 \rangle + \langle 1|w|1 \rangle \langle 2|v|2 \rangle \right). \]
The outcome is

\[-\langle 1|v|2\rangle \langle 2|v|2\rangle + \langle 2|v|2\rangle \langle 2|w|1\rangle - \frac{\langle 1|v|1\rangle 1\langle 1|w|1\rangle}{2} - \frac{2|v|2\rangle \langle 2|w|2\rangle}{2}\]

\[+ \left( v \leftrightarrow w \right).\]

The remaining non-vanishing coefficient, $c_{2,3}^{(21)}$, is obtained in a similar way, using eq. (5.12). The outcome is

\[c_{2,3}^{(21)} = -\frac{1}{4(k_1 \cdot k_2)^2} \left( \langle 1|w|1\rangle \langle 2|v|1\rangle + \langle 1|w|2\rangle \langle 2|v|1\rangle + \langle 1|w|1\rangle \langle 2|v|2\rangle - \langle 1|w|2\rangle \langle 2|v|2\rangle - \langle 1|v|2\rangle \langle 1|w|1\rangle - \langle 1|v|2\rangle \langle 1|w|2\rangle \right) + \left( v \leftrightarrow w \right). \tag{5.24}\]

The reduced polynomial $\Delta_{21}^R$ is given by

\[
\Delta_{21}^R = c_{2,1}^{(21)} + c_{2,3}^{(21)} + c_{2,5}^{(21)} (q + k_2) \cdot k_1 + \frac{(c_{2,1}^{(21)} - c_{2,3}^{(21)} - c_{2,5}^{(21)}(q \cdot k_2)}{2} + \frac{(c_{2,5}^{(21)} - c_{2,3}^{(21)} - c_{2,1}^{(21)}(1|q|2)}{2} + \frac{(c_{2,3}^{(21)} - c_{2,5}^{(21)} - c_{2,1}^{(21)}(2|q|1)}{2}}. \tag{5.25}\]

The cut (02)

The computation of the coefficients of the cut (02) proceeds along the lines described above. We define the basis using

\[K_1^{\mu} = k_2^{\mu}, \quad K_2^{\mu} = k_1^{\mu} + \frac{1}{2}(\gamma^{\mu}|2\rangle + \frac{2|\gamma^{\mu}|1\rangle}{2}, \tag{5.26}\]

and

\[e_1^{\mu} = K_1^{\mu}, \quad e_2^{\mu} = (K_1^{\mu} + K_2^{\mu}), \quad e_3^{\mu} = \frac{\langle e_1|\gamma^{\mu}|e_2\rangle}{2}, \quad e_4^{\mu} = \frac{\langle e_2|\gamma^{\mu}|e_1\rangle}{2}. \tag{5.27}\]

The solutions of the cut are:

\[q_+^{(02)} = x_1 e_1 + t e_3 - \frac{m^2}{2(k_1 \cdot k_2)} t e_4, \quad q_-^{(02)} = x_1 e_1 - \frac{m^2}{2(k_1 \cdot k_2)} t e_3 + t e_4. \tag{5.28}\]

The non-vanishing coefficients are

\[c_{2,1}^{(02)} = -\frac{\langle 2|v|2\rangle \langle 2|w|2\rangle}{2(k_1 \cdot k_2)^2} \tag{5.29}\]

\[c_{2,3}^{(02)} = \frac{\langle 2|v|2\rangle \langle 2|w|1\rangle + \langle 2|v|1\rangle \langle 2|w|2\rangle + \langle 2|v|2\rangle \langle 2|w|2\rangle}{2(k_1 \cdot k_2)^2} \tag{5.30}\]

\[c_{2,5}^{(02)} = \frac{\langle 1|w|2\rangle \langle 2|v|2\rangle + \langle 1|v|2\rangle \langle 2|w|2\rangle + \langle 2|v|2\rangle \langle 2|w|2\rangle}{2(k_1 \cdot k_2)^2}, \tag{5.31}\]

while the reduced residue reads as follows

\[\Delta_{02}^R = c_{2,1}^{(02)} q \cdot k_1 + \left( c_{2,1}^{(02)} + c_{2,3}^{(02)} + c_{2,5}^{(02)} \right) q \cdot k_2 + \frac{c_{2,1}^{(02)} + c_{2,5}^{(02)}(1|q|2)}{2} + \frac{c_{2,1}^{(02)} + c_{2,3}^{(02)}(2|q|1)}{2}. \tag{5.32}\]
The cut (2)

We parametrize the single cut solution (3.18) in the basis (5.12),

\[ q_+^{(2)} = -k_2 + x_1 e_1 + \frac{m^2}{(2 k_1 \cdot k_2)} \frac{1}{x_1} e_2. \]  

(5.33)

