On 4D, $\mathcal{N} = 1$ Massless Gauge Superfields of Higher Superspin: Half-Odd-Integer Case

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ABSTRACT

We present an alternative method to explore the off-shell component structure of theories that describe half-integer super-helicities $Y = s + 1/2$ (where $s$ is any positive integer). We use it to derive the component action, component SUSY transformation laws, and count the component-level degrees of freedom involved. This counting will give us clues about $\mathcal{N} = 2$ representations. The foundation of the process relies on the superfield equations of motion, generated by variation of a superspace action expressed in terms of prepotentials. Using this approach we reproduce the half-integer super-helicity superspace action using unconstrained superfields.
1 Introduction

In the preceding paper [1] we discussed the case of integer super-helicity theories. In this complementary paper, the corresponding program is carried out for the case of half-integer super-helocities. Following the same strategy as in [1], we use representation theory as a guideline to dictate the proper type of superfields we should consider for the construction of the theory.

Under our restriction\(^4\) using auxiliary superfields with lower spins than the main gauge superfield, unlike the integer case, we recover two different formulations for the description of the highest possible super-helicity, in agreement with the results in [2]. We will verify that although they describe the same physical system on-shell, they don’t have the same off-shell structure and they involve different numbers of degrees of freedom. From that point of view, we can say that these two theories are not equivalent off-shell, meaning that there is no 1-1 mapping between the two.

After the construction of the superspace action in terms of unconstrained superfields, we use the equations of motion and their properties, such as the Bianchi identities, to define the various components, derive the component action and their SUSY-transformation laws. We also do a counting of the off-shell degrees of freedom. A simple counting argument provides a supporting case for pairs of \(\mathcal{N} = 1\) theories noted before [5] to create \(\mathcal{N} = 2\) irreducible higher spin representations.

This paper is organized as follows: In section 2 we quickly review the representation theory of the little group of the 4\(D\), \(\mathcal{N} = 1\) Super-Poincaré group for a half-integer superspin/helicity system. In section 3 we focus on the massless case and illustrate how the gauge transformation of the superfield emerges. In section 4, using the invariance of the physical degrees of freedom as a guideline, we build the superspace action of the theory and prove that it describes the desired super-helicity. As mentioned before there are two ways to do this and we will present both. The next section 5 we discuss the off-shell components for both of these theories. Using the equations of motion of the superspace action we define the off-shell components, obtain the component action in a diagonal form and explicit expressions for their SUSY-transformation laws.

2 Irreducible Representations

Following the review of the representation theory presented in [1],[3] we conclude that the two cases of massive and massless have the following properties discussed below.

\(^4\)The formalism in [2] does permit violations of this condition.
2.1 Massive Case

For the massive case, the superfield which describes a real irreducible representation of half-integer superspin $Y = s + 1/2$ (and it is the highest superspin that it can describe) is a real bosonic superfield $H_{\alpha(s)\dot{\alpha}(s)}$ with $s$ undotted symmetrized indices and $s$ dotted symmetrized indices and must satisfy the constraints

$$D^2 H_{\alpha(s)\dot{\alpha}(s)} = 0$$
$$\bar{D}^2 H_{\alpha(s)\dot{\alpha}(s)} = 0$$
$$D^\gamma H_{\gamma\alpha(s-1)\dot{\alpha}(s)} = 0$$
$$\partial^{\gamma\dot{\gamma}} \Phi_{\gamma\alpha(n-1)\dot{\gamma}(m-1)} = 0$$
$$\Box H_{\alpha(s)\dot{\alpha}(s)} = m^2 H_{\alpha(s)\dot{\alpha}(s)}$$

(1)

Equivalently there is a chiral superfield $W_{\alpha(s+1)\dot{\alpha}(s)}$ defined as

$$W_{\alpha(s+1)\dot{\alpha}(s)} = \frac{1}{(s+1)!} \bar{D}^2 D^{(\alpha(s+1)H_{\alpha(s)\dot{\alpha}(s)}}$$

(2)

with

$$\bar{D}_\beta W_{\gamma\alpha(s)\dot{\alpha}(s)} = 0, \text{ chiral}$$
$$\partial^{\beta\dot{\beta}} W_{\beta\alpha(s)\dot{\beta}(s-1)} = 0$$
$$\Box W_{\alpha(s+1)\dot{\alpha}(s)} = m^2 W_{\alpha(s+1)\dot{\alpha}(s)}$$

(3)

The spin content of this supermultiplet is $j = s + 1, s + 1/2, s + 1/2, s$.

2.2 Massless Case

For the massless case, the half-integer super-helicity representation is described by a chiral superfield $F_{\alpha(2s+1)}$ with $2s + 1$ symmetrized undotted indices and no dotted indices. It must satisfy the constraints

$$\bar{D}_\gamma F_{\alpha(2s+1)} = 0, \text{ chiral}$$
$$D^\beta F_{\beta\alpha(2s)} = 0$$

(4)

and the helicity content is $h = s + 1, s + 1/2$

3 Massless limit and Redundancy

Now that we know the proper building blocks for the two irreducible representations we impose the convenient feature that the massless limit of the massive representation gives the massless representation plus other sectors that decouple.
In order for something like this to occur, we should be able to construct $F_{\alpha(2s+1)}$ out of the remaining objects after the limit of the massive theory has been taken. Given the chirality properties of $F$ and $W$ and their index structure we can guess a mapping that could do the trick.

$$F_{\alpha(2s+1)} \sim \partial_{(\alpha_{2s+1}} \dot{\alpha}_s \cdots \partial_{\alpha_{s+2}} \dot{\alpha}_1 \bar{D}^2 D_{\alpha_{s+1}} H_{\alpha(s)} \dot{\alpha}(s)$$  \hspace{1cm} (5)

As it is explained in [1] that identification is problematic because $F$ is the object that carries the physical gauge-invariant degrees of freedom and not $H$ and also the degrees of freedom of $F$ and $H$ don’t match. The way out of that is to introduce a redundancy and identify $H_{\alpha(s)} \dot{\alpha}(s)$ with $H_{\alpha(s)} \dot{\alpha}(s) + R_{\alpha(s)} \dot{\alpha}(s)$.

The redundancy has to respect the physical (propagating) degrees of freedom of $F$ and leave them unchanged. Hence

$$\partial_{(\alpha_{2s+1}} \dot{\alpha}_s \cdots \partial_{\alpha_{s+2}} \dot{\alpha}_1 \bar{D}^2 D_{\alpha_{s+1}} R_{\alpha(s)} \dot{\alpha}(s) = 0$$  \hspace{1cm} (6)

The most general solution to this is

$$R_{\alpha(s)} \dot{\alpha}(s-1) = \frac{1}{s!} D_{(\alpha_s} \bar{L}_{\alpha(s-1))} \dot{\alpha}(s) - \frac{1}{s!} \bar{D}_{(\dot{\alpha}_s} L_{\alpha(s) \dot{\alpha}(s-1))}$$  \hspace{1cm} (7)

This redundancy will be the gauge transformation of the superfield $H$

4 The Superspace Action

Using the equivalency class characterized by $H$ and the redundancy $R$ we attempt to construct a superspace action that will describe the irreducible representation of half-integer super-helicity. For that $H$ must have mass dimension zero and the action must involve four covariant derivatives.

The most general action is

$$S = \int d^8 z \ a_1 H^{\alpha(s)} \dot{\alpha}(s) D^\gamma \bar{D}^2 D_\gamma H_{\alpha(s)} \dot{\alpha}(s)$$

$$+ a_2 H^{\alpha(s)} \dot{\alpha}(s) \{ D^2, \bar{D}^2 \} H_{\alpha(s)} \dot{\alpha}(s)$$

$$+ a_3 H^{\alpha(s)} \dot{\alpha}(s) D_{\alpha_s} \bar{D}^2 D^\gamma H_{\gamma \alpha(s-1) \dot{\alpha}(s)} + c.c.$$  \hspace{1cm} (8)

$$+ a_4 H^{\alpha(s)} \dot{\alpha}(s) D_{\alpha_s} \bar{D}_{\dot{\alpha}_s} D^\gamma \bar{D}^\gamma H_{\gamma \alpha(s-1) \dot{\alpha}(s-1)} + c.c.$$  

The goal is to have a gauge invariant action $\delta_G S = 0$, meaning the action respects the equivalence between $H$ and $H + R$ and therefore the physical degrees of freedom.

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5The sum of the indices of $W$ is the number of the undotted indices of $F$

6There the argument was for integer super-helicities, but it can be repeated for the half-integer case

7$R$ must be real since $H$ is real

8Its highest spin component is a propagating boson

9The action must be quadratic to $H$ and dimensionless
described by that action are invariant (gauge invariance). The strategy to do this is to pick the free parameters in a special way. If this is not possible then we introduce auxiliary superfields, compensators and/or put constraints on the parameter $L$ of the redundancy (gauge parameter). We demand the compensators introduced, if necessary, will not contain degrees of freedom with spin higher or equal to that in the main gauge superfield, therefore they must have fewer indices than the main object $H$.

The deformation of the action is:

$$
\delta_G S = \int d^8 z \left[ (-2a_1 + 2\frac{s + 1}{s}a_3 + 2a_4)D^2 \bar{D}_\alpha H^{\alpha(s)\dot{\alpha}(s)}
+ (-2a_3 - \frac{s + 1}{s}a_4)D^\alpha s \bar{D}_\gamma H^{\gamma\alpha(\dot{\alpha}(s-1))} \right]
+ D^\alpha s + 1 \Lambda_{\alpha(s+1)\dot{\alpha}(s-1)}

+ D^{\alpha s - 1} L_{\alpha(s)\dot{\alpha}(s-1)} + \bar{D}^{\dot{\alpha}s - 1} J_{\alpha(s-1)\dot{\alpha}(s-3)}

+ c.c. \right]
$$

Notice that because of the D-algebra we have the freedom to add terms like $D^\alpha s + 1 \Lambda_{\alpha(s+1)\dot{\alpha}(s-1)}$ and $\bar{D}^{\dot{\alpha}s - 1} J_{\alpha(s-1)\dot{\alpha}(s-3)}$ which identically vanish and they don’t effect the result.

Obviously we can not set the variation of the action to zero just by picking values for the $a$’s without setting them all to zero, but we can introduce compensators with proper mass dimensionality and index structure. There are two different ways to do that

- (I) Choose coefficients to kill the last two terms ($a_2 = a_4 = 0$) and introduce a compensator that cancels the first term

- (II) Choose coefficients to kill the first two terms

$$(-2a_1 + 2\frac{s + 1}{s}a_3 + 2a_4 = 0, -2a_3 - \frac{s + 1}{s}a_4, a_2 = 0)$$

and introduce a compensator to cancel the last term

These two different approaches will lead to the two different formulations of half-integer super-helicity, mentioned above.

### 4.1 Case (I) - Transverse theory

For case (I) we find

$$a_2 = a_4 = 0$$
\[ \delta_G S = \int d^8 z \left[ (-2a_1 + \frac{2s + 1}{s} a_3) D^2 D_{\alpha_s} H^{(s)}(s) + 2a_3 D^\alpha_\gamma \bar{D}_\gamma H^{\gamma\alpha(s-1)}(s-1) \right] \left( \bar{D}^2 L_{(s)}(s-1) + D^{\alpha_{s+1}} \Lambda_{(s+1)}(s-1) \right) \] 

This suggests us to introduce a fermionic compensator \( \chi_{(s)}(s) \) which transforms like \( \delta_G \chi_{(s)}(s-1) = \bar{D}^2 L_{(s)}(s-1) + D^{\alpha_{s+1}} \Lambda_{(s+1)}(s-1) \). So in order to obtain invariance we add to the action two new pieces: The coupling term of \( H \) with \( \chi \) and the kinetic energy terms for \( \chi \). The full action takes the form

\[ S = \int d^8 z \ a_1 H^{(s)}(s) D^\gamma \bar{D}^2 D_{\gamma} H_{(s)}(s) + a_3 H^{(s)}(s) D_{\alpha_s} \bar{D}^\gamma D^\gamma H^{\gamma\alpha(s-1)}(s-1) + c.c. \]

\[ - (2a_1 - \frac{2s + 1}{s} a_3) H^{(s)}(s) D_{\alpha_s} \bar{D}^\gamma D^\gamma \chi_{(s-1)}(s-1) + c.c. \]

\[ + 2a_3 H^{(s)}(s) D_{\alpha_s} \bar{D}_{\alpha_s} D^\gamma \chi^{(s-1)}(s-1) + c.c. \]

\[ + b_1 \chi^{(s)}(s-1) D^2 \chi_{(s)}(s-1) + c.c. \]

\[ + b_2 \chi^{(s)}(s-1) \bar{D}^2 \chi_{(s)}(s-1) + c.c. \]

\[ + b_3 \chi^{(s)}(s-1) \bar{D}_{\alpha_s} D_{\alpha_s} \chi_{(s)}(s-1) + b_4 \chi^{(s)}(s-1) D_{\alpha_s} \bar{D}_{\alpha_s} \chi_{(s)}(s-1) \]

and it has to be invariant under

\[ \delta_G H_{(s)}(s) = \frac{1}{s!} D_{(s)} \bar{L} \Lambda_{(s)}(s-1) = \frac{1}{s!} \bar{D}_{(s)} \Lambda_{(s)}(s-1) \] (12a)

\[ \delta_G \chi_{(s)}(s-1) = \bar{D}^2 L_{(s)}(s-1) + D^{\alpha_{s+1}} \Lambda_{(s+1)}(s-1) \] (12b)

