Fast and secure key distribution using mesoscopic coherent states of light.

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This work shows how two parties A and B can securely share sequences of random bits at optical speeds. A and B possess true-random physical sources and exchange random bits by using a random sequence received to cipher the following one to be sent. A starting shared secret key is used and the method can be described as an unlimited one-time-pad extender. It is demonstrated that the minimum probability of error in signal determination by the eavesdropper can be set arbitrarily close to the pure guessing level. Being based on the M-ry encryption protocol this method also allows for optical amplification without security degradation, offering practical advantages over the BB84 protocol for key distribution.

INTRODUCTION

Physical cryptography can create schemes providing two users, at distinct locations, with on-demand copies of a secure sequence of random bits of arbitrary length and at fast rates. These schemes could be of high value for commercial systems operating over long distances. Based on physical laws instead of mathematical complexities, communication with perfect secrecy could be guaranteed over an insecure channel in Vernam’s sense of a one-time-pad. Technology advances, therefore, such as enhanced computational power, should not affect the security of these schemes. The BB84 quantum protocol for key distribution [1], the paradigm among protocols of this type, has been in use in short distance applications [2] but not in long distance networks. One fundamental reason is that the same no-cloning theorem that guarantees its security level prohibits the signal amplification necessary in long-haul communication links. No practical alternative quantum scheme using quantum repeaters or entangled states has yet been proposed although theoretical studies exist [3]. Other practical impediments are the slow speed of the photon sources and the large recovery time of single photon detectors.

Recently, Yuen proposed a [4] a ciphering scheme utilizing an M-ry bases system that was implemented for data encryption [5, 6]. Ref. [6] introduces the M-ry scheme and presents its first prototype-level implementation. Ref. [6] gives a more complete description of these systems. Basically, in these cryptographic prototypes, known as $\alpha \eta$ ($\alpha$ standing for coherence and $\eta$ for efficiency) systems, the quantum noise inherent to coherent states forces different measurement results for the eavesdropper and the legitimate users that use a shared key in their measurements. This noise will increase the observational uncertainty preponderantly for the eavesdropper, Eve (E), rather than Alice (A) and Bob (B), the legitimate users. Although this noise is irreducible by nature to all observers, the knowledge of the key allows A and B to discard this noise while it points to the correct information. The very simple idea behind this is that, for each bit, the noise inherent to the stated and generated at the emitter is distributed without control among the output ports in Eve’s measurement apparatus while A and B use the key to select a single output port where the noise does not practically affect bit readings.

In this work a key distribution method is presented that also utilizes an M-ry bases ciphering scheme similar to the one described in [5] for the purpose of data encryption. Each basis in the M-ry set of bases defines two orthogonal states whereas these bases are non-orthogonal among themselves [6]. In these schemes a starting shared secret key is assumed between A and B. The phrase “key distribution” is being used here to denote that one party sends to the other random bits created by a truly random physical process. The exchange of random bits between A and B is done in such a way that the quantum noise of the light does not allow E to obtain the final random sequence shared by A and B. In contrast, a classical key expansion method could mean a process to generate mathematically—e.g., by one-way functions—two identical sets of random bits, one for each user, from a set of shared starting bits. Stream-ciphers, for example, generate a stream of pseudo-random bits from a starting key. However, this deterministic process produces correlations that can be detected by the eavesdropper. Known-plaintext attacks are particularly useful to exploit these correlations in classical cryptography. In the M-ry data encryption scheme, a stream cipher is used to generate the running key and the quantum noise of light protects against the correlations.

The key distribution method presented in this work uses physical sources to guarantee the true randomness of signals. As in the data encryption scheme, the quantum noise of light provides the ultimate basic protection against signal identification. After presenting a set of basic conditions to be obeyed by the system and the physical resources needed for A and B, the key distribution

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protocol will be described step-by-step. Each step will be followed by a brief description of its possible implementation using the described physical resources. Very briefly, these protocol steps describe how A and B succeed in sending new random sequences of bits from one to the other securely through judicious use of the quantum noise of light. This security is achieved by using a correct combination of average number of photons per bit and number of ciphering bases \( M \), as will be shown. The bit encoding mechanism and the associated physical protection will then be discussed and a measure of the minimum probability of error forced by the system on the eavesdropper will be achieved. After showing that the system obeys the established conditions, conclusions will be presented.

**BASIC CONDITIONS**

First, a set of conditions will be defined to specify the boundaries within which the problem has to be solved:

I) The eavesdropper is allowed to have full access to the random signal sequence being generated. Granting full access to the signal should be understood as similar to an opaque attack or, giving Eve a perfect quantum copy of the signal sequence. Anyway, Eve does not need to subtly tap the channel to obtain the signals. Eve could perform arbitrary measurements on this sequence or she could generate as many realistic (imperfect) copies as she wants. The unrestricted access to the signal sequence is the best (idealized) possible condition given to the eavesdropper.

II) Eve samples all signals near the source, such that energy loss does not affect her data.

It will be initially assumed that all parties have similar detectors; the simplest possible assumption would be of noiseless detectors with efficiency 1. However, it will be shown that although the eavesdropper needs high signal resolution to distinguish between two closest bases in the \( M \)-ry system and precision to identify a sent basis, the legitimate users do not need such strict conditions. Therefore, the detectors used by A and B can be less efficient. It will be demonstrated how one can implement a secure key distribution system where the minimum bit-by-bit eavesdropping probability of error can be arbitrarily set at the pure guessing value of 1/2.

As will be shown, the protection of this scheme does not rely on an intrusion detection mechanism, but instead on the measurement advantage enjoyed by A and B over the eavesdropper, thanks to the knowledge of the key.

