Principles of Einstein–Finsler Gravity and Perspectives in Modern Cosmology

Sergiu I. Vacaru *

University "Al. I. Cuza" Iaşi, Science Department,
54 Lascar Catargi street, Iaşi, Romania, 700107

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Abstract

We study the geometric and physical foundations of Finsler gravity theories with metric compatible connections defined on tangent bundles, or (pseudo) Riemannian manifolds. There are analyzed alternatives to Einstein gravity (including theories with broken local Lorentz invariance) and shown how general relativity and modifications can be equivalently re–formulated in Finsler like variables. We focus on prospects in modern cosmology and Finsler acceleration of Universe.

All known formalisms are outlined - anholonomic frames with associated nonlinear connection structure, the geometry of the Levi–Civita and Finsler type connections, all defined by the same metric structure, Einstein equations in standard form and/or with nonholonomic/ Finsler variables – and the following topics are discussed: motivation for Finsler gravity; generalized principles of equivalence and covariance; fundamental geometric/ physical structures; field equations and nonholonomic constraints; equivalence with other models of gravity and viability criteria.

Einstein–Finsler gravity theories are elaborated following almost the same principles as in the general relativity theory but extended to Finsler metrics and connections. Gravity models with anisotropy can be defined on (co) tangent bundles or on nonholonomic pseudo–Riemannian manifolds. In the second case, Finsler geometries can be modelled as exact solutions in Einstein gravity. Finally, some examples of generic off–diagonal metrics and generalized connections, defining anisotropic cosmological Einstein–Finsler spaces are analyzed; certain criteria for Finsler accelerating evolution are formulated.

*sergiu.vacaru@uaic.ro, Sergiu.Vacaru@gmail.com
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1 Introduction

During last 30 years, the experimental data and existing methodology and phenomenology of particle physics, and gravity, imposed an interpretation doctrine that models of Finsler like spacetimes (with metrics and connections depending on ”velocity/momenta”) are subjected to strong experimental restrictions. Such theories were not included in the standard paradigm of modern physics (see respective arguments in Refs. [18, 124]).

Nevertheless, there are various theoretical arguments [52, 115, 48, 31, 84, 47, 123] that any quantum gravity model positively results in nonlinear dispersion relations depending on velocities/momenta. Anisotropic quasi-classical Finsler configurations originating from quantum gravity are not obligatory restricted for some inflationary cosmological models and may have important contributions to dark energy and dark matter in Universe. This constrains us to investigate Finsler type spacetimes both in quantum gravity theories and modern cosmology.

In a survey [101] oriented to non-experts in Finsler geometry (but researches in particle physics and gravity) we discussed in details and formulated well defined criteria how the Finsler geometry methods and theories

\footnote{those studies did not include all fundamental geometric/physical objects in Finsler geometry/ gravity, for instance, the nonholonomic structure, nonlinear connections, N-connections, and new types of linear connections which are adapted to N-connections, the possibility to model (pseudo) Finsler geometric models as exact solutions in Einstein gravity etc}
Almost all classical gravitational effects are described in the framework of General Relativity (GR). Recently, it was proposed that certain exceptions in theoretical cosmology may be related to the dark matter and dark energy problems and some approaches were formulated for “nonmetric” Finsler gravity models. Here we note that the existence of anisotropies and inhomogeneities has become a conventional feature in cosmology physics. A number of cosmological models with Finsler metrics have been elaborated by now, and this number seems to grow rapidly. It is also expected that small corrections with violations of equivalence principle and local Lorentz invariance have to be considered in low energy limits within general approaches to quantum gravity, see a series of works related to Finsler geometry and methods and references therein.

Finsler gravity models have been considered in the past, for instance, with the aim to compute possible post–Riemannian corrections, to eliminate black hole singularities and propose a new spacetime axiomatics (70, 71, 72, 69, 8), or to develop locally anisotropic inflation scenarios (117) (in our case, in a metric compatible form). In recent years there has been an increased interest in classical and quantum gravity theories with broken local Lorentz invariance and related methods of Finsler geometry (48, 31, 52, 58, 22, 2, 84, 47, 59, 123, 20, 19, 8). There are several reasons that justify this interest, among them to study quantum gravity theories and prove that the extensions of GR to higher dimensions provide the best routes to unification of gravity with the interactions of particle physics. Following quantum arguments, we suppose that such extra dimension coordinates may be of velocity/momentum type. In an alternative approach,

A pair (V,N), where V is a manifold and N is a nonintegrable distribution on V, is called a nonholonomic manifold. Modeling Finsler like geometries in Einstein gravity, we have to consider that V is a four dimensional (pseudo) Riemannian spacetime when the Levi–Civita connection is correspondingly deformed to a linear connection adapted to a N–connection structure defined by a nonintegrable (2 + 2)–splitting (nonholonomic distribution). The boldface symbols will be used for nonholonomic manifolds/ bundles and geometric objects on such spaces, as we discuss in details.
it is possible to model Finsler like geometries on Einstein manifolds by introducing nonholonomic frames and deformations of connections completely determined by certain off–diagonal coefficients of metrics \[95, 101, 120\].

There are two general classes of Finsler type gravity theories with very different implications in physics, mechanics and cosmology. The first class originates from E. Cartan works on Finsler geometry [25], see further geometrical developments and applications in [77, 8, 57, 63, 64, 15, 17, 61, 62, 89, 90, 92, 120, 101]. In those works a number of geometric and physical constructions and Finsler geometry methods were considered for both types of metric compatible or noncompatible connections. The most related to "standard physics" constructions were elaborated for the metric compatible Cartan and canonical distinguished connections (in brief, d–connections, see details in [64, 101]) following geometric and physical principles which are very similar to those used for building the general relativity theory [115].

In the second class of theories, there are Finsler geometry and gravity models derived for the Berwald and Chern d–connections which are not metric compatible, see details in [30, 11, 27, 28, 29]; summaries of results and applications to (non) standard physical theories are given in Part I of [120] and in Refs. [98, 101].

The article is organized as follows: In section 2, we provide physical motivations for Finsler gravity theories and explain in brief how fundamental Finsler geometric objects are defined of tangent bundles and in Einstein gravity. In section 3, we consider the gravitational field equations for Finsler–Einstein gravity. We briefly discuss how generic off–diagonal solutions can be constructed in exact form and provide some examples. Two classes of cosmology diagonal and off–diagonal solutions on tangent bundles modeling Finsler acceleration of Universe are constructed and analyzed in section 4. Finally, in section 5 we outline the approach and formulate conclusions. In Appendices, we provide some local formulas and examples of general cosmological solutions.

### 2 (Pseudo) Finsler Spacetimes

In this section, the most important geometric constructions and physical motivations for Finsler gravity are summarized. We shall follow the system of notations proposed in [101, 98, 120] (where details and more complete...
lists of references are provided\footnote{in this work, we do not aim to provide an exhaustive bibliography (it is not possible to list and discuss tenths of monographs and thousands of articles on Finsler geometry and generalizations and various extensions of the general relativity theory; we can not discuss a number of papers on implications of locally anisotropic cosmological models etc; following our purposes, we cite the main results which are closed to metric compatible models of Finsler gravity and cosmology}}.

Finsler geometry is not familiar to most physicists. It is often regarded as a very sophisticate geometrical model and it takes a big effort to get used with such concepts and methods. Nevertheless, there are strong arguments that the quantum gravity theory is "almost sure" of a Finsler type and that one could be important applications in modern gravity and cosmology theories. Certain classes of metric compatible Finsler gravity theories can be elaborated in a form very similar to the Einstein and/or Einstein–Cartan gravity. The main difference is that for locally anisotropic models we have to extend the constructions on tangent bundles and/or on nonholonomic (pseudo) Riemannian manifolds, and adapt the geometric objects with respect to nonlinear connections (N–connections). This imposes us to work with new classes of linear connections being adapted to N–connections (called distinguished connections, in brief, d–connections).

There is a special interest for Finsler geometry/gravity models containing in some local limits the usual Minkowski spacetime metric (in SR) and when the metric on total space\footnote{of a tangent bundle, or on a corresponding pseudo–Riemannian manifold enabled with nonholonomic distributions determined by N–connection} defines completely a metric compatible "canonical" d–connection, for instance, the Cartan d–connection (see below subsection 2.2 for definitions and main properties). Even such theories contain nonholonomically induced torsions with coefficients determined by some off–diagonal coefficients of metric, all constructions can be redefined equivalently, via nonholonomic deformations, for the Levi–Civita connection (when the resulting theoretic scheme is not N–adapted). The main differences between a Finsler gravity and the general relativity theory is that the Ricci, and Einstein, tensors are determined by different types of linear connections; we shall discuss this issue in section 3.

### 2.1 Physical motivations for Finsler gravity theories

The first example of a Finsler metric \cite{40} was given by G. Riemann \cite{75} who instead of quadratic line elements considered forth order forms, see historical remarks in \cite{77,63,64,11} and, in relation to standard and non–standard physical theories, in \cite{101,98,120,95,92}. In this subsection,
we examine how Finsler-like nonlinear line elements can be generated by deformations of standard Minkowski metrics in special relativity, SR (similar arguments were presented in \[20, 50, 52\]).

2.1.1 Violations of local Lorentz symmetry and Finslerian Hessians

Finsler metrics can be generated if instead of the Lorentz transforms in SR there are considered nonlinear generalizations, restrictions of symmetries and/or deformations of the Minkowski metric, see examples in \[8, 7, 19, 20, 21, 47, 50\]. We provide a simple construction when anisotropic metrics are used for modeling light propagation in anisotropic media (ether) and/or for small perturbations in quantum gravity.

In SR, a Minkowski metric \(\eta_{ij} = \text{diag}[-1, +1, +1, +1]\), (for \(i = 1, 2, 3, 4\)), defines a quadratic line element,

\[
 ds^2 = \eta_{ij} dx^i dx^j = -(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2. \quad (1)
\]

The light velocity \(c\) is contained in \(x^1 = ct\), where \(t\) is the time like coordinate. Along a light ray \(x^i(\varsigma)\), parametrized by a real smooth parameter \(0 \leq \varsigma \leq \varsigma_0\), when \(ds^2/d\varsigma^2 = 0\), we can define a "null" tangent vector field \(y^i(\varsigma) = dx^i/d\varsigma\), with \(d\tau = dt/d\varsigma\). Under general coordinate transforms \(x^i' = x^i(x^k)\), we have \(\eta_{ij} \rightarrow g_{ij}(x^k)\); the condition \(ds^2/d\varsigma^2 = 0\) holds always for propagation of light, i.e. \(g_{ij} y^i y^j = 0\). We can write for some classes of coordinate systems (for simplicity, omitting priming of indices and considering that indices of type \(\hat{i}, \hat{j}, ... = 2, 3, 4\))

\[
 c^2 = g^{-\hat{i}j}(x^i) y^\hat{i} y^\hat{j} / \tau^2. \quad (2)
\]

This formula holds also in GR, when the local coordinates on a (pseudo) Riemannian manifold are chosen such a way that the coefficients of metric \(g^-\hat{i}j(x^i)\) are constrained to be a solution of Einstein equations. For certain local constructions in vicinity of a point \(x^i_{(0)}\), we can omit the explicit dependence on \(x^i\) and consider only formulas derived for \(y^i\).

In anisotropic media (and/or modeling spacetime as an ether model), we can generalize the quadratic expression \(g^{-\hat{i}j}(x^i)y^\hat{i} y^\hat{j}\) to an arbitrary non-linear one \(\tilde{F}^2(y^\hat{i})\) subjected to the condition of homogeneity that \(\tilde{F}(\beta y^\hat{i}) = \beta \tilde{F}(y^\hat{i})\), for any \(\beta > 0\). The formula for light propagation (2) transforms into \(c^2 = \tilde{F}^2(y^\hat{i})/\tau^2\). Small deformations of the Minkowsky metric (2) can be parametrized in the form \(\tilde{F}^2(y^\hat{i}) \approx \left(\eta_{ij} y^\hat{i} y^\hat{j}\right)^\tau + q_{ij\hat{1}...\hat{r}} y^{\hat{1}...\hat{r}}, \) for
We can generalize the coefficients of $\tilde{F}$ by introducing additional dependencies on $x^i$, when $F^2(x^i, y^j) \approx (g^\hat{i}\hat{j}(x^k)y^\hat{i}y^\hat{j})^r + q_{i_1\ldots i_2r}^1\ldots y^{i_2r}$, and consider instead of (1) and (2) certain generalized nonlinear homogeneous relations (with $F(x^i, \beta y^j) = \beta F(x^i, y^j)$, for any $\beta > 0$), when

$$ds^2 = F^2(x^i, y^j)$$

$$\approx -(cdt)^2 + g^\hat{i}\hat{j}(x^k)y^\hat{i}y^\hat{j} \left[ 1 + \frac{1}{r} \frac{q_{i_1\ldots i_2r}^1\ldots y^{i_2r}}{(g^\hat{i}\hat{j}(x^k)y^\hat{i}y^\hat{j})^r} \right] + O(q^2).$$

A nonlinear element $ds^2 = F^2(x^i, y^j)$ is usually called by physicists a "Finslerian metric". In the bulk of geometric books [77, 63, 64, 15, 11], the function $F$ is considered to be a fundamental (and/or generating) Finsler function satisfying the condition that the Hessian

$$F = \frac{1}{2} \frac{\partial F^2}{\partial y^i \partial y^j}$$

is not degenerate\(^7\). Physical implications of Finsler type deformations of SR were analyzed in a series of works, see [19, 17, 18, 69, 59, 31, 47, 50].

### 2.1.2 Nonlinear dispersion relations

The nonlinear quadratic element (3) results in a nonlinear dispersion relation between the frequency $\omega$ and the wave vector $k^i$ of a light ray (see details, for instance, in [52]),

$$\omega^2 = c^2 \left[ g^\hat{i}\hat{j}k^i k^j \right]^2 \left[ 1 - \frac{1}{r} \frac{q_{i_1\ldots i_2r}^1\ldots y^{i_2r}}{(g^\hat{i}\hat{j}k^i k^j)^{2r}} \right],$$

where, for simplicity, we consider such a relation in a fixed point $x^k = x^k(0)$, when $g^\hat{i}\hat{j}(x^k) = g^\hat{i}\hat{j}$ and $q_{i_1\ldots i_2r}^1\ldots y^{i_2r} = q_{i_1\ldots i_2r}^1\ldots y^{i_2r}$. The coefficients $q_{i_1\ldots i_2r}^1\ldots y^{i_2r}$ should be computed from a model of quantum gravity, or from a well defined Finsler like modification of the general relativity theory.

\(^7\)It is positively definite for models of Finsler geometry; for pseudo–Finsler configurations [65, 128, 112, 114], this condition is not imposed.
For a locally anisotropic spacetime ether, i.e. in a modified classical model of gravity with broken local Lorentz invariance, the coefficients for dispersions of type (5) should be measured following some generalized Michelson–Morley experiments when light rays propagate according to a Riemannian/ Finsler metric.

Dispersion relations should be parametrized and computed differently for theories with nonlocal interactions and noncommutative variables. Nevertheless, the form (5) is a very general one which can be obtained in various Finsler like and extra dimension models even the values of coefficients $q_{i_1 i_2 ... i_r}$ depend on the class of exact solutions of certain generalized gravitational equations, types of classical and quantum models etc.

2.1.3 Osculating Finsler geometry and very special/general relativity

The Finslerian line element

$$ds = F^2(x^i, y^k) = (\eta_{ij}dx^i dx^j)^{(1-b)/2} (n_k dx^k)^b$$

was proposed by Bogoslovsky [20, 21] for the study of local anisotropies, where the vector $n^k$ is a null spurionic vector field that determines the direction of the "eternal" motion’s 4–velocity and can be selected as $n^i = (1,0,0,1)$ and $y^k = dx^k/ds$. The line element determines for a point–like particle a mass–tensor $m_{ij} = (1 - b)m(\delta_{ij} + bn_i n_j)$ and, for the canonical momentum $p_k = m \frac{\partial F}{\partial y^k}$, the nonlinear dispersion relation

$$\eta^{ij}p_ip_j = m^2(1-b)^2 \left( \frac{n^i p_i}{n(1-b)} \right)^{2b/(b+1)}.$$

An osculating (pseudo) Finsler space/manifold $F_n, \text{dim} F_n = n$, are defined on a (pseudo) Riemannian manifold $M$ via sections $y^l(x^k)$ of $TM$ by an effective metric (see details in Refs. [50, 51]; here we emphasize that we use a different system of denotations)

$$f_{ij}(x^k) = F g_{ij}(x^k, y^l(x^k))$$

the process of osculation relates the velocity field uniquely to the spacetime points $x \in M$. The osculating effective metric $Fg$ can be used for constructing the Einstein field equations

$$E_{ij}(x, y(x)) = \kappa T_{ij}(x, y(x)),$$
where \( \kappa = \text{const.} \), \( E_{ij} \) is the standard Einstein tensor for the Levi–Civita connection \( \nabla \) computed for the "pseudo–Riemannian" \( f_{ij} \) and \( T_{ij} \) is an energy–momentum tensor (for cosmological applications, it is taken that for a general imperfect fluid with velocity \( y'(x^k) \).

The Finsler line element (6) can be generalized for an arbitrary (pseudo) Riemannian background metric \( g_{ij}(x^k) \), when \( \eta_{ij} \to g_{ij}(x^k), b \to b(x) \), and

\[
F^2(x, y) = (g_{ij}u^iu^j)^{1-b(x)}/2(n_ku^k)h(x). 
\]

This induces a quite sophisticated relation between \( g_{ij}(x^k) \) and \( f_{ij}(x^k) \). For such a model, \( f_{ij} \) play the role of a "physical" spacetime locally anisotropic ether (in [51]) called cosmological fluid), which models an effective Lorentz violation. The pseudo–Riemannian metric \( g_{ij} \) represents the gravitational "potential". In the case of a general fluid, the co–moving observers live on a different Riemannian space to the one of a tilted observer; different observers are related in the context of Finsler geometry, as it is considered in [18, 51].

The main question we put here is that if some relations of type (6), (7) and/or (9) really define a Finsler geometry? Our remark is based on the fact that for such models the Finsler nonlinear and/or linear connections are not involved (excepting [86] where the Cartan, Finsler type, linear connection is used). In our opinion, the constructions for the so–called osculator Finsler manifold with Bogoslovsky, or another types line elements, define a generalized Brans–Dicke type generalization of Einstein gravity (see original works [23, 24], applications to cosmology [39] and, for higher dimension spaces, [12], see also references therein) when instead of a scalar field \( \varphi(x) \) certain more sophisticated constructions are used for \( y'(x^k) \). To geometrize such models into terms of a Finsler space we have to define a nonlinear connection (for instance, induced canonically by generating function (51)) and work with a linear connection adapted to this nonlinear connection, instead of \( \nabla \).

