Constrained-Order Prophet Inequalities

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Abstract

Free order prophet inequalities bound the ratio between the expected value obtained by two parties each selecting one value from a set of independent random variables: a “prophet” who knows the value of each variable and may select the maximum one, and a “gambler” who is free to choose the order in which to observe the values but must select one of them immediately after observing it, without knowing what values will be sampled for the unobserved variables. It is known that the gambler can always ensure an expected payoff at least 0.669…times as great as that of the prophet. In fact, even if the gambler uses a threshold stopping rule, meaning there is a fixed threshold value such that the gambler rejects every sample below the threshold and accepts every sample above it, the threshold can always be chosen so that the gambler-to-prophet ratio is at least \(1 - \frac{1}{e} \approx 0.632\ldots\) In contrast, if the gambler must observe the values in a predetermined order, the tight bound for the gambler-to-prophet ratio is 1/2.

In this work we investigate a model that interpolates between these two extremes. We assume there is a predefined set of permutations of the set indexing the random variables, and the gambler is free to choose the order of observation to be any one of these predefined permutations. Surprisingly, we show that even when only two orderings are allowed — namely, the forward and reverse orderings — the gambler-to-prophet ratio improves to \(\psi^{-1} = \frac{1}{2} (\sqrt{5} - 1) = 0.618\ldots\) the inverse of the golden ratio. As the number of allowed permutations grows beyond 2, a striking “double plateau” phenomenon emerges: after increasing from 0.5 to \(\psi^{-1}\) when two permutations are allowed, the gambler-to-prophet ratio achievable by threshold stopping rules does not exceed \(\psi^{-1} + o(1)\) until the number of allowed permutations grows to \(O(\log n)\). The ratio reaches \(1 - \frac{1}{\varphi} - \varepsilon\) for a suitably chosen set of \(O(\poly(\varepsilon^{-1}) \cdot \log n)\) permutations and does not exceed \(1 - \frac{1}{\varphi}\) even when the full set of \(n!\) permutations is allowed.
1 Introduction

When one is planning under uncertainty, the order in which decisions must be made can powerfully influence the quality of the eventual outcome. It is therefore advantageous for a decision maker to be able to control the order of the decisions they will face. For example, a new Ph.D. student might prefer to know if they will be offered a position in the research group of their top-choice advisor before having to decide whether to accept a position with their second-choice advisor, but they might not have the freedom to dictate the order in which those two decisions are made.

Optimal stopping theory furnishes a theoretical basis for quantifying the benefit of exercising control over the order of one’s decisions in uncertain environments. In the classical optimal stopping problem there is a sequence of independent random variables $X_1, \ldots, X_n$ with known distributions, and one aims to design and analyze a stopping rule $\tau$ that maximizes the expected value of the sample selected by $\tau$, namely $E X_\tau$.

Optimal stopping with order selection, introduced by Hill (1983), posits that the decision maker can choose the order in which to observe the random variables and the time at which to stop the permuted sequence. In other words, the aim is now to design a permutation $\pi: [n] \to [n]$ and a stopping rule $\tau$ adapted to the sequence $X_{\pi(1)}, \ldots, X_{\pi(n)}$, such that $E X_\tau$ is maximized. (In principle one could also consider a model in which the permutation $\pi$ is specified adaptively, i.e. the value $\pi(k)$ is allowed to depend on $X_{\pi(1)}, \ldots, X_{\pi(k-1)}$.

One of the main results of Hill (1983), Theorem 3.11, shows that the optimal adaptive ordering is no better than the optimal non-adaptive one.)

Prophet inequalities, which bound the ratio between the expected value selected by an optimal stopping rule (metaphorically, a “gambler”) and the expectation of the maximum value in the sequence (metaphorically, a “prophet”), are a fruitful way of analyzing the effectiveness of different types of stopping rules. For optimal stopping without order selection, Krengel and Sucheston (1977, 1978) famously proved that the gambler’s expected value is always at least one-half that of the prophet, and the ratio 1/2 in this bound is the best possible. Samuel-Cahn (1984) showed that the optimal ratio 1/2 remains attainable if the gambler is constrained to use a threshold stopping rule, which sets a fixed threshold $\theta$ and commits to reject every sample less than $\theta$ and to stop whenever it encounters a sample strictly greater than $\theta$.

For the optimal stopping problem with order selection, bounds on the ratio between the gambler’s and prophet’s expected values are known as free order prophet inequalities. The first free order prophet inequality was proven by Yan (2011), who showed that the gambler-to-prophet ratio is always at least $1 - \frac{1}{e} = 0.632 \ldots$. This bound was later shown to be attainable even if the gambler is constrained to observe the values in uniformly-random order (Esfandiari et al., 2015) and to use a threshold stopping rule (Correa et al., 2019)\(^1\). Furthermore, $1 - \frac{1}{e}$ is asymptotically the best possible ratio attainable by threshold stopping rules (Kleinberg and Kleinberg, 2018), even if the distributions of $X_1, \ldots, X_n$ are identical. However, general stopping rules can do strictly better: the optimal factor in the free-order prophet inequality is known to be between 0.669 \ldots and 0.745 \ldots, and closing the gap between these two bounds is a major open question.

\(^1\)To achieve gambler-to-prophet ratio $1 - \frac{1}{e}$ using a threshold stopping rule, it is vital that our definition of threshold stopping rule treats its behavior at time $t$ as being unconstrained if the value $X_t$ is equal to the threshold $\theta$: the requirement of stopping at time $t$ is only triggered if $X_t > \theta$. For a stricter definition of threshold stopping rule that requires stopping whenever $X_t \geq \theta$ and continuing otherwise, Esfandiari et al. (2015) showed that 1/2 is the best achievable gambler-to-prophet ratio.
1.1 Our contributions

To summarize the foregoing discussion, the gap between the gambler-to-prophet ratios attainable with or without order selection formalizes, and quantifies, the advantage that a decision maker gains by being able to control the order in which decisions are made under uncertainty. But how much control over the ordering is needed to gain this advantage? Our paper initiates an in-depth exploration of that question. We introduce constrained-order prophet inequalities, in which there is a predefined set $\Pi \subseteq S_n$ of permutations of $[n]$, and the gambler is allowed to reorder the random variables $X_1, \ldots, X_n$ using any permutation in $\Pi$ before running a stopping rule.

**Definition 1.1.** A non-empty set of permutations $\Pi \subseteq S_n$ is said to satisfy a constrained-order prophet inequality with factor $\alpha$ if for every $n$-tuple of distributions $X_1, \ldots, X_n$ supported on the non-negative reals, there is a permutation $\pi \in \Pi$ and a stopping rule $\tau$ adapted to $X_{\pi(1)}, X_{\pi(2)}, \ldots, X_{\pi(n)}$, such that

$$\mathbb{E}X_\tau \geq \alpha \cdot \mathbb{E}\left[\max_{1 \leq i \leq n} X_i\right].$$

The prophet ratio of $\Pi$, $\text{PR}(\Pi)$, is the supremum of all $\alpha$ such that $\Pi$ satisfies a constrained-order prophet inequality with factor $\alpha$. Constrained-order threshold prophet inequalities and the threshold prophet ratio $\text{TPR}(\Pi)$ are defined similarly, but allowing the gambler to optimize only over threshold stopping rules rather than all stopping rules.

In the extreme cases where $\Pi$ has only one element or $\Pi$ is the entire permutation group $S_n$, one recovers the definitions of prophet inequality and free-order prophet inequality, respectively. Constrained-order prophet inequalities interpolate between these two extremes and allow us to gain deeper insight into how and why optimizing the order of decisions leads to better outcomes for optimal stopping rules. Interestingly, our first main result shows that much of the benefit of order selection can be gained by choosing the better of just two permutations, corresponding to the forward and reverse orderings.

**Theorem 1.1.** Let $\Pi = \{\iota, \rho\}$, where $\iota$ and $\rho$ are the permutations of $[n]$ that place its elements in forward and reverse order, respectively. The threshold prophet ratio of $\Pi$ is $\text{TPR}(\Pi) = \varphi^{-1} = \frac{1}{2} \left(\sqrt{5} - 1\right) = 0.618\ldots$.

One may interpret this result as lending support to the intuition that most of the benefit of order selection stems from the ability to schedule a high-risk high-reward option (e.g. a value that is $1/\epsilon$ with probability $\epsilon$, else zero) in the first half of the decision sequence, leaving safer options for afterward.

Given the large quantitative gain in $\text{TPR}(\Pi)$ when $\Pi$ is enlarged to contain the reverse ordering of the input sequence, one might expect to extract additional significant gains when $\Pi$ is allowed to contain three permutations, or an even larger constant number of them. Surprisingly, we show this is not the case: to exceed the golden-ratio bound at all one needs a super-constant number of permutations, and to exceed it by any constant $\epsilon > 0$ one needs a logarithmic number of them.

**Theorem 1.2.** If $n \geq 3$ and $|\Pi| < \sqrt{\log n}$ then $\text{TPR}(\Pi) \leq \varphi^{-1}$. For any $\epsilon > 0$, if $|\Pi| < \log_{1/\epsilon}(n)$ then $\text{TPR}(\Pi) \leq \varphi^{-1} + O(\epsilon)$.

