Photon-mediated interactions: a scalable tool to create and sustain entangled many-body states

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Generation and sustenance of entangled many-body states is of fundamental and applied interest. Recent experimental progress in the stabilization of two-qubit Bell states in superconducting quantum circuits using an autonomous feedback scheme [1] has demonstrated the effectiveness and robustness of driven-dissipative approaches, i.e. engineering a fine balance between driven-unitary and dissipative dynamics. Despite the remarkable theoretical and experimental progress in those approaches for superconducting circuits, no demonstrably scalable scheme exists to drive an arbitrary number of spatially separated qubits to a desired entangled quantum many-body state. Here we propose and study such a scalable scheme, based on engineering photon-mediated interactions, for driving a register of spatially separated qubits into multipartite entangled states. We demonstrate how generalized W-states can be generated with remarkable fidelities and the entanglement sustained for an indefinite time. The protocol is primarily discussed for a superconducting circuit architecture but is ideally realized in any platform that permits controllable delivery of coherent light to specified locations in a network of Cavity QED systems.

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driven to a desired steady state by merely tuning the common drive frequency. Earlier schemes either rely on internal transitions of multi-level qubits \[^1\] or on multiple drives \[^2\] \[^3\] that are not demonstrably generalizable to more than two qubits. Finally, we analyze the fault-tolerance of the method to phase and amplitude noise of the drive parameters, as well as the non-uniformity of qubit and cavity parameters. In addition to a nontrivial scalable extension of QBE schemes, we introduce a scheme for finely tailoring the entangled steady states, consisting in breaking the lattice symmetry with highly versatile non-equilibrium drives.

The underlying idea to photon-mediated many-body state generation is quite general as illustrated in Fig. 1. Consider an engineered electromagnetic environment in the form of an interconnected network of cavities, described by the retarded and the Keldysh components of the photon Green’s function, \(G^{R}_{\text{ph}}(x,x';\omega)\) and \(G^{K}_{\text{ph}}(x,x';\omega)\), and \(N\) qubits with splitting \(\omega_q\) connected to this ‘photonic backbone’ at arbitrary locations \(x_i\). The cavities forming the electromagnetic environment are in addition driven by external light sources at frequency \(\omega_d\) with amplitudes \(c_i^d\). Under such a situation, the entire photonic background can be integrated out to obtain (a) effective qubit-qubit interactions of the form \(G^{K}_{\text{ph}}(x_i,x_j;\omega_q)\sigma^+_i\sigma^-_j + \text{h.c.}\), (b) local Zeeman fields of the form \(c_i^d\sigma_i^z\), (c) a non-unitary evolution resulting in non-equilibrium transition rates governed by the photon’s Keldysh Green’s function.

A one-dimensional model. To demonstrate the general approach discussed above for a specific case, we consider a one-dimensional array of \(N\) microwave optical cavities with frequency \(\omega_i\) which are coupled together capacitively and each one hosting a superconducting qubit with frequency \(\omega_q\). We shall consider both the cases of open and periodic boundary conditions (identifying site \(N\) with site 0). Each cavity can be driven (or not) by a classical monochromatic microwave source at frequency \(\omega_d\), see Fig. 2. We work in a regime where \(\omega_c\), \(\omega_q\) and \(\omega_d\) are far detuned (typically on the order of GHz) such that there are few qubit and photonic excitations present in the system. The total Hamiltonian is that of a one-dimensional driven Jaynes-Cummings lattice model studied before in Ref. \[^2\] \[^3\],

\[
H = H_\sigma + H_{\sigma,a} + H_a ,
\]

where \(H_\sigma\), \(H_{\sigma,a}\), and \(H_a\) are respectively the qubit, the Jaynes-Cummings light-matter coupling, and the driven cavity-photon Hamiltonians (we set \(\hbar = 1\)),

\[
H_\sigma = \sum_i \omega_d \sigma^z_i / 2, \quad H_{\sigma,a} = g \sum_i [a^+_i \sigma^-_i + \text{h.c.}] ,
\]

\[
H_a = \sum_i [\omega_c a^+_i a_i - J(a^+_i a_{i+1} + \text{h.c.}) + 2\epsilon_i^d \cos(\omega_d t + \Phi_i) (a_i + a^+_i)] .
\]

