Polynomial Ensembles and Pólya Frequency Functions

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Abstract

We study several kinds of polynomial ensembles of derivative type which we propose to call Pólya ensembles. These ensembles are defined on the spaces of complex square, complex rectangular, Hermitian, Hermitian antisymmetric and Hermitian anti-self-dual matrices, and they have nice closure properties under the multiplicative convolution for the first class and under the additive convolution for the other classes. The cases of complex square matrices and Hermitian matrices were already studied in former works. One of our goals is to unify and generalize the ideas to the other classes of matrices. Here, we consider convolutions within the same class of Pólya ensembles as well as convolutions with the more general class of polynomial ensembles. Moreover, we derive some general identities for group integrals similar to the Harish–Chandra–Itzykson–Zuber integral, and we relate Pólya ensembles to Pólya frequency functions. For illustration, we give a number of explicit examples for our results.

Keywords  Probability measures on matrix spaces · Sums and products of independent random matrices · Polynomial ensembles · Additive convolution ·
1 Introduction

In 2012, it was observed that the spectral statistics of certain products of independent random matrices may be calculated explicitly at finite matrix dimension [2,6]. This breakthrough did not only lead to a new and fast development on products of random matrices, see, e.g., [4] for a review, but also on related topics like sums of random matrices [12,33]. Originally, the whole development started with products of independent Ginibre and Jacobi (truncated unitary) matrices due to their simplicity and their field of applications, e.g., for the local spectral statistics at finite and infinite matrix dimension, see [1–3,5–7,14,23,31,34–36]. While the results and proofs for these ensembles relied on the particular form of the ensembles, it soon became clear that there is some common structure in the background which leads to unified proofs as well as to further generalizations. For instance, in [34], the notion of a polynomial ensemble was introduced, and it was shown that this yields a convenient framework to investigate the multiplication of a random matrix by an independent Ginibre or Jacobi (truncated unitary) matrix [31,32,34]. Motivated by these findings, the concept of a polynomial ensemble of derivative type was introduced in [29,30]. These matrix ensembles have several important properties. First, they are isotropic (also called bi-unitarily invariant or rotationally invariant), and second, they depend on a single one-point weight, cf. Definition 2.3 (2). Third, they generalize the results about the multiplication by Ginibre and Jacobi matrices to a larger subclass of polynomial ensembles. Interestingly, this concept does not only lead to a unified perspective on many of the preceding results, but it also includes several further prominent examples of complex (non-Hermitian) random matrix ensembles.

Shortly after the appearance of these results, it was noted in [33] that the notion of a polynomial ensemble of derivative type may also be adapted to investigate sums of independent Hermitian random matrices. Our main aim is to extend these results to further symmetry classes of random matrices, namely complex rectangular matrices, Hermitian antisymmetric matrices and Hermitian anti-self-dual matrices. As we shall see, all these classes can be dealt with in a similar way by using the appropriate multivariate transforms from harmonic analysis. For the additive convolution on the space of Hermitian matrices and the multiplicative convolution on the space of complex square matrices, one needed the matrix-variate Fourier transform and the spherical transform, i.e., the multivariate counterparts of the univariate Fourier and Mellin transform, respectively. We will generalize this idea by identifying the appropriate matrix version of the Hankel transform [40, Chapter 10.22(v)], which is intimately related to the addition of isotropically distributed random vectors, for the above-mentioned classes of matrices. This is our first major goal.

Note that all five kinds of convolutions considered in the present work are ensembles of Dyson index $\beta = 2$ because they are related to either compact Lie algebras or
complex matrices. The reason why one can easily deal with all five classes in a similar way is the knowledge of certain group integrals involved, namely the Harish–Chandra–Itzykson–Zuber integral [19,24], the Gelfand–Naïmark integral [16] and the Berezin–Karpelevich integral [9,18], which have essentially the same structure. For random matrix ensembles corresponding to the Dyson indices $\beta = 1$ or $\beta = 4$, explicit results for such group integrals are only known for very small matrix dimensions but not in general.

We have two further goals. The second goal is to explore the connection of the class of polynomial ensembles of derivative type to the class of Pólya frequency functions [43,44], a notion from classical analysis [28]. There exists some related work in this direction in representation theory and ergodic theory [11,13,28,39,42,45], but it seems that this connection has not been explored yet from the viewpoint of random matrix theory, i.e., as regards the associated singular value and eigenvalue distributions at finite matrix dimension. The relation between Pólya frequency functions (with certain differentiability and integrability properties) and polynomial ensembles of derivative type will be essentially bijective for the cases of complex square matrices (with multiplication) and Hermitian random matrices (with addition), cf. Theorem 2.9, while we will only establish a certain injective relationship for the other matrix classes under consideration, cf. Theorem 2.10. Anyway, we suggest to call these polynomial ensembles of derivative type by the shorter name Pólya ensembles.

Our third goal is to investigate some generalizations of the above-mentioned group integrals, see also [17,20,41]. We obtain a number of new and highly non-trivial examples of such identities which arise naturally from our approach. Incidentally, these identities for group integrals also play a central role with respect to our second goal.

The present work is organized as follows. In Sect. 2, we introduce the necessary notation and state our main results. In Sect. 3, we introduce the univariate and multivariate transforms from harmonic analysis, which are central to our approach. Section 4 is devoted to the proofs of our main results. We discuss and summarize our findings in Sect. 5.

2 Notation and Main Results

2.1 Matrix Spaces

We introduce the relevant matrix spaces, the corresponding group actions as well as the sets of probability measures invariant under these actions, and we recall the relations of these probability measures to the induced probability measures on the eigenvalues or singular values.

We adapt the notation from our previous works [29,30] to our current purposes. Let $O(n)$, $U(n)$ and $\text{USp}(2n)$ denote the classical groups of orthogonal, unitary and unitary symplectic matrices, and let $o(n)$, $u(n)$ and $\text{usp}(2n)$ denote the associated classical Lie algebras.

We are interested in the following matrix spaces, with $n \in \mathbb{N}$:
(1) $G := \text{GL}(n, \mathbb{C})$ is the group of invertible complex $n \times n$ matrices, endowed with the action of the group $\hat{K} := U(n) \times U(n)$ via $(k_1, k_2).g := k_1 g k_2^*$.

(2) $H_2 := \text{iu}(n) = \text{Herm}(n)$ is the linear space of Hermitian $n \times n$ matrices, endowed with the action of the group $K_2 := U(n)$ via $k.y := k k^*$.

(3a) For fixed $v \in \mathbb{N}_0$, $\text{Mat}_\mathbb{C}(n, n + v)$ is the linear space of complex $n \times (n + v)$ rectangular matrices, endowed with the action of the group $U(n) \times U(n + v)$ via $(k_1, k_2).y := k_1 y k_2^*$. We identify an element $y \in \text{Mat}_\mathbb{C}(n, n + v)$ with a chiral Hermitian matrix

$$\begin{bmatrix} 0 & y \\ y^* & 0 \end{bmatrix} \in \text{Herm}(2n + v) \quad (2.1)$$

and an element $k = (k_1, k_2) \in U(n) \times U(n + v)$ with a block matrix

$$\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \in U(2n + v). \quad (2.2)$$

With these identifications, the group action may be written as $k.y = k y k^*$. Write $M_v$ and $\hat{K}_v$ for the spaces of matrices in (2.1) and (2.2).

(3b) $H_1 := i\sigma(2n)$ or $H_1 := i\sigma(2n + 1)$ and $H_4 := i\text{usp}(2n)$ are the linear spaces of Hermitian antisymmetric and Hermitian anti-self-dual matrices, endowed with the action of the groups $K_1 := O(2n)$ or $K_1 := O(2n + 1)$ and $K_4 := \text{USp}(2n)$ via $k.y := k y k^*$.

Moreover, we also need the linear space of real diagonal $n \times n$ matrices, $D \simeq \mathbb{R}^n$, and the group of positive-definite diagonal $n \times n$ matrices, $A \simeq \mathbb{R}_+^n$, each endowed with the natural action of the symmetric group $\mathbb{S}$ of all permutations of order $n$ on the diagonal elements.

When it is possible to consider several of these cases simultaneously, we write $M$ for the matrix space and $K$ for the associated group. Note that the dependence on $n$ is implicit. When the need arises to make it explicit, we write $M(n)$ instead of $M$, $K(n)$ instead of $K$, etc.

In (2) and (3b), we have used the index $\beta = 1, 2, 4$ to underline the connection to the field of real, complex and quaternion numbers, respectively; yet, it has to be distinguished from the level repulsion that corresponds to Dyson index 2 in all cases. The multiplication by $i$ is convenient since it leads to matrices with real eigenvalues. However, it is clear that our results can easily be translated into results for real antisymmetric, anti-Hermitian and anti-Hermitian anti-self-dual (or quaternion anti-Hermitian) matrices, respectively. We treat the case $\beta = 2$ separately because it turns out to be simpler than the other cases.

As reference measures on the matrices spaces $G$, $M_v$, $H_\beta$, $A$ and $D$, we use the flat Lebesgue measures on the linearly independent matrix entries, typically denoted by $dg$ for $G$, by $dy$ for $M_v$ and $H_\beta$, and by $da$ for $A$ and $D$. For the groups $\hat{K}$, $\hat{K}_v$, $K_\beta$ as well as $U(n)$, we take the normalized Haar measures, always denoted by $d^*k$. Occasionally, we also use the Haar measure $d^*g = dg / \det(gg^*)^n$ on $G = \text{GL}(n, \mathbb{C})$, with $g^*$ the Hermitian adjoint of $g$.

We always ignore sets of measure zero. Thus, the spaces $G$ and $M_0$ are essentially the same. However, we prefer to use different notations to reflect the different group
operations on these spaces, namely matrix multiplication on \( G \) and matrix addition on \( M_0 \). Thus, when we speak about random matrices on \( G \) or on \( M_0 \), we will be interested in their products and sums, respectively.

### 2.2 Matrix Densities and Spectral Densities

Let \( M \) and \( K \) be as in Sect. 2.1. As functions on \( M \), we consider integrable functions invariant under the action of \( K \), i.e.,

\[
L^{1,K}(M) = \left\{ f_M \in L^1(M) \mid f_M(k.m) = f_M(m) \forall m \in M, \, k \in K \right\}. \tag{2.3}
\]

In the following, we indicate the space on which the function (or density) is defined by a subscript, e.g., \( f_G, f_{M_0}, f_{H_0}, \ldots \). The invariance of the functions is called \( K \)-invariance with respect to the respective group \( K = \hat{K}, \hat{K}_v, K_\beta, \mathbb{S} \). Note that this \( K \)-invariance amounts to bi-unitary invariance for the spaces \( \text{GL}(n, \mathbb{C}) \) and \( \text{Mat}_C(n, n+v) \), to conjugation invariance for the spaces \( H_\beta \), and to permutation invariance or symmetry for the spaces \( D \) and \( A \). The subset of all probability densities in the set \( L^{1,K}(M) \) will be denoted by \( L^1_{\text{Prob}}(M) \).

For each matrix space \( M \) as in (1) – (3), the \( K \)-invariant probability densities are in one-to-one correspondence with the induced spectral densities, i.e., the induced symmetric probability densities of the eigenvalues (for \( M = H_2 \) or the (nonzero) squared singular values (for \( M = G, M_0, H_1, H_3 \)). Let us describe these correspondences by bijective mappings \( I_M \) which associate with each \( K \)-invariant matrix density the induced spectral density. It turns out convenient to consider these mappings not only on the sets \( L^{1,K}_{\text{Prob}}(M) \) of all \( K \)-invariant probability densities, but also, via linear extension, on the larger sets \( L^{1,K}(M) \) of all \( K \)-invariant integrable functions. Let the Vandermonde determinant be defined by

\[
\Delta_n(a) = \prod_{1 \leq b < c \leq n} (a_c - a_b) = \det[a_i^{k-1}]_{i,k=1,...,n}, \quad a_1, \ldots, a_n \in \mathbb{R}. \tag{2.4}
\]

Then, the mappings \( I_M \) are as follows:

1. Almost every matrix \( y \in H_2 \) has \( n \) distinct eigenvalues \( a_1, \ldots, a_n \in \mathbb{R} \), and \( I_{H_2} : L^{1,K_2}(H_2) \rightarrow L^{1,\mathbb{S}}(D) \) with

\[
(I_{H_2}f_{H_2})(a) = C_{H_2} f_{H_2}(a) \Delta_n^2(a) =: f_D(a), \quad a \in D. \tag{2.5}
\]

2. Almost every matrix \( g \in G \) has \( n \) distinct squared singular values \( a_1, \ldots, a_n > 0 \), and \( I_G : L^{1,K}(G) \rightarrow L^{1,\mathbb{S}}(A) \) with

\[
(I_Gf_G)(a) = C_G f_G(\sqrt{a}) \Delta_n^2(a) =: f_A(a), \quad a \in A. \tag{2.6}
\]

3. Almost every matrix \( y \in M \in \{M_0, H_1, H_3\} \) has \( n \) distinct squared singular values \( a_1, \ldots, a_n > 0 \), which are the squares of the nonzero eigenvalues \( \pm \sqrt{a_1}, \ldots, \pm \sqrt{a_n} \), and \( I_M : L^{1,K}(M) \rightarrow L^{1,\mathbb{S}}(A) \) with
\[(\mathcal{I}_M f_M)(a) = C_M \left( \det a \right)^{\nu} f_M(\iota_M(a)) \Delta_n^{2}(a) =: f_A(a), \quad a \in A, \quad (2.7)\]

with \(\nu \in \mathbb{N}_0 \cup \{ \pm \frac{1}{2} \}\) and \(\iota_M\) as defined below.

