IMPROVED FINITE DIFFERENCE RESULTS FOR THE CAPUTO TIME-FRACTIONAL DIFFUSION EQUATION

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Abstract. We begin with a treatment of the Caputo time-fractional diffusion equation, by using the Laplace transform, to obtain a Volterra intego-differential equation where we may examine the weakly singular nature of this convolution kernel. We examine this new equation and utilize a numerical scheme that is derived in parallel to the L1-method for the time variable and a usual fourth order approximation in the spatial variable. The main method derived in this paper has a rate of convergence of $O(k^2 + h^4)$ for $u(x,t) \in C^6(\Omega) \times C^2[0,T]$, which improves previous estimates by a factor of $k^\alpha$. We also present a novel alternative method for a first order approximation in time, which allows us to relax our regularity assumption to $u(x,t) \in C^6(\Omega) \times C^1[0,T]$, while exhibiting order of convergence slightly less than $O(k^{1+\alpha})$ in time. This allows for a much wider class of functions to be analyzed which was previously not possible under the L1-method. We present numerical examples demonstrating these results and discuss future improvements and implications by using these techniques.

1. Introduction

Fractional differential equations have been of great interest to various fields in physics, engineering, and mathematics over the past several decades, as seen in [10,11] and many others. Many applications of fractional diffusion equations are studied due to their physical applications, we refer to [10-16] for a small survey of relevant and related works. In their 2014 article [16], Zhang et al. established a numerical scheme for the one-dimensional time-fractional order diffusion equation with initial and boundary conditions

$$\mathcal{D}_t^\alpha u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t) + f(x,t), \quad x \in \Omega, \quad t \in [0,T], \quad (1)$$

$$u(x,0) = \phi(x), \quad x \in [0,1] \text{ and } u(0,t) = u(1,t) = 0,$$

with $\alpha \in (0,1)$ order Caputo fractional time derivative defined by

$$\mathcal{D}_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,s)}{\partial s} (t-s)^{-\alpha} \, ds.$$
where $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$. Various authors have placed various hypotheses on $\phi$ and $f$ in their analysis, see [1,2,15,16]. This problem was solved numerically in [16] on the domain $[0,1] \times [0,T]$ with numerical accuracy of order $O(k^{2-\alpha} + h^4)$ by application of a 4th order spatial and a 2nd order time scheme, where $k$ denotes the time mesh size and $h$ denotes the space mesh size, with a constant that depends on $T^\alpha$. The 2nd order time scheme is the so-called L1-method, which has been studied extensively in previous works, see [16] for further discussion.

In section 2 we will transform (1) into its equivalent form

$$u(x,t) = \phi(x) + \left(a_{1-\alpha}(t) * \left(\frac{\partial^2 u}{\partial x^2} + f\right)\right)(x,t),$$

(2)

where $a_{1-\alpha}(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, by application of the Laplace transform and $*$ denotes convolution as defined in Section 2. We will use the same 4th order discrete space operator as in [16], and we construct two time discretizations for functions $g(t) \in C^1[0,T]$ and for $g(t) \in C^2[0,T]$. Thanks to the use of Laplace transform, we are able to relax the regularity assumption to $g(t) \in C^1[0,T]$, under such weaker regularity setting, our analysis shows that the order of convergence is $O(k+h^4)$ but numerical experiences show a better convergence rate as $O(k^{1+\alpha}+h^4)$. In addition, with the same regularity assumptions as in [16] ($g(t) \in C^2[0,T]$), we can modify the scheme such that it provides an optimal convergence rate as $O(k^2 + h^4)$.

Existence, uniqueness, and monotonicity results were established in [3] by A. Friedman for the solution of a generalization of equation (2), see Corollary 1 of [2, p.143]. Applications are referenced as well in [2, p.146-147]. More recently, M. Stynes et al were able to obtain existence and uniqueness for the solution of a generalization of equation (2) in [12], see Theorem 2.1 of [12] for further discussion.

The remainder of the paper is organized as follows. Section 2 presents the numerical preliminaries and presents the existence and uniqueness of a solution to this newly transformed equation. Section 3 defines the numerical schemes and establishes the necessary lemmas for our a priori error estimates. Section 4 contains the statements of our main theorems presented in this paper guaranteeing the convergence and stability of our method. Section 5 presents some numerical examples illustrating our results, where we observe order $O(k^2)$ in time convergence for the $C^2[0,T]$ scheme, and slightly less than order $O(k^{1+\alpha})$ in time convergence for the $C^1[0,T]$ scheme. Finally, we conclude our findings in Section 6.
2. Preliminaries

We will begin by showing that (1) and (2) are equivalent by application of the Laplace transform under the hypotheses of Theorem A from [12], which we state below.

Let \( \{ (\lambda_i, \psi_i) : i = 1, 2, \ldots \} \) be the eigenvalues and eigenfunctions for the Sturm-Liouville two-point boundary value problem

\[
\mathcal{L}\psi_i = -p \frac{\partial^2 \psi_i}{\partial x^2} + c \psi_i = \lambda_i \psi_i \quad \text{on} \ (0, 1), \quad \psi_i(0) = \psi_i(1) = 0,
\]

where the eigenfunctions are normalized by requiring \( \| \psi_i \|_2 = 1 \) for all \( i \). Define the fractional power \( \mathcal{L}^\gamma \) of the operator \( \mathcal{L} \) for each \( \gamma \in \mathbb{R} \) with corresponding domain

\[
D(\mathcal{L}^\gamma) = \left\{ g \in H^2_0(0, 1) : \sum_{i=1}^{\infty} \lambda_i^{2\gamma} |(g, \psi_i)| < \infty \right\} \subset L^2(0, 1).
\]

Further, we will use the Sobolev space norm

\[
\| g \|_{\mathcal{L}^\gamma} = \left( \sum_{i=1}^{\infty} \lambda_i^{2\gamma} |(g, \psi_i)| \right)^{1/2}, \quad \text{for all} \quad g \in D(\mathcal{L}^\gamma).
\]

**Theorem (A [3, p.1061]).** Let \( \phi \in D(\mathcal{L}^{5/2}) \), \( f(\cdot, t) \in D(\mathcal{L}^{5/2}) \), \( f_i(\cdot, t) \in D(\mathcal{L}^{5/2}) \), and \( f_{tt}(\cdot, t) \in D(\mathcal{L}^{5/2}) \) for each \( t \in (0, T] \) with

\[
\| f(\cdot, t) \|_{\mathcal{L}^{5/2}} + \| f_i(\cdot, t) \|_{\mathcal{L}^{1/2}} + t^\rho \| f_{tt}(\cdot, t) \|_{\mathcal{L}^{1/2}} \leq C_1
\]

for all \( t \in (0, T] \) and some constant \( \rho < 1 \) where \( C_1 \) is a constant independent of \( t \). Then, (1) has a unique solution \( u \) that satisfies the initial and boundary conditions pointwise, and there exists a constant \( C \) such that

\[
\left| \frac{d^k u}{dx^k} \right| \leq C \quad \text{for} \quad k=0,1,2,3,4 \quad (3)
\]

\[
\left| \frac{d^l u}{dt^l} \right| \leq C(1 + t^{\alpha-l}) \quad \text{for} \quad l=0,1,2. \quad (4)
\]

**Lemma 2.1.** Assume the hypotheses of Theorem A. Then the function \( u = u(x,t) \) satisfies (1) if and only if it satisfies (2).

