The Toda\textsubscript{2} chain.

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Abstract

We show that a natural discretisation of Virasoro algebra yields a quantum integrable model which is the Toda chain in the second Hamiltonian structure.

1 Introduction

It is well known that the classical Virasoro–Poisson bracket algebra is closely related to the so-called exchange algebra [2, 5]. The latter admits a straightforward discretisation which allows one to define, on the classical level, a discretised Virasoro-like theory [3]. We observe that this theory coincides with the classical Toda chain in the second Hamiltonian structure (the Toda\textsubscript{2} chain), c.f. [1, 4]. As usual in the context of Toda lattice, the $N$-particle model can be formulated by means of a $N \times N$ single Lax matrix or through a product of $2 \times 2$ Lax matrices, both giving rise to the same spectral curve.

The quantisation of the exchange algebra is straightforward and provides a definition of the Toda\textsubscript{2} chain which we study using the $2 \times 2$ Lax matrix formalism. The quantum Lax matrix $l_{0n}$ we obtain is not ultra-local. As a consequence, we need to resort to the Freidel-Maillet formalism [7, 8] so as to obtain the generating function of the quantum conserved quantities. This requires the introduction of supplementary scalar matrices $M_0$ (5.9) and $\tilde{M}_0$ (5.16) satisfying certain compatibility conditions. The most general solutions we find, exhibit a non-trivial dependence on 3 free parameters. The Toda\textsubscript{2} chain arises as a specialisation of these parameters.

We choose an appropriate set of canonical operators to represent the Toda\textsubscript{2} algebra. Then, by means of a suitable gauge transformation, we show that we can relate the generating function of the quantum conserved quantities to a new transfer matrix solely built in terms of ultralocalised Lax operators. This step brings us back to the standard formalism. We single out three special cases of the ultralocalised Lax matrix we obtained: the Toda\textsubscript{2} chain, the $q$-Toda chain and the $q$-oscillator model which coincides with the QTASEP stochastic model [9].

The paper is organised as follows. In Section 2 we recall two ways of constructing the classical Virasoro algebra in the continuum. In Section 3 we remind the construction of a natural discretisation of the classical Virasoro algebra. Then, in Section 4 we connect these results with the classical Toda chain endowed with the second Hamiltonian structure. In Section 5 we construct the quantum counterpart of this model, the so-called Toda\textsubscript{2} chain. We show that although the resulting model is non-ultralocal, its quantum integrability can be described within the Freidel-Maillet scheme [7, 8]. Finally, in Section 6 we show that the Toda\textsubscript{2} chain transfer matrix ultralocalises, namely can be expressed in terms of an auxiliary transfer matrix associated with an ultralocal monodromy matrix.

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2 Classical Virasoro in the continuum

Let $u$ be a 1-periodic function on $\mathbb{R}$ that is endowed with the Virasoro Poisson bracket:

$$\{u(x), u(y)\} = \frac{1}{2}[u(x) + u(y)]\delta_1'(x-y) + \frac{1}{2}\delta_1^{(3)}(x-y),$$

(2.1)

where $\delta_1$ stands for the 1-periodic Dirac Comb. Then, the re-scaled Fourier components of $u$

$$u(x) = -48\pi^2 \sum_{n \in \mathbb{Z}} \left[\mathcal{L}_n - \frac{\delta_{n,0}}{24}\right] e^{2i\pi nx}$$

(2.2)

satisfy the classical Virasoro Poisson algebra

$$48i\pi \{\mathcal{L}_n, \mathcal{L}_m\} = \mathcal{L}_{n+m}(n-m) + \frac{n^3-n}{12}\delta_{n+m,0}.$$ 

(2.3)

The latter is obtained from the Virasoro algebra satisfied by the $L_n$'s by taking the $c \to \infty$ scaling limit:

$$\mathcal{L}_n \to c\mathcal{L}_n \quad \text{and} \quad [\cdot, \cdot] \to 48i\pi c^{-1}\{\cdot, \cdot\}.$$ 

Given $u$ endowed with the above bracket, consider the ordinary differential equation on $\mathbb{R}$

$$f''(x) + u(x)f(x) = 0.$$ 

(2.4)

It was shown in [2, 5] that (2.4) admits a basis of solutions $\{\xi^1(x), \xi^2(x)\}$, whose Wronskian is normalised to 1:

$$\det \begin{pmatrix} \xi^1(x) & \xi^2(x) \\ (\xi^1)'(x) & (\xi^2)'(x) \end{pmatrix} = 1,$$

(2.5)

and such that the vector valued row function $\xi(x) = (\xi^1(x), \xi^2(x))$ has the Poisson bracket:

$$\{\xi(x) \circ \xi(y)\} = \xi(x) \circ \xi(y) \cdot [r^+\theta(x-y) + r^-\theta(y-x)]$$

(2.6)

where $r^\pm$ are the $\mathfrak{sl}_2$ solutions to the classical Yang-Baxter equation

$$r^\pm = \pm [\sigma^\pm \otimes \sigma^\mp + 4\sigma^\pm \otimes \sigma^\mp].$$

Here $\sigma^\pm$ refer to the Pauli matrices

$$\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$ 

Since $\xi^1, \xi^2$ solve (2.4), one can reconstruct the potential $u(x)$ in terms of $\xi^1(x)$ and $\xi^2(x)$ by a Wronskian type relation

$$u(x) = \det \begin{pmatrix} (\xi^1)'(x) & (\xi^2)'(x) \\ (\xi^1)''(x) & (\xi^2)''(x) \end{pmatrix}.$$ 

