Extrapolation in weighted classical and grand Lorentz spaces. Application to the boundedness of integral operators

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Abstract
We establish weighted extrapolation theorems in classical and grand Lorentz spaces. As a consequence we have the weighted boundedness of operators of Harmonic Analysis in grand Lorentz spaces. We treat both cases: diagonal and off-diagonal ones.

Keywords
Weighted extrapolation · Lorentz spaces · Grand Lorentz spaces · Maximal functions · Calderón–Zygmund operator · Fractional integrals

Mathematics Subject Classification
42B20 · 42B25 · 42B35 · 46B30 · 47B38

1 Introduction
Our aim is to introduce new weighted grand Lorentz spaces and to derive Rubio de Francia’s weighted extrapolation results in these spaces. The obtained results are applied to get the boundedness of operators of Harmonic Analysis in weighted grand Lorentz spaces. To derive the boundedness of operators we rely on a weighted extrapolation theorem in the classical Lorentz spaces which has an independent interest. To get the latter result we first prove weighted extrapolation statements for Banach function spaces. Rubio de Francia’s extrapolation theory gives powerful tools in the study mapping properties of integral operators in weighted function spaces. One of
the important properties of the $A_p$ weights is the extrapolation theorem announced by Rubio de Francia [38], and given with a detailed proof in [39]. The first version of the extrapolation theorem says that if for some $p_0$, a sublinear operator is bounded in $L^{p_0}$ for all $w \in A_{p_0/\lambda}$ with $1 \leq \lambda < \infty$ and $\lambda \leq p_0 \leq \infty$, then it is bounded in $L^p_w$ for all $p$, $\lambda < p < \infty$, and for all $w \in A_{p/\lambda}$. There exists long list of papers which deals with different proofs of this theorem, generally speaking, in various function spaces and related topics (see e.g., [9,12] and references cited therein).

2 Preliminaries

Let $(X, d, \mu)$ be a quasi-metric measure space with a quasi-metric $d$ and measure $\mu$. A quasi-metric $d$ is a function $d: X \times X \to [0, \infty)$ which satisfies the following conditions:

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) for all $x, y \in X$, $d(x, y) = d(y, x)$;
(iii) there is a positive constant $\kappa$ such that $d(x, y) \leq \kappa(d(x, z) + d(z, y))$ for all $x, y, z \in X$.

In what follows we will assume that the balls $B(x, r) := \{y \in X; d(x, y) < r\}$ are measurable with positive $\mu$ measure for all $x \in X$ and $r > 0$.

If $\mu$ satisfies the doubling condition, i.e., there is a positive constant $D_\mu$ such that for all $x \in X$ and $r > 0$,

$$\mu(B(x, 2r)) \leq D_\mu \mu(B(x, r)), \quad (1)$$

then we say that $(X, d, \mu)$ is a space of homogeneous type (SHT). Throughout the paper we will assume that $(X, d, \mu)$ is an SHT.

For the definition, examples and some properties of an SHT see, e.g., the paper [34] and the monographs [7,41].

Throughout the paper, when we deal with an SHT, we will assume the class of continuous functions is dense in $L^1(X)$.

For a given quasi-metric measure space $(X, d, \mu)$ and $q$ satisfying $1 \leq q \leq \infty$, we will denote by $L^q = L^q(X, \mu)$ the Lebesgue space equipped with the standard norm. Let $f$ be a $\mu-$ measurable function on $X$ and let $1 \leq p < \infty$, $1 \leq s \leq \infty$. Suppose that $w$ is a weight function on $X$, i.e. $w$ is $\mu-$ a.e. positive and locally integrable on $X$. We say that $f$ belongs to the weighted Lorentz space $L^{p,s}_w(X)$ ($L^{p,s}_w$ shortly) if

$$\|f\|_{L^{p,s}_w} = \left\{ \begin{array}{ll} \left( \int_0^\infty (w \{x \in X : |f(x)| > \tau\})^{s/p} \tau^{s-1} d\tau \right)^{1/s}, & \text{if } 1 \leq s < \infty, \\
\sup_{s > 0} \left( w(\{x \in X : |f(x)| > s\})^{1/p} \right), & \text{if } s = \infty \end{array} \right.,$$

is finite, where

$$wE := \int_E w(x)d\mu(x).$$
It is easy to see that $L_{w}^{p,s}(X)$ coincides with the weighted Lebesgue space $L_{w}^{p}(X)$. Denote by $f_{w}^{*}$ a weighted non-increasing rearrangement of $f$ with respect to the measure $d\nu = wd\mu$. Then by integration by parts it can be checked that (see also [22]):

$$
\| f \|_{L_{w}^{p,s}} = \begin{cases}
\left( \frac{s}{p} \int_{0}^{\infty} \left( \frac{1}{p} f_{w}^{*}(t) \right)^{s} \frac{dt}{t} \right)^{1/s}, & \text{if } 1 \leq s < \infty, \\
\sup_{t>0} \left( \frac{1}{p} f_{w}^{*}(t) \right), & \text{if } s = \infty,
\end{cases}
$$

Now we list some useful properties of Lorentz spaces (see e.g., [5,22] (Ch. 6), [27]):

(i) If $1 \leq p < \infty$ and $1 \leq s \leq \infty$, then $L_{w}^{p,s}(X)$ is a Banach space with the norm

$$
\| f \|_{L_{w}^{p,s},w} = \begin{cases}
\left( \int_{0}^{\infty} \left( \frac{1}{p} f_{w}^{**}(t) \right)^{s} \frac{dt}{t} \right)^{1/s}, & \text{if } 1 \leq s < \infty, \\
\sup_{t>0} t^{s} f_{w}^{**}(t), & \text{if } s = \infty,
\end{cases}
$$

which is equivalent to $\| \cdot \|_{L_{w}^{p,s}}$, where

$$
f_{w}^{**}(t) = \int_{0}^{t} f_{w}^{*}(\tau) d\tau;
$$

(ii) $\| \chi_{E} \|_{L_{w}^{p,s}} = (wE)^{1/p}$;

(iii) If $1 \leq p < \infty$, $s_{2} \leq s_{1}$, then $L_{w}^{p,s_{2}} \hookrightarrow L_{w}^{p,s_{1}}$ with the embedding constant $C_{p,s_{1},s_{2}}$ depending only on $p, s_{1}$ and $s_{2}$;

(iv) There is a positive constant $C_{p,s}$ such that

$$
C_{p,s}^{-1} \| f \|_{L_{w}^{p,s}} \leq \sup_{\| h \|_{L_{w}^{p',s'}} \leq 1} \left| \int_{X} f(x)h(x)w(x)d\mu(x) \right| \leq C_{p,s} \| f \|_{L_{w}^{p,s}}
$$

for every $f \in L_{w}^{p,s}$, where $p' = p/(p-1)$, $s' = s/(s-1)$.

(v) (Hölder’s inequality) Let $\frac{1}{p} = \frac{1}{p_{1}} + \frac{1}{p_{2}}$, $\frac{1}{s} = \frac{1}{s_{1}} + \frac{1}{s_{2}}$. Then

$$
\| f_{1} f_{2} \|_{L_{w}^{p,s}} \leq C \| f_{1} \|_{L_{w}^{p_{1},s_{1}}} \| f_{2} \|_{L_{w}^{p_{2},s_{2}}}
$$

for all $f \in L_{w}^{p_{1},s_{1}}$ and $f_{2} \in L_{w}^{p_{2},s_{2}}$, where $C = C_{p,s,s_{1},s_{2}}$;

(vi)

$$
\| f^{1/q_{0}} \|_{L_{w}^{p,s}}^{q_{0}} = \| f \|_{L_{w}^{p,q_{0}s/q_{0}}}
$$

for $f \in L_{w}^{p,q_{0}s/q_{0}}$ with $q_{0}$ satisfying the condition $p/q_{0} > 1, s/q_{0} > 1$. 
Taking property (iv) into account we have the following statement:

**Proposition 1** There is a positive constant \( C_{p,s} \) such that

\[
C_{p,s}^{-1} \| f \|_{L^p_w, s} \leq \sup_{\| w^{-1}h \|_{L^p_w, s'} \leq 1} \left| \int_X f(x) h(x) d\mu(x) \right| \leq C_{p,s} \| f \|_{L^p_w, s},
\]

(2)

with the same constant \( C_{p,s} \) as in (iv). Hence, the Köthe dual space \( (L^p_w, s)' \) of \( L^p_w \)

with respect to the measure space \((X, \mu)\) (not with respect to the measure space \((X, \nu)\),

where \( d\nu = wd\mu \)) is given by the norm equivalent to the quasi-norm \( \| w^{-1} f \|_{L^p_w, s'}(X) \).

**Remark 1** In the sequel constants of the type \( C_{p,s}, \ldots \), depending on parameters \( p, s, \ldots \)
(for example, on parameters of Lorentz spaces) and having the property

\[
\sup_{0 \leq \varepsilon, \eta, \ldots < \sigma_0} C_{p-s, \eta-s, \ldots} \equiv C < \infty;
\]

where \( \sigma_0 \) is a small positive constant, will be denoted simply by \( C \).

For example, the constants from (iii), (iv), (v) and (2) have such a property (see e.g., [22]).

Condition (3) is satisfied, for example, if the mappings \((p, s, \ldots) \mapsto C_{p,s, \ldots}\) are continuous with respect to \( p, s, \ldots \).

Let \( X \) be bounded (i.e., it is contained in some ball) and let \( w \) be a weight on \( X \).
In this case, \( w \) is integrable on \( X \). By the definition the weighted Iwaniec–Sbordone space \( L^{p,\theta}_w(X) \)

is defined with respect to the norm:

\[
\| f \|_{L^{p,\theta}_w(X)} = \sup_{0 < \varepsilon < \theta - p - 1} \varepsilon^{\theta - p - \varepsilon} \| f \|_{L^{p-\varepsilon}_w(X)}, \quad 1 < p < \infty, \quad \theta > 0.
\]

The space \( L^{p,\theta}_w(X) \) for \( w \equiv 1 \) and \( X = \Omega \), where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \),

was introduced in [24] for \( \theta = 1 \) and in [13] for \( \theta > 0 \).

For structural properties of \( L^{p,\theta}_w(X) \) spaces and mapping properties of operators of Harmonic Analysis in these spaces we refer, e.g., to the monograph [31] and references cited therein.

