0 Introduction

Throughout this paper, $k$ is a field, $R$ is an algebra over $k$, and $H$ is a Hopf algebra over $k$. We say that $R\#\sigma H$ is the crossed product of $R$ and $H$ if $R\#\sigma H$ becomes an algebra over $k$ by multiplication:

$$(a\#h)(b\#g) = \sum_{h,g} a(h_1 \cdot b)\sigma(h_2, g_1)\#h_3g_2$$

for any $a, b \in R, h, g \in H$, where $\Delta(h) = \sum h_1 \otimes h_2$ (see, [2, Definition 7.1.1].)

Let $lpd(RM)$, $lid(RM)$ and $lfd(RM)$ denote the left projective dimension, left injective dimension and left flat dimension of left $R$-module $M$, respectively. Let $lgD(R)$ and $wD(R)$ denote the left global dimension and weak dimension of algebra $R$, respectively.

Crossed products are very important algebraic structures. The relation between homological dimensions of algebra $R$ and crossed product $R\#\sigma H$ is often studied. J.C.Mcconnell and J.C.Robson in [4, Theorem 7.5.6] obtained that

$$rgD(R) = rgD(R * G)$$

for any finite group $G$ with $|G|^{-1} \in k$, where $R * G$ is skew group ring. It is clear that every skew group ring $R * G$ is a crossed product $R\#\sigma kG$ with trivial $\sigma$. Zhong

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Yi in [9] obtained that the global dimension of crossed product $R \ast G$ is finite when the global dimension of $R$ is finite and some other conditions hold.

In this paper, we obtain that the global dimensions of $R$ and the crossed product $R\#_\sigma H$ are the same; meantime, their weak dimensions are also the same, when $H$ is finite-dimensional semisimple and cosemisimple Hopf algebra.

1 The homological dimensions of modules over crossed products

In this section, we give the relation between homological dimensions of modules over $R$ and $R\#_\sigma H$.

If $M$ is a left (right) $R\#_\sigma H$-module, then $M$ is also a left (right ) $R$-module since we can view $R$ as a subalgebra of $R\#_\sigma H$.

**Lemma 1.1** Let $R$ be a subalgebra of algebra $A$.

(i) If $M$ is a free $A$-module and $A$ is a free $R$-module, then $M$ is a free $R$-module;
(ii) If $P$ is a projective left $R\#_\sigma H$-module, then $P$ is a projective left $R$-module;
(iii) If $P$ is a projective right $R\#_\sigma H$-module and $H$ is a Hopf algebra with invertible antipode, then $P$ is a projective right $R$-module;
(iv) If $P_M : \cdots P_n \overset{d_n}{\rightarrow} P_{n-1} \cdots \rightarrow P_0 \overset{d_0}{\rightarrow} M \rightarrow 0$ is a projective resolution of left $R\#_\sigma H$-module $M$, then $P_M$ is a projective resolution of left $R$-module $M$;
(v) If $P_M : \cdots P_n \overset{d_n}{\rightarrow} P_{n-1} \cdots \rightarrow P_0 \overset{d_0}{\rightarrow} M \rightarrow 0$ is a projective resolution of right $R\#_\sigma H$-module $M$ and $H$ is a Hopf algebra with invertible antipode, then $P_M$ is a projective resolution of right $R$-module $M$.

**Proof.** (i) It is obvious.

(ii) Since $P$ is a projective $R\#_\sigma H$-module, we have that there exists a free $R\#_\sigma H$-module $F$ such that $P$ is a summand of $F$. It is clear that $R\#_\sigma H \cong R \otimes H$ as left $R$-module, which implies that $R\#_\sigma H$ is a free $R$-module. Thus it follows
from part (i) that $F$ is a free $R$-module and $P$ is a summand of $F$ as $R$-module. Consequently, $P$ is a projective $R$-module.

(iii) By [2, Corollary 7.2.11], $R\#_{\sigma}H \cong H \otimes R$ as right $R$-module. Thus $R\#_{\sigma}H$ is a free right $R$-module. Using the method in the proof of part (i), we have that $P$ is a projective right $R$-module.