(2.7)

This observation allows one to recast (2.4) in a determinantal form

$$\det \begin{pmatrix} f(x) & \xi^1(x) & \xi^2(x) \\ f'(x) & (\xi^1)'(x) & (\xi^2)'(x) \\ f''(x) & (\xi^1)''(x) & (\xi^2)''(x) \end{pmatrix} = 0.$$ 

In fact, one may turn things the other way around and consider the algebra (2.6) as a starting point. Then defining $u$ in terms of $\xi$ through (2.7), one recovers the Virasoro–Poisson bracket eq. (2.1), see [2].
3 Classical Virasoro on the lattice

The algebra \( \{ \xi_n, \xi_m \} = \xi_n \otimes \xi_m \cdot [r^+ \theta(n-m) + r^- \theta(m-n)] \) only if \( \theta(n) = 1 \) if \( n > 0 \), \( \theta(n) = 0 \) if \( n < 0 \) and \( \theta(0) = 1/2 \).

Define row vector variables \( \xi_n = (\xi^1_n, \xi^2_n) \) indexed by sites \( n \) of a lattice and satisfying the natural lattice version of exchange algebra eq. (2.6)

\[
\{ \xi_n \otimes \xi_m \} = \xi_n \otimes \xi_m \cdot [r^+ \theta(n-m) + r^- \theta(m-n)]
\]

where the Heaviside step function is defined as \( \theta(n) = 1 \) if \( n > 0 \), \( \theta(n) = 0 \) if \( n < 0 \) and \( \theta(0) = 1/2 \).

The algebra (2.6) corresponds to an equivalent formulation of the Virasoro–Poisson bracket. One of the advantages of such way of writing things is that this algebra can be naturally put on the lattice, as it was shown in [3]. We now recall this construction

This separability has nonetheless a price in that the resulting subalgebra \( \{ \xi_n \}_{n \in \mathbb{Z}} \) is cubic as opposed to the quadratic nature of the \( W \)-algebra. The \( S_n \) algebra was first obtained in [6, 9]. It may be considered as a lattice deformation of the Virasoro algebra. Indeed, setting \( S_n = 1 + \frac{\Delta}{2} u_\Delta(n\Delta) \), and imposing the scaling in the \( \Delta \to 0^+ \) limit,

\[
u_\Delta(n\Delta) \to u(x) \quad \text{and} \quad \{ \cdot, \cdot \} \to 16 \{ \cdot, \cdot \}_0
\]

where \( \{ \cdot, \cdot \}_0 \) denotes the bracket in the continuum that appeared in Section 2, one gets that (3.6) goes to (2.1).

4 The classical Toda\(_2\) chain

By analogy with the continuum case, one may interpret the quantities \( \xi^1_n \) and \( \xi^2_n \) as two independent solutions of a three term linear recursion which can be put in the form

\[
\begin{vmatrix}
v_n & \xi^1_n & \xi^2_n \\
v_{n+1} & \xi^1_{n+1} & \xi^2_{n+1} \\
v_{n+2} & \xi^1_{n+2} & \xi^2_{n+2}
\end{vmatrix} = 0,
\]
or, equivalently,
\[
W_n^{(1)} v_{n+2} - W_n^{(2)} v_{n+1} + W_n^{(1)} v_n = 0 .
\] (4.1)

It appears convenient to recast eq. (4.1) in terms of new quantities
\[
Q_n = \frac{1}{W_n^{(1)}}, \quad P_n = \frac{W_n^{(2)}}{W_n^{(1)}},
\] (4.2)

which have simpler Poisson brackets than the \( W_n^{(n)} \)'s:
\[
\{ Q_n, Q_m \} = Q_n Q_m (\delta_{n+1,m} - \delta_{n,m+1}) ,
\] (4.3)
\[
\{ Q_n, P_m \} = -2Q_n P_m (\delta_{n,m} - \delta_{n+1,m}) ,
\] (4.4)
\[
\{ P_n, P_m \} = -4Q_n^2 \delta_{n,m} + 4Q_n^2 \delta_{n+1,m} .
\] (4.5)

Then, eq. (4.1) takes the form
\[
Q_n v_{n+1} - P_n v_n + Q_{n-1} v_{n-1} = 0 .
\] (4.6)

When imposing periodic boundary conditions for the \( P_n \)'s and \( Q_n \)'s: \( P_{n+N} = P_n \) and \( Q_{n+N} = Q_n \) and quasi-periodic ones for the \( v \)'s \( v_{N+n} = \mu v_n \), the linear system eq. (4.6) can be put in an \( N \times N \) matrix form:
\[
L(\mu) \cdot \vec{v} = \begin{pmatrix}
-P_1 & Q_1 & 0 & \cdots & 0 & \mu^{-1} Q_N \\
Q_1 & -P_2 & Q_2 & 0 & \cdots & 0 \\
0 & Q_2 & -P_3 & Q_3 & 0 & \vdots \\
\vdots & 0 & \ddots & \ddots & \ddots & 0 \\
0 & 0 & & & & Q_{N-1} \\
\mu Q_N & 0 & \cdots & 0 & Q_N & -P_N
\end{pmatrix} \begin{pmatrix}
v_1 \\
v_2 \\
v_3 \\
\vdots \\
v_{N-1} \\
v_N
\end{pmatrix} = 0 .
\] (4.7)

Note that under periodic boundary conditions for the \( Q_n, P_n \)'s, one has to understand the Kronecker symbols in (4.3)-(4.5) modulo \( N \), viz. \( \delta_{n,m} = 1 \) if and only if \( n - m \in NZ \).

