TOPOLOGY OF QUASIPERIODIC FUNCTIONS ON THE PLANE

I. DYNNIKOV AND S. NOVIKOV

ABSTRACT. This article describes a topological theory of quasiperiodic functions on the plane. The development of this theory was started (in different terminology) by the Moscow topology group in early 1980s. It was motivated by the needs of solid state physics, as a partial (nongeneric) case of Hamiltonian foliations of Fermi surfaces with multivalued Hamiltonian function [1]. The unexpected discoveries of their topological properties that were made in 1980s [2, 3] and 1990s [4, 5, 6] have finally led to nontrivial physical conclusions [7, 8] along the lines of the so-called geometric strong magnetic field limit [9]. A very fruitful new point of view comes from the reformulation of that problem in terms of quasiperiodic functions and an extension to higher dimensions made in 1999 [10]. One may say that, for single crystal normal metals put in a magnetic field, the semiclassical trajectories of electrons in the space of quasimomenta are exactly the level lines of the quasiperiodic function with three quasiperiods that is the dispersion relation restricted to a plane orthogonal to the magnetic field. General studies of the topological properties of levels of quasiperiodic functions on the plane with any number of quasiperiods were started in 1999 when certain ideas were formulated for the case of four quasiperiods [10]. The last section of this work contains a complete proof of these results based on the technique developed in [21, 22]. Some new physical applications of the general problem were found recently [11].

1. Quasiperiodic functions

Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ denote the $n$-dimensional torus, $\nu: \mathbb{R}^n \to \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ the standard projection.

We say that a real smooth function $\varphi(y) = \varphi(y^1, \ldots, y^k)$ on the $k$-plane $\mathbb{R}^k$ is quasiperiodic with $n$ quasiperiods (frequencies) if it can be represented in the form $\varphi(y) = f(x(y))$:

$$\varphi = f \circ \nu \circ \iota,$$

where $\iota: \mathbb{R}^k \to \mathbb{R}^n$ is an affine imbedding:

$$x^s = a^s_i y^i + x^s_0,$$

$f = f(x): \mathbb{T}^n \to \mathbb{R}$ is some smooth function, and $n \geq 2$ is the minimal possible integer for which such a function $f$ and an affine imbedding $\iota$ exist. Here

---

The work of I. Dynnikov was supported in part by Russian Foundation for Basic Research (grant no. 02-01-00659); the work of S. Novikov was supported in part by the Council of the Russian Academy of Science (grant “Mathematical methods of nonlinear dynamics”).
2 I. DYNNIKOV AND S. NOVIKOV

s = 1, . . . , n and r = 1, . . . , k. In the theory of quasicrystals, people call the space $\mathbb{R}^k$ (where $k = 2, 3$) the physical space and the space $\mathbb{R}^n$ the superspace. Every $n$-periodic function $f(x)$ generates a family of descendants, which are obtained by varying the initial vector $x_0 = (x_0^1, \ldots, x_0^n)$ in the superspace $\mathbb{R}^n$. Any two descendants $\varphi_1(y), \varphi_2(y)$ of the same function $f$ are said to be related. They have the same frequencies and obtain the following property: for any $\varepsilon > 0$, there is a shift $y \mapsto y + a$ in the physical space such that the shifted function $\varphi_2(y + a)$ is $\varepsilon$-close to $\varphi_1(y)$:

$$|\varphi_2(y + a) - \varphi_1(y)| < \varepsilon \quad \forall y \in \mathbb{R}^k.$$ 

Any linear function $\lambda : \mathbb{R}^k \to \mathbb{R}$ of the form $\lambda(y) = \ell(x(y))$ or $\lambda = \ell \circ \iota$, where the linear function $\ell : \mathbb{R}^n \to \mathbb{R}$ belongs to the dual (or “reciprocal”) lattice $(\mathbb{Z}^n)^*$ (i.e., we have $\ell(\mathbb{Z}^n) \subset \mathbb{Z}$) is called a frequency of $\varphi$. The set of all frequencies form a free abelian group with $n$ natural generators $\lambda_1 = \ell^1 \circ \iota, \ldots, \lambda_1 = \ell^n \circ \iota$ where the functions $\ell^s(x) = x^s$, $s = 1, \ldots, n$, are dual to the basic periods. We call this group the group of frequencies. It is a free abelian subgroup $\Gamma^*$ of the dual vector space $\mathbb{R}^*$, and it is the same for the whole family of related quasiperiodic functions (descendants of the same $n$-periodic function $f$).

Analytically, any $n$-periodic function can be presented in the form of a trigonometric series

$$f(x) = \sum_{\ell \in (\mathbb{Z}^n)^*} c_\ell \exp \left(2\pi i \ell(x)\right)$$

Therefore, any quasiperiodic function can be presented in a similar form:

$$\varphi(y) = \sum_{\lambda \in \Gamma^*} b_\lambda \exp \left(2\pi i \lambda(y)\right) = \sum_m b_m \exp \left(2\pi i \sum_{s=1}^n m_s \lambda_s(y)\right),$$

where $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$. By definition, the set of basic frequencies $\lambda_s$ generates the space $\mathbb{R}^k$ over the field $\mathbb{R}$.

For the space $\mathbb{R}^k$ endowed with a Euclidean metric there is a natural identification $\mathbb{R}^k \cong (\mathbb{R}^k)^*$, so the subgroup of frequencies can be treated as a subgroup $\Gamma \cong \Gamma^* \subset \mathbb{R}^k$ in the physical space $\mathbb{R}^k$.

There is an affine symmetry semigroup associated with each family of related quasiperiodic functions. By definition, this semigroup $\tilde{G}$ consists of all affine transformations

$$g : \mathbb{R}^k \to \mathbb{R}^k$$

of the physical space such that

$$g(\Gamma^*) \subset \Gamma^*,$$

where $\Gamma^*$ is treated as a subset of the group of translations: $\Gamma^* \subset \mathbb{R}^k$. For the Euclidean space $\mathbb{R}^k$ we define the symmetry group $G \subset \tilde{G}$ consisting only of isometries $g$ such that $g(\Gamma^*) = \Gamma^*$. 
This group satisfies the general definition of a quasicrystallographic group introduced by S. Novikov and A. Veselov in 1980s in order to answer the question: what is the symmetry of quasicrystals (see [13])? According to that definition the intersection $G \cap \mathbb{R}^k \subset \text{Iso}(\mathbb{R}^k)$ of a quasicrystallographic group with the subgroup of translations should be a finitely generated free abelian group. In our case it is exactly the group $\Gamma^*$. The definition allows the “rotational” quotient group $G/(G \cap \mathbb{R}^k) \subset O_k$ to be infinite. S. Piunikhin studied these groups for $k=2, 3$ in a series of works (see [13]).

Example 1. Consider the two-dimensional case, $k=2$. Let $\theta$ be a unimodular complex number, $|\theta|=1$, $\theta = \exp(i\psi)$, satisfying the equation

$$P(\theta) = \theta^n + a_1\theta^{n-1} + \ldots + a_{n-1}\theta + 1 = 0,$$

where all coefficients are integer-valued, $a_s \in \mathbb{Z}$, and we have $a_s = a_{n-s}$. The complex numbers (or real two-vectors) $\lambda_1 = 1, \lambda_2 = \theta, \ldots, \lambda_n = \theta^{n-1}$ generate a group of frequencies $\Gamma^* \subset \mathbb{C} = \mathbb{R}^2$ with nontrivial rotational symmetry $g \rightarrow g \exp(i\psi)$. It is easy to find such a polynomial $P$ with a root at $\theta = \exp(i\psi)$, where the ratio $\psi/2\pi$ is irrational. There are very complicated quasicrystallographic groups for $k=3$ (see [13]).

2. Quasiperiodic functions in analysis, geometry, and dynamical systems motivated by natural sciences

2.1. Quasiperiodic functions on the real line. Consider the case $k=1$. In the XIX century, one-dimensional quasiperiodic functions with $n$ quasiperiods appeared in the theory of completely integrable Hamiltonian systems of the classical mechanics with $n$ degrees of freedom. According to so-called Liouville’s theorem, the integrability follows from the existence of $n$ smooth independent pairwise commuting integrals of motion. If their common level sets are compact, then the time dependence of the space coordinates along a trajectory can be described by quasiperiodic functions $x^r(t)$ with (at most) $n$ quasiperiods. So, all studies of perturbations of completely integrable systems should start with quasiperiodic unperturbed background. A lot of fundamental work has been done in this area (see [12]).

