INNER FUNCTIONS ON QUOTIENT DOMAINS RELATED TO THE POLYDISC

MAINAK BIOWMIK AND POORNENDU KUMAR

Abstract. Inner functions are the backbone of holomorphic function theory. This paper studies the inner functions on quotient domains of the open unit polydisc, $\mathbb{D}^d$, arising from the group action of finite pseudo-reflection groups. Such quotient domains are known to be biholomorphic to the proper image $\theta(\mathbb{D}^d)$ of $\mathbb{D}^d$ under certain polynomial map $\theta : \mathbb{D}^d \to \theta(\mathbb{D}^d)$. The main contributions of this paper are as follows:

1. We show that the closed algebra generated by inner functions on $\theta(\mathbb{D}^d)$ forms a proper subalgebra of $H^\infty(\theta(\mathbb{D}^d))$, the algebra of bounded holomorphic functions on $\theta(\mathbb{D}^d)$. This in particular shows that Marshall’s theorem, which states that the algebra generated by rational inner functions on the disc is the algebra of bounded analytic functions on $\mathbb{D}$, does not hold in these domains. En route, we also shed some light on the Shilov boundary corresponding to $H^\infty(\theta(\mathbb{D}^d))$.

2. The set of all rational inner functions on $\theta(\mathbb{D}^d)$ is found.

3. The set of all rational inner functions on $\theta(\mathbb{D}^d)$ is shown to be dense in the norm-unit ball of $H^\infty(\theta(\mathbb{D}^d))$ with respect to the uniform compact-open topology, thereby proving the Carathéodory approximation result.

4. As an application of the Carathéodory approximation theorem, we approximate holomorphic functions on $\theta(\mathbb{D}^d)$ that are continuous in the closure of $\theta(\mathbb{D}^d)$ by convex combinations of rational inner functions in the $L^2$-norm, thereby obtaining a version of the Fisher’s theorem.

1. Introduction

Just like the function theory on the unit polydisc and on the Euclidean unit ball caught the fancy of complex analysts a few decades ago [38, 41], and the recent works [8, 9, 23, 22], the function theory on a class of domains emerging from a group action on various bounded symmetric domains is a theme of great research at present [2, 5, 11, 14, 20, 30, 32]. In this article, we study the bounded holomorphic functions on quotient domains related to the unit polydisc.

Fix a positive integer $d > 1$. A pseudo-reflection is a linear map $\sigma : \mathbb{C}^d \to \mathbb{C}^d$ such that rank$(I_d - \sigma) = 1$ and $\sigma^n = I_d$ for some $n \in \mathbb{N}$. A group $G$ generated by pseudo-reflections is called a pseudo-reflection group. A domain $\Omega$ is said to be $G$-invariant if under the action $\sigma \cdot z = \sigma^{-1}(z)$ of $G$, the domain $\Omega$ remains invariant. This action induces a natural action

$$(\sigma \cdot f)(z) = f(\sigma^{-1} \cdot z) = f(\sigma(z))$$

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of $G$ on complex-valued functions on $\Omega$. A function $f$ is said to be $G$-invariant if $\sigma \cdot f = f$ for each $\sigma \in G$. For example the permutation group $S_d$, acting on $\mathbb{C}^d$ by permuting the co-ordinates, is a pseudo-reflection group which keeps $\mathbb{D}^d$ invariant. The symmetric polynomials are some examples of $S_d$-invariant functions in this case.

The famous works of Chevalley [12], Shephard and Todd [42] establish that for a finite linear group $G$ and a $G$-invariant domain $\Omega$, the quotient topological space $\Omega/G$ is the image $\theta(\Omega)$ under a polynomial map (called as basic polynomial map associated to $G$) $\theta(z) = (\theta_1(z), \ldots, \theta_d(z))$ for $z \in \Omega$ if and only if $G$ is a pseudo-reflection group. Although the map $\theta$ is not unique but the degrees of $\theta_j$’s are unique for $G$ up to order. Also, $\theta_j$’s are homogeneous algebraically independent polynomials.

In this article, we shall consider $\Omega = \mathbb{D}^d$ and the pseudo-reflection group $G$ to be one of the imprimitive pseudo-reflection groups $G(m, t, d)$, appeared in the classification of finite pseudo-reflection groups, defined as follows [40]. Let $m$ and $t$ be positive integers such that $t$ divides $m$. Let $\alpha = e^{\frac{2\pi i}{m}}$, and $G(m, t, d)$ consist of the maps

$$z = (z_1, \ldots, z_d) \rightarrow (\alpha^{\nu_1}z_{\sigma(1)}, \ldots, \alpha^{\nu_d}z_{\sigma(d)})$$

where $\sigma \in S_d$ and $\nu_1, \ldots, \nu_d$ are integers whose sum is divisible by $t$. The group $G(m, t, d)$ is a pseudo-reflection group of order $\frac{mtd}{\gcd(mt, t)}$. Note that the choice $(m, t) = (1, 1)$ gives $G(1, 1, d) = S_d$. The basic polynomials associated to $G$ are the following:

$$\theta_j(z) = E_j(z_1^m, \ldots, z_d^m) \text{ for } j = 1, \ldots, d - 1,$$

$$\theta_d(z) = [E_d(z_1^m, \ldots, z_d^m)]^\frac{1}{m} = (z_1 \ldots z_d)^\frac{1}{m},$$

where $E_1, \ldots, E_d$ are the elementary symmetric polynomials. For fixed $m, d$ and $t$, we shall write $G$ to denote the group $G(m, t, d)$. From now, we shall work on $\theta(\mathbb{D}^d)$ which is biholomorphic to the quotient domain $\mathbb{D}^d/G(m, t, d)$. The quotient domains $\Omega/G(m, t, d)$ have been studied in [7, 40, 20] when $\Omega = \mathbb{B}_d$ or $\mathbb{D}^d$. In particular, the quotient domain $\mathbb{D}^d/S_d$ is known as the symmetrized polydisc and it is a non-convex but polynomially convex domain, see [16]. It is naturally associated with the famous spectral interpolation problem, see [15].

The basic polynomial map $\theta$ is proper on $\mathbb{D}^d$ and it extends as proper map of same multiplicity from a neighbourhood of $\overline{\mathbb{D}^d}$ to a neighbourhood of $\overline{\theta(\mathbb{D}^d)}$. Also, the Shilov boundary $\mathbb{T}^d$ of $\mathbb{D}^d$ with respect to the polydisc algebra $\mathcal{A}(\mathbb{D}^d)$ is the same as $\theta^{-1}(\partial \theta(\mathbb{D}^d))$ where $\partial \theta(\mathbb{D}^d)$ is the Shilov boundary of $\overline{\theta(\mathbb{D}^d)}$ with respect to the uniform algebra $\mathcal{A}(\theta(\mathbb{D}^d))$ of continuous functions on $\overline{\theta(\mathbb{D}^d)}$ which are holomorphic in $\theta(\mathbb{D}^d)$; see [29]. Therefore, $\partial \theta(\mathbb{D}^d) = \theta(\mathbb{T}^d)$. We shall use $p = (p_1, \ldots, p_d)$ as co-ordinates in $\theta(\mathbb{D}^d)$.

Let $H^\infty(\theta(\mathbb{D}^d))$ be the Banach algebra of all bounded holomorphic functions on $\theta(\mathbb{D}^d)$ with sup-norm. If $g \in H^\infty(\mathbb{D}^d)$, the Banach algebra bounded analytic functions on $\mathbb{D}^d$, then the radial limit

$$g^*(\zeta_1, \ldots, \zeta_d) := \lim_{r \to 1^-} g(r\zeta_1, \ldots, r\zeta_d)$$

exists almost everywhere in $\mathbb{T}^d$ with respect the normalized Haar measure $\nu$ on $\mathbb{T}^d$. For every $f \in H^\infty(\theta(\mathbb{D}^d))$, $f \circ \theta$ is in $H^\infty(\mathbb{D}^d)$ having a radial limit. Therefore, the
boundary values of \( f \) exist and the boundary value function \( f^* \) is given by
\[
f^*(q_1, \ldots, q_d) = \lim_{r \to 1} f(r^{m_1} q_1, r^{2m_2} q_2, \ldots, r^{(d-1)m_{d-1}} q_{d-1}, r^{dm_d} q_d)
\]
for almost every (with respect to \( \mu \)) \( q = (q_1, \ldots, q_d) \in \partial \theta(\mathbb{D}^d) \) where \( \mu \) is the push-forward of the measure \( \nu \) on \( \mathbb{T}^d \) under the map \( \theta \). Here we are using the explicit expressions for \( \theta_j \)'s in terms of the elementary symmetric polynomials. Also, it is easy to check that \( f \mapsto f^* \) is an isometric embedding of \( H^\infty(\theta(\mathbb{D}^d)) \) into \( L^\infty(\partial \theta(\mathbb{D}^d), \mu) \).