The large \( x_1 \) expansion in eq. (3.19) is obtained from the large \( x_1 \) expansion of the subtraction coefficients

\[ \frac{\Delta_{012}^R}{D_0 + D_1, +} \bigg|_{x_1 \to \infty} = b_{1,0}^{(2|01)} + O \left( \frac{1}{x_1} \right) + \Omega(x_1), \]

\[ \frac{\Delta_{01}^R}{D_1, +} \bigg|_{x_1 \to \infty} = b_{1,0}^{(2|1)} + O \left( \frac{1}{x_1} \right) + \Omega(x_1), \]

\[ \frac{\Delta_{02}^R}{D_0, +} \bigg|_{x_1 \to \infty} = b_{1,0}^{(2|0)} + O \left( \frac{1}{x_1} \right) + \Omega(x_1), \]  

(5.34)

and from the polynomial division of the ratio

\[ \frac{N_+}{D_0 + D_1, +} = a_{1,0}^{(2)} + O \left( \frac{1}{x_1} \right). \]  

(5.35)

The tadpole coefficients is

\[ c_{1,0}^{(2)} = a_{1,0}^{(2)} - b_{1,0}^{(2|01)} - b_{1,0}^{(2|1)} - b_{1,0}^{(2|0)} = \frac{1}{8(k_1 \cdot k_2)^2} \left( \langle 1|w|2 \rangle \langle 2|v|1 \rangle - \langle 1|w|1 \rangle \langle 2|v|1 \rangle \right) 
+ \langle 1|w|1 \rangle \langle 2|v|2 \rangle + 3 \langle 1|w|2 \rangle \langle 2|v|2 \rangle + 3 \langle 2|v|2 \rangle \langle 2|w|1 \rangle 
- \langle 1|v|2 \rangle \langle 1|w|1 \rangle \right) + \left( v \leftrightarrow w \right). \]  

(5.36)

6. Extended decomposition

In the previous sections we assumed to deal with a renormalizable theory, where the rank \( r \) of the numerator can not be greater than the number of external legs \( n \). In this section we extend the integrand decomposition to the case where the rank \( r \) can become larger than \( n \).

In general the residue of an \( m \)-point function is a multivariate polynomial in \( \mu^2 \) and the ISP’s characterizing the residue. Each monomial has to be irreducible and its maximum rank has to be at most \( (m + r - n) \). In the following we list the irreducible monomial entering each cut. For later convenience we give the decomposition of \( g^{\mu\nu} \) in terms of the basis (2.6) and of the vectors (2.7),

\[ g^{\mu\nu} = (e_1^{\mu} e_2^{\nu} + e_2^{\mu} e_1^{\nu}) - (e_3^{\mu} e_4^{\nu} + e_4^{\mu} e_3^{\nu}), \]  

(6.1)

\[ g^{\mu\nu} = (e_1^{\mu} e_2^{\nu} + e_2^{\mu} e_1^{\nu}) + \frac{v^{\mu} v^{\nu}}{v^2} + \frac{v_1^{\mu} v_1^{\nu}}{v_1^2}. \]  

(6.2)
• **Quintuple cut, $(ijkℓm)$** – The only irreducible monomial is $\mu^2$. Indeed the residue of the quintuple cut does not have ISP’s, thus the allowed monomials are $(\mu^2)^\alpha$. Moreover from eq. (6.1)

$$
(\mu^2)^\alpha = \left[D_i + m_i^2 - p_i^2 - 2(q \cdot p_i) - q^2\right]^\alpha
$$

$$
= \left[D_i + m_i^2 - p_i^2 - 2(q \cdot p_i) - 2(q \cdot e_1)(q \cdot e_2) + 2(q \cdot e_3)(q \cdot e_4)\right]^\alpha
$$

$$
= \text{constant terms} + \text{RSP’s},
$$

(6.3)

where the abbreviation “RSP’s” means “reducible scalar products”. This relation allows to express all the powers of $\mu^2$ in terms of a particular one, $(\mu^2)^{\alpha_0}$. As in the renormalizable case we choose $\alpha_0 = 1$ in order to decouple the contribution of the pentagons from the computation of the coefficients of the boxes.

• **Quadruple cut, $(ijkℓ)$** – The irreducible monomials are

$$
(\mu^2)^\alpha ((q + p_i) \cdot v_\perp)^\beta \quad \text{with } \beta = 0, 1 \text{ and } \alpha = 0, 1, 2, \ldots.
$$

(6.4)

Eq. (6.2) implies

$$
((q + p_i) \cdot v_\perp)^2 = v_\perp^2 \left(q^2 - 2((q + p_i) \cdot e_1)((q + p_i) \cdot e_2) - \frac{(q + p_i) \cdot v}{v_\perp}^2\right)
$$

$$
= \text{constant terms} + \text{terms in } \mu^2 + \text{RSP’s},
$$

therefore the terms with $\beta \geq 2$ are reducible.

• **Triple cut, $(ijk)$** – In this case the irreducible monomials are

$$
(\mu^2)^\alpha ((q + p_i) \cdot e_{3,4})^\beta \quad \text{with } \alpha, \beta = 0, 1, 2, \ldots.
$$

(6.5)

The monomials containing both $e_3$ and $e_4$ are reducible. Indeed from eq. (6.3)

$$
((q + p_i) \cdot e_3)((q + p_i) \cdot e_4) = \text{constant terms} + \text{terms in } \mu^2 + \text{RSP’s}.
$$

• **Double cut, $(ij)$** – The irreducible monomials are of the type

$$
(\mu^2)^\alpha ((q + p_i) \cdot e_{3,4})^\beta ((q + p_i) \cdot e_2)^\gamma \quad \text{with } \alpha, \beta, \gamma = 0, 1, 2, \ldots.
$$

(6.6)

As in the previous case, the monomials depending on both $e_3$ and $e_4$ are reducible.

