Distributed Stochastic Nonsmooth Nonconvex Optimization

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Abstract—Distributed consensus optimization has received considerable attention in recent years; several distributed consensus-based algorithms have been proposed for (nonsmooth) convex and (smooth) nonconvex objective functions. However, the behavior of these distributed algorithms on nonconvex, nonsmooth and stochastic objective functions is not understood. This class of functions and distributed setting are motivated by several applications, including problems in machine learning and signal processing.

This paper presents the first convergence analysis of the decentralized stochastic subgradient method for such classes of problems, over networks modeled as undirected, fixed, graphs.

Index Terms—Distributed Subgradient Methods, Nonsmooth Optimization, Nonconvex Optimization, Optimization for Machine Learning.

I. INTRODUCTION

We consider the following nonsmooth unconstrained nonconvex optimization problem over a network of $n$ agents:

$$\min_{\theta \in \mathbb{R}^m} F(\theta) := \sum_{i=1}^{n} f_i(\theta),$$

where $f_i : \mathbb{R}^m \rightarrow \mathbb{R}$ is the cost function of agent $i$, known only to agent $i$. We make no assumptions about the smoothness or convexity of $F$; each $f_i$ is only assumed to be locally Lipschitz continuous, and thus in general it is nonconvex and nonsmooth. Furthermore, we assume that agent $i$ does not have access to the (elements of the) subgradient of its own $f_i$, but only unbiased stochastic estimates of the elements of the subgradient are available. Agents are connected through a communication network, modeled as a connected, undirected graph. No specific topology is assumed for the graph (such as star or hierarchical structure). In this setting, agents seek to cooperatively solve Problem (1) by exchanging information with their immediate neighbors in the network.

This class of problems and distributed setting arises naturally from many applications in different fields, including signal processing, statistical data analysis, machine learning, and engineering. For instance data may be collected and stored across different nodes and networks; and loss functions, regularizers, or risk measures that are nonsmooth are increasingly utilized in statistical data analysis [1], [2]. Alternatively, consider training Deep Neural Net architectures on data existing at different centers that may communicate across a network, or using distributed memory parallel architectures with high latency. These problems often involve large volumes of data and result in a loss function that is the finite sum of typically nonsmooth functions, due to the presence of rectified linear units, max-pooling, and other activations, or nonsmooth loss functions [3]. Clusters incorporating CPU cores each with its own distributed memory are common tools available to solve such problems [4]. Each worker (e.g., core) has access to its own storage of memory, and can communicate data to other workers as needed.

A. Related works

We are not aware of any result on the convergence of decentralized schemes for stochastic, nonsmooth, nonconvex problems in the form (1). There is a vast literature on distributed algorithms for deterministic (nonsmooth) convex problems; see, e.g., the tutorial papers [5], [6, Ch. 2 & 3], the earlier works [7], [8], and references therein. Distributed methods for nonconvex optimization have also received attention [9], [10], [11], [12], [13]. The schemes in [11], [12], [13] are applicable to unconstrained smooth nonconvex optimization, with [11] handling also compact constraints while [9], [10] can handle objectives with additive nonsmooth convex functions. Distributed algorithms for stochastic optimization problems over networks were proposed in [14], [15], [16], [17]; we group these papers as follows. The work [14] studied the effects of stochastic subgradient errors on the convergence of the distributed gradient projection algorithm [17] applied to convex, smooth, constrained optimization over undirected graphs. A distributed projected stochastic gradient algorithm (resp. distributed stochastic approximation algorithm) involving random gossip between agents and decreasing stepsize was studied in [15] for nonconvex, smooth, constrained optimization (resp. [16]); to deal with the nonconvexity of the objective, the analysis in [15] relies on stochastic approximation techniques introduced in [18]. Finally, [17] studied the effect of additive i.i.d. noise to the iterates of the push-sum gradient algorithm [8] applied to (deterministic) nonconvex, smooth optimization over digraphs.