2.2. Quasiperiodic functions in the theory of nonlinear PDE. Completely integrable PDE systems of the theory of solitons give rise to quasiperiodic functions with $k > 1$. There are very famous $(1+1)$ PDE systems such as KdV ($u_t = 6uu_x + u_{xxx}$) or sine-Gordon ($u_{tt} = u_{xx} + \sin\{u(x,t)\}$), which are completely integrable by the so-called inverse scattering transform method for rapidly decreasing initial values. A countable number of continuous families of exact smooth real “finite-gap” solutions of these equations were discovered in 1970s (see [12]). These solutions are quasiperiodic functions in $x, t$, and depend on many parameters $a, a'$:

$$u(x,t) = F(xU + Vt + U_0; a)$$
for KdV, and
\[
\exp \left( i u(x,t) \right) = F'(U' x + V' t + U'_0; a')
\]
for sine-Gordon. Here \( u(x,t) \) is real in both cases, \( F, F' \) are \( n \)-periodic smooth functions in \( n \) variables (i.e., smooth functions on the real \( n \)-torus). They can be expressed through special functions, namely, theta-functions of a hyperelliptic Riemann surface of genus \( n \). \( U, U', V, V' \) are the \( n \)-vectors of periods of some Abelian differentials of the second kind (see [3]). Let us mention that, for the sine-Gordon system, the function \( u = 1/i \log F' \) is generically a multi-valued function on the “real” \( n \)-torus imbedded in the complex \( 2n \)-dimensional Jacoby torus associated with a complex hyperelliptic Riemann surface. Here we have \( k = 2 \). For famous completely integrable \((2+1)\) systems (like KP, and others) one comes to quasiperiodic solutions of the form \( u(x,y,t) \), which are quasiperiodic functions in \( k = 3 \) physical variables. When studying the dependence of the solution on so-called higher times one may arrive at any value of \( k \).

2.3. Quasiperiodic functions and quasicrystals. Completely different examples come from solid state physics. In 1980s a new type of 2D and 3D media was discovered. People named them “quasicrystals”. The optical analysis of the location of atoms gave an evidence for the group of frequencies incompatible with an ordinary crystal structure. For example, for \( k = 2 \), the observed group of frequencies \( \Gamma^* \) might be generated by the 5th roots of unity:
\[
\lambda_r = \eta^r \in \Gamma^*, \quad r = 0, 1, 2, 3, \quad \eta^5 = 1, \quad P(\eta) = 0,
\]
where
\[
P(z) = z^4 + z^3 + z^2 + z + 1.
\]

Recall that our extension of the idea of symmetry allows the rotational symmetry to be even infinite.

There are two mathematical models of quasicrystals. Let us think of atoms in the physical space \( \mathbb{R}^k \) as being located in a discrete set of points \( x_A \) such that there exists a couple of positive “radii” \( \rho_1, \rho_2 \) with the properties:

a. We have \( |x_A - x_{A'}| \geq \rho_2 \) for all pairs \( A, A' \) with \( A \neq A' \);

b. For every point \( x \in \mathbb{R}^k \), there exists a point \( x_A \) such that \( |x_A - x| \leq \rho_1 \).

We call this set of points quasiperiodic if the distribution \( \sum_A \delta(x - x_A) \) can be decomposed into a Fourier series with finitely generated free abelian group of frequencies \( \Gamma^* \).

In another model, our physical space \( \mathbb{R}^k \) is endowed with a “quasiperiodic tiling”. This means the following:

a. The space is covered by countably many polytopes \( P_B, \mathbb{R}^k = \bigcup_B P_B \), where \( P_B \cap P_{B'} \) is a face for any pair \( B, B' \).

b. Up to shift, there is only a finite number of different polytopes \( P_1, \ldots, P_q \) among them.

c. Let us associate some constant \( c_q \) with every polytope \( P_q \) and consider a function that is equal to \( c_j \) everywhere in the interior of any \( P_B \).
obtained from $P_j$ by a shift. We obtain a piecewise constant function $c(x)$ in $x \in \mathbb{R}^k$ defined (at a full measure set) by our tiling and the choice of the constants $c_j$. The tiling is said to be quasiperiodic if, for every choice of constants $c_j$, the function $c(x)$ is quasiperiodic, \textit{i.e.}, can be presented in the form of a trigonometric series with finitely generated free abelian group $\Gamma^*$ of frequencies.

There is a famous tiling of the plane $\mathbb{R}^2$ by rhombi of two types: one with angles $\pi/5$ and $4\pi/5$, and the other with angles $2\pi/5$ and $3\pi/5$. It is called the Penrose tiling. This tiling is quasiperiodic, which was discovered a few years later after Penrose’s original work (see the history and details of the subject in [13]). An interesting idea of “local rules” was developed by physicists and mathematicians in order to explain the growth of quasicrystals in terms of tilings. In this model, the atoms are located at the vertices of the tiles.

Both models can be obtained from the following construction. Let a “superlattice” $\Gamma$ of full rank be given in the superspace $\mathbb{R}^n$, and the superspace be presented as the direct sum $\mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^{n-k}$, where $\mathbb{R}^k$ is the physical subspace. Let $p : \mathbb{R}^n \to \mathbb{R}^k$ and $q : \mathbb{R}^n \to \mathbb{R}^{n-k}$ be the natural projections. Fix a finite $(n-k)$-polyhedron $D \subset \mathbb{R}^{n-k}$ and consider the “tubular $D$-neighborhood” $D_q = q^{-1}(D) \subset \mathbb{R}^n$ of the physical subspace $\mathbb{R}^k \subset \mathbb{R}^n$. Assume that the boundary of the polyhedron $D$ is disjoint from $q(\Gamma)$, or equivalently, $\partial D_q \cap \Gamma = \emptyset$. Then the set of points

$$p(\Gamma \cap D_q) \subset \mathbb{R}^k$$

is quasiperiodic in the sense of the definition given above.

By taking a certain polytope decomposition of the space $\mathbb{R}^n$ associated with the lattice $\Gamma$ and the polyhedron $D$, one obtains a quasiperiodic tiling of $\mathbb{R}^k$ whose tiles are the intersections of $\mathbb{R}^k$ with the $n$-cells of the decomposition (see survey article [13]).

Very interesting examples of nontrivial symmetry groups come from the superspace $\mathbb{R}^4$ endowed with the Minkovski metric and a lattice $\Gamma \cong \mathbb{Z}^4$ such that the physical subspace $\mathbb{R}^2$, which is spacelike (\textit{i.e.} Euclidean), is invariant under some lattice-preserving mapping from the group $O(3,1)$.

The superspaces $\mathbb{R}^{l,m}$, where $l + m = n$, might also appear in interesting cases.

2.4. Quasiperiodic functions in the theory of conductivity. Here we describe the situation that is the main motivation for our topological and dynamical theory.

For every single crystal normal metal, we have a lattice $\Gamma$ in the physical space $\mathbb{R}^3$. However, our geometrical constructions will live in a completely different space, namely the 3-torus of quasimomenta $T^3$, which is the quotient space of the dual 3-space $(\mathbb{R}^3)^* \cong \mathbb{R}^3$ by the dual (reciprocal) lattice $\Gamma^* \cong \mathbb{Z}^3$. The “Bloch” states of quantum electrons are parameterized by pairs $(m,p)$, where $p$ is a point in the space of quasimomenta, $p \in T^3 = \mathbb{R}^3 / \Gamma^*$, and $m$ is a natural number, which is the index of a branch of the dispersion relation $f(p) =$
\[ \epsilon_m(p) : \mathbb{T}^3 \to \mathbb{R}. \]

In what follows we will always deal with just one branch only, so we drop the index \( m \) in the notation. We assume that \( f(p) = \epsilon(p) \) is a Morse function on the 3-torus or, in other words, a three-periodic Morse function on the covering Euclidean space \( \mathbb{R}^3 \). At zero temperature all electrons occupy the “Dirac sea” \( \epsilon(p) \leq \epsilon_F \) where the “Fermi energy” \( \epsilon_F \) depends on the number of free electrons in the metal. We assume that \( \epsilon_F \) is a regular value for the Morse function \( f = \epsilon(p) \). At low temperatures we are dealing only with “excited” electrons nearby the Fermi level \( \epsilon(p) = \epsilon_F \).

The Fermi level looks geometrically as a two-dimensional surface \( M_F \subset \mathbb{T}^3 \) in the space of quasimomenta. This surface is nonsingular and homologous to zero in the 3-torus. Let us assume that it is connected.

The topological rank \( r(M_F) \) of the Fermi surface is defined as the rank of the image of the first homology group of \( M_F \) under the mapping \( i_* : H_1(M_F, \mathbb{Z}) \to H_1(\mathbb{T}^3, \mathbb{Z}) \cong \mathbb{Z}^3 \) induced by the inclusion \( i : M_F \hookrightarrow \mathbb{T}^3 \). Since \( i_*(H_1(M_F)) \) is a sublattice in \( \mathbb{Z}^3 \), we always have \( r \in \{0, 1, 2, 3\} \).

For example, the topological rank of the Fermi surface of lithium is equal to zero (in this case, the Fermi surface looks like a topological 2-sphere), whereas it is equal to three for copper, gold, platinum, and some other noble metals. For gold, for example, the genus of the Fermi surface is equal to four.

The problem that we will consider is most difficult when the topological rank of the Fermi surface is maximal possible, i.e., equal to three. One can easily show that the genus of the Fermi surface must be greater than or equal to the topological rank.

Interesting dynamical phenomena occur in the presence of a magnetic field. In the semiclassical approximation, an electron, which is considered as a point in the space of quasimomenta, moves along constant energy lines in the plane \( \mathbb{R}^2_{B, p_0} \) orthogonal to the magnetic field \( B \) and passing through the initial position \( p_0 \) of the quasimomentum.

