Extended Absolute Parallelism Geometry

Nabil. L. Youssef and A. M. Sid-Ahmed

Department of Mathematics, Faculty of Science,
Cairo University, Giza, Egypt

nlyoussef2003@yahoo.fr, nyoussef@frcu.eun.eg
amrsidahmed@gmail.com, amrs@mailer.eun.eg

Abstract. In this paper, we study Absolute Parallelism (AP-) geometry on the tangent bundle $TM$ of a manifold $M$. Accordingly, all geometric objects defined in this geometry are not only functions of the positional argument $x$, but also depend on the directional argument $y$. Moreover, many new geometric objects, which have no counterpart in the classical AP-geometry, emerge in this different framework. We refer to such a geometry as an Extended Absolute Parallelism (EAP-) geometry.

The building blocks of the EAP-geometry are a nonlinear connection assumed given a priori and $2n$ linearly independent vector fields (of special form) defined globally on $TM$ defining the parallelization. Four different $d$-connections are used to explore the properties of this geometry. Simple and compact formulae for the curvature tensors and the $W$-tensors of the four defined $d$-connections are obtained, expressed in terms of the torsion and the contortion tensors of the EAP-space.

Further conditions are imposed on the canonical $d$-connection assuming that it is of Cartan type (resp. Berwald type). Important consequences of these assumptions are investigated. Finally, a special form of the canonical $d$-connection is studied under which the classical AP-geometry is recovered naturally from the EAP-geometry. Physical aspects of some of the geometric objects investigated are pointed out and possible physical implications of the EAP-space are discussed, including an outline of a generalized field theory on the tangent bundle $TM$ of $M$.¹

Keywords: Parallelizable manifold, Absolute Parallelism, Extended Absolute Parallelism, Metric $d$-connection, Canonical $d$-connection, $W$-tensor, Field equations, Cartan type, Berwald type.

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0. Introduction

The geometry of parallelizable manifolds or the Absolute Parallelism geometry (AP-geometry) ([4], [11], [12], [14], [17]) has many advantages in comparison to Riemannian geometry. Unlike Riemannian geometry, which has ten degrees of freedom (corresponding to the metric components for \( n = 4 \)), AP-geometry has sixteen degrees of freedom (corresponding to the number of components of the four vector fields defining the parallelization). This makes AP-geometry a potential candidate for describing physical phenomena other than gravity. Moreover, as opposed to Riemannian geometry, which admits only one symmetric linear connection, AP-geometry admits at least four natural (built-in) linear connections, two of which are non-symmetric and three of which have non-vanishing curvature tensors. Last, but not least, associated with an AP-space there is a Riemannian structure defined in a natural way. Thus, AP-geometry contains within its geometric structure all the mathematical machinery of Riemannian geometry. Accordingly, a comparison can be made between the results obtained in the context of AP-geometry and general relativity, which is based on Riemannian geometry.

The geometry of the tangent bundle \((TM, \pi, M)\) of a smooth manifold \(M\) is very rich. It contains a lot of geometric objects of theoretical interest and of a great importance in the construction of various geometric models which have proved very useful in different physical theories. Examples of such theories are the general theory of relativity, particle physics, relativistic optics and others.

In this paper, we study AP-geometry in a context different from the classical one. Instead of dealing with geometric objects defined on the manifold \(M\), as in the case of classical AP-space, we are dealing with geometric objects defined on the tangent bundle \(TM\) of \(M\). Accordingly, all geometric objects considered are, in general, not only functions of the positional argument \(x\), but also depend on the directional argument \(y\).

The paper is organized in the following manner. In section 1, following the introduction, we give a brief account of the basic concepts and definitions that will be needed in the sequel. The definitions of a \(d\)-connection, \(d\)-tensor field, torsion, curvature, \(hv\)-metric and metric \(d\)-connection on \(TM\) are recalled. We end this section by the construction of a (unique) metric \(d\)-connection on \(TM\) which we refer to as the natural metric \(d\)-connection. In section 2, we introduce the Extended Absolute Parallelism (EAP-) geometry by assuming that \(TM\) is parallelizable [3] and equipped with a nonlinear connection. The canonical \(d\)-connection is then defined, expressed in terms of the natural metric \(d\)-connection. In analogy to the classical AP-geometry, two other \(d\)-connections are introduced: the dual and the symmetric \(d\)-connections. We end this part with a comparison between the classical AP-geometry and the EAP-geometry. In section 3, we carry out the task of computing the different curvature tensors of the four defined \(d\)-connections. They are expressed, in a relatively compact form, in terms of the torsion and the contortion tensors of the EAP-space. All admissable contractions of these curvature tensors are also obtained. In section 4, we introduce and investigate the different \(W\)-tensors corresponding to the different \(d\)-connections defined in the EAP-space, which are again expressed in terms of the torsion and the contortion tensors. In sections 5 and 6, we assume that the canonical \(d\)-connection is of Cartan and Berwald type respectively. Some interesting results are obtained, the most important of which is
that, in the Cartan type case, the given nonlinear connection is not independent of the vector fields forming the parallelization, but can be expressed in terms of their vertical counterparts. In section 7, we further assume that the canonical $d$-connection is both of Cartan and Berwald type. We show that, under this assumption, the classical AP-geometry is recovered, in a natural way, from the EAP-geometry. In section 8, we end this paper with some concluding remarks which reveal possible physical applications of the EAP-space; among them is an outline of a generalized field theory on the tangent bundle $TM$ of $M$, based on Euler-Lagrange equations \[8\] applied to a suitable scalar Lagrangian.

1. Fundamental Preliminaries

In this section we give a brief account of the basic concepts and definitions that will be needed in the sequel. Most of the material covered here may be found in \[8, 9\] with some slight modifications.

Let $M$ be a paracompact manifold of dimension $n$ of class $C^\infty$. Let $\pi : TM \to M$ be its tangent bundle. If $(U, x^\mu)$ is a local chart on $M$, then $(\pi^{-1}(U), (x^\mu, y^a))$ is the corresponding local chart on $TM$. The coordinate transformation on $TM$ is given by:

$$x^\mu' = x^\mu(x^\nu), \quad y^a' = p^a_\alpha y^\alpha,$$

$\mu = 1, ..., n; \ a = 1, ..., n; \ p^a_\alpha = \frac{\partial y^a'}{\partial y^\alpha} = \frac{\partial y^a'}{\partial x^\nu}$ and $\det(p^a_\alpha) \neq 0$. The paracompactness of $M$ ensures the existence of a nonlinear connection $N$ on $TM$ with coefficients $N^a_\alpha(x, y)$. The transformation formula for the coefficients $N^a_\alpha$ is given by

$$N^a_\alpha' = p^a_\beta p^\beta_\alpha N^a_\alpha + p^a_\alpha p^\alpha_\beta y^\beta,$$

(1.1)

where $p^a_\alpha = \frac{\partial y^a'}{\partial x^\nu}$. The nonlinear connection leads to the direct sum decomposition

$$T_u(TM) = H_u(TM) \oplus V_u(TM), \ \forall u \in TM \setminus \{0\}.$$  \hspace{1cm} (1.2)

Here, $V_u(TM)$ is the vertical space at $u$ with local basis $\dot{\alpha} := \frac{\partial}{\partial y^\alpha}$, whereas $H_u(TM)$ is the horizontal space at $u$ associated with $N$ supplementary to the vertical space $V_u(TM)$. The canonical basis of $H_u(TM)$ is given by

$$\delta_\mu := \partial_\mu - N^a_\mu \dot{\alpha},$$

(1.3)

where $\partial_\mu := \frac{\partial}{\partial x^\nu}$. Now, let $(dx^\alpha, \delta y^a)$ be the basis of $T^*_u(TM)$ dual to the adapted basis $(\delta_\alpha, \dot{\alpha})$ of $T_u(TM)$. Then

$$\delta y^a := dy^a + N^a_\alpha dx^\alpha$$

(1.4)

and

$$dx^\alpha(\delta_\beta) = \delta^\alpha_\beta, \ dx^\alpha(\dot{\alpha}) = 0; \ \delta y^a(\delta_\beta) = 0, \ \delta y^a(\dot{\alpha}) = \delta^a_\beta.$$  \hspace{1cm} (1.5)

Any vector field $X \in \mathfrak{X}(TM)$ is uniquely decomposed in the form $X = hX + vX$, where $h$ and $v$ are respectively the horizontal and the vertical projectors associated with the decomposition \[1.2\]. In the adapted frame $(\delta_\nu, \dot{\alpha})$, $hX = X^\alpha \delta_\alpha$ and $vX = X^\alpha \dot{\alpha}$.  

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Definition 1.1. A nonlinear connection $N^\alpha_\mu$ is said to be homogeneous if it is positively homogeneous of degree 1 in the directional argument $y$.

Definition 1.2. A $d$-connection on $TM$ is a linear connection on $TM$ which preserves by parallelism the horizontal and vertical distribution: if $Y$ is a horizontal (vertical) vector field, then $DXY$ is a horizontal (vertical) vector field, for all $X \in \mathfrak{X}(TM)$.

Consequently, a $d$-connection $D$ on $TM$ has only four coefficients. The coefficients of a $d$-connection $D = (\Gamma^\alpha_\mu, C^\alpha_\mu_\nu)$ are defined by

\[
D_{\delta \nu} \delta_\mu =: \Gamma^\alpha_\mu \delta_\alpha, \quad D_{\delta \nu} \delta_\beta =: \Gamma^\alpha_\mu \delta_\alpha, \quad D_{\delta \nu} \delta_\nu =: C^\alpha_\mu \delta_\alpha, \quad D_{\delta \nu} \delta_\nu =: C^\alpha_\mu \delta_\alpha. \tag{1.6}
\]

The transformation formulae of a $d$-connection are given by:

\[
\Gamma'^{\alpha'}_{\mu' \nu'} = p^\alpha_\mu p^\nu_\nu \Gamma^\alpha_\mu + p^\alpha_\nu p^\nu_\mu \Gamma^\alpha_\mu, \quad \Gamma'^{\alpha'}_{\nu' \mu'} = p^\alpha_\nu p^\nu_\nu \Gamma^\alpha_\mu + p^\alpha_\nu p^\nu_\mu \Gamma^\alpha_\mu;
\]

\[
C'^{\alpha'}_{\mu' \nu' \rho'} = p^\alpha_\rho p^\rho_\nu C^\rho_\nu C^\nu_\mu, \quad C'^{\alpha'}_{\nu' \mu' \rho'} = p^\alpha_\rho p^\rho_\nu C^\nu_\mu C^\nu_\mu.
\]

A comment on notation: Both Greek indices \{\alpha, \beta, \mu, ...\} and Latin indices \{a, b, c, ...\}, as previously mentioned, take values from the same set \{1, ..., n\}. It should be noted, however, that Greek indices are used to denote horizontal counterpart, whereas Latin indices are used to denote vertical counterpart. Einstein convention is applied on both types of indices.

Definition 1.3. A $d$-tensor field $T$ on $TM$ of type $(p, r; q, s)$ is a tensor field on $TM$ which can be locally expressed in the form

\[
T = T^{u_1 \ldots u_p r} \partial_{u_1} \otimes \ldots \otimes \partial_{u_p r} \otimes dx^{v_1} \otimes \ldots \otimes dx^{v_q s},
\]

where $u_i \in \{\alpha_i, a_i\}$, $v_j \in \{\beta_j, b_j\}$,

\[
\partial_{u_i} \in \{\partial_{\alpha_i}, \partial_{a_i}\}, \quad dx^{v_j} \in \{dx^{\beta_j}, \delta y^{b_j}\}, \quad i = 1, ..., p + r; \quad j = 1, ..., q + s,
\]

so that the number of $\alpha_i$'s = $p$, the number of $a_i$'s = $r$, the number of $\beta_j$'s = $q$ and the number of $b_j$'s = $s$.

Let $T = T^{\alpha \beta_\mu} \delta_\alpha \otimes \delta_\beta \otimes dx^\beta \otimes dy^b$ be a $d$-tensor field of type $(1, 1; 1, 1)$. Let $X \in \mathfrak{X}(TM)$ be such that $X = hX + vX = X^\mu \delta_\mu + X^c \delta_c$. Then, by the properties of a $d$-connection, we have

\[
D^h_X T := D_{hX} T = (X^\mu T^{\alpha \beta_\mu}) \delta_\alpha \otimes \delta_\beta \otimes dx^\beta \otimes dy^b,
\]

where

\[
T^{\alpha \beta_\mu} = \delta_\mu T^{\alpha \beta_\mu} + T^{\alpha \beta_\mu} \Gamma^\mu_\epsilon + T^{\alpha \beta_\mu} \Gamma^\mu_\epsilon - T^{\alpha \beta_\mu} \Gamma^\mu_\epsilon - T^{\alpha \beta_\mu} \Gamma^\mu_\epsilon. \tag{1.7}
\]

Similarly,

\[
D^v_X T := D_{vX} T = (X^c T^{\alpha \beta\epsilon_\mu}) \delta_\alpha \otimes \delta_\beta \otimes dx^\beta \otimes dy^b,
\]

where

\[
T^{\alpha \beta\epsilon_\mu} = \delta_c T^{\alpha \beta_\mu} + T^{\alpha \beta_\mu} C^\epsilon_\mu + T^{\alpha \beta_\mu} C^\epsilon_\mu - T^{\alpha \beta_\mu} C^\epsilon_\mu - T^{\alpha \beta_\mu} C^\epsilon_\mu. \tag{1.8}
\]

It is evident that (1.7) and (1.8) can be written for any $d$-tensor field of arbitrary type.
Definition 1.4. The two operators $D^h_X$ (denoted locally by $|$) and $D^v_X$ (denoted locally by $||$) are called respectively the horizontal ($h$-) and vertical ($v$-) covariant derivatives associated with the $d$-connection $D$.

Definition 1.5. The torsion $T$ of a $d$-connection $D$ on $TM$ is defined by

$$T(X,Y) := D_X Y - D_Y X - [X,Y]; \ \forall X,Y \in \mathfrak{X}(TM). \quad (1.9)$$

For getting the local expression for $T$, we first recall that

$$[\delta_\mu, \delta_\nu] = R^a_{\mu\nu} \dot{\delta}_a; \ \ [\delta_\mu, \dot{\delta}_b] = (\dot{\delta}_b N^a_\mu) \dot{\delta}_a,$$

where

$$R^a_{\mu\nu} := \delta_\nu N^a_\mu - \delta_\mu N^a_\nu \quad (1.10)$$

is the curvature of the nonlinear connection.

