NEW NUMERICAL SCHEME OF ATANGANA-BALEANU FRACTIONAL INTEGRAL: AN APPLICATION TO GROUNDWATER FLOW WITHIN LEAKY AQUIFER

J.D. DJIDA, I. AREA, AND A. ATANGANA

Abstract. Many physical problems can be described using the integral equations known as Volterra equations. There exists quite a number of analytical methods that can handle these equations for linear cases. For non-linear cases, a numerical scheme is needed to obtain an approximation. Recently a new concept of fractional differentiation was introduced using the Mittag-Leffler function as kernel, and the associate fractional integral was also presented. Up to this point there is no numerical approximation of this new integral in the literature. Therefore, to accommodate researchers working in the field of numerical analysis, we propose in this paper a new numerical scheme for the new fractional integral. To test the accuracy of the new numerical scheme, we first revisit the groundwater model within a leaky aquifer by reverting the time classical derivative with the Atangana-Baleanu fractional derivative in Caputo sense. The new model is solved numerically by using this new scheme.

1. Introduction

The concept of fractional differentiation with non-singular and non-local kernel has been suggested recently and is becoming a hot topic in the field of fractional calculus. The concept was tested in many fields including chaotic behaviour, epidemiology, thermal science, hydrology and mechanical engineering. The numerical approximation of this differentiation was also proposed in [3]. In the recent decade, the integral equations were revealed to be great mathematical tools to model many real world problems in several fields of science, Technology and Engineering. In many research papers under some conditions (see e.g. [5] [6] [7] and references therein), it was proven that there is equivalence between a given differential equation and its integral equation associate.

Recently [6] proposed a method based on a semi-discrete finite difference approximation in time and Galerkin finite element method in space. In this work we propose a new numerical approximation of Atangana-Baleanu integral which is the summation of the average of the given function and its fractional integral in Riemann-Liouville sense. This numerical scheme will be validate by solving the partial differential equation describing
the subsurface water flowing within a confined aquifer model with the new derivative with fractional order in time component.

This paper is organized as follows: In Section 2, for the convenience of the reader, we recall some definitions and properties of fractional Calculus within the scope of Atangana-Baleanu. In Section 3 a numerical approach of Atangana-Baleanu derivative with fractional order is introduced. In Section 4 as application of the new numerical approximation of Atangana-Baleanu fractional integral. We study analytically and numerically the model of groundwater flow within a leaky aquifer based upon the Mittag-Leffler function. Finally, Section 5 is dedicated to our perspectives and conclusions.

2. SOME DEFINITIONS OF THE ATANGANA-BALEANU FRACTIONAL DERIVATIVE AND INTEGRAL

We recall the definitions of the new derivative with non singular kernel and integral introduced by Atangana and Baleanu in the senses of Caputo and Riemann-Liouville derivatives [1, 4].

Let \((a, b) \subset \mathbb{R}\) and let \(u\) be a function of the Hilbert space \(L^2(a, b)\). We define by \(u'\) the derivative of \(u\) as distribution on \((a, b)\).

**Definition 2.1.** The Sobolev space of order 1 in \((a, b)\) is defined as
\[
H^1(a, b) = \{u \in L^2(a, b) \mid u' \in L^2(a, b)\}.
\]

**Definition 2.2.** Let \(\alpha \in (0, 1)\) and a function \(u \in H^1(a, b), b > a\). The Atangana-Baleanu fractional derivative in Caputo sense of order \(\alpha\) of \(u\) with a based point \(a\) is defined as
\[
(2.1) \quad ABCD^\alpha_t u(t) = B(\alpha) \int_a^t u'(s) E_\alpha \left[ -\alpha \frac{1}{1-\alpha} (t-s)^\alpha \right] ds,
\]
where \(B(\alpha)\) has the same properties as in Caputo and Fabrizio case, and is defined as
\[
B(\alpha) = 1 - \alpha + \frac{\alpha}{\Gamma(\alpha)},
\]
\(E_{\alpha,\beta}(\lambda z^\alpha)\) is the Mittag-Leffler function, defined in terms of a series as the following entire function
\[
(2.2) \quad E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\lambda z^\alpha)^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \quad \lambda < \infty, \text{ and } \beta > 0,
\]
\(\lambda = -\alpha(1-\alpha)^{-1}\).

The Mittag-Leffler functions appear in the solution of linear and nonlinear fractional differential equations [3]. The above definition is very helpful to discuss real world problems and will also have a great advantage when using the Laplace transform with initial condition. Now let us recall the definition of the Atangana-Baleanu fractional derivative in the Riemann-Liouville sense.
Definition 2.3. Let $\alpha \in (0,1)$ and a function $u \in H^1(a,b), b > a$. The Atangana-Baleanu fractional derivative in the Riemann-Liouville sense of order $\alpha$ of $u$ is defined as

$$ABRD_t^\alpha u(t) = \frac{B(\alpha)}{1-\alpha} \frac{d}{dt} \int_a^t u(s)E_{\alpha} \left[-\frac{\alpha}{1-\alpha} (t-s)^{\alpha} \right] ds.$$ 

Notice that, when the function $u$ is constant, we get zero.

