Thick simplices and quasicategories

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Dedicated to André Joyal on the occasion of his 70th birthday

1 Thick simplices

In the study of quasicategories, an important role is played by the simplicial sets

$$\Delta^n = \cosk_0 \Delta^n.$$ 

In other words, $\Delta^n$ is the simplicial set whose $k$-simplices are the sequences

$$(i_0, \ldots, i_k) \in \{0, \ldots, n\}^{k+1}.$$ 

We call these simplicial sets the thick simplices. They go by different names in the literature: Rezk [3] denotes them $E(n)$, while Joyal and Tierney [2] use the notation $\Delta'[n]$.

The thick simplex $\Delta^k$ is the nerve of the groupoid obtained by localizing the category $[k]$, that is, adjoining inverses for all non-identity morphisms. (This is the groupoid whose objects are the vertices of $\Delta^k$, and such that the set of morphisms from $i$ to $j$ has exactly one element for each pair of vertices $i$ and $j$.)

A $k$-simplex of $\Delta^n$ is non-degenerate if $i_{j-1} \neq i_j$ for all $1 \leq j \leq k$; $\Delta^n$ has $(n+1)n^k$ non-degenerate $k$-simplices. For $n = 0$, $\Delta^0 = \Delta^0$.

The thick 1-simplex $\Delta^1$ has two non-degenerate simplices $\delta_k = (0, 1, 0, 1, \ldots)$ and $\delta_k = (1, 0, 1, 0, \ldots)$ in each dimension $k \geq 0$, and $\Delta^1$ may be identified with the infinite-dimensional sphere $S^\infty$.

Recall that the horn $\Lambda_i^n \subset \Delta^n$ is the simplicial set obtained by removing the face $\partial_i \Delta^n$ from the boundary $\partial \Delta^n$ of $\Delta^n$.

If $X_\bullet$ is a simplicial set, form the coend

$$X_\bullet \times_\Delta \Delta^\bullet = \int_{n\in \Delta} X_n \times \Delta^n.$$
(This simplicial set is denoted $k_iX_\bullet$ by Joyal and Tierney [2].) In particular, we have the thick horns and the thick boundary

$$
\Lambda^n_i = \wedge^n_i \times^\Delta \Delta^* \subset \partial \Delta^n = \partial \Delta^n \times^\Delta \Delta^*.
$$

A simplex $(i_0, \ldots, i_k)$ lies in $\partial \Delta^n$ if and only if $\{i_0, \ldots, i_k\}$ is a proper subset of the set $\{0, \ldots, n\}$ of vertices of $\Delta^n$, and in $\Lambda^n_i$ if and only if $\{i_0, \ldots, i_k\}$ is a proper subset of the set $\{0, \ldots, n\}$ of vertices of $\Delta^n$ not equal to $\{0, \ldots, i, \ldots, n\}$.

Consider the following cofibrations (inclusions) of simplicial sets:

a) the inclusion of the boundary

$$
\mu_n : \partial \Delta^n \hookrightarrow \Delta^n;
$$

b) a thickened form of this inclusion

$$
\mu'_n : \partial \Delta^n \cup \Delta^n \hookrightarrow \Delta^n;
$$

c) the inclusion of a horn

$$
\lambda_{n,i} : \wedge^n_i \hookrightarrow \Delta^n;
$$

d) a thickened form of this inclusion

$$
\lambda'_{n,i} : \Lambda^n_i \cup \Delta^n \hookrightarrow \Delta^n.
$$

Recall the following result.

**Proposition 1** (Proposition 1.20, Joyal and Tierney [2])  **The inclusion**

$$
\mu'_n : \partial \Delta^n \cup \Delta^n \hookrightarrow \Delta^n
$$

**is a trivial cofibration of simplicial sets.**

The proof of this proposition comes down to the statement that the geometric realization of $\partial \Delta^n \cup \Delta^n$ is contractible, and ultimately, that $\Delta^n$ is contractible for all $n \geq 0$. This is the case because $\Delta^n$ is the nerve of a groupoid equivalent to the trivial group.

Let $X_\bullet$ be a simplicial set. The proposition immediately implies, by induction on the simplices of $X_\bullet$, that the inclusion

$$
X_\bullet \hookrightarrow X_\bullet \times^\Delta \Delta^*
$$

is a trivial cofibration.