Here,

\[\nu \in \mathbb{N}\text{ is given and } \iota_M(a) := \begin{bmatrix} 0 & \sqrt{\det a} \Pi_{n,n+\nu} \end{bmatrix} \text{ for } M = M_{\nu},\]

\[\nu := -\frac{1}{2} \quad \text{and} \quad \iota_M(a) := \sqrt{\det a} \otimes \tau_2 \quad \text{for } M = H_1 = \text{io}(2n),\]

\[\nu := \frac{1}{2} \quad \text{and} \quad \iota_M(a) := \begin{bmatrix} \sqrt{\det a} \otimes \tau_2 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{for } M = H_1 = \text{io}(2n+1),\]

\[\nu := \frac{1}{2} \quad \text{and} \quad \iota_M(a) := \sqrt{\det a} \otimes \tau_3 \quad \text{for } M = H_4 = \text{iusp}(2n),\]

where \(\Pi_{j,k}\) \((j \leq k)\) is the projection onto the first \(j\) out of \(k\) rows, the square root \(\sqrt{\det a}\) is taken component-wise, \(\tau_2 := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\) and \(\tau_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\) are the second and third Pauli matrix, and

\[x \otimes y := \begin{pmatrix} x_{11}y & x_{12}y & \cdots \\ x_{21}y & x_{22}y & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}\]

is the Kronecker product of two matrices \(x = (x_{ij})\) and \(y = (y_{ij})\). The constants in Eqs. (2.5)–(2.7) are given by

\[C_{H_2} = \frac{1}{n!} \prod_{j=0}^{n-1} \pi_j \quad \text{for } C_G = C_{n,0}, \quad C_{M_{\nu}} = C_{n,\nu}^{*}, \quad C_{H_1} = C_{n,v}^{*}, \quad C_{H_4} = \frac{C_{n,v}^{*}}{2n(n-1)}, \quad (2.8)\]

where

\[C_{n,v}^{*} = \frac{1}{n!} \prod_{j=0}^{n-1} \frac{\pi^{2j+v+1}}{\Gamma[j+v+1]j!} \quad (2.9)\]

and \(\Gamma\) denotes the Gamma function. These constants may be found using the techniques from Chapter 3 in [22], for example.

Let us emphasize that the \(K\)-invariance of the matrix functions \(f_M\) is crucial for the bijectivity of the mappings \(\mathcal{I}_M\). Furthermore, the mappings \(\mathcal{I}_M\) remain bijective when restricted to probability densities. Finally, the reader familiar with [29] should be warned that the names of the operators \(\mathcal{I}_M\) follow a different logic than in [29].

In particular, the \(K\)-invariant functions on \(M\) correspond to symmetric functions of \(n\) eigenvalues or squared singular values. This observation will also be important in Sect. 3, where we introduce the multivariate transforms central to our approach.
2.3 Convolutions

On the linear matrix spaces $M = H_1, H_2, H_4, M_\nu$, the additive convolution is defined by

$$(f_M \ast h_M)(y) = \int_M f_M(y') h_M(y - y') \, dy' \quad (y \in M) \quad (2.10)$$

for $f_M, h_M \in L^1(M)$, and on the matrix group $M = G$, the multiplicative convolution is defined by

$$(f_M \odot h_M)(g) = \int_M f_M(g') h_M((g')^{-1}g) \, d^*g' \quad (g \in G) \quad (2.11)$$

for $f_M, h_M \in L^1(M)$, where $d^*g' = dg'/\det(g'g'^*)^n$ denotes the Haar measure on $G$.

In terms of random matrices, these convolutions describe the density of the sum or product of two independent random matrices $X_1$ and $X_2$ with the densities $f_M$ and $h_M$. For instance, in the additive case, we have

$$\mathbb{E}(\phi(X_1 + X_2)) = \int_M \int_M \phi(y_1 + y_2) f_M(y_1) h_M(y_2) \, dy_2 \, dy_1$$

$$= \int_M \int_M \phi(y) f_M(y_1) h_M(y - y_1) \, dy \, dy_1$$

for all nonnegative measurable functions $\phi$, where we have made the change of variables $y = y_1 + y_2$, $dy = dy_2$. This shows that $X_1 + X_2$ has the density (2.10). In the multiplicative case, the claim follows by a similar calculation using the change of variables $g = g_1g_2$, $d^*g = d^*g_2$.

When $f_M$ and $h_M$ are additionally $K$-invariant, their convolution is also $K$-invariant. It is then natural to consider the convolution at the level of the spectral densities. We denote these induced convolutions by

$$f_A \circledast h_A := I_G(I_G^{-1}f_A \circledast I_G^{-1}h_A) \quad (f_A, h_A \in L^{1,S}(A)) \quad (2.12)$$

for $M = G$, by

$$f_D \ast h_D := I_{H_2}(I_{H_2}^{-1}f_D \ast I_{H_2}^{-1}h_D) \quad (f_D, h_D \in L^{1,S}(D)) \quad (2.13)$$

for $M = H_2$, and by

$$f_A \circledast_\nu h_A := I_M(I_M^{-1}f_A \circledast I_M^{-1}h_A), \quad (f_A, h_A \in L^{1,S}(A)) \quad (2.14)$$

for $M \in \{M_\nu, H_1, H_4\}$, with $\nu \in \mathbb{N} \cup \{\pm \frac{1}{2}\}$ defined as in (2.7).

In Eq. (2.14), the problem arises that for $\nu = +\frac{1}{2}$, there are two different choices for $M$, namely $M = i\sigma(2n + 1)$ and $M = i\upsilon(2n)$. However, as we will see in Sect. 3, both choices lead to the same convolution $\ast_{1/2}$ on the space $L^{1,S}(A)$.
We will use the induced convolutions $\ast$, $\ast_\nu$, primarily for $n = 1$, where they reduce to convolutions on the spaces $L^1(\mathbb{R}_+)$, $L^1(\mathbb{R})$ and $L^1(\mathbb{R}_+)$, respectively. While the first two convolutions are simply the ordinary multiplicative and additive convolutions on $\mathbb{R}_+$ and $\mathbb{R}$, respectively, the third convolution is closely related to the Hankel convolution; see the comments at the beginning of Sect. 3.

**Remark 2.1 (Products of Rectangular Matrices)** As already mentioned, the multiplicative convolution on $G$ may be used to study products of independent bi-invariant random square matrices. One could also consider products of independent bi-invariant random rectangular matrices. However, such products can always be traced back to products of independent bi-invariant random square matrices with modified but related densities, see [23]. This statement is not true for sums of $K$-invariant rectangular random matrices.

For instance, given two independent bi-invariant random matrices $g_1 \in \mathbb{C}^{n \times (n+\nu_1)}$ and $g_2 \in \mathbb{C}^{(n+\nu_1) \times (n+\nu_2)}$ with $\nu_1, \nu_2 \in \mathbb{N}_0$, we may write

$$g_1 = \hat{g}_1 \Pi_{n,n+\nu_1} k_1 \quad \text{and} \quad \Pi_{n,n+\nu_1} k_1 g_2 = \hat{g}_2 \Pi_{n,n+\nu_2} k_2$$

with $\hat{g}_1, \hat{g}_2 \in \mathbb{C}^{n \times n}$ bi-invariant, $k_1 \in U(n + \nu_1)$, $k_2 \in U(n + \nu_2)$ Haar distributed, and all of them independent, to obtain a representation $g_1 g_2 = \hat{g}_1 \hat{g}_2 \Pi_{n,n+\nu_2} k_2$ for the product.

In contrast to that, given two independent bi-invariant random matrices $y_1, y_2 \in \mathbb{C}^{n \times (n+\nu)}$, it does not seem possible in general to find a representation $y_1 + y_2 = (\hat{y}_1 + \hat{y}_2) \Pi_{n,n+\nu} k$ with $\hat{y}_1, \hat{y}_2 \in \mathbb{C}^{n \times n}$ bi-invariant, $k \in U(n + \nu)$ Haar distributed, and all of them independent.

### 2.4 Polynomial Ensembles

Polynomial ensembles were introduced by Kuijlaars and co-authors [34]. They have a simple algebraic structure which occurs in many prominent random matrix ensembles. In a previous work [30], we identified a subset of these polynomial ensembles which is closed under the multiplicative convolution (2.12) on $G$. This behavior is in general not true for general polynomial ensembles. After that, a subset of polynomial ensembles with similar properties was investigated for the additive convolution (2.13) on $H_2$ in [33].

The main purpose of this subsection is to introduce similar subsets for the additive convolution (2.14) on the spaces $M_\nu$, $H_1$ and $H_4$ of rectangular matrices, Hermitian antisymmetric matrices and Hermitian anti-self-dual matrices. For comparison and for later use, we also briefly describe the existing results for the other classes. Thus, we define three subsets of polynomial ensembles.

For an interval $I \subset \mathbb{R}$ and a (measurable) subset $\mathcal{N} \subset \mathbb{R}$, let

$$L^1(I(\mathcal{N})) = \left\{ f \in L^1(\mathcal{N}) \mid \text{for all } \kappa \in I : \int_{\mathcal{N}} |x|^\kappa |f(x)| \, dx < \infty \right\}.$$
Definition 2.2 (Polynomial ensembles)

(i) See [34]. Let $n \in \mathbb{N}$ and $\mathcal{N} \subset \mathbb{R}$ be a subset. A probability measure $\mu$ on $\mathcal{N}^n$ is called the polynomial ensemble on $\mathcal{N}^n$ associated with the one-point weights $w_1, \ldots, w_n \in L^1_{[1,n]}(\mathcal{N})$ if it has a Lebesgue density of the form

$$p(a) = C_n[w] \Delta_n(a) \det[w_b(a_c)]_{b,c=1,\ldots,n} \geq 0, \quad a \in \mathcal{N}^n,$$

(2.15)

where $C_n[w] > 0$ is the normalization constant. When $\mathcal{N} = \mathbb{R}$ or $\mathcal{N} = \mathbb{R}_+$, we also call $\mu$ a polynomial ensemble on $D$ or $A$, in line with our identifications $D \simeq \mathbb{R}^n$ and $A \simeq \mathbb{R}^n_+$.

(ii) For $M \in \{G, H_2, M_\nu, H_1, H_4\}$, a probability measure on $M$ with a density $f_M \in L^1_{\text{Prob}}(M)$ is called a polynomial ensemble on $M$ if $\mathcal{I}_M f_M$ is the density of a polynomial ensemble on $D$ or $A$ (i.e., if the induced spectral density is a polynomial ensemble on $D$ or $A$).

In part (ii), we also write $f_M = \text{PE}_M(w_1, \ldots, w_n)$ for the density, where $w_1, \ldots, w_n$ are the one-point weights for $\mathcal{I}_M f_M$.

For each of our matrix spaces $M \in \{G, H_2, M_\nu, H_1, H_4\}$, there exists a subclass of polynomial ensembles with nice closure properties under the respective convolution. We baptize them Pólya ensembles due to their close relation to Pólya frequency functions (as discussed further below).

Definition 2.3 (Pólya ensembles)

(1) A polynomial ensemble on $H_2$ is called a Pólya ensemble on $H_2$ if

$$w_j(x) = \left(-\frac{\partial}{\partial x}\right)^{j-1}\omega(x) \quad \text{for all } x \in \mathbb{R} \text{ and } j = 1, \ldots, n$$

(2.16)

with

$$\omega \in L^1_{H_2}(\mathbb{R}) := \left\{ f \in L^1(\mathbb{R}) \left| f \text{ being nonnegative and } (n - 1)\text{-times differentiable}ight.\right. $$

and for all $j = 0, \ldots, n - 1$:

$$\frac{\partial^j f}{\partial x^j}(x) \in L^1_{[1,n]}(\mathbb{R}) \right\}.$$

(2) A polynomial ensemble on $G$ is called a Pólya ensemble on $G$ if

$$w_j(x) = \left(-x\frac{\partial}{\partial x}\right)^{j-1}\omega(x) \quad \text{for all } x \in \mathbb{R}_+ \text{ and } j = 1, \ldots, n$$

(2.17)

with

$$\omega \in L^1_G(\mathbb{R}_+) := \left\{ f \in L^1(\mathbb{R}_+) \left| f \text{ being nonnegative and } (n - 1)\text{-times differentiable}ight.\right. $$

and for all $j = 0, \ldots, n - 1$:

$$x^j \frac{\partial}{\partial x} f(x) \in L^1_{[1,n]}(\mathbb{R}_+) \right\}.$$
For $M \in \{M_v, H_1, H_4\}$, a polynomial ensemble on $M$ is called a Pólya ensemble on $M$ if

$$w_j(x) = \left( x^v \frac{\partial}{\partial x} x^{v-1} \right)^{j-1} \omega(x) \text{ for all } x \in \mathbb{R}_+ \text{ and } j = 1, \ldots, n \quad (2.18)$$

with

$$\omega \in L^1_M(\mathbb{R}_+) \subseteq \left\{ f \in L^1(\mathbb{R}_+) \bigg| f \text{ being nonnegative and } (2n-1)\text{-times differentiable, } \right.$$

for all $j = 0, \ldots, n-1$:

$$\left( x^v \frac{\partial}{\partial x} x^{v-1} \right)^j f(x) \in L^1_{[1,n]}(\mathbb{R}_+) \text{ and }$$

for all $l = 0, \ldots, n-2$:

$$\lim_{x \to 0} x^{v+1} \frac{\partial}{\partial x} x^v \left( \frac{\partial}{\partial x} x^{v+1} \right)^l f(x) = 0 \bigg\}.$$

In all cases (1)–(3), we also write $f_M = PE_M(\omega)$ for the density.