A function \( f \) in \( H^\infty(\theta(\mathbb{D}^d)) \) is said to be inner if \( f^* \) is unimodular almost everywhere with respect to \( \mu \) on \( \partial \theta(\mathbb{D}^d) \). A rational inner function \( f \) on \( \theta(\mathbb{D}^d) \) is a rational function with poles off \( \partial \theta(\mathbb{D}^d) \) which is also inner. An important family of examples of a rational inner functions on the symmetrized bidisc, \( G_2 \) which is biholomorphic to \( \mathbb{D}^2/G(1, 1, 2) \), is
\[
\varphi_\alpha(p_1, p_2) = \frac{2\alpha p_2 - p_1}{2 - \alpha p_1}
\]
for each \( \alpha \) in the unit circle \( \mathbb{T} \). See [2, 3, 5] for its importance. Rational inner functions appear as solutions to the Pick-Nevanlinna interpolation problem, see [25, 24, 31].

We first ask whether the algebra generated by inner functions is large enough to give us any advantage. In this direction, we have the following result.

**Theorem A.** The closed algebra generated by the inner functions in \( H^\infty(\theta(\mathbb{D}^d)) \) is a proper subalgebra of \( H^\infty(\theta(\mathbb{D}^d)) \).

The proof requires us to find a new characterization of inner functions on \( \theta(\mathbb{D}^d) \). We also show that unlike the classical case of \( \mathbb{D} \), the Shilov boundary of \( H^\infty(\theta(\mathbb{D}^d)) \) is a proper subset of the set of all restrictions of complex homomorphisms on \( L^\infty(\partial \theta(\mathbb{D}^d), \mu) \). These are the contents of Section 2.

It may be a surprise that in spite of Theorem A we have approximation results. We begin with a structure of rational inner functions on \( \theta(\mathbb{D}^d) \). This is our Theorem B.

**Theorem B.** Given a rational inner function \( f \) on \( \theta(\mathbb{D}^d) \), there exist a non-negative integer \( k \), \( \tau \in \mathbb{T} \) and a polynomial \( g \) with no zero in \( \theta(\mathbb{D}^d) \) such that
\[
\tag{1.1}
f(p_1, \ldots, p_{d-1}, p_d) = \tau p_d^k \frac{g \left( \frac{p_1}{p_d}, \frac{p_2}{p_d}, \ldots, \frac{p_{d-1}}{p_d}, \frac{1}{p_d} \right)}{g (p_1, \ldots, p_{d-1}, p_d)}
\]
Conversely, any rational function of the form (1.1) is inner. Moreover, any inner function \( f \in \mathcal{A}(\theta(\mathbb{D}^d)) \) is a rational function of the form (1.1) with the additional property that \( g \) has no zero in \( \theta(\mathbb{D}^d) \).

This result was proved in the case of the polydisc by Rudin (Theorem 5.2.5 in [38]) and in the case of the bounded symmetric domains by Korányi-Vági [28]. We cannot apply the Korányi-Vági result to \( \theta(\mathbb{D}^d) \) as it is not a bounded symmetric domain. The crux of our proof lies in being able to choose the polynomial \( Q \) in Rudin’s theorem in a certain way.

Rational inner functions on the unit disc have been greatly studied for their usefulness. A classical theorem of Carathéodory says that any holomorphic self map of \( \mathbb{D} \) can be approximated uniformly on compact subsets of \( \mathbb{D} \) by rational inner functions.
It is natural to ask for Carathéodory approximation theorem in quotient domains. We prove such an approximation result in these domains. This in turn enables us to prove a theorem on approximation by convex combinations of rational inner functions in the $L^2$ sense.

**Theorem C.** Any function in the norm unit ball of $H^\infty(\theta(\mathbb{D}^d))$ can be approximated by convex combinations of rational inner functions in $\mathcal{A}(\theta(\mathbb{D}^d))$ with respect to the $L^2$-norm on $\partial \theta(\mathbb{D}^d)$ equipped with the measure $\mu$.

Building on the success of the previous results, a natural progression is to explore their generalization in the operator-valued setting. The concept of inner functions can also be extended to this context, where the modulus-one condition is replaced by values in the space of isometries. Proving these results in the operator-valued setting required drawing on ideas from function-theoretic operator theory as well as operator theory on these domains. However, these tools are not available for all quotient domains. Fortunately, the domain $\mathbb{G}_2$ possesses a rich function theory and operator theory. Utilizing the Pick-Nevanlinna interpolation theory, in Section 4.1 we prove the Carathéodory approximation theorem for operator-valued setting on $\mathbb{G}_2$.

2. **Algebra generated by inner functions**

Marshall [35] a half-century ago proved that the algebra generated by inner functions on the disc is $H^\infty(\mathbb{D})$. From this it follows that the inner functions separate the points in the maximal ideal space of $H^\infty(\mathbb{D})$. After that, Kon showed that these results do not hold for $H^\infty(\mathbb{D}^d)$ when $d > 1$; see [26, 27]. Naturally, we can ask these questions in our setting of quotient domains. In this section, we answer them in a negative direction for $\theta(\mathbb{D}^d)$ for $d > 1$. The main theorem of this section is the following. The results of this section are motivated by [26, 27, 37].

**Theorem 2.1.** The closed algebra generated by inner functions in $H^\infty(\theta(\mathbb{D}^d))$ is a proper subalgebra of $H^\infty(\theta(\mathbb{D}^d))$.

To establish this, we shall rely on a set of lemmas and propositions. Some of these lemmas can be deduced using the results from the polydisc, while others require their own distinct treatment. Furthermore, certain propositions hold independent interest and importance, even beyond their relevance to the theorem. Let $M_1$ and $M_2$ be the maximal ideal spaces of $H^\infty(\theta(\mathbb{D}^d))$ and $H^\infty(\mathbb{D}^d)$ respectively. The maximal ideal spaces $M_1$ and $M_2$ are endowed with the weak$^*$ topologies inherited from the norm-unit balls of the continuous duals of the Banach spaces $H^\infty(\theta(\mathbb{D}^d))$ and $H^\infty(\mathbb{D}^d)$ respectively. Also, we denote the maximal ideal spaces of $L^\infty(\partial \theta(\mathbb{D}^d), \mu)$ and $L^\infty(\mathbb{T}^d, \nu)$ by $X_1$ and $X_2$ respectively. Define two maps $\tau_j : X_j \to M_j$ for $j = 1, 2$ in the following way:

$$\tau_1(m) = m|_{H^\infty(\theta(\mathbb{D}^d))} \text{ for all } m \in X_1,$$

$$\tau_2(m) = m|_{H^\infty(\mathbb{D}^d)} \text{ for all } m \in X_2.$$ 

For $f \in H^\infty(\theta(\mathbb{D}^d))$, the Gelfand transform $\hat{f}$ of $f$ is a continuous function from $M_1$ to $\mathbb{C}$ given by

$$\hat{f}(m) = m(f) \text{ for each } m \in M_1.$$

Similarly, the Gelfand transforms of functions in \( L^\infty(\partial \theta(\mathbb{D}^d), \mu) \), \( L^\infty(\mathbb{T}^d, \nu) \) and \( H^\infty(\mathbb{D}^d) \) are defined on their respective maximal ideal space. See [18] for more details.

A closed boundary for \( H^\infty(\theta(\mathbb{D}^d)) \) is a closed subset \( \mathcal{C} \) of the maximal ideal space of \( H^\infty(\theta(\mathbb{D}^d)) \) such that

\[
\|f\|_\infty = \sup \{|\varphi(f)| : \varphi \in \mathcal{C}\}
\]

for all \( f \) in \( H^\infty(\theta(\mathbb{D}^d)) \). The Shilov boundary of the uniform algebra \( H^\infty(\theta(\mathbb{D}^d)) \) is the smallest closed boundary of \( H^\infty(\theta(\mathbb{D}^d)) \). Let \( \partial \) denote the Shilov boundary for \( H^\infty(\theta(\mathbb{D}^d)) \). For any \( m \in X_2 \) we can define, \( \hat{m} : H^\infty(\theta(\mathbb{D}^d)) \to \mathbb{C} \) such that

\[
\hat{m}(\psi) = m(\psi \circ \theta).
\]

Then \( \hat{m} \in X_1 \). In a similar way, every member of \( M_2 \) gives rise to a unital complex homomorphism in \( M_1 \). The following proposition characterizes the inner functions on the bidisc in terms of the homomorphisms in \( \tau_2(X_2) \). It is due to Kon.

**Proposition 2.2** ([28]). A function \( u \) in \( H^\infty(\mathbb{D}^d) \) is inner if and only if \( |\hat{u}(\Phi)| = 1 \) for all \( \Phi \in \tau_2(X_2) \).

**Lemma 2.3.** A function \( f \) in \( H^\infty(\theta(\mathbb{D}^d)) \) is inner if and only if \( |\hat{f}(\Psi)| = 1 \) for all \( \Psi \in \tau_1(X_1) \).

**Proof.** If \( f \) is inner in \( H^\infty(\theta(\mathbb{D}^d)) \), then it straightforward that \( |\hat{f}(\psi)| = 1 \) for all \( \psi \in \tau_1(X_1) \). For the converse part, assume that \( |\hat{f}(\Psi)| = 1 \) for all \( \Psi \in \tau_1(X_1) \). It is enough to show that \( f \circ \theta \) is inner function in \( \mathbb{D}^d \). Let \( m \in X_2 \). Then,

\[
|\tau_2(m)(f \circ \theta)| = |m|_{H^\infty(\mathbb{D}^d)}(f \circ \theta) = |\tau_1(\hat{m})(f)| = |\hat{f}(\tau_1(\hat{m}))| = 1.
\]

Since \( m \in X_2 \) is arbitrary, Proposition 2.2 implies that \( f \circ \theta \) is an inner function. We note from the proof that the whole of \( \tau_1(X_1) \) may not be needed for the converse part.