B. Contributions

In this work, we introduce the first provably convergent distributed stochastic subgradient method solving Problem (1), over undirected graphs. The proposed algorithm can be considered as an extension of [15], in presenting the same setting of stochastic approximation for modeling the sequence of iterates, however, with the objective function not assumed to be continuously differentiable.
II. ASSUMPTIONS AND PRELIMINARIES

A. System model

We will assume that \( F(\theta) \) is continuous and subdifferentially regular [19]. We shall refer to the subgradient operator \( \partial f(\cdot) \) of any regular function \( f(\cdot) \) as the Clarke subgradient [20], defined, at a point \( \bar{x} \), as the convex hull of,

\[
\left\{ \frac{f(x_k) - f(\bar{x})}{x_k - \bar{x}}, x_k \to \bar{x} \right\}
\]

We note that by Rademacher’s Theorem [19] it holds that a subdifferentially regular function is continuously differentiable almost everywhere. Thus it can be said that \( F(\theta) \) is equal to a selection of one of a possibly infinite set of continuously differentiable functions.

The communication network of the agent is modeled as a fixed undirected graph \( G := (V, E) \) with vertices \( V \triangleq \{1, \ldots, I\} \) and \( E := \{(i, j) \mid i, j \in V\} \) representing the agents and communication links, respectively. We assume that the graph \( G \) is strongly connected.

Each agent \( i \) has access to and controls an estimate of the primal variables \( x_i^\nu \). We define the graph matrix \( L = I - W \) where \( W = A \otimes I \) with \( A \) satisfying \( A_{ij} > 0 \) for \( i \neq j \) if \( (i, j) \in E \) and \( A_{ii} = 0 \) otherwise.

We assume that \( L \) is double stochastic. The eigenvalues of \( L \) are real and can be sorted in a nonincreasing order \( 1 = \lambda_1(L) \geq \lambda_2(L) \geq \ldots \geq \lambda_n(L) \geq -1 \).

Defining,

\[
\beta \triangleq \max\{\|\lambda_2(L)\|, \|\lambda_n(L)\|\}
\]

we shall make the following assumption.

**Assumption 1.** It holds that,

\[ \beta < 1 \]

In addition, we assume that each agent \( i \) does not have access to the entire subgradient of its function, i.e., \( \partial f_i(x_i) \), but only has access to a stochastic oracle estimating some element of this set. In particular, we will assume that the following assumptions are made when using any noisy subgradient estimates \( y_i \) evaluated at \( x_i \).

**Assumption 2.** Each agent \( i \) has access to an oracle that returns \( y_i \) which may be written as,

\[
y_i = g_i + \delta M_i
\]

where \( g_i \in \partial f_i(x_i) \) and \( \delta M_i \) is a Martingale difference stochastic noise, and,

- \( \mathbb{E}[\delta M_i] = 0 \)
- \( \mathbb{E}[\|\delta M_i\|^2] \leq R \)
- \( \text{For every realization } \|y_i\| \leq B. \)

where \( R, B \in \mathbb{R}^+ \) are some constants.

Finally we make an assumption about the structure of the points of nonsmoothness. In particular, we consider that each \( f_i \) is defined to be the maximum of a set of smooth functions. Furthermore the set of activities, i.e., the active smooth function corresponding to the value of \( f_i(\cdot) \) at \( x \) does not significantly change across \( x \) in neighborhoods of arbitrarily small size for almost all \( x \). It can be seen that this assumption holds for the standard problems arising in estimation and data science.

**Assumption 3.** Each \( f_i(\cdot) \) can be defined as,

\[
f_i(x) = \max_{j \in C_i} f_{i,j}(x).
\]

It holds that \( f_{i,j} \) has Lipschitz continuous first derivatives, and the Lipschitz constants across all \( i \) and \( j \) are bounded uniformly by \( L \).

