A Doubly Optimistic Strategy for Safe Linear Bandits

Abstract

We propose a doubly optimistic strategy for the safe-linear-bandit problem, DOSLB. The safe linear bandit problem is to optimise an unknown linear reward whilst satisfying unknown round-wise safety constraints on actions, using stochastic bandit feedback of reward and safety-risks of actions. In contrast to prior work on aggregated resource constraints, our formulation explicitly demands control on roundwise safety risks.

Unlike existing optimistic-pessimistic paradigms for safe bandits, DOSLB exercises supreme optimism, using optimistic estimates of reward and safety scores to select actions. Yet, and surprisingly, we show that DOSLB rarely takes risky actions, and obtains $\tilde{O}(d\sqrt{T})$ regret, where our notion of regret accounts for both inefficiency and lack of safety of actions. Specialising to polytopal domains, we first notably show that the $\sqrt{T}$-regret bound cannot be improved even with large gaps, and then identify a slackened notion of regret for which we show tight instance-dependent $O(\log^2 T)$ bounds. We further argue that in such domains, the number of times an overly risky action is played is also bounded as $O(\log^2 T)$.

1 Introduction

The (stochastic) linear bandit problem is a canonical setting that uses structure to navigate the exploration-exploitation problem in a large action space [Aue02; DHK08; RT10; APS11]. In a series of rounds, indexed by $t$, a learner for the linear bandit problem must choose vector $x_t \in \mathcal{K} \subset \mathbb{R}^d$ as their action, upon doing which they observe a noisy reward $R_t = \langle \theta^*, x_t \rangle + \eta_t$, where $\theta^*$ is a latent parameter. The goal of the learner is to obtain high reward over a time horizon, as measured by the performance regret $\sum (\theta^*, x^*) - (\theta^*, x_t)$, where $x^*$ maximises the inner products over vectors in $\mathcal{K}$. The problem is generic, well-studied, and has applications to diverse settings, including recommendation systems, clinical treatments, and engineering design [DM12; RKD21; Gor+20].

Of course, most application domains such as the above have constraints accompanying the reward maximisation objectives. For instance, when assigning treatments, we need to ensure that the extent of side-effects is well controlled. In the standard linear bandit setting, the set $\mathcal{K}$ of actions plays the role of enforcing these constraints. However, this set is assumed to be available a priori, which is an unrealistic assumption in a multitude of interesting cases. The recent literature has thus seen a shift into modeling unknown constraints that are stochastically observed. We present several motivating examples in the following.

Drug Trial. Indeed, take the case of designing drug cocktails to treat some condition - here, both the efficacy and the side-effects of a treatment are a priori unknown, to be figured out through the trial. In this context, we study linear bandits under unknown linear safety constraints: we are given known risk thresholds $\alpha^i$, and try to enforce that $\langle a^i, x_t \rangle \leq \alpha^i$ for latent constraint vectors $a^i$. Observationally, along with the (noisy) reward, we also observe noisy values of the constraint levels, i.e. $\langle a^i, x_t \rangle + \gamma^i_t$ for centered noise $\gamma^i_t$. In the context of clinical trials, $a^i$ may measure the side-effect on a particular system, such as the liver or heart function scores, with the $j$th coordinate $a^i_j$...
measuring the effect of a unit amount of the $j$th drug on the same. The known levels $\alpha^i$ may be interpreted as levels of function that we need to ensure, and the stochasticity of the scores is due to patient variability. To highlight the safety aspect of such applications, we observe that the latent constraints must be satisfied in a round-wise rather than an aggregate way - in a clinical trial, it is necessary to assure that most patients are given safe treatments, and it is not enough to meet some average safety goal that switches between playing very unsafe and very ineffective actions.

**Engineering Design.** In engineering design, there are various factors for component design in order to guarantee a safe and durable operation of the system. In particular, as described in [NAS08], each component of spaceflight hardware has a limit load $\alpha_i$, which is the nominal load value this component is supposed to bare in operation. Beyond the limit load, there are test load $\alpha^t_i$ and design load $\alpha^{design}_i$ satisfying $\alpha^t_i < \alpha^{test}_i < \alpha^{design}_i$, where the design load is deliberately set large to pass the test, and the test load is large enough to guarantee safe operation for loads slightly above the limit. Nevertheless, the optimal operating point is calculated with the nominal values, that is, the limit load, while the system is allowed and guaranteed to operate well under higher loads. In such a scenario, the constraint vector $a^i$ maps the control loads to average loads on component $i$, $E[L^i] = \langle a^i, x^i \rangle$, and the $i$th constraint is such that the average load does not exceed the limit load $\langle a^i, x^i \rangle \leq \alpha^i$. Note that the constraint violation, as in the drug trial example, should be penalized in a per-round sense; while slight violation of the constraint is allowed, due to the redundancy in the design.

In both examples, we would like to seek a performance measurement that smoothly penalize the infeasibility in a per-round sense, tolerating slight constraint violations. Surprisingly, such scenario has not been thoroughly addressed by the literature in safe bandits.

**Prior Works.** Prior works in this context can be grouped into two categories– those that deal with aggregated constraints and other more recent works that account for per-round constraint. As noted in recent work [PGBJ21; CGS22] the per-round constraint requirement is stringent, and prior strategies that utilize aggregate constraints for this problem [BKS13; BLS14; AD14; ADL16; AD16] are not compatible. On the other hand, prior works accounting for per-round safety exclusively rely upon optimistic-pessimistic strategies to ensure that these per-round constraints are met with high probability for each round [AAT19; MAAT21; PGBJ21]. While the safety property of these algorithms is strong, they suffer from significant disadvantages in efficacy due to their assumptions. The pessimistic exploration requires the assumption that the learner is a priori given a safe point $x^*$, and further knows the safety-margin $M^* = \min_i(\alpha^i - \langle a^i, x^* \rangle)$. This margin is used to construct a local pessimistic safe region around $x^*$—points that with high probability must be safe—which then enables local exploration (see §3.2). This assumption is strong, and limits the applicability of the methods, since $(x^*, M^*)$ need to be known to implement them. Further, the regret guarantees of the methods also depend on $M^*$, and when the methods are relaxed to remove this dependence, they suffer from $\Omega(T^{2/3})$ regret in terms of rewards [AAT19, Thm. 3]. See §A for for more details.

**Proposed DOSLB Algorithm.** The main theme of our paper is a doubly optimistic approach to the safe linear bandit problem - we describe and analyse an aggressive strategy, the ‘doubly-optimistic safe linear bandits (DOSLB)’ algorithm, that uses optimistic estimates of both the reward and safety scores of actions to enable fast exploration, whilst ensuring that in most rounds, the amount of safety violation is well controlled. While we note that doubly optimistic approaches have previously appeared in the context of aggregate constraints [e.g. AD14] and for safe multi-armed bandits [CGS22], ours is the first work employing a doubly-optimistic approach in per-round safety constrained linear bandits.

**Proposed Notion for Regret.** The principal benefit of the doubly-optimistic approach is a considerably improved regret in terms of reward at the cost of a worse, but well-controlled, increase in safety violations. Concretely, we argue that the DOSLB algorithm ensures round-wise safety in the sense of the (pseudo-)regret without assuming access to a known safe point.

$$\text{Regret}_T := \sum_{t=1}^{T} \max_{i} \left( \max_{\theta_+} \langle \theta^*, x^* - x_t \rangle, \max_i \langle a^i, x_t \rangle - \alpha^i \right) = \tilde{O}(d \sqrt{T}).$$

Further, in the important special case of a polytopal action space (i.e. for linear programs), a mildly relaxed version of the above regret can be controlled to the much smaller $O((d \log T)^2)$.

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\[1\] Chen et al. [CGS22] have recently proposed a related notion for multi-armed bandits
Notice that $\text{Regret}_T$ sums over a point-wise non-negative loss, since any feasible point must have $(\theta^*, x^* - x_t) \geq 0$, while any infeasible point has $\max_i ((a^i, x_t) - \alpha^i) > 0$. This has the effect of washing out any possible (unfair) advantage gained by playing an overly unsafe action, and thus ensures that the net round-wise extent of constraint violation is well controlled. The above result should be compared with the guarantees offered by the optimistic-pessimistic approach of [AAT19; MAAT21; PGBJ21]. These methods ensure that the safety regret $\sum_t \max_i ((a^i, x_t) - \alpha^i)_+$ is exactly 0, but (with weak assumptions), suffer an efficacy regret of $\sum_t (\theta^*, x^* - x_t) = \Omega(T^{2/3})$. The DOSLB method improves the efficacy regret to $O(d\sqrt{T})$, but at the cost of worsening the safety regret to the same order. Further, in polytopal domains, it improves the efficacy regret to $O(\log^2 T)$, whilst controlling a relaxed version of the safety regret to the same order. We additionally show that the number of rounds in which a significant constraint violation occurs is similarly small.

Justification. The DOSLB strategy forms an assumption-free, effective alternative to optimistic-pessimistic strategies where the matrix $A$ is provided guidelines for calculating adverse symptoms, their quantification is often subjective. The number of rounds in which a significant constraint violation occurs is similarly small.

Technical Novelty. The general analysis of the DOSLB scheme relies on established constructions of confidence sets for the reward vector $\theta^*$, and the safety vectors $a^i$. The main technical novelty of our approach lies in the analysis of the polytopal domains, wherein we use both analytic and geometric approaches to describe the behaviour of DOSLB. In particular, we use a geometric argument to show the action $x_t$ picked by DOSLB satisfies a set of $d$ ‘empirical versions’ of constraints with equality, and use a dual analysis to lower bound the estimation error-level in rounds where these constraints have suboptimal noiseless solutions. This leads to polylogarithmic regret bounds via existing analyses of the cumulative error-scale in linear bandits.

2 Problem Setup

For naturals $a, b$, let $[a : b] := \{a, \ldots, b\}$. $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ denote the Euclidean inner-product and $\ell_2$-norm in $\mathbb{R}^d$ respectively, while for a matrix $V > 0$, $\|z\|_V := \sqrt{\langle z, Vz \rangle}$.

Setting The stochastic linear bandit problem under linear safety constraints is parameterised by an objective vector $\theta^*$, $K$ known constraint vectors $\{a^i\}_{i \in [1:K]}$, and $U$ unknown constraint vectors $\{a^i\}_{i \in [K+1:K+U]}$, all of which lie in $\mathbb{R}^d$, and $M = K + U$ constraint levels $\{\alpha^i\}$. Together with a compact convex set $\mathcal{X} \subset \mathbb{R}^d$, these define the principal linear program of relevence,

$$\max_{x \in \mathcal{X}} (\theta^*, x) \text{ s.t. } Ax \leq \alpha,$$

where the matrix $A$ and vector $\alpha$ stack the corresponding linear constraints. We shall assume that the above program is feasible, and admits a unique optimal feasible solution $x^*$.

We assume that the constraint levels $\{\alpha^i\}_{i \in [1:M]}$, as well as the known constraint vectors $\{a^i\}_{i \in [1:K]}$ are revealed to the learner, while the objective and the unknown constraint vectors $\{a^i\}$ are hidden.
We define the known domain as
\[ \mathcal{K} := \{ x \in \mathcal{X} : \forall i \in [1 : K], \langle a^i, x \rangle \leq \alpha^i \}, \]
and the feasible set
\[ \mathcal{S} := \{ x \in \mathcal{K} : \forall i \in [K + 1 : M], \langle a^i, x \rangle \leq \alpha^i \}. \]

Play The learning problem proceeds in rounds, indexed by \( t \). In each round, the learner chooses an action \( x_t \in \mathcal{K} \), and receives the noisy reward feedback \( r_t \) and safety feedback \( \{ s^i_t \}_{i=1:M} \)
\[
    r_t = \langle \theta^*, x_t \rangle + \gamma^i_t, \quad \text{and} \quad s^i_t = \langle a^i, x_t \rangle + \gamma^i_t.
\]
where the various \( \gamma^i_t \) are conditionally centered sub-Gaussian noises, which need not be independent. The information set of the learner at time \( t \) is \( \mathcal{H}_{t-1} := \{ (x_t, r_t, \{ s^i_t \}_{i=1:M})_{r < t} \} \), and \( x_t \) must be adapted to the filtration induced by \( \mathcal{H}_{t-1} \). We note that \( \{ s^i_t \}_{i=1:K} \) is redundant feedback, and is included for notational convenience. The matrix \( X_{1:t} = [x_1, \ldots, x_t]^T \) and the vectors \( R_{1:t} = [r_1, \ldots, r_t]^T, S_{1:t}^i = [s^i_1, \ldots, s^i_t]^T \) arise by stacking the action and feedback.

Regret The feedback above allows the learner to accumulate knowledge about the latent vectors \( \theta^* \) and \( \{ a^i \}_{i=1:[K+1:M]} \), and the challenge of the problem lies in balancing the exploration to gain fresh information without violating constraints by too much, and the exploitation of accumulated information to gain a reward close to \( \langle \theta^*, x^* \rangle \). We measure the performance of the learner via the following pseudo-regret, previously mentioned in (*)
\[
    \text{Regret}_T = \sum_{t=1}^T \max \left( \langle \theta^*, x^* - x_t \rangle, \max_i (a^i, x_t - \alpha^i) \right).
\]

Assumptions We conclude by explicitly noting the assumptions we make on the problem parameters. These assumptions are standard in the linear bandit literature [APS11].

- **Assumption 2.1.** (Bounded-domain) \( \mathcal{X} \) is bounded. WLOG, \( \| x \|_2 \leq 1 \) for all \( x \in \mathcal{X} \).

- **Assumption 2.2.** (Bounded-parameter) \( \| \theta^* \| \leq 1, \| a^i \| \leq 1 \) for all \( i \in [m] \).

- **Assumption 2.3.** (Sub-Gaussian noise) For all \( t \in [T] \), the noises in the reward and constraint observations are conditionally 1-sub-Gaussian, that is \( \forall i \in [0 : M] \), and \( F_t := \sigma(\mathcal{H}_{t-1}, x_t) \)
\[
    \mathbb{E}[\gamma^i_t | F_t] = 0, \quad \mathbb{E}[\exp(\xi \eta^i_t) | F_t] \leq \exp(\xi^2 / 2) \forall \xi \in \mathbb{R}.
\]

3 Method: Optimism in the face of Uncertain Constraints and Rewards

The optimism principle is broadly successful in bandit problems [LS20], and in particular for stochastic linear bandits [APS11]. In the latter, this is instantiated through the ‘OFUL’ algorithm, which constructs consistent confidence sets \( C_t \) for the parameter \( \theta^* \), and chooses an \( x_t \) that maximises the optimistic reward \( \max_{\theta \in C_t} \langle \theta, x \rangle \). The success of the method comes from the property that if an \( x_t \) that is far from optimal is picked, then the direction along it must be strongly under-explored, and so each \( x_t \) is either near-optimal or allows the algorithm to best refine its knowledge. This prior success of the optimism principle forms the basis of our interest in the same for the case of safe linear bandits. Our method is similarly based on confidence sets for both the reward and constraint vectors. We begin by describing the confidence sets we will use, and then describe the main algorithm.