We want to observe that the differential operator in the first large round brackets in part (3) satisfies $x^v \partial_x x^{1-v} \partial_x = \partial_x x^{v+1} \partial_x x^{-v}$. Furthermore, it is equal to $y^{(2v-1)/2} \left[ \partial_y^2 - (4v^2 - 1)/(4y^2) \right] y^{(1-2v)/4}$ with $y = \sqrt{x}$, and it simplifies to $y^{(2v-1)/2} \partial_y y^{(1-2v)/2}/4$ for $v = \pm 1/2$. The latter simplification is related to the fact that the Bessel functions in the corresponding Hankel transform (3.7) reduce to the trigonometric functions $\cos(z)/\sqrt{2}$ and $\sin(z)/\sqrt{2}$, respectively.

Note that we use the abbreviation $PE_M$ both for polynomial ensembles and for Pólya ensembles. However, this is unlikely to cause confusion, since the definitions coincide for $n = 1$ and the numbers of parameters are different for $n > 1$.

Pólya ensembles on $G$ and on $H_2$ were already introduced and investigated in [29, 30] and in [33], respectively. In those works, they were called polynomial ensembles of derivative type. Our motivation to call all these ensembles Pólya ensembles will become clear in Theorem 2.9.

The Pólya ensembles cover quite a lot of the classical random matrix ensembles [37], see also [29,30,33]. We name only a few examples here.

**Examples 2.4 (”Classical” Pólya Ensembles)**

(a) **Gaussian ensembles** For $M = M_v$ or $M = H_\beta$ and $\varepsilon > 0$, the matrix density $q_{M,\varepsilon}(y) \propto e^{-\text{tr}(y y^y)/(2\varepsilon)}$ defines a Pólya ensemble on $M$. The underlying weight function $\omega$ is $\omega_\varepsilon(x) \propto x^v e^{-x/\varepsilon}$ for $M = M_v$, $H_1$, $H_4$ and $\omega_\varepsilon(x) \propto e^{-x^2/(2\varepsilon)}$ for $M = H_2$. For $M = G$, the log-normal density $\omega_\varepsilon(x) \propto x^{-1} e^{-(\log x)^2/(2\varepsilon)}$ gives rise to a Pólya ensemble on $M$. Similarly to the Gaussian distribution in classical probability theory, these ensembles show up in connection with the heat kernel on the respective matrix spaces, see, e.g., Chapter XII.5 in [25]. For this reason, it seems appropriate to call them Gaussian ensembles.

(b) The Pólya ensemble on $H_2$ with $\omega(x) = e^{-(x-\alpha)^2/2}$ is the Gaussian unitary ensemble (GUE) with shift $\alpha \in \mathbb{R}$, with density $p_{H_2}(y) \propto \exp[-\text{tr}(y - \alpha \mathbb{1}_n)^2/2]$. 

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(c) Consider the classical Laguerre ensemble on $A \simeq \mathbb{R}_+^n$ given by
\[
p_A(a) \propto \det a^\nu \exp\{-\text{tr } a^\nu \Theta(a)\} |\Delta_n(a)|^2 \quad (a \in \mathbb{R}^n),
\] (2.19)
where $\nu > -1$ and $\Theta(y)$ denotes the Heaviside step function for matrices, which is 1 for $y \in H_2$ positive-definite and 0 otherwise. This ensemble induces Pólya ensembles on $H_2$ and on $G$ and, for $\nu$ an integer, also on $M_{\nu}$. On $H_2$, we get the Pólya ensemble with the density $p_{H_2}(y) \propto \det y^\nu \exp\{-\text{tr } y^\nu \Theta(y)\}$, which is also called the induced Laguerre ensemble. The corresponding weight function is $\omega(x) = x^{n+\nu-1} \exp\{-x \Theta(x)\}$, with $\Theta(x)$ the ordinary Heaviside step function on $\mathbb{R}$.

On the space $G$, we get the Pólya ensemble with the density $p_G(g) \propto \det(gg^*)^{\nu} \exp\{-\text{tr } gg^*\}$, which is also known as the induced Ginibre ensemble. Now, the corresponding weight function is $\omega(x) = x^\nu \exp\{-x\}$. On $M_{\nu}$, we get the Pólya ensemble with the density $p_{M_{\nu}}(y) \propto \exp\{-\text{tr } yy^*/2\}$, which is also known as the chiral Gaussian unitary ensemble or chiral Wishart ensemble and which is a special case of the Gaussian ensemble discussed in (a). This time the underlying weight function is also $\omega(x) = x^\nu \exp\{-x\}$. In contrast to that, for $\mu \neq \nu$, the density (2.19) does not induce a Pólya ensemble on $M_{\mu}$ in general.

Note that the weight functions in the above Pólya ensembles are different, since different differential operators are applied to create the joint probability density (2.19).

(d) Further examples of Pólya ensembles on $G$ are the induced Jacobi ensemble ($p_G(g) \propto \det(gg^*)^\nu \det(\mathbb{I}_n - gg^*)^\mu \Theta(\mathbb{I}_n - gg^*)$) and the induced Cauchy-Lorentz ensemble ($p_G(g) \propto \det(gg^*)^\nu \det(\mathbb{I}_n + gg^*)^{-2n-\nu-\mu}$) with the weight functions $\omega(x) = x^\nu (1 - x)^{n+\mu-1} \Theta(1 - x)$ and $\omega(x) = x^\nu/(1 + x)^{n+\nu+\mu+1}$, respectively, with $\nu, \mu > -1$.

2.5 Convolution Theorems

The polynomial ensembles on $M$ are special in that the relevant multivariate transform is of determinantal form (see Theorem 3.3), and the Pólya ensembles on $M$ are even more special in that the multivariate transform factorizes (see Corollary 3.4). The following convolution theorems are simple consequences of these observations. Here, $w_1, \ldots, w_n$ and $\omega$ are suitable weight functions as in Definitions 2.2 and 2.3, respectively.

Theorem 2.5 (Convolution of Pólya Ensembles and Polynomial Ensembles)

1. See [33]. $\text{PE}_{H_2}(w_1, \ldots, w_n) * \text{PE}_{H_2}(\omega) = \text{PE}_{H_2}(w_1 * \omega, \ldots, w_n * \omega)$.
2. See [30]. $\text{PE}_G(w_1, \ldots, w_n) \otimes \text{PE}_G(\omega) = \text{PE}_G(w_1 \otimes \omega, \ldots, w_n \otimes \omega)$.
3. For $M \in \{M_{\nu}, H_1, H_4\}$, we have
\[
\text{PE}_M(w_1, \ldots, w_n) * \text{PE}_M(\omega) = \text{PE}_M(w_1 * \nu, \omega, \ldots, w_n * \nu \omega).
\]
Corollary 2.6 (Convolution of Pólya Ensembles)

(1) See [33]. \( \text{PE}_{H_2}(\omega) \ast \text{PE}_{H_2}(\chi) = \text{PE}_{H_2}(\omega \ast \chi) \).
(2) See [30]. \( \text{PE}_{G}(\omega) \circ \text{PE}_{G}(\chi) = \text{PE}_{G}(\omega \circ \chi) \).
(3) For \( M \in \{ M_\nu, H_1, H_4 \} \), we have \( \text{PE}_M(\omega) \ast \text{PE}_M(\chi) = \text{PE}_M(\omega \ast_\nu \chi) \).

The convolutions on the left-hand sides are the matrix convolutions introduced in (2.10) and (2.11), while the univariate convolutions on the right sides are special cases (for \( n = 1 \)) of the induced convolutions described in (2.12)–(2.14). The proofs of the theorem and its corollary are given in Sect. 4.

Let us emphasize that in terms of random matrices, Corollary 2.6 gives the distribution of the sum or product of two independent random matrices \( X_1 \) and \( X_2 \) on \( M \) whose distributions are Pólya ensembles on \( M \). A similar remark applies to Theorem 2.5.

Finally, as a consequence of Corollary 2.6 and the obvious associativity of all three univariate convolutions, each of the three classes of Pólya ensembles with the corresponding convolution becomes a semigroup when adding the appropriate neutral element (the Dirac measure in the zero matrix for the additive convolution and the Haar distribution on the unitary group for the multiplicative convolution). Moreover, it has a natural action on the set of all polynomial ensembles on the respective matrix space, as shown by Theorem 2.5.

2.6 Relation to Pólya Frequency Functions

We next address the question which weight functions \( \omega \) (as in Definition 2.3) give rise to Pólya ensembles. Since this question is trivial for \( n = 1 \), we shall always assume that \( n \geq 2 \). Our main result shows that for \( M = H_2 \) and \( M = G \), these weight functions are closely related to Pólya frequency functions [28,43–45], thereby justifying our name “Pólya ensembles.” For \( M \in \{ M_\nu, H_1, H_4 \} \), the relation is less perfect, but we can at least provide a construction by which Pólya frequency functions give rise to many (in fact, infinitely many) examples of Pólya ensembles on \( M \).

Let us recall the definition of a Pólya frequency function.

Definition 2.7 (Pólya Frequency Functions, see [43–45])

(a) Let \( N \in \mathbb{N} \). A measurable function \( f \) is called a Pólya frequency function of order \( N \) if

\[
\Delta_n(x) \Delta_n(y) \det f(x_b - y_c)_{b,c=1,...,n} \geq 0 \tag{2.20}
\]

for all \( n = 1, \ldots, N \) and all \( x, y \in \mathbb{R}^n \).
(b) When Eq. (2.20) holds for any integer $n \in \mathbb{N}$, $f$ is called a Pólya frequency function of infinite order.

Pólya frequency functions (also called totally positive functions) play a role in approximation theory, where they give rise to translation-invariant totally positive kernels. The corresponding theory is mainly due to Schoenberg and to Karlin, based on earlier work by Pólya. See Chapter 7 in [28] for background information.

Some examples of Pólya frequency functions can be found in [28] and [13]. Let us highlight a few examples important in random matrix theory.

**Examples 2.8 (Pólya Frequency Functions)**

(a) The Gaussian density $f(x) = e^{-(x-\alpha)^2/2}$ with $\alpha \in \mathbb{R}$ is a Pólya frequency function of infinite order.

(b) The Heaviside step function $f(x) = \Theta(x)$ is a Pólya frequency function of infinite order because all combinations which do not satisfy the interlacing condition $y_1 < x_1 < y_2 < x_2 < \cdots < y_n < x_n$ vanish.

(c) The Heaviside step function with a monomial $f(x) = x^\nu \Theta(x)$ and $\nu > N - 2$ is a Pólya frequency function of order $N$ because of the identity

\[ \det[(x_b - y_c)^\nu \Theta(x_b - y_c)]_{b,c=1,...,n} = \left( \prod_{j=0}^{n-1} \frac{\Gamma[v + 1]}{j! \Gamma[v - j + 1]} \right) \times \Delta_n(x) \Delta_n(y) \int_{K_2} \det(x - ky^*)^{v-n+1} \Theta(x - ky^*)d^*k \]  

(2.21)

for all $n \leq N$, see [31, Theorem 2.3]. This function is even a Pólya frequency function of infinite order for any integer $\nu \geq 0$ though for less obvious reasons, cf. Eq. (2.24).

(d) The function $f(x) = \exp[-e^{-x}]$ (which is closely related to the density of the Gumbel distribution) is a Pólya frequency function of infinite order, as can be checked by the Harish–Chandra integral

\[ \det[\exp[-e^{-(x_b - y_c)}]]_{b,c=1,...,n} = \left( \prod_{j=0}^{n-1} \frac{1}{j!} \right) \times \frac{\Delta_n(e^x) \Delta_n(e^y)}{\det e^{(n-1)x}} \int_{K_2} \exp[-\text{tr}ke^{-x}k^*e^y]d^*k, \]  

(2.22)

compare Eq. (3.9), where $e^x$, $e^y$, etc. are defined component-wise and interpreted as diagonal matrices when necessary.

