Completeness of general $pp$-wave spacetimes and their impulsive limit

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Abstract

We investigate geodesic completeness in the full family of $pp$-wave (or Brinkmann) spacetimes in their extended and impulsive forms. This class of geometries contains the recently studied gyratonic $pp$-waves, modelling the exterior field of a spinning beam of null particles, as well as $N$-fronted waves with parallel rays, which generalize classical $pp$-waves by allowing for a general wave surface. The problem of geodesic completeness reduces to the question of completeness of trajectories on a Riemannian manifold under an external force field. Building upon respective recent results, we derive completeness criteria in terms of the spatial asymptotics of the profile function in the extended case. In the impulsive case, we use a fixed-point argument to show that, irrespective of the behaviour of the profile function, all geometries in the class are complete.

Keywords: $pp$-waves, gyratons, geodesic completeness, low regularity, impulsive limit

(Some figures may appear in colour only in the online journal)

1. Introduction

Since the seminal work of Penrose [Pen65a], singularities in general relativity are usually understood as the presence of incomplete causal geodesics, i.e. geodesics which cannot be extended to all values of their parameter. In this work, we study geodesic completeness for a
large class of spacetimes admitting a covariantly constant null vector field, forming the well-known \( pp \)-wave subclass of the \( Kundt \) family of non-twisting shear-free and non-expanding geometries [\( \text{Kun61, Kun62} \)]. This subfamily includes, e.g. gyratonic \( pp \)-waves [\( \text{FF05, FZ06} \)], representing the exterior vacuum field of spinning particles moving with the speed of light, which may serve as an interesting toy model in high-energy physics [\( \text{YZF07} \)], but also N-fronted waves with parallel rays (NPWs): a generalization of classical \( pp \)-waves allowing for an \( n \)-dimensional Riemannian manifold \( N \) as the wave surface [\( \text{CFS03} \)]. Remarkably, this family of exact spacetimes has already been described by the original Brinkmann form [\( \text{Bri25} \)] of the \( pp \)-wave metric, given in equation (1), below.

We will consider geodesic completeness both in the extended case, i.e. where the profile functions are smooth, as well as in the impulsive case, i.e. when the metric functions are strongly concentrated and of a distributional nature. The problem of completeness in this class of spacetimes can be reduced to a purely Riemannian problem, namely the question of completeness of the motion on a Riemannian manifold under the influence of a time- and velocity-dependent force. In the extended case we generalize recent results ([\( \text{CRS12, CRS13} \)]) to the case at hand to provide completeness statements subject to conditions on the spatial fall-off of the profile function. We also consider the case of impulsive waves in our class, which are models of short but violent bursts of gravitational radiation emitted by a spinning (beam of) ultrarelativistic particle(s). These models are also interesting from a purely mathematical point of view, since they are examples of geometries of low regularity, which has attracted growing attention recently, e.g. [\( \text{CG12, KSSV15, Min15, Säm16, Sbi15} \)]. Our main result here is that all impulsive geometries in the full class of \( pp \)-waves are geodesically complete irrespective of the spatial asymptotics of the profile function. This result confirms the effect (previously noted in similar situations: see, for example, [\( \text{PV99, SS12} \)]) that the influence the spatial characteristics of the profile function exert on the behaviour of the geodesics is wiped out in the impulsive limit. In this way, we prove a large class of interesting non-smooth geometries to be non-singular in view of the Penrose definition.

This article is structured as follows. In section 2 we introduce the full \( pp \)-wave (or Brinkmann) metric and summarize its basic geometric properties along with relevant subclasses and special cases which have been treated extensively in the literature.

2. The spacetime metric

In this section we introduce the full \( pp \)-wave (or Brinkmann metric) ([\( \text{Bri25} \)]) and review some of its basic geometric properties along with relevant subclasses and special cases which have been treated extensively in the literature.
To begin with, let \((N, h)\) be a smooth connected \(n\)-dimensional Riemannian manifold. We consider the spacetime \((M = N \times \mathbb{R}^2, g)\) where the line element is given by

\[
d s^2 = h_{ij} \, d x^i \, d x^j - 2 \, u \, d u \, d x^2 + \mathcal{H}(x, u) \, d u^2 + 2 \, A_i(x, u) \, d u \, d x^i. \tag{1}
\]

Here, \(x = x^i = (x^1, \ldots , x^n)\) are coordinates on \(N\) and \(u, r\) are global coordinates on \(\mathbb{R}^2\). Moreover \(\mathcal{H} : N \times \mathbb{R} \to \mathbb{R}\) and \(A_i : N \times \mathbb{R} \to \mathbb{R}\) are smooth functions. We fix a time orientation on \(M\) by defining the null vector field \(\partial_r\) to be future-directed.

Some immediate geometric properties of the spacetime (1) are the following: \(\partial_r\) is the generator of the null hypersurfaces of constant \(u\), \(P(u_0) = \{ u = u_0 \} \cong N \times \mathbb{R}\), i.e. \(\partial_r = -\nabla u = -\nabla u\) and it is covariantly constant. The latter property is the defining condition for \(pp\)-waves (plane-fronted waves with parallel rays), and this is why we will refer to (1) as the full \(pp\)-wave (see, for example, [GP09], p 324 and section 18.5).

The null geodesic generators of \(P(u_0)\) form a non-expanding, shear-free and twist-free congruence and the family of \(n\)-dimensional space-like submanifolds \(N \times \{ u_0 \}\) orthogonal to \(\partial_r\) has the interpretation of wave surfaces. Consequently, we will also refer to \(N\) as the spatial surface, and to \(h\) and \(x^i\) as the spatial metric and coordinates, respectively.

Using the coordinates \((x^i, u, r)\), the inverse metric takes the form

\[
g^{i\nu} = \begin{pmatrix} h^{ij} & 0 & g^{iu} \\ 0 & 0 & -1 \\ g^{ri} & -1 & g^{rr} \end{pmatrix}, \tag{2}
\]

where \(h^{ij}\) denotes the components of the inverse spatial metric \(h^{-1}\) on \(N\), and we have \(g^{i\nu} = -\mathcal{H} + h^{ij} A_i A_j\) and \(g^{i\nu} = h^{ij} A_j\). The non-vanishing Christoffel symbols then are

\[
\Gamma_{uu}^r = g^{i\nu} \left( A_{i, u} - \frac{1}{2} h_{i, r} \right) - \frac{1}{2} \mathcal{H}_{u, r},
\]

\[
\Gamma_{uj}^r = \frac{1}{2} g^{i\nu} \left( A_{i, j} - A_{j, i} \right) - \frac{1}{2} h_{j, r}, \quad \Gamma_{jk}^r = -A_{[j]k},
\]

\[
\Gamma_{uu}^i = h^{ik} \left( A_{k, u} - \frac{1}{2} h_{k, r} \right), \quad \Gamma_{uj}^i = \frac{1}{2} h^{ik} \left( A_{k, j} - A_{j, k} \right), \quad \Gamma_{jk}^i = \Gamma^{(N)ij},
\]

where \(\Gamma^{(N)ij}\) denotes the Christoffel symbols of the Riemannian metric \(h\) on \(N\), \(\|\) stands for the covariant derivative of \(h\), and \(\_\|\) and \(\_\cdot\) denote derivatives with respect to the \(i\)-th spatial direction and with respect to \(u\), respectively.

From the vanishing of all Christoffel symbols of the form \(\Gamma_{iu}^u\), we immediately find that for any geodesic \(\gamma(s) = (x^i(s), u(s), r(s))\) in (1) we have \(u = 0\). Hence, there are geodesics with \(u(s) = u_0\) that are entirely contained in the null hypersurface \(P(u_0)\) and thus are either space-like or the null generators of \(P(u_0)\). Observe that in case \(A_i = 0\), the \(r\)-component of these geodesics also becomes affine, \(r(s) = r_0 + r_0 s\). All other geodesics may be rescaled to take the form

\[
\gamma(s) = (x^i(s), s, r(s)). \tag{4}
\]

The coordinate function \(u\) is increasing along any future-directed causal curve \(c(t) = (x^i(t), u(t), r(t))\) since \((u \circ c) = g(\nabla u, \dot{c}) = \dot{u} \geq 0\) and it is even strictly increasing along any future-directed time-like curve. Hence, \((M,g)\) is chronological. Moreover, \(u\) is strictly increasing along all future-directed causal geodesics of the form (4). So, in case \(A_i = 0\) there is no closed null geodesic segment and the spacetime is even causal.
The non-vanishing components of the Ricci tensor for (1) are
\[ R_{ij} = R^{(N)}_{ij}, \quad R_{ii} = h^{mn} A_{[m,i][n]}, \quad R_{uu} = h^{mn} \left( A_{m,u} A_{u} + \frac{1}{2} H_{i[mn]} \right) + h^{ij} h^{mn} A_{[k,m]} A_{[l,n]}, \]

(5)
where the square brackets, as usual, denote antisymmetrization. The Ricci scalar of (1) corresponds to that of the transverse space, i.e. \( R = R^{(N)} \). Also, the metric (1) belongs to the class of VSI (vanishing scalar invariant) spacetimes if \( N \) is flat.

Next, we employ the algebraic Petrov classification ([OPP13, PS13]) to the \( pp \)-wave metric (1). We project the Weyl tensor onto the natural null frame
\[ k = \partial_{r}, \quad l = \frac{1}{2} \mathcal{H} \partial_{r} + \partial_{\alpha}, \quad m_{(i)} = m_{(i)}^{j} (A_{i} \partial_{r} + \partial_{\alpha}), \]

(6)
where \( h_{ij} \), \( m_{(i)}^{j} \), \( m_{(i)}^{k} \) = \( \delta_{ij} \) and find that the highest boost-weight irreducible components \( \Psi_{2p}^{\prime} \), \( \Psi_{1T} \), and \( \tilde{\Psi}_{0}^{\prime} \) vanish and the Brinkmann spacetimes are thus necessarily at least of algebraic type II with \( k = \partial_{r} \) being a double-degenerated null direction, in fact \( \Psi_{2p}^{\prime} = 0 \) and (1) is of type II(d). More precisely, without employing any field equations, the non-vanishing Weyl scalars are
\[ \Psi_{2S} = \frac{1}{n(n+1)} R^{(N)}, \]

(7)
\[ \tilde{\Psi}_{2T}^{\prime} = \frac{1}{n} m_{(i)}^{j} \left( R^{(N)}_{ij} - \frac{1}{n} h_{ij} R^{(N)} \right), \]

(8)
\[ \tilde{\Psi}_{2T}^{\prime} = m_{(i)}^{j} m_{(j)}^{k} m_{(k)}^{l} C^{(N)}_{ijkl}, \]

(9)
\[ \Psi_{3T} = \frac{1}{n} m_{(i)}^{j} h^{kl} A_{[k,j][l]}, \]

(10)
\[ \tilde{\Psi}_{3T}^{\prime} = m_{(i)}^{j} m_{(j)}^{k} m_{(k)}^{l} \left( A_{[k,j][l]} - \frac{1}{n-1} h^{mn} (h_{ij} A_{[k,m][l]} - h_{ik} A_{[j,m][l]} \right), \]

(11)
\[ \Psi_{4T} = m_{(i)}^{j} m_{(j)}^{k} \left( -\frac{1}{2} H_{i[ij]} A_{[j,m][l]} + h^{mn} A_{[m,j]} A_{[n,l]} \right) \]
\[ -\frac{1}{n} h_{ij} h^{kl} \left( -\frac{1}{2} H_{[ik]l} + A_{k,i[l]} + h^{mn} A_{[m,k]} A_{[n,i]} \right). \]

(12)
The boost-weight zero components \( \Psi_{2S}, \tilde{\Psi}_{2T}^{\prime}, \tilde{\Psi}_{2T}^{\prime} \) are entirely governed by the properties of the transverse space \( N \); the components of boost-weight \(-1, \Psi_{3T}, \) and \( \tilde{\Psi}_{0}^{\prime} \), are governed by the off-diagonal terms \( A_{i} \); while the boost-weight \(-2 \) component \( \Psi_{4T} \) depends on all the metric functions \( \mathcal{H} \) and \( A_{i} \). The conditions under which the geometry (1) becomes algebraically more special are summarized in table 1.

