1. Introduction

Let $G$ be a split semisimple algebraic group over $\mathbb{Q}$, $\mathfrak{g}$ be the Lie algebra of $G$ and $U_q(\mathfrak{g})$ be the corresponding quantized enveloping algebra. Lusztig has introduced
canonical bases for finite-dimensional $U_q(\mathfrak{g})$-modules. About the same time Kashiwara introduced in \cite{Kashiwara} crystal bases as a natural framework for parametrizing bases of finite-dimensional $U_q(\mathfrak{g})$-modules. It was shown in \cite{Lusztig} that Kashiwara’s crystal bases are the limits as $q \to 0$ of Lusztig’s canonical bases. Later, in \cite{Kashiwara} Kashiwara introduced a new combinatorial concept: crystals. Kashiwara’s crystals generalize the crystal bases and provide a natural framework for their study.

In this paper we study geometric crystals and unipotent crystals which are algebro-geometric analogues of the crystals and the crystal bases respectively.

Our approach of the “geometrization” of combinatorial objects comes back to the works of Lusztig. On the one hand, Lusztig constructed for each reduced decomposition $i$ of the longest element $w_0$ of the Weyl group $W$ of $G$ a parametrization $\psi^i$ of the canonical basis $B$ by the cone $(\mathbb{Z}_{\geq 0})^{l_0}$, where $l_0$ is the length $w_0$. On the other hand, in his study of the positive elements of $G$ Lusztig constructed for each $i$ a birational isomorphism $\pi^i : \mathbb{A}^{l_0} \cong U$, where $U$ is the unipotent radical of the Borel subgroup of $G$. Moreover, Lusztig observed that for any two decompositions $i, i'$ of $w_0$, the piecewise-linear transformation $(\psi^{i'})^{-1} \circ \psi^i$ of $(\mathbb{Z}_{\geq 0})^{l_0}$ is similar to the birational transformation $(\pi^{i'})^{-1} \circ \pi^i$ of $\mathbb{A}^{l_0}$ (see \cite{Lusztig}, \cite{Lusztig}).

This analogy was further developed in a series of papers \cite{Lusztig}, \cite{Lusztig}, \cite{Lusztig}, \cite{Lusztig}. The main idea of these papers is to introduce the notion of the “tropicalization” – a set of formal rules for the passage from birational isomorphisms to piecewise-linear maps, and to study piecewise-linear automorphisms of $B$ using the “tropicalization” of explicit birational isomorphisms $\pi^i : \mathbb{A}^{l_0} \cong U$.

Another geometric approach to constructing crystal bases is suggested in the recent work of Braverman and Gaitsgory \cite{BravermanGaitsgory}. Within this approach the crystal bases and, in fact, actual bases in finite-dimensional $G$-modules, are constructed in terms of perverse sheaves on the affine Grassmannian of $G$.

In the present paper we go in the opposite direction – we study geometric crystals in their own right and “geometrize” some of the mentioned above piecewise-linear structures of the crystal bases.

Let $I$ be the set of vertices of the Dynkin diagram of $G$, and $T$ be a maximal torus of $G$. The structure of a geometric crystal on an algebraic variety $X$ consists of a rational morphism $\gamma : X \to T$ and a compatible family $e_i : G_m \times X \to X$, $i \in I$, of rational actions of the multiplicative group $G_m$ on $X$. We show that such a structure induces a rational action of the Weyl group $W$ on $X$. Surprisingly, many interesting rational actions of $W$ come from geometric crystals. For example, the natural action of $W$ on Grothendieck’s simultaneous resolution $\hat{G} \to G$ comes from a structure of a geometric crystal on $\hat{G}$. Also all the examples of the action of $W$ in \cite{Lusztig}, come from geometric crystals. Another application of geometric crystals is a construction for any $SL_n$-crystal built on $(X, \gamma)$ of a trivialization: a $W$-equivariant isomorphism $X \cong \gamma^{-1}(e) \times T$.

It is also interesting that the Langlands dual group $L^*G$ emerges when we reconstruct Kashiwara’s combinatorial crystals out of positive geometric crystals (see section 2.5). The presence of $L^*G$ in the “crystal world” has been noticed in \cite{Kashiwara}. The combinatorial results of \cite{BerensteinKazhdan} also involve $L^*G$.

A number of statements in the paper are almost immediate. We call them Lemmas. Proofs of all the Lemmas from sections 2, 3 and 4 are left for the reader.

The material of the paper is organized as follows.
In section 2 we introduce geometric crystals and formulate their main properties.

In section 3 we introduce unipotent crystals as an algebro-geometric analogue of crystal bases for \( U_q(\mathfrak{g}) \)-modules, where \( \mathfrak{g} \) is the Lie algebra of \( \mathcal{L} \). A unipotent crystal on an irreducible algebraic variety \( X \) consists of a rational action of \( \mathcal{L} \) on \( X \) and a rational \( \mathcal{U} \)-equivariant morphism \( \mathfrak{f} : X \to \mathcal{G}/\mathcal{U} \). The unipotent crystals have a number of properties characteristic of the crystal bases. One of the main properties is a natural product of unipotent crystals which is an analogue of tensor product of crystal bases. We prove that the category of unipotent crystals is strict monoidal with respect to this product. We also define the notion of a dual unipotent crystal. One of the main results of this section is a construction for any unipotent crystal on \( X \), of an induced geometric crystal \( \mathcal{X} \). We also give a closed formula for the action of \( \mathcal{W} \) on this induced geometric crystal \( \mathcal{X} \).

In section 4 we study the standard unipotent crystals which are the \( \mathcal{U} \)-orbits \( BwB/B \subset G/B \). We obtain a decomposition of any standard crystal \( BwB/B \) into the product of 1-dimensional crystals corresponding to any reduced decomposition of \( w \in \mathcal{W} \). This is a geometrization of Kashiwara’s results. We also construct \( \mathcal{W} \)-invariant functions on certain standard unipotent crystals. These functions are important for the study of \( \gamma \)-functions of representations \( \mathcal{L} \) (see [7]).

In section 5 we collect proofs of the results from sections 2, 3 and 4.

In section 6 we study the restrictions to a standard Levi subgroup \( L \subset G \) of standard unipotent \( \mathcal{G} \)-crystals \( BwB/B \). In particular, for each \( w \in \mathcal{W} \) we construct a rational morphism \( p_w : BwB/B \to L/(U \cap L) \). In the case when \( p_w \) is a birational isomorphism with its image, we study the direct image of \( W_L \)-invariant functions under \( p_w \). In the special case when \( G = GL(m+n), L = GL(m) \times GL(n) \), and \( w = w_0^m \cdot w_0 \), it is possible to write explicitly the corresponding action of the group \( W_L \) on the image of \( p_w \), and the \( W_L \)-equivariant trivialization. This simplification of the structure of the crystal on \( BwB/B \) is used in [7] for the prove that Piatetsky-Shapiro’s \( \gamma \)-function to \( GL(m) \times GL(n) \) is equal to the one introduced in [7]. We expect that in general the action of \( W_L \) on \( BwB/B \), and the trivialization of the corresponding crystal would be easy to describe, and that this description can help in the study of \( \gamma \)-functions on \( L \).

In the Appendix we collect necessary definitions and results related to Kashiwara’s crystals. We refer to these crystals as combinatorial in order to distinguish between them and their geometric counterparts.

The notion of a geometric crystal was introduced by the first author who noticed that the formulas for the \( \mathcal{W} \)-action which appear in the definition of \( \gamma \)-functions for the group \( GL_2 \times GL_3 \) (see [2]) can be interpreted as a “geometrization” of the \( \mathcal{W} \)-action for the free combinatorial crystal defined earlier by the first author. We want to express our gratitude to Alexander Braverman for his help in different stages of this work and in particular for the definition of the notion of degree in section 2.4. We are also grateful to Yuval Flicker for numerous remarks about the paper. Special thanks are due to Charles Cochet for correcting errors in the first version of the paper.

2. Definitions and main results on geometric crystals
2.1. General notation. Let $G$ be a split reductive algebraic group over $\mathbb{Q}$ and $T \subset G$ a maximal torus. We denote by $\Lambda^\vee$ and $\Lambda$ the lattices of co-characters and characters of $T$ and by $\langle \cdot, \cdot \rangle$ the evaluation pairing $\Lambda^\vee \times \Lambda \to \mathbb{Z}$.

Let $B$ be a Borel subgroup containing $T$. Denote by $I$ the set of vertices of the Dynkin diagram of $G$; for any $i \in I$ denote by $\alpha_i \in \Lambda$ the simple root $\alpha_i : T \to \mathbb{G}_m$, and by $\alpha_i^\vee \in \Lambda^\vee$ the simple coroot $\alpha_i^\vee : \mathbb{G}_m \to T$.

Let $B^- \subset G$ be the Borel subgroup containing $T$ such that $B \cap B^- = T$. Denote by $U$ and $U^-$ respectively the unipotent radicals of $B$ and $B^-$. For each $i \in I$ we denote by $U_i \subset U$ and $U_i^- \subset U^-$ the corresponding simple root subgroups and denote by $\xi_i : U \to U_i, \xi_i^- : U^- \to U_i^-$ the canonical projections.

We fix a family of isomorphisms $x_i : \mathbb{G}_a \rightarrow U_i, y_i : \mathbb{G}_a \rightarrow U_i^-, i \in I$, such that

\begin{equation}
(2.1) \quad x_i(a)y_i(a') = y_i \left( \frac{a'}{1 + aa'} \right) \alpha_i^\vee (1 + aa') x_i \left( \frac{a}{1 + aa'} \right).
\end{equation}

Clearly, each isomorphism $y_i$ is uniquely determined by $x_i$.

Remark. Each pair $x_i, y_i$ defines a homomorphism $\phi_i : SL_2 \rightarrow G$:

$$
\phi_i \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = y_i \left( \frac{c}{a} \right) \alpha_i^\vee (a) x_i \left( \begin{array}{c} b \\ d \end{array} \right).$$

Denote by $\tilde{U}$ and $\tilde{U}^-$ respectively the spaces $Hom(U, \mathbb{G}_a)$ and $Hom(U^-, \mathbb{G}_a)$. For each $i \in I$ define $\chi_i \in \tilde{U}, \chi_i^- \in \tilde{U}^-$ by

$$
\chi_i(u_+) = x_i^{-1}(\xi_i(u_+)), \quad \chi_i^-(u_-) = y_i^{-1}(\xi_i^-(u_-))
$$

for $u_+ \in U^-, u_- \in U$. By definition, the family $\chi_i, i \in I,$ is a basis in the vector space $\tilde{U}$, and the family $\chi_i^-, i \in I,$ is a basis in the vector space $\tilde{U}^-$. For any $\chi \in \tilde{U}$, and any $\chi^- \in \tilde{U}^-$ define functions $\chi, \chi^- : U^- \cdot T \cdot U \to \mathbb{G}_a$ by

$$
\chi^-(u - tu_+) = \chi^-(u), \quad \chi(u - tu_+) = \chi(u_+).
$$

The Weyl group $W = Norm_G(T)/T$ of $G$ is generated by simple reflections $s_i \in W, i \in I$. The group $W$ acts on the lattices $\Lambda, \Lambda^\vee$ by $s_i(\lambda) = \lambda - (\alpha_i^\vee, \lambda)\alpha_i$ for $\lambda \in \Lambda, s_i(\lambda^\vee) = \lambda^\vee - (\lambda^\vee, \alpha_i)\alpha_i^\vee$ for $\lambda^\vee \in \Lambda^\vee$.

Let $l : W \rightarrow \mathbb{Z}_{\geq 0}$ ($w \mapsto l(w)$) be the length function. For any sequence $i = (i_1, \ldots, i_l) \in I^l$ we write $w(i) = s_{i_l} \cdots s_{i_1}$. A sequence $i \in I^l$ is called reduced if the length of $w(i)$ is equal to $l$. For any $w \in W$ we denote by $R(w)$ the set of all reduced sequences $i$ such that $w(i) = w$. We denote by $w_0 \in W$ the element of the maximal length in $W$.

For $i \in I$ define $\overline{s_i} \in G$ by

\begin{equation}
(2.2) \quad \overline{s_i} = x_i(-1)y_i(1)x_i(-1) = \phi_i \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right).
\end{equation}

Each $\overline{s_i}$ belongs to $Norm_G(T)$ and is a representative of $s_i \in W$. It is well-known (5) that the elements $\overline{s_i}, i \in I,$ satisfy the braid relations. Therefore we can associate to each $w \in W$ its standard representative $\overline{w} \in Norm_G(T)$ in such a way that for any $(i_1, \ldots, i_l) \in R(w)$ we have:

\begin{equation}
(2.3) \quad \overline{w} = \overline{s_{i_1}} \cdots \overline{s_{i_l}}.
\end{equation}
2.2. **Geometric pre-crystals and geometric crystals.** Let $X$ and $Y$ be algebraic varieties over $\mathbb{Q}$. Denote by $R(X,Y)$ the set of all rational morphisms from $X$ to $Y$.

For any $f \in R(X,Y)$ denote by $\text{dom}(f) \subset X$ the maximal open subset of $X$ on which $f$ is defined; denote by $f_{\text{reg}} : \text{dom}(f) \to Y$ the corresponding regular morphism. We denote by $\text{ran}(f) \subset Y$ the closure of the constructible set $f_{\text{reg}}(\text{dom}(f))$ in $Y$. For any regular morphism $f : X' \to Y$, where $X' \subset X$ is a dense subset we denote by $[f] : X' \to Y$ the corresponding rational morphism. Note that $[f_{\text{reg}}] = f$ and $\text{dom}(f_{\text{reg}}) = \text{dom}(f)$ for any $f \in R(X,Y)$.

It is easy to see that for any irreducible algebraic varieties $X,Y,Z$ and rational morphisms $f : X \to Y$, $g : Y \to Z$ such that $\text{dom}(g)$ intersects $\text{ran}(f)$ non-trivially, the composition $(f,g) \mapsto g \circ f$ is well-defined and is a rational morphism from $X$ to $Z$.

We denote by $\mathcal{V}$ be the category whose objects are irreducible algebraic varieties and arrows are dominant rational morphisms.

For any algebraic group $H$ we call a rational action $\alpha : H \times X \to X$ **unital** if $\text{dom}(\alpha) \supset \{e\} \times X$.

**Definition.** Let $X \in \mathcal{O}(\mathcal{V})$ and $\gamma$ be a rational morphism $X \to T$. A geometric $G$-pre-crystal (or simply a geometric pre-crystal) on $(X,\gamma)$, is a family $e_i : \mathbb{G}_m \times X \to X$, $i \in I$, of unital rational actions of the multiplicative group $\mathbb{G}_m : (c,x) \mapsto e_i(x)$, such that $\gamma(e_i(x)) = \alpha_i^\gamma(c)\gamma(x)$.

**Remark.** Geometric pre-crystals are analogues of free combinatorial pre-crystals (see Appendix). Under this analogy, the variety $X$ corresponds to the set $\mathcal{B}$, the maximal torus $T$ corresponds to the lattice $\Lambda^\mathcal{V}$, the rational morphism $\gamma : X \to T$ corresponds to the map $\tilde{\gamma} : \mathcal{B} \to \Lambda^\mathcal{V}$ and rational actions $e_i$ of $\mathbb{G}_m$ on $X$ corresponds to bijections $\tilde{e}_i : \mathcal{B} \to \mathcal{B}$. We will make this analogy precise in section 4.3.

Given a geometric pre-crystal $X$, and a reduced sequence $i = (i_1, \ldots, i_l) \in I^l$, we define a rational morphism $e_i : T \times X \to X$ by

$$
(t,x) \mapsto e_i^l(x) = e_{i_1}^{\alpha_{i_1}^{(1)}}(t) \circ \cdots \circ e_{i_l}^{\alpha_{i_l}^{(l)}}(t)(x),
$$

where $\alpha^{(k)} = s_{i_k}s_{i_{k-1}} \cdots s_{i_{k+1}}(\alpha_{i_k})$, $k = 1, \ldots, l$ are the associated positive roots.

**Definition.** A geometric pre-crystal $X$ on $(X,\gamma)$ is called a geometric crystal if for any $w \in W$, and any $i,i' \in R(w)$, one has:

$$
e_i = e_{i'}.
$$

**Lemma 2.1.** The relations (2.5) are equivalent to the following relations between $e_i, e_j$ for $i,j \in I$:

$$
e_i e_j e_i = e_j e_i e_i
$$

if $\langle \alpha_i^\vee, \alpha_j \rangle = 0$;

$$
e_i e_j e_i = e_j e_i e_i
$$

if $\langle \alpha_i^\vee, \alpha_j \rangle = \langle \alpha_i^\vee, \alpha_i \rangle = -1$;

$$
e_i^\vee e_j e_i e_j = e_j e_i e_j e_i
$$

if $\langle \alpha_i^\vee, \alpha_j \rangle = -2, \langle \alpha_j^\vee, \alpha_i \rangle = -1$;

$$
e_i^\vee e_j e_i e_j e_i = e_j e_i e_j e_i e_i = e_j e_i e_j e_i e_i
$$

if $\langle \alpha_i^\vee, \alpha_j \rangle = -3, \langle \alpha_j^\vee, \alpha_i \rangle = -1$;

$$
e_i^\vee e_j e_i e_j e_i e_i e_j = e_j e_i e_j e_i e_i e_j
$$

if $\langle \alpha_i^\vee, \alpha_j \rangle = -4, \langle \alpha_j^\vee, \alpha_i \rangle = -1$. 


if \( \langle \alpha'_i, \alpha_j \rangle = -3, \langle \alpha'_j, \alpha_i \rangle = -1 \).

**Lemma 2.2.** Let \( \mathcal{X} \) be a geometric pre-crystal. For any \( \lambda \in \Lambda, i \in I \), the formula \( x \mapsto e_i^{\lambda(\gamma(x))}(x) \) defines a rational morphism \( X \to X \). This is a birational isomorphism \( X \cong X \) if and only if \( \langle \alpha'_i, \lambda \rangle \in \{-2, 0\} \). In the latter case the inverse morphism is given by the formula

\[
x \mapsto \begin{cases} e_i^{\lambda(\gamma(x))}(x), & \text{if } \langle \alpha'_i, \lambda \rangle = -2; \\ e_i^{\lambda(\gamma(x))'}(x), & \text{if } \langle \alpha'_i, \lambda \rangle = 0. 
\end{cases}
\]

**Remarks.**
1. The relations \((2.4) - (2.9)\) are multiplicative analogues of the Verma relations in the universal enveloping algebra \( U(\mathfrak{g}) \) (see [15], Proposition 39.3.7).
2. An analogue of the relations \((2.4), (2.7)\) for a combinatorial \( U \) the universal enveloping algebra

**Proposition 2.3.** The correspondence \( W \times X \to X \) defined by

\[
(w, x) \mapsto w(x) = e_i^{\gamma(x)^{-1}}(x)
\]

is a rational unital action of \( W \) on \( X \).

**Remark.** The formula \( s_i(x) = e_i^{\gamma(x)^{-1}}(x) \) is a multiplicative analogue of \((1.1)\) in the Appendix.

### 2.3. The geometric crystal \( X_{\omega_0} \)

Let \( B_{\omega_0}^- := U^- \cap B^- \). The natural inclusion \( B_{\omega_0}^- \hookrightarrow G \) induces the open inclusion \( \jmath_0 : B_{\omega_0}^- \hookrightarrow G/B \). Let \( \gamma : G/B \to T \) be the rational morphism defined by \( \gamma = pr_T \circ [\jmath_0]^{-1} \), where \( pr_T : B^- \to T = B^-/U^- \) is the natural projection.