### 2.1.4 A Finsler metric does not define a complete tangent bundle geometry

Any (pseudo) Riemannian geometry on (for our purposes, we consider any necessary smooth class) manifold \( M \) is determined by a metric field \( g = g_{ij}(x)e^i \otimes e^i \), where \( x = \{ x^i \} \) label the local coordinates and the coefficients of a symmetric tensor \( g_{ij}(x) \) are defined with respect to a general nonholonomic co-frame \( e^i = e^i_j(x)dx^j \). There is on \( M \) a second fundamental

\[8 We follow the system of notations from [101, 120, 95, 92] when "underlined", "primed" and other type indices are used in order to distinguish, for instance, the local coordi-
geometric object, the Levi–Civita connection, $\nabla = \{\nabla_i\}$ (parametrized locally by coefficients $\Gamma^i_{jk}$ for a 1–form $\Gamma^i_j = \Gamma^i_{jk}(x)dx^k$) which is completely defined by a set $\{g_{ij}\}$ if and only if we impose two basic conditions:

1) metric compatibility, $\nabla_k g_{ij} = 0$;
2) zero torsion, $\mathcal{T}^i = \nabla^i e^j = de^i + \Gamma^i_j \wedge e^j = 0$,

where $\wedge$ is the anti–symmetric product forms (see, for instance, [66]).

Contrary to (pseudo) Riemannian geometry completely determined by a quadratic linear form (a metric), a Finsler metric (3), or (9) does not state a geometric spacetime model in a self–consistent and complete form. An element $ds = F(x^i, y^j)$ and Hessian $g_{ij}(x^i, y^j)$ (4) do not define completely any metric and connections structures on the total space $TM$ of a tangent bundle $(TM, \pi, M)$, where $\pi$ is a surjective projection (see [64, 11]).

We have to introduce additional suppositions in order to elaborate a "well–defined" spacetime model generated by $F(x^i, y^j)$, i.e. a Finsler gravity theory. Such a locally anisotropic gravity is determined by three fundamental geometric objects on $TM$ and $TTM$, a metric structure, $^Fg$, a nonlinear connection, $^FN$, and a linear connection, $^FD$, which is adapted to $^FN$.

How to define completely a Finsler spacetime we discuss in next section.

2.2 Finsler geometry on nonholonomic tangent bundles/ manifolds

Let us consider the main concepts and fundamental geometric objects which are necessary for Finsler geometry, and gravity, models on nonholonomic tangent bundles/ manifolds.

2.2.1 Fundamental geometric objects in Finsler geometry

Nonlinear connections:

A rigorous analysis of nonlinear connection structures $^FN$ was many times omitted by physicists in their works on Finsler gravity and and analyzes of experimental restrictions on "velocity" dependent theories [18, 124].

We put a left label $^F$ in order to emphasize (if necessary) that some objects are introduced for a Finsler geometry model.
This geometric/physical object is less familiar to researchers working in particle physics and cosmology and it is confused with nonlinear realizations of connections for generalized gauge theories.

A nonlinear connection (N–connection) $N$ can be defined as a Whitney sum (equivalently, a nonholonomic distribution)

$$TTM = hTM \oplus vTM,$$

with a conventional splitting into horizontal ($h$), $hTM$, and vertical ($v$), $vTM$, subspaces. It is given locally by a set of coefficients $N = \{N^a_i\}$, where

$$N = N^a_i(u)dx^i \otimes \frac{\partial}{\partial y^a},$$

There is a frame (vielbein) structure which is linear on N–connection coefficients and on partial derivatives $\partial_i = \partial/\partial x^i$ and $\partial_a = \partial/\partial y^a$ and, respectively, theirs duals, $dx^i$ and $dy^a$,

$$e^\nu = (e_i = \partial_i - N^a_i \partial_a, e_a = \partial_a),$$

$$e^\mu = (e^i = dx^i, e^a = dy^a + N^a_i dx^i).$$

The vielbeins (13) satisfy the nonholonomy relations

$$[e_\alpha, e_\beta] = e_\alpha e_\beta - e_\beta e_\alpha = w^\gamma_{\alpha\beta} e_\gamma$$

with (antisymmetric) nontrivial anholonomy coefficients $w^b_{ia} = \partial_a N^b_i$ and $w^a_{ji} = \Omega^a_{ij}$, where

$$\Omega^a_{ij} = e_j (N^a_i) - e_i (N^a_j)$$

are the coefficients of N–connection curvature. The holonomic/integrable frames are selected by the integrability conditions $w^\gamma_{\alpha\beta} = 0$.

For a $TM$ endowed with a generating Finsler function $F$, we can introduce a homogeneous Lagrangian $L = F^2$. There is the canonical (Cartan’s) N–connection with coefficients

$$c_{N^a_i} = \frac{\partial G^a}{\partial y^{a+n+i}},$$

10 coordinates $u = (x, y)$ on an open region $U \subset TM$ are labelled in the form $u^a = (x^i, y^a)$, with indices of type $i, j, k, ... = 1, 2, ...n$ and $a, b, c... = n + 1, n + 2, ...n + n$; on $TM$, $x^i$ and $y^a$ are respectively the base coordinates and fiber (velocity like) coordinates; we use boldface symbols for spaces (and geometric objects on such spaces) enabled with N–connection structure.
for

\[ G^a = \frac{1}{4} F^{ab} \left( \frac{\partial^2 L}{\partial y^{n+i} \partial x^k} y^{n+k} - \frac{\partial L}{\partial x^i} \right) \],

(17)

where \( F^{ab} \) is inverse to \( F_{ab} \) \([11] \).

We can consider nonhomogeneous regular Lagrangians, \( L : TM \supset U \to \mathbb{R} \), with nondegenerate Hessians \( L_{ij}(x^i, y^j) = \frac{1}{2} \frac{\partial^2 L}{\partial y^i \partial y^j} \) and generalize the constructions for the Lagrange geometry (see details in Refs. \([63, 101, 120]\); for "pseudo" configurations, this mechanical analogy is a formal one, with some "imaginary" coordinates). In this case, the corresponding semi–spray configuration is defined by nonlinear geodesic equations which are equivalent to the Euler–Lagrange equations for \( L \).

For simplicity, in this work we shall restrict our considerations only to physical models with some effective Finsler type generating function \( L = F^2 \). It is always possible to introduce Finsler type variables on nonholonomic vector bundles / manifolds by corresponding frame and coordinate transforms \([105]\).

For any set \( N^a_i \), we can chose a well–defined \( \tilde{F} \) and corresponding frame coefficients \( e^a_{\alpha'} \) and (inverse) \( e^\alpha_{a'} \) when \( e_\alpha \to e^\alpha_{a'} e_a, \) transform \( N^a_i \) into respective \( \tilde{N}^a_i \) (for instance, \( N^a_i = e^a_{\alpha'} e'_\alpha \tilde{N}^\alpha_i \), or using more general types of transforms). So, we can work equivalently with any convenient set \( N^a_i \) which may be redefined as a "canonical" set \( \tilde{F} \tilde{N}^a_i = \tilde{N}^a_i \). For simplicity, we shall write the coefficients of a \( N \)-connection in general form, \( N^{\alpha}_{\beta \gamma} = \{ N^a_i \} \), if that will not result in ambiguities.

In our works devoted to applications of Lagrange–Finsler geometry methods in modern gravity and string theory \([101, 120, 95, 105]\) related to standard physics models, we considered \( N \)-connections not only on tangent bundles but also on nonholonomic manifolds (see definition in footnote 2). In

\footnote{Respective contractions of \( h \)- and \( v \)-indices, \( i, j, \ldots \), and \( a, b, \ldots \), are performed following the rule: we write an up \( v \)-index as a as \( a = n+i \) and contract it with a low index \( i = 1, 2, \ldots n \); on total spaces of even dimensions, we can write \( y^i \) instead of \( y^{n+i} \), or \( y^a \). Here we also note that the spacetime signature may be encoded formally into certain systems of frame (vielbein) coefficients and coordinates, some of them being proportional to the imaginary unity \( i \), when \( i^2 = -1 \). For a local tangent Minkowski space of signature \((-+, +, +, +)\), we can chose \( e_{\alpha'} = i \partial / \partial u^{\alpha'} \), where \( i \) is the imaginary unity, \( i^2 = -1 \), and write \( e_\alpha = (i \partial / \partial u^\alpha, \partial / \partial u^\beta, \partial / \partial u^\gamma, \partial / \partial u^\delta) \).

Euclidean coordinates with \( i \) were used in textbooks on relativity theory (see, for instance, \([53, 60]\)). Latter, they were considered for analogous modelling of gravity theories as effective Lagrange mechanics, or Finsler like, geometries \([101, 95]\). If such formal complex coordinates and frame components can be defined (this is still not a complexification of classical spacetimes concepts and related geometric mechanics), we may use respectively the terms pseudo–Euclidean, pseudo–Riemannian, pseudo–Finsler spaces etc. The term "pseudo–Finsler" was also introduced in a different form, more recently, for some analogous gravity models (see, for instance, \([123]\) and has certain relation to a similar one used in mathematical books \([16]\).}
such approaches, it is considered a general manifold $V$, instead of $TM$, when the Whitney sum (10), existing naturally on vector/tangent bundles, is introduced as a nonholonomic distribution on $V$ with conventional h– and v–splitting into (holonomic and nonholonomic variables, respectively, distinguished by coordinates $x^i$ and $y^a$),

\[ TV = hV \oplus vV. \tag{18} \]

A N–anholonomic manifold (or tangent bundle; in brief, we shall write respectively bundle and manifold; we can consider similarly vector bundles) is a nonholonomic manifold enabled with N–connection structure (18). The properties of a N–anholonomic bundle/manifold are determined by N–adapted bases (12) and (13). A geometric object is N–adapted (equivalently, distinguished), i.e. it is a d–object, if it can be defined by components adapted to the splitting (18) (one uses terms d–vector, d–form, d–tensor). For instance, a d–vector is represented as $X = X^a e_a = X^i e_i + X^a e_a$ and a one d–form $\tilde{X}$ (dual to $X$) is represented as $\tilde{X} = X^a e^a = X_i e^i + X_a e^a$.

Lifts of base metrics on total spaces:

The most known procedure to extend $g_{ab}$ (4) to a metric in $TM$ is the so–called Sasaki type lift when the Hessian metric is considered in N–adapted form both of the h– and v–components metric,

\[ F g = g_{\alpha \beta} (u) \, du^\alpha \otimes du^\beta, \]

\[ = g_{ij} (u) \, dx^i \otimes dx^j + g_{ab} (u) \, e^a \otimes e^b, \]

\[ e^a = dy^a + e^a_{N^i} (u) \, dx^i, \quad du^a = (dx^i, dy^a). \]

Similarly, we can define a metric structure for an even dimensional N–anholonomic manifold $V$ (this condition is satisfied for any $TM$) endowed with a h–metric $g_{ij}$, on $hV$, and a given set of N–connection coefficients $N^a_i$.

Using arbitrary frame transforms with coefficients $e^\alpha_{\alpha'} (u)$, we can transform the total Finsler metric (19) into a "general" one $g = g_{\alpha' \beta'} e^{\alpha'} \otimes e^{\beta'}$ on $TM$, where $g_{\alpha' \beta'} = e^\alpha_{\alpha'} e^\beta_{\beta'}$, $F g_{\alpha \beta}$ and $e^{\alpha'} = e^\alpha_{\alpha'} (u) \, e^\alpha$, for $e^a = (dx^i, \, e^a)$. A metric d–tensor (d–metric) is with $n+n$ splitting. Such a decomposition, adapted to a chosen N–connection structure, can be preserved by corresponding parametrizations of components of matrices $e^\alpha_{\alpha'}$.

\[ \text{A geometric object can be redefined equivalently for arbitrary frame and coordinate systems; nevertheless, the N–adapted constructions allow us to preserve a prescribed h– and v–splitting.} \]
when $g_{\alpha'\beta'} = [g_{ij}, h_{ab}, N_i^a]$. Haven redefined the coordinates and 1–frame coefficients, we can express

$$
g = g_{ij}(x, y)dx^i \otimes dx^j + h_{ab}(x, y)e^a \otimes e^b, \tag{20}
$$

$$
e^a = dy^a + N_i^a(x, y)dx^i.
$$

With respect to a local dual coordinate frame, the metric (20) is parametrized in the form

$$
g = g_{\alpha\beta}(u) du^\alpha \otimes du^\beta, \tag{21}
$$

where

$$
g_{\alpha\beta} = \begin{bmatrix}
g_{ij} + N_i^aN_j^bh_{ab} & N_j^eh_{ae} \\
N_j^eh_{be} & h_{ab}
\end{bmatrix}. \tag{22}
$$

Such a metric is generic off–diagonal because, in general, it cannot be diagonalized by coordinate transforms. Here we emphasize that the values $N_i^a(u)$ should not be identified as certain gauge fields in a Kaluza–Klein theory if we do not consider compactifications on coordinates $y^a$. In Finsler like theories, a set $\{N_i^a\}$ defines a N–connection structure, with elongated partial derivatives (12). In Kaluza–Klein gravity they are linearized on $y^a$, $N_i^a = A_{a}^{i}(x)y^{b}$, when $A_{a}^{i}(x)$ are treated as some Yang–Mills potentials inducing certain covariant derivatives.

For simplicity, hereafter we shall work on a general nonholonomic space $V$ enabled with N–connection $N$ splitting (12) and resulting N–adapted base and co–base (12) and (13). Such a space is also endowed with a symmetric metric structure $g$ (of necessary local Euclidean or pseudo–Euclidean signature) which can be parametrized in the form (20) (or, equivalently, (21)). Any metric $g$ on $V$ can be represented equivalently in the form $Fg$ (19) after corresponding frame and coordinate transforms for a well defined generating function $F(x, y)$. It is always possible to introduce on a (pseudo) Riemannian manifold/ tangent bundle $V$ some local Finsler like variables when the metric is parametrized in a Sasaki type form. We shall write $V = TM$ when it will be necessary to emphasize that the constructions are defined explicitly for tangent bundles.

**Distinguished connections, theirs torsions and curvatures:**

For any d–metric of type (19) and/or (20), we can construct the Levi–Civita connection $\nabla = \{\Gamma_{\beta_\gamma}^\alpha\}$ on $V$ in a standard form. Nevertheless, this connection is not used in Finsler geometry and generalizations. The problem is that $\nabla$ is not compatible with a N–connection splitting, i.e. under parallel transports with $\nabla$, it is not preserved the Whitney sum (18).
In order to perform geometric constructions with h–/v–splitting, it was introduced the concept of distinguished connection (in brief, d–connection). By definition, such a d–connection $\mathbf{D} = (h\mathbf{D}, v\mathbf{D})$ is a linear one preserving under parallelism the N–connection structure on $\mathbf{V}$. The N–adapted components $\Gamma^{\alpha}_{\beta\gamma}$ of a d–connection $\mathbf{D}$ are defined by equations $D_{\alpha}e_{\beta} = \Gamma^{\gamma}_{\alpha\beta}e_{\gamma}$ and parametrized in the form $\Gamma^{\gamma}_{\alpha\beta} = \left( L_{jk}^{i}, L_{bk}^{a}, C_{jc}^{i}, C_{bc}^{a} \right)$, where $D_{\alpha} = (D_{i}, D_{a})$, with $h\mathbf{D} = (L_{jk}^{i}, L_{bk}^{a})$ and $v\mathbf{D} = (C_{jc}^{i}, C_{bc}^{a})$ defining the covariant, respectively, h– and v–derivatives.

The simplest way to perform computations with a d–connection $\mathbf{D}$ is to associate it with a N–adapted differential 1–form

$$\Gamma^{\alpha}_{\beta} = \Gamma^{\alpha}_{\beta\gamma}e_{\gamma}, \quad (23)$$

when the coefficients of forms and tensors (i.e. d–tensors etc) are defined with respect to (13) and (12). In this case, we can apply the well known formalism of differential forms as in general relativity [66]. It also allows us to elaborate an N–adapted differential/integral calculus for Finsler spaces and generalizations. For instance, torsion of $\mathbf{D}$ is defined/computed

$$T^{\alpha} = D\epsilon^{\alpha} = d\epsilon^{\alpha} + \Gamma^{\alpha}_{\beta\gamma}e_{\gamma}, \quad (24)$$

see formulas (A.1) in Appendix, for explicit values of coefficients $T^{\alpha} = \{T^{\alpha}_{\beta\gamma}\}$.

Similarly, using the d–connection 1–form (12), one computes the curvature of $\mathbf{D}$ (d–curvature)

$$R^{\alpha}_{\beta} = D\Gamma^{\alpha}_{\beta} = d\Gamma^{\alpha}_{\beta} - \Gamma^{\gamma}_{\beta\gamma} \wedge \Gamma^{\alpha}_{\gamma} = R^{\alpha}_{\beta\gamma\delta}e_{\gamma} \wedge e_{\delta}, \quad (25)$$

see formulas (A.2) for h–v–adapted components, $R^{\alpha}_{\beta} = \{R^{\alpha}_{\beta\gamma\delta}\}$.

The Ricci d–tensor $Ric = \{R_{\alpha\beta}\}$ is defined in a standard form by contracting respectively the components of (A.2), $R_{\alpha\beta} = R^{\alpha}_{\gamma\beta\delta}$. The h–/ v–components of this d–tensor, $R_{\alpha\beta} = \{R_{ij}, R_{ia}, R_{ai}, R_{ab}\}$, are

$$R_{ij} = R_{ijk}, \quad R_{ia} = -R_{kia}, \quad R_{ai} = R_{aib}, \quad R_{ab} = R_{abc}, \quad (26)$$

see explicit coefficients formulas (A.2). Here we emphasize that for an arbitrary d–connection $\mathbf{D}$, this tensor is not symmetric, i.e. $R_{\alpha\beta} \neq R_{\beta\alpha}$.

In order to define the scalar curvature of a d–connection $\mathbf{D}$, we have to use a d–metric structure $g$ (20) on $\mathbf{V}$, or $TM$,

$$^gR = g^{\alpha\beta}R_{\alpha\beta} = g^{ij}R_{ij} + h^{ab}R_{ab}, \quad (27)$$

16
with \( R = g^{ij}R_{ij} \) and \( S = h^{ab}R_{ab} \) are respectively the h– and v–components of scalar curvature.

For any d–connection \( D \) in Finsler geometry, and generalizations, the Einstein d–tensor is (by definition)

\[
E_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta} R.
\]  

This d–tensor is also not symmetric and, in general, \( D_{\alpha}E^{\alpha\beta} \neq 0 \). Such a tensor is very different from that for the Levi–Civita connection \( \nabla \) which is symmetric and with zero covariant divergence, i.e. \( E_{\alpha\beta} = E_{\beta\alpha} \) and \( \nabla_{\alpha}E^{\alpha\beta} \neq 0 \), where \( E_{\alpha\beta} \) is computed using \( \Gamma^{\alpha}_{\beta\gamma} \).

### 2.2.2 Notable connections for Finsler spaces

In general, it is possible to define on \( V \) two independent fundamental geometric structures \( g \) and \( D \) which are adapted to a given \( N \). In Part I of [120], we studied various models of Lagrange/Finsler affine spaces and Finsler gravity with nonmetricity field \( Q = Dg \neq 0 \). Such theories are very different from standard theories of physics. It is not clear the physical meaning of nonmetricity \( Q \) and there are a number of problems with definition of spinors, Dirac operators, conservation laws etc on such generalized nonholonomic spaces, see discussions in [101, 120, 95].

For applications in modern physics, it is more convenient to work with a d–connection \( D \) which is metric compatible satisfying the condition \( Dg = 0 \). There is an infinite number of d–connections which are compatible to a metric \( g \). A special interest presents a subclass of such metrics which are completely defined by \( g \) in a unique \( N \)–adapted form following a well defined geometric principle.

#### The canonical d–connection:

In our works on Finsler gravity, we used the so–called canonical d–connection \( \hat{D} \) (on spaces of even dimensions it is called the h–/v–connection, such connections were studied geometrically in [63, 64] on vector/tangent bundles and for generalized Lagrange–Finsler geometry).