By a direct substitution in formula (1.9), we obtain

Proposition 1.6. In the adapted basis $(\delta_\alpha, \dot{\delta}_a)$, the torsion tensor $T$ of a $d$-connection $D = (\Gamma^\alpha_{\mu\nu}, \Gamma^\alpha_{ba}, C^\alpha_{\mu c}, C^a_{bc})$ is characterized by the following $d$-tensor fields with the local coefficients $(\Lambda^\alpha_{\mu\nu}, R^a_{\mu\nu}, C^\alpha_{\mu c}, P^a_{\mu c}, T^a_{bc})$ defined by:

$$hT(\delta_\mu, \delta_\mu) =: \Lambda^\alpha_{\mu\nu} \delta_\alpha; \quad vT(\delta_\nu, \delta_\mu) =: R^a_{\mu\nu} \dot{\delta}_a$$

$$hT(\partial c, \delta_\mu) =: C^\alpha_{\mu c} \delta_\alpha; \quad vT(\partial c, \delta_\mu) =: P^a_{\mu c} \dot{\delta}_a; \quad vT(\partial c, \dot{\delta}_b) =: T^a_{bc} \dot{\delta}_a,$$

where

$$\Lambda^\alpha_{\mu\nu} := \Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\nu\mu}; \quad P^a_{\mu c} := \dot{\delta}_c N^a_\mu - \Gamma^a_{\nu c}; \quad T^a_{bc} := C^a_{bc} - C^a_{eb} \quad (1.11)$$

Throughout the paper we shall use the notation $T = (\Lambda^\alpha_{\mu\nu}, R^a_{\mu\nu}, C^\alpha_{\mu c}, P^a_{\mu c}, T^a_{bc})$.

Corollary 1.7. The torsion tensor $T = (\Lambda^\alpha_{\mu\nu}, R^a_{\mu\nu}, C^\alpha_{\mu c}, P^a_{\mu c}, T^a_{bc})$ of a $d$-connection $D$ vanishes if

$$\Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\nu\mu}; \quad R^a_{\mu\nu} = C^\alpha_{\mu c} = 0; \quad \dot{\delta}_c N^a_\mu = \Gamma^a_{\nu c}; \quad C^a_{bc} = C^a_{eb}.$$

Definition 1.8. The curvature tensor $R$ of a $d$-connection $D$ is given by

$$R(X,Y)Z := D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z; \ \forall X,Y,Z \in \mathfrak{X}(TM).$$

By definition of a $d$-connection, it follows that $R(X,Y)Z$ is determined by eight $d$-tensor fields, six of which are independent due to the fact that $R(X,Y) = -R(Y,X)$. We set

$$R(\delta_\mu, \delta_\nu)\delta_\beta =: R^a_{\beta\mu\nu} \delta_a; \quad R(\delta_\mu, \delta_\nu)\dot{\delta}_b =: R^a_{\beta\mu\nu} \dot{\delta}_a,$$

$$R(\partial c, \delta_\nu)\delta_\beta =: P^a_{\beta\mu c} \delta_a; \quad R(\partial c, \delta_\nu)\dot{\delta}_b =: P^a_{\beta\mu c} \dot{\delta}_a,$$

$$R(\partial c, \partial c)\delta_\beta =: S^a_{\beta\mu c} \delta_a; \quad R(\partial c, \partial c)\dot{\delta}_b =: S^a_{\beta\mu c} \dot{\delta}_a.$$

Throughout the paper we shall use the notation $R = (R^a_{\beta\mu\nu}, R^a_{\beta\mu\nu}, P^a_{\beta\mu c}, P^a_{\beta\mu c}, S^a_{\beta\mu c}, S^a_{\beta\mu c})$.  


Theorem 1.9. The curvature $R$ of a $d$-connection $D = (\Gamma^\alpha_{\mu\nu}, \Gamma^\alpha_{b\mu}, C^\alpha_{\mu c}, C^a_{bc})$ is characterized by the $d$-tensor fields with local coefficients:

(a) $R^\alpha_{\beta\mu\nu} = \delta^\alpha_{\mu} \Gamma^\beta_{\nu\mu} - \delta^\beta_{\nu} \Gamma^\alpha_{\mu\mu} + \Gamma^\mu_{\beta\mu} \Gamma^\alpha_{\nu\mu} - \Gamma^\mu_{\beta\nu} \Gamma^\alpha_{\mu\mu} + C^\alpha_{\beta d} R^d_{\mu\nu}$,

(b) $R^a_{b\mu\nu} = \delta^a_b \Gamma^\lambda_{\beta\mu\nu} - \delta^\beta_{\nu} \Gamma^a_{b\mu} + \Gamma^\nu_{b\nu} \Gamma^a_{\mu\mu} - \Gamma^\nu_{b\mu} \Gamma^a_{\nu\nu} + C^a_{bd} R^d_{\mu\nu}$,

(c) $P^\alpha_{\beta\nu c} = \partial_{\beta} \Gamma^\alpha_{\nu c} - C^\alpha_{\beta c\nu} + C^\alpha_{\beta d} P^d_{\nu c}$,

(d) $P^a_{b\nu c} = \partial_{\beta} \Gamma^a_{\nu c} - C^a_{b c\nu} + C^a_{bd} P^d_{\nu c}$,

(e) $S^\alpha_{\beta c d} = \partial_{\beta} C^\alpha_{c d} - \partial_{d} C^\alpha_{c \beta} + C^\alpha_{c e} C^e_{c d} - C^\alpha_{c \beta} C^e_{e d}$,

(f) $S^a_{b c d} = \partial_{c} C^a_{b d} - \partial_{b} C^a_{c d} + C^a_{b e} C^e_{c d} - C^a_{c e} C^e_{b d}$.

Corollary 1.10. The curvature tensor $R = (R^\alpha_{\beta\mu\nu}, R^a_{b\mu\nu}, P^\alpha_{\beta\nu c}, P^a_{b\nu c}, S^\alpha_{\beta c d}, S^a_{b c d})$ of a $d$-connection $D$ vanishes iff

$$R^\alpha_{\beta\mu\nu} = R^a_{b\mu\nu} = P^\alpha_{\beta\nu c} = P^a_{b\nu c} = S^\alpha_{\beta c d} = S^a_{b c d} = 0.$$

Definition 1.11. An $hv$-metric on $TM$ is a covariant $d$-tensor field $G := hG + vG$ on $TM$, where $hG := g_{\alpha\beta} dx^\alpha \otimes dx^\beta$, $vG := g_{ab} \delta y^a \otimes \delta y^b$ such that:

$$g_{\alpha\beta} = g_{\beta\alpha}, \quad \det(g_{\alpha\beta}) \neq 0; \quad g_{ab} = g_{ba}, \quad \det(g_{ab}) \neq 0. \quad (1.12)$$

The inverses of $(g_{\alpha\beta})$ and $(g_{ab})$, denoted by $(g^{\alpha\beta})$ and $(g^{ab})$ respectively, are given by

$$g^{\alpha\beta} g_{\beta\beta} = \delta^\alpha_{\beta}, \quad g^{ab} g_{ba} = \delta^a_b. \quad (1.13)$$

Definition 1.12. A $d$-connection $D$ on $TM$ is said to be metric or compatible with the metric $G$ if $D_X G = 0$, $\forall X \in \mathfrak{X}(TM)$.

In the adapted frame $(\delta^\alpha, \hat{\partial}^\alpha)$, the above condition can be expressed locally in the form:

$$g_{\alpha\beta|\mu} = g_{\alpha\beta||\mu} = g_{ab|\mu} = g_{ab||\mu} = 0. \quad (1.14)$$

We have the following Theorem [8]:

Theorem 1.13. There exists a unique metrical $d$-connection $\hat{D} = (\hat{\Gamma}^\alpha_{\mu\nu}, \hat{\Gamma}^\alpha_{b\mu}, \hat{C}^\alpha_{\mu c}, \hat{C}^a_{bc})$ on $TM$ with the properties that

(a) $\hat{\Lambda}^\alpha_{\mu\nu} = \hat{\Gamma}^\alpha_{\mu\nu} - \Gamma^\alpha_{\mu\nu} = 0, \quad \hat{T}^a_{bc} = \hat{C}^a_{bc} - C^a_{bc} = 0$,

(b) $\hat{\Gamma}^a_{b\nu} := \hat{\partial}_b N^a_{\nu} + \frac{1}{2} g^{ac}(\delta^\nu_{\mu} g_{bc} - g_{dc} \hat{\partial}_b N^d_{\nu} - g_{bd} \hat{\partial}_c N^d_{\nu}), \quad \hat{C}^\alpha_{\mu c} := \frac{1}{2} g^{\alpha\beta} \hat{\partial}_c g_{\mu\beta}$.

In this case, the coefficients $\hat{\Gamma}^\alpha_{\mu\nu}$ and $\hat{C}^\alpha_{bc}$ are necessarily of the form

$$\hat{\Gamma}^\alpha_{\mu\nu} := \frac{1}{2} g^{\alpha\beta}(\delta_{\mu} g_{\nu\beta} + \delta_{\nu} g_{\mu\beta} - \delta_{\beta} g_{\mu\nu}), \quad \hat{C}^\alpha_{bc} := \frac{1}{2} g^{ab}(\hat{\partial}_b g_{a\nu} + \hat{\partial}_c g_{a\nu} - \hat{\partial}_d g_{a\nu}).$$
Definition 1.14. The $d$-connection $\tilde{D} = (\tilde{\Gamma}^a_{\mu\nu}, \tilde{\Gamma}^a_{b\nu}, \tilde{C}^a_{\mu\nu}, \tilde{C}^a_{b\nu})$ defined in Theorem 1.13 will be referred to as the natural metric $d$-connection. The $h$- and $v$-covariant derivatives with respect to the natural metric $d$-connection $\tilde{D}$ will be denoted by $\partial\bigg|_{\partial}$ and $\partial\bigg|\bigg|$, respectively.

2. Extended Absolute Parallelism Geometry (EAP-geometry)

In this section, we study AP-geometry in a context different from the classical one. Instead of dealing with geometric objects defined on the manifold $M$, we will be dealing with geometric objects defined on the tangent bundle $TM$ of $M$. Many new geometric objects, which have no counterpart in the classical AP-geometry, emerge in this different framework. Moreover, the basic geometric objects of the new geometry acquire a richer structure compared to the corresponding basic geometric objects of the classical AP-geometry (See Table 2).

As in the previous section, $M$ is assumed to be a smooth paracompact manifold of dimension $n$. This insures the existence of a nonlinear connection on $TM$ so that the decomposition (1.2) induced by the nonlinear connection holds.

We assume that $\lambda, i = 1, ..., n$, are $n$ vector fields defined globally on $TM$. In the adapted basis $(\delta, \delta')$, we have $\lambda = h\lambda + v\lambda = \lambda^a\delta_a + \lambda^a\delta'$. We further assume that the $n$ horizontal vector fields $h\lambda$ and the $n$ vertical vector fields $v\lambda$ are linearly independent. This implies, in particular, that the $n$ vector fields $\lambda_i$ themselves, are linearly independent. Moreover, we have

$$\lambda^a\lambda^b = \delta^a_b, \quad \lambda^a\lambda^b = \delta^a_b, \quad \lambda^a\lambda^b = \delta^a_b, \quad \lambda^a\lambda^b = \delta^a_b,$$

(2.1)

where $(\lambda^a_i)$ and $(\lambda^b_i)$ denote the inverse matrices of $(\lambda^a_i)$ and $(\lambda^a_i)$ respectively.

We refer to the above space, which we denote by $(TM, \lambda)$, as an Extended Absolute Parallelism (EAP-) geometry which is characterized by the existence of $2n$ linearly independent vector fields defined globally on $TM$.

The Latin indices $\{i, j\}$ will be used for numbering the $n$ vector fields (mesh indices). Einstein convention is applied on the mesh indices (which will always be written in lower position) as well as the component indices. In the sequel, to simplify notations, we will use the symbol $\lambda$ without the subscript $i$ to denote any one of the vector fields $\lambda_i$ ($i = 1, ..., n$). The index $i$ will appear only when summation is performed.

Let us define

$$g_{\alpha\beta} := \lambda^a_i\lambda^b_i, \quad g_{a b} := \lambda^a_i\lambda^b_i.$$

(2.2)

Then, clearly,

$$G = g_{\alpha\beta}dx^\alpha \otimes dx^\beta + g_{a b}\delta y^a \otimes \delta y^b$$

is an $hv$-metric on $TM$. Moreover, in view of (2.1), the inverse of the matrices $(g_{\alpha\beta})$ and $(g_{a b})$ are given by $(g^{\alpha\beta})$ and $(g^{a b})$ respectively, where

$$g^{\alpha\beta} = \lambda^a_i\lambda^b_i, \quad g^{a b} = \lambda^a_i\lambda^b_i.$$

(2.3)
Now, let \( \hat{D} = (\hat{\Gamma}^\alpha_{\mu\nu}, \hat{\Gamma}^\alpha_{bc}, \hat{C}^{\alpha}_{\mu c}, \hat{C}^{\alpha}_{bc}) \) be the natural metric \( d \)-connection defined by Theorem 1.13, where \( g_{\mu\nu} \) and \( g_{ab} \) are the metric tensors given by (2.2).