Define \( A_i(x) := \{j \in C_i : f_i(x) = f_{i,j}(x)\} \). For each \( i \) and every \( x \), either:

- \( \forall x_j \rightarrow x \) with \( x_j \neq x \), \( A_i(x) \neq A_i(x_j) \), or,
- \( \exists D \) such that for all \( x \), for all \( z \in B_\delta(x, D) \), it holds that \( A_i(x) = A_i(z) \), where \( B_\delta(x, D) \) is the open ball centered at \( x \) with radius \( D \).

The assumption implies, in particular that there exists a set \( Z_i \) of zero measure with respect to \( \mathbb{R}^n \) such that all the points satisfying the first condition are contained in \( Z_i \).

B. Some Examples

Consider training a deep neural network, which results in an objective function that is a composition of nested functions and activations, with a sum additive loss function at the final exterior, with training data \( z \), e.g.,

\[
F(\theta) = l(\phi(\theta, z), z),
\]

where, for instance, \( l \) could be an l1 loss, e.g., \( l(\phi(\theta, z), z) = \|\phi(\theta, z) - z\|_1 \), \( \phi_3 \) and \( \phi_4 \) could be sigmoids, i.e., \( \phi_j(\theta, z) = \frac{1}{1 + e^{-\theta_j z_j}} \), where we use the subscript \([·,]_j\) to indicate the components in the index set \( J \) of the vector inside, \( \phi_5 \) a Rectified Linear Unit, i.e., \( \phi_5(x) = \max(0, [\theta_j \cdot \|z\|_J]) \) and \( \phi_2, \phi_3, \phi_4 = \max(\phi_3, \phi_4) \). Notice the function is summable, but clearly nonconvex and nonsmooth, and also non-separable in variables (thus presenting no viable closed form prox solution).

Other examples of nonconvex nonsmooth functions can be found in, e.g., [21]. They include robust phase retrieval, covariance matrix estimation, blind deconvolution, sparse PCA and conditional value at risk. Note that all but the last one are immediately given as a sum of functions across data, thus if data is distributed across a network the setting applies. Conditional value at risk, if evaluated with sample average approximation, with the data on the different samples distributed, also becomes a summable distributed optimization problem.

III. PRELIMINARIES AND ALGORITHM

Define \( x \) to be the stack of vectors \( \{x_i\} \) and problem,

\[
\min_{x \in \mathbb{R}^n} F_d(x) = \sum_{i=1}^{n} f_i(x_i),
\]

to be an auxiliary optimization problem to facilitate the analysis of solving problem (1).

We present the Algorithm for this paper as Algorithm 1.

The primary step of the algorithm, given by (4) can be also given as,

\[
x^{\nu+1} = ((I_n - W) \otimes I_m) (x^{\nu} - \gamma^{\nu} y^{\nu}).
\]
We first present a necessary standing assumption for this section.

**Assumption 5.** For every agent $i$, the iterates $x_{(i)}^r$ are bounded almost surely.

Alternatively, one can introduce a compact set on which the iterates are constrained to lie in.

### A. Consensus

**Lemma 1.** The iterates reach consensus, i.e., for all $i$,

$$
\lim_{k \to \infty} \left\| x_k - x_{(i)}^r \right\| = 0
$$

**Proof.** Same as in [15, Lemma 1].

### B. Differential Inclusion and Stochastic Approximation

Let $G(\theta) = \partial F(\theta)$. The differential flow defined for the sequential subgradient method for minimizing $F(\theta)$ with arbitrarily small stepsizes is given by,

$$
\theta(t) \in -G(\theta) \tag{7}
$$

The update to $\bar{x}$ is given by,

$$
\bar{x}^{k+1} = (11^T \otimes I) ((I - W) \otimes I) (x^r - \gamma^r y) = \bar{x}^k - \frac{1}{\gamma} \sum_{i=1}^{m} (g_{i,k}(x_{(i)}^r) + \delta M_{i,k}) \tag{8}
$$

where

$$
g_{i,k}(x_{(i)}^r) \in \{ \nabla f_{i,j}(x_{(i)}^r) : j \in A_i(x_{(i)}^r) \}
$$

holds almost surely by Assumption [3].