Moreover, if the vacuum Einstein field equations (with \( \Lambda = 0 \)) are employed some of the conditions are satisfied identically (cf. (5) and table 2).

The spacetimes (1) were originally considered by Brinkmann in the context of conformal mappings of Einstein spaces. Over the decades, starting with Peres [Per59], several special cases of (1) have been used as exact models of spacetimes with gravitational waves.

The most prominent case arises when the wave surface \( N \) is flat \( \mathbb{R}^{2} \) and the off-diagonal terms \( A_{i} \) vanish. In fact, writing \( x^{i} = (x, y) \), these ‘classical’ \( pp \)-waves,
\[ ds^{2} = dx^{2} + dy^{2} - 2du dr + \mathcal{H}(x, y, u) du^{2}, \]

(13)
have become textbook examples of exact gravitational wave spacetimes (see, for example, [GP09, Chapter 17]). The Petrov type is N (see table 1) without the application of the field
Table 1. The conditions refining possible algebraic (sub)types of the Brinkmann geometry (1). No field equations are employed. The relations II(b), III(b) are identities for \( n = 2 \) and II(c) for \( n = 2, 3 \), respectively. When II(abc) are satisfied simultaneously, the transverse space has to be flat.

| (Sub)type | Condition |
|-----------|-----------|
| II(a)     | \( R_{ij}^{(N)} = 0 \) |
| II(b)     | \( R_{ij}^{(N)} = \frac{1}{n} h_{ij} R_{ij}^{(N)} \) |
| II(c)     | \( C_{ij}^{(N)} = 0 \) |
| II(d)     | always |
| III       | II(abcd) |
| III(a)    | \( h^{[i} A_{i,\{j\}|k|l]} = 0 \) |
| III(b)    | \( A_{i,\{j\}|k|l]} = \frac{1}{n-1} h^m_{ij} (h_{il} A_{i,\{m\}|j|k]} - h_{ik} A_{i,\{j\}|l|m]} \) |
| N         | III(ab) |
| O         | \( -\frac{1}{2} h^{[i} h_{ij} A_{i,\{j\}|k|l]} + A_{i,\{j\}|k|l]} + h^m_{ij} A_{i,\{m\}|j|k]} + h^m_{ij} A_{i,\{j\}|l|m]} = 0 \) |

Table 2. The algebraic structure of the spacetimes (1) restricted by Einstein’s vacuum field equations. The algebraic type becomes, in general, II(abcd). Moreover, the condition II(c) is identically satisfied in \( n = 2, 3 \) and III(b) in \( n = 2 \), respectively. Condition II(c) necessarily leads to a flat transverse space.

| (Sub)type | Condition |
|-----------|-----------|
| II(a)     | always |
| II(b)     | always |
| II(c)     | \( C_{ij}^{(N)} = 0 \) |
| II(d)     | always |
| III       | II(abcd) |
| III(a)    | always |
| III(b)    | \( A_{i,\{j\}|k|l]} = 0 \) |
| N         | III(ab) |
| O         | \( -\frac{1}{2} h^{[i} h_{ij} A_{i,\{j\}|k|l]} + A_{i,\{j\}|k|l]} + h^m_{ij} A_{i,\{m\}|j|k]} + h^m_{ij} A_{i,\{j\}|l|m]} = 0 \) |

equations, hence in any theory of gravity. Moreover, the Ricci tensor simplifies to \( R_{\mu \nu} = -(1/2) \partial_i H_{ij} \) (see (5)) and its source can be interpreted as any type of null matter or radiation. The vacuum Einstein equations reduce to the two-dimensional (flat) Laplacian so that (13) with harmonic \( H \) represents pure gravitational waves. Such solutions are most conveniently written using the complex coordinate \( \zeta = x + iy \) with \( H(\zeta, u) = F(\zeta, u) + F(\zeta, u) \) and \( F \) a combination of terms of the form.
with arbitrary ‘profile functions’ \( \alpha_m, \beta_m, \mu \) of \( u \). Here, the inverse-power terms represent pp-waves generated by sources with multipole structure (see, for example, \([\text{PG98}]\)) moving along the axis, which is clearly singular. Hence, it has to be removed from the spacetime which now has spatial part \( N = \mathbb{R}^2 \setminus \{0\} \). The same is true for the extended Aichelburg–Sexl \([\text{AS71}]\) solution represented by the logarithmic term. Finally, the polynomial terms are non-singular with \( \beta_2 \) representing plane waves (see below). The higher-order terms \( (m \geq 3) \) lead to unbounded curvature at infinity and display chaotic behaviour of geodesics \( \text{e.g.} \ [\text{PV98}] \) consequently seem to be physically less relevant.

In the special case of \( \mathcal{H} \) being quadratic in \( x, y \), one arrives at plane waves, i.e.

\[
F(\zeta, u) = \sum_{m=1}^{\infty} \alpha_m(u) \zeta^{-m} - \mu(u) \log \zeta + \sum_{m=2}^{\infty} \beta_m(u) \zeta^m
\]

where \( \mathcal{H}_{ij} \) is a real symmetric \( (2 \times 2) \)-matrix-valued function on \( \mathbb{R} \). Here, the curvature tensor components are constant along the wave surfaces and the spacetime is Ricci-flat provided the trace of the profile function vanishes, \( \mathcal{H}_i^i = 0 \), in which case we speak of (purely) gravitational plane waves. While being complete by virtue of the linearity of the geodesic equations, plane waves ‘remarkably’ fail to be globally hyperbolic \( \text{[Pen65b]} \) due to a focussing effect of the null geodesics. This phenomenon has been investigated thoroughly for gravitational plane waves in a series of papers by Ehrlich and Emch \( \text{[EE92b, EE92a, EE93]} \); see also \([\text{BEE96, Chapter 13]}\), who were able to determine their precise position on the causal ladder: They are causally continuous, but not causally simple. In this analysis, however, the high degree of symmetries of the flat wave surface was extensively used and the ‘stability’ of the respective properties of plane waves within larger classes of solutions remained obscure.

Partly to clarify these matters Flores and Sánchez, in part together with Candela, in a series of papers \( \{[\text{CFS03, CFS04, FS03, FS06}]\) introduced more general models which they called (general) plane-fronted waves. Indeed, they generalize pp-waves \((13)\) by replacing the flat two-dimensional wave surface by an arbitrary \( n \)-dimensional Riemannian manifold \( N \), i.e. they are defined as \((1)\) with vanishing off-diagonal terms \( A_i \),

\[
dx^2 = h_{ij} dx^i dx^j - 2 du dr + \mathcal{H}(x,u) du^2.
\]

Motivated by the geometric interpretation given above, and following \([\text{SS12]}\), we call these models \( N \)-fronted waves with parallel rays \( \text{(NPWs)} \). The Petrov type of these geometries is now \( \text{II}(d) \) (see table 1) and the vacuum field equations become \( \Delta_H \mathcal{H} = h^0 \mathcal{H}_{ij} = 0 \), i.e. the Laplace equation for \( \mathcal{H} \) on \( (N, h) \) which, in addition, has to be Ricci-flat. Hence, vacuum NPWs are necessarily of Petrov type \( \text{II(abd)} \) and are of type \( N \) if and only if \( (N, h) \), in addition, is conformally flat, and hence flat. It turns out that the behaviour of \( \mathcal{H} \) at spatial infinity is decisive for many of the global properties of NPWs with quadratic behaviour marking the critical case: NPWs are causal, but not necessarily distinguishing: they are strongly causal if \( -\mathcal{H} \) behaves, at most, quadratically at spatial infinity\(^4\) and they are globally hyperbolic if \( -\mathcal{H} \) is subquadratic and \( N \) is complete. Similarly, the global behaviour of geodesics in NPWs is governed by the behaviour of \( \mathcal{H} \) at spatial infinity. The respective results will be discussed in the next section together with the stronger and newer results of \([\text{CRS12, CRS13]}\).

A complementary generalization of pp-waves \((13)\) was considered by Bonnor \( \text{[Bon70]} \) and independently by Frolov and his collaborators in \([\text{FF05, FIZ05, FZ06, YZF07]}\). Here, the

\(^4\) For precise definitions of these conditions, see section 3, below.
\( n \)-dimensional transverse space \( N \) is considered to remain flat, but non-trivial off-diagonal terms \( A_i(x, u) \) are allowed to obtain gyratonic pp-waves
\[
dx^2 = \delta_{ij}dx^idx^j - 2du^2 + \mathcal{H}(x, u)du^2 + 2A_i(x, u)du^i.
\]
(17)

Physically, this geometry represents a spinning null beam of pure radiation, first called gyraton in [FF05]. By flatness of the wave surface, the Petrov type is at least III and in case of vacuum solutions at least III(a) and N if \( n = 2 \) (see tables 1, 2). In [Bon70] the metric (17) was matched to an ‘interior’ non-vacuum region, where the spinning source of the gravitational waves was given phenomenologically by an energy-momentum tensor of the form \( T_{\mu \nu} = \varrho \) and \( T_{\mu \nu} = j^\mu \). In the surrounding vacuum region, the metric functions \( \mathcal{H} \) and \( A_i \) are restricted by the vacuum Einstein equations following from (5) for \( R_{\mu \nu} = 0 \) and \( R_{uu} = 0 \):
\[
\delta^{mn}A_{[m,i]n} = 0, \quad -\frac{1}{2}\delta^{mn}\mathcal{H}_{,mn} + (\delta^{mn}A_{m,n})_{,u} + \delta^{kl}\delta^{mn}A_{[k,m]}A_{j,l,n} = 0,
\]
(18)
where in addition the ‘Lorenz’ gauge \( \delta^{mn}A_{m,n} = 0 \) can be applied. In [FF05, FIZ05] explicit solutions have been calculated, all displaying fall-off-like inverse powers of \( |x| \) if \( n > 2 \) and a logarithmic behaviour if \( n = 2 \). Moreover, the weak field approximation in the presence of a gyratonic source, reflected in the only non-trivial energy-momentum tensor components \( T_{uu} \) and \( T_{uv} \), gives physical meaning to the metric functions in the entire spacetime. In particular, \( \mathcal{H} \) determines the mass-energy density and \( A_i \) correspond to the angular momentum density of the source.

In four dimensions \((n = 2)\), the off-diagonal terms in (17) can be removed locally by a coordinate transformation which, however, obscures the global (topological) properties of the spacetime, cf. e.g. [FIZ05, PSS14]. In higher dimensions \((n > 2)\), the off-diagonal terms in (17) can anyway not be removed even locally due to the lack of coordinate freedom.

The spinning character of the four-dimensional gyratonic pp-waves (17) was emphasized in [FIZ05, PSS14] employing transverse polar coordinates \( x = \rho \cos \varphi, y = \rho \sin \varphi \) with the identifications \( A_1 = -J\rho^{-1}\sin \varphi, A_2 = J\rho^{-1}\cos \varphi \) which gives
\[
dx^2 = d\rho^2 + \rho^2d\varphi^2 - 2du^2 + \mathcal{H}(\rho, \varphi, u)du^2 + 2J(\rho, \varphi, u)dud\varphi.
\]
(19)
This can also be understood as the specific form of the full Brinkmann metric (1) with \( x^1 = \rho, x^2 = \varphi \) and \( dh^2 = d\rho^2 + \rho^2d\varphi^2 \). An important explicit example is given by the axially symmetric gyraton accompanied by a pp-wave generated by a monopole, namely
\[
dx^2 = d\rho^2 + \rho^2d\varphi^2 - 2du^2 - 2\mu(u)\ln \rho du^2 + \chi(u)dud\varphi,
\]
(20)
where \( \mu(u) \) determines the mass density in the Aichelburg–Sexl-like logarithmic term, and the angular momentum density is determined by \( \chi(u) \).

