GLOBAL LARGE SMOOTH SOLUTIONS FOR 3-D
HALL-MAGNETOHYDRODYNAMICS

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(Communicated by Fanghua Lin)

ABSTRACT. In this paper, the global smooth solution of Cauchy’s problem of incompressible, resistive, viscous Hall-magnetohydrodynamics (Hall-MHD) is studied. By exploring the nonlinear structure of Hall-MHD equations, a class of large initial data is constructed, which can be arbitrarily large in $H^3(\mathbb{R}^3)$. Our result may also be considered as the extension of work of Lei-Lin-Zhou [15] from the second-order semilinear equations to the second-order quasilinear equations, because the Hall term elevates the Hall-MHD system to the quasilinear level.

1. Introduction. In this paper we consider the following incompressible, resistive, viscous Hall-MHD equations

$$\begin{cases}
u u_t + u \cdot \nabla u + \nabla p = \nu \Delta u + b \cdot \nabla b, \\b_t + u \cdot \nabla b - b \cdot \nabla u + \eta \nabla \times ((\nabla \times b) \times b) = \mu \Delta b, \\
\nabla \cdot u = 0, \quad \nabla \cdot b = 0, \\
|\nabla| u = |u|, \quad |\nabla| b = |b|, \quad
\end{cases}\tag{1}$$

on the domain $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^3$, where $u = (u_1, u_2, u_3)^T, b = (b_1, b_2, b_3)^T \in \mathbb{R}^3$ denote the fluid velocity and magnetic fields respectively. The scalars $p, \nu, \mu, \eta$ are the pressure, viscosity, magnetic diffusivity, Hall effect coefficient respectively ($\nu, \mu$ and $\eta$ are positive constants). $u_0$ and $b_0$ are the initial data satisfying

$$\nabla \cdot u_0 = \nabla \cdot b_0 = 0.\tag{2}$$

Equation 1 is important to describe some physical phenomena, e.g., space plasmas, star formation, neutron stars and dynamo, see for [16, 2, 13, 14, 19, 22, 25] and references therein. In the case $\eta = 0$, the Hall-MHD equation reduces to the standard MHD equation which has been extensively researched, and there exists a lot of excellent works, see instances, [21, 10, 9, 17, 15, 18, 20, 27].

2010 Mathematics Subject Classification. Primary: 35A01, 35A02, 35A09; Secondary: 35E15, 35Q35.

Key words and phrases. Incompressible Hall-MHD equations, large data, global smooth solution.

The first author is supported by Education Department of Hunan Province, general Program(grant No.1703709), and Hunan Provincial Key Laboratory of Intelligent Processing of Big Data on Transportation, Changsha University of Science and Technology, Changsha; 410114, China.

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For Hall-MHD equations, some newly developments have been made. For example, Chae et al. in [3] proved global smooth solutions of three-dimensional Hall-MHD equation with small initial data in \((H^3(\mathbb{R}^3))^3 \times (H^3(\mathbb{R}^3))^3\). Chae and Lee improved their results under weaker smallness assumptions on the initial data, see [4] for details. There are other prominent works for small solutions for Hall-MHD equations, for examples, [1, 11, 26, 6, 7, 8]. These results of the global well-posedness of the three-dimensional Hall-MHD system under the smallness condition on the initial data in the deterministic case requires positive diffusion on both velocity and magnetic fields equations. However, with noise, zero viscosity is allowed; Yamazaki and Moha in [26] proved the global well-posedness of the three dimensional stochastic Hall-MHD system with zero viscosity under the smallness condition on the initial data. However, none of results are known for Hall-MHD equations for general initial data without smallness conditions. Under a class of large initial data, we found some results for incompressible Navier-Stokes equations and incompressible standard MHD equations, see [10, 9, 15, 18, 27] for details. Those motivate us to study the global well-posedness of Cauchy’s problem of Hall-MHD equations with large initial data. But the Hall term heightens the level of nonlinearity of the standard MHD system from a second-order semilinear to a second-order quasilinear level, significantly making its qualitative analysis more difficult. To the author’s knowledge, it’s quite rare to prove the existence of large, smooth, global solutions for quasilinear system. Using the large initial data constructed in [15], the author is very fortunately to go through these difficulties by combining nonlinear structures and commutator energy estimates for resistive, viscous Hall-MHD equations.

The aim of this paper is to prove the existence of a unique, global smooth solution of Hall-MHD equations with initial data being arbitrarily large in \(H^3(\mathbb{R}^3)\). Our result completely drops the smallness condition on the initial data.

Before we state our main results, we first give some notations. Let \(\chi(x) \in C_0^\infty(\mathbb{R}^3)\) be a cut off function satisfying \(|\chi(x)| \leq 2\) and
\[
\chi(x) \equiv 1, \quad \text{for } |x| \leq 1; \quad \chi(x) \equiv 0, \quad \text{for } |x| \geq 2,
\]
\(|\nabla^k \chi(x)| \leq 2, \quad 0 \leq k \leq 5. \tag{3}\]

Denote
\[
\chi_{M_0}(x) := \chi\left(\frac{x}{M_0}\right). \tag{4}\]

Here \(M_0\) is a positive constant. Let \(v_0\) be that constructed by Lei et al. [15], and it has the following properties
\[
\nabla \cdot v_0 = 0, \quad \nabla \times v_0 = \sqrt{-\Delta} v_0, \tag{5}\]
\[
\text{supp}\hat{v}_0 \subseteq \{\xi|1 - \delta \leq |\xi| \leq 1 + \delta\}, \quad 0 < \delta \leq \frac{1}{2}, \tag{6}\]
\[
||\hat{v}_0||_{L^1} \leq M_1, \quad |\nabla^k v_0| \leq \frac{M_2}{1 + |x|}, \quad 0 \leq k \leq 5, \tag{7}\]

where \(M_1, M_2\) are positive constants, \(\hat{v}_0\) is the Fourier transform of \(v_0\) and the operator \(\sqrt{-\Delta}\) is defined through the Fourier transform
\[
\sqrt{-\Delta} f(\xi) = |\xi| \hat{f}(\xi).\]