• **Single cut, $(i)$** – The irreducible monomials read as follows

$$
(\mu^2)^\alpha ((q + p_i) \cdot e_{1,2})^\beta ((q + p_i) \cdot e_3)^\gamma ((q + p_i) \cdot e_4)^\delta \quad \text{with } \alpha, \beta, \gamma, \delta = 0, 1, \ldots
$$

(6.7)

Eq. (6.1) allows to write

$$
((q + p_i) \cdot e_1)((q + p_i) \cdot e_2) = ((q + p_i) \cdot e_3)((q + p_i) \cdot e_4)
$$

$$
+ \text{constant terms} + \text{terms in } \mu^2 + \text{RSP’s}.
$$

Therefore the terms containing both $e_1$ and $e_2$ do not enter the parametrization of the residue.
The residues $\Delta$ presented in section 2 are the most general polynomials with $r \leq n$ satisfying these requirements. Here we show, as an example, their extension to the case $r \leq n + 1$. In this case, the decomposition of the numerator has to be extended as follows:

$$
N(q, \mu^2) = \sum_{i<j<k}^{\text{r}} \Lambda_{ijk\ell m}(q, \mu^2) \prod_{h \neq i,j,k,\ell,m}^{\text{n-1}} D_h + \sum_{i<j}^{\text{r}} \Lambda_{ij}(q, \mu^2) \prod_{h \neq i,j}^{\text{n-1}} D_h + \sum_{i}^{\text{n-1}} \Lambda_i(q, \mu^2) \prod_{h \neq i}^{\text{n-1}} D_h , \tag{6.8}
$$

where the polynomials $\Lambda$ are defined as,

$$
\Lambda_{ijk\ell m}(q, \mu^2) = \Delta_{ijk\ell m}(q, \mu^2) ,
$$

$$
\Lambda_{ij}(q, \mu^2) = \Delta_{ij}(q, \mu^2) + c_{4,5}^{(ij)} \mu^4 (q + p_1) \cdot v_\perp ,
$$

$$
\Lambda_{ijk}(q, \mu^2) = \Delta_{ijk}(q, \mu^2) + c_{3,14}^{(ijk)} \mu^4 + c_{3,10}^{(ijk)} \mu^2 ((q + p_1) \cdot e_3)^2 + c_{3,11}^{(ijk)} ((q + p_1) \cdot e_4)^2 + c_{3,12}^{(ijk)} ((q + p_1) \cdot e_3)^4 + c_{3,13}^{(ijk)} ((q + p_1) \cdot e_4)^4 ,
$$

$$
\Lambda_{ij}(q, \mu^2) = \Delta_{ij}(q, \mu^2) + \mu^2 (c_{2,10}^{(ij)} (q + p_1) \cdot e_2 + c_{2,11}^{(ij)} (q + p_1) \cdot e_3 + c_{2,12}^{(ij)} (q + p_1) \cdot e_4 + c_{2,13}^{(ij)} ((q + p_1) \cdot e_2)^3 + c_{2,14}^{(ij)} ((q + p_1) \cdot e_3)^3 + c_{2,15}^{(ij)} ((q + p_1) \cdot e_4)^3 + c_{2,16}^{(ij)} ((q + p_1) \cdot e_2)^2 ((q + p_1) \cdot e_3) + c_{2,17}^{(ij)} ((q + p_1) \cdot e_2)^2 ((q + p_1) \cdot e_4) + c_{2,18}^{(ij)} ((q + p_1) \cdot e_2)(q + p_1) \cdot e_3)^2 + c_{2,19}^{(ij)} ((q + p_1) \cdot e_2)((q + p_1) \cdot e_4)^2 ,
$$

$$
\Lambda_i(q, \mu^2) = \Delta_i(q, \mu^2) + c_{1,5}^{(i)} ((q + p_1) \cdot e_1)^2 + c_{1,6}^{(i)} ((q + p_1) \cdot e_2)^2 + c_{1,7}^{(i)} ((q + p_1) \cdot e_3)^2 + c_{1,8}^{(i)} ((q + p_1) \cdot e_4)^2 + c_{1,9}^{(i)} ((q + p_1) \cdot e_1)(q + p_1) \cdot e_1) + c_{1,11}^{(i)} ((q + p_1) \cdot e_1)((q + p_1) \cdot e_3) + c_{1,12}^{(i)} ((q + p_1) \cdot e_2)((q + p_1) \cdot e_2) + c_{1,13}^{(i)} ((q + p_1) \cdot e_2)((q + p_1) \cdot e_4) + c_{1,14}^{(i)} \mu^2 + c_{1,15}^{(i)} ((q + p_1) \cdot e_3)((q + p_1) \cdot e_4) , \tag{6.9}
$$