By definition, \( \hat{D} \) is with vanishing horizontal and vertical components of torsion and satisfies the conditions \( \hat{D}g = 0 \), see explicit component formulas (A.3) and (A.1). From many points of view, on a nonholonomic space \( V \), \( \hat{D} \) is the "best" \( N \)–adapted analog of the Levi–Civita connection \( \nabla \). We have the distortion relation

\[
\nabla = \hat{D} + \hat{Z},
\]  

(29)
when both linear connections \( \nabla = \{ \Gamma^\alpha_{\beta \gamma} \} \) and \( \hat{\nabla} = \{ \hat{\Gamma}^\alpha_{\beta \gamma} \} \) and the distorting tensor \( \hat{Z} = \{ \hat{Z}^\alpha_{\alpha \beta} \} \) are uniquely defined by the same metric tensor \( g \). The coefficient formulas are given in Appendix, see (A.4) and (A.5). The connection \( \hat{\nabla} \) is with nontrivial torsion (the coefficients \( T^i_{ja} \) and \( \hat{T}^a_{ji} \) are not, in general, zero, see (A.1)) but such \( d \)-torsions are nonholonomically induced by \( N \)-connection coefficients and completely determined by certain off–diagonal \( N \)-terms in (22). All geometric constructions can be performed equivalently and redefined in terms of both connections \( \nabla \) and \( \hat{\nabla} \) using (29).

The connection \( \hat{\nabla} \) and various types of \( d \)-connections \( D = (hD, vD) \) with \( h \)- and \( v \)-covariant derivatives, \( hD = (L^i_{jk}, L^a_{bk}) \) and \( vD = (C^i_{jc}, C^a_{bc}) \) can be defined on vector bundles and on (pseudo) Riemann spaces of arbitrary dimensions, alternatively to \( \nabla \).

On spaces of even dimension, we can define \( d \)-connections when a couple \( (L^i_{jk}, C^i_{jc}) \) is respectively identical to \( (L^a_{bk}, C^a_{bc}) \) and consider covariant derivatives stated completely by \( hD = \{ L^i_{jk} \} \) and \( vD = \{ C^i_{jc} \} \). A general Finsler \( d \)-connection is of type \( \mathcal{F}D = \{ L^i_{jk}, C^i_{jc} \} \) with coefficients \( L^i_{jk} \) and \( C^i_{jc} \) determined by a generating Finsler function \( F \) and a \( N \)-connection \( N = \{ N^a_i \} \) and some arbitrary and/or induced nonholonomically torsion fields. In general, \( \mathcal{F}D \) is not metric compatible, i.e., \( \mathcal{F}D g = \mathcal{F}Q \neq 0 \).

**Nonmetric Finsler \( d \)-connections:**

There were considered three types of such notable connections (see details and references in \[63, 64, 11\] and, on physical applications, \[101, 120, 98\]). In general, it is possible to define following different geometric principles an infinite number of metric noncompatible or compatible \( d \)-connections in Finsler geometry and generalizations. The first (metric noncompatible) one was the Berwald \( d \)-connection \( \mathcal{F}D = \mathcal{B}D = \{ L^a_{bk} = \partial N^a_k / \partial y^b, C^a_{bc} = 0 \} \). It is completely defined by the \( N \)-connection structure and \( \mathcal{B}D g = \mathcal{B}Q \neq 0 \).

Then, it was introduced the Chern \( d \)-connection \( \mathcal{C}D \) (latter rediscovered by Rund \[77\]), \( \mathcal{F}D = \mathcal{C}D = \hat{L}^i_{jk}, C^i_{jc} = 0 \), where \( \hat{L}^i_{jk} \) is given by the first formula in (A.3). This \( d \)-connection is torsionless, \( \mathcal{C}T = 0 \), but (in general) metric noncompatible, \( \mathcal{C}D \) is \( \mathcal{C}Q \neq 0 \). Recently, some authors attempted to elaborate Chern–Finsler, or Berwald–Finsler cosmological models and certain modifications of Einstein gravity using such metric noncompatible \( d \)-connections \[26, 27, 28, 29, 54, 55\]. Really, geometrically, the Chern \( d \)-connection is a "nice" one, but with a number of problems for applications in standard models of physics (with difficulties in...
defining spinors and Dirac operators, conservation laws, quantization etc, see discussions and critical remarks in [115, 101, 98]).

Experts on Finsler geometry also know about the Hashiguchi d–connection

\[ F^D = H^D = \{ L^i_{jk} = \partial N^i_k / \partial y^j, \quad H^i_{jc} \}, \]

where \( H^i_{jc} = \frac{1}{2} F g^{id} (e_c F g_{bd} + e_e F g_{cd} - e_d F g_{bc}) \), for \( e_c = \partial / \partial y^c \) and a given \( F g_{bd} \) [4], see details, for instance, in [64]. It contains both nontrivial torsion and nonmetricity components, all completely defined by the N–connection and Hessian structure, and does not have perspectives in standard models of physics.

We studied the Lagrange/Finsler–affine theories with very general torsion and nonmetric structures, provided examples of exact solutions and discussed physical implications of models in Part I of monograph [120].

A preferred metric compatible Cartan d–connection:

Historically, it was the first d–connection introduced in Finsler gravity [25]. It is given by local coefficients

\[ F^D = C^D = \{ C^i_{jk}, \quad C^a_{bc} \}, \]

where

\[ C^i_{jk} = \frac{1}{2} F g^{ir} \left( e_k F g_{jr} + e_j F g_{kr} - e_r F g_{jk} \right), \]
\[ C^a_{bc} = \frac{1}{2} F g^{ad} \left( e_c F g_{bd} + e_e F g_{cd} - e_d F g_{bc} \right). \]

The Cartan d–connection (30) is metric compatible, \( F^D F g = 0 \), but with nontrivial torsion \( ^cT \neq 0 \) (the second property follows from formulas (24) and (A.1) redefined for \( C^D \)). The nontrivial torsion terms are induced nonholonomically by a fundamental Finsler function \( F \) via \( F g_{bd} \) [4] and Cartan’s N–connection \( ^cN^a_i \) [16]. This torsion is very different from the well known torsion in Einstein–Cartan, or string/gauge gravity models, because in the Cartan–Finsler case we do not need additional field equations for the torsion fields. The torsion \( ^cT \), similarly to \( \hat{T} \), is completely defined by a (Finsler) d–metric structure.

A very important property of \( C^D \) is that it defines also a canonical almost symplectic connection [57, 64], see details and recent applications to quantization of Lagrange–Finsler spaces and Einstein/brane gravity in [96, 97, 104, 105, 68]. Perhaps, the Cartan d–connection is the “best” one for physical applications in modern physics of Finsler and Lagrange geometry and related anholonomic deformation method, see additional arguments in [115, 101, 98, 120, 104]. The d–connections \( ^cD \) and \( \tilde{D} \) allow us to work in N–adapted form in Finsler classical and quantum gravity theories keeping all geometric and physical constructions to be very similar to those for the Levi–Civita connection \( \nabla \).
In Finslerian theories of gravity, it is possible to work geometrically with very different types of \(d\)-connections because there are always some transformations of the Cartan \(d\)-connection into, for instance, any mentioned above (or more general ones) notable Finsler connections, of Berwald, Chern or Hashiguchi types. Nevertheless, the main issue is that for what kind of \(d\)-connections we can formulate Einstein/Dirac/ Yang–Mills etc equations which are well defined, self–consistent and with important physical implications.

In our approach, for gravity models on nonholonomic manifolds/bundles of arbitrary dimension, we give priority to the canonical \(\hat{D}\) (here we note also that the Einstein equations for this \(d\)-connection can be integrated in very general forms, [112, 92, 95, 120, 114]). On nonholonomic spaces of even dimension (in particular, on \(TM\)), \(\hat{D}\) transforms into the so–called canonical \(h\)- \(v\)–connection, which for Finsler d–metrics (19) can be nonholonomically deformed to be just the Cartan \(d\)–connection \(cD\). We note that the gravitational field equations for \(cD\) also can be integrated in very general forms, see formula (60).

### 3 Field Equations for Finsler Theories of Gravity

In this section, we outline the theory of Einstein–Finsler spaces. There are formulated the Einstein equations for the canonical \(d\)-connection and/or Cartan \(d\)-connection. We analyze the class of theories extending the four dimensional general relativity to metric compatible Finsler gravities on tangent bundles and/or nonholonomic manifolds of eight dimensions. Considering nonholonomic distributions with three shells of anisotropy (nonholonomic splitting of dimensions with base of dimension 2 and higher order shells 2+2+2), we prove that for certain generic off–diagonal ansatz the Einstein–Finsler equations can be integrated in very general forms. The section ends with a discussion of principles of general relativity and their extension to metric compatible Finsler gravity theories.

Some remarks on former models of Finsler gravity:

Different models of gravitational theory of gravity with Finsler metrics and nonmetricity were elaborated and analyzed periodically beginning the first geometric models with the Berwald and Chern \(d\)-connections (see, for instance, [74] and some recent applications, [20, 27, 28, 29]; examples with Berwald–Moore metrics and generalizations are in [42, 19, 6]). A rigorous study of metric compatible and noncompatible Finsler gravities with various
types of connections and metrics was provided in a series of works [43, 44, 45, 46, 85], see references therein and Appendix to [63]. There were developed some concepts of Finsler Kaluza–Klein theories [13, 14, 5] but they are very different from the standard approaches with compactified extra coordinates (we briefly discussed the problem in relation to formula (22)). In Finsler gravities, not all "velocities" can be compactified. Here we mention also that there are different ideas how to formulate "gauge" Finsler gravity models, see [8, 64, 110, 111] and references therein.

Generalized gravitational field equations for connections with torsion and/or nonmetricity were postulated in various metric–affine/ gauge/ string / brane theories etc beginning the second part of previous century (see history, reviews of results and exhaustive references in [41, 79, 120, 101, 98, 110]). Perhaps, in the framework of Finsler gravity theories, the most related to the standard paradigm of modern physics are the Einstein equations\textsuperscript{13} for the canonical $\hat{D}$–connection $\hat{D}$. On $TM$, such equations were similarly written for the $h$–/v–connection, the Cartan/almost Kähler $d$–connection and various other $d$–connections in Lagrange–Finsler gravity and generalizations. The key result was that the Ricci and Einstein $d$–tensors, of type (26) and (28), where constructed in N–adapted form. This allows us to elaborate following general geometric and variational principles (similarly to general relativity but with a corresponding N–adapted calculus) various classes of Finsler–Lagrange and nonholonomic manifold gravity theories.

In our approach, we suggested the idea that N– and $d$–connections and Finsler like variables can be formally introduced in Einstein gravity and generalizations by considering nonholonomic distributions and associated frames on (pseudo) Riemannian manifolds. The surprising thing was that various types of gravitational field equations, in Einstein gravity and modifications, can be integrated in very general off–diagonal forms using Finsler geometry methods, see reviews of results [112, 92, 95, 120, 114]. It was possible to elaborate an unified geometric approach to various types of commutative and noncommutative nonholonomic gravity and Ricci flow theories on bundle and manifold spaces [103, 106, 107, 94, 122, 102, 108, 109, 14].

\textsuperscript{13}Such equations were written, in general, on vector bundles [63, 64], where they were studied geometrically (there were not considered solutions of the gravitational field equations in those books); in our works, on nonholonomic manifolds of arbitrary odd or even dimensions, superbundles and/or noncommutative generalizations, we provided further developments in geometry and classical and quantum physics, with exact solutions and quantum methods, see reviews of results in [120, 92, 101, 95, 104].

\textsuperscript{14}Such constructions can not be performed for Finsler geometry/gravity theories with nonmetricity based on the Chern $d$–connection, even there were proposed some variants of gravitational nonmetric field equations, see [1]. It is not clear how to introduce spinors...
3.1 Einstein equations for distinguished connections

Having prescribed a N–connection \( N \) and d–metric \( g \) structures on a N–anholonomic manifold \( V \), for any metric compatible d–connection \( D \), we can compute the Ricci \( R_{\alpha\beta} \) and Einstein \( E_{\alpha\beta} \) d–tensors. The N–adapted gravitational field equations \([63, 64, 92, 120, 101]\) are

\[
E_{\alpha\beta} = \Upsilon_{\alpha\beta},
\]

where the source \( \Upsilon_{\alpha\beta} \) has to be defined in explicit form following certain explicit models of "locally anisotropic" gravitational and matter field interactions. In h–v–components, the coefficients of d–tensors are computed with respect to (co) frames \([12]\) and \([13]\).

**Einstein equations in h–/v–components:**

The equations \([31]\) can be distinguished in the form

\[
R_{ij} - \frac{1}{2}(R + S)g_{ij} = \Upsilon_{ij},
\]

\[
R_{ab} - \frac{1}{2}(R + S)h_{ab} = \Upsilon_{ab},
\]

\[
R_{ai} = \Upsilon_{ai},
\]

where \( R_{ai} = R_{aib}^b \) and \( R_{ia} = R_{ikb}^k \) are defined by formulas for d–curvatures \([A.2]\) containing d–torsions \([A.1]\). For theories with arbitrary torsions \( T \), we have to complete such equations with additional algebraic or dynamical ones (for torsion’s coefficients) like in the Einstein–Cartan, gauge, or string theories with torsion. On generalized Finsler spaces, such constructions should be in N–adapted forms (see details in Part I of \([120]\)).

For a metric compatible d–connection \( D \) which is completely defined by a d–metric structure \( g \), the corresponding system \([32]–[34]\) is very similar to that for the usual Einstein gravity. The difference is that \( R_{ai} \neq R_{ia}, \nabla \neq D \) (even, it is possible to define some preferred systems of references when for some very general metric ansatz both linear connections, with different conservation law, are represented by the same coefficients). Elaborating geometric/gravity models on \( TM \), containing in the limit \( D \to \nabla \) the Einstein gravity theory on \( M \), we should consider that equations \([32]\) define a generalization of \( R_{ij} - \frac{1}{2}Rg_{ij} = \Upsilon_{ij} \) for \( \nabla = \{ \Gamma^i_{jk} \} \) and a well defined procedure of "compactification" on fiber coordinates \( y^a \).

and the Dirac equations on metric noncompatible spaces and how to construct general exact solutions for nonmetric gravitational Finsler field equations.
Gravitational field equations for $\hat{D}$ and $cD$

Because the canonical d–connection $\hat{D}$ is completely defined by $g_{\beta\delta}$, the corresponding analog of equations (31), for $D = \hat{D}$, written in the form

$$\hat{R}_{\beta\delta} - \frac{1}{2}g_{\beta\delta}^{\quad s} R = \hat{\Upsilon}_{\beta\delta},$$

(35)
can be constructed to be equivalent to the Einstein equations for $\nabla$. This is possible if $\hat{\Upsilon}_{\beta\delta} = \text{matter } \Upsilon_{\beta\delta} + z \Upsilon_{\beta\delta}$ are derived in such a way that they contain contributions from 1) the N–adapted energy–momentum tensor (defined variationally following the same principles as in general relativity but on $V$) and 2) the distortion of the Einstein tensor in terms of $\hat{Z}$ (29), i.e. $\hat{E}_{\beta\delta} = E_{\alpha\beta} + z \hat{E}_{\beta\delta}$, for $z \hat{E}_{\beta\delta} = z \Upsilon_{\beta\delta}$. The value $z \hat{E}_{\beta\delta}$ is computed by introducing $\hat{D} = \nabla - \hat{Z}$ into (A.2) and corresponding contractions of indices in order to find the Ricci d–tensor and scalar curvature.

The system of equations (A.2) can be integrated in very general forms, see explicit constructions in subsection 3.1.2. Such solutions can be considered also in general relativity if we impose additionally the condition that

$$\hat{L}^c_{\alpha j} = e_a (N^c_j), \quad \hat{C}^{ij}_c = 0, \quad \Omega^a_{\ ji} = 0,$$

(36)
for $\Upsilon_{\beta\delta} \rightarrow \kappa T_{\beta\delta}$ (matter energy–momentum in Einstein gravity) if $\hat{D} \rightarrow \nabla$. We emphasize here that if the constraints (36) are satisfied the tensors $\hat{T}^{\gamma}_{\alpha \beta}$ (36) and $Z^{\gamma}_{\alpha \beta}$ (A.5) are zero. For such configurations, we have $\hat{\Gamma}^{\gamma}_{\alpha \beta} = \Gamma^{\gamma}_{\alpha \beta}$, with respect to (12) and (13), see (A.4), even $\hat{D} \neq \nabla$.

In Finsler geometry/gravity models, the constraints (36) are not obligatory. On $TM$, and any even dimensional $V$, it is possible to perform such frame deformations when $\hat{D} \rightarrow cD$. So, the Einstein equations for the Cartan d–connection also can be integrated in very general forms.

### 3.1.1 Einstein–Finsler spaces

An Einstein space (manifold) is defined in standard form by a Levi–Civita connection $\nabla = \{ \Gamma^{\gamma}_{\alpha \beta} \}$ satisfying the conditions

$$\iota R_{ij} = \lambda g_{ij},$$

(37)
for a (pseudo) Riemannian manifold $M$ endowed with metric $g_{ij}$, where $\lambda$ is the cosmological constant.

In this work, we study cosmological models for Finsler gravity on $TM$ (in general, on any $V$), extending Einstein gravity on $M$ with generalizations of
for some classes of metric compatible \(d\)-connection. Because in general relativity \(\dim M = 4\), and \(\dim TM = 8\) (for realistic gravity models with local anisotropy), we can restrict our considerations for parametrization of local coordinates of type \(v^\alpha = (x^i, y^a)\). In order to apply the anholonomic frame method for constructing exact solutions from higher dimension gravity [112], we should introduce “three shells of anisotropy” [15].

For a \(d\)-connection \(\mathbf{D}\) which is metric compatible with \(\mathbf{g}\) on \(\mathbf{V}\), we can consider generalizations of (37) in the form

\[
R_{ij} = h_\lambda(u)g_{ij}, \quad (38)
\]

\[
R_{a_0 a_0 b} = 0_v \lambda(u)h_{a_0 a_0 b}, \quad (39)
\]

\[
R_{1_a 1_b} = 1_v \lambda(u)h_{1_a 1_b}, \quad (40)
\]

\[
R_{2_a 2_b} = 2_v \lambda(u)h_{2_a 2_b}, \quad (41)
\]

\[
R_{a_0 i} = R_{i a_0} = 0, \quad (42)
\]

\[
R_{2_a 1_a} = R_{1_a 2_a} = 0, \quad (43)
\]

where \(h_\lambda(u)\) and \(0_v \lambda(u)\), \(1_v \lambda(u)\), \(2_v \lambda(u)\) are respectively the so-called locally anisotropic \(h-\) and \(0_v-, 1_v-, 2_v-\) polarized gravitational “constants”. Such polarizations should be defined for certain well defined constraints on matter and gravitational field dynamics, lifts on tangent bundles, corrections from quantum gravity or any extra dimension gravitational theory.

An Einstein–Finsler space is defined by a triple \([\mathbf{N}, \mathbf{g}, \mathbf{D}]\) with a metric compatible (Finsler type) \(d\)-connection \(\mathbf{D}\) subjected to the condition to be a solution of equations (38)–(43).