Our main result is as follows.
Theorem 1.1. Consider Cauchy’s problem 1-2. Suppose that
\[ u_0 = u_{01} + \chi M_0 u_{02}, \quad b_0 = b_{01} + \chi M_0 b_{02}, \]
with
\[ \nabla \cdot u_{02} = \nabla \cdot b_{02} = 0, \]
\[ u_{02} = \alpha_1 v_0, \quad b_{02} = \alpha_2 v_0, \]
where \( \chi M_0, v_0 \) are stated as above. \( \alpha_1, \alpha_2 \) are two real constants. Then there exist constants \( M_0 \gg 1 \) depending on \( M_1, M_2, \alpha_1, \alpha_2, \mu, \nu, \eta \) such that Cauchy’s problem 1-2 has a unique, global smooth solution provided that
\[ \|u_{01}\|_{H^3} + \|b_{01}\|_{H^3} \leq M_0^{-\frac{3}{2}}. \]

Remark 1. For \( \|u_0\|_{L^\infty} + \|b_0\|_{L^\infty} \leq M_1, \)
\[ \|u_0\|_{H^3} + \|b_0\|_{H^3} \leq \left( M_0^{-\frac{3}{2}} + (|\alpha_1| + |\alpha_2|) \sum_{k=0}^3 \frac{M_2}{M_0^k} \right), \]
and the constant \( M_1, M_2 \) can be arbitrary large, thus our initial data can be arbitrary large.

Remark 2. In the limiting case \( \delta = 0, \nabla \times u_{02} = u_{02}, \nabla \times b_{02} = b_{02}, \) and the flow, magnetic field are called Beltrami flow and force-free fields respectively. Let us also mention that the magnetic energy achieves the minimum value for force-free fields, one can refer [24] for details.

Remark 3. We throughout use a notation \( C \). It may be different from line to line, but it is a universal positive constant in this paper.

The proof of Theorem 1.1 is based on a perturbation argument along with a standard cut-off technique, and the perturbation is as large as the initial data. Compared with the standard MHD equations, a part of the nonlinearities may not be small for Hall-MHD equations (see 29 and 33). Fortunately, by combining the nonlinear structure of the term and commutator estimates, these terms can be estimated carefully.

This paper is organized as follows: In section 2, we introduce commutator estimates and give some estimates of some quadratic terms. Section 3 is devoted to prove the global existence and uniqueness of large smooth solutions for Hall-MHD equations.

2. Preliminaries. In this section, we first give some notations. Let \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \) be a multi index, \( |\alpha| = \sum_{i=1}^3 \alpha_i \), \( \partial = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3}) \) and \( \partial^\alpha = (\partial_{x_1}^{\alpha_1}, \partial_{x_2}^{\alpha_2}, \partial_{x_3}^{\alpha_3}) \).

Let \( m \) be a positive integer and \( r > 0 \). We use the notation \( \|g\|_{H^m(|x| \leq r)} \) to denote the Sobolev norm localized in bounded domain \( \{ x \in \mathbb{R}^3 | |x| \leq r \} \), that is,
\[ \|g\|_{H^m(|x| \leq r)} := \sum_{0 \leq |\alpha| \leq m} \left( \int_{|x| \leq r} |\partial^\alpha g|^2 \, dx \right)^{\frac{1}{2}}. \]
Next, we introduce the commutator estimate.
Lemma 2.1. [3] Let \( m \) be a positive integer, \( h, v \in H^m(\mathbb{R}^3) \). The following commutator estimate
\[
\sum_{|\alpha|\leq m} ||\partial^\alpha (hv) - (\partial^\alpha h)v||_{L^2} \leq C (||h||_{H^{m-1}} ||v||_{L^\infty} + ||h||_{L^\infty} ||v||_{H^m})
\] (13)
holds.

Let \( f, g \) satisfy
\[
\begin{cases}
  f_t - \nu \Delta f = 0, \\
  t = 0 : f = u_{02},
\end{cases}
\] (14)
and
\[
\begin{cases}
  g_t - \mu \Delta g = 0, \\
  t = 0 : g = b_{02}.
\end{cases}
\] (15)
Therefore, we have
\[ f = e^{\nu t \Delta} u_{02}, \quad g = e^{\mu t \Delta} b_{02}. \]

Lemma 2.2. Let \( f, g \) be defined in 14, 15. It holds
\[
\begin{align*}
\nabla \cdot f &= 0, \quad \nabla \times f = \sqrt{-\Delta} f, \\
\nabla \cdot g &= 0, \quad \nabla \times g = \sqrt{-\Delta} g,
\end{align*}
\]
\[
|\nabla^k f| \leq \left| \frac{\alpha_1 |M_2|}{1 + |x|} e^{-\frac{\mu t}{4}} \right|, \quad |\nabla^k g| \leq \left| \frac{\alpha_2 |M_2|}{1 + |x|} e^{-\frac{\mu t}{4}} \right|, \quad 0 \leq |k| \leq 5.
\]

Proof. By
\[ \nabla \cdot v_0 = 0, \quad \nabla \times v_0 = \sqrt{-\Delta} v_0, \]
\[ f = e^{\nu t \Delta} u_{02}, \quad g = e^{\mu t \Delta} b_{02}, \]
we can deduce that
\[
\begin{align*}
\nabla \cdot f &= 0, \quad \nabla \times f = \sqrt{-\Delta} f, \\
\nabla \cdot g &= 0, \quad \nabla \times g = \sqrt{-\Delta} g.
\end{align*}
\]
We choose a \( C^\infty(\mathbb{R}^3) \) cut-off function \( \alpha(\xi) \) such that \( \alpha \equiv 1 \) on the support of \( v_0 \), and \( \alpha(\xi) \equiv 0 \) if \( |\xi| \geq 1 + 2\delta \) or \( |\xi| \leq 1 - 2\delta \). Then we have
\[
\begin{align*}
f(t, x) &= \alpha_1 e^{-\frac{\mu t}{4}} F^{-1} \left( e^{-\nu (|\xi|^2 - \frac{1}{2}) t} \alpha(\xi) \right) \ast v_0, \\
g(t, x) &= \alpha_2 e^{-\frac{\mu t}{4}} F^{-1} \left( e^{-\nu (|\xi|^2 - \frac{1}{2}) t} \alpha(\xi) \right) \ast v_0.
\end{align*}
\]
In a result, we get
\[
|\nabla^k f| \leq \left| \frac{\alpha_1 |M_2|}{1 + |x|} e^{-\frac{\mu t}{4}} \right|, \quad |\nabla^k g| \leq \left| \frac{\alpha_2 |M_2|}{1 + |x|} e^{-\frac{\mu t}{4}} \right|, \quad 0 \leq |k| \leq 5.
\]
\[ \square \]