\(i\) runs from site 0 to \(N - 1\). The qubits are two-level systems described by the usual Pauli matrices and \(\sigma^\pm = (\sigma^x \pm \text{i}\sigma^y) / 2\). \(\epsilon_i^d\) and \(\Phi_i\) are respectively the cavity-dependent amplitude and phase of the ac microwave drives. Zero-temperature equilibrium is achieved by setting all drive amplitudes to zero, \(\epsilon_i^d = 0\). Without loss of generality, we assume that the detuning between cavity and qubit frequency \(\Delta \equiv \omega_q - \omega_c > 0\) and typical experimental parameters give \(g/\Delta \sim 10^{-1}\). \(J\) is the nearest-neighbour photon hopping amplitude between cavities and can be made on the order of \(g\).

In addition to the driven-unitary dynamics, the system is subject to two main sources of dissipation: cavity loss of photons with a decay rate \(\kappa\) and qubit relaxation at a rate \(\gamma\) which are both typically on the order of 10-100 MHz. We also include qubit dephasing with a pure dephasing rate \(\gamma_{\phi}\) which can typically be made an order of magnitude smaller than \(\gamma\) \[^4\] \[^5\].

We eliminate the time-dependence in \(H_a\) by working in the frame rotating at \(\omega_d\) and dropping the terms rotating at higher frequencies. In the rest of the Hamiltonian \[^1\], this also amounts to replacing \(\omega_q\) by \(\Delta_q \equiv \omega_q - \omega_d\) and \(\omega_c\) by \(-\Delta_c \equiv \omega_c - \omega_d\). Once expressed in the eigenbasis of the (undriven) coupled cavity system, \(H_a\) is given by

\[
H_a = \sum_k (\omega_k - \omega_d) a^+_k a_k + (\epsilon_k^d a^+_k a_k + \text{h.c.}) ,
\]

where the photonic dispersion relation

\[
\omega_k \equiv \omega_c - 2J \cos(k)
\]

yields the following spectral function per mode \(k\)

\[
\rho_k(\omega) = -\frac{1}{\pi} \text{Im} \frac{1}{\omega - \omega_k + \text{i}\kappa/2} .
\]

With periodic boundary conditions, the translational symmetry of the undriven system model imposes the set of quasi-momenta \(k = n \pi/N, n = 0 \ldots N - 1\). Open
boundary conditions break this symmetry. In Eq. (11) we incorporated the phase $\Phi_i$ in $\epsilon^d_i$ which can henceforth be complex. Note that the precise coupling between the cavities dictates the dispersion relation $\omega_k$, however the Hamiltonian (11) is far more general than the one in Eq. (5) that is based on a nearest-neighbor tight-binding model, as the former can describe any complex “photonic backbone”.

**Effective dissipative XY model.** We treat the light-matter interaction to second-order perturbation theory in $g/\Delta$ by means of a Schrieffer-Wolff transformation which maps $H \rightarrow e^X H e^{X^\dagger}$ where

$$X \equiv g \sum_k \left[ \frac{a_k \sigma^+_k}{\omega_k - \omega_k} - \text{h.c.} \right]. \quad (7)$$

We obtain a transverse-field isotropic XY-model coupled to a bath of photons

$$H_\sigma = \sum_i h_i \cdot \frac{\sigma^z_i}{2} - \frac{J}{2} \left( \frac{g}{\Delta} \right)^2 \left[ \sigma^x_i \sigma^x_{i+1} + \sigma^y_i \sigma^y_{i+1} \right], \quad (8)$$

$$H_{\sigma \sigma} = \sum_i \left( \frac{g}{\Delta} \right)^2 \sigma^z_i \left( \Delta a_i^d \sigma^+_i + \epsilon^d_i \sigma^d_i + \epsilon^d_i \sigma^*_i \right), \quad (9)$$

with $h_i^x = 2 \text{Re}(\epsilon^d_i)(g/\Delta)$, $h_i^y = -2 \text{Im}(\epsilon^d_i)(g/\Delta)$, $h_i^z \approx \Delta_q$. The spin-spin couplings stem from photon-mediated interactions between the cavities. The effective magnetic field $h_i$ is mainly oriented along the $z$ direction but we shall see that, while they break the integrability of the model, the $x$ and $y$ components of the emergent Zee- man field play a crucial role to the scheme below. In equilibrium, $H_\sigma$ experiences a phase transition from an equilibrium zero-temperature bath, the Fermi Golden rule yields the transition rates between the ground state $|0\rangle$ and any state $|k\rangle$ in the one-excitation manifold, $H_\sigma$ simply is the separable state $|0\rangle \equiv |↓ \ldots \downarrow \rangle$.

