DEGREE-2 ABEL MAPS AND HYPERELLEPTIC CURVES

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Abstract. In this paper we resolve the degree-2 Abel map for nodal curves. Our results are based on a previous work of the authors reducing the problem of the resolution of the Abel map to a combinatorial problem via tropical geometry. As an application, we characterize when the (symmetrized) degree-2 Abel map is not injective, a property that, for a smooth curve, is equivalent to the curve being hyperelliptic.

MSC (2020): 14H10, 14H40, 14T90.
Keywords: Algebraic curve, hyperelliptic curve, tropical curve, Abel map.

1. Introduction

This paper is dedicated to the construction of an explicit resolution of the degree-2 Abel-Jacobi map for a regular smoothing of a nodal curve. For a smooth curve $C$, the degree-$d$ Abel map is an important morphism taking a $d$-tuple of points on $C$, to the associated invertible sheaf on the curve (tensored with a fixed invertible sheaf on the curve). When the fixed invertible sheaf is $\mathcal{O}_C(dP_0)$ for a point $P_0$ on $C$, the map is usually called Abel-Jacobi map. This map encodes many important properties of the curve. For instance, the degree-2 Abel map detects when the curve is hyperelliptic. More precisely, a smooth curve is hyperelliptic if and only the degree-2 Abel map is not injective, and in this case the curve is endowed with a $g^1_2$, which can be identified with the fiber of the Abel map (up to the action of the symmetric group). In this paper we investigate how we can extend the above description to singular curves.

We construct an explicit resolution of the degree-2 Abel-Jacobi map using the results in [1], where the general problem of resolving Abel maps is reduced to checking a certain combinatorial property of the tropical Abel map. More precisely, this translates into the problem of showing the existence of a compatibility between the polyhedral structures of the tropical Jacobian of a curve and the product of the relevant tropical curve. This is the result contained in [1, Theorem A]. In degree 2, this yields a very explicit combinatorial condition describing how to blow up the domain of the geometric Abel map to get a resolution. This is summarized in Theorem 3.4.

Let $\pi: \mathcal{C} \to B$ be a regular smoothing of a curve $C$ with a section $\sigma$ through its smooth locus, and $\mu$ be a polarization on $C$. We denote by $\mathcal{T}_\mu$ the Esteves compactified Jacobian parametrizing $(\sigma, \mu)$-quasistable torsion-free rank-1 sheaves on $C$ (see [7]). As usual, we write $\mathcal{C}^2 := \mathcal{C} \times_B \mathcal{C}$. Let $\mathcal{L}$ be an invertible sheaf on $\mathcal{C}/B$ of degree-$(k + 2)$. We define the degree-2 Abel (rational) map $\alpha^2_{\mathcal{L}}$ as

$$\alpha^2_{\mathcal{L}}: \mathcal{C}^2 \to \mathcal{T}_\mu$$

$$(Q_1, Q_2) \mapsto [\mathcal{L}|_{\pi^{-1}(Q_1)}(-Q_1 - Q_2)].$$

Our main result holds when $\mu$ is the trivial degree-0 polarization and $\mathcal{L}$ is the trivial sheaf $\mathcal{O}_C$. An important ingredient to describe the resolution of the degree-2 Abel-Jacobi map is the notion of tail of a nodal curve. A subcurve of a nodal curve is a $\delta$-tail if the subcurve and its complementary curves are connected and intersect each other in $\delta$ nodes.
**Theorem (Theorem 1.7).** Let $Z_1, \ldots, Z_N$ be the 2-tails and the 3-tails of $C$ which do not contain $\sigma(0)$. Consider the sequence of blowups

$$
\widetilde{C}_N^2 \xrightarrow{\phi_N} \cdots \xrightarrow{\phi_2} \widetilde{C}_1^2 \xrightarrow{\phi_1} \widetilde{C}_0^2 \xrightarrow{\phi_0} C^2
$$

where $\phi_0$ is the blowup of $C^2$ along its diagonal subscheme and $\phi_i$ is the blowup of $\widetilde{C}_{i-1}^2$ along the strict transform of the divisor $Z_i \times Z_i$ of $C^2$ via $\phi_0 \circ \cdots \circ \phi_{i-1}$. Then the rational map

$$
\alpha^2_{\Delta C} \circ \phi_0 \circ \cdots \circ \phi_N : \widetilde{C}_N^2 \dashrightarrow \mathcal{T}_\mu
$$

is a morphism, i.e., it is defined everywhere.

Next we investigate the relation between the degree-2 Abel map and hyperelliptic (nodal) curves. More precisely, we study when the (symmetrized) degree-2 Abel map is not injective. The upshot is that this happens exactly when the curve has a simple torsion-free rank-1 sheaf of degree 2 with non-negative degree over every component of the curve and at least two sections. We call a curve satisfying all these condition a pseudo-hyperelliptic curve. It is easy to see that if a curve is hyperelliptic, then it is pseudo-hyperelliptic.

It is worth noticing that a variation of the condition of hyperelliptic curve was already given by Caporaso in [4]. She introduced and study the notion of weakly-hyperelliptic curve, which is the condition of the existence of a balanced degree-2 invertible sheaf on a curve with at least 2 sections. Again, if a curve is hyperelliptic, then it is weakly-hyperelliptic. We study the relation between weakly-hyperelliptic and pseudo-hyperelliptic.

**Theorem (Theorem 4.20).** Let $C$ be a curve with no separating nodes. The following properties hold.

1. The curve $C$ is pseudo-hyperelliptic if and only if, for some (every) regular smoothing $C \rightarrow B$ of $C$, the symmetrized degree-2 Abel map of $C$ is not injective.

2. If $C$ is stable and weakly-hyperelliptic, then $C$ is pseudo-hyperelliptic.

### 2. Preliminaries

Throughout the paper, we will use the notations introduced in [3 Sections 2 and 3] and [1 Section 3]. In this section we just recall some basic definitions and constructions.

Given a graph $\Gamma$, we denote by $V(\Gamma)$ and $E(\Gamma)$ the sets of vertices and edges of $\Gamma$. Given a subset $V \subset V(\Gamma)$, we set $V^c = V(\Gamma) \setminus V$. For an orientation $\Gamma^+$ on $\Gamma$, we denote by $s(e)$ and $t(e)$ the source and target of an edge $e \in E(\Gamma)$. Given subsets $V, W \subset V(\Gamma)$, we let $E(V, W)$ be the set of edges of $\Gamma$ connecting a vertex of $V$ with a vertex of $W$. A refinement of a graph $\Gamma$ is a graph obtained by inserting a non-negative number $n_e$ (depending on $e$) of vertices in the interior of each edge $e$ of $\Gamma$. If a vertex $v$ of the refinement is inserted in the interior of an edge $e$ of $\Gamma$, we say that $v$ is a vertex over $e$.

A metric graph is a pair $(\Gamma, \ell)$ where $\Gamma$ is a graph and $\ell$ is a function $\ell : E(\Gamma) \rightarrow \mathbb{R}_{>0}$, called the length function. Let $(\Gamma, \ell)$ be a metric graph. If $\Gamma^+$ is an orientation on $\Gamma$, we define the tropical curve $X$ associated to $(\Gamma^+, \ell)$ as

$$
X = \left( \bigcup_{e \in E(\Gamma)} I_e \cup V(\Gamma) \right) / \sim
$$

where $I_e = [0, \ell(e)] \times \{ e \}$ and $\sim$ is the equivalence relation generated by $(0, e) \sim s(e)$ and $(\ell(e), e) \sim t(e)$. We usually just write $e$ to represent $I_e$ in $X$, and denote by $e^\circ$ the interior of $e$. We say that $(\Gamma, \ell)$ is a model of the tropical curve $X$. We will identify tropical curves with isometric models.

Let $\Gamma$ be a graph and define $\ell : E(\Gamma) \rightarrow \mathbb{R}$ by $\ell(e) = 1$ for every $e \in E(\Gamma)$. We denote by $X_\Gamma$ the tropical curve induced by the metric graph $(\Gamma, \ell)$. 
Let $X$ be a tropical curve and $\Gamma$ be a graph. A divisor on $X$ (respectively, on $\Gamma$) is a function $D: X \to \mathbb{Z}$ (respectively, $D: V(\Gamma) \to \mathbb{Z}$) such that $D(p) \neq 0$ only for finitely many points $p \in X$. Given a divisor $D$ on $X$, we define the support of $D$ as the set of points $p$ of $X$ such that $D(p) \neq 0$ and denote it by $\text{supp}(D)$. A polarization on $X$ (respectively, on $\Gamma$) is a function $\mu: X \to \mathbb{R}$ (respectively, $\mu: V(\Gamma) \to \mathbb{R}$) such that $\mu(p) \neq 0$ only for finitely many points $p \in X$ and such that $\sum_{p \in X} \mu(p)$ (respectively, $\sum_{v \in V(\Gamma)} \mu(v)$) is an integer, called the degree of the polarization $\mu$.

Given a point $p_0$ in $X$, (respectively, a vertex $v_0 \in V(\Gamma)$), a divisor $D$ on $X$ (respectively, $D$ on $\Gamma$) is called $(p_0, \mu)$-quasistable (respectively, $(v_0, \mu)$-quasistable) if:

$$\sum_{p \in Y} (D(p) - \mu(p)) + \frac{\delta_Y}{2} \geq 0$$

for every tropical subcurve $Y$ of $X$ (respectively, every subset $Y \subseteq V(\Gamma)$), where the inequality is strict if $p_0 \notin Y \neq X$. Here, $\delta_Y$ is the number of tangent direction outgoing from $Y$ in the case of a tropical curve (see [3] Section 3.1 for the precise definition), while it is equal to $|E(Y,Y^c)|$ in the case of a graph.