**THE KEY DISTRIBUTION PROTOCOL**

**Basic physical resources**

The basic resources necessary for implementation of this key distribution protocol are sketched in Fig. 1. Two stations, A and B, are represented where the optical channel can be either free space or a fiber channel. Both sides have identical resources to operate as emitter or receiver. The OM’s are optical modulator systems performing polarization or phase modulation on mesoscopic coherent pulses of light. Each party also possesses a fast speed physical random generator (PhRG) that produces binary outputs \( R \). PCI is a PC controlled interface card that can generate \( M \) voltage levels. PBS is a polarizing beam splitter that is followed by two detectors constituting the detection system in the case of polarized signals. In the case of phase modulation, a phase sensitive detection system should be used.

**The protocol**

Each of the seven protocol steps will be stated briefly (in italics) and for a more complete illustration of the scheme, a short description of one possible way to implement each step will follow:

1. Parties A and B share an initial secret random sequence (of length \( L_0 \)) of bits \( K_0 \).

How A and B will share this initial sequence is, of course, an important matter. Although current cryptography can provide enough security for sharing the short sequence \( K_0 \) at this moment, it may be vulnerable to the evolution of computational power. Just as an example, \( K_0 \) could be obtained in a secure way within a few years through the slow but proved secure BB84 key distribution system. The use of satellites to distribute quantum
keys have been under study (See Ref. [5] and references therein) and developments in this direction can be expected to produce fruits in a near future. Unless proved otherwise, the expected rates of this quantum key distribution will be low. The scheme studied in this paper aims to create a fast distribution rate once a short sequence \textbf{K}_0 have been obtained - even through a slow process.

2. Party \textbf{A} generates a sequence (of length \textit{L}_0) of true random bits \textbf{R}. This sequence of bits \textbf{R} can be obtained from the binary output of the physical random generator (PhRG) as voltages \textit{V}_R = \textit{V}_0 \text{ or \textit{V}_1} that are going to be associated with bits 0 and 1. A possible visualization of such a process could be the voltage outputs \textit{V}_i (i = 1, 2, \cdots) within a short time window \textit{\Delta}t, around \textit{\tau}_i, produced by a fast light detector, shot-noise limited, illuminated by a coherent light beam. The sign \textit{i} of these pulses, \textit{sign}_i = \langle \textit{V}_i - \textit{\overline{V}_i} \rangle / \langle \textit{\overline{V}_i} \rangle, where \textit{\overline{V}_i} is the average pulse voltage, will feed a binary voltage source to provide the random bit sequence \textbf{R}

3. \textbf{A} sends to \textbf{B} the random sequence \textbf{R} (\equiv \textbf{R}_1) of length \textit{L}_0 in blocks of size \textit{K}_M. Ciphering each of these blocks uses \textbf{K}_M bits from \textbf{K}_0. The number of blocks to be ciphered in \textit{L}_0 is \textit{L}_0/\textit{K}_M. A coherent state carrier is used with intensity \langle \textit{n} \rangle / \text{bit}. In order to generate each cipher basis \textit{k} (= 0, 1, \cdots, \textit{M} − 1), \textit{K}_M (= \log_2 \textit{M}) bits are used from the random sequence of bits \textbf{K}_0 (\textit{e.g.}, \textit{k} = \textit{b}2^{\textit{K}_M} + \textit{b}(\textit{K}_M−1)2^{\textit{M}−2} + \cdots b(\textit{K}_1)2^{0}). In other words, each \textit{k} basis of the \textit{M} = 2^{\textit{K}_M} set will be randomly defined by \textit{K}_M bits taken from \textbf{K}_0. Each \textit{k} will be used to cipher a block sequence of size \textit{K}_M from \textbf{R}_1. Ciphering \textbf{R}_1 in blocks of size \textit{K}_M keeps the length of the transmitted bits constant and equal to \textit{L}_0 (See Ref. [11]).

From the experimental point of view, the signals provided by the PhRG and by the running key \textbf{K}_0 define voltage levels to be applied by the PCI to the optical modulator OM. Each voltage \textit{V}_k generated is associated with a specific basis of the \textit{M}-ry scheme. The pulsed mesoscopic coherent state at the input (see Fig. 1) can be seen as a linearly polarized state of light. Orthogonal polarizations define bits 0 or 1. The input pulse is modified by the action of the OM into a state (\textit{e.g.}, elliptically polarized light) \textbf{Y}(\textbf{R}, \textbf{K}_0) that is sent to \textbf{B}. Without the modulation given by \textit{V}_k the output signal would show the sequence \textbf{R} of orthogonally linearly polarized states (bits 0 and 1) on a single basis. The \textit{V}_k modulation converts these signals to a non-orthogonal set of \textit{M}-ry states. A similar line of reasoning applies to phase modulated signals, where phases 0 and \pi provide the two bits.

4. By knowing the sequence of bits \textbf{K}_0, \textbf{B} demodulates the received sequence obtaining \textbf{R}_1.

At the receiving station, by applying the shared key \textbf{K}_0 Bob demodulates the changes introduced by \textbf{A} and reads the resulting true random stream \textbf{R}_1 of orthogonally polarized light states. \textbf{A} and \textbf{B} now share a fresh sequence of random bits \textbf{R}_1.

5. \textbf{Bob} obtains a fresh random sequence \textbf{R}_2 from his PhRG and sends it to \textbf{A}, ciphering the sequence in blocks of size \textit{K}_M. Ciphering bits are taken from the earlier sequence received \textbf{R}_1.

Each sequence of bits, of length \textit{K}_M, from \textbf{R}_1 define the ciphering basis for \textit{K}_M fresh bits in \textbf{R}_2. By knowing \textbf{R}_1, \textbf{A} reads \textbf{R}_2 with perfection. The first cycle is complete.