The functions $\Delta$, appearing already in the case $r \leq n$, were given in Eqs. (2.8)–(2.13). We observe that the polynomial residues of 4-, 3-, 2-, and 1-point function acquire a richer structure, and a 0-point coefficient $c_0$ does appear. The latter coefficient is needed for the complete reconstruction of the integrand, but it is spurious. Indeed it multiplies a
scaleless integral, which vanish in dimensional regularization. The coefficient $c_0$ is not cut-constructible but it can be computed by inverting eq. (6.8) in correspondence to any value of $(q, \mu^2)$ not annihilating any propagator. It can be shown that the 0-point coefficient is present in the decomposition of rank-two 1-point integrals only. In the case of higher point integrals this term is absent, owing to mutual cancellation between different 1-point polynomials $\Lambda_i$ parametrised in terms of a common single-cut vector basis $\{e_1, e_2, e_3, e_4\}$.

We remark that according to our new algorithm, also in the case $r \leq n+1$ the residues of the 5-point functions and the spurious 4-point coefficients are not needed. Moreover, since the 0-point coefficient is spurious, the spurious coefficients of the 1-point functions are not needed as well. The other coefficients can be computed performing quadruple, triple, double and single cuts, using the series expansions described in section 3 and selecting the appropriate terms of the series. The new coefficients are obtained including higher order contributions in the expansions. The nice features of the method hereby discussed are not spoiled by the presence of higher rank numerators, and the coefficients of the 4-point functions do not affect the determination of the lower-point coefficients. We checked the validity of our procedure reconstructing the integrands of up to seventh rank 6-point functions.

The one loop $n$-point amplitude is obtained upon integration of Eq.(6.8). The outcome reads,

$$ A_n + \delta A_n, \quad (6.10) $$

where $A_n$ is given in eq. (2.16) and the new contribution $\delta A_n$ is

$$\begin{align*}
\delta A_n &= \sum_{i<j<k}^{n-1} c_{3,14}^{ijk} I_{ijk}[\mu^4] \\
&+ \sum_{i<j}^{n-1} \left\{ c_{2,13}^{ij} I_{ij}[((q + p_i) \cdot e_2)^3] + c_{2,10}^{ij} I_{ij}[\mu^2((q + p_i) \cdot e_2)] \right\} \\
&+ \sum_i^{n-1} \left\{ c_{1,14}^{i} I_i[\mu^2] + c_{1,15}^{i} I_i[((q + p_i) \cdot e_3)((q + p_i) \cdot e_4)] \right\}.
\end{align*}\quad (6.11)$$

The integral $I_{ij}[((q + p_i) \cdot e_2)^3]$ can be obtained from the analytic expression of the rank-3 bubble given in section 4 of [44]. The integrals $I_i[\mu^2], I_i[((q + p_i) \cdot e_3)((q + p_i) \cdot e_4)], I_{ij}[\mu^2((q + p_i) \cdot e_2)]$, and $I_{ijk}[\mu^4]$ are computed in Appendix B.

7. Conclusions

In this paper we presented a procedure for the semi-analytic reduction of one-loop scattering amplitudes in dimensional regularization, which exploits the asymptotic behavior of the integrand-decomposition.

The algorithm is based on a partial reconstruction of the numerator, where the coefficients of the master integrals are determined through a simplified integrand-reduction. Whenever necessary, the complete integrand reconstruction can be achieved as well.
The analytic informations allow to avoid the computation of 5-point coefficients and of the spurious 4-point ones. Moreover the 4-point non-spurious coefficients do not enter the determination of the lower-point ones. The integrand reduction algorithm is indeed required only for the coefficients of 3-, 2-, and 1-point functions.

The asymptotic expansion makes both the numerator and the subtraction terms separately polynomial. Therefore the subtraction of higher-point residues can be omitted during the reduction and replaced by coefficient-level corrective terms. The latter can be determined a priori from the Laurent expansion of the expression of the integrand-subtraction terms. Therefore the actual reduction algorithm applies only to the terms involving the numerator, whose reconstruction is achieved by polynomial division. The coefficients of the 3-, 2-, and 1-point functions are finally obtained as trivial combinations of the coefficients coming out of the polynomial division and the corrective coefficients.

This method exploits as much as possible the known analytic structure of the integrand, hence it relies on the analytic structure of the numerator and its asymptotic expansions, used as input. It has been implemented and tested in MATHEMATICA and in C++, using the polynomial division. The semi-analytic implementation in C++ has been designed as a reduction library to be linked to codes like GOSSAM and FORMCALC which generate analytic expressions for the integrands, as well as to any package providing the tensor-structure of the integrand.

We also discussed the extension of integrand-reduction methods to theories allowing for integrands with powers of the loop momentum larger then the number of denominators. We explicitly presented the extended polynomials in the case of powers larger by one unit than the number of denominators. The advantages of the method hereby discussed are not spoiled by the presence of higher rank numerators.

We are confident that the investigation of the asymptotic regimes can ameliorate the integrand decomposition of scattering amplitudes beyond one-loop as well.