Recall that if the gambler is allowed to order the elements arbitrarily, the best possible threshold prophet ratio is $\text{TPR}(S_n) = 1 - \frac{1}{e} + o(1)$. Our third and final main result shows that a logarithmic number of permutations are sufficient to get within $\epsilon$ of this bound, and that a quadratic number of permutations suffice to match the $1 - \frac{1}{e}$ bound exactly.
**Theorem 1.3.** For every \( n \in \mathbb{N} \) and \( \varepsilon > 0 \), there is a set \( \Pi \) consisting of \( O(\text{poly}(\varepsilon^{-1}) \cdot \log n) \) permutations such that \( \text{TPR}(\Pi) > 1 - \frac{1}{e} - \varepsilon \). There is also a set \( \Pi \) consisting of \( O(n^2) \) permutations such that \( \text{TPR}(\Pi) \geq 1 - \frac{1}{e} \).

Taken together, our results give a nearly complete answer to the question: for a given \( \alpha \), what is the smallest \( m \) such that there exists an \( m \)-element set of permutations whose threshold prophet ratio is at least \( \alpha \)? The answer is:

1. \( m = 1 \) for \( \alpha \leq \frac{1}{2} \) (Krengel and Sucheston, 1977, 1978);
2. \( m = 2 \) for \( \frac{1}{2} < \alpha \leq \varphi^{-1} \) (Theorems 1.1 and 1.2);
3. \( m = \Theta(\log n) \) for \( \varphi^{-1} < \alpha < 1 - \frac{1}{e} \), with the constant inside the \( \Theta(\cdot) \) depending on the value of \( \alpha \) (Theorem 1.3);
4. \( m = O(n^2) \) for \( \alpha = 1 - \frac{1}{e} \) (Theorem 1.3);
5. there is no set of permutations whose threshold prophet ratio is greater than \( 1 - \frac{1}{e} \) (Kleinberg and Kleinberg, 2018); see also Proposition 5.1 below.

It would be desirable, of course, to gain a similarly comprehensive understanding of the smallest set of permutations needed to achieve a given prophet ratio (rather than threshold prophet ratio), but at present this seems out of reach: even determining the prophet ratio of the full permutation group, \( \text{PR}(S_n) \), is a major open problem as it amounts to determining the best possible constant in the free order prophet inequality. For now, the most we can say is that our results imply lower bounds on the prophet ratio \( \text{PR}(\Pi) \) for the permutation sets \( \Pi \) we study, due to the trivial observation that \( \text{PR}(\Pi) \geq \text{TPR}(\Pi) \) for every set \( \Pi \).

In summary, then, the message of our paper is as follows: a gambler solving an optimal stopping problem using a threshold stopping rule can achieve significant gains if they are allowed to control the order in which they observe the values in the sequence, but most of these gains can be achieved just by choosing the better of the forward or reverse ordering, and nearly all of the gains can be achieved by choosing among a logarithmic number of predefined permutations.

### 1.2 Relation to the prophet secretary problem

Although we have motivated and presented our results in terms of free-order and constrained-order prophet inequalities, i.e. a setting in which the gambler chooses the order in which values are observed, our results also have a bearing on the prophet secretary problem due to their method of proof. In the prophet secretary problem (Esfandiari et al., 2015; Azar et al., 2018; Ehsani et al., 2018) the gambler observes \( n \) independent random variables in *uniformly random order*, and once again the question is what gambler-to-prophet ratio can be guaranteed. The best bounds currently known, due to Correa et al. (2019), show that the answer is at least \( 0.669 \ldots \) and at most \( \sqrt{3} - 1 = 0.732 \ldots \).

The constrained-order prophet inequalities asserted in Theorems 1.1 to 1.3 are all proven by constructing a small set of permutations, \( \Pi \), and analyzing the performance of a threshold stopping rule when the order in which values are observed is drawn uniformly at random from \( \Pi \). Thus, all of our results can also be interpreted as constructing “pseudo-random” distributions over permutations that have small support size, but ensure a gambler-to-prophet ratio (for threshold stopping rules) that is nearly as good as what can be achieved in the prophet secretary problem when the order of observation is sampled uniformly at random.
2 Preliminaries

This section presents definitions, notations, and conventions that will be used throughout the paper.

Let $X_1, X_2, \ldots, X_n$ be any $n$-tuple of independent random variables supported on the non-negative reals. A stopping rule adapted to the sequence $X_1, \ldots, X_n$ is a random variable $\tau$ taking values in $[n] \cup \{\perp\}$, such that for all $i \in [n]$ the event $\{\tau = i\}$ is measurable with respect to the $\sigma$-field generated by $X_1, \ldots, X_i$. One interprets $\tau$ as defining the time at which a gambler selects a value from the sequence — with $\tau = \perp$ denoting the event that no value is selected — and the measurability condition expresses the notion that the gambler must decide whether or not to select $i$ after having seen only the values $X_1, \ldots, X_i$. For a stopping rule $\tau$, the random variable $X_\tau$ is defined as follows: if $\tau = i \in [n]$ then $X_\tau = X_i$, and if $\tau = \perp$ then $X_\tau = 0$. If $\pi : [n] \to [n]$ is a permutation, a stopping rule $\tau$ is called $\pi$-adapted if it is adapted to the sequence $X_{\pi(1)}, \ldots, X_{\pi(n)}$, in other words, the event $\{\tau = \pi(i)\}$ is measurable with respect to the $\sigma$-field generated by $X_{\pi(1)}, X_{\pi(2)}, \ldots, X_{\pi(n)}$.

A stopping rule $\tau$ is a threshold stopping rule with threshold $\theta$ if it never selects a value strictly less than $\theta$, and it always selects the first value strictly greater than $\theta$. In other words, $\tau$ must satisfy the following constraints for all $i \in [n]$:

$$X_i < \theta \Rightarrow \tau \neq i, \quad X_i > \theta \Rightarrow \tau \leq i.$$  

Note that the tie-breaking behavior of a threshold stopping rule is unconstrained: when $X_i = \theta$, then $\tau = i$ is allowed but not required\(^2\). If the distributions of $X_1, \ldots, X_n$ have no point-masses, the event $X_i = \theta$ has probability zero so the tie-breaking convention is inconsequential. To reduce from the general case to the case where tie-breaking is inconsequential, we adopt the following artificial but convenient convention. We assume that in addition to the random variables $X_1, \ldots, X_n$, there is an auxiliary sequence of random variables $\tilde{X}_1, \ldots, \tilde{X}_n$, each uniformly distributed in $[0, 1]$, independent of $X_1, \ldots, X_n$ and mutually independent of one another. These auxiliary values are used for tie-breaking as follows. The pairs $((X_i, \tilde{X}_i))_{i=1}^n$ are regarded as elements of $\mathbb{R} \times [0, 1]$ under the lexicographic ordering. For every $\tilde{\theta} \in [0, 1]$, the threshold stopping rule with threshold $(\theta, \tilde{\theta})$ is defined as

$$\tau = \min\{i \in [n] \mid (X_i, \tilde{X}_i) \geq (\theta, \tilde{\theta})\}$$  

where, again, the relation $<$ is interpreted lexicographically. We make the following observations.

1. For all $i \in [n]$, the event $(X_i, \tilde{X}_i) = (\theta, \tilde{\theta})$ has probability zero.

2. For any $i \in [n]$ and $p \in (0, 1)$, we can find $(\theta, \tilde{\theta})$ such that $\Pr((X_i, \tilde{X}_i) > (\theta, \tilde{\theta})) = p$.

3. Similarly, if $(X_*, \tilde{X}_*)$ denotes the $(\mathbb{R} \times [0, 1])$-valued random variable that is the lexicographic maximum of $((X_i, \tilde{X}_i))_{i=1}^n$, then for any $p \in (0, 1)$ we can find $(\theta, \tilde{\theta})$ such that $\Pr((X_*, \tilde{X}_*) > (\theta, \tilde{\theta})) = p$.

In short, by treating the sequence of $n$ random variables as taking values in $\mathbb{R} \times [0, 1]$ (lexicographically ordered) and treating the threshold as belonging to the same set, we can make the same continuity and no-tie-breaking assumptions that are always justified in the case of point-mass-free distributions, but without having to assume the distributions of $X_1, \ldots, X_n$ have no point masses.

To avoid cumbersome notation in what follows, when discussing threshold stopping rules we will still denote the threshold by $\theta$ rather than $(\theta, \tilde{\theta})$ and we’ll use the notation $X_i < \theta$ or $X_* < \theta$ as shorthand for

\(^2\)As noted in the introduction, this tie-breaking convention is not universal. For example Esfandiari et al. (2015) adopt the stricter convention that $X_i = \theta$ implies $\tau = i$. 


Definition 1.1. The threshold prophet ratio of a sequence \((X, \tilde{X})\) under the point-mass-free assumption extends to the case of general distributions, depending on the conventions set forth in this section to justify that the results derived under the point-mass-free assumption extend to the case of general distributions.

For a threshold \((\theta, \tilde{\theta})\) and a permutation \(\pi\), the stopping rule \(\tau(\pi, \theta, \tilde{\theta})\) selects the earliest element of the sequence \(X_{\pi(1)}, \ldots, X_{\pi(n)}\) such that \((X_{\pi(i)}, \tilde{X}_{\pi(i)}) \geq (\theta, \tilde{\theta})\). More precisely, define \(\tau(\pi, \theta, \tilde{\theta})\) as follows. If \((X_i, \tilde{X}_i) < (\theta, \tilde{\theta})\) for all \(i \in [n]\) then \(\tau(\pi, \theta, \tilde{\theta}) = \perp\). Otherwise, \(\tau(\pi, \theta, \tilde{\theta}) = \pi(i_{\min})\) where \(i_{\min}\) is the minimum \(i\) such that \((X_{\pi(i)}, \tilde{X}_{\pi(i)}) \geq (\theta, \tilde{\theta})\). We will use the notation \(X_{\pi, \theta}\) as shorthand for \(X_{\tau(\pi, \theta, \tilde{\theta})}\).