**Theorem 2.1.** There exists a unique \( d \)-connection \( D = (\Gamma^\alpha_{\mu\nu}, \Gamma^\alpha_{bc}, C^\alpha_{\mu c}, C^\alpha_{bc}) \) such that

\[
\lambda^\alpha|\mu = \lambda^\alpha||c = \lambda^\alpha|\mu = \lambda^\alpha||c = 0, \tag{2.4}
\]

where \| and || are the h- and v-covariant derivatives with respect to \( D \). Consequently \( D \) is a metric \( d \)-connection. It is given by

\[
\begin{align*}
\Gamma^\alpha_{\mu\nu} &:= \tilde{\Gamma}^\alpha_{\mu\nu} + \lambda^\alpha\lambda^\mu\nu, \\
\Gamma^\alpha_{bc} &:= \tilde{\Gamma}^\alpha_{bc} + \lambda^\alpha\lambda^b\gamma^c.
\end{align*}
\tag{2.5}
\]

**Relation** (2.4) **will be called the AP-condition** (as in the classical AP-geometry).

**Proof.** First, it is clear that \( D \) is a \( d \)-connection on \( TM \). We next prove that \( \lambda^\alpha|\mu = 0 \). We have

\[
\begin{align*}
\lambda^\alpha|\nu &= \delta^\alpha_\nu \lambda^\alpha + \lambda^\alpha\Gamma^\alpha_{\mu\nu} = \delta^\alpha_\nu \lambda^\alpha + \lambda^\mu(\tilde{\Gamma}^\alpha_{\mu\nu} + \lambda^\alpha\lambda^\mu\nu) \\
&= (\delta^\alpha_\nu \lambda^\alpha + \tilde{\Gamma}^\alpha_{\mu\nu}\lambda^\mu) - (\lambda^\mu\lambda^\nu)\lambda^\alpha = \lambda^\alpha|\nu - \lambda^\alpha|\nu = 0.
\end{align*}
\]

The rest is proved in a similar manner. \( \Box \)

**Definition 2.2.** The \( d \)-connection \( D = (\Gamma^\alpha_{\mu\nu}, \Gamma^\alpha_{bc}, C^\alpha_{\mu c}, C^\alpha_{bc}) \) defined in Theorem 2.1 will be referred to as the **canonical** \( d \)-connection of the EAP-space.

In analogy to the classical AP-space, the torsion tensor of the canonical \( d \)-connection will be called the torsion tensor of the EAP-space.

**Theorem 2.3.** The canonical \( d \)-connection \( D \) can be expressed explicitly in terms of the \( \lambda \)'s only in the form:

\[
\begin{align*}
\Gamma^\alpha_{\mu\nu} &= \lambda^\alpha(\delta^\nu_{\mu} \lambda^\mu), \\
\Gamma^\alpha_{bc} &= \lambda^\alpha(\delta^\nu_{\mu} \lambda^\mu), \\
C^\alpha_{\mu c} &= \lambda^\alpha(\partial^\gamma_{\mu} \lambda^\mu), \\
C^\alpha_{bc} &= \lambda^\alpha(\partial^\gamma_{\mu} \lambda^\mu).
\end{align*}
\tag{2.7}
\]

**Proof.** Since \( \lambda^\alpha|\nu = 0 \), it follows that \( \delta^\nu_{\mu} \lambda^\alpha = -\lambda^\alpha\Gamma^\alpha_{\mu\nu} \). Multiplying by \( \lambda^\mu \), we get \( \lambda^\mu(\delta^\nu_{\mu} \lambda^\alpha) = -\Gamma^\alpha_{\mu\nu} \) so that, by (2.1), \( \Gamma^\alpha_{\mu\nu} = \lambda^\alpha(\delta^\nu_{\mu} \lambda^\mu) \). The other formulae are derived in a similar manner. \( \Box \)

By Theorem 2.1 and Theorem 2.3, we have

**Corollary 2.4.** The natural metric \( d \)-connection \( \hat{D} \) can be expressed explicitly in terms of the \( \lambda \)'s only in the form

\[
\begin{align*}
\hat{\Gamma}^\alpha_{\mu\nu} &= \lambda^\alpha(\delta^\nu_{\mu} \lambda^\mu - \lambda^\mu_{\mu\nu}), \\
\hat{\Gamma}^\alpha_{bc} &= \lambda^\alpha(\delta^\nu_{\mu} \lambda^\mu - \lambda^\mu_{\mu\nu}); \\
\hat{C}^\alpha_{\mu c} &= \lambda^\alpha(\partial^\gamma_{\mu} \lambda^\mu - \lambda^\mu_{\mu\nu}), \\
\hat{C}^\alpha_{bc} &= \lambda^\alpha(\partial^\gamma_{\mu} \lambda^\mu - \lambda^\mu_{\mu\nu}).
\end{align*}
\tag{2.9}
\]

\[8\]
Definition 2.5. The contortion tensor of an EAP-space is defined by
\[ C(X, Y) := D_Y X - \hat{D}_Y X; \quad \forall X, Y \in \mathfrak{X}(TM), \]
where \( D \) is the canonical \( d \)-connection and \( \hat{D} \) is the natural metric \( d \)-connection.

In the adapted basis \((\delta_{\mu}, \tilde{\delta}_{\alpha})\), the contortion tensor is characterized by the following \( d \)-tensor fields:
\[
C(\delta_{\mu}, \delta_{\nu}) := \gamma^\alpha_{\mu\nu}, \quad C(\delta_{\mu}, \tilde{\delta}_{\alpha}) := \gamma^\alpha_{\mu\beta}, \quad C(\tilde{\delta}_{\nu}, \delta_{\mu}) := \gamma^\alpha_{\nu\mu}, \quad C(\tilde{\delta}_{\alpha}, \tilde{\delta}_{\beta}) := \gamma^\alpha_{\beta\gamma}.
\]

Throughout the paper we shall use the notation \( C(\delta_{\mu}, \delta_{\nu}), C(\delta_{\mu}, \tilde{\delta}_{\alpha}), C(\tilde{\delta}_{\nu}, \delta_{\mu}), C(\tilde{\delta}_{\alpha}, \tilde{\delta}_{\beta}) \) vanishes iff \( \gamma^\alpha_{\gamma}, C^\alpha_{\delta}, \gamma^\alpha_{\delta} \).

By definition of the canonical \( d \)-connection and \((2.11)\), the contortion tensor can be expressed explicitly in terms of the \( \lambda \)'s only in the form:
\[
\gamma^\alpha_{\mu\nu} = \lambda^\alpha_{\mu \gamma} \gamma^\gamma_{\nu}, \quad \gamma^\alpha_{\mu\beta} = \lambda^\alpha_{\mu \beta}, \quad \gamma^\alpha_{\nu\mu} = \lambda^\alpha_{\nu \gamma} \gamma^\gamma_{\mu}, \quad \gamma^\alpha_{\beta\gamma} = \lambda^\alpha_{\beta \gamma}. \quad (2.12)
\]

Proposition 2.6. Let \( \gamma_{\alpha\mu\nu} := g_{\alpha\lambda} \gamma^{\lambda}_{\mu\nu}, \gamma_{\alpha\beta\mu} := g_{\alpha\lambda} \gamma^{\lambda}_{\beta\mu}, \gamma_{\alpha\mu\nu} := g_{\alpha\lambda} \gamma^{\lambda}_{\mu\nu}, \gamma_{\alpha\beta\gamma} := g_{\alpha\lambda} \gamma^{\lambda}_{\beta\gamma}. \) Then each of the above defined \( d \)-tensor fields is skew-symmetric in the first pair of indices. Consequently, \( \gamma^\alpha_{\alpha\mu} = \gamma^\alpha_{\mu\alpha} = \gamma^\alpha_{\alpha\nu} = \gamma^\alpha_{\nu\alpha} = 0. \)

Proof. We have
\[
\gamma_{\alpha\mu\nu} + \gamma_{\mu\alpha\nu} = \lambda^\alpha_{\mu \gamma} \gamma^\gamma_{\nu} + \lambda^\alpha_{\mu \gamma} \gamma^\gamma_{\nu} = (\lambda^\alpha_{\mu \gamma}) \gamma^\gamma_{\nu} = g_{\alpha\mu} \gamma^\gamma_{\nu} = 0.
\]
The rest is proved analogously.

A simple calculation gives

Proposition 2.7. Let \( T = (\Lambda^\alpha_{\mu\nu}, R^\alpha_{\mu\nu}, A_{\mu\nu}, P^a_{\mu\nu}, T^a_{\mu\nu}) \) and \( C = (\gamma^\alpha_{\mu\nu}, \gamma^\alpha_{\mu\beta}, \gamma^\alpha_{\nu\mu}, \gamma^\alpha_{\beta\gamma}) \) be the torsion and the contortion tensors of the EAP-space respectively. Then the following relations hold:
\[
\Lambda^\alpha_{\mu\nu} = \gamma^\alpha_{\mu\nu} - \gamma^\alpha_{\nu\mu}, \quad P^a_{\mu\nu} = -\gamma^a_{\mu\nu} + \hat{P}^a_{\mu\nu}, \quad A_{\mu\nu} = \gamma^\alpha_{\mu\nu} + \hat{C}^\alpha_{\mu\nu}, \quad T^a_{\nu\mu} = \gamma^a_{\nu\mu} - \gamma^a_{\mu\nu}. \quad (2.13)
\]
Consequently,
\[
\Lambda^\alpha_{\mu} = \gamma^\alpha_{\mu} =: C^\alpha_{\mu}, \quad T^a_{\mu} = \gamma^a_{\mu} =: C^a_{\mu}. \quad (2.14)
\]

Remark 2.8. It can be shown, in analogy to the classical AP-space \[2\], that
\[
\gamma_{\alpha\mu\nu} = \frac{1}{2}(\Lambda_{\alpha\mu\nu} + \Lambda_{\nu\alpha\mu} + \Lambda_{\mu\alpha\nu}), \quad \gamma_{\alpha\beta\gamma} = \frac{1}{2}(T_{\alpha\beta\gamma} + T_{\alpha\gamma\beta} + T_{\beta\gamma\alpha}). \quad (2.15)
\]
where \( \Lambda_{\alpha\mu\nu} := g_{\alpha\lambda} \Lambda^\lambda_{\mu\nu} \) and \( T_{\alpha\beta\gamma} := g_{\alpha\lambda} T^d_{\beta\gamma}. \)

By \((2.13)\) and \((2.15)\), \( \Lambda^\alpha_{\mu\nu} \) (resp. \( T^a_{\nu\mu} \)) vanishes iff \( \gamma^\alpha_{\mu\nu} \) (resp. \( \gamma^a_{\mu\nu} \)) vanishes.

Definition 2.9. Let \( D = (\Gamma^\alpha_{\mu\nu}, \Gamma^a_{\mu\nu}, \gamma^\alpha_{\mu\nu}, \gamma^a_{\mu\nu}) \) be the canonical \( d \)-connection.
(a) The dual d-connection $\tilde{D} = (\tilde{\Gamma}_{\mu\nu}^\alpha, \tilde{\Gamma}_b^a, \tilde{C}_m^a, \tilde{C}_b^c)$ is defined by
$$\tilde{\Gamma}_{\mu\nu}^\alpha := \Gamma_{\mu\nu}^\alpha, \quad \tilde{\Gamma}_b^a := \Gamma_b^a; \quad \tilde{C}_m^a := C_m^a; \quad \tilde{C}_b^c := C_b^c.$$  \hspace{1cm} (2.16)

(b) The symmetric d-connection $\hat{D} = (\hat{\Gamma}_{\mu\nu}^a, \hat{\Gamma}_b^a, \hat{C}_m^a, \hat{C}_b^c)$ is defined by
$$\hat{\Gamma}_{\mu\nu}^a := \frac{1}{2}(\Gamma_{\mu\nu}^a + \Gamma_{\nu\mu}^a), \quad \hat{\Gamma}_b^a := \Gamma_b^a; \quad \hat{C}_m^a := C_m^a; \quad \hat{C}_b^c := \frac{1}{2}(C_b^c + C_c^b).$$ \hspace{1cm} (2.17)

We shall denote the horizontal (vertical) covariant derivative of $\tilde{D}$ and $\hat{D}$ by $\tilde{\nabla}$ (\nabla) and $\tilde{\nabla}$ (\nabla) respectively.

It follows immediately from the above definition that
$$\lambda^\alpha ||c = \lambda^\alpha ||c = 0, \quad \lambda^\alpha |\mu = \lambda^\alpha |\mu = 0; \quad (2.18)$$
$$\lambda^\alpha \lambda^\alpha = \lambda^\beta \Lambda_{\beta\mu}^\alpha, \quad \lambda^\alpha \lambda^\alpha = \frac{1}{2} \lambda^\beta \Lambda_{\beta\mu}^\alpha, \quad \lambda^\alpha \lambda^\alpha = \frac{1}{2} \lambda^\beta T^{\alpha}_{\beta}, \quad \lambda^\alpha \lambda^\alpha = \frac{1}{2} \lambda^\beta T^{\alpha}_{\beta}. \quad (2.19)$$

As easily checked, we also have

**Proposition 2.10.** The covariant derivatives of the metric $G$ with respect to the dual and symmetric d-connections $\tilde{D}$ and $\hat{D}$ are given respectively by:

$$g_{\alpha\beta|\mu} = \Lambda_{\alpha\beta\mu}^\mu + \Lambda_{\beta\alpha\mu}^\mu, \quad g_{\alpha\beta||c} = g_{\alpha\beta|c} = 0, \quad g_{\alpha\beta||c} = T_{abc} + T_{bac}; \quad (2.20)$$

$$g_{\alpha\beta|\mu} = \frac{1}{2} g_{\alpha\beta|\mu}, \quad g_{\alpha\beta||c} = g_{\alpha\beta|c} = 0, \quad g_{\alpha\beta|c} = \frac{1}{2} g_{\alpha\beta|c}. \quad (2.21)$$

Consequently, $\tilde{D}$ and $\hat{D}$ are non-metric connections.

We end this section with the following tables.

**Table 1: Fundamental connections of the EAP-space**

| Connection | Coefficients | Covariant derivative | Torsion | Metricity |
|------------|-------------|---------------------|---------|-----------|
| Natural    | \((\hat{\Gamma}_{\mu\nu}^{\alpha}, \hat{\Gamma}_b^a, \hat{C}_m^a, \hat{C}_b^c)\) | \(\nabla\) | \((0, R_{\mu\nu}^{\alpha}, \hat{C}_m^a, \hat{C}_b^c, 0)\) | metric |
| Canonical  | \((\Gamma_{\mu\nu}^{\alpha}, \Gamma_b^a, C_m^a, C_b^c)\) | \(\nabla\) | \((\Lambda_{\mu\nu}^\alpha, P_{\mu\nu}^\alpha, C_m^a, C_b^c, T_b^a)\) | metric |
| Dual       | \((\Gamma_{\mu\nu}^{\alpha}, \Gamma_b^a, C_m^a, C_b^c)\) | \(\nabla\) | \((-\Lambda_{\mu\nu}^\alpha, P_{\mu\nu}^\alpha, C_m^a, C_b^c, -T_b^a)\) | non-metric |
| Symmetric  | \((\Gamma_{\mu\nu}^{\alpha}, \Gamma_b^a, C_m^a, C_b^c)\) | \(\nabla\) | \((0, R_{\mu\nu}^{\alpha}, C_m^a, C_b^c, 0)\) | non-metric |

The next table gives a comparison between the classical AP-space and the EAP-space. We shall refer to the Riemannian connection in the classical AP-space and the natural metric d-connection in the EAP-space simply as the metric connection. Moreover, we consider only the metric and the canonical connections in both spaces. We also set $L_{\mu\nu}^a := \frac{1}{2} g^{ac}(d\nu g_{bc} - g_{de} \partial_b N_{\nu}^d - g_{bd} \partial_c N_{\nu}^d)$. 

