Entanglement witnesses arising from Choi type positive linear maps

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Abstract

We construct optimal PPTES witnesses to detect $3 \otimes 3$ PPT entangled edge states of type $(6, 8)$ constructed recently by Kye and Osaka (2012 \textit{J. Math. Phys.} \textbf{53} 052201). To do this, we consider positive linear maps which are variants of the Choi type map involving complex numbers, and examine several notions related to the optimality of those entanglement witnesses. Throughout the discussion, we suggest a method to check the optimality of entanglement witnesses without the spanning property.

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1. Introduction

The notion of quantum entanglement plays a key role in the current study of quantum information and quantum computation theory. There are two main criteria to distinguish entanglement from separable states: the PPT criterion [2, 3] tells us that the partial transpose of a separable state is positive, that is, positive semi-definite. The converse is not true in general by a work of Woronowicz [4] who gave an example of a $2 \otimes 4$ PPT entangled state. Such examples were also given in [2, 5] for the $3 \otimes 3$ cases, in the early eighties. Another criterion was given by Horodecki et al [6] using positive linear maps between matrix algebras, and this was formulated as the notion of entanglement witnesses [7]. An equivalent formulation is obtained by using the duality theory [8] between the positivity of linear maps and separability of block matrices, through the Jamiołkowski–Choi (JC) isomorphism [9, 10] Through this isomorphism, an entanglement witness is just a positive linear map which is not completely positive. We refer to [11, 12] for systematic approaches to the duality using the JC isomorphism.
For the convenience of readers, we recall some notions on positive linear maps. Important classes of positive linear maps from $M_m$ to $M_n$ come from elementary operators together with the transpose map:

\[ \phi_V : X \rightarrow V^*XV, \quad \phi^V : X \rightarrow V^*X^V, \]

where $V$ is an $m \times n$ matrix with the Hermitian conjugate $V^*$. The convex sums of the first (respectively second) types are said to be completely positive (respectively completely copositive) linear maps, and the convex sums of completely positive linear maps and completely copositive linear maps are said to be decomposable positive linear maps. If a positive linear map is not decomposable, we call it indecomposable. Note that for a completely positive map $\phi$, $(I \otimes \phi)(A)$ is positive for any positive operator $A$ on the combined system, where $I$ denotes the identity map on the first system. This is the reason why completely positive linear maps are used as quantum operations and channels.

For a linear map $\phi$ from the $C^*$-algebra $M_m$ of all $m \times m$ matrices into $M_n$, the Choi matrix $C_\phi$ of $\phi$ is given by

\[ C_\phi = \sum_{i,j} e_{ij} \otimes \phi(e_{ij}) \in M_m \otimes M_n, \]

where $e_{ij} = |i\rangle \langle j|$ is the usual matrix unit in $M_m$. The correspondence $\phi \mapsto C_\phi$ is called the JC isomorphism. It is known that $\phi$ is positive if and only if $C_\phi$ is block-positive, and $\phi$ is completely positive if and only if $C_\phi$ is positive. For a linear map $\phi : M_m \rightarrow M_n$ and a block matrix $A \in M_m \otimes M_n$, we define the bilinear pairing by

\[ \langle A, \phi \rangle = \text{Tr} (AC_\phi^*). \]

It turns out that $A$ is separable if and only if $\langle A, \phi \rangle \geq 0$ for each positive map $\phi$, and $A$ is of PPT if and only if $\langle A, \phi \rangle \geq 0$ for each decomposable positive map $\phi$. Therefore, each entangled state $A$ is detected by a positive linear map $\phi$ in the sense that $\langle A, \phi \rangle < 0$, and each PPT entangled state is detected by an indecomposable positive linear map. A positive linear map is said to be an optimal entanglement witness if it detects a maximal set (in the sense of inclusion) of entangled states, and an optimal PPTES witness if it detects a maximal set of PPT entangled states, as was introduced in [13]. See [14] for comments on the terminology. For a given entangled state, it is easy to find an entanglement witness to detect it, as suggested in [13]. However, it is not clear at all how to construct an optimal entanglement witness to detect a given entangled state. The primary purpose of this paper is to construct optimal PPTES witnesses which detect PPT entangled edge states constructed in [1]. Note that the states considered in [1] are the first examples of two qutrit PPT entangled edge states of type $(6, 8)$, whose existence had been a long standing question [15]. We also suggest a method to check the optimality of entanglement witnesses without the so-called spanning property. Of course, we know that $W$ is an optimal entanglement witness if $W$ has the spanning property [13], that is the set $\{\xi, \eta \} \in \mathbb{C}^m \otimes \mathbb{C}^n : \langle \xi, \eta | W | \xi, \eta \rangle = 0$ spans the whole space $\mathbb{C}^m \otimes \mathbb{C}^n$. We say that $W$ has the co-spanning property if a partial transpose of $W$ has the spanning property.

For non-negative real numbers $a, b, c$ and $-\pi \leq \theta \leq \pi$, we consider the map $\Phi[a, b, c; \theta]$ acting in $M_3$ defined by

\[ \Phi[a, b, c; \theta](X) = \begin{pmatrix} ax_{11} + bx_{22} + cx_{33} & -e^{i\theta}x_{12} & -e^{-i\theta}x_{13} \\ -e^{-i\theta}x_{21} & cx_{11} + ax_{22} + bx_{33} & -e^{i\theta}x_{23} \\ -e^{i\theta}x_{31} & -e^{-i\theta}x_{32} & bx_{11} + cx_{22} + ax_{33} \end{pmatrix}, \]
for $X \in M_3$. Note that the Choi matrix $C_\phi$ of the map $\Phi[a, b, c; \theta]$ is given by

$$W[a, b, c; \theta] = \begin{pmatrix}
a & -e^{i\theta} & 
- \theta & b & 
- e^{-i\theta} & c & a
\end{pmatrix}.$$ 

The map of the form $\Phi[a, b, c; 0]$ and its variants have been investigated by many authors in various contexts, as was summarized in [14]. We just note here that $W[a, b, c; 0]$ is separable if and only if it is of PPT [1]. On the other hand, many interesting examples of PPT states are of the form $W[a, b, c; \theta]$. For example, the PPT entangled states [5] considered in the early 1980s are just $W[1, b, 1/b, \pi]$ which turn out to be PPT entangled edge states of type $(6, 7)$. These states were reconstructed systematically from indecomposable positive linear maps together with other types of PPT entangled edge states [16]. The PPT entangled edge states of type $(6, 8)$ constructed in [1] are given by $W[e^{i\theta} + e^{-i\theta}, b, 1/b; \theta]$ for $-\pi/3 < \theta < \pi/3$ and $\theta \neq 0$. Recently, the authors [17] analyzed $W[a, b, c; \pi]$ to understand the boundary structures between separable and entangled PPT states.

For indecomposable positive linear maps, some attention needs to be paid to the actual meaning of the term ‘optimal’ as was pointed out in [14]. We denote by $\mathbb{P}_1$ the convex cone of all positive linear maps. Recall that a positive map $\phi$ is an optimal (respectively co-optimal) entanglement witness if and only if the smallest face of $\mathbb{P}_1$ determined by $\phi$ has no completely positive (respectively completely copositive) map. We also note that $\phi$ has the spanning property (respectively co-spanning property) if and only if the smallest exposed face of $\mathbb{P}_1$ determined by $\phi$ has no completely positive (respectively completely copositive) map. We say that $\phi$ is bi-optimal if it is both optimal and co-optimal. The term bi-spanning is defined similarly.