**Proof.** We use the convolution theorem (see Chapter 6, Section 1.3 of [4, p.135])

\[
\mathcal{L}(a * b) = \mathcal{L}(a) \mathcal{L}(b) \quad \text{if} \quad (a * b)(t) = \int_0^t a(t-s)b(s) \, ds
\]

and the facts

\[
\mathcal{L}(a_{\alpha})(z) = z^{\alpha-1} \quad \text{and} \quad \mathcal{L}(h')(z) = z \mathcal{L}(h)(z) - h(0)
\]
to obtain
\[ \mathcal{L}(\partial_t^\alpha u(x,t)) = (z\mathcal{L}(u(x,\cdot))(z) - \phi(x))z^{\alpha - 1} \]

Applying the Laplace transform to equation (1) we obtain after some algebra,
\[
(z\mathcal{L}(u(x,\cdot))(z) - \phi(x))z^{\alpha - 1} = \left[ \mathcal{L} \left( \frac{\partial^2 u}{\partial x^2}(x,\cdot) \right) + \mathcal{L}(f(x,\cdot))(z) \right]
\]

By inverting the Laplace transform, we get the equivalent Volterra integral equation
\[ u(x,t) = \phi(x) + a_1 - \alpha * \left( \frac{\partial^2 u}{\partial x^2} + f \right)(x,t). \]

Since the steps are reversible and our formal manipulations are valid by Theorem A, then the result follows.

**Remark 2.2.** The manipulations in the prior lemma use the assumptions from Theorem A in order to guarantee our a priori estimates that are derived in sections 3 and 4. We note that a similar existence and uniqueness theorem can be derived under the assumptions detailed in [3], but the a regularity of the solution that results is insufficient for our finite difference methods.

With the equivalence established between (1) and (2), we next provide the finite difference schemes that are used and the resulting a priori error estimates in the following sections. The existence and uniqueness of a solution to (2) is presented in Appendix A. We now examine the consistency, stability, and convergence of multiple numerical schemes for (2) based on the regularity of the solution in the time-variable.

### 3. Fully Discretized Numerical Schemes

In [1, 15, 16], a fully discrete scheme was derived for the L1-method in the time variable and analyzed as such. By utilizing the Laplace transform, we are able to derive an equation with a different integral kernel than the fractional derivative operator as defined before. Therefore, we will derive two convergent numerical schemes for this newly transformed equation for both a first and second-order approximation to (2) in time. The schemes are defined by the degree of regularity that will be assumed, therefore we will construct a first-order accurate scheme for functions that are $C^1[0,T]$ in time and a second-order accurate scheme for functions that are $C^2[0,T]$ in time. From there, we will
utilize a spatial operator that was defined in [16] which is fourth-order accurate in the spatial variable and hence we will arrive at the fully discrete equations.

We will use the notations and state key results from [16] that extend to our work. Divide the time interval \([0,T]\) into \(N\) intervals where \(0 = t_0 < t_1 < \ldots < t_N = T\). Denote the time steps as

\[
\tau_n = t_n - t_{n-1}, \quad 1 \leq n \leq N,
\]

and the mesh of the partition

\[
\tau_{max} = \max_{1 \leq j \leq N} \tau_j.
\]

We will derive the numerical results for any temporal mesh provided, see Theorems 3.1, 3.2, 4.3, and 4.6 for these results. Having established the unique solution to (2) in section 2.1, we shall denote the grid function by

\[
v = \{v_i : 0 \leq i \leq M\}, \text{ where } M > 0, \quad h = \frac{1}{M}, \text{ and } x_i = ih,
\]

and the grid operator

\[
\mathcal{H}_h v_i = \begin{cases} \frac{1}{12}(v_{i+1} + 10v_i + v_{i-1}), & 1 \leq i \leq M - 1, \\ v_i, & i = 0 \text{ or } i = M. \end{cases} \quad (5)
\]

By applying \(\mathcal{H}_h\) to equation (2) we see that when \(i = 0\) or \(i = M\),

\[
\mathcal{H}_h u(x_i, t_n) = u(x_i, 0) + \frac{1}{\Gamma(\alpha)} \int_0^{t_n} (t_n - s)^{\alpha - 1} \left( \frac{\partial^2 u}{\partial x^2}(x_i, s) + f(x_i, s) \right) ds,
\]

and when \(1 \leq i \leq M - 1\),

\[
\begin{align*}
\mathcal{H}_h u(x_i, t_n) &= \frac{1}{12} u(x_{i-1}, t_n) + \frac{10}{12} u(x_i, t_n) + \frac{1}{12} u(x_{i+1}, t_n) \\
&= \mathcal{H}_h \left[ u(x_i, 0) + \frac{1}{\Gamma(\alpha)} \int_0^{t_n} (t_n - s)^{\alpha - 1} \left( \frac{\partial^2 u}{\partial x^2}(x_i, s) + f(x_i, s) \right) ds \right] \\
&= \mathcal{H}_h u(x_i, 0) + \frac{1}{\Gamma(\alpha)} \int_0^{t_n} (t_n - s)^{\alpha - 1} \left( \mathcal{H}_h \frac{\partial^2 u}{\partial x^2}(x_i, s) + \mathcal{H}_h f(x_i, s) \right) ds.
\end{align*}
\]

We present the discretization in the space variable for (2), which was used in [16].
Lemma (4.1 of [16]). Let \( g(x) \) and \( \xi(s) \) be functions such that \( g(x) \in C^6[x_{i-1}, x_{i+1}] \) and \( \xi(s) = 5(1 - s)^3 - 3(1 - s)^5 \), then

\[
\frac{g''(x_{i+1}) + 10g''(x_i) + g''(x_{i-1})}{12} = \frac{g(x_{i+1}) - 2g(x_i) + g(x_{i-1})}{h^2} + \frac{h^4}{360} \int_0^1 [g^{(6)}(x_i - sh) + g^{(6)}(x_i + sh)]\xi(s) \, ds.
\]

3.1. A \( C^1[0,T] \) in Time Scheme. The following is an analogue of Lemma 2.1 of [16].

**Theorem 3.1.** For \( 0 < \alpha < 1 \) and for \( g(t) \in C^1[0,T] \), it follows that

\[
\int_0^{t_n} g(s)(t_n - s)^{\alpha-1} \, ds = \sum_{k=1}^{n} \frac{g(t_{k-1}) + g(t_k)}{2} \int_{t_{k-1}}^{t_k} (t_n - s)^{\alpha-1} \, ds + R^n_t,
\]

where

\[
|R^n_t| \leq (\tau_n + \tau_{\max}) \frac{T_\alpha}{2\alpha} \max_{0 \leq t \leq t_n} |g'(t)|.
\]

**Proof.** We begin by writing the integral as

\[
\int_0^{t_n} g(s)(t_n - s)^{\alpha-1} \, ds = \int_0^{t_{n-1}} g(s)(t_n - s)^{\alpha-1} \, ds + \int_{t_{n-1}}^{t_n} g(s)(t_n - s)^{\alpha-1} \, ds.
\]