If $U = \{ U_\alpha \}$ is an open cover of a space $X$, and $A$ is a sheaf of abelian groups on $X$, let $\check{C}^\bullet(U, A)$ be the Čech-complex

$$
\check{C}^k(U, A) = \prod_{\alpha_0, \ldots, \alpha_k} \Gamma(U_{\alpha_0} \cap \cdots \cap U_{\alpha_k}, A),
$$
and let $\tilde{C}^*(U, A) \subset \tilde{C}^*(U, A)$ be the subcomplex of antisymmetric Čech-cochains. Serre [4, §20, Proposition 2] proves that the inclusion of complexes

$$\tilde{C}^*(U, A) \hookrightarrow \tilde{C}^*(U, A)$$

is a quasi-isomorphism. Proposition 1 may be viewed as a non-abelian analogue of this result.

## 2 Expansions and inner expansions

Our goal is to refine Proposition 1, in such a way as to yield more combinatorial proofs of certain basic results in the theory of quasicategories.

**Definition 2** An expansion $S \hookrightarrow T$ is a map of simplicial sets such that there exists a filtration

$$S = S_0 \subset S_1 \subset S_2 \subset \cdots \subset T$$

satisfying the following conditions:

a) $T = \bigcup_k S_k$;

b) there are maps $\bigwedge^n_k \rightarrow S_{k-1}$ such that the following diagram is a pushout square:

$$\begin{array}{ccc}
\bigwedge^n_k & \rightarrow & S_{k-1} \\
\downarrow & & \downarrow \\
\Delta^n_k & \rightarrow & S_k
\end{array}$$

The expansion is **inner** if $0 < i_k < n_k$ for all $k$.

Every expansion is a trivial cofibration. The first part of the following proposition is thus a sharpening of Proposition 1.

**Proposition 3**

a) The inclusion

$$\iota_n : \partial \Delta^n \cup \Delta^n \hookrightarrow \Delta^n$$

is an expansion.

b) If $0 < i < n$, the inclusion

$$\iota_{n,i} : \Delta_i^n \cup \Delta^n \hookrightarrow \Delta^n$$

is an inner expansion.
Proof  We start with the proof of Part a). Partition the non-degenerate simplices of $\Delta^n$ into the following disjoint subsets: a $k$-simplex $s = (i_0 \ldots i_k)$ of $\Delta^n$ has

a) type $P_k$ if it is contained in $\partial \Delta^n \cup \Delta^n$;

b) type $Q_{k,m}$, $0 \leq m < n$, if there exists $m \geq 0$ (which is necessarily uniquely determined by $s$) such that

i) $i_j = j$ for $0 \leq j \leq m$, and

ii) $\{i_{m+1}, \ldots, i_n\} = \{m, \ldots, n\}$;

c) type $R_k$ otherwise.

Write $Q_k$ for the unions of the types $Q_{k,m}$ over all values of $m$.

We show that $\Delta^n$ is obtained by adjoining the simplices of type $Q_{k,m}$ to $\partial \Delta^n \cup \Delta^n$ in order first of increasing $k$, then of decreasing $m$. (The order in which the simplices are adjoined within the sets $Q_{k,m}$ is unimportant.)

Given a simplex $s = (i_0, \ldots, i_k)$ of type $R_k$, let $m$ be the largest integer such that $i_j = j$ for $j < m$. Thus

$$s = (0, \ldots, m-1, i_m, \ldots, i_k),$$

and $i_m \neq m$. The infimum $\ell$ of the set $\{i_m, \ldots, i_k\}$ equals $m$: it cannot be any larger, or the simplex would have type $P_k$, and it cannot be any smaller, or the simplex would have type $Q_{k,\ell}$. Define the simplex

$$s' = (0, \ldots, m, i_m, \ldots, i_k)$$

of type $Q_{k+1,m}$. We have $s = \partial_m s'$.

If $m$ occurs more than once in the sequence $\{i_m, \ldots, i_k\}$, then all faces of the simplex $s'$ are either degeneracies of $(k-1)$-simplices, or of type $P_k$ or $Q_k$. If $m$ does occur just once in this sequence, say $i_\ell = m$, then it is still the case that all faces of the simplex $s'$ other than $\partial_m s'$ and $\partial_{\ell+1} s'$ are either degeneracies of $(k-1)$-simplices, or of type $P_k$ or $Q_k$. The face

$$\partial_{\ell+1} s' = (0, \ldots, m, i_m, \ldots, \hat{i}_\ell, \ldots, i_k)$$

is of type $R_k$: it is a face of a simplex of type $Q_{k+1,m'}$, where $m' > m$.

In any case, all faces of $s'$ are simplices of type $P_j$ or $Q_j$, $j \leq k$, and simplices of type $Q_{k+1,m'}$, where $m' > m$. This completes the proof of Part a).

