Hopf algebras of canonical commutation relations

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Abstract

Given a Heisenberg algebra $A$ of canonical commutation relations modelled over an infinite-dimensional nuclear space, a Hopf algebra of its quantum deformations is also an algebra of canonical commutation relations whose Fock representation recovers some non-Fock representation of $A$.

1 Introduction

By virtue of the well-known Stone–von Neumann uniqueness theorem, all irreducible representations of the canonical commutation relations (henceforth the CCR) of finite degree of freedom are equivalent. On the contrary, the infinite-dimensional CCR possess many non-equivalent irreducible representations (see [1] for a survey). Here, we restrict our consideration to the CCR modelled over a nuclear space. They include the CCR of finite degrees of freedom, but we focus on the infinite-dimensional CCR. In particular, this is the case of field theory [5].

Let $A$ be the Heisenberg algebra of the CCR modelled over a nuclear space. Since $A$ is a Lie algebra, one can associate to $A$ a Hopf algebra, regarded as an algebra of $q$-deformed CCR (see [3] for the case of finite-dimensional CCR). We show that this Hopf algebra is the enveloping algebra of another CCR algebra $A_{q,c}$. Moreover, $A$ and $A_{q,c}$ possess the same set of representations. Herewith, operators of the Fock representation of $A_{q,c}$ carry out some non-Fock representation of $A$.

2 The nuclear CCR

Let us recall the notion of a nuclear space (see, e.g., [4]). Let a complex vector space $V$ be provided with a countable set of non-degenerate Hermitian forms $\langle \cdot | \cdot \rangle_k$, $1, \ldots$, such that

$$\langle v | v \rangle_1 \leq \cdots \leq \langle v | v \rangle_k \leq \cdots$$

for all $v \in V$. Let $V$ be complete in the topology defined by the set of norms $\| \cdot \|_k^{1/2} = \langle \cdot | \cdot \rangle_k$. Then $V$ is called a countably Hilbert space. Let $V_k$ denote the completion of $V$ with respect
to the norm $\|\cdot\|_k$. There is the chain of injections $V_1 \supset V_2 \supset \cdots \supset V_k \supset \cdots$, and $V = \cap V_k$.

Let $T_m^k$, $m \leq n$, be a prolongation of the map $V_n \supset V \ni v \mapsto v \in V \subset V_m$ to the continuous map of $V_n$ onto a dense subset of $V_m$. A countably Hilbert space $V$ is called a nuclear space if, for any $m$, there exists $n$ such that $T_n^m$ is a nuclear map, i.e.,

$$T_n^m(v) = \sum_i \lambda_i \langle v | v_n^i \rangle v_n^i v_m^i,$$

where: (i) $\{v_n^i\}$ and $\{v_m^i\}$ are bases for the Hilbert spaces $V_n$ and $V_m$, respectively, (ii) $\lambda_i \geq 0$, and (iii) the series $\sum \lambda_i$ converges. Note that a Hilbert space is not nuclear, unless it is finite-dimensional.

Let $V$ be a real nuclear space provided with still another non-degenerate Hermitian form $\langle \cdot | \cdot \rangle$, which is separately continuous. This form makes $V$ to a separable pre-Hilbert space. Let us consider the group $G$ of the triples $(v_1, v_2, \lambda)$ of elements $v_1, v_2$ of $V$ and complex numbers $\lambda$ of unit modulus which are subject to multiplications

$$(v_1, v_2, \lambda)(v_1', v_2', \lambda') = (v_1 + v_1', v_2 + v_2', \exp[i\langle v_2, v_1' \rangle] \lambda \lambda').$$  \hspace{1cm} (1)

It is a Lie group whose group space is a nuclear manifold modelled over

$$W = V \oplus V \oplus \mathbb{R}. \hspace{1cm} (2)$$

Let us denote $T(v) = (v, 0, 0)$ and $P(v) = (0, v, 0)$. Then the multiplication law (2) takes the form

$$T(v)T(v') = T(v + v'), \quad P(v)P(v') = P(v + v'), \quad P(v)T(v') = \exp[i\langle v | v' \rangle]T(v')P(v).$$ \hspace{1cm} (3)

Written in this form, $G$ is called the Weyl CCR group.

