On some properties of symplectic Grothendieck polynomials

Eric Marberg*HKUST
br.pawlowski@gmail.com

Brendan PawlowskiUniversity of Southern California

Abstract

Grothendieck polynomials, introduced by Lascoux and Schützenberger, are certain $K$-theory representatives for Schubert varieties. Symplectic Grothendieck polynomials, described more recently by Wyser and Yong, represent the $K$-theory classes of orbit closures for the complex symplectic group acting on the complete flag variety. We prove a transition formula for symplectic Grothendieck polynomials and study their stable limits. We show that each of the $K$-theoretic Schur $P$-functions of Ikeda and Naruse arises from a limiting procedure applied to symplectic Grothendieck polynomials representing certain “Grassmannian” orbit closures.

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1 Introduction

Let \( n \) be a positive integer. The \( K \)-theory ring of the variety \( \text{Fl}_n \) of complete flags in \( \mathbb{C}^n \) is isomorphic to a quotient of a polynomial ring [11] \( \mathbb{Z}[\beta][x_1, x_2, \ldots] \). Under this correspondence, the Grothendieck polynomials \( G_w \) represent the classes of the structure sheaves of Schubert varieties. The results in this paper concern a family of symplectic Grothendieck polynomials \( G_{\text{Sp}z} \) which similarly represent the \( K \)-theory classes of the orbit closures of the complex symplectic group acting on \( \text{Fl}_n \).

The Grothendieck polynomials \( G_w \) lie in \( \mathbb{Z}[\beta][x_1, x_2, \ldots] \), where \( \beta, x_1, x_2, \ldots \) are commuting indeterminates, and are indexed by elements \( w \) of the group \( S_\infty \) of permutations of the positive integers \( \mathbb{P} := \{1, 2, 3, \ldots\} \) with finite support. Lascoux and Schützenberger first defined these polynomials in [13, 15]. Setting \( \beta = 0 \) transforms Grothendieck polynomials to Schubert polynomials, which represent the Chow classes of Schubert varieties.

Lenart [17], extending work of Lascoux [14], proved a “transition formula” expressing any product \( x_k G_w \) as a finite linear combination of Grothendieck polynomials; the Bruhat order on \( S_\infty \) controls which terms appear. A nice corollary of Lenart’s result is that the set of Grothendieck polynomials form a \( \mathbb{Z}[\beta] \)-basis for the polynomial ring \( \mathbb{Z}[\beta][x_1, x_2, \ldots] \) (see Corollary 3.3).

For \( w \in S_\infty \) and \( m \geq 0 \), let \( 1^m \times w \in S_\infty \) denote the permutation sending \( i \mapsto i \) for \( i \leq m \) and \( m + i \mapsto m + w(i) \) for \( i > m \). The stable Grothendieck polynomial of \( w \) is then given by

\[
G_w := \lim_{m \to \infty} \mathcal{G}_{1^m \times w} \in \mathbb{Z}[\beta][[x_1, x_2, \ldots]].
\] (1.1)

Results of Fomin and Kirillov [4] show that this limit converges in the sense of formal power series to a well-defined symmetric function. Despite its name, \( G_w \) is a power series rather than a polynomial.

Of particular interest are the stable Grothendieck polynomials \( G_\lambda := G_{w_\lambda} \) where \( w_\lambda \) is the Grassmannian permutation associated to an integer partition \( \lambda \) (see [11]). The \( G_\lambda \)’s represent structure sheaves of Schubert varieties in a Grassmannian [2] and are natural “\( K \)-theoretic” generalizations of Schur functions. One can deduce from the transition formula for \( \mathcal{G}_w \) that \( G_w \) is an \( \mathbb{N}[\beta] \)-linear combination of \( G_\lambda \)’s, and the Hecke insertion algorithm of [3] leads to a combinatorial description of the coefficients in this expansion.

The symplectic Grothendieck polynomials \( \mathcal{G}_{\text{Sp}z} \) are a second family of polynomials in \( \mathbb{Z}[\beta][x_1, x_2, \ldots] \), which now represent the \( K \)-theory classes of the orbit closures of the complex symplectic group acting on \( \text{Fl}_n \) for even \( n \). They are indexed by elements \( z \) of the set \( I^{\text{FPF}}_{\infty} \) of bijections \( z : \mathbb{P} \to \mathbb{P} \) such that \( z \) is a fixed-point-free involution of the positive integers with “finite support.”) Wyser and Yong first considered these polynomials in [23], but their definition differs from ours by a minor change of variables. Setting \( \beta = 0 \) gives the fixed-point-free involution Schubert polynomials studied in [6, 9, 23].

Our first main result, Theorem 5.8, is an analogue of Lenart’s transition
formula for symplectic Grothendieck polynomials. This is somewhat more complicated than Lenart’s identity, involving multiplication of $G_{z_k}^{sp}$ by two indeterminates $x_k$ and $x_{z(k)}$; the corresponding proof is also more involved. Nevertheless, there is a surprising formal similarity between the two transition equations. A variant of Bruhat order again plays a key role.

This paper is a sequel to [20], where we showed that the natural analogue of the stable limit (1.1) for symplectic Grothendieck polynomials defines a symmetric formal power series $GP_z^{sp}$ for each $z \in I_{\text{FPF}}^\infty$. Results of the first author [19] show that $GP_z^{sp}$ is a finite $N[\beta]$-linear combination of Ikeda and Naruse’s $K$-theoretic Schur $P$-functions $GP_\lambda^{sp}$. Here we prove an important related fact: each $GP_\lambda$ occurs as $GP_z^{sp}$ where $z_\lambda \in I_{\text{FPF}}^\infty$ is the FPF-Grassmannian involution corresponding to $\lambda$. See Theorem 4.17 for the precise statement.

Every symmetric power series in $Z[\beta][x_1, x_2, \ldots]$ can be written as a possibly infinite $Z[\beta]$-linear combination of stable Grothendieck polynomials. Results in [5, 22] imply that each $K$-theoretic Schur $P$-function $GP_\lambda$ is a (possibly infinite) sum of $G_\mu$’s with coefficients in $N[\beta]$. One application of the preceding paragraph, which does not seem to follow from prior work, is a proof that these sums are always finite; see Corollary 4.18.

A brief outline of the rest of this article is as follows. Section 2 covers some background material on permutations, divided difference operators, and Grothendieck polynomials. In Section 3 we review Lenart’s transition formula for $G_w$ and then prove its symplectic analogue. Section 4 finally, contains our results on symplectic stable Grothendieck polynomials.

2 Preliminaries

This section includes a few preliminaries and sets up most of our notation. We write $N = \{0, 1, 2, \ldots\}$ and $P = \{1, 2, 3, \ldots\}$ for the sets of nonnegative and positive integers, and define $[n] := \{1, 2, \ldots, n\}$ for $n \in N$. Throughout, the symbols $\beta, x_1, x_2, \ldots$ denote commuting indeterminates.

2.1 Permutations

For $i \in P$, define $s_i = (i, i + 1)$ to be the permutation of $P$ interchanging $i$ and $i + 1$. These simple transpositions generate the infinite Coxeter group $S_\infty := \langle s_i : i \in P \rangle$ of permutations of $P$ with finite support, as well as the finite subgroups $S_n := \langle s_1, s_2, \ldots, s_{n-1} \rangle$ for each $n \in \mathbb{P}$. The length of $w \in S_\infty$ is $\ell(w) := \{((i, j) \in P \times P : i < j \text{ and } w(i) > w(j))\}$. This finite quantity is also the minimum number of factors in any expression for $w$ as a product of simple transpositions.

We represent elements of $S_\infty$ in one-line notation by identifying a word $w_1 w_2 \cdots w_n$ that has $\{w_1, w_2, \ldots, w_n\} = [n]$ with the permutation $w \in S_\infty$ that has $w(i) = w_i$ for $i \in [n]$ and $w(i) = i$ for all integers $i > n$. 3
2.2 Divided difference operators

Let \( L = \mathbb{Z}[\beta][x_1^{\pm 1}, x_2^{\pm 1}, \ldots] \) denote the ring of Laurent polynomials in the variables \( x_1, x_2, \ldots \) with coefficients in \( \mathbb{Z}[\beta] \). Given \( i \in \mathbb{P} \) and \( f \in L \), write \( s_if \) for the Laurent polynomial formed from \( f \) by interchanging the variables \( x_i \) and \( x_{i+1} \). This operation extends to a group action of \( S_\infty \) on \( L \). For \( i \in \mathbb{P} \), the divided difference operators \( \partial_i \) and \( \partial_i^{(\beta)} \) are the maps \( L \to L \) given by

\[
\partial_i f = \frac{x_i - s_i f}{x_i - x_{i+1}} \quad \text{and} \quad \partial_i^{(\beta)} f = \partial_i((1 + \beta x_{i+1})f) = -\beta f + (1 + \beta x_i)\partial_i f. \tag{2.1}
\]

Both operators preserve the subring of polynomials \( \mathbb{Z}[\beta][x_1, x_2, \ldots] \subset L \).

Some identities are useful for working with these maps. All formulas involving \( \partial_i^{(\beta)} \) reduce to formulas involving \( \partial_i \) on setting \( \beta = 0 \). Fix \( i \in \mathbb{P} \) and \( f, g \in L \). Then

\[
\partial_i^{(\beta)}(fg) = s_i f \cdot (\partial_i^{(\beta)}g + \beta g) + \partial_i^{(\beta)}f \cdot g \tag{2.2}
\]

and we have \( \partial_i f = 0 \) and \( \partial_i^{(\beta)} f = -\beta f \) if and only if \( s_i f = f \), in which case

\[
\partial_i(fg) = f \cdot \partial_i g \quad \text{and} \quad \partial_i^{(\beta)}(fg) = f \cdot \partial_i^{(\beta)} g. \tag{2.3}
\]

Moreover, one has \( \partial_i \partial_i = 0 \) and \( \partial_i^{(\beta)} \partial_i^{(\beta)} = -\beta \partial_i^{(\beta)} \). Both families of operators satisfy the usual braid relations for \( S_\infty \), meaning that we have

\[
\partial_i^{(\beta)} \partial_j^{(\beta)} = \partial_j^{(\beta)} \partial_i^{(\beta)} \quad \text{and} \quad \partial_i^{(\beta)} \partial_{i+1}^{(\beta)} \partial_i^{(\beta)} = \partial_{i+1}^{(\beta)} \partial_i^{(\beta)} \partial_{i+1}^{(\beta)} \tag{2.4}
\]

for all \( i, j \in \mathbb{P} \) with \( |i - j| > 1 \). If \( w \in S_\infty \) then we can therefore define

\[
\partial_w := \partial_{i_1} \partial_{i_2} \cdots \partial_{i_t} \quad \text{and} \quad \partial_w^{(\beta)} := \partial_{i_1}^{(\beta)} \partial_{i_2}^{(\beta)} \cdots \partial_{i_t}^{(\beta)}
\]

where \( w = s_{i_1} s_{i_2} \cdots s_{i_t} \) is any reduced expression, i.e., a minimal length factorization of \( w \) as a product of simple transpositions.

