TITLE: Co-Dimension One Area-Minimizing Currents with $C^{1,\alpha}$ Tangentially Immersed Boundary.

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ABSTRACT: We introduce and study co-dimension one area-minimizing locally rectifiable currents $T$ with $C^{1,\alpha}$ tangentially immersed boundary: $\partial T$ is locally a finite sum of orientable co-dimension two submanifolds which only intersect tangentially with equal orientation. We show that any such $T$ is supported in a smooth hypersurface near any point on the support of $\partial T$ where $T$ has tangent cone which is a hyperplane with constant orientation but non-constant multiplicity. We also introduce and study co-dimensional one area-minimizing locally rectifiable currents $T$ with boundary having co-oriented mean curvature: $\partial T$ has generalized mean curvature $H_{\partial T} = h\nu_T$ with $h$ a real-valued function and $\nu_T$ the generalized outward pointing unit normal of $\partial T$ with respect to $T$.

KEYWORDS: Currents; Area-minimizing; Boundary Regularity.

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1 Introduction

Our goal in this work is to generalize the boundary regularity theory for co-dimension one area-minimizing locally rectifiable currents established in [8] and [20]. To this end, we introduce two definitions. Suppose $T$ is an $n$-dimensional area-minimizing locally rectifiable current in $\mathbb{R}^{n+1}$. First, we say $T$ has $C^{1,\alpha}$ tangentially immersed boundary with $\alpha \in (0,1]$ if $\partial T$ is locally a finite sum of orientable $(n-1)$-dimensional (embedded) submanifolds which meet only tangentially with equal orientation; see Definition 3.1. Second, we say the boundary of $T$ has co-oriented mean curvature if $\partial T$ has generalized mean curvature $H_{\partial T} = h\nu_T$ for $h$ a real-valued function and $\nu_T$ the generalized outward pointing unit normal of $\partial T$ with respect to $T$; see Definition 4.1, in addition to Lemma 3.1 of [4] and (2.9) of [3] for the existence of $\nu_T$.

In order to proceed, we briefly describe the now classical results of [8] and [20]. Suppose $T$ is an $n$-dimensional area-minimizing locally rectifiable current in $\mathbb{R}^{n+1}$, and suppose $x$ is in the support of $\partial T$. First, [8] shows that if $\partial T$ near $x$ corresponds to integrating over an orientable
(n − 1)-dimensional $C^{1,\alpha}$ submanifold with multiplicity one and $\alpha \in (0, 1)$, then one of the following two occurs: $T$ corresponds to integrating (with multiplicity one) over a $C^{1,\alpha}$ hypersurface-with-boundary, with boundary the support of $\partial T$; the support of $T$ near $x$ is an analytic minimal hypersurface $M$ containing the support of $\partial T$ near $x$, and $T$ corresponds to integrating over $M$ with multiplicities $\theta, (\theta - 1)$ for some positive integer $\theta$ respectively over the two $C^{1,\alpha}$ regions of $M$ determined by the support of $\partial T$. This result is extended by [20]. If instead $\partial T$ near $x$ corresponds to integrating over an orientable $(n - 1)$-dimensional $C^{1,\alpha}$ submanifold with multiplicity a positive integer $m$ and $\alpha \in (0, 1)$, then one of the following two occurs: near $x$ we have that $T$ corresponds to integrating over a disjoint union of $C^{1,\alpha}$ hypersurfaces-with-boundary each with some multiplicity, each hypersurface having boundary $\partial T$; the support of $T$ near $x$ is an analytic minimal hypersurface $M$ containing the support of $\partial T$ near $x$, and $T$ corresponds to integrating over $M$ with multiplicities $(m + \theta), \theta$ for some positive integer $\theta$ respectively over the two $C^{1,\alpha}$ regions of $M$ determined by the support of $\partial T$.

We now describe our main results, the first of which is the most important:

**Theorem 3.18.** Suppose $T$ is an $n$-dimensional area-minimizing locally rectifiable current in $\mathbb{R}^{n+1}$ with $C^{1,\alpha}$ tangentially immersed boundary where $\alpha \in (0, 1]$. Suppose $x$ is in the support of $\partial T$, and that $T$ at $x$ has a tangent cone $C$ which is a hyperplane with constant orientation but non-constant multiplicity. Then the support of $T$ near $x$ is the graph of a smooth solution to the minimal surface equation $u$ off the hyperplane supporting the (now unique) tangent cone $C$, and the orientation vector of $T$ near $x$ corresponds to the upward pointing unit normal of the graph of $u$.

We say that an $n$-dimensional current $C$ in $\mathbb{R}^{n+1}$ is a hyperplane with constant orientation but non-constant multiplicity if (after rotation) $C$ corresponds to integrating over $\{(y_1, \ldots, y_n) \in \mathbb{R}^n : y_n > 0\}$ with orientation $e_{n+1}$ and multiplicity $(m + \theta)$ and over $\{(y_1, \ldots, y_n) \in \mathbb{R}^n : y_n < 0\}$ with orientation $e_{n+1}$ and multiplicity $\theta$ for positive integers $m, \theta$; see Definition

If $\partial T$ is $C^{1,1}$ tangentially immersed with Lipschitz co-oriented mean curvature, we conclude $\partial T$ is regular:
Theorem 5.12: Suppose $T$ is an $n$-dimensional area-minimizing locally rectifiable current in $\mathbb{R}^{n+1}$ with $C^{1,1}$ tangentially immersed boundary and where $\partial T$ has co-oriented mean curvature $H_{\partial T} = h\nu_T$ with $h$ a Lipschitz real-valued function and $\nu_T$ the generalized outward pointing unit normal of $\partial T$ with respect to $T$. If $x$ is in the support of $\partial T$ and $T$ at $x$ has a tangent cone which is a hyperplane with constant orientation but non-constant multiplicity, then $\partial T$ near $x$ corresponds to integrating over an $(n-1)$-dimensional submanifold, which is $C^{2,\alpha}$ for any $\alpha \in (0, 1)$, with multiplicity.

In this case, [8] and [20] imply that $T$ near $x$ is supported in an analytic hypersurface $M$ containing the support of $\partial T$ near $x$, and $T$ corresponds to integrating over $M$ with multiplicities $(m + \theta)$, $\theta$ for some positive integers $m, \theta$ respectively over the two $C^{2,\alpha}$ regions of $M$ determined by the support of $\partial T$.

Finally, we have the following very geometric partial regularity theorem:

Theorem 5.13: Suppose $T$ is an $n$-dimensional area-minimizing locally rectifiable current in $\mathbb{R}^{n+1}$ with $C^{1,1}$ tangentially immersed boundary and where $\partial T$ has co-oriented mean curvature $H_{\partial T} = h\nu_T$ with $h$ a Lipschitz real-valued function and $\nu_T$ the generalized outward pointing unit normal of $\partial T$ with respect to $T$. Suppose $x$ is in the support of $\partial T$, and that near $x$ the support of $T$ equals a finite union of $C^1$ hypersurfaces-with-boundary. Then the support of $T$ near $x$ is the finite union of $C^{1,1}$ hypersurfaces-with-boundary which pairwise meet only at common boundary points.

The better way to understand Theorem 5.13 is through the contrapositive: at any point $x$ in the support of $\partial T$ near which the support of $T$ does not equal a finite union of $C^{1,1}$ hypersurfaces-with-boundary which pairwise meet only at common boundary points, the support of $T$ near $x$ must have extremely complicated structure; perhaps for example infinite topology at $x$.

Before we discuss our main results with more detail, we describe a partial boundary regularity result given by the author in [15]. For convenience, we state this result here as Theorem 3.5, as it will be crucial to the proofs. We describe it loosely now:
Theorem 3.5 (Theorem 2.1 of [15]): Suppose $T$ is an $n$-dimensional area-minimizing locally rectifiable current in $\mathbb{R}^{n+1}$ with $C^{1,\alpha}$ tangentially immersed boundary where $\alpha \in (0, 1]$. Suppose $x$ is in the support of $\partial T$, and that $T$ at $x$ has a tangent cone $C$ which is a hyperplane with constant orientation but non-constant multiplicity. Then near $x$ a large portion of the support of $T$ can be written as the graph of a $C^{1,\frac{n}{n+6}}$ function defined over a large region of the hyperplane supporting $C$; this region is large enough to conclude that $C$ is the unique tangent cone of $T$ at $x$.

We now discuss Theorems 3.18, 5.12, 5.13 in more detail.

1.1 Tangentially Immersed Boundaries

Throughout this section, let $T$ be an $n$-dimensional area-minimizing locally rectifiable current over $\mathbb{R}^{n+1}$ with $C^{1,\alpha}$ tangentially immersed boundary, where $\alpha \in (0, 1]$. We also suppose that 0 is in the support of $\partial T$, and that $T$ at 0 has a tangent cone $C$ which is integrating over $\{(y_1, \ldots, y_n) \in \mathbb{R}^n : y_n > 0\}$ with orientation $e_{n+1}$ and multiplicity $(m + \theta)$ and over $\{(y_1, \ldots, y_n) \in \mathbb{R}^n : y_n < 0\}$ with orientation $e_{n+1}$ and multiplicity $\theta$ for positive integers $m, \theta$.

We now discuss Theorem 3.18 which concludes that the support of $T$ near 0 is the graph of $u$ a smooth solution to the minimal surface equation, and the orientation vector $\ast \vec{T}$ of $T$ near 0 is the upward pointing unit normal of the graph of $u$. It will be evident that Theorem 3.18 is a natural consequence of the partial boundary regularity result Theorem 3.5 (Theorem 2.1 of [15]). To prove Theorem 3.18 we must first prove that having unique tangent cone a hyperplane with constant orientation but non-constant multiplicity (see Definition 3.3) is an open condition along $\partial T$:

Lemma 3.6: For all $x$ in the support of $\partial T$ near 0, we have that $T$ has at $x$ a unique tangent cone which is a hyperplane with constant orientation but non-constant multiplicity.

Having stated Lemma 3.6 it is then expected that Theorem 3.18 is true. Lemma 3.6 is the linchpin of the present work, and so we discuss it in detail. The proof of Lemma 3.6 is a generalization of the proof of Theorem 8 of [16], where special cases of the present results appear in the context of two-dimensional solutions to the c-Plateau problem in space; see §1.4. In
turn, the techniques used in \cite{16} (and presently) are similar to those used by the author in \cite{10}, \cite{11}, \cite{13} to study the two-valued minimal surface equation, a degenerate second-order PDE first introduced in \cite{19} to produce examples of co-dimension one $C^{1,\alpha}$ stable branched minimal immersions.

The proof of Lemma 3.6 follows by analyzing the cross-sections of $T$ across affine planes \{$z\}$ $\times$ $\mathbb{R}^2$ for $z \in \mathbb{R}^{n-1}$ near the origin. For simplicity, we describe the proof in the case $n = 2$. First, Theorem 3.5 implies (after rescaling) that the support of $T$ in the bored-out unit ball

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 : \delta < |(x_2, x_3)|, |x| < 1\}$$

is the graph of a function defined off the region in the horizontal plane

$$\{y = (y_1, y_2) \in \mathbb{R}^2 : \delta < |y_2|, |y| < 1\},$$

for some small $\delta > 0$. We now suppose for contradiction $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ is a point in the support of $\partial T$ with $|\tilde{x}| < \delta$ and so that $T$ at $\tilde{x}$ does not have unique tangent cone which is a plane with constant orientation but non-constant multiplicity. It follows (see Theorem 3.4) that every tangent cone $C$ of $T$ at $\tilde{x}$ must be a sum of half-planes with constant orientation after rotation meeting along a common line; see Definition 3.3. As $T$ has $C^{1,\alpha}$ tangentially immersed boundary, the spine of any such $C$ is a fixed line, namely the tangent line of $\partial T$ at $x$, which is close to the $x_1$-axis depending on how close $\tilde{x}$ is to the origin.

We now consider the cross-section of the support of $T$ along the affine plane \{$\tilde{x}_1\}$ $\times$ $\mathbb{R}^2$. In the region $\{x = (\tilde{x}_1, x_2, x_3) \in \mathbb{R}^3 : \delta < |(x_2, x_3)|, |x| < 1\}$ this cross-section is by Theorem 3.5 two curves $\Gamma_1, \Gamma_2$ which are respectively $C^1$ close to the line segments

$$\{(\tilde{x}_1, t, 0) \in \mathbb{R}^3 : t \in (\delta, 1 - |\tilde{x}_1|)\} \text{ and } \{(\tilde{x}_1, t, 0) \in \mathbb{R}^3 : t \in (-\delta, -1 + |\tilde{x}_1|)\}.$$  

In particular, the orientation vector $*\tilde{T}$ of $T$ (the generalized unit normal vector field of $T$) satisfies $*\tilde{T} \approx e_3$ along $\Gamma_1, \Gamma_2$.

On the other hand, assuming (without loss of generality) that $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ is the only point in the support of $\partial T$ on the cross-section \{$\tilde{x}_1\}$ $\times$ $\mathbb{R}^2$, and that $\tilde{x}_1$ is a regular value of the function $f(x_1, x_2, x_3) = x_1$ over the support of $T$ away from the support of $\partial T$ (by Sard’s theorem and
interior regularity of co-dimension one area-minimizing locally rectifiable currents), then we essentially argue that \( \Gamma_1, \Gamma_2 \) are contained respectively in smooth curves \( \tilde{\Gamma}_1, \tilde{\Gamma}_2 \) meeting only at \( \tilde{x} \). Moreover, \( \ast \tilde{T} \) is smooth over \( \tilde{\Gamma}_1, \tilde{\Gamma}_2 \). However, as every tangent cone of \( T \) at \( \tilde{x} \) is a sum of half-planes with constant orientation after rotation (see Definition 3.3) with spine close to the \( x_1 \)-axis (as noted above), then considering the continuity of \( \ast \tilde{T} \) along \( \tilde{\Gamma}_1 \) and \( \tilde{\Gamma}_2 \) yields a contradiction; essentially, we contradict that \( \ast \tilde{T} \) is continuous over \( \tilde{\Gamma}_2 \), due to orientation.

Having done most of the work in proving Lemma 3.6, then the proof of Theorem 3.18 is a relatively short proof by induction based on the number of \((n-2)\)-dimensional \( C^{1,\alpha} \) submanifolds in the decomposition of \( \partial T \) near 0.

### 1.2 Tangentially Immersed Boundaries with Co-Oriented Mean Curvature

We describe in this subsection the proofs of Theorems 5.12, 5.13. Throughout this subsection suppose \( T \) is an \( n \)-dimensional area-minimizing locally rectifiable current over \( \mathbb{R}^{n+1} \) with \( C^{1,1} \) tangentially immersed boundary and so that \( \partial T \) has co-oriented mean curvature \( H_{\partial T} = h \nu_T \) where \( h \) is a Lipschitz real-valued function and \( \nu_T \) is the generalized outward pointing unit normal of \( \partial T \) with respect to \( T \). Suppose as well that 0 is in the support of \( \partial T \).

We begin by discussing Theorem 5.12 which concludes that if \( T \) at 0 has tangent cone which is a hyperplane with constant orientation but non-constant multiplicity, then \( \partial T \) is regular near 0. We prove Theorem 5.12 using the Hopf boundary point lemma at half-regular singular points of \( \partial T \). We roughly describe such points in the following lemma:

**Lemma 3.19:** If 0 is a singular point of \( \partial T \), then for any \( \rho > 0 \) there exists \( x \) in the singular set of \( \partial T \) with \( |x| < \rho \) and a non-empty open set \( U \subset \mathbb{R}^{n+1} \) so that \( x \in \partial U \) and the support of \( \partial T \) in \( U \) is a union of disjoint non-empty \((n-1)\)-dimensional submanifolds; we call such an \( x \) a half-regular point.

The proof of Lemma 3.19 is by induction on the number of distinct submanifolds in the decomposition of \( \partial T \) near 0, if 0 is a singular point. A version of Lemma 3.19 appears as Lemma 1 of [16] in the context of two-dimensional solutions to the \( c \)-Plateau problem in space; see §1.4.
The proof of Theorem 5.12 then follows through naturally. Suppose for contradiction that $T$ at $0$ has unique tangent cone $C$ which is a hyperplane with constant orientation but non-constant multiplicity, but that $0$ is a singular point of $\partial T$. Theorem 3.18 implies that $T$ near $0$ is supported in the graph of a function $u$ defined off the hyperplane supporting $C$, and the orientation vector of $T$ near $0$ corresponds to the upward pointing unit normal of the graph of $u$. Since $0$ is a singular point of $\partial T$, then we can consider a half-regular point $x$ in the singular set of $\partial T$ with $x$ sufficiently close to the origin. Then there is a nonempty open set $U \subset \mathbb{R}^{n+1}$ so that $x \in \partial U$ and the support of $\partial T$ in $U$ is a union of disjoint non-empty $(n - 1)$-dimensional submanifolds contained in the graph of $u$. The proof of Theorem 5.12 then proceeds by applying the Hopf boundary point lemma to $\partial T$ in $U$ at $x$, using $H_{\partial T} = hu_T$.

We now discuss Theorem 5.13, which concludes that if near $0$ the support of $T$ equals a finite union of $C^1$ hypersurfaces-with-boundary, then near $0$ the support of $T$ equals a finite union of $C^{1,1}$ hypersurfaces-with-boundary which pairwise meet only at common boundary points. The proof of Theorem 5.13 is very geometric, and technical, similar to the proof of Lemma 3.6.

Suppose now that near the $0$ the support of $T$ is a finite union of $C^1$ hypersurfaces-with-boundary, and consider the set

$$W = \left\{ x \in \text{support of } T : \begin{array}{l} x \text{ is not in the support of } \partial T \\
or \ T \text{ at } x \text{ has a tangent cone} \\
\text{which is a hyperplane with constant orientation but non-constant multiplicity} \end{array} \right\}.$$  

Interior regularity for co-dimension one area-minimizing locally rectifiable currents, Theorem 5.12 as well as the classical boundary regularity given by [8] and [20] imply that $W$ is a smooth embedded hypersurface. Then an analysis similar to the proof of Lemma 3.6 using Sard’s theorem, shows that $W$ near the origin decomposes into finitely many connected components, each of which is a $C^{1,1}$ hypersurface-with-boundary. To this end, the proof of Theorem 5.13 closely mirrors the proof of Theorem 9 of [16], which again is a version of Theorem 5.13 in the context of two-dimensional solutions to the $c$-Plateau problem in space; see §1.4.
1.3 Future Work

We expect to relax the assumption of Theorems 5.12,5.13 that $T$ has $C^{1,1}$ tangentially immersed boundary to $T$ having more generally $C^{1,\alpha}$ tangentially immersed boundary with $\alpha \in (0,1]$. For this, we nevertheless expect it necessary to assume that $\partial T$ has co-oriented mean curvature $H_{\partial T} = h\nu_T$ with $h$ a Lipschitz real-valued function. Indeed, we suspect that to generalize Theorems 5.12,5.13, we must show the following:

**Conjecture:** Suppose $T$ is an $n$-dimensional area-minimizing locally rectifiable current in $\mathbb{R}^{n+1}$ with $C^{1,\alpha}$ tangentially immersed boundary where $\alpha \in (0,1)$, and that $\partial T$ has co-oriented mean curvature $H_{\partial T} = h\nu_T$ with $h$ a Lipschitz real-valued function. Also suppose $x$ is in the support of $\partial T$, and that $T$ at $x$ has a tangent cone which is a hyperplane with constant orientation but non-constant multiplicity. Then $T$ near $x$ has $C^{1,1}$ tangentially immersed boundary.

We suspect the proof of this conjecture is standard regularity for elliptic systems, but many details are needed; we leave this for future work. Note that more general versions of Theorems 5.12,5.13 (namely, if we only assume $T$ has $C^{1,\alpha}$ tangentially immersed boundary with $\alpha \in (0,1]$) hold in case $n = 2$, for two-dimensional area-minimizing currents in space. For this, see Theorems 6.10,6.11.

It remains to investigate the geometric structure of co-dimension one area-minimizing locally rectifiable currents $T$ with $C^{1,\alpha}$ tangentially immersed boundary where $\alpha \in (0,1]$ near points $x$ in the support of $\partial T$ so that $T$ at $x$ has a tangent cone which is a sum of half-hyperplanes with constant orientation after rotation (see Definition 3.3). To this end, we perhaps expect to at least assume $\partial T$ has co-oriented mean curvature $H_{\partial T} = h\nu_T$ with $h$ Lipschitz. An very optimistic conjecture is that the support of $T$ near such an $x$ is a finite union of $C^1$ hypersurfaces-with-boundary, so that then Theorem 5.13 applies. It may at least be possible to show that $T$ has unique tangent cone at every $x$ in the support of $\partial T$. Much more work is needed.

There is no hope to extend the present results in the case of general $n$-dimensional area-minimizing currents in $\mathbb{R}^{n+k}$. It is well-known that the boundary regularity of [8],[20] do not hold in higher co-dimensions.
Furthermore, we refer the reader to §1.2 of [15], which gives examples showing that not even the partial boundary regularity result Theorem 3.5 (Theorem 2.1 of [15]) holds in higher co-dimensions, or even for merely stable two-dimensional currents in space.

1.4 Applications

We now discuss the thread problem and the $c$-Plateau problem, both of which yield categories of area-minimizing currents for which the present results are relevant.

First, we describe the thread problem as defined in [5]. We say that $T$, an $n$-dimensional locally rectifiable current in $\mathbb{R}^{n+1}$, is a minimizer of the thread problem with respect to $\Gamma$, an $(n-1)$-dimensional locally rectifiable current in $\mathbb{R}^{n+1}$, if $M_U(T) \leq M_U(S)$ whenever $U \subset \mathbb{R}^{n+1}$ is an open set, $M_U$ is the usual mass on currents in $U$, and $S$ is an $n$-dimensional locally rectifiable current in $\mathbb{R}^{n+1}$ such that the support of $S - T$ is contained in $U$ and $M_U(\partial S - \Gamma) = M_U(\partial T - \Gamma)$.

Naturally, minimizers of the thread problem are area-minimizing; see (2) of [5]. Theorem 2.3 of [5] states that if $T$ is a minimizer of the thread problem with respect to $\Gamma$, then the free boundary $\Sigma = \partial T - \Gamma$ has away from the support of $\Gamma$ co-oriented mean curvature $H_\Sigma = \frac{1}{\lambda_\Sigma} \nu_T$ where $\lambda_\Sigma$ is a positive number; see in particular (2.8) of [5]. Also, Theorem 3.4 of [5] concludes in case $n = 2$ that $T$ away from the support of $\Gamma$ has $C^{1,1}$ tangentially immersed boundary. The (tangentially immersed) regularity of the free boundary is unknown for general dimensions $n > 2$.

Second, we describe the $c$-Plateau problem as defined in [12]. We say that $T$, an $n$-dimensional locally rectifiable current in $\mathbb{R}^{n+1}$, is a solution to the $c$-Plateau problem with respect to $\Gamma$, an $(n-1)$-dimensional locally rectifiable current in $\mathbb{R}^{n+1}$, if $\partial T = \Sigma + \Gamma$ where $\Sigma, \Gamma$ have disjoint supports and

$$M_U(T) + cM_U(\partial T) \frac{n}{n-1} \leq M_U(S) + cM_U(\partial S) \frac{n}{n-1}$$

whenever $U \subset \mathbb{R}^{n+1}$ is an open set and $S$ is an $n$-dimensional locally rectifiable current in $\mathbb{R}^{n+1}$ such that $(\partial S - \Gamma), \Gamma$ have disjoint supports and the support of $S - T$ is contained in $U$. 

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Solutions to the $c$-Plateau problem are not only area-minimizing, they solve the thread problem; see Theorem 5.2 of [12]. However, it follows directly from the definition that if $T$ is a solution to the $c$-Plateau problem with respect to $\Gamma$, then the free boundary $\Sigma = \partial T - \Gamma$ has co-oriented mean curvature $H_\Sigma = -\frac{1}{c(n)}M_\nu(\partial T, n)\nu_T$; see (6.1) of [12]. In case $n = 2$, we can use Theorem 3.4 of [5] to conclude that $T$ away from the support of $\Gamma$ has $C^{1,1}$ tangentially immersed boundary; see Theorem 4 of [16]. The (tangentially immersed) regularity of the free boundary is unknown for general dimensions $n > 2$.