The set of all permutations of \([n]\) is denoted by \(S_n\). Two permutations that are important in this work are the identity permutation \(\iota\) and its reverse, \(\rho(k) = n+1-k\). Recall that for a non-empty subset \(\Pi \subseteq S_n\), the terms constrained-order (threshold) prophet inequality and (threshold) prophet ratio were defined above, in Definition 1.1. The threshold prophet ratio of \(\Pi\) is denoted by \(\text{TPR}(\Pi)\).

3 Using the forward and reverse permutations

In this section, we let \(\Pi = \{\iota, \rho\}\) and prove that \(\text{TPR}(\Pi) = \varphi\).

Theorem 3.1. For every \(n\)-tuple of independent random variables \(X_1, \ldots, X_n\), there exists a threshold \(\theta\) such that

\[
\mathbb{E}_\pi \left[ \mathbb{E} X_{\pi, \theta} \right] \geq \varphi^{-1} \cdot \mathbb{E} X_n,
\]

where the outer expectation on the left side is over a uniformly random choice of \(\pi \in \Pi = \{\iota, \rho\}\).

Proof. For a given threshold \(\theta\), let \(p = \Pr(X_\iota \geq \theta)\). We have

\[
\mathbb{E} X_\iota \leq \theta + \mathbb{E}[(X_\iota - \theta)^+] \leq \theta + \sum_{i=1}^{n} \mathbb{E}[(X_i - \theta)^+].
\]

For the random variable \(X_{\tau} = X_{\pi, \theta}\), we have

\[
\mathbb{E} X_{\tau} = p\theta + \mathbb{E}[(X_{\tau} - \theta)^+] = p\theta + \sum_{i=1}^{n} c_i \mathbb{E}[(X_i - \theta)^+]
\]

where

\[
c_i = \frac{1}{2} \left( \prod_{j=1}^{i-1} \Pr(X_j < \theta) + \prod_{j=i+1}^{n} \Pr(X_j < \theta) \right)
\]

denotes the probability that no element is selected before the stopping rule observes \(X_i\), given that the sequence is observed in forward or reverse order with equal probability. Letting

\[
a_i = \prod_{j=1}^{i-1} \Pr(X_j < \theta), \quad b_i = \prod_{j=i+1}^{n} \Pr(X_j < \theta),
\]

we have

\[
a_i b_i \geq \prod_{j=1}^{n} \Pr(X_j < \theta) = 1 - p
\]

\[
c_i = \frac{1}{2}(a_i + b_i) \geq (a_i b_i)^{1/2} \geq \sqrt{1 - p},
\]
Lemma 3.2. For all \( \varepsilon > 0 \) there exists a sequence of three independent random variables \( X_1, X_2, X_3 \) such that \( X_1 \) and \( X_3 \) are identically distributed, and for every threshold stopping rule \( \tau \) we have
\[
\mathbb{E} X_\tau < (\varphi^{-1} + \varepsilon) \cdot \mathbb{E} X_*.
\] (10)

Proof. Fix a parameter \( \delta > 0 \) to be determined later, and suppose \( X_1 \) and \( X_3 \) are uniformly distributed in \([1 - \delta, 1]\) while \( X_2 \) is equal to \((\sqrt{5} - 1)/\delta \) with probability \( \delta \), otherwise \( X_2 = 0 \). Then
\[
\mathbb{E} X_* > \delta \cdot (\sqrt{5} - 1)/\delta + (1 - \delta) \cdot (1 - \delta) > \sqrt{5} - 2\delta.
\] (11)
If \( \tau \) is a threshold stop rule with threshold \( \theta \) there are three cases to consider. When \( \theta < 1 - \delta \), we have \( \mathbb{E} X_\tau = 1 - \frac{\varphi}{\delta}. \) When \( \theta > 1 \) we have \( \mathbb{E} X_\tau = \sqrt{5} - 1 \). In both of these cases, \( \mathbb{E} X_\tau < (\varphi^{-1} + \varepsilon) \cdot \mathbb{E} X_* \) provided \( \delta \) is sufficiently small. The remaining case is when \( 1 - \delta \leq \theta \leq 1 \). In this case, let \( r = (1 - \theta)/\delta \), so that \( \Pr(X_1 > \theta) = r \). Then
\[
\mathbb{E} X_\tau \leq r \cdot 1 + (1 - r)\delta \cdot (\sqrt{5} - 1)/\delta + (1 - r)(1 - \delta) \cdot 1
\leq r + (1 - r)(\sqrt{5} - 1) + (r - r^2) = (\sqrt{5} - 1) + (3 - \sqrt{5})r - r^2.
\] (12)
The right side of (12) is maximized when \( r = \frac{3}{2}(3 - \sqrt{5}) \), when it equals \((\sqrt{5} - 1) + \frac{1}{2}(14 - 6 \sqrt{5}) = \frac{1}{2}(5 - \sqrt{5})\). Hence,
\[
\mathbb{E} X_\tau \leq \frac{1}{2}(5 - \sqrt{5}) = \varphi^{-1} \cdot \sqrt{5}.
\] (13)
Combining (11) with (13) we see that the conclusion of the lemma holds, as long as \( \varepsilon \) is chosen small enough that \((\varphi^{-1} + \varepsilon) \cdot (\sqrt{5} - 2\delta) \geq \varphi^{-1} \cdot \sqrt{5} \).

Corollary 3.3. Suppose \( \Pi \) is a non-empty set of permutations of \( [n] \) and \( i, j, k \in [n] \) are three distinct indices such that \( \pi^{-1}(j) \) is between \( \pi^{-1}(i) \) and \( \pi^{-1}(k) \) for all \( \pi \in \Pi \). Then \( \text{TPR}(\Pi) \leq \varphi^{-1} \).

Proof. Define an \( n \)-tuple of independent random variables \( X_1, \ldots, X_n \) by specifying that the distributions of \( X_i, X_j, X_k \) are identical to the distributions of \( X_1, X_2, X_3 \) specified in Lemma 3.2 above, and for \( \ell \notin \{i, j, k\} \) let the distribution of \( X_\ell \) be identically zero. For any \( \pi \in \Pi \), a \( \pi \)-adapted threshold stopping rule with threshold \( \theta > 0 \) will skip \( X_\ell \) for every \( \ell \notin \{i, j, k\} \), and it will observe the values \( X_i, X_j, X_k \) at times \( \pi^{-1}(i), \pi^{-1}(j), \pi^{-1}(k) \). Note that the distributions of these three variables and the order in which they are observed are identical to the distributions \( X_1, X_2, X_3 \) specified in Lemma 3.2, so the inequality (10) will be satisfied. As \( \tau \) is an arbitrary threshold stopping rule, and \( \varepsilon > 0 \) is arbitrarily small, we conclude that \( \Pi \) does not satisfy a constrained-order prophet inequality with factor \( \alpha \) for any \( \alpha > \varphi^{-1} \).
Theorem 3.1. Let $\Pi = \{\iota, \rho\}$, if $n \geq 3$, we have $\text{TPR}(\Pi) = \varphi^{-1}$.

Proof. A threshold stopping rule that is allowed to select the better of $\iota$ and $\rho$ can do no worse than one which samples one of the two permutations uniformly at random. Therefore, Theorem 3.1 implies $\text{TPR}(\Pi) \geq \varphi^{-1}$. On the other hand, any three distinct indices $i < j < k$ in $[n]$ satisfy the condition in Corollary 3.3, implying that $\text{TPR}(\Pi) \leq \varphi^{-1}$. \hfill $\square$

4 Beating the golden-ratio bound requires many permutations

In this section we show that $\text{TPR}(\Pi) \leq \varphi^{-1}$ whenever $|\Pi| < \sqrt{\log n}$ and that $\text{TPR}(\Pi) \leq \varphi^{-1} + O(\varepsilon)$ whenever $\varepsilon > 0$ and $|\Pi| < \log_{1/\varepsilon}(n)$. This was stated as Theorem 1.2 in the introduction, and is proven by combining Lemmas 4.1 and 4.2 below.

Corollary 3.3 presented a necessary condition for a permutation set $\Pi$ to satisfy $\text{TPR}(\Pi) > \varphi^{-1}$: for every three indices $i, j, k$ there exists $\pi \in \Pi$ such that $\pi^{-1}(j)$ does not lie between $\pi^{-1}(i)$ and $\pi^{-1}(k)$. It turns out this condition guarantees $|\Pi| > \log \log n$ (Kesselheim et al., 2015) but it does not imply any stronger lower bound than that doubly-logarithmic one. To prove stronger lower bounds on $|\Pi|$, we begin by generalizing Corollary 3.3.

Definition 4.1. If $\Pi$ is a set of permutations of $[n]$, we say an index $j \in [n]$ is $\varepsilon$-centered with respect to $\Pi$ if there exists a probability distribution $p$ on $[n] \setminus \{j\}$ such that for every $\pi \in \Pi$ with inverse permutation $\sigma = \pi^{-1}$, the sets $\{i \mid \sigma(i) < \sigma(j)\}$ and $\{i \mid \sigma(i) > \sigma(j)\}$ both have measure greater than $\frac{1}{2} - \varepsilon$ under $p$. In the special case $\varepsilon = 0$, we shall say that $j$ is centered with respect to $\Pi$.