The Lie algebra of the nuclear Lie group $G$ is the above mentioned Heisenberg algebra $A$. It is generated by the Hermitian elements $I, \phi(v), \pi(v), v \in V$, which obey the commutation relations

$$[\phi(v), I] = [\pi(v), I] = [\phi(v), \phi(v')] = [\pi(v), \pi(v')] = 0, \hspace{1cm} (4)$$

$$[\pi(v), \phi(v')] = -i\langle v | v' \rangle I. \hspace{1cm} (5)$$

Given a countable orthonormal basis $\{v_i\}$ for the pre-Hilbert space $V$, the CCR (4) – (5) take the form

$$[\phi(v_j), \phi(v_k)] = [\pi(v_k), \pi(v_j)] = 0, \quad [\pi(v_j), \phi(v_k)] = -i\delta_{jk} I.$$

One also introduces the creation and annihilation operators

$$a^\pm(v) = \frac{1}{\sqrt{2}}[\phi(v) \mp i\pi(v)]. \hspace{1cm} (6)$$

They obey the conjugation rule $(a^\pm(v))^* = a^\mp(v)$ and the commutation relations

$$[a^-(v), a^+(v')] = \langle v | v' \rangle I, \quad [a^+(v), a^+(v')] = [a^-(v), a^-(v')] = 0.$$
3 Hopf algebras of the CCR

Let us consider the tensor algebra $\otimes W$ of the vector space $W$ generated by elements $\phi(v)$, $\pi(v)$ and $I$. It is provided with a unique Hopf algebra structure, characterized by the comultiplication

$$\Delta(w) = w \otimes 1 + 1 \otimes w, \quad w \in W,$$

the counit $\epsilon(w) = 0$, the antipode $S(w) = -w$, and the universal matrix $R = 1 \otimes 1$. It is a cocommutative quasi-triangular Hopf algebra, called the classical Hopf algebra.

Let $A$ be the enveloping algebra of the Heisenberg CCR algebra $A$. It is the quotient of the tensor algebra $\otimes W$ by the commutation relations (4) – (5), written with respect to the tensor product $\otimes$, and by the relation

$$I \otimes I = I. \quad (7)$$

The $A$ inherits the structure of the classical Hopf algebra on $\otimes W$. We denote it $B_{cl}(A)$.

Now let us consider the quotient $A_{q,c}$ of the tensor algebra $\otimes W$ by the relations (4), (7) and the commutation relations

$$[\pi(v), \phi(v')] = -i\langle v|v' \rangle \frac{q^c I - q^{-c} I}{c(q - q^{-1})}, \quad (8)$$

where $q$ and $c$ are strictly positive real numbers. Due to the relation (7), the right-hand side of the relations (8) is well defined on $\otimes W$, and we have

$$[\pi(v), \phi(v')] = -i\langle v|v' \rangle \frac{q^c - q^{-c}}{c(q - q^{-1})} I. \quad (9)$$

Hence, $A_{q,c}$ is the enveloping algebra of the Heisenberg CCR algebra $A_{q,c}$ given by the commutation relations (4) and (9). This CCR algebra is modelled over the same nuclear space $V$, but provided with the Hermitian form

$$\langle v|v' \rangle_{q,c} = C_{q,c} \langle v|v' \rangle, \quad C_{q,c} = \frac{q^c - q^{-c}}{c(q - q^{-1})}. \quad (10)$$

