Saturated magnetic field amplification at supernova shocks

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ABSTRACT
Cosmic ray streaming instabilities at supernova shocks are discussed in the quasi-linear diffusion formalism which takes into account the feedback effect of wave growth on the cosmic ray streaming motion. In particular, the non-resonant instability that leads to magnetic field amplification in the short wavelength regime is considered. The linear growth rate is calculated using kinetic theory for a streaming distribution. We show that the non-resonant instability is actually driven by a compensating current in the background plasma. The non-resonant instability can develop into a non-linear regime generating turbulence. The saturation of the amplified magnetic fields due to particle diffusion in the turbulence is derived analytically. It is shown that the evolution of parallel and perpendicular cosmic ray pressures is predominantly determined by non-resonant diffusion. However, the saturation is determined by resonant diffusion which tends to reduce the streaming motion through pitch angle scattering. The saturated level can exceed the mean background magnetic field.

Key words: radiation mechanisms: general – shock waves – cosmic rays – ISM: magnetic fields – supernova remnants.

1 INTRODUCTION
Diffusive shock acceleration (DSA) is regarded as the preferred mechanism for the acceleration of the Galactic cosmic rays (CRs) (Drury 1983). It is commonly believed that supernova (SN) shocks are responsible for acceleration of high-energy CRs at least up to the ‘knee’ (∼4 × 1015 eV) of the CR spectrum (Hillas 2006). In the DSA model, particles gain energy by bouncing back and forth across the shock. Although particles only gain a small amount of energy in each crossing, they can be accelerated to very high energy through many crossings provided they can be trapped in the acceleration region for a sufficiently long time. The acceleration is rather efficient and naturally leads to a power-law energy distribution. Despite these advantages, there have been two long-standing problems. First, the standard DSA theory predicts the maximum energy well below the ‘knee’ (Bell 2004). The maximum energy achievable is limited by both the Bohm approximation, in which the particle’s gyroradius must not exceed the mean free path to scattering, and the condition that the gyroradius is smaller than the shock’s width (otherwise the particle would escape from the acceleration region) (Bell 2004; Zirakashvili, Puskin & Völk 2008). For a magnetic field of order 10−10 T and typical parameters of SN shocks, one has the maximum energy ∼5 × 1014 eV. Secondly, turbulence is required for effective scattering in the acceleration region so that the particles can be trapped (Skilling 1975; Bell 1978; Lagage & Cesarsky 1983a,b). So far, in the standard DSA theory, one generally postulates that Alfvén turbulence can be generated by CRs themselves through resonant interactions, even though in practice the growth of Alfvén waves due to CRs is known to be ineffective.

There is growing interest in the possibility that the magnetic field at the shock may be amplified due to CR-induced instability in the non-resonant regime (Bell & Lucek 2001; Bell 2004). Bell (2004) showed a non-resonant form of the instability driven by a CR current can outgrow the familiar resonant form, leading to magnetic amplification. A strong magnetic field at shocks reduces the gyroradius and this raises the maximum energy to which particles can be accelerated. X-ray observations of young supernova remnants (SNRs) suggest that the magnetic field strength near the shocks is much higher than that in the interstellar medium (ISM) and is a strong function of the shock speed (Vink & Laming 2003; Völk, Ksenofontov & Berezhko 2008). The presence of strong magnetic fields may explain the lack of strong TeV gamma-ray fluxes, as shown from High Energy Stereoscopic System (HESS) observations (Völk et al. 2008). The non-thermal X-rays are well described by synchrotron spectra. A stronger magnetic field in the emission region implies that a lower electron energy is required. This would lead to a lower TeV gamma-ray flux from inverse Compton scattering.

The non-resonant growth of Alfvén waves due to CR streaming has been discussed in both the magnetohydrodynamics (MHD) formalism (Bell & Lucek 2001; Bell 2004; Zirakashvili et al. 2008) and the kinetic formalism (Melrose 2005; Reville, Kirk & Duffy 2006; Amato & Blasi 2009). In both formalisms, the instability is shown to exist in the linear regime. To test the magnetic field amplification model against observations, one needs to determine the saturated magnetic field accurately. The saturation of the

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instability cannot be determined in linear theory as the reaction of the instability on the CR momentum distribution is not automatically included in the linear calculation. Although there are numerical simulations of the instability that extend to the non-linear regime, different saturation levels have been predicted (Niemic et al. 2008; Riquelme & Spitkovsky 2009). In this paper, the instability is discussed in the quasi-linear formalism in which the reaction of the instability on the CR distribution can be included self-consistently. So, in this formalism one can estimate the saturation analytically, with both non-resonant and resonant diffusion processes considered. The treatment of the non-resonant diffusion presented here is similar to that used for the fire hose instability (Davidson 1972). We emphasize the major difference between the CR streaming instability and the fire hose instability. The former is caused by streaming motion and the latter is due to a pressure anisotropy with excess of parallel pressure over the perpendicular pressure (with respect to the mean magnetic field). To some extent, the streaming instability also resembles the Weibel instability—a non-resonant, purely growing mode driven by anisotropy in the particle distribution (Weibel 1959).

In Section 2, the kinetic theory of CR streaming instabilities is discussed with emphasis on the non-resonant instability. Quasi-linear diffusion driven by the non-resonant instability is discussed in both the short and long wavelength approximations in the non-resonant regime in Section 3 and in the resonant regime in Section 4. Application to SN shocks is discussed in Section 5.

2 COSMIC RAY STREAMING INSTABILITIES

We outline the kinetic theory of CR streaming instabilities including both the usual resonant instability and the non-resonant instability and focus particularly on the latter. Our treatment builds on other recent discussions of linear kinetic theory of the CR-induced non-resonant instability (Reville et al. 2006; Amato & Blasi 2009). For convenience, we assume a single species of CRs with charge $q$ and mass $m$.

