How to bosonize fermions with non-linear energy dispersion

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We develop a systematic method to treat the effect of non-linearity in the energy dispersion on the usual bosonization result for the single-particle Green’s function of fermions in arbitrary dimension. The leading corrections due to the quadratic term in the energy dispersion are explicitly calculated. In the Chern-Simons theory of the half-filled quantum Hall systems curvature is shown to be essential.

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For many years the bosonization technique has been successfully used to study one-dimensional Fermi systems beyond perturbation theory [1]. Motivated by experimental evidence for non-Fermi liquid behavior in high-temperature superconductors and quantum Hall systems, the generalization of this approach to arbitrary dimensions \(d\) has recently received a lot of attention [2,3]. However, although bosonization is non-perturbative in the sense that infinitely many Feynman diagrams are summed via an underlying Ward identity [10], it hinges on one essential approximation: the linearization of the non-interacting energy dispersion \(\epsilon_k\). Namely, measuring wave-vectors with respect to coordinate systems centered at points \(k^\alpha\) (see Fig. 1), we may write \(\epsilon_{k^\alpha + \mathbf{q}} = \epsilon_{k^\alpha} + v^\alpha \cdot \mathbf{q} + \frac{\alpha^2}{2m^\alpha}\), where \(v^\alpha\) is the Fermi velocity and \(m^\alpha\) is the effective mass close to \(k^\alpha\). Conventional bosonization sets \(\frac{\alpha^2}{2m^\alpha} = 0\). Haldane [11] has speculated that it should be possible to develop a perturbation theory around the non-perturbative bosonization solution of the linearized theory, using \(\frac{\alpha^2}{2m^\alpha}\) as small parameter. An attempt to construct such an expansion has recently been made by Khveshchenko [2]. However, so far his method has not been proven to be useful in practice. To the best of our knowledge, an explicit calculation of the effect of finite \(m^\alpha\) on the usual bosonization solution for the single-particle Green’s function does not exist. In the present work we shall solve this problem in arbitrary \(d\) by means of our functional bosonization approach [7–9]. We then use our method to study the relevance of curvature in the Chern-Simons theory of the half-filled Landau level [13]. This problem has recently been examined by several authors with conflicting results [3,12,14]. We hope that our work will clarify the issue.

For simplicity let us first consider a system of spinless fermions with Landau interaction parameters \(f_{\mathbf{q}kk'}\), where \(\mathbf{q}\) is the momentum transfer between two particles with initial momenta \(k\) and \(k'\). We start by partitioning momentum space into a finite number of sectors \(K^\alpha\), \(\alpha = 1, \ldots, n\), which depend on cutoffs \(\Lambda^\alpha\) and \(\lambda^\alpha\) as shown in Fig. 1. Assuming that the \(f_{\mathbf{q}kk'}\) are dominated by \(|\mathbf{q}| \lesssim q_c\), we choose \(\Lambda^\alpha, \lambda^\alpha \gg q_c\). On the other hand, the sectors must be sufficiently small so that the local curvature of the Fermi surface is constant within a given sector. Furthermore, \(f_{\mathbf{q}kk'}\) should not change appreciably when \(k\) and \(k'\) are restricted to given sectors, so that we may replace \(f_{\mathbf{q}kk'} \to f_{\mathbf{q}kk'}^{\alpha}\). In particular, for Fermi surfaces with constant curvature and interactions \(f_{\mathbf{q}}\) that are independent of \(k\) and \(k'\), there is not need any more for introducing several sectors as long as we can handle the curvature problem. In this case we formally identify the entire momentum space with a single sector.

FIG. 1. Definition of the sectors \(K^\alpha\) and cutoffs \(\Lambda^\alpha\) and \(\lambda^\alpha\). The thick solid line is the Fermi surface, and the dots are the origins \(k^\alpha\) of local coordinate systems. For sectors that intersect the Fermi surface we choose \(k^\alpha\) such that \(\epsilon_{k^\alpha} = \mu\), but in general \(\epsilon_{k^\alpha} \neq \mu\). By construction the union \(\bigcup_\alpha K^\alpha\) covers all degrees of freedom in the system.