3.1 Confidence Sets

The confidence sets are adapted from the work of Abbasi-Yadkori et al. [APS11], which are built using analysis of the noise scales of regularised least squares (RLS) estimators. For a choice of \( \lambda > 0 \), we define \( V = \lambda I \) and \( V_t = V + \sum_{\tau=1}^t x_{\tau} x_{\tau}^T \). The RLS estimates of the parameters are
\[
    \hat{\theta}_t = (X_{1:t}^T X_{1:t} + \lambda I)^{-1} X_{1:t}^T R_{1:t}, \quad \hat{a}^i_t = (X_{1:t}^T X_{1:t} + \lambda I)^{-1} X_{1:t}^T S_{1:t}^i.
\]
Let $\sqrt{\beta_{t}(\delta)} := \sqrt{\frac{1}{\delta} \log \left( \frac{\lambda^1}{1+\delta} \det(V_t)^{1/2} \det(M)^{-1/2} \right) + \lambda^{1/2}}$. The relevant confidence ellipsoids are

$$C_t^0(\delta) := \{ \theta : \| \theta - \hat{\theta}_{t-1} \|_{V_{t-1}} \leq \sqrt{\beta_{t-1}(\delta)} \}$$

(1)

$$C_t^i(\delta) := \{ a^i : \| \tilde{a}^i - \hat{a}^i \|_{V_{t-1}} \leq \sqrt{\beta_{t-1}(\delta)} \} , \quad i \in [1 : K]$$

(2)

The set $\| z \|_{V_t}$ is small for $z$ that the previous $x_t$s are poorly aligned with, and so the confidence sets are wider in underexplored directions. The main property of the confidence sets is their consistency.

**Lemma 3.1.** (Variant of [APS11, Thm.2]) Assume 2.2, 2.3. Then, for all $\lambda > 0, \delta \in (0, 1)$,

$$\mathbb{P} ( \forall t \geq 0, i \in [1 : M] : \theta^* \in C_t^0(\delta), \alpha^i \in C_t^i(\delta) ) \geq 1 - \delta.$$  

3.2 The DOSLB algorithm

With the above in hand, we introduce our main algorithm, Doubly-Optimistic Safe Linear Bandit (DOSLB) (Algorithm 1). The scheme maintains the confidence sets $C_t^0$ for $\theta^*$ and $C_t^i$ for the $\alpha^i$ as in (1). Using the latter, it constructs an optimistic permissible set of points $x$ that are safe according to at least one choice of constraints in the confidence sets, concretely,

$$\tilde{S}_t := \{ x \in K : \forall i \in [1 : M], \min_{\tilde{a}^i \in C_t^i(\delta)} \langle \tilde{a}^i, x \rangle \leq \alpha^i \}.$$  

(2)

The set $\tilde{S}_t$ consists of all possible actions that may plausibly be safe given the past knowledge. The action $x_t$ is selected optimistically from this permissible set, i.e., as

$$(x_t, \hat{\theta}_t) \in \arg \max_{x \in \tilde{S}_t, \theta \in C_t^0} \langle \theta, x \rangle.$$  

(3)

The doubly optimistic approach for safe linear bandits enjoys similar benefits as the standard setting, in that if an action $x_t$ that is either overly inefficient or overly unsafe is selected, then it must be the case that the direction along it is strongly underexplored, in the sense of having small $V_t$-norm, and thus the resulting feedback would allow the algorithm to strongly refine its knowledge. As a result, the scheme either improves its future performance, or it must already be playing well, and so enjoys strong regret control.

The optimistic construction of the permissible set is the main distinction between our doubly-optimistic approach, and prior optimistic-pessimistic approaches, which instead construct a conservative permissible set around a point known in advance to be very safe ($\S$1). This optimistic $\tilde{S}$ enables significantly more aggressive exploration, and thus leads to improved efficiency performance, as analysed below. Of course, this is naturally accompanied by a decay in the safety performance, and we show via subsequent regret analysis that this cost is well controlled for the DOSLB algorithm.

4 Regret Bounds For General Convex Domains

We proceed to state and contextualise our main result for general domains.

**Theorem 4.1.** Under assumptions 2.1, 2.2, and 2.3, DOSLB($\lambda, \delta$) with $\lambda \geq 1$ and $\delta > 0$ satisfies the following regret bound with probability at least $1 - \delta$

$$\text{Regret}_T \leq 4 \sqrt{T d \log \left( 1 + \frac{T}{\lambda d} \right)} \left( \lambda^{1/2} + \frac{1}{2} \log \left( \frac{U + 1}{\delta} \right) + \frac{d}{4} \log \left( 1 + \frac{T}{\lambda d} \right) \right).$$  

(4)

The proof proceeds similarly to the argument for standard bandits. We argue that under the event that the confidence sets are consistent, in any round $t$, the instantaneous regret satisfies

$$\max(\theta^*, x_t - x^*) , \max_{i} (\alpha^i, x_t - \alpha^i) \leq \rho_t := \sqrt{\beta_{t}(\delta)}\| x_t \|_{V_{t-1}}^{-1}.$$  

(5)
Regret is further bounded as $\sum \rho_t \leq \sqrt{T} \sum \rho_t^2$, after which we invoke standard bounds using the matrix-determinant lemma to show $\sum \rho_t^2 = O(\sqrt{\beta T d \log T})$. A detailed argument is left to §C.

**Lower Bound.** Choosing $\lambda = O(1)$ in the statement yields a regret bound of $\tilde{O}(d \sqrt{T})$. This bound is minimax tight up to logarithmic terms, as demonstrated by an $\Omega(d \sqrt{T})$ lower bound of Shamir [Sha15] for unconstrained linear bandits. We further refine this lower bound in the following section.

The resulting bound of $O(\sqrt{T})$ should be compared to the $\Omega(T^{2/3})$ behaviour of generic efficacy regret bounds under optimistic-pessimistic approaches. Thus, the doubly-optimistic approach results in much stronger reward performance, which arises due to the more aggressive exploration it uses. Of course, this is accompanied by a non-trivial safety violation (which the pessimistic approach entirely avoids), but this remains controlled to $O(\sqrt{T})$ as in the above statement. Additionally, the result above is assumption free in the sense that an action known to be safe a priori is not required to execute the scheme. We finally point out that the dependence of the result on the number of unknown constraints $U$ is mild in the above bounds.

5 Logarithmic Regret Bounds for Polytopal Domains

We now turn to the important case of polytopal domains. Formally, we make the following assumptions.

**Assumption 5.1.** (Polytopal domain) The known domain $\mathcal{K}$ forms a polytope contained in the unit ball.

**Assumption 5.2.** (Non-degenerate LP) All the basic feasible solutions of the noiseless linear program $Ax \leq \alpha$ are non-degenerate.

The polytopal setting is of broad relevance, since it models an online approach to the canonical linear programming problem with some unknown constraints. The non-degenerate LP assumption is standard in linear programming literature to avoid endless corner-case discussion. In standard linear bandits, the polytopal assumption above yields instance-dependent logarithmic regret bounds for large $T$ [DHK08, APS11]. Such results arise from two observations. First, it is (implicitly) argued that due to the polytopal structure of the domain, the OFUL algorithm must play on the extreme points of the domain. Since these form a discrete set, there exists a $\Delta > 0$, the gap of the problem, such that any suboptimal extreme point has optimality gap at least $\Delta$. Analysis proceeds by arguing that any time a suboptimal $x_t$ is played, the error-bound $\rho_t$ must exceed $\Delta$, and regret is bounded as $\sum \rho_t^2 / \Delta$.

**Impossibility of logarithmic regret.** While the above logarithmic bounds are desirable, these are obstructed in safe bandits since we do not know the feasible polytope $\mathcal{S}$, but only have noisy estimates of it. This noise smears out the extreme points of the feasible set, and forces $\Omega(\sqrt{T})$ regret.

**Theorem 5.3.** For every $T, d$, there exists a safe linear bandit instance with polytopal $\mathcal{K}$ such that
- Every extreme point of $\mathcal{K}$ and $\mathcal{S}$ has an $\Omega(1)$ gap, in the sense of §5.2.
- Every algorithm must suffer $\text{Regret}_T = \Omega(d \sqrt{T})$ on this instance.

We highlight that the regret bound above holds even under large gaps, in contrast to $\sqrt{T}$ minimax lower bounds for standard linear bandits, which need to send the gaps to 0 at the rate $1/\sqrt{T}$.

The above result demonstrates a fundamental hardness in the safe linear bandit problem. Nevertheless, the polytopal structure does induce a discreteness in the action space, and thus enable gap-dependent analyses. To elucidate this, we define the following notion of a ‘relaxed regret,’ which introduces a slack in the constraint violation penalty.

**Definition 5.4.** We define the $\epsilon$-relaxed regret as

$$\text{Regret}_T^\epsilon := \sum_{t \leq T} \max \left( \langle \theta^*, x_t^* - x_t \rangle, \max_i (\langle a_i^t, x_t \rangle - \alpha - \epsilon)_+ \right).$$

Importantly, the above definition introduces a relaxation in the constraint values, but not in the objective. An effective algorithm thus may play in an $\epsilon$-neighbourhood of $\mathcal{S}$, but must ensure that in the long run, the rewards obtained must be at least that of $x^*$. This relaxation is in line with the broad
theme of the paper of showing that doubly-optimistic methods have extremely favourable efficacy performance, whilst at the cost of well-controlled safety violations. As previously discussed, such an approach is desirable in scenarios where ineffective solutions have high implicit costs.

Our main result in this section offers an instance-dependent logarithmic regret bound for the relaxed regret. Our results are analogous to the standard linear bandit case, and will be developed in a similar way. We first use a geometric argument to associate the behaviour of the DOSLB algorithm with size-$d$ subsets of $[1:M]$. This discreteness lets us define non-trivial gaps, and the logarithmic regret bounds are developed by arguing that suboptimal play must be accompanied by a noise scale larger than these gaps. We shall now proceed to develop this theory in detail, and then state our main results.

5.1 Structural Behaviour of DOSLB

We introduce the key notions of Basic Index Sets and association. For a set $J \subset [1:M]$ we use $A(J)$ to denote the matrix with rows $a^j$ for $j \in J$, and similarly $\alpha(J)$ for the vector $(\alpha^j)_{j \in J}$.

**Definition 5.5.** A Basic Index Set (BIS) is a set $I \subset [1:M]$ such that $|I| = d$. A BIS $I$ and a point $x \in \mathcal{K}$ are associated, denoted $I \sim x$, if $A(I)x = \alpha(I)$, and is further called optimally-associated if $I \sim x^*$. Finally, we define $I^* := \{\text{BISs } I : I \sim x^*\}$, and say that BISs not in $I^*$ are suboptimal.

BISs may be associated with single points, sections of an affine subspace, or with no points, depending on the properties of the linear system $A(I)x = \alpha(I)$ and $K$. We shall say that a BIS $I$ is consistent if there exists $x : I \sim x$, and that otherwise it is inconsistent.

**Example 5.6.** In $\mathbb{R}^2$, consider the domain $\mathcal{K} = \{x_2 \geq 0, x_1 \geq x_2, x_1 + x_2 \leq 4\}$, unknown constraint $x_2/2 \leq 0.55$, and $\theta^* = (0.1, 1)^T$. This corresponds to $K = 3, U = 1$, and the constraint vectors $a^1 = (0, -1)^T, a^2 = (-1, 1)^T/\sqrt{2}, a^3 = (1, 1)^T/\sqrt{2}, a^4 = (0, 1/2)^T; \alpha = (0, 0, 2\sqrt{2}, 0.55)$. There are $\binom{4}{2} = 6$ BISs, of which $\{1, 4\}$ is inconsistent. $x^* = (2.9, 1.1)$, and the BIS $\{3, 4\} \sim x^*$.

The action of DOSLB depends on $A$ through the noisy estimates of its entries. We define the following notion to capture the variability incurred in play due to this.

**Definition 5.7.** A BIS $I$ and a point $x \in \mathcal{K}$ are said to be noisily associated at time $t$, denoted $I \sim^t x$, if for every $i \in I$, there exists a $\tilde{a}^i \in C^i_{t-1}$ such that $\langle \tilde{a}^i, x \rangle = \alpha^i$.

To illustrate this, consider that in Example 5.6, it may be the case that $\tilde{a}^4 = (1/2, 1/2) \in C^4_{t-1}$. This would mean that $\{1, 4\} \sim^t (1, 1, 0)$, since the latter solves $-x_2 = 0, (x_1 + x_2)/2 = 0.55$.

The main structural result is as follows. This statement identifies BISs as a crucial discrete structure underlying the behaviour of DOSLB, and is crucial to our subsequent gap analysis.

**Lemma 5.8.** The action of DOSLB is always noisily associated with some BIS, i.e., $\forall t, \exists I_t \sim^t x_t$.

5.2 Gaps

The main results show that since $x^*$ is always permissible w.h.p., if DOSLB plays some $x_t$ that is noisily associated with only suboptimal BISs, then the noise scale at $x_t$ must necessarily be large. We identify three notions of gaps with each BIS that quantify this largeness. Structurally, these gaps take the form of analytic-separation, where the separation typically measures the difference in the values of two linear programs, while the geometric spread captures how well the constraint vectors of a BIS can express the objective of this program. To this end, we first define a common notion of ‘spread.’

