Relationship of the Hennings and Chern-Simons Invariants For Higher Rank Quantum Groups

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The Hennings invariant for the small quantum group associated to an arbitrary simple Lie algebra at a root of unity is shown to agree with the Chern-Simons (aka Jones-Witten or Reshetikhin-Turaev) invariant for the same Lie algebra and the same root of unity on all integer homology three-spheres, at roots of unity where both are defined. This partially generalizes the work of Chen, et al. ([CYZ12, CKS09]) which relates the Hennings and Chern-Simons invariants for SL(2) and SO(3) for arbitrary rational homology three-spheres.

Introduction

In [CKS09] Chen, Kuppum, and Srinivasan show that the Chern-Simons invariant (which they call the Jones-Witten invariant and is also called the Reshetikhin-Turaev invariant) associated to the Lie group SO(3) at odd levels of the SO(3) theory (in the sense of Dijkgraaf and Witten [DW90], corresponding to the quantum group at odd primitive roots of unity with root lattice representations) and the Hennings invariant associated to the corresponding quantum group have a tight relationship: The Hennings invariant of a manifold is the Chern-Simons invariant times the order of the first homology (or zero if it is infinite order). This result is new in the case of rational homology three-spheres (i.e. when the order is finite), in which case each invariant determines the other. This result was generalized by Chen, Yu and Zhang in [CYZ12] to the same result for the SL(2) invariant at integer levels (corresponding to the same quantum group at even roots of
unity with all representations). Notice in each case that this implies the invariants agree exactly on integral homology spheres.

One would like to generalize this result to arbitrary quantum groups, which is to say from the Chern-Simons point of view to arbitrary complex simple Lie groups, or at least complex simply-connected simple Lie groups. This is not primarily because the higher-rank invariants are interesting in themselves; indeed, both the Hennings and CS invariants are so complicated it is difficult to imagine calculating either invariant beyond lens spaces and perhaps SL(3). However, one expects from physics that the CS invariant is telling us deep geometric information, and it is reasonable to guess that if the Hennings invariant is so deeply connected to the CS invariant then it too is telling us something geometric (e.g., perturbative aspects of Chern-Simons theory). A property of the CS and Hennings invariants that applies to the rank one case may be an artifact of the algebraic simplicity of this case, while a property of a wide range of examples is likely to be connected with the underlying geometry of the situation, and a proof that works in a general case might bring us closer to revealing that geometry.

There are two problems with generalizing the claim in the above two papers to higher rank quantum groups (from here on we will be working entirely with quantum groups, relegating mentions of Lie groups themselves to motivation). The first is that their argument relies heavily on a complete knowledge of the center of the small quantum group, which is not known for higher rank and seems like a very deep problem (though important progress is being made on it, e.g. Lachowska ([Lac03])). The second is that the result as stated seems unlikely to be true! If every nilpotent element of the center of the quantum group squares to zero, as in the rank one case, the result still holds, but there is no reason to believe this is the case, and if it is not the relationship between the invariants may be quite complicated. However, this issue does not come up with integral homology spheres.

Section One expands on one author’s previous paper [Saw06b] to extend and complete the restricted quantum group so that it is a topological ribbon Hopf algebra. It also introduces a grading on the algebra introduced by Habiro in [Hab08] and developed by Habiro and Lê in [HL16] which is here called left degree (We follow [Hab08] in calling the left-degree-zero component the even part, but this is distinct from the use of the word “even” in [CYZ12]).

Section Two reviews the universal invariant of tangles, focusing on Habiro’s ([Hab06]) notion of bottom tangles. It also reviews the small quantum group at a root of unity. The fundamental new result here is the map $\Phi$ from the topological ribbon Hopf algebra extending and completing the restricted quantum group to the ribbon Hopf algebra called the small quantum group, and Proposition 2.1 which says that it intertwines the universal invariant of tangles. Essentially, the universal invariant is the same for the generic quantum group and the quantum group at roots of unity. Section Two also reviews the representation theory, center and invariant functionals, proving that the even part of the center of the completion of the restricted quantum group is spanned by elements $z_\theta$ constructed from quantum traces. The two main results of this section together give a complete description of the image of the even part of the center under $\Phi$ as the span of these $z_\theta$. 

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Section Three defines the invariant functionals from which the Hennings and Chern-Simons three-manifold invariants are constructed. It uses results of [Hab08] to prove the key result that the universal invariant of a bottom tangle with zero linking matrix is in the even part of the tensor product of the completion of the restricted quantum group. Thus when the tangle has one open component its universal invariant is an infinite linear combination of the central elements $z_θ$, and its universal invariant in the small quantum group is a finite linear combination of their images. This is the key insight from which the proof of the main result follows.

Integral homology three-spheres are obtained by surgery on the closure of zero linking matrix tangles with twists added. Thus the computation of any three-manifold invariant constructed from the universal invariant is reduced to a computation involving the underlying invariant functional and these $z_θ$. This computation in fact agrees for any such three-manifold invariant.

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1 Quantum Groups and their Completions

1.1 Large Quantum Group

This subsection defines an extended version of the standard quantum group over the ring of Laurent polynomials in a way that contains the various standard specializations, restricts coherently to roots of unity, and permits defining link invariants. In this it extends work of one author ([Saw06b]), but the bulk of the material is standard and largely comes from Lusztig ([Lus90a, Lus90b, Lus93]) and sticks closely to Lusztig’s notation, most of it appearing in a more expository fashion in Chari and Pressley ([CP94]). Chen et al ([CYZ12]) define $\hat{U}^g_\nu$ which is our $U_q(\mathfrak{sl}_2)$ with $\nu = q^{1/2}$, $E = E_1$, $F = F_1$ and $K^2 = K_1$. Habiro and Lê ([HL16]) define $U_\nu(\mathfrak{g})$ exactly as our $U_q(\mathfrak{g})$ with $\nu = q$ and with the opposite comultiplication and antipode, in each case extended from the ring $A_q$ to the field $\mathbb{Q}(q)$.

1.1.1 Lie algebra and Cartan matrix

Let $\{a_{ij}\}_{i,j=1}^n$ be a Cartan matrix and $\{d_i\}_{i=1}^n$ be a minimal sequence of integers such that $d_ia_{ij}$ is positive definite. Let $\mathfrak{h}$ be an $n$ dimensional Euclidean space and let $\alpha_i \in \mathfrak{h}^*$ for $1 \leq i \leq n$, called simple roots, be such that $\langle \alpha_i, \alpha_j \rangle = d_ia_{ij}$ (thus the inner product is normalized so short roots have length squared 2). Let $\Lambda_r$, the root lattice, be the lattice in $\mathfrak{h}^*$ generated by the $\alpha_i$. Let $\Lambda$, the weight lattice, be the lattice in $\mathfrak{h}^*$ whose inner product with each $\alpha_i/d_i$ is an integer. Let $L$ be the smallest integer such that $L$ times any inner product of weights is an integer. Finally let the Weyl group $\mathcal{W}$ be the group generated by reflections of $\mathfrak{h}^*$ about the hyperplanes $\{x \in \mathfrak{h}^* \mid \langle x, \alpha_i \rangle = 0 \}$. $\mathcal{W}$ preserves $\Lambda_r \subset \Lambda$. See Humphreys ([Hum72]) for detail on Lie algebras and their representations.

The orbit of $\{\alpha_i\}$ under the Weyl group is the set of roots $\Delta \in \Lambda_r$. Each element of $\Delta$ is either a nonnegative or nonpositive combination of simple roots. Call $\Delta^+$ the...
set $\beta_1, \ldots, \beta_N$ of positive roots. Let $\rho \in \Lambda$ be half the sum of the positive roots. The translated action of the Weyl group on $\mathfrak{h}^*$ is given by $\lambda \mapsto \sigma(\lambda + \rho) - \rho$ for each $\sigma \in \mathcal{W}$.

Let $\mathfrak{g}$ be the complex, simple Lie algebra associated to $a_{ij}$ with Cartan subalgebra $\mathfrak{h}$.

1.1.2 Quantum integers and base rings

Let $A_s = \mathbb{Z}[s, s^{-1}]$ be the ring of Laurent polynomials in an indeterminant $s$ and let $A_q = \mathbb{Z}[q, q^{-1}]$ be embedded in $A_s$ via $q = s^{\lambda_i}$. Let $q^i = q^{d_i}$, and also write $q^{\beta_j} = q^{i}$ for any positive root $\beta_j$ of the same length as $\alpha_i$.