We showed the positivity for the latter two examples in a way which already connects the problem to group integrals and random matrix theory. We will return to this idea in Sect. 2.7. Alternatively, one can derive many of these examples with the help of the complete characterization of Pólya frequency functions of infinite order via their...
Laplace transforms [28,45]. As discussed in these references, the Laplace transform of an integrable Pólya frequency function $f$ of infinite order is of the form

$$
\int_{-\infty}^{\infty} f(x)e^{-sx}\,dx = C \exp[\gamma s^2 - \delta s] \prod_{j=1}^{\infty} \frac{\exp[\delta_j s]}{1 + \delta_j s}, \quad C > 0, \quad \gamma \geq 0,
$$

$$
\delta, \delta_j \in \mathbb{R}, \quad 0 < \gamma + \sum_{j=1}^{\infty} \delta_j^2 < \infty, \text{ and } \min_{\delta_j > 0} \left\{ \frac{1}{\delta_j} \right\} > -\Re s > \max_{\delta_j < 0} \left\{ \frac{1}{\delta_j} \right\},
$$

(2.23)

when the support of $f$ is contained in $\mathbb{R}$ and of the form

$$
\int_{0}^{\infty} f(x)e^{-sx}\,dx = C \exp[-\delta s] \prod_{j=1}^{\infty} \frac{1}{1 + \delta_j s}, \quad C > 0,
$$

$$
\delta, \delta_j \geq 0, \quad 0 < \sum_{j=1}^{\infty} \delta_j < \infty, \quad \text{ and } \min_{\delta_j > 0} \left\{ \frac{1}{\delta_j} \right\} > -\Re s,
$$

(2.24)

when the support of $f$ is contained in $[0, \infty]$. In particular, the reciprocal Laplace transform of $f$ is an entire function in either case. Unfortunately, there is no such explicit characterization for Pólya frequency functions of a finite order, although they may yield some interesting random matrix ensembles. For instance, similarly as in Example 2.8(c), the function $f(x) = x^\nu e^{-x} \Theta(x)$ is a Pólya frequency function of order $N$ for any $\nu > N - 2$. The Laplace transform of this example is given by

$$
\int_{0}^{\infty} x^\nu e^{-(1+s)x} dx = \Gamma(\nu + 1)/(1 + s)^{\nu + 1},
$$

which is not of the form (2.23) or (2.24) when $\nu$ is not an integer. Since integrable Pólya frequency functions of a fixed finite order $N$ are closed under additive convolution [28,45], we can at least say that if $f$ is an integrable function such that

$$
\int_{-\infty}^{\infty} f(x)e^{-sx}\,dx = C \exp[\nu_j \delta_j s] \prod_{j=1}^{\infty} \frac{\exp[\nu_j \delta_j s]}{(1 + \delta_j s)^{\nu_j}}, \quad C > 0, \quad \nu_j, \gamma \geq 0,
$$

$$
\delta, \delta_j \in \mathbb{R}, \quad 0 < \gamma + \sum_{j=1}^{\infty} \nu_j \delta_j^2 < \infty, \quad \text{ and } \min_{\delta_j > 0} \left\{ \frac{1}{\delta_j} \right\} > -\Re s > \max_{\delta_j < 0} \left\{ \frac{1}{\delta_j} \right\},
$$

(2.25)
\[
\int_0^\infty f(x) e^{-sx} \, dx = C \exp[-\delta s] \prod_{j=1}^\infty \frac{1}{(1 + \delta_j s)^{\nu_j}}, \quad C > 0, \quad \nu_j \geq 0,
\]
\[
\delta, \delta_j \geq 0, \quad 0 < \sum_{j=1}^\infty \nu_j \delta_j < \infty, \quad \text{and} \quad \min_{\delta_j > 0} \left\{ \frac{1}{\delta_j} \right\} > -\text{Re} s
\]
(2.26)

then \( f \) is a Pólya frequency function of order \( N \) with \( N - 1 \) smaller than all non-integer exponents \( \nu_j \). The problem is that Eqs. (2.25) and (2.26) are not exhaustive. For example, the Pólya frequency functions of order \( N = 1 \) are the positive functions, and the Pólya frequency functions of order \( N = 2 \) are the positive and log-concave functions, see [28].

For fixed \( n \geq 2 \), we will prove Theorems 2.9 and 2.10 in Sect. 4.2. Before we state our results, let us recall the main problem. Any function \( \omega \) with suitable differentiability and integrability properties as in Definition 2.3 defines a “signed” Pólya ensemble on \( M \), meaning that the associated density as in Eq. (2.15) (but without the prefactor) is not necessarily nonnegative. The hard question is: What are the necessary or sufficient conditions on \( \omega \) such that this density is also nonnegative and hence gives rise to a random matrix ensemble after proper normalization? The following theorem answers this question completely for Pólya ensembles on \( H_2 \) and on \( G \).

**Theorem 2.9** (Relation to Pólya Frequency Functions)

1. Let \( \omega \in L^1_{H_2}(\mathbb{R}) \) with \( \omega \neq 0 \). Then, \( \omega \) gives rise to a Pólya ensemble on \( H_2 \) if and only if it is a Pólya frequency function of order \( n \).
2. Let \( \omega \in L^1_G(\mathbb{R}^+) \) with \( \omega \neq 0 \). Then, \( \omega \) gives rise to a Pólya ensemble on \( G \) if and only if \( \tilde{\omega}(x) := \omega(e^{-x})e^{-x} \) is a Pólya frequency function of order \( n \).

Thus, Pólya ensembles on \( H_2 \) or on \( G \) are closely related to Pólya frequency functions for which the associated function \( \omega \) is an element of \( L^1_{H_2}(\mathbb{R}) \) or \( L^1_G(\mathbb{R}^+) \), respectively. In fact, in view of Corollary 3.4, the function \( \omega \) is determined by the corresponding Pólya ensemble on \( H_2 \) or on \( G \) up to a scalar factor. Thus, the relations just described become bijections if we restrict ourselves to normalized Pólya frequency functions (i.e., Pólya frequency functions of Lebesgue integral 1) from the respective classes.

The first statement in Theorem 2.9 is closely related to several similar results in [27,28], although we have not been able to find a result entailing the precise formulation given above. Moreover, the first statement is well known in the situation where \( \omega \) is assumed to define a Pólya ensemble on \( H_2 \) for any \( n \in \mathbb{N} \), see [13,39,42]. The relation claimed in the second statement was implicitly used in a previous work of ours [30] to show that the analytic continuation of matrix product ensembles in the number of Ginibre factors does not always yield a probability density.

For \( M \in \{ H_1, H_4, M_\nu \} \), we do not have a complete characterization, but we have at least the following partial result.
Theorem 2.10 (Sufficient condition for Pólya ensembles on $M \in \{H_1, H_4, M_v\}$) Let $M \in \{H_1, H_4, M_v\}$, and let $\tilde{\omega} \in L^1_{H_4}(\mathbb{R})$ be a Pólya frequency function of order $n$ with $\tilde{\omega} \neq 0$ and support contained in $[0, \infty[$. Then, the function
\[
\omega(x) = \frac{1}{\Gamma[v + 1]} \int_0^\infty \left(\frac{x}{y}\right)^v \exp\left[-\frac{x}{y}\right] \tilde{\omega}(y) \frac{dy}{y} \in L^1_M(\mathbb{R}_+) \tag{2.27}
\]
gives rise to a Pólya ensemble on $M$.

This theorem shows that even Pólya ensembles on $M$ are closely related to Pólya frequency functions with support contained in $\mathbb{R}_+$, in the sense that the latter give rise to a large number of examples. In particular, Eqs. (2.24) and (2.26) in combination with Eq. (2.27) yield many ensembles of this kind. Theorem 2.10 can be found by noticing that the Hankel transform of $\omega$ (as in Sect. 3) is equal to the Laplace transform of $\tilde{\omega}$. Moreover, Theorem 2.10 should be compared with [26, Corollary 5.2], which provides a related characterization for certain infinite-dimensional random matrices.

Unfortunately, the construction in Theorem 2.10 does not produce all Pólya ensembles on $M$. For example, to recover the Gaussian ensemble on $M$ (see Example 2.4(a)), which corresponds to the weight function $\omega(x) = x^v e^{-x}$, we would have to choose $\tilde{\omega}(y) = \Gamma[v + 1] \delta(y - 1)$, which is a distribution and not a function. Even worse, the weights $\omega$ arising in Theorem 2.10 satisfy $\omega(x) > 0$ for all $x > 0$, but there also exist examples of Pólya ensembles on $M$ without this property:

Example 2.11 (Example of a Pólya ensemble on $M_0$ beyond Theorem 2.10) Set $\omega(x) := e^{-\frac{1}{2x}} 1_{(0,\alpha)}(x)$, where $\alpha \in (0, \frac{1}{2})$. Then, it is straightforward to show that $\omega$ satisfies the conditions of Definition 2.3(3) (including $p_{M_0}(\omega) \geq 0$) with $n = 2$ and $v = 0$ and hence defines a Pólya ensemble on $M_0$.

We conclude this subsection with some applications of Theorems 2.9 and 2.10:

Examples 2.12 (Non-Trivial Pólya Ensembles)

(a) Since the deformed Gumbel density $\omega(x) = \exp[-e^{-x} - \alpha x]$ with $\alpha > 0$ is a Pólya frequency function of infinite order [13], it defines a Pólya ensemble on $H_2$. Moreover, applying Theorem 2.9 (2) to this density, we recover the Pólya ensemble on $G$ induced by some Laguerre ensemble, compare Example 2.4(c).

(b) The combination of the Gaussian density as a Pólya frequency function (Pólya ensemble on $H_2$) and Theorem 2.9 (2) yields the log-normal distribution $\omega(x) = x^{-1} \exp[-(\ln x - \alpha)^2/(2\sigma^2)]$. The associated Pólya ensemble on $G$ was already identified as a particular form of a Muttalib–Borodin ensemble [10,15,38] in [29, 30].

(c) The function $\omega(x) = \cosh^{-\mu}(x)$ is a Pólya frequency function of infinite order for any real number $\mu > 0$, see also [13] for $\mu = 1, 2$, and hence gives rise to a Pólya ensemble on $H_2$. The joint probability density of the eigenvalues associated with $\omega(x)$ takes the form
\[
\Delta_n(x) \det[((-\partial_x)^{a-1} \omega(x_j))] \propto \Delta_n(x) \Delta_n(e^{2x}) \prod_{j=1}^n \frac{\exp[-(n-1)x_j]}{\cosh^{a+n-1}(x_j)}. \tag{2.28}
\]
Thence, it yields a particular Muttalib–Borodin ensemble \([10,15,38]\). With the aid of Theorem 2.9 (2), one can show that for \(\mu > n - 1\), it corresponds to a particular form of a Cauchy–Lorentz ensemble as a Pólya ensemble on \(G\).

(d) The Pólya ensemble on \(H_2\) corresponding to the induced Jacobi ensemble (a Pólya ensemble on \(G\), see Example 2.4 (d)), is associated with the Pólya frequency function \(\omega(x) = \exp[-ax] \sinh^{\mu}(x) \Theta(x)\) with \(\alpha > \mu > n - 2\). This function also yields a Muttalib–Borodin ensemble with an exponential function in the second Vandermonde determinant. Moreover, in combination with Eq. (2.27), we obtain a non-trivial Pólya ensemble on \(M_\nu\) associated with the function

\[
\hat{\omega}(x) = \int_0^\infty \left(\frac{x}{y}\right)^\nu \exp\left[-\frac{x}{y} - \alpha y\right] \sinh^{\mu}(y) \frac{dy}{y} .
\]

(2.29)

(e) Using Eq. (2.27) with the weight function \(\tilde{\omega}\) for the induced Laguerre ensemble on \(H_2\), we find that \(\omega(x) = x^{(\mu + \nu)/2} K_{\mu - \nu}(2\sqrt{x})\) with \(\mu > n - 2\) and \(K_j\) the modified Bessel function of the second kind gives rise to a Pólya ensemble on \(M_\nu\).

2.7 Identities Involving Group Integrals

Another nice feature of Pólya ensembles is that they satisfy interesting identities involving group integrals.

Theorem 2.13 (Group Integrals and Pólya Ensembles)

(1) Let \(p_{H_2} = \text{PE}_{H_2}(\omega)\). Then,

\[
\Delta_n(y) \Delta_n(x) \int_{K_2} p_{H_2}(y - k x k^*) d^*k = \frac{1}{n! C_{H_2}} \frac{\det[\omega(y_b x_c^{-1})]_{b,c=1,...,n}}{(\mathcal{F}\omega(0))^n} \quad (2.30)
\]

for almost all \(x, y \in D\), with \(C_{H_2}\) as in Eq. (2.8) and \(\mathcal{F}\omega\) the Fourier transform as in Eq. (3.13).