In the following section, we give conditions on parameters $a, b, c$ and $\theta$ for which the map $\Phi[a, b, c; \theta]$ is a positive linear map. Next, in section 3 we characterize for each fixed $\theta$ the facial structures of the three-dimensional convex body of positive maps $\Phi[a, b, c; \theta]$. From these facial structures, it is clear that some of positive maps are not optimal, and/or not co-optimal. We examine the spanning and co-spanning properties for the family of maps $\Phi[a, b, c; \theta]$ in section 4, and check various notions of optimality for all possible cases in section 5. To do this, we suggest a more efficient method to check the optimality of an entanglement witness when it does not have the spanning property. In the section 6, we find optimal entanglement witnesses to detect PPT entangled edge states [1] of type $(6, 8)$. We conclude this paper by reporting that our constructions give counter-examples to the SPA conjecture [18].

From now on, each vector in $\mathbb{C}^m$ may be considered as an $m \times 1$ matrix, and we denote by $\bar{x}$ and $x^*$ the complex conjugate and the Hermitian conjugate of $x$, respectively. In this notation, rank 1 matrix $|x\rangle\langle y|$ is written by $xy^*$.

2. Positivity

To begin with, we first find the conditions for complete positivity and complete copositivity. We note that the map $\Phi[a, b, c; \theta]$ is completely positive if and only if $W[a, b, c; \theta]$ is positive
if and only if the following $3 \times 3$ matrix:

$$P[a, \theta] = \begin{pmatrix} a & -e^{-i\theta} & -e^{i\theta} \\ -e^{i\theta} & a & -e^{-i\theta} \\ -e^{-i\theta} & -e^{i\theta} & a \end{pmatrix}$$

is positive. We mention again that ‘positivity’ of matrices means the positive semi-definiteness, throughout this paper. We see that the polynomial

$$\det P[a, \theta] = a^3 - 3a - (e^{3i\theta} + e^{-3i\theta})$$

$$= [a - (e^{i\theta} + e^{-i\theta})][a^2 + a(e^{i\theta} + e^{-i\theta}) + (e^{2i\theta} + e^{-2i\theta} - 1)]$$

has the following three real zeros:

$$q_{(\theta-\frac{\pi}{3})} = e^{i(\theta-\frac{2\pi}{3})} + e^{-i(\theta-\frac{2\pi}{3})}, \quad q_{\theta} = e^{i\theta} + e^{-i\theta}, \quad q_{(\theta+\frac{2\pi}{3})} = e^{i(\theta+\frac{2\pi}{3})} + e^{-i(\theta+\frac{2\pi}{3})}.$$  

We denote by

$$p_\theta = \max\{q_{(\theta-\frac{\pi}{3})}, q_{\theta}, q_{(\theta+\frac{\pi}{3})}\}.$$  

Then we see that $1 \leq p_\theta \leq 2$ for each $\theta$ (see figure 1), and $P[a, \theta]$ is positive if and only if

$$a \geq p_\theta,$$  

(1)

if and only if $\Phi[a, b, c; \theta]$ is completely positive. We note that $p_\theta = 2$ if and only if $\theta = 0, \pm 2\pi / 3$, and $p_\theta = 1$ if and only if $\theta = \pm \pi / 3, \pm \pi$. It is easy to see that the map $\Phi[a, b, c; \theta]$ is completely copositive if and only if

$$bc \geq 1.$$  

(2)

**Theorem 2.1.** The map $\Phi[a, b, c; \theta]$ is completely positive if and only if condition (1) holds, and completely copositive if and only if condition (2) holds.

In order to obtain a necessary condition for the positivity of $\Phi[a, b, c; \theta]$, we note that a linear map $\phi$ is positive if and only if $(zz^*, \phi) \geq 0$ for each product vector $z = x \otimes y \in \mathbb{C}^n \otimes \mathbb{C}^n$. Here, $z$ is considered as a column vector, and so $zz^*$ belongs to $M_n \otimes M_n$. We write

$$z = x \otimes y = (x_1y_1, x_1y_2, x_1y_3; x_2y_1, x_2y_2, x_2y_3; x_3y_1, x_3y_2, x_3y_3)^t.$$  

By a direct calculation, we see that the pairing $(zz^*, \Phi[a, b, c; \theta])$ is equal to

$$a(x_1y_1^2 + x_2y_2^2 + x_3y_3^2) + b(x_1y_3^2 + x_2y_1^2 + x_3y_2^2) + c(x_2y_2^2 + x_3y_3^2 + x_1y_1^2) - e^{i\theta}x_1y_1x_2y_2 - e^{-i\theta}x_1y_1x_3y_3 - e^{i\theta}x_2y_2x_3y_3 - e^{-i\theta}x_2y_2x_1y_1 - e^{i\theta}x_3y_3x_1y_1.$$  

From now on, we suppose that $\Phi[a, b, c; \theta]$ is positive, and put the product vectors

$$(1, 1, 1)^t \otimes (1, 1, 1)^t, \quad (e^{i\frac{2\pi}{3}}, 1, 1)^t \otimes (1, 1, 1)^t, \quad (e^{-i\frac{2\pi}{3}}, 1, 1)^t \otimes (1, 1, 1)^t$$  

Figure 1. The graph of $y = p_\theta$. 


in the above quantity, to obtain the following necessary condition:
\[ a + b + c \geq p_0. \] (3)

We also take product vectors
\[
\begin{align*}
(\sqrt{e^{-t^2}}, t, 0)^\dagger \otimes (\sqrt{t}, 1, 0)^\dagger, \\
(\sqrt{e^{-t^2}}, t, 0)^\dagger \otimes (\sqrt{t}, e^{i2\pi/3}, 0)^\dagger, \\
(\sqrt{e^{-t^2}}, t, 0)^\dagger \otimes (\sqrt{t}, e^{-i2\pi/3}, 0)^\dagger
\end{align*}
\]
for \( t \geq 0 \), to obtain the condition \( 2a t^2 + c t + b t^3 \geq 2r^2 \) for each \( t \geq 0 \) if and only if
\[ a \leq 1 \implies bc \geq (1 - a)^2. \] (4)

Therefore, we obtain necessary conditions (3) and (4) for the positivity of \( \Phi[a, b, c; \theta] \).

Now, we proceed to show that two conditions (3) and (4) are sufficient for positivity. Note
that \( \Phi[a, b, c; \theta] \) is positive if and only if the matrix
\[
\begin{pmatrix}
(a|x|^2 + b|y|^2 + c|z|^2) & -e^{i\theta}xy & -e^{i\theta}xz \\
-e^{i\theta}yx & (a|x|^2 + b|y|^2 + c|z|^2) & -e^{i\theta}yz \\
-e^{i\theta}zx & -e^{i\theta}zy & (b|x|^2 + c|y|^2 + a|z|^2)
\end{pmatrix}
\] (5)
is positive semi-definite for any \((x, y, z) \in \mathbb{C}^3\). We first consider the determinant
\[
\begin{align*}
(a|x|^2 + b|y|^2 + c|z|^2)(c|x|^2 + a|y|^2 + b|z|^2)(b|x|^2 + c|y|^2 + a|z|^2) &
- (e^{i\theta} + e^{-i\theta})|xy|z^2 \\
-(a|x|^2 + b|y|^2 + c|z|^2)|yz|^2 &
- (c|x|^2 + a|y|^2 + b|z|^2)|zx|^2 \\
&
- (b|x|^2 + c|y|^2 + a|z|^2)|xy|^2.
\end{align*}
\]
We may replace \(|x|^2, |y|^2 \) and \(|z|^2 \) by non-negative \(x, y \) and \(z\) to obtain
\[
F(x, y, z) := (ax + by + cz)(cx + ay + bz)(bx + cy + az) - (e^{i\theta} + e^{-i\theta})xyz \\
- (ax + by + cz)yz - (cx + ay + bz)zx - (bx + cy + az)xy.
\]
Now, we check that all the \(2 \times 2\) principal minors are non-negative. For example, the third
\(2 \times 2\) minor is
\[
M_1 := (cx + ay + bz)(bx + cy + az) - yz,
\]
and we have
\[
M_1 \geq \frac{1}{by + cz}F(0, y, z) = (ay + bz)(cy + az) - yz = acy^2 + (a^2 + bc - 1)yz + abz^2.
\]
This is non-negative when \( a \geq 1 \). If \( 0 \leq a \leq 1 \), then we use condition (4) to see easily that
this quadratic form is non-negative for each \( y, z \geq 0 \). In the same way, we see that all \(2 \times 2\)
minors are non-negative, and \( F(x, y, z) \geq 0 \) whenever one of \(x, y\) or \(z\) is zero.