6. \textbf{A} and \textbf{B} continue to exchange random sequences as described in the first cycle.

Subsequent cycles can be performed and in each cycle, blocks of size \textit{K}_M are ciphered to keep the total length in each cycle constant and equal to \textit{L}_0. \textbf{A} and \textbf{B} can then share sequences of random bits obtained from the PhRGs. A shared random sequence can be used to restart a cycle by \textbf{A} or \textbf{B} whenever an interruption occurs.

7. \textbf{A} and \textbf{B} apply information reconciliation and privacy amplification to distill a final sequence of bits. The process of privacy amplification discards bits in the sequence and, consequently, destroys the short-ranged bit-cipher correlations due to the block ciphering. As the PhRGs present no bit correlation, the final shared random sequence will present a similar statistical property.

These steps describe the protocol without discussing security aspects. However, being a physical protocol, it would be incomplete without specifying \langle \textit{n} \rangle and \textit{M}. These parameters have to be provided, under the initial conditions presented, and a quantitative measure of the security level associated with them has to be derived. This is the subject of the following sections.

**BIT ENCODING AND THE PHYSICAL PROTECTING MECHANISM**

The physical protecting mechanism in this case is the same as that on which the \textit{cay} systems are based. Although it has already been described, with examples, in Ref. [5], it will be presented here and discussed to clarify the security provided by the quantum noise of light to this key distribution system. A bit-by-bit proof, based on a Positive Operator Valued Measured theory (POVM), will follow. The choice of a POVM demonstration relies on its generality once the wave function or the density matrix that represents the physical process is chosen. The resulting analysis carries the information content in the density matrix and has broad validity. This is particularly useful because the eavesdropper should be allowed to use any technology or attack (beam-splitter, cloning, homodyne measurements and so on) and a general protection cannot be based on particular threat models.

The security analysis to be presented covers both polarization and phase modulation of optical signals. In the case of free-space implementation, the coherent states defining each bit are two orthogonal modes of polariza-
tion. In the phase ciphering, two modes separated by a phase of \( \pi \) could be used. In the polarization case the running key specifies a polarization basis from a set of \( M \) uniformly spaced two-mode bases spanning a great circle on the Poincaré sphere. Fig. 2 sketches the \( M \)-ry ciphering protocol as implemented in the \( a \sigma \) systems \[\mathbf{2}\] where closest bits are mostly distinct from each other. In this key distribution scheme, the same \( M \)-ry scheme is utilized. Each basis represents a polarization state and its antipodal state at an angle \( \pi \) from it (bits 0 and 1). The mapping of the stream of bits onto points of the Poincaré sphere is the key to be shared by A and B. It points precisely to the basis being used at each bit emission. Each \( k \)-basis is defined by the Poincaré angles \( \Theta_k \) and \( \Phi_k \). The number of bases \( M \) chosen should be such that the uncertainties caused by the quantum noise of light on the polarization angles leads to a large error. This can be understood in a variety of ways; for example, by directly writing the manifold of two-state \( \{ |\Psi(\Theta_k, \Phi_k)\rangle \} \) bases in Cartesian \((x, y)\) coordinates fixed at the OM physical axes (chosen at 45° from the horizontal) gives

\[
|\Psi(\Theta_k, \Phi_k)\rangle = |\alpha\gamma(\Theta_k, \Phi_k)\rangle_x \otimes |\alpha\delta(\Theta_k, \Phi_k)\rangle_y , \tag{1}
\]

where \( \alpha \) is the coherent amplitude and \( \gamma \) and \( \delta \) are the projections on \( x \) and \( y \).

\[
\begin{align*}
\gamma &= \left[1 + i\right]e^{i\Phi_k/2} \cos(\Theta_k/2) + \left[1 - i\right]e^{-i\Phi_k/2} \sin(\Theta_k/2), \\
\delta &= \left[1 + i\right]e^{i\Phi_k/2} \cos(\Theta_k/2) + \left[1 - i\right]e^{-i\Phi_k/2} \sin(\Theta_k/2).
\end{align*}
\]

For example, on a great circle set by \( \Theta_k = \Theta_0 = \Theta_0 \), the overlap \( \langle \Psi(\Theta_k, \Phi_k)|\Psi(\Theta_0, \Phi_0)\rangle \) between states \( k \) \((\Phi_k = \Phi_0)\) and \( p \) \((\Phi_p = \Phi_0)\) gives

\[
|\langle \Psi(\Phi_k)|\Psi(\Phi_p)\rangle|^2 = e^{-2\langle n\rangle \left[1 - \cos\frac{\Phi_k - \Phi_p}{2}\right]}, \tag{2}
\]

This will define the polarization angle uncertainty produced by the shot noise associated with the coherent states. For large \( \langle n\rangle \) the periodic functions in Eq. 2 can be expanded around \( \Phi_p \), as \( \Phi_k \sim \Phi_p + \Delta \Phi \), giving \( |\langle \Psi(\Phi_k, \Phi_0)|\Psi(\Phi_p, \Phi_0)\rangle|^2 \simeq \exp\left[-\Delta \Phi^2/(2\sigma^2)\right]. \) \( \sigma^2 = 1/\langle n\rangle \) is the uncertainty associated with the Poincaré angle. This uncertainty is directly associated with light’s shot noise and cannot be overcome regardless of one’s precision capabilities. Without knowing the precise basis sent (or angle), \( E \) cannot obtain the bit that is sent. Her measurement of the polarization angle becomes uncertain by the uncorrelated noise \( \mathbf{2} \) in the two axes \((\langle n_1 n_2 \rangle = \langle n_1 \rangle \langle n_2 \rangle) \). It will be shown that this noise can be used judiciously to prevent an eavesdropper from accessing the information while the legitimate receiver \( B \) can control it. This access is given by the knowledge of the key: the legitimate receiver projects the received signal completely onto one of the physical axes of the receiving system (e.g., the PBS in Fig. 1) and this way the associated noise becomes irrelevant to his binary determination (See Refs. \[\mathbf{2}\] for experimental results).