The first integral on the right hand side is rewritten as

\[
\int_0^{t_{n-1}} g(s)(t_n - s)^{\alpha-1} \, ds = \sum_{k=1}^{n-1} \int_{t_{k-1}}^{t_k} g(s)(t_n - s)^{\alpha-1} \, ds
\]

\[
= \sum_{k=1}^{n-1} \int_{t_{k-1}}^{t_k} \left( g(s) - \frac{g(t_{k-1}) + g(t_k)}{2} \right) (t_n - s)^{\alpha-1} \, ds
\]

\[
+ \int_{t_{k-1}}^{t_k} \left( \frac{g(t_{k-1}) + g(t_k)}{2} \right) (t_n - s)^{\alpha-1} \, ds
\]

\[
= \sum_{k=1}^{n-1} \left( \frac{g(t_{k-1}) + g(t_k)}{2} \right) \int_{t_{k-1}}^{t_k} (t_n - s)^{\alpha-1} \, ds + (R_1)^n,
\]

where

\[
(R_1)^n = \sum_{k=1}^{n-1} \int_{t_{k-1}}^{t_k} \left( g(s) - \frac{g(t_{k-1}) + g(t_k)}{2} \right) (t_n - s)^{\alpha-1} \, ds.
\]
By utilizing the Taylor expansion of \( g(s) \) for \( s \in (0, t_{n-1}) \),
\[
| (R_1)^n | \leq \max_{0 \leq t \leq t_{n-1}} | g'(t) | \sum_{k=1}^{n-1} \int_{t_{k-1}}^{t_k} \left| t_k - s - \frac{\tau_k}{2} \right| (t_n - s)^{\alpha-1} ds
\]
\[
\leq \frac{\tau_{\text{max}}}{2} \max_{0 \leq t \leq t_{n-1}} | g'(t) | \int_0^{t_{n-1}} (t_n - s)^{\alpha-1} ds
\]
\[
= \frac{\tau_{\text{max}}}{2} \max_{0 \leq t \leq t_{n-1}} | g'(t) | \left( \frac{t_n^\alpha}{\alpha} - \frac{t_n^\alpha}{\alpha} \right)
\]
\[
\leq \frac{\tau_{\text{max}}}{2\alpha} \max_{0 \leq t \leq t_{n-1}} | g'(t) |.
\]

In a similar manner, by the Taylor expansion of \( g(s) \) for \( s \in (t_{n-1}, t_n) \), we have
\[
\left| g(s) - \left( g(t_{n-1}) + \frac{g(t_n)}{2} \right) \right| \leq \frac{\tau_n}{2} \max_{t_{n-1} \leq t \leq t_n} | g'(t) |, \ t_{n-1} < s < t_n.
\]

Therefore, the approximation error in the interval \([t_{n-1}, t_n]\) satisfies
\[
| (R_2)^n | = \left| \int_{t_{n-1}}^{t_n} \left( g(s) - \frac{g(t_{n-1}) + g(t_n)}{2} \right) (t_n - s)^{\alpha-1} ds \right|
\]
\[
\leq \frac{\tau_n}{2\alpha} \max_{t_{n-1} \leq t \leq t_n} | g'(t) |.
\]

Finally, since
\[
R_t^n = (R_1)^n + (R_2)^n = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \left( g(s) - \frac{g(t_{k-1}) + g(t_k)}{2} \right) (t_n - s)^{\alpha-1} ds,
\]
by combining the error estimates (8) and (9), we have the desired result. \( \square \)

We remark that under these assumptions, we may obtain a first order accurate scheme for \( g(t) \in C^1[0, T] \). The L1-method requires the function \( g(t) \in C^2[0, T] \) based on a Taylor series argument, so the condition for the L1-method cannot be relaxed to allow \( g(t) \in C^1[0, T] \) due to the nature of the Caputo Fractional Derivative. In section 5, we will see that this scheme exhibits superconvergence for this \( C^1[0, T] \) scheme. Define
\[
a_k^\alpha = \frac{1}{\Gamma(\alpha)} \int_{t_k-1}^{t_k} (t_n - s)^{\alpha-1} ds,
\]
\[
= \frac{1}{\Gamma(\alpha+1)} \left[ (t_n - t_{k-1})^\alpha - (t_n - t_k)^\alpha \right].
\]
We define
\[
 f_{1-\alpha}(x, t) = \int_0^t \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} f(x, s) \, ds
\]  
(11)
to help succinctly denote the forcing function term in the approximate equation. By applying the \( H_h \) operator, Lemma 4.1 of [16], Lemma 2.2, and the previously stated discretization to (2), we have the fully discretized approximate equation for \( u^n_i \approx u(x_i, t_n) \)
\[
 H_h u^n_i = H_h \phi(x_i) + H_h f_{1-\alpha}(x_i, t_n) + \sum_{k=1}^n \frac{a^n_k}{2h^2} \left[u^k_{i+1} - 2u^k_i + u^k_{i-1}\right] + \frac{a^n_{k+1}}{2h^2} \left[u^{k-1}_{i+1} - 2u^{k-1}_i + u^{k-1}_{i-1}\right],
\]
which is to be solved for \( \{u^n_i\}_{n=0,1,...,N, i=0,1,...,M} \).

3.2. A \( C^2[0,T] \) in Time Scheme. We begin our findings in this section by establishing a second order method in time, with the restriction of \( g(t) \in C^2[0,T] \).

**Theorem 3.2.** For \( 0 < \alpha < 1 \) and for \( g(t) \in C^2[0,T] \), it follows that
\[
 \int_0^{t_n} g(s)(t_n - s)^{\alpha-1} \, ds = \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \left(\left(1 - \frac{t_k - s}{\tau_k}\right) g(t_k) + \left(\frac{t_k - s}{\tau_k}\right) g(t_{k-1})\right) (t_n - s)^{\alpha-1} \, ds + R^n_t,
\]
(13)
where
\[
 |R^n_t| \leq \frac{(\tau^2_{\max} + \tau^2_{\min}) T^\alpha}{8\alpha} \max_{0 \leq t \leq t_{n-1}} |g''(t)|.
\]

**Proof.** We begin with the Taylor expansions of \( g(s) \) at the points \( s = t_k \), where \( s \in [t_{k-1}, t_k], t_k \in [0, t_n] \) for each \( k = 0, 1, 2, ..., N \),
\[
g(s) = g(t_k) + (s - t_k)g'(t_k) + \frac{(s - t_k)^2}{2}g''(t_k) + O((s - t_k)^3)
\]
ge\( g(t_{k-1}) = g(t_k) - \tau_k g'(t_k) + \frac{\tau_k^2}{2} g''(t_k) + O(\tau_k^3). \\
Hence, we may combine the above in the following manner:
\[
g(s) - \left(\left(1 - \frac{t_k - s}{\tau_k}\right) g(t_k) + \left(\frac{t_k - s}{\tau_k}\right) g(t_{k-1})\right) = \left(\frac{(s - t_k)^2 - \tau_k(s - t_k)}{2}\right) g''(t_k) + O((t_k - s)^3)
\]
We now rewrite the integral as
\[
\int_0^{t_n} g(s)(t_n-s)^{\alpha-1} \, ds = \int_0^{t_{n-1}} g(s)(t_n-s)^{\alpha-1} \, ds + \int_{t_{n-1}}^{t_n} g(s)(t_n-s)^{\alpha-1} \, ds.
\]

