On the generalized unitary parasupersymmetry algebra of Beckers-Debergh

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Abstract

An appropriate generalization of the unitary parasupersymmetry algebra of Beckers-Debergh to arbitrary order is presented in this paper. A special representation for realizing of the even arbitrary order unitary parasupersymmetry algebra of Beckers-Debergh is analyzed by one dimensional shape invariance solvable models, 2D and 3D quantum solvable models obtained from the shape invariance theory as well. In particular in the special representation, it is shown that the isospectrum Hamiltonians consist of the two partner Hamiltonians of the shape invariance theory.
1 Introduction

Supersymmetry (symmetry between the fermionic and bosonic degrees of freedom) has important role [1, 2] in analyzing of the quantum mechanical systems, since it can study remarkable properties including the degeneracy structure of the energy spectrum, the relations among the energy spectra of the various Hamiltonians and etc.. In particular in this theory, the energy eigenvalues are necessarily non-negative and the energy of non-zero (zero) ground state is related to the broken (unbroken) supersymmetry. For the first time, Rubakov and Spiridonov [3] extended the supersymmetry which is called parasupersymmetry and it describes an essential symmetry between bosons and parafermions. Then, it was realized that the parasupersymmetry presented in Ref. [3] is of order $p = 2$ and Khare [4, 5] generalized the Rubakov-Spiridonov (R-S) parasupersymmetry to arbitrary order $p \geq 1$:

$$Q_{p+1}^1 Q_1^p + Q_1^{p+1} Q_1^p + \cdots + Q_1^p Q_1^p = 2p Q_1^{p-1} H,$$

$$Q_1^{p+1} = 0,$$

$$[H, Q_1] = 0. \tag{1}$$

Here, $Q_1$ and $H$ stand for a parasupercharge and the bosonic Hamiltonian. The relations (1) for $p = 1$ and $p = 2$ are reduced to the supersymmetry algebra and the Rubakov-Spiridonov (R-S) parasupersymmetry algebra [3], respectively. In addition, the relations (1) describe the unitary parasupersymmetry algebra of arbitrary order $p$, since $Q_1^\dagger$ is the Hermitian conjugate of the parasupercharge $Q_1$. So, it is evident that similar relations are satisfied under interchange of $Q_1$ and $Q_1^\dagger$ in (1).

Before the extension of the R-S parasupersymmetry algebra to arbitrary order $p$, for the $p = 2$ case remarkable and interesting discussions had been made [6-9]. For instance, the motion of a spin-1 particle along the z-axis in the presence of a magnetic field can be described by a parasupersymmetric Hamiltonian which is obtained by the simple harmonic oscillator and Morse potentials [10], and also the problem was generalized [11] for the spin-$\frac{p}{2}$ particles. Meanwhile, in the context of quantum field theory, the R-S parasupersymmetry algebra of order $p = 2$ leads to the infinite bosonic and parafermionic variables [12]. In a special formulation of the R-S parasupersymmetric quantum mechanics [13], the Hamiltonian for which the energy spectrum cannot be negative is expressed in terms of an explicit function of the parasupercharge $Q_1$. However, in general for the Rubakov-Spiridonov parasupersymmetric theory of arbitrary order the bosonic Hamiltonian cannot be obtained directly in terms of the parasupercharge $Q_1$, and the energy eigenvalues are not necessarily non-negative. Moreover, there is no connection between the non-zero (zero) ground-state energy and the broken (unbroken) parasupersymmetry. Also, it has been shown [4, 14] that there are $p - 1$ other conserved parasupercharges and $p$ bosonic constants in the R-S parasupersymmetric theory of arbitrary order $p$.

In an early work, Infeld and Hull [15] studied the factorization and algebraic solutions of the bound state problems and later, Gendenshtein et.al. considered the subject in the framework of the shape invariance symmetry as an important aspect of solvability of wide range of the 1D quantum mechanical models. It must be mentioned that the
factorization and the shape invariance symmetry have obtained a very helpful approach in the representation of the supersymmetry theory [16-24]. Recently, most of the one dimensional shape invariant solvable quantum mechanical models have been classified into two bunches. The first bunch [23] includes models for which the shape invariance parameter is the main quantum number $n$. On the other hand, in the second bunch [11] the shape invariance parameter of the models is the secondary quantum number $m$. Meanwhile, it has been shown in Ref. [11] that the R-S parasupersymmetry algebra of arbitrary order $p$ is realized by the shape invariant quantum mechanical models so that the algebra can be represented by the quantum mechanical states of the models. For realizing the algebra, it has also been shown that the bosonic Hamiltonian involves $p + 1$ isospectrum Hamiltonians. This fact has been studied in detail for the second bunch of the shape invariant models in Ref. [11]. In fact, $p + 1$ isospectrum Hamiltonians are obtained by adding $p$ appropriate constants to the factorized Hamiltonians \( \frac{1}{2}A(1)B(1), \frac{1}{2}A(2)B(2), \ldots, \frac{1}{2}A(p)B(p) \), and by adding the corresponding constant of the last Hamiltonian to its partner Hamiltonian \( \frac{1}{2}B(p)A(p) \) as well. Here, the operators $B(l)$ and $A(l); l = 1, 2, \ldots, p$ are the raising and lowering operators of the quantum states of the shape invariance theory, respectively. One of the other successes of the R-S parasupersymmetry theory of arbitrary order $p$ is the fact that it can be realized [26-28] by the 2D and 3D solvable quantum mechanical models obtained from the shape invariance symmetry.