**Definition 5.9.** For a set $\mathcal{L} \subset [1:M]$, define $\mathcal{L}_K = \mathcal{L} \cap [1 : K], \mathcal{L}_U = \mathcal{L} \cap [K + 1 : M]$. For $\mathcal{L} \subset [1:M], \mathcal{M} \subset [1 : K]$, and $v \in \mathbb{R}^d$, define the program $\Pi(v; \mathcal{L}, \mathcal{M})$ as

$$\min_{\pi, \rho, \sigma} \langle \pi, (\alpha(\mathcal{L})) \rangle + \langle \rho, (\alpha(\mathcal{L}_K)) \rangle + \langle \sigma, (\alpha(\mathcal{M})) \rangle$$

$$s.t. \quad \pi^T A(\mathcal{L}_U) + \rho^T A(\mathcal{L}_K) + \sigma^T A(\mathcal{M}) = v^T, \sigma \geq 0.$$

We define the spread of $v$ with respect to $\mathcal{L}$ as $\text{spread}(v; \mathcal{L}, \mathcal{M}) := \inf\{1 + \|\pi\|_1 : \pi \in \mathbb{R}^{[\mathcal{L}_U]} | s.t. \exists \rho, \sigma \text{ so that } (\pi, \rho, \sigma) \text{ optimise } \Pi(v; \mathcal{L}, \mathcal{M})\}$. 

\[\text{spread}(v; \mathcal{L}, \mathcal{M}) := \inf\{1 + \|\pi\|_1 : \pi \in \mathbb{R}^{[\mathcal{L}_U]} | s.t. \exists \rho, \sigma \text{ so that } (\pi, \rho, \sigma) \text{ optimise } \Pi(v; \mathcal{L}, \mathcal{M})\}\]
The program $\Pi$ arises in the proofs as the dual of a linear program with objective $v$. The spread essentially controls the change in the value of the primal under unit perturbations in \{a^i\}_{i \in \mathcal{L}} and $v$.

**Example** Take $\mathcal{L} = \{2, 4\}, \mathcal{M} = \{1\}$ in Ex. 5.6. We have $\mathcal{L}_K = \{2\}, \mathcal{L}_U = \{4\}$, and

\[
\Pi(\theta^*; \{2, 4\}, \{1\}) = \min 0.55\pi \text{ s.t. } \rho = -0.1, \pi/2 + \rho - \sigma = 1, \sigma \geq 0
\]

\[
\text{spread}(\theta^*; \{2, 4\}, \{1\}) = \inf \{1 + |\pi| : \pi \in \{2.2\}\} = 3.2.
\]

**Consistent BISs** The first two notions of gap mainly deal with consistent BISs.

**Definition 5.10.** For a consistent BIS $I$, we define the efficiency separation of a BIS $I$ as $\delta(I) := \max_{x \sim x}(0, (\theta^*, x^* - x))$, and the efficiency gap of $I$ as

\[
\Delta_I := \frac{\delta(I)}{\text{spread}(\theta^*; I, [1 : K] \setminus I)}.
\]

$\delta(I)$ measures how close the points associated with $I$ can get to the optimal value, while the spread of $\theta^*$ measures how strongly a unit bounded noise can perturb value of any $x \sim I$. $\Delta_I$ thus measures how large the noise needs to be in order to make some $x \sim I$ look better than $x^*$ to DOSLB.

**Definition 5.11.** For a BIS $I$, and $k \in [1 : M] \setminus I$, the feasibility separation of $I$ with respect to $k$ is

\[
\gamma(I; k) := \inf \{\langle a^k, x \rangle - \alpha^k : x \text{ s.t. } I \sim x\}.
\]

We further define the feasibility gap of a consistent BIS $I$ as

\[
\Gamma_I := \max_{k \in [1 : M] \setminus I} \frac{\gamma(I; k)}{\text{spread}(\alpha^k; I, [1 : K] \setminus \{I \cup \{k\})\}}.
\]

$\Gamma_I$ measures how much noise is needed for a BIS with only unsafe associated points to look safe instead. The index $k$ in the separation $\gamma(I; k)$ is a particular constraint whose violation is measured, and the associated spread of $-\alpha^k$ appears since constraints are taken in the $\leq$ form.

Note that for any consistent suboptimal BIS $I$, at least one of $\Delta_I$ and $\Gamma_I$ must be positive - indeed, if not, then $I$ is associated with a safe point of value $\geq \langle \theta^*, x^* \rangle$, contradicting its suboptimality.

Finally, for the sake of notational convenience, for an inconsistent BIS $I$, we set $\Delta_I = \Gamma_I = 0$.

**Example** In Ex.5.6, $\{2, 4\} \sim (1.1, 1.1)$, and has $\Delta(\{2, 3\}) = 0.18$, since it has unit spread. Further, $\Gamma_{\{2, 4\}} = 0$. Similarly, $\{2, 3\} \sim (2, 2)$, has $\Delta_{\{2, 3\}} = 0$, but is unsafe, with $\Gamma_{\{2, 3\}} = 0.45$.

**Inconsistent BISs** While there are no points associated with inconsistent BISs, due to the noisy knowledge about some of the constraints, DOSLB may hallucinate points noisily associated with them in the permissible set - for instance, in Ex.5.6, if $\tilde{a}^4 = (1/2, 1/2) \in C^4_{\{1, 0\}}$, then the point $(1.1, 0)$ is noisily associated with $\{1, 4\}$. The inconsistency gap measures how strong the noise has to be in order to allow such hallucination. This is measured with respect to some constraint, which may either lie in $I$ (if the inconsistency is due to the linear system $A(I)x = \alpha(I)$ not having solutions), or outside of it (if the inconsistency arises due to the linear system having solutions, but not in $K$).

**Definition 5.12.** Let $I$ be an inconsistent BIS. We define the outer and inner consistency separation of $\overline{I} \subset I$ with respect to $\overline{J} \subset [1 : K] \setminus I, k \in [1 : M] \setminus (\overline{I} \cup \overline{J})$ as

\[
\overline{\lambda}_I(\overline{I}; \overline{J}, k) := \max 0, \inf_{x \in \mathcal{X}(\overline{I}, \overline{J})} \langle a^k, x \rangle - \alpha^k,
\]

\[
\overline{\Delta}_I(\overline{I}; \overline{J}, k) := -\min 0, \sup_{x \in \mathcal{X}(\overline{I}, \overline{J})} \langle a^k, x \rangle - \alpha^k]\|\{k \in I\},\text{ where}
\]

\[
\mathcal{X}(\overline{I}, \overline{J}) := \{x \text{ s.t. } A(\overline{I})x = \alpha(\overline{I}), A(\overline{J}) \leq \alpha(\overline{J})\}.
\]

The consistency gap of $I$ is defined as

\[
\Lambda_I := \max_{\overline{I} \subset I, \overline{J} \subset [1 : K] \setminus I} \max_{k \in [1 : M] \setminus (\overline{I} \cup \overline{J})} \left(\frac{\overline{\lambda}_I(\overline{I}; \overline{J}, k)}{\text{spread}(a^k; \overline{I}, \overline{J})}, \frac{\overline{\Delta}_I(\overline{I}; \overline{J}, k)}{\text{spread}(a^k; \overline{I}, \overline{J})}\right).
\]

8
Intuition behind the definition: suppose instead of enforcing all of the constraints in $I$ with equality and membership in $K$, we only enforced a subset of these, $\tilde{I}, \tilde{J}$, such that $\mathcal{X}(\tilde{I}, \tilde{J}) \neq \emptyset$. The inconsistency of $I$ means that for some $(\tilde{I}, \tilde{J})$, there is a constraint $k$ violated by all points in $\mathcal{X}(\tilde{I}, \tilde{J})$. This gives a restricted ‘safety gap’ for $I$, and $\Delta_I$ maximises over all of these restricted gaps. If $k \in I$, then this constraint must be met with equality, and possibly $\bar{\lambda} = 0$ but $\underline{\lambda} > 0$.

We note that every inconsistent BIS has positive consistency gap - otherwise $\bar{\lambda}_I(\tilde{I}, \tilde{J}, k)$ and $\Delta_I(\tilde{I}, \tilde{J}, k)$ would be zero for every $\tilde{I}, \tilde{J}, k$, indicating that $I$ has an associated point, contradicting inconsistency.

**Example** In Ex.5.6, the BIS $\{1, 4\}$ is inconsistent. For this, $\mathcal{X}([1, \{2, 3\}) = \{x_2 = 0\}$ and since $\{4\} \in \{1, 4\}, \lambda([1]; \{2, 3\}, 4) = 1.1 = \Delta_{\{1, 4\}}$.

**Gap of the Problem**

**Definition 5.13.** The effective gap of a safe linear bandit problem with polytopal $K$ is defined as $\Xi := \min_I \max \{\Delta_I, \Gamma_I, \Lambda_I\}$.

**Example** Figure 1 depicts the separations in the setting of Example 5.6. For this example, $\Xi = 0.18$.

### 5.3 Results

**NOTE:** All results in this section hold with high probability under assumptions 2.1.2, 2.2.3, 5.1, 5.4.2.

The gaps of §5.2 determine noise scales required in order for DOSLB to play an $x_t$ that is associated with a suboptimal BIS, as is captured below using the estimation error-scale $\rho_t$ from (5).

**Lemma 5.14.** For all $t$, the actions $x_t$ of DOSLB($\lambda, \delta$) satisfy that if $I_t \neq x^*$, then $\rho_t \geq \Xi$.

This result is of independent interest, since it yields the fact that suboptimal BISs are rarely played.

**Theorem 5.15.** The actions of DOSLB($\lambda, \delta$) satisfy $\sum_{t \leq T} (1 - I_t \sim x^*) = O((d \log T)^2/\Xi^2)$.

To extend the above observation to a regret bound, we show the following structural result.

**Lemma 5.16.** If $\{I : I \sim x_t\} \subset \mathcal{I}^*$, then $\langle \theta^*, x_t - x^* \rangle \geq 0$. If $\rho_t < \epsilon$, then $\max_t \langle a^i, x_t \rangle - \alpha_t < \epsilon$.

Coupling Lemma 5.14 and Lemma 5.16 yields the main result of this section.

**Theorem 5.17.** DOSLB($\lambda, \delta$) enjoys the following regret bound.

$$\text{Regret}_T = O \left( (d \log T)^2 \left( \frac{1}{\Xi} + \frac{1}{\epsilon} \right) \right).$$

Further, the number of times the safety violation exceeds $\epsilon$ is bounded as

$$\sum_t \mathbb{1} \{ \max_t \langle a^i, x_t \rangle - \alpha^i > \epsilon \} = O \left( \frac{d^2 \log^2 T}{\epsilon^2} \right).$$

**Remarks** The results above indicate that doubly optimistic play in the safe linear bandit setting enjoys very favourable properties. Firstly, observe that neither the parameter $\epsilon$, nor the polytopal assumption is required in the definition of the DOSLB scheme, and so the method adapts to polytopal structure. Further, we note that since the relaxed regret does not slacken the objective value, the result implies that the DOSLB method incurs only $O(d^2 \log^2 T)$ regret in terms of efficacy with respect to the optimal point $x^*$. Finally, we observe that not only the net safety violation beyond $\epsilon$, but also the total number of times that an $\epsilon$-unsafe point is played is logarithmically bounded, indicating that the method consistently plays in the vicinity of the feasible set.

The bound above behaves inversely with respect to the three gaps, as well as with respect to $\epsilon$. The following lower bound argues, via a reduction to the safe multi-armed bandit problem [CGS22], that the dependence on $\min(\Delta_I, \Gamma_I)$ in the above is tight for consistent algorithms (§D.5).
Theorem 5.18. For any $\Xi < \epsilon$, and any consistent algorithm, there exists an instance of the safe linear bandit problem such that for all suboptimal BISs $I$, $\Lambda_I = 0$, $\max(\Delta_I, \Gamma_I) \geq \Xi$, and $\text{Regret}_T^* = \Omega((\log T)/\Xi)$.

6 Simulations

We verify the theoretical study above with simulations over Example 5.6, and study the relative performance of DOCLB and the optimistic-pessimistic method Safe-LTS [MAAT21]. These implementations are based on the following relaxation of Algorithm 1.

6.1 Computationally Feasible Relaxation

A well-known barrier to implementing Algorithm 1 is that even if all constraints were known, the program (3) is non-convex [DHK08]. In our case, this is further complicated by the fact that the set $\tilde{S}_t$ needs to be determined, which too is computationally subtle.

We approach these issues by constructing box confidence sets. Observe that the $C_i^t$ are defined using level sets of $\|\tilde{a} - \hat{a}^i\|_{V_t} = \|(\tilde{a} - \hat{a}^i)V_{1/2}^t\|_2$. We replace these sets by

$$C_i^{t, \infty} := \{\tilde{a} : \|(\tilde{a} - \hat{a}^i)V_{1/2}^t\|_{\infty} \leq \sqrt{\beta_{t-1}}\},$$

with a similar change for $C_0^{t, \infty}$. Since $\|z\|_2 \leq \|z\|_{\infty} \leq \sqrt{d}\|z\|_2$, these boxes contain the original $C_i^t$ of (1), and running DOCLB with these worsens regret bounds by at most $O(\sqrt{d})$. The principal advantage of using $C_i^{t, \infty}$ lies in the fact that the box-confidence sets are polytopes. Due to this, the values of $\hat{\theta} \in C_0^{t, -1, \infty}$ and $\hat{\alpha}^i \in C_i^{t, -1, \infty}$ that are active for the optimistic action $x_t$ must lie at the extreme points of these sets. Since each set has only $2d$ extreme points, this allows us to determine $x_t$ by solving $(2d)^{U+1}$ convex programs, which is computationally feasible so long as $U$ is small. The following simulations implement exactly this strategy, which is described in detail in §E.

6.2 Experimental results

Example 5.6. We report means over 6 runs for parameters $T = 10^4$, $\lambda = 1$, $\delta = 0.01$, and Gaussian noise of variance 0.1. Figure 2a reports the behaviour of $\text{Regret}_T$ as well as $\text{Regret}_T^*$ for $\epsilon = 0.01$. Figure 2b reports the number times that $x_t$ was not associated with the optimal BIS $\{3, 4\}$.

![Figure 2: Results for Ex.5.6.](image)

Optimistic vs. Pessimistic Comparison. [AAT19] and [MAAT21] and propose pessimistic constructions of the permissible set. To benchmark our method, we test the performance of the safe-LTS method of [MAAT21] in the running example 5.6. In safe-LTS, pessimism arises from the fact that for an action to be feasible, it must meet every constraint generated from the confidence set of constraint parameters. Please see §A for a detailed exposition of the method. We note that we do not evaluate the safe-LUCB method of [AAT19], since this has been shown to consistently underperform safe-LTS [MAAT21], as we verified in our pilot runs.
Fig. 3 compares the behaviour of the raw efficacy regret ($\sum \langle \theta^*, x^* - x_t \rangle$) (top) and the raw safety regret $\sum \max_i (\langle a^i, x_t \rangle - \alpha^i)$ (bottom) of safe-LTS and DOSLB (mean over 6 runs). As expected, safe-LTS suffers from 0 safety regret, since it plays in a pessimistic set. However, we observe that the safety-regret of DOSLB is well controlled, and further, that it has considerably stronger efficacy performance than the conservative safe-LTS, demonstrating the practical strength of doubly-optimistic approaches.