Define

$$ [n]^i = (q^{n^i} - q^{-n^i})/(q^{i} - q^{-i}) \in A_q $$

$$ [n]_i! = [n]_i \cdot [n-1]_i \cdots [1]_i \in A_q $$

$$ [m]_i^n = [m]_i!/( [n]_i! [m-n]_i! ) \in A_q. $$

1.1.3 The full Cartan subalgebra

Define the algebra

$$ U_0(\mathfrak{g}, \text{full}, \mathcal{A}_s) = \text{Map}(\Lambda, \mathcal{A}_s) $$

(from here on drop the reference to $\mathfrak{g}$ as it and the Cartan matrix will remain fixed throughout) the space of set-theoretic maps. Define $K_\gamma, K_\pm^i, [K_i; m], [K_i; m, n] \in U_0(\text{full}, \mathcal{A}_s)$ for $\gamma \in \Lambda$, $1 \leq i \leq n$ and $m \in \mathbb{N}$ by

$$ K_\gamma(\lambda) = s^{L(\lambda, \gamma)} $$

$$ K_\pm^i = K_{\pm \alpha_i} $$

$$ [K_i; m](\lambda) = \left[ (\langle \lambda, \alpha_i \rangle + m)/d_i \right]_i $$

$$ [K_i; m, n](\lambda) = \left[ (\langle \lambda, \alpha_i \rangle + m)/d_i \right]_i^n. $$

Define $U_0(\text{res}, \mathcal{A}_s)$ to be the subalgebra generated by $K_\pm^i$ (or equivalently $K_\theta$ for $\theta \in \Lambda_r$) and $[K_i; m, n]_i$, define $U_0(\text{res}, \mathcal{A}_q)$ to be the subalgebra over $\mathcal{A}_q$ generated by the same elements, define $U_0(\text{res}, \text{weight}, \mathcal{A}_s)$ to be the subalgebra generated by $K_\gamma$ for $\gamma \in \Lambda$ and $[K_i; m, n]_i$, and finally define $U_0(\text{nonres}, \mathcal{A}_q)$ to be generated by $K_\pm^i$ and $[K_i; m]_i$. The Cartan subalgebra of the nonrestricted quantum group [CP94][Sec. 9.2] and of the restricted quantum group [CP94][Sec. 9.3] clearly map homomorphically onto $U_0(\text{nonres}, \mathcal{A}_q)$ and $U_0(\text{res}, \mathcal{A}_q)$ respectively, and it is easy to check that it has no kernel (any polynomial in the generators is clearly nonzero on sufficiently large weights).
1.1.4 The full Hopf algebra Define \( U(\text{full}, \mathcal{A}_s) \) to be the algebra over \( \mathcal{A}_s \) generated by \( \{E_i, F_i\}_{i=1}^n \) and \( \text{Map}(\Lambda, \mathcal{A}_s) \) subject to relations

\[
E_i f(\lambda) = f(\lambda - \alpha_i) E_i, \quad F_i f(\lambda) = f(\lambda + \alpha_i) F_i, \quad (1.3)
\]

\[
E_i F_j - F_j E_i = \delta_{ij} [K_i; 0], \quad (1.4)
\]

\[
\sum_{r=0}^{1-a_{ij}} (-1)^r \binom{1-a_{ij}}{r} E_i^{1-a_{ij}-r} E_j E_i^r = 0 \quad \text{if } i \neq j,
\]

\[
\sum_{r=0}^{1-a_{ij}} (-1)^r \binom{1-a_{ij}}{r} F_i^{1-a_{ij}-r} F_j F_i^r = 0 \quad \text{if } i \neq j. \quad (1.5)
\]

\( U_0(\text{full}, \mathcal{A}_s) \) is naturally a subalgebra of \( U(\text{full}, \mathcal{A}_s) \). Define \( U_{\pm}(\text{full}, \mathcal{A}_s) \) and \( U_{\pm}(\text{full}, \mathcal{A}_q) \) to be the subalgebras over the given rings generated by \( \{E_i\} \) and \( \{F_i\} \) respectively. Define \( U(\text{nonres}, \mathcal{A}_q) \) to be the subalgebra generated by \( U_0(\text{nonres}, \mathcal{A}_q) \) and \( \{E_i, F_i\} \). In each case define the degree on this algebra, a grading by \( \Lambda_r \) sending \( E_i, f(\lambda) \) and \( F_i \) to \( \alpha_i, 0 \) and \( -\alpha_i \).

Two key facts about more standard specializations readily extend to this larger algebra. First, it is immediate from Equation 1.3 and Equation 1.4 that any element of \( U(\text{full}) \) or \( U(\text{nonres}) \) can be written as a linear combination of products \( FKE \), where \( F \in U_- \), \( K \in U_0 \) and \( E \in U_+ \). Second, following [Lus90a], the braid group of the Weyl group acts as automorphisms of \( U(\text{full}) \), and a choice of reduced factorization of the longest element of the Weyl group gives a map of the Weyl group into the braid group which in turn gives an ordering \( \beta_1, \ldots, \beta_N \) of \( \Delta^+ \) and elements \( E_{\beta_j}, F_{\beta_j} \in U_{\pm} \) homogeneous respectively of degree \( \pm \beta_j \). Further, writing \( \vec{r} = (r_1, \ldots, r_N) \) with \( r_i = 0, 1, 2, \ldots \), one can show that a basis for \( U_{\pm} \) is given by \( E_{\beta_1}^{r_1} \cdots E_{\beta_N}^{r_N} \) and \( F_{\beta_1}^{r_1} \cdots F_{\beta_N}^{r_N} \) respectively.

From these two facts it follows that \( U(\text{nonres}, \mathcal{A}_q) \) is in fact the nonrestricted specialization of the quantum group associated to \( \mathfrak{g} \) in [CP94][Sec. 9.2].

1.1.5 Divided powers and the restricted specialization Of course \( U(\text{full}, \mathcal{A}_s) \) can be extended to the field of fractions over \( \mathcal{A}_s \), and over that field one can define the so-called divided powers

\[
E_{\beta_j}^{(r)} = E_{\beta_j}^r / [q]_{\beta_j}! \quad (1.6)
\]

and likewise for \( F \). Define \( U(\text{full}, \text{divided}, \mathcal{A}_s) \) to be the subalgebra generated over \( \mathcal{A}_s \) by the divided powers Equation 1.6 and \( \text{Map}(\Lambda, \mathcal{A}_s) \). Once again \( U_{\pm}(\text{full}, \text{divided}, \mathcal{A}_s) \) have as bases

\[
E_{\beta_1}^{(r_1)} \cdots E_{\beta_N}^{(r_N)}, \quad F_{\beta_1}^{(r_1)} \cdots F_{\beta_N}^{(r_N)} \quad (1.7)
\]

and because [CP94][Sec. 9.3] argues

\[
E_i^{(r)} F_i^{(s)} = \sum_{0 \leq t \leq r, s} F_i^{(s-t)} [K_i; 2t - s - r, t] E_i^{(r-t)}
\]

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it again follows that $U(\text{full, divided, } A_s)$ is spanned by
\[ F^{(r)} K E^{(i)} \] (1.8)
where $K \in \text{Map}(\Lambda, A_s)$. Call this algebra $U_{\text{full}}$.

Similarly $U(\text{res, divided, } A_q)$ is the subalgebra over $A_q$ generated by the divided powers Equation 1.6 and $U_0(\text{res, } A_q)$, and $U(\text{res, divided, weight, } A_s)$ is the subalgebra generated by the divided powers and $U_0(\text{res, weight, } A_s)$. In each case the algebra is spanned by Equation 1.8 with $K$ an element of the associated Cartan subalgebra. In particular as above one can check that the restricted quantum group $U_q^{\text{res}} [\text{CP94}][9.3]$ is isomorphic to $U(\text{res, divided, } A_q)$. Henceforth call this algebra $U_q$ for simplicity. Also define $U_{\text{weight}} = U(\text{res, divided, weight, } A_s)$, the restricted specialization of what [CP94] calls the simply connected version of the quantum group.

