Simulations of physical regular black holes in fluids

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Abstract

On the premise of emphasizing the importance of energy conditions for regular black holes, we propose a method to remedy those models which break the energy conditions, e.g., the Bardeen and Hayward black holes. Likewise, we prove a no-go theorem for conformally related regular black holes, which states that the energy conditions can never be met in this class of black holes. In order to seek the evidences of distinguishing the regular black holes from the traditional black holes with singularities at their centers, we resort the aid of analogue gravity as a tool to mimic the physical regular black holes in a fluid. The equations of state for the fluid are carried out by an asymptotic analysis associated with a numerical method, which provides a modus operandi for experimental observations, in particular, the conditions under which one can simulate the physical regular black holes in the fluid.

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1 Introduction

As is well-known, Einstein’s general relativity lacks [1] the ultraviolet (UV) completeness which is reflected [2] in the singular solutions of Einstein’s equations at the classical level and in the non-renormalizability at the quantum level. The regular black holes (RBHs) [3, 4] which have no curvature singularities at the centers spring up in order to launch a challenge to the UV incompleteness at the classical level. This challenge was originated in changing the perspective of vacuum [5, 6] and was implemented through various approaches, such as introducing nonlinear matters [7], deforming the commutative property of spaces [8], regularizing the singularity by quantum effects [9, 10], transforming to the alternative theories of gravity [11–13], etc. Meanwhile, the apparent differences that are able to be tested between RBHs and singular BHs (SBHs) have motivated a lot of studies [14–16]. Whether we can distinguish RBHs from SBHs by theoretical and experimental evidences becomes a critical point of the research program in the field of RBHs. Since RBHs are widely considered to be related to quantum physics, the discovery of RBHs in the universe will certainly provide a new hope to search for quantum gravity.

The construction of RBHs is separated into two streams. The first one starts with establishing BH metrics by certain mathematical rules [17–19] that guarantee the finite curvatures at BH centers, and then endues these metrics with physical meanings, e.g., using some instructive methods [20], one can establish the theory of RBHs by writing down the actions of matter fields. The second stream draws support from some physical theories or phenomena at first, e.g., the asymptotic safety [9], the existence of a finite length scale [8], etc., and then derives the corrected metrics which give rise to finite curvatures. Nevertheless, among all the constructed RBHs, more than a few models break physical conditions or conjectures, in particular, the weak energy condition associated with the second law of BH mechanics [21], the dominate energy condition [22] relevant to the causal structure of spacetime, and the limited curvature conjecture [23] which states that the curvature invariants should be bounded by some universal value, see also the collections in Ref. [24]. In other words, not all existing RBHs and their extensions are physical.

In comparison, the analogue gravity as a tool of gaining insights into general relativity has shown [25] its significance which makes a great leap from passively waiting for signals from external galaxies to actively studying BHs in ground laboratories. Among various manifestations of analogue gravity, acoustic BHs (ABHs) not only have a long history [26], but also an active status in the current researches [27–29]. Following our previous work [30] in which we have proposed a new method to construct acoustic regular BHs (ARBHs), we shall explore ARBHs on the premise of the energy conditions of their astronomical counterparts in the present work. Our aim is to investigate the physical RBHs with the help of analogue gravity and try to find the apparent evidences or phenomena in order to distinguish RBHs from SBHs.

The paper is organized as follows. In Sec. 2, we clarify what the physical RBHs mean by discussing the energy conditions. In Sec. 3, we are going to propose a remedy to those RBHs that break the energy conditions, including the Bardeen and Hayward BHs, and other extensions. Sec. 4 is dedicated to conformally related RBHs, where we shall prove a no-go theorem under two general situations. In Sec. 5 we simulate the physical RBHs in a fluid by exhibiting the properties of flows, which may have a guiding significance for experiments. Two specific models are discussed in detail in terms of the asymptotic analysis associated with the numerical method. The results
show that the equations of state (EoS) of flows have locally polytropic behaviors. Inspired by
the locally polytropic behaviors, we address a question whether it is possible to establish a RBH
that possesses a globally polytropic EoS in Sec. 6. Secs. 7 and 8 cover two extended topics
on cylindrical RBHs and lower dimensional RBHs with the polar symmetry. The conclusions
with future outlooks are summarized in Sec. 9. The appendices are dedicated to the detailed
investigations on the differential inequalities in Apps. A and B, the calculation of Eq. (8) in App.
C, the regular conditions of \( n \)-dimensional RBHs in App. D, and the asymptotic analysis of solving
nonlinear differential equations in App. E.

2 Physical regular black holes

Let us first consider the simplest case of RBHs whose metrics are spherically symmetric and
of the following form,

\[
g_{\mu\nu} = \text{diag}\{-f, f^{-1}, \xi^2, \xi^2 \sin^2 \theta\}, \quad f := 1 - \frac{2M}{\xi} \frac{\sigma(\xi, x_i)}{\xi},
\]

where \( f \) is shape function, \( \xi \) is radial coordinate, and \( \sigma \) is dimensionless and may contain several
parameters \( x_i, \; i = 1, \ldots, N \), such as mass, charge, etc. Moreover, these parameters must appear
in \( \sigma \) via the combinations, \( \xi^{n_0} x_1^{n_1} \cdots x_N^{n_N} \), which are also dimensionless. If every combination
includes a non-zero \( n_0 \), one can reduce one parameter and obtain \( N - 1 \) independent dimensionless
parameters by the Buckingham \( \pi \) theorem [31].

From the mathematical perspective, the regularity of curvature invariants at BH centers de-
mands the limit: \( \sigma \sim O(\xi^n) \) with \( n \geq 3 \), and the asymptotic flatness requires the limit: \( \sigma \sim O(\xi^m) \)
with \( 0 \leq m < 1 \) [32]. When \( m = 1 \), \( f \) may converge to a non-zero and non-unit constant, such
that the spacetime is Ricci flat, \( R = 0 \), at infinity. The general properties of the shape function
are demonstrated by illustrations in Ref. [18].

2.1 Energy conditions

From the physical perspective, the constructed RBHs should not violate the known physical
laws, among which the energy conditions play an important role [33,34]. The energy conditions can
be formulated into three classes by different approaches, geometric, physical, and effective ones [35].
If the mechanism of constructing RBHs does not change the gravitational part of Einstein’s
equations, e.g., the gravitational field of nonlinear electric or magnetic monopoles [7, 36], the
three classes of definitions are equivalent to each other. In this situation, the energy-momentum
tensor can be represented via Eq. (1) and Einstein’s equations as follows,

\[
T^\mu_\nu := \frac{1}{8\pi} G^\mu_\nu = \text{diag}\left\{ -\frac{M\sigma'}{4\pi\xi^2}, -\frac{M\sigma'}{4\pi\xi^2}, -\frac{M\sigma''}{8\pi\xi}, -\frac{M\sigma''}{8\pi\xi} \right\},
\]

where the prime denotes the derivative with respect to \( \xi \). Since \( T^\xi_\xi = T^\xi_\xi \), there is no need [37] to
formally distinguish the definitions of energy densities inside and outside the outermost horizon.
We define the energy density \( \epsilon \) and pressures \( p_\xi \) and \( p_\perp \) by the diagonal components of \( T^\mu_\nu \),

\[
\epsilon := \frac{M\sigma'}{4\pi\xi^2}, \quad p_\xi := -\frac{M\sigma'}{4\pi\xi^2}, \quad p_\perp := -\frac{M\sigma''}{8\pi\xi}.
\]
Thus, the energy conditions can be cast [38] in terms of $\sigma$ and its derivatives,

$$\begin{align*}
\text{WEC:} & \quad \sigma' \geq 0 \cup \xi\sigma'' \leq 2\sigma', \\
\text{NEC:} & \quad \xi\sigma'' \leq 2\sigma', \\
\text{SEC:} & \quad \sigma'' \leq 0 \cup \xi\sigma'' \leq 2\sigma', \\
\text{DEC:} & \quad \sigma' \geq 0 \cup -2\sigma' \leq \xi\sigma'' \leq 2\sigma',
\end{align*}$$

(4)

where WEC denotes weak energy condition, NEC null energy condition, SEC strong energy condition, and DEC dominant energy condition.

It is not difficult from the energy conditions in Eq. (4) to find that the NEC, $\xi\sigma'' \leq 2\sigma'$, must be maintained, otherwise the other three conditions will be broken. In other words, $\xi\sigma'' \leq 2\sigma'$ is such an inequality that all the energy conditions must satisfy; furthermore, this differential inequality can be solved by the Grönwall-Bellman lemma [39] and its solution gives $\sigma \leq \sigma_0\xi^3$, where $\sigma_0 := \lim_{\xi \to 0} \sigma/\xi^3$ is a positive constant, see also App. A. As a counterexample, let us see the widely discussed model [19] with $\sigma = \exp[-q^2/(2M\xi)]$, where $q$ is charge, in which this inequality is not satisfied because of $\sigma_0 = 0$. Therefore, the model of Ref. [19] is suggested to be ruled out from the physical RBHs, so are its extensions [40] because the matters generating such RBHs break all the energy conditions.

Next, the first inequality in the WEC and DEC, $\sigma' \geq 0$, provides a solution, $\sigma \geq 0$, under the boundary condition $\sigma(\xi)|_{\xi=0} = 0$, i.e., $\sigma$ is a non-negative and monotone increasing function of $\xi$. It is known that the non-minimal Wu-Yang monopole [41–45] is a counterexample because its $\sigma$ function, $\sigma = -\xi^3 (Q^2 - 2M\xi)/(2M(2qQ^2 + \xi^4))$, where $Q$ is also charge parameter, is not monotone and not strictly positive, either. Thus, the WEC of the Wu-Yang monopole is broken, so is the DEC. In addition, the breaking of $\sigma' \geq 0$ may also lead to other problems in the construction of RBHs. For instance, when $\sigma$ is bell shaped, $\sigma = 4\exp(-q^2/\xi^2) - \exp(-2q^2/\xi^2)$, the corresponding BH is of two horizons and all the curvature invariants are finite, but the extreme BH radius is the maximum of horizons and the temperature is divergent as the extreme horizon approaches to zero.

The SEC implies an attractive interaction due to the Landau-Raychaudhuri equation [46], i.e., when the affine parameter increases, the expansion scalar of a family of neighboring time-like geodesics is shrinking because of the condition, $-(\epsilon + p_\xi + 2p_\perp) \propto \sigma'' \leq 0$. Therefore, the violation of the SEC leads to a repulsive interaction. However, this violation is nothing to be concerned about because it is consistent with [37] the thought that the SEC of a RBH must be violated near the RBH center. Moreover, the zeros of equation, $\sigma''(\xi_*) = 0$, separate the spacetime into different types of interactions, which will be discussed in the concrete examples later.

When compared with the other energy conditions, the DEC has its particularity reflected in the inequality, $\xi\sigma'' \geq -2\sigma'$, which can be visualized from the Ricci curvature, $R \propto \xi\sigma'' + 2\sigma'$, i.e., the negative Ricci curvature violates [47] the DEC. However, the differential inequality, $\xi\sigma'' \geq -2\sigma'$, gives the solution, $\xi\sigma \geq 0$, under the boundary conditions $\sigma(\xi)|_{\xi=0} = \sigma'(\xi)|_{\xi=0} = 0$. This solution is trivial and provides no more constraints to $\sigma$ function. In practice, one does not need to verify all the energy conditions. If the DEC is not destroyed, the other conditions except the SEC must be maintained simultaneously. Therefore, checking the DEC is enough to guarantee the WEC and NEC. As to the SEC, we do not need to check it individually for RBHs because it is
maintained beyond a RBH central region and should be violated within a RBH central region due to repulsive interactions. As a result, if the DEC is maintained for a RBH, the all energy conditions are ensured.