In 1990 another formulation of the unitary parasupersymmetry algebra of order $p = 2$ was introduced by Beckers and Debergh [29] as follows:

\[
\begin{align*}
[Q_1, [Q_1^\dagger, Q_1]] &= 2 Q_1 H \\
[Q_1^\dagger, [Q_1, Q_1^\dagger]] &= 2 Q_1^\dagger H \\
Q_1^3 &= Q_1^{\dagger 3} = 0 \\
[H, Q_1] &= [H, Q_1^\dagger] = 0.
\end{align*}
\]

The appropriate form of the parasupersymmetric Hamiltonian presented in the Eqs. (2) has been constructed in Ref. [29] by using the generators of the simple harmonic oscillator, and the algebra (2) has been represented by the quantum states of the simple harmonic oscillator. But, a generalization of the Beckers-Debergh (B-D) parasupersymmetry algebra to arbitrary order $p \geq 2$ which is satisfied by the quantum mechanical systems has not been found yet. In general, the R-S parasupersymmetry algebra has been much more successful than the B-D parasupersymmetry algebra. Nevertheless, for the B-D parasupersymmetry algebra of order $p = 2$, it has been performed useful discussions. For example, Mostafazadeh has proved that [30] for both Rubakov-Spiridonov and Beckers-Debergh formulations of the parasupersymmetric quantum mechanics of order $p = 2$, the degeneracy structure of the energy spectrum can be derived using a thorough analysis of the parasupersymmetry algebra. He showed that the result is independent of the details of the Hamiltonian, for example, the degeneracy structure is not related to the dimension of the coordinate manifolds that the Hamiltonian is defined on it. Moreover, it has been shown that in general the Rubakov-Spiridonov (R-S) and Beckers-Debergh (B-D) systems possess identical degeneracy structures [30]. Also, similar to the Witten index of the supersymmetric quantum mechanics, for the two
kinds of \((p = 2)\) parasupersymmetry systems (B-D and R-S) a new set of topological invariants has been obtained \([30, 31]\).

In this paper it is intended to generalize the unitary parasupersymmetry algebra of B-D. Also, we introduce a special representation of the B-D unitary parasupersymmetry algebra of even arbitrary order by 1D shape invariance solvable models, and some 2D and 3D quantum solvable models as well. Meanwhile, the B-D unitary parasupersymmetry algebra of even arbitrary order \(p = 2k\) is introduced with \(2k\) independent conserved parasupercharges.

\section{Towards an appropriate generalization of the B-D parasupersymmetry algebra}

In order to generalize the B-D unitary parasupersymmetry algebra, we shall take into account the parafermionic operators \(b\) and \(b^\dagger\) of arbitrary order \(p\) which have been used by Khare \([4, 5]\). In fact in moving from statistics to parastatistics, the parafermionic operators are considered as the following \((p + 1) \times (p + 1)\) matrices:

\[(b)_{i,j} = C_j \delta_{i,j+1} \quad (b^\dagger)_{i,j} = C_i \delta_{i+1,j} \quad i,j = 1,2,...,p+1 \tag{3}\]

where the coefficients \(C_j\) are given by

\[C_j := \sqrt{j(p-j+1)} = C_{p-j+1} \quad j = 1,2,...,p+1. \tag{4}\]

It can be easily shown that

\[C_1 C_2 ... C_p = p! \tag{5}\]

Using the parafermionic operators \(b\) and \(b^\dagger\), one may obtain a spin-\(\frac{p}{2}\) representation for the group SU(2). Indeed, by defining the operators \(J_+\) and \(J_3\) as:

\[J_+ := b^\dagger \quad J_- := b \quad J_3 := \frac{1}{2} [b^\dagger, b] = \text{diag} \left( \frac{p}{2}, \frac{p}{2} - 1, ..., \frac{p}{2} + 1, -\frac{p}{2} \right), \tag{6}\]

the following commutation relations corresponding to the Lie algebra SU(2) may be derived

\[[J_+, J_-] = 2 J_3 \quad [J_3, J_\pm] = \pm J_\pm. \tag{7}\]

It is noticed that the commutation relation of the parafermionic operators \(b\) and \(b^\dagger\) is proportional to the third component of the spin-\(\frac{p}{2}\) representation of the Lie group SU(2).

One of the well-known and important properties of the parafermionic operators is

\[b^{p+1} = b^{p+1} = 0. \tag{8}\]

Now it can be verified that the parafermionic operators \(b\) and \(b^\dagger\) of order \(p\) possess the following new multilinear structural relations \((p \geq 2)\):
\[ \sum_{l=0}^{p} (-1)^{l+1} C_l^p b^{p-l} b^l b^l = 2 \delta_{p,2} (-1)^p b^{p-1} \]  
\[ \sum_{l=0}^{p} (-1)^{l+1} C_l^p b^{p-l} b^l b^l = 2 \delta_{p,2} b^{p-1} \]

where the constants \( C_l^p; \ l = 0, \cdots, p \) are the Newton binomial expansion coefficients. Using the Baker-Hausdorff formula for two arbitrary operators \( A \) and \( B \) which is given by

\[ e^{\lambda A} B e^{-\lambda A} = B + \frac{\lambda}{1!} [A, B] + \frac{\lambda^2}{2!} [A, [A, B]] + \cdots, \]

the structural relations (9) between the operators \( b \) and \( b^\dagger \) may be written down as

\[ [b, [b, \cdots, [b, [b^\dagger, b]] \cdots]] = 2 \delta_{p,2} (-1)^p b^{p-1} \]

\[ \text{ }(p - 1) - \text{times} \]  
\[ [b^\dagger, [b^\dagger, \cdots, [b^\dagger, [b, b^\dagger]] \cdots]] = 2 \delta_{p,2} b^{p-1}. \]

\[ \text{ }(p - 1) - \text{times} \]