For \( i \in I \) let \( \varphi_i : G/B \to \mathbb{G}_a \) be the rational function given by \( \varphi_i := \bar{\gamma}_i \circ [\jmath_0]^{-1} \), where \( \bar{\gamma}_i : B^- \cdot U \hookrightarrow \mathbb{G}_a \) is the regular function defined in section 2.1

**Lemma 2.4.** For each \( i \in I \) we have:

(a) \( \varphi_i \neq 0 \);

(b) \( \frac{\varphi_i(x) \cdot a(x)}{\varphi_i(x)} = \frac{1}{\varphi_i(x)} + a \);

(c) \( \gamma(x_i(a) \cdot x) = \alpha'_i(1 + a \varphi_i(x)) \gamma(x) \).

For each \( i \in I \) define a rational morphism \( e_i : G_m \times G/B \to G/B \) by the formula

\[
e_i^c(x) = x_i \left( \frac{c - 1}{\varphi_i(x)} \right) \cdot x .
\]

**Remark.** Lemma \(2.4\) implies the equality \( \varphi_i(e_i^c(x)) = c^{-1} \varphi_i(x) \) for \( i \in I \).

**Theorem 2.5.** The morphisms \( e_i, i \in I \), define a geometric crystal on \((G/B, \gamma)\)

We denote this crystal by \( X_{\omega_0} \). This is one of our main examples of geometric crystals.

**Remark.** The geometric crystal \( X_{\omega_0} \) is an analogue of the free combinatorial \( W \)-crystal \( B_{\omega_0} \), and the rational functions \( \varphi_i : G/B \to \mathbb{G}_m \) are analogues of the functions \( -\varphi_i \) on \( B_{\omega_0} \) (see Appendix).
2.4. Positive geometric crystals and their tropicalization. Let $T'$ be an algebraic torus over $\mathbb{Q}$. We denote by $X^*(T')$ and $X_*(T')$ respectively the the lattices of characters and co-characters of $T'$, and by $\langle \cdot, \cdot \rangle$ the canonical pairing $X_*(T') \times X^*(T') \to \mathbb{Z}$.

Let $\mathcal{L}(T')$ be the set of formal loops $\phi : \mathbb{G}_m \to T'$. In particular, $\mathcal{L}(\mathbb{G}_m) = \mathcal{L}(c)$, where $\mathcal{L}(c)$ is the set of all Laurent series in the variable $c$.

Remark. That for any $\mu \in \mathcal{L}(T')$ we have $\mu \circ \phi \in \mathcal{L}_0(c)$, where $\mathcal{L}_0(c) \subset \mathcal{L}(c)$ is the set of all invertible Taylor series in the variable $c$.

Clearly, $\mathcal{L}_0(T')$ has a natural structure of an irreducible pro-algebraic variety.

**Lemma 2.6.** The multiplication map $X_*(T') \times \mathcal{L}_0(T') \to \mathcal{L}(T')$ is a bijection.

This defines a surjective map $\deg_{T'} : \mathcal{L}(T') \to X_*(T')$.

**Remarks.**

1. If $T' = \mathbb{G}_m$ then $\deg_{\mathbb{G}_m} : \mathcal{L}(c) \to \mathbb{Z}$ is the valuation map which associates to a non-zero Laurent series $\phi(c)$ its lowest degree. By definition, $\mathcal{L}_0(c) = \deg_{\mathbb{G}_m}^{-1}(0)$.

2. For any co-character $\lambda \in X_*(T')$, the pre-image $\deg_{T'}^{-1}(\lambda) \subset \mathcal{L}(T')$ naturally isomorphic to $\mathcal{L}_0(T')$. Hence it makes sense to talk about generic points of $\deg_{T'}^{-1}(\lambda)$.

For any $f \in \mathbb{R}(T', T'')$ and any $\lambda \in X_*(T')$, $\mu \in X_*(T'')$ let $U_f(\lambda, \mu)$ be the set of all $\phi \in \deg_{T'}^{-1}(\lambda)$ such that $f \circ \phi \in \deg_{T''}^{-1}(\mu)$.

**Lemma 2.7.** Let $f \in \mathbb{R}(T', T'')$. For any $\lambda \in X_*(T')$ there is a unique $\mu \in X_*(T'')$ such that the set $U_f(\lambda, \mu)$ is dense in $\deg_{T''}^{-1}(\mu)$.

This allows to define a map $\deg(f) : X_*(T') \to X_*(T'')$ by $\deg(f)(\lambda) = \mu$, where $\mu \in X_*(T'')$ is determined in Lemma 2.6.

We call this map the degree of $f$. It is easy to check that $\deg(f)$ is a piecewise-linear map. Note that $\deg(f)$ is linear if $f$ is a group homomorphism.

**Definition.**

(a) A rational function $f$ on a torus $T'$ is called positive if it can be written as a ratio $f = f'/f''$, where $f'$ and $f''$ are linear combinations of characters with positive integer coefficients.

(b) For any two algebraic tori $T', T''$ we call a rational morphism $f \in \mathbb{R}(T', T'')$ positive if for any character $\mu : T'' \to \mathbb{G}_m$ the composition $\mu \circ f$ is a positive rational function on $T'$.

Denote by $\mathbb{R}_+(T', T'')$ the set of positive rational morphisms from $T'$ to $T''$.

**Remark.** Any homomorphism of algebraic tori is positive.

**Lemma 2.8.** For any two positive morphisms $f \in \mathbb{R}_+(T', T''), g \in \mathbb{R}_+(T'', T''')$ the composition $g \circ f \in \mathbb{R}_+(T', T''')$ is well-defined.

Therefore, we can consider a category $\mathcal{T}_+$ whose objects are algebraic tori, and the arrows are positive rational morphisms.

For any $f \in \mathbb{R}(T', T'')$ and any $\lambda \in X_*(T'), \mu \in X_*(T'')$ let $U_{f, \lambda}$ be the set of all $\phi \in \deg_{T'}^{-1}(\lambda)$ such that $f \circ \phi \in \mathcal{L}(T'')$.

We say that a formal loop $\phi \in \mathcal{L}(T')$ is positive rational if $\phi \in \mathbb{R}_+(\mathbb{G}_m, T')$.

**Lemma 2.9.** Let $f : T' \to T''$ be a positive morphism. Then for any $\lambda \in X_*(T')$ and any formal positive rational loop $\phi \in \deg_{T'}^{-1}(\lambda)$ we have

$$\deg_{T''} \circ f \circ \phi = \deg(f)(\lambda) .$$
Corollary 2.10. For any algebraic tori $T', T''$, and any $f \in \mathbb{R_+(T', T'')}$, we have
\begin{equation}
\deg(g \circ f) = \deg(g) \circ \deg(f).
\end{equation}

This implies that there is a functor $\text{Trop} : \mathcal{T}_+ \rightarrow \text{Set}$ such that $\text{Trop}(T') = X_*(T')$ and $\text{Trop}(T' \rightarrow T'') = (\deg(f) : X_*(T') \rightarrow X_*(T''))$.

Following [1], we call the functor $\text{Trop}$ tropicalization.

Remark. If $f : T' \rightarrow T''$ is not positive then the assertion of Lemma 2.4 is not true even if $f$ is a birational isomorphism. Moreover, one can find a birational isomorphism $f : T' \rightarrow T''$ such that $\text{(2.12)}$ does not hold for the pair $(f, g)$, where $g = f^{-1} : T'' \rightarrow T'$. Consider, for example, the case when $T' = T'' = \mathbb{G}_m$, $f(c) = c - 1, g(c) = c + 1$.

Definition. Let $\mathcal{X}$ be a geometric pre-crystal on $(X, \gamma)$. A birational isomorphism $\theta : T' \rightarrow X$ for some algebraic torus $T'$ is called a positive structure on $\mathcal{X}$ if the following conditions are satisfied.
1. The rational morphism $\gamma \circ \theta : T' \rightarrow T$ is positive.
2. For each $i \in I$ the rational morphism $e_i : \mathbb{G}_m \times T' \rightarrow T'$ given by
\begin{equation}
e_i(\theta(c, t')) = \theta^{-1}(e_i(\theta(t')))\end{equation}
is positive.

Remark. In all our examples of positive structures $\theta$ the composition $\gamma \circ \theta : T' \rightarrow T$ is a group homomorphism.

We say that two positive structures $\theta, \theta'$ are equivalent if the rational morphisms $\theta^{-1} \circ \theta'$ and $\theta'^{-1} \circ \theta$ are positive.

Recall from the Appendix that a combinatorial pre-crystal consists of a set $B$, a map $\hat{\gamma} : B \rightarrow \Lambda'$ and compatible collection of partial bijections $\check{e}_i : B \rightarrow B$.

For any positive structure $\theta$ on a geometric pre-crystal $\mathcal{X}$ built on $(X, \gamma)$ define for $i \in I$ the $\mathbb{Z}$-action $\check{e}_i^\bullet : \mathbb{Z} \times X_*(T') \rightarrow X_*(T')$ by the formula
\begin{equation}
\check{e}_i^\bullet = \text{Trop}(e_i ; \theta),
\end{equation}
where $e_i : \mathbb{G}_m \times T' \rightarrow T'$ is the positive morphism defined by $\text{(2.13)}$.

Also denote $\hat{\gamma}_{\theta} := \text{Trop}(\gamma \circ \theta) : X_*(T') \rightarrow \Lambda'$.

Obviously, the map $\check{e}_i^\bullet : X_*(T') \rightarrow X_*(T')$ is a bijection for $i \in I$. The bijections $\check{e}_i := \check{e}_i^1, i \in I$, define a free combinatorial pre-crystal $\text{Trop}_\theta(\mathcal{X})$ on $(X_*(T'), \hat{\gamma}_{\theta})$.

Theorem 2.11. Let $\mathcal{X}$ be a geometric crystal, and let $\theta$ be a positive structure on $\mathcal{X}$. Then $\text{Trop}_\theta(\mathcal{X})$ is a free combinatorial $W$-crystal.

We call this free combinatorial pre-crystal $\text{Trop}_\theta(\mathcal{X})$ the tropicalization of $\mathcal{X}$ with respect to the positive structure $\theta$.

For a reduced sequence $i = (i_1, \ldots, i_l)$ define a morphism $\theta_i : (\mathbb{G}_m) \rightarrow G/B$ by
\begin{equation}
\theta_i(c_1, \ldots, c_l) := x_{i_1}(c_1) \cdots x_{i_l}(c_l) \cdot \overline{s_{i_1}} \cdots \overline{s_{i_{l-1}}} \cdots \overline{s_{i_l}} \cdot B.
\end{equation}

The following theorem was proved in [4] in a slightly different form.

Theorem 2.12. For each $i \in R(w_0)$ the morphism $\theta_i$ is a positive structure on the geometric crystal $X_{w_0}$, and the tropicalization of $X_{w_0}$ with respect to $\theta_i$ is equal to the free combinatorial $W$-crystal $B_i$. All these positive structures $\theta_i$ are equivalent to each other.
Remark. In [14] the Lusztig introduced morphisms \( \theta^i : (G_m)^{l(w_0)} \to G/B \):
\[
\theta^i(c_1, \ldots, c_l) = y_{i_1}(c_{i_1}) \cdots y_{i_l}(c_{i_l}) \cdot B,
\]
and proved that these morphisms are related to each other in the same way as the corresponding parametrizations of the canonical basis for \( U_q(\mathfrak{g}) \). We discuss similar morphisms \( \pi^i \) in section 1.3.

It was shown in [4] that the morphisms \( \theta^i : (G_m)^{l(w_0)} \to G/B, \; i \in R(w_0) \) are also positive structures on \( X_{w_0} \), and, moreover, these positive structures are equivalent to \( \theta_i, i \in R(w_0) \).

2.5. Trivialization of geometric crystals. Let \( \mathcal{X} \) be a geometric \( G \)-crystal built on \( (X, \gamma) \). Without loss of generality we may assume that \( \gamma \) is a regular surjective morphism \( X \to T \). Denote by \( X_0 = \gamma^{-1}(e) \) the fiber over the unit \( e \in T \).

By definition of the action of \( W \) on \( X \) (see Proposition 2.3, \( w(x_0) = x_0 \) for \( x_0 \in X_0, \; w \in W \).

Definition. A trivialization of a geometric crystal \( \mathcal{X} \) is a \( W \)-invariant rational projection \( \tau : X \to X_0 \).

It is easy to see that the following formula defines a trivialization for any geometric \( SL_2 \)-crystal \( \mathcal{X} \):
\[
\tau(x) = e_{i_1}^{\omega_1(x)} \cdot \cdots \cdot e_{i_l}^{\omega_l(x)} (x)
\]
where \( I = \{1\} \) and \( \omega_1 \) is the only fundamental weight of \( T \).

In the case when \( G = SL_3 \) we can construct two different trivializations \( \tau, \tau' : X \to X_0 \) for any geometric \( SL_3 \)-crystal \( \mathcal{X} \). These trivializations are given by
\[
\tau(x) = e_1 e_2 e_1 e_2 e_1 (x),
\]
\[
\tau'(x) = e_2 e_1 e_2 e_1 e_2 (x),
\]
where \( \omega_1, \omega_2 \in \Lambda \) are the fundamental weights.

Warning. The morphism \( X \to X_0 : x \mapsto e_{i_1}^{\omega_1(x)} \cdot e_{i_2}^{\omega_2(x)} (x) \) is not \( W \)-invariant.

Actually for any \( r > 1 \) and any geometric \( SL_{r+1} \)-crystal \( \mathcal{X} \) we can construct two trivializations \( \tau, \tau' : X \to X_0 \) of \( \mathcal{X} \) in the following way. Let \( \omega_1, \omega_2, \ldots, \omega_r \in \Lambda \) be the fundamental weights ordered in the standard way and \( \omega_i(x) = \omega_i(\gamma(x)) \) for \( i = 1, \ldots, r \).

For every geometric \( SL_{r+1} \)-crystal \( \mathcal{X} \) define a morphism \( \tau : X \to X_0 \) by the formula
\[
\tau(x) = \left( e_{-\frac{1}{\omega_1(x)}} e_{-\frac{1}{\omega_2(x)}} \ldots e_{-\frac{1}{\omega_r(x)}} \right) \left( e_{\frac{1}{\omega_1(x)}} e_{\frac{1}{\omega_2(x)}} \ldots e_{\frac{1}{\omega_r(x)}} \right) \cdots \left( e_{\frac{1}{\omega_2(x)}} e_{\frac{1}{\omega_3(x)}} \right) \left( e_{\frac{1}{\omega_1(x)}} \right) (x).
\]

Theorem 2.13. This morphism \( \tau : X \to X_0 \) is a trivialization of \( \mathcal{X} \).

The formula for the second trivialization \( \tau' : X \to X_0 \) is obtained from \( \tau \) by applying the automorphism of the Dynkin diagram which exchanges \( i \) with \( r+1-i \).

We do not know whether trivializations of geometric \( G \)-crystals exist for other reductive groups. We expect that for the unipotent crystals one can find a trivialization.
3. Unipotent crystals

3.1. Definition of unipotent crystals and their product. As we have seen, geometric crystals are geometric analogues of free combinatorial $W$-crystals. Our next task is to introduce a geometric analogue of crystal bases.

**Definition.** A $U$-variety $X$ is a pair $(X, \alpha)$, where $X \in \text{Ob}(\mathcal{V})$ and $\alpha : U \times X \to X$ is a rational unital $U$-action on $X$ such that the restriction of $\alpha$ to each $U_i \times X$, $i \in I$, is a rational action $U_i \times X \to X$.

For $U$-varieties $X, Y$ we say that a rational morphism $f : X \to Y$ is $U$-morphism if it commutes with the $U$-actions.

It is well-known that the multiplication in $G$ induces a birational isomorphism $B^- \times U \to G$. Denote by $g$ the inverse birational isomorphism:

$$g : G \to B^- \times U.$$ (3.1)

Let $\pi^- : G \to B^-$ and $\pi : G \to U$ be the rational morphisms defined by

$$\pi^- = pr_1 \circ g, \pi = pr_2 \circ g.$$ By definition, $\text{dom}(\pi^-) = \text{dom}(\pi) = B^- \cdot U$.

Passing to the quotient, we obtain a birational isomorphism $G/U \to B^-$. Therefore, the natural left action of $U$ on $G/U$ defines a left $U$-action $\alpha_{B^-} : U \times B^- \to B^-$. This action satisfies for $u \in U, b \in B^-$:

$$\alpha_{B^-}(u, b) = \pi^-(u \cdot b) = u \cdot b \cdot (\pi(u \cdot b))^{-1}.$$ (3.2)

In particular, the pair $B^- := (B^-, \alpha_{B^-})$ is a $U$-variety.

**Lemma 3.1.** For $u \in U^-, t \in T$ one has:

$$\pi(x_i(a) \cdot u \cdot t) = x_i((a^{-1} + \chi^i(u))^{-1} \cdot \alpha_i(t^{-1}))$$ (3.3)

**Lemma 3.2.**

(a) For $b = u \cdot t$, $u \in U^-, t \in T$, $a \in \mathbb{G}_a$ and $i \in I$ we have:

$$\alpha_{B^-}(x_i(a), b) = x_i(a) \cdot b \cdot x_i \left( -\frac{a}{1 + a \chi^i(u) \alpha_i(t)} \right).$$ (3.4)

(b) Every $U$-orbit in $B^-$ is the intersection of $B^-$ with a $U \times U$-orbit in $G$.

**Definition.** A **unipotent crystal** is a pair $(X, f)$, where $X$ is a $U$-variety and $f : X \to B^-$ is a $U$-morphism.

We denote by $U - \text{Cryst}$ the category whose objects are unipotent $G$-crystals and arrows are dominant rational morphisms.

For any $(X, f_X), (Y, f_Y) \in \text{Ob}(U - \text{Cryst})$ define a rational morphism $\alpha : U \times X \times Y \to X \times Y$ by the formula:

$$\alpha(u, (x, y)) := (u(x), (\pi(u \cdot f_X(x))) (y)),$$ (3.5)

where $u(x) = \alpha(u, x)$. We will often write $u(x, y)$ instead of $\alpha(u, (x, y))$.

**Theorem 3.3.**

(a) The morphism $\alpha : U \times X \times Y \to X \times Y$ defined above is a rational $U$-action on $X \times Y$.

(b) Let $m : B^- \times B^- \to B^-$ be the multiplication morphism. Let $f = f_{X \times Y} : X \times Y \to B^-$ be the rational morphism defined by $f = m \circ (f_X \times f_Y)$. Then $f_{X \times Y}$ is a $U$-morphism.
We denote the $U$-variety $(X \times Y, \alpha_{X \times Y})$ by $X \times_f Y$. According to the Theorem the pair $(X \times_f Y, f_{X \times Y})$ is a unipotent crystal. We call it the product of $(X, f_X)$ and $(Y, f_Y)$ and denote it by $(X, f_X) \times (Y, f_Y)$.

**Remark.** Product of unipotent crystals is analogous to the tensor product of Kashiwara’s crystals defined in [10]. This analogy is made precise in sections 3.3 and 3.4 below.