2.3 Grothendieck polynomials

The following definition of Grothendieck polynomials originates in \cite{4}.

**Theorem-Definition 2.1** (Fomin and Kirillov \cite{4}). There exists a unique family \( \{ \mathcal{G}_w \}_{w \in S_\infty} \subset \mathbb{Z}[\beta][x_1, x_2, \ldots] \) with \( \mathcal{G}_{n \cdots 321} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}^{1} \) for all \( n \in \mathbb{P} \) and such that \( \partial_i^{(\beta)} \mathcal{G}_w = \mathcal{G}_{ws_i} \) for \( i \in \mathbb{P} \) with \( w(i) > w(i+1) \).

Note that it follows that \( \partial_i^{(\beta)} \mathcal{G}_w = -\beta \mathcal{G}_w \) if \( w(i) < w(i+1) \).

**Example 2.2.** The Grothendieck polynomials for \( w \in S_3 \) are

\[
\begin{align*}
\mathcal{G}_{123} &= 1, & \mathcal{G}_{132} &= x_1 + x_2 + \beta x_1 x_2, & \mathcal{G}_{312} &= x_1^2, \\
\mathcal{G}_{213} &= x_1, & \mathcal{G}_{231} &= x_1 x_2, & \mathcal{G}_{321} &= x_1^2 x_2.
\end{align*}
\]
We typically suppress the parameter \( \beta \) in our notation, but for the moment write \( \mathcal{G}_w^{(i)} = \mathcal{G}_w \) for \( w \in S_\infty \). The Schubert polynomial \( \mathcal{G}_w \) of a permutation \( w \in S_\infty \) (see [18] Chapter 2) is then \( \mathcal{G}_w^{(0)} \). The polynomials \( \{\mathcal{G}_w\}_{w \in S_\infty} \) are a \( \mathbb{Z}[\beta] \)-basis for \( \mathbb{Z}[\beta][x_1, x_2, \ldots] \) [18] Proposition 2.5.4 so the Grothendieck polynomials are linearly independent.

Some references use the term “Grothendieck polynomial” to refer to the polynomials \( \mathcal{G}_w^{(-1)} \). One loses no generality in setting \( \beta = -1 \) since one can show by downward induction on permutation length that

\[
(-\beta)^{i(w)}\mathcal{G}_w^{(\beta)} = \mathcal{G}_w^{(-1)}(-\beta x_1, -\beta x_2, \ldots).
\]

Thus, it is straightforward to translate formulas in \( \mathcal{G}_w^{(-1)} \) to formulas in \( \mathcal{G}_w^{(\beta)} \).

2.4 Symplectic Grothendieck polynomials

Let \( \Theta : \mathbb{P} \rightarrow \mathbb{P} \) be the map sending \( i \mapsto i - (-1)^i \), so that \( \Theta = (1, 2)(3, 4)(5, 6) \cdots \). Define \( I^\text{FPF}_\infty := \{ w^{-1}\Theta w : w \in S_\infty \} \). The elements of \( I^\text{FPF}_\infty \) are the involutions of the positive integers that have no fixed points and that agree with \( \Theta \) at all sufficiently large values of \( i \). We represent elements of \( I^\text{FPF}_\infty \) in one-line notation by identifying a word \( z_1 z_2 \cdots z_n \), satisfying \( \{ z_1, z_2, \ldots, z_n \} = [n] \) and \( z_i = j \) if and only if \( z_j = i \neq j \), with the involution \( z \in I^\text{FPF}_\infty \) that has \( z(i) = z_i \) for \( i \in [n] \) and \( z(i) = \Theta(i) \) for \( i > n \).

The symplectic analogues of \( \mathcal{G}_w \) introduced below were first studied by Wyser and Yong in a slightly different form; see [24] Theorems 3 and 4]. The characterization given here combines [20] Theorem 3.10 and Proposition 3.11.

**Theorem-Definition 2.3** ([20] [23]). There exists a unique family \( \{\mathcal{G}_z^{Sp}\}_{z \in I^\text{FPF}_\infty} \subset \mathbb{Z}[\beta][x_1, x_2, \ldots] \) with \( \mathcal{G}_z^{Sp} = \prod_{1 \leq i < j \leq n-1} (x_i + x_j + \beta x_i x_j) \) for all \( n \in 2\mathbb{P} \) and such that \( \partial_i^{(\beta)} \mathcal{G}_z^{Sp} = \mathcal{G}_{z_i z_i}^{Sp} \), for \( i \in \mathbb{P} \) with \( i + 1 \neq z(i) \neq z(i+1) \neq i \).

The elements of this family are the symplectic Grothendieck polynomials described in the introduction. If \( i \in \mathbb{P} \) is such that \( z(i) < z(i+1) \) or \( i + 1 = z(i) > z(i+1) = i \) then \( \partial_i^{(\beta)} \mathcal{G}_z^{Sp} = -\beta \mathcal{G}_z^{Sp} \) [20] Proposition 3.11.

**Example 2.4.** The polynomials for \( z \in I^\text{FPF}_4 := \{ w\Theta w^{-1} : w \in S_4 \} \) are

\[
\mathcal{G}_z^{Sp} = \begin{cases}
1, & z = 2143 \\
1 + x_1 x_2 + \beta x_1 x_2, & z = 3124 \\
1 + x_1 x_2 + x_1 x_3 + 2\beta x_1 x_2 x_3 + \beta x_1^2 x_2 + \beta x_1^2 x_3 + \beta^2 x_1^2 x_2 x_3, & z = 3214
\end{cases}
\]

Setting \( \beta = 0 \) transforms \( \mathcal{G}_z^{Sp} \) to the fixed-point-free involution Schubert polynomials \( \mathcal{G}_z^{Sp} \), studied in [6] [7] [8] [23]. Since the family \( \{\mathcal{G}_z^{Sp}\}_{z \in I^\text{FPF}_\infty} \) is linearly independent, \( \{\mathcal{G}_z^{Sp}\}_{z \in I^\text{FPF}_\infty} \) is also linearly independent.

We need one other preliminary result concerning the polynomials \( \mathcal{G}_z^{Sp} \). The symplectic Rothe diagram of an involution \( z \in I^\text{FPF}_\infty \) is the set of pairs

\[
D^\text{Sp}(z) := \{(i, z(j)) : (i, j) \in [n] \times [n] \text{ and } z(i) > z(j) < i < j\}.
\]
An element $z \in I^\text{FPP}_\infty$ is Sp-dominant if $D_{\text{Sp}}(z) = \{(i+j,j) \in \mathbb{P} \times [k] : 1 \leq i \leq \mu_j \}$ for a strict partition $\mu = (\mu_1 > \mu_2 > \cdots > \mu_k > 0)$. This condition holds, for example, when $z = n \cdots 321$ for any $n \in 2\mathbb{P}$.

**Theorem 2.5** ([20] Theorem 3.8]). If $z \in I^\text{FPP}_\infty$ is Sp-dominant then

$$\mathfrak{S}^\text{Sp}_2 = \prod_{(i,j) \in D_{\text{Sp}}(z)} (x_i + x_j + \beta x_i x_j).$$

### 3 Transition equations

Lenart [17] derives a formula expanding the product $x_k \mathfrak{S}_v$ for $k \in \mathbb{P}$ and $v \in S_\infty$ in terms of other Grothendieck polynomials. In this section, we prove a similar identity for symplectic Grothendieck polynomials.

#### 3.1 Lenart’s transition formula

We recall Lenart’s formula to motivate our new results. Given $v \in S_\infty$ and $k \in \mathbb{P}$, define $P_k(v)$ to be the set of all permutations in $S_\infty$ of the form

$$w = v(a_1, k) (a_2, k) \cdots (a_p, k) (k, b_1) (k, b_2) \cdots (k, b_q)$$

where $p, q \in \mathbb{N}$ and $a_1 < \cdots < a_p < a_1 < b_1 < b_2 < \cdots < b_q < b_1$, and the length increases by exactly one upon multiplication by each transposition. Differing slightly from the convention in [17], we allow the case $p = q = 0$ so $w \in P_k(v)$. Given $w \in P_k(v)$ define $\epsilon_k(w, v) = (-1)^p$. This notation is well-defined since $p$ can be recovered from $w \in P_k(v)$ as the number of indices $i < k$ with $v(i) \neq w(i)$.

**Theorem 3.1** ([17] Theorem 3.1]). If $v \in S_\infty$ and $k \in \mathbb{P}$ then

$$(1 + \beta x_k) \mathfrak{S}_v = \sum_{w \in P_k(v)} \epsilon_k(w, v) \beta^{\ell(w) - \ell(v)} \mathfrak{S}_w.$$

The cited theorem of Lenart applies to the case when $\beta = -1$, but this is equivalent to the given identity for generic $\beta$ by (2.4).

**Example 3.2.** Taking $v = 13452 \in S_\infty$ and $k = 3$ in Theorem 3.1 gives

$$(1 + \beta x_3) \mathfrak{S}_{13452} = \mathfrak{S}_{13452} + \beta \mathfrak{S}_{13542} - \beta \mathfrak{S}_{14352} - \beta^2 \mathfrak{S}_{14352} + \beta^2 \mathfrak{S}_{34152} + \beta^3 \mathfrak{S}_{34512} + \beta^3 \mathfrak{S}_{34251} + \beta^4 \mathfrak{S}_{34521}.$$

This reduces to [17] Example 3.9] on setting $\beta = -1$.

Lenart’s formula implies that $x_k \mathfrak{S}_v$ is a finite $\mathbb{Z}[\beta]$-linear combination of $\mathfrak{S}_w$’s. By starting with $v = 1$ so that $\mathfrak{S}_v = 1$, we deduce that any monomial in $\mathbb{Z}[\beta][x_1, x_2, \ldots]$ is a finite linear combination of Grothendieck polynomials. Since these functions are also linearly independent, the following holds:

**Corollary 3.3.** The set $\{\mathfrak{S}_w \}_{w \in S_\infty}$ is a $\mathbb{Z}[\beta]$-basis for $\mathbb{Z}[\beta][x_1, x_2, \ldots]$. 


Remark 3.4. This corollary is nontrivial since $S_w$ is an inhomogeneous polynomial of the form $S_w + (\text{terms of degree greater than } \ell(w) \text{). Since } \{S_w\}_{w \in S}^\infty \text{ is a } \mathbb{Z}\text{-basis for } \mathbb{Z}[x_1, x_2, \ldots], \text{ it follows that any polynomial in } \mathbb{Z}[\beta][x_1, x_2, \ldots] \text{ can be inductively expanded in terms of Grothendieck polynomials. However, it is not clear a priori that such an expansion will terminate in a finite sum.}

For $v, w \in S^\infty$, write $v \widesim w$ if $\ell(w) = \ell(v) + 1$ and $v^{-1}w = (i, j)$ is a transposition for some positive integers $i < j$. It is well-known that if $w \in S^\infty$ and $i, j \in \mathbb{P}$ are such that $i < j$, then $w \widesim w(i, j)$ if and only if $w(i) < w(j)$ and no integer $e$ has $i < e < j$ and $w(i) < w(e) < w(j)$.

