On Antipodes Of Hom-Hopf algebras

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Abstract
In the recent definition of Hom-Hopf algebras the antipode $S$ is the relative Hom-inverse of the identity map with respect to the convolution product. We observe that some fundamental properties of the antipode of Hopf algebras and Hom-Hopf algebras, with the original definition, do not hold generally in the new setting. We show that the antipode is a relative Hom-anti algebra and a relative anti-coalgebra morphism. It is also relative Hom-unital, and relative Hom-counital. Furthermore if the twisting maps of multiplications and comultiplications are invertible then $S$ is an anti-algebra and an anti-coalgebra map. We show that any Hom-bialgebra map between two Hom-Hopf algebras is a relative Hom-morphism of Hom-Hopf algebras. Specially if the corresponding twisting maps are all invertible then it is a Hom-Hopf algebra map. If the Hom-Hopf algebra is commutative or cocommutative we observe that $S^2$ is equal to the identity map in some sense. At the end we study the images of primitive and group-like elements under the antipode.

1 Introduction

The examples of Hom-Lie algebras were first appeared in $q$-deformations of algebras of vector fields, such as Witt and Virasoro algebras [AS, CKL, CZ]. The concept of Hom-Lie algebras generalizes the one for Lie algebras where the Jocobi identity is twisted by a homomorphism [HLS, LS]. Hom-associative algebras were introduced and studied in [MS1]. Moreover Hom-coalgebras and Hom-bialgebras were studied in [MS2, MS3, Ya2, Ya3, Ya4]. In the last years, many classical algebraic concepts have been extended to the framework of Hom-structures. For examples see [HLS, GW, PSS, HSS, AEM, GMMP, GR, CQ, CS, ZZ]. The Hom-Hopf algebras first introduced in [MS2] and [MS3]. In these works they defined a Hom-Hopf algebra $H$ to be a Hom-bialgebra $(H, \mu, \alpha, \eta, \Delta, \beta, \varepsilon)$, endowed with a map $S : H \rightarrow H$, where it is the inverse of the identity map $\text{Id}_H$ with respect to the convolution product $\ast$, i.e,

$$S \ast \text{Id} = \text{Id} \ast S = \eta \circ \varepsilon.$$ 

This definition of antipode is the same as the one for Hopf algebras. The universal enveloping algebra of a Hom-Lie algebra introduced in [Ya4]. It has been shown that it is
a Hom-bialgebra. However it is not a Hom-Hopf algebra in the sense of [MS2], since it is shown in [LMT] that the antipode is not an inverse of the identity map with respect to the convolution product. This was a motivation to change definition of the antipode such that a Hom-Hopf algebra is a Hom-bialgebra which satisfies a weakened condition. For every $h \in H$ there exists $k \in \mathbb{N}$ satisfying the weakened condition

$$\alpha^k(S \ast \text{Id})(h) = \alpha^k(\text{Id} \ast S)(h) = \eta \circ \varepsilon(h).$$

This naturally suggests to change the definition of invertible elements of a Hom-algebra $A$ as being elements $a \in A$ such that there exists $b \in A$ and $k \in \mathbb{N}$ where $\alpha^k(ab) = \alpha^k(ba) = 1_A$. This means the antipode is the relative Hom-inverse of the identity map. In this paper we study this recent notion of Hom-Hopf algebras. More precisely by Definition 2.8 a Hom-Hopf algebra in the new setting is a Hom-bialgebra endowed with a unital, counital, anti-algebra and anti-coalgebra map $S : H \rightarrow H$ which is relative Hom-inverse of the identity map $\text{Id}_H$, and it commutes with $\alpha$. The Hom-Hopf algebras in Examples ?? and 2.17 satisfy the conditions of both definitions. The set of group-like elements and primitive elements are important to study Hopf type objects. The group-like elements gives a relation between Hom-Hopf algebras and Hom-groups while primitive elements connects Hom-Hopf algebras to Hom-Lie algebras. The authors in [LMT] showed that the set of group-like elements in a Hom-Hopf algebra is a Hom-group where the inverse elements are given by the antipode. In Example 2.16 we introduce a Hom-bialgebra containing a group-like element which does not have any inverse. Therefore it does not have any antipode or Hom-Hopf algebra structure. The main aim of this paper is to find out if one removes the important conditions unitality, counitality, anti-algebra map, anti coalgebra map, and $S \circ \alpha = \alpha \circ S$, from Definition 2.8 and only sticks with the relative Hom-invertibility condition of $S$, then how much of these properties can be recovered and what are the other properties of the antipode. To investigate this, we consider a Hom-bialgebra endowed with a map $S$ which is a relative Hom-inverse of the identity map. First we need to find out the relations between relative Hom-inverse elements with respect to the convolution product in Proposition 3.2. In Propositions 3.3 and 3.5 we show that the antipode is a relative Hom-anti-algebra and a relative Hom-anti-coalgebra morphism. It is also shown in Propositions 3.7 and 3.8 that the antipode is relative Hom-unital and relative Hom-counital. Furthermore if the twisting maps $\alpha$ and $\beta$ are invertible then $S$ is an anti-algebra and an anti-coalgebra map. Then in Proposition 3.11 we prove that any Hom-bialgebra map between two Hom-Hopf algebras is a relative Hom-morphism of Hom-Hopf algebras. By Corollary 3.12 if the corresponding twisting maps are all invertible then it is a Hom-Hopf algebra map. Furthermore we observe that if $\alpha = \beta$ then $S$ commutes with powers of $\alpha$. Later we study $S^2$ for commutative and cocommutative Hom-Hopf algebras. In these cases we prove that $S^2$ is equal to the identity map in some sense. If $\alpha$ and $\beta$ are invertible then $S^2 = 1_A$. At the end we study the images of primitive and group-like elements under the antipode.

**Notations:** In this paper all (Hom)-algebras, (Hom)-colagebras, (Hom)-bialgebras and (Hom)-Hopf algebras are defined on a field $\mathbb{K}$. All tensor products $\otimes$ are on a field $\mathbb{K}$. We
denote the set of natural numbers by \( \mathbb{N} \).

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# 2 Hom-Hopf algebras

In this section we recall the basics of Hom-algebras, Hom-coalgebras, Hom-bialgebras and Hom-Hopf algebras. To understand these structures we introduce some examples.