In theories with symmetric metrics, \(g_{\alpha\beta} = g_{\beta\alpha}\), for a variational calculus on coefficients of metrics parametrized in the form (20), we can always

---

[15] Parametrization I with “anisotropic shells” for higher order anisotropic extensions, see details in Ref. [112], when \(TM = hTV \oplus 0_v TV \oplus 1_v TV \oplus 2_v TV\), for local coordinates \(u^1 = (u^0, u^1) = (u^0, u^{0_a}, u^{1_a})\), \(u^{2} = (u^{1_a}, u^{2_a}) = (u^0, u^{0_a}, u^{1_a}, u^{2_a})\), with \(i, j, \ldots = 1, 2, \ldots, n; \quad 0_a, 0_b, \ldots = n + 1, \ldots, n + m\); \(1_a, 1_b, \ldots = n + m + 1, \ldots, n + m + \ldots + 1_m; \quad 2_a, 2_b, \ldots = n + m + 1, \ldots, n + m + \ldots + 1_m, n + m + \ldots + 1_m + \ldots + 2_m\); For explicit examples, we shall consider \(n = 4\) and \(m = 1_m = 2_m = 2\), when \(\dim M = 2\), or \(\dim V = 8\).

Parametrization II is for \(n = 4, m = 4, 1_m = 0\), with trivial ”shall” \((u^1)\) and local coordinates \(u^a = (x^i, y^a)\), for \(i = 1, 2, 3, 4\) and consequently \(a = 5, 6, 7, 8\) when on tangent bundles 5 can be contracted to 1, 6 to 2 and so on.

Parametrization III is for \(n = 2\) and \(m = 2\), when \(\dim M = 2, \dim V = 4\), with indices for local coordinates running values \(i = 1, 2\) and \(a = 3, 4\).
diagonalize the source in (31), with respect to certain N–adapted frames, and consider generalized Einstein equations of type

$$\mathbf{R}^{\alpha}_{\beta} = \Psi^{\alpha}_{\beta} (44)$$

for $\Psi^{\alpha}_{\beta} = \text{diag}[1, \Psi]$. Such equations can be solved in very general form for the canonical d–connection $\tilde{\mathbf{D}}$ and certain nonholonomic restrictions to the Levi–Civita connection $\nabla$, see [112, 120, 90]. For a particular types of sources $\Psi^{\alpha}_{\beta}$ determined by polarized cosmological constants, the corresponding subclasses of solutions define Einstein–Finsler spaces. The equations (44) are more general than the system (38)–(43). Nevertheless, for nonholonomic distributions with splitting of dimensions of type $n+2+2+\ldots$, we can always define certain N–adapted frames when such systems of partial equations became equivalent for corresponding redefinitions of polarization functions.

In a general context, we can consider that an Einstein–Finsler space is determined by a set of solutions of (44) with given sources and for a triple $[\mathbf{N}, \mathbf{g}, \mathbf{D}]$ when $\mathbf{D} = \tilde{\mathbf{D}}$, or $\mathcal{D}$, and/or any higher order generalizations of such d–connections.

3.1.2 Metric ansatz and partial differential equations

**Third order anisotropic ansatz:**

Any metric (20) can be reparametrized in a form with three shell anisotropy,

$$\mathbf{g} = g_{ij}(x)dx^i \otimes dx^j + h_{ab}(x, 0y)\mathbf{e}^a \otimes \mathbf{e}^b$$

$$+ h_{1ab}(x, 0y, 1y)\mathbf{e}^1 \otimes \mathbf{e}^b + h_{2ab}(x, 0y, 1y, 2y)\mathbf{e}^2 \otimes \mathbf{e}^b,$$

$$\mathbf{e}^0 = dy^0 + N_i^0(0u)dx^i,$$

$$\mathbf{e}^1 = dy^1 + N_i^1(1u)dx^i + N_0^1(1u)\mathbf{e}^0,$$

$$\mathbf{e}^2 = dy^2 + N_i^2(2u)dx^i + N_0^2(2u)\mathbf{e}^0 + N_1^2(2u)\mathbf{e}^1,$$

for $x = \{x^i\}, 0y = \{y^0\}, 1y = \{y^1\}, 2y = \{y^2\}$, when the vertical indices and coordinates split in the form $y = [0y, 1y, 2y]$, or $y^a = [y^0, y^1, y^2]; 0u = (x, 0y), 1u = (0u, 1y), 2u = (1u, 2y), u_0 = (x, y^0), u_1 = (0u, y^1), u_2 = (1u, y^2)$. There is a very general ansatz of this form (with Killing symmetry on $y^8$, when the metric coefficients do not depend on variable $y^8$; it is convenient to write $y^8 = 0v$, 25
\( y^5 = 1v, y^7 = 2v \) and introduce parametrization of \( N \)-coefficients via \( n \)-and \( w \)-functions) defining exact solutions of (44),

\[
\begin{align*}
\text{sol } g &= g_i(x^k)dx^i \otimes dx^i + h \cdot o_{o_i}(x^k, 0v) e^{0a} \otimes e^{0a} \\
&+ h \cdot 1_a(u^{0a}, 1v) e^{1a} \otimes e^{1a} + h \cdot 2_a(u^{1a}, 2v) e^{2a} \otimes e^{2a},
\end{align*}
\]

\[
\begin{align*}
e^3 &= dy^3 + w_i(x^k, 0v)dx^i, & e^4 &= dy^4 + n_i(x^k, 0v)dx^i, \\
e^5 &= dy^5 + w \cdot \beta (u^{0a}, 1v)du^\beta, & e^6 &= dy^6 + n \cdot \beta (u^{0a}, 1v)du^\beta, \\
e^7 &= dy^7 + w \cdot 1_\beta (u^{1a}, 2v)du^1_\beta, & e^8 &= dy^8 + n \cdot 1_\beta (u^{1a}, 2v)du^1_\beta.
\end{align*}
\]

In Theorem 1.1 of [112] (we should consider those results for three shells and trivial \( \omega \)-coefficients), there are stated explicit conditions on \( w \)-and \( n \)-coefficients and \( \Upsilon \)-sources, for arbitrary dimensions and in very general forms, when an ansatz (46) generates exact solutions of equations of systems of equations of type (38)–(43). In this subsection, we analyze three different types of \( d \)-connections when the corresponding ansatz induces separation of equations in (44). Following such constructions, we can prove the integrability in very general form of field equations in general gravity and metric compatible Finsler generalizations of gravity. Some explicit cosmological solutions in Finsler gravity will be analyzed in section 4.

Separation of equations for the canonical \( d \)-connection:

Let us consider the ansatz (46) for dimensions \( n = 2 \) and \( m = 2, \), \( m = 0 \), when the source is parametrized in the form

\[
\Upsilon_{\alpha \beta} = \text{diag}(\Upsilon_\gamma; \Upsilon_1 = \Upsilon_2 = \Upsilon_2(x^k); \Upsilon_3 = \Upsilon_4 = \Upsilon_4(x^k, y^3)).
\]

Computing the corresponding coefficients of \( d \)-connection \( \hat{D} \) following formulas (A.3) and introducing them respectively into (A.2) (26), we express the gravitational field equations (35) in the form (10)

\[
\begin{align*}
\hat{R}_1^1 &= \hat{R}_2^2 \\
&= -\frac{1}{2g_1g_2} \left[ 2g_{12}^2(2g_2 - (g_1)^2) + g_{12} \cdot (g_1)^2 \right] = \Upsilon_2(x^k), \\
\hat{R}_3^3 &= \hat{R}_4^4 = -\frac{1}{2h_3h_4} \left[ h_4^2(2h_4 - (h_3)^2) + h_3^2(h_4) \right] = \Upsilon_4(x^k, y^3),
\end{align*}
\]

\[\text{the details of such computations can be found in Part II of } [120] \text{ and in } [112].\]
\[ \tilde{R}_{3k} = \frac{w_k}{2h_4} [h_{4}^{**} - \left( \frac{h_1^*}{2h_3} \right)^2 - \frac{h_2^*}{2h_3} h_1^*] + \frac{h_1^*}{4h_3} \left( \frac{\partial_k h_3}{h_3} + \frac{\partial_k h_4}{h_4} \right) - \frac{\partial_k h_3^*}{2h_4} = 0 \] (50)

\[ \tilde{R}_{4k} = \frac{h_4^*}{2h_3} n_k^{**} + \left( \frac{h_4^*}{h_3} h_3 - \frac{3}{2} h_4^* \right) \frac{n_k^*}{2h_3} = 0. \] (51)

In the above formulas, we denote the partial derivatives in the form \( a^* = \partial a/\partial x^1, a' = \partial a/\partial x^2, a^* = \partial a/\partial y^3 \).

The system (48)–(51) is nonlinear and with partial derivatives. Nevertheless, the existing separation of equations (we should not confuse with separation of variables which is a different property) allows us to construct very general classes of exact solutions (depending on conditions if certain partial derivatives are zero, or not). For any prescribed \( \Upsilon_2(x^k) \), we can define \( g_1(x^k) \) (or, inversely, \( g_2(x^k) \)) for a given \( g_2(x^k) \) (or, inversely, \( g_1(x^k) \)) as an explicit, or non–explicit, solution of (48) by integrating two times on \( h \)–variables. Similarly, taking any \( \Upsilon_4(x^k, y^3) \), we solve (49) by integrating one time on \( y^3 \) and defining \( h_3(x^k, y^3) \) for a given \( h_4(x^k, y^3) \) (or, inversely, by integrating two times on \( y^3 \) and defining \( h_4(x^k, y^3) \) for a given \( h_3(x^k, y^3) \)).

Haven determined the values \( g_i(x^k) \) and \( h_{\alpha}(x^k, y^3) \), we can compute the coefficients of N–connection: The functions \( w_j(x^k, y^3) \) are solutions of algebraic equations (50). Finally, we have to integrate two times on \( y^3 \) in order to obtain \( n_j(x^k, y^3) \). Such general solutions depend on integration functions depending on coordinates \( x^k \). In physical constructions, we have to consider well defined boundary conditions for such integration functions.

**Equations for the h-v / Cartan d–connection:**

In N–adapted frames, the h–v d–connection \( \tilde{\Gamma} \) is determined by coefficients \( \tilde{\Gamma}^a_{\beta\gamma} = \left( \tilde{L}^a_{\beta\gamma}, \tilde{C}^a_{\beta\gamma} \right) \), where

\[ \tilde{L}^a_{\beta\gamma} = \frac{1}{2} g^{ab} (e_k g_{jh} + e_j g_{kh} - e_h g_{jk}), \quad \tilde{C}^a_{\beta\gamma} = \frac{1}{2} h^{ab} (e_b h_{ec} + e_c h_{eb} - e_e h_{bc}), \] (52)

are computed for a d–metric \( \mathbf{g} = [g_{ij}, h_{ab}] \). Via frame transforms to a (pseudo) Finsler metric \( F \mathbf{g} \), \( g_{\alpha'\beta'} = e^\alpha_{\alpha'} e^\beta_{\beta'} \) \( F \mathbf{g}_{\alpha\beta} \), we can define such N–adapted frames when the coefficients of \( \tilde{\Gamma} \) are equal to the coefficients of the Cartan d–connection \( \mathbf{D} = \{ c \mathbf{L}^a_{\beta\gamma}, c \mathbf{C}^a_{\beta\gamma} \} \).

For dimensions \( n = 2 \) and \( 0 m = 2, 1 m = 2 m = 0 \), the gravitational field equations (14) for \( \tilde{\Gamma}^\gamma_{\alpha\beta} \) (52) and source (17), transform into an exactly integrable system of partial differential equations when the coefficients of (20) are stated by an ansatz \( g_{\alpha\beta} = \text{diag}[g_i(x^k), h_{\alpha}(x^i, v)] \) and
\( N^3_k = w_k(x^i, v), N^4_k = n_k(x^i, v). \) The first two equations of such a system are completely equivalent, respectively, to (48) and (49),

\[
\tilde{R}_1 = \tilde{R}_2 = \hat{R}_1 = \hat{R}_2 = - \Upsilon_2(x^k), \quad \hat{R}_3 = \hat{R}_4 = - \Upsilon_4(x^k, y^3),
\]

but instead of (50), (51) we get, correspondingly,

\[
\tilde{R}_{3j} = \frac{h^*_3}{2h_3} w^*_j + A^* w_j + B_j = 0, \quad \tilde{R}_{4i} = - \frac{h^*_3}{2h_3} n^*_i + \frac{h^*_4}{2} K_i = 0,
\]

where \( A = \left( \frac{h^*_3}{2h_3} + \frac{h^*_4}{2h_4} \right), B_k = \frac{h^*_3}{2h_4} \left( \frac{\partial_k g_1}{2g_1} - \frac{\partial_k g_2}{2g_2} \right) - \partial_k A, \quad K_1 = - \frac{1}{2} \left( \frac{g^*_1}{g_2 h_3} + \frac{g^*_2}{g_2 h_4} \right), \quad K_2 = \frac{1}{2} \left( \frac{g^*_2}{g_1 h_3} - \frac{g^*_1}{g_2 h_4} \right).
\]

The system (53)–(56) also has the property of separation of equations. In this case (having defined \( h_a(x^i, v) \)), we can compute \( n_k(x^i, v) \) integrating the equation (56) on \( y^3 = v \) and, respectively, solving an usual first order differential equation (55), on \( y^3 = v \), considering \( x^i \) as parameters. Prescribing a generating function \( F(x^i, v) \), such a solution given by data \( g_i(x^k), h_a(x^i, v), \) and \( N^3_k = w_k(x^i, v), N^4_k = n_k(x^i, v) \) can be represented equivalently as a (pseudo) Finsler space. To associate such a \( h–v \)-configuration to a real Finsler geometry is convenient to work with sets of local carts on \( TM, \) or \( V, \) when the quadratic algebraic system for \( e^{\alpha}_{\alpha'} \) has well defined real solutions.

**Equations for the 3d order anisotropic canonical \( d \)-connections:**

The Einstein equations for the canonical \( d \)-connection \( \hat{D} \) computed for ansatz (46) and source

\[
\Upsilon^{3\alpha}_{3\beta} = \text{diag}[\Upsilon_{3\gamma}, \Upsilon_1 = \Upsilon_2 = \Upsilon_2(x^k); \Upsilon_3 = \Upsilon_4 = \Upsilon_4(x^k, 0v); \Upsilon_5 = \Upsilon_6(u^0_{\alpha}, 1v); \Upsilon_7 = \Upsilon_8 = \Upsilon_8(u^{1\alpha}, 2v)]
\]

are equivalent to

\[
\tilde{R}_1 = \tilde{R}_2 = - \Upsilon_2(x^k), \quad \tilde{R}_3 = \tilde{R}_4 = - \Upsilon_4(x^k, 0v),
\]
\[ \tilde{R}_5^2 = \tilde{R}_6^6 = - \frac{1}{2h_5 h_6} \left[ \partial^2_{1_v 1_v} h_6 - \frac{(\partial_{1_v} h_6)^2}{2h_6} - \frac{(\partial_{1_v} h_5)(\partial_{1_v} h_6)}{2h_5} \right] \]
\[ = - \Upsilon_6(u^{\alpha}, 1_v), \]
\[ \tilde{R}_7^5 = \tilde{R}_8^8 = - \frac{1}{2h_7 h_8} \left[ \partial^2_{2_v 2_v} h_8 - \frac{(\partial_{2_v} h_8)^2}{2h_8} - \frac{(\partial_{2_v} h_7)(\partial_{2_v} h_8)}{2h_7} \right] \]
\[ = - \Upsilon_8(u^{\alpha}, 2_v), \]
\[ \tilde{R}_{3k} = \frac{w_k}{2h_4} \left[ h_4^{**} - \frac{(h_4^*)^2}{2h_4} - \frac{h_3^* h_4^*}{2h_3} \right] \]
\[ + \frac{h_4^*}{4h_4} \left( \frac{\partial h_3}{h_3} + \frac{\partial h_4}{h_4} \right) - \frac{\partial_{1_v} h_4^*}{2h_4} = 0, \]
\[ \tilde{R}_{4k} = \frac{h_4^*}{2h_3} n_k^{**} + \left( \frac{h_4 h_4^* - 3 h_4^*}{2h_3} \right) \frac{n_k^*}{2h_3} = 0, \]
\[ \tilde{R}_{5^0\alpha} = \frac{1}{2h_6} \left[ \partial^2_{1_v 1_v} h_6 - \frac{(\partial_{1_v} h_6)^2}{2h_6} - \frac{(\partial_{1_v} h_5)(\partial_{1_v} h_6)}{2h_5} \right] \]
\[ + \frac{\partial_{1_v} h_6}{4h_6} \left( \frac{\partial_{0\alpha} h_5}{h_5} + \frac{\partial_{0\alpha} h_6}{h_6} \right) - \frac{\partial_{0\alpha} \partial_{1_v} h_6}{2h_6} = 0, \]
\[ \tilde{R}_{6^0\alpha} = \frac{h_6}{2h_5} \partial^2_{1_v 1_v} n_{0\alpha} + \left( \frac{h_6}{h_5} \partial_{1_v} h_5 - \frac{3}{2} \partial_{1_v} h_6 \right) \frac{\partial_{1_v} n_{0\alpha}}{2h_5} = 0, \]
\[ \tilde{R}_{7^1\alpha} = \frac{2w_1}{2h_4} \left[ \partial^2_{2_v 2_v} h_8 - \frac{(\partial_{2_v} h_8)^2}{2h_8} - \frac{(\partial_{2_v} h_7)(\partial_{2_v} h_8)}{2h_7} \right] \]
\[ + \frac{(\partial_{2_v} h_8)}{4h_8} \left( \frac{\partial_{1\alpha} h_7}{h_7} + \frac{\partial_{1\alpha} h_8}{h_8} \right) - \frac{\partial_{1\alpha} \partial_{2_v} h_8}{2h_8} = 0, \]
\[ \tilde{R}_{8^1\alpha} = \frac{h_8}{2h_7} \partial^2_{2_v 2_v} n_{1\alpha} + \left( \frac{h_8}{h_7} \partial_{2_v} h_7 - \frac{3}{2} \partial_{2_v} h_8 \right) \frac{\partial_{2_v} n_{1\alpha}}{2h_8} = 0, \]

where partial derivatives, for instance, are \( \partial_{1_v} = \partial / \partial 1_v = \partial / \partial y^5 \), \( \partial_{2_v} = \partial / \partial 2_v = \partial / \partial y^7 \), and \( N_{5^0\alpha} = \frac{1}{w_{0\alpha} (u^{\alpha}, 1_v), N_{6^0\alpha} = \frac{1}{n_{0\alpha} (u^{\alpha}, 1_v), N_{7^1\alpha} = 2 n_{1\alpha} (u^{\alpha}, 2_v), N_{8^1\alpha} = 2 n_{1\alpha} (u^{\alpha}, 2_v).} \)

**General solutions for Finsler gravity:**

The system of equations \([59]\) with three shell anisotropy (and its first shell restriction given by \([48]-[51]\)) can be solved in general form following
the results of the mentioned above Theorem 1.1 from [112]. We omit in this work such cumbersome formulas but give an explicit example of anisotropic generalization of Friedman–Robertson–Walker (FRW) solutions depending on time and three velocity coordinates in subsection 4.2.