3. Geodesic completeness

In this section we discuss geodesic completeness of the full pp-wave metric (1) and its various special cases in the extended, i.e. smooth case before turning to the impulsive limit in subsequent sections. We will see that all these questions can be reduced to a purely Riemannian question about the completeness of trajectories on \( N \) under a specific force field.

The explicit form of the system of geodesic equations for a curve \( \gamma(s) = (x^i(s), u(s), r(s)) \), see (3), is
\[
\ddot{x}^i = -\Gamma^{ijk}\dot{x}^j\dot{x}^k - h^{ik}(A_{k\ell} - A_{k,j}\dot{u})\dot{x}^\ell\dot{u} - \frac{1}{2}h^{ik}(2A_{k\ell,u} - \mathcal{H}_{,\ell})\dot{u}^2,
\]
(21)
\[ \ddot{u} = 0, \quad \ddot{r} = A_{(ii)} \dddot{x}^{i} - (g^{ii}(A_{ij} - A_{ji}) - \mathcal{H}_{ij})\dddot{x}^{j} - \left( g^{ii} \left( A_{iuv} - \frac{1}{2} \mathcal{H}_{i,v} \right) - \frac{1}{2} \mathcal{H}_{i,u} \right) u^{2}. \] 

We observe (again) that the equation for \( u \) is trivial and that the equation for \( r \) decouples from the rest of the system and can simply be integrated once the \( x \)-equations are solved. Finally, the \( x \)-equations are the equations of motion on the Riemannian manifold \( N \) under an external force term depending on time and velocity. Hence, the basic result on the form of the geodesics of the Brinkmann metric (1) is the following (see [CFS03], proposition 3.1):

**Proposition 3.1:** (Form of the geodesics). Let \( \gamma = (x', u, r): (-a, a) \rightarrow M \) be a curve on \( M \) with constant 'energy' \( E_{\gamma} = g(\dot{\gamma}, \dot{\gamma}) \) which assumes the data

\[ \gamma(0) = (x^{0}, u_{0}, r_{0}),\dot{\gamma}(0) = (k^{0}, \dot{u}_{0}, \dot{r}_{0}). \] 

Then, \( \gamma \) is a geodesic if the following conditions hold true:

(a) \( u \) is affine, i.e. \( u(s) = u_{0} + s \dot{u}_{0} \) for all \( s \in (-a, a) \),

(b) \( x' \) solves

\[ D_{x}x' = -h^{jk}(A_{k,j} - A_{j,k})x'^{k} \dot{u}_{0} - \frac{1}{2} h^{jk}(2A_{k,u} - \mathcal{H}_{k,u})u_{0}^{2}, \] 

where \( D \) denotes the covariant derivative of \((N, h), \) and

(c) \( r \) is given by

\[ r(s) = r_{0} - \frac{1}{2 \dot{u}_{0}} \int_{0}^{s} (E_{\gamma} - h(\dot{x}(\sigma), \dot{x}(\sigma)) - \dot{u}_{0}^{2} \mathcal{H}(x(\sigma), u(\sigma)) \right. \]

\[ \left. - 2 \dot{u}_{0} A_{u}(x(\sigma), u(\sigma)) \dot{x}(\sigma))d\sigma. \] 

Here, we have used the explicit form of \( E_{\gamma} \) in the last condition. The key fact is now that completeness essentially depends on completeness of the solutions to (25). Note that in most cases we will use a rescaling to achieve the form (4) of the geodesics, which amounts to setting \( u_{0} = 0 \) and \( \dot{u}_{0} = 1 \) in equations (25) and (26).

**Corollary 3.2:** (Basic condition for completeness). The spacetime (1) is complete if all inextendible solutions of (25) are complete.

Although there are also some results in case of an incomplete spatial manifold (see the discussion after corollary 3.4 below) we assume for the moment \((N, h)\) to be complete. The question of completeness of solutions of equations such as (25) has, if only in special cases, been addressed in the 'classical' literature. To begin with, we observe that in the special case of NPWs (16) the equation of motion (25) reduces to

\[ D_{x}x' = \frac{1}{2} \nabla_{x} \mathcal{H}(x, s) \] 

(\( \nabla_{x} \) denoting the gradient on \((N, h)\), i.e. to the equation of motion on \( N \) under the influence of a time-dependent potential. First results on the completeness of NPWs, mainly restricted to the case of autonomous \( \mathcal{H} \), i.e. \( \mathcal{H} = \mathcal{H}(x) \) independent of \( u \), were derived in [CFS03, section 3]. In fact, it follows from, for example, [AM78], theorem 3.7.15 that an NPW (16) is complete if \( \mathcal{H} \) is autonomous and controlled by a positively complete function at infinity, i.e. if there exists some arbitrarily fixed \( \bar{x} \in N \) and some positive constant \( \mathcal{R} \) such that
\[ \mathcal{H} = \mathcal{H}(x) \leq - V(d(x, \bar{x})) \quad \text{for all } x \in N \text{ with } d(x, \bar{x}) \geq \mathcal{R}, \]  
\tag{28}

where \( d \) denotes the Riemannian distance function on \((N, h)\) and \( V : [0, \infty) \to \mathbb{R} \) is \( C^2 \) with \( \int_0^\infty \frac{dx}{\sqrt{e - V(x)}} = +\infty \) for one (hence any) \( e > V(x) \) and all \( x \). Consequently, autonomous NPWs are complete if \( \mathcal{H} \) grows at most quadratically at spatial infinity, i.e. if \( \exists \ x \in N, \mathcal{R} > 0 \) such that
\[ \mathcal{H} = \mathcal{H}(x) \leq C d^2(x, x) \quad \text{for all } x \text{ with } d(x, x) \geq \mathcal{R}, \]  
\tag{29}

for some constant \( C > 0 \). Of course, this result extends immediately to \textit{sandwich}\(^5\) NPWs, which grow at most quadratically at spatial infinity. Additionally, the case of plane NPWs is easily settled, that is, (16) with \((N, h)\) flat and quadratic non-autonomous \( \mathcal{H} \), i.e.
\[ \mathcal{H}(x, u) = h(A(u)x, x), \]  
\tag{30}

where \( A \) is (at least) a continuous map from \( \mathbb{R} \) into the space of real symmetric \((n \times n)\)-matrices. Here, completeness follows from the global existence of solutions to linear ODEs generalizing the case of plane waves (15).

More substantial results on non-autonomous NPWs have been given in [CRS12] based on recent results on the completeness of trajectories of equations like (25) and (27) in [CRS13] (for more general and somewhat sharper results see [Min15]). Since we will also use these statements in our discussion of the general case (1) we recall in the following the key notions and theorems. We say that a (time-dependent) tensor field \( X \) on the projection \( \pi : N \times \mathbb{R} \to N \) grows at most linearly in \( N \) along finite times if for all \( T > 0 \) there exists \( x \in N \) and constants \( A_T, C_T > 0 \) such that
\[ |X|_{(x, s)} \leq A_T d(x, \bar{x}) + C_T \quad \forall (x, s) \in N \times [-T, T] \]  
\tag{31}

with \( | \cdot | \) and \( d \) the norm and the distance function of \( h \), respectively. Analogously, we define the notions of \textit{at most quadratic growth} along finite times and \textit{boundedness along finite times}, where in the special case of functions we use the estimate (31) without norm. Now given a smooth \((1, 1)\)-tensor field \( F \) and a smooth vector field \( X \) on \( \pi \) we consider the second-order ODE
\[ D^t \gamma (s) = F_{\gamma(t), t} \gamma (s) + X_{\gamma(t), t} \]  
\tag{32}

and the special case when \( X \) is derived from a potential, i.e.
\[ D^t \gamma (s) = F_{\gamma(t), t} \gamma (s) - \nabla_s V(\gamma(s), s) \]  
\tag{33}

with \( V \) a smooth function on \( N \times \mathbb{R} \). Then, we have

\textbf{Theorem 3.3} (Theorems 1, 2 in [CRS13]). \textit{Let \((N, h)\) be a connected, complete Riemannian manifold. If the self-adjoint part \( S \) of \( F \) is bounded in \( N \) along finite times then}

\(1\) all inextendible solutions of (32) are complete provided \( X \) grows at most linearly in \( N \) along finite times, and

\(2\) all inextendible solutions of (33) are complete provided that \(-V\) and \( \frac{\partial V}{\partial s} \) grow at most quadratically in \( N \) along finite times.

Observe that one may also apply theorem 3.3(1) to equation (33) in which case one has to assume that \( \nabla_s V \) grows at most linearly along finite times. Provided that we are in the non-autonomous case, this condition is logically independent of the condition of theorem 3.3(2).

\(^5\) We call a spacetime (1) a sandwich wave if \( \mathcal{H} \) and \( A_t \) vanish outside some bounded \( u \)-interval.
Hence, in the case of NPWs, which amounts to setting $F = 0$ and $X = -\nabla_i V = \nabla_i \mathcal{H}$, one obtains different types of results based on either of these conditions, see [CRS12, CRS13]. Explicitly we have

**Corollary 3.4** (Completeness of NPWs and classical pp-waves). NPW spacetimes (16) and, in particular, classical pp-wave spacetimes (13) with complete wave surface $N$ are complete provided that either

1. $\nabla_i \mathcal{H}$ grows at most linearly along finite times, or
2. $\mathcal{H}$ and $\frac{\partial \mathcal{H}}{\partial u}$ grow at most quadratically along finite times, or
3. $\mathcal{H}(x, u) \leq \beta_0(u)$ and $\left| \frac{\partial \mathcal{H}}{\partial u}(x, u) \right| \leq \alpha_0(u)(\beta_0(u) - \mathcal{H}(x, u))$ for some continuous real functions $\alpha_0$, $\beta_0$ and all $(x, u) \in N \times \mathbb{R}$.

Condition (3) is, however, not derived from theorem 3.3 but due to [CRS12], Cor. 3.3 and, again, logically independent of the other conditions. A physically interesting consequence of condition (2), which actually generalizes the above results on autonomous and sandwich NPWs of quadratic growth, is that it provides stability of completeness of plane waves within the class of NPWs with quadratic behaviour of $\mathcal{H}$ (see [CRS12], Rem. 3.5).

Observe that physically reasonable models of classical gravitational pp-waves, as discussed below equation (14), possess a non-complete wave surface and hence corollary 3.4 does not apply in this case. However, the geodesics will still be ‘complete at infinity’ since the asymptotic conditions of corollary 3.4 hold true for the multipole as well as for the logarithmic terms in (14). However, the geodesics could leave the exterior region ‘at the inside’ proceeding to the matter region. This behaviour clearly has to be considered as physically reasonable. Also, mathematically, completeness of the trajectories of (32), (33) on incomplete Riemannian manifolds is subject to very strong conditions, see for example, [Gor70]: a sufficient condition, for example, is that $\mathcal{H}$ is proper and bounded from below, which certainly does not hold in our case. Note that this applies to wave surfaces of the form $N = \mathbb{R}^n \setminus \{0\}$ as well as to those of the form $\mathbb{R}^n$ with a (closed) ball removed. In the latter case one would, of course, match the solution to some non-vacuum interior region inside the ball. The situation is, of course, completely analogous in the case of NPWs.

Turning now to the general case, i.e. to the quest for completeness of the full pp-wave geometry (1) we more extensively make use of the power of theorem 3.3. Indeed $F_j = -\hbar^2 (A_{k,j} - A_{j,k})$ is no longer vanishing, but still its self-adjoint part satisfies $S = 0$ so that theorem 3.3 puts no restriction on $A_{k,j}$. On the other hand, $X = -\frac{1}{2}\hbar^2 (2A_{k,u} - \mathcal{H}, k)$ and we can no longer write $X$ as the gradient of a potential. So we cannot use condition (2) and have to exclusively resort to theorem 3.3(1). In this way we obtain

**Corollary 3.5** (Completeness of the Brinkmann metric). The full pp-wave spacetime (1) is complete if $N$ is complete and $\nabla_i \mathcal{H}$ and $\hbar^2 A_{k,u}$ grow at most linearly along finite times.