Lemma 4.1. If $\Pi$ is a non-empty set of permutations of $[n]$ and there exists an index $j \in [n]$ that is $\varepsilon$-centered with respect to $\Pi$, then $\text{TPR}(\Pi) \leq \varphi^{-1} + O(\varepsilon)$.

The proof, which is deferred to Appendix A, generalizes the proof of Corollary 3.3. The value of $X_j$ is defined to be $(\sqrt{5} - 1)/\delta$ with probability $\delta$, else $X_j = 0$, whereas the remaining distributions are all tightly concentrated around 1. Threshold stopping rules face a dilemma: if the threshold is set high enough that there is a reasonable probability of not stopping before $X_j$, then there must be a reasonable probability that the gambler also doesn’t stop after $X_j$ and comes away empty-handed. The prophet faces no such dilemma: she can pick $X_j$ when it is non-zero, and otherwise she can pick a value that is nearly equal to 1.

Lemma 4.2. If $\Pi$ is a set of fewer than $\sqrt{\log n}$ permutations of $[n]$ then there exists an index $j$ that is $\varepsilon$-centered with respect to $\Pi$. If $\Pi$ is a set of fewer than $\log_{1/e}(n)$ permutations of $[n]$ for some $\varepsilon > 0$ then there exists an index $j$ that is $\varepsilon$-centered with respect to $\Pi$.

Proof. Let $\sigma_1, \sigma_2, \ldots, \sigma_m$ be an enumeration of the permutations whose inverse belongs to $\Pi$. For any pair of distinct indices $i \neq j$ in $[n]$ define a vector $v_{ij} \in \pm 1]^m$ by specifying that for all $k \in [m]$

$$ (v_{ij})_k = \begin{cases} -1 & \text{if } \sigma_k(i) < \sigma_k(j) \\ 1 & \text{if } \sigma_k(i) > \sigma_k(j). \end{cases} $$

(14)

For any probability distribution $p$ on $[n] \setminus j$, if we use $q_k(p)$ to denote the measure of the set $\{i \mid \sigma_k(i) < \sigma_k(j)\}$ under $p$ and $q(p)$ to denote the vector $(q_1(p), \ldots, q_m(p))$, then we have

$$ \sum_{i \neq j} p(i)v_{ij} = 1 - 2q(p) $$

(15)
where \( \mathbf{1} \) denotes the vector \((1, 1, \ldots, 1)\). The criterion that \( j \) is centered with respect to \( \Pi \) is equivalent to the existence of a distribution \( p \) such that \( \mathbf{q}(p) = \frac{1}{2} \cdot \mathbf{1} \). Similarly, \( j \) is \( \epsilon \)-centered with respect to \( \Pi \) if and only if there is a distribution \( p \) such that \( \mathbf{q}(p) \in \left( \frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon \right)^m \). Using (15) we now see that \( j \) is centered with respect to \( \Pi \) if and only if 0 is a convex combination of the vectors in the set \( V_j = \{ \mathbf{v}_i \mid i \neq j \} \). Similarly, \( j \) is \( \epsilon \)-centered with respect to \( \Pi \) if and only if the convex hull of \( V_j \) intersects the hypercube \(( -2\epsilon, 2\epsilon)^m \).

To finish proving the first part of the lemma, we must show that if \( m < \sqrt{\log n} \), then for some index \( j \), \( 0 \) is a convex combination of the vectors in \( V_j \). Contrapositively, we will assume that for every \( j \), \( 0 \) is not in the convex hull of \( V_j \), and we will deduce from this assumption that \( m^2 \geq \log n \). By the separating hyperplane theorem, our assumption that \( 0 \) is not in the convex hull of \( V_j \) implies there is a vector \( \mathbf{w}_j \) such that \( \langle \mathbf{w}_j, \mathbf{v}_{ij} \rangle > 0 \) for all \( i \neq j \). Note that \( \mathbf{v}_{ij} = -\mathbf{v}_{ji} \), so for all \( i \neq j \),

\[
\langle \mathbf{w}_j, \mathbf{v}_{ij} \rangle > 0 > \langle \mathbf{w}_i, \mathbf{v}_{ij} \rangle,
\]

where the second inequality follows because \( \langle \mathbf{w}_i, \mathbf{v}_{ji} \rangle > 0 \) by our assumption on \( \mathbf{w}_i \). Now consider the linear threshold functions defined by \( f_i(\mathbf{x}) = \text{sgn}(\langle \mathbf{w}_i, \mathbf{x} \rangle) \) for each \( i \in [n] \). Equation (16) says that \( f_i(\mathbf{v}_{ij}) \neq f_j(\mathbf{v}_{ij}) \). Recalling that \( \mathbf{v}_{ij} \in \{\pm 1\}^m \), this means the restrictions of \( f_1, \ldots, f_n \) to \{\pm 1\}^m are pairwise distinct. There are fewer than \( 2^m \) distinct linear threshold functions on \{\pm 1\}^m (Cover, 1965), hence \( m^2 \geq \log n \).

To finish proving the second part of the lemma, we must show that if \( m < \log_2(n) \), then for some index \( j \), the convex hull of \( V_j \) and the hypercube \(( -2\epsilon, 2\epsilon)^m \) intersect. Contrapositively, we will assume that for every \( j \), the minimum \( \infty \)-norm of the vectors in the convex hull of \( V_j \) is at least \( 2\epsilon \). From this assumption we will deduce that \((1/\epsilon)^m \geq n \). Minimizing the \( \infty \)-norm of vectors in the convex hull of \( V_j \) is equivalent to solving the following linear program, whose dual is presented alongside it.

\[
\begin{align*}
\min \quad & r \\
\text{s.t.} \quad & r - \sum_{i \neq j} v_{ij} p_i \geq 0 & \forall k \in [m] \\
& r + \sum_{i \neq j} v_{ij} p_i \geq 0 & \forall k \in [m] \\
& \sum_{i \neq j} p_i = 1 \\
& p_i \geq 0 & \forall i \in [n] \setminus j
\end{align*}
\]

\[
\begin{align*}
\max \quad & z \\
\text{s.t.} \quad & z + \sum_{k=1}^{m} v_{ijk} (y_k - x_k) \leq 0 & \forall i \in [n] \setminus j \\
& \sum_{k=1}^{m} (y_k + x_k) = 1 \\
& x_k, y_k \geq 0 & \forall k \in [m]
\end{align*}
\]

Our assumption is that for each \( j \), the optimum of the primal LP is at least \( 2\epsilon \), which means that the optimum of the dual LP is also at least \( 2\epsilon \). Let \( \mathbf{x}, \mathbf{y}, z \) denote a feasible dual solution with \( z \geq 2\epsilon \), and let \( \mathbf{w}_j = \mathbf{x} - \mathbf{y} \). The first dual constraint, combined with the inequality \( z \geq 2\epsilon \), implies \( \langle \mathbf{w}_j, \mathbf{v}_{ij} \rangle \geq 2\epsilon \) for all \( i \in [n] \setminus j \). Using the fact that \( x_k, y_k \geq 0 \) implies \( x_k + y_k \geq |x_k - y_k| \), we see that the second dual constraint implies \( ||\mathbf{w}_j||_1 \leq 1 \). Thus, there exist vectors \( \mathbf{w}_1, \ldots, \mathbf{w}_n \) in the \( L_1 \) unit ball, such that for all \( i \neq j \) in \([n]\), \( \langle \mathbf{w}_j, \mathbf{v}_{ij} \rangle \geq 2\epsilon \). Recalling that \( \mathbf{v}_{ji} = -\mathbf{v}_{ij} \), we may combine the inequalities

\[
2\epsilon \leq \langle \mathbf{w}_j, \mathbf{v}_{ij} \rangle, \quad 2\epsilon \leq \langle \mathbf{w}_i, \mathbf{v}_{ji} \rangle
\]

to obtain

\[
4\epsilon \leq \langle \mathbf{w}_j - \mathbf{w}_i, \mathbf{v}_{ij} \rangle \leq \|\mathbf{w}_j - \mathbf{w}_i\|_1 \cdot \|\mathbf{v}_{ij}\|_\infty = \|\mathbf{w}_j - \mathbf{w}_i\|_1
\]

(17)

Summarizing, the \( L_1 \) unit ball contains \( n \) vectors \( \mathbf{w}_1, \ldots, \mathbf{w}_n \), and the pairwise distances between these vectors (in \( L_1 \)) are at least \( 4\epsilon \). The \( L_1 \) balls of radius \( 2\epsilon \) centered at these vectors are pairwise disjoint, and all of them are contained in the ball of radius \( 1 + \epsilon \) centered at \( 0 \), so the combined volume of the \( n \) balls of radius \( 2\epsilon \) must not exceed the volume of the radius\((1 + \epsilon)\) ball. Since \( 1 + \epsilon < 2 \), this implies \((2\epsilon)^m \cdot n < 2^m \), hence \( n < (1/\epsilon)^m \) as claimed. \( \square \)
5 Achieving the optimal threshold prophet ratio

Recall that when one is free to order the elements arbitrarily, threshold stopping rules satisfy a prophet inequality with factor $1 - \frac{1}{e}$ but (asymptotically) no greater, i.e. $\text{TPR}(S_n) = 1 - \frac{1}{e} + o(1)$. In this section we construct a small set of permutations that achieves this bound, and an even smaller set that comes arbitrarily close. The constructions make use of pairwise independence and almost pairwise independence; see Definition 5.1 below.

The fact that $\text{TPR}(S_n) = 1 - \frac{1}{e} + o(1)$ is implicit in (Correa et al., 2019; Kleinberg and Kleinberg, 2018); we prove the following in Appendix B for the sake of making our exposition self-contained.