(iv) and (v) can be obtained by part (ii) and (iii). □

Lemma 1.2 (i) Let $R$ be a subalgebra of $A$. If $M$ is a flat right (left) $A$-module and $A$ is a flat right (left) $R$-module, then $M$ is a flat right (left) $R$-module;

(ii) If $F$ is a flat left $R\#_{\sigma}H$-module, then $F$ is a flat left $R$-module;

(iii) If $F$ is a flat right $R\#_{\sigma}H$-module and $H$ is a Hopf algebra with invertible antipode, then $F$ is a flat right $R$-module;

(iv) If

$\mathcal{F}_M : \cdots F_n \xrightarrow{d_n} F_{n-1} \cdots \rightarrow F_0 \xrightarrow{d_0} M \rightarrow 0$

is a flat resolution of left $R\#_{\sigma}H$-module $M$, then $\mathcal{F}_M$ is a flat resolution of left $R$-module $M$;

(v) If

$\mathcal{F}_M : \cdots F_n \xrightarrow{d_n} F_{n-1} \cdots \rightarrow F_0 \xrightarrow{d_0} M \rightarrow 0$

is a flat resolution of right $R\#_{\sigma}H$-module $M$ and $H$ is a Hopf algebra with invertible antipode, then $\mathcal{F}_M$ is a flat resolution of $M$;

Proof. (i) We only show part (i) in the case which $M$ is a right $A$-module and $A$ is a right $R$-module; the other cases can similarly be shown. Let

$0 \rightarrow X \xrightarrow{f} Y$

be an exact left $R\#_{\sigma}H$-module sequence. By assumptions,

$0 \rightarrow A \otimes_R X \xrightarrow{A \otimes f} A \otimes_R Y$

and

$0 \rightarrow M \otimes_A (A \otimes_R X) \xrightarrow{M \otimes (A \otimes f)} M \otimes_A (A \otimes_R Y)$

are exact sequences. Obviously,

$M \otimes_A (A \otimes_R X) \cong M \otimes_R X \quad \text{and} \quad M \otimes_A (A \otimes_R Y) \cong M \otimes_R Y$
as additive groups. Thus

\[ 0 \rightarrow M \otimes_R X \xrightarrow{M \otimes f} M \otimes_R Y \]

is an exact sequence, which implies \( M \) is a flat \( R \)-module.

(ii)-(v) are immediate consequence of part (i) \( \Box \)

The following is a immediate consequence of Lemma 1.1 and 1.2.

**Proposition 1.3** (i) If \( M \) is a left \( R#_\sigma H \)-module, then
\[ \text{lpd}(RM) \leq \text{lpd}(R#_\sigma H M) \]

(ii) If \( M \) is a right \( R#_\sigma H \)-module and \( H \) is a Hopf algebra with invertible antipode, then
\[ \text{rpd}(MR) \leq \text{rpd}(M R#_\sigma H) \]

(iii) If \( M \) is a left \( R#_\sigma H \)-module, then
\[ \text{lfd}(RM) \leq \text{lfd}(R#_\sigma H M) \]

(iv) If \( M \) is a right \( R#_\sigma H \)-module and \( H \) is a Hopf algebra with invertible antipode, then
\[ \text{rfd}(MR) \leq \text{rfd}(M R#_\sigma H) \]

**Lemma 1.4** Let \( H \) be a finite-dimensional semisimple Hopf algebra, and let \( M \) and \( N \) be left \( R#_\sigma H \)-modules. If \( f \) is an \( R \)-module homomorphism from \( M \) to \( N \), and
\[ \bar{f}(m) = \sum \gamma^{-1}(t_1)f(\gamma(t_2)m) \]
for any \( m \in M \), then \( \bar{f} \) is an \( R#_\sigma H \)-module homomorphism from \( M \) to \( N \), where \( t \in f_H^* \) with \( \epsilon(t) = 1 \), and \( \gamma \) is a map from \( H \) to \( R#_\sigma H \) sending \( h \) to \( 1#h \).