Let $X$ be a tropical curve and $p_0$ be a point of $X$. Let $\mu$ be a polarization on $X$. Recall that in an equivalence class of a divisor on a tropical curve there is just one $(p_0, \mu)$-quasistable divisor (see [3] Theorem 5.6]). For a degree-$d$ divisor $D$ on $X$, we denote by $q_0(D)$ the unique $(p_0, \mu)$-quasistable divisor on $X$ which is equivalent to $D$. Given an oriented model $(\Gamma, \ell)$ of $X$, for every edge $e \in E(\Gamma)$ and every real number $t \in [0, \ell(e)]$, we let $p_{e,t}$ the point on $e$ at distance $t$ from the source of $e$.

Given a tropical curve $X$, we let $J^\text{trop}_{p_0,\mu}(X)$ be the tropical Jacobian parametrizing $(p_0, \mu)$-quasistable divisors on $X$. Recall that $J^\text{trop}_{p_0,\mu}(X)$ is homeomorphic to the usual tropical Jacobian (see [3] Theorem 5.10]. We set $X^2 := X \times X$. Given a divisor $D^\dagger$ on $X$, we define the tropical Abel map

$$\alpha^\text{trop}_{2,D^\dagger} : X^2 \to J^\text{trop}_{p_0,\mu}(X) \quad (p_1, p_2) \mapsto [D^\dagger - p_1 - p_2],$$

where $[-]$ denotes the class of a divisor in the tropical Jacobian. Alternatively, the map $\alpha^\text{trop}_{2,D^\dagger}$ takes $(p_1, p_2)$ to the unique $(p_0, \mu)$-quasistable divisor in the class of $D^\dagger - p_1 - p_2$.

**Remark 2.1.** Let $X$ be a tropical curve with a point $p_0 \in X$. Let $\Gamma$ be a model of $X$. Let $\mu$ be a polarization on $X$ induced by a polarization on $\Gamma$ and $D$ a degree-$d$ divisor on $X$. We let $\hat{\Gamma}$ the minimal refinement of $\Gamma$ such that $\text{supp}(D) \subset V(\hat{\Gamma})$. We denote by $D$ the divisor on $\hat{\Gamma}$ induced by $D$. We call the pair $(\hat{\Gamma}, D)$ on $\hat{\Gamma}$ the combinatorial type of $D$. By [3] Proposition 5.3, the degree-$d$ divisor $D$ on $X$ is $(p_0, \mu)$-quasistable if and only if $\hat{\Gamma}$ is obtained by inserting at most one vertex in the interior of each edge of $\Gamma$ and $D$ is $(p_0, \mu)$-quasistable on $\hat{\Gamma}$.

### 3. Degree-2 Abel maps

Let $C$ be a nodal curve over an algebraically closed filed $k$. A subcurve $Z$ of $C$ is a reduced union of components of $C$. Given a subcurve $Z$ of $C$, we let $Z^c := C \setminus Z$. Throughout this section we will fix a regular smoothing $\pi: C \to B$ of a nodal curve $C$ with a section $\sigma: B \to C$ of $\pi$ through its smooth locus. We denote by $C^2 := C \times_B C$.

Let $\mu$ be a degree-$k$ polarization on $C$. We denote by $\mathcal{J}_\mu^\text{trop}$ the sheaves parametrizing $(\sigma, \mu)$-quasistable torsion-free rank-1 sheaves on the curves of the family $\pi$ (see [7] for more details). Let $\mathcal{L}$ be an invertible sheaf on $C/B$ of degree-$(k + 2)$. As in [2], we define the
degree-2 Abel (rational) map $\alpha_\mu^2$ as
\[
\alpha_\mu^2: \mathcal{C}^2 \to \mathcal{J}_\mu, \\
(Q_1, Q_2) \mapsto [C_{\pi^{-1}(\pi(Q_1))}(-Q_1 - Q_2)].
\]

We let $\Gamma$ be the dual graph of $C$ and $X_\Gamma$ be the tropical curve induced by $\Gamma$ (with unitary lengths). Given an invertible sheaf $\mathcal{L}$ on $\mathcal{C}$, we denote by $D^1_{\mathcal{L}}$ the divisor on $\mathcal{C}$ given by the multidegree of $\mathcal{L}|_C$. We also let $D^2_{\mathcal{L}}$ be the divisor on $X_\Gamma$ induced by $D^1_{\mathcal{L}}$.

Given a point $N = (N_1, N_2)$ of $\mathcal{C}^2$, where $N_i$ is a node of $C$, we will consider the following two ways of blowing up $\mathcal{C}^2$ locally around $N$. If $N_1 \in Z_1 \cap Z_1^\Gamma$ and $N_2 \in Z_2 \cap Z_2^\Gamma$ for subcurves $Z_1$ and $Z_2$ of $C$, we can consider the blowups $\phi: C^2_\phi \to C^2$ and $\phi': C^2_{\phi'} \to C^2$ respectively along $Z_1 \times Z_2$, or along $Z_1^\Gamma \times Z_2^\Gamma$. The first one is also equivalent to the blowup along $Z_1^\Gamma \times Z_2$, and the second one is equivalent to the blowup along $Z_1^\Gamma \times Z_2^\Gamma$. In both cases, the inverse image of $N$ is isomorphic to $\mathbb{P}^1_k$. The situation is illustrated in Figure 1 where $\text{st}_\phi$ and $\text{st}_{\phi'}$ applied to a divisor of $\mathcal{C}^2$ denote the strict transform of this divisor. These blowups induce a dual picture on the product $X^2_\Gamma$: we illustrate the relation between these blowups and the dual picture in Figure 2.

![Figure 1](image)

**Figure 1.** The two types of blowup of $\mathcal{C}^2$ around $(N_1, N_2)$.  

**Theorem 3.1.** Let $\pi: C \to B$ be a regular smoothing of a nodal curve $C$ with smooth components. Let $\sigma: B \to C$ be a section of $\pi$ through its smooth locus. Let $\mu$ be a polarization of degree $k$ over the family and $\mathcal{L}$ be an invertible sheaf on $C$ of degree $k + 2$. Let $(N_1, N_2)$ be a point of $\mathcal{C}^2$, with $N_i$ a node of $C$. Let $Z_1$ and $Z_2$ be subcurves of $C$ such that $N_1 \in Z_1 \cap Z_1^\Gamma$ and $N_2 \in Z_2 \cap Z_2^\Gamma$. Let $e_1$ and $e_2$ be the edges in the dual graph $\Gamma$ of $C$ that correspond to $N_1$ and $N_2$, where $e_i$ is oriented from $Z_i$ to $Z_i^\Gamma$. Consider the divisor $D_{x,y} = D^1_{\mathcal{L}} - p_{e_1,x} - p_{e_2,y}$ on $X_\Gamma$, for some $x, y \in [0, 1]$. 

1. If the combinatorial type of $\text{qs}(D_{x,y})$ is constant on the sets
   \[
   \{(x,y); 0 < x < y < 1\} \text{ and } \{(x,y); 0 < y < x < 1\},
   \]
   then the blowup of $\mathcal{C}^2$ along $Z_1 \times Z_2$ resolves the Abel map $\alpha_\mu^2$ locally around the point $(N_1, N_2)$.

2. If the combinatorial type of $\text{qs}(D_{x,y})$ is constant on the sets
   \[
   \{(x,y); 0 < x < 1 - y < 1\} \text{ and } \{(x,y); 0 < 1 - y < x < 1\},
   \]
   then the blowup of $\mathcal{C}^2$ along $Z_1 \times Z_2^\Gamma$ resolves the Abel map $\alpha_\mu^2$ locally around the point $(N_1, N_2)$.  

restrict our attention to a special class of subcurves, called tails, particular, we have $\alpha_2$ of $C$ as in Figure 3, where $0$ and $\eta$ consider a map $\rho$ we have

4.1. Local resolutions. Throughout this section we will fix a regular smoothing $\pi: C \to B$ of a nodal curve $C$ with a section $\sigma: B \to C$ of $\pi$ through its smooth locus. We will perform blowups of $C^2$ along divisors of type $Z \times Z$, where $Z$ is a subcurve of the special fiber $C$. Actually, we will restrict our attention to a special class of subcurves, called tails.

\begin{center}
\includegraphics[width=0.5\textwidth]{figure2.png}
\end{center}

\textbf{Figure 2.} The sets $\{(x, y): 0 < x < 1 - y < 1\}$ and $\{(x, y): 0 < 1 - y < x < 1\}$ and the corresponding blow-up.

(3) If the combinatorial type of $qs(D_{x,y})$ is constant on the set $\{(x, y); 0 < x, y < 1\}$, then the Abel map $\alpha_2^{\ast}$ is defined at the point $(N_1, N_2)$. 

Proof. Items (1) and (2) follow directly from [1, Theorem 5.4]. Let us prove Item (3). Let $N = (N_1, N_2) \subset C^2$. Let $\phi: X \to C^2$ and $\phi': Y \to C^2$ be the blowups respectively along $Z_1 \times Z_2$ and $Z_1 \times Z_2^c$ (see Figure 1). By items (1) and (2), we know that $\alpha_2^{\ast} \circ \phi$ and $\alpha_2^{\ast} \circ \phi'$ are defined respectively over the inverse images $\phi^{-1}(N) \cong \mathbb{P}_k^1$ and $\phi'^{-1}(N) \cong \mathbb{P}_k^1$. Let $y_0$ be the distinguished point on $\phi^{-1}(N)$ given by $y_0 = st_{\phi}(Z_1 \times Z_2) \cap st_{\phi}(Z_1 \times Z_2^c) \cap st_{\phi}(Z_1^c \times Z_2^c)$.