In summary, we list the requirements for a physical RBH from the perspective of energy conditions. If a RBH with the metric Eq. (1) is physical, its $\sigma$-function is of the following behaviors:

- $\sigma$ is a non-negative and monotone increasing function of $\xi$, where $\xi \in [0, \infty)$;
- $\sigma$ must be bounded by $\sigma_0 \xi^3$ from above, i.e., $\sigma \leq \sigma_0 \xi^3$, where $\sigma_0 := \lim_{\xi \to 0} \sigma/\xi^3$ and $\sigma_0$ must be positive.

These two conditions are necessary but not sufficient for a RBH to be physical, see App. A for the detailed explanation. In the next section, we shall show that some well-known examples, such as the Bardeen and Hayward BHs, comply with these two conditions, but their corresponding energy conditions are broken. To solve this problem, we provide a phenomenological approach to restore their energy conditions.

3 A remedy to regular black holes with defects

The problem of Hayward BHs depicted [17] by $\sigma = \xi^3/(\xi^3 + q^3)$, where $q$ is charge, is the violation of the DEC beyond the domain $\xi \leq 2^{1/3}q$, even if this $\sigma$ satisfies the two items above. The reason is demonstrated in App. A. However, the problem can be solved by a deformation of $\sigma$ function,

$$\sigma = \frac{M^{\mu-3}\xi^3}{\xi^\mu + q^\mu}, \quad (5)$$

where $M^{\mu-3}$ is introduced for balancing the dimension. The DEC requires $2 < \mu \leq (\sqrt{145} - 7)/2 \approx 2.52$, with which the modified Hayward BH Eq. (5) is physical in the whole domain of $\xi$, $\xi \in [0, \infty)$. Alternatively, the dimensionless $\sigma$ can be established by a parameterization, $\sigma = (\xi/l)^3/[1+(\xi/l)^\mu]$, where $l$ is a parameter with the length dimension.

The similar procedure can be applied to the Bardeen BH, which gives rise to the deformed $\sigma$ function,

$$\sigma = \frac{M^{3\mu/2-3}\xi^3}{(\xi^\mu + q^\mu)^{3/2}}. \quad (6)$$

For the modified Bardeen model, the DEC gives rise to $4/3 < \mu \leq (\sqrt{113} - 7)/2 \approx 1.82$.

In fact, we can construct a general $\sigma$ function,

$$\sigma = \frac{M^{\mu-3}\xi^3}{(\xi^\mu + q^\mu)^\nu}, \quad (7)$$

which will meet all the energy conditions if the parameters $\mu$ and $\nu$ take the values in the following intervals (see App. C for the derivation),

$$\frac{2}{\nu} < \mu \leq \frac{1}{2}\sqrt{\frac{49\nu + 96}{\nu}} - \frac{7}{2} \quad \text{when} \quad \frac{2}{5} < \nu \leq 3; \quad (8a)$$

$$\frac{2}{\nu} < \mu \leq \frac{3}{\nu} \quad \text{when} \quad \nu > 3. \quad (8b)$$
It is not difficult to verify that the RBHs with Eq. (7) are physical because the numerator plays a decisive role, $\sigma \sim O(\xi^3)$ when $\xi \to 0$, and the asymptotic flatness is maintained simultaneously, i.e., $f \to 1$, since the power of the denominator, $\mu\nu$, is greater than 2 when $\xi \to \infty$.

Nevertheless, we cannot remedy all the RBHs with defects in the energy conditions by simply changing the power of the radial coordinate. If $\sigma$ is not a rational function, for instance, $\sigma = \exp[-\frac{q^2}{(2M\xi)}]$, this model cannot be repaired. On the other hand, although the RBH obtained by quantum corrections, e.g. Refs. [9,10], can be remedied by the above phenomenological method, it will lose the original motivation of quantum corrections. Let us take the RG-improved Schwarzschild BH [48] as an example which is motivated by the theory of gravitational asymptotic safety [49,50]. The shape function reads

$$f = 1 - \frac{2G(r)M}{r}, \quad G(r) = \frac{G_0 r^3}{r^3 + \omega G_0 (r + \gamma G_0 M)},$$

(9)

where $G(r)$ is the running Newton constant and it plays the similar role to that of the $\sigma$-function, $G_0$ is identified with the experimentally observed value of Newton’s constant, and $\omega$ and $\gamma$ are two positive parameters. This RG-improved BH is regular from the perspective of finite curvatures, but it breaks the DEC because $G(r)$ violates $-2G' \leq rG''$ beyond a certain value $r_0$, where $r_0$ is determined by a positive root of the algebraic equation,

$$-6\gamma^2 G_0^3 M^2 \omega + 3\gamma G_0 M r^3 - 8\gamma G_0^2 M r \omega - 3G_0 r^2 \omega + r^4 = 0.$$ 

Thus, according to our remedy used above, we change $r^3$ to $r^\mu$ and multiply $M^{3-\mu}$ for balancing the dimension in the denominator of $G(r)$,

$$\tilde{G}(r) = \frac{G_0 r^3}{M^{3-\mu} + \omega G_0 (r + \gamma G_0 M)},$$

(10)

and determine that the DEC requires $0 \leq \mu \leq (\sqrt{145} - 7)/2 \approx 2.52$. However, such a modification loses the original motivation of the RG-improvement, which can be understood from the distance scale $\lambda$ that provides the relevant cutoff for the Newton constant. Using Eq. (10) and the formula [48], $\tilde{G}(r) = G_0 \lambda^2/(G_0 \omega + \lambda^2)$, we obtain

$$\lambda^2 = \frac{G_0 \omega r^3}{M^{3-\mu} + r^3 + G_0 \omega (r + \gamma G_0 M)},$$

(11)

and give the asymptotic behaviors at zero and infinity, respectively,

$$\lambda^2 \xrightarrow{r \to 0} \frac{r^3}{\gamma G_0 M}, \quad \lambda^2 \xrightarrow{r \to \infty} -G_0 \omega,$$

(12)

where the second one violates the original asymptotic requirement, $\lambda \xrightarrow{r \to \infty} r$.

In addition, we shall demonstrate that the conformally related RBHs cannot be repaired, either, by proving a no-go theorem in the next section.
4 No-go theorem for conformally related regular black holes

We shall discuss two classes of conformally related regular black holes, where one class is the conformally related Schwarzschild-type black holes and the other class is the astronomical counterparts of the ARBHs with unit speed of sound.

4.1 Conformally related Schwarzschild-type black holes

We claim that one cannot establish a scale factor $\Omega$ that regularizes the Schwarzschild BH and makes the metric satisfy the DEC at the same time. To specify our statement, let us first write down the spacetime metric of conformally related Schwarzschild BHs \[51\] first,

$$\tilde{g}_{\mu\nu} = \Omega g_{\mu\nu}, \quad g_{\mu\nu} = \text{diag} \left\{ -\left(1 - \frac{1}{\xi}\right), \left(1 - \frac{1}{\xi}\right)^{-1}, \xi^2, \xi^2 \sin^2 \theta \right\}, \quad (13)$$

where the scale factor is set to be $\Omega = \exp\left[ S(\xi) \right] > 0$ and $2M = 1$ is chosen for the discussions in this subsection. The metric being regularized implies that the corresponding curvature invariants are finite in the whole spacetime, in particular, at the BH center. Next, instead of observing the Kretschmann scalar $K$, we concentrate on the contraction of two Weyl tensors $W_{\mu\nu\alpha\beta}$ and $W^{\mu\nu\alpha\beta}$, where $W := W_{\mu\nu\alpha\beta}W^{\mu\nu\alpha\beta}$, and call it Weyl curvature in the following. Because of the Ricci decomposition $[52, 53]$, $W = K - 2R_2 + R^2 / 3$, the Kretschmann scalar and Weyl curvature are equivalent for diagnosing the singularity in the four dimensional spacetime, where $R_2 := R_{\mu\nu}R^{\mu\nu}$ is the contraction of two Ricci tensors and $R := g^{\mu\nu}R_{\mu\nu}$ is Ricci scalar. The Weyl curvature corresponding to the metric Eq. (13) reads

$$W = 12 e^{-2S(\xi)} \frac{\xi^6}{\xi^6}, \quad (14)$$

which is finite at the BH center if $e^{-S(\xi)}$ converges to zero not slower than $\xi^3$. When $e^{-S(\xi)}$ converges to zero in the order of $\xi^3$, i.e., $e^{-S(\xi)} \sim O(\xi^3)$, $S(\xi)$ diverges positively and its first derivative must be negative. In contrast, the asymptotic flatness requires $\Omega \to 1$ as $\xi \to \infty$, i.e., $S(\xi)$ must converge to zero at infinity. Summarizing the above properties of $S(\xi)$, we find that $\Omega^{-1} = e^{-S}$ is a bounded function on the whole non-negative axis of $\xi$.

The energy conditions of the conformally related Schwarzschild BH depicted by Eq. (13) should be defined inside and outside the horizon, respectively, because the relation $T^t_t = T^\xi_\xi$ no longer holds. In other words, the constraint $T^t_t = T^\xi_\xi$ breaks the finiteness of the Weyl curvature. This can be understood easily by solving $G^t_t = G^\xi_\xi$ as a differential equation of $S(\xi)$, which provides a solution $e^{-S} = c_2(\xi + 2c_1)^2$ converging to zero slower than $\xi^3$, where $c_1$ and $c_2$ are two integration constants. Consequently, the energy conditions inside and outside the horizon will be different, and should be treated separately. The energy density and pressures are defined inside the horizon ($\xi < 1$) by

$$e^{\text{in}} := -\frac{1}{8\pi} G^t_t, \quad p^{\text{in}}_\xi := \frac{1}{8\pi} G^{t\xi}_t, \quad p^{\text{in}}_t := \frac{1}{8\pi} G^{t\theta}_\theta; \quad (15)$$
while outside the horizon \((\xi > 1)\) by

\[
\varepsilon^{\text{out}} := -\frac{1}{8\pi} G_t^t, \quad p^{\text{out}}_\xi := \frac{1}{8\pi} G_\xi^\xi, \quad p^t := \frac{1}{8\pi} G^\theta_\theta, \quad (16)
\]

where \(G_t^t, G_\xi^\xi\) and \(G^\theta_\theta\) are components of the Einstein tensor calculated by the metric Eq. (13). Thus, the DEC will be reduced to four differential inequalities in terms of \(S(\xi)\) and its derivatives \(S'(\xi)\) and \(S''(\xi)\) in the range of \(\xi < 1\) or \(\xi > 1\). Among all the differential inequalities, \(\varepsilon^{\text{in}} + p^{\text{in}}_\xi \geq 0\) and \(\varepsilon^{\text{out}} + p^{\text{out}}_\xi \geq 0\) provide the same differential inequality,

\[
(S')^2 - 2S'' \geq 0, \quad \xi \in [0, 1) \cup (1, \infty). \quad (17)
\]

Multiplying both sides of this inequality by a non-negative factor \(e^{-S/2}\), we arrive at

\[
e^{-S/2} [(S')^2 - 2S''] = \frac{d^2}{d\xi^2} (4e^{-S/2}) \geq 0, \quad (18)
\]

from which we can conclude that \(e^{-S/2}\) is a convex function in the range of \(\xi \in [0, 1) \cup (1, \infty)\). However, the finiteness of curvature invariants and asymptotic flatness of the metric demand that \(e^{-S/2}\) is bounded, therefore \(e^{-S/2}\) must be a constant,\(^1\) which is obviously contradictory to the asymptotic behavior of the Weyl curvature at \(\xi \to 0, e^{-S} \sim O(\xi^n)\) with \(n \geq 3\).

