There is no square-complementary graph of girth 6

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Abstract

A graph is square-complementary (squco, for short) if its square and complement are isomorphic. We prove that there is no squco graph of girth 6, thus answering a question asked by Milanič et al. [Discrete Math., 2014, to appear], and leaving $g = 5$ as the only possible value of $g$ for which the existence of a squco graph of girth $g$ is unknown.

1 Introduction

Given two graphs $G$ and $H$, we say that $G$ is the square of $H$ (and denote this by $G = H^2$) if their vertex sets coincide and two distinct vertices $x$, $y$ are adjacent in $G$ if and only if $x$, $y$ are at distance at most two in $H$. Squares of graphs and their properties are well-studied in literature (see, e.g., Section 10.6 in the monograph [3]). A graph $G$ is said to be square-complementary (squco for short) if its square is isomorphic to its complement. That is, $G^2 \cong \overline{G}$, or, equivalently, $G \cong \overline{G^2}$. The question of characterizing squco graphs was posed by Seymour Schuster at a conference in 1980 [10]. Since then, squco graphs were studied in the context of graph equations in terms of operators such as the line graph and complement (see [1,2,4–6,9]). The entire set of solutions of some of these equations was found (see for example [1] and references quoted therein). However, the set of solutions of the equation $G^2 \cong \overline{G}$ remains unknown, despite several attempts to describe it (see for example [2, 5, 8]). The problem of determining all squco graphs was also posed as Open Problem No. 36 in Prisner’s book [9].

Examples of squco graphs are $K_1$, $C_7$, and a cubic vertex-transitive bipartite squco graph on 12 vertices, known as the Franklin graph (see Fig. 1).

![Figure 1: The Franklin graph.](image)

The following two propositions, due to Baltić et al. [2] (and partially due to Capobianco and Kim [5]), summarize the results regarding the connectivity, radius, and diameter of squco graphs.

**Proposition 1.** Every squco graph is connected and has no cut vertices.

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Proposition 2. If \( G \) is a nontrivial squco graph, then \( \text{rad}(G) = 3 \) and \( 3 \leq \text{diam}(G) \leq 4 \). Moreover, if \( G \) is regular, then \( \text{diam}(G) = 3 \).

It is not known whether a squco graph of diameter 4 exists. In the paper \([8]\), several other questions regarding squco graphs were posed, and a summary of the known necessary conditions for squco graphs was given. Among them is the following result expressing a condition on the girth. (Recall that the girth of a graph \( G \) is the length of a shortest cycle in \( G \), or \( \infty \) if \( G \) is acyclic.)

Proposition 3. If \( G \) is a nontrivial squco graph with girth at least 7, then \( G \) is the 7-cycle.

This proposition leaves only 5 possible values for the girth \( g \) of a squco graph \( G \), namely \( g \in \{3, 4, 5, 6, 7\} \). The case \( g = 7 \) is completely characterized by Proposition 2. Baltić et al. \([2]\) and Capobianco and Kim \([5]\) asked whether there exists a squco graph of girth 3. An affirmative answer to this question was provided in \([8]\) by a squco graph on 41 vertices with a triangle (namely, the circulant \( C_{41}(\{4, 5, 8, 10\}) \)). As shown by the Franklin graph, there also exists a squco graph of girth 4. The questions regarding the existence of squco graphs of girth 5 or 6 were left as open questions in \([8]\). In this note, we answer one of them, by proving that there is no squco graph of girth 6. This leaves \( g = 5 \) as the only possible value of \( g \) for which the existence of a squco graph of girth \( g \) is unknown.

We briefly recall some useful definitions. Given two vertices \( u \) and \( v \) in a connected graph \( G \), we denote by \( d_G(u, v) \) the distance in \( G \) between \( u \) and \( v \) (that is, the number of edges on a shortest \( u \)-\( v \) path). For a positive integer \( i \), we denote by \( N_i(v, G) \) the set of all vertices \( u \) in \( G \) such that \( d_G(u, v) = i \), and by \( N_{\geq i}(v, G) \) the set of all vertices \( u \) in \( G \) such that \( d_G(u, v) \geq i \).

We use standard graph terminology \([7]\).}

2 The result

Theorem 1. There is no squco graph of girth 6.