We treat the non-linearities in $H_{\sigma \sigma}$ by decomposing the photonic field into mean field plus bosonic fluctuations:

$$a_i \equiv \bar{a}_i + d_i \quad \text{with} \quad \bar{a}_i \simeq \frac{\epsilon^d_i}{\Delta_c + i k/2} \simeq \frac{\epsilon^d_i}{\Delta_c}.$$

(10)

Neglecting those terms that are quadratic in the fluctuations and that couple to the qubits, the light sector reduces to

$$H_d = \sum_k (\omega_k - \omega_\Delta) d_k^d d_k^\dagger, \quad (11)$$

$$H_{\sigma \sigma} = \left( \frac{g}{\Delta} \right)^2 \sum_i \sigma^z_i \left[ (\Delta \bar{a}_i + \epsilon^d_i) d_i^\dagger + \text{h.c.} \right]. \quad (12)$$

Our goal is to identify non-trivial entangled eigenstates of the spin chain $H_\sigma$ and design a protocol which, starting from the ground state $|0\rangle$, achieves an effective cooling to any of these excited states. For simplicity, we shall aim at those states carrying only one excitation (the so-called one-excitation manifold) and neglect higher excited states. This truncation of the spectrum holds if the higher excitation manifolds are not significantly occupied. We shall later come back to this assumption and validate it. The truncated spin-chain Hamiltonian reads [27]

$$H_\sigma = \sum_k E_k |k\rangle \langle k| + \left( \frac{g}{\Delta} \right) (\epsilon^d_i |k\rangle \langle 0| + \text{h.c.}) \quad (13)$$

where the states $|k\rangle$ are the eigenstates of the undriven spin chain (i.e. for $\epsilon^d_i = 0 \forall i$) with a dispersion relation

$$E_k \equiv \epsilon_k - \omega_\Delta, \quad \epsilon_k \equiv \omega_k - 2J \left( \frac{g}{\Delta} \right)^2 \cos(k). \quad (14)$$

The $|k\rangle$’s are single excitations of quasi-momentum $k$, entangled over the whole chain. For example, $k = 0$ corresponds to the $N$-qubit W state $|k = 0\rangle \equiv \frac{1}{\sqrt{N}} \sum_j |i\rangle$ where $|i\rangle \equiv |↓ \ldots \uparrow \downarrow \ldots \downarrow \rangle$ indicates one excitation located at site $i$.

Let us first discuss the case of open boundary conditions for which the absence of translational and space-reversal symmetry generically yields a fully non-degenerate spectrum. Perturbation theory in $(g/\Delta)(\epsilon^d_i/\Delta_q)$ yields the following eigenstates of $H_\sigma$:

$$|0\rangle \simeq |0\rangle - \left( \frac{g}{\Delta} \right) \sum_k \frac{\epsilon^d_k}{\Delta_q} |k\rangle, \quad \tilde{E}_0 \simeq -\left( \frac{g}{\Delta} \right)^2 \sum_k |\epsilon^d_k|^2/\Delta_q, \quad (15)$$

$$|\tilde{k}\rangle \simeq |k\rangle + \left( \frac{g}{\Delta} \right) \frac{\epsilon^d_k}{\Delta_q} |0\rangle, \quad \tilde{E}_k \simeq \sum_k \epsilon_k \left( \frac{g}{\Delta} \right)^2 |\epsilon^d_k|^2/\Delta_q. \quad (16)$$

The above corrections to the undriven eigenstates are crucial for the success of the two-photon cooling mechanism presented below.