In this subsection, we provide the general solution of equations (53)–(56) for the h–v/Cartan d–connection. It can be written for a d–metric $g = [g_{ij}, h_{ab}]$ (20) with coefficients computed in the form

$$
\begin{align*}
\epsilon_i & = \epsilon_1 \psi(x^i), \text{ for } \epsilon_1 \psi^{\bullet\bullet} + \epsilon_2 \psi'' = \Upsilon_2(x^k); \\
\epsilon_3 & = \epsilon_3 \frac{0}{h(x^i)} [f^*(x^i, v)]^2 |\varsigma(x^i, v)|, \\
\text{for } \varsigma & = \epsilon_3 \frac{0}{h(x^i)} \int dv \ U_4(x^k, v) \times f^*(x^i, v) [f^*(x^i, v) - 0 f(x^i)], \\
\text{and } h_4 & = \epsilon_4 [f(x^i, v) - 0 f(x^i)]^2,
\end{align*}
$$

$$
\begin{align*}
w_j & = 0w_j(x^i) \exp \left\{ - \int_0^v \left[ \frac{2h_3 A^*}{h_3^*} \right]_{v\rightarrow v_1} dv_1 \right\} \times \\
& \int_0^v dv_1 \left[ \frac{h_3 B_j}{h_3^*} \right]_{v\rightarrow v_1} \exp \left\{ - \int_0^{v_1} \left[ \frac{2h_3 A^*}{h_3^*} \right]_{v\rightarrow v_1} dv_1 \right\}, \\
n_i & = 0n_i(x^k) + \int dv h_3 K_i;
\end{align*}
$$

Such solutions with $h_3^* \neq 0$ and $h_4^* \neq 0$ are determined by generating functions $f(x^i, v), f^* \neq 0$, and integration functions $0 f(x^i), 0 h(x^i), 0 w_j(x^i)$ and $0 n_i(x^k)$; the coefficients $A$ and $B_j, K_i$ are determined by formulas (57).

### 3.2 Principles of general relativity and Finsler gravity

The goal of this section is to show how the theoretic scheme and principles for GR can be naturally extended to a class of metric compatible Finsler gravity theories.

#### 3.2.1 Why classical and quantum gravity theories can be modelled as Finsler gravity and generalizations?

At present, all classical gravitational phenomena are completely described the standard GR (one may be some exceptions for dark energy and

\footnote{we can prove that such coefficients determine general solutions by straightforward computations being similar to those for the canonical d–connections, see details in Refs. [120, 112]}

30
dark matter theories; we shall discuss the problem in section 4. However, it is generally accepted that the incompatibility between GR and quantum theory which should be treated in a more complete theory of quantum gravity (QG) which is under elaboration. It is supposed also that from any general theory (string/brane gravity, commutative and noncommutative gauge or other generalizations, various quantum models etc) GR is reproduced with small corrections in the classical limit. Such small corrections necessarily violate the equivalence principle in GR [33]. This also results in violation of local Lorenz invariance both in special relativity (SR) and GR (we explained in section 2.1 how nonlinear dispersion relations from any QG model result in Finsler type metrics depending on velocity/momentum variables). A general conclusion is that any model of QG must incorporate modifications of GR.

In quasi–classical limits of gravity theories, there are analyzed possible violations of the (universality) free falls of point particles and gravitational redshift (for light). For instance, this can be derived from scalar–tensor models related also to low energy limits of string theory. There are also very general constructions related to metric–affine and gauge models of gravity, including additional dynamical tensor fields like torsion and nonmetricity, spontaneous symmetry breaking scenarios in string theories and/or kinematical level violations of Lorentz invariance, see details and references in [76, 56, 41, 33]. In connection to such approaches, the Finsler scheme is considered to be an alternative one when the spacetime metric is generalized to a Finsler one [52].

We argue that Finsler geometry is not only one example explaining possible violations of the equivalence principle and local Lorentz invariance, but provides a general geometric formalism incorporating all other schemes. Let us provide some details and motivations. Finsler like nonlinear line elements (3) are derived from very general arguments on nonlinear dispersion for any broken local Lorentz invariance. Nevertheless, the form for such a Berwald–Moore Finsler metric, or any other ones (of Rindler, Kropina, Asanov, Bogoslovsky types etc, see a number of examples in Refs. 57, 64, 11, 74, 8, 19, 20, 26, 28, 29, 47, 48, 59, 123), is not preserved if we consider frame/coordinate transforms of models with small violations derived from Einstein spaces with general covariance on ”base” manifold $M$. Instead of a Finsler Hessian $F g_{ab}(x, y) = \frac{1}{2} \frac{\partial F^2}{\partial y^a \partial y^b}$ (4), and any lift to a Finsler metric $F g(x, y)$ (19), we have to work with a general metric $g^{\alpha \beta}(x, y)$ (20) on $TM$. Such geometric models are not obligatory homogeneous as those defined by a fundamental Finsler function $F(x, y)$ but always can be related to some
homogeneous ones via nonholonomic transforms. Various types of generalized metrics \( g_{\alpha\beta}(x, y) \) can be derived in low energy limits of string theory \([89, 90]\); noncommutative geometry and gravity \([92, 107]\); various deformation quantization, A–brane, nonholonomic loop and gauge like models of QG \([96, 97, 104, 105, 4, 110, 111]\); exact solutions with generic off–diagonal metrics and/or generalized connections in Einstein gravity and generalizations \([95, 120, 112]\); nonholonomic Ricci flow theory and Riemann–Finsler geometries \([103, 94, 107]\) and in geometric mechanics based on generalized Lagrange–Finsler and Hamilton–Cartan geometries \([63, 64, 61, 62]\).

In another turn, prescribing nonholonomic distributions and associated frame structures, Finsler like variables can be introduced on Einstein manifolds, on tangent bundles, or any general type of metric–affine, generalized Eisenhart (with nonsymmetric metrics), nonholonomic Riemann–Cartan and other spaces \([120, 98, 108, 109, 102, 101]\). For applications of classical and quantum geometry methods in generalized gravity theories modelled on a nonholonomic manifold \( V \), on \( TM \) or its dual \( T^*M \), we can work with a Finsler metric \( Fg(x, y) \) \([19]\) or with a general one \( g_{\alpha\beta}(x, y) \) \([20]\) and/or certain particular ansatz.

As a matter of principle, any geometric/physical model with a Finsler type \( Fg \) can be equivalently re–defined into a similar one with \( g \), or inversely. The main difference between the Finsler and ”non–Finsler” geometry/gravity theories on nonholonomic bundles/manifolds is that in the first case we have to adapt all geometric constructions to a ”new” fundamental geometric object, i.e. to the N–connection structure \( N \) \([18]\). Such an object was not considered in standard approaches to general relativity and for (pseudo) Riemannian spacetimes (event it is possible to define \( N \)–connections on Einstein manifolds endowed with off–diagonal metrics and prescribed nonholonomic \( 2 + 2 \) splitting \([95, 101, 120]\)). We emphasize that any generating Finsler function \( F(x, y) \) induces canonically a \( N \)–connection \( FN^\alpha_i = cN^\alpha_i \) \([16]\) which via general frame transforms can be related to a general one, \( N \). A Finsler like gravity theory is distinguished by existence of a \( N \)–connection fundamental object to which all constructions are adapted.

### 3.2.2 Minimal extensions of Einstein gravity to Finsler theories

Different researches in geometry and physics work with different concepts of Finsler space. In order to avoid ambiguities, let us state what rules we follow in this paper in order to elaborate a class of Finsler gravity theories which seem to be most closed to the modern paradigm of standard physics.

A (pseudo) Finsler geometry model is canonically defined by a (funda-
mental Finsler function $F(x,y)$ on a N–anholonomic manifold $V$ (in particular, on a tangent bundle $TM$) with prescribed h–/ v–splitting. Such a model is completely defined on $TV$ if there are formulated certain geometric principles for generating by $F$, in a unique form, a triple of fundamental geometric objects $(^F N, ^F g, ^F D)$. Via frame transforms, any such triple transforms into a general one $(N, g, D)$, with a decompositions $\nabla = D + Z$, when a $d$–connection $D$, a Levi–Civita connection $\nabla$ and a distorting tensor $Z$ are uniquely determined by $N$ and $g$. Inversely, we can always introduce Finsler variables (with left up $F$–label) for any data $(N, g, D)$. For simplicity, we shall omit hereafter the label $F$ is that will not result in ambiguities.

There are two general classes of Finsler geometries: 1) metric compatible, which can be included into (or minimally extending) standard theories, when $Dg = 0$, and 2) metric noncompatible (generating nonstandard models), when $Dg = Q \neq 0$.

For any given $F$, and/or $g$, there is a unique canonical $d$–connection $\hat{D}$ (A.3) when $\hat{D}g = 0$ and $h$- and $v$–torsions vanish but the general nonzero torsion $\hat{T}$ is induced by $N$ and $g$. On spaces of odd dimension, $\hat{D}$ can be transformed into the $h$– $v$ d–connection $\tilde{D}$ is determined by a couple of coefficients $\tilde{\Gamma}^\alpha_{\beta\gamma} = \left(\tilde{L}^a_{bk}, \tilde{C}^a_{bc}\right)$ (52). Via frame transforms, we can relate this $d$–connection to the Cartan $d$–connection $^C D = \{^C L^i_{jk}, ^C C^a_{bc}\}$ (30) for Finsler geometry, which also defines a canonical almost symplectic d–connection, see details in [57, 64, 101, 104]. We can work on a nonholonomic space $V$ with any $D, \hat{D}$ and/or $^C D$ in N–adapted form when all geometric constructions are similar to those for the Levi–Civita connection $\nabla$. Here we emphasize that in Finsler geometry the connection $\nabla$ does not play a preferred role (like in Riemannina geometry) because it is not adapted to the N–connection structure.

The key issues for elaborating a Finsler generalization of Einstein gravity are to introduce on $V$ (in particular on $TM$) a metric $g$ with pseudo–Euclidean signature and to decide what type of metric compatible $d$–connection $D$ will be used for postulating the field (Einstein–Finsler) equations. For physically viable Finsler gravity theories, any generalized Finsler fundamental geometric objects $g$ and $D$ should contain as particular cases certain Einstein gravity solutions. We proved some important results [112, 95, 120, 101] that for $\hat{D}$ and/or $\tilde{D} \leftrightarrow ^C D$, the Einstein equations can be integrated in very general forms. Imposing the “zero torsion” constraints (36), when $\hat{D} \to \nabla$, we restrict the integral varieties to define general solutions in Einstein gravity and its higher dimension generalizations.
3.2.3 Principles of Einstein–Finsler relativity

Finsler configurations in general relativity:

Finsler variables and the canonical d–connection $\hat{D} = \nabla - \hat{Z}$ (29), (similarly, $\tilde{D}$ and/or $\hat{\mathcal{D}}$) can be introduced in general relativity if nonholonomic 2+2 splitting are considered for a nonholonomic pseudo–Riemannian spacetime $\mathbf{V}$ [101, 104]. All geometric and physical objects and fundamental equations can be re–expressed in terms of $\hat{D}$ and N–adapted variables. Such a formal Finsler gravity satisfies all axioms introduced for the Einstein gravity theory. So, alternatively to well known tetradic, spinor, Ashtekar and other variables in general relativity, we can introduce nonholonomic/Finsler variables. The distorting tensor $\hat{Z}$ define a ”nonholonomic” source like we postulated in (35). To construct generic off–diagonal solutions in general relativity is convenient to work with $\hat{D}$ and/or $\tilde{D}$. Certain models of Fedosov deformation quantization and/or A–brain quantization of Einstein gravity can be performed using almost Kähler/Finsler variables with $\hat{\mathcal{D}}$ [?, 96, 97, 104].

Minimal Finsler extensions on $TM$ of the standard model

Special relativity and Finsler metrics: The concept of flat Minkowski spacetime, with pseudo–Euclidean signatures, and postulates for SR are related to the Maxwell electromagnetic field theory. They where formulated following Michelson–Morley type experiments with constant speed of light. The most important symmetry is that of Lorentz (pseudo–rotation) and Poincare (with translations) invariance with respect to certain linear group transforms. Locally, such fundamental properties are considered in GR.

Any possible contributions from QG result in nonlinear dispersions for light rays of type (5) and nonlinear quadratic elements (3). In order to explain such physical effects and elaborate generalized models of classical and quantum theories there were considered various generalizations/restrictions of symmetries in SR [19, 20, 21, 7, 8, 22, 48, 47, 50, 59] when the Minkowski metric $\eta_{ij} = [-1,1,1,1]$ transform into a Finsler type $g_{ij}(y)$ depending locally on velocity/momentum type coordinates $y^i$. It is not clear from general physical arguments why certain models of broken Lorentz invariance should have priorities with respect to another ones (there are experimental evidences for broken internal symmetries in particle theory but not of space–time symmetries). Perhaps, in a ”minimal way” we can say that $\eta_{ij} \rightarrow g_{ij}(y)$ is similar in some lines with generalizations of SR to GR, $\eta_{ij} \rightarrow g_{ij}(x)$, but in our case we may have an additional curvature determined by fibers of a
The co/tangent bundle, in general, by metrics of form \( g_{ij}(x, y) \). Such a metric should be a solution of the Einstein–Finsler equations \((44)\) and may possess some nonholonomically deformed Lorentz symmetries, with local nonlinear dependence on \( y^i \).

Our main conclusion is that Finsler like modifications of SR result not only in certain violations of local spacetime symmetries (with new types of group/algebras transforms etc derived from certain kinetic or symmetry arguments) but request also new classes of spacetime geometries (with a nontrivial N–connection and d–connection and related Riemann curvature and Ricci tensor depending on fiber variables \( y^i \)). We get from SR a Finsler type gravity model on (co) tangent bundle even we try to keep the base manifold \( M \) to be a Minkowski spacetime.

**Generalized equivalence principle:** In Newtonian theory of gravity, the experimental data show that the gravitational force on a body is proportional to its inertial mass. This supports a fundamental idea that all bodies are influenced by gravity and, indeed, all bodies fall precisely the same way in gravitational fields. Because motion is independent of the nature of the bodies, the paths of freely falling bodies define a preferred set of curves in spacetime just as in special relativity the paths in spacetime of inertial bodies define a preferred set of curves, independent of the nature or the bodies.

The world lines of freely falling bodies in a gravitational field are simply the geodesics of the (curved) spacetime metric. This suggests the possibility of ascribing properties of the gravitational field to the structure of spacetime itself. Because nonlinear dispersions from Minkowski spacetime can be associated to metrics of type \( g_{ij}(y) \), and GR to metrics of type \( g_{ij}(x) \), we can consider a generalized equivalence principle on Finsler spacetimes with metrics of type \( F g_{ij}(x, y) \). We may preserve the ideas of Universality of Free Fall and the Universality of the Gravitational Redshift in a Finsler type spacetime modelled by data \((N, g, D)\). In such a locally anisotropic spacetime, the paths of freely falling bodies are not usual geodesics but certain nonlinear (semi–spray) ones which are different from auto–parallels of \( D \), see details on such geometries in \([64, 11]\) and references therein. This is not surprising because a Finsler geometry is with nonholonomically constrained (gravitational) variables.

Working with metric compatible d–connections completely determined by the metric and N–connection structures, we can establish a 1–1 correspondence between one type of preferred curves (semi–sprays) and auto–parallels of \( D \). This way we can encode equivalently the experimental (curvature de-
viation) data with respect to both types of congruences. In all important physical equations for a Finsler gravitational and matter fields, the connection $D$ (for canonical constructions, it is used $\tilde{D}$, $\hat{D}$ and/or $\tilde{cD}$) is contained. Such a $d$-connection can be used for constructing the Dirac, d’Alambert and other important operators which allows us to compute the light and particle propagation in a Finsler spacetime. We have deviations from the SR and GR geometries because of nonholonomic constraints and dependencies of geometric/physical objects on ”velocity” type coordinates.

**Generalized Mach principle:** The Einstein gravity theory was formulated using a second much less precise set of ideas which goes under the name of Mach’s principle. In SR and in pre–relativity notions of spacetime, the geometric structure of spacetime is given once and for all and is unaffected by the material bodies that may be present. In particular, the properties of “inertial motion“ and “non–rotating“ are not influenced by matter in the universe. Mach and a number of earlier philosophers and scholars (for instance, Riemann considered both types of quadratic and nonlinear, Finsler type, line elements) found this idea unsatisfactory. Mach supposed that all matter in the universe should contribute to the local definition of ”non–acceleration” and ”non–rotating”. Einstein accepted this idea and was strongly motivated to formulate a theory where, unlike SR, the structure of spacetime is influenced by the presence of matter. In GR, such purposes were achieved only partially. Perhaps, the constructions related to osculating Finsler geometry outlined in section 2.1.3 see details in Refs. [51, 50], go more far to models of curved spacetime ether with a more sophisticate intrinsic structure. Nevertheless, even in that approach the ideas of Mach principle are not completely realized in a gravity theory being not involved into constructions the N–connection and d–connection structures. With respect to Finsler gravity theories on (co) tangent bundles derived from quantum nonlinear dispersion we can consider a generalized Mach principle that quantum energy/motion should contribute to spacetime, i.e. the structure of spacetime is influenced by the presence of quantum world. This influence is encoded both into the nonholonomic structure and via coefficients of $(N, g, D)$ into energy–momentum tensors for matter fields imbedded self–consistently in spacetime ether with moving coordinates $y^a$.

**Einstein–Finsler spacetimes and gravitational equations:** New theories of locally anisotropic spacetime and gravitation state the following: The intrinsic, observer–independent, properties of Finsler spacetime are described by a Finsler generating functions which canonically determine the N–connection, d–metric and d–connection fundamental geometric
objects in a metric compatible form as in GR but for N–adapted constructions. However, a Finsler spacetime metric need not have a standard form with Hessian parametrization, see [1] and [19] (let say of Berwald–Moor, Kropina, Randers, Bogoslovsky or other types), it has in Finsler geometry [64, 11, 57, 77, 15]. Via general frame/coordinate transforms such Finsler metrics, and various types of linear connections, transform from one to another. We define a Finsler gravity model and its fundamental gravitational equations on a N–anholonomic manifold \( \mathcal{V} \), including GR and SR (as certain particular classes of solutions) following the same principles as in Einstein theory but in N–adapted form for a fixed canonical metric compatible d–connection (a Finsler d–connection) which is different from the Levi–Civita connection.

Confronting Finsler theories with high precision experiments in gravity, it is considered that Finsler type deviations from the Minkowski metrics of quantum origin are restricted with an accuracy of \( 10^{-16} \) [52, 124] even QG seems to be positively of Finsler type. It is not clear yet how such restrictions may be related to modern dark energy and dark matter problems (even there is a series or works on Finsler cosmology based, in the bulk, on metric noncompatible d–connections [5, 9, 26, 27, 28, 29, 55], see also some metric compatible, or almost compatible, approaches in [117, 50, 86]). For Finsler like cosmological configurations in general relativity [114, 101] such restrictions do not exist.

**Principle of general covariance:** In GR, this is a natural consequence from that fact that spacetime models are constructed on (pseudo) Riemannian manifolds. So, the geometric and physical constructions do not depend on frames of reference (observers) and coordinate transforms. In definition of Finsler geometry models the concept of manifold is also involved (in certain approaches such manifolds are tangent/vector bundle spaces). So, the principles of general covariance has to be extended on \( \mathcal{V} \), or \( TM \). We can introduce certain preferred systems of reference and adapted coordinate transforms when a fixed \( h–v \)-decomposition is preserved/distinguished but this is a property of some particular classes of solutions of the Einstein–Finsler equations. In general, we can not distinguish between triples of data \((F_N, F_g, F_D)\) and \((N, g, D)\). We can use any parametrizations of Finsler data which are necessary for certain construction in a model of classical or quantum gravity. In an extended for Finsler spaces principle of generalized covariance (for instance, for the canonical d–connection), there are included distortion relations of type \( \hat{D} = \nabla - \hat{Z} \) (29). So, we can describe geometrical and physical models equivalently both in terms of \( \hat{D} \) and
because such connections are defined by the same metric structure.