We are interested in the Matsubara Green’s function \(G(k)\), which can be represented as a Grassmannian functional integral in the usual way [13,16]. We eliminate the Grassmann fields by means of a Hubbard-Stratonovich transformation involving bosonic auxiliary fields \(\phi_\alpha^\beta\) associated with the sectors. After the standard transformations [7,9] the Green’s function can be exactly written as \(G(k) = \langle \hat{G}\rangle_{kk} > S_{eff}\), where \(\hat{G}\) is an infinite matrix in...
momentum- and frequency space, with matrix elements given by 
\[ \hat{G}^{-1} \]_{kk'} = [G_0^{-1}]_{kk'} - [\hat{V}]_{kk'}. \] Here \[ [G_0]_{kk'} = \delta_{kk'}G_0(k), \] with \[ G_0(k) = i\omega_k - \epsilon_k + \mu \] \(^{-1}\), and the infinite matrix \( \hat{V} \) is defined by 
\[ [\hat{V}]_{kk'} = \sum_q \Theta^\alpha(k)V^{\alpha}_{k-q}, \]
with \( V^{\alpha}_q = \frac{i}{2}\phi^\alpha_q \). Here \( \beta \) is the inverse temperature, \( \mu \) is the chemical potential, and the cutoff function \( \Theta^\alpha(k) \) is unity if \( k \in K^\alpha \), and vanishes otherwise. Let us emphasize that the above construction includes the case that we identify the entire momentum space with a single sector: then the \( \alpha \)-sums run over a single term \( \alpha = 1 \) with \( \Theta^\alpha(k) = 1 \), and it is convenient to choose \( k^0 = \mu \). The symbol \( < \ldots >_{S_{eff}} \) denotes functional averaging with respect to the effective action \( S_{eff} = S_2 + S_{kin} \), where 
\[ S_2 = \frac{V}{2\beta} \sum_q \sum_\alpha [f_q^{-1}]^{\alpha'}_\alpha \phi^\alpha_q^\dagger \phi^\alpha_q, \quad \text{and} \quad S_{kin} = -Tr\ln[1 - \hat{G}_0(\hat{V})]. \]
Here \( V \) is the volume of the system and \( \int_q \) is a matrix in the sector indices, with elements 
\[ [f_q]^{\alpha'}_\alpha = f_q^{\alpha'\alpha}. \]
For \( q_e \ll \Lambda^\alpha, \lambda^\alpha \) the matrix \( \hat{G} \) is approximately block-diagonal, with blocks labelled by the sector index \( \alpha \). Shifting \( k = k^0 + q \) and choosing \( |q| \ll \Lambda^\alpha, \lambda^\alpha \), we have 
\[ \hat{G}(k^0 + q, \omega_q) = (\hat{G}_0^q[q, \omega_q])_{\alpha\alpha'} >_{S_{eff}}, \]
where the Fourier transform \( G^\alpha(r, r', \tau, \tau') \) of \( [G^\alpha]_{qq'} \) satisfies
\[ -\partial_\tau - e^\alpha(P_\tau) + \mu - V^{\alpha}(r, \tau) \] \[ \times G^\alpha(r, r', \tau, \tau') = \delta(r - r')\delta^{\tau}(\tau - \tau'). \]
(1)
Here \( P_\tau = -i\nabla_\tau \) is the momentum operator (we use units where \( \hbar = 1 \), \( e^\alpha(q) \equiv e^{\epsilon_{k^0}+q} \), and \( \delta(r) = \frac{1}{2\beta} \sum_n e^{-i\omega_n r} \) is the antiperiodic \( \delta \)-function. The potential \( V^{\alpha}(r, \tau) \equiv \sum_q e^{i(q \cdot r - \omega_m \tau)}V^\alpha_q \) is the Fourier transform of the Hubbard-Stratonovich field \( V^\alpha_q = \check{\Phi}^\alpha_q \). Eq.\( \text{(1)} \) together with the boundary condition that \( G^\alpha(r, r', \tau, \tau') \) should be antiperiodic in \( \tau \) and \( \tau' \) uniquely determines \( G^\alpha \). Hence, to calculate the Green’s function of the many-body system, we first need to solve Eq.\( \text{(1)} \) for fixed \( V^{\alpha}(r, \tau) \), and then average the result with respect to the effective action \( S_{eff} \). Note that for \( \frac{1}{m^\alpha} \neq 0 \) the action \( S_{eff} \) is not Gaussian. The leading non-Gaussian corrections can be easily calculated \( \text{[9]} \).