Definition 2.4. The Atangana-Baleanu fractional integral of order $\alpha$ with base point $a$ is defined as

$$ABI_t^\alpha u(t) = \frac{1-\alpha}{B(\alpha)} u(t) + \frac{\alpha}{B(\alpha) \Gamma(\alpha)} \int_a^t u(s)(t-s)^{\alpha-1} ds.$$ 

3. Numerical approach of the Atangana-Baleanu integral with fractional order

In this section, we will start by given the discretization for Riemann-Liouville fractional integral. Next, following the same idea as, we introduce a numerical scheme to discretize the temporal fractional integral, and give the corresponding error analysis which will be used to give the numerical solution of the Atangana-Baleanu fractional integral and also to obtain the solution of the modified groundwater flow within a leaky aquifer.

Let $f \in C^2(a,b)$. Then $\int_a^b f(x)dx$ could be discretized as follows as

$$\int_a^b f(x)dx = \sum_{j=0}^n \int_{t_j}^{t_{j+1}} \frac{f(t_{j+1}) + f(t_j)}{2} dy = \sum_{j=0}^n \frac{f(t_{j+1}) + f(t_j)}{2} \int_{t_j}^{t_{j+1}} dy = \sum_{j=0}^n \frac{f(t_{j+1}) + f(t_j)}{2} (t_{j+1} - t_j).$$ 

On the other hand

$$\int_a^b f(x)dx = \sum_{j=0}^n \int_{t_j}^{t_{j+1}} f(t_{j+1}) dy = \sum_{j=0}^n f(t_{j+1})(t_{j+1} - t_j).$$

With the Riemann-Liouville fractional integral we have the following.

We choose $t \in [0,T]$, the fractional order is denoted by $\alpha \in (0,1)$, the step $\tau = \frac{T}{n}$, $(n \in \mathbb{N})$, and the grid points are $t_k, k \in \{0,1,2,\ldots,n\}$, where $t_k = k\tau$. Thus for $k \in \{0,1,2,\ldots,n\}$, we have

$$R^\alpha I_0^\alpha f(t_k) = \frac{1}{\Gamma(\alpha)} \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} f(y)(t_k - y)^{\alpha-1} dy.$$

If we consider the following discretization for the function $f$.

$$f(x) = \frac{f(x_{i+1}) + f(x_i)}{2},$$
by replacing (3.4) into (3.3) we obtain
\[
\int_0^t \mathcal{I}_t^\alpha \left( f(t) \right) = \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \frac{f(t_{j+1}) + f(t_j)}{2} (t_k - s)^{\alpha - 1} ds,
\]
\[
= \frac{\tau^\alpha}{\Gamma(\alpha + 1)} \sum_{j=0}^{k-1} \frac{f(t_{j+1}) + f(t_j)}{2} \left[ (k - j)^\alpha - (k - j + 1)^\alpha \right] + R_{k,\alpha}.
\]
Using the same approach as in [10], the error is given as
\[
R_{k,\alpha} = \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \frac{f(y) - f(t_{j+1}) + f(t_j)}{(t_k - y)^{1-\alpha}} - \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k-1} \frac{f(y) - \frac{f(t_{j+1}) - f(t_j)(t_{j+1} - t_j)}{(t_{j+1} - t_j)}}{(t_k - y)^{1-\alpha}}
\]
\[
= \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k-1} \frac{(t_{j+1} - t_j)}{(t_k - y)^{1-\alpha}} (f(y) - f'(\xi)) dy, \quad t_j < \xi < t_{j+1}.
\]
From the Taylor series at the point \( \xi \)
\[
f(y) = f(\xi) + y f'(\xi) + \cdots
\]
we can approximate
\[
f(y) - f'(\xi) \approx f(\xi) + y f'(\xi) - f'(\xi) = f(\xi) - f'(\xi)(y - 1).
\]
Now taking the norm, we get
\[
\| f(y) - f'(\xi) \| = \| f(\xi) - f'(\xi)(y - 1) \| \leq M,
\]
since the function \( f \) is differentiable. Hence
\[
\| R_{k,\alpha} \| = \frac{\tau}{\Gamma(\alpha + 1)} M t_k^\alpha.
\]
**Lemma 3.1.** Let \( f \in C^2[0, T] \). The Atangana-Baleanu fractional integral is given by
\[
\mathcal{A}_0^t \mathcal{I}_t^\alpha \left( f(t_k) \right) = \frac{1 - \alpha}{B(\alpha)} f(t_k) + \frac{\alpha \tau^\alpha}{B(\alpha) \Gamma(\alpha + 1)} \sum_{j=0}^{k-1} b_j^\alpha f(t_{j+1}) - f(t_{j+1}) + R_{k,\alpha},
\]
where \( |R_{k,\alpha}| \leq K t_k^\alpha \), \( k = 1, 2, 3, \ldots, n \), and \( b_j^\alpha = (j + 1)^\alpha - j^\alpha, \quad j = 0, 1, 2, \ldots, n \).