The above Jacobi matrix is typical of the Lax matrix of the Toda chain. Furthermore, the quantities \( \text{tr} [L^n(\mu)] \) are in involution. Indeed, one can check that the Poisson bracket of \( L \) is of the form
\[
\{ L_1(\mu_1) \otimes L_2(\mu_2) \} = [\delta_{12}, L_1(\mu_1)] - [\delta_{21}, L_2(\mu_2)]
\] with
\[
L_1(\mu) = \text{id} \otimes L(\mu) \quad \text{and} \quad L_2(\mu) = L(\mu) \otimes \text{id}
\]
ensuring the Poisson commutativity in question. We find
\[
\delta_{12} = - (r_{12} - a_{12}) \cdot L_2(\mu_2) - L_2(\mu_2) \cdot (r_{12} + a_{12})
\] where, using the elementary \( N \times N \) matrices \( E_{ij} \), one has
\[
r_{12} = \sum_{j>i}^N \left[ \frac{2\mu_2}{\mu_1 - \mu_2} E_{ij} \otimes E_{ji} + \frac{2\mu_1}{\mu_1 - \mu_2} E_{ij} \otimes E_{ij} \right] + \sum_{i=1}^N \frac{\mu_1 + \mu_2}{\mu_1 - \mu_2} E_{ii} \otimes E_{ii} ,
\] (4.8)
\[
a_{12} = \frac{1}{2} \sum_{j>i}^N \left[ E_{ii} \otimes E_{jj} - E_{jj} \otimes E_{ii} \right].
\] (4.9)

Explicitly, we have
\[
\{ L_1(\mu_1) \otimes L_2(\mu_2) \} = [ -2r_{12} L_1(\mu_1) L_2(\mu_2) ] + (2a_{12}) L_1(\mu_1) L_2(\mu_2) + L_1(\mu_1) L_2(\mu_2) (2a_{12})
- 2L_1(\mu_1) a_{12} L_2(\mu_2) - 2L_2(\mu_2) a_{12} L_1(\mu_1)
\] (4.10)
Hence, the finite difference equation (4.6) is an integrable system belonging to the Toda family. In fact, the bracket given in eqns (4.3)-(4.5) is the well known second Poisson bracket of the Toda chain [1, 4].

The spectral curve of the model is then defined as

$$0 = \det_N [L(\mu) + \lambda \text{id}] = (-1)^{N+1} \prod_{a=1}^N Q_a \cdot \{\mu + \mu^{-1}\} + p_N(\lambda) \quad (4.11)$$

where $p_N$ is a monoic polynomial in $\lambda$ of degree $N$.

Alternatively, one can embrace the classical integrability of the model by means of the $2 \times 2$ Lax matrix

$$l_n(\lambda) = \begin{pmatrix} \lambda - P_n & -1 \\ Q_n^2 & 0 \end{pmatrix} \quad (4.12)$$

Since we will provide a thorough discussion of the integrability in the quantum case, we leave aside the details of the construction of the $2 \times 2$ monodromy matrix and the one of the generating function of conserved quantities. The bottom line is that, in such a case, the monodromy matrix takes the form

$$T_N(\lambda) = l_N(\lambda)l_{N-1}(\lambda) \cdots l_1(\lambda) \quad (4.13)$$

The associated spectral curve takes the form

$$0 = \mu^{-1} \cdot \det_2 [T_N(\lambda) - \mu \text{id}] = \mu + \prod_{a=1}^N Q_a^2 \cdot \mu^{-1} - \text{tr}[T_N(\lambda)] \quad (4.14)$$

An explicit calculation shows that it should hold $p_N(\lambda) = \text{tr}[T_N(\lambda)]$. Thus, upon noticing that $\prod_{a=1}^N Q_a$ is a conserved quantity and hence plays the role of a scalar quantity, one may rescale $\mu \mapsto \mu(-1)^N \prod_{a=1}^N Q_a$ in the monodromy matrix based spectral curve so as to get a full identification of the two spectral curves introduced above.

One can represent $\zeta_n^{1,2}$ in terms of canonical Darboux coordinates $\{x_n, X_m\} = 4 \delta_{nm}$:

$$\zeta_n^1 = e^{-\frac{x}{2}} \prod_{a=1}^n e^{\frac{1}{2} X_a}, \quad \zeta_n^2 = e^{-\frac{x}{2}} \prod_{a=1}^n \sum_{b=1}^n e^{\frac{1}{2} X_b} \cdot e^{x_a} \prod_{b=1}^{n-1} e^{-\frac{1}{2} X_b} \quad (4.15)$$

In terms of these canonical coordinates, one has

$$Q_n^2 = e^{-X_{n+1}} e^{x_n-x_{n+1}}, \quad P_n = e^{-X_n} e^{x_n-x_{n+1}} \quad (4.16\text{-}17)$$

Within this parametrisation and upon setting

$$v_n = f_\Delta(n\Delta) \quad X_n = 2\Delta \pi_\Delta(n\Delta) \quad x_n = \varphi_\Delta(n\Delta) \quad (4.18)$$

consider the continuum limit

$$N\Delta \to 1 \quad \text{and} \quad \left\{ \begin{array}{l} \pi_\Delta(n\Delta) \to \pi(x) \\ \varphi_\Delta(n\Delta) \to \varphi(x) \end{array} \right. \quad \text{so that} \quad \{\varphi(x), \pi(x)\} = 2\delta(x-y) \quad (4.19)$$