Solving Eq.\( \text{(1)} \) for finite \( m^\alpha \) is more difficult. Let us make the ansatz
\[ G^\alpha(r, r', \tau, \tau') = G^\alpha_0(r, r', \tau, \tau') e^{\Phi^\alpha(r, \tau) - \Phi^\alpha(r', \tau')}. \]
(2)
To satisfy the boundary conditions, we require that \( \Phi^\alpha(r, \tau) \) should be periodic in \( \tau \), while \( G^\alpha_0(r, r', \tau, \tau') \) should be antiperiodic in \( \tau \) and \( \tau' \). The crucial observation is now that we obtain an exact solution of Eq.\( \text{(1)} \) by choosing \( \Phi^\alpha \) and \( G^\alpha_0 \) such that
\[ [-\partial_\tau - \xi^\alpha(P_\tau)] \Phi^\alpha(r, \tau) = V^{\alpha}(r, \tau) + \frac{[P_\tau \Phi^\alpha(r, \tau)]^2}{2m^\alpha}, \]
(3)
where \( \xi^\alpha(q) = e^\alpha(q) - e^\alpha(0) \), and \( u^\alpha(r, \tau) = P_\tau \Phi^\alpha(r, \tau) \).

Differential equations of the type (3) are called eikonal equations, and appear in many fields of physics, such as geometrical optics, quantum mechanical scattering theory, and relativistic quantum field theories \( \text{[7]} \). In the limit \( \frac{1}{m^\alpha} \rightarrow 0 \) the eikonal equation (3) is linear and can be solved exactly via Fourier transformation. The solution has first been discussed by Schwinger \( \text{[8]} \), see also Refs. \( \text{[8, 9]} \). Furthermore, in this case the velocity \( u^\alpha(r, \tau) \) vanishes, so that \( G^\alpha_0(r, r', \tau, \tau') = G^0_0(r - r', \tau - \tau') \), where the Fourier transform of \( G^0_0(r, \tau) \) is given by \( G^0_0(q) \equiv G^0_0(0) \equiv G^0_0(\mathbf{k}^0 + q, \omega_n) \). For finite \( m^\alpha \), Eq.\( \text{(3)} \) describes the motion of a fermion under the influence of a space- and time-dependent random velocity \( u^\alpha(r, \tau) \). At the first sight it seems that this problem is just as difficult to solve as the original Eq.\( \text{(1)} \).

However, perturbation theory in terms of the derivative potential \( u^\alpha(r, \tau) \cdot P_\tau \) in Eq.\( \text{(4)} \) is less infrared singular than perturbation theory in terms of the original random potential \( V^\alpha(r, \tau) \) in Eq.\( \text{(1)} \). Moreover, for large effective mass \( m^\alpha \) the random velocity \( u^\alpha(r, \tau) \) is a small parameter which formally justifies the perturbative treatment of the derivative potential.