In other words, there exists no such a conformal factor \(\Omega\) that can regularize the Schwarzschild BH and guarantee the energy conditions simultaneously. This conclusion can be extended to the conformally related Schwarzschild-type BHs with singularity at \(\xi = 0\). For such a BH with the metric,

\[
g_{\mu\nu} = \text{diag}\{-f, f^{-1}, \xi^2, \xi^2 \sin^2 \theta\}, \quad f = 1 - \frac{\sigma(\xi)}{\xi}, \quad (19)
\]

where \(\sigma(\xi)/\xi\) is of a unique pole at \(\xi = 0\) and goes to zero as \(\xi \to \infty\), there does not exist a conformal factor \(\Omega\) that satisfies the following two conditions simultaneously:

- The Weyl curvature of metric \(\Omega g_{\mu\nu}\) is finite in \(\xi \in [0, \infty)\), where \(\Omega \to 1\) at \(\xi \to \infty\);

- The DEC calculated by \(\Omega g_{\mu\nu}\) is not broken.

This is the so-called no-go theorem for the conformally related Schwarzschild-type BHs that belong to conformally related RBHs.

The proof exactly follows the case of conformally related Schwarzschild BHs. First, the Weyl curvature of metric Eq. (19) reads

\[
W = \frac{e^{-2S}}{3\xi^6} \left[\xi (\xi \sigma'' - 4\sigma') + 6\sigma\right]^2. \quad (20)
\]

Assuming that \(\xi = 0\) is the \(d\)-th order pole of \(\sigma\), we can find an asymptotic relation,

\[
e^{-S} \sim O(\xi^{3+n}), \quad n \geq d \geq 0, \quad (21)
\]

which ensures that the Weyl curvature is finite. When \(\xi \to \infty\), the asymptotic flatness demands \(e^{-S} \to 1\). Moreover, the conditions \(\varepsilon^{\text{in}} + p^{\text{in}}_\xi \geq 0\) and \(\varepsilon^{\text{out}} + p^{\text{out}}_\xi \geq 0\) provide exactly the same differential inequality as Eq. (17) from which we can obtain that \(e^{-S}\) is a convex function. Nevertheless, the combination of convex and boundness of \(e^{-S}\) leads to a contradiction to the asymptotic relation Eq. (21). Therefore, our statement is proved.

\(^1\)If a differentiable and convex function is bounded on \(\mathbb{R}\), it must be a constant, see e.g. Ref. [54].
4.2 Astronomical counterparts of the ARBHs with unit speed of sound

On the premise that the speed of sound is set to be unity, we have proposed [30] a general method to construct ARBHs in a fluid, where the metrics are of similarity to the ones of conformally related BHs [51]. However, as we have noted in Ref. [30], the astronomical counterparts of simulated ARBHs under a certain parameterization violate the DEC, i.e., the ARBHs we constructed hardly have any physical counterparts in the universe. Thus, it is natural to ask if we can find such a way that the DEC for the astronomical counterparts of the ARBHs is able to be repaired and consequently the astronomical counterparts of the ARBHs may have a chance to be detected in the universe.

Here the case is different from that in the above subsection, where the unregularized metric $g_{\mu\nu}$ has no singularity for the former while it has for the latter. Moreover, the ARBHs are different from the conformally related BHs because their conformal factors are proportional to the energy density of fluids and not constrained by any dynamical equations.

Let us discuss in detail by following the strategy used in Ref. [30], which is opposite to that of Sec. 4.1. We first write down [30] the metric of acoustic RBHs with the spherical symmetry,

$$\tilde{g}_{\mu\nu} = \rho \text{diag}\{-f, f^{-1}, r^2, r^2 \sin^2 \theta\}, \quad f = 1 - v^2,$$

where $r$ is radial coordinate in fluids, $\rho$ is mass density and $v$ radial velocity of fluids. The density and velocity are related [30] by $v = A/(\rho r^2)$, where $A$ is a positive constant. The radial velocity $v$ is supposed to be positive, otherwise the density will be negative. Moreover, since we have set the speed of sound to be unity, $v$ is dimensionless and $A$ has the same dimension as that of $\rho r^2$.

Considering the similarity between Eq. (13) and Eq. (22) and dealing with $\rho$ as a scale factor, $\rho := e^{S(r)}$, we can write the Weyl curvature,

$$W = \frac{4A^4 e^{-6S}}{3r^{12}} (2r^2 S'' - r^2 S'' + 10rS' + 15)^2,$$

where the prime denotes the derivative with respect to $r$. The regularity at $r = 0$ demands

$$e^{-S} \sim O(r^n), \quad n \geq 2.$$

Namely, $\rho$ diverges because $\rho \propto r^{-n}$, and $v$ converges to a constant [30] in order to make sure that the Weyl curvature is finite when $r \to 0$. As mentioned in the second paragraph of this subsection, if $\rho$ was removed from Eq. (22), the remaining metric would still give rise to finite curvature invariants everywhere, which is different from the situation in Sec. 4.1. In addition, the asymptotic flatness requires $\rho \to 1$ or $v \sim O(r^{-2})$ as $r \to \infty$, i.e., $v$ is a monotone decreasing function at infinity.

Similar to the discussion in Sec. 4.1, we suppose that the acoustic metric Eq. (22) directly corresponds to a spacetime metric. Thus, the energy density and pressures corresponding to the astronomical matter generating astronomical BHs must be defined inside and outside the horizon separately, otherwise the equation, $G^t_{\,t} = G^r_{\,r}$, gives a false solution, $S(r) = c_4 - 2 \ln (r + 2c_3)$, where $c_3$ and $c_4$ are integration constants. If this solution is asked to be consistent with the regularity Eq. (24), we have $c_3 = 0$, which leads to the result that $f$ degenerates to a constant,

\[\text{We distinguish } r \text{ from } \xi \text{ — the radial coordinate of astronomical BHs.}\]
$1 - A^2 c_4^2$, such that the corresponding metric is no longer a BH solution. We then deduce that the forms of density and pressure inside the horizon must be different from those outside. Next, following the proof process in Sec. 4.1, we derive the inequality from two similar inequalities, $\epsilon^{\text{in}} + p_r^{\text{in}} \geq 0$ and $\epsilon^{\text{out}} + p_r^{\text{out}} \geq 0$,

\[(S')^2 - 2S'' \geq 0, \quad r \in \{r | r > 0, \ v \neq 1\}, \quad (25)\]

which is similar to Eq. (17). Therefore, we conclude that $e^{-S/2}$ is a convex function in the domain $r \in \{r | r > 0, \ v \neq 1\}$, which contradicts to the regular condition Eq. (24) and asymptotic flatness. In other words, even if $\tilde{g}_{\mu\nu}/\rho$ is regular in the sense of finite curvatures, the regularized metric $\tilde{g}_{\mu\nu}$ cannot meet the DEC. That is, the DEC is violated in the astronomical counterparts of the ARBHs with unit speed of sound.

5 Construction of physical RBHs in fluids

Now let us go to the simulation of physical RBHs in a fluid. Our aim is to construct the spacetime of physical RBHs with the spherical symmetry by acoustic waves, and make certain of the conditions under which the physical RBHs can be realized in the fluid, i.e., find the equations of state in the fluid.

We start with the general stationary acoustic metric [25],

\[ds^2 = \frac{\rho}{c} \left[ -(c^2 - v^2) d\tau^2 + \left( \delta_{ij} + \frac{v^i v^j}{c^2 - v^2} \right) dx^i dx^j \right], \quad (26)\]

where $\rho$ and $v^i$ are the mass density and velocity of fluids, respectively, $c := \sqrt{|\partial p/\partial \rho|}$ denotes the local speed of sound and $p$ the pressure. We shall employ Eq. (26) to simulate physical RBHs by providing the equations of state.

First of all, we suppose that the fluid is spherically symmetric and its velocity contains only radial component, i.e., $v^i = \{v_r(r), \ 0, \ 0\}$, so Eq. (26) reduces to the following form in the spherical coordinates,

\[ds^2 = \rho c \left[ - \left( 1 - \frac{v_r^2}{c^2} \right) d\tau^2 + \left( 1 - \frac{v_r^2}{c^2} \right)^{-1} \frac{dr^2}{c^2} + \frac{v_r^2}{c^2} d\Omega^2 \right], \quad (27)\]

where $d\Omega^2 := d\theta^2 + \sin^2 \theta d\phi^2$. Using the solution of continuity equation,

\[\rho = \frac{A}{r^2 v_r}, \quad (28)\]

to replace $v_r$ and defining a new variable,

\[\xi^2 := \frac{r^2 \rho}{c}, \quad (29)\]

we rewrite the acoustic metric,

\[ds^2 = -F d\tau^2 + H d\xi^2 + \xi^2 d\Omega^2, \quad (30)\]
which is supposed to equal the astronomical metric formally, where $F$ and $H$ are defined by

$$F := c \rho - \frac{A^2}{r^4 c \rho}, \quad H := \frac{4 r^4 c^4 \rho^4}{(r^4 c^2 \rho^2 - A^2) [r pc' - c (r p' + 2 \rho)]^2}.$$  \hspace{1cm} (31)

Here the prime denotes the derivative with respect to $r$. Note that our $F$ and $H$ are slightly different from those in Ref. [55] where they were represented in terms of $\xi$ and $v$. The aim of our expression is to derive analytical expressions for density $\rho$ and pressure $p$ of the fluid.

Secondly, we impose the condition, $FH = 1$, i.e., the simulated metric has only one shape function, which will lead to the differential equation,

$$4c^3 \rho^3 = \left[ r \rho c' - c (r p' + 2 \rho) \right]^2.$$  \hspace{1cm} (32)

Now we solve $c$ from the first equation of Eq. (31),

$$c = \frac{\beta}{\rho}, \quad \beta := \frac{F}{2} + \sqrt{\left( \frac{F}{2} \right)^2 + \frac{A^2}{r^4}},$$  \hspace{1cm} (33)

where the negative root has been ignored due to positive $c$ and $\rho$. Substituting $c = \beta/\rho$ into Eq. (32), we obtain

$$\frac{\rho'}{\rho} = -\frac{1}{r} + \frac{\beta'}{2\beta} \pm \frac{\sqrt{\beta}}{r},$$  \hspace{1cm} (34)

where there are no rules to select any one of the two solutions at this moment, and then derive the density analytically,

$$\rho_\pm = \rho_0 \frac{\sqrt{\beta}}{r} \exp \left( \pm \int \frac{\sqrt{\beta}}{r} \, dr \right).$$  \hspace{1cm} (35)

where $\rho_0$ is integration constant. In addition, using Eqs. (33) and (35) together with the definition of $c$, we compute the pressure,

$$p_\pm = p_0 - \frac{\beta^2}{\rho_\pm} + 2 \int \frac{\beta \rho'}{\rho_\pm} \, dr,$$  \hspace{1cm} (36)

where $p_0$ is integration constant. Eqs. (35) and (36) give the equation of state for the fluid.