Approximating the bath of photon fluctuations by an equilibrium zero-temperature bath, the Fermi Golden rule yields the transition rates between the ground state $|0\rangle$ and any state $|k\rangle$ in the one-excitation manifold,

$$\Gamma_{0\rightarrow k} = 2\pi \sum_q \Lambda_{kq} \rho_q(\omega_\Delta + \tilde{E}_0 - \tilde{E}_k), \quad (17)$$

where the contribution of the photonic mode $q$ to the transition is $\Lambda_{kq} = |\langle k|h_{bd,q}|0\rangle|$, in which $h_{\sigma,d,k}$ is the projected Hamiltonian defined by $H_{\sigma,d} = \sum_k h_{\sigma,d,k} a_k^d d_k^\dagger + \text{h.c.}$,

$$h_{\sigma,d,k} = \left( \frac{g}{\Delta} \right)^2 \sum_{k'} (\Delta \bar{a}_{k-k'} + \epsilon^d_{k-k'}) \frac{\sigma^z_{i}}{\sqrt{N}}. \quad (18)$$

We obtain

$$\Lambda_{kq} = \frac{2}{N} \left( 1 + \frac{\Delta}{\Delta_c} \right) \frac{1}{\Delta_q} \left( \frac{g}{\Delta} \right)^3 \sum_{k'} \epsilon^d_{k-k'} \epsilon^d_{k'+q} \quad (19)$$
for all \(k\) and \(q\) in \([0, 2\pi]\). The integration over the photon-fluctuation degrees of freedom also yields Lamb-shift corrections of the energy levels, but this effect does not play any substantial role in our scheme, see below Eq. (23).

The populations of eigenstates, \(n_0\) and \(n_k\), are the solutions of the following rate equations

\[
\frac{dn_0}{dt} = \sum_q n_q \gamma - n_0 \Gamma_{0 \rightarrow q},
\]

\[
\frac{dn_k}{dt} = -n_k \gamma + n_0 \Gamma_{0 \rightarrow k} + \gamma \phi \sum_q (n_q - n_k).
\]

Independently of the initial conditions, these equations have a unique non-equilibrium steady-state solution and, after transient dynamics, the state \(|k\rangle\) is achieved with fidelity

\[
n_k^{NESS} = \frac{1}{1 + N\gamma_\phi/\gamma} \frac{\Gamma_{0 \rightarrow k} + (\gamma_\phi/\gamma) \sum_q \Gamma_{0 \rightarrow q}}{\gamma + \sum_q \Gamma_{0 \rightarrow q}}.
\]

Equation (22) together with Eq. (17) transparently elucidate how to achieve the preparation of a given entangled state \(|k\rangle\) with high fidelity \(n_k^{NESS}\). The protocol consists in maximizing the transition rate \(\Gamma_{0 \rightarrow k}\) to make it the largest of all rates. This is performed by optimally tuning the drive frequency \(\omega_d\) such that \(\rho_\phi(\omega_d + \tilde{E}_0 - \tilde{E}_k)\) in Eq. (17) hits the maximum of the Lorentzian which is on the order of \(1/\kappa\). This is whenever there is at least one mode \(q_0\) with \(\Lambda_{kq_0} > 0\) and the optimum \(\omega_d\) is the solution of the energy-conservation equation

\[
\omega_d = \omega_0 + \omega_c/2 - J \cos(q_0) + \left(\frac{\gamma}{\Delta}\right)^2 \left[J \cos(k) + \frac{1}{2} \sum_{q \neq k} |\epsilon_q|^2 / \Delta_q \right].
\]

Note that when this resonance condition is satisfied, the Lamb-shift correction of \(\tilde{E}_q\) vanishes.

Equation (22) sets an upper bound on the fidelities, \(n_k^{NESS} \leq n_{max} = (\gamma + \gamma_\phi)/(\gamma + N\gamma_\phi)\), which highlights the necessity of working with qubits that have an intrinsic dephasing rate much smaller than their decay rate. The success of the protocol also relies on the resolving power of the spectral width of the photon density of states (\(\sim \kappa\)) i.e. the precision with which the photon fluctuations can target the state \(|k\rangle\) without exciting other eigenstates close in energy. For instance, when driving the first site only, this requirement translates into the condition \(\kappa < \delta \tilde{E}_k \sim 2J|\omega_d/\Delta|^2|\cos(k \pm 2\pi/N) - \cos(k)|\).