Then, eq.(4.6) yields

$$f''(x) - \left[ \left(\pi(x) - \frac{1}{2} \varphi'(x)\right)^2 + \left(\pi(x) - \frac{1}{2} \varphi'(x)\right)' \right] f(x) = 0 \quad (4.20)$$

Hence, $p(x) = \pi(x) - \frac{1}{2} \varphi'(x)$ is the standard chiral Coulomb field in CFT and eqs.(4.11-14) are lattice analogues of the standard representation of the solutions of eq.(4.20) by classical vertex operator and screening.
5  The quantum Toda$_2$ chain

In the quantum case the exchange algebra for the row vectors $\xi_n = (\xi_n^1, \xi_n^2)$ reads

$$\xi_n \otimes \xi_m = \xi_m \otimes \xi_n \cdot \text{P} [ R^+(q) \theta_q (n-m) + R^-(q) \theta_q (m-n) ]$$  (5.1)

where P is the permutation matrix on $\mathbb{C}^2 \otimes \mathbb{C}^2$,

$$R^+(q) = \begin{pmatrix} q^{1/2} & 0 & 0 & 0 \\ 0 & q^{-1/2} & q^{1/2} - q^{-1/2} & 0 \\ 0 & 0 & q^{-1/2} & 0 \\ 0 & 0 & 0 & q^{1/2} \end{pmatrix} \quad \text{and} \quad R^-(q) = \text{P} R^+(q^{-1}) \text{P} .$$

Finally, $\theta_q$ is the $q$-deformation of the Heaviside function:

$$\theta_q(x) = 0 \quad \text{if} \quad x < 0 , \quad \theta_q(x) = 1 \quad \text{if} \quad x > 0 \quad \text{and} \quad \theta_q(0) = \frac{1}{q^{2} + q^{-2}} .$$  (5.2)

Since the R-matrix is independent of the lattice spacing $\Delta$, it can be computed in the continuous CFT theory [5]. The parameter $q$ is related to the central charge by:

$$c = 1 + 6 \left( b + \frac{1}{b} \right)^2 , \quad q = e^{i\pi b^2} , \quad \tilde{q} = e^{i\pi b} .$$

They form a dual pair in the sense of Faddeev’s modular double theory. The range of $b$ is the real axis $b > 1$ for $c > 25$, the unit circle $0 < \text{Arg} \ b < \pi/2$ for $1 < c < 25$ and the imaginary axis $\text{Im} \ b > 1$ for $c < 1$. In the following, it appears convenient to choose the below parametrisation of $b$

$$b^2 = \frac{\omega_1}{\omega_2} .$$  (5.3)

Also, in our further considerations, we will consider any value of $\omega_1, \omega_2$, viz. $b$, without limiting ourselves to the values giving rise to a natural CFT interpretation.

We define as in [3] the quantum $W$-algebra. Let

$$W_n^p = q^{\xi_n^1 \xi_{n+p}^2} - q^{\xi_n^2 \xi_{n+p}^1} , \quad p = 1, 2 .$$

Then the quantum $W$-algebra reads

$$W_n^{(1)} W_m^{(1)} = q^{\delta_n + \delta_m} W_m^{(1)} W_n^{(1)} ,$$

$$W_n^{(1)} W_m^{(2)} = q^{\delta_n + \delta_m} W_m^{(1)} W_n^{(2)} ,$$

$$W_n^{(2)} W_m^{(2)} = q^{\delta_n + \delta_m} W_m^{(2)} W_n^{(2)} + (q^{2} - q^{-2}) \delta_n \delta_m .$$

The quantum analogues $P_n, Q_n$ of the classical variables given in eq. (4.12) are constructed as

$$Q_n = [W_n^{(1)}]^{-1} , \quad P_n = Q_{n-1} W_n^{(2)} W_{n-1}^{(1)} = W_{n-1}^{(2)} Q_{n-1} Q_n ,$$

and enjoy the commutation relations

$$Q_n Q_m = q^{2 \delta_n} Q_m Q_n ,$$

$$P_n P_m = P_m P_n + (q^{2} - q^{-2}) (Q_n \delta_{n+1,m} - Q_m \delta_{n,m+1}) ,$$

$$P_n Q_m = q^{2 \delta_n} Q_m P_n .$$

The quantum Lax matrix takes a form analogous to the classical case

$$L_{0n}(\lambda) = \begin{pmatrix} \lambda - P_n & -1 \\ Q_n & 0 \end{pmatrix} .$$  (5.4)
where 0 indexes the auxiliary space. Since the model is non-ultralocal, we must use the Freidel-Maillet scheme to encode, in an integrable fashion, the commutation relations for the quantum Lax matrix:

$$A_{12}(\lambda_1, \lambda_2) 1_{n}(\lambda_1) 1_{2n}(\lambda_2) = 1_{2n}(\lambda_2) 1_{1n}(\lambda_1) D_{12}(\lambda_1, \lambda_2), \quad (5.5)$$

$$1_{1n}(\lambda_1) 1_{2,n+1}(\lambda_2) = 1_{2,n+1}(\lambda_2) C_{12}(\lambda_1, \lambda_2) 1_{1n}(\lambda_1), \quad (5.6)$$

$$1_{2n}(\lambda_2) 1_{1,n+1}(\lambda_1) = 1_{1,n+1}(\lambda_1) B_{12}(\lambda_1, \lambda_2) 1_{2n}(\lambda_2). \quad (5.7)$$