The equations of motion of the superfields are the variation of the action with respect to the superfield

\[ T_{(s)}(s) = \frac{\delta S}{\delta H_{(s)}(s-1)} \]

and the invariance of the action gives the following Bianchi Identities

\[ \bar{D}_{\alpha_s} T_{(s)}(s) - \bar{D}^2 G_{(s)}(s-1) = 0 \] (14a)

\[ \frac{1}{(s + 1)!} D_{(s+1)} G_{(s)}(s-1) = 0 \] (14b)

The Bianchi identities fix all the coefficients

\[ a_3 = 0, \quad b_3 = 0 \]

\[ b_1 = - \frac{s + 1}{s} a_1, \quad b_4 = 2a_1 \]

\[ b_2 = 0 \]
and the final form of the action is:

\[
S = \int d^8z \left\{ c H^{\alpha(s)\dot{\alpha}(s)} D^\gamma D_\gamma H_{\alpha(s)\dot{\alpha}(s)} \\
-2c H^{\alpha(s)\dot{\alpha}(s)} D_\alpha D^2 \chi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\
-\frac{s+1}{s} c \chi^{\alpha(s)\dot{\alpha}(s-1)} D^2 \chi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\
+ 2c \chi^{\alpha(s)\dot{\alpha}(s-1)} D_{\dot{\alpha}} \bar{D} \chi_{\alpha(s-1)\dot{\alpha}(s)} \right\} 
\]

(15)

The expressions for the equations of motion are:

\[
T_{\alpha(s)\dot{\alpha}(s)} = 2c D^\gamma D_\gamma H_{\alpha(s)\dot{\alpha}(s)} \\
+ \frac{2c}{s!} \left( D_{(\dot{\alpha}s)} \bar{D} \chi_{(s-1)\dot{\alpha}(s)} - \bar{D}_{(\dot{\alpha}s)} D^2 \chi_{(s-1)\dot{\alpha}(s)} \right) \quad (16a)
\]

\[
G_{\alpha(s)\dot{\alpha}(s-1)} = -2c D^2 \bar{D} \chi_{\alpha(s)\dot{\alpha}(s)} - 2c \frac{s+1}{s} D^2 \chi_{\alpha(s)\dot{\alpha}(s-1)} \\
+ \frac{2c}{s!} D_{(\dot{\alpha}s)} \bar{D} \chi_{(s-1)\dot{\alpha}(s)} \quad (16b)
\]

where \( c \) is a free overall parameter that can be absorbed in the definition of the superfields but for the moment we’ll leave it as it is and fix it later when we define the components.

The above action is the same as the transversely-linear theory presented in [2] if we solve the constraints and express it in terms of the prepotential, but now we have an alternate understanding why we have to consider these types of superfields in order to construct the action and why they have these gauge transformation.

Before we do anything else we must first prove that indeed this action describes the desired representation. Using the equations of motion we can now prove that a chiral superfield \( F_{\alpha(2s+1)} \) exists and satisfies the following Bianchi identity

\[
D^{\alpha_2 s+1} F_{\alpha(2s+1)} = \\
= \frac{1}{c(2s)!} \partial_{\alpha_2} \dot{\alpha}_s \ldots \partial_{\alpha_{s+1}} \dot{\alpha}_1 T_{\alpha(s)\dot{\alpha}(s)} \\
+ \frac{i}{2c} \frac{s}{2s+1} \frac{B}{B+\Delta(2s)!} D_{(\alpha_2} \bar{D} \partial_{\alpha_{2s-1}} \dot{\alpha}_s \ldots \partial_{\alpha_{s+1}} \dot{\alpha}_1 G_{\alpha(s)}\dot{\alpha}(s-1) \\
+ \frac{1}{2c} \frac{s}{2s+1} \frac{1}{(2s)!} D_{(\alpha_2} \partial_{\alpha_{2s-1}} \dot{\alpha}_s \ldots \partial_{\alpha_{s+1}} \dot{\alpha}_1 G_{\alpha(s-1)}\dot{\alpha}(s) \\
+ \frac{i}{2c} \frac{s}{2s+1} \frac{\Delta}{B+\Delta(2s)!} D_{(\alpha_2} \bar{D} \partial_{\alpha_{2s-1}} \dot{\alpha}_s \ldots \partial_{\alpha_{s+1}} \dot{\alpha}_1 T_{\alpha(s)}\dot{\alpha}(s) 
\]

(17)

where

\[
F_{\alpha(2s+1)} = \frac{1}{(2s+1)!} \bar{D}^2 D_{(\alpha_2 s+1} \partial_{\alpha_2} \dot{\alpha}_s \ldots \partial_{\alpha_{s+1}} \dot{\alpha}_1 H_{\alpha(s)}\dot{\alpha}(s)
\]

and that proves that on-shell where \( T_{\alpha(s)\dot{\alpha}(s)} = G_{\alpha(s)\dot{\alpha}(s-1)} = 0 \), we find the desired constraints to describe a super-helicity \( Y = s + 1/2 \) system. The constants \( B \) and \( \Delta \) are only constrained by \( B + \Delta \neq 0 \).

7
Like in the integer super-helicity case, this action and superfield configuration are not unique, but a simple representative of a two parameter family of equivalent theories. To see that we perform redefinitions of the superfields. Dimensionality and index structure allow us to do the following redefinition of $\chi$

$$\chi_{\alpha(s)\dot{\alpha}(s-1)} \rightarrow \chi_{\alpha(s)\dot{\alpha}(s-1)} + z\bar{D}^{\dot{\alpha}} H_{\alpha(s)\dot{\alpha}(s)}$$ (18)

where $z$ is a complex parameter. This operation will generate an entire class of actions and transformation laws which all are related by the above redefinition.

The generalized action is

$$S = \int d^8 w \ e \ H^{\alpha(s)\dot{\alpha}(s)} D^\gamma \bar{D}^2 D_\gamma H_{\alpha(s)\dot{\alpha}(s)}$$

$$-2c \left[ 1 + \frac{s+1}{s} \right] H^{\alpha(s)\dot{\alpha}(s)} \bar{D}_{\dot{\alpha}s} D^2 \chi_{\alpha(s)\dot{\alpha}(s-1)} + c.c.$$ 

$$-2c\bar{z} \ H^{\alpha(s)\dot{\alpha}(s)} D_{\alpha s} \bar{D}_{\dot{\alpha}s} D^\gamma \chi_{\alpha(s-1)\dot{\alpha}(s-1)} + c.c.$$ 

$$-2c\bar{z} \left[ 1 + \frac{s+1}{s} \right] H^{\alpha(s)\dot{\alpha}(s)} D_{\alpha s} \bar{D}^2 D^\gamma H_{\gamma\alpha(s-1)\dot{\alpha}(s)} + c.c.$$ 

$$-c|z|^2 H^{\alpha(s)\dot{\alpha}(s)} D_{\alpha s} \bar{D}_{\dot{\alpha}s} D^\gamma \bar{D}^\gamma H_{\gamma\alpha(s-1)\dot{\alpha}(s-1)} + c.c.$$ 

$$-\frac{s+1}{s} c \chi_{\alpha(s)\dot{\alpha}(s-1)} D^2 \chi_{\alpha(s)\dot{\alpha}(s-1)} + c.c.$$ 

$$+2c \chi^{\alpha(s)\dot{\alpha}(s-1)} D_{\alpha s} \bar{D}^{\dot{\alpha}s} \chi_{\alpha(s-1)\dot{\alpha}(s)}$$

and the generalized transformation laws are

$$\delta_G H_{\alpha(s)\dot{\alpha}(s)} = \frac{1}{s!} D_{(\alpha s} \bar{L}_{\alpha(s-1)\dot{\alpha}(s)} - \frac{1}{s!} \bar{D}_{(\dot{\alpha}s} L_{\alpha(s)\dot{\alpha}(s-1)})$$ (20a)

$$\delta_G \chi_{\alpha(s)\dot{\alpha}(s-1)} = \left[ 1 + \frac{s+1}{s} \right] \bar{D}^2 L_{\alpha(s)\dot{\alpha}(s-1)} - \frac{z}{s!} \bar{D}^{\dot{\alpha}s} D_{(\alpha s} \bar{L}_{\alpha(s-1)\dot{\alpha}(s)}$$ 

$$+ D^{\alpha s+1} A_{\alpha(s+1)\dot{\alpha}(s-1)}$$ (20b)

4.2 Case (II) - Longitudinal theory

For case (II) we obtain the conditions

$$a_1 = c, \quad a_2 = 0$$

$$a_3 = \frac{s(s+1)}{2s+1} c, \quad a_4 = -\frac{s^2}{2s+1} c$$

and we have to introduce a fermionic compensator $\chi_{\alpha(s-1)\dot{\alpha}(s-2)}$ which transforms like

$$\delta_G \chi_{\alpha(s-1)\dot{\alpha}(s-2)} = \bar{D}^{\dot{\alpha}s-1} D^{\alpha s} L_{\alpha(s)\dot{\alpha}(s-1)} + \frac{s-1}{s} D^{\alpha s} \bar{D}^{\dot{\alpha}s-1} L_{\alpha(s)\dot{\alpha}(s-1)}$$

$$+ \bar{D}^{\dot{\alpha}s-2} J_{\alpha(s-1)\dot{\alpha}(s-3)}$$

8
and couples with the term $\tilde{D}^{\hat{a}}D^{\gamma}D^{\hat{b}}H_{\gamma\alpha(s-1)\hat{a}\hat{b}(s-2)}$

So in order to achieve invariance we add to the action two new pieces, the coupling term of $H$ with $\chi$ and the kinetic energy terms for $\chi$. The full action takes the form

$$S = \int d^8 z \ c \ H^{\alpha(s)\hat{a}(s)}D^{\gamma}D^{\hat{a}}D_\gamma H^{\alpha(s)\hat{a}(s)}$$

$$+ \frac{s(s+1)}{2s+1} c \ H^{\alpha(s)\hat{a}(s)}D_{\alpha_s} \tilde{D}^{\hat{a}}D^{\gamma}H_{\gamma\alpha(s-1)\hat{a}(s)} + c.c.$$

$$- \frac{s^2}{2s+1} c \ H^{\alpha(s)\hat{a}(s)}D_{\alpha_s} \tilde{D}^{\hat{a}}D^{\gamma}H_{\gamma\alpha(s-1)\hat{a}(s)} + c.c.$$

$$- \frac{s^2}{2s+1} c \ H^{\alpha(s)\hat{a}(s)}\tilde{D}_{\alpha_s}D_{\alpha_s-1} \chi_{\alpha(s-1)\hat{a}(s-2)} + c.c.$$

$$+ b_1 \chi^{\alpha(s-1)\hat{a}(s-2)}D^2 \chi_{\alpha(s-1)\hat{a}(s-2)} + c.c.$$

$$+ b_2 \chi^{\alpha(s-1)\hat{a}(s-2)}\tilde{D}^2 \chi_{\alpha(s-1)\hat{a}(s-2)} + c.c.$$

$$+ b_3 \chi^{\alpha(s-1)\hat{a}(s-2)}\tilde{D}^{\hat{a}_{s-1}}D_{\alpha_{s-1}} \chi_{\alpha(s-2)\hat{a}(s-1)}$$

$$+ b_4 \chi^{\alpha(s-1)\hat{a}(s-2)}D_{\alpha_{s-1}} \tilde{D}^{\hat{a}_{s-1}} \chi_{\alpha(s-2)\hat{a}(s-1)}$$

and it has to be invariant under

$$\delta_G H_{\alpha(s)\hat{a}(s)} = \frac{1}{s!} D_{(\alpha_s} \tilde{L}_{\alpha(s-1))]\hat{a}(s)} - \frac{1}{s!} D_{(\hat{a}_s} \tilde{L}_{\alpha(s-1))]\hat{a}(s)}$$

$$\delta_G \chi_{\alpha(s-1)\hat{a}(s-2)} = \tilde{D}^{\hat{a}_{s-1}}D^\alpha L_{\alpha(s)\hat{a}(s-1)} + \frac{s-1}{s} D^\alpha \tilde{D}^{\hat{a}_{s-1}} L_{\alpha(s)\hat{a}(s-1)}$$

$$+ \tilde{D}^{\hat{a}_{s-2}} J_{\alpha(s-1)\hat{a}(s-2)}$$

The equations of motion of the superfields are

$$T_{\alpha(s)\hat{a}(s)} = \frac{\delta S}{\delta H^{\alpha(s)\hat{a}(s)}} \hspace{1cm} G_{\alpha(s-1)\hat{a}(s-2)} = \frac{\delta S}{\delta \chi_{\alpha(s-1)\hat{a}(s-2)}}$$

and satisfy the Bianchi Identities

$$\tilde{D}^{\hat{a}_s} T_{\alpha(s)\hat{a}(s)} + \frac{1}{s!(s-1)!} D_{(\alpha_s} \tilde{D}^{\hat{a}_{s-1}} G_{\alpha(s-1))\hat{a}(s-2)}$$

$$+ \left[ \frac{s-1}{s} \right] s! (s-1)! D_{(\hat{a}_{s-1}} D_{\alpha_s} G_{\alpha(s-1))\hat{a}(s-2)} = 0$$

$$\tilde{D}^{\hat{a}_{s-2}} G_{\alpha(s-1)\hat{a}(s-2)} = 0 \hspace{1cm} \tilde{D}^2 G_{\alpha(s-1)\hat{a}(s-2)} = 0$$

which fix all free coefficients to the following values:

$$b_1 = 0 \hspace{1cm} b_2 = \frac{s^2(s+1)}{(2s+1)(s-1)c}$$

$$b_4 = 0 \hspace{1cm} b_3 = \frac{2s^2}{2s+1}c$$

9
The superspace action takes the final form
\[ S = \int d^8 z \ c \ H^{(\alpha)\bar{\alpha}(s)} D\gamma D^2 D_\gamma H_{\alpha(s)\bar{\alpha}(s)} \]
\[ + \left[ \frac{s(s + 1)}{2s + 1} \right] c \ H^{(\alpha)\bar{\alpha}(s)} D_\alpha \bar{D} D\gamma H_{\gamma\alpha(s-1)\bar{\alpha}(s)} + c.c. \]
\[ - \left[ \frac{s^2}{2s + 1} \right] c \ H^{(\alpha)\bar{\alpha}(s)} D_\alpha \bar{D} D\gamma \bar{\gamma} H_{\gamma\alpha(s-1)\bar{\alpha}(s-1)} + c.c. \]
\[ - \left[ \frac{s^2}{2s + 1} \right] c \ H^{(\alpha)\bar{\alpha}(s)} \bar{D}_\alpha D_\alpha \bar{D} \bar{\gamma} H_{\gamma \alpha(s-1)\bar{\gamma}(s-1)} + c.c. \]
\[ + \left[ \frac{s^2(s + 1)}{2s + 1} \right] c \ \chi^{(s-1)\bar{\alpha}(s-2)} \bar{D}^2 \chi_{\alpha(s-1)\bar{\alpha}(s-2)} + c.c. \]
\[ + \left[ \frac{s^2}{2s + 1} \right] c \ \chi^{(s-1)\bar{\alpha}(s-2)} \bar{D} \bar{\alpha}_{s-1} D_\alpha \bar{\alpha}_{s-1} \bar{\chi}_{\alpha(s-2)\bar{\alpha}(s-1)} \]

and the equations of motion are
\[ T_{\alpha(s)\bar{\alpha}(s)} = 2c D\gamma D^2 D_{\gamma} H_{\alpha(s)\bar{\alpha}(s)} \]
\[ + \frac{2c}{s!} \left[ \frac{s(s + 1)}{2s + 1} \right] D_{(\alpha s} \bar{D} D^2 D\gamma H_{\gamma\alpha(s-1)\bar{\alpha}(s)} \]
\[ + \frac{2c}{s!} \left[ \frac{s(s + 1)}{2s + 1} \right] \bar{D}_{(\bar{\alpha} s} \bar{D} D^2 D\gamma H_{\gamma\alpha(s)\bar{\alpha}(s-1)} \]
\[ - \frac{2c}{s! s!} \left[ \frac{s^2}{2s + 1} \right] D_{(\alpha s} \bar{D} \bar{D}_{\gamma} D\gamma H_{\gamma\alpha(s-1)\bar{\gamma}(s-1)} \]
\[ - \frac{2c}{s! s!} \left[ \frac{s^2}{2s + 1} \right] \bar{D}_{(\bar{\alpha} s} \bar{D} \bar{D}_{\gamma} D\gamma H_{\gamma\alpha(s)\bar{\gamma}(s-1)} \]
\[ - \frac{2c}{s! s!} \left[ \frac{s^2}{2s + 1} \right] \bar{D}_{(\bar{\alpha} s} \bar{D} \bar{D}_{\gamma} H_{\gamma\alpha(s-1)\bar{\gamma}(s-1)} \]
\[ - \frac{2c}{s! s!} \left[ \frac{s^2}{2s + 1} \right] D_{(\alpha s} \bar{D} \bar{D}_{a_{s-1}} \chi_{\alpha(s-1)\bar{\alpha}(s-2)} \]
\[ - \frac{2c}{s! s!} \left[ \frac{s^2}{2s + 1} \right] D_{(\alpha s} \bar{D} \bar{D}_{a_{s-1}} \bar{\chi}_{\alpha(s-2)\bar{\alpha}(s-1)} \]

\[ G_{\alpha(s-1)\bar{\alpha}(s-2)} = 2c \left[ \frac{s^2(s + 1)}{2s + 1} \right] \bar{D}^\alpha_{a_{s-1}} D^\alpha \bar{D}^\gamma \bar{\gamma} H_{\alpha(s)\bar{\alpha}(s)} \]
\[ + \frac{2c}{(s! s! (s+1) (s+1))} \left[ \frac{s^2(s + 1)}{2s + 1} \right] \bar{D}^2 \chi_{(\alpha(s-1)\bar{\alpha}(s-2)} \]
\[ + \frac{2c}{(s-1)! s! (2s + 1)} \left[ \frac{s^2}{2s + 1} \right] \bar{D}^\alpha_{a_{s-1}} \bar{D} \bar{\alpha}_{s-1} \chi_{\alpha(s-2)\bar{\alpha}(s-1)} \]

Using the equations of motion we can now prove that a chiral superfield \( F_{(2s+1)} \) exist and satisfies the following identity
\[ D^{\alpha_{2s+1}} F_{(2s+1)} = \frac{1}{2c} \frac{\partial}{\partial_{\alpha_{2s}} \ldots \partial_{\alpha_{s+1}}} T_{\alpha(s)\bar{\alpha}(s)} \]

where
\[ F_{(2s+1)} \equiv \frac{1}{(2s + 1)!} \bar{D}^2 \bar{D}_{(\alpha_{2s+1}} \partial_{\alpha_{2s}} \ldots \partial_{\alpha_{s+1}} \bar{\alpha}_{s+1} H_{\alpha(s)\bar{\alpha}(s)} \]
and that proves that in the on-shell theory where \( T_\alpha(s)\dot{\alpha}(s-1) = G_\alpha(s)\dot{\alpha}(s-1) = 0 \) we obtain the desired constraints to describe a super-helicity \( Y = s + 1/2 \) system.

Unlike the previous theories of half-integer and integer super-helicity, we can not perform any local redefinitions of the superfields because of the difference in their index structure. So the above action is unique.

5 Projection and Components

The superspace actions derived above in terms of unconstrained objects will be the starting point for our component discussion. We will use the method described in [1] to derive the field structure of the theory, the component action and their SUSY-transformations laws.

5.1 Component structure for Transverse theories (I)

The two superfields \( T_\alpha(s)\dot{\alpha}(s), G_\alpha(s)\dot{\alpha}(s-1) \) in (16) have mass dimensionality \([T_\alpha(s)\dot{\alpha}(s)] = 2, [G_\alpha(s)\dot{\alpha}(s-1)] = 3/2\) and satisfy the Bianchi identities and their consequences:

\[
\begin{align*}
\bar{D}\dot{\alpha}_s T_\alpha(s)\dot{\alpha}(s) - \bar{D}^2 G_\alpha(s)\dot{\alpha}(s-1) &= 0 \Leftrightarrow \bar{D}^2 T_\alpha(s)\dot{\alpha}(s) = 0 \quad \text{reality} \\
1/(s+1)! D_{(\alpha_k+1)} G_\alpha(s)\dot{\alpha}(s-1) &= 0 \Leftrightarrow D^2 G_\alpha(s)\dot{\alpha}(s-1) = 0
\end{align*}
\]

These identities constrained most of the components of superfields \( T \) and \( G \) and only few of them remain to play the role of off-shell auxiliary components. So just by looking at them we immediately see the structure of auxiliary fields:

\[
\begin{align*}
\bar{D}^{\dot{\alpha}_{s-1}} G_\alpha(s)\dot{\alpha}(s-1), & \quad \bar{D}(\dot{\alpha}_s G_\alpha(s)\dot{\alpha}(s-1)), \quad T_\alpha(s)\dot{\alpha}(s), \quad D^{\dot{\alpha}_s} G_\alpha(s)\dot{\alpha}(s-1) \quad \text{for bosons} \\
G_\alpha(s)\dot{\alpha}(s-1), & \quad D_{(\alpha_k)} \bar{D}^{\dot{\alpha}_{s-1}} G_\alpha(s-1)\dot{\alpha}(s) \quad \text{for fermions}
\end{align*}
\]

The next step is to express the action in terms of \( T \) and \( G \)

\[
S = \int d^8 z \left\{ \frac{1}{2} H^\alpha(s)\dot{\alpha}(s) T_\alpha(s)\dot{\alpha}(s) + \frac{1}{2} \chi^\alpha(s)\dot{\alpha}(s-1) G_\alpha(s)\dot{\alpha}(s-1) + \text{c.c.} \right\}
\]

\[
= \int d^4 x \left\{ \frac{1}{2} \bar{D}^2 D^2 \left( H^\alpha(s)\dot{\alpha}(s) T_\alpha(s)\dot{\alpha}(s) \right) + \frac{1}{2} \bar{D}^2 D^2 \left( \chi^\alpha(s)\dot{\alpha}(s-1) G_\alpha(s)\dot{\alpha}(s-1) \right) + \text{c.c.} \right\}
\]

and then to distribute the covariant derivatives.

11
5.1.1 Fermions

After the distribution of D’s and the usage of Bianchi identities we derive for the fermionic Lagrangian:

\[
\mathcal{L}_F = \frac{1}{2} \tilde{D} \tilde{D} \alpha_{s+1} H^\alpha(s) \hat{\alpha}(s) \left| \frac{1}{(s+1)!} \tilde{D}(\hat{\alpha}_{s+1} T(\alpha(s)) \hat{\alpha}(s)) \right|
\]

\[
+ \frac{1}{2} \left( \frac{s}{s+1} \tilde{D} \tilde{D} \gamma H^\alpha(s) \hat{\alpha}(s-1) + \tilde{D}^2 \chi^\alpha(s) \hat{\alpha}(s-1) \right) \left| \tilde{D} \alpha_{s} T(\alpha(s)) \hat{\alpha}(s) \right|
\]

\[
+ \frac{1}{2s+1} \tilde{D} \tilde{D} \alpha_{s} \gamma^\alpha(s-1) \hat{\alpha}(s-1) \left| \frac{1}{s!} \tilde{D}(\hat{\alpha}_s D^\alpha(s) \hat{\alpha}(s-1)) \right|
\]

\[
- \frac{1}{2s+1} \tilde{D} \tilde{D} \gamma^\alpha(s-1) \hat{\alpha}(s-2) \left| \tilde{D} \alpha_{s-1} D^\alpha(s) \hat{\alpha}(s-1) \right|
\]

\[
+ \frac{1}{2} \tilde{D}^2 \tilde{D}^2 \chi^\alpha(s-1) |G(\alpha(s)) \hat{\alpha}(s-1)|
\]

\[
+ c.c.
\]

\[
\frac{1}{(s+1)!} \tilde{D}(\hat{\alpha}_{s+1} T(\alpha(s)) \hat{\alpha}(s)) = \]

\[
= \frac{2ic}{(s+1)!^2} \frac{1}{\tilde{D}^2 \tilde{D} \alpha_{s+1} H(\alpha(s)) \hat{\alpha}(s))}
\]

\[
- \frac{s}{(s+1)!} \frac{1}{s+1} \partial(\hat{\alpha}_{s+1}) \left[ \tilde{D}^2 \tilde{D}^2 \gamma H \hat{\alpha}(s-1)) \hat{\alpha}(s)) - \frac{s+1}{s} \tilde{D}^2 \chi \hat{\alpha}(s-1)) \hat{\alpha}(s)) \right]
\]

\[
\tilde{D} \alpha_{s-1} D^\alpha(s) \hat{\alpha}(s-1) = i \frac{s+1}{s} \partial \tilde{D} \alpha_{s-1} \left[ G(\alpha(s-1)) + 2c \tilde{D}^2 \tilde{D}^\alpha(s) \hat{\alpha}(s) \hat{\alpha}(s) \right]
\]

\[
+ \frac{s+1}{s} \tilde{D}^2 \chi \hat{\alpha}(s-1)) \hat{\alpha}(s)) \right]
\]

\[
- \frac{2ic}{(s-1)!} \frac{s^2-1}{s^2} \partial(\hat{\alpha}_{s-1}) \tilde{D} \tilde{D} \gamma \chi \hat{\alpha}(s-2)) \hat{\alpha}(s-1)) \hat{\alpha}(s) \hat{\alpha}(s-1))
\]

\[
\tilde{D} \alpha_{s} T(\alpha(s)) \hat{\alpha}(s) = \frac{2ic}{(s+1)!} \frac{1}{\tilde{D} \tilde{D} \alpha_{s+1} H(\alpha(s)) \hat{\alpha}(s))}
\]

\[
+ \frac{2ic}{s!} \frac{s+1}{s} \partial(\hat{\alpha}_{s}) \left[ \tilde{D}^2 \tilde{D} \gamma H \hat{\alpha}(s-1)) \hat{\alpha}(s)) - \frac{s+1}{s} \tilde{D}^2 \chi \hat{\alpha}(s-1)) \hat{\alpha}(s)) \right]
\]

\[
+ \frac{2ic}{s!} \frac{s^2-1}{s^2} \partial(\hat{\alpha}_{s-1}) \tilde{D} \tilde{D} \gamma \chi \hat{\alpha}(s-2)) \hat{\alpha}(s-1)) \hat{\alpha}(s) \hat{\alpha}(s-1))
\]

\[
+ \frac{1}{s!} \tilde{D}(\alpha_{s}) \tilde{D} \alpha_{s} \tilde{G}(\alpha(s-1)) \hat{\alpha}(s)
\]

\[
- \frac{i}{s!} \frac{s+1}{s} \partial(\hat{\alpha}_{s}) \tilde{G}(\alpha(s-1)) \hat{\alpha}(s)
\]
We observe that in all the above expressions and in the fermionic Lagrangian there are some specific combinations that appear repeatedly. So let us define

\[
\frac{1}{(s+1)!} \bar{D}^2 D_{(\alpha s+1)H_\alpha(s)} \tilde{\alpha}(s) \equiv N_1 \psi_\alpha(s+1) \tilde{\alpha}(s)
\]