Now, we show that \( F(x, y, z) \geq 0 \) in the region \{(x, y, z) : x, y, z > 0\}. First, we note that
all of \(\partial F/\partial x, \partial F/\partial y\) and \(\partial F/\partial z\) are quadratic forms associated with the
following symmetric matrices:
\[
\begin{pmatrix}
P & R & Q \\
R & Q & S \\
Q & S & R
\end{pmatrix}, \quad
\begin{pmatrix}
R & Q & S \\
Q & S & R \\
S & R & Q
\end{pmatrix}, \quad
\begin{pmatrix}
Q & S & R \\
S & R & Q \\
R & Q & P
\end{pmatrix},
\]
where
\[
\begin{align*}
P &= 3abc, \\
Q &= a^2 c + b^2 a + c^2 b - c, \\
R &= a^2 b + b^2 c + c^2 a - b, \\
2S &= a^3 + b^3 + c^3 + 3abc - 3a - (e^{i\theta} + e^{-i\theta}).
\end{align*}
\]
That is, \( \partial F / \partial x(x, y, z) \) is expressed by
\[
\frac{\partial F}{\partial x}(x, y, z) = \begin{pmatrix} P & R & Q \\ R & Q & S \\ Q & S & R \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},
\]
and similarly for \( \partial F / \partial y \) and \( \partial F / \partial z \). In the case of \( a \geq 1 \), we have
\[
P + Q + R \geq 3bc + (b^2 + c^2) + bc(b + c) \geq 0,
\]
and the equality holds if and only if \( b = c = 0 \). In the case of \( 0 \leq a < 1 \), we have \( bc \geq (1-a)^2 \) by (4). Hence, it follows that
\[
P + Q + R = abc + (b + c)[a^2 + a(b + c) + bc - 1]
\[\geq abc + (b + c)[a^2 + 2a(1-a) + (1-a)^2 - 1] = abc,
\]
and the equality holds if and only if \( a = 0, bc = 1 \). Consequently, we have \( P + Q + R \geq 0 \) for all cases.

First, we consider the case \( P + Q + R = 0 \), which can happen in the following two cases:
- \( a = 0, bc = 1 \),
- \( a \geq 1, b = c = 0 \).

In the first case, we already know that the map is completely copositive. In the second case, we have \( a \geq p_0 \) by (3), and so the map is completely positive. Therefore, if \( P + Q + R = 0 \), then we see that \( \Phi[a, b, c; 0] \) is positive.

Now, we assume that \( P + Q + R > 0 \). In this case, we have \( abc \neq 0 \) except for the following two cases:
- \( ab \neq 0 \) and \( c = 0 \),
- \( ac \neq 0 \) and \( b = 0 \).

For the case of \( ab \neq 0 \) and \( c = 0 \), we have \( a \geq 1 \) from condition (4), and so
\[
F(x, y, z) = b(a^2 - 1)(x^2 y + y^2 z + z^2 x) + ab^2(x^2 z + y^2 x + z^2 y)
\[+ [a^3 + b^3 - 3a - (e^{3i\theta} + e^{-3i\theta})]xyz
\[\geq [3b(a^2 - 1) + 3ab^2 + a^3 + b^3 - 3a - (e^{3i\theta} + e^{-3i\theta})]xyz
\[= [(a + b)^3 - 3(a + b) - (e^{3i\theta} + e^{-3i\theta})]xyz,
\]
which is non-negative by condition (3). Similarly, one can show that \( F(x, y, z) \) is non-negative for the case of \( ac \neq 0 \) and \( b = 0 \).

Now, we consider the case of \( abc \neq 0 \). In this case, the coefficients of \( x^2 \), \( y^2 \) and \( z^2 \) in the polynomial \( F \) are positive, and so there exists a sufficiently large cube \( R = \{(x, y, z) : 0 \leq x, y, z \leq M\} \) such that \( F(x, y, z) \geq 0 \) outside of \( R \). Furthermore, we already know that \( F(x, y, z) \geq 0 \) if \( yz = 0 \). Therefore, it suffices to show that the local minima of \( F \) in the region \( \{(x, y, z) : x, y, z > 0\} \) are non-negative.

If \( (x, y, z) \) is a nontrivial common solution of \( \partial F / \partial x, \partial F / \partial y \) and \( \partial F / \partial z \), then it is also a nontrivial solution of homogeneous quadratic equation given by
\[
\begin{pmatrix}
P + Q + R & Q + R + S & Q + R + S \\
Q + R + S & P + Q + R & Q + R + S \\
Q + R + S & Q + R + S & P + Q + R
\end{pmatrix}.
\]
This means that a nontrivial common solution \( (x, y, z) \) of \( \partial F / \partial x, \partial F / \partial y \) and \( \partial F / \partial z \) satisfies
\[
(x \ y \ z) \begin{pmatrix}
P + Q + R & Q + R + S & Q + R + S \\
Q + R + S & P + Q + R & Q + R + S \\
Q + R + S & Q + R + S & P + Q + R
\end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}
\[= (P + Q + R)(x^2 + y^2 + z^2) + 2(Q + R + S)(xy + yz + zx) = 0.
\]
If \( P = S \), then common solutions are on the plane \( x + y + z = 0 \), and so there is no nonzero common solution in the region \( \{ (x, y, z) : x, y, z > 0 \} \). Therefore, we may assume \( P \neq S \). In this case, the \( 2 \times 2 \) minors are not zero, and so the rank of the matrix (6) is not less than 2. We also note that the determinant of this matrix (6) is

\[(P - S)^2(3Q + 3R + P + 2S) = (P - S)^2[(a + b + c)^3 - 3(a + b + c) - (e^{3i\theta} + e^{-3i\theta})] \geq 0\]

by condition (3). If \( P > S \), we see that the matrix (6) is positive semi-definite. Thus, if equation (7) is satisfied for some nontrivial \((x, y, z)\), matrix (6) must be singular and of rank 2. Therefore, a common solution must belong to the one-dimensional kernel space of the matrix (6). Consequently, all common solutions are of the form \((x, x, x)\). Now, consider the case \( P < S \). In this case, we see that the common solutions satisfy

\[(P + Q + R)(x^2 + y^2 + z^2) + 2(Q + R + S)(xy + yz + zx)\]
\[= (P + Q + R)(x + y + z)^2 + 2(S - P)(xy + yz + zx) = 0\]

which is impossible in the region \( \{ (x, y, z) : x, y, z > 0 \} \). All in all, we see that if \( F \) takes a local minimum at \((x, y, z)\) with \( x, y, z > 0 \) then \( x = y = z \). We note that

\[
\frac{1}{x^3} F(x, x, x) = a^3 + b^3 + c^3 + 3(a^2b + a^2c + b^2c + b^2a + c^2a + c^2b + 2abc) - (e^{3i\theta} + e^{-3i\theta}) - 3(a + b + c)
\]

which is non-negative by (3) when \( x \neq 0 \). This completes the proof for the following

**Theorem 2.2.** The map \( \Phi[a, b, c; \theta] \) is positive if and only if both conditions (3) and (4) hold.

### 3. Facial structures

For each \( \theta \), we denote by \( \Gamma^\theta \) the three-dimensional convex body determined by (3) and (4) (see figure 2). The facial structures of the convex body \( \Gamma^\theta \) has been analyzed in [14] for the case of \( \theta = 0 \). Facial structures of \( \Gamma^\theta \) are similar to those of \( \Gamma^0 \) except for several differences. We first consider the case \( 1 < p_\theta < 2 \). In this case, the convex body \( \Gamma^\theta \) has the following four two-dimensional faces:

- \( f^\theta_{ab} = \{(a, b, c) : c = 0, a + b \geq p_\theta, a \geq 1\} \),
- \( f^\theta_{bc} = \{(a, b, c) : b = 0, a + c \geq p_\theta, a \geq 1\} \),
- \( f^\theta_{ac} = \{(a, b, c) : a = 0, a + c \geq p_\theta, b \geq 1\} \),
- \( f^\theta_{ca} = \{(a, b, c) : b = 0, a \geq p_\theta, c \geq 1\} \).
The shape of the face $f_{abc}$ shrinks to a single point.

