Bosonic superfluid transport in a quantum point contact

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We present a microscopic theory of heat and particle transport of a weakly interacting, low temperature Bose-Einstein condensate in a quantum point contact. We show that the presence of gapless phonon modes in the condensate yields a contact resistance at zero temperature and a corresponding nonzero DC conductance. This originates from the tunneling process that condensate elements are coherently converted into phonon excitations and vice versa, yielding a linear current-bias relation. As a consequence, we predict zero thermopower and Lorenz number at zero temperature, a breakdown of the bosonic Wiedemann-Franz law. These effects are found to dominate the transport properties up to temperatures of the order of the chemical potential, with a leading power law temperature dependence behavior for heat transport coefficients. The consequences on heat and particle transport measurements in bosonic two-terminal setups should be readily observable in existing experiments.

A mesoscopic system connected to reservoirs, and its description within the Landauer paradigm, is one of the most important example of nonequilibrium quantum many-body physics \cite{1}. For a long time, such a system has mainly been discussed in electron systems, which revealed nontrivial outcomes absent in bulk systems such as conductance quantization \cite{2}, current noise \cite{3}, and fluctuation relation \cite{4}. Recently, atomic quantum gases emerged as an alternative to investigate mesoscopic physics \cite{5,6} in particular single-mode quantum point contacts (QPC) \cite{7}. QPCs are the corner stone of mesoscopic quantum devices, as building blocks for quantum coherent devices such as quantum dots and interferometers. Their counterparts in atomic gases opens the perspective of complex, quantum coherent ’atom-tronic’ devices \cite{8,11} featuring controlled interactions \cite{12} and the possibility to use fermionic or bosonic statistics.

In contrast with superconductors which are charged, atomic quantum gases are charge neutral. As a result, in the superfluid phase, there always exist gapless collective modes which are not present in charged systems due to the Anderson-Higgs mechanism \cite{13,14}. The fact that the dominant low-energy excitations of superconductors are not collective but of pair-breaking type has spectacular consequences such as multiple Andreev reflections (MARs) in superconductor-normal interfaces \cite{15} or QPCs \cite{16}, yet little is known concerning the role played by gapless collective excitations for quantum transport in charge-neutral superfluids. In the case of a Bose-Einstein condensate (BEC), relevant to two-terminal systems of cold atoms \cite{6,17}, as well as superfluid helium in nanopores \cite{18}, a hydrodynamic description correctly predicts the Josephson dynamics \cite{19,22} and dissipation associated with topological excitations in wide junctions, but fails in the case of a QPC.

In this Letter, we present a microscopic theory of transport of interacting neutral bosons through a single-mode QPC. We calculate the heat and particle currents using a combination of the tunneling Hamiltonian and Bogoliubov theory of interacting condensates \cite{23,28}. This theory accounts for the crucial contribution of the gapless phonon-like Bogoliubov excitations in the reservoirs, which were discarded in previous approaches \cite{29,32}. As a result, we predict a DC current obeying Ohm’s law even at zero temperature with a fully superfluid system, with an associated conductance which is not quantized and can reach values much larger than the conductance quantum. The associated Seebeck coefficient and Lorenz number are zero at absolute zero, with cubic and quadratic leading temperature dependences, respectively. These findings are in sharp contrast with the case of charged superconductors \cite{15}, as well as with effective hydrodynamic models of superfluid transport at finite temperature \cite{30} or one-dimensional reservoirs models \cite{33}. Accounting for the thermodynamics of harmonically trapped BECs, we predict the experimental consequences for two-terminal bosonic systems.

\textit{Model.}— We consider a two-terminal transport system, with two macroscopic reservoirs \cite{34} filled by a weakly-interacting BEC and connected by a mesoscopic channel, as shown in figure 1. We describe the channel as a single mode QPC, which is adequate when its width and lengths are much shorter than the coherence length of the BEC. In such a case, the details of the motion of atoms in the constriction become irrelevant \cite{16,35}, which allows one

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1}
\caption{(a) Model used for the calculation, with $T_i$, $\mu_i$ the temperatures and chemical potential in reservoir $i$, $t$ the transmission amplitude for single atoms across the contact and particle and entropy currents $I_i$ and $I_2$ respectively. (b) Mechanism leading to a finite DC conductance at zero-temperature: a coherent tunnel event is accompanied by phonon emission in the reservoirs.}
\end{figure}
to start with the following tunneling Hamiltonian [23–25 36, 37].

\[ H = H_L + H_R + t [\hat{\psi}_L^\dagger(0)\hat{\psi}_R(0) + \hat{\psi}_R^\dagger(0)\hat{\psi}_L(0)] , \]

where \( H_L (R) \) and \( \hat{\psi}_L (R) (x) \) are the grand Hamiltonian of a Bose gas and field operator of the reservoir \( L (R) \), respectively. The last two terms in Eq. (1) describe exchanges of particles between the reservoirs, where \( t \) is the tunneling amplitude. In the two-terminal system, the particle and heat current operators can respectively be expressed as \( I_1 = -\hat{N}_L \) and \( I_2 = -\hat{N}_R \), where \( \hat{N}_i \) is the number operator in the reservoir \( i \) and we set \( h = k_B = 1 \) if not otherwise specified.