Receiver \( B \) can even support moderate misalignments of his bases system because whenever most of the light falls into one of his detectors this would indicate the correct bit. In contrast, for Eve, apart from the uncertainty caused by the noise, even a small misalignment will give her an incorrect basis. Furthermore, her measurement system needs high resolution and precision to obtain reliable data for analysis. The number of bases \( N_\sigma \) within \( \sigma \) is \( N_\sigma = M\sigma/\pi = M/(\pi\sqrt{\langle n\rangle}) \). The system should be designed, as it will be shown, such that \( N_\sigma \) covers a reasonable number of adjacent bases.

Phase modulation of the signals can be utilized by creating two pulses delayed by a fixed amount of time and introducing a phase difference \( \phi_i \) between them to represent bits 0 or 1 (e.g., \( \phi_0 = 0 \) and \( \pi \)). An extra phase difference \( \phi_i \) is provided by the \( K_M \) shared bits. At the receiver, these pulses can be made to interfere and by subtracting the phase \( \phi_i \), \( B \) can recover each random bit sent. Formally, this phase encoding could be written starting from a coherent state \( |\alpha\rangle \) that is split into a two-mode coherent state \( |\Psi_0\rangle = \alpha/\sqrt{2}|1\rangle \otimes |\alpha/\sqrt{2}\rangle_2 \). Bit encoding using the two-mode state, represented by annihilation operators \( a_1, a_2 \), can be done by

\[
|\Psi_b\rangle = e^{-i\phi_i\sqrt{2}}|\Psi_0\rangle = |e^{-i\phi_i\sqrt{2}}\frac{\alpha}{\sqrt{2}}|_1 \otimes |e^{i\phi_i\sqrt{2}}\frac{\alpha}{\sqrt{2}}|_2, \tag{3}
\]

where \( J_z = (a_1^\dagger a_1 - a_2^\dagger a_2)/2 \). This phase modulation can also be interpreted as a relative one, with the zero reference taken at one of the states . A crucial ingredient in the security demonstration is that the modulation operations have to be unitary or energy conserving. In this way, the input energy associated to each pulse will have to be distributed between the two modes. Precise information about the energy content in each mode is not needed, but one is assured that all energy is being accounted for in the demonstration. Although losses are unavoidable in real systems, this condition also reflects the fact that technical losses are expected to decrease with advances in technology and so they can be considered asymptotically negligible. Therefore, for a modulation system that is not energy conserving in principle, the following demonstration does not apply.
In the phase modulation case, one can associate an index \( \nu \), in general, to the ciphering angle \( \phi_\nu \) to represent a possible applied modulation. This index \( \nu \) could represent a discrete or a continuous variable determined by a general distribution. In this \( M \)-ry scheme \( \nu = k \).

A ciphered bit in the two-mode state will be written

\[
|\Psi_{b\nu}\rangle = e^{-iJ_k(\phi_\nu+\phi_\nu)}|\Psi_0\rangle
\]

\[
= |e^{-i(\phi_\nu+\phi_\nu)/2\alpha/\sqrt{2}}| e^{i(\phi_\nu+\phi_\nu)/2\alpha/\sqrt{2}}\rangle, \quad (4)
\]

where \( \phi_\nu = (0, \pi) \) specifies the bit being “sent” and \( \phi_\nu \) is the ciphering phase. The overlap of \(|\Psi_{b\nu}\rangle\) and \(|\Psi_{b\mu}\rangle\) leads to an equation similar to \( (2) \).

EAVESDROPPER’S MINIMUM PROBABILITY OF ERROR

To show that this key distribution scheme is secure two basic points have to be demonstrated:
1) For a fresh bit sent, the minimum probability of error \( P_e \) that an eavesdropper can achieve in the bit determination must be guaranteed to be arbitrarily close to 1/2.
2) The use of a given random sequence \( 2 \times 2 \times K_M \), one time as a “message” and as a cipher for the fresh random sequence, still allows one to set \( P_e \to 1/2 \).

As a starting point for the first part of the demonstration, the density matrix \( \rho \) for all possible two-mode states resulting from ciphering a bit \( b \) is written as

\[
\rho_b = \frac{1}{L} \int_0^L P_{\phi_\nu}|\Psi_{b\nu}\rangle \langle \Psi_{b\nu}| \, d\nu,
\]

where \( L \) is the space spanned by \( \nu \) and \( P_{\phi_\nu} \) describes a general phase distribution. The optimal POVM for discriminating between \( \rho_0 \) and \( \rho_1 \) (or \( \Delta \rho = \rho_1 - \rho_0 \)), in the polarization case, was first applied in Ref. [6].

Calling \( \Pi_1 \) and \( \Pi_0 \) (\( \Pi_1 + \Pi_0 = 1 \)) the projectors over eigenstates with the positive and negative eigenvalues of \( \Delta \rho \), the probability of error \( P_e \) is

\[
P_e = \text{Tr}[\Pi_1 \rho_1 + \Pi_0 \rho_0],
\]

where \( \rho_1 \) and \( \rho_0 \) are \( a-priori \) probabilities to find a state in \( \rho_1 \) or \( \rho_0 \), respectively. \( P_e \) defines the minimum probability of error that is caused by a wrong choice of bases by Eve when she tries to determine a bit sent. Of course, error levels higher than the one given by Eq. \( (6) \) can be found but the interest here is to find Eve’s lower bound of error in a bit-by-bit determination.