In the paper [35] (see also [31], Ch. 14) it was introduced the generalized weighted grand Lebesgue space on the interval \((0, 1)\) as follows: we say that \( f \in A^{p,\theta}_w \), \( 0 < p < \infty \), if

\[
\| f \|_{A^{p,\theta}_w} = \sup_{0 < \varepsilon < \varepsilon_0} \left( \varepsilon^{\theta} \int_0^1 |f^*(t)|^{p-\varepsilon} w(t) dt \right)^{1/(p-\varepsilon)} < \infty,
\]

where \( f^* \) is the decreasing rearrangement of \( f \) with respect to the Lebesgue measure on \((0, 1)\). In the same paper the boundedness of the Hardy–Littlewood maximal operator
was established in $A_{\infty}^{p,\theta}$ (see also [25], [14] for related topics). Here $g^*$ is decreasing rearrangement of $g$ with respect to Lebesgue measure and $\varepsilon_0$ is defined as follows:

$$\varepsilon_0 = \begin{cases} p - 1, & \text{if } p > 1, \\ p, & \text{if } p \leq 1. \end{cases}$$

Now we introduce weighted grand Lorentz space. For a measurable function $f$ and a weight function $w$ on $X$, we let

$$\|f\|_{L_{w}^{p,s,\theta}} = \sup_{0 < \varepsilon < p^{-1}} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|f\|_{L_{w}^{p-\varepsilon,s}},$$

where $1 < p < \infty$, $1 \leq s \leq \infty$.

Let $1 < p < \infty$. A weight function $w$ defined on $X$ belongs to the Muckenhoupt class $A_p(X)$ if

$$[w]_{A_p(X)} := \sup_B \left( \frac{1}{\mu(B)} \int_B w(x) \, d\mu(x) \right)^{\frac{p-1}{p-1}} \left( \frac{1}{\mu(B)} \int_B w^{1-p'}(x) \, d\mu(x) \right)^{p-1} < \infty,$$

where the supremum is taken over all balls $B \subset X$.

Further, we say that $w \in A_1(X)$ if

$$(Mw)(x) \leq Cw(x), \quad \text{for } \mu\text{-a.e. } x, \quad (4)$$

where $M$ is the Hardy–Littlewood maximal operator defined on $X$, i.e.,

$$Mg(x) = \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |g(y)| \, d\mu(y).$$

We denote by $[w]_{A_1}$ the best possible constant in (4).

The class of weights $A_{\infty}$ is the union of classes $A_p$, $1 \leq p < \infty$. Further (see [21] and [23]),

$$[w]_{A_{\infty}} := \sup_B \left( \frac{1}{\mu(B)} \int_B w \, d\mu \right) \exp \left( \frac{1}{\mu(B)} \int_B \log w^{-1} \, d\mu \right).$$

There exists also another $A_{\infty}$ characteristic due to [15]:

$$[w]_{A_{\infty}}^W := \sup_B \frac{1}{w(B)} \int_B M(w \chi_B) \, d\mu.$$

It can be checked (see also [23]) that
\[ [w]_{A_p}^W \leq C_{\kappa,\mu} [w]_{A_p} \leq \overline{C}_{\kappa,\mu} [w]_{A_p} \]

with some structural constants \( C_{\kappa,\mu} \) and \( \overline{C}_{\kappa,\mu} \).

Let \( 1 < p, q < \infty \). Suppose that \( \rho \) is \( \mu \)-a.e. positive function such that \( \rho^q \) is locally integrable. We say that \( \rho \in A_{p,q}(X) \) if

\[ [\rho]_{A_{p,q}} := \sup_B \left( \frac{1}{\mu(B)} \int_B \rho^q \, d\mu \right)^{\frac{q}{p'}} < \infty, \]

where the supremum is taken over all balls \( B \in X \).

If \( p = q \), then we denote \( A_{p,q} \) by \( A_p \). The next relation can be checked immediately

\[ [\rho]_{A_{p,q}} = [\rho^q]_{A_{1+q/p'}} , \quad 1 < p \leq q < \infty . \]

In particular, this equality for \( p = q \) has the form

\[ [\rho]_{A_p} = [\rho^p]_{A_p} , \quad 1 < p < \infty . \]

Since the Lebesgue differentiation theorem holds in \( (X, d, \mu) \), it can be checked that

\[ [w]_{A_p} \geq 1 ; \quad [\rho]_{A_{p,q}} \geq 1 . \]

Due to Hölder’s inequality the following monotonicity property of \( A_p \) classes holds:

\[ [w]_{A_q} \leq [w]_{A_p} , \quad 1 \leq p \leq q < \infty . \quad (5) \]

It can be also verified that

\[ [w]_{A_p}(X) = \left[ w^{\frac{1}{1-p'}} \right]^{p-1} A_{p'}(X) . \quad (6) \]

Further, let \( 1 < p < \infty \) and \( 1 \leq s \leq \infty \). We say that a weight function \( w \) belongs to the class \( A(p, s) \) if there is a positive constant \( C \) such that

\[ \| \chi_B \|_{L_{w}^{p,s}(X)} \| w^{-1} \chi_B \|_{L_{w^{-q'}}_{s'}(X)} \leq C \mu(B) . \]

The class of weights \( A(p, s) \) was introduced in [6] in Euclidean spaces. In the same paper (see also [16]) it was shown that \( w \in A_{p,s} \) if and only if \( w \in A_p \) provided that \( 1 < s \leq \infty \).

Let us recall that the following Buckley-type estimate holds for the Hardy-Littlewood maximal operator \( M \) defined on an \( SHT \):

\[ \| M \|_{L_w^p(X) \rightarrow L_w^p(X)} \leq \overline{C} p' [w]_{A_p(X)}^{1/(p-1)} , \quad 1 < p < \infty , \quad (7) \]
where \([w]_{A_p(X)}\) is the \(A_p\) characteristic of a weight \(w\) defined on \(X\) (see [23]). For example, if \(X\) is an interval in \(\mathbb{R}\), then we can take \(\overline{c} = 2\). According to [23] the constant \(\overline{c}\) is defined as follows (see also [30])

\[
\overline{c} = 32\kappa^{D_\mu} (2\theta)^{D_\mu} (1 + \tau_{\kappa,\mu}),
\]

where

\[
\tau_{\kappa,\mu} = 6 \left( 32\kappa^4 (4\kappa + 1) \right)^{D_\mu},
\]

\(\theta = 4\kappa^2 + \kappa\), \(D_\mu\) is the constant defined by (1), \(\kappa\) is the triangle inequality constant for the quasi-metric \(d\).

In fact, in [23] the authors established more general bound for \(\|M\|_{L^p_w(X)}\) involving \(A_\infty\) characteristic but for our aims it suffices to apply estimate (7).

In what follows we use standard notation from Banach space theory and operator theory. Let \(L^0(\mu) = L^0(X, \mu)\) be the space of (equivalence classes of) \(\mu\)-measurable real-valued functions. A Banach space \(E\) is said to be a Banach function space (BFS shortly) on \(X\) if the following properties are satisfied:

(i) \(\|f\|_E = 0\) if and only if \(f = 0\) \(\mu\)–a.e.;
(ii) \(|g| \leq |f| \mu - a.e.\) implies that \(\|g\|_X \leq \|f\|_X\);
(iii) if \(0 \leq f_j \uparrow f \mu - a.e.\), the, \(\|f_j\|_E \uparrow \|f\|_E\);
(iv) if \(\chi_F \in L^0(\mu)\) is such that \(\mu(F) < \infty\), then \(\chi_F \in E\);
(v) if \(\chi_F \in L^0(\mu)\) is such that \(\mu(F) < \infty\), then \(\int_F f d\mu \leq C_F \|f\|_E\) for all \(f \in E\) and with some positive constant \(C_F\).

For a BFS \(E\) it is defined Köthe dual (or associated) space \(E'\) consists of all \(f \in L^0(\mu)\)

\[
\|f\|_{E'} = \sup \left\{ \int_X fg d\mu : \|g\|_E \leq 1 \right\} < \infty.
\]

It is known that the space \(E'\) is a Banach function space (see e.g., [2], Theorem 2.2). In Banach function spaces the Hölder inequality holds (see, e.g., [2], Theorem 2.4):

\[
\int_X |fg| d\mu \leq \|f\|_E \|g\|_{E'}.
\]

For a Banach space \(E\) and \(0 < p < \infty\), the \(p\)-convexification of \(E\) is defined as follows:

\[
E^p = \{ f : |f|^p \in E \}.
\]

\(E^p\) can be equipped with the quasi-norm \(\|f\|_{E^p} = \||f|^p\|_{E}^{1/p}\). It can be observed that if \(1 \leq p < \infty\), then \(E^p\) is a Banach space as well. For \(1 \leq p < \infty\) and BFSs \(E\) and \(F\), we have that \(E^{1/p} = F\) if and only if \(E = F^p\).
In [12] it was proved the following quantitative variant of the Rubio de Francia’s [38] extrapolation theorem (see also [29] for related topics):

**Theorem A** (Diagonal case) Let \((X, d, \mu)\) be an SHT. Suppose that for some family \(F\) of pairs of non-negative measurable functions \((f, g)\), for some \(p_0 \in [1, \infty)\) and all \((f, g) \in F\) and \(w \in A_{p_0}(X)\) the inequality

\[
\left( \int_X g^{p_0} w \, d\mu \right)^{\frac{1}{p_0}} \leq C N \left( [w]_{A_{p_0}(X)} \right) \left( \int_X f^{p_0} w \, d\mu \right)^{\frac{1}{p_0}}
\]

holds, where \(N\) is a non-decreasing function and the constant \(C\) does not depend on \((f, g)\) and \(w\). Then for any \(p, 1 < p < \infty, w \in A_p(X)\) and all \((f, g) \in F\) we have

\[
\left( \int_X g^p w \, d\mu \right)^{\frac{1}{p}} \leq C K(w, \|M\|, p, p_0) \left( \int_X f^{p_0} w \, d\mu \right)^{\frac{1}{p_0}},
\]

where the positive constant \(C\) is the same as in (10), and

\[
K(w, \|M\|, p, p_0) = \begin{cases} 
N \left( [w]_{A_p(X)} \left( 2\|M\|_{L^p_w(X) \to L^{p_0}_w(X)} \right)^{p_0 - p} \right), & p < p_0, \\
N \left( [w]_{A_{p_0}(X)} \left( 2\|M\|_{L^{p_0}\left(L^{1/p}_w(X) \to L^{1/p_0}_w\right)} \right)^{\frac{p - p_0}{p_0 - p}} \right), & p > p_0.
\end{cases}
\]