2.1 CR streaming motion

To model CR streaming at velocity $v_{\text{CR}}$ along the mean magnetic field, we consider a class of streaming distributions in momentum space given by

$$f(u_1, u_\perp) = n_{\text{CR}} \left[ 1 + (2\pi + 1) \frac{v_{\text{CR}}}{u} \left( \frac{u_1}{u} \right)^{2\pi - 1} \right] g(u) \frac{1}{4\pi u^2},$$

where $u_1$ and $u_\perp$ are the non-dimensional momenta (normalized by $mc$) parallel and perpendicular to the mean magnetic field, respectively, $\sigma$ is an integer $\geq 1$, $n_{\text{CR}}$ is the CR number density, $v$ is the CR’s velocity written as a function of $u_\parallel$ and $u_\perp$, $u = (u_\parallel^2 + u_\perp^2)^{1/2}$, and

$$g(u) = \begin{cases} \frac{p - 1}{u_1} \left[ 1 - \left( \frac{u_1}{u_2} \right)^{p - 1} \right]^{-1} \left( \frac{u}{u_1} \right)^{-p}, & u_1 \leq u \leq u_2, \\ 0, & \text{otherwise.} \end{cases}$$

For the standard DSA one has $p = 2$ (Bell 1978; Drury 1983) and when the non-linear effect on DSA is included, $p$ deviates from this canonical value (Eichler 1984). The distribution with $\sigma = 1$ corresponds to that used in Melrose (1986). The distribution (1) implies

$$n_{\text{CR}} = 4\pi \int_0^\infty u_\perp du_\perp \int_{-\infty}^\infty du_1 f(u_1, u_\perp) = 2\pi \int_0^\infty \sin \alpha d\alpha \int_{0}^{\infty} u^2 du \ f(u \cos \alpha, u \sin \alpha),$$

(3)

where $\cos \alpha = u_1/u$. In the second expression in (3), one chooses $(u, \alpha)$ in place of $(u_1, u_\perp)$ as independent variables. It can be verified that averaging the parallel velocity $v_1$ over the distribution (1) gives the streaming velocity $v_{\text{CR}}$, which is independent of the choice of the parameter $\alpha$. The CR current is then given by

$$J_{\text{CR}} = q n_{\text{CR}} v_{\text{CR}}.$$ 

The presence of streaming CRs affects the background plasma in two ways, due to their charge density and their current density, respectively. The background plasma must have a charge density and a current density that are equal and opposite to those of the CRs. This requires that the electrons and ions (assumed to be protons) have different charge densities, $n_1 \neq n_p$, and that they move relative to each other with streaming velocities $v_1 \neq v_p$, which are assumed to be along the guiding magnetic field. The neutralization conditions require

$$e(n_e - n_p) = q n_{\text{CR}} v_{\text{CR}}.$$

(4)

These properties of the background plasma drive the non-resonant instability attributed to the CRs.

2.2 Dispersion relation

A formal procedure to derive the dispersion relation involves separating the plasma response tensor into that for a background plasma, denoted by $K_\alpha$, plus that for the CR component, denoted by $\Delta K_{ij}$. The background plasma can be regarded as a cold, magnetized plasma, while the CR component is described by the distribution (1). Assume that the gyrofrequency of CRs is $\Omega = |q| B/m$, where $B$ is the mean magnetic field. A useful approximation is $k_\perp v_\perp/\Omega \ll 1$, where $k_\perp$ is the perpendicular wavenumber, $\Omega = \Omega/\gamma$ and $\gamma = (1 + v^2)^{1/2}$ is the Lorentz factor of CRs. The approximation implies that in the response tensor, only the first gyroharmonics terms are important. For the background plasma, one assumes $v_\perp^2 \ll c^2$ and the low-frequency approximations, $\omega \ll \Omega$, $\omega \ll |k_\perp v_\perp|$ and $\omega \ll \Omega$, where $\Omega$ is the gyrofrequency of ions in the background plasma. The background magnetic field is assumed to be along the 3-axis. Since $K_{33} \propto \Omega^2/c^2$ can be set to $\infty$, only the $2 \times 2$ components of the response tensor are relevant. For the background plasma, these components can be written as

$$K_{11} = K_{22} = \frac{c^2}{\omega^2},$$

(5)

$$K_{12} = -K_{21} = i \frac{c^2 \Omega}{v_\perp^3} \frac{n_e - n_p}{2m_i} + \frac{k_\perp^2 (n_e v_\perp - n_p v_p)}{\omega \omega_i},$$

(6)

where $n_i = k_i/c/\omega_i$, $v_\parallel = \Omega_i/\omega_i$ is the Alfvén speed, $\omega_i$ is the ion plasma frequency of the background plasma. Using the neutralization conditions (4), (6) can be rewritten in the form

$$K_{12} = -K_{21} = -i n^2 \chi_0 \omega_i, \quad \chi_0 = \frac{\mu_0 J_{\text{CR}}}{2k_\perp B}.$$

(7)

The CR components are

$$\Delta K_{11} = \Delta K_{22} = -i n^2 \chi_0, \quad \Delta K_{12} = -\Delta K_{21} = -i n^2 \chi_0,$$

(8)

where

$$\chi_0 = \frac{\pi}{2} A \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx \ x^3 (1 - x^2) H(1 - x^2).$$

(9)
where \( x_0 = A \int x^p x^{p+2\sigma - 3} \left[ (2 + x^2) \ln \frac{1 + x}{1 - x} + 2x^2 \right] - 4 \sum_{s=1}^{\sigma-1} x^{1-2s} \right] \) with \( (x_1, \sigma_x) = b(1/\mu, 1/\mu), \sigma_b = \Omega/|k_1|c \) and \( H(x) \) the step function.

Equations (9) and (10) correspond to the dissipative and reactive part respectively of the response tensor due to CRs. In the strong magnetic field limit \( x > 1 \), the dissipative part is zero. When \( x \ll 1 \), both dissipative and reactive terms contribute to the dispersion. The dissipative part is due to CRs in the resonance \( x = b/\mu \approx \mu \) as discussed extensively in the context of the resonant instability (see discussion below) (Kulsrud & Pearce 1969; Melrose & Wentzel 1970; Skilling 1975; McKenzie & Vold 1982a; Bagguley). The dispersion relation takes the following form

\[
\omega^2 = \frac{2}{k_1^2} + 2|\chi_h| \left\{ \left( \chi_h^{(0)} + \chi_h \right)^2 \right\}^{1/2} k_1^2 v_A^2.
\]

In the \( \chi_h \to 0 \) and \( \chi_h \to 0 \) limit \( \chi_h^{(0)} \) must also be zero), (12) with the lower sign reproduces the usual Alfvén mode dispersion \( \omega = |k_1| v_A \) and with the upper sign the fast mode dispersion \( \omega = k_1 v_A \).