In practice, $F$ as a function of $\xi$ corresponds to the shape function of the physical RBH we are going to simulate in the fluid. The relation between $\xi$ and $r$, Eq. (29), can be represented [55] by the nonlinear differential equation,

$$A^2 [\xi(r)]^4 + F[\xi(r)]^6 [\xi'(r)]^2 - r^8 [\xi'(r)]^4 = 0,$$  \hspace{1cm} (37)

or equivalently by

$$A^2 \xi^4 \left[ \frac{dr(\xi)}{d\xi} \right]^4 + F(\xi) \xi^2 [r(\xi)]^6 \left[ \frac{dr(\xi)}{d\xi} \right]^2 - [r(\xi)]^8 = 0,$$  \hspace{1cm} (38)

which does not have analytical solutions generally.\footnote{We note that Eq. (37) is more suitable for the numerical treatment for specific models in Sec. 5.1 and Sec. 5.2, while Eq. (38) is more suitable for the asymptotic analysis made below.} Furthermore, the variables associated with the fluid can be written in terms of the following functions of $r$,

$$c = \frac{r^2 \xi'}{\xi^2}, \quad v = \frac{A}{r^2 \xi}, \quad \rho = \xi', \quad p' = \xi'' \left( \frac{r^2 \xi'}{\xi^2} \right)^2.$$  \hspace{1cm} (39)
We note from Eq. (39) that there exists a special position $r_c$ which is a stationary point of both $\rho$ and $p$, and this point is determined by $\xi''(r_c) = 0$ because $\rho'$ and $p'$ are proportional to $\xi''$. The detailed discussions will be carried out numerically in the concrete examples below.

Before studying specific models, we give an asymptotic analysis to Eq. (38) and provide general properties of solutions at $r \to 0$ and $r \to \infty$, respectively.

For the simulated RBH at $r \to 0$, we obtain an asymptotic $F$ with the help of Ref. [32],

$$ F \sim 1 - \frac{R(0)}{12} \xi^2, \quad (40) $$

when $\xi$ approaches to zero. Here $R(0)$ is the limit of Ricci scalars at $\xi = 0$. Using the method of dominant balance [56, 57] and the boundary condition $\xi(r)|_{r=0} = 0$, we find the asymptotic solution of Eq. (38) when $\xi$ approaches to zero (see App. E for the details),

$$ \xi \sim c_6 \exp \left( -\frac{\sqrt{A}}{r} \right), \quad (41) $$

where $c_6$ is an integration constant. From Eqs. (33), (40), and (41) we derive the asymptotic forms of $\beta$, the density, and the pressure, respectively,

$$ \beta \sim \frac{A}{r^2}, \quad \rho_\pm \sim \rho_0 \frac{\sqrt{A} e^{\pm \sqrt{A}/r}}{r^2}, \quad p_\pm \sim -\frac{A^{3/2} e^{\pm \sqrt{A}/r}}{r^2 \rho_0}, \quad (42) $$

which do not depend on specific RBH models at the leading order. On the other hand, if we substitute Eq. (41) directly into Eq. (39), we are able to fix $c_6$ and rule out the redundant root by comparing with Eq. (42). As a result, we obtain $c_6 = \rho_0$ and know that the solution with subscript “+” is physical, see App. E for the details. Thus, we shall omit the subscript “+” in $\rho$ and $p$ for simplifying the notations in the following.

We can observe from Eq. (42) that the pressure of fluids at $r \to 0$ must be divergent, while the density converges to zero. Further, according to the first equation of Eq. (39), we obtain the speed of sound,

$$ c \sim \sqrt{A} e^{\sqrt{A}/r}, \quad (43) $$

which is divergent at $r \to 0$. In other words, if the maximum speed of sound exists [58], there must be a cutoff $r_0$, such that the speed of sound is regularized. This will be particularly important in the numerical calculation later.

Combining the density and pressure in Eq. (42), we give the equation of state around the BH center,

$$ p = -\frac{16}{\rho} \left[ W_0 \left( -\frac{\sqrt{A}}{2} \sqrt{\frac{\rho}{\rho_0}} \right) \right]^4, \quad (44) $$

where $W_0(\cdot)$ is Lambert $W$ function. To make Eq. (44) more intuitive, we perform the asymptotic expansion and give the leading order for $\rho \to 0$,

$$ p = -\frac{A \rho}{\rho_0^2}. \quad (45) $$

Here the relation between $\rho$ and $p$ around $r = 0$ is linear.
In contrast, for the simulated RBH at \( r \to \infty \), we have \( F \sim 1 - 2M/\xi^n \), \( 0 < n \leq 1 \) from Eq. (31), and the asymptotic solution, \( \xi = c_7 r \), where \( c_7 \) is constant, see App. E. Thus, we obtain \( \beta \sim 1 - 2M/(c_7 r)^n \) due to Eq. (33). The density and pressure can be solved from Eqs. (35) and (36) with the plus subscript,

\[
\rho \sim \rho_0 e^{M/[n(c_7 r)^n]}, \quad p \sim p_0 - \frac{(1 - 4n)}{\rho_0} e^{-M/[n(c_7 r)^n]},
\]

where we have kept the leading term of \( p \) valid by adding \( n \neq 1/4 \). The limits of the density and pressure are \( \rho \to \rho_0 \) and \( p \to p_0 - (1 - 4n)/\rho_0 \), respectively. In particular, the approximate equation of state at infinity reads

\[
p = p_0 - \frac{1 - 4n}{\rho},
\]

which is polytropic, more precisely, it describes a similar thermal process to that of the Chaplygin gas [59]. If \( n = 1/4 \), the equation of state becomes \( p = p_0 - 1/(2\rho) \) approximately.

Up to the present discussions in this section we focus on the simulation of the RBHs depicted by Eq. (1) in terms of acoustic analogy. In Sec. 5.1 and Sec. 5.2, we are going to simulate two physical RBHs where the DEC is maintained, as to the details of the two models, see Sec. 3 for the first one and Ref. [20] for the second one.

### 5.1 Remedied Bardeen model

We simulate the remedied Bardeen BH by taking \( \mu = 3/2, M = 1, \) and \( q = 1 \) in Eq. (6). The solution of Eq. (37) obtained numerically with the boundary condition, \( \xi(0.2) = 0.2 \), is exhibited in Fig. 1, where the two horizons are \( r_- \approx 0.35 \) and \( r_+ \approx 1.39 \), or \( \xi_- \approx 1.69 \) and \( \xi_+ \approx 14.34 \) equivalently. The initial point starting at \( r_0 = 0.2 \) instead of 0 is based on the possibility of the existence of maximum speed of sound, and such a setting can avoid dealing with divergent speed of sound, velocity of fluid, and pressure in numerical calculations. Furthermore, the critical point that separates the spacetime into different types of interactions is determined by \( \sigma''(\xi_*) = 0 \), see

![Figure 1: \( \xi(r) \)](image-url)
Eq. (4) and its following discussions about the SEC, i.e. \( \xi^* = \frac{2^{2/3}}{3} \approx 1.59 < \xi_- \), which is located inside the inner horizon.

In Fig. (2) we show the speed of sound and velocity of fluid, and highlight their difference by using the Mach number, \( \mathcal{M} := v/c \). We note that the Mach number is located in the range of \( \mathcal{M} \in [0.8, 1.2] \) between the inner and outer horizons, which indicates that the transonic phenomenon happens. The similar phenomenon was also noticed \([55]\) in the traditional BHs with singularities. As a matter of fact, the existence of horizons for the acoustic model described by Eq. (27) separates the spacetime into different domains by the signs of \( c^2 - v^2 \). For the simulated BHs with one horizon, the fluid inside the horizon flows with the transonic phenomenon. For the simulated BHs with two horizons, the transonic flow is sandwiched between the two horizons.

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Figure 2: \( v(r) \) and \( c(r) \). The two curves are almost overlapped. For the details, see the small picture on \( \mathcal{M}(r) \).

Generally, the Mach number can be computed with the help of Eq. (39),

\[
\mathcal{M} = \frac{A\xi^2}{r^4(\xi')^2} = \frac{1}{z + \sqrt{1 + z^2}}, \quad z := \xi^2 F(\xi)/(2A). \tag{48}
\]

Since \( F(\xi) < 0 \) as long as \( \xi \in (\xi_-, \xi_+) \), we have \( z < 0 \) between the two horizons; meanwhile, we find that the Mach number is constrained by the following inequality,\(^4\)

\[
1 < \mathcal{M} < \frac{1}{z_{\text{min}} + \sqrt{1 + z_{\text{min}}^2}}. \tag{49}
\]

For the remedied Bardeen model, the minimum of \( z \) can be calculated numerically, \( z_{\text{min}} \approx -0.033 \), under our setting, \( A = M = q = 1 \), \( \rho_0 = 1 \), and \( p_0 = 0 \), and this minimum corresponds to \( r \approx 0.773 \). Thus, the Mach number is of the range, \( 1 < \mathcal{M} < 1.034 \).

We also provide the numerical calculations of Eqs. (33), (35) and (36) with subscript “+” for the density and pressure in Figs. 3a and 3b, respectively, and the EoS in Fig. 3c. From Figs. 3a

\(^4\)The function \( (z + \sqrt{1 + z^2})^{-1} \) is positive and monotone decreasing because its derivative is negative, \( -1 + z/\sqrt{z^2 + 1} < 0 \), and its limit at \( z = 0 \) equals one.
and 3b, we can see that there are a global maximum of $\rho$ and a global maximum of $p$ located between the two horizons, $r_c = 0.489$. This point plays a special role in Fig. 3c because it shows a sharp discontinuity of the EoS. In addition, the curve of the EoS ends at the green dot, where its values of $\rho$ and $p$, $\rho \approx 0.509$ and $p \approx 49240.914$, are estimated numerically when $r$ approaches to 500 as the infinity of our numerical calculations.

![Graphs](image)

(a) $\rho(r)$  
(b) $p(r)$

(c) Equation of state

Figure 3: Numerical solutions for the remedied Bardeen model, where $A = M = q = 1$, $\rho_0 = 1$, and $p_0 = 0$.

### 5.2 RBHs associated with nonlinear electromagnetic fields

Now let us turn to the model associated with nonlinear electromagnetic fields [20], whose $\sigma$ is a rational and sigmoid function,

$$\sigma = \frac{\xi^3}{(\xi + q)^3} \text{ with } q \geq 0,$$

(50)

where $\xi$ denotes the radial coordinate in the BH. Note that $\sigma$ is non-negative, monotone increasing in the whole domain of $\xi$, i.e., $\sigma \geq 0$ and $\sigma' = q\xi^2/(q + \xi)^2 \geq 0$; meanwhile $\sigma$ is bounded by $\sigma \leq \xi^3/q^3$ because of $\sigma_0 = 1/q^3$. The critical point of this model can be obtained by solving
\( \sigma''(\xi) = 0 \), which gives \( \xi_* = q \). Moreover, the existence of horizons demands \( q \leq 4 \xi_{\text{Sch}}/27 \), i.e., the critical point \( \xi_* \) is not greater than \( 4 \xi_{\text{Sch}}/27 \), where \( \xi_{\text{Sch}} = 2M \) is the Schwarzschild horizon radius.

The numerical results are shown in Fig. 4, where \( M = 1/2 \), \( q = 0.1 \), and \( A = 1 \) are set. Since \( p \) and \( c \) are divergent at \( r = 0 \), see Eq. (42) and Eq. (43), we have performed a cutoff for the lower boundary by setting \( r_0 = 0.2 \) as we did for the remedied Bardeen model.