Acknowledgments

We would like to thank Simon Badger, Gudrun Heinrich, Giovanni Ossola and Thomas Reiter for interesting discussions and feedback on the manuscript. This work was supported by the Alexander von Humboldt Foundation, in the framework of the Sofja Kovaleskaja Award, endowed by the German Federal Ministry of Education and Research.
A. Numerical examples

In this appendix we present two modest numerical applications of the hereby discussed algorithm. We compare the results with the ones obtained using the standard $d$-dimensional integrand reduction implemented in SAMURAI. A comprehensive comparison between the two algorithms is beyond the scope of this paper. We only intend to show the potential benefits arising from a lighter algorithm, which requires less coefficients and which uses subtraction at the coefficient-level rather than at the integrand-level.

**Rank six 6-point integrand**

We consider a six-point integrand of rank-6:

\[
\mathcal{I}_{6,6} = \frac{\mathcal{N}(q, \mu^2)}{\prod_{i=0}^{6} D_i}, \quad \mathcal{N}(q, \mu^2) = \prod_{i=1}^{6} (q \cdot r_i)
\]  

(A.1)

The momenta appearing the denominator $D_i$ are

\[
\begin{align*}
p_0^\mu &= (0, 0, 0, 0) \\
p_1^\mu &= (-56.6251094805816, 0, 0, -56.6251094805816) \\
p_2^\mu &= (-113.250218961163, 0, 0, 0) \\
p_3^\mu &= (-68.5281885958052, 33.5, 15.9, 25) \\
p_4^\mu &= (-48.768869887140, 21, 31.2, 25.3) \\
p_5^\mu &= (-27.9148705889889, 11, 13.2, 22)
\end{align*}
\]  

(A.2)
while all the masses are assumed to be vanishing. The momenta \( r_i \) are given by

\[
\begin{align*}
  r_1^\mu &= (1.30, 5.10, 0.50, 0.40) \\
  r_2^\mu &= (0.80, 1.00, 2.30, 2.50) \\
  r_3^\mu &= (1.90, 3.20, 1.77, 2.11) \\
  r_4^\mu &= (3.03, 1.05, 2.33, 1.77) \\
  r_5^\mu &= (3.56, 5.30, 3.09, 2.34) \\
  r_6^\mu &= (7.08, 1.98, 5.30, 4.55). 
\end{align*}
\]  (A.3)

The integrated result is given as a series in \( \epsilon = (4 - d)/2 \),

\[
\frac{(2\pi\mu^2)^{(4-d)}}{i\pi^2} \int d^d \bar{q} \mathcal{I}_{6,6} = \frac{a_{-2}}{\epsilon^2} + \frac{a_{-1}}{\epsilon^1} + a_0 + \mathcal{O}(\epsilon). 
\]  (A.4)

In Table I we show the numerical values of the coefficients \( a_i \) computed with the algorithm implemented in \textsc{samurai} and the new one. The two algorithms are in good agreement, but the new algorithm requires the determination of 386 coefficients, instead of the 461 required by the standard reduction.

We estimate the quality of the reconstruction of the numerator using the \textit{global} \((N = N)-\text{test}\) described in Section 3.4.1 of [18]. For this integrand, the reconstruction of the new algorithm is two digits more accurate than the one performed by \textsc{samurai}.

\textbf{Fermion-mass dependence of the 4-photons amplitude}

Because of the presence of products of denominators in Eq. (2.4), the numerical integrand reconstruction may become unstable if the internal masses are larger then the kinematic invariants [18, 27]. A simple example where such a situation may occur is the four-photon scattering in QED,

\[
0 \rightarrow \gamma(k_1, h_1) \gamma(k_2, h_2) \gamma(k_3, h_3) \gamma(k_4, h_4), 
\]  (A.5)

where \( k_i \) \((h_i)\) denotes the momentum (helicity) of the corresponding particle. We define \( s \equiv (k_1 + k_2)^2 \). The leading-order process proceeds via fermionic-loop, and in the case of a single fermion of mass \( m \), there are three independent helicity amplitudes

\[
\mathcal{A}(\pm, \pm, \pm, \pm), \quad \mathcal{A}(\pm, \pm, -, +), \quad \mathcal{A}(\pm, \pm, -, -),
\]  (A.6)

which are known analytically [45]. Mutual cancellations among contributions from different MI’s render the numerical evaluation of these amplitudes unstable, in particular when the ratio \( m^2/s \) becomes large. In this Appendix we explore the behaviour of the new algorithm in this kinematic regime, by comparing it to the one implemented in \textsc{samurai}. We consider the phase space point

\[
\begin{align*}
  k_1^\mu &= (-7.0, 0.0, 0.0, -7.0) \\
  k_2^\mu &= (-7.0, 6.1126608202785198, -0.8284979592001092, 3.3089226083172685) \\
  k_3^\mu &= (7.0, 6.1126608202785198, -0.8284979592001092, 3.3089226083172685) \\
  k_4^\mu &= (-7.0, -6.1126608202785198, 0.8284979592001092, -3.3089226083172703),
\end{align*}
\]  (A.7)

and we vary the numerical value of \( m \). In Figure II we plot the relative difference \( \delta \), defined as