In the case of $p_\theta = 1$, the face $f_{abc}^\theta$ shrinks to a single point $(1 \ 0 \ 0)$. In order to figure out the shape of the face $f_{abc}^\theta$ (see figure 3), we modify the parametrization in [19] and put

$$a_\theta(t) = (p_\theta - 1) \cdot \frac{(1-t)^2}{1-t^2} + (2-p_\theta) = 1 - \frac{(p_\theta - 1)t}{1-t+t^2},$$

$$b_\theta(t) = (p_\theta - 1) \cdot \frac{t^2}{1-t+t^2},$$

$$c_\theta(t) = (p_\theta - 1) \cdot \frac{1}{1-t+t^2},$$

for $0 < t < \infty$. Then we have

$$a_\theta(t) + b_\theta(t) + c_\theta(t) = p_\theta, \quad 0 \leq a_\theta(t) \leq 1, \quad b_\theta(t)c_\theta(t) = (1-a_\theta(t))^2.$$ 

If $p_\theta = 2$, then this face touches the face $f_{bc}$ at the point $(0, 1, 1)$ which corresponds to a completely copositive map. On the other hand, if $1 < p_\theta < 2$, then this face does not touch the face $f_{bc}$.

The convex body $\Gamma^\theta$ also has the following one-dimensional faces:

- $e^\theta_a = \{(a, 0, 0) : a \geq p_\theta\}$,
- $e^\theta_b = \{(1, b, 0) : b \geq p_\theta - 1\}$,
- $e^\theta_c = \{(1, 0, c) : c \geq p_\theta - 1\}$,
- $e^\theta_{ab} = \{(a, b, 0) : a + b = p_\theta, \ 1 \leq a \leq p_\theta\}$,
- $e^\theta_{ac} = \{(a, 0, c) : a + c = p_\theta, \ 1 \leq a \leq p_\theta\}$,
- $e^\theta_{bc} = \{(1-s, s, s/t) : (p_\theta - 1)t(1-t+t^2)^{-1} \leq s \leq 1\}$ for $t > 0$.

We note that $e^\theta_t$ is the line segment from the point $(a_\theta(t), b_\theta(t), c_\theta(t))$ to the point $(0, 1, 1/t)$, and lies on the surface $bc = (1-a)^2$ for $0 \leq a < 1$. We also note that $e^\theta_t$ shrinks to a single point $(0, 1, 1)$ if $p_\theta = 2$. It remains to list up zero-dimensional faces $v_{(a, \beta, \gamma)} = \{(a, \beta, \gamma)\}$ as follows:

- $v(0, p_\theta, 0)$,
- $v(1, 0, p_\theta-1)$, $v(1, p_\theta-1, 0)$,
- $v(a_\theta(t), b_\theta(t), c_\theta(t))$ for $t > 0$,
- $v(0, 1, 1/t)$ for $t > 0$.

Figure 3. Picture of the face $f_{abc}^\theta$ in the plane $a+b+c = p_\theta$. 

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It \( p_0 = 1 \), then all of the following faces:

\[
\begin{align*}
    f_{abc}^0, & \quad e_{ab}^0, \quad e_{ac}^0, \quad v(p_0,0,0), \quad v(0,0,p_0-1), \quad v(1,0,p_0-1), \quad v(1,p_0-1,0), \quad v(0,1,0) \quad v(0,1,0,\alpha(t),0,0,0,0)
\end{align*}
\]

shrink to the single point \((1,0,0)\). Furthermore, the face \( e_i^0 \) connects the two points \((1,0,0)\) and \((0,1,1)\) which represent completely positive and completely copositive maps, respectively. Therefore, for \( p_0 = 1 \), each positive linear map \( \Phi[a, b, c; \theta] \) is decomposable. Since we are interested in indecomposable cases, we assume \( p_0 > 1 \) throughout this paper.

By the exactly same argument as in [14], we have the following.

- Interior points of \( f_{abc}^0, f_{ac}^0, f_{bc}^0, e_{ab}^0, e_{ac}^0 \), and \( e_{bc}^0 \) are neither optimal nor co-optimal.
- Interior points of \( f_{abc}^0, f_{ac}^0, f_{bc}^0 \) are not optimal.
- Interior points of \( e_{ab}^0 \) and \( v(0,1,1,0) \) are not co-optimal.
- \( v(1,0,0) \) and \( v(1,1,0) \) do not have the spanning property.

We recall that if two positive maps \( \phi_1 \) and \( \phi_2 \) determine a common smallest face, then they are interior points of the common face and they have the same spanning and optimality properties.

### 4. Spanning properties

In this section, we determine which positive linear maps have the spanning property and/or the co-spans the whole space

\[
\text{co-spanning property. We remind the readers that we are assuming that}
\]

Therefore, the matrix (5) is singular with

\[
(2) 2 < a < \text{a} < (4) 1 \]

If \( 0 < a = 1 \neq a \) and \( b = (1 - a)^2, a = b + c = p_0 \),

\[
(3) a = 0, \quad b = 1, \quad c = p_0.
\]

We recall [20] that \( \phi \in \mathbb{P}_1 \) has the spanning property if and only if the set

\[
P[\phi] := \{ z = \xi \otimes \eta : \langle zz^*, \phi \rangle = 0 \}
\]

spans the whole space \( \mathbb{C}^m \otimes \mathbb{C}^n \), where

\[
\langle zz^*, \phi \rangle = \langle \xi \xi^* \otimes \eta \eta^* \rangle = \text{Tr}(\phi(\xi \xi^*)\eta \eta^*) = \langle \phi(\xi \xi^*)\eta \eta^* \rangle.
\]

Therefore, we see that \( \xi \otimes \eta \in P[\phi] \) if and only if \( \phi(\xi \xi^*)\eta \eta^* = 0 \). In order to determine the set \( P[\Phi[a, b, c; \theta]] \), we first find vectors \( (x, y, z) \in \mathbb{C}^3 \) such that the matrix (5) is singular. In other words, we look for \( (x, y, z) \) for which \( F(x, y, z) = 0 \). The only possibility for \( F(x, y, z) = 0 \) with \( xyz \neq 0 \) is \( F(x, x, x) = 0 \), and this happens only when the equality holds in (3).

Now, we consider case (i). In this case, we see that \( F(x, y, z) = 0 \) holds only if \( xyz = 0 \). We first consider the case \( z = 0 \), for which we have

\[
F(x, y, z) = |a| \sqrt{x}^2 + (a^2 + bc - 1)xy + aby^2)(bx + cy) = a(\sqrt{c}x - \sqrt{b}y)^2 (bx + cy).
\]

Therefore, the matrix (5) is singular with \( z = 0 \) if and only if \( (x, y, 0) = (b^{1/4} \alpha, c^{1/4} \beta, 0) \) with complex numbers \( \alpha, \beta \) with modulus 1.