Here are four points that need to be demonstrated.

- The oscillation of Mach numbers at the left tail in Fig. 4b arises from our computational accuracy;
- The critical point \( r_c \) at which \( \rho \) and \( p \) take their maxima is no long stuck between the two horizons, see Figs. 4c and 4d, which is different from the case in the remedied Bardeen model;
- The green dot in Fig. 4e denotes the end of EoS, and corresponds to \( \rho \approx 13.322 \) and \( p \approx 48416.215 \);
- The upper boundary of Mach numbers can be estimated by a formula similar to Eq. (49), \( \mathcal{M} < 1.010 \), where \( z \) reaches its global minimum \( z_{\text{min}} = -0.010 \) at \( r = 0.209 \).

6 Polytropic equations of state

Motivated by the polytropic behaviors of equations of state in Sec. 5, we try to find out whether the fluid with polytropic equations of state can be used to simulate physical RBHs. For this purpose, we suppose that the fluid is barotropic and has a polytropic equation of state, \( p = \tilde{B} \rho^\gamma \), where \( \gamma \) is a real and nonzero number, its valid domain will be determined below, and \( \tilde{B} \) is a constant. The local speed of sound can be calculated,

\[
c = B \rho^{(\gamma-1)/2}, \quad \tilde{B} := (\tilde{B} \gamma)^{1/2} > 0.
\]

Meanwhile, the fluid should satisfy the continuity equation which provides the same relation between \( \rho \) and \( v \) as Eq. (28). Then, substituting Eq. (51) and Eq. (28) into Eq. (27), we give the metric in terms of \( \rho \) as follows:

\[
g_{\mu\nu} = \frac{\rho^{(3-\gamma)/2}}{B} \left\{ -B^2 f \rho^{\gamma-1}, \ f^{-1}, \ r^2, \ r^2 \sin^2 \theta \right\}, \quad f = 1 - \frac{A^2}{B^2 r^4 \rho^{\gamma+1}},
\]

where \( f \) is the shape function.

Now we investigate whether the above metric can mimic the RBHs with metric Eq. (1). To this end, we use the new variable \( \xi \) defined by Eq. (29), which takes the following form when Eq. (51) is considered,

\[
\xi^2 = r^2 \rho^{(3-\gamma)/2}.
\]

When \( r \) is replaced by \( \xi \), the metric Eq. (52) becomes

\[
g_{\mu\nu} = \text{diag} \left\{ -B f \rho^{(\gamma+1)/2}, \ f^{-1} \rho^{(3-\gamma)/2} \frac{\xi^2}{B(\xi')^2}, \xi^2, \xi^2 \sin^2 \theta \right\},
\]
where $\xi' := \frac{d\xi}{dr}$. Therefore, the condition, $\gamma \xi \xi = -1$, leads to the following differential equation of $\rho$,

$$[(\gamma - 3)r \rho' - 4\rho]^2 = 16B\rho^{(\gamma+5)/2},$$

Figure 4: Numerical solutions for the model associated with nonlinear electromagnetic fields, where $q = 0.1, A = 1, \rho_0 = 1$, and $p_0 = 0$ are set.
whose general solution is

\[ \rho^{-(\gamma+1)/4} = \pm \sqrt{B} + c_8 r^{(\gamma+1)/(3-\gamma)}, \]  

where \( c_8 \) is integration constant. Furthermore, we note that the asymptotic behavior of \( \rho \) is \( \rho \to B^{-2/(\gamma+1)} \) as \( r \to 0 \) if \( -1 < \gamma < 3 \); while for \( \gamma < -1 \cup \gamma > 3 \), it is \( \rho \to c_8^{-4/(\gamma+1)} r^{-4/(3-\gamma)} \). Thus, the asymptotic behaviors of Weyl curvatures for these two cases are

\[ W \sim O(r^{-12}), \quad -1 < \gamma < 3; \]  
\[ W \sim O \left( r^{-(16(\gamma-1)/(\gamma-3))} \right), \quad \gamma < -1 \cup \gamma > 3. \]  

The asymptotic relation associated with \( -1 < \gamma < 3 \) shows that the Weyl curvature inevitably has a singular point at \( r = 0 \); as to the case of \( \gamma < -1 \cup \gamma > 3 \), the regularity requires \( 1 \leq \gamma < 3 \), which contradicts to \( \gamma < -1 \cup \gamma > 3 \). As a result, the fluid with polytropic equations of state cannot simulate the RBHs with the metric Eq. (1).

Let us turn to the investigation under what conditions the metric Eq. (52) describes a RBH solution in the whole spacetime. We calculate the Weyl curvature of the metric Eq. (52) directly,

\[ W = \frac{\rho^{\gamma-9}}{12B^2r^{12}} \left\{ 2A^2(\gamma + 5)r^2 \rho^2 + 2A^2 r \rho \left[ (3\gamma + 17)\rho' - 2r\rho'' \right] + 60A^2 \rho^2 \right. 
\]  
\[ - B^2 \left( \gamma^2 - 4\gamma + 3 \right) r^6 \rho^{\gamma+1} \rho'' - 2B^2(\gamma - 1)r^5 \rho^{\gamma+2} (r\rho'' - \rho') \left. \right\}^2. \]  

In order to give the conditions just mentioned, we make an asymptotic ansatz [1], \( \rho \sim r^{-n} \) as \( r \to 0 \), and substitute it into Eq. (58), where \( n \) is a real and positive number. We find that the square root of Weyl curvatures consists of the following two terms,

\[ r^{-(\gamma-3)n/2-2} \quad \text{and} \quad r^{(\gamma+5)n/2-6}, \]

where we have omitted relevant coefficients. The regularity at \( r = 0 \) provides two inequalities,

\[ n \geq \frac{12}{\gamma + 5}, \quad \gamma \in (-5, 1]; \quad (60a) \]
\[ n \geq -\frac{4}{\gamma - 3}, \quad \gamma \in (1, 3). \quad (60b) \]

On the other hand, the asymptotic flatness requires

\[ \rho \to B^{-2/(\gamma+1)}, \]  
\[ \quad \text{when} \quad r \to \infty. \]

As a summary, we conclude that the metric Eq. (52) describes a RBH if Eqs. (60) and (61) are satisfied.

We take the Chaplygin gas as an example, where \( \gamma = -1 \) in Eq. (52) and the density has the form,

\[ \rho = \rho_0 \frac{l^3}{r^3} \left( 1 + \frac{l^2}{r^2} \right), \]  
\[ \quad \text{where} \quad l \text{ is introduced for balancing the length dimension.} \]

The corresponding Weyl curvature is

\[ W = \frac{4r^4 \left[ A^2 (4l^2 r^2 + l^4) + B^2 r^4 \left( 8l^2 r^2 + 8l^4 + 3r^4 \right) \right]^2}{3B^2 l^{12} \rho_0^4 (l^2 + r^2)^6}, \]  

where \( A \) and \( B \) are constants.
which has an asymptotic relation,
\[ W \sim \frac{4A^4r^4}{3B^{216} \rho_0^4} + O(r^5), \quad (64) \]
when \( r \to 0 \). Thus, the Weyl curvature is regular. Moreover, the bracket of the denominator of Eq. (63) is an algebraic quadratic function of \( r \), but it has no real roots since \( l \in \mathbb{R} \) and \( B \neq 0 \), which consequently indicates that the Weyl curvature is finite on the non-negative axis.

It is time for us to investigate the energy conditions for astronomical counterpart of the metric Eq. (52). We study the vacuum equation, \( T^t_t = T^r_r \), in order to clarify whether we have to define the energy density and pressure inside and outside the horizon, respectively. The vacuum equation leads to a second-order nonlinear differential equation of \( \rho \),
\[ 4\rho [2(\gamma - 1)\rho' + (\gamma - 3)r\rho''] - r(\gamma - 3)(\gamma + 5)(\rho')^2 = 0, \quad (65) \]
whose general solution is
\[ \rho = c_1 r^{(\gamma-3)/4} (3 - \gamma - 4c_9 r^{(\gamma+1)/(\gamma-3)})^{-4/(\gamma+1)}, \quad (66) \]
where \( c_9 \) and \( c_{10} \) are integration constants. Furthermore, the asymptotic analysis at \( r \to 0 \) gives us two situations,
\[ \rho \sim O(1), \quad W \sim O(r^{-12}), \quad \text{for} \quad -1 < \gamma < 3; \quad (67a) \]
\[ \rho \sim O(r^{4/(\gamma-3)}), \quad W \sim O(r^{-16(\gamma-1)/(\gamma-3)}), \quad \text{for} \quad \gamma < -1. \quad (67b) \]
None of them can realize a regular Weyl curvature at \( r = 0 \). Namely, we have to discuss the energy conditions inside and outside a horizon separately.

The definitions of energy density \( \epsilon \) and radial pressure \( p_r \) depend on the number of horizons \( n \), meanwhile the horizons separate the spacetime into \( n + 1 \) regions. If we start from the region outside the outermost horizon and denote that area by 1, then for the region with odd number \( n \in 2\mathbb{N} + 1 \), the energy density and radial pressure are defined by
\[ \epsilon_{\text{odd}} = -\frac{G^t_t}{8\pi}, \quad p_{r_{\text{odd}}} = \frac{G^r_r}{8\pi}; \quad (68) \]
while for even \( n \in 2\mathbb{N} \), they are defined by
\[ \epsilon_{\text{even}} = -\frac{G^r_r}{8\pi}, \quad p_{r_{\text{even}}} = \frac{G^t_t}{8\pi}. \quad (69) \]
It can be verified that the model Eq. (62) violates the DEC because there is no intersection between \( \epsilon \geq |p_r| \) and \( \epsilon \geq |p_t| \), no matter the number of horizons is odd or even. That is, the Chaplygin gas cannot be used to mimic an astronomical counterpart. In fact, the inverse problem, i.e. to construct \( \rho \) from the energy conditions, is rather complicated since the DEC leads to four second-order nonlinear differential inequalities which are hard to be dealt with. Therefore, we stop searching for the models with polytropic EoS and leave it for future studies.
7 Cylindrical regular black holes and their equatorial sections

In this section, we study the RBHs with a cylindrical symmetry. The metric can be cast in the following,

\[ ds^2 = -f dt^2 + f^{-1} d\zeta^2 + \xi^2 d\phi^2 + e^{2\zeta/\xi} dz^2, \quad f = 1 - 2M/\xi, \tag{70} \]

where \( \zeta \) is a real function of \( \xi \). Instead of analyzing the Weyl curvature, we are going to analyze the Ricci scalar because of its simplicity,

\[ R = \frac{4M\zeta'}{\xi} + \frac{4M\sigma''}{\xi} + \frac{2M\sigma''}{\xi} - 2\zeta'^2 - \frac{2\zeta'}{\xi} - 2\zeta'', \tag{71} \]

where the prime means the derivative with respect to \( \xi \). The Ricci scalar is regular at \( \xi = 0 \) if \( \sigma \) and \( \zeta \) have the asymptotic forms, \( \sigma \sim O(\xi^m) \) and \( \zeta \sim O(\xi^n) \), where \( m \geq 3 \) and \( n \geq 2 \) or \( n = 0 \).