It is seen that the multilinear expressions in the left-hand sides of the relations (11) are described in terms of the parafermionic operators \( b \) and \( b^\dagger \) in the right-hand sides. Actually, the multilinear relations (11) between the parafermionic operators \( b \) and \( b^\dagger \) propose similar relations between the parasupercharges of the parasupersymmetric quantum mechanics of order \( p \). The relations (8) and (11) indicate that there exist the conserved parasupercharges \( Q_1 \) and \( Q_1^\dagger \) of order \( p \), and the bosonic Hamiltonian so that they generate the following parasupersymmetry algebra:

\[ [Q_1, [Q_1, \cdots, [Q_1, [Q_1^\dagger, Q_1]] \cdots]] = 2(-1)^p Q_1^{p-1} H \]

\[ \text{ }(p - 1) - \text{times} \]  
\[ [Q_1^\dagger, [Q_1^\dagger, \cdots, [Q_1^\dagger, [Q_1, Q_1^\dagger]] \cdots]] = 2 Q_1^{1+p-1} H \]

\[ \text{ }(p - 1) - \text{times} \]

\[ Q_1^{p+1} = Q_1^{1+p+1} = 0 \]

\[ [H, Q_1] = 0 \]

\[ [H, Q_1^\dagger] = 0. \]
It can be easily verified that the relations (12) for any arbitrary $p$ are closure under Hermitian conjugation. Meanwhile, the algebraic relations (2) are a special case ($p = 2$) of the relations (12). Therefore, the relations (12) is an appropriate generalization of the B-D unitary parasupersymmetry algebra with the bosonic Hamiltonian $H$ and the parafermions of arbitrary order $p$ i.e. $Q_1$ and $Q_1^\dagger$.

3 A special representation for realizing of the even arbitrary order unitary parasupersymmetry algebra of B-D by quantum solvable models

In this section we analyze a special representation for the B-D quantum mechanical unitary parasupersymmetry algebra of even arbitrary order $p = 2k$ by wide range of the 1D, 2D and 3D solvable models. In fact, the 1D models are the solvable models obtained from the two approaches of the factorization with respect to the main and secondary quantum numbers i.e. $n$ and $m$. On the other hand, the 2D and 3D models representing the algebraic relations (12) for $p = 2k$ are some known quantum mechanical models on the homogeneous manifolds $SL(2, c)/GL(1, c)$ and the group manifolds $SL(2, c)$. Now in order to obtain the mentioned representations, we only introduce two bunches of the shape invariance models which have been classified before [11, 23].

In master function theory, a function $A(x)$ which is at most of second order in terms of $x$, and a non-negative weight function $W(x)$ defined in an interval $(a, b)$ may be chosen so that $(1/W(x))(d/dx)(A(x)W(x))$ is a polynomial of at most first order. For a given master function $A(x)$ and its corresponding weight function $W(x)$, it has been shown that the eigenvalue equations of the one dimensional partner Hamiltonians corresponding to the first bunch of the superpotentials, which are obtained from the factorization with respect to the main quantum number $n$, will be [23]

$$B(n)A(n)\psi_n(\theta) = E(n)\psi_n(\theta)$$
$$A(n)B(n)\psi_{n-1}(\theta) = E(n)\psi_{n-1}(\theta),$$

where the variable $\theta$ is introduced by means of solving the following first order differential equation

$$d\theta = \frac{dx}{A(x)}.$$  

The energy spectrum $E(n)$ and the wave function $\psi_n(\theta)$ are given by

$$E(n) = \frac{n}{4\left[\left(\frac{A(x)W'(x)}{W(x)}\right)' + nA''(x)\right]^2} \left(\frac{4\left(\frac{A(x)W'(x)}{W(x)}\right)}{W(x)}\right)^2$$

$$\times \left(nA^2(0) - A(0)\left(\frac{A(x)W'(x)}{W(x)}\right)\right).$$
\[- \left( \frac{AW'}{W} \right)(0) \left( \frac{A''(x)}{W} \right)(0) - 2A'(0) \left( \frac{A(x)W''(x)}{W(x)} \right)' \times \left( 2 \left( \frac{A(x)W'(x)}{W(x)} \right)' + nA''(x) \right) + n^2 A'(0) \left( A''(0) - 2A'(0)A(0) \right) \left( nA''(0) + 4 \left( \frac{A(x)W'(x)}{W(x)} \right)' \right) - 10nA(0)A''(x) \left( \frac{A(x)W'(x)}{W(x)} \right)^2 \right] \]

\[\psi_n(\theta) = \left[ \frac{a_n}{\sqrt{W(x)}} \left( \frac{d}{dx} \right)^n \left( A^n(x)W(x) \right) \right]_{x=x(\theta)}. \quad (16)\]