\[
\delta \equiv \left| \frac{\mathcal{A}_{\text{num}} - \mathcal{A}_{\text{ana}}}{\mathcal{A}_{\text{ana}}} \right|,
\]  (A.8)

as a function of \( m^2/s \). \( \mathcal{A}_{\text{ana}} \) \((\mathcal{A}_{\text{num}})\) is the analytical \((\text{numerical})\) value of the amplitude \( \mathcal{A} \). For each helicity configuration the new algorithm seems to be less affected by this kind of inaccuracy than the one currently implemented in \textsc{samurai}.
Figure 1: Leading order contribution to the amplitude $A$. For each helicity configuration we plot the quantity $\delta$, eq. (A.8), as a function of $m^2/s$. The numerical evaluation of $A$ has been performed using either SAMURAI or the new algorithm.

B. Higher-rank integrals

In this appendix we compute the higher-rank master integrals appearing eq. (6.11),

$$I_i[\mu^2], \quad I_i[((q + p_i) \cdot e_3)((q + p_i) \cdot e_4)], \quad I_{ij}[\mu^2((q + p_i) \cdot e_2)], \quad I_{ijk}[\mu^4]. \quad (B.1)$$

The strategy is outlined in the Appendix I of [46]. The integrals (B.1) are obtained by projections, namely contracting appropriate tensors with the covariant decomposition of suitably chosen tensor integrals.

For later convenience we split the $d$-dimensional metric tensor $\hat{g}^{\mu\nu}$ into the 4-dimensional part, $g^{\mu\nu}$, an the $(-2\epsilon)$-dimensional part, $\bar{g}^{\mu\nu}$. We have

$$\hat{g}^{\mu\nu} = g^{\mu\nu} + \bar{g}^{\mu\nu}, \quad \bar{g}^{\mu\nu}\bar{g}_{\mu\nu} = -2\epsilon, \quad \bar{g}_{\mu\nu}q^{\mu}q^{\nu} = -\mu^2, \quad \bar{g}_{\mu\nu}k^{\mu} = 0, \quad (B.2)$$

for each 4-dimensional vector $k^{\mu}$.
**Integral** $I_i[\mu^2]$  

The covariant decomposition of a rank-2 tadpole is  
\[
I_i^{\mu \nu} = \int d^d \bar{q} \frac{(\bar{q} + p_i)\mu (\bar{q} + p_i)\nu}{D_i} = \hat{g}^{\mu \nu} A_{00} \tag{B.3}
\]

After the contraction with $\hat{g}^{\mu \nu}$ we have  
\[
I_i[\mu^2] = \int d^d \bar{q} \frac{\mu^2}{D_i} = 2 \epsilon A_{00} = \frac{i \pi^2 m_i^4}{2} + \mathcal{O}(\epsilon). \tag{B.4}
\]

The analytic expression of $A_{00}$ and of the other tensor coefficients appearing in this section can be found in [44].

**Integral** $I_i[(q + p_i) \cdot e_3][(q + p_i) \cdot e_4]$  

The expression of this integral can be obtained by contracting the covariant decomposition (B.3) with $e_3^\mu e_4^\nu$ and by using the relation $e_3 \cdot e_4 = -1$. The outcome is  
\[
I_i[(q + p_i) \cdot e_3][(q + p_i) \cdot e_4] = \int d^d \bar{q} \frac{(\bar{q} + p_i) \cdot e_3 ((q + p_i) \cdot e_4)}{D_i} = (e_3 \cdot e_4) A_{00}
\]
\[
= - \frac{m_i^2 I_i[\mu^2]}{4} + \mathcal{O}(\epsilon). \tag{B.5}
\]

**Integral** $I_{ij}[\mu^2 ((q + p_i) \cdot e_2)]$  

The covariant decomposition of a rank-3 bubble reads as follows  
\[
I_{ij}^{\mu \rho} = \int d^d \bar{q} \frac{(\bar{q} + p_i)^\mu (\bar{q} + p_i)^\rho (\bar{q} + p_i)^\nu}{D_i D_j D_k} = \left( \hat{g}^{\mu \nu} p_{ij}^\rho + \hat{g}^{\mu \rho} p_{ji}^\nu + \hat{g}^{\rho \nu} p_{ji}^\mu \right) B_{001}
\]
\[
+ p_{ij}^\mu p_{ji}^\nu p_{ji}^\rho B_{111}, \tag{B.6}
\]

with $p_{ji} \equiv p_j - p_i$. After the contraction with $\hat{g}^{\mu \nu} e_2^\rho$ we have  
\[
I_{ij}[\mu^2 ((q + p_i) \cdot e_2)] = \int d^d \bar{q} \frac{\mu^2 ((q + p_i) \cdot e_2)}{D_i D_j} = 2 \epsilon (p_{ji} \cdot e_2) B_{001}
\]
\[
= \frac{i \pi^2}{12} (p_{ji} \cdot e_2) (p_{ji}^2 - 2m_i^2 - 4m_j^2) + \mathcal{O}(\epsilon). \tag{B.7}
\]