\[
\left\{ \bar{D}^2 \bar{D}^{\tilde{\alpha} s} H_\alpha(s) \tilde{\alpha}(s) + \frac{s+1}{s} D^2 \chi_\alpha(s) \tilde{\alpha}(s-1) \right\} \equiv N_2 \psi_\alpha(s) \tilde{\alpha}(s-1)
\]

\[
\bar{D}^{\tilde{\alpha} s-1} D^{\alpha s} \chi_\alpha(s) \tilde{\alpha}(s-1) \equiv N_3 \psi_\alpha(s-1) \tilde{\alpha}(s-2)
\]

where \( N_1, N_2, N_3 \) are normalization constants to be fixed later. Putting everything together we have for the Lagrangian

\[
\mathcal{L}_F = \mathcal{G}^{\alpha(s) \tilde{\alpha}(s-1)} \left( \frac{1}{2e} \frac{s}{s+1} \frac{1}{s!} D_{(\alpha s) \bar{D}^{\tilde{\alpha} s} \mathcal{G}_{\alpha(s-1)} \tilde{\alpha}(s)} + \frac{i}{4e} \frac{1}{s!} \partial_{(\alpha s) \bar{D}^{\tilde{\alpha} s} \mathcal{G}_{\alpha(s-1)}} \tilde{\alpha}(s) \right) + \text{c.c.}
\]

\[
+2ic \left| N_1 \right|^2 \bar{\psi}^{\alpha(s) \tilde{\alpha}(s+1)} \mathcal{G}^{\alpha(s+1) \tilde{\alpha}(s)} \psi_\alpha(s+1) \tilde{\alpha}(s)
\]

\[
-2ic \frac{s}{s+1} N_1 N_2 \psi^{\alpha(s+1) \tilde{\alpha}(s)} \partial_{\alpha(s+1) \tilde{\alpha}(s)} \psi_\alpha(s) \tilde{\alpha}(s-1) + \text{c.c.}
\]

\[
-2ic \frac{2s+1}{(s+1)^2} N_2 \left| \mathcal{G}^{\alpha(s+1) \tilde{\alpha}(s)} \partial_{\alpha(s+1) \tilde{\alpha}(s)} \psi_\alpha(s) \tilde{\alpha}(s-1) \right|
\]

\[
+2ic \frac{s-1}{s} N_2 N_3 \psi^{\alpha(s) \tilde{\alpha}(s-1)} \partial_{\alpha(s-1) \tilde{\alpha}(s-1)} \psi_\alpha(s) \tilde{\alpha}(s-2) + \text{c.c.}
\]

\[
-2ic \left( \frac{s-1}{s} \right)^2 \left| N_3 \right|^2 \bar{\psi}^{\alpha(s-2) \tilde{\alpha}(s-1)} \mathcal{G}^{\alpha(s-1) \tilde{\alpha}(s-1)} \psi_\alpha(s-1) \tilde{\alpha}(s-2)
\]

The first term in the Lagrangian is the algebraic kinetic energy term of two auxiliary fields and the rest of the terms are exactly the structure of a theory that describes helicity \( h = s + 1/2 \) \[3\]\(^{10}\). To have an exact match we choose coefficients

\[
c = 1, \quad N_2 = -\frac{1}{\sqrt{2}}, \quad N_1 = \frac{1}{\sqrt{2}}, \quad N_3 = -\frac{1}{\sqrt{2} \cdot s - 1}
\]

So the fields that appear in the fermionic action are defined as:

\[
\beta_\alpha(s) \tilde{\alpha}(s-1) \equiv \mathcal{G}^{\alpha(s) \tilde{\alpha}(s-1)}
\]

\[
\psi_\alpha(s+1) \tilde{\alpha}(s) \equiv \frac{\sqrt{2}}{(s+1)!} \bar{D}^2 D_{(\alpha s+1)H_\alpha(s)} \tilde{\alpha}(s)
\]

\[
\psi_\alpha(s) \tilde{\alpha}(s-1) \equiv \sqrt{2} \left\{ \bar{D}^2 \bar{D}^{\tilde{\alpha} s} H_\alpha(s) \tilde{\alpha}(s) + \frac{s+1}{s} D^2 \chi_\alpha(s) \tilde{\alpha}(s-1) \right\}
\]

\[
\psi_\alpha(s-1) \tilde{\alpha}(s-2) \equiv -\sqrt{2} \left( \frac{s-1}{s} \right) \bar{D}^{\tilde{\alpha} s-1} D^{\alpha s} \chi_\alpha(s) \tilde{\alpha}(s-1)
\]

\(^{10}\)Using the conventions of [4]
The Lagrangian is

\[ L_F = \rho^{\alpha(s)\dot{\alpha}(s-1)} \beta^{(s+1)}_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \]
\[ + i \bar{\psi}^{\alpha(s+1)} \dot{\alpha}(s) \gamma^{\alpha_{s+1} \dot{\alpha}_{s+1}} \psi_{\alpha(s+1)\dot{\alpha}(s)} \]
\[ + i \left[ \frac{s}{s+1} \right] \bar{\psi}^{(s+1)\dot{\alpha}(s)} \partial_{\alpha_{s+1} \dot{\alpha}} \psi_{\alpha(s+1)\dot{\alpha}(s-1)} + c.c. \]
\[ - i \left[ \frac{2s+1}{(s+1)^2} \right] \bar{\psi}^{(s-1)\dot{\alpha}(s)} \gamma^{\alpha} \partial_{\dot{\alpha}} \psi_{\alpha(s-1)\dot{\alpha}(s-1)} \]
\[ + i \bar{\psi}^{(s-2)\dot{\alpha}(s)} \partial_{\dot{\alpha}} \psi_{\alpha(s-2)\dot{\alpha}(s-2)} + c.c. \]
\[ - i \bar{\psi}^{(s-2)\dot{\alpha}(s)} \partial_{\dot{\alpha}} \psi_{\alpha(s-2)\dot{\alpha}(s-2)} \]  

and the gauge transformations of the fields are

\[ \delta G \rho_{\alpha(s)\dot{\alpha}(s-1)} = 0 \]
\[ \delta G \psi_{\alpha(s+1)\dot{\alpha}(s)} = \frac{1}{s!(s+1)!} \partial_{\alpha_{s+1} \dot{\alpha}} \xi_{\alpha(s)\dot{\alpha}(s-1)} \]
\[ \delta G \beta_{\alpha(s)\dot{\alpha}(s-1)} = 0 \]
\[ \delta G \psi_{\alpha(s-1)\dot{\alpha}(s-2)} = \frac{s-1}{s} \partial_{\dot{\alpha}} \xi_{\alpha(s-1)\dot{\alpha}(s-1)} \]
\[ \text{with } \xi_{\alpha(s)\dot{\alpha}(s-1)} = -i\sqrt{2} \bar{D}^{2} L_{\alpha(s)\dot{\alpha}(s-1)} \]

5.1.2 Bosons

For the bosonic action we follow exactly the same procedure. The fields that appear in the action are defined as:

\[ U_{\alpha(s)\dot{\alpha}(s-2)} = D^{\alpha_{s-1}} G_{\alpha(s)\dot{\alpha}(s-1)} \]
\[ u_{\alpha(s)\dot{\alpha}(s)} = \frac{1}{2s!} \left\{ D_{\alpha(s)\dot{\alpha}} G_{\alpha(s)\dot{\alpha}(s-1)} - \bar{D} (\dot{\alpha}) G_{\alpha(s)\dot{\alpha}(s-1)} \right\} \]
\[ v_{\alpha(s)\dot{\alpha}(s)} = -i \frac{1}{2s!} \left\{ D_{\alpha(s)\dot{\alpha}} G_{\alpha(s)\dot{\alpha}(s-1)} + \bar{D} (\dot{\alpha}) G_{\alpha(s)\dot{\alpha}(s-1)} \right\} \]
\[ A_{\alpha(s)\dot{\alpha}(s)} = T_{\alpha(s)\dot{\alpha}(s)} + \frac{s}{2s+1} \left\{ D_{\alpha(s)\dot{\alpha}} G_{\alpha(s)\dot{\alpha}(s-1)} - \bar{D} (\dot{\alpha}) G_{\alpha(s)\dot{\alpha}(s-1)} \right\} \]
\[ S_{\alpha(s)\dot{\alpha}(s-1)} = \frac{1}{2} \left\{ D^{\alpha_{s}} G_{\alpha(s)\dot{\alpha}(s-1)} + \bar{D} \bar{D} \bar{G}_{\alpha(s)\dot{\alpha}(s-1)} \right\} \]
\[ P_{\alpha(s-1)\dot{\alpha}(s-1)} = -i \frac{1}{2} \left\{ D^{\alpha_{s}} G_{\alpha(s-1)\dot{\alpha}(s-1)} - \bar{D} \bar{D} G_{\alpha(s-1)\dot{\alpha}(s-1)} \right\} \]
\[ h_{\alpha(s+1)\dot{\alpha}(s+1)} = \frac{1}{2} \frac{1}{(s+1)!} \left[ D_{\alpha_{s+1}} D_{\dot{\alpha}_{s+1}} H_{\alpha(s)\dot{\alpha}(s)} \right] \]
\[ h_{\alpha(s-1)\dot{\alpha}(s-1)} = \frac{1}{2} \frac{s}{(s+1)!} \left[ D^{\alpha_{s}} H_{\alpha(s)\dot{\alpha}(s)} \right] \]
\[ + \frac{1}{s+1} \left( D^{\alpha_{s}} \bar{H}_{\alpha(s)\dot{\alpha}(s)} + \bar{D} \bar{D} \bar{H}_{\alpha(s)\dot{\alpha}(s)} \right) \]
the gauge transformations are

\[ \begin{align*}
\delta_G U_{\alpha(s)} \dot{\alpha}(s-2) &= 0, \\
\delta_G u_{\alpha(s)} \dot{\alpha}(s) &= 0, \\
\delta_G s_{\alpha(s-1)} \dot{\alpha}(s-1) &= 0, \\
\delta_G v_{\alpha(s)} \dot{\alpha}(s) &= 0, \\
\delta_G p_{\alpha(s-1)} \dot{\alpha}(s-1) &= 0
\end{align*} \]

(41)

\[ \delta_G h_{\alpha(s+1)} \dot{\alpha}(s+1) = \frac{1}{(s + 1)^2} \partial_{(\alpha_{s+1} (\dot{\alpha}_{s+1}) \zeta(s)) \dot{\alpha}(s)} \]

\[ \delta_G h_{\alpha(s-1)} \dot{\alpha}(s-1) = \frac{s}{(s + 1)^2} \partial^{\alpha \dot{\alpha}} \zeta(s) \dot{\alpha}(s) \]

where

\[ \zeta(s) \dot{\alpha}(s) = \frac{i}{2s!} \left( D_{\alpha\beta} \tilde{L}(\alpha(s-1)) \dot{\alpha}(s) + \tilde{D}(\dot{\alpha}_{s} L_{\alpha(s)} \dot{\alpha}(s-1)) \right) \]

and the Lagrangian

\[ L_B = \frac{1}{4} \left[ \frac{s - 1}{s + 1} \right] U^{\alpha(s)} \dot{\alpha}(s-2) U_{\alpha(s)} \dot{\alpha}(s-2) + c.c. \]

\[ + \left[ \frac{s}{2} \right] \alpha(s) u_{\alpha(s)} \dot{\alpha}(s) \]

\[ - \left[ \frac{s}{2} \right] \alpha(s) v_{\alpha(s)} \dot{\alpha}(s) \]

\[ + \left[ \frac{s}{2} \right] \alpha(s) A_{\alpha(s)} \dot{\alpha}(s) \]

\[ - \left[ \frac{s}{2} \right] \alpha(s-1) s_{\alpha(s-1)} s_{\alpha(s-1)} \dot{\alpha}(s-1) \]

\[ - \left[ \frac{s}{2} \right] \alpha(s) p_{\alpha(s-1)} \dot{\alpha}(s-1) \]

\[ + h^{\alpha(s+1)} \dot{\alpha}(s+1) \square h_{\alpha(s+1)} \dot{\alpha}(s+1) \]

(42)

\[ - \left[ \frac{s}{2} \right] h^{\alpha(s+1)} \dot{\alpha}(s+1) \partial_{\dot{\alpha}_{s+1} \dot{\alpha}_{s+1}} \partial^{\gamma \dot{\gamma}} h_{\gamma \alpha(s)} \dot{\alpha}(s) \]

\[ + [s(s + 1)] h^{\alpha(s+1)} \dot{\alpha}(s+1) \partial_{\dot{\alpha}_{s+1} \dot{\alpha}_{s+1}} \partial_{\dot{\alpha}_{s+1}} h_{\alpha(s-1)} \dot{\alpha}(s-1) \]

\[ - [(s + 1)(2s + 1)] h^{\alpha(s-1)} \dot{\alpha}(s-1) \square h_{\alpha(s-1)} \dot{\alpha}(s-1) \]

\[ - \left[ \frac{s}{2} \right] h^{\alpha(s-1)} \dot{\alpha}(s-1) \partial_{\dot{\alpha}_{s-1} \dot{\alpha}_{s-1}} \partial^{\gamma \dot{\gamma}} h_{\gamma \alpha(s-2)} \dot{\alpha}(s-2) \]

gives rise to the theory of helicity \( h = s + 1 \) as expected.

