2D supergravity in \( p+1 \) dimensions

Henrik Gustafsson\(^a\) and Ulf Lindström\(^{a,b}\)

\(^{a)\} \text{Institute of Theoretical Physics, University of Stockholm}
\hspace{1em} \text{Box 6730, S-113 85 Stockholm SWEDEN}

\(^{b)\} \text{Institute of Physics, University of Oslo}
\hspace{1em} \text{Box 1048,}
\hspace{1em} \text{N-0316 Blindern, Oslo, NORWAY}

ABSTRACT

We describe new \( N \)-extended 2D supergravities on a \((p+1)\)-dimensional (bosonic) space. The fundamental objects are moving frame densities that equip each \((p+1)\)-dimensional point with a 2D "tangent space". The theory is presented in a \([p+1,2]\) superspace. For the special case of \( p = 1 \) we recover the 2D supergravities in an unusual form. The formalism has been developed with applications to the string-parton picture of \( D \)-branes at strong coupling in mind.

\(^{1}\text{e-mail: henrik@physto.se}

\(^{2}\text{e-mail: ul@physto.se}
1 Introduction

Diffeomorphism invariant models where the “gravity” fields $e_{\underline{a}}^{\underline{m}}$ may be non-invertible moving frames arise in several different contexts. One example is the Chern-Simons description of $3D$ gravity discussed by Witten [1] and others [2]. In this paper we are concerned with the case when $e_{\underline{a}}^{\underline{m}}$ relate the $(p + 1)$-dimensional space-time manifold coordinatized by $\xi^{\underline{m}}$ to a lower $d$-dimensional “tangent space”. Such a situation arises in the description of the tensionless, $T \to 0$ limit of the fundamental bosonic [3], supersymmetric [4], and spinning string [5]. It has also been discussed more recently in the context of a strong coupling limit of $D$-branes [6].

The $T \to 0$ limit of the spinning string has been given a superspace description in terms of a “null” superspace [7] where a $2D$ supergravity based on non-invertible $e_{\underline{a}}^{\underline{m}}$’s is introduced. This corresponds to $p = 1$ and $d = 1$ above. For the limit of $D$-branes the relevant bosonic dimensions are $p + 1$ and $d = 2$, and it is this case which shall concern us below.

Using the bosonic description in [6] as our starting point, we construct superspace supergravities based on a superspace with $p + 1$ bosonic and $2N$ fermionic coordinates. The basic fields transform as densities and the space-time field content of the superfields is reduced via constraints. These constraints take a form which is unfamiliar from the usual $2D$ supergravity point of view, but one which generalizes that used in [7]. Following standard superspace supergravity procedures, we use a Wess-Zumino gauge to display the physical content of the model. In this gauge we solve the Bianchi identities and find the component relations that determine the vector derivative components in terms of the spinor derivative ones. The component transformations are found from the superspace ones, both for the supergravity fields and for scalar matter fields. Finally, as examples, we present $\sigma$-models based on this supergravity for certain $N$. In fact, the requirements on the Lagrangian limit the number of supersymmetries to $N = 1, 2$, if the full su-

To discriminate between $(p, q)$ superspace, $(p$ left movers and $q$ right movers), and our superspace, we denote the latter by $[p + 1, 2]$. 

1
perspace measure is used. Some of these models are expected to be relevant for supersymmetrization of the models that describe the strong coupling limit of $D$-branes.

The plan of the paper is as follows: Section 2 contains the basic definitions of our supergravities. In Section 3 we present the component relations that follow from solving the Bianchi identities and in Section 4 we derive the transformations. The discussion is exhaustive for $N = 1$. For higher $N$ we give the lower components. Section 6 contains the $\sigma$-model actions in superspace as well as in components, and Section 7 contains our conclusions. We have collected some useful superspace relations used in our derivations in an Appendix where we also explain our conventions.

2 Basics

In this section we define the new $[p+1,2]$ superspace supergravity. It should be compared to the standard $N = 1, 2D$ superspace supergravity as described in, e.g., [8] or [9], and to [10, 11] for higher $N$.

The fundamental supergravity objects are

$$\nabla_{i\pm} = E_{i\pm}^{m} \partial_{m} + E_{i\pm}^{j+} \partial_{j+} + E_{i\pm}^{j-} \partial_{j-} + \omega_{i\pm} M \equiv E_{i\pm}^{M} \partial_{M} + \omega_{i\pm} M,$$

(1)

where $\xi_{m}$, $m = 0, \ldots, p$ are bosonic coordinates, $\theta^{\pm}$, $i = 1, \ldots, N$ are fermionic coordinates, $\partial_{m} \equiv \partial/\partial \xi_{m}$, $\partial_{i\pm} \equiv \partial/\partial \theta^{i\pm}$, $M$ is the 2D Lorentz generator and $M \in \{m, i+, i-\}$. Occasionally we will also use the “tangent space” indices $A \in \{+, =, i+, i-\}$. The operators in (1) obey the constraints

$$\{\nabla_{i+}, \nabla_{j-}\} + \Gamma_{(i+\nabla_{j-})} = \delta_{ij} RM,$$

$$\{\nabla_{i\pm}, \nabla_{j\pm}\} + \Gamma_{(i\pm\nabla_{j\pm})} = \pm 2i\delta_{ij} \nabla_{\pm},$$