**Proposition 5.1.** As $n \to \infty$, the threshold prophet ratio $\text{TPR}(S_n)$ converges to $1 - \frac{1}{e}$ from above.

To design sets of permutations that achieve, or approach, the $1 - \frac{1}{e}$ bound, we use (almost) pairwise independent permutations, a notion we now define.

**Definition 5.1.** A distribution over permutations $\sigma \in S_n$ is pairwise independent if for every pair of distinct indices $i \neq j$ in $[n]$, the pair $(\sigma(i), \sigma(j))$ is distributed uniformly over $\{(a, b) \in [n] \times [n] \mid a \neq b\}$. It is $(\varepsilon, \delta)$-almost pairwise independent if for every pair of distinct indices $i \neq j$, the distribution of the pair $\left(\frac{\sigma(i)}{\varepsilon}, \frac{\sigma(j)}{\varepsilon}\right)$ is $\delta$-close, in total variation distance, to the uniform distribution on $\left[\frac{1}{\varepsilon}\right] \times \left[\frac{1}{\varepsilon}\right]$, where $\left[\frac{1}{\varepsilon}\right]$ denotes the set $\{1, 2, \ldots, \left[\frac{1}{\varepsilon}\right]\}$.

**Lemma 5.2.** For prime $n$, there exists a set $\Pi$ of $n(n-1)$ permutations such that the uniform distribution over $\Pi$ is pairwise independent. For any $\varepsilon, \delta > 0$ such that $1/\varepsilon$ is an integer, if $n$ is an integer multiple of $1/\varepsilon$ and $\varepsilon n \geq 2/\delta$, then there exists a set $\Pi$ of $O\left((\frac{1}{\varepsilon})^2 \log n\right)$ permutations such that the uniform distribution over $\Pi$ is $(\varepsilon, \delta)$-almost pairwise independent.

The proof is deferred to Appendix B. The first part is proven using the permutations $\sigma(k) = ak+b \pmod{n}$ for all $a \in [n-1]$ and $b \in [n]$. The second part is proven using the probabilistic method to show that a random $\Pi \subset S_n$ of the given cardinality has positive probability of being $(\varepsilon, \delta)$-almost pairwise independent. Explicit constructions using $\varepsilon$-biased sets (Naor and Naor, 1993; Ta-Shma, 2017) can achieve $|\Pi| = O\left((\frac{1}{\varepsilon^2})^{2+o(1)} \log n\right)$.

**Theorem 5.3.** Suppose $\sigma$ is a random permutation of $[n]$ and $\pi = \sigma^{-1}$. If $\sigma$ is drawn from a pairwise independent distribution then there exists a threshold $\theta$ such that

$$\mathbb{E}_{\pi} \left[ \mathbb{E} X_{\pi, \theta} \right] \geq \left(1 - \frac{1}{e}\right) \cdot \mathbb{E} X_s. \quad (18)$$

If $\sigma$ is drawn from an $(\varepsilon, \varepsilon^2)$-almost pairwise independent distribution then there exists a threshold $\theta$ such that

$$\mathbb{E}_{\pi} \left[ \mathbb{E} X_{\pi, \theta} \right] \geq \left(1 - \frac{1}{e} - O(\varepsilon)\right) \cdot \mathbb{E} X_s. \quad (19)$$

**Proof.** For a given threshold $\theta$, let $p = \Pr(X_s \geq \theta)$. We have

$$\mathbb{E} X_s \leq \theta + \mathbb{E} \left[ (X_s - \theta)^+ \right] \leq \theta + \sum_{i=1}^{n} \mathbb{E} \left[ (X_i - \theta)^+ \right]. \quad (20)$$

For the random variable $X_T = X_{\pi, \theta}$, we have

$$\mathbb{E} X_T = p \theta + \mathbb{E} \left[ (X_T - \theta)^+ \right] = p \theta + \sum_{i=1}^{n} c_i \mathbb{E} \left[ (X_i - \theta)^+ \right] \quad (21)$$
Theorem 3.1

Let \( q_j = \Pr(X_j < \theta) \) for each \( j \in [n] \). For any set \( S \subset [n] \), let \( p_{k,i}(S) = \Pr(\pi((k-1)) = S \mid \pi(k) = i) \) denote the conditional probability that \( S \) is exactly equal to the set of elements observed before \( X_i \), given that \( \pi(k) = i \).

\[
\prod_{\ell=1}^{k-1} \Pr(X_{\pi(\ell)} < \theta \mid \pi(k) = i) = \sum_{S \subset [n]} p_{k,i}(S) \prod_{j \in S} q_j \geq \prod_{S \subset [n]} \left( \prod_{j \in S} q_j \right)^{\rho_{k,i}(S)} = \prod_{j \in [n], \pi(j) = k} \sum_{S \subset [n], \pi \in S} p_{k,i}(S). \tag{23}
\]

The exponent \( \sum_{S \subset [n], \pi \in S} p_{k,i}(S) \) on the right side is equal to \( \Pr(\sigma(j) < k \mid \sigma(i) = k) \). Hence,

\[
c_i \geq \sum_{k \in [n]} \Pr(\pi(k) = i) \prod_{j \neq k} p_{\sigma(j), k} = \prod_{j \in [n]} q_j^{\Pr(\sigma(j) < k \mid \sigma(i) = k)}. \tag{24}
\]

If \( \sigma \) is pairwise independent then \( \Pr(\pi(k) = i) = \frac{1}{n} \) and for all \( j \), \( \Pr(\sigma(j) < k \mid \sigma(i) = k) = \frac{k-1}{n-1} \). Hence, if we let \( q = \prod_{j=1}^n q_j \), then

\[
c_i \geq \frac{1}{n} \sum_{k=1}^n \left( \prod_{j \neq k} q_j \right)^{\frac{k-1}{n-1}} \geq \frac{1}{n} \left( 1 + q^\frac{1}{n} + \cdots + q^\frac{n-1}{n} \right). \tag{25}
\]

If we set \( \theta \) so that \( q = \frac{1}{e} \), then Lemma B.3 in Appendix B shows that the right side of (25) is greater than \( 1 - \frac{1}{e} \). Also, note that

\[
p = \Pr(X_\ast \geq \theta) = 1 - \Pr(X_\ast < \theta) = 1 - \prod_{j=1}^n q_j = 1 - q = 1 - \frac{1}{e}.
\]

Having shown that \( p = 1 - \frac{1}{e} \) and that \( c_i > 1 - \frac{1}{e} \) for all \( i \), we may substitute these bounds into (21) and conclude that

\[
\mathbb{E} X_\ast \geq \left( 1 - \frac{1}{e} \right) \theta + \left( 1 - \frac{1}{e} \right) \sum_{i=1}^n \mathbb{E} [(X_i - \theta)^+] \geq \left( 1 - \frac{1}{e} \right) \mathbb{E} X_\ast. \tag{26}
\]

Now we turn to the case that the distribution of \( \sigma \) is \((\epsilon, \delta)\)-almost pairwise independent for \( \delta = \epsilon^2 \). In that case, we group the time steps \( k \in [n] \) into “buckets” of \( \epsilon n \) consecutive steps; the bucket containing the time step when \( X_i \) is observed is numbered \( b(i) = \left\lceil \frac{\sigma(i)}{\epsilon n} \right\rceil \). The counterpart of (22) is the following inequality:

\[
c_i \geq \sum_{u=1}^{1/\epsilon} \Pr(b(i) = u) \cdot \prod_{\ell=1}^{\epsilon n} \Pr(X_{\pi(\ell)} < \theta \mid b(i) = u) \tag{27}
\]

The inequality is valid because the left side is the probability that no value greater than \( \theta \) is observed before \( X_i \) is observed, and the right side is the probability no value greater than \( \theta \) is observed in buckets 1, 2, \ldots, \( b(i) \). For any set \( S \subset [n] \), let \( \tilde{p}_{u,i}(S) = \Pr(\pi([\epsilon n u]) = S \mid b(i) = u) \) denote the conditional probability that \( S \) is exactly equal to the set of elements observed in buckets 1, 2, \ldots, \( u \), given that \( b(i) = u \). Analogously to (23), we can use the AM-GM inequality to derive

\[
\prod_{\ell=1}^{\epsilon n} \Pr(X_{\pi(\ell)} < \theta \mid b(i) = u) \geq \prod_{j \in [n]} q_j^{\sum_{S \subset [n], \pi \in S} \tilde{p}_{u,i}(S)}. \tag{28}
\]
The exponent of $q_j$ on the right side is equal to $\Pr(b(j) \leq u \mid b(i) = u)$. If the distribution of $\sigma$ is $(\varepsilon, \delta)$-almost pairwise independent, this probability is at most $\varepsilon u + \delta$ and $\Pr(b(i) = u) \geq \varepsilon - \delta$. Therefore,

$$c_i \geq (\varepsilon - \delta) \cdot \sum_{i=1}^{1/\varepsilon} q^{q^{\varepsilon+\delta}} = \left(1 - \frac{\varepsilon}{2}\right) \cdot q^{\varepsilon+\delta} \cdot \left[1 + q^\varepsilon + q^{2\varepsilon} + \cdots + q^{1-\varepsilon}\right].$$  \hspace{1cm} (29)

If $q = \frac{1}{2}$ and $\delta = e^2$ then the factors $1-\frac{\varepsilon}{2}$ and $q^{\varepsilon+\delta}$ are both $1+O(\varepsilon)$ and the factor $e \left(1 + q^\varepsilon + q^{2\varepsilon} + \cdots + q^{1-\varepsilon}\right)$ is at least $1 - \frac{1}{e}$, again by Lemma B.3. Hence, $c_i \geq 1 - \frac{1}{e} - O(\varepsilon)$. As before, $p = 1-q = 1 - \frac{1}{e}$, and the lemma follows by substituting these bounds for $p$ and $c_i$ into (21) and comparing with (20).