We now turn to the proof of Part b). Partition the non-degenerate simplices of $\Delta^n$ into the following disjoint subsets: a $k$-simplex $s = (i_0 \ldots i_k)$ of $\Delta^n$ has
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a) type $P_k$ if it is contained in $\Delta^n \cup \Delta^n$;
b) type $Q_{k,m}$, $m \geq 0$ even, if it is not of type $P_k$, has the form

$$(i_0, \ldots, i_{j-1}, i_{j}, j, j', \ldots, i_{j+m+2}, \ldots, i_k),$$

and $i_{j+m+2}$ does not equal $j'$;
c) type $\tilde{Q}_{k,m}$, $m \geq 0$, if it is not of type $P_k$, has the form

$$(i_0, \ldots, i_{j-1}, i_{j}, j, j', \ldots, i_{j+m+2}, \ldots, i_k),$$

and $i_{j+m+1}$ equals neither $j$ nor $j'$;
d) type $R_k$ otherwise.

Write $Q_k$ and $\tilde{Q}_k$ for the unions of the types $Q_{k,m}$ and $\tilde{Q}_{k,m}$ over all values of $m$. We show that $\Delta^n$ is obtained by adjoining the simplices of type $Q_k$ and $\tilde{Q}_k$ in order of increasing $k$, with the simplices of $Q_k$ adjoined before those of $\tilde{Q}_k$. (The order in which the simplices are adjoined within the sets $Q_k$ and $\tilde{Q}_k$ is unimportant.)

Let $s = (i_0, \ldots, i_k)$ be a simplex of type $R_k$, and let $j' = i_j$. We will assign a simplex $s'$ of type $Q_{k+1}$ or $\tilde{Q}_{k+1}$ such that $s$ is an inner face of $s'$.

First consider the case in which $i_{j-1} = j$. Let $m$ be the largest even number such that $s$ has the form

$$(i_0, \ldots, i_{j-2}, i_{j}, j, j', \ldots, i_{j+m+1}, \ldots, i_k),$$

and define $s'$ to be the simplex

$$s' = (i_0, \ldots, i_{j-2}, i_{j}, j, j', \ldots, i_{j+m+1}, \ldots, i_k)$$

of type $Q_{k+1,m}$. The condition $0 < j < n$ implies that $s = \partial_{j+m+1}s'$ is an inner face of $s'$: otherwise we would have $k = j + m$, and $\{i_0, \ldots, i_k\} = \{i_0, \ldots, i_j\}$, which is a proper subset of $\{0, \ldots, n\}$.

If $i_{j-1} \neq j$, we let $m$ be the largest natural number such that $s$ has the form

$$(i_0, \ldots, i_{j-1}, i_{j}, j, j', \ldots, i_{j+m+1}, \ldots, i_k),$$

and define $s'$ to be the simplex

$$s' = (i_0, \ldots, i_{j-1}, i_{j}, j, j', \ldots, i_{j+m+1}, \ldots, i_k)$$
of type $\hat{Q}_{k+1,m}$. The condition $0 < j < n$ again implies that $s = \partial s'$ is an inner face of $s'$.

Every face of a simplex $s'$ of type $Q_{k+1}$ other than $\partial_{k+m+1}s'$, is either a degeneracy of a $(k - 1)$-simplex, or of type $P_k$, or type $\hat{Q}_k$.

Likewise, every face of a simplex $s'$ of type $\hat{Q}_{k+1,m}$, $m > 0$, other than $\partial s'$, is either a degeneracy of a $(k - 1)$-simplex, or of type $P_k$, or type $Q_k$ or type $\hat{Q}_k$.

It remains to consider a simplex $s'$ of type $\hat{Q}_{k+1,0}$. The face $\partial s'$, $\ell > j$, is either a degeneracy of a $(k - 1)$-simplex, or of type $P_k$, or type $Q_k$. The face $\partial s'$, $\ell < j$, is either a degeneracy of a $(k - 1)$-simplex, or the face $\partial_{j+1}s''$ of the simplex

$$s'' = (i_0, \ldots, \hat{i}_\ell, \ldots, i_{j-1}, j, j', j, j+2, \ldots, i_k)$$

of type $Q_{k+1,0}$. This completes the proof of the proposition.

Corollary 4  If $n > 1$, the inclusion $\Lambda^n_i \hookrightarrow \Lambda^n$ is an inner expansion for all $0 \leq i \leq n$.