Finally, we come to discuss completeness of gyrotropic pp-waves (17). In this case, the wave surface is flat and so we only have to deal with the asymptotics of the metric functions.
Corollary 3.6 (Completeness of gyratons). Any gyratonic pp-wave (17) with $N = \mathbb{R}^n$ and $\mathcal{H}_x$, as well as $A_{k,u}$ growing at most linearly along finite times is complete.

Now, the asymptotics of the explicit gyratonic pp-waves of [FF05, FIZ05] (see section 2, p.7) imply that $\mathcal{H}_x$ and $A_{k,u}$ even decay or only grow logarithmically for large $x$. However, physically reasonable models are, again, singular on the axis (e.g. (20)) or should be matched to some interior matter region so that the wave surface is $\mathbb{R}^n$ without a point or $\mathbb{R}^n$ with a ball removed and hence incomplete. So, again, we obtain for such ‘gravitational’ gyratons only ‘completeness at infinity’, but the geodesics could leave the exterior region ‘at the inside’ proceeding into the matter region. Once again, this behaviour is to be considered as physically perfectly reasonable.

4. The impulsive limit

In this section we turn our focus to impulsive versions of the Brinkmann metric (1). Generally, impulsive gravitational waves model short, but violent pulses of gravitational or other radiation. In particular, in his seminal work [Pen72], Penrose has considered impulsive pp-waves, that is, spacetimes of the form (13) with

$$\mathcal{H}(x, u) = H(x) \delta(u),$$

(34)

where $\delta$ denotes the Dirac function and $H$ is a function of the spatial variables only. Since then, various methods of constructing impulsive gravitational waves with or without cosmological constant have been introduced; for an overview see, for example, [GP09, Chapter 20]. In particular, impulsive gravitational waves have been found to arise as ultrarelativistic limits of Kerr–Newman and other static spacetimes which make them interesting models for quantum scattering in general relativistic spacetimes.

More generally, impulsive NPWs (iNPWs), i.e. (16) with (34), have been considered in [SS12, SS15]. In all these models, which are impulsive versions of special cases of (1) with the off-diagonal terms $A_i$ vanishing, the field equations put no restriction on the $u$-behaviour of the profile function $\mathcal{H}$ (see section 2). Hence, the most straightforward approach to impulsive waves in this class of solutions is, indeed, to view them as impulsive limits of sandwich waves with an ever shorter, but stronger profile function which precisely leads to (34).

However, here we are mainly interested in impulsive versions of the full pp-wave spacetimes (1) which, in particular, includes impulsive versions of gyratonic pp-waves (17). Here, the situation is more subtle, as detailed in [FF05, FIZ05], where such geometries have been considered along with their extended versions. A more detailed discussion of four-dimensional geometries with a flat transverse space in the form (19) was given recently in [PSS14, section 7.] Since this discussion also applies to the general case and leads to our model of the impulsive full pp-wave metric, we briefly recall it here. To begin with, we introduce the convenient quantity

$$\omega(\rho, \varphi, u) \equiv \frac{J_{\rho}(\rho, \varphi, u)}{2\rho},$$

(35)

such that the vacuum field equations take the form (with $\triangle$ denoting the flat Laplacian)

$$\omega,\varphi = 0, \quad \omega,\rho = 0, \quad \Re \mathcal{H} = 4 \omega^2 + \frac{2}{\rho^2} J_{u\varphi},$$

(36)
implying $\omega = \omega(u)$ which corresponds to a rigid rotation. Relation (35) immediately gives

$$J = \omega(u)\rho^2 + \chi(u, \varphi),$$

(37)

where $\chi(u, \varphi)$ is an arbitrary $2\pi$-periodic function in $\varphi$. Taking (37) and a suitable ansatz for $\mathcal{H}$,

$$\mathcal{H} = \omega^2(u)\rho^2 + 2\omega(u)\chi(u, \varphi) + \mathcal{H}_0(u, \rho, \varphi),$$

(38)

the remaining field equation in (36) becomes

$$\Delta \mathcal{H}_0 = \rho^{-2} \Sigma, \quad \text{with} \quad \Sigma(u, \varphi) \equiv 2(\chi_{,u\varphi} - \omega \chi_{,\varphi}).$$

(39)

Removing the rigid rotation by the natural global gauge $\omega = 0$, and using the splitting $\mathcal{H}_0(u, \rho, \varphi) = \mathcal{H}_0(\rho, \varphi)\chi_H(u)$, $\chi(u, \varphi) = \tilde{\chi}(\varphi) \chi_J(u) + \Phi(\varphi)$, we obtain $\Sigma(u, \varphi) = 2\tilde{\chi}(\varphi)\chi_{J,u}(u)$ and equation (39) takes the form

$$\Delta \mathcal{H}_0(\rho, \varphi)\chi_H(u) = \frac{2}{\rho^2} \tilde{\chi}(\varphi) \chi_{J,u}(u).$$

(41)

If $\Sigma = 0$, i.e. $\tilde{\chi}(\varphi) = \text{const.}$, equation (41) reduces to $\Delta \mathcal{H}_0 = 0$ and there is no restriction on the $u$-dependence of $\mathcal{H}_0$ and $J$. In particular, the energy profile $\chi_H(u)$ and the angular momentum density profile $\chi_J(u)$ can be taken independently of each other. In [PSS14], it was demonstrated that the curvature is proportional to $\chi_H$ and $\chi_{J,u}$ which leads to impulsive waves by setting $\chi_H(u)$ to be proportional to the Dirac $\delta$, but using a box-like profile for $\chi_J(u)$.

However, in the case when $\Sigma \neq 0$ there occurs a coupling of the profile functions. Indeed, the supports of $\chi_H(u)$ and $\chi_{J,u}(u)$ have to coincide since otherwise both sides of (41) have to vanish individually, leading to the vanishing of $\tilde{\chi}(\varphi)$ and hence $\Sigma$. In particular, the box profile in the angular momentum density $\chi_J(u)$ leads to two Dirac deltas in the energy density.

Moreover, we can, of course, combine such a coupled solution with specific homogeneous solutions. Hence, it is most natural and physically relevant to prescribe a general box-like profile for the angular momentum density and a delta-like profile for the energy density of the form

$$\mathcal{H}(x, u) = H(x)\delta_{a,b}(u), \quad \mathcal{A}(x, u) = a_i(x)\partial_{L_i}(u),$$

(42)

where we define (see figure 1)

$$\delta_{a,b}(u) = \alpha \delta(u) + \beta \delta(u - L), \quad \text{and} \quad \partial_{L_i}(u) = \frac{1}{L_i}(\Theta(u) - \Theta(u - L)).$$

(43)

Here, $\alpha$, $\beta$, and $L > 0$ are some constants, $\delta$ denotes the Dirac measure and $\Theta$ is the Heaviside function. This ansatz covers the coupled case ($\alpha = 1/L, \beta = -1/L$) as well as all
the models studied in [YZF07] (p-gyraton: \( \alpha = 0 = \beta \), AS-gyraton: \( a_i = 0 = \beta \), a-gyraton: \( \alpha > 0 \), \( \beta = 0 \), b-gyraton: \( \alpha = 0 \), \( \beta > 0 \)), which all arise from specific combinations of homogeneous solutions.

So the impulsive full pp-wave metric we will consider in the rest of our work is explicitly given by

\[
\text{d} s^2 = h_{ij} \text{d} x^i \text{d} x^j - 2 \text{d} u \text{d} \tau + H(x) \delta_{\alpha,\beta}(u) \text{d} u^2 + 2a_i(x) \partial_{\tau}(u) \text{d} u \text{d} x^i. \tag{44}
\]

Of course, off the wave zone (given by \( u \in [0, L] \)) the spacetime is just the product of the Riemannian wave surface \((N, h)\) with flat \( \mathbb{R}^2 \). From now on we will assume \((N, h)\) to be complete and call \( M_0 = N \times \mathbb{R}^2 \) the background of the impulsive wave (44), which is then complete as well. From (7)–(12) we immediately observe that the components of boost-weight \(-1\) and of \(-2\), namely,

\[
\Psi_{3\gamma} = \frac{1}{n} m^i_{(i} h^{i|j|} a_{|j|;i} \partial_{\tau} L, \tag{45}
\]

\[
\Psi_{3\gamma} = m^i_{(i} m^j_{|j|} m^k_{|k|} \left( a_{|k|;i} - \frac{1}{n-1} h^{\alpha\beta} (h_{ij} a_{|k;\tau|} - h_{\alpha\beta} a_{|k;\tau|}) \right) \partial_{\tau} L, \tag{46}
\]

\[
\Psi_{4\ell} = m^i_{(i} m^j_{|j|} \left( -\frac{1}{2} h_{|i|j|} \delta_{\alpha,\beta} a_{|i|;|j|} \delta_{L^{-1},|L^{-1}|} + h^{\alpha\beta} a_{|i|;|j|} \partial_{\tau} L^2 \right. \\

\left. - \frac{1}{n} h_{ij} h^{ij} \left( -\frac{1}{2} h_{|i||j|} \delta_{\alpha,\beta} a_{|i|;|j|} \delta_{L^{-1},|L^{-1}|} + h^{\alpha\beta} a_{|i|;|j|} \partial_{\tau} L^2 \right) \right), \tag{47}
\]

are only non-trivial in the wave zone while, in general, the rest of the spacetime corresponds to the type-D background.

5. Completeness of the impulsive limit

First, we review previous results on the completeness of impulsive gravitational waves. In the simplest case of classical impulsive pp-waves, i.e. (13) with (34), the spacetime is flat Minkowski space off the single wave surface \( \{ u = 0 \} \) where the curvature is concentrated. Consequently, the geodesics for impulsive pp-waves have been derived in the physics literature (see, for example, [FPV88]) by matching the geodesics of the background on either side of the wave in a heuristic manner—the geodesic equation contains nonlinear terms, ill-defined in distribution theory. This approach, in particular, leaves it open whether the geodesics cross the wave surface at all.

In [KS99a, KS99b] this question has been answered in the affirmative using a regularization approach within the theory of nonlinear distributional geometry ([GKOS01, Chapter 4]) based on algebras of generalized functions ([Col85]). In this way, a completeness result for all impulsive pp-waves, i.e. for all smooth profile functions \( H \), was achieved although this aspect was not emphasized in the original works. Observe that this contrasts with the completeness results in the extended case (corollary 3.4) where the spatial asymptotics of \( \mathcal{H} \) enter decisively. Moreover, this approach in a limiting process establishes that the geodesics in the entire spacetime are, indeed, the straight line geodesics of the background which are refracted by the impulse to become broken and possibly discontinuous.

More generally, in [SS12, SS15] geodesics in iNPWs, i.e. (16) with (34), were investigated. Again, using a regularization approach it was proven that if the wave surface \( N \) is complete, then the iNPW is geodesically complete irrespective of the behaviour of the profile function \( H \). Again, this is in contrast to the extended case where the completeness
depends crucially on the spatial asymptotic behaviour of the profile function $H$ (see section 3). Moreover, the geodesics in the limit again are geodesics of the background, which are refracted by the impulse (see also [FIZ05] for a heuristic argument). More precisely, using a fixed-point argument it was shown in [SS12] that in the regularized iNPW

$$dx^2 = h_{ij}dx^i dx^j - 2dudr + \delta_\varepsilon(u)H(x)du^2,$$  \hspace{1cm} (48)

with $\delta_\varepsilon$ a standard mollifier\(^6\), geodesics are complete in the following sense: for each point $p \in M$ ‘in front’ of the impulsive wave, i.e. $u < 0$ and each tangent direction $v \in T_p M$, we consider the geodesic $\gamma_v$ of (48) starting in $p$ into direction $v$. If $\gamma_v$ reaches the regularized wave zone given by $|u| \leq \varepsilon$, then there is an $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the geodesic $\gamma_v$ passes through the regularized wave zone and continues as a (complete) geodesic of the background ‘behind’ the impulsive wave. This result has been rephrased in the language of nonlinear distributional geometry in [SS15], which allows one to omit the reference to the initial data in the final completeness statement.