(2) Let \(p_{G} = \text{PE}_{G}(\omega)\). Then,

\[
\Delta_n(y) \Delta_n(-x^{-1}) \int_{U(n)} p_{G}(x^{-1/2} y^{1/2}) d^*k = \frac{1}{n! C_{G}} \frac{\det[\omega(y_b x_c^{-1})]_{b,c=1,...,n}}{\prod_{j=1}^n \mathcal{M}\omega(j)} \quad (2.31)
\]

for almost all \(x, y \in A\), with \(x^{1/2}, y^{1/2}\) defined component-wise, \(C_{G}\) as in Eq. (2.8) and \(\mathcal{M}\omega\) the Mellin transform as in Eq. (3.15).

(3) For \(M \in \{M_\nu, H_1, H_4\}\), let \(p_M = \text{PE}_M(\omega)\). Then,

\[
\Delta_n(y) \Delta_n(x) \int_{K} p_{M}(t_M(y) - k t_M(x) k^*) d^*k = \frac{C_{M(1)}^n}{n! C_{M}} \frac{\det}{\prod_{j=1}^n \mathcal{M}\omega(j)} \int_{K(1)} p_{M(1)}(t_{M(1)}(y) - k t_{M(1)}(x) k^*) d^*k \quad (2.32)
\]

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for almost all \( y, x \in A \), where \( C_M \) is as in Eq. (2.8), \( M(1) \), \( K(1) \) and \( C_{M(1)} \) are the matrix space, the matrix group and the constant for \( n = 1 \), and \( p_{M(1)} = PE_{M(1)}(\omega) \) (with the same weight function \( \omega \)).

**Remark** By the invariance of the matrix densities, the above identities continue to hold if we replace the diagonal matrices \( x \) and \( y \) inside the group integrals by general matrices \( x \) and \( y \) in \( H_2 \), \( G \) or \( M \), respectively, with the same eigenvalues or squared singular values.

Theorem 2.13 will be proved in Sect. 4.1 and also plays an important role in the proofs of Theorems 2.9 and 2.10.

The identities are far from trivial in quite a few cases, see the following examples.

**Examples 2.14 (Some Group Integrals)** For a differentiable function \( F: B \to \mathbb{R} \) with first derivative \( F'(x) \neq 0 \) for all \( x \in B \subset \mathbb{R} \), we define the abbreviation

\[
\tilde{F}(y) := \frac{1}{\det F'(y)} \frac{\det F(y) \otimes 1_n - 1_n \otimes F(y)}{y \otimes 1_n - 1_n \otimes y}
\]

\[
:= \prod_{j<k} \frac{(F(a_k) - F(a_j))^2}{(a_k - a_j)^2} = \frac{\Delta_n(F(a))}{\Delta_n(a)} \tag{2.33}
\]

for all \( y \in H_2 \) with eigenvalues \( a_1, \ldots, a_n \in B \).

(a) The Pólya ensemble on \( G \) defined by the weight \( \omega(x) = \exp[-(\ln x - \omega)^2/(2\sigma^2)] \) (a Muttalib–Borodin ensemble) has the matrix density

\[
p_G(g) \propto \tilde{\ln}(gg^*) \det(gg^*)^{\alpha/\sigma^2} \exp[-\text{tr}(\ln gg^*)^2/(2\sigma^2)] \tag{2.34}
\]

and satisfies the identity

\[
\Delta_n(y) \Delta_n(x^{-1}) \int_{U(n)} p_G(x^{-1/2}ky^{1/2})d^*k
\]

\[
\propto \det \left[ \exp \left[ -(\ln y_b - \ln x_c - \omega)^2/(2\sigma^2) \right] \right]_{b,c=1,\ldots,n}. \tag{2.35}
\]

(b) The Pólya ensemble on \( H_2 \) defined by the weight \( \omega(x) = \cosh^{-\mu}(x) \) (another Muttalib–Borodin ensemble) has the matrix density

\[
p_{H_2}(y) \propto \exp(2y) \frac{\exp[-(n-1)\text{tr}y]}{\det[\cosh^{\mu+n-1}y]} \tag{2.36}
\]

(where the matrix in the determinant is defined by spectral calculus) and satisfies the identity

\[
\Delta_n(y) \Delta_n(x) \int_{K_2} p_{H_2}(y - kxx^*)d^*k \propto \det \left[ \cosh^{-\mu}(y_b - x_c) \right]_{b,c=1,\ldots,n}. \tag{2.37}
\]
3 Multivariate Transforms

To motivate our approach, let us begin with an exposition of the Hankel transform tailored to our needs. Fix $\nu \in \mathbb{N}_0$. Then, for $f \in L^1(\mathbb{C}^{1+\nu})$, its Fourier transform is defined by

$$(\mathcal{F}f)(y) = \int_{\mathbb{C}^{1+\nu}} f(x) e^{i(x^*y+y^*x)} \, dx, \quad y \in \mathbb{C}^{1+\nu}.$$  \hfill (3.1)

Now, suppose that $f$ is additionally unitarily invariant, i.e., $f(Ux) = f(x)$ for any $U \in U(1 + \nu)$ or (equivalently) $f(x) = F(\|x\|)$ with $\|x\|$ the Euclidean norm. Then, $\mathcal{F}f$ is also unitarily invariant, and it is natural to express everything in terms of the squared radii $r := \|x\|^2$ and $s := \|y\|^2$. To this end, set

$$\tilde{f}(r) := \frac{\pi^{\nu+1}}{\Gamma(\nu+1)} f(\sqrt{r}e_1)r^{-\nu} \quad (r > 0),$$ \hfill (3.2)

which is the induced density of the squared radius $r := \|x\|^2$ in case $f$ is a probability density. By unitary invariance, it suffices to compute $(\mathcal{F}f)(\sqrt{s}e_1)$ for $y = \sqrt{s}e_1$, with $e_1 := (1, 0, \ldots, 0) \in \mathbb{C}^{1+\nu}$. After a moderate calculation, one finds that

$$(\mathcal{F}f)(\sqrt{s}e_1) = \Gamma(\nu+1) \int_0^{\infty} \tilde{f}(r) \frac{J_\nu(2\sqrt{rs})}{(rs)^{\nu/2}} \, dr =: (\mathcal{H}_\nu \tilde{f})(s),$$ \hfill (3.3)

where $J_\nu$ is the Bessel function of parameter $\nu$ and $\mathcal{H}_\nu$ is (a version of) the Hankel transform. Let us note that the Hankel transform is usually defined a bit differently in the literature, so that it becomes an involution. Also, let us note that if $f$, $g \in L^1(\mathbb{C}^{1+\nu})$ are unitarily invariant, then $f \ast g \in L^1(\mathbb{C}^{1+\nu})$ is also unitarily invariant, and the well-known relation $\mathcal{F}(f \ast g) = \mathcal{F}f \cdot \mathcal{F}g$ for the Fourier transform translates into the relation $\mathcal{H}_\nu(f \ast g) = \mathcal{H}_\nu \tilde{f} \cdot \mathcal{H}_\nu \tilde{g}$ for the Hankel transform.

Indeed, in view of the identification $\mathbb{C}^{1+\nu} \simeq \text{Mat}_\mathbb{C}(1, 1 + \nu) \simeq M_\nu(1) =: M$, the radial density $\tilde{f}$ is nothing else than the spectral density $\mathcal{I}_M(f)$ introduced in Sect. 2.2, and the relation for the Hankel transform just derived takes the form

$$\mathcal{H}_\nu(f_A \ast_\nu g_A) = (\mathcal{H}_\nu f_A) \cdot (\mathcal{H}_\nu g_A),$$ \hfill (3.4)

with $f_A$, $g_A$ and $\ast_\nu$ as in Eq. (2.14). By similar reasoning, Eq. (3.4) also holds for the matrix spaces $H_1$ and $H_4$, again with $n = 1$ and the parameter $\nu$ as in (2.7).

We will now develop a similar approach for the matrix spaces $M$ listed in Sect. 2.1, where it seems natural to express $K$-invariant functions in terms of their eigenvalues or squared singular values, and hence in terms of the induced spectral densities as introduced in Sect. 2.2.
For the Hermitian matrix spaces $M \in \{H_1, H_2, H_4, M_v\}$ and $f_M \in L^1(M)$, the Fourier transform is given by

$$(\mathcal{F} f_M)(x) := \int_M f_M(y) \exp(i t r x y) \, dy, \quad x \in M.$$

When $f_M$ is additionally $K$-invariant, then $\mathcal{F} f_M$ is also $K$-invariant. Suppose that $s = \text{diag}(s_1, \ldots, s_n) \in \mathbb{C}^{n \times n}$ with $s_i \neq s_j$ for $i \neq j$ such that the respective integrals exist in the Lebesgue sense. Then, the Fourier transform simplifies to

$$(\mathcal{F} f_M)(s) = \int_M f_M(y) \left( \int_{K_2} \exp[i t r y s k^*] d^* k \right) \, dy = \prod_{j=0}^{n-1} \prod_{j=0}^{n-1} \Gamma[j + v + 1] \int_A f_A(a) \frac{\det[J_v(2\sqrt{a_b s_c})/(a_b s_c)^{v/2}]}{\Delta_n(-s) \Delta_n(a)} da =: (\mathcal{F} f_D)(s)$$

with $f_D := \mathcal{I}_{H_2}(f_H) \in L^{1,S}(D)$ for $M = H_2$ and to

$$(\mathcal{F} f_M)(\imath_M(s)) = \int_M f_M(y) \left( \int_K \exp[i t r y l_M(s)] d^* k \right) \, dy = \prod_{j=0}^{n-1} \Gamma[j + v + 1] \int_A f_A(a) \frac{\det[J_v(2\sqrt{a_b s_c})/(a_b s_c)^{v/2}]}{\Delta_n(-s) \Delta_n(a)} da =: (\mathcal{F} f_A)(s)$$

with $f_A := \mathcal{I}_{M}(f_M) \in L^{1,S}(A)$ for $M \in \{H_1, H_4, M_v\}$. Here, $v$ and $\imath_M(s)$ are defined similarly to (2.7). Furthermore, for the space $G$ and $f_G \in L^{1,K}(G)$, we may consider the spherical transform defined by

$$(S f_G)(s) = \int_G f_G(g) \left( \int_{U(n)} \prod_{j=1}^{n} \det(\Pi_{j,n} k g s k^* \Pi_{j,n}^*) s_j - s_{j+1} - 1 d^* k \right) \, dg = \prod_{j=0}^{n-1} \prod_{j=0}^{n-1} \Gamma[j + v + 1] \int_A f_A(a) \frac{\det[a_b^{s_c - (n+1)/2}]}{\Delta_n(s) \Delta_n(a)} da =: (\mathcal{M} f_A)(s),$$

where $s_{n+1} := \frac{n-1}{2}$ and $f_A := \mathcal{I}_{G}(f_G) \in L^{1,S}(A)$.

To obtain the simplifications in Eqs. (3.6), (3.7) and (3.8), we have used the Harish–Chandra–Itzykson–Zuber integral [19,24] for $K_2 = U(n)$,

$$\int_{K_2} \exp[i t r y s k^*] d^* k = \prod_{j=0}^{n-1} \frac{\det[\exp[i a_b s_c]]_{b,c=1,\ldots,n}}{\Delta_n(is) \Delta_n(a)}.$$
the Harish–Chandra integral [19] for \( K = K_1, K_4 \) and the Berezin–Karpelevich integral [9,18] for \( K = K_v \),

\[
\int_K \exp \left[ \iota \text{tr} k \ast a \iota M(s) \right] d^* k = \prod_{j=0}^{n-1} \Gamma[j+\nu+1] j! \frac{\det \left[ J_{v} (2 \sqrt{a_b s_c}) / (a_b s_c)^{v/2} \right]_{b,c=1,\ldots,n}}{\Delta_n(-s) \Delta_n(a)},
\]

and the Gelfand–Na˘ımark integral [16] for \( U(n) \),

\[
\int_{U(n)} \prod_{j=1}^{n} \det (\Pi_{j,n} k \ast \Pi_{j,n}^*)^{s_j-s_{j+1}} d^* k = \prod_{j=0}^{n-1} j! \frac{\det [a_b^{\nu+(n+1)/2}]_{b,c=1,\ldots,n}}{\Delta_n(s) \Delta_n(a)},
\]

with \( s_{n+1} := \frac{2}{v} \) as above. Recall that we work with the induced spectral densities, so that the Jacobians resulting from the diagonalization of the matrices are absorbed in \( f_D \) and \( f_A \), respectively.

In view of the preceding results, we make the following definition:

**Definition 3.1 (Multivariate Transforms)** The induced transforms in (3.6), (3.7) and (3.8) are called the multivariate Fourier transform, the multivariate Hankel transform and the multivariate Mellin transform, respectively.