(2)
where $R$ is a curvature superfield. These constraints define the vector derivatives
\[
\nabla_\pm \equiv e_\pm^m \partial_m + \chi^i_\pm \partial_i^+ + \chi^i_- \partial_i^- + \omega^i_\pm M, \tag{3}
\]
and integration by parts leads to the relations
\[
(1 - N)(\nabla_{i+} \Gamma_{j-} + \nabla_{j+} \Gamma_{i-}) = \pm i \delta_{ij} \left(1 \cdot \frac{\nabla_{\pm}}{2}\right),
\]
\[
\nabla_{i+} \Gamma_{j-} + \nabla_{j-} \Gamma_{i+} = 0. \tag{4}
\]
The “connection” terms are given by
\[
\Gamma_{i\pm} \equiv \frac{1}{(3-2N)} \left( \partial_m E_{i\pm}^m + \partial_j E_{i\pm}^j + \partial_j - E_{i\pm}^j \pm \frac{1}{2} \omega_{i\pm} \right) \equiv \frac{1}{(3-2N)} \left(1 \cdot \frac{\nabla_{i\pm}}{2}\right). \tag{5}
\]
All fields are superfields and depend $\xi^m, \theta^i$ and $\theta^i$. The $\theta$’s transform as (weight $-\frac{1}{4}$) densities under $\xi$ diffeomorphisms. Diffeomorphisms, $(\sigma^m)$, Supersymmetry, $(\epsilon^i)$ and Lorentz, $(\Lambda)$, transformations are coded into the superfield $K$ defined by
\[
K \equiv \partial_m \sigma^m + \epsilon^i \partial_i^+ + \epsilon^i_- \partial_i^- + \Lambda M, \tag{6}
\]
and the transformations of the derivatives in (5) are given by
\[
\delta \nabla_{i\pm} = [\nabla_{i\pm}, K] + \frac{1}{2(N-2)} (1 \cdot \frac{\hat{K}}{2}) \nabla_{i\pm}, \tag{7}
\]
\[
\delta \nabla_{\pm} = [\nabla_{\pm}, K] + \frac{1}{(N-2)} (1 \cdot \frac{\hat{K}}{2}) \nabla_{\pm}. \tag{8}
\]
where
\[
(1 \cdot \frac{\hat{K}}{2}) \equiv \partial_m \sigma^m - \partial_{i+} \epsilon^i + \partial_{i-} \epsilon^i. \tag{9}
\]
These lead to the appropriate transformations for densities of weights $\frac{1}{4}$ and $\frac{1}{2}$ respectively. For $N = 2$ we have to constrain the transformations to be supervolume preserving, i.e. $\left(1 \cdot \frac{\hat{K}}{2}\right) = 0$. The relations (5), (7) and (8) were
\footnote{One is inclined to call the corresponding terms in (3) torsion terms, but the density character of the $\nabla$’s makes “connections” more appropriate.}
found by allowing arbitrary coefficients for the density terms, and matching
the resulting component expressions to the bosonic case. Finally we mention
that the $\Gamma_\pm$'s could be absorbed into fully covariant derivatives that feel the
density character of the objects they act on. We do that for $N = 2$ in Section
5.

3 Components

In this section we find the full component content of the $N = 1$ theory and
the first few components for higher $N$.

To display the physical content of the theory it is convenient to work in
a Wess-Zumino (WZ) gauge which we define as follows:

$$\nabla_i \alpha| = \partial_i \alpha,$$

$$[\nabla_i \alpha, \nabla_j \beta] + \Gamma_{i\alpha} \nabla_j \beta| - \Gamma_{j\beta} \nabla_i \alpha| = 0, \quad \alpha, \beta \in \{+, -\}, \quad (10)$$

where $|$ denotes “the $\theta$-independent part of”.

We define components by projection and use the same notation for the
supergravity superfields and their lowest components:

$$e_{\pm} \equiv e_{\pm}|,$$

$$\chi_{\pm}^{i\alpha} \equiv \chi_{\pm}^{i\alpha}|,$$

$$\omega_{\pm} \equiv \omega_{\pm}|.$$

$$R \equiv R|,$$

$$\rho_{i \pm} \equiv \nabla_{i \pm} R|.$$

$$ (11)$$

From (12) we obtain the relations

$$\pm i N E_{\pm}^M = E_{i \pm}^N \left( \partial_N E_{i \pm}^M \right) + \frac{1}{(3-2N)} \left( \partial_N E_{i \pm}^N \right) E_{i \pm}^M$$

$$\pm \left( \frac{1}{2} + \frac{1}{(3-2N)} \right) \omega_{i \pm} E_{i \pm}^M,$$

$$\pm i \omega_{\pm} = E_{i \pm}^N \partial_N \omega_{i \pm} + \frac{1}{(3-2N)} \left( \partial_N E_{i \pm}^N \right) \omega_{i \pm}. \quad (12)$$

Using (12) and additional relations that follow from (12) and (10) in conjunction
with the Bianchi identities we obtain relations for the components (in
WZ-gauge). The lowest components of the vector derivative are determined
in terms of the spinor components:

$$\partial_{i \pm} E_{j \pm}^{l \pm} = \pm i \delta_{ij} \chi_{\pm}^{l \pm} - \Gamma_{i \pm} \delta_{l j}.$$
\[ \partial_{\pm} E_{j \pm} = \pm i \delta_{ij} \chi_\pm^{\pm}, \]
\[ \partial_{\pm} E_{j \pm}^{\pm} = 0, \]
\[ \partial_{\pm} E_{j \pm}^{\mp} = -\Gamma_{i \pm} |\delta_j^l, \]
\[ \partial_{\pm} E_{j \pm}^{m} = \pm i \delta_{ij} e_\pm^{\pm}, \]
\[ \partial_{\pm} E_{j \pm}^{m} = 0, \]
\[ \partial_{\pm} \omega_{j \pm} = \pm i \delta_{ij} \omega_\pm, \]
\[ \partial_{\pm} \omega_{j \mp} = \frac{1}{2} \delta_{ij} R, \]

where
\[ \Gamma_{i \pm} = \pm \frac{i}{2 (2 - N)} |\delta_{ij} \chi_\pm^{\pm}. \]