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### Table 2: Comparison between classical AP-geometry and EAP-geometry

|                      | Classical AP-geometry | EAP-geometry |
|----------------------|-----------------------|--------------|
| Underlying space     | \( M \)               | \( TM \)     |
| Building blocks      | \( \lambda^\alpha(x) \) | \( N_\mu^\alpha(x, y), \lambda(x, y) = (\lambda^\alpha(x, y), \lambda^\alpha(x, y)) \) |
| Metric               | \( g_{\mu\nu} = \lambda^\alpha_{, i} \lambda^{\alpha, i} \) | \( G = (g_{\mu\nu}, g_{ab}); \) |
|                      | \( g_{\mu\nu} = \lambda^\alpha_{, i} \lambda^{\alpha, i} \) | \( g_{\mu\nu} = \lambda^\alpha_{, i} \lambda^{\alpha, i} \) |
| Metric connection    | \( \Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\rho} \left( \partial_\rho g_{\mu\nu} + \partial_\nu g_{\mu\rho} - \partial_\mu g_{\nu\rho} \right) \) | \( \hat{\Gamma}^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\rho} \left( \delta_\rho g_{\mu\nu} + \delta_\nu g_{\mu\rho} - \delta_\mu g_{\nu\rho} \right) \) |
|                      | \( \Gamma^\alpha_{\mu\nu} = \lambda^\alpha_{, i} \lambda^{\alpha, i} \) | \( \hat{\Gamma}^\alpha_{\mu\nu} = \lambda^\alpha_{, i} \lambda^{\alpha, i} \) |
|                      | \( \Gamma^\alpha_{\mu\nu} = \lambda^\alpha_{, i} \lambda^{\alpha, i} \) | \( \hat{\Gamma}^\alpha_{\mu\nu} = \lambda^\alpha_{, i} \lambda^{\alpha, i} \) |
| AP-condition         | \( \lambda^\alpha_{|\mu} = 0 \) | \( \lambda^\alpha_{|\mu} = \lambda^\alpha_{||c} = 0, \) |
|                      | \( \lambda^\alpha_{|\mu} = 0 \) | \( \lambda^\alpha_{|\mu} = \lambda^\alpha_{||c} = 0, \) |
| Torsion              | \( \Lambda^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\nu\mu} \) | \( T = (\Lambda^\alpha_{\mu\nu}, R^\alpha_{\mu\nu}, C^\alpha_{\mu\nu}, P^\alpha_{\mu\nu}, T^a_{bc}); \) |
|                      | \( \Lambda^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\nu\mu} \) | \( \Lambda^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\nu\mu}; \) \( R^\alpha_{\mu\nu} = \delta_\nu N^\alpha_{\mu} - \delta_\mu N^\alpha_{\nu}, \) |
| Contortion           | \( \gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} - \hat{\Gamma}^\alpha_{\mu\nu} \) | \( C = (\gamma^\alpha_{\mu\nu}, \gamma^\alpha_{\mu}^\alpha_{, \beta}, \gamma^\alpha_{\mu}^\beta_{, \alpha}, \gamma^\alpha_{\mu}^\alpha_{, \beta}; \) |
|                      | \( \gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} - \hat{\Gamma}^\alpha_{\mu\nu}; \) \( \hat{\gamma}^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} - \hat{\Gamma}^\alpha_{\mu\nu}, \) \( \gamma^\alpha_{\mu} = C^\alpha_{\mu} - \hat{C}^\alpha_{\mu}, \) \( \gamma^\alpha_{\mu} = C^\alpha_{\mu} - \hat{C}^\alpha_{\mu}; \) |
| Basic vector         | \( C_\mu = \Lambda^\alpha_{\mu\alpha} = \gamma^\alpha_{\mu\alpha} \) | \( B = (C_\mu, C^a); \) |
|                      | \( C_\mu = \Lambda^\alpha_{\mu\alpha} = \gamma^\alpha_{\mu\alpha}; \) \( C_\alpha = T^d_{ad} = \gamma^d_{ad} \) |
3. Curvature tensors in the EAP-space

Let \((TM, \lambda)\) be an EAP space. Let \(\tilde{D}, \tilde{D}\) and \(\tilde{D}\) be the natural metric \(d\)-connection, the dual \(d\)-connection and the symmetric \(d\)-connection respectively. The curvature tensors of the four \(d\)-connections will be denoted respectively by \(R, \tilde{R}, \tilde{R}\) and \(\tilde{R}\). In this section, we carry out the task of calculating the curvature tensors together with their contractions.

**Lemma 3.1.** The following commutation formulae hold:

(a) \(\lambda^\alpha_{\mu|\nu} - \lambda^\alpha_{\mu|\nu} = R_{\beta\mu\nu}^\alpha \lambda^\beta - \Lambda^\beta_{\nu\mu} \lambda^\alpha_{\beta|\nu} - R_{\nu\mu}^d \lambda^\alpha_{|d} \)

(b) \(\lambda^\alpha_{\nu|c} - \lambda^\alpha_{\nu|c} = P_{\beta\nu\epsilon}^\alpha \lambda^\beta - C^\beta_{\nu\epsilon} \lambda^\alpha_{|\beta} - P_{\nu\epsilon}^d \lambda^\alpha_{|d} \)

(c) \(\lambda^\alpha_{||b|c} - \lambda^\alpha_{||c|b} = S_{dbc}^\alpha \lambda^d - T_{bc}^\alpha \lambda^c \)

(d) \(\lambda^\alpha_{|\nu|\mu} - \lambda^\alpha_{|\nu|\mu} = R_{\nu\mu\beta}^d \lambda^\beta - \Lambda^\beta_{\mu\nu} \lambda^\alpha_{\beta|\mu} - R_{\mu\nu}^d \lambda^\alpha_{|d} \)

(e) \(\lambda^\alpha_{|\nu|c} - \lambda^\alpha_{|\nu|c} = P_{\nu\epsilon\beta}^c \lambda^\epsilon - C^\epsilon_{\nu\beta} \lambda^\alpha_{|\epsilon} - P_{\nu\epsilon}^d \lambda^\alpha_{|d} \)

(f) \(\lambda^a_{|\beta|\nu} - \lambda^a_{|\beta|\nu} = S_{\alpha\beta\nu}^d \lambda^d - T_{bc}^\alpha \lambda^c \)

In view of (2.1), Corollary 1.10 and Theorem 2.1, Lemma 3.1 directly implies that

**Theorem 3.2.** The curvature tensor \(\tilde{R} = (R_{\beta\mu\nu}^\alpha, R_{\nu\mu\beta}^d, P_{\beta\nu\epsilon}^\alpha, P_{\nu\epsilon\beta}^c, S_{\alpha\beta\epsilon}^d, S_{\beta\nu\epsilon}^d)\) of the canonical \(d\)-connection vanishes identically.

**Corollary 3.3.** The following identities hold:

\[
\Lambda_{\beta\mu|\alpha}^{\alpha} = (C_{\beta|\mu} - C_{\mu|\beta}) + C_{\epsilon} \Lambda_{\beta\mu}^{\epsilon} + \mathcal{S}_{\beta\mu\alpha} R_{\beta\mu\beta}^{\alpha} C_{\alpha}^\alpha
\]

(3.1)

\[
T_{bc|\nu}^d = (C_{\beta|\nu} - C_{\nu|\beta}) + C_{d} T_{bc}^\alpha
\]

(3.2)

**Proof.** The Bianchi identities [9] applied to the canonical \(d\)-connection \(D\) give

\[
\mathcal{S}_{\beta,\mu,\nu} (\Lambda_{\beta\mu|\nu}^{\alpha} + \Lambda_{\mu\nu}^{\epsilon} \Lambda_{\beta\epsilon}^{\alpha} + R_{\beta\mu}^{\epsilon} C_{\nu\epsilon}^\alpha) = 0
\]

(3.3)

and

\[
\mathcal{S}_{b,c,d} (T_{bc|\nu}^a + T_{cd}^e T_{be}^a) = 0,
\]

(3.4)

where the notation \(\mathcal{S}_{\beta,\mu,\nu}\) denotes a cyclic permutation on the indices \(\beta, \mu, \nu\) and summation. (3.1) and (3.2) are obtained by setting \(\alpha = \nu\) and \(a = d\) in (3.3) and (3.4) respectively.

By applying the commutation formula (c) of Lemma 3.1 with respect to the dual and symmetric \(d\)-connections respectively, taking into account (2.1) and (2.18), we obtain

\[
\tilde{S}_{\beta\nu\epsilon}^\alpha = \tilde{S}_{\beta\nu\epsilon}^\alpha = 0.
\]

This could be also deduced from Theorem 1.9 (e) and Theorem 3.2, noting that \(C_{\mu\nu}^\alpha = \tilde{C}_{\mu\nu}^\alpha = \tilde{C}_{\mu\nu}^\alpha\).
In Theorems 3.4, 3.5 and 3.6 below, concerning the curvature tensors of the d-connections $\hat{D}, \widehat{D}$ and $\widetilde{D}$, we will make use of Theorem 3.2, namely that the curvature tensors of the canonical d-connection vanish identically.

**Theorem 3.4.** The curvature tensors of the natural metric d-connection $\hat{D}$ can be expressed in the form:

(a) $\hat{R}_{\beta\mu\nu}^\alpha = (\gamma_{\beta\mu|\nu}^\alpha - \gamma_{\beta\nu|\mu}^\alpha) + (\gamma_{\beta\nu|\epsilon}^\alpha - \gamma_{\beta\epsilon|\nu}^\alpha) - \gamma_{\beta\epsilon}^\alpha \Lambda_{\nu\epsilon}^\mu - \gamma_{\beta\nu}^\alpha R_{\epsilon \nu}^d$.

(b) $\hat{R}_{b\mu\nu}^a = (\gamma_{b\mu|\nu}^a - \gamma_{b\nu|\mu}^a) + (\gamma_{b\nu|\epsilon}^a - \gamma_{b\epsilon|\nu}^a) - \gamma_{b\epsilon}^a \Lambda_{\nu\epsilon}^\mu - \gamma_{b\nu}^a R_{\epsilon \nu}^d$.

(c) $\hat{P}_{\beta\nu e}^c = (\gamma_{\beta|\nu}^c - \gamma_{\beta|\epsilon}^c) + (\gamma_{\beta|\epsilon}^c - \gamma_{\beta|\epsilon}^c) - \gamma_{\beta\epsilon}^c C_{\nu\epsilon}^c - \gamma_{\beta\nu}^c P_{\epsilon \nu}^d$.

(d) $\hat{P}_{b\nu e}^c = (\gamma_{b|\nu}^c - \gamma_{b|\epsilon}^c) + (\gamma_{b|\epsilon}^c - \gamma_{b|\epsilon}^c) - \gamma_{b\epsilon}^c C_{\nu\epsilon}^c - \gamma_{b\nu}^c P_{\epsilon \nu}^d$.

(e) $\hat{S}_{\beta c} = (\gamma_{\beta|c}^c - \gamma_{\beta|\epsilon}^c) + (\gamma_{\beta|\epsilon}^c - \gamma_{\beta|\epsilon}^c) - \gamma_{\beta\epsilon}^c T_{\epsilon \epsilon}^d$.

(f) $\hat{S}_{b e} = (\gamma_{b|\epsilon}^c - \gamma_{b|\epsilon}^c) + (\gamma_{b|\epsilon}^c - \gamma_{b|\epsilon}^c) - \gamma_{b\epsilon}^c T_{\epsilon \epsilon}^d$.

Consequently,

(g) $\bar{R}_{\beta\mu} := \hat{R}_{\beta\mu}^\alpha = (\gamma_{\beta\mu|\alpha}^\alpha - C_{\beta|\mu}^\alpha) - C_{\epsilon\mu}^\alpha \Lambda_{\mu\epsilon}^\alpha - \gamma_{\beta\mu}^\alpha R_{\mu\epsilon}^d$.

(h) $\bar{\mathcal{R}} := g^{\beta\mu} \hat{R}_{\beta\mu} = \frac{1}{2} \Omega_{\alpha\mu}^\mu (C_{\alpha|\mu}^\mu - C_{\alpha\mu}^\mu) - C_{\mu|\mu}^\mu + \gamma_{\alpha\mu}^\alpha \Lambda_{\mu\epsilon}^\mu - \gamma_{\beta\mu}^\alpha R_{\mu\epsilon}^d$.

(i) $\bar{P}_{\beta\epsilon} := \hat{P}_{\beta\epsilon}^c = (\gamma_{\beta|\epsilon}^c - \gamma_{\beta|\epsilon}^c) + (\gamma_{\beta|\epsilon}^c - \gamma_{\beta|\epsilon}^c) - \gamma_{\beta\epsilon}^c C_{\epsilon\epsilon}^c - \gamma_{\beta\epsilon}^c P_{\epsilon\epsilon}^d$.

(j) $\bar{P}_{b\epsilon} := \hat{P}_{b\epsilon}^c = (\gamma_{b|\epsilon}^c - \gamma_{b|\epsilon}^c) + (\gamma_{b|\epsilon}^c - \gamma_{b|\epsilon}^c) - \gamma_{b\epsilon}^c C_{\epsilon\epsilon}^c - \gamma_{b\epsilon}^c P_{\epsilon\epsilon}^d$.