**The equations of motion and conservation laws:** The conservation law \( \nabla_i T^{ij} = 0 \) is a consequence of the Bianchi relations and involve the idea that in GR the Einstein’s equations alone actually implies the geodesic hypothesis (that the world lines of test bodies are geodesics of the spacetime metric). This demonstrates an important self-consistency of Einstein’s equation with the basic principles for constructing the general relativity theory. Note however, that bodies which are “large” enough to feel the tidal forces of the gravitational field will deviate from geodesic motion. Such deviations may be caused by certain nonholonomic constraints on the dynamics of gravitational fields. The equations of motion of such bodies in GR also can be found from the condition \( \nabla_a T^{ab} = 0 \).

For a Finsler d–connection \( D \), even if it is metric compatible, \( D_\alpha \Upsilon^{\alpha\beta} \neq 0 \), which is a consequence of non–symmetry of the Ricci and Einstein d–tensors, see explanations for formula (28) and generalized Bianchi identities. Such a property is also related to nonholonomic constraints on the dynamics of Finsler gravitational fields. It is not surprising that the “covariant divergence” of source does not vanish even for \( \hat{D}, \tilde{D} \) and/or \( cD \). Using distorting relations of type \( \hat{D} = \nabla - \hat{Z} \) (29), we can always compute \( \hat{D}_a \Upsilon^{\alpha\beta} \) from \( \nabla_i T^{ij} \) for matter fields moving in a canonical Finsler spacetime following principles minimally generalizing those for general relativity as we explained above. In this case, the conservation law are more sophisticate by nonholonomic constraint but nevertheless it is possible to compute effective nonholonomic tidal forces of locally anisotropic gravitational fields when auto-parallels of \( \hat{D} \) deviated from nonlinear geodesic (semi–spray) configurations.

**Axiomatics for the Einstein–Finsler gravity:** A constructive–axiomatic approach to GR was proposed in 1964 by J. Ehlers, F. A. E. Pirani and A. Schild [37] (the so–called EPS axioms). In a series of publications in the early 1970’s and further developments, see original results and references in [35, 36, 73, 125, 32, 69, 82, 83, 80, 81, 10, 65], it was elaborated the concept of EPS spacetime as a physically motivated geometric model of spacetime geometry. Within that framework the spacetime geometry was derived from a small number of assumptions on the propagation of light rays and on the motion of test particles. That axiomatic approach led to a common belief that the underlying geometry of the spacetime can be only pseudo–Riemannian which lead to the paradigmatic concept of ”Lorentzian 4–manifold” in GR.

An axiomatic approach to Finsler gravity theory was proposed in [70, 71]; it was formulated also a minimal set of axioms for Finsler geometry [57].
Tavakol and N. Van den Berg [87] have shown that physically interesting Finslerian spacetimes can exist in which the EPS axioms hold identically (a recent analysis of EPS for Finsler relativity is provided in [3]).

Among the conclusions of work [87] we find "... in principle many of the assumptions made here can be dropped. Even though the resulting Finsler spaces will no longer satisfy EPS conditions identically, nevertheless it would be interesting to study them as examples of even more general geometrical frameworks". In another turn, a rigorous analysis of Finsler like gravity theories, including recent developments and possible applications in modern cosmology and astrophysics, is important to be performed from the viewpoint of EPS axiomatic system and compatibility with experimental data and modern standard high energy theories.

We consider that it is not possible to elaborate a general EPS type system for all Finsler gravity theories and generalizations. Such constructions are very general ones, with nonmetricity and nonstandard physical models. For Einstein–Finsler spaces, the EPS axioms can be extended, for instance, on $TM$ when $\overline{\mathbf{D}}$ and/or $\mathbf{D}$ are used for definition of auto–parallels and propagation of light on nonholonomic manifolds.

4 Accelerating Cosmology as Finsler Evolution

In this section we show that the acceleration expansion of the present matter–dominated universe may be generated along with the evolution of Finsler space in velocity type dimensions. A diagonal approximation for solutions of Einstein–Finsler equations will be analyzed quantitatively so as to exhibit explicitly some patterns of the accelerating expansion scenario. Then, two examples of exact off–diagonal solutions associated with cosmological evolution scenarios will be constructed and discussed. We shall prove that solitonic nonholonomic deformations induced by velocity type variables can modify scenarios of acceleration in real Universe (with usual four dimensional spacetime coordinates).
4.1 Diagonal accelerating Finsler universes

We consider a prime metric of a \((n + 1 + 3)\)-dimensional spacetime, with \(n = 4\) and \(m = 1 + 3\), with time like coordinate \(y^5 = t\),

\[
\sigma g = \varepsilon_1 dx^1 \otimes dx^1 + \frac{h_a^2(t)}{1 - h_k(h_r)^2} d h_r \otimes d h_r + h_a^2(t)(h_r)^2 d h \theta \otimes d h \theta \\
+ h_a^2(t)(h_r)^2 \sin^2 h \theta d h \varphi \otimes d h \varphi - dt \otimes dt + \frac{v_a^2(t)}{1 - v_k(v_r)^2} d v_r \otimes d v_r \\
+ v_a^2(t)(v_r)^2 d v \theta \otimes d v \theta + v_a^2(t)(v_r)^2 \sin^2 v \theta d v \varphi \otimes d v \varphi.
\]

This diagonal ansatz is a particular case of (45) with coordinates \(x^i, y^a\), for \(i, j, \ldots = 1, 2, 3, 4\) and \(a, b, \ldots = 5, 6, 7, 8\), and coefficients of metric \(\sigma g_{\alpha \beta} = \text{diag}(\sigma g_i, \sigma h_a)\) parametrized in the form:

- spherical \(h\)-coordinates: \(x^1 = x^1, x^2 = h_r, x^3 = h \theta, x^4 = h \varphi\);
- spherical \(v\)-coordinates: \(y^5 = t, y^6 = v_r, y^7 = v \theta, y^8 = v \varphi\);

and

\[
\sigma g_1 = \varepsilon_1 = \pm 1, \quad \sigma g_2 = \frac{h_a^2(t)}{1 - h_k(h_r)^2}, \\
\sigma g_3 = h_a^2(t)(h_r)^2, \quad \sigma g_4 = h_a^2(t)(h_r)^2 \sin^2 h \theta, \\
\sigma h_5 = -1, \quad \sigma h_6 = \frac{v_a^2(t)}{1 - v_k(v_r)^2}, \\
\sigma h_7 = v_a^2(t)(v_r)^2, \quad \sigma h_8 = v_a^2(t)(v_r)^2 \sin^2 v \theta,
\]

for trivial \(N\)-connection coefficients, \(\sigma N_i^a = 0\). In this case, \(\hat{\nabla} = \nabla\).

4.1.1 Diagonal cosmological equations for pseudo–Finsler metrics

The metric (45), for \(\varepsilon_1 = 0\), describes two types (conventional horizontal and vertical, spherical ones) evolutions with time variable \(t\) of two universes with respective constant curvatures \(h_k\) and \(v_k\). To derive cosmological solutions in a most simple form is convenient to consider that the \(h\)-subspace is of "velocity" type with radial coordinate \(0 < h_r < 1\) taken for the light velocity \(c = 1\). The coefficients \(\sigma h_a\) define a usual FRW type metric in \(v\)-subspaces. The values \(h_a^2(t)\) and \(v_a^2(t)\) are respective \(h\)- and \(v\)-scale factors. In such models, the \(h\)-coordinates are dimensionless (we can introduce a standard dimension, for instance, by multiplying on Planck length) and the \(v\)-coordinates are usual ones, with dimension of length.

Assuming that the matter content in this pseudo–Finsler spacetime is taken to be a perfect fluid, we can write the Einstein equations (48 )–(51) in
local coordinate base,

\[ 4 \frac{v a^* h a^*}{v a} h a^* + 2 \left( \frac{h a^*}{h a} \right)^2 + \frac{h k}{(h a)^2} \right] + \left[ \frac{(v a^*)^2}{v a} + \frac{v k}{(v a)^2} \right] = \frac{8}{3} \pi G \rho. \]  

\[ 4 \frac{h a^{**}}{h a} + 2 \frac{v a^{**}}{v a} + 6 \left( \frac{h a^*}{h a} \right)^2 + \frac{h k}{(h a)^2} \right] + \left[ \frac{(v a^*)^2}{v a} + \frac{v k}{(v a)^2} \right] = -8 \pi G v p. \]

\[ \frac{h a^{***}}{h a} + \frac{v a^{***}}{v a} + 2 \left( \frac{h a^*}{h a} \right)^2 + \frac{h k}{(h a)^2} \right] + \left[ \frac{(v a^*)^2}{v a} + \frac{v k}{(v a)^2} \right] = -\frac{8}{3} \pi G h p. \]

In the above formulas, the right "dot" means derivative on time coordinate \( t \) and \( G \) and \( \rho \) are respectively the formal gravitational constant and the energy density in the total (tangent) space. The values \( h p \) and \( v p \) are, correspondingly, the pressures in the \( h \)- and \( v \)-spaces. We assume simple equations of matter states of type \( h p = h \omega h \rho \) and \( v p = v \omega v \rho \) for some constant state parameters \( h \omega \) and \( v \omega \). The conservation law \( \nabla \alpha T^{\alpha \beta} = 0 \) for \( T^{\alpha \beta} = \text{diag}[ h \rho, h p; \ldots; v \rho, v p; \ldots] \) gives rise to

\[ \rho \approx h a^{-4(1 + h \omega)} \times v a^{-3(1 + v \omega)}. \]

For simplicity, we may assume \( h k = 0 \) and study the evolution of the scale factors \( h a(t) \) and \( v a(t) \) following certain approximations in equations \((62)\).

### 4.1.2 Diagonal scale evolution and velocity type dimensions

In general, a Finsler gravity dynamics is with generic off–diagonal metrics and generalized connections. Such nonlinear systems may result in non–perturbative effects and instability even for small off–diagonal metric terms etc. Nevertheless, for some nonolonomic configurations, we can compute some diagonal contributions into the "real" Universe metric from "velocity" type coordinates.
Radiation–dominated diagonal Finsler universe:

We define such an universe following conditions \( h_P = 0 \) and \( v_P = \frac{1}{3} v_P \), when \( h_a = \text{const} \) is accepted as a solution. For such configurations, the third equation in (62) is a consequence of the first two ones when \( v_a(t) \) must be a solution of the system

\[
\left( \frac{v_a^*}{v_a} \right)^2 + \frac{v_k}{v_a^2} = \frac{8}{3} \pi G \rho,
\]

\[
2 \frac{v_a^{***}}{v_a} + \left( \frac{v_a^*}{v_a} \right)^2 + \frac{v_k}{v_a^2} = -\frac{8}{3} \pi G \rho.
\]

The source \( \frac{8}{3} \pi G \rho \) of such equations is determined by generalized gravitational constant \( G \) and \( \rho \) matter density in total spacetime. By straightforward computations, we can show that the constant \( h_a \)-solution is stable under small perturbations of scale factors \( h_a(t) \) and \( v_a(t) \). This means that we can retrieve the ordinary evolution of radiation–dominated Finsler universe with a total spacetime model. Here we note that for a matter–dominated Finsler configuration with \( h_P = v_P = 0 \) there is not a solution with \( h_a = \text{const} \) unless \( \rho = 0 \).

Matter–dominated diagonal Finsler universe:

There are solutions as in the standard FRW cosmology (in our case, for the \( v \)-part) with \( h_a = \text{const} \), when the matter in the ”velocity” space provides negative pressure \( h_P = -\frac{1}{2} \rho \), when \( v_P = 0 \). Such conditions may be realistic if we associate point like non–relativistic particles in \( v \)-space certain extended objects (let say, strings) with additional velocity variables when the pressure is provided in such a strange manner. The Friedman–Finsler equations (62) transform into

\[
\left( \frac{v_a^*}{v_a} \right)^2 + \frac{v_k}{v_a^2} = \frac{8}{3} \pi G \rho,
\]

\[
2 \frac{v_a^{***}}{v_a} + \left( \frac{v_a^*}{v_a} \right)^2 + \frac{v_k}{v_a^2} = 0,
\]

which allows us to find general solutions for \( v_a(t) \).

Different extension rates in \( h \)– and \( v \)–subspaces:

For simplicity, we can assume \( h_k = v_k = 0 \) and that some constants \( h_\omega \) and \( v_\omega \) determine \( h_P = h_\omega h_P \) and \( v_P = v_\omega v_P \), i.e. the equations of
states in a matter like dominated Finlser universe. The difference between the \( v \)- and \( h \)- expansion rates is expressed as

\[
\beta(t) := \left(1 - 3 v \omega + 2 h \omega\right) v a - \left[1 + 3 v \omega - 4 h \omega\right] h a,
\]

where \( v a \) and \( h a \) are the volume of \( (3+4) \)-dimensional space like total pseudo–Finsler subspace if signature \( \varepsilon_1 = 1 \) in (62). It follows from this formula that the difference \( \beta(t) \) decreases (grows) as the volume \( v a \) grows (decreases). Here we note that \( h a \) has a limit corresponding to the maximal velocity of light.

Similarly, it is possible to consider the difference between \( v \)- and \( h \)-expansions \( \beta(t) \) for a radiation–dominated Finsler universe with \( h p = \frac{1}{3} \rho \) and \( v p = 0 \),

\[
\beta(t) := 2 h a \approx \frac{1}{(v a)^{3/4} (h a)^{1/4}} \approx \frac{1}{V o l_{3+4}}.
\]

We conclude that if \( v a \) is growing, the expansion rate of the \( h \)-spaces drops to zero. So, the constant \( h a \) solution is stable for the radiation dominated Finsler universe.

It is possible to consider a more general matter dominated Finsler universe with \( h \omega = v \omega \neq \frac{1}{3} \) when

\[
\beta(t) \approx \frac{v a}{v a} - \frac{h a}{h a} \approx \frac{1}{(v a)^{3/4} (h a)^{1/4}} \approx \frac{1}{V o l_{3+4}}.
\]

If the total volume \( v a \) is growing, the expansion rates of the \( h \)- and \( v \)-spaces tend to approach each other, i.e. the limited \( h \)-volume, because of finite speed of light, limits the three–space in the \( v \)-part. If for an inverse decreasing of \( v a \), with one expanding and another collapsing subspaces, then \( |v a^*/v a| \), or \( |h a^*/h a| \), becomes large and larger. This results in an accelerating expansion. For collapsing \( h \)-space with velocity types coordinates we can induce an accelerating expansion of ”our” inverse modelled by this pseudo–Finsler model as the \( v \)-subspace. We analyze below more details on such models of Finsler–acceleration.

4.1.3 Accelerating diagonal expansion generated by Finsler evolution

We explore analytically the possibility to generating Finsler type accelerating expansions via evolution of \( h \)-space with velocity coordinates. For
simplicity, we consider $h_k = v_k = 0$ and trivial equations of states with $h_{\bar{P}} = v_{\bar{P}} = 0$. For such conditions, the last two equations in (62) became

\begin{align*}
\dot{v}H + \frac{5}{2} (vH)^2 + 2 \dot{v}H - \left(\frac{hH}{vH}\right)^2 & = 0, \quad (63) \\
\dot{h}H - \frac{1}{2} (vH)^2 + \dot{v}H \frac{hH}{vH} + 3 \left(\frac{hH}{vH}\right)^2 & = 0,
\end{align*}

where the respective effective Hubble $h$- and $v$-“constants” are $h_H := h_{a^*}/h_a$ and $v_H := v_{a^*}/v_a$. These equations impose the corresponding conditions for accelerating ($v_{a^{**}}/v_a > 0$), or decelerating ($v_{a^{**}}/v_a < 0$) of ”our” three dimensional $v$–subspace,

acceleration: $h_H > \left(1 + \sqrt{\frac{5}{2}}\right) h_H := +H h_H$,

or $h_H > \left(1 - \sqrt{\frac{5}{2}}\right) h_H := -H h_H$;

deceleration: $-H v_H < h_H < +H v_H$.

To investigate the correlation between $h$– and $v$–subspaces is useful to introduce the fraction–function $\gamma(t) := \frac{hH}{vH}$ and see the behavior of $d\gamma/dt$ for different values of $\gamma$ and some ”critical” values of this function, which for our dimensions $n = 4$ and $m = 1 + 3$ are defined

attracting: $attH := -1 + 1/\sqrt{2}$; repelling: $repH := -1 - 1/\sqrt{2}$.

It is always satisfied the condition

$attH < -H < repH < 0 < 1 < +H$,

i. e. there are two ”attractors” determined by $\gamma = attH$ and $\gamma = 1$ and one ”repeler” for $\gamma = repH$. From equations (63), one follows the conditions

$\gamma^* > 0$, for $\gamma < attH$, $repH < \gamma < 1$;

$\gamma^* < 0$, for $attH < \gamma < repH$, $\gamma > 1$.

For Finsler universes, there are four kinds of evolution processes depending of a initial value $\gamma = ^{o}\gamma$ :

acceleration and, then, deceleration, $^{o}\gamma > +H$;
always deceleration, $repH < ^{o}\gamma < +H$;

deceleration and, then acceleration, $-H < ^{o}\gamma < repH$;
always acceleration, $^{o}\gamma < -H$.

\textsuperscript{18}Models with nonzero $h_k$ and/or $v_k$ offer a number of interesting possibilities. In the next subsections, we shall investigate examples with nontrivial N–connection and Riemannian and scalar curvatures.
A realistic for our universe is the third condition above, when \(-H < ^{\circ}\gamma < repH\). Such a scenario states that initially the Finsler universe is in the region \((-H, repH)\) when the \(h\)-space collapses and our three dimension space in the \(v\)-part decelerates. As \(\gamma\) passes \(-H\) in the collapsing process of "velocity" \(h\)-coordinates, our "real" three dimensional space begins to accelerate.

More realistic scenarios can be modelled by numeric methods, or by deriving analytically more general solutions with off–diagonal interactions (see next subsection).

4.2 Off–diagonal anisotropic Finsler acceleration

More realistic Finsler type cosmological models can be elaborated for generic off–diagonal metrics and with nontrivial N–connection and canonical/ Cartan d–connections. We study below such generalizations.

4.2.1 Examples of off–diagonal cosmological solutions

We construct in explicit form two classes of such solutions defining certain models of four dimensional, 4-d, and 8-d Finsler spacetimes.

A pseudo–Finsler 4-d off–diagonal toy cosmology:

Let us consider a four dimensional (4-d) metric\[\tilde{g}_2 = h a^2(t) \, d^4h \quad \tilde{g}_3 = -1, \tilde{h}_4 = h a^2(t) \, (h r)^2, \tilde{h}_5 = h a^2(t) \, (h r)^2 \sin^2 h \theta, \]

which is contained as a particular case of 8-d ansatz \([61]\), when \(\varepsilon_1 = 0, \, a = 0\) an, for simplicity, \(h k = v k = 0\). We use this metric for a prime cosmological model in variables \(u^\hat{a} = (h r, t, h \theta, h \varphi)\), with \(x^i = (h r, t)\) and \(y^\hat{a} = (h \theta, h \varphi)\), for \(i, j, \ldots = 2, 3\) and \(a, b, \ldots = 4, 5\) (such a model describes evolution in time \(t\) of a \(h\)- subpace for certain conditions, and sources, analyzed in subsection \([4.1]\)).