Let us define,

$$
\bar{Y}^k = \frac{1}{\gamma} \sum_{i=1}^{m} (g_{i,k}(x_{(i)}^r) + \delta M_{i,k})
$$

$$
= \frac{1}{\gamma} \sum_{i=1}^{m} (g_{i,k}(\bar{x}^k) - g_{i,k}(x_{(i)}^r) + g_{i,k}(x_{(i)}^r) + \delta M_{i,k})
$$

$$
= g_{k}(\bar{x}^k) + \frac{1}{\gamma} \sum_{i=1}^{m} (g_{i,k}(x_{(i)}^r) - g_{i,k}(x_{(i)}^r) + \delta M_{k})
$$

$$
= g_{k}(\bar{x}^k) + \beta_k + \delta M_{k}
$$

Let $m(t)$ be the smallest integer greater than $t$. Let

$$
M^0(t) = \sum_{k=0}^{m(t)-1} \gamma^k \delta M_{k}
$$

and

$$
B^0(t) = \sum_{k=0}^{m(t)-1} \gamma^k \beta_k
$$

### C. Convergence

We recall the following Theorem, arising as [22, Theorem 5.6.3]

**Theorem 1.** Given a stochastic process,

$$
x_{k+1} = x_k + \gamma^r Y^k
$$

$$
Y^k = \gamma_k \bar{Y}^k + \beta_k + \delta M_{k}
$$

Define $M^0$ and $B^0$ as above.

**Assume,**

- $\mathbb{E}[Y^k] < \infty$
• \[ \lim_{k} \sup_{j \geq k} \max_{0 \leq t \leq T} \| M^0(jT + t) - M^0(jT) \| = 0, \quad \text{and}, \]
\[ \lim_{k} \sup_{j \geq k} \max_{0 \leq t \leq T} \| B^0(jT + t) - B^0(jT) \| = 0 \]
with probability one.

• \[ \lim_{\Delta \to 0} \lim_{k} \sup_{m(k + \Delta) \geq j \geq k} \frac{\| \gamma_j - \gamma_k \|}{\gamma_k} = 0 \]

• \( g_k(X) \) is continuous, \( G(X) \) is upper semicontinuous and,
\[ \lim_{k,j \to \infty} \text{dist}\left\{ \frac{1}{j} \sum_{l=k}^{k+j-1} g_l(X), G(X) \right\} = 0 \]

• \( X^k \) is bounded with probability one.

Then almost surely, limits of convergent subsequences of \( X^k \) are trajectories of the differential inclusion, \( X \in -G(X) \)
in some bounded internally chain transitive set and \( X^k \) converges to this set.

**Proof.** [22] Theorem 5.6.3 and 5.2.1]

We now apply this theorem to the process \( \hat{x}^k \) given by (3).

**Theorem 2.** [7] Applies to \( \hat{x}^k \) for the differential inclusion defined by (1), i.e., almost surely, limits of \( \hat{x}^k \) are trajectories of (4) and \( \hat{x}^k \) converges to an invariant set of this DI.

**Proof.** We shall see that the assumptions of Theorem 1 are satisfied for this process.

It holds that \( \mathbb{E}[X^k] < \infty \) and \( \hat{x}^k \) are bounded with probability one by assumption. Furthermore,
\[ \lim_{k} \sup_{j \geq k} \max_{0 \leq t \leq T} \| M^0(jT + t) - M^0(jT) \| = 0 \] w.p.1
by standard arguments regarding the Martingale difference noise (see, e.g., the proof of [22] Theorem 5.2.1]).

Next we have that, by Assumption 3 and the definition of \( g_i,k \),
\[ \| g_i,k(x^k_{(i)}) - g_i,k(\hat{x}^k) \| \leq L\| x^k_{(i)} - \hat{x}^k \| \to 0 \]
for all \( i \), thus \( \beta_k \to 0 \) implying, together with the step-size conditions, that
\[ \lim_{k} \sup_{j \geq k} \max_{0 \leq t \leq T} \| B^0(jT + t) - B^0(jT) \| = 0 \] w.p.1

Finally, we know that \( g_i,k(\cdot) \) are continuous and \( G(\cdot) \) is upper semicontinuous by definition.