For the algebras labeled res and nonres, define a second grading, the left degree, valued in $\Lambda_r/2\Lambda_r$ (or $\Lambda/2\Lambda$ if labeled by weight). It assigns the weights $\rho\alpha$, $0, -\lambda$, $0$, and $0$ to $E^{(r)}$, $F^{(r)}$, $K\lambda$, $[K_i, m]$, and $[K_i; m, n]$ respectively. The degree 0 subalgebra of each of these algebras is called even or ev, so for instance $U_q^{\text{ev}} = U(\text{res, divided, ev, } A_q)$.

**Remark 1.1** Habiro and Habiro and Lê define even with the degree on $E$ and $F$ reversed, because they are using the opposite comultiplication. The term even here should be treated as identical to theirs, and the results about the even subalgebra here are, apart from details about the ring, algebra and completions used, duplicates of results in those papers.

### 1.2 Topological Ribbon Hopf Algebra

Topological Hopf algebras have a variety of definitions such as Bonneau & Sternheimer and Habiro & Lê ([BS05, HL16]), but here it will be useful to give a slightly more general one.

**1.2.1 Uniform Structure** A countable collection $\mathcal{I}$ of ideals of a ring $A$ or submodules of a module $U$ whose intersection is empty defines a uniform structure as in Bourbaki [Bou98] (often called the $\mathcal{I}$-adic topology) by defining the collection of entourages (analogues of $\epsilon$-neighborhoods) for the uniform structure to be the set $\mathcal{I}$ of all finite intersections of elements of $\mathcal{I}$. A sequence is Cauchy if for each $I \in \mathcal{I}$ there is a point after which all entries differ by elements of $I$, and a function between such spaces is uniformly continuous if for each $I$ in the range there is a finite collection of ideals/submodules in the domain whose intersection is mapped into $I$. $\bar{A}$ and $\bar{U}$ are the completion of $A$ and $U$ with respect to this uniform structure, and may usually best be thought of as consisting of infinite sums in $A$ or $U$ where for each $I \in \mathcal{I}$ all terms are eventually in $I$.

If $U$ is a module over $A$ and for each ideal $I \in \mathcal{I}_A$ the submodule $IU$ is in $\mathcal{I}_U$, then call the uniform structure on $U$ consistent and note that the action of $A$ on $U$ extends to an action of $\bar{A}$ on $\bar{U}$.

If $U$ and $V$ are consistent modules over $A$ then $U \otimes_A V$ has a uniform structure consisting of the submodule $I_U \otimes_A V + U \otimes_A I_V$ for each $I_U \in \mathcal{I}_U$ and $I_V \in \mathcal{I}_V$. 
The completion of the tensor product of completions is the same as the completion of the tensor product, and therefore completion is a monoidal functor from the category of modules with uniform structures to the category of completed modules with the completed tensor product, and there is a natural transformation from the identity functor to this functor. When speaking of completed rings and modules use the ordinary tensor product symbol to represent the completed tensor product.

A topological Hopf algebra is a Hopf algebra $U$ over a ring $A$ with a consistent uniform structure and uniformly continuous maps $M: \overline{U} \otimes \overline{U} \to \overline{U}$, $i: A \to \overline{U}$, $\Delta: \overline{U} \to \overline{U} \otimes \overline{U}$, $\epsilon: \overline{U} \to A$, and $S: \overline{U} \to \overline{U}$, all tensor products understood to be completed, that satisfy all the usual relations of a Hopf algebra, as enumerated in Chari & Pressley [CP94]. A topological ribbon Hopf algebra is a topological Hopf algebra $U$ together with an element $R \in \overline{U}^{\otimes 2}$ called an $R$ matrix and ribbon element $g \in \overline{U}$ satisfying the usual axioms of a ribbon Hopf algebra as enumerated in Reshetikhin & Turaev, Chari & Pressley, Bakalov & Kirillov, and Turaev ([RT90, CP94, BK01, Tur94]).

1.2.2 The Cartan topological Hopf algebra For $A = A_q$ or $A = A_s$ let $\mathcal{I}_A$ consist of all ideals $I_{[k]} = [k]_\Lambda A$ for $i \in \{1, \ldots, n\}$ and $k \in \mathbb{Z}$, and let $\mathcal{A}_q$ and $\mathcal{A}_s$ be the completions (1.2.1) with respect to this structure. For every module $U$ over $A$ or $A_q$ assume that all submodules $I_{[k]} = [k]_\Lambda U$ are included in its uniform structure so that it is consistent.

For each $\lambda \in \Lambda$ let $I_\lambda$ be the ideal of $U_{\text{full},0} = \text{Map}(\Lambda, A_s)$ consisting of all maps which are zero on $\lambda$. Then $\mathcal{I}_{U_{\text{full},0}}$ consisting of all $I_\lambda$ and $I_{[k]}$ is a consistent uniform structure on $U_{\text{full},0}$. It is natural for each $\lambda \in \Lambda$ to write $\lambda$ for the element of $U_{\text{full},0}$ sending $\gamma \in \Lambda$ to $\delta_{\lambda, \gamma}$, so that

$$\lambda \cdot \gamma = \delta_{\lambda, \gamma} \lambda.$$ (1.9)

and an arbitrary element of $U_{\text{full},0}^{\otimes m}$ can be written

$$\sum_{\lambda_1, \ldots, \lambda_m \in \Lambda} c_{\lambda_1, \ldots, \lambda_m} \lambda_1 \otimes \cdots \otimes \lambda_m$$

with $c_{\lambda_1, \ldots, \lambda_m}$ in $A_s$.

With the uniform topology generated by $\mathcal{I}_{U_{\text{full},0}}$ above $U_{\text{full},0}$ is a topological Hopf algebra dual to the group Hopf algebra on $\Lambda$ via $S(\lambda) = -\lambda$ and $\Delta(\lambda) = \sum_{\gamma + \gamma' = \lambda} \gamma \otimes \gamma'$ (this restricts to a literal Hopf algebra structure on $U_0(\text{res})$ and $U_0(\text{nonres})$).

1.2.3 The full topological Hopf algebra For each $\lambda$, $U_{\text{full}} I_\lambda U_{\text{full}}$ is an ideal of $U_{\text{full}}$ and $U_{\text{full}} I_{[k]} U_{\text{full}}$ is a consistent uniform structure on $U_{\text{full}}$. Define $\mathcal{I}_q$ to be the consistent uniform structure on $U_q$ consisting of $I_{[k]}$, and $\mathcal{I}_{\text{weight}}$ the consistent uniform structure on $U_{\text{weight}}$ consisting of all $I_{[k]}$.

Then each element of the completion $U_{\text{full}}^{\otimes m}$ can be written in the form

$$\sum_\alpha c_\alpha [k]_\Lambda ! (F(\bar{r}_1) \otimes \cdots \otimes F(\bar{r}_m)) (\lambda_1 \otimes \cdots \otimes \lambda_m) (E(\bar{s}_1) \otimes \cdots \otimes E(\bar{s}_m))$$ (1.10)
where \( \alpha = (k, i, \vec{r}_1, \ldots, \vec{r}_m, \lambda_1, \ldots, \lambda_m, \vec{s}_1, \ldots, \vec{s}_m) \) and for each \( k, \lambda_1, \ldots, \lambda_m \) only finitely many of the corresponding \( c_{\alpha} \in A_s \) are nonzero. Each element of the completion \( \mathcal{U}_q^{\otimes m} \) and \( \mathcal{U}_{weight}^{\otimes m} \) can be written

\[
\sum_{\alpha} [k]_i! \left( F(\vec{r}_1) \otimes \cdots \otimes F(\vec{r}_m) \right) H_{\alpha}(E(\vec{s}_1) \otimes \cdots \otimes E(\vec{s}_m))
\]

(1.11)

where \( \alpha = (k, i, \vec{r}_1, \ldots, \vec{r}_m, \vec{s}_1, \ldots, \vec{s}_m) \) and each \( H_{\alpha} \) is respectively a polynomial in tensor products of \( K_{\theta}[, K_i; m, n] \) for \( \theta \in \Lambda \) with coefficients in \( A_q \) or tensor products of \( K_{\lambda}, [K_i; m, n] \) for \( \lambda \in \Lambda \) with coefficients in \( A_s \), and for each \( k \) only finitely many \( H_{\alpha} \) are nonzero.