The matrices $A_{12}(\lambda_1, \lambda_2)$, $B_{12}(\lambda_1, \lambda_2)$, $C_{12}(\lambda_1, \lambda_2)$ and $D_{12}(\lambda_1, \lambda_2)$ take the form

$$A_{12}(\lambda_1, \lambda_2) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{\lambda_2 - \lambda_1}{\lambda_2 q^2 - \lambda_1} & \frac{\lambda_1 (q^2 - 1)}{\lambda_2 q^2 - \lambda_1} & 0 \\
0 & \frac{\alpha}{\lambda_2 q^2 - \lambda_1} & \frac{\lambda_2 (q^2 - 1)}{\lambda_2 q^2 - \lambda_1} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},$$

$$D_{12}(\lambda_1, \lambda_2) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{\lambda_2 - \lambda_1}{\lambda_2 q^2 - \lambda_1} & \frac{\lambda_1 (q^2 - 1)}{\lambda_2 q^2 - \lambda_1} & 0 \\
0 & \frac{\alpha}{\lambda_2 q^2 - \lambda_1} & \frac{\lambda_2 (q^2 - 1)}{\lambda_2 q^2 - \lambda_1} & 0 \\
\frac{\lambda_1 (\lambda_2 - \lambda_1) (q^2 - 1)}{\lambda_2 q^2 - \lambda_1} & -\frac{\lambda_2 (\lambda_2 - \lambda_1) (q^2 - 1)}{\lambda_2 q^2 - \lambda_1} & 1
\end{pmatrix},$$

$$C_{12}(\lambda_1, \lambda_2) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & -\left(q^{3/2} - q^{1/2}\right) & 0 \\
0 & 0 & q^2 & 0 \\
0 & 0 & \lambda_2 (q^2 - 1) & 1
\end{pmatrix},$$

$$B_{12}(\lambda_1, \lambda_2) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & q^2 & 0 & 0 \\
0 & -\left(q^{3/2} - q^{-1/2}\right) & 1 & 0 \\
0 & \lambda_1 (q^2 - 1) & 0 & 1
\end{pmatrix}.$$
where $d_1, d_2, d_3$ are arbitrary constants. We denote this specific solution of (5.8) as

$$G_0(\lambda) = \begin{pmatrix} 1 & q^{\frac{3}{2}}d_2d_3\lambda \\ (1 - q^2)\lambda & q^{-\frac{1}{2}} + q^{\frac{3}{2}}d_1\lambda + q^{\frac{3}{2}}d_2d_3\lambda^2 \end{pmatrix}. \quad (5.12)$$

Thus, we shall focus below on the monodromy matrix

$$T_{0N}(\lambda) = \hat{1}_{0N}(\lambda) \hat{1}_{0N-1}(\lambda) \cdots \hat{1}_{02}(\lambda) \hat{1}_{01}(\lambda) \quad (5.13)$$

where $\hat{1}_{0n}(\lambda) = 1_{0n}(\lambda)G_0(\lambda)$.

Thus, the generating function of conserved quantities is expressed as

$$\text{tr}_0[\hat{T}_{0N}(\lambda)\hat{M}_0(\lambda)] \quad (5.14)$$

The exchange algebra (5.5)-(5.7) and the equation (5.8) put together ensure that the above monodromy matrix satisfies the quadratic algebra

$$\mathcal{A}_{12}(\lambda_1, \lambda_2)\mathcal{T}_{1N}(\lambda_1)\mathcal{B}_{12}(\lambda_1, \lambda_2)\mathcal{T}_{2N}(\lambda_2) = \mathcal{T}_{2N}(\lambda_2)\mathcal{C}_{12}(\lambda_1, \lambda_2)\mathcal{T}_{1N}(\lambda_1)\mathcal{D}_{12}(\lambda_1, \lambda_2) \quad (5.15)$$

where

$$\mathcal{D}_{12}(\lambda_1, \lambda_2)\mathcal{B}_{12}(\lambda_1, \lambda_2)\mathcal{M}_1(\lambda_1) = \mathcal{M}_1(\lambda_1)\mathcal{C}_{12}(\lambda_1, \lambda_2)\mathcal{M}_2(\lambda_2)\mathcal{A}_{12}(\lambda_1, \lambda_2) \quad (5.16)$$

The most general solution to (5.15) takes the form

$$\hat{M}_0(\lambda) = \hat{\alpha} \begin{pmatrix} 1 & \beta \lambda \\ \tilde{\beta} \lambda & q^2 + \delta \lambda + \tilde{\beta}^2 \lambda^2 \end{pmatrix} \quad (5.16)$$

for arbitrary constants $\hat{\alpha}, \beta, \tilde{\alpha}, \tilde{\beta}$. The solution of interest to our analysis is obtained by taking

$$\hat{\alpha} = q^{-1}, \quad \beta = q^{\frac{3}{2}}d_2d_3, \quad \tilde{\beta} = (1 - q^2), \quad \delta = q^{\frac{5}{2}}d_1 \quad (5.17)$$

This specific solution is motivated by eq.(5.5) below. It takes the factorised form $\hat{G}_0(\lambda)q^{-\sigma\hat{\delta}}$, where

$$\hat{G}_0(\lambda) = \begin{pmatrix} 1 & q^{-\frac{1}{2}}d_2d_3\lambda \\ (1 - q^2)\lambda & q^{-\frac{1}{2}} + q^{\frac{3}{2}}d_1\lambda + q^{-\frac{1}{2}}d_2d_3\lambda^2 \end{pmatrix} \quad (5.18)$$

Thus, the generating function of conserved quantities is expressed as

$$\tau(\lambda) = \text{tr}_0[\hat{T}_{0N}(\lambda)\hat{G}_0(\lambda)q^{-\sigma\hat{\delta}}] \quad (5.19)$$

These considerations show that the Toda$_2$ chain is a quantum integrable non-ultralocal model that we could already study at this level. However in the next section we will introduce quantum analogues of the Darboux coordinates eqs. (4.10), (4.17) which have the advantage of ultra-localising the model.