**Proof.** (see, the proof of [2, Theorem 7.4.2]) For any \( a \in R, h \in H, m \in M \), we see that
\[
\bar{f}(am) = \sum \gamma^{-1}(t_1)f((t_2 \cdot a)\gamma(t_3)m) \\
= \sum \gamma^{-1}(t_1)(t_2 \cdot a)f(\gamma(t_3)m) \\
= \sum a\gamma^{-1}(t_1)f(\gamma(t_2)m) \\
= a\bar{f}(m)
\]
and
\[ \bar{f}(\gamma(h)m) = \sum \gamma^{-1}(t_1)f(\gamma(t_2)\gamma(h)m) \]
\[ = \sum \gamma^{-1}(t_1)f(\sigma(t_2, h_1)\gamma(t_3 h_2)m) \text{ by } [2, \text{ Definition 7.1.1}] \]
\[ = \sum \gamma^{-1}(t_1)\sigma(t_2, h_1)f(\gamma(t_3 h_2)m) \]
\[ = \sum \gamma(h_1)\gamma^{-1}(t_1 h_2)f(\gamma(t_2 h_3)m) \]
\[ = \sum \gamma(h)\gamma^{-1}(t_1)f(\gamma(t_2)m) \text{ since } \sum h_1 \otimes t_1 h_2 \otimes t_2 h_3 = \sum h \otimes t_1 \otimes t_2 \]
\[ = \gamma(h)\bar{f}(m) \]

Thus \( \bar{f} \) is an \( R^\# \sigma H \)-module homomorphism. \( \square \)

In fact, we can obtain a functor by Lemma 1.4. Let \( R^\# \sigma H \mathcal{M} \) denote the full subcategory of \( R \mathcal{M} \); its objects are all of left \( R^\# \sigma H \)-modules and its morphisms from \( M \) to \( N \) are all of \( R \)-module homomorphisms from \( M \) to \( N \). For any \( M, N \in \text{ob} R^\# \sigma H \mathcal{M} \) and \( R \)-module homomorphism \( f \) from \( M \) to \( N \), we define that

\[ F : R^\# \sigma H \mathcal{M} \longrightarrow R^\# \sigma H \mathcal{M} \]

such that

\[ F(M) = M \quad \text{and} \quad F(f) = \bar{f}, \]

where \( \bar{f} \) is defined in Lemma 1.4. It is clear that \( F \) is a functor.

**Lemma 1.5** Let \( H \) be a finite-dimensional semisimple Hopf algebra, and let \( M \) and \( N \) be right \( R^\# \sigma H \)-modules. If \( f \) is an \( R \)-module homomorphism from \( M \) to \( N \), and

\[ \bar{f}(m) = \sum f(m\gamma^{-1}(t_1))\gamma(t_2) \]
for any \( m \in M \), then \( \bar{f} \) is an \( R\#_\sigma H \)-module homomorphism from \( M \) to \( N \), where \( t \in \int_H \) with \( \epsilon(t) = 1 \), \( \gamma \) is a map from \( H \) to \( R\#_\sigma H \) sending \( h \) to \( 1\# h \).

**Proof.** (see, the proof of [2, Theorem 7.4.2]) For any \( a \in R, h \in H, m \in M \), we see that

\[
\bar{f}(ma) = \sum f(ma\gamma^{-1}(t_1))\gamma(t_2) = \sum \sum f(m\gamma^{-1}(t_1)(t_2 \cdot a))\gamma(t_3) = \sum \sum f(m\gamma^{-1}(t_1))(t_2 \cdot a)\gamma(t_3) = \sum f(m\gamma^{-1}(t_1))\gamma(t_2)a = \bar{f}(m)a
\]

and

\[
\bar{f}(m\gamma(h)) = \sum f(m\gamma(h)\gamma^{-1}(t_1))\gamma(t_2) = \sum f(m\gamma(h_1)\gamma^{-1}(t_1h_2))\gamma(t_2h_3) = \sum f(m\gamma^{-1}(t_1)\sigma(t_2, h_1))\gamma(t_3h_2) = \sum f(m\gamma^{-1}(t_1))\gamma(t_2)\gamma(h) = \bar{f}(m))\gamma(h)
\]

Thus \( \bar{f} \) is an \( R\#_\sigma H \)-module homomorphism. \( \Box \)

**Proposition 1.6** Let \( H \) be a finite-dimensional semisimple Hopf algebra.

(i) If \( P \) is a left (right) \( R\#_\sigma H \)-modules and a projective left (right) \( R \)-module, then \( P \) is a projective left (right) \( R\#_\sigma H \)-module;

(ii) If \( E \) is a left (right) \( R\#_\sigma H \)-modules and an injective left (right) \( R \)-module, then \( E \) is an injective left (right) \( R\#_\sigma H \)-module;

(iii) If \( F \) is a left (right) \( R\#_\sigma H \)-modules and a flat left (right) \( R \)-module, then \( F \) is a flat left (right) \( R\#_\sigma H \)-module.