1.5 Outline

We begin in §2 by listing the notation we shall use. Next, in §3 we carefully define what it means for $T$ to have $C^{k,\alpha}$ tangentially immersed boundary for $k \geq 1$ in Definition 3.1, and eventually state and prove Theorem 3.18; to this we also include an appendix, where we prove a technical result Lemma A.2 needed to prove Lemma 3.6. In §4 we define what it means for $\partial T$ to have co-oriented mean curvature in Definition 4.1, and give a few basic related results. Finally, in §5 we state and prove Theorems 5.12,5.13.

2 Notation

We list basic notation and terminology we shall use throughout.

- $\mathbb{N}, \mathbb{R}$ will denote the natural and real numbers respectively. We shall let $n \in \mathbb{N}$ with $n \geq 2$. In this section we will let $\tilde{n} \in \{1, \ldots, n\}$.
- We shall typically write points $x = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}$. Depending on context, we shall let
  \[ \mathbb{R}^{\tilde{n}} = \{(x_1, \ldots, x_{\tilde{n}}, 0, \ldots, 0) \in \mathbb{R}^{n+1} : x_1, \ldots, x_{\tilde{n}} \in \mathbb{R}\}. \]
  We shall typically write $z = (z_1, \ldots, z_{n-1}) = (z_1, \ldots, z_{n-1}, 0, 0, 0) \in \mathbb{R}^{n-1}$ and $y = (y_1, \ldots, y_n) = (y_1, \ldots, y_n, 0, 0) \in \mathbb{R}^n$. For $z = (z_1, \ldots, z_{n-1}) \in \mathbb{R}^{n-1}$ we will often make use of the affine plane
  \[ \{z\} \times \mathbb{R}^2 = \{(z_1, \ldots, z_{n-1}, x_n, x_{n+1}) \in \mathbb{R}^{n+1} : x_n, x_{n+1} \in \mathbb{R}\}. \]
We will let 0 denote the zero vector in different dimensions, depending on context. Hence, we will also make use of 
\( \{0\} \times S^1 = \{(0, \tilde{x}_n, x_{n+1}) \in \mathbb{R}^{n+1} : (x_n, x_{n+1}) \in S^1 \} \).

- Let \( e_1, \ldots, e_{n+1} \) be the standard basis vectors for \( \mathbb{R}^{n+1} \).

- For \( A \subseteq \mathbb{R}^{n+1} \), let \( \text{clos}\ A \) be the closure of \( A \). \( U \subseteq_o \mathbb{R}^{n+1} \) shall denote \( U \) is open in \( \mathbb{R}^{n+1} \). For \( U, \tilde{U} \subseteq_o \mathbb{R}^{n+1} \), we write \( \tilde{U} \subset\subset U \) if \( \text{clos}\ \tilde{U} \) is compact with \( \text{clos}\ \tilde{U} \subset U \).

- We shall let \( B_\rho(x) \) be the open ball in \( \mathbb{R}^{n+1} \) of radius \( \rho > 0 \) centered at \( x \). For \( x \in \tilde{n} \), we write \( B_{\tilde{\rho}}(x) = B_\rho(x) \cap \tilde{n} \).

- For \( x \in \mathbb{R}^{n+1} \) and \( \lambda > 0 \), we let \( \eta_{x, \lambda} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \) be the map 
  \[ \eta_{x, \lambda}(\tilde{x}) = \frac{\tilde{x} - x}{\lambda} \] . We shall often make use of \( \eta_{x, 1} \), which is translation to the right by \( x \).

- For two sets \( A, B \subseteq \mathbb{R}^{n+1} \), we denote the Hausdorff distance between \( A \) and \( B \) by \( \text{dist}_H(A, B) \).

- For \( U \subseteq_o \mathbb{R}^{n+1} \), we say \( M \subset U \) is a hypersurface-with-boundary (embedded) in \( U \) if \( M \) is a hypersurface (embedded) in \( U \) and \( M \) attains its topological boundary \( \partial M = (\text{clos}\ M) \setminus M \) in the strong sense in \( U \): \( M \) near every point of \( (\partial M) \cap U \) is diffeomorphic to the graph of a smooth function defined over a half-hyperplane. We similarly define a \( C^1 \), or more generally a \( C^{k, \alpha} \) for \( \alpha \in (0, 1] \) and \( k \in \mathbb{N} \), hypersurface-with-boundary.

- We let \( \ast : \wedge_n \mathbb{R}^{n+1} \to \mathbb{R}^n \) be the Hopf map 
  \[ \ast \left( \sum_{i=1}^{n+1} x_i (-1)^{i-1} e_1 \wedge \ldots \wedge e_{i-1} \wedge e_{i+1} \wedge \ldots \wedge e_{n+1} \right) = \sum_{i=1}^{n+1} x_i e_i. \]
  Note that \( \ast(e_1 \wedge \ldots \wedge e_n) = (-1)^n e_{n+1} \).

- We shall let \( D \) denote differentiation generally over \( \mathbb{R}^{n+1} \) or \( \tilde{n} \), depending on context. In the proof of Lemma \[\text{we will let } \tilde{D} \text{ denote differentiation over } \mathbb{R}^{n-1}, \text{ for emphasis.} \]

- \( \mathcal{H}^\tilde{n} \) shall denote \( \tilde{n} \)-dimensional Hausdorff measure in \( \mathbb{R}^{n+1} \). We also let \( \omega_{\tilde{n}} = \mathcal{H}^\tilde{n}(B^\tilde{n}_1(0)) \).
A smooth Jordan arc is a curve \( \gamma \in C([a,b]; \mathbb{R}^{n+1}) \cap C^\infty((a,b); \mathbb{R}^{n+1}) \) which is injective over \((a,b)\); we may have \( \gamma(a) = \gamma(b) \). A smooth closed Jordan curve is an injective curve \( \gamma \in C(S^1; \mathbb{R}^{n+1}) \). A continuous Jordan arc is a curve \( \gamma \in C([a,b]; \mathbb{R}^{n+1}) \) injective over \((a,b)\), and a continuous closed Jordan curve is an injective curve \( \gamma \in C(S^1; \mathbb{R}^{n+1}) \).

We now give notation related to currents in \( \mathbb{R}^{n+1} \). For a thorough introduction to currents, see [6], [18].

- Recall that \( D_{\tilde{n}}(U) \) denotes for \( U \subseteq_o \mathbb{R}^{n+1} \) the smooth \( \tilde{n} \)-forms compactly supported in \( U \).
- For a current in \( U \subseteq_o \mathbb{R}^{n+1} \) and \( f : U \to \mathbb{R}^{n+1} \), we denote \( f\#T \) the push-forward current of \( T \) by \( f \); we shall frequently make use of \( \eta_{x,\lambda}\#T \).
- We say a current \( C \) is a cone if \( \eta_{0,\lambda}\#C = C \) for every \( \lambda > 0 \).
- Given an orientable \( \tilde{n} \)-dimensional submanifold \( M \subset \mathbb{R}^{n+1} \), we denote \( \langle M \rangle \) the associated multiplicity one current, given an orientation.
- Denote by \( E_{\tilde{n}} \) the \( \tilde{n} \)-dimensional current in \( \mathbb{R}^{n+1} \) given by \( E_{\tilde{n}}(\omega) = \int_{\mathbb{R}^{n+1}} \langle \omega, e_1 \wedge \ldots \wedge e_{\tilde{n}} \rangle \, d\mathcal{H}_{\tilde{n}} \) for \( \omega \in D_{\tilde{n}}(\mathbb{R}^{n+1}) \).
- For \( U \subseteq_o \mathbb{R}^{n+1} \) and \( T \) an \( \tilde{n} \)-dimensional current in \( U \), we let \( \mu_T \) denote the associated mass measure of \( T \). This is given for \( \bar{U} \subseteq_o U \) by \( \mu_T(\bar{U}) = \sup_{\omega \in D_{\tilde{n}}(\bar{U}), \|\omega\| \leq 1} T(\omega) \). As usual, we set \( \text{spt} T = \text{spt} \mu_T \).
- For a \( \mu_T \)-measurable set, we let \( T \ll A \) denote the restriction current \( (T \ll A)(\omega) = \int_A \langle \omega, \bar{T} \rangle \, d\mu_T \) for \( \omega \in D_{\tilde{n}}(U) \), where \( \bar{T} \) is the orientation vector of \( T \).

Given \( x \in U \), we denote the density of \( T \) at \( x \) by \( \Theta_T(x) = \lim_{\rho \to 0} \frac{\mu_T(B_\rho(x))}{\omega_{\tilde{n}}(B_\rho)} \) for \( \omega_{\tilde{n}} = \mathcal{H}_{\tilde{n}}(B_\rho(0)) \), wherever this limit exists.

- Given \( U \subseteq_o \mathbb{R}^{n+1} \), we let \( I_{\tilde{n},\text{loc}}(U) \) be the set of \( \tilde{n} \)-dimensional currents \( T \) so that \( T, \partial T \) are respectively \( \tilde{n} \)- and \( (\tilde{n} - 1) \)-rectifiable integer multiplicity.
For $T \in I_{\tilde{n}, \text{loc}}$ we let $T_x T$ denote the approximate tangent space of $T$ for the $\mu_T$-almost-every $x \in U$ such that this space exists; naturally, we let $T_x^\perp T$ denote the orthogonal complement of $T_x T$ in $\mathbb{R}^{n+1}$.

- For $T \in I_{\tilde{n}, \text{loc}}(U)$, we denote $\delta T$ to be the first variation of mass, given by
  \[
  \delta T(X) = \int \mathrm{div}_T X \, d\mu_T
  \]
  for $X \in C^1_c(U; \mathbb{R}^{n+1})$.

We say that $T$ has mean curvature $H_T : U \to \mathbb{R}^{n+1}$ if $H_T$ is $\mu_T$-measurable and if

\[
\delta T(X) = \int X \cdot H_T \, d\mu_T
\]

for every $X \in C^1_c(U; \mathbb{R}^{n+1})$.

- For $T \in I_{\tilde{n}, \text{loc}}(U)$, we let $\text{reg} T$ denote the regular set of $T$: the set of $x \in \text{spt} T$ so that there is a $\rho > 0$ such that $T \subset B_\rho(x) = \theta[M]$ for $\theta \in \mathbb{N}$ and $M$ an $\tilde{n}$-dimensional orientable (embedded) $C^1$ submanifold of $B_\rho(x)$. We define the singular set $\text{sing} T = \text{spt} T \setminus \text{reg} T$.

- We say $T \in I_{\tilde{n}, \text{loc}}(U)$ is area-minimizing if $\mu_T(\tilde{U}) \leq \mu_R(\tilde{U})$ whenever $\tilde{U} \subset U$ and $R \in I_{\tilde{n}, \text{loc}}(U)$ with $\partial R = \partial T$ and $\text{spt}(T - R) \subset \tilde{U}$.

- For $T \in I_{\tilde{n}, \text{loc}}(U)$ area-minimizing there is, by Lemma 3.1 of [4] (see as well (2.10) of [4]), a $\mu_{\partial T}$-measurable vectorfield $\nu_T : U \to \mathbb{R}^{n+1}$ satisfying $|\nu_T| \leq 1$ for $\mu_{\partial T}$-almost-everywhere so that

\[
\delta T(X) = \int \nu_T \cdot X \, d\mu_{\partial T}
\]

for every $X \in C^1_c(U; \mathbb{R}^{n+1})$. We call $\nu_T$ the generalized outward pointing normal of $\partial T$ with respect to $T$. Note that since

\[
|\delta T(X)| \leq \int |X \wedge \partial T| \, d\mu_{\partial T}
\]

by Lemma 3.1 of [4] (see also (2.9) of [5]) for $X \in C^1_c(U; \mathbb{R}^{n+1})$, we conclude $\nu_T(x) \in T_x^\perp \partial T$ for $\mu_{\partial T}$-almost-every $x \in U$. 

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3 Tangentially Immersed Boundary

We begin by defining precisely what it means to have $C^{k,\alpha}$ tangentially immersed boundary. The goal of this section is to prove Theorem 3.18. In the subsequent sections, §4,5, we will also assume our boundaries have co-oriented mean curvature; see Definition 4.1. The first part of this section will consist of giving more general lemmas as well as recalling some needed previous results.

**Definition 3.1** Let $U \subset_o \mathbb{R}^{n+1}$, $k \in \mathbb{N}$, and $\alpha \in (0,1]$. We define $\text{T}_m^{k,\alpha}(U)$ to be the set of area-minimizing $T \in I_{n,\text{loc}}(U)$ so that $\partial T$ is locally $C^{k,\alpha}$ tangentially immersed: for every $x \in \text{spt} \partial T$ there is $\rho > 0$, an orthogonal rotation $Q$, and $N \in \mathbb{N}$ so that

$$\partial T \cap B_\rho(x) = (-1)^n \sum_{\ell=1}^N m_\ell \left[ (\eta_{-x,1} \circ Q \circ \Phi_{T,\ell})_\# (E_{n-1} \cap B_{\rho}^{n-1}(0)) \right] \cap B_\rho(x),$$

where for each $\ell = 1,\ldots,N$ we have $m_\ell \in \mathbb{N}$, and $\Phi_{T,\ell} \in C^{k,\alpha}(B_{\rho}^{n-1}(0); \mathbb{R}^{n+1})$ is the map

$$\Phi_{T,\ell}(z) = (z, \varphi_{T,\ell}(z), \psi_{T,\ell}(z)),$$

where $\varphi_{T,\ell}, \psi_{T,\ell} \in C^{k,\alpha}(B_{\rho}^{n-1}(0))$ satisfy

$$\varphi_{T,\ell}(0) = \psi_{T,\ell}(0) = 0 \text{ and } D\varphi_{T,\ell}(0) = D\psi_{T,\ell}(0) = 0.$$

Observe that we could define what it means for a current to have $C^{k,\alpha}$ tangentially immersed boundary in general. But we include the requirement that $T \in \text{T}_m^{k,\alpha}(U)$ must be area-minimizing for future brevity. Observe that if $T \in \text{T}_m^{k,\alpha}(U)$ then the approximate tangent space $T_x \partial T$ exists for every $x \in \text{spt} \partial T$.

In order to state and prove the results of this section, we need the following lemma, leading to Definition 3.3.

**Lemma 3.2** Suppose $C \in I_{n,\text{loc}}(\mathbb{R}^{n+1})$ is an area-minimizing cone with

$$\partial C = m(-1)^n Q_\# E^n$$

for some $m \in \mathbb{N}$ and an orthogonal rotation $Q$. Then $C$ is of one of the following two forms:
(1) There is \( N \in \{1, \ldots, m\} \) and distinct orthogonal rotations \( Q_1, \ldots, Q_N \) about \( \mathbb{R}^{n-1} \) so that

\[
C = \sum_{k=1}^{N} m_k (Q \circ Q_k)_\#(E^n \mbox{ lin } \{ y \in \mathbb{R}^n : y_n > 0 \}),
\]

where \( m_1, \ldots, m_N \) are positive integers with \( \sum_{k=1}^{N} m_k = m \).

(2) There is \( \theta \in \mathbb{N} \) so that

\[
C = Q_\# \left( (m + \theta) E^n \mbox{ lin } \{ y \in \mathbb{R}^n : y_n > 0 \} + \theta E^n \mbox{ lin } \{ y \in \mathbb{R}^n : y_n < 0 \} \right).
\]

**Proof:** By the boundary regularity for area-minimizing currents of \([8]\) and \([20]\), we get that the density \( \Theta_C(x) \) is constant for \( x \in \mathbb{R}^{n-1} \). The lemma then follows from Theorem 5.1 of \([4]\). \( \square \)

We thus give the following definition:

**Definition 3.3** Suppose \( C \in I_{n,\text{loc}}(\mathbb{R}^{n+1}) \) If \( C \) is as in (1) of Lemma 3.2 (with \( Q, Q_1, \ldots, Q_k \) orthogonal rotations, \( m, N, m_1, \ldots, m_N \in \mathbb{N} \)), then we say that \( C \) is a sum of half-hyperplanes with constant orientation after rotation. If \( C \) is as in (2) of Lemma 3.2 (with \( Q \) an orthogonal rotation and \( m, \theta \in \mathbb{N} \)), then we say that \( C \) is a hyperplane with constant orientation but non-constant multiplicity.

The first step in proving Theorem 3.18, the main result of this section, is to show that \( T \in T_{n,\text{loc}}^{1,\alpha}(U) \) has tangent cones at every point \( x \in \text{spt} \partial T \). We also categorize the tangent cones, using Lemma 3.2. To do this requires proving a monotonicity formula.

**Theorem 3.4** Let \( U \subseteq \mathbb{R}^{n+1} \), \( \alpha \in (0, 1] \), and suppose \( T \in T_{n,\text{loc}}^{1,\alpha}(U) \). For every \( x \in \text{spt} T \) the following hold:

(a) For \( 0 < \sigma < \rho \leq \text{dist}(x, \partial U) \)

\[
\frac{\mu_T(B_r(x))}{r^n} + \frac{1}{n} \int_{B_r(x)} \left( \frac{1}{|\tilde{x} - x|^n} - \frac{1}{r^n} \right) (\tilde{x} - x) \cdot \nu_T \, d\mu_T(\tilde{x}) \bigg|_{r=\sigma}^{\rho} = \int_{B_{\rho}(x) \setminus B_{\sigma}(x)} \frac{|\text{proj}_{T}(\tilde{x} - x)|^2}{|\tilde{x} - x|^{n+2}} \, d\mu_T(\tilde{x})
\]

where \( \nu_T \) is the generalized outward pointing unit normal of \( \partial T \) with respect to \( T \).
(b) The density \( \Theta_T(x) \) of \( T \) at \( x \) exists.

(c) There exists an area minimizing oriented tangent cone of \( T \) at \( x \). Every tangent cone of \( T \) at \( x \) is either a sum of half-hyperplanes with constant orientation after rotation or a hyperplane with constant orientation but non-constant multiplicity; see Definition 3.3.

Proof: We wish to apply Theorem 3.3 and Corollary 3.5 of [4]. For this, we need only to check that for \( \rho \in (0, \text{dist}(x, \partial U)) \)

\[
\int_{B_\rho(x)} \frac{|\text{proj}_{T^\perp}(\tilde{x} - x)|}{|\tilde{x} - x|^{n+1}} \, d\mu_{\partial T}(\tilde{x}) < \infty.
\]

This readily follows from \( \partial T \) being \( C^{1,\alpha} \) tangentially immersed. From this we get (a),(b) and that \( T \) has an area-minimizing oriented tangent cone at \( x \), with every tangent cone \( C \) of \( T \) at \( x \) being area-minimizing. The remainder of (c) follows from Lemma 3.2 and \( T \in \text{TI}^{1,\alpha}_{n,\text{loc}}(U) \). ∎

Given Definition 3.3 and having established the existence and structure of the tangent cones along the boundary of any \( T \in \text{TI}^{1,\alpha}_{n,\text{loc}}(U) \), it is convenient now to give Theorem 2.1 of [15], a partial boundary regularity result. Note that the assumption that \( T \in \text{TI}^{1,\alpha}_{n,\text{loc}}(U) \) is not needed in Theorem 2.1 of [15], but only that \( \partial T \) consists of a finite sum of \((n-1)\)-dimensional \( C^{1,\alpha} \) submanifolds which meet tangentially, with same orientation, at the boundary point considered. Nevertheless, for convenience we give here the more restrictive version.

**Theorem 3.5** For every pair \( m, \theta \in \mathbb{N} \) and \( \alpha \in (0,1] \) there is \( \delta = \delta(n,m,\theta,\delta) \in (0,1) \) so that the following holds:

Let \( U \subseteq \mathbb{R}^{n+1} \) and suppose \( T \in \text{TI}^{1,\alpha}_{n,\text{loc}}(U) \) with \( 0 \in \text{spt} \partial T \). Also suppose that \( T \) at \( 0 \) has tangent cone

\[
\mathbb{C} = (m + \theta)\mathbb{E}^n \mathbb{L} \{ y \in \mathbb{R}^n : y_n > 0 \} + \theta \mathbb{E}^n \mathbb{L} \{ y \in \mathbb{R}^n : y_n < 0 \}.
\]

Then, for \( \rho \in (0, \text{dist}(0, \partial U)) \) sufficiently small (depending on \( T \)), and with \( \beta = \frac{\alpha}{4n+6} \), there is a function

\[
u \in \left\{ \begin{array}{ll}
C^\infty(\{ y \in B^n_\rho(0) : |y_n| > (\delta/\rho)^\beta |y|^{1+\beta} \}) \\
C^{1,\beta}(\{ y \in B^n_\rho(0) : |y_n| \leq (\delta/\rho)^\beta |y|^{1+\beta} \})
\end{array} \right.
\]

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with $u(0) = 0$ and $Du(0) = 0$ so that

$$T \cdot B_\rho(0) \cap \left( \{ y \in B_n(0) : |y_n| \geq (\delta/\rho)\beta |y|^{1+\beta} \} \times \mathbb{R} \right)$$

$$= \left[ (m + \theta) F_\#(E^n \cdot \{ y \in B_n(0) : y_n \geq (\delta/\rho)\beta |y|^{1+\beta} \}) + \theta F_\#(E^n \cdot \{ y \in B_n(0) : y_n \leq -(\delta/\rho)\beta |y|^{1+\beta} \}) \right]$$

$$\cdot B_\rho(0) \cap \left( \{ y \in B_n(0) : |y_n| \geq (\delta/\rho)\beta |y|^{1+\beta} \} \times \mathbb{R} \right)$$

where $F(y) = (y, u(y))$.

**Proof:** This is Theorem 2.1 of [15]; note that we have reversed the roles of $m, \theta$ found therein. $\square$

We can now proceed to the main result of this section, Theorem 3.18. Recall that the brunt of proving Theorem 3.18 involves giving the following Lemma 3.6. In fact, Lemma 3.6 will be instrumental to the subsequent results. On account of this, we give a careful proof of Lemma 3.6. This proof is technical, but very geometric.

**Lemma 3.6** Let $U \subseteq \mathbb{R}^{n+1}$, $\alpha \in (0, 1]$, and suppose $T \in TI_{n,loc}^1(U)$ with $0 \in \text{spt } \partial T$. If $T$ at $0$ has a tangent cone which is a hyperplane with constant orientation but non-constant multiplicity (see Definition 3.3), then there is $\rho \in (0, \text{dist}(0, \partial U))$ so that $T$ at every $x \in \text{spt } \partial T \cap B_\rho(0)$ has a unique tangent cone which is a hyperplane with constant orientation but non-constant multiplicity.

Furthermore suppose $T_0 \partial T = \mathbb{R}^{n-1}$. Then $\rho \in (0, \text{dist}(0, \partial U))$ can also be chosen so that for $\mathcal{H}^{n-1}$-almost-every $z \in B_n(0)$ we have

$$\text{spt } T \cap \left( \{ z \} \times \mathbb{R}^2 \right) \cap B_\rho(0) = \Gamma \cup L$$

where $\Gamma$ is a continuous Jordan arc, smooth away from spt $\partial T$, with endpoints in $\partial B_\rho(0)$, and $L$ is a finite disjoint collection of smooth closed Jordan curves with $L \cap (\Gamma \cup \text{spt } \partial T) = \emptyset$.

