Phase Object Tomography Reconstruction Based on high-order Transport of Intensity Equation

Hong Cheng¹, Xiaolong Zhang¹, Tianxiang Wang¹ and Xiaotian Zhu¹

¹ School of Electronic Information Engineering, Anhui University, Hefei, 230601, China

*Corresponding author’s e-mail: chenghong@ahu.edu.cn

Abstract. The phase of the object contains the depth, shape, refractive index and other information of the object surface, which is more important than the intensity. As a classical phase retrieval algorithm, the Transport of Intensity Equation (TIE) can directly obtain the phase calculated from the known intensity. In this paper, a new algorithm is proposed to reconstruct the three-dimensional phase information of an object by tomography. The algorithm obtains high-precision phase input through the high-order TIE, and then reconstructs the three-dimensional phase information of the object by using Fourier slice theorem backprojection tomography. The experimental results show that the algorithm can solve the problem of low phase accuracy caused by intensity differential approximation constraints, and can obtain high-precision 3D phase reconstruction results of objects.

1. Introduction

Different from the traditional two-dimensional imaging, three-dimensional imaging carries more information obtained from the real world, so it has more extensive applications [1]. The phase includes the depth, shape, refractive index and other information of the object surface, which is more important than the intensity, but the phase information cannot be directly measured by optical instruments [2]. As a classical phase retrieval algorithm, the Transport of Intensity Equation (TIE) has the characteristics of non-interference and non-iteration, and is widely used in three-dimensional reconstruction, microscopic imaging and other fields [3-5].

Optical projection tomography (OPT) is a high-resolution three-dimensional reconstruction technology, which is widely used in medical and biomedical fields [6-8]. Fourier slice tomography is a classical OPT algorithm [9]. Memarzadeh et al, proposed a non-interference tomography reconstruction algorithm for phase objects [10]. This method uses the traditional TIE to calculate the projection phase of each angle, and then uses the multiplication technique to reconstruct the three-dimensional shape of the phase object. In this paper, the finite difference of intensity is used to approximate the intensity differential when TIE is used to solve the phase. Usually, this approximate solution of phase is not accurate enough. At the same time, multiplication technique can only reconstruct some simple objects with smooth edges, and its robustness is poor. All these factors will affect the accuracy of the final 3D reconstruction.

In this paper, a phase object tomography reconstruction algorithm based on high-order TIE Equation is proposed. First, high-order TIE is used to solve the high-precision phase, and then Fourier slice theorem is used to reconstruct the final three-dimensional phase of the object. The algorithm can not only solve the problem of low phase accuracy caused by intensity differential approximation constraints, but also obtain high-precision 3D phase reconstruction results of objects.
2. Transport of Intensity Equation

TIE shows the mathematical relationship between the phase of the light field and the axial intensity differential component. The required phase distribution can be reconstructed from the multi-plane intensity by TIE. The TIE formula in the traditional monochromatic coherent light field is as follows:

$$-\nabla (I(x, y, z_0)\nabla \phi(x, y, z_0)) = k \left. \frac{\partial I(x, y, z)}{\partial z} \right|_{z=z_0}$$  \hspace{1cm} (1)

Where $\lambda$ is the wavelength, $k$ is the wavenumber, $k = 2\pi / \lambda$, $z_0$ is the propagation distance, and $\partial I(x, y, z = z_0) / \partial z$ is the axial intensity differential, that is, the axial variation of intensity, which cannot be directly obtained by measurement. It can be approximately estimated by the finite difference of the two defocused images.

$$\left. \frac{\partial I(x, y, z)}{\partial z} \right|_{z=z_0} \approx \frac{I(x, y, z_0 + \Delta z) - I(x, y, z_0 - \Delta z)}{2\Delta z}$$  \hspace{1cm} (2)

Bring formula (2) into formula (1), we can obtain:

$$\phi(x, y, z_0) = -k \mathcal{F}^{-1} \left[ \frac{2\pi^2 (f_x^2 + f_y^2)}{2\Delta z} \mathcal{F} \left[ \frac{\partial I(x, y, z = z_0)}{\partial z} \right] \right]$$  \hspace{1cm} (3)

Where $\mathcal{F}$ and $\mathcal{F}^{-1}$ represent Fourier transform and inverse Fourier transform.