Using the second approach of the discretization (3.2), we obtain the following approximation
\[
\mathcal{A}_0^t \mathcal{I}_t^\alpha \left( f(t_k) \right) = \frac{1 - \alpha}{B(\alpha)} f(t_k) + \frac{\alpha \tau^\alpha}{B(\alpha) \Gamma(\alpha + 1)} \sum_{j=0}^{k-1} y_j (t_{j+1}) \left[ (k - j)^\alpha - (k - j - 1)^\alpha \right] + \tilde{R}_{k,\alpha},
\]
where
\[
\tilde{R}_{k,\alpha} = \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \frac{f(y) + f(t_{j+1})}{(t_k - y)^{1-\alpha}} dy,
\]
and
\[ |\tilde{R}_{k,\alpha}| \leq \frac{\tau}{\Gamma(\alpha)} t_k^\alpha \max_{0 \leq t \leq t_k} |f'(t)|. \]

If the function \( f \) is differentiable such that its Atangana-Baleanu fractional integral exists then
\[
\frac{d}{dt} \quad _0^A \mathcal{I}_t^\alpha (f(t)) = \frac{1 - \alpha}{B(\alpha)} f'(t) + \frac{\alpha}{B(\alpha) \Gamma(\alpha)} \int_0^t f(y)(t - y)^{\alpha - 1} dy.
\]
Then for \( k = 0, 1, 2, 3, \ldots, n \) the Atangana-Baleanu integral can be decomposed as follows
\[
(3.11) \quad _0^A \mathcal{I}_t^\alpha (f(t)) = \frac{1 - \alpha}{B(\alpha)} (f(t_k) - f(t_{k-1})) + \Psi(\alpha) \int_0^{t_k} \frac{f(s)}{(t_k - s)^{1-\alpha}} ds - \Psi(\alpha) \int_0^{t_k-1} \frac{f(s)}{(t_k - s)^{1-\alpha}} ds
\]
\[
= \frac{1 - \alpha}{B(\alpha)} (f(t_k) - f(t_{k-1})) + \Psi(\alpha) \int_0^{\tau} \frac{f(y) - f(\tau)}{(t_k - y)^{1-\alpha}} dy + \Psi(\alpha) \int_0^{\tau} \frac{f(y)}{(t_k - y)^{1-\alpha}} dy
\]
\[
= \Psi(\alpha) \int_0^{\tau} \frac{f(y) - f(\tau)}{(t_k - y)^{1-\alpha}} dy + \_0^A \mathcal{I}_t^\alpha (\beta(t_{k-1})),
\]
where \( \Psi(\alpha) = \frac{\alpha}{B(\alpha) \Gamma(\alpha)}. \) Nevertheless
\[
\int_0^{\tau} \frac{f(y)}{(t_k - y)^{1-\alpha}} dy = \int_0^{\tau} \frac{f(y)}{(t_k - y)^{1-\alpha}} dy + \int_0^{\tau} \frac{f(y) - f(\tau)}{(t_k - y)^{1-\alpha}} ds,
\]
where also
\[
\int_0^{\tau} \frac{f(y) - f(\tau)}{(t_k - y)^{1-\alpha}} ds = \int_0^{\tau} \frac{f(\xi)(y - \tau)}{(t_k - y)^{1-\alpha}} ds, \quad y < \xi < \tau.
\]
Let us now assume that \( f \) is two times differentiable on \([0, T]\). Then, we obtain
\[
|\Psi(\alpha) \int_0^{\tau} \frac{f(y) - f(\tau)}{(t_k - y)^{1-\alpha}} dy| \leq \frac{\tau^{1+\alpha}}{B(\alpha) \Gamma(\alpha)} b_{k-1}^{\alpha} \max_{0 \leq t \leq \tau} |f''(t)|.
\]
With the above relation in hand, we can conclude that
\[
\_0^A \mathcal{I}_t^\alpha (\beta(t_{k-1})) = \frac{1 - \alpha}{B(\alpha)} \beta(t_{k-1}) + \frac{\tau^\alpha}{B(\alpha) \Gamma(\alpha)} \sum_{j=1}^{k-1} b_{j-1}^{\alpha} \beta(t_{j-1}) + R'_{k,\alpha}.
\]
Here we have
\[
|R'_{k,\alpha}| \leq \frac{\tau^\alpha}{B(\alpha) \Gamma(\alpha)} t_{k-1}^{\alpha} \max_{0 \leq \xi \leq t_k} |f''(\xi)|, \quad 0 \leq \xi < t_k.
\]
Therefore, we have the following relationship
\[
(3.12) \quad \_0^A \mathcal{I}_t^\alpha f(t_{k-1}) = \frac{1 - \alpha}{B(\alpha)} \beta(t_{k-1}) + \frac{\tau^\alpha}{B(\alpha) \Gamma(\alpha)} \left(b_{k-1}^{\alpha} f(\tau) + \sum_{j=1}^{k-1} b_{j-1}^{\alpha} [f(t_{j-1}) - f(t_{j-1})] \right) + R^2_{k,\alpha},
\]
where \( |R^2_{k,\alpha}| < c_\alpha b_{k-1}^{\alpha} \tau^{\alpha+1} + c_\alpha \tau^2 b_{k-1}^{\alpha} \).
Finally the equation above can be reformulated as
\[(3.13)\]
\[\begin{aligned}
\mathcal{I}^\alpha_0 t^\alpha f(t_{k-1}) &= \frac{1 - \alpha}{B(\alpha)} \left( f(t_k - t_{k-1}) \right) + \frac{\tau^\alpha}{B(\alpha) \Gamma(\alpha)} \left\{ f(t_k) + \sum_{j=1}^{k-1} b_j^k - (b_{j-1}^k) f(t_{k-j}) \right\} + R_{k,\alpha},
\end{aligned}\]
where indeed \(|R_{k,\alpha}| \leq c_\alpha b_{k-1}^\alpha \tau^{\alpha+1}\).