Let $x_1, x_2$ be any two points on $\phi'^{-1}(N)$. We know that $\alpha_2^\ast \circ \phi'$ is defined at $x_1$ and $x_2$. For $i = 1, 2$, consider a map $\rho_i: \text{Spec} k[[t]] \to Y$ such that $\rho_i(0) = x_i$ and $\rho_i(\eta)$ is contained in $st_{\phi'}(Z_1 \times Z_2^c)$, as in Figure 3 where 0 and $\eta$ are the special and generic points of $\text{Spec} k[[t]]$, respectively. In particular, we have $\alpha_2^\ast \circ \phi'(x_1) = \alpha_2^\ast \circ \phi(0)$, where $\phi_i = \phi' \circ \rho_i$: $\text{Spec} k[[t]] \to C^2$. By construction, we can lift $\phi_i$ to maps $\rho_i: \text{Spec} k[[t]] \to X$ such that $\rho_i(0) = \rho_2(0) = y_0$. By the same reasoning, we have $\alpha_2^\ast \circ \phi(y_0) = \alpha_2^\ast \circ \rho_i(0)$ for $i = 1, 2$. Then we get:

$$\alpha_2^\ast \circ \phi'(x_1) = \alpha_2^\ast \circ \phi(y_0) = \alpha_2^\ast \circ \phi'(x_2).$$

Hence $\alpha_2^\ast \circ \phi'$ contracts all fibers of $\phi'$. Moreover, arguing as in the proof of [1, Corollary III 11.4], we have $\phi'_i O_Y \cong \mathcal{O}_{C^2}$, since $\phi'$ is birational and $C^2$ is normal. Hence by the Rigidity Lemma (see [1, Lemma 1.15, pag.12]) the map $\alpha_2^\ast \circ \phi'$ factors through $\phi'$, so $\alpha_2^\ast$ is defined at $N$.

\[\square\]

4. The resolution of the degree-2 Abel map

4.1. Local resolutions. Throughout this section we will fix a regular smoothing $\pi: C \to B$ of a nodal curve $C$ with a section $\sigma: B \to C$ of $\pi$ through its smooth locus. We will perform blowups of $C^2$ along divisors of type $Z \times Z$, where $Z$ is a subcurve of the special fiber $C$. Actually, we will restrict our attention to a special class of subcurves, called tails.
Definition 4.1. A δ-tail of a nodal curve $C$ is a connected subcurve $Z$ such that $Z^c$ is connected and $|Z \cap Z^c| = \delta$.

Proposition 4.2. Let $\mu$ be a polarization of degree $k$ and $\mathcal{L}$ be an invertible sheaf of degree $k + 2$ over $\mathcal{C}/\mathcal{B}$. Assume that the components of $C$ are smooth. Consider a point $\mathcal{N} = (N_1, N_2) \in \mathcal{C}^2$, where $N_1, N_2$ are nodes of $C$, with $N_1 = Z \cap Z^c$ for a 1-tail $Z$ of $C$. Then the degree-2 Abel map $\alpha_2^\sharp : \mathcal{C}^2 \to \overline{\mathcal{M}}_\mu$ is defined at $\mathcal{N}$.

Proof. Let $\Gamma$ be the dual graph of $C$ and $X = X_\Gamma$ the associated tropical curve with edges of unitary lengths. Let $v_0$ be the vertex of $\Gamma$ corresponding to $P_0 = \sigma(0)$, and $p_0 \in X$ be the point corresponding to $v_0$. We let $e_1$ and $e_2$ be the edges of $\Gamma$ corresponding to $N_1$ and $N_2$. The tropical Abel map $\alpha_{2, \mathcal{D}_\mathcal{L}^\dagger}^{\trop} : X^2 \to \overline{\mathcal{M}}^\mu_{p_0, \mu}$ takes a pair $(p_{e_1, t_1}, p_{e_2, t_2})$, for real numbers $t_1, t_2 \in (0, 1)$, to the class of the divisor on $X$ given by:

$$\alpha_{2, \mathcal{D}_\mathcal{L}^\dagger}^{\trop}(p_{e_1, t_1}, p_{e_2, t_2}) = [\mathcal{D}_\mathcal{L}^\dagger - p_{e_1, t_1} - p_{e_2, t_2}].$$

We define the divisor $\mathcal{P} = p_{e_1, t_1} - p_{e_2, t_2}$ on $X$. Since $N_1 = Z \cap Z^c$ for a 1-tail $Z$ of $X$, we have that the graph obtained from $\Gamma$ by removing the edge $e_1$ is not connected. Hence the divisor $\mathcal{P}$ on $X$ is principal. So we can write:

$$\alpha_{2, \mathcal{D}_\mathcal{L}^\dagger}^{\trop}(p_{e_1, t_1}, p_{e_2, t_2}) = [\mathcal{D}_\mathcal{L}^\dagger - p_{e_2, t_2}],$$

where $\mathcal{D}_\mathcal{L}^\dagger = \mathcal{D}_\mathcal{L}^\dagger - p_{e_1, 0}$ (which is a divisor on $X$ induced by a divisor on $\Gamma$). So we reduce ourselves to the case of the degree-1 Abel map. As explained in [1, Lemma 5.10] and in the proof of [1, Theorem 5.8], the combinatorial type of the quasistable divisor on $X$ equivalent to $\mathcal{D}_\mathcal{L}^\dagger - p_{e_2, t_2}$ is independent of $t_2$. Hence the combinatorial type of the quasistable divisor on $X$ equivalent to $\mathcal{D}_\mathcal{L}^\dagger - p_{e_1, t_1} - p_{e_2, t_2}$ is independent of the pairs $t_1, t_2 \in (0, 1)$. By Theorem 4.1(3), we deduce that the Abel map $\alpha_2^\sharp$ is already defined at $(N_1, N_2)$. □

Proposition 4.3. Let $\mu$ be a polarization of degree $k$ and $\mathcal{L}$ an invertible sheaf of degree $k + 2$ over $\mathcal{C}/\mathcal{B}$. Assume that the components of $C$ are smooth. Let $Z$ be a 2-tail of $C$ and write $\{N_1, N_2\} = Z \cap Z^c$. Consider the point

$$\mathcal{N} = (N_1, N_2) \in (Z \cap Z^c) \times (Z \cap Z^c) \subset \mathcal{C}^2.$$  

Let $\phi : \mathcal{C}^2 \to \mathcal{C}^2$ be the blowup of $\mathcal{C}^2$ with center $Z \times Z$. Then the rational map

$$\overline{\alpha}_2^\sharp : \mathcal{C}^2 \xrightarrow{\phi} \mathcal{C}^2 \xrightarrow{\alpha_2^\sharp} \overline{\mathcal{M}}_\mu$$

is defined along the rational curve $\phi^{-1}(\mathcal{N}) \cong \mathbb{P}_k^1$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3}
\caption{The blowups $\phi$ and $\phi'$.}
\end{figure}
Proof. We can keep the set-up of Proposition \[4.2\] The tropical Abel map $\alpha_{2,D_L}^{\text{trop}}: X^2 \to J_{p_0,\mu}^{\text{trop}}$ is as in Equation (1). Assume that $t_1 > t_2$. We define the divisor on $X$:

$$\mathcal{P} = p_{e,0} - p_{e,t_1} - p_{e,t_2} + p_{e,t_1 + t_2}.$$ 

Since $\{N_1, N_2\} = Z \cap Z^c$ for a 2-tail $Z$ of $C$, we have that the graph obtained from $\Gamma$ by removing the edges $e_1, e_2$ is not connected. Hence $\mathcal{P}$ is a principal divisor. Then we have

$$\alpha_{2,D_L}^{\text{trop}}(p_{e_1,t_1}, p_{e_2,t_2}) = [\mathcal{D}^1 - p_{e_1,t}],$$

where $t = t_1 - t_2$ and $\mathcal{D}^1 = D^1_L - p_{e_2,0}$ (which is a divisor induced by a divisor on $\Gamma$). So we reduce ourselves to the case of the degree-1 Abel map: as explained in \[1, \text{Lemma 5.10}\] and in the proof of \[1, \text{Theorem 5.8}\], the combinatorial type of the quasistable divisor on $X$ equivalent to $D^1_L - p_{e_1,t}$ is independent of $t$. Hence the combinatorial type of the quasistable divisor on $X$ equivalent to $D^1_L - p_{e_1,t_1} - p_{e_2,t_2}$ is independent of $(t_1, t_2)$ whenever $t_1 > t_2$. A similar reasoning can be done for the case $t_1 < t_2$. Hence, using Theorem \[5.1\] (1), we conclude that the blowup along $Z \times Z$ gives rise to a resolution of $\alpha_L^2$ locally around $N = (N_1, N_2)$. □

**Proposition 4.4.** Let $\mu$ be a polarization of degree $k$ and $\mathcal{L}$ an invertible sheaf of degree $k + 2$ over $C/B$. Assume that the components of $C$ are smooth. Consider the point $N = (N, N) \in C^2$, for a node $N$ of $C$. Let $\phi: \mathcal{C}^2 \to \mathcal{C}^2$ be the blowup of $\mathcal{C}^2$ with center the diagonal subscheme of $\mathcal{C}^2$. Then the rational map

$$\rho^2: \mathcal{C}^2 \xrightarrow{\phi} \mathcal{C}^2 \xrightarrow{\alpha^2} \mathcal{J}_{\mu}$$

is defined along the rational curve $\phi^{-1}(N) \cong \mathbb{P}^1_k$.

Proof. We can keep the set-up of Proposition \[4.2\] The tropical Abel map $\alpha_{2,D_L}^{\text{trop}}: X^2 \to J_{p_0,\mu}^{\text{trop}}$ is as in Equation (1), where $e := e_1 = e_2$. Assume that $t_1 + t_2 < 1$. We define the principal divisor on $X$:

$$\mathcal{P} = p_{e,0} - p_{e,t_1} - p_{e,t_2} + p_{e,t_1 + t_2}.$$ 

Then we have

$$\alpha_{2,D_L}^{\text{trop}}(p_{e,t_1}, p_{e,t_2}) = [\mathcal{D}^1 - p_{e,t}],$$

where $\mathcal{D}^1 = D^1_L - p_{e,0}$ and $t = t_1 + t_2$. So we reduce ourselves to the case of the degree-1 Abel map: as explained in \[1, \text{Lemma 5.10}\] and in the proof of \[1, \text{Theorem 5.8}\], the combinatorial type of the quasistable divisor on $X$ equivalent to $D^1_L - p_{e,t}$ is independent of $t$. Hence the combinatorial type of the quasistable divisor on $X$ equivalent to $D^1_L - p_{e,t_1} - p_{e,t_2}$ is independent of $(t_1, t_2)$ whenever $t_1 + t_2 < 1$.