### 5.1.3 Off-shell degrees of freedom

Let us count the bosonic degrees of freedom:
| fields            | d.o.f | redundancy | net        |
|------------------|-------|------------|------------|
| $h_{\alpha(s+1)\dot{\alpha}(s+1)}$ | $(s+2)^2$ | $(s+1)^2$   | $s^2 + 2s + 3$ |
| $h_{\alpha(s-1)\dot{\alpha}(s-1)}$ | $s^2$ | $(s+1)^2$   | $s^2$ |
| $u_{\alpha(s)}\dot{\alpha}(s)$   | $(s+1)^2$ | 0          | $(s+1)^2$ |
| $v_{\alpha(s)}\dot{\alpha}(s)$   | $(s+1)^2$ | 0          | $(s+1)^2$ |
| $A_{\alpha(s)}\dot{\alpha}(s)$   | $(s+1)^2$ | 0          | $(s+1)^2$ |
| $U_{\alpha(s)}\dot{\alpha}(s-2)$ | $2(s+1)(s-1)$ | 0          | $2(s+1)(s-1)$ |
| $S_{\alpha(s-1)}\dot{\alpha}(s-1)$ | $s^2$ | 0          | $s^2$ |
| $P_{\alpha(s-1)}\dot{\alpha}(s-1)$ | $s^2$ | 0          | $s^2$ |
| **Total**        |       |            | $8s^2 + 8s + 4$ |

and the same counting for the fermionic degrees of freedom:

| fields            | d.o.f | redundancy | net        |
|------------------|-------|------------|------------|
| $\psi_{\alpha(s+1)\dot{\alpha}(s)}$ | $2(s+2)(s+1)$ | $2(s+1)s$ | $4s^2 + 4s + 4$ |
| $\psi_{\alpha(s)}\dot{\alpha}(s-1)$ | $2(s+1)s$ | $2(s+1)s$ |
| $\psi_{\alpha(s-1)}\dot{\alpha}(s-2)$ | $2s(s-1)$ | 0          | $2(s+1)s$ |
| $\rho_{\alpha(s)}\dot{\alpha}(s-1)$ | $2(s+1)s$ | 0          | $2(s+1)s$ |
| $\beta_{\alpha(s)}\dot{\alpha}(s-1)$ | $2(s+1)s$ | 0          | $2(s+1)s$ |
| **Total**        |       |            | $8s^2 + 8s + 4$ |

5.1.4 SUSY-transformation laws

The last thing left in order to complete the component picture, is to find the SUSY-
transformation laws of the fields. They can be calculated by the action of the SUSY-
generators on the specific component. In terms of the covariant derivatives we obtain

$$
\delta S_{\text{Component}} = - \left( \epsilon^\beta D_{\beta} + \bar{\epsilon}^\bar{\beta} \bar{D}_{\bar{\beta}} \right)_{\text{Component}}
$$

For the dynamical fields (they have non-zero gauge transformation) the redundancy allows
us to ignore all the terms that have the same structure as their gauge transformation law
because of the identification

$$
\delta S_{\{\text{Dynamical field}\}} \sim \delta S_{\{\text{Dynamical field}\}} + \partial (\delta S\zeta)
$$

With all that in mind, the transformation of the fermionic fields are:

$$
\delta S\rho_{\alpha(s)}\dot{\alpha}(s-1) = \frac{s}{s+1} \epsilon(\alpha_s) \left[ S_{\alpha(s-1)\dot{\alpha}(s-1)} + iP_{\alpha(s-1)\dot{\alpha}(s-1)} \right] + \bar{\epsilon}^{\dot{\alpha}_s} \left[ u_{\alpha(s)\dot{\alpha}(s)} - iv_{\alpha(s)\dot{\alpha}(s)} \right] + \frac{s-1}{s} \epsilon(\alpha_{s-1}) U_{\alpha(s)\dot{\alpha}(s-2)}
$$

(43)
\[ \delta S \beta_\alpha(s) \hat{\alpha}(s-1) = \frac{s}{s+1} \frac{1}{s!} \epsilon(\alpha_\alpha A) \hat{\alpha}(s-1) \]

\[ - \frac{2s}{(s+1)!} \frac{i}{s!} \epsilon(\alpha_\gamma A) \hat{\gamma}(s-1) \]

\[ + \frac{2s}{s!} \epsilon(\alpha_\alpha \hat{\gamma} v_\gamma(s-1)) \hat{\alpha}(s-1) \]

\[ - \frac{i}{s!} \epsilon(\alpha_\alpha \hat{\gamma} u_\gamma(s-1)) \hat{\alpha}(s-1) \]

\[ + \frac{1}{s!} \epsilon(\alpha_\alpha \hat{\gamma} v_\gamma(s-1)) \hat{\alpha}(s-1) \]

\[ + \frac{i}{s!} \epsilon(\alpha_\alpha \hat{\gamma} U_\alpha(s-1)) \hat{\alpha}(s-1) \]

\[ + \frac{i}{s!} \epsilon(\alpha_\alpha \hat{\gamma} \hat{\alpha}(s-1)) \hat{\alpha}(s-1) \]

\[ + \frac{2s}{s!} \epsilon(\alpha_\alpha \hat{\gamma} \hat{\beta} \hat{\gamma}(s-1)) \hat{\alpha}(s-1) \]

\[ + \frac{2(s-1)^2}{s!} \epsilon(\alpha_\alpha \hat{\gamma} \hat{\alpha}(s-1)) \hat{\alpha}(s-1) \]

\[ \delta S \psi_\alpha(s+1) \hat{\alpha}(s) = \frac{\sqrt{2} i}{(s+1)!} \frac{s}{s+1} \frac{1}{s!} \epsilon(\alpha_{s+1} \hat{\gamma} h_\gamma(s)) \hat{\alpha}(s) \]

\[ - \frac{i}{\sqrt{2}(s+1)!} \epsilon(\alpha_{s+1} \hat{\gamma} h_\gamma(s)) \hat{\alpha}(s) \]

\[ + \frac{1}{2 \sqrt{2}} \frac{2s+1}{s+1} \frac{1}{s!} \epsilon(\alpha_{s+1} A) \hat{\alpha}(s) \]

\[ \delta S \psi_\alpha(s) \hat{\alpha}(s-1) = -\frac{1}{2 \sqrt{2}} \frac{s}{s+1} \frac{1}{s!} \epsilon(\alpha_\alpha \hat{\gamma} h_\gamma(s)) \hat{\alpha}(s) \]

\[ + \frac{1}{s!} \epsilon(\alpha_\alpha \hat{\gamma} h_\gamma(s)) \hat{\alpha}(s) \]

\[ + \frac{i}{s!} \epsilon(\alpha_{s-1} U_\alpha(s)) \hat{\alpha}(s-2) \]

\[ - \frac{i s}{\sqrt{2}} \epsilon(\alpha_{s-1} \hat{\gamma} h_\gamma(s)) \hat{\alpha}(s) \]

\[ - \frac{i s}{\sqrt{2}} \epsilon(\alpha_{s-1} \hat{\gamma} h_\gamma(s-1)) \hat{\alpha}(s-1) \]

\[ - \frac{i s}{\sqrt{2}} \epsilon(\alpha_{s-1} \hat{\gamma} h_\gamma(s-2)) \hat{\alpha}(s-2) \]}
and the SUSY-transformation laws for the bosonic fields are:

\[
\delta_S A_{\alpha(s)} \hat{a}_s = -\frac{i\sqrt{2}}{(s+1)!} \hat{\epsilon}^{s+1} \hat{\partial}^{s+1} (\hat{a}_{s+1} \psi_{\alpha(s+1)} \hat{a}_s) + c.c. \\
+ \frac{i\sqrt{2}}{s!} \frac{s^2}{(2s+1)(s+1)} \epsilon(\hat{\alpha}_s) \hat{\partial}^\gamma \hat{\psi}_{\alpha(s)} \hat{\gamma}(\hat{a}_{s+1}(\hat{a}_{s+1} \bar{\rho}_{\alpha(s-1)})) + c.c. \\
- \frac{i}{s!(s+1)!} 2s + 1 \epsilon^{s+1} \hat{\partial}_{(\hat{\alpha}_s \hat{\alpha}_{s+1} \bar{\rho}_{\alpha(s-1)})} \hat{\gamma}(\hat{a}_{s+1}(\hat{a}_{s+1} \bar{\rho}_{\alpha(s-1)})) + c.c. \\
- \frac{i\sqrt{2}}{s! (s+1)!} \frac{s}{2s+1} \epsilon(\hat{\alpha}_s) \hat{\partial}_{(\hat{\alpha}_s \hat{\alpha}_{s+1} \bar{\rho}_{\alpha(s-1)})} \hat{\gamma}(\hat{a}_{s+1}(\hat{a}_{s+1} \bar{\rho}_{\alpha(s-1)})) + c.c. \\
+ \frac{i\sqrt{2}}{s! (s+1)!} \frac{s}{2s+1} \epsilon(\hat{\alpha}_s) \hat{\partial}_{(\hat{\alpha}_s \hat{\alpha}_{s+1} \bar{\rho}_{\alpha(s-1)})} \hat{\gamma}(\hat{a}_{s+1}(\hat{a}_{s+1} \bar{\rho}_{\alpha(s-1)})) + c.c. \\
- \frac{i\sqrt{2}}{s! (s+1)!} \frac{s}{2s+1} \epsilon(\hat{\alpha}_s) \hat{\partial}_{(\hat{\alpha}_s \hat{\alpha}_{s+1} \bar{\rho}_{\alpha(s-1)})} \hat{\gamma}(\hat{a}_{s+1}(\hat{a}_{s+1} \bar{\rho}_{\alpha(s-1)})) + c.c. \\
\]

(47)

\[
\delta_S (u_{\alpha(s)} \hat{a}_s + i\nu_{\alpha(s)} \hat{a}_s) = -\frac{i\sqrt{2}}{s!} \epsilon(\hat{\alpha}_s) \hat{\partial}^\gamma \hat{\psi}_{(\hat{\gamma} \alpha(s-1))} \hat{\gamma}(\hat{a}_{s+1}(\hat{a}_{s+1} \bar{\rho}_{\alpha(s-1)})) + c.c. \\
+ \frac{i\sqrt{2}}{s! (s+1)!} \frac{s^2}{2s+1} \epsilon(\hat{\alpha}_s) \hat{\partial}^\gamma \hat{\psi}_{\alpha(s)} \hat{\gamma}(\hat{a}_{s+1}(\hat{a}_{s+1} \bar{\rho}_{\alpha(s-1)})) + c.c. \\
+ \frac{i\sqrt{2}}{s! (s+1)!} \frac{s}{2s+1} \epsilon(\hat{\alpha}_s) \hat{\partial}_{(\hat{\alpha}_s \hat{\alpha}_{s+1} \bar{\rho}_{\alpha(s-1)})} \hat{\gamma}(\hat{a}_{s+1}(\hat{a}_{s+1} \bar{\rho}_{\alpha(s-1)})) + c.c. \\
- \frac{i\sqrt{2}}{s! (s+1)!} \frac{s}{2s+1} \epsilon(\hat{\alpha}_s) \hat{\partial}_{(\hat{\alpha}_s \hat{\alpha}_{s+1} \bar{\rho}_{\alpha(s-1)})} \hat{\gamma}(\hat{a}_{s+1}(\hat{a}_{s+1} \bar{\rho}_{\alpha(s-1)})) + c.c. \\
+ \frac{i\sqrt{2}}{s! (s+1)!} \frac{s}{2s+1} \epsilon(\hat{\alpha}_s) \hat{\partial}_{(\hat{\alpha}_s \hat{\alpha}_{s+1} \bar{\rho}_{\alpha(s-1)})} \hat{\gamma}(\hat{a}_{s+1}(\hat{a}_{s+1} \bar{\rho}_{\alpha(s-1)})) + c.c. \\
\]

(48)
\[ \delta s U_{\alpha(s-1)\bar{\alpha}(s-2)} = i\sqrt{2} \epsilon^{\bar{\alpha} s - 1} \partial^{\alpha s + 1} \psi_{\alpha(s+1)\bar{\alpha}(s)} \]
\[ + \frac{i\sqrt{2}}{s!} \frac{2s + 1}{s(s + 1)} \epsilon^{\bar{\alpha} s - 1} \partial_{\alpha s} \bar{\psi}_{\alpha(s-1)\bar{\alpha}(s)} \]
\[ + \frac{i\sqrt{2}}{s!} \epsilon(\alpha_s \partial^\gamma \bar{\psi}_{\gamma\alpha(s-1)\bar{\alpha}(s-1)} \gamma \bar{\alpha}(s-2) \]
\[ + \frac{i\sqrt{2}}{s!} \epsilon(\alpha_{s-1} \partial^\gamma \bar{\psi}_{\gamma\alpha(s-2)\bar{\alpha}(s-1)} \gamma \bar{\alpha}(s-1)) \]
\[ - \frac{i\sqrt{2}}{s!(s-1)!} \frac{s + 1}{s} \epsilon^{\bar{\alpha} s - 1} \partial_{\alpha(s+1)\bar{\alpha}(s-1)} \bar{\psi}_{\alpha(s-1)\bar{\alpha}(s-2)} \]
\[ - \frac{i}{s!} \frac{1}{s + 1} \epsilon(\alpha_s \partial^\gamma \bar{\psi}_{\gamma\alpha(s-1)\bar{\alpha}(s-2)}) \]
\[ - \frac{i}{(s + 1)!} \epsilon^{\bar{\alpha} s - 1} \partial_{\alpha(s+1)\bar{\alpha}(s-1)} \bar{\psi}_{\alpha(s-1)\bar{\alpha}(s-2)} \]
\[ + \frac{i}{s!} \frac{1}{s + 1} \epsilon^{\bar{\alpha} s - 1} \partial_{\alpha s} \bar{\psi}_{\alpha(s-2)\bar{\alpha}(s-1)} \gamma \bar{\alpha}(s-2) \]
\[ - \frac{i\sqrt{2}}{(s-1)!} \frac{s + 1}{s} \epsilon^{\bar{\alpha} s - 1} \partial_{\alpha s} \bar{\psi}_{\alpha(s-2)\bar{\alpha}(s-1)} \gamma \bar{\alpha}(s-2) \]
5.2 Component structure for Longitudinal theories (II)