Now let us consider the energy conditions of this type of black holes. When \( \zeta = 0 \), \( G_{tt} = G_{\xi\xi} \) is valid in the regions inside and outside a horizon. The energy density and pressures can be carried out only in terms of \( \sigma \) and its derivatives,

\[ \epsilon = \frac{M}{8\pi\xi^3} (\xi\sigma' - \sigma), \quad p_\xi = \frac{M}{8\pi\xi^3} (\sigma - \xi\sigma'), \quad p_\phi = -\frac{M}{8\pi\xi^3} [\xi (\xi\sigma'' - 2\sigma') + 2\sigma], \quad p_z = -\frac{M\sigma''}{8\pi\xi}. \tag{72} \]

Thus the DEC is

\[ \xi (\xi\sigma'' - \sigma') = \sigma, \quad \xi (\xi\sigma'' + \sigma') \geq \sigma. \tag{73} \]

Note that the equality comes from \( \epsilon \geq p_\phi \cap p_z \geq -\epsilon \) and can be used to fix \( \sigma, \sigma = c_{11}\xi + c_{12}\ln(\xi) \), where \( c_{11} \) and \( c_{12} \) are integration constants. However, the corresponding Ricci scalar, \( R = 2c_{12}M/\xi^2 \), is singular unless the metric is trivial, \( c_{12} = 0 \). In other words, the metric Eq. (70) with \( \zeta = 0 \) can never represent a physical RBH.

If omitting the flow along the \( z \) direction, i.e., considering only the equatorial section (ES), we reduce the DEC to be

\[ \epsilon \geq p_z \geq -\epsilon, \quad \epsilon \geq p_\phi \geq -\epsilon, \quad \epsilon \geq 0, \tag{74} \]

where the contribution from \( p_z \) has been ignored. Therefore, we have the following three differential inequalities,

\[ \xi^2\sigma'' + \sigma \geq \xi\sigma', \quad \xi^2\sigma'' + 3\sigma \leq 3\xi\sigma', \quad \xi\sigma' \geq \sigma. \tag{75} \]

We take the modified Hayward BH as an example, see Eq. (5), but here we discuss its cylindrical counterpart, \( \sigma = M^{\alpha-3}/(q^\alpha + \xi^\alpha) \). When substituting this \( \sigma \) into the above inequalities, we then obtain

\[ 2q^\alpha \geq (\alpha - 2)\xi^\alpha, \quad (\alpha + 2)q^\alpha \geq (\alpha - 2)\xi^\alpha, \quad [2q^\alpha + (\alpha - 2)\xi^\alpha]^2 \geq [\alpha(\alpha + 8) - 16]q^\alpha\xi^\alpha, \tag{76} \]
which hold for $\xi \in [0, \infty)$ if $-4(\sqrt{2} + 1) \leq \alpha \leq 4(\sqrt{2} - 1)$.

To simulate the equatorial section of cylindrical regular black holes, we take $\alpha = 3/2$ in
\(\sigma = M^{\alpha - 3} \xi^3 / (q^{\alpha} + \xi^{\alpha})\), then we can determine the nonlinear differential equation, see Eq. (12) of Ref. [55], that describes the relation between the radial coordinate of black holes ($\xi$) and that of the simulation in fluids ($r$),
\[
(\xi')^4 - f(\xi)\xi^2 (\xi')^2 - A^2 \xi^6 = 0, \quad f(\xi) = 1 - \frac{2M^{-1/2} \xi^2}{q^{3/2} + \xi^{3/2}},
\]
where the prime stands for the derivative with respect to $r$. From Eq. (77), we can find the asymptotic relations of $\xi$ and $r$,
\[
\xi^+ \sim c_{13} e^{\pm r}, \quad \text{as} \quad \xi \to 0, \quad \xi^- \sim \frac{4}{(\sqrt{Ar} \pm c_{14})^2}, \quad \text{as} \quad \xi \to \infty,
\]
where $c_{13}$ and $c_{14}$ are integration constants. Meanwhile, since the physical variables of fluids can be represented by $\xi$ and its derivatives, see Eq. (39),
\[
c = \frac{\xi'}{\xi^2}, \quad v = \frac{A \xi}{\xi'}, \quad \rho = \frac{\xi'}{\xi}, \quad p' = \frac{(\xi')^2}{\xi^6} \left[ \xi \xi'' - (\xi')^2 \right],
\]
their asymptotic behaviors for the case with the positive sign are
\[
c^+ \to \frac{e^{-r}}{c_{13}}, \quad v^+ \to A, \quad \rho^+ \to 1, \quad p'^+ \to 0,
\]
when $r \to 0$, and
\[
c^+ \to -\frac{\sqrt{A}}{2} \left( \sqrt{Ar} + c_{14} \right), \quad v^+ \to -\frac{\sqrt{A}}{2} \left( \sqrt{Ar} + c_{14} \right), \quad \rho^+ \to -\frac{2\sqrt{A}}{\sqrt{Ar} + c_{14}}, \quad p'^+ \to \frac{A^2}{2},
\]
when $r \to \infty$. The phenomenon of transonic flows occurs outside the horizon and the Mach number converges to one, $M \to 1$, as $r$ approaches to infinity. The numerical analysis is shown in Fig. 5, where we have adopted the setting, $M = 1/2$, $q = 1/2$, and $A = 1$, and chosen $r_H = 0$ which corresponds to $\xi(0) \approx -3.837$. The upper boundary of Mach numbers is determined numerically, $M \leq 1.124$, the maximum is reached at $r \approx 0.796$.

For the case of $\zeta \neq 0$, the vacuum equation, $G^\xi_\xi = \zeta^2$, gives rise to
\[
\zeta'^2 + \zeta'' = 0,
\]
which is valid for an arbitrary $\zeta \in [0, \infty)$. The solution, $\zeta = \ln(\xi - c_{15}) + c_{16}$, where $c_{15}$ and $c_{16}$ are integration constants, does not meet the regular condition. Namely, we have to separate the discussion inside the horizon from that outside the horizon. However, both $p^+_{\zeta} \geq -\epsilon_{\text{out}}$ and $p^+_{\zeta} \geq -\epsilon_{\text{in}}$ provide the same inequality,
\[
\zeta'^2 + \zeta'' \leq 0,
\]

Figure 5: Numerical solutions for the ES of repaired cylindrical Hayward-like models with $M = q = 0.5$ and $A = 1$, where $r_c \approx -1.352$, $r_0 \approx -3.675$ which is determined by $\xi(r_0) = 0$, and $r_\infty \approx 1.684$ which is determined by a very large value of $\xi$ in the numerical calculation.
which can be solved when we multiple \( \exp(\zeta) \) on its both sides,

\[
\frac{d^2}{d\xi^2} (e^\zeta) \leq 0.
\]  

(85)

This inequality indicates that \( \exp(\zeta) \) is a concave function in \( \xi \in [0, \xi_H) \cup (\xi_H, \infty) \), thus \( \zeta \) is a concave function because the logarithm of a non-negative and concave function is concave.\(^5\) This can also be seen from Eq. (84), \( \zeta'' \) is nonpositive because \( (\zeta')^2 \) is nonnegative. Nevertheless, \( \zeta \) is bounded in order to meet the condition of finite curvatures, which contradicts to the concavity of \( \zeta \). In other words, the metric Eq. (70) can in no way represent a physical RBH. In contrast, the 2D BHs with the polar symmetry can be regarded as cylindrical BHs with the \( z \) direction being suppressed. In the following section, we are going to investigate the 2D polar-symmetric RBHs and their simulations in fluids.

8 The simulations of lower dimensional regular black holes

The 2D simulation is rather different from the 3D case discussed in the above section. At first, the Weyl curvature tensor vanishes identically for any \((2+1)D\) spacetimes, thus the Weyl scalar is no longer an appropriate candidate for analyzing the curvature divergence. Secondly, the regular condition is closely related to the dimension of spacetime. When one studies the RBHs in a \((2+1)D\) spacetime, the criteria for shape functions will change, see App. D. Thirdly, the form of acoustic line elements also depends on dimension, in particular the prefactor \([25]\). Moreover, the 2D analogue BHs in a fluid are rather interesting and relatively easy to be realized in a laboratory. Let us start considering a circularly symmetric BH,

\[
\bar{g}_{ij} = \text{diag}\{-f, f^{-1}, \xi^2\}, \quad f = 1 - \mu \sigma(\xi),
\]  

(86)

where \( \mu \) is mass-like parameter. According to App. D, such a metric is curvature regular at the BH center if \( \sigma \sim O(\xi^n), \ n \geq 2, \) as \( \xi \to 0, \) and asymptotic flat if \( \sigma \to 0 \) as \( \xi \to \infty. \) In order to construct an example that satisfies these conditions, we make an ansatz,

\[
\sigma(\xi) = \frac{\xi^\alpha}{\xi^\beta + q^\beta} \quad \text{with} \quad \alpha, \beta \geq 0,
\]  

(87)

and substitute it into Kretschmann scalars. We find that the Kretschmann scalar is regular at the BH center if \( \alpha \geq 2 \) and that the asymptotic flatness is satisfied if \( \beta > \alpha. \) Next, given a particular \( \alpha \) we are going to fix \( \beta \) by the DEC. For instance, taking \( \alpha = 2, \) we reduce the DEC to the two inequalities,

\[
(\beta + 2)q^\beta \geq (\beta - 2)\xi^\beta, \quad 4q^{2\beta} + (\beta - 2)^2\xi^{2\beta} \geq [\beta(\beta + 4) - 8]q^\beta \xi^\beta.
\]  

(88)

Because the left hand sides of the two inequalities are positive, the two inequalities hold for all non-negative \( \xi \) and \( q \) if their right hand sides are non-positive, i.e., the DEC is satisfied in the whole spacetime. Following this idea, we find \( 0 < \beta \leq 2\sqrt{3} - 2 \approx 1.46. \) However, this result

\(^5\)Suppose \( y(x) \geq 0 \) and \( y''(x) < 0, \) thus \( d^2 \ln(y)/dx^2 = (-y'^2 + yy'')/y^2 \) is negative, i.e., \( \ln(y) \) is concave.
contradicts to the asymptotic flatness. In fact, for the case of $\alpha \geq 2$, the positive energy condition $\epsilon \geq 0$ leads to $\alpha \geq \beta$ in $\xi \in [0, \infty)$, while the asymptotic flatness requires $\beta > \alpha$. No intersections exist.

If relaxing the asymptotic flatness, we replace it with the Ricci flatness, $R = 0$, at infinity. Then substituting the ansatz Eq. (87) into the Ricci scalar, we obtain

$$R/\mu = \frac{(\beta - 3)(\beta - 2)\xi^{2\beta} + [12 - \beta(\beta + 5)]q^{\beta} \xi^\beta + 6q^{2\beta}}{(\xi^\beta + q^\beta)^3}. \tag{89}$$

Since $\beta$ is non-negative, the power of $\xi$ in the denominator is larger than that in the numerator, thus $R$ vanishes as $\xi$ approaches to infinity. In other words, if $\alpha = 2$, the metric with Eq. (87) automatically satisfies the condition of Ricci flatness. As a result, the model Eq. (87) together with the Ricci flatness is regular and satisfies the energy conditions, that is, it is a physical RBH.

Furthermore, before we focus on the analogue in a fluid, we make a note on the toy model we constructed. The causal structure of the 2D RBH with Eq. (87) is exotic. Since the power of $\xi$ in the numerator of $\sigma$ is larger than that of the denominator, i.e., $\alpha = 2$ and $0 < \beta \leq 2\sqrt{3} - 2 \approx 1.46$, $\sigma$ is an increasing function with respect to $\xi$.