The level \( \theta^2 \) relations for the vector derivative components relate them to lower ones:\(^5\)

\[ \partial_{\pm} e_\pm^{\pm} = \pm i \delta_{ij} \left( \frac{N - 1}{N - 2} \right) \chi_\pm^{\pm} e_\pm^{\pm}, \]
\[ \partial_{\pm} e_\pm^{\mp} = \pm i \delta_{ij} \left( \chi_\pm^{\pm} e_\pm^{\pm} - \frac{1}{(2 - N)} \chi_\pm^{\mp} e_\pm^{\mp} \right), \]
\[ \partial_{\pm} \chi_\pm^{\pm} = \pm i \left( \frac{N - 1}{N - 2} \right) \chi_\pm^{\pm} \chi_\pm^{\pm}, \]
\[ \partial_{\pm} \chi_\pm^{\pm} = \delta_i^j \left( \frac{1}{2} \partial_{me_\pm^{\pm}} = \pm i \chi_\pm^{k \mp} \chi_\pm^{k \mp} \right), \]
\[ \partial_{\pm} \chi_\pm^{\pm} = -\frac{i}{2N} \delta_i^j R \mp \frac{i}{(2 - N)} \chi_\pm^{\pm} \chi_\pm^{\pm} \mp i \chi_\pm^{\mp} \chi_\pm^{\mp}, \]
\[ \partial_{\pm} \chi_\pm^{\pm} = \frac{i}{2} \chi_\pm^{\mp} R \pm i \left( \frac{N - 1}{N - 2} \right) \chi_\pm^{\pm} \chi_\pm^{\pm}, \]
\[ \partial_{\pm} \chi_\pm^{\pm} = \left( \frac{2 + N(N - 2)}{2N(N - 2)} \right) \chi_\pm^{\mp} R \pm \frac{i}{(2 - N)} \chi_\pm^{\mp} \chi_\pm^{\mp} \]
\[ \mp i \chi_\pm^{\mp} \chi_\pm^{\mp} \pm \frac{i}{N} p_{\mp}. \]

The level \( \theta^2 \) spinor derivative components cannot all be determined for \( N > 1 \). For \( N = 1 \) we find:

\[ \partial_{\pm} \partial_{\pm} E_\pm^{\pm} = \frac{1}{2} \chi_\pm^{\pm} - \chi_\pm^{\mp} - \frac{1}{2} \chi_\pm^{\mp} - \chi_\pm^{\pm}, \]

\(^5\)In spite of their seemingly divergent character for \( N = 2 \), these relations are applicable for that case too, provided one sets \( \chi_\pm^{\mp} = \chi_\pm^{\pm} = 0 \), see below.
\[
\begin{align*}
\partial_+ \partial_- E^\mp_\pm &= \mp \frac{i}{2} \partial_m e^m_{\pm} \mp 1 \chi^\mp_\pm \chi^\mp_\pm, \\
\partial_+ \partial_- E^\mp_\pm &= \mp \chi^\pm_\pm e^\pm_\pm \mp 1 \chi^\pm_\pm \chi^\pm_\pm, \\
\partial_+ \partial_- \omega_\pm &= \mp \frac{1}{4} \chi^\pm_\pm R \mp \frac{1}{2} \chi^\pm_\pm \omega_\mp \mp \chi^\pm_\pm \omega^\pm_\pm \pm \rho_{\pm}. \tag{16}
\end{align*}
\]

Using this result and the equations (13) gives the level \(\theta\) relations for \(\Gamma_\pm\),

\[
\begin{align*}
\partial_\pm \Gamma_\pm &= \pm \frac{i}{2} \left( \partial_m e^m_\pm \mp \omega_\pm \mp \chi^\mp_\pm \chi^\mp_\pm \right), \\
\partial_\pm \Gamma_\mp &= \mp \frac{1}{4} R \mp \frac{1}{2} \chi^\pm_\pm \omega^\pm_\pm \mp \frac{1}{2} \chi^\pm_\pm \omega^\pm_\pm \pm \rho_{\pm}. \tag{17}
\end{align*}
\]

The relations (15) and (16) were determined using the lowest dimension Bianchi identities. Applying the Bianchi identity to \([\nabla_+, \nabla_-]\) for \(N = 1\) confirms these relations and leads to the constraint

\[
\partial_m \left( e^m_{[-+]} \right) = 0. \tag{18}
\]

and an expression for the \(\theta^2\) component of \(R\),

\[
\frac{1}{4} \partial_+ \partial_- R = -\frac{1}{2} e^m_{[\pm]} \partial_m \omega_\pm + \omega_\pm \omega_\pm + \Gamma_{\pm \rho_{\pm}}. \tag{19}
\]

Note that \(\rho_{\pm}\) is an independent field. For higher \(N\) the constraint (18) is still valid but new relations for the higher components of \(R\) are found. In particular, \(\rho_{i\pm}\) is not independent for \(N > 1\).

The \(\theta^2\) components of the vector derivative can be related to lower ones using (2), (13) and (16). For \(N = 1\) these are all the components. For higher \(N\), the constraints lead to additional relations between higher \(\theta\) components, which we omit.