We repeat the same steps for the second formulation of half-integer super-helicity theories. The superspace action (25) can be expressed like

\[ S = \int d^{8}z \left\{ \frac{1}{2} H^{\alpha(s)\dot{\alpha}(s)} T_{\alpha(s)\dot{\alpha}(s)} \right. \\
+ \frac{1}{2} \chi^{\alpha(s-1)\dot{\alpha}(s-2)} G_{\alpha(s-1)\dot{\alpha}(s-2)} + c.c. \right\} \]

\[ = \int d^{4}x \left. \left\{ \frac{1}{2} D^{2}\tilde{D}^{2} \left( H^{\alpha(s)\dot{\alpha}(s)} T_{\alpha(s)\dot{\alpha}(s)} \right) \\
+ \frac{1}{2} D^{2}\tilde{D}^{2} \left( \chi^{\alpha(s-1)\dot{\alpha}(s-2)} G_{\alpha(s-1)\dot{\alpha}(s-2)} \right) + c.c. \right\} \]

where \( T, G \) are defined by (26)

5.2.1 Fermions

For the fermionic Lagrangian we have

\[ \mathcal{L}_{F} = \]

\[ = \frac{1}{2} \frac{1}{(s+1)!} D^{2}\tilde{D}^{2}(\dot{\alpha}_{s+1} H^{\alpha(s)\dot{\alpha}(s)}) | \frac{1}{(s+1)!} D_{\dot{\alpha}_{s+1}} T_{\alpha(s)\dot{\alpha}(s)} | \\
+ \left( \frac{1}{2} \frac{1}{s+1} D^{2}\tilde{D}^{2} \dot{H}^{\alpha(s)\dot{\alpha}(s-1)} - \frac{1}{2} \frac{1}{s!} D^{\alpha_{s}} \tilde{D}^{\dot{\alpha}_{s} - 1} \chi^{\alpha(s-1)\dot{\alpha}(s-2)} \right) \\
- \frac{i}{2} \frac{1}{s!} D^{\alpha_{s}} \partial_{\gamma} \dot{H^{\gamma\alpha(s-1)\dot{\alpha}(s-1)}} | \frac{1}{s!} D_{\alpha_{s}} \tilde{D}_{\dot{\alpha}_{s-1}} G_{\alpha(s-1)\dot{\alpha}(s-2)} | \\
+ \left( \frac{i}{2} \frac{1}{s} \frac{1}{s-1} \frac{1}{s!} D_{\dot{\gamma}} \chi^{\alpha(s-2)\dot{\alpha}(s-2)} \right) | \frac{1}{(s-1)!} D^{\alpha_{s-1}} \tilde{D}_{\dot{\alpha}_{s-1}} G_{\alpha(s-1)\dot{\alpha}(s-2)} | \\
+ \left( \frac{i}{2} \frac{1}{s} \frac{1}{s+1} \partial_{\alpha_{s}} \dot{H}^{\beta\gamma \alpha(s-2)\dot{\alpha}(s-1)} \right) | \frac{1}{(s-1)!} D^{\alpha_{s-1}} \tilde{D}_{\dot{\alpha}_{s-1}} G_{\alpha(s-1)\dot{\alpha}(s-2)} | \\
+ \left( \frac{i}{2} \frac{1}{s} \frac{1}{s+1} \partial_{\alpha_{s}} H^{\alpha(s)\dot{\alpha}(s)} \right) | D_{\dot{\alpha}_{s-1}} G_{\alpha(s-1)\dot{\alpha}(s-2)} | \\
+ \frac{1}{2} D^{2}\chi^{\alpha(s-1)\dot{\alpha}(s-2)} | D^{2} G_{\alpha(s-1)\dot{\alpha}(s-2)} | \\
+ c.c. \]

20
We can prove the following identities for $T$ and $G$:

\[
\frac{1}{(s+1)!} \hat{D}(\hat{\alpha}_{s+1} T_{(s)} \hat{\alpha}(s)) = \\
= \frac{2ic}{(s+1)!} \partial^{\hat{\alpha}_{s+1}} \left\{ \frac{1}{(s+1)!} \hat{D}^{2} \partial^{\hat{\alpha}} (\hat{\alpha}_{s+1} H_{(s)} \hat{\alpha}(s)) \right\} \tag{55}
\]

\[
- \frac{2ic}{s! (s+1)! (2s+1)(s+1)} \partial^{\hat{\alpha}_{s}} \left\{ \begin{aligned}
&\hat{D}^{2} \partial^{\hat{\gamma}} H_{\hat{\alpha}(s-1)} \hat{\alpha}(s) \\
+ &\frac{i(s+1)}{s!} \partial^{\hat{\alpha}} \partial^{\hat{\gamma}} H_{\hat{\alpha}(s-1)} \hat{\alpha}(s-1) \\
+ &\frac{s+1}{s!(s-1)!} \hat{D}(\hat{\alpha}_{s-1} \hat{\alpha}(s)) \hat{\alpha}(s-1) \\
\end{aligned} \right\} \tag{56}
\]

\[
\frac{1}{s!(s-1)!} \hat{D}(\hat{\alpha}_{s} \hat{D}(\hat{\alpha}_{s-1} G_{(s)} \hat{\alpha}(s-2))) = \\
= - \frac{2ic}{s! (s+1)(s+1)} \partial^{\hat{\alpha}_{s}} \left\{ \begin{aligned}
&\hat{D}^{2} \partial^{\hat{\gamma}} H_{\hat{\alpha}(s-1)} \hat{\alpha}(s) \\
+ &\frac{i(s+1)}{s!} \hat{D}(\hat{\alpha}_{s} \partial^{\hat{\gamma}} H_{\hat{\alpha}(s-1)} \hat{\alpha}(s-1)) \\
+ &\frac{s+1}{s!(s-1)!} \hat{D}(\hat{\alpha}_{s-1} \hat{\alpha}(s)) \hat{\alpha}(s-1) \\
\end{aligned} \right\} \tag{56}
\]

\[
\frac{1}{(s-1)!} \hat{D}(\hat{\alpha}_{s-1} \hat{D}(\hat{\alpha}_{s-1} \hat{G}_{(s-2)} \hat{\alpha}(s-1))) = \\
= - \frac{s}{s+1} \hat{D}^{2} G_{\hat{\alpha}(s-1)} \hat{\alpha}(s-2) \\
+ \frac{i}{s!(s-1)! (s+1)^{2}} \partial^{\hat{\alpha}_{s-1}} \hat{G}_{\hat{\alpha}(s-2)} \hat{\alpha}(s-1) \\
- \frac{2ic}{(2s+1)(s+1)} \partial^{\hat{\alpha}_{s-1} \hat{\alpha}_{s-1}} \left\{ \begin{aligned}
&\hat{D}^{2} \hat{D} \hat{\alpha}(s) \hat{\alpha}(s-1) \\
+ &\frac{i(s+1)}{s!} \hat{D}(\hat{\alpha}_{s} \hat{\alpha}) \hat{\alpha}(s) \hat{\alpha}(s-1) \\
+ &\frac{s+1}{s!(s-1)!} \hat{D}(\hat{\alpha}_{s-1} \hat{\alpha}(s)) \hat{\alpha}(s-2) \\
\end{aligned} \right\} \tag{57}
\]

\[
+ \frac{2ic}{(s+1)^{2} (s-1)!} \partial^{\hat{\alpha}_{s-1} \hat{\alpha}_{s-1}} \left\{ \begin{aligned}
&i\hat{D}^{2} \partial^{\hat{\gamma}} H_{\hat{\beta} \hat{\gamma} \hat{\alpha}(s-2)} \hat{\alpha}(s-1) \\
+ &\frac{1}{s!(s-1)!} \hat{D}^{2} \hat{D}(\hat{\alpha}_{s-1} \hat{\alpha}(s)) \hat{\alpha}(s-2) \\
\end{aligned} \right\} \tag{57}
\]
Let us define the following fields

\[ \frac{1}{(s+1)!} \hat{D}^{2} \hat{D}(\alpha_{s+1} H_{\alpha(s)} \hat{\alpha}(s)) \equiv N_{1} \psi_{\alpha(s+1)} \hat{\alpha}(s) \]

\[ D^{2} \hat{D}^{\alpha s} H_{\alpha(s)} \hat{\alpha}(s) + \frac{i(s+1)}{s!} D(\alpha_{s} \partial^{\gamma} H) \gamma_{\alpha(s-1)} \hat{\gamma}(s-1) \]
\[ + \frac{s+1}{s!(s-1)!} D(\alpha_{s} \hat{D}(\alpha_{s-1} \chi_{\alpha(s-1)} \hat{\alpha}(s-2))) \]
\[ \equiv N_{2} \psi_{\alpha(s)} \hat{\alpha}(s-1) \]

\[ i \hat{D}^{\delta} \partial^{\gamma} H_{\gamma_{\alpha(s-1)} \hat{\beta} \hat{\gamma} \hat{\alpha}(s-2)} \]
\[ + \frac{1}{(s-1)!} \hat{D}^{\alpha s-1} D(\alpha_{s-1} \chi_{\alpha(s-2)} \hat{\alpha}(s-1)) \]
\[ \equiv N_{3} \psi_{\alpha(s-1)} \hat{\alpha}(s-2) \]

Putting everything together, the component Lagrangian takes the form

\[ \mathcal{L}_{F} = 2ic|N_{1}|^{2} \bar{\psi}^{\alpha(s)} \hat{\alpha}(s+1) \partial^{\alpha s+1} \alpha_{s+1} \psi_{\alpha(s)} \hat{\beta}(s) \]
\[ - 2ic \frac{s^{2}}{(2s+1)(s+1)} N_{1} N_{2} \bar{\psi}^{\alpha(s+1)} \hat{\alpha}(s) \partial_{\alpha_{s+1}} \chi_{\alpha(s)} \hat{\alpha}(s-1) + c.c. \]
\[ - 2ic \frac{s^{2}}{(2s+1)(s+1)} N_{2}^{2} \bar{\psi}^{\alpha(s)} \hat{\alpha}(s) \partial_{\alpha_{s}} \chi_{\alpha(s)} \hat{\alpha}(s-1) \]
\[ - 2ic \frac{s(s-1)}{(2s+1)(s+1)} N_{2} N_{3} \bar{\psi}^{\alpha(s)} \hat{\alpha}(s-1) \partial_{\alpha_{s}} \chi_{\alpha(s-1)} \hat{\alpha}(s-2) + c.c. \]
\[ - 2ic \left( \frac{s-1}{s+1} \right)^{2} N_{3}^{2} \bar{\psi}^{\alpha(s-2)} \hat{\alpha}(s-1) \partial_{\alpha_{s-1}} \chi_{\alpha(s-1)} \hat{\alpha}(s-2) \]
\[ + \frac{1}{2c} \frac{(2s+1)(s-1)}{s^{2}(s+1)^{2}} G_{\alpha(s)} \hat{\alpha}(s-1) \]
\[ \left( D^{2} G_{\alpha(s)} \hat{\alpha}(s-2) - \frac{i s - 1}{2 s + 1} \frac{1}{(s-1)!} \partial_{\alpha_{s-1}} \hat{D}(\alpha_{s-1}) \hat{G}_{\alpha(s)} \hat{\alpha}(s-1) \right) \]
\[ + c.c. \]

The last term in the Lagrangian is the algebraic kinetic energy term of two auxiliary fields and the rest of the terms are exactly the structure of a theory that describes helicity \( h = s + 1/2 \). To have an exact match we choose coefficients

\[ c = 1, \quad N_{2} = - \frac{1}{\sqrt{2}} \frac{2s+1}{s} \]
\[ N_{1} = \frac{1}{\sqrt{2}}, \quad N_{3} = \frac{1}{\sqrt{2}} \frac{s+1}{s-1} \]
So the fields that appear in the fermionic action are defined as:

\[ \rho_{\alpha(s-1)\dot{\alpha}(s-2)} \equiv G_{\alpha(s-1)\dot{\alpha}(s-2)} \]

\[ \beta_{\alpha(s-1)\dot{\alpha}(s-2)} \equiv \left\{ \begin{array}{l}
d - \frac{i}{2} \frac{s - 1}{s + 1} (s - 1)! \partial_{(\alpha s - 1) \dot{\alpha}(s-2)} G_{\alpha(s-1)\dot{\alpha}(s-2)} \\
\end{array} \right\} \]

\[ \psi_{\alpha(s+1)\dot{\alpha}(s)} \equiv \frac{\sqrt{s}}{(s + 1)!} \bar{D}^2 D_{(\alpha s + 1) H(s)\dot{\alpha}(s)} \]

\[ \psi_{\alpha(s)\dot{\alpha}(s-1)} \equiv -\sqrt{s} \frac{s}{2s + 1} \left\{ \bar{D}^2 \bar{D}_{\alpha(s)\dot{\alpha}(s)} + \frac{i(s + 1)}{s} \partial_{(\alpha s + 1) \dot{\alpha}(s-1)} H_{\alpha(s-1)\dot{\alpha}(s-1)} + \frac{s + 1}{s! (s - 1)!} \partial_{(\alpha s - 1) \dot{\alpha}(s-2)} \right\} \]

\[ \psi_{\alpha(s-1)\dot{\alpha}(s-2)} \equiv \sqrt{s} \frac{s - 1}{2s + 1} \left\{ i \bar{D} \gamma \partial_{\alpha(s-1) \dot{\alpha}(s-2)} \right\} + \frac{1}{(s - 1)!} \bar{D}^{\dot{\alpha}(s-2)} D_{(\alpha s - 1) \dot{\alpha}(s-2)} \psi_{\alpha(s-1)\dot{\alpha}(s-1)} \]