Note that the prime symbol indicates the derivative with respect to \(x\). Moreover, the explicit forms of the raising and lowering operators corresponding to the main quantum number \(n\) are, respectively:

\[B(n) = \frac{d}{d\theta} + \frac{1}{2} \left[ nA'(x) + \frac{A(x)W'(x)}{W(x)} \right. \]
\[\left. + \frac{A''(0) \left( \frac{A(x)W'(x)}{W(x)} \right)' - A'(x) \left( \frac{AW'}{W(x)} \right)(0)}{\left( \frac{A(x)W'(x)}{W(x)} \right)' + nA''(x)} \right]_{x=x(\theta)} \]

\[A(n) = -\frac{d}{d\theta} + \frac{1}{2} \left[ nA'(x) + \frac{A(x)W'(x)}{W(x)} \right. \]
\[\left. + \frac{A'(0) \left( \frac{A(x)W'(x)}{W(x)} \right)' - A'(x) \left( \frac{AW'}{W(x)} \right)(0)}{\left( \frac{A(x)W'(x)}{W(x)} \right)' + nA''(x)} \right]_{x=x(\theta)} \quad (17)\]

The change of variable \(x = x(\theta)\) is substituted in the relations (16) and (17) by solving the first order differential equation (14). Now, by choosing the suitable normalization coefficients \(a_n\) for the wave functions \(\psi_n(\theta)\), one may write down the shape invariance equations (13) as the raising and lowering relations:

\[B(n)\psi_{n-1}(\theta) = \sqrt{E(n)}\psi_n(\theta) \]
\[A(n)\psi_n(\theta) = \sqrt{E(n)}\psi_{n-1}(\theta). \quad (18)\]

The potentials like Coulomb, Rosen-Morse I, Rosen-Morse II and Eckart are involved in the first bunch of the solvable models.

In master function theory, the eigenvalue equations of the one dimensional partner Hamiltonian corresponding to the second bunch of the superpotentials which are obtained from the factorization with respect to the secondary quantum number \(m\) are given by [11].
In the factorization equations (19), the variable $\theta$ is introduced by solving the following first order differential equation

$$d\theta = \frac{dx}{\sqrt{A(x)}}.$$ \hspace{1cm} (20)

The energy spectrum and the eigenfunctions of the partner Hamiltonians obtained from the factorization with respect to the secondary quantum number $m$ are

$$E(n, m) = -(n - m + 1) \left[ \frac{A(x)W'(x)}{W(x)} \right]' + \frac{1}{2} (n + m)A''(x)$$ \hspace{1cm} (21)

$$\psi_{n,m}(\theta) = \left[ \frac{a_{n,m}}{A^{(2m-1)/4}(x)W^{1/2}(x)} \left( \frac{d}{dx} \right)^{n-m} \left( A^n(x)W(x) \right) \right]_{x=x(\theta)},$$ \hspace{1cm} (22)

where $m = 0, 1, 2, \cdots, n$. The explicit forms of the raising and lowering operators corresponding to the secondary quantum number $m$ are given by

$$B(m) = \frac{d}{d\theta} - \left[ \frac{A(x)W'(x)}{2W(x)} + \frac{m-1}{4} A'(x) \right]_{x=x(\theta)} \sqrt{A(x)},$$

$$A(m) = -\frac{d}{d\theta} - \left[ \frac{A(x)W'(x)}{2W(x)} + \frac{m-1}{4} A'(x) \right]_{x=x(\theta)} \sqrt{A(x)}.$$ \hspace{1cm} (23)

This time the change of variable $x = x(\theta)$ is substituted in the Eqs. (22) and (23) by solving the first order differential equation (20). Similar to the Eqs. (18), one may write down the raising and lowering relations corresponding to the secondary quantum number $m$ on the wave functions $\psi_{n,m}(\theta)$ by using the factorization equation (19) as:

$$B(m)\psi_{n,m-1}(\theta) = \sqrt{E(n, m)}\psi_{n,m}(\theta),$$

$$A(m)\psi_{n,m}(\theta) = \sqrt{E(n, m)}\psi_{n,m-1}(\theta).$$ \hspace{1cm} (24)

The potentials like 3D harmonic oscillator, Scarf I, Scarf II, Natanzon and generalized Pöschl-Teller are included in the second bunch of the solvable models. Moreover, simple harmonic oscillator and Morse potentials belong to both of the solvable models.

Now let us analyze a special realization of the B-D unitary parasupersymmetry algebra of even arbitrary order $p = 2k$ by means of the one dimensional quantum mechanical solvable models which are obtained from the factorization with respect to the secondary quantum number $m$. Similar procedure can be made by means of
the one dimensional quantum mechanical models which are obtained from the shape invariance with respect to the main quantum number $n$. In order to realize the B-D unitary parasupersymmetry algebra of even arbitrary order $p = 2k$, one may define the parafermionic generators $Q_1$ and $Q_1^\dagger$ of order $p = 2k$, and the bosonic operator $H$ as the following $(2k+1) \times (2k+1)$ matrices:

\begin{align}
(Q_1)_{l,l'} & := \delta_{l+1,l'} A(m), \quad l = \text{odd} \\
(Q_1)_{l,l'} & := \delta_{l+1,l'} B(m), \quad l = \text{even} \\
(Q_1^\dagger)_{l,l'} & := \delta_{l,l'+1} A(m), \quad l = \text{odd} \\
(Q_1^\dagger)_{l,l'} & := \delta_{l,l'+1} B(m), \quad l = \text{even}
\end{align}