The first integral on the right hand side is rewritten as
\[
\int_0^{t_{n-1}} g(s)(t_n-s)^{\alpha-1} \, ds = \sum_{k=1}^{n-1} \int_{t_{k-1}}^{t_k} g(s)(t_n-s)^{\alpha-1} \, ds
\]
\[
= \sum_{k=1}^{n-1} \int_{t_{k-1}}^{t_k} \left( g(s) - \left( 1 - \frac{t_k - s}{\tau_k} \right) g(t_k) + \left( \frac{t_k - s}{\tau_k} \right) g(t_{k-1}) \right) (t_n-s)^{\alpha-1} \, ds
\]
\[
= \sum_{k=1}^{n-1} \int_{t_{k-1}}^{t_k} \left( 1 - \frac{t_k - s}{\tau_k} \right) g(t_k) + \left( \frac{t_k - s}{\tau_k} \right) g(t_{k-1}) \right) (t_n-s)^{\alpha-1} \, ds + (R_1)^n,
\]
where then
\[
(R_1)^n = \sum_{k=1}^{n-1} \int_{t_{k-1}}^{t_k} \left( g(s) - \left( 1 - \frac{t_k - s}{\tau_k} \right) g(t_k) + \left( \frac{t_k - s}{\tau_k} \right) g(t_{k-1}) \right) (t_n-s)^{\alpha-1} \, ds.
\]

We remark that since \( s \in [t_{k-1}, t_k] \) for each \( k \), then we have
\[
| (s - t_k)^2 - \tau_k(s-t_k) | = | (s - t_k)(s-t_k - \tau_k) | = | (s-t_k)(s-t_{k-1}) | \leq \frac{\tau_k^2}{4}
\]
for each \( k \). By utilizing the Taylor expansion of \( g(s) \) for \( s \in (0, t_{n-1}) \), and by neglecting the higher order terms,
\[
| (R_1)^n | \leq \max_{0 \leq t \leq t_{n-1}} |g''(t)| \sum_{k=1}^{n-1} \int_{t_{k-1}}^{t_k} \left| \frac{(s-t_k)^2 - \tau_k(s-t_k)}{2} \right| (t_n-s)^{\alpha-1} \, ds
\]
\[
\leq \frac{\tau_{max}^2}{8} \max_{0 \leq t \leq t_{n-1}} |g''(t)| \int_{t_{n-1}}^{t_n} (t_n-s)^{\alpha-1} \, ds
\]
\[
\leq \frac{\tau_{max}^2}{8} \max_{0 \leq t \leq t_{n-1}} |g''(t)| \left( \frac{t_n^{\alpha}}{\alpha} - \frac{t_{n-1}^{\alpha}}{\alpha} \right)
\]
\[
\leq \frac{\tau_{max}^2}{8\alpha} \max_{0 \leq t \leq t_{n-1}} |g''(t)|. \tag{14}
\]
For a uniform mesh, \( \tau_{max} = \tau_n = \tau \), we have the result \( |(R_1)^n| \leq C\tau^2 \). For the remaining integral term from \([t_{n-1}, t_n] \), the same argument is used as for the interval \([0, t_{n-1}] \). Therefore, the approximation error in
the interval $[t_{n-1}, t_n]$ satisfies
\[
|(R_2)^n| = \left| \int_{t_{n-1}}^{t_n} \left( g(s) - \left( 1 - \frac{t_k - s}{\tau_k} \right) g(t_k) + \left( \frac{t_k - s}{\tau_k} \right) g(t_{k-1}) \right) \right| \\
\times (t_n - s)^{a-1} ds \\
\leq \frac{\tau^2_n}{8} \max_{t_{n-1} \leq t \leq t_n} |g''(t)| \int_{t_{n-1}}^{t_n} (t_n - s)^{a-1} ds \\
\leq \frac{\tau^2_n}{8} \max_{t_{n-1} \leq t \leq t_n} |g''(t)| \left( \frac{\tau_n^a}{\alpha} - 0 \right) \\
\leq \frac{\tau^{2+\alpha}_n}{8\alpha} \max_{t_{n-1} \leq t \leq t_n} |g''(t)|. \tag{15}
\]
Finally, since
\[
R^n_t = (R_1)^n + (R_2)^n = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \left( g(s) - \frac{g(t_{k-1}) + g(t_k)}{2} \right) (t_n - s)^{a-1} ds,
\]
by combining the error estimates (14) and (15), we have the desired result. \hfill \square

We then arrive at a fully discretized equation for $u^n_i \approx u(x_i, t_n)$ by recalling the definition of $a^n_k$ from (10) and by setting
\[
b^{n}_{1,k} = \frac{1}{\Gamma(\alpha)} \int_{t_{k-1}}^{t_k} \left( 1 - \frac{t_k - s}{\tau_k} \right) (t_n - s)^{a-1} ds \\
b^{n}_{2,k} = \frac{1}{\Gamma(\alpha)} \int_{t_{k-1}}^{t_k} \left( \frac{t_k - s}{\tau_k} \right) (t_n - s)^{a-1} ds.
\]
The resulting fully discretized equation is as follows:
\[
\mathcal{H}_h u^n_i = \mathcal{H}_h \phi(x_i) + \mathcal{H}_h f_{1-\alpha}(x_i, t_n) \\
+ \sum_{k=1}^{n} \left( b^{n}_{1,k} \left[ u^{k}_{i+1} - 2u^{k}_{i} + u^{k}_{i-1} \right] + b^{n}_{2,k} \left[ u^{k-1}_{i+1} - 2u^{k-1}_{i} + u^{k-1}_{i-1} \right] \right),
\]
which is to be solved for $\{u^n_i\}_{n=0,1,...,N, \ i=0,1,...,M}$.

4. Error Estimates

Before we establish stability and convergence of the numerical methods used, we will make use of the definitions in [1, p.202]. Let
\[
\mathcal{V}_h = \{v = (v_0, v_1, ..., v_M) | v_0 = v_M = 0 \}. 
\]
IMPROVED F. D. RESULTS FOR THE CAPUTO TIME-FRACTIONAL DIFF. EQUATION

For any grid functions $v, w \in \mathcal{V}_h$, we will define the following:

- $L_2$ norm: $\|v\|_h = \sqrt{\langle v, v \rangle_h}$
- $H^1$ semi-norm: $\|\delta_x v\|_h = \sqrt{\sum_{i=1}^{M} (\delta_x v_{i-1})^2}$
- $H^1$ norm: $\|v\|_{1,h} = \sqrt{\|v\|_h^2 + \|\delta_x v\|_h^2}$

Where $\|H_h v\|_h$ and $\|\delta_x^2 v\|_h$ are defined in a similar manner. By applying Lemma 4.2 of [16], then

$$\|v\|_h \leq \frac{1}{\sqrt{6}} \|\delta_x v\|_h.$$

Following [16], define

$$\langle v, w \rangle_A = h \sum_{i=1}^{M} (\delta_x v_{i-1/2} \cdot \delta_x w_{i-1/2}) - \frac{h^2}{12} h \sum_{i=1}^{M-1} \delta_x^2 v_i \cdot \delta_x^2 w_i,$$

and

$$\|v\|_A = \sqrt{\langle v, v \rangle_A}.$$

They further go on to show that, by Lemma 4.3 of [16],

$$-h \sum_{i=1}^{M-1} (H_h v_i) \cdot \delta_x^2 w_i = \langle v, w \rangle_A,$$

which establishes that $\|\cdot\|_A$ and $\|\delta_x\|_h$ are equivalent.