Lemma 2.3. Set \( \tilde{f} := \chi_{M_0} f, \tilde{g} := \chi_{M_0} g \). Let \( f, g, \chi_{M_0} \) be defined by 14, 15 and 4 respectively. Then we have
\[
||\tilde{f}||_{W^{5,\infty}} + ||\tilde{g}||_{W^{5,\infty}} \leq C \left( ||\alpha_1 |M_1| e^{-\frac{\mu t}{4}} + ||\alpha_2 |M_1| e^{-\frac{\mu t}{4}} \right), \]
(16)
\[
||\tilde{f} \times (\nabla \times \tilde{f})||_{H^3} + ||\tilde{g} \times (\nabla \times \tilde{g})||_{H^3}
\leq C \left( \alpha_1^2 e^{-\frac{\mu t}{4}} + \alpha_2^2 e^{-\frac{\mu t}{4}} \right) \left( \delta M_0^2 M_1^2 + M_0^{-1} M_1^2 \right),
\] (17)
\[
\int_0^\infty \|\tilde{f} \times \tilde{g}\|_{H^3}(t) \, dt \leq CM_0^3 M_1^2 \delta.
\] (18)

**Proof.** Firstly, we have \(|\nabla^k \chi_{M_0}| \leq CM_0^{-k}, k \leq 5\). Then
\[
\|\tilde{f}\|_{W^{5,\infty}} = \|\chi_{M_0} f\|_{W^{5,\infty}} \leq \|\chi_{M_0}\|_{W^{5,\infty}} \|f\|_{W^{5,\infty}} \leq C \|f\|_{W^{5,\infty}}.
\] (19)

Using \(\tilde{f} = e^{-\nu t|\xi|^2} \tilde{u}_0\) and supp \(\tilde{u}_0 \subseteq \{\xi|1-\delta \leq |\xi| \leq 1+\delta\}, 0 < \delta \leq \frac{1}{2}\), we get
\[
\|f\|_{W^{5,\infty}} \leq \left\|\left(1 + |\xi|\right)^5 \tilde{f}\right\|_{L_t^1} \leq C \left\|e^{-\nu t|\xi|^2} \tilde{u}_0\right\|_{L_t^1} \leq C |\alpha_1| M_1 e^{-\frac{\nu t}{2}}.
\]

Similarly, we have
\[
\|g\|_{W^{5,\infty}} \leq C |\alpha_2| M_1 e^{-\frac{\nu t}{2}}.
\] (20)

Adding 20 to 19, we obtain
\[
\|\tilde{f}\|_{W^{5,\infty}} + \|\tilde{g}\|_{W^{5,\infty}} \leq C \left(|\alpha_1| M_1 e^{-\frac{\nu t}{2}} + |\alpha_2| M_1 e^{-\frac{\nu t}{2}}\right).
\]

Secondly, we notice the fact
\[
\nabla \times (\chi_{M_0} f) = \nabla \chi_{M_0} \times f + \chi_{M_0} \nabla \times f,
\]
\[
\nabla \times (\chi_{M_0} g) = \nabla \chi_{M_0} \times g + \chi_{M_0} \nabla \times g.
\]

Thus, we get
\[
\|\tilde{f} \times (\nabla \times \tilde{f})\|_{H^3} + \|\tilde{g} \times (\nabla \times \tilde{g})\|_{H^3}
= \|\chi_{M_0} f \times (\nabla \times (\chi_{M_0} f))\|_{H^3} + \|\chi_{M_0} g \times (\nabla \times (\chi_{M_0} g))\|_{H^3}
\leq C \|\chi_{M_0}^2\|_{H^3} (\|f \times (\nabla \times f)\|_{W^{3,\infty}} + \|g \times (\nabla \times g)\|_{W^{3,\infty}})
+ C \|\nabla (\chi_{M_0}^2)\|_{W^{3,\infty}} (\|f\|_{H^3}^2 + \|g\|_{H^3}^2).
\]

We calculate that
\[
\|\chi_{M_0}^2\|_{H^3} \leq C \sum_{i=0}^3 M_0^{-i} M_0^3 \leq CM_0^3,
\]
\[
\|\nabla (\chi_{M_0}^2)\|_{W^{3,\infty}} \leq C \sum_{i=0}^3 M_0^{-i-1} \leq CM_0^{-1}.
\] (21)

For \(f \times f = 0, g \times g = 0\), then we have
\[
\|f \times (\nabla \times f)\|_{W^{3,\infty}} + \|g \times (\nabla \times g)\|_{W^{3,\infty}}
= \|f \times (\nabla \times f - f)\|_{W^{3,\infty}} + \|g \times (\nabla \times g - g)\|_{W^{3,\infty}}
\leq C \left\|(1 + |\xi|)^3 \tilde{f}\right\|_{L_t^1} + C \left\|(1 + |\xi|)^3 \tilde{g}\right\|_{L_t^1}
\leq C \delta \left(\|f\|_{L_t^1}^2 + \|g\|_{L_t^1}^2\right)
\leq CM_0^2 \delta (\alpha_1^2 e^{-\frac{\nu t}{2}} + \alpha_2^2 e^{-\frac{\nu t}{2}}).
\] (22)
Noticing that \( \text{supp}[\hat{f}|^2, \text{supp}[\hat{g}|^2 \subseteq \{\xi|\xi| \leq 2 + 2\delta\}, 0 < \delta \leq \frac{1}{2} \), we derive that
\[
|||f||^2||_{H^3} + |||g||^2||_{H^3} \leq C(|||f||^2||_{L^3} + |||g||^2||_{L^3})\leq C \left( ||f||^2_{L^3} + ||g||^2_{L^3} \right) \leq C \left( \alpha_1^2 e^{-\frac{4\delta}{3}} M_2^2 + \alpha_2^2 e^{-\frac{4\delta}{3}} M_2^2 \right). \tag{23}
\]