4. Application to groundwater flow within leaky aquifer based upon Atangana-Baleanu fractional derivative

The concept of groundwater flow within the geological formation is a very complex physical problem and has attracted the attention of several scholars from different branches of sciences and technology. In particular the model portraying the movement of this subsurface water within the medium called leaky aquifer. In this section, we focus our attention on the model based on differentiation with non-local and non-singular kernel. To do this we consider the time derivative to be the time fractional derivative based on the Mittag-Leffler function. The new model will be analysed analytically and numerically. For analytical investigation, we shall focus on the analysis of existence and uniqueness of the solution of the new model. Then we apply the new numerical scheme to derive the numerical solution of the new model.

4.1. Analytical solution of the flow within the leaky aquifer based upon Atangana-Baleanu fractional derivative. Let \(\Omega = (a, b)\) be an open and bounded subset of \(\mathbb{R}^n\) \((n \geq 1)\), with boundary \(\partial \Omega\). For a given \(\alpha \in (0, 1)\), and a function \(\varphi(r,t) \in H^1(\Omega) \times [0,T]\), which represents the head, we seek \(\varphi\) such that the flow of water within the leaky aquifer is governed by
\[(4.1)\]
\[\begin{aligned}
\beta^2 \quad ABC \mathcal{D}^\alpha_t \varphi &= \partial_{rr} \varphi + \frac{1}{r} \partial_r \varphi - \frac{\varphi}{\varphi^2},
\end{aligned}\]
where \(\varphi = KDc\) and \(\beta^2 = Sc\varphi^{-2}\), \(S\) denotes the coefficient of storage, \(K\) denotes the conductivity.

The problem \((4.1)\) of groundwater flow within the leaky aquifer will be approached analytically and solved numerically using the implicit scheme.

4.2. Existence and uniqueness of the solution of the problem. In the following, we discuss the existence and uniqueness of solutions of the direct problem.
\[(4.2)\]
\[\begin{aligned}
\mathcal{D}^\alpha_t \varphi(t, r) &= g(t, \varphi(t, r)),
\varphi(r_c, 0) &= h(r) \quad \text{on} \quad [0,T] \times \partial \Omega,
\varphi(r_c, t) &= \varphi_c \quad \text{in} \quad \{0\} \times \Omega,
\end{aligned}\]
for the initial datum \(\varphi_c \in L^\infty(\Omega), \varphi_c > 0\), and where
\[(4.3)\]
\[g(r, t, \varphi(r, t)) = \beta^{-2} \left( \partial_{rr} \varphi(t, r) + \frac{1}{r} \partial_r \varphi(t, r) - \frac{\varphi(t, r)}{\varphi^2} \right).\]
Given \( \varphi_c \in L^\infty(\Omega) \), \( \varphi_c > 0 \), a solution of (4.2) is a positive function \( \varphi \in H^1(\Omega) \times [0, T] \) such that if we apply the fractional integral defined in (2.4) to (4.2) it yields

\[
\varphi(t, r) = \varphi(0) + \frac{1 - \alpha}{B(\alpha)} g(t, \varphi(t, r)) + \frac{\alpha}{B(\alpha) \Gamma(\alpha)} \int_0^t g(s, \varphi(s, r))(t - s)^{\alpha - 1} ds,
\]

for all \( t \in [0, T] \).