The enveloping algebra $A_{q,c}$ admits both the structure of the classical Hopf algebra $B_{cl}(A_{q,c})$ and the Hopf algebra $B(A_{q,c})$, which differs from the classical one in the comultiplication law

$$\Delta(\phi(v)) = \phi(v) \otimes q^{cI/2} + q^{-cI/2} \otimes \phi(v), \quad \Delta(\pi(v)) = \pi(v) \otimes q^{cI/2} + q^{-cI/2} \otimes \pi(v), \quad \Delta(I) = I \otimes 1 + 1 \otimes I.$$
One can think of $B(A_{q,c})$ as being a Hopf algebra of the $q$-deformed CCR. It is readily observe that, if $c = 1$, the CCR algebras $A$ and $A_{q,1}$ coincide for any $q$, but the Hopf algebra $B(A_{q,1})$ differs from the classical one $B_{cl}(A_{q,1}) = B_{cl}(A)$. If $q = 1$, then $A_{1,c} = A$ and $B(A_{1,c}) = B_{cl}(A)$ for any $c$.

Since the Hopf algebra $B(A_{q,c})$ is the enveloping algebra of the CCR algebra $A_{q,c}$, its representations are determined in full by representations of $A_{q,c}$. Let us compare the representations of the CCR algebras $A$ and $A_{q,c}$.

### 4 Representations of the nuclear CCR

The CCR group $G$ contains two Abelian subgroups $T$ and $P$. Following the representation algorithm in [2], we first construct representations of the nuclear Abelian group $T$ [3].

Its cyclic strongly continuous unitary representation $\rho$ in a Hilbert space $(E, \langle \cdot, \cdot \rangle_E)$ with a (normed) cyclic vector $\theta \in E$ defines the complex function

$$Z(v) = \langle \rho(T(v))\theta|\theta \rangle_E$$

on $V$. This function is continuous and positive-definite, i.e., $Z(0) = 1$ and

$$\sum_{i,j} Z(v_i - v_j)c_ic_j \geq 0$$

for any finite set $v_1, \ldots, v_m$ of elements of $V$ and arbitrary complex numbers $c_1, \ldots, c_m$. By virtue of the well-known Bochner theorem, such a function on a nuclear space $V$ is the Fourier transform

$$Z(v) = \int \exp[i\langle v, u \rangle]\mu$$

of a positive measure $\mu$ of total mass 1 on the topological dual $V'$ of $V$. Then the above mentioned representation $\rho$ of $T$ can be given by the operators

$$T_Z(v)f(u) = \exp[i\langle v, u \rangle]f(u)$$

in the Hilbert space $L^2(V', \mu)$ of classes of $\mu$-equivalent square integrable complex functions $f(u)$ on $V'$. The cyclic vector $\theta$ of this representation is the $\mu$-equivalence class $\theta \approx_{\mu} 1$ of the constant function $f(u) = 1$. Conversely, every positive measure $\mu$ of total mass 1 on the dual $V'$ of $V$ (and, consequently, every continuous positive-definite function $Z(v)$ on $V$) defines a cyclic strongly continuous unitary representation (12) of the nuclear group $T$. We agree to call $Z$ a generating function of this representation. One can show that distinct generating functions $Z$ and $Z'$ determine equivalent representations $T_Z$ and $T_{Z'}$ (12) of $T$ in the Hilbert spaces $L^2(V', \mu)$ and $L^2(V', \mu')$ iff they are the Fourier transform of equivalent measures on $V'$.
The representation $T_Z$ \((12)\) of the group $T$ can be extended to the CCR group $G$ if the measure $\mu$ possesses the following property. Let $u_v, v \in V$, denote an element of $V'$ given by the condition
\[
\langle v', u_v \rangle = \langle v| v \rangle, \quad \forall v' \in V.
\tag{13}
\]
These elements form the image of the monomorphism $V \to V'$ determined by the Hermitian form $\langle \cdot | \cdot \rangle$ on $V$. Let the measure $\mu$ in \((11)\) remain equivalent under translations $u \mapsto u + u_v$ of $V'$ by any element $u_v$ of $V \subset V'$, i.e.,
\[
\mu(u + u_v) = a^2(v, u)\mu(u), \quad \forall u_v \in V \subset V',
\tag{14}
\]
where a function $a(v, u)$ is square $\mu$-integrable and strictly positive almost everywhere on $V'$. This function fulfils the relations
\[
a(0, u) = 1, \quad a(v + v', u) = a(v, u)a(v', u + u_v).
\tag{15}
\]
A measure on $V'$ obeying the condition \((14)\) is called translationally quasi-invariant. Let the generating function $Z$ of a cyclic strongly continuous unitary representation of the nuclear group $T$ be the Fourier transform \((11)\) of such a measure $\mu$ on $V'$. Then the representation \((12)\) of $T$ is extended to the representation of the nuclear CCR group $G$ in the Hilbert space $L^2(V', \mu)$ by operators
\[
P_Z(v)f(u) = a(v, u)f(u + u_v).
\tag{16}
\]
Moreover, one can show that if $\mu'$ is a $\mu$-equivalent positive measure of total mass 1 on $V'$, it is also translationally quasi-invariant and provides an equivalent representation of $G$.