Combining inequalities 21, 22 and 23, we get
\[
|||\hat{f} \times (\nabla \times \hat{f})|||_{H^3} + |||\hat{g} \times (\nabla \times \hat{g})|||_{H^3} \leq C \left( \alpha_1^2 e^{-\frac{4\delta}{3}} + \alpha_2^2 e^{-\frac{4\delta}{3}} \right) \left( \delta M_0^2 M_2^2 + M_0^{-1} M_2^2 \right).
\]

In what follows, we will estimate \( \int_0^t |||\hat{f} \times \hat{g}|||_{H^3}(\tau)d\tau \). On one hand,
\[
|||\hat{f} \times \hat{g}|||_{H^3} = \|||\chi_{M_0} f \times (\chi_{M_0} g)|||_{H^3} \leq \|||\chi_{M_0} f|||_{H^3} \|||g|||_{W^{3,\infty}}. \tag{24}
\]

On the other hand, \( \text{supp} \hat{\chi} \subseteq \{\xi|\xi| \leq 2 + 2\delta\}, 0 < \delta \leq \frac{1}{2} \). Then we have
\[
|||\hat{f} \times \hat{g}|||_{H^3} \leq C M_0^3 |||\hat{f} \times \hat{g}|||_{L^3}. \tag{25}
\]

Calculate
\[
\hat{f} \times g = \alpha_1 \alpha_2 \int_{\mathbb{R}^3} e^{-\nu|\xi-\eta|^2} v_0(\xi - \eta) \times e^{-\mu|\eta|^2} v_0(\eta) d\eta
\]
\[
= \frac{1}{2} \alpha_1 \alpha_2 \int_{\mathbb{R}^3} \left( e^{-\nu|\xi-\eta|^2 + \mu|\eta|^2} - e^{-\nu|\xi-\eta|^2 + \nu|\eta|^2} \right) v_0(\xi - \eta) \times v_0(\eta) d\eta,
\]
and
\[
e^{-\nu|\xi-\eta|^2 + \mu|\eta|^2} - e^{-\nu|\xi-\eta|^2 + \nu|\eta|^2} \leq C e^{-\frac{\delta}{2}(\xi - \eta|^2 + |\eta|^2)} \frac{|\xi - \eta|^2 - |\eta|^2|}{|\xi - \eta|^2 + |\eta|^2}. \tag{26}
\]
In the support of \( \hat{v}_0(\xi - \eta) \times \hat{v}_0(\eta) \), we have
\[
\frac{|\xi - \eta|^2 - |\eta|^2|}{|\xi - \eta|^2 + |\eta|^2} \leq 10\delta. \tag{27}
\]
Therefore, we conclude that
\[
\int_0^\infty |||\hat{f} \times \hat{g}|||_{H^3}(t)dt \leq C M_0^3 M_2^2 \delta.
\]

Then we complete the proof of Lemma 2.3.

\[\square\]

3. The proof of Theorem 1.1. In this section, we will prove Theorem 1.1 using a perturbation argument along with a standard cut-off technique.
Proof of Theorem 1.1. Let $\tilde{f} = \chi_{M_0} f, \tilde{g} = \chi_{M_0} g$, and $u = U + \tilde{f}, b = B + \tilde{g}$. Then $U, B$ satisfy

$$U_t - \nu \Delta U + \nabla \left( p + \frac{1}{2} |\tilde{f}|^2 - \frac{1}{2} |\tilde{g}|^2 \right)$$

$$= -U \cdot \nabla U - \tilde{f} \cdot \nabla U - U \cdot \nabla \tilde{f} + B \cdot \nabla B + \tilde{g} \cdot \nabla B + B \cdot \nabla \tilde{g} + F, \quad (28)$$

where

$$F := \tilde{f} \times (\nabla \times \tilde{f}) - \tilde{g} \times (\nabla \times \tilde{g}) - \nu \Delta \chi_{M_0} f + 2\nu \nabla \cdot (\nabla \chi_{M_0} f),$$

$$G := \nabla \times (\tilde{f} \times \tilde{g}) - \mu \Delta \chi_{M_0} g + 2\mu \nabla \cdot (\nabla \chi_{M_0} g) + \frac{1}{2} f \cdot \nabla \chi_{M_0} g + \frac{1}{2} g \cdot \nabla \chi_{M_0} f.$$

In what follows, we will derive some energy estimates of $U$ and $B$.

**Step 1. Energy inequalities of $B$.** Operating Equation 29 with $\partial^\alpha, |\alpha| \leq 3$, and taking $L^2$ inner product with $\partial^\alpha B$, we get

$$\frac{1}{2} \frac{d}{dt} \|\partial^\alpha B\|_{L^2}^2 + \mu \|\partial^\alpha \nabla B\|_{L^2}^2$$

$$= -\sum_{1 \leq |\beta| \leq |\alpha|} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) \left( \int_{\mathbb{R}^3} \partial^\beta U \cdot \nabla \partial^{\alpha-\beta} B \partial^\alpha B dx \right) + \int_{\mathbb{R}^3} \partial^\beta \tilde{f} \cdot \nabla \partial^{\alpha-\beta} B \partial^\alpha B dx$$

$$- \int_{\mathbb{R}^3} \partial^\alpha (U \cdot \nabla \tilde{g}) \partial^\alpha B dx + \sum_{1 \leq |\beta| \leq |\alpha|} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) \int_{\mathbb{R}^3} \partial^\beta B \cdot \nabla \partial^{\alpha-\beta} U \partial^\alpha B dx$$

$$+ \sum_{1 \leq |\beta| \leq |\alpha|} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) \int_{\mathbb{R}^3} \partial^\beta \tilde{g} \cdot \nabla \partial^{\alpha-\beta} U \partial^\alpha B dx + \int_{\mathbb{R}^3} \partial^\alpha \left( B \cdot \nabla \tilde{f} \right) \partial^\alpha B dx$$