Let us first consider the eikonal equation (3). Although it is impossible to solve this non-linear partial differential equation exactly, we can obtain the solution as series in powers of \( V^\alpha \) \( \text{[7]} \). At this point it is convenient to work in Fourier space. Defining \( \Phi^\alpha(q, \tau) = \sum_q e^{i(q \cdot r - \omega_m \tau)}\check{\Phi}^\alpha_q \) and \( \Psi^\alpha_q = [i\omega_m - \xi^\alpha(q)]\Phi^\alpha_q \), the eikonal can be written as
\[ \Phi^\alpha(r, \tau) - \Phi^\alpha(r', \tau') = \sum_q \mathcal{J}^\alpha_q(r, r', \tau, \tau') \Psi^\alpha_q, \]
(5)
where the functional \( \Psi^\alpha_q \) satisfies for \( q \neq 0 \) the non-linear integral equation
\[ \Psi^\alpha_q = V^\alpha_q + \sum_{q', q''} \delta_{q, q'} \gamma^\alpha_{q''} q'' \Psi^\alpha_q \Psi^\alpha_{q''}, \]
(7)
with the kernel given by \( \gamma^\alpha_{q''} = \frac{q''^2}{2m^\alpha}G^{0}_0(q')G^{0}_0(q'') \). Here \( G^0_0(q) = [i\omega_m - \xi^\alpha(q)]^{-1} \) is a bosonic Matsubara Green’s function with energy dispersion given by the excitation energy \( \xi^\alpha(q) \) \( \text{[10]} \). The \( q = 0 \)-term requires a special treatment. From the definition \( \Psi^\alpha_q = [i\omega_m - \xi^\alpha(q)]\Phi^\alpha_q \) it is clear that \( \Psi^\alpha_0 = 0 \). Iterating Eq.\( \text{(4)} \), we obtain a series in powers of the random potential, \( \Psi^\alpha_q = \sum_{n=1}^\infty \Psi^\alpha_{n,q} \) for \( q \neq 0 \)
\[ \Psi^\alpha_{n,q} = \sum_{q_1 \ldots q_n} \delta_{q_1 q} \ldots \delta_{q_n q} C^\alpha_{n}(q_1 \ldots q_n)V^\alpha_{q_1} \ldots V^\alpha_{q_n}, \]
(8)
with \( C^\alpha_{n} \propto (1/m^\alpha)^{n-1} \). The first two vertices are \( C^\alpha_{1}(q_1) = 1 \) and \( C^\alpha_{2}(q_1 q_2) = \gamma^\alpha_{q_1 q_2} \). Having solved Eq.\( \text{(4)} \)
to a certain order in $V^\alpha$, we know also the random velocity $\mathbf{u}^\alpha(r, \tau)$ in Eq. (3) to the same order in $V^\alpha$. In Fourier space Eq. (4) is equivalent with the Dyson equation

$$G^\alpha_1(\mathbf{q}, \mathbf{q}') = \delta_{\mathbf{q}, \mathbf{q}'} G^\alpha_0(\mathbf{q}) + G^\alpha_0(\mathbf{q}) \sum_{\mathbf{q}''} D^\alpha_{\mathbf{q}, \mathbf{q}'} G^\alpha_2(\mathbf{q}'', \mathbf{q}'),$$

(9)

where the matrix elements of the derivative potential are $D^\alpha_{\mathbf{q}, \mathbf{q}'} = \Psi^\alpha_{\mathbf{q} - \mathbf{q}''} \lambda^\alpha_{\mathbf{q}''} \Psi^\alpha_{\mathbf{q}''}$. Iteration of Eq. (8) generates an expansion of $G^\alpha_1$ in powers of the derivative potential. We would like to emphasize that we are not simply expanding in powers of $\frac{1}{\beta V}$. Because the Gaussian propagator of the $V^\alpha$-field is proportional to the screened interaction $f^{RPA, \alpha}$ within random-phase approximation (RPA) [3, 4], the effective expansion parameter is proportional to $f^{RPA, \alpha}/m^\alpha$. This will become obvious shortly.

