Gauge theories on noncommutative Kähler manifolds

1 Yoshiaki Maeda, 2 Akifumi Sako, 3 Toshiya Suzuki and 3 Hiroshi Umetsu
1 Tohoku Forum for Creativity, Tohoku University, 2-1-1, Katahira, Aoba-Ku, Sendai, Miyagi, 980-8577, Japan
2 Department of Mathematics, Faculty of Science, Tokyo University of Science, 1-3 Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan
3 Kushiro National College of Technology, 2-32-1 Otanoshike-nishi, Kushiro, Hokkaido 084-0916, Japan
E-mail: sako@rs.tus.ac.jp

Abstract. We study gauge theories on noncommutative homogeneous Kähler manifolds. To make the noncommutative manifolds, we use the deformation quantization with separation of variables for Kähler manifolds. We construct models of noncommutative gauge theories that are connected with usual Yang-Mills theories in the commutative limits. It is expected that the models connecting to commutative gauge theories are uniquely determined. As examples, we give noncommutative CP^N and noncommutative CH^N and gauge theories on them. Some kinds of gauge symmetry breaking and topological symmetry breaking by noncommutative deformations are observed by concrete geometrical calculations.

1. Motivation
One of the most important problems in physics is to formulate a quantum gravity theory. String theories are the strongest candidates of the quantum gravity theory. However the string theories themselves have not been completed until now. It means that many string theories are defined as perturbative formulations, and we have to formulate them as non-perturbative formulations. There are some approaches to make non-perturbative string theories; string field theories, M theory, Matrix models, and so on, but they have never get the total win yet. IKKT(Type IIB) matrix theory is one of the candidates of non-perturbative string theory[1]. This matrix theory contains noncommutative gauge theories as classical solutions, for example, a gauge theory on the the Moyal plane, the one on the fuzzy CP^1, and so on. The gauge theories on the Moyal spaces have been studied in detail, but research of gauge theories on the noncommutative CP^N or more generally gauge theories on noncommutative homogeneous Kähler manifolds are not so much as the ones on Moyal spaces. One reason is difficulty in constructing concretely calculable models of noncommutative gauge theories on such spaces.

In this article, we introduce concretely calculable gauge theories on noncommutative CP^N and CH^N. Noncommutative deformations of the manifolds are performed by the deformation quantization. This method has an advantage that we can calculate geometrical quantities in the similar way of classical geometry, because all objects appearing in deformation quantizations are interpreted as formal power series of quantities on classical manifolds. Using this advantage, we can discuss various phenomena in the noncommutative field theories. As an example, we observe a gauge symmetry and topological symmetry breaking by a noncommutative deformation.
2. Deformation quantization of Kähler manifolds

Noncommutative spaces in this article are constructed by using the deformation quantization defined as follows. Let $M$ be a smooth manifold.

**Definition 2.1** Let $\mathcal{F}$ be a set of formal power series of $\hbar \in \mathbb{R}$:

$$\mathcal{F} := \{ f \mid f = \sum_k f_k \hbar^k, \text{ where } f_k \text{ is a smooth function on } M \}.$$

A star product of $f, g \in \mathcal{F}$ is defined as

$$f * g = \sum C_k(f, g) \hbar^k,$$

such that the product satisfies the following conditions.

(i) $*$ is associative product.

(ii) $C_k$ is a bidifferential operator.

(iii) $C_0$ and $C_1$ is defined as

$$C_0(f, g) = fg, \quad C_1(f, g) - C_1(g, f) = i\{f, g\}.$$

where $\{f, g\}$ is the Poisson bracket.

(iv) $f * 1 = 1 * f = f$.

There is a star product called a star product with separation of variables, which is defined on Kähler manifolds. A Kähler manifold has a Kähler potential $\Phi$ which introduces a Kähler 2-form $\omega$ and a metric $g$ as

$$\omega = ig_{ij} dz^i \wedge d\bar{z}^j, \quad g_{ij} = \frac{\partial^2 \Phi}{\partial z^i \partial \bar{z}^j}.$$

$*$ is called a star product with separation of variables when

$$a * f = af$$

for locally holomorphic function $a$ and

$$f * b = fb$$

for locally anti-holomorphic function $b$. Karabegov found a way to construct this star product with separation variables $*$ for arbitrary Kähler manifolds [2] (see also [3, 4]). Let us review how to construct the star product $*$.

