WE WAVELETS AND TRIEBEL TYPE OSCILLATION SPACES

PENGTAO LI, QIXIANG YANG, AND BENTUO ZHENG

Abstract. We apply wavelets to identify the Triebel type oscillation spaces with the known Triebel-Lizorkin-Morrey spaces \( \dot{F}_{p,q}^{\gamma_1,\gamma_2}(\mathbb{R}^n) \). Then we establish a characterization of \( \dot{F}_{p,q}^{\gamma_1,\gamma_2}(\mathbb{R}^n) \) via the fractional heat semigroup. Moreover, we prove the continuity of Calderón-Zygmund operators on these spaces. The results of this paper also provide necessary tools for the study of well-posedness of Navier-Stokes equations.

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1. Introduction

We state briefly the history of Triebel-Lizorkin spaces and their Morrey type generalization. Triebel-Lizorkin spaces $\dot{F}^s_{p,q}(\mathbb{R}^n)$ were first introduced by H. Triebel and can be seen as generalizations of many standard function spaces such as Lebesgue spaces $L^p$ and Sobolev spaces. In the research of harmonic analysis and partial differential equations, Triebel-Lizorkin spaces play an important role. In recent decades, $\dot{F}^s_{p,q}(\mathbb{R}^n)$ have attracted great attention of many mathematicians, and a lot of work has been done. We refer the readers to Triebel [26, 27] for an overview of Triebel-Lizorkin spaces and their applications.

D. Yang and his collaborators are pioneers on the study of Triebel type Morrey spaces. By applying Hausdorff capacity and Littlewood-Paley theory, Yang-Yuan [32] introduced a new class of function spaces $\dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n)$ with $p \in (0, \infty)$ which generalize many classical function spaces. For example, $\dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n) = \dot{F}^{s}_{p,q}(\mathbb{R}^n)$, where $Q_\alpha(\mathbb{R}^n)$ are the spaces introduced by Essén-Janson-Peng-Xiao [9]. For more information, we refer to Yuan-Sickel-Yang [34].

Our aim is to study a class of mean oscillation spaces with Triebel-Lizorkin norm by wavelets and semigroup. In this paper, the Triebel type oscillation spaces $\dot{F}^{\gamma_1,\gamma_2}_{p,q}(\mathbb{R}^n)$ are defined as

$$\sup_Q |Q|^{\frac{1}{p}} \inf_{P \in \mathcal{S}_{p,q,f}} \|\varphi_Q(f - P_Q f)\|_{\dot{F}^{\gamma_1,\gamma_2}_{p,q}} < +\infty,$$

where the supremum is taken over all cubes $Q$ and $\mathcal{S}_{p,q,f}$ denotes the set of all polynomials satisfying certain conditions. Details can be found in Definition 2.2. In Theorem 2.3, we give a wavelet characterization of these spaces. As a consequence, $\dot{F}^{\gamma_1,\gamma_2}_{p,q}(\mathbb{R}^n)$ coincide with $\dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n)$ introduced by Yang-Yuan [32]. Theorem 2.3 implies that Calderón-Zygmund operators are bounded on $\dot{F}^{\gamma_1,\gamma_2}_{p,q}(\mathbb{R}^n)$. Moreover, our wavelet characterization is independent of the choice of wavelet bases. See Corollary 2.5 and Theorem 2.9 respectively.

It is well-known that for any $f \in BMO(\mathbb{R}^n)$, the Poisson integral $P_t(f)$ gives a harmonic extension of $f$ to the tent space on $\mathbb{R}^{n+1}_+$. This result gives a relation between function spaces on $\mathbb{R}^n$ and the ones on $\mathbb{R}^{n+1}_+$. The well-posedness of fluid equations needs often such characterizations of function spaces. In 2001, Koch-Tataru [11] obtained a semigroup characterizations of $BMO(\mathbb{R}^n)$. In 2007, by Hausdorff capacity, Xiao [40] gave a semigroup characterization of $Q_\alpha(\mathbb{R}^n)$. Li-Zhai [14] further developed the idea of [11, 40] and obtained a semigroup characterization of $Q_\alpha^0(\mathbb{R}^n)$. We refer the readers to Cannone [5, 6], Li-Xiao-Yang [13], Lin-Yang [17] and Miao-Yuan-Zhang [20] for further information.
In Section 3, we introduce tent type spaces $F_{p,q,m,m'}^γ$ and $F_{r,∞}$ defined on $\mathbb{R}^{n+1}$ and study some properties of these spaces. In Section 4, via fractional heat semigroup, we establish a relation between the functions in $F_{p,q}^γ(\mathbb{R}^n)$ and $F_{p,q}^γ$. Furthermore, the index $n$ is defined as follows:

$$F_{p,q,m,m'}^γ = F_{p,q,m}^γ \cap F_{p,q}^γ \cap F_{p,q,m}^γ \cap F_{p,q,m'}^γ$$

$$= X_1 \cap X_2 \cap X_2 \cap X_4.$$

Actually, Theorem 4.1 is not a simple generalization of the results in [11, 14, 30]. In the above mentioned spaces, $BMO$, $Q_q$ and $Q'_q$ are all $F_{p,q}^γ(\mathbb{R}^n)$ spaces with $p = q = 2$. For the cases $p \neq q$ with $p, q \neq 2$, the Fourier transform is not valid. To overcome this difficulty, we apply a new method. Let $\{\Phi_{j,k}^γ(x)\}_{j,k} \in \Lambda_n$ be a wavelet basis. Let $Q$ be any cube and

$$f(x) = \sum_{(j,k) \in \Lambda_n} a_{j,k}^γ \Phi_{j,k}^γ(x) \in F_{p,q}^γ(\mathbb{R}^n).$$

Based on the relation between $j$ and the radius of $Q$, we decompose the function $F(x, t) = e^{i\Lambda^β} f(x)$ into several parts such that every part belongs to some $X_i$. Such decomposition reflects the local structures of the space and the frequency very well. A semigroup characterization of $F_{p,q}^γ(\mathbb{R}^n)$ can be obtained easily.

Our characterization has a distinct advantage when we apply it to the well-posedness of fluid equations. Roughly speaking, for $F(x, t) \in F_{p,q,m,m'}^γ$, the four parts of $\|F\|_{F_{p,q,m,m'}^γ}$ have different meanings:

- the norms $\|F\|_{X_1}$ and $\|F\|_{X_2}$ denote the $L^∞$-parts of $F(x, t)$,
- the norms $\|F\|_{X_3}$ and $\|F\|_{X_4}$ denote the $L^p$-parts of $F(x, t)$.

Furthermore, the index $m$ represents the regularities for the variable $x$. Compared with the results in [11, 14, 30], if $m$ becomes bigger, the elements in $F_{p,q,m,m'}^γ$ have higher regularities. Moreover, Riesz operators are continuous on $F_{p,q,m,m'}^γ$. We will also use such characterization to study the well-posedness of Navier-Stokes equations in another paper.

The rest of this paper is organized as follows. In Section 2, we present some preliminary knowledge, notations and terminology. Then we give a wavelet characterization of $F_{p,q}^γ(\mathbb{R}^n)$ and prove Calderón-Zygmund operators are bounded on $F_{p,q}^γ(\mathbb{R}^n)$. In Section 3, we introduce Triebel type tent spaces. In the last section, we establish first a relation between $F_{p,q}^γ(\mathbb{R}^n)$ and $F_{p,q,m,m'}^γ$. Then, we prove the continuity of Riesz operators on $F_{p,q,m,m'}^γ$. 

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2. Triebel type oscillation spaces $F^{s_1,s_2}_{p,q}$

In this paper, the symbols $\mathbb{Z}$ and $\mathbb{N}$ denote the sets of all integers and natural numbers, respectively. For $n \in \mathbb{N}$, $\mathbb{R}^n$ is the $n$-dimensional Euclidean space, with Euclidean norm denoted by $|x|$ and Lebesgue measure denoted by $dx$. $\mathbb{R}^{n+1}$ is the upper half-space $\{(t,x) \in \mathbb{R}^{n+1} : t > 0, x \in \mathbb{R}^n\}$ with Lebesgue measure $dt dx$. $B(x,r)$ denotes the ball in $\mathbb{R}^n$ with center $x$, radius $r$ and volume $|B|$. Denote by $Q$ a cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes. The volume and side length of $Q$ are denoted by $|Q|$ and $l(Q)$, respectively.

For convenience, the positive constants $C$ may change and usually depend on the dimension $n$, $\alpha$, $\beta$ and other parameters. The Schwartz class of rapidly decreasing functions and its dual will be denoted by $\mathcal{S}^{(1)}(\mathbb{R}^n)$ and $\mathcal{S}^{(1)}(\mathbb{R}^n)$, respectively. For a function $f \in \mathcal{S}^{(1)}(\mathbb{R}^n)$, $\widehat{f}$ means the Fourier transform of $f$.

2.1. Wavelets. In this paper, we use real valued tensor product orthogonal wavelets $\Phi^\epsilon(x)$ which will be Daubechies wavelets or classical Meyer wavelets. Daubechies wavelets are only used in Section 2.2 and Meyer wavelets will be used throughout this paper. If $\Phi^\epsilon(x)$ is a Daubechies wavelet, we assume that there exists a sufficiently big integer $m_0$ which is greater than some constant depending on the index of the relative Triebel type oscillation spaces such that

1. $\forall (\epsilon \in \{0, 1\}^n, \Phi^\epsilon(x) \in C_0([-2^M, 2^M]^n)$;
2. For any $\epsilon \in E_n$, $\Phi^\epsilon(x)$ has the vanishing moments up to the order $m_0 - 1$.

We state some preliminaries on classic Meyer wavelets. Let $\Psi^0(\xi) \in C_0^\infty([-\frac{8\pi}{3}, \frac{8\pi}{3}])$ be an even function satisfying

$$
\begin{cases}
\Psi^0(\xi) \in [0, 1], \\
\Psi^0(\xi) \equiv 1, \text{ if } |\xi| \leq \frac{2\pi}{3}.
\end{cases}
$$

Let $\Omega(\xi) = ((\Psi^0(\xi))^2 - (\Psi^0(\xi))^2)^{\frac{1}{2}}$. Then $\Omega(\xi) \in C_0^\infty([-\frac{8\pi}{3}, \frac{8\pi}{3}])$ is an even function satisfying:

$$
\begin{cases}
\Omega(\xi) \equiv 0, \text{ if } |\xi| \leq \frac{2\pi}{3}; \\
\Omega^2(\xi) + \Omega^2(2\xi) = \Omega^2(\xi) + \Omega^2(2\pi - \xi) = 1, \text{ if } \xi \in \left[\frac{2\pi}{3}, \frac{4\pi}{3}\right].
\end{cases}
$$

Let $\Psi^1(\xi) = \Omega(\xi)e^{-\frac{i\xi}{2}}$. For any $\epsilon = (\epsilon_1, \cdots, \epsilon_n) \in \{0, 1\}^n$, let the Fourier transform of $\Phi^\epsilon(x)$ be $\hat{\Phi^\epsilon}(\xi) = \prod_{i=1}^n \Psi^{\epsilon_i}(\xi_i)$. 

For $j \in \mathbb{Z}, k \in \mathbb{Z}^n$, let $\Phi^\epsilon_{jk}(x) = 2^{\frac{n\epsilon}{2}} \Phi(2^j x - k)$. In this paper, we denote

$$
E_n := \{0, 1\}^n \setminus \{0\},
F_n := \{(e, k) : e \in E_n, k \in \mathbb{Z}^n\},
\Lambda_n = \{(e, j, k), e \in E_n, j \in \mathbb{Z}, k \in \mathbb{Z}^n\}.
$$

For further information about wavelets, we refer the reader to Meyer [18], Wojtaszczyk [28] and Yang [33]. The following result is well-known.

**Lemma 2.1.** \{$\Phi^\epsilon_{j,k}(x)\}_{(e,j,k) \in \Lambda_n}$ is an orthogonal basis in $L^2(\mathbb{R}^n)$.

For function $f(x), \forall \epsilon \in \{0, 1\}^n$ and $k \in \mathbb{Z}^n$, denote by $f^\epsilon_{j,k} = \langle f(x), \Phi^\epsilon_{j,k}(x) \rangle$ the wavelet coefficients of $f$. Let

$$
P_j f(x) = \sum_{k \in \mathbb{Z}^n} f^0_{j,k} \Phi_{j,k}(x)\text{ and } f_j(x) = Q_j f(x) = \sum_{(e,j,k) \in F_n} f^\epsilon_{j,k} \Phi^\epsilon_{j,k}(x).
$$

By Lemma 2.1, we can see that $P_j$ and $Q_j$ are two projection operators on $L^2(\mathbb{R}^n)$. In fact, for any two functions $u$ and $v$, we have

$$
u v = \sum_{j < 0} P_{j-3} u Q_j v + \sum_{j \in \mathbb{Z}} Q_j u Q_j v + \sum_{0 < j - f \leq 3} Q_j u Q_j v
$$

$$+ \sum_{0 < j - f \leq 3} Q_j u Q_j v + \sum_{j \in \mathbb{Z}} Q_j u P_{j-3} v

(2.1)
$$

**2.2. Characterization via Daubechies wavelets.** Now we introduce a class of Triebel type oscillation spaces. Let $\varphi(x) \in C^\infty_0 (B(0, \sqrt{n}))$ be such that $\varphi(x) = 1$ for $x \in B(0, \sqrt{n})$. Let $Q(x_0, r)$ be a cube with sides parallel to the coordinate axis, centered at $x_0$ and with side length $r$. To simplify the notation, sometimes, we denote $Q = Q(r) = Q(x_0, r)$ and let $\varphi_Q(x) = \varphi(\frac{x - x_0}{r})$.