7 Discussion

The DOSLB algorithm represents an alternative to optimistic-pessimistic methods for safe linear bandits that has strong efficacy performance, while maintaining safety in the sense of $\text{Regret}_T$. It also adapts well to polytopal structure in the problem, as demonstrated by logarithmic bounds on $\text{Regret}_T$. These properties are achieved without using strong assumptions a la the pessimistic methods.

In a broader sense, which of these methods to use for a particular problem is a policy question, which asks that we balance the loss of efficacy that comes with conservative approaches and the unsafety associated with the aggressive doubly-optimistic method. Further, the method raises wider questions of whether there are schemes that can vary the trade-off between these extremes - for instance, a natural approach might be to play aggressively in order to determine the optimal BIS, and then play conservatively in its vicinity whilst refining estimates of it - such a proposition is particularly attractive for large $\Xi$. Similarly, we may ask if there are alternatives to $\text{Regret}_T$ that both better capture the safety requirements and are well controlled in polytopal problems.

We believe that the safe linear bandit problem, and associated reinforcement learning problems, form an exciting area with scope for both novel solution concepts and algorithms that may both inform, and be informed by, policy level principles.
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A A More Detailed Look at Related Work

There are two broad families of bandit problems with unknown constraint that have been studied in the literature.

The first of these are aggregate constraint problems, which are often used to model resource consumption [AD14; ADL16; BKS13; BLS14]. Here, each constraint is associated with a notion of regret \( S^r_T = \sum (a^i, x_t) - \alpha^r \), and the overall regret is measured as \( \max((\max_s S^r_T), \sum (\theta, x^r - x_t)) \). As noted by Pacchiano et al.[PGBJ21] and Chen et al.[CGS22], the main disadvantage of this formulation from our perspective arises from the fact that constraint violations are aggregated. This functionally means that it is okay for effective algorithms to alternate between actions that have large reward & poor safety, and actions that have poor reward but good safety. Indeed, this behaviour is exhibited quite explicitly in prior work [CGS22]. Such behaviour intrinsically does not model scenarios such as clinical trials or engineering design, since in these settings the interest is in maintaining the safety for most instances. We note that the setting of conservative bandits [WSLS16] also has aggregate constraints, but now measured with respect to the performance of a baseline strategy. Similar problems have also been studied from an optimisation point of view [e.g. YNW17].

More germane to us are round-wise constraints, which are needed to model safety scenarios. Chen et al. [CGS22] have recently studied this problem in the setting of multi-armed bandits, using a doubly optimistic approach and analysing a soft regret similarly to us. Indeed, one may view our work as a considerable generalisation of the same to the linear bandit setting.

In contrast, other existing work has concentrated on hard round-wise constraints, i.e., enforcing that, with high probability, all rounds satisfy that \( \langle a^i, x_t \rangle \leq \alpha^i \). We give brief summaries of the main work in this area.

Unknown round-wise constraints were first considered for linear bandits by Amani et al.[AAT19], who consider the case of a single feedback \( \langle \theta^r, x_t \rangle + \gamma_t \), and impose the constraint \( \theta^r \frac{B x_t}{C} \leq \alpha \) for some given matrix \( B \). Their method, Safe-LUCB, can however be generalised to a constraint of the form \( \langle a^i, x \rangle \leq \alpha \) under our feedback model (where the constraint value is also observed). Amani et al. propose a two phase optimistic pessimistic strategy. The algorithm assumes that a safe point \( x^r \) is given to it, along with a number \( M^r \) such that \( C - \langle a^i, x^r \rangle \geq M^r \). This effectively yields a small region \( D^r \) around \( x^r \) that is known to be safe. The algorithm first executes a pure exploration phase of length roughly \( (d \log T)/(M^r)^2 \), where points are selected from \( D^r \) randomly, and used to refine knowledge of \( A^1 \) and \( \theta^r \). In the second phase, the algorithm constructs a confidence set in the same manner as \( C_t^0, C_t^1 \) and then produces the pessimistic permissible set \( P_t = \{ x : \forall a^i \in C_{t-1}^0 \langle a^i, x \rangle \leq \alpha \} \). The point \( x_t \) is selected optimistically from this set as \( \hat{\theta}_t, x_t = \arg \max \hat{\theta}_{t-1}^{C_{t-1}^0}, x_{t-1} \in P_t \langle \hat{\theta}_t, x_t \rangle \). Under the assumption that \( M^r > 0 \), the authors show an (efficacy) regret guarantee of \( O((M^r)^{-1} + \sqrt{T}) \). If, on the other hand, \( M^r \) is not available (or is small), a regret bound can be derived by instead running the exploration phase for time \( T^{2/3} \) and similar regret.

Subsequently, Moradipari et al.[MAAT21] considered the same scenario as us (but with only a single unknown constraint), and extended the pessimistic approach of Amani et al. [AAT19] in the following way: the pure exploration phase is eliminated, and instead the algorithm, Safe-LTS, constructs the pessimistic safe set \( P_t \) as above, and uses Thompson sampling to select the action \( x_t \) - roughly, the method samples the point \( \tilde{\theta}_t = \hat{\theta}_t + KV_t^{-1/2} \beta_t \eta_t \), where \( \eta_t \) is centred noise, and chooses \( x_t = \arg \max_{x \in P_t} \langle \tilde{\theta}_t, x_t \rangle \). This scheme is analysed under the assumption that \( \alpha > 0 \), which effectively means that the origin can serve the role of \( x^r \) from Safe-LUCB (and \( \alpha \) itself behaves as \( M^r \)). The authors show (efficacy) regret bounds of \( O((\sqrt{d} T)^{2/3}) \). Note that, while ostensibly of order \( \sqrt{T} \), the algorithm bears an uncomfortable dependence on the constraint level \( \alpha \) which fundamentally arises for the same reasons as the dependence on \( M^r \) in the safe-LUCB regret analysis, namely the need to demonstrate a reasonably large a priori safe set near the origin to enable initial exploration.

Empirically, Moradipari et al. observe that safe-LTS consistently outperforms safe-LUCB. This is attributed to the fact that safe-LUCB suffers from poor expansion of the pessimistic permissible set [MAAT21, Fig. 7, 8], which safe-LTS evidently corrects due to a stronger randomisation in its exploration. We note that a similar Thompson-sampling based action selection is quite viable in the
doubly-optimistic approach as well, and it is an interesting question as to whether this shows similar advantages.

We finally briefly mention the recent work of Pacchiano et al.\cite{PGBJ21}. This work lifts the action space to policies, i.e., distributions over $x_t$, and demands that at each time $t$, the algorithm produces a policy $\pi_t$ such that $\mathbb{E}_{x \sim \pi_t} \langle a, x \rangle \leq \alpha_t$, while the goal is to maximise the reward $\mathbb{E}_{x \sim \pi_t} \langle \theta^*, x \rangle$, and the optimal safe policy is called $\pi^*$. Actual points are selected, of course, by sampling from $\pi_t$, and noisy version of the corresponding inner products are given as feedback. The principal observation in light of the safety criterion is that while ostensibly round-wise, this study is in fact fundamentally enforcing only an aggregate constraint. This arises since the optimal policy $\pi^*$ is only required to be safe in expectation, and so may place non-trivial mass on unsafe points. Indeed, such behaviour has been demonstrated both theoretically and empirically in the case of multi-armed bandits\cite{CGS22}.

### B Preliminaries

We give an expanded discussion of Assumptions 2.1, 2.2 and 2.3, and discuss a standard result controlling $\sum \|x_t\|_{V_t^{-1}}$.

#### B.1 A closer look at the assumptions

The assumptions made in the main text are slightly simplified version of standard assumptions from the literature on linear bandits.

**Bounded Domain** Assumption 2.1 is used chiefly to ensure that the underlying optimisation problem of interest has finite value. Quantitatively, this may be replaced with a generic bound $\|x\| \leq L$ instead without appreciably changing the study. The principal way this affects DOSLB is via the choice of the regulariser $\lambda$ - concretely, the validity of Lemma requires that $\lambda \geq \max_i \|x_i\|$, which may be ensured by setting $\lambda \geq L^2$. Indeed, this fact underlies our choice of $\lambda = 1$ in the main results. A second aspect that is affected by the quantity $L$ is that the upper bound of Lemma B.1 would instead read $\log(1 + TL^2/\lambda d)$ instead of $\log(1 + T/\lambda d)$, which mildly affects some logarithmic terms in the regret bounds.

**Bounded Parameters** Assumption 2.2 is largely without loss of generality - indeed, if we had a bound $\|\theta^\ast\|, \max_i \|a^i\| \leq S$ instead, the only change required is that the confidence set radius $\beta_t$ would instead need to be set as $\sqrt{\beta_t(\delta; S)} = \sqrt{\beta_t(\delta; 1)} + (S - 1)\sqrt{\lambda}$, i.e., only the additive $\sqrt{\lambda}$ term in $\sqrt{\beta_t}$ from the main text would need adjustment. We note that in general, the norm bounds on the various $a^i$ and $\theta^*$ need not agree, and it is in fact possible to adapt to their norms without prior knowledge of the same, by setting distinct $\beta_t^i$s for each $a^i$, and using the techniques of the recent work of Gales et al.\cite{GSJ22}.

**SubGaussianity** While the subGaussianity condition can also be relaxed (for instance, linear bandits with heavy tailed noise have been studied\cite{SYKL18}), it yields significant technical convenience whilst remaining quite a generic setting. In Assumption 2.3, we concretely assume that the noise is conditionally 1-subGaussian. This may be relaxed to conditionally $R$-subGaussing. This too can be handled with a small change in $\beta_t$ to

$$\sqrt{\beta_t(\delta; R)} = R \sqrt{\frac{1}{2} \log \left( \frac{(U + 1)\det(V_t)^{1/2} \det(\lambda I)^{-1/2}}{\delta} \right)} + \lambda^{1/2}. $$

This change is somewhat stronger than the corresponding change induced by altering $\|\theta^*\|$ and $\|a^i\|$, since the scaling is now applied to the first term of $\beta_t$, which grows with $t$ unlike the constant $\sqrt{\lambda}$ penalty.

**Overall Confidence Radius with General Parameters** To sum up, under the generic conditions $\|x\| \leq L, \|\theta^*\| \leq S, \|a^i\| \leq S$, and $R$-subGaussianity of $\{\gamma_t\}$, the entirety of our following analysis will go through, but with the blown up confidence radii

$$\beta_t(\delta; L, S, R) = R \sqrt{\frac{1}{2} \log \left( \frac{(U + 1)\det(V_t)^{1/2} \det(\lambda I)^{-1/2}}{\delta} \right)} + SL^{1/2},$$
and under the condition \( \lambda \geq L \). This results in roughly an increase in the regret bounds of a factor of at most \( \max(R, S) \), along with a potential increase in the logarithmic terms to \( \log(1 + TL^2/\delta) \) instead of \( \log(1 + T/\delta) \). For the remainder of our analysis, we shall stick to the default parameters \( R = S = L = 1 \).

### B.2 Controlling the Cumulative Norm of Actions

We conclude the preliminaries with the following generic statement, which holds due to a couple of applications of the matrix-determinant lemma. The result is standard - see the discussions of Abbasi-Yadkori et al [APS11; Lemma 11] for historical discussions.

**Lemma B.1.** Suppose that for all \( t \), \( \|x_t\|_2 \leq 1 \), and let \( \lambda \geq 1 \). Then for any \( T \),

\[
\sum_{t=1}^{T} \|x_t\|_{V^{-1}}^2 \leq 2 \log \left( \frac{\det(V_T)}{\det(V)} \right) \leq d \log \left( 1 + \frac{T}{\lambda d} \right).
\]

**Proof.** First notice that since \( V_t = V_{t-1} + x_t x_T^T \), by the matrix-determinant lemma,

\[
\det(V_t) = \det(V_{t-1}) \det(I + V_{t-1}^{-1/2} x_t x_T (V_{t-1}^{-1/2})^T) = \det(V_{t-1})(1 + \|x_t\|_{V_{t-1}^{-1}}^2),
\]

and induction yields

\[
\det(V_T) = \det(V) \prod_{i=1}^{T} (1 + \|x_t\|_{V_{t-1}^{-1}}^2).
\]

Now, notice that since \( V_{t-1} \succeq \lambda I \) for each \( t \), it follows that \( \|x_t\|_{V_{t-1}^{-1}} \leq \|x_t\|/\lambda \leq 1 \). But for \( z \in (0, 1), z \leq 2 \log(1+z) \), which implies that

\[
\sum_{t=1}^{T} \|x_t\|_{V_{t-1}^{-1}}^2 \leq 2 \sum_{t=1}^{T} \log(1 + \|x_t\|_{V_{t-1}^{-1}}^2) = 2 \log \frac{\det(V_T)}{\det(V)}.
\]

Finally, note that since \( V_T \) is positive definite, by an application of the AM-GM inequality, \( \det(V_T) \leq (\text{trace}(V_T))/d^d \), and further, \( \text{trace}(V_T) = d\lambda + \sum_t \|x_t\|_2^2 \leq \lambda + T \). Further observing that \( \det(V) = \lambda^d \), we conclude that

\[
\log \frac{\det(V_T)}{\det(V)} \leq d \log \left( \frac{(d\lambda + T)/d}{\lambda} \right) = d \log \left( 1 + \frac{T}{d\lambda} \right).
\]

### C Proofs of Upper Bounds

This section details all the proofs of regret bounds, as well as the structural results from §5.2.

#### C.1 Proofs Omitted from §4

We now proceed to argue the main result for general domains.

**Proof of Theorem 4.1.** Recall from Lemma 3.1 that with probability at least \( 1 - \delta \), \( \theta^* \in C_0^t \) and \( a^i \in C_i^t \) for each \( i \) and each \( t \). We shall argue that the bound holds given the same in order to show the result.

Since the bounds resulting in this analysis will be repeatedly reused in the subsequent proofs, we extract the same into an explicit statement.