We want to use the contraction mapping theorem, so for this purpose we need to build a closed set \( \varepsilon \) of \( H^1(\Omega) \times [0, T] \) such that the nonlinear operator \( g \) be a contraction which maps \( \varepsilon \) into itself.

So we first show that \( g \) is a contraction mapping.

**Proposition 4.1.** The nonlinear operator \( g \in H^1(\Omega) \times [0, T] \) is locally Lipschitz.

**Proof.** We consider two bounded functions \( \varphi \) and \( \vartheta \) in \( H^1(\Omega) \times [0, T] \). Then,

\[
\|g(\varphi) - g(\vartheta)\|_{H^1} = \beta^{-2} \|\partial_r \varphi - \partial_r \vartheta + r^{-1}(\partial_r \varphi - \partial_r \vartheta) - \omega^{-2}(\varphi - \vartheta)\|.
\]

We have by the triangular inequality

\[
\|g(\varphi) - g(\vartheta)\|_{H^1} \leq \beta^{-2} \left\{ \|\partial_r \varphi - \partial_r \vartheta\| + \|r^{-1}(\partial_r \varphi - \partial_r \vartheta) - \omega^{-2}(\varphi - \vartheta)\| \right\}.
\]

As the derivative operator satisfy the Lipschitz conditions in \( H^1 \), hence there exist two positive constants \( c_1 \) and \( c_2 \) such that

\[
\|g(\varphi) - g(\vartheta)\|_{H^1} \leq \beta^{-2} c_1 \|\varphi - \vartheta\| + \beta^{-2} c_2 \|r^{-1}\| \|\varphi - \vartheta\| - \beta^{-2} \omega^{-2} \|\varphi - \vartheta\|,
\]

\[
\leq C \|\varphi - \vartheta\|,
\]

where \( C = |\beta^{-2}(c_1 + c_2 \|r^{-1}\| - \omega^{-2})| < 1 \). Then \( g \) is a contraction mapping. \( \Box \)

**Theorem 4.2.** For a given initial datum \( \varphi_c \in L^\infty(\Omega) \), \( \varphi_c > 0 \), there exists an unique positive solution \( \varphi \) of (4.2) on \( H^1(\Omega) \times [0, T] \), for all \( t < T < \infty \), and

\[
\lim_{t \to T} \|\varphi(t, r)\|_{L^\infty} \to \infty.
\]

**Proof.** We shall follow the idea of [12]. Since the nonlinear operator \( g \) is locally Lipschitz, for \( \bar{\varphi}_c = \|\varphi_c\|_{H^1} \) there exists \( C_{\bar{\varphi}_c} \) such that \( 0 < C_{\bar{\varphi}_c} < \infty \) and

\[
\|g(\varphi) - g(\vartheta)\|_{H^1} \leq C_{\bar{\varphi}_c} \|\varphi - \vartheta\|_{H^1}.
\]

Let \( T_1 > 0 \) be a constant such that \( T_1 < \frac{1}{C_{\bar{\varphi}_c}} \).

Set

\[
\varepsilon = \{ \varphi \in H^1(\Omega) \times [0, T]; \|\varphi\|_{H^1} \leq \bar{\varphi}_c, \text{for all } t \in [0, T_1] \},
\]

endowed with the norm

\[
\|\varphi\|_\varepsilon = \sup_{0 \leq t \leq T_1} \|\varphi\|_{H^1},
\]

then \( \varepsilon \) is a closed convex subset of \( H^1(\Omega) \times [0, T] \).
Now we consider the following associated problem of (4.4) defined on \( \varepsilon \),

\[
\psi(\varphi) = \varphi_c + \frac{1 - \alpha}{B(\alpha)} g(t, \varphi(t, r)) + \frac{\alpha}{B(\alpha) \Gamma(\alpha)} \int_0^t g(s, \varphi(s, r))(t - s)^{\alpha - 1}ds.
\]

For all \( \varphi \in \varepsilon \) and \( t \geq 0 \) we have

\[
\|\psi(\varphi)\|_\varepsilon = \sup_{0 \leq t \leq T_1} \left\| \varphi + \frac{1 - \alpha}{B(\alpha)} g(t, \varphi(t, r)) + \Psi(\alpha) \int_0^t g(s, \varphi(s, r))(t - s)^{\alpha - 1}ds \right\|_{H^1} \\
\leq \|\varphi_c\|_{H^1} + \left\| \frac{1 - \alpha}{B(\alpha)} g(t, \varphi(t, r)) \right\|_{H^1} + \frac{\alpha}{B(\alpha) \Gamma(\alpha)} \int_0^t \|g(s, \varphi(s, r))\|_{H^1} ds \\
\leq \|\varphi_c\|_{H^1} + \int_0^t \|g(s, \varphi(s, r))\|_{H^1} ds.
\]

