SOME RESULTS ON MULTITHRESHOLD GRAPHS

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Abstract. Jamison and Sprague defined a graph \( G \) to be a \( k \)-threshold graph with thresholds \( \theta_1, \ldots, \theta_k \) (strictly increasing) if one can assign real numbers \((r_v)_{v \in V(G)}\), called ranks, such that for every pair of vertices \( v, w \), we have \( vw \in E(G) \) if and only if the inequality \( \theta_i \leq r_v + r_w \) holds for an odd number of indices \( i \). When \( k = 1 \) or \( k = 2 \), the precise choice of thresholds \( \theta_1, \ldots, \theta_k \) does not matter, as a suitable transformation of the ranks transforms a representation with one choice of thresholds into a representation with any other choice of thresholds. Jamison asked whether this remained true for \( k \geq 3 \) or whether different thresholds define different classes of graphs for such \( k \), offering $50 for a solution of the problem. Letting \( C_t \) for \( t > 1 \) denote the class of \( 3 \)-threshold graphs with thresholds \( -1, 1, t \), we prove that there are infinitely many distinct classes \( C_t \), answering Jamison’s question. We also consider some other problems on multithreshold graphs, some of which remain open.

1. Introduction

Multithreshold graphs were introduced by Jamison and Sprague \cite{JamisonSprague} as a generalization of the well-studied threshold graphs, first introduced by Chvátal and Hammer \cite{ChvatalHammer}. Given real numbers \( \theta_1, \ldots, \theta_k \) with \( \theta_1 < \theta_2 < \cdots < \theta_k \), we say that a simple graph \( G \) is a \( k \)-threshold graph with thresholds \( \theta_1, \ldots, \theta_k \) if there exist real numbers \((r_v)_{v \in V(G)}\), called ranks, such that for every pair of distinct vertices \( v, w \in V(G) \), we have \( vw \in E(G) \) if and only if the inequality \( \theta_i \leq r_v + r_w \) holds for an odd number of indices \( i \). (Equivalently, adopting the convention that \( \theta_{k+1} = \infty \), we want \( vw \in E(G) \) if and only if \( r_v + r_w \in [\theta_{2i-1}, \theta_{2i}) \) for some \( i \).) In this case, we call \( r \) a \((\theta_1, \ldots, \theta_k)\)-representation of \( G \).

We will abbreviate this notation by saying that \( G \) is \((\theta_1, \ldots, \theta_k)\)-threshold to mean that \( G \) is \( k \)-threshold with thresholds \( \theta_1, \ldots, \theta_k \). When \( k = 1 \), we obtain the classical threshold graphs.

In the case of the classical threshold graphs, it is clear that the exact choice of threshold does not matter: by appropriately rescaling the vertex ranks, any \( \theta \)-threshold graph is seen to also be a \( \theta' \)-threshold graph. The same observation holds for \( k = 2 \): any ranks witnessing that \( G \) is \((\theta_1, \theta_2)\)-threshold can be transformed, via an appropriate affine transformation, into ranks witnessing that \( G \) is \((\theta_1', \theta_2')\)-threshold.

At the 2019 Spring Sectional AMS Meeting in Auburn, Jamison asked whether this phenomenon continues for higher \( k \), and specifically whether it still holds when \( k = 3 \). Observing that an affine transformation of the vertex ranks still uses up two “degrees of freedom” and let us express any \((\theta_1, \theta_2, \theta_3)\)-threshold graph as a \((-1, 1, t)\)-threshold graph for some \( t \), his question can be phrased as follows.

Question 1 (Jamison). Do there exist real numbers \( t, t' > 1 \) such that the class of \((-1, 1, t)\)-threshold graphs and the class of \((-1, 1, t')\)-threshold graphs differ?
Jamison offered a $50 bounty for an answer to this question. In this paper, we answer the question in the affirmative: letting $C_t$ denote the class of $(-1, 1, t)$-threshold graphs, we prove in Section 2 that there are infinitely many distinct classes $C_t$.