By [MS1], a Hom-associative algebra \( A \) over a field \( \mathbb{K} \) is a \( \mathbb{K} \)-vector space with a bilinear map \( m : A \otimes A \rightarrow A \), called multiplication, and a linear homomorphism \( \alpha : A \rightarrow A \) satisfying the Hom-associativity condition

\[
m \circ (m \otimes \alpha) = m \circ (\alpha \otimes m).
\]

In terms of elements \( a, b, c \in A \), this can be written as \( \alpha(a)(bc) = (ab)\alpha(c) \). The Hom-associativity property in terms of a commutative diagram is

\[
\begin{array}{c}
A \otimes A \otimes A \\
\downarrow \alpha \otimes m \\
A \otimes A
\end{array} \xrightarrow{m \otimes \alpha} \begin{array}{c} A \otimes A \\
\downarrow m \\
A
\end{array}
\]

A Hom-associative algebra \( A \) is called unital if there exists a linear map \( \eta : k \rightarrow A \) where \( \alpha \circ \eta = \eta \), and

\[
m \circ (\text{Id} \otimes \eta) = m \circ (\eta \otimes \text{Id}) = \alpha.
\]

The unit element of \( A \) is \( \eta(1_k) = 1_A \). These conditions in terms of an element \( a \in A \) can be written as \( \alpha(1_A) = 1 \) and \( a1_A = 1_Aa = \alpha(a) \). The unitality condition in terms of a commutative diagram is

\[
\begin{array}{c}
A \xrightarrow{\eta \otimes \text{Id}} A \otimes A \\
\downarrow m \\
A
\end{array} \xleftarrow{\eta \otimes \text{Id}} \begin{array}{c} A \otimes A \\
\downarrow m \\
A
\end{array} \xrightarrow{\alpha} \begin{array}{c} A \\
\downarrow \alpha \\
A
\end{array}
\]

In many examples \( \alpha \) is an algebra map, i.e., \( \alpha(xy) = \alpha(x)\alpha(y) \) for all \( x, y \in A \). When \( \alpha = \text{Id} \), then we obtain the definition of associative algebras.
Example 2.1. Let $A$ be an algebra with multiplication $m : A \otimes A \to A$, and $\alpha : A \to A$ be an algebra map. We twist the multiplication of $A$ by $\alpha$ to obtain a new multiplication $m_\alpha(x, y) = m(\alpha(x), \alpha(y))$. Then $(A, m_\alpha, \alpha)$ is a Hom-algebra.

Example 2.2. This example is a special case of the last example in [MS2]. In this example we define a 2-dimensional Hom-algebra $A$ with a basis $B = \{e_1, e_2\}$. We define the multiplication by

$$m(e_1, e_1) = e_1, \quad m(e_1, e_2) = m(e_2, e_1) = e_2, \quad m(e_2, e_2) = e_2.$$ 

We set $\alpha(e_1) = 2e_1 - e_2$ and $\alpha(e_2) = e_2$. This Hom-algebra is unital and commutative with the unit element $\eta(1) = e_1$.

An element $x$ in an unital Hom-associative algebra $(A, \alpha)$ is called Hom-invertible [LMT], if there exists an element $x^{-1}$ and a non-negative integer $k \in \mathbb{N}$ such that

$$\alpha^k(xx^{-1}) = \alpha^k(x^{-1}x) = 1.$$ 

The element $x^{-1}$ is called a Hom-inverse and the smallest $k$ is the invertibility index of $x$. The Hom-inverse may not be unique if it exists. However the authors in [LMT] showed that the unit element $1_A$ is Hom-invertible, the product of any two Hom-invertible elements is Hom-invertible and every inverse of a Hom-invertible element is Hom-invertible.

For two Hom-algebras $(A, \mu, \alpha)$ and $(A', \mu', \alpha')$ a linear map $f : A \to A'$ is called a Hom-algebra morphism if

$$f(xy) = f(x)f(y), \quad \text{and} \quad f(\alpha(x)) = \alpha'(f(x)), \quad \forall x, y \in A.$$

Now we recall the dual notion of a Hom-algebra which is called a Hom-coalgebra [MS2], [MS3]. A Hom-coalgebra is a triple $(A, \Delta, \beta)$, where $C$ is a $K$-vector space, $\Delta : C \to C \otimes C$ is linear map, called comultiplication, and $\beta : C \to C$ a linear map satisfying the Hom-coassociativity condition,

$$(\Delta \otimes \beta) \circ \Delta = (\beta \otimes \Delta) \circ \Delta.$$ 

If we use the Sweedler notation $\Delta(c) = c^{(1)} \otimes c^{(2)}$, then the coassociativity condition can be written as

$$\beta(c^{(1)}) \otimes c^{(2)(1)} \otimes c^{(2)(2)} = c^{(1)(1)} \otimes c^{(1)(2)} \otimes \beta(c^{(2)}).$$

The coassociativity property in terms of a commutative diagram is the dual of the one for the Hom-associativity of Hom-algebras as follows;

$$\begin{array}{c}
C \otimes C \otimes C \xrightarrow{\Delta \otimes \beta} C \otimes C \\
\beta \otimes \Delta \downarrow \quad \quad \quad \quad \quad \Delta \downarrow \\
C \otimes C \xleftarrow{\Delta} C
\end{array}$$
A Hom-coassociative coalgebra is said to be counital if there exists a linear map $\varepsilon : C \to K$ where
$$(\text{Id} \otimes \varepsilon) \circ \Delta = (\varepsilon \otimes \text{Id}) \circ \Delta = \beta.$$  
This means
$$c^{(1)} \varepsilon(c^{(2)}) = \varepsilon(c^{(1)}) c^{(2)} = \beta(c).$$  
Furthermore the map $\beta$ is counital, i.e, $\varepsilon(\beta(c)) = \varepsilon(c)$. The counitality condition in terms of a commutative diagram is

Moreover if the map $\beta$ is a coalgebra map then we have $\Delta \circ \beta = (\beta \otimes \beta) \circ \Delta$.

**Example 2.3.** This example is a special case of the last example in [MS2]. The Hom-algebra introduced in Example 2.2 is a Hom-coalgebra by
$$\Delta(e_1) = e_1 \otimes e_1, \quad \Delta(e_2) = e_1 \otimes e_2 + e_2 \otimes e_1 - 2e_2 \otimes e_2,$$$$
\varepsilon(e_1) = 1, \quad \varepsilon(e_2) = 0.$$  
We set $\beta(e_1) = e_1 + e_2$ and $\beta(e_2) = e_2$.

Let $(C, \Delta, \varepsilon, \beta)$ and $(C', \Delta', \varepsilon', \beta')$ be two Hom-coalgebras. A morphism $f : C \to C'$ is called a Hom-coalgebra map if for all $x \in C$ we have
$$f(x)^{(1)} \otimes f(x)^{(2)} = f(x^{(1)}) \otimes f(x^{(2)}), \quad f \circ \beta = \beta' \circ f.$$  
A $(\alpha, \beta)$-Hom-bialgebra is a tuple $(B, m, \eta, \alpha, \Delta, \varepsilon, \beta)$ where $(B, m, \eta, \alpha)$ is a Hom-algebra and $(B, \varepsilon, \Delta, \beta)$ is a Hom-coalgebra where $\Delta$ and $\varepsilon$ are morphisms of Hom-algebras, that is

i) $\Delta$ is a Hom-algebra map, $\Delta(hk) = \Delta(h)\Delta(k)$, which is
$$(hk)^{(1)} \otimes (hk)^{(2)} = h^{(1)}k^{(1)} \otimes h^{(2)}k^{(2)}, \quad \forall \ h, k \in B,$$

ii) $\Delta$ is unital; $\Delta(1) = 1 \otimes 1$.

iii) $\varepsilon$ is a Hom-algebra map; $\varepsilon(xy) = \varepsilon(x)\varepsilon(y)$.

iv) $\varepsilon$ is unital; $\varepsilon(1) = 1$.

v) $\varepsilon(\alpha(x)) = \varepsilon(x)$.

