The inverse conductivity problem with limited data and applications

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Abstract. This paper describes recent uniqueness results in inverse problems for semiconductor devices and in the inverse conductivity problem. We remind basic inverse problems in semiconductor theory and outline use of an adjoint equation and a proof of uniqueness of piecewise constant doping profile. For the inverse conductivity problem we give a first uniqueness proof when the Dirichlet-to-Neumann map is given at an arbitrarily small part of the boundary of a three-dimensional domain.

1. On inverse problems for semiconductors
In so-called Slotboom variables, after scaling, a typical semiconductor device is described by the elliptic quasilinear system

\[ \lambda^2 \Delta V = \delta^2 (e^V u - e^{-V} v) - C, \]

\[ \text{div} J_n = \delta^4 Q(V, u, v)(uv - 1), \quad J_n = \mu_n q n_i e^V \nabla u \text{ in } \Omega, \]

\[ \text{div} J_p = -\delta^4 Q(V, u, v)(uv - 1), \quad J_p = -\mu_p q n_i e^{-V} \nabla v, \]

and the boundary conditions

\[ V = U + V_{bi}, \quad u = e^{-U}, \quad v = e^U \text{ on } \Gamma_D. \]

\[ \partial_n V = 0, \quad J_n \cdot \nu = J_p \cdot \nu = 0 \text{ on } \Gamma_N \]

Here \( V \) is the electrostatic potential, \( u, v \) are Slotboom concentrations of negatives charges (electrons) and of positive charges (holes), \( \lambda^2 \) is a small constant (between \( 10^{-3} \) and \( 10^{-6} \)), \( \delta^2 = n_i \) (intrinsic density of electrons), \( Q \) is recombination-regeneration rate, \( q \) is elementary charge, \( \mu_n, \mu_p \) are mobilities of electrons and holes, \( V_{bi} \) is equilibrium potential, \( U \) is applied potential. \( \Omega \) (semiconductor device) is a Lipschitz domain in \( \mathbb{R}^n, n = 1, 2, 3 \), \( \nu \) is (outer) unit normal, \( \Gamma_D, \Gamma_N \) is a partition of the boundary \( \partial \Omega \) into open parts. By using Fixed Point Theorems in \( H^1(\Omega) \) one can show existence of a weak solution to (1), (2). Uniqueness is known only for small \( U \) [3], [4]. An advantage of the form (1) of the stationary drift diffusion system is that it has an equilibrium solution

\[ V = V_0, u = 1, v = 1, \text{ for } U = 0, \]
where $V_0$ solves
\[ \lambda^2 \Delta V_0 = \delta^2 (e^{V_0} - e^{-V_0}) - C \text{ in } \Omega, \] (3)
and satisfies the boundary conditions
\[ V_0 = V_{bi} \text{ on } \Gamma_D, \quad \partial_{\nu} V_0 = 0 \text{ on } \Gamma_N. \]

The linearization $V_1, u_1, v_1$ at the equilibrium is the solution to the system of linear elliptic equations
\[ \lambda^2 \Delta V_1 = \delta^2 (e^{V_0} V_1 + e^{-V_0} V_1 + e^{V_0} u_1 - e^{-V_0} v_1), \]
\[ \text{div}(e^{V_0} \nabla u_1) = C_1 Q(1, V_0, 1, 1)(u_1 + v_1) \text{ in } \Omega, \]
\[ \text{div}(e^{-V_0} \nabla v_1) = C_2 Q(1, V_0, 1, 1)(u_1 + v_1). \] (4)

where $C_1, C_2$ are some (small) constants, with the boundary data
\[ V_1 = U_1, \quad u_1 = -U_1, \quad v_1 = U_1 \text{ on } \Gamma_D, \]
\[ \partial_{\nu} V_1 = \partial_{\nu} u_1 = \partial_{\nu} v_1 = 0 \text{ on } \Gamma_N. \] (5)

An important characteristic of a semiconductor device is doping profile $C = C(x)$. $C$ is not known and one can try to determine it by using additional data about $V, u, v$. Due to complexity of the system (1), (2) and some flexibility with testing experiments one can look for $C$ from the linearized system (4), (5). Even this system looks too complicated, so one considers practically valuable particular cases.