The multivariate transforms are normalized in such a way that

\[
\mathcal{F} f_D(0) = \int_{H_2} f_{H_2}(y) dy, \quad \mathcal{H}_v f_A(0) = \int_{M} f_{M}(y) dy, \quad \mathcal{M} f_A(s^{(0)}) = \int_{G} f_{G}(g) dg,
\]

where the functions are as in (3.6), (3.7) and (3.8) and \( s^{(0)} := (j-1) + (n+1)/2 \), \( j = 1, \ldots, n \). Our motivation for calling the multivariate transforms \( \mathcal{F}, \mathcal{M} \) and \( \mathcal{H}_v \) is that, for \( n = 1 \), they reduce to the univariate Fourier, Hankel and Mellin transform defined by

\[
\mathcal{F} f(s) = \int_{-\infty}^{\infty} f(x) \exp[\iota x s] dx, \quad f \in L^1(\mathbb{R}),
\]

\[
\mathcal{H}_v f(s) = \Gamma[v+1] \int_{0}^{\infty} f(x) \frac{J_v(2 \sqrt{x s})}{(sx)^{v/2}} dx, \quad f \in L^1(\mathbb{R}^+_v),
\]

\[
\mathcal{M} f(s) = \int_{0}^{\infty} f(x) x^{s-1} dx, \quad f \in L^1(\mathbb{R}^+_v),
\]

respectively. Note that Eq. (3.14) coincides with Eq. (3.3).

Also note that, by Eqs. (3.6) and (3.7), the induced transforms \( \mathcal{F} \) and \( \mathcal{H}_v \) are essentially the ordinary Fourier transform (3.5) restricted to the matrices \( s \) and \( \iota M(s) \) with \( s \) diagonal, which is sufficient by \( K \)-invariance. Thus, in particular, they inherit the injectivity of the ordinary Fourier transform. The transform \( \mathcal{M} \) in (3.8) is also injective on \( L^{1,S}(A) \), but for less simple reasons, see, e.g., [21,29]. Finally, let us note that all the transforms in Definition 3.1 may be inverted fairly explicitly (possibly after appropriate regularization) either as a consequence of the Fourier inversion formula.
The multivariate transforms in Definition 3.1 have the important property that they convert the induced convolutions in Eqs. (2.12)–(2.14) into multiplications. For the Fourier transform and the Hankel transform, this follows from the corresponding property of the Fourier transform (3.5). For the spherical transform, see, e.g., [21, Proof of Lemma IV.3.2].

**Theorem 3.2** (Multiplication Theorems for the Convolutions)

1. For \( f_D, h_D \in L^{1,S}(D) \), \( \mathcal{F}[f_D \ast h_D] = \mathcal{F}f_D \mathcal{F}h_D \).
2. For \( f_A, h_A \in L^{1,S}(A) \), \( \mathcal{M}[f_A \circ h_A] = \mathcal{M}f_A \mathcal{M}h_A \).
3. For \( f_A, h_A \in L^{1,S}(A) \), \( \mathcal{H}_\nu[f_A \ast \nu h_A] = \mathcal{H}_\nu f_A \mathcal{H}_\nu h_A \).

Similarly to the univariate transforms, these multiplication theorems come in handy for studying the three matrix convolutions and often allow for explicit results in special examples. In particular, they are crucial for identifying the subsets of polynomial ensembles which are closed under these convolutions, see Sect. 2.4.

The reason why polynomial ensemble and Pólya ensembles are so special becomes clear when taking the multivariate transforms 3.1.

**Theorem 3.3** (Multivariate Transforms of Polynomial Ensembles)

1. See [33]. For \( p_{H_2} = \text{PE}_{H_2}(w_1, \ldots, w_n) \) and \( p_D := \mathcal{T}_{H_2}(p_{H_2}) \),

\[
\mathcal{F}p_D(s) = C_n[w] \left( \prod_{j=1}^n j! \right) \frac{\det[\mathcal{F}w_b(s_c)]_{b,c=1,\ldots,n}}{\Delta_n(t,s)}. \tag{3.16}
\]

2. See [30]. For \( p_G = \text{PE}_{G}(w_1, \ldots, w_n) \) and \( p_A := \mathcal{T}_{G}(p_G) \),

\[
\mathcal{M}p_A(s) = C_n[w] \left( \prod_{j=1}^n j! \right) \frac{\det[\mathcal{M}w_b(s_c - (n - 1)/2)]_{b,c=1,\ldots,n}}{\Delta_n(s)}. \tag{3.17}
\]

3. For \( M \in \{ M_3, H_1, H_4 \} \), \( p_M = \text{PE}_{M}(w_1, \ldots, w_n) \) and \( p_A := \mathcal{T}_{M}(p_M) \),

\[
\mathcal{H}_\nu p_A(s) = C_n[w] \left( \prod_{j=1}^n j! \frac{\Gamma[j + \nu]}{\Gamma[1 + \nu]} \right) \frac{\det[\mathcal{H}_\nu w_b(s_c)]_{b,c=1,\ldots,n}}{\Delta_n(-s)}. \tag{3.18}
\]

**Proof** For all three statements we have to plug Eq. (2.15) into the second lines of Eqs. (3.6), (3.7) and (3.8). The integrals can be performed by applying Andréief’s integration theorem [8] and by identifying the entries in the remaining determinant with the univariate transforms in Eqs. (3.13), (3.14) and (3.15), which concludes the proof.

\[\square\]
Corollary 3.4 (Multivariate Transforms of Pólya Ensembles)

(1) See [33]. For \( p_{H_2} = PE_{H_2}(\omega) \) and \( p_D := \mathcal{I}_{H_2}(p_{H_2}) \),

\[
\mathcal{F} p_D(s) = \prod_{j=1}^{n} \frac{\mathcal{F}_\omega(s_j)}{\mathcal{F}_\omega(0)} \quad \text{and} \quad C_n[\omega] = \prod_{j=1}^{n} \frac{1}{j! \mathcal{F}_\omega(0)}. \tag{3.19}
\]

(2) See [30]. For \( p_G = PE_G(\omega) \) and \( p_A := \mathcal{I}_G(p_G) \),

\[
\mathcal{M} p_A(s) = \prod_{j=1}^{n} \frac{\mathcal{M}_\omega(s_j - (n - 1)/2)}{\mathcal{M}_\omega(j)} \quad \text{and} \quad C_n[\omega] = \prod_{j=1}^{n} \frac{1}{j! \mathcal{M}_\omega(j)}. \tag{3.20}
\]

(3) For \( M \in \{ M_\nu, H_1, H_4 \} \), \( p_M = PE_M(\omega) \) and \( p_A := \mathcal{I}_M(p_M) \),

\[
\mathcal{H}_\nu p_A(s) = \prod_{j=1}^{n} \frac{\mathcal{H}_\nu \omega(s_j)}{\mathcal{H}_\nu \omega(0)} \quad \text{and} \quad C_n[\omega] = \prod_{j=1}^{n} \frac{\Gamma[1 + \nu]}{j! \Gamma[j + \nu] \mathcal{H}_\nu \omega(0)}. \tag{3.21}
\]

Proof For the proof, we have to combine Definition 2.2 of the Pólya ensembles, Theorem 3.3 and the relations

\[
\mathcal{F} \left( -\frac{\partial}{\partial x} \omega(x); s_c \right) = ts_c \mathcal{F}_\omega(s_c), \\
\mathcal{M} \left( -x \frac{\partial}{\partial x} \omega(x); s_c \right) = s_c \mathcal{M}_\omega(s_c), \\
\mathcal{H}_\nu \left( x^\nu \frac{\partial}{\partial x} x^{\nu-1} \frac{\partial}{\partial x} \omega(x); s_c \right) = -s_c \mathcal{H}_\nu \omega(s_c), \tag{3.22}
\]

for suitable functions \( \omega \). These relations can be proven via integrations by parts. For the Hankel transform \( \mathcal{H}_\nu \), one needs that the boundary terms vanish, which follows from our integrability assumptions at infinity and from the limit condition in Definition 2.3 (3) at the origin.

After inserting Eq. (3.22) into Eqs. (3.16)–(3.18), we may pull those factors out of the determinants in the numerators which are independent of the index \( b \). This leaves us with Vandermonde determinants \( \Delta_n(s) \) multiplied by products of the functions \( \mathcal{F}_\omega(s_j), \mathcal{M}_\omega(s_j - (n - 1)/2), \) and \( \mathcal{H}_\nu \omega(s_j) \), respectively. The Vandermonde determinants cancel with those in the denominators. Since the densities are normalized, we may use Eq. (3.12) to fix the constants \( C_n[\omega] \). This finishes the proof. \( \Box \)

Remark 3.5 Corollary 3.4 may be generalized to non-normalized and even to signed Pólya ensembles. Here, the respective multivariate transform is still proportional to the product of the univariate transforms, and the corresponding normalizing constants may be determined by means of the latter formulas in (3.19)–(3.21).
4 Proofs of the Main Results

With the help of the multivariate transforms calculated in Theorem 3.3 and Corollary 3.4, it is easy to prove Theorem 2.5 and Corollary 2.6:

Proof of Theorem 2.5 Since the proofs of the three parts are very similar, we give the proof for part (3) only. Let $f_M := \text{PE}_M(\omega_1, \ldots, \omega_n)$ and $h_M := \text{PE}_M(\omega)$, and let $f_A$ and $h_A$ denote the associated induced densities as in (2.7). By $K$-invariance, we may work with the induced transforms and apply Theorem 3.2:

$$H_\nu(I_M(f_M \ast h_M))(s) = H_\nu(f_A \ast h_A)(s) = H_\nu f_A(s) H_\nu h_A(s). \quad (4.1)$$

After using Theorem 3.3 and Corollary 3.4, we can push the factors from the transform of the Pólya ensemble into the determinant from the transform of the polynomial ensemble, thereby obtaining

$$H_\nu(I_M(f_M \ast h_M))(s) \propto \det[H_\nu(\omega b \ast \nu(\omega))(sc)]_{b,c=1,\ldots,n} \Delta_n(-s)^n. \quad (4.2)$$

For the entries of the resulting determinant, we can apply Theorem 3.2 for the univariate case $n = 1$:

$$H_\nu(I_M(f_M \ast h_M))(s) \propto \det[H_\nu(\omega b \ast \nu(\omega))(s_c)]_{b,c=1,\ldots,n} \Delta_n(-s). \quad (4.3)$$

In the end, we apply the uniqueness theorem by which a $K$-invariant matrix density is determined by its multivariate transform.

Proof of Corollary 2.6 The proof works along the same lines as that of Theorem 2.5, only that we may use Corollary 3.4 for both ensembles and, hence, have no determinant. Additionally, we need to check that the conditions in Definition 2.3 are preserved under convolution. For brevity, we confine ourselves to a rough outline of the argument for part (3). The integrability and differentiability conditions can be checked using the relations

$$\int_{R_+} (F \ast \nu G) = \int_{R_+} F \cdot \int_{R_+} G \quad (4.4)$$

and

$$(\partial_x x^{v+1} \partial_x x^{-v})(F \ast \nu G) = (\partial_x x^{v+1} \partial_x x^{-v} F) \ast \nu G. \quad (4.5)$$

The extra limit condition then follows from the observation that, given the other conditions, $\lim_{x \to 0} (x^{v+1} \partial_x x^{-v} F)(x) = 0$ is equivalent to $\int_0^\infty (\partial_x x^{v+1} \partial_x x^{-v} F)(x) \, dx = 0$.

For the proof of Theorem 2.9, we need some preparations. In order to show that an integrable function $f$ is a Pólya frequency function of order $N$, it suffices to check the condition (2.20) for $n = 1$ and $n = N$. 

\[ Springer \]
Lemma 4.1 (Sufficient Condition for Pólya Frequency Function of Order \( N \)) Let the function \( f : \mathbb{R} \rightarrow \mathbb{R} \) be integrable and suppose that the condition (2.20) holds for \( n = 1 \) and \( n = N \). Then, \( f \) is a Pólya frequency function of order \( N \).

**Proof** By way of induction, it suffices to show that if a nonnegative and integrable function \( f \) satisfies condition (2.20) for \( n = N \), then it also satisfies condition (2.20) for \( n = N - 1 \). In doing so, we may assume that \( f \neq 0 \), because the claim is trivial otherwise. Thus, there exists a real number \( z_\ast \in \mathbb{R} \) with \( f(z_\ast) > 0 \).

We fix \( x_1 < \cdots < x_{N-1}, y_1 < \cdots < y_{N-1} \). Then, we know that for any \( x_N \geq x_{N-1} \) and \( y_N \geq y_{N-1} \),

\[
\hat{D} := \det \begin{bmatrix} f(x_j - y_k) \end{bmatrix}_{j,k=1,\ldots,N} \geq 0, \tag{4.6}
\]

and we must show that

\[
\det \begin{bmatrix} f(x_j - y_k) \end{bmatrix}_{j,k=1,\ldots,N-1} \geq 0. \tag{4.7}
\]

To this end, we expand the determinant (4.6) in the last row and the last column,

\[
\hat{D} = f(x_N - y_N) \det A + \sum_{j,k=1}^{N-1} (-1)^{j+k-1} f(x_j - y_N) f(x_N - y_k) \det A^{[j:k]}. \tag{4.8}
\]

Here, \( A \) denotes the \((N-1) \times (N-1)\) matrix in Eq. (4.7), and \( A^{[j:k]} \) denotes the \((N-2) \times (N-2)\) matrix obtained from \( A \) by removing the \( j \text{'}th \) row and the \( k \text{'}th \) column. Note that the matrix \( A \) and, hence, the maximum \( K := \max_{j,k=1,\ldots,n} | \det A^{[j:k]} | \) do not depend on \( x_N \) and \( y_N \).