For distinct integers $i, j \in \mathbb{P}$, let $t_{ij}$ be the linear operator, acting on the right, with $S_w t_{ij} = S_{w(i,j)}$ for $w \in S^\infty$. We can restate Theorem 3.1 as the following identity:

**Theorem 3.5.** Fix $v \in S^\infty \text{ and } k \in \mathbb{P}$. Suppose

\[ 1 \leq j_1 < j_2 < \cdots < j_p < k < l_q < \cdots < l_2 < l_1 \]

are the integers such that $v \widesim v(j, k)$ and $v \widesim v(k, l)$. Then

\[ (1 + \beta x_k) \left[ S_v \cdot (1 + \beta t_{j_1}) \cdots (1 + \beta t_{j_p}) \right] = S_v \cdot (1 + \beta t_{k_1}) \cdots (1 + \beta t_{k_q}). \]

**Proof.** After setting $\beta = -1$, this is a slight generalization of [17, Corollary 3.10] (which is the main result of [14]), and has nearly the same proof. Let $J = \{j_1, j_2, \ldots, j_p\}$ and $L = \{l_1, l_2, \ldots, l_q\}$. For subsets $E = \{e_1 < e_2 < \cdots < e_m\} \subset J$ and $F = \{f_n < \cdots < f_2 < f_1\} \subset L$ define $t_{E,k}, t_{k,F} \in S^\infty$ by

\[ t_{A,k} = (e_1, k)(e_2, k) \cdots (e_m, k) \quad \text{and} \quad t_{k,B} = (k, f_1)(k, f_2) \cdots (k, f_n). \]

One has $\ell(v t_{E,k}) = \ell(v) + |E|$ and $\ell(v t_{k,F}) = \ell(v) + |F|$ for all choices of $E \subset J$ and $F \subset L$. Hence, by Theorem 3.1, we must show that

\[ \sum_{E \subset J} \sum_{w \in P_k(v t_{E,k})} e_k(w, v t_{E,k}) \beta^{\ell(w) - \ell(v)} S_w = \sum_{F \subset L} \beta^{|F|} S_{v t_{k,F}}. \]  

(3.1)

Each permutation $w$ indexing the sum on the left can be written as

\[ w = v(i_1, k)(i_2, k) \cdots (i_m, k)(i_{m+1}, k) \cdots (i_{n}, k)(k, i_{n+1}) \cdots (k, i_r) \]

for some indices with $i_1 < i_2 < \cdots < i_m > i_{m+1} > \cdots > i_n$ and $i_{n+1} > \cdots > i_r > k$ and $\{i_1, i_2, \ldots, i_m\} \subset J$. Here, the set indexing the outer sum on the left side of (3.1) is $E = \{i_1, i_2, \ldots, i_m\}$. If $n > 0$ then each such $w$ appears twice with opposite associated signs $e_k(w, v t_{E,k})$; the two appearances correspond to $E = \{i_1, \ldots, i_m\}$ and $E = \{i_1, \ldots, i_{m-1}\}$. The permutations $w$ that arise with $n = 0$, alternatively, are exactly the elements $v t_{k,F}$ for $F \subset L$, so (3.1) holds. □
3.2 Fixed-point-free Bruhat order

For each involution \( z \in I_{\infty}^{\text{FPF}} \), let

\[
\ell_{\text{FPF}}(z) = |\{(i, j) \in \mathbb{P} \times \mathbb{P} : z(i) > z(j) < i < j\}|. \tag{3.2}
\]

One can check that if \( z \in I_{\infty}^{\text{FPF}} \) and \( i \in \mathbb{P} \) then

\[
\ell_{\text{FPF}}(s_i z s_i) = \begin{cases} 
\ell_{\text{FPF}}(z) + 1 & \text{if } z(i) < z(i + 1) \\
\ell_{\text{FPF}}(z) & \text{if } i + 1 = z(i) > z(i + 1) = i \\
\ell_{\text{FPF}}(z) - 1 & \text{if } i + 1 \neq z(i) > z(i + 1) \neq i.
\end{cases} \tag{3.3}
\]

It follows by induction that

\[
\ell_{\text{FPF}}(z) = \min \{ \ell(w) : w \in S_\infty \text{ and } w^{-1} \Theta w = z \}.
\]

For \( y, z \in I_{\infty}^{\text{FPF}} \), we write \( y \prec_F y \) if \( \ell_{\text{FPF}}(z) = \ell_{\text{FPF}}(y) + 1 \) and \( z = tyt \) for a transposition \( t \in S_\infty \). The transitive closure of this relation is the Bruhat order on \( I_{\infty}^{\text{FPF}} \) from \([9, \S 4.1]\). One can give a more explicit characterization of \( \prec_F \):

**Proposition 3.6 ([7, Proposition 4.9]).** Suppose \( y \in I_{\infty}^{\text{FPF}} \), \( i, j \in \mathbb{P} \), and \( i < j \).

(a) If \( y(i) < i \) then \( y \prec_F (i, j)y(i, j) \) if and only if these properties hold:

- Either \( y(i) < i < j < y(j) \) or \( y(i) < y(j) < i < j \).
- No integer \( e \) has \( i < e < j \) and \( y(i) < y(e) < y(j) \).

(b) If \( j < y(j) \) then \( y \prec_F (i, j)y(i, j) \) if and only if these properties hold:

- Either \( y(i) < i < j < y(j) \) or \( i < j < y(i) < y(j) \).
- No integer \( e \) has \( i < e < j \) and \( y(i) < y(e) < y(j) \).

**Remark 3.7.** Let \( y \in I_{\infty}^{\text{FPF}} \) and \( i < j \) and \( t = (i, j) \in S_\infty \). The cases when \( y \prec_F tyt \) correspond to the following pictures, in which the edges indicate the cycle structure of the relevant involutions restricted to \( \{i, j, y(i), y(j)\} \):

\[
\begin{align*}
y &= \begin{array}{c} \circ \end{array} \xleftarrow{i} \begin{array}{c} \circ \end{array} \xleftarrow{j} \begin{array}{c} \circ \end{array} \prec_F \begin{array}{c} \circ \end{array} \xleftarrow{i} \begin{array}{c} \circ \end{array} \xleftarrow{j} \begin{array}{c} \circ \end{array} = tyt, \\
y &= \begin{array}{c} \circ \end{array} \xleftarrow{i} \begin{array}{c} \circ \end{array} \xleftarrow{j} \begin{array}{c} \circ \end{array} \prec_F \begin{array}{c} \circ \end{array} \xleftarrow{i} \begin{array}{c} \circ \end{array} \xleftarrow{j} \begin{array}{c} \circ \end{array} = tyt, \\
y &= \begin{array}{c} \circ \end{array} \xleftarrow{i} \begin{array}{c} \circ \end{array} \xleftarrow{j} \begin{array}{c} \circ \end{array} \prec_F \begin{array}{c} \circ \end{array} \xleftarrow{i} \begin{array}{c} \circ \end{array} \xleftarrow{j} \begin{array}{c} \circ \end{array} = tyt.
\end{align*}
\]

3.3 Symplectic transitions

For distinct \( i, j \in \mathbb{P} \), define \( u_{ij} \) to be the linear operator with \( \Theta^\text{Sp}_z u_{ij} = \Theta^\text{Sp}_{z(i,j)} u_{ij} \) for \( z \in I_{\infty}^{\text{FPF}} \). One cannot hope for a symplectic version of Theorem 3.1 since products of the form \( (1 + \beta x_k) \Theta^\text{Sp}_z \) may fail to be linear combinations of symplectic Grothendieck polynomials. There is an analogue of Theorem 3.5 however:
Theorem 3.8. Fix $v \in \mathcal{I}_\infty^{\text{FF}}$ and $j, k \in \mathbb{P}$ with $v(k) = j < k = v(j)$. Suppose

$$1 \leq i_1 < i_2 < \cdots < i_p < j < k < l_q < \cdots < l_2 < l_1 \tag{3.4}$$

are the integers such that $v \prec_F (i,j) v(i,j)$ and $v \prec_F (k,l) v(k,l)$. Then

$$(1 + \beta x_j)(1 + \beta x_k)
\left[\mathcal{G}^S_v \cdot (1 + \beta u_{i_1 j})(1 + \beta u_{i_2 j}) \cdots (1 + \beta u_{i_p j})\right] \tag{3.5}$$

is equal to

$$\mathcal{G}^S_v \cdot (1 + \beta u_{k l_1})(1 + \beta u_{k l_2}) \cdots (1 + \beta u_{k l_q}). \tag{3.6}$$

This is a generalization of [7, Theorem 4.17], which one recovers by subtracting $\mathcal{G}^S_v$ from (3.5) and (3.6), dividing by $\beta$, and then setting $\beta = 0$. These results belong to a larger family of similar formulas related to Schubert calculus; see also [1, 12, 21]. Before giving the proof, we present one example.

Example 3.9. If $v = (1, 2)(3, 5)(4, 8)(6, 7) \in \mathcal{I}_\infty^{\text{FF}}$ and $(j, k) = (3, 5)$, then we have $\{i_1 < i_2 < \cdots < i_p\} = \{2\}$ and $\{l_1 > l_2 > \cdots > l_q\} = \{6, 8\}$ and Theorem 3.8 is equivalent, after a few manipulations, to the claim that $\sum_{i, j} \mathcal{G}^S_v \cdot (1 + \beta u_{i_1 j})(1 + \beta u_{i_2 j}) \cdots (1 + \beta u_{i_p j})$ is equal to $\mathcal{G}^S_v \cdot (1 + \beta u_{k l_1})(1 + \beta u_{k l_2}) \cdots (1 + \beta u_{k l_q})$.

Proof of Theorem 3.8. The proof is by downward induction on $\ell_{\text{FF}}(v)$. As a base case, suppose $v = n \cdots 321 \in \mathcal{I}_\infty$ where $n \in 2\mathbb{P}$, so that $j = n + 1 - k$. Then $p = 0$, $q = 1$, $l_1 = n + 1$, and the theorem reduces to the claim that

$$(x_j + x_{n+1-j} + \beta x_j x_{n+1-j}) \mathcal{G}^S_{v_{321-N}} = \mathcal{G}^S_{v_{321}}$$

for $w := (k, n + 1)v(k, n + 1)$. This follows from Theorem 2.5 since $w$ is $S$-dominant with $D^S(w) = D^S(v) \cup \{(n + 1 - j, j)\}$.