In unipolar case (no holes) one lets $v_1 = 0, Q = 0$, then the second equation in (4) decouples into
\[ \text{div}(e^{V_0} \nabla u_1) = 0 \text{ in } \Omega, \quad u_1 = -U_1 \text{ on } \Gamma_D, \quad \partial_{\nu} u_1 = 0 \text{ on } \Gamma_N, \] (6)

where we also used the boundary conditions (5).

Let $\Gamma_1$ be a part of $\Gamma_D$ and $\Gamma_0 = \Gamma_D \setminus \Gamma_1$. In applications one can prescribe various $U_1$ on $\Gamma_0$ and measure the total flux of $u_1$ across $\Gamma_1$. So we formulate

**Inverse Problem U1**
Find $C$ from the mapping
\[ U_1 \to \int_{\Gamma_1} \partial_{\nu} u_1, \quad U_1 = 0 \text{ on } \Gamma_1, \quad U_1 \in H^{1/2}(\Gamma_0). \] (7)

It is realistic to find only the so called $p-n$ junction, or equivalently a domain $D$ with
\[ C = 1 - 2\chi_D. \] (8)

For theoretical purposes we can assume that we are given flux at any point of $\Gamma_1$, not only a total flux. This leads to the following

**Inverse Problem U2**
Find $C$ from the mapping
\[ U_1 \to \partial_{\nu} u_1 \text{ on } \Gamma_1. \]

At present there are no uniqueness results for both inverse problems U1, U2.

Smallness of $\lambda$ and absence of theoretical results for the inverse problem suggest letting $\lambda = 0$ in (3) and correspondingly
\[ e^{V_0} - e^{-V_0} = \delta^2 C. \] (9)

Then Inverse Problem U1 simplifies to

**Inverse Problem S1**
Find $D$ entering the boundary value problem
\[
div(a \nabla u_1) = 0 \text{ on } \Omega,
\]
\[
u = 0 \text{ on } \Gamma_1, \ u = U_1 \text{ on } \Gamma_0, \ \partial_v u_1 = 0 \text{ on } \Gamma_N,
\]
with
\[
a = a_+ - a_- \chi_D, \ 0 < a_+ - a_- < 1 < a_+ , \ a_- , a_+ \text{ given constants,}
\]
from the mapping
\[
U_1 \rightarrow \int_{\Gamma_1} \partial_v u.
\]

In bipolar case one uses the full system (4), (5) with $u_1 = v_1 = 0$ on $\Gamma_D$. Difficulties with uniqueness (and stability) stimulate study of flipped bipolar case which models laser-beam induced current [4], [6]. Laser beam source of density $f(x)$ results in the system (4), (5) with the source term
\[
div(e^{V_0} \nabla u_1) = C_1 Q(V_0, 1, 1, x)(u_1 + v_1) + f,
\]
\[
div(e^{-V_0} \nabla v_1) = C_2 Q(V_0, 1, 1, x)(u_1 + v_1) + f \text{ in } \Omega,
\]
with the boundary value data
\[
u = v = 0 \text{ on } \Gamma_D, \ \partial_v u_1 = \partial_v v_1 = 0 \text{ on } \Gamma_N.
\]

As in the unipolar case one can measure the combined total flux
\[
\int_{\Gamma_1} (C_1^{-1} e^{V_0} \partial_v u_1 - C_2^{-1} e^{-V_0} \partial_v v_1).
\]

So we arrive at

**Inverse Problem S2** Find $V_0$ entering the boundary value problem (13), (14) from the mapping
\[
f \rightarrow \int_{\Gamma_1} (C_1^{-1} e^{V_0} \partial_v u_1 - C_2^{-1} e^{-V_0} \partial_v v_1).
\]

**Adjoint Problem**
Inverse problems S1, S2 are hard to handle directly, since we do not have sufficient boundary or interior data. A way out is to use an ”adjoint” inverse problem where instead of functionals we are given extra boundary data. This device is widely used in the inverse option pricing problem (Dupire’s equation) where it enables us to obtain theoretical and numerical results. Also in [6] they used an adjoint boundary value problem in numerical solution of the inverse problem S2.