The reasoning is similar for $t_1 + t_2 > 1$: we just consider

$$\mathcal{P} = p_{e,1} - p_{e,t_1} - p_{e,t_2} + p_{e,t_1 + t_2 - 1},$$

$\mathcal{D}^1 = D^1_L - p_{e,1}$ and $t = t_1 + t_2 - 1$, so that

$$\alpha_{2,D_L}^{\text{trop}}(p_{e,t_1}, p_{e,t_2}) = [\mathcal{D}^1 - p_{e,t}].$$

By Theorem \[5.1\] (2), we deduce that the blowup along the diagonal subscheme of $\mathcal{C}^2$ gives rise to a resolution of $\alpha_L^2 \circ \phi$ locally around $N = (N, N)$. □
4.2. The resolution of the Abel-Jacobi map. Our main goal is to give a complete resolution of the degree-2 Abel-Jacobi map of any nodal curve, namely the map taking a pair \((Q_1, Q_2)\) of points on a curve \(C\) to \(O_C(2P_0 - Q_1 - Q_2)\) for a given smooth point \(P_0\) of \(C\). This is done in Theorem 4.7. Before we need two results.

**Lemma 4.5.** Let \(Z\) be a \(\delta\)-tail of a curve \(C\).

1. If \(Z \cap Z^c \subset Z^c\) for some tail \(Z^c\) of \(C\), then either \(Z \subset Z^c\) or \(Z^c \subset Z^c\).
2. If \(|Z \cap Z^c \cap Z^c| = \delta - 1\) for some tail \(Z^c\) of \(C\), then one of the following conditions holds:
   \[Z \subset Z^c, \quad Z^c \subset Z, \quad Z^c \subset Z^c, \quad Z^c \subset Z^c.\]

**Proof.** See [10, Lemma 2.4].

**Lemma 4.6.** Let \(P_0\) be a smooth point of \(C\). Let \(T = (Z_1, \ldots, Z_h)\) be a sequence of tails of \(C\), where \(Z_i\) is a \(k_i\)-tail with \(k_i \in \{2, 3\}\) and \(P_0 \notin Z_i\). Consider the sequence of blowups
\[
\phi_T: C^2 \overset{\phi_0}{\rightarrow} C^2 \overset{\phi_2}{\rightarrow} C^2 \overset{\phi_1}{\rightarrow} C^2 \overset{\phi_0}{\rightarrow} C^2
\]
where \(\phi_0\) is the blowup of \(C^2\) along its diagonal subscheme and \(\phi_i\) is the blowup of \(C^2\) along the strict transform of the divisor \(Z_i \times Z_i\) of \(C^2\) via \(\phi_i \circ \cdots \circ \phi_i\). Then \(\phi_T\) is independent of the ordering of the sequence \(T\).

**Proof.** Assume that a permutation \(T'\) of \(T\) gives rise to a blowup \(\phi_{T'}\) different from \(\phi_T\). This implies that, locally at a point \(N = (N_1, N_2)\) with \(N_1, N_2\) distinct nodes of \(C\), the blowups \(\phi_T\) and \(\phi_{T'}\) are different. We can assume that locally at \(N\), the blowup \(\phi_T\) has center \(Z_i \times Z_i\) and the blowup \(\phi_{T'}\) has center \(Z_j \times Z_j\), for \(i, j \in \{1, \ldots, h\}\), so that we have \(Z_i \neq Z_j\) and
\[
\{N_1, N_2\} \subset Z_i \cap Z_i \cap Z_j \cap Z_j.
\]
By Lemma 4.5, one of the following conditions holds:
\[Z_i \subset Z_j, \quad Z_j \subset Z_i, \quad Z^c \subset Z^c, \quad Z^c \subset Z^c.
\]
Let \(C_1, C_1', C_2, C_2'\) be the components of \(C\) such that \(N_1 \in C_1 \cap C_1'\) and \(N_2 \in C_2 \cap C_2'\). Since \(\phi_T\) and \(\phi_{T'}\) are different locally at \(N\), we can assume, without loss of generality, that \(C_1 \cup C_2 \subset Z_i\) and \(C_1' \cup C_2' \subset Z_j\). Then we have
\[
C_1 \subset Z_i \cap Z_j, \quad C_2 \subset Z_i \cap Z_j, \quad C_2' \subset Z_j \cap Z_j.
\]
On the other hand:
\[N_1 \in C_1 \cap C_1', \quad Z_j \subset Z_j.
\]
This contradicts Equation (2).

**Theorem 4.7** (Degree-2 Abel-Jacobi map). Let \(\pi: C \to B\) be a regular smoothing of a nodal curve \(C\). Let \(\sigma: B \to C\) be a section of \(\pi\) through its smooth locus and \(\mu\) be the trivial degree-0 polarization. Let \(Z_1, \ldots, Z_N\) be the 2-tails and the 3-tails of \(C\) which do not contain \(\sigma(0)\). Consider the sequence of blowups
\[
\tilde{C}^2_N \overset{\phi_N}{\rightarrow} \cdots \overset{\phi_2}{\rightarrow} \tilde{C}^2_1 \overset{\phi_1}{\rightarrow} \tilde{C}^2_0 \overset{\phi_0}{\rightarrow} C^2
\]
where \(\phi_0\) is the blowup of \(C^2\) along its diagonal subscheme and \(\phi_i\) is the blowup of \(\tilde{C}^2_i\) along the strict transform of the divisor \(Z_i \times Z_i\) of \(C^2\) via \(\phi_0 \circ \cdots \circ \phi_i\). Then the rational map
\[
\phi_0^2 \circ \phi_0 \circ \cdots \circ \phi_N: \tilde{C}^2_N \to \tilde{J}_\mu^J
\]
is a morphism, i.e., it is defined everywhere.
Before proving the theorem, we need to recall a result in [10] describing how to convert the sheaf $O_C(2P_0 - Q_1 - Q_2)$ into a $(\sigma, \mu)$-quasistable sheaf, where $P_0, Q_1, Q_2$ are smooth points of the nodal curve $C$. We will give the graph-theoretical equivalent of this result, which suits better with our purposes. More precisely, given a graph $\Gamma$ and vertices $v_0, v_1, v_2 \in V(\Gamma)$, we will describe the $(v_0, \mu)$-quasistable divisor on $\Gamma$ equivalent to $2v_0 - v_1 - v_2$ (see Theorem [4,10]). Let $\Gamma$ be a graph. Given a subset $V \subset V(\Gamma)$, we denote by $\Gamma(V)$ the subgraph of $\Gamma$ whose set of vertices is $V$ and whose edges are the edges of $\Gamma$ connecting two (possibly coinciding) vertices of $V$.

**Definition 4.8.** A hemisphere of $\Gamma$ is a subset $H \subset V(\Gamma)$ such that $\Gamma(H)$ and $\Gamma(H^c)$ are connected subgraphs of $\Gamma$. A $\delta$-hemisphere of $\Gamma$ is a hemisphere $H$ such that $|E(H, H^c)| = \delta$.

We denote by $\mathcal{H}_{\Gamma, \delta}$ the set of $\delta$-hemispheres of $\Gamma$. Given subsets $V, W \subset V(\Gamma)$, we define:

$$\mathcal{H}_{\Gamma, \delta}(V, W) := \{H \in \mathcal{H}_{\Gamma, \delta} | V \subset H^c \text{ and } W \subset H\}.$$

Let $S$ be a finite set. We say that a set $\mathcal{H}$ of subsets of $S$ is union-closed (respectively, intersection-closed) if $H_1 \cup H_2 \in \mathcal{H}$ (respectively, $H_1 \cap H_2 \in \mathcal{H}$) for every $H_1, H_2 \in \mathcal{H}$. We note that every non-empty intersection-closed set has a unique minimal element and every non-empty union-closed set has a unique maximal element.

**Proposition 4.9.** Let $\Gamma$ be a graph and $v_0, v_1, v_2$ vertices of $\Gamma$. Then the sets $\mathcal{H}_{\Gamma, 1}(v_0, v_1)$ and $\mathcal{H}_{\Gamma, 2}(v_0, \{v_1, v_2\})$ are union-closed and intersection-closed.

**Proof.** See [5] Lemma 4.3.4 and [10] Section 3 and Proposition 3.1. □

**Definition 4.10.** Given subsets $V, W \subset V(\Gamma)$, we say that $W$ is $V$-free if $E(V, V^c) \cap E(W, W^c) = \emptyset$.

**Remark 4.11.** A 1-hemisphere $H$ is $H'$-free for every $\delta$-hemisphere $H' \neq H, H^c$. If $H_1, H_2 \subset V(\Gamma)$ are $V$-free hemispheres, for some $V \subset V(\Gamma)$, then $H_1 \cap H_2$ and $H_1 \cup H_2$ are also $V$-free.

**Definition 4.12.** Let $\Gamma'$ be a subdivision of $\Gamma$. Let $V$ be a subset of $V(\Gamma')$. We say that an edge $e \in E(\Gamma)$ is fully contained in $V$ if $V$ contains the vertices incident to $e$ and all the vertices over $e$. Note that when $\Gamma' = \Gamma$, this simply means that $e \in E(V, V)$.

**Lemma 4.13.** Let $H_2$ and $H_3$ be a 2-hemisphere and a 3-hemisphere. Write $E(H_2, H_3^c) = \{f_1, f_2\}$ and $E(H_3, H_2^c) = \{e_1, e_2, e_3\}$. Assume that the intersection $H = H_2 \cap H_3$ is non-empty and properly contained in $H_2$ and $H_3$. Assume that $H_2 \cup H_3 \neq V(\Gamma)$. Then, up to reordering the indices, one of the following properties hold:

1. $H$ is a 2-hemisphere such that $E(H, H^c) = \{f_1, e_1\}$, with $f_1$ fully contained in $H_3$ and $e_1$ fully contained in $H_2$, while $f_2$ is fully contained in $H_3^c$ and $e_2, e_3$ are fully contained in $H_2^c$.