To obtain the Green’s function of the many-body system, we need to average Eq. (3) with respect to the effective action $S_{eff}$. Because averaging restores translational invariance, we may set $r' = \tau' = 0$ and calculate $G^\alpha_2(r, \tau) = \langle G^\alpha_2(r, 0, \tau) \rangle_{S_{eff}}$, where

$$G^\alpha_2(r, \tau) = \frac{\langle \delta G^\alpha_1(r, 0, \tau) \delta e^{\Phi^\alpha(r, \tau) - \Phi^\alpha(0, 0)} \rangle_{S_{eff}}}{\langle e^{\Phi^\alpha(r, \tau) - \Phi^\alpha(0, 0)} \rangle_{S_{eff}}}.$$

(11)

Here $\delta X = X - \langle X \rangle_{S_{eff}}$. We now perform the averaging perturbatively. Note that the fermionic degrees of freedom have been completely eliminated, so that the perturbation theory is formulated in terms of the bosonic field $V^\alpha$. Following Refs. [3, 4], we calculate $Q^\alpha(r, \tau)$ via a linked cluster expansion. In this way we obtain an expansion $Q^\alpha(r, \tau) = \sum_{n=1}^{\infty} Q^\alpha_n(r, \tau)$, with

$$Q^\alpha_n(r, \tau) = \sum_{q_1, \ldots, q_n} \delta_{\mathbf{q}_1, \ldots, \mathbf{q}_n} W^\alpha_{n}(\mathbf{q}_1 \ldots \mathbf{q}_n) \times J^\alpha_n(\mathbf{q}_1, \mathbf{r}, \tau) J^\alpha_n(\mathbf{q}_2, \mathbf{r}, \tau) \ldots J^\alpha_n(\mathbf{q}_n, \mathbf{r}, \tau) \ldots$$

(12)

where $J^\alpha_n(\mathbf{r}, \tau) = J^\alpha_n(\mathbf{r}, 0, \tau, 0)$, and the vertices $W^\alpha_n$ can be calculated perturbatively in powers of our small parameter $f^{RPA, \alpha}/m^\alpha$. At the leading tree-level (where bosonic loops are neglected) the vertex $W^\alpha_n$ is proportional to $(f^{RPA, \alpha}/m^\alpha)^{n-1}$. Already the first term $Q^\alpha_1(r, \tau)$ contains non-trivial effects due to the finiteness of $\frac{1}{\beta V}$. Hence, to study the relevance of curvature, it is sufficient to calculate $Q^\alpha_1(r, \tau)$ at tree-level, in which case $W^\alpha_1$ is approximated by $-\frac{1}{\beta V} f^{RPA, \alpha}$. This amounts to averaging in Eq. (3) with respect to $S_{eff}$ in Gaussian approximation [3, 4], and yields $Q^\alpha_1(r, \tau) = R^\alpha_1 - S^\alpha_1(r, \tau)$, with $R^\alpha_1 = S^\alpha_1(0, 0)$ and $S^\alpha_1(r, \tau) = \frac{1}{\beta V} \sum_q f^{RPA, \alpha}_q \cos(\mathbf{q} \cdot \mathbf{r} - \omega_m \tau)$. (13)

Note that $\xi^\alpha(-\mathbf{q}) = -\xi^\alpha(\mathbf{q}) + \frac{9}{m^\alpha}$, so that for finite $m^\alpha$ the integrand in Eq. (13) has only simple poles. In contrast, for $m^\alpha = 0$ the denominator in Eq. (13) gives rise to a double pole, which leads to rather peculiar features in the analytic structure of the Green’s function in $d > 1$ [3, 4]. It is also not difficult to calculate the next order in $f^{RPA, \alpha}/m^\alpha$. Then one should retain the vertex $W^\alpha_2(q, q_2)$ at tree-level (in which case it is approximated by $-\frac{1}{\beta V} f^{RPA, \alpha}_q f^{RPA, \alpha}_{q_2}$ and include one-loop corrections to $W^\alpha_1(q_1)$ (which lead to a small renormalization of the RPA-interaction in Eq. (3)).