$$+ \int_{\mathbb{R}^3} \partial^\alpha G \partial^\alpha B dx + T_1 + T_2 + T_3, \quad (30)$$

where

$$T_1 = -\int_{\mathbb{R}^3} U \cdot \nabla \partial^\alpha B \partial^\alpha B dx - \int_{\mathbb{R}^3} \tilde{f} \cdot \nabla \partial^\alpha B \partial^\alpha B dx$$

$$= -\int_{\mathbb{R}^3} u \cdot \nabla \partial^\alpha B \partial^\alpha B dx$$

$$= 0, \quad (31)$$

and

$$T_2 = \int_{\mathbb{R}^3} B \cdot \nabla \partial^\alpha U \partial^\alpha B dx + \int_{\mathbb{R}^3} \tilde{g} \cdot \nabla \partial^\alpha U \partial^\alpha B dx$$

$$= \int_{\mathbb{R}^3} b \cdot \nabla \partial^\alpha U \partial^\alpha B dx, \quad (32)$$
and

\[
I = -\eta \int_{\mathbb{R}^3} \partial^\alpha B \cdot \partial^\alpha (\nabla \times ((\nabla \times B) \times B)) \, dx \\
- \eta \int_{\mathbb{R}^3} \partial^\alpha B \cdot \partial^\alpha (\nabla \times ((\nabla \times B) \times \tilde{g})) \, dx \\
- \eta \int_{\mathbb{R}^3} \partial^\alpha B \cdot \partial^\alpha (\nabla \times ((\nabla \times \tilde{g}) \times B)) \, dx \\
- \eta \int_{\mathbb{R}^3} \partial^\alpha B \cdot \partial^\alpha (\nabla \times ((\nabla \times \tilde{g}) \times \tilde{g})) \, dx
\]

\(\tag{33}\)

\(\begin{aligned}
&:= I_1 + I_2 + I_3 + I_4.
\end{aligned}\)

Firstly, we have

\[
\left| \int_{\mathbb{R}^3} \partial^\alpha G \partial^\alpha B \, dx \right| = \left| \int_{\mathbb{R}^3} \partial^\alpha (\nabla \times (\tilde{f} \times \tilde{g}) + 2\nu \nabla \cdot (\nabla \chi_{M_0} g) - \nu \Delta \chi_{M_0} g \\
+ \frac{1}{2} f \cdot \nabla \chi_{M_0}^2 g - \frac{1}{2} g \cdot \nabla \chi_{M_0} f) \cdot \partial^\alpha B \, dx \right|
\leq C \|\tilde{f} \times \tilde{g}\|_{H^3} \|\nabla B\|_{H^3} + \|\nabla \chi_{M_0} g\|_{H^3} \|\nabla B\|_{H^3}
+ C \left( \|f \cdot \nabla \chi_{M_0}^2 g\|_{W^{3, \frac{3}{2}}} + \|g \cdot \nabla \chi_{M_0} f\|_{W^{3, \frac{3}{2}}} \right) \|B\|_{W^{3, 6}}
+ C \|\Delta \chi_{M_0} g\|_{W^{3, \frac{3}{2}}} \|B\|_{W^{3, 6}}
\leq C \left( \|\tilde{f} \times \tilde{g}\|_{H^3} + M_0^{-1} \|g\|_{H^3} \|_{\{x|\leq 2 M_0\}} \right) \|\nabla B\|_{H^3}
+ C M_0^{-2} \|g\|_{W^{3, \frac{3}{2}} (\{x|\leq 2 M_0\})} \|\nabla B\|_{H^3}
+ C M_0^{-1} \|f \otimes g\|_{W^{3, \frac{3}{2}} (\{x|\leq 2 M_0\})} \|\nabla B\|_{H^3}
\leq C \left( \|\tilde{f} \times \tilde{g}\|_{H^3} + M_0^{-\frac{5}{2}} M_2 |\alpha_2| e^{-\frac{\mu}{\nu}} \right) \|\nabla B\|_{H^3}
+ C |\alpha_1 \alpha_2| M_0^{-\frac{5}{2}} M_2 e^{-\frac{\mu_1 + \mu_2}{2}} \|\nabla B\|_{H^3}.
\] \(\tag{34}\)

Next, we need to estimate \(I_1, I_2, I_3\) and \(I_4\). For

\[
I_1 = -\eta \int_{\mathbb{R}^3} \partial^\alpha B \cdot \partial^\alpha (\nabla \times ((\nabla \times B) \times B)) \, dx
\]

\[
= \eta \int_{\mathbb{R}^3} \partial^\alpha (\nabla \times B) \cdot \partial^\alpha ((\nabla \times B) \times B) \, dx
\]

\[
= \eta \int_{\mathbb{R}^3} \{ \partial^\alpha ((\nabla \times B) \times B) - \partial^\alpha (\nabla \times B) \times B \} \cdot \partial^\alpha (\nabla \times B) \, dx,
\]

using Lemma 2.1, we deduce that

\[
|I_1| \leq C \eta \sum_{|\alpha| \leq 3} \|\partial^\alpha ((\nabla \times B) \times B)\|_{L^2} \|\partial^\alpha ((\nabla \times B) \times B) - \partial^\alpha (\nabla \times B) \times B\|_{L^2}
\leq C \eta \|\nabla B\|_{H^3} \|\nabla \times B\|_{H^3} \|\nabla B\|_{L^\infty} + \|\nabla \times B\|_{L^\infty} \|B\|_{H^3}
\leq C \eta \|\nabla B\|_{H^3}^2 \|B\|_{H^3}.
\] \(\tag{35}\)

We calculate

\[
I_2 = -\eta \int_{\mathbb{R}^3} \partial^\alpha B \cdot \partial^\alpha (\nabla \times ((\nabla \times B) \times \tilde{g})) \, dx
\]
Combining inequalities 37 and 38, we get thus,

\[ \int \eta \int R (\nabla \times B) \cdot (\nabla \times (\nabla \times B) \times \tilde{g}) \, dx \]

and

\[ \int \eta \int R (\nabla \times B) \cdot (\nabla \times (\nabla \times B) \times \tilde{g}) \, dx = 0. \]