2. $H$ is a 3-hemisphere such that $E(H, H^c) = \{f_1, e_1, e_2\}$, with $f_1$ fully contained in $H_3$ and $e_1, e_2$ fully contained in $H_2$, while $f_2$ is fully contained in $H_3^c$ and $e_3$ is fully contained in $H_2^c$.

**Proof.** By the hypothesis, the sets $H, H_2 \setminus H, H_3 \setminus H$ and $H_2^c \cap H_3^c$ are nonempty and form a partition of $V(\Gamma)$. Since $H_2$ is connected, we have that $E(H, H_3 \setminus H)$ is nonempty. However $E(H, H_3 \setminus H) \subseteq E(H_2, H_3^c) = \{f_1, f_2\}$. We assume that $f_1 \in E(H, H_3 \setminus H)$. Arguing in a similar manner, using that $H_3^c$ is connected, we have that $f_2 \in E(H_2 \setminus H, H_2^c \cap H_3^c)$. Even more, we can conclude that $e_1 \in E(H, H_2 \setminus H)$ and $e_3 \in E(H_3 \setminus H, H_2^c \cap H_3^c)$ (see Figure [3]). Since $H_2$ is a 2-hemisphere, we have that

$$E(H, H_3 \setminus H) = \{f_1\}, \quad E(H_2 \setminus H, H_3 \setminus H) = \emptyset,$$

$$E(H, H_2^c \cap H_3^c) = \emptyset, \quad E(H_2 \setminus H, H_2^c \cap H_3^c) = \{f_2\}.$$

Hence $E(H_3, H_2^c) = E(H, H_2 \setminus H) \cup E(H_3 \setminus H, H_2^c \cap H_3^c)$. Therefore, the edge $e_2$ only has two possibilities: it belongs to either $E(H_3 \setminus H, H_2^c \cap H_3^c)$ or $E(H, H_2 \setminus H)$. In the former case $H$ satisfies the conditions in item (1), while in the latter case it satisfies the conditions in item (2).
we have a sequence of nested 3-hemispheres

\[ \exists j < i \quad \text{such that } H_j = H \cup (H_{j+1} \setminus H_j) \]

is connected and \( H \) and \( H_j \) are connected by a single edge, hence each \( H \) and \( H_j \) must be connected. The same reasoning holds for \( H_2 \setminus H \) and \( H_2 \setminus H_0 \), using the fact that \( H_2 \setminus H_0 \) is connected. Hence \( H' = (H_3 \setminus H) \cup (H_2 \setminus H) \cup (H_2 \setminus H_0) \) is connected, which means that \( H \) is a hemisphere. \( \square \)

We let \( H_{2,1} \) be the minimal element of \( \mathcal{H}_{G,2}(v_0, \{v_1, v_2\}) \) (which exists and is unique by Proposition 4.9). Define

\[ \mathcal{H}^\text{free}_{G,2}(v_0, \{v_1, v_2\}) = \{H_{2,1}, \ldots, H_{2,m_2}\}, \]

where \( H_{2,i} \) is the minimal element of the set of hemispheres of \( \mathcal{H}_{G,2}(v_0, \{v_1, v_2\}) \) that are \( H_{2,j} \)-free for every \( j < i \leq m_2 \) and containing \( H_{2,i-1} \). The hemisphere \( H_{2,i} \) exists and is unique since the set

\[ \{ H \in \mathcal{H}_{G,2}(v_0, \{v_1, v_2\}) \mid H_{2,i-1} \subset H \text{ is } H_{2,j} \text{-free for } j = 1, \ldots, i - 1 \} \]

is intersection-closed by Proposition 4.9 and Remark 4.11. Notice that we have a sequence of nested 2-hemispheres

\[ H_{2,1} \subset H_{2,2} \subset \cdots \subset H_{2,m_2}. \]

We let \( \mathcal{H}'_{G,3}(v_0, \{v_1, v_2\}) \) be the subset of \( \mathcal{H}_{G,3}(v_0, \{v_1, v_2\}) \) of the hemispheres that are \( H \)-free for every \( H \in \mathcal{H}^\text{free}_{G,2}(v_0, \{v_1, v_2\}) \).

**Proposition 4.14.** The subset \( \mathcal{H}'_{G,3}(v_0, \{v_1, v_2\}) \) is intersection-closed.

*Proof.* See \( [10] \) Proposition 3.5. \( \square \)

We let \( H_{3,1} \) be the minimal element of \( \mathcal{H}'_{G,3}(v_0, \{v_1, v_2\}) \) and define

\[ \mathcal{H}^\text{free}_{G,3}(v_0, \{v_1, v_2\}) = \{H_{3,1}, \ldots, H_{3,m_3}\}, \]

where \( H_{3,i} \) is the minimal element of the set of hemispheres \( \mathcal{H}'_{G,3}(v_0, \{v_1, v_2\}) \) that are \( H_{3,j} \)-free for every \( j < i \leq m_3 \) and containing \( H_{3,i-1} \). As before, we have that \( \mathcal{H}^\text{free}_{G,3} \) is well-defined. Notice that we have a sequence of nested 3-hemispheres

\[ H_{3,1} \subset H_{3,2} \subset \cdots \subset H_{3,m_3}. \]

**Remark 4.15.** Let \( k = 2, 3 \). Notice that we have a natural orientation on every edge \( e \in E(H_{k,i}, H_{k,i}) \) such that \( s(e) \in H_{k,i} \) and \( t(e) \in H_{k,i} \). Moreover, if \( e \in E(H_{k,i}, H_{k,i}) \), then \( t(e) \in H_{k,i+1} \).
Finally we set:

$$\mathcal{F}_Γ(v_0, v_1, v_2) = \mathcal{H}_{Γ,1}(v_0, v_1) \uplus \mathcal{H}_{Γ,1}(v_0, v_2)$$

$$\uplus \mathcal{H}_{Γ,2}^{\text{free}}(v_0, \{v_1, v_2\}) \uplus \mathcal{H}_{Γ,3}^{\text{free}}(v_0, \{v_1, v_2\}).$$

Notice that the same 1-hemisphere could belong in both $\mathcal{H}_{Γ,1}(v_0, v_1)$ and $\mathcal{H}_{Γ,1}(v_0, v_2)$.

For every subset $V ⊂ V(Γ)$, we let $\text{div}(V)$ be the principal divisor on $Γ$ given by:

$$\text{div}(V) = \sum_{e ∈ E(V, V')} (s(e) − t(e)),$$

where the orientation is chosen such that $s(e) ∈ V$ for every $e ∈ E(V, V')$. Let $v_0, v_1, v_2$ be vertices of $Γ$ and $μ$ the trivial degree-0 polarization on $Γ$. The following result tells us how to find the $(v_0, μ)$-quasistable divisor equivalent to the divisor $2v_0 − v_1 − v_2$.

**Theorem 4.16.** Let $Γ$ be a graph and $v_0, v_1, v_2$ vertices of $Γ$. Then

$$2v_0 − v_1 − v_2 − \sum_{V ∈ \mathcal{F}_Γ(v_0, v_1, v_2)} \text{div}(V)$$

is the $(v_0, μ)$-quasistable divisor equivalent to $2v_0 − v_1 − v_2$.

**Proof.** See [10, Theorem 5.3].

Before going on with the proof of Theorem 4.17 we need to introduce a divisor on a tropical curve $X$ attached to a hemisphere of its underlying graph.

**Definition 4.17.** Let $X$ be a tropical curve and $(Γ, ℓ)$ be a model of $X$. Let $v_0$ be a vertex of $Γ$. For a hemisphere $H$ of $Γ$, consider the orientation on an edge $e ∈ E(H, H')$ from the vertex of $e$ contained in $H$ to the vertex of $e$ contained in $H'$ (see Remark 4.15). We define the divisor

$$\mathcal{P}_H = \sum_{e ∈ E(H, H')} p_e,0 − \sum_{e ∈ E(H, H')} p_e,ℓ(e).$$

**Remark 4.18.** Notice that if $ℓ(e) = ℓ(e')$ for every $e, e' ∈ E(H, H')$, then $\mathcal{P}_H$ is a principal divisor on $X$.

**Proof of Theorem 4.17** We can assume that all the components of $C$ are smooth. Indeed, the general case follows from the case in which the components of $C$ are smooth arguing as in the last part of the proof of [11, Theorem 5.8], and using [10, Theorem 1.3].

Since quasistability is an open property by [12, Proposition 34], it is enough to check that the global blowup of $C^2$ described in the statement is a blowup resolving the Abel map $α^{2,0}_{O_C}: C^2 → \mathcal{F}_μ$ locally around any point $N = (N_1, N_2)$ of $C^2$ for $N_1$ a node of $C$.

If $N_1 = Z ∩ Z'$ for a 1-tail $Z$ of $C$, the result follows from Proposition 4.2. If $N_1 = N_2$, the result follows from Proposition 4.3. So we will assume, through the rest of the proof, that $N_1 ≠ N_2$ and neither $N_1$ nor $N_2$ disconnects $C$.

We will use Theorem 5.1. Let $Γ_0$ be the dual graph of $C$, and let $X = X_{Γ_0}$, namely, the tropical curve whose underlying graph is $Γ_0$ with all unitary lengths. Let $v_0$ be the vertex of $Γ_0$ corresponding to $P_0 = σ(0)$, and $p_0 ∈ X$ be the point corresponding to $v_0$. We have $D_{O_C} = 2p_0$.

The tropical Abel map $α^{\text{top}}_{2,D_{O_C}}: X^2 → J^{\text{top}}_{p_0,μ}$ takes a pair $(p_{e_1,t_1}, p_{e_2,t_2})$, for edges $e_1, e_2 ∈ E(Γ_0)$ and real numbers $t_1, t_2 ∈ [0, 1]$, to:

$$α^{\text{top}}_{2,D_{O_C}}(p_{e_1,t_1}, p_{e_2,t_2}) = [2p_0 − p_{e_1,t_1} − p_{e_2,t_2}].$$

Let $(Γ, ℓ)$ be the model of $X$ such that $Γ$ is the refinement of $Γ_0$ obtained by inserting vertices $v_{e_1}, v_{e_2}$ in the interior of $e_1, e_2$, respectively, with $ℓ([s(e_i), v_{e_i}]) = t_i$. We let $K_{2,1}$ be the minimal element of $\mathcal{H}_{Γ,2}^{\text{free}}(v_0, \{v_{e_1}, v_{e_2}\})$ and $K_{3,1}$ be the minimal element of $\mathcal{H}_{Γ,3}^{\text{free}}(v_0, \{v_{e_1}, v_{e_2}\})$. 
We have three cases to consider.