Next, consider the calculation of $G^\alpha_1$ and $G^\alpha_2$. Because the derivative potential removes possible infrared singularities, we may use the conventional impurity diagram technique [21]. For the average $G^\alpha_1 = \langle G^\alpha_2 \rangle$ we calculate the irreducible self-energy in self-consistent Born approximation. By expanding the self-energy (and not directly the Green’s function) we take into account an infinite number of terms in the iteration of the Dyson equation (3). A truncation at a finite order would lead to unphysical multiple poles in the Fourier transform $G^\alpha_2(q)$ of $G^\alpha_2(r, \tau)$. We obtain $[G^\alpha_1(q)]^{-1} = [G^\alpha_0(q)]^{-1} - \Sigma^\alpha(q)$, where, to leading order in $f^{RPA, \alpha}/m^\alpha$,

$$\Sigma^\alpha(q) = \frac{1}{\beta V} \sum_q f^{RPA, \alpha}_q G^\alpha_0(q) G^\alpha_0(-q') \frac{\mathbf{q} \cdot \mathbf{q}'}{m^\alpha} \times \left\{ \frac{\mathbf{q} \cdot \mathbf{q}'}{m^\alpha} G^\alpha_1(q + q') + \frac{\mathbf{q}^2}{2m^\alpha} [G^\alpha_1(q + q') - G^\alpha_1(q - q')] \right\}.$$ 

(14)

Similarly, we use perturbation theory to calculate $G^\alpha_2$. The leading contribution to Eq. (13) is of order $f^{RPA, \alpha}/m^\alpha$, and yields for the Fourier transform $G^\alpha_2(q) = G^\alpha_1(q) Y^\alpha(q)$, with

$$Y^\alpha(q) = -\frac{1}{\beta V} \sum_q f^{RPA, \alpha}_q G^\alpha_0(q') G^\alpha_0(-q') \times \left\{ \frac{\mathbf{q}^2}{m^\alpha} G^\alpha_1(q + q') + \frac{\mathbf{q} \cdot \mathbf{q}'}{m^\alpha} [G^\alpha_1(q + q') - G^\alpha_1(q - q')] \right\}.$$ 

(15)

Eqs. (13), (14) and (15) are the main result of this work. Higher order corrections involve at least an additional power of $f^{RPA, \alpha}/m^\alpha$. For $\frac{\beta V}{m^\alpha} = 0$ we have $G^\alpha_2 = 0$ and $G^\alpha_1 = G^\alpha_0$. In $d = 1$ we reproduce then the well-known bosonization solution for the Tomonaga-Luttinger model. Furthermore, direct expansion of our result for $G^\alpha_2(r, \tau)$ to first order in $f^{RPA, \alpha}$ exactly reproduces the so-called GW-approximation for the self-energy [22], with non-linear energy dispersion. If we set $\Sigma^\alpha_1 = \langle Y^\alpha \rangle = 0$, we would have obtained a discrepancy with the GW self-energy, because for finite $m^\alpha$ the exponentiation $e^{Q^\alpha}$ of
the perturbation series is not quite correct. In a sense, we have exponentiated "too much", so that it is necessary to introduce correction terms in the prefactor. However, even in $d = 1$ these corrections can be calculated perturbatively, because the extra powers of $q'\tilde{\alpha}$ in the numerator of Eqs. (14) and (13) remove the infrared divergencies.