One may say that this is a Hamiltonian system on the 3-torus of quasimomenta with Poisson bracket

\[ \{p_j, p_l\} = \frac{\epsilon}{c} B_{jl} = \frac{\epsilon}{c} \varepsilon_{jq} B^q \]

and Hamiltonian \( f = \epsilon(p) \):

\[ \frac{dp_j}{dt} = \{p_j, \epsilon(p)\}, \]

so the motion preserves the energy and a linear Casimir of the Poisson bracket. The level sets of this Casimir are planes orthogonal to the magnetic field. The trajectories can be treated as leaves of the Hamiltonian foliation on the Fermi surface given by the equation \( \omega = 0 \) where \( \omega \) is the following closed 1-form:

\[ \omega = \sum_j B^j dp_j |_{M_F}. \]

According to the “strong magnetic field limit” principle worked out by I. Lifshitz, M. Azbel, M. Kaganov, and V. Peschanski in early 1960s, all essential properties of the electrical conductivity in the presence of a reasonably
strong uniform magnetic field \( B \) should follow from the structure of the dynamical system on the Fermi surface described above (see [9, 13, 16, 17]). For ordinary normal metals (like gold, for example) one may use this approximation for magnetic fields strong enough in the human sense (like \( 1 \text{ Tesla} < |B| < 10^3 \text{ Tesla} \) for low temperatures; recall: \( 1 \text{ Tesla} = 10^4 \text{ Gauss} \)). If the magnetic field is too strong, then the semiclassical approximation will not be valid. If the magnetic field is too weak, then the electron quasimomentum drift will be too slow, and the distance that the quasimomentum passes for the characteristic time of the electron free motion will become insufficient to affect the observable conductivity.

However, in 1960s the study of the just mentioned dynamical system was only started. Some conceptual mistake was then made in [14] and further investigation was stopped, and resumed only many years later in works [1, 2, 3, 4, 6, 18, 19, 20].

What is crucial for us here is following:

the electron trajectories coincide with connected components of the level curves \( \epsilon(p) = \epsilon_F \) of the function \( \epsilon \) restricted to the planes orthogonal to the magnetic field \( B \); in other words, they are connected components of the level curves of functions that form a family of related quasiperiodic functions with three quasiperiods.

In work [10] an extension of these studies to a larger number of quasiperiods was started. In particular, some new ideas and results were formulated for the case \( n = 4 \). The present work contains the first complete proof of those (properly corrected) statements. The proof is based on the topological technique developed in [21, 22].

Let a constant Poisson bracket \( B_{jk} \) of rank two be given on the \( n \)-torus. Then every Hamiltonian \( f(p) = \epsilon(p) : \mathbb{T}^n \to \mathbb{R} \) defines a Hamiltonian system whose trajectories are exactly the level lines \( \epsilon(p) = \text{const} \) of the restriction of the function \( \epsilon \) to the planes \( \mathbb{R}_{B,a}^2 \) defined as follows. There exist exactly \( n - 2 \) independent linear Casimirs \( K_1, \ldots, K_{n-2}, K_j(p) = K_j^l p_l \), such that \( \{p_s, K_j\} = 0 = K_j^l B_{sl} \). We put

\[
\mathbb{R}_{B,a}^2 = \{K_1 = a_1, \ldots, K_{n-2} = a_{n-2}\},
\]

where \( a = (a_1, \ldots, a_{n-2}) \). So our trajectories are exactly the levels of quasiperiodic functions on the two-planes \( \mathbb{R}_{B,a}^2 \), which form the family of descendants of the \( n \)-periodic function \( \epsilon(p) \). They depend on the constants \( a_1, \ldots, a_{n-2} \).

Topological study of this problem is the central part of this article.

Modern experimental technology allows to construct surfaces with a variety of prescribed small fluctuations. In particular, it is possible to make a quasiperiodic construction with any number of quasiperiods. It presents us a two-dimensional weak quasiperiodic electric potential \( V(x, y) \). In a strong magnetic field \( B \) electrons move along the surface. After averaging we obtain
a slow motion along the level curves $V(x, y) = \text{const}$. These studies, experimental and theoretical, were done originally for periodic potentials with $n \leq 2$ periods only (see [25]), but it was pointed out in work [11] that quasiperiodic potentials can also appear here; new predictions were made for quasiperiodic cases with three and four quasiperiods based on the topological results obtained in a series of works of the present authors ([10] [22] [20]).

3. Topology and dynamics of quasiperiodic functions on the plane: the case of three quasiperiods. The electrical conductivity in metals

We address the following general question. How may the level lines $\varphi = \text{const}$ of a quasiperiodic function $\varphi$ on the plane with $n$ quasiperiods look like? In a generic situation, such a level line is a one-dimensional submanifold of $\mathbb{R}^2$, i.e. a union of curves. We will call these curves “trajectories” because in our studies they have been appearing as semiclassical electron trajectories on the Fermi surface in the presence of a magnetic field since early 1980s when this problem was posed in work [1] as a problem of topology and dynamical systems. It corresponds to the case of three quasiperiods only. Some of those curves may be closed in $\mathbb{R}^2$ (compact) and others nonclosed in $\mathbb{R}^2$ (open). Let us ask the following questions.

**Question 1:** Is the size of the compact trajectories uniformly bounded (for a fixed level of $\varphi$)?

**Question 2:** Do the open trajectories have some nice asymptotic behavior?

The first results were obtained in work [2]. It became clear in the second half of 1980s that the proper form of Question 2 is the following: does any open trajectory have a “strong asymptotic direction” in $\mathbb{R}^2$, i.e., lie in a strip of uniformly bounded width and passes through the strip “from $-\infty$ to $+\infty$”? This specification of the problem was made in article [3]. In work [4] the results of [2] were improved accordingly to the new formulation of the problem. The important breakthrough was made in work [6], but for a long time there was no applications. Physical applications were found later in works [7] [8].

In the physically important case $n = 3$, the positive answer to our Question 1 follows easily from a quite elementary argument. For $n > 3$ it is more difficult, and it will be discussed later. Question 2 is highly nontrivial already for $n = 3$. As mentioned above, the asymptotic behavior of open electron trajectories, i.e., of open connected components of a level line of a quasiperiodic function with three quasiperiods, was studied in [2] [4] [6]. It became finally clear after work [6] that for the family of related quasiperiodic function corresponding to a “typical” direction of the magnetic field (which is regarded as the direction of a plane $\mathbb{R}^2 \subset \mathbb{R}^3$), either their level lines do not have open components at all or the open components all have a strong asymptotic direction. The latter means that each open curve has a parametrization $\gamma(t)$ such that the following
holds for some nonzero two-vector \((x_1, y_1)\):

\[
\gamma(t) = (x_0, y_0) + t \cdot (x_1, y_1) + O(1).
\]

Below we will explain precisely what ‘a typical direction’ means, and provide references to the papers containing proofs of the corresponding results. In the first work [2] completed in the note [4], this type of result was obtained for the special case of small perturbations of a magnetic field having “rational” direction. We shall return to this special case in the next section where we discuss the quasiperiodic functions with four quasiperiods.

Applications of this studies to explaining the electrical conductivity in a strong magnetic field are presented in works [7, 8]. They are based on the results of the Lifshitz school of 1960s. Physicists calculated the contribution of individual trajectories of simple types to the conductivity tensor. These calculations have become a part of textbooks (see [9, 15]). In the case when all trajectories are compact the conductivity components orthogonal to the magnetic field \(B\) decrease as \(|B|^{-1}\) or \(|B|^{-2}\) when \(|B|\) grows while the direction of \(B\) remains fixed. Some special examples of open trajectories lying in finitely wide strips were found at the same time and their contribution to the conductivity was calculated. As pointed out in [7, 8], one can easily extend the just mentioned calculation to the case of general trajectories of the same type. The projection to the plane orthogonal to \(B\) of the part of the conductivity tensor contributed by such a trajectory has two eigenvalues one of which is zero and the other nonzero. Since the contribution of closed trajectories tends to zero when \(|B|\) grows, the observable conductivity tensor for a strong enough \(B\) will depend on open trajectories only.

However, the observable physical conductivity tensor is formed by the contributions of all electron trajectories as the sum of them. What conclusion about this tensor can be made from the qualitative dynamical properties of that system, which was defined on the quantum level?

In order to obtain a nontrivial new physical result, one needs more than the theorems explicitly formulated in [6]. But luckily an additional crucial property also holds for our dynamical system, and this can be extracted from the proofs of the main theorems of works [2, 6]. The property, which we call topological resonance, implies the following for the behavior of trajectories.
For a “typical” family \( \{ \varphi_a \} \) of related quasiperiodic functions \( \varphi_a(p) = \epsilon(p) |_{\mathbb{R}^2_{B,a}} \), the strong asymptotical direction \( \eta_B \) of non-compact trajectories is the same for all trajectories. Moreover, there exist an integral two-plane \( \mu \subset \mathbb{R}^3 \) (i.e., a plane generated by two reciprocal lattice vectors, \( \mu \cap \mathbb{Z}^3 \approx \mathbb{Z}^2 \)) such that \( \eta_B \) has the direction of the intersection of \( \mu \) with the plane orthogonal to the magnetic field:

\[
\eta_B \in \mu \cap \mathbb{R}^2_B.
\]

This integer plane \( \mu \) is locally rigid, i.e., it remains unchanged under small variations of the direction of the magnetic field.

The topological resonance property of our dynamical system makes possible serious applications. It was missed in the classical works of physicists, and a conceptual mistake was made in [14], where calculations led to a result contradicting to this property. This mistake was revealed and corrected only in works [7, 8, 18].