**Remark 2** By (6) and (7) we have that

\[
K(w, \|M\|, p, p_0) \leq K(w, p, p_0),
\]

where

\[
K(w, p, p_0) = \begin{cases} 
N \left( \frac{2c}{p} \left( p_0 - p \right) \left( 2\|M\|_{L^p_w(X) \to L^{p_0}_w(X)} \right)^{p_0 - p} \right), & p < p_0, \\
N \left( \frac{2c}{p} \left( \frac{p - p_0}{p_0} \right) \left[ w \right]_{A_p(X)} \right), & p > p_0.
\end{cases}
\]

**Theorem B** (Off-diagonal case) Let \((X, d, \mu)\) be an SHT. Suppose that for pairs of non-negative measurable functions \((f, g)\) \(\in F\), \(p_0 \in [1, \infty), q_0 \in (0, \infty),\) and all \(w \in A_{p_0,q_0}(X)\) we have

\[
\left( \int_X g^{q_0} w^{q_0} \, d\mu \right)^{\frac{1}{q_0}} \leq C N \left( [w]_{A_{p_0,q_0}(X)} \right) \left( \int_X f^{p_0} w^{p_0} \, d\mu \right)^{\frac{1}{p_0}}.
\]

where \(N\) is non-decreasing function and the constant \(C\) does not depend on \((f, g)\) and \(w\). Then for all \(p, 1 < p < \infty,\) and \(q, 0 < q < \infty,\) such that

\[
\frac{1}{q_0} - \frac{1}{q} = \frac{1}{p_0} - \frac{1}{p},
\]
and all \( w \in A_{p,q}(X) \) the inequality

\[
\left( \int_X g^q w^q \, d\mu \right)^{\frac{1}{q}} \leq C K(w, \|M\|, p, q, p_0, q_0) \left( \int_X f^p w^p \, d\mu \right)^{\frac{1}{p}},
\]

is fulfilled where \( C \) is the same constant as in (11) and

\[
K(w, \|M\|, p, q, p_0, q_0) = \begin{cases} 
N \left( [w]_{A_{p,q}(X)} \left( 2 \|M\|_{L^{\gamma q}_w(X) \rightarrow L^{\gamma q}_w(X)} \right)^{\gamma(q_0-q)}, q < q_0, \\
N \left( [w]_{A_{p,q}(X)} \left( 2 \|M\|_{L^{\gamma p'_w} w^{-p'}(X) \rightarrow L^{\gamma p'_w} w^{-p'}(X)} \right)^{\gamma(q-q_0)}, q > q_0, 
\right) \end{cases}
\]

with

\[
\gamma := \frac{1}{q_0} + \frac{1}{p_0}'.
\]

**Remark 3** From (7) and the equality \([w^q]_{A_{1+q/p'}} = [w]_{A_{p,q}}\) it follows the following estimate:

\[
K(w, \|M\|, p, q, p_0, q_0) \leq K(w, p, q, p_0, q_0),
\]

where

\[
K(w, p, q, p_0, q_0) = \begin{cases} 
N \left( 2\overline{c} \left( 1 + \frac{p'}{q} \right) \right)^{\gamma(q_0-q)} [w^q]_{A_{1+q/p'}(X)}, q < q_0, \\
N \left( 2\overline{c} \left( 1 + \frac{q}{p'} \right) \right)^{\gamma(q_0-q)} [w^q]_{A_{1+q/p'}(X)}, q > q_0, 
\end{cases}
\]

and \( \overline{c} \) is defined by (8).

Taking the estimate

\[
[w]_{A_{p,q}} = [w^q]_{A_{1+q/p'}}
\]

and Remark 3 into account, Theorem B can be reformulated as follows:

**Theorem B’** Let \( 0 < q_0 < \infty \). Assume that for some family \( F \) of pairs of non-negative functions \((f, g)\), for \( p_0 \in [1, \infty)\), and for all \( w \in A_{1+q_0/(p_0)'} \) the inequality

\[
\left( \int_X g^{q_0}(x) w(x) \, d\mu \right)^{\frac{1}{q_0}} \leq C N \left( [w]_{A_{1+q_0/(p_0)'}} \right) \left( \int_X f^{p_0}(x) w^{p_0/q_0}(x) \, d\mu \right)^{\frac{1}{p_0}}
\]

(13)
holds, where \( N \) is a non-decreasing function and the constant \( C \) does not depend on \((f, g)\) and \(w\). Then for all \( 1 < p < \infty, 0 < q < \infty \) such that

\[
\frac{1}{q_0} - \frac{1}{q} = \frac{1}{p_0} - \frac{1}{p},
\]

for all \( w \in A_{1+q/p'} \) and all \((f, g) \in \mathcal{F}(X \times Y)\) we have

\[
\left( \int_X g^q(x)w(x) \, d\mu \right)^{\frac{1}{q}} \leq CK(w, p, q, p_0, q_0) \left( \int_X f^p(x)w^{p/q}(x) \, d\mu \right)^{\frac{1}{p}},
\]

where \( C \) is the same constant as in (13),

\[
K(w, p, q, p_0, q_0) = \begin{cases} 
N \left[ \left( 2c \left( 1 + \frac{p}{q} \right) \right)^{\gamma(q_0-q)} \left[ w \right]_{A_1+\frac{q}{p'}}(X) \right], & q < q_0, \\
N \left[ \left( 2c \left( 1 + \frac{q}{p} \right) \right)^{\frac{q(q_0-q)}{q_0-1}} \left[ w \right]_{A_1+\frac{q}{p'}}(X) \right], & q > q_0,
\end{cases}
\]

with \( c \) and \( \gamma \) defined by (8) and (12), respectively.

Finally we mention that in the sequel under the symbol \( f(t) \approx g(t) \) we mean that there is a positive constants \( c \) independent of \( t \) such that \( \frac{1}{c} f(t) \leq g(t) \leq cf(t) \).

### 3 Extrapolation in Banach function spaces

One of our aims in this paper is to establish weighted extrapolation in Banach function spaces (BFS shortly) defined on an SHT. This will enable us to get quantitative estimates in the case of weighted Lorentz spaces \( L_{w}^{p,s}(X) \) which will be applied to get appropriate results in grand Lorentz spaces and consequently, the boundedness of operators of Harmonic Analysis in these spaces.

We say that a BFS denoted by \( E \) belongs to \( M(X) \) if the maximal operator \( M \) is bounded in \( E \).

For extrapolation results on BFSs we refer to [8,10,20] (see also [9] for related topics). It should be emphasized that in [20] the author studied weighted extrapolation problem in mixed norm spaces.

Before formulating the main results recall that according to Remark 1 we denote by \( C \) constants depending on \( p, s, \ldots, \) and having property (3).

**Theorem 1** (Diagonal case) Let \( \mathcal{F} \) be a family of pairs \((f, g)\) of measurable non-negative functions \( f, g \) defined on \( X \). Suppose that there is a positive constant \( C \) such that for some \( 1 < p_0 < \infty \), for every \( w \in A_{p_0}(X) \) and all \((f, g) \in \mathcal{F}, \) the one-weight inequality holds
\[
\left( \int_X g^{p_0}(x)w(x)\,d\mu(x) \right)^{\frac{1}{p_0}} \leq CN \left( [w]_{A_{p_0}} \right) \left( \int_X f^{p_0}(x)w(x)\,d\mu(x) \right)^{\frac{1}{p_0}}, \tag{16}
\]

where \(N(\cdot)\) is a non-negative and non-decreasing function. Suppose that \(E\) is a BFS and that there exists \(1 < q_0 < \infty\) such that \(E^{1/q_0}\) is again a BFS. If \((E^{1/q_0})' \in \mathcal{M}(X)\), then for any \((f, g) \in \mathcal{F}\) with \(\|g\|_E < \infty\),

\[
\|g\|_E \leq 4CK(\|M\|_{(E^{1/q_0})'}, q_0, p_0)\|f\|_E,
\]

where \(K\) is defined as follows:

\[
K \left( \|M\|_{(E^{1/q_0})'}, q_0, p_0 \right) = \begin{cases} 
N \left( (2c(q_0)')^{p_0-q_0} \|M\|_{(E^{1/q_0})'} \right), & q_0 < p_0, \\
N \left( 2c_{q_0} \frac{q_0-p_0}{q_0-1} \|M\|_{(E^{1/q_0})'} \right), & q_0 > p_0
\end{cases}
\]

and \(C\) is the same as in (16).

**Theorem 2** (Off-diagonal case) Let \(\mathcal{F}\) be a family of pairs \((f, g)\) of measurable non-negative functions \(f, g\) on \(X\). Suppose that for some \(1 \leq p_0, q_0 < \infty\) and for every \(w \in A_{1+q_0/(p_0)'}(X)\) and \((f, g) \in \mathcal{F}\), the one-weight inequality holds

\[
\left( \int_X g^{q_0}(x)w(x)d\mu(x) \right)^{\frac{1}{q_0}} \leq CN \left( [w]_{A_{1+q_0/(p_0)'}}(X) \right) \left( \int_X f^{p_0}(x)w^{p_0/p_0}(x)d\mu(x) \right)^{\frac{1}{p_0}}, \tag{18}
\]

with a positive constant \(C\) independent of \((f, g)\) and \(w\), and with some positive non-decreasing function \(N(\cdot)\). Suppose that \(E\) and \(\overline{E}\) are BFSs such that there exist \(1 < \bar{p}_0 < \infty, 1 < \bar{q}_0 < \infty\) satisfying the conditions

\[
\frac{1}{\bar{p}_0} - \frac{1}{\bar{q}_0} = \frac{1}{p_0} - \frac{1}{q_0}, \tag{19}
\]

\(\overline{E}(X)^{1/\bar{q}_0}, E(X)^{1/\bar{p}_0}\) are BFSs

and

\[
\left( \overline{E}(X)^{1/\bar{q}_0} \right)' = \left( E(X)^{1/\bar{p}_0} \right)' \left[ \overline{E}(X)^{1/\bar{q}_0} \right]. \tag{21}
\]

If \((\overline{E}^{1/\bar{q}_0}(X))' \in \mathcal{M}(X)\), then for any \((f, g) \in \mathcal{F}\), we have

\[
\|g\|_{\overline{E}} \leq 4C \left( \overline{K}(\|M\|, \bar{p}_0, \bar{q}_0, p_0, q_0) \right)^{\bar{q}_0} \|f\|_E,
\]

where the constant \(C\) is the same as in (18).
with \( \gamma \) defined by (12).