**Proposition 3.4.** Product of unipotent crystals is associative.

**Remark.** The above results define a strict monoidal structure on the category $U-\text{Cryst}$.

The product of unipotent crystals is not commutative in general. For any family $(X_k, f_k) \in \text{Ob}(U-\text{Cryst})$, $k = 1, \ldots, l$ we denote by $\prod_{k=1}^l (X_k, f_k)$ the product $(X_1, f_1) \times \cdots \times (X_l, f_l)$.

**Example.** Denote by $U$ the pair $(U, \alpha_U)$, where $\alpha_U : U \times U \to G$ is the left action. Let $pr_e^U : U \to B^-$ be the projection on the unit $e$. Clearly, $(U, pr_e^U) \in \text{Ob}(U-\text{Cryst})$. Similarly, denote by $G$ the pair $(G, \alpha_G)$, where $\alpha_G : U \times G \to U$ is the left action. Clearly, $(G, \pi^-) \in \text{Ob}(U-\text{Cryst})$.

**Proposition 3.5.** The birational isomorphism $g$ induces the isomorphism in $U-\text{Cryst}$:

$$(G, \pi^-) \cong (B^-, \text{id}_{B^-}) \times (U, pr_e^U).$$

More generally, taking the right quotient by any subgroup $U' \subset U$, we obtain the birational isomorphism $g_{U'} : G/U' \cong B^- \times U/U'$ which induces the isomorphism in $U-\text{Cryst}$:

$$(G/U', \pi_{U'}^-) \cong (B^-, \text{id}_{B^-}) \times (U/U', pr_e^{U'/U'})$$

where $\pi_{U'}^- = pr_1 \circ g_{U'}$.

The following result shows that the unipotent crystal $(G, \pi^-)$ is universal.

**Lemma 3.6.** For every $U$-equivariant rational morphism $f_U : U \to B^-$ there is an element $\tilde{w} \in \text{Norm}_G(T)$ such that

$$f_U(u) = \pi^-(u \cdot \tilde{w})$$

for every $u \in U$.

3.2. From unipotent $G$-crystals to unipotent $L$-crystals. Throughout this section we fix a subset $J$ of $I$. Let $L = L_J$ be the Levi subgroup of $G$ generated by $T$ and by $U_j, U_j^-$, $j \in J$.

Let $P := L \cdot U$ and $P^- := U^- \cdot L$. By definition, $P$ and $P^-$ are parabolic subgroups of $G$ such that $P \supset B$, $P^- \supset B^-$, and $P \cap P^- = L$.

Let $U_P$ and $U_P^-$ be the respectively the unipotent radicals of $P$ and $P^-$. Let $U_L = L \cap U$, $U_L^- = L \cap U^-$. Then $U_L$ and $U_L^-$ are the opposite unipotent radicals of $L$.

Denote $B_L^- := B^- \cap L$, and $B_L = B \cap L$. The open inclusion $B_L^- \hookrightarrow L/U_L$ induces a rational action of $U_L$ on $B_L^-$ which we denote by $\alpha_L : U_L \times B_L^- \to B_L^-$. Let $\pi^- = p_L^- : B^- \to B_L^-$ be the canonical projection. By definition, $p_L^-$ commutes with the rational action of $U_L$. In particular, $(B^-, p_L^-) \in \text{Ob}(U_L-\text{Cryst})$.

For any $U$-variety $X$ we denote by $X|_L$ the $U_L$-variety obtained by the restriction of the $U$-action $\alpha : U \times X \to X$ to $U_L$. 


Lemma 3.7. The mapping $\langle X, f_X \rangle \mapsto \langle X, f_X \rangle|_L := (X|_L, p_L^X \circ f_X)$ defines a functor of monoidal categories $|_L : U - \text{Cryst} \to U_L - \text{Cryst}$.

We call the unipotent $L$-crystal $(X, f_X)|_L$ the restriction of $(X, f_X)$ to $L$.

3.3. From unipotent $G$-crystals to geometric $L$-crystals. For each unipotent $G$-crystal $(X, f)$ define a morphism $\gamma = \gamma_X : X \to T$ by $\gamma = pr_T \circ f$. For each $i \in I$ define the function $\varphi_i = \varphi_i^X$ by

$$\varphi_i := \chi_i \circ f.$$ 

Let $\text{supp}(X, f)$ be set of all those $i \in I$ for which $\varphi_i^X \neq 0$. We call this set the support of the unipotent crystal $(X, f_X)$. For $i \in \text{supp}(X, f)$ define the morphism $e_i : \mathbb{G}_m \times X \to X$ by

$$e_i^X(x) = x_i \left( \frac{c - 1}{\varphi_i(x)} \right) (x).$$

It is easy to see that each $e_i$ is a rational action of $\mathbb{G}_m$ on $X$.

Theorem 3.8. For any $(X, f) \in \text{Ob}(U - \text{Cryst})$ the actions $e_i : \mathbb{G}_m \times X \to X$, $i \in \text{supp}(X, f)$ define a geometric $L_J$-crystal on $(X, \gamma_X)$, where $J = \text{supp}(X, f)$.

We denote this geometric $L_J$-crystal by $X_{\text{ind}}$ and call it the geometric crystal induced by $(X, f)$.

Remark. For the unipotent crystal $(G/B, [\gamma_0]^{-1})$ (defined in section 2.3) Theorem 3.8 specializes to Theorem 2.5.

Examples.
1. For the unipotent $G$-crystal $(B^-, \text{id}_{B^-})$ the actions $e_i : \mathbb{G}_m \times B^- \to B^-$, $i \in I$, are given by

$$e_i^B(b) = x_i \left( \frac{c - 1}{\varphi_i(b)} \right) \cdot b \cdot x_i \left( \frac{c - 1}{\varphi_i(b)} \right).$$

2. For the unipotent crystal $(G, \pi^-)$ the actions $e_i : \mathbb{G}_m \times G \to G$, $i \in I$, are given by

$$e_i^G(g) = x_i \left( \frac{c - 1}{\chi_i(g)} \right) \cdot g.$$

Lemma 3.9. For $(X, f_X), (Y, f_Y) \in \text{Ob}(U - \text{Cryst})$ put $(Z, f_Z) := (X, f_X) \times (Y, f_Y)$, where $Z = X \times Y$, and let $Z_{\text{ind}}$ be the geometric crystal on $(Z, \gamma_Z)$ induced by $(Z, f_Z)$. We have:

(a) $\gamma_Z = m \circ (\gamma_X \circ \gamma_Y)$, where $m : T \times T \to T$ is the multiplication morphism.
(b) For each $i \in I$, $(x, y) \in Z$:

$$\varphi_i^Z(x, y) = \varphi_i^X(x) + \frac{\varphi_i^Y(y)}{\alpha_i(\gamma_X(x))}.$$ 

which implies that $\text{supp}(Z, f_Z) = \text{supp}(X, f_X) \cup \text{supp}(Y, f_Y)$.

(c) For any $i \in \text{supp}(Z, f_Z)$ the action $e_i : \mathbb{G}_m \times Z \to Z$ is given by the formula: $e_i^Z(x, y) = (e_i^X(x), e_i^Y(y))$, where

$$e_1 = \frac{\varphi_i(x) \alpha_i(\gamma(x)) + \varphi_i(y)}{\varphi_i(x) \alpha_i(\gamma(x)) + \varphi_i(y)}, \quad e_2 = \frac{\varphi_i(x) \alpha_i(\gamma(x)) + \varphi_i(y)}{\varphi_i(x) \alpha_i(\gamma(x)) + c^{-1} \varphi_i(y)}.$$

Remark. The formula (3.11) for the action of $e_i$ on $X \times Y$ is analogous to the formula in [10], section 1.3, for the tensor product of Kashiwara’s crystals.
3.4. **Positive unipotent crystals and duality.** Let \((X, f) \in \text{Ob}(U - \text{Cryst})\), and let \(T'\) be an algebraic torus of the same dimension as \(X\). A birational isomorphism \(\theta : T' \rightarrow X\) is called a **positive structure** on \((X, f)\) if the following two conditions are satisfied.

1. The isomorphism \(\theta\) is a positive structure on the induced geometric \(L_{J'}\)-crystal \(X_{\text{ind}}\), where \(J = \text{supp}(X, f)\).
2. For any \(i \in J\) the function \(\varphi_i^X \circ \theta\) on \(T'\) is positive.

**Theorem 3.10.** Let \((X, f_X), (Y, f_Y) \in \text{Ob}(U - \text{Cryst})\) and \(\theta_X : T' \rightarrow X, \theta_Y : T'' \rightarrow Y\) be respectively the positive structures. Then the birational isomorphism \(\theta_{X,Y} := \theta_X \times \theta_Y\) is a positive structure on the product \((X, f_X) \times (Y, f_Y)\).

For a geometric pre-crystal \(X\) on \((X, \gamma)\) denote by \(\gamma^* : X \rightarrow T\) the morphism \(\gamma^*(x) = (\gamma(x))^{-1}\) and consider the **dual geometric pre-crystal** \(X^*\) on \((X, \gamma^*)\) by defining \((e_i^*)^c(x) = e_i^{-1}(x)\).

Given a geometric pre-crystal \(X\) on \((X, \gamma)\), for any morphism \(\theta : T' \rightarrow X\) define a morphism \(\theta^* : T' \rightarrow X\) as the composition of \(\theta\) with the inverse \(^{-1} : T' \rightarrow T'\).

The following fact is obvious.

**Lemma 3.11.** For any geometric crystal \(X\) the dual geometric pre-crystal \(X^*\) is a geometric crystal. For any positive structure \(\theta\) on \(X\) the morphism \(\theta^*\) is a positive structure on \(X^*\).

**Remark.** Duality in geometric crystals is an analogue of duality in Kashiwara’s crystals (see [16] and the Appendix below). More precisely, the tropicalization of \(X^*\) with respect to the positive structure \(\theta^*\) is the free combinatorial \(W\)-crystal dual to the tropicalization \(X\) with respect to \(\theta\).

Let \((X, f)\) be a unipotent crystal. Define a morphism \(f^* : X \rightarrow B\) by \(f^*(x) = (f(x))^{-1}\) and a rational morphism \(\alpha^* : U \times X \rightarrow X\) by

\[\alpha^*(u, x) := \pi \left( u \cdot f^*(x) \right) (x) .\]

**Proposition 3.12.** The morphism \(\alpha^*\) is a rational action of \(U\) on \(X\) and \(f^*\) is a \(U\)-morphism with respect to this action.

Denote the pair \((X, \alpha^*)\) by \(X^*\). Then the pair \((X^*, f^*)\) is a unipotent crystal. We call it the **dual unipotent crystal** of \((X, f)\) and denote it by \((X, f)^*\).

**Remark.** It follows from the definition that the mapping \((X, f) \mapsto (X, f)^*\) is an involutive functor \(* : U - \text{Cryst} \rightarrow U - \text{Cryst}\).

**Theorem 3.13.** For any unipotent crystals \((X, f_X), (Y, f_Y)\) the permutation of the factors \((12) : X \times Y \rightarrow Y \times X\) induces an isomorphism of unipotent crystals:

\[(X, f_X) \times (Y, f_Y)^* \simeq (Y, f_Y) \times (X, f_X)^* .\]

**Remark.** This theorem implies that the functor \(* : U - \text{Cryst} \rightarrow U - \text{Cryst}\) reverses the monoidal structure on \(U - \text{Cryst}\).

**Theorem 3.14.** Let \((X, f)\) be a unipotent crystal and \(X\) the induced geometric crystal. Then the geometric crystal induced by \((X, f)^*\) is equal to \(X^*\).
3.5. **Diagonalization of products of unipotent crystals.** Let \( v \) be the regular morphism \( Bu_0 B \rightarrow U \) defined by \( v(uut^{-1}) = uu' \) for any \( u, u' \in U, \ t \in T \).

By definition, \( v \) is two-sided \( U \)-equivariant.

**Remark.** For \( G = SL_2 \) the map \( v : G \rightarrow U \) is given by

\[
v \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & \frac{a-b}{c} \\ 0 & 1 \end{pmatrix}
\]

We call a unipotent crystal \((X, f_X)\) non-degenerate if \( ran(f_X) \) intersects \( Bu_0 B \) non-trivially.

For any non-degenerate unipotent crystal \((X, f_X) \in Ob(U - \text{Cryst})\) and any \((Y, f_Y) \in Ob(U - \text{Cryst})\) define the rational morphism \( F = F_{X,Y} : X \times Y \rightarrow X \times Y \) by \( F(x, y) = (x, v_x(y)) \), where \( v_x := v(f_X(x)) \). Clearly, \( F \) is a birational isomorphism \( X \times Y \cong X \times Y \), and its inverse is given by \( F^{-1}(x, y) = (x, (v_x)^{-1}(y)) \).

Denote by \( \delta = \delta_{X \times Y} : U \times X \times Y \rightarrow X \times Y \) the diagonal action of \( U \):

\[
(u, (x, y)) \mapsto (u(x), u(y)).
\]

**Proposition 3.15.** Let \((X, f)\) be a non-degenerate unipotent crystal, and \((Y, f_Y)\) be any unipotent crystal. Then

\[
(3.13) \quad F \circ \alpha = \delta \circ (\text{id}_U \times F).
\]

In other words the action \( \alpha = \alpha_{X \times Y} : U \times X \times Y \rightarrow X \times Y \) is diagonalized by \( F \).

3.6. **Unipotent action of the Weyl group.** For each \( J \subset I \) let \( W_J \) be the subgroup of \( W \) generated by \( s_j, j \in J \). By definition, \( W_J \) is the Weyl group of the standard Levi subgroup \( L = L_J \). So we sometimes denote \( W_J \) by \( W_L \).

**Lemma 3.16.** Let \((X, f)\) be a unipotent \( G \)-crystal, \( J = supp(X, f) \) and \( X_{\text{ind}} \) be the induced geometric \( L_J \)-crystal. Then the formula \( \underline{\underline{\phi}}(b) \) defines rational action of \( W_J \) on \( X \).

We call this action of \( W_J \) on \( X \) the unipotent action of \( W_J \) on \((X, f_X)\) (or, simply, unipotent action of \( W_J \) on \( X \)).

Recall from section \( \text{[3.3]} \) that \((B^-, \text{id}_{B^-})\) is a unipotent crystal.

**Lemma 3.17.** The unipotent action of \( W \) on \( B^- \) satisfies for \( i \in I \):

\[
(3.14) \quad s_i(b) = x_i(a) \cdot b \cdot (x_i(a))^{-1},
\]

where \( a = \frac{1 - \alpha_i(\gamma(b))}{\varphi(b) \alpha_i(\gamma(b))} \).

**Remark.** This lemma implies that \( dom(s_i) = \{ b \in B : x_i^{-1}(b) \neq 0 \} \).

For each \( w \in W \) let \( supp(w) \) be the minimal subset \( J \subset I \) such that \( w \in W_J \).

For example, \( supp(s_i) = \{ i \} \).

**Proposition 3.18.** For each \( w \in W \) there is a rational morphism \( u_w : B^- \rightarrow U \) such that:

(a) For any \( w' \in W \) such that \( supp(w') \supset supp(w) \) each rational \( U \)-orbit of the form \( t \cdot UwU \cap B^- \), \( t \in T \) intersects \( dom(u_w) \) non-trivially.

(b) The unipotent action of \( W \) on \( B^- \) is given by

\[
(3.15) \quad (w, b) \mapsto w(b) = u_w(b) \cdot b \cdot (u_w(b))^{-1}.
\]

For a any unipotent crystal \((X, f_X)\) and any \( w \in W \) with \( supp(w) \subset supp(X, f_X) \) we define a rational morphism \( u_w^X : X \rightarrow U \) by \( u_w^X = u_w \circ f_X \).
Theorem 3.19. Let \((X, f_X) \in \text{Ob}(U - \text{Cryst})\) such that \(\text{supp}(X, f_X) = J\). Then the crystal action of \(W_J\) on \(X\) is given by

\[
w(x) = (u^X_w(x))\ (x)
\]

for \(w \in W_J\).

Proposition 4.20. Let \((X_k, f_{X_k}) \in \text{Ob}(U - \text{Cryst})\), \(k = 1, \ldots, l\), and let \(J = \bigcup_k \text{supp}(X_k, f_{X_k})\). Then the unipotent action of \(W_J\) on \(X = X_1 \times \cdots \times X_l\) is given by

\[
w(x_1, \ldots, x_l) = (u(x_1), \ldots, u(x_l))
\]

for \(w \in W\), where \(u = u^X_w(x_1 \ldots x_l)\) as in \((3.16)\).

4. Examples of unipotent crystals

4.1. Standard unipotent crystals. For \(w \in W\) let \(O(w) := U \bar{w} U / U\). Clearly, \(O(w)\) is a left \(U\)-orbit in \(G/U\). Define \(U(w) := U \cap \bar{w} U \bar{w}^{-1}\).

Lemma 4.1. The mapping \(U \to O(w)\) defined by \(u \mapsto u \bar{w} U\) for \(u \in U\) is left \(U\)-equivariant and surjective. It induces the isomorphism of homogeneous \(U\)-varieties:

\[
\tilde{\eta}^w : U/U(w) \cong O(w).
\]

Remark. Under the natural map \(G \to G/B\) each orbit \(O(w)\) is identified with the corresponding Schubert cell. In particular, \(\dim O(w) = \dim U/U(w) = l(w)\).

For \(w \in W\) define \(U^w := U \cap B^{-w^{-1}} B^{-}\) and \(B_w^- := U \bar{w} U \cap B^{-}\).

Example. For \(G = SL_2\), \(w = w_0\) the sets \(U^w\) and \(B_w^-\) consist respectively of the matrices of the form:

\[
\begin{pmatrix}
1 & c \\
0 & 1
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
c & 0 \\
1 & c^{-1}
\end{pmatrix},
\]

\(c \in \mathbb{C}^\times\).

Denote by \(j^w : U^w \to U/U(w)\) the morphism induced by the natural inclusion \(U^w \hookrightarrow U\), and by \(j_w : B_w^- \to O(w)\) the morphism induced by the natural inclusion \(B_w^- \hookrightarrow U \bar{w} U\).

By definition, the restrictions of \(\pi\) and \(\pi^-\) to \(B^- \cdot U\) are the regular projections \(B^- \cdot U \to U\) and \(B^- \cdot U \to B^-\) respectively. Since \(B^- \bar{w}^{-1} B^- \subset B^- \cdot U\) we can define a regular morphism \(\eta^w : U^w \to B^-\) by

\[
\eta^w(u) = \pi^-(u \bar{w})
\]

for \(u \in U^w\).

Proposition 4.2.

(a) The morphisms \(j_w\) and \(j^w\) are the open inclusions \(B_w^- \hookrightarrow O(w)\) and \(U^w \hookrightarrow U/U(w)\) respectively.

(b) The morphism \(\eta^w\) is a biregular isomorphism \(U^w \cong B^-\). The inverse isomorphism \(\eta_w = (\eta^w)^{-1} : B_w^- \cong U^w\) is given by

\[
(4.1)
\eta_w(b) = \pi(\bar{w} \cdot b^{-1})^{-1}.
\]

(c) The following diagram is commutative:

\[
\begin{array}{ccc}
U/U(w) & \xrightarrow{\tilde{\eta}^w} & O(w) \\
\uparrow j^w & & \uparrow j_w \\
U^w & \xrightarrow{\eta^w} & B_w^-
\end{array}
\]

\[
(4.2)
\]
The birational isomorphisms \([j^w]\) and \([j_w]\) define for any \(w \in W\) rational unital \(U\)-actions \(\alpha^w : U \times U^w \to U^w\) and \(\alpha_w : U \times B_w^- \to B^-\) respectively. We denote by 
\[ U^w := (U^w, \alpha^w) \text{ and } B_w^- := (B_w, \alpha_w) \]
the corresponding \(U\)-varieties.