Now let us turn to the simulation. From Eq. (37), we obtain the relation between $\xi$ and $r$,

$$\left(\xi'\right)^4 - f(\xi)\xi^2(\xi')^2 - A^2\xi^6 = 0. \tag{90}$$

For a specific model, $\alpha = 2$ and $\beta = 1$, i.e., $\sigma = \xi^2/(\xi + q)$, we find the asymptotic solutions,

$$\xi_0^\pm(r) \sim c_{17}e^{\pm r}, \quad \text{as} \quad \xi \to 0, \tag{91}$$

and

$$\xi_\infty^\pm(r) \sim \frac{4}{(\mp kr + c_{18})^2}, \quad \text{as} \quad \xi \to \infty, \tag{92}$$

where $k := \sqrt{(\sqrt{4A^2 + \mu^2} - \mu)/2}$, and $c_{17}$ and $c_{18}$ are integration constants. Since the case with the positive sign corresponds to the positive correlation between $\xi$ and $r$, we would like to select it as the candidate for the simulation. Meanwhile, we note from Eq. (92) that there is a movable singularity in the asymptotic solution as $\xi \to \infty$. In other words, $\xi_\infty^+(r)$ diverges at a finite $r_\infty$ which also depends on the choice of $\xi_\infty^+(0)$. To estimate the value of $r_\infty$, we apply the condition $\xi_\infty^+(0) = \xi_H$ which determines the integration constant $c_{18} = 2\sqrt{\sqrt{2} - 1}$ and $r_\infty = 2\sqrt{(\sqrt{2} - 1)/(\sqrt{4A^2 + \mu^2} - \mu)}$. Here $\xi_H$ denotes the horizon radius of the astronomical counterpart depicted by Eq. (86).

Similarly, by applying Eq. (80) we find the asymptotic behaviors for the case with the positive sign around $r = 0$,

$$c_0^+ \to \frac{e^{-r}}{c_{17}}, \quad v_0^+ \to A, \quad \rho_0^+ \to 1, \quad p_0^+ \to 0, \tag{93}$$

and in the limit of $r \to \infty$,

$$c_\infty^+ \to -\frac{1}{2}k(kr + c_{17}), \quad v_\infty^+ \to -\frac{A(kr + c_{17})}{2k}, \quad \rho_\infty^+ \to -\frac{2k}{kr + c_{17}}, \quad p_\infty^+ \to \frac{k^4}{2}. \tag{94}$$
The phenomenon of transonic flows occurs outside the horizon and the Mach number converges to a constant $\mathcal{M} \to A/k^2$ as $r$ approaches to infinity. The numeric analysis is shown in Fig. 6, where we have adopted the setting $A = 1$, $q = 1/2$, and $\mu = 1/2$, together with the condition $\xi^+(0) = 1 + \sqrt{2}$. The upper boundary of Mach numbers is determined numerically, $\mathcal{M} \leq (1 + \sqrt{17})/4$.

Figure 6: Numerical solutions for the 2D repaired Hayward model with $\mu = q = 0.5$ and $A = 1$, where $r_0 \approx -2.203$ which is determined by $\xi(r_0) = 0$, and $r_{\infty} \approx 1.030$ which is determined by a very large value of $\xi$. 
9 Conclusions and outlooks

As an extension of our previous work [30], in the present paper, we analyze two specific questions on RBHs: The first is how to remedy astronomical RBHs with defects on energy conditions; and the second is how to simulate physical RBHs through acoustic gravity. We emphasize that the research strategies of the previous and present works are completely opposite. In the previous work [30], we first construct an acoustic metric which is regular at the acoustic BH center, then we investigate the energy conditions of the astronomical counterpart of the acoustic BH. In the present work, we first remedy an astronomical metric to meet the energy conditions and the condition of regular curvatures, and then we simulate it in a fluid.

The energy conditions of astronomical BHs occupy a fundamental and decisive position among all the physical constraints in the BH physics. They determine whether a black hole is observable in the universe or just imaginary in an academic paper. On the premise of fully considering the energy conditions, we have remedied several widely known RBHs whose energy conditions were broken. The procedure we proposed in the present paper works for a broad class of RBHs with defects, in particular, for those models whose shape functions are the rational fraction function with respect to the radial coordinate. In addition, we have demonstrated that two types of conformally related RBHs can never meet the energy conditions by proving a no-go theorem.

Although the analogue gravity is widely regarded as a tool of gaining insight into general relativity [25], the first simulation of Schwarzschild and Reissner-Nordström black holes was not realized in the acoustic gravity until 2021 [55]. Prior to this, the acoustic counterparts did not mimic the astronomical BHs at least up to a conformal factor [60], which would be shown in the investigations of desired phenomena. For instance, the quasinormal modes (QNMs) of BHs would be affected [61] by conformal factors. We hope such an analogue in a fluid completely mimics astronomical BHs. To this end, starting with the physical RBHs, we have constructed their counterparts in acoustic gravity. Our ultimate goals focus on the guidance on simulation of physical RBHs in a fluid and the possibility to distinguish the RBHs from singular BHs. Our guidance on simulation has been illustrated by the equations of state, see Figs. 3c, 4e, and 6e, by the speed of sound and velocity of flow, see Figs. 2, 4b, and 6b. Moreover, the similarities and differences are listed below when we compare our RBHs with the RN BH [55] in the aspects of velocities and Mach numbers, and in the aspects of densities and pressures for 3D models, equatorial sections of 3D models, and 2D models.

**Velocity and Mach number in 3D models:** The velocity of flow is divergent at \( r = 0 \) and tends to zero as \( r \to \infty \) for the RBHs considered in Sec. 5, thus the Mach number converges to unity as \( r \to 0 \) and vanishes as \( r \to \infty \). For the RN BH [55], the velocity of flow is finite at \( r = 0 \) and converges to a constant as \( r \to \infty \), thus the Mach number vanishes at \( r = 0 \) and becomes a finite number as \( r \to \infty \).

**Density and pressure in 3D models:** \( \rho(r) \) and \( p(r) \) for our RBHs in Sec. 5 have maximums at the same point of \( r \), which is related to the zero of \( \xi''(r) \). This property leads to a sharp discontinuity in the plot of EoS, which does not occur for the RN BH. Meanwhile, the pressures of our RBHs are divergent negatively at \( r = 0 \), while they are positively infinite at this point for the RN BH.
Velocity, Mach number, density, and pressure in equatorial sections of 3D models: The velocity, Mach number, density, and pressure for the equatorial section of the 3D RBH constructed in Sec. 7 exhibit similar configurations to those of a RN naked singularity [55], which implies that the regularity of our model does not appear in the simulation of equatorial sections. Note that we have applied the method [55] to turn a non-compact dimension into a compact one in the construction of the relation between $\xi$ and $r$.

Velocity and Mach number in 2D models: The velocity and Mach number of our 2D RBH constructed in Sec. 8 differ greatly from those of the 2D RN BH. At first, our RBH may have only one horizon; then, the transonic flow of our RBH occurs outside the horizon. For the 2D RN BH, it has two horizons and its transonic flow occurs between the inner and outer horizons.

Density and pressure in 2D models: The differences between our 2D RBH constructed in Sec. 8 and the 2D RN BH [55] are obvious in the density and pressure. For our 2D RBH, $\rho(r)$ and $p(r)$ are no longer divergent at $r = 0$ and the EoS becomes smooth in the whole domain of $r$, see Figs. 6c, 6d, and 6e. Meanwhile, the behaviors of the two variables in the 2D RBH differ from those in the equatorial sections of 3D models, despite the fact that we use the identical compactification in both situations. This difference represents the fact that the dimensions of RBHs affect the properties of flow simulations.

Finally, we summarize that the EoS of fluid can be used to simulate the physical RBHs. Meanwhile, we show that the acoustic analogues of RBHs have apparently different features from those of singular BHs, such as the RN BH, and that the differences are indeed caused by singularities. In other words, the acoustic gravity can be applied as a tool to study astronomic RBHs, which also offers a theoretical basis to investigate more phenomena of astronomic RBHs in a fluid.

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A The differential inequalities

Here we provide the derivations of solving the differential inequalities used in this paper, in particular, in Sec. 2. We start with solving the differential inequality, $\xi\sigma'' \leq 2\sigma'$, with the boundary conditions, $\sigma(0) = \sigma'(0) = 0$. It can be rewritten as

$$\frac{d}{d\xi} (3\sigma - \xi\sigma') \geq 0. \quad (95)$$

After considering the boundary conditions, we obtain $3\sigma - \xi\sigma' \geq 0$. Next, multiplying $\xi^{-4}$ on its both sides, we derive

$$3\xi^{-4}\sigma - \xi^{-3}\sigma' = \frac{d}{d\xi} (-\xi^{-3}\sigma) \geq 0. \quad (96)$$
If we define \( \sigma_0 := \lim_{\xi \to 0} \sigma / \xi^3 \), we arrive at the solution,

\[
\sigma \leq \sigma_0 \xi^3,
\]

which can also be obtained by directly using the differential form of the Grönwall-Bellman lemma [39].

Similarly, we can obtain \( \sigma \geq 0 \) from \( \sigma' \geq 0 \) when the boundary condition, \( \sigma(0) = 0 \), is considered.

We emphasize that the differential inequalities’ solutions provide the necessary condition for a RBH to meet the energy conditions, but not the sufficient one. For example, taking a bell-shaped function,

\[
\sigma = \frac{1}{\left( \xi - \frac{1}{2\xi} \right)^4 + 1},
\]

we can see that the inequality’s solution, \( \sigma \geq 0 \), is satisfied but the WEC and DEC are violated because of \( \sigma' \not\geq 0 \). However, if \( \sigma \geq 0 \) is broken, then the WEC and DEC must be violated. In addition, the Hayward BH gives us another example, that is, it satisfies \( 0 \leq \sigma \leq \sigma_0 \xi^3 \), but breaks the DEC. The reason comes from the characteristics of differential inequalities, i.e., a differential inequality signifies that all functions satisfying this differential inequality must be bounded by its solution, while the functions bounded by the solution may not necessarily meet the original differential inequality.

\[\text{B Local properties of the differential inequalities}\]

Now we give an explanation from the energy conditions by analyzing the local properties of a physical RBH at its center, that is, why a physical RBH cannot have a flat or an AdS core around its center. The similar discussion can be found in Ref. [24].

Summarizing the energy conditions Eq. (4), we have four differential inequalities in total,

\[
\sigma' \geq 0, \quad 2\sigma' - \xi\sigma'' \geq 0, \quad 2\sigma' + \xi\sigma'' \geq 0, \quad \sigma'' \leq 0.
\]

Supposing \( \xi \ll 1 \), we expand \( \sigma \) by an asymptotic series,

\[
\sigma = \xi^3 \sum_{n=0}^{\infty} a_n \xi^n,
\]

meanwhile, we have the property, \(|a_n \xi^n| \gg |a_{n+1} \xi^{n+1}|\), as \( \xi \) approaches to zero. Then substituting
the above series into Eq. (98), we obtain

\( \sigma' \geq 0 : \sum_{n=0}^{\infty} (n+3)a_n \xi^{n+2} \geq 0, \)  

\( 2\sigma' \geq \xi \sigma'' : \sum_{n=1}^{\infty} n(n+3)a_n \xi^{n+2} \leq 0, \)  

\( -2\sigma' \leq \xi \sigma'' : \sum_{n=0}^{\infty} (n+3)(n+4)a_n \xi^{n+2} \geq 0, \)  

\( \sigma'' \leq 0 : \sum_{n=0}^{\infty} (n+2)(n+3)a_n \xi^{n+1} \leq 0. \)  

If \( a_0 \neq 0 \) and \( a_1 \neq 0 \), the leading terms of Eqs. (100a) and (100c) lead to \( a_0 > 0 \), which inevitably violates Eq. (100d), and the leading term of Eq. (100b) gives \( a_1 < 0 \). If \( a_0 = 0 \), the DEC must be broken because Eq. (100a) and Eq. (100b) lead to the results that are contradictory to each other. In other words, the analogous power extension in Ref. [20] may have no meanings from the perspective of energy conditions. If \( a_0 \neq 0 \) and \( a_1 = 0 \), the leading term of Eq. (100b) gives \( a_m < 0 \), \( m > 1 \), where \( m \) is the ordinal number of the first non-zero term.