In this method of deformation quantization, a star product is constructed as a formal power series of differential operators. Let $L_f$ be a differential operator corresponding to a left $*$ multiplication by $f$:

$$L_f g := f * g.$$

Then $L_f$ has the following form:

$$L_f = \sum_{n=0}^{\infty} \hbar^n A_n,$$

where $A_n$ is given by

$$A_n = a_{n,\alpha}(f) \prod_i \left( D^i \right)^{\alpha_i}, \quad (D^i := g^{ij} \partial_j).$$
We use \( \partial_i = \frac{\partial}{\partial z^i} \) for simplicity. We also use \( D^i := g^{ij} \partial_j \), where \( \partial_i = \frac{\partial}{\partial \bar{z}^i} \), in the following. It is required that \( L_f \) satisfies

\[
L_f 1 = f \ast 1 = f, \\
L_f (L_g h) = f \ast (g \ast h) = (f \ast g) \ast h = L_{L_f g} h.
\]

\( L_f \) which has the properties described above is determined by the following condition:

\[
[L_f, \partial_i \Phi + \hbar \partial_i] = 0,
\]

and \( A_0 = f \). This condition is equivalent to the recursion relations:

\[
[A_n, \partial_i \Phi] = [\partial_i, A_{n-1}].
\]

If one obtains the operator \( L_{\bar{z}^i} \) (\( L_{\bar{z}^i} f = \bar{z}^i \ast f \)), \( L_f \) is given by

\[
L_f = \sum_{\alpha} \frac{1}{\alpha!} \left( \frac{\partial}{\partial \bar{z}} \right)^{\alpha} f (L_{\bar{z}} - \bar{z})^\alpha.
\]

Here, \( \alpha \) is a multi-index, \( \alpha = (\alpha_1, \ldots, \alpha_m) \). It is not easy to derive explicit expressions of star products in all order of \( \hbar \) by solving the recursion relation.

3. Gauge theory on noncommutative homogeneous Kähler manifolds

3.1. Derivations

Field theories on noncommutative spaces can be constructed based on the derivations of the algebra \( C^\infty(M)[[\hbar]] \) with its star product, where derivation \( d \) are linear operators satisfying the Leibniz rule, i.e. \( d(f \ast g) = d(f) \ast g + f \ast d(g) \). In commutative space, vector fields are derivations. However first order differential operators in noncommutative space do not satisfy the Leibniz rule in general, because star products depend on the local position. \( L \) is an inner derivations when \( P \in C^\infty(M)[[\hbar]] \) exists such that \( L(f) = [P, f]_\ast := P \ast f - f \ast P \), for any \( f \in C^\infty(M)[[\hbar]] \).

Note that \([P, f]_\ast \) includes higher derivative terms of \( f \) for a generic \( P \). So, it turns out that inner derivations corresponding to vector fields play an important role, when we construct physical field theories. What is the vector field having such the property. Here is an answer.

**Proposition 3.1** [Müller-Bahns and N. Neumaier] [6, 7] In deformation quantization defined above, the inner derivations given as vector fields are the Killing vector fields \( L_a \).

An important point to read the later half of this article is that the first order differential operators satisfy the Leibniz rule are the Killing vector fields.

3.2. Noncommutative gauge theory

In the following, we consider \( U(n) \) gauge theories for simplicity. All following results can be applied for any matrix groups. Proofs of theorems in this section are given in [6]

We introduce a noncommutative \( U(n) \) transformations as a deformation of the unitary transformations. If \( g \in U(n) \), then \( g \dagger g = I \), where \( g \dagger \) is the hermitian conjugate of \( g \) and \( I \) is the identity matrix.

As a natural extension, we define \( G := M_n(C^\infty[[\hbar]]) \) such that for \( U = \sum_{k=0}^\infty \hbar^k U^{(k)} \) and
\[
U^\dagger = \sum_{k=0}^{\infty} \hbar^k U^{(k) \dagger} \in G,
\]

\[
U^\dagger \ast U = \sum_{n=0}^{\infty} \hbar^n \sum_{m=0}^{n} U^{(m) \dagger} \ast U^{(n-m)} = I.
\]

For arbitrary \( U^{(0)} : M \rightarrow U(n) \), this condition has solutions which are determined recursively at each order.