Note that \((U^w, \eta^w)\) and \((B_w, \text{id}_w)\) are unipotent \(G\)-crystals. We call these unipotent crystals standard.

**Lemma 4.3.** For each \(w \in W\) the isomorphism \(\eta^w\) induces the isomorphism of unipotent crystals \((U^w, \eta^w) \to \to (B_w, \text{id}_w)\).

Note that the \(U\)-action on \(B_w^-\) is given by \([s]\). In order to describe the \(U\)-action on \(U^w\) we need to introduce more notation.

By Proposition \([4.2]\) the multiplication morphism \(U \times U \to U\) induces the birational isomorphism \(U^w \times U^w \to U^w\). Denote by \(\pi^w : U \to U^w\) the composition of the inverse of this isomorphism with \(pr_2\).

By definition, the action \(U \times U^w \to U^w\): \((u, x) \mapsto u(x)\) is given by:

\[
\pi^w(u \cdot x) = (\pi^w(u \cdot x))^{-1}.
\]

**Lemma 4.4.** For any \(w, w' \in W\) such that \(l(w) = l(w) + l(w')\) and \(x \in U^w\), \(y \in U^w\), \(u \in U\) we have:

\[
\pi^{w'}(u \cdot x \cdot y) = \pi^w(\pi^w(u \cdot x) \cdot y)
\]
or, equivalently, \(u(x \cdot y) = u(x) (\pi^w(u \cdot x)(y))\).

Next, we compute the action \([4.3]\) explicitly in terms of the isomorphism \(\eta^w\).

**Proposition 4.5.** For each \(w \in W\), \(u \in U\), and \(x \in U^w\) we have:

\[
\pi^w(u \cdot x) = \pi (\text{Ad} (\pi(u \cdot b)) \cdot \pi^{-1}(b \cdot b^{-1}))
\]
where \(b = \eta^w(x)\).

4.2. **Multiplication of standard unipotent crystals.** Define a new associative multiplication \(\ast\) on the set \(W\) by

\[
w \ast w' = w w'
\]
if \(l(w w') = l(w) + l(w')\) for \(w, w' \in W\) and \(s_i \ast s_i = s_i\) for all \(i\).

Under this new operation \(\ast\) is identified with the standard multiplicative monoid of the degenerate Hecke algebra of \(W\). We denote this monoid by \((W, \ast)\).

**Lemma 4.6.** For any \(w, w' \in W\) the multiplication morphism \(G \times G \to G\) induces a dominant rational morphism

\[
\pi^{w', w} : U^{w'} \times U^w \to U^{w \ast w'}.
\]

This is a birational isomorphism if and only if \(w \ast w' = w w'\).

It is well-known (see e.g., \([5]\), Theorems 1.2, 1.3) that for any \(w, w' \in W\) such that \(w \ast w' = w w'\) the multiplication morphism \(G \times G \to G\) induces the open inclusion

\[
B_w^- \times B_{w'}^- \to B_{w \ast w'}^-.
\]

The following result deals with a generalization \([4.10]\) of to any pair \(w, w'\).
Proposition 4.7. For any $w, w' \in W$:
(a) The intersection of $B_w^- \cdot B_{w'}^-$ with $B_{ww'}^-$ is a dense subset of $B_{ww'}^-$. 
(b) There exists an algebraic sub-torus $\overline{T}_{w, w'} \subset T$ such that the restriction of the multiplication morphism $G \times G \to G$ to $B_w^- \times B_{w'}^-$ is a dominant rational morphism
\begin{equation}
B_w^- \times B_{w'}^- \to B_{ww'}^- \cdot \overline{T}_{w, w'}^- .
\end{equation}
This is a birational isomorphism if and only if $l(w \ast w') = l(w) + l(w') - \dim \overline{T}_{w, w'}^- .$
(c) $\overline{T}_{w, w'} = \{ e \}$ if and only if $w \ast w = ww'$.
(d) $\overline{T}_{\alpha_i, \alpha_i} = \alpha_i^\vee (G_m)$ for each $i \in I$.
(e) For any $w, w', w'' \in W$ one has: $\overline{T}_{w, w''} \cdot \overline{T}_{w', w''} = \overline{T}_{ww', w''} \cdot (w'')^- (\overline{T}_{w, w'})$.

Corollary 4.8. For any $w, w' \in W$ the morphism $\mathbb{L}_{w, w'}$ induces the morphism in $U - \text{Cryst}$: $(B_w^-, \text{id}_{w}) \times (B_{w'}^-, \text{id}_{w'}) \to (B_{ww''}^-, \overline{T}_{w, w'}, \text{id}_{B_{ww''}^- \cdot \overline{T}_{w, w'}})$.

Next, we compute a lower bound for each $\overline{T}_{w, w'}^-$. For each $w \in W$ let us consider a homomorphism of tori $T \to T$ defined by $t \mapsto w(t) \cdot t^{-1}$. Denote by $T_w$ the image of this homomorphism. Clearly, $T_w$ is a sub-torus of $T$ such that
\[ X_*(T_w) = (\Lambda^w)^- \bigcap \bigoplus_l \mathbb{Z}_{\alpha_l^\vee}, \]
where $\Lambda^w = \{ \lambda \in \Lambda : w(\lambda) = \lambda \}$. This implies that $X_*(T_w)$ has a $\mathbb{Z}$-basis of certain (not necessarily simple) coroots.

Proposition 4.9. For any $w, w' \in W$ we have $\overline{T}_{w, w'}^- \subset T_{(w \ast w')^-}^{-1} \cdot w \ast w'$.

From now on we will freely use the notation of section 3.2. For any standard Levi subgroups $L \subset L' \subset G$ define the elements $w_{L', L}, w_{L', L'} \in W$ by
\begin{equation}
w_{L', L} := w_{L'}^0 \cdot w_{L, L}^0, \quad w_{L', L'} := w_{L'}^0 \cdot w_{L'}^0 .
\end{equation}
Clearly, $w_{L', L} \cdot w_{L', L} = e$ and $l(w_{L', L}) = l(w_{L, L}) + l(w_{L'}^0) - l(w_{L}^0)$.

Note that $U_{L'}(w_{L', L}) = U_L$ and $U_{L'}(w_{L', L'}) = \text{Ad}_{w_{L', L'}}^{-1}(U_L)$.

It turns out that for $w = w_{L', L}$ the formula 3.2 simplifies when $u \in U_L$.

Proposition 4.10. Let $L \subset L'$ be standard Levi subgroups of $G$. Then for any $u \in U_L$, $x \in U_{w_{L', L}^-}$ we have:
\[ \pi_{w_{L', L}^-}^w(u \cdot x) = \text{Ad}_{w_{L', L}^-}^{-1}(\pi(u \cdot \eta_{w_{L', L}^-}^w(x))) . \]

For each $w \in W$ let $I(w)$ be the set of all $i \in I$ such that $w(\alpha_i) = \alpha_i$ for some $i' \in I$. Note that an element $w' \in W$ belongs to $W_{I(w)}$ if and only if $\text{Ad} \overline{w'}(U_{w'}) = U_{ww'w'^{-1}}$.

Lemma 4.11. Let $w \in W, w', w'' \in W_{I(w)}$. Then
\[ \text{Ad} \overline{w'} \circ \pi_{w''}^w|_{U_{w''}} = \pi_{ww'w'^{-1}}|_{U_{ww'w'^{-1}}} \circ \text{Ad} \overline{w''} . \]

Theorem 4.12. Let $L \subset L'$ be standard Levi subgroups of $G$, and let $w \in W$ be any element such that $\overline{w_{L', L}^-} \cdot \overline{w}$ centralizes $U_L$. Then for any $w' \in W_{I(w)}$ we have:
(a) The morphism $f_{w, w'} : U_{w_{L', L}^-} \times U_{w'}^r \to U$ given by $(x, y) \mapsto x \cdot \text{Ad} \overline{w'}(y)$ is a dominant rational morphism $U_{w_{L', L}^-} \times U_{w'}^r \to U_{ww'w'^{-1} \ast w_{L', L}^-}$. This is a birational isomorphism if and only if $(ww'w'^{-1}) \ast w_{L', L}^- = (ww'w'^{-1}) \cdot w_{L', L}^-$. 

Corollary 4.13.

(a) The restriction \((\mathcal{U}^{w_{m,n}, \eta_{w_{m,n}}})|_{L_m}\) is isomorphic in \(U_{L_m} - \text{Cryst}\) to

\[
\prod_{l=1}^{n}(\mathcal{U}^{C_m, \eta_{C_m}})|_{L_m}.
\]

(b) The restriction \((\mathcal{U}^{w_{m,n}, \eta_{w_{m,n}}})|_{L'_n}\) is isomorphic in \(U_{L'_n} - \text{Cryst}\) to

\[
\prod_{k=1}^{m}(\mathcal{U}^{C'_n, \eta_{C'_n}})|_{L'_n}.
\]

4.3. \(W\)-invariant functions on standard unipotent \(G\)-crystals. For each \(w \in W\) let \(\hat{U}(w)\) be the set of all \(\chi \in \hat{U}\) such that \(\chi(\overline{w^{-1}}u\overline{w}) = \chi(u)\) for all \(u \in U(w)\).

Clearly, \(\hat{U}(e) = \hat{U}(w_0) = \hat{U}\).

For any \(\chi \in \hat{U}\) define a regular function \(\chi^w : BwB \rightarrow \mathbb{G}_a\) by

\[
\chi^w_{\chi}(g) = \chi(\overline{w^{-1}}g).
\]

Let \(\rho^i \in X_*(T^{ad})\) be the co-character of \(T^{ad} = T/Z(G)\) such that \((\rho^i, \alpha_i) = 1\) for \(i \in I\). Define the anti-automorphism \(i\) of \(G\) by \(i(g) = Ad \rho^i(-1)(g^{-1})\).

By definition, \(i(t) = t^{-1}\) for \(t \in T\) and \(i \circ x_i = x_i\), \(i \circ y_i = y_i\) for all \(i \in I\).

It is easy to see that \(i(\overline{w}) = \overline{w^{-1}}\) for any \(w \in W\) and \(\chi \circ i = \chi\) for any \(\chi \in \hat{U}\).

For any \(\chi, \chi' \in \hat{U}\) and any \(w \in W\) define a regular function \(f^w_{\chi, \chi'}\) on \(BwB\) by

\[
f^w_{\chi, \chi'} : \chi^w_{\chi'}(g) = \chi^w + \chi^w_{\chi'} \circ i.
\]

Let \(L = L_J\) be any standard Levi subgroup of \(G\). For any \(\chi = \sum_{i \in I} a_i \chi_i \in \hat{U}\) let \(\chi^L \in \hat{U}\) be defined by

\[
\chi^L := \sum_{i \in I \setminus J} a_i \chi_i.
\]

By definition, \(\chi^L|_{U_L} \equiv 0\).

Theorem 4.14. For any \(\chi \in \hat{U}(w_{L,G})\) and any \(t \in Z(L)\) the restriction of the function \(f^w_{\chi, \chi^L}: L \cdot Bw_{L,G} - \text{Cryst}\) is invariant under the unipotent action of \(W\) on \(t \cdot Bw_{L,G}\).
Next, we describe for each $w \in W$ a basis for the subspace $\hat{U}(w) \subset \hat{U}$. For each $w \in W$ define a bijective map $\zeta_w : I(w) \to I(w^{-1})$ by $w(\alpha_i) = \alpha_{\zeta_w(i)}$. This $\zeta_w$ is a partial bijection $I \to I$ (see the Appendix) with $\text{dom}(\zeta) = I(w)$ and $\text{ran}(\zeta) = I(w^{-1})$.

For each $i \in I$ we define the set $\mathbf{o}_w(i)$ as follows. For $i \in \text{dom}(\zeta) \cup \text{ran}(\zeta)$ we put

$$\mathbf{o}_w(i) := \{ \ldots, (\zeta_w)^{-2}(i), (\zeta_w)^{-1}(i), i, \zeta_w(i), (\zeta_w)^2(i), \ldots \}$$

(where for each $k \in \mathbb{Z}$ the power $(\zeta_w)^k$ is considered to be a partial bijection $I \to I$).

For $i \notin \text{dom}(\zeta) \cup \text{ran}(\zeta)$ we define $\mathbf{o}_w(i) := \{ i \}$.

For any $J \subset I$ let $\chi_J \in \hat{U}$ be defined by $\chi_J = \sum_{j \in J} \chi_j$.

**Lemma 4.15.** For each $i \in I$ the function $\chi_{\mathbf{o}_w(i)}$ belongs to $\hat{U}(w)$, and the set of all $\chi_{\mathbf{o}_w(i)}$, $i \in I$, is a basis in the vector space $\hat{U}(w)$.

Recall that the unipotent action of $W_J$ was defined in section 3.6.

**Theorem 4.16.**

(a) For each $\chi \in \hat{U}(w_G,L)$ the restriction of $\chi$ to $U^{w,G,L}$ is invariant under the unipotent action of $W_J$ on the unipotent L-crystal $(U^{w,G,L}, \eta^{w,G,L})|_{W_J}$.

(b) For each $\chi \in \hat{U}(w_{L,G})$ the restriction of the function $\chi_{w,L,G}$ to $B_{w,L,G}^-$ is invariant under the unipotent action of $W_J$ on the unipotent L-crystal $(B_{w,L,G}^-, \id_{w_{L,G}})|_{B_{w,L,G}^-}$.

### 4.4. Positive structures on standard unipotent crystals.

For any $i \in I$ define the morphism $\pi_i : \mathbb{G}_m \to B^-$ by the formula $\pi_i(c) := y_i\left( \frac{1}{\alpha_i} \right) \alpha_i^c(c)$. Clearly, $\pi_i$ is a birational isomorphism $\mathbb{G}_m \rightarrowtail B_-^{\alpha_i}$. The isomorphism $\pi_i : \mathbb{G}_m \rightarrowtail B_-^{\alpha_i}$ defines a structure of $U$-variety on $\mathbb{G}_m$ and a unipotent crystal on $\mathbb{G}_m$ which we denote by $(\mathbb{G}_m, \pi_i)$.

For each sequence $i = (i_1, \ldots, i_l)$ define the regular morphism $\pi_i : (\mathbb{G}_m)^l \to B^-$ by the formula

$$\pi_i(c_1, \ldots, c_l) = \pi_{i_1}(c_1) \cdots \pi_{i_l}(c_l).$$

For any sequence $i = (i_1, \ldots, i_l) \in I^l$ define $w_* (i) := s_{i_1} \cdots s_{i_l}$.

**Proposition 4.17.** For any sequence $i = (i_1, \ldots, i_l) \in I^l$ we have:

(a) There exists a sub-torus $\hat{T}_i \subset T$ such that $[\pi_1]$ is a dominant rational morphism $(\mathbb{G}_m)^l \to B_{w_*(i)}^- \cdot \hat{T}_i$. The morphism $[\pi_1]$ is a birational isomorphism if and only if

$$l = l(w_*(i)) + \dim \hat{T}_i.$$

(b) The rational morphism $[\pi_1]$ induces the morphism in $U - \text{Cryst}$:

$$(\mathbb{G}_m, \pi_1) \times \cdots \times (\mathbb{G}_m, \pi_1) \to \left( B_{w_*(i)}^- \cdot \hat{T}_i, \id_{B_{w_*(i)}^-} \cdot \hat{T}_i \right).$$

(c) If the equality (4.14) holds then $[\pi_1]$ is a positive structure on the unipotent crystal $\left( B_{w_*(i)}^- \cdot \hat{T}_i, \id_{B_{w_*(i)}^-} \cdot \hat{T}_i \right)$.

(d) There is an inductive formula for the computation of $\hat{T}_i$:

$$\hat{T}_i = \alpha_i^c((\mathbb{G}_m), \text{ and for any sequence } i = (i_1, \ldots, i_l) \in I^l \text{ and any } i \in I \text{ one has }$$

$$\hat{T}_{(i,i)} = \left\{ \begin{array}{ll} s_i(\hat{T}_i), & \text{if } l(w_*(i)s) = l(w_*(i)) + 1, \\ s_i(\hat{T}_i) \cdot \hat{T}_i, & \text{if } l(w_*(i)s) = l(w_*(i)) - 1. \end{array} \right.$$
Remarks.
1. For each \( w \in W, i \in R(w) \), the tropicalization of the geometric crystal induced by \((B^-_w, id)_w\) with respect to \( \theta^i\) is equal to the corresponding free combinatorial crystal \( B_i \) which was constructed in [10] as a product of 1-dimensional crystals \( B_{i_1}, \ldots, B_{i_l} \). By the construction, \( B_i \) is a free \( W_{[i]} \)-crystal, where \( [i] = \{i_1, \ldots, i_l\} \).
2. For any dominant \( \lambda^\vee \in \Lambda^\vee \) the image of Kashiwara’s embedding \((\mathfrak{4})\) can be described as follows (see e.g. \([4]\)). Let \( \chi = \sum_{i \in I} \chi_i \in \bar{U} \). For any \( i \in R(w_0) \) let \( \theta_i = id_T \times [\pi_i]: T \times (\mathbb{G}_m)^{l_0}_w \rightarrow T \cdot B^-_{w_0} \) be the positive structure on \( T \cdot B^-_{w_0} \). Then the image of the embedding \( B(V_{\lambda^\vee}) \hookrightarrow B_i = \mathbb{Z}^{l_0} \) is
\[
\{ b \in B_i : \text{Trop}_{\theta_i} (f^\vee_{w_0} (\lambda^\vee, b) \geq 0) \}.
\]

For each \( i = (i_1, \ldots, i_l) \in I^l \) define the regular morphism \( \pi^i : (\mathbb{G}_m)^l \rightarrow U \) by
\[
\pi^i(c_1, \ldots, c_l) = x_{i_1}(c_{i_1}) \cdots x_{i_l}(c_{i_l}).
\]

Proposition 4.18. For any sequence \( i = (i_1, \ldots, i_l) \in I^l \) we have:
(a) The morphism \( \pi^i \) induces a dominant rational morphism \((\mathbb{G}_m)^l \rightarrow U^{(w, (i))^{-1}} \).
If \( i \in R(w) \) for some \( w \in W \) then \( \pi^i \) is an open inclusion \((\mathbb{G}_m)^l(w) \hookrightarrow U_w \).
(b) If the sequence \( i \) belongs to \( R(w) \) for some \( w \in W \) then the birational isomorphism \( \pi^i : (\mathbb{G}_m)^l(w) \hookrightarrow U^{w^{-1}} \) is a positive structure on the unipotent crystal \((U^{w^{-1}}, \eta^{w^{-1}})\).

4.5. Duality and symmetries for standard unipotent crystals. It is easy to see that the inverse \(-1 : B^- \rightarrow B^- \) induces the isomorphism \( B^-_w \cong B^-_{w^{-1}} \) for each \( w \in W \).

Lemma 4.19. The inverse in \( B^- \) induces the isomorphism of unipotent crystals
\[
(B^-_w, id)_w \cong (B^-_{w^{-1}}, id)_{w^{-1}}
\]
for each \( w \in W \). In particular, the unipotent crystal \( (B^-_w, id)_w \) is self-dual if \( w^2 = e \).