Now let $v \in \mathcal{I}_\infty^{\text{FF}} \setminus \{\Theta\}$ and $j, k \in \mathbb{P}$ be arbitrary with $v(k) = j < k = v(j)$. It is helpful to introduce some relevant notation. Define

$$\Pi^-(v, j, k) := \mathcal{G}^S_v \cdot (1 + \beta u_{i_1 j})(1 + \beta u_{i_2 j}) \cdots (1 + \beta u_{i_p j})$$

and let $\text{Asc}^-(v, j, k) = \{i_1, i_2, \ldots, i_p\}$ and $\text{Asc}^+(v, j, k) = \{l_1, l_2, \ldots, l_q\}$ where the indices $i_1, i_2, \ldots, i_p$ and $l_1, l_2, \ldots, l_q$ are as in (3.4). For each nonempty subset $A = \{a_1 < a_2 < \cdots < a_m\} \subset \text{Asc}^-(v, j, k)$, define

$$\tau_A^-(v, j, k) = \sigma v \sigma^{-1}, \quad \text{where } \sigma := (a_1, a_2, \ldots, a_m, j) \in S_\infty.$$ 

For each nonempty subset $B = \{b_m < \cdots < b_2 < b_1\} \subset \text{Asc}^+(v, j, k)$, define

$$\tau_B^+(v, j, k) = \sigma v \sigma^{-1}, \quad \text{where } \sigma := (b_1, b_2, \ldots, b_m, k) \in S_\infty.$$ 

For empty sets, we define $\tau_\emptyset^+(v, j, k) = v$. It then follows from Proposition 3.6 that $\ell_{\text{FF}}(\tau_\emptyset^+(v, j, k)) = \ell_{\text{FF}}(v) + |S|$ for all choices of $S$, and we have

$$\Pi^+(v, j, k) = \sum_{S \subset \text{Asc}^+(v, j, k)} \beta^{|S|} \mathcal{G}^S_v \tau_\emptyset^+(v, j, k) \tag{3.7}$$
If we represent elements of $I^\text{FPF}_\infty$ as arc diagrams, i.e., as perfect matchings on the positive integers with an edge for each 2-cycle, then the elements $\tau^\pm_S(v, j, k)$ can be understood as follows. The arc diagram of $\tau^+_S(v, j, k)$ is formed from $v$ by cyclically shifting up the endpoints $S \cup \{j\}$. The arc diagram of $\tau^-_S(v, j, k)$ is formed from $v$ by cyclically shifting down the endpoints $\{k\} \cup S$.

Suppose the theorem holds for a given $v \in I^\text{FPF}_\infty \setminus \{\Theta\}$ in the sense that $(1 + \beta x_j)(1 + \beta x_k)\Pi^-(w, j, k) = \Pi^+(w, j, k)$ for all choices of $v(k) = j < k = v(j)$. Let $d \in \mathbb{P}$ be any positive integer with $d + 1 \neq v(d) > v(d + 1) \neq d$ and set

$$w := s_d v s_d \in I^\text{FPF}_\infty.$$

Choose integers $j, k \in \mathbb{P}$ with $v(k) = j < k = v(j)$; note that we cannot have $j = d < d + 1 = k$. In view of the first paragraph, it is enough to show that

$$(1 + \beta x_j)(1 + \beta x_k)\Pi^-(w, j, k) = \Pi^+(w, j, k),$$

where $j' = s_d(j)$ and $k' = s_d(k)$. There are seven cases to examine:

- **Case 1:** Assume that $d + 1 < j$. We must show that

  $$(1 + \beta x_j)(1 + \beta x_k)\Pi^-(w, j, k) = \Pi^+(w, j, k).$$

  It suffices by (2.3) to prove that $\partial_d^{(\beta)} \Pi^+(v, j, k) = \Pi^+(w, j, k)$. The + form of this claim is straightforward from Proposition 3.6 and (3.7); in particular, it holds that $\text{Asc}^+(w, j, k) = \text{Asc}^+(v, j, k)$. For the other form, there are four subcases to consider:

  1a) Assume that $d, d + 1 \notin \text{Asc}^-(v, j, k)$. Since $v(d) > v(d + 1)$, it is again straightforward from Proposition 3.6 and (3.7) to show that $\text{Asc}^-(w, j, k) = \text{Asc}^-(v, j, k)$ and $\partial_d^{(\beta)} \Pi^-(v, j, k) = \Pi^-(w, j, k)$.

  1b) Assume that $d \in \text{Asc}^-(v, j, k)$ and $d + 1 \notin \text{Asc}^-(v, j, k)$. Then

  $$\text{Asc}^-(w, j, k) = \text{Asc}^-(v, j, k) \setminus \{d\} \cup \{d + 1\}$$

  and $d + 1 \neq \tau^-_S(v, j, k)(d) > \tau^-_S(v, j, k)(d + 1) \neq d$ for all subsets $S \subset \text{Asc}^-(v, j, k)$, so we again have $\partial_d^{(\beta)} \Pi^-(v, j, k) = \Pi^-(w, j, k)$.

  1c) Assume that $d \notin \text{Asc}^-(v, j, k)$ and $d + 1 \in \text{Asc}^-(v, j, k)$. This can only occur if $d < k < v(d)$, so we have

  $$\text{Asc}^-(w, j, k) = \text{Asc}^-(v, j, k) \setminus \{d + 1\} \cup \{d\}. $$

  From here, we deduce that $\partial_d^{(\beta)} \Pi^-(v, j, k) = \Pi^-(w, j, k)$ by an argument similar to the one in case (1b).

  1d) Assume that $d, d + 1 \in \text{Asc}^-(v, j, k)$. Three situations are possible for the relative order of $d, d + 1, v(d)$, and $v(d + 1)$. First suppose $v(d + 1) < d < d + 1 < v(d)$. Then

  $$\text{Asc}^-(w, j, k) = \text{Asc}^-(v, j, k) \setminus \{d\}$$

  (3.8)
and every \( i \in \text{Asc}^-(v, j, k) \) with \( i < d \) must have \( v(d) < v(i) < k \). It follows that if \( S \subset \text{Asc}^-(v, j, k) \) then

\[
\partial_d^{(\beta)} \mathcal{O}_{\tau_\Delta^+}^{S}(v,j,k) = \begin{cases} 
\mathcal{O}_{\tau_\Delta^+}^{S}(v,j,k) & \text{if } d, d + 1 \in S \\
-\beta \cdot \mathcal{O}_{\tau_\Delta^+}^{S}(v,j,k) & \text{if } d \notin S, \ d + 1 \in S \\
\mathcal{O}_{\tau_\Delta^+}^{S}(v,j,k) & \text{if } d, d + 1 \notin S \\
\mathcal{O}_{\tau_\Delta^+}^{S}(v,j,k) & \text{if } d \in S, \ d + 1 \notin S.
\end{cases}
\]  

(3.9)

If we have \( v(d + 1) < v(d) < d \) or \( j < v(d + 1) < v(d) < k \) then (3.8) and (3.9) both still hold and follow by similar reasoning. Combining these identities with (3.7) gives \( \partial_d^{(\beta)} \Pi^-(v, j, k) = \Pi^-(w, j, k) \).

We conclude from this analysis that \( \partial_d^{(\beta)} \Pi^*(v, j, k) = \Pi^*(w, j, k) \).

- **Case 2:** Assume that \( d + 1 = j \), so that \( k < v(j - 1) \). We must show that

\[
(1 + \beta x_{j-1})(1 + \beta x_k)\Pi^-(w, j - 1, k) = \Pi^+(w, j - 1, k).
\]

It follows from (2.2) that \( \partial_{j-1}^{(\beta)} ((1 + \beta x_j)(1 + \beta x_k)\Pi^-(v, j, k)) \) is equal to

\[
(1 + \beta x_{j-1})(1 + \beta x_k)\partial_{j-1}^{(\beta)} \Pi^-(v, j, k) - \beta \cdot \Pi^+(v, j, k)
\]

and it is easy to see that \( \partial_{j-1}^{(\beta)} \Pi^-(v, j, k) = \Pi^-(w, j - 1, k) \). Thus, it suffices to show that

\[
(\partial_{j-1}^{(\beta)} + \beta)\Pi^+(v, j, k) = \Pi^+(w, j - 1, k). \tag{3.10}
\]

It follows from Proposition 3.6 that \( \text{Asc}^+(w, j - 1, k) = \{ l \in \text{Asc}^+(v, j, k) : l < v(j - 1) \} \cup \{ v(j) \} \). We deduce that if \( S \subset \text{Asc}^+(v, j, k) \) then

\[
\partial_{j-1}^{(\beta)} \mathcal{O}_{\tau_\Delta^+}^{S}(v,j,k) = \begin{cases} 
\mathcal{O}_{\tau_\Delta^+}^{S}(w, j - 1, k) & \text{if } S \subset \text{Asc}^+(w, j - 1, k) \\
-\beta \cdot \mathcal{O}_{\tau_\Delta^+}^{S}(v,j,k) & \text{otherwise.}
\end{cases}
\]

If \( v(j) \in S \subset \text{Asc}^+(w, j - 1, k) \) then \( \tau_\Delta^+(w, j - 1, k) = \tau_\Delta^+(v, j, k) \). Combining these identities with (3.7) shows the needed claim (3.10).

- **Case 3:** Assume that \( d = j \), so that either \( v(j + 1) < j < j + 1 < k \) or \( j < j + 1 < v(j + 1) < k \). We must show that

\[
(1 + \beta x_{j+1})(1 + \beta x_k)\Pi^-(w, j + 1, k) = \Pi^+(w, j + 1, k).
\]

It follows from (2.2) that \( \partial_j^{(\beta)} ((1 + \beta x_j)(1 + \beta x_k)\Pi^-(v, j, k)) \) is equal to

\[
(1 + \beta x_{j+1})(1 + \beta x_k)(\partial_j^{(\beta)} + \beta)\Pi^-(v, j, k).
\]
It is easy to deduce that \( \partial_d^{(\beta)} \Pi^+(v, j, k) = \Pi^+(w, j + 1, k) \) from Proposition 3.6, so it suffices to show that
\[
(\partial_d^{(\beta)} + \beta) \Pi^-(v, j, k) = \Pi^-(w, j + 1, k). \tag{3.11}
\]

First assume \( v(j + 1) < j < j + 1 < k \). Then every \( i \in \text{Asc}^-(v, j, k) \) with \( i < v(j + 1) \) must have \( j + 1 < v(i) < k \), and \( \text{Asc}^-(w, j + 1, k) \) is equal to \( \{ i \in \text{Asc}^-(v, j, k) : v(j + 1) < v(i) \} \) if \( v(j + 1) \leq i < j \} \cup \{ j \} \).

As in Case 2, we deduce that if \( S \subset \text{Asc}^-(v, j, k) \) then
\[
\partial_d^{(\beta)} \mathcal{S}_G^{(\tau_G(w, j, k))} = \begin{cases} 
\mathcal{S}_G^{(\tau_G(w, j + 1, k))} & \text{if } S \subset \text{Asc}^-(w, j + 1, k) \\
-\beta \cdot \mathcal{S}_G^{(\tau_G(v, j, k))} & \text{otherwise.}
\end{cases} \tag{3.12}
\]

On the other hand, if \( j \in S \subset \text{Asc}^-(w, j + 1, k) \) then
\[
\tau_G(w, j + 1, k) = \tau_G(w, j, k). \tag{3.13}
\]

Combining these identities with (3.6) gives (3.11) as desired. Alternatively, if we have \( j < j + 1 < v(j + 1) < k \), then

\[
\text{Asc}^-(w, j + 1, k) = \{ i \in \text{Asc}^-(v, j, k) : v(j + 1) < v(i) < k \} \cup \{ j \}
\]

and we deduce by similar reasoning that the identities (3.12) and (3.13) both still hold, so (3.11) again follows.