Our functional bosonization approach can be generalized to include transverse gauge fields $\tilde{A}_0$, so that we have now a powerful non-perturbative method for studying the relevance of curvature in the Chern-Simons theory for the half-filled Landau level [13]. To leading order in the relevant small parameter (see below), we simply need to replace $f_q^{\text{RPA},\alpha}$ in Eqs. (13), (14) and (15) by the corresponding propagator $f_q^{\text{CS},\alpha}$ of the transverse gauge field [13, 14]. In the most important parameter regime $|\omega_m| \ll v_F^2|q| \ll v_F^2\kappa$ (here $\kappa = e^2m^*/\epsilon$ is the Thomas-Fermi screening wave-vector in $d = 2$, $v_F^2$ is the effective Fermi velocity, $m^*$ is the effective mass, and $\epsilon$ is the dielectric constant) $f_q^{\text{CS},\alpha}$ can be written as

$$f_q^{\text{CS},\alpha} = \frac{2\pi}{m^*|\omega_m|/|v_F|^2} \frac{1 - (\kappa^2 \cdot \hat{q})^2}{q^2} + |q|/q_c,$$

(16)

where we have used the Coulomb gauge and assumed a circular Fermi surface. Here $\kappa^2 = \frac{\kappa^2}{|q|}$, $\hat{q} = \frac{\kappa^2}{|q|}$, and $q_c = (\hat{\phi}/v_F^2)^2/\kappa$, with $\hat{\phi} = 2$. Substituting Eq. (16) into Eq. (13) and rescaling the variables [8], we find $Q^2(\vec{r},\tau) = gF(\tilde{\tau},\tilde{\tau}_\perp\tau;g)$, where $g = q_c/(\hat{\phi}/v_F^2)$ is a dimensionless parameter, $\tilde{\tau}_\perp\tilde{\tau}_\parallel$ is the component of the dimensionless vector $g\tilde{\tau}$ parallel [perpendicular] to $\kappa^2$, $\tilde{\tau} = v_F^2q\tau$, and $F(\tilde{\tau}_\parallel\tilde{\tau}_\perp,\tilde{\tau};g)$ is a dimensionless function that can be explicitly written down as a three-dimensional integral. Linearization of the energy dispersion corresponds to setting $g = 0$ in the integrand before doing the integration. Then it is easy to show that $F(\tilde{\tau}_\parallel,0,0;0) \sim \ln |\tilde{\tau}|$ for $|\tilde{\tau}| \rightarrow \infty$, implying an algebraic singularity in the momentum distribution [13], just like in a Luttinger liquid. On the other hand, evaluation of Eq. (13) with non-linear energy dispersion corresponds to keeping $g$ finite in the integrand. For small $g$ we find in this case $F(\tilde{\tau}_\parallel,0,0;g) \sim -cg$ for $|\tilde{\tau}_\parallel| \rightarrow \infty$, where $c > 0$ is a numerical constant. Thus, the algebraic singularity in the momentum distribution is an artifact of the linearization. This gives support to the arguments put forward in Ref. [13], and seems to agree with the experimental fact that half-filled quantum Hall systems have a sharp Fermi surface [14]. We have convinced ourselves that in the one-dimensional Tomonaga-Luttinger model a finite value of $\frac{1}{m^*}$ does not destroy the algebraic singularity in the momentum distribution. Thus, the relevance of curvature is a specific property of gauge fields.

The precise form of the Green’s function predicted by Eqs. (13), (14) and (15) for the half-filled Landau level will be discussed elsewhere. Because by construction the expansion of these expressions to first order in the interaction exactly reproduces lowest order perturbation theory, the Green’s function will certainly not be of the Fermi liquid type [13]. For small $g$ it can be justified to neglect the higher-order terms in the eikonal expansion and in the perturbative calculation of $\Sigma^{\alpha}_1$ and $Y^{\alpha}$. Using the estimates of Ref. [13], we find $g \approx 0.6$ in the experimentally relevant regime [24]. We are thus forced to conclude that for an accurate quantitative comparison with experiments higher-order terms have to be retained. We have profited from discussions with L. Bartosch, K. Schönhammer, W. Metzner, and D. V. Khveshchenko. The work of G. C. was supported by the Division of Material Science, U.S. Department of Energy under contract No. DE-AC02-76CH00016.
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