Let’s construct a gauge theory on deformed homogeneous Kähler manifold \( \mathcal{G} / \mathcal{H} \) whose covariant derivatives are determined by inner derivations corresponding to the Killing vector fields.

In a \( \mathcal{G} / \mathcal{H} \), there are the holomorphic Killing vector fields \( \mathcal{L}_a = \zeta^i_a(z) \partial_i + \zeta^{\bar{i}}_a(\bar{z}) \partial_{\bar{i}} \) corresponding to the isometry group \( \mathcal{G} \),

\[
[\mathcal{L}_a, \mathcal{L}_b] = i f_{abc} \mathcal{L}_c,
\]

where \( f_{abc} \) is a structure constant of the Lie algebra of \( \mathcal{G} \). As we saw in the previous subsection, \( \mathcal{L}_a \) satisfies the Leibniz rule.

\[
\mathcal{L}_a(f \ast g) = (\mathcal{L}_a f) \ast g + f \ast (\mathcal{L}_a g).
\]

The Killing vectors are normalized here as

\[
\eta^{ab} \zeta^i_a \zeta^j_b = g^{ij}, \quad \eta^{ab} \zeta^{\bar{i}}_a \zeta^{\bar{j}}_b = 0, \quad \eta^{ab} \zeta^i_a \zeta^{\bar{j}}_b = 0,
\]

where \( \eta^{ab} \) is the inverse of the Killing form of the Lie algebra of \( \mathcal{G} \).

In the commutative homogeneous Kähler manifolds, \( M = \mathcal{G} / \mathcal{H} \), let us introduce a gauge connection \( A^{(0)}_a \) given as

\[
A^{(0)}_a = \zeta^\mu_a A_\mu = \zeta^i_a A_i + \zeta^{\bar{i}}_a A_{\bar{i}},
\]

where \( A_\mu \) or its complex expressions \( A_i \) and \( A_{\bar{i}} \) are usual gauge fields on \( M \). The curvature is also defined as

\[
F^{(0)}_{ab} := \mathcal{L}_a A^0_b - \mathcal{L}_b A^0_a - i[A^{(0)}_a, A^{(0)}_b] - i f_{abc} A^{(0)}_c,
\]

where \([A, B] = AB - BA\). \( F^{(0)}_{ab} \) is related to the curvature of \( A_\mu \), \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \), as

\[
F^{(0)}_{ab} = \zeta^\mu_a \zeta^\nu_b F_{\mu\nu}.
\]

It is shown easily that

\[
\eta^{ac} \eta^{bd} F^{(0)}_{ab} F^{(0)}_{cd} = g^{\rho\sigma} g^{\mu\nu} F_{\rho\mu} F_{\sigma\nu}.
\]

These fields \( A^{(0)}_a \) and \( F^{(0)}_{ab} \) are introduced to be convenient when we use the Killing vector fields as derivations to construct gauge theories.

Now, we consider a noncommutative deformation of the gauge theory. The formal power series of the gauge connections are written as

\[
A_a := \sum_{k=0}^{\infty} \hbar^k A^{(k)}_a.
\]
The gauge transformation is defined by
\[ A_a \rightarrow A'_a = iU^{-1} \ast \mathcal{L}_a U + U^{-1} \ast A_a \ast U. \]

Let us define the curvature of \( A_a \) by
\[ F_{ab} := \mathcal{L}_a A_b - \mathcal{L}_b A_a - i[A_a, A_b]_* - i f_{abc} A_c. \]

We can show that the following lemma.

**Lemma 3.2** \( F_{ab} \) transforms covariantly:
\[ F_{ab} \rightarrow F'_{ab} = U^{-1} \ast F_{ab} \ast U. \]

Using this lemma, we obtain the gauge invariant action.

**Theorem 3.3** A gauge invariant action for the gauge field is given by
\[ S_g := \int_{\mathcal{G}/\mathcal{H}} \mu_g \, \text{tr} \left( \eta^{ac} \eta^{bd} F_{ab} \ast F_{cd} \right), \]
where \( \mu_g \) is a trace density.