These algebras admit a topological Hopf algebra structure via the topological Hopf structure on the Cartan subalgebra (1.2.2) and

\[
\epsilon(E_i) = \epsilon(F_i) = 0 \\
S(E_i) = -E_i K_i^{-1} \quad S(F_i) = -K_i F_i \\
\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i \quad \Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i
\]

extending this to divided powers (1.1.5) in the obvious way. The proof is the same as given in [CP94][Sec. 6.5].

### 1.2.4 The full topological ribbon Hopf algebra

Define \( R_0 \in \mathcal{U}_{full,0} \otimes^2 \) by

\[
R_0 = \sum_{\lambda, \gamma \in \Lambda} q^{\langle \lambda, \gamma \rangle} \lambda \otimes \gamma
\]

(1.12)

(notice here coefficients in \( A_s \) are essential). Define \( R \in \mathcal{U}_{full} \otimes^2 \), by

\[
R = R_0 \sum_{\vec{r}} b_{\vec{r}} E(\vec{r}) \otimes F(\vec{r})
\]

(1.13)

where

\[
b_{\vec{r}} = \prod_{j=1}^{N} q_{\beta_j} \left( \frac{r_j (r_j + 1)}{2} (1 - q_{\beta_j}^{2}) \right)^{r_j [r_j]_{\beta_j} \beta_j^{-1}}.
\]

\( R \) is quasitriangular and together with the ribbon element \( g = K_{2\rho} \) defines a ribbon structure.

### 2 Invariants of Tangles

#### 2.1 Bottom Tangles and the Universal Invariant

The universal invariant was first defined by Lawrence and Hennings ([Law89, Hen96]). Our discussion of bottom tangles and the universal invariant follows [Hab06], with terminology and definitions most closely matching [HL16].
2.1.1 Adjoint action and invariants In a topological Hopf algebra the adjoint action
of \( a \) on \( U \) is given by
\[
\text{ad}_a(x) = \sum_j a_j' x S(a_j'')
\]
where \( \Delta a = \sum_j a_j' \otimes a_j'' \), and generalizes to
an action on \( U^\otimes m \) by applying \( \text{ad}_a \otimes_m \) to \( \Delta^m(a) \). \( U^\otimes m, \text{inv} \) is the subspace of elements on
which this action is trivial, i.e. \( \text{ad}_a x = \epsilon(a) x \).

Check that \( U_q^{\text{ev}} \) is a module for the adjoint action of \( U_{\text{full}} \). In particular, if \( \lambda \in \Lambda \) and \( x \in U_q \) of degree \( \theta \in \Lambda_r \) then
\[
\text{ad}_\lambda(x) = \delta_{\lambda,\theta} x. \tag{2.1}
\]

Define \( U^*_{\text{inv}} \) to be the set of uniformly continuous linear functionals \( U \to \mathbb{A} \) invariant
under the coadjoint action of \( U \) on its dual (i.e., \( \psi \) such that \( \psi(\text{ad}_a(x)) = \epsilon(a) \psi(x) \)),
whose elements are called invariant functionals. These are exactly the uniformly continuous functionals \( \psi \) such that
\( \psi(ab) = \psi(b S^2(a)) \) for \( a, b \in U \). In this case \( \phi(a) = \psi(g^{-1} a) \)
is tracial in the sense that \( \phi(ab) = \phi(ba) \), and indeed any tracial functional \( \phi \) gives an
invariant functional \( \psi(a) = \phi(ga) \), so tracial and invariant functionals are in one-to-one
 correspondence.

2.1.2 Framed tangles and bottom tangles A framed tangle is a smooth embedding
of \( m \) oriented unit intervals and \( n \) oriented circles (open and closed components) in
\( D = \mathbb{R} \times [-1, 1] \times [0, 1] \) with endpoints of the intervals lying on \( \mathbb{R} \times \{0\} \times \{0\} \) and
\( \mathbb{R} \times \{0\} \times \{1\} \) together with a smoothly varying normal vector at each point of each
component which is pointing in the positive \( x \)-direction at each endpoint. Framed tangles
are considered up to isotopy of \( D \) leaving the endpoints of the interval fixed. A framed
bottom tangle is a framed tangle with the initial and final endpoint of the the \( j \)th interval
at the points \( (2j, 0, 0) \) and \( (2j - 1, 0, 0) \). A projection of a framed tangle is a projection
of an isotopy representative onto the \( xz \) plane so that the framing is never normal
to the projection, the map of each component into the plane is an immersion, and
self-intersections are all transverse double points and not at critical points of \( z \). Two
projections represent the same isotopy class of a framed tangle if and only they can be
connected by planar isotopy and the framed Reidemeister moves of Figure 1 discussed
by Burde and Zieschang in unframed form and Trace in framed form ([BZ86, Tra83]).

If \( T \) and \( S \) are two tangles, write \( T \otimes S \) for the tangle formed by placing \( T \) to the left of
\( S \), and write \( TS \) for the tangle formed by placing \( T \) above \( S \) and rescaling \( z \) by a factor
of \( 1/2 \), assuming the open components of \( T \) intersect the bottom \( (\mathbb{R} \times \{0\} \times \{0\}) \) in
the same locations and corresponding orientations as the open components of \( S \) intersect
the top, so that the result is again a tangle.
2.1.3 The universal invariant  Suppose $U$ is a topological ribbon Hopf algebra (¶1.2.1) with quasitriangular $R = \sum_{k \in K} a_k \otimes b_k$ with each $a_k, b_k \in U$ (for each $I \in I$, $a_k$ and $b_k$ are in $I$ except for finitely many $k \in K$) and ribbon element $g$. Write $R^{-1} = \sum_{k \in K} \bar{a}_k \otimes \bar{b}_k = \sum_{k \in K} S(a_k) \otimes b_k = \sum_{k \in K} a_k \otimes S^{-1}(b_k)$. The universal invariant of a tangle $T$ with each closed component labeled by an invariant functional $\Gamma_U(T)$ is defined as follows. First define it for tangles with all open components. Choose a projection and an ordering of the components, and place a bead at each local extreme point of the height ($z$-direction) of each component where the component’s orientation is pointing to the right, and a bead on each strand of each crossing. A state is an assignment of an element of the index set $K$ of the sum defining $R$ to each crossing of the projection. For each state, put elements of $U$ on each bead as follows. Put $g$ on each bead at a minimum and $g^{-1}$ on each bead at a maximum. Put $a_k$ on the bead on the lower strand and $\bar{b}_k$ on the bead of the upper strand of each positively oriented crossing, where $k$ is the element of $K$ associated to that crossing by the state, similarly put $\bar{a}_k$ and $b_k$ on the lower and upper strands of each negative crossing. If both strands are pointing to the right, apply $S^2$ to the strand pointing up and to the right. Combine the beads by sliding together and replacing adjacent beads labeled $b$ and $a$ in the direction of the orientation with the with a single bead labeled $ab$ until each component has a single bead. Using the ordering of the components the elements of $U$ on each bead determine an element of $U \otimes^m$, and summing over all states gives an element $\Gamma_U(T)$ of the completion $U \otimes^m$ which we call the universal invariant of the tangle. If there are closed components, choose a projection $P$ of an open tangle and an ordering of its components so that $P(C \otimes C \otimes \cdots \otimes C \otimes 1 \otimes \cdots \otimes 1)$ is a projection of $T$ with the first $k$ components becoming the closed components, where $C$ is the right pointing cup at the bottom left of Figure 2 and 1 is the tangle with one vertical component oriented up or down. Then $\Gamma(T) = (\phi_1 \otimes \cdots \otimes \phi_k \otimes 1 \otimes \cdots \otimes 1)\Gamma(P)$ where the $\phi_i$ are the invariant functionals labeling the closed components of $T$.

Habiro shows

**Theorem 2.1 ([Hab06]):** If a framed tangle $T$ has $m$ open components and all closed components are labeled by invariant functionals then the quantity $\Gamma_U(T)$ computed above is an element of $U \otimes^m$ which is independent of the choices made in the construction (except for the obvious effect of reordering the open components) and of ambient isotopy. Further, concatenating two tangles in the $x$ direction gives an invariant which is the tensor product of the two invariants, and concatenating compatible tangles in the $z$ direction (gluing open components appropriately) gives an invariant which is the product of the
invariants. Finally, if \(T\) is a bottom tangle and the components are ordered left to right then \(\Gamma_U(T) \in U^{\otimes m, \text{inv}}\).