Proof. Suppose for a contradiction that \( G \) is a squco graph of girth 6. First, we observe that if \( x \) is a vertex of \( G \), then there are no edges in any of sets \( N_i(x, G) \) for \( i = 1, 2 \) and no two distinct vertices in \( N_1(x, G) \) have a common neighbor in \( N_2(x, G) \). Let \( k = \Delta(G) \) be the maximum degree of \( G \), and let \( w \) be a vertex of degree \( k \). Since the only squco graphs with maximum degree at most 2 are \( K_1 \) and \( C_7 \) \([8]\), we have \( k \geq 3 \).

We consider two cases.

Case 1. \( w \) has a neighbor of degree at least three.

Let \( v \) be a neighbor of \( w \) of degree at least three, and let \( p \) and \( q \) be two neighbors of \( v \) other than \( w \). If one of them, say \( p \), is of degree at least 3, then \( p \) has at least two neighbors in \( N_2(v, G) \) and thus \( \Delta(G') \geq |N_1(q, G')| \geq k + 1 \), contrary to the fact that \( G' \cong G \). Hence, both \( p \) and \( q \) are of degree 2. (Notice that Proposition 1 excludes the possibility of having degree 1 vertices.) Let \( a \) and \( b \) be the unique neighbors of \( p \) and \( q \) in \( N_2(v, G) \), respectively. The set \( N_3(v, G) \) is nonempty, because radius of \( G \) is 3 by Proposition 2. Vertices \( a \) and \( b \) must be adjacent to all of vertices in \( N_2(v, G) \), otherwise \( \Delta(G') \geq \max(|N_1(p, G')|, |N_1(q, G')|) \geq k + 1 \), contrary to the fact that \( G' \cong G \). To avoid a 4-cycle in \( G \), we conclude that \( |N_3(v, G)| = 1 \). But now, the degree of \( v \) in \( G' \) is 1, which implies that \( G' \) has a cut vertex, contrary to the fact that \( G \) is squco and Proposition 1.

Case 2. All neighbors of \( w \) are of degree at most two.

In this case, all neighbors of \( w \) are of degree exactly two. In particular, \( |N_2(w, G)| = |N_1(w, G)| = k \geq 3 \). Now we will show that every vertex \( x \) from \( N_2(w, G) \) is of degree at least
Let $x \in N_2(w, G)$, and let $y$ be the unique neighbor of $x$ in $N_1(w, G)$. Vertex $x$
has at least $|N_3(w, G)| - 1$ neighbors in $N_3(w, G)$, since otherwise $|N_1(y, \overline{G^2})| \geq k + 1$. This
implies that any two vertices from $N_2(w, G)$ (the size of $N_2(w, G)$ is at least 3) have at least
$|N_3(w, G)| - 2$ common neighbors in $N_3(w, G)$. This bounds $|N_3(w, G)| \leq 3$, otherwise we would
have a 4-cycle.

Suppose $|N_3(w, G)| = 3$. To each of the three pairs of vertices in $N_3(w, G)$, associate, if possible,
their common neighbor in $N_2(w, G)$. Because, each vertex in $N_2(w, G)$ is connected to at least two vertices in $N_3(w, G)$, it is surely associated with some pair. If $|N_2(w, G)| \geq 4$
then some two vertices from $N_2(w, G)$ are associated with the same pair and we get a 4-cycle, a
contradiction. We thus have $|N_1(w, G)| = |N_2(w, G)| = k \leq 3$ and $|N_{\geq 4}(w, G)| = 0$ (otherwise
we would have a vertex of degree at least $4 > k$ in $\overline{G^2}$). This implies that our graph has at most
ten vertices. All squco graphs with at most 11 vertices are known \[8\]: none of them has girth 6. Hence this is a contradiction with $G$ having girth 6.

Suppose $|N_3(w, G)| = 2$. If $k \leq 4$, then we our graph has no more than 11 vertices, which
is not possible. Hence $k \geq 5$. There must be at least $2k - 1$ vertices of degree two in $G$ (all $k$
vertices in $N_1(w, G)$; at most one of $k$ vertices in $N_2(w, G)$ has both vertices from $N_3(w, G)$ for
neighbors, otherwise we have a 4-cycle as before). In $\overline{G^2}$ at most $k + 3$ of them are of degree
two, because every vertex in $N_1(w, G)$ will be connected to all but one vertex in $N_2(w, G)$ in $\overline{G^2}$,which is a contradiction, because $k \geq 5$.

The last possibility is that $|N_3(w, G)| = 1$, but then $w$ would be of degree 1 in $\overline{G^2}$, again a
contradiction. This completes the proof.$\Box$

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