(25a)

\begin{align}
(H)_{l,l'} & := \delta_{l+l'} H_l, \quad l, l' = 1, 2, \ldots, 2k + 1
\end{align}

(25c)

where $m = 1, 2, \ldots, n$. It is evident that by choosing the definitions (25a) and (25b) for $Q_1$ and $Q_1^\dagger$, the relations (12c) are satisfied automatically. To satisfy the Eqs. (12a) and (12b), by using the definitions (25) in them we obtain

\begin{align}
\left(\sum_{l=1}^{2k-1} (-1)^{l+1} C_l^{2k}\right) (A(m) B(m))^k A(m) &= 2 (A(m) B(m))^{k-1} A(m) H_{2k} \\
\left(\sum_{l=1}^{2k} (-1)^{l+1} C_l^{2k}\right) (B(m) A(m))^k B(m) &= 2 (B(m) A(m))^{k-1} B(m) H_{2k+1}
\end{align}

(26a)

\begin{align}
\left(\sum_{l=1}^{2k} (-1)^{l+1} C_l^{2k}\right) (B(m) A(m))^k B(m) &= 2 (B(m) A(m))^{k-1} B(m) H_1 \\
\left(\sum_{l=0}^{2k-1} (-1)^{l+1} C_l^{2k}\right) (A(m) B(m))^k A(m) &= 2 (A(m) B(m))^{k-1} A(m) H_2.
\end{align}

(26b)

Considering the following identities

\begin{align}
\sum_{l=0}^{2k-1} (-1)^{l+1} C_l^{2k} &= \sum_{l=1}^{2k} (-1)^{l+1} C_l^{2k} = 1,
\end{align}

then the Eqs. (26) give the following results

\begin{align}
H_1 &= H_{2k+1} = \frac{1}{2} A(m) B(m) \\
H_2 &= H_{2k} = \frac{1}{2} B(m) A(m).
\end{align}

(27)
Meanwhile, by using the definitions (25), the Eqs. (12d) and (12e) lead to the following relations, respectively

\[ A(m) H_{2l} = H_{2l-1} A(m) \]
\[ B(m) H_{2l+1} = H_{2l} B(m), \quad l = 1, 2, \cdots, k \] (28a)

\[ A(m) H_{2l} = H_{2l+1} A(m) \]
\[ B(m) H_{2l-1} = H_{2l} B(m), \quad l = 1, 2, \cdots, k. \] (28b)

It is noticed that the Eqs. (27) satisfy the relations (28a) and (28b), and additionally, in order to determine the remaining components of the bosonic Hamiltonian \( H \) it is sufficient to substitute the relations (27) in the recursion Eqs. (28a) and (28b). Then, one may obtain consistent solutions which are the same for the Eqs. (28a) and (28b) as:

\[ H_{2l-1} = \frac{1}{2} A(m) B(m), \quad l = 1, 2, \cdots, k + 1 \] (29a)
\[ H_{2l} = \frac{1}{2} B(m) A(m), \quad l = 1, 2, \cdots, k. \] (29b)

The recent result declares that the isospectrum Hamiltonians of the B-D unitary par supersymmetry theory of even arbitrary order \( p = 2k \) are the two partner Hamiltonians \( \left( \frac{1}{2} A(m) B(m) \right)((k + 1) - \text{times}) \) and \( \frac{1}{2} B(m) A(m)(k - \text{times}) \) of the shape invariance theory with the energy spectrum \( \frac{1}{2} E(n, m) \).

Clearly, the following \((2k + 1) \times 1\) columns matrix

\[ \Psi(\theta) = \begin{pmatrix} 
\psi_{n,m-1}(\theta) \\
\psi_{n,m}(\theta) \\
\psi_{n,m-1}(\theta) \\
\psi_{n,m}(\theta) \\
\vdots \\
\psi_{n,m-1}(\theta) 
\end{pmatrix}_{(2k+1) \times 1} \] (30)

as the basis represent the B-D unitary parasupersymmetry algebra of even arbitrary order \( p = 2k \). The eigenvalue equation of the bosonic Hamiltonian \( H \) is written down as

\[ H \Psi(\theta) = E(n, m) \Psi(\theta). \] (31)

The representation of the parafermionic generators \( Q_1 \) and \( Q_1^\dagger \) on the basis (30) by using the Eqs. (24) has the following forms
\[
Q_1 \Psi(\theta) = \sqrt{E(n, m)} \begin{pmatrix}
\psi_{n,m-1}(\theta) \\
\psi_{n,m}(\theta) \\
\psi_{n,m-1}(\theta) \\
\vdots \\
\psi_{n,m}(\theta) \\
0
\end{pmatrix}, \quad Q_1^\dagger \Psi(\theta) = \sqrt{E(n, m)} \begin{pmatrix}
0 \\
\psi_{n,m}(\theta) \\
\psi_{n,m-1}(\theta) \\
\vdots \\
\psi_{n,m}(\theta) \\
\psi_{n,m-1}(\theta)
\end{pmatrix}.
\] (32)

It is easily seen that the states \(Q_1 \Psi(\theta)\) and \(Q_1^\dagger \Psi(\theta)\) \((l = 1, 2, \ldots, p = 2k)\) are eigenfunctions of the bosonic Hamiltonian \(H\).