1. We have that \( E(v_{e_i}, \{v_{e_i}\}^c) \cap E(K_{2,1}, K_{2,1}^c) \neq \emptyset \), for every \( i = 1, 2 \).
2. We have that
   \[
   E(v_{e_1}, \{v_{e_1}\}^c) \cap E(K_{2,1}, K_{2,1}^c) \neq \emptyset.
   \]
   \[
   E(v_{e_2}, \{v_{e_2}\}^c) \cap E(K_{2,1}, K_{2,1}^c) = \emptyset.
   \]

   We distinguish 3 subcases:
   
2.a) There exists a 3-hemisphere \( K_3 \) containing \( v_{e_1} \) and \( v_{e_2} \) and not containing \( v_0 \) such that
   \[
   E(v_{e_1}, \{v_{e_1}\}^c) \cap E(K_3, K_3^c) \neq \emptyset, \text{ for every } i = 1, 2.
   \]
2.b) Every 3-hemisphere \( K_3 \) containing \( v_{e_1} \) and \( v_{e_2} \) and not containing \( v_0 \) satisfies the condition
   \[
   E(v_{e_2}, \{v_{e_2}\}^c) \cap E(K_3, K_3^c) = \emptyset
   \]
   and there exists a 3-hemisphere \( K_3' \) such that
   \[
   E(v_{e_1}, \{v_{e_1}\}^c) \cap E(K_3', K_3'^c) \neq \emptyset.
   \]
2.c) Every 3-hemisphere \( K_3 \) containing \( v_{e_1} \) and \( v_{e_2} \) and not containing \( v_0 \) satisfies the condition
   \[
   E(v_{e_2}, \{v_{e_2}\}^c) \cap E(K_3, K_3^c) = \emptyset, \text{ for every } i = 1, 2.
   \]
   Notice that there are no other subcases to consider, because if the two conditions:
   \[
   E(v_{e_2}, \{v_{e_2}\}^c) \cap E(K_3, K_3^c) \neq \emptyset.
   \]
   \[
   E(v_{e_1}, \{v_{e_1}\}^c) \cap E(K_3, K_3^c) = \emptyset
   \]
   hold for some 3-hemisphere \( K_3 \), then, by Lemma 4.13, we have that \( K_{2,1} \cap K_3 \) is a 3-
   hemisphere (because \( K_{2,1} \) is minimal) and it would satisfy the condition in case (2.a).

3. We have
   \[
   E(v_{e_i}, \{v_{e_i}\}^c) \cap E(K_{2,1}, K_{2,1}^c) = \emptyset, \text{ for every } i = 1, 2.
   \]

   We distinguish 3 subcases:
   
3.a) We have that \( E(v_{e_i}, \{v_{e_i}\}^c) \cap E(K_{3,1}, K_{3,1}^c) \neq \emptyset, \text{ for every } i = 1, 2. \)
3.b) We have that
   \[
   E(v_{e_1}, \{v_{e_1}\}^c) \cap E(K_{3,1}, K_{3,1}^c) \neq \emptyset.
   \]
   \[
   E(v_{e_2}, \{v_{e_2}\}^c) \cap E(K_{3,1}, K_{3,1}^c) = \emptyset
   \]
3.c) We have that \( E(v_{e_i}, \{v_{e_i}\}^c) \cap E(K_{3,1}, K_{3,1}^c) = \emptyset, \text{ for every } i = 1, 2. \)

   We discuss the above cases. Case (1) follows from Proposition 4.3.

Case (2.a). We assume that the orientation of \( e_1 \) and \( e_2 \) satisfies the condition \( s(e_1), s(e_2) \in K_3. \)

Recall that \( t_i = f([s(e_i), v_{e_i}]). \)

We consider the refinement \( \Gamma' \) of \( \Gamma_0 \) by adding two vertices over each edge. Of course, \( \Gamma' \) is a

refinement of \( \Gamma \). We denote by \( \psi \) the natural function

\[
\psi: E(\Gamma') \to E(\Gamma_0)
\]

taking an edge \( e \) of \( \Gamma' \) to the edge \( f \) of \( \Gamma_0 \) if \( e \) is obtained by subdividing \( f \).

For \( i = 1, 2 \), we already have the vertex \( v_{e_i} \) over the edge \( e_i \), so we will only add another vertex \( v_{e_i}' \). As illustrated in Figure 5 if \( t_1 < t_2 \), the vertices over \( e_1 \) will be ordered as follows:

\[
\begin{align*}
  s(e_1) & \quad v_{e_1} & \quad v_{e_1}' & \quad t(e_1) \\
  s(e_2) & \quad v_{e_2}' & \quad v_{e_2} & \quad t(e_2)
\end{align*}
\]

\[
\begin{align*}
  s(e_1) & \quad v_{e_1}' & \quad v_{e_1} & \quad t(e_1) \\
  s(e_2) & \quad v_{e_2} & \quad v_{e_2}' & \quad t(e_2)
\end{align*}
\]

\textbf{Figure 5.}
Throughout the proof of case (2a), we will assume that $t_1 < t_2$, leaving to the reader the case $t_1 > t_2$. Our goal will be to find a length function $\ell'$ on $\Gamma'$ so that $(\Gamma', \ell')$ is a model of $X$, and the divisors $\mathcal{P}_H$ for every $H \in \mathcal{F}_\Gamma(v_0, v_{e_1}, v_{e_2})$ are principal on $X$. This allows us to conclude the proof. Indeed, using Remark 2.1 and Theorem 4.16 we get that the divisor

\begin{equation}
\mathcal{D}_{t_1, t_2} := 2p_0 - p_{e_1, t_1} - p_{e_2, t_2} + \sum_{H \in \mathcal{F}_\Gamma(v_0, v_{e_1}, v_{e_2})} \mathcal{P}_H
\end{equation}

is $(p_0, \mu)$-quasistable. Hence Theorem 5.1 (1) tells us that the blowup illustrated on the left hand side of Figure 8 with $Z_1 = Z_2 = Z$, where $Z$ is the 3-tail of $C$ induced by $K_3 \cap V(\Gamma)$, gives rise to a resolution of the Abel map $\alpha_{3, \mathfrak{C}}^2$ locally at $(N_1, N_2)$. This is the blowup locally around $(N_1, N_2)$ prescribed by the global blowup in the statement of Theorem 4.7.

We proceed with the construction of the length function $\ell'$. We write

\begin{equation}
\begin{align*}
\mathcal{H}^{\text{free}}_{t_1, t_2}(v_0, \{v_{e_1}, v_{e_2}\}) &= \{H_{2,1}, \ldots, H_{2,m_2}\} \\
\mathcal{H}^{\text{free}}_{t_1, t_2}(v_0, \{v_{e_1}, v_{e_2}\}) &= \{H_{3,1}, \ldots, H_{3,m_3}\}.
\end{align*}
\end{equation}

We define a sequence $f_1, f_2, \ldots, f_k$ of edges of $\Gamma_0$ as illustrated in Figure 8 with $f_1 = e_1$ and $f_2 \in E(H_{2,1}, H_{3,1}^c)$, and where the other edges are chosen as follows. Assume that $t_1 < t_2$. The edges of the sequence satisfy

\begin{equation}
f_{2i+1}, f_{2i+2} \in \psi(E(H_{2,3i+1}, H_{2,3i+1}^c)), \quad f_{2i+1}, f_{2i+2} \in \psi(E(H_{2,3i+2}, H_{2,3i+2}^c))
\end{equation}

Notice that if $k$ is odd with $k = 2k' + 1$, then

\begin{equation}
\psi(E(H_{2,3i+1}, H_{2,3i+1}^c)) = \psi(E(H_{2,3i+2}, H_{2,3i+2}^c)) = \psi(E(H_{2,3i+3}, H_{2,3i+3}^c))
\end{equation}

for every $i \geq k'$. If $k$ is even with $k = 2k'$, then for every $i \geq k'$ we have

\begin{equation}
\psi(E(H_{2,3i}, H_{2,3i}^c)) = \psi(E(H_{2,3i+1}, H_{2,3i+1}^c)) = \psi(E(H_{2,3i+2}, H_{2,3i+2}^c)).
\end{equation}

Now we consider the 3-hemispheres. If $\psi(E(H_{3,1}, H_{3,1}^c))$ and $\psi(E(H_{2,1}, H_{2,1}^c))$ are disjoint for every $i$, let us define a length function $\ell'$ on the set of edges of $\Gamma'$ so that $(\Gamma', \ell')$ is a model for $X$. We will assume that $k = 2k'$ is even (see Figure 6 also see Figure 7 for the case $t_1 > t_2$), leaving to the reader to work out the other case (see Figure 8). For every $e \in E(\Gamma')$, we define:

\begin{equation}
\ell'(e) = \begin{cases}
\frac{1-t_1}{2} & \text{if } e \in E(H_{2,3i+1}, H_{2,3i+1}^c) \text{ or } e \in E(H_{2,3i+2}, H_{2,3i+2}^c) \text{ for some } i < k' \\
t_1 & \text{if } \psi(e) = f_k \text{ and } e \notin E(H_{2,3k'-2}^c, H_{2,3k'-2}^c) \cup E(H_{2,3k'-1}^c, H_{2,3k'-1}^c) \\
1/3 & \text{otherwise.}
\end{cases}
\end{equation}

For every edge $f \in E(\Gamma')$ we have that $\sum_{e \in \psi^{-1}(f)} \ell'(e) = 1$. Indeed if $f = f_i$, then the sum of the lengths will be $t_1 + \frac{1-t_1}{2} + \frac{1-t_1}{2} = 1$, while if $f \notin \{f_1, \ldots, f_k\}$, then the 3 edges in $\psi^{-1}(f)$ will have length $1/3$. So $(\Gamma', \ell')$ is a model of $X$. Notice that the divisors $\mathcal{P}_{H_{3,i}}$ and $\mathcal{P}_{H_{2,i}}$ are principal divisors by Remark 4.18. This conclude the proof in this case.