In contrast, based on Ref. [32] we know

\[ \sigma = \xi^3 \sum_{n=0}^{\infty} \frac{\xi^n R^{(n)}(0)}{2M(n+3)(n+4)n!} \sim \frac{\xi^3 R(0)}{24M} + \frac{\xi^4 R'(0)}{40M} + O(\xi^5), \]  

as \( \xi \) approaches to zero. Thus the case of \( a_0 > 0 \) and \( a_1 < 0 \) implies

\[ R(0) > 0 \quad \text{and} \quad R'(0) < 0. \]  

For \( a_1 = 0 \) but \( a_2 \neq 0 \), we obtain \( R(0) > 0 \) and \( R''(0) < 0 \), i.e. \( \xi = 0 \) is a local maximum. Moreover, if a RBH is Ricci flat at its center, \( R(0) = 0 \), or it has an AdS core, \( R(0) < 0 \), its energy conditions must be violated around the center.

C The calculation of Eq. (8)

In order to derive the conditions given by Eq. (8), we substitute Eq. (7) into the DEC of Eq. (4). Since the DEC holds for \( \xi \in [0, \infty) \) and \( q \in [0, \infty) \), we obtain

\[ 3 - \mu \nu \geq 0, \quad (\mu \nu - 4)(\mu \nu - 3) \geq 0, \quad 24 - \mu(\mu + 7)\nu \geq 0. \]  

Meanwhile, the asymptotic flatness demands

\[ 3 - \mu \nu < 1. \]  

Then, combining all above inequalities and considering the positiveness of all parameters, we finally arrive at Eq. (8).
D The $d$-dimensional regular black holes

We calculate the regular conditions which have been applied in Sec. 8. We first write down the $d$-dimensional metric with the spherical symmetry \[62,\]
\[
ds^2 = -f dt^2 + f^{-1}d\xi^2 + \xi^2 d\Omega_{d-2}^2,
\]
(105)
where the unique shape function is \( f = 1 - \mu\sigma(\xi)/\xi^{d-3} \) and \( \mu \) is a mass-like parameter. As we did in Ref. \[32,\] we are going to use the following three curvatures,
\[
R = \frac{\mu}{\xi^{d-2}} (2\sigma' + \xi\sigma''),
\]
(106)
\[
W = \frac{(d-3)\mu^2}{(d-1)\xi^{2d-2}} \left[ (d-2)(d-1)\sigma - 2(d-2)\xi\sigma' + \xi^2\sigma'' \right]^2,
\]
(107)
\[
E = \frac{2\mu^2}{d\xi^{2d-4}} [(d-2)\sigma' - \xi\sigma'']^2,
\]
(108)
to represent \( \sigma \) and its derivatives. With the help of the relations,
\[
W = K - \frac{4R_2}{d-2} + \frac{2R^2}{(d-1)(d-2)}, \quad E = \frac{4R_2}{d-2} - \frac{4R^2}{d(d-2)},
\]
(109)
we arrive at
\[
\sigma = \frac{\xi^{d-1}}{(d-2)(d-1)d\mu} \left[ (d-2)R + s_2(d-1)\sqrt{2dE} + s_1d\sqrt{\frac{d-1}{d-3}W} \right],
\]
(110)
\[
\sigma' = \frac{\xi^{d-2}}{2d\mu} \left( 2R + s_2\sqrt{2dE} \right),
\]
(111)
\[
\sigma'' = \frac{\xi^{d-3}}{d\mu} \left[ (d-2)R - s_2\sqrt{2dE} \right],
\]
(112)
where \( s_{1,2} = \pm \) are two signs which are not much important for the discussion of the finiteness of curvatures. Meanwhile, it is not difficult to see that the metric Eq. (105) has regular curvatures if \( \sigma \) has asymptotic relation \( \sigma \lesssim O(\xi^{d-1}) \).

E Asymptotic solutions of the differential equation

We analyze the local properties of the differential equation Eq. (38),
\[
A^2\xi^4r'(\xi)^4 + F(\xi)\xi^2r(\xi)^6r'(\xi)^2 - r(\xi)^8 = 0,
\]
(113)
at the two boundaries, \( \xi \to 0 \) and \( \xi \to \infty \), by means of the dominant balance \[56, 57,\]. Here \( \xi \in [0, \infty) \) is the radial coordinate of RBHs, while \( r \) is radial coordinate of fluids with the spherical symmetry.

First of all, we consider the asymptotic of the shape function as \( \xi \to 0 \),
\[
F \sim F_0 := 1 - \frac{R(0)}{12}\xi^2.
\]

According to the method of dominant balance, we can separate the discussions into three situations.

In the first case, we have

\[ A^2 \xi^4 r'(\xi)^4 \sim -F_0 \xi^2 r(\xi)^6 r'(\xi)^2, \quad (114a) \]

\[ A^2 \xi^4 r'(\xi)^4 \gg r(\xi)^8, \quad F_0 \xi^2 r(\xi)^6 r'(\xi)^2 \gg r(\xi)^8, \quad (114b) \]

where Eq. (114a) gives the solution,

\[ r_{\pm}^2 = -\frac{2}{A} \sqrt{-1 + \frac{R(0)}{12} \xi^2} + \frac{2}{A} \tan^{-1} \left[ \sqrt{-1 + \frac{R(0)}{12} \xi^2} \right] - 2 \tilde{c}_1, \quad (115) \]

and \( \tilde{c}_1 \) is integration constant. This solution becomes complex as \( \xi \to 0 \), which contradicts the physical setting that \( r(\xi) \) must be real.

In the second case, the asymptotic relations become

\[ F_0 \xi^2 r(\xi)^6 r'(\xi)^2 \sim r(\xi)^8, \quad (116a) \]

\[ F_0 \xi^2 r(\xi)^6 r'(\xi)^2 \gg A^2 \xi^4 r'(\xi)^4, \quad r(\xi)^8 \gg A^2 \xi^4 r'(\xi)^4. \quad (116b) \]

The solution of Eq. (116a) is

\[ r_{\pm} = \frac{\xi \sqrt{3R(0)}}{6 \pm 6 \sqrt{1 - \frac{R(0)}{12} \xi^2}}. \quad (117) \]

However, \( r_+ \) is not consistent with the asymptotic assumption depicted by Eq. (116b). For \( R(0) \neq 0 \), we have a divergent limit as \( \xi \to 0 \),

\[ \lim_{\xi \to 0} \frac{A^2 \xi^4 r_+^2(\xi)^4}{F_0 \xi^2 r(\xi)^6 r_+^2(\xi)^2} \to \infty. \quad (118) \]

As to \( r_- \), we note that it becomes complex if \( R(0) < 0 \), thus this second case should also be ruled out.

In the third case, we suppose

\[ A^2 \xi^4 r'(\xi)^4 \sim r(\xi)^8, \quad (119a) \]

\[ A^2 \xi^4 r'(\xi)^4 \gg F_0 \xi^2 r(\xi)^6 r'(\xi)^2, \quad r(\xi)^8 \gg F_0 \xi^2 r(\xi)^6 r'(\xi)^2. \quad (119b) \]

Eq. (119a) provides the solution,

\[ r_{\pm} = -\frac{\sqrt{A}}{\tilde{c}_2 \pm \ln(\xi)}, \quad \text{or} \quad \xi = \tilde{c}_3 \exp \left( \pm \frac{\sqrt{A}}{r_{\pm}} \right), \quad (120) \]

where \( \tilde{c}_2 \) and \( \tilde{c}_3 \) are integration constants. This solution is consistent with the the asymptotic assumption Eq. (119b) because we have

\[ \lim_{\xi \to 0} \frac{F_0 \xi^2 r(\xi)^6 r'(\xi)^2}{A^2 \xi^4 r'(\xi)^4} = \lim_{\xi \to 0} \frac{1}{\tilde{c}_2 \pm \ln(\xi)} = 0. \quad (121) \]
However, we exclude $r_-$ from physical solutions due to $r_- < 0$ when $\xi \sim 0^+$. Next, we turn to the asymptotic solutions as $\xi \to \infty$. The shape function then becomes

$$F \sim F_\infty := 1 - 2M\xi^{-n},$$

where $0 < n \leq 1$. Similarly, the discussion can also be separated into three situations.

In the first case, we have

$$\xi^2 F_\infty r(\xi)^6 r'(\xi)^2 \sim r(\xi)^8, \quad (122a)$$

$$\xi^2 F_\infty r(\xi)^6 r'(\xi)^2 \gg A^2 \xi^4 r'(\xi)^4, \quad r(\xi)^8 \gg A^2 \xi^4 r'(\xi)^4. \quad (122b)$$

The asymptotic assumption Eq. (122a) gives two solutions,

$$r_\pm = \tilde{c}_4 \left(\sqrt{\xi^n/(2M)} + \sqrt{\xi^n/(2M) - 1}\right)^{\pm 2/n} \sim \tilde{c}_5 \xi^{\pm 1}, \quad (123)$$

where $\tilde{c}_4$ and $\tilde{c}_5$ are integration constants. It can be verified that $r_-$ contradicts to the asymptotic assumption Eq. (122b), while $r_+$ does not.

In the second case, we suppose

$$A^2 \xi^4 r'(\xi)^4 \sim r(\xi)^8, \quad (124a)$$

$$A^2 \xi^4 r'(\xi)^4 \gg \xi^2 r(\xi)^6 F_\infty r'(\xi)^2, \quad r(\xi)^8 \gg \xi^2 r(\xi)^6 F_\infty r'(\xi)^2. \quad (124b)$$

The first relation leads to

$$r_\pm = -\sqrt{A} \frac{1}{\tilde{c}_6 \pm \ln(\xi)}, \quad (125)$$

where $\tilde{c}_6$ is integration constant. We can see that $r_\pm$ converge to zero as $\xi \to \infty$. This implies that the transformation between $r$ and $\xi$ is not injective, which is obvious because both $\xi = 0$ and $\xi = \infty$ map to the single point $r = 0$. As a result, we eliminate this case.

In the last case, the asymptotic assumption contains

$$A^2 \xi^4 r'(\xi)^4 \sim -\xi^2 r(\xi)^6 F_\infty r'(\xi)^2, \quad (126a)$$

$$A^2 \xi^4 r'(\xi)^4 \gg r(\xi)^8, \quad \xi^2 r(\xi)^6 F_\infty r'(\xi)^2 \gg r(\xi)^8, \quad (126b)$$

whose solutions are inevitably complex. Thus, this case is not in our consideration.

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