Proof  The action of the symmetric group $S_{n+1}$ on the simplicial set $\Delta^n$ induces a transitive permutation of the subcomplexes $\Lambda^n_i$. Thus, it suffices to establish the result when $0 < i < n$, in which case we may factor the inclusion $\Lambda^n_i \hookrightarrow \Lambda^n$ as the inner expansion

$$\Lambda^n_i \hookrightarrow \Lambda^n_i \cup \Delta^n$$

followed by the inner expansion $\lambda_{n,i} : \Lambda^n_i \cup \Delta^n \hookrightarrow \Delta^n$.

3  A Künneth theorem

Let $\Delta^{m,n} = \Delta^m \times \Delta^n$ be the prism. There are two different thick prisms. The first of these is the product of two thick simplices, which is itself a thick simplex:

$$\Delta^m \times \Delta^n \simeq \Delta^{m+n+m+n}.$$ 

In fact, the vertices of the product are the pairs $(i,j)$, where $0 \leq i \leq m$ is a vertex of $\Delta^m$, and $0 \leq j \leq n$ is a vertex of $\Delta^n$. This simplicial set contains the simplicial set

$$\Delta^{m,n} = \Delta^{m,n} \times \Delta^\bullet,$$

which is the union of $(m+n) \choose m$ copies of the thick simplex $\Delta^{m+n}$. The following proposition is an analogue of the Künneth theorem.
**Proposition 5** The inclusion $\Delta^{m,n} \hookrightarrow \Delta^m \times \Delta^n$ is an inner expansion.

**Proof** If $S \hookrightarrow T$ is an expansion of simplicial sets such that all of the simplices adjoined to $S$ in order to obtain $T$ have dimension greater than 1, then by Corollary 4, $S \times_\Delta \Delta^* \hookrightarrow T \times_\Delta \Delta^*$ is an inner expansion. Thus, to prove the proposition, it suffices to prove that the inclusion $\Delta^{m,n} \hookrightarrow \Delta^{mn+m+n}$ is an expansion.

Partition the non-degenerate $k$-simplices of $\Delta^{mn+m+n}$ into the following disjoint subsets: a $k$-simplex $s = ((i_0, j_0), \ldots, (i_k, j_k))$ of $\Delta^{mn+m+n}$ has

1. type $P_k$ if it is contained in $\Delta^{m,n}$;
2. type $Q_k$ if it is not of type $P_k$, and $i_0 = j_0 = 0$;
3. type $R_k$ otherwise.

We show that $\Delta^{mn+m+n}$ is obtained from $\Delta^{m,n}$ by adjoining the simplices of type $Q_k$ in order of increasing $k$. (The order in which the simplices are adjoined within the sets $Q_k$ is unimportant.)

If $s = ((i_0, j_0), \ldots, (i_k, j_k))$ is a simplex of type $R_k$, let $s'$ be the simplex $s' = ((0, 0), (i_0, j_0), \ldots, (i_k, j_k)) = (0, i_0, \ldots, i_k) \times (0, j_0, \ldots, j_k)$ of type $Q_{k+1}$. Thus $s = \partial_0 s'$, while every other face of $s'$ has type either $P_k$ or $Q_k$. This completes the proof.

4 Application to the theory of quasicategories

If $S_*$ and $X_*$ are simplicial sets, let $\text{Map}(S, X)$ be the set of simplicial maps from $S_*$ to $X_*$. If $i : S \hookrightarrow T$ is a cofibration of simplicial sets (that is, if $S_k \to T_k$ is a monomorphism for all $k \geq 0$), let $\langle i \mid f \rangle$ be the induced function from $\text{Map}(T, X)$ to $\text{Map}(S, X) \times_{\text{Map}(T, X)} \text{Map}(T, Y)$. 


We write \( \langle i \mid X \rangle \) in place of \( \langle i \mid f \rangle \), when \( f \) is the unique simplicial map from the simplicial set \( X_* \) to the terminal simplicial set \( \Delta^0 \).

**Definition 6** A simplicial set \( X_* \) is

- a **quasicategory** if the functions \( \langle \lambda_{n,i} \mid X \rangle \) are surjective for \( 0 < i < n \);
- a **Kan complex** if the functions \( \langle \lambda_{n,i} \mid X \rangle \) are surjective for \( n > 0 \) and \( 0 \leq i \leq n \).

A simplicial map \( f : X_* \rightarrow Y_* \) is a **trivial fibration** if the functions \( \langle \mu_n \mid f \rangle \) are surjective for \( n \geq 0 \).

If \( X_* \) is a simplicial set, denote by \( G(X)_* \) the simplicial set with \( n \)-simplices

\[
G(X)_n = \text{Map}(\Delta^n, X).
\]

The inclusions \( \Delta^n \hookrightarrow \Delta^m \) induce a natural simplicial map from \( G(X)_* \) to \( X_* \), with image \( G(X)_* \). A functor similar to \( G \) was introduced by Rezk [3], in his study of complete Segal spaces.