5. Proofs

5.1. Proofs of results in section 2

Proof of Theorem 2.3. Since theorem 2.3 is a particular case of Theorem 3.8, we refer to the proof of Theorem 3.8 in section 5.2. \( \square \)

Proof of Proposition 2.3. Clearly, \( s_i : X \rightarrow X \) is a birational involution. Thus, it suffices to prove that for any \( w_1, w_2 \in W \) satisfying \( l(w_1w_2) = l(w_1) + l(w_2) \) one has:
\[
w_1(w_2(x)) = (w_1w_2)(x).
\]

As it follows from definition of \( e_w : T \times X \rightarrow X \) we have
\[
e^t_{w_1w_2} = e^t_{w_1w_2} = e^t_{w_1}(t) \cdot e^t_{w_2},
\]
for all \( w_1, w_2 \in W \) satisfying \( l(w_1w_2) = l(w_1) + l(w_2) \). Note that for any \( w \in W \) we have
\[
\gamma(e^t_w(x)) = t \cdot w(t^{-1}) \cdot \gamma(x).
\]
This implies that \( \gamma(w(x)) = \gamma(e^t_w(x)^{-1}(x)) = w(\gamma(x)) \). Therefore, for any \( w_1, w_2 \in W \) such that \( l(w_1w_2) = l(w_1) + l(w_2) \) we have
\[
w_1(w_2(x)) = e^{\gamma(w_2(x))^{-1}}_{w_1}(e^{\gamma(w_2(x))^{-1}}_{w_2}(x))
\]
the equalities

Lemma 5.1. (5.1)

Proof of Theorem 2.13 By definition, for any \(T \times T' \to T'\) by twisting with \(\theta\), that is, \((w, t') \to \theta^{-1}(w(\theta(t')))\). It is easy to see that \(T\) acts on \(T'\) by a positive birational isomorphisms. Therefore, applying the functor \(T\) to the action \(W \times T' \to T'\) we obtain an action of \(W\) on \(X_s(T')\).

Theorem 2.11 is proved.

Proof of Theorem 2.13 By definition, for \(i, j \in I = \{1, 2, \ldots, r\}\) we have:

\[\omega_j(e^c_j(x)) = e^c \omega_i(x)\]

Thus \(\tau(e^c_j(x)) = \tau(x)\) and therefore \(\tau(s_1(x)) = \tau(x)\).

Let us compute \(\tau(e^c_j(x))\) for \(j > 1\). The computation is based on the following obvious statement.

Lemma 5.1. For each \(j = 2, \ldots, r\) we have:

\[(5.1) \quad \tau(e^c_j(x)) = (e^c_{j/r} \circ \tau_j) (e^c_j(x))\]

where

\[(5.2) \quad \tau_j(z) = \begin{pmatrix} \omega_{j+1}(z) & \omega_j(z) & \cdots & \omega_2(z) & \omega_1(z) \\ e_1 & e_2 & \cdots & e_2 & e_1 \end{pmatrix} \begin{pmatrix} \omega_{j+1}(z) \\ e_1 \omega_{j+1}(z) & \omega_j(z) & \cdots & \omega_2(z) & \omega_1(z) \\ e_1 & e_2 & \cdots & e_2 & e_1 \end{pmatrix} (z),
\]

\[e^c_{j/r}(z) = \begin{pmatrix} \omega_{j+1}(z) & \omega_j(z) & \cdots & \omega_2(z) & \omega_1(z) \\ e_1 & e_2 & \cdots & e_2 & e_1 \end{pmatrix} \begin{pmatrix} \omega_{j+1}(z) \\ e_1 \omega_{j+1}(z) & \omega_j(z) & \cdots & \omega_2(z) & \omega_1(z) \\ e_1 & e_2 & \cdots & e_2 & e_1 \end{pmatrix} (z),\]

and we use the convention \(\omega_{r+1}(x) \equiv 1\) and \(e^c_{(r/r)} = 1\).

Substituting \(c = \frac{1}{\omega_j(x)} = \frac{\omega_{j-1}(x) \omega_{j+1}(x)}{\omega_j(x)}\) into (5.1) we obtain
\[
\tau(s_j(x)) = (e^c_{j/r}) \circ \tau_j(s_j(x))\]

This implies that all we have to do to prove Theorem 2.13 is to check that

\[(5.3) \quad \tau_j(s_j(x)) = \tau_j(x)\]

for \(j = 2, \ldots, r\).

Let us compute first \(\tau_j(e^c_j(x)), j \geq 2\). Substituting \(z = e^c_j(x)\) in (5.2) and using the equalities \(\omega_k(z) = e^c_k \omega_j(x)\) and \(\tau^{-1} \circ e^c_j = e^c_j \circ \tau^{-1}\) we obtain
\[
\tau_j(e^c_j(x)) = \begin{pmatrix} \omega_{j+1}(x) \\ e_1 \omega_{j+1}(x) & \omega_j(x) & \cdots & \omega_2(x) & \omega_1(x) \\ e_1 & e_2 & \cdots & e_2 & e_1 \end{pmatrix} \begin{pmatrix} \omega_{j+1}(x) \\ e_1 \omega_{j+1}(x) & \omega_j(x) & \cdots & \omega_2(x) & \omega_1(x) \\ e_1 & e_2 & \cdots & e_2 & e_1 \end{pmatrix} (e^c_j \circ \tau_j^{-1}(x)).
\]

Substituting \(c = \frac{\omega_{j-1}(z) \omega_{j+1}(z)}{\omega_j(z)}\) into (5.2), we obtain
\[
\tau_j(s_j(x)) = \begin{pmatrix} \omega_{j+1}(x) & \omega_j(x) & \cdots & \omega_2(x) & \omega_1(x) \\ e_1 \omega_{j+1}(x) & \omega_j(x) & \cdots & \omega_2(x) & \omega_1(x) \\ e_1 & e_2 & \cdots & e_2 & e_1 \end{pmatrix} \begin{pmatrix} \omega_{j+1}(x) \\ e_1 \omega_{j+1}(x) & \omega_j(x) & \cdots & \omega_2(x) & \omega_1(x) \\ e_1 & e_2 & \cdots & e_2 & e_1 \end{pmatrix} (\tau_j^{-1}(x)).
\]

In order to finish the proof of Theorem 2.13 we will use the following result.

Lemma 5.2. For any \(j = 1, 2, \ldots, r\) the following relation holds.

\[(5.4) \quad (e^c_1 \cdots e^c_j)(e^c_j \cdots e^c_1) e^c_j = (e^c_1 \cdots e^c_j)(e^c_i \cdots e^c_1)\]
Lemma is proved. □

This identity is true because the morphism \( \pi \) for any \( \pi \)

Proof. Induction in \( j \). For \( j = 1 \) the identity \( e_j^\varepsilon e_j^{\varepsilon'} = e_j^{\varepsilon'} \) is true.

Now let \( j > 1 \). Using the commutation of \( e_j \) with each of \( e_1, \ldots, e_{j-2} \) we rewrite the left hand side of (5.4) as:

\[
(\epsilon_j^c \cdots \epsilon_{j-1}^c)(\epsilon_j^e \cdots \epsilon_{j-2}^e)\epsilon_j^c e_{j-1}^{\varepsilon'}.
\]

Applying the to above expression the basic relation (2.4) written in the form

\[
e_j^c e_{j-1}^c e_j^{\varepsilon'} = e_{j-1}^c e_j^c e_{j-1}^{\varepsilon'},
\]

we see that the left hand side of (5.4) equals:

\[
(\epsilon_j^c \cdots \epsilon_{j-1}^c)(\epsilon_j^e \cdots \epsilon_{j-2}^e)\epsilon_j^{\varepsilon'} e_{j-1}^{\varepsilon'} = (\epsilon_j^c \cdots \epsilon_{j-1}^c)(\epsilon_j^e \cdots \epsilon_{j-2}^e)\epsilon_j^{\varepsilon'} e_{j-1}^{\varepsilon'}.
\]

Finally, applying the inductive hypothesis (5.4) with \( j-1 \), we obtain

\[
(\epsilon_j^c \cdots \epsilon_{j-1}^c)(\epsilon_j^e \cdots \epsilon_{j-2}^e)\epsilon_j^c e_{j-1}^{\varepsilon'} = (\epsilon_j^c \cdots \epsilon_{j-1}^c)(\epsilon_j^e \cdots \epsilon_{j-2}^e)\epsilon_j^{\varepsilon'} e_{j-1}^{\varepsilon'}.
\]

Lemma is proved.

We see that (5.4) is a special case of Lemma (5.2) with \( \varepsilon = \frac{\omega_j(x)}{\omega_{j-1}(x)} \) \( e' = \frac{\omega_{j+1}(x)}{\omega_j(x)} \). Theorem (2.3) is proved. □

5.2. Proof of results in section 3

Proof of Theorem 3.3

Lemma 5.3. For any unipotent crystal \((X, f)\) the following identity holds.

\[
\pi(u' \cdot f(u(x))) \cdot \pi(u \cdot f(x)) = \pi(\pi(u') \cdot f(x))
\]

for any \( u, u' \in U, x \in X \).

Proof. Denote \( b = f(x) \). Note that \( f(u(x)) = u(b) = u \cdot b \cdot \pi(u \cdot b)^{-1} \). Then the identity (5.5) can be rewritten as

\[
\pi(u' \cdot b \cdot \pi(u \cdot b)^{-1}) \cdot \pi(u \cdot b) = \pi(u' \cdot b)
\]

This identity is true because the morphism \( \pi : G \to U \) is right \( U \)-equivariant.

Lemma is proved.

One can easily see that (5.5) implies that \( (u' u)(x, y) = u'(u(x, y)) \) for all \( u, u' \in U, (x, y) \in X \times Y \).

Thus, the formula (5.5) defines an \( U \)-action.

Part (a) is proved. Prove (b). We need the following fact.

Lemma 5.4. For \( u \in U, b, b' \in B^- \) we have

\[
\pi(u \cdot b \cdot b') = \pi(\pi(u \cdot b) \cdot b')
\]

Proof. It suffices to take \( u = x_i(a) \). In this case Lemma (3.1) implies that for \( u, u' \in U^- \), \( t, t' \in T \) we have:

\[
\pi(x_i(a) \cdot u \cdot tu' \cdot t') = \pi(x_i(a) \cdot u \cdot tu^{-1} \cdot t') = x_i((a^{-1} + \chi_i^{-1}(u)tu^{-1})^{-1} \cdot \alpha_i(t^{-1}))
\]

\[
= x_i((a^{-1} + \chi_i^{-1}(u + \alpha_i(t^{-1}) \chi_i(u')^{-1} \cdot \alpha_i(t^{-1})) \cdot \alpha_i(t^{-1}))
\]

\[
= x_i((a'^{-1} + \chi_i^{-1}(u')^{-1} \cdot \alpha_i(t^{-1})) = \pi(x_i(a') \cdot u't')
\]

where \( a' = (a^{-1} + \chi_i^{-1}(u))^{-1} \cdot \alpha_i(t^{-1}) \). Lemma is proved.

Part (b) is proved, and Theorem (3.3) is also proved. □
Proof of Proposition 3.5 Let \((X_1, f_{X_1}), (X_2, f_{X_2}), (X_3, f_{X_3})\) be unipotent crystals and let
\[
(X_{12,3}, f_{X_{12,3}}) := ((X_1, f_{X_1}) \times (X_2, f_{X_2})) \times (X_3, f_{X_3}) ,
\]
\[
(X_{1,23}, f_{X_{1,23}}) := (X_1, f_{X_1}) \times ((X_2, f_{X_2}) \times (X_3, f_{X_3})).
\]
For any \(x_k \in X_k, k = 1, 2, 3\) we have
\[
f_{X_{12,3}}((x_1, x_2), x_3) = f_{X_1}(x_1) \cdot f_{X_2}(x_2) \cdot f_{X_3}(x_3) = f_{X_{1,23}}(x_1, (x_2, x_3)),
\]
that is, \(f_{X_{12,3}} = f_{X_{1,23}}\).

It suffices to prove that \(X_{12,3} = X_{1,23}\), that is, these two \(U\)-actions on \(X_1 \times X_2 \times X_3\) are equal.

For \(x_k \in X_k, k = 1, 2, 3\) denote \(b_k := f_{X_k}(x_k)\).

Let us write the \(U\)-actions respectively on \(X_{12,3}, X_{1,23}\):
\[
\begin{align*}
(u(x_1, x_2), x_3) &= (u(x_1, x_2), \pi(u \cdot b_1 \cdot b_2)(x_3)) \\
&= (u(x_1), \pi(u \cdot b_1)(x_2), \pi(u \cdot b_1 \cdot b_2)(x_3)). \\
(u(x_1, x_2), x_3) &= (u(x_1), \pi(u \cdot b_1)(x_2), \pi(u \cdot b_1 \cdot b_2)(x_3)).
\end{align*}
\]

It follows from Lemma 5.4 that \(u((x_1, x_2), x_3) = u(x_1, (x_2, x_3))\).

This proves Proposition 3.5. \(\square\)

Proof of Proposition 3.6 It follows from the definition (3.4) of \(g\) and the definition of \(\pi\) that for \(g \in G, u \in U\) one has:
\[
g(u \cdot g) = f_G(u \cdot g) \cdot \pi(u \cdot g).
\]

Note that \(f_G(u \cdot g) = u \cdot (f_G(g))\) and \(\pi(u g) = \pi(u f_G(g)) \cdot \pi(g)\). Thus, Proposition 3.6 follows from Theorem 3.3. \(\square\)

Proof of Theorem 3.8 In the view of Lemma 3.1, it suffices to prove the relations (2.5) and (2.9).

First, we are going to prove the relations (2.6) and (2.7). For \(\varepsilon \in \{0, -1\}\) let
\[
(I \times I)_{\varepsilon} = \{(i, j) \in I \times I : i \neq j\} \text{ and } \langle \alpha_i', \alpha_j \rangle = \langle \alpha_j', \alpha_i \rangle = \varepsilon.
\]

Note that if \(G\) is simply-laced, then \((I \times I)_0 \cup (I \times I)_{-1} \cup \Delta(I) = I \times I\).

Lemma 5.5.
(a) For any \((i, j) \in (I \times I)_0\) we have
\[
(5.7) 
\]
(b) For any \((i, j) \in (I \times I)_{-1}\) we have
\[
(5.8) 
\]

Proof. Clear. \(\square\)

Proposition 5.6. Let \((X, f)\) be a unipotent \(G\)-crystal and let \(X_{ind}\) be the induced geometric crystal. For any \(i, j \in (I \times I)_0 \cup (I \times I)_{-1}\) such that \(\varphi_i \neq 0, \varphi_j \neq 0\), one has:

(a) if \((i, j) \in (I \times I)_0\) then
\[
(5.9) 
\]

(b) if \((i, j) \in (I \times I)_{-1}\) then
\[
(5.10) 
\]
(b) If \((i, j) \in (I \times I)_0\) then there exist rational functions \(\varphi_{ij}, \varphi_{ji}\) on \(X\) such that:

\[
\varphi_i(x)\varphi_j(x) = \varphi_{ij}(x) + \varphi_{ji}(x)
\]

and

\[
\varphi_{ij}(e^c_j(x)) = \varphi_{ij}(x), \quad \varphi_{ij}(e^{c_1}_j(x)) = e^{-1}\varphi_{ij}(x),
\]

\[
\varphi_{ji}(e^c_i(x)) = \varphi_{ji}(x), \quad \varphi_{ji}(e^{c_1}_i(x)) = e^{-1}\varphi_{ji}(x)
\]

(c) We have

\[
\varphi_j(x)\varphi_i(e^c_j(x)) = c\varphi_{ij}(x) + \varphi_{ji}(x), \quad \varphi_i(x)\varphi_j(e^c_i(x)) = c\varphi_{ji}(x) + \varphi_{ij}(x)
\]

Proof. It suffices to prove the proposition in the assumption that \(I = \{i, j\}\), that is, when \(G\) is semisimple of types \(A_1 \times A_1\) and \(A_2\) respectively. The part (a) follows.

Let us prove (b). It suffices to analyze the case when \(G = GL_3\). Due to Lemma 3.10 it suffices to prove the statement only for \(X = G\), \(f_X = \pi^{-}\). In this case \(I = \{i, j\}\), and we set \(i := 1, j := 2\) in the standard way.

It is easy to see that \(\varphi_k(g) = \Delta_k^{-}(g) = \Delta_k^{-}(g)\) for \(k = 1, 2\), where

\[
\Delta_1(g) = g_{11}, \quad \Delta_1'(g) = g_{21}, \quad \Delta_2(g) = \det \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \quad \Delta_2'(g) = \det \begin{pmatrix} g_{11} & g_{12} \\ g_{31} & g_{32} \end{pmatrix}.
\]

Furthermore, is easy to see that the actions \(e_1, e_2 : G_m \times B^- \to B^-\) are given by the formula 5.10:

\[
e_k^c(g) = x_k \left( (c-1) \frac{\Delta_k^{-}(g)}{\Delta_k^{-}(g)} \right) \cdot g
\]

for \(k = 1, 2\).

Define the functions \(\varphi_{12}, \varphi_{21}\) on \(G\) by:

\[
\varphi_{12}(g) = \frac{\Delta_1''}{\Delta_1'}, \quad \varphi_{21}(g) = \frac{\Delta_2''}{\Delta_2'},
\]

where

\[
\Delta_1''(g) = g_{31}, \quad \Delta_2''(g) = \det \begin{pmatrix} g_{21} & g_{22} \\ g_{31} & g_{32} \end{pmatrix}
\]

It is easy to see that

\[
\Delta_1'(g)\Delta_1''(g) = \Delta_1(g)\Delta_1''(g) + \Delta_1'(g)\Delta_2(g).
\]

This identity implies 5.11. Furthermore, it is easy to see that

\[
\Delta_k(e^c_l(g)) = e^{c_1}K_k(g), \quad \Delta_k' (e^c_l(g)) = \Delta_k'(g)
\]

for \(k, l \in \{1, 2\}\) and \(\Delta_k'(e^c_l(g)) = \Delta_k(g)\) for \(k = 1, 2\). This implies 5.11. Part (b) is proved.

Part (c) easily follows from (b). □

In order to prove the relations 2.10 for any \((i, j) \in (I \times I)_0\) we compute the left hand side and the right hand side of 2.10. By definition

\[
e_i^c e_j^c(x) = x_i(a_1)x_j(a_2)(x),
\]
where $a_1 = \frac{c_1 - 1}{\varphi_1(e_{j_1^2}(x))} = \frac{c_1 - 1}{\varphi_j(x)}$, and $a_2 = \frac{c_2 - 1}{\varphi_j(x)}$. Analogously
\[
e_{j_1^2} e_{i_1}^c(x) = x_j(a_2)x_j(a_1)(x).
\]
Thus, using the relation (5.7), we see that the left and the right hand sides of (2.6) are equal. This proves all the relations (2.6).