### 2.1.4 Properties of the universal invariant

Natural operations on tangles correspond to natural operations on the Hopf algebra. In particular, if one doubles one open component of a tangle (the framing makes this unambiguous), the invariant of the doubled tangle is obtained from the invariant of the original by applying \(\Delta\) to the label on the doubled component and putting the first factor on the rightmost of the two new components (according to the orientation) and the second on the leftmost. Reversing the orientation of an open component has the effect of applying \(S\) to that component, and then multiplying it on the left by \(g^{-1}\) if the original component started on the bottom and on the right by \(g\) if the original component ended on the top. These relationships can be checked on generating tangles and are preserved by concatenating tangles horizontally and vertically. This is shown in [Hab06].

Putting this together gives a point that will be used crucially in the proof of Proposition 3.1. Suppose \(B\) is a bottom tangle, and \(\Gamma(B) = \sum x_i \otimes y_i\) with \(x_i \in U\) and \(y_i \in U^{m-1}\). Suppose that \(T\) is a tangle consisting of an open component on the left going from the top to the bottom which is first in the ordering, followed by a series of \(k\) bottom components (i.e whose endpoints are arranged as for a bottom tangle), followed by \(m\) copies of the a vertical strand down and a vertical strand up. Suppose that \(\Gamma(T) = \sum a_j \otimes b_j \otimes 1\) where \(a_j \in U\) corresponds the first component, \(b_j \in U^{\otimes k}\) to the bottom components, and \(1 \in U^{\otimes m-1}\) corresponds to the vertical components. Suppose that \(T'\) is \(T\) with the first component doubled and the orientation of the rightmost of the two new component reversed. Notice that the concatenation \(BT'\) is now consistent. Then

\[
\Gamma(BT') = \sum_{ij} \text{ad}_{a_j}(x_i) \otimes b_j \otimes y_i \tag{2.2}
\]

by the definition of adjoint and the properties in the previous paragraph. Thus the natural adjoint action on Hopf algebras corresponds to doubling a component and reversing the second of the resulting components. The analogous statement is true if several components are doubled and reversed in this fashion.

The sequel will also frequently mention the universal invariants \(r\) of the bottom tangle representing the positive twist, \(r^{-1}\) of the bottom tangle representing the negative twist, and \(C\) of the clasp tangle, all as illustrated in Figure 3. The first two are elements of the center \(U^{\text{inv}}\), the third of \(U^{\otimes 2, \text{inv}}\). The closure of the clasp tangle is called the Hopf link, and if each component is labeled by an invariant functional its universal invariant is a scalar in \(\overline{A}\).

### 2.2 Small Quantum Group

The discussion of the small quantum group follows Lusztig ([Lus93, Lus90b]), with occasional small additions from [Saw06b].
2.2.1 Small quantum group  
Now restrict the generic $q$ to a root of unity. Specifically, let $l$ be a positive integer and consider the homomorphism $A_s \rightarrow Q(s)$, where $s$ is a primitive $lL$th root of unity (i.e. satisfies the $lL$th cyclotomic polynomial) given by $s \mapsto s$. Since $|k|! = 0$ in $Q(s)$ for large enough $k$, this map extends continuously to $A_s$. Define $U_s = U_{\text{full}} \otimes Q(s)$.

For each $i \leq n$ let $l_i$ be $l/\gcd(l, d_i)$ and let $l'_i$ be $l_i$ or $l_i/2$ according to whether $l_i$ is odd or even. Likewise define $l_\beta$ and $l'_\beta$ for positive roots $\beta$, and define $l'$ to be $l$ or $l/2$ according to the parity of $l$. Then $l'_\beta$ is the least natural number such that $[l'_\beta]_\beta = 0$ and $l_\beta$ is the least such that $q^{l_\beta}$ is a unit. Define $\Lambda_l = \{ \lambda \in \Lambda \mid \langle \lambda, \gamma \rangle \in l\mathbb{Z}, \forall \gamma \in \Lambda \}$. Notice in $U_s$, $E_{l_\beta} = F_{l_\beta} = 0$ and $K_\lambda = 1$ when $\lambda \in \Lambda_l$ (in particular $K_{l_i}^{l_i} = 1$). Therefore in $U_s$ there are only finitely many distinct ideals $I_{\lambda}$ and only finitely many distinct ideals $I_{[k]}$, so the extension map completes by continuity to a map $\Phi: U_{\text{full}} \rightarrow U_s$.

Define $u_q$ to be the image of $U_q \subset U_{\text{full}}$ (and hence of $U_q$ under $\Phi$, an algebra over $Q(s^L) = Q(q)$). Likewise define $u_s$ to be the image of $U_{\text{weight}} \subset U_{\text{full}}$. Because extension is not exact these are not the same as the enrichment of $U_q$ and $U_{\text{weight}}$ by $Q(q)$ and $Q(s)$ respectively. In fact by the previous paragraph $u_q$ and $u_s$ are finite dimensional. The left degree and the even subalgebras $u_{q}^{\text{ev}}$ and $u_{s}^{\text{ev}}$ are well-defined. It is straightforward to check that $u_q$ and $u_s$ are Hopf subalgebras.

2.2.2 Small ribbon Hopf algebra  
The Hopf algebra $u_s$ admits a ribbon (not just topological) Hopf algebra structure with

$$\Phi(R_0) = \frac{1}{|\Lambda/\Lambda_l|} \sum_{[\lambda], [\gamma] \in \Lambda/\Lambda_l} q^{-\langle \lambda, \gamma \rangle} K_\lambda \otimes K_\gamma$$

$$R = \Phi(R) = \Phi(R_0) \sum_{\{\vec{r}, \vec{t} \prec l'_i\}} b_{\vec{r}} E_{(\vec{r})} \otimes F_{(\vec{t})} \in u_{s}^{\otimes 2}$$

where $b_{\vec{r}}$ is as in Equation 1.13 and $g = K_{2\beta} = \Phi(K_{2\beta}) \in u_q$.

Remark 2.1  
$u_q$ is essentially the same algebra as the one defined by [Lus93] and other authors. However, this $R$ matrix is not the same as the one defined by Rosso ([Ros90]) and [Lus93], which is in $u_q$ not $u_s$. For most purposes these $R$ matrices function interchangeably, but here being the image of the $R$-matrix for the large quantum group is a crucial property, while being in $u_q$ will not prove important.
Proposition 2.1 if $T$ is a bottom tangle (¶ 2.1.2) with all closed components labeled by elements of $U_s^{*\text{inv}}$, then

\[ \Phi(\Gamma_{U_{\text{full}}}(T)) = \Gamma_u(T) \]  

(2.3)

where the invariant functionals are interpreted as the restriction to $u_s$ on the right side and the pullback by $\Phi$ on the left.

Proof: This is immediate from the fact that $\Phi$ is a homomorphism which preserves the ribbon structure. \qed

2.3 Representations, Invariant Functionals and the Center

2.3.1 Representations From [Ros90, CP94] every finite-dimensional representation of $U_q$ extended to an algebra over $\mathbb{C}(q)$ is a finite direct sum of finite-dimensional simultaneous eigenspaces for \{\(K_i\}\} with $E_i$ and $F_i$ acting nilpotently. More precisely it is a sum of simple representations which are each a tensor products of a one-dimensional representation with the extension to $\mathbb{C}(q)$ of a Weyl module $V_\lambda$, a representation of $U_q$ over $A_q$ indexed by weight $\lambda$ in the Weyl chamber $\Lambda^+$, i.e. \(\lambda \in \Lambda\) satisfying $\langle \lambda + \rho, \alpha_i \rangle > 0$ for all $i$. One can write $V_\lambda = \bigoplus_{\mu \in \Lambda} V_{\lambda,\mu}$ where $K_i v = q_i^{\langle \alpha_i, \mu \rangle} v$ for $v \in V_{\lambda,\mu}$ and $E_\beta^{(r)} : V_{\lambda,\mu} \to V_{\lambda,\mu+r\beta}$, $F_\beta^{(r)} : V_{\lambda,\mu} \to V_{\lambda,\mu-r\beta}$, and the only $\mu$ with nonzero contributions are in Weyl orbits of $\mu \in \Lambda^+$ with $\lambda - \mu$ a sum of positive roots. In fact the Weyl modules extend naturally to $U_{\text{full}}$ and $\overline{U}_{\text{full}}$, while the nontrivial one-dimensional representations straightforwardly do not. More generally

Lemma 2.1 If $\lambda_1, \ldots, \lambda_m \in \Lambda^+$ then the action of $U_q^{\otimes m}$ on $V_{\lambda_1} \otimes V_{\lambda_2} \otimes \cdots \otimes V_{\lambda_m}$ extends to an action of $\overline{U}_{\text{full}}^{\otimes m}$ on the corresponding module over $\overline{A}_s$. In particular $\Gamma_{U_{\text{full}}}(T)$ for $T$ a bottom tangle acts on any tensor product of finite-dimensional representations.