- **Case 4:** Assume that \( j < d \) and \( d + 1 < k \). We must show that
\[
(1 + \beta x_j)(1 + \beta x_k) \Pi^-(w, j, k) = \Pi^+(w, j - 1, k).
\]

It suffices by (2.3) to prove that \( \partial_d^{(\beta)} \Pi^+(v, j, k) = \Pi^+(w, j, k) \). There are three subcases to consider:

1. **(4a)** If \( j < v(d) < k \) or \( j < v(d + 1) < k \) or \( v(d + 1) < j < k < v(d) \) then the desired identities are straightforward from Proposition 3.6.

2. **(4b)** Assume that \( v(d + 1) < v(d) < j \). In this case it is easy to see that \( \partial_d^{(\beta)} \Pi^+(v, j, k) = \Pi^+(w, j, k) \) and if \( v(d + 1) \notin \text{Asc}^-(v, j, k) \), then we likewise deduce that \( \partial_d^{(\beta)} \Pi^-(v, j, k) = \Pi^-(w, j, k) \). Suppose instead that \( v(d + 1) \in \text{Asc}^-(v, j, k) \). We then also have \( v(d) \in \text{Asc}^-(v, j, k) \), but no \( i \in \text{Asc}^-(v, j, k) \) is such that \( v(d + 1) < i < v(d) \), and

\[
\text{Asc}^-(w, j, k) = \text{Asc}^-(v, j, k) \setminus \{ v(d + 1) \}.
\]

It follows that if \( S \subset \text{Asc}^-(v, j, k) \) then
\[
\partial_d^{(\beta)} \mathcal{S}_G^{(\tau_G(w, j, k))} = \begin{cases} 
\mathcal{S}_G^{(\tau_G(v(d), j, k))} & \text{if } v(d), v(d + 1) \in S \\
-\beta \cdot \mathcal{S}_G^{(\tau_G(v, j, k))} & \text{if } v(d) \notin S, \ v(d + 1) \in S \\
\mathcal{S}_G^{(\tau_G(w, j, k))} & \text{if } v(d + 1) \notin S.
\end{cases}
\]

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Combining this with (3.7) gives \( \partial_d^{(\beta)}\Pi^-(v, j, k) = \Pi^-(w, j, k) \).

(4c) Assume that \( k < v(d+1) < v(d) \). This is the mirror image of (4b) and we get \( \partial_d^{(\beta)}\Pi^+(v, j, k) = \Pi^+(w, j, k) \) by symmetric arguments.

- **Case 5:** Assume that \( d + 1 = k \), so that either \( j < k - 1 < k < v(k-1) \) or \( j < v(k-1) < k - 1 < k \). We must show that
  \[
  (1 + \beta x_j)(1 + \beta x_{k-1})\Pi^-(w, j, k-1) = \Pi^+(w, j, k-1).
  \]
  It follows from (2.2) that \( \partial_{k-1}^{(\beta)}((1 + \beta x_j)(1 + \beta x_k)\Pi^-(v, j, k)) \) is equal to
  \[
  (1 + \beta x_j)(1 + \beta x_k)\Pi^-(v, j, k) - \beta \cdot \Pi^+(v, j, k)
  \]
  and it is easy to deduce that \( \partial_{k-1}^{(\beta)}\Pi^-(v, j, k) = \Pi^-(w, j, k-1) \). It therefore suffices to show that \( (\partial_{k-1}^{(\beta)} + \beta)\Pi^+(v, j, k) = \Pi^+(w, j, k-1) \). The required argument is the mirror image of Case 3; we omit the details.

- **Case 6:** Assume that \( d = k \). We must show that
  \[
  (1 + \beta x_j)(1 + \beta x_{k-1})\Pi^-(w, j, k+1) = \Pi^+(w, j, k+1).
  \]
  It follows from (2.2) that \( \partial_k^{(\beta)}((1 + \beta x_j)(1 + \beta x_k)\Pi^-(v, j, k)) \) is equal to
  \[
  (1 + \beta x_j)(1 + \beta x_{k+1})(\partial_k^{(\beta)} + \beta)\Pi^-(v, j, k)
  \]
  and it is easy to see that \( \partial_k^{(\beta)}\Pi^+(v, j, k) = \Pi^+(w, j, k+1) \). Thus, it suffices to show that \( (\partial_k^{(\beta)} + \beta)\Pi^-(v, j, k) = \Pi^-(w, j, k+1) \). The required argument is the mirror image of Case 2; we omit the details.

- **Case 7:** Finally, assume that \( k < d \). We must show that
  \[
  (1 + \beta x_j)(1 + \beta x_k)\Pi^-(w, j, k) = \Pi^+(w, j, k).
  \]
  It suffices by (2.3) to prove that \( \partial_k^{(\beta)}\Pi^+(v, j, k) = \Pi^+(w, j, k) \). The required argument is the mirror image of Case 1; we omit the details.

This case analysis completes our inductive proof. \( \square \)

**Corollary 3.10.** Suppose \( v \in I_\infty^{\text{FPF}} \) and \( j, k \in \mathbb{P} \) have \( j < k = v(j) \). Then
\[
(1 + \beta x_j)(1 + \beta x_k)\mathfrak{G}^\text{Sp}_v \in \mathbb{Z}[\beta]\text{-span}\{\mathfrak{G}^\text{Sp}_z : z \in I_\infty^{\text{FPF}}\}.
\]

**Proof.** It follows by induction from Theorem 3.8 that \( (1 + \beta x_j)(1 + \beta x_k)\mathfrak{G}^\text{Sp}_y \) is a possibly infinite \( \mathbb{Z}[\beta]\)-linear combination of \( \mathfrak{G}^\text{Sp}_z \)'s. This combination must be finite by Corollary 3.3 since no Grothendieck polynomial \( \mathfrak{G}_w \) appears in the expansion of \( \mathfrak{G}^\text{Sp}_y \) and \( \mathfrak{G}^\text{Sp}_z \) for distinct \( y, z \in I_\infty^{\text{FPF}} \) by [20] Theorem 3.12. \( \square \)

A visible descent of \( z \in I_\infty^{\text{FPF}} \) is an integer \( i \) such that \( z(i+1) < \min\{i, z(i)\} \).
Corollary 3.11. Let $k \in \mathbb{P}$ be the last visible descent of $z \in I_{\infty}^{\text{FPF}}$. Define $l$ to be the largest integer with $k < l$ and $z(l) < \min\{k, z(k)\}$, and set

$$v = (k, l)z(k, l) \quad \text{and} \quad j = v(k).$$

Let $1 \leq i_1 < i_2 < \cdots < i_p < j$ be the integers with $v \trianglerighteq F(i, j)v(i, j)$. Then

$$\beta G_{\text{Sp}}z = (1 + \beta x_j)(1 + \beta u_{i_1})\cdots (1 + \beta u_{i_p}) - G_{\text{Sp}}v.$$

Note that one could rewrite the right side without using any minus signs.

Proof. It suffices by Theorem 3.8 to show that $\text{Asc}^+(v, j, k) = \{l\}$. This is precisely [9, Lemma 5.2], but also follows as a self-contained exercise.

4 Stable Grothendieck polynomials

The limit of a sequence of polynomials or formal power series is defined to converge if the coefficient sequence for any fixed monomial is eventually constant.

Given $n \in \mathbb{N}$ and $w \in S_\infty$, write $1^n \times w \in S_\infty$ for the permutation that maps $i \mapsto i$ for $i \leq n$ and $i + n \mapsto w(i) + n$ for $i \in \mathbb{P}$. The stable Grothendieck polynomial of $w \in S_\infty$ is defined as the limit

$$G_w := \lim_{n \to \infty} \Theta_{1^n \times w}^v.\quad (4.1)$$

Remarkably, this always converges to a well-defined symmetric function [2, §2]. Given $n \in \mathbb{N}$ and $z \in I_{\infty}^{\text{FPF}}$, we similarly write $(21)^n \times z \in I_{\infty}^{\text{FPF}}$ for the involution mapping $i \mapsto i - (-1)^i$ for $i \leq 2n$ and $i + 2n \mapsto z(i) + 2n$ for $i \in \mathbb{P}$. Following [20], the symplectic stable Grothendieck polynomial of $z \in I_{\infty}^{\text{FPF}}$ is defined as

$$GP_{\text{Sp}}z := \lim_{n \to \infty} \Theta_{(21)^n \times z}^{\text{Sp}}.\quad (4.2)$$

The next lemma is a consequence of [20, Theorem 3.12 and Corollary 4.7]:

Lemma 4.1 (20). The limit (4.2) converges for all $z \in I_{\infty}^{\text{FPF}}$. Moreover, the resulting power series $GP_{\text{Sp}}z$ is the image of $\Theta_{z}^{\text{Sp}}$ under the linear map $\mathbb{Z}[\beta][x_1, x_2, \ldots] \to \mathbb{Z}[\beta][[x_1, x_2, \ldots]]$ with $\Theta_w \mapsto G_w$ for $w \in S_\infty$.

It follows that $GP_{\text{Sp}}z$ is also a symmetric function. These power series have some stronger symmetry properties, which we explore in this section.

4.1 $K$-theoretic Schur functions

Besides permutations and involutions, there is also a notion of stable Grothendieck polynomials for partitions, though these would more naturally be called $K$-theoretic Schur functions. The precise definition is as follows.
If \( \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0) \) is an integer partition, then a \textit{set-valued tableau} of shape \( \lambda \) is a map \( T : (i, j) \mapsto T_{ij} \) from the Young diagram 
\[
D_\lambda := \{(i, j) \in \mathbb{P} \times \mathbb{P} : j \leq \lambda_i \}
\]
to the set of finite, nonempty subsets of \( \mathbb{P} \). For such a map, define
\[
x^T := \prod_{(i, j) \in D_\lambda} \prod_{k \in T_{ij}} x_k \quad \text{and} \quad |T| := \sum_{(i, j) \in D_\lambda} |T_{ij}|.
\]
A set-valued tableau \( T \) is \textit{semistandard} if one has \( \max(T_{ij}) \leq \min(T_{i+1,j}) \) and \( \max(T_{ij}) < \min(T_{i+1,j}) \) for all relevant \((i, j) \in D_\lambda \).

Let \( \text{SetSSYT}(\lambda) \) denote the set of semistandard set-valued tableaux of shape \( \lambda \).