**Proof.** (i) Let

\[
X \xrightarrow{f} Y \rightarrow 0
\]
be an exact sequence of left (right) $R\#_{\sigma}H$-modules and $g$ be a $R\#_{\sigma}H$-module homomorphism from $P$ to $Y$. Since $P$ is a projective left (right) $R$-module, we have that there exists a $R$-module homomorphism $\varphi$ from $P$ to $X$, such that

$$f\varphi = g.$$ 

By Lemma 1.4 and 1.5, there exists a $R\#_{\sigma}H$-module homomorphism $\bar{\varphi}$ from $P$ to $X$ such that

$$f\bar{\varphi} = g.$$
Thus \( P \) is a projective left (right ) \( R#_\sigma H \)-module.

Similarly, we can obtain the proof of part (ii).

(iii) Since \( F \) is a flat left (right ) \( R \)-module, we have the character module \( \text{Hom}_\mathbb{Z}(F, \mathbb{Q}/\mathbb{Z}) \) of \( F \) is a injective left (right ) \( R#_\sigma H \)-module. By part (ii), \( \text{Hom}_\mathbb{Z}(F, \mathbb{Q}/\mathbb{Z}) \) is a injective left (right ) \( R#_\sigma H \)-module. Thus \( F \) is a flat left (right ) \( R#_\sigma H \)-module. ✷

**Proposition 1.7** Let \( H \) be a finite-dimensional semisimple Hopf algebra. Then for left (right ) \( R#_\sigma H \)-modules \( M \) and \( N \),

\[
\text{Ext}^n_{R#_\sigma H}(M, N) \subseteq \text{Ext}^n_R(M, N),
\]

where \( n \) is any natural number.

**Proof.** We view the \( \text{Ext}^n(M, N) \) as the equivalent classes of \( n \)- extension of \( M \) and \( N \) (see, [8, Definition 3.3.7]). We only prove this result for \( n = 1 \). For other cases, we can similarly prove. We denote the equivalent classes in \( \text{Ext}^1_{R#_\sigma H}(M, N) \) and \( \text{Ext}^1_R(M, N) \) by \( [E] \) and \( [F]' \), respectively, where \( E \) is an extension of \( R#_\sigma H \)-modules \( M \) and \( N \), and \( F \) is an extension of \( R \)-modules \( M \) and \( N \). We define a map

\[
\Psi : \text{Ext}^1_{R#_\sigma H}(M, N) \to \text{Ext}^1_R(M, N), \quad \text{by sending } [E] \text{ to } [E]'.
\]

Obviously, \( \Psi \) is a map. Now we show that \( \Psi \) is injective. Let

\[
0 \to M \xrightarrow{f} E \xrightarrow{g} N \to 0 \quad \text{and} \quad 0 \to M \xrightarrow{f'} E' \xrightarrow{g'} M \to 0
\]

are two extensions of \( R#_\sigma H \)-modules \( M \) and \( N \), and they are equivalent in \( \text{Ext}^1_R(M, N) \). Thus there exists \( R \)-module homomorphism \( \varphi \) from \( E \) to \( E' \) such that

\[
\varphi f = f' \quad \text{and} \quad \varphi g = g'.
\]

By lemma 1.4 , there exists \( R#_\sigma H \)-module homomorphism \( \tilde{\varphi} \) from \( E \) to \( E' \) such that

\[
\tilde{\varphi} f = f' \quad \text{and} \quad \tilde{\varphi} g = g'.
\]

Thus \( E \) and \( E' \) is equivalent in \( \text{Ext}^1_{R#_\sigma H}(M, N) \), which implies that \( \Psi \) is injective. □
Lemma 1.8 For any \( M \in \mathcal{M}_{R \#_\sigma H} \) and \( N \in \mathcal{M}_{R \#_\sigma H} \), there exists an additive group homomorphism
\[
\xi : M \otimes_R N \to M \otimes_{R \#_\sigma H} N
\]
by sending \((m \otimes n)\) to \( m \otimes n\), where \( m \in M, n \in N\).