We consider a regime of weak interaction and high degeneracy, and describe the BECs in the reservoirs with the Bogoliubov theory [38, 39]. To construct a theory of heat and particle transport, we adopt the Keldysh formalism [40, 41], allowing for a description of the dynamics in the presence of arbitrary bias and tunneling amplitude. Under fixed chemical potential bias \( \Delta \mu \) and temperature bias \( (\Delta T) \), the currents at time \( \tau \) can be written as [42]

\[ I_q(\tau) = I_0 \delta_{q1} \sin \phi(\tau) + \sum_n c^{2n} \sin(\sigma n) I_{n,q}, \]

\[ I_{n,q} = \sum_m \int d\omega \pi \text{Re} \left[ \omega^{n-1} \hat{g}^R_{L}(\omega)\hat{T}_{A}^R(\omega)\hat{g}^R_{R}(\omega + m\Delta\mu)\hat{T}_{A}^L(\omega) + (\omega - \Delta\mu)^{n-1} \hat{g}^R_{L}(\omega)\hat{g}^R_{R}(\omega + m\Delta\mu)\hat{T}_{A}^L(\omega)\hat{g}^R_{R}(\omega + m\Delta\mu) \right], \]

where \( \phi(\tau) = \Delta\mu \tau, \hat{g}^R_{L(R)}, \hat{g}^A_{L(R)}, \) and \( \hat{g}^L_{L(R)} \) are 2 × 2 representation of retarded, advanced, and lesser Green’s functions in the reservoir \( L(R) \), which are conventionally employed in a BEC system [40, 13]. Here, \( I_0 = -\frac{2\mu}{\tau} \) with the chemical potential \( \mu \) and the interaction coupling constant \( g \) describes the contribution by the condensate, which can also be obtained through the Gross-Pitaevskii equation. It is responsible for the well known Josephson-plasma oscillation occurring under a small chemical potential bias [19, 21], but it does not contribute to the heat current.

The higher harmonics \( I_{n,q} \) contain all contributions from the excitations within the reservoirs, which depend on the renormalized tunneling matrices \( \hat{T}_{A}^R(\omega) \), satisfying the following relations:

\[ \hat{T}_{A}^R(\tau, \tau') = \sum_n \int \frac{d\omega}{2\pi} e^{-i\omega \tau + i(\omega + n\Delta\mu)\tau'} \hat{T}_{mn}^R(\omega), \]

\[ \hat{T}_{A}^R(\tau, \tau') = i\overline{\lambda}(\tau)\delta(\tau - \tau') + \int d\tau_1 d\tau_2 \overline{\lambda}(\tau)\hat{T}_{mn}^R(\tau - \tau_1)\times\hat{T}_{mn}^R(\tau_1 - \tau_2)\hat{T}_{A}^R(\tau_2, \tau'), \]

with \( \hat{T}_{mn}^R(\omega) \equiv \hat{T}_{A}^R(\omega + m\Delta\mu, B + n\Delta\mu) \) and \( \overline{\lambda}(\tau) = t \left[ e^{-i\phi(\tau)} 0 0 e^{i\phi(\tau)} \right] \). As in superconducting contacts [23, 24], the renormalized tunneling matrix represents the coherent sum of all processes by which an atom can traverse the channel. However, due to the absence of a gap in boson Green’s function, the behavior of \( \hat{T}_{A}^R(\omega) \) and currents in a Bose gas are distinct from ones in fermionic superconductors. To focus on effects of phonon contributions directly related to symmetry breaking, below, we use the phonon approximation for Green’s function [42, 44]. Results—We calculate the DC current as a function of a finite applied bias at zero temperature, by directly solving the equation [3] up to high order in \( t \). This describe the experimentally relevant situation [45] where all AC components are averaged out. To this end, we point out that such a contribution can be obtained by the replacement for lesser Green’s function,

\[ \hat{g}^<\omega + n\Delta\mu \rightarrow -\frac{2\pi i\mu}{g} \delta(\omega + n\Delta\mu) \left[ \begin{array}{c} 1 \n 1 \\ 1 \end{array} \right]. \]

Here, we note that the above does not depend on the bosonic distribution function \( n(\omega) = 1/(\omega^2 + T - 1) \), in contrast to Green’s functions of \( k \neq 0 \) components.