In this case, (5) is given by

\[
\begin{pmatrix}
\frac{\sqrt{b}}{\sqrt{c}} & -e^{-i \theta} a \beta \sqrt{1 - a} & 0 \\
-e^{-i \theta} a \beta \sqrt{1 - a} & \sqrt{c} & 0 \\
0 & 0 & b \sqrt{b} + c \sqrt{c}
\end{pmatrix}
\]
and the kernel is \((\alpha e^{i\theta} \sqrt{1 - a}, \beta \sqrt{b}, 0)\). In the same way, we see that \(z\) belongs to \(P[\Phi[a, b, c, \theta]]\) if and only if \(z\) is one of the following:

\[
\begin{align*}
z_1[\alpha, \beta] &= (b^{1/4} \alpha, c^{1/4} \beta, 0)^t \otimes (\bar{a} \alpha e^{-i\theta} \sqrt{1 - a}, \bar{b} \sqrt{b}, 0)^t, \\
z_2[\alpha, \beta] &= (c^{1/4} \beta, 0, b^{1/4} \alpha)^t \otimes (\bar{b} \sqrt{b}, 0, \bar{a} \alpha e^{-i\theta} \sqrt{1 - a})^t, \\
z_3[\alpha, \beta] &= (0, b^{1/4} \alpha, c^{1/4} \beta)^t \otimes (0, \bar{a} \alpha e^{-i\theta} \sqrt{1 - a}, \bar{b} \sqrt{b})^t,
\end{align*}
\]

(9)

with complex numbers \(a, \beta\) with modulus 1. It is clear that the above vectors do not span the whole space if \(a = 1\), which implies \(bc = 0\).

Now, let us consider the case \(0 < a < 1\). We take \(\beta_1 = 1, \beta_2 = -1,\) and \(\beta_3 = i\), and consider the \(9 \times 9\) matrix whose columns are nine vectors \(z_k[\alpha, \beta_\ell]\) for \(k, \ell = 1, 2, 3\). Then the determinant of \(M\) is given by

\[
|\det M| = |64 b^2 c^4 e^{-3i\theta} i(1 + e^{-3i\theta})|
\]

which is nonzero, since \(\theta \neq \pm \pi/3, \pm \pi,\) and \(a < 1\) implies that \(bc \neq 0\). Therefore, we conclude that \(\Phi[a, b, c, \theta]\) has the spanning property if and only if \(a < 1\) for case (i). It is clear that it does not have the co-spanning property from the discussion of the facial structures in the previous section.

Now, we consider case (ii). First of all, we note that product vectors in (9) already belong to \(P[\Phi[a, b, c, \theta]]\), and so we see that \(\Phi[a, b, c, \theta]\) has the spanning property if \(a < 1\). We will see in the following section that \(\Phi[1, 0, p_0 - 1; \theta]\) and \(\Phi[1, p_0 - 1, 0; \theta]\) are optimal, but do not have the spanning property. We look for other product vectors in \(P[\Phi[a, b, c, \theta]]\) to check if \(\Phi[a, b, c, \theta]\) has the co-spanning property. If \(x = (x_1, x_2, x_3)^t\) with \(|x_1| = |x_2| = |x_3|\), then \(\Phi[a, b, c, \theta](xx^\top)\) is given by

\[
\begin{pmatrix}
|x_1|^2 p_0 & -e^{i\theta} x_1 \bar{x}_2 & -e^{-i\theta} x_1 \bar{x}_3 \\
-e^{i\theta} x_2 \bar{x}_1 & |x_2|^2 p_0 & -e^{-i\theta} x_2 \bar{x}_3 \\
-e^{i\theta} x_3 \bar{x}_1 & -e^{-i\theta} x_3 \bar{x}_2 & |x_3|^2 p_0
\end{pmatrix}
\]

for which

- \((x_1, x_2 e^{2i/3}, x_3 e^{-i/3})^t\) is a kernel element if \(-\pi \leq \theta \leq -\pi/3;\)
- \((x_1, x_2, x_3)^t\) is a kernel element if \(-\pi/3 \leq \theta \leq \pi/3;\)
- \((x_1, x_2 e^{-2i/3}, x_3 e^{i/3})^t\) is a kernel element if \(\pi/3 \leq \theta \leq \pi;\)

Therefore, we see that \(z \in P[\Phi[a, b, c, \theta]]\) if and only if \(z\) is any of the vectors in (9), or one of the following form:

\[
w[\alpha, \beta, \gamma] = (\alpha, \beta, \gamma)^t \otimes (\bar{\alpha}, \bar{\beta}, e^{-i\theta} e^{2i\theta/3})^t \quad \text{if} \quad -\pi < \theta < -\pi/3,
\]

\[
w[\alpha, \beta, \gamma] = (\alpha, \beta, \gamma)^t \otimes (\bar{\alpha}, \bar{\beta}, \gamma)^t \quad \text{if} \quad -\pi/3 < \theta < \pi/3,
\]

\[
w[\alpha, \beta, \gamma] = (\alpha, \beta, \gamma)^t \otimes (\bar{\alpha}, \bar{\beta}, \bar{\gamma} e^{-2i\theta/3})^t \quad \text{if} \quad \pi/3 < \theta < \pi,
\]

(10)

with \(|\alpha| = |\beta| = |\gamma|\). Now, we take product vectors in \(P[\Phi[a, b, c, \theta]]\) as follows:

\[
z_1[1, 1], z_1[1, -1], z_2[1, 1], z_2[1, -1], z_3[1, 1], z_3[1, -1], w[1, 1, 1], w[1, -1, 1],
\]

\[
w[1, i, -i].
\]

(11)

We consider the \(9 \times 9\) matrix whose columns are the partial conjugates of the above nine vectors, then the determinant is given as follows:

\[
|\det M| = \begin{cases}
16 \sqrt{3} b^{9/4} c^{3/4} |(\sqrt{b} - \sqrt{c} e^{i(\theta + 2\pi/3)})^3 (1 + e^{3i\theta})| & \text{if} \quad -\pi < \theta < -\pi/3, \\
16 \sqrt{3} b^{9/4} c^{3/4} |(\sqrt{b} - \sqrt{c} e^{i\theta})^3 (1 + e^{3i\theta})| & \text{if} \quad -\pi/3 < \theta < \pi/3, \\
16 \sqrt{3} b^{9/4} c^{3/4} |(\sqrt{b} - \sqrt{c} e^{i(\theta - 2\pi/3)})^3 (1 + e^{3i\theta})| & \text{if} \quad \pi/3 < \theta < \pi.
\end{cases}
\]
We note that $\det M = 0$ implies $b = c$ and $\theta = 0, \pm 2\pi/3$, which is not possible by the assumption $p_0 < 2$. Therefore, partial conjugates of the product vectors in (11) span the whole space $C^3 \otimes C^3$, and $W[a, b, c; \theta]$ has the co-spanning property for case (ii).