For a non-negative Borel measure \( \eta \) on \( \mathbb{T}^d \) and a function \( f \) in \( L^1(\mathbb{T}^d, \nu) \), we shall denote the real measure \( fd\nu - d\eta \) by \( \eta_f \). Let \( \mathcal{P} \) be the Poisson kernel of \( \mathbb{D}^d \). See [38] for more details on Poisson kernel.

**Lemma 2.4.** Let \( f \) be a positive, \( G \)-invariant and lower semi-continuous function on \( \mathbb{T}^d \) such that \( f \in L^1(\mathbb{T}^d, \nu) \). Then there exists a positive measure \( \eta \) on \( \mathbb{T}^d \) which is singular with respect to \( \nu \) such that

\[
P[\eta_f] = \text{Re}(g)
\]

for some \( G \)-invariant holomorphic function \( g \) on \( \mathbb{D}^d \), where \( P[\eta_f] \) is the Poisson integral of the real Borel measure \( \eta_f \) on \( \mathbb{T}^d \).

**Proof.** We use Theorem 2.4.2 of [38] to obtain a non-negative measure \( \omega \) on \( \mathbb{T}^d \) which is singular with respect to \( \nu \) and \( P[\omega_f] = \text{Re}(u) \) for some holomorphic function \( u \) in \( \mathbb{D}^d \). For \( \sigma \in G \), we consider the pull-back \( \sigma^* \omega \) of \( \omega \) under the homeomorphism \( \sigma \) on \( \mathbb{T}^d \) and define a new measure \( \eta \) on \( \mathbb{T}^d \) as follows:

\[
\eta(E) := \frac{1}{|G|} \sum_{\sigma \in G} \sigma^* \omega(E) = \frac{1}{|G|} \sum_{\sigma \in G} \omega(\sigma(E))
\]
for each $\omega$-measurable subset $E$ of $\mathbb{T}^d$. It is easy to observe that the support of the measure $\eta$ is the union of the supports of the measures $\sigma^*\omega$ for all $\sigma$ in $G$. using change of variable for the maps $\sigma : \mathbb{T}^d \to \mathbb{T}^d$ we observe that supports of the measures $\sigma^*\omega$ have $\nu$ measure zero. Hence $\eta$ is singular with respect to $\nu$.

Again $P[\omega] = \text{Re}(u)$ implies that for every $\sigma \in G$,

$$\text{Re } u(\sigma(z)) = P[f](\sigma(z)) - P[\omega](\sigma(z))$$

$$= \int_{\mathbb{T}^d} \mathcal{P}(\sigma(z), \zeta) f(\zeta) d\nu(\zeta) - \int_{\mathbb{T}^d} \mathcal{P}(\sigma(z), \zeta) d\eta(\zeta). \tag{2.1}$$

Suppose,

$$\sigma(z) = \left(e^{\frac{2\pi i}{m} \nu_1 z_{\beta(1)}}, \ldots, e^{\frac{2\pi i}{m} \nu_d z_{\beta(d)}}\right)$$

for some $\beta \in S_d$ and the integers $\nu_j$ are as in the definition of the group $G = G(m, t, d)$. Then by the change of variable

$$\zeta_j = e^{\frac{2\pi i}{m} \nu_j \zeta_{\beta(j)}}$$

for each $j = 1, \ldots, d$ and the invariance of the measure $d\nu$ under this transformation we get

$$\mathcal{P}(\sigma(z), \zeta) f(\zeta) d\nu(\zeta) = \prod_{j=1}^{d} \frac{1 - |z_{\beta(j)}|^2}{|1 - e^{\frac{2\pi i}{m} \nu_j z_{\beta(j)} \zeta_j}|^2} f(\zeta) d\nu(\zeta)$$

$$= \int_{\mathbb{T}^d} \mathcal{P}(\sigma(z), \zeta) d\nu(\zeta) = P[f](z).$$

The penultimate equality holds as $f$ is $G$-invariant. Finally, note that by our construction the measure $\eta$ is also $G$-invariant and hence by the same change of variable trick as above shows that

$$\int_{\mathbb{T}^d} \mathcal{P}(\sigma(z), \zeta) d\eta(\zeta) = P[\eta](z).$$

Thus from (2.1) we conclude that

$$\text{Re } u(\sigma(z)) = \text{Re } u(z) = P[\eta](z).$$

This completes the proof.

We shall denote the subalgebra of all $G$-invariant holomorphic functions in $H^\infty(\mathbb{D}^d)$ by $H^\infty(\mathbb{D}^d)^G$.

**Lemma 2.5.** Given a positive, bounded and $G$-invariant lower semi-continuous function $\psi$ on $\mathbb{T}^d$, there exists a bounded $G$-invariant holomorphic function $f$ on $\mathbb{D}^d$ such that

$$\psi = |f^*| \quad \text{a.e. } (\nu)\text{on } \mathbb{T}^d,$$

where $f^*$ is the radial limit of $f$ which is defined a.e. $(\nu)$ in $\mathbb{T}^d$. 
Proof. Since \( \psi \) is a lower-semicontinuous function, it attains its minimum on \( \mathbb{T}^d \). After adding a positive constant we can make it bigger than 1. Hence, we shall assume that \( \psi > 1 \). Thus \( \log \psi \) is well defined and satisfies all the conditions of Lemma 2.4. Thus we get a positive measure \( \eta \) on \( \mathbb{T}^d \) which is singular with respect to \( \nu \), and a \( G \)-invariant holomorphic function \( g \) on \( \mathbb{D}^d \) such that

\[
P[\eta_{\log(\psi)}] = \text{Re}(g).
\]

Set \( f = \exp(g) \). By the definition of \( f \), it is \( G \)-invariant and holomorphic on \( \mathbb{D}^d \). Now we shall show that \( f \) is bounded. To that end, note that

\[
\log |f| = \text{Re}(g) = P[\eta_{\log(\psi)}] = P[\log(\psi)] - P[\eta]\cdot
\]

As \( \eta \) is a non-negative measure, we get

\[
P[\eta_{\log(\psi)}] \leq P[\log(\psi)].
\]

This implies that \( \log |f| \leq P[\log(\psi)] \). Since \( P[\log(\psi)] \) is bounded, \( \log |f| \) is bounded. Thus, \( f \) is bounded and hence \( f \in H^\infty(\mathbb{D}^d)^G \). Applying Theorem 2.3.1 of [38] to the function \( f \) and the measure \( \eta \), we get that \( \psi = |f^*| \).

Consider the maximal ideal spaces \( M_0 \) and \( X_0 \) of the algebras \( H^\infty(\mathbb{D}) \) and \( L^\infty(\mathbb{T}) \), respectively. It is well known that the map \( \tau_0 : X_0 \to M_0 \), defined as \( \tau_0(m) = m|_{H^\infty(\mathbb{D})} \) for all \( m \in X_0 \), is a well-known homeomorphism from \( X_0 \) onto the Shilov boundary of \( H^\infty(\mathbb{D}) \), see [21]. In the subsequent theorem, we shall establish that, unlike the classical case of the unit disc, the Shilov boundary is a proper subset of the set comprising of all restrictions of complex homomorphisms on \( L^\infty(\partial(\mathbb{D}^d), \mu) \).

Theorem 2.6. The map \( \tau_1 : X_1 \to M_1 \) is continuous and \( \partial \) is a proper subset of \( \tau_1(X_1) \).

Proof. Take a countable dense subset \( \{ \mathbf{p}^{(n)} = (p_1^{(n)}, \ldots, p_d^{(n)}) : n \in \mathbb{N} \} \) of \( \partial(\mathbb{D}^d) \). For each \( n \), define

\[
A_n = \{(e^{ims}p_1^{(n)}, \ldots, e^{i(d-1)ms}p_d^{(n)}, e^{is/\tau}p_d^{(n)} : s \in [0, 2\pi]\}
\]

Each \( A_n \) is a compact set. We consider tubular type open neighbourhoods \( A_{j,n} \) of the set \( A_n \) such that \( \mu(A_{j,n}) < \frac{1}{j^{2n+1}} \) for every \( j \in \mathbb{N} \). Moreover, we choose these neighbourhoods in such a way that, if \( (p_1, \ldots, p_d) \in A_{j,n} \), then

\[
(e^{ims}p_1, \ldots, e^{i(d-1)ms}p_d, e^{is/\tau}p_d) \in A_{j,n}
\]

for all \( s \in [0, 2\pi] \). Take, \( E_j = \cup_{n \in \mathbb{N}} A_{j,n} \). Clearly, \( \mu(E_j) \leq \frac{1}{2j} \) and \( E_j \) is an open dense subset of \( \partial\mathbb{D}^d \).