(k) $\bar{S}_{\epsilon} := \hat{S}_{\epsilon} = (\gamma_{\epsilon|\epsilon}^c - \gamma_{\epsilon|\epsilon}^c) + (\gamma_{\epsilon|\epsilon}^c - \gamma_{\epsilon|\epsilon}^c) - \gamma_{\epsilon\epsilon}^c T_{\epsilon\epsilon}^d$.

(l) $\bar{S} := \hat{S} = (\gamma_{\epsilon|\epsilon}^c - \gamma_{\epsilon|\epsilon}^c) + (\gamma_{\epsilon|\epsilon}^c - \gamma_{\epsilon|\epsilon}^c) - \gamma_{\epsilon\epsilon}^c T_{\epsilon\epsilon}^d$.

where $\Omega_{\beta\mu}^\alpha := \gamma_{\beta\mu}^\alpha + \gamma_{\beta\mu}^\alpha$, $\Omega_{\nu\epsilon}^\alpha := \gamma_{\nu\epsilon}^\alpha + \gamma_{\nu\epsilon}^\alpha$.

**Proof.** We prove (a) and (c) only. The other formulae of the first part are proved in a similar manner. The second part is obtained directly by applying the suitable contractions.

(a) We have

$$\hat{R}_{\beta\mu\nu}^\alpha = \delta_{\mu}^{\alpha} \hat{\Gamma}_{\beta\nu}^\alpha - \delta_{\nu}^{\alpha} \hat{\Gamma}_{\beta\mu}^\alpha + \hat{\Gamma}_{\beta\nu}^\alpha \hat{\Gamma}_{\epsilon\mu}^\alpha - \hat{\Gamma}_{\epsilon\nu}^\alpha \hat{\Gamma}_{\mu\epsilon}^\alpha + \hat{C}_{\beta\mu}^\alpha R_{\epsilon \nu}^d$$

$$= \delta_{\mu}^{\alpha} (\Gamma_{\beta\nu}^\alpha - \gamma_{\beta\nu}^\alpha) - \delta_{\nu}^{\alpha} (\Gamma_{\beta\mu}^\alpha - \gamma_{\beta\mu}^\alpha) + (\Gamma_{\epsilon\mu}^\alpha - \gamma_{\epsilon\mu}^\alpha) (\Gamma_{\mu\epsilon}^\alpha - \gamma_{\mu\epsilon}^\alpha) - (\Gamma_{\beta\mu}^\alpha - \gamma_{\beta\mu}^\alpha)$$

$$= \hat{R}_{\beta\mu\nu}^\alpha + (\beta_{\mu\nu}^\alpha - \gamma_{\beta\nu}^\alpha (\Gamma_{\epsilon\mu}^\alpha - \gamma_{\epsilon\mu}^\alpha) - (\delta_{\mu}^{\alpha} \gamma_{\beta\nu}^\alpha + \beta_{\nu\epsilon}^\alpha \Gamma_{\epsilon\mu}^\alpha - \gamma_{\epsilon\mu}^\alpha \Gamma_{\beta\nu}^\alpha)$$

$$- \gamma_{\beta\nu}^\alpha R_{\epsilon \nu}^d + (\gamma_{\beta\mu}^\alpha + \gamma_{\beta\mu}^\alpha)$$

$$= (\gamma_{\beta\mu\nu}^\alpha - \gamma_{\beta\nu\mu}^\alpha) + (\gamma_{\beta\mu}^\alpha \Gamma_{\epsilon\mu}^\alpha - \gamma_{\beta\mu}^\alpha \Gamma_{\epsilon\nu}^\alpha) - \gamma_{\beta\epsilon}^\alpha \Lambda_{\epsilon \nu}^\mu - \gamma_{\beta\nu}^\alpha R_{\epsilon \nu}^d.$$
(c) We have
\[ \tilde{P}^\alpha_{\beta\nu c} = \partial_{\nu} \tilde{\Gamma}^\alpha_{\beta\nu} - \partial_{\beta} \tilde{\gamma}^\alpha_{\nu} + \tilde{C}^\alpha_{\beta\nu c} P^{d}_{\nu c} \]
\[ = \partial_{\nu} \tilde{\Gamma}^\alpha_{\beta\nu} - \partial_{\beta} \tilde{\gamma}^\alpha_{\nu} + (\tilde{C}^\alpha_{\beta\nu} - \tilde{\gamma}^\alpha_{\beta\nu})P^{d}_{\nu c} + \tilde{C}^\alpha_{\beta\nu c} P^{d}_{\nu c} \]
\[ = \tilde{P}^\alpha_{\beta\nu c} = \tilde{P}^\alpha_{\beta\nu c} = \tilde{C}^\alpha_{\beta\nu c} P^{d}_{\nu c} \]
\[ \tilde{\gamma}^\alpha_{\beta\nu} - \tilde{\gamma}^\alpha_{\beta\nu} + (\tilde{C}^\alpha_{\beta\nu} - \tilde{\gamma}^\alpha_{\beta\nu})P^{d}_{\nu c} + \tilde{C}^\alpha_{\beta\nu c} P^{d}_{\nu c} \]

**Theorem 3.5.** The non-vanishing curvature tensors of the dual d-connection \( \tilde{D} \) can be expressed in the form:

(a) \( \tilde{R}^\alpha_{\beta\mu\nu} = \Lambda^\alpha_{\mu\beta\nu} + \mathcal{S}_{\beta\mu\nu} C^\alpha_{\beta\mu\nu} \)

(b) \( \tilde{R}^a_{\beta\mu\nu} = R^a_{\mu\nu} T^a_{\beta\mu} \)

(c) \( \tilde{P}^a_{\beta\mu c} = \Lambda^a_{\mu\beta c} + \Lambda^a_{\mu c} \)

(d) \( \tilde{T}^a_{\beta\mu c} = T^a_{\beta\mu c} + T^a_{\beta\mu c} \)

(e) \( \tilde{S}^a_{\beta c d} = T^a_{\beta c d} - T^a_{\beta c d} \)

Consequently,

(f) \( \tilde{R}^\alpha_{\beta\mu} := \tilde{R}^\alpha_{\beta\mu c} = - C^\alpha_{\beta\mu} + \mathcal{S}_{\beta\mu c} C^\alpha_{\beta\mu c} \)

(g) \( \tilde{\gamma}^\alpha_{\beta\mu} := \tilde{\gamma}^\alpha_{\beta\mu c} = C^\alpha_{\beta\mu c} + \Lambda^\alpha_{\beta\mu c} \)

(h) \( \tilde{P}^\alpha_{\beta\mu c} = C^\alpha_{\beta\mu c} + \Lambda^\alpha_{\beta\mu c} \)

(i) \( \tilde{T}^a_{\beta\mu c} = T^a_{\beta\mu c} + T^a_{\beta\mu c} \)

(j) \( \tilde{S}^a_{\beta c d} := \tilde{S}^a_{\beta c d} = - C^a_{\beta c d} \)

(k) \( \tilde{S}^a_{\beta c d} := \tilde{S}^a_{\beta c d} = - C^a_{\beta c d} \)

**Proof.** (b) is a consequence of the commutation formula (d) of Lemma 3.1 applied to the dual d-connection, taking into account (2.1), (2.18) and (2.19). (b) could be also obtained from Theorem 1.3 (b) and Theorem 3.2, noting that \( \Gamma^a_{\beta\mu c} = \tilde{\Gamma}^a_{\beta\mu c} \) and \( C^a_{\beta\mu c} = T^a_{\beta\mu c} + \tilde{C}^a_{\beta\mu c} \). We next prove (a) and (c) of the first part. The second part follows immediately by applying the suitable contractions.
(a) We have
\[ R^{\alpha}_{\beta\mu\nu} = \delta_{\nu} \tilde{\Gamma}^{\alpha}_{\beta\mu} - \delta_{\mu} \tilde{\Gamma}^{\alpha}_{\beta\nu} + \tilde{\Gamma}^{\alpha}_{\beta\mu} \Gamma_{\nu\varepsilon} - \tilde{\Gamma}^{\alpha}_{\beta\nu} \Gamma_{\mu\varepsilon} + \tilde{C}^{\alpha}_{\beta\nu} R_{\mu\nu} \\
= \delta_{\nu} \Gamma^{\alpha}_{\beta\mu} - \delta_{\mu} \Gamma^{\alpha}_{\beta\nu} + \Gamma^{\alpha}_{\beta\mu} \Gamma_{\nu\varepsilon} - \Gamma^{\alpha}_{\beta\nu} \Gamma_{\mu\varepsilon} + C^{\alpha}_{\beta\nu} R_{\mu\nu} \\
= \{ \delta_{\nu} \Gamma^{\alpha}_{\mu\beta} + \Gamma^{\alpha}_{\mu\beta}(\Lambda_{\nu\varepsilon} + \Lambda_{\nu\varepsilon}) \} - \{ \delta_{\mu} \Gamma^{\alpha}_{\nu\beta} + \Gamma^{\alpha}_{\nu\beta}(\Lambda_{\mu\varepsilon} + \Lambda_{\mu\varepsilon}) \} + C^{\alpha}_{\beta\nu} R_{\mu\nu} \\
= (R^{\alpha\beta}_{\mu\nu} - C^{\alpha\beta}_{\mu\nu} R^{\alpha}_{\mu\nu} + \beta_{\nu\beta} \Gamma^{\alpha}_{\mu\nu} + \Gamma^{\alpha}_{\nu\beta} \Gamma^{\beta}_{\mu\varepsilon}) - (R^{\alpha\beta}_{\nu\mu} - C^{\alpha\beta}_{\nu\mu} R^{\alpha}_{\nu\mu} + \beta_{\mu\beta} \Gamma^{\alpha}_{\nu\mu} + \Gamma^{\alpha}_{\nu\beta} \Gamma^{\beta}_{\nu\varepsilon}) + C^{\alpha}_{\beta\nu} R_{\mu\nu} \\
= \beta_{\nu\beta} \Lambda^{\alpha}_{\mu\nu} + \Gamma^{\alpha}_{\nu\beta} \Lambda^{\mu}_{\nu\varepsilon} - \Gamma^{\alpha}_{\nu\beta} \Lambda^{\mu}_{\nu\varepsilon} - \Gamma^{\alpha}_{\mu\beta} \Lambda^{\nu}_{\mu\varepsilon} + \gamma_{\beta\mu\nu} C^{\alpha}_{\beta\nu} R^{\alpha}_{\mu\nu} \\
= \beta_{\nu\beta} \Lambda^{a}_{\mu\nu} + \gamma_{\beta\mu\nu} C^{\alpha}_{\beta\nu} R^{\alpha}_{\mu\nu} \]

(c) We have
\[ \tilde{P}^{\alpha}_{\beta\mu c} = \hat{\partial}_{\varepsilon} \tilde{\Gamma}^{\alpha}_{\beta\mu\varepsilon} - \tilde{C}^{\alpha}_{\beta\varepsilon\mu} + \tilde{C}^{\alpha}_{\varepsilon\beta\mu} = \hat{\partial}_{\varepsilon} \Gamma^{\alpha}_{\beta\mu\varepsilon} - C^{\alpha}_{\beta\varepsilon\mu} + C^{\alpha}_{\varepsilon\beta\mu} P_{\mu\varepsilon} \\
= (\hat{\partial}_{\varepsilon} \Gamma^{\alpha}_{\beta\mu\varepsilon} - C^{\alpha}_{\beta\varepsilon\mu} + C^{\alpha}_{\varepsilon\beta\mu} P_{\mu\varepsilon}) + \hat{\partial}_{\varepsilon} \Lambda^{\alpha}_{\beta\mu\varepsilon} + (C^{\alpha}_{\beta\varepsilon\mu} - C^{\alpha}_{\varepsilon\beta\mu}) \\
= P^{\beta\mu}_{\alpha\varepsilon} + \hat{\partial}_{\varepsilon} \Lambda^{\alpha}_{\beta\mu\varepsilon} + C^{\alpha}_{\beta\mu\varepsilon} C^{\alpha}_{\varepsilon\mu\varepsilon} - C^{\alpha}_{\beta\mu\varepsilon} C^{\varepsilon}_{\beta\mu\varepsilon} \\
= \beta_{\varepsilon\beta} \Lambda^{\alpha}_{\mu\nu} + \gamma_{\beta\mu\varepsilon} C^{\alpha}_{\beta\mu\varepsilon} \]

Theorem 3.6. The non-vanishing curvature tensors of the symmetric $d$-connection $D$ can be expressed in the form

(a) \[ \tilde{R}^{\alpha}_{\beta\nu\mu} = \frac{1}{2} \left( \Lambda^{\alpha}_{\beta\mu\nu} - \Lambda^{\alpha}_{\beta\mu\nu} \right) + \frac{1}{4} \left( \Lambda^{\alpha}_{\beta\nu\nu} - \Lambda^{\alpha}_{\beta\mu\nu} \right) + \frac{1}{2} \left( \Lambda^{\varepsilon}_{\beta\mu\nu} \Lambda^{\alpha}_{\beta\nu\varepsilon} \right) , \]

(b) \[ \tilde{R}^{\alpha}_{\beta\nu\mu} = \frac{1}{2} \tilde{R}^{\alpha}_{\beta\nu\mu} , \]

(c) \[ \tilde{P}^{\alpha}_{\beta\varepsilon\mu} = \frac{1}{2} \tilde{P}^{\alpha}_{\beta\varepsilon\mu} \]

(d) \[ \tilde{P}^{\alpha}_{\beta\varepsilon\mu} = \frac{1}{2} \tilde{P}^{\alpha}_{\beta\varepsilon\mu} \]

(e) \[ \tilde{S}^{a}_{bcd} = \frac{1}{4} (T^{a}_{bc|d} - T^{a}_{bd|c}) + \frac{1}{4} (T^{c}_{bd} T^{a}_{de} - T^{c}_{bd} T^{a}_{de}) + \frac{1}{2} (T^{e}_{de} T^{a}_{ce}) . \]

Consequently,

(f) \[ \tilde{R}^{\alpha}_{\beta\nu\mu} := \tilde{R}^{\alpha}_{\beta\nu\mu} = \frac{1}{2} \tilde{R}^{\alpha}_{\beta\nu\mu} - \frac{1}{4} (C^{\alpha}_{\beta\nu\mu} + C^{\alpha}_{\beta\mu\nu}) , \]

