Gas laser energy characteristics with different active element cross section geometry

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Abstract. The models for estimating the contribution of the cross section geometry to the active medium gain of the gas laser have been considered. A testing of our method for the solving of the Helmholtz equation with different boundary conditions have been fulfilled. The next step is the complication of the model taking into account the distribution of the field intensity in the resonator.

1. Introduction
Lasers are widely used over half a century by man. It is used in the life sciences [1-4], ecology [5], at the transmission of information [6-9], etc. The gas lasers are more preferable in the measurements, since they have the high coherence of the radiation. Let us consider the question of interest to us at the example of He-Ne lasers. Most of them have the cylindrical active elements. The question arises is this geometry optimal from the energy point of view.

The first model for estimating the cross section geometry contribution to the active medium gain was extremely simple [10]. The electromagnetic field distribution was not taken into account. Only the phenomenological relationship was used as the amplification at the discharge axis is inversely proportional to the distance from the axis to the wall of the active element. Despite the simplified approach this model withstood the experimental test for the rectangular cross section [11]. It was assumed that the approach is valid and for other cross sections in the He-Ne laser.

With the cross section was constant along the length the geometric part of the laser gain coefficient and has the form [10]:

\[ k = \int_V k_0 f(\vec{r})dV / S_0, \]  

where the function \( f \), describing the spatial distribution of the medium gain coefficient, satisfies the Helmholtz equation in the region \( V \):

\[ \Delta f(\vec{r}) + k^2 f(\vec{r}) = 0 \]  

with boundary condition

\[ f|_{r} = u, \]
here $\Gamma$ is the boundary of the region in which the solution is sought, $u$ is some known function defining the boundary conditions and, from physical considerations, continuous on $\Gamma$, $S_0$ is the tube cross-section square, $k_0$ is the gain at the system axis.

The specificity of our work of previous years was that we had to carry out approximate calculations, because the models were usually not analytical. Modern methods and computing systems must bring us closer to more accurate and possibly qualitatively different results.

2. Homogeneous boundary conditions
We considered at first the homogeneous boundary conditions in the form
$$ f \bigg|_{\Gamma} = 0 \tag{4} $$

We proposed in [12] the method for finding of this equation approximate solution in the cylindrical coordinates $(r, \phi, z)$ (with independence from the coordinate $z$):
$$ f^n(r, \phi) = \zeta_0(\lambda r, \phi) + \sum_{k=1}^{n} a_k \zeta_k(\lambda r, \phi) \tag{5} $$

where $\zeta_0(\lambda r, \phi)$ = $J_0(\lambda r)$ and $\zeta_k(\lambda r, \phi)$ = $J_k(\lambda r)$cos($k\phi$) or $\zeta_k(\lambda r, \phi)$ = $J_k(\lambda r)$sin($k\phi$), $J_0(\lambda r)$, $J_k(\lambda r)$ – Bessel functions of order $k$ (with normalization of the function $f$ to 1 at $r = 0$). The function $f^n$ exactly satisfies to the equation (2), and approximately to the boundary condition (4). We will understand under the approximate fulfillment of the boundary condition (3) the following: let’s choose some $N$ points $\xi_1, \xi_2, \ldots, \xi_N$ at the boundary $\Gamma$ and require the minimal sum of the function (5) squares at these points:
$$ \sum_{j=1}^{N} (f^n(\xi_j))^2 \rightarrow \min, \quad \xi_j \in \Gamma. \tag{6} $$

Our method allows us to obtain the expressions for the coefficients $a_{nk}$, as well as to find the eigenvalues $\lambda$ of the equations (2), (4) with high accuracy at relatively low computational complexity.

3. The method testing and the optimal cross section finding
We tested our method at first for the cross sections that allows the analytical solution — a rectangle, a circle, an ellipse. The calculation results by our method give excellent agreement in the magnitudes of the gain coefficient and the parameter $\lambda$ [12]. Then we began to study the different cross sections in the search of gain coefficient increasing.