Now, for sufficiently large \( z \in \mathbb{R} \), set \( x_N := z + z_\ast \) and \( y_N := z \) with \( z_\ast \) as above. We define the auxiliary function \( h(z) := \sum_{j=1}^{N-1} f(x_j - z) + \sum_{k=1}^{N-1} f(z + z_\ast - y_k) \). The integrability of \( f \) carries over to \( h \). Hence, there exist real numbers \( z_m > \max\{x_{N-1} - z_\ast, y_{N-1} \} \) such that \( K^{1/2} h(z_m) < 1/m \) for all \( m \in \mathbb{N} \) and \( \lim_{m \to \infty} z_m = \infty \). It then follows from Eq. (4.8) that

\[
f(z_\ast) \det A \geq \hat{D} - K \sum_{j,k=1}^{N-1} f(x_j - z_m) f(z_m + z_\ast - y_k)
\]

\[
\geq \hat{D} - K (h(z_m))^2 \geq \hat{D} - \frac{1}{m^2} \tag{4.9}
\]

for all \( m \in \mathbb{N} \). Since \( \hat{D} \geq 0 \) and \( f(z_\ast) > 0 \), this implies \( \det A \geq 0 \), and the proof is complete. \( \square \)

To the best of our knowledge, Lemma 4.1 is a new result. It is helpful when checking whether a density is a Pólya frequency function, especially when proving some statements of our Theorem 2.9.
4.1 Proof of Theorem 2.13

Since the proofs of parts (1)–(3) are very similar, we provide the full details for the most complicated part (3) only and confine ourselves to rough sketches for parts (1) and (2).

(3) For any $\varepsilon > 0$, let $q_{M, \varepsilon}$ be the Gaussian density on $M$ as in Example 2.4(a), and set $p_{M, \varepsilon} := p_M * q_{M, \varepsilon}$. Then, by basic properties of the convolution, we have

$$\lim_{\varepsilon \to 0} \int_M \left| (p_M - p_{M, \varepsilon})(\iota_M(y) - \tilde{x}) \right| d\tilde{x} = 0$$

for any fixed $y \in A$, with $\iota_M(y)$ as in Eq. (2.7). By the change of variables $\tilde{x} \to k\iota_M(x)k^*$, where $k \in K$ and $x \in A$, it follows that

$$\lim_{\varepsilon \to 0} \int_A \int_K \left| (p_M - p_{M, \varepsilon})(\iota_M(y) - k\iota_M(x)k^*) \right| d^* k \left| \det s \right|^\nu |\Delta_n(s)|^2 d\tilde{x} = 0 .$$

Thus, for fixed $y \in A$, since convergence in $L^1$ implies almost sure convergence along some subsequence, we may find a sequence $(\varepsilon_m)_{m \in \mathbb{N}}$ of positive numbers such that

$$\lim_{m \to \infty} \varepsilon_m = 0$$

and

$$\lim_{m \to \infty} \int_K \left| (p_M - p_{M, \varepsilon_m})(\iota_M(y) - k\iota_M(x)k^*) \right| d^* k = 0$$

for almost all $x \in A$.

We now prove (2.32) with $p_M$ replaced by $p_{M, \varepsilon}$. Fix $\varepsilon > 0$ and $x, y \in A$, and set $p_A := \mathcal{I} p_M$ and $p_{A, \varepsilon} := \mathcal{I} p_{M, \varepsilon}$. Then, $(\mathcal{H}_v p_A, s) = (\mathcal{H}_v p_A)(s) \cdot \prod_{j=1}^n e^{-\varepsilon s_j}$ for all $s \in \mathbb{R}^n$, where the first factor is bounded and the second factor is integrable even after multiplication by $|\Delta_n(s)|^2$. Thus, multivariate Fourier inversion (recall from Eq. (3.7) that the Hankel transform arises from the Fourier transform (3.5) in the matrix space $M$) yields

$$p_{M, \varepsilon}(z) = \hat{C} \int_A \left( \int_K \exp[-i\text{tr}(\tilde{k}\iota_M(s)k^*)]d^* k \right) (\det s)^\nu |\Delta_n(s)|^2 ds ,$$

where $z \in M$ and $\hat{C}$ is the normalizing constant in the inverse Fourier transform. Replacing $z$ with $\iota_M(y) - k\iota_M(x)k^*$, integrating with respect to $k$ and exchanging the order of integration, it follows that

$$\int_K p_{M, \varepsilon}(\iota_M(y) - k\iota_M(x)k^*) d^* k \propto \int_A (\mathcal{H}_v p_{A, \varepsilon})(s) \times \left( \int_K \int_K \exp[-i\text{tr}(\iota_M(y) - k\iota_M(x)k^*)\tilde{k}\iota_M(s)k^*)] d^* k d^* \tilde{k} \right) (\det s)^\nu |\Delta_n(s)|^2 ds .$$
By the invariance of the Haar measure under the translation \( k \to \tilde{k} k \), the inner double integral in (4.14) factorizes into

\[
\int_K \exp[t \text{tr} \iota_M(x) k^* \iota_M(s)] d^s k \int_K \exp[-t \text{tr} \tilde{k} \iota_M(y) \tilde{k}^* \iota_M(s)] d^s \tilde{k},
\]

where both integrals are Berezin–Karpelevich integrals \([9, 18]\). Hence, using Eq. (3.10), it follows that

\[
\Delta_n(y) \Delta_n(x) \int_K p_{M, \varepsilon}(\iota_M(y) - k \iota_M(x) k^*) d^s k \propto \int_A (\mathcal{H} p_{A, \varepsilon})(s) \times \det \left[ J_v(2 \sqrt{x_b s_c}) \right]_{b, c = 1, \ldots, n} \det \left[ J_v(2 \sqrt{y_b s_c}) \right]_{b, c = 1, \ldots, n} (s)^v \, ds.
\]

Since the multivariate Hankel transform \( \mathcal{H}_v p_{A, \varepsilon}(s) = \prod_{j=1}^n (\mathcal{H}_v \omega)(s_j) e^{-\varepsilon s_j} \) factorizes by Eqs. (3.19) and (3.21), we can apply Andréief’s integration theorem \([8]\) to obtain

\[
\Delta_n(y) \Delta_n(x) \int_K p_{M, \varepsilon}(\iota_M(y) - k \iota_M(x) k^*) d^s k \propto \det \left[ \int_0^\infty (\mathcal{H}_v \omega)(s) e^{-\varepsilon s} J_v(2 \sqrt{x c s}) J_v(2 \sqrt{y c s}) s^v \, ds \right]_{b, c = 1, \ldots, n}.
\]

Using (4.17) for \( n = 1 \), we see that the integral inside the determinant in (4.17) is proportional to the group integral of the expression \( p_{M(1), \varepsilon}(\iota_{M(1)}(y_b) - k \iota_{M(1)}(x_c) k^*) \) over the set \( K(1) \).

The normalizing constant in (2.32) results from the normalizing constants in the inverse Fourier transform and the Berezin–Karpelevich integral and is hence independent of \( p_{M, \varepsilon} \). Instead of bookkeeping all constants, it can be determined by choosing \( p_M(y) \propto \exp[-\text{tr}(y^* y)/2] \) to be Gaussian, where the group integrals in Eq. (2.32) may be evaluated explicitly using (3.10). This completes the proof of Eq. (2.32) when \( p_M \) is replaced with \( p_{M, \varepsilon} \).

Finally, for \( y \) and \( x \) as in (4.12), we may deduce Eq. (2.32) by letting \( m \to \infty \) in the corresponding result for \( p_{M, \varepsilon m} \). Then, the integral of \( p_{M, \varepsilon} \) converges to that of \( p_M \) by (4.12), while the integral of \( p_{M(1), \varepsilon} \) converges to that of \( p_{M(1)} \) due to the continuity of \( p_{M(1)}(s) \) for \( n > 1 \). (For \( n = 1 \), Eq. (2.32) is trivial.)

(1) This proof is almost the same. We must only replace \( \iota_M \) with the identity, use the Harish–Chandra–Itzykson–Zuber integral (3.9) instead of the Berezin–Karpelevich integral (3.10) and note that the integral over \( K_2(1) = U(1) \) reduces to a constant by commutativity.

(2) This proof goes along the same ideas as for part (3). First of all, we argue that it is sufficient to prove (2.31) for \( p_{G, \varepsilon} := p_G \otimes q_\varepsilon \) instead of \( p_G \), where \( q_\varepsilon \) is the multivariate “log-normal” density on \( G \) as in Example 2.4(a). Then, we insert the
spherical inversion formula \[21, \text{Theorem IV.7.5}\]
\[
p_{G, \varepsilon}(g) \propto \int_{\mathbb{R}^n} (S_{p_{G, \varepsilon}})(n \mathbb{1} + is) \varphi(gg^*, -n \mathbb{1} - is) |\Delta_n(is)|^2 \, ds
\] (4.18)

with \( \mathbb{1} := (1, \ldots, 1) \in \mathbb{R}^n \) on the left side in Eq. (2.31) and interchange the group integral with the integral over \( s \). The spherical function
\[
\varphi(gg^*, s) := \int_{U(n)} \prod_{j=1}^{n} \det(\Pi_{j,n}kgg^*k^n\Pi_{j,n}^*)^{s_j-s_j+1} d^*k
\] (4.19)

(where \( s_{n+1} := -\frac{n+1}{2} \)) satisfies the functional equation \[21, \text{Prop. IV.2.2}\]
\[
\int_{U(n)} \varphi(x^{-1/2}kyk^*x^{-1/2}, s) d^*k = \varphi(x^{-1}, s)\varphi(y, s).
\] (4.20)

Then, we use the explicit representation (3.11) of the spherical function due to Gelfand and Naĭmark [16]. The Vandermonde determinants cancel out, and since the spherical transform \( S_{p_{G, \varepsilon}} \) is of the form (3.20), we can again apply Andréief’s integration theorem [8]. Setting \( n = 1 \), the entries of the resulting determinant can be recognized as inverse Mellin transforms, yielding Eq. (2.31).

\[\blacksquare\]

**Remark 4.2** The identities in Theorem 2.13 do not rely on the positivity of the weights, as follows from the preceding proofs. Thus, they are valid for signed densities as well.

### 4.2 Proof of Theorems 2.9 and 2.10

For \( n = 1 \), the claims are trivial, since both the nonnegativity of the density (2.15) and the Pólya frequency property reduce to the nonnegativity of the function \( \omega \). Hence, we assume that \( n > 1 \).

**Proof of Theorem 2.9** (1) Let \( \omega \in L^1_{H_2}(\mathbb{R}) \) define a Pólya ensemble on \( H_2 \), and without loss of generality suppose \( \omega \) to be normalized, i.e., \( \int_{-\infty}^{\infty} \omega(x') \, dx' = 1 \). Then, we know that
\[
p_{H_2(1)}(x) = \omega(x) \geq 0 \quad \text{and} \quad p_{H_2}(y) \geq 0
\] (4.21)

for all \( x \in \mathbb{R} \) and \( y \in H_2 \). Due to the group integral (2.30), we also have
\[
\Delta_n(x)\Delta_n(y) \det [\omega(y_b - x_c)]_{b,c=1,\ldots,n} = n! C_{H_2} \Delta_n^2(y) \int_{K_2} p_{H_2}(y - kxk^*) \, d^*k \geq 0
\] (4.22)

for almost all \( y, x \in D \). By continuity, the nonnegativity extends to all \( y, x \in D \). Combining these two nonnegativity properties for \( N = 1 \) and \( N = n \) with the integrability of \( \omega \), we know from Lemma 4.1 that \( \omega \) is a Pólya frequency function of order \( n \).
Conversely, let \( \omega \in L_{H_2}^1(\mathbb{R}) \) be a Pólya frequency function of order \( n \) with \( \omega \neq 0 \). Then, we know, compare, e.g., Theorem 2 in Ref. [27], that
\[
\Delta_n(x) \det \left[ (-\partial_{x_c})^{b-1} \omega(x_c) \right]_{b,c=1,\ldots,n} \geq 0
\]
for all \( x \in D \). We still have to check that the normalizing constant \( C_n[\omega] \) does not vanish. But this follows from Eq. (3.19) and the observation that \( \omega \) is nonnegative and does not vanish identically; see also Remark 3.5. Thus, \( \omega \) gives rise to a Pólya ensemble on \( H_2 \).