(g) \[ \tilde{R} := g^{\beta\nu} \tilde{R}^{\alpha}_{\beta\nu\mu} = \frac{1}{2} \tilde{R} - \frac{1}{4} \Lambda^{\alpha\varepsilon} \Lambda^{\varepsilon}_{\beta\mu\nu} , \]

(h) \[ \tilde{P}^{\alpha}_{\beta\varepsilon\mu} := -\tilde{P}^{\alpha}_{\beta\varepsilon\mu} = \frac{1}{2} \tilde{P}^{\alpha}_{\beta\varepsilon\mu} \]

(i) \[ \tilde{P}^{\alpha}_{\beta\varepsilon\mu} := \tilde{P}^{\alpha}_{\beta\varepsilon\mu} = \frac{1}{2} \tilde{P}^{\alpha}_{\beta\varepsilon\mu} \]

(j) \[ \tilde{S}^{a}_{bd} := \tilde{S}^{a}_{bd} = \frac{1}{2} \tilde{S} - \frac{1}{4} (C^{a}_{b} T^{a}_{de} + T^{a}_{de} T^{a}_{de}) \]

(k) \[ \tilde{S} := g^{bd} \tilde{S}^{a}_{bd} = \frac{1}{2} \tilde{S} - \frac{1}{4} T^{a}_{de} T^{a}_{de} . \]

Proof. Similar to the proof of Theorems 3.4 and 3.5. \]
4. Wanas tensors (W-tensors)

The Wanas tensor, or simply the W-tensor, in the classical AP-geometry, is a tensor which measures the non-commutativity of covariant differentiations of the parallelization vector fields $\lambda$ with respect to the dual connection:

$$W^\alpha_{\beta\mu} := \lambda_\beta(\lambda^\alpha_{\nu|\mu} - \lambda^\alpha_{\mu|\nu})$$  (4.1)

This tensor explicitly contains the curvature and torsion tensors. The W-tensor was first defined by M. Wanas [12] and has been used by F. Mikhail and M. Wanas [5] to construct a geometric theory unifying gravity and electromagnetism (GFT: generalized field theory). The scalar Lagrangian function of the GFT is obtained by double contractions of the tensor $W^{\alpha\beta}_{\nu\mu}$. The symmetric part of the field equation obtained contains a second-order tensor representing the material distribution. This tensor is a pure geometric, not a phenomenological, object. The skew part of the field equation gives rise to Maxwell-like equations. The use of the W-tensor has thus aided to construct a geometric theory via one single geometric entity, which Einstein was seeking for [1]. Various significant applications (e.g. [12], [13], [15]) have supported such a theory. Recently, the authors of this paper investigated the most important properties of this tensor in the context of classical AP-geometry [17]. The W-tensor was also studied by the present authors in the context of generalized Lagrange spaces [16].

It should be noted that the W-tensor can be defined only in the context of AP-geometry and its generalized versions (cf. e.g. [16], [17]), since it is defined only in terms of the vector fields $\lambda$'s.

Due to its importance in physical applications, we are going to investigate the properties of the W-tensor in the present section. The W-tensor (4.1) can be generalized in the context of the EAP-geometry as follows.

**Definition 4.1.** Let $(TM, \lambda)$ be an EAP-space. For a given d-connection $D = (\Gamma^\alpha_{\mu\nu}, \Gamma^a_{\mu\nu}, C^\alpha_{\mu c}, C^a_{\mu c})$, the W-tensor is given by

$$W = (W^\alpha_{\beta\mu}, W^a_{\beta\mu}, W^\alpha_{\beta|\nu}, W^a_{\beta|\nu}, W^\alpha_{\beta|\nu|\nu}, W^a_{\beta|\nu|\nu})$$

(a) the hhh-tensor $W^\alpha_{\beta\mu}$ is defined by the formula

$$\lambda^\alpha_{|\nu|\mu} - \lambda^\alpha_{|\mu|\nu} = \lambda^\alpha W^\alpha_{\nu\mu},$$

(b) the hhv-tensor $W^a_{\beta\mu}$ is defined by the formula

$$\lambda^a_{|\nu|\mu} - \lambda^a_{|\mu|\nu} = \lambda^d W^a_{\nu\mu},$$

(c) the vhh-tensor $W^\alpha_{\beta|\nu}$ is defined by the formula

$$\lambda^\alpha_{|\nu|\nu} - \lambda^\alpha_{||\nu|\nu} = \lambda^e W^\alpha_{\nu|\nu},$$

(d) the vhh-tensor $W^a_{\beta|\nu}$ is defined by the formula

$$\lambda^a_{|\nu|\nu} - \lambda^a_{||\nu|\nu} = \lambda^d W^a_{\nu|\nu}.$$
(e) the vvh-tensor $W^\alpha_{\beta bc}$ is defined by the formula

$$\lambda^\alpha |||c|-\lambda^\alpha ||c|b = \lambda^\alpha W^\alpha_{ebc},$$

(f) the vvv-tensor $W^\alpha_{bed}$ is defined by the formula

$$\lambda^\alpha ||c|d|e = \lambda^\alpha W^\alpha_{edc},$$

where “|” and “||” are the $h$- and $v$-covariant derivatives with respect to the given $d$-connection $D$.

Theorem 2.1 together with (2.1), directly implies that the $W$-tensors of the canonical $d$-connection vanish identically.

In view of Theorem 3.4, we obtain

**Theorem 4.2.** The $W$-tensors corresponding to the natural metric $d$-connection $\tilde{D}$ are given by:

(a) $\tilde{W}^\alpha_{\beta\nu\mu} = (\gamma^\alpha_{\beta\nu|\mu} - \gamma^\alpha_{\beta\nu|b}) + (\gamma^\epsilon_{\beta\nu}, \gamma^\alpha_{\mu\nu} - \gamma^\alpha_{\beta\mu}, \gamma^\epsilon_{\nu\epsilon}) - \gamma^\alpha_{\beta\epsilon} \Lambda^\epsilon_{\nu\mu},$

(b) $\tilde{W}^\alpha_{\nu\beta\mu} = (\gamma^\alpha_{\nu\beta|\mu} - \gamma^\alpha_{\nu\beta|\nu}) + (\gamma^\nu_{\nu\beta|d} - \gamma^\nu_{\nu\beta|\nu}) - \gamma^\alpha_{\nu\beta} \Lambda^\epsilon_{\nu\mu},$

(c) $\tilde{W}^\alpha_{\beta\epsilon|\nu} = (\gamma^\alpha_{\beta\epsilon|\nu} - \gamma^\alpha_{\beta\epsilon|\nu}) + (\gamma^\epsilon_{\beta\nu} \gamma^\epsilon_{\nu\epsilon} - \gamma^\epsilon_{\beta\epsilon} \gamma^\epsilon_{\nu\epsilon}) + (\gamma^\epsilon_{\nu\epsilon} \gamma^\epsilon_{\nu\epsilon} - \gamma^\epsilon_{\beta\epsilon} \gamma^\epsilon_{\nu\epsilon}),$

(d) $\tilde{W}^\alpha_{\nu\epsilon|\beta} = (\gamma^\alpha_{\nu\epsilon|\beta} - \gamma^\alpha_{\nu\epsilon|\nu}) + (\gamma^\epsilon_{\nu\epsilon} \gamma^\epsilon_{\nu\epsilon} - \gamma^\epsilon_{\beta\epsilon} \gamma^\epsilon_{\nu\epsilon}) + (\gamma^\epsilon_{\nu\epsilon} \gamma^\epsilon_{\nu\epsilon} - \gamma^\epsilon_{\beta\epsilon} \gamma^\epsilon_{\nu\epsilon}),$

(e) $\tilde{W}^\alpha_{\nu|b} = \tilde{S}^\alpha_{\nu\beta c},$

(f) $\tilde{W}^\alpha_{ced} = \tilde{S}^\alpha_{ced}.$

**Proof.** We prove (c) only. The other formulae are derived in a similar manner. By definition, we have

$$\lambda^\epsilon \tilde{W}^\alpha_{\nu|e} = \lambda^\epsilon \tilde{P}^\alpha_{\nu|e} - \tilde{C}^\epsilon_{\nu|e} \lambda^\alpha_{\epsilon|\nu} - \tilde{P}^d_{\nu|e} \lambda^\alpha_{\epsilon|d}. $$

Consequently, by (2.11), (2.12) and (2.13), we obtain

$$\tilde{W}^\alpha_{\beta\nu\mu} = \tilde{P}^\alpha_{\beta\nu\mu} - (\lambda^\alpha_{\beta\nu|\mu})(C^\alpha_{\nu|e} - \gamma^\epsilon_{\nu|e}) - (\lambda^\alpha_{\beta\nu|\mu})(P^\alpha_{\nu|e} + \gamma^d_{\nu|e}),$$

$$= \tilde{P}^\alpha_{\beta\nu\mu} + C^\alpha_{\nu|e} \gamma^\epsilon_{\nu|\mu} + P^d_{\nu|e} \gamma^\epsilon_{\nu|\mu} - (\gamma^\epsilon_{\nu|e} \gamma^\epsilon_{\nu|\mu}) - \gamma^\epsilon_{\nu|e} \gamma^\epsilon_{\nu|\mu},$$

$$= (\gamma^\alpha_{\beta\nu|\mu} - \gamma^\alpha_{\beta\nu|\nu}) + (\gamma^\epsilon_{\beta\nu} \gamma^\epsilon_{\epsilon|\nu} - \gamma^\epsilon_{\beta\epsilon} \gamma^\epsilon_{\epsilon|\nu}) + (\gamma^\epsilon_{\epsilon|\nu} \gamma^\epsilon_{\epsilon|\nu} - \gamma^\epsilon_{\beta\epsilon} \gamma^\epsilon_{\epsilon|\nu}),$$

Since $\lambda^\alpha_{\tilde{\mu}} = \lambda^\alpha_{\tilde{\nu}} = \lambda^\alpha_{\tilde{\epsilon}} = 0$, it follows, by definition, that

$$\tilde{W}^\alpha_{\beta\nu\mu} = \tilde{W}^\alpha_{\beta\nu|b} = \tilde{W}^\alpha_{\beta\nu\mu} = \tilde{W}^\alpha_{\beta\nu|\nu} = \tilde{W}^\alpha_{\beta\nu|\nu} = 0.$$

Proceeding as in Theorem 4.2 taking into account Theorem 3.5 and Theorem 3.6, we have the following
Theorem 4.3. The non-vanishing $W$-tensors corresponding to the dual $d$-connection $\bar{D}$ are given by:

(a) $\bar{W}_{\beta \nu \mu}^\alpha = \Lambda_{\nu \mu}^{\beta | \beta} + \Lambda_{\nu \mu}^{\epsilon} \Lambda_{\beta | \epsilon}^{\alpha} + \mathcal{S}_{\nu, \mu, \beta} C_{\beta a}^\alpha R_{a \nu \mu}^\alpha$,
(b) $\bar{W}_{\beta \nu \mu}^\alpha = \Lambda_{\nu \beta || \gamma}^\alpha$,
(c) $\bar{W}_{\beta \nu \mu}^a = T_{\beta \nu \mu}^a$,
(d) $\bar{W}_{\beta \nu \mu}^a = T_{\beta \nu \mu}^a + T_{\beta \mu \nu}^e T_{\beta e}^a$.

Theorem 4.4. The non-vanishing $W$-tensors corresponding to the symmetric $d$-connection $\hat{D}$ are given by:

(a) $\hat{W}_{\beta \nu \mu}^\alpha = \hat{R}_{\beta \nu \mu}^\alpha$,
(b) $\hat{W}_{\beta \nu \mu}^a = \frac{1}{2} \hat{W}_{\beta \nu \mu}^a$,
(c) $\hat{W}_{\beta \nu \mu}^a = \frac{1}{2} \hat{W}_{\beta \nu \mu}^a$,
(d) $\hat{W}_{\beta \nu \mu}^a = \hat{S}_{\beta \nu \mu}^a$.

It is clear by the above theorem that the $W$-tensors corresponding to the symmetric $d$-connection give no new $d$-tensor fields.

Remark 4.5. The $W$-tensors corresponding to a given $d$-connection can be also defined covariantly in the form

$$\lambda_{|\beta | \nu} - \lambda_{|\beta | \mu} = \lambda_{|\gamma | \nu} W_{|\beta \nu \mu}^\alpha,$$

with similar expressions for the other counterparts. These expressions give the same formulae (up to a sign) for the $W$-tensors obtained in Theorems 4.2, 4.3 and 4.4.

Proposition 4.6. The following identities hold:

(a) $\mathcal{G}_{\beta, \mu, \nu} \hat{W}_{\beta \nu \mu}^\alpha = \mathcal{G}_{\beta, \mu, \nu} \bar{W}_{\beta \nu \mu}^\alpha = \mathcal{G}_{\beta, \mu, \nu} R_{\mu \beta}^\alpha C_{\nu a}^\alpha$
(b) $\mathcal{G}_{\beta, \mu, \nu} \tilde{W}_{\beta \nu \mu}^\alpha = 2 \mathcal{G}_{\beta, \mu, \nu} R_{\mu \beta}^\alpha C_{\nu a}^\alpha$

Proof. By Theorem 4.2 we have

$$\mathcal{G}_{\beta, \mu, \nu} \hat{W}_{\beta \nu \mu}^\alpha = \mathcal{G}_{\beta, \mu, \nu} (\gamma_{|\beta | \nu}^\alpha - \gamma_{|\beta | \mu}^\alpha) + \mathcal{G}_{\beta, \mu, \nu} (\gamma_{|\beta | \nu}^\alpha \Lambda_{|\nu | \mu}^\epsilon + \gamma_{|\beta | \epsilon}^\alpha \gamma_{|\epsilon | \mu}^\alpha)$$
$$= \mathcal{G}_{\beta, \mu, \nu} (\gamma_{|\beta | \nu}^\alpha - \gamma_{|\beta | \mu}^\alpha + \gamma_{|\beta | \epsilon}^\alpha \gamma_{|\epsilon | \mu}^\alpha) + \mathcal{G}_{\beta, \mu, \nu} (\gamma_{|\beta | \nu}^\alpha \Lambda_{|\nu | \mu}^\epsilon + \gamma_{|\beta | \epsilon}^\alpha \gamma_{|\epsilon | \mu}^\alpha)$$
$$= \mathcal{G}_{\beta, \mu, \nu} R_{\mu \beta}^\alpha C_{\nu a}^\alpha$$

where in the last step we have used (3.3). The proof of the other part of (a) is achieved by applying the first Bianchi identity to the symmetric $d$-connection taking into account Theorem 4.4 (a) together with the fact that $C_{\nu a}^\alpha = C_{\nu a}^\alpha$. The proof of (b) is carried out in a similar manner, again by using (3.3), taking into consideration Theorem 4.3 (a).
Summing up, the EAP-space has three distinct $W$-tensors (corresponding to the natural metric, dual and symmetric $d$-connections), each with six counterparts. Eight only out of the eighteen are independent, four coincide with the corresponding curvature tensors and four vanish identically.