**Proof of Theorem 1** We use the arguments from the proof of Theorem 3.2 in [20]. Take \( q_0 \) so that the conditions of the theorem are satisfied. By using Theorem A together with Remark 2 we have that for any \( w \in A_1(X) \),

\[
\left( \int_X g^{q_0}(x)w(x)d\mu(x) \right)^{\frac{1}{q_0}} \leq C K(w, q_0, p_0) \left( \int_X f^{q_0}(x)w(x)d\mu(x) \right)^{\frac{1}{q_0}},
\]

where the constant \( C \) is the same as in (16) and

\[
K(w, q_0, p_0) = \begin{cases} 
N \left( \left( 2\bar{c}(q_0) \right)^{p_0-q_0} [w]^{(q_0-1)(p_0-q_0)}_{A_1(X)} \right), & q_0 < p_0, \\
N \left( 2\bar{c}q_0 \right)^{\frac{q_0-p_0}{q_0-1}} [w]^{q_0-p_0}_{A_1(X)}, & q_0 > p_0.
\end{cases}
\]

Let now \( F = E^{1/q_0} \). Then following to the Rubio de Francia’s algorithm ( [38]), for any non-negative measurable functions \( h \), we define

\[
\mathcal{R}h(x) = \sum_{k=0}^{\infty} \frac{M^kh(x)}{2^k\|M\|_{F'}} \quad x \in X,
\]

where \( M \) is the Hardy-Littlewood maximal operators defined on \( X \); \( M^k \) is \( k \)-th iteration of \( M \) with \( M^0h = h \). It is easy to check that

\[
h(x) \leq \mathcal{R}h(x); \quad \|\mathcal{R}h\|_{F'} \leq 2\|h\|_{F'}; \quad [\mathcal{R}h]_{A_1(X)} \leq \|M\|_{F'}. \tag{22}
\]

Further, from the definition of the Köthe dual space, there exists a non-negative \( \mu \)-measurable function \( h \in F'(X) \) with \( \|h\|_{F'(X)} \leq 1 \) such that

\[
\|g\|_{E}^{q_0} = \|g^{q_0}\|_{F} \leq 2 \int_X |g(x)|^{q_0}h(x)d\mu(x).
\]

Further, by the first inequality of (22) we have that

\[
\int_X |g(x)|^{q_0}h(x)d\mu(x) \leq \int_X |g(x)|^{q_0}(\mathcal{R}h)(x)d\mu(x).
\]
To apply Theorem A we show that
\[
\int_X |g(x)|^{q_0} (Rh)(x) \, d\mu(x) < \infty.
\]

This is true because the first and second inequalities of (22) with Hölder’s inequality yield that
\[
\int_X (g(x))^{q_0} (Rh)(x) \, d\mu(x) \leq \|g^{q_0}\|_F \|Rh\|_{F'} \leq 2 \|g\|_E^{q_0} \|h\|_{F'} \leq 2 \|g\|_E^{q_0} < \infty.
\]

Further, by the third inequality of (22) we have that \( Rh \in A_1(X) \). Consequently,
\[
\|g\|_E^{q_0} \leq 2 \int_X g^{q_0} h \, d\mu \leq 2 \int_X g^{q_0} (Rh) \, d\mu \leq 2CK^{q_0} (Rh, q_0, p_0) \int_X f^{q_0} (Rh) \, d\mu
\]
\[
\leq 2CK^{q_0} (Rh, q_0, p_0) \|f\|_E^{q_0} \|h\|_{F'} \leq 4CK^{q_0} (Rh, q_0, p_0) \|f\|_E^{q_0} \|h\|_{F'},
\]

where
\[
K(Rh, q_0, p_0) = \begin{cases} 
N \left( (2\sigma(q_0)')^{p_0-q_0} [Rh]_{A_1(X)}^{(q_0)-1}(p_0-q_0) \right), & q_0 < p_0, \\
N \left( 2\sigma(q_0)^{q_0-p_0} [Rh]_{A_1(X)} \right), & q_0 > p_0.
\end{cases}
\]

Thus, applying the third estimate of (22) we find that
\[
\|g\|_E \leq 4CK \left( \|M\|_{(E^{1/q_0})'}, q_0, p_0 \right) \|f\|_E
\]

with \( K(\|M\|_{(E^{1/q_0})'}, q_0, p_0) \) defined by (17).

This complete the proof of the theorem.

\[ \square \]

**Proof of Theorem 2.** Choose \( \tilde{p}_0, \tilde{q}_0 \) so that \( p_0 \leq \tilde{p}_0 < \infty, q_0 \leq \tilde{q}_0 < \infty \), and conditions (19), (20) and (21) are satisfied.

Applying Theorem B’ we have that for any \( w \in A_1 \),
\[
\left( \int_X g^{q_0}(x)w(x) \, d\mu(x) \right)^{\frac{1}{q_0}} \leq CK(w, \tilde{p}_0, \tilde{q}_0, p_0, q_0) \left( \int_X f^{p_0}(x)w^{\frac{p_0}{q_0}}(x) \, d\mu(x) \right)^{\frac{1}{p_0}}
\]

\[ \text{Birkhäuser} \]
holds, where \( N \) is a non-decreasing function and the constant \( C \) is the same as in (18), and

\[
K(w, \tilde{\varrho}_0, \tilde{\varrho}_0, p_0, q_0) = \begin{cases} 
N \left[ 2^C \left( 1 + \frac{\tilde{\varrho}_0}{\varrho_0} \right) \right] \varphi(q_0 - \tilde{\varrho}_0), & \tilde{\varrho}_0 < q_0 \\
N \left[ 2^C \left( 1 + \frac{\tilde{\varrho}_0}{\varrho_0} \right) \right] \varphi(q_0 - \tilde{\varrho}_0), & \tilde{\varrho}_0 > q_0.
\end{cases}
\]

Let now \( \tilde{F} = E^{1/\tilde{\varrho}_0} \) and \( F = E^{1/\varrho_0} \). Then following again to the Rubio de Francia’s algorithm, for any non-negative measurable function \( h \), we introduce

\[
\mathcal{R}h(x) = \sum_{k=0}^{\infty} \frac{M^k h(x)}{2^k \| M \|_{\tilde{F}'}}, \quad x \in X,
\]

where, as before, \( M \) is the Hardy-Littlewood maximal operators defined on \( X \). Further, it can be checked that

\[
h(x) \leq \mathcal{R}h(x); \quad \| \mathcal{R}h \|_{\tilde{F}'} \leq 2 \| h \|_{\tilde{F}'}; \quad [\mathcal{R}h]_{A_1(X)} \leq \| M \|_{\tilde{F}'}.
\]

Let us take now non-negative \( \mu - \) measurable function \( h \in \tilde{F}'(X) \) with \( \| h \|_{\tilde{F}'(X)} \leq 1 \) such that

\[
\| g \|_{E}^{\tilde{\varrho}_0} = \| g \|_{\tilde{F}^{\tilde{\varrho}_0}} = \| g \|_{\tilde{F}'}^{\tilde{\varrho}_0} \leq 2 \int_X g^{\tilde{\varrho}_0}(y) h(y) d\mu(y) \leq 2 \int_X g^{\tilde{\varrho}_0}(y) (\mathcal{R}h)(y) d\mu(y).
\]

The latter estimate follows from the first inequality in (23). Further, observe that Hölder’s inequality and the second estimate of (23) yield that

\[
\int_X (g(x))^{\tilde{\varrho}_0} (\mathcal{R}h)(x) d\mu(x) \leq 2 \| g \|_{\tilde{F}'}^{\tilde{\varrho}_0} \| \mathcal{R}h \|_{\tilde{F}'} \leq 4 \| g \|_{\tilde{F}'}^{\tilde{\varrho}_0} = 4 \| g \|_{E}^{\tilde{\varrho}_0} < \infty.
\]

By using the fact that \( \mathcal{R}h \in A_1(X) \), Hölder’s inequality, Theorem B’ and the third inequality of (23) we find that

\[
\| g \|_{E}^{\tilde{\varrho}_0} \leq 2 \int_X (g(x))^{\tilde{\varrho}_0} \mathcal{R}h(x) d\mu(x)
\]

\[
\leq 2C (K(\mathcal{R}h, \tilde{\varrho}_0, \tilde{\varrho}_0, p_0, q_0))^{\tilde{\varrho}_0} \left( \int_X (f(x))^{\tilde{\varrho}_0} (\mathcal{R}h(x))^{\tilde{\varrho}_0} d\mu(x) \right)^{\tilde{\varrho}_0/p_0}
\]

\[
\leq 2C (K(\mathcal{R}h, \tilde{\varrho}_0, \tilde{\varrho}_0, p_0, q_0))^{\tilde{\varrho}_0} \| f \|_{\tilde{F}'}^{\tilde{\varrho}_0} \| \mathcal{R}h \|_{\tilde{F}'}^{\tilde{\varrho}_0/p_0}
\]

\[
= 2C (K(\mathcal{R}h, \tilde{\varrho}_0, \tilde{\varrho}_0, p_0, q_0))^{\tilde{\varrho}_0} \| f \|_{E}^{\tilde{\varrho}_0} \| \mathcal{R}h \|_{\tilde{F}'}
\]
\[
\leq 4C \left( K(Rh, \tilde{p}_0, \tilde{q}_0, p_0, q_0) \right)^{\frac{q_0}{\tilde{q}_0}} \| f \|_{E} \| h \|_{F'}
\]

where \( K(Rh, \tilde{p}_0, \tilde{q}_0, p_0, q_0) \) is given by

\[
K(Rh, \tilde{p}_0, \tilde{q}_0, p_0, q_0) = \begin{cases} \\
N \left[ \left( 2^r \left( 1 + \frac{R_h \cdot (p_0)'}{q_0} \right) \right)^{\frac{\gamma(q_0 - \tilde{q}_0)}{\| R_h \|_{A_1}}} \right], & \tilde{q}_0 < q_0, \\
N \left[ \left( 2^r \left( 1 + \frac{q_0}{(p_0)'} \right) \right)^{\frac{\gamma(q_0 - q_0)}{\| R_h \|_{A_1}}} \right], & \tilde{q}_0 > q_0.
\end{cases}
\]

Further, by virtue of the third inequality of (23) we have that

\[
K(Rh, \tilde{p}_0, \tilde{q}_0, p_0, q_0) \leq \tilde{K}(\| M \|, \tilde{p}_0, \tilde{q}_0, p_0, q_0).
\]

Finally we get the desired result. \( \square \)

### 4 Weighted extrapolation in Lorentz spaces

In this section we prove weighted extrapolation results for weighted Lorentz spaces. Initially let us recall the following result regarding the boundedness of \( M \) in weighted Lorentz spaces (see [6] for \( \mathbb{R}^n \) and [16] for an \( SHT \):

**Theorem C** Let \( 1 < p, s < \infty \). Then \( M \) is bounded in \( L_w^{p,s}(X) \) if and only if \( w \in A_p(X) \).