Consider the set \( Z_k = \{ \varphi \in X_1 : \varphi(\chi_{E_k}) = 1 \} \) where \( \chi_{E_k} \) is the indicator function of the set \( E_k \). Then the Gelfand transform of \( \chi_{E_k} \) satisfies the condition \( \hat{\chi}_{E_k} = \chi_{Z_k} \).

For \( f \in L^\infty(\partial(\mathbb{D}^d), \mu) \), the Gelfand transform \( \hat{f} \) is in \( C(X_1) \). Define a continuous linear functional on \( C(X_1) \) given by \( \hat{f} \mapsto \int_{\partial(\mathbb{D}^d)} f d\mu \). Then by the Riesz representation theorem, there exists a unique regular Borel measure \( \hat{\mu} \) such that

\[
\int_{\partial(\mathbb{D}^d)} f d\mu = \int_{X_1} \hat{f} d\hat{\mu}.
\]
Therefore
\[ \hat{\mu}(Z_k) = \int_{\mathcal{X}_1} \chi_{Z_k} \, d\hat{\mu} = \int_{\mathcal{X}_1} \hat{\chi}_{E_k} \, d\hat{\mu} = \int_{\partial \theta(\mathbb{D}^d)} \chi_{E_k} \, d\mu = \mu(E_k) \leq \frac{1}{2k}. \]

Let \( f \in H^\infty(\partial(\mathbb{D}^d)) \). Define, \( L_f : \partial \theta(\mathbb{D}^d) \to \mathbb{R} \) by
\[
L_f(p) = \text{ess sup}_{s \in [0,2\pi]} |f(e^{ims} p_1, \ldots, e^{i(d-1)ms} p_{d-1}, e^{idms/t} p_d)| = \sup_{0 < r < 1} \left\{ \sup_s |f(r^m e^{ims} p_1, \ldots, r^{m(d-1)} e^{i(d-1)ms} p_{d-1}, r^{m/t} e^{idms/t} p_d)| \right\}.
\]

Then \( L_f \) can be shown to be lower semi-continuous on \( \partial \theta(\mathbb{D}^d) \) using a similar argument used in Theorem 3.5.2 in [18]. Now suppose that
\[ \sup_{\tau_{\hat{\mu}}(Z_k)} |\hat{f}| = \text{ess sup}_{E_k} |f^*| \leq 1. \]

For a.e. \( \mu(p) \in E_k \), \( L_f(p) \leq 1 \). Since \( L_f \) is lower semi-continuous and \( E_k \) is open dense in \( \partial \theta(\mathbb{D}^d) \), for all \( p \in \partial \theta(\mathbb{D}^d) \), \( L_f(p) \leq 1 \) and hence \( \|f\|_\infty \leq 1 \). Therefore, \( \tau_1(Z_k) \) is a closed boundary for \( H^\infty(\theta(\mathbb{D}^d)) \). We can construct \( E_j \)'s such that \( E_j \supset E_{j+1} \) and so, \( Z_j \supset Z_{j+1} \). \( Z_j \)'s being non empty compact subsets, by the finite intersection property we have,
\[ E = \bigcap_{j=1}^\infty Z_j \neq \phi. \]

Also,
\[ \tau_1(E) = \bigcap_{j=1}^\infty \tau_1(Z_j). \]

We know that \( \tau_1(Z_j) \)'s are closed boundaries for \( H^\infty(\theta(\mathbb{D}^d)) \) and hence \( \tau_1(E) \) is so. Note that,
\[ \hat{\mu}(E) = \lim_{j \to \infty} \hat{\mu}(Z_j) = 0. \]

Also, the closed support of \( \hat{\mu} \) is \( \mathcal{X}_1 \) (This can be easily seen from Lemma 9.1 in [18]). So, we have shown that there exists a nowhere dense subset \( E \) of \( \mathcal{X}_1 \) with \( \hat{\mu}(E) = 0 \) such that \( \tau_1(E) \) is a closed boundary for \( H^\infty(\theta(\mathbb{D}^d)) \).

Consider a subset \( A \subset \partial \theta(\mathbb{D}^d) \setminus E_1 \) with \( \mu(A) > 0 \). Then \( \theta^{-1}(A) \) is \( G \)-invariant subset of positive \( \nu \) measure in \( \mathbb{T}^d \).

Now we take the lower semi continuous function \( \psi \) given by
\[
\psi = \begin{cases} 
1, & \text{on } \theta^{-1}(A) \\
2, & \text{on } \theta^{-1}(E_1) \\
0, & \text{else}.
\end{cases}
\]

By Lemma 2.5 there exists a \( G \)-invariant bounded holomorphic function \( g \) such that
\[
|g^*| = \begin{cases} 
1, & \text{on } \theta^{-1}(A) \\
2, & \text{on } \theta^{-1}(E_1).
\end{cases}
\]

This means that there exists \( f \in H^\infty(\theta(\mathbb{D}^d)) \) such that
\[
|f^*| = \begin{cases} 
1, & \text{on } A \\
2, & \text{on } E_1.
\end{cases}
\]
Therefore, $|\hat{f}| = 2$ on the Shilov boundary of $H^\infty(\theta(\mathbb{D}^d))$. But $|\hat{f}|$ is not identically 2 on $\tau_1(X_1)$. Indeed, consider

$$Z_A = \{ \varphi \in X_1 : \hat{\chi}_A(\varphi) = 1 \} \text{ and } Z_{E_1} = \{ \varphi \in X_1 : \hat{\chi}_{E_1}(\varphi) = 1 \}.$$  

We can find two unimodular measurable functions $g_1$ and $h$ on $\partial \theta(\mathbb{D}^d)$ such that

$$f^* = 2g_1\chi_{E_1} + h\chi_A.$$  

For $\varphi \in Z_{E_1}$, we have $\varphi|_{H^\infty(\theta(\mathbb{D}^d))} \in \tau_1(Z_{E_1})$ and

$$|\hat{f}(\varphi|_{H^\infty(\theta(\mathbb{D}^d))})| = |\varphi(f^*)| = 2|\varphi(g_1)\varphi(\chi_{E_1}) + \varphi(h)\varphi(\chi_A)| = 2$$

as $|\varphi(h)| = |\varphi(g_1)| = 1$ and $\varphi(\chi_A) = 0$. So, $|\hat{f}| = 2$ on $\tau_1(Z_{E_1})$. We know that $\tau_1(Z_{E_1})$ is a closed boundary for $H^\infty(\theta(\mathbb{D}^d))$. Hence $|\hat{f}| = 2$ on the Shilov boundary of $H^\infty(\theta(\mathbb{D}^d))$. Similarly, $|\hat{f}| = 1$ on $\tau_1(Z_A)$. Since $\mu(A) > 0$, $\tau_1(Z_A) \neq \phi$ and so $\tau_1(Z_A) \cap \partial = \phi$. Therefore, $\partial$ is a proper subset of $\tau_1(X_1)$. \hfill \Box

It is worth noting that in the proof of the theorem above not only confirms this result but also provides additional insights. Specifically, it demonstrates the existence of a nowhere dense subset in the maximal ideal space of $L^\infty(\partial \theta(\mathbb{D}^d), \mu)$, which serves as a closed boundary for $H^\infty(\theta(\mathbb{D}^d))$. The theorem mentioned above, along with the following proposition, will play a crucial role in proving the main theorem.

**Proposition 2.7.** The Gelfand transforms of the inner functions in $H^\infty(\theta(\mathbb{D}^d))$ cannot separate points of the maximal ideal space $M_1$.

**Proof.** The proof of this theorem is standard once we have Theorem 2.6. Indeed, if possible, suppose that the Gelfand transforms of the inner functions in $H^\infty(\theta(\mathbb{D}^d))$ separate points of $\tau_1(X_1)$. Take a proper compact subset $K \subset \tau_1(X_1)$ and let $\Psi \in \tau_1(X_1) \setminus K$. Then for each $\Phi_\alpha \in K$, by our assumption, there exists an inner function $I_\alpha$ such that

$$I_\alpha(\Psi) \neq I_\alpha(\Phi_\alpha).$$  

By Lemma 2.3, both $I_\alpha(\Psi)$ and $I_\alpha(\Phi_\alpha)$ are unimodular. So, without loss of generality, we can assume that

$$I_\alpha(\Psi) = 1 \quad \text{and} \quad \Re(I_\alpha(\Phi_\alpha)) < 1.$$  

By continuity, there exists a neighbourhood $N_\alpha$ of $\Phi_\alpha$ such that $\Re(I_\alpha(\Phi)) < 1$ for all $\phi$ in $N_\alpha$. Since $K$ is a compact set, there exists $r$ such that $K \subset \cup_{j=1}^r N_{\alpha_j}$. Hence, for each $j = 1, 2, \ldots, r$, we have $I_\alpha(\Psi) = 1$. Also,

$$\inf_{1 \leq j \leq r} \Re(I_{\alpha_j}(\Phi)) < 1 \quad \text{for all } \Phi \in K.$$  

Consider the holomorphic function $g = 1 + I_{\alpha_1} + \cdots + I_{\alpha_r}$ in $H^\infty(\theta(\mathbb{D}^d))$. Note that,

$$\hat{g}(\Psi) = \hat{I}(\Psi) + \hat{I}_{\alpha_1}(\Psi) + \cdots + \hat{I}_{\alpha_r}(\Psi) = r + 1 = \|\hat{g}\|.$$  