The only unknown phase value $\phi(x, y, z_0)$ can be obtained by formula (3). However, the finite difference results obtained by using the two intensity images have large errors, which will affect the subsequent phase retrieval results.

3. Tomography Reconstruction Algorithm Based on high-order TIE

It is mentioned in the introduction that the accuracy of object phase retrieval will directly affect the accuracy of object phase tomographic reconstruction. In view of this influence factor, in this paper, a phase object tomography reconstruction algorithm based on high-order TIE is proposed, the flow of the algorithm is shown in Fig. 1. The algorithm first uses intensity images with different defocus distances to solve the projection phase of the required angle through high-order TIE, and then brings the projection phase of different angles into Fourier slice tomography algorithm to reconstruct the final phase three-dimensional result.
3.1. Phase Retrieval by high-order TIE
In order to break through the limitation of finite difference, the higher-order term in Taylor expansion of axial differential can be approximately estimated by increasing light intensity measurement [11-12]. An estimate of $\hat{I}_0$ can be obtained by suitable linear combinations of strength measurements $I(x, y, z_j)$ on different planes, i.e.:

$$\frac{\partial \hat{I}_0}{\partial z} = \frac{1}{N\Delta z} \sum_{j=-N}^{N} a_j I(x, y, z_j) \tag{4}$$

$$z_j = z_0 \pm j\Delta z, \ j = 1, 2, \ldots, N$$

Where $j$ represents serial numbers of different plane intensities, $N$ represents the order of TIE, and $a_j$ represents the weight coefficient, and $\frac{\partial \hat{I}_0}{\partial z}$ must meet the following conditions:

$$\lim_{\Delta z \to 0} \left( \frac{\partial \hat{I}_0}{\partial z} \right) = \lim_{\Delta z \to 0} \frac{\partial^2 I(x, y, z = z_0)}{\partial z^2}, \varepsilon^2 = \left( \left. \frac{\partial \hat{I}_0}{\partial z} \right|_{\partial z} \right)^2 = \min$$

$$\left\{ \right\}$$

Where $\left\{ \right\}$ represents the average value and $\varepsilon^2$ takes the minimum value. Expand $\frac{\partial \hat{I}_0}{\partial z}$ by Taylor Theorem, by making

$$\sum_{j=-N}^{N} a_j = 0, \ \sum_{j=-N}^{N} a_j = 1, \ \sum_{j=-N}^{N} a_j^2 = \min$$

Satisfying equation (6) and using Legendre multipliers can be easily found the coefficients $a_j$

$$a_j = \frac{3j}{N(N+1)(2N+1)}$$

The weight of each plane is calculated from Equation (7) and then substituted into Equation (4) to solve the axial strength differential. Considering the compromise between the complexity of the subsequent experiment and the phase retrieval accuracy, in the subsequent experiment, we use 5 intensity diagrams to approximately estimate the intensity differential (i.e. N=2).

3.2. Fourier slice reconstruction
The projection phase results with higher accuracy for different angles can be obtained through section 3.1, and then the final three-dimensional results can be reconstructed by using Fourier slice theorem[13]. Fourier slice theorem is as follows:

$$G(\omega, \theta) = \int_{-\infty}^{\infty} g(\rho, \theta) e^{-j2\pi\rho \omega} d\rho = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi(x\cos \theta + y\sin \theta)} dx dy$$

$$\tag{8}$$

Where $\rho = x\cos \theta + y\sin \theta$. Let $u = \omega \cos \theta, \nu = \omega \sin \theta$, and $F(u, v) = \mathcal{F}_{2D} \{ f(x, y) \}$, and have

$$R \{ f(x, y) \} = g(\rho, \theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x\cos \theta + y\sin \theta - \rho) dx dy$$

$$\tag{9}$$

Equation (9) is the Radon transformation of $f(x, y)$[17]. Then it can be seen from (8)-(9) that

$$f(x, y) = \mathcal{F}_{2D}^{-1} \{ G(\omega, \theta) \} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \omega |G(\omega, \theta)e^{j2\pi\rho} d\omega d\rho$$

$$\tag{10}$$

Where $\mathcal{F}_{2D}^{-1}$ is a two-dimensional inverse Fourier transform and $|\omega|$ is a one-dimensional filtering function.