### 2.3 Streaming instabilities

Instability occurs when \( \omega \) has an imaginary part of the appropriate sign. An imaginary part can arise in two different ways, which we refer to as resonant and non-resonant instabilities. Writing \( \omega \to \omega^0 + i \Gamma \) in (12) and assuming \( \Gamma \ll \omega^0 \), the resonant instability is described by equating the small imaginary terms, giving \( \Gamma = \chi_h k_1^2 v_A^2 / 2 \omega^0 \). The value of \( \chi_h \) is determined by the CR distribution. A resonant instability can develop for both Alfvén and fast-mode waves if streaming CRs are in resonance with the wave considered. For Alfvén waves (\( \omega = |k_1| v_A \)), the growth rate is

\[
\Gamma \approx \frac{1}{2} \chi_h |k_1| v_A.
\]

The polarization of both modes is generally elliptical and becomes approximately circular for nearly parallel propagation \( |k_1/k_i| \ll 1 \). In each mode, waves of both sense of polarization can grow.

On neglecting \( \chi_n \), (12) with the upper sign becomes a real equation, because the quantity inside the square root is a sum of squares. For the negative sign (Alfvén waves), \( \omega^2 \) can be negative, and one of the two solutions corresponds to an intrinsically growing wave. This is identified as the non-resonant instability. For \( |k_1/k_i| \ll 1 \), the condition for this non-resonant instability to occur is \( \chi_h^{(0)} + \chi_h > 1 \). The growth rate is

\[
\Gamma \approx \sqrt{2} \left( |\chi_h^{(0)} + \chi_h| - 1 \right)^{1/2} |k_1| v_A.
\]

In principle, this can be satisfied either due to \( |\chi_h| > 1 \) or to \( |\chi_h^{(0)}| > 1 \). For \( \sigma = 1 \), the integral \( \chi_h \) given by (10) can be calculated exactly, i.e.

\[
\chi_h = \frac{3 \mu_0 J_{CR} (b - 1) / p - 1}{4 k_1 B x^{p-1}} \int x^{p} \left[ (1 - x^2) \ln \frac{1 + x}{1 - x} + 2x \right] - 4 \sum_{s=1}^{\sigma-1} x^{1-2s} \right] \cdot
\]

\[
= \frac{3 \mu_0 J_{CR}}{16 k_1 B x_1} \left\{ 2 \left[ x_1 \left( 1 + x_1^2 \right) - x_2 \left( 1 + x_2^2 \right) \right] \right\}
\]

\[
- \left( x_1^2 - 1 \right)^{1/2} \ln \left( \frac{1 + x_1}{1 - x_1} \right) + \left( x_2^2 - 1 \right)^{1/2} \ln \left( \frac{1 + x_2}{1 - x_2} \right) \right\}. \quad (15)
\]

The second equality in (15) is obtained for \( p = 2 \). The condition \( |\chi_h| > 1 \) cannot be satisfied for \( x_1 \ll 1 \). Thus, only the second possibility \( \chi_h^{(0)} \) is relevant.

For \( x_2 \ll x_1 \ll 1 \), one finds

\[
\chi_h \approx \frac{\mu_0 J_{CR} x_1^2}{4 k_1 B}.
\]

so that the contribution from the CR current is much smaller than the contribution due to the compensating current in the background plasma. It follows that in treating the non-resonant instability, one can neglect the direct contribution of the CRs, described by \( \chi_h^{(0)} \), in comparison with the indirect contribution, described by \( \chi_h^{(0)} \). The growth rate (14) can be written in the following approximate form

\[
\Gamma \approx \sqrt{2} \left( \chi_h^{(0)} - 1 \right)^{1/2} k_1 v_A = \frac{\mu_0 J_{CR}}{k_1 B} \left( \frac{\mu_0 J_{CR}}{k_1 B} - 1 \right)^{1/2} k_1 v_A. \quad (17)
\]

Non-resonant instability requires

\[
|J_{CR}| > \frac{k_1 B}{\mu_0}. \quad (18)
\]

An explanation for why the direct contribution from the CR current is so much smaller than the indirect contribution from the background plasma is that in the limit \( k_1 r_g \gg 1 \), most of CRs move rather rigidly. Their only role in this limit is to induce the compensating current in the background plasma that drives the instability. It is appropriate to point out here that a small fraction of CRs can satisfy the resonant condition \( (\mu = x) \) and that resonant instability due to these resonant CRs is insignificant compared to the non-resonant instability due to the compensating current. However, the resonant interactions between these CRs and waves with \( k_1 \gg 1/r_g \) can lead to resonant diffusion that has feedback effects on CR streaming motion (cf. Section 4).

### 3 QUASI-LINEAR DIFFUSION

To determine saturation, we calculate the back reaction of wave growth on CRs in the quasi-linear diffusion theory (Shapiro & Shevchenko 1964; Davidson 1972). We assume that the non-resonant instability discussed in Section 2 can develop well into the non-linear regime, producing turbulence. The growing turbulence causes CRs to diffuse in momentum space and this in turn reduces the anisotropy, suppressing the instability.

We calculate the particle diffusion in the weak turbulence approximation. The distribution of CRs in momentum space can be written as \( f(u, u) = f(u_{||}, u_{\perp}) \) and \( f^{(1)}(u, u) \), where \( f^{(1)} \) is a linear function of the electric and magnetic fields of the wave and \( F \) is the mean distribution, which is assumed to be axisymmetric with respect to the mean magnetic field, \( B \), and as the zeroth-order approximation it can be taken to be (1). The response of particles to wave growth is determined by including quadratic terms of the electric and magnetic fields of waves. The diffusion equation for
\[ F \text{ can be obtained from the Vlasov equation (the derivation is outlined in Appendix A). We assume } k \perp r_s \ll 1 \text{ so that only the first gyroharmonics terms are relevant. This condition is generally satisfied for waves propagating nearly parallel to the mean background magnetic field.} \]

### 3.1 Non-resonant diffusion equation

For a non-resonant instability, it is relevant to consider the approximation that the resonant width is much larger than the growth rate, i.e., \( |k_\perp v_\parallel - \Omega/\gamma| \gg \Gamma \), where \( \Gamma > 0 \) is the growth rate and for convenience the frequency is set to zero. The diffusion equation can be expanded in \( \Gamma / |k_\perp v_\parallel - \Omega/\gamma| \). Let \( \delta B_\parallel \) be the spatially Fourier-transformed magnetic fluctuations arising from the instability. The energy of magnetic fluctuations in an elementary phase volume \( dk/(2\pi)^3 \) is \( |\delta B_\parallel|^2 / 2\mu_0 \). The expansion yields the following approximate diffusion equation (Appendix A):

\[
\frac{\partial F}{\partial t} \approx \frac{1}{2V} \int \frac{dk}{(2\pi)^3} \frac{\Gamma |\delta B_\parallel|^2}{B_\parallel^2} \left[ u_\parallel \frac{\partial}{\partial u_\parallel} \Phi_1 \frac{\partial}{\partial u_\parallel} - (\Phi_1 + \Phi_2) \frac{u_\perp}{u_\parallel} \frac{\partial}{\partial u_\perp} \right] F, \quad (19)
\]

\[
\Phi_1 = \frac{1}{u_\perp^2 - b^2}, \quad \Phi_2 = \frac{u_\perp^2 + b^2}{(u_\perp^2 - b^2)^2}, \quad (20)
\]

with \( b = B/B_* \), \( B_* = |k_\parallel| mc/|q| \) and \( V \) the volume of the region considered.