The resulting current-bias relations are shown in figure [2], for various tunneling amplitudes normalized by the density of states in the reservoirs at the chemical potential \( \rho(\mu) \), as in the case of fermionic transport [23]. Remarkably, in all cases we observe a linear current-bias relation for sufficiently low bias, even though at zero temperature the system is fully superfluid. This contrasts with the case of charged, fermionic superconductors, which show large non linearies in the low bias regime associated with MARs [15]. This also differs from the case with one-dimensional reservoirs described by the Luttinger liquid, where power-law current-bias relations are obtained [33].

The microscopic mechanism underlying the emergence of a finite conductance at zero temperature is the tunnel-
ing of one atom from one reservoir to the other accompanied by the coherent emission of a phonon. Like in superconductors, the DC current appears due the coupling between the coherent, periodic evolution at frequency \(\Delta \mu\) with a dissipation channel. While in superconductors, this channel is a gapped pair-breaking process, yielding MARs and the associated highly non-linear current-bias relation, here the phonons provide a gapless dissipation channel, and thus a contribution at the linear response level.

By numerically solving the equation \([3]\), we systematically investigate the conductance emerging in the linear regime as a function of the tunneling amplitude. The result is shown as solid line in figure \([2a]\). Remarkably, even for moderate tunneling amplitudes of the order of \(0.4/\rho(\mu)\), the conductance reaches \(10/\hbar\), in contrast with the upper bound of \(1/\hbar\) per mode allowed for the Fermi liquid. In the low tunneling regime, conductance shows a quadratic dependence on \(t\) (dashed line in figure \([2b]\)), suggesting that the linear response theory accurately captures the physics.

We now focus on the linear response regime, where analytic treatments are allowed. This is achieved by the replacement \(T R^{(A)}(\tau, \tau') \rightarrow \hbar(\tau)\delta(\tau - \tau')\). The currents can be expressed in terms of an Onsager matrix

\[
\begin{pmatrix}
I_1 \\
I_2
\end{pmatrix} = \begin{pmatrix}
L_{11} & L_{12} \\
T L_{12} & L_{22}
\end{pmatrix} \begin{pmatrix}
\Delta \mu \\
\Delta T
\end{pmatrix},
\]

Here, transport coefficients \((L_{11}, L_{12}, L_{22})\) depend on temperature \(T\). They are proportional to \(t^2\), by hypothesis.

The conductance is expressed as \(G = L_{11} = \frac{e^2 T}{h} \left[ \frac{1}{2} + \frac{T^2}{2 \pi^2 c^2} \right] \) with the speed of sound \(c\). It features a zero temperature part inversely proportional to the interaction strength, which arises due to the gapless modes in the reservoirs, and a contribution quadratic in temperature which originates from the incoherent tunneling of Bogoliubov quasi-particles. The latter could be obtained by a direct application of the Landauer theory to the free bosonic quasi-particle gas. Importantly, the zero temperature part dominates over the thermal component up to \(T \sim \mu/(na^3)^{1/4} \gg \mu\), where \(\sqrt{na^2}\) is the small parameter in the Bogoliubov theory. The decreasing dependence of the zero temperature conductance with interactions can be interpreted as reflecting the impedance mismatch between the coherent superfluid current oscillating at frequency \(\Delta \mu\) in the contact, and the reservoirs response at the corresponding frequency which is more phase-like as interactions increase.

We also find no zero-temperature contribution to heat transport. As a result, the Lorenz number \(L = \frac{k_B^2}{e^2} - \frac{L_{12}^2 T L_{11}}{T L_{12}}\) and Seebeck coefficient \(S = L_{12}/L_{11}\) which characterize the relation between entropy and particle currents, go to zero as temperature is reduced. The leading temperature dependence is quadratic and cubic for the Lorenz number and Seebeck coefficient, respectively. As a result, the bosonic version of the Wiedemann-Franz law, stating that for free Bosons the ratio tends to \(4\pi^2/5\) at low temperature \([30]\), is necessarily violated. While this is expected on general grounds in a superfluid, a simple analysis based on a hydrodynamic description misses the zero-temperature contribution to \(L_{11}\) and predicts that the Wiedemann-Franz law is obeyed \([30]\).

Figure 3 illustrates these results, where the conductance (on panel a), Seebeck coefficient and Lorenz numbers (on panel b) are presented as a function of \(T/\mu\). There we have used \(tp(\mu) = 0.2\), \(\mu = 400\,\text{nK}\), the scattering length \(a = 25\,\text{nm}\), and a density in each reservoir of \(10^{-19}\,\text{m}^{-3}\), consistent with experimentally relevant Bose-Einstein condensates of \(^6\text{Li}\) dimers \([8,40]\). Even for this case representing relatively large interactions, the zero temperature, phonon contribution dominates for the whole range of temperature. Note that the phonon approximation for Green’s function fails for \(T \gtrsim \mu\), so we cannot extrapolate our results to the high temperature regime covered in particular in \([30]\).