4.1. Consistency, Stability, and Convergence Results. With the preliminaries established in section 2, we will present the main theorems of this paper. We begin with deriving the consistency of the schemes (12) and (16) and then the stability for each. With both these proofs, we are able to assert the convergence of each scheme, which is then demonstrated in the next section.

**Theorem 4.1.** Let $\{u^n_i\}_{0 \leq i \leq M, 1 \leq n \leq N}$ be the solution of the approximate scheme (12), with a uniform grid used in the spatial domain. Further, let $\phi, f(\cdot, t), f_t(\cdot, t), f_{tt}(\cdot, t) \in D(C^{3/2})$ for each $t \in (0, T]$. Then, $u$ is a unique solution to (2) with resulting approximation error

$$\|u(x_i, t_j) - u^n_i\|_A \leq \frac{T^\alpha}{\Gamma(\alpha + 1)} \left( \frac{h^4}{180} \left\| \frac{\partial^6 u}{\partial x^6} \right\|_\infty + \left( \frac{\tau_n + \tau_{max}}{2} \right) \left\| \frac{\partial u}{\partial t} \right\|_\infty \right).$$

(17)
Further, by applying Theorem 3.1 to (6), we have

$$
(R_x)^n(x_i, t_n) = \frac{h^4}{360} \int_0^1 \left[ \frac{\partial^6 u}{\partial x^6}(x_i - sh, t_n) + \frac{\partial^6 u}{\partial x^6}(x_i + sh, t_n) \right] ds (18)
$$

for all $t \in [0, 1]$. We may then bound $(R_x)^n(x_i, t_n)$ by

$$
\left| (R_x)^n(x_i, t_n) \right| = \left| \frac{h^4}{360} \int_0^1 \left( \frac{\partial^6 u}{\partial x^6}(x_i - sh, t_n) + \frac{\partial^6 u}{\partial x^6}(x_i + sh, t_n) \right) ds \right|
$$

By applying Lemma 4.1 of [16] to (6), we see that

$$
\mathcal{H}_h u(x_i, t_n) = \mathcal{H}_h u(x_i, 0) + \frac{1}{\Gamma(\alpha)} \int_0^{t_n} (t_j - s)^{\alpha-1} \mathcal{H}_h u_{xx}(x_i, s) + \mathcal{H}_h f(x_i, s) ds
$$

$$
= \frac{1}{\Gamma(\alpha)} \int_0^{t_n} \left( u(x_{i+1}, s) - 2u(x_i, s) + u(x_{i-1}, s) + h^2(R_x)^n(x_i, s) \right) ds
$$

+ \mathcal{H}_h f_{1-\alpha}(x_i, t_j)).

Further, by applying Theorem 3.1 to (4), we have

$$
\mathcal{H}_h u_i^n = \mathcal{H}_h \phi(x_i) + \mathcal{H}_h f_{1-\alpha}(x_i, t_n)
$$

$$
+ \sum_{k=1}^{n} \left( \frac{a_k^h}{2h^2} [u_{i+1}^k - 2u_i^k + u_{i-1}^k] + \frac{a_k^h}{2h^2} [u_{i+1}^{k-1} - 2u_i^{k-1} + u_{i-1}^{k-1}] \right)
$$

$$
+ \frac{1}{\Gamma(\alpha)} \int_0^{t_n} ((R_x)^n(x_i, s) + (R_t)^n(x_i, s)) (t_n - s)^{\alpha-1} ds.
$$

Finally, we see that the approximation error is

$$
\|u(x_i, t_n) - u^n_i\|_A = \left\| \frac{1}{\Gamma(\alpha)} \int_0^{t_n} ((R_x)^n(x_i, s) + (R_t)^n(x_i, s)) (t_n - s)^{\alpha-1} ds \right\|_A
$$

$$
\leq \frac{T^\alpha}{\Gamma(\alpha + 1)} (\| (R_x)^n \|_A + \| (R_t)^n \|_A)
$$

$$
\leq \frac{T^\alpha}{\Gamma(\alpha + 1)} \left( \frac{h^4}{180} \left\| \frac{\partial^6 u}{\partial x^6} \right\|_\infty + \left( \frac{\tau_n + \tau_{\text{max}}}{2} \right) \left\| \frac{\partial u}{\partial t} \right\|_\infty \right).
$$

\[\square\]
We will remark that as $\alpha \to 0$ then $\frac{T^\alpha}{\Gamma(1 + \alpha)} \to 1$. Also, as $\alpha \to 1$ then $\frac{T^\alpha}{\Gamma(1 + \alpha)} \to T$. The following corollary is immediate from the previous theorem.

**Corollary 4.1.1.** Under a uniform partition of the time domain where $\tau_n = \tau$ for all $n$, then the approximation error of (12) is $O(h^4 + \tau)$.

We also have a theorem asserting the stability of the discrete scheme and derives the corresponding error equations of the scheme:

**Theorem 4.2.** Suppose $\{u^n_i\}_{0 \leq i \leq M, 1 \leq n \leq N}$ is the solution of the difference scheme (12). Then, for any size temporal mesh described before, the discrete difference scheme (12) is unconditionally stable to $f$ and $\phi$, where

$$
\|u^n\|^2_A \leq \|\phi\|^2_A + \frac{T^\alpha}{\Gamma(\alpha + 1)} \max_{1 \leq l \leq N} \|\mathcal{H}_hf^l\|^2_h
$$

**Proof.** Recall that

$$a^n_k = \frac{1}{\Gamma(\alpha)} \int_{t_{k-1}}^{t_k} (t_n - s)^{\alpha - 1} ds,
= \frac{1}{\Gamma(\alpha + 1)} [(t_n - t_{k-1})^\alpha - (t_n - t_k)^\alpha].
$$

We consider the scheme (12) after combining the initial and boundary conditions. By omitting the residual term $R^n_i$ and by substituting the exact solution $U^n_i$ with its approximation $u^n_i$ into (12), we have:

$$\mathcal{H}_hu^n_i = \mathcal{H}_hu_i^0 + \sum_{k=1}^n a^n_k \left( \frac{\delta_x^2 u^n_k + \delta_x^2 u^{k-1}}{2} + \mathcal{H}_hf^n_i \right).$$