We shall also need the first few components of a scalar superfield \(X\) (in WZ-gauge),

\[
\begin{align*}
X &\equiv X|, \quad \Psi_{i\pm} \equiv \partial_{i\pm} X|, \\
\mathcal{F}_{i\pm j\pm} &\equiv \partial_{i\pm} \partial_{j\pm} X|, \quad \mathcal{F}_{i\pm j\mp} \equiv \partial_{i\pm} \partial_{j\mp} X|. \tag{20}
\end{align*}
\]

Note that the density character of \(\theta\) leads to \(\Psi\) and \(\mathcal{F}\) being densities.
4 Transformations

In this section we present the transformations of the component fields in WZ-gauge. To stay in this gauge the transformation superfield $K$ must fulfil

$$0 = \delta \nabla = \left[ \nabla, K \right] + \frac{1}{2(2 - N)} (1 \cdot \tilde{K}) \nabla.$$

This constrains the various transformation parameters in (3), and leads to the following component relations for $K$:

$$\left( 1 \cdot \tilde{K} \right) = \frac{1}{2} (2 - N) \partial_{\underline{m}} \sigma_{\underline{m}} + i(N - 1) \left( \epsilon^{i+} \chi_{++}^{i+} - \epsilon^{-} \chi_{-}^{i-} \right)$$

$$\nabla_i \chi = \pm \frac{1}{2} e^{i+} \nabla_{\pm} + \frac{1}{2} e^{i+} R M$$

$$+ \frac{1}{2} \left( i \epsilon^{+} \chi_{-}^{i-} - \epsilon^{-} \chi_{++}^{i+} \pm \Lambda + \frac{1}{2} \partial_{\underline{m}} \sigma_{\underline{m}} \right) \partial_{\pm}.$$

In particular, for $N = 2$, where $(1 \cdot \tilde{K}) = 0$, we find $\chi_{++}^{+} = \chi_{-}^{-} = 0$.

Under $(p + 1)$-dimensional diffeomorphisms the components transform as specified by their density weights, and under Lorentz transformations according to their Lorentz charge. The local supersymmetry transformations of the supergravity fields are found from (3) using (22), (A.1), (A.2) and the component relations.

They are:

$$\delta e_{\pm} = \mp i \epsilon^{\pm} \chi_{\pm}^{\pm} e_{\pm} \mp i \epsilon^{\mp} (2 \chi_{\pm}^{\pm} e_{\pm} - \chi_{\mp}^{\mp} e_{\pm})$$

$$\delta \chi_{\pm}^{\pm} = \partial_{\pm} \epsilon^{\pm} - \epsilon^{\mp} \left( \frac{1}{2} \partial_{\pm} e_{\pm} \pm i \chi_{\pm}^{\mp} \chi_{\pm}^{\pm} \pm \omega_{\pm} \right) + \frac{3}{2} \epsilon^{\mp} \chi_{\pm}^{\pm} \chi_{\pm}^{\mp}$$

$$+ \frac{1}{2} \left( i \epsilon^{+} \chi_{-}^{i-} - \epsilon^{-} \chi_{++}^{i+} \mp \Lambda + \frac{1}{2} \partial_{\underline{m}} \sigma_{\underline{m}} \right) \partial_{\pm}.$$

$$\delta \chi_{\pm}^{\mp} = \partial_{\pm} \epsilon^{\mp} - \epsilon^{\pm} \left( \frac{1}{2} \partial_{\pm} e_{\pm} \pm i \chi_{\pm}^{\pm} \chi_{\pm}^{\pm} \right) + \frac{3}{2} \epsilon^{\pm} \chi_{\pm}^{\pm} \chi_{\pm}^{\pm}$$

$$+ \frac{3}{2} \epsilon^{\mp} \chi_{\pm}^{\pm} \chi_{\pm}^{\pm}$$

$$\delta \omega_{\pm} = -\epsilon^{\pm} \left( \pm i \chi_{\mp}^{\mp} \omega_{\pm} + \chi_{\pm}^{\mp} R \right) - \frac{1}{2} \partial_{\pm} \omega_{\pm} \mp \frac{1}{2} \rho_{\pm}$$

$$\delta R = -\epsilon^{+} \rho_{++} - \epsilon^{-} \rho_{--}.$$
\[ \delta \chi^a_{\alpha} = \nabla^a e^\alpha - \frac{i}{2N} R \left( \bar{e}^i \gamma_a \right)^\alpha + 2i \left( \bar{e} \gamma^b \chi^a \right) \chi^i_{\alpha} \]
\[ - \frac{i}{2} \left( \bar{e} \gamma^b \chi^a \right) \chi^i_{\alpha} - ie^{i\alpha} \left( \bar{e} \gamma^b \chi^a \right), \]
\[ \delta \omega_a = \frac{i}{2N} \left( \bar{e} \gamma_a \gamma^5 \rho \right) \right. + i \left( \bar{e} \gamma_a \chi^i \right) \omega_a + i \left( \bar{e} \gamma_a \chi^i \right) \omega_a e^{bc} \]
\[ + \frac{1}{2N(N-2)} \left( \bar{e} \gamma_a \chi^i \right) R - \frac{1+iN(N-2)}{N(N-2)} \left( \bar{e} \gamma^5 \chi_a \right) R, \]
\[ \delta R = - \bar{e} \rho, \quad (24) \]

where
\[ \nabla^a e^\alpha = e^m_a \left( \partial^m e^\alpha + \frac{1}{4} \Gamma^m_{mn} e^\alpha + \omega^m_e e^\alpha \right), \quad (25) \]

with \( \omega_a = e^m_a \omega^m_a \) the full spin-connection, including torsion, and \( \Gamma^p_{mn} \) the \((p+1)\)-dimensional connection. To obtain the supercovariant form of the \( \chi^a_{\alpha} \) transformation in (24) we have employed a generalized metricity condition on \( e^m_a \),
\[ \nabla_m e^m_a = \partial_m e^m_a + \Gamma^m_{mn} e^m_a - \frac{1}{2} \Gamma^m_{mn} e^m_a + \omega^m e^m_a = 0, \quad (26) \]

which for \( p = 1 \) is equivalent to the ordinary condition \( \nabla_m e^m_a = 0 \).