**Proof:** The proof of the first part of the lemma essentially requires us to prove the second part. To this end, we analyze the cross-sections of the support of $T$ perpendicular to $T_0 \partial T$, in hopes that we can apply Lemma...
Assume (after rotation) that \( T \) at 0 has tangent cone

\[
\mathbb{C} = (m^\mathbb{C} + \theta^\mathbb{C}) \mathbb{E}^n \sqcap \{ y \in \mathbb{R}^n : y_n > 0 \} + \theta^\mathbb{C} \mathbb{E}^n \sqcap \{ y \in \mathbb{R}^n : y_n < 0 \}
\]

for \( m^\mathbb{C}, \theta^\mathbb{C} \in \mathbb{N} \). Also assume \( \rho \in (0, \text{dist}(0, \partial U)) \) is sufficiently small so that the following three occur:

First, by Definition 3.1 for some \( N, m_1, \ldots, m_N \in \mathbb{N} \)

\[
\partial T \sqcap B_\rho(0) = (-1)^n \sum_{\ell=1}^N m_\ell \Phi_{T,\ell \#}(\mathbb{E}^{n-1} \sqcap B^{n-1}_\rho(0)) \sqcap B_\rho(0)
\]

where \( \Phi_{T,\ell}(z) = (z, \varphi_{T,\ell}(z), \psi_{T,\ell}(z)) \) with \( \varphi_{T,\ell}, \psi_{T,\ell} \in C^{1,\alpha}(B^{n-1}_\rho(0)) \) satisfy
\( \varphi_{T,\ell}(0) = \psi_{T,\ell}(0) = 0 \) and \( D\varphi_{T,\ell}(0) = D\psi_{T,\ell}(0) = 0 \).

Second, for each \( z \in B^{n-1}_\rho(0) \) and \( \ell \in \{1, \ldots, N\} \) there is an orthogonal rotation \( Q^z_\ell \) so that

\[
\|Q^z_\ell - I\| < \frac{1}{8} \quad \text{and} \quad Q^z_\ell(\mathbb{R}^{n-1}) = T_{\Phi_{T,\ell}(z)} \Phi_{T,\ell}(B^{n-1}_\rho(0)).
\]

Third, by Theorem 3.5 we can assume that with \( \beta = \frac{\alpha}{4n+6} \)

\[
T \sqcap B_\rho(0) \cap \left( \{ y \in B^n_\rho(0) : |y_n| \geq (\delta/\rho)^\beta |y|^{1+\beta} \} \times \mathbb{R} \right) =
\[
\left[ (m^\mathbb{C} + \theta^\mathbb{C}) F_{\#}(\mathbb{E}^n \sqcap \{ y \in B^n_\rho(0) : y_n \geq (\delta/\rho)^\beta |y|^{1+\beta} \})
\right.
\]

\[
\left. + \theta^\mathbb{C} F_{\#}(\mathbb{E}^n \sqcap \{ y \in B^n_\rho(0) : y_n \leq -(\delta/\rho)^\beta |y|^{1+\beta} \}) \right]
\]

\[
\sqcap B_\rho(0) \cap \left( \{ y \in B^n_\rho(0) : |y_n| \geq (\delta/\rho)^\beta |y|^{1+\beta} \} \times \mathbb{R} \right)
\]

where \( F(y) = (y, u(y)) \) for a function

\[
u \in \left\{ C^\infty(\{ y \in B^n_\rho(0) : |y_n| > (\delta/\rho)^\beta |y|^{1+\beta} \})
\right.
\]

\[
\left. C^{1,\beta}(\{ y \in B^n_\rho(0) : |y_n| \geq (\delta/\rho)^\beta |y|^{1+\beta} \}) \right\}
\]

satisfying \( u(0) = 0 \) and \( Du(0) = 0 \).
To prove the first part of the lemma, suppose for contradiction there is \( z \in B_{\rho/4}^{-1}(0) \) so that (after relabeling) \( T \) at \( \Phi_{T,1}(z) \) has a tangent cone which is of a sum of half-hyperplanes with constant orientation after rotation (as in Definition 3.3). Fixing such a \( z \in B_{\rho/4}^{-1}(0) \), our goal is to contradict Lemma A.2. We begin by identifying the centers of the disjoint balls we will use in applying Lemma A.2. For this, choose \( N_z \in \{1, \ldots, N\} \) and \( N_z^{(1)} \in \{1, \ldots, N_z\} \) so that by Theorem 3.4 (after relabeling) the following three occur:

- \( \{\Phi_{T,\ell}(z)\}_{\ell=1}^{N_z} \) is the set of distinct points in \( \{\Phi_{T,\ell}(z)\}_{\ell=1}^{N_z} \).
- \( T \) at \( \Phi_{T,\ell}(z) \) for each \( \ell \in \{1, \ldots, N_z^{(1)}\} \) has a tangent cone
  \[
  C_\ell = \sum_{k=1}^{N_{C_\ell}} m_\ell^{C_\ell} (Q_\ell^k \circ Q_{\ell,k})_\# \left( \mathbb{E}^n \setminus \{y \in \mathbb{R}^n : y_n > 0\} \right)
  \]
  where \( N_{C_\ell}, m_1^{C_\ell}, \ldots, m_{N_{C_\ell}}^{C_\ell} \in \mathbb{N} \) and \( Q_{\ell,1,\ldots, Q_{\ell,N_{C_\ell}}} \) are distinct orthogonal rotations about \( \mathbb{R}^{n-1} \). Recall that the orthogonal rotations \( Q_\ell^k \) satisfy (3.7).
- \( T \) at \( \Phi_{T,\ell}(z) \) for each \( \ell \) in the possibly empty set \( \{N_z^{(1)} + 1, \ldots, N_z\} \) has tangent cone
  \[
  C_\ell = (Q_\ell^k \circ Q_{\ell,1})_\# \left( (m_\ell^{C_\ell} + \theta_\ell^{C_\ell}) \mathbb{E}^n \setminus \{y \in \mathbb{R}^n : y_n > 0\} + \theta_\ell^{C_\ell} \mathbb{E}^n \setminus \{y \in \mathbb{R}^n : y_n < 0\} \right)
  \]
  where \( m_\ell^{C_\ell}, \theta_\ell^{C_\ell} \in \mathbb{N} \) and \( Q_{\ell,1} \) is an orthogonal rotation about \( \mathbb{R}^{n-1} \).

Next, we choose the correct disjoint balls using the points \( \{\Phi_{T,\ell}(z)\}_{\ell=1}^{N_z} \), so that we may apply Lemma A.2. First, for any \( x \in \mathbb{R}^{n+1} \) and \( \sigma > 0 \) denote the bored-out ball:

\[
\hat{B}_\sigma(x) = B_\sigma(x) \setminus \{\hat{x} \in \mathbb{R}^{n+1} : |\text{proj}_{\{0\} \times \mathbb{R}^2}(\hat{x} - x)| \leq \sigma/2\}.
\]

Choose for each \( \ell = 1, \ldots, N_z \) a \( \sigma_\ell > 0 \) sufficiently small so that \( \{B_{\sigma_\ell}(\Phi_{T,\ell}(z))\}_{\ell=1}^{N_z} \) is a disjoint collection of balls with \( B_{\sigma_\ell}(\Phi_{T,\ell}(z)) \subset B_{\rho/2}(0) \), and so that the following two occur:
First, for each $\ell \in \{1, \ldots, N_z^{(1)}\}$ by Theorem 1 of [17] and (3.7), (3.9), (3.11)
\[ \eta_{\Phi_{T,\ell}(z),1} \# \mathcal{T} \setminus \hat{B}_{\sigma_{\ell}}(0) \]
\[ = \sum_{j=1}^{k} \left( Q_{\ell,k}^{y} \circ Q_{\ell,k}^{-1} \right) \# \left( \mathcal{E}^{n} \setminus \{ y \in B_{\sigma_{\ell}}^{n}(0) : y < \sigma_{\ell}/4 \} \right) \]
\[ \setminus \hat{B}_{\sigma_{\ell}}(0) \]
(3.12)
where $F_{j,k}^{\ell}(y) = (y, u_{j,k}^{\ell}(y))$ for $u_{j,k}^{\ell} \in C^{\infty}(\{ y \in B_{\sigma_{\ell}}^{n}(0) : y > \sigma_{\ell}/4 \})$. For each $k \in \{1, \ldots, N_{\ell}\}$ and $j, j' \in \{1, \ldots, m_{k}^{\gamma_{\ell}}\}$, the graphs
\[ \text{graph}(y \in B_{\sigma_{\ell}}^{n}(0) : y > \sigma_{\ell}/4) u_{j,k}^{\ell} \quad \text{and} \quad \text{graph}(y \in B_{\sigma_{\ell}}^{n}(0) : y > \sigma_{\ell}/4) u_{j',k}^{\ell} \]
are either equal or disjoint. Meanwhile, if $k, \tilde{k} \in \{1, \ldots, N_{\ell}\}$ with $k \neq \tilde{k}$ then
\[ Q_{\ell,k}(\text{graph}(y \in B_{\sigma_{\ell}}^{n}(0) : y > \sigma_{\ell}/4) u_{j,k}^{\ell}) \cap Q_{\ell,\tilde{k}}(\text{graph}(y \in B_{\sigma_{\ell}}^{n}(0) : y > \sigma_{\ell}/4) u_{j',k}^{\ell}) = \emptyset \]
for any $j \in \{1, \ldots, m_{k}^{\gamma_{\ell}}\}$ and $\tilde{j} \in \{1, \ldots, m_{\tilde{k}}^{\gamma_{\ell}}\}$.

Second, for each $\ell \in \{N_z^{(2)} + 1, \ldots, N_z\}$ and with $\beta = \frac{\alpha}{4n+6}$
\[ \eta_{\Phi_{T,\ell}(z),1} \# \mathcal{T} \setminus \{ x \in B_{\sigma_{\ell}}^{n}(0) : |x_{n}| > (\delta/\sigma_{\ell})^{\beta} \text{proj}_{R^{n}} x^{1+\beta} \} \]
\[ = (Q_{\ell,1} \circ Q_{\ell,1}) \# \left( (m_{C_{\ell}}^{\gamma_{\ell}} + \theta_{\ell}^{C_{\ell}}) F_{\#}^{\ell} \left( \mathcal{E}^{n} \setminus \{ y \in B_{\sigma_{\ell}}^{n}(0) : y > (\delta/\sigma_{\ell})^{\beta} |y|^{1+\beta} \} \right) \right. \]
\[ + \left. \theta_{\ell}^{C_{\ell}} F_{\#}^{\ell} \left( \mathcal{E}^{n} \setminus \{ y \in B_{\sigma_{\ell}}^{n}(0) : y < -(\delta/\sigma_{\ell})^{\beta} |y|^{1+\beta} \} \right) \right] \]
\[ \setminus \{ x \in B_{\sigma_{\ell}}^{n}(0) : |x_{n}| > (\delta/\sigma_{\ell})^{\beta} \text{proj}_{R^{n}} x^{1+\beta} \} \]
(3.13)
where $F^{\ell}(y) = (y, u^{\ell}(y))$ for
\[ u^{\ell} \in \left\{ C^{\infty}(\{ y \in B_{\sigma_{\ell}}^{n}(0) : |y_{n}| > (\delta/\sigma_{\ell})^{\beta} |y|^{1+\beta} \}) \right. \]
\[ \left. \cup C^{1,\beta}(\{ y \in B_{\sigma_{\ell}}^{n}(0) : |y_{n}| > (\delta/\sigma_{\ell})^{\beta} |y|^{1+\beta} \}) \right. \]

In order to apply Lemma A.2 and obtain a contradiction, we must consider the support of $T$ over a cross-section $\{ \tilde{z} \} \times R^{2}$ with suitably chosen $\tilde{z} \in R^{n-1}$ near $z$. To this end, define the set
\[ \mathcal{T} = \left\{ \begin{array}{l}
\tilde{z} \in B_{\rho}^{n-1}(0) : \text{and} \quad *T(\tilde{x}) \notin R^{n-1} \quad \text{for all} \\
x \in (\text{spt } T \setminus \text{spt } \partial T) \cap (\text{clos } B_{\rho}(0)) \end{array} \right\} \]
(3.14)
Observe that $\ast \tilde{T}(x) \notin \mathbb{R}^{n-1}$ for $x \in \text{reg} T$ implies that $T, T \cap (\{0\} \times \mathbb{R}^2)$ is a one-dimensional subspace in $\mathbb{R}^{n+1}$. Therefore $\mathcal{H}^{n-1}(T) = 0$ by Sard’s theorem as well as interior regularity for co-dimension one area-minimizing currents. Fix $\tilde{z} \in B_{\min(\sigma_1, \ldots, \sigma_N)}(z) \cap T$.

We conclude $\text{spt} T \cap (\text{clos} B_{\rho}(0)) \cap (\{\tilde{z}\} \times \mathbb{R}^2)$ is a disjoint union of smooth Jordan arcs with endpoints only at $\partial B_{\rho}(0)$ or $\text{spt} \partial T$, together with a disjoint collection of smooth closed Jordan curves (we shall soon be more precise). This will allow us to apply Lemma \text{A.2} In order to do so, let

$$V \in C\left( (\text{spt} T \setminus \text{spt} \partial T) \cap (\text{clos} B_{\rho}(0)) \cap (\{\tilde{z}\} \times \mathbb{R}^2); \{0\} \times S^1 \right)$$

be defined for $x \in (\text{spt} T \setminus \text{spt} \partial T) \cap (\text{clos} B_{\rho}(0)) \cap (\{\tilde{z}\} \times \mathbb{R}^2)$ by

$$V(x) = \frac{\text{proj}_{(0) \times \mathbb{R}^2} \ast \tilde{T}(x)}{||\text{proj}_{(0) \times \mathbb{R}^2} \ast \tilde{T}(x)||}.$$

Then by (3.8), (3.12), (3.13), (3.14) we have

$$\text{spt} T \cap (\text{clos} B_{\rho}(0)) \cap (\{\tilde{z}\} \times \mathbb{R}^2) \setminus K_z = \left[ \Gamma_1 \cup \Gamma_2 \cup \left( \bigcup_{\ell=1}^{N} \bigcup_{k=1}^{N_{\ell}} \bigcup_{j=1}^{G_{\ell}} \gamma_{j,k}^{\ell} \right) \right. \left. \cup \left( \bigcup_{\ell=N_{\ell}+1}^{N} (G_{\ell} \cup g_{\ell}) \right) \right] \cup L \setminus K_z$$

where

$$K_z = \bigcup_{\ell=1}^{N} \{ x \in B_{\sigma_{\ell}}(\Phi_{T,\ell}(z)) : |\text{proj}_{(0) \times \mathbb{R}^2}(x - \Phi_{T,\ell}(z))| < \sigma_{\ell}/2 \}$$

and the following hold:

1. $\Gamma_1, \Gamma_2$ are Jordan arcs so that by (3.8) (for $\delta > 0$ sufficiently small) with $\tilde{B}_{\rho}(0)$ as in (3.11),

$$\text{spt} T \cap (\text{clos} \tilde{B}_{\rho}(0)) \cap (\{\tilde{z}\} \times \mathbb{R}^2) = (\Gamma_1 \cup \Gamma_2) \cap (\text{clos} \tilde{B}_{\rho}(0)) \cap (\{\tilde{z}\} \times \mathbb{R}^2)$$

We can parameterize $\Gamma_1, \Gamma_2$ by arc-length so that

$$\Gamma_1 \in \left\{ C([0, \mathcal{H}^1(\Gamma_1)]; (\text{clos} B_{\rho}(0)) \cap (\{\tilde{z}\} \times \mathbb{R}^2)) \right\}$$

$$C^\infty ((0, \mathcal{H}^1(\Gamma_1)); B_{\rho}(0) \cap (\{\tilde{z}\} \times \mathbb{R}^2))$$
and
\[
\Gamma_2 \in \begin{cases} C([0, \mathcal{H}^1(\Gamma_2)]; (\text{clos} B_\rho(0)) \cap \{[\tilde{z}] \times \mathbb{R}^2\}) \\ C^\infty((0, \mathcal{H}^1(\Gamma_2)); B_\rho(0) \cap \{[\tilde{z}] \times \mathbb{R}^2\}) \end{cases}
\]
\[
\Gamma_1(0), \Gamma_2(0) \in \partial B_\rho(0).
\]
\[
\Gamma_1([0, \mathcal{H}^1(\Gamma_1))] \cap \Gamma_2([0, \mathcal{H}^1(\Gamma_2))] = \emptyset
\]
(for this, see (3.16) below), while \(\Gamma_1(\mathcal{H}^1(\Gamma_1)) = \Phi_{T,\ell_1}(z_1)\) and \(\Gamma_2(\mathcal{H}^1(\Gamma_2)) = \Phi_{T,\ell_2}(z_1)\) for some \(\ell_1, \ell_2 \in \{1, \ldots, N\}\). Thus, \(\Gamma_1(\mathcal{H}^1(\Gamma_1)), \Gamma_2(\mathcal{H}^1(\Gamma_2)) \in K_2\) by (3.7).

With this parameterization, we have in the sense of Definition A.1 that \(\Gamma_1, V\) are positively oriented while \(\Gamma_2, V\) are negatively oriented, by (3.8). The density of \(T\) also satisfies by (3.8)

\[
\Theta_T(x) = m^C + \theta^C \quad \text{for} \quad x \in \Gamma_1([0, \mathcal{H}^1(\Gamma_1))]
\]
\[
\Theta_T(x) = \theta^C \quad \text{for} \quad x \in \Gamma_2([0, \mathcal{H}^1(\Gamma_2))]
\]

(2) For each \(\ell = 1, \ldots, N_2^{(1)}\) we have that \(\{\gamma_{j,k}^\ell\}_{j=1,k=1}^{m_\ell, \ell} \subset C\) is a collection of Jordan arcs so that by (3.12)

\[
spt T \cap \hat{B}_{\sigma_\ell}(\Phi_{T,\ell}(z)) \cap ([\tilde{z}] \times \mathbb{R}^2) = \bigcup_{k=1}^{N_\ell} \bigcup_{j=1}^{m_\ell, \ell} \gamma_{j,k}^\ell \cap \hat{B}_{\sigma_\ell}(\Phi_{T,\ell}(z)).
\]

We can parameterize each arc

\[
\gamma_{j,k}^\ell \in \begin{cases} C([0, \mathcal{H}^1(\gamma_{j,k}^\ell)]; (\text{clos} B_\rho(0)) \cap \{[\tilde{z}] \times \mathbb{R}^2\}) \\ C^\infty((0, \mathcal{H}^1(\gamma_{j,k}^\ell)); B_\rho(0) \cap \{[\tilde{z}] \times \mathbb{R}^2\}) \end{cases}
\]

by arc-length so that

\[
\gamma_{j,k}^\ell(\mathcal{H}^1(\gamma_{j,k}^\ell)) \in \{x \in B_{\sigma_\ell}(\Phi_{T,\ell}(z)) : |\text{proj}_{[0]} \times \mathbb{R}^2(x - \Phi_{T,\ell}(z))| < \sigma_\ell/2\}.
\]

Meanwhile, \(\gamma_{j,k}^\ell(0) \in (\partial B_1(0)) \cup K_2\) with \(\gamma_{j,k}^\ell \cap (\partial B_{\sigma_\ell}(\Phi_{T,\ell}(z)) \cap ([\tilde{z}] \times \mathbb{R}^2) = \emptyset\). For each \(k \in \{1, \ldots, N_\ell\}\) the images \(\gamma_{j,k}^\ell(\mathcal{H}^1(\gamma_{j,k}^\ell))\) and \(\gamma_{j,k}^\ell(0, \mathcal{H}^1(\gamma_{j,k}^\ell))\) are either equal or
disjoint for each \( j, \tilde{j} \in \{1, \ldots, m^{C_\ell}_k \} \), by (3.12). On the other hand, the images \( \gamma^{C_\ell}_{j,k}((0, \mathcal{H}^1(\gamma^{C_\ell}_{j,k}))) \) and \( \gamma^{C_\ell}_{\tilde{j},k}((0, \mathcal{H}^1(\gamma^{C_\ell}_{\tilde{j},k}))) \) are disjoint if \( k \neq \tilde{k} \).

With this parameterization we have that \( \gamma^{C_\ell}_{j,k}, V \) are positively oriented (as in Definition A.1), by (3.12).

(3) For each \( \ell \) in the (possibly empty) set \( \{N_z^{(1)} + 1, \ldots, N_z\} \) we have that \( G^\ell, g^\ell \) are Jordan arcs so that by (3.13)

\[
\text{spt } T \cap \tilde{B}_{\sigma_\ell}(\Phi_{T,\ell}(z)) \cap (\{ \tilde{z} \} \times \mathbb{R}^2) = (G^\ell \cup g^\ell) \cap \tilde{B}_{\sigma_\ell}(\Phi_{T,\ell}(z)).
\]

We can parameterize each arc

\[
G^\ell \in C([0, \mathcal{H}^1(G^\ell)]; (\text{clos } B_\rho(0)) \cap (\{ \tilde{z} \} \times \mathbb{R}^2))
\]

and

\[
g^\ell \in C([0, \mathcal{H}^1(g^\ell)]; (\text{clos } B_\rho(0)) \cap (\{ \tilde{z} \} \times \mathbb{R}^2))
\]

by arc-length so that

\[
G^\ell(\mathcal{H}^1(G^\ell)), g^\ell(\mathcal{H}^1(g^\ell)) \in \{ x \in B_{\sigma_\ell}(\Phi_{T,\ell}(z)) : \text{proj}_{\{0\} \times \mathbb{R}^2}(x - \Phi_{T,\ell}(z)) < \sigma_\ell/2 \}.
\]

Meanwhile, \( G^\ell(0), g^\ell(0) \in (\partial B_1(0)) \cup K_z, \)

\[
G^\ell \cap (\partial B_{\sigma_\ell}(\Phi_{T,\ell}(z))) \cap (\{ \tilde{z} \} \times \mathbb{R}^2) \neq \emptyset,
\]

and

\[
g^\ell \cap (\partial B_{\sigma_\ell}(\Phi_{T,\ell}(z))) \cap (\{ \tilde{z} \} \times \mathbb{R}^2) \neq \emptyset.
\]

Moreover, the images \( G^\ell((0, \mathcal{H}^1(G^\ell))) \) and \( g^\ell((0, \mathcal{H}^1(g^\ell))) \) are disjoint (see (3.17) below).

With this parameterization \( G^\ell, V \) are positively oriented while \( g^\ell, V \) are negatively oriented, by (3.13). The density of \( T \) also satisfies

\[
\Theta_T(x) = m^{C_\ell} + \theta^{C_\ell} \text{ for } x \in G^\ell((0, \mathcal{H}^1(G^\ell))),
\]

\[
\Theta_T(x) = \theta^{C_\ell} \text{ for } x \in g^\ell((0, \mathcal{H}^1(g^\ell))).
\]

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(4) $L$ is a disjoint union of smooth closed Jordan curves with

$$L \subset \{ x \in B_\rho(0) : |\text{proj}_{(0) \times \mathbb{R}^2} x| < \rho/2 \} \cap (\{ \tilde{z} \} \times \mathbb{R}^2) \setminus \left( \bigcup_{\ell=1}^{N_z} B_{\sigma_\ell}(\Phi_{T,\ell}(z)) \right)$$

and so that

$$L \cap \left[ \Gamma_1 \cup \Gamma_2 \cup \left( \bigcup_{\ell=1}^{N_x} \bigcup_{j=1}^{N_{\ell}} \bigcup_{k=1}^{m_{\ell}^j} \gamma_{j,k}^{T,\ell} \right) \cup \left( \bigcup_{\ell=1}^{N_x} (G^\ell \cup g^\ell) \right) \right] = \emptyset.$$

Setting

$$G = \Gamma_1 \cup \Gamma_2 \cup \left( \bigcup_{\ell=1}^{N_x} \bigcup_{j=1}^{N_{\ell}} \bigcup_{k=1}^{m_{\ell}^j} \gamma_{j,k}^{T,\ell} \right) \cup \left( \bigcup_{\ell=1}^{N_x} (G^\ell \cup g^\ell) \right)$$

means that we can apply Lemma A.2(b) with $V$ as in (3.15) in order to contradict that $\Gamma_1, V$ are positively oriented while $\Gamma_2, V$ are negatively oriented. This shows the first part of the theorem.