**Definition 4.2** ([2]). The \textit{stable Grothendieck polynomial} of a partition \( \lambda \) is
\[
G_\lambda := \sum_{T \in \text{SetSSYT}(\lambda)} \beta^{|T| - |\lambda|} x^T \in \mathbb{Z}[[x_1, x_2, \ldots]].
\]

This definition sometimes appears in the literature with the parameter \( \beta \) set to \( \pm 1 \), but if we write \( G_\lambda^{(\beta)} = G_\lambda \) then \(-\beta)^{|\lambda|} G_\lambda^{(\beta)} = G_\lambda^{(\beta)}(-\beta x_1, -\beta x_2, \ldots) \).

Setting \( \beta = 0 \) transforms \( G_\lambda \) to the usual Schur function \( s_\lambda \). For example, if \( \lambda = (1) \) then \( G_{(1)} = s_{(1)} + \beta s_{(1,1)} + \beta^2 s_{(1,1,1)} + \ldots \).

The functions \( G_\lambda \) are related to \( G_w \) for \( w \in S_\infty \) by the following result of Buch [2]. For a partition \( \lambda \) with \( k \) parts, define \( w_\lambda \in S_\infty \) to be the permutation with \( w_\lambda(i) = i + \lambda_{k+1-i} \) for \( i \in [k] \) and \( w_\lambda(i) < w_\lambda(i+1) \) for all \( i > k \).

**Theorem 4.3** ([2] Theorem 3.1). If \( \lambda \) is any partition then \( G_{w_\lambda} = G_\lambda \).

Write \( \mathscr{P} \) for the set of all partitions.

**Theorem 4.4** ([3] Theorem 1). If \( w \in S_\infty \) then \( G_w \in \mathbb{N}[\beta] \text{-span} \{G_\lambda : \lambda \in \mathscr{P} \} \).

A symplectic analogue of Theorem 4.4 is already given in [19]. Our goal in the rest of this section is to prove a symplectic analogue of Theorem 4.4.

### 4.2 Stabilization

We refer to the linear map \( \mathbb{Z}[\beta][x_1, x_2, \ldots] \to \mathbb{Z}[\beta][[x_1, x_2, \ldots]] \) with \( \Theta_w \mapsto G_w \) as \textit{stabilization}. It will be useful in the next two sections to have a description of this operation in terms of divided differences.

As in Section 2.2, let \( \mathcal{L} = \mathbb{Z}[\beta][x_1^{\pm 1}, x_2^{\pm 1}, \ldots] \). For \( i \in \mathbb{P} \), write \( \pi_i^{(\beta)} \) for the \textit{isobaric divided difference operator} defined by the formula
\[
\pi_i^{(\beta)} f = \partial_i^{(\beta)}(x_i f) = f + x_{i+1}(1 + \beta x_i) \partial_i f \quad \text{for} \quad f \in \mathcal{L}. \tag{4.3}
\]

We have \( \pi_i^{(\beta)} f = f \) if and only if \( s_i f = f \), in which case \( \pi_i^{(\beta)}(fg) = f \cdot \pi_i^{(\beta)} g \). These operators are idempotent with \( \pi_i^{(\beta)} \pi_i^{(\beta)} = \pi_i^{(\beta)} \) for all \( i \in \mathbb{P} \), and we have
\[
\pi_i^{(\beta)} \pi_j^{(\beta)} = \pi_j^{(\beta)} \pi_i^{(\beta)} \quad \text{and} \quad \pi_i^{(\beta)} \pi_{i+1}^{(\beta)} = \pi_i^{(\beta)} \pi_{i+1}^{(\beta)} = \pi_{i+1}^{(\beta)} \pi_i^{(\beta)}. \tag{4.4}
\]
for all $i, j \in \mathbb{P}$ with $|i - j| > 1$. For $w \in S_\infty$ we can therefore define

$$\pi_w^{(\beta)} = \pi_{i_1}^{(\beta)} \pi_{i_2}^{(\beta)} \cdots \pi_{i_\ell}^{(\beta)}$$

where $w = s_{i_1} \cdots s_{i_\ell}$ is any reduced expression.

Given $f \in \mathbb{Z}[\beta][[x_1, x_2, \ldots]]$ and $n \in \mathbb{N}$, write $f(x_1, \ldots, x_n)$ for the polynomial obtained by setting $x_{n+1} = x_{n+2} = \cdots = 0$ and let $w_n = n \cdots 321 \in S_n$.

**Proposition 4.5.** If $v \in S_n$ then $\mathfrak{G}_{1N \times v}(x_1, \ldots, x_n) = \pi_{w_n}^{(\beta)} \mathfrak{G}_v$ for all $N \geq n$.

**Proof.** Fix $v \in S_n$ and define $\tau_N := \pi_1^{(\beta)} \pi_2^{(\beta)} \cdots \pi_{n-1}^{(\beta)}$. We then have

$$\tau_{n+1} \mathfrak{G}_v = \partial_1^{\beta} \partial_2^{(\beta)} \cdots \partial_n^{(\beta)} (x_1 x_2 \cdots x_n \mathfrak{G}_v).$$

Let $u = (v_1 + 1)(v_2 + 2) \cdots (v_n + 1)1 \in S_{n+1}$. Since

$$x_1 x_2 \cdots x_n \mathfrak{G}_v = x_1 x_2 \cdots x_n \partial_{v_{n+1}} \mathfrak{G}_{w_n} = \partial_{v_1}^{(\beta)} \partial_{v_2}^{(\beta)} \cdots \partial_{v_{n+1}}^{(\beta)} \mathfrak{G}_{w_n} = \mathfrak{G}_u,$$

it follows that $\tau_{n+1} \mathfrak{G}_v = \mathfrak{G}_{11 \times v}$ and $\mathfrak{G}_{1N \times v} = \tau_{n+N} \cdots \tau_{n+2} \tau_{n+1} \mathfrak{G}_v$ for all $N \in \mathbb{P}$. Define $r_n(f) := f(x_1, x_2, \ldots, x_n)$. Then (4.3) implies that

$$r_n(\pi_i^{(\beta)} f) = \begin{cases} r_n(f) & \text{if } n \leq i \\ \pi_i^{(\beta)} r_n(f) & \text{if } i < n \end{cases}$$

(4.5)

so $r_n(\tau_{N} f) = \tau_n r_N(f)$ for $n \leq N$ and $r_n(\tau_{N} f) = \tau_N r_n(f)$ for $N < n$. Since $r_n(\mathfrak{G}_v) = \mathfrak{G}_v$ and $(\tau_n)^n = \pi_{w_n}^{(\beta)}$, we have $r_n(\mathfrak{G}_{1N \times v}) = \pi_{w_n}^{(\beta)} \mathfrak{G}_v$ for $N \geq n$. \qed

**Corollary 4.6.** If $v \in S_\infty$ and $z \in I_\infty^{\mathbb{P}}$ then

$$G_v = \lim_{N \to \infty} \pi_{w_N}^{(\beta)} \mathfrak{G}_v \quad \text{and} \quad GP_{z}^{\mathbb{P}} = \lim_{N \to \infty} \pi_{w_N}^{(\beta)} \mathfrak{G}_z^{\mathbb{P}}.$$

**Proof.** These identities are clear from Lemma 4.1 and Proposition 4.5 \qed

For any polynomials $x$ and $y$, let

$$x \oplus y := x + y + \beta xy \quad \text{and} \quad x \odot y := \frac{x - y}{1 + \beta y}$$

(4.6)

For integers $0 < a \leq b$, define

$$\partial_{b \setminus a}^{(\beta)} := \partial_{b-1}^{(\beta)} \partial_{b-2}^{(\beta)} \cdots \partial_a^{(\beta)} \quad \text{and} \quad \partial_{b \setminus a}^{(\beta)} := \partial_{b-1}^{(\beta)} \partial_{b-2}^{(\beta)} \cdots \partial_a^{(\beta)}$$

so that $\partial_{a \setminus a} = \partial_{a \setminus a}^{(\beta)} = 1$. Finally, let $\Delta_{m,n}^{(\beta)}(x) := \prod_{j=2}^n (1 + \beta x_{m+j})^{-1}$.

**Lemma 4.7.** If $m \in \mathbb{N}$ and $n \in \mathbb{P}$ then

$$\partial_{1 \setminus w}^{(\beta)} f = \partial_{1 \setminus w} \Delta_{m,n}^{(\beta)}(x) f = \sum_{w \in S_n} w \left( \frac{f}{\prod_{1 \leq i < j \leq n} x_{m+i} \odot x_{m+j}} \right)$$

where in the last sum $S_n$ acts by permuting the variables $x_{m+1}, x_{m+2}, \ldots, x_{m+n}$. 16
Proof. The second equality is \cite{18} Proposition 2.3.2. The first equality follows by induction: the base case when \( n = 1 \) holds by definition, and if \( n > 1 \) then \( \partial^{(\beta)}_{1 \times w_n} = \partial^{(\beta)}_{(m+n) \times (m+1)} \partial^{(\beta)}_{1 \times w_{n-1}} \) and the desired identity is easy to deduce using the fact that \( \partial^{(\beta)}_{b \times a} f = \partial_{b, a} ((1 + \beta x_k) \cdots (1 + \beta x_{a+2}) (1 + \beta x_{a+1}) f) \).

For any integer sequence \( \lambda = (\lambda_1, \lambda_2, \ldots) \) with finitely many nonzero terms, define \( x^\lambda := x_{\lambda_1} x_{\lambda_2} \cdots \). Let \( \delta_n := (n-1, n-2, \ldots, 2, 1, 0) \) for \( n \in \mathbb{P} \).

**Lemma 4.8.** If \( n \in \mathbb{P} \) then \( \pi^{(\beta)} w_n f = \partial^{(\beta)} w_n (x^{\delta_n} f) \) for all \( f \in \mathcal{L} \).

**Proof.** The expression \( w_n = (s_1)(s_2s_1)(s_3s_2s_1) \cdots (s_{n-1} \cdots s_3s_2s_1) \) is reduced and one can check, noting that \( \partial^{(\beta)}_1 (x_1 x_2 \cdots x_n f) = x_1 x_2 \cdots x_n \partial^{(\beta)} f \) for \( i < n \), that \( \partial^{(\beta)}_{n-1} \cdots \partial_2 \partial_1 (x^{\delta_n} f) = x^{\delta_n-2} x^{\delta_n-1} \cdots \partial_2 \partial_1 (x^{\delta_n} f) \). The lemma follows by induction from these identities. \qed

**Corollary 4.9.** If \( \lambda \) is a partition then \( G_\lambda = \lim_{n \to \infty} \pi^{(\beta)} w_n (x^\lambda) \).

**Proof.** Apply Lemmas 4.7 and 4.8 to \cite{10} Eq. (2.14)], for example. \qed

### 4.3 K-theoretic Schur P-functions

The natural symplectic analogues of Theorems 4.3 and 4.4 involve shifted versions of the symmetric functions \( G_\lambda \), which we review here.