To prove the relations (2.7) for any $(i, j) \in (I \times I)_-{1}$ we compute the left hand side and the right hand side of (2.6). By definition
\[
e_{i_1}^c e_{j_1^2} e_{i_1}^c(x) = x_i(a_1)x_j(a_2)x_i(a_3)(x),
\]
where
\[
a_3 = \frac{c_2 - 1}{\varphi_j(x)}, a_2 = \frac{c_1 c_2 - 1}{\varphi_j(e_{i_1^2}(x))} = \frac{\varphi_i(x)(c_1 c_2 - 1)}{e_2 \varphi_j(x) + \varphi_j(x)},
\]
and
\[
a_1 = \frac{c_1 - 1}{\varphi_j(e_{j_1^2} e_{i_1}^c(x))} = \frac{(c_1 - 1)(c_2 \varphi_j(x) + \varphi_j(x))}{\varphi_i(x)(c_1 \varphi_j(x) + \varphi_j(x))}.
\]
Similarly,
\[
e_{j_1^2} e_{i_1}^c e_{i_1}^c(x) = x_j(a_1')x_i(a_2')x_j(a_3')(x),
\]
where
\[
a_3' = \frac{c_2 - 1}{\varphi_j(x)}, a_2' = \frac{c_1 c_2 - 1}{\varphi_j(e_{i_1^2}(x))} = \frac{\varphi_j(x)(c_1 c_2 - 1)}{e_1 \varphi_j(x) + \varphi_j(x)},
\]
and
\[
a_1' = \frac{c_1 - 1}{\varphi_j(e_{j_1^2} e_{i_1}^c(x))} = \frac{(c_1 - 1)(c_2 \varphi_j(x) + \varphi_j(x))}{\varphi_i(x)(c_2 \varphi_j(x) + \varphi_j(x))}.
\]
It is easy to see that
\[
a_1'a_2' = a_2a_3, a_2'a_1' = a_1a_2, a_2' = a_1 + a_3.
\]

Thus it follows from (5.8) that
\[
e_{i_1}^c e_{j_1^2} e_{i_1}^c(x) = x_i(a_1)x_j(a_2)x_i(a_3)(x) = x_j(a_1')x_i(a_2')x_j(a_3')(x) = e_{i_1}^c e_{j_1^2} e_{i_1}^c(x)
\]

This proves all the relations (2.7).

Thus, we have proved Theorem 3.8 for all simply-laced reductive groups $G$. In particular, the relations (2.5) hold for such groups.

It remains to prove the relations (2.6) and (2.7) for any group $G$ by deducing these relations from the relations (2.6) for a certain simply-laced group $G'$ containing $G$.

Without loss of generality we may assume that $G$ is adjoint semisimple.

It is well-known that there is an adjoint semisimple simply-laced group $G'$, an outer automorphism $\sigma : G' \to G'$ and an injective group homomorphism $f : G \hookrightarrow G'$ such $f(G) = (G')^\sigma$. In what follows we identify $G$ with it’s image $f(G)$.

Moreover, one can always choose an outer automorphism $\sigma$ in such a way that $\sigma$ preserves a chosen Borel subgroup $B' \subset G'$ and a maximal torus $T' \subset B'$, and satisfies $B = (B')^\sigma, T = (T')^\sigma$. Let $(B')^-$ be the opposite Borel subgroup containing $T'$. Then $(B')^-$ is also $\sigma$-invariant and $B^- = ((B')^-)^\sigma$. Also the automorphism $\sigma$ induces an injection $\sigma_* : \Lambda^\vee \to (\Lambda')^\vee$.

Let $I'$ be the Dynkin diagram of $G'$. It is easy to see that for any $i \in I$ there exists a subset $\tau(i) \subset I'$ such that $\alpha_i = \sum_{i' \in \tau(i)} \alpha_i^{\tau(i)}$ and that for any $i', j' \in \tau(i)$ the
subgroups $U_i, U_{i'}$ commute. This implies that the Weyl group $W$ is the subgroup of the Weyl group $W'$ of $G'$ and its generators $s_i, i \in I$, are given by:

$$s_i = \prod_{i' \in \tau(i)} s_{i'}.$$ 

One can always choose the homomorphisms $x_{i'} : \mathbb{G}_m \to U_{i'}$, $y_{i'} : \mathbb{G}_m \to U_{i'}$, $i' \in I'$ in such a way that

$$x_i(a) = \prod_{i' \in \tau(i)} x_{i'}(a), \quad y_i(a) = \prod_{i' \in \tau(i)} y_{i'}(a)$$

for any $i \in I$. This implies that $\mathbf{s}_i = \prod_{i' \in \tau(i)} \mathbf{s}_{i'}$.

Due to Lemma 3.6. we may assume without loss of generality that $X = G$. Let $X'_{\text{ind}}$ be the geometric $G$-crystal induced by $(G, \text{id}_G)$ and let $X''_{\text{ind}}$ be the geometric $G'$-crystal induced by $(G', \text{id}_{G'})$. We denote by $e_i : \mathbb{G}_m \times \mathbb{C} \to G$, $i \in I$, the actions (3.8) of $\mathbb{G}_m$, and by $e_i' : \mathbb{G}_m \times G' \to G'$, $i' \in I'$ the corresponding actions of $\mathbb{G}_m$. As it follows from (2.4), the transformations $e_i'$ commute with $e_{i'}$ for any $i \in I, i', j' \in \tau(i)$.

For each $i \in I$ let us fix a linear ordering $\sigma(i)$ of $\tau(i)$. It is easy to see that for any reduced sequence $i = (i_1, \ldots, i_l) \in I^l$ the sequence $\sigma(i) := (\tau(i_1); \ldots; \tau(i_l))$ is also reduced.

Let $e_i : T \times G \to G$ and $e_i'_{\sigma(i)} : T' \times G' \to G'$ be morphisms as in (2.4).

The following lemma is obvious.

**Lemma 5.7.** For each reduced sequence $i = (i_1, \ldots, i_l) \in I^l$ we have:

(i) $\text{dom}(e_i) = \text{dom}(e'_{\sigma(i)}) \cap (T \times G)$.

(ii) $e'_{\sigma(i)}|_{T \times G} = e_i$.

The lemma implies that for any reduced sequences $i, j \in I^l$ satisfying $w(i) = w(j)$ we have $e_i = e_i'_{\sigma(i)}|_{T \times G} = e_j'_{\sigma(j)}|_{T \times G} = e_j$.

This proves all the relations 2.5 for geometric $G$-crystals.

**Theorem 5.8.** is proved.

**Proof of Theorem 5.10.** As it follows from Lemma 3.9 that:

(a) The morphism $\gamma_{X \times Y} \circ \theta_{X \times Y} : T' \times T'' \to T$ is positive.

(b) Each function $\varphi^X_{i, Y}$ is positive.

(c) Each action $e_i : \mathbb{G}_m \times X \times Y \to X \times Y$ is positive.

**Theorem 5.10.** is proved.

**Proof of Proposition 3.12.** Without loss of generality we may assume that $X$ is a subset of $B^-$ or even $X = B^-$ so that $f_X$ is the identity map $X \to B^-$. Then it is easy to see that $\alpha^* (u, b) = (u(b^{-1}))^{-1}$. This implies that $\alpha^*$ is an action of $U$. Furthermore, $f^*(b) = b^{-1}$. Thus

$$f^*(\alpha^*(u, b)) = u(b^{-1}) = u(f^*(b)).$$

Proposition 3.12 is proved.

**Proof of Proposition 3.13.** It is easy to see that the formula 3.13 is equivalent to the equation

$$uv(f(x)) = v(f(u(x)) \pi(f(x))$$

for $x \in B^-, u \in U$. $v$ is two-sided $U$-equivariant this identity is equivalent to:

$$v(u \cdot f(x)) = v(f(u(x)) \cdot \pi(f(u(x))).$$
This identity is true since
\[ f(u(x))\pi(f(u(x))) = u(f(x))\pi(f(u(x))) = u \cdot f(x). \]

Proposition 5.14 is proved. □

Proof of Theorem 3.14 We need the following obvious result.

Lemma 5.8. For any unipotent crystal (X, f) let \( \varphi_i^* = x_i^-((f_X(x))^{-1}) \) be the function on \( X \) defined by \( \chi \) for the dual unipotent crystal \( (X, f)^* \), \( i \in I \). Then \( \varphi_i^*(x) = -\varphi_i(x)\alpha_i(\gamma(x)). \)

Using (3.3), we obtain \( \alpha^*(x_1(a), x) = x_1((a^{-1} + \varphi_i^*(x))^{-1}\alpha_i(\gamma^*(x)^{-1})) (x) \).

Substituting \( a = \frac{c - 1}{\varphi_i^*(x)} \) into this expression for \( \alpha^* \), we obtain the action of the corresponding multiplicative group of the induced dual crystal. On the other hand, simplifying the right hand side, we obtain
\[ \alpha^* \left( x_i \left( \frac{c - 1}{\varphi_i^*(x)} \right), x \right) = x_i \left( \frac{c - 1}{-c\varphi_i^*(x)\alpha_i(\gamma^*(x))} \right) (x) = x_i \left( \frac{c^{-1} - 1}{\varphi_i(x)} \right) (x) = c_i^{c - 1}(x) \]
which proves Theorem 3.14. □

Proof of Proposition 3.18 Denote by \( T_r \) the set of regular elements of \( T \) and by \( B_{rss}^- \) the pre-image \( pr_T^{-1} \) which is the set of all regular semisimple elements in \( B^- \). This is an open dense subset. Note that \( B_{rss}^- \) intersects non-trivially each \( B_{w,t}^- \), \( t \in T_r \).

We define the rational morphism \( u_w : B_{rss}^- \rightarrow U_w \) inductively. Define \( u_w := pr_w, \)
\[ u_{w_i} (b) := x_i \left( \frac{1 - \alpha_i(\gamma(b))}{\varphi_i(b)\alpha_i(\gamma(b))} \right) \]
for \( i \in I \), and for any \( w', w'' \in W \) such that \( l(w'w'') = l(w') + l(w'') \), define:
\[ u_{w'w''} (b) := u_{w'} (u_{w''} (b) \cdot b \cdot u_{w''} (b)^{-1}) \cdot u_{w''} (b). \]

Lemma 5.9. For each \( w \in W \) the morphism \( u_w \) does not depend on the choice of the expression (5.14) and satisfies (3.16).

Proof. It is easy to see that \( u_w \) satisfies (3.16). Since the centralizer in \( U \) of any element \( b \in B_{rss}^- \) is trivial we see that \( u_w \) does not depend on a choice of the expression (5.14).

Lemma is proved. □

Furthermore, we can repeat the same definitions for any sub-variety of the form \( B_{w,t}^- \cap B_{rss}^- \). In this case for any \( i \in \text{supp}(w') \) the rational morphism \( u_{w_i} : B_{w,t}^- \cap B_{rss}^- \rightarrow U_{w_i}^* \), is well-defined and it is given by the analogous formula.

Proposition 3.18 is proved.

5.3. Proof of results in section □

Proof of Proposition 4.12 Follows from [4], Theorem 4.7 with \( u = e, v = w \). □

Proof of Proposition 4.13 Let us express the action of \( U \) on \( U_w \) as the conjugation of the action of \( U \) on \( B_w^- \) with the isomorphism \( \eta_w \):
\[ u(x) = \eta_w (u(b)) , \]
where \( b = \eta_w^*(x) \). Using (3.2), (4.1) and the right \( U \)-equivariance of \( \pi \), we obtain:
\[ u(x) = \pi \left( \pi(u(b) \cdot b^{-1} u^{-1})^{-1} \right) = u \cdot \left( \pi(\pi(u(b) \cdot b^{-1})^{-1} \right). \]
Using \( \pi(w \cdot x) = \pi(w \cdot \pi(u \cdot b) \cdot b^{-1}) \cdot x = \pi(w \cdot \pi(u \cdot b) \cdot b^{-1}) \cdot x \).

Finally,

\[ b^{-1} x = b^{-1} \eta_w(b) = b^{-1} \cdot (\pi(w \cdot b^{-1}))^{-1} = \pi^{-1} \cdot \pi((w ? b^{-1}) \cdot \pi(w \cdot b^{-1}) \cdot x) \cdot x \]

Proposition is proved. \( \square \)

\textbf{Proof of Proposition 4.7} Prove (a) and (b). If \( w \ast w' = w w' \) then the statement follows from (4.6). The general statement reduces to the case when \( w = w' = s_i \) for \( i \in \mathbb{I} \). Let us use the following identity in \( SL_2 \):

\[
\begin{pmatrix} c & 0 \\ 1 & c^{-1} \end{pmatrix} \cdot \begin{pmatrix} c' & 0 \\ 1 & c^{-1} \end{pmatrix} = \begin{pmatrix} d & 0 \\ 1 & d^{-1} \end{pmatrix}, \begin{pmatrix} d' & 0 \\ 0 & d^{-1} \end{pmatrix},
\]

for any \( c, c', d, d' \in \mathbb{G}_m \) satisfying \( d' c = 1 + dd', dd' = cc' \neq -1 \). This implies that \( B_s^{-1} \cdot B_s^{-1} = B_s^{-1} \cdot \alpha_s'(\mathbb{G}_m) \) and that \( B_s^{-1} \cdot B_s^{-1} \) contains the set \( B_s^{-1} \setminus \{ \pi_i(-1) \} \), where \( \pi_i : \mathbb{G}_m \rightarrow \mathbb{G}_m \) is the biregular isomorphism defined in section 4.3.

Parts (a) and (b) are proved. Parts (c)-(e) easily follow.

Proposition 4.7 is proved. \( \square \)

\textbf{Proof of Proposition 4.9} For \( w \in W \) define an action of \( T \) on \( G \) by

\[ (t, g) \mapsto Ad_w t(g) = w(t) \cdot g \cdot t^{-1}. \]

Lemma 5.10. For any \( w_1, w_2 \in W \) one has \( Ad_{w_1} T(U \bar{w}_2 U) = U \bar{w}_2 U \cdot T_{w_2^{-1}w_1}. \)

Proof. It suffices to show that \( Ad_w T(U \bar{w}_2 U) = U \bar{w}_2 U \cdot T_{w_2^{-1}w_1}. \) By definition, we have:

\[ Ad_{w_1} t(\bar{w}_2) = w_1(t) \cdot \bar{w}_2 \cdot t^{-1} = \bar{w}_2 \cdot (w_2^{-1}w_1)(t) \cdot t^{-1} \]

for \( t \in T \). Lemma is is proved. \( \square \)

Furthermore, we have \( B_w^{-1} \cdot B_w^{-1} \subset U \bar{w}_2 U \cdot U \bar{w}_2 U \). Thus \( Ad_{w_2} T(B_w^{-1} \cdot B_w^{-1}) = B_{w_2}^{-1} \cdot B_{w_2}^{-1}. \) On the other hand, \( Ad_{w_2} T(B_{w_2}^{-1}) = B_{w_2}^{-1} \cdot T_{w_2^{-1}w_2} \cdot w_2. \)

Therefore, by Proposition 4.7 (a), \( B_{w_2}^{-1} \cdot T_{w_2^{-1}w_2} \subset B_{w_2}^{-1} \cdot T_{w_2^{-1}w_2}. \)

Proposition 4.9 is proved. \( \square \)

\textbf{Proof of Proposition 4.10} Denote temporarily \( w := w_L \), \( b := \eta_w(x) = \pi^{-1}(\pi(x)) \) and \( u' = Ad \pi^{-1}(\pi(u \cdot x)). \)

Furthermore, we have by \( \text{14.3} \):

\[ \pi(u \cdot x) = \pi(w \cdot \pi^{-1}(\pi(w \cdot b^{-1}))). \]

It is easy to see that \( u' \in U \) because \( \pi(u \cdot b) \in U_\mathbb{L} \). Note that \( \bar{w} \cdot b^{-1} \in \bar{w}U \bar{w}^{-1}U. \)

Thus \( \pi^{-1}(\bar{w} \cdot b^{-1}) \in \bar{w}U \bar{w}^{-1} \cap U^- = \bar{w}(U \cap L) \bar{w}^{-1} \subset \bar{w}U \bar{w}^{-1}. \) The latter set is the unipotent radical of \( wPw^{-1} \). This implies that \( Ad u'(\pi^{-1}(\bar{w} \cdot b^{-1})) \in U^- \) and

\[ \pi(u \cdot x) = \pi(Ad u'(\pi^{-1}(\pi(w \cdot b^{-1})))) \cdot u' = u'. \]

Proposition 4.10 is proved. \( \square \)

\textbf{Proof of Theorem 4.12} Part (a) follows from Lemma 4.10. Prove (b). It suffices to show that \( f_w \cdot w' \) commutes with the action of \( U_\mathbb{L} \). By definition, the action of \( U_\mathbb{L} \) on the product \( (U^{w', \mathbb{L}}, \eta_{w', \mathbb{L}}) \times (U^w, \eta^w) \) is given by (see 3.3):

\[ u(x, y) = (u(x), \pi u \eta_{w', \mathbb{L}}(x))(y) \]

For \( x \in U^{w', \mathbb{L}}, y \in U^w \) and \( u \in U_\mathbb{L} \) we have

\[ u(f_w \cdot w'(x, y)) = u(x \cdot y') = u(x) \cdot u'(y'), \]
where \( y' = \text{Ad} \, \varpi(y) \) and \( u' = \pi_{wL,L}(u \cdot x) \). By Proposition 5.10, \( u' = \text{Ad} \, \varpi_{L,L}(u'') \), where \( u'' = \pi(u \cdot \eta^w(x)) \).

It is easy to see that
\[
  u' \cdot y' = \varpi_{L,L} \cdot u'' \cdot \varpi_{L,L}^{-1} \cdot \varpi \cdot y \cdot \varpi^{-1} = \text{Ad} \, \varpi(u'' \cdot y)
\]
because \( \varpi_{L,L}^{-1} \cdot \varpi \) commutes with \( u'' \in U_L \).

Furthermore,
\[
  u'(y') = u' \cdot y' \cdot \pi_{wLw^{-1}}(u'')^{-1} = \text{Ad} \, \varpi(u''(y))
\]
because \( \pi_{wLw^{-1}}(u'') = \pi_{wLw^{-1}}(Ad \, \varpi(u''y)) = Ad \, \varpi(\pi w'') \) by Lemma 4.11 applied with \( u'' = w''_t \cdot w'. \)

Finally,
\[
  u(f_{w,w'}(x,y)) = u(x) \cdot \text{Ad} \, \varpi(u''(y)) = f_{w,w'}(u(x), u''(x))(y) = f_{w,w'}(u(x), y).
\]

Theorem 4.14 is proved. \( \square \)

**Proof of Theorem 4.14** For each \( w \in W \), \( \chi \in \widehat{U} \), we have (see (5.12)):
\[
  \chi^w(u'gu) = \chi^w(u'g) + \chi(u)
\]
for all \( g \in t \cdot \varpi_{L,L}u \), \( t \in T \), \( u' \in \text{Norm}_U(V(w)) \) and \( u \in U \).

**Proof.** Note that \( \chi^w(u'gu) = \chi^w(u'g) + \chi(u) \) for all \( u \). So we prove the lemma in the assumption that \( u = 1 \).