For the second part of the theorem, our aim is to apply Lemma A.2(b). Choose any $z \in B_{\rho/4}^{-1}(0) \cap \mathcal{T}$ with $\mathcal{T}$ as in (3.14). Take again $N_z \in \{1, \ldots, N\}$ so that (after relabeling) $\{\Phi_{T,\ell}(z)\}_{\ell=1}^{N_z}$ is the collection of distinct points in $\{\Phi_{T,\ell}(z)\}_{\ell=1}^{N_z}$. Again, we choose a disjoint collection of balls $\{B_{\sigma_\ell}(\Phi_{T,\ell}(z))\}_{\ell=1}^{N_z}$ with $B_{\sigma_\ell}(\Phi_{T,\ell}(z)) \subset B_{\rho/2}(z)$ and $\sigma_\ell > 0$ sufficiently small so that (3.13) holds now for each $\ell \in \{1, \ldots, N_z\}$.

In this case we conclude

$$\text{spt } T \cap (\text{clos } B_\rho(0)) \cap (\{z\} \times \mathbb{R}^2) = \Gamma_1 \cup \Gamma_2 \cup \left( \bigcup_{\ell=1}^{N_x} (G^\ell \cup g^\ell) \right) \cup L$$

where the following hold:

- $\Gamma_1, \Gamma_2$ are as in (1) above, in the proof of the first part of the theorem, but of course with $\tilde{z} = z$.

- For each $\ell \in \{1, \ldots, N_z\}$, the arcs $G^\ell, g^\ell$ are as in (3) above, but of course with $\tilde{z} = z$. In this case we further conclude by (3.13) that $G^\ell(\mathcal{H}^1(G^\ell)) = g^\ell(\mathcal{H}^1(g^\ell)) = \Phi_{T,\ell}(z)$.
• $L$ is a disjoint collection of smooth closed Jordan curves as in (4) above, disjoint from the arcs $\Gamma_1, \Gamma_2, \{G^\ell, g^\ell\}_{\ell=1}^{N_z}$.

With $V$ as in (3.15) (but of course again with $\tilde{z} = z$) and

$$G = \Gamma_1 \cup \Gamma_2 \cup \bigcup_{\ell=1}^{N_z} (G^\ell \cup g^\ell),$$

then we may apply Lemma A.2(b), since $\Gamma_1, V$ are positively oriented while $\Gamma_2, V$ are negatively oriented.

Assume for contradiction that, in applying Lemma A.2(b), we conclude

$$G = \Gamma \cup \bigcup_{\ell=1}^{N_z^{(2)}} L_\ell$$

where $\Gamma \subset (\text{clos } B_\rho(0)) \cap (\{z\} \times \mathbf{R}^2)$ is a continuous Jordan arc, smooth away from the points $\{\Phi_{T,\ell}(z)\}_{\ell=1}^{N_z}$ and with endpoints in $\partial B_\rho(0)$.

Meanwhile, for each $\ell \in \{1, \ldots, N_z^{(2)}\}$ we have that $L_\ell \subset B_{\rho/2}(z) \cap (\{z\} \times \mathbf{R}^2)$ is a continuous closed Jordan curve, smooth away from the points $\{\Phi_{T,\ell}(z)\}_{\ell=1}^{N_z}$. Furthermore, $\Gamma, L_1, \ldots, L_{N_z^{(2)}}$ have pairwise disjoint images.

Consider $L_1$, then $L_1 \cap \{\Phi_{T,\ell}(z)\}_{\ell=1}^{N_z} \neq \emptyset$ by Lemma A.2(b). We can suppose (after relabeling) that $\Phi_{T,1}(z) \in L_1$. This exactly means $G^1, g^1 \subset L_1$. On the other hand, $G^1((0, \mathcal{H}^1(G^1)))$ and $g^1((0, \mathcal{H}^1(g^1)))$ are disjoint by (3) above.

This implies that (after relabeling) $\Phi_{T,2}(z) \in L_1$ with $G^1 = g^2$ (see the proof of Lemma A.2, particularly in concluding (A.3),(A.4)). We therefore have $G^2 \subset L_1$ as well.

Arguing iteratively, we conclude there is $N_z^{L_1} \in \{2, \ldots, N_z\}$ so that (after relabeling)

$$G^1 = g^2$$

$$
\vdots
$$

$$G^{N_z^{L_1}-1} = g^{N_z^{L_1}}$$

$$G^{N_z^{L_1}} = g^1.$$
But then (3.17) implies

\[
m^1 + \theta^1 = \theta^2 \\
\vdots \\
m^{N_L^1} + \theta^{N_L^1} = \theta^{N_L^1} \\
m^{N_L^1} + \theta^{N_L^1} = \theta^1,
\]

which gives

\[
\theta^1 < m^1 + \theta^1 = \theta^2 \leq \ldots \leq \theta^{N_L^1} < m^{N_L^1} + \theta^{N_L^1} = \theta^1,
\]
a contradiction. We conclude the second part of the theorem. \(\Box\)

We are now ready to state and prove our first main result.

**Theorem 3.18** Let \(U \subseteq \mathbb{R}^{n+1}, \alpha \in (0, 1], \) and \(T \in \mathcal{T}_1^{1,\alpha}(U).\) Suppose \(x \in \text{spt}\, \partial T\) and that \(T\) at \(x\) has tangent cone which is a hyperplane with constant orientation but non-constant multiplicity (as in Definition 3.3). Then there is a \(\rho \in \text{dist}(0, \text{dist}(x, \partial U))\) and a solution to the minimal surface equation \(u \in C^\infty(B^n_\rho(0))\) with \(u(0) = 0\) and \(Du(0) = 0\) such that

\[
\text{spt} \, T \cap B_\rho(x) = \eta_{x,1}(Q(\text{graph}B^n_\rho(0))) \cap B_\rho(x)
\]

for an orthogonal rotation \(Q\). The orientation vector for \(T\) if \(\bar{x} \in \text{spt} \, T \cap B_\rho(x),\) is given by

\[
\ast\vec{T}(\bar{x}) = Q \left( \left( \frac{-Du}{\sqrt{1 + |Du|^2}}, \frac{1}{\sqrt{1 + |Du|^2}} \right) \right)_{\text{proj}_{\mathbb{R}^n}}.
\]

**Proof:** Suppose (after translation) that \(0 \in \text{spt}\, \partial T,\) and that \(T\) at \(0\) has a tangent cone which is a hyperplane with constant orientation but non-constant multiplicity. Choose \(\rho \in (0, \text{dist}(0, \partial U))\) so that Lemma 3.6 holds, and so that (after rotation)

\[
\partial T \pitchfork B_\rho(x) = (-1)^n \sum_{\ell=1}^N m_\ell \Phi_{T,\ell}(E^n_{\rho}(B_{\rho}^{n-1}(0)) \pitchfork B_\rho(0)
\]

for \(N, m_1, \ldots, m_N \in \mathbb{N}\) where \(\Phi_{T,\ell}(z) = (z, \varphi_{T,\ell}(z), \psi_{T,\ell}(z))\) with \(\varphi_{T,\ell}, \psi_{T,\ell} \in C^{1,\alpha}(B_{\rho}^{n-1}(0))\) satisfying \(\varphi(0) = \psi(0) = 0\) and \(D\varphi(0) = D\psi(0) = 0\) for each \(\ell \in \{1, \ldots, N\}.\) We proceed by induction on \(N.\)
If $N = 1$, then $\text{spt} \partial T \cap B_\rho(0) \subset \text{reg} \partial T$. We conclude the theorem in this case by [8] and [20], since $T$ at 0 has tangent cone which is a hyperplane with constant orientation but non-constant multiplicity.

Now suppose $N \geq 2$. We can of course assume $0 \in \text{sing} \partial T$. Define the set

$$M = \left( (\text{spt} T \setminus \text{sing} T) \cup (\text{spt} \partial T \setminus \bigcap_{\ell=1}^{N} \{ \Phi_{T,\ell}(z) : z \in B_{\rho}^{n-1}(0) \}) \right) \cap B_\rho(0).$$

We claim $M$ is a smooth orientable hypersurface, smoothly oriented by $*\vec{T}$. For this, consider $x \in M$, then there are three possibilities. First, $x \in \text{reg} T \cap B_\rho(0)$, in which case $M$ near $x$ is a smooth orientable hypersurface, smoothly oriented by $\vec{T}$. Second, $x \in \text{reg} \partial T \cap B_\rho(0)$, in which case by the first part of Lemma 3.6 and the boundary regularity given by [8], [20] there is $\sigma > 0$ such that $M \cap B_\sigma(x)$ is a smooth orientable hypersurface, smoothly oriented by $*\vec{T} \in C^{\infty}(M \cap B_\sigma(x); \mathbb{R}^{n+1})$. Third, $x \in \text{sing} \partial T \cap B_\rho(0)$, in which case by induction there is again $\sigma > 0$ such that $M \cap B_\sigma(x)$ is a smooth orientable hypersurface, smoothly oriented by $*\vec{T} \in C^{\infty}(M; \mathbb{R}^{n+1})$.

Consider the current $[M]$, where $M$ is oriented by $*\vec{T}|_M \in C^{\infty}(M; \mathbb{R}^{n+1})$. Note that $\text{spt}[M] \cap B_\rho(0) = \text{spt} T \cap B_\rho(0)$ (by, for example, Lemma 3.6 and Theorem 3.5). We claim $\partial [M] \llcorner B_\rho(0) = 0$. To see this, observe that

$$\text{spt} \partial [M] \cap B_\rho(0) \subset \bigcap_{\ell=1}^{N} \{ \Phi_{T,\ell}(z) : z \in B_{\rho}^{n-1}(0) \} \cap B_\rho(0)$$

by interior regularity for area-minimizing currents, [8], [20], and induction. By the constancy theorem (see 4.1.31 of [9]) we conclude that for each $\ell \in \{1, \ldots, N\}$

$$\partial [M] \llcorner B_\rho(0) = m_{\ell}^{M} \Phi_{T,\ell\#}(E^{n-1} \llcorner B_{\rho}^{n-1}(0)) \llcorner B_\rho(0)$$

for some integer $m_{\ell}^{M}$.  

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Recall that we have assumed $0 \in \text{sing} \partial T$. Thus, there is a $z \in B_{\rho}^{n-1}(0)$ so that (after relabeling) $\Phi_{T,1}(z) \in B_{\rho}(0) \setminus \bigcap_{\ell=1}^{N} \Phi_{T,\ell}(B_{\rho}^{n-1}(0))$. This implies we must have $m_{1}^{M} = 0$, so that $\partial[M] \cap B_{\rho}(0) = 0$ as claimed.

Now, $[M]$ is area-minimizing by Lemma 33.4 of [15]. Recall as well that $\text{spt}[M] \cap B_{\rho}(0) = \text{spt} T \cap B_{\rho}(0)$. We conclude (using, for example, Theorem 1 of [17]) along with the fact that $T$ has tangent cone at 0 which is a hyperplane with constant orientation but non-constant multiplicity) that $0 \in \text{reg}[M]$. We conclude the theorem, by induction. $\square$

In the remainder of this section we give two lemmas that we shall need in the subsequent sections. We give these lemmas now, as they regard $T \in \text{TI}^{1,\alpha}_{n,\text{loc}}(U)$. First, Lemma 3.19 shows the existence of half-regular points along the boundary of $T \in \text{TI}^{1,\alpha}_{n,\text{loc}}$ near any singular point of $\partial T$. Half-regular points were used in studying the $c$-Plateau problem (see §1.4), and in that context appeared in Lemma 1 of [16]; see as well Lemma 6.5 forthcoming. We shall make use of half-regular points in proving Theorem 5.12.

**Lemma 3.19** Let $U \subset \mathbb{R}^{n+1}$, $\alpha \in (0,1]$, and $T \in \text{TI}^{1,\alpha}_{n,\text{loc}}(U)$. Suppose $x \in \text{sing} \partial T$ and that $\rho \in (0,\text{dist}(x, \partial U))$ is as in Definition [3.1], so that

$$
\partial T \cap B_{\rho}(x) = (-1)^{n} \sum_{\ell=1}^{N} m_{\ell} \left[ (\eta_{-x,1} \circ Q \circ \Phi_{T,\ell})_{\#}(\mathbb{E}^{n-1} \cap B_{\rho}^{n-1}(0)) \right] \cap B_{\rho}(x)
$$

for $N, m_{1}, \ldots, m_{N} \in \mathbb{N}$, an orthogonal rotation $Q$, and $\Phi_{T,\ell} \in C^{1,\alpha}(B_{\rho}^{n-1}(0); \mathbb{R}^{n+1})$ for each $\ell = 1, \ldots, N$. Then there is $z \in B_{\rho}^{n-1}(0)$, a radius $\sigma \in (0, \rho - |z|]$, and a non-empty set $\mathcal{N} \subseteq \{1, \ldots, N\}$ so that:

- $\Phi_{T,\ell}(B_{\rho}^{n-1}(z)) \subset \text{reg} \partial T$ for each $\ell \in \mathcal{N}$,
- $\Phi_{T,\ell}(B_{\rho}^{n-1}(z)) \cap \Phi_{T,\ell}(B_{\rho}^{n-1}(z)) = \emptyset$ for some $\ell, \tilde{\ell} \in \mathcal{N}$,
- $\cap_{\ell \in \mathcal{N}} \Phi_{T,\ell}(\partial B_{\rho}^{n-1}(z)) = \emptyset$.

With this in mind, we say that any point

$$
x \in \bigcap_{\ell \in \mathcal{N}}(\eta_{-x,1} \circ Q \circ \Phi_{T,\ell})(\partial B_{\rho}^{n-1}(z))
$$

is half-regular.
Proof: Suppose (after translation) $0 \in \text{sing} \partial T$. Also assume
\( \rho \in (0, \text{dist}(0, \partial U)) \) is such that (after rotation)
\[
\partial T \mathbin{\#} B_\rho(0) = (-1)^n \sum_{t=1}^{N} m_t \Phi_{T,t}(E^{n-1} \mathbin{\#} B^{n-1}_\rho(0)) \mathbin{\#} B_\rho(0).
\]

We now argue by induction on $N \geq 2$.

First, suppose $N = 2$. Since $0 \in \text{sing} \partial T$, then there is $z \in B^{n-1}_\rho(0) \setminus \{0\}$ so that $\Phi_{T,1}(z) \neq \Phi_{T,2}(z)$. We can thus find $\sigma \in (0, |z|]$ so that $\Phi_{T,1}^\sigma(z) \neq \Phi_{T,2}(z)$ for each $z \in B^{n-1}_\sigma(z)$, but there is $x \in \Phi_{T,1}(\partial B^{n-1}(z)) \cap \Phi_{T,2}(\partial B^{n-1}(z))$. This proves the case $N = 2$.

Second, suppose $N \geq 3$. Since $0 \in \text{sing} \partial T$ we can suppose (after relabeling) there is $z^1 \in B^{n-1}_{\rho/2}(0)$ and $\sigma_1 \in (0, |z^1|]$ so that $\Phi_{T,1}(z) \neq \Phi_{T,2}(z)$ for each $z \in B^{n-1}_{\sigma_1}(z^1)$ while $\Phi_{T,1}(\partial B^{n-1}_{\sigma_1}(z^1)) \cap \Phi_{T,2}(\partial B^{n-1}_{\sigma_1}(z^1)) = \emptyset$. We now consider the two cases: $\Phi_{T,\ell}(z^1) \in \text{reg} \partial T$ for each $\ell = 1, \ldots, N$; $\Phi_{T,\ell_1}(z^1) \in \text{sing} \partial T$ for some $\ell_1 \in \{1, \ldots, N\}$.

- Suppose $\Phi_{T,\ell}(z^1) \in \text{reg} \partial T$ for each $\ell = 1, \ldots, N$.

Then we can find $\sigma_2 \in (0, \sigma_1]$ such that $\Phi_{T,\ell}(B^{n-1}_{\sigma_2}(z^1)) \subset \text{reg} \partial T$ for each $\ell = 1, \ldots, N$ but so that there is $z^2 \in \partial B^{n-1}_{\sigma_2}(z^1)$ with $\Phi_{T,\ell_2}(z^2) \in \text{sing} \partial T$ for some $\ell_2 \in \{1, \ldots, N\}$. This leads to two sub-cases:

  - Suppose $\Phi_{T,\ell_2}(z^2) \in \bigcap_{\ell=1}^{N} \Phi_{T,\ell}(\partial B^{n-1}_{\sigma_2}(z^1))$.

  We thus let $z = z^1$, $\sigma = \sigma_2$, and $x = \Phi_{T,1}(z^2) = \ldots = \Phi_{T,N}(z^2)$.

  Then we have $\Phi_{T,\ell}(B^{n-1}_{\sigma_2}(z)) \subset \text{reg} \partial T$ for each $\ell \in \{1, \ldots, N\}$.

  Furthermore, since $\Phi_{T,1}(z) = \Phi_{T,1}(z_1) \neq \Phi_{T,2}(z_1) = \Phi_{T,2}(z)$, then $\Phi_{T,1}(B^{n-1}_{\sigma_2}(z)) \cap \Phi_{T,2}(B^{n-1}_{\sigma_2}(z)) = \emptyset$. Since $x \in \bigcap_{\ell=1}^{N} \Phi_{T,\ell}(\partial B^{n-1}_{\sigma_2}(z))$, then we conclude the lemma in this case with $N = \{1, \ldots, N\}$.

  - Suppose $\Phi_{T,\ell_2}(z^2) \notin \bigcap_{\ell=1}^{N} \Phi_{T,\ell}(\partial B^{n-1}_{\sigma_2}(z^1))$.

    Recall that $\Phi_{T,\ell} \in C^{1,\alpha}(B^{n-1}_\rho; \mathbb{R}^{n+1})$ for each $\ell = 1, \ldots, N$. Then $\Phi_{T,\ell_2}(z^2) \in \text{sing} \partial T$ and $\Phi_{T,\ell_2}(z^2) \notin \bigcap_{\ell=1}^{N} \Phi_{T,\ell}(\partial B^{n-1}_{\sigma_2}(z^1))$ imply
that, by induction, we can find \( z \in B^n_{\rho}(0) \) (in fact, with \( z \) close to \( z^2 \)), a radius \( \sigma \in (0, \rho - |z|] \), and a non-empty set \( \mathcal{N} \subseteq \{1, \ldots, N\} \) with \( \ell_{z^2} \in \mathcal{N} \) such that there is a half-regular \( x \in \cap_{\ell \in \mathcal{N}} \Phi_{T, \ell}(\partial B^n_{\sigma}(z)) \) as required.

- Suppose \( \Phi_{T, \ell_{z^2}}(z^1) \in \text{sing} \partial T \) for some \( \ell_{z^2} \in \{1, \ldots, N\} \).

Since \( \Phi_{T,1}(z^1) \neq \Phi_{T,2}(z^1) \), then either \( \Phi_{T, \ell_{z^2}}(z^1) \neq \Phi_{T,1}(z^1) \) or \( \Phi_{T, \ell_{z^2}}(z^1) \neq \Phi_{T,2}(z^1) \). Since \( \Phi_{T, \ell_{z^2}}(z^1) \in \text{sing} \partial T \), then this case also holds by induction.

We thus conclude the lemma. □

The next lemma will be convenient for the proof of Theorem \( \ref{5.13} \).

**Lemma 3.20** Let \( U \subseteq_\alpha \mathbb{R}^{n+1} \), \( \alpha \in (0, 1] \), and suppose \( T \in T^1_{1\cdot \text{loc}}(U) \). Suppose \( x \in \text{spt} \partial T \) and there exists \( \rho \in (0, \text{dist}(x, \partial U)) \) so that

\[
\text{spt} T \cap B_\rho(x) = \bigcup_{a=1}^A (\text{clos} \ M_a) \cap B_\rho(x)
\]

for \( C^1 \) hypersurfaces-with-boundary \( M_1, \ldots, M_a \) in \( B_\rho(x) \). Then there is \( \sigma \in (0, \rho) \) so that

\[
\text{spt} T \cap B_\sigma(x) = \bigcup_{\{a \in \{1, \ldots, A\} : x \in \text{clos} \ M_a\}} (\text{clos} \ M_a) \cap B_\sigma(x),
\]

and \( T_x \partial T \subset T_x M_a \) for each \( a \in \{1, \ldots, A\} \) with \( x \in \text{clos} \ M_a \).

**Proof:** Choose \( \sigma \in (0, \rho) \) so that \( B_\sigma(x) \cap \text{clos} \ M_a = \emptyset \) for each \( a \in \{1, \ldots, A\} \) with \( x \notin \text{clos} \ M_a \), then \( \text{spt} T \cap B_\sigma(x) = \bigcup_{\{a \in \{1, \ldots, A\} : x \in \text{clos} \ M_a\}} (\text{clos} \ M_a) \cap B_\sigma(x) \).

Suppose for contradiction \( x \in \text{clos} \ M_a \) but \( T_x \partial T \notin T_x M_a \). Let \( C \) be any tangent cone of \( T \) at \( x \), then Theorem \( \ref{3.34} \) implies that either \( C \) is a sum of half-hyperplanes, each containing \( T_x \partial T \), with constant orientation after rotation, or \( C \) is a hyperplane, containing \( T_x \partial T \), with constant orientation but non-constant multiplicity. Then \( T_x \partial T \notin T_x M_a \) implies we can find \( \tilde{x} \in T_x M_a \setminus \{0\} \) so that \( \{t \tilde{x} : t \in \mathbb{R}\} \cap \text{spt} C = \{0\} \). However, the fact that \( M_a \) is a \( C^1 \) hypersurface-with-boundary with \( 0 \in \text{clos} \ M_a \) and Theorem 5.4.2 of \( \ref{6} \) imply that either \( \{t \tilde{x} : t \geq 0\} \subset \text{spt} C \), \( \{t \tilde{x} : t \leq 0\} \subset \text{spt} C \), or \( \{t \tilde{x} : t \in \mathbb{R}\} \subset \text{spt} C \) (the last occurs if, for example, if \( x \in M_a \)), giving a contradiction. □
4 Boundaries With Co-Oriented Mean Curvature

In this section we define what it means for a co-dimension one area-minimizing locally rectifiable current \( T \) to have boundary with co-oriented mean curvature. We also give two basic results that we will need in §6, but are also of independent interest; these results will follow straight from well-known theory.

**Definition 4.1** Let \( U \subseteq_o \mathbb{R}^{n+1} \), and suppose \( T \in I_{n,loc}(U) \) is area-minimizing. Then we say \( \partial T \) has co-oriented mean curvature if \( \partial T \) has mean curvature \( H_{\partial T} = h\nu_T \) for \( h : U \to \mathbb{R} \) a \( \mu_{\partial T} \)-locally integrable function, where \( \nu_T : U \to \mathbb{R}^{n+1} \) is the generalized outward pointing normal of \( \partial T \) with respect to \( T \); this means that

\[
\int \text{div}_{\partial T} X \, d\mu_{\partial T} = \int X \cdot (h\nu_T) \, d\mu_{\partial T}
\]

for all \( X \in C^1_c(U; \mathbb{R}^{n+1}) \).

The assumption that \( T \) is area-minimizing in Definition 4.1 is merely to guarantee the existence of the generalized outward pointing unit normal \( \nu_T \) of \( \partial T \) with respect to \( T \); see Lemma 3.1 of [4] and (2.10) of [5].

We now give the following boundary regularity result, which follows directly from [8], [20].

**Theorem 4.2** Let \( U \subseteq_o \mathbb{R}^{n+1} \) and suppose \( T \in I_{n,loc}(U) \) is area-minimizing, and that \( \partial T \) has locally bounded co-oriented mean curvature; meaning \( H_{\partial T} = h\nu_T \) with \( h \in L^\infty_{\text{loc}}(\mu_{\partial T}) \). If \( x \in \text{reg} \partial T \), then there is \( \rho > 0 \) such that \( (\text{spt} \partial T) \cap B_\rho(0) \) is an \((n-1)\)-dimensional \( C^{1,\alpha} \) submanifold for any \( \alpha \in (0,1) \). Furthermore, one of the following holds:

1. \( T \) at \( x \) has unique tangent cone which is a sum of half-hyperplanes with constant orientation after rotation; see Definition 3.3.