We considered in [13] the tube cross sections in the form of regular $n$-gons for $n = 3, 5, 6, 8$. We can consider other polygons, for example, a regular nonagon. If the side of a regular nonagon is $a$, and the polar coordinate system origin is taken at the center of the nonagon and the polar axis is directed across one of its vertices, then the gain coefficient will be equal to:
\[ k = \frac{4k_0\tan(\pi/9)}{9a^2} \left\{ \int_{2\pi/9}^{2\pi/9} \frac{p(\cos(\phi-\pi/9))}{\cos(\phi-\pi/9)} \, d\phi \right\} + \int_{2\pi/9}^{2\pi/9} \frac{p(\cos(\phi-\pi/3))}{\cos(\phi-\pi/3)} \, d\phi + \left\{ \int_{2\pi/9}^{2\pi/9} \frac{p(\cos(\phi-5\pi/9))}{\cos(\phi-5\pi/9)} \, d\phi \right\} + \left\{ \int_{2\pi/9}^{2\pi/9} \frac{p(\cos(\phi-11\pi/9))}{\cos(\phi-11\pi/9)} \, d\phi \right\} + \left\{ \int_{2\pi/9}^{2\pi/9} \frac{p(\cos(\phi-17\pi/9))}{\cos(\phi-17\pi/9)} \, d\phi \right\} \right\} 
\]

The calculation results by our method give the value of \( k = 0.429k_0 \) for the gain coefficient which corresponds to the general conclusions of [13].

We investigated further [14] the parabolic polygons - these were polygons with sides from parabolas, and the parabolic foci coincided and were located in the center of the figure. We considered a parabolic bigon, a parabolic square, and a parabolic hexagon — these were, respectively, figures formed by two specularly reflected parabolas; four parabolas turned relative to each other by an angle \( \pi/2 \) and six parabolas rotated relative to each other by angle \( \pi/3 \).

Great hopes were pinned on the figures in the form of a triangle, in which circular sectors are located at the vertices, and a circle with four symmetrically arranged small circles, and whose centers of small circles lie on a large circle. We applied our method and investigated the gain coefficient of these figures at the different ratios of the radii of the small circle to the side of the triangle and the radius of the large circle. Unfortunately, the earlier assumptions that additional circles would increase the gain coefficient did not materialize - the gain coefficient decreased monotonically with increasing radius of small circles.

4. Generalization to inhomogeneous boundary conditions and verification of the modified algorithm

In the real physical problems, inhomogeneous boundary conditions arise in the form (3), rather than (4), so we generalized our method for the solving of the equations (2) - (3). As it is known [15-17], if the parameter \( \lambda \) in equations (2), (3) does not coincide with any of the eigenvalues of equations (2), (4), then the solution of the equations (2), (3) exists and only one, and if it coincides with one of the eigenvalues of equations (2), (4), then the solution may not exist, and if it exists, then it is not unique. In the case of inhomogeneous boundary conditions, the parameter \( \lambda \) is determined by the physics of the problem. Therefore, in the modified algorithm, in order to find a solution of the Helmholtz equation with inhomogeneous boundary conditions, it is necessary to omit the step of finding \( \lambda \), and immediately find the values of the coefficients \( a_k \). In the modified algorithm, we require the minimal sum of this expression:

\[ \sum_{j=1}^{N} (f^n(\xi_j) - u(\xi_j))^2 \rightarrow \min, \quad \xi_j \in \Gamma \]  

(8)

and instead of formula (5) we find the expansion coefficients of the boundary function (at the same time we shift the numbering of functions - i.e. \( \zeta(\lambda r, \phi) = J_0(\lambda r) \) and \( \zeta(\lambda r, \phi) = J_1(\lambda r)\cos((k-1)\phi) \) or \( \zeta(\lambda r, \phi) = J_{k,1}(\lambda r)\sin((k-1)\phi) \)).
\[ u(r, \phi) = \sum_{k=1}^{n} a_k \zeta_k(\lambda r, \phi); \quad (r, \phi) \in \Gamma. \]  

(9)

The algorithm itself for finding the coefficients \( a_k \) does not change as compared with the homogeneous case.

We checked the modification of our method for cross sections of different shapes with inhomogeneous boundary conditions for which we can obtain an analytical solution - a circle, a rectangle, an ellipse. Let us give an example for a rectangle with sides \( a \) and \( b \) - choose the Cartesian coordinate system origin in the center of the rectangle, and direct the axes parallel to the sides. Choose numbers \( \alpha \) and \( \beta \) such that \( \lambda^2 = \alpha^2 + \beta^2 \), does not coincide with any of the numbers \( \lambda_{n,m} \), such that \( \lambda_{n,m}^2 = n^2 \pi^2/a^2 + m^2 \pi^2/b^2 \) \( (n=1,2,3..., m=1,2,3...) \) – these are the eigenvalues of the task (2), (4).