But since \( g \) is locally Lipschitz for all \( s \in [0, T_1] \), it follows that

\[
\|\psi(\varphi)\|_\varepsilon \leq \|\varphi_c\|_{H^1} + \int_0^t \left( C_{\varphi_c} \|\varphi(s, r)\|_{H^1} + c_3 \right) ds.
\]

Thus for all \( \varphi_1, \varphi_2 \in \varepsilon \)

\[
\|\psi(\varphi_1) - \psi(\varphi_2)\|_\varepsilon = \sup_{0 \leq t \leq T_1} T_1 C_{\varphi_c} \|\varphi_1 - \varphi_2\|_\varepsilon.
\]

This shows that \( \psi \) is a contraction mapping in \( \varepsilon \). Thus \( \psi \) has a fixed point which is a solution to (4.2).

Now let us show that the problem (4.2) has an unique solution.

Let \( \varphi_1, \varphi_2 \in H^1 \) be two solutions of (4.2) and let \( \varphi = \varphi_1 - \varphi_2 \). Then

\[
\varphi = \frac{1 - \alpha}{B(\alpha)} \left( g(t, \varphi_1(t, r)) - g(t, \varphi_2(t, r)) \right) + \frac{\alpha}{B(\alpha) \Gamma(\alpha)} \int_0^t \left( g(s, \varphi_1(s, r)) - g(s, \varphi_2(s, r)) \right) ds.
\]

By thanking the norm on both sides,

\[
\|\varphi\| \leq \frac{1 - \alpha}{B(\alpha)} \|g(t, \varphi_1(t, r)) - g(t, \varphi_2(t, r))\| + \frac{\alpha}{B(\alpha) \Gamma(\alpha)} \int_0^t \|g(s, \varphi_1(s, r)) - g(s, \varphi_2(s, r))\| ds \\
\leq T_1 C_{\varphi_c} \int_0^t \|\varphi_1(s, r)\|_{H^1} ds.
\]

By the Gronwall inequality \[12\], the result follows.

4.3. Exact solution of the flow within the leaky aquifer based upon Atangana-Baleanu fractional derivative. In the model here discussed for water flow in the leaky aquifer, the head \( \varphi(r, t) \), which appears in (4.1), is assumed to be governed by the one-dimensional time-fractional differential equation involving the Atangana-Baleanu fractional derivative.
Applying Laplace transform to (4.1), the fundamental solution \( \tilde{\varphi}(r,p) \), \( p \in \mathbb{N} \) results to be:

\[
-\beta^2 \mathcal{L}_t \left[ ABC D_t^\alpha \varphi \right](p) + \mathcal{L}_t \left[ \partial_r \varphi \right](p) + \mathcal{L}_t \left[ \frac{1}{r} \partial_r \varphi \right](p) - \varpi^{-2} \mathcal{L}_t [\varphi](p) = 0,
\]

where \( \mathcal{L}_t := \tilde{\varphi} \) denotes the Laplace transform. Replacing each term by its value, we get

\[
-\frac{B(\alpha) \beta^2 [p^\alpha \varphi - p^{\alpha-1} \varphi_c]}{(1 - \alpha) p^\alpha + \alpha} + \partial_{rr} \tilde{\varphi} + \frac{1}{r} \partial_r \tilde{\varphi} - \varpi^{-2} \tilde{\varphi} = 0.
\]

Hence the following differential equation in the form holds

(4.6)

\[
r^2 \partial_{rr} \tilde{\varphi} + r \partial_r \tilde{\varphi} - qr^2 \tilde{\varphi} = 0,
\]

with \( \varphi_c \approx 0 \), where

\[
q = \frac{B(\alpha) \beta^2 p^\alpha}{(1 - \alpha) p^\alpha + \alpha} + \varpi^{-2}.
\]

Since \( q \) is positive, the exact solution of the differential equation (4.6) is given in terms of Bessel function of the first kind, \( J_0 \) and modified kind, \( K_0 \) as

(4.7)

\[
\tilde{\varphi}(r,p) = AJ_0(r \sqrt{q}) + BK_0(r \sqrt{q}),
\]

where \( A \) and \( B \) are the constants and \( J_0, K_0 \) respectively given as

\[
J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(k + \nu + 1)} \left( \frac{z}{2} \right)^{2k+\nu} \quad \text{and} \quad K_\nu(z) = \frac{\pi}{2} (-i)^\nu \left( \frac{-J_\nu(z) + J_{-\nu}(z)}{\sin(\nu \pi)} \right).
\]

Using the boundary condition in (4.7), we obtain \( B = 0 \), then the solution is reduced to

(4.8)

\[
\tilde{\varphi}(r,p) = AJ_0(r \sqrt{q}) = A \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(k + \nu + 1)} \left( \frac{r \sqrt{q}}{2} \right)^{2k+\nu}.
\]

Due to the difficulties to obtain the inverse Laplace transform of (4.8), we therefore propose to obtain the approximate solution of (4.2) by using the proposed numerical approximation of the Atangana-Baleanu integral.