Moreover, there are disjoint orientable \( C^{1,\alpha} \)

hypersurfaces-with-boundary \( M_1, \ldots, M_N \) in \( B_\rho(x) \), with

\[
(\partial M_\ell) \cap B_\rho(x) = (\text{spt} \partial T) \cap B_\rho(x) \quad \text{and} \quad m_1, \ldots, m_N \in \mathbb{N} \quad \text{so that}
\]

\[
T \cap B_\rho(x) = \sum_{\ell=1}^N m_\ell[M_\ell];
\]
each $M_\ell$ is oriented so that $\partial [M_\ell] \cdot B_\rho(x) = \langle [\text{spt } \partial T] \cdot B_\rho(x) \rangle$ for each $\ell = 1, \ldots, N$, where $[\text{spt } \partial T] \cdot B_\rho(x)$ has orientation $\partial T$. Furthermore, no two $M_\ell$ can meet tangentially at any point of $(\text{spt } \partial T) \cap B_\rho(x)$. Thus, for any $\tilde{x} \in (\text{spt } \partial T) \cap B_\rho(x)$ we have

$$\nu_T(\tilde{x}) = \frac{\sum_{\ell=1}^N m_\ell \nu_{M_\ell}(\tilde{x})}{\sum_{\ell=1}^N m_\ell}$$

where $\nu_{M_\ell}$ is the outward pointing unit normal of $(\text{spt } \partial T) \cap B_\rho(x)$ with respect to $M_\ell$ for each $\ell = 1, \ldots, N$.

If in addition $h|_{(\text{spt } \partial T) \cap B_\rho(x)} \in C^{k, \alpha}((\text{spt } \partial T) \cap B_\rho(x), \mathbb{R}^{n+1})$ with $k \in \{0\} \cup \mathbb{N}$ (or $C^\infty$, analytic), then $(\text{spt } \partial T) \cap B_\rho(x)$ is $C^{2+k, \alpha}$ (respectively $C^\infty$, analytic) and each $M_\ell$ is $C^{2+k, \alpha}$ (respectively $C^\infty$, analytic).

(2) $T$ at $x$ has unique tangent cone which is a hyperplane with constant orientation but non-constant multiplicity; see Definition 3.3.

Furthermore, there is an orientable analytic minimal hypersurface $M$ in $B_\rho(x)$ containing $(\text{spt } \partial T) \cap B_\rho(x)$ and $m, \theta \in \mathbb{N}$ so that

$$T \cdot B_\rho(x) = (m + \theta) [M^+] + \theta [M^-]$$
$$= m [M^+] + \theta [M]$$

where we have the following: each $M^* \subset M$ is an orientable $C^{3, \alpha}$ hypersurface-with-boundary in $B_\rho(x)$, with $(\partial M^*) \cap B_\rho(x) = (\text{spt } \partial T) \cap B_\rho(x)$; each of $[M], [M^*]$ is oriented by $\tilde{T}$; $M \cap B_\rho(x) = (M^+ \cup \text{spt } \partial T \cup M^-) \cap B_\rho(x)$; $M^*, \text{spt } \partial T$ are pairwise disjoint in $B_\rho(x)$. In this case, $\nu_T(\tilde{x})$ for $\tilde{x} \in (\text{spt } \partial T) \cap B_\rho(x)$ is the outward pointing unit normal of $\text{spt } \partial T$ with respect to $M^*$.

If in addition $h|_{(\text{spt } \partial T) \cap B_\rho(x)} \in C^{k, \alpha}((\text{spt } \partial T) \cap B_\rho(x), \mathbb{R}^{n+1})$ for $k \in \{0\} \cup \mathbb{N}$ (or $C^\infty$, analytic), then $(\text{spt } \partial T) \cap B_\rho(x)$ is $C^{2+k, \alpha}$ (respectively $C^\infty$, analytic), and each $M^*$ is a $C^{2+k, \alpha}$ (respectively $C^\infty$, analytic) hypersurface-with-boundary.
Proof: Since \( \partial T \) has locally bounded mean curvature, then standard regularity theory for \( C^1 \) solutions to the mean curvature system (see for example §6.8 of [9]) implies that for \( x \in \text{reg} \, \partial T \) there is \( \rho \in (0, \text{dist}(x, \partial U)) \) so that \((\text{spt} \, \partial T) \cap B_\rho(x)\) is an \((n - 1)\)-dimensional \( C^{1,\alpha} \) submanifold for any \( \alpha \in (0, 1) \). If in addition \( h|_{(\text{spt} \, \partial T) \cap B_\rho(x)} \in C^{k,\alpha}((\text{spt} \, \partial T) \cap B_\rho(x), \mathbb{R}^{n+1}) \) for \( k \in \{0\} \cup \mathbb{N} \) (or \( C^\infty \), analytic), then \((\text{spt} \, \partial T) \cap B_\rho(x)\) is \( C^{k+2} \) (respectively \( C^\infty \), analytic; see again §6.8 of [9]).

The remainder of the theorem then follows from the boundary regularity theory for co-dimension one area-minimizing currents, given by [8],[20]. \( \square \)

We end this section by giving the following monotonicity formula.

**Theorem 4.3** Let \( U \subseteq_o \mathbb{R}^{n+1} \) and suppose \( T \in I_{n, \text{loc}}(U) \) is area-minimizing. Also suppose \( \partial T \) has co-oriented mean curvature \( H_{\partial T} = h \nu_T \). If \( x \in U \), \( R \in (0, \text{dist}(x, \partial U)) \), and for some \( \alpha \in (0, 1] \) and \( \Lambda \geq 0 \) we have

\[
\frac{1}{\alpha} \int_{B_\rho(x)} |h| \, d\mu_{\partial T} \leq \Lambda (\rho/R)^{\alpha-1} \mu_{\partial T}(B_\rho(x)) \quad \text{for all } \rho \in (0, R),
\]

then

\[
e^{-\Lambda \rho^{1-\alpha}} \frac{\mu_{\partial T}(B_\rho(x))}{\rho^{n-1}} - e^{-\Lambda \sigma^{1-\alpha}} \frac{\mu_{\partial T}(B_\sigma(x))}{\sigma^{n-1}}
\]

\[
\geq \int_{B_\rho(x) \setminus B_\sigma(x)} \frac{|\text{proj}_{T_\perp}(\tilde{x} - x)|^2}{|\tilde{x} - x|^{n+1}} \, d\mu_{\partial T}(\tilde{x})
\]

whenever \( 0 < \sigma < \rho \leq R \).

Proof: This is the usual monotonicity formula applied to \( \partial T \); see for example Theorem 17.6 of [18]. \( \square \)

## 5 Tangentially Immersed Boundaries with Co-Oriented Mean Curvature.

We now study co-dimension one area-minimizing currents with boundary being both \( C^{1,1} \) tangentially immersed and having co-oriented Lipschitz mean curvature. The first main result we wish to show is that the
boundary $\partial T$ of any such current $T$ is regular near any point $x \in \text{spt } \partial T$ such that $T$ at $x$ has tangent cone which is a hyperplane with constant orientation but non-constant multiplicity; this is Theorem 5.12. For this, we must prove Lemma 5.1, which shows that if $x \in \text{spt } \partial T$ is half-regular (see Lemma 3.19), then every tangent cone of $T$ at $x$ must be a sum of half-hyperplanes with constant orientation after rotation. Theorem 5.12 then follows from Theorem 3.18. We then give Theorem 5.13, which states that near any $x \in \text{sing } \partial T$, either $T$ near $x$ exhibits a reasonable amount of regularity or $\text{spt } \partial T$ near $x$ must be extremely irregular.

As mentioned in §1.3, we wish in the future to prove the results of this section for $T \in T_{n,\text{loc}}^{1,1}(U)$ with boundary having co-oriented Lipschitz mean curvature, but with $\alpha \in (0,1]$ more generally.

**Lemma 5.1** Let $U \subseteq \mathbb{R}^{n+1}$, and suppose $T \in T_{n,\text{loc}}^{1,1}(U)$ where $\partial T$ has co-oriented mean curvature $H_{\partial T} = h\nu_T$ with $h : U \to \mathbb{R}$ Lipschitz. For any $x \in \text{sing } \partial T$, there is $\rho \in (0, \text{dist}(x,\partial U))$ so that for any half-regular $x \in \text{sing } \partial T \cap B_\rho(0)$ (see Lemma 3.19), every tangent cone of $T$ at $x$ is the sum of half-hyperplanes with constant orientation after rotation (see Definition 3.3).

**Proof:** Suppose (after translation) $0 \in \text{sing } \partial T$, and choose $\rho \in (0, \text{dist}(0,\partial U))$ so that (after rotation)

$$\partial T \lhd B_\rho(0) = (-1)^n \sum_{\ell=1}^N m_\ell \Phi_{T,\ell}(\mathbb{E}^{n-1} \lhd B_\rho^{n-1}(0)) \lhd B_\rho(0)$$

for $N, m_1, \ldots, m_N \in \mathbb{N}$ and $\Phi_{T,\ell} \in C^{1,1}(B_\rho^{n-1}(0); \mathbb{R}^{n+1})$ is the map $\Phi_{T,\ell}(z) = (z, \varphi_{T,\ell}(z), \psi_{T,\ell}(z))$ where $\varphi_{T,\ell}, \psi_{T,\ell} \in C^{1,1}(B_\rho^{n-1}(0))$ satisfy $\varphi_{T,\ell}(0) = \psi_{T,\ell}(0) = 0$ and $D\varphi_{T,\ell}(0) = D\psi_{T,\ell}(0) = 0$ for each $\ell = 1, \ldots, N$. We also choose $\rho \in (0, \text{dist}(x,\partial U))$ sufficiently small depending on $\epsilon = \epsilon(n, \partial T) > 0$, to be chosen later, so that

$$\|D\varphi_{T,\ell}\|_{C(B_\rho^{n-1}(0))}, \|D\psi_{T,\ell}\|_{C(B_\rho^{n-1}(0))} < \epsilon. \quad (5.2)$$

Suppose $x \in \text{spt } \partial T \cap B_{\rho/3}(0)$ is a half-regular point. Thus, by Lemma 3.19 there is $z \in B_{\rho/3}^{n-1}(0)$ and $\sigma \in (0, \rho/3 - |z|)$ is such that (after relabeling)

$$x \in \Phi_{T,1}(\partial B_{\sigma}^{n-1}(z)) \cap \Phi_{T,2}(\partial B_{\sigma}^{n-1}(z)), \quad \Phi_{T,1}(B_{\sigma}^{n-1}(z)), \Phi_{T,2}(B_{\sigma}^{n-1}(z)) \subset \text{reg } \partial T, \text{ and } \Phi_{T,1}(B_{\sigma}^{n-1}(z)) \cap \Phi_{T,2}(B_{\sigma}^{n-1}(z)) = \emptyset. \quad (5.3)$$

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Suppose for contradiction that $T$ at $x$ has tangent cone $\mathbb{C}$ which is a hyperplane with constant orientation but non-constant multiplicity. In fact, (5.2) implies

\[(5.4) \quad C = Q^T_{\theta} \mathbb{L} \{ (m + \theta) \mathbb{E}^n : y_n > 0 \} + \partial \mathbb{E}^n \{ y \in \mathbb{R}^n : y_n < 0 \} \]

for some $m, \theta \in \mathbb{N}$, a rotation $Q$ about $\mathbb{R}^{n-1}$, and $Q^T_{\theta}$ an orthogonal rotation with $\|Q^T_{\theta} - I\| < \epsilon$ for some $\epsilon = c(n) > 0$ (in fact, $Q^T_{\theta}(\mathbb{R}^{n-1}) = T_{x} \partial T$).

Taking $z$ closer to $x$ in that direction, we can also assume $z \in B_{p/3}(0)$ and $\sigma \in (0, p/3 - |z|]$ are such that (5.3) continues to hold while

\[(5.5) \quad \text{spt}((Q^{-1} \circ \eta_{x,1})_{\#} T) \cap B_{3\sigma}(0) = \left( \text{graph}_{B_{3\sigma}(0)}(u) \right) \cap B_{3\sigma}(0) \quad \text{and} \quad \text{spt}(y_n \circ \eta_{x,1})_{\#} T(y, u(y)) = \frac{-Du(y)}{1 + |Du(y)|^2} \quad \text{for} \quad (y, u(y)) \in B_{3\sigma}(0)\]

where $u \in C^\omega(B_{3\sigma}(0))$ with $u(0) = 0$; this follows by Theorem 5.18 and (5.4), in particular using $\|Q^T_{\theta} - I\| < \epsilon$ for some $\epsilon = c(n) > 0$ where we choose $\epsilon = \epsilon(n) > 0$ sufficiently small in (5.2). Note that while we may have $Du(0) \neq 0$, we do have by (5.4) that $|Du(0)| < \epsilon$ for some $c = c(n) > 0$.

For $\ell = 1, 2$ define for $z \in B_{2\sigma}^{-1}(0)$

\[(5.6) \quad s^{(\ell)}(z) = (Q^{-1} \circ \eta_{x,1} \circ \Phi_{T,\ell})(z + \text{proj}_{\mathbb{R}^n \cdot x} \cdot e_n)\]

Observe that $s^{(1)}, s^{(2)} \in C^{1,1}(B_{p/3}(0))$, as $\Phi_{T,1}, \Phi_{T,2} \in C^{1,1}(B_{p/3}(0); \mathbb{R}^{n+1})$.

Since $Q$ is a rotation about $\mathbb{R}^{n-1}$, then by (5.5) (if $\epsilon = \epsilon(n) > 0$ is sufficiently small in (5.2)) we have for each $\ell = 1, 2$

\[(Q^{-1} \circ \eta_{x,1} \circ \Phi_{T,\ell})(B_{2\sigma}^{-1}(\text{proj}_{\mathbb{R}^{n-1} \cdot x})) = \{ (z, s^{(\ell)}(z), u(z, s^{(\ell)}(z))) : z \in B_{2\sigma}^{-1}(0) \} \subset B_{3\sigma}(0)\]

We as well define for each $\ell = 1, 2$ and $z \in B_{2\sigma}^{-1}(0)$

\[(5.7) \quad \begin{align*}
    h^{(\ell)}(z) &= \hat{h}((\eta_{-x,1} \circ \Phi)(z, s^{(\ell)}(z), u(z, s^{(\ell)}(z)))), \\
    Du^{(\ell)}(z) &= (Du)(z, s^{(\ell)}(z)), \\
    u^{(\ell)}(z) &= (Du)(z, s^{(\ell)}(z)) \quad \text{for} \quad i \in \{1, \ldots, n\}, \\
    D^2u^{(\ell)}(z) &= (D^2u)(z, s^{(\ell)}(z)), \\
    s^{(\ell)}(z) &= \frac{Du^{(\ell)}(z)}{1 + |Du^{(\ell)}(z)|^2} \quad \text{and} \\
    \partial_j^{(\ell)}(z) &= (e_j, s^{(\ell)}(z), u^{(\ell)}(z) + u^{(\ell)}(z) s^{(\ell)}(z)) \quad \text{for} \quad j \in \{1, \ldots, n - 1\},
\end{align*}\]
where for emphasis we let \( \overrightarrow{D} \) be the gradient over \( \mathbb{R}^{n-1} \). Also consider for \( \ell = 1, 2 \) the downward pointing unit normal of the graph of \( s^{(\ell)} \) within the graph of \( u \); this is given by

\[
\nu^{(\ell)} = \frac{\left( \overrightarrow{D}s^{(\ell)}, -1, 0 \right) + \left( \frac{\overrightarrow{D}s^{(\ell)}}{1 + |Du^{(\ell)}|^2} \right) (Du^{(\ell)}, 1)}{\sqrt{1 + |Ds^{(\ell)}|^2 - \frac{(Ds^{(\ell)}, 1) (Du^{(\ell)}, 1)}{1 + |Du^{(\ell)}|^2}}}.
\]

(5.8)

Let \( g \in C^\infty(\mathbb{R}^{n-1} \times \mathbb{R}^n) \) be the \((n-1) \times (n-1)\) matrix

\[
g(p, q) = I + (1 + q_n^2) pp^T + q_n \left( \left( \text{proj}_{\mathbb{R}^{n-1}} q \right) p^T + \text{proj}_{\mathbb{R}^{n-1}} q \right) p^T
\]

(5.9)

for \( p \in \mathbb{R}^{n-1} \) and \( q \in \mathbb{R}^n \), where \( I \) is the \((n-1) \times (n-1)\) identity matrix. Also let \( g^{ij} \in C^\infty(\mathbb{R}^{n-1} \times \mathbb{R}^n) \) for \( i, j \in \{1, \ldots, n-1\} \) be so that \( g^{ij}(p, q) \) is the \( ij \)-th entry of \( g(p, q)^{-1} \) for \( p \in \mathbb{R}^{n-1} \) and \( q \in \mathbb{R}^n \) (\( g \) is generally invertible, but by (5.2), (5.4), (5.5), (5.6) we may if we like restrict \( g \) to \(|p|, |q| \) small).

For \( \ell = 1, 2 \) let \( S^{(\ell)} = \{(z, s^{(\ell)}(z), u(z, s^{(\ell)}(z))) : z \in B_{2\sigma}^{n-1}(0)\} \). Since \( H_{\partial T} = hv_T \), then (5.2), (5.3), (5.4), (5.5), (5.6) imply that for \( \ell = 1, 2 \)

\[
\sum_{i,j=1}^{n-1} g^{ij}(Ds^{(\ell)}, Du^{(\ell)}) \text{proj}_{\mathbb{T}_{s^{(\ell)}(z)}}^{z} s^{(\ell)} D_{s^{(\ell)}} \partial_{j}^{(\ell)} = -h^{(\ell)} \nu^{(\ell)}
\]

for \( z \in B_{2\sigma}^{n-1}(z - \text{proj}_{\mathbb{R}^{n-1}} x) \), where we have used the notation (5.7), (5.8), (5.9). This implies, using that \( s^{(\ell)} \perp \partial_{j}^{(\ell)} \) for each \( j \in \{1, \ldots, n-1\} \),

\[
\sum_{i,j=1}^{n-1} g^{ij}(Ds^{(\ell)}, Du^{(\ell)}) \left( \partial_{j}^{(\ell)} \cdot D_{s^{(\ell)}} \nu^{(\ell)} \right) = h^{(\ell)}.
\]

(5.10)

Next, we compute by (5.7), (5.8)

\[
\partial_{j}^{(\ell)} \cdot D_{s^{(\ell)}} \nu^{(\ell)} = \partial_{j}^{(\ell)} \cdot \frac{\partial}{\partial z_i} \frac{\left( \overrightarrow{D}s^{(\ell)}, -1, 0 \right) + \left( \frac{\overrightarrow{D}s^{(\ell)}}{1 + |Du^{(\ell)}|^2} \right) (Du^{(\ell)}, 1)}{\sqrt{1 + |Ds^{(\ell)}|^2 - \frac{(Ds^{(\ell)}, 1) (Du^{(\ell)}, 1)}{1 + |Du^{(\ell)}|^2}}}.
\]

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Since \( \partial_j^{(\ell)} \perp (D_s^{(\ell)}, -1, 0), (-Du^{(\ell)}, 1) \)

\[
\partial_j^{(\ell)} \cdot D_{\partial_j^{(\ell)}} = \frac{\partial_j^{(\ell)} \cdot \left( ((D_s^{(\ell)}, 0, 0) + \left( \frac{(D_s^{(\ell)}, -1) \cdot Du^{(\ell)}}{1 + |Du^{(\ell)}|^2} \right) (-D^2u^{(\ell)}(e_i, s_i^{(\ell)}), 0)) \right)}{\sqrt{1 + |D_s^{(\ell)}|^2 - \frac{|(D_s^{(\ell)}, -1) \cdot Du^{(\ell)}|^2}{1 + |Du^{(\ell)}|^2}}}
\]

using the notation from (5.7). We simplify this to get

\[
\partial_j^{(\ell)} \cdot D_{\partial_j^{(\ell)}} = \frac{s_{ij}^{(\ell)} - \left( \frac{(D_s^{(\ell)}, -1) \cdot Du^{(\ell)}}{1 + |Du^{(\ell)}|^2} \right) \left( (e_j, s_j^{(\ell)}) D^2u^{(\ell)}(e_i, s_i^{(\ell)}) \right)}{\sqrt{1 + |D_s^{(\ell)}|^2 - \frac{|(D_s^{(\ell)}, -1) \cdot Du^{(\ell)}|^2}{1 + |Du^{(\ell)}|^2}}}
\]

Using this in (5.10), we conclude for \( \ell = 1, 2 \) and \( z \in B_{\sigma}^{n-1}(z - \text{proj}_{\mathbb{R}^{n-1}} x) \)

\[
(5.11) \quad \sum_{i,j=1}^{n-1} a_{ij}(z, s^{(\ell)}, D_s^{(\ell)}) s_i^{(\ell)} + b(z, s^{(\ell)}, D_s^{(\ell)}) = 0
\]

where we define the functions \( a_{ij} \in C^\infty(B_{2\sigma}^n(0) \times \mathbb{R}^{n-1}) \) and the Lipschitz function \( b : B_{2\sigma}^n(0) \times \mathbb{R}^{n-1} \to \mathbb{R} \) given by

\[
a_{ij}(y, p) = g^{ij}(p, Du(y))
\]

\[
b(y, p) = - \sum_{i,j=1}^{n-1} g^{ij}(p, Du(y)) \left( \frac{(p, -1) \cdot Du(y)}{1 + |Du(y)|^2} \right) \left( (e_j, p_j) D^2u(y)(e_i, p_i) \right) - h((\eta_{x,1} \circ Q)(y, u(y))) \sqrt{1 + |p|^2 - \frac{(p, -1) \cdot Du(y)^2}{1 + |Du(y)|^2}}
\]

for \( y \in B_{2\sigma}^n(0) \) and \( p \in \mathbb{R}^{n-1} \).

Subtracting (5.11) across \( \ell = 1, 2 \) we get for \( z \in B_{\sigma}^{n-1}(z - \text{proj}_{\mathbb{R}^{n-1}} x) \)

\[
\sum_{i,j=1}^{n-1} \int_0^1 \frac{d}{dt} \left[ a_{ij}(z, ts^{(1)}, tD_s^{(1)}), (1 - t) s_i^{(1)} + (1 - t) s_i^{(1)} \right] dt + \int_0^1 \frac{d}{dt} b(z, ts^{(1)}, tD_s^{(1)}), (1 - t) s_i^{(1)} + (1 - t) D_s^{(1)}) = 0,
\]

using that \( b \) is Lipschitz. This gives that \( s = s^{(2)} - s^{(1)} \) is a solution to the equation over \( B_{\sigma}^{n-1}(z - \text{proj}_{\mathbb{R}^{n-1}} x) \) given by

\[
\sum_{i,j=1}^{n-1} A_{ij} D_i D_j s + B \cdot Ds + C s = 0
\]

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where for $z \in B_{2\sigma}^{n-1}(0)$

$$A_{ij}(z) = \sum_{i,j=1}^{n-1} \int_0^1 a_{ij}(z, ts^{(2)} + (1-t)s^{(1)}, tDs^{(2)} + (1-t)Ds^{(1)}) \, dt$$

$$B(z) = \sum_{i,j=1}^{n-1} \int_0^1 \frac{\partial a_{ij}}{\partial p} \bigg|_{(z, ts^{(2)} + (1-t)s^{(1)}, tDs^{(2)} + (1-t)Ds^{(1)})} (ts_{ij}^{(2)} + (1-t)s_{ij}^{(1)}) \, dt$$

$$+ \int_0^1 \frac{\partial b}{\partial p} \bigg|_{(z, ts^{(2)} + (1-t)s^{(1)}, tDs^{(2)} + (1-t)Ds^{(1)})} dt$$

$$C(z) = \sum_{i,j=1}^{n-1} \int_0^1 \frac{\partial a_{ij}}{\partial s} \bigg|_{(z, ts^{(2)} + (1-t)s^{(1)}, tDs^{(2)} + (1-t)Ds^{(1)})} (ts_{ij}^{(2)} + (1-t)s_{ij}^{(1)}) \, dt$$

$$+ \int_0^1 \frac{\partial b}{\partial s} \bigg|_{(z, ts^{(2)} + (1-t)s^{(1)}, tDs^{(2)} + (1-t)Ds^{(1)})} dt.$$ 

By definition (see (5.3)) as well as the definitions after (5.11), the coefficients $A_{ij}$ are uniformly elliptic (again, we may use (5.3) with $\epsilon = \epsilon(n, \partial T)$ sufficiently small). Since $u \in C^\infty(B_{2\sigma}^n(0))$, the functions $s^{(2)}, s^{(1)} \in C^{1,1}(B_{2\sigma}^{n-1}(0))$, and $h$ is Lipschitz, then the coefficients $B, C$ are bounded in $B_{2\sigma}^{n-1}(0)$. On the other hand, $T \in TI_{n,loc}(U)$ and (5.6) give $s(0) = 0$ and $Ds(0) = 0$. But by (5.3) (after relabeling, if necessary) $s^{(1)}(z) < s^{(2)}(z)$ so that $s(z) > 0$ for each $z \in B_{2\sigma}^{n-1}(z - \text{proj}_{R^{n-1}} x)$. This contradicts the Hopf boundary point lemma (see for example Lemma 3.4 of [7]). □

Having shown Lemma 5.1, then the proof of the first main result of this section is relatively short.