We now restate and prove Theorem 1.3.

**Theorem 5.4.** For every $n \in \mathbb{N}$ and $\varepsilon > 0$, there is a set $\Pi$ consisting of $O(\text{poly}(e^{-1}) \cdot \log n)$ permutations such that $\text{TPR}(\Pi) > 1 - \frac{1}{e} - \varepsilon$. There is also a set $\Pi$ of $O(n^2)$ permutations such that $\text{TPR}(\Pi) \geq 1 - \frac{1}{e} - \varepsilon$.

**Proof.** If $n$ is prime, Lemma 5.2 combined with Theorem 5.3 show that there exists a set $\Pi$ of $(n(n-1))$ that satisfies $\text{TPR}(\Pi) \geq 1 - \frac{1}{e}$. If $n$ is composite, we make use of a “padding lemma” (Lemma B.4 in Appendix B) that says that for any $n$ satisfying $\Pi \subseteq S_n$, we can transform a set of at most $n$ permutations into a set of at most $n+1$ permutations, $\Pi_n \subseteq S_n$, such that $\text{TPR}(\Pi_n) \geq \text{TPR}(\Pi)$. Taking $n$ to be a prime between $n$ and $2n$ (Tchebychev, 1852), the padding lemma implies there is a set $\Pi_n$ of fewer than $4n^2$ permutations satisfying $\text{TPR}(\Pi) \geq 1 - \frac{1}{e}$. Similarly, for $e > 0$, we can choose $k \in \mathbb{N}$ such that $e/2 < \frac{1}{k} \leq e$. If $2 \leq n \leq 2k^3$ then $n^2 \leq 4k^6 \leq 256e^{-6}$, and we have shown the existence of $\Pi \subseteq S_n$ with $|\Pi| = O(n^2) = O(e^{-6})$ and $\text{TPR}(\Pi) \geq 1 - \frac{1}{e}$. Otherwise, $n > 2k^3$ so there is a multiple of $k$ between $n$ and $2n$. Denoting this number by $N$, and observing that the constraint $e'N \geq 2/\delta'$ is satisfied when $e' = 1/k$ and $\delta' = (e')^2$, we apply Lemma 5.2 and Theorem 5.3 to deduce the existence of a set $\Pi_N$ of $O(k^6 \log N)$ permutations of $[N]$ such that $\text{TPR}(\Pi_N) \geq 1 - \frac{1}{e} - \varepsilon$; then we use the padding lemma to deduce the existence of $\Pi_n \subseteq S_n$ such that $|\Pi_n| = O(k^6 \log N) = O(e^{-6} \log n)$ and $\text{TPR}(\Pi_n) \geq 1 - \frac{1}{e} - \varepsilon$.

# A Deferred proofs from Section 4

In this section we restate and prove Lemma 4.1.

**Lemma A.1.** If $\Pi$ is a non-empty set of permutations of $[n]$ and there exists an index $j \in [n]$ that is $\varepsilon$-centered with respect to $\Pi$, then $\text{TPR}(\Pi) \leq \varphi^{-1} + O(\varepsilon)$.

**Proof.** Suppose $j$ is $\varepsilon$-centered with respect to $\Pi$, and let $p$ be a probability distribution on $[n] \setminus \{j\}$ such that for every permutation $\sigma$ whose inverse belongs to $\Pi$, the sets $\{i \mid \sigma(i) < \sigma(j)\}$ and $\{i \mid \sigma(i) > \sigma(j)\}$ have measure at least $\frac{1}{2} - \varepsilon$ under $p$. Define the distributions of $X_1, X_2, \ldots, X_n$ as follows. For a small positive number $\delta$ to be determined later, the value of $X_j$ is $(\sqrt{5} - 1)/\delta$ with probability $\delta$, and otherwise $X_j = 0$. For every $i \in [n] \setminus \{j\}$, $X_i$ has cumulative distribution function $F_i$ satisfying

$$F_i(t) = \Pr(X_i \leq 1 - t) = \begin{cases} 1 & \text{if } t < 0 \\ \exp\left(-\frac{pt}{\delta}\right) & \text{if } 0 \leq t \leq 1 \\ 0 & \text{if } t > 1. \end{cases}$$
In other words, $1 - X_i = Y_i \land 1$ where $Y_i$ is exponentially distributed with rate parameter $\frac{p(i)}{\theta}$ and the notation $a \land b$ denotes the minimum of $a$ and $b$.

First, observe that $X_s$ is equal to $(\sqrt{5} - 1)/\delta$ with probability $\delta$, and otherwise

$$X_s = \max_{i \neq j}(X_j) = 1 - \min_{i \neq j}(Y_i \land 1) \geq 1 - j \setminus Y_i.$$

The minimum of independent exponential random variables with rates $r_1, \ldots, r_n$ is exponential with rate $r_1 + \cdots + r_n$. Hence, $\min_{i \neq j}(Y_i) = 1$ is exponentially distributed with rate $\frac{1}{\delta} \sum_{i \neq j} p(i) = \frac{1}{\delta}$, and its expected value is $\delta$. Consequently,

$$\mathbb{E}[X_s | X_j = 0] \geq \mathbb{E}\left[1 - \bigwedge_{i \neq j} Y_i\right] = 1 - \delta$$

and the prophet’s unconditional expected value satisfies

$$\mathbb{E}X_s = \delta \cdot \frac{\sqrt{5} - 1}{\delta} + (1 - \delta) \cdot \mathbb{E}[X_s | X_j = 0] \geq \sqrt{5} - 1 + (1 - \delta)^2 > \sqrt{5} - 2\delta. \quad (30)$$

Now we turn to analyzing the expected value obtained by a threshold stopping rule $\tau$ with threshold $\theta$, assuming $\tau$ is $\pi$-adapted for some $\pi \in \Pi$ with inverse permutation $\sigma = \pi^{-1}$. If $\theta < 1$ then $\tau = 1$ and $X_\tau = X_{\pi(1)} \leq 1$ so $\mathbb{E}X_\tau \leq 1$. If $\theta > 1$ then $\tau \in \{j, \perp\}$ and $X_\tau = X_j$, so $\mathbb{E}X_\tau = \mathbb{E}X_j = \sqrt{5} - 1$. In both of these cases, $\mathbb{E}X_\tau < \varphi^{-1} \cdot \mathbb{E}X_s$ provided $\delta < 0.07$. The remaining case to consider is when $0 \leq \theta \leq 1$. In that case let $I_0 = \{i \mid \sigma(i) < \sigma(j)\}$ denote the set indexing the values appearing before $X_j$ in the sequence $X_{\pi(1)}, \ldots, X_{\pi(n)}$, and let $I_1 = \{i \mid \sigma(i) > \sigma(j)\}$ denote the set indexing the values appearing after $X_j$ in that sequence. If $q_0, q_1$ denote the probabilities of the events $\max\{X_i \mid i \in I_0\} < \theta$ and $\max\{X_i \mid i \in I_1\} < \theta$, respectively, then the gambler’s expected value satisfies

$$\mathbb{E}X_\tau \leq (1 - q_0) \cdot 1 + q_0 \cdot \delta \cdot \frac{\sqrt{5} - 1}{\delta} + q_0 \cdot (1 - \delta) \cdot (1 - q_1) \cdot 1 < 1 - q_0 q_1 + (\sqrt{5} - 1)q_0, \quad (31)$$

where the first inequality is justified because $X_\tau \leq 1$ if $\tau = i$ for any $i \in [n] \setminus j$ and $X_\tau = 0$ if $\tau = \perp$. Writing $\theta = 1 - \delta s$, we have $\Pr(X_i < \theta) = \exp(-sp(i))$ for $i \neq j$ and therefore

$$q_0 = \prod_{i \in I_0} \Pr(X_i < \theta) = \exp\left(-s \sum_{i \in I_0} p(i)\right) < \exp\left(-\left(\frac{1}{2} - \epsilon\right)s\right)$$

$$q_0 q_1 = \prod_{i \in I_0 \cup I_1} \Pr(X_i < \theta) = \exp\left(-s \sum_{i \neq j} p(i)\right) = \exp(-s).$$

Substituting these into (31) we find that

$$\mathbb{E}X_\tau < 1 - \exp(-s) + (\sqrt{5} - 1)\exp\left(-\left(\frac{1}{2} - \epsilon\right)s\right). \quad (32)$$