We need to calculate the quantitative upper bound of the norm of maximal operator in weighted Lorentz spaces.

**Proposition 2** Let \( 1 < p, s < \infty \) and let \( w \in A_p(X) \). Then the following estimate holds:

\[
\| Mf \|_{L_w^{p,s}} \leq C 2^{1/p(\varepsilon_0)^{-1}} \left[ p[w]_{A_{p-\varepsilon_0}} + (p - \varepsilon)[w]_{A_{p+\varepsilon_0}} \right],
\]

where \( C \) is a structural constant and

\[
\varepsilon_0 = \frac{p - 1}{1 + \tau_{\kappa, \mu}[w]_{A_p}}, \quad (24)
\]

with \( \tau_{\kappa, \mu} \) defined in (8).

**Proof** Let \( w \in A_p \). Then \( w \in A_{p-\varepsilon_0} \) with \( \varepsilon_0 \) defined by (24) (see, e.g. [23]). By monotonicity property of \( A_p \) classes we have that \( w \in A_{p+\varepsilon_0} \). Hence by (7):

\[
\| M \|_{L_w^{p-\varepsilon_0}(X) \rightarrow L_w^{p+\varepsilon_0, \infty}(X)} \leq \tilde{c}(p - \varepsilon_0)'[w]_{A_{p-\varepsilon_0}(X)}
\]
and
\[
\|M\|_{L_w^{p+\varepsilon_0}(X)\to L_w^{p+\varepsilon_0,\infty}(X)} \leq C[w]_{A_{p+\varepsilon_0}}(X),
\]
where \(\overline{c}\) is defined by (8).

Consequently, by virtue of the Marcinkiewicz interpolation theorem in Lorentz spaces (see [40, Ch. V]) we find that
\[
\|M\|_{L_w^{p,s}(X)\to L_w^{p,s}(X)} \leq \frac{2^{1/p}M_0^{-1}}{\varepsilon_0} \left[ p\|M\|_{L_w^{p-\varepsilon_0}(X)\to L_w^{p-\varepsilon_0,\infty}(X)} + (p-\varepsilon_0)\|M\|_{L_w^{p+\varepsilon_0}(X)\to L_w^{p+\varepsilon_0,\infty}(X)} \right].
\]
This implies that
\[
\|M\|_{L_w^{p,s}(X)\to L_w^{p,s}(X)} \leq C^{2/p}\varepsilon_0^{-1} \left[ p[w]_{A_{p+\varepsilon_0}} + (p-\varepsilon_0)[w]_{A_{p-\varepsilon_0}} \right].
\]

The next statement will be useful for us:

**Proposition 3** Let \(1 < p, s < \infty\) and let \(w \in A_p(X)\). Then the following estimate holds:
\[
\|w^{-1}Mf\|_{L_w^{p',s'}} \leq C^{2/p'}(\varepsilon_0)^{-1} \left[ p'[w]_{A_{p-\varepsilon_0}} + (p-\varepsilon_0)'[w]_{A_{p+\varepsilon_0}} \right] \|w^{-1}f\|_{L_w^{p',s'}},
\]
where \(\varepsilon_0\) is defined by (24).

**Proof** Let \(w \in A_p\). Then, \(w \in A_{p-\varepsilon_0}, w \in A_{p+\varepsilon_0}\), where \(\varepsilon_0\) is defined by (24). Hence,
\[
w^{1-(p-\varepsilon_0)'} \in A_{(p-\varepsilon_0)'}; \quad w^{1-(p+\varepsilon_0)'} \in A_{(p+\varepsilon_0)'}.
\]

Consequently, by (7),
\[
\|M\|_{L_w^{(p-\varepsilon_0)'}(X)\to L_w^{(p-\varepsilon_0)'}(X)} \leq \overline{c}(p-\varepsilon_0) \left[ w^{1-(p-\varepsilon_0)'} \right]^{1/((p-\varepsilon_0)'-1)}_{A_{(p-\varepsilon_0)'}(X)}
\]
and
\[
\|M\|_{L_w^{(p+\varepsilon_0)'}(X)\to L_w^{(p+\varepsilon_0)'}(X)} \leq \overline{c}(p+\varepsilon_0) \left[ w^{1-(p+\varepsilon_0)'} \right]^{1/((p+\varepsilon_0)'-1)}_{A_{(p+\varepsilon_0)'}(X)}.
\]

We can rewrite these estimates as follows:
\[
\|w^{-1}Mf\|_{L_w^{(p-\varepsilon_0)'}(X)} \leq C_1(w, p, \varepsilon_0) \|w^{-1}f\|_{L_w^{(p-\varepsilon_0)'}(X)}
\]
and
\[ \| w^{-1} Mf \|_{L_w^{(p+x_0)'}(X)} \leq C_2(w, p, \varepsilon_0) \| w^{-1} f \|_{L_w^{(p+x_0)'}(X)}, \]
where
\[ C_1(w, p, \varepsilon_0) = \bar{c}(p - \varepsilon_0) \left[ w^{1-(p-\varepsilon_0)'} \right]^{1/((p-\varepsilon_0)'-1)}_{A_{(p-\varepsilon_0)'}(X)} \]
and
\[ C_2(w, p, \varepsilon_0) = \bar{c}(p + \varepsilon_0) \left[ w^{1-(p+\varepsilon_0)'} \right]^{1/((p+\varepsilon_0)'-1)}_{A_{(p+\varepsilon_0)'}(X)} \]
with the constant \( \bar{c} \) defined by (8).

By using Marcinkiewicz interpolation theorem for Lorentz spaces (see [40], Ch. V) with respect to sublinear operator
\[ Tf = w^{-1} Mf \]
we get
\[ \| w^{-1} Mf \|_{L_w^{p,r}'}(X) \leq C(w, p, \varepsilon) \| w^{-1} f \|_{L_w^{p,r}'}(X), \]
where \( 1 < r < \infty \) and
\[ C(w, p, \varepsilon) = C_2^{1/p'} \varepsilon_0^{-1} \left[ p'C_1(w, p, \varepsilon) + (p - \varepsilon_0)'C_2(w, p, \varepsilon) \right]. \]
Here we used the fact that
\[ \frac{1}{p'} = \frac{1-t}{(p-\varepsilon)'} + \frac{t}{(p+\varepsilon)'}, \quad 0 < t < 1. \]
Taking \( r = s' \) we get the desired result. \( \square \)

**Theorem 3** (Diagonal case) Let \( \mathcal{F} \) be a family of pairs \((f, g)\) of measurable non-negative functions \( f, g \) defined on \( X \). Suppose that for some \( 1 \leq p_0 < \infty \), for every \( w \in A_{p_0}(X) \) and all \((f, g) \in \mathcal{F}\), the one-weight inequality
\[ \left( \int_X g^{p_0}(x) w(x) \, d\mu(x) \right)^{1/p_0} \leq CN \left( [w]_{A_{p_0}} \right) \left( \int_X f^{p_0}(x) w(x) \, d\mu(x) \right)^{1/p_0} \]  
(25)
holds with a positive non-decreasing function \( N(\cdot) \) and some positive constant \( C \) which does not depend on \((f, g)\) and \( w \). Then for any \( 1 < p, s < \infty \), for all \((f, g) \in \mathcal{F}\) and any \( w \in A_{p}(X)\),
\[ \| g \|_{L^p,w} \leq 4CK_1 (\| M \|_{L^{(p/q_0)'}, q_0, s}) \| f \|_{L^p,w}, \]

where the constant \( C \) is the same as in (25) and

\[ K_1 (\| M \|_{L^{(p/q_0)'}, (s/q_0)'}, p, s) = \begin{cases} \quad N \left( 2\tilde{C}(q_0) \right)_{p_0 - q_0} M \left( \left( \frac{q_0}{q_0 - q_0} \right)^{\frac{1}{p_0 - q_0}} \right), & q_0 < p_0, \\ \quad N \left( 2\tilde{C}(q_0) \right)_{q_0 - q} M \left( \left( \frac{q_0}{q_0 - q_0} \right)^{\frac{1}{q_0 - p}} \right), & q_0 > p_0, \end{cases} \]

with non-decreasing \( N \) and \( q_0 \in (1, p) \), and

\[ \tilde{L}^{(p/q_0)'}, (s/q_0)' = \left\{ f : X \mapsto \mathbb{R} : \left\| \frac{1}{w} f \right\|_{L^{(p/q_0)'}, (s/q_0)'} < \infty \right\}. \]

**Proof** Let 1 < \( p < \infty \) and let \( w \in A_p(X) \). Then \( w \in A_{p - \varepsilon_0}(X) \) with \( \varepsilon_0 \) equal to the expression given by (24). Take \( q_0 \) so that \( p - \varepsilon_0 < p/q_0 \). Then by monotonicity property of Muckenhoupt classes, \( w \in A_{p/q_0} \). Due to Proposition 3 we find that

\[ \| w^{-1} Mf \|_{L^{(p/q_0)', (s/q_0)'}} \]

\[ \leq C2^{1/(p/q_0)}(\varepsilon_0)^{-1} \left[ \left( \frac{p}{q_0} \right)_{q_0 - q_0} \left[ w \right]_{A_{p/q_0}} + \left( \frac{p}{q_0} - \varepsilon \right)_{q_0 - q_0} \left[ w \right]_{A_{p/q_0 + \varepsilon_0}} \right] \| w^{-1} f \|_{L^{(p/q_0)', (s/q_0)'}}. \]

Since \( E = L^{p,s}_w \), the result follows from Proposition 1 and Theorem 1. \( \square \)

**Remark 4** Taking the proof of Theorem 3 into account we find that

\[ \| M \|_{\tilde{L}^{(p/q_0)', (s/q_0)'} \leq C \left( p, q_0, \varepsilon_0, \left[ w \right]_{A_{p/q_0 - \varepsilon_0}}, \left[ w \right]_{A_{p/q_0 + \varepsilon_0}} \right), \]

where

\[ \sup_{0 < \varepsilon < \delta_0} C \left( p - \varepsilon, q_0, \varepsilon_0, \left[ w \right]_{A_{p - \varepsilon/q_0 - \varepsilon_0}}, \left[ w \right]_{A_{p - \varepsilon/q_0 + \varepsilon_0}} \right) < \infty \]

for some small positive number \( \delta_0 \).