The algebra map property of $\Delta$ in terms of commutative diagrams is

$$B \otimes B \xrightarrow{\Delta m} B \otimes B$$

$$\Delta \otimes \Delta$$

$$B \otimes B \otimes B \otimes B \xrightarrow{id \otimes \tau \otimes id} B \otimes B \otimes B \otimes B$$
Here the linear map $\tau : B \otimes B \rightarrow B \otimes B$ is given by $\tau(h \otimes k) = k \otimes h$.

The map $\epsilon$ being an algebra morphism in terms of a commutative diagram means

\[
\begin{array}{ccc}
B \otimes B & \xrightarrow{m} & B \\
\downarrow{\epsilon \otimes \epsilon} & & \downarrow{\epsilon} \\
k \otimes k & \xrightarrow{} & k
\end{array}
\]

$k \otimes k = k$

**Remark 2.4.** It can be proved that $\Delta$ and $\epsilon$ are morphisms of unital Hom-algebras if and only if $m$ and $\eta$ are morphisms of Hom-coalgebras.

**Example 2.5.** The 2-dimensional Hom-algebra in Example 2.2 is a $(\alpha, \beta)$-Hom-bialgebra by the coalgebra structure given in Example 2.3. See [MS2].

**Example 2.6.** Let $(B, m, \eta, \alpha, \Delta, \epsilon)$ be a bialgebra and $\alpha : B \rightarrow B$ be a bialgebra map. Then $(B, m_\alpha = \alpha \circ m, \alpha, \eta, \Delta_\alpha = \Delta \circ \alpha, \epsilon, \alpha)$ is a $(\alpha, \alpha)$-Hom-bialgebra.

Let $(B, m, \eta, \alpha, \Delta, \epsilon, \beta)$ and $(B', m', \eta', \alpha', \Delta', \epsilon', \beta')$ be two Hom-bialgebras. A morphism $f : B \rightarrow B'$ is called a map of Hom-bialgebras of it is both morphisms of Hom-algebras and Hom-coalgebras. Let $(B, m, \eta, \alpha, \Delta, \epsilon, \beta)$ be a Hom-algebra. The authors in [MS2], [MS3], showed that $(\text{Hom}(B, B), \star, \gamma)$ is an unital Hom-algebra with $\star$ is the convolution product

$$f \star g = m \circ (f \otimes g) \circ \Delta,$$

and $\gamma \in \text{Hom}(B, B)$ is defined by $\gamma(f) = \alpha \circ f \circ \beta$. The unit is $\gamma \circ \epsilon$. Similarly if $(A, m, \eta, \alpha)$ and $(C, \epsilon, \Delta, \beta)$ are a Hom-algebra and a Hom-coalgebra, respectively, then $(\text{Hom}(C, A), \star, \gamma)$ is an unital Hom-algebra where $\star$ is the convolution product.

Here we recall the original definition of Hom-Hopf algebras.

**Remark 2.7.** The notion of Hom-Hopf algebras first was appeared in [MS2] and [MS3] as follows. A $(\alpha, \beta)$-Hom-bialgebra $(H, m, \eta, \alpha, \Delta, \epsilon, \beta)$ with an antipode $S : H \rightarrow H$ is called a $(\alpha, \beta)$-Hom-Hopf algebra. A map $S$ is called antipode if it is an inverse of the identity map $\text{Id} : H \rightarrow H$ in the Hom-associative algebra $\text{Hom}(H, H)$ with respect to the multiplication given by the convolution product, i.e. $S \star \text{Id} = \text{Id} \star S = \eta \circ \epsilon$. In fact for all $h \in H$ we have

$$S(h^{(1)})h^{(2)} = h^{(1)}S(h^{(2)}) = \epsilon(h)1.$$  

This is the same as usual definition of an antipode for Hopf algebras. The following properties of antipode of Hom-Hopf algebras with this definition were proved in [CG] and [MS2]. For all $x, y \in H$ we have:

i) If $\alpha = \beta$ then $S \circ \alpha = \alpha \circ S$.

ii) The antipode $S$ of a Hom-Hopf algebra is unique.

iii) $S$ is anti-algebra map, i.e, $S(xy) = S(y)S(x)$.
iv) $S$ is anti-coalgebra map, i.e, $S(x^{(1)}) \otimes S(x^{(2)}) = S(x^{(2)}) \otimes S(x^{(1)})$.

v) $S$ is unital, i.e, $S(1) = 1$.

vi) $S$ is counital, i.e, $\varepsilon(S(x)) = \varepsilon(x)$.

In this paper we use the recent notion of Hom-Hopf algebras introduced in [LMT].

**Definition 2.8.** [LMT] Let $(B, m, \eta, \alpha, \Delta, \varepsilon, \beta)$ be a $(\alpha, \beta)$-Hom-bialgebra. An anti-algebra, anti-coalgebra morphism $S : B \to B$ is said to be an antipode if

a) $S \circ \alpha = \alpha \circ S$.

b) $S \circ \eta = \eta$ and $\varepsilon \circ S = \varepsilon$.

c) $S$ is a relative Hom-inverse of the identity map $\text{Id} : B \to B$ for the convolution product, i.e, for any $x \in B$, there exists $k \in \mathbb{N}$ such that

$$
\alpha^k \circ (S \otimes \text{Id}) \circ \Delta(x) = \alpha^k \circ (\text{Id} \otimes S) \circ \Delta(x) = \eta \circ \varepsilon(x).
$$

(2.1)

A $(\alpha, \beta)$-Hom-bialgebra with an antipode is called a $(\alpha, \beta)$-Hom-Hopf algebra.

One notes that Definition 2.8(c) in terms of Sweedler notation can be written as follows:

$$
\alpha^k(S(x^{(1)})x^{(2)}) = \alpha^k(x^{(1)}S(x^{(2)})) = \varepsilon(x)1_B.
$$

(2.2)

**Remark 2.9.** There are some differences between the old definition of Hom-Hopf algebras in Remark 2.7 and the recent one in Definition 2.8. The Definition 2.8(a) in the special case of $\alpha = \beta$ is followed by the definition of Hom-Hopf algebras in Remark 2.7(i). Also Definition 2.8(b) is the result of the old definition in Remark 2.7(vi). Furthermore the antipodes of Hom-Hopf algebras in Definition 2.8 are the relative Hom-inverse of the identity map whereas the antipode in Remark 2.7 is actually the inverse of the identity map. Finally the antipode in Remark 2.7 is unique however the antipode in Definition 2.8 is not necessarily unique. In fact the authors in [LMT] proved that if $S$ and $S'$ are two antipodes for the Hom-Hopf algebra $H$ in the sense of Definition 2.8 then for every $x \in H$ there exist $k \in \mathbb{N}$ where

$$
\alpha^{k+2} \circ S \circ \beta^2(x) = \alpha^{k+2} \circ S' \circ \beta^2(x).
$$

In special case when $\alpha$ and $\beta$ are both invertible then $S = S'$ and the antipode is unique.