Proof: Define the action of $U_{\text{full},0}$ on $V_{\lambda,\mu}$ by sending $\lambda$ to multiplication by $\delta_{\lambda,\mu}$. This extends the action of $U_q$ on any $V_{\lambda}$ to an action of $U_{\text{full}}$, and thus of $U_{\text{full}}^{\otimes m}$ on $V_{\lambda_1} \otimes V_{\lambda_2} \otimes \cdots \otimes V_{\lambda_m}$. $U_{\text{full}}$ has a uniform topology generated by $I_{[k],i}$ and $I_{\lambda}$, the endomorphism space has the uniform topology generated by $I_{[k],i}$. If $X$ is an element of the form Equation 1.10 then $(\gamma_1 \otimes \cdots \otimes \gamma_m) X$ gets sent to zero by the homomorphism for all but finitely many sequences $\gamma_1, \ldots, \gamma_m$, so the homomorphism is uniformly continuous and therefore extends to the completions. \qed

2.3.2 Representations of the small quantum group Passing to a root of unity, one can extend the module $V_\lambda$ over $A_s$ or $A_q$ to a module over $\mathbb{Q}(s)$ or $\mathbb{Q}(q)$ and it becomes a representation of the image of $U_s$, $u_s$ and $u_q$, all with the same weight decomposition and all also called $V_\lambda$. Tensor products of such representations are representations of the
appropriate tensor power of algebra (without need of completion) and therefore of the
original algebra through $\Delta$. If $\lambda$ is in the Weyl alcove $\Lambda^+$, i.e. it is in the Weyl chamber
$\Lambda^+$ and satisfies $\langle \lambda + \rho, \phi \rangle \leq l'$ where $\phi$ is the long positive root if $\max(d_i)|l'$ and the
short positive root otherwise, then it is a simple representation and is the orbit of the
highest weight vector in $u_s$ and $u_q$ [Saw06b].

2.3.3 Quantum traces In particular for $\lambda \in \Lambda^+$, define $qtr_\lambda(a) = \text{tr}_\lambda(K_{2\rho}a)$ where $\text{tr}_\lambda$
is the trace in the representation $V_\lambda$ (¶ 2.3.1). This is an invariant functional, and thus
the universal invariant (¶ 2.1.3) for $U_{\text{full}}$ is defined for bottom tangles (¶ 2.1.2) where
some closed components are labeled by $qtr_\lambda$.

2.3.4 The Drinfel’d map Recall $C \in U^{\otimes 2,\text{inv}}$ is the universal invariant (¶ 2.1.3) of the
clasp tangle in Figure 3. In particular note that for $U_{\text{full}}$
\begin{equation}
C = \sum_{\vec{r}, \vec{s}} b_{\vec{r}}b_{\vec{s}} q^{-\langle \theta_{\vec{r}}, \theta_{\vec{s}} \rangle} \sum_\gamma E(\vec{r}) K_{\gamma} \otimes F(\vec{s}) K_{-\gamma} E(\vec{s}) \label{eq:2.4}
\end{equation}
\begin{equation}
= \sum_{\vec{r}, \vec{s}} b_{\vec{r}}b_{\vec{s}} q^{-\langle \theta_{\vec{r}}, \theta_{\vec{s}} \rangle} \sum_\lambda E(\vec{r}) K_{\gamma} \otimes F(\vec{s}) K_{2\lambda - \gamma} E(\vec{s}) \label{eq:2.5}
\end{equation}

$C$ defines an algebra homomorphism called the Drinfeld map from invariant functionals
to the center sending each $\psi \in U^{\text{inv}}$ to
\begin{equation}
D[\psi] = (1 \otimes \psi)(C) \in U^{\text{inv}}.
\end{equation}

Notice in the special case of $U_{\text{full}}$ the image of $D$ is invariant elements of the even part
of $U_{\text{weight}}$.

2.3.5 The even center If $\lambda \in \Lambda^+$ define $z_\lambda \in U_{\text{full}}^{\text{inv}}$ or in $u_q^{\text{inv}}$ by
\begin{equation}
z_\lambda = D[qtr_\lambda]. \label{eq:2.6}
\end{equation}
In fact by Equation 2.4 $z_\lambda \in U_{\text{weight}}^{\text{ev, inv}}$ and if $\theta \in \Lambda_r \cap \Lambda^+$ then $z_\theta \in U_q^{\text{ev, inv}}$. Define
$Z = U^{\text{inv}}$ and $Z^{\text{ev}} = U_q^{\text{inv, ev}}$.

In particular in $u_s$, $z_\lambda = (1 \otimes qtr_\lambda) \Gamma u_s(C) = (1 \otimes qtr_\lambda) \Phi(U_{\text{full}}(C)) = \Phi(z_\lambda)$.

2.3.6 The Harish-Chandra map Equation 1.11 implies any element of $U_q$ can be written
as a sum $F(\vec{r}) H_{\vec{r}, \vec{s}} E(\vec{s})$ with $H_{\vec{r}, \vec{s}} \in U_{q,0}$ and for each $k, i$ all but finitely many
$\vec{r}, \vec{s}$ have $H_{\vec{r}, \vec{s}}$ in $I_{[k]}$. Thus one gets the uniformly continuous Harish-Chandra map
$HC: U_q \to U_{q,0}$ which sends each element of $U_q$ to $H_{0,0}$. It is easy to check that this
map is an algebra homomorphism on the center. De Concini and Kac [DCK90] prove
that this map acting on $U_q \otimes A_q C(q)$ (which [CP94] calls the rational form) is a bijection
from the center to the translated Weyl-group-invariant portion of the Cartan subalgebra
(¶ 1.1.1).
Lemma 2.2 The left degree 0 part $Z^\text{ev}$ of the center $Z$ of $U_q$ is spanned by $\{ z_\theta \}_{\theta \in \Lambda_r}$. The left degree 0 part $\overline{Z}^\text{ev}$ of the center $\overline{Z}$ of $U_q$ is spanned topologically by the $\{ z_\theta \}_{\theta \in \Lambda_r}$.

Proof: Since the Harish-Chandra map preserves left degree (1.1.5) it is also a bijection when restricted to a map between the even parts of the center and of the translated Weyl-invariant Cartan algebra of the rational form. Check following Jantzen [Jan96] that $HC(z_\lambda) = \sum_\mu \dim_\lambda(\mu)K_{2\mu}$, where $\dim_\lambda(\mu)$ is the dimension of the weight $\mu$ subspace of $V_\lambda$. Clearly for each $\theta \in \Lambda_r \cap \Lambda^+$ one can form a linear combination $z_\theta'$ of $z_\mu$ for $\mu \leq \theta$ such that $HC(z_\theta') = \sum_{\sigma \in W} K_{\sigma(\theta)}$, which implies that the $z_\theta'$ and hence $z_\theta$ span the even part of the center of the rational form. Since each $z_\theta \in Z^\text{ev}$, it follows that $Z^\text{ev}$ is spanned by these elements in $U_q$.