Therefore, we have

\[ |I_2| \leq C\eta ||\nabla B||_{H^1} ||\nabla B||_{H^2} ||\tilde{g}||_{W^{3,\infty}} \leq C\eta ||\nabla B||_{H^1} ||B||_{H^3} ||\tilde{g}||_{W^{3,\infty}}. \] (36)

To estimate \( I_3 \), we divide it into two parts

\[ I_3 = -\eta \int \eta \int R (\nabla \times ((\nabla \times \tilde{g}) \times B)) \, dx \]

\[ = -\eta \sum_{|\beta| \leq |\alpha| - 1} \left( \alpha \beta \right) \int \eta \int R (\nabla \times ((\nabla \times \tilde{g}) \times \partial^\beta B)) \, dx \]

\[ = I_{31} + I_{32}. \]

Applying Hölder inequality, we easily infer

\[ |I_{31}| \leq C\eta ||B||_{H^3} ||\tilde{g}||_{W^{5,\infty}}. \] (37)

Moreover,

\[ I_{32} = -\eta \int \eta \int R (\nabla \times ((\nabla \times \tilde{g}) \times \partial^\alpha B)) \, dx \]

\[ = \eta \int \eta \int R (\nabla \times (\nabla \times \tilde{g}) \cdot \nabla) \partial^\alpha B \, dx - \eta \int \eta \int R (\nabla \times (\nabla \times \tilde{g})) \partial^\alpha B \, dx \]

\[ = \frac{\eta}{2} \int \eta \int R (\nabla \times (\nabla \times \tilde{g})) ||\partial^\alpha B||^2 \, dx - \frac{\eta}{2} \int \eta \int R (\nabla \times (\nabla \times \tilde{g})) \, dx \]

\[ = -\frac{\eta}{2} \int \eta \int R (\nabla \times (\nabla \times \tilde{g})) \, dx \]

thus,

\[ |I_{32}| \leq C\eta ||\tilde{g}||_{W^{2,\infty}} ||B||_{H^3}^2. \] (38)

Combining inequalities 37 and 38, we get

\[ |I_3| \leq C\eta ||\tilde{g}||_{W^{5,\infty}} ||B||_{H^3}^2. \] (39)
As for $I_4$, we can write
\[
I_4 = -\eta \int_{\mathbb{R}^3} \partial^\alpha B \cdot \partial^\alpha (\nabla \times ((\nabla \times \tilde{g}) \times \tilde{g})) \, dx
\]
\[
= \eta \int_{\mathbb{R}^3} \partial^\alpha (\nabla \times B) \cdot \partial^\alpha ((\nabla \times \tilde{g}) \times \tilde{g}) \, dx.
\]
We thus have
\[
|I_4| \leq C\eta \|\nabla B\|_{H^3} \|\nabla \times \tilde{g}\|_{H^3}. \tag{40}
\]

**Step 2. Energy inequalities of $U$.** Operating Equation 28 with $\partial^\alpha, |\alpha| \leq 3$, and taking $L^2$ on Equation 28 yields
\[
\frac{1}{2} \frac{d}{dt} \|\partial^\alpha U\|^2_{L^2} + \mu \|\partial^\alpha \nabla U\|^2_{L^2}
\]
\[
= - \sum_{1 \leq |\beta| \leq |\alpha|} \left( \frac{\alpha}{\beta} \right) \left( \int_{\mathbb{R}^3} \partial^\beta U \cdot \nabla \partial^{\alpha-\beta} U \partial^\alpha U \, dx + \int_{\mathbb{R}^3} \partial^\beta \tilde{f} \cdot \nabla \partial^{\alpha-\beta} U \partial^\alpha U \, dx \right)
\]
\[
- \int_{\mathbb{R}^3} \partial^\alpha (U \cdot \nabla \tilde{f}) \partial^\alpha U \, dx + \sum_{1 \leq |\beta| \leq |\alpha|} \left( \frac{\alpha}{\beta} \right) \int_{\mathbb{R}^3} \partial^\beta B \cdot \nabla \partial^{\alpha-\beta} B \partial^\alpha U \, dx
\]
\[
+ \sum_{1 \leq |\beta| \leq |\alpha|} \left( \frac{\alpha}{\beta} \right) \int_{\mathbb{R}^3} \partial^\beta \tilde{g} \cdot \nabla \partial^{\alpha-\beta} B \partial^\alpha U \, dx + \int_{\mathbb{R}^3} \partial^\alpha (B \cdot \nabla \tilde{g}) \partial^\alpha U \, dx
\]
\[
+ \int_{\mathbb{R}^3} \partial^\alpha F \partial^\alpha U \, dx - \int_{\mathbb{R}^3} \partial^\alpha \nabla \left( p + \frac{1}{2} |\tilde{f}|^2 - \frac{1}{2} |\tilde{g}|^2 \right) \partial^\alpha U \, dx + T_3 + T_4,
\]
where
\[
T_3 := - \int_{\mathbb{R}^3} U \cdot \nabla \partial^\alpha U \partial^\alpha U \, dx - \int_{\mathbb{R}^3} \tilde{f} \cdot \nabla \partial^\alpha U \partial^\alpha U \, dx
\]
\[
= - \int_{\mathbb{R}^3} u \cdot \nabla \partial^\alpha U \partial^\alpha U \, dx
\]
\[
= 0, \tag{41}
\]
and
\[
T_4 := \int_{\mathbb{R}^3} B \cdot \nabla \partial^\alpha B \partial^\alpha U \, dx + \int_{\mathbb{R}^3} \tilde{g} \cdot \nabla \partial^\alpha B \partial^\alpha U \, dx
\]
\[
= \int_{\mathbb{R}^3} b \cdot \nabla \partial^\alpha B \partial^\alpha U \, dx.
\]
Combining $T_2$ and $T_2$, we have
\[
T_2 + T_4 = 0. \tag{42}
\]
Firstly, we have
\[
\left| \int_{\mathbb{R}^3} \partial^\alpha F \partial^\alpha U \, dx \right| = \left| \int_{\mathbb{R}^3} \partial^\alpha (\tilde{f} \times (\nabla \times \tilde{f}) - \tilde{g} \times (\nabla \times \tilde{g})) - \nu \Delta \chi_{M_0} f \right.
\]
\[
\left. + 2\nu \nabla \cdot ((\nabla \chi_{M_0} f)) \cdot \partial^\alpha U \, dx \right|
\]
\[
\leq C \left( ||\tilde{f} \times (\nabla \times \tilde{f})||_{H^3} + ||\tilde{g} \times (\nabla \times \tilde{g})||_{H^3} \right) ||\nabla U||_{H^3}
\]
\[
+ C ||\nabla \chi_{M_0} f||_{H^3} ||\nabla U||_{H^3} + C ||\Delta \chi_{M_0} f||_{W^{3,6}} ||U||_{W^{3,6}}
\]
\[
\leq C \left( ||\tilde{f} \times (\nabla \times \tilde{f})||_{H^3} + ||\tilde{g} \times (\nabla \times \tilde{g})||_{H^3} \right) ||\nabla U||_{H^3}.
Since