### 3.2 Perpendicular and parallel kinetic energy

It is of interest to examine the evolution of the parallel and perpendicular kinetic energy by evaluating the rate of change in the average \( u_*^2 \) and \( u_\perp^2 \) (over the mean distribution \( F \)). The averaging involves singular terms \( 1/(u_\parallel \pm b)^n \) with \( n = 1, 2, 3 \). The \( n \geq 2 \) terms can be avoided using

\[
\Phi_2 = -\Phi_1 + \frac{u_\perp^2}{b} \frac{\partial \Phi_1}{\partial b}, \quad (21)
\]

\[
\frac{\partial \Phi_1}{\partial u_\parallel} = -\frac{u_\parallel}{b} \Phi_1, \quad (22)
\]

\[
\frac{\partial \Phi_2}{\partial u_\parallel} = 3\frac{u_\parallel}{b} \Phi_1 - \frac{u_\perp^2}{b} \frac{\partial}{\partial b} \Phi_1, \quad (23)
\]

Multiplying (19) by \( u_*^2 \) and \( u_\perp^2 \) respectively, and integrating them over \( 2\pi u_\perp du_\perp du_\parallel \) divided by \( n_{CR} \), one obtains

\[
\frac{d(u_*^2)}{dt} = \frac{1}{2\tau} \left[ - \left( u_*^2 - 4u_\perp^2 \right) \Phi_1 + 2 \frac{\partial}{\partial b} \left( 2u_*^2 - u_\perp^2 \right) \Phi_1 \right] - \frac{1}{b} \frac{\partial}{\partial b} \frac{1}{b} \frac{\partial}{\partial b} \left( u_\perp^2 \Phi_3 \right), \quad (24)
\]

\[
\frac{d(u_\perp^2)}{dt} = \frac{1}{2\tau} \left[ - \left( u_\perp^2 - 4u_*^2 \right) \Phi_1 + 2 \frac{\partial}{\partial b} \left( 2u_\perp^2 - u_*^2 \right) \Phi_1 \right] + \frac{1}{b} \frac{\partial}{\partial b} \frac{1}{b} \frac{\partial}{\partial b} \left( u_*^2 \Phi_3 \right) , \quad (25)
\]

where

\[
\tau = \frac{1}{V} \int \frac{dk}{(2\pi)^3} \frac{2\pi |\delta B_\parallel|^2}{B_\parallel^2} \quad (26)
\]

The ratio of the parallel to perpendicular pressures is determined by the same angular averages as the ratio of \( |u_*^2| \) to \( |u_\perp^2| \). It follows that the ratio of equations (24) and (25) effectively determines how the ratio of the parallel to perpendicular pressures evolves.

### 3.3 Long wavelength approximation

In the long wavelength approximation \( 1/k_\parallel \gg r_s \), one has \( b \gg |u_*| \) and

\[
\Phi_1 \approx -\Phi_2 \approx - \frac{1}{b^2} \quad (27)
\]

The rates of change in the average of \( u_*^2 \) and \( u_\perp^2 \) is given by

\[
\frac{d(u_*^2)}{dt} = - \left( 2(u_*^2) - (u_\perp^2) \right) \frac{1}{\tau}, \quad (28)
\]

\[
\frac{d(u_\perp^2)}{dt} = (u_\perp^2) \frac{1}{\tau}, \quad (29)
\]

where \( \tau \) has the same form as (26) but with \( B_\parallel \) replaced by \( B \). Thus, one has \( 1/\tau = \delta(\delta B^2/B^2)/dt \). One may compare the non-resonant diffusion to the fire hose instability. Similar to the fire hose instability, a growing wave causes the parallel kinetic energy to decrease and the perpendicular energy to increase with time. For the CR streaming distribution (1), one can show that the right-hand sides of (28) and (29) are zero. In the long wavelength regime, an instability can arise from both dissipative (resonant) and reactive (non-resonant) effects. As we focus on the non-resonant instability in the short wavelength regime, quasi-linear diffusion in the long wavelength regime is not discussed further.

### 3.4 Short wavelength approximation

The right-hand side of equations (24) and (25) can be evaluated in a general case in which the particle distribution is isotropic in pitch angles. The angular integration in (24) and (25) leads to

\[
\frac{d(u_*^2)}{dt} = \frac{1}{\tau} \left( 2 + x \ln \left| \frac{1 - x}{1 + x} \right| \right), \quad (30)
\]

\[
\frac{d(u_\perp^2)}{dt} = - \frac{1}{\tau} \left( 2 + 2x^2 + x^3 \ln \left| \frac{1 - x}{1 + x} \right| \right), \quad (31)
\]

where one uses the following integral

\[
I_{2x} = \int \frac{\mu^{2x}}{2x - 1} + x^{2x-1} \ln \left| \frac{1 - x}{1 + x} \right| . \quad (32)
\]

\[\]
The second expression in (32) is obtained by retaining the Cauchy principal value. Surprisingly, equations (30) and (31) have an opposite sign compared to their counterpart in the long wavelength regime (equations 28 and 29). Although both the fire hose instability and the instability discussed here are driven by anisotropy in the particle distribution in momentum space, there is an important difference between the two. The fire hose instability is driven by anisotropy in kinetic energy with \( \langle u_1^2 \rangle > \langle u_2^2 \rangle /2 \). By contrast, the streaming instability is due to the CR streaming motion (\( u_1 \neq 0 \)), which is also called streaming anisotropy, and the instability can develop even when the CR pressure is isotropic. With the specific choice of the distribution (1), one can show that \( \langle u_1^2 \rangle = \langle u_2^2 \rangle /2 \).