For $0 < p, q \leq \infty$ and $\gamma_1, \gamma_2 \in \mathbb{R}$, let $m_0 = m_{p,q}^{\gamma_1,\gamma_2}$ be a sufficiently big positive real number. For arbitrary function $f(x)$, let $S_{p,q,j}^{\gamma_1,\gamma_2}$ be the set of polynomial functions $P_{Q,f}(x)$ such that $\forall |a| \leq m_0$,

$$
\int x^a \varphi_Q(x)(f(x) - P_{Q,f}(x))dx = 0.
$$

**Definition 2.2.** Given $0 < p < \infty, 0 < q \leq \infty$ and $\gamma_1, \gamma_2 \in \mathbb{R}$, Triebel type oscillation spaces $F_p^{\gamma_1,\gamma_2}(\mathbb{R}^n)$ are defined as:

$$
\sup_{\text{cube } Q} \frac{|Q|^{\frac{\gamma_1}{p} - \frac{1}{p}}}{P_{Q,f} \in S_{p,q,j}^{\gamma_1,\gamma_2}} \inf_{P_{Q,f} \in S_{p,q,j}^{\gamma_1,\gamma_2}} \|\varphi_Q(f - P_{Q,f})\|_{F_p^{\gamma_1,\gamma_2}} < +\infty,
$$

where the supremum is taken over all the cubes in $\mathbb{R}^n$.

Let $0 < p, q \leq \infty$ and $\gamma_1, \gamma_2 \in \mathbb{R}$. There exists a sufficiently big integer $m_{p,q}^{\gamma_1,\gamma_2}$ such that regular Daubechies can characterize such spaces. We call
a Daubechies wavelets $\Phi^\varepsilon(x)$ regular if there exist two integers $m_0 \geq m_{p,q}^\gamma$ and $M$ such that
\begin{equation}
\forall \varepsilon \in [0, 1]^n, \Phi^\varepsilon(x) \in C_0^{m_0}([-2^M, 2^M]^n);
\end{equation}
\begin{equation}
\forall \varepsilon \in E_n, \int x^\varepsilon \Phi^\varepsilon(x)dx = 0, \forall |\alpha| \leq m_0.
\end{equation}
Using Daubechies wavelets, we have the following wavelet characterization of $F_{p,q}^{\gamma_1, \gamma_2}$:

**Theorem 2.3.** Given $0 < p < \infty, 0 < q \leq \infty$ and $\gamma_1, \gamma_2 \in \mathbb{R}$, $f(x) = \sum_{(\varepsilon, j, k) \in \Lambda_x} a_{j, k}^\varepsilon \Phi_{j, k}(x) \in F_{p,q}^{\gamma_1, \gamma_2}$ if and only if
\begin{equation}
\sup_{Q} \|Q\|^{-\frac{1}{p}} \left\| \sum_{(\varepsilon, j, k) \in \Lambda_q^\varepsilon} 2^{\varepsilon j (\gamma_1 + \frac{2}{q})} |a_{j, k}^\varepsilon| \chi(2^j \cdot -k) \right\|_{L_p} < +\infty,
\end{equation}
where the supremum is taken over all the dyadic cubes $Q$ in $\mathbb{R}^n$.

**Proof.** we prove first that $f(x) \in F_{p,q}^{\gamma_1, \gamma_2}$ implies $f(x)$ satisfies (2.5). For any dyadic cube $Q$ with center $x_Q$ and side length $l(Q)$, there exists a cube $\hat{Q}$, parallel to the coordinate axis, centered at $x_Q$ and with side length $2^{M+2}l(Q)$. By definition of $\varphi_Q(x)$ and (2.4), for such $\hat{Q}$ and $Q_{j,k} \subset Q$, $x \in \hat{Q}$, we have $f(x) = \varphi_Q(x) f(x)$ and
\begin{align*}
\int (\varphi_Q(x) P_{Q,f}(y)) \Phi_{j,k}(y)dy = \int P_{\hat{Q},f}(y) \Phi_{j,k}(y)dy = 0.
\end{align*}
Hence, for any $\varepsilon \in E_n$ and $Q_{j,k} \subset Q$, we have
\begin{align*}
\langle f, \Phi_{j,k}^\varepsilon \rangle = \langle \varphi_Q(f - P_{\hat{Q},f}), \Phi_{j,k} \rangle.
\end{align*}
By wavelet characterization of Triebel-Lizorkin spaces, we have
\begin{align*}
\left\| \sum_{(\varepsilon, j, k) \in \Lambda_q^\varepsilon} 2^{\varepsilon j (\gamma_1 + \frac{2}{q})} |a_{j, k}^\varepsilon| \chi(2^j \cdot -k) \right\|_{L_p} < +\infty,
\end{align*}
where $Q_{j,k} \subset Q$, $t \in \Lambda_x^\varepsilon$, $\hat{Q} \subset 2^n$ and $\hat{Q}$ is a subset of $\cup_{i=1}^{2^n} e E_n$.

Conversely, for any cube $Q$, there exist $2^n$ dyadic cubes $Q_i$ such that $2^{M+2}l(Q_i) \leq l(Q) \leq 2^{M+1}l(Q)$ and
\begin{align*}
\phi_Q(x) f(x) = \phi_Q(x) \sum_{i=1}^{2^n} \sum_{Q_{j,k} \subset Q_i} f_{j,k}^\varepsilon \Phi_{j,k}(x).
\end{align*}
Hence
\begin{align*}
\|\phi_Q f\|_{F_{p,q}^{\gamma_1, \gamma_2}} & \leq \|\phi_Q \sum_{i=1}^{2^n} \sum_{Q_{j,k} \subset Q_i} f_{j,k}^\varepsilon \Phi_{j,k}\|_{F_{p,q}^{\gamma_1, \gamma_2}} \\
& \leq C \|\sum_{i=1}^{2^n} \sum_{Q_{j,k} \subset Q_i} f_{j,k}^\varepsilon \Phi_{j,k}\|_{F_{p,q}^{\gamma_1, \gamma_2}},
\end{align*}
where $C$ is a constant.
If (2.5) holds, we get (2.2).

**Remark 2.4.** If \( \gamma_2 = \frac{n}{p} \), \( \dot{F}^{\gamma_1, \gamma_2}_{p, q} \) becomes the Triebel-Lizorkin space \( \dot{F}^\gamma_{p, q} \). Moreover, if \( \gamma_2 > \frac{n}{p} \), for any \( f \in \dot{F}^{\gamma_1, \gamma_2}_{p, q} \), the wavelet coefficients of \( f \) are all zero. Hence \( f \) is only a polynomial.

By Theorem 2.3 we can identify a function with its wavelet coefficients. That is to say,

\[
\sum_{(\epsilon, j, k) \in \Lambda_n} a^\epsilon_{j, k} \Phi^\epsilon_{j, k}(x) \in \dot{F}^{\gamma_1, \gamma_2}_{p, q} \text{ if and only if } \{a^\epsilon_{j, k}\}_{(\epsilon, j, k) \in \Lambda_n} \text{ satisfy (2.5).}
\]

Further, it is easy to check the following results about Triebel type oscillation spaces.

**Corollary 2.5.** Given \( 0 < p < \infty, 0 < q \leq \infty \) and \( \gamma_1, \gamma_2 \in \mathbb{R} \).

(i) The definition of \( \dot{F}^{\gamma_1, \gamma_2}_{p, q} \) is independent of the choice of \( \phi \).

(ii) \( \dot{F}^{\gamma_1, \gamma_2}_{p, q}(\mathbb{R}^n) \) are Banach spaces for \( 1 \leq p < \infty \) and \( 1 \leq q \leq \infty \).

2.3. **Calderón-Zygmund operators.** Now we introduce some preliminaries about Calderón-Zygmund operators. See [18, 19]. For \( x \neq y \), let \( K(x, y) \) be a smooth function such that there exists a sufficiently large \( N_0 \leq m_0 \) satisfying that

\[
|\partial^\alpha_x \partial^\beta_y K(x, y)| \leq \frac{C}{|x - y|^{(|\alpha| + |\beta|)}}, \forall |\alpha| + |\beta| \leq N_0.
\]

**Definition 2.6.** A linear operator

\[
Tf(x) = \int K(x, y)f(y)dy
\]

is said to be a Calderón-Zygmund operator if

(i) \( T \) is continuous from \( C^1(\mathbb{R}^n) \) to \( (C^1(\mathbb{R}^n))' \);

(ii) the kernel \( K \) satisfies (2.6);

(iii) \( Tx^\alpha = T^*x^\alpha = 0, \forall \alpha \in \mathbb{N}^n \).

We denote by \( CZO(N_0) \) the set of all operators satisfying (i), (ii) and (iii).

From (2.6), we can see that the kernel \( K(\cdot, \cdot) \) may have high singularity on the diagonal \( x = y \). According to Schwartz kernel theorem, \( K(\cdot, \cdot) \) is a distribution in \( S'(\mathbb{R}^{2n}) \). For any \( (\epsilon, j, k), (\epsilon', j', k') \in \Lambda_n \), let

\[
a^{\epsilon}_{j, k, \epsilon', j', k'} = \langle K(\cdot, \cdot), \Phi^\epsilon_{j, k}\Phi^{\epsilon'}_{j', k'} \rangle.
\]

If \( T \in CZO(N_0) \), its kernel \( K(\cdot, \cdot) \) and \( \{a^{\epsilon}_{j, k, \epsilon', j', k'}\} \) satisfy the following relations. We refer the reader to Meyer [18], Meyer-Yang [19] and Yang [33] for the proofs.
Lemma 2.7. (i) If \( T \in \text{CZO}(N_0) \), then the coefficients \( \{a_{j,k,j',k'}^{\epsilon,\epsilon'}\}_{(\epsilon,j,k),(\epsilon',j',k') \in \Lambda_n} \) satisfy the following condition:

\[
|a_{j,k,j',k'}^{\epsilon,\epsilon'}| \leq C 2^{\frac{-j-j'}{2^j + 2^{-j'}}} \left( \frac{2^{-j} + 2^{-j'}}{2^{-j} + 2^{-j} + |k2^{-j} - k'2^{-j}|} \right)^{N_0}. 
\]

(ii) If \( \{a_{j,k,j',k'}^{\epsilon,\epsilon'}\}_{(\epsilon,j,k),(\epsilon',j',k') \in \Lambda_n} \) satisfy (2.7), then

\[
K(x,y) = \sum_{(\epsilon,j,k) \in \Lambda_n} \sum_{(\epsilon',j',k') \in \Lambda_n} a_{j,k,j',k'}^{\epsilon,\epsilon'} \Phi_{\epsilon,j,k}(x) \Phi_{\epsilon',j',k'}^{\epsilon'}(y)
\]

in the sense of distributions and for any small positive real number \( \delta \), \( T \in \text{CZO}(N_0 - \delta) \).

For \( A > 0 \) and a sequence \( f = \{f_j\} \), we define the vector-valued maximal function \( M_A(f) \) as

\[
M_A(f)(x) = \left( \sum_j M(|f_j|^A)(x) \right)^{1/A}.
\]

For \( \{a_{j,k}^{\epsilon}\}_{(\epsilon,j,k) \in \Lambda_n} \), we set

\[
f_j = \sum_{(\epsilon,j,k) \in \Lambda_n} 2^{(j+\frac{q}{2})} |a_{j,k}^{\epsilon}| \chi(2^j x - k).
\]

Let

\[
g_{j,k}^{\epsilon} = \begin{cases} 
\sum_{\epsilon',k'} \frac{2^{j+\frac{q}{2}} |a_{j,k,j',k'}^{\epsilon,\epsilon'}|}{(1 + |k' - 2^{-j} k|)^{n+\gamma}}, & j \geq j', k \in \mathbb{Z}^n; \\
\sum_{\epsilon',k'} \frac{2^{j+\frac{q}{2}} |a_{j,k,j',k'}^{\epsilon,\epsilon'}|}{(1 + |k - 2^{-j} k'|)^{n+\gamma}}, & j < j', k \in \mathbb{Z}^n.
\end{cases}
\]

Yang [33] obtained the following result.