\[
p = (-\Delta)^{-1} \text{div} (u \cdot \nabla u - b \cdot \nabla b)
\]

\[
= \sum_{i,j} (-\Delta)^{-1} \partial_i \partial_j (u_i U_j - b_i B_j) + (-\Delta)^{-1} \nabla \cdot \left( U \cdot \nabla \tilde{f} - B \cdot \nabla \tilde{g} \right)
\]

\[
+ (-\Delta)^{-1} \nabla \cdot \left( \tilde{f} \times (\nabla \times \tilde{f}) - \tilde{g} \times (\nabla \times \tilde{g}) \right) - \frac{1}{2} |\tilde{f}|^2 + \frac{1}{2} |\tilde{g}|^2,
\]

then we have

\[
\Pi := \sum_{|\alpha| \leq 3} \int_{\mathbb{R}^3} \partial^\alpha \nabla \left( p + \frac{1}{2} |\tilde{f}|^2 - \frac{1}{2} |\tilde{g}|^2 \right) \partial^\alpha U dx
\]

\[
\leq \sum_{|\alpha| \leq 3} \int_{\mathbb{R}^3} \partial^\alpha \nabla (-\Delta)^{-1} \nabla \partial_j (u_i U_j - b_i B_j) \partial^\alpha \nabla \cdot U dx
\]

\[
+ \sum_{|\alpha| \leq 3} \int_{\mathbb{R}^3} \partial^\alpha \nabla (-\Delta)^{-1} \nabla \cdot \left( U \cdot \nabla \tilde{f} - B \cdot \nabla \tilde{g} \right) \partial^\alpha U dx
\]

\[
+ \sum_{|\alpha| \leq 3} \int_{\mathbb{R}^3} \partial^\alpha \nabla (-\Delta)^{-1} \nabla \cdot \left( \tilde{f} \times (\nabla \times \tilde{f}) - \tilde{g} \times (\nabla \times \tilde{g}) \right) \partial^\alpha U dx.
\]

Using Hölder inequality and Calderon-Zygmund estimate in \cite{23}, then we get

\[
\Pi \leq \left( |u \otimes U|_{W^{3, \frac{3}{2}}} + |b \otimes B|_{W^{3, \frac{3}{2}}} \right) \|f \cdot \nabla \chi_{M_0}\|_{W^{3, 3}}
\]

\[
+ \|U \cdot \nabla \tilde{f} - B \cdot \nabla \tilde{g}\|_{L^3}\|U\|_{H^3}
\]

\[
+ \|\tilde{f} \times (\nabla \times \tilde{f}) - \tilde{g} \times (\nabla \times \tilde{g})\|_{H^3}\|U\|_{H^3}.
\]

Furthermore, we derive that

\[
\Pi \leq C \left[ \|U\|_{H^3}^2 + \|\tilde{f}\|_{W^{3, 6}} \|U\|_{H^3} \right] \|\nabla \chi_{M_0}\|_{W^{3, \infty}} \|f\|_{W^{3, 3}(M_0 \leq |x| \leq 2M_0)}
\]

\[
+ \|H\|_{H^3} \cdot \|\nabla \chi_{M_0}\|_{W^{3, \infty}} \|f\|_{W^{3, 3}(M_0 \leq |x| \leq 2M_0)}
\]

\[
+ \|\tilde{g}\|_{W^{3, 2}} \|H\|_{W^{3, 6}} \|\nabla \chi_{M_0}\|_{W^{3, \infty}} \|f\|_{W^{3, 3}(M_0 \leq |x| \leq 2M_0)}
\]

\[
+ \left( \|\nabla \tilde{f}\|_{W^{3, \infty}} \|U\|_{H^3} + \|\nabla \tilde{g}\|_{W^{3, \infty}} \|B\|_{H^3} \right) \|U\|_{H^3}
\]

\[
+ \|\nabla \tilde{f}\|_{H^3} \|\nabla \times \tilde{f}\|_{H^3} + 2 \|\tilde{g}\|_{H^3} \|\nabla \times \tilde{g}\|_{H^3} \|U\|_{H^3}
\]

\[
+ \|\nabla B\|_{H^3} \|\nabla \times \tilde{g}\|_{H^3} + \|B\|_{H^3}^2 \|\tilde{g}\|_{W^{5, \infty}} \|U\|_{H^3}^2
\]

\[
\leq C \left[ \|U\|_{H^3}^2 + \|U\|_{H^3} + \|B\|_{H^3} \right] \|\alpha_1 M_2 M_0^{-1} e^{-\frac{\mu t}{2}}
\]

\[
+ \|\alpha_2 M_2 M_0^{-1} e^{-\frac{\mu t}{2}}\|\nabla B\|_{H^3} \cdot \|\alpha_1 M_2 M_0^{-1} e^{-\frac{\mu t}{2}}\|B\|_{H^3} \|U\|_{H^3}
\]

\[
+ \left( \|\alpha_1 M_2 M_0^{-1} e^{-\frac{\mu t}{2}}\|U\|_{H^3} + \|\alpha_2 M_2 M_0^{-1} e^{-\frac{\mu t}{2}}\|B\|_{H^3} \right) \|U\|_{H^3}
\]