**Proposition 2.10.** Any Hom-Hopf algebra $(H, m, \eta, \alpha, \Delta, \varepsilon, \beta, S)$ in the sense of Remark 2.7 which satisfies the extra condition $S \circ \alpha = \alpha \circ S$, is a Hom-Hopf algebra in the sense of Definition 2.8 where $k = 1$ for all elements $x \in H$.

**Proof.** By Remark 2.7 the antipode $S$ is an unital, counital, anti-algebra, and anti-coalgebra map. Therefore for $k = 1$ it satisfies all the conditions of Hom-Hopf algebras in Definition 2.8. 

**Corollary 2.11.** If for Hom-Hopf algebra $(H, m, \eta, \alpha, \Delta, \varepsilon, \beta, S)$ in the sense of Remark 2.7 satisfies $\alpha = \beta$ then $H$ is a Hom-Hopf algebra in the sense of Definition 2.8.
Proof. If $\alpha = \beta$ then $\alpha$ is a map of Hom-bialgebras and by Remark 2.7 we have $\alpha \circ S = S \circ \alpha$. Now the result is followed by the previous Proposition.

A Hom-Hopf algebra is called commutative if it is commutative as Hom-algebra and it is called cocommutative if it cocommutative as Hom-coalgebra.

**Example 2.12.** Let $(H, m, \eta, \alpha, \Delta, \varepsilon, \beta, S)$ and Let $(K, m', \eta', \alpha', \Delta', \varepsilon', \beta', S')$ be two Hom-Hopf algebras. Then $H \otimes K$ is also a Hom-Hopf algebra by multiplication $m \otimes m'$, unit $\eta \otimes \eta'$, and $\alpha \otimes \alpha': H \otimes H \rightarrow H \otimes H$, the coproduct $\Delta \otimes \Delta'$ and counit $\varepsilon \otimes \varepsilon'$ and the linear map $\beta \otimes \beta': H \otimes H \rightarrow H \otimes H$.

**Definition 2.13.** Let $(H, m, \eta, \alpha, \Delta, \varepsilon, \beta, S)$ be a $(\alpha, \beta)$-Hom-Hopf algebra. An element $h \in H$ is called a group-like element if $\Delta(h) = h \otimes h$ and $\beta(h) = h$. Therefore $\varepsilon(h) = 1$.

One notes that the authors in [LMT] introduced group-like elements with condition $\varepsilon(h) = 1$ which in fact implies $\beta(h) = h$. Therefore their definition is equivalent to the one in this paper. However we preferred to have $\beta(h) = h$ as definition and similar as ordinary Hopf algebras the condition $\varepsilon(h) = 1$ is the result of the fact that $h$ is a group-like element. The notion of Hom-groups introduced in [LMT] and studied in [H]. For any Hom-group $(G, \alpha)$, the author in [H] introduce the Hom-group algebra $K G$.

**Proposition 2.15.** For any Hom-group $(G, \alpha)$, the Hom-group algebra $K G$ is a $(\alpha, \text{Id})$-Hom-Hopf algebra.

Proof. We define the coproduct by $\Delta(g) = g \otimes g$, counit by $\varepsilon(g) = 1$, and the antipode by $S(g) = g^{-1}$. Since $\beta = \text{Id}$ one verifies that $K G$ is a $(\alpha, \text{Id})$-Hom-bialgebra. Since for any Hom-group we have $\alpha(g) g^{-1} = \alpha(g^{-1})$ then $S(\alpha(g)) = \alpha(S(g))$. The unit element of $K G$ is $1_G$ and therefore $S(1_G) = 1_G^{-1} = 1_G$. Furthermore $\varepsilon(S(g)) = \varepsilon(g^{-1}) = 1$. Finally if the invertibility index of $g \in G$ is $k$ then

$$\alpha^k(S(g)g) = \alpha^k(g^{-1}g) = 1.$$  

One notes that all elements $g \in K G$ are group-like elements. Also $K G$ is a cocommutative Hom-Hopf algebra. If $G$ is an abelian Hom-group then $K G$ is a commutative Hom-Hopf algebra. The authors in [LMT], proved that set of group-like elements of a Hom-Hopf algebra is a Hom-group. In the Hom-bialgebra structure of $K G$ one can define the comultiplication by $\Delta(g) = \alpha(g) \otimes \alpha(g)$ to obtain a $(\alpha, \alpha)$-Hom-bialgebra.

**Example 2.16.** (Hom-bialgebra of quantum matrices) In this example we study a 4-dimensional Hom-bialgebra which is not a Hom-Hopf algebra. First we recall the construction of quantum matrices from [ES], [K], [M], [S]. Let $q \in \mathbb{K}$
where $q \neq 0$ and $q^2 \neq -1$. Let $\mathcal{O}_q(M_2) = \mathbb{K}[a, b, c, d]$ be the polynomial algebra with variables $a, b, c, d$ satisfying the following relations

\[
ab = q^{-1}ba, \quad bd = q^{-1}db, \quad ac = q^{-1}ca, \quad cd = q^{-1}dc
\]

\[
bc = cb, \quad ad - da = (q^{-1} - q)bc.
\]

Clearly $\mathcal{O}_q(M_2)$ is not commutative except $q = 1$.

We define a coproduct as follows.

\[
\Delta(a) = a \otimes a + b \otimes c, \quad \Delta(b) = a \otimes b + b \otimes d
\]

\[
\Delta(c) = c \otimes a + d \otimes c, \quad \Delta(d) = c \otimes b + d \otimes d
\]

If we consider the elements of $\mathcal{O}(M_2)$ as $2 \times 2$ matrices with entries in $\mathbb{K}$ then

\[
\begin{bmatrix}
\Delta(a) & \Delta(b) \\
\Delta(c) & \Delta(d)
\end{bmatrix}_{\mathcal{O}(M_2)} = \begin{bmatrix}
a & b \\
c & d
\end{bmatrix} \otimes \begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\]

This comultiplication is not cocommutative. We define the counit by

\[
\varepsilon(a) = \varepsilon(d) = 1, \quad \varepsilon(b) = \varepsilon(c) = 0.
\]

This coproduct and counit defines a bialgebra structure on $\mathcal{O}(M_2)$. Now we explain the Hom-bialgebra structure from [Ya1]. We define a bialgebra map $\alpha : \mathcal{O}(M_2) \rightarrow \mathcal{O}(M_2)$ by

\[
\alpha(a) = a, \quad \alpha(b) = \lambda b, \quad \alpha(c) = \lambda^{-1}c, \quad \alpha(d) = d.
\]

where $\lambda \in \mathbb{K}$ is any invertible element. In fact

\[
\alpha \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} \alpha(a) & \alpha(b) \\ \alpha(c) & \alpha(d) \end{bmatrix} = \begin{bmatrix} a & \lambda b \\ \lambda^{-1}c & d \end{bmatrix}
\]