The matter field transformations are
\[ \delta X = -e^{+i} \Psi_{i+} - e^{-i} \Psi_{i-}, \]
\[ \delta \Psi_{i\pm} = \mp ie^{\pm} \partial^i X - e^{+j} \mathcal{F}_{j+i\pm} - e^{-j} \mathcal{F}_{j-i\pm} \]
\[ + \frac{i}{2} \left( e^{j+} \chi^+_{j+} - e^{-j} \chi^+_{j-} \right) \Psi_{i\pm} \]
\[ \mp ie^{\pm} \chi^\pm_{j\pm} \Psi_{j\pm} \mp ie^{\pm} \chi^\pm_{j\pm} \Psi_{j\pm}. \quad (27) \]

The covariant form of these transformations read
\[ \delta X = -\bar{e} \Psi, \]
\[ \delta \Psi^\alpha = i \left( \gamma^m \right)^\alpha \partial_m X + i \left( \bar{e} \gamma^i \right) \left( \bar{\chi}^a \Psi \right) \]
\[ - \frac{i}{2} \left( \bar{e} \gamma^b \chi^a \right) \Psi^\alpha - e^{\beta} \mathcal{F}_{\beta j,\alpha}. \quad (28) \]

We will not need \( \delta \mathcal{F} \) in general. For \( N = 1 \) it is
\[ \delta \mathcal{F} = -ie^{-} \partial^+ \Psi_{+} - ie^{+} \partial^+ \Psi_{-} - \lambda^+ \Psi_{+} - \lambda^- \Psi_{-} - l^m \partial^m X \]
\[ = i\bar{e} \gamma^m \partial^m X - \lambda \Psi - l^m \partial^m X, \quad (29) \]

where \( \lambda^\pm \equiv \partial^+ \partial^- \epsilon^\pm \) and \( l^m \equiv \partial^+ \partial^- \sigma^m \).

Using the transformations in (24) we verify that the \( e_m^a \) constraint (18) is supersymmetric.
5 Actions

In this section we discuss actions coupling the supergravity to matter fields. Since the full superspace measure has weight $N/2$, a Lagrangian has to have weight $1 - N/2$. For $N > 2$ this becomes negative which cannot be achieved using a weight 0 scalar field and the available operators (which have positive weight). In this case one has to resort to integration over invariant subspaces, using techniques described in, e.g., [12]. We will not treat those cases, and hence the discussion below is restricted to $N = 1, 2$.

A general locally supersymmetric action for $N = 1$ is

$$S = \int d^{p+1} \xi d^2 \theta \mathcal{L} (X, \nabla X).$$

(30)

To evaluate the component version of $S$, we need the following relation:

$$\int d^{p+1} \xi d^2 \theta \mathcal{L} = \int d^{p+1} \xi \left( \nabla_+ \nabla_- + \frac{1}{2} \chi^+ \gamma^+ \nabla_- \right) \mathcal{L} |$$

$$= - \int d^{p+1} \xi \left( \nabla_- \nabla_+ - \frac{1}{2} \chi^- \gamma^- \nabla_+ \right) \mathcal{L} |$$

(31)

A general $\sigma$-model action is

$$S = \int d^{p+1} \xi d^2 \theta \sum_{\mu, \nu} X^\mu \nabla_- X^\nu \mathcal{E}_{\mu \nu}(X),$$

(32)

where $\mu, \nu = 1, \ldots, D$ and $\mathcal{E}_{\mu \nu} = G_{\mu \nu} + B_{\mu \nu}$ is the sum of the $D$-dimensional target space metric and antisymmetric tensor field. The component version of (32) is

$$S = \frac{1}{4} \int d^{p+1} \xi \left\{ \partial_+ X \cdot \partial_- X + 2 \chi^+ \partial_+ X \cdot \Psi_+ + 2 \chi^- \partial_- X \cdot \Psi_-$$

$$+ i \Psi_+ \cdot \partial_- \Psi_+ - i \Psi_- \cdot \partial_+ \Psi_- - 2 (\Psi_+ \cdot \Psi_-) \left( \chi^+ \chi^- \right)$$

$$+ \mathcal{F} \cdot \mathcal{F} - A_m \partial_\mu \left( e_+^m e_-^m \right) \right\}$$

$$= \frac{1}{4} \int d^{p+1} \xi \left\{ \eta^{ab} e_+^m e_-^m \partial_+ \nabla_\mu X \cdot \partial_\mu X + 2 \chi_a \gamma^a \gamma^a \Psi \cdot \partial_- X$$

$$+ i \bar{\Psi} \cdot \gamma^a \partial_\mu \Psi - \frac{1}{2} \left( \bar{\Psi} \cdot \Psi \right) \left( \chi_a \gamma^a \gamma^a \right) \right\}$$

For simplicity we set $\mathcal{E}_{\mu \nu} = \eta_{\mu \nu}$. A non trivial $\mathcal{E}$ will give rise to target space curvature, connection and torsion terms of the usual $\sigma$-model type.
\[ F^{\alpha \beta} \cdot F_{\alpha \beta} - e^{a b} A_m \partial_n (e^m a c^b) \{ . \] (33)

where we have included the covariant version and taken care of the constraint
\[ e^{a b} \partial_n (e^m a c^b) = 0 \] from (18) using a Lagrange multiplier \( A_m \). The last term
is invariant under local supersymmetry provided that \( A_m \) transforms as a
singlet. To see the invariance of the action under diffeomorphisms, note that
the only field that is not a density is \( X \).