Proof. It is trivial. \( \square \)

Proposition 1.9 If \( M \) is a right \( R \#_\sigma H \)-module and \( N \) is a left \( R \#_\sigma H \)-module, then there exists additive group homomorphism
\[
\xi_* : \text{Tor}^R_n(M, N) \to \text{Tor}^{R \#_\sigma H}_n(M, N)
\]
such that \( \xi_*([z_n]) = [\xi(z_n)] \), where \( \xi \) is the same as in Lemma 1.8.

Proof. Let
\[
\mathcal{P}_M : \cdots P_n \overset{d_n}{\to} P_{n-1} \cdots \to P_0 \overset{d_0}{\to} M \to 0
\]
is a projective resolution of right \( R \#_\sigma H \)-module \( M \), and set
\[
T = - \otimes_{R \#_\sigma H} N \quad \text{and} \quad T^R = - \otimes_R N.
\]
We have that
\[
T \mathcal{P}_M : \cdots T(P_n) \overset{Td_n}{\to} T(P_{n-1}) \cdots \to T(P_1) \overset{Td_1}{\to} TP_0 \to 0
\]
and
\[
T^R \mathcal{P}_M : \cdots T^R(P_n) \overset{T^{Rd_n}}{\to} T^R(P_{n-1}) \cdots \to T^R(P_1) \overset{T^{Rd_1}}{\to} T^R(P_0) \to 0
\]
are complexes. Thus \( \xi \) is a complex homomorphism from \( T^R \mathcal{P}_M \) to \( T \mathcal{P}_M \), which implies that \( \xi_* \) is an additive group homomorphism. \( \square \)

2 The global dimensions and weak dimensions of crossed products

In this section we give the relation between homological dimensions of \( R \) and \( R \#_\sigma H \).
Lemma 2.1 If \( R \) and \( R' \) are Morita equivalent rings, then

(i) \( \text{rgD}(R) = \text{rgD}(R') \);
(ii) \( \text{lgD}(R) = \text{lgD}(R') \);
(iii) \( wD(R) = wD(R') \).

Proof. It is an immediate consequence of [1, Proposition 21.6, Exercise 22.12] \( \square \)

Theorem 2.2 Let \( H \) be a finite-dimensional semisimple Hopf algebra,

(i) \( \text{rgD}(R^\#_{\sigma}H) \leq \text{rgD}(R) \);
(ii) \( \text{lgD}(R^\#_{\sigma}H) \leq \text{lgD}(R) \);
(iii) \( wD(R^\#_{\sigma}H) \leq wD(R) \).

Proof. (i) When \( \text{lgD}(R) \) is infinite, obviously part (i) holds. Now we assume \( \text{lgD}(R) = n \). For any left \( R^\#_{\sigma}H \)-module \( M \), and a projective resolution of left \( R^\#_{\sigma}H \)-module \( M \):

\[
\mathcal{P}_M : \quad \cdots P_n \xrightarrow{d_n} P_{n-1} \cdots \xrightarrow{d_0} P_0 \xrightarrow{d_0} M \to 0,
\]

we have that \( \mathcal{P}_M \) is also a projective resolution of left \( R \)-module \( M \) by Lemma 1.1. Let \( K_n = \ker d_n \) be syzygy \( n \) of \( \mathcal{P}_M \). Since \( \text{lgD}(R) = n \), \( \text{Ext}^{n+1}_R(M, N) = 0 \) for any left \( R \)-module \( N \) by [8, Corollary 3.3.6]. Thus \( \text{Ext}^1_R(K_n, N) = 0 \), which implies \( K_n \) is a projective \( R \)-module. By Lemma 1.6 (i), \( K_n \) is a projective \( R^\#_{\sigma}H \)-module and \( \text{Ext}^{n+1}_{R^\#_{\sigma}H}(M, N) = 0 \) for any \( R^\#_{\sigma}H \)-module \( N \). Consequently,

\[ \text{lgD}(R^\#_{\sigma}H) \leq n = \text{lgD}(R) \quad \text{by [8, Corollary 3.3.6]} . \]