By multiplying both sides by $-2h\delta_x^2 u^n_i$ and summing over each $i$, then

$$2 \|u^n\|^2_A + \sum_{k=1}^{n-1} a^n_k \left( \|\delta_x^2 u^k\|^2_h + \|\delta_x^2 u^{k-1}\|^2_h \right) = 2 < u^0, u^n >_A - 2 \sum_{k=1}^n a^n_k < \mathcal{H}_hf, \delta_x^2 u^n >_h
\leq \left( \|u^0\|^2_A + \|u^n\|^2_A \right) + \sum_{k=1}^n a^n_k \left( \|\mathcal{H}_hf^n\|^2_h + \|\delta_x^2 u^n\|^2_h \right)
\Rightarrow \|u^n\|^2_A \leq \|\phi\|^2_A + \sum_{k=1}^n a^n_k \max_{1 \leq l \leq N} \|\mathcal{H}_hf^l\|^2_h 1 \leq n \leq N.
$$

Finally, since $\sum_{k=1}^n a^n_k = \frac{T^\alpha}{\Gamma(\alpha + 1)}$ when $n = N$, we see the result holds. \qed
To further see the convergence of the numerical scheme, denote $e^n_i := u(x_i, t_n) - u^n_i$. The error equations are then obtained:

$$H_h e^n_i = \sum_{k=1}^{n} a^n_k \delta^2_{x} e^n_i + a^n_i R^n_i$$  

(19)

$$e^n_0 = e^n_M = 0, 1 \leq n \leq N$$

$$e^0_i = 0, 0 \leq i \leq M.$$  

By applying (18) and by applying the previous stability analysis, we have the immediate error convergence result

$$\|e^n\|_A \leq \|e^0\|_A + \frac{T^\alpha}{\Gamma(\alpha + 1)} \|R^n_i\|_h$$

$$\leq \frac{T^\alpha}{\Gamma(\alpha + 1)} \left( \frac{h^4}{180} \left\| \frac{\partial^6 u}{\partial x^6} \right\|_\infty + \left( \frac{\tau_n + \tau_{\max}}{2} \right) \left\| \frac{\partial u}{\partial t} \right\|_\infty \right)^2,$$

$$\|e^n\|_A \leq \sqrt{\frac{T^\alpha}{\Gamma(\alpha + 1)} \left( \frac{h^4}{180} \left\| \frac{\partial^6 u}{\partial x^6} \right\|_\infty + \left( \frac{\tau_n + \tau_{\max}}{2} \right) \left\| \frac{\partial u}{\partial t} \right\|_\infty \right)}.$$

That is, the scheme (12) is both stable and consistent, hence it is convergent, see [7-9] for further details. Therefore, by [3, theorem 2.1] we have the following immediate results:

**Theorem 4.3.** Let $\{u^n_i\}_{0 \leq i \leq M, 1 \leq n \leq N}$ be the solution of the approximate scheme (12), with a uniform grid used in the spatial domain and any grid spacing used in the temporal direction. Further, let $\phi, f(\cdot, t), f_t(\cdot, t), f_{tt}(\cdot, t) \in D(L^{9/2})$ for each $t \in (0, T]$. Then, it holds for some $C > 0$

$$\|u(x, t_n) - u^n_i\|_A \leq \sqrt{\frac{T^\alpha}{\Gamma(\alpha + 1)}} C \left( h^4 + \tau_{\max} \right), \quad 1 \leq n \leq N.$$  

(20)

We also have a corollary detailing the use of a truncation of the exact solution to generate the data at $u(x, t_1)$.

**Corollary 4.3.1.** Let $u_{h,1}(x, t_1) = \phi(x) + \frac{\phi''(x) t^\alpha}{\Gamma(\alpha + 1)} + (f * a_{1-\alpha})(x, t_1)$.

Then, the truncation error

$$\|e_{h,1}\|_{\infty} = \|u(x, t_1) - u_{h,1}(x, t_1)\|_{\infty} \leq C \tau_{\max} \left( \|\phi^{(4)}(x)\|_{\infty} + \|f(x, t_1)\|_{\infty} \right)$$

Proof. Consider the exact solution $u(x, t)$ of (2) which is generated from (25). That is,

$$u(x, t) = \phi(x) + \frac{\phi''(x) t^\alpha}{\Gamma(\alpha + 1)} + \frac{\phi^{(4)}(x) t^{2\alpha} \Gamma(1/2)}{4^\alpha \Gamma(\alpha + 1/2)} + O(t^{3\alpha}) \phi^{(6)}(x) + (f * a_{1-\alpha})(x, t)$$

$$+ ((f * a_{1-\alpha}) * a_{1-\alpha})(x, t) + O(t^{3\alpha}).$$
Theorem 4.5. The approximation error of (16) is

\[ \|e_{h,1}\|_\infty = \frac{\phi(t^2) t^{2\alpha} \Gamma(1/2)}{4\alpha \Gamma(\alpha + 1/2)} + \left( (f * a_{1-\alpha}) * a_{1-\alpha} \right) (x, t_1) \]

\[ \leq C t_1^{2\alpha} \|\phi(x)\|_\infty + \left\| \int_0^{t_1} \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} \left( \int_0^s \frac{(s-v)^{\alpha-1} f(x,v)}{\Gamma(\alpha)} dv \right) ds \right\| \]

\[ \leq C t_1^{2\alpha} \|\phi(x)\|_\infty + \left\| f(x,t_1) \right\|_\infty \left\| \int_0^{t_1} \frac{s^\alpha (t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} ds \right\| \]

\[ = C t_1^{2\alpha} \|\phi(x)\|_\infty + \left\| f(x,t_1) \right\|_\infty \left\| \frac{1}{4\alpha \Gamma(\alpha + 1/2)} \right\|_\infty \]

\[ \leq C t_1^{2\alpha} (\|\phi(x)\|_\infty + \left\| f(x,t_1) \right\|_\infty). \]

\[ \square \]

Remark. By letting \( t_1 = \tau, \phi(x) = 0 \), and where \( f(x, \tau) = (\tau + O(\tau^{1+\alpha})) X(x) \), we have the truncation error in corollary 4.3.1 after neglecting the terms of order \( O(\tau^{1+\alpha}) \):

\[ \|e_{h,1}\|_\infty = \| u(x, t_1) - u_{h,1}(x, t_1) \|_\infty \leq C t_1^{1+2\alpha} \| X(x) \|_\infty. \]

We have a similar set of results for the numerical scheme (16) for functions \( g(t) \in C^2[0,T] \). Beginning with the consistency results, we will provide each theorem as follows:

Theorem 4.4. Let \( \{ u^n \}_{0 \leq i \leq M, 1 \leq n \leq N} \) be the solution of the approximate scheme (16), with a uniform grid used in the spatial domain. Further, let \( \phi, f, f_i, f_u \in D(\mathcal{L}^{3/2}) \) for each \( t \in (0,T) \). Then, \( u \) is a unique solution to (2), with resulting approximation error

\[ \| u(x, t_j) - u^n_i \|_A \leq \frac{T^\alpha}{\Gamma(\alpha + 1)} \left( \frac{h^4}{180} \left\| \frac{\partial^6 u}{\partial x^6} \right\|_\infty + \left( \frac{\tau_n^2 + \tau_{\max}^2}{8} \right) \left\| \frac{\partial^2 u}{\partial t^2} \right\|_\infty \right). \]

(21)

Proof. The proof is identical to Theorem 4.1 and is omitted. \[ \square \]

Therefore, under a uniform partition of the time domain, the approximation error of (16) is \( O(h^4 + \tau^2) \).