If we consider the first bunch of the shape invariance models, we will be able to construct the parafermionic generators by means of the raising and lowering operators \(B(n)\) and \(A(n)\), therefore we can obtain the bosonic Hamiltonian \(H\) including two independent partner components \(\frac{1}{2}A(n)B(n)\) and \(\frac{1}{2}B(n)A(n)\) with the same energy spectrum \(E(n)\). In this case, the basis of the representation of the B-D unitary parasupersymmetry algebra of even arbitrary order \(p = 2k\) is constructed by the eigenfunctions \(\psi_n(\theta)\) and \(\psi_{n-1}(\theta)\).

In Ref. [27] the following eigenvalue equations have been obtained by using the shape invariance equations (19)

\[
L_+ L_- \psi_{n,m}(\theta, \phi) = E(n, m) \psi_{n,m}(\theta, \phi),
\]

\[
L_- L_+ \psi_{n,m-1}(\theta, \phi) = E(n, m) \psi_{n,m-1}(\theta, \phi),
\] (33)

where the explicit differential forms of the operators \(L_+\) and \(L_-\) are given by

\[
L_+ = e^{i\phi} \left( \frac{\partial}{\partial \theta} + i \frac{A'(x)}{2\sqrt{A(x)}} \frac{\partial}{\partial \phi} \right) - \left[ \frac{1}{2\sqrt{A(x)}} \left( \frac{A(x)W'(x)}{W(x)} \right) + \frac{1}{4\sqrt{A(x)}} A'(x) \right]_{x=x(\theta)}.
\]

\[
L_- = e^{-i\phi} \left( -\frac{\partial}{\partial \theta} + i \frac{A'(x)}{2\sqrt{A(x)}} \frac{\partial}{\partial \phi} \right) - \left[ \frac{1}{2\sqrt{A(x)}} \left( \frac{A(x)W'(x)}{W(x)} \right) - \frac{1}{4\sqrt{A(x)}} A'(x) \right]_{x=x(\theta)}.
\] (34)

Now taking into account the following operators

\[
L_3 = -i \frac{\partial}{\partial \phi},
\]

\[
I = 1
\] (35)
then, the operators $L_+, L_-, L_3$ and $I$ constitute the Lie algebra $\mathfrak{gl}(2, c)$ i.e.

$$[L_+, L_+] = -A''(x)L_3 - \left(\frac{A(x)W'(x)}{W(x)}\right) I$$

$$[L_3, L_\pm] = \pm L_\pm$$

$$[L, I] = 0. \quad (36)$$

The change of variable $x = x(\theta)$, which is used in the Eqs. (34), is obtained by solving the differential Eq. (20). It has been shown in Ref. [27] that the Casimir of the generators $L_+, L_-, L_3$ and $I$ is the corresponding Hamiltonian of the charged particle on the homogeneous manifolds $SL(2, c)/GL(1, c)$ in the presence of magnetic monopole with degeneracy group $GL(2, c)$. The wave functions $\psi_{n,m}(\theta, \phi)$, which represent the Lie algebra $\mathfrak{gl}(2, c)$ as

$$L_+ \psi_{n,m-1}(\theta, \phi) = \sqrt{E(n, m)} \psi_{n,m}(\theta, \phi)$$

$$L_- \psi_{n,m}(\theta, \phi) = \sqrt{E(n, m)} \psi_{n,m-1}(\theta, \phi)$$

$$L_3 \psi_{n,m}(\theta, \phi) = m \psi_{n,m}(\theta, \phi)$$

$$I \psi_{n,m}(\theta, \phi) = \psi_{n,m}(\theta, \phi), \quad (37)$$

describe the two dimensional quantum states of the charged particle on the homogeneous manifolds $SL(2, c)/GL(1, c)$ in the presence of the magnetic monopole and they are given by

$$\psi_{n,m}(\theta, \phi) = e^{i m \phi} \psi_{n,m}(\theta). \quad (38)$$

Now it can be easily shown that the quantum states $\psi_{n,m}(\theta, \phi)$ also represent the B-D unitary parasupersymmetry algebra of arbitrary order $p = 2k$. In order to show the mentioned fact it is sufficient to define the parafermionic generators $Q_1$ and $Q_1^\dagger$ of order $p = 2k$, and the bosonic operator $H$ as

$$\left(1\right)^{l,l'}_{\text{II}} := \delta_{l,l+1'} \ L_-, \quad l = \text{odd}$$

$$\left(1\right)^{l,l'}_{\text{II}} := \delta_{l,l+1'} \ L_+, \quad l = \text{even} \quad (39a)$$

$$\left(1\right)^{l,l'}_{\text{II}} := \delta_{l,l+1} \ L_-, \quad l = \text{odd}$$

$$\left(1\right)^{l,l'}_{\text{II}} := \delta_{l,l+1} \ L_+, \quad l = \text{even} \quad (39b)$$

$$\left(1\right)^{l,l'}_{\text{II}} := \delta_{l,l'} \ H_1, \quad l,l' = 1, 2, \ldots, 2k+1. \quad (39c)$$

Once again, the relations (39a) and (39b) satisfy the relation (12c) automatically. Using the definitions (39a), (39b) and (39c), the Eqs. (12a) and (12b) lead to the following results
The Eqs. (12d) and (12e) by using the definitions (39), and the Eqs. (40) lead to

\[ H_{2l-1} = \frac{1}{2} L_- L_+ \quad l = 1, 2, \ldots, k + 1 \]
\[ H_{2l} = \frac{1}{2} L_+ L_- \quad l = 1, 2, \ldots, k. \]  