The gauge invariance of the action is obtained by the above lemma and the cyclic symmetry of the trace density. The existence of trace density, \( \int_M f \ast g \mu_g = \int_M g \ast f \mu_g \), is guaranteed [5].

We expect that this is the unique action of noncommutative Yang-Mills theory connecting with commutative Yang-Mills theory in the commutative limit in our noncommutative homogeneous Kähler manifolds.

Other fields can be formulated with the gauge covariant way. For example, let us introduce a complex scalar field \( \phi = \sum_k \phi^{(k)} \mathcal{H}^k \) and its hermitian conjugate scalar field \( \phi^\dagger \) whose gauge transformation as the fundamental representation of the gauge group is given by
\[ \phi \rightarrow \phi' = U^{-1} \ast \phi, \quad \phi^\dagger \rightarrow \phi'^\dagger = \phi^\dagger \ast U. \]

The covariant derivative for this scalar field is defined by
\[ \nabla_a \phi := \mathcal{L}_a \phi - i A_a \ast \phi, \]
Using these, a gauge invariant action is obtained.

**Theorem 3.4** Gauge invariant action is given by
\[ S_\phi = \int_{\mathcal{G}/\mathcal{H}} \mu_g \left\{ \eta^{ab} \nabla_a \phi^\dagger \ast \nabla_b \phi + V(\phi^\dagger \ast \phi) \right\}, \]
where \( V \) is a potential as a function of one variable.

4. **Noncommutative deformation of \( \mathbb{C}P^N \)**

As examples of above models, we study noncommutative \( \mathbb{C}P^N \) in this section and \( \mathbb{C}H^N \) in the next section. The detail derivations of the following results are found in [8].
4.1. Noncommutative $\mathbb{C}P^N$

Let $z^i$ ($i = 1, 2, \ldots, N$) be inhomogeneous coordinates of $\mathbb{C}P^N$. Then, the Kähler potential of $\mathbb{C}P^N$ is given by

$$
\Phi = \ln (1 + |z|^2), \quad (|z|^2 = \sum_i z^i \bar{z}^i).
$$

The complex metric $(g_{ij})$ is derived from the Kähler potential as

$$
ds^2 = 2 g_{ij} dz^i d\bar{z}^j, \quad g_{ij} = \partial_i \partial_j \Phi = \frac{(1 + |z|^2) \delta_{ij} - z^i \bar{z}^j}{(1 + |z|^2)^2}.
$$

Its inverse metric $(\tilde{g}^{ij})$ is given as

$$
\tilde{g}^{ij} = (1 + |z|^2) (\delta_{ij} + z^i \bar{z}^j).
$$

Solving the recursion relation (1), we obtain $L_{z^i}$:

$$
L_{z^i} = z^i + h D^i + \sum_{n=2}^{\infty} \sum_{m=2}^{n} a_m^{(n)} \partial_{j_1} \Phi \cdots \partial_{j_{m-1}} \Phi D^{j_1} \cdots D^{j_{m-1}} D^i
$$

$$
= z^i + \sum_{m=1}^{\infty} \alpha_m(h) \partial_{j_1} \Phi \cdots \partial_{j_{m-1}} \Phi D^{j_1} \cdots D^{j_{m-1}} D^i.
$$

Here $\alpha_m(t)$ is defined as

$$
\alpha_2(t) = \sum_{n=2}^{\infty} t^n a_2^{(n)} = \sum_{n=2}^{\infty} t^n = \frac{t^2}{1 - t},
$$

$$
\alpha_m(t) = t^m \prod_{n=1}^{m-1} \frac{1}{1 - nt} = \frac{\Gamma(1 - m + \frac{1}{2})}{\Gamma(1 + \frac{1}{2})}, \quad (m \geq 2).
$$

This $\alpha_m(t)$ is a generating function of the Stirling number of the second kind $S(n, k)$.