Define the **marked alphabet** to be the totally ordered set of primed and unprimed integers \( \mathbb{M} := \{ 1' < 1 < 2' < 2 < \ldots \} \), and write \( |i'| := |i| = i \) for \( i \in \mathbb{P} \). If \( \lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_k > 0) \) is a strict partition, then a **shifted set-valued tableau** of shape \( \lambda \) is a map \( T: (i, j) \mapsto T_{i,j} \) from the shifted diagram

\[
SD_\lambda := \{(i, i+j-1) \in \mathbb{P} \times \mathbb{P} : 1 \leq j \leq \lambda_i \}
\]

to the set of finite, nonempty subsets of \( \mathbb{M} \). Given such a map \( T \), define

\[
x^T := \prod_{(i,j) \in SD_\lambda} \prod_{k \in T_{i,j}} x_{|k|} \quad \text{and} \quad |T| := \sum_{(i,j) \in SD_\lambda} |T_{i,j}|.
\]

A shifted set-valued tableau \( T \) is **semistandard** if for all relevant \( (i, j) \in SD_\lambda \):

(a) max(\( T_{i,j} \)) \leq \min(\( T_{i,j+1} \)) and \( T_{i,j} \cap T_{i,j+1} \subset \{1, 2, 3, \ldots \} \).

(b) max(\( T_{i,j} \)) \leq \min(\( T_{i+1,j} \)) and \( T_{i,j} \cap T_{i+1,j} \subset \{1', 2', 3', \ldots \} \).

In such tableaux, an unprimed number can appear at most once in a column, while a primed number can appear at most one in a row. Let SetSSMT(\( \lambda \)) denote the set of semistandard shifted set-valued tableaux of shape \( \lambda \).

**Definition 4.10** (\cite{10}). The **K-theoretic Schur P-function** of a strict partition \( \lambda \) is the power series \( GP_\lambda := \sum_T \beta^{|T|} x^T \) where the summation is over tableaux \( T \in \text{SetSSMT}(\lambda) \) with no primed numbers in any position on the main diagonal.
This definition is due to Ikeda and Naruse \[10\], who also show that each $GP_\lambda$ is symmetric in the $x_i$ variables \[10\] Theorem 9.1. Setting $\beta = 0$ transforms $GP_\lambda$ to the classical Schur $P$-function $P_\lambda$.

**Proposition 4.11.** If $\lambda$ is a strict partition with $r$ parts then

$$GP_\lambda = \lim_{n \to \infty} \pi_{w_n}^{(\beta)} \left( x^\lambda \prod_{i=1}^r \prod_{j=i+1}^n \frac{x_i \oplus x_j}{x_t} \right)$$

where we set $x \oplus y := x + y + \beta xy$ as in (4.6).

**Proof.** As in (4.6), set $x \ominus y := x - y + \frac{\beta - 1}{1 + \beta y}$. Fix a strict partition $\lambda$ with $r$ parts. Ikeda and Naruse’s first definition of $GP_\lambda$ (see \[10\] Definition 2.1) is

$$GP_\lambda = \lim_{n \to \infty} \frac{1}{(n-r)!} \sum_{w \in S_n} w \left( x^\lambda \prod_{i=1}^r \prod_{j=i+1}^n \frac{x_i \oplus x_j}{x_t} \right) \left( \prod_{r+1 \leq i < j \leq n} x_i \ominus x_j \right). \ (4.7)$$

We can rewrite this as

$$GP_\lambda = \lim_{n \to \infty} \sum_{w \in S_{n-r}} w \left( x^\lambda \prod_{i=1}^r \prod_{j=i+1}^n \frac{x_i \oplus x_j}{x_t} \right) \left( \prod_{r+1 \leq i < j \leq n} x_i \ominus x_j \right) \ (4.8)$$

where $S_{n-r}$ acts on the variables $x_{r+1}, x_{r+2}, \ldots, x_n$. Lemma 4.7 implies that

$$1 = \sum_{w \in S_{n-r}} w \left( \prod_{r+1 \leq i < j \leq n} x_i \ominus x_j \right) \text{ since the left side is } \delta_{1 \times w_{n-r}}^{(\beta)} 1_{r \times w_{n-r}} = 1.$$ 

Multiplying the right side of (4.8) by this expression gives

$$GP_\lambda = \lim_{n \to \infty} \sum_{w \in S_n} w \left( \prod_{1 \leq i < j \leq n} x_i \ominus x_j \cdot x^\delta \prod_{i=1}^r \prod_{j=i+1}^n \frac{x_i \oplus x_j}{x_t} \right) \ (4.9)$$

which is equivalent to the desired formula by Lemmas 4.7 and 4.8.

### 4.4 Grassmannian formulas

We are ready to state the main new results of this section. Fix $z \in I_\infty^{\mathbb{F}_2}$. The *symplectic code* of $z$ is the sequence of integers

$$\epsilon^{s_p}(z) = (c_1, c_2, \ldots), \text{ where } c_i := |\{ j \in \mathbb{P} : z(i) > z(j) < i < j \}|.$$

The *symplectic shape* $\lambda^{s_p}(z)$ of $z$ is the transpose of the partition sorting $\epsilon^{s_p}(z)$.

For example, if $n \in 2\mathbb{P}$ and $z = n \cdots 321 \in I_\infty^{\mathbb{F}_2}$ then

$$D^{s_p}(z) = \{(i, j) \in \mathbb{P} \times \mathbb{P} : j < i \leq n - j \},$$

$$c^{s_p}(z) = (0, 1, 2, \ldots, \frac{n}{2} - 1, \frac{n}{2} - 1, \ldots, 2, 1, 0, 0, \ldots),$$

$$\lambda^{s_p}(z) = (n - 2, n - 4, n - 6, \ldots, 2).$$
Proposition 4.14. Suppose \( n \) visible descent \( \lambda \in I_{\infty} \) for some integers \( 1 \leq x \leq \infty \) where \( x \). This means that \( y(i) = i \) if \( z(i) = i \pm 1 \). In the sequel, we set \( \text{dearc}(z) = y \).

The operation \( \text{dearc} \) is easy to understand in terms of the arc diagram \( \{ \{ i, z(i) \} : i \in \mathbb{P} \} \) of \( z \in I_{\infty}^{\text{FFP}} \). The arc diagram of \( \text{dearc}(z) \) is formed from that of \( z \) by deleting each edge \( \{ i < j \} \) with \( e < z(e) \) for all \( i < e < j \).

Recall that \( i \) is a visible descent of \( z \in I_{\infty}^{\text{FFP}} \) if \( z(i + 1) < \min \{ i, z(i) \} \).

Definition 4.12 ([2, §4]). An element \( z \in I_{\infty}^{\text{FFP}} \) is \( \text{FFP-Grassmannian} \) if

\[
\text{dearc}(z) = (\phi_1, n + 1)(\phi_2, n + 2) \cdots (\phi_r, n + r)
\]

for a sequence of integers \( 1 \leq \phi_1 < \phi_2 < \cdots < \phi_r \leq n \). In this case, one has

\[
\lambda^{\text{Sp}}(z) = (n - \phi_1, n - \phi_2, \ldots, n - \phi_r)
\]

by [2, Lemma 4.16], and \( n \) is the last visible descent of \( z \).

We allow \( r = 0 \) in this definition; this corresponds to the FPF-Grassmannian involution \( \Theta \in I_{\infty}^{\text{FFP}} \) with \( \text{dearc}(\Theta) = 1 \). For a given strict partition \( \lambda \) with \( r < n \) parts, there is exactly one FPF-Grassmannian involution \( z \in I_{\infty}^{\text{FFP}} \) with shape \( \lambda^{\text{Sp}}(z) = \lambda \) and last visible descent \( n \).

Example 4.13. The involution \( z = 47816523 = (1, 4)(2, 7)(3, 8)(5, 6) \in I_{\infty}^{\text{FFP}} \) is FPF-Grassmannian with \( \text{dearc}(z) = (2, 7)(3, 8) \) and \( \lambda^{\text{Sp}}(z) = (4, 3) \).

Define \( \pi_{a}^{(b)} := \pi_{b-1}^{(b)} \pi_{b-2}^{(b)} \cdots \pi_{a}^{(b)} \) for \( 0 < a \leq b \), with \( \pi_{i}^{(b)} \) given by [1, 3].

Proposition 4.14. Suppose \( z \in I_{\infty}^{\text{FFP}} - \{ \Theta \} \) is FPF-Grassmannian with last visible descent \( n \) and shape \( \lambda^{\text{Sp}}(z) = (n - \phi_1, n - \phi_2, \ldots, n - \phi_r) \), so that

\[
\text{dearc}(z) = (\phi_1, n + 1)(\phi_2, n + 2) \cdots (\phi_r, n + r)
\]

for some integers \( 1 \leq \phi_1 < \phi_2 < \cdots < \phi_r \leq n \). Then

\[
\Theta_{\text{Sp}_{z}} = \pi_{\phi_1}^{(\beta)} \pi_{\phi_2}^{(\beta)} \cdots \pi_{\phi_r}^{(\beta)} \left( x^{\lambda^{\text{Sp}}(z)} \prod_{i=1}^{n} \prod_{j=i+1}^{n} \frac{x_{i} \oplus x_{j}}{x_{i}} \right)
\]

where \( x_{i} \oplus x_{j} := x_{i} + x_{j} + \beta x_{i}x_{j} \).

We need two lemmas to prove this proposition.

Lemma 4.15. If \( a \leq b \) then \( \phi_{b-a}^{(b)} (x_{a}^{e}) = (-\beta)^{b-a-e} \) for \( e \in \{0, 1, 2, \ldots, b - a\} \).
Proof. Since $\partial_t^{(3)}(1) = -\beta$, it is enough to check that $\partial_{b \setminus a}^{(3)}(x_{a}^{b-a}) = 1$. As

$$\partial_a^{(3)}(x_{a}^{b-a}) = -\beta x_{a}^{b-a} + (1 + \beta x_a) \partial_a(x_{a}^{b-a})$$

we have $\partial_{b \setminus a}^{(3)}(x_{a}^{b-a}) = (-\beta x_a)^{b-a} + (1 + \beta x_a) \partial_{b \setminus (a+1)}^{(3)}(\partial_a x_{a}^{b-a})$. By induction

$$\partial_{b \setminus (a+1)}^{(3)}(\partial_a x_{a}^{b-a}) = \partial_{b \setminus (a+1)}^{(3)} \left( \sum_{i=0}^{b-a-1} x_a^{i} x_{a+1}^{b-a-1-i} \right) = \sum_{i=0}^{b-a-1} (-\beta x_a)^i$$

so the lemma follows.