To complete the proof we must bound the right side of (32) above by $\varphi^{-1} + O(\epsilon)$ when $s \geq 0$. The derivative of the right side is

$$\exp(-s) - (\sqrt{5} - 1)(\frac{1}{2} - \epsilon)\exp\left(-\left(\frac{1}{2} - \epsilon\right)s\right) = \exp\left(-\left(\frac{1}{2} + \epsilon\right)s\right) - \varphi^{-1} + (\sqrt{5} - 1)\epsilon \cdot \exp\left(-\left(\frac{1}{2} - \epsilon\right)s\right)$$

which is negative for all $s \geq 1$, provided that $\epsilon < \frac{1}{110}$. Assume henceforth that $\epsilon < \frac{1}{110}$, as otherwise the inequality asserted by the lemma, $\text{TPR}(\Pi) \leq \varphi^{-1} + O(\epsilon)$, is trivially satisfied due to the $O(\epsilon)$ term.
derivative calculation above shows that the right side of (32) is a decreasing function of \( s \geq 1 \), implying that its maximum value on the interval \( s \in [0, \infty) \) is attained when \( s \in [0, 1] \). For \( s \) in this range, if we let \( q = \exp(-\frac{1}{2}s) \), then
\[
\exp(-\left(\frac{1}{2} - \varepsilon\right)s) = q \cdot e^{\varepsilon s} \leq q \cdot e^{\varepsilon} \leq q + O(\varepsilon).
\]
Hence we can rewrite (32) as
\[
\mathbb{E}X_{\tau} < 1 - q^2 + (\sqrt{5} - 1)q + O(\varepsilon). \tag{33}
\]
The right side of (33) is a quadratic function of \( q \) that is maximized when \( q = \varphi^{-1} \) and \( 1 - q^2 = q = \varphi^{-1} \). Therefore,
\[
\mathbb{E}X_{\tau} < \sqrt{5}\varphi^{-1} + O(\varepsilon). \tag{34}
\]
Combining (30) with (34), the bound \( \mathbb{E}X_{\tau} \leq (\varphi^{-1} + O(\varepsilon)) \cdot \mathbb{E}X_s \) follows, which confirms that \( \text{TPR}(\Pi) \leq \varphi^{-1} + O(\varepsilon) \). \( \square \)

\section{Deferred proofs from Section 5}

This section contains proofs of propositions and lemmas that were mentioned in Section 5 whose proofs were deferred.

\textbf{Proposition B.1.} As \( n \to \infty \), the threshold prophet ratio \( \text{TPR}(S_n) \) converges to \( 1 - \frac{1}{e} \) from above.

\textbf{Proof.} By Theorem 1.3, \( \text{TPR} \geq 1 - \frac{1}{n} \), so we need only show that \( \text{TPR} \leq 1 - \frac{1}{e} + o(1) \) as \( n \to \infty \). To this end, let \( H \gg 1 \) be an arbitrarily large number and consider a sequence of random variables \( X_1, \ldots, X_n \) drawn i.i.d. from a distribution whose cumulative distribution function \( F \) is given by

\[
F(x) = \begin{cases} 
0 & \text{if } x < 1 \\
\left( H - \frac{1}{(e-2)H} \right)(x-1) & \text{if } 1 \leq x \leq 1 + \frac{1}{H} \\
1 - \frac{1}{(e-2)H} & \text{if } 1 + \frac{1}{H} < x < H + 1 \\
1 - \frac{H-1}{(e-2)H} & \text{if } H + 1 \leq x \leq H + 1 + \frac{1}{H} \\
1 & \text{if } x > H + 1 + \frac{1}{H}.
\end{cases} \tag{35}
\]

In words, each \( X_i \) is sampled from a mixture of two uniform distributions, on the intervals \([1, 1 + \frac{1}{H}]\) and \([H + 1, H + 1 + \frac{1}{H}]\), with the mixture weights being \( 1 - \frac{1}{(e-2)H} \) and \( \frac{1}{(e-2)H} \), respectively.

Since the variables \( X_1, \ldots, X_n \) are identically distributed, reordering them has no effect on the performance of stopping rules, so we merely need to show that if \( \tau \) is a threshold stopping rule adapted to the sequence \( X_1, \ldots, X_n \) then \( \mathbb{E}X_{\tau} \leq (1 - \frac{1}{e} + o(1)) \cdot \mathbb{E}X_s \), where the \( o(1) \) term vanishes as \( n \to \infty \).

Let \( p = \frac{1}{(e-2)H} \) denote the probability that a sample \( X_i \) exceeds \( H \). The prophet’s expected value satisfies
\[
\mathbb{E}X_s = 1 + (1 - (1-p)^n)H + O(\frac{1}{H}) \geq \frac{e - 1}{e - 2} - O(\frac{1}{H}) \tag{36}
\]
where the second inequality follows from:
\[
(1 - (1-p)^n)H > \frac{1}{e - 2} - \frac{1}{2(e-2)^2H^2}.
\]
Now consider a threshold stopping rule $\tau$ that sets a threshold $\theta$ such that $F(\theta) = 1 - q$; in other words, any given element of the sequence $X_1, \ldots, X_n$ has probability $q$ of exceeding the threshold. Then, $\tau = \bot$ with probability $(1 - q)^n$. Furthermore, conditional on $\tau < \bot$, the events $X_\tau \in [H + 1, H + 1 + \frac{1}{\tau}]$ and $X_\tau \in [1, 1 + \frac{1}{\tau}]$ have conditional probabilities $\min[p/q, 1]$ and $1 - \min[p/q, 1]$, respectively. Thus,

$$\mathbb{E}X_\tau \leq (1 - (1 - q)^n) \left( \frac{\ell}{n} H + 1 + \frac{1}{\tau} \right).$$

Let $r = q/n$. Ignoring $O(\frac{1}{\tau})$ terms that vanish as $H \to \infty$ and $O(1/n)$ terms that vanish as $n \to \infty$ we have

$$\mathbb{E}X_\tau \approx (1 - e^{-r}) \cdot \left( 1 + \frac{1}{(1-e)^r} \right)$$

$$\mathbb{E}X_\tau \approx (1 - e^{-r}) \cdot \left( \frac{e - 2}{e - 1} + \frac{1}{(e - 1)^r} \right).$$

(37)

The right side of (37) is maximized at $r = 1$, where it equals $1 - \frac{1}{e}$.

Lemma B.2. For prime $n$, there exists a set $\Pi$ of $n(n-1)$ permutations such that the uniform distribution over $\Pi$ is pairwise independent. For any $\epsilon, \delta > 0$ such that $1/\epsilon$ is an integer, if $n$ is an integer multiple of $1/\epsilon$ and $en \geq 2/\delta$, then there exists a set $\Pi$ of $O((\frac{1}{\epsilon})^2 \log n)$ permutations such that the uniform distribution over $\Pi$ is $(\epsilon, \delta)$-almost pairwise independent.

Proof. If $n$ is equal to a prime number $p$, for any integers $a, b$ such that $a$ is not divisible by $p$, the function $x \mapsto ax + b \pmod{p}$ is a permutation of $[p]$. If $(a, b)$ are sampled uniformly at random from $[p-1] \times [p]$, this defines a pairwise independent permutation distribution. The reason is that for any $(i, j)$ and $(k, \ell)$ in $[p] \times [p]$ such that $i \neq j$, $k \neq \ell$, the system of linear congruences

$$ai + b \equiv k, \quad aj + b \equiv \ell \pmod{p}$$

has the unique solution $a \equiv (k - \ell)(j - i)^{-1}, b \equiv k - ai \pmod{p}$.

For $\epsilon, \delta > 0$ such that $1/\epsilon$ and $en$ are integers and $en \geq 2/\delta$, we use the probabilistic method to prove the existence of a multiset $\Pi$ of $m = O((\epsilon \delta)^{-2} \log n)$ permutations such that the uniform distribution on $\Pi$ is $(\epsilon, \delta)$-almost pairwise independent. In fact, we will prove that the multiset obtained by drawing $m$ uniformly-random samples, with replacement, satisfies this property with positive probability. Define the function $b(k) = \left\lfloor \frac{k}{en} \right\rfloor$, mapping $[n]$ to $[\frac{1}{\epsilon}]$. The $(\epsilon, \delta)$-almost pairwise independence property asserts that when $\sigma$ is a random sample from the distribution, for any $i \neq j$ the distribution of the pair $(b(\sigma(i)), b(\sigma(j)))$ is $\delta$-close to the uniform distribution on $[\frac{1}{\epsilon}] \times [\frac{1}{\epsilon}]$ in total variation distance. For any pair $(u, v) \in [\frac{1}{\epsilon}] \times [\frac{1}{\epsilon}]$, a uniformly random $\sigma \in S_n$ satisfies:

$$\Pr((b(\sigma(i)), b(\sigma(j))) = (u, v)) = \begin{cases} \frac{\epsilon n}{n} \cdot \frac{\epsilon n - 1}{n - 1} & \text{if } u = v \\ \frac{\epsilon n}{n} \cdot \frac{\epsilon n}{n - 1} & \text{if } u \neq v. \end{cases}$$

(38)

In both cases we have

$$\Pr((b(\sigma(i)), b(\sigma(j))) = (u, v)) > \frac{\epsilon n}{n} \cdot \frac{\epsilon n - 1}{n} = \epsilon \left( 1 - \frac{1}{n} \right) \geq \epsilon^2 \left( 1 - \frac{\delta}{2} \right)$$

(39)

where the last inequality is a consequence of $\epsilon n \geq 2/\delta$. 

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Bernstein’s Inequality (Bernstein, 1946) ensures that

$$\Pr\left(\sum_{i=1}^{m} Z_{i} > \frac{m}{2}e^{2}\delta\right) < \exp\left(-\frac{\frac{1}{3}m^{2}e^{2}\delta^{2}}{m^{2} + \frac{1}{6}me^{2}\delta}\right) = \exp\left(-\frac{me^{2}\delta^{2}}{8(1 + \delta/6)}\right).$$

(40)

Since $\delta < 1$, the right side can be made less than $(\varepsilon/n)^{2}$ by setting $m \geq 18(\varepsilon\delta)^{-2} \log(n/\varepsilon)$. Note that our assumption that $n$ is divisible by $1/\varepsilon$ implies $n/\varepsilon \leq n^{2}$, hence $\log(n/\varepsilon) \leq 2 \log n$. Thus, drawing $m = 36(\varepsilon\delta)^{-2} \log n$ samples suffices to make $\Pr\left(\sum_{i=1}^{m} Z_{i} > \frac{m}{2}e^{2}\delta\right)$ less than $(\varepsilon/n)^{2}$.