For a strong enough \( B \), the direction \( \eta_B \) is a zero eigenvector of the projection of the conductivity tensor to the plane orthogonal to the magnetic field. The integral plane \( \Pi \subset \mathbb{R}^3 \) is directly observable by measuring the zero eigenvector \( \eta_B \) for two or more magnetic fields \( B \) close to each other.

We refer the reader to recent article [18] for a more detailed physical discussions.

Let us describe the picture topologically. Consider all our objects in the universal covering space \( \mathbb{R}^3 \) with the reciprocal lattice \( \Gamma^* \subset \mathbb{R}^3 \) and the three-periodic Fermi surface

\[
\widehat{M}_F = \nu^{-1}(M_F) \subset \mathbb{R}^3
\]

covering the compact one

\[
M_F \subset \mathbb{T}^3.
\]

(Recall that \( \nu \) stays for the standard projection \( \mathbb{R}^3 \to \mathbb{T}^3 \).) The electron trajectories in the covering space are connected components of the intersections of the three-periodic Fermi surface with planes orthogonal to \( B \). Let \( M_0(B) \) be the closure of the union of all compact trajectories, and let \( L(B) \) be the closure of its complement in the Fermi surface:

\[
L(B) = \overline{M_F \setminus M_0(B)}.
\]

Let \( L_i(B) \) be the connected components of \( L(B) \). In the typical case, \( L(B) \) is a compact two-manifold with boundary

\[
\partial L(B) = \bigcup_{i,s} \beta_{is},
\]

where

\[
\partial L_i(B) = \bigcup_s \beta_{si}.
\]
All boundary curves $\beta_{sl}$ are saddle connection cycles. In the typical case, we may assume that every cycle $\beta_{sl}$ joins a saddle critical point to itself, since all the other cases have positive codimension in the appropriate functional space. (In particular, we assume that there is no rational linear dependence between the components of $B$, and that the Hamiltonian foliation defined by $\omega = \sum B^i dp_i |_{M_F} = 0$ has only Morse singularities and does not have saddle connections between different saddles.)

The part $M_0$ of the Fermi surface can be presented as the union of “cylinders” $Z_q$, $M_0 = \bigcup_q Z_q$, whose interior consists of regular compact trajectories and “bases” are either saddle connection cycles or isolated points (centers). There are finitely many such cylinders, and they are obviously compact. This immediately implies a positive answer to Question 1 posed in the beginning of this section:

the size of all compact trajectories is uniformly bounded.

We call the pieces $L_l$ of the Fermi surface the carriers of open trajectories. By construction, every open trajectory (in $T^3$) is contained in one of the carriers, and, in the generic case, is everywhere dense in it. Let $D^2_{ls} \subset \mathbb{R}^2_{B,a}$ be planar two-discs orthogonal to the magnetic field such that $\partial D^2_{ls} = \beta_{ls}$. We define the “closure” $N_l$ of every carrier $L_l$ as follows:

$$N_l = M_l \cup \left( \bigcup_s D^2_{ls} \right).$$

By construction we also have

$$N_l \cap N_{l'} = \emptyset$$

for $l \neq l'$.

We call our system stable topologically completely integrable if the genus of each surface $N_l$ is equal to one, and this picture is stable under arbitrary small enough perturbations of the magnetic field.

We call the system chaotic if the genus of some $N_l$ is greater than one. According to the main theorem of [6], the latter situation is always topologically unstable.

According to [7, 8, 18], what is important for physical applications, is the following topological resonance property of our system. In the stable topologically completely integrable case, all the closures $N_l$ of the carriers of open trajectories have the same up to sign nonzero homology class:

$$[N_l] = \pm \mu \in H_2(T^3, \mathbb{Z}), \quad \mu \neq 0,$$

which is an indivisible element of the group $H_2(T^3, \mathbb{Z}) \cong \mathbb{Z}^3$. The number of the tori $N_l \subset T^3$ is even because the sum of their homology classes is equal to the class of Fermi surface, which is zero. (Note that every homologically nontrivial connected closed nonselfintersecting two-manifold $M \subset T^3$ always represent an indivisible homology class. Any two such submanifolds with empty intersection represent the same homology class up to sign.) The class $\mu \in H_2(T^3, \mathbb{Z}) \cong \mathbb{Z}^3$ is presented by three relatively prime integers $\mu(B) = (m_1, m_2, m_3)$. 
The integral vector $\mu(B)$ remains unchanged under small perturbations of $B$. Therefore, there is an open set on the sphere $S^2$ with the same $\mu(B)$. This set as well as the integral vector $\mu$ is an observable characteristics of our system, and it can be found experimentally by measuring the conductivity tensor in the presence of strong enough magnetic fields having generic directions. The stable topologically integrable case occurs for all directions $B/|B| \in S^2$ of the magnetic field from an everywhere dense open subset of $S^2$. It was proved in [21, 22] (by two different methods) that this picture may be not valid for directions $B/|B| \in S^2$ of the magnetic field from a nonempty subset whose codimension is at least one.

In terms of quasiperiodic functions, we can say that “non-typical” functions $\varphi_a(p) = \epsilon(p)|_{K_{B,a}^2}$ with three quasiperiods, i.e., such that open connected components of their level sets don’t have a strong asymptotic direction, all lie in a subset that has codimension one (in some natural sense). Examples of level lines with chaotic behavior in the case $n = 3$ were constructed in [21]. We call such level lines strongly chaotic trajectories. Interesting attempts were made in order to find physical properties of the conductivity in these cases. For some special examples it was done in work [24] but in general the stochastic properties of these trajectories are unknown. A. Maltsev formulated the following conjecture.

**Conjecture 1.** The contribution of strongly chaotic trajectories to the conductivity tensor tends to zero when $|B|$ grows (remaining in a reasonable range), which includes the conductivity in the direction of the magnetic field itself.

Previously, S. Tsarev constructed a “weakly chaotic” example (unpublished, see work [21]). In his case, there is a rational dependence between the components of $B$, and there is just one carrier of open trajectories in our sense, which coincides with the Fermi surface. However, the closure of any trajectory in $\mathbb{T}^3$ is not the whole surface, but just a half of it, which is homeomorphic to a 2-torus with two holes. The holes are not homologically trivial in $\mathbb{T}^3$, so they are not regarded as closed in $\mathbb{R}^3$. In Tsarev’s example, the nonclosed level lines still have an asymptotic direction in a weaker sense,

$$\gamma(t) = (x_0, y_0) + t \cdot (x_1, y_1) + o(t),$$

but the projection of any trajectory to a straight line perpendicular to $(x_1, y_1)$ is unbounded.

It is interesting to look at the behavior of trajectories in the special (nongeneric) case of the Fermi surface

$$\epsilon(p) = \cos(p_1) + \cos(p_2) + \cos(p_3) = 0.$$

Examples of this type were investigated numerically and analytically in works [21, 24]. There are chaotic trajectories for the set of magnetic fields whose Hausdorff dimension is presumably equal to some $\alpha$ with

$$1 < \alpha < 2.$$
There are many (in fact, infinitely many) different stable topologically completely integrable zones on the sphere \( S^2 \) having different integral characteristics \( \mu(B) \in \mathbb{Z}^3 \). We call this type of examples generic symmetric levels, see below.

We conjecture the following.

**Conjecture 2.** (i) For a generic connected smooth two-manifold \( M_F \subset \mathbb{T}^3 \) homologous to zero, the set of chaotic directions of the magnetic field has Hausdorff dimension less than one in \( S^2 \).

(ii) For a generic 1-parametric smooth family \( M_{F,t} \subset \mathbb{T}^3 \) of such Fermi surfaces this set has Hausdorff dimension less than two.

A detailed investigation of this problem containing the proofs of all topological statements needed for physical applications found in [7, 8] is performed in [21, 22]. Special attention is paid there to one-parametric families of Fermi surfaces that are levels of the same Morse function \( f : \mathbb{T}^3 \rightarrow \mathbb{R} \):

\[
M_c = \{ f(p) = c \}.
\]

It is proved that, for any \( B \) from a stability zone, open trajectories live on the levels \( M_c \) from a connected interval \( c_1(B) \leq c \leq c_2(B) \) of the real line. For a \( B \) away from the stability zones, the strongly chaotic behavior might appear only on a single level \( c(B) \in \mathbb{R} \). As a corollary we obtain the following result:

For a function \( f=\epsilon(p) \) with symmetry \( \epsilon(p+p_0) = -\epsilon(p) \), where \( p_0 \in \mathbb{T}^3 \) is some shift, strongly chaotic trajectories cannot appear on the levels \( M_c \) with \( c \neq 0 \) because otherwise they must appear on \( M_{-c} \), too, for the same \( B \), which is impossible. In such a case we call the level \( c = 0 \) a generic symmetric level.

According to our conjecture, the Hausdorff dimension of the set of \( B/|B| \in S^2 \) for which the strongly chaotic behavior occurs on such a level is equal to some \( \alpha < 2 \).

The surface \( \sum_{j=1}^{3} \cos(p_j) = 0 \) gives an example of a generic symmetric level with \( p_0 = (\pi, \pi, \pi) \).

Some more details about chaotic trajectories and stability zones for the case of three quasiperiods will be given below. They will be needed for proving our main result about quasiperiodic functions with four quasiperiods (see the next section).