We also study some other questions involving multithreshold graphs. Say that $G$ is a $k$-threshold graph if there exist real numbers $\theta_1 < \cdots < \theta_k$ such that $G$ is a $(\theta_1, \ldots, \theta_k)$-threshold graph. Jamison and Sprague [3] proved that for every graph $G$, there is some $k$ such that $G$ is a $k$-threshold graph. Thus, we may define the threshold number $\Theta(G)$ of a graph $G$ to be the smallest nonnegative $k$ such that $G$ is a $k$-threshold graph.

It is natural to compare the parameter $\Theta(G)$ to other graph parameters involving threshold graphs. Cozzens and Leibowitz [2] define the threshold dimension $t(G)$ of a graph $G$ to be the smallest nonnegative integer $k$ such that $G$ can be expressed as the union of $k$ threshold graphs. Since the complement of a threshold graph is a threshold graph, we can also view $t(\overline{G})$ as the smallest nonnegative $k$ such that $G$ can be expressed as the intersection of $k$ threshold graphs.

Doignon observed, in a personal communication with the authors of [4], that any $2$-threshold graph is the intersection of two threshold graphs, hence $t(\overline{G}) \leq 2$ whenever $\Theta(G) \leq 2$. This observation suggests a possible converse:

**Question 2** (Jamison). Replacing $t(\overline{G})$ with $t(G)$, does $\Theta(G) \leq 2$ imply any bound on $t(G)$?

**Question 3** (Jamison). Is $\Theta(G)$ bounded by any function of $t(\overline{G})$ or of $t(G)$?

Question 2 has a brief answer. For any graph $G$ and positive integer $p$, let $pG$ be the disjoint union of $p$ copies of $G$. The graph $pK_2$ evidently has $t(G) = p$, since $2K_2$ is a forbidden induced subgraph for a threshold graph; on the other hand, $pK_2$ is a $(-1, 1)$-threshold graph, as witnessed by giving the endpoints $u_i$ and $v_i$ of the $i$th edge ranks $r(u_i) = -2i$ and $r(v_i) = 2i$. Hence there are graphs with $\Theta(G) = 2$ for which $t(G)$ is arbitrarily large.

In Section 3, we partially answer by proving that there are graphs with $t(\overline{G}) = 3$ for which $\Theta(G)$ is arbitrarily large. Finally, in Section 4, we discuss some remaining open problems about multithreshold graphs, along the lines of the questions considered in this paper.

## 2. Distinct Families of $(-1, 1, t)$-Threshold Graphs

To facilitate proofs about multithreshold graphs, we introduce some notational conventions. Given a multithreshold representation of a graph $G$, the weight of an edge or non-edge $uv$ is the sum of the ranks of $u$ and $v$. When it is understood which multithreshold representation we are working with, we will omit the function $r$ and simply write $v$ to stand for the rank of the vertex $v$. (Hence, the weight of an edge $uv$ will simply be written as $u + v$.)

For positive integers $p$, let $G_p = pK_2$.

**Lemma 1.** For any $p \geq 2$ and any $t > 2p - 3$, the graph $G_p$ is a $(-1, 1, t)$-threshold graph.

**Proof.** Write $t = (1 + 2\epsilon)(2p - 3)$ with $\epsilon > 0$. Letting $a_i, b_i$ be the endpoints of the $i$th edge for $i = 1, \ldots, p$, observe that the following ranks yield a $(-1, 1, t)$-threshold representation of $G_p$.

...
and by the previous claim we have
\[ a_i = -(1 + \epsilon)(i - 1) \text{ for } i = 1, \ldots, p, \]
\[ b_i = (1 + \epsilon)(i - 1) \text{ for } i = 1, \ldots, p. \]
Evidently \( a_i + b_i = 0 \) for every edge \( a_i b_i \). On the other hand, any nonadjacent pair of vertices has a weight whose absolute value is at least \( 1 + \epsilon \), hence does not fall into the interval \([-1, 1]\), and whose value is at most \((1 + \epsilon)(p - 1) + (1 + \epsilon)(p - 2) = (1 + \epsilon)(2p - 3) < t\), hence does not fall into the interval \([t, \infty)\). Hence, this is a \((-1, 1, t)\)-representation of \( G \).

□

Computational experiments suggest that this bound is sharp: that \( G_p \) is not \((-1, 1, t)\)-threshold for any \( t \leq 2p - 3 \). Lacking a formal proof of this sharpness, we prove a weaker statement.