Lemma 2.8. ([33], Chapter 5, Lemma 3.2) For any \( \gamma > \frac{n}{A} + 1 \) and \( x \in Q_{j,k} \), we have

\[
g_{j,k}^{\epsilon} = \begin{cases} 
CM_A(f_j)(x), & \text{if } j \geq j'; \\
C 2^{\frac{A}{A-\gamma} - n} M_A(f_j)(x), & \text{if } j < j'.
\end{cases}
\]

Following the idea of [19, 33, 34], we can prove that the Calderón-Zygmund operators are bounded on \( \dot{F}^{\gamma_1,\gamma_2}_{p,q} \). For completeness, we give the proof. In fact, for all \( (\epsilon, j, k) \in \Lambda_n \), denote

\[
\tilde{g}_{\epsilon,j,k}^{\epsilon} = \sum_{(\epsilon',j',k') \in \Lambda_n} a_{j,k,j',k'}^{\epsilon,\epsilon'} g_{j',k'}^{\epsilon'}.
\]

By Lemma 2.7, the boundedness of Calderón-Zygmund operators on \( \dot{F}^{\gamma_1,\gamma_2}_{p,q} \) is equivalent to the following theorem.
Theorem 2.9. Given $0 < p, q \leq \infty$ and $\gamma_1, \gamma_2 \in \mathbb{R}$. There exists sufficiently big $N_0$ such that $\{a_{\epsilon, \epsilon}^{\varepsilon, \varepsilon}(\epsilon, j, k), (\epsilon', j', k')\in \Lambda_\alpha\}$ satisfies \((2.7)\). If $\{g_{\epsilon, \epsilon}^{\varepsilon, \varepsilon}(\epsilon, j, k) \in \mathbb{R}, \{\tilde{g}_{\epsilon, \epsilon}^{\varepsilon, \varepsilon}(\epsilon, j, k) \in \mathbb{R}\} \in \mathcal{F}_{p_1, q_1}$. Consequently, Calderón-Zygmund operators are bounded on $\mathcal{F}_{p_1, q_1}$.

Proof. Let $Q$ be any dyadic cube with $|Q| = 2^{-n_0}$. For $\tau = 1$, denote by $Q_\tau$ the dyadic cube satisfying $Q \subset Q_\tau$ and $|Q_\tau| = 2^{n_\tau}|Q|$. Specially, $Q_0 = Q$. If $l \in \mathbb{Z}^n$ and $Q_{j, k} \subset 2^{-|l|}l + Q_\tau$, we denote $Q_{j, k} \subset S_{j, l}$. Then we have

$$
\tilde{g}_{\epsilon, \epsilon}^{\varepsilon, \varepsilon} = \sum_{\epsilon' \in \Lambda_\alpha} a_{\epsilon, \epsilon}^{\varepsilon, \varepsilon}(\epsilon, j, k) S_{\epsilon', \epsilon}^{\varepsilon, \varepsilon}(\epsilon, j, k)
$$

We will prove

$$
I_Q = |Q|^{1/p - 1/2} \left\| \sum_{j=-\log_2 r} \sum_{Q_{j, k} \subset Q_\tau} 2^{j(\gamma_1 + \frac{3}{2})} \left| \tilde{g}_{\epsilon, \epsilon}^{\varepsilon, \varepsilon}(\epsilon, j, k) \chi(2^j x - k) \right|^{1/q} \right\|_p < \infty.
$$

By Hölder’s inequality, for $\delta$ small enough, we have

$$
I_Q \leq |Q|^{1/p - 1/2} \left\| \sum_{j=-\log_2 r} \sum_{Q_{j, k} \subset Q_\tau} 2^{j(\gamma_1 + \frac{3}{2})} \sum_{r \geq 0 \in \mathbb{Z}^n} a_{\epsilon, \epsilon}^{\varepsilon, \varepsilon}(\epsilon, j, k) S_{\epsilon', \epsilon}^{\varepsilon, \varepsilon}(\epsilon, j, k) \chi(2^j x - k) \frac{1}{P} \right\|_p
$$

(1) For $\tau = 0$,

$$
I_Q \leq |Q|^{1/p - 1/2} \left\| \sum_{j=-\log_2 r} \sum_{Q_{j, k} \subset Q_\tau} 2^{j(\gamma_1 + \frac{3}{2})} (1 + |l|)^{n_\delta} \sum_{r \geq 0 \in \mathbb{Z}^n} a_{\epsilon, \epsilon}^{\varepsilon, \varepsilon}(\epsilon, j, k) S_{\epsilon', \epsilon}^{\varepsilon, \varepsilon}(\epsilon, j, k) \chi(2^j x - k) \frac{1}{P} \right\|_p.
$$

If $|l| < 8^n$, by the boundedness of Calderón-Zygmund operator on $\mathcal{F}_{p_1, q_1}$, we have

$$
I_Q \leq |Q|^{1/p - 1/2} \left\| \sum_{j=-\log_2 r} \sum_{Q_{j, k} \subset Q_\tau} 2^{j(\gamma_1 + \frac{3}{2})} (1 + |l|)^{n_\delta} \sum_{r \geq 0 \in \mathbb{Z}^n} a_{\epsilon, \epsilon}^{\varepsilon, \varepsilon}(\epsilon, j, k) S_{\epsilon', \epsilon}^{\varepsilon, \varepsilon}(\epsilon, j, k) \chi(2^j x - k) \frac{1}{P} \right\|_p < \infty.
$$

If $|l| > 8^n$, because

$$
|a_{\epsilon, \epsilon}^{\varepsilon, \varepsilon}| \leq 2^{-j} (2^{-j} + 2^{-j} + 2^{-j}) \left(2^{-j} + 2^{-j} + 2^{-j} + 2^{-j} \right)^{n_\delta},
$$

we have
We can get
\[
I_Q \leq \left\| \sum_{j \geq -\log_2 r} \sum_{Q \subset Q_r} 2^{j(y_1+\frac{\sigma}{2})} \sum_{|l| > 8^n} (1 + |l|)^{\epsilon + \delta} \int_{\mathbb{R}} \sum_{j \geq j_f} \sum_{Q \subset Q_r} \sum_{Q_{j,k}} a_{j,k,j',k'}^{e,e'} \left( \sum_{j \geq -\log_2 r} \sum_{Q \subset Q_r} \sum_{Q_{j,k}} a_{j,k,j',k'}^{e,e'} \left| \chi(2^j x) \right|^q \right) \right\|_p
\]
\[
+ \left\| \sum_{j \geq j_f} \sum_{Q \subset Q_r} \sum_{Q_{j,k}} a_{j,k,j',k'}^{e,e'} \left( \sum_{j \geq -\log_2 r} \sum_{Q \subset Q_r} \sum_{Q_{j,k}} a_{j,k,j',k'}^{e,e'} \left| \chi(2^j x) \right|^q \right) \right\|_p
\]
\[
:= I_1 + I_2.
\]

For \(I_1\),
\[
\left| a_{j,k,j',k'}^{e,e'} \right| \leq 2^{-|j-f|(|y_1+\frac{\sigma}{2})} \left( 1 + \left| 2^{j-f} k - k' \right| \right)^{-(n+N_0)}
\]
\[
\leq 2^{-|j-f|(|y_1+\frac{\sigma}{2})} \left( 1 + \left| 2^{j-f} k - k' \right| \right)^{-(n+N_0)/2}.
\]

On the other hand, \(Q_{j,k} \in S_{0j}\) implies \(Q_{j,k'} = 2^{-j} l + Q_{j,k}\). It is easy to see that \(2^{j_f-j} - k' - 2^{j_f-j} |l| \leq 2^{j_f-j} |l| \). Then
\[
I_1 \leq \left\| \sum_{j \geq -\log_2 r} \sum_{Q \subset Q_r} \sum_{Q_{j,k}} 2^{j(y_1+\frac{\sigma}{2})} \sum_{|l| > 8^n} (1 + |l|)^{\epsilon + \delta} \left| \chi(2^j x) \right|^q \right\|_p
\]
\[
\leq \left\| \sum_{j \geq -\log_2 r} \sum_{Q \subset Q_r} \sum_{Q_{j,k}} 2^{j(y_1+\frac{\sigma}{2})} \sum_{j > j_f} 2^{(j-j') \gamma} 2^{-q(j-j')(|y_1+\frac{\sigma}{2})} \left| \chi(2^j x) \right|^q \right\|_p
\]
\[
\leq \left\| \sum_{j \geq -\log_2 r} \sum_{Q \subset Q_r} \sum_{Q_{j,k}} 2^{j(y_1+\frac{\sigma}{2})} \sum_{j > j_f} 2^{(j-j') \gamma} \left| \chi(2^j x) \right|^q \right\|_p
\]
\[
\leq \left\| \sum_{j \geq -\log_2 r} \sum_{Q \subset Q_r} \sum_{Q_{j,k}} 2^{j(y_1+\frac{\sigma}{2})} \sum_{j > j_f} 2^{j(y_1+\frac{\sigma}{2})} \left| \chi(2^j x) \right|^q \right\|_p
\]
\[
\leq \left\| \sum_{j \geq -\log_2 r} \sum_{Q \subset Q_r} \sum_{Q_{j,k}} 2^{j(y_1+\frac{\sigma}{2})} \left| \chi(2^j x) \right|^q \right\|_p.
\]

We can get
\[
I_1 \leq \left\| \sum_{j \geq -\log_2 r} \sum_{Q \subset Q_r} \sum_{Q_{j,k}} 2^{j(y_1+\frac{\sigma}{2})} \left| g_{j,k,j',k'}^{e,e'} \chi(2^j x - k') \right|^q \right\|_p
\]
\[
\leq \left\| \sum_{j \geq -\log_2 r} \sum_{Q \subset Q_r} \sum_{Q_{j,k}} 2^{j(y_1+\frac{\sigma}{2})} \left| g_{j,k,j',k'}^{e,e'} \chi(2^j x - k') \right|^q \right\|_p
\]
\[
\leq \left\| g \right\|_{L^q_{j,k,j'}^{\gamma,\frac{\sigma}{2}}},
\]
where
\[
f_{j} = 2^{j(y_1+\frac{\sigma}{2})} \sum_{Q_{j,k} \subset Q_{j'}} \left| g_{j,k,j',k'}^{e,e'} \chi(2^j x - k') \right|
\]
For $I_2$, because $j < j'$, we have
\[
|e_{j, k', j', k}^{\epsilon, \epsilon'}| \lesssim 2^{-j-j'k' + \epsilon + N_0} \left( \frac{2^{-j+2jk'}}{2^{-j+2jk' + 2j-k' - 2 \epsilon}} \right)^{n+N_0} \lesssim 2^{-(j-j')(\frac{2}{N} + N_0)} \left( 1 + |k - 2^{j-j'k'}| \right)^{-n+N_0}.
\]

Let $x_0$ be the center of $Q$.
\[
|k - 2^{-j-j'k'}| = 2^{-j} |2^{-j}k - x_0 - 2^{-j_0}l + x_0 + 2^{-j_0}l - 2^{-j}k| \\
\geq 2^{-j} (2^{-j_0} |l| - 2^{-j_0}) \\
\geq 2^{-j} |l|.
\]

Let
\[
f_{j'} = 2^{j'(y_1 + \frac{j}{2})} \sum_{Q, j, k \in S_{\alpha_j}} |g_{j, k}^{\epsilon, \epsilon'}\chi(2^{j'} x - k').
\]

Hence we get
\[
I_2 \leq \sum_{|l| > 8^n} \left[ \sum_{j \geq 0} \sum_{Q, j', k < Q} 2^{q_2 j'(y_1 + \frac{j}{2})} |Q_j|^{\frac{2}{p-1}} \left\| \sum_{j \geq 0} \sum_{Q, j', k \in Q} |g_{j, k}^{\epsilon, \epsilon'}| \left( \frac{|Q_j|^{\frac{1}{p'}}}{(1 + |2^{-j-j'k'} - k|)^{\frac{1}{p'}}} \chi(2^{j'} x - k) \right) \right\|_p \right]
\leq \sum_{|l| > 8^n} \left[ \sum_{j \geq 0} \sum_{Q, j', k \in Q} 2^{q_2 j'(y_1 + \frac{j}{2})} 2^{2q_2 (j-j')(\frac{j}{2} + N_0)} \left\| M_A(f_{j'})(x) \chi(2^{j'} x - k) \right\|_p \right]
\leq \left\| g \right\|_{F^{q_2, p}_{j', j_0}}.
\]

(2) For $\tau \geq 1$, we can get for $Q, j, k \in Q$ and $Q_{j', k'} = 2^{r-j_0} l + 2 r Q$, $j' = j_0 - r$. it is easy to see that $j > j'$ for this case. If $|l| < 8^n$ and $j > j'$,
\[
|e_{j, k', j', k}^{\epsilon, \epsilon'}| \leq 2^{-j-j'k' + \epsilon + N_0} \left( \frac{2^{-j+2jk'}}{2^{-j+2jk' + 2j-k' - 2 \epsilon}} \right)^{n+N_0} \lesssim 2^{-(j-j')(\frac{2}{N} + N_0)} \left( 1 + |k - 2^{j-j'k'}| \right)^{-n+N_0}.
\]