4. **The stable topological complete integrability for** \( n = 4 \) **quasiperiods**

Let us consider now the case of \( n = 4 \) (or more) quasiperiods. For every direction \( \Pi \) of two-planes in \( \mathbb{R}^n \) (i.e., a two-dimensional vector subspace \( \Pi \subset \mathbb{R}^n \)), the original \( n \)-periodic Morse function \( f : \mathbb{T}^n \rightarrow \mathbb{R} \) defines a family of descendants \( \{ \varphi_a(y) \} \) on the family affine two-planes \( \mathbb{R}^{2}_{\Pi,a} \subset \mathbb{R}^n \) having direction \( \Pi \).

We call the level \( \{ f = c \} \) of the function \( f \) topologically completely integrable (TCI) for the direction \( \Pi \) if, for each \( \varphi_a \) from the family, all regular connected
components of the level line $\varphi_a(y) = c$ are either compact or have a strong asymptotic direction. We call this level stable TCI if this property remains unchanged under small perturbations of the function $f$ and the direction $\Pi$, which is a point in the Grassmanian manifold $G_{n,2}$.

We say that the Stable TCI level satisfies the topological resonance condition (for given $\Pi$) if there exists an integral hyperplane $\mu \subset \mathbb{R}^n$, $\mu \cap \mathbb{Z}^n \cong \mathbb{Z}^{n-1}$, such that all open regular trajectories have the same asymptotical direction $\eta_\Pi$ that coincides with the direction of the straight line $\mu \cap \Pi \cong \mathbb{R}$. Since $\mu$ is integral, it must remain unchanged under small perturbations of anything.

Let us point out that even the “trivial case” $n = 2$ is meaningful (as a subject of the elementary Morse theory on the 2-torus): for a generic double-periodic function on the plane there exists a level $f = c$ with a connected component presenting a nontrivial indivisible homology class $\mu \in H_1(T^2, \mathbb{Z})$. All other components of every level are either homologically trivial or homologous to $\pm \mu$. For Morse functions with exactly four critical points and critical values $c_0 < c_1 < c_2 < c_3$, all levels $f = c$, with $c_1 < c < c_2$, have exactly two connected components, whose homology classes are $\pm \mu$. All other nonsingular levels are either compact or empty.

**Question:** Consider famous real nonsingular quasiperiodic finite-gap solutions of the KdV equation

$$u(x,t) = 2\partial_x^2 \log \Theta(xU + tV + U_0) + c_\Gamma$$

with an arbitrary number of quasiperiods (or gaps). Are the levels $u(x,t) = c$ always stable topologically completely integrable or they can be chaotic? How to find their strong asymptotic direction and their integer-valued characteristic $\mu$?

P. Grinevich pointed out to us that, for real smooth finite-gap solutions of the KdV equation, the $n$-periodic function $f = 2\partial_\eta^2 \log \Theta(\eta_1, \ldots, \eta_n) + c_\Gamma$ on the $\eta$-space is always a Morse function on the real $n$-torus with $2^n$ critical points, simply because it can be reduced to the form $f = \sum_{j=1}^{2^n} \alpha_j \sin x_j$ by a diffeomorphism of the torus isotopic to the identity. There is a canonical lattice in the $\eta$-space generated by the so-called $a$-cycles which are the real finite gaps of the 1D Schrödinger operator on a hyperelliptic “spectral” Riemann surface $\Gamma$. The real constants $\alpha_j$ depend on the spectrum. As a conclusion, we get the following:

For generic real nonsingular two-gap solutions of the KdV equation, there exist a critical value $c_{\text{cr}}$ such that all constant speed levels $u(x,t) = c$,

$$c_1 = c_\Gamma - c_{\text{cr}} < c < c_\Gamma + c_{\text{cr}} = c_2$$

are periodic perturbations of a family of straight lines with integral direction $m_1 : m_2$ on the plane with lattice. This direction is locally rigid, but globally depends on the constants $\alpha_j$. All other levels are either compact or empty. We call it the topological speed of the solution.
The computational studies of this problem for finite-gap solutions are now being investigated numerically.

We concentrate now on the case \( n = 4 \) quasiperiods. Work [10] presents an idea of the proof that, for every generic Morse function \( f \) and a noncritical generic level \( f = c \), there exists an open everywhere dense set of two-plane directions \( \Pi \in G_{4,2} \) for which the level \( f = c \) is stable TCI. The proof of this theorem requires the use of some extension of the results of work [22]. Here we make some corrections to the statement of [10] and provide a complete proof.

We believe that a generic level is stable TCI for all directions \( \Pi \) from a subset \( S \subset G_{4,2} \) whose measure is full in \( G_{4,2} \). However, we don’t have an idea how to prove this conjecture. For \( n > 4 \) nothing like that is expected.

Now we start a detailed investigation of the case \( n = 4 \). Even Question 1 of the previous section presents a difficulty here. It is possible that a single level set of a quasiperiodic function with four quasiperiods contains a family of compact components without an upper bound for their size. An example can be constructed easily.

However, we are going to show that there is an open everywhere dense open set of quasiperiodic functions \( \varphi \) with four quasiperiods such that the level lines \( \varphi = \text{const} \) have the same qualitative behavior as those in the typical case of three quasiperiods. The precise formulation is as follows.

**Theorem 1.** There exists an open everywhere dense subset \( S \subset C^\infty(T^4) \) of 4-periodic functions \( f \) and an open everywhere dense subset \( X_f \subset G_{4,2} \) depending on \( f \) such that any level \( M^3_f = \{ f = c \} \) of \( f \) is stable TCI (or does not contain open trajectories at all) for any \( \Pi \in X_f \).

Moreover, for any regular open trajectory, the remainder term \( O(1) \) in (2) as well as the diameter of any compact trajectory are bounded from above by a constant \( C \) not depending on the affine plane \( \mathbb{R}^2_{\Pi,a} \) containing the trajectory, provided that the level \( c \) and the direction \( \Pi \) are fixed.

Let us make a remark about notation and terminology. Once we switched to the case of four quasiperiods, our problem is no longer relevant to the discussed above physical model of conductivity in normal metals in the presence of a magnetic field. So we change the notation for the coordinates in \( \mathbb{R}^n \) from \( p_l \), which we used for quasimomentum, to more customary, \( x_l, l = 0, 1, 2, 3 \), and don’t longer think of the lattice \( \mathbb{Z}^4 \subset \mathbb{R}^4 \) as the one dual to some physical lattice. We think of the “magnetic field” \( B \) as a linear mapping from \( \mathbb{R}^4 \) to \( \mathbb{R}^2 \) (or from \( \mathbb{R}^3 \) to \( \mathbb{R} \) in the \( n = 3 \) case) such that \( \Pi = \ker(B) \). Thus, by \( \mathbb{R}^2_{\Pi,a} \) we mean the two-plane \( B^{-1}(a) \), where \( a \in \mathbb{R}^2 \). In the case \( n = 3 \) we may also think of \( B \) as a vector perpendicular to the plane \( \Pi \). However, we keep calling connected components of the intersections of \( M^3_c \) with the two-planes \( \mathbb{R}^2_{\Pi,a} \) trajectories, just for briefness.

We start by recalling results of [21, 22] for the three-dimensional case in the form needed to prove our theorem. Let \( B : \mathbb{R}^3 \to \mathbb{R} \) be a linear function of irrationality degree three, i.e., of the form \( B(x) = B_1 x_1 + B_2 x_2 + B_3 x_3 \), where \( B_1, B_2, B_3 \) are reals linearly independent over \( \mathbb{Z} \), and let \( f : T^3 \to \mathbb{R} \) be a
generic smooth function. By ‘generic’ we mean that $f$ does not satisfy certain conditions that have codimension $\geq 1$. However, we shall pay attention to codimension one singularities as, in order to deal with the four-dimensional case, we are going to consider one-parametric families of three-dimensional pictures.

We abuse notation by using the same letter $f$ for the lift of $f$ to the covering 3-space $\mathbb{R}^3$. We use notation $M^2_c$ for the level set $f^{-1}(c)$ in $\mathbb{T}^3$ and $\hat{M}^2_c$ for its cover in $\mathbb{R}^3$. By $\gamma_{a,c}$ we denote the whole intersection of $\hat{M}^2_c$ with the plane $\mathbb{R}^2_{\Pi,a} = B^{-1}(a)$. So, the trajectories that we are studying are regular connected components of $\gamma_{a,c}$ or their projections to $\mathbb{T}^3$.

First of all, consider closed trajectories on $M^2_c$. Notice that, since we assumed $B$ to be of maximal irrationality degree, a trajectory in $\mathbb{R}^3$ is closed if and only if so is its image in $\mathbb{T}^3$. Without the assumption on $B$ this may be not true, since a closed trajectory in $\mathbb{T}^3$ may then be non-homologous to zero, in which case its cover in $\mathbb{R}^3$ consists of infinite “periodic” trajectories treated as “open” in the physical applications.

For a generic $f$, compact trajectories on every $M^2_c$ form finitely many cylinders whose bases are either saddle connections or extrema of the restriction $B|_{M^2_c}$. Obviously, the length of compact trajectories is bounded from above by some constant.