Lemma 2. For integer \( p \geq 4 \), if \( G_p \) is a \((-1, 1, t)\)-threshold graph then \( t > 2p - 5 \).

Proof. View the edges whose weight lies in \([-1, 1]\) as colored red and view the edges whose weight lies in \([t, \infty)\) as colored yellow. Since the yellow edges form a threshold graph and \( 2K_2 \) is a forbidden induced subgraph for threshold graphs, there is at most one yellow edge in \( G_p \). Let \( a_1 b_1, a_2 b_2, \ldots, a_p b_p \) be the edges of \( G_p \).

By symmetry, we may assume that \( a_i \leq b_i \) for each \( i \) and that \( b_1 \leq \cdots \leq b_p \). This implies that \( b_p \) has the largest rank of all vertices and, thus, if there is a yellow edge, then that edge is \( a_p b_p \).

Let \( q = p \) if \( a_p b_p \) is red, and otherwise let \( q = p - 1 \), so that all edges \( a_1 b_1, \ldots, a_p b_q \) are red.

Claim 1: \( a_k < a_j \) whenever \( j < k \leq q \). If not, then there exist \( j < k \) with \( a_k \geq a_j \) and \( b_j \leq b_k \). Hence \( a_k + b_j \geq a_j + b_j \geq -1 \),

and

\[ a_k + b_j \leq a_k + b_k < 1, \]

which contradicts the fact that the edge \( a_k b_j \) is absent.

It follows that the intervals \([a_i, b_i] \) are nested, with \([a_1, b_1] \subset [a_2, b_2] \subset \cdots \subset [a_q, b_q] \).

Claim 2: \( a_j + b_k \geq 1 \) and \( a_k + b_j < -1 \) whenever \( j < k \leq q \). Using the previous claim, we have \( a_j + b_k \geq a_k + b_k \geq -1 \), hence \( a_j + b_k \geq 1 \) since otherwise the edge \( a_j b_k \) should be present. Similarly, since \( b_j \leq b_k \), we have \( a_k + b_j \leq a_k + b_k < 1 \), hence \( a_k + b_j < -1 \) since otherwise the edge \( a_k b_j \) should be present.

Claim 3: \( b_j - a_j \geq 2(j - 1) \) for all \( j \in \{q\} \). We prove this by induction on \( j \).
When \( j = 1 \) this is just the assumption that \( b_j \geq a_j \). Assuming it holds for \( j - 1 \), we prove that it holds for \( j \). Observe that

\[
(b_j - a_j) - (b_{j-1} - a_{j-1}) = (a_{j-1} + b_j) - (a_j + b_{j-1}),
\]

and by the previous claim we have \( a_{j-1} + b_j \geq 1 \) and \( a_j + b_{j-1} \leq -1 \), so that

\[
b_j - a_j \geq (b_{j-1} - a_{j-1}) + 2 \geq 2(j - 2) + 2 = 2(j - 1).
\]
Claim 4: \( b_j \geq j - 3/2 \) for all \( j \in [q] \). This follows immediately from the inequalities
\[
\begin{align*}
b_j - a_j & \geq 2j - 2, \\
b_j + a_j & \geq -1.
\end{align*}
\]

Having established these claims, we now complete the proof. If \( q = p \), then Claim 4 gives \( b_{p-1} \geq p-5/2 \) and \( b_p \geq p-3/2 \), so to avoid the unwanted edge \( b_{p-1}b_p \), it is necessary that \( b_{p-1} + b_p < t \), which requires \( (p - 5/2) + (p - 3/2) = 2p - 4 < t \).

If \( q = p-1 \), then Claim 4 gives \( b_{p-1} \geq p-5/2 \), and since \( a_p + b_p \geq t \) with \( a_p \leq b_p \), we have \( b_p \geq t/2 \). Hence, to avoid the unwanted edge \( b_{p-1}b_p \) it is necessary that \( (p - 5/2) + t/2 < t \), which implies \( 2p - 5 < t \).

\[\Box\]

Corollary 3. For each \( k \geq 3 \), the graph \( G_{2^k} \) is \((-1,1,2^{k+1})\)-threshold but not \((-1,1,t)\)-threshold for any \( t \leq 2^k \).