A strongly continuous unitary representation $T_Z$ \((12)\), $P_Z$ \((16)\) of the nuclear CCR group $G$ implies a representation of its Lie algebra $A$ by (unbounded) operators
\[
I = 1, \quad \phi(v)f(u) = \langle v, u \rangle f(u), \quad \pi(v)f(u) = -i(\delta_v + \eta(v, u))f(u),
\tag{17}
\]
\[
\delta_v f(u) = \lim_{\alpha \to v} \alpha^{-1}[f(u + \alpha u_v) - f(u)], \quad \alpha \in \mathbb{R},
\]
\[
\eta(v, u) = \lim_{\alpha \to 0} \alpha^{-1}[a(\alpha v, u) - 1],
\tag{18}
\]
in the same Hilbert space $L^2(V', \mu)$. With the aid of the formulas
\[
\delta_v \delta_{v'} = \delta_v \delta_{v'}, \quad \delta_v(\eta(v', u)) = \delta_{v'}(\eta(v, u)),
\]
\[
\delta_v = -\delta_{-v}, \quad \delta_v(\langle v', u \rangle) = \langle v'| v \rangle,
\]
\[
\eta(0, u) = 0, \quad \forall u \in V', \quad \delta_v \theta = 0, \quad \forall v \in V,
\]
derived from the relations \((13)\), it is easily justified that the operators \((17)\) fulfil the Heisenberg CCR \((4)\).
Gaussian measures exemplify a physically relevant class of translationally quasi-invariant measures on the dual $V'$ of a nuclear space $V$. The Fourier transform of a Gaussian measure reads

$$Z(v) = \exp \left[ -\frac{1}{2} M(v) \right],$$

where $M(v)$ is a seminorm on $V'$ called the covariance form. Let $\mu_K$ denote a Gaussian measure on $V'$ whose Fourier transform is the generating function

$$Z_K = \exp[-\frac{1}{2} M_K(v)]$$

with the covariance form $M_K(v) = \langle K^{-1}v|K^{-1}v \rangle$, where $K$ is a bounded invertible operator in the Hilbert completion $\tilde{V}$ of $V$ with respect to the Hermitian form $\langle .| . \rangle$. The Gaussian measure $\mu_K$ is translationally quasi-invariant:

$$\mu_K(u + u_v) = a^2_K(v, u) \mu_K(u),$$

$$a_K(v, u) = \exp[-\frac{1}{4} M_K(Cv) - \frac{1}{2} \langle Cq, u \rangle],$$

where $C = KK^*$ is a bounded Hermitian operator in $\tilde{V}$.