Also for $\Phi \in K$, we have

$$|\hat{g}(\Phi)| = |\hat{I}(\Phi) + \hat{I}_{\alpha_1}(\Phi) + \cdots + \hat{I}_{\alpha_r}(\Phi)| \leq r + 1.$$
Now, if $|\hat{g}(\Phi)| = r+1$, then $\hat{I}_{\alpha_1}(\Phi) + \cdots + \hat{I}_{\alpha_r}(\Phi) = r$. Thus, $\sum_{j=1}^{r} \text{Re} \left( \hat{I}_{\alpha_j}(\Phi) \right) = r$, which is a contradiction as
\[
\inf_{1 \leq j \leq r} \text{Re} \left( \hat{I}_{\alpha_j}(\Phi) \right) < 1.
\]

Therefore, for each $\Phi \in K$, we have $|\hat{g}(\Phi)| < r+1$ and $|\hat{g}(\Psi)| = r+1$. This implies that $K$ can not be a closed boundary of $H^\infty(\theta(\mathbb{D}^d))$. Since $K$ is an arbitrary proper compact subset of $\tau_1(X_1)$, we must have that $\tau_1(X_1)$ is the Shilov boundary for $H^\infty(\theta(\mathbb{D}^d))$. This is not possible because of Theorem 2.6. Hence, The Gelfand transforms of inner functions cannot separate points of $\tau_1(X_1)$ and hence the points of $M_1$.  

Now, we are ready to prove the main Theorem of this section.

**Proof of Theorem 2.1.** Let $\mathcal{A}$ be the closed algebra generated by inner functions in $H^\infty(\theta(\mathbb{D}^d))$. Suppose that $\mathcal{A} = H^\infty(\theta(\mathbb{D}^d))$. Let $m_1$ and $m_2$ be two distinct elements in $M_1$. Then there exists $f \in H^\infty(\theta(\mathbb{D}^d))$ such that $m_1(f) \neq m_2(f)$. If for every inner function $I$, $m_1(I) = m_2(I)$ then $m_1 = m_2$ on $\mathcal{A}$ as the inner functions generate it. This is a contradiction. Hence the Gelfand transforms of the inner functions separate points of $M_1$. This contradicts Proposition 2.7. Hence $\mathcal{A}$ is a proper subalgebra of $H^\infty(\theta(\mathbb{D}^d))$.

### 3. Structure of Rational Inner Functions

The goal of this section is to study the rational inner functions on $\theta(\mathbb{D}^d)$. Often, $z$ is used in this section to represent $(z_1, z_2, \ldots, z_d)$. For a polynomial $\xi \in \mathbb{C}[z]$ with degree $n = (n_1, \ldots, n_d)$, the polynomial $\tilde{\xi}$ is defined as
\[
\tilde{\xi}(z) = z^n \xi \left( \frac{1}{z} \right)
\]
where $z^n = z_1^{n_1} z_2^{n_2} \cdots z_d^{n_d}$ and $\frac{1}{z} = \left( \frac{1}{z_1}, \frac{1}{z_2}, \ldots, \frac{1}{z_d} \right)$. We shall start with a few algebraic lemmas which will be used in the proof of the main result of this section.

**Lemma 3.1.** Let $\xi$ and $\eta$ be two polynomials in $\mathbb{C}[z]$ such that $\xi(0,0) \neq 0$ and $\eta(0,0) \neq 0$. Then the following hold:

1. $\tilde{\xi} = \xi$ and $\tilde{\eta} = \eta$.
2. $\xi \eta = \tilde{\xi} \tilde{\eta}$.

Moreover, if $\xi$ and $\eta$ are two distinct irreducible polynomials, then the following are also true:

3. If $\xi$ divides $\tilde{\xi}$, then $\xi$ is a unimodular scalar multiple of $\tilde{\xi}$ and $\tilde{\eta}$.
4. If $\xi$ divides $\tilde{\eta}$ then $\eta$ is a non-zero scalar scalar multiple of $\tilde{\xi}$.

**Proof.** The first two parts are obvious. So, we shall prove (3) and (4). Suppose $\xi$ divides $\tilde{\xi}$. Then there exists $\psi \in \mathbb{C}[z]$ such that $\tilde{\xi} = \psi \xi$. Thus $\tilde{\xi} = \tilde{\psi} \xi$. Hence, by part (1), we have $\xi = \tilde{\psi} \xi$. Since $\xi$ is irreducible, it follows that $\psi$ is a unimodular scalar.
Finally, suppose that \( \xi \) divides \( \tilde{\eta} \). Then there exists \( \psi \in \mathbb{C}[z] \) such that \( \tilde{\eta} = \psi \xi \). This gives that

\[
\eta = \tilde{\eta} = \psi \xi.
\]

Since \( \eta \) is irreducible, \( \psi \) must be a non-zero scalar. This proves (4). \( \square \)

The following lemma about \( G \)-invariant rational functions on \( \mathbb{D}^d \) is crucial for us.

**Lemma 3.2.** Let \( f \) be a \( G \)-invariant rational function on \( \mathbb{D}^d \). Then \( f \) can be expressed as a quotient of two \( G \)-invariant polynomials.

**Proof.** If \( f \) is the identically zero function, then there is nothing to prove. So, we assume \( f \) to be non-zero. Since \( f \) is a rational function, there exist polynomials \( \xi \) and \( \eta \) in \( \mathbb{C}[z] \) which are co-prime such that

\[
f = \frac{\xi}{\eta}.
\]

Let \( \sigma \in G \). Then \( \sigma \cdot f = \frac{\sigma \xi}{\sigma \eta} \). Since \( f \) is \( G \)-invariant, we have

\[
\frac{\xi}{\eta} = f = \sigma \cdot f = \frac{\sigma \cdot \xi}{\sigma \cdot \eta}.
\]

Thus, \( \xi(\sigma \cdot \eta) = \eta(\sigma \cdot \xi) \). Since \( \xi \) and \( \eta \) are co-prime, \( \xi \) divides \( \sigma \cdot \xi \) and \( \eta \) divides \( \sigma \cdot \eta \).

Also, note that the total degrees of \( \psi \) and \( \sigma \cdot \psi \) are same for any polynomial \( \psi \in \mathbb{C}[z] \). Thus,

\[
\sigma \cdot \xi = \lambda_\sigma \xi \quad \text{and} \quad \sigma \cdot \eta = \mu_\sigma \eta
\]

for some non-zero scalars \( \lambda_\sigma \) and \( \mu_\sigma \). From equation (3.1), it follows that \( \lambda_\sigma = \mu_\sigma \).

The map

\[
\sigma \mapsto \lambda_\sigma
\]

defines a group homomorphism from \( G \) to the multiplicative group \( \mathbb{C} \setminus \{0\} \). But \( G \) being a finite group, it is actually a homomorphism from \( G \) to \( \mathbb{T} \).

If we show that this group homomorphism is the trivial one, then \( \sigma \cdot \xi = \xi \) and \( \sigma \cdot \eta = \eta \), proving that \( \xi \) and \( \eta \) are \( G \)-invariant polynomials. Since the group \( G \) is generated by finitely many pseudo-reflections, it is enough to prove that these generators are mapped to 1 by the aforementioned homomorphism. Let \( \sigma \) be such a pseudo-reflection. Then \( \text{Ker}(I_d - \sigma) \) is a \( d - 1 \) dimensional subspace of \( \mathbb{C}^d \) and so, there is a linear polynomial \( L_\sigma \) in \( \mathbb{C}[z] \) such that

\[
\text{Ker}(I_d - \sigma) = \{ z \in \mathbb{C}^d : L_\sigma(z) = 0 \}.
\]

If possible, suppose that \( \lambda_\sigma \neq 1 \). Then \( \sigma \cdot \xi = \lambda_\sigma \xi \) gives

\[
\xi(\sigma(z)) = \lambda_\sigma \xi(z). \tag{3.2}
\]

For \( z \in \text{Ker}(I_d - \sigma) \), \( \sigma(z) = z \) and so, by (3.2), \( \xi(z) = 0 \). Therefore, \( L_\sigma \) divides \( \xi \). Similarly, \( L_\sigma \) also divides \( \eta \). But this is not possible as \( \xi \) and \( \eta \) are co-prime with each other. So, we must have \( \lambda_\sigma = 1 \) and hence the above group homomorphism is trivial. \( \square \)
Lemma 3.3. Let \( \xi \) be a polynomial in \( \mathbb{C}[z] \). Then \( \tilde{\xi} \) can be written as

\[
\frac{\tau \tilde{\xi}_1 \tilde{\xi}_2 \ldots \tilde{\xi}_{j_l}}{\xi_1 \xi_2 \ldots \xi_{j_l}}
\]

for some \( \tau \in \mathbb{T} \) and \( j_1, j_2, \ldots, j_l \) such that \( \tilde{\xi}_1, \tilde{\xi}_2 \ldots \tilde{\xi}_{j_l} \) and \( \xi_1, \xi_2 \ldots \xi_{j_l} \) are co-prime.