### 6 Concrete realisation and ultra-localisation

One can represent the quantum operators $\xi_n$ in a way paralleling the classical construction (4.16). One first introduces the local Hilbert spaces $\mathfrak{h}_n \simeq L^2(\mathbb{R})$ and constructs the full Hilbert space as $\mathfrak{h} = \otimes_{n=1}^\infty \mathfrak{h}_n \simeq L^2(\mathbb{R}^N)$. Then, on each $\mathfrak{h}_n$ one introduces a pair of canonically conjugate operators $x_n, \xi_n$ such that $[x_n, \xi_n] = i$ and builds from them the Weyl pair

$$\{e^{\frac{i}{\sqrt{2}}x_n, e^{\omega_1x_n}} \quad \text{so that} \quad e^{\frac{i}{\sqrt{2}}x_n e^{\omega_1x_n} e^{\frac{i}{\sqrt{2}}x_n}, q = e^{\frac{i}{\sqrt{2}}x_n}. \quad \text{\(8\)}}$$
Written in these coordinates, the quantum Lax operator
\[ T \]
where the matrix
\[ T \]
can be recast in terms of a transfer matrix built up from a monodromy matrix having an ultralocal structure.

Then,
\[ \xi^1_n = e^{-\frac{\pi}{2}x_n} \prod_{a=1}^{n} e^{\frac{\lambda}{2}x_n} \quad \text{and} \quad \xi^1_n = e^{-\frac{\pi}{2}x_n} \sum_{a=1}^{n} e^{\frac{\omega}{2}x_n} \cdot e^{-\frac{2\omega}{2}x_n} \cdot \prod_{b=1}^{n} e^{\frac{\lambda}{2}x_b} \]  

provides one with a representation on \( \mathfrak{g} \) of the algebra \([5.1]\).

In terms of these canonical operators, one has
\[ Q_n^2 = q^2 e^{-\omega_1 x_{n+1}} e^{\frac{2\omega}{2}(x_n-x_{n+1})}, \quad P_n = e^{-\omega_1 x_n} + e^{\frac{2\omega}{2}(x_n-x_{n+1})}. \]

Written in these coordinates, the quantum Lax operator \( L_{0n}(\lambda) \) is obviously not-ultralocal and reads
\[ L_{0n}(\lambda) = \begin{pmatrix} \lambda - e^{-\omega_1 x_n} e^{\frac{2\omega}{2}(x_n-x_{n+1})} & -1 \\ q^2 e^{-\omega_1 x_{n+1}} e^{\frac{2\omega}{2}(x_n-x_{n+1})} & 0 \end{pmatrix}. \]

In the following, we will show that the generating function of conserved quantities given in \([5.19]\) can be recast in terms of a transfer matrix built up from a monodromy matrix having an ultralocal structure.

More precisely, define the ultralocal Lax matrix
\[ L_{0n}(\lambda) = \begin{pmatrix} \lambda - e^{-\omega_1 x_n} e^{\frac{2\omega}{2}(x_n-x_{n+1})} & -1 \\ q^2 e^{-\omega_1 x_{n+1}} e^{\frac{2\omega}{2}(x_n-x_{n+1})} & 0 \end{pmatrix}. \]

This ultralocal Lax matrix satisfies the usual Yang-Baxter equation with quantum (twisted) \( R \)-matrix
\[ R_{12}(\lambda_1, \lambda_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{q^2-1}{q^2-\lambda_1} & \frac{q^2-1}{q^2-\lambda_2} & 0 \\ 0 & \frac{q^2-1}{q^2-\lambda_1} & \frac{q^2-1}{q^2-\lambda_2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]

Let
\[ \tau_{\text{loc}}(\lambda) = \text{tr}[L_{0n}(\lambda) \cdots L_{01}(\lambda)], \]
be the associated transfer matrix and let \( \kappa_2 \) parameterise \( d_2 \) as \( d_2 = e^{-\frac{2\pi}{2} \kappa_2} \). Then, it holds
\[ d_2^N \cdot q \cdot V \cdot \tau(q^{-2}d_2^{-1}1) \cdot V^{-1} = \tau_{\text{loc}}(\lambda). \]

where
\[ V = \prod_{n=1}^{N} V_n \quad \text{and} \quad V_n = e^{\frac{2\pi i}{q^2} \kappa_2 x_n} \cdot e^{i\kappa_2 x_n}. \]

In order to establish the result, first observe that one can transform the non-ultralocal monodromy matrix \( T_{0N}(\lambda) \) by doing local gauge transformations:
\[ T_{0N}(\lambda) \bar{G}(\lambda) q^{-\sigma_5} = N_{0N+1} \bar{L}_{0N}(\lambda) \cdots \bar{L}_{02}(\lambda) \bar{L}_{01}(\lambda) N_{01}^{-1} q^{-\sigma_5} \]

where
\[ \bar{L}_{0n}(\lambda) = N_{0n+1} \tilde{L}_{0n}(\lambda) N_{0n} \quad \text{for} \quad n = 2, \ldots, N \quad \text{and} \quad \bar{L}_{01}(\lambda) = \bar{L}_{01}(\lambda) \big|_{d_1 \rightarrow q^{-2}d_1}. \]