Now, we consider case (iii). In this case, we note that $\Phi[a, b, c; \theta]$ is completely copositive, and so it does not satisfy the co-spanning property. Note that the matrix (5) is given by

$$\begin{pmatrix}
b |y|^2 & -e^{i\theta}x \bar{y} & 0 \\
-e^{-i\theta}x \bar{y} & c |x|^2 & 0 \\
0 & 0 & b |x|^2 + c |y|^2
\end{pmatrix},$$

and the kernel is $(x, e^{-i\theta}yb, 0)^t$. Therefore, the following vectors:

$$w_1[\alpha, \beta] = (\alpha, \beta, 0)^t \otimes (\bar{a}, e^{i\theta} \bar{b}b, 0)^t,$$

$$w_2[\alpha, \beta] = (\beta, 0, \alpha)^t \otimes (e^{i\theta} \bar{b}b, 0, \bar{a})^t,$$

$$w_3[\alpha, \beta] = (0, \beta, \alpha)^t \otimes (0, \bar{a}, e^{i\theta} \bar{b}b)^t$$

belong to $P[\Phi[a, b, c; \theta]]$. We see that the set of the following vectors:

$$w_1[1, 1], w_1[1, -1], w_1[1, i], w_2[1, 1], w_2[1, -1], w_2[1, i], w_3[1, 1], w_3[1, -1], w_3[1, i]$$

span the whole space $C^3 \otimes C^3$ because the determinant of $9 \times 9$ matrix $M$ whose columns are the above vectors is given by

$$| \det(M) | = 64b^3 |1 + b^3 e^{3i\theta}|,$$

which is nonzero by the assumption $p_0 > 1$. Therefore, we see that the map $\Phi[a, b, c; \theta]$ has the spanning property for case (iii).

It remains to consider case (iv). In this case, the map $\Phi[a, b, c; \theta]$ never has the spanning property, since $\Phi[p_0, 0, 0; \theta]$ is completely positive. To check the co-spanning property, we first consider the interior points of the two-dimensional face $f_{ab}^\theta$ on the plane $a + b + c = p_0$. In this case, the only possible product vectors in $P[\Phi[a, b, c; \theta]]$ are of form (10). It is clear that the partial conjugates of them do not span the whole space, and so $\Phi[a, b, c; \theta]$ does not have the co-spanning property. In the following section, we will see that they are not co-optimal. Finally, we consider the line segment $e_{ac}^\theta$ between two points $(p_0, 0, 0)$ and $(1, 0, p_0 - 1)$. We note that the smallest exposed face $F$ containing $\Phi[1, 0, p_0 - 1; \theta]$ is bigger than $e_{ac}^\theta$. We have already shown that $\Phi[1, 0, p_0 - 1; \theta]$ has the co-spanning property, and so $F$ has no completely copositive map. This shows that the line segment $e_{ac}^\theta$ has the co-spanning property. See [21] for the case of $\theta = 0$.

**Theorem 4.1.** Suppose that the map $\Phi[a, b, c; \theta]$ is positive, and $1 < p_0 < 2$. Then, we have the following.

(i) $\Phi[a, b, c; \theta]$ has the spanning property if and only if

$$0 \leq a < 1, \quad bc = (1 - a)^2.$$

(ii) $\Phi[a, b, c; \theta]$ has the co-spanning property if and only if

$$2 - p_0 \leq a < 1, \quad bc = (1 - a)^2, \quad a + b + c = p_0,$$

holds or

$$1 \leq a \leq p_0, \quad bc = 0, \quad a + b + c = p_0.$$

We summarize the results in terms of faces for $1 < p_0 < 2$ as follows:

- $e_{ab}^\theta$, $v_{(a_0, 0, 0; \theta)}$, and $v_{(0, b_0, c)}$ have the spanning property.
- $v_{(1, 0, p_0 - 1; \theta)}$ and $v_{(1, p_0 - 1, 0; \theta)}$ do not have the spanning property. It should be checked if they are optimal or not.
- $e_{ab}^\theta$, $e_{ac}^\theta$, $v_{(0, p_0 - 1, 0; \theta)}$, $v_{(1, 0, p_0 - 1; \theta)}$, and $v_{(0, b_0, c)}$ have the co-spanning property.
- $f_{abc}^\theta$ does not have the co-spanning property. It should be checked if the interior points of $f_{abc}^\theta$ are co-optimal or not.
5. Optimality without the spanning property

It remains to check the optimality of $\Phi[1, 0, p_0; \theta]$ and $\Phi[1, p_0-1, 0; \theta]$, and co-optimality of interior points of the face $f_{abc}^0$. We recall that $\Phi[1, 0, 1; 0]$ and $\Phi[1, 1, 0; 0]$, which are usually called the Choi map, are extremal when $\theta = 0$ by [22], and so they are optimal. In order to check the optimality of a positive map $\phi$, we first find all extremal completely positive maps $\phi_v$ in the smallest exposed face determined by $\phi$, and check if they belong to the smallest face determined by $\phi$. Recall that $\phi_v$ is the completely positive map given by

$$\phi_v(X) = V^*XV, \quad X \in M_m,$$

where $V$ is an $m \times n$ matrix. We also recall [20] that $\phi_v$ belongs to the smallest exposed face determined by $\phi$ if and only if $V$ is orthogonal to $P[\phi]$, where we identify an $m \times n$ matrix and a vector in $\mathbb{C}^m \otimes \mathbb{C}^n$.

We proceed to show that $\Phi[1, p_0 - 1, 0; \theta]$ is optimal. We first consider the case $-\pi/3 < \theta < \pi/3$. In this case, $P[\Phi[1, p_0 - 1, 0; \theta]]$ consists of

$$(1, 0, 0)^t \otimes (0, 0, 1)^t, \quad (0, 1, 0)^t \otimes (0, 0, 1)^t, \quad (0, 0, 1)^t \otimes (1, 0, 0)^t,$$

with complex numbers $\alpha, \beta$ and $\gamma$ of modulus one, from (9) and (10). Each vector orthogonal to all of these vectors is of the form

$$(\xi, 0, 0; \eta, 0; \eta, 0, \xi)^t, \quad \xi + \eta + \xi = 0,$$

and the Choi matrix of the completely positive map associated with this vector is given by

$$V[\xi, \eta, \zeta] = \begin{pmatrix}
|\xi|^2 & \cdots & \xi\bar{\eta} & \cdots & \xi\zeta \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\eta\bar{\xi} & \cdots & |\eta|^2 & \cdots & \eta\zeta \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\zeta\bar{\xi} & \cdots & \zeta\bar{\eta} & \cdots & |\zeta|^2
\end{pmatrix}.$$

In order to show the optimality of $\Phi[1, p_0 - 1, 0; \theta]$, we show that if

$$W[1, p_0 - 1, 0; \theta] = pV[\xi, \eta, \zeta]$$

is block-positive for a non-negative $p > 0$ with $\xi + \eta + \xi = 0$ then $\xi = \eta = \xi = 0$. First, we take product vectors $z_t = (\sqrt{\rho}e^{i\theta}, t, 0)^t \otimes (\sqrt{\rho}, 1, 0)$ for $t > 0$, then we have

$$\langle z_t z_t^*, W[1, p_0 - 1, 0; \theta] - pV[\xi, \eta, \zeta] \rangle = t^2(t(p_0 - 1) - p|\xi|e^{i\theta} + \eta)^2 \geq 0$$

for all $t > 0$ if and only if $\eta = -e^{-i\theta}\xi$. This implies $\zeta = (e^{-i\theta} - 1)\xi$ from the relation $\xi + \eta + \zeta = 0$. Now, we take product vectors $u_t = (\sqrt{\rho}e^{-i\theta}, t)^t \otimes (\sqrt{\rho}, 1, 0)$ for $t > 0$. Then we have

$$\langle u_t u_t^*, W[1, p_0 - 1, 0; \theta] - pV[\xi, -e^{-i\theta}\xi, (e^{-i\theta} - 1)\xi] \rangle$$

$$= t^2(t(p_0 - 1) - p\xi^2(1 - 2 \cos \theta)^2) \geq 0$$

for all $t > 0$ if and only if $|\xi|^2(1 - 2 \cos \theta)^2 = 0$. Since $(1 - 2 \cos \theta) \neq 0$ for $|\theta| < \pi/3$, we conclude that $\xi = 0$. Consequently, we have $\xi = \eta = \xi = 0$, and this completes the proof of the optimality of $\Phi[1, p_0 - 1, 0; \theta]$. For other ranges of $\theta$, we have the similar argument.