**Lemma C.1.** Suppose that for all \( t \) and for all \( i \in [1 : M] \), it holds that \( \theta^* \in C_0^i \) and \( a^i \in C_i^t \). Then the action of DOSLB is such that for every \( t \), and for every \( i \),

\[
\langle \theta^*, x^* \rangle - \langle \theta^*, x_t \rangle \leq \rho_t,
\]

\[
\langle a^i, x_t \rangle - \alpha^i \leq \rho_t,
\]

where

\[
\rho_t := 2 \sqrt{\beta_t} \|x_t\|_{V_{t-1}^{-1}}.
\]
Proof. Since each \( a^i \in C^i_t \), it follows that for each \( t, S \subseteq \tilde{S}_t \) - indeed, for any \( x \in S \), the vectors \( \tilde{a}^i = a^i \in C^i_t \) can serve as witnesses for its inclusion in \( \tilde{S}_t \). Further, since \( \theta^* \in C^0_t \), we find that for any \( t \),

\[
\langle \hat{\theta}_t, x_t \rangle = \max_{\theta \in C^0_t, x \in \tilde{S}_t} \langle \theta, x \rangle \\
\geq \langle \theta^*, x^* \rangle.
\]

As a result,

\[
\langle \theta^*, x^* \rangle - \langle \theta^*, x_t \rangle = \langle \theta^*, x^* \rangle - \langle \hat{\theta}_t, x_t \rangle \\
\leq \langle \theta^* - \hat{\theta}_t, x_t \rangle \\
\leq \| \theta^* - \hat{\theta}_t \|_{V_{t-1}} \| x_t \|_{V_{t-1}}^{-1}
\]

where the first inequality is due to the above argument, and the second is an application of Cauchy-Schwarz inequality. But, notice that \( \| u - v \|_{V_t} \) is a metric, and both \( \theta^* \) and \( \hat{\theta}_t \) lie within \( \| \|_{V_t} \)-distance of \( \sqrt{T} \) of the RLS estimate \( \hat{\theta}_t \). We conclude by the triangle inequality that

\[
\langle \theta^*, x^* \rangle - \langle \theta^*, x_t \rangle \leq 2\sqrt{\beta_t} \| x_t \|_{V_{t-1}^{-1}} = \rho_t.
\]

A similar argument can be carried out for the constraint violation. For an \( i \in [K+1 : M] \), let \( \tilde{a}^i_t \) be such that \( \langle \tilde{a}^i_t, x_t \rangle \leq \alpha^i \), which is known to exist since \( x_t \in \tilde{S}_t \). Then observe that

\[
\langle a^i, x_t \rangle - \alpha^i = \langle a^i - \tilde{a}^i_t, x_t \rangle + \langle \tilde{a}^i_t, x_t \rangle - \alpha^i \\
\leq \langle a^i - \tilde{a}^i_t, x_t \rangle \\
\leq \| a^i - \tilde{a}^i_t \|_{V_t} \| x_t \|_{V_{t-1}}^{-1} \\
\leq 2\sqrt{\beta_t} \| x_t \|_{V_{t-1}^{-1}},
\]

where the last inequality again uses that both \( a^i, \tilde{a}^i_t \) are in \( C^i_{t-1} \), which is again a ball of \( V_t \)-norm radius \( \sqrt{\beta_t} \) around \( \tilde{a}^i_t \).

The conclusion of Theorem 5.3 is now forthcoming. Indeed, combining the bounds (8) and (7) of Lemma C.1, we have shown (5) from the main text,

\[
\forall t : \max_i (\langle \theta^*, x^* - x_t \rangle, \max_i (\langle a^i, x_t \rangle - \alpha^i)) \leq \rho_t.
\]

Therefore,

\[
\text{Regret}_T \leq \sum_t \rho_t = \sum_t 2\sqrt{\beta_t} \| x_t \|_{V_{t-1}^{-1}} \\
\leq 2\sqrt{\beta_T} \sum_t \| x_t \|_{V_{t-1}^{-1}} \\
\leq 2\sqrt{\beta_T} \cdot \sqrt{T} \cdot \sqrt{\sum_t \| x_t \|_{V_{t-1}^{-1}}^2},
\]

where we have used that \( \beta_T \) is non-decreasing and the Cauchy-Schwarz inequality. The claim follows on bounding the final inequality using Lemma B.1, and plugging in \( \beta_T \) to get

\[
\text{Regret}_T \leq 4\sqrt{T} \left( \sqrt{\frac{1}{2} \log \left( \frac{(U + 1) \det(V_t)^{1/2} \det(\lambda I)^{-1/2}}{\delta} \right)} + \sqrt{\lambda} \right) \sqrt{d \log \left( 1 + \frac{T}{\lambda d} \right)}.
\]

C.2 Proofs Omitted From §5

This section deals with proofs of structural claims of the behaviour of DOSLB on polytopal domains, and then arguments that use this property and the gaps to develop logarithmic relaxed-regret upper bounds.
C.2.1 Structural Properties of DOSLB

We show the main result of §5.1, namely that any point that DOSLB plays must be noisily associated with some BIS. To this end, we first characterise the behaviour of DOSLB relative to polytopes contained in the permissible set. Before stating the same, recall that an extreme point of a polytope (and indeed a closed convex set), is any point that is not contained on a line joining two other points in the polytope. Further, each extreme point of a polytope must satisfy at least \( d \) constraints with equality. For a polytope \( P \), we will denote its extreme points as \( \mathcal{E}_P \).

**Lemma C.2.** Suppose that \( P \) is a polytope such that \( P \subset \tilde{S}_t \). If DOSLB plays in \( P \), then \( x_t \) must be an extreme point of \( P \), i.e., \( x_t \in P \implies x_t \in \mathcal{E}_P \).

Let us first argue that the claim follows from the above Lemma.

**Proof of Lemma 5.8.** For a choice of \( \{ \tilde{a}^i \}_{i \in [1:M]} \) such that for each \( i \), \( \tilde{a}^i \in C^i_{t-1} \), define the polytope

\[
P(\{ \tilde{a}^i \}) = \{ x : \forall i, \langle \tilde{a}^i, x \rangle \leq \alpha^i \}.
\]

Now, observe that

\[
\tilde{S}_t = \bigcup_{\{ \tilde{a}^i \in C^i_{t-1} \}_{i \in [1:M]}} \{ \forall x : \langle \tilde{a}^i, x \rangle \leq \alpha^i \} = \bigcup_{\{ \tilde{a}^i \in C^i_{t-1} \}_{i \in [1:M]}} P(\{ \tilde{a}^i \}),
\]

i.e., \( \tilde{S}_t \) can be decomposed as a union of polytopes. But then the selected point \( x_t \) must lie in one of these polytopes, say \( P^* \).

Now, we have that \( P^* \subset \tilde{S}_t \), and \( x_t \in P^* \), and so by Lemma C.2, \( x_t \) must be an extreme point of \( P^* \). But this implies that there are at least \( d \) values \( i(1), \ldots, i(d) \in [1:M] \) such that \( \langle \tilde{a}^i, x_t \rangle = \alpha^i \) for every \( i \in \{i(1), \ldots, i(d)\} \), showing the claim. \( \square \)

It remains to show the preceding Lemma.

**Proof of Lemma C.2.** Suppose that \( x_t \in P \). Then, due to the optimistic choice, there also exists some \( \hat{\theta}_t \in C^0_{t-1} \) such that

\[
(\hat{\theta}_t, x_t) \in \arg \max_{\hat{\theta} \in C^0_{t-1}, x \in P} \langle \hat{\theta}, x \rangle.
\]

Notice also that \( x_t \) is a solution of the linear program \( \max_{x \in P} \langle \hat{\theta}_t, x \rangle \), and so lies on the boundary of \( P \). Similarly, \( \hat{\theta}_t \) lies on the boundary of \( C^0_{t-1} \). We need to argue that \( x_t \) in fact be an extreme point of \( P \), i.e., it does not lie in the interior of some face of dimension \( \geq 1 \) of \( P \).

For this, first suppose for the sake of contradiction that \( x_t \) lies in the interior of some 1-dimensional face of \( P \), say \( F \). Let \( u \) be the direction of variation of \( F \). Then it must hold that \( \langle \hat{\theta}_t, u \rangle = 0 \), else \( \langle \hat{\theta}_t, x_t + \varepsilon u \rangle \) would exceed \( \langle \hat{\theta}_t, x_t \rangle \) for some small choice of \( \varepsilon \). Now, let us rotate the domain so that \( u \) is directed along one coordinate axis, and project onto the 2D subspace spanned by the (orthogonal) directions \( u \) and \( \hat{\theta}_t \). Next, rescale the vectors so that both \( u \) and \( \hat{\theta}_t \) have norm 1, and finally translate the polytope so that the \( u \)th component of \( x_t \) is 0. Notice that the projection of an ellipsoid is an ellipsoid, and so doing the same transformations to \( C^0_{t-1} \) produces a 2-dimensional convex confidence ellipsoid \( D \).

Let us relabel the axes of the resulting system as \( u_1 \) and \( u_2 \). In the resulting coordinate system, \( \hat{\theta} = (0, 1) \), and \( F \) is a line segment of the form \( \{ u_1 \in [p, q], u_2 = r \} \), where \( p < 0 < q, r = \langle \hat{\theta}_t, x_t \rangle / ||\hat{\theta}_t|| \) and \( x_t = (0, r) \). Observe that \( \hat{\theta}_t \) must lie on the boundary of \( D \). We shall argue that there is some other \( z \in F \) and some other \( \phi \in D \) such that \( \langle z, \phi \rangle > r \), which violates the assumption.

We first take the case of \( r > 0 \). Observe that if any point of \( D \) has \( u_2 \) coordinate greater than 1, then we immediately have a contradiction, since then for such a point \( \phi, \langle \phi, x_t \rangle > \langle \hat{\theta}_t, x_t \rangle \). But,
since \( \hat{\theta}_{t} = (0, 1) \in D \), it follows that the ellipse \( D \) is tangent to \( u_2 = 1 \). But this means that for small \( \varepsilon \), \( D \) must contain points \( \phi_{\varepsilon} = (\varepsilon, 1 - f(\varepsilon)) \) where \( 0 \leq f(\varepsilon) = O(\varepsilon^2) \). But this implies a contradiction - indeed, take \( \varepsilon > 0 \), and consider \( \phi_{\varepsilon} = (\varepsilon^{1/2}, r) \). Then \( \phi_{\varepsilon} \in F \) for small enough \( \varepsilon \), and \( \langle \phi_{\varepsilon}, \phi_{\varepsilon} \rangle - r = \varepsilon^{3/2} - rf(\varepsilon) \). Since \( f(\varepsilon) = O(\varepsilon^2) \), this is positive for small enough \( \varepsilon \), demonstrating a contradiction.

If \( r < 0 \), the same argument can be run mutatis mutandis - now \( D \) must lie above the line \( u_2 = 1 \), but still be tangent to it, and we can develop points of the form \( (\varepsilon, 1 - f(\varepsilon)) \) for \( 0 \leq f = O(\varepsilon^2) \) in \( D \), and the analogous inner product \( \langle \phi_{\varepsilon}, \phi_{\varepsilon} \rangle - r = \varepsilon + rf(\varepsilon) \) which is again positive for small enough \( \varepsilon \).

Finally, we have the case \( r = 0 \), wherein \( x_t \) lies at the origin. But in this case any point in \( D \) of non-zero \( u_1 \) coordinate serves as a contradiction (since either \( p, 0 \) or \( 0, q \) will yield a positive inner product).

Together, the above paragraphs imply that \( x_t \) cannot lie on the interior of an edge of \( P \). But this argument generalises to the interior of any non-trivial face. Indeed, since \( \hat{\theta}_{t} \) must be orthogonal to the affine subspace formed by this face, we can argue that there must be a point in the interior of a 1-D face (that forms a boundary of the larger face) that must also attain the optimal value for \( \langle \hat{\theta}_{t}, x \rangle \), and then run the above argument for this point. It follows that \( x_t \) cannot lie in the interior of any non-trivial face of \( P \).

The above argument is not restricted to confidence ellipsoids of the form of \( \S 3.1 \), but extends to any \( C_t \) with a smooth and convex boundary. Indeed, this further extends to convex \( C_t \) with continuous boundaries, barring the case where \( \hat{\theta}_{t} \) is itself the extreme point of a polytope (with large curvature at \( \hat{\theta}_{t} \)). In such a case the property that \( f(\varepsilon) = O(\varepsilon^2) \) does not hold, and a more global argument may be needed. One attack may pass through the use of continuous noise, in which case the confidence sets would almost surely not produce any extreme points that are orthogonal to the faces of a polytope (since such directions lie in a union of a finite number of dimension \( d - 1 \) affine subspaces, which in turn is Lebesgue null), and so we may almost surely avoid this disadvantage case.

### C.2.2 Instance-Dependent Logarithmic Bounds

We now turn our attention to the results of \( \S 5.3 \). The main technical meat of this section lies in the proofs of Lemma 5.14 and Lemma 5.16. In the following, we will show the Theorems using these key lemmata, and leave their proof to the following section \( \S C.2.3 \).

**Proof of Theorem 5.15.** Let \( t \) be such that a non-optimally BIS is associated with \( x_t \). Then by Lemma 5.14, it must hold that \( \rho_t \geq \Xi \). But, then observe that

\[
\sum_t (1 - I\{I_t \sim x^*\}) = \sum_t I\{\rho_t > \Xi\} \leq \sum_t \frac{\rho_t^2}{\Xi^2}
\]

\[
\leq 4\Xi^{-2}(\sqrt{\beta T})^2 \sum_t \|x_t\|^2_{V_t^{-1}}
\]

\[
\leq \frac{8}{\Xi^2} \left( d^2 \log^2(1 + T/\lambda d) + 2d\sqrt{\lambda} \log(1 + T/\lambda d) \right),
\]

where the first inequality follows because for any non-negative \( z \), \( I\{z > 1\} \leq z \) \( I\{z > 1\} \leq z \), the second inequality uses the definition of \( \rho_t \) and the fact that \( \beta_t \) is non-decreasing, and the final inequality uses the bound of Lemma B.1, and bounds \( (\sqrt{\beta T})^2 \) by using \( (u + v)^2 \leq 2u^2 + 2v^2 \).