Let
\[
f_{j'} = 2^{j'(y_1 + \frac{j}{2})} \sum_{Q, j, k \in S_{\alpha_j}} |g_{j, k}^{\epsilon, \epsilon'}\chi(2^{j'} x - k').
\]
Hence

\[
I_Q \lesssim \sum_{|\ell| < 8^N} (1 + |\ell|)^{\alpha + \delta} \sum_{r \geq 1} 2^{r(\alpha + \delta)} \| \sum_{j \in \Lambda \cap Q} 2^{q(j + \frac{\tau}{2})} \left| \frac{1}{2} \sum_{|\ell| < 8^N} 2^{-r(\alpha + \delta)} \sum_{j \in \Lambda \cap Q} 2^{q(j + \frac{\tau}{2})} \right|^q \chi(2^j x - k) \left\|^q \right|_p
\]

\[
\lesssim \sum_{r \geq 1} 2^{2r(\alpha + \delta)} (1 + |\ell|)^{\alpha + \delta} \| \sum_{j \in \Lambda \cap Q} 2^{q(j + \frac{\tau}{2})} \left| \frac{1}{2} \sum_{|\ell| < 8^N} 2^{-r(\alpha + \delta)} \sum_{j \in \Lambda \cap Q} 2^{q(j + \frac{\tau}{2})} \right|^q \chi(2^j x - k) \left\|^q \right|_p
\]

\[
(\mathcal{M}(f_j)(x))^q \chi(2^j x - k) \left\|^q \right|_p
\]

\[
\lesssim \|g\|_{F^{1,2}_{p,q}}.
\]

If \(|\ell| > 8^N\), for \(Q_{j', k'} = 2^{j'} - j_0 + 1\),

\[
|2^{j'} k - k'| = 2^{j'} |2^{-j} k - 2^{-j} k'| \geq 2^{j + j_0} |\ell|.
\]

Then

\[
|d_{j,k,k'}^{\epsilon'}| \lesssim 2^{-j_0} (2^{j + j_0} |\ell|)^{\alpha + N_0/2} (1 + |2^{-j} k - k'|)^{N_0/2}.
\]

Let

\[
f_j' = 2^{j(j + \frac{\tau}{2})} \sum_{Q_{j', k} \in S_{j', k}} |g_{j', k'}^{\epsilon'}| \chi(2^{j'} x - k).
\]

So

\[
I_Q \lesssim \sum_{r \geq 1} 2^{2r(\alpha + \delta)} (1 + |\ell|)^{\alpha + \delta} \| \sum_{j \in \Lambda \cap Q} 2^{q(j + \frac{\tau}{2})} \left| \frac{1}{2} \sum_{|\ell| < 8^N} 2^{-r(\alpha + \delta)} \sum_{j \in \Lambda \cap Q} 2^{q(j + \frac{\tau}{2})} \right|^q \chi(2^j x - k) \left\|^q \right|_p
\]

\[
\lesssim \sum_{r \geq 1} 2^{2r(\alpha + \delta)} (1 + |\ell|)^{\alpha + \delta} \| \sum_{j \in \Lambda \cap Q} 2^{q(j + \frac{\tau}{2})} \left| \frac{1}{2} \sum_{|\ell| < 8^N} 2^{-r(\alpha + \delta)} \sum_{j \in \Lambda \cap Q} 2^{q(j + \frac{\tau}{2})} \right|^q \chi(2^j x - k) \left\|^q \right|_p
\]

\[
(\mathcal{M}(f_j')(x))^q \chi(2^{j'} x - k) \left\|^q \right|_p
\]

\[
\lesssim \|g\|_{F^{1,2}_{p,q}}.
\]

\[\square\]

For \(i = 1, 2\) and two regular orthogonal wavelets basis \(\{\Phi_{j,k}^{i,e}(x)\}_{(e,j,k) \in \Lambda_n}\), \(\forall (e,j,k),(e',j',k') \in \Lambda_n\), denote \(a_{j,k,j',k'}^{e,e'} = \langle \Phi_{j,k}^{1,e}, \Phi_{j',k'}^{2,e'} \rangle\). We know that \(\{a_{j,k,j',k'}^{e,e'}\}_{(e,j,k),(e',j',k') \in \Lambda_n}\) satisfies the condition (2.7). According to Theorem 2.9 Lemma 2.3 is also true for Meyer wavelets. The reader can also find a proof of the following result in [34].
Lemma 2.10. The wavelet characterization in Lemma 2.3 is also true for Meyer wavelets.

3. Triebel type tent spaces

For the rest of this paper, we only use classical tensorial Meyer wavelets. In this section, we introduce two classes of Triebel type tent spaces which will be used in the well-posedness of Navier-Stokes equations. We would like to remind the readers that, for wavelets \( \{ \Phi_{j,k} : (\varepsilon, j, k) \in \Lambda_n \} \), \( 2^j \) represents the range of frequency \( \xi \) and \( 2^{-j}k \) represents the range of position \( x \) in some sense.

3.1. Preliminaries relative to semigroup. Throughout this section, we denote \( N > 0 \) a fixed sufficient big real number. For fixed \( \beta > 0 \), we may choose a radial \( \phi \in \mathcal{S}(\mathbb{R}^n) \) such that there exists \( C_{\beta} > 0 \) satisfying

(i) \( \int_{\mathbb{R}^n} x^\gamma \phi(x) dx = 0 \) for all \( \gamma \in \mathbb{N}^n \);

(ii) \( \int_0^{\infty} (\hat{\phi}(t|\xi|)) \frac{dt}{t} = 1 \) for all \( \xi \neq 0 \);

(iii) \( \int_0^{\infty} \phi(t|\xi|) e^{-\frac{t}{2}} \frac{dt}{t} = \frac{1}{c_{\beta}} \).

See [10], Lemma 1.1 and [18] Chap. 3, §2.

Define \( \phi_0^\beta(x) = t^{-\frac{n}{2}} \phi(t^{-\frac{1}{2}} x) \). Then \( \hat{\phi}_0^\beta(\xi) = \hat{\phi}(t^{-\frac{1}{2}} |\xi|) \), and hence

\[
\hat{f}(\xi) = C_{\beta} \int_0^{\infty} \hat{\phi}(t|\xi|) e^{-\frac{t}{2}} \hat{f}(\xi) \frac{dt}{t} = C_{\beta} \int_0^{\infty} \hat{\phi}(t|\xi|) e^{-\frac{t}{2}} \hat{f}(\xi) \frac{dt}{t}.
\]

Since

\[
f(t, x) := e^{-t(\Delta)^{\beta}} f(x) = K_t^\beta \ast f(x).
\]

we have

\[
(3.1) \quad f(x) = C_{\beta} \int_0^{\infty} \int_{\mathbb{R}^n} f(t, x-y) \phi_0^\beta(y) dy \frac{dt}{t} := \pi_\phi f(\cdot, x).
\]

For \( (\varepsilon, j, k) \in \Lambda_n \), let \( a^\varepsilon_{j,k}(t) = \langle f(t, \cdot), \Phi_{j,k}^\varepsilon \rangle \) and \( a^\varepsilon_j = \langle f, \Phi_j^\varepsilon \rangle \). Then

\[
f = \sum_{(\varepsilon, j, k) \in \Lambda_n} a^\varepsilon_{j,k} \Phi_{j,k}^\varepsilon \text{ and } f(t, \cdot) = \sum_{(\varepsilon, j, k) \in \Lambda_n} a^\varepsilon_{j,k}(t) \Phi_{j,k}^\varepsilon.
\]

We first express \( a^\varepsilon_{j,k}(t) \) by using \( a^\varepsilon_{j',k'} \). If \( f(t, x) = K_t^\beta \ast f(x) \), then

\[
a^\varepsilon_{j,k}(t) = \sum_{\varepsilon', j' \leq 3^j} a^\varepsilon_{j',k'} \langle K_t^\beta \Phi_{j',k'}^\varepsilon, \Phi_{j,k}^\varepsilon \rangle
\]

\[
= \sum_{\varepsilon', j' \leq 3^j} a^\varepsilon_{j',k'} \int e^{-|t|^{3/4}} \hat{\Phi}_{j',k'}^{\varepsilon'}(2^{-j'}|\xi|) \hat{\Phi}_{j,k}^{\varepsilon}(2^{-j} |\xi|) e^{-i(2^{-j'}k' - 2^{-j}k)|\xi|} d\xi
\]

\[
= \sum_{\varepsilon', j' \leq 3^j} a^\varepsilon_{j',k'} \int e^{-|t|^{3/4}} \hat{\Phi}_{j',k'}^{\varepsilon'}(2^{-j'}|\xi|) \hat{\Phi}_{j,k}^{\varepsilon}(2^{-j} |\xi|) e^{-i(2^{-j'}k' - 2^{-j}k)|\xi|} d\xi.
\]

Applying integration by parts, we could control \( a^\varepsilon_{j,k}(t) \) by \( \{ a^\varepsilon_{j',k'} \} \) as follows.
Lemma 3.1. There exists a fixed small constant $\hat{c} > 0$ depending only on $\beta$ and the support of $\Phi_{j,k}$ such that

(i) For $t 2^{2j} \geq 1$,
\[ |a_{j,k}^e(t)| \leq C e^{-t 2^{j\beta}} \sum_{\epsilon', j'-j' \leq 3, k'} |a_{j', k'}^e|(1 + |2^{j'-j} k' - k|)^{-N}; \]

(ii) For $0 \leq t 2^{2j} \leq 1$,
\[ |a_{j,k}^e(t)| \leq C \sum_{j'-j \leq 3} \sum_{\epsilon', k'} |a_{j', k'}^e|(1 + |2^{j'-j} k' - k|)^{-N}. \]

Furthermore, if $f$ is obtained by (3.1), then we could express $a_{j,k}^e$ by $\{a_{j', k'}^e(t)\}$ as follows:
\[
a_{j,k}^e = \int_{\mathbb{R}_{+}^{n+1}} \sum_{(\epsilon', k') \in \Lambda_n} a_{j', k'}^e(t) (\phi_{j'}^\beta) \Phi_{j', k'}(x) \Phi_{j,k}(x) dx dt. \]

Similarly, we apply integration by party to obtain the following estimation.

Lemma 3.2.
\[
|a_{j,k}^e| \leq C \sum_{j'-j \leq 3} \int_0^\infty \left( \max\{t 2^{j\beta}, t^{-1} 2^{-j\beta}\}\right)^{-N} \sum_{(\epsilon', k') \in \Lambda_n} \frac{|a_{j', k'}^e(t)|}{(1 + |2^{j'-j} k' - k|)^{2N}} dt.
\]

3.2. Tent spaces $\mathbb{F}_{\gamma_1, \gamma_2}^{p,q,m,m'}$ and $\mathbb{F}_{r,\infty}^\gamma$. For any $a(t, x)$ defined on $\mathbb{R}_{+}^{n+1}$, by wavelet theory there exists a family $\{a_{j,k}^e(t)\}_{(\epsilon, k) \in \Lambda_n}$ such that
\[
a(t, x) = \sum_{(\epsilon, k) \in \Lambda_n} a_{j,k}^e(t) \Phi_{j,k}(x).
\]

Given $\gamma_1, \gamma_2 \in \mathbb{R}$, $1 < p < \infty$, $m \in \mathbb{R}$ and $m' > 0$, $\gamma$-Triebel-Lizorkin-Morrey spaces are defined as follows:
\[
\mathbb{F}_{\gamma_1, \gamma_2}^{p,q,m,m'} = \mathbb{F}_{p,q,m}^{\gamma_1, \gamma_2} \cap \mathbb{F}_{p,q}^{\gamma, \gamma_2, \gamma_1} \cap \mathbb{F}_{p,q,m}^{\gamma, \gamma_2, \gamma_1} \cap \mathbb{F}_{p,q,m'}^{\gamma, \gamma_2, \gamma_1},
\]
where the spaces of $L^\infty$ type norm on $t$ are defined as follows:

(i) $f \in \mathbb{F}_{p,q,m}^{\gamma_1, \gamma_2, \gamma_1}(\mathbb{R}_{+}^{n+1})$, if
\[
|Q_{1/2}^{j_{1/2}} f| \left\| \left( \sum_{j \geq \max \left( -\log t, -\frac{\log \log t}{\log 2} \right) \right) \sum_{Q \subset Q_{j}} 2^q |P_{j,k}(t)| \chi(2^j x - k) \right\|_p < \infty.
\]

(ii) $f \in \mathbb{F}_{p,q}^{\gamma, \gamma_2, \gamma_1}(\mathbb{R}_{+}^{n+1})$, if
\[
|Q_{1/2}^{j_{1/2}} f| \left\| \left( \sum_{-\log t < j < 0} \sum_{Q \subset Q_{j}} 2^q |P_{j,k}(t)| \chi(2^j x - k) \right) \right\|_p < \infty.
\]

the spaces of integration type norm on $t$ are defined as follows:

(i) $f \in \mathbb{F}_{p,q,m,m'}^{\gamma_1, \gamma_2, \gamma_1}(\mathbb{R}_{+}^{n+1})$, if
\[
|Q_{1/2}^{j_{1/2}} f| \left\| \left( \sum_{(\epsilon, k) \in \Lambda_n} 2^q |P_{j,k}(t)| t^{q \gamma / (q+1)} \right) \right\|_p < \infty.
\]
Proof. For Lemma 3.3.