Now, since \( x^k \) is a stochastic process with nonzero noise for all \( k \), it holds that there is a set of dense probability measure such that for all \( k > 0 \), \( x^k_{(i)} \notin Z \) and \( \hat{x}^k \notin Z \).

This implies that for all \( i \), since \( \| x^k_{(i)} - \hat{x}^k \| \to 0 \), that Assumption 3 implies,
\[ \lim_{k,j \to \infty} \sum_{l=k}^{k+j-1} \frac{1}{j} (A_i(\hat{x}^k) \neq A_i(x^k_{(i)})) \to 0 \]
and thus the fourth condition of the Theorem has been shown, and the results follow.

**D. Properties of Limit Points**

The previous sections showed that asymptotically \( \hat{x}^k \) are trajectories of the differential inclusion [7]. The proof of [22] Theorem 5.2.1] concludes that in the case of the presence of a compact constraint or an ODE instead of a DI, limit points of the sequence are thus stationary points of (1). In [21] Theorem 4.2] the argument was extended for an unconstrained nonsmooth function satisfying certain properties.

**Theorem 3.** [27] Theorem 4.2] If it holds that,

• The set of stationary points of (1) is dense, and
• For any trajectory \( z(t) \) of the DI (7), it holds that if \( z(0) \) is not stationary, there exists a \( T \) such that \( F(z(t)) < F(z(0)) \) for \( t \in (0, T] \).

then every limit point of \( \hat{x}^k \) is critical for \( F(\cdot) \) and the function values \( F(\hat{x}^k) \) converge.

By Assumption 3 the first condition holds. Second, note that the same Assumption defines a Whitney \( C^1 \)-stratification of the graph of \( F \) and thus by [21] Theorem 5.8] the second condition holds as well.

Finally by \( \| x^k_{(i)} - \hat{x}^k \| \to 0 \) we have that this convergence theorem holds for the individual iterates as well.

**V. Numerical Results**

We simulated Algorithm 1 on training a neural net architecture for the MNIST data set. We used a nonconvex loss function with an 11-regularizer, and two layers that included a softmax and a relu operator, with 100 nodes in the inner layer. Specifically, with \( \theta = (w, v, b, c) \) the parameters, \( A \) the training data and \( y \) the labels,
\[ f_i(\theta) = \| \phi(\theta, A_i) - y \|_1 + \lambda \| \theta \|_1 \]
\[ \phi(\theta, A) = \frac{1}{1 + e^{-\sum_j w_{ij} \phi_i(\theta, A_i) + y_j}} \]
\[ \psi(\theta, A_i) = \max(0, \sum_j u_{ij} A_i[i] + c_k) \]
We ran 8500 iterations using 50 agents with randomly generated connections at 50% for each potential edge. Each agent sampled 1% of its apportioned data set uniformly at each iteration to perform the update. We use \( \alpha = 0.1 \).

We show the results in Figure 1. We see that the iterates appear to be near-consensus. Furthermore, the norm of the (sub)gradients, evaluated at the average iterate among the agents is monotonically decreasing, along with the objective value (also evaluated at the average iterate). Given that diminishing step-sizes must be used, the convergence is slow.

**VI. Conclusion**

This paper presents an advancement in the state of the art for analysis of decentralized optimization schemes in extending the available convergence theory to nonsmooth, nonconvex problems, using stochastic updates. Using ideas from the control consensus literature and stochastic approximation theory, we were able to prove convergence of a simple procedure for a standard auxiliary problem, and bound the distance of its solution to a solution of the original problem. We demonstrated the efficacy of the procedure on a standard example in training DNN architecture. As this just begins the chapter in the analysis of such problems, there is considerable scope for future research extensions.
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Fig. 1. Consensus error, (sub)gradient norm at the average of the iterates, and the objective value per iteration.