\( P(\phi_\nu=k) \) randomly establishes the index \( k \) associated with discrete phase values \( \phi_\nu \) in the ciphering wheel shown in Fig. \( 2 \), where adjacent bits to a given \( k \) are mostly distinct from the \( k \)-th bit. For this implementation the location of the two-state bases are given by

\[
\phi_k = \pi \left[ k \frac{1 - (-1)^k}{M} \right], \quad k = 0, 1, ..., M - 1.
\]

For equal \( a-priori \) probabilities \( p_1 = p_0 = 1/2 \), Eq. \( (6) \) reduces to

\[
P_e = \frac{1}{2} \text{Tr}[\Pi_0 \rho_1 + \Pi_1 \rho_0] = \frac{1}{2} \left( 1 - \text{Tr}[\Pi_1 \Delta \rho] \right) = \frac{1}{2} \left( 1 - 2 \sum_j \lambda_j \right), \quad (8)
\]

where \( \lambda_j \) are the positive eigenvalues to be obtained from

\[
\Delta \rho = \frac{1}{M} \sum_{\nu=k=0}^{M-1} e^{-iJ_k \phi_\nu} (|\Psi_1\rangle \langle \Psi_1| - |\Psi_0\rangle \langle \Psi_0|) e^{iJ_k \phi_\nu}. \quad (9)
\]

Eq. \( (9) \) can be expanded as

\[
\Delta \rho = \sum_{q=-\infty}^{\infty} \sum_{q'=-\infty}^{\infty} \Delta \rho_{q,q'} (|\Phi_q\rangle \langle \Phi_q| - |\Phi_{q'}\rangle \langle \Phi_{q'}|), \quad (10)
\]

where

\[
\Delta \rho_{q,q'} = -2ie^{-|\alpha|^2} \sqrt{I_{2|q|}(|\alpha|^2) I_{2|q'|}} \text{sin}[(q'-q)\pi/2] e^{i(q'-q)\pi/2} \frac{1}{M} \sum_{k=0}^{M-1} e^{i\phi_k(q'-q)}, \quad (11)
\]

From the positive eigenvalues of Eq. \( (11) \), the minimum probability of error, Eq. \( (8) \), can be calculated.

Eq. \( (8) \) can be expanded in several ways, the adopted expansion uses the angular momentum basis \( |J,q\rangle = |J - q\rangle \otimes |J + q\rangle \), that is a natural basis to deal with angular rotations.

Assuming that \( k \) values have uniform probability of occurrence one can show that the number of \( a-priori \) probabilities for the number of occurrence of even- \( k \) or odd- \( k \) lines, given \( M \), is

\[
p_{\text{even}-k}(M) = \frac{1 - (-1)^M + 2M}{4M},
\]

\[
p_{\text{odd}-k}(M) = \frac{-1 + (-1)^M + 2M}{4M}. \quad (14)
\]

For simplicity and without loss of generality, let us adopt bases even in \( M \), where \( p_{\text{even}-k}(M) = p_{\text{odd}-k}(M) = 1/2 = p_1 = p_2 \), to show numerical examples. Figure \( 8 \) shows the minimum probability of error as a function of the number of ciphering levels \( M \). \( P_e \) goes very fast to the asymptotic pure-guessing limit of \( 1/2 \) as \( M \) increases. It is then shown that the minimum probability of error \( P_e \to 1/2 \), at a fixed average number of photons \( |\alpha|^2 \), can be achieved by increasing the number of bases.
FIG. 3: \( P_E^r \) as a function of \( M \) for \(|\alpha|^2 \equiv \langle n \rangle = 1, 10, 100, \) and 1000.

\( M \) adequately. This demonstrates that in this scheme an eavesdropper cannot obtain the individual bits sent, regardless the precision of her devices. The physical origin causing this impossibility for Eve to obtain an angle \( \phi_k \) and therefore the associated bit, rests in the source emission itself. Although a deterministic angle can be applied to the modulator the resulting light output is probabilistic in nature and in single shot measurements with mesoscopic states the resulting field does not carry this applied angle \( \phi_k \) but presents a distinct \( \phi'_k \). This is a Nature’s fact that cannot be changed regardless the measurement applied (homodyne, heterodyne, and so on). This completes the first part of the demonstration.

For the second part of the demonstration, one has to show how repetitions of the cipher to encode the distinct bits generated in the random process increase the resolution achievable by the eavesdropper over the signal sent. One should recall that cipher repetition are used in the block ciphering described in step 3 of the protocol. As discussed in \([11]\), these repetitions are not necessary because the ciphering procedure can be randomized bit-by-bit through use of a stream cipher using the \( K_M \) sequences as seed keys.

One could ignore this perfectly possible randomization an calculate a much more drastic case, one where both cipher and bit were repeated \( r \)-times. This can be seen as an overestimated upper bound for the actual situation because each random bit sent is a fresh bit in the sequence being sent. As photon numbers in distinct coherent pulses of same amplitude fluctuates in an uncorrelated way, in \( r \)-repetitions of a signal the resolution achieved for extraction of this signal increases with the number of repetitions. This is quantitatively obtained from the \( r \)-product of Eq. \([2]\): 

\[
P(r; k|p) = P(k|p)^{\otimes r} \propto \exp \left[ -r \Delta \Phi^2/(2\sigma^2) \right].
\]

This Gaussian process gives the standard deviation \( \sigma' = \sigma/\sqrt{r} \) associated with the angle uncertainty in a measurement process. This uncertainty \( \sigma' \) is equivalent to the one obtained from a single shot measurement with the photon number \( r\langle n \rangle \). In other words, a single shot using \( r \)-times the laser power will give the same signal resolution for a bit reading as the \( r \)-repeated sequence with \( \langle n \rangle \). Consequently, for a fixed \( M \), the \( r \)-repetition of the random sequence then reduces \( P_E^r \) from \( P_E^r(\langle n \rangle) \) to \( P_E^r(r\langle n \rangle) \).