Let us construct the representation of the CCR algebra $A$ determined by the generating function $Z_K$ (20). Substituting the function (21) into the formula (18), we find

$$\eta(v, u) = -\frac{1}{2} \langle Cv, u \rangle.$$  

Hence, the operators $\phi(v)$ and $\pi(v)$ (17) take the form

$$\phi(v) = \langle v, u \rangle, \quad \pi(v) = -i(\delta_v - \frac{1}{2} \langle Cv, u \rangle).$$

Accordingly, the creation and annihilation operators (1) read

$$a^\pm(v) = \frac{1}{\sqrt{2}} \left[ \mp \delta_v \pm \frac{1}{2} \langle Cv, u \rangle + \langle v, u \rangle \right].$$

In particular, let us put $K = \sqrt{2} \cdot 1$. Then the generating function (20) takes the form

$$Z_F(v) = \exp[-\frac{1}{4} \langle v|v \rangle],$$

and determines the Fock representation of the CCR algebra $A$ by the operators

$$\phi(v) = \langle v, u \rangle, \quad \pi(v) = -i(\delta_v - \langle v, u \rangle),$$

$$a^+(v) = \frac{1}{\sqrt{2}} \left[ -\delta_v + 2\langle v, u \rangle \right], \quad a^-(v) = \frac{1}{\sqrt{2}} \delta_v.$$
Note that the Fock representation up to an equivalence is characterized by the existence of a cyclic vector \( \theta \) such that
\[
a^-(v)\theta = 0, \quad \forall v \in V.
\] (25)
An equivalent condition is that there exists the particle number operator \( N \) possessing a lower bounded spectrum. This operator is defined by the conditions
\[
[N, a^\pm(v)] = \pm a^\pm(v)
\]
up to a summand \( \lambda \). With respect to a countable orthonormal basis \( \{v_k\} \), it is given by the sum
\[
N = \sum_k a^+(v_k)a^-(v_k).
\]
A glance at the expression (23) shows that the condition (25) does not hold, unless \( Z_K \) is \( Z_F \) (24). For instance, the particle number operator in the representation (23) reads
\[
N = \sum_j a^+(v_j)a^-(v_j) = \sum_j \left[ -\delta_{ij}\delta_{v_j} + C^j_k(v_k, u)\partial v_j + \left( \delta_{km} - \frac{1}{4}C^j_kC^j_m(v_k, u)(v_m, u) - (\delta_{jj} - \frac{1}{2}C^j_j) \right] \right].
\]
One can show that this operator is defined and is lower bounded only if the operator \( C \) is a sum of the scalar operator \( 2 \cdot 1 \) and a nuclear operator in \( \tilde{V} \). For instance, the generating function
\[
Z_c(v) = \exp\left[ -\frac{c^2}{2} \langle v|v \rangle \right], \quad c^2 \neq \frac{1}{2},
\]
determines a non-Fock representation of the nuclear CCR.

At the same time, the non-Fock representation (22) of the CCR algebra (3) is the Fock representation
\[
\phi_K(v) = \phi(v) = \langle v, u \rangle,
\]
\[
\pi_K(v) = \pi(S^{-1}v) = -i(\delta^K_v - \frac{1}{2}\langle v, u \rangle), \quad \delta^K_v = \delta_{S^{-1}v},
\]
of the CCR algebra \( \{\phi_K(v), \pi_K(v), I\} \), where
\[
[\phi_K(v), \pi_K(v)] = i\langle K^{-1}v|K^{-1}v' \rangle I.
\]
Bearing in mind this fact, turn now to the CCR algebra \( A_{q,c} \) in Section 3. Comparing the commutation relations (5) and (9), one can show that, given a representation \( \rho \) of the CCR algebra \( A \), the CCR algebra \( A_{q,c} \) admits a representation \( \rho_{q,c} \) by the operators
\[
\rho_{q,c}(\phi(v)) = \rho(\phi(v)), \quad \rho_{q,c}(\pi(v)) = \rho(\pi(C_{q,c}v)), \quad \rho_{q,c}(I) = \rho(I) = 1,
\]
where \( C_{q,c} \) is the real number given by the expression (10). For instance, if \( \rho \) is the Fock representation of the CCR algebra \( A \), the representation \( \rho_{q,c} \) is not equivalent to the Fock representation of the CCR algebra \( A_{q,c} \), unless \( V \) is finite-dimensional.
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