Using these results, star products among $z^i$ and $\bar{z}^j$ are obtained as

$$
z^i \ast z^j = z^i z^j, \quad z^i \ast \bar{z}^j = z^i \bar{z}^j, \quad \bar{z}^i \ast z^j = \bar{z}^i z^j, \quad \bar{z}^i \ast \bar{z}^j = \bar{z}^i \bar{z}^j, \quad (2)
$$

$$
z^i \ast \bar{z}^j = \bar{z}^i z^j + h \delta_{ij} (1 + |z|^2)_{2F_1} \left( 1, 1; 1 - 1/h; -|z|^2 \right)
$$

$$
+ \frac{h}{1 - h} \bar{z}^i z^j (1 + |z|^2)_{2F_1} \left( 1, 2; 2 - 1/h; -|z|^2 \right). \quad (3)
$$

Furthermore, we can derive an explicit star product for arbitrary functions $f$ and $g$:

$$
f \ast g = \sum_{n=0}^{\infty} \frac{\alpha_n(h)}{n!} g_{j_1 k_1} \cdots g_{j_n k_n} \left( D^{j_1} \cdots D^{j_n} f \right) \left( D^{k_1} \cdots D^{k_n} g \right).
$$

This star product on $\mathbb{C}P^N$ is characterized by a function of $h$, $\alpha_n(h)$. 

4.2. Inner derivations and gauge theory on noncommutative $\mathbb{C}P^N$

Let us study differentials in a noncommutative $\mathbb{C}P^N$. The isometry group of this $\mathbb{C}P^N$ is $SU(N+1)$. In the following, we give concrete expressions of the Killing vectors corresponding to the generators of $su(N+1)$, the Lie algebra of $SU(N+1)$. We use homogeneous coordinates of $\mathbb{C}P^N$:

$$\{ \xi^A | A = 0, 1, \cdots, N \} = \{ \xi^0, \xi^i | i = 1, 2, \ldots, N \},$$

and inhomogeneous coordinates on the chart of $\xi^0 \neq 0$:

$$z^i = \frac{\xi^i}{\xi^0}, \quad \bar{z}^i = \frac{\xi^i}{\bar{\xi}^0}, \quad (i = 1, 2, \ldots, N).$$

Generators $(T_a)_{AB}$ of $su(N+1)$ in the fundamental representation satisfy the following relations,

$$[T_a, T_b] = if_{abc}T_c, \quad \text{Tr} \ T_a = 0,$$

$$\text{Tr} \ T_aT_b = \delta_{ab},$$

$$(T_a)_{AB}(T_a)_{CD} = \delta_{AD}\delta_{BC} - \frac{1}{N+1}\delta_{AB}\delta_{CD},$$

where $f_{abc}$ is the structure constant of $su(N+1)$, $a = 1, 2, \ldots, N^2 + 2N$, and $A, B = 0, 1, \ldots, N$.

As you can find it from the fact that Kähler potential is given by $\Phi = \ln(1 + |z|^2) = \ln |\xi|^2$, the isometry $SU(N+1)$ transformation with the homogeneous coordinates is given by

$$\delta \xi^A = i\theta^a(T_a)_{AB}\xi^B, \quad \delta \bar{\xi}^A = -i\theta^a(T_a)_{BA},$$

and its Lie derivative is given by

$$\mathcal{L}_a = -(T_a)_{AB} \left( \xi^B \frac{\partial}{\partial \xi^A} - \bar{\xi}^A \frac{\partial}{\partial \bar{\xi}^B} \right),$$

$$[\mathcal{L}_a, \mathcal{L}_b] = if_{abc}\mathcal{L}_c.$$  

The generators of the isometry $SU(N+1)$ is given as

$$\mathcal{L}_a = \zeta^A_a \partial_A + \bar{\zeta}^A_a \partial_{\bar{A}} = (T_a)_{00} \left( z^i \partial_i - \bar{z}^i \partial_{\bar{i}} \right) + (T_a)_{0i} \left( z^i \bar{z}^\bar{j} \partial_j + \bar{\partial}_{\bar{i}} \right) + (T_a)_{ij} \left( -z^j \partial_i + \bar{z}^j \partial_{\bar{i}} \right),$$

and

$$\zeta^A_a := (T_a)_{00} z^i + (T_a)_{0j} z^j \bar{z}^\bar{i} - (T_a)_{i0} - (T_a)_{ij} \bar{z}^j,$$

$$\bar{\zeta}^A_a := -(T_a)_{00} \bar{z}^i + (T_a)_{0i} \bar{z}^j + (T_a)_{j0} \bar{z}^j \bar{z}^i + (T_a)_{ji} \bar{z}^j.$$