Theorem 4.5. Suppose \( \{ u^n \}_{0 \leq i \leq M, 1 \leq n \leq N} \) is the solution of the difference scheme (16). Then, for any size temporal mesh described before, the discrete difference scheme (16) is unconditionally stable to \( f \) and \( \phi \), where

\[ \| u^n \|_A \leq \| \phi \|_A + \frac{T^\alpha}{\Gamma(\alpha + 1)} \max_{1 \leq i \leq N} \| \mathcal{H} f_i \|_h^2 \]
Proof. The proof is identical to Theorem 4.3 and is omitted. \qed

Following our stability result, the error equations are then obtained:

\[
\mathcal{H}_h \varepsilon^n_i = \sum_{k=1}^{n} a_k^n \delta^2 x^n_i + R^n_i
\]

(22)

\[
\varepsilon^n_0 = \varepsilon^n_M = 0, \ 1 \leq n \leq N
\]

\[
\varepsilon^0_i = 0, \ 0 \leq i \leq M.
\]

In a similar manner to the convergence result for the scheme (12), we also have an error convergence result for (16)

\[
\| \varepsilon^n \|_A^2 \leq \| \varepsilon^0 \|_A^2 + \frac{T^\alpha}{\Gamma(\alpha + 1)} \| R^n_i \|_h^2 \\
\leq \frac{T^\alpha}{\Gamma(\alpha + 1)} \left( \frac{h^4}{180} \left\| \frac{\partial^6 u}{\partial x^6} \right\|_\infty + \left( \frac{\tau^2_n + \tau^2_{\text{max}}}{8} \right) \left\| \frac{\partial^2 u}{\partial t^2} \right\|_\infty \right)^2
\]

\[
\| \varepsilon^n \|_A \leq \sqrt{\frac{T^\alpha}{\Gamma(\alpha + 1)}} \left( \frac{h^4}{180} \left\| \frac{\partial^6 u}{\partial x^6} \right\|_\infty + \left( \frac{\tau^2_n + \tau^2_{\text{max}}}{8} \right) \left\| \frac{\partial^2 u}{\partial t^2} \right\|_\infty \right).
\]

Hence, the schemes (12) and (16) are both stable and consistent, hence they are both convergent. Therefore, by [3, theorem 2.1] we have the following immediate results:

**Theorem 4.6.** Let \( \{ u^n_i \mid 0 \leq i \leq M, 1 \leq n \leq N \} \) be the solution of the approximate scheme (16), with a uniform grid used in the spatial domain and any grid spacing used in the temporal direction. Further, let \( \phi, f(\cdot, t), f_t(\cdot, t), f_{tt}(\cdot, t) \in D(L^{9/2}) \) for each \( t \in (0, T) \). Then, it holds for some \( C > 0 \)

\[
\| u(x_i, t_n) - u^n_i \|_A \leq \sqrt{\frac{T^\alpha}{\Gamma(\alpha + 1)}} C \left( h^4 + \tau^2_{\text{max}} \right), \quad 1 \leq n \leq N.
\]

These results imply that under the same regularity assumptions in [16], we may improve our order of convergence by a factor of \( \alpha \). Further, we may also relax these regularity assumptions to have \( g(t) \in C^1[0, T] \) while preserving an order of convergence of \( O(k) \) in the time variable, which is not possible with the L1-method. In the next section we shall consider a simple numerical experiment that illustrates our theoretical results.
5. Numerical Experiment

We will consider the following test problem for our numerical experiments:

\[ u(x, t) = \sin(\pi x)t^2, \quad u(0, t) = u(1, t) = 0, \quad \phi = u(x, 0) = 0, \]
\[ f_{1-\alpha}(x, t) = \sin(\pi x) \left[ t^2 + \frac{2\pi^2 t^{\alpha+2}}{\Gamma(\alpha + 3)} \right] = a_{1-\alpha}(t) \ast f(x, t), \]

which will satisfy \( u(x, t) \in C^1[0, T], C^2[0, T] \) in time. We will define \( M \) to be the number of partitions of the spatial domain, \( E_2(M, N) \) to be the maximum error attained over the total mesh for a uniform mesh for functions in \( C^2[0, T] \), and \( \text{rate}_2 = \log_2 \left( \frac{E_2(M, N)}{E_2(M, N/2)} \right) \). Therefore, for \( M = 25 \) and \( T = 1 \), we have the following:

| \( \alpha \) | N   | \( E_1(M, N) \)     | rate \( \text{rate}_1 \) |
|------------|-----|---------------------|--------------------------|
| 0.05       | 10  | 0.0786              | *                        |
|            | 20  | 0.0370              | 1.089                    |
|            | 40  | 0.0179              | 1.046                    |
|            | 80  | 0.0087              | 1.046                    |
|            | 160 | 0.0042              | 1.0477                   |
| 0.25       | 10  | 0.0370              | *                        |
|            | 20  | 0.0157              | 1.2388                   |
|            | 40  | 0.0067              | 1.222                    |
|            | 80  | 0.0029              | 1.2329                   |
|            | 160 | 0.0012              | 1.2398                   |
| 0.5        | 10  | 0.0122              | *                        |
|            | 20  | 0.0046              | 1.4884                   |
|            | 40  | 0.0017              | 1.4294                   |
|            | 80  | 6.276e-4            | 1.4517                   |
|            | 160 | 2.2705e-4           | 1.4668                   |
| 0.75       | 10  | 0.0078              | *                        |
|            | 20  | 0.0014              | 2.4316                   |
|            | 40  | 2.7567e-4           | 2.3859                   |
|            | 80  | 9.1448e-5           | 1.5919                   |
|            | 160 | 2.9572e-5           | 1.6287                   |
| 0.95       | 10  | 0.0046              | *                        |
|            | 20  | 7.76962e-4          | 2.6739                   |
|            | 40  | 1.1287e-4           | 2.7832                   |
|            | 80  | 1.5854e-5           | 2.8317                   |
|            | 160 | 3.7211e-6           | 2.0911                   |

The results for using the \( C^2[0, T] \) scheme for the same test problem is as follows:
The above table shows that for various values of $\alpha$, the error estimate improves with an increase in the number of space and time steps used in the mesh partitioning while preserving a rate of convergence of $O(h^4+k^2)$ as expected. As a result, our method exhibits a better rate of convergence overall, under the same regularity assumptions. By corollary 4.3.1 if we instead replace $u(x_i,t_1)$ with its approximation derived from the exact solution, we instead have the following improved results for a small amount of time steps due to the truncation error. For this example, we have $u(x_i,t_1) = f_{1-\alpha}(x_i,t_1)$ These results are summarized in the following table:

| $\alpha$ | N   | $E_2(M,N)$   | rate$_2$ |
|----------|-----|--------------|----------|
| 0.05     | 10  | 0.0003       | *        |
|          | 20  | 9.5503e-5    | 1.8065   |
|          | 40  | 2.5878e-5    | 1.8838   |
|          | 80  | 6.5375e-6    | 1.9849   |
|          | 160 | 1.6291e-6    | 2.0047   |
| 0.25     | 10  | 0.0011       | *        |
|          | 20  | 0.0003       | 1.8854   |
|          | 40  | 7.8046e-5    | 1.9375   |
|          | 80  | 1.9597e-5    | 1.9937   |
|          | 160 | 4.8246e-6    | 2.0222   |
| 0.5      | 10  | 0.0014       | *        |
|          | 20  | 0.0004       | 1.9483   |
|          | 40  | 9.4512e-5    | 1.9847   |
|          | 80  | 2.3281e-5    | 2.0214   |
|          | 160 | 5.6626e-6    | 2.0396   |
| 0.75     | 10  | 0.0016       | *        |
|          | 20  | 0.0004       | 1.9801   |
|          | 40  | 9.9404e-5    | 2.0014   |
|          | 80  | 2.4403e-5    | 2.0262   |
|          | 160 | 5.9182e-6    | 2.0439   |
| 0.95     | 10  | 0.0016       | *        |
|          | 20  | 0.0004e-4    | 1.9974   |
|          | 40  | 0.0001e-5    | 2.0067   |
|          | 80  | 2.5303e-5    | 2.0165   |
|          | 160 | 6.1781e-6    | 2.0341   |
Numerical Error for $u(x, t) = \sin(\pi x)t^2$ with truncation error for $u(x, t_1)$

| $\alpha$ | $N$ | $E_1(M,N)$ | rate$_1$ |
|---------|-----|------------|---------|
| 0.05    | 10  | 0.0840     | *       |
|         | 20  | 0.0370     | 1.1821  |
|         | 40  | 0.0179     | 1.046   |
| 0.25    | 10  | 0.0435     | *       |
|         | 20  | 0.0157     | 1.4744  |
|         | 40  | 0.0067     | 1.222   |
| 0.5     | 10  | 0.0189     | *       |
|         | 20  | 0.0046     | 2.0227  |
|         | 40  | 0.0017     | 1.4294  |
| 0.75    | 10  | 0.0079     | *       |
|         | 20  | 0.0014     | 2.4636  |
|         | 40  | 2.7567e-4  | 2.3859  |
| 0.95    | 10  | 0.0050     | *       |
|         | 20  | 7.76962e-4 | 2.6739  |
|         | 40  | 1.1287e-4  | 2.7832  |

6. Conclusion

We have shown that using the Laplace transform on the Caputo fractional derivative can preserve the maximum accuracy of an estimate, with some improvements depending on the value of $\alpha$. By a Taylor Series expansion to approximate the convolution integral, we may assert that one can design a scheme that more accurately approximates this problem for certain values of $\alpha$ over the desired meshes. Further, utilizing this Taylor Series expansion argument, we are able to derive a scheme that only requires the function to be in $C^1(0, T]$ for the time variable, which allows for a wider class of functions. We also present a scheme for $C^2[0, T]$ functions that parallels the L1-method, as seen in [1,3,15,16], which has error of $O(k^2)$ in time. This novel result improves over previous results, which guarantee an error of $O(k^{2-\alpha})$ in time for the same regularity assumption.
Appendix A. Existence and Uniqueness of a Solution to \([2]\)

Consider the Hilbert space \(L^2(0, 1)\) and let \(\sigma(A)\) denote the spectrum of the operator \(A = -\frac{\partial^2}{\partial x^2}\) which is a strictly positive self-adjoint operator on the dense subspace \(H^2_0(0, 1)\). The operator valued equation \((\lambda I - A)(X) = 0\) has the solution

\[
(\lambda I - A)(X) = (\lambda X - A(X)) = 0
\]

\[
= X'' + \lambda X = 0
\]

\[
\Rightarrow X_\lambda(x) = \sin(\sqrt{\lambda}x), \text{ with Eigenvalues } \lambda_n = (n\pi)^2.
\]

Now, let

\[
\delta_A = \inf_{y \neq 0, \ y \in H^2_0((0,1))} \frac{\langle Ay, y \rangle}{\langle y, y \rangle} = \pi^2.
\]

It is easy to see that \(a_{1-\alpha}(t) = \frac{\Gamma(\alpha)}{\Gamma(\alpha-1)}\) is positive, decreasing on \((0, \infty)\), and \(a_{1-\alpha} \in C(0, \infty) \cap L^1(0, 1)\). Therefore, we may apply Theorem 4.1 from \([3]\) to see that the operator \(S(t)\) defined as

\[
S(t)x_0 = \int_{\delta_A}^{\infty} S_\lambda(t)dE_\lambda x_0 \quad (x_0 \in L^2(0, 1)),
\]

is the fundamental solution of \([2]\), as defined in \([3]\). Here, \(S_\lambda = S_\lambda(t)\) is the solution of the scalar equation

\[
S_\lambda(t) = 1 - \lambda \int_s^t a_{1-\alpha}(t - \tau)S_\lambda(\tau) d\tau, \quad (24)
\]

and \(E_\lambda\) is the resolution of the identity for \(A\) and because the operator-valued function \(S = S(t)\) is a fundamental solution, \(S \in L^1((0, T]; B(L^2(0, 1)))\), and for almost all \(t \in [0, T]\)

\[
S(t) = I - A \int_0^t a_{1-\alpha}(t - \tau)S(\tau) d\tau
\]

where \(B(L^2(0, 1))\) is the space of all bounded linear operators of \(L^2(0, 1)\) and \(I\) is the identity operator. If \(\phi(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) \in H^2_0(0, 1)\)
then \( \sum_{n=1}^{\infty} |a_n|^2 n^2 < \infty \), and

\[
A\phi(x) = \int_{\delta_A}^{\infty} \lambda dE_\lambda \phi(x) = \sum_{n=1}^{\infty} \lambda_n a_n \sin(n\pi x) = \sum_{n=1}^{\infty} (\pi n)^2 a_n \sin(n\pi x).
\]

Let \( \hat{a}_{1-\alpha}(s) = \mathcal{L}(a_{1-\alpha}(t)) \). Define \( g(s) = s\hat{a}_{1-\alpha}(s) = s(s^{-\alpha}) = s^{1-\alpha} \).

We may calculate \( S_\lambda \) using the following from [3]:

\[
S_\lambda = \mathcal{L}^{-1}\left( \frac{1}{s + \lambda g(s)} \right) = \mathcal{L}^{-1}\left( \frac{1}{s + \lambda s^{1-\alpha}} \right) = \mathcal{L}^{-1}\left( \frac{s^{-1}}{1 + \lambda s^{-\alpha}} \right) = E_\beta(-\lambda t^\alpha),
\]

where \( E_\beta \) is the well known Mittag-Leffler function, \( E_\beta(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1 + n\beta)} \), see Theorem 6.1.1 of [2] for more details. Now, from the above calculations,

\[
S(t)\phi(x) = \int_{\delta_A}^{\infty} S_\lambda(t) dE_\lambda \phi(x) = \sum_{n=1}^{\infty} S_{\lambda n}(t) a_n \sin(n\pi x)
\]

\[
= \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{(-\lambda_n t^\alpha)^m}{\Gamma(m\alpha + 1)} a_n \sin(n\pi x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) \sum_{m=0}^{\infty} \frac{(-\lambda_n t^\alpha)^m}{\Gamma(m\alpha + 1)}.
\]

Then

\[
u(x, t) = S(t)\phi(x) + (S * f)(x, t)
\]

is, in closed form, the unique solution of (2), ensured by Theorem A.

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