For \( N = 2 \) it is convenient to work with complex objects. We define
\[
\begin{align*}
\nabla_\pm & \equiv \nabla^1_\pm + i \nabla^2_\pm, \\
\bar{\nabla}_\pm & \equiv \nabla^1_\pm - i \nabla^2_\pm,
\end{align*}
\] (34)

and
\[
\chi^{\pm}_\alpha \equiv \chi^{1\alpha}_\pm + i \chi^{2\alpha}_\pm, \quad \bar{\chi}^{\pm}_\alpha \equiv \chi^{1\alpha}_\pm - i \chi^{2\alpha}_\pm.
\] (35)

Since for \( N = 2 \) the superdiffeomorphisms are restricted to be super-volume
preserving, \( 1 \cdot \tilde{K} = 0 \), we must put \( \chi^{\pm}_\pm = \bar{\chi}^{\pm}_\pm = 0 \). In fact this may be
viewed as a superconformal gauge choice utilizing the transformations
\[
\delta \chi^{\pm}_\pm = \eta^{\pm},
\] (36)

where \( \eta^{\pm} \) is a complex spinor parameter. To stay in this gauge we must
require that the supersymmetry transformations of \( \chi^{\pm}_\pm \) be accompanied by
compensating superconformal transformation (36).

We also introduce “hatted” derivatives
\[
\hat{\nabla}_\pm = \nabla_\pm + 4w \Gamma_\pm, \quad \hat{\nabla}_\pm = \nabla_\pm - 2w (1 \cdot \nabla_\pm)
\] (37)

where \( w \) is the density weight of the object \( \hat{\nabla} \) is acting on. The constraint
algebra then simplifies to that of ordinary \( N = 2 \) supergravity in 2D [10, 11].

Covariantly (anti-)chiral superfields \(( \Phi ) \Phi \) are defined by
\[
\hat{\nabla}_\pm \Phi = \hat{\nabla}_\pm \Phi = 0.
\] (38)

This form of the action is a direct supersymmetrization of the strong coupling limit
of the Born-Infeld action, as described in [13].
The (hatted covariant) components of the chiral multiplet are defined by
\[
\begin{align*}
\Phi| &= \varphi, \quad \hat{\nabla}_\pm \Phi| = \psi_\pm, \quad \hat{\nabla}_+ \hat{\nabla}_- \Phi| = F, \\
\bar{\Phi}| &= \bar{\varphi}, \quad \hat{\nabla}_\pm \bar{\Phi}| = \bar{\psi}_\pm, \quad \hat{\nabla}_+ \hat{\nabla}_- \bar{\Phi}| = \bar{F}.
\end{align*}
\] (39)

The (anti)chiral measure is \(d^2 \bar{\theta} \mathcal{L}_{\text{chir}} = \hat{\nabla}_+ \hat{\nabla}_- \mathcal{L}_{\text{chir}}\). For a general Lagrangian \(\mathcal{L}\) the full \(N=2\) superspace measure is
\[
\int d^{p+1} \xi d^4 \theta \mathcal{L} = \int d^{p+1} \xi d^2 \bar{\theta} \hat{\nabla}_+ \hat{\nabla}_- \mathcal{L} |_{\theta=0} = \int d^{p+1} \xi \left( \hat{\nabla}_\pm \hat{\nabla}_- + Y \right) \hat{\nabla}_+ \hat{\nabla}_- \mathcal{L} |_{\theta=0},
\] (40)
where the coefficient \(Y\) is determined below.

An action for the chiral multiplet is found by choosing \(\mathcal{L} = \bar{\Phi} \Phi\),
\[
\begin{align*}
S &= \frac{1}{8} \int d^{p+1} \xi d^4 \theta \mathcal{L} = \int d^{p+1} \xi \left\{ \right. 2 \partial_+ \varphi \partial_-=\varphi \\
&\quad + \frac{i}{4} \left( \tilde{\psi}_+ \partial_+=\psi_+ - \tilde{\psi}_- \partial_+=\psi_- - \partial_+=\tilde{\psi}_+ \psi_+ + \partial_+=\tilde{\psi}_- \psi_- \right) \\
&\quad + \tilde{\chi}_+ \psi_+ \partial_+=\varphi - \chi_+ \psi_+ \partial_+=\varphi + \chi_- \psi_- \partial_+=\varphi + \chi_- \psi_+ \partial_-=\varphi \\
&\quad - \frac{1}{2} \left( \chi_+ \tilde{\chi}_- \psi_+ \psi_- + \chi_- \tilde{\chi}_+ \psi_+ \psi_- \right) + \frac{1}{8} \tilde{F} \tilde{F} - A_{m} \partial_+ (e_+^m e_-^m) \left\} \right.
\] (41)
where we again have incorporated the constraint (18) that follows by matching the \(\partial_\pm\) coefficients in (A.4). Furthermore, in contrast to \(N=1\), matching the \(\partial_\pm \equiv \partial_1^\pm + i \partial_2^\pm\) coefficients we can solve for \(\rho_\pm\) in terms of the other fields,
\[
\frac{1}{8} \rho_\pm = \pm \partial_\pm \chi_+ \tilde{\chi}_- \pm \frac{1}{2} \chi_+ \tilde{\chi}_- (\partial_\pm e_+^m e_-^m) + \chi_+ \tilde{\chi}_- \omega_+^\pm,
\] (42)
and similarly for \(\bar{\rho}_\pm\). The \(M\) part of (A.4) gives relations for the \(\theta \bar{\theta}\)-components of \(R\),
\[
\begin{align*}
- \frac{1}{4} \hat{\nabla}_+ \hat{\nabla}_- R| - \frac{1}{4} \hat{\nabla}_+ \hat{\nabla}_- R| - \frac{1}{2} R^2 \\
= 2 e_+^m \partial_+ \omega_+^m + 4 \omega_+ \omega_- + \bar{\chi}_+ \chi_- + R - \bar{\chi}_+ \chi_- - R
\end{align*}
\] (43)
Following [14] we have determined the coefficient \(Y\),
\[
Y = 2 \chi_+ \tilde{\chi}_-, \quad (44)\]
by requiring the terms in the component action (41) containing auxiliary fields, $F$, $\bar{F}$, to be symmetric in barred and unbarred quantities.