Let $U$ be the set of $c$ such that $\gamma_{a,c}$ has unbounded connected components for some $a$. In other words, $c \in U$ if and only if $M^2_c$ contains open trajectories that are not saddle connections. The following picture, which was sketched in the previous section, can be extracted from work [22]:

[Diagram representation of the mentioned case]
The set $U$ is either a closed interval, $U = [c_-, c_+]$, or just one point, $U = \{c_0\}$.
If $U = [c_-, c_+]$ is a nontrivial interval, then for any $c \in U$, there is a (unique) family of two-tori $T^2_{c,1}, \ldots, T^2_{c,2k}$ (with $k$ depending on $c$) imbedded in $\mathbb{T}^3$ such that

1. Each $T^2_{c,i}$ consists of the closure of some open trajectory on $M^2_c$ and a few (may be zero) planar disks perpendicular to $B$;
2. Every open trajectory is contained by whole in one of the tori $T^2_{c,i}$;
3. All the tori $T^2_{c,i}$ define the same up to sign nonzero homology class $\mu$ in $H_2(\mathbb{T}^3, \mathbb{Z})$;
4. For all but finitely many $c$, the tori $T^2_{c,i}$ are pairwise disjoint, and, in this case, a sufficiently small variation of $c$ causes small deformation of the tori. For the exceptional $c$’s they can be made disjoint by a small perturbation. At such a $c$, a couple of tori is born or killed.

All the picture is stable in this case, which means that after a small enough perturbation of $f$, the interval $U$ and the family of tori $T^2_{c,i}$ are perturbed slightly. In particular, the homology class $\mu$ fixed.

Remark 1. In the setting of papers [21, 22], the function $f$ was assumed to be fixed and the point of concern was the dependence of the behavior of our dynamical system on the magnetic field $B$ and on the level of the function $f$. The stability of the whole picture under small perturbations of the function $f$ was not discussed. However, the arguments of those works can be easily modified in order to prove such stability. Indeed, one of the key observations in [21, 22] is that, locally, the qualitative behavior of the trajectories (including the existence of a strong asymptotic direction) depends only on finitely many parameters, which are certain critical values of the “height” function $B(x)$ restricted to the surface $\{f = \text{const}\}$. (For instance, the existence of strongly chaotic examples was proved in [21] by specifying the combinatorial structure of the surface and particular values of the parameters.) It is easy to see that those parameters behave nicely under small perturbations of $f$, so extending the arguments of [21, 22] to this, more general type of perturbations requires almost no additional work.

Let us describe the three-dimensional picture in more details. For a generic level surface $M^2 \subset \mathbb{T}^3$, the structure of trajectories on $M^2$ is as follows. Compact trajectories form a few open cylinders whose “ends” approach either an extremum point of the function $f|_{M^2}$ or a saddle connection cycle, see Fig. 11. The rest of the surface (if not empty) consists of an even number of two-tori with or without holes, and each hole is a saddle connection cycle. Each hole
can be glued up by a planar disk perpendicular to the vector $B$. We denote the obtained surface by $N$. The preimage $\tilde{N} \subset \mathbb{R}^3$ of $N$ under the projection $\nu : \mathbb{R}^3 \to \mathbb{T}^3$ is a family of finitely deformed periodic “wrapped” planes in $\mathbb{R}^3$, see Fig. 2.

What happens to $\tilde{N}$ when the surface $M^2$ changes? Small deformations of the surface $M^2$ cause just small deformations of the tori and their covering planes. Suppose we have a generic 1-parametric family of surfaces $M^2(t)$. This means that we consider a generic 1-parametric family of functions $f_t : \mathbb{T}^3 \to \mathbb{R}$, and for each $t$, the surface $M^2(t)$ is defined by the equation $f_t(x) = \text{const}$.

When the parameter $t$ varies, the connected components of $N$ are just deformed while they stay apart from each other. However, eventually two tori can collide and disappear or, on the contrary, a pair of tori can be born. This occurs when $M^2(t)$ traverses a subset which has codimension one in a natural sense. It is not important here whether $M^2(t)$ is the family of level surface of a single function or an arbitrary generic one-parametric family of surfaces.

The generic tori collision was described in [22]. It was assumed in [22] that the family of surfaces $M^2(t)$ is the family of level surfaces of the same function, $M^2(t) = \{x \in \mathbb{T}^3 \mid f(x) = t\}$. However, the argument is exactly the same for an arbitrary generic family of surfaces.

The following two types of tori collision are possible in the generic case.

(1) A cylinder of closed trajectories with bases attached to two different components of $N$ collapses. The corresponding codimension-one condition has the following form: two different saddles get joined by a saddle
connection. This causes an “interaction” of pairs of open trajectories lying on the collided tori, which turns them into infinitely many closed trajectories, see Fig. 3.

Figure 3. Collapse of a cylinder

(2) A Morse-type surgery occurs on $M^2$ that results in a one-handle added to the surface. The behavior of trajectories is shown in Fig. 4.

Figure 4. A Morse surgery destroying open trajectories

It is important to note here that whenever $N$ consists of just two tori and $M^2(t)$ passes a singularity of one of the two types mentioned above, then all open trajectories get destroyed, so that $N$ is empty right after the critical
event. However, the following is not proven to be impossible in a generic one-parametric family of surfaces: bearing and canceling of a pair of tori occurs alternatingly at moments $t_1, t_2, t_3, \ldots$ so that the sequence $(t_n)$ converges to some $t_*$ and an ergodic regime occurs on $M^2(t_*)$. The latter means that there is an open trajectory on $M^2(t_*)$ whose closure has genus more than one (actually, it should then be equal to three). The existence of such ergodic regimes was proven in [21, 22], and it was shown only that such regimes satisfy a codimension one condition. However, we will not need to deal with ergodic regimes in order to prove our result.

Now we turn to the case when $M^2(t) = M^2_t$ is the family of level surfaces of a generic function on $\mathbb{T}^3$.

Consider the restriction of the function $f$ to the plane $\mathbb{R}_{II,a}^2$ for some $a$. Let $V \subset \mathbb{R}_{II,a}^2$ be the union of all compact components of $\gamma_{a,c}$ over all $c$, and $W \subset \mathbb{R}_{II,a}^2$ the union of all unbounded components of $\gamma_{a,c}$. We have $\mathbb{R}_{II,a}^2 = V \cup W$, $V \cap W = \emptyset$, $V$ is open. Notice: connected components of $V$ are not necessarily bounded. Let $V_1, V_2, \ldots$ be the connected components of $V$. It is easy to see that $f$ is constant on $\partial V_i$ for any $i$.

For $x \in \mathbb{R}_{II,a}^2$ we put

$$f(x) = \begin{cases} f(x) & \text{if } x \in W, \\ f(\partial V_i) & \text{if } x \in V_i. \end{cases}$$

By doing so for all $a$, we obtain a new function $\bar{f} : \mathbb{R}^3 \to \mathbb{R}$. We use the following notation:

$$N_c = \{ x \in \mathbb{T}^3 ; \bar{f}(x) = c \}.$$

The function $f$ and its level sets $N_c$ have the following properties.

**Lemma 1.** The function $\bar{f}$ is a well defined continuous function on $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$.

If $U = \{ c_0 \}$, then $\bar{f} \equiv c_0$.

If $U = [c_-, c_+]$, where $c_- < c_+$, then for all but finitely many $c \in (c_-, c_+)$, we have

$$N_c = \bigcup_i \mathbb{T}^2_{c,i}.$$

For all $c \in [c_-, c_+]$ a small regular neighborhood of $N_c$ is homeomorphic to the union of a few copies of $\mathbb{T}^2 \times [0, 1]$.

**Proof.** In the case $U = \{ c_0 \}$ our claim is trivial.

Assume that $U = [c_-, c_+]$ with $c_- < c_+$. By construction, $\mathbb{T}^2_{c,i} \cap M_c$ consists of open trajectories, thus, we have $\bar{f}(x) \equiv c$ on $\mathbb{T}^2_{c,i} \cap M_c$. The whole torus $\mathbb{T}^2_{c,i}$ is obtained from $\mathbb{T}^2_{c,i} \cap M_c$ by attaching disks each of which lies in the plane $\mathbb{R}_{II,a}^2$ for some $a$. The boundary of such a disk is a part of a singular unbounded component of the level set of $f|_{\mathbb{R}_{II,a}^2}$. By construction, we have $\bar{f} \equiv c$ in such a disk. Therefore, we always have $\mathbb{T}^2_{c,i} \subset N_c$. 

Let us look at what happens with the tori $T^2_{c,i}$ when $c$ varies. For all but finitely many $c$ the tori $T^2_{c,i}, T^2_{c,j}$ are disjoint if $i \neq j$. Moreover, for any $c \neq c'$ and any $i, j$, the tori $T^2_{c,i}, T^2_{c',j}$ are always disjoint.

When $c$ varies, tori $T^2_{c,i}$ are continuously deformed except at a few values of $c$, where one of the following happens: 1) two tori collide and then disappear; 2) two tori are newly born. The latter event is opposite to the first one.

Let us describe torus collision in more detail. At the moment of the collision we have a closed domain $W$ in $T^3$ that has the form of the manifold $T^2 \times [0, 1]$ in which some intervals $x \times [0, 1]$ are collapsed to a point. There may be just one such points $x$ or a closed disk $D^2 \subset T^2$ of such points. The first case corresponds to an index one or index two Morse critical point of $f$, whereas the latter corresponds to a degenerate cylinder of closed trajectories.