It can be verified that $\alpha$ is a bialgebra morphism. One notes that $\varepsilon \circ \alpha = \varepsilon$. Now we use $\alpha$ to twist both product and coproduct of $\mathcal{O}(M_2)$ as explained in Example 2.6 to obtain a $(\alpha, \alpha)$- Hom-bialgebra $\mathcal{O}_q(M_2)_\alpha$. Therefore the coproduct of $\mathcal{O}_q(M_2)_\alpha$ is

\[
\Delta(a) = a \otimes a + b \otimes c, \quad \Delta(b) = \lambda a \otimes b + \lambda b \otimes d
\]

\[
\Delta(c) = \lambda^{-1}c \otimes a + \lambda^{-1}d \otimes c, \quad \Delta(d) = c \otimes b + d \otimes d
\]

In fact
\[
\begin{bmatrix}
\Delta(a) & \Delta(b) \\
\Delta(c) & \Delta(d)
\end{bmatrix}_{O_q(M_2)_\alpha} = \begin{bmatrix}
a & \lambda b \\
\lambda^{-1}c & d
\end{bmatrix} \otimes \begin{bmatrix}
a & \lambda b \\
\lambda^{-1}c & d
\end{bmatrix}
\]

Now we consider quantum determinant element

\[\det_q = \mu_\alpha(a, d) - q^{-1}\mu_\alpha(b, c) \in O_q(M_2)_\alpha.\]

One notes that

\[\det_q = \alpha(a)d - q^{-1}\alpha(b)c = ad - q^{-1}\alpha(b)(\lambda^{-1}c) = ad - q^{-1}bc.\]

Similarly \(\alpha(\det_q) = \det_q\). Therefore

\[\Delta_{O_q(M_2)_\alpha}(\det_q) = \Delta(\det_q) = \Delta(\det_q).\]

It is shown in [K] and [S] that \(\Delta(\det_q) = \det_q \otimes \det_q\) which means \(\det_q\) is a group-like element. Also \(\varepsilon_{O(M_2)_\alpha} = \varepsilon_{O(M_2)}\). Therefore \(\varepsilon(ad - q^{-1}bc) = 1\). Then \(\det_q\) is a group-like element of Hom-bialgebra \(O_q(M_2)_\alpha\). Since the set of group-like elements of a Hom-Hopf algebras is a Hom-group [LMT], then every group-like element is relative Hom-invertible. However \(\det_q\) is not clearly relative Hom-invertible by definition of \(\alpha\). Therefore \(O_q(M_2)\) is not a Hom-Hopf algebra.

**Example 2.17.** The 2-dimensional bialgebra \(H\), in Example 2.5, is a \((\alpha, \beta)\)-Hom-Hopf algebra by

\[S(e_1) = e_1, \quad S(e_2) = e_2.\]

It is straightforward to check that \(S(h^{(1)}h^{(2)} = h^{(1)}S(h^{(2)}) = \varepsilon(h)1\) for all \(h \in H\). Therefore it is a Hom-Hopf algebra in the sense of Remark 2.7. Since \(S = \text{Id}\) then \(S \circ \alpha = \alpha \circ S\) and \(H\) is also Hom-Hopf algebra in the sense of Definition 2.8.

### 3 Properties of antipodes

In this section we study the properties of antipods for Hom-Hopf algebras. We remind that we are using Definition 2.8. We need the following basic properties of convolution product for later results.

**Remark 3.1.** Let \((H, m, \eta, \alpha, \Delta, \varepsilon, \beta)\) be a \((\alpha, \beta)\)-Hom-bialgebra. We consider the convolution Hom-algebra \(\text{Hom}(H, H)\). If \(f, g \in \text{Hom}(H, H)\), then the authors in [LMT], showed that

i) \(\alpha^n(f \ast g) = \alpha^n f \ast \alpha^n g\).

ii) \(f \ast (\eta \circ \varepsilon) = \alpha \circ f \circ \beta = (\eta \circ \varepsilon) \ast f\).

The following Proposition will play an important role for the further results in this paper.
Let \( k \) and \( \beta \).

**Proof.** For \( x \) with \( xy \) in the seventh equality, unitality of the Hom-associativity in the fifth equality, the Hom-coassociativity in the sixth equality, the previous proposition shows that the relative Hom-inverses are unique in some sense.

The previous proposition shows that the relative Hom-inverses are unique in some sense. In fact if \( \alpha \) and \( \beta \) are invertible then \( f = g \).

Let \( k = \max(k', k'') \). We ignore the composition sign for easier computation. We have

\[
\alpha^{k+2} f \beta^2 = \alpha' (\alpha^{k+1} f \beta) \beta
\]

\[
= (\alpha^{k+1} f \beta) \ast (\eta \varepsilon) = (\alpha^{k+1} f \beta) \ast \alpha^k (\varphi \ast g)
\]

\[
= (\alpha^{k+1} f \beta) \ast (\alpha^k \varphi \ast \alpha^k g) = (\alpha^k f \beta) \ast \alpha^k \varphi \ast \alpha^k g
\]

\[
= (\alpha^k f \ast \alpha^k \varphi) \ast \alpha^{k+1} g \beta = \alpha^k (f \ast \varphi) \ast \alpha^{k+1} g \beta
\]

\[
= \alpha^k (\eta \varepsilon) \ast \alpha^{k+1} g \beta = \eta \varepsilon \ast \alpha^{k+1} g \beta
\]

\[
= \alpha^{k+2} g \beta^2.
\]

We used the Remark 3.1(ii) in the second equality, the Remark 3.1(i) in the fourth equality, the Hom-associativity in the fifth equality, the Hom-coassociativity in the sixth equality, the Remark 3.1(i) in the seventh equality, unitality of \( \alpha' \) in the eighth equality, and the Remark 3.1(ii) in the last equality.

The previous proposition shows that the relative Hom-inverses are unique in some sense. In fact if \( \alpha \) and \( \beta \) are invertible then \( f = g \).

**Proposition 3.3.** Let \((H, m, \eta, \alpha, \Delta, \varepsilon, \beta, S)\) and \((A, m', \eta', \alpha', \Delta', \varepsilon', \beta', S')\) be \((\alpha, \beta)\) and \((\alpha', \beta')\)-Hom-bialgebras and \(\text{Hom}(H, A)\) be the convolution Hom-algebra. If \( f, g, \varphi \in \text{Hom}(H, A) \) where \( f \) and \( g \) are the relative Hom-inverse of \( \varphi \), then for every \( x \in H \) there exists \( k \in \mathbb{N} \) where

\[
\alpha^{k+2} f \beta^2(x) = \alpha^{k+2} g \beta^2(x).
\]

**Proof.** For \( x \in H \) there exist \( k', k'' \in \mathbb{N} \) where

\[
\alpha^{k'} (f \ast \varphi)(x) = \alpha^{k'} (\varphi \ast f)(x) = \varepsilon(x) 1 = \eta \circ \varepsilon(x),
\]

and

\[
\alpha^{k''} (g \ast \varphi)(x) = \alpha^{k''} (\varphi \ast g)(x) = \varepsilon(x) 1 = \eta \circ \varepsilon(x).
\]