Let $u_1^*$ be the solution to the boundary value problem
\[
div(a \nabla u_1^*) = 0 \text{ on } \Omega,
\]
\[
\partial_v u_1^* = 0 \text{ on } \Gamma_N, \ u_1^* = 0 \text{ on } \Gamma_0,
\]
\[
u_1^* = 1 \text{ on } \Gamma_1.
\]

**Lemma 1.1** The data of the inverse problem S1 uniquely determine $\partial_v u_1^* = g_1^*$ on $\Gamma_0$. Moreover, there is a constant $C$ such that
\[
\|g_1^*\|_{H^{-1/2}(\Gamma_0)} \leq CF
\]
where $F$ is the norm of the linear functional (7) in $H^{1/2}_{\Gamma_0}(\Gamma_0)$.  

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Proof. Using the definition of a weak solution to the boundary value problem (17), (18) with the test function $u_1$ we get

$$- \int_{\Omega} a \nabla u_1^* \cdot \nabla u_1 + \int_{\partial \Omega} a \partial_{\nu} u_1^* u_1 = 0.$$ 

Similarly the definition of a weak solution to the boundary value problem (10) with the test function $u_1^*$ yields

$$- \int_{\Omega} a \nabla u_1^* \cdot \nabla u_1^* + \int_{\partial \Omega} a \partial_{\nu} u_1^* u_1^* = 0.$$ 

Therefore

$$\int_{\partial \Omega} a (\partial_{\nu} u_1^* u_1 - \partial_{\nu} u_1 u_1^*) = 0,$$

and using boundary conditions (18),(2) we obtain

$$\int_{\Gamma_0} a g_1^* U_1 = \int_{\Gamma_1} a \partial_{\nu} u_1.$$

The standard elliptic estimate

$$\| \partial_{\nu} u_1 \|_{(-\frac{1}{2})}(\Gamma_1) \leq C \| U_1 \|_{(\frac{1}{2})}(\Gamma_0)$$

and (20) imply that

$$\int_{\Gamma_0} a g_1^* U_1 \leq C \| U_1 \|_{(\frac{1}{2})}(\Gamma_0).$$

This bound implies (19).

This lemma shows that the data of the inverse problem $S1$ uniquely and in a stable way determine the data $g_1^*$ for the following inverse problem.

**Inverse Problem S1**

Find $D$ entering the boundary value problem (17), (18) from the additional boundary data

$$\partial_{\nu} u_1^* = g_1^* \text{ on } \Gamma_0.$$ 

Let $u_1^*, v_1^*$ solve the boundary value problem

$$\text{div}(a \nabla u_1^*) - C_1 Q u_1^* + C_1 Q v_1^* = 0 \text{ on } \Omega,$$

$$\text{div}(a^{-1} \nabla v_1^*) - C_2 Q v_1^* + C_2 Q v_1^* = 0 \text{ on } \Omega,$$

with the mixed boundary data

$$\partial_{\nu} u_1^* = \partial_{\nu} v_1^* = 0, \text{ on } \Gamma_N, u_1^* = v_1^* = 0, \text{ on } \Gamma_0,$$

$$u_1^* = v_1^* = 1 \text{ on } \Gamma_1 \quad (22)$$

**Lemma 1.2** The data of the inverse problem uniquely determine $u_1^* - v_1^*$ on $\Omega$. Moreover, there is a constant $C$ such that

$$\| u_1^* - v_1^* \|_{(0)}(\Omega) \leq C F_1$$

where $F_1$ is the norm of the linear functional (16) in $L_2(\Omega)$. 


Inverse Problem S2*  
Find $a$ entering the boundary value problem (21), (22) from the additional boundary data

$$u_1^* - v_1^* = g_1^* \text{ on } \Omega.$$ 

Let $D(1), D(2)$ generate equal data to S1, then for corresponding solutions $u(1), u(2)$ to (17), (11), (18)

$$\text{div}(a(j)\nabla u(; j)) = 0 \text{ in } \Omega, \quad a(j) = a_+ - a_- \chi_{D(j)},$$

$$\partial_\nu u(; j) = 0 \text{ on } \Gamma_N, \quad u(; j) = 0 \text{ on } \Gamma_0, \quad u(; j) = 1 \text{ on } \Gamma_1,$$ (24)

we have

$$\partial_\nu u(1) = \partial_\nu u(2) \text{ on } \Gamma_0.$$ (25)

We introduce the following notation

$$\partial_j u^i(x^0) = \lim_j \partial_j u(x), \quad x^0 \in \partial D, \quad x \to x^0, \quad x \in D.$$ 

Similarly, $\partial_j u^e$ denotes limit from outside $D$.