As is well known, classical impulsive $pp$-waves, and more generally non-expanding as well as expanding impulsive gravitational waves propagating in constant curvature backgrounds, have also been described by a continuous form of the metric (see, for example, [GP09, Chapter 20]). Actually, these metrics are locally Lipschitz continuous and hence the geodesics equations possess locally bounded, but possibly discontinuous, right-hand sides. Employing the solution concepts of Carathéodory and Filippov ([Fil88]), respectively, these systems of ODEs have been recently investigated leading to the following results: the geodesics are complete and of $C^1$-regularity in classical $pp$-waves ([LSŠ14]), non-expanding ([PSSŠ15]) and expanding ([PSSŠ16]) impulsive waves propagating on Minkowski, de Sitter, and anti-de Sitter backgrounds. However, so far no continuous form of the impulsive full $pp$-wave metric or merely of the gyratonic $pp$-wave metric has been found.

Geodesic completeness for non-expanding impulsive gravitational waves in (anti-)de Sitter space has also been proven in the distributional picture in [SSLP16] using a regularization approach and a fixed-point argument in a spirit similar to the present article. Finally, a proof of geodesic completeness of impulsive gyratonic $pp$-waves has been sketched in [PSŠ14], section VIII.

In the following we provide our main result, which establishes completeness of the impulsive full $pp$-wave metric (44) using a regularization approach.

To begin with, we consider the regularized metric

$$dx^2 = h_{ij}dx^i dx^j - 2dudr + H(x)\delta_{\varepsilon,0}^{\varepsilon}(u)du^2 + 2a_i(x)\delta_{\varepsilon}^{\varepsilon}(u)du dx^i,$$  \hspace{1cm} (49)

where we have regularized the profile functions replacing $\delta$ by a standard mollifier

$$\delta_\varepsilon(x) = \frac{1}{\varepsilon} \phi\left(\frac{x}{\varepsilon}\right)$$  \hspace{1cm} (50)

with $\phi$ a smooth function supported in $[-1, 1]$ with unit integral. Moreover, we have regularized the Heaviside function by the primitive of $\delta_\varepsilon$, i.e. replacing $\Theta$ by

$$\Theta_\varepsilon(x) = \int_{-1}^{x} \delta_\varepsilon(t) dt.$$  \hspace{1cm} (51)

\(^6\) For a precise definition, see (50), below.
More explicitly, we set
\[ \delta_{\alpha,\beta}(u) = \alpha \delta_{\alpha}(u) + \beta \delta_{\beta}(u - L), \quad \text{and} \quad \vartheta_{L}(u) = \frac{1}{L}(\Theta_{L}(u) - \Theta_{L}(u - L)). \] (52)

In the following we will prove that any geodesic in the regularized impulsive full \( pp \)-wave metric (49) that reaches the wave zone given by \( \{ u \in [-\varepsilon, L + \varepsilon] \} \) will pass through it, provided that \( \varepsilon \) is small enough. This will lead to our main result on the completeness of the impulsive full \( pp \)-wave metric (44) which we state at the end of this section. In the final section 6, we will relate these complete geodesics to the geodesics of the background.

To begin with, we give the explicit form of the geodesic equations for the metric (49).

Observe that in the present case the \( u \)-equation is also trivial and hence we may use a rescaling as in (4) to write any geodesic not parallel to, or contained in the impulsive wave surface \( \{ u = 0 \} \) as
\[ \gamma_{\varepsilon}(s) = \left( x_{\varepsilon}^{i}(s), s, r_{\varepsilon}(s) \right). \] (53)

Now we obtain, cf. (21)
\[ \dot{r}_{\varepsilon} = \vartheta_{L}(a_{i||j}) \dot{x}_{\varepsilon}^{j} = -(g_{\varepsilon}^{\alpha l}(a_{i\alpha l} - a_{\alpha l}, l) - H_{j} \delta_{\alpha,\beta}) \dot{x}_{\varepsilon}^{l} - \left( g_{\varepsilon}^{\alpha l}(a_{i\alpha l} - a_{\alpha l}, l - \frac{1}{2}H_{j} \delta_{\alpha,\beta}) - \frac{1}{2}H (\delta_{\alpha,\beta}) u \right). \] (54)

\[ \ddot{x}_{\varepsilon}^{l} = -\nabla_{\varepsilon}^{ij} \dot{x}_{\varepsilon}^{i} \dot{x}_{\varepsilon}^{j} - \vartheta_{L} h^k (a_{k,j} - a_{j,k}) \dot{x}_{\varepsilon}^{k} - \frac{1}{2}h^k (2a_k \delta_{\varepsilon}^{l} - H_{j} \delta_{\varepsilon}^{l} + H_{j,k} \delta_{\varepsilon}^{k}), \] (55)

where \( g_{\varepsilon}^{\alpha l} = h^k a_k \vartheta_{L}. \)

As in the extended case, the \( r \)-equation can simply be integrated once the \( x \)-equations are solved and completeness of the geodesics is determined by completeness of the solutions to the spatial equations (see proposition 3.1.) The latter again take the form of the equations of motion on the Riemannian manifold \( (N, h) \), now with an external force term depending on time, velocity and the regularization parameter \( \varepsilon \). More explicitly, we may rewrite equation (55) in the form, cf. (32)
\[ D_{\dot{x}_{\varepsilon}}(x_{\varepsilon})(s) = F_{\varepsilon}(x_{\varepsilon}(s)) \dot{x}_{\varepsilon}(s) + X^{\varepsilon}(x_{\varepsilon}(s)), \] (56)

where \( D \) denotes the connection on \( (N, h) \) and we have set \( F_{\varepsilon} = -\vartheta_{L} h^k (a_{k,j} - a_{j,k}) \) and \( X^{\varepsilon} = -\frac{1}{2}h^k (2a_k \delta_{\varepsilon}^{l} - H_{j} \delta_{\varepsilon}^{l} + H_{j,k} \delta_{\varepsilon}^{k}). \)

Now, for fixed \( \varepsilon \), the solutions will be complete by corollary 3.5 provided \( H \) and \( a_{i} \) show a suitable asymptotic behaviour. But here we aim at a result for general \( H \) and \( a_{i} \), and so we have to take a different approach. Indeed, for fixed \( \varepsilon \) by ODE theory we have a local solution \( x_{\varepsilon} \) for any initial condition taken at say the ‘left’ boundary of the wave zone \( u = -\varepsilon \) (see figure 2). However, the time of existence of such a solution will, in general, depend upon the regularization parameter \( \varepsilon \) and could shrink to zero if \( \varepsilon \to 0 \). We will prove that this is not the case. More precisely, applying a fixed-point argument we will show that such solutions \( x_{\varepsilon} \) have a uniform (in \( \varepsilon \)) lower bound \( \eta \) on their time of existence which will (at least for small \( \varepsilon \)) allow them to cross the regularization region of the first \( \delta \)-spike, i.e. \( |u| \lesssim \varepsilon \). Once they reach \( u = \varepsilon \) they are subject to equations (55) with only the \( \vartheta_{L} \)-terms being non-trivial. In other words, we have to deal with (56) with \( X^{\varepsilon} \) vanishing. In this situation we may now apply the completeness results established in section 3. More precisely, we appeal to theorem 3.3(1) whose assumptions hold anyway in our case and we

\footnote{We will deal with these (simple) geodesics separately.}
obtain that the solution \( x_c \) will continue at least until it reaches the regularization region of the second \( \delta \)-spike at \( u = L - \varepsilon \). However, there we can reapply our fixed-point argument to secure that \( x_c \) reaches \( u = +L \) and then clears the entire wave zone to enter the background region ‘behind’ the wave.

We will now state and prove the fixed-point argument. To simplify notations we will, instead of dealing with equation (55) directly, consider the following model initial value problem:

\[
\ddot{x} = F_1(x_c, \dot{x}_c) + F_2(x_c)\delta_c + F_3(x_c, \dot{x}_c)\frac{1}{L}\Theta_c, \tag{57}
\]

\[
x_c(-\varepsilon) = x_0^\varepsilon, \quad \dot{x}_c(-\varepsilon) = \dot{x}_0^\varepsilon. \tag{58}
\]

Additionally, we will write \( x_c(\varepsilon) \) or briefly \( (x_c)_\varepsilon \) to denote nets (sequences). Now, we have

**Proposition 5.1.** Let \( F_1, F_2 \in C^\infty(\mathbb{R}^n, \mathbb{R}^n) \), \( F_2 \in C^\infty(\mathbb{R}^n, \mathbb{R}^n) \), let \( x_0, \dot{x}_0 \in \mathbb{R}^n \), let \( (x_0)_\varepsilon \), \( (\dot{x}_0)_\varepsilon \), in \( \mathbb{R}^n \) such that \( x_0^\varepsilon \rightarrow x_0 \) and \( \dot{x}_0^\varepsilon \rightarrow \dot{x}_0 \) for \( \varepsilon \searrow 0 \) and let \( b, c > 0 \). Define \( I_1 = \{x \in \mathbb{R}^n : |x - x_0| \leq b\} \), \( I_2 = \{x \in \mathbb{R}^n : |x - x_0| \leq c + K\|F_2\|_{\infty, I} \} \) and \( I_3 = I_1 \times I_2 \), where \( K \) is a bound on the \( L^1 \)-norm of \( \Theta_c \). Furthermore, set

\[
\eta = \min \left(1, \frac{b}{C_1}, \frac{c}{C_2}, \frac{L}{2}\right), \tag{59}
\]

where

\[
C_1 = 2 + |x_0| + \|F_1\|_{\infty, I} + K\|F_2\|_{\infty, I} + \frac{K}{L}\|F_3\|_{\infty, I}. \quad \text{and}
\]

\[
C_2 = 1 + \|F_1\|_{\infty, I} + \frac{K}{L}\|F_3\|_{\infty, I}. \quad \text{Finally, let} \quad \varepsilon_0' \quad \text{be such that} \quad |x_0^\varepsilon - x_0| \leq \eta \quad \text{and} \quad |\dot{x}_0^\varepsilon - \dot{x}_0| \leq \eta \quad \text{for all} \quad 0 < \varepsilon \leq \varepsilon_0'. \quad \text{Then, the initial value problem (57), (58) has a unique solution} \quad x_c \quad \text{on} \quad I \quad \text{with} \quad (x_c(\varepsilon), \dot{x}_c(\varepsilon)) \subseteq I_3.
\]

**Proof.** We aim at applying Weissinger’s fixed-point theorem ([Wei52]) to the solution operator

\[
A_{\varepsilon}(x)(t) := x_0^\varepsilon + \dot{x}_0^\varepsilon(t + \varepsilon)
+ \int_{-\varepsilon}^{t} \int_{-\varepsilon}^{\sigma} (F_1(x(\sigma), \dot{x}(\sigma)) + F_2(x(\sigma))\delta_c(\sigma) + F_3(x(\sigma), \dot{x}(\sigma))\Theta_c(\sigma)) d\sigma \, ds
\]

on the complete metric space

\[
X_c := \{x \in C^1([-\varepsilon, \eta - \varepsilon]) : (x, \dot{x})([-\varepsilon, \eta - \varepsilon]) \subseteq I_3\},
\]

where we use the norm \( \|x\|_{C^1} = \|x\| + \|\dot{x}\| \).