4.4. Numerical analysis of the ground water flow within the leaky aquifer based upon Atangana-Baleanu fractional derivative. To achieve this, we revert the fractional differential equation to the fractional integral equation using the link between the Atangana-Baleanu derivative and the Atangana-Baleanu integral to obtain

(4.9)

\[
\varphi(r,t) - \varphi(r_c,0) = AB \int_0^t I^\alpha_0 g(r,t,\varphi(r,t)) dt.
\]

For some positive and large integers \( M = N = 350 \), the grid sizes in space and time is denoting respectively by \( \xi = 1/M \) and \( \tau = 1/N \). The grid points in the space interval \([0,1]\) are the numbers \( r_i = i\xi, \ i = 1,2,\ldots, M \) and the grid points in the time interval \([0,1]\) are the numbers \( t_k = k\tau, \ k = 0,1,2,\ldots, N \). The value of the function \( \varphi \) at the grid points are denoted by \( \varphi^k_i = \varphi(r_i,t_k) \).
Using the implicit finite differences method, a discrete approximation of \( g(r, t, \varphi(r, t)) \) given by (4.3) can be obtained as follows

\[
(4.10) \quad g(r_i, t_k, \varphi(r_i, t_k)) = \frac{\beta - 2}{\xi^2} \left\{ \left( 1 + \frac{1}{2i} \right) \varphi_{i+1}^k - \left( 2 + \frac{\xi^2}{\omega^2} \right) \varphi_i^k + \left( 1 - \frac{1}{2i} \right) \varphi_{i-1}^k \right\}.
\]

In order to use the numerical approximation proposed in this work, discrete solution of (4.2) is then given as follows

\[
(4.11) \quad \varphi_{i+1}^k - \varphi_0^c = AB T^\alpha_t g(r_i, t_k, \varphi(r_i, t_{k+1})).
\]

\[
(4.12) \quad AB T^\alpha_t g = \frac{1 - \alpha}{B(\alpha)} g(r_i, t_{k+1}, \varphi(r_i, t_{k+1})) + \Psi(\alpha) \sum_{j=0}^k \frac{b_j^k}{2} \left( g(r_i, t_{k-j}, \varphi_i(r_i, t_{k-j})) + g(r_i, t_{k-j+1}, \varphi(r_i, t_{k-j+1})) \right).
\]

Therefore the numerical approximation can be given as follows

\[
(4.13) \quad \varphi_{i+1}^k - \varphi_0^c = \frac{(1 - \alpha)}{B(\alpha)} \frac{\beta - 2}{\xi^2} \left\{ \left( 1 + \frac{1}{2i} \right) \varphi_{i+1}^k - \left( 2 + \frac{\xi^2}{\omega^2} \right) \varphi_i^k + \left( 1 - \frac{1}{2i} \right) \varphi_{i-1}^k \right\} + \Psi(\alpha) \frac{\beta - 2}{\xi^2} \frac{\sum_{j=0}^k \frac{b_j^k}{2} \left( \left( 1 + \frac{1}{2i} \right) \varphi_{i+1}^{k-j} - \left( 2 + \frac{\xi^2}{\omega^2} \right) \varphi_i^{k-j} + \left( 1 - \frac{1}{2i} \right) \varphi_{i-1}^{k-j} \right)}{2}.
\]

In order obtain the plots of the solution given by (4.13), we shall consider 350 equidistant nodes in \([0, 1]\). For the value \( \alpha = 1/2 \), the approximate solution found by using the numerical method of the Atangana-Baleanu integral proposed is obtained. Figure 1 shows the approximate solution for the different time step \( k = 0 \), \( k = 20 \) and \( k = 50 \) respectively.

![Figure 1](image1.png)

**Figure 1.** Numerical solution of (4.13) for the specific value of the parameters \( \alpha = 1/2 \), and different time step \( k = 0 \), \( k = 20 \) and \( k = 50 \) respectively.

Moreover, to have an overview of the variation of flow or the behaviour of the function \( \varphi \) in a finite time in terms of the parameter \( \alpha \), we consider two different time steps \( k = 0 \) and \( k = 50 \). For this purpose, Figure 2 gives us the approximate result using the
method proposed. We would like to notice that for larger number of nodes in $[0, 1]$ the better approximated solution is obtained in the whole interval.