Since the current in (1) is \( \propto \mu^{3} \), it does not contribute to the average \( \langle Y \rangle \) if \( Y \) is proportional to an even power of \( \mu \).

The instability can lead to pressure anisotropy with perpendicular pressure increasing. In the approximation \( x \ll 1 \), one has \( d \langle u_2^2 \rangle /dt \approx 2/\tau \) and \( d \langle u_1^2 \rangle /dt \approx -2/3 \tau \). This gives

\[
\frac{d}{dt} (2 \langle u_1^2 \rangle - \langle u_2^2 \rangle) = \frac{10}{3} \tau^{-1} > 0.
\]

One should emphasize here that development of such anisotropy in kinetic energy is driven by the streaming motion of CRs.

From (19), one may calculate the rate of change in the streaming speed due to non-resonant diffusion. One starts with a particle distribution with streaming motion similar to (1) but without specifying the specific form of \( g(p) \). Consider the case \( \sigma = 1 \), i.e. the streaming component is \( \propto \mu \). As in Section 4.4, in the averaging, one can carry out the angular integration first. This gives

\[
d\langle v_i \rangle \approx \frac{1}{4\tau} \left\{ \begin{array}{l}
\frac{\langle (\alpha + \beta)^2 \rangle I_2 - (2 - 8\beta^2 + 3\beta^4) I_4}{u^2} \\
- 3\beta^2 (1 + \beta^2) I_4 + \frac{1}{x} \frac{\partial}{\partial x} \left[ -3 I_2 + (1 + 5\beta^2) I_4 \right] \\
+ \beta^2 (6 - 7\beta^2) I_6 - 3 \beta^4 I_4 \\
+ \frac{1}{x} \frac{\partial}{\partial x} \left[ x^2 I_2 - \beta^2 I_6 - (1 - \beta^2) I_4 \right]
\end{array} \right\},
\]

where \( I_{2n} \) is given by (32). Since \( dv_{\text{CR}}/dt = d\langle v_i \rangle /dt \), in the limit \( x \ll 1 \), one has

\[
dv_{\text{CR}} \approx - \frac{3v_{\text{CR}}}{\tau} \left\{ \begin{array}{l}
\frac{1}{u^2} (-70 + 28\beta^2 + 3\beta^4) \\
\frac{1}{5} (1 - 3\beta^2 + \beta^4)
\end{array} \right\}.
\]

One obtains \( dv_{\text{CR}} /dt \approx 0 \), contrary to what one would expect. In Section 4, cf. equation (40), we show that resonant diffusion has dominant effects on the evolution of CR streaming motion, which ensures \( dv_{\text{CR}} /dt < 0 \).

### 4 Resonant Diffusion

Waves generated from the non-resonant CR streaming instability can interact with CRs in resonance at \( \mu = x \). Although the fraction of CRs in the resonance is small, we show that diffusion through such resonant interactions has a dominant effect on the CR streaming motion. Provided that the phase speed of the wave is \( \ll c \mu \), resonant diffusion is equivalent to pitch angle scattering, in which particles change their direction of motion without gaining or losing energy. Since the non-resonant instability is in the regime \( x \ll 1 \), resonant scattering involves mainly CRs with pitch angles \( \alpha \sim \pi/2 \pm x \).

Since one deals with purely growing waves that can be regarded as a quasi-mode with \( \omega \to 0 \), the assumption of low phase speed \( \ll c \mu \) is valid even for particles with pitch angles near \( \pi/2 \). The standard formalism involves writing down the diffusion equation involving pitch angles only (Melrose 1986). Using \( F(u, \mu) = F(u_1, u_\perp) \) with the replacements \( u_1 = u\mu \) and \( u_\perp = u\sqrt{1 - \mu^2} \) and retaining only the pitch angle scattering part in (A12), one obtains

\[
\frac{dF(u, \mu)}{dr} = \frac{\partial}{\partial \mu} \left[ (1 - \mu^2)D \frac{\partial F(u, \mu)}{\partial \mu} \right]
\]

\[
D = \frac{\pi \Omega \delta B^2}{4 \gamma B^2},
\]

where \( \delta B^2 / \mu_0 = k_r \int \frac{dk \perp \delta B_\perp^2}{(2\pi)^2} \left\langle \frac{1}{\mu_0} \right\rangle \]

is the density of magnetic energy at the resonant wavenumber \( k_r \equiv \Omega / \mu_0 u \). Using (1), it can be shown that the resonant diffusion does not contribute to changes in \( \langle u_1^2 \rangle \) and \( \langle u_2^2 \rangle \). Thus, the rate of change in parallel and perpendicular pressures is determined by non-resonant diffusion.

The rate of change in the streaming speed is

\[
dv_{\text{CR}} /dt = -Dv_{\text{CR}},
\]

where

\[
D = \frac{1}{2} \int_0^1 d\mu (1 - \mu^2) D \approx \frac{\pi^2}{4 \gamma} \frac{\delta B^2}{\Omega B^2} \frac{1}{\mu_0 k_0 g_0},
\]

where \( \delta B \) is the set value at \( \mu \sim \pi / 2 \). The approximation in (39) is obtained by writing \( \delta B^2 = (2\pi^2 \Omega B^2 k_0 g_0) / (2\pi^2 k_0 g_0) \), i.e. one assumes that the magnetic fluctuations are centred at the wavenumber \( k_0 \). Here \( g_0 \) is the gyroradius of CRs with an average Lorentz factor \( \gamma_0 = \langle \gamma \rangle \).

Since the rate of the change in streaming motion due to resonant diffusion is \( dv_{\text{CR}} /dt \propto \Omega / \gamma \) and the corresponding rate due to non-resonant diffusion is \( dv_{\text{CR}} /dt \propto \gamma \Omega / \gamma \), one has

\[
\left| \frac{dv_{\text{CR}} /dt}{dv_{\text{CR}} /dt} \right| \approx 5 \times 10^{-2} \frac{\gamma \Omega}{\gamma} \frac{\delta B^2}{\Omega B^2} \frac{1}{\mu_0 k_0 g_0}.
\]

Here \( \delta B \) is the typical amplitude of the magnetic fluctuations generated by the non-resonant instability. The right-hand side is larger than 1 for \( k_0 g_0 \gg 20 \Omega / \gamma \), where one assumes \( \delta B \sim \delta B_0 \) and \( k_0 / B \sim k_0 g_0 \). This condition is satisfied for the typical parameters relevant for SN shocks (cf. Section 5). If one writes \( \delta B \propto \exp(\Gamma r) \) and takes the limit \( r \gg 1 / \Gamma \), equation (38) can be solved to yield

\[
\frac{\delta B^2}{B^2} \approx \frac{12 \Gamma \gamma}{\pi \Omega} \ln \left( \frac{\mu_0}{\gamma_0} \right).
\]