The quadratic forms of $\zeta^A_a$ and $\bar{\zeta}^A_a$ become the metric,

$$\zeta^A_a \zeta^B_a = -(1 + |z|^2)(\delta_{ij} + z^i \bar{z}^j) = -g^{ij},$$

$$\zeta^A_a \bar{\zeta}^B_a = 0, \quad \zeta^A_a \bar{\zeta}^B_a = 0.$$

For the noncommutative $\mathbb{C}P^N$ trace density $\mu_g$ is given by the Riemannian volume form. Thus, the Yang-Mills type action is constructed as

$$S_g := \int_{\mathbb{C}P^N} \sqrt{g}\,dz^1 \cdots dz^N \, d\bar{z}^\bar{1} \cdots d\bar{z}^\bar{N} \, \text{tr} \left( F_{ab} * F_{cd} \bar{\eta}^{ac} \eta^{bd} \right),$$

where $\text{tr}$ is trace for gauge group $G$. All classical calculations of this Yang-Mills theory can be done by using above Killing vectors and concrete expressions of the star products.
5. Noncommutative deformation of $\mathbb{C}H^N$

Similar to the $\mathbb{C}P^N$, a noncommutative deformation of $\mathbb{C}H^N$ is constructed.

As similar to (2)-(3), star products between inhomogeneous coordinates are given as

\[ z^i \ast \bar{z}^j = z^i \bar{z}^j, \]
\[ z^i \ast \bar{z}^j = z^i \bar{z}^j, \]
\[ \bar{z}^i \ast z^j = \bar{z}^i z^j, \]
\[ \bar{z}^i \ast \bar{z}^j = \bar{z}^i \bar{z}^j, \]
\[ \bar{z}^i \ast z^j = \bar{z}^i z^j + \hbar \delta_{ij} (1 - |z|^2) F_1 (1, 1; 1 + 1/\hbar; |z|^2) \]
\[ - \frac{\hbar}{1 + \hbar} \bar{z}^i z^j (1 - |z|^2) F_1 (1, 2; 2 + 1/\hbar; |z|^2). \]

The explicit representation of the star product with separation of variables on $\mathbb{C}H^N$ is given by

\[ L_{\bar{z}^i} = \bar{z}^i + \sum_{m=1}^{\infty} (-1)^{m-1} \beta_m (\hbar) \partial_{\bar{z}^1} \Phi \cdots \partial_{\bar{z}^m} \Phi \bar{D}^m \cdots \bar{D} \bar{D}^{m-1} \bar{D}, \]

with

\[ \beta_m (t) = (-1)^m \alpha_m (-t) = \frac{\Gamma(1/t)}{\Gamma(n + 1/t)}. \]

The generators of the isometry $SU(1, N)$ in the inhomogeneous coordinates are

\[ \mathcal{L}_\alpha = \zeta^i_{\alpha} \partial_i + \bar{\zeta}^i_{\alpha} \partial_i = (T_a)_{00} (z^i \partial_i - \bar{z}^i \partial_i) + (T_a)_{0i} (z^j \partial_j - \partial_i) \]
\[ + (T_a)_{i0} (-\partial_i + \bar{z}^i z^j \partial_j) + (T_a)_{ij} (-z^j \partial_i + \bar{z}^i \partial_j), \]

and

\[ \zeta^i = (T_a)_{00} z^i + (T_a)_{0j} z^j \bar{z}^i - (T_a)_{0i} - (T_a)_{ij} z^j, \]
\[ \bar{\zeta}^i = -(T_a)_{00} \bar{z}^i - (T_a)_{0j} \bar{z}^j - (T_a)_{ij} \bar{z}^j + (T_a)_{ji} \bar{z}^i. \]

Using these data, we can construct a noncommutative gauge theory on $\mathbb{C}H^N$ in the similar way in the previous section.

6. Spontaneous gauge symmetry breaking

As an application of the above gauge theories, let us consider gauge symmetry breaking on noncommutative homogeneous Kähler manifolds $M (= \mathcal{G}/\mathcal{H})$.