In Theorem 1.3 we let $\Omega = \{x : 0 < x_k < l_k, k < n, -1 < x_n < 0\}, n = 2, 3$ and $D(j) = \{x : d(x'; j) < x_n < 0\}$ where $d(; j)$ are Lipschitz functions, $-1 < d(; j) \leq 0, j = 1, 2$. Let $\Gamma_0 = \{x : x_n = -1\} \cap \partial \Omega, \quad \Gamma_1 = \{x : 0 < x_k < l_k, k < n, x_n = 0\}$ where $l_k$ are some numbers, $0 < l_k1 < l_k$. We assume that $\Gamma_1 \subset \partial D(j)$.

**Theorem 1.3** Let us assume that

$$\partial_\nu u^i(; j) < 0 \text{ on } \partial D(j) \cap \Omega, \quad j = 1, 2.$$ (26)

Then the equality (25) implies that $D(1) = D(2)$.

We will comment on the condition (26). While this condition is expected for at least convex domains $D$ we do not have a complete proof yet. In applied situations constant $a_+$ in (11) is large and $a_+ - a_-$ is small (their ratio can be avove $10^3$) [4]. For solutions to (17), (18) it can be shown that $u_1^*$ converges to 0 in $\Omega \setminus D$ and on $D$ to the harmonic function $w$ with $w = 1$ on $\Gamma_1, \quad w = 0$ on $\partial_\nu w = 0$ on $\Gamma_N \cap \partial D$ as $a_+ \to +\infty, \quad a_+ - a_- \to 0$. By maximum principles, $\partial_\nu w < 0$ on $\partial D \cap \Omega$. Hence by elliptic estimates we expect $\partial_\nu u^i$ to converge to $\partial_\nu w$ on $\partial D \cap \Omega$ and hence (26) to hold for large $a_+$ and small $a_+ - a_-$. Proofs and precise statements will be published later.

**Lemma 1.4** Under the conditions of Theorem 1.3

$$\int_{\partial D(1)} v \partial_\nu u^i(; 1) = \int_{\partial D(2)} v \partial_\nu u^i(; 2)$$ (27)

for any function $v$ which is $H^1$ and harmonic near $(\bar{D}(1) \cup \bar{D}(2)) \cap \Omega$ in $\Omega, v = 0$ on $\Gamma_1$ and $\partial_\nu v = 0$ on $\Gamma_N$ near $(\bar{D}(1) \cup \bar{D}(2))$.

Proof is similar to [9], Lemma 4.3.7.

**Proof of Theorem 1.3.** Let us assume that $D(1)$ and $D(2)$ are different. If one of this domains is contained in another domain, then domains must coincide by results of Alessandrini [9], section 4.3. So we may assume that the sets $D(1) \setminus \bar{D}(2), D(2) \setminus \bar{D}(1)$ are not void.

Let us introduce the notation:

$$\Gamma_{1e} = (\partial D(1) \setminus \bar{D}(2)) \cap \Omega, \quad \Gamma_{1i} = \partial D(2) \cap D(1),$$
Nachman [13] demonstrated uniqueness of \( \partial \Omega \cap \partial D^* \). Sylvester and Uhlmann in their fundamental paper proposed the idea of using complex exponential solutions to demonstrate uniqueness in the Schrödinger equation. Kohn and Vogelius, 1985, showed uniqueness of the boundary reconstruction (of all existing partial derivatives of \( a \)) and hence uniqueness of piecewise analytic \( a \). For special geometry (\( \partial \Omega \setminus \Gamma \) is a part of the union of two parallel planes) uniqueness was shown.

2. Uniqueness from local Dirichlet-to-Neumann map

The inverse conductivity problem with many boundary measurements consists of recovery of conductivity coefficient \( a \) (principal part) of an elliptic equation in a domain \( \Omega \subset \mathbb{R}^n \), from the Neumann data given for all Dirichlet data (Dirichlet-to-Neumann map). Calderon proposed the idea of using complex exponential solutions to demonstrate uniqueness in the linearized inverse conductivity problem. Sylvester and Uhlmann in their fundamental paper [14] proved global uniqueness of \( a \) in the three-dimensional case. In the two-dimensional case Nachman [13] demonstrated uniqueness of \( a \in C^2(\Omega) \) and Astala and Päivärinta [1] showed uniqueness of \( a \in L_\infty(\Omega) \).