The dependence of \( P_E^r \) can be calculated as a function of \( \langle n \rangle \) and \( M \) for arbitrary numbers. Therefore, the system can be designed to a desired security level \( P_E^r \); through the correct choice of \( \langle n \rangle \) and \( M \). As a numerical example consider, say, \( M = 32 \) (or \( K_M = 5 \) bits) with \( \langle n \rangle = 100 \) to achieve \( P_E^r = 0.476 \) in a single shot (see Fig. 3). To guarantee the same security level \( (P_E^r = 0.476) \), due to the \( r = 2 \times K_M = 10 \) repetitions, one should use \( M = 90 \) \((K_M \sim 7) \) corresponding to \( \langle n \rangle = 10 \times 100 = 1000 \). The conclusion is general regardless of the specific numerical example. Proper scaling can be done for other intensity levels adequate for the sensitivity of the detection system. Although this is an overestimated calculation it is adequate for our purposes to show that the protection level can be increased according it is needed. The alternate encryption described in \([11]\) reduces this overhead substantially because each level used to cipher each bit is close to have occurred in a truly random.

It has been shown that the transmission stages A→B and B→A can be made secure under individual bit attacks. The quantity \( P_E^r \) can be connected to the bit-error-rate probability and entropy measurements such as mutual information or relative entropy, can be directly derived from it.

The following section discusses some aspects of attacks on this scheme. As there are no ciphertext or known-plaintext involved in the transmission of random bits, attacks on the transmitted random sequences to obtain the key \( K_0 \) have to start considering a guessing probability of 1/2 for each bit or 1/2\(^L_0\) for any complete sequence sent. Considering that for each new bit sent the random noise may produces \( \sim N_\sigma \) possible outcomes in a measurement process, the number of possibilities to be considered grows exponentially depending on \( L_0 \) and \( N_\sigma \). This indicates that an unsurmountable computational problem would occur for a large number of bits sent.

### EVE’S RECORD, CORRELATIONS AND DRAWBACKS

Next, one has to show that the security level calculated also holds under the basic conditions already defined: I) The eavesdropper is allowed to have full access to the random signal sequence being generated. II) She would work near the source to avoid signal losses.
Bit-cipher block correlations

In general, any degree of correlation in a bit sequence can be explored by Eve to decrease her degree of uncertainty. While use of true random numbers eliminate intrinsic correlations in the generation process the $K_M$ block cipher utilized introduces a short range bit-cipher correlation. The increased number of levels $M$ utilized was aimed to increase Eve’s bit-by-bit probability of error to the guessing levels even under the block ciphering used. Although Eve cannot obtain the random bits sent from the measured signals one may argue that some correlation may be detected due to the bit-cipher correlation. As one can see from the given numerical examples, the condition $N > K_M$ can be easily applied to the communication scheme with mesoscopic states. This condition assures that the uncertainty produced by the noise overcomes the amount of knowledge associated to these correlations. In a linear algebraic system it corresponds to a number of unknowns larger than the number of available equations. Nevertheless, information reconciliation and privacy amplification distill a random sequence about which Eve has a negligible amount of information and also has the additional effect of destroying the short bit-cipher correlations. Basically, the fact that the information known by the legitimate parties differs from that obtained by the eavesdropper is what allows A and B to achieve the secrecy goal.

Homodyne and delay line

Homodyne and heterodyne techniques can be also utilized by Eve to obtain the signals with better precision than a direct detection measurement. With knowledge of the key sequence Bob and Alice always utilize the proper quantum measurement basis for their optimal binary detection, assuring a resolution superior to the one obtained by Eve. In principle, if the seed key is available to Eve, after her records were created, she could apply rotations over the recorded signals to obtain the correct sequence. This has to be performed in a sequential operation due to the lack of structure in the PhRGS generation of random numbers. This differs from performing a mathematical operation with a deterministic function to obtain a result $x_{n-1}$ given $x_n$, up to $x_0$. Nevertheless, assuming that this inversion is possible in principle, Eve could obtain the random sequence once the starting key is available. This will be equivalent to a futuristic optical delay line that could be tapped on demand and Eve could wait as long as necessary until the shared starting key is made available to her and only then perform her measurements perfectly mimicking Bob’s measurement over her copy of the signals. With the key, E does not need the same resolution as before and the applied rotations would lead her close enough to the correct axis orientation and to bit identification. Therefore, the shared starting key has to be protected at all times. This is the fundamental drawback of this system, although it is not important in the current state of technology. In essence, assuming a protected starting key, this method can be described as a one-time-pad extender.

The futuristic delay line discussed led to the protection of the starting key at all-times because there are no in-principle impediments for creation of such a device. One could also invoke a perfect photon number cloning so that Eve could obtain multiple perfect copies of each signal sent. Perfect copies would allow Eve to obtain the correct bit sequence but perfect cloning violates the no-cloning theorem. Realistic amplification schemes with linear optical amplifiers introduce spontaneous noise that decreases Eve’s signal resolution. Imperfect cloning using realistic number distribution consisting of mixtures of stimulated and spontaneous thermal processes are akin to linear optical amplification and degrades signal resolution.