4. Experiment
According to the algorithm proposed in this paper, the phase retrieval results of the algorithm are
tested first. We simulate a phase-only three-dimensional object as shown in Fig. 2.

![Fig. 2 Initial three-dimensional object](image)

The original phase in the 0 degree direction of the object is shown in Fig. 3. The size of the image is 256pixel * 256pixel, the wavelength is $\lambda = 532\text{nm}$, the defocus distance is $\Delta z = 50\text{um}$, and the propagation distance is $z = 0.2\text{mm}$. In Fig. 4, (a)-(e) are five intensity diagrams with distances obtained by angular spectrum propagation. Fig. 5 (a) is a phase result solved by a conventional TIE using the three intensity images (b), (c) and (d) in Fig. 4, and Fig. 5 (b) is a phase result solved by a higher-order TIE using the five intensity images (a)-(e) in Fig. 4.

![Fig. 3 Initial phase result](image)

![Fig. 4 Intensity diagrams with different defocusing distances](image)

![Fig. 5 (a) Traditional TIE phase retrieval results (b) High-order TIE phase retrieval results](image)
In order to show the accuracy of the retrieval phase results more intuitively, the normalized gray values of the vertical center section line of the traditional TIE retrieval phase, the high-order TIE retrieval phase and the original phase are selected and compared. As shown in Fig. 6, where that blue curve represent the original phase, the yellow curve represents the high-order TIE retrieval phase, and the red represents the vertical centerline gray value of the conventional TIE retrieval phase. As can be seen from Fig. 6, that improve order TIE method can effectively ensure the accuracy of the retrieval phase and has higher accuracy than the traditional TIE retrieval result.

![Fig.6 Gray value comparison of vertical center section line](image)

In order to quantitatively express the accuracy of the recovered phase, the following error calculation formula is defined:

\[
E = \sqrt{\frac{\sum (\phi - \varphi_0)}{M \times N}}
\]

Here \(\phi\) and \(\varphi_0\) respectively represent the recovered phase and the original phase, \(M\) and \(N\) respectively represent the number of rows and columns of the image size. Fig 5 (a) and Fig 5 (B) are calculated with errors of 0.3409 and 0.3287, respectively, from the original phase. It can also be seen from the error value that compared with the traditional TIE retrieval results, the high-order TIE retrieval results have higher accuracy.

Then the 3D reconstruction results of the algorithm are tested. The experiment of solving phase projection in the above paper is set as the phase solving process of high-order TIE in the 0-degree direction, and the process is extended to 60 different directions. The object of Fig. 2 is rotated 360 degrees to obtain projections in different directions, with 5 projections in each direction. Fig. 7(A) shows the partial projection phases in 60 different directions calculated by high-order TIE, with an interval of 6 degrees between different directions.
Fig. 7 Part of the projection phase in different directions and reconstruction slice results

The projection phase is brought into Fourier slice theorem to reconstruct 256 slices of objects. Fig 7 (B) shows partially reconstructed slice images, and then the 256 slices are superimposed together to obtain the final reconstructed three-dimensional phase object, as shown in Fig 8 (A). Fig. 8 (B) is a three-dimensional result reconstructed by multiplication technique (MT). It can be clearly seen that there is a large error between the reconstructed object and the real object by multiplication technique.

(a) Reconstruction results of the algorithm in this paper  
(b) Reconstruction results of MT

Fig. 8 Comparison of 3D reconstruction results

In order to compare the accuracy of the reconstruction results more intuitively, we respectively compared the reconstructed images of the 120th slice of objects in three different situations, as shown in Fig. 9. The slice image reconstructed by multiplication technique (MT) is shown in the Fig (a), the slice image reconstructed by the method in this paper is shown in the Fig (b), and the simulated original slice image is shown in the Fig (a). Obviously, (b) has better reconstruction accuracy and effect than (a). In order to further quantitatively analyze the accuracy of reconstruction results, the correlation coefficients of (a), (b) in Fig. 9 and the original slice (c) are compared, and the correlation coefficients are expressed as:

$$R = \frac{\sum_{m,n} (A(m,n) - \bar{A})(B(m,n) - \bar{B})}{\sqrt{\sum_{m,n} (A(m,n) - \bar{A})^2 \sum_{m,n} (B(m,n) - \bar{B})^2}}$$  \hspace{1cm} (12)