There is a known hypothesis that the Dirichlet-to-Neumann map given at any (nonvoid open) part \( \Gamma \) of the boundary also uniquely determines conductivity coefficient or potential in the Schrödinger equation. Kohn and Vogelius, 1985, showed uniqueness of the boundary reconstruction (of all existing partial derivatives of \( a \)) and hence uniqueness of piecewise analytic \( a \). For special geometry (\( \partial \Omega \setminus \Gamma \) is a part of the union of two parallel planes) uniqueness was shown.
by Ikehata, Makrakis, and Nakamura [7] by using periodic complex geometrical optics solutions. In smooth case Bukhgeim and Uhlmann [5] made use of Carleman estimates (with linear phase function) to show that the Neumann data on a sufficiently large part \( \Gamma \) of the boundary given for all Dirichlet data on the whole boundary uniquely determine the solution. They used quadratic phase function and demonstrated uniqueness of the solution from Neumann data on (possibly small) \( \Gamma \) for all Dirichlet data on a complementary part.

We give a simple proof of this hypothesis when the Dirichlet-to-Neumann map is given on arbitrary part \( \Gamma \) of \( \partial \Omega \) while on the remaining part \( \Gamma_0 \) one has homogeneous Dirichlet or Neumann data. Our restrictive assumption is that \( \Gamma_0 \) is a part of a plane or of a sphere.

Let \( \Omega \) be a domain in \( \mathbb{R}^3 \) with Lipschitz boundary. We consider the conductivity equation

\[-\text{div}(a \nabla u) = 0 \text{ in } \Omega.\]  

(29)

Let \( B_0 \) be a ball in \( \mathbb{R}^3 \). Let \( \Omega \) be a subdomain of \( B_0 \). Let \( \Gamma_0 = \partial B_0 \cap \partial \Omega \) and \( \Gamma = \partial \Omega \setminus \Gamma_0 \). We will assume that \( \Gamma_0 \neq \partial B_0 \). We will assume that \( a \in C^2(\Omega) \), \( \partial_\nu a = 0 \) on \( \Gamma_0 \), \( a > 0 \) on \( \Omega \). We define local Dirichlet-to-Neumann map for (29) as \( \Lambda(a; D, \Gamma) \) on \( \Gamma \), \( g_0 = 0 \) on \( \partial \Omega \setminus \Gamma \), \( u = g_0 \) on \( \partial \Omega \). Similarly we define the local Neumann-to-Dirichlet map \( \Lambda(a; N, \Gamma) \).

**Theorem 2.1** [10] The equalities

\[\Lambda(a_1; D, \Gamma) = \Lambda(a_2; D, \Gamma) \text{ or } \Lambda(a_1; N, \Gamma) = \Lambda(a_2; N, \Gamma)\]

implies that \( a_1 = a_2 \).

Now we consider the plane case and assume that \( \Omega \) is a bounded simply connected domain in \( \mathbb{R}^2 \) with \( C^2 \)-boundary.

**Theorem 2.2** (Astala, Lassas, Päivärinta, [2])

Let \( a_1, a_2 \in L_\infty(\Omega) \). Let \( \Gamma \) be any nonvoid open arc of \( \partial \Omega \).

Then equalities

\[\Lambda(a_1; D, \Gamma) = \Lambda(a_2; D, \Gamma), \quad \Lambda(a_1; N, \Gamma) = \Lambda(a_2; N, \Gamma),\]

imply that \( a_1 = a_2 \) in \( \Omega \).

**Outline of proof of Theorem 2.1**

The well known substitution \( u = a^{-\frac{1}{2}} v \) transforms the conductivity equation into the Schrödinger equation \(-\Delta u + cu = 0\). Using the Kelvin transform we can assume that \( \Gamma_0 \) is a part of the plane \( \{x_3 = 0\} \) and \( \Omega \subset \{x_3 < 0\} \).