We are left to consider the case in which $\psi(E(H_{3,1}, H_{3,1}^c))$ and $\psi(E(H_{2,1}, H_{2,1}^c))$ have a common edge for some $i$. In this case, this edge must be $f_k$.

We claim that $k$ is odd. First we prove that $H_{3,1}$ contains the vertices of $\Gamma'$ incident to $f_{2i}$ and the vertices over $f_{2i}$ for every $i$. Let us denote by $e_2, f, f_k$ the edges of $\psi(E(H_{3,1}, H_{3,1}^c))$. The intersection $H_{3,1} \cap H_{2,1}$ cannot be a 2-hemisphere, otherwise we would contradict the minimality of $H_{2,1}$. Indeed, the fact that $E(v_{e_2}, \{v_{e_2}\}^c) \cap E(H_{3,1}, H_{3,1}^c) \neq \emptyset$ implies that $E(v_{e_2}, \{v_{e_2}\}^c) \cap E(H_{3,1} \cap H_{2,1}, (H_{3,1} \cap H_{2,1})^c) \neq \emptyset$,
so $H_{3,1} \cap H_{2,1} \subseteq H_{2,1}$. By Lemma 4.13, we see that $\psi(E(H_{3,1} \cap H_{2,1}, (H_{3,1} \cap H_{2,1})^c)) = \{e_2, f_1\}$, with $e_2$ and $f$ fully contained in $H_{2,1}$ and $f_2$ fully contained in $H_{3,1}$.

We now iterate the reasoning. Intersecting $H_{3,1} \cap H_{2,3}$ (see Figure 8), we must have that $e_2, f \in \psi(E(H_{3,1} \cap H_{2,3}, (H_{3,1} \cap H_{2,3})^c))$, hence, by Lemma 4.13, $H_{3,1} \cap H_{2,3}$ is a 3-hemisphere, and $f_3$ is fully contained in $H_{3,1}$ (as neither $f_2$ nor $f_3$ is fully contained in $H_{3,1}$). Considering $H_{3,1} \cap H_{2,4}$, we have that $f_4$ must be fully contained in $H_{3,1}^c$, and iterating this process we see that $f_2i$ is fully contained in $H_{3,1}^c$ for every $i = 1, \ldots, [\frac{k}{2}]$. So $k$ must be odd and we write $k = 2k' + 1$.

As illustrated in Figure 8 for every $e \in E(\Gamma')$, we define:

$$\ell'(e) = \begin{cases} 
\frac{1-t_1}{2} & \text{if } e \in E(H_{2,3i+1}, H_{2,3i+2}) \text{ or } E(H_{2,3i+1}, H_{2,3i+2}) \text{ for some } i < k', \\
t_1 & \text{if } e \in E(H_{2,3i}, H_{2,3i+1}) \text{ or } t(e) = v_1, \\
1 - t_2 & \text{if } e \in E(H_{2,3i+1}, H_{2,3i+1}) \text{ for any } i, \\
t_2 - t_1 & \text{if } e \in E(H_{2,3i+2}, H_{2,3i+2}) \text{ for any } i, \\
t_1 & \text{if } e \in E(H_{2,3i}, H_{2,3i}) \text{ for any } i, \\
1/3 & \text{if } \psi(e) \notin \psi(E(H_{j,i}, H_{j,i}^c)) \text{ for any } j = 2, 3 \text{ and } i.
\end{cases}$$

The remaining edges can be assigned lengths in a such way that $\sum_{e \in \psi^{-1}(f)} \ell'(e) = 1$ for every $f \in E(\Gamma_0)$, so $(\Gamma', \ell')$ is a model of $X$. Again, by Remark 4.18, the divisors $\mathcal{P}_{H_{2,i}}$ and $\mathcal{P}_{H_{3,i}}$ are principal divisors, finishing the proof.

Case (2.b). Consider the refinement $\Gamma'$ of $\Gamma_0$ by adding one vertex over each edge. Notice that $\Gamma'$ is a refinement of $\Gamma$. Let $H_{2,i}$ and $H_{3,j}$ be defined as in Equation (4). Let $k$ be the integer such that $|\psi(E(H_{2,i}, H_{2,i})) \cap \psi(E(H_{2,i+1}, H_{2,i+1}))| = 1$ for every $i \leq k - 1$, and

$$\psi(E(H_{2,k+2i+1}, H_{2,k+2i+1}^c)) = \psi(E(H_{2,k+2i+2}, H_{2,k+2i+2}^c)), \text{ for } i \geq 0.$$
If $k$ is even, we define the length $\ell'$ on $\Gamma'$ as follows:

$$
\ell'(e) = \begin{cases} 
1 - t_1 & \text{if } e \in E(H_{2,2i+1}, H_{2,2i+1}) \text{ with } i = 0, \ldots, \left\lfloor \frac{m_1-1}{2} \right\rfloor \\
t_1 & \text{if } e \in E(H_{2,2i}, H_{2,2i}) \text{ with } i = 1, \ldots, \left\lceil \frac{m_1}{2} \right\rceil \text{ or } t(e) = v_1 \\
1 - t_1 & \text{if } e \in E(H_{3,2i}, H_{3,2i}) \text{ with } i = 1, \ldots, \left\lceil \frac{m_2}{2} \right\rceil \\
t_1 & \text{if } e \in E(H_{3,2i+1}, H_{3,2i+1}) \text{ with } i = 0, \ldots, \left\lfloor \frac{m_2-1}{2} \right\rfloor \\
1/2 & \text{if } \psi(e) \notin \psi(E(H_{r,i}, H_{r,i}^c)) \text{ for any } r = 2, 3 \text{ and } i.
\end{cases}
$$

The remaining edges can be assigned lengths in a such way that $\sum_{e \in \psi^{-1}(f)} \ell'(e) = 1$ for every $f \in E(\Gamma_0)$, so $(\Gamma', \ell')$ is a model of $X$. 

Figure 8. Attributing lengths to the edges of $\Gamma'$, for $t_1 < t_2$ and $k$ odd. In this case, $v_{e_2}$ is contained in $H_{2,i}$ for every $i = 1, \ldots, m_2$.

Figure 9. Attributing lengths to the edges of $\Gamma'$ for $t_1 > t_2$ and $k$ odd. In this case, $v_{e_2}$ is contained in $H_{2,i}$ for every $i = 1, \ldots, m_2$. 

When \( k \) is odd the situations is similar (see Figure [10]): the unique difference is that we define \( \ell'(e) = 1 - t_1 \) for \( e \in E(H_{3,2i+1}, H_{3,2i+1}^c) \) and \( \ell'(e) = t_1 \) for \( e \in E(H_{3,2i}, H_{3,2i}^c) \).

![Figure 10. Attributing lengths to the edges of \( \Gamma' \) for \( k \) odd.](image)

As in Case (2.a), we have that the combinatorial type of the divisor \( D_{t_1,t_2} \) defined in Equation (3) does not depend on \( 0 < t_1, t_2 < 1 \). Hence Theorem 3.1 (3) ensures that the Abel map \( \alpha_{\mathbb{C}}^2 \) is already defined at \((N_1, N_2)\), as given by the global blowup in the statement of Theorem 4.7.

Case (2.c). This case is the same as Case (2.b) except that \( \psi(E(H_{2,i}, H_{2,i}^c)) \cap \psi(E(H_{3,j}, H_{3,j}^c)) = \emptyset \) for every \( i = 1, \ldots, m_2 \) and \( j = 1, \ldots, m_3 \). So we can freely assign lengths to the edges in \( E(H_{3,j}, H_{3,j}^c) \). The conclusion is the same as in Case (2.b).

Case (3.a). This case follows the same steps in Case (2.a): the difference is that \( k = 0 \) and the sequence of edges \( f_1, \ldots, f_k \) is empty. The conclusion is the same as in Case (2.a).

Case (3.b). This case follows the same steps in Case (2.b): the difference is that \( k = 0 \). The conclusion is the same as in Case (2.b).

Case (3.c). In this case, we do not have to further refine \( \Gamma_0 \) as \( E(H, H^c) \) does not contain any edge incident to \( v_{e_1} \) or \( v_{e_2} \), for every \( H \) in \( \mathcal{F}_T(v_0, v_{e_1}, v_{e_2}) \). So \( P_H \) is principal on \( X \) for every \( H \in \mathcal{F}_T(v_0, v_{e_1}, v_{e_2}) \). As in the previous cases, using Remark 2.1 and Theorem 4.16, the divisor

\[
D_{t_1,t_2} := 2p_0 - p_{e_1,t_1} - p_{e_2,t_2} + \sum_{H \in \mathcal{F}_T(v_0, v_{e_1}, v_{e_2})} P_H
\]

on the tropical curve \( X \) is \((p_0, \mu)\)-quasistable and equivalent to \( 2p_0 - p_{e_1,t_1} - p_{e_2,t_2} \). Since the combinatorial type of \( D_{t_1,t_2} \) is independent of \( t_1 \) and \( t_2 \), it follows from Theorem 3.1 (3) that the Abel map \( \alpha_{\mathbb{C}}^2 \) is already defined at \((N_1, N_2)\), as prescribed by the global blowup in the statement of Theorem 4.7. \( \square \)
Given a regular smoothing \( f : C \to B \) of a curve, consider the blowup \( \tilde{C} \to C \) giving rise to a resolution of the degree-2 Abel map \( \alpha^2_{\tilde{C}} \), as in Theorem [4.17]. Since the locus we are blowing up is invariant under the natural action of \( S_2 \) on \( C^2 \), we can take the quotient
\[
\text{Sym}^2(\tilde{C}) = \tilde{C}/S_2.
\]
We thus obtain a map:
\[
(5) \quad \beta_2 : \text{Sym}^2(\tilde{C}) \to \mathcal{J}_\mu(C)
\]
resolving the rational “symmetrized” Abel map \( \text{Sym}(C^2) \to \mathcal{J}_\mu(C) \).