**Proof of Theorem 5.17.** Using Lemma 5.16, we first note that since any time a \( \epsilon \)-infeasible point is picked, \( \rho_t > \epsilon \), the proof of Theorem 5.15 can be repeated verbatim with \( \Xi \) replaced by \( \epsilon \) to get

\[
\sum_t I\{t : \max_i \langle a^i, x_t \rangle - \alpha^i > \epsilon\} \leq \frac{8d^2 \log^2(1 + T/\lambda d) + 16d\sqrt{\lambda} \log(1 + T/\lambda d)}{\epsilon^2}.
\]
So, we need to show the main regret bound. To this end, recall from Lemma C.1 that
\[
\forall t : \max_i (\langle \theta^*, x^* - x_t \rangle, \max_i (\langle a^i, x_t \rangle - \alpha^i)) \leq \rho_t.
\]

For notational conciseness, we define the following sets of time indices
\[
T_{s, \leq} = \{ t : x_t \text{ is such that } I_t \sim x^*, \text{ and } \max_i \langle a^i, x_t \rangle - \alpha^i \leq \epsilon \},
\]
\[
T_{s, >} = \{ t : x_t \text{ is such that } I_t \sim x^*, \text{ and } \max_i \langle a^i, x_t \rangle - \alpha^i > \epsilon \},
\]
\[
T_{s} = [1 : T] \setminus (T_{s, \leq} \cup T_{s, >}).
\]

Observe that by Lemma 5.16, for \( t \in T_{s, \leq} \), the algorithm incurs no instantaneous regret. Further, for \( t \in T_{s, >} \), by the same Lemma, it must hold that \( \rho_t > \epsilon \), and that for \( t \in T_{s} \), it must hold that \( \rho_t \geq \Xi \).

With the above observations, the conclusion is drawn as below.
\[
\text{Regret}_T^2 \leq \sum_t \rho_t (\mathbb{I}\{t \in T_{s, >}\} + \mathbb{I}\{t \in T_{s}\}),
\]
\[
\leq \sum_t \rho_t \mathbb{I}\{\rho_t > \epsilon\} + \sum_t \rho_t \mathbb{I}\{\rho_t \geq \Xi\},
\]
\[
\leq \sum_t \frac{\rho_t^2}{\epsilon} + \sum_t \frac{\rho_t^2}{\Xi},
\]
\[
\leq \left(\frac{1}{\Xi} + \frac{1}{\epsilon}\right) \left(8d^2 \log^2 (1 + T/d\lambda) + 16d\sqrt{\lambda} \log (1 + T/\lambda d)\right),
\]

where the first line zeros out the regret for \( t \in T_{s, \leq} \), and then upper bounds the instantaneous regret over the other rounds by \( \rho_t \). The second line invokes the above lower bounds on \( \rho_t \) over the \( T \)s, and the remainder follows as in the proof of Thm. 5.15.

C.2.3 Proofs of Key Lemmata from §5.3

We begin with the proof of Lemma 5.16, which is both simpler, and serves to demonstrate the basic technique being used. We then describe three auxiliary results that together imply Lemma 5.14, and then proceed to argue the same.

**Rounds in which \( x_t \) is associated to an optimally-associated BIS have \( \theta \) efficiency loss**

**Proof of Lemma 5.16.** We first show that if \( x_t \) is \( \epsilon \)-unsafe, then \( \rho_t \) must exceed \( \epsilon \). This follows trivially upon recalling Lemma C.1 - indeed, since \( x_t \in \bar{S}_t \), there exist \( \tilde{a}^i \in C_{t-1}^i \) such that \( \langle \tilde{a}^i, x_t \rangle \leq \alpha^i \). But, since \( a^i \in C_t^i \), we know that for every \( i \)
\[
| \langle a^i, x_t \rangle - \langle \tilde{a}^i, x_t \rangle | \leq 2\sqrt{\beta_t} \| x_t \|_{V_{t-1}} = \rho_t.
\]

But, since \( x_t \) is \( \epsilon \)-unsafe, there exists at least one \( i \) such that \( \langle a^i, x_t \rangle \geq \alpha^i + \epsilon \). Plugging in this \( i \) and the bounds on the inner products discussed, we conclude that \( \alpha^i + \epsilon - \alpha^i = \epsilon \leq \rho_t \).

We now move to arguing that if \( I_t \sim x^* \), then \( \langle \theta^*, x_t \rangle \geq \langle \theta^*, x^* \rangle \). To this end, fix \( I_t = I \), and consider the following linear program
\[
\max_{x \in \mathbb{R}^d} \langle \theta^*, x \rangle \quad \text{s.t.} \quad A(I)x = \alpha(I) \tag{10}
\]

Notice that the above program is feasible - indeed, since \( I \sim x^* \), \( x^* \) is the unique feasible point for the same. Therefore, it’s value is \( \text{OPT} := \langle \theta^*, x^* \rangle \). Further, since the program is feasible and bounded, so is its dual program of (10)
where the final line uses the characterisation of $\delta A$. 

Therefore, the program reduces to

\[
\max_{\tilde{A}(I) \in A_{t-1}(I), \delta \in C_{t-1}^d} \max_{x \in \mathbb{R}^d} \langle \tilde{\theta}, x \rangle \\
\text{s.t.} \quad \tilde{A}(I)x = \alpha(I).
\]

Indeed, the outer maximum allows us to scan over all $x$ that are noisily associated with $I$, and all $\tilde{\theta}$ in the confidence set, and the optimistic selection rule demands that we choose an $x_t$ that maximises these (subject of course to the assumption that $I_t \sim x_t$).

Now, for any $\tilde{\theta} \in C_{t-1}^d$, define $\delta \theta = \tilde{\theta} - \theta^*$, and similarly define $\delta A = \tilde{A} - A$. We may equivalently view the program above as

\[
\max_{\delta A, \delta \theta, x} \langle \theta^* + \delta \theta, x \rangle \\
\text{s.t.} \quad (A(I) + \delta A(I))x = \alpha(I).
\]

Now, express $x = (x - x^*) + x^*$. The objective above can be resolved as $\langle \theta^*, x^* \rangle + \langle \theta^*, x - x^* \rangle + \langle \delta \theta, x \rangle$, while the constraint works out to

\[
A(I)(x - x^*) = -\delta A(I)x,
\]

upon recalling that $A(I)x^* = \alpha(I)$. But, recall that $\theta^* = A(I)^\top \pi$. Thus,

\[
\langle \theta^*, x - x^* \rangle = \langle \pi, A(I)(x - x^*) \rangle = -\langle \pi, \delta A(I)x \rangle.
\]

Therefore, the program reduces to

\[
\max_{\delta A, \delta \theta, x} \text{OPT} + \langle \delta \theta, x \rangle - \langle \pi, \delta A(I)x \rangle.
\]

Of course, there exists a choice of $\delta \theta$ and $\delta A$ that together with $x_t$ maximise the above.

Notice that, since $a^i \in C_{t-1}$ for each $i$, the setting $\delta A(I) = 0$ is feasible for the above program. As a result, $\delta A_t$ and $x_t$ satisfy that $\langle \pi, \delta A_t(x_t) \rangle \leq 0$.

Now, finally, we observe that

\[
\langle \theta^*, x_t \rangle = \langle \theta^*, x^* \rangle + \langle \theta^*, x_t - x^* \rangle \\
= \langle \theta^*, x^* \rangle + \langle \pi, A(I)(x_t - x^*) \rangle \\
= \langle \theta^*, x^* \rangle - \langle \pi, \delta A_t(x_t) \rangle \\
\geq \langle \theta^*, x^* \rangle,
\]

where the final line uses the characterisation of $\delta A_t(I)$ and $x_t$ offered above. 

With the above in place, let us recap the approach of the above argument. For a BIS $I$, we isolated a noiseless program (10) that identified points associated with it, and used it’s dual (11) to identify a convenient way to express $\theta^*$ in terms of the active constraints $A(I)$. We then formulated a noisy version of the program (12) that the solution $x_t$ must solve, given that it is noisily associated with $I$. We then re-expressed this in terms of the differences $\tilde{a}^i - a^i$ in (13), and then using the optimality conditions of the two noiseless programs, we eliminated the constraint from (13) to obtain
an unconstrained (in $x$) program (14), from which we could infer properties for $\tilde{a}_i^t - a_i^t$ for $i \in I$. These were finally used to relate the value of the point associated with $I$ (i.e. $x^*$) to the value of $x_t$.

We shall repeatedly use the above structure to argue Lemma 5.14.

**Association with a non-optimal BIS implies that the noise scale must be large**

We shall develop the proof of Lemma 5.14 via three auxiliary statements, which handle the three gaps separately.

**Efficacy Gap**

**Lemma C.3.** Assume that all of the confidence sets are consistent. If $x_t$ is noisily associated to some $I_t$ such that $I_t \neq x^*$, then $\rho_t \geq \Delta_t$.

**Proof.** Set $I = I_t$. Without loss of generality, we may assume that $\Delta_t > 0$. Denote $J_t := [1 : K] \setminus I$. Let $x^t$ be any solution to

$$\max \langle \theta^*, x \rangle \quad \text{s.t.} \quad A(I)x = \alpha(I), \ A(J_t) \leq \alpha(J_t).$$

Due to the consistency of $I$, this program is feasible and bounded. Therefore, so is the dual program

$$\min_{\pi, \rho, \sigma} \langle \pi, \alpha(I_U) \rangle + \langle \rho, \alpha(I_K) \rangle + \langle \rho, \alpha(J_t) \rangle$$

$$\text{s.t.} \quad \pi^T A(I_U) + \rho^T A(I_K) + \sigma^T A(J_t) = (\theta^*)^T.$$

Further, by complementary slackness, since $x^t$ optimises the primal, we can choose $(\pi, \rho, \sigma)$ such that

$$\forall j \in J_U, \sigma^T A(J_U) x^t = \alpha(J_U).$$

Fix such a $(\pi, \rho, \sigma)$.

Now we proceed as in the proof of Lemma 5.16. Since $x_t$ is noisily associated to $I$, there must be $\tilde{a}_i^t \in C_i^t$ for $i \in I$ such that $\langle \tilde{a}_i^t, x_t \rangle = \alpha^t$. Recalling the notation $A_{t-1}$ from the proof of Lemma 5.16, $x_t$ must be a solution to

$$\max_{\tilde{A}(I_U) \in A_{t-1}(I), \tilde{A} \in C_i^t} \max_{x \in \mathbb{R}^d} \langle \tilde{\theta}, x \rangle$$

$$\text{s.t.} \quad \tilde{A}(I_U)x = \alpha(I_U), \quad A(I_K)x = \alpha(I_K) \quad \text{(18)}$$

where in the final line we are adding known constraints that $x_t$ must satisfy in any case.

Expressing $\tilde{A}(I_U) = A(I_U) + \delta A(I_U)$ and $\tilde{\theta} = \theta^* + \delta \theta$, we may express the above as

$$\max_{\delta A, \delta \theta, \delta x} \langle \theta^* + \delta \theta, x \rangle$$

$$\text{s.t.} \quad (A(I_U) + \delta A(I_U))x = \alpha(I_U) \quad \text{(20)}$$

$$A(I_K)x = \alpha(I_K) \quad \text{(21)}$$

Now, express $x = (x - x^t) + x^t$. The objective above can be resolved as $\langle \theta^*, x^t \rangle + \langle \theta^*, x - x^t \rangle + \langle \delta \theta, x \rangle$, while the constraint on $\tilde{A}$ works out to

$$A(I)(x - x^t) = -\delta A(I)x,$$

upon recalling that $A(I)x^t = \alpha(I)$. But, recall that $\theta^* = A(I)^T \pi + A(J_t)^T \sigma$. Thus,

$$\langle \theta^*, x - x^t \rangle = \langle \pi, A(I_U)(x - x^t) \rangle + \langle \rho, A(I_K)(x - x^t) \rangle + \langle \sigma, A(J_t)(x - x^t) \rangle$$

$$= -\langle \pi, \delta A(I)x \rangle + \langle \sigma, A(J_t)(x - x^t) \rangle.$$

Therefore, the program reduces to

$$\max_{\delta A, \delta \theta, x} \text{OPT} - \delta I + \langle \delta \theta, x \rangle - \langle \pi, \delta A(I)x \rangle + \langle \sigma, A(J_t)(x - x^t) \rangle.$$

(22)
In particular, there exists a choice of \( \delta \theta_t \) and \( \delta A_t \) that together with \( x_t \) maximise the above.

But notice that \( \delta \theta_t \) and the rows of \( \delta A_t \) must each lie in a ball of \( V_t \)-norm \( 2\sqrt{\beta_t} \) around the RLS estimates \( 0 \). As a result, we have

\[
| \langle \delta \theta_t, x_t \rangle | \leq 2\sqrt{\beta_t} \| x_t \|_{V_t^{-1}} = \rho_t
\]

\[
| \langle \pi, \delta A_t, x_t \rangle | \leq \rho_t \| \pi \|_1.
\]

Finally, we note that

\[
\sigma^T A(J_I)(x_I - x^t) = \sigma^T A(J_I)x_I - \sigma^T \alpha(J_I(x^t)) \leq 0,
\]

since \( \sigma \geq 0, A(J_I)x_I \leq \alpha(J_I) \), and since we chose \( \sigma \) such that \( \sigma^T A(J_I)x^t = \sigma^T \alpha(J_I) \). Therefore, the final term in the objective is nonpositive. It then follows that the value of this program is upper bounded by

\[
\text{OPT} - \delta(I) + \rho_t(1 + \| \pi \|_1).
\]

But, notice that due to the consistency of the confidence sets, \( x^* \in S_t \) and \( \theta^* \in C^t_{I-1} \). Therefore, the value of the noisy program determining \( x_t \) must exceed \( \text{OPT} = \langle \theta^*, x^* \rangle \), since otherwise the algorithm would have preferred \( x^* \) to \( x_t \). We conclude that

\[
\text{OPT} \leq \text{OPT} - \delta(I) + \rho_t(1 + \| \pi \|_1) \iff \rho_t \geq \delta(I)/(1 + \| \pi \|_1).
\]

Now, for each optimiser \((\pi, \rho, \sigma)\) of the dual, there is a corresponding optimiser \( x^t \) of the primal that satisfies complementary slackness. Therefore, varying over the \( x^t \), we may choose a \( \pi \) that maximises the denominator in the above. But, by definition, this value is simply the spread, and we conclude that

\[
\rho_t \geq \frac{\delta(I)}{\text{spread}(\theta^*; I, J_I)} = \Delta_I.
\]

\( \square \)

**Safety Gap**

**Lemma C.4.** Assume that all of the confidence sets are consistent. If \( x_t \) is noisily associated to some \( I_t \), such that \( I_t \neq x^* \), then \( \rho_t \geq \Gamma_{I_t} \).