The supersymmetry transformations for the chiral- and antichiral component fields are

$$\delta \varphi = -\frac{1}{2} \epsilon^+ \psi_+ - \frac{1}{2} \epsilon^- \psi_-,$$
$$\delta \psi_\pm = \mp i \epsilon^\pm \partial_\pm \varphi \mp \frac{1}{2} \epsilon^\pm \lambda^\pm \psi_\mp \pm \frac{1}{2} \epsilon^\mp F,$$
$$\delta F = -i \epsilon^- \partial_\pm \psi_+ - i \epsilon^+ \partial_\mp \psi_- - \frac{1}{2} \lambda^+ \psi_+ - \frac{1}{2} \lambda^- \psi_- - l^m \partial_m \varphi,$$ (45)

and

$$\delta \bar{\varphi} = -\frac{1}{2} \epsilon^+ \bar{\psi}_+ - \frac{1}{2} \epsilon^- \bar{\psi}_-,$$
$$\delta \bar{\psi}_\pm = \mp i \epsilon^\pm \bar{\partial}_\pm \bar{\varphi} \mp \frac{1}{2} \epsilon^\pm \bar{\lambda}^\pm \bar{\psi}_\mp \pm \frac{1}{2} \epsilon^\mp \bar{F},$$
$$\delta \bar{F} = -i \epsilon^- \bar{\partial}_\pm \bar{\psi}_+ - i \epsilon^+ \bar{\partial}_\mp \bar{\psi}_- - \frac{1}{2} \bar{\lambda}^+ \bar{\psi}_+ - \frac{1}{2} \bar{\lambda}^- \bar{\psi}_- - \bar{l}^m \bar{\partial}_m \bar{\varphi},$$ (46)

where $\epsilon^\pm = \epsilon^\pm_1 + i \epsilon^\pm_2$, $\lambda^\alpha = \bar{\partial}_+ \partial_- \epsilon^\alpha |$ and $l^m = \partial_+ \partial_- \sigma^m$.

A more general $N = 2$ action is

$$\int d^{p+1}x d^4\theta K(\Phi, \bar{\Phi}).$$ (47)

Here the target space geometry is determined by a single potential function $K$ leading to a restricted Kähler geometry. As is most easily seen from an analysis of the bosonic content of (32), (33), (41) and (47), the non-degenerate case $p = 1$ leads to the usual 2D supergravity-matter couplings. The relation is via field-redefinitions that reintroduce the determinant of the zweibein.

6 Discussion

We have presented $[p + 1, 2]$ supergravities for $N \in \{1, 2\}$. As mentioned, we could allow for a larger range of $N$, but then the actions have to be constructed as integrals over invariant subspaces. We may likewise extend the treatment to $N = (p, q)$ supergravities. The most direct example leads to a straightforward generalization of the $(p, 0)$ supergravities of [13, 16].
In the previous Section we mentioned that for \( p = 1 \) we recover the standard 2D supergravities via field redefinitions. We thus have a novel description of those theories. This description may sometimes be advantageous, e.g., when discussing the measure in \( N = 2 \).

We find the supergravities presented intrinsically interesting as examples of non-standard geometries, but they were developed with one particular application in mind. The \( T \to 0 \) limit of the Born-Infeld action corresponds to \( D \)-branes at very large values of the fundamental string coupling. As shown in \( \text{[3]} \), the \( D \)-brane world volume becomes foliated by string world sheets in this limit. Since the discussion in \( \text{[3]} \) is purely bosonic and the fundamental string is supersymmetric, we wanted to confirm this parton picture by supersymmetrizing the model. This is presented in \( \text{[13]} \), based on the results reported on here.

**Acknowledgements**

We are grateful to Martin Roček for comments and discussions. The work of UL was supported in part by NFR grant No. F-AA/FU 04038-312 and by NorFA grant No. 96.55.030-O.

**Appendix**

In this Appendix we collect some useful relations that were used in the derivation of the component relations in the text. It also contains our conventions.

The following (WZ-gauge) relations are needed in evaluating the component Lagrangian and in deriving the transformations:

\[
\nabla_{i} \nabla_{j,|} = \pm i \delta_{ij} \left( \chi_{\pm}^{\mp \partial_{t_{\pm}} + \chi_{\pm}^{\mp \partial_{\tau_{\mp}} + e_{\pm}^{m} \partial_{m} + \omega_{\pm}^M} \right)
\]

\[
\nabla_{i} \nabla_{j,|} = \pm \frac{i}{2(2-N)} \chi_{\pm}^{i \chi_{j,\mp}} + \partial_{i} \partial_{j,\mp},
\]

\[
\nabla_{i} \nabla_{j,|} = \pm \frac{i}{2(2-N)} \chi_{\pm}^{i \chi_{j,\mp}} + \frac{1}{2} \delta_{ij} RM + \partial_{i} \partial_{j,\mp}
\]

(A.1)
and

\[
\begin{aligned}
\nabla_\pm \nabla_{i\pm} &= \partial_\pm \partial_{i\pm} + \chi_\pm^j \nabla_j \nabla_{i\pm} \\
&\quad + \chi_\pm^j \nabla_j - \nabla_{i\pm} \pm \frac{1}{2} \Omega_\pm \partial_{i\pm},
\end{aligned}
\]

\[
\begin{aligned}
\nabla_\pm \nabla_{i\mp} &= \partial_\pm \partial_{i\mp} + \chi_\pm^j \nabla_j \nabla_{i\mp} \\
&\quad + \chi_\pm^j \nabla_j - \nabla_{i\mp} \mp \frac{1}{2} \Omega_\pm \partial_{i\mp}.
\end{aligned}
\] (A.2)