Corollary 4. The classes \( C_{2^k} \) for \( k \geq 3 \) are pairwise distinct.

Corollary 5. For all \( t > 1 \), there exist 2-threshold graphs that are not \((-1,1,t)\)-threshold graphs.

3. Threshold number versus threshold dimension

In this section, we partially answer Question 3 by proving that there exist graphs with \( t(G) = 3 \) for which \( \Theta(G) \) is arbitrarily large. We will require the following result of Cozzens and Leibowitz [2] concerning the threshold dimension of complete multipartite graphs:

Theorem 6 (Cozzens–Leibowitz [2]). For positive integers \( m_1 \leq \cdots \leq m_p \), the complete \( p \)-partite graph \( K_{m_1,\ldots,m_p} \) has threshold dimension \( t(K_{m_1,\ldots,m_p}) = m_{p-1} \).

Let \( G = pK_3 \). Applying Theorem 6 with all \( m_i = 3 \) shows that \( t(G) = 3 \). Therefore, to show that \( \Theta(G) \) can be arbitrarily large for graphs with \( t(G) = 3 \), it suffices to prove the following theorem.

Theorem 7. If \( pK_3 \) is a \( k \)-threshold graph, then \( p \leq \binom{k+2}{3} \). In particular, \( \Theta(pK_3) \geq \frac{1}{2}(6p)^{1/3} \).

Note that the first part of the theorem is stronger than the second part, which is obtained using a crude lower bound on \( \binom{k+2}{3} \).

To prove this theorem, we will use a lemma stated in terms of edge colorings (not necessarily proper) induced by a threshold representation. Given a \( (\theta_1,\ldots,\theta_k) \)-representation of \( pK_3 \), we assign colors \( 1,\ldots,\lfloor k/2 \rfloor \) to the edges of \( pK_3 \) by giving edge \( e \) color \( i \) if its weight lies in the interval \( [\theta_{2i-1},\theta_{2i}) \). (By the definition of a \( (\theta_1,\ldots,\theta_k) \)-representation, for each edge there is exactly one such \( i \).)

Now, given an edge coloring, we can view each triangle as inducing a multiset of colors on its edges (for example, we consider “2 red edges and 1 yellow edge” and “1 red edge and 2 yellow edges” as different multisets, despite having the same underlying set).

Lemma 8. In a \( (\theta_1,\ldots,\theta_k) \)-representation of \( pK_3 \), no two triangles have the same multiset of colors appearing on their edges.
Before proving the lemma, we show how the proof of the theorem follows immediately.

**Proof of Theorem 7.** When \( p = 1 \), both parts of the theorem clearly hold, since \( k \geq 1 \) is required. For \( p \geq 2 \), observe that \( pK_3 \) has an induced \( 2K_2 \) and thus is not a threshold graph; thus, we may assume \( k \geq 2 \).

Assume \( pK_3 \) has a \((\theta_1, \ldots, \theta_k)\)-representation. By Lemma 8, no two triangles have the same multiset of colors on their edges. Hence, by the pigeonhole principle, the number of triangles is at most the number of size-3 multisets from \( \{1, \ldots, k\} \), which by the standard stars-and-bars argument is \( \binom{k+2}{3} \). Since \( k \geq 2 \), we have \( \binom{k+2}{3} \leq \binom{2k}{3} \leq \left(\frac{2k}{3}\right)^3 / 6 \). Rearranging \( p \leq \left(\frac{2k}{3}\right)^3 / 6 \) gives the desired inequality. \( \Box \)

**Proof of Lemma 8.** Suppose to the contrary that two triangles \( x_1, x_2, x_3 \) and \( y_1, y_2, y_3 \) have the same multiset of colors on their edges. Without loss of generality, we may assume that:

- \( x_1 \leq x_2 \leq x_3 \),
- \( y_1 \leq y_2 \leq y_3 \), and
- \( x_1 \leq y_1 \).