Given $\lambda \in \Lambda$ consider the right ideal in $U_q$ generated by $\{ E_i \}$ and by $k - \lambda(k)$ for each $k \in \overline{U}_{q,0}$. $U_q$ acts on the left on the quotient by this ideal. Notice that $z \in \overline{Z}$ acts by multiplication by the scalar $\lambda(\text{HC}(z))$, since it acts on the image of 1 as that. On the other hand if $\langle \alpha_i, \lambda \rangle = n > 0$ then the image of $F_i^{n+1}$ in this module is easily recognized to be a highest weight module, and therefore $Z$ acts on it by multiplication by $\sigma_i \lambda(\text{HC}(z))$, where $\sigma_i \lambda = \lambda - (n+1)\alpha_i$ is the image of $\lambda$ under the appropriate generator of the translated Weyl group. This is to say that $\lambda$ and $\gamma$ agree on $\text{HC}(z)$ if they are linked by an element of the translated Weyl group. Since $K \in \overline{U}_{q,0} = 0$ if all $\lambda(K) = 0$ for $\lambda \in \Lambda$ (because this is true for $U_{q,0}$) this implies that $HC(\overline{Z})$ is invariant under the translated Weyl group action.

Therefore modulo any $I_{[k]}$, $HC(z)$ for $z \in \overline{Z}^\text{ev}$ is equivalent to an invariant element of $U_{q,0}$, which is $HC(z')$ for $z'$ a linear combination of $\{ z_\theta \}$, and therefore $z$ is equivalent to $z'$ modulo $I_{[k]}$. So $\overline{Z}^\text{ev}$ is in fact the completion of $Z^\text{ev}$.

Remark 2.2 The argument above readily generalizes to show that the even part of the center of $U_{q, \text{weight}}$ is spanned by $z_\lambda$ for $\lambda \in \Lambda$. This fact will not be important in this paper.

3 Chern-Simons and Hennings Invariants

For this section fix a positive integer $l$ and hence roots of unity $s$ and $q$ (2.2.1). Most of this section will work with $U_{s, u_s}$ and $u_{q, s}$, but in some cases conclusions will be drawn about $U_{\text{full}}$.

3.1 The CS and Hennings Invariant

3.1.1 The Chern-Simons invariant functional Let $\Lambda_H$ be a sublattice of $\Lambda$ containing $\Lambda_r$ (e.g. $\Lambda/\Lambda_r$ is cyclic except in the case $D_4$, in that case do not consider the case $\Lambda_H = \Lambda_r$). Let $\Lambda_H^\prime$ be the intersection of the Weyl alcove with $\Lambda_H$, i.e. the set of all $\lambda \in \Lambda_H^+$ such that $\langle \lambda + \rho, \phi^\prime \rangle \leq l'$ where $\phi^\prime$ is the long positive root if $\max(d_i)|l'$ and the short positive
root otherwise. Define the invariant functional

\[ \text{cs}_{H} = \sum_{\lambda \in \Lambda_{H}^{l}} \text{qtr}_{\lambda}(1)\text{qtr}_{\lambda}. \] (3.1)

This expression can be interpreted as an invariant functional on \( U_{\text{full}}, U_{u} \) or \( u_{u} \), and the map \( \Phi \) intertwines these functionals. The quantum Racah formula [Saw06b][Cor. 8] and shows that if \( \phi \) is a linear combination of quantum traces \( \text{qtr}_{\lambda} \) for \( \lambda \in \Lambda_{H}^{l} \) then in \( U_{u} \) the handle-slide condition holds

\[ [\phi \text{cs}_{H}](x) = \phi(1)\text{cs}_{H}(x) \] (3.2)

which on the level of invariants says that any component labeled by a linear combination of quantum traces in the sublattice can slide through a component labeled by \( \text{cs}_{H} \) as in Figure 4(a). Consider any pair \( l, H \) such that the invariant \( \text{cs}_{H}(D[\text{cs}_{H}])\text{cs}_{H}(C) \) of the Hopf link with both components labeled by \( \text{cs}_{H} \) is nonzero. Equation 3.2 then implies that \( \text{cs}_{H}(r^{\pm 1}) \) are nonzero, as in Figure 4(b). [Saw02, Saw06a, Saw06b] identify a large array of pairs for which this holds, including all the cases corresponding to levels and Lie groups identified by Dijkgraaf and Witten [DW90] as admitting a Chern-Simons field theory. In particular this holds when \( l \) is divisible by \( 2 \max(d_{i}) \) and \( H = \{0\} \). In any such case

\[ \text{cs}_{H}^{\otimes m}(\Gamma_{u_{u}}(T)) \text{cs}_{H}(r)^{-\sigma_{-}}\text{cs}_{H}(r^{-1})^{-\sigma_{-}} \] (3.3)

where \( \sigma_{\pm} \) are the number of positive and negative eigenvalues of the linking matrix, is an invariant of the three-manifold given by surgery on the closure of an open bottom tangle \( T \).

Observe that \( \text{qtr}_{\lambda} \) can be defined as an invariant functional on the small and large quantum group and that these are preserved by \( \Phi \). Thus \( \text{cs}_{H} \) can be interpreted as an invariant functional on either quantum group. In particular \( \text{cs}_{H}^{\otimes m}(\Gamma_{U_{\text{full}}}(T)) \) makes sense in \( \mathcal{A}_{u} \), and when extended to \( \mathbb{C}(s) \) agrees with \( \text{cs}_{H}^{\otimes m}(\Gamma_{u_{u}}(T)) \) and thus Equation 3.3 with this replacement defines the same three-manifold invariant.

3.1.2 The Hennings invariant functional

Let \( \tilde{r}_{\max} = (l'_{\beta_{1}} - 1, \ldots, l'_{\beta_{N}} - 1) \) and \( \theta_{\max} = \sum_{j}(l'_{\beta_{j}} - 1)\beta_{j} \). Notice that in \( u_{u} \) and \( u_{q} \) we have \( E_{i}E_{i}(\tilde{r}_{\max}) = E_{i}(\tilde{r}_{\max})E_{i} = 0 \) for all \( i \).
The small quantum group \( u_s \) has a left integral \([Hen96]\) which is an invariant functional
\[
\text{hen}(F(\vec{r})K_{\theta}E(\vec{s})) = \begin{cases} 
1 & \text{if } \vec{r} = \vec{s} = \vec{r}_{\max} \text{ and } \theta = \theta_{\max} \\
0 & \text{else.}
\end{cases}
\]
with again the handleslide property the property that
\[
[\phi \text{hen}](x) = \phi(1) \text{hen}(x)
\]
for \( \phi \in u_s^* \). An easy calculation gives that the invariant of the Hopf link \( \text{hen}(\mathcal{D}[\text{hen}]) \), and therefore \( \text{hen}(r^{\pm 1}) \) are all nonzero, and thus once again
\[
\text{hen}^{\otimes m}(\Gamma_{u_s}(T)) \text{hen}(r)^{-\sigma_+} \text{hen}(r^{-1})^{-\sigma_-}
\]
is an invariant of the three-manifold given by surgery on the closure of \( T \).

### 3.2 The Universal Invariant of ZLMTs

A zero linking matrix tangle or ZLMT is a bottom tangle in which the self-linking number of each open component and the linking number between each pair of components are all zero, and in which each closed component is labeled by an invariant functional (if there are no closed components call it an open ZLMT). On the one hand, this section argues that this condition restricts the possible values of the universal invariant (\( \S 2.1.3 \)) to even elements of \( \mathcal{U}_q \). On the other hand, this invariant can be used to compute the CS and Hennings invariants (\( \S 3.1.1, \S 3.1.2 \)) of homology three-spheres.

Writing out a state of the universal \( \Gamma_{\text{full}} \) invariant of a projection of a ZLMT, the algebra of the tensor \( R_0 \) of Equation 1.12 and its relation with the other elements of the state are quite straightforward and one sees readily that the result is a tensor product of elements of \( \mathcal{U}_q^{\text{ev}} \). More care is required to see that the universal invariant is in fact even in the sense of the previous section.

**Proposition 3.1** If \( T \) is an open zero linking matrix tangle with \( m \) components, then \( \Gamma_{\text{full}}(T) \in \mathcal{U}_q^{\text{ev,inv,}\otimes m} \).

**Proof:**

According to [Hab06][Cor. 9.13], any open ZLMT can be written as a product \( WB^{\otimes k} \), where \( B^{\otimes k} \) is a horizontal product of the Borromean tangle \( B \) in Figure 5, and \( W \) is a horizontal and vertical product of the tangles \( 1_b, \mu_b, \eta_b, \gamma_+, \gamma_-, \psi_{b,b}, \psi_{b,b}^{-1} \) in Figure 6. Since horizontal products correspond under \( \Gamma_{\text{full}} \) to tensor product (which preserves evenness) and vertical product to product in the quantum group, it suffices to check that \( \Gamma_{\text{full}} \), of each of the tangles above, as an operator on \( \mathcal{U}_q^{\otimes m} \), preserves \( \mathcal{U}_q^{\text{ev,}\otimes m} \).