(2) We pursue the same ideas as in the proof of part (1). Consider a Pólya ensemble \( p_G \) on \( G \) associated with the normalized weight \( \omega \in L_G^1(\mathbb{R}_+) \). The counterparts of Eqs. (4.21) and (4.22) are
\[
p_G(1)(x) = \omega(x) \geq 0 \quad \text{and} \quad p_G(g) \geq 0
\]
for all \( x \in \mathbb{R}_+ \) and \( g \in G \) and
\[
\Delta_n(x) \Delta_n(y) \det \left[ \omega \left( \frac{y_b}{x_c} \right) \right]_{b,c=1,\ldots,n} = n! \ C_G \left( \prod_{j=1}^n \mathcal{M}_j \right) \frac{\Delta_n^2(y) \Delta_n^2(x)}{\det(x^n-1)} \int_{\text{U}(n)} p_G(x^{-1/2}k y^{1/2}) \, d^n k \geq 0
\]
for almost all \( y, x \in A \). Again by continuity, the nonnegativity extends to all \( x, y \in A \). Setting \( \tilde{\omega}(x) = \omega(e^{-x})e^{-x} \), we find that \( \tilde{\omega} \neq 0 \) is integrable, nonnegative and satisfies
\[
\Delta_n(x) \Delta_n(y) \det \left[ \tilde{\omega} \left( y_b - x_c \right) \right]_{b,c=1,\ldots,n} = \Delta_n(x) \Delta_n(y) \det \left[ \omega \left( e^{x_c-y_b} \right) e^{x_c-y_b} \right]_{b,c=1,\ldots,n} \geq 0
\]
for all \( x, y \in D \), since the products \( \Delta_n(x) \Delta_n(y) \) and \( \Delta_n(e^{-x}) \Delta_n(e^{-y}) \) have the same sign. Therefore, \( \tilde{\omega}(x) \) is a Pólya frequency function of order \( n \) by Lemma 4.1.

Conversely, when we start from a Pólya frequency function \( \tilde{\omega}(x) = \omega(e^{-x})e^{-x} \) of order \( n \) with \( \omega \in L_G^1(\mathbb{R}_+) \) and \( \omega \neq 0 \), it follows similarly to Eq. (4.23) that
\[
\Delta_n(y) \det \left[ (-y_c \partial_{x_c})^{b-1} \omega(x_c) \right]_{b,c=1,\ldots,n} = \Delta_n(-e^{-x}) \det \left[ (-\partial_{x_c})^{b-1} \tilde{\omega}(x_c) \right]_{b,c=1,\ldots,n} \geq 0
\]
for all \( y = e^{-x} \in A \), since \( \Delta_n(-e^{-x}) \) has the same sign as \( \Delta_n(x) \). This time the normalizing constant \( C_n[\omega] \) is given by Eq. (3.20); see Remark 3.5. The positivity and integrability conditions on \( \omega \) immediately tell us that \( \mathcal{M}_j \omega(j) > 0 \) for all \( j = 1, \ldots, n \), so that \( C_n[\omega] \neq 0 \). This implies that \( \omega \) gives rise to a Pólya ensemble on \( G \). \( \square \)
Proof of Theorem 2.10 Let \( \tilde{\omega} \in L^1_{H_2}(\mathbb{R}) \) with support contained in \([0, \infty[\), and let \( \omega \) be defined as in Eq. (2.27). The weight \( \tilde{\omega} \) is an \( L^1 \)-function on \( \mathbb{R}_+ \), and for any \( x > 0 \), the function \( \tilde{\omega}(y) \exp[-x/y]x^y/y^{y+1} \) as well as its derivatives \( \partial_1^l [\tilde{\omega}(y) \exp[-x/y]x^y/y^{y+1}], l = 1, \ldots, 2n-2 \), is integrable in \( y \). Thus, it is easy to see that the derivatives of \( \omega \) up to order \( 2n-2 \) exist. Moreover, \( \omega \) is positive because the integrand is positive.

In the next step, we check that the integrability conditions in the definition of the set \( L^1_M(\mathbb{R}_+) \), see Definition 2.3 (c). For this purpose, we will repeatedly use the identity

\[
\left( x^v \frac{\partial}{\partial x} x^{1-v} \frac{\partial}{\partial x} \right)^l \omega(x) = \frac{1}{\Gamma[v+1]} \int_0^\infty \left( \frac{x}{y} \right)^v \exp \left[ -\frac{x}{y} \right] \left[ \left( -\frac{\partial}{\partial y} \right)^l \tilde{\omega}(y) \right] \frac{dy}{y} \tag{4.28}
\]

for \( l = 0, \ldots, n-1 \), which can be proven via integration by parts. We also recall the operator identity \( x^v \partial_x x^{1-v} \partial_x = \partial_x x^{v+1} \partial_x x^{-v} \) and that the boundary terms vanish because of the definition of the set \( L^1_{H_2}(\mathbb{R}) \), see Definition 2.3 (a), i.e., \( \lim_{y \to 0} \tilde{\omega}^{(l)}(y) = 0 \) for all \( l = 0, \ldots, n-2 \). Using (4.28), we have

\[
\int_0^\infty \left| x^{v-1} \left( x^v \frac{\partial}{\partial x} x^{1-v} \frac{\partial}{\partial x} \right)^l \omega(x) \right| dx \\
\leq \frac{1}{\Gamma[v+1]} \int_0^\infty \int_0^\infty \left| x^{v-1} \left( \frac{x}{y} \right)^v \exp \left[ -\frac{x}{y} \right] \left[ \left( -\frac{\partial}{\partial y} \right)^l \tilde{\omega}(y) \right] \right| \frac{dy}{y} dx \\
= \frac{\Gamma[v+\kappa]}{\Gamma[v+1]} \int_0^\infty y^{v-1} \left| \left( -\frac{\partial}{\partial y} \right)^l \tilde{\omega}(y) \right| dy < \infty \tag{4.29}
\]

for any \( \kappa \in [1, n] \) and \( l = 0, \ldots, n-1 \).

Now, we check whether the boundary conditions in the definition of \( L^1_M(\mathbb{R}_+) \) are satisfied. For this purpose, we let \( \tilde{\omega}^{(l)} \) denote the \( l \)th derivative of \( \tilde{\omega} \). We choose an auxiliary parameter \( \gamma \in ]2/3, 1[ \). Then, for \( x > 0 \) small enough, using Eq. (4.28) in the first step, we have the estimate

\[
\left| x^{v+1} \frac{\partial}{\partial x} x^v \left( \frac{\partial}{\partial x} x^{v+1} \frac{\partial}{\partial x} \right)^l \omega(x) \right| \\
= \frac{1}{\Gamma[v+1]} \left| \int_0^\infty \left( \frac{x}{y} \right)^{v+1} \exp \left[ -\frac{x}{y} \right] \tilde{\omega}^{(l)}(y) \frac{dy}{y} \right| \\
= \frac{1}{\Gamma[v+1]} \left| \int_0^\infty y^{v} \exp [-y] \tilde{\omega}^{(l)} \left( \frac{x}{y} \right) dy \right| \\
\leq \frac{1}{\Gamma[v+1]} \left( \int_0^{x^v} y^{v+2} \exp [-y] \tilde{\omega}^{(l)} \left( \frac{x}{y} \right) \frac{dy}{y^2} \right) \\
+ \left| \int_{x^v}^\infty y^{v} \exp [-y] \tilde{\omega}^{(l)} \left( \frac{x}{y} \right) dy \right| \\
\leq \frac{1}{\Gamma[v+1]} \left( \int_0^{x^v} y^{v+2} \exp [-y] \tilde{\omega}^{(l)} \left( \frac{x}{y} \right) \frac{dy}{y^2} \right)
\]
In the present work, we have extended the notion of a polynomial ensemble of derivative type to the classes of complex rectangular matrices, Hermitian antisymmetric complete the proof of Theorem 2.10. The initial inequality holds because the density associated with \( \tilde{\omega}_0(\lambda) \) is integrable for any \( l = 0, \ldots, n - 1 \) and \( \gamma > 2/3 \geq 1/(v + 2) \) for any \( v \geq -1/2 \). The second term vanishes because \( \lim_{\lambda \to 0} \tilde{\omega}_0(\lambda) = 0 \) for any \( l = 0, \ldots, n - 2 \). Thus, we have

\[
\lim_{x \to 0} \left| x^{v+1} \frac{\partial}{\partial x} \frac{1}{x^v} \left( \frac{\partial}{\partial x} x^{v+1} \frac{\partial}{\partial x} x^v \right)^l \omega_\nu(x) \right| = 0 \tag{4.31}
\]

for all \( l = 0, \ldots, n - 2 \).

At last, we show that the density associated with \( \omega_\nu \) is indeed nonnegative. To this end, we establish a relation between the densities on \( A \) corresponding to \( \omega_\nu \) and \( \tilde{\omega}_0 \). For \( x \in A \) with \( x_1, \ldots, x_n \) pairwise different, we have

\[
0 \leq \int_A \left( \int_{K_\nu} \exp[-\text{tr} x a^{-1} k^+] \text{det} a \right) \frac{\text{det} x^v \text{det} a^{v+n} \Delta_n(a)}{\text{det} a^{v+n} \Delta_n(a)} \text{det} \left[ \left( -\frac{\partial}{\partial a_c} \right)^{-1} \tilde{\omega}(a_c) \right] \text{det} \left[ \left( -\frac{\partial}{\partial a_c} \right)^{-1} \tilde{\omega}(a_c) \right] \text{det} a \text{da} 
\]

\[
= \prod_{j=1}^n \frac{j!}{\Delta_n(x)} \int_{A} \text{det} \left[ e^{-x_b/a_c} \right]_{b,c=1,\ldots,n} \text{det} \left[ \left( -\frac{\partial}{\partial a_c} \right)^{-1} \tilde{\omega}(a_c) \right]_{b,c=1,\ldots,n} \text{det} a \text{da} 
\]

\[
= \prod_{j=1}^n \frac{j! \Gamma[v+1]}{\Delta_n(x)} \text{det} \left[ \left( x_c^v \frac{\partial}{\partial x_c} x_c^{v+1} \frac{\partial}{\partial x_c} x^v \right)^{-1} \tilde{\omega}(x_c) \right]_{b,c=1,\ldots,n} . \tag{4.32}
\]

The initial inequality holds because the density associated with \( \tilde{\omega} \) is nonnegative and the other factors are positive. In the next steps, we use the Harish–Chandra–Itzykson–Zuber integral (3.9) in combination with the relation \( \Delta_n(-a^{-1}) = \text{det} (a^{1-n}) \Delta_n(a) \), which shifts the exponent of the determinant \( \text{det} a \), the Andréief identity [8] and Eq. (4.28), respectively.

Apart from the normalizing constant, the last line in (4.32) is exactly the joint probability density of the squared singular values of a Pólya ensemble on \( M_v \) divided by \( |\Delta_n(x)|^2 \). Finally, the relation \( \mathcal{H}_v \omega_\nu(0) = \int_{0}^{\infty} \tilde{\omega}_0(x) \text{dx} > 0 \) shows that the corresponding normalizing constant \( C_n[\omega] \) is indeed positive; see also Remark 3.5. This completes the proof of Theorem 2.10.

5 Conclusions and Outlook

In the present work, we have extended the notion of a polynomial ensemble of derivative type to the classes of complex rectangular matrices, Hermitian antisymmetric
matrices and Hermitian anti-self-dual matrices. We have shown that these ensembles have nice closure properties with respect to additive convolution (Theorem 2.5 and Corollary 2.6), thereby extending previous results for complex square matrices [29] and complex Hermitian matrices [33]. In fact, by using the appropriate multivariate transforms from harmonic analysis, all these classes of matrices may be handled in a unified way.

Furthermore, for each of these classes of matrices, we have addressed the question which weight functions give rise to polynomial ensembles of derivative type. We have shown in Theorems 2.9 and 2.10 that these weight functions are closely related to Pólya frequency functions [43–45]. For this reason, we propose the shorter name Pólya ensembles for the resulting ensembles.

Another main result is the group integrals in Theorem 2.13 which generalize known group integrals [17,20,41] and which seem to be of independent interest. Moreover, on the practical side, the relation to Pólya frequency functions yields a multitude of examples of Pólya ensembles and associated group integral identities. Some of them are highly non-trivial and go beyond the classical results in random matrix theory.

As is evident from our proofs, both the convolution theorems and the group integral identities are intimately related to the fact that the relevant multivariate transform factorizes. Furthermore, we would like to emphasize that these results do not really require the positivity of the matrix densities. They also hold for signed matrix densities as long as the integrability and differentiability conditions on the underlying weight function are satisfied. The question is whether these results can be extended even beyond these conditions. A natural candidate is the signed measures such that the relevant multivariate transforms factorize, but a closer description seems to require the language of distributions.

Interestingly, the cases of real antisymmetric matrices of odd dimension and of Hermitian anti-self-dual matrices lead to exactly the same results. From a group theoretical perspective, this has to be expected since the roots are the same apart from their length. Thus, the random matrix ensembles show the same eigenvalue statistics, but the eigenvector statistics should be different.

Finally, it is crucial for our results that the group integrals involved in the multivariate transforms admit explicit expressions and that the ensembles correspond to the Dyson index $\beta = 2$. How our results can be extended to other symmetry classes of matrices, e.g., real-symmetric matrices or Hermitian self-dual matrices, is still an open problem.

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