The Lagrangian is

\[ \mathcal{L}_F = \rho^{\alpha(s)\dot{\alpha}(s-1)} \beta_{\alpha(s)\dot{\alpha}(s-1)} + \text{c.c.} \]

\[ + i \psi^{\alpha(s)\dot{\alpha}(s-1)} \partial_{\alpha s + 1} \psi_{\alpha(s)\dot{\alpha}(s-1)} \]

\[ + i \left( \frac{s}{s + 1} \right) \psi^{\alpha(s)\dot{\alpha}(s-1)} \partial_{(\alpha s + 1) \dot{\alpha}(s-1)} \psi_{\alpha(s)\dot{\alpha}(s-1)} + \text{c.c.} \]

\[ - i \left( \frac{2s + 1}{2(s + 1)^2} \right) \bar{\psi}^{\alpha(s-1)\dot{\alpha}(s)} \partial_{\alpha s} \bar{\psi}_{\alpha(s)\dot{\alpha}(s-1)} \]

\[ + i \bar{\psi}^{\alpha(s)\dot{\alpha}(s-1)} \partial_{\alpha s \dot{\alpha}(s-1)} \bar{\psi}_{\alpha(s)\dot{\alpha}(s-1)} \]

\[ - i \bar{\psi}^{\alpha(s-2)\dot{\alpha}(s-1)} \partial_{\alpha s} \psi_{\alpha(s-1)\dot{\alpha}(s-1)} \]

and the gauge transformations of the fields are

\[ \delta_G \rho_{\alpha(s)\dot{\alpha}(s-1)} = 0 \]

\[ \delta_G \psi_{\alpha(s)\dot{\alpha}(s-1)} = \frac{1}{s! (s + 1)!} \partial_{(\alpha s + 1) \dot{\alpha}(s-1)} \psi_{\alpha(s)\dot{\alpha}(s-1)} \]

\[ \delta_G \beta_{\alpha(s)\dot{\alpha}(s-1)} = 0 \]

\[ \delta_G \psi_{\alpha(s)\dot{\alpha}(s-1)} = \frac{1}{s!} \partial_{(\alpha s) \dot{\alpha}(s-1)} \xi_{\alpha(s)\dot{\alpha}(s-1)} \]

\[ \delta_G \psi_{\alpha(s-1)\dot{\alpha}(s-2)} = \frac{s - 1}{s} \partial_{\alpha s \dot{\alpha}(s-1)} \xi_{\alpha(s)\dot{\alpha}(s-1)} \]

with \( \xi_{\alpha(s)\dot{\alpha}(s-1)} = -i \sqrt{2} \bar{D}^2 L_{\alpha(s)\dot{\alpha}(s-1)} \)
5.2.2 Bosons

For the bosonic Lagrangian we do the same. The fields that appear in the action are defined as:

\[
\begin{align*}
A_\alpha(s)\dot{\alpha}(s) & \equiv T_\alpha(s)\dot{\alpha}(s) \\
U_\alpha(s)\dot{\alpha}(s-2) & \equiv \frac{1}{s!} D_{(\alpha_s G_\alpha(s-1))}\dot{\alpha}(s-2) \\
u_{\alpha(s-1)}\dot{\alpha}(s-1) & \equiv \frac{1}{2(s-1)!} \left\{ \tilde{D}_{(\dot{\alpha}_{s-1} G_\alpha(s-1)\dot{\alpha}(s-2))} - D_{(\alpha_{s-1} \tilde{G}_\alpha(s-2))}\dot{\alpha}(s-1) \right\} \\
v_{\alpha(s-1)}\dot{\alpha}(s-1) & \equiv -\frac{i}{2(s-1)!} \left\{ \tilde{D}_{(\dot{\alpha}_{s-1} G_\alpha(s-1)\dot{\alpha}(s-2))} + D_{(\alpha_{s-1} \tilde{G}_\alpha(s-2))}\dot{\alpha}(s-1) \right\} \\
S_{\alpha(s-2)}\dot{\alpha}(s-2) & \equiv \frac{1}{2} \left\{ D^\alpha_{s-1} G_\alpha(s-1)\dot{\alpha}(s-2) + \tilde{D}^\alpha_{s-1} \tilde{G}_\alpha(s-2)\dot{\alpha}(s-1) \right\} \\
P_{\alpha(s-2)}\dot{\alpha}(s-2) & \equiv -\frac{i}{2} \left\{ D^\alpha_{s-1} G_\alpha(s-1)\dot{\alpha}(s-2) - \tilde{D}^\alpha_{s-1} \tilde{G}_\alpha(s-2)\dot{\alpha}(s-1) \right\} \\
h_{\alpha(s+1)}\dot{\alpha}(s+1) & \equiv \frac{1}{2(s+1)!^2} \left[ D_{(\alpha_{s+1}, \tilde{D}_{\dot{\alpha}_{s+1}})} H_\alpha(s)\dot{\alpha}(s) \right] \\
h_{\alpha(s-1)}\dot{\alpha}(s-1) & = -\frac{1}{2(s+1)(s+1)} \left[ D^\alpha_{s+1}, \tilde{D}^\alpha_{s+1} \right] H_\alpha(s)\dot{\alpha}(s) \\
& \quad -\frac{1}{(s+1)(s+1)} \frac{1}{(s-1)!} \left( D_{(\alpha_{s-1}, \tilde{\alpha}_\alpha(s-2))}\dot{\alpha}(s-1) \right. \\
& \quad \left. -\tilde{D}_{(\dot{\alpha}_{s-1}, \chi_\alpha(s-1)\dot{\alpha}(s-2))} \right) \\
\end{align*}
\]

the gauge transformations are

\[
\begin{align*}
\delta_G U_\alpha(s)\dot{\alpha}(s-2) & = 0, \quad \delta_G A_\alpha(s)\dot{\alpha}(s) = 0 \\
\delta_G u_{\alpha(s-1)}\dot{\alpha}(s-1) & = 0, \quad \delta_G S_{\alpha(s-2)}\dot{\alpha}(s-2) = 0 \\
\delta_G v_{\alpha(s-1)}\dot{\alpha}(s-1) & = 0, \quad \delta_G P_{\alpha(s-2)}\dot{\alpha}(s-2) = 0 \\
\delta_G h_{\alpha(s+1)}\dot{\alpha}(s+1) & = \frac{1}{(s+1)!^2} \partial_{(\alpha_{s+1}, \tilde{\alpha}_\alpha(s))}\dot{\alpha}(s) \\
\delta_G h_{\alpha(s-1)}\dot{\alpha}(s-1) & = \frac{s}{(s+1)!^2} \partial^\alpha_{s+1} \zeta_\alpha(s)\dot{\alpha}(s) \\
\end{align*}
\]

where

\[
\zeta_\alpha(s)\dot{\alpha}(s) = \frac{i}{2s!} \left( D_{(\alpha_s L_\alpha(s-1))}\dot{\alpha}(s) + \tilde{D}_{(\dot{\alpha}_s L_\alpha(s)\dot{\alpha}(s-1))} \right) 
\]
and the bosonic Lagrangian is
\[
\mathcal{L}_B = -\frac{1}{4} \left[ \frac{(2s + 1)(s - 1)}{s^2(s + 1)} \right] U^\alpha(s)\dot{\alpha}(s-2)U_{\dot{\alpha}(s)} + \text{c.c.}
\]
\[
+ \frac{1}{8} \left[ \frac{2s + 1}{s + 1} \right] A^\alpha(s)\dot{\alpha}(s)A_{\dot{\alpha}(s)}
\]
\[
- \frac{1}{2} \left[ \frac{2s + 1}{s^2} \right] u^{\alpha(s-1)}\dot{\alpha}(s-1)u_{\alpha(s-1)}\dot{\alpha}(s-1)
\]
\[
- \frac{1}{2} \left[ \frac{2s + 1}{s^2} \right] v^{\alpha(s-1)}\dot{\alpha}(s-1)v_{\alpha(s-1)}\dot{\alpha}(s-1)
\]
\[
- \frac{1}{2} \left[ \frac{(2s + 1)(s - 1)^2}{s^3} \right] S^\alpha(s-2)\dot{\alpha}(s-2)S_{\dot{\alpha}(s-2)}
\]
\[
+ \frac{1}{2} \left[ \frac{(s - 1)^2}{s^3} \right] P^\alpha(s-2)\dot{\alpha}(s-2)P_{\dot{\alpha}(s-2)}
\]
\[
+ h^{\alpha(s+1)}\dot{\alpha}(s+1)\Box h_{\alpha(s+1)}\dot{\alpha}(s+1)
\]
\[
- \left[ \frac{s + 1}{2} \right] h^{\alpha(s+1)}\dot{\alpha}(s+1)\partial_{\alpha s+1}\dot{\alpha}_{s+1} \gamma^\gamma h_{\gamma\alpha(s)}\dot{\alpha}(s)
\]
\[
+ \left[ s(s + 1) \right] h^{\alpha(s+1)}\dot{\alpha}(s+1)\partial_{\alpha s+1}\dot{\alpha}_{s+1} \partial_{\alpha s}\dot{\alpha}(s+1)
\]
\[
- \left[ (s + 1)(2s + 1) \right] h^{\alpha(s+1)}\dot{\alpha}(s+1)\Box h_{\alpha(s)}\dot{\alpha}(s+1)
\]
\[
- \left[ \frac{(s + 1)(s - 1)^2}{2} \right] h^{\alpha(s-1)}\dot{\alpha}(s-1)\partial_{\alpha s-1}\dot{\alpha}_{s-1} \gamma^\gamma h_{\gamma\alpha(s-2)}\dot{\alpha}(s-2)
\]
and gives rise to the theory of helicity \( h = s + 1 \) as expected

5.2.3 Off-shell degrees of freedom

Let us count the bosonic degrees of freedom

| fields | d.o.f | redundancy | net |
|--------|-------|------------|-----|
| \( h_{\alpha(s+1)}\dot{\alpha}(s+1) \) | \((s + 2)^2\) | \((s + 1)^2\) | \(s^2 + 2s + 3\) |
| \( h_{\alpha(s-1)}\dot{\alpha}(s-1) \) | \(s^2\) | 0 | \(s^2\) |
| \( u_{\alpha(s-1)}\dot{\alpha}(s-1) \) | \(s^2\) | 0 | \(s^2\) |
| \( v_{\alpha(s-1)}\dot{\alpha}(s-1) \) | \(s^2\) | 0 | \(s^2\) |
| \( A_{\alpha(s)}\dot{\alpha}(s) \) | \((s + 1)^2\) | 0 | \((s + 1)^2\) |
| \( U_{\alpha(s)}\dot{\alpha}(s-2) \) | \(2(s + 1)(s - 1)\) | 0 | \(2(s + 1)(s - 1)\) |
| \( S_{\alpha(s-2)}\dot{\alpha}(s-2) \) | \((s - 1)^2\) | 0 | \((s - 1)^2\) |
| \( P_{\alpha(s-2)}\dot{\alpha}(s-2) \) | \((s - 1)^2\) | 0 | \((s - 1)^2\) |

and the same counting for the fermionic degrees of freedom

| | | | |
|---|---|---|---|
| Total | | | 8s^2 + 4 |
5.2.4 SUSY-transformation laws

The explicit expressions for the SUSY-transformation laws of the fields can be found in the same way as for case (I). For the fermionic fields:

\[ \delta S \rho_{\alpha(s-1)\dot{a}(s-2)} = -\epsilon^{\alpha s} U_{\alpha(s)\dot{a}(s-2)} \]

\[ + \left[ \frac{s-1}{s} \right] \frac{1}{(s-1)!} \epsilon_{(a_{s-1})} \left[ S_{\alpha(s-2)\dot{a}(s-2) + iP_{\alpha(s-2)\dot{a}(s-2)}} \right] \]

\[ -\epsilon^{\alpha s_{-1}} \left[ u_{\alpha(s-1)\dot{a}(s-1)} + iv_{\alpha(s-1)\dot{a}(s-1)} \right] \]

\[ \delta S \beta_{\alpha(s-1)\dot{a}(s-2)} = \]

\[ = \frac{i s^2}{2 s + 1} \epsilon^{\alpha s_{-1}} \partial^{\alpha s} A_{\alpha(s)\dot{a}(s)} \]

\[ + \frac{s^2}{2 s + 1} \epsilon^{\alpha s_{-1}} \partial^{\alpha s} \dot{\alpha} h_{\alpha(s+1)\dot{a}(s+1)} \]

\[ - 2s \epsilon^{\alpha s_{-1}} \partial h_{\alpha(s-1)\dot{a}(s-2)} \]

\[ - \frac{s(s-1)^2}{2 s + 1} \frac{1}{(s-1)!} \epsilon^{\alpha s_{-1}} \partial^{\alpha s_{-1}} \dot{a}_{s_{-1}} \beta_{\alpha(s-2)} \]

\[ - \frac{i}{(s-1)!} \epsilon^{\alpha s_{-1}} \partial^{\alpha s_{-1}} U_{\alpha(s)\dot{a}(s-2)} \]

\[ + \frac{s - 2}{s - 1} \frac{i}{(s - 2)!} \epsilon^{\alpha s_{-2}} \partial^{\alpha s_{-2}} U_{\beta(s-4)\dot{b}(s-3)} \]

\[ + \frac{1}{2 s + 1} \frac{i}{(s - 1)!} \epsilon^{\alpha s_{-1}} \partial^{\alpha s_{-1}} U_{\alpha(s-2)\dot{a}(s)} \]

\[ + \frac{(s - 1)(2s^2 + 2s + 1)}{2s(s + 1)} \frac{i}{(s - 1)!} \epsilon^{\alpha s_{-1}} \partial^{\alpha s_{-1}} S_{\alpha(s-2)\dot{a}(s-2)} \]