Let \( k = \max(k', k'') \). We ignore the composition sign for easier computation. We have

\[
\alpha^{k+2} f \beta^2 = \alpha' (\alpha^{k+1} f \beta) \beta
\]

\[
= (\alpha^{k+1} f \beta) \ast (\eta \varepsilon) = (\alpha^{k+1} f \beta) \ast \alpha^k (\varphi \ast g)
\]

\[
= (\alpha^{k+1} f \beta) \ast (\alpha^k \varphi \ast \alpha^k g) = (\alpha^k f \beta) \ast \alpha^k \varphi \ast \alpha^k g
\]

\[
= (\alpha^k f \ast \alpha^k \varphi) \ast \alpha^{k+1} g \beta = \alpha^k (f \ast \varphi) \ast \alpha^{k+1} g \beta
\]

\[
= \alpha^k (\eta \varepsilon) \ast \alpha^{k+1} g \beta = \eta \varepsilon \ast \alpha^{k+1} g \beta
\]

\[
= \alpha^{k+2} g \beta^2.
\]

We used the Remark 3.1(ii) in the second equality, the Remark 3.1(i) in the fourth equality, the Hom-associativity in the fifth equality, the Hom-coassociativity in the sixth equality, the Remark 3.1(i) in the seventh equality, unitality of \( \alpha' \) in the eighth equality, and the Remark 3.1(ii) in the last equality.

The previous proposition shows that the relative Hom-inverses are unique in some sense. In fact if \( \alpha \) and \( \beta \) are invertible then \( f = g \).
Let $k = \max(k', k'') + 2$. Then
\[
\alpha^k(M \ast N)(x, y) = \alpha^k[M(x^{(1)}, y^{(1)})N(x^{(2)}, y^{(2)})] \\
= \alpha^k([x^{(1)}y^{(1)}][S(y^{(2)})S(x^{(2)})]) \\
= [\alpha^k(x^{(1)})\alpha^k(y^{(1)})]\alpha^k(S(y^{(2)})S(x^{(2)})) \\
= \alpha^{k+1}(x^{(1)})[\alpha^k(y^{(1)})\alpha^{k-1}(S(y^{(2)})S(x^{(2)}))] \\
= \alpha^{k+1}(x^{(1)})[\alpha^k(y^{(1)})\alpha^{k-1}(S(y^{(2)})S(x^{(2)})[\alpha^k(S(x^{(2)})))] \\
= \alpha^{k+1}(x^{(1)})[\alpha^k(y^{(1)})\alpha^{k-1}(S(y^{(2)})\alpha^k(S(x^{(2)}))] \\
= \alpha^{k+1}(x^{(1)})[\alpha^k(y^{(1)})\alpha^{k-1}(S(y^{(2)})\alpha^k(S(x^{(2)}))] \\
= \alpha^{k+1}(x^{(1)})\alpha^{k+1}(S(x^{(2)})) \varepsilon(y) \\
= \alpha^{k+1}(\varepsilon(x)1)\varepsilon(y) \\
= \varepsilon(x)\varepsilon(y)1.
\]

We used the Hom-associativity property in the fourth equality, the Hom-unitality in ninth equality. Therefore $N(x, y) = S(y)S(x)$ is a relative Hom-inverse of $M(x, y) = xy$. Now for $xy \in H$ there exists $n \in \mathbb{N}$ where
\[
\alpha^n(S(x^{(1)})x^{(2)}) = \alpha^n(x^{(1)}S(x^{(2)})) = \varepsilon(x)1_B.
\]

Then
\[
\alpha^n(P \ast M)(x, y) = \alpha^n(P(x^{(1)}, y^{(1)})M(x^{(2)}, y^{(2)})) \\
= \alpha^n[S(x^{(1)}y^{(1)}x^{(2)}y^{(2)}] = \alpha^n[S((xy)^{(1)})(xy)^{(2)}] = \varepsilon(xy)1.
\]

Therefore $P(x, y) = S(xy)$ is a relative Hom-inverse of $M(x, y) = xy$. Then $S(xy)$ and $S(y)S(x)$ are both relative Hom-inverse of the multiplication $M(x, y) = xy$ in $\text{Hom}(H \otimes H, H)$ with respect to the convolution product. Therefore by Proposition 3.2 there exists $K \in \mathbb{N}$ such that
\[
\alpha^{K+2} \circ P \circ \beta^2_{H \otimes H}(x, y) = \alpha^{K+2} \circ N \circ \beta^2_{H \otimes H}(x, y).
\]

By Example 2.12 we have $\beta_{H \otimes H} = \beta_H \otimes \beta_H$ and therefore we obtain the result. 

The relation 3.1 is called the relative Hom-anti algebra map property of $S$.

**Corollary 3.4.** Let $(H, m, \eta, \alpha, \Delta, \varepsilon)$ be a $(\alpha, \beta)$-Hom-bialgebra, where $\alpha$ is multiplicative and $\alpha$ and $\beta$ are invertible. If $H$ is endowed with a linear map $S : H \rightarrow H$ where $S$ is a relative Hom-inverse of the identity map $\text{Id} : H \rightarrow H$ in $\text{Hom}(H, H)$, then $S$ is an anti-algebra map.
Proof. By previous Proposition we have \( \alpha^{K+2} \circ P \circ \beta^{2}_{H \otimes H}(x, y) = \alpha^{K+2} \circ N \circ \beta^{2}_{H \otimes H}(x, y) \).

Since \( \alpha \) and \( \beta \) are invertible then \( P = N \) or \( S(xy) = S(y)S(x) \).

Similarly we have the following proposition.

**Proposition 3.5.** Let \( (H, m, \eta, \alpha, \Delta, \varepsilon, \beta) \) be a Hom-bialgebra where \( \beta \) is a coalgebra map. Assume \( H \) is endowed with a map \( S : H \to H \) where \( S \) is a relative Hom-inverse of the identity map \( \text{Id} : H \to H \) with respect to the convolution product. Let \( P(x) = \Delta(S(x)) \), \( N = \tau(S \otimes S)\Delta \), and \( \Delta(x) = x^{(1)} \otimes x^{(2)} \) where \( \tau(x, y) = (y, x) \) for all \( x, y \in H \). Then \( P \) and \( N \) are both Hom-relative inverse of the comultiplication \( \Delta \) in \( \text{Hom}(H, H \otimes H) \) with respect to the convolution product and for every \( x \in H \) there exists \( k \in \mathbb{N} \) such that

\[
\alpha^{k+2}[S(\beta^{2}(x))^{(1)}] \otimes \alpha^{k+2}[S(\beta^{2}(x))^{(2)}] = \alpha^{k+2}[S(\beta^{2}(x^{(2)}))] \otimes \alpha^{k+2}[S(\beta^{2}(x^{(1)}))].
\] (3.2)

**Proof.** Using proposition 3.2 and Example 2.12, the proof is similar to the previous Proposition. \( \square \)

The relation 3.2 is called the relative Hom-anti coalgebra map property of \( S \). In special case we have the following.