Now, we show that an interior point $\Phi[1, b, c; \theta]$, with $b + c = p_0 - 1$, of the face $f_{abc}^0$ is not co-optimal. In the case of $\theta = 0$, this is clear. We first consider the case
\(-\pi/3 < \theta < \pi/3\) \((\theta \neq 0)\). To prove our assertion, we consider the completely copositive linear map whose Choi matrix is given by

\[
W[0, 1; 0] = \begin{pmatrix}
-1 & -1 & -1 & -1 & -1 & -1 \\
-1 & 1 & -1 & -1 & -1 & -1 \\
-1 & 1 & 1 & -1 & -1 & -1 \\
-1 & 1 & 1 & 1 & -1 & -1 \\
-1 & 1 & 1 & 1 & 1 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 \\
\end{pmatrix},
\]

and look for a small positive number \(p > 0\) so that \(W[1, b, c; \theta] - pW[0, 1; 0]\) is block-positive. We note that

\[
W[1, b, c; \theta] - pW[0, 1; 0] = |e^{i\theta} - p|W\left[\frac{1}{|e^{i\theta} - p|}, \frac{b - p}{|e^{i\theta} - p|}, \frac{c - p}{|e^{i\theta} - p|}; \theta'\right],
\]

where \(\theta'\) is the argument of \(e^{i\theta} - p\). First, we take the positive number \(t_0\) so that \(e^{i\theta} - t_0 = |e^{i\theta} - t_0|e^{i\pi/3}\), and then pick a positive number \(p < t_0\) (see figure 4).

Now, it is clear that \(|e^{i\theta} - p| < 1\), and thus condition 4 is automatically satisfied. We also see that the argument \(\theta'\) of \(e^{i\theta} - p\) satisfies the condition \(\theta' < \pi/3\). Then we have that

\[
\frac{1 + b - p + c - p}{|e^{i\theta} - p|} = \frac{p_0 - 2p}{|e^{i\theta} - p|} = \frac{(e^{i\theta} - p) + (e^{i\theta} - p)}{|e^{i\theta} - p|} = \frac{|e^{i\theta} - p|(|e^{i\theta} + e^{-i\theta})}{|e^{i\theta} - p|} = p_0.
\]

Therefore, condition 3 is also satisfied, and interior points of \(\Phi[a, b, c; \theta]\) are not co-optimal. The other cases for \(\theta\) are similar. We summarize the results in table 1.

We note that the bi-optimality automatically implies indecomposability, and so we see that \(\nu_{(1,0,p_0-1)}, \nu_{(1,0,-1,0)}\) and \(\nu_{(a,b,c,t),(t,c)}\) give rise to an indecomposable map. This can be seen directly. We note that

\[
\langle W[p_{\pi-\theta}, t, 1/t; \pi - \theta], \Phi[a, b, c; \theta]\rangle = 3(ap_{\pi-\theta} + bt + c/t - 2).
\]

Assume \(bc = (1 - a)^2\) and take \(t = (c/b)^{1/2}\), to obtain

\[
\langle W[p_{\pi-\theta}, t, 1/t; \pi - \theta], \Phi[a, b, c; \theta]\rangle = 3(ap_{\pi-\theta} + 2\sqrt{bc} - 2) = 3a(p_{\pi-\theta} - 2).
\]
We note that $p_{\pi-\theta} < 2$ if and only if $\theta \neq \pm \pi/3, \pm \pi$. Therefore, we see that $\Phi[a, b, c; \theta]$ is an indecomposable positive map whenever the condition

$$0 < a \leq 1, \quad a + b + c \geq p_0, \quad bc = (1 - a)^2, \quad \theta \neq \pm \pi/3, \pm \pi$$

holds. Recall that we have already seen that positivity of $\Phi[a, b, c; \theta]$ implies decomposability in the case of $\theta = \pm \pi/3, \pm \pi$ for which $p_0 = 1$.

6. Detecting PPT edge states

We proceed to find an optimal entanglement witness which detects the PPT entangled edge state $W[p_0, b, 1/b; \theta]$ for $-\pi/3 < \theta < \pi/3$ with $\theta \neq 0$ and $b > 0$ [1]. We note that if we put $z = (1, 0, 0; 0, 1, 0; 0, 0, 1)^t$ and

$$w_1 = (0, \sqrt{b}, 0; 1/\sqrt{b} e^{i\theta}, 0, 0, 0, 0)^t,$$

$$w_2 = (0, 0, 0; 0, 0, 1/\sqrt{b} e^{i\theta}, 0)^t,$$

$$w_3 = (0, 0, 1/\sqrt{b} e^{i\theta}; 0, 0, 0; \sqrt{b}, 0, 0)^t,$$

then we see that

$$\langle z z^* , W[p_0, b, 1/b; \theta] \rangle = \langle w_i w_i^*, W[p_0, b, 1/b; \theta]^r \rangle = 0.$$

Therefore, the most natural candidate is

$$W = \begin{pmatrix}
1 - \alpha & b - \beta & \cdots & 1 + e^{-i\theta} & \cdots & 1 + e^{i\theta} \\
\cdot & -1/b - \gamma & \cdots & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdots & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdots & \cdot & \cdots & \cdot \\
1 + e^{i\theta} & \cdots & \cdots & 1 - \alpha & \cdots & 1 + e^{-i\theta} \\
\cdot & \cdot & \cdots & b - \beta & \cdots & \cdot \\
\cdot & \cdot & \cdots & \cdot & b - \beta & \cdot \\
\cdot & \cdot & \cdots & \cdot & \cdot & 1/b - \gamma \\
1 + e^{-i\theta} & \cdots & \cdots & 1 + e^{i\theta} & \cdots & 1 - \alpha
\end{pmatrix},$$

which is equal to

$$2 \cos(\theta/2) W \left[ \frac{1 - \alpha}{2 \cos(\theta/2)}, \frac{1/b - \gamma}{2 \cos(\theta/2)}, \frac{b - \beta}{2 \cos(\theta/2)}, \frac{\pi - \theta}{2} \right]$$

because $1 + e^{-i\theta} = 2 \cos(\theta/2) e^{-i\theta/2}$.

To begin with, we note that

$$\frac{1}{3} \langle W[p_0, 1/b, b, \theta], W \rangle = p_0 (1 - \alpha) + 1/b (b - \beta) + b (1/b - \gamma) - p_0 - 2 = -p_0 \alpha - b \beta - b \gamma < 0.$$
We search for \( \alpha, \beta, \gamma > 0 \) so that \( W \) is bi-optimal. To do this, we look for \( \alpha, \beta \) and \( \gamma \) satisfying the conditions

\[
2 \cos(\theta/2)(2 - p_{\pi - \theta/2}) < 1 - \alpha < 2 \cos(\theta/2), \quad b - \beta > 0, \quad 1/b - \gamma > 0,
\]

\[
(1 - \alpha) + (b - \beta) + (1/b - \gamma) = 2p_{\pi - \theta/2} \cos(\theta/2),
\]

\[
(b - \beta)(1/b - \gamma) = [2\cos(\theta/2) - (1 - \alpha)]^2,
\]

from theorem 4.1. For simplicity, we put

\[
t := \cos(\theta/2), \quad \tilde{\alpha} = 1 - \alpha, \quad \tilde{\beta} = b - \beta, \quad \tilde{\gamma} = \frac{1}{b} - \gamma.
\]