Let \( g \in U \varpi_{L,L}U \). Express \( g \) as \( g = u_1 \varpi_{L,L}u_2 \), where \( u_1 \in V(w) \), \( u_2 \in U \). Then \( \chi^w(g) = \chi^w(u_1 \varpi_{L,L}u_2) = \chi(u_2) \) because \( \varpi^{-1}V(w) \varpi \subset U^- \). Furthermore, for each \( u' \in \text{Norm}_U(V(w)) \) we have: \( u'g = u'u_1 \varpi_{L,L}u_2 = u'_1 \varpi_{L,L}u_2 \), where \( u'_1 = \text{Ad} \, u'(u_1) \in V(w) \). Finally,
\[
  \chi^w(u'g) = \chi^w(u'_1 \varpi_{L,L}u_2) = \chi^w(u'1 \varpi_{L,L}u_2) = \chi^w(\varpi_{L,L}^{-1}u' \varpi_{L,L}u_2) = \chi(u).
\]

Lemma is proved. \( \square \)

**Proposition 5.12.** For any \( \chi \in \widehat{U}(w_{L,G}) \) and any \( t \in Z(L) \) the function \( f^{w_{L,G}}_{\chi, \chi^t} \) satisfies
\[
  f^{w_{L,G}}_{\chi, \chi^t}(u'gu) = \chi(u) + \chi(u') + f^{w_{L,G}}_{\chi, \chi^t}(g)
\]
for \( u, u' \in U \), \( g \in t \cdot U \varpi_{G,L} \). In particular, \( f^{w_{L,G}}_{\chi, \chi^t} \) is invariant under the adjoint action of \( U \) on \( t \cdot U \varpi_{G,L}U \).

**Proof.** Throughout the proof we denote for shortness \( w := w_{L,G} \). In this case we have \( U(w) = U_L \), \( V(w) = U_P \) (where \( P = U \cdot B \)) which implies that \( \text{Norm}_U(V(w_{L,G})) = U \).

The following result is obvious.

**Lemma 5.13.** Let \( \chi \in \widehat{U}(w) \). Then for any \( u \in U \) and \( t \in Z(L) \) we have:
\[
  \chi^t(u) = \chi(u) - \chi^t(u).
\]
Furthermore, let $t \in Z(L)$, $g \in t \cdot U \varpi U$, $u', u \in U$. Then
\[ f_{\chi}^{w_{L,G}}(u' gu) = \chi^w(u' gu) + (\chi^L)^{-1}(\iota(u)\iota(g)\iota(u')) \]
Applying Lemma 5.11 twice, we obtain
\[ f_{\chi}^{w_{L,G}}(u' gu) = \chi^w(u' gu) + \chi^w(g) + (\chi^L)^{-1}(\iota(u')\iota(g)) \]
By by Lemma 5.13 we have $\chi^L(\iota(u')) = \chi^L(u')$ we obtain
\[ f_{\chi}^{w_{L,G}}(u' gu) = \chi^w(u') + \chi^w(g) + \chi(u) + (\chi^L)^{-1}(\iota(g)) \]
Proposition 5.12 is proved.

Proof of Theorem 4.16. Recall that $W_L = W_J$ denotes the Weyl group of $L = L_J$.

Proposition 5.14. For any $\chi \in \hat{U}(w_{L,G})$, $t \in T$ we have
\[ \chi^w_{L,G}(u' gu) = \chi^w(u' gu) + \chi^w(g) \]
for $u' \in U_L, u \in U, g \in t \cdot U \varpi U$. In particular, $\chi^w_{L,G}$ is invariant under the adjoint action of $U_L$ on $t \cdot U \varpi U$ for $t \in Z(L)$.

Proof. Taking into account that $\Norm_U(V(w_{L,G})) = U$, we rewrite 5.10
\[ \chi^w_{L,G}(u' gu) = \chi^w(u' gu) + \chi^w(g) = \chi^w(u' gu) + \chi^w(g) \]
Proposition 5.14 is proved.

Proof of Theorem 4.16(b). We have $\supp(w_{L,G}) = I$. By 3.14, for each $j \in J$ and generic $b_- \in B^-_{w_{L,G}}$ we have $s_j(b_-) = \Ad u(b_-)$ for some $u \in U$. This implies that for all $j \in J$ the restriction $\chi^w_{L,G} |_{B^-_{w_{L,G}}}$ is $s_j$-invariant. Theorem 4.16(b) is proved.

Recall from section 4.11 that the $U$-variety $U^{w_{G,L}}$ equipped with the isomorphism $\eta^{w_{G,L}} : U^{w_{G,L}} \cong B^-_{w_{G,L}}$ is an unipotent $G$-crystal.

Proof of Theorem 4.16(a). The proof is based on the following property of the biradial isomorphism $\eta_w : B^-_{w} \cong U_w$ which is defined in section 4.11

Lemma 5.15. For any $\chi \in \hat{U}$ and any $w \in W$ we have:
\[ \chi(\eta_w(b)) = \chi^w_{L}(b) \]
for $b \in B^-_w$

Proof. It is easy to see that for any $\chi \in \hat{U}$ and $u \in U$ we have $\chi(u^{-1}) = -\chi(u)$. It is also clear that for any $\chi \in \hat{U}$, $g \in U^- \cdot T \cdot U$ we have:
\[ \chi(t_0gt_0^{-1}) = -\chi(g) \]
where $t_0 = \rho^\vee(-1) \in T^{ad}$ as in section 4.13
Furthermore, by definition of $\varpi$, one has
\[ t_0\varpi t_0^{-1} = \varpi^{-1} \]
Proposition 5.14 and the property of the mapping $b$ in Lemma 6.2. For each $\Ad U$ we see that the right hand side of (5.17) is $S$-dense in each $W$, and this sub-monoid $(\text{say that two such subsets} S,S$ in $W)$, the monoid $((W,\circ)$ is generated by all $\mathcal{J}_i$ defined in section 4.2 the birational isomorphism $[\pi_T]^{-1} \circ \eta_w^{-1} \circ \pi^I$ and its inverse are positive isomorphisms $T' \sim T'$, where $T' := (\mathbb{G}_m)^{\ell(w)}$. This follows from the results of [3], sections 4 and 5.

Proposition 4.18 is proved.

Proof of Proposition 4.18. Part (a) follows from (4.7), part (b) follows from Corollary 4.8, part (c) follows from Theorem 3.10, and part (d) is clear.

Proof of Proposition 4.17. Part (a) is immediate. So we prove (b). Due to Lemma 4.3 and Proposition 4.17(c), it suffices to prove that for any $i \in R(w)$ and $i' \in R(w^{-1})$ the birational isomorphism $[\pi_T]^{-1} \circ \eta_w^{-1} \circ \pi^I$ and its inverse are positive isomorphisms $T' \sim T'$, where $T' := (\mathbb{G}_m)^{\ell(w)}$. This follows from the results of [3], sections 4 and 5.

Proposition 4.18 is proved.

6. PROJECTIONS OF BRUHAT CELLS TO THE PARABOLIC SUBGROUPS

6.1. General facts on projections. Recall that we have defined in section 4.2 the monoid structure $(W,\ast)$ on $W$. Clearly, for any standard Levi subgroup $L = L_J$, the monoid $(W_J,\ast)$ is a sub-monoid of $W$ under the operation $\ast$ defined in section 4.1 and this sub-monoid $(W_J,\ast)$ is generated by all $s_j, j \in J$.

Lemma 6.1. The correspondence 

$$s_i \mapsto [s_i] = \begin{cases} s_i, & \text{if } i \in J, \\ e, & \text{if } i \in I \setminus J, \end{cases}$$

extends to the homomorphism of monoids $[\cdot] : (W,\ast) \to (W_J,\ast)$.

Similarly to the projection $p_L^L : B^- \to B^-_L$ defined in section 5.2 let $p^+ = p_J^L$ be the natural projection $U \to U_L$.

There is a natural equivalence relations between constructible subsets of $G$. We say that two such subsets $S, S' \subset G$ are equivalent if their intersection $S \cap S'$ is dense in each $S$ and $S'$. In this case we denote $S \equiv S'$.

Lemma 6.2. For each $w \in W$ one has $p^+(U^w) \equiv U[w].$
Proposition 6.4. For any $\tilde{p}$ there is an algebraic sub-torus $T(w)$.

Proof. We proceed by the induction in $l(w)$. If $l(w) = 1$, that is, $w = s_i$ for some $i \in I$ then

\[ p^+(U^{s_i}) = \begin{cases} U^{s_i}, & \text{if } i \in J, \\ \{e\}, & \text{if } i \in I \setminus J. \end{cases} \]

Let us now assume that $l(w) > 1$. Using Proposition 4.2 and the fact that $p^+$ is a group homomorphism, we obtain for any $i = (i_1, \ldots, i_l) \in R(w)$:

\[ p^+(U^w) = p^+(U^{s_i_1}U^{s_i_2} \cdots U^{s_i_l}) = p^+(U^{s_i_1}) \cdot p^+(U^{s_i_2}) \cdots p^+(U^{s_i_l}) = U^{[s_i_1] \cdots [s_i_l]} = U^{[w]} \]

Lemma 4.2 is proved.

The computation of $p_L^-(B_{w^0})$ is a little more complicated.

Proposition 6.3. For any $w \in W$ we have:

(a) The intersection of $p_L^-(B_{w^0})$ with $B_{[w]}$ is a dense subset of $B_{[w]}$;

(b) There is an algebraic sub-torus $T_w \subset T$ such that $p_L^-(B_{w^0}) \equiv B_{[w]} \cdot T_w$.

(c) The torus $\tilde{T}_w$ satisfies: $\tilde{T}_w = \{e\}$ if and only if $w \in W_J$, $\tilde{T}_w = \alpha_i(\mathbb{G}_{m})$ for $i \in I \setminus J$, and $T_{w_{w'}} = [w']^{-1}(\tilde{T}_w) \cdot T_{w'} \cdot \tilde{T}_{[w],[w']}$ for any $w, w'$ such that $w \cdot w' = w^0$.

Proof. It follows from Proposition 4.3 that for any $w, w' \in W$ satisfying $w \cdot w' = w^0$ we have:

\[ p_L^-(B_{w^0}) \equiv p_L^-(B_{w^1} \cdot B_{w^2}) = p_L^-(B_{w^1}) \cdot p_L^-(B_{w^2}) = B_{[w]} \cdot T_w \cdot B_{[w]} \cdot T_{w'} \]

This proves (b) and (c). Let us prove (a) now. Denote $B_{[w]}^0 := p_L^-(B_{w^0}) \cap B_{[w]}$. We will proceed by induction in $l(w)$. If $w = e$, the statement is obvious. Let now $w \neq e$. Let us express $w = w'w''$, where $w' \neq e$, $w'' \neq e$ and $l(w) = l(w') + l(w'')$. Thus, the inductive hypothesis implies that $B_{[w]}^0$ is dense in $B_{[w]}^0$, and $B_{[w']}^0 := p_L^-(B_{w''}) \cap B_{[w']}^0$ is dense in $B_{[w']}^0$.

Then (4.4) implies that

\[ p_L^-(B_{w^0}) \supset p_L^-(B_{w^1} \cdot B_{w''}) \supset p_L^-(B_{w'}) \cdot p_L^-(B_{w''}) \supset \tilde{B}_{[w']} \cdot \tilde{B}_{[w'']} \]

But $\tilde{B}_{[w']} \cdot \tilde{B}_{[w'']}$ is a dense subset of $B_{[w']} \cdot B_{[w'']}$. Finally, Proposition 4.3(a) implies that the latter set contains a dense subset of the variety $B_{[w'] \cdot [w'']} = B_{[w^0]}$.

This proves part (a). Proposition is proved.

It is well-known that the set $G_0 = U^P \cap T$ is open in $T$. Denote by $p = p_L$ the natural projection $G_0 \to P$. By definition, $p$ commutes with the action of $B_L \times B$, and the restriction of $p$ to $B^- \subset G_0$ is the natural projection $p_L^- : B^0 \to B_L^-$. Our next task is to describe the image of each reduced Bruhat cell $U^wU$ under the projection $p$.

Recall that in section 4.2 we defined for each $w \in W$ the sub-torus $T_w \subset T$.

Proposition 6.4. For any $w \in W$ we have:

(a) $U^wU \cdot T_{w^{-1}} \subset p(U^wU \cap G_0)$.

(b) The restriction $p^w = p_Lw$ of $p$ to $U^wU$ is a dominant rational morphism $p^w : U^wU \to U^wU \cdot T_w$. 
(c) \( T_{[w]^{-1}w} \subset \tilde{T}_w \).

**Proof.** Prove (b). The multiplication in \( G \) induces the open inclusions \( B_w^{-} \times U \hookrightarrow U\overline{w}U \cap G_0, B_w^{-} \times U_L \hookrightarrow U\overline{w}U_L \cap G_0. \) Furthermore,

\[
\text{Lemma 5.10 to the above identity, we obtain}
\]

\[
(6.2) \quad p(B_w^{\ominus} \cdot U) = p_L^-(B_w^{\ominus}) \cdot U, \quad p(B_w^{-} \cdot U_L) = p_L^- (B_w^{\ominus}) \cdot U_L.
\]

Therefore, it follows from Lemma 6.2 that

\[
p(U\overline{w}U) = p_L^-(B_w^{\ominus}) \cdot U \equiv B_{[w]}^{-} \cdot \tilde{T}_w \cdot U = B_{[w]}^{-} \cdot U \equiv \tilde{U}_{[w]} U \cdot \tilde{T}_w.
\]

Part (b) is proved.

Let us prove (a) now. Recall that \( p \) is \( U_L \times U \)-equivariant, and \( U_{[w]} U = U_L[w] U \). Note that \( p_L^-(B_w^{\ominus}) \) intersects \( B_{[w]}^{-} \) non-trivially. Thus

\[
U_{[w]} U \subset p(U\overline{w}U \cap G_0)
\]

Recall that the action \( Ad_w : T \times G \to G \) is defined in the proof of Proposition 6.4. Clearly, both \( P \) and the reduced Bruhat cell \( U\overline{w}U \) are invariant under this action, and the morphism \( p : G_0 \to P \) commutes with this action. Thus applying Lemma 5.11 to the above identity, we obtain

\[
Ad_w T(U_{[w]} U) = U_{[w]} U \cdot T_{[w]^{-1}w} \subset p(U\overline{w}U \cap G_0).
\]

Part (a) is proved. Part (c) follows.

**Proposition 6.5.** For any \( w \in W \) we have:

(a) for each \( u_- \in U_{P_w}^- \) the morphism \( q_{u_-} \) is a section of \( p_w \), that is,

\[
p_w \circ q_{u_-} = \text{id}_{U_{[w]} U \cdot T_{[w]^{-1}w}}.
\]

(b) The variety \( U_{P_w}^- \) is invariant under the adjoint action of \( T_{[w]^{-1}w} \cdot U_L([w]) \).

(c) Let \( q \) be a morphism \( U_{P_w}^- \times U_{[w]} U \cdot T_{[w]^{-1}w} \to U\overline{w}U \) defined by \( q(u_-, g) = q_{u_-}(g) \). Then \( q \) is an inclusion.

**Proof.** Prove (a). Indeed, \( V([w]) \subset U_L \). Thus \( q_{u_-} \) is a section of \( p_w \) because \( p_w \) is \( U_L \times B \)-equivariant.

Prove (b). It is easy to see that \( U_L([w]) = \text{Norm}_{U_L}([w]) \). Thus, for \( u_- \in U_{P_w}^- \) and any \( u \in \text{Norm}_{U_L}([w]), t \in T_{[w]^{-1}w} \) we have \( Ad_w t(u \cdot u_- \cdot [w] \cdot u^{-1}) \in U\overline{w}U \).

On the other hand, \( Ad_w t(u \cdot u_- \cdot [w] \cdot u^{-1}) = (tu) \cdot u_- \cdot (tu)^{-1} \cdot [w] \). This implies that \( (tu) \cdot u_- \cdot (tu)^{-1} \in U_{P_w}^- \).

Prove (c). We proceed by the contradiction. Assume that \( q \) is not injective. That is, we assume that there are elements \( u_-, u'_- \in U_{P_w}^- \) such that \( u'_- \neq u_- \) and

\[
q_{u'_-}(w) = q_{u_-}(ut).
\]
for some $u' \in V([w])$, $u \in U$ and $t \in T$. In other words,

$$u'_- \cdot [w] = u'_- \cdot u_- \cdot [w] \cdot u \cdot t .$$

Denote $\tilde{u}_- := Ad \ u'_- $ Clearly, $\tilde{u}_- \in U'_L$ because $u' \in U_L$. Furthermore, let us factorize $Ad \ [w](u) = u_+ \cdot u''_-$, where $u_+ \in U([w])$, and $u''_- \in U'_L$. Thus, the above identity can be rewritten as

$$(\tilde{u}_-)^{-1} \cdot u'_- = u'_+ \cdot u''_- \cdot [w](t) .$$

This implies that $t = e$, $u''_- = e$, $u'_+ \cdot u''_- = e$, and $u'_- = \tilde{u}_-$. In particular, $u' = (u_+)^{-1} \in V([w]) \cap U([w]) = U_L([w]) = \{e\}$.

In its turn, this implies that $u'_- = u_-$ which contradicts to the original assumption.

\[\Box\]

**Definition.** We call an element $w \in W$ $L$-special (or simply special) if $l(w) = l([w]) + \dim \ T_{[w]}^{-1} w$.

**Theorem 6.6.** For any special element $w \in W$ we have:

(a) The set $U_{P,w}$ consists of a single element $u_- = u_{P,w}^-$. This element $u_-$ is centralized by the group $T[[w]] \cdot U_L([w])$. The inclusion $q_{u_-}$ is dense.

(b) $P_w = q_{u_-}^{-1}$. In particular, $P_{w} : U \pi U \to U[[w]] \cdot T[[w]]^{-1} w$ is a birational isomorphism.

(c) $T_w = T[[w]]^{-1} w$.

(d) The composition $P_L \circ \eta^w$ is a birational isomorphism $U^w \tilde{\to} B_{[w]}^- \cdot T[[w]]^{-1} w$.

**Proof.** Follows from Proposition 6.5.

\[\Box\]

### 6.2. Example of the projection for $G = GL_{m+n}$

Let us fix two positive integers $m \leq n$. Let $G = GL_{m+n}$ and $L = L_{m,n} = GL_m \times GL_n \subset GL_{m+n}$. In this case $P = P_{m,n} = L \cdot B$ is the corresponding maximal parabolic subgroup which consists of those matrices $g \in GL_{m+n}$ which have zeros in the $m \times n$-rectangle in the lower left corner.

We use the standard labeling of the Dynkin diagram of type $A_{m+n-1}$: $I = \{1, 2, \ldots, m + n - 1\}$. Then the Weyl group $W$ is naturally identified with the symmetric group $S_{m+n}$ via $s_i \mapsto (i, i + 1)$.

Note that the longest element $w_0$ of $W$ acts on simple roots by $w_0(\alpha_i) = -\alpha_i$ for $i = 1, \ldots, m + n - 1$, where $i^* = m + n - i$.

The corresponding element $w_{L,G} = w_{m,n} \in W$ admits a factorization (6.9).

For $i \leq j$ denote by $s_{ij}$ the reflection about the root $\alpha_i + \cdots + \alpha_j$. Clearly, $s_{ij}$ is the transposition $(i, j + 1) \in W = S_{m+n}$.

Denote $\sigma_{m,n} := [w_{m,n}]^{-1} w_{m,n}$. It is easy to see that

$$\sigma_{m,n} = \prod_{i=1}^{m} s_{i,i^*} ,$$

that is, $\sigma_{m,n}$ is a product of exactly $m$ commuting reflections in $W$. This implies that $\dim \ T_{\sigma_{m,n}} = m = l([w_{m,n}]) - l([w_{m,n}])$. Thus $w_{m,n}$ is special.

Let $q_{m,n}$ be the dense inclusion $U_L[[w_{m,n}]] \cdot T_{\sigma_{m,n}} \to U[[w_{m,n}]]$ prescribed by Theorem 6.5.
Let $\omega_1, \ldots, \omega_{m+n}$ be the fundamental weights for $GL_{m+n}$. For each $w \in S_{m+n}$ the weight $w(\omega_k)$ is naturally identified with a $k$-element subset of $\{1, \ldots, m+n\}$.