Phase measurement bounds

Even without reference to a particular setup, it is worth comparing the obtained numerical results with the bounds on phase measurements imposed by the smallest detectable phase shifts usually considered (See 

The phase interval of interest to probe for the best resolution region where the phase measurements lapse below shot noise condition. In the case of interest in this paper, each signal pulse will be set randomly in one basis chosen among $M$ bases. The minimum interval between bases is $\Delta \phi_{\text{min}} = \pi/M$. A uniform scan on the semicircle ($\pi$) with maximum efficiency will find the particular basis sent within a fraction $\Delta \phi_{\text{min}} / \pi = 1/M$ of the pulse containing $N$ photons. One could therefore impose the condition for indistinguishability of the sent phase using the lowest limit given by $\Delta \phi_H$:

$$\Delta \phi_H > \Delta \phi_{\text{min}}.$$  \hspace{1cm} (15)

Writing $\Delta \phi_H$ to take into account the optimized fraction of the pulse, one has

$$\Delta \phi_H = \frac{1}{(1/M)N},$$  \hspace{1cm} (16)

and therefore

$$M > \sqrt{\pi N}.$$  \hspace{1cm} (17)

This condition is satisfied under the conditions exemplified in Fig. 1 in agreement with obtained results.
Speed costs and other aspects

The main cost for the security obtained in this scheme is the $M$-ry number of levels needed. An increased number of levels demands an increased number of bits to provide the necessary resolution and a wider dynamic range for the waveform generators to provide such modulations at high speeds. For example, in case one assumes the overestimated $2K_M$ repetition this will decrease the bit output rate from, say, 10GHz, to 10GHz/$2K_M$. For a nonrepeated cipher, if one uses $M = 1000$ ($K_M \sim 10$), and $\langle n \rangle = 10^4$, the number of levels covered $N_\sigma = M/\langle \pi \sqrt{\langle n \rangle} \rangle \sim 3.18$. To keep the same number of levels $N_\sigma$ covered under the $2 \times K_M$ repetition, equivalent to a single shot with the intensity $2 \times K_M$ higher, the number of levels necessary is $M_{\text{new}} \sim 4472$. This reduces the speed to 10GHz/$2K_{M\text{new}} = 0.4$GHz from 10GHz. Although this large increase in the number of levels is not necessary, it emphasizes the overhead involved when one increases the number of levels $M$.

Exploratory physical attacks by the eavesdropper, such as injection of a strong signal to detect the weak reflections from the surface of the OM modulator, and from these to obtain the modulation applied, can be easily detected by signal splitting. Specific analysis for the physical attacks could be applied on a case-by-case basis. The robustness of the signals under signal jamming by an enemy, for example, may be of interest for some applications. In this case, one could superpose on the ciphering levels phase and amplitude modulations and even utilize emission at distinct wavelengths to provide a set of conditions that the legitimate parties could use to extract the signals. Again, specific issues require case-by-case responses. These questions are not related to the security aspects of interest here. In the same way, computer attacks [10] and general attacks outside the optical channel are not the focus this work.

Signal amplification

The presented key distribution scheme aims to defeat Eve’s actions at the source, where no losses occurred. Amplification processes always degrade signal resolution for Eve or Bob. As Eve is already defeated at the source, she cannot obtain any improvement through amplification. On the contrary, the knowledge of the key allows Bob to distinguish signals as long as he has a good signal-to-noise ratio. Therefore, amplification is possible for the legitimate receivers because they need a smaller resolution degree than Eve. It works as long as A and B can identify signals in orthogonal bases. A and B apply simple binary decisions to distinguish between orthogonal bases while Eve, on the other hand, needs high resolution to distinguish between adjacent levels of the $M$-ry scheme. This is why an increased noise for A and B due to the amplification process does not have the same effect on Eve’s measurement. With the signal protected at the source, noise created by amplifiers can be acceptable for A and B until their error bit rate exceeds a toleration level.

Numerical simulations indicate that A and B can utilize amplifiers up to distances of $\sim 500$km before signal regeneration becomes necessary. These simulations consider the spontaneous decay in amplifiers onto the mode to be amplified as well as onto the mode orthogonal to the carrier. These added noise sources degrade signal resolution for the users and increase the bit error rate. Other error sources exist such as acoustic and thermal fluctuations but they occur in a much slower time scale.

It is important to observe that any advantage obtained by A and B over E, at the source, leads to an increased communication distance for A and B. Although B can support losses as long as this system presents a low bit error rate, his superiority over Eve decreases and ceases at the moment where Eve can have an equal or larger amount of information than Bob [17]. These differences can be estimated by calculating $P_e^A(0)/P_e^B$, where $P_e^A(0)$ is Eve’s error probability at the source and $P_e^B$ is calculated after the amplification stages. Ideally, $P_e^A(0)/P_e^B \gg 1$. Therefore, any randomization that can be further introduced by Alice at the source amplifying Eve’s uncertainty will reflect as an increased range for secure communication [17]. Randomization can be increased by several means; however, increasing the randomization level usually has a cost associated to the process that has to be weighted with respect to the gain to be achieved.

Summarizing, signal amplification is essential for Internet and this key distribution scheme can be tailored for some of these applications.

MUTUAL INFORMATION

The concept of mutual information $I(X : Y | X)$, concerning the amount of information on X giving the observable Y, allows one to extract basic information on this key distribution system. The minimum probability of error for Eve, bit-by-bit, derived in Eq. 4, $P^E_e$, gives the bit-error-rate for Eve in the binary entropy $H(R | Y_e | R_e)$. The mutual information $I_e(R | Y_e(R)) = H(R) - H(R | Y_e(R)) = 1 + P^E_e \log_2 P^E_e + (1 - P^E_e) \log_2 (1 - P^E_e)$ describes the amount of information Eve could obtain in a bit-by-bit attack on $R$. Fig. 4 shows this dependence as a function of M and some values of $\langle n \rangle$. $H(R) = 1/\text{per bit}$ (or $L_0$ for a perfect random stream of length $L_0$). Although an individual bit-by-bit attack is only a limited attack from Eve it shows the increasingly difficulty the attacker finds as a function of the variables used.