Choose indices \( \alpha, \beta, \gamma \in \{1, \ldots, k\} \) so that \( x_1x_2 \) has weight in \( [\theta_\alpha, \theta_{\alpha+1}) \), \( x_1x_2 \) has weight in \( [\theta_\beta, \theta_{\beta+1}) \), and \( x_2x_3 \) has weight in \( [\theta_\gamma, \theta_{\gamma+1}) \). (For convenience, we will adopt the convention that \( \theta_{k+1} = \infty \).) Observe that \( x_1 \leq x_2 \leq x_3 \) forces \( \alpha \leq \beta \leq \gamma \).

Say an edge with weight in \( [\theta_\alpha, \theta_{\alpha+1}) \) is red, an edge with weight in \( [\theta_\beta, \theta_{\beta+1}) \) is yellow, and an edge with weight in \( [\theta_\gamma, \theta_{\gamma+1}) \) is pink. (It is possible that some of these thresholds coincide, in which case an edge may be, say, both red and yellow.)

Since \( y_1 \leq y_2 \leq y_3 \) and the \( y \)-edges have the same multiset of colors as the \( x \)-edges, the colors of the \( y \)-edges must agree with the colors of the corresponding \( x \)-edges:

- \( x_1x_2 \) and \( y_1y_2 \) are red,
- \( x_1x_3 \) and \( y_1y_3 \) are yellow,
- \( x_2x_3 \) and \( y_2y_3 \) are pink.

Now we will derive our contradiction using the absence of the \( x_iy_j \)-edges.

**Claim 1:** \( y_2 < x_2 \). If instead \( x_2 \leq y_2 \), then we have
\[
\theta_\alpha \leq x_1 + x_2 \leq x_1 + y_2 \leq y_1 + y_2 < \theta_{\alpha+1},
\]
forcing a red \( x_1y_2 \)-edge, a contradiction.

**Claim 2:** \( x_3 < y_3 \). If instead \( y_3 \leq x_3 \), then since \( y_2 < x_2 \), we have
\[
\theta_\beta \leq y_2 + y_3 \leq y_2 + x_3 < x_2 + x_3 < \theta_{\beta+1},
\]
forcing a pink \( y_2x_3 \)-edge, a contradiction.

Now since \( x_3 < y_3 \), we have
\[
\theta_\gamma \leq x_1 + x_3 \leq y_1 + x_3 < y_2 + y_3 < \theta_{\gamma+1},
\]
forcing a yellow \( y_1x_3 \)-edge, again a contradiction. This completes the proof. \( \Box \)
After being informed of a preliminary version of the results in Section 2, Jamison (personal communication) suggested studying the class $D = \bigcap_{t > 1} C_t$, where $C_t$ is the class of $(-1,1,t)$-threshold graphs.

Intuition suggests that perhaps $D$ is related somehow to the class of 2-threshold graphs. Since $G_p$ is a 2-threshold graph for all $p$, Lemma 2 implies that not all 2-threshold graphs lie in the class $D$. On the other hand, since all 2-threshold graphs satisfy $t(G) \leq 2$, and since Theorem 6 implies that $t(2K_3) = 3$, we see that $2K_3$ is not a 2-threshold graph; however, $2K_3 \in D$, as the ranking in Figure 1 can easily be verified to be a $(-1,1,t)$-representation for $2K_3$ whenever $\epsilon$ is sufficiently small (in terms of $t$). Thus, $2K_3 \in D$ but $2K_3$ is not 2-threshold; the two classes are incomparable.

Open Question 1. Is there a nice characterization of the class $D$?

While the results in Section 4 imply that there are at least countably many distinct classes $C_t$, it is not clear whether there are countably many distinct classes or uncountably many distinct classes. Indeed, it seems plausible that $C_t \neq C_{t'}$ whenever $t, t'$ are distinct real numbers exceeding 1.

Open Question 2. Are there uncountably many distinct classes $C_t$?

Open Question 3. Are there distinct real numbers $t, t' > 1$ such that $C_t = C_{t'}$?

Theorem 7 only partially answers Question 3 which seeks a bound of $\Theta(G)$ in terms of $t(G)$ or $t(G)$. In particular, the following questions remain open:

Open Question 4. Is $\Theta(G)$ bounded on the class of graphs $G$ with $t(G) = 2$?

Open Question 5. Is $\Theta(G)$ bounded by any function of $t(G)$?

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