Let \( f \in \mathbb{F}_{p,q,m}^{\gamma_1,\gamma_2,JV} (\mathbb{R}_+^{n+1}) \), if

\[
|Q|^{-\frac{n-\beta}{m}} \left( \sum_{(\epsilon,j,k) \in A_n} 2^{q_j(y_1+\frac{s}{2}+2m\epsilon)} \int_0^{2^{-2\beta}r} |a_{jk}^\epsilon (t)|^q r^m dt \right)^{\frac{1}{q}} < \infty.
\]

Then, we define \( t \)-Bloch spaces and \( t-L^\infty \) spaces which can be applied to the proof of boundedness of the bilinear operators. For \( \gamma_1 \in \mathbb{R} \) and \( \tau > 0 \), we say that \( a(t,\cdot) \) belongs to \( t \)-Bloch space \( \mathbb{F}_{r,\infty}^{\gamma_1} \) if

\[
\sup_{(\epsilon,j,k) \in A_n} \left\{ \sup_{t \geq 1} (t2^{2\beta}\lambda 2^{\frac{n}{2}} j a_{jk}^\epsilon (t)) + \sup_{0 < t \leq 1} 2^{\frac{n}{2}} 2^{\gamma_1} 2^{\frac{n}{2}} |a_{jk}^\epsilon (t)| \right\} < \infty.
\]

We say that \( a(t,\cdot) \) belongs to \( t-L^\infty \) space \( \mathbb{F}_{0,\infty}^{\gamma_1} \) if

\[
\sup_{t > 0} \sup_{j \in \mathbb{Z}, k \in \mathbb{Z}_n} |a(t,\cdot), \Phi_{jk}^0(x)| < \infty.
\]

The following lemma can be obtained immediately,

**Lemma 3.3.** Given \( 1 < p, q < \infty \), \( \gamma_1, \gamma_2 \in \mathbb{R} \), \( m > p \) and \( m', \tau > 0 \),

(i) If \( m > 0 \), \( \mathbb{F}_{p,q,m,m'}^{\gamma_1,\gamma_2} \subset \mathbb{F}_{\gamma_1,\gamma_2}^{\gamma_1,\gamma_2} \).

(ii) If \( -2\beta \tau < \gamma < 0 < \beta \), then \( \mathbb{F}_{r,\infty}^{\gamma} \subset \mathbb{F}_{0,\infty}^{\gamma} \), where \( a(t, x) \in \mathbb{F}_{0,\infty}^{\gamma} \) if \( t^{-\gamma/2} 2^{\epsilon/2} |a_{jk}^\epsilon (t)| \leq 1 \).

**Proof.** We divide the proof into two cases.

Case 1: \( t2^{2\beta} \geq 1 \). Because \( f \in \mathbb{F}_{p,q,m}^{\gamma_1,\gamma_2,JV} \),

\[
|Q|^{-\frac{n-\beta}{m}} t^m \left( \sum_{j \geq \max \left\{ -\log_2 r, -\frac{\log_2 t}{m/2} \right\} \sum_{\epsilon,j,k} 2^{q_j(y_1+\frac{s}{2}+2m\epsilon)} |a_{jk}^\epsilon (t)|^q \chi(2^j x - k) \right)^{\frac{1}{q}} < \infty.
\]

Fix \( j, k \). We have

\[
|Q|^{-\frac{n-\beta}{m}} t^m \left( \int 2^{p_j(y_1+\frac{s}{2}+2m\epsilon)} |a_{jk}^\epsilon (t)|^p \chi(2^j x - k) dx \right)^{1/p} \leq |Q|^{-\frac{n-\beta}{m}} t^m 2^{\gamma_1} 2^{\epsilon} 2^{s/2} |a_{jk}^\epsilon (t)| 2^{-jn/p}.
\]

Let \( r = 2^{-j} \). We can get

\[
2^{-\gamma_2} 2^{jn/p} 2^{\gamma_1} 2^{jn/2} (t2^{2\beta} m)^m |a_{jk}^\epsilon (t)| 2^{-jn/p} \leq 1,
\]

that is,

\[
|a_{jk}^\epsilon (t)| \leq 2^{\gamma_2 - \gamma_1 - \frac{\epsilon}{2}} (t2^{2\beta})^{-m/2}.
\]

Case 2: \( 0 < t2^{2\beta} \leq 1 \). Because \( f \in \mathbb{F}_{p,q,m}^{\gamma_1,\gamma_2,JV} \), we can obtain that

\[
|Q|^{-\frac{n-\beta}{m}} \left( \sum_{-\log_2 r < j < -\log_2 \frac{t}{2}} \sum_{\epsilon,j,k} 2^{q_j(y_1+\frac{s}{2})} |a_{jk}^\epsilon (t)|^q \chi(2^j x - k) \right)^{1/q} \leq 1.
\]

This implies that for fixed \( \epsilon \) and \( k \),

\[
|a_{jk}^\epsilon (t)| \leq 2^{-j(y_1-\gamma_2)} 2^{-jn/2}.
\]
4. Semigroup characterization and Riesz operators

In this section, we characterize $\dot{F}^\gamma_{p,q} (\mathbb{R}^n)$ by Triebel type tent spaces $\mathcal{P}^\gamma_{p,q,m,m'}$. And we prove the continuity of Riesz operators on these spaces.

4.1. Semigroup characterization of $\dot{F}^\gamma_{p,q}$. For any dyadic cube $Q$, with side length $r$, we denote $\tilde{Q}$, the dyadic cube which contains $Q$, with side length $2^7 r$. For all $w \in \mathcal{Z}$, we write $\tilde{Q}_w := 2^7 r w + \tilde{Q}$, and say $(e', k') \in S^w_{\tilde{Q}}$, if $(e', j', k') \in \Lambda_n$ and $Q_{e', j', k'} \subset \tilde{Q}_w$. We have

**Theorem 4.1.** Given $\gamma_1, \gamma_2 \in \mathbb{R}$, $1 < p < m < \infty$, $\gamma_1 - \gamma_2 < 0 < \beta$, $m' > 0$ and $\tau + \frac{\gamma_1 - \gamma_2}{2\beta} > 0$.

(i) If $f \in \dot{F}^\gamma_{p,q}$, then $f \ast K^\beta_t \in \mathcal{P}^\gamma_{p,q,m,m'}$;

(ii) The operator $\pi_\delta$ is a bounded and surjective operator from $\mathcal{P}^\gamma_{p,q,m,m'}$ to $\dot{F}^\gamma_{p,q}$.

**Proof.** (1) We divide the proof into four parts.

**Part I.** $K^\beta_t \ast f \in \mathcal{P}^\gamma_{p,q,m,m'}$. Because

$$|a_{e, j,k}^e(t)| \leq e^{-c2^\beta |t|} \sum_{e', j', k' \leq 1} \frac{|a_{e', j', k'}^e|}{(1 + |2^j r k' - k|)^N},$$

we can get

$$f_{e_j} = \sum_{e', k'} 2^{|2^j r k' - k|} |a_{e', j', k'}^e| \chi(2^j r x - k).$$
We can get
\[
\sum_{e',k'} |a_{e',k'}|^r (1 + |2^{-j} k' - k|)^{-N} 
= 2^{-j}(\gamma + \frac{3}{2} + 2\mu \beta) \sum_{e',k'} 2^{j}(\gamma + \frac{3}{2} + 2\mu \beta) |a_{e',k'}| 
\leq 2^{-j}(\gamma + \frac{3}{2} + 2\mu \beta) M_A(f_j)(x), \quad x \in Q_{jk}.
\]

Notice that for fixed $j, k$, the number of $Q_{jk}$ such that $x \in Q_{jk}$ is 1. Then
\[
P_{\gamma_1, \gamma_2}^{\gamma_1, \gamma_2} 
\leq |Q_j|^{\gamma_1 - 1} \left\| \sum_{j' \geq \max \{-\log_2 r, -\log_2 t\}} \sum_{e',k'} 2^{j'}(\gamma + \frac{3}{2} + 2\mu \beta) |a_{e',k'}| e^{-ct2^j} \right\|_p 
\leq \left\| \sum_{j' \geq \max \{-\log_2 r, -\log_2 t\}} |f_j(x)|^{\gamma} \right\|_p 
\leq \left\| \sum_{j' \geq \max \{-\log_2 r, -\log_2 t\}} |f_j(x)|^{\gamma} \right\|_p.
\]

It is easy to see that
\[
\left( \sum_{j' \geq \max \{-\log_2 r, -\log_2 t\}} |f_j(x)|^{\gamma} \right)^{1/q} 
\leq \left( \sum_{j' \geq \max \{-\log_2 r, -\log_2 t\}} \sum_{e',k'} 2^{j'}(\gamma + 2\mu \beta) |a_{e',k'}| |\chi(2^{-j'} x - k')|^{\gamma} \right)^{1/q} 
\leq \left( \sum_{j' \geq \max \{-\log_2 r, -\log_2 t\}} 2^{j'}(\gamma + 2\mu \beta) \sum_{e',k'} |a_{e',k'}| |\chi(2^{-j'} x - k')| \right)^{1/q}.
\]

Hence we can get $P_{\gamma_1, \gamma_2}^{\gamma_1, \gamma_2} \lesssim \|f\|_{P_{\gamma_1, \gamma_2}^{\gamma_1, \gamma_2}}$.

**Part II.** $K_t^\beta \ast f \in P_{\gamma_1, \gamma_2}^{\gamma_1, \gamma_2}$.  
Similarly we have
\[
|a_{e',k'}^{e'}(t)| \leq e^{-ct2^j} \sum_{e',k'} |a_{e',k'}^{e'}| (1 + |2^{-j} k' - k|)^N.
\]
Hence

\[ II^{\gamma_1,\gamma_2}_{p,q} = |Q|^{\frac{1}{p}} \left\| \left( \sum_{-\log_2 r < j < -\frac{\log_2 Q}{2^\gamma r}} \sum_{Q_{j,k} \subset Q_r} 2^{q(j+\frac{1}{2})} |d_{j,k}^r| q \chi(2^j x - k) \right)^{1/q} \right\|_p \]

\[ \leq |Q|^{\frac{1}{p}} \left\| \left( \sum_{-\log_2 r < j < -\frac{\log_2 Q}{2^\gamma r}} \sum_{Q_{j,k} \subset Q_r} 2^{q(j+\frac{1}{2})} e^{-c q t^2 2^{\gamma r}} \chi(2^j x - k) \right)^{1/q} \right\|_p \]

\[ \times \left( \sum_{-\log_2 r < j < -\frac{\log_2 Q}{2^\gamma r}} \sum_{Q_{j,k} \subset Q_r} \frac{|a_{j,k}^r|}{(1+|2^j x - k|)^q} \right)^{1/q} \]

\[ \leq |Q|^{\frac{1}{p}} \left\| \left( \sum_{-\log_2 r < j < -\frac{\log_2 Q}{2^\gamma r}} \sum_{Q_{j,k} \subset Q_r} 2^{q(j+\frac{1}{2})} e^{-c q t^2 2^{\gamma r}} \chi(2^j x - k) \right)^{1/q} \right\|_p \]

\[ \times \sum_{|j'-j| \leq 1} \left( \sum_{Q_{j',k'} \subset Q_{j,k}} \frac{|a_{j',k'}^r|}{(1+|2^j x - k|)^q} \right)^{1/q} \]

\[ \leq |Q|^{\frac{1}{p}} \left\| \left( \sum_{-\log_2 r < j < -\frac{\log_2 Q}{2^\gamma r}} \sum_{Q_{j,k} \subset Q_r} 2^{q(j+\frac{1}{2})} e^{-c q t^2 2^{\gamma r}} \chi(2^j x - k) \right)^{1/q} \right\|_p \]

\[ \times \sum_{|j'-j| \leq 1} \left( \sum_{Q_{j',k'} \subset Q_{j,k}} \frac{2^{q(j+\frac{1}{2})} |a_{j',k'}^r|}{(1+|2^j x - k|)^q} \right)^{1/q} \chi(2^j x - k) \right\|_p. \]