The action functional (4) on an open subset in $M (= \mathcal{G}/\mathcal{H})$ does not have gauge symmetry, i.e. for $D \subset M, D \neq M$,

\[ S'_g|_{D} - S_g|_{D} = \int_{D} \mu_g \text{tr} U^{-1} \ast \left( \eta^o \bar{\eta}^{bd} \mathcal{F}_{ab} * \mathcal{F}_{cd} \right) \ast U - \int_{D} \mu_g \text{tr} \left( \eta^o \bar{\eta}^{bd} \mathcal{F}_{ab} * \mathcal{F}_{cd} \right) \neq 0. \]

Since trace density $\mu_g$ does not have the cyclic symmetry on the subset $D$, i.e. $\int_{D} \mu_g f * g \neq \int_{D} \mu_g g * f$. This phenomenon is not special one appearing in our gauge theory. However owing to the concrete expressions of $*$-products we can observe what happen by the gauge symmetry breaking.
For example, let us observe that a topological invariant receives quantum corrections in the $\mathbb{C}P^1 \cong S^2$. Consider $\int_{S^2} \mu_g \text{tr} L_\alpha \epsilon_{abc} F_{bc}$. This is rewritten as $\int_{S^2} \text{tr} \partial_\mu (\zeta_\mu^\alpha \epsilon_{abc} F_{bc})$, so we find this is a topological term. This is zero on commutative $S^2 \cong \mathbb{C}P^1$ since the Stokes’ theorem, but we can show that this is non zero on the noncommutative $\mathbb{C}P^1$ in general and estimate the value of it concretely by using explicit expression of the star products (3) [9].

7. Summary and discussions

In this article, we introduced noncommutative homogeneous Kähler manifolds by using the Karabegov’s star products with separation of variables, and we constructed gauge theories on them. The gauge theories are connected with the ordinary Yang-Mills theory on commutative homogeneous Kähler manifolds in the commutative limit. In our noncommutative manifolds, it is expected that such gauge theories are uniquely determined, because the inner derivations given as vector fields are only the Killing vector fields. Owing to the explicit expressions of the star products of $\mathbb{C}P^N$ and $\mathbb{C}H^N$, we can calculate physical quantities in the gauge theories, concretely. As an example, we observed that noncommutative deformation breaks gauge symmetry and topological symmetry. Someone might think we should omit this type of symmetry breaking. However, if we remind that the Seiberg-Witten transformation between a noncommutative gauge theory and a commutative gauge theory with background $B$ field [10], we might realize such situations as real physics. Anyway further studying these phenomena are left for future as an important subjects.

Acknowledgments

Y.M. was supported in part by JSPS KAKENHI No.23340018 and No.22654011, and A.S. was supported in part by JSPS KAKENHI No.23540117.

[1] Ishibashi N, Kawai H, Kitazawa Y and Tsuchiya A 1997 A Large N reduced model as superstring Nucl. Phys. B 498 467 (Preprint arXiv:hep-th/9612115)
[2] Karabegov A V 1996 Deformation quantizations with separation of variables on a Kahler manifold Commun. Math. Phys. 180 745 (Preprint arXiv:hep-th/9508013)
[3] Karabegov A V 1996 On deformation quantization, on a Kahler manifold, associated to Berezin’s quantization Funct. Anal. Appl. 30 142
[4] Karabegov A V 2011 An explicit formula for a star product with separation of variables Preprint arXiv:1106.4112 [math.QA]
[5] Karabegov A V 1998 On the canonical normalization of a trace density of deformation quantization Lett. Math. Phys. 45 217
[6] Maeda Y, Sako A, Suzuki T and Umetsu H 2014 Gauge theories in noncommutative homogeneous Kähler manifolds J. Math. Phys. 55 092301 (Preprint arXiv:1403.5727 [hep-th])
[7] Müller-Bahns M and Neumaier N 2004 Invariant Star Products of Wick Type: Classification and Quantum Momentum Mappings Lett. Math. Phys. 70 1
[8] Sako A, Suzuki T and Umetsu H 2012 Explicit Formulas for Noncommutative Deformations of $\mathbb{C}P^N$ and $\mathbb{C}H^N$ J. Math. Phys. 53 073502 (Preprint arXiv:1204.4030 [math-ph])
[9] Sako A, Suzuki T and Umetsu H in preparation
[10] Seiberg N and Witten E 1999 String theory and noncommutative geometry JHEP 9909 032 (Preprint arXiv:hep-th/9908142)