**Integral** $I_{ijk}[\mu^4]$  

We start from the decomposition  
\[
I_{ijk}^{\mu \nu \rho \sigma} = \int d^d \bar{q} \frac{(\bar{q} + p_i)^\mu (\bar{q} + p_i)^\nu (\bar{q} + p_i)^\rho (\bar{q} + p_i)^\sigma}{D_i D_j D_k} = \left( \hat{g}^{\mu \nu} \hat{g}^{\rho \sigma} + \hat{g}^{\mu \rho} \hat{g}^{\nu \sigma} + \hat{g}^{\mu \sigma} \hat{g}^{\nu \rho} \right) C_{0000}
\]
\[
+ \text{(rank-4 tensors containing } p_{ji}, p_{jk} \text{)} \quad \tag{B.8}
\]

After the contraction with $\hat{g}^{\mu \nu} \hat{g}^{\rho \sigma}$ the tensors containing $p_{ji}, p_{jk}$ vanishes and we get  
\[
I_{ijk}[\mu^4] = \int d^d \bar{q} \frac{\mu^4}{D_i D_j D_k} = 4 \epsilon (\epsilon - 1) C_{0000}
\]
\[
= \frac{i \pi^2}{6} \left[ p_{jk}^2 + p_{ji}^2 + p_{ki}^2 \right] - m_i^2 - m_j^2 - m_k^2 + \mathcal{O}(\epsilon). \tag{B.9}
\]
References

[1] F. Cachazo, P. Svrcek, and E. Witten, MHV vertices and tree amplitudes in gauge theory, JHEP 09 (2004) 006, [hep-th/0403047].

[2] R. Britto, F. Cachazo, and B. Feng, New Recursion Relations for Tree Amplitudes of Gluons, Nucl. Phys. B715 (2005) 499–522, [hep-th/0412308].

[3] Z. Bern, L. J. Dixon, D. C. Dunbar, and D. A. Kosower, One-Loop n-Point Gauge Theory Amplitudes, Unitarity and Collinear Limits, Nucl. Phys. B425 (1994) 217–260, [hep-ph/9403226].

[4] R. Britto, F. Cachazo, and B. Feng, Generalized unitarity and one-loop amplitudes in N = 4 super-Yang-Mills, Nucl. Phys. B725 (2005) 275–305, [hep-th/0412103].

[5] G. Ossola, C. G. Papadopoulos, and R. Pittau, Reducing full one-loop amplitudes to scalar integrals at the integrand level, Nucl.Phys. B763 (2007) 147–169, [hep-ph/0609007].

[6] G. Ossola, C. G. Papadopoulos, and R. Pittau, Numerical evaluation of six-photon amplitudes, JHEP 0707 (2007) 085, [arXiv:0704.1271].

[7] R. K. Ellis, W. T. Giele, and Z. Kunszt, A Numerical Unitarity Formalism for Evaluating One-Loop Amplitudes, JHEP 03 (2008) 003, [arXiv:0708.2393].

[8] W. T. Giele, Z. Kunszt, and K. Melnikov, Full one-loop amplitudes from tree amplitudes, JHEP 0804 (2008) 049, [arXiv:0801.2237].

[9] R. Ellis, W. T. Giele, Z. Kunszt, and K. Melnikov, Masses, fermions and generalized D-dimensional unitarity, Nucl.Phys. B822 (2009) 270–282, [arXiv:0806.3467].

[10] C. F. Berger and D. Forde, Multi-Parton Scattering Amplitudes via On-Shell Methods, Ann.Rev.Nucl.Part.Sci. 60 (2010) 181–205, [arXiv:0912.3534].

[11] R. Britto, Loop Amplitudes in Gauge Theories: Modern Analytic Approaches, J.Phys.A A44 (2011) 454006, [arXiv:1012.4493]. 34 pages. Invited review for a special issue of Journal of Physics A devoted to 'Scattering Amplitudes in Gauge Theories'.

[12] R. Ellis, Z. Kunszt, K. Melnikov, and G. Zanderighi, One-loop calculations in quantum field theory: from Feynman diagrams to unitarity cuts, [arXiv:1105.4319].

[13] G. Passarino and M. J. G. Veltman, One Loop Corrections for $e^+ e^- \rightarrow \mu^+ \mu^-$ in the Weinberg Model, Nucl. Phys. B160 (1979) 151.

[14] G. van Oldenborgh and J. Vermaseren, New Algorithms for One Loop Integrals, Z.Phys. C46 (1990) 425–438.

[15] P. Mastrolia and G. Ossola, On the Integrand-Reduction Method for Two-Loop Scattering Amplitudes, JHEP 1111 (2011) 014, [arXiv:1107.6041].

[16] S. Badger, H. Frellesvig, and Y. Zhang, Hepta-Cuts of Two-Loop Scattering Amplitudes, [arXiv:1202.2019].

[17] G. Ossola, C. G. Papadopoulos, and R. Pittau, CutTools: a program implementing the OPP reduction method to compute one-loop amplitudes, JHEP 03 (2008) 042, [arXiv:0711.3596].