Let

\[ f_j^w = \sum_{Q_{j,k} \subset Q_{j,k}^w} 2^{q(j+\frac{1}{2})} |d_{j,k}^r| q \chi(2^j x - k') \]

We can get

\[ II^{\gamma_1,\gamma_2}_{p,q} \]

\[ \leq |Q|^{\frac{1}{p}} \left\| \left( \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \left( \sum_{-\log_2 r < j < -\frac{\log_2 Q}{2^\gamma r}} \sum_{Q_{j,k} \subset Q_r} 2^{q(j+\frac{1}{2})} \right) \right)^{1/q} \right\|_p \]

\[ \times \sum_{|j'-j| \leq 1} \left( \sum_{Q_{j',k'} \subset Q_{j,k}} 2^{q(j'+\frac{1}{2})} |a_{j',k'}^r| q \chi(2^{j'} x - k') \right)^{1/q} \]

\[ \leq |Q|^{\frac{1}{p}} \left\| \left( \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \left( \sum_{-\log_2 r < j < -\frac{\log_2 Q}{2^\gamma r}} \sum_{Q_{j,k} \subset Q_r} 2^{q(j+\frac{1}{2})} \right) \right)^{1/q} \right\|_p \]

\[ \times \left( \sum_{Q_{j',k'} \subset Q_{j,k}^w} 2^{q(j'+\frac{1}{2})} |a_{j',k'}^r| q \chi(2^{j'} x - k') \right)^{1/q} \chi(2^{j'} x - k) \right\|_p. \]

By Hölder’s inequality, we obtain

\[ \left( \sum_{Q_{j',k'} \subset Q_{j,k}^w} |a_{j',k'}^r| q \chi(2^{j'} x - k') \right)^q \leq \left( \sum_{Q_{j',k'} \subset Q_{j,k}^w} |a_{j',k'}^r| q \chi(2^{j'} x - k') \right)^{q-1} \]

\[ \leq 2^{n(q-j)(q-1)} \left( \sum_{Q_{j',k'} \subset Q_{j,k}^w} |a_{j',k'}^r| q \chi(2^{j'} x - k') \right) \]

\[ \leq \left( \sum_{Q_{j',k'} \subset Q_{j,k}^w} |a_{j',k'}^r| q \chi(2^{j'} x - k') \right). \]
where we have used the fact that $|j - j'| \leq 1$. By the above estimate, we get

$$
II_{p,q}^{\gamma_1,\gamma_2} f \leq |Q|^{\frac{1}{2^q} - \frac{1}{2}} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \left\| \left( \sum_{-\log_2 r < j < -\log_2 \frac{r}{2^N}} \sum_{Q \in Q_j} 2^{q(j - j')(\gamma_1 + \frac{\gamma}{2})} \prod_{Q_j \subset Q} \right) \right\|_p
$$

$$
\times 2^{q(j+\gamma_1+\gamma_2)} \left( \sum_{Q_j \subset Q} |a_{p_j,k}^e|^q \chi(2^{j} x - k) \right)^{1/q} \chi(2^{j} x - k) \left\| \right\|_p
$$

$$
\leq |Q|^{\frac{1}{2^q} - \frac{1}{2}} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \left\| \left( \sum_{-\log_2 r < j < -\log_2 \frac{r}{2^N}} \sum_{Q_j \subset Q} 2^{q(j - j')(\gamma_1 + \frac{\gamma}{2})} \prod_{Q_j \subset Q} \right) \right\|_p
$$

$$
\leq \|f\|_{II_{p,q}^{\gamma_1,\gamma_2}}.
$$

**Part III.** $K^B_{\gamma} \ast f \in II_{p,q,m}^{\gamma_1,\gamma_2,III}.

$$
III_{p,q,m}^{\gamma_1,\gamma_2} f = |Q|^{\frac{1}{2^q} - \frac{1}{2}} \left\| \left( \sum_{(e, j, k) \in \Lambda_0^e} 2^{q(j+\gamma_1+\frac{\gamma}{2})} \int_{2^{-2^q} r < t < 2^{2^q} r} |a_{p_j,k}^e|^q r^{q m} dt \chi(2^{j} x - k) \right)^{1/q} \right\|_p
$$

$$
\leq |Q|^{\frac{1}{2^q} - \frac{1}{2}} \left\| \left( \sum_{(e, j, k) \in \Lambda_0^e} 2^{q(j+\gamma_1+\frac{\gamma}{2})} \int_{2^{-2^q} r < t < 2^{2^q} r} e^{-ct^{2^q} + \frac{\gamma}{2}} r^{q m} dt \chi(2^{j} x - k) \right)^{1/q} \right\|_p
$$

$$
\leq |Q|^{\frac{1}{2^q} - \frac{1}{2}} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \left\| \left( \sum_{(e, j, k) \in \Lambda_0^e} 2^{q(j+\gamma_1+\frac{\gamma}{2})} \int_{2^{-2^q} r < t < 2^{2^q} r} e^{-ct^{2^q} + \frac{\gamma}{2}} r^{q m} dt \chi(2^{j} x - k) \right)^{1/q} \right\|_p
$$

It is easy to see that

$$
\int_{2^{-2^q} r < t < 2^{2^q} r} e^{-ct^{2^q} + \frac{\gamma}{2}} r^{q m} dt \leq 1.
$$

Then let

$$
f^w_j(x) = \sum_{Q_j \subset Q^{\gamma_2}_j} 2^{q(j+\gamma_1+\frac{\gamma}{2})} |a_{p_j,k}^e|^q \chi(2^{j} x - k).
$$
We can get

\[
\begin{align*}
\sum_{j, j' \leq 1} \left| \sum_{Q_{j, j'} \subset Q_{i, k} \cap \mathbb{Z}^{n}} \frac{1}{Q_{j, j'}} \left( \sum_{k', l' \leq 1} \sum_{Q_{k', l'} \subset Q_{i, k} \cap \mathbb{Z}^{n}} 2^{\alpha(j + \frac{\alpha}{2})} \chi(2^j x - k') \right)^{\frac{1}{q}} \right| \lesssim \| f \|_{L^{q}(\mathbb{R}^n)}.
\end{align*}
\]

**Part IV.** \( K_{\delta}^{\psi} * f \in L^{q}(\mathbb{R}^n) \)

Similar to the proof of Part III, we can obtain that

\[
\begin{align*}
IV^{\psi}_{p, q, m, m'} &= \| f \|_{L^{q}(\mathbb{R}^n)}.
\end{align*}
\]

(2) Now we prove

\[
\pi_{\phi, f}(t, x) : \mathbb{R}^{\psi}_{p, q, m, m'} \to F^{\psi}_{p, q}.
\]
For the dyadic cube $Q$, and $j \geq -\log_2 r$, Hólder's inequality implies that

$$|a^e_{jk}|^q \lesssim \left| \sum_{\mathcal{A}_0 \subset \mathcal{A}^n_0} 2^{q(j+\frac{1}{2})} |a^e_{jk}|^q \chi(2^j x - k) \right|_{L^p}^{1/q}.$$
We can get

\[
M_1 \leq \left\| Q_j^{\frac{1}{2}} \left( \sum_{(e,j,k) \in \Lambda^n_0} 2^{q_j(y_1+\frac{2}{p})} \sum_{|j-j'| \leq 1} \sum_{e',k'} \left( 1 + |2^{j-j'} k' - k| \right)^{-N} \left( \int_0^{2^j} (t^{2^{j-j'}})^{-N} 2^{-n j'/(2^{j-j'} y_1-m_0)} \frac{dt}{t} \right)^{\frac{p}{2}} \chi(2^j x - k) \right) \right\|_p
\]

\[
\leq \left\| Q_j^{\frac{1}{2}} \left( \sum_{(e,j,k) \in \Lambda^n_0} 2^{q_j(y_1+\frac{2}{p})} \sum_{|j-j'| \leq 1} \sum_{e',k'} \left( 1 + |2^{j-j'} k' - k| \right)^{-N} \left( \int_{2^{-j-j'}}^{2^j} (t^{2^{j-j'}})^{-N} 2^{-n j'/(2^{j-j'} y_1-m_0)} \frac{dt}{t} \right)^{\frac{p}{2}} \chi(2^j x - k) \right) \right\|_p
\]

\[
\leq \left\| \left( \sum_{(e,j,k) \in \Lambda^n_0} 2^{q_j(y_1+\frac{2}{p})} \sum_{|j-j'| \leq 1} \sum_{e',k'} \left( 1 + |2^{j-j'} k' - k| \right)^{-N} \right) Q_{j,k} \left( 2^{-n j'/(2^{j-j'} y_1-m_0)} \right) \right\|_p
\]

\[
\leq \left\| \left( \sum_{(e,j,k) \in \Lambda^n_0} 2^{q_j(y_1+\frac{2}{p})} \sum_{|j-j'| \leq 1} \sum_{e',k'} \left( 1 + |2^{j-j'} k' - k| \right)^{-N} \right) \right\|_p \leq 1.
\]

For \( M_2 \), we can get

\[
M_2 \leq \left\| Q_j^{\frac{1}{2}} \left( \sum_{(e,j,k) \in \Lambda^n_0} 2^{q_j(y_1+\frac{2}{p})} \sum_{|j-j'| \leq 1} \sum_{e',k'} \left( 1 + |2^{j-j'} k' - k| \right)^{-N} \left( \int_2^{2^j} (t^{2^{j-j'}})^{-N} |a_{j,k'}^{e'}(t)| \frac{dt}{t} \right) \right) \right\|_p.
\]

Let

\[
b_{j,k'}^{e'} = \int_2^{2^j} (t^{2^{j-j'}})^{-N} |a_{j,k'}^{e'}(t)| \frac{dt}{t}.
\]

We can get

\[
\sum_{|j-j'| \leq 1} \sum_{e',k'} \left( 1 + |2^{j-j'} k' - k| \right)^{-N} \left( \int_2^{2^j} (t^{2^{j-j'}})^{-N} |a_{j,k'}^{e'}(t)| \frac{dt}{t} \right) =: \sum_{|j-j'| \leq 1} \sum_{e',k'} \left( 1 + |2^{j-j'} k' - k| \right)^{-N} b_{j,k'}^{e'}
\]

\[
\leq 2^{-j' (y_1+2m_0+\frac{2}{p})} \sum_{e',k'} \frac{2^{q_j(y_1+2m_0+\frac{2}{p})} |b_{j,k'}^{e'}|}{(1+|2^{j-j'} k' - k|)^y}
\]

\[
\leq 2^{-j' (y_1+2m_0+\frac{2}{p})} M_A(f_j')(x), \ x \in Q_{j,k},
\]
We obtain that

\[
M_2 \leq |Q_r|^{\frac{2}{2-\beta}} \left( \sum_{(e,k) \in \Lambda_{0}} \sum_{|j| \leq 1} 2^{qj(y_1+\frac{2}{2-\beta}j)} \right)^{\frac{1}{q}}
\]

where

\[
f_j(x) = \sum_{e',k'} 2^{j(y_1+2\beta + \frac{2}{2-\beta}j)} |b_{j,e',k'}^e| \chi(2^j x - k').
\]

Then

\[
M_2 \leq |Q_r|^{\frac{2}{2-\beta}} \left( \sum_{(e,k) \in \Lambda_{0}} \sum_{|j| \leq 1} 2^{qj(y_1+\frac{2}{2-\beta}j)} \right)^{\frac{1}{q}}
\]

Finally we obtain

\[
M_2 \leq |Q_r|^{\frac{2}{2-\beta}} \sum_{w \in \mathbb{Z}^n} \left( \sum_{j \geq -\log r_w} \sum_{Q_j \subset Q_r} \right)^{\frac{1}{q}} \leq \|f\|_{p,\gamma,\tilde{\gamma}^{(n)}}.
\]

Now we deal with \(M_3\).

\[
M_3 \leq |Q_r|^{\frac{2}{2-\beta}} \left( \sum_{(e,k) \in \Lambda_{0}} \sum_{|j| \leq 1} (1 + |2^{-j} k' - k|)^{-N} \right)^{\frac{1}{q}}
\]

Let

\[
b_{j,e',k'}^e = \int_0^{2^{-2j}} (t2^{j} \beta)^N |a_{j,e',k'}^e(t)| \frac{dt}{t}.
\]

We obtain that

\[
\sum_{e',k'} (1 + |2^{-j} k' - k|)^{-N} \left( \int_0^{2^{-2j}} (t2^{j} \beta)^N |a_{j,e',k'}^e(t)| \frac{dt}{t} \right) \leq 2^{-j(y_1+\frac{2}{2-\beta}j)} \sum_{e',k'} \frac{2^{j(y_1+\frac{2}{2-\beta}j)} |b_{j,e',k'}^e|}{(1 + |2^{-j} k' - k|)^{\tilde{\gamma}^N}}
\]

where

\[
f_j = \sum_{e',k'} 2^{j(y_1+\frac{2}{2-\beta}j)} |b_{j,e',k'}^e| \chi(2^j x - k').
\]
So
\[
M_3 \lesssim |Q_1|^{\frac{2}{p} - \frac{1}{q}} \sum_{l \in \mathbb{Z}^n} (1 + |w|)^{-N} \left\| \left( \sum_{j \geq -\log_2 r} 2^{q'j} (\gamma_1 + \frac{q}{2}) \sum_{\mathcal{Q}_{x',k'} \subset \mathcal{Q}_l} |b_{j,k'}^{\varepsilon} q' (2j'x - k')^{1/q} \right)^{1/p} \right\|_p.
\]