**Proof.** Fix a \( k \) such that \( \gamma(I; k) > 0 \), and set \( J_{I,k} = [1 : K] \setminus (I \cup \{k\}) \). Consider the program

\[
\max \langle -\alpha^k, x \rangle \quad \text{s.t.} \quad A(I_U)x = \alpha(I_U), A(I_K)x = \alpha(I_K), A(J)x \leq \alpha(J).
\]

This program has value \(-\alpha^k - \gamma(I; k)\). Let \( x^t \) be some optimiser of the same.

As before, consider the dual program

\[
\min_{\pi, \rho, \sigma} \langle \pi, \alpha(I_U) \rangle + \langle \rho, \alpha(I_K) \rangle + \langle \rho, \alpha(J_{I,k}) \rangle
\]

\[
\text{s.t.} \quad \pi^T A(I_U) + \rho^T A(I_K) + \sigma^T A(J_{I,k}) = (-\alpha^k)^T.
\]

This program is feasible and bounded, since the primal is, and so by complementary slackness, we may choose \((\pi, \rho, \sigma)\) that optimise the same such that \( \sigma^T A(J_{I,k})x = \sigma^T \alpha(J_{I,k}) \).

Now, consider the noisy version of the same program

\[
\max_{\tilde{a}^k \in C^t_{I-1}, \tilde{a}i \in C^t_{I-1}} \langle -\tilde{a}^k, x \rangle \quad \text{s.t.} \quad \tilde{A}(I_U)x = \alpha(I_U), A(I_K)x = \alpha(I_K), A(J_I)x \leq \alpha(J_I).
\]

Since \( I \sim_t x_t, x_t \) must be feasible for this problem.

Now express the variable \( x \) as \( (x - x^t) + x^t \), and write \( \tilde{a}^k = a^k + \delta a^k, \tilde{a} \tilde{i} = a^i + \delta a^i, \tilde{A} = A + \delta A \).

Since \( A(I)x^t = \alpha(I), \sigma^T A(J_{I,k})x^t = \alpha(J_{I,k}) \) and \( \sigma \geq 0 \), it follows that for any feasible \( x \)

\[
A(I_U)(x - x^t) = -\delta A(I_U)x
\]

\[
A(I_K)(x - x^t) = 0
\]

\[
\sigma^T A(J_{I,k})(x - x^t) \leq 0.
\]
Using this, we can write for any feasible \( x, \tilde{a}^k \) that
\[
\langle -\tilde{a}^k, x \rangle = \langle -a^k, x^I \rangle + \langle -a^k, x - x^I \rangle + \langle -\delta a^k, x \rangle
\]
\[
= (-\alpha^k - \gamma(I; k)) + \langle \delta a^k, x \rangle + \langle \pi, A(I_U)(x - x^I) \rangle + \langle \rho, A(I_K)(x - x^I) \rangle
\]
\[
+ \langle \sigma, A(J_{I,k})(x - x^I) \rangle
\]
\[
\leq (-\alpha^k - \gamma(I; k)) + \langle -\delta a^k, x \rangle - \langle \pi, \delta A(I_U)x \rangle.
\]

But, for \( x = x_t \), we know that \( |\langle \delta a^k, x \rangle| \leq \rho_t \). It thus follows that \( \forall \tilde{a}^k \in C_{I_t}^k \),
\[
\langle -\tilde{a}^k, x_t \rangle \leq -\alpha^k - \gamma(I; k) + \rho_t(1 + \|\pi\|_1).
\]

However, since \( x_t \) is played, it must hold that there exists \( \tilde{a}^k : \langle \tilde{a}^k, x_t \rangle \leq \alpha^k \iff \langle -\tilde{a}^k, x_t \rangle \geq -\alpha^k \) - otherwise \( x_t \) would not have been permissible. We immediately conclude that
\[
-\alpha^k \leq -\alpha^k - \gamma(I; k) + \rho_t(1 + \|\pi\|_1)
\]
\[
\iff \rho_t \geq \frac{\gamma(I; k)}{1 + \|\pi\|_1}.
\]

Now, varying \( x^I \) as in the proof of Lemma C.3, we can vary over all dual solutions \( (\pi, \rho, \sigma) \) in order to minimise the denominator of the above lower bound. We conclude that
\[
\rho_t \geq \frac{\gamma(I; k)}{\text{spread}(-a^k; I, J_{I,k})}.
\]

Finally, since this lower bound holds no matter what \( k \) is chosen, we may maximise over the same to conclude that \( \rho_t \geq \Gamma_I \).

\[\square\]

\textbf{Consistency Gap}

\textbf{Lemma C.5.} Assume that all of the confidence sets are consistent. If \( x_t \) is noisily associated to some \( I_t \) such that \( I_t \not\supseteq x^* \), then \( \rho_t \geq \Lambda_{I_t} \).

\textbf{Proof.} Again set \( I = I_t \), and wlog assume that \( \Lambda_{I_t} > 0 \).

Fix \( \bar{I} \subset I, \bar{J} \subset [1 : K] \setminus I \) such that \( X(\bar{I}, \bar{J}) \not= \emptyset \).

We now split the analysis depending on which of \( X \) and \( \bar{X} \) are positive.

\textbf{Case I} First consider some \( k \not\in I \cup \bar{J} \) such that \( \overline{\chi_I}(\bar{I}; \bar{J}, k) > 0 \). This means that for points in \( X(\bar{I}, \bar{J}) \), the constraint \( \langle a^k, x \rangle \leq \alpha^k \) cannot be met.

Consider the linear program
\[
\max \langle -a^k, x \rangle \quad \text{s.t.} \quad A(I_U)x = \alpha(I_U), A(I_K)x = \alpha(I_K), A(\bar{J})x \leq \alpha(\bar{J}).
\]

This program is feasible and bounded, and by definition has value \( -\alpha^k - \overline{\chi_I}(\bar{I}; \bar{J}, k) \). Let \( x^0 \) be some optimiser of the same.

Again, by duality, we may choose a \( (\pi, \rho, \sigma) \) such that \( \sigma \geq 0 \),
\[
\pi^\top A(I_U) + \rho^\top A(I_K) + \sigma^\top A(\bar{J}) = (-a^k)^\top,
\]
and
\[
\sigma^\top A(\bar{J})x^0 = \sigma^\top \alpha(\bar{J}).
\]

Fixing this, we again consider the noisy program
\[
\max_{\tilde{a}^k \in C_{I_{t-1}}^k, \bar{a}^k \in C_{\bar{I}_{t-1}}^k} \langle -\tilde{a}^k, x \rangle \quad \text{s.t.} \quad A(I_U)x = \alpha(I_U), A(I_K)x = \alpha(I_K), A(J_{I,k})x \leq \alpha(\bar{J}).
\]

Since \( I \sim_t x_t, x_t \) must be feasible for this problem.
Following the same steps as in the proof of Lemma C.4, we conclude that \(-a^k, x_t\) \leq -\alpha^k - \lambda_t(\tilde{I}; \tilde{J}, k) + \rho_t(1 + ||\pi||_1), and due to the permissibility of \(x_t, \langle -a^k, x_t \rangle \geq -\alpha^k\), yielding that

\[
\rho_t \geq \frac{\lambda_t(\tilde{I}; \tilde{J}, k)}{1 + ||\pi||_1}.
\]

Finally, varying \(x_0\), and through this the associated dual optima, we may replace the lower bound by the spread of \(-a^k\), giving that

\[
\rho_t \geq \frac{\lambda_t(\tilde{I}; \tilde{J}, k)}{\text{spread}(-a^k; \tilde{I}, \tilde{J})}.
\]

**Case II** Next, we consider a \(k\) such that \(\lambda_t(\tilde{I}; \tilde{J}, k) > 0\). Notice that in this case, \(k \in I\), and so since \(I \sim x_t\), it must be the case that for some noisy \(\hat{a}^k \in C^k_{t-1}, \langle \hat{a}^k, x_t \rangle = \alpha^k\), which can be interpreted as the joint constraint \(\langle \hat{a}^k, x_t \rangle \leq \alpha^k\) and \(\langle \hat{a}^k, x_t \rangle \geq \alpha^k\). The positivity of \(\lambda\) signals that the latter is hard to meet because points in \(\mathcal{X}(\tilde{I}, \tilde{J})\) have \(\langle a^k, x \rangle \leq \alpha^k - \lambda_t\).

Consequently, we study the linear program

\[
\max \langle a^k, x \rangle \text{ s.t. } A(\tilde{I}_U)x = \alpha(\tilde{I}_U), A(\tilde{I}_K)x = \alpha(\tilde{I}_K), A(\tilde{J})x = \alpha(\tilde{J}).
\]

The resulting noisy program can be analysed in the same way as the previous case - it has value at most \(\alpha^k - \lambda_t(\tilde{I}; \tilde{J}, k)\), and via a similar analysis as above, there is some \(\hat{a}^k \in C^k_{t-1}\) such that

\[
\alpha^k \leq \langle \hat{a}^k, x_t \rangle \leq \alpha^k - \lambda_t(\tilde{I}; \tilde{J}, k) + \|\pi\|_1 \rho_t,
\]

but where \(\pi\) is now such that there exist \(\rho\) and \(\sigma \geq 0\) that together optimise the dual program \(\Pi(+a^k; \tilde{I}, \tilde{J})\). It follows that

\[
\rho_t \geq \frac{\lambda_t(\tilde{I}; \tilde{J}, k)}{\text{spread}(a^k; \tilde{I}, \tilde{J})}.
\]

**Summing up** Of course, since the above inequalities hold for every \(k\), we conclude that

\[
\rho_t \geq \max \left( \frac{\lambda_t(\tilde{I}; \tilde{J}, k)}{\text{spread}(-a^k; \tilde{I}, \tilde{J})}, \frac{\lambda_t(\tilde{I}; \tilde{J}, k)}{\text{spread}(a^k; \tilde{I}, \tilde{J})} \right).
\]

Maximising this lower bound over \(\tilde{I}, \tilde{J}, k\) gives the conclusion \(\rho_t \geq \lambda_t\)

\[\square\]

**Overall result**

With the above in hand, we can show the main result.

**Proof of Lemma 5.14.** Suppose \(I_t : I_t \not\sim x^*\). Then by the previous three Lemmata, it must hold that

\[
\rho_t \geq \max(\Delta_{I_t}, \Gamma_{I_t}, \Lambda_{I_t}) \geq \Xi.
\]

\[\square\]

**D** **Lower bounds**

**D.1 Lower Bound for Safe Regret Without Relaxation**

We first show Theorem 5.3, which explicitly constructs a set of instances with large \(\Xi\) such that any algorithm must suffer \(\Omega(\sqrt{T})\) regret on at least one of them.

**Proof of Theorem 5.3. Basic construction in 1-dimension** Let us first consider the case of \(d = 1\). We take the setting \(K = \{x \leq 1, -x \leq 1\}\), and \(\theta^* = 1\). We further introduce a single safety constraint of the form \(\alpha^3 x \leq 1/4\), where \(\alpha^3 \in \{\alpha^3_x, \alpha^3_+\}\), where \(\alpha^3_x = (1 - \varepsilon)/2, \alpha^3_+ = (1 + \varepsilon)/2\), and where \(\varepsilon > 0\) will be chosen by below. The noises \(\gamma^0, \gamma^3\) are taken as independent standard Gaussian.
Description of the setting. Notice that this setting is polytopal - indeed, the known domain is $[-1, 1]$, described by two linear constraints. Further, there are three BISs, each of which is full rank, and has the associated points $x^1 = 1, x^2 = -1, x^3 \in \left\{ \frac{1}{2(1 + \varepsilon)} \right\}$.

For either choice of $a^3$, the BIS $\{3\}$ is optimal.

The BIS $\{1\}$ is unsafe, and has
\[
\gamma(\{1\}; 3) \geq \frac{1}{4} - \frac{1 + \varepsilon}{2} = \frac{1 - 2\varepsilon}{4}; \text{ saf-spread}(\{1\}; 3) \leq \frac{3 + \varepsilon}{2} \Rightarrow \Gamma_{\{1\}} \geq \frac{1 - 2\varepsilon}{6 + 2\varepsilon} \geq \frac{1 - 2\varepsilon}{8}.
\]

The BIS $\{2\}$ is ineffective, and has
\[
\lambda_{\{2\}} \geq 1 + \frac{1}{2(1 + \varepsilon)}, \text{ eff-spread}(\{2\}) = 2 \quad \Rightarrow \quad \Delta_{\{2\}} \geq \frac{3 + \varepsilon}{4 + 4\varepsilon} \geq \frac{3}{8}.
\]

Both of these are bounded below by $1/8$ so long as $\varepsilon \leq 1/9$. Thus,
\[
\varepsilon \leq 1/9 \Rightarrow \Xi \geq 1/8 = \Omega(1).
\]

Similarity of the settings under $a_{3+}^3$. Now, denote the law of the feedback under $a_{3+}^3$ and $a_{3-}^3$ as $P_+$ and $P_-$. Notice that no matter what $x_t$s are played, it holds that
\[
D(P_+(r_t, s_t^3)\|P_-(r_t, s_t^3)|x_t = x) = \frac{(\varepsilon x)^2}{2} \leq \varepsilon^2/2,
\]
where we have used the fact that $r_t$ are identical under the two laws, while $s_t^3$ are respectively $N(a_{3+}^3 x_t, 1)$ random variables, and so have KL divergence $(a_{3+}^3 x_t - a_{3-}^3 x_t)^2/2$.

This in turn means that for any sequence of $x_t$s,
\[
D(P_+(\mathcal{H}_T)\|P_-^{\mathcal{H}_T}) \leq T\varepsilon^2/2.
\]

Bounding the regret. We now follow the approach of Chapter 24 of Lattimore & Szepesvári.

Define
\[
x_+ = \frac{1}{2(1 + \varepsilon)}, x_- = \frac{1}{2(1 - \varepsilon)}; \text{ and } x_{av} = (x_+ + x_-)/2 = \frac{1}{2(1 - \varepsilon^2)}.
\]

or any action $x_t$, define $z_t := 2\{x_t \geq x_{av}\} - 1$.