**Theorem 5.12** Let $U \subset R^{n+1}$ and suppose $T \in TI_{n,loc}^{1,1}(U)$ where $\partial T$ has co-oriented mean curvature $H_{\partial T} = hT$ with $h : U \to R$ Lipschitz. If $x \in \text{spt} \partial T$ and $T$ at $x$ has tangent cone which is a hyperplane with constant orientation but non-constant multiplicity (as in Definition 3.3), then $x \in \text{reg} \partial T$.

**Proof:** Suppose for contradiction $x \in \text{sing} \partial T$ and that $T$ at $x$ has tangent cone which is a hyperplane with constant orientation but non-constant multiplicity. By Theorem 3.18 there is $\rho \in (0, \text{dist}(x, \partial U))$ so that $T$ at every $\tilde{x} \in B_{\rho}(x) \cap \text{spt} \partial T$ has unique tangent cone which is a hyperplane with constant orientation but non-constant multiplicity. However, by

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Lemma 3.19 we can find a half-regular \( x \in B_\rho(x) \cap \text{sing}\ T \). This contradicts Lemma 5.1. \( \square \)

We are not ready to prove the final main result of this section.

**Theorem 5.13** Let \( U \subseteq \mathbb{R}^{n+1} \) and suppose \( T \in \text{TI}^{1,1}_{n,\text{loc}}(U) \) where \( \partial T \) has co-oriented mean curvature \( H_{\partial T} = h\nu_T \) with \( h : U \to (0, \infty) \) Lipschitz. Suppose \( x \in \text{spt}\ \partial T \) and that there exists \( \rho \in (0, \text{dist}(x, \partial U)) \) and \( C^1 \) hypersurfaces-with-boundary \( M_1, \ldots, M_A \) in \( B_\rho(x) \) so that either:

1. \( \text{spt}\ T \cap B_\rho(x) = \bigcup_{a=1}^A (\text{clos}\ M_a) \cap B_\rho(x) \), or
2. \( \text{spt}\ T \cap B_\rho(x) \subseteq \bigcup_{a=1}^A (\text{clos}\ M_a) \cap B_\rho(x) \) and \( T_x \partial T \not\subset T_x M_a \) for each \( a \in \{1, \ldots, A\} \) such that \( x \in \text{clos}\ M_a \).

Then there is \( \sigma \in (0, \rho) \) and \( B \in \{1, \ldots, 2\Theta_T(x)\} \) so that

\[
\text{spt}\ T \cap B_\sigma(x) = \bigcup_{b=1}^B (\text{clos}\ W_b) \cap B_\sigma(x)
\]

for orientable \( C^{1,1} \) hypersurfaces-with-boundary \( W_1, \ldots, W_B \) in \( B_\sigma(x) \). For each \( b \in \{1, \ldots, B\} \) we have \( x \in \partial W_b \) and \( W_b \cap \text{spt}\ \partial T \subset \text{reg}\ \partial T \). Furthermore, for each \( b, \bar{b} \in \{1, \ldots, B\} \) we have

\[(\text{clos}\ W_b) \cap (\text{clos}\ W_{\bar{b}}) \cap B_\sigma(x) \subseteq (\partial W_b) \cap (\partial W_{\bar{b}}) \cap B_\sigma(x).\]

Theorem 5.13 essentially appears as Theorem 9 of [16] in the context of two-dimensional solutions to the \( c \)-Plateau problem; see §1.4. The proof of Theorem 5.13 follows closely the proof of Theorem 9 of [16], with both some simplifications (primarily due to Theorem 3.18) to the argument and some more subtle analysis due to the more general present setting.

**Proof:** Observe that by Lemma 3.20 it suffices to show (2). This we do in what follows.

Suppose (after translation) that \( 0 \in \text{spt}\ \partial T \) and (after rotation) \( T_0 \partial T = \mathbb{R}^{n-1} \). Also suppose (by scaling) that \( U = B_1(0) \) and

\[
\text{spt}\ T \cap B_1(0) \subseteq \bigcup_{a=1}^A (\text{clos}\ M_a) \cap B_1(0)
\]

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with \(0 \in \text{clos} M_a\) for each \(a = 1, \ldots, A\). Observe that for any \(\epsilon > 0\) there is
\[
\rho = \rho(\epsilon, \{M_a\}_{a=1}^A) \epsilon (0, 1)
\] so that for every \(a = 1, \ldots, A\)

\[
\text{dist}_H(T_x M_a, T_0 M_a) < \epsilon \quad \text{for each} \quad x \in \text{clos} M_a \cap B_\rho(0).
\]

Recall that we are assuming (1) holds, so that \(\{0\} \times \mathbb{R}^2 = T_0^a \partial T \not\subset T_0 M_a\) for each \(a = 1, \ldots, A\). Hence, if \(\epsilon = \epsilon(n, \{T_0 M_a\}_{a=1}^A) > 0\) is sufficiently small, then

\[
(5.14)
\]

with \(\rho = \rho(\epsilon, \{M_a\}_{a=1}^A) \epsilon (0, 1)\) implies \(\{0\} \times \mathbb{R}^2 \not\subset T_x M_a\) for each \(x \in \text{clos} M_a \cap B_\rho(0)\) and \(a = 1, \ldots, A\).

By definition of \(T \in \mathcal{T}^{1,1}_{n,loc}(B_1(0))\), for any \(\epsilon > 0\) there is \(\rho = \rho(\epsilon, \partial T) \epsilon (0, 1)\) so that

\[
\text{dist}_T(B_\rho(0)) = (-1)^n \sum_{\ell \leq 1} m_\ell^T \# \left( E^{n-1} \setminus B_\rho^{-1}(0) \right) \setminus B_\rho(0)
\]

where \(\Phi_{T,\ell}(z) = (z, \varphi_{T,\ell}(z), \psi_{T,\ell}(z))\) with \(\varphi_{T,\ell}, \psi_{T,\ell} \in C^{1,1}(B_\rho^{-1}(0))\) satisfying \(\phi_{T,\ell}(0) = \psi_{T,\ell}(0) = 0\) and \(D\phi_{T,\ell}(0) = D\psi_{T,\ell}(0) = 0\) as well as

\[
\|D\varphi_{T,\ell}\|_{C(B_\rho^{-1}(0))}, \|D\psi_{T,\ell}\|_{C(B_\rho^{-1}(0))} < \epsilon
\]

for each \(\ell \in \{1, \ldots, N\}\); using \((5.14)\) we can also take \(\rho = \rho(\epsilon, \partial T) \epsilon (0, 1)\) so that for each \(x \in \text{spt} \partial T \cap B_\rho(0)\) we have \(T_x^a \partial T \not\subset T_x M_a\) for each \(a \in \{1, \ldots, A\}\) with \(x \in \text{clos} M_a\).

We shall proceed by induction on \(N \geq 1\) in \((5.15)\). If \(N = 1\), then \(0 \in \text{reg} \partial T\) and the theorem follows by Theorem 4.2. So we assume \(N \geq 2\), and again by Theorem 4.2 assume that \(0 \in \text{sing} \partial T\). Our first goal is to define a smooth hypersurface \(W\) in \(B_1(0)\). We will then show that \(W\) near 0 decomposes into finitely many connected components \(\{W_b\}_{b=1}^B\) with each \(W_b\) a \(C^{1,1}\) hypersurface-with-boundary as required. Before we define \(W\), we make three sets of observations using Theorem 5.12.

First, \(0 \in \text{sing} \partial T\) and Theorem 5.12 imply that every tangent cone of \(T\) at 0 is a sum of half-hyperplanes with constant orientation after rotation. In fact, since \(T_0 \partial T = \mathbb{R}^{n-1}\) then every tangent cone \(C\) of \(T\) at 0 is of the form

\[
(5.16)
\]

\[
C = \sum_{k=1}^{N_\ell} m_k^C Q_k^\# \left( E^n \setminus \{ y \in \mathbb{R}^n : y_n > 0 \} \right)
\]

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for distinct orthogonal rotations $Q_1, \ldots, Q_{N^C}$ about $\mathbb{R}^{n-1}$, and where $N^C, m_1^C, \ldots, m_{N^C}^C \in \mathbb{N}$ satisfy $\sum_{k=1}^{N^C} m_k^C = \sum_{\ell=1}^N m_\ell$.

Second, Theorem 1 of [17] and (5.16) imply there exists $\delta_0 = \delta_0(n) \in (0, 1)$ with the following property: for any $\delta \in (0, \delta_0)$ and $\mathcal{C}$ a tangent cone of $T$ at 0, there is $\rho = \rho(\delta, T, \mathcal{C}) \in (0, 1)$ so that

$$\tag{5.17} T \mathbf{L} \{ x \in B_\rho(0) : |\text{proj}_{\{0\}\times \mathbb{R}^2} x | \geq \delta \rho \}$$

where for each $k \in \{1, \ldots, N^C\}$ we have that $N_k^C, m_{1,k}^C, \ldots, m_{N_k^C,k}^C \in \mathbb{N}$ satisfy $\sum_{j=1}^{N_k^C} m_{j,k}^C = m_k^C$; for each $j \in \{1, \ldots, N_k^C\}$ the map $F_{j,k}$ is given by $F_{j,k}(y) = (y, u_{j,k}(y))$ for $y \in B_\rho^n(0)$ with $y_n > \delta \rho/2$ where

$$u_{j,k} \in C^\infty(\{ y \in B_\rho^n(0) : y_n > \delta \rho/2 \}) \text{ with } \| Du_{j,k} \| _{C(\{ y \in B_\rho^n(0) : y_n > \delta \rho/2 \})} < c \delta$$

for $c = c(n)$; for each $j \in \{1, \ldots, N_k^C - 1\}$ we have $u_{j,k}(y) < u_{j+1,k}(y)$ for each $y \in B_\rho^n(0)$ with $y_n > \delta \rho/2$.

Third, by Theorem 5.4.2 of [6] we have that

$$\tag{5.18} \text{spt } \mathcal{C} \subseteq \bigcup_{\{a \in \{1, \ldots, A\} : \text{x is clos } M_a \}} T_x M_a$$

for any tangent cone $\mathcal{C}$ of $T$ at (any) $x \in \text{spt } T \cap B_1(0)$. In case $x \in \text{reg } T \cap B_1(0)$, then $T_x T = T_x M_a$ for some $a \in \{1, \ldots, A\}$, and (5.17) further implies $\text{dist}_H(T_x T, T_0 M_a) < \epsilon$ for some $a \in \{1, \ldots, A\}$. If $\mathcal{C}$ is a tangent cone of $T$ at 0, then $\text{spt } \mathcal{C} \subseteq \bigcup_{a=1}^A T_0 M_a$, as $0 \in \text{clos } M_a$ for each $a \in \{1, \ldots, A\}$ (see the paragraph containing (5.14)).

Now define the set

$$W = (\text{spt } T \cap B_1(0) \setminus \text{spt } \partial T) \cup \begin{cases} x \in \text{spt } \partial T \cap B_1(0) : & T \text{ has tangent cone at } x \\ & \text{which is a hyperplane with constant orientation} \\ & \text{but non-constant multiplicity} \end{cases}.$$
By Theorem 3.18 the (topological) boundary of $W$ in $B_1(0)$ satisfies

$$\partial W \cap B_1(0) = \left\{ x \in \text{spt} \partial T \cap B_1(0) : \begin{array}{c}
T \text{ has tangent cone at } x \\
\text{which is a sum of half-hyperplanes with constant orientation after rotation}
\end{array} \right\}. \tag{5.19}$$

We claim that $W$ is a smooth hypersurface. We prove this in what follows.

First, if $x \in \text{spt} \partial T \cap B_1(0)$ and $T$ at $x$ has a tangent cone consisting of a hyperplane with constant orientation but non-constant multiplicity, then $W$ near $x$ is a smooth hypersurface by Theorem 3.18 (or Theorem 5.12).

Second, if $x \in \text{spt} T \cap B_1(0) \setminus \text{spt} \partial T$, then standard interior regularity for area-minimizing currents shows $x \in \text{reg } T$. To see this more clearly, let $C$ be a tangent cone of $T$ at $x$. Then $C$ is area-minimizing with $\partial C = 0$, and thus the singular set of $C$ must satisfy Federer’s alternatives; see §37 of [18], in particular see Theorem 37.7 of [18]. We thus conclude $C$ must be a hyperplane with multiplicity by (5.18) (for this, we can use a proof by induction on the number of distinct planes $T_aM_a$ with $x \in \text{clos } M_a$).

It follows that $x \in \text{reg } T$ (using, for example, Theorem 1 of [17]), and thus $W$ near $x$ is a smooth hypersurface.

Our goal now is to show that $W$ near $0$ decomposes into finitely many connected components, each of which is a $C^{1,1}$ hypersurface-with-boundary containing the origin. To do this, we fix in what follows $\epsilon = \epsilon(n, \{T_0M_a\}_{a=1}^A) > 0$ and $\delta = \delta(n, \epsilon) \in (0, \delta_0)$ to be chosen later. Fix as well any tangent cone $C$ of $T$ at $0$, and consider $\rho = \rho(\epsilon, \{M_a\}_{a=1}^A, \partial T, \delta, T, C)$ so that (5.14)-(5.17) hold.

Consider any $x \in W \cap B_{\rho/2}(0)$, and let $z = \text{proj}_{\mathbb{R}^{n-1}} x$. We claim $x \in \gamma((0,1))$ for a Jordan arc

$$\gamma \in \left\{ C([0,1]; (\text{clos } W) \cap \{z\} \times \mathbb{R}^2) \cap \text{clos } B_\rho(0)) \right\} \tag{5.20}$$

$$\text{with } \gamma(0) \in (\partial W) \cap B_\rho(0) \text{ while } \gamma(1) \in \partial B_\rho(0).$$

To see this, observe that (5.18) implies that for each $\tilde{x} \in W$ we have $T_{\tilde{x}}W = T_{\tilde{x}}M_a$ for some $a \in \{1, \ldots, A\}$. Thus, $\{0\} \times \mathbb{R}^2 \notin T_{\tilde{x}}W$ for each $\tilde{x} \in W \cap B_\rho(0)$ by (5.14) (if $\epsilon = \epsilon(n, \{T_0M_a\}_{a=1}^A) > 0$ is sufficiently small, and
\( \rho \in (0, 1) \) is chosen depending on \( \epsilon, \{M_a\}_{a=1}^A \). Sard’s theorem thus implies \( x \in \gamma((0, 1)) \) for a Jordan arc as claimed in (5.20), although it remains to show \( \gamma(0) \in (\partial W) \cap B_\rho(0) \) while \( \gamma(1) \in \partial B_\rho(0) \); we show this in what follows.

First, suppose for contradiction \( \gamma(0), \gamma(1) \in (\partial W) \cap B_\rho(0) \). Let \( \gamma(0) = x^1 \) and \( \gamma(1) = x^2 \). Thus we conclude by (5.19) that \( T \) has at \( x^d \) for \( d = 1, 2 \) a tangent cone

\[
C_{x^d} = \sum_{k=1}^{N_{x^d}} m_{k}^C \sigma_{x^d}^d Q_{x^d}^d \left( E^n \cap \{ y \in \mathbb{R}^n : y_n > 0 \} \right)
\]

for \( N_{x^d}, m_1^C, \ldots, m_{N_{x^d}}^C \in \mathbb{N} \), an orthogonal rotation \( Q_{x^d}^d \) with \( \| Q_{x^d}^d - I \| \leq \epsilon \) for some \( c = c(n) > 0 \) by (5.15), and distinct orthogonal rotations \( Q_{x^d,1}^d, \ldots, Q_{x^d,N_{x^d}}^d \) about \( \mathbb{R}^{n-1} \). Choose \( \sigma \in (0, \rho) \) so that

\[
B_\sigma(x^1) \cap B_\sigma(x^2) = \emptyset \quad \text{and} \quad B_\sigma(x^1), B_\sigma(x^2) \subset B_\rho(0).
\]

By Theorem 1 of [17] (as in (5.17)) we can also choose \( \sigma \in (0, \rho) \) sufficiently small so that for each \( d = 1, 2 \)

\[
((Q_{x^d})^{-1} \circ \eta_{x^d,1})_# T \{ \tilde{x} \in B_\sigma(0) : |\text{proj}_{(0) \times \mathbb{R}^2} \tilde{x}| \geq \sigma/2 \}
\]

(5.21)

\[
= \sum_{k=1}^{N_{x^d}} \sum_{j=1}^{N_{x^d}} m_{j,k} \sigma_{x^d}^d(Q_{x^d})_#(F_{j,k})_# \left( E^n \cap \{ y \in B_\sigma^n(0) : y_n > \sigma/4 \} \right)
\]

\[
\{ \tilde{x} \in B_\sigma(0) : |\text{proj}_{(0) \times \mathbb{R}^2} \tilde{x}| \geq \sigma/2 \}
\]

for \( \{N_1^C, \ldots, N_{N_{x^d}}^C\}, \{m_j^C, j=1,k=1\} \subset \mathbb{N} \) and \( F_{j,k}(y) = (y, u_j,k(y)) \) for \( y \in B_\sigma^n(0) \) with \( y_n > \sigma/4 \) where \( u_j,k \in C^\infty(\{ y \in B_\sigma^n(0) : y_n > \sigma/4 \}) \).

Furthermore, for each \( k \in \{1, \ldots, N_{x^d}^C \} \) and \( j \in \{1, \ldots, N_{x^d}^C - 1 \} \) we have that \( u_{j,k} < \gamma((0, 1)) \) for each \( y \in B_\sigma^n(0) \) with \( y_n > \sigma/4 \).

Since \( \text{proj}_{\mathbb{R}^{n-1}} x^d = z \) and \( \| Q_{x^d}^d - I \| < \epsilon \), then we conclude (as \( \epsilon = \epsilon(n, \{T_0 M_a\}_{a=1}^A) > 0 \) can be chosen small depending on \( n \)) that for each \( d = 1, 2 \)

\[
(\eta_{x^d,1} \circ \sigma_{x^d}^d)(\{ \tilde{x} \in B_\sigma(0) : |\text{proj}_{(0) \times \mathbb{R}^2} \tilde{x}| \geq \sigma/2 \}) \cap \gamma((0, 1)) \neq \emptyset,
\]

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since \( B_\sigma(x^1) \cap B_\sigma(x^2) = \emptyset \) and \( \gamma(0) = x^1, \gamma(1) = x^2 \). Thus, we can find a Jordan arc
\[
g \in C^\infty([0, \mathcal{H}^1(g)]; \gamma((0, 1)))
\]
parameterized by arc-length so that
\[
g(0) \in (\eta_{-x, 1} \circ Q \mathbb{x}^3)\{\{\tilde{x} \in B_\sigma(0) : |\text{proj}_{(0) \times \mathbb{R}^2} \tilde{x}| < \sigma/2\}\} \text{ while } g(\mathcal{H}^1(g)) \in (\eta_{-x, 2} \circ Q \mathbb{x}^2)\{\{\tilde{x} \in B_\sigma(0) : |\text{proj}_{(0) \times \mathbb{R}^2} \tilde{x}| < \sigma/2\}\}.
\]
Define by (5.14), (5.18) the unit-vector field \( V \in C^\infty(g([0, \mathcal{H}^1(g)]); \{0\} \times S^1) \) given for \( \tilde{x} \in g([0, \mathcal{H}^1(g)]) \) by
\[
V(\tilde{x}) = \frac{\text{proj}_{(0) \times \mathbb{R}^2} \mathcal{T}(\tilde{x})}{|\text{proj}_{(0) \times \mathbb{R}^2} \mathcal{T}(\tilde{x})|}.
\]
Then (5.21) with \( d = 1 \) implies \( g, V \) are negatively oriented as in Definition A.1 while (5.21) with \( d = 2 \) implies \( g, V \) are positively oriented, giving a contradiction. We conclude at least one \( x^d \in \partial B_\rho(0) \).

Second, suppose for contradiction both \( x^1, x^2 \in \partial B_\rho(0) \). Now parameterize
\[
\gamma : [0, \mathcal{H}^1(\gamma)] \to (\text{clos} W) \cap (\{z\} \times \mathbb{R}^2) \cap \text{clos} B_\rho(0)
\]
by arc-length so that \( \gamma(0) = x^1 \) and \( \gamma(\mathcal{H}^1(\gamma)) = x^2 \). Consider by (5.14), (5.18)
\[
V \in C(\gamma([0, \mathcal{H}^1(\gamma)]); \{0\} \times S^1) \cap C^\infty(\gamma((0, \mathcal{H}^1(\gamma))); \{0\} \times S^1)
\]
given for \( \tilde{x} \in \gamma([0, \mathcal{H}^1(\gamma)]) \) by
\[
V(\tilde{x}) = \frac{\text{proj}_{(0) \times \mathbb{R}^2} \mathcal{T}(\tilde{x})}{|\text{proj}_{(0) \times \mathbb{R}^2} \mathcal{T}(\tilde{x})|}.
\]
Then \( \gamma(0) = x^1 \in \partial B_\rho(0) \), (5.17), and \( z = \text{proj}_{\mathbb{R}^{n-1}} x \) with \( x \in B_{\rho/2}(0) \) imply that \( \gamma, V \) are positively oriented (again as in Definition A.1). Contrarily \( \gamma(\mathcal{H}^1(\gamma)) = x^2 \in \partial B_\rho(0) \) and (5.17) (along with \( z = \text{proj}_{\mathbb{R}^{n-1}} x \) and \( x \in B_{\rho/2}(0) \)) imply that \( \gamma, V \) are negatively oriented. This gives a contradiction, and so we conclude (5.20).

We now show that \( W \) near 0 decomposes into finitely many connected components, using (5.17). For this, consider any connected component \( W_1 \).
of $W \cap (B_{\rho/4}^{n-1}(0) \times (-\rho/8, \rho/8)^2)$. Applying (5.21) to any $x \in W_1$, we can assume by (5.17)

$$Q_1 \left( \text{graph}_{y \in B_{\rho}^n(0), y_n \geq \delta \rho} u_{1,1} \right) \cap \left( B_{\rho/4}^{n-1}(0) \times (-\rho/8, \rho/8)^2 \right) \subset W_1.$$  

Suppose (after rotation about $\mathbb{R}^{n-1}$) that $Q_1 = I$. We claim this is the only graph from (5.17) which is contained in $W_1$. We show this in what follows.

First, suppose for contradiction and without loss of generality that

$$Q_2 \left( \text{graph}_{y \in B_{\rho}^n(0), y_n \geq \delta \rho} u_{1,2} \right) \cap \left( B_{\rho/4}^{n-1}(0) \times (-\rho/8, \rho/8)^2 \right) \subset W_1$$

Since $W_1$ is connected, then there is a curve $g \in C([0, 1]; W_1)$ with

$$g(0) \in \text{graph}_{y \in B_{\rho}^n(0), y_n \geq \delta \rho} u_{1,1} \text{ while } g(1) \in Q_2(\text{graph}_{y \in B_{\rho}^n(0), y_n \geq \delta \rho} u_{1,2}).$$

By (5.11) and (5.15), for each $t \in [0, 1]$ there is $a_t \in \{1, \ldots, A\}$ so that $	ext{dist}(T_{g(t)}W, T_0 M_{a_t}) < \epsilon$. However, if we choose $\epsilon = \epsilon(n, \{T_0 M_{a_t}\}_{a_t=1}^A) > 0$ sufficiently small and subsequently $\rho \in (0, 1)$ sufficiently small (so far depending on $\epsilon, \{M_{a_t}\}_{a_t=1}^A, \partial T$) in (5.14) and (5.15), then we conclude since $W$ is a smooth hypersurface that $T_0 M_{a_t} = T_0 M_{a_0}$ for each $t \in [0, 1]$.