Observe as in Figure 5 that \( B \) can be written as a product of the other generating tangels, and thus it suffices to check the others. Straightforwardly \( \Gamma(1_b) \) sends \( x \) to \( x \), \( \Gamma(\eta_b) = 1 \) and \( \Gamma(\mu_b) \) sends \( x \otimes y \) to \( xy \), and thus all preserve evenness.

Compute using Equation 2.2 that
\[
\Gamma(\psi_{b,b})(x \otimes y) = \sum_i \text{ad}_{a_i}(x) \otimes \text{ad}_{b_i}(y)
\]
where $R = \sum_i a_i \otimes b_i$. Since (¶ 2.1.1) the adjoint action preserves evenness this is even. Thus the same is true of $\psi_{b,b}^{-1}$.

Compute using Equation 2.2 that

$$
\Gamma(\gamma_{-})(x) = \sum_{\vec{r},\vec{s}} b_{\vec{r}} b_{\vec{s}} \sum_{\lambda} q^{(\rho,\theta_{\vec{r}}) + (\theta_{\vec{s}} - \lambda,\theta_{\vec{r}})} F_{(\vec{r})} K_{2\lambda - \theta_{\vec{s}}} E_{(\vec{s})} \otimes \text{ad}_y(x)
$$

where $y = F_{(\vec{s})} \lambda E_{(\vec{r})}$. By Equation 2.1 and the fact that the adjoint action preserves evenness this operation preserves evenness. A corresponding argument shows that $\Gamma(\gamma_{+})(x)$ is even if $x$ is.

**Theorem 3.1** $T$ is a mixed ZLMT with closed components labeled by invariant functionals on $U_q$ or $U_{\text{full}}$ then $\Gamma_{U_{\text{full}}}(T) \in U_{q}^{\text{ev,inv,} \otimes m}$. If the components are labeled by invariant functionals on $u_q$ or $u_\text{full}$ (¶ 2.1.1) then $\Gamma_{u_\text{full}}(T) \in u_{q}^{\text{ev,inv,} \otimes m}$. If $T$ has a single open component then in either case it is a linear combination (in the first case possibly infinite) of $\{z_{\theta}\}_{\theta \in \Lambda_r}$.

**Proof:** The first sentence follows from Proposition 3.1 because applying an invariant functional times $r^n$ to one factor of $U_{q}^{\text{ev,inv,} \otimes m}$ maps into $U_{q}^{\text{ev,inv,} \otimes m - 1}$. The second sentence follows from this, Proposition 2.1 and the fact that $\Phi$ maps $U_{q}^{\text{ev,inv,} \otimes m}$ to $u_{q}^{\text{ev,inv,} \otimes m}$. The third sentence follows from Lemma 2.2.

### 3.3 Equality of Invariants on Homology Spheres

In this section assume that $l$ and $H$ are such that $\text{cs}_H(\gamma^{\pm 1})$ are nonzero (¶ 3.1.1).
Lemma 3.1 Let $\lambda \in \Lambda$. Then
\[
c_{H}(z_{\lambda}r^{\pm 1})h_{n}(r^{\pm 1}) = h_{n}(z_{\lambda}r^{\pm 1})c_{H}(r^{\pm 1}). \tag{3.6}
\]

**Proof:** Recall that since $r^{-1}$ is central it acts on $V_{\lambda}$ (§ 2.3.1) as a multiple of the identity, so that $q_{t_{\lambda}}(r^{-1} \cdot)$ is a multiple of $q_{t_{\lambda}}$. Thus by Equation 3.2 and the fact that $\Delta r = (r \otimes r)C$
\[
q_{t_{\lambda}}(r^{-1})c_{H}(r) = [q_{t_{\lambda}}(r^{-1} \cdot) c_{H}] (r) = [q_{t_{\lambda}} \otimes c_{H}(r \cdot)](C) = c_{H}(z_{\lambda}r).
\]

The same is true replacing $h_{n}$ for $c_{H}$, and Equation 3.4 for Equation 3.2. Multiplying the resulting equations by $h_{n}(r)$ and $c_{H}(r)$ gives
\[
c_{H}(z_{\lambda}r)h_{n}(r) = c_{H}(r)q_{t_{\lambda}}(r^{-1})h_{n}(r) = c_{H}(r)h_{n}(z_{\lambda}r).
\]

The same argument works replacing $r$ and $r^{-1}$, using the inverse of the clasp element, and replacing $\lambda$ with the highest weight $\lambda^{*}$ of the dual representation. □

Lemma 3.2 Let $T$ be a ZLMT with every component but one closed off and labeled by a quantum functional on $u_{s}$. Let $\epsilon = \pm 1$. Then
\[
\frac{c_{H}[r^{*}\Gamma_{u_{s}}(T)]}{c_{H}[r^{*}]} = \frac{h_{n}[r^{*}\Gamma_{u_{s}}(T)]}{h_{n}[r^{*}]} \tag{3.7}
\]

**Proof:**
By Theorem 3.1 $\Gamma_{u_{s}}(T)$ is a linear combination of $z_{\theta}$ for $\theta \in \Lambda_{r}$. Thus Equation 3.6 holds with $z_{\lambda}$ replaced by $\Gamma_{u_{s}}(T)$, and dividing by the nonzero quantities gives Equation 3.7. □

**Proposition 3.2** Let $T$ be an open ZLMT with $m$ components and let $\epsilon_{1}, \ldots, \epsilon_{m}$ be nonzero integers with $|\epsilon| = 1$ for all $1 \leq i \leq m$. Then
\[
\frac{c_{H}^{\otimes m}[(r^{\epsilon_{1}} \otimes r^{\epsilon_{2}} \otimes \cdots \otimes r^{\epsilon_{m}})\Gamma_{u_{s}}(T)]}{c_{H}(r^{\epsilon_{1}})c_{H}(r^{\epsilon_{2}}) \cdots c_{H}(r^{\epsilon_{m}})} = \frac{h_{n}^{\otimes m}[(r^{\epsilon_{1}} \otimes r^{\epsilon_{2}} \otimes \cdots \otimes r^{\epsilon_{m}})\Gamma_{u_{s}}(T)]}{h_{n}(r^{\epsilon_{1}})h_{n}(r^{\epsilon_{2}}) \cdots h_{n}(r^{\epsilon_{m}})} \tag{3.8}
\]

**Proof:** For each $0 \leq k \leq m$ let $A_{k}$ be the left-hand side of Equation 3.8 with the $c_{H}$ replaced by $h_{n}$ in each factor in the top and bottom from the first to through the $k$th. Thus the left hand side corresponds to $A_{0}$ and the right hand side to $A_{m}$. So it suffices
to prove that $A_{k-1} = A_k$ for $1 \leq k \leq m$. If $T_k$ is $T$ with component $j$ for $j < k$ closed and labeled by $\text{cs}_H(r^{e_j} \cdot \cdot)$ and component $j$ component for $j > k$ closed and labeled by $\text{hen}(r^{e_j} \cdot \cdot)$, then $A_k = A\text{hen}[r^{e_k} \Gamma(T_k)] / A\text{hen}[r^{e_k}]$ and $A_{k-1} = \text{Acs}_H[r^{e_k} \Gamma(T_k)] / \text{Acs}_H[r^{e_k}]$

Thus $A_k = A_{k-1}$ is exactly Equation 3.7.

**Theorem 3.2** When $l$ and $H$ are such that the Chern-Simons invariant is defined (i.e. if $\text{cs}_H(r^{\pm 1} \neq 0)$, The Hennings invariant of an integral homology three-sphere $M$ is the Chern-Simons invariant of $M$.

**Proof:** $M$ can be obtained by (integral) surgery in $S^3$ on the a framed link whose linking matrix has all zeros except for $\pm 1$ along the diagonal. The invariant of such a link is a product of invariant functionals applied to $(r^{e_1} \cdot \cdot \cdot r^{e_m} \Gamma_U(T))$ where $T$ is an open ZLMT. Therefore the Chern-Simons invariant of $M$ is given by the left hand side of Equation 3.8 and the Hennings invariant is given by the right-hand side.

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