5. Cartan-type case

A drawback in the construction of the EAP-space is the fact that the nonlinear connection is assumed to exist a priori, independently of the vector fields $\lambda$’s defining the parallelization. It would be more natural and less arbitrary if the nonlinear connection were expressed in terms of these vector fields. In this case, all geometric objects of the EAP-space will be defined solely in terms of the building blocks of the space. Below, we impose an extra condition on the canonical $d$-connection, namely, being of Cartan type. The outcome of such a condition is the accomplishment of our requirement, besides many other interesting results.

We first recall the definition of a Cartan type $d$-connection [7], [8].

**Definition 5.1.** A $d$-connection $D = (\Gamma^\alpha_{\mu\nu}, \Gamma^a_{\mu c}, C^a_{\mu c}, C^a_{bc})$ on $TM$ is said to be of Cartan type if

$$D^h_v C = 0; \quad D^v_X C = vX; \quad \forall X \in \mathfrak{X}(TM),$$

where $C = y^a \hat{\partial}_a$ is the Liouville vector field.

Locally, the above conditions are expressed in the form

$$y^a|_\mu = 0, \quad y^a|_c = \delta^a_c,$$  \hspace{1cm} (5.1)

or, equivalently,

$$N^a_\mu = y^b \Gamma^a_{b \mu}, \quad y^b C^a_{bc} = 0.$$ \hspace{1cm} (5.2)

**Proposition 5.2.** If a $d$-connection $D$ is of Cartan type, then the following identities involving the torsions and curvatures hold:

$$R^a_{\mu\nu} = y^b R^a_{b\nu\mu}, \quad P^a_{\mu c} = y^b P^a_{b\mu c}, \quad T^a_{bc} = y^d S^a_{dcb}.$$ \hspace{1cm} (5.3)

**Proof.** Follows by applying the commutation formulae (d), (e) and (f) of Lemma 3.1 to the vector field $y^a$. \hfill $\square$

**Theorem 5.3.** Let $(TM, \lambda)$ be an EAP-space. Assume that the canonical $d$-connection $D$ is of Cartan type. Then we have:

(a) The expression $y^b(\lambda^a \partial_\mu \lambda_b)$ represents the coefficients of a nonlinear connection which coincides with the given nonlinear connection $N^a_\mu$: $N^a_\mu = y^b(\lambda^a \partial_\mu \lambda_b)$\footnote{A similar expression is found in “ArXiv: 0801.1132 [gr-qc]]”, but in a completely different situation.}

(b) $R^a_{\mu\nu} = P^a_{\mu c} = T^a_{bc} = 0$.

Consequently, $C^a_{bc}$ is symmetric, $\gamma^a_{bc} = 0$ and $C^a_{bc} = \widehat{C}^a_{bc} = \hat{C}^a_{bc} = \tilde{C}^a_{bc}$. 

Proof. We have

\( \gamma^a_{\mu \nu} = 0. \) Consequently, \( \hat{\Gamma}^a_{\mu \nu} = \Gamma^a_{\mu \nu}. \)

\( P^a_{\mu \nu} = 0. \)

\( \tilde{R}^a_{b \mu \nu} = \tilde{P}^a_{b \mu \nu} = \tilde{S}^a_{b c d} = 0, \)

\( R^a_{\mu \nu} = P^a_{\mu \nu} = T^a_{b c} = \gamma^a_{\nu \mu} = \gamma^a_{\mu \nu} = 0, \)

Taking into consideration Proposition 4.6 and Theorems 3.4, 3.5, 3.6, 4.2, 4.3 and 4.4.

**Corollary 5.4.** If the canonical d-connection is of Cartan type, then \( \tilde{D}, \hat{D} \) and \( \check{D} \) are also of Cartan type.

In what follows, we assume that the canonical d-connection is of Cartan type. The next two results are immediate consequences of the fact that

\( R^a_{\mu \nu} = P^a_{\mu \nu} = T^a_{b c} = \gamma^a_{\nu \mu} = \gamma^a_{\mu \nu} = 0, \)

The following relations hold:

\( \tilde{R}^a_{b \mu \nu} = \tilde{P}^a_{b \mu \nu} = \tilde{S}^a_{b c d} = 0, \)
\(\tilde{R}^\alpha_{\beta\mu\nu} = \Lambda^\alpha_{\nu\mu|\beta}\)

\(\tilde{R}^a_{b\mu\nu} = \tilde{R}^a_{b\mu\nu} = \tilde{P}^a_{b\mu\nu} = \tilde{S}^a_{bcd} = \tilde{S}^a_{bcd} = 0\).

\(\tilde{W}^\alpha_{\beta\mu\nu} = \tilde{R}^\alpha_{\beta\mu\nu},\)

\(\tilde{W}^a_{b\mu\nu} = \tilde{W}^a_{b\mu\nu} = \tilde{W}^a_{bcd} = \tilde{W}^a_{bcd} = \tilde{W}^a_{bcd} = 0,\)

\(\tilde{W}^a_{\beta\mu\nu} = \tilde{W}^a_{\beta\mu\nu} = \tilde{W}^a_{\beta\mu\nu} = 0.\)

Consequently, \(\tilde{W}^\alpha_{\beta\mu\nu}, \tilde{W}^a_{\beta\mu\nu}\) and \(\tilde{W}^a_{\beta\mu\nu}\) satisfy the first Bianchi identity of the Riemannian curvature tensor.

**Theorem 5.6.** The independent non-vanishing W-tensors are given by:

(a) \(\tilde{W}^\alpha_{\beta\mu\nu} = (\gamma^\alpha_{\beta\nu}) + (\gamma^\epsilon_{\beta\nu}) - (\gamma^\epsilon_{\beta\mu} - \gamma^\epsilon_{\beta\nu})\)

(b) \(\tilde{W}^a_{\beta\mu\nu} = \Lambda^a_{\nu\mu|\beta} + \Lambda^\epsilon_{\nu\mu} \Lambda^a_{\beta\epsilon},\)

(c) \(\tilde{W}^a_{\beta\mu\nu} = \Lambda^a_{\mu\beta|\nu} - \Lambda^\epsilon_{\mu\beta} \Lambda^a_{\nu\epsilon} - \Lambda^\epsilon_{\nu\mu} \Lambda^a_{\beta\epsilon}.\)

To sum up, the assumption that the canonical d-connection being of Cartan type implies that all the geometric objects defined in the EAP-space are expressed in terms of the vector fields \(\lambda\)'s only. The curvature of the nonlinear connection vanishes and the three other defined d-connections of the EAP-space, namely the dual, symmetric and the natural metric d-connections, are also of Cartan type. Moreover, there are only seven non-vanishing curvature tensors and only three independent non-vanishing W-tensors, some of which have simpler expressions than that obtained in the general case. The EAP-space becomes richer as new relations among its various geometric objects - which are not valid in the general case - emerge. Accordingly, the EAP-space in this case becomes more tangible, thus more suitable for physical applications.

We end this section with the following table.

**Table 3: EAP-geometry under the Cartan type case**

| Connection   | Coefficients | Torsion       | Curvature               |
|--------------|--------------|---------------|-------------------------|
| Canonical    | \((\Gamma^\alpha_{\mu\nu}, \hat{\partial}_b N^\alpha_{\nu}, C^\alpha_{\mu\nu}, C^\alpha_{\beta\mu})\) | \((\Lambda^\alpha_{\nu\mu|\beta}, 0, C^\alpha_{\mu\nu}, 0, 0)\) | \((0, 0, 0, 0, 0)\) |
| Dual         | \((\Gamma^\alpha_{\nu\mu}, \hat{\partial}_b N^\alpha_{\nu}, C^\alpha_{\mu\nu}, C^\alpha_{\beta\mu})\) | \((-\Lambda^\alpha_{\nu\mu}, 0, C^\alpha_{\mu\nu}, 0, 0)\) | \((\tilde{R}^\alpha_{\beta\mu\nu}, 0, \tilde{P}^\alpha_{\beta\mu\nu}, 0, 0, 0)\) |
| Symmetric    | \((\Gamma^\alpha_{\mu\nu}, \hat{\partial}_b N^\alpha_{\nu}, C^\alpha_{\mu\nu}, C^\alpha_{\beta\mu})\) | \((0, 0, C^\alpha_{\mu\nu}, 0, 0)\) | \((\tilde{R}^\alpha_{\beta\mu\nu}, 0, \tilde{P}^\alpha_{\beta\mu\nu}, 0, 0, 0)\) |
| Natural      | \((\tilde{\Gamma}^\alpha_{\mu\nu}, \hat{\partial}_b N^\alpha_{\nu}, C^\alpha_{\mu\nu}, C^\alpha_{\beta\mu})\) | \((0, 0, 0, C^\alpha_{\mu\nu}, 0, 0)\) | \((\tilde{R}^\alpha_{\beta\mu\nu}, 0, \tilde{P}^\alpha_{\beta\mu\nu}, 0, \tilde{S}^\alpha_{\beta\mu\nu}, 0)\) |
6. Berwald-type case

In this section we assume that the canonical $d$-connection $D$ is of Berwald type [8], [10]. The consequences of this assumption are investigated.

**Definition 6.1.** A $d$-connection $D = (\Gamma^\alpha_{\mu\nu}, \Gamma^a_{bp}, C^\alpha_{\mu c}, C^a_{\mu c})$ on $TM$ is said to be of Berwald type if

\[ \dot{\partial}_b N^a_\mu = \Gamma^a_{bp}; \quad C^\alpha_{\mu c} = 0. \] (6.1)

**Proposition 6.2.** If the canonical $d$-connection $D$ is of Cartan type such that $C^\alpha_{\mu c} = 0$, then it is of Berwald type.

**Theorem 6.3.** Let $(TM, \lambda)$ be an EAP-space. Assume that the canonical $d$-connection $D$ is of Berwald type. Then we have:

(a) $P^a_{\mu b} = 0$

(b) $\lambda_\mu$ are functions of the positional argument $x$ only. Consequently, so are $g_{\mu\nu}$.

(c) $\dot{\omega}^\alpha_{\mu c} = 0$. Consequently, $\gamma^\alpha_{\mu c} = 0$.

(d) The coefficients $\Gamma^\alpha_{\mu\nu}$ and $\dot{\Gamma}^\alpha_{\mu\nu}$ are functions of the positional argument $x$ only and are given respectively by

\[ \Gamma^\alpha_{\mu\nu}(x) = (\lambda^\alpha_i \partial_\nu \lambda_\mu_i)(x), \quad \dot{\Gamma}^\alpha_{\mu\nu}(x) = \frac{1}{2} g^{\alpha\epsilon}(\partial_\mu g_{\nu\epsilon} + \partial_\nu g_{\mu\epsilon} - \partial_\epsilon g_{\mu\nu})(x). \]

(e) $\Lambda^\alpha_{\mu\nu}$ and $\gamma^\alpha_{\mu\nu}$ are functions of the positional argument $x$ only.

(f) $\gamma^a_{bp} = \dot{P}^a_{bp} = 0$. Consequently, $\Gamma^a_{bp} = \dot{\Gamma}^a_{bp}$.

**Proof.** The proof is straightforward except for the relation $\gamma^a_{bp} = 0$, which can be proved in exactly the same manner as (f) of Theorem 5.3.

**Corollary 6.4.** If the canonical $d$-connection $D$ is of Berwald type, then $\hat{D}$, $\tilde{D}$ and $\hat{D}$ are also of Berwald type.

In what follows, we assume that the canonical $d$-connection is of Berwald type. The next two results are immediate consequences of the fact that

\[ P^a_{\mu c} = C^\alpha_{\mu c} = \gamma^\alpha_{\mu c} = \gamma^a_{bp} = 0, \]

taking into account Theorems 3.4 3.5 3.6 4.2 4.3 and 4.4.

**Proposition 6.5.** The following relations hold:

(a) $\tilde{P}^a_{bp\nu} = \dot{\gamma}^a_{bd} P^d_{p\nu}, \quad \tilde{P}^a_{\beta c} = \dot{\omega}^a_{\beta ed} = 0$,

(b) $\tilde{P}^a_{\beta p\mu} = \Lambda^\alpha_{\mu \beta \mid \beta}, \quad \tilde{P}^a_{\beta c} = \tilde{P}^a_{\beta p\mu} = 0, \quad \tilde{P}^a_{bp\mu} = W^a_{bp\mu}$.
(c) \( \tilde{W}_{b\nu}^a = \tilde{W}_{\beta c}^a = \tilde{W}_{\beta c}^a = \tilde{W}_{\beta c}^a = 0. \)

**Theorem 6.6.** The independent non-vanishing \( W \)-tensors are given by:

(a) \( \check{W}_{\alpha \beta \nu \mu}^a = (\gamma_{\beta \nu}^{\alpha} - \gamma_{\beta \nu}^{\alpha}) + (\gamma_{\beta \nu}^{\alpha} \gamma_{\nu \mu}^{\alpha} - \gamma_{\beta \nu}^{\alpha} \gamma_{\nu \mu}^{\alpha}) - \gamma_{\beta \nu}^{\alpha} \Lambda_{\nu \mu}^{\epsilon}, \)

(b) \( \check{W}_{\alpha \beta \nu}^a = \gamma_{\alpha \beta \nu}^{a}, \)

(c) \( \tilde{W}_{\alpha \beta \nu \mu}^a = \Lambda_{\alpha \nu \beta}^{a} + \Lambda_{\alpha \nu}^{a} \Lambda_{\nu \beta}^{a}, \)

(d) \( \tilde{W}_{\alpha \beta \nu}^a = T_{\alpha \beta \nu}^a + T_{\alpha \beta \nu}^a. \)

To sum up, the assumption that the canonical \( d \)-connection being of Berwald type implies that most of the purely horizontal geometric objects of the EAP-space become functions of the positional argument \( x \) only and coincide with the corresponding geometric objects of the classical AP-space. Moreover, the three other defined \( d \)-connections turn out also to be of Berwald type. Finally, in this case, there are twelve non-vanishing curvature tensors and four independent \( W \)-tensors.