To begin with, we show that \( A_{\varepsilon} \) maps \( X_c \) to itself. Indeed, for \( x \in X_c \) we have

\[
\|A_{\varepsilon}(x)(t) - x_0\| \leq |x_0^\varepsilon - x_0| + \eta(|\dot{x}_0^\varepsilon - \dot{x}_0| + |x_0|) + \eta^2\|F_1\|_{\infty, I}
+ \eta\|F_2\|_{\infty, I} \|\delta_c\|_{L^1} + \eta\frac{1}{L}\|F_3\|_{\infty, I} \|\Theta_c\|_{C^0}
\leq \eta C_1 \leq b, \quad \text{and}
\]

\[
\left| \frac{d}{dt}A_{\varepsilon}(x)(t) - \dot{x}_0 \right| \leq |\dot{x}_0^\varepsilon - \dot{x}_0| + \eta \left(\|F_1\|_{\infty, I} + \frac{K}{L}\|F_3\|_{\infty, I} \right) + K\|F_2\|_{\infty, I} \tag{63}
\]

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Moreover for \( x, y \in X_\varepsilon \), we have

\[
|A^a_\varepsilon (x) - A^a_\varepsilon (y)| \leq \text{Lip}(F_1, I_\varepsilon) \|x - y\|_{c^1} \frac{\eta^{2n}}{(2n)!} + \text{Lip}(F_2, I_\varepsilon) K \|x - y\| \frac{\eta^{2n-1}}{(2n - 1)!}
\]

\[
+ \text{Lip}(F_3, I_\varepsilon) \frac{K}{L} \|x - y\|_{c^1} \frac{\eta^{2n-2}}{(2n - 2)!}
\]

\[
+ \text{Lip}(F_3, I_\varepsilon) \frac{K}{L} \|x - y\|_{c^1} \frac{\eta^{2n-1}}{(2n - 1)!}.
\]

(64)

where \( \text{Lip}(F_i, I_j) \) denotes a Lipschitz constant for \( F_i \) on \( I_j \). So we have

\[
\|A^a_\varepsilon (x) - A^a_\varepsilon (y)\|_{c^1} \leq C \frac{\eta^{2n}}{(2n)!} \|x - y\|_{c^1}
\]

(65)

and since \( \sum_{n=0}^{\infty} \frac{\eta^{2n}}{(2n)!} \) converges, we obtain a unique fixed point of \( A_\varepsilon \) on \( X_\varepsilon \). Hence a net of unique solutions \( x_\varepsilon \) of (57), (58) defined on \([-\varepsilon, \eta - \varepsilon]\), which together with its derivatives is uniformly bounded in \( \varepsilon \).

We will now detail the procedure envisaged prior to proposition 5.1 to obtain our main result. Fix a point

\[
p = (x_\varepsilon, u_\varepsilon, r_\varepsilon) \text{ in } M
\]

lying say ‘before’ the wave zone\(^8\), i.e. \( u_\varepsilon < 0 \), and a vector \( v \) in \( T_p M \). In the following we will, most of the time, simplify notations by omitting the index from the \( x \)-component as we have done in the model initial value problem (57), (58) and write, for example, \( x_\varepsilon \) instead of \( x_\varepsilon^i \). Now, without loss of generality, we may assume \( \varepsilon \) to be so small that \( p \) also lies ‘before’ the regularized wave zone, i.e. in \([u < -\varepsilon]\) and hence in a the region of \((M, g)\) which coincides with the background spacetime \( M_0 = N \times \mathbb{R}^2 \) ‘before’ the impulse. Now we consider the geodesic \( \gamma(s) = (x(s), u(s), r(s)) \) starting at \( p \) in direction \( v \) in the background spacetime \( M_0 \) which we will from now on call our ‘seed geodesic’. By virtue of the geodesic equations in the background \( M_0 \)

\[
D_x \dot{x} = 0, \quad \ddot{u} = 0, \quad \ddot{r} = 0,
\]

(66)

we see again that \( u(s) \) is affine and hence it suffices to consider the case of strictly increasing \( u(s) \). Indeed, otherwise the ‘seed geodesic’ will never reach the (regularized) wave zone, being either confined to the null surface \( P(u_\varepsilon) \) (see section 2), or even moving away from the wave zone and hence in any case being (forward) complete. So we may, without loss of generality, write the seed geodesic in the form (4), i.e. \( \gamma(s) = (x(s), u(s), r(s)) \) or briefly as \( \gamma(s) = (x(s), r(s)) \).

Now \( \gamma \) will reach the wave zone of the impulsive wave, i.e. \( s = 0 \) in finite time and it is convenient to introduce the data of \( \gamma \) at this instance as

\(^8\) The entire argument is precisely the same in the ‘time-reflected’ case when \( p \) is assumed to lie ‘behind’ the wave zone, i.e. in \([u > L]\).
Now, we start to think of the ‘seed geodesic’ \( \gamma \) also as being a geodesic in the regularized spacetime (49). In fact, it will reach the regularized wave zone at \( u = -\varepsilon \) with data

\[
\gamma (-\varepsilon) = (x_0^\varepsilon, \varepsilon, r_0^\varepsilon), \quad \dot{\gamma} (-\varepsilon) = (\dot{x}_0^\varepsilon, 1, \dot{r}_0^\varepsilon).
\]

(68)

Using this data we solve the initial value problem for the geodesics in the regularized spacetime (49), that is, we consider the system (54), (55) with data (68). Now, by smoothness of the ‘seed geodesic’ \( \gamma \) the data (68) converges to the data (67), in particular:

\[
x_0^\varepsilon \to x_0 \quad \text{and} \quad \dot{x}_0^\varepsilon \to \dot{x}_0,
\]

and we may apply proposition 5.1 to obtain a solution \( x_\varepsilon \) of (55), (68) on \((-\infty, \varepsilon)\), provided \( \varepsilon \leq \eta/2 \). Hence, we also obtain a solution \( r_\varepsilon \) of (54) with data (68), hence a geodesic \( \gamma_\varepsilon \) which coincides with the ‘seed geodesic’ \( \gamma \) up to \( s = -\varepsilon \) and exists until it leaves the regularized first \( \delta \)-spike at \( s = \varepsilon \) and we denote the corresponding data by

\[
\gamma_\varepsilon (\varepsilon) = (x_\varepsilon^\varepsilon, \varepsilon, r_\varepsilon^\varepsilon), \quad \dot{\gamma}_\varepsilon (\varepsilon) = (\dot{x}_\varepsilon^\varepsilon, 1, \dot{r}_\varepsilon^\varepsilon).
\]

(70)

As discussed earlier in \([\varepsilon, L - \varepsilon]\), the geodesic equation (55) reduces to

\[
\ddot{x}_j^\varepsilon = -\Gamma^{ij}_k x_k^\varepsilon \ddot{x}_j^\varepsilon - \frac{1}{L} h^{k} (a_{k,j} - a_{j,k}) \dot{x}_j^\varepsilon,
\]

(71)

whose right-hand side is actually independent of \( \varepsilon \). However, we have to solve (71) with the \( \varepsilon \)-dependent data (70). Anyway, by theorem 3.3(1) we obtain a solution \( x_\varepsilon \) which extends our prior solution from \( s = \varepsilon \) up to \( s = L - \varepsilon \), and since by proposition 5.1 the data \( x_\varepsilon^\varepsilon \) and \( \dot{x}_\varepsilon^\varepsilon \) are uniformly bounded (in \( \varepsilon \)), the solutions will be uniformly bounded as well. In particular, this applies to the data at \( s = L - \varepsilon \):

\[
x_\varepsilon (L - \varepsilon) = x_\varepsilon^\varepsilon, \quad \dot{x}_\varepsilon (L - \varepsilon) = \dot{x}_\varepsilon^\varepsilon.
\]

(72)

We now wish to reapply proposition 5.1 on the interval \([L - \varepsilon, L - \varepsilon + \eta]\) and so we need the data (72) to even converge. This however, follows from continuous dependence of solutions to ODEs once we have established that the data (70) converges, which we do next.

**Lemma 5.2.** Let \((x_\varepsilon)_{\varepsilon}\), given by proposition 5.1. Then

(i) \( \sup_{t \in [-\varepsilon, \varepsilon]} |x_\varepsilon (t) - x_0| = O(\varepsilon) \),

(ii) \( \dot{x}_\varepsilon (\varepsilon) \to \dot{x}_0 - F_2 (x_0) \) as \( \varepsilon \to 0 \).

**Proof.** (i) On \([-\varepsilon, \varepsilon]\), the solution \( x_\varepsilon \) is given by proposition 5.1 and can be expressed by

\[
x_\varepsilon (t) = x_0^\varepsilon + \dot{x}_0^\varepsilon (t + \varepsilon)
\]

\[
+ \int_{-\varepsilon}^t \int_{-\varepsilon}^s F_1 (x_\varepsilon (\sigma), \dot{x}_\varepsilon (\sigma)) + F_2 (x_\varepsilon (\sigma)) \dot{\delta}_\varepsilon (\sigma)
\]

\[
+ F_3 (x_\varepsilon (\sigma), \dot{x}_\varepsilon (\sigma)) \delta_\varepsilon (\sigma) d\sigma ds.
\]

(73)

Consequently,

\[
|x_\varepsilon (t) - x_0| \leq |x_0^\varepsilon - x_0| + 2\varepsilon (|\dot{x}_0^\varepsilon - \dot{x}_0| + |\dot{x}_0|)
\]

\[
+ 4\varepsilon^2 \|F_1\|_{\infty, b} + 2\varepsilon \|F_2\|_{\infty, K} + 4\varepsilon^2 K \|F_3\|_{\infty, t},
\]

(74)

where we used the uniform boundedness of \((x_\varepsilon)_{\varepsilon}\) and \((\dot{x}_\varepsilon)_{\varepsilon}\) established in proposition 5.1.
To obtain (ii) we differentiate (73), insert $t = \varepsilon$, and then we estimate
\[
|k_2(\varepsilon) - \dot{x}_0 - F_2(x_0)| \leq |k'_0 - \dot{x}_0| + 2\varepsilon\|F_1\|_{\infty,d_3} + \int_{-\varepsilon}^{\varepsilon} (F_2(x_2(s)) - F_2(x_0))\delta(s)\,ds + 2K\varepsilon\|F_3\|_{\infty,d_3}
\]
where we have used that $\int_{-\varepsilon}^{\varepsilon} \delta(s)\,ds = 1$ in the first inequality and (i) to see that the first term in the final line converges to zero as $\varepsilon \searrow 0$.

We will explicitly give the limit in (ii) for our case in section 6, below. For the time being we are in the position to reapply proposition 5.1 to obtain a solution $\gamma_\varepsilon = (x_\varepsilon, r_\varepsilon)$ to (55), (54) with data (72) on the domain $[L - \varepsilon, L + \varepsilon]$, again provided that $\varepsilon \leq \eta/2$. Moreover, $x_\varepsilon$ is uniformly bounded in $\varepsilon$ together with its derivative which, in particular, applies to the data at $s = L + \varepsilon$:
\[
x_\varepsilon(L + \varepsilon) = x_\varepsilon^L, \quad \dot{x}_\varepsilon(L + \varepsilon) = \dot{x}_\varepsilon^L.
\] (75)

But now we have reached the background spacetime ‘behind’ the regularized wave zone and the solutions just obtained can be continued as solutions $x_\varepsilon$ of the background geodesic equations (66) with data (75). By completeness of the background $M_0$ these solutions extend to all positive values of their parameter. Now, inserting this solution into the geodesic equation’s $r$-component (54), we obtain also a forward complete solution $r_\varepsilon$. Hence, together we have obtained a complete smooth geodesic $\gamma_\varepsilon$, which coincides with the ‘seed geodesic’ $\gamma$ on $(-\infty, -\varepsilon)$ and with a background geodesic for $u \geq L + \varepsilon$. Note, however, that in the background ‘behind’ the regularized wave zone, $\gamma_\varepsilon$ does not coincide with a single geodesic of the background since the data (75), which we feed into the background geodesic equation (66) at $s = L + \varepsilon$, depends on $\varepsilon$. Therefore, the global geodesic $\gamma_\varepsilon$ for $s \geq L + \varepsilon$ coincides with a background geodesic starting at $L + \varepsilon$ with data $x_\varepsilon^L, \dot{x}_\varepsilon^L$, and $r_\varepsilon(L + \varepsilon) = r_\varepsilon^L, \dot{r}_\varepsilon(L + \varepsilon) = \dot{r}_\varepsilon^L$.