**Theorem 4** (Off-diagonal case) Let \( \mathcal{F} \) be a family of pairs \( (f, g) \) of measurable non-negative functions \( f, g \) on \( X \). Suppose that for some \( 1 < p_0, q_0 < \infty \) and for every \( w \in A_{1+q_0/(p_0)}(X) \) and \( (f, g) \in \mathcal{F} \), the one-weight inequality

\[ \left( \int_X g^{q_0}(x) w(x) d\mu(x) \right)^{\frac{1}{q_0}} \leq \frac{1}{q_0} \left( \int_X f^{p_0}(x) w^{p_0/q_0}(x) d\mu(x) \right)^{\frac{1}{p_0}} \]

holds with a positive constant \( C \) independent of \((f, g)\) and \( w \), and some non-decreasing positive function \( N(\cdot) \). Suppose that \( 1 < p, q, r, s < \infty \) are chosen so that

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Then for all $w \in A_{1+p/q'}$ and all $(f, g) \in \mathcal{F}$ we get

$$\|g\|_{L^q_w, r}(X) \leq K(\|M\|, p, q, r, s)\|w^{1 - \frac{1}{p}} f\|_{L^{p,s}_w(X)},$$

where

$$K(\|M\|, p, q, r, s) = C \begin{cases} N \left[ (2\varepsilon \left( 1 + \frac{p_0}{q_0} \right) )^{\gamma(q_0 - \tilde{q}_0)} \|M\|^{1 + \frac{\gamma p_0 (q_0 - \tilde{q}_0)}{(p_0)'} } (L_{q'}^w)^{1/q_0} \right], & \tilde{q}_0 < q_0, \\
N \left[ (2\varepsilon \left( 1 + \frac{\tilde{q}_0}{p_0} \right) )^{\gamma(q_0 - \tilde{q}_0)} \|M\|^{1 + \frac{\gamma p_0 (q_0 - \tilde{q}_0)}{(p_0)'} } (L_{q'}^w)^{1/q_0} \right], & \tilde{q}_0 > q_0. \end{cases}$$

with $\gamma$ defined by (12) and $\tilde{q}_0$ is defined so that

$$1 < \tilde{q}_0 < \frac{qp'}{p' + q - \varepsilon_0 p'}$$

(26)

with $\varepsilon_0$ defined by (24).

**Proof** Let $1 < p, r, q, s < \infty$ are chosen so that the conditions of the theorem are fulfilled. Suppose that $w \in A_{1+p/q'}(X)$. Then the openness property of Muckenhoupt classes yields that $w \in A_{1+p/q'-\varepsilon_0}(X)$, where $\varepsilon_0$ is defined by (24) but replaces $p$ by $1 + q/p'$.

Choose $\tilde{p}_0$ and $\tilde{q}_0$ so that

$$\frac{1}{p_0} - \frac{1}{q_0} = \frac{1}{\tilde{p}_0} - \frac{1}{\tilde{q}_0}$$

and that (26) holds. In this case, $w \in A_{q/\tilde{q}_0}$ and $\tilde{q}_0 < q$. Hence, by Proposition 3 we find that

$$\|w^{-1} Mf\|_{L_w^{(q/\tilde{q}_0)'},(r/\tilde{q}_0)'} \leq C(q, r, q_0, [w]_{A_{q/\tilde{q}_0-r_0}}, [w]_{A_{q/\tilde{q}_0-r_0}})\|w^{-1} f\|_{L_w^{(q/\tilde{q}_0)'},(r/\tilde{q}_0)'},$$

with

$$\sup_{0<\varepsilon<\varepsilon_0} C(q - \varepsilon, r, q_0, [w]_{A_{q-\varepsilon}/\tilde{q}_0-r_0}, [w]_{A_{q-\varepsilon}/\tilde{q}_0+r_0}) < \infty.$$
Observe now that
\[
\tilde{p}_0 \left( \frac{p}{\tilde{q}_0} \right)' = \left( \frac{q}{\tilde{q}_0} \right)' ; \quad \tilde{q}_0 \left( \frac{s}{\tilde{p}_0} \right)' = \left( \frac{r}{\tilde{q}_0} \right)'
\]
which, on the other hand, implies that
\[
\left\| w^{\frac{1}{q} - \frac{1}{p}} \right\|_{L^{p,r,\tilde{q}_0,\tilde{p}_0}_w} = \left\| w^{-1} f \right\|_{L^{q,r,\tilde{q}_0,\tilde{p}_0}_w},
\]
where
\[
\tilde{L}^{p,r,\tilde{q}_0,\tilde{p}_0}_w = \left[ \left( (L^{p,r}_w)^{1/\tilde{p}_0} \right)^{\tilde{p}_0/\tilde{q}_0} \right] ; \quad L^{q,r,\tilde{q}_0,\tilde{p}_0}_w = \left[ \left( (L^{q,r}_w)^{1/\tilde{q}_0} \right)^{\tilde{q}_0/\tilde{p}_0} \right].
\]

Due to Proposition 1 and Theorem 2 the result follows. \qed

**Remark 5** The proof of Theorem 4 yields that
\[
\left\| M \right\|_{L^{q,r,\tilde{p}_0,\tilde{q}_0}_w} \leq C (q, q_0, \tilde{q}_0, \varepsilon_0, w),
\]
where
\[
\sup_{0 < \varepsilon < \delta_0} C (q - \varepsilon, q_0, \tilde{q}_0, \varepsilon_0, w) < \infty
\]
for some small positive number \(\delta_0\).

## 5 Extrapolation in grand Lorentz spaces

Applying statements proven in Sect. 4 we have the following results regarding grand Lorentz spaces:

**Theorem 5** (Diagonal case) Let \( w \) be integrable weight on \( X \) and let \( \mathcal{F} \) be a family of pairs \( (f, g) \) of measurable non-negative functions \( f, g \) defined on \( X \). Suppose that for some \( 1 \leq p_0 < \infty \), for every \( w \in A_{p_0}(X) \) and all \( (f, g) \in \mathcal{F} \), the one-weight inequality holds
\[
\left( \int_X g^{p_0}(x) w(x) d\mu(x) \right)^{1/p_0} \leq CN \left( [w]_{A_{p_0}} \right) \left( \int_X f^{p_0}(x) w(x) d\mu(x) \right)^{1/p_0}
\]
with some positive constant \( C \) which does not depend on \( (f, g) \) and \( w \), and positive non-decreasing function \( N(\cdot) \). Then for any \( 1 < p < \infty \), \( 1 \leq s < \infty \), \( \theta > 0 \), \( w \in A_{p}(X) \) and for all \( (f, g) \in \mathcal{F} \),
\[ \|g\|_{L^p_w} \leq C \|f\|_{L^p_w}, \]

with the positive constant \( C \) independent of \((f, g)\).

**Proof** Let \( w \in A_p \). By Hölder’s inequality and the fact that \( w \) is integrable on \( X \) it is enough to prove that

\[ \sup_{0 < \varepsilon < \sigma_0} \varepsilon^{p-q} \|g\|_{L^{p-q}w} \leq C \sup_{0 < \varepsilon < \sigma_0} \varepsilon^{p-q} \|f\|_{L^{p-q}w} \]

for all \((f, g) \in \mathcal{F}\) and for some positive constant \( \sigma_0 \).

Observe that Theorem 3 and Remark 4 yield that

\[ \sup_{0 < \varepsilon < \sigma_0} \varepsilon^{p-q} \|g\|_{L^{p-q}w} \leq C(w, p, s, \varepsilon) \varepsilon^{p-q} \|f\|_{L^{p-q}w} \]

for all \((f, g) \in \mathcal{F}\) and all \( w \in A_{p-\varepsilon} \) with \( 0 < \varepsilon < \sigma_0 \), where

\[ \sup_{0 < \varepsilon < \sigma_0} C(w, p, s, \varepsilon) < \infty. \]

\( \square \)

**Theorem 6** (Off-diagonal case) Let \( w \) be an integrable weight on \( X \). Let \( \mathcal{F} \) be a family of pairs \((f, g)\) of measurable non-negative functions \( f, g \) on \( X \). Suppose that for some \( 1 < p_0 \leq q_0 < \infty \) and for every \( w \in \text{A}_{1+q_0/(p_0)'}(X) \) and \((f, g) \in \mathcal{F}\), the one-weight inequality holds

\[ \left( \int_X g^{q_0}(x)w(x)\,d\mu(x) \right)^{\frac{1}{q_0}} \leq CN \left( |w|_{\text{A}_{1+q_0/(p_0)'}(X)} \right) \left( \int_X f^{p_0}(x)w^{p_0/q_0}(x)\,d\mu(x) \right)^{\frac{1}{p_0}} \]

with a positive constant \( C \) independent of \((f, g)\) and \( w \), and some non-decreasing positive function \( N(\cdot) \). Suppose that \( 1 < p, q, r, s < \infty \) are chosen so that

\[ \frac{1}{p} - \frac{1}{q} = \frac{1}{p_0} - \frac{1}{q_0}; \quad \frac{1}{s} - \frac{1}{r} = \frac{1}{p_0} - \frac{1}{q_0}. \]

Then for all \( w \in \text{A}_{1+q/p'} \) and all \((f, g) \in \mathcal{F}\), we have

\[ \|g\|_{L^q_w} \leq C \|f\|_{L^{p_0}_w}, \]

where the positive constant \( C \) is independent of \((f, g)\).

**Proof** Since \( X \) is bounded, by Hölder’s inequality we have that

\[ \|g\|_{L^{q_0}_w} \leq C \sup_{0 < \varepsilon < \sigma_0} \varepsilon^{p-q} \|g\|_{L^{p-q}_w}. \]
Let us set:

\[ \Psi(x) := \Phi(x^\theta), \quad \Phi(x) := \left[ \frac{x - q}{1 - A(x - q)} + p \right]^{1-(x-q)A} \]

with a number \( A \) defined by

\[ A := \frac{1}{p_0} - \frac{1}{q_0} = \frac{1}{p} - \frac{1}{q} = \frac{1}{s} - \frac{1}{r}. \]

It is easy to check that

\[ \Psi(x) \approx x^{q\theta/p}, \quad x \to 0. \]

Hence, it suffices to show that

\[ \sup_{0 < \varepsilon < \varepsilon_0} \Psi(\varepsilon) \frac{1}{1-\varepsilon} \|g\|_{L^{q-r,s,r}} \leq C \sup_{0 < \eta < \eta_0} \eta^{\frac{q}{p-\eta}} \|f\|_{L^{p-\eta,s}} \]

for all \((f, g) \in \mathcal{F}\) and for some positive constant \( \varepsilon_0 \), where \( \varepsilon_0 \in (0, q - 1) \).

Here \( \eta \) and \( \varepsilon \) satisfy the condition:

\[ \frac{1}{p-\eta} - \frac{1}{q-\varepsilon} = A, \]

and \( \eta_0 \) is chosen so that if \( \varepsilon \in (0, \varepsilon_0) \), then \( \eta \in (0, \eta_0) \).