By Hölder’s inequality, we get
\[
|b_{j,k'}^{\varepsilon} q'| = \left( \int_0^{2^{-2j'\beta}} (t^{2j\beta})^N |a_{k,k'}^\varepsilon (t)|^{|\beta|q} \right)^q \\
\leq \left( \int_0^{2^{-2j'\beta}} (t^{2j\beta})^{qm'} |a_{k,k'}^\varepsilon (t)|^{q'\beta} \right) \left( \int_0^{2^{-2j'\beta}} (t^{2j\beta})^{N-qm'} \frac{dt}{t} \right)^{q-1} \\
\leq \int_0^{2^{-2j'\beta}} (t^{2j\beta})^{qm'} |a_{k,k'}^\varepsilon (t)|^{q'\beta} \frac{dt}{t}
\]

Hence
\[
M_3 \lesssim |Q_1|^{\frac{2}{p} - \frac{1}{q}} \sum_{l \in \mathbb{Z}^n} (1 + |w|)^{-N} \left\| \left( \sum_{j \geq -\log_2 r} 2^{q'j} (\gamma_1 + \frac{q}{2} + 2m'\beta) \int_0^{2^{-2j'\beta}} t^{qm'} |a_{k,k'}^\varepsilon (t)|^{q'\beta} \right)^{1/q} \right\|_p \\
\lesssim \|f\|_{L^{p,q,m,m'}}.
\]

4.2. Continuity of Riesz operators on $\mathbb{R}^{\gamma_1,\gamma_2}_{p,q,m,m'}$. For Riesz operators $R_l$ ($l = 1, \cdots, n$), $(\varepsilon, j, k), (\varepsilon', j', k') \in \Lambda_n$, denote
\[
ad_{j,k}^{\varepsilon,\varepsilon',l} = \langle \Phi_{j,k}^{\varepsilon}, R_l \Phi_{j',k'}^{\varepsilon'} \rangle.
\]

If $|j - j'| \geq 2$, then $a_{j,k}^{\varepsilon,\varepsilon',l} = 0$. Similar to the proof in the Lemma 2.9, we can verify that the Riesz transforms $R_l$, $l = 1, \cdots, n$, are continuous on $\mathbb{R}^{\gamma_1,\gamma_2}_{p,q,m,m'}$. See also [2, 19, 33].

**Theorem 4.2.** Given $1 < p, q < \infty$, $\gamma_1, \gamma_2 \in \mathbb{R}$, $m > p$ and $m' > 0$. The Riesz transforms $R_1, R_2, \cdots, R_n$ are bounded on $\mathbb{R}^{\gamma_1,\gamma_2}_{p,q,m,m'}$.

**Proof.**
\[
R_l g(x) = \sum_{(\varepsilon, j, k) \in \Lambda_n} g_{j,k}^{\varepsilon} (t) R_l \Phi_{j,k}^{\varepsilon} (x) \\
= \sum_{(\varepsilon, j, k) \in \Lambda_n} b_{j,k}^{\varepsilon} (t) \Phi_{j,k}^{\varepsilon} (x),
\]

where
\[
b_{j,k}^{\varepsilon} (t) = \langle R_l g(t, \cdot), \Phi_{j,k}^{\varepsilon} \rangle = \sum_{|j-j'| \leq 1} \sum_{|e,e'\varepsilon|} a_{j,k,j',k'}^{\varepsilon,e} g_{j,k'}^{e'} (t).
\]

By Lemma 2.7, we have
\[
|a_{j,k,j',k'}^{\varepsilon,e} | \lesssim 2^{-|j-j'|(\frac{q}{2} + N_0)} \left( \frac{2^{-j} + 2^{-j'}}{2^{-j} + 2^{-j'} + 2^{-\gamma_1k - 2^{-\gamma_2k'}}} \right)^{n+N_0}.
\]

We divide the proof into four parts.

**Step 1:** $(R_l g)(t, x) \in \mathbb{R}^{\gamma_1,\gamma_2}_{p,q,m}$, $l = 1, 2, \cdots, n$. 

We can see that
\[
P_{p,q,m}^{1,2} = \left| Q \right|^{\frac{2}{m}} \left\| \left( \sum_{j \geq \max_{r \in (\log_2 r, -\frac{\log_2 r}{2})} |Q|_j \chi(2^j x - k) \right) \right\|_{p}^{1/4} \left( \sum_{j \geq \max_{r \in (\log_2 r, -\frac{\log_2 r}{2})} |Q|_j \chi(2^j x - k) \right)^{1/4} \right\|_{p}^{1/4}
\]
\[
\leq \left| Q \right|^{\frac{2}{m}} \left\| \left( \sum_{j \geq \max_{r \in (\log_2 r, -\frac{\log_2 r}{2})} |Q|_j \chi(2^j x - k) \right) \right\|_{p}^{1/4} \left( \sum_{j \geq \max_{r \in (\log_2 r, -\frac{\log_2 r}{2})} |Q|_j \chi(2^j x - k) \right)^{1/4} \right\|_{p}^{1/4}
\]
Let
\[
f_j = \sum_{\epsilon', \epsilon''} 2^{j(\gamma_1 + \frac{\epsilon}{2} + 2\beta)} |g_{\epsilon, \epsilon''_{j,k}}(t)| \chi(2^j x - k').
\]
Then we get
\[
P_{p,q,m}^{1,2}(t) \leq \left| Q \right|^{\frac{2}{m}} \left\| \left( \sum_{j \geq \max_{r \in (\log_2 r, -\frac{\log_2 r}{2})} |Q|_j \chi(2^j x - k) \right) \right\|_{p}^{1/4} \left( \sum_{j \geq \max_{r \in (\log_2 r, -\frac{\log_2 r}{2})} |Q|_j \chi(2^j x - k) \right)^{1/4} \right\|_{p}^{1/4}
\]
\[
\leq \left| Q \right|^{\frac{2}{m}} \left\| \left( \sum_{j \geq \max_{r \in (\log_2 r, -\frac{\log_2 r}{2})} |Q|_j \chi(2^j x - k) \right) \right\|_{p}^{1/4} \left( \sum_{j \geq \max_{r \in (\log_2 r, -\frac{\log_2 r}{2})} |Q|_j \chi(2^j x - k) \right)^{1/4} \right\|_{p}^{1/4}
\]
\[
\times \left( \sum_{\epsilon', \epsilon''_{j,k} \subset Q^o_{\epsilon}} \left| g_{\epsilon', \epsilon''_{j,k}}(t)|\chi(2^j x - k') \right| \right)^{1/4} \right\|_{p}^{1/4}.
\]
By Hölder’s inequality, we obtain
\[
P_{p,q,m}^{1,2}(t) \leq \left| Q \right|^{\frac{2}{m}} \left\| \left( \sum_{j \geq \max_{r \in (\log_2 r, -\frac{\log_2 r}{2})} |Q|_j \chi(2^j x - k) \right) \right\|_{p}^{1/4} \left( \sum_{j \geq \max_{r \in (\log_2 r, -\frac{\log_2 r}{2})} |Q|_j \chi(2^j x - k) \right)^{1/4} \right\|_{p}^{1/4}
\]
\[
\times \sum_{\epsilon', \epsilon''_{j,k} \subset Q^o_{\epsilon}} \left| g_{\epsilon', \epsilon''_{j,k}}(t)|\chi(2^j x - k') \right| \right\|_{p}^{1/4}.
\]
\[
\leq \left\| g \right\|_{L^2(Q_{p,q,m})} + \left\| g \right\|_{L^2(Q_{p,q,m})}.
\]
Step II: \((R_G(t, x) \in B^p_{\alpha, q})^{II}, l = 1, 2, \ldots, n.\)

For this term, let the radius \(r \) of \(Q_r \) be \(2^{-j_0}.\)

\[
I_{p,q,Q_r}^{\gamma_1, \gamma_2} = |Q_r|^{\frac{2}{p} - \frac{2}{\alpha}} \left\| \left( \sum_{j_0 < j < -\log_2 t/2\beta} \sum_{Q_{jk} \subset Q_r} 2^{q(j+\frac{2}{p})} |a_{j,k}^e(t)|^{q} \chi(2^j x - k) \right)^{1/q} \right\|_p.
\]

\[
\leq |Q_r|^{\frac{2}{p} - \frac{2}{\alpha}} \left\| \left( \sum_{j_0 < j < -\log_2 t/2\beta} \sum_{Q_{jk} \subset Q_r} 2^{q(j+\frac{2}{p})} \right) \right\|_p.
\]

Let

\[
f_j = 2^j(\gamma_1 + \frac{2}{q}) \sum_{Q_{jk'} \subset Q_{jk'}} |g_{j,k'}^e(t)| \chi(2^j x - k').
\]

We have

\[
I_{p,q,Q_r}^{\gamma_1, \gamma_2} \leq |Q_r|^{\frac{2}{p} - \frac{2}{\alpha}} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \left\| \left( \sum_{j_0 < j < -\log_2 t/2\beta} \sum_{Q_{jk} \subset Q_r} 2^{q(j+\frac{2}{p})} \sum_{|j-j'| \leq 1} (M_A(f_j}(x)(x))^{q} \chi(2^j x - k) \right)^{1/q} \right\|_p.
\]

Because \(|j-j'| \leq 1 \) and \(j_0 < j < -\log_2 t/2\beta,\) we have \(j' \geq -\log_2 t - 1 \) and \(j' \leq -\log_2 t/2\beta + 1.\) On the other hand, The facts \(Q_{jk} \subset Q_r,\) and \(Q_{j', k'} \subset Q_{jk}\) imply that \(Q_{j', k'} \subset Q_{jr}.\) We obtain that

\[
I_{p,q,Q_r}^{\gamma_1, \gamma_2} \leq |Q_r|^{\frac{2}{p} - \frac{2}{\alpha}} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \left\| \left( \sum_{j_0 < j < -\log_2 t/2\beta} \sum_{Q_{jk} \subset Q_r} 2^{q(j+\frac{2}{p})} \left( \sum_{(j', k') \subset Q_{jk}} |g_{j,k'}^e(t) \chi(2^j x - k') \right)^{1/q} \right) \right\|_p.
\]

\[
=: M_1 + M_2,
\]

where

\[
M_1 = |Q_r|^{\frac{2}{p} - \frac{2}{\alpha}} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \left\| \left( \sum_{j_0 < j < -\log_2 t/2\beta} \sum_{Q_{jk} \subset Q_r} 2^{q(j+\frac{2}{p})} \left( \sum_{(j', k') \subset Q_{jk}} |g_{j,k'}^e(t) \chi(2^j x - k') \right)^{1/q} \right) \right\|_p.
\]
and

\[ M_2 = |Q|^{\frac{2}{p'}} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \left\| \sum_{j' \geq \max\{ -\log_2 r_w, -\frac{\log_2 l}{2p'} \}} 2^{q_j'(y_1 + \frac{q}{2})} \left( \sum_{(e', k') : Q_{e', k'} \subset Q_j} |g^e_{j', k'}(t)| \chi_{2^j x - k'} \right)^{q_j'/q} \right\|_p \]

Obviously \( M_1 \lesssim \|g\|_{p,q,n}. \) For \( M_1, \) if \( j' \geq \max\{ -\log_2 r_w, -\frac{\log_2 l}{2p'} \}, \) then \((2^{2j} \beta)^{mq} \geq 1\) and

\[ M_2 \lesssim |Q|^{\frac{2}{p'}} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \left\| \sum_{j' \geq \max\{ -\log_2 r_w, -\frac{\log_2 l}{2p'} \}} 2^{q_j'(y_1 + \frac{q}{2})} (t2^{2j} \beta)^{mq} \right\|_p \]

\[ \lesssim |Q|^{\frac{2}{p'}} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \left\| \sum_{j' \geq \max\{ -\log_2 r_w, -\frac{\log_2 l}{2p'} \}} 2^{q_j'(y_1 + \frac{q}{2})} \chi_{2^j x - k'} \right\|_p \]