Let's set the boundary condition (3) in the form \( u(x, y) = \cos(\alpha x) \cdot \cos(\beta y) \). It is easy to verify that then the solution of equations (2), (3) will be \( f(x, y) = \cos(\alpha x) \cdot \cos(\beta y) \), and this solution is unique. Then you can find the gain coefficient by the formula (1):

\[ k = 4k_0 \sin(\alpha a / 2) \sin(\beta b / 2) / (\alpha \beta). \]  

(10)

On the other hand, we can apply our method. If we can find the expansion coefficients \( a_k \) for the function \( u(x, y) = \cos(\alpha x) \cdot \cos(\beta y) \) in (9), then we can build function \( f^n(x, y) = \sum_{k=1}^{n} a_k \zeta_k(\lambda r, \phi) \) and calculate laser gain coefficient:

\[ k = \frac{1}{S_0} \int_{S} f^n(r, \phi) r dr d\phi = \frac{1}{S_0} \sum_{k=1}^{n} a_k \int_{S} k_0 \zeta_k(\lambda r, \phi) r dr d\phi. \]  

(11)

The results of the gain coefficients for different \( a, b, \alpha, \beta \) calculated by the formulas (10) and (11) match with high precision. Also, the calculation results for the gain coefficients for the ellipse and the circle coincide too.

5. Conclusions

Our method testing for the solving of the Helmholtz equation with different boundary conditions has been met our expectations. However, we have not yet found a cross section that gives the maximum gain coefficient value. In the future, we plan to complicate the model under consideration, taking into account the distribution of the field intensity in the resonator.

References

[1] Nepomnyashchaya E K, Velichko E N, Pleshakov I V, Aksenov E T and Savchenko E A 2017 J. Physics: Conference Series \textbf{841} 012020

[2] Nepomnyashchaya E, Velichko E, Aksenov E and Bogomaz T 2018 \textit{Biophotonics: Photonic Solutions for Better Health Care VI}. \textbf{10685} 106852F

[3] Putintseva M V Aksenov E T, Korikov C C and Velichko E N 2018 \textit{J. Physics: Conference Series} \textbf{1124} 031021

[4] Savchenko E, Velichko E, Nepomnyashchaya E, Dubo D and Tsybin O 2017 \textit{J. Physics: Conference Series} \textbf{917} 042010

[5] Voronina E I, Privalov V E and Shemanin V G 2004 \textit{Tech. Phys. Lett.} \textbf{30} 14 - 17

[6] Bisyarin M A, Chapalo I E, Kotov O I and Petrov A V 2018 \textit{J. Opt. Soc. Am. B} \textbf{35} 1990-1999

[7] Kotov O I, Bisyarin M A, Liokumovich L B, Hartog A H and Ushakov N A 2016 \textit{Appl. Optics} \textbf{55} 5041-5051
[8] Chapalo I E, Kotov O I and Medvedev A V 2014 Proc. SPIE 9203 92030B
[9] Bisyarin M A Liokumovich L B, Hartog A H and Ushakov N A 2017 Appl. Optics 56 354-364
[10] Privalov V E and Fridrikhov S A 1968 Russian J. Appl. Phys. 38 2080-2084
[11] Privalov V E and Khodovoy V A 1974 Opt. and Spectroscopy 37 797-799
[12] Kozhevnikov V A and Privalov V E 2018 St. Petersburg Polytechnical State University J Phys. and Matem. 11 77 – 87
[13] Kozhevnikov V A and Privalov V E 2018 Russian Phys. J. 61 913-917
[14] Kozhevnikov V A and Privalov V E 2018 Russian Phys. J. 61 1861-1867
[15] Vladimirov V S 1099 Equations of mathematical physics (Moscow: Nauka)
[16] Abramowitz M and Stegun I A 1972 Handbook of Mathematical Functions: with Formulas, Graphs, and Mathematical Tables (National Bureau of Standards: Tenth Printing)
[17] Higham N J 2002 Accuracy and stability of numerical algorithms 2nd ed. (Philadelphia: Society for Industrial and Applied Mathematics)