The gauge matrices \( N_{0n} \) are chosen as
\[ N_{0n} = \begin{pmatrix} 1 & -q^{-\frac{1}{2}e^{-\frac{2\pi}{2} x_n}} \\ 0 & e^{-\omega_1 x_n} e^{-\frac{2\pi}{2} x_n} \end{pmatrix}, \quad \text{so that} \quad N_{0n}^{-1} = \begin{pmatrix} 1 & q^{-\frac{1}{2}e^{\omega_1 x_n}} \\ 0 & e^{\frac{2\pi}{2} x_n} e^{\omega_1 x_n} \end{pmatrix}. \]
The gauge transform of $\mathfrak{L}_{0n}(\lambda)$ already takes the ultralocal form

$$
\mathfrak{L}_{0n+1}\mathfrak{L}_{0n}(\lambda) = \left( \lambda - e^{-\omega_1 x_n} \begin{bmatrix} -q^{-1/2} \lambda + (q^{-1/2} - 1)e^{-\omega_1 x_n} \\ -1 \end{bmatrix} \right).
$$

One still needs to deal with the matrix $G_0(\lambda)$:

$$
G_{0n}(\lambda) \equiv \mathfrak{N}_{0n}^{-1} G_0(\lambda) \mathfrak{N}_{0n} = \begin{bmatrix} 1 + (q^{-\frac{1}{2}} - q^\frac{1}{2})\lambda e^{\omega_1 x_n} & [G_{0n}(\lambda)]_{12} \\ (1 - q^2)\lambda e^{2\omega_1 x_n} e^{\omega_1 x_n} & [G_{0n}(\lambda)]_{22} \end{bmatrix}.
$$

Above, we agree upon

$$
[G_{0n}(\lambda)]_{12} = [q^{-1} + q^2 d_1 \lambda + q^3 d_2 d_3 \lambda^2 - q^{-\frac{1}{2}} + q^2 d_2 d_3 \lambda e^{-\omega_1 x_n} - (q^{-1} - q)\lambda e^{-\omega_1 x_n}] e^{-\frac{\pi x}{2}} e^{\omega_1 x_n},
$$

$$
[G_{0n}(\lambda)]_{22} = q^{-\frac{1}{2}} + q^2 d_1 \lambda + q^2 d_2 d_3 \lambda^2 - \lambda(q^2 - q^2) e^{\omega_1 x_n}.
$$

Hence, all-in-all, one obtains

$$
\mathcal{L}_{0n}(\lambda) = \begin{bmatrix} q^2 \lambda - e^{-\omega_1 x_n} & -q^2 \lambda [1 + q d_1 e^{-\omega_1 x_n} + q^2 d_2 d_3 e^{-2\omega_1 x_n}] e^{-\frac{\pi x}{2}} e^{\omega_1 x_n} \\ q^{\frac{1}{2}} e^{\frac{\pi x}{2}} e^{\omega_1 x_n} & -1 + q^2 \lambda d_2 d_3 e^{-\omega_1 x_n} \end{bmatrix}.
$$

Then straightforward algebra and the use of periodic boundary conditions $\mathfrak{N}_0 N_{01} = \mathfrak{N}_{01}$ leads to

$$
\text{Tr} \left[ \mathfrak{N}_0 N_{01} \mathcal{L}_{0N}(\lambda) \cdots \mathcal{L}_{02}(\lambda) \mathcal{L}_{01}(\lambda) \mathfrak{N}_0 q^{-\sigma_0^5} \right] = q^{-1} \cdot \text{Tr} \left[ \mathcal{L}_{0N}(\lambda) \cdots \mathcal{L}_{01}(\lambda) \right].
$$

The equality can be obtained by first taking explicitly the matrix products on the auxiliary space of $\mathfrak{N}_0^{-1}$, $\mathcal{L}_{0N}(\lambda) \cdots \mathcal{L}_{02}(\lambda)$ and $\mathcal{L}_{01}(\lambda) \mathfrak{N}_0 q^{-\sigma_0^5}$ and then by taking the trace. Finally one compares this with the result of a similar calculation carried out on the level of the rhs expression.

It now only remains to observe that it holds

$$
d_2 \sigma_0^5 q^{\frac{1}{2}} \sigma_0^5 \mathfrak{N}_n \mathcal{L}_{0n} \left( q^{-2} \mathfrak{L}_{0n+1}(\lambda) \cdot \mathfrak{N}_n^{-1} \sigma_0^5 q^{-\frac{1}{2}} \sigma_0^5 \right) = \mathcal{L}_{0n}(\lambda) .
$$

**Local Hamiltonians.** Upon expanding the transfer matrix into powers of $\lambda$:

$$
\mathfrak{t}_{\text{loc}}(\lambda) = \text{tr} \left[ \mathcal{L}_{0N}(\lambda) \cdots \mathcal{L}_{01}(\lambda) \right] = \sum_{j=0}^{N} (-1)^j \cdot \lambda^{N-j} \cdot H_j ,
$$

one gets a family of commuting Hamiltonians $\{H_1, \ldots, H_N\}$. There are three interesting limiting cases.