Step III: \((Rg)(t, x) \in L^p_{\gamma_1, \gamma_2, \mu}, l = 1, 2, \ldots, n.\)

\[ III_{p,q,m}^{\gamma_1, \gamma_2} \lesssim \left\| Q \right\|^{\frac{2}{p'}} \sum_{(r, j) \in \Lambda_{N_0}^n} 2^{q_j'(y_1 + \frac{q}{2}) + 2mq} \int_{2^{2j} \beta}^{2^{2j+1} \beta} t^{mq} \left| \int_{-\log_2 r_w}^{0} \frac{dt}{t} \chi_{2^j x - k} \right|^{1/q} \right\|_p \]

\[ \lesssim \left\| Q \right\|^{\frac{2}{p'}} \sum_{(r, j) \in \Lambda_{N_0}^n} 2^{q_j'(y_1 + \frac{q}{2}) + 2mq} \int_{2^{2j} \beta}^{2^{2j+1} \beta} t^{mq} \left( \sum_{e', k'} \frac{|g^e_{j', k'}(t)|}{(1 + |2^j x - k|^{p/m} \chi_{2^j x - k})} \right)^{q_j'/q} \chi_{2^j x - k} \right\|_p \]

\[ \lesssim \left\| Q \right\|^{\frac{2}{p'}} \sum_{(r, j) \in \Lambda_{N_0}^n} 2^{q_j'(y_1 + \frac{q}{2}) + 2mq} \left( \int_{2^{2j} \beta}^{2^{2j+1} \beta} t^{mq} \left| g^e_{j', k'}(t) \right|^q \frac{dt}{t} \right)^{1/q} \chi_{2^j x - k} \right\|_p \]

Let

\[ |b^e_{j', k'}| = \int_{2^{2j} \beta}^{2^{2j+1} \beta} t^{mq} \left| g^e_{j', k'}(t) \right|^q \frac{dt}{t}. \]

Because \(|j - j'| \leq 1,\)

\[ |b^e_{j', k'}| \lesssim \int_{2^{2j} \beta}^{2^{2j+1} \beta} t^{mq} \left| g^e_{j', k'}(t) \right|^q \frac{dt}{t}. \]
We can get

\[
III_{p,q,m}^{y_1, y_2} \lesssim |Q_1|^{\frac{p}{p+1}} \left\| \left( \sum_{(e, j, k) \in \mathbb{A}_0} 2q_j (y_1 + \frac{1}{4} + 2\eta_j) \sum_{|j| \leq 1} \frac{|b_{j,k}^{e'}|}{|1 + [2j_{k,k}' k'-k]|^{\eta_k} \chi(2j x - k)} \right)^{1/q} \right\|_p
\]

\[
\lesssim |Q_1|^{\frac{p}{p+1}} \left\| \left( \sum_{(e, j, k) \in \mathbb{A}_0} 2q_j (y_1 + \frac{1}{4} + 2\eta_j) \sum_{|j| \leq 1} \frac{|b_{j,k}^{e'}|}{|1 + [2j_{k,k}' k'-k]|^{\eta_k} \chi(2j x - k)} \right)^{1/q} \right\|_p
\]

Let

\[
g_{f'} = \sum_{Q_{f',k'} \subset Q_r} 2^{f' (y_1 + \frac{1}{4} + 2\eta_j)} |b_{f',k'}^{e'}(t)|^{1/q} \chi(2j x - k').
\]

Then

\[
III_{p,q,m}^{y_1, y_2} \lesssim |Q_1|^{\frac{p}{p+1}} \left\| \left( \sum_{j \geq - \log_2 r_u} (M_n g_{f'}(x))^{q_j} \right)^{1/q} \right\|_p
\]

\[
\lesssim |Q_1|^{\frac{p}{p+1}} \left\| \left( \sum_{j \geq - \log_2 r_u} |g_{f'}(x)|^q \right)^{1/q} \right\|_p
\]

\[
\lesssim |Q_1|^{\frac{p}{p+1}} \left\| \left( \sum_{j \geq - \log_2 r_u} 2q_j (y_1 + \frac{1}{4} + 2\eta_j) \sum_{Q_{f',k'} \subset Q_r} |b_{f',k'}^{e'}(t)|^{1/q} \chi(2j x - k') \right)^{1/q} \right\|_p.
\]

For fixed $j'$, there exist only one $Q_{f',k'}$ such that $x \in Q_{f',k'}$. Then

\[
\left( \sum_{Q_{f',k'} \subset Q_r} |b_{f',k'}^{e'}(t)|^{1/q} \chi(2j x - k') \right)^{q_j}
\]

\[
\lesssim \sum_{Q_{f',k'} \subset Q_r} |b_{f',k'}^{e'}(t)| \chi(2j x - k').
\]

Hence

\[
III_{p,q,m}^{y_1, y_2} \lesssim |Q_1|^{\frac{p}{p+1}} \left\| \left( \sum_{j \geq - \log_2 r_u} 2q_j (y_1 + \frac{1}{4} + 2\eta_j) \sum_{Q_{f',k'} \subset Q_r} |b_{f',k'}^{e'}(t)| \chi(2j x - k') \right)^{1/q} \right\|_p
\]

\[
\lesssim |Q_1|^{\frac{p}{p+1}} \left\| \left( \sum_{j \geq - \log_2 r_u} 2q_j (y_1 + \frac{1}{4} + 2\eta_j) \sum_{Q_{f',k'} \subset Q_r} \int_{2^{2j} 2^{f'-2\eta_j}} \int_{2^{2j} 2^{f'-2\eta_j}} \int_{2^{2j} 2^{f'-2\eta_j}} \sum_{Q_{f',k'} \subset Q_r} |g_{f'}^{e'}(t)| q_j^{1/q} \chi(2j x - k') \right)^{1/q} \right\|_p
\]

\[
=: M_1 + M_2,
\]

where

\[
M_1 = |Q_1|^{\frac{p}{p+1}} \left\| \left( \sum_{j \geq - \log_2 r_u} 2q_j (y_1 + \frac{1}{4} + 2\eta_j) \sum_{Q_{f',k'} \subset Q_r} \int_{2^{2j} 2^{f'-2\eta_j}} \int_{2^{2j} 2^{f'-2\eta_j}} \int_{2^{2j} 2^{f'-2\eta_j}} |g_{f'}^{e'}(t)| \chi(2j x - k') \right)^{1/q} \right\|_p
\]
and

$$M_2 = |Q|^{\frac{2a}{p} - \frac{1}{p}} \sum_{w \in \mathbb{Z}^n} \frac{1}{(1 + |w|)^\alpha} \left( \sum_{j' \geq -\log_2 r_w} 2^{j'q(y_1 + \frac{\beta}{2} + 2m\beta)} \sum_{Q_{f', k'} \subset Q_w} \int_{2^{-2j'\beta} - 2q}^{2^{-j'\beta}} t^{\alpha q} |g_{f', k'}(t)|^{q} \frac{dt}{t} \chi(2^{j'}x - k') \right)^{\frac{1}{q}}_p.$$  

It is easy to see that $M_1 \leq \|g\|_{L^{2, \gamma_2, IV_{\mu}}_{p, q, m}}$. For $M_2$, because $t < 2^{2j\beta}$, $(2^{2j\beta})^m \leq (t2^{j\beta})^m$. Then

$$M_2 \leq |Q|^{\frac{2a}{p} - \frac{1}{p}} \sum_{w \in \mathbb{Z}^n} \frac{1}{(1 + |w|)^\alpha} \left( \sum_{j' \geq -\log_2 r_w} 2^{j'q(y_1 + \frac{\beta}{2} + 2m\beta)} \sum_{Q_{f', k'} \subset Q_w} \int_{2^{-2j'\beta} - 2q}^{2^{-j'\beta}} t^{\alpha q} |g_{f', k'}(t)|^{q} \frac{dt}{t} \chi(2^{j'}x - k') \right)^{\frac{1}{q}}_p \leq \|g\|_{L^{2, \gamma_2, IV_{\mu}}_{p, q, m}}.$$

**Step IV:** $(Rg)(t, x) \in L^{2, \gamma_2, IV_{\mu}}_{p, q, m}$, $l = 1, 2, \ldots, n$.

$$IV_{\mu}^{\gamma_1, \gamma_2}_{p, q, m} = |Q|^{\frac{2a}{p} - \frac{1}{p}} \left( \sum_{(e, k) \in \mathbb{A}_0} 2^{qj(y_1 + \frac{\beta}{2} + 2m\beta)} \int_0^{2^{-2j\beta}} t^{\alpha q} |g_{e, k}(t)|^{q} \frac{dt}{t} \chi(2^jx - k) \right)^{1/q}_p \leq |Q|^{\frac{2a}{p} - \frac{1}{p}} \left( \sum_{(e, k) \in \mathbb{A}_0} 2^{qj(y_1 + \frac{\beta}{2} + 2m\beta)} \sum_{|j - j'| \leq 1} \int_0^{2^{-2j\beta}} t^{\alpha q} \left( \sum_{e', k'} \frac{|g_{e', k'}(t)|^{q}}{(1 + 2^{|j-j'|}k - k')^{\mu + \eta_0}} \right) \frac{dt}{t} \chi(2^jx - k) \right)^{1/q}_p \leq |Q|^{\frac{2a}{p} - \frac{1}{p}} \left( \sum_{(e, k) \in \mathbb{A}_0} 2^{qj(y_1 + \frac{\beta}{2} + 2m\beta)} \sum_{|j - j'| \leq 1} \int_0^{2^{-2j\beta}} t^{\alpha q} \frac{1}{(1 + 2^{|j-j'|}k - k')^{\mu + \eta_0}} \frac{dt}{t} \chi(2^jx - k) \right)^{1/q}_p.$$

because $0 < t < 2^{-2j\beta}$ and $|j - j'| \leq 1$, $2^{-2j\beta} \leq 2^{-2j'\beta + 2\beta}$. We can get

$$IV_{\mu}^{\gamma_1, \gamma_2}_{p, q, m} \leq |Q|^{\frac{2a}{p} - \frac{1}{p}} \left( \sum_{(e, k) \in \mathbb{A}_0} 2^{qj(y_1 + \frac{\beta}{2} + 2m\beta)} \sum_{|j - j'| \leq 1} \sum_{e', k'} \int_0^{2^{-2j\beta}} t^{\alpha q} |g_{e', k'}(t)|^{q} \frac{dt}{t} \chi(2^jx - k) \right)^{1/q}_p.$$

Let

$$b_{f, k}^{e'} = \int_0^{2^{-j\beta}} t^{\alpha q} |g_{f, k}(t)|^{q} \frac{dt}{t}.$$
Then

\[ IV_{p,q,m'}^{\gamma_1,\gamma_2} \lesssim |Q|^{\frac{2\gamma}{p}} \left( \sum_{(e,j,k) \in \Lambda_0} 2^{qj(\gamma_1 + \frac{\beta}{2} + 2m')} \sum_{|j-j'| \leq 1} |b_{g,f,x}|^{1/q} \right) \left( \sum_{|j-j'| \leq 1} \chi(2^j x - k) \right)^{1/q} \]

\[ \lesssim |Q|^{\frac{2\gamma}{p}} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \left( \sum_{j \geq -\log_2 r_w} \left| M_A(g f)(x) \right|^q \right)^{1/q} \]

\[ \lesssim |Q|^{\frac{2\gamma}{p}} \left( \sum_{Q, s' \subset Q} \left| b_{g,f,x}^{1/q} \chi(2^j x - k') \right|^q \right)^{1/q} \]

Take

\[ g' = 2^{j(\gamma_1 + \frac{\beta}{2} + 2m')} \sum_{Q, s' \subset Q} |b_{g,f,x}^{1/q} \chi(2^j x - k')| \]

We obtain

\[ IV_{p,q,m'}^{\gamma_1,\gamma_2} \lesssim |Q|^{\frac{2\gamma}{p}} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \left( \sum_{j \geq -\log_2 r_w} \left| M_A(g f)(x) \right|^q \right)^{1/q} \]

\[ \lesssim |Q|^{\frac{2\gamma}{p}} \left( \sum_{Q, s' \subset Q} \left| b_{g,f,x}^{1/q} \chi(2^j x - k') \right|^q \right)^{1/q} \]

\[ \lesssim |Q|^{\frac{2\gamma}{p}} \left( \sum_{Q, s' \subset Q} \left( \int_0^{2^{-\gamma j} + 2m'} 2^{j(\gamma_1 + \frac{\beta}{2} + 2m')} \right) \right)^{1/q} \]

It is easy to see that

\[ IV_{p,q,m'}^{\gamma_1,\gamma_2} \lesssim |Q|^{\frac{2\gamma}{p}} \sum_{w \in \mathbb{Z}^n} (1 + |w|)^{-N} \left( \sum_{j \geq -\log_2 r_w} \left| M_A(g f)(x) \right|^q \right)^{1/q} \]

\[ \lesssim |Q|^{\frac{2\gamma}{p}} \left( \sum_{Q, s' \subset Q} \left| b_{g,f,x}^{1/q} \chi(2^j x - k') \right|^q \right)^{1/q} \]

\[ \lesssim \|g\|_{p,q,m'} + \|g\|_{p,q,m'}^{\gamma_1,\gamma_2,j} \]
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