An off–diagonal anisotropic dynamics in the space of "velocities" can be modelled by nonholonomic deformations with \(\eta\)–polarizations \(\eta^\hat{i} = \eta_i(x^\hat{k})\) and \(\eta_a = \eta_a(x^\hat{k}, h \theta)\) and N–connection coefficients \(N^3_i = w_i(x^\hat{k}, h \theta)\) and \(N^5_i = v_i(x^\hat{k}, h \theta)\). For the "prime" metric

\[\tilde{g}_2 = h a^2(t), \tilde{g}_3 = -1, \tilde{h}_4 = h a^2(t) \, (h r)^2, \tilde{h}_5 = h a^2(t) \, (h r)^2 \sin^2 h \theta, \]
we define \[ \bar{\mathbf{g}} = \left[ \bar{g}_{i\bar{j}} \right] \rightarrow \mathbf{g} = \left[ g_i = \eta_i \bar{g}_{i\bar{j}} \right] \], with a "target" metric

\[ 4 d \mathbf{g} = g_i^d x^i \otimes dx^i + \bar{h}\alpha (dy^\bar{\alpha} + N_i^\bar{\alpha} dx^i) \otimes (dy^\bar{\alpha} + N_i^\bar{\alpha} dx^i), \] (66)

constrained to be a cosmological solution of equations (53)–(56) for \( \bar{D} \). For simplicity, we consider with respect to \( N \)-adapted frames a source with constant coefficients \( \bar{\alpha} = \text{diag}[\gamma_1; \gamma_2 = \gamma_3 = \text{const}; \gamma_4 = \gamma_5 = \text{const}] \) transforming for a "diagonal" limit into \( T^{\alpha\beta} = \text{diag}[h_{\bar{p}} h_{\bar{p}} ...; v_{\bar{p}} ...] \) used for generating a metric (62).

The coefficients of such a new solution (66) are of type (60) generated from (65) by polarization functions and N-connection coefficients\(^{19}\):

\[ \eta_2 = e^{\psi_0 h_{r,t}} h_{a,2}(t), \quad \eta_3 = e^{\psi_0 h_{r,t}} \text{ for } \psi_0 - \psi'' = \gamma_2; \]
\[ \eta_4 = [f(*) h_{r,t}(h_\theta)]^2 \left[ \psi_0 h_{r,t}(h_\theta) \right], \] for \( \zeta = 1 - \gamma_4 \int d^h \theta f(*) h_{r,t}(h_\theta) \left[ f(*) h_{r,t}(h_\theta) - f^0 h_{r,t}(h_\theta) \right], \]
\[ \eta_5 = [f(*) h_{r,t}(h_\theta) - f^0 h_{r,t}(h_\theta)]^2; \]

\[ w^r_j = \omega w^r_j(h_{r,t}) \exp \left\{ - \int_0^{h_\theta} \left[ \frac{2\eta_4 A^*}{\eta_2^*} \right] v_1 \right\} \times \int_0^{h_\theta} dv_1 \left[ \frac{\eta_4 B_j}{\eta_4^*} \right] v_1 \exp \left\{ - \int_0^{v_1} \left[ \frac{2\eta_4 A^*}{\eta_2^*} \right] v_1 \right\}, \]
\[ n_i = \omega n_i h_{r,t} + \int d^h \theta \eta_4 h_{a,2}(h_{r,t}) K_i, \]

where the coefficients of type (57) are computed for polarization functions,

\[ A = \left( \frac{\eta_2^*}{2\eta_4} \right)^* + \frac{\eta_5^*}{2\eta_5} \], \[ B_k = \frac{\eta_5^*}{2\eta_5} \left( \frac{\partial_k g_3}{g_2} - \frac{\partial_k g_3}{g_3} \right) - \partial_k A, \]
\[ K_2 = -\frac{1}{2} \left( \frac{g_3^*}{g_3 h_5} + \frac{g_3^*}{g_3 h_5} \right), \quad K_3 = \frac{1}{2} \left( \frac{g_3^*}{g_2 h_4} - \frac{g_3^*}{g_3 h_5} \right). \] (68)

In formulas (67) and (68), the partial derivatives are written in brief in the form \( \eta_4 = \partial \eta_4 / \partial h_\theta \), \( g_3 = \partial g_3 / \partial h_r \), \( g_3 = \partial g_3 / \partial t \) and \( f(* h_{r,t}) \), \( \omega w^r_j(h_{r,t}) \),

\(^{19}\)to simplify formulas, we chose corresponding parametrizations for generating/ integration functions.
$n_r(h_r, t)$ are integration functions to be determined by fixing some boundary/initial conditions on the dynamics of gravitational variables in the space of "velocities".

Putting together the above coefficients, we find the metric for our 4–d toy Finsler cosmological model

$$
\mathbf{g} = e^{\psi(h_r,t)} \left( d^2 h_r \otimes d^2 h_r - dt \otimes dt \right)
$$

$$
+ [f^*(h_r,t, h_\theta)]^2 \left[ \left( h_r(t) \right)^2 a^2(h_r) \right] \delta h_\theta \otimes \delta h_\theta
$$

$$
+ [\dot{f}(h_r,t) - 0 f(h_r,t)]^2 a^2(h_r) \sin^2 h_\theta \delta h_\varphi \otimes \delta h_\varphi,
$$

$$
\delta h_\theta = d^2 h_\theta + w_2(h_r,t, h_\theta) d^2 h_r + w_3(h_r,t, h_\theta) d^2 h_r,
$$

$$
\delta h_\varphi = d^2 h_\varphi + n_2(h_r,t, h_\theta) d^2 h_r + n_3(h_r,t, h_\theta) d^2 h_r,
$$

with the coefficients defined by data (67). Such a metric defined an off–diagonal Finsler inhomogeneous model in the $h$–subspace. In our "real" Universe it may contribute via a nontrivial $e^{\psi(h_r,t)}$ before time like $dt$; such a solution should be imbedded into a 8–d Finsler spacetime.

**A class of inhomogeneous off–diagonal 8–d Finsler cosmologies:**

Following the geometric method of constructing exact solutions in extra dimensional spacetime [112], we can generalize the metric (66) with coefficients (67) to define exact cosmological solutions of the system (59) for a total 8-d Finsler spacetime. We consider a source (58) with constant coefficients modeling on $h$– and $v$–subspaces perfect fluid matter/radiation with simple equations of states as we approximated in previous sections. The 8-d ansatz for such off–diagonal metrics is of type

$$
8d \mathbf{g} = \varepsilon_1 dx^1 \otimes dx^1 + \eta_1 \tilde{g} \dot{d} x^\hat{1} \otimes dx^\hat{1} + \eta_4 \tilde{h} \dot{d} x^\hat{4} \otimes (dy^4 + w_7 dx^4)
$$

$$
+ h_5(dy^5 + w_\alpha(u^\alpha, 1v)du^\alpha) \otimes (dy^5 + w_\alpha(u^\alpha, 1v)du^\alpha)
$$

$$
+ h_6(dy^6 + n_\alpha(u^\alpha, 1v)du^\alpha) \otimes (dy^6 + n_\alpha(u^\alpha, 1v)du^\alpha)
$$

$$
+ h_7(dy^7 + w_\alpha(u^{1\alpha}, 2v)du^{1\alpha}) \otimes (dy^7 + w_\alpha(u^{1\alpha}, 2v)du^{1\alpha})
$$

$$
+ h_8(dy^8 + n_\alpha(u^{1\alpha}, 2v)du^{1\alpha}) \otimes (dy^8 + n_\alpha(u^{1\alpha}, 2v)du^{1\alpha}),
$$

where coordinates and respective indices are parametrized

$$
u^\alpha = (x^1, x^\hat{1} = (h_r, t), x^4 = y^4 = h_\theta);
$$

$$
u^{1\alpha} = (u^\alpha, y^{1\alpha} = (y^5 = 1v = y^6 = h_\varphi));
$$

$$
u^{2\alpha} = (u^{2\alpha}, y^{2\alpha} = (y^7 = 2v = y^8 = h_\varphi));
$$
and $h_6 = \eta_0 h_6 + \eta_6$ (for $h_6$ known for given $h_6$ and $\eta_0 h_6$) and $n_\alpha(u^\alpha, v) = (n_\xi(x^\xi, \theta, h)\eta_0 h_6, n_\alpha(u^\alpha, v)$, if $\alpha > 3$) when the values $\eta_0 h_6, \eta_4 h_6, \eta_6 h_6$ being former $\eta_5 h_5$ in (67), $w_7$ and $n_7$ are given by coefficients of metrics (66) and (65).

From the class of general solutions, we can extract a subclass of 3-d solitonic configurations $\xi = \xi(t, h, \theta, v_r)$ from the $h$–subspace, depending on time and velocity type coordinates, inducing small perturbations in the $v$–subspace. Such anisotropic on velocities 8–d metrics are written

$$sol \mathbf{g} = \epsilon_1 dx^1 \otimes dx^1 + e^{\psi(h_r, t)} h a^2(t) d h_r \otimes d h_r + \eta_4(h_r, t, h) h a^2(t)(h_r)^2 \delta^h(\theta) \otimes \delta^h(\theta) d h_r + \eta_5(h_r, t, h) h a^2(t)(h_r)^2 \sin^2 h \delta^h \phi[\xi] \otimes \delta^h \phi[\xi]$$

$$-e^{\psi(h_r, t)} dt \otimes dt + (1 + \varepsilon \omega_5[\xi]) v a^2(t) \delta^v r \phi[\xi] \otimes \delta^v r \phi[\xi]$$

$$+ v a^2(t)(v r)^2 d v \theta \otimes d v \theta + v a^2(t)(v r)^2 \sin^2(\theta) d v \phi \otimes d v \phi,$$

where $\delta^h(\theta) = d h_r + w_2(h_r, t, h) d h_r + w_3(h_r, t, h) d t$, $\delta^v r \phi[\xi] = d v_r + \varepsilon \omega_3[\xi] d t + \varepsilon \omega_4[\xi] d h_r$, $\delta^h \phi[\xi] = d \theta + \varepsilon \omega_3[\xi] d t + \varepsilon \omega_4[\xi] d h_r$,

for values $\varepsilon, \eta_4, w_2, w_3$ and $\eta_5$ determined in formulas (67) and the coefficients depending functionally on $[\xi]$, for a 3-d solitonic function $\xi = \xi(t, h, \theta, v_r)$ (for instance, being a solution of (1.1) as we computed the end of Appendix B, see formulas (B.2)). Such 3-d solitons were considered in our works on propagation of black holes in extra dimensional spacetimes and on local anisotropic black holes in noncommutative gravity [118, 92, 113]. Solitonic configurations can be stable and propagate from the space of "velocities" into "real" universe for various models of Finsler cosmology.

For $\varepsilon \to 0$, the solutions (70) transform into the 8-d metric (61) with possible nonholonomic generalizations containing solutions of type (66) with coefficients (67).

### 4.2.2 Solitonic contribution to N–connections and anisotropic expansion and acceleration

The pseudo–Finsler cosmological model described by (70) is generically off–diagonal. The solitonic deformation $\xi$ contributes both to diagonal and off–diagonal terms of metric. We can fix a nonholonomic (co) frame of reference,

$$e^\alpha = (dx^1, d h r, \delta^h \theta, \delta^h \phi, dt, \delta^v r, d v \theta, d v \phi),$$

48
for an observer in a point \((v_{r0}, v_{\theta0}, v_{\varphi0})\), for simplicity, considering that the "velocity" space is with \(h_T = 0\), \(h_\theta\) with one anisotropic velocity \(h_\varphi\). There are two effective scaling parameters

\[
h_\tilde{a}(\tau) = (1 + \varepsilon \chi(\tau)) h_a(\tau); \quad v_\tilde{a}(\tau) = (1 + \varepsilon \varpi_5[\xi(\tau)]) v_a(\tau),
\]

when we approximate \(e^{\psi(h_{r0}, \tau)} = 1 + \varepsilon \chi(\tau)\) with the solitonic function \(\xi(\tau)\) taken for a redefined time like variable \(\tau(t)\) when \(d\tau = e^{\psi(0,t)} dt\).

Let us introduce

\[
h_{\tilde{H}}* : = h_\tilde{a}*/ h_a = h_H + \varepsilon \chi^*, \quad \text{for} \quad h_H = h_a*/ h_a,
\]

\[
v_{\tilde{H}}* : = v_{\tilde{a}}*/ v_a = v_H + \varepsilon \varpi_5^*, \quad \text{for} \quad v_H = v_a*/ v_a,
\]

where \(\chi^* = \partial \chi / \partial \tau\). The conditions \((63)\) for acceleration, in our case modified by a solitonic function both for off–diagonal and diagonal terms of metric, are redefined in the form \(h_H \rightarrow h_{\tilde{H}}, v_H \rightarrow v_{\tilde{H}}\) and \(\partial / \partial t \rightarrow \partial \tau\). This introduces additional, solitonic type, correlations between \(h–\) and \(v–\) subspaces via an additional nonholonomic deformation of fraction–function,

\[
\tilde{\gamma}(\tau) := h_{\tilde{H}} / v_{\tilde{H}} \approx \gamma(\tau) + \varepsilon (\chi - \varpi_5)^*.
\]

Conclusions about Finsler–solitonic off–diagonal acceleration, or deceleration, should be drawn from behavior of functions \(\tilde{\gamma}^*(\tau)\) and \(\tilde{\gamma}(\tau)\).

Respectively, the solitonic versions of equations \((63)\) and \((64)\) are

\[
\tilde{\gamma}^* > 0, \quad \text{for} \quad \tilde{\gamma} < h_{\text{att}} H, \quad repH < \tilde{\gamma} < 1;
\]

\[
\tilde{\gamma}^* < 0, \quad \text{for} \quad h_{\text{att}} H < \tilde{\gamma} < repH, \quad \tilde{\gamma} > 1,
\]

and (the four types of evolution processes depend on a initial value \(\tilde{\gamma} = \gamma^*\))

- acceleration and, then, deceleration, \(\gamma^* > +H\); always deceleration, \(repH < \gamma^* < +H\);
- deceleration and, then acceleration, \(-H < \gamma^* < repH\); always acceleration, \(\gamma^* < -H\).

A solitonically modified directly observable universe is \(-H < \gamma^* < repH\).

Such a condition is very sensitive with respect to possible diagonal or off–diagonal perturbations from the space of velocities. This follows from the facts that the conditions of acceleration for \(\gamma\) and \(\tilde{\gamma} = \gamma + \varepsilon (\chi - \varpi_5)^*\) are, in general, different. Small modifications proportional to \(\varepsilon (\chi - \varpi_5)^*\) may transfer, for instance, an accelerating configuration into decelerating, and inversely.
4.3 Discussion on Finsler acceleration

In this section we have investigated the scenario of producing the accelerating expansion of the present universe via evolving small velocity type diagonal and off–diagonal nonholonomic deformations of Finsler metrics. For a radiation–dominated cosmological model, such as the model of our early universe, we obtain stable configurations with static velocity type coordinates. In this case, the existence of Finsler type velocity coordinates may have no significant influence on 3–d observable space. Here we note that on the contrary, diagonal solutions with static extra dimensions does not exist for the present matter–dominated cosmologies. This way we preserve concordance between observations and current theories regarding the early universes with radiation–domination and primordial nucleosynthesis. The diagonalized metrics for Finsler type universes implies the stability of configurations in the radiation–dominated models. Certain configurations are with increasing positive expansion rate representing expansion.

There are four classes of evolution for the matter–dominated cases, as we derived from our quantitative analysis of both types diagonal and off–diagonal solutions. Cosmological models that decelerate first and than accelerate are included into the schemes. Therefore the accelerating Finsler expansion of the present universe may be described in our locally anisotropic scenario. Nevertheless, certain small solitonic type deformations from the velocity type subspace may modify substantially the character of acceleration of universe and conditions of stability or instability.

The violation of local Lorentz invariance is a general feature in theories with geometric/physical objects depending on velocity type coordinates. In general, such theories are beyond the standard model in particle physics. The transition to standard physics can be associated to some processes in the early universe, when the "velocity" component is finally compactified, or by an another nonlinear mechanism, and became almost not observable. In Einstein–Finsler theories, we conclude that velocity type extra dimensions may manifest itself as some "effective" dark energy/ matter.

In Finsler gravity, the nonholonomic geometric structure and the evolution patter of velocity type dimensions play an important role in cosmological scenarios. In this work, we studied the simplest models with Finsler type exact solutions. Other theories with more realistic and richer structure are worthy of being further investigated. Such a research will have to include also models with generic off–diagonal solutions in Finsler variables in Einstein gravity [114] and inflation and dark energy/matter theories for Einstein–Finsler spaces.
5 Discussion and Conclusions

5.1 Summary

Recently, Finsler modifications of Einstein gravity have received certain increased attention due to combined motivation coming from quantum gravity and extra dimension/ noncommutative gravity theories (i. e. theories with nonlinear dispersions and related violations of Lorentz symmetry), exact solutions with off–diagonal metrics and nonholonomic interactions and geometric mechanics and analogous gravity. There is a special interest in modern cosmology related to possible implications of Finsler geometry methods and locally anisotropic gravity theories for explaining dark energy and dark matter effects.

Among numerous alternatives to general relativity, the Finsler–Cartan like gravity theories, and related Lagrange–Hamilton generalizations, have some some important distinguished properties. The most important one is that Finsler spacetimes are with nonlinear connection (N–connection) structure, defined as a nonholonomic distribution on a vector/tangent bundle, or, for instance, on a pseudo–Riemannian manifold. For canonical constructions, three fundamental geometric objects (the metric tensor and nonlinear and linear connection structures) of a standard Finsler geometry/ gravity are induced by a fundamental/ generating function on a nonholonomic (co) tangent bundle/ manifold.

For elaborating a Finsler gravity theory, it is very important to decide for what type of linear connections (in Finsler geometry, there are used distinguished connections, d–connections) we are going to formulate the gravitational field equations. In general, there are two classes of d–connections (and, respectively, Finsler gravity theories): The first one consists from metric compatible d–connections (for instance, the Cartan d–connection and/or the canonical d–connection) and the second one is with metric non–compatible d–connections (for instance, the Chern d–connection; for non–Finsler gravity theories, such models were reviewed in [11], and for Lagrange–Finsler generalization in [120, 101]). The well known Levi–Civita connection is not the "best one" for a Finsler geometry/gravity because this connection is not adapted to the N–connection structure. Finsler type d–connections are positively with non–trivial torsion structure (excepting the Chern d–connection which instead of this is not metric compatible).

In a subclass of metric compatible Finsler gravity theories, the nontrivial torsion is canonically induced by nonholonomic deformations and related to certain generic off–diagonal coefficients of the metric tensor. Such grav-
ity models can be constructed following the same geometric and physical
principles as in the Einstein gravity, but on tangent bundles (with metrics and connections depending on "velocities") and for certain classes of
Finsler connections. We also can model locally anisotropic interactions
on (pseudo) Riemannian spaces enabled with nonholonomic distributions
with associated N–connection splitting. It is possible to formulate well
defined conditions when Finsler configurations are modelled as exact solutions
in general relativity and higher dimension generalizations, see details in
[112, 114, 92, 95, 101, 120].