Now, suppose $a^3 = a_{3+}^3$. Then if $z_t = +1$, then the player must incur a safety regret of at least $\frac{1 + \varepsilon}{2(1 + \varepsilon)} - \frac{x}{2} \geq \frac{\varepsilon}{4}.$

Similarly, if $a^3 = a_{3-}^3$, and $z_t = -1$, then the player must incur an efficacy regret of at least $1(x_- - x_{av}) = \frac{\varepsilon}{2(1 - \varepsilon^2)} \geq \frac{\varepsilon}{4}.$

But, since the divergence between the entire trajectories in the situation $+$ and $-$ is bounded as $T\varepsilon^2/2$, it holds that
\[
\mathbb{P}_+(\sum_{t=1}^T \mathbb{I}\{Z_t = +1\} \geq T/2) + \mathbb{P}_-(\sum_{t=1}^T \mathbb{I}\{Z_t = -1\} \geq T/2) \geq \frac{1}{2} \exp(-T\varepsilon^2/2),
\]
and so in at least one of the two cases, the learner must incur expected regret at least
\[
\max(\mathbb{E}_+[\text{Regret}_T], \mathbb{E}_-[\text{Regret}_T]) \geq \frac{T\varepsilon^2}{8} e^{-T\varepsilon^2/2}.
\]

Now, for $T \geq 100$, we choose $\varepsilon(T) = 1/\sqrt{T}$. Since this is at most $1/9$, the gaps in this problem are large. However, by the above computation, the average regret over at least one of the two settings is at least $\frac{T\varepsilon^2}{8e^{\varepsilon^2}} \geq \sqrt{T}/32$.

Extending to arbitrary $d$ The same construction can be tensorised over $d$-dimensions by taking the known constraints $\{x_i \leq 1, -x_i \leq 1\}$ and the unknown constraints $\{a_{3+}^3 x_i \leq 1/2\}$, with
We introduce a notation with \(d\) arms and \(d\) Bernoulli bandits. For any algorithm that is consistent over the class of Bernoulli bandits, we can derive a lower bound through reduction to safe MAB and apply the asymptotic instance-dependent lower bound for safe MAB.

Note that restricting ourselves to the class of consistent algorithms is not losing too much of generality. This restriction is essentially requiring the learning algorithm to have a sub-polynomial regret rate, so that it is competitive enough to the SOTA. Lower bounds for (standard) MAB is typically stated on this class to rule out algorithms that are overly naive.

The next theorem, which is Proposition 6 in [CGS22], states an asymptotic lower bound for consistent algorithms over Bernoulli bandits.

**Theorem D.2.** Any algorithm that is consistent over the class of Bernoulli bandits must suffer

\[
\text{Regret}_T = \Omega \left( \frac{\log T}{\max\{\Delta_{\text{MAB}}, \Gamma_{\text{MAB}}\}} \right)
\]

Based on the lower bound for safe MAB, we introduce more definitions about the safe LB problem and derive a lower bound through reduction to safe MAB.

**Definition D.1.** (Consistent MAB algorithm) An MAB algorithm is called consistent over a class of MAB problems, if for any problem instance in such class, and for any \(p \in (0, 1)\)

\[
\lim_{T \to \infty} \frac{\mathbb{E}[\text{Regret}_T]}{T^p} = 0
\]

Note that restricting ourselves to the class of consistent algorithms is not losing too much of generality. This restriction is essentially requiring the learning algorithm to have a sub-polynomial regret rate, so that it is competitive enough to the SOTA. Lower bounds for (standard) MAB is typically stated on this class to rule out algorithms that are overly naive.

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**Theorem D.2.** Any algorithm that is consistent over the class of Bernoulli bandits must suffer

\[
\text{Regret}_T = \Omega \left( \frac{\log T}{\max\{\Delta_{\text{MAB}}, \Gamma_{\text{MAB}}\}} \right)
\]

Based on the lower bound for safe MAB, we introduce more definitions about the safe LB problem and derive a lower bound through reduction to safe MAB.

**Definition D.3.** (Benign Domain) We call a domain \(X' \subseteq \mathbb{R}^d\) benign, if \(X' \) is a bounded polytope, and all of the extreme points of \(X'\) are non-degenerate. We denote the set of extreme points of a Benign domain \(X\) by \(\mathcal{E}(X')\).

**Definition D.4.** (LP-easy problem) We call a safe LB problem LP-easy, if the domain is benign, the optimal feasible solution \(x^*\) is unique, and \(x^* \in \mathcal{E}(X')\).

**Definition D.5.** (Consistent safe LB algorithm) We call a safe LB problem consistent over the class of LP-easy problems with sub-Gaussian observation noise, if for any \(x \in X' : x \neq x^*\), and any \(p \in (0, 1)\)

\[
\lim_{T \to \infty} \frac{\mathbb{E}[N_{x,T}]}{T^p} = 0
\]
We next verify that the observations are indeed with means \( \mu \). We take the output \( x_t \) at each time step \( t \). The regret lower bound we provide for safe MAB algorithms, is on Bernoulli bandits, hence the last result is that the observation noises. We show that SMAB solves the problem.

Now we are ready to introduce a reduction from a safe LB algorithm to a safe MAB algorithm, on the probability simplex domain.

Consider the following safe LB problem \( SLB \), where we apply an algorithm \( SLB \)

\[
\max_{x \in S_{d-1}} \langle \theta^*, x \rangle \\
\text{s.t. } \langle \zeta^*, x \rangle \leq \alpha
\]

At each time step \( t \), take the output \( x_t \) of \( SLB \) and sample \( i_t \sim x_t \) in the following sense:

\[
P(i_t = k) = x_{t,k}, \forall k \in [d]
\]

We take the \( i_t \) as the output of our derived MAB algorithm \( SMAB \) on the induced safe MAB problem \( SMAB \). We describe the induced safe MAB problem \( SMAB \) as follows:

For each arm \( i \in [d] \), the reward parameter \( \mu_i = \langle \theta^*, e_i \rangle \), where \( e_i \) is the \( i \)th basis vector of \( \mathbb{R}_d \); the safety risk parameter \( \nu_i = \langle \zeta^*, e_i \rangle \); the safety level \( \alpha \) stays the same as \( SLB \).

We next verify that the observations are indeed with means \( \mu_i \) and \( \nu_i \) for arm \( i \). The observation at time step \( t \) is \( r_t = \langle \theta^*, x_t \rangle + \eta_t \) for the reward and \( s_t = \langle \zeta^*, x_t \rangle + \gamma_t \) for the safety risk. Now assume that \( i_t = j \), we can calculate the deviation of the observation from the expectation as follows:

\[
\mu_j - r_t = \mu_j - \langle \theta^*, x_t \rangle + \langle \theta^*, x_t \rangle - r_t
\]

\[
= \langle \theta^*, e_j \rangle - \langle \theta^*, x_t \rangle - \eta_t
\]

\[
= \sum_{k=1}^d \theta^*_k (e_{j,k} - x_{t,k}) - \eta_t
\]

Note that

\[
\mathbb{E}[\mu_j - r_t] = \mathbb{E}[\langle \theta^*, e_j \rangle - \langle \theta^*, x_t \rangle - \eta_t]
\]

\[
= \mathbb{E}[\mathbb{E}[\langle \theta^*, e_j \rangle - \langle \theta^*, x_t \rangle | x_t]] - 0
\]

\[
= \mathbb{E}[\langle \theta^*, x_t - x_t \rangle]
\]

\[
= 0
\]

so the observation is unbiased, which fits into the MAB assumptions.

Since \( e_{j,k} - x_{t,k} \in [-1, 1] \), we conclude that \( e_{j,k} - x_{t,k} \) is 1-sub-Gaussian. Hence \( \mu_j - r_t \) is \( \sqrt{2(||\theta^*||_2 + 1)} \leq 2\)-sub-Gaussian. The same result holds for the observation of safety risk.

The regret lower bound we provide for safe MAB algorithms, is on Bernoulli bandits, hence the last step in designing the problem is to force observations \( r_t \) and \( s_t \) to be in \( \{0, 1\} \). We further restrict that \( \theta^* \geq 0, \zeta^* \geq 0 \), since \( \langle \theta^*, x_t \rangle, \langle \zeta^*, x_t \rangle \in [0, 1] \), we can always make the observation Bernoulli.

Up to now, we show that \( SMAB \) is a valid bandit problem, with \( d \) arms and 2-sub-Gaussian observation noises. We show that \( SMAB \) is a consistent MAB algorithm over the class of such bandit problems. Note that under the aforementioned reduction, the regret of the safe LB problem is equal to the regret of the induced safe MAB problem. We assume our safe LB algorithm to be consistent (with respect to the LP-easy problem we construct). We apply the regret lower bound for MAB, and get the following:

\[
\text{Regret}_T = \Omega \left( \frac{\log T}{\max \{\Delta_{MAB}, \Gamma_{MAB} \}} \right)
\]

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Lastly note that the definition of efficiency and feasibility gaps in the LB and MAB problems are different. The last step here is to design the $\theta^*$, $\zeta^*$ and $\alpha$ parameters such that the former one is a multiplicative of the latter one. We are free to choose any value as an design of these parameters. We take a special case as an example, take $d = 3,$ $\theta^* = (1/2, \sqrt{3}/4, 3/4),$ $\zeta^* = (0, 0, 1)$ and $\alpha = 0.5.$ Then we know that $e_1$ is optimal, $e_2$ is feasible but less efficient, $e_3$ is infeasible but more efficient. The gaps of the induced MAB problem are $\Delta_{MAB} = \frac{2 - \sqrt{3}}{4}$ and $\Gamma_{MAB} = 1/2;$ while the gaps of the LB problem are $\Delta_{LB} = \frac{2 - \sqrt{5}}{4}$ and $\Gamma_{LB} = 0.5.$ In this case, the gaps in these two problems coincide.

Now we are ready for the Theorem 5.18.

**Theorem D.7.** Let $S_{d-1} = \{x \in \mathbb{R}^d : \|x\|_1 = 1, x \geq 0\}$ be the domain, consider the following safe linear bandit problem with Bernoulli observation:

$$\max_{x \in S} \langle \theta^*, x \rangle$$

$s.t. \langle \zeta^*, x \rangle \leq \alpha$

where $\theta^*$ and $\zeta^*$ are the unknown reward and safety risk parameters respectively. Then for any consistent algorithm,

$$\text{Regret}_T = \Omega \left( \frac{\log T}{\max\{\Delta, \Gamma\}} \right)$$

## E Computationally Efficient Implementations

As discussed in §6.1, we utilize a relaxation to box confidence sets in the implementation of DOSLB. The original confidence set stated in Algorithm 1 is:

$$C_i^t := \{\hat{a} : \|(\hat{a} - \hat{a})V_t^{1/2}\|_2 \leq \sqrt{\beta_t}\}.$$  

We consider two relaxations to this, the $L_\infty$ box and the $L_1$ box, as follows:

$$C_{i,\infty}^t := \{\hat{a} : \|(\hat{a} - \hat{a})V_t^{1/2}\|_\infty \leq \sqrt{\beta_t}\},$$

$$C_{i,1}^t := \{\hat{a} : \|(\hat{a} - \hat{a})V_t^{1/2}\|_1 \leq \sqrt{d}\beta_t\},$$

such that $C_i^t \subset C_{i,\infty}^t$ and $C_i^t \subset C_{i,1}^t,$ since $L_2$ ball with radius $r$ is contained in $L_1$ ball with radius $r$ and $L_\infty$ ball with radius $\sqrt{dr}.$ Take any $\hat{a} \in C_{i,\infty}^t,$ $\|(\hat{a} - \hat{a})V_t^{1/2}\|_\infty \leq \sqrt{\beta_t} \implies \|(\hat{a} - \hat{a})V_t^{1/2}\|_2 \leq \sqrt{d}\beta_t,$ and the same holds for the $L_1$ relaxation as well. Thus replacing $C_i^t$ by $C_{i,\infty}^t$ or $C_{i,1}^t,$ the only change in analysis will be from $\rho_t$ to $\rho_t = \sqrt{d}\rho_t.$ Hence running DOSLB with these worsens regret bounds by at most $O(\sqrt{d})$ (and the relaxed regret bound by at most $O(d)$), see §C.1 and §C.2.2.

In the following, we present the simulation result of executing the DOSLB algorithm with $L_\infty$ and $L_1$ confidence set relaxation. The problem instance is the same as Ex. 5.6, and the setting is the same as §6.2. We average the results for 30 independent realizations, and mark the standard deviation with shadow.

These two relaxations show similar performance in the efficacy regret and safety regret, in that both prefer to pick more efficient but slightly unsafe arms. The difference is that $L_\infty$-DOSLB has lower safety violations, with lower efficacy gains, compared to $L_1$-DOSLB. Our conjecture on the empirical performance is that $L_1$-DOSLB has a larger magnitude in terms of the $V_t$-norm, and hence encourages a more aggressive play than the $L_\infty$ one. Due to our regret formulation, we utilize the $L_\infty$ relaxation throughout the simulation, but $L_1$ might be preferred in other application scenarios with different performance metric.²

## F Experimental Results

We restate the experimental results in §6.2, averaged for 30 independent realizations, with uncertainty marked with shadow.

²The code of implementation in §E and F can be found here: https://github.com/ctromg/DOSLB.git
We observe that both plots show behaviour consistent with our theoretical results. Fig 5a exhibits the regret, relaxed regret and the upper bound we derive. The parameters we picked here are \( \lambda = 1, \delta = 0.01, \epsilon = 0.01 \). It is clear that the regret is well controlled under the corresponding upper bounds. Another thing worth noticing is that the logarithmic bound might be vacuous for small time horizon, as has already been observed by bandit literature. Fig 5b shows the number of time steps where the non-optimal BIS is picked. This value is relatively small, and we present the plot with log scale in the y-axis.

The comparison between DOSLB and Safe-LTS is shown in Fig 6. DOSLB uses an optimistic construction of confidence set, thus enjoys a higher efficacy with a slight deficiency in the safety score. Another interesting observation is that Safe-LTS bears a much larger variance, due to the sampling nature of the method.
In Fig 7, we show an example where Safe-LTS almost fails, while DOSLB’s performance is still consistent. The only change from the above example is to set $\alpha = 0.1$ instead of 1.1. Fig 8 further shows that the time when non-optimal BIS is picked by Safe-LTS is almost linear, and hence its regret in terms of both efficacy and safety are almost linear. To be specific, Safe-LTS plays close to $(0, 0)$ almost all the time. Our conjecture is that the feasible set is relatively small, and the conservative construction of permissible set restricts Safe LTS from exploring the space thoroughly, hence it can only play a very safe action $(0, 0)$ and its noisy associated versions.