### 5 Magnetic Fields in SNRs

There are a number of SNRs whose post-shock magnetic fields have been estimated from X-ray observations and found to be in the range \( 10^{-3} \), much higher than the typical value in the ISM \( \sim 10^{-10} \) (Vink & Laming 2003; Bamba et al. 2005; Völk, Berezhko & Ksenofontov 2005; Tanaka et al. 2008). Although one cannot rule out the possibility that such strong field is due to compression of strong pre-existing magnetic fields in the stellar wind of the progenitor star, the observations favour the interpretation that the magnetic fields are due to an instability at the shock. Observations show that the magnetic field strength strongly depends on the...
Magnetic field amplification at SN shocks

5.1 SN shocks

We consider a forward shock travelling at a velocity $v_s$. The shock undergoes initially a free expansion phase and after sweeping up sufficient mass, it enters a Sedov phase. In the Sedov phase, the velocity is given by $v_s = 2R_s/\tau_t$, where $R_s$ is the shock radius. The total kinetic energy of the shock is $E = (4/3)R_s^3\rho_0 v_s^2$, where $\rho_0$ is the density of the ISM. On eliminating $R_s$ in favour of $t$, the shock velocity becomes

$$v_s = \left(\frac{6E}{125\rho_0}\right)^{1/5} t^{-3/5}. \quad (42)$$

One can show that development of the non-resonant instability is faster in the early Sedov phase. The maximum growth rate is $\propto J_{cr}^{1/2}$, which can be written in terms of the ram pressure in the upstream region. Thus, one expects the growth rate to be a strong function of the shock speed. Since the shock radius can be expressed as $R_s = (3\eta_p/\rho_0 v_s^2)$, where $\eta_p \approx 1$ is the acceleration efficiency. We consider precursor CRs with the initial streaming velocity $v_{cr0} \sim v_s$. Since $P_{cr} = n_{cr}mc^2(u/v)\eta_{cr}$, one has $J_{cr} = 3n_{cr}q\rho_0 v_s^2 c n_{cr}/(mcu_s \ln(u_s/\rho))$, where $(u/s) \approx c u_s$ and $u_s$ is the average momentum given by

$$u_s = \int_0^\infty u g(u)du = u_t \frac{p - 1}{p - 2} \frac{1 - (u_t/u_s)^{2-p}}{(u_t/u_s)^{1-p}}. \quad (43)$$

The average Lorentz factor of CRs is $\gamma_s = (1 + u_s^2)^{1/2}$. For $p = 2$ and $u_s^{2-p} \gg u_t$, one has $u_s = u_t \ln(u_s/\rho)$. Therefore, the faster the shock speed the higher the growth rate.

5.2 Saturated magnetic field versus shock speed

One may estimate the maximum magnetic field, $B_{max}$, of the wave by equating the magnetic pressure to the ram pressure, $P_{cr} = P_{cr}$. This leads to $B_{max} \approx (2\mu_0\eta_\rho) v_s$, or

$$\delta B_{max} \approx 3 \times 10^{-7} \eta_\rho^{1/2} \left(\frac{n_{cr}}{10^{10} \text{ m}^{-3}}\right)^{1/2} \left(\frac{v_s}{5 \times 10^6 \text{ m s}^{-1}}\right) \Omega / \gamma_t. \quad (44)$$

Equation (44) is a rather optimistic estimate and calculation of the saturation due to the feedback effects of the instability on the streaming anisotropy generally leads to a lower saturation level than (44).

One can calculate the saturated magnetic field due to the diminishing of the streaming anisotropy from (41). This involves calculation of the cut-off streaming speed, denoted by $v_{cr}$, at which the instability turns off. From (18), one obtains

$$v_{cr} \approx \frac{5c}{24\eta_p\rho_0 v_s^2} k_B T_{cr0}, \quad (45)$$

where $U_B = B^2/2\mu_0$ is the energy density of the mean magnetic field. At a given parallel wavenumber $k_s$, wave growth stops when the streaming speed is reduced to $v_{cr} \leq v_{cr}$. One may assume $v_{cr} \sim v_s$. Equation (45) gives

$$\frac{v_{cr0}}{v_{cr}} \approx 5 \times 10^5 \eta_p \left(\frac{v_s}{5 \times 10^6 \text{ m s}^{-1}}\right)^3 \left(\frac{10^{-10} \text{ T}^2}{B^2}\right)^2 \left(\frac{n_{cr}}{10^9 \text{ m}^{-3}}\right). \quad (46)$$

For convenience, one assumes here that CRs and ions in the ISM are all protons.

The growth rate (17) is estimated to be $\Gamma \approx (6\eta_p k_B T_{cr0})^{1/2} \sim (v_s/c)^{1/2} \Omega / \gamma_t$. From (41), one obtains

$$\delta B^2 \approx \frac{12}{\pi} \left(\frac{6\eta_p}{c} \frac{v_s}{c}\right)^{3/2} \left(\frac{k_B T_{cr0}}{c}\right)^{3/2} \ln \left(\frac{v_{cr0}}{v_{cr}}\right). \quad (47)$$

Equation (47) implies that significant amplification $\delta B^2/B^2 \gg 1$ is possible if one assumes a large $k_B T_{cr0}$. However, one should point out that the quasi-linear theory is based on the weak turbulence assumption that requires the relevant wave fluctuations be small. None the less, since the saturation mechanism discussed is quite generic, i.e. the saturation is attributed to the reduction in CR streaming, one expects that extrapolation of equation (47) to the $\delta B^2/B^2 \gg 1$ regime would still provide a reasonable estimate for the saturation. To estimate the upper limit to the saturation one assumes that the Larmor radius is limited by the scale length of the acceleration region, which is assumed to be the shock’s radius, $R_s$. The characteristic wavenumber $k_0$ is limited by the condition (18), which gives $k_0 \sim (\rho_0 c \rho_{cr})/v_s$, where $\rho_{cr}$ is the plasma frequency of CRs. This corresponds to $(k_0 \rho_{cr})_{max} \sim 5 \times 10^3$. For $n_{cr} = 0.1, v_s = 5 \times 10^6 \text{ m s}^{-1}$ and $k_0 \rho_{cr} = 10^2$, one has $\delta B^2/B^2 \sim 39$. The saturated magnetic field can exceed the background field by a modest factor. If the CR pressure is a fixed fraction of the ram pressure, one has $n_{cr} \rho_0 \sim v_s^2$. Since the shock radius can be expressed as $R_s = (3E/4\rho_0 v_s^2)^{1/3}$, one has $k_0 R_s \sim 1/\lambda_s$, where $\lambda_s \approx v_{cr}$. Equation (47) implies that the energy density of the saturated magnetic field increases with the shock speed as $\delta B^2 \propto v_s^2$. Remarkably, this predicts the same shock-speed scaling as the one from the equipartition argument, i.e. the magnetic pressure equal to the CR pressure (cf. equation 44).