**Corollary 3.6.** Let \( (H, m, \eta, \alpha, \Delta, \varepsilon, \beta) \) be a \((\alpha, \beta)\)-Hom-bialgebra, where \( \beta \) is coalgebra morphism and \( \alpha \) and \( \beta \) are invertible. Assume \( H \) is endowed with a linear map \( S : H \to H \) where \( S \) is a relative Hom-inverse of the identity map \( \text{Id} : H \to H \) with respect to the convolution product. Then \( S \) is an anti-coalgebra map.

**Proposition 3.7.** Let \( (H, m, \eta, \alpha, \Delta, \varepsilon, \beta) \) be a Hom-bialgebra, endowed with a map \( S : H \to H \) where \( S \) is a relative Hom-inverse of the identity map \( \text{Id} : H \to H \) with respect to the convolution product. Then there exists \( k \in \mathbb{N} \) such that

\[
\alpha^{k+1}(S(1)) = 1.
\] (3.3)

**Proof.** We apply relative Hom-invertibility of \( S \) for \( h = 1 \). So there exist \( k \in \mathbb{N} \) where

\[
1 = \varepsilon(1)1 = \alpha^{k}(\text{Id} \ast S)(1) = \alpha^{k}(1S(1)) = \alpha^{k+1}(S(1)).
\]

\( \square \)

The condition 3.3 is called the relative Hom-unitality property of \( S \).

**Proposition 3.8.** Let \( (H, m, \eta, \alpha, \Delta, \varepsilon, \beta) \) be a Hom-bialgebra, endowed with a map \( S : H \to H \) where \( S \) is a relative Hom-inverse of the identity map \( \text{Id} : H \to H \) with respect to the the convolution product. Then there exists \( k \in \mathbb{N} \) such that

\[
\varepsilon(\alpha^{k}(S(h))) = 1 \varepsilon(h).
\] (3.4)
Proof. For any $h \in H$ there exists $k \in \mathbb{N}$ such that

$$\varepsilon(h)1 = \alpha^k(S(h^{(1)})h^{(2)}).$$

Therefore

$$\varepsilon(\varepsilon(h)1) = \varepsilon(\alpha^k(S(h^{(1)})h^{(2)})).$$

Since $\varepsilon$ is unital and it commutes with $\alpha$ then

$$1\varepsilon(h) = \alpha^k(\varepsilon(S(h^{(1)})h^{(2)})) = \alpha^k(\varepsilon(h^{(1)})\varepsilon(h^{(2)})).$$

Therefore

$$1\varepsilon(h) = \alpha^k(\varepsilon(S(h^{(1)})h^{(2)})) = \alpha^k \varepsilon(S(h)) = \varepsilon(\alpha^k(S(h))).$$

\[\square\]

The condition \[(3.4)\] is called the relative Hom-counitality property of $S$.

**Lemma 3.9.** Let $(H, m, \eta, \alpha, \Delta, \varepsilon, \beta, S)$ be a Hom-Hopf algebra, $(A, m', \eta', \alpha')$ be a Hom-algebra and $f : H \rightarrow A$ be a Hom-algebra map. Then $f \circ S$ is a relative Hom-inverse of $f$ in Hom($H, A$).

**Proof.** We show that $f \circ S$ is a relative Hom-inverse of $f$ in Hom($H, A$). For any $h \in H$ there exist $k \in \mathbb{N}$ where $\alpha^k(S(h^{(1)})h^{(2)}) = \alpha^k(h^{(1)}S(h^{(2)})) = \varepsilon(x)1$. Therefore

$$\alpha^k((f \circ S) \ast f)(h) = \alpha^k(f(S(h^{(1)}))f(h^{(2)})) = \alpha^k(f(S(h^{(1)})h^{(2)}))$$

$$= f(\alpha^k(S(h^{(1)})h^{(2)})) = f(\varepsilon(h)1) = \varepsilon(h)1.$$  

Similarly since $\alpha^k(h^{(1)}S(h^{(2)})) = \varepsilon(x)1$ we have $\alpha^k(f \ast (f \circ S))(h) = \varepsilon(h)1$.  

\[\square\]

**Lemma 3.10.** Let $(H, m, \eta, \alpha, \Delta, \varepsilon, \beta, S)$ be a Hom-Hopf algebra, $(C, \Delta', \varepsilon', \beta')$ be a Hom-coalgebra and $f : C \rightarrow H$ be a Hom-coalgebra map. Then $S \circ f$ is a relative Hom-inverse of $f$ in Hom($C, H$).

**Proof.** Similar to the previous Lemma.  

\[\square\]

**Proposition 3.11.** Let $(H, m, \eta, \alpha, \Delta, \varepsilon, \beta, S)$ and $(K, m', \eta', \alpha', \Delta', \varepsilon', \beta', S')$ be $(\alpha, \beta)$ and $(\alpha', \beta')$-Hom-Hopf algebras. If $f : H \rightarrow K$ is a map of Hom-bialgebras then there exists $K \in \mathbb{N}$ such that

$$\alpha^K \circ (f \circ S) \circ \beta^2(h) = \alpha^K \circ (S' \circ f)\beta^2(h).$$  

\[(3.5)\]

**Proof.** By the previous Lemmas $f \circ S$ and $S' \circ f$ are the relative Hom-inverse of $f$ in Hom($H, K$). Then the result is followed by Proposition 3.2.  

\[\square\]

The condition \[(3.5)\] is called the relative Hom-Hopf algebra map property of $S$. As a special case of the previous Proposition we have the following result.
Corollary 3.12. Let \((H, m, \eta, \alpha, \Delta, \varepsilon, \beta, S)\) and \((K, m', \eta', \alpha', \Delta', \varepsilon', \beta', S')\) be Hom-Hopf algebras where \(\alpha, \beta, \alpha', \beta'\) are invertible. Then any Hom-bialgebra map \(f : H \to K\) is a Hom-Hopf algebra map, i.e.,

\[ f \circ S = S' \circ f. \]

Corollary 3.13. Let \((H, m, \eta, \alpha, \Delta, \varepsilon, \beta, S)\) be Hom-Hopf algebra where \(\alpha = \beta\). Then there exists \(k \in \mathbb{N}\) such that

\[ \alpha^k(\alpha \circ S)\alpha^2(h) = \alpha^k(S \circ \alpha)\alpha^2. \quad (3.6) \]

If \(\alpha\) is invertible then \(\alpha \circ S = S \circ \alpha\).

Proof. Since \(\alpha = \beta\) then \(\alpha\) is a map of Hom-bialgebras and the result is followed by Proposition 3.11. \(\square\)

Here we summarize some of the results in this section.