**Definition 4.19.** Let \( C \) be a curve. We say that \( C \) is pseudo-hyperelliptic if it has a simple torsion-free rank-1 sheaf \( I \) of degree 2 with non-negative degree over every component such that \( h^0(C, I) \geq 2 \).

Recall that a curve is weakly-hyperelliptic if it has a degree-2 balanced invertible sheaf (see [11] for more details). If a stable curve is hyperelliptic, then it is weakly-hyperelliptic.

**Theorem 4.20.** Let \( C \) be a curve with no separating nodes. The following properties hold.

1. \( C \) is pseudo-hyperelliptic if and only if, for some (every) regular smoothing \( C \to B \) of \( C \), the map \( \beta_2 : \text{Sym}^2(C) \to \mathcal{J}_\mu(C) \) is not injective.
2. if \( C \) is stable and weakly-hyperelliptic, then \( C \) is pseudo-hyperelliptic.
3. if \( C \) is stable and has a simple torsion-free rank-1 sheaf \( I \) of degree 2 with non-negative degree over every component such that \( h^0(C, I) \geq 2 \), then \( I \) is invertible.

**Proof.** If \( C \) has a rational component \( E \) such that \( |E \cap E^c| \leq 2 \) then it is easy to see that \( C \) is pseudo-hyperelliptic and weakly-hyperelliptic, and that \( \beta_2 \) is not injective. So, we will assume that \( C \) is stable.

Assume that \( C \) is stable and pseudo-hyperelliptic. Let \( I \) be a torsion-free rank-1 sheaf satisfying the condition in Definition [4.19]. Let \( \mathbb{P}(I) := \text{Proj}(\text{Sym}(I)) \to C \) be the semistable modification of \( C \) where we add a rational curve over the nodes of \( C \) where \( I \) is not locally free. We consider the invertible sheaf \( L := \mathcal{O}_{\mathbb{P}(I)}(1) \), so that we have \( I = f_*(L) \) and \( L \) has degree 1 on the exceptional components (see [8] Section 5)). Then, \( L \) has non-negative degree on every component of \( \mathbb{P}(I) \) and \( h^0(\mathbb{P}(I), L) \geq 2 \).

We will apply [4] Theorem 5.9] to \( \mathbb{P}(I) \) and \( L \). We have two cases. In the first case, there is a component \( C_0 \) of \( \mathbb{P}(I) \) satisfying the following property. Let \( Z_1, \ldots, Z_n \) be the connected components of \( C_0^c \). Then
\[
(6) \quad h^0(L|_{C_0}) \geq 2, \quad L|_{C_0} = \mathcal{O}_{C_0}, \quad L|_{Z_i} = \mathcal{O}_{Z_i}(C_0 \cap Z_i), \quad |C_0 \cap Z_i| = 2.
\]
This means that the component \( C_0 \) is not exceptional, because \( L \) has degree 2 on \( C_0 \). Moreover, \( L \) has degree 0 on every other component, which implies that \( I \) is an invertible sheaf, hence \( L = I \) and \( \mathbb{P}(I) = C \). We can consider smooth points \( q_1, q_2, q_1', q_2' \) of \( C \) lying over \( C_0 \) such that
\[
L|_{C_0} \cong \mathcal{O}_{C_0}(q_1 + q_2) \cong \mathcal{O}_{C_0}(q_1' + q_2').
\]
By Condition [4], we have that \( L \cong \mathcal{O}_C(q_1 + q_2) \cong \mathcal{O}_C(q_1' + q_2') \), hence \( \mathcal{O}_C(2p_0 - q_1 - q_2) \cong \mathcal{O}_C(2p_0 - q_1' - q_2') \), which means that \( \beta_2(q_1 + q_2) = \beta_2(q_1' + q_2') \), where \( \beta_2 \) is the map in Equation [5] for some (every) regular smoothing of \( C \).

In the second case, there are two components \( C_1 \) and \( C_2 \) of \( \mathbb{P}(I) \) such that \( (C_1, C_2) \) is a special \( B \)-pair (in the sense of [5] Definition 5.8]). By [4] Theorem 5.9], we have
\[
\deg L|_{C_1} = \deg L|_{C_2} = 1, \quad L|_{(C_1 \cup C_2)^c} \cong \mathcal{O}_{(C_1 \cup C_2)^c}.
\]
Notice that, if one between \( C_1 \) and \( C_2 \) is exceptional, then the other must be exceptional as well, and in particular this implies that \( I \) is not simple, which is a contradiction. We deduce that \( I \) is an invertible sheaf, hence \( L = I \) and \( \mathbb{P}(I) = C \). We can repeat the argument used in the first case,
now taking \( q_1, q'_1 \in C_1 \) and \( q_2, q'_2 \in C_2 \). We leave the details to the reader. Notice that we proved (3) and the “only if” part of (1).

Now assume that there is a regular smoothing \( C \to B \) of \( C \) such that \( \beta_2 \) is not injective. We have different cases to consider.

In the first case, there are smooth points \( q_1, q_2, q'_1, q'_2 \) of \( C \) such that \( \beta_2(q_1 + q_2) = \beta_2(q'_1 + q'_2) \). This means that there exists an invertible sheaf \( T \) on \( C \) of type \( T = \mathcal{O}_C(\sum a_i C_i) \), where \( a_i \in \mathbb{Z} \) and \( C_i \) are the components of \( C \), such that

\[
\mathcal{O}_C(2p_0 - q_1 - q_2) \cong \mathcal{O}_C(2p_0 - q'_1 - q'_2) \otimes T.
\]

We deduce that

\[
\mathcal{O}_C(q'_1 + q'_2 - q_1 - q_2) \cong T.
\]

Let \( \Gamma \) be the dual graph of \( C \). Notice that \( \Gamma \) has no separating edge. If \( v_1, v_2, v'_1, v'_2 \) are the vertices of \( \Gamma \) corresponding to the components containing the points \( q_1, q_2, q'_1, q'_2 \), we have that \( v'_1 + v'_2 - v_1 - v_2 \) is a principal divisor on \( \Gamma \). Let \( f : V(\Gamma) \to \mathbb{Z} \) be the rational function on \( \Gamma \) such that \( \text{div}(f) = v'_1 + v'_2 - v_1 - v_2 \) (notice that \( v'_i \neq v_j \) for every \( i, j = 1, 2 \) because \( \Gamma \) has no separating edge). We denote by \( Z \) the subcurve of \( C \) corresponding to the vertices of \( \Gamma \) where \( f \) attains its minimum. In particular, \( q_1, q_2 \in Z, q'_1, q'_2 \in Z^c \) and \( |Z \cap Z^c| = 2 \). Moreover we have \( T|_Z = \mathcal{O}_Z(-Z \cap Z^c) \), which implies that

\[
\mathcal{O}_Z(Z \cap Z^c) \otimes \mathcal{O}_C(-q_1 - q_2)|_Z \cong (\mathcal{O}_C(q'_1 + q'_2 - q_1 - q_2) \otimes T^{-1})|_Z \cong \mathcal{O}_C|_Z = \mathcal{O}_Z.
\]

Define \( L := \mathcal{O}_C(q_1 + q_2) \). We see that \( L \) satisfies \( h^0(L, C) \geq 2 \) (indeed \( L \) has the trivial section that vanishes only over \( q_1, q_2 \) and a section that vanishes on the whole \( Z^c \)). Thus \( C \) is pseudo-hyperelliptic in the sense of Definition 4.19.

In the second case, we have nodes \( n, n' \) and smooth points \( q, q' \) of \( C \) such that \( \beta_2(n + q) = \beta_2(n' + q') \). Let \( \tilde{C} \) be the semistable modification of \( C \) obtained by adding an exceptional component over each node of \( C \). Then, there exists a twister \( T \) on \( \tilde{C} \) such that

\[
\mathcal{O}_{\tilde{C}}(2p_0 - \tilde{n} - q) \cong \mathcal{O}_{\tilde{C}}(2p_0 - \tilde{n}' - q') \otimes T
\]

where \( \tilde{n} \) and \( \tilde{n}' \) are any smooth points of \( \tilde{C} \) lying over the exceptional component over \( n \) and \( n' \).

Arguing as before, we see that \( L := \mathcal{O}_C(n + q) \) satisfies \( h^0(L, \tilde{C}) \geq 2 \), and hence \( h^0(f_*(L), C) \geq 2 \). Thus \( C \) is pseudo-hyperelliptic in the sense of Definition 4.19.

In the third case, we have a node \( n \) and smooth points \( q, q'_1, q'_2 \) such that \( \beta_2(n + q) = \beta_2(q'_1 + q'_2) \). This case is not possible, since the sheaf represented by \( \beta_2(q'_1 + q'_2) \) is invertible, while the one represented by \( \beta_2(n + q) \) is not.

The remaining cases are the following ones:

- \( \beta(n_1 + n_2) = \beta(q'_1 + q'_2) \);
- \( \beta(n_1 + n_2) = \beta(n' + q') \);
- \( \beta(n_1 + n_2) = \beta(n'_1 + n'_2) \).

where \( n_1, n_2, n', n'_1, n'_2 \) are nodes of \( C \), and \( q, q'_1, q'_2 \) smooth points. All these cases are done in a similar manner as the second case: first, we change \( C \) by a suitable semistable modification \( f : \tilde{C} \to C \) and find a line bundle \( L \) such that \( h^0(L, \tilde{C}) \geq 2 \). This implies that \( h^0(f_*(L), C) \geq 2 \) which proves that \( C \) is pseudo-hyperelliptic in the sense of Definition 4.19.

Finally, item (2) of the statement readily follows by [4] Theorem 5.9.

\[ \square \]

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