6 CONCLUSIONS AND DISCUSSION

We develop a kinetic version of the CR streaming instability of Bell (2004), who treated it using MHD. We show that the non-resonant growth is not due to the CRs themselves, but rather to a current in the background plasma that neutralizes the current due to streaming CRs. Saturation of the CR streaming instability is discussed in the quasi-linear theory. Growth of the non-resonant instability leads to particle diffusion that suppresses the instability. In the quasi-linear diffusion formalism, the saturated magnetic field can be calculated analytically. We consider both resonant and non-resonant diffusion, with the latter being treated in a similar way to that used for the fire hose instability ( Shapiro & Shevchenko 1964; Davidson 1972). In our model, the CR pressure is set to a fixed fraction of the ram pressure at the shock front. It is shown that evolution of the parallel and perpendicular pressures is due to non-resonant diffusion. The direction of the evolution is that the parallel energy increases and perpendicular energy decreases. This is in contrast to the fire hose instability which causes the perpendicular pressure to increase and parallel pressure to decrease. One should note the difference in the sources of free energy that drive the instabilities; the free energy for the fire hose instability is an excess of parallel pressure while the free energy for the non-resonant instability considered here is...
streaming motion of CRs. It is shown that saturation is determined by resonant diffusion which reduces the streaming motion. The saturated magnetic fluctuations can exceed the background field provided that the shocks are sufficiently fast and saturation occurs at short wavelength \( k_0 \ll 1 \).

There have been numerical simulations of the streaming instability in both the MHD approximation and kinetic theory, which confirm rapid growth of the instability under the physical conditions of young SN shocks. However, these simulations predict different saturation levels. One of the main difficulties in numerical simulation of the instability is the huge difference between the CR number density and background plasma density, with \( n_{\text{CR}}/n_0 \sim 10^{-5} \). A particular approximation is usually adopted to make numerical simulation of such system feasible. For example, in Niemiec et al. (2008)'s numerical model, the density ratio is set artificially to a large value. Because the existing numerical models use different approximations, and it is unrealistic to make meaningful comparison among their results. MHD simulations generally predict a saturation at strong magnetic fields (Bell 2004; Zirakashvili et al. 2008), while kinetic particle-in-cell simulations appear to predict much lower saturated magnetic fields (Niemiec et al. 2008; Riquelme & Spitkovsky 2009). The analytical calculation presented here predicts a relatively modest amplification that gives rise to magnetic field fluctuations similar to the level predicted by the recent kinetic simulations (Riquelme & Spitkovsky 2008).

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REFERENCES

Achterberg A., 1983, A&A, 119, 274
Amato E., Blasi P., 2009, ApJ, 692, 173
Bamba A., Yamazaki R., Yoshida T., Terasawa T., Koyama K., 2005, ApJ, 621, 793
Bell A. R., 1978, MNRAS, 182, 147
Bell A. R., 2004, ApJ, 605, 445
Bell A. R., Lucz G., 2001, MNRAS, 321, 433
Davidson R. C., 1972, Methods in Nonlinear Plasma Theory. Academic Press, New York
Drury L. O’C., 1983, Rep. Prog. Phys., 46, 973
Eichler D., 1984, ApJ, 277, 429
Hillas A. M., 2006, J. Phys. Conf. Ser. 47, 168
Kulsrud R., Pearce W. P., 1969, ApJ, 156, 445
Lagage P. O., Cesarsky C. J., 1983b, A&A, 125, 249
Lagage P. O., Cesarsky C. J., 1983a, A&A, 118, 223
Machabeli G. Z., Luo Q., Melrose D. B., Vladmirov S., 2000, MNRAS, 312, 51
McKenzie J. F., Vlung H. J., 1982a, A&A, 116, 191
McKenzie J. F., Vlung H. J., 1982b, 17th Int. Cosmic Ray Conf., V ol. 9, 110
Melrose D. B., 1986, Instabilities in Space and Laboratory Plasmas. Cambridge University Press, Cambridge
Melrose D. B., 2005, AIP Proc., 781, 135
Melrose D. B., Wentzel D. G., 1970, ApJ, 457, 476
Niemiec J. A., Pohl M., Strumia T., Ishikawa K. I., 2008, ApJ, 684, 1174
Revilie B., Kirk J. G., Duffy P., 2006, Plasma Phys. Control. Fusion, 48, 1741
Riquelme M. A., Spitkovsky A., 2009, ApJ, 694, 626
Shapiro V. D., Shchepchenko V. I., 1964, Sov. Phys. JETP 18, 1109
Skilling J., 1975, MNRAS, 172, 557
Tanaka T. et al., 2008, ApJ, 685, 988
Vink J., 2008, AIP Proc., 1085, 169
Vink J., Laming J. M., 2003, ApJ, 584, 758
Vlung H., Berezhko E. G., Ksenofontov L. T., 2008, A&A, 490, 515
Weibel E. S., 1959, Phys. Rev. Lett., 2, 83
Zirakashvili V. N., Ptuskin V. S., Vlung H. J., 2008, ApJ, 678, 255

APPENDIX A: QUASI-LINEAR DIFFUSION

Using the similar method discussed in Davidson (1972), we write the distribution function as \( f \approx f^{(0)} + f^{(1)} \) with \( f^{(1)} \) a linear function of electric and magnetic fluctuations, \( \delta E \) and \( \delta B \), of the wave. A formal approach to determine the feedback effect of the wave on the particle distribution involves calculation of time dependence of the distribution averaged over the random phase, denoted by \( F \equiv \langle f^{(0)} \rangle \), where \( \langle \cdot \cdot \cdot \rangle \) is the random phase average.