Proof. Write \( \xi \) as \( \xi_1 \xi_2 \ldots \xi_r \) for some irreducible polynomials \( \xi_1, \ldots, \xi_r \in \mathbb{C}[z] \). Then, by part (2) of Lemma 3.1, we have

\[
\tilde{\xi} = \tilde{\xi}_1 \tilde{\xi}_2 \ldots \tilde{\xi}_r.
\]

So,

\[
\frac{\tilde{\xi}}{\xi} = \frac{\tilde{\xi}_1 \tilde{\xi}_2 \ldots \tilde{\xi}_r}{\xi_1 \xi_2 \ldots \xi_r}.
\]

Note that \( \tilde{\xi} \) and \( \xi \) can have common factors. If \( \xi_j \) divides \( \tilde{\xi}_j \) for some \( j \), then by part (3) of Lemma 3.1, \( \tilde{\xi}_j / \xi_j \in \mathbb{T} \). Since \( \xi_j \)'s are irreducible, any such common factor is divisible by \( \xi_i \) for some \( i \). Without loss of generality, we can assume that \( \xi_1 \) divides \( \tilde{\xi}_2 \). Then by part (4) of Lemma 3.1, \( \tilde{\xi}_2 = \beta \xi_1 \). Also \( \tilde{\xi}_2 = \beta \xi_1 \). So, in this case, we have

\[
\frac{\tilde{\xi}}{\xi} = \frac{\tilde{\xi}_1 (\beta \xi_1) \ldots \tilde{\xi}_r}{\xi_1 (\beta \xi_1) \ldots \xi_r} = \frac{\beta \tilde{\xi}_3 \ldots \tilde{\xi}_r}{\beta \xi_3 \ldots \xi_r}.
\]

After such cancellations, we shall end up with

\[
\frac{\tilde{\xi}}{\xi} = \frac{\tau \tilde{\xi}_1 \tilde{\xi}_2 \ldots \tilde{\xi}_{j_l}}{\xi_1 \xi_2 \ldots \xi_{j_l}}
\]

for some \( \tau \in \mathbb{T} \) and \( j_1, j_2, \ldots, j_l \) such that \( \tilde{\xi}_1, \tilde{\xi}_2 \ldots \tilde{\xi}_{j_l} \) and \( \xi_1, \xi_2 \ldots \xi_{j_l} \) are co-prime. This completes the proof.

Now, we are ready to prove the main theorem about the structure of rational inner functions on \( \theta(\mathbb{D}^d) \). This is Theorem 3.4. We restate it for the reader's convenience.

Theorem 3.4. Given a rational inner function \( f \) on \( \theta(\mathbb{D}^d) \), there exist a non-negative integer \( k \), \( \tau \in \mathbb{T} \) and a polynomial \( g \) with no zero in \( \theta(\mathbb{D}^d) \) such that

\[
f(p_1, \ldots, p_{d-1}, p_d) = \tau p_d^k \frac{g \left( \frac{p_{d-1}}{p_d}, \frac{p_{d-2}}{p_d}, \ldots, \frac{p_1}{p_d}, \frac{1}{p_d} \right)}{g(p_1, \ldots, p_{d-1}, p_d)}
\]

Conversely, any rational function of the form (3.3) is inner. Moreover, any inner function \( f \in \mathcal{A}(\theta(\mathbb{D}^d)) \) is a rational function of the form (3.3) with the additional property that \( g \) has no zeros in \( \theta(\mathbb{D}^d) \).
Proof. Let $f$ be a rational inner function on $\theta(\mathbb{D}^d)$. Then $f \circ \theta : \mathbb{D}^d \to \mathbb{T}$ is a $G$-invariant rational inner function on $\mathbb{D}^d$. Invoke Theorem 5.2.5 in [38] to get a polynomial $\xi$ with no zeros in $\mathbb{D}^d$, a $\tau_1 \in \mathbb{T}$, and $n = (n_1, n_2, \ldots, n_d)$ such that

$$f \circ \theta(z) = \tau_1 z^n \frac{\tilde{\xi}(z)}{\xi(z)}. \quad (3.4)$$

Applying Lemma 3.3, we can find a polynomial $\psi$ such that $f \circ \theta = \tau z^n \frac{\tilde{\psi}}{\psi}$ for some $\tau \in \mathbb{T}$ with $\tilde{\psi}$ and $\psi$ being co-prime.

Since $\xi$ has no zeros in $\mathbb{D}^d$, $\psi$ cannot have any zero in $\mathbb{D}^d$. Therefore $z^n \tilde{\psi}$ and $\psi$ are co-prime. By applying Lemma 3.2 we conclude that $z^n \tilde{\psi}$ and $\psi$ are both $G$-invariant polynomials. Note that, if $\varphi$ is a $G$-invariant polynomial, then by Chevally-Shephard-Todd theorem there exists a polynomial $\gamma \in \mathbb{C}[z]$ such that $\varphi(z) = \gamma \circ \theta(z)$. If $\gamma(z) = \sum_{\alpha} a_{\alpha} z^{\alpha}$, then

$$\varphi(z) = \sum_{\alpha} a_{\alpha} \prod_{j=1}^{d} \theta_j(z)^{\alpha_j}.$$

Recall that, $\theta_j(z) = E_j(z_1^m, \ldots, z_d^m)$ for $1 \leq j \leq d - 1$ and $\theta_d(z) = (z_1 \ldots z_d)^{m/t}$. Now, if $\varphi$ is not a constant, then $\gamma$ is non-constant polynomial and suppose the degree of $\gamma$ is $(r_1, \ldots, r_d)$. Therefore the degree of $\varphi$ will be of the form:

$$(n, \ldots, n) \text{ with } n = m \sum_{j=1}^{d-1} r_j + \frac{m}{t} r_d. \quad (3.5)$$

Suppose $\ell = (\ell, \ldots, \ell)$ is the degree of $\psi$ and it is of the form $(3.5)$ with $\ell = m \sum_{j=1}^{d-1} s_j + \frac{m}{t} s_d$. Then the degree of $z^n \tilde{\psi}$ is $(n_1 + \ell, n_2 + \ell, \ldots, n_d + \ell)$. Since $z^n \tilde{\psi}$ is a $G$-invariant polynomial, for every $k$, $n_k + \ell = m \sum_{j=1}^{d-1} r_j + \frac{m}{t} r_d$, for some non-negative integers $r_1, \ldots, r_d$. Hence, $n$ is of the form $(3.5)$ and we rename it as $n = (\kappa, \ldots, \kappa)$ where $\kappa = m \sum_{j=1}^{d-1} (r_j - s_j) + \frac{m}{t} (r_d - s_d)$.
Since $\psi$ is a $G$-invariant polynomial, by Chevally-Shephard-Todd theorem there exists a polynomial $g$ such that $\psi = g \circ \theta$. Now,

$$f \circ \theta(z) = \tau z^{n} \frac{g \circ \theta(z)}{g \circ \theta(z)} = \tau z^{n} z^{\ell} \frac{g \circ \theta(\frac{1}{z_1}, \frac{1}{z_2}, \ldots, \frac{1}{z_d})}{g \circ \theta(z_1, z_2, \ldots, z_d)}$$

$$= \tau z^{n} z^{\ell} \frac{g \left(\theta_1(\frac{1}{z}), \ldots, \theta_d-1(\frac{1}{z}), (\frac{1}{z_1 z_2 \ldots z_d})^{m/t}\right)}{g \left(p_1, \ldots, p_{d-1}, p_d\right)}$$

$$= \tau z^{n+\ell} \frac{g \left(\frac{p_{d-1}}{p_d}, \frac{p_{d-2}}{p_d}, \ldots, \frac{p_1}{p_d}, \frac{1}{p_d}\right)}{g \left(p_1, \ldots, p_{d-1}, p_d\right)}.$$

Therefore,

$$f(p_1, \ldots, p_{d-1}, p_d) = \tau p_d^{\ell_0} \frac{g \left(\frac{p_{d-1}}{p_d}, \frac{p_{d-2}}{p_d}, \ldots, \frac{p_1}{p_d}, \frac{1}{p_d}\right)}{g \left(p_1, \ldots, p_{d-1}, p_d\right)}$$

where $\ell_0 = m \sum_{j=1}^{d-1} r_j + r_d$.

Conversely, consider a function $f$ on $\theta(\mathbb{D}^d)$ of the form (3.3). From [38, Theorem 5.2.5], it can be seen that $g$ has no zeros in $\theta(\mathbb{D}^d) \cup \partial \theta(\mathbb{D}^d)$ except possibly on a subset of $\partial \theta(\mathbb{D}^d)$ of $\mu$ measure zero. For $(p_1, \ldots, p_d) \in \partial \theta(\mathbb{D}^d)$, we have $p_j = p_{d-j}p_d$ for $1 \leq j \leq d - 1$ and $|p_d| = 1$. Therefore, for almost every point $(p_1, \ldots, p_d)$ in $\partial \theta(\mathbb{D}^d)$, $|f(p_1, \ldots, p_d)| = 1$.

Hence $f$ is inner.