\[ - \frac{(s - 1)(2s^2 + 4s + 3)}{2s(s + 1)(2s + 1)} \frac{1}{(s - 1)!} \epsilon^{\alpha s_{-1}} \partial^{\alpha s_{-1}} P_{\alpha(s-2)\dot{a}(s-2)} \]

\[ - \frac{(s - 2)(3s + 2)}{2s(s + 1)} \frac{i}{(s - 2)!} \epsilon^{\alpha s_{-2}} \partial^{\alpha s_{-2}} P_{\alpha(s-2)\dot{b}(s-3)} \]

\[ + \frac{(s - 2)(s + 2)}{2s(s + 1)} \frac{1}{(s - 2)!} \epsilon^{\alpha s_{-2}} \partial^{\alpha s_{-2}} P_{\alpha(s-2)\dot{b}(s-3)} \]

\[ - \frac{1}{2 s + 1} \frac{i}{(s - 1)!} \epsilon^{\beta} \partial^{\alpha s_{-1}} \dot{a}_{s_{-1}} \beta_{\alpha(s-2)} \]

\[ - \frac{1}{2 s + 1} \frac{1}{(s - 1)!} \epsilon^{\beta} \partial^{\alpha s_{-1}} \dot{a}_{s_{-1}} \beta_{\alpha(s-2)} \]
\[
\delta S\psi_{\alpha(s+1)}\dot{\alpha}(s) = \frac{\sqrt{2}i}{(s+2)!}\xi^{\alpha_{s+2}}\partial_{(\alpha_{s+2}}\dot{\alpha}_{s+1)\hat{h}_{\alpha(s+1)}}\dot{\alpha}(s+1) \\
- \frac{1}{\sqrt{2}}\frac{s + 1}{s + 1}i\epsilon(\alpha_{s+1})\partial^{\gamma\gamma}h_{\gamma\alpha(s)}\dot{\gamma}\dot{\alpha}(s) \\
+ \frac{1}{2\sqrt{2}}\frac{s + 1}{s + 1}i\epsilon(\alpha_{s+1})\epsilon(A_{\alpha(s)})\dot{\alpha}(s) \\
\]

\[
\delta S\psi_{\alpha(s)}\dot{\alpha}(s-1) = \frac{1}{\sqrt{2}}\frac{s + 1}{s + 1}\epsilon(\alpha_s)\left[-u_{\alpha(s-1)}\dot{\alpha}(s-1) + i\nu_{\alpha(s-1)}\dot{\alpha}(s-1)\right] \\
+ \frac{1}{s - 1}i\epsilon(\alpha_s)\left[\tilde{U}_{\alpha}(s)\dot{U}(s-2)\right] \\
- \frac{1}{\sqrt{2}}\frac{s + 1}{s + 1}\epsilon(\alpha_s)A_{\alpha(s)}\dot{\alpha}(s) \\
- \frac{i\epsilon(\alpha_{s+1})\dot{\alpha}_{s+1}h_{\alpha(s+1)}\dot{\alpha}(s+1)}{\sqrt{2}s!} \\
+ \frac{i\epsilon(\alpha_{s+1})\dot{\alpha}_{s+1}h_{\alpha(s+1)}\dot{\alpha}(s+1)}{\sqrt{2}s!} \\
\]

\[
\delta S\psi_{\alpha(s-1)}\dot{\alpha}(s-2) = \frac{1}{\sqrt{2}}\frac{(2s + 1)(s - 1)}{s^2(s + 1)}\epsilon(\dot{\alpha}_{s-1})h_{\alpha(s-1)}\dot{\alpha}(s-1) \\
- \frac{i}{\sqrt{2}}\frac{(2s + 1)(s - 1)}{s^2(s + 1)}\epsilon(\dot{\alpha}_{s-1})h_{\alpha(s-1)}\dot{\alpha}(s-1) \\
- \frac{1}{\sqrt{2}}\frac{(s - 1)^2(2s + 1)}{s^2(s + 1)}\epsilon(\alpha_{s-1})S_{\alpha(s-2)}\dot{\alpha}(s-2) \\
+ \frac{i}{\sqrt{2}}\frac{(s - 1)^2}{s^2(s + 1)}\epsilon(\alpha_{s-1})P_{\alpha(s-2)}\dot{\alpha}(s-2) \\
+ \frac{i}{\sqrt{2}}\frac{(s - 1)^2}{s^2(s + 1)}\epsilon(\alpha_{s-1})\partial^{\gamma\gamma}h_{\gamma\alpha(s-2)}\dot{\gamma}\dot{\alpha}(s-2) \\
\]

and the SUSY-transformation laws for the bosonic fields are:

\[
\delta S\epsilon_{\alpha(s)}\dot{\alpha}(s) = \frac{i\sqrt{2}}{(s + 1)!}\epsilon(\dot{\alpha}_{s+1})\epsilon(\alpha_{s+1})\epsilon(\alpha_{s+1})\dot{\alpha}(s) + c.c. \\
+ \frac{i\sqrt{2}}{s!}\frac{s}{(s + 1)(2s + 1)}\epsilon(\dot{\alpha}_{s+1})\partial^{\gamma\gamma}\gamma_{\alpha(s)}\dot{\gamma}\dot{\alpha}(s-1) + c.c. \\
+ \frac{i\sqrt{2}}{(s + 1)!s!}\frac{s}{s + 1}\epsilon(\dot{\alpha}_{s+1})\partial_{(\alpha_{s+1})}\dot{\alpha}(s) + c.c. \\
+ \frac{i\sqrt{2}}{s!}\frac{s}{s + 1}\epsilon(\dot{\alpha}_{s+1})\partial_{(\alpha_{s+1})}\dot{\alpha}(s) + c.c. \\
+ \frac{i\sqrt{2}}{s!}\frac{s}{s + 1}\epsilon(\dot{\alpha}_{s+1})\partial_{(\alpha_{s+1})}\dot{\alpha}(s) + c.c. \\
\]

\[
- \frac{i}{s!}\frac{s}{s + 1}\epsilon(\dot{\alpha}_{s+1})\partial_{(\alpha_{s+1})}\dot{\alpha}(s) + c.c. \\
- \frac{i}{s!}\frac{s}{s + 1}\epsilon(\dot{\alpha}_{s+1})\partial_{(\alpha_{s+1})}\dot{\alpha}(s) + c.c. \\
\]

27
\[
\delta S U_{\alpha(s)\dot{\alpha}(s-2)} = \frac{1}{s!} \epsilon(\alpha, \beta) \delta (s-1) \dot{\alpha}(s-2) \\
- \frac{i}{s(s-1)!} \epsilon^{\alpha\delta \beta} \partial_{\alpha_s} (\dot{\alpha}_{s-1} \beta_{s-1}) \dot{\alpha}(s-2) \\
+ \frac{s - 2}{s!} \epsilon(\alpha_s, \beta_s) \partial_{\alpha_{s-1}} \beta_{s-1} \dot{\alpha}(s-3) \\
+ \frac{i}{s(s-2)!} \epsilon(\alpha, \beta) \delta_{s-1} \beta_{s-1} \dot{\alpha}(s-1) \\
-i\sqrt{2} \frac{s^2}{2s + 1} \epsilon^{\alpha s} \partial \gamma_{s-1} \gamma_{s+1} \psi_{\alpha(s+1)} \dot{\alpha}(s) \\
- i\sqrt{2} \frac{s}{s!} \frac{1}{s + 1} \epsilon^{\alpha s} \partial_{\alpha_s} \psi_{\alpha(s-1)} \dot{\alpha}(s) \\
+ \frac{i\sqrt{2}}{s(s-1)!} \epsilon^{\alpha s} \partial_{\alpha_{s-1}} \psi_{\alpha(s-1)} \dot{\alpha}(s-2) \\
\]

\[
\delta S \left( u_{\alpha(s-1)} \dot{\alpha}(s-1) + iv_{\alpha(s-1)} \dot{\alpha}(s-1) \right) = \\
= i\sqrt{2} \frac{s^2}{2s + 1} \epsilon^{\alpha \delta \gamma} \partial_{\alpha_s} \gamma_{s+1} \psi_{\alpha(s+1)} \dot{\alpha}(s) \\
+ i\sqrt{2} \frac{s}{s!} \frac{1}{s + 1} \epsilon^{\alpha s} \partial_{\alpha_s} \gamma_{s+1} \psi_{\alpha(s+1)} \dot{\alpha}(s) \\
- i\sqrt{2} \frac{s}{s(s-1)!} \frac{1}{(s+1)(s-1)!} \epsilon(\alpha_{s-1}, \beta_{s-1}) \dot{\alpha}(s-1) \\
- i\sqrt{2} \frac{s}{s^2} \frac{1}{(s+1)(s-1)!} \epsilon(\alpha_{s-1}, \gamma_{s+1}) \dot{\alpha}(s-2) \\
+ i\sqrt{2} \frac{s^2}{s(s+1)(s-1)!} \epsilon(\alpha_{s-1}, \gamma_{s+1}) \dot{\alpha}(s-2) \\
\]

\[
\delta S \left( S_{\alpha(s-2)} \dot{\alpha}(s-2) + iP_{\alpha(s-2)} \dot{\alpha}(s-2) \right) = \\
= \epsilon^{\alpha \delta \gamma} \partial_{\alpha_s} \gamma_{s+1} \psi_{\alpha(s+1)} \dot{\alpha}(s-2) \\
+ \frac{i}{2} \frac{s - 1}{s + 1} \frac{1}{(s-1)!} \epsilon(\alpha_{s-1}, \beta_{s-1}) \dot{\alpha}_{s-1} \partial_{\alpha_s} \gamma_{s+1} \psi_{\alpha(s+1)} \dot{\alpha}(s-2) \\
- \frac{i}{(s-1)!} \epsilon^{\alpha \delta \gamma} \partial_{\alpha_s} \gamma_{s+1} \psi_{\alpha(s+1)} \dot{\alpha}(s-2) \\
- \frac{i}{s(s-2)!} \epsilon(\alpha_{s-2}, \gamma_{s+1}) \rho_{s+1} \partial_{\alpha_{s-1}} \beta_{s-2} \dot{\alpha}(s-1) \\
- i\sqrt{2} \frac{s}{s(s-1)!} \epsilon^{\alpha \delta \gamma} \partial_{\alpha_s} \gamma_{s+1} \psi_{\alpha(s+1)} \dot{\alpha}(s-2) \\
\]

(70)
\[ \delta S_{h \alpha(s+1)\dot{\alpha}(s+1)} = \frac{1}{\sqrt{2(s+1)!}} \epsilon(\alpha_{s+1} \bar{\psi}_\alpha(s))\dot{\alpha}(s+1) + \text{c.c.} \]  

(73)

\[ \delta S_{h \alpha(s-1)\dot{\alpha}(s-1)} = \frac{1}{\sqrt{2(s+1)!}} \epsilon(\alpha_{s-1} \bar{\psi}_\alpha(s-2))\dot{\alpha}(s-1) + \text{c.c.} \]  

(74)

6 Hints for \( \mathcal{N} = 2 \)

The massless irreducible representations of 4D, \( \mathcal{N} = 2 \) Super-Poincaré group for super-helicity \( Y \) describe helicities \( \lambda = Y + 1, \lambda = Y + 1/2, \lambda = Y + 1/2, \lambda = Y \). At least on-shell that looks like the direct sum of two \( \mathcal{N} = 1 \) massless irreducible representations, one describing super-helicity \( Y + 1/2 \) and the other one describing super-helicity \( Y \). Therefore one will be tempted to try to combine the theory of integer super-helicity \( Y = s \) presented in [5] with one of the theories of half-integer super-helicity \( Y = s + 1/2 \) presented here in order to construct an \( \mathcal{N} = 2 \) representation. The question is which pair [integer, half-integer(I)] or [integer, half-integer(II)] will be the one to give the \( \mathcal{N} = 2 \) representation. In an attempt to find the answer the authors of [5], by trial and error concluded that the answer was [integer, half-integer(I)].

The counting of the degrees of freedom argument provides a very simple explanation why this is the case. The integer theory has exactly the same degrees of freedom as the half-integer (I) theory. This is a sign that if we add together the two theories then in principle we can have a second direction of supersymmetry that will map the bosons (fermions) of one theory to the fermions (bosons) of the other theory. This can only happen if the number of bosons and fermions match exactly, as they do. Therefore we can construct an irreducible representation of 4D, \( \mathcal{N} = 2 \) Super-Poincaré group. Also in the same manner we can understand why a possible pair of integer theory with half-integer (II) theory can never work.

7 Summary

We continue the program started in [1] for the case of half-integer super-helicity. There are two classes of theories that describe the same physical system but they will turn out to have different off-shell structure. We reproduce the superspace action for both of them in terms of unconstrained superfields, following the redundancy guideline as was force on us by the representation theory of the Super-Poincaré group.
For Transverse theories this action is a representative of a bigger two parameter family of actions that are all equivalent and they are related by superfield redefinitions. That is not the case for Longitudinal theories, were the action is unique and no local redefinitions of the superfields can be done.

Finally, using the equations of motion generated by the superspace action we define the components of the theory. We derive the component action in diagonal form and calculate the susy transformation laws for each one of them. A counting of the off-shell degrees of freedom for transverse theories will give the same number as the theory of integer super-helicity and therefore explains why they can be combined to give an $\mathcal{N} = 2$ irreducible representation. The same counting for Longitudinal theories will also prove that 1) they can not be used together with integer super-helicity theories to make $\mathcal{N} = 2$ representations and 2) Off-shell Longitudinal theories are not equivalent to Transverse theories since there can be no 1-1 mapping between the two off-shell.

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