For any $w_1, w_2 \in S_{m+n}$ let $\Delta_{w_1(\omega), w_2(\omega)} : G \to \mathbb{A}^1$ be the minor located in the intersection of rows indexed by $w_1(\omega_1)$ and columns indexed by $w_2(\omega_i)$.

**Proposition 6.7.** The variety $U_L[w_{m,n}]U_L : T_{\sigma_{m,n}}$ consists of all $g \in B_L[w_{m,n}]B_L$ satisfying for $i = 1, \ldots, m+n-1$:

\[
\Delta_{[w_{m,n}](\omega_i), \omega_i}(g) = \begin{cases} 
\Delta_{\omega_i \omega_i}(g), & \text{if } i \in [m;n], \\
\Delta_{w_{m,n}(\omega_i^*), \omega_i^*}(g), & \text{if } i \notin [m;n].
\end{cases}
\]

**Proof.** Let us describe the sub-torus $T_{\sigma_{m,n}}$ of $T$. We have

\[
T_{\sigma_{m,n}} = \prod_{i=1}^{m} T_{s_{i,i^*}}.
\]

That is, $T_{\sigma_{m,n}}$ can be described in $T$ by the equations $(i = 1, \ldots, m+n-1)$:

\[
\Delta_{\omega_i, \omega_i}(t) = \begin{cases} 
\Delta_{\omega_i \omega_i}(t), & \text{if } i \in [m;n], \\
\Delta_{\omega_i^* \omega_i^*}(t), & \text{if } i \notin [m;n].
\end{cases}
\]

On the other hand, for any $w \in W_J, t \in T$ one has

\[
U_L w U_L t = \{ g \in B_L w B_L : \Delta_{w(\omega_i), \omega_i}(g) = \Delta_{\omega_i(\omega_i), \omega_i}(t), i = 1, \ldots, m+n-1 \}.
\]

Proposition 6.7 is proved. 

**Proposition 6.8.** For any $\chi \in \hat{U}(w_{m,n})$, $u' \in U_L, u \in U, t \in T$ we have:

\[
\chi^{w_{m,n}}(q_{m,n}(u'[w_{m,n}] \sigma_{m,n}(t) \cdot t^{-1} \cdot u)) = \chi(u') + \chi(u) - \delta_{m,n} \alpha_m(t) \cdot \chi(x_m(1)).
\]

**Proof.** For any $1 \leq i \leq m \leq j$, $w \in W_J$ let $s_{ij} \in \text{Norm}_G(T)$ be the representative of $s_{ij}$ defined by

\[
\tilde{s}_{ij} := w^{-1} s_{m} w
\]

for any $w \in W_J$ such that $w^{-1} s_{m} w = s_{ij}$.

We need the following obvious refinement of (6.3).

**Lemma 6.9.** $\overline{[w_{m,n}]^{-1}}^{-1} = \prod_{i=1}^{m} \tilde{s}_{i,i^*}$.

For any $1 \leq i \leq m$ choose an element $w_i \in W$ such that $s_{i,i^*} = w_i^{-1} s_m w_i$. By (2.2), we have:

\[
\tilde{s}_{i,i^*}^{-1} = u_i^* u_i^{-1} u_i^* u_i^+,
\]

where $u_i^+ = \overline{w_i^{-1} x_m (-1) w_i}, u_i^- = \overline{w_i^{-1} y_m (1) w_i}$.

Clearly, $u_i^+ \in U_P \cap \text{Ad} \overline{w_{m,n}^{-1}(U_P)}$, and $u_i^- \in U_P \cap \text{Ad} \overline{w_{m,n}^{-1}(U_P)}$. It is also clear that $u_i^*$ commutes with each $u_j^*$ whenever $i \neq j$, where $\varepsilon, \varepsilon' \in \{+, -\}$.

Denote $u^z := \prod_{i=1}^{m} u_i^z$ for $\varepsilon \in \{+, -\}$.

**Lemma 6.10.** The only element of $U_P \overline{w_{m,n}}$ is equal to $\text{Ad} \overline{w_{m,n}^{-1}}(u^+)^{-1}$, and

\[
u_{-} \cdot [w_{m,n}] = \overline{w_{m,n}} \cdot u^e \cdot u^+.
\]
Proof. Let $u_{-} := \text{Ad} w_{m,n}(u^{+})^{-1}$. Clearly, $u_{-} \in U_{P}$.

Next, let us rewrite the equation from Lemma 8.14 $w_{m,n}^{-1}w_{m,n} = u^{+} \cdot u^{-} \cdot u^{+}$ or, equivalently, $w_{m,n}^{-1}w_{m,n} = u_{1}^{-1} \cdot u_{1}^{-1} \cdot u_{1}^{-1}$, where $u_{1}^{-1} = \text{Ad} w_{m,n}(u^{+})^{-1}$. Clearly, $u_{1}^{-1} \in U_{P}$.

Thus, $u_{-} \in U_{P} \cap U_{w_{m,n}}U[w_{m,n}]^{-1} = U_{P,w_{m,n}}$.

Lemma 6.10 is proved. \hfill \Box

Since $q_{m,n}$ is $U_{L} \times U$-equivariant it suffices to prove 0.15 in the case when $u = u^{'} = e$. Using Lemma 8.14 we obtain:

\[
\chi^{w_{m,n}}(q_{m,n}(Ad_{w_{m,n}} t([w_{m,n}]))) = \chi^{w_{m,n}}(Ad_{w_{m,n}} t(q_{m,n}([w_{m,n}])))
\]

\[
\chi^{w_{m,n}}(Ad_{w_{m,n}} t([w_{m,n}]u^{-}u^{+})) = \chi(u_{-}u^{+} \cdot t^{-1}) = \chi(tu_{1}^{-1}) .
\]

Let us compute $\chi(tu_{1}^{-1})$ using the fact that $u_{1}^{-1} \in [U,U]$ for $i = 1, \ldots, m - 1$:

\[
\chi(tu_{1}^{-1}) = \sum_{i=1}^{m} \chi(tu_{1}^{-1}) = \chi(tu_{1}^{-1}) = -\delta_{m,n} \chi(x_{m}(t))
\]

The last equality holds because $tu_{1}^{-1} \in [U,U]$ for $m \neq n$, and $tu_{1}^{-1} = tx_{m}(-1)^{-1} = x_{m}(-a_{m}(t))$ if $m = n$.

Proposition is proved. \hfill \Box

Let $L_{1} = L_{m,n} := (G_{m})^{2m} \times GL_{m,n} \subset L_{m,n}$. By definition, $[w_{m,n}] = w_{L_{1},L_{m,n}}$. It is easy to see that $[w_{m,n}](T_{m,n}) \subset Z(L_{1})$.

Recall from Proposition 5.12 that for any $\chi \in \hat{U}$ the function $f^{[w_{m,n}]}_{\chi,\chi_{L_{1}}}$ on $Z(L_{1}) \cdot U_{L_{1}}[w_{m,n}]U_{L}$ defined by 4.13 is proper under the action of $U_{L} \times U_{L}$.

**Corollary 6.11.** For any $\chi \in \hat{U}(w_{m,n}) \cap \hat{U}([w_{m,n}])$, and $g \in U_{L}[w_{m,n}]U_{L} \cdot t$, $t \in T_{m,n}$, one has:

\[
\chi^{w_{m,n}}(q_{m,n}(g)) = f^{[w_{m,n}]}_{\chi,\chi_{L_{1}}}(g) - \delta_{m,n}a_{m} \cdot t_{m}^{-1} .
\]

where $a_{m}$ is the coefficient of $\chi_{m}$ in the expansion $\chi = \sum_{i=1}^{m+n-1} a_{i} \chi_{i}$.

**Remark.** It is well-known that $U_{P,w_{m,n}}$ is abelian in the above example. One can expect that for any $G$ and any standard parabolic subgroup $P = L \cdot U_{P}$ the element $w_{L,G}$ is special if and only if $U_{P}$ is abelian.

On the other hand, there is an example of a standard parabolic subgroup in the simple group $G$ of the type $D_{4}$ with non-abelian $U_{P}$ and non-special $w_{L,G}$.

We use the following labeling of the Dynkin diagram of type $D_{4}$: $I = \{0,1,2,3\}$, where 0 stands for the “center” of the Dynkin diagram. Let $J = \{1,2,3\}$, and $L := L_{J}$. Then $w_{0}^{J} = s_{1}s_{2}s_{3}$, and $w_{L,G} = s_{0}s_{1}s_{2}s_{3}s_{0}s_{1}s_{2}s_{3}s_{0}$. It is easy to see that $[w_{L,G}] = w_{0}^{J}$. Thus, $l([w_{L,G}]) - l([w_{L,G}]) = 9 - 3 = 6 > \dim T[w_{L,G}]^{-1}w_{L,G}$, which implies that $w_{L,G}$ is not special.

Moreover, $p_{w}$ is not a birational isomorphism because $\dim U[w_{L,G}]U = 21$, $\dim U[w_{L,G}]U = 15$, and $\dim T[w_{L,G}] \leq 4$.\hfill \Box
7. Appendix: Combinatorial pre-crystals and W-crystals

In this section we recall Kashiwara’s framework (see [10]).

**Definition.** A partial bijection of sets \( f : A \to B \) is a bijection \( A' \to B' \) of subsets \( A' \subset A, B' \subset B \). We denote the subset \( A' \) by \( \text{dom}(f) \) and the subset \( B' \) by \( \text{ran}(f) \).

**Remark.** A partial bijection \( f : A \to B \) is an embedding \( A \hookrightarrow B \) if and only if \( \text{dom}(f) = A \).

The inverse \( f^{-1} \) of a partial bijection \( f : A \to B \) is the inverse bijection \( \text{ran}(f) \to \text{dom}(f) \). Composition \( g \circ f \) of partial bijections \( f : A \to B, g : B \to C \) is naturally a partial bijection with \( \text{dom}(g \circ f) = \text{dom}(f) \cap f^{-1}(\text{ran}(f) \cap \text{dom}(g)) \) and \( \text{ran}(g \circ f) = g(\text{ran}(f) \cap \text{dom}(g)) \). In particular, for any partial bijection \( f : B \to B \) and \( n \in \mathbb{Z} \) the \( n \)-th power \( f^n \) is a partial bijection \( B \to B \).

Note that for any partial bijection \( f : A \to B \) the composition \( f^{-1} \circ f \) is the partial identity bijection \( \text{id}_{\text{dom}(f)} : A \to A \).

**Definition.** Let \( B \) be a set \( \tilde{\gamma} : B \to \Lambda^\vee \) be a map. We call a family of partial bijections combinatorial pre-crystal (or simply pre-crystal) on \((B, \tilde{\gamma})\) if
\[
\tilde{\gamma}(\tilde{e}_i(b)) = \tilde{\gamma}(b) + \alpha_i^\vee
\]
for \( b \in \text{dom}(\tilde{e}_i), i \in I \).

With any pre-crystal \( B \) we define partial bijections \( \tilde{s}_i : B \to B, i \in I \), by
\[
\tilde{s}_i(b) = \tilde{e}_i^{-1}(\tilde{\gamma}(b), \alpha_i^\vee) \tag{7.1}
\]
In fact \( \tilde{s}_i \) are partial involutions: \( \tilde{s}_i^2 = \text{id}_{\text{dom}(\tilde{s}_i)} \).

**Examples.**
1. Fix \( i \in I \). Denote by \( \tilde{\gamma}_i : \mathbb{Z} \to \Lambda^\vee \) the map \( n \mapsto n\alpha_i^\vee \). We define a pre-crystal \( B_i \) on \((\mathbb{Z}, \tilde{\gamma}_i)\) in the following way. We define a partial bijection \( \tilde{e}_i : \mathbb{Z} \to \mathbb{Z} \) by \( \tilde{e}_i(n) = n + 1 \) and for any \( j \neq i \) we define \( \text{dom}(\tilde{e}_j) = \text{ran}(\tilde{e}_j) = \emptyset \). This defines a pre-crystal on \((\mathbb{Z}, \tilde{\gamma}_i)\) which we denote by \( B_i \) (see [10], Example 1.2.6).
2. The lattice \( \Lambda^\vee \) is a pre-crystal with \( \tilde{\gamma} = \text{id}_{|\Lambda^\vee} \) and \( \tilde{e}_i(\lambda^\vee) = \lambda^\vee + \alpha_i^\vee \) for \( \lambda^\vee \in \Lambda^\vee, i \in I \).
3. For any pre-crystal \( B \) on \((B, \tilde{\gamma})\) we denote by \( \tilde{\gamma}' \) the map \( \Lambda^\vee \times B \to \Lambda^\vee \) given by \( \tilde{\gamma}'(\lambda^\vee, b) = \lambda^\vee + \tilde{\gamma}(b) \). One can define a pre-crystal on \((\Lambda^\vee \times B, \tilde{\gamma}_A \times \tilde{\gamma})\) by \( \tilde{e}_i(\lambda^\vee, b) = (\lambda^\vee, \tilde{e}_i(b)) \).
4. For any pre-crystal \( B \) on \((B, \tilde{\gamma})\) put \( \gamma^* := -\tilde{\gamma} \). Define also the partial bijections \( \tilde{e}_i^* := \tilde{e}_i^{-1} \). The collection \( \tilde{e}_i^*, i \in I \), defines the structure of a pre-crystal on \((B, \gamma^*)\). This pre-crystal is called dual to \( B \) and is denoted by \( B^* \).

**Remarks.**
1. In the [10] the partial bijection \( \tilde{e}_i^{-1} \) was denoted by \( \tilde{f}_i \).
2. Our pre-crystals correspond to Kashiwara’s crystals for \( \mathfrak{g} \).

**Definition.** A pre-crystal \( B \) is called free if each \( \tilde{e}_i \) is a bijection \( B \to B \).

**Definition.** A morphism of pre-crystals \( f : B' \to B \) is a partial bijection \( f : B' \to B \) such that \( \text{dom}(f) = B' \), \( \tilde{\gamma} = \tilde{\gamma}' \circ f \), and the partial bijections \( \tilde{e}_i' : B' \to B', i \in I \), are obtained from the partial bijections \( e_i : B \to B, i \in I \), by \( \tilde{e}_i' = f^{-1} \circ \tilde{e}_i \circ f \).

**Definition.** We say that a pre-crystal \( B \) is a combinatorial \( W \)-crystal (or simply \( W \)-crystal) if for any \( i \in I \) we have: \( \text{dom}(\tilde{s}_i) = B \), and the involutions \( \tilde{s}_i \) satisfy the braid relations.
It is clear that a structure of a $W$-crystal on $(B, \gamma)$ defines an action of $W$ on $B$ in such a way that for any $i \in I$ $s_i \in W$ acts by $\tilde{s}_i : B \to B$.

**Remarks.**
1. The condition that $\text{dom}(\tilde{s}_i) = B$ is equivalent to:

\begin{equation}
\inf \{n : b \in \text{dom}(\tilde{e}_i^n)\} \leq - (\gamma(b), \alpha_i) \leq \sup \{n : b \in \text{dom}(\tilde{e}_i^n)\}
\end{equation}

for all $b \in B$, $i \in I$.

2. The pre-crystal on $(\Lambda^\vee, \text{id}_{\Lambda^\vee})$ defined above is a free $W$-crystal.

3. For any $W$-crystal the structure map $\tilde{\gamma} : B \to \Lambda^\vee$ is $W$-equivariant.

We denote by $\text{Pre} - \text{Cryst}$ the category such that the objects are pre-crystals and arrows are morphisms of pre-crystals. We denote by $W - \text{Cryst}$ the full subcategory of $\text{Pre} - \text{Cryst}$ whose objects are $W$-crystals. Note that each morphism $f : B \to B'$ in $W - \text{Cryst}$ is an injective $W$-equivariant map $B \to B'$.

**Example.** Let $V$ be a finite-dimensional $U_q(\mathfrak{g})$-module, and let $B = B(V)$ be a crystal basis for $V$ (see [3]). Denote by $\tilde{\gamma} : B \to \Lambda^\vee$ the weight grading. Then the partial bijections $\tilde{e}_i, \tilde{f}_i : B \to B$ define a structure of a pre-crystal on $(B, \tilde{\gamma})$. It is a deep result of [11] that this is a $W$-crystal.

We denote this $W$-crystal by $B(V)$.

**Remark.** For $B = B(V)$ all $e_i, \tilde{e}_i^{-1}$ are nilpotent, that is, $\text{dom}(\tilde{e}_i^n) = \text{dom}(\tilde{e}_i^{-n}) = \emptyset$ for some $n > 0$, and $- (\gamma(b), \alpha_i) = \inf \{n : b \in \text{dom}(\tilde{e}_i^n)\} + \sup \{n : b \in \text{dom}(\tilde{e}_i^n)\}$ for $b \in B(V)$, $i \in I$. Clearly, this identity implies the inequalities (7.2), that is, $\text{dom}(\tilde{s}_i) = B(V)$ for each $i \in I$.

There are examples of free $W$-crystals, i.e., in which each $\tilde{e}_i$ is torsion-free. Among them are the free $W$-crystals $B_i$ from [10], section 2.2 built on $(\mathbb{Z}^{(w_0)}, \gamma_i)$, $i \in R(w_0)$. Kashiwara has constructed each $B_i$ as the tensor product of $B_{i_1}, \ldots, B_{i_l}$, so that $B_i$ is equipped with the family of functions $\tilde{\varphi}_i^l : \mathbb{Z}^{(w_0)} \to \mathbb{Z}$, $i \in I$, satisfying $\tilde{\varphi}_i^l(\tilde{e}_i(b)) = \tilde{\varphi}_i^l(b) + 1$ for $b \in \mathbb{Z}^{(w_0)}$, $i \in I$.

Let $R(w_0)$ be the category whose objects are reduced sequences $i \in R(w_0)$ and for each pair of objects $i, i' \in R(w_0)$ there is exactly one arrow $i \to i'$.

A functor $F : R(w_0) \to W - \text{Cryst}$ such that $F(i) = B_i$ was defined in [1]. Moreover it was shown in [3] that $\tilde{\varphi}_i^l = \varphi_i^l \circ F(i \to i')$ for any $i, i' \in R(w_0)$.

Therefore, the free $W$-crystal $B_{w_0}$ is well-defined and is equipped with the family of functions $\tilde{\varphi}_i : B_{w_0} \to \mathbb{Z}$, $i \in I$, such that $\tilde{\varphi}_i(\tilde{e}_i(b)) = \tilde{\varphi}_i(b) + 1$.

**Remark.** Let $\Lambda^\vee$ be the simple finite-dimensional $U_q(\mathfrak{g})$-module with the lowest weight $\lambda$ (e.g., $\Lambda^\vee$ is an anti-dominant co-weight), and let $B(V_{\Lambda^\vee})$ be the corresponding combinatorial crystal. It follows from the results of Kashiwara that there is a morphism of $W$-crystals

\begin{equation}
f_{\lambda^\vee} : B(V_{\lambda^\vee}) \to \Lambda^\vee \times B_{w_0}
\end{equation}

such that $\tilde{\varphi}_i(f_{\lambda^\vee}(b)) = \sup \{n : b \in \text{dom}(\tilde{e}_i^{-n})\}$ for $b \in B(V_{\lambda^\vee})$, $i \in I$. 

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