We complete the proof of part (i).
We can similarly show part (ii) and part (iii). \( \square \)

Theorem 2.3 Let \( H \) be a finite-dimensional semisimple and cosemisimple Hopf algebra. Then

(i) \( \text{rgD}(R) = \text{rgD}(R^\#_{\sigma}H) \);
(ii) \( \text{rgD}(R) = \text{rgD}(R^\#_{\sigma}H) \);
(iii) \( wD(R) = wD(R^\#_{\sigma}H) \).
Proof. (i) By dual theorem (see, [2, Corollary 9.4.17]), we have \((R\#\sigma H)\#H^*\) and \(R\) are Morita equivalent algebras. Thus \(\lg D(R) = \lg D((R\#\sigma H)\#H^*)\) by Lemma 2.1 (i). Considering Theorem 2.2 (i), we have that

\[
\lg D((R\#\sigma H)\#H^*) \leq \lg D(R\#\sigma H) \leq \lg D(R).
\]

Consequently,

\[
\lg D(R) = \lg D(R\#\sigma H).
\]

Similarly, we can prove (ii) and (iii). \(\square\)

**Corollary 2.4** Let \(H\) be a finite-dimensional semisimple Hopf algebra.

(i) If \(R\) left (right ) semi-hereditary, then so is \(R\#\sigma H\);

(ii) If \(R\) is von Neumann regular, then so is \(R\#\sigma H\).

Proof. (i) It follows from Theorem 2.2 and [8, Theorem 2.2.9].

(ii) It follows from Theorem 2.2 and [8, Theorem 3.4.13]. \(\square\)

By the way, part (ii) of Corollary 2.4 give one case about the semiprime question in [2, Question 7.4.9]. That is, If \(H\) is a finite-dimensional semisimple Hopf algebra and \(R\) is a von Neumann regular algebra (notice that every von Neumann regular algebra is semiprime ), then \(R\#\sigma H\) is semiprime.

**Corollary 2.5** Let \(H\) be a finite-dimensional semisimple and cosemisimple Hopf algebra. Then

(i) \(R\) is semisimple artinian iff \(R\#\sigma H\) is semisimple artinian;

(ii) \(R\) is left (right ) semi-hereditary iff \(R\#\sigma H\) is left (right ) semi-hereditary;

(iii) \(R\) is von Neumann regular iff \(R\#\sigma H\) is von Neumann regular.

Proof. (i) It follows from Theorem 2.3 and [8, Theorem 2.2.9].

(ii) It follows from Theorem 2.3 and [8, Theorem 2.2.9].

(iii) It follows from Theorem 2.3 and [8, Theorem 3.4.13]. \(\square\)

If \(H\) is commutative or cocommutative, then \(S^2 = id_H\) by [7]. Consequently, by [6, Proposition 2 (c)], \(H\) is semisimple and cosemisimple iff the character \(ch_{ark}\) of \(k\) does not divides \(dim H\). Considering Theorem 2.3 and Corollary 2.5, we have:
Corollary 2.6  Let $H$ be a finite-dimensional commutative or cocommutative Hopf algebra. If the character $\text{char} k$ of $k$ does not divides $\dim H$, then

(i) $\text{rg} D(R) = \text{rg} D(R \#_{\sigma} H)$;
(ii) $\text{rg} D(R) = \text{rg} D(R \#_{\sigma} H);$ 
(iii) $wD(R) = wD(R \#_{\sigma} H);$ 
(iv) $R$ is semisimple artinian iff $R \#_{\sigma} H$ is semisimple artinian;
(v) $R$ is left (right ) semi-hereditary iff $R \#_{\sigma} H$ is left (right ) semi-hereditary;
(vi) $R$ is von Neumann regular iff $R \#_{\sigma} H$ is von Neumann regular.

Since group algebra $kG$ is a cocommutative Hopf algebra, we have that

$$\text{rg} D(R) = \text{rg} D(R * G).$$

Thus Corollary 2.6 implies in [4, Theorem 7.5.6].

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