Finally, it remains to deal with the geodesics which start at points $p$ with $u_p \in [0, L)$. To begin with, if $u_p = 0$, we start within the first impulsive surface in some specified direction $v \in T_pM$. If $v$ is tangent to the null hypersurface $P(0)$ then the corresponding geodesic will be either null or space-like, but will, in any case, stay entirely within $P(0)$ and thus have a trivial $u$-component (see section 2). However, then an inspection of the geodesic equation (21) reveals that the $x$-equation coincides with the geodesic equation on the complete Riemannian manifold $(N, h)$ and hence its solution is complete. Feeding this solution into the $r$-equation (which again simplifies drastically) we obtain completeness. In the case where $v$ is transversal to $P(0)$ there is a ‘seed geodesic’ with data (67) coinciding with $p$ and $v$ and we have already covered this case. Precisely the same argument applies in the ‘time-reflected’ case to all points $p$ with $u_p = L$, i.e. which lie on the impulsive surface of the second spike.

Finally, for all points $p$ with $u_p \in (0, L)$ we may assume that $\varepsilon$ is so small that $u_p \in (\varepsilon, L - \varepsilon)$, hence that $p$ lies in the ‘intermediate’ region where the geodesic equations (71) are independent of $\varepsilon$. In case $v \in T_pM$ is tangential to $P(u_p)$ the geodesic again stays entirely in the hypersurface $P(u_p)$ and is complete by theorem 3.3(1). In the case where $v$ is transversal to $P(u_p)$, again, by ‘time symmetry’ we have to only discuss the case of an increasing $u$-component. So once more by theorem 3.3(1), the geodesic will reach
and we may apply proposition 5.1 since the data at this instant will converge to the data of the corresponding solution of (71) at $s = L$.

Summing up, we have proved our main result.

**Theorem 5.3.** Given a point $p$ in the regularized impulsive full pp-wave spacetime (49) and $v \in T_p M$, then there exists $\varepsilon_0$ such that the maximal unique geodesic $\gamma_\varepsilon$ starting in $p$ in direction $v$ is complete, provided $\varepsilon \leq \varepsilon_0$.

We now briefly discuss the case of profile functions $H$ and $a_i$ in the metric (49) possessing poles, making it necessary to remove them from the spacetime, or likewise the case that the exterior solution (49) is matched to some interior non-vacuum solution for small $x$ and at least some $u$-interval. Recall from section 2 that such situations occur in physically interesting models and that this leads to an incomplete wave surface $N$. In such a case our method still applies, but with some restrictions. Indeed, if a ‘seed geodesic’ $\gamma$ hits the wave zone at $u = 0$ sufficiently far away from the poles or the matching surface to an interior solution we may first apply proposition 5.1 with the constants $b$ and $c$ chosen to be so small that the problematic region is omitted. Then, in the ‘intermediate region’ $u \in [\varepsilon, L - \varepsilon]$ we can estimate the solution $x_\varepsilon$ in terms of the data of the ‘seed geodesic’ and the right-hand side of (71), hence independently of $\varepsilon$, which again makes it possible to avoid the problematic region. This finally applies as well to the second application of proposition 5.1 in the interval $[L - \varepsilon, L + \varepsilon]$. On the other hand, for ‘seed geodesics’ aiming too closely at poles or the matching boundary to an interior solution, completeness cannot be guaranteed, which is actually in complete agreement with physical expectations.

Finally, to end this section we prove an additional boundedness result for the global geodesics $\gamma_\varepsilon$ of theorem 5.3. Indeed, local uniform boundedness of the $x$-component $x_\varepsilon$ (and of its derivative) follows directly from proposition 5.1 and the fact that in the ‘intermediate region’ $u \in [\varepsilon, L - \varepsilon]$ the geodesic equation is actually $\varepsilon$-independent. However, we also obtain local uniform boundedness of the $r$-component and hence of $\gamma_\varepsilon$ itself, as follows from the next statement.
Lemma 5.4 (Uniform boundedness of \( r \)). The \( r \)-component \( r_\varepsilon \) of any complete geodesic \( \gamma_\varepsilon \) of theorem 5.3 is locally uniformly bounded in \( \varepsilon \).

Proof. We first consider the first spike and hence let \( s \in [-\varepsilon, \varepsilon] \). Then

\[
|\dot{r}_\varepsilon(s)| \leq |\dot{r}_0 - \dot{r}_0| + |\dot{r}_0| + |\dot{r}_0 - \dot{r}_0| + |\dot{r}_0| + \int_{-\varepsilon}^{\varepsilon} |\dot{r}_\varepsilon(\sigma)| \, d\sigma \, ds,
\]

where for the last term we use equation (54) and estimate

\[
\int_{-\varepsilon}^{\varepsilon} \int_{-\varepsilon}^{\varepsilon} \dot{r}_\varepsilon(\sigma) \, d\sigma \, ds \leq \frac{K}{L} \|Da\|4\varepsilon^2(C')^2
\]

\[
+ \frac{\|h^{-1}\|}{\alpha} \frac{K^2}{L^2} \|Da\|4\varepsilon C' + \|DH\| \times (\alpha + \beta)K2\varepsilon C'
\]

\[
+ \frac{\|h^{-1}\|}{\alpha} \frac{K^2}{L^2} 4\varepsilon + \|DH\| (\alpha + \beta)D\varepsilon^2 + 2\|DH\| (\alpha + \beta)\|\rho\|,
\]

where \( C' = |\dot{x}_0| + c + K\|F_2\|_\infty \). Since \( \varepsilon \leq 1 \), this is bounded independently of \( \varepsilon \). Here, the norms are \( L^\infty \)-norms over the compact set \( I \subseteq \mathbb{R}^n \) given by proposition 5.1.

In the interval \([L - \varepsilon, L + \varepsilon]\), we may argue in precisely the same way. Finally, for the intervals \((-\infty, -\varepsilon]\), \([\varepsilon, L - \varepsilon]\), and \([L + \varepsilon, \infty)\) one uses the continuous dependence of solutions to ODEs on the initial conditions and the convergence of the data of the ‘seed geodesics’ \( r(\varepsilon) \) of (67) to \( r(0) = r_0 \).

6. Limits

In this final section we investigate the limiting behaviour of the complete regularized geodesics \( \gamma_\varepsilon \) provided by theorem 5.3 as the regularization parameter \( \varepsilon \) goes to zero. Physically, this amounts to calculating the geodesics of the impulsive full \( pp \)-wave metric (44). In fact, this is only interesting if the geodesics are not parallel to, or contained in the wave zone \( u \in [0, L] \). So let \( \gamma = (s, x(s), r(s)) \) again be a ‘seed geodesic’ starting ‘in front’ of the wave zone with increasing \( s \). To simplify notations we will also briefly write \( \gamma = (x, r) \) and denote the data at the first impulsive surface by \( \gamma(0) = (x_0, r_0) \) and \( \gamma(0) = (\bar{x}_0, \bar{r}_0) \), respectively. Clearly motivated by the procedure which leads to the completeness result in section 5 we define the limiting geodesic \( \tilde{\gamma} \) by (see also figure 3)

\[
\gamma(s) := \begin{cases} 
\gamma(s) & (s < 0), \\
\gamma^+(s) & (0 \leq s < L), \\
\gamma^{++}(s) & (s \geq L), 
\end{cases}
\]

(77)

where \( \gamma^+ = (x^+, r^+) \) is a geodesic between the spikes, i.e. a solution of

\[
D_x^i \dot{x}^i = -\frac{1}{L} \hbar^2 (a_{i,j} - a_{j,i}) \dot{x}^j, \quad \tilde{r} = \frac{1}{L} a_{i,j} \dot{x}^i \dot{x}^j - \left( \hbar^2 \frac{1}{L} a_{i,j} (a_{i,j} - a_{j,i}) \right) \tilde{x}^i,
\]

with initial data (cf. (70))

\[
\gamma^+(0) = (x_0^+, \bar{r}_0^+) = \lim_{\varepsilon \to 0} \gamma_\varepsilon(\varepsilon) = \lim_{\varepsilon \to 0} (x_\varepsilon^+, \bar{r}_\varepsilon^+),
\]

(79)
Figure 3. The limiting behaviour of the complete geodesics $\gamma_\varepsilon$ with two values of the regularization parameter $0 < \varepsilon_2 < \varepsilon_1$ exemplified. The $x$-components of the seed and limiting geodesics are drawn in bold black, the regularized geodesics given by proposition 5.1 are drawn dotted in green $(x_{\varepsilon_1})$ and blue $(x_{\varepsilon_2})$, respectively. Note that $x_\varepsilon(\varepsilon)$ converges to $x(0)$ for $\varepsilon \searrow 0$ and similarly, $x_\varepsilon(L + \varepsilon) \to x^+(L)$.

\[
\dot{\gamma}^+(0) = \left(\dot{x}_0^+, \dot{r}_0^+\right) = \lim_{\varepsilon \to 0} \dot{\gamma}_\varepsilon(\varepsilon) = \lim_{\varepsilon \to 0} \left(\dot{x}_\varepsilon^+, \dot{r}_\varepsilon^+\right),
\]

where of course $\gamma_\varepsilon$ is the global geodesic of theorem 5.3 associated with the ‘seed’ $\gamma$. Furthermore, $\gamma^{++} = (x^{++}, r^{++})$ is a background geodesic ‘behind’ the wave zone, i.e. $\gamma^{++}$ solves (66) with initial data (cf. (75))

\[
\gamma^{++}(L) = \left(x_0^{++}, r_0^{++}\right) = \lim_{\varepsilon \to 0} \gamma(L + \varepsilon) = \lim_{\varepsilon \to 0} \left(x_\varepsilon^{++}, r_\varepsilon^{++}\right),
\]

and

\[
\dot{\gamma}^{++}(L) = \left(\dot{x}_0^{++}, \dot{r}_0^{++}\right) = \lim_{\varepsilon \to 0} \dot{\gamma}(L + \varepsilon) = \lim_{\varepsilon \to 0} \left(\dot{x}_\varepsilon^{++}, \dot{r}_\varepsilon^{++}\right).
\]

Now we turn to the explicit calculation of the limits of the data (79) (80), i.e. the behaviour of the limiting geodesic at the first spike.

**Proposition 6.1.** Let $\gamma_\varepsilon = (x_\varepsilon, r_\varepsilon)$ given by theorem 5.3 with ‘seed’ $\gamma$ as above. Then,

\[
x_0^+ = x_0, \quad (x_0^+)’ = (x_0)’ - \frac{1}{2} h^k(x_0) \left(\frac{2 a_k(x_0)}{L} - \alpha H_{,k}(x_0)\right),
\]

\[
r_0^+ = r_0 + \frac{\alpha}{2} H(x_0),
\]

\[
\dot{r}_0^+ = \dot{r}_0 + \alpha H_{,j}(x_0) \left(\frac{\dot{x}_j(x_0)}{2} - \frac{1}{8} h^{ik}(x_0) \left(\frac{2 a_k(x_0)}{L} - \alpha H_{,k}(x_0)\right) + h^{ik}(x_0) a_k(x_0)\right) + \frac{1}{2L^2} h^{ik}(x_0) a_k(x_0) a_j(x_0).
\]
Proof. We only sketch these overly technical calculations. First, \( x_0^+ = x_0 \) is a direct consequence of lemma 5.2(i). To obtain \( \dot{x}_0^+ \) by lemma 5.2(ii) we only have to read off \( F_2(x_0) \) from (55).