**Figure 2.** Approximate solution of (4.13) for initial time step $k = 0$ and $\alpha = 1/2$—blue— and $\alpha = 9/10$—orange— (graph in the left). Approximate solution of (4.13) for time step $k = 50$ and $\alpha = 1/2$—blue— and $\alpha = 9/10$—orange— (graph in the right)

Next in Figure 3 we show the approximate solution of (4.13) by using the numerical approximation of the Atangana-Baleanu method proposed for different time steps $k = 0$ and $k = 50$ for $\alpha = 1/2$ and $\alpha = 9/10$ in $[0, 1]$.

**Figure 3.** Approximate solution of (4.13) for $\alpha = 1/2$, $k = 0$ —blue— and $k = 50$—orange— (graph in the left). Approximate solution of (4.13) for $\alpha = 9/10$, $k = 0$ —blue— and $k = 50$—orange— (graph in the right).

5. Conclusions

The new scheme of the fractional integral in the sense Atangana-Baleanu has been proposed in this work. The error analysis of the novel scheme was successfully presented and the error obtained shows that the scheme is highly accurate. A new model of groundwater flowing within a leaky aquifer was suggested using the concept of fractional differentiation based on the generalised Mittag-Leffler function in order to fully introduce into mathematical formulation the complexities of the physical problem as the flow taking place in a very heterogeneous medium. The Mittag-Leffler operator provide more natural observed fact than the more used power law. The new model was analysed, as the uniqueness and the existence of the solution was investigated with care. To further access the accuracy of the proposed numerical scheme, we solved the new model numerically using this suggested scheme. Some simulations have been presented for different values of fractional order. We strongly believe that this numerical scheme will be applied in
many fields of science, technology and engineering for those problems based on the new fractional calculus.

REFERENCES

[1] A. Atangana. *On the new fractional derivative and application to nonlinear Fisher’s reaction-diffusion equation*. Applied Mathematics and Computation, 1, 273, pp. 948–956, (2016).

[2] I. Area, J. D. Djida, J. Losada, and Juan J. Nieto. *On Fractional Orthonormal Polynomials of a Discrete Variable*. Discrete Dynamics in Nature and Society, vol. 2015, Article ID 141325, 7 pages, 2015. doi:10.1155/2015/141325

[3] A. Atangana, and J. J. Nieto. *Numerical solution for the model of RLC circuit via the fractional derivative without singular kernel*. Advances in Mechanical Engineering, vol.7, no.10, pp. 1–7, (2015).

[4] A. Atangana and D. Baleanu. *New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model*. Thermal Science, 20(2), pp. 763–769, (2016).

[5] M. Meerschaert. M and C. Tadjeran. *Finite difference approximations for fractional advection-dispersion flow equations*, Journal of Computational and Applied Mathematics 1, 172, pp. 65–77 (2004),

[6] P. Zhuang, Liu, F, I. Turner and V. Anh. *Galerkin finite element method and error analysis for the fractional cable equation*, Numerical Algorithms 72(2), pp. 447–466, (2016).

[7] Z. Hao, W. Cao, and G. Lin. *A second-order difference scheme for the time fractional substantial diffusion equation*, Journal of Computational and Applied Mathematics, 313, pp. 54–69, (2017).

[8] I. Podlubny. Fractional differential equations. Academic Press, Inc., San Diego, CA, (1999).

[9] R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh. *A new definition of fractional derivative*, Journal of Computational and Applied Mathematics, 264, pp. 65–70 (2014).

[10] P. Wang, C. Huang, and L. Zhao. *Point-wise error estimate of a conservative difference scheme for the fractional Schrödinger equation*, Journal of Computational and Applied Mathematics, 306, pp. 231–247, (2016).

[11] J.A Gallegos, M.A Duarte-Mermoud. *Boundedness and convergence on fractional order systems*, Journal of Computational and Applied Mathematics, 296, pp. 815–826, (2016).

[12] Song-Mu Zheng. *Nonlinear Evolution Equations*. Chapman and Hall/CRC monographs and surveys in pure and applied mathematics, 133, (2004).

[13] O. I. Marichev. Handbook of Integral Transforms of Higher Transcendental Functions: Theory and Algorithmic Tables. Ellis Horwood & Halsted Press, New York, 1983.

(Djida) **African Institute for Mathematical Sciences (AIMS)**, P.O. Box 608, Limbe Crystal Gardens, South West Region, Cameroon.

E-mail address, Djida: jeandaniel.djida@aims-cameroon.org

(Area) **Departamento de Matemática Aplicada II, E.E. Aeronáutica e do Espazo, Universidade de Vigo, Campus As Lagoas s/n**, 32004 Ourense, Spain.

E-mail address, Area: area@uvigo.es

(Atangana) **Institute for Groundwater Studies, Faculty of Natural and Agricultural Sciences, University of the Free State**, 9301, Bloemfontein, South Africa.

E-mail address, Atangana: abdonatangana@yahoo.fr