The presented key distribution process intends to perform a distribution between two users of a fresh sequence
of random numbers generated by one or more PhRG’s. While the final shared sequence is truly random, as it was thoroughly discussed, each sequence was obtained using the former sequence to cipher it. The whole process was then dependent on the secrecy of the first key sequence $K_0$ (of length $L_0$). As long as one chain in the sequence is not compromised, the whole sequence results secure. In other words, although A and B may share an arbitrarily long sequence of random numbers, the security of the process relies in a single element of the chain. This is distinct from processes described as fresh key generation in the conventional nomenclature where one aims to create keys that are unconditionally secure even after the starting key is compromised. This goal has not yet been achieved outside from single photon protocols and it is a constant matter of study \[15\].

This intuitive dependence of the security of the chained sequence of keys on a single element of the chain can be formalized as follows. Using the definition of mutual information utilized above,

$$I(R : Y(R)) = H(R) - H(R|Y(R)),$$  \hfill (18)

and writing the difference between the mutual informations between B and E at each cycle of length $L_0$:

$$\Delta I = I_B - I_E = [H(R) - H(R|Y_B(R))] - [H(R) - H(R|Y_E(R))] = H(R|Y_E(R)) - H(R|Y_B(R)).$$  \hfill (19)

From the fact that, at best, $H(R|Y_B(R)) = 0$ and $H(R|Y_E(R)) = L_0$ in the first cycle, one sees that $\Delta I \leq L_0$, what shows that the uncertainty may be kept at the first sequence $L_0$ with a level given by $L_0$ itself. The conditions that Bob reads without errors the random numbers sent $(H(R|Y_B(R)) = 0)$ in a sequence of length $L_0$ and that the secrecy of the shared key for Eve is perfect $(H(R|Y_E(R)) = L_0)$ are basic and reasonable assumptions. To complete the derivation, one may write the chaining rule

$$H(L_1, L_2, \ldots, L_n|Y_E) = \sum_{i=1}^{n} H(L_i|Y, L_1, \ldots, L_{i-1})$$

and

$$H(L_1|Y_E) + H(L_2|Y_E, L_1) + H(L_3|Y_E, L_1, L_2) + \cdots (20)$$

One can see from this rule that in the first round where $L_1$ (or $\{R_1, \cdots, R_{K_M}\}$) is ciphered with $K_0$, unknown to Eve, Eve’s entropy knowing $Y_E$ is $H(L_1|Y_E) = K_0$, as long as no information on $K_0$ is leaked to her. However, in case Eve knows $L_1$ and $Y_E$ she could know $L_2$, that is to say $H(L_2|Y_E, L_1) = 0$ because this provides her with the same information that Bob has, and so on with the subsequent terms. Therefore $\Delta I = K_0$ or, in other words, Eve’s uncertainty is $K_0$ (length $L_0$). In case Eve is unable to obtain $L_1$, $L_2$, this result does not apply, of course, and Eve’s knowledge on the whole sequence of transmitted random numbers is null. This analysis stands for any length $L_0$, that has to be tailored according one needs. For $L_0 = 10^9$, as an example, implies that to break the sequence Eve has to order correctly $10^9$ bits, what gives her a probability for success of $1/(2^{10^9})$. Privacy amplification steps further decreases $P_e^{K_0}$ as well as destroys short range correlations in original random streams. Even periodic replacements of the starting keys can be introduced and the whole process can be tailored to any advances in computational capabilities.

Quantum noise

One should understand that the inherent quantum noise in the system sometimes invalidates conventional techniques one could use in a noiseless system. For example, in the case of a classical $M$-ry ciphering, or ciphering with intense coherent signals, Alice could send repeatedly a given bit $r$ to Bob fixed in any basis of the $M$-ry detection system. Bob measures successively $r_1 = r, r_2 = r, r_3 = r, \ldots$. The following properties then holds: 1) Any pair of $r$ obeys $r \oplus r = 0$. Therefore, 2) If a message bit $x$ is ciphered as $y = x \oplus r$ by Alice, Bob is able to recover $x$ using any obtained $r$ by performing $y \oplus r = x$. This holds because $r$ is obtained noiseless and, therefore, $r \oplus r = 0$. Classical one-time pad applications are “noiseless”. Quite distinctly, when using a mesoscopic coherent state, a repeatedly sent bit $r$ will be read by a receiver that does not know the basis to be used, as $r_1 = r + \Delta_1, r_2 = r + \Delta_2, \ldots$, where $\Delta_j$ express the effect of the noise in the channel. In this case, 1) Any pair of $r$ obeys $r_i \oplus r_j = \Delta_1 + \Delta_2, \ldots, r \oplus r = 0$; 2) If a message bit $x$ is ciphered as $y = (x \oplus r)_i$ by Alice and Bob has $r_j$, he obtains $y \oplus r_j = x + \Delta_i + \Delta_j$ instead of $x$. This only stresses that the knowledge of the bases used to transmit the random sequences or keys is a crucial step for this key distribution system. One should realize that the channel noise, inherent to the light field, has its source at the emitter itself, ignoring all technical noises eventually present.
CONCLUSIONS

It has been demonstrated that a fast key distribution scheme between two stations can be implemented where the practical physical limitations are set by the speed of the electro-optic modulators and acquisition electronics available through current technology. A secure random sequence obtained can be utilized in “Vernam’s one-time-pad” sense, for applications that demand unconditional security. Fundamentally, the system allows for signal amplification, as in the αη systems. The system works as long as the receiver has a good signal-to-noise ratio after the last amplification stage and the occurred losses do not give him an information level worst than Eve’s. The possibility of amplification paves the way for long distance key-distribution protocols protected by the quantum noise of light, offering a practical advantage over single-photon protocols.

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