The interior of the domain $W$ is filled by compact trajectories and, by construction, the function $\tilde{f}$ is constant inside $W$. Thus, $W$ is a connected component of some $N_c$, since $W$ is squeezed between the two collided tori. We call such a $W$ *pseudotorus*.

So, we have the following picture. The decomposition of $T^3$ into the union of (connected components of) $N_c$ over all $c$ is nothing else but a trivial fibration over $S^1$ with fibre $T^2$, with a few fibres replaced by pseudotori.

Schematically this is shown in Fig. 5.

\[\text{Figure 5. A family of tori with a few replaced by pseudotori}\]

Now we return to the four-dimensional case. Let $\Pi \in G_{4,2}$ be a two-plane defined by a linear mapping $B : \mathbb{R}^4 \rightarrow \mathbb{R}^2$. By $\mathbb{R}^2_{\Pi,a,b}$ we denote the affine plane $B^{-1}(a, b) \subset \mathbb{R}^4$, and by $M^4_{\leq c}$ (respectively, $M^4_{\geq c}$) the subset of $T^4$ defined by the inequality $f(x) \leq c$ (respectively, $f(x) \geq c$).

Let $N \subset T^4$ be a submanifold (or, more generally, a subset). We say that $N$ is *essentially below* (respectively, *essentially above*) $M^4_c$ if for any $a, b \in \mathbb{R}$, the intersection $\tilde{N} \cap \mathbb{R}^2_{\Pi,a,b}$ is disjoint from all unbounded components of $M^4_{\geq c} \cap \mathbb{R}^2_{\Pi,a,b}$.
(respectively, of $\tilde{M}_4^{k} \cap \mathbb{R}_{\Pi,a,b}$). Thus, the property of $N$ to be essentially below $M_3^c$ depends on $\Pi$.

The following two facts are proved by analogy with the 3D case.

**Lemma 2.** If $N$ is essentially below or essentially above $M_3^c$ for a given $\Pi$ then this remains true after a small perturbation of $\Pi$, $f$, and $c$.

**Lemma 3.** If there exists a homologically nontrivial 3-torus $N$ which is essentially above or essentially below $M_3^c$, then the assertion of Theorem 1 is true for these specific $f$, $\Pi$, and $c$.

Thus, in order to prove Theorem 1 it suffices to show that for every where dense set of pairs $(f, \Pi)$, and for each $c$, there exists a homologically nontrivial 3-torus $N \subset T^4$ which is essentially below or essentially above $M_3^c$.

Let $B = (\ell_1, \ell_2)$ be a couple of linear functions on $\mathbb{R}^4$ such that

1. the function $\ell_1$ is rational, i.e., $\ell_1 \in (\mathbb{Z}^4)^*$;
2. the restriction of $\ell_2$ to the integral three-plane $\ell_1 = 0$ has irrationality degree three.

Obviously, the set of 2-planes $\Pi = \text{ker } B$ defined by $\ell_1, \ell_2$ of this form is everywhere dense in $G_{4,2}$.

Without loss of generality, we assume that $\ell_1(x) = x_0$, $\ell_2(x) = H_1x_1 + H_2x_2 + H_3x_3$, where $x = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4$, rank$_\mathbb{Z}(H_1, H_2, H_3) = 3$. We consider the 4-torus $T^4$ as a one-parametric family of three-tori $T^3_t = \{x_0 = t\}$.

For any $t \in [0, 1]$, we deal with the restrictions $f_t$ and $\ell_2,t$ of respectively $f$ and $\ell_2$ to $T^3_t$ as in the three-dimensional case. We introduce $U_t = [c_t-, c_t+]$, $f_t$, $N_{t,c}$ as before.

Let us consider the dependence of the interval $U_t$ on $t$. The endpoints $c_t\pm$ of the interval $U_t$ are continuous functions of $t$. Moreover, in the regions where $c_{t+} > c_{t-}$ these functions are piecewise smooth. This follows from the fact that locally, near a generic $t$, they are defined by a condition of the form: two saddles on $M^2_{t,c_t\pm}$ are connected by a separatrix. Here we call such intervals stability zones. To every stability zone there corresponds an integral vector $\mu \in H_1(T^3, \mathbb{Z}) = \mathbb{Z}^3$, which we call the label of the zone.

Figure 6 shows how the functions $c_{t\pm}$ may look like in the generic case. It is possible that at some $t$ we have $c_{t+} = c_{t-}$, see Fig. 6a). This may occur at the boundary of a stability zone or at $t$ such that the open trajectories in $M^3_t$ are chaotic. For all such $t$ we have $c_{t+} = c_{t-} = c_0(t)$, where $c_0(t)$ is a piecewise smooth function of $t$. It is defined locally by a condition of the form: the sum of “heights” of certain saddles equals to zero, see [21, 22].

It is most likely that any “chaotic” $t$ must be an accumulating point of an infinite sequence of stability zones. In other words, it cannot happen that the equality $c_{t+} = c_{t-}$ holds everywhere in a nontrivial interval $(t_1, t_2)$. However, this does not follow directly from the previous works [21, 22], and will not be used here.
Figure 6. Functions $c_{t\pm}$ in the generic case.
Lemma 4. The equality
\[ \min_t c_{t+} = \max_t c_{t-} \]
does not hold for a generic \( f \).

Proof. We should consider the following three cases.

Case 1. For all \( t \) we have \( c_{t+} > c_{t-} \). Then there exists a smooth periodic function \( g(t) \) such that \( c_{t+} > g(t) > c_{t-} \). Condition (3) will not hold if we disturb the function \( f \) in the following way: \( f(t, x_1, x_2, x_3) \mapsto f(t, x_1, x_2, x_3) + \varepsilon g(t) \), where \( |\varepsilon| \) is sufficiently small. So, (3) impose a codimension one condition on \( f \) in this case.

Case 2. For two different \( t = t_1, t_2 \) we have \( c_{t+} = c_{t-} \). Then condition (3) will not hold after an arbitrary perturbation \( f \mapsto f + \varepsilon g(t) \), where \( g(t) \) is an arbitrary function with \( g(t_1) \neq g(t_2) \).

Case 3. There is exactly one \( t = t_0 \) such that \( c_{t+} = c_{t-} = c_0 \) holds, and we have \( c_{t+} > c > c_{t-} \) for all \( t \neq t_0 \). Let us take \( t \) close to \( t_0 \). The interval \([c_{t-}, c_{t+}]\) is then small, which means that, when \( c \) varies from \((c_{t-} - \delta)\) to \((c_{t+} + \delta)\) with \( \delta > 0 \), a pair of tori \( N_{t,c} \) is born at \( c = c_{t-} \) and then almost immediately destroyed at \( c = c_{t+} \). Therefore, there are two cylinders of closed trajectories on \( M_{t,c} \) of very small height.

Now let us fix \( c = c_0 \) and vary \( t \). When \( t \) approaches \( t_0 \) from the left, say, we will have two closed trajectory cylinders that get degenerate at the moment \( t = t_0 \). When \( t \) passes \( t_0 \), two cylinders must appear again. The main point here is that those, new, cylinders appear from the same pair of degenerate cylinders. Indeed, the pair of degenerate cylinders that we obtain when \( t \) approaches \( t_0 \) from the left cuts \( M_{t_0,c_0} \) into two tori. Since the irrationality degree of \( t_2 \) is equal to three, one can show that there may no other degenerate cylinder on \( M_{t_0,c_0} \).

Thus, we have the following picture. When \( t \) goes from \( t_0 - \delta \) to \( t_0 + \delta \), two closed trajectory cylinders degenerate and then regenerate again. Let \( h_1(t) \), \( h_2(t) \) be there heights. So, we not only have \( h_1(t_0) = h_2(t_0) = 0 \), but also \( h'_1(t_0) = h'_2(t_0) = 0 \), which impose a codimension two condition on the function \( f \).

By construction we have

Lemma 5. For any \( t \) and \( c > c_{t+} \) (respectively, \( c < c_{t-} \)), the torus \( T^3_t \) is essentially below (respectively, essentially above) \( M^3_t \).

According to Lemma 4 only the following two cases are possible: 1) \( \min_t c_{t+} < \max_t c_{t-} \); 2) \( \min_t c_{t+} > \max_t c_{t-} \). In Case 1, for any \( c \) we have either \( c > \min_t c_{t+} \) or \( c < \max_t c_{t-} \), and by Lemma 4 we are done.

So, it remains to consider Case 2, \( \min_t c_{t+} > \max_t c_{t-} \). This inequality means, in particular, that the intervals \( U_t \) have a nontrivial intersection \( U = \cap_t U_t = [c_-, c_+] \), and we have just one stability zone, which covers the whole circle \( S^1 \). This is illustrated in Fig. 6c.).
We define a function $\tilde{f} : \mathbb{T}^4 \to \mathbb{R}$ as in the three-dimensional case by considering intersections $M^3_c \cap \mathbb{R}^2_{a,b}$, and introduce notation
\[ N_c = \{ x \in \mathbb{T}^4 ; \tilde{f}(x) = c \} . \]
By construction, $\tilde{f}_t$ coincides with the restriction of $\tilde{f}$ to $\mathbb{T}^3_t$, and we have
\[ N_c = \bigcup_t N_{t,c} . \]
For almost any $c, t$, the intersection $N_c$ with $\mathbb{T}^3_t$ is either empty or consists of 2-tori, and all those tori have the same up to sign homology class $\alpha \in H_2(\mathbb{T}^3, \mathbb{Z})$. In this case, the whole torus $\mathbb{T}^4$ has the structure of a trivial $\mathbb{T}^2$-bundle over $\mathbb{T}^2$ with a 1-parametric family of fibres replaced by pseudotori. The function $\tilde{f}$ is constant over each fibre, so it can be considered as a function on $\mathbb{T}^2$.