**Proposition 7** \( G(G(X))_* \cong G(G(X))_* \cong G(X)_* \).

**Proof** In order to prove that \( G(G(X))_* \) is isomorphic to \( G(X)_* \), it suffices to show that for all \( k, n \geq 0 \),

\[
\text{Map}(\Delta^k, \Delta^n) \cong \text{Map}(\Delta^k, \Delta^n).
\]

Since \( \Delta^k \) is the nerve of the groupoid obtained by localizing the category \([k]\), we see that \( \text{Map}(\Delta^k, \Delta^n) \) may be identified with the set of functors from the localization of \([k]\) to the localization of \([n]\). But a functor from the localization of \([k]\) to a groupoid \( G \) determines, and is determined by, a functor from \([k]\) to \( G \), i.e. by a \( k \)-simplex of the nerve \( N_* G \) of \( G \).

Applying the functor \( G_n \) to the composition of morphisms

\[
G(X)_* \rightarrow G(X)_* \rightarrow X_*,
\]

we obtain a factorization of the identity map of \( G(X)_n \):

\[
G(G(X))_n \cong G(X)_n \rightarrow G(G(X))_n \rightarrow G(X)_n.
\]

Since the functor \( G_n \) is a finite limit, it takes monomorphisms to monomorphisms. Thus the morphism from \( G(G(X))_n \) to \( G(X)_n \) is a monomorphism, and hence, since it has a section, an isomorphism.
The following theorem assembles results of Joyal [1], Joyal and Tierney [2] and Rezk [3]. Our approach is to reduce the proof to Proposition 3. Although these results are not new in the context of quasicategories, the approach developed here allows the development of a theory of quasicategories in the setting where sets are replaced by analytic spaces (or **Lie quasicategories**, by analogy with Lie groupoids, of which they are a generalization).

**Theorem 8**  Let $X$ be a quasicategory.

- The simplicial sets $G(X)_\bullet$ and $G(X)_\bullet$ are Kan complexes.
- The simplicial map $G(X)_\bullet \to G(X)_\bullet$ is a trivial fibration.

**Proof**  The simplicial set $G(X)_\bullet$ is a Kan complex by Corollary 4.

For $n > 0$, consider the assertions

$A_n$: for all $0 \leq i \leq n$, the function $G(X)_n \to \wedge_{n,i}(G(X))$ is surjective; and

$B_n$: for all $0 \leq i \leq n$, the function $G(X)_n \to \wedge_{n,i}(G(X)) \times \wedge_{n,i}(G(X))$ is surjective.

We will establish these assertions for all $n$ by induction: it will follow that $G(X)_\bullet$ is a Kan complex.

$A_1$ is clear. If $A_n$ holds, the composition

$$G(X)_n \times \wedge_{n,i}(G(X)) \to G(X)_n \times \wedge_{n,i}(G(X)) \times \wedge_{n,i}(G(X)) \to G(X)_n \to G(X)_n$$

is surjective, establishing $B_n$.

Suppose that $T$ is a finite simplicial set and $S \hookrightarrow T$ is an expansion obtained by attaching simplices of dimension at most $n - 1$ to $S$. If $B_{n-1}$ holds, the function $\langle S \hookrightarrow T \mid G(X) \to G(X) \rangle$ is surjective. In particular, applying this argument to the expansion $\Delta^0 \hookrightarrow \wedge^n_0$, we see that the function

$$\wedge_{n,i}(G(X)) \to G(X)_0 \times G(X)_0 \wedge_{n,i}(G(X)) \cong \wedge_{n,i}(G(X))$$

is surjective. In the commuting diagram

$$\begin{array}{ccc}
G(X)_n & \longrightarrow & \wedge_{n,i}(G(X)) \\
\downarrow & & \downarrow \\
G(X)_n & \longrightarrow & \wedge_{n,i}(G(X))
\end{array}$$
the solid arrows are surjective, hence so is the bottom arrow, establishing $A_n$, and completing the induction.

We may now prove that $G(X) \to G(X)$ is a trivial fibration. We must show that for each $n \geq 0$, the function

$$G(X)_n \to M_n(G(X)_n) \times_{M_n(G(X))} G(X)_n$$

is a surjection. By Proposition 7, this function may be identified with the function

$$G(G(X))_n \to M_n(G(G(X))_n) \times_{M_n(G(G(X))} G(X)_n,$$

and hence with $\langle \mu_n | G(X) \rangle$. As $\mu_n$ is an expansion and $G(X)$ is a Kan complex, the result follows.

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