Choosing $\delta = \delta(n, \epsilon) \in (0, \delta_0)$ sufficiently small in (5.17), and now $\rho = \rho(\epsilon, \{M_{a_t}\}_{a_t=1}^A, \partial T, \delta, T, C) \in (0, 1)$, we conclude $T_0 M_{a_0} = \mathbb{R}^n$. This implies that $Q_2$ must be either the identity or the rotation about $\mathbb{R}^{n-1}$ by $\pi$. Since $Q_1, \ldots, Q_{nc}$ are distinct rotations about $\mathbb{R}^{n-1}$ by (5.16), then $Q_2$ must be the rotation about $\mathbb{R}^{n-1}$ by angle $\pi$. We thus conclude by (5.17) that $\ast \tilde{T}(g(0)) \cdot e_{n+1} > 0$ while $\ast \tilde{T}(g(1)) \cdot e_{n+1} < 0$. On the other hand, $\ast \tilde{T}(g(t)) \cdot e_{n+1} \ast \tilde{T}(g(t)) \cdot e_{n+1} \equiv 0$ for each $t \in [0, 1]$. This gives a contradiction.

Second, suppose for contradiction

$$\left( \text{graph}_{y \in B_{\rho}^n(0), y_n \geq \delta \rho} u_{1,1} \right) \cap \left( B_{\rho/4}^{n-1}(0) \times (-\rho/8, \rho/8)^2 \right) \subset W_1$$

(as assumed above, with $Q_1 = I$) while

$$\left( \text{graph}_{y \in B_{\rho}^n(0), y_n \geq \delta \rho} u_{2,1} \right) \cap \left( B_{\rho/4}^{n-1}(0) \times (-\rho/8, \rho/8)^2 \right) \subset W_1$$

as well. Take $g \in C([0, 1]; W_1)$ a curve with

$$g(0) \in \text{graph}_{y \in B_{\rho}^n(0), y_n \geq \delta \rho} u_{1,1} \text{ while } g(1) \in \text{graph}_{y \in B_{\rho}^n(0), y_n \geq \delta \rho} u_{2,1}.$$
Next, we show each $g(t) \in \gamma^t \in \{ C([0,1]; (\text{clos } W) \cap (\{ \text{proj}_{\mathbb{R}^{n-1}} g(t) \} \times \mathbb{R}^2) \cap \text{clos } B_\rho(0)) \),
\gamma^t \in \{ \text{proj}_{\mathbb{R}^{n-1}} g(t) \} \times \mathbb{R}^2 \cap B_\rho(0) \}
with g(t) \in \gamma^t((0,1))$, and where $\gamma^t(0) \in (\partial W) \cap B_\rho(0)$ while $\gamma^t(1) \in \partial B_\rho(0)$. If $\rho = \rho(\epsilon, \{ M_0 \} \in T_0, \delta, T, \mathbb{C}) \in (0, 1)$ are chosen sufficiently small, then (5.11)-(5.18) again imply $\text{dist}_H(T_{g(t)} W, \mathbb{R}^n) < \epsilon$ for each $t \in [0, 1]$. We as well conclude
\[
\gamma^t((0,1)) \cap \{ x \in B_\rho(0) : |\text{proj}_{\mathbb{R}^n} x| \geq \delta \rho \} = (\text{graph}_{(y \in B^n_\rho(0); y_n \geq \delta \rho)} u_{j^t,1} \} \cap (\{ \text{proj}_{\mathbb{R}^{n-1}} g(t) \} \times \mathbb{R}^2)
\]
for some $j^t \in \{1, \ldots, N^n_1 \}$. Since $W$ is a smooth hypersurface and $j \in \{1, \ldots, N^n_1 - 1 \}$ we have that $u_{j,1}(y) < u_{j+1,1}(y)$ for each $y \in B^n_\rho(0)$ with $y_n \geq \delta \rho$, then the choice of $j^t$ is continuous in $t$. Finally, $j^0 = 1$ while $j^1 = 2$ gives a contradiction.

Fixing our choice of $\rho \in (0, 1)$, we conclude by (5.17) and (5.20) that $W \cap (B^{-1}_\rho(0) \times (-\rho/8, \rho/8)^2)$ decomposes precisely into connected components $\{ W_{j,k} \}_{j=1, k=1}^{N^n_1, N^n_1}$, where for each $k = 1, \ldots, N^n_1$ and $j = 1, \ldots, N^n_1$
\[
Q_k \left( \text{graph}_{(y \in B^n_\rho(0); y_n \geq \delta \rho)} u_{j,1} \right) \cap (B^{-1}_\rho(0) \times (-\rho/8, \rho/8)^2) \subset W_{j,k},
\]
while on the other hand
\[
Q_k \left( \text{graph}_{(y \in B^n_\rho(0); y_n \geq \delta \rho)} u_{j,1} \right) \cap W_{j,k} = \varnothing
\]
if $k \in \{1, \ldots, N^n_1 \}$ with $\tilde{k} \neq k$, or if $k = k$ but $\tilde{j} \in \{1, \ldots, N^n_1 \}$ with $\tilde{j} \neq j$.

Next, we show each $W_{j,k}$ is a $C^{1,1}$ hypersurface-with-boundary, with $0 \in \partial W_{j,k}$. Consider without loss of generality $W_{1,1}$. We claim
\[
(\partial W_{1,1}) \cap (B^{-1}_\rho(0) \times (-\rho/8, \rho/8)^2) = \{ (z, w_{1,1}(z)) : z \in B^{-1}_\rho(0) \}
\]
where $w_{1,1} \in C^{1,1}(B^{-1}_\rho(0))$. For this, suppose for contradiction there are $x^1, x^2 \in (\partial W_{1,1}) \cap (B^{-1}_\rho(0) \times (-\rho/8, \rho/8)^2)$ with $x^1 \neq x^2$ but so that

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We conclude by (5.20), (5.22), (5.23) that there is a Jordan arc

\[ \gamma \in \left\{ \mathcal{C}([0, 1]; (\text{clos } W_{1,1}) \cap (\{z\} \times \mathbb{R}^2)) \right\} \]

with \( \gamma((0, 1)) = W_{1,1} \cap (\{z\} \times \mathbb{R}^2) \) and

\[ \gamma(0) \in \partial W_{1,1} \cap (B_{\rho/4}^n(0) \times (-\rho/8, \rho/8)^2) \text{ while } \gamma(1) \in \partial (B_{\rho/4}^n(0) \times (-\rho/8, \rho/8)^2). \]

Since \( x^1 \neq x^2 \), then either \( \gamma(0) \neq x^1 \) or \( \gamma(0) \neq x^2 \). However, if for example \( \gamma(0) \neq x^1 \), then we can choose by (5.19) and Theorem 1 of [17] a \( \sigma \in (0, \text{dist}(x^1, \gamma)) \) so that we get as in (5.21) with \( d = 1 \). However, then (5.21) and \( \sigma < \text{dist}(x^1, \gamma) \) imply \( \gamma((0, 1)) \neq W_{1,1} \cap (\{z\} \times \mathbb{R}^2) \), which is a contradiction.

We conclude that \( x^1, x^2 \in (\partial W_{1,1}) \cap (B_{\rho/4}^n(0) \times (-\rho/8, \rho/8)^2) \) with \( \text{proj}_{\mathbb{R}^n-1} x^1 = \text{proj}_{\mathbb{R}^n-1} x^2 \) must satisfy \( x^1 = x^2 \). By (5.20) and (5.22) we further conclude

\[ (\partial W_{1,1}) \cap (B_{\rho/4}^n(0) \times (-\rho/8, \rho/8)^2) = \{(z, w_{1,1}(z)) : z \in B_{\rho/4}^n(0)\} \]

for some function \( w_{1,1} : B_{\rho/4}^n(0) \to \mathbb{R}^2 \). By (5.19), (5.19) there is for each \( z \in B_{\rho/4}^n(0) \) an \( \ell_z \in \{1, \ldots, N\} \) such that \( (z, w_{1,1}(z)) = \Phi_{T, \ell_z}(z) \). We now show \( w_{1,1} \in C^{1,1}(B_{\rho/4}^n(0); \mathbb{R}^2) \). Fix \( z \in B_{\rho/4}^n(0) \), we argue two cases.

First, suppose there is \( \ell \in \{1, \ldots, N\} \) so that \( \Phi_{T, \ell}(z) \neq (z, w_{1,1}(z)) \). Then there is \( \sigma \in (0, 1 - |(z, w_{1,1}(z))|) \) so that by (5.15), after relabeling,

\[ \partial T \triangledown B_{\sigma}(\{(z, w_{1,1}(z))\}) = \sum_{\ell=1}^{N_z} m_{\ell} \Phi_{T, \ell, \#} \left( E^{n-1} \triangledown B_{\sigma}^{n-1}(z) \right) \triangledown B_{\sigma}(\{(z, w_{1,1}(z))\}) \]

where \( N_z \in \{1, \ldots, N - 1\} \). Furthermore, by (5.15) we have \( T_{(z, w_{1,1}(z))} \triangledown B_{\sigma}(\{(z, w_{1,1}(z))\}) \in C^{1,1}(B_{\sigma}^{n-1}(z); \mathbb{R}^2) \).

Second, suppose \( \Phi_{T, \ell}(z) = (z, w_{1,1}(z)) \) for each \( \ell \in \{1, \ldots, N\} \). Since for each \( \tilde{z} \in B_{\rho/4}^n(0) \) we have \( (\tilde{z}, w_{1,1}(\tilde{z})) = \Phi_{T, \ell, \#}(\tilde{z}) \) for some \( \ell \in \{1, \ldots, N\} \), then by
Definition 3.1 it follows that \( Dw_{1,1}(z) \) exists and
\[
Dw_{1,1}(z) = (D\varphi_{T,1}(z), D\psi_{T,1}(z)) = \ldots = (D\varphi_{T,N}(z), D\psi_{T,N}(z))
\]
with \( \varphi_{T,1}, \ldots, \varphi_{T,N} \) and \( \psi_{T,1}, \ldots, \psi_{T,N} \) as in (5.15).

The two cases together with Definition 3.1 imply \( w_{1,1} \in C^{1,1}(B_{\rho/4}(0); \mathbb{R}^2) \).

Next, we show \( W_{1,1} \) is a \( C^{1,1} \) hypersurface-with-boundary, using [2]. For this, we claim
\[
(5.24) \quad \frac{\mathcal{H}^n(W_{1,1} \cap B_{\rho/8}(0))}{\omega_n(\rho/8)^n} < \frac{1 + c\delta}{2}
\]
where \( \omega_n = \mathcal{H}^n(B_1(0)) \) and \( c = c(n) > 0 \). For this, first observe that by (5.17), (5.22), (5.23) we can compute
\[
\frac{\mathcal{H}^n(W_{1,1} \cap \{ x \in B_{\rho/8}(0) : |\text{proj}_{\{0\} \times \mathbb{R}^2} x| \geq \delta \rho \})}{\omega_n(\rho/8)^n} \leq \frac{\sqrt{1 + c_1\delta^2}}{2}
\]
for some \( c_1 = c_1(n) \). Meanwhile, using Theorem 3.4 we can compute
\[
\frac{\mathcal{H}^n(W_{1,1} \cap \{ x \in B_{\rho/8}(0) : |\text{proj}_{\{0\} \times \mathbb{R}^2} x| \leq \delta \rho \})}{\omega_n\rho^n} \leq c_2\delta
\]
for some \( c_2 = c_2(n) \). These two calculations together show (5.24).

Consider the varifold \( |W_{1,1}| \) associated to \( W_{1,1} \). By (5.20) and (5.22) (with \( z = 0 \)) we have \( 0 \in \text{spt} W_{1,1} \). Observe that \( W_{1,1} \) is stationary in \( B_{\rho/8}(0) \setminus \{(z, w_{1,1}(z)) : z \in B_{\rho/4}^{n-1}(0)\} \), where \( w_{1,1} \in C^{1,1}(B_{\rho/4}^{n-1}(0); \mathbb{R}^2) \).

Choosing \( \delta = \delta(n, \epsilon) \in (0, \delta_0) \) sufficiently small depending on \( n \), then (5.24) and [2] imply that \( W_{1,1} \) is a \( C^{1,1} \) hypersurface-with-boundary in \( B_{\rho/16}(0) \).

Finally, \( W_{1,1} \cap \text{spt} \partial T \subseteq \text{reg} \partial T \) follows by Theorem 5.12 \( \square \)

6 Tangentially Immersed Boundaries in Space.

In this section we discuss two-dimensional area-minimizing currents in \( \mathbb{R}^3 \) with \( C^{1,\alpha} \) tangentially immersed boundaries having co-oriented Lipschitz
mean curvature. In this setting, the main results of the previous section can be sharpened. In particular, Theorems 5.12, 5.13 hold for \( T \in \mathbf{T}_{1,\alpha}^{1,\alpha}(U) \) for \( \alpha \in (0,1] \) more generally; see respectively Theorems 6.10, 6.11.

To begin, we give the following lemma which will, in Theorem 6.11, allow us to give a more general version of Theorem 5.13.

**Lemma 6.1** Let \( U \subseteq_o \mathbb{R}^3 \), \( \alpha \in (0,1] \), and suppose \( T \in \mathbf{T}_{1,\alpha}^{1,\alpha}(U) \). Suppose \( x \in \text{spt} \partial T \) and there exists \( \rho \in (0,\text{dist}(x,\partial U)) \) so that

\[ \text{spt} T \cap B_\rho(x) \subseteq \bigcup_{\alpha=1}^A (\text{clos } M_\alpha) \cap B_\rho(x) \]

for \( C^1 \) hypersurfaces-with-boundary \( M_1, \ldots, M_a \) in \( B_\rho(x) \). Then there is \( \sigma \in (0,\rho) \) and \( A \subseteq \{1, \ldots, A\} \) so that

\[ \text{spt} T \cap B_\sigma(x) \subseteq \bigcup_{\alpha \in A} (\text{clos } M_\alpha) \cap B_\sigma(x), \]

and \( x \in \text{clos } M_\alpha \) with \( T_x \partial T \subset T_x M_\alpha \) for each \( \alpha \in A \).

**Proof:** Suppose (after translation) \( 0 \in \text{spt} \partial T \) and that (after rotation) \( T_0 \partial T = \mathbf{R} \). We can also suppose \( \rho \in (0,\text{dist}(0,\partial U)) \) is such that (after relabeling) \( \text{spt} T \cap B_\rho(0) \subseteq \bigcup_{\alpha=1}^A (\text{clos } M_\alpha) \cap B_\rho(0) \) with \( 0 \in \text{clos } M_\alpha \) for each \( \alpha \in \{1, \ldots, A\} \). Define

\[ A = \{ a \in \{1, \ldots, A\} : \mathbf{R} \subset T_0 M_\alpha \}. \]

We claim there is \( \sigma \in (0,\rho) \) so that

\[ \text{spt} T \cap B_\sigma(0) \subseteq \bigcup_{\alpha \in A} (\text{clos } M_\alpha) \cap B_\sigma(0). \]

It suffices to discount the following two scenarios:

First, suppose for contradiction there exists a sequence

\[ \{x^d\}_{d=1}^\infty \in \text{spt} \partial T \cap B_\rho(0) \setminus \bigcup_{\alpha \in A} \text{clos } M_\alpha \]

so that \( x^d \to 0 \). For each \( d \in \mathbb{N} \) let \( \mathbb{C}_{x^d} \) be a tangent cone of \( T \) at \( x^d \). By Theorem 5.4.2 of \( [6] \) and Theorem 3.4 we have

\[ T_{x^d} \partial T \subseteq \text{spt} \mathbb{C}_{x^d} \subseteq \bigcup_{\{a \in \{1, \ldots, A\} \setminus \mathbb{A} : x^d \in \text{clos } M_\alpha \} \setminus \mathbb{A}} T_{x^d} M_\alpha. \]
Recall that $0 \in \text{clos } M_a$ for each $a \in \{1, \ldots, A\}$. Since for each $a \in \{1, \ldots, A\} \setminus \mathcal{A}$ we have that $M_a$ is a $C^1$ hypersurface-with-boundary with $R \notin T_0 M_a$, then we conclude there is $\delta > 0$ small so that for all sufficiently large $d$ we have $\text{dist}_H(R, T_x \partial T) \geq \delta$. This contradicts $T \in \mathcal{T}_{2, \text{loc}}^1(U)$.

Second, suppose there exists a sequence

$$\{x^d\}_{d=1}^{\infty} \subset \text{spt } T \cap B_\rho(0) \setminus \left(\text{spt } \partial T \cup \bigcup_{a \in \mathcal{A}} \text{clos } M_a \right);$$

we can suppose $\{x^d\}_{d=1}^{\infty} \subset \text{clos } M_{a_0}$ for some fixed $a_0 \in \{1, \ldots, A\} \setminus \mathcal{A}$. By Theorem 5.4.2 of [6] and interior regularity of area-minimizing currents

(6.2) $T_{x^d} T = T_{x^d} M_a$ for some $a \in \{1, \ldots, A\} \setminus \mathcal{A}$ with $x^d \in \text{clos } M_a$.

For each $d \geq 1$ let $\rho_d = 2|x^d|$, so $\rho_d > 0$.

By taking a subsequence and relabeling, we can suppose that $C$ is a tangent cone of $T$ at 0 with $\eta_{0, \rho_d} T \to C$. By Theorem 5.4.2 of [6] and Theorem 3.4 we again have $\text{spt } C \subset \bigcup_{a=1}^{A} T_0 M_a$. In fact,

(6.3) $\text{spt } C \subset \bigcup_{a=1}^{A} T_0 M_a$.

To see this, suppose for contradiction $\tilde{x} \in \text{spt } C \setminus \bigcup_{a \in \mathcal{A}} T_0 M_a$. Then $\text{spt } C \subset \bigcup_{a=1}^{A} T_0 M_a$, the constancy theorem (see Theorem 26.27 of [18]), and $\text{spt } \partial C = R$ imply there is $\tilde{\rho} > 0$ and $a \in \{1, \ldots, A\} \setminus \mathcal{A}$ so that $\text{spt } C \cap B_{\tilde{\rho}}(\tilde{x}) = T_0 M_a \cap B_{\tilde{\rho}}(\tilde{x})$. However, $R \notin T_0 M_a$ contradicts Theorem 3.4 and so we conclude (6.3).

Fix $\delta = \delta(\{T_0 M_a\}_{a=1}^{A}) \in (0, \delta_0)$ to be chosen later, with $\delta_0 > 0$ as in Theorem 1 of [17]. Recall that $M_1, \ldots, M_A$ are $C^1$ hypersurfaces-with-boundary, and $R \subset T_0 M_a$ for each $a \in \mathcal{A}$ while $R \notin T_0 M_a$ for each $a \in \{1, \ldots, A\} \setminus \mathcal{A}$. If we choose $\delta = \delta(\{T_0 M_a\}_{a=1}^{A}) \in (0, \delta_0)$ sufficiently small, then (6.2) implies that for all sufficiently large $d \in \mathbb{N}$

$$\text{dist}_H(T_{x^d} T, T_0 M_a) \geq \delta \quad \text{for each } a \in \mathcal{A}.$$ 

On the other hand, Theorem 1 of [17], $\eta_{0, \rho_d} T \to C$, and (6.3) imply that for all sufficiently large integers $d$

$$\text{dist}_H(T_x T, T_0 M_a) < \delta \quad \text{for some } a \in \mathcal{A}$$.
for each $x \in B_{\rho_d}(0)$ with $|\text{proj}_{(0) \times \mathbb{R}^2} x| > \delta \rho_d$; see for example (5.16) in the proof of Theorem 5.13. Since $\rho_d = 2|x^d|$, then we conclude

\begin{equation}
|\text{proj}_{(0) \times \mathbb{R}^2} x_d| \leq \delta \rho_d
\end{equation}

for all sufficiently large $d \in \mathbb{N}$.

Now consider $x^d = \text{proj}_{T_0 M_{a_0}} x^d + \text{proj}_{T_0 M_{a_0}} x^d$. Since $\mathbb{R} \notin T_0 M_{a_0}$, then there exists $\delta_{a_0} > 0$ so that for each $x \in T_0 M_{a_0}$ we have $|\text{proj}_{(0) \times \mathbb{R}^2} x| \geq \delta_{a_0}|x|$; thus

$|\text{proj}_{(0) \times \mathbb{R}^2} \text{proj}_{T_0 M_{a_0}} x^d| \geq \delta_{a_0} |\text{proj}_{T_0 M_{a_0}} x^d|.$

Since $M_{a_0}$ is a $C^1$ hypersurface-with-boundary and $x^d \in \text{clos} M_{a_0}$ for each $d \in \mathbb{N}$, then for all sufficiently large $d \in \mathbb{N}$

$|\text{proj}_{T_0 M_{a_0}} x^d| \leq \frac{\delta_{a_0}}{2} |\text{proj}_{T_0 M_{a_0}} x^d|.$

Since $\rho_d = 2|x^d| \leq 2|\text{proj}_{T_0 M_{a_0}} x^d| + 2|\text{proj}_{T_0 M_{a_0}} x^d|$ then

$|\text{proj}_{T_0 M_{a_0}} x^d| \geq \frac{\rho_d}{2 + \delta_{a_0}}.$

Using these three computations together, we conclude for all sufficiently large $d \in \mathbb{N}$

$|\text{proj}_{(0) \times \mathbb{R}^2} x_d| \geq |\text{proj}_{(0) \times \mathbb{R}^2} \text{proj}_{T_0 M_{a_0}} x^d| - |\text{proj}_{(0) \times \mathbb{R}^2} \text{proj}_{T_0 M_{a_0}} x^d|

\geq \delta_{a_0} |\text{proj}_{T_0 M_{a_0}} x^d| - \frac{\delta_{a_0}}{2} |\text{proj}_{T_0 M_{a_0}} x^d| \geq \frac{\delta_{a_0} \rho_d}{4 + 2 \delta_{a_0}}.$

This contradicts (6.4), choosing $\delta = \delta\left(\{T_0 M_{a}\}^A_{a=1}\right) \in (0, \delta_0)$ small. $\square$

We now give a different definition of half-regular points, differing slightly from that given by Lemma 3.19. This definition of half-regular points appears in Lemma 1 of [16], in the setting of two-dimensional solutions to the $c$-Plateau problem in space; see §1.4.

**Lemma 6.5** Let $U \subset \mathbb{R}^3$, $\alpha \in (0, 1]$, and $T \in \mathcal{T}^{1,\alpha}_{2,\text{loc}}(U)$. For any $x \in \text{sing} \partial T$ and $\rho \in (0, \text{dist}(x, \partial U))$, there is $x \in \text{sing} \partial T$ and $\sigma \in (0, \rho - |x - x|)$ so that

$\partial T \subset B_{\sigma}(x) = \sum_{\ell=1}^N m_{\ell}\left[\eta_{-x, \ell} \circ Q \circ \Phi_{T, \ell} \circ (\mathbb{E} \subset (-\sigma, \sigma))\right] \subset B_{\sigma}(x);$. 