Finally, suppose $f \in A(\theta(\mathbb{D}^d))$ is inner. Then $f \circ \theta \in A(\mathbb{D}^d)$ is an inner function. Thus, by Theorem 5.2.5 in [38], $f \circ \theta$ is of the form (3.4) such that $g$ has no zeros in $\mathbb{D}^d$. Again, applying an argument similar to above, we get $f$ to be of the form (3.3) as well as the fact that the polynomial in the denominator has no zeros in $\theta(\mathbb{D}^d)$. This completes the proof.

Remark 3.5. One consequence of the structure of rational inner functions is that there are no non-constant rational inner functions dependent solely on the variables $p_1, p_2, \ldots, p_{d-1}$.

4. APPROXIMATION BY INNER FUNCTIONS

This section aims to discuss some approximation results by rational inner functions. We first show that the set of rational inner functions is dense in the norm-unit ball of $H^{\infty}(\theta(\mathbb{D}^d))$ with respect to the compact-open topology. The idea of the proof stems from of Theorem 5.5.1 in [38]. In recent work [76, Section 7], the density of rational inner functions in $\theta(\mathbb{D}^d)$ leads to deep result about projectivity of Hilbert modules over $A(\theta(\mathbb{D}^d))$ in certain category.
Theorem 4.1. Any holomorphic function \( f : \theta(\mathbb{D}^d) \to \overline{\mathbb{D}} \) can be approximated (uniformly on compact subsets) by rational inner functions in \( \mathcal{A}(\theta(\mathbb{D}^d)) \).

Proof. Choose \( \epsilon > 0 \). Let us fix a compact subset \( S \) of \( \theta(\mathbb{D}^d) \). Since the map \( \theta \) is a proper map, \( K = \theta^{-1}(S) \) is a compact subset of \( \mathbb{D}^d \). The function \( f \circ \theta \) is \( G \)-invariant and holomorphic in \( \mathbb{D}^d \). Also, for \( z \in K \), \( \theta(\sigma(z)) = \theta(z) \in K \) i.e., \( \sigma(z) \in K \) for every \( \sigma \in G \).

We choose a polynomial \( P \) such that \( |P| < 1 \) on \( \overline{\theta(\mathbb{D}^d)} \) and

\[
|f \circ \theta(z) - P(z)| < \epsilon \quad \text{for } z \in K.
\]

Consider

\[
P_G(z) = \frac{1}{|G|} \sum_{\sigma \in G} P(\sigma(z)).
\]

Clearly, \( |P_G| < 1 \) and

\[
|f \circ \theta(z) - P_G(z)| = \left| \frac{1}{|G|} \sum_{\sigma \in G} f \circ \theta(\sigma(z)) - P(\sigma(z)) \right| < \epsilon \quad (4.1)
\]

for \( z \) in \( K \). Note that \( P_G \) is a \( G \)-invariant polynomial and so, from the proof of Theorem 3.3 the degree of \( P_G \) is of the form \( \mathbb{N} \). Let \( n = (n_1, \ldots, n_d) \) be the degree of \( P_G \). Now we choose a monomial

\[
g(z) = e^{i\gamma}(z_1 z_2 \ldots z_d)^N
\]

where the positive integer \( N \geq n \) is such that \( m \) divides \( N \) with

\[
|g(z)| < \epsilon \quad \text{and} \quad |P_G(z)g(z)P_G\left(\frac{1}{\bar{z}}\right)| < 1 \quad \text{on } K. \quad (4.2)
\]

Since \( N \geq n \), \( Q(z) = g(z)P_G\left(\frac{1}{\bar{z}}\right) \) is a monomial times \( P_G(z) \). We show that \( Q \) is a \( G \)-invariant polynomial. To see this, let \( \sigma \in G \) be such that

\[
\sigma(z) = \left( e^{\frac{2\pi i \nu_1}{m}} z_{\eta(1)}, \ldots, e^{\frac{2\pi i \nu_d}{m}} z_{\eta(d)} \right)
\]

for some non-negative integers \( \nu_1, \ldots, \nu_d \) whose sum is divisible by \( t \) and \( \eta \in S_d \). Then

\[
g(\sigma(z)) = e^{i\gamma} e^{\frac{2\pi i}{m}(\nu_1 + \cdots + \nu_d)N}(z_1 \ldots z_d)^N = g(z)
\]

as \( m \) divides \( N \). Now, suppose \( P_G(z) = \sum_\alpha a_\alpha z^\alpha \). Then

\[
P_G\left(\frac{1}{\bar{z}}\right) = \sum_\alpha \frac{a_\alpha}{z_1^{\alpha_1} \ldots z_d^{\alpha_d}}
\]

where \( \alpha = (\alpha_1, \ldots, \alpha_d) \) are multi-indices with non-negative integers \( \alpha_j \) for all \( j \) and \( a_\alpha \)'s are zero except for finitely many \( \alpha \). Therefore,

\[
Q(\sigma(z)) = g(z) \sum_\alpha \frac{a_\alpha e^{\frac{2\pi i}{m}(\nu_1 \alpha_1 + \cdots + \nu_d \alpha_d)}}{z_{\eta(1)}^{\alpha_1} \ldots z_{\eta(d)}^{\alpha_d}}
\]

Again,

\[
P_G(z) = P_G(\sigma(z)) = P_G\left( e^{\frac{2\pi i \nu_1}{m}} z_{\eta(1)}, \ldots, e^{\frac{2\pi i \nu_d}{m}} z_{\eta(d)} \right)
\]
implies that
\[
P_G\left(\frac{1}{z}\right) = P_G\left(e^{\frac{2\pi i \nu_1}{m} \frac{1}{z(1)}}, \ldots, e^{\frac{2\pi i \nu_d}{m} \frac{1}{z(\eta(d))}}\right)
= \sum_{\alpha} \bar{\alpha}^{\nu_1} e^{\frac{2\pi i \nu_1 \cdot \cdots \cdot \nu_d \cdot \bar{\alpha}}{m \cdot z(1) \cdots z(\eta(d))}}.
\]
Thus \(Q(\sigma(z)) = Q(z)\). Consider the function
\[
\psi_\epsilon(z) = \frac{g(z) + P_G(z)}{1 + Q(z)}.
\]
Clearly, \(\psi_\epsilon\) is a \(G\)-invariant rational function. Note that, on \(T^d\), \(|Q| < 1\) as \(|P_G| < 1\) and so, the denominator of \(\psi_\epsilon\) is non-vanishing in \(\overline{D}_d\) which also shows that \(\psi_\epsilon \in \mathcal{A}(D_d)\).
Also, for \(z \in T^d\) it easy clear that \(w = P_G(z) \in D, \overline{w} = P_G(\frac{1}{z})\) and \(|g(z)| = 1\) and so \(|\psi_\epsilon(z)| = 1\). Moreover, for \(z \in K\)
\[
|f \circ \theta(z) - \psi_\epsilon(z)| \leq |f \circ \theta(z) - P_G(z)| + |P_G(z) - \psi_\epsilon(z)|
< \epsilon + |P_G(z) - g(z) + P_G(z)| + |Q(z)| < 5\epsilon \quad \text{using (4.1) and (4.2)}.
\]
Since the numerator and the denominator of the holomorphic function \(\psi_\epsilon\) are \(G\)-invariant holomorphic polynomials by the Chevally-Shephard-Todd theorem, there exists a rational function \(\Psi_\epsilon\) on \(\theta(D_d)\) such that
\[
\psi_\epsilon = \Psi_\epsilon \circ \theta.
\]
It is clear that \(\Psi_\epsilon\) is a rational inner function in \(\mathcal{A}(\theta(D_d))\) and
\[
|f - \Psi_\epsilon| < 5\epsilon
\]
on the compact set \(S\). This completes the proof. \(\square\)

A landmark theorem of Fisher \[17\] says that the uniform closure of the convex hull of finite Blaschke products is the closed norm-unit ball of the disc algebra. In \[39\], Rudin showed that the closed (with respect to sup-norm) convex hull of the inner functions on the polydisc is the norm-unit ball of the polydisk algebra. He proved it in a more general setting, namely, on compact abelian groups (note that \(T^d\) is a compact abelian group). But the Shilov boundary \(\partial \theta(D_d)\) of \(\theta(D_d)\) is not necessarily orientable. For example, when \(G = G(1, 1, d)\), the Shilov boundary \(\partial \theta(D_d)\) of associated quotient domain \(\theta(D_d)\) which is the symmetrized polydisk, is non-orientable for each even number \(d\), see \[33\] and hence \(\partial \theta(D_d)\) cannot have a topological group structure. Therefore, Rudin’s arguments cannot be applied for such quotient domains to get a Fisher type approximation result. Instead, here we give an \(L^2\)-norm approximation on \(\partial \theta(D_d)\). Such a result is known for the \(d\)-dimensional Euclidean ball in \[11\].