For \( r_0^+ \) and \( \dot{r}_0^+ \) we use the integral equation for \( r_0 \), with the integral equation for \( \dot{r}_0 \) inserted, together with the identity \( \int_{-\varepsilon}^{\varepsilon} \dot{r}_0(s) \int_{-\varepsilon}^{\varepsilon} \tilde{c}_e(\sigma) \, d\sigma \, ds = \frac{\varepsilon}{2} \) and lemma 5.2. We only detail this in case of \( r_0^+ \):

\[
r_0^+(\varepsilon) = r_0^+ + 2\varepsilon \dot{r}_0^+ + \frac{1}{L} \int_{-\varepsilon}^{\varepsilon} \int_{-\varepsilon}^{\varepsilon} \theta_D \dot{x}_0^+ \dot{x}_e \, d\sigma \, ds - \frac{1}{L} \int_{-\varepsilon}^{\varepsilon} \int_{-\varepsilon}^{\varepsilon} h \theta D \dot{x}_0^+ \dot{x}_e \, d\sigma \, ds \nonumber
\]

\[
+ \int_{-\varepsilon}^{\varepsilon} \int_{-\varepsilon}^{\varepsilon} DH \delta_{\alpha,\beta} \dot{x}_e \, d\sigma \, ds - \frac{1}{L} \int_{-\varepsilon}^{\varepsilon} \int_{-\varepsilon}^{\varepsilon} h a^2 \theta \delta_{\alpha,\beta} \dot{x}_e \, d\sigma \, ds \nonumber
\]

\[
+ \frac{2L}{\varepsilon} \int_{-\varepsilon}^{\varepsilon} \int_{-\varepsilon}^{\varepsilon} h a^2 \theta \delta_{\alpha,\beta} \dot{x}_e \, d\sigma \, ds + \frac{1}{2} \int_{-\varepsilon}^{\varepsilon} \int_{-\varepsilon}^{\varepsilon} H(\delta_{\alpha,\beta}) \delta \, d\sigma \, ds,
\]

where clearly I-V are \( O(\varepsilon) \). For example, \( ||II|| \leq 2\varepsilon ||DH|| ||\alpha||K (||x_0||^2 + K_2 ||F_2||_{\infty,L}) \). Finally, we estimate VI:

\[
|VI| = \frac{\alpha}{2} H(x_0) \leq \frac{1}{2} \int_{-\varepsilon}^{\varepsilon} \int_{-\varepsilon}^{\varepsilon} \left| H(x_0(\sigma)) - H(x_0) \right| |(\delta_{\alpha,\beta}(\sigma))_\delta| \, d\sigma \, ds
\]

\[
\leq \left( \sup_{\sigma \in [-\varepsilon,\varepsilon]} \left| H(x_0(\sigma)) - H(x_0) \right| \right) ||\alpha|| ||\beta||,
\]

where we used that \( \int_{-\varepsilon}^{\varepsilon} \int_{-\varepsilon}^{\varepsilon} \delta_{\alpha,\beta}(\sigma) \, d\sigma \, ds = 1 \) and \( \sup_{\sigma \in [-\varepsilon,\varepsilon]} \left| H(x_0(\sigma)) - H(x_0) \right| \) converges to zero by lemma 5.2.

The most difficult case is \( r_0^+ \), since \( \dot{r}_0^+ \) is not uniformly bounded in \( \varepsilon \). However, \( \dot{r}_0^+(\varepsilon) \) converges, as can be seen in the following. As above, write \( \dot{r}_0^+(\varepsilon) = r_0^+ + I_0^+ + II^0 + III^0 + IV^0 + V^0 + VI^0 \). Then, \( I_0^+ + II^0 + III^0 + IV^0 + V^0 + VI^0 \) one inserts the integral equation for \( \dot{x}_0^+ \) and uses that \( \int_{-\varepsilon}^{\varepsilon} \delta_{\alpha,\beta}(s) \int_{-\varepsilon}^{\varepsilon} \delta_{\alpha,\beta}(\sigma) \, d\sigma \, ds = \frac{\varepsilon^2}{4} \). Finally, \( IV^0 \) and \( V^0 \) can be handled similarly and for \( VI^0 \) one uses integration by parts to obtain \( \frac{\alpha}{2} \int_{-\varepsilon}^{\varepsilon} \int_{-\varepsilon}^{\varepsilon} H(\delta_{\alpha,\beta}) \delta \, d\sigma \, ds = \frac{\alpha}{2} \int_{-\varepsilon}^{\varepsilon} \delta H \dot{x}_0^+ \, ds \), which can be handled as \( III^0 \).

We see from the explicit expressions given in proposition 6.1 that the limiting geodesic \( \tilde{\gamma} \) displays the following behaviour at the first spike: the \( x \)-component is continuous with a finite jump in its velocity, since \( \dot{x}_0^+ = x_0 \) (in general). On the other hand, the \( r \)-component itself is discontinuous, suffering a finite jump, and the same is also true for its derivative. This behaviour is correlated with the fact that while \( \dot{r}_0^+ \) is not uniformly bounded on \( [-\varepsilon, \varepsilon] \), its value when leaving the regularization strip \( \dot{r}_0^+(\varepsilon) \) is nevertheless uniformly bounded in \( \varepsilon \).

Now, we may analogously calculate the limits at the second spike to obtain

**Proposition 6.2.** Let \( \gamma_c = (x_c, r_c) \) given by theorem 5.3 with 'seed' \( \gamma \) as above. Then,

\[
x_0^{++} = x^+(L),
\]

\[
(\dot{x}_0^{++})^+ = (\dot{x}^+(L))^+ - \frac{1}{2} \lambda^2 \left( \frac{2a_0(x^+(L))}{L} - \beta \right),
\]

where

\[
\beta = \frac{\alpha}{2} \int_{-\varepsilon}^{\varepsilon} \int_{-\varepsilon}^{\varepsilon} H(\delta_{\alpha,\beta}) \delta \, d\sigma \, ds - \frac{\alpha}{2} \int_{-\varepsilon}^{\varepsilon} \delta H \dot{x}_0^+ \, ds.
\]
\[ n_0^+ = r^+(L) + \frac{\beta}{2} H(x^+(L)), \]
\[ \dot{n}_0^+ = \dot{r}^+(L) \quad (88) \]
\[ + \beta H, (x^+(L)) \left( \frac{\dot{x}^+(L)}{2} - \frac{1}{2} H(x^+(L)) \frac{2 a_k(x^+(L))}{L} - \beta H, (x^+(L)) \right) \]
\[ + h^k(x^+(L)) a_k(x^+(L)) + \frac{1}{2 L^2} h^k(x^+(L)) a_k(x^+(L)) a_j(x^+(L)). \quad (89) \]

Finally, we may prove the actual convergence result, saying that the regularized complete geodesics \( \gamma \) of theorem 5.3 converge to the limiting geodesics \( \gamma \) of (77), consisting of appropriately matched geodesics of the complete background and the ‘intermediate’ region.

**Theorem 6.3.** Let \( \gamma = (x, r) \) be the complete geodesic of theorem 5.3 with ‘seed’ \( \gamma \) as above. Then, \( \gamma \) converges to the limiting geodesic \( \dot{\gamma} = (\dot{x}, \dot{r}) \) of (77) in the following sense:

1. \( x_e \to \ddot{x} \) locally uniformly, \( \dot{x}_e \to \dot{x} \) as distribution and uniformly on compact intervals not containing \( t = 0 \) or \( t = L \).
2. \( r_e \to \ddot{r} \) as distribution and in \( \mathcal{C}^1 \) on compact intervals not containing \( t = 0 \) or \( t = L \).

Observe that the notions of convergence in the theorem are optimal given the regularity of the limits: \( \ddot{x} \) is discontinuous at \( s = 0 \) and \( s = L \) so uniform convergence can only hold on bounded intervals not containing these two points. The same reasoning applies to \( \ddot{r} \) and its derivative.

**Proof.** Let \( T > 0 \); then, on \([-T, -\varepsilon]\] the regularized geodesic \( \gamma_e \) is equal to the ‘seed geodesic’ \( \gamma \) since both solve the same initial value problem.

Now, on \( [\varepsilon, T] \) with \( T \leq L - \varepsilon \) (the first spike), \( \gamma_e \) and \( \gamma^+ \) solve the same ODE, i.e. (78), but with different initial conditions. By continuous dependence of the solutions of ODEs on initial data we have for \( t \in [\varepsilon, T] \)

\[ \max(|\gamma_e(t) - \gamma^+(t)|, |\dot{\gamma}_e(t) - \dot{\gamma}^+(t)|) \leq \max(|\gamma_e(\varepsilon) - \gamma^+(\varepsilon)|, |\dot{\gamma}_e(\varepsilon) - \dot{\gamma}^+(\varepsilon)|) e^{LT}, \quad (90) \]

where \( L \) is a Lipschitz constant for the (\( \varepsilon \)-independent) right-hand side of (78) (on some suitable bounded set). Now, we can insert \( \gamma^+(0) \) to obtain

\[ |\gamma_e(\varepsilon) - \gamma^+(\varepsilon)| \leq |\gamma_e(\varepsilon) - \gamma^+(0)| + |\gamma^+(0) - \gamma^+(\varepsilon)| \to 0, \quad (91) \]

since \( \gamma^+(0) = \lim_{\varepsilon \to 0} \gamma^+(\varepsilon) \) and \( \gamma^+ \) is continuous. Analogously, we insert \( \gamma^+(0) \) to obtain

\[ |\dot{\gamma}_e(\varepsilon) - \dot{\gamma}^+(\varepsilon)| \leq |\dot{\gamma}_e(\varepsilon) - \dot{\gamma}^+(0)| + |\dot{\gamma}^+(0) - \dot{\gamma}^+(\varepsilon)| \to 0, \quad (92) \]

since \( \dot{\gamma}^+(0) = \lim_{\varepsilon \to 0} \dot{\gamma}^+(\varepsilon) \) and by the continuity of \( \dot{\gamma}^+ \). This gives uniform convergence of \( (\gamma_e) \) on any compact interval not containing \( t = 0 \) (in its interior).

On \( [-\varepsilon, \varepsilon] \), lemma 5.2 yields the convergence of \( (x_e) \) to \( x_0 \) and to establish the global distributional convergence of \( (r_e) \), it remains only to consider \( (r_e) \) on \([-\varepsilon, \varepsilon]\). So let \( \phi \in \mathcal{D}(\mathbb{R}) \) and by lemma 5.4 there is a \( C > 0 \) such that \( |r_e(t)| \leq C \) and thus

\[ \left| \int_{-\varepsilon}^{\varepsilon} (r_e(s) - \dot{r}(s)) \phi(s) ds \right| \leq 2\varepsilon C \|\phi\|_{\infty} \to 0. \quad (93) \]
Finally, the second spike, i.e. $[L - \varepsilon, L + \varepsilon]$, and behind the wave zone, i.e. $(L + \varepsilon, \infty)$, can be handled analogously.

7. Conclusion

In this contribution we have provided completeness results both for the extended as well as for the impulsive case of full pp-waves. This class of geometries allows for an arbitrary $n$-dimensional Riemannian manifold $N$ as a wave surface and for non-trivial off-diagonal terms in the metric (encoding the internal spin of the source), hence includes (as special cases) classical pp-waves, NPWs, and gyratons alike. In the extended case, we have generalized the results on NPWs by providing a sufficient criterion for completeness in terms of the spatial asymptotics of the metric functions with a certain (local) uniformity with respect to proper time. In the impulsive case we have employed a regularization approach to prove that all these geometries are complete (provided the spatial profile functions are smooth). This confirms earlier results saying that the effect of the spatial asymptotics of the metric functions on completeness is wiped out in the impulsive limit. Finally, we have explicitly derived the geodesics in the impulsive case in terms of a matching of corresponding background geodesics. This result, in particular, allows us to derive the particle motion in the field of specific ultrarelativistic particles possessing an internal spin, opening the door to applications in quantum scattering and high-energy physics.

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