For $(i, j, u, v) \in [n] \times [n] \times [\frac{1}{\varepsilon}] \times [\frac{1}{\varepsilon}]$ with $i \neq j$, define $S(i, j, u, v) \subset S_{n}$ to be the set of permutations $\sigma$ such that $(b(\sigma(i)), b(\sigma(j))) = (u, v)$. Let $\mathcal{E}(i, j, u, v)$ denote the event that $S(i, j, u, v)$ contains $me^{2}(1 - \delta)$ or fewer of the permutations $\sigma_{1}, \ldots, \sigma_{m}$, and let $\mathcal{E} = \bigcup_{i,j,u,v} \mathcal{E}(i, j, u, v)$. Our calculation using Bernstein’s Inequality showed that for $m \geq 36(\varepsilon\delta)^{-2} \log n$, we have $\Pr(\mathcal{E}(i, j, u, v)) < (\varepsilon/n)^{2}$. Taking the union bound over all $n(n - 1)/e^{2}$ choices of $i, j, u, v$, we conclude that $\Pr(\mathcal{E}) < 1$. Therefore, the complementary event $\overline{\mathcal{E}}$ occurs with positive probability. When $\overline{\mathcal{E}}$ occurs, we claim the uniform distribution over $\{\sigma_{1}, \ldots, \sigma_{m}\}$ is $(\varepsilon, \delta)$-almost pairwise independent. To verify this, consider any $i \neq j$ and let $D_{ij}$ denote the distribution of $(b(\sigma(i)), b(\sigma(j)))$ when $\sigma$ is drawn randomly from $\{\sigma_{1}, \ldots, \sigma_{m}\}$. Let $U$ denote the uniform distribution on $[\frac{1}{\varepsilon}] \times [\frac{1}{\varepsilon}]$. We have

$$\|D_{ij} - U\|_{TV} = \sum_{u=1}^{1/\varepsilon} \sum_{v=1}^{1/\varepsilon} \left(\left(U(S(i, j, u, v)) - D_{ij}(S(i, j, u, v))\right)^{+}\right) < \left(\frac{1}{2}\right)^{2} \left(\varepsilon^{2} - \frac{me^{2}(1 - \delta)}{m}\right) = \delta$$

as required by the definition of $(\varepsilon, \delta)$-almost pairwise independence. \hfill \Box

**Lemma B.3.** If $k \in \mathbb{N}$ and $r^{k} \geq \frac{1}{\varepsilon}$ then $\frac{1}{k+1} \left(1 + r + \cdots + r^{k}\right) \geq 1 - \frac{1}{e}$.

**Proof.** The left side is an increasing function of $r$, so it suffices to prove the inequality when $r^{k} = \frac{1}{e}$. We begin by noting that if the hypothesis had been $r^{k+1} = \frac{1}{e}$ rather than $r^{k} = \frac{1}{e}$, the lemma would follow easily by comparing the sum to an integral.

$$\frac{1}{k+1} \left(1 + r + \cdots + r^{k}\right) > \frac{1}{k+1} \int_{0}^{k+1} r^{t} dt = \left. \frac{1}{k+1} r^{t+1} \right|_{t=0}^{t=k+1} = 1 - \frac{1}{e}.$$

We do not know of any comparably simple argument that works when $r^{k} = \frac{1}{e}$. Instead, we define the function

$$f(k) = \frac{1}{k+1} \frac{1 - e^{-\frac{k+1}{r}}}{1 - e^{-\frac{1}{r}}}$$

and argue that $f(k) \geq 1 - \frac{1}{e}$ for all $k \geq 1$, by proving that $f(k)$ is monotonically decreasing in $k$ and that $\lim_{k \to \infty} f(k) = 1 - \frac{1}{e}$. The computation of $\lim_{k \to \infty} f(k)$ follows from the facts that $1 - e^{-\frac{k+1}{r}} \to 1 - \frac{1}{e}$ and
\[(k + 1)(1 - e^{-1/k}) \to 1 \text{ as } k \to \infty.\] The monotonicity claim follows by expanding the domain of \(f\) from the natural numbers to the real numbers and taking the derivative with respect to \(k\).

\[
\frac{df}{dk} = \frac{-e^{k+1} k^2 - k^2 + e^k ((k + 1)(k - 2) + 1) + e^{k+1} (k^2 + k + 1)}{e (e^k - 1)^2 (k + 1)^2 k^2}
\]

The denominator is positive for all \(k \geq 1\) so it suffices to prove that the numerator in non-positive which is equivalent to showing:

\[
e^{\frac{k}{k^2} + e^{k+1} \geq \frac{(e + 1)(k + 1)^2}{k^2} - (e + 3)(k + 1) + (e + 1)\]

To prove the last inequality for all \(k \geq 1\), we make use of the following quadratic lower bounds on \(e^x\) which can be shown using elementary calculus\(^3\).

\[
e^x \geq 1 + x + \frac{1}{e} x^2, \text{ for all } x \in [-1, 0]
\]

\[
e^x \geq \frac{e}{2} (x^2 + 1) \text{ for all } x \geq 1
\]

Notice that \(-1 \leq -\frac{1}{k} \leq 0\) and \(\frac{k+1}{k^2} \geq 1\) for \(k \geq 1\) so the above inequalities apply.

\[
e^{\frac{1}{k^2}} + e^{\frac{k}{k^2}} \geq \frac{1}{k} + \frac{1}{e} \left(\frac{k+1}{k}\right)^2 + \frac{e}{2} \left(\frac{k^2}{k} + 1\right)
= \frac{(1 + \frac{1}{e} + e)(k + 1)^2 - (e + 3)(k + 1) + \frac{e}{2} + 1}{k^2}
\geq \frac{(1 + e)(k + 1)^2 - (e + 3)(k + 1) + (e + 1)}{k^2}
\]

where the last inequality follows from \(\frac{1}{e}(k + 1)^2 + \frac{e}{2} + 2 \geq e + 1\) for \(k \geq 1\). \(\square\)

**Lemma B.4** (Padding Lemma). If \(N \in \mathbb{N}\) and \(\Pi_N \subseteq S_N\) is a set of \(m\) permutations such that \(\text{TPR}(\Pi_N) \geq \alpha\), then for all \(n \leq N\) there is a set \(\Pi_n \subseteq S_n\) of at most \(m\) permutations such that \(\text{TPR}(\Pi_n) \geq \alpha\).

**Proof.** For any permutation \(\pi \in S_N\) let \(\pi_{|n}\) denote the unique permutation of \([n]\) that can be expressed as the composition \(\pi \circ f\) for some monotonically increasing \(f : [n] \to [N]\). (This \(f\) maps the elements of \([n]\) to the elements of \(\pi^{-1}([n])\) in increasing order.) If \(\Pi \subseteq S_N\) satisfies \(|\Pi| = m\) and \(\text{TPR}(\Pi) \geq \alpha\) then the set \(\Pi_n = \{\pi_{|n} \mid \pi \in \Pi\}\) certainly satisfies \(|\Pi_n| \leq m\); we claim it also satisfies \(\text{TPR}(\Pi_n) \geq \alpha\). To see why, consider any independent non-negative-valued random variables \(X_1, \ldots, X_n\). Extend this to a sequence \(X_1, \ldots, X_N\) by defining \(X_i = 0\) for \(n < i \leq N\). Since \(\text{TPR}(\Pi) \geq \alpha\) there is a \(\pi \in \Pi\) and a threshold \(\theta\) such \(\mathbb{E}X_{\pi,\theta} \geq \alpha \cdot \mathbb{E}X_n\). (Note that \(\max(X_1, \ldots, X_N) = \max(X_1, \ldots, X_N)\) so it is immaterial whether \(X_i\) is interpreted as referring to the former or the latter quantity.) Let \(\pi' = \pi_{|n}\), and note that \(\pi' \in \Pi_n\). For any threshold \((\theta, \tilde{\theta}) \in \mathbb{R} \times [0, 1]\) the stopping rule \(\tau(\pi, \theta, \tilde{\theta})\) stops at the earliest element in the sequence \(X_{\pi(1)}, X_{\pi(2)}, \ldots, X_{\pi(n)}\) such that \((X_{\pi(i)}, X_{\pi(i)}) \geq (\theta, \tilde{\theta})\), whereas \(\tau(\pi', \theta, \tilde{\theta})\) stops at the earliest element such that \((X_{\pi(i)}, \bar{X}_{\pi(i)}) \geq (\theta, \tilde{\theta})\) and \(\pi(i) \in [n]\). These two elements are the same unless \(\theta = 0\) and \(X_{\pi,\theta} = 0\),

\(^3\)Both are Taylor expansion approximations of \(e^x\) around 0 and 1 respectively.
because when \( \pi(i) \notin [n] \) we have \( X_{\pi(i)} = 0 \) by construction. Therefore, \( X_{\pi', \theta} \geq X_{\pi, \theta} \) pointwise, and we find that

\[
\mathbb{E} X_{\pi', \theta} \geq \mathbb{E} X_{\pi, \theta} \geq \alpha \cdot \mathbb{E} X^*,
\]

as desired. \( \square \)

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