Recall that, \(H^\infty(\mathbb{D}_d)^G\) is the collection of all \(G\)-invariant bounded analytic functions on \(\mathbb{D}_d\). Since the radial limits of functions in \(H^\infty(\mathbb{D}_d)^G\) exist almost everywhere on \(T^d\), we identify the \(H^\infty(\mathbb{D}_d)\) functions with their radial limits and consequently as bounded
\[ \nu \text{-measurable functions on } \mathbb{T}^d. \text{ Denote by } \mathcal{Q}^G \text{ the closed convex hull (with respect to the } L^2(\nu) \text{-norm on } \mathbb{T}^d \text{) of the } G \text{-invariant rational inner functions in } \mathcal{A}(\mathbb{D}^d). \]

**Theorem 4.2.** Any function in the closed norm-unit ball of \( H^\infty(\theta(\mathbb{D}^d)) \) can be approximated by convex combinations of rational inner functions in \( \mathcal{A}(\theta(\mathbb{D}^d)) \) in \( L^2(\partial \theta(\mathbb{D}^d), \mu) \), where the measure \( \mu \) on the Shilov boundary \( \partial \theta(\mathbb{D}^d) \) is the push-forward by \( \theta \) of \( \nu \) on \( \mathbb{T}^d \).

**Proof.** Note that any function \( g \) in \( H^\infty(\theta(\mathbb{D}^d)) \) can be identified with the \( G \)-invariant function \( g \circ \theta \) in \( H^\infty(\mathbb{D}^d) \) with
\[
\|g\|_{H^\infty(\theta(\mathbb{D}^d))} = \|g \circ \theta\|_{H^\infty(\mathbb{D}^d)}.
\]
Therefore, it is enough to prove that any function in the closed norm-unit ball of \( H^\infty(\mathbb{D}^d)^G \) can be approximated by convex combination of \( G \)-invariant rational inner functions in \( \mathcal{A}(\mathbb{D}^d) \) in \( L^2(\mathbb{T}^d, \nu) \).

If possible, suppose that there exists \( f \) in the closed norm-unit ball of \( H^\infty(\mathbb{D}^d)^G \) such that \( f \notin \mathcal{Q}^G \). Then by the Hahn-Banach separation theorem, there exist \( g \in L^2(\mathbb{T}^d, \nu) \) and real numbers \( \gamma_1 \) and \( \gamma_2 \) such that
\[
\text{Re} \int_{\mathbb{T}^d} fg d\nu(\zeta) < \gamma_1 < \gamma_2 < \text{Re} \int_{\mathbb{T}^d} hg d\nu(\zeta) \tag{4.3}
\]
for every \( h \in \mathcal{Q}^G \). Invoke Theorem 4.1 to get a sequence \( \{u_k\} \) of \( G \)-invariant rational inner functions in \( \mathcal{A}(\mathbb{D}^d) \) such that \( u_k \to f \) uniformly on compact subsets of \( \mathbb{D}^d \). Let \( \psi \) be a polynomial in \( z_1, \ldots, z_d \) and \( \overline{z}_1, \ldots, \overline{z}_d \). Then, the Cauchy integral formula gives the following:
\[
\int_{\mathbb{T}^d} u_j(\zeta) \psi(\zeta) d\nu(\zeta) \longrightarrow \int_{\mathbb{T}^d} f(\zeta) \psi(\zeta) d\nu(\zeta).
\]
We know that the polynomials in \( z_1, \ldots, z_d \) and \( \overline{z}_1, \ldots, \overline{z}_d \) are dense in \( L^1(\mathbb{T}^d, \nu) \). Hence
\[
\int_{\mathbb{T}^d} u_j(\zeta) \psi(\zeta) d\nu(\zeta) \longrightarrow \int_{\mathbb{T}^d} f(\zeta) \psi(\zeta) d\nu(\zeta)
\]
for every \( \psi \in L^1(\mathbb{T}^d, \nu) \). But \( g \in L^2(\mathbb{T}^d, \nu) \) and hence \( g \in L^1(\mathbb{T}^d, \nu) \). Therefore, we have
\[
\int_{\mathbb{T}^d} u_j(\zeta) g(\zeta) d\nu(\zeta) \longrightarrow \int_{\mathbb{T}^d} f(\zeta) g(\zeta) d\nu(\zeta).
\]
Since \( u_j \in \mathcal{Q}^G \), equation (4.3) implies that
\[
\text{Re} \int_{\mathbb{T}^d} f(\zeta) g(\zeta) d\nu(\zeta) \leq \gamma_1 < \gamma_2 \leq \text{Re} \int_{\mathbb{T}^d} f(\zeta) g(\zeta) d\nu(\zeta)
\]
which is a contradiction. Thus, \( f \in \mathcal{Q}^G \). This completes the proof. \qed
4.1. A Comment for operator-valued case. So far, we have discussed inner functions and used them to approximate the scalar-valued functions. The next step is to explore these concepts in the context of operator-valued setting. What we have done for scalar-valued case cannot be directly applied to the operator-valued case, requiring us to adopt a different approach.

Carathéodory’s approximation theorem for the open unit disc has been proven using the Pick-Nevanlinna interpolation problem, as shown in [19], which extends seamlessly to the matrix-valued case. Recently, this theorem has also been proven using dilation methods [4] and the Herglotz Representation theorem [6]. The dilation theory and Pick-Nevanlinna interpolation theory have not been explored for all quotient domains, but a version of Herglotz representation has been established; see [6]. However, the method to prove matrix-valued Carathéodory approximation via Herglotz representation theory does not work in several variables.

Nevertheless, Pick-Nevanlinna interpolation is known for a particular quotient domain, namely, the symmetrized bidisc $\mathbb{G}_2$. Recently, it has been established that if the Carathéodory approximation theorem holds for the matrix-valued case, then the result can be extended to the operator-valued case using approximation arguments. Therefore, we will focus on discussing the Carathéodory approximation theorem for the matrix-valued case.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be points in $\Omega \subset \mathbb{C}^2$ and $A_1, A_2, \ldots, A_n$ be points in the closed operator-norm unit ball of $M_{M \times N}(\mathbb{C})$. The Pick-Nevanlinna interpolation problem asks necessary and sufficient conditions for the existence of a function $\psi$ which is analytic in $\Omega$ with $\|\Psi(z)\| \leq 1$ for all $z \in \Omega$ and interpolates the data i.e.,

$$\Psi(\lambda_j) = A_j \quad (j = 1, 2, \ldots, n).$$

When $\Omega = \mathbb{D}$, the problem was solved over a century ago, [34, 36]. Moreover, it was also shown that a solvable data on $\mathbb{D}$ always has a rational inner solution. This problem is studied on various domains. Recently, this problem has been studied by Agler-Young [3] and Bhattacharyya-Sau [5] in the symmetrized bidisc. We take advantage of the Pick-Nevanlinna interpolation problem as a tool to leverage to prove the approximation results on $\mathbb{G}_2$. Let us denote the Shilov boundary of $\mathbb{G}_2$ by $\partial \mathbb{G}_2$.

**Definition 4.3.** We say that a rational map $\Phi = ((\Phi_{ij})) : \mathbb{G}_2 \to M_{M \times N}(\mathbb{C})$ is

1. iso-inner if $\Phi(p_1, p_2)^*\Phi(p_1, p_2) = I_N$ a.e. on $\partial \mathbb{G}_2$;
2. coiso-inner if $\Phi(p_1, p_2)\Phi(p_1, p_2)^* = I_M$ a.e. on $\partial \mathbb{G}_2$.

In [25], it is shown that any solvable data with initial nodes in $\mathbb{G}_2$ and the final nodes in the set of square matrices has rational inner solution. We notice that similar outcome can be obtained for rectangular matrices as final nodes and which is the following:

**Theorem 4.4.** A solvable matrix Pick-Nevanlinna problem with initial nodes in $\mathbb{G}_2$ and the final nodes in the closed operator-norm unit ball of $M_{M \times N}(\mathbb{C})$ has a rational iso-inner or coiso-inner solution.

Having Theorem 4.4 on $\mathbb{G}_2$ in hand, we have the following approximation result.
Theorem 4.5. Any holomorphic function $f : G_2 \to M_{M \times N}(\mathbb{C})$ with $\|f(p_1, p_2)\| \leq 1$ for all $(p_1, p_2) \in G_2$, can be approximated (uniformly on compact subsets) by matrix-valued rational iso-inner or coiso-inner functions.

Proof. We sketch the proof. Choose a countable dense subset $\{\lambda_1, \lambda_2, \ldots\}$ of $G_2$ and set up the interpolation problem $\{\lambda_j \to f(\lambda_j)\}_{j=1}^n$, for each $n \geq 1$. Since the data is solvable by $f$, according to Proposition 4.4, there exists a rational iso-inner or coiso-inner function $f_n$ on $G$ such that

$$f_n(\lambda_j) = f(\lambda_j) \quad \text{for all } j = 1, 2, \ldots, n.$$  

Note that each $f_n$ is bounded by 1. Thus, the family

$$\mathcal{F} := \{f_n : n \in \mathbb{N}\}$$

is uniformly bounded. By Montel’s Theorem, there exists a subsequence of $\mathcal{F}$ that converges uniformly on each compact subset of $G_2$. An application of Arzela-Ascoli theorem completes the proof. \hfill \square

We shall end with the following remark.

Remark 4.6. In general, an analogue of Theorem 4.4 on a quotient domain $\theta(\mathbb{D}^d)$, is an open problem.

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DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF SCIENCE, BANGALORE 560012, INDIA
Email address: mainakb@iisc.ac.in

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MANITOBA, WINNIPEG R3T 2N2, CANADA
Email address: poornendu.kumar@umanitoba.ca