\[
+ \left( \alpha_1^2 e^{-\frac{\mu t}{2}} + \alpha_2^2 e^{-\frac{\mu t}{2}} \right), \left( \delta M_0^2 M_2^2 + M_0^{-1} M_2^3 \right) \|U\|_{H^3} \right].
\]
In a result, we get
\[ \Pi \leq C \left( |\alpha_1| (M_1 + M_2) e^{-\frac{\mu t}{4}} \|U\|_{H^3}^2 + |\alpha_1| M_2 e^{-\frac{\mu t}{4}} \|B\|_{H^3}^2 \right) \\
+ C |\alpha_2| M_1 e^{-\frac{\mu t}{4}} \|U\|_{H^3} \|B\|_{H^3} \\
+ C \left( \alpha_2^2 e^{-\frac{\mu t}{4}} + \alpha_2^2 e^{-\frac{\mu t}{4}} \right) \left( \delta M_0^2 M_1^2 + M_0^{-1} M_2^2 \right) \|U\|_{H^3} \\
+ C |\alpha_1 \alpha_2| M_0^{-\frac{3}{2}} M_2^2 e^{-\frac{\mu t}{4}} \|\nabla B\|_{H^3}. \] (44)

**Step 3. Energy estimates of \( U, B \).** Gathering above estimates in Step 1 and Step 2, we obtain
\[
\frac{1}{2} \frac{d}{dt} \left( \|U\|_{H^3}^2 + \|B\|_{H^3}^2 \right) + \frac{\nu}{2} \|\nabla U\|_{H^3}^2 + \frac{\mu}{2} \|\nabla B\|_{H^3}^2 \leq C \sum_{i=1}^{8} J_i,
\]
where
\[
J_1 = (\|U\|_{H^3} + (1 + \eta) \|B\|_{H^3}) (\|\nabla U\|_{H^3}^2 + \|\nabla B\|_{H^3}^2), \\
J_2 = \left( \|\tilde{f}\|_{W^{4,\infty}} + (1 + \eta) \|\tilde{g}\|_{W^{4,\infty}} \right) (\|U\|_{H^3}^2 + \|B\|_{H^3}^2), \\
J_3 = \left( \|\tilde{f} \times \tilde{g}\|_{H^3} + M_0^{-\frac{3}{2}} M_2^2 |\alpha_2| e^{-\frac{\mu t}{4}} + |\alpha_1 \alpha_2| M_0^{-\frac{3}{2}} M_2^2 e^{-\frac{(\mu+\eta)t}{4}} \right) \|\nabla B\|_{H^3}, \\
J_4 = (\|f \times \tilde{g}\|_{H^3} + |\alpha_1| (M_1 + M_2) e^{-\frac{\mu t}{4}} \|U\|_{H^3}^2, \\
J_5 = |\alpha_1| M_2 e^{-\frac{\mu t}{4}} \|B\|_{H^3}^2 + |\alpha_2| M_1 e^{-\frac{\mu t}{4}} \|U\|_{H^3} \|B\|_{H^3}, \\
J_6 = \left( \alpha_1^2 e^{-\frac{\mu t}{4}} + \alpha_2^2 e^{-\frac{\mu t}{4}} \right) \left( \delta M_0^2 M_1^2 + M_0^{-1} M_2^2 \right) \|U\|_{H^3}, \\
J_7 = |\alpha_1 \alpha_2| M_0^{-\frac{3}{2}} M_2^2 e^{-\frac{(\mu+\eta)t}{4}} \|\nabla B\|_{H^3} + \eta \|\tilde{g}\|_{W^{3,\infty}} \|B\|_{H^3}^2, \\
J_8 = \left( \|\tilde{f} \times (\nabla \times \tilde{f})\|_{H^3} + |\tilde{g} \times (\nabla \times \tilde{g})|_{H^3} \right) \|U\|_{H^3}, \\
J_9 = \eta \|\nabla B\|_{H^3} \|B\|_{H^3} |\tilde{g}|_{W^{3,\infty}} + M_0^{-\frac{3}{2}} M_2^2 |\alpha_1| e^{-\frac{\mu t}{4}} \|\nabla U\|_{H^3}. \\
\]

Using Lemma 2.3, Young’s inequality and \( \delta^{-\frac{1}{2}} \geq M_0 \gg 1 \), we derive that
\[
\frac{d}{dt} \left( \|U\|_{H^3}^2 + \|B\|_{H^3}^2 \right) + \left( \frac{\nu}{2} - C \|U\|_{H^3} - C \|B\|_{H^3} \right) \|\nabla U\|_{H^3}^2 \\
+ \left( \frac{\mu}{2} - C \|U\|_{H^3} - C \|B\|_{H^3} \right) \|\nabla B\|_{H^3}^2 \leq C \left( e^{-\frac{\mu t}{4}} + e^{-\frac{\mu t}{4}} \right) (\|U\|_{H^3}^2 + \|B\|_{H^3}^2) + C \left( M_0^{-1} + \delta^2 M_0^3 \right) \left( e^{-\frac{\mu t}{4}} + e^{-\frac{\mu t}{4}} \right) \] (45)

for some constant \( C \) depending on \( M_1, M_2, \mu, \nu, \eta, \alpha_1, \alpha_2 \).

For \( t \in [0, \infty) \), we assume that
\[
\|U(t)\|_{H^3}^2 + \|B(t)\|_{H^3}^2 \leq \frac{\min\{\mu, \nu\}}{4C}.
\]

In case \( t = 0 \), the above estimate holds. Applying differential inequality 45, Gronwall’s inequality and \( \delta \leq M_0^{-2} \), we have
\[
\|U(t)\|_{H^3} + \|B(t)\|_{H^3} \leq M_0^{-\frac{1}{2}}. \] (46)
In a result,
\[ ||U(t)||_{H^3} + ||B(t)||_{H^3} \leq M_0^{\frac{1}{2}} \]
for all \( t \in [0, \infty) \). Therefore, we complete the proof of Theorem 1.1.

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Received for publication March 2019.

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