Theorem 3.14. Let \((H, m, \eta, \alpha, \Delta, \varepsilon, \beta)\) be a Hom-bialgebra where \(\alpha\) and \(\beta\) are morphisms of algebra and coalgebra, respectively. Assume \(H\) is endowed with a map \(S : H \to H\) where \(S\) is a relative Hom-inverse of the identity map \(\text{Id} : H \to H\) with respect to the convolution product. Then \(S\) is a relative Hom-anti-algebra map, a relative Hom-anti-coalgebra map, relative Hom-unital, and relative Hom-counital, i.e., there exists \(k \in \mathbb{N}\) such that

- \(i)\ \alpha^{k+2}(S(\beta^2(x))\beta^2(y)) = \alpha^{k+2}(S(\beta^2(y))S(\beta^2(x))).\)
- \(ii)\ \alpha^{k+2}[S(\beta^2(x))^2] \otimes \alpha^{k+2}[S(\beta^2(x))^2] = \alpha^{k+2}[S(\beta^2(x^2))] \otimes \alpha^{k+2}[S(\beta^2(x^1))].\)
- \(iii)\ \alpha^{k+1}(S(1)) = 1.\)
- \(iv)\ \varepsilon(\alpha^k(S(h))) = 1 \varepsilon(h).\)
- \(v)\ If \(\alpha = \beta\) then \(\alpha^k(\alpha \circ S)\alpha^2(h) = \alpha^k(S \circ \alpha)\alpha^2.\)

Furthermore if \(\alpha\) and \(\beta\) are invertible then \(S\) is morphisms of algebras and coalgebras, respectively, and \(\alpha \circ S = S \circ \alpha\).

Proposition 3.15. Let \((H, m, \eta, \alpha, \Delta, \varepsilon, \beta, S)\) be a commutative Hom-Hopf algebra. Then for every \(x \in H\) there exist \(k \in \mathbb{N}\) where

\[ \alpha^{k+2} \circ S^2 \circ \beta^2(x) = \alpha^{k+2} \circ \text{Id} \circ \beta^2(x). \]

Proof. We show that \(S^2\) is a relative Hom-inverse of \(S\). For any \(h \in H\) there exist \(k \in \mathbb{N}\) where \(\alpha^k(S(h^{(1)})h^{(2)}) = \alpha^k(h^{(1)}S(h^{(2)})) = \varepsilon(x)1\). Therefore

\[ \alpha^k(S \ast S^2)(h) = \alpha^k[S\{h^{(1)}S(h^{(2)})\}] \]

\[ = \alpha^k[S[S(h^{(2)})h^{(1)}]] = \alpha^k[S[h^{(1)}S(h^{(2)})]] \]

\[ = S[\alpha^k[h^{(1)}S(h^{(2))}]] = S(\varepsilon(h)1) = \varepsilon(h)1. \]
We used the anti-algebra map property of $S$ in the second equality, commutativity of $H$ in the third equality, commutativity of $S$ and $\alpha$ in the fourth equality, and the unitality of $S$ in the last equality. Similarly it can be shown that $\alpha^k(S^2 \star S)(h) = \varepsilon(h)1$. Therefore $S^2$ and the identity map $\text{Id}_H$ are both relative Hom-inverse of $S$. Now the result is followed by Proposition 3.2.

As a special case of previous Proposition, we have the following result.

**Corollary 3.16.** If $(H,m,\eta,\alpha,\Delta,\varepsilon,\beta,S)$ is a commutative Hom-Hopf algebra with invertible $\alpha$ and $\beta$ then

$$S^2 = \text{Id}.$$  

**Proposition 3.17.** Let $(H,m,\eta,\alpha,\Delta,\varepsilon,\beta,S)$ be a cocommutative Hom-Hopf algebra. Then for every $x \in H$ there exist $k \in \mathbb{N}$ where

$$\alpha^{k+2} \circ S^2 \circ \beta^2(x) = \alpha^{k+2} \circ \text{Id} \circ \beta^2(x).$$

**Proof.** We show that $S^2$ is a relative Hom-inverse of $S$. For any $h \in H$ there exist $k', k'' \in \mathbb{N}$ where $\alpha^{k'}(S(h(1))h(2)) = \alpha^{k'}(h(1)S(h(2))) = \varepsilon(x)1$, and $\alpha^{k''}(S(S(h(1))h(2))) = \alpha^{k''}(h(1)S(S(h(2)))) = \varepsilon(x)1$. Let $k = \max(k',k'')$.

Therefore

$$\alpha^k(S \star S^2)(h) = \alpha^k[S(h(1))S^2(h(2))]$$

$$= \alpha^k[S(h(2))S(S(h(1)))] = \alpha^k[S(h(1))S(S(h(2)))]$$

$$= \alpha^k(\varepsilon(S(h))1) = \varepsilon(S(h))\alpha^k(1) = \varepsilon(h)1.$$  

We used the anti-coalgebra map property of $S$ in the second equality, cocommutativity of $H$ in the third equality, and the counitality of antipode in fifth equality. Similarly $\alpha^k(S^2 \star S)(h) = \varepsilon(h)1$. Therefore $S^2$ and the identity map $\text{Id}_H$ are both Hom-relative inverse of $S$. Then the result is followed by Proposition 3.2. \hfill \square

As a special case of the previous Proposition we have the following result.

**Corollary 3.18.** If $(H,m,\eta,\alpha,\Delta,\varepsilon,\beta,S)$ is a cocommutative Hom-Hopf algebra with invertible $\alpha$ and $\beta$ then

$$S^2 = \text{Id}.$$  

**Proposition 3.19.** Let $(H,m,\eta,\alpha,\Delta,\varepsilon,\beta,S)$ be a Hom-Hopf algebra and $h \in H$ a primitive element, i.e, $\Delta(h) = 1 \otimes h + h \otimes 1$. Then there exists $k \in \mathbb{N}$ where

$$\alpha^{k+1}(S(h)) = -\alpha^{k+1}(h).$$  

(3.7)
Proof. There exists \( k \in \mathbb{N} \) such that \( \alpha^k(S(h^{(1)})h^{(2)}) = \alpha^{k'}(h^{(1)}S(h^{(2)})) = \varepsilon(x)1 \). Therefore

\[
\alpha^k(hS(1)) + \alpha^k(1S(h)) = \varepsilon(h)1.
\]

Since \( S \) is unital and for any \( x \in H \), we have \( 1x = \alpha(x) \), then

\[
\alpha^{k+1}(h) + \alpha^{k+1}(h) = \varepsilon(h)1.
\]

By [LMT], for any primitive element \( h \), we have \( \varepsilon(h) = 0 \). Therefore \( \alpha^{k+1}(S(h)) = -\alpha^{k+1}(h) \).

\[
\begin{align*}
\text{Proposition 3.20.} \quad &\text{Let } (H, m, \eta, \alpha, \Delta, \varepsilon, \beta, S) \text{ be a Hom-Hopf algebra and } h \in H \text{ a group-like element, i.e, } \Delta(h) = h \otimes h. \text{ Then there exists } k \in \mathbb{N} \text{ where} \\
&\alpha^k(S(h)h) = \alpha^k(hS(h)) = 1. \quad (3.8)
\end{align*}
\]

Proof. There exists \( k \in \mathbb{N} \) such that \( \alpha^k(S(h^{(1)})h^{(2)}) = \alpha^{k'}(h^{(1)}S(h^{(2)})) = \varepsilon(x)1 \). Then

\[
\alpha^k(S(h)h) = \alpha^k(hS(h)) = \varepsilon(h)1.
\]

Now the result is followed by the fact that \( \varepsilon(h) = 1 \).

By previous Proposition the relative Home inverse of a group-like element \( h \) is \( S(h) \).

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