Figure 7 illustrates the structure of level lines of $\tilde{f}$ viewed as a function on $\mathbb{T}^2$. The preimage of a generic point is a 2-torus imbedded in $\mathbb{T}^4$, and all those 2-tori are “parallel”. Whenever the function $\tilde{f}$ has an extremum on the line $t = \text{const}$, the preimage of the critical point is a pseudotorus.
Including a pseudotorus into a 1-parametric family of 2-tori does not change the topological type of the union of the tori. Indeed, as we mentioned above, a small regular neighborhood of a pseudotorus in $T^3$ is homeomorphic to $T^2 \times (0,1)$. In other words, attaching collars $T^2 \times (0,1)$ to a pseudotorus again gives $T^2 \times (0,1)$.

We conclude the following from this.

**Lemma 6.** For almost all $c$ each connected component of $N_c$ will be homeomorphic to $T^3$. The exceptions are those $c$ that are critical values of the function $f$ on $T^2$.

For a generic $f$ we obtain a generic Morse function $\overline{f}$ on $T^2$. For such a function, there must be an interval $[c_1, c_2]$ such that, whenever we have $c \in [c_1, c_2]$, the level line $\overline{f} = c$ contains a closed curve non-homologous to zero in $T^2$. The preimage $N_c$ of this level line in $T^4$ is a 3-torus non-homologous to zero. Thus, we get the following.

**Lemma 7.** There exist $c_1, c_2$ such that $c_1 < c_2$ and both $N_{c_1}$ and $N_{c_2}$ contain a connected component homeomorphic to $T^3$ and non-homologous to zero.

It remains to notice that whenever $c > c_1$ the hypersurface $N_{c_1}$ is essentially below $M^3_c$, and whenever $c < c_2$ the hypersurface $N_{c_2}$ is essentially above $M^3_c$. Thus, for all $c$ we have a non-homologous to zero 3-torus which is either essentially above or essentially below $M^3_c$, and we are done in Case 2.

**References**

[1] S. P. Novikov. Hamiltonian formalism and a multivalued analog of Morse theory. (Russian) Uspekhi Mat. Nauk 37 (1982), no. 5, 3–49; translation in Russian Math. Surveys 37 (1982); [http://genesis.mi.ras.ru/~snovikov/74.zip](http://genesis.mi.ras.ru/~snovikov/74.zip).

[2] A. V. Zorich. Novikov’s problem on semiclassical motion of electron in homogeneous magnetic field close to rational. (Russian) Uspekhi Mat. Nauk 39 (1984), no. 5, 235–236; translation in Russian Math. Surveys 39 (1984).

[3] S. P. Novikov. Quasiperiodic structures in topology. Topological methods in modern mathematics (Stony Brook, NY, 1991), 223–233, Publish or Perish, Houston, TX, 1993.

[4] I. A. Dynnikov. A proof of S. P. Novikov’s conjecture for the case of small perturbations of rational magnetic fields. (Russian) Uspekhi Mat. Nauk 47 (1992), no. 3(285), 161–162; translation in Russian Math. Surveys 47 (1992), no. 3, 172–173.

[5] I. A. Dynnikov. S. P. Novikov’s problem on the semiclassical motion of an electron. (Russian) Uspekhi Mat. Nauk 48 (1993), no. 2(290), 179–180; translation in Russian Math. Surveys 48 (1993), no. 2, 173–174.

[6] I. A. Dynnikov. A proof of the conjecture of S. P. Novikov on the semiclassical motion of an electron. (Russian) Mat. Zametki 53 (1993), no. 5, 57–68; translation in Math Notes 53 (1993), no. 5–6, 495–501.

[7] S. P. Novikov and A. Ya. Maltsev. Topological quantum characteristics observed in the investigation of the conductivity in normal metals. (Russian) Pis’ma Zh. Eksp. Teor. Fiz. 63 (1996), no. 10, 809–813; translation in JETP Letters 63 (1996), no. 10, 855–860.

[8] S. P. Novikov and A. Ya. Maltsev. Topological phenomena in normal metals. (Russian) Uspekhi Phys. Nauk 41 (1998), no. 3, 231–239; [arXiv:cond-mat/9709007](https://arxiv.org/abs/cond-mat/9709007).

[9] I. Lifshitz, M. Ya. Azbel, M. I. Kaganov. Elektronnaya teoriya metallov. (Russian) Nauka, Moscow, 1971; translation: Electron theory of metals, Consultants Bureau, New York, 1973.
TOPOLOGY OF QUASIPERIODIC FUNCTIONS ON THE PLANE

[10] S. P. Novikov. Levels of quasiperiodic functions on a plane, and Hamiltonian systems. (Russian) Uspekhi Mat. Nauk 54 (1999), no. 5(329), 147–148; translation in Russian Math. Surveys 54 (1999), no. 5, 1031–1032; \texttt{arXiv:math-ph/9909032}

[11] A. Ya. Maltsev. Quasiperiodic functions theory and the superlattice potentials for a two-dimensional electron gas. Journal of Mathematical Physics 45, no. 3 (March 2004), 1128–1149; \texttt{arXiv:cond-mat/0302014}

[12] V. I. Arnold, S. P. Novikov (editors), Encyclopedia Mathematical Sciences, Dynamical Systems, vol 4: Completely Integrable Systems and Symplectic Geometry. Springer Verlag, second edition (revised), 2001:
1. V. Arnold, A. Givental;
2. B. Dubrovin, I. Krichever, S. Novikov.

[13] T. Q. T. Le, S. Piunikhin, and V. Sadov. The geometry of quasicrystals. (Russian) Uspekhi Mat. Nauk 48 (1993), no. 1(289), 41–102; translation in Russian Math. Surveys 48 (1993), no. 1, 37–100.

[14] I. M. Lifshitz, V. G. Peschanski. Halvanomagnetic characteristics of metals with open Fermi surfaces, II. (Russian) Zh. Eksp. Teor. Fiz. 38 (1960), 188–193; translation in JETP.

[15] A. A. Abrikosov. Osnovy teorii metallov. (Russian) Nauka, Moscow, 1987; translation: Fundamentals of theory of metals, North-Holland, Amsterdam, 1988.

[16] C. Kittel, Quantum theory of solids. John Wiley & Sons Inc., New York, London, 1963.

[17] J. M. Ziman. Principles of the theory of solids. Cambridge Univ. Press, 1972.

[18] A. Ya. Maltsev and S. P. Novikov. Dynamical Systems, Topology, and Conductivity in Normal Metals. Journal of Statistical Physics 115 no. 1, April 2004, 31-46; \texttt{arXiv:cond-mat/0312708}

[19] A. Ya. Maltsev and S. P. Novikov. Quasiperiodic functions and dynamical systems in quantum solid state physics. Dedicated to the 50th anniversary of IMPA. Bull. Braz. Math. Soc. (N.S.) 34 (2003), no. 1, 171–210; \texttt{arXiv:math-ph/0301033}

[20] A. Ya. Maltsev, S. P. Novikov. Topology, Quasiperiodic functions and the transport phenomena. Topology in condensed matters, M. I. Monastyrsky (ed.), Springer Verlag, 2004, to appear; \texttt{arXiv:cond-mat/0312710}

[21] I. A. Dynnikov. Semiclassical motion of the electron. A proof of the Novikov conjecture in general position and counterexamples. Solitons, geometry, and topology: on the crossroad, 45–73, Amer. Math. Soc. Transl. Ser. 2, 179, Amer. Math. Soc., Providence, RI, 1997

[22] I. A. Dynnikov. The geometry of stability regioines in Novikov’s problem on the semiclassical motion of the electron. (Russian) Uspekhi Mat. Nauk, 54 (1999), no. 1, 21–60.

[23] R. De Leo. Numerical Analysis of the Novikov Problem of a Normal Metal in a Strong Magnetic Field. SIAM Journal of Applied Dynamical Systems 2:4 (2003), 517–545.

[24] I. A. Dynnikov and A. Ya. Maltsev. Topological characteristics of electron spectra in monocrystals. (Russian) Zh. Eksp. Teor. Fiz. 112:1 (1997), 371–378; translation in JETP 85:1 (1997), 205–208.

[25] C. W. J. Beenaker. Guiding-center-drift resonance in a periodically modulated two-dimensional electron gas. Phys. Rev. Lett. 62 (1989), no. 17, 2020–2023.

DEPT. OF MECH. AND MATH., MOSCOW STATE UNIVERSITY, MOSCOW 119992 GSP-2, RUSSIA
E-mail address: dynnikov@mech.math.msu.su

LANDAU INSTITUTE FOR THEORETICAL PHYSICS, KOZYGINA STR. 2, MOSCOW 119334, RUSSIA; IPST, UNIVERSITY OF MARYLAND, COLLEGE PARK, MD 20742, USA
E-mail address: novikov@glue.umd.edu