Where $A(m,n)$ and $B(m,n)$ are images of size $m \times n$, $\bar{A}$ and $\bar{B}$ are the mean values of $A(m,n)$ and $B(m,n)$, respectively, the correlation coefficients of (a) and (c) in Fig. 9 are 0.6138, and the correlation coefficients of (b) and (c) in Fig. 9 are 0.9280. From the quantitative experimental results, it can be seen that the method proposed in this paper improves the accuracy of reconstructed slices and finally improves the accuracy of 3D reconstruction results.
5. Conclusion
In this paper, a phase object tomography reconstruction algorithm based on high-order TIE is proposed. The algorithm approximates the high-order Taylor expansion in intensity differential by adding light intensity measurement, thus improving the accuracy of Phase retrieval. At the same time, the phase object is reconstructed by Fourier slice tomography. The experimental results show that the algorithm can solve the problem of low phase accuracy caused by approximate estimation of intensity differential by finite difference, and can obtain high-precision three-dimensional phase reconstruction results of the object.

Acknowledgments
This paper is supported and sponsored by the National Natural Science Foundation of China (Nos.61501001,61300021), Natural Science Foundation of Anhui Province (No. 2008085MF209), Natural Science Project of Anhui Higher Education Institutions of China (No. KJ2019ZD04, KJ2020ZD02).

References
[1] Chen, N., Zuo, C., Byoungho, L. (2019)3D imaging based on depth measurement.J. Infrared and Laser Engineering, 48(6): 603013.
[2] Cheng, H.,lv,Q, Q., Zhang, W., J., et al.(2017)Phase Retrieval Method Based on Liquid Crystal on Silicon Tunable-Lens.J. Chinese Journal of Laser, 44(3): 0304001.
[3] Cheng, H.,Xiong, B.L., et al.(2019)Phase Retrieval Based on Registration Progressive Compensation Algorithm.J.Acta Photonica Sinica, 48(4):410002-0410002.
[4] Cheng, H., Gao, Y.L., et al.(2018)Non-interference Phase Retrieval Algorithm with Two Wavelength Illumination.J. Acta Photonica Sinica, 47(4): 407002-0407002.
[5] Lu, X.Y., Zhao, C.L., Cai, Y.J.(2020)Research Progress on Methods and Applications for Phase Reconstruction Under Partially Coherent Illumination.J.Chinese Journal of Laser, 47(5): 0500016.
[6] Du, W.H., Fei, C., Liu, J.L., et al. (2020)Optical Projection Tomography Using a Commercial Microfluidic System.J. Micromachines, 11(3): 293.
[7] Li, Y.C., Liu, A., Li, G.Y., et al. (2018)Optical Projection Tomography Technique and Its Research
Progress. J. Chinese Journal of Laser, 45(3): 0307012

[8] Funamizu, H., Aizu, Y. (2018) Three-dimensional quantitative phase imaging of blood coagulation structures by optical projection tomography in flow cytometry using digital holographic microscopy. J. Journal of biomedical optics, 24(3): 1-6.

[9] Hawraa, H.A., Hameed, M.A. (2020) 3D image reconstruction from its 2D projection - a simulation study. J. Progress in industrial Ecology, 14(1).

[10] Memarzadeh, S., Nehmetallah, G.T., Banerjee, P.P. (2014) Noninterferometric tomographic reconstruction of 3D static and dynamic amplitude and phase objects. In: Sensing Technologies and Applications. Ohio. pp. 91170M-91170M-9.

[11] Zheng, S.L., Xue, B.D., Xue, W.F., et al. (2012) Transport of intensity phase imaging from multiple noisy intensities measured in unequally-spaced planes. J. Optics Express, 20 (2): 972-985.

[12] Zhong, J.S., Claus, R.A., Dauwels, J, et al. (2014) Transport of intensity phase imaging by intensity spectrum fitting of exponentially spaced defocus planes. J. Optics Express, 22(9): 10661-10674.

[13] Rafael, C.G, Richard, E.W. (2010) Digital Image Processing. Electronic Industry Press BeiJing. pp. 233-240.