We end this section with the following table (compare with Table 3).

**Table 4: EAP-geometry under the Berwald type case**

| Connection  | Coefficients | Torsion         | Curvature          |
|------------|--------------|-----------------|--------------------|
| **Canonical** | (\( \Gamma_{\alpha \mu \nu}^{a}, \hat{\partial}_{\beta} \Gamma_{\nu}^{a}, 0, \hat{C}_{bc}^{a} \)) | (\( \Lambda_{\alpha \mu \nu}^{a}, R_{\alpha \mu \nu}^{a}, 0, 0, T_{\alpha}^{a} \)) | (0, 0, 0, 0, 0) |
| **Dual**   | (\( \Gamma_{\alpha \mu \nu}^{a}, \hat{\partial}_{\beta} \Gamma_{\nu}^{a}, 0, \hat{C}_{cb}^{a} \)) | (\( -\Lambda_{\alpha \mu \nu}^{a}, R_{\alpha \mu \nu}^{a}, 0, 0, -T_{\alpha}^{a} \)) | (\( \tilde{R}_{\alpha \mu \nu}^{a}, \tilde{R}_{\alpha \mu \nu}^{a}, 0, \tilde{\tilde{R}}_{\alpha \beta \nu}^{a}, 0, \tilde{R}_{bc}^{a} \)) |
| **Symmetric** | (\( \Gamma_{(\mu \nu)}^{a}, \hat{\partial}_{\beta} \Gamma_{(\nu}^{a}, 0, \hat{C}_{(bc)}^{a} \)) | (0, \( R_{\mu \nu}^{a}, 0, 0, 0 \)) | (\( \check{R}_{\beta \mu \nu}^{a}, \check{R}_{\beta \mu \nu}^{a}, 0, \check{C}_{bc}^{a} \)) |
| **Natural** | (\( \Gamma_{(\mu \nu)}^{a}, \hat{\partial}_{\beta} \Gamma_{(\nu}^{a}, 0, \hat{C}_{bc}^{a} \)) | (0, \( \check{R}_{\mu \nu}^{a}, 0, 0, 0 \)) | (\( \check{\check{R}}_{\beta \mu \nu}^{a}, \check{\check{R}}_{\beta \mu \nu}^{a}, 0, \check{\check{R}}_{bc}^{a} \)) |

**7. Recovering the classical AP-space**

We now assume that the canonical \( d \)-connection \( D \) is both Cartan and Berwald type. In view of Proposition 6.2, this condition is equivalent to the (apparently weaker) condition that \( D \) is of Cartan type and \( C_{\mu \nu}^{a} = 0 \). We show that in this case the classical AP-space emerges, in a natural way, as a special case from the EAP-space. We refer to this condition as the \( \text{CB} \)-condition.

\[ \text{CB-condition:} \ N_{\mu}^{a} = y^{b} \Gamma_{\mu b}^{a}, \quad y^{b} C_{bc}^{a} = 0; \quad C_{\mu \nu}^{a} = 0. \]

As easily checked, we have

**Theorem 7.1.** Assume that the \( \text{CB} \)-condition holds. Then
The four defined \(d\)-connections of the EAP-space coincide up to the \(hh\)-coefficients. Moreover, these \(hh\)-coefficients are functions of the positional argument \(x\) only and are identical to the coefficients of the corresponding four defined connections in the classical AP-space.

The torsion of the canonical \(d\)-connection and the contortion of the EAP-space are functions of the positional argument \(x\) only and are given by

\[
T = (\Lambda^\alpha_{\mu\nu}, 0, 0, 0); \quad C = (\gamma^\alpha_{\mu\nu}, 0, 0, 0)
\]

The three non-vanishing curvature tensors are functions of the positional argument \(x\) only and are given by

\[
\begin{align*}
(a) \quad \hat{R}^\alpha_{\beta\mu\nu} &= (\gamma^\alpha_{\beta\nu |\mu} - \gamma^\alpha_{\beta\mu |\nu}) + (\gamma^\epsilon_{\beta\nu} \gamma^\alpha_{\epsilon\mu} - \gamma^\epsilon_{\beta\mu} \gamma^\alpha_{\epsilon\nu}) + \gamma^\alpha_{\beta\epsilon} \Lambda^\epsilon_{\mu\nu} \\
(b) \quad \tilde{R}^\alpha_{\beta\mu\nu} &= \Lambda^\alpha_{\nu\mu |\beta} \\
(c) \quad \hat{R}^\alpha_{\beta\mu\nu} &= \frac{1}{2} (\Lambda^\alpha_{\beta\nu |\mu} - \Lambda^\alpha_{\beta\mu |\nu}) + \frac{1}{4} (\Lambda^\epsilon_{\beta\nu} \Lambda^\alpha_{\epsilon\mu} - \Lambda^\epsilon_{\beta\mu} \Lambda^\alpha_{\epsilon\nu}) + \frac{1}{2} (\Lambda^\epsilon_{\beta\epsilon} \Lambda^\alpha_{\epsilon\mu})
\end{align*}
\]

There is only one \(W\)-tensor which is a function of the positional argument \(x\) only and is given by

\[
\tilde{W}^\alpha_{\beta\mu\nu} = \Lambda^\alpha_{\nu\mu |\beta} + \Lambda^\epsilon_{\nu\mu} \Lambda^\alpha_{\beta\epsilon}
\]

All other \(W\)-tensors vanish identically, or coincide with the corresponding curvature tensors.

Consequently, the fundamental geometric objects of the EAP-space coincide with the corresponding fundamental geometric objects of the classical AP-space [17].

**Corollary 7.2.** If the canonical \(d\)-connection \(D\) satisfies the CB-condition, then \(\tilde{D}, \hat{D}\) and \(\hat{D}\) also satisfy the CB-condition.

We end this section with the following two tables which summarize the geometry of the EAP-space under the CB-condition.

**Table 5: Fundamental connections of EAP-space under the CB-condition**

| Connection     | Coefficients          | hh-Coefficients               |
|----------------|-----------------------|-------------------------------|
| Canonical      | \((\Gamma^\alpha_{\mu\nu}, \partial_{\beta}N^\alpha_{\nu}, 0, C^a_{bc})\) | \(\Gamma^\alpha_{\mu\nu}(x) = (\Lambda^\alpha_{\nu\mu})L^\alpha_{\mu}(x)\) |
| Dual           | \((\tilde{\Gamma}^\alpha_{\mu\nu}, \partial_{\beta}N^\alpha_{\nu}, 0, C^a_{bc})\) | \(\tilde{\Gamma}^\alpha_{\mu\nu}(x) = \Lambda^\alpha_{\nu\mu}(x)\) |
| Symmetric      | \((\hat{\Gamma}^\alpha_{\mu\nu}, \partial_{\beta}N^\alpha_{\nu}, 0, C^a_{bc})\) | \(\hat{\Gamma}^\alpha_{\mu\nu}(x) = \Lambda^\alpha_{\nu\mu}(x)\) |
| Natural        | \((\check{\Gamma}^\alpha_{\mu\nu}, \partial_{\beta}N^\alpha_{\nu}, 0, C^a_{bc})\) | \(\check{\Gamma}^\alpha_{\mu\nu}(x) = \frac{1}{2} g^\alpha_{\tau\nu} (\partial_{\mu}g_{\nu\tau} + \partial_{\nu}g_{\mu\tau} - \partial_{\tau}g_{\mu\nu})(x)\) |
Table 6: EAP-geometry under the CB-condition

| Connection   | Coefficients                                      | Torsion       | Curvature       |
|--------------|---------------------------------------------------|---------------|-----------------|
| Canonical    | $(\Gamma^\alpha_{\mu\nu}, \dot{\partial}_b N^{a}_{\nu}, 0, C^{a}_{bc})$ | $(\Lambda^\alpha_{\mu\nu}, 0, 0, 0, 0)$ | $(0, 0, 0, 0, 0)$ |
| Dual         | $(\Gamma^\alpha_{\nu\mu}, \dot{\partial}_b N^{a}_{\nu}, 0, C^{a}_{bc})$ | $(-\Lambda^\alpha_{\mu\nu}, 0, 0, 0, 0)$ | $(\tilde{R}^\alpha_{\beta\mu\nu}, 0, 0, 0, 0)$ |
| Symmetric    | $(\Gamma^\alpha_{(\mu\nu)}, \dot{\partial}_b N^{a}_{\nu}, 0, C^{a}_{bc})$ | $(0, 0, 0, 0, 0)$ | $(\tilde{R}^\alpha_{\beta\mu\nu}, 0, 0, 0, 0)$ |
| Natural      | $(\Gamma^\alpha_{\mu\nu}, \dot{\partial}_b N^{a}_{\nu}, 0, C^{a}_{bc})$ | $(0, 0, 0, 0, 0)$ | $(\tilde{R}^\alpha_{\beta\mu\nu}, 0, 0, 0, 0)$ |

Some comments on the CB-condition:

(a) It should be noted that the non-vanishing of the purely vertical tensors $\lambda^a$, $C^a_{bc}$, and $g_{ab}$ and the $h\nu$-coefficients $\Gamma^a_{b\mu}$ of the canonical $d$-connection may represent extra degrees of freedom which do not exist in the classical AP-context. Moreover, these vertical geometric objects still depend on the directional argument $y$. However, they actually do not contribute to the EAP-geometry under the CB-condition. This is because the torsion and the contortion tensors in this case have only one non-vanishing counterpart each, namely the purely horizontal components $\Lambda^\alpha_{\mu\nu}$ and $\gamma^\alpha_{\mu\nu}$ respectively.

(b) One reading of Theorem 7.1 is roughly that the “projection” of the geometric objects of the EAP-space on the base manifold $M$ can be identified with the classical AP-geometry. The distinction which appears between the two geometries is due to the fact that the geometric objects of the EAP-space live in the double tangent bundle $TTM \rightarrow TM$ and not in the tangent bundle $TM \rightarrow M$. Consequently, it can be said, roughly speaking, that the classical AP-space is a copy of the EAP-space equipped with the CB-condition, viewed from the perspective of the base manifold $M$.

8. Concluding remarks

In the present article, we have constructed and developed a parallelizable structure analogous to the AP-geometry on the tangent bundle $TM$ of $M$. Four linear connections, depending on one a priori given nonlinear connection, are used to explore the properties of this geometry. Different curvature tensors characterizing this structure, together with their contractions, are computed. The different $W$-tensors are also derived. Extra conditions are imposed on the canonical $d$-connection, the consequences of which are investigated. Finally, a special form of the canonical $d$-connection is introduced under which the EAP-geometry reduces to the classical AP-geometry.

On the present work, we have the following comments:
(1) Existing theories of gravity suffer from some problems connected to recent observed astrophysical phenomena, especially those admitting anisotropic behavior of the system concerned (e.g. the flatness of the rotation curves of spiral galaxies). So, theories in which the gravitational potential depends on both position and direction may be needed. Such theories are to be constructed in spaces admitting this dependence; a potential candidate is the EAP-space. This is one of the aims motivating the present work.

(2) One possible physical application of the EAP-geometry would be the construction of a generalized field theory on the tangent bundle $TM$ of $M$. This could be achieved by a double contraction of the purely horizontal $W$-tensor $W^{\alpha \beta \mu \nu}$ and the purely vertical $W$-tensor $W_{ab \, cd}$ to obtain respectively the “horizontal” scalar Lagrangian $H := |\lambda| H := |\lambda| g^{\alpha \beta} H_{\alpha \beta}$, where $|\lambda| := \det(\lambda)$ and

$$H_{\alpha \beta} := \Lambda^\nu_{\alpha} \Lambda^\epsilon_{\beta} - C_\alpha C_\beta$$

and the “vertical” scalar Lagrangian $V := ||\lambda|| V := ||\lambda|| g^{ab} V_{ab}$, where $||\lambda|| := \det(\lambda)$ and

$$V_{ab} := T^d_{ea} T^e_{db} - C_a C_b.$$  

The field equations are obtained by the use of the Euler-Lagrange equations

$$\frac{\partial H}{\partial \lambda_\beta} - \frac{\partial}{\partial x^e} \left( \frac{\partial H}{\partial \lambda_\beta, e} \right) - \frac{\partial}{\partial y^e} \left( \frac{\partial H}{\partial \lambda_\beta, e} \right) = 0, \quad \text{(horizontal form)}$$

$$\frac{\partial V}{\partial \lambda_b} - \frac{\partial}{\partial x^e} \left( \frac{\partial V}{\partial \lambda_b, e} \right) - \frac{\partial}{\partial y^e} \left( \frac{\partial V}{\partial \lambda_b, e} \right) = 0. \quad \text{(vertical form)}$$

The resulting field equations could be compared with those derived by M. Wanas [12] and R. Miron [6].

(3) Among the advantages of the classical AP-geometry are the ease in calculations and the diverse and its thorough applications [14]. For these reasons, we have kept, in this work, as close as possible to the classical AP-case. However, the extra degrees of freedom in our EAP-geometry have created an abundance of geometric objects which have no counterpart in the classical AP-geometry. Since the physical meaning of most of the geometric objects of the classical AP-structure is clear, we hope to attribute physical meaning to the new geometric objects appearing in the present work. The physical interpretation of the geometric objects existing in the EAP-space and not in the AP-geometry may need deeper investigation. The study of the Cartan type case, due to its simplicity, may be our first step in tackling the general case.

(4) In conclusion, we hope that physicists working in AP-geometry would divert their attention to the study of EAP-geometry and its consequences, due to its wealth, relative simplicity and its close resemblance (at least in form) to the classical AP-geometry. We believe that the extra degrees of freedom offered by the EAP-geometry may give us more insight into the infrastructure of physical phenomena studied in the context of classical AP-geometry and thus help us better understand the theory of general relativity and its connection to other physical theories.
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