Therefore, the operator \( H \) of the B-D unitary parasupersymmetry algebra of arbitrary order \( p = 2k \) has two Hamiltonian components \( \frac{1}{2} L_- L_+ ((k+1) \text{ times}) \) and \( \frac{1}{2} L_+ L_- (k \text{ times}) \) on the homogeneous manifolds \( SL(2,c)/GL(1,c) \) with the same energy spectrum \( \frac{1}{2} E(n,m) \). The representation basis of the parasupersymmetry algebra in terms of the quantum states \( \psi_{n,m}(\theta, \phi) \) which describe the motion of the particle on the homogeneous manifolds \( SL(2,c)/GL(1,c) \) has the following form

\[
\Psi(\theta, \phi) = \begin{pmatrix}
\psi_{n,m-1}(\theta, \phi) \\
\psi_{n,m}(\theta, \phi) \\
\psi_{n,m-1}(\theta, \phi) \\
\psi_{n,m}(\theta, \phi) \\
\vdots \\
\psi_{n,m-1}(\theta, \phi)
\end{pmatrix}_{(2k+1) \times 1}.
\]  

(42)

The eigenvalue equation of the parasupersymmetric Hamiltonian is

\[ H \Psi(\theta, \phi) = E(n, m) \Psi(\theta, \phi). \]  

(43)

The representation of the parafermionic generators \( Q_1 \) and \( Q_1^\dagger \) by using the representation of the Lie algebra \( gl(2,c) \) given in the relations (37) is the relations (32) and the only difference is the fact that we must substitute the quantum states \( \psi_{n,m-1}(\theta, \phi) \) and \( \psi_{n,m}(\theta, \phi) \) instead of \( \psi_{n,m-1}(\theta) \) and \( \psi_{n,m}(\theta) \), respectively.

By choosing the generators of the Lie algebra \( gl(2,c) \) in terms of three variables, given in Ref. \[28\], we have taken into account the solvable quantum models on the group manifolds \( SL(2,c) \). In this case, like the two dimensional models, it can be also shown that the three dimensional solvable quantum models on the group manifolds \( SL(2,c) \) described in Ref. \[28\] represent the B-D unitary parasupersymmetry algebra of arbitrary order \( p = 2k \).

4 B-D unitary parasupersymmetry algebra with \( p = 2k \) conserved parasupercharges

For the B-D unitary parasupersymmetry algebra of even arbitrary order \( p = 2k \) introduced by the relations (12), it can be shown that there are \( 2k - 1 \) independent
conserved parasupercharges in addition to $Q_1$. We denote these parasupercharges by $Q_2, Q_3, \cdots, Q_{2k}$. They and their Hermitian conjugates are defined as (In this section, we only follow the discussion for the second bunch of the 1D shape invariance models):

\[
(Q_r)_{ll'} := \delta_{l+1,l'} A(m) \quad r \neq l \quad l = \text{odd}
\]

\[
(Q_r)_{ll'} := -\delta_{l+1,l'} A(m) \quad r = l
\]

\[
(Q_r^\dagger)_{ll'} := \delta_{l,l'+1} A(m) \quad r \neq l' \quad l = \text{odd}
\]

\[
(Q_r^\dagger)_{ll'} := -\delta_{l,l'+1} A(m) \quad r = l'
\]

\[
(Q_r^\dagger)_{ll'} := \delta_{l,l'+1} B(m) \quad r \neq l' \quad l = \text{even}
\]

\[
(Q_r^\dagger)_{ll'} := -\delta_{l,l'+1} B(m) \quad r = l'.
\]

where $r = 2, 3, \cdots, 2k$. In this section, $Q_1$ is considered as the relations (25). Once more, the bosonic Hamiltonian $H$ is defined as the relation (25c). The algebraic relations (12c) are satisfied automatically for the definitions (44). The equations (26a) and (26b) are also obtained from the equations (12a) and (12b) for the parasupercharges (44a) and (44b), respectively. Meanwhile, the algebraic relations (12d) and (12e) lead to the equations (28) again. Therefore, the solutions (29) are obtained for the bosonic Hamiltonian $H$ once again. Each of $2k$ conserved parasupercharges along with a bosonic Hamiltonian $H$ satisfy separately the parasupersymmetric algebraic relations. So, we have ($r = 1, 2, \ldots, 2k$):

\[
-C_0^{2k} Q_r^{2k} Q_r^\dagger + C_1^{2k} Q_r^{2k-1} Q_r^\dagger Q_r - \cdots - C_{2k}^{2k} Q_r^\dagger Q_r^{2k} = 2Q_r^{2k-1}H
\]

\[
-C_0^{2k} Q_r^{12k} Q_r + C_1^{2k} Q_r^{12k-1} Q_r Q_r^\dagger - \cdots - C_{2k}^{2k} Q_r Q_r^{12k} Q_r^\dagger = 2Q_r^{12k-1}H
\]

\[
Q_r^{2k+1} = Q_r^{12k+1} = 0
\]

\[
[H, Q_r] = 0
\]

\[
[H, Q_r^\dagger] = 0.
\]

Similar situation is occurred in the R-S arbitrary order parasupersymmetry algebra.