Theorem 4 and Remark 5 yield

\[ \sup_{0 < \varepsilon < \varepsilon_0} \Psi(\varepsilon) \frac{1}{1-\varepsilon} \|g\|_{L^{q-r,s,r}(X)} \leq C \sup_{0 < \eta < \eta_0} C(w, p-\eta, s, q-\varepsilon, r)\eta^{\frac{q}{p-\eta}} \|f\|_{L^{p-\eta,s}(X)} \]

\[ \leq C\|f\|_{L^{p,s}(X)}. \]

Observe that here \( 0 < \varepsilon < \varepsilon_0 \) if and only if \( 0 < \eta < \eta_0 \). Finally, since \( \Phi(\varepsilon) \approx \varepsilon^{q\theta/p} \)

we have the desired result. \( \square \)

### 6 Applications of extrapolation results in grand Lorentz spaces

Based on extrapolation results we get the boundedness of integral operators of Harmonic Analysis in grand Lorentz spaces. In this section we will assume that \( X \) is bounded. We denote by \( \mathcal{D}(X) \) the class of bounded functions on \( X \).

#### 6.1 Maximal, fractional and singular integral operators

Let \( K \) be the Calderón-Zygmund operator defined on an SHT, i.e., \( K \) satisfies the following conditions (see, e.g., \([1,7]\)):
(i) $K$ is linear and bounded in $L^p(X)$ for every $p \in (1, \infty)$;
(ii) there is a measurable function $k : X \times X \mapsto \mathbb{R}$ such that for every $f \in D(X)$,
\[ Kf(x) = \int_X k(x, y) f(y) d\mu(y), \]
for a.e. $x \not\in \text{supp } f$, where $D(X)$ is the class of bounded functions with compact supports defined on $X$.
(iii) the kernels $k$ and $k^*$ (here $k^*(x, y) := k(y, x)$) satisfy the following pointwise Hörmander’s condition: there are positive constants $C$, $\beta$ and $A > 1$ such that
\[ |k(x_0, y) - k(x, y)| \leq C \frac{d(x_0, x)\beta}{\mu(B(x_0, 2d(x_0, y)))d(x_0, y)\beta} \]
holds for every $x_0 \in X, r > 0, x \in B(x_0, r), y \in X \setminus B(x_0, Ay)$;
(iv) there is a positive constant $C$ such that for all $x, y \in X$,
\[ |k(x, y)| \leq \frac{C}{\mu(B(x, 2d(x, y)))}. \]

The operator $K$ (see, e.g., [13,37] and references therein) is bounded in $L^{p_0}_w(X)$ for $1 < p_0 < \infty$ and $w \in A_{p_0}(X)$. Moreover, the following estimate holds:\n\[ \|Kf\|_{L^{p_0}_w(X)} \leq C_0([w]_{A_{p_0}(X)}) \|f\|_{L^{p_0}_w(X)}, \quad f \in D(X), \]
where $C_0([w]_{A_{p_0}(X)})$ is a constant depending on $[w]_{A_{p_0}(X)}$ such that the mapping $x \mapsto C_0(x)$ is non-decreasing.

In the next statement by the symbol $I_\alpha$ will be denoted the fractional integral operator defined by
\[ I_\alpha f(x) = \int_X K_\alpha(x, y) f(y) d\mu(y), \quad x \in X, \]
where
\[ K_\alpha(x, y) = \begin{cases} \mu(B_{xy})^{\alpha-1}, & x \neq y, \\ \mu\{x\}, & x = y, \mu\{x\} > 0, \end{cases} \]
$0 < \alpha < 1, B_{xy} := B(x, d(x, y))$.

It is known that (see [26] and [16]) the following inequality holds:
\[ \|I_\alpha(fw^\alpha)\|_{L^q_w(X)} \leq C \|f\|_{L^{p,s}_w(X)}, \quad f \in L^{p,s}_w(X), \]
where $1 < p < \frac{1}{\alpha}, q = \frac{p}{1-\alpha p}, 1 < s < \infty$ and $w \in A_{1+q/p'}$. 

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Together with $I_\alpha$ we are interested in the related fractional maximal operator

$$M_\alpha f(x) = \sup_{B \ni x} \frac{1}{\mu(B)^{1-\alpha}} \int_B |f(y)|d\mu(y), \quad 0 < \alpha < 1.$$ 

The following pointwise estimate holds for $f \geq 0$

$$M_\alpha f(x) \leq C_\alpha I_\alpha f(x), \quad (27)$$

where $C_\alpha$ is a positive constant independent of $f$ and $x$.

To prove the statements of this subsection we need the following lemma:

**Lemma 1** Let $1 < p, s < \infty$ and let $\theta > 0$. Then there is a positive constant $C$ such that for all balls $B$ and all $f \in L^p_w(B)$,

$$\|f\|_{L^p,\theta}(B) \leq C w(B)^{-1/p} \|f\|_{L^p_w(B)} \|\chi_B\|_{L^p,\theta}(B).$$

**Proof** By using properties (ii) and (v) of the Lorentz spaces (see Sect. 2) with respect to the exponents:

$$\frac{1}{p - \varepsilon} = \frac{1}{p} + \frac{\varepsilon}{p(p - \varepsilon)}; \quad \frac{1}{s} = \frac{1}{s_1} + \frac{1}{s_2},$$

where $\varepsilon \in (0, p - 1], p < s_1$, we have

$$\|f\|_{L^p,\theta}(B) = \sup_{0 < \varepsilon \leq p - 1} \varepsilon^{\theta/p} \|f\|_{L^{p-\varepsilon,\theta}(B)}$$

$$\leq C \sup_{0 < \varepsilon \leq p - 1} \varepsilon^{\theta/p} \|f\|_{L^p_w(B)} \|\chi_B\|_{L^p,\theta}(B)^{1/\theta}$$

$$\leq C \|f\|_{L^p_w(B)} \sup_{0 < \varepsilon < p - 1} \varepsilon^{\theta/p} w(B)^{p(\theta + 1)/\theta}$$

$$\leq C \|f\|_{L^p_w(B)}^{-1/p} \|\chi_B\|_{L^p,\theta}(B).$$

\[\Box\]

**Theorem 7** Let $w$ be an integrable weight on $X$ and let $1 < p, s < \infty$. Suppose that $\theta > 0$. Then $M$ is bounded in $L^{p,\theta}(B)$ if and only if $w \in A_p$.

**Proof** Sufficiency follows from Theorem 5. That is why we show only Necessity. Suppose that $M$ is bounded in $L^{p,\theta}(B)$. Take a ball $B$ and non-negative $f \in L^p_w(B)$. By Lemma 1 we have that

$$\|\chi_B f\|_{L^p,\theta}(B) \leq C w(B)^{-1/p} \|\chi_B f\|_{L^p_w(B)} \|\chi_B\|_{L^p,\theta}(B).$$
with a positive constant $C$ independent of $B$ and $f$. Since the pointwise inequality
\[ \frac{1}{\mu(B)} \int_B |f(y)| d\mu \leq M(f \chi_B)(x) \]
holds for $x \in B$, then we have that
\[ \| Mf \|_{L^p,w(B)^s,\theta} \geq \frac{1}{\mu(B)} \int_B \| \chi_B \|_{L^p,w(B)^s,\theta} \left( \frac{1}{\mu(B)} \int_B |f(y)| d\mu(y) \right) \]
\[ = \| \chi_B \|_{L^p,w(B)^s,\theta} \left( \frac{1}{\mu(B)} \int_B |f(y)| d\mu(y) \right). \]
Consequently, taking the boundedness of $M$ into account we find that
\[ \left( \frac{1}{\mu(B)} \int_B |f(y)| d\mu(y) \right) \| \chi_B \|_{L^p,w(I)^s,\theta} \leq C w(B)^{-1/p} \| f \|_{L^p,w(I)} \| \chi_B \|_{L^p,w(I)}. \]
Now choosing $f = \chi_B w^{1-p'}$ we conclude that $w \in A_p(X)$. \qed

**Theorem 8** Let $w$ be an integrable weight on $X$ and let $1 < p < \infty$. Suppose that $\theta > 0$ and $w \in A_p$. Then there is a positive constant $C$ such that for all $f \in D(X)$, the inequality
\[ \| Kf \|_{L^p,w(I)^s,\theta} \leq C \| f \|_{L^p,w(I)^s,\theta} \]
holds. Further, if $K = H$ is the Hilbert transform on $I := (0, 1)$:
\[ Hf(x) = (p,v) \int_0^1 \frac{f(t)}{x-t} dt, \]
then from the boundedness of $H$ in $L^p,w(I)^s,\theta$ it follows that $w \in A_p(I)$.

**Proof** The first part (sufficiency) of the statement follows immediately from Theorem 5; that is why we prove the second part of the theorem (necessity).

We follow [28]. First we show that there is a positive constant $C$ such that for all intervals $J, J' \subset I$, the following inequality holds:
\[ \| \chi_J \|_{L^p,w(I)^s,\theta} \leq C \| \chi_J' \|_{L^p,w(I)^s,\theta}, \quad (28) \]
where \( J := (a, b) \) with \( b - a \leq 1/4 \), and

\[
J' = \begin{cases} 
(b, 2b - a) & \text{if } (b, 2b - a) \subset I, \\
(2a - b, a) & \text{if } (2a - b, a) \subset I \text{ and } (b, 2b - a) \cap I^c \neq \emptyset.
\end{cases}
\]

Indeed, without loss of generality suppose that \( J' = (b, 2b - a) \). Then for \( f = \chi_J' \),

\[
\|Hf\|_{L^{p,\theta}(J')} \geq \frac{1}{2}\|\chi_J\|_{L^{p,\theta}(I)}.
\]

On the other hand, observe that

\[
\|f\|_{L^{p,\theta}(J')} = \|\chi_J\|_{L^{p,\theta}(I)}.
\]

Consequently, due to the boundedness of \( H \) we have (28).

Arguing now as in the proof of Theorem 7 for intervals \( J \) and \( J' \) and by using Lemma 1 we get the condition \( w \in A^p(I) \).

\[\square\]

**Theorem 9** Suppose that \( 0 < \alpha < 1 \) and let \( 0 < p, s < 1/\alpha \). Let \( w \) be an integrable weight on \( X \), and let \( \theta > 0 \). We set \( q = \frac{p}{1-\alpha p}, r = \frac{s}{1-\alpha s} \). Then the following statements are equivalent:

(i) There is a positive constant \( C \) such that for all \( f \in L^{p,s,\theta}_w \),

\[
\|I_{\alpha}(w^{\alpha} f)\|_{L^{q,r,q\theta/p}_w} \leq C \|f\|_{L^{p,s,\theta}_w};
\]

(ii) There is a positive constant \( C \) such that for all \( f \in L^{p,s,\theta}_w \),

\[
\|M_{\alpha}(w^{\alpha} f)\|_{L^{q,r,q\theta/p}_w} \leq C \|f\|_{L^{p,s,\theta}_w};
\]

(iii) \( w \in A^{1+q/p'} \).