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for \( N, m_1, \ldots, m_N \in \mathbb{N} \), an orthogonal rotation \( Q \), and 
\( \Phi_{T,\ell} \in C^{1,\alpha}((-\sigma,\sigma); \mathbb{R}^3) \) for each \( \ell = 1, \ldots, N \) is the map 
\[
\Phi_{T,\ell}(z) = (z, \varphi_{T,\ell}(z), \psi_{T,\ell}(z)) \quad \text{where} \quad \varphi_{T,\ell}, \psi_{T,\ell} \in C^{1,\alpha}((-\sigma,\sigma))
\]
\[
\varphi_{T,\ell}(0) = \psi_{T,\ell}(0) = 0 \quad \text{and} \quad D\varphi_{T,\ell}(0) = D\psi_{T,\ell}(0) = 0.
\]

Moreover, one of the two following occurs:

- For each \( \ell, \tilde{\ell} \in \{1, \ldots, N\} \) the images 
  \[
  \Phi_{T,\ell}(0, \sigma) \quad \text{and} \quad \Phi_{T,\tilde{\ell}}(0, \sigma)
  \]
  are either equal or disjoint. Moreover \( \Phi_{T,\ell}(0, \sigma) \cap \Phi_{T,\tilde{\ell}}(0, \sigma) = \emptyset \) for some \( \ell, \tilde{\ell} \in \{1, \ldots, N\} \).

- For each \( \ell, \tilde{\ell} \in \{1, \ldots, N\} \) the images 
  \[
  \Phi_{T,\ell}(-\sigma, 0) \quad \text{and} \quad \Phi_{T,\tilde{\ell}}(-\sigma, 0)
  \]
  are either equal or disjoint. Moreover \( \Phi_{T,\ell}(-\sigma, 0) \cap \Phi_{T,\tilde{\ell}}(-\sigma, 0) = \emptyset \) for some \( \ell, \tilde{\ell} \in \{1, \ldots, N\} \).

We say that such a point \( x \in \text{sing} \partial T \) is half-regular.

**Proof:** We can either modify the proof of Lemma 3.19 or more directly merely apply the proof of Lemma 1 of [16]. \( \square \)

Our aim is now to show that Theorem 5.12 holds for \( T \in \mathbf{T}_{1,\alpha}^{1,\alpha}(U) \) with \( \alpha \in (0,1] \) more generally if \( U \subseteq_o \mathbb{R}^3 \). The first step is to give the more general version of Lemma 5.1.

**Lemma 6.6** Let \( U \subseteq_o \mathbb{R}^3 \), \( \alpha \in (0,1] \), and suppose \( T \in \mathbf{T}_{2,\alpha}^{1,\alpha}(U) \), where \( \partial T \) has co-oriented mean curvature \( H_{\partial T} = h\nu_T \) with \( h : U \to \mathbb{R} \) Lipschitz. If 
\( x \in \text{sing} \partial T \) is half-regular (as in Lemma 6.5), then every tangent cone of \( T \) at \( x \) is a sum of half-planes with constant orientation after rotation (as in Definition 3.3 with \( n = 2 \)).
**Proof:** Suppose for contradiction that (after translation) 0 ∈ sing∂T is half-regular and that T at 0 has tangent cone

\[(6.7) \quad C = (m + \theta)E^2 \sqcup \{ y \in \mathbb{R}^2 : y_2 > 0 \} + \theta E^2 \sqcup \{ y \in \mathbb{R}^2 : y_2 < 0 \}\]

where \(m, \theta \in \mathbb{N}\). Suppose as well that \(\rho \in (0, \text{dist}(0, \partial U))\) is such that

\[
\partial T \sqcup B_\rho(0) = \sum_{\ell=1}^{N} m_\ell \Phi_{T,\ell\#}(E^1 \sqcup (-\rho, \rho)) \sqcup B_\rho(0)
\]

as in Lemma 6.5 and where (without loss of generality, and after relabeling)

\[(6.8) \quad \text{for each } \ell, \ell' \in \{1, \ldots, N\} \text{ the images } \Phi_{T,\ell}((0, \rho)) \text{ and } \Phi_{T,\ell'}((0, \rho)) \]

are either equal or disjoint, while \(\Phi_{T,1}((0, \rho)) \cap \Phi_{T,2}((0, \rho)) = \emptyset\).

We can also suppose by Theorem 3.18 there exists a solution to the minimal surface equation \(u \in C^\infty(B_\rho^2(0))\) with \(u(0) = 0\) and \(Du(0) = 0\) so that

\[
\text{spt } T \cap B_\rho(0) = \left(\text{graph}_{B_\rho^2(0)} u\right) \cap B_\rho(0),
\]

where the orientation vector for \(T\) if \(x \in \text{spt } T \cap B_\rho(0)\) is given by

\[(6.9) \quad * \vec{T}(x) = \left(\frac{-Du}{\sqrt{1 + |Du|^2}}, \frac{1}{\sqrt{1 + |Du|^2}}\right)\bigg|_{\text{proj}_{\mathbb{R}^2} x}.
\]

For \(\ell = 1, 2\) let \(\sigma_\ell\) be the curve parameterized by arc-length with image \(\Phi_{T,\ell}([0, \rho])\). Let \(P : \mathbb{R}^3 \to \mathbb{R}^3\) be the map \(P(x) = (x_2, -x_1, 0)\). We conclude by (6.7), (6.8), (6.9) that there is an \(\epsilon > 0\) so that for each \(\ell = 1, 2\)

\[
\sigma''_\ell(t) = -h(\sigma_\ell(t)) \frac{P(\sigma'_\ell(t)) - (P(\sigma'_\ell(t)) \cdot * \vec{T}(\sigma_\ell)) * \vec{T}(\sigma_\ell)}{\|P(\sigma'_\ell(t)) - (P(\sigma'_\ell(t)) \cdot * \vec{T}(\sigma_\ell)) * \vec{T}(\sigma_\ell)\|}
\]

for each \(t \in (0, \epsilon)\). Since \(h\) is Lipschitz while

\[
\sigma_1(0) = \sigma_2(0) = 0 \text{ and } \sigma'_1(0) = \sigma'_2(0) = e_1,
\]

then we conclude by uniqueness of ODEs that \(\sigma_1 = \sigma_2\). This contradicts \(\Phi_{T,1}((0, \rho)) \cap \Phi_{T,2}((0, \rho)) = \emptyset\). □

As in the the higher dimensional case, Lemma 6.6 readily implies the following regularity theorem:

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Theorem 6.10 Let $U \subset \alpha \subset \mathbb{R}^3$, $\alpha \in (0,1]$, and suppose $T \in \mathbf{T}^{1,\alpha}_{\text{loc}}(U)$ where $\partial T$ has co-oriented mean curvature $H_{\partial T} = h_{\partial T}$ with $h : U \to \mathbb{R}$ Lipschitz. If $T$ at $x \in \text{spt} \partial T$ has tangent cone a plane with constant orientation but non-constant multiplicity (as in Definition 7.3 with $n = 2$), then $x \in \text{reg} \partial T$.

Proof: Follows exactly as the proof of Theorem 5.12 where we now use Lemma 6.6. $\square$

Finally, we give a more general version of Theorem 5.13 in case $n = 2$:

Theorem 6.11 Let $U \subset \alpha \subset \mathbb{R}^3$, $\alpha \in (0,1]$, and suppose $T \in \mathbf{T}^{1,\alpha}_{\text{loc}}(U)$ where $\partial T$ has co-oriented mean curvature $H_{\partial T} = h_{\partial T}$ with $h : U \to \mathbb{R}$ Lipschitz. Suppose $x \in \text{spt} \partial T$ and that there exists $\rho \in (0, \text{dist}(x, \partial U))$ and $C^1$ hypersurfaces-with-boundary $M_1, \ldots, M_{\Lambda}$ in $B_{\rho}(x)$ so that

$$\text{spt} \, T \cap B_{\rho}(x) \subseteq \bigcup_{a=1}^{\Lambda}(\text{clos} \, M_a) \cap B_{\rho}(x).$$

Then there is $\sigma \in (0, \rho)$ and $\mathcal{B} \in \{1, \ldots, 2\Theta(x)\}$ so that

$$\text{spt} \, T \cap B_{\sigma}(x) = \bigcup_{b=1}^{\mathcal{B}}(\text{clos} \, W_b) \cap B_{\sigma}(x),$$

for orientable $C^{1,\alpha}$ hypersurfaces-with-boundary $W_1, \ldots, W_{\mathcal{B}}$ in $B_{\sigma}(x)$. For each $b \in \{1, \ldots, \mathcal{B}\}$ we have $x \in \partial W_b$ and $W_b \cap \text{spt} \partial T \subset \text{reg} \partial T$. Furthermore, for each $b, b' \in \{1, \ldots, \mathcal{B}\}$ we have

$$(\text{clos} \, W_b) \cap (\text{clos} \, W_{b'}) \cap B_{\sigma}(x) \subseteq (\partial W_b) \cap (\partial W_{b'}) \cap B_{\sigma}(x).$$

Proof: We follow the proof of Theorem 5.13 where we use Lemmas 6.10, 6.11 in place of Lemmas 5.20, 5.12. The only other significant difference is in the argument following (5.24); we describe this difference. At that analogous point in the argument, we show there is $\sigma \in (0, \rho)$ sufficiently small so that $\text{spt} \, T \cap B_{\sigma}(x) = \bigcup_{b=1}^{\mathcal{B}}(\text{clos} \, W_b) \cap B_{\sigma}(x)$ where $W_1, \ldots, W_{\mathcal{B}}$ are pairwise disjoint smooth surfaces with topological boundary $(\partial W_b) \cap B_{\rho}(x)$ a $C^{1,\alpha}$ Jordan arc through $x$ for each $b \in \{1, \ldots, \mathcal{B}\}$. For each $b \in \{1, \ldots, \mathcal{B}\}$ we have that (5.24) holds as well, more specifically that for each $b \in \{1, \ldots, \mathcal{B}\}$ we have $\frac{H^2(W_b \cap \partial W_b(x))}{\pi \sigma^2} < \frac{1+\delta}{2}$, and that $|W_b|$ the varifold associated to $W_b$ is stationary in $B_{\sigma}(x) \setminus \partial W_b$. We conclude in this case by [3] that each $W_1, \ldots, W_{\mathcal{B}}$ is a $C^{1,\alpha}$ surface-with-boundary. $\square$
Appendix

In this section we give the technical Lemma A.2, which we need for the proof of Theorem 3.18. We work in $\mathbb{R}^2$, so that for $x = (x_1, x_2) \in \mathbb{R}^2$ we have that $B_\rho(x)$ is the disk of radius $\rho > 0$ centered at $x$. First, we make the following useful definition:

**Definition A.1** Suppose $\gamma \in C([0, \mathcal{H}^1(\gamma)]; \mathbb{R}^2) \cap C^\infty((0, \mathcal{H}^1(\gamma)); \mathbb{R}^2)$ is a Jordan arc parameterized by arc length; write $\gamma(t) = (\gamma_1(t), \gamma_2(t))$. Also suppose $V \in C^\infty((0, \mathcal{H}^1(\gamma)); S^1)$. Then we say $\gamma, V$ are positively oriented if $V(\gamma(t)) = (-\gamma_2'(t), \gamma_1'(t))$ for each $t \in (0, \mathcal{H}^1(\gamma))$. Contrarily, we say $\gamma, V$ are negatively oriented if $V(\gamma(t)) = (\gamma_2'(t), -\gamma_1'(t))$ for each $t \in (0, \mathcal{H}^1(\gamma))$.

The following general lemma about curves then is proved geometrically:

**Lemma A.2** Suppose $\{B_{\sigma_\ell}(x_\ell)\}_{\ell=1}^N$ is a collection of pairwise disjoint balls with $B_{\sigma_\ell}(x_\ell) \subset B_{1/2}(0) \subset \mathbb{R}^2$ for each $\ell \in \{1, \ldots, N\}$. Then (a) and (b) hold:

(a) With $N^{(1)} \in \{1, \ldots, N\}$ and $N_1, \ldots, N_{N^{(1)}} \in \mathbb{N}$, suppose we have a collection of Jordan arcs

$$\Gamma_1, \Gamma_2, \{\{\gamma_k^{(\ell)}\}_{k=1}^{N_{\ell}}\}_{\ell=1}^{N^{(1)}}, \{G^{(\ell)}_\ell, g^{(\ell)}_\ell\}_{\ell=N^{(1)}+1}^N,$$

with images that are pairwise either equal or meet only at mutual endpoints, and let $\mathcal{E}$ be the collection of endpoints of these arcs; we allow for the collection $\{G^{\ell}_\ell, g^{\ell}_\ell\}_{\ell=N^{(1)}+1}^N$ to be empty. With

$$\mathcal{G} = \Gamma_1 \cup \Gamma_2 \cup \left( \bigcup_{\ell=1}^{N^{(1)}} \bigcup_{k=1}^{N_{\ell}} \gamma_k^{(\ell)} \right) \cup \left( \bigcup_{\ell=N^{(1)}+1}^{N} G^{\ell}_\ell \cup g^{\ell}_\ell \right)$$

suppose there is $V \in C^\infty(\mathcal{G} \setminus \mathcal{E}; S^1)$ so that the following hold:

(1) The curves $\Gamma_1, \Gamma_2$ satisfy

$$\mathcal{G} \cap (\text{clos } B_1(0)) \setminus B_{1/2}(0) = (\Gamma_1 \cup \Gamma_2) \cap (\text{clos } B_1(0)) \setminus B_{1/2}(0).$$

We parameterize

$$\Gamma_1 \in C([0, \mathcal{H}^1(\Gamma_1)]; \text{clos } B_1(0)) \cap C^\infty((0, \mathcal{H}^1(\Gamma_1)); B_1(0))$$
and 
\[ \Gamma_2 \in C([0, \mathcal{H}^1(\Gamma_2)]; \text{clos } B_1(0)) \cap C^\infty((0, \mathcal{H}^1(\Gamma_2)); B_1(0)) \]

by arc-length and so that \( \Gamma_1(0), \Gamma_2(0) \in \partial B_1(0) \) while 
\[ \Gamma_1(\mathcal{H}^1(\Gamma_1)), \Gamma_2(\mathcal{H}^1(\Gamma_2)) \in \bigcup_{\ell=1}^N B_{\sigma_\ell/2}(x_\ell). \]

The images \( \Gamma_1([0, \mathcal{H}^1(\Gamma_1)) \) and \( \Gamma_2([0, \mathcal{H}^1(\Gamma_2)) \) are disjoint.

(2) For each \( \ell \in \{1, \ldots, N^{(1)}\} \)
\[ \mathcal{G} \cap (\text{clos } B_{\sigma_\ell}(x_\ell)) \setminus B_{\sigma_\ell/2}(x_\ell) = \bigcup_{k=1}^{N_\ell} \gamma^\ell_k \cap (\text{clos } B_{\sigma_\ell}(x_\ell)) \setminus B_{\sigma_\ell/2}(x_\ell). \]

We parameterize for each \( k \in \{1, \ldots, N_\ell\} \)
\[ \gamma^\ell_k \in C([0, \mathcal{H}^1(\gamma^\ell_k)]; \text{clos } B_1(0)) \cap C^\infty((0, \mathcal{H}^1(\gamma^\ell_k)); B_1(0)) \]

by arc-length so that \( \gamma^\ell_k(\mathcal{H}^1(\gamma^\ell_k)) \in B_{\sigma_\ell/2}(x_\ell) \). Meanwhile, 
\( \gamma^\ell_k(0) \in (\partial B_1(0)) \cup \bigcup_{k=1}^N B_{\sigma_\ell}(x_\ell) \) and \( \gamma^\ell_k(0) \cap \partial B_{\sigma_\ell}(x_\ell) \neq \emptyset. \)

With this parameterization \( \gamma^\ell_k, V \) are positively oriented.

(c) For each \( \ell \) in the (possibly empty) set \( \{N^{(1)} + 1, \ldots, N\} \)
\[ \mathcal{G} \cap (\text{clos } B_{\sigma_\ell}(x_\ell)) \setminus B_{\sigma_\ell/2}(x_\ell) = (G^\ell \cup g^\ell) \cap (\text{clos } B_{\sigma_\ell}(x_\ell)) \setminus B_{\sigma_\ell/2}(x_\ell). \]

We parameterize
\[ G^\ell \in C([0, \mathcal{H}^1(G^\ell)]; \text{clos } B_1(0)) \cap C^\infty((0, \mathcal{H}^1(G^\ell)); B_1(0)) \]

and
\[ g^\ell \in C([0, \mathcal{H}^1(g^\ell)]; \text{clos } B_1(0)) \cap C^\infty((0, \mathcal{H}^1(g^\ell)); B_1(0)) \]

by arc-length so that \( G^\ell(\mathcal{H}^1(G^\ell)), g^\ell(\mathcal{H}^1(g^\ell)) \in B_{\sigma_\ell/2}(x_\ell) \). Meanwhile, 
\( G^\ell(0), g^\ell(0) \in (\partial B_1(0)) \cup \bigcup_{\ell=1}^N B_{\sigma_\ell/2}(x_\ell) \) along with \( G^\ell \cap \partial B_{\sigma_\ell}(x_\ell) \neq \emptyset \) and \( g^\ell \cap \partial B_{\sigma_\ell}(x_\ell) \neq \emptyset. \) The images 
\( G^\ell((0, \mathcal{H}^1(G^\ell))) \) and \( g^\ell((0, \mathcal{H}^1(g^\ell))) \) are disjoint.

With this parameterization \( G^\ell, V \) are positively oriented, while \( g^\ell, V \) are negatively oriented.
Then $\Gamma_1, V$ and $\Gamma_2, V$ are either both positively oriented or both negatively oriented.

(b) Consider now a collection of Jordan arcs

$$\Gamma_1, \Gamma_2, \{G^\ell, g^\ell\}_{\ell=1}^N$$

with images that are pairwise either equal or meet only at mutual endpoints, and let $\mathcal{E}$ be the collection of endpoints of these arcs. Let

$$\mathcal{G} = \Gamma_1 \cup \Gamma_2 \cup \bigcup_{\ell=1}^N G^\ell \cup g^\ell$$

and suppose $V \in C^\infty(\mathcal{G} \setminus \mathcal{E}; S^1)$ is such that the following hold:

1. $\Gamma_1$ and $\Gamma_2$ satisfy (a)(1) above. We also assume $\Gamma_1, V$ are positively oriented and $\Gamma_2, V$ are negatively oriented.

2. For each $\ell \in \{1, \ldots, N\}$

   $$\mathcal{G} \cap (\text{clos } B_{\sigma}(x^\ell)) = (G^\ell \cup g^\ell) \cap (\text{clos } B_{\sigma}(x^\ell)).$$

   We parameterize

   $$G^\ell \in C([0, \mathcal{H}(G^\ell)]; \text{clos } B_1(0)) \cap C^\infty((0, \mathcal{H}(G^\ell)); B_1(0))$$

   and

   $$g^\ell \in C([0, \mathcal{H}(g^\ell)]; \text{clos } B_1(0)) \cap C^\infty((0, \mathcal{H}(g^\ell)); B_1(0))$$

   by arc-length so that $G^\ell(\mathcal{H}(G^\ell)) = g^\ell(\mathcal{H}(g^\ell)) = x^\ell$. Meanwhile, $G^\ell(0), g^\ell(0) \in (\partial B_1(0)) \cup \{x_1, \ldots, x_N\}$. The images $G^\ell((0, \mathcal{H}(G^\ell)))$ and $g^\ell((0, \mathcal{H}(g^\ell)))$ are disjoint.

   With this parameterization $G^\ell, V$ are positively oriented, while $g^\ell, V$ are negatively oriented.

Then either

$$\mathcal{G} = \Gamma \text{ or } \mathcal{G} = \Gamma \cup \bigcup_{\ell=1}^{N^{(2)}} L_\ell$$

where the following hold:
• Γ is a continuous Jordan arc with endpoints \( \Gamma_1(0), \Gamma_2(0) \), smooth away from the collection of points \( \{x_1, \ldots, x_N\} \), and so that the images of \( \Gamma_1, \Gamma_2 \) are contained in the image of \( \Gamma \).

• In the latter case, \( N^{(2)} \in \mathbb{N} \) and for each \( \ell \in \{1, \ldots, N^{(2)}\} \) we have that \( L_{\ell} \) is a continuous closed Jordan curve, smooth away from \( \{x_1, \ldots, x_N\} \), and with \( L_{\ell} \cap \{x_1, \ldots, x_N\} \neq \emptyset \). The curves \( \Gamma, L_1, \ldots, L_{N^{(2)}} \) are pairwise disjoint.

**Proof:** To prove (a), suppose for contradiction (and without loss of generality) that \( \Gamma_1, V \) are positively oriented, while \( \Gamma_2, V \) are negatively oriented.

Observe more specifically that \( \Gamma_2, V \) are negatively oriented and \( \Gamma_2(0) \in \partial B(0) \). On the other hand, the following hold:

\[
\Gamma_1([0, \mathcal{H}^1(\Gamma_1)]) \neq \Gamma_2([0, \mathcal{H}^1(\Gamma_2)]);
\]

\( \gamma_k^\ell, V \) are positively oriented;

\( \gamma_k^\ell(\mathcal{H}^1(\gamma_k^\ell)) \in B_{\sigma_{\ell}/2}(x_\ell) \) for each \( \ell \in \{1, \ldots, N^{(1)}\}, k \in \{1, \ldots, N_\ell\} \);

\( G^\ell, V \) are positively oriented

\( G^\ell(\mathcal{H}^1(G^\ell)) \in B_{\sigma_{\ell}/2}(x_\ell) \) for each \( \ell \in \{N^{(1)} + 1, \ldots, N\} \).

Since \( \Gamma_2(\mathcal{H}^1(\Gamma_2)) \in \bigcup_{\ell=1}^{N} B_{\sigma_{\ell}/2}(x_\ell) \), then we must have (after relabeling) \( \Gamma_2 = g_{N^{(1)}+1} \); in particular we must have \( N^{(1)} < N \).

Now consider \( G_{N^{(1)}+1} \). Since \( G_{N^{(1)}+1}, V \) are positively oriented and \( G^\ell(\mathcal{H}^1(G_{N^{(1)}+1})) \in B_{\sigma_{\ell}/2}(x_{N^{(1)}+1}) \), then we cannot have \( G_{N^{(1)}+1} = \gamma_k^\ell \) for any \( \ell \in \{1, \ldots, N^{(1)}\}, k \in \{1, \ldots, N_\ell\} \). However, observe that:

\[
G_{N^{(1)}+1}(0) \in (\partial B(0)) \cup \bigcup_{\ell=1}^{N} B_{\sigma_{\ell}/2}(x_\ell);
\]

\( G_{N^{(1)}+1} \cap \partial B_{\sigma_{N^{(1)}+1}}(x_{N^{(1)}+1}) = \emptyset; \)

\( G_{N^{(1)}+1} \neq g_{N^{(1)}+1} \).

We conclude that either \( G_{N^{(1)}+1} = \Gamma_1 \) or (after relabeling) \( G_{N^{(1)}+1} = g_{N^{(1)}+2} \). By considering \( G_{N^{(1)}+2} \) and arguing iteratively, we see that there is
\( \tilde{N} \in \{1, \ldots, N\} \) so that (after relabeling)

\[
\begin{align*}
\Gamma_2 &= g_{N(1)+1}, \\
G_{N(1)+1} &= g_{N(1)+2}, \\
& \quad \vdots \\
G_{N-1} &= g_{N-1}, \\
G_{\tilde{N}} &= \Gamma_1.
\end{align*}
\]

(A.3)

Now consider the collection \( \{(\gamma^{\ell N}_k)_{k=1}^{N(1)}\}_{\ell=1}^{N(1)} \), in particular consider \( \gamma^1_1 \). Since \( \gamma^1_1, V \) are positively oriented and \( \gamma^1_1(\mathcal{H}^1(\gamma^1_1)) \in B_{x_1} \), then we conclude \( \tilde{N} \in \{1, \ldots, N-1\} \) and that (after relabeling) \( \gamma^1_1 = g^{\tilde{N}+1} \). By considering \( G^{\tilde{N}+1} \) and arguing iteratively, we see that (after relabeling)

\[
\begin{align*}
\gamma^1_1 &= g^{\tilde{N}+1}, \\
G^{\tilde{N}+1} &= g^{\tilde{N}+2}, \\
& \quad \vdots \\
G^{N-1} &= g^{N}.
\end{align*}
\]

(A.4)

However, the fact that \( G^N \neq \gamma^\ell_k \) for each \( \ell \in \{1, \ldots, N(1)\}, k \in \{1, \ldots, N_\ell\} \) together with (A.3),(A.4) give a contradiction.

The proof of (b) follows similarly. \( \square \)

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