Let \( \xi = (\xi_1, \xi_2, \xi_3) \), \( \xi^* = (\xi_1, \xi_2, -\xi_3) \).

\[e(1) = (\xi_1^2 + \xi_2^2)^{-\frac{1}{2}}(\xi_1, \xi_2, 0), \quad e(3) = (0, 0, 1),\]

and \( e(1), e(2), e(3) \) be an orthonormal basis in \( \mathbb{R}^3 \) with coordinates \((x_1e, x_2e, x_3e)_e\). We define

\[\zeta(1) = (\xi_1e - \tau \xi_3, i|\xi|\left(\frac{1}{4} + \tau^2\right)^{\frac{1}{2}}, \xi_3 + \tau \xi_1)_e,\]

\[\zeta(2) = (\xi_1e + \tau \xi_3, -i|\xi|\left(\frac{1}{4} + \tau^2\right)^{\frac{1}{2}}, \xi_3 + \tau \xi_1)_e,\]

where \( \tau \) is a positive real number. Then \( \zeta(1) \cdot \zeta(1) = \zeta(2) \cdot \zeta(2) = 0.\)
Let us extend $c_1, c_2$ onto $\mathbb{R}^3$ as even functions of $x_3$. It is known that there are almost exponential solutions
\[ e^{i\zeta(1) \cdot x}(1 + w_1), \quad e^{i\zeta(2) \cdot x}(1 + w_2), \]
\[ \|w_1\|_2(B_0) + \|w_2\|_2(B_0) \to 0 \text{ as } \tau \to \infty, \]
to the equations
\[ -\Delta u_1 + c_1 u_1 = 0, \quad -\Delta u_2 + c_2 u_2 = 0 \text{ in } \mathbb{R}^3. \] (30)
We define $f^*(x_1, x_2, x_3) = f(x_1, x_2, -x_3)$ and we let
\[ u_1(x) = e^{i\zeta(1) \cdot x}(1 + w_1(x)) - e^{i\zeta^*(1) \cdot x}(1 + w_1^*(x)), \]
\[ u_2(x) = e^{i\zeta(2) \cdot x}(1 + w_2(x)) - e^{i\zeta^*(2) \cdot x}(1 + w_2^*(x)). \] (31)

By (30), (31), and known orthogonality relations
\[ 0 = \int_{\Omega} c(x)(e^{i(\zeta(1) + \zeta(2)) \cdot x}(1 + w_1(x))(1 + w_2(x)) - e^{i(\zeta^*(1) + \zeta^*(2)) \cdot x}(1 + w_1^*(x))(1 + w_2(x)) - e^{i(\zeta(1) + \zeta^*(2)) \cdot x}(1 + w_1(x))(1 + w_2^*(x)) + e^{i(\zeta^*(1) + \zeta(2)) \cdot x}(1 + w_1^*(x))(1 + w_2^*(x))) dx \]
due to (31). Using definition of $\zeta(j)$ we conclude that
\[ \int_{\Omega} c(x)(e^{i\xi \cdot x}(1 + w_1(x))(1 + w_2(x)) - e^{i(\xi_{1x} x_1 - 2\tau \xi_{1x} x_3)}(1 + w_1^*(x))(1 + w_2(x)) - e^{i(\xi_{1x} x_1 + 2\tau \xi_{1x} x_3)}(1 + w_1(x))(1 + w_2^*(x)) + e^{i\xi^* \cdot x}(1 + w_1^*(x))(1 + w_2^*(x))) dx = 0. \]

Now $\tau \to \infty$. By the Riemann-Lebesgue Lemma limits of
\[ \int_{\Omega} c(x)e^{i(\xi_{1x} x_1 - 2\tau \xi_{1x} x_3)} dx, \quad \int_{\Omega} c(x)e^{i(\xi_{1x} x_1 + 2\tau \xi_{1x} x_3)} dx \]
as $\tau \to \infty$ are zero provided $\xi_{1x} \neq 0$. So
\[ \int_{\Omega} c(x)(e^{i\xi \cdot x} + e^{i\xi^* \cdot x}) dx = \int_{\mathbb{R}^3} c(x)e^{i\xi \cdot x} dx = 0 \]
for any $\xi \in \mathbb{R}^3$. By uniqueness of the inverse Fourier transformation $c = 0$, and hence $c = 0$ and $c_1 = c_2$.

The main remaining open question is of course how obtain uniqueness from local Dirichlet-to-Neumann map when $\Gamma_0$ is an arbitrary surface.

**Acknowledgments**

The author is grateful to the referee for raising several important questions and informing about the paper [7]. This work is in part supported by the NSF grant DMS 04-05976.
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