- **$q$-Toda.** For $d_2 = d_3 = 0$, one gets the $q$-Toda chain

$$
H_1^{\text{Toda}} = \sum_{n=1}^{N} \left( 1 + q^{-1} d_1 e^{-\frac{\pi x}{2}} (x_n - x_{n-1}) \right) e^{-\omega_1 x_n} .
$$

- **Toda$_2$.** For $d_1 = d_3 = 0$, one gets the per se Toda$_2$ chain. We find

$$
H_1^{\text{Toda}_2} = \sum_{n=1}^{N} \left( e^{-\omega_1 x_n} + d_2 e^{-\frac{\pi x}{2}} (x_n - x_{n-1}) \right) .
$$

$$
H_2^{\text{Toda}_2} - \frac{1}{2} (H_1^{\text{Toda}_2})^2 = -\frac{1}{2} \sum_{n=1}^{N} \left( e^{-2\omega_1 x_n} + d_2 e^{-\frac{\pi x}{2}} (x_n - x_{n-1}) \right) .
$$
In the absence of a quantum analogue of eq. (4.10) and a suitable definition of $\text{tr}_q$, it is not possible to define the family of quantum Hamiltonians of Toda$_2$ directly in terms of the big quantum Lax matrix $L(\mu)$. However for $H_1^\text{Toda}_2$ we obviously have

$$H_1^\text{Toda}_2|_{d_2=1} \equiv -\text{tr}_q[L(\mu)] = \sum_{n=1}^{N} P_n.$$  

We easily find the correct definition of $\text{tr}_q[L^2(\mu)]$ by imposing the commutativity with $\text{tr}_q[L(\mu)]$ and the fact that in the classical limit $q \to 1$, $\text{tr}_q[L^2(\mu)]$ should reduce to $\text{tr}[L^2(\mu)]$,

$$\text{tr}_q[L^2(\mu)] = \sum_{n=1}^{N} \left\{ P_n^2 + (q^{3/2} + q^{-1/2})Q_n^2 \right\}.$$  

Inserting the ultralocal parameterisation of $P_n, Q_n^2$, we discover that

$$\left[ H_2^\text{Toda}_2 - \frac{1}{2}(H_1^\text{Toda}_2)^2 \right]|_{d_2=1} = -\frac{1}{2} \text{tr}_q[L^2(\mu)]$$

These formulae support the evidence that the quantum Toda$_2$ model is indeed obtained by the choice of parameters $d_1 = 0, d_2 = 1, d_3 = 0$ in the above family of models.

• $q$-oscillator. When $d_1 = -q^{-1}, d_2 = 1$ and $d_3 = 0$, the local Lax operator given above can also be rewritten in terms of $q$-oscillator.

Let

$$a_n = (1 - e^{-\omega_1 x_n})e^{-\frac{3}{2}x_n}, \quad a_n^* = e^{\frac{3}{2}x_n}, \quad e^{-\omega_1 x_n} = q^{2n}$$

then the $q$-oscillators algebra is satisfied

$$a_n a_n^* = 1 - q^{2n}, \quad a_n^* a_n = 1 - q^{2n-2}$$

$$a_n q^{2n} = q^{2n+2} a_n, \quad a_n^* q^{2n} = q^{2n-2} a_n^*$$

The local Lax operator then becomes

$$L_{0n}(\lambda) = \begin{pmatrix} \lambda - q^{2n} & q^2 \lambda \ a_n \\ -q^{-2} a_n^* & -1 \end{pmatrix}.$$  

Choose a complete set of left Eigenstates associated with the $q$-boson oscillator algebra $\{ v_n^{(k)} \}_{k \geq 0}$, viz.

$$v_n^{(k)} a_n = (1 - q^{-2k})v_n^{(k-1)}, \quad v_n^{(k)} a_n^* = v_n^{(k+1)}, \quad v_n^{(k)} q^{2n} = q^{-2k}v_n^{(k)}.$$  

Next consider $\omega_n$ to be the left Eigenstate of $a_n$ with eigenvalue $q^{-2}$:

$$\omega_n a_n = q^{-2} \omega_n \quad \text{so that} \quad \omega_n = \sum_{k=0}^{\infty} \frac{q^{-2k}}{(1-q^{-2}) \cdots (1-q^{-2k})} v_n^{(k)}.$$  

$\omega_n$ is an Eigenstate of the sum of the matrix elements of each column of the $q$-oscillator Lax matrix (6.13) with eigenvalue $\lambda - 1$. This property readily follows from the identity: $-q^{-2} \omega_n a_n^* = \omega_n (q^{2n} - 1)$. Consequently, $\Omega = \otimes_{n=1}^{N} \omega_n$ is a left Eigenstate of the Hamiltonian

$$H_1 = \sum_{n=1}^{N} \left\{ a_n a_n^* + q^{2D_n} \right\}$$  

with eigenvalue $N$. So, if we define a scalar product on each local space such that $\langle \omega_n, v_n^{(k)} \rangle = 1$, $\forall k$, then the Hamiltonian $\mathcal{H} = -N\mathcal{D}$ defines a stochastic model [9].
Conclusion

In the present work we have constructed the appropriate quantisation of the Toda chain endowed with the second Hamiltonian structure. This model appears as a natural lattice discretisation of the quantum Virasoro algebra and may appear in the future as a convenient way to build explicit representations of this algebra, in the continuum, for arbitrary values of the central charge. Our construction builds on the quantum ultra-localisation of the non-ultra local algebra appearing naturally in the context of the Toda chain in the second Hamiltonian formulation.

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