Then we have \( \sqrt{3}/2 < t < 1 \) and \( p_{\pi - \theta/2} = t + \sqrt{3(1-t^2)} \). Now, we only need to look for \( \tilde{\alpha}, \tilde{\beta} \) and \( \tilde{\gamma} \) satisfying the conditions

\[
2t(2 - t - \sqrt{3(1-t^2)}) < \tilde{\alpha} < 2t,
\]

\[
\tilde{\beta} + \tilde{\gamma} = 2t(2 - \sqrt{3(1-t^2)}) - \tilde{\alpha}, \quad \tilde{\beta}\tilde{\gamma} = (2t - \tilde{\alpha})^2, \quad \tilde{\beta} > 0, \quad \tilde{\gamma} > 0.
\]

It is easy to see that

\[
1 < t + \sqrt{3(1-t^2)} < \sqrt{3},
\]

for \( \sqrt{3}/2 < t < 1 \). Therefore, if we choose \( \tilde{\alpha} \) satisfying condition (12) for each \( \sqrt{3}/2 < t < 1 \), then we see that

\[
\tilde{\beta} + \tilde{\gamma} > 0 \quad \text{and} \quad \tilde{\beta}\tilde{\gamma} > 0
\]

in (13) and

\[
(\tilde{\beta} + \tilde{\gamma})^2 - 4\tilde{\beta}\tilde{\gamma} = -[\tilde{\alpha} - 2t(2 - t - \sqrt{3(1-t^2)})][3\tilde{\alpha} - 2t(2 + t + \sqrt{3(1-t^2)})] > 0.
\]

Consequently, we can find \( \tilde{\beta} \) and \( \tilde{\gamma} \) as positive roots of the quadratic equation

\[
x^2 - [2t(2 - \sqrt{3(1-t^2)}) - \tilde{\alpha}]x + (2t - \tilde{\alpha})^2 = 0.
\]

This completes the proof of the following

**Theorem 6.1.** For each \( 0 < |\theta| < \pi/3 \) and \( b > 0 \), let \( \tilde{\alpha} \) be a positive number satisfying condition (12) with \( t = \cos(\theta/2) \), and let \( \tilde{\beta} \) and \( \tilde{\gamma} \) be roots of the quadratic equation (14). Then

\[
\frac{2 \cos(\theta/2)}{3(\tilde{\alpha} + \tilde{\beta} + \tilde{\gamma})} W \left[ \frac{\tilde{\alpha}}{2 \cos(\theta/2)}, \frac{\tilde{\beta}}{2 \cos(\theta/2)}, \frac{\tilde{\gamma}}{2 \cos(\theta/2)}, \pi - \frac{\theta}{2} \right]
\]

is an optimal PPTES witness which detects \( W[p_0, b, 1/b, \theta] \).

We note that a general method had been suggested in [13] to construct entanglement witnesses detecting a given entangled state. If we follow this method for \( W[p_0, b, 1/b, \theta] \), then we obtain the above \( W \) with \( \alpha = \beta = \gamma \). But, it turns out that this method does not give us an optimal PPTES witness in general. In fact, one can show that if \( W \) is an optimal PPTES witness with positive \( \alpha = \beta = \gamma \), then we have the restriction

\[
b + \frac{1}{b} \leq 2 - \sqrt{3} + (6\sqrt{3} - 6)^{1/2} \quad \text{and} \quad \cos(\theta/2) \leq \frac{1}{8}(3 + \sqrt{21})
\]

This is why we consider the above \( W \) with different \( \alpha, \beta \) and \( \gamma \) to obtain more general results.
7. Conclusion

We determined the positivity of linear maps with four parameters, which are similar to the Choi map, but involve complex entries. We also determined their optimality, co-optimality, spanning property and co-spanning property. In this way, we found parameterized examples of indecomposable positive linear maps with the bi-spanning property. They are optimal PPTES witnesses, which are nondecomposable optimal entanglement witnesses (‘nd-OEW’s) in the sense of [13]. Optimality is not so easy to determine for a given positive linear map because we do not know the whole facial structure of the convex cone $P_1$ consisting of all positive maps. The spanning property is stronger than optimality and relatively easy to check. We suggest a general method to check the optimality of a positive map $\phi$. We first find all extremal completely positive maps in the smallest exposed face of $P_1$ containing $\phi$, and check if they belong to the smallest face containing $\phi$.

The optimal PPTES witnesses we constructed detect two qutrit PPT entangled edge states of type (6, 8) considered in [1], whose existence had been a long standing question [15]. We report here one more interesting byproduct of our work. Our constructions give counter-examples to the conjecture [18] regarding the structural physical approximations (SPAs), which claims that the SPA of an optimal entanglement witness is separable. Several authors [23–25] checked recently various kinds of entanglement witnesses to support the conjecture. In a forthcoming paper [26], the authors consider the SPA conjecture in a systematic way. We introduce the notions of positive type and copositive type for entanglement witnesses depending on the distances to the positive part and the copositive part. We will show that if the SPA of an entanglement witness is separable then the witness must be of copositive type, and so the SPA conjecture can only hold for witnesses of copositive type. Our construction in the paper [26] shows that the SPA conjecture does not hold even in cases of copositive types.

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References

[1] Kye S-H and Osaka H 2012 J. Math. Phys. 53 052201
[2] Choi M-D 1982 Operator Algebras and Applications. Proc. Symp. Pure Math. (Kingston, Ontario, 14 July–2 August 1980) vol 38 part 2 (Providence, RI: American Mathematical Society) pp 583–90
[3] Peres A 1996 Phys. Rev. Lett. 77 1413–15
[4] Woronowicz S L 1976 Rep. Math. Phys. 10 165–83
[5] Størmer E 1982 Proc. Am. Math. Soc. 86 402–4
[6] Horodecki M, Horodecki P and Horodecki R 1996 Phys. Lett. A 223 1–8
[7] Terhal B M 2000 Phys. Lett. A 271 319–26
[8] Eom M-H and Kye S-H 2000 Math. Scand. 86 130–42
[9] Choi M-D 1975 Linear Algebra Appl. 10 285–90
[10] Jamiołkowski A 1974 Rep. Math. Phys. 5 415–24
[11] Skowronek L, Størmer E and Życzkowski K 2009 J. Math. Phys. 50 062106
[12] Życzkowski K and Bengtsson I 2004 Open Syst. Inform. Dyn. 11 3–42
[13] Lewenstein M, Kraus B, Cirac I I and Horodecki P 2000 Phys. Rev. A 62 052310
[14] Ha K-C and Kye S-H 2012 Optimality for indecomposable entanglement witnesses Phys. Rev. A 86 034301
[15] Sanpera A, Bruß D and Lewenstein M 2001 Phys. Rev. A 63 050301
[16] Ha K-C and Kye S-H 2005 J. Phys. A: Math. Gen. 38 9039–50
[17] Ha K-C and Kye S-H 2012 Open Syst. Inform. Dyn. 19 1250009
[18] Korbicz J K, Almeida M L, Bae J, Lewenstein M and Acin A 2008 Phys. Rev. A 78 062105
[19] Ha K-C and Kye S-H 2011 Phys. Rev. A 84 024302
[20] Kye S-H 2012 Facial structures for various notions of positivity and applications to the theory of entanglement arXiv:1202.4255
[21] Choi H-S and Kye S-H 2012 J. Korean Math. Soc. 49 623–39
[22] Choi M-D and Lam T-T 1977 Math. Ann. 231 1–18
[23] Augusiak R, Bae J, Czekaj Ł and Lewenstein M 2011 J. Phys. A: Math. Theor. 44 185308
[24] Chruściński D and Pytel J 2011 J. Phys. A: Math. Theor. 44 165304
[25] Qi X and Hou J 2012 Phys. Rev. A 85 022334
[26] Ha K-C and Kye S-H 2012 The structural physical approximations and optimal entanglement witnesses J. Math. Phys. at press