Analysis of trapped systems— Compared to electronic systems, quantum gas two-terminal setups, for which our model directly applies, feature finite-size reservoirs. Transport is studied by direct measurements of the reservoirs atom number and temperature, under the hypothesis that thermal equilibrium is achieved within reservoirs at each point in time \([4]\). We now consider two harmonically trapped, three-dimensional BECs as reservoirs connected by a QPC and calculate the time evolution of atom number and temperature differences between the reservoirs \(\Delta N\) and \(\Delta T\), which results from the combined effects of the transport properties of the channel and the thermodynamics of the reservoirs. It reads
\[ \frac{d}{d\tau} \left( \frac{\Delta N}{\kappa} \right) = - \left( \frac{1}{S_{\text{eff}}} \frac{S_{\text{eff}}}{L + S_{\text{eff}}} \right) \left( \frac{\Delta N}{\kappa} \right), \quad (8) \]

where \( \kappa \) is the compressibility, \( \tau_0 = \kappa/(2L_{11}) \) is the transport time for mass transport. We introduce the analog of the Lorenz number for the reservoirs \( l \equiv C_{\mu}/(\kappa T) \) \(- (\alpha/\kappa)^2 \), and the effective Seebeck coefficient \( S_{\text{eff}} \equiv \alpha/\kappa - S \) with the heat capacity at constant \( \mu \) \( C_{\mu} \), and the dilatation coefficient \( \alpha \) (see [42] for details).

The resulting evolution is presented in figure 4 for a molecular BEC of \(^6\)Li dimers at \( T = 150\text{mK} \) [4] [10], with the reservoirs and channel parameters yielding \( \tau_0 \sim 10s \), similar to the case of a single mode conductor for fermionic atoms [7], since the large conductance of the BEC is compensated by its much larger compressibility.

The initial relative atom number difference, without temperature bias (figure 4(a)), decays in the time scale \( \tau_0 \). The low value of the heat conductance shows up in the slower decay of an initial temperature difference, without atom number difference, where our parameters yield \( L = 2.9k_B \), and \( L/l = 9.6 \times 10^{-2} \). While this decay is slow, very efficient thermometry is available for BECs which should allow for an experimental observation [37].

The thermo-electric responses, resulting either in the build up of a temperature difference as a result of the current running in the contact (figure 4(a)) or an atom number difference building up as a result of heat current (figure 4(b)) are negligibly small with \( S_{\text{eff}} = 7.9 \times 10^{-3} k_B \), due to the combination of the low Seebeck coefficient and the thermodynamics of the trapped BEC having \( \alpha/\kappa = 0.22 k_B \). This is similar to Fermi gases [45] where such a combination is also observed [10]. For box rather than harmonic traps as reservoirs, the thermo-electric response is larger and the decay time of both atom-number and temperature biases is of the order of \( \tau_0 \) [22]. In the absence of a thermo-electric response, the ratio of particle to temperature decay times is equal to the ratio of Lorenz number to analogue Lorenz number for the reservoirs [50].

Discussion — Our work elucidates the relation between the dissipative current and the superfluid character of the BEC. The QPC implements a coherent coupling of the superfluid current with the phonon modes, in the spirit of the celebrated Landau argument on superfluidity. Even though the zero temperature DC component does not carry any entropy, as demonstrated by the zero value of the Seebeck coefficient, it cannot be identified with a superfluid current. This also illustrates the difficulty in modeling the dynamics of superfluids in mesoscopic structures using effective two-fluid models: in particular QPCs behave very differently from super-leaks in helium, and the fountain effect is not expected [51]. Charge neutral fermionic superfluids also feature a gapless mode resulting from the \( U(1) \) symmetry breaking. While we cannot extrapolate the bosonic results derived here to the fermionic case, the Bogoliubov theory is adequate for the far molecular side of the BEC-BCS crossover. The \( 1/g \) dependence of the zero temperature contribution suggests that it should become less relevant as the system approaches the Feshbach resonance. This is confirmed by the good agreement between low temperature transport experiments at unitarity and MAR theory observed in [45].

A direct generalization of our theory could describe spin transport in spinor BECs, or include spin-orbit coupling in the reservoirs and the contact. It could also include several channels in the QPC to reflect more accurately the experimental situations. Our theory also predicts the AC Josephson effect, in particular the heat current oscillations at \( 2\Delta \mu \) encountered in superconductors [52], which have never been studied with cold atoms, and will be described separately. The same formalism also predicts the current noise spectrum [12]. While the current noise in the two-terminal setup has never been observed in quantum gas experiments, we expect it to become accessible in future generations of experimental setups [53].

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