To complement the analytic approach, we have performed numerical simulations which (i) compute exactly the full spectrum of the spin chain \(H_\sigma\) in Eq. 3 with \(N = 5\) and open boundary conditions (ii) determine the rates between all the eigenstates (iii) solve for the steady-state populations. In Figure 3 we have used a simple drive profile where only the first cavity is driven \((\epsilon^d = \epsilon^d_{0,0})\) and obtained fidelities \(n^{NESS} \sim n_{max} = 0.73\) in the steady state. The crosses show the numerical results obtained after neglecting second and higher order excitation manifolds. The excellent agreement with the results obtained with the full spectrum justifies this approximation that we used in the analytics. We have also tested the robustness of this approach against site-to-site inhomogeneities of the different parameters and found no qualitative difference for \(\delta \omega_c/\omega_c \sim 10^{-2}, \delta \omega_\kappa/\omega_\kappa \sim 10^{-4}, \delta g/\kappa \sim 10^{-4}\) and \(\Delta /J \sim 10^{-2}\).

Let us now consider the case of periodic boundary conditions for which the undriven system is space-translational and space-reversal invariant. Such a symmetry results in degeneracies between the eigenstates \(|k\rangle\) and \(|2\pi - k\rangle\) of the undriven spin chain (except for \(k = 0\) and \(k = \pi\)). A symmetry-breaking drive profile will generically lift the degeneracy of the spectrum and, importantly, the emergent eigenstates will strongly depend on the particular drive profile. To exemplify this point, let us start by driving the first cavity only: \(\epsilon^d = \epsilon^d_{\pi,0}\). Second-order degenerate perturbation theory in \((g/\Delta)(\epsilon^d/\sqrt{N}\Delta_q)\) lifts the degeneracy in the subspaces spanned by \(|k\rangle\) and \(|2\pi - k\rangle\). To lowest order, the eigenstates are

\[
|k_\pm\rangle \equiv |k\rangle \pm |2\pi - k\rangle \sqrt{2}
\]

for all \(k \in [0, \pi]\) complemented with \(|0_+\rangle \equiv |0\rangle\) and \(|\pi_+\rangle \equiv |\pi\rangle\) (for \(N\) even) and one obtains the rates \(\Gamma_{0 \rightarrow k_\pm} = 2\pi \Lambda^2_{k_\pm} \sum_q \rho_q(\omega_d + \tilde{E}_0 - \tilde{E}_{k_\pm})\) with

\[
\Lambda_{k_-} = 0 \quad \text{and} \quad \Lambda_{k_+} = 2\sqrt{2} \frac{\Delta}{N^2} \left[1 + \left(\frac{\Delta}{\Delta_c}\right)^2 \left(\frac{\gamma}{\Delta}\right)^3 \epsilon^q_2\right]
\]
for all $|k_-\rangle$ and $|k_+\rangle$ except for $|0\rangle$ or $|\pi\rangle$ in which case $\Lambda_{k_1}$ is reduced by a factor $\sqrt{2}$. Figure 4 shows that the $|k_+\rangle$ states can be obtained with remarkable fidelities.

Importantly, in the case of a generic driving profile $\epsilon^d_k$, we find that the relative weights of $|k\rangle$ and $|2\pi - k\rangle$ that enter Eq. (24) are now controlled by the ratio $\epsilon^d_k/\epsilon^{d}_{2\pi-k}$. Therefore, such a non-equilibrium symmetry-breaking scenario offers a highly flexible control over the target entangled state by simply engineering the drive profile $\epsilon^d_k$.

**Discussion.** In this Letter, we proposed a scalable and robust protocol, hinging on engineering light mediated interactions to achieve multipartite entangled states of quantum matter. Our system is a large-scale open quantum system consisting of a lattice of cavity-qubits subject to drive and dissipation. We first showed this experimentally realizable system to map to a driven-dissipative version of an XX spin chain. We proposed to use the non-equilibrium drive as a symmetry-breaking term, offering a highly flexible control over the eigenstates of this spin chain. By optimally driving the system, one can target, realize and sustain indefinitely eigenstates in the one-excitation manifold with remarkable fidelities. This also provides a reliable way to switch from one eigenstate to another. More generally, this method and its possible generalization to higher dimensional lattices and to entangled states in higher order excitation manifolds has promising implications in manipulating and transmitting quantum information over large distances and long time scales.

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