In addition to the relations (45c), the parasupercharges $Q_1, Q_2, \cdots, Q_{2k}$ and their Hermitian conjugates satisfy a generalised form of the relations (45c) which are given by
\[ Q_1^{a_1} Q_2^{a_2} \cdots Q_2^{a_{2k}} = 0 \]  
\[ Q_1^{a_1} Q_2^{a_2} \cdots Q_2^{a_{2k}} \dagger = 0, \]  
\[ (46a) \]

where \( a_1 + a_2 + \cdots + a_{2k} = 2k + 1 \). Furthermore, there exist \((2k + 1)\) independent bosonic constants as

\[ (I_1)_{ll'} := \delta_{l,l'} \]
\[ (I_s)_{ll'} := \begin{cases} \delta_{l,l'} & l \neq s \\ -\delta_{l,l'} & l = s \end{cases} \]
\[ s = 2, 3, \ldots 2k + 1 \]
\[ (47) \]

which commute with the bosonic Hamiltonian \( H \):

\[ [H, I_s] = 0 \quad s = 1, 2, 3, \ldots, 2k + 1. \]
\[ (48) \]

It is also noticed that the commutation relations of the parasupercharges and the bosonic constants are closure, that is,

\[ [I_s, Q_r] = \sum_{l=1}^{l=2k} d_l Q_l \quad s = 1, 2, 3, \ldots, 2k + 1 \]  
\[ r = 1, 2, 3, \ldots, 2k \]  
\[ (49) \]

where the coefficients \( d_l \) are constants. Similar relations exist for the Hermitian conjugate of the parasupercharges which are obtained by taking Hermitian conjugate of the relations (49). The bosonic constants \( I_s \) and the parasupercharges \( Q_r \) satisfy the mixed multilinear relations which are a generalisation of the relations (45a) and (45b). For example, one may introduce the B-D unitary parasupersymmetry algebra of order \( p = 2 \) with two parasupercharges and three bosonic constants:

\[ Q_1 = \begin{pmatrix} 0 & A(m) & 0 \\ 0 & 0 & B(m) \\ 0 & 0 & 0 \end{pmatrix} \quad Q_1^\dagger = \begin{pmatrix} 0 & 0 & 0 \\ B(m) & 0 & 0 \\ 0 & A(m) & 0 \end{pmatrix} \]
\[ Q_2 = \begin{pmatrix} 0 & A(m) & 0 \\ 0 & 0 & -B(m) \\ 0 & 0 & 0 \end{pmatrix} \quad Q_2^\dagger = \begin{pmatrix} 0 & 0 & 0 \\ B(m) & 0 & 0 \\ 0 & -A(m) & 0 \end{pmatrix} \]
\[ I_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad I_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \]
\[ H = \frac{1}{2} \begin{pmatrix} A(m)B(m) & 0 & 0 \\ 0 & B(m)A(m) & 0 \\ 0 & 0 & A(m)B(m) \end{pmatrix}, \]
\[ (50) \]
The mixed multilinear relations which are a generalisation of the relation (45a) with $k = 1$, are given by

$$Q_1^2 Q_2 = Q_1 Q_2^2 = 0$$
$$[I_2, Q] = 2 Q_2$$
$$[I_3, Q] = Q_1 - Q_2$$
$$[I_s, I_{s'}] = [I_s, H] = 0 \quad s, s' = 1, 2, 3.$$  \hspace{1cm} (51)

The mixed multilinear relations which are a generalisation of the relation (45a) with $k = 1$, are given by

$$-C_0^2 Q_r^\dagger Q_r^\dagger I_2 + C_1^2 Q_r Q_r^\dagger Q_r - C_2^2 Q_r^\dagger Q_r^2 I_3 = 2 Q_r H \quad r \neq r'$$
$$-C_0^2 Q_r Q_r^\dagger Q_r^\dagger I_2 + C_1^2 Q_r Q_r^\dagger Q_r - C_2^2 Q_r^\dagger Q_r Q_r I_3 = 2 Q_r H \quad r \neq r'$$
$$-C_0^2 Q_r Q_r^\dagger Q_r^\dagger I_2 + C_1^2 Q_r Q_r^\dagger Q_r - C_2^2 Q_r^\dagger Q_r Q_r = 2 Q_r H \quad r \neq r'.$$  \hspace{1cm} (52)

Note that when $r = r'$, the multilinear relations are the same as the relations (45a) with $k = 1$. The generalised mixed multilinear relations of the relations (45b) with $k = 1$ are obtained by taking Hermitian conjugate of the relations (52).

5 Conclusions

In this paper an appropriate generalization of the B-D unitary parasupersymmetry algebra of arbitrary order $p$ is presented. It is shown that in a special approach the generalization for even arbitrary order $p = 2k$ can be represented by the one dimensional shape invariance quantum models. In this approach, the partner Hamiltonians of the shape invariance theory are the isospectrum components of the bosonic Hamiltonian in the B-D unitary parasupersymmetric theory. Also, as we mentioned before, the 2D and 3D quantum models on the homogeneous manifolds $SL(2, c)/GL(1, c)$ and the group manifolds $SL(2, c)$ obtained from the shape invariance approach with respect to the secondary quantum number $m$ realize the B-D unitary parasupersymmetry algebra of arbitrary order $p = 2k$. At the same time, the bosonic Hamiltonian $H$ for the 2D and 3D models has two isospectrum components. Meanwhile, the B-D unitary parasupersymmetry algebra of even arbitrary order $p = 2k$ is generalised to a B-D unitary parasupersymmetry algebra in the presence of the 2$k$ conserved parasupercharges with the mixed multilinear relations.

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