In our approach, all classes of such metric compatible models of gravity,
on nonholonomic (co) tangent and/or (pseudo) Riemannian manifolds, are
called Einstein–Finsler theories. Via nonholonomic transforms/ deformations of geometric structures, and introducing nonhomogeneous generating
functions, the Einstein–Finsler constructions can be generalized to Einstein–
Lagrange ones and various modifications of gravity.

5.2 New perspectives in gravity and cosmology and Finsler
geometry methods

It seems that Finsler like gravity theories on (co) tangent bundles (with metrics and connections depending on velocity/ momentum type variables)
are natural consequences of all models of quantum gravity, see physical
arguments and a review of recent results provided in [52, 47, 48]. The
principle of general covariance in classical Einstein gravity results in very
general quantum nonlinear dispersions of Finsler and generalize Finsler type.
It is not the case to postulate from the very beginning that such a Finsler
spacetime should be of any special Randers/Berwald–Moor/Bogoslovsky –
Finsler type with fixed line elements like those, for instance, taken for in Very
Special Relativity, osculating Finsler manifold, or Berwald/ Chern gravity
models.

To elaborate a self–consistent geometric/ physical approach the most im-
portant step is when we decide what type of Finsler connections we chose for
deriving (geometrically and/or following a N–adapted variational calculus)
generalizations of Einstein equations. The second (also very important) step
is to construct exact solutions and formulate a procedure of quantization of
such a generalized Finsler gravity theory. In certain limits, we may analyze
if some classes of metrics from very special relativity, or with another type
of anisotropy, can be included in such a scheme.

There are strong arguments to conclude that the quantum gravity the-
ory is "almost sure" of generalized Finsler type, formulated on a corre-
spondingly quantized (co) tangent bundle. In certain classical limits such a quantum theory is described by nonholonomic gravity configurations on (pseudo) Riemannian–Finsler spacetimes and possible observable effects in modern cosmology and quantum physics. A brief list of items to be studied in classical and quantum gravity theories with nonholonomic distribution is:

1. We can consider generating fundamental Finsler, or Lagrange (on cotangent bundles, respectively, Cartan, or Hamilton; in general, of higher order, see Refs. [90, 91, 119, 121, 61, 62, 116, 34]) functions. Lifts, for instance, of Sasaki type, allow us to define canonical (Finsler type and generalizations) metric, \( F_g \), and N–connection, \( ^cN \), structures. Following the Riemann–Finsler general covariance principle, we can apply any type of frame/coordinate transforms and work equivalently with various parametrizations of metrics and nonlinear and linear connections.

2. From the class of infinite number of metric compatible and noncompatible linear (Finsler type, or generalized) connections, we can always work with the canonical d–connection \( \hat{\mathcal{D}} \). For even dimensions, a preferred role is played by the Cartan d–connection, \( ^cD \), which is metric compatible and completely defined by \( F_g \) and \( ^cN \). This way, we do not involve into physical models any difficulties/sophistications related to the nonmetricity geometry and fields. We can consider and/or derive in certain limits, various types of Finsler–Lagrange (super) string, gauge, nonholonomic Clifford/spinor, Finsler–affine and/or noncommutative gravity theories [120, 88, 91, 93, 92, 89, 90].

3. The gravitational field equations in Einstein–Finsler gravity can be formulated for \( \hat{\mathcal{D}} \), or \( ^cD \). For corresponding classes of nonholonomic constraints, we can extract the usual Einstein equations for the Levi–Civita connection \( \nabla \). Such equations can be solved in very general forms for the Einstein and Finsler gravity theories, and various noncommutative/ supersymmetric etc generalizations, following the anholonomic deformation method [95, 92, 120, 101, 112]. From general classes of exact off–diagonal solutions, it is possible to extract some subclasses of physical importance (describing locally anisotropic black hole/ellipsoid/torus configurations, cosmological inhomogeneous and locally anisotropic solutions, solitons etc, see reviews or results and references in [118, 117, 114, 101, 120, 95]).

4. To elaborate realistic models of Finsler like inflation, dark energy and dark matter scenarios is important to construct minimal extensions
of the Friedman and Robertson–Walker (FRW) metric. Such cosmological models are, in general, with inhomogeneous and anisotropic metrics. We have to construct exact cosmology solutions in Finsler gravity and analyze how such generalized metrics and connections may explain observational data from modern cosmology.

5. The fundamental geometric objects $^{c}D$ and $^{F}g$, for a fixed $^{c}N$, define canonical almost Kähler models of Finsler–Lagrange, Hamilton–Cartan, Einstein gravity and various generalizations. Such theories can be quantized applying a nonholonomically generalized Fedosov method \[99, 100, 68, 110, 68\], following the A–brane formalism \[104\], and developing a two–connection perturbative approach to the Einstein and gauge gravity theories \[110, 111\]. Quantum Einstein and/or Finsler–Lagrange gravity theories should be formulated to have in certain quasi–classical limits different terms with locally violated Lorentz invariance, anomalies, formal renormalization properties etc. It is also possible to construct models limiting locally relativistic and covariant theories. There are also two another very important properties of the Cartan d–connection (which do not exist for the Chern/Berwald and other metric noncompatible d–connections).

Of course, as already mentioned, the issues on classical and quantum gravity models related to modern cosmology and astrophysics can not be summarized in an article. We invite readers to consult the previous sections and presented there references.

5.3 Concluding remarks

To avoid repetition, in this concluding section we do not attempt to summarize all of the issues and application we discussed. This is because as, in most cases, the preliminary insight gained and perspectives can not be summarized in a sentence or two and this would not be very helpful to readers. However, we have encountered two key issues: 1) The Einstein–Finsler gravity can be formulated following the same principles as general relativity but on certain nonholonomic bundle/manifold spaces and corresponding generalized Finsler connections (which are also uniquely defined by the coefficients of metric tensor in a metric compatible form). 2) Analyzing possible implications of quantum gravity and related Lorentz violations in Early Universe and present day cosmology, we derived very easy that the dynamics (in general, with nonholonomic constraints) in the space of velocities contributes substantially to stability and acceleration/deceleration
stadies of cosmological models. So, a study of Finsler type inflation and dark energy and dark matter scenarios provides a new stimulus for further research and learning with a number of interesting results.

Specifically, the bulk of models of Finsler gravity and cosmology, and related concepts of anisotropic spacetimes with violation of Lorentz symmetry, which have been exploited in modern literature, are for metric non-compatible connections (for instance, the Chern, or Berwald, d–connections). Certain preferred Finsler metrics (for instance, Berwald–Moor, Bogoslovsky, Randers etc) considered for generalizations of cosmological scenarios are not form/parametrization independent and can be transformed from one to another types by nonholonomic deformations of Finsler connections and general frame/coordinates transforms. Before a Finsler gravity theory is formulated following some fundamental principles, with corresponding type of generalized gravitational field equations, it is not clear what type of special Finsler type metrics can be derived and chosen for physical applications.

The main purpose of this work was to show how the main postulates for the general relativity theory can be extended on nonholonomic tangent bundles/ manifolds. It was provided a self–consistent scheme for formulating Finsler gravity models and fundamental physical equations in a form most closed to standard particle physics. Applying the anholonomic deformation method, it was possible to construct new classes of exact cosmological solutions with generic off–diagonal metrics. We also analyzed possible scenarios for Finsler acceleration of Universe.

Finally, it is worth mentioning an "orthodox" approach with Finsler like and/or almost Kähler variables when the cosmological solutions are derived for generic off–diagonal metrics in general relativity. Following this approach, we may conclude that there are not modifications of Einstein gravity at classical level and that all accelerating and anisotropic effects in our days cosmology are consequences of certain nonlinear off–diagonal classical gravitational and matter field interactions. Considering (in Finsler variables) nonholonomically deformed FRW universes, we may model the bulk of dark energy and dark matter physics. Such exact cosmological solutions can be constructed in explicit form \[114\]. Nevertheless, we have to work with canonical Finsler gravity models on (co) tangent bundles if quantum effects are taken into consideration. Further developments will be provided in our papers under elaboration.

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A Formulas for N–adapted Coefficients

For convenience, we present some important formulas in Finsler geometry, generalizations and applications in modern gravity, see details and proofs in \[64 \ 92 \ 95 \ 101 \ 120]. Introducing the respective h–v–components of the d–connection 1–form (23), for a d–connection \(D = \{\Gamma^\alpha_{\beta\gamma}\}\), we get the N–adapted coefficients \(T^\alpha_{\beta\gamma} = \{T^i_{jk}, T^i_{ja}, T^a_{ji}, T^a_{bi}, T^a_{bc}\}\) of torsion d–tensor (d–torsion) (24),

\[
T^i_{jk} = L^i_{jk} - L^i_{kj}, \quad T^i_{ja} = -T^i_{aj} = C^i_{ja}, \quad T^a_{ji} = \Omega^a_{ji},
\]

\[
T^a_{bi} = -T^a_{ib} = \partial N^a_i / \partial y^b - L^i_{bi}, \quad T^a_{bc} = C^a_{bc} - C^a_{cb}. \tag{A.1}
\]

A N–adapted differential form calculus allows us to derive the formulas for h–v–components of curvature d–tensor (25) of a d–connection (23), for a d–connection (21),

\[
R^i_{hjk} = e_k L^i_{hj} - e_j L^i_{hk} + L^m_{ij} L^l_{mk} - L^m_{hk} L^l_{mj} - C^i_{ha} \Omega^a_{kj},
\]

\[
R^a_{bja} = e_k L^a_{bj} - e_j L^a_{bk} + L^r_{ja} L^l_{ik} - L^r_{bk} L^l_{cj} - C^a_{bc} \Omega^c_{kj}, \tag{A.2}
\]

\[
R^i_{jka} = e_a L^i_{jk} - D_k C^a_{ja} + C^i_{jb} T^b_{ka},
\]

\[
R^c_{bka} = e_a L^c_{bk} - D_k C^c_{ba} + C^c_{bd} T^k_{ca},
\]

\[
R^a_{jbc} = e_c C^a_{jb} + C^h_{jc} C^b_{hc} - C^h_{ja} C^b_{hc},
\]

\[
R^a_{bcd} = e_d C^a_{bc} - a C^a_{bd} + C^c_{bc} C^e_{ed} - C^c_{bd} C^a_{ce}.
\]

The values (A.1) and (A.2) can be computed in explicit form for the canonical d–connection \(\bar{\Gamma}^\alpha_{\beta\gamma} = \{\bar{L}^i_{jk}, \bar{L}^a_{bj}, \bar{C}^i_{jc}, \bar{C}^a_{bc}\}\) with

\[
\bar{L}^i_{jk} = \frac{1}{2} g^{ir} (e_k g_{jr} + e_j g_{kr} - e_r g_{jk}), \tag{A.3}
\]

\[
\bar{L}^a_{bk} = e_b (N^a_k) + \frac{1}{2} h^{ac} \left( e_k h_{bc} - h_{dc} e_b N^d_k - h_{db} e_c N^d_k \right),
\]

\[
\bar{C}^i_{jc} = \frac{1}{2} g^{ik} e_c g_{jk}, \quad \bar{C}^a_{bc} = \frac{1}{2} h^{ad} (e_c h_{bd} + e_b h_{cd} - e_d h_{bc}).
\]

For any d–metric \(g\) on a N–anholonomic manifold \(V\), \(\hat{D} = \{\hat{\Gamma}^\gamma_{\alpha\beta}\}\) satisfies the condition \(\hat{D} g = 0\) vanishing of ”pure” horizontal and vertical torsion coefficients, i.e. \(\hat{T}^i_{jk} = 0\) and \(\hat{T}^a_{bc} = 0\), see formulas (A.1). We emphasize that, in general, \(\hat{T}^i_{ja}, \hat{T}^a_{ji}, \) and \(\hat{T}^a_{bi}\) are not zero, but such nontrivial components of torsion are induced by coefficients of an off–diagonal metric \(g_{\alpha\beta}\) (22).
Any geometric construction for the canonical $d$–connection $\hat{\mathbf{D}} = \{\hat{\Gamma}_\gamma^\alpha\}$ can be re–defined equivalently into a similar one with the Levi–Civita connection $\nabla = \{\Gamma^\gamma_\alpha\}$ following formulas

$$\Gamma^\gamma_\alpha\beta = \hat{\Gamma}^\gamma_\alpha\beta + \hat{Z}^\gamma_\alpha\beta, \quad (A.4)$$

where N–adapted coefficients of connections, $\Gamma^\gamma_\alpha\beta$ and $\hat{\Gamma}^\gamma_\alpha\beta$, and the distortion tensor $\hat{Z}^\gamma_\alpha\beta$ are determined in unique forms by the coefficients of a metric $g^\alpha\beta$. The N–adapted components of the distortion tensor $\hat{Z}^\gamma_\alpha\beta = \{Z^a_{jk}, Z^i_{bk}, Z^a_{jk}, Z^i_{jk}, Z^a_{jb}, Z^i_{jb}, Z^i_{ab}\}$ are

$$Z^a_{jk} = -\hat{C}^i_{jk}g_hg^i_{ab} - \frac{1}{2}\Omega^a_{jk}, \quad Z^i_{bk} = \frac{1}{2}\Omega^c_{jk}h^c_{gb}g^i_j - \xi^i_{jk}\hat{C}_h^j, \quad (A.5)$$

$$Z^a_{jb} = \pm\xi_{cd}\hat{T}^c_{jb}, \quad Z^i_{bk} = \frac{1}{2}\Omega^a_{jk}h^c_{gb}g^i_j + \xi^i_{jk}\hat{C}_h^j, \quad Z^i_{jk} = 0.$$ 

for $\xi^i_{jk} = \frac{1}{2}(\delta^i_j\delta^h_k - g_{jk}g^i_h), \quad \pm\xi^a_{cd} = \frac{1}{2}(\delta^a_c\delta^b_d + h_{cd}h^a_b)$ and $\hat{T}^c_{ja} = L^c_{ja} - e_a(N^c_j)$.

B A Class of General Cosmological Solutions

For zero N–connection coefficients $N^a_i$, with $i, j, ... = 1, 2, 3, 4$ and $a, b, ... = 5, 6, 7, 8$, we can chose such solutions for $h_a$ when (69) have certain limits to the diagonal cosmological metric (61). Such a very general off–diagonal, inhomogeneous and locally anisotropic cosmological dynamics, with one Killing symmetry vector $\partial/\partial y^a = \partial/\partial \phi^a$ (a similar class of solutions can

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be generated if as $y^8$ we take $w \theta$ for $y^7 = w \varphi$) is determined by coefficients

$$h_5 = \frac{0}{5} h(u^0, 0) \left[ \partial_{iv}^2 f(u^0, 0, 1) \right]^2 | \frac{1}{1} \varsigma(u^0, 1, v) |,$$

$$h_6 = \left[ \frac{1}{1} f(u^0, 1) - \frac{0}{9} f(u^0, 0) \right]^2;$$

$$w_{0} = - \partial_{0} \varsigma(u^0, 1) / \partial_{iv} | \frac{1}{1} \varsigma(u^0, 1, v) |,$$

$$n_{0} = 1 \left[ n_{i0}(u^0) + 2 n_{0}(u^0) \right] \times$$

$$\int d^1 v \left[ \frac{1}{1} \varsigma(u^0, 1) | \partial_{iv}^2 f(u^0, 1, v) |^2 \right]$$

$$\left[ \frac{1}{1} f(u^0, 1) - \frac{0}{9} f(u^0, 0) \right]^{-3},$$

for $1 \varsigma = \frac{0}{1} \varsigma(u^0) - \frac{1}{1} \varsigma u^0 \int d^1 v \times$$

$$\left[ \partial_{iv}^2 f(u^0, 1, v) \right]^2 | \frac{1}{1} f(u^0, 1, v) | - \frac{0}{9} f(u^0, 0) \right];$$

$$h_7 = \frac{0}{2} h(u^1, 0) \left[ \partial_{iv}^2 f(u^1, 2, v) \right]^2 | \frac{2}{2} \varsigma(u^1, 2, v) |,$$

$$h_8 = \left[ \frac{2}{2} f(u^1, 2, v) - \frac{0}{2} f(u^1, 0) \right]^2;$$

$$w_{1} = - \partial_{1} \varsigma(u^1, 0, 2) / \partial_{iv} | \frac{2}{2} \varsigma(u^1, 0, 2) |,$$

$$n_{1} = 1 \left[ n_{i1}(u^1) + 2 n_{1}(u^1) \right] \int d^2 v \left[ \partial_{iv}^2 f(u^1, 0, 2) \right] \times$$

$$\left[ \partial_{iv}^2 f(u^1, 2, v) \right]^2 \left[ \frac{2}{2} f(u^1, 2, v) - \frac{0}{2} f(u^1, 0) \right]^3,$$

for $1 \varsigma = \frac{0}{2} \varsigma(u^1) - \frac{2}{2} \varsigma u^1 \int d^2 v \left[ \partial_{iv}^2 f(u^1, 0, 2) \right] \times$$

$$\left[ \partial_{iv}^2 f(u^1, 2, v) \right]^2 \left[ \frac{2}{2} f(u^1, 2, v) - \frac{0}{2} f(u^1, 0) \right]^3.$$
\[ \left[ \partial_{v} \frac{1}{2} f(x^1, h_r, t, h_{\theta}, v_r) \right] \]
\[ \frac{1}{2} f(x^1, h_r, t, h_{\theta}, v_r) - \frac{1}{2} f(x^1, h_r, t, h_{\theta}) \];
for any generation \( f(x^1, h_r, t, h_{\theta}, v_r) \) functions and a small parameter \( \varepsilon \)
\[ \varepsilon \left( x^1, h_r, t, h_{\theta}, v_r \right) = \varepsilon a(t) \left( 1 + \varepsilon \xi(x^1, h_r, t, h_{\theta}, v_r) \right) \]
and \( w_{1\beta} = 0, n_{1\beta} = 0, h_7 = v a^2(t) (v_r)^2, h_8 = v a^2(t) (v_r)^2 \sin^2 v_{\theta} \).

For simplicity, we can fix \( n_{0\beta} = 0 \) and that \( f(x^1, h_r, t, h_{\theta}, v_r) \) induces \( \xi = \xi(t, h_{\theta}, v_r) \) as a solution of any three dimensional solitonic (nonlinear wave) equation, for instance, of type
\[
\frac{\partial^2 \xi}{\partial (v_r)^2} + \epsilon (\xi' + 6 \xi \xi^* + \xi^{**})^* = 0, \quad \epsilon = \pm 1, \tag{B.1}
\]
where \( \xi' = \partial \xi / \partial t \) and \( \xi^* = \partial \xi / \partial h_{\theta} \). Such solitons are stable and generate solitonic configurations for the metric and nontrivial N–connection coefficients (for \( 0, 1, 2, \ldots, 5 \)),
\[
\tilde{\eta}_5 = \tilde{\eta}_5[\xi] \sim 1 + \varepsilon \tilde{\omega}_5[\xi], \tilde{\eta}_6 = \tilde{\eta}_6[\xi] \sim 1 + \varepsilon \tilde{\omega}_6[\xi], \quad \text{and}
\]
\[ \tilde{\omega}_3 \rightarrow \varepsilon \tilde{\omega}_3[\xi], \tilde{\omega}_4 \rightarrow \varepsilon \tilde{\omega}_4[\xi], \tilde{n}_3 \rightarrow \varepsilon \tilde{n}_3[\xi], \tilde{n}_4 \rightarrow \varepsilon \tilde{n}_4[\xi] \]
where, for simplicity, we fixed the boundary conditions to have only functional dependence on \( \xi \) and vanishing values if \( \varepsilon \to 0 \).

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