Independent of the gauge, we have the following commutation relations

\[
\begin{aligned}
\left[ \nabla_{i\pm} , \nabla_\mp \right] &= -2 \Gamma_{i\pm} \nabla_\pm \pm \frac{i}{N} (\nabla_j \Gamma_j) \nabla_{i\pm} \\
&\quad - \frac{i}{2N} R \nabla_{i\mp} \mp (\nabla_{i\mp} R + \Gamma_{i\mp} R) M,
\end{aligned}
\]

\[
\begin{aligned}
\left[ \nabla_{i\pm} , \nabla_\mp \right] &= -2 \Gamma_{i\pm} \nabla_\mp \pm \frac{i}{N} (\nabla_j \Gamma_j) \nabla_{i\pm} \\
&\quad + \frac{i}{N} (\nabla_\mp R + \Gamma_\mp R) \nabla_{i\pm}.
\end{aligned}
\] (A.3)

and

\[
\begin{aligned}
\left[ \nabla_+ , \nabla_- \right] &= \left\{ - \frac{R}{2N^2} \Gamma_+ + \frac{i}{N} (\nabla_+ \Gamma_+ \nabla_-) + \frac{1}{N} \Gamma_+ (\nabla_- \Gamma_+) \\
&\quad + \frac{i}{N} (\nabla_+ \Gamma_+) \nabla_- + \frac{1}{N} (\nabla_- \Gamma_+ \nabla_+) + \frac{1}{N} \Gamma_+ (\nabla_- \Gamma_+) \nabla_+ \\
&\quad + \frac{i}{N} (\nabla_+ \Gamma_+) \nabla_- + \frac{2i}{N} (\nabla_- \Gamma_+ \nabla_+) \nabla_- \\
&\quad + \frac{1}{N} \left\{ \frac{5}{2} \Gamma_+ \nabla_+ R + 6 \Gamma_+ \Gamma_+ R + R \nabla_- \Gamma_+ \right\} \\
&\quad - \frac{1}{2} R^2 + \frac{1}{2} \left[ \nabla_- , \nabla_+ \right] R \right\} M.
\end{aligned}
\] (A.4)

Spinors in a \((p + 1)\)-dimensional space-time with non-invertible moving frames \(e^a_\pm\) of rank \(d_a = 0, \ldots, d_1 = 1, m = 0, \ldots, p\) are introduced by prescribing the Clifford algebra

\[
\{ \gamma^a, \gamma^b \} = 2 \eta^{ab} \quad \Rightarrow \quad \{ \gamma^m, \gamma^n \} = 2 g^{mn} \equiv e^m_a e^n_b \eta^{ab},
\] (A.5)

where \(\gamma^m \equiv e^m_a \gamma^a\) and \(\eta^{ab} = (-1, 1)\). For our case, \(d = 2\), we use a real representation for the gamma matrices, \((\gamma_2)^\alpha_\beta = (i\sigma^2, \sigma^1)\) and \((\gamma_5)^\alpha_\beta = (\sigma^3)\).

The spinor indices are raised and lowered by \(C_{\alpha\beta} = C^{\alpha\beta} = i\sigma^2\), according to
\( \chi^\alpha = C^{\alpha\beta} \chi_\beta \) and \( \chi_\alpha = \chi^\beta C_{\beta\alpha} \). Since the 2D Lorentz group is \( SO(1,1) \), which has only 1-dimensional representations, it is convenient to work in a basis of helicity eigenstates. Then a spinor index \( \alpha \) takes the values \{\(+,−\)\} (helicity \( \pm \frac{1}{2} \)), and a (tangent) vector index \( a \) takes the values \{\(+,=\)\} (helicity \( \pm 1 \)) and are equivalent to light-cone components: \( (v^a \gamma_a)^\pm = \pm v_\pm \). The Lorentz generator \( M \) act on spinors and vectors as

\[
[M, \chi_\pm] = \pm \frac{1}{2} \chi_\pm, \quad [M, v_\pm] = \pm v_\pm.
\]
References

[1] E. Witten, *Nucl. Phys.* B311 (1988) 46

[2] J. H. Horne and E. Witten, *Phys. Rev. Lett.* 62 (1989) 501; U. Lindström and M. Roček *Phys. Rev. Lett.* 62 (1989) 2905.

[3] A. Karlhede and U. Lindström, *Class. Quantum Grav.* 3 (1986) L73.

[4] U. Lindström, B. Sundborg and G. Theodoridis *Phys. Lett.* 253B (1991) 319.

[5] U. Lindström, B. Sundborg and G. Theodoridis *Phys. Lett.* 258B (1991) 331.

[6] U. Lindström and R. von Unge, *Phys. Lett.* B403 (1997) 233.

[7] U. Lindström and M. Roček, *Phys. Lett.* 271B (1991) 79.

[8] S. J. Gates Jr. and H. Nishino, *Class. Quantum Grav.* 3 (1986) 391.

[9] M. Roček, P. van Nieuwenhuizen and S. C. Zhang, *Annals of Phys.* 172 (1985) 348.

[10] P. S. Howe and G. Papadopoulos, *Class. Quantum Grav.* 4 (1987) 11.

[11] S. J. Gates Jr., L. Liu and N. Oerter, *Phys. Lett.* 218B (1989) 33.

[12] U. Lindström and M. Roček, *Commun. Math. Phys.* 128 (1990) 191.

[13] H. Gustafsson and U. Lindström, “A Picture of D-branes at strong coupling II. Spinning Partons.” University of Stockholm preprint USITP-98-13, (1998), hep-th/9807064.

[14] M. T. Grisaru and M. E. Wehlau, *Nucl. Phys.* B457 (1995) 219.

[15] R. Brooks, F. Muhammad and S. J. Gates Jr., *Nucl. Phys.* B268 (1986) 599.
[16] M. Evans and B. A. Ovrut, *Phys. Lett.* **186B** (1987) 134.