Lemma 4.16. If $a \leq b$ and $s_i f = f$ for $a < i < b$, then $\pi^{(3)}_{b \setminus a}(f) = \partial_{b \setminus a}^{(3)}(x_{a}^{b-a} f)$. Proof. Assume $a < b$. It holds by induction that

$$\pi^{(3)}_{b \setminus a}(f) = \pi^{(3)}_{b \setminus (a+1)}(\pi^{(3)}_a f) = \partial_{b \setminus (a+1)}^{(3)} \left( \sum_{i=0}^{b-a-1} \pi^{(3)}_a \cdot \partial_{b \setminus (a+1)}^{(3)}(f) \right).$$

Since $\partial_a^{(3)}(x_{a}^{b-a} f) = x_{a+1}^{b-a-1} \left( \pi^{(3)}_a f + \beta x_a f \right) + x_a f \cdot \partial_a^{(3)}(x_{a}^{b-a-1})$, we have

$$\pi^{(3)}_{b \setminus a}(f) = \partial_{b \setminus (a+1)}^{(3)}(x_{a}^{b-a} f) - x_a f \left( \beta \cdot \partial_{b \setminus (a+1)}^{(3)}(x_{a+1}^{b-a-1}) + \partial_{b \setminus (a+1)}^{(3)}(x_{a}^{b-a-1}) \right).$$

From here, it suffices to show that $\beta \cdot \partial_{b \setminus (a+1)}^{(3)}(x_{a+1}^{b-a-1}) + \partial_{b \setminus (a+1)}^{(3)}(x_{a}^{b-a-1}) = 0$ and this is immediate from Lemma 4.15.

Proof of Proposition 4.14. Setting $\beta = 0$ recovers Lemma 4.18; the proof for generic $\beta$ is similar. Let $\Psi_{n,r}(x) = \prod_{i=1}^{n} \prod_{i=1}^{r} x_i \otimes x_{x_i}$. Then $x^{\lambda^{\psi}(z)} \Psi_{n,r}(x)$ is symmetric in $x_{r+1}, x_{r+2}, \ldots, x_n$. For any $j \in [r]$, the expression

$$\theta_j := \pi^{(3)}_{\phi_j \setminus \lambda} \pi^{(3)}_{\phi_j \setminus \lambda}(j+1) \cdots \pi^{(3)}_{\phi_r \setminus \lambda} \left( x^{\lambda^{\psi}(z)} \Psi_{n,r}(x) \right)$$

is symmetric in $x_j, x_{j+1}, \ldots, x_{\phi_j}$ since if $i \in \{j, j+1, \ldots, \phi_j - 1\}$ then either $i = \phi_j - 1$ and $\pi_{\phi_j}^{(3)} \theta_j = \theta_j$ or $i < \phi_j - 1$ and

$$\pi_{\phi_j}^{(3)} \theta_j = \pi_{\phi_j}^{(3)} \pi_{\phi_j \setminus \lambda}^{(3)} \theta_j = \pi_{\phi_j \setminus \lambda}^{(3)} \pi_{\phi_{j+1}}^{(3)} \pi_{\phi_j \setminus \lambda} \theta_j = \pi_{\phi_j \setminus \lambda}^{(3)} \theta_{j+1} = \theta_j$$

by the braid relations for $\pi_{\phi_j}^{(3)}$ and induction. Using Theorem 2.3 we can rewrite

$$x^{\lambda^{\psi}(z)} \Psi_{n,r}(x) = x_1^{1-\phi_1} x_2^{2-\phi_2} \cdots x_r^{r-\phi_r} \prod_{i=1}^{r} \prod_{i=1}^{n} x_i \otimes x_{x_i}$$

where $w \in F_{\infty}^{PF}$ is the Sp-dominant involution satisfying $d_{\text{arc}}(w) = (1, n+1)(2, n+2) \cdots (r, n+r)$. Hence by Lemma 4.16 we have

$$\pi_{\phi_1 \setminus \lambda}^{(3)} \pi_{\phi_2 \setminus \lambda}^{(3)} \cdots \pi_{\phi_r \setminus \lambda}^{(3)} \left( x^{\lambda^{\psi}(z)} \Psi_{n,r}(x) \right) = \partial_{\phi_1 \setminus \lambda}^{(3)} \partial_{\phi_2 \setminus \lambda}^{(3)} \cdots \partial_{\phi_r \setminus \lambda}^{(3)} \left( \Psi_{w}^{\psi} \right).$$

It is straightforward from Theorem-Definition 2.3 to show that this is $\Psi_{w}^{\psi}$. □
We can now prove the obvious identity suggested by the notation \(GP^\text{Sp}_{\infty}\):

**Theorem 4.17.** If \(z \in I^\text{FPF}_{\infty}\) is FPF-Grassmannian then \(GP^\text{Sp}_z = GP^\text{Sp}_{\lambda^z(x)}\).

**Proof.** Assume \(z \in I^\text{FPF}_{\infty}\) is as in Proposition 4.14 Then

\[
\pi_{\mu}^{(\beta)} \ast^\text{Sp} = \pi_{\mu}^{(\beta)} \cdot \pi_{\mu}^{(\beta)} \cdot \cdots \cdot \pi_{\mu}^{(\beta)} (x^\lambda y^z (x) \Psi_{n,r}(x)) = \pi_{\mu}^{(\beta)} (x^\lambda y^z (x) \Psi_{n,r}(x))
\]

so \(GP^\text{Sp}_z = \lim_{n \to \infty} \pi_{\mu}^{(\beta)} \ast^\text{Sp} = GP^\text{Sp}_{\lambda^z(x)}\) by Corollary 4.10 and Proposition 4.11 \(\square\)

Let \(P_{\text{strict}}\) denote the set of strict partitions.

**Corollary 4.18.** If \(\lambda \in P_{\text{strict}}\) then \(GP^\text{Sp}_\lambda \in \mathbb{N}[\beta]\)-span \(\{G_\mu : \mu \in P\}\).

**Proof.** If \(\lambda \in P_{\text{strict}}\) then there is an FPF-Grassmannian \(z \in I^\text{FPF}_{\infty}\) with \(\lambda^z(x) = \lambda\), and [20, Corollary 4.7] shows that \(GP^\text{Sp}_z \in \mathbb{N}[\beta]\)-span \(\{G_\mu : w \in S_z\}\). The corollary therefore follows from Theorems 4.4 and 4.17 \(\square\)

There is a “stable” version of the transition equation for \(\ast^\text{Sp}\). Let \(S_Z\) denote the group of permutations of \(Z\) with finite support. Write \(\Theta_Z\) for the permutation of \(Z\) with \(i \mapsto i - (-1)^i\) and let

\[
I^\text{FPF}_Z = \{w \cdot \Theta_Z \cdot w^{-1} : w \in S_Z\}.
\]

Define \(\ell^\text{FPF}(z)\) for \(z \in I^\text{FPF}_{\infty}\) by modifying the formula 4.12 to count pairs \((i, j) \in Z \times Z\) then \(\ell^\text{FPF}(\Theta_Z) = 0\) and 4.13 still holds. We again write \(y \ll z\) for \(y, z \in I^\text{FPF}_Z\) if \(\ell^\text{FPF}(z) = \ell^\text{FPF}(y) + 1\) and \(z = t y z\) for a transposition \(t \in S_Z\).

Identify \(I^\text{FPF}_{\infty}\) with the subset of \(z \in I^\text{FPF}_Z\) with \(z(i) = \Theta_Z(i)\) for all \(i \leq 0\). Let \(\sigma : Z \to Z\) be the map \(i \mapsto i + 2\). Conjugation by \(\sigma\) preserves \(I^\text{FPF}_Z\), and every \(z \in I^\text{FPF}_Z\) has \(\sigma^n z \sigma^{-n} \in I^\text{FPF}_{\infty}\) for all sufficiently large \(n \in \mathbb{N}\). We define

\[
GP^\text{Sp}_z := \lim_{n \to \infty} GP^\text{Sp}_{\sigma^n z \sigma^{-n}} \quad \text{for} \quad z \in I^\text{FPF}_{\infty}.
\]

Also let \(GP^\text{Sp}_{u_{ij}} := GP^\text{Sp}_{(i)z(i,j)}\) for \(i < j\) and extend by linearity. In this context, \(u_{ij}\) is a formal symbolic operator, not a well-defined linear map.

**Corollary 4.19.** Fix \(v \in I^\text{FPF}_Z\) and \(j, k \in Z\) with \(v(k) = j < k = v(l)\). Suppose

\[
i_1 < i_2 < \cdots < i_p < j < k < l_q < \cdots < l_2 < l_1
\]

are the integers such that \(v \ll (i, j) v(i, l)\) and \(v \ll (k, l) v(k, l)\). Then

\[
GP^\text{Sp}_v (1 + \beta u_{i_1,j}) \cdots (1 + \beta u_{i_p,j}) = GP^\text{Sp}_v (1 + \beta u_{k_1,l}) \cdots (1 + \beta u_{k_q,l}).
\]

**Proof.** Define \(\text{Asc}^{-}(v, j, k) = \{i_1, i_2, \ldots, i_p\}\) and \(\text{Asc}^{+}(v, j, k) = \{l_1, l_2, \ldots, l_q\}\). If \(m \in \mathbb{N}\) is sufficiently large then \(\text{Asc}^{\pm}(\Theta^{2m} v, 2m + j, 2m + k) = 2m + \text{Asc}^{\pm}(v, j, k)\), so we obtain this result by taking the limit of Theorem 3.8 \(\square\)

The preceding corollary is a \(K\)-theoretic generalization of [9, Theorem 3.6]. The latter result has an “orthogonal” variant given by [8, Theorem 3.2].

21
Corollary 4.20. Let \( k \in \mathbb{P} \) be the last visible descent of \( z \in \mathcal{I}_{\infty}^{\text{FPF}} \). Define \( v \in \mathcal{I}_{\infty}^{\text{FPF}} \) as in Corollary 3.11 and let \( I = \{ i_1 < i_2 < \cdots < i_p \} \) be the (possibly nonpositive) integers with \( i < j := v(k) \) and \( v \leq_F (i, j) v(i, j) \). Then

\[
GP_{\mathcal{I}_{\infty}^{\text{FPF}}} z = \sum_{\emptyset \neq A \subset I} \beta^{|A| - 1} GP^\mathcal{P}_{v} u_{A_j}
\]

where if \( A = \{ a_1 < a_2 < \cdots < a_q \} \subset I \) then \( u_{A_j} := u_{a_1} u_{a_2} \cdots u_{a_q} \).

Proof. The proof is the same as for Corollary 3.11 now using Corollary 4.19.

This gives a positive recurrence for \( GP_{\mathcal{I}_{\infty}^{\text{FPF}}} z \). We expect that one could use this recurrence and the inductive strategy in [1, 9, 16] to prove the following theorem. However, a direct bijective proof is already available in [19]:

Theorem 4.21 ([19 Theorem 1.9]). If \( z \in \mathcal{I}_{\infty}^{\text{FPF}} \) then

\[
GP_{\mathcal{I}_{\infty}^{\text{FPF}}} z \in \mathbb{N}[\beta]-\text{span}\{GP_{\lambda} : \lambda \in \mathcal{P}_{\text{strict}} \}.
\]

Combining Theorems 4.17 and 4.21 gives this corollary:

Corollary 4.22. If \( z \in \mathcal{I}_{\infty}^{\text{FPF}} \) then

\[
GP_{\mathcal{I}_{\infty}^{\text{FPF}}} z \in \mathbb{N}[\beta]-\text{span}\{GP_{\mathcal{I}_{\infty}^{\text{FPF}}} y : y \in \mathcal{I}_{\infty}^{\text{FPF}} \text{ is FPF-Grassmannian} \}.
\]

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