**Proof** First we will show that (iii) \( \Rightarrow \) (i). Let \( w \in A^{1+q/p'} \). Since

\[
\frac{1}{p} - \frac{1}{q} = \frac{1}{s} - \frac{1}{r} = \alpha,
\]

due to Theorem 6 we get

\[
\|I_{\alpha} f\|_{L^{q,r,q\theta/p}_w(X)} \leq C \|w^{\frac{1}{q} - \frac{1}{p}} f\|_{L^{p,s,\theta}_w(X)} = C \|w^{-\alpha} f\|_{L^{p,s,\theta}_w(X)}
\]

provided that the right-hand side norm is finite.
The latter inequality is equivalent to

$$\|I_\alpha (w^\alpha f)\|_{L^q_w,r,q^0/p(X)} \leq C \|f\|_{L^p_w}.$$  

Since (i) $\Rightarrow$ (ii) by the pointwise inequality (27), it suffices to show that (ii) $\Rightarrow$ (iii).

We follow the arguments of the proof of Theorem 3.1 from [36].

Observe that (ii) is equivalent to the inequality

$$\|M_\alpha (f w^\alpha)\|_{L^q_w,r,\psi(x)} \leq c \|f\|_{L^p_w}.$$  

(29)

where

$$\|g\|_{L^q_w,r,\psi(x)} := \sup_{0<\varepsilon<q-1} \psi(\varepsilon) \frac{1}{q-\varepsilon} \|g\|_{L^q_w,r}(X),$$

$$\psi(t) := \varphi(t^0), \quad \varphi(t) := \left[ \frac{t-q}{1-\alpha(t-q)} + p \right]^{1-(t-q)\alpha}.$$

This follows from the fact that $\varphi(t) \approx t^{q/p}$ as $t \to 0$.

Let (ii) (i.e., equivalently (29)) holds. Let us take a ball $B \subset X$ and $f = \chi_B w^{-\alpha-p'/q}$. Then for $x \in B$, we get that

$$M_\alpha (w^\alpha f)(x) \geq \frac{1}{\mu(B)^{1-\alpha}} \int_B w^\alpha f d\mu = \frac{1}{\mu(B)^{1-\alpha}} \int_B w^{-p'/q} d\mu.$$

Hence,

$$\|M_\alpha (w^\alpha f)\|_{L^q_w,r,\psi(x)} \geq \mu(B)^{\alpha-1} \left( \int_X w^{-p'/q} d\mu \right) \|f\|_{L^q_w,r,\psi(x)}.$$

Further, by Lemma 1 we find that

$$\mu(B)^{\alpha-1} \left( \int_X w^{-p'/q} d\mu \right) \|f\|_{L^q_w,r,\psi(x)} \leq c \|f\|_{L^p_w} \leq c \|w(B)\| \left( \int_B |f(y)|^p w(y) d\mu(y) \right)^{1/p} \|\chi_B\|_{L^p_w} \leq c \|w(B)\| \left( \int_B w^{-p'/q} d\mu \right)^{1/p} \|\chi_B\|_{L^p_w},$$

$$= c w(B)^{-\frac{1}{p}} \left( \int_B w^{-p'/q} \right)^{1/p} \|\chi_B\|_{L^p_w}.$$
It is easy to see that there is a number \( \eta_B \) depending on \( B \) such that \( 0 < \eta_B \leq p - 1 \) and

\[
\mu(B)^{\alpha-1} w(B)^{\frac{1}{p}} \left( \int_B w^{-p'/q} \, d\mu \right)^{\frac{1}{p'}} \| \chi_B \|_{L^q_{w, \psi(x)}}(X) \leq c \left( \eta_B^\theta w(B)^{1/p-\eta_B} \right)^{\frac{1}{p-\eta_B}}.
\]

For such an \( \eta_B \) we choose \( \varepsilon_B \) so that

\[
\frac{1}{p-\eta_B} - \frac{1}{q-\varepsilon_B} = \alpha.
\]

Then \( 0 < \varepsilon_B < q - 1 \) and

\[
\mu(B)^{\alpha-1} w(B)^{\frac{1}{p}} - \frac{1}{p-\eta_B} \psi(B)^{\frac{1}{q}} w(B)^{\frac{1}{q-\varepsilon_B}} \left( \int_B w^{-p'/q} \, d\mu \right)^{\frac{1}{p'}} \leq C.
\]

Observe that since \( \psi(t) \approx t^{\theta(1+\alpha q)} \) for small positive \( t \), we have that

\[
\eta_B^{\theta} \psi(B)^{\frac{1}{q-\varepsilon_B}} \eta_B^{\frac{1}{q-\varepsilon_B}} \approx \eta_B^{\theta} \psi(B)^{\frac{1}{q-\varepsilon_B}} \eta_B^{\frac{1}{q-\varepsilon_B}} \approx \eta_B^{\theta} \psi(B)^{\frac{1}{q-\varepsilon_B}} \eta_B^{\frac{1}{q-\varepsilon_B}} \approx 1
\]

and also,

\[
\frac{1}{p} - \frac{1}{p-\eta_B} + \frac{1}{q-\varepsilon_B} = \frac{1}{p} - \alpha = \frac{1}{q}.
\]

Finally, we have that

\[
\mu(B)^{\alpha-1} w(B)^{\frac{1}{q}} \left( \int_B w^{-p'/q} \right)^{1/p'} \leq C.
\]

The theorem has been proved. \( \square \)

### 6.2 Commutators

We say that a function \( b \) defined on \( X \) belongs to \( BMO \) if
\[ \|b\|_{BMO} = \sup_B \frac{1}{\mu(B)} \int_B |b(x) - b_B| d\mu(x) < \infty, \]

where \( b_B = \frac{1}{\mu(B)} \int_B b(y) d\mu(y) \).

Let \( b \in BMO(X) \), \( m \in \mathbb{N} \cup \{0\} \) and let

\[ K_m^b f(x) = \int_X [b(x) - b(y)]^m k(x, y) f(y) d\mu(y), \]

where \( k \) is the Calderón-Zygmund kernel.

It is known (see [37]) that if \( 1 < r < \infty \) and \( w \in A_{\infty} \), then the one-weight inequality

\[ \| K_m^b f \|_{L^r_w(X)} \leq C \| b \|_{BMO(X)} \| M^{m+1} f \|_{L^r_w(X)}, \quad f \in \mathcal{D}(X), \]

holds, where \( M^{m+1} \) is the Hardy–Littlewood maximal operator iterated \( m + 1 \) times.

Based on extrapolation result in grand Lebesgue spaces we have

**Theorem 10** Let \( X \) be bounded and let \( 1 < p, s < \infty, \theta > 0 \). Then there is a positive constant \( C \) such that for all \( f \in \mathcal{D}(X) \) and all \( w \in A_p(X) \),

\[ \| K_m^b f \|_{L^{p,w}_{\theta}(X)} \leq C \| M^{m+1} f \|_{L^{p,w}_{\theta}(X)}, \quad f \in \mathcal{D}(X). \]

Further, for \( b \in BMO(X) \), let

\[ I_{\alpha,b}^m f(x) = \int_X [b(x) - b(y)]^m K_{\alpha}(x, y) d\mu(y), \quad 0 < \alpha < 1, \]

\[ T_{\alpha,b}^m f(x) = \int_X |b(x) - b(y)|^m K_{\alpha}(x, y) d\mu(y), \quad 0 < \alpha < 1. \]

It is easy to see that, for \( f \geq 0 \), \( |I_{\alpha,b}^m f(x)| \leq T_{\alpha,b}^m f(x) \). In the same paper [3] the authors showed that if \( 1 < p < \infty, 0 < \alpha < 1, m \in \mathbb{N} \cup \{0\}, w \in A_{\infty}(X), b \in BMO(X) \), then there is a constant \( C \equiv C_{\alpha,m,p,k,\mu} \) such that

\[ \int_X |T_{\alpha,b}^m f(x)|^p w(x) d\mu(x) \leq CN([w]_{A_{\infty}}) \| b \|_{BMO(X)}^{mp} \int_X [M_{\alpha}(M^m f)(x)]^p w(x) d\mu(x) \]

for some non-decreasing function \( N \).

Based on this result and appropriate extrapolation theorem we have the following statement:
Theorem 11 Let \(1 < p, s < \infty, m \in \mathbb{N} \cup \{0\}\) and let \(\theta > 0\). Suppose that \(X\) is bounded and that \(w \in A_p(X)\). Then there is a positive constant \(C\) such that
\[
\|I_{m, b}^{\alpha} f\|_{L^{p(s, \theta)}_w (X)} \leq C\|M_\alpha (M^m f)\|_{L^{p(s, \theta)}_w (X)}, \quad f \in \mathcal{D}(X).
\]

Corollary 1 Under the conditions of Theorem 11 we have that there is a positive constant \(C\) such that for all \(f \in \mathcal{D}(X),\)
\[
\|I_{m, b}^{\alpha} f\|_{L^{p(s, \theta)}_w (X)} \leq C\|f\|_{L^{p(s, \theta)}_w (X)}.
\]

7 Further remarks

In this section we do some remarks regarding the results obtained in this paper.

Remark 6 If we define grand Lorentz spaces with respect to the quasi-norm
\[
\|f\|_{L^{p(s, \theta)}_w} = \sup_{0 < \varepsilon < \sigma} \varepsilon^{\frac{\theta}{p}} \|f\|_{L^{p-\varepsilon, s}_w},
\]
where \(\sigma\) is a small positive constant, then the sufficiency part of Theorems 5–11 remain true even for unbounded \(X\).

Remark 7 Let \(\varphi\) be a positive increasing function on \((0, p - 1]\) such that \(\lim_{x \to 0} \varphi(x) = 0\). Let us define the grand Lorentz space with respect to the quasi-norm:
\[
\|f\|_{L^{p(s, \varphi)}_w} = \sup_{0 < \varepsilon < \sigma} \varphi(\varepsilon)^{\frac{1}{p-\varepsilon}} \|f\|_{L^{p-\varepsilon, s}_w}.
\]
Then again the results of this paper remains valid for such spaces.

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