A1 General formalism

From the Vlasov equation, one obtains

\[
\frac{dF}{dr} + q \left( \left( \delta E + v \times \delta B \right) \cdot \frac{\partial f^{(1)}}{\partial p} \right) = 0, \quad (A1)
\]

where \( d/dr = \delta/\delta t + v, \delta/\delta x, v \) is the drift velocity. Let \( \xi = \langle \delta E, \delta B, f^{(1)} \rangle \) and the corresponding spatial Fourier transform be \( \hat{\xi}_k = \langle \delta E_k, \delta B_k, f^{(1)}_k \rangle \), defined by

\[
\hat{\xi}_k(t) = \int \frac{dk}{(2\pi)^3} \xi_k(t) \exp{(ik \cdot x)}, \quad (A2)
\]

where we assume the Wentzel–Kramers–Brillouin (WKB) approximation for the time dependence

\[
\xi_k(t) \propto \exp{-i\psi_k(t)}, \quad \psi_k(t) = \int_0^t \left( \omega_k(t) + i\Gamma(t) \right) dt. \quad (A3)
\]

Here, \( \omega_k \) is the wave frequency and \( \Gamma \) is the growth rate. One adopts the convention \( \omega_{\perp} = -\omega_k, \xi_{\perp} = \xi_k^{\star} \).

In cylindrical coordinates, the velocity can be expressed as \( v = (v_\perp \cos \phi, v_\perp \sin \phi, v_z) \), where \( \phi \) is the gyrophase. The 3-axis is assumed to be along the magnetic field. For a transverse wave propagating parallel to the magnetic field, one has \( \delta E = \delta E_{\parallel} + (i\omega/k) \delta B_{\perp} \) and \( \delta B = \delta B_{\parallel} + \delta B_{\perp} \). In the limit \( k \ll v_\perp/\Omega \ll 1 \), the perturbation of the distribution can be written as

\[
F^{(1)} = \frac{1}{2} \left( f_+ e^{i\phi} + f_- e^{-i\phi} \right). \quad (A4)
\]

From the linearized Vlasov equation, one obtains

\[
f_k = \frac{\pm q\delta B_k}{\omega + i\Gamma - k_0 v_\parallel \pm \Omega/\gamma} \times \left[ 
\begin{array}{c}
\frac{\partial}{\partial \rho_\parallel} \\
\frac{\partial}{\partial \rho_\parallel}
\end{array}
\right]
\begin{array}{c}
v_\parallel \\
v_\parallel + (\omega - i\Gamma) k_0
\end{array}
\right] F. \quad (A5)
\]

Substituting (A4) and (A5) into (A1), one derives

\[
\frac{dF}{dr} = -\frac{q}{4iV} \int \frac{dk}{(2\pi)^3} \left[ \frac{v_\parallel}{\rho_1} \left( v_\perp - \frac{\omega - i\Gamma}{k_0} \right) \frac{\partial}{\partial \rho_\perp} F \right.
\]

\[
\left. - \left( v_\parallel - \frac{\omega - i\Gamma}{k_0} \right) \frac{1}{\rho_1} \frac{\partial}{\partial \rho_\perp} F \right. \times (\delta B^*_k f_+ - \delta B_-)
\]

\[
\times \left( \delta B^*_k f_+ - \delta B_- \right), \quad (A6)
\]

where one uses

\[
\langle \delta B_k \cdot \delta B'_k \rangle = \frac{(2\pi)^3}{V} |\delta B_k|^2 \delta(k + k'). \quad (A7)
\]
With $|\delta B_{||}|^2 = |\delta B_{\perp}|^2 = |\delta B_{\perp}|^2$, one obtains
\[
\frac{dF}{dt} = -\frac{q}{2iV} \int \frac{dk}{(2\pi)^3} |\delta B_k|^2 \times \left[ \frac{v_\perp}{p_{||}} \left( v_{||} - \left( \frac{\omega + i\Gamma}{k_{||}} \right) \frac{\partial}{\partial p_{||}} p_{\perp} \right) \right] \times \left[ \frac{\omega + i\Gamma - k_{||} v_{||}}{\left( \omega + i\Gamma - k_{||} v_{||} \right)^2 - \Omega^2 / \gamma^2} \frac{\partial}{\partial p_{\perp}} \right] F. \tag{A8}
\]
For practical purposes, we set $\omega \sim 0$ here.

### A2 Non-resonant diffusion

The quasi-linear diffusion equation in the long wavelength approximation can be obtained using the following expansion in $1/\Omega$:
\[
\frac{\omega + i\Gamma - k_{||} v_{||}}{\left( \omega + i\Gamma - k_{||} v_{||} \right)^2 - \Omega^2 / \gamma^2} \approx \frac{k_{||} v_{||}}{\Omega^2 - \Omega^2 / \gamma^2}, \tag{A9}
\]
with $\tilde{\Omega} = \Omega / \gamma$ Machabeli et al. (2000). We consider a more general expansion in $\Gamma / |k_{||} v_{||} - \tilde{\Omega}| \gg 1$:
\[
\frac{i\Gamma - k_{||} v_{||}}{\left( i\Gamma - k_{||} v_{||} \right)^2 - \Omega^2} \approx \frac{1}{k_{||}^2 v_{||}^2 - \Omega^2} \times \left( i\Gamma - k_{||} v_{||} - \frac{2i\Gamma k_{||}^2 v_{||}^2}{k_{||}^2 v_{||}^2 - \Omega^2} \right). \tag{A10}
\]
Integration on the right-hand side of (A.8) includes terms $\propto k_{||}$ or $\propto 1/k_{||}$. For turbulent spectra with symmetry $k_{||} \rightarrow -k_{||}$, one has
\[
\int_{-\infty}^{\infty} dk_{||} \text{ (terms } \propto k_{||} \text{ or } \propto 1/k_{||} \text{)} = 0, \tag{A11}
\]
and only terms $\propto i\Gamma$ remain.

### A3 Resonant diffusion

The standard procedure to write down the resonant diffusion involves setting $\Gamma = 0$. Equation (A.8) can be expressed in the following familiar form
\[
\frac{dF}{dt} = \left( \frac{\partial}{\partial p_{||}} D_{||} + \frac{\partial}{\partial p_{\perp}} D_{\perp} \right) p_{\perp} \left( \frac{\partial}{\partial p_{||}} + \frac{\partial}{\partial p_{\perp}} D_{\perp} \right), \tag{A12}
\]
where $D_{\alpha\beta}$ are the diffusion coefficients.

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