[18] P. Mastrolia, G. Ossola, T. Reiter, and F. Tramontano, Scattering AMplitudes from Unitarity-based Reduction Algorithm at the Integrand-level, JHEP 1008 (2010) 080, [arXiv:1006.0710].
[19] C. Berger, Z. Bern, L. Dixon, F. Febres Cordero, D. Forde, et. al., An Automated Implementation of On-Shell Methods for One-Loop Amplitudes, Phys.Rev. D78 (2008) 036003, arXiv:0803.4180.

[20] W. Giele and G. Zanderighi, On the Numerical Evaluation of One-Loop Amplitudes: The Gluonic Case, JHEP 0806 (2008) 038, arXiv:0805.2152.

[21] A. Lazopoulos, Multi-gluon one-loop amplitudes numerically, arXiv:0812.2996.

[22] J.-C. Winter and W. T. Giele, Calculating gluon one-loop amplitudes numerically, arXiv:0902.0094.

[23] A. van Hameren, C. Papadopoulos, and R. Pittau, Automated one-loop calculations: A Proof of concept, JHEP 0909 (2009) 106, arXiv:0903.4665.

[24] G. Bevilacqua, M. Czakon, M. Garzelli, A. van Hameren, Y. Malamos, et. al., NLO QCD calculations with HELAC-NLO, Nucl.Phys.Proc.Suppl. 205-206 (2010) 211–217, arXiv:1007.4918.

[25] S. Badger, B. Biedermann, and P. Uwer, NGLuon: A Package to Calculate One-loop Multi-gluon Amplitudes, Comput.Phys.Commun. 182 (2011) 1674–1692, arXiv:1011.2900.

[26] G. Bevilacqua, M. Czakon, M. Garzelli, A. van Hameren, A. Kardos, et. al., HELAC-NLO, arXiv:1110.1499.

[27] V. Hirschi, R. Frederix, S. Frixione, M. V. Garzelli, F. Maltoni, et. al., Automation of one-loop QCD corrections, JHEP 1105 (2011) 044, arXiv:1103.0621.

[28] G. Cullen, N. Greiner, G. Heinrich, G. Luisoni, P. Mastrolia, et. al., Automated One-Loop Calculations with GoSam, arXiv:1111.2034.

[29] S. Agrawal, T. Hahn, and E. Mirabella, FormCalc 7, arXiv:1112.0124.

[30] T. Hahn, Feynman Diagram Calculations with FeynArts, FormCalc, and LoopTools, PoS ACAT2010 (2010) 078, arXiv:1006.2231.

[31] R. Britto and B. Feng, Integral Coefficients for One-Loop Amplitudes, JHEP 02 (2008) 095, arXiv:0711.4284.

[32] R. Britto, B. Feng, and P. Mastrolia, Closed-Form Decomposition of One-Loop Massive Amplitudes, Phys. Rev. D78 (2008) 025031, arXiv:0803.1988.

[33] D. Forde, Direct extraction of one-loop integral coefficients, Phys. Rev. D75 (2007) 125019, arXiv:0704.1835.

[34] W. B. Kilgore, One-loop Integral Coefficients from Generalized Unitarity, arXiv:0711.5015.

[35] S. D. Badger, Direct Extraction Of One Loop Rational Terms, JHEP 01 (2009) 049, arXiv:0806.4609.

[36] D. Maitre and P. Mastrolia, S0M, a Mathematica Implementation of the Spinor-Helicity Formalism, Comput. Phys. Commun. 179 (2008) 501–574, arXiv:0710.5559.

[37] A. van Hameren, Multi-gluon one-loop amplitudes using tensor integrals, JHEP 0907 (2009) 088, arXiv:0905.1008.

[38] G. Heinrich, G. Ossola, T. Reiter, and F. Tramontano, Tensorial Reconstruction at the Integrand Level, JHEP 1010 (2010) 105, arXiv:1008.2441.
[39] F. Cascioli, P. Maierhofer, and S. Pozzorini, *Scattering Amplitudes with Open Loops*, arXiv:1111.5206.

[40] R. Pittau, A simple method for multi-leg loop calculations, *Comput. Phys. Commun.* 104 (1997) 23–36, hep-ph/9607309.

[41] Z. Bern and A. G. Morgan, Massive Loop Amplitudes from Unitarity, *Nucl. Phys.* B467 (1996) 479–509, hep-ph/9511336.

[42] K. Melnikov and M. Schulze, NLO QCD corrections to top quark pair production in association with one hard jet at hadron colliders, *Nucl.Phys.* B840 (2010) 129–159, arXiv:1004.3284.

[43] J. A. M. Vermaseren, *New features of FORM*, math-ph/0010025.

[44] A. Denner and S. Dittmaier, Reduction schemes for one-loop tensor integrals, *Nucl.Phys.* B734 (2006) 62–115, hep-ph/0509141.

[45] G. Gounaris, P. Porfyriadis, and F. Renard, The $\gamma\gamma \rightarrow \gamma\gamma$ process in the standard and SUSY models at high-energies, *Eur.Phys.J.* C9 (1999) 673–686, hep-ph/9902230.

[46] Z. Bern and G. Chalmers, Factorization in one loop gauge theory, *Nucl.Phys.* B447 (1995) 465–518, hep-ph/9503236.