Tropical Dynamic Programming for Lipschitz Multistage Stochastic Programming

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Abstract

We present an algorithm called Tropical Dynamic Programming (TDP) which builds upper
and lower approximations of the Bellman value functions in risk-neutral Multistage Stochastic
Programming (MSP), with independent noises of finite supports.

To tackle the curse of dimensionality, popular parametric variants of Approximate Dy-
namic Programming approximate the Bellman value function as linear combinations of basis
functions. Here, Tropical Dynamic Programming builds upper (resp. lower) approximations
of a given value function as min-plus linear (resp. max-plus linear) combinations of ”basic
functions”. At each iteration, TDP adds a new basic function to the current combination
following a deterministic criterion introduced by Baucke, Downward and Zackeri in 2018 for
a variant of Stochastic Dual Dynamic Programming.

We prove, for every Lipschitz MSP, the asymptotic convergence of the generated approx-
imating functions of TDP to the Bellman value functions on sets of interest. We illustrate
this result on MSP with linear dynamics and polyhedral costs.

1 Introduction

In this article we study multistage stochastic optimal control problems in the hazard-decision
framework (hazard comes first, decision second). Starting from a given state $x_0$, a decision maker
observes the outcome $w_1$ of a random variable $W_1$, then decides on a control $u_0$ which induces
a known cost $c_0^w(x_0, u_0)$ and the system evolves to a future state $x_1$ from a known dynamic:
$x_1 = f_{x_0}^w(x_0, u_0)$. Having observed a new random outcome, the decision maker makes a new
decision based on this observation which induces a known cost, then the system evolves to a
known future state, and so on until $T$ decisions have been made. At the last step, there are
constraints on the final state $x_T$ which are modeled by a final cost function $\psi$. The decision maker
aims to minimize the average cost of her decisions.

Multistage Stochastic optimization Problems (MSP) can be formally described by the following
optimization problem

$$
\min_{(X,U)} \mathbb{E} \left[ \sum_{t=0}^{T-1} c_t^{w_{t+1}}(X_t, U_t) + \psi(X_T) \right],
$$

s.t. $X_0 = x_0$ given, $\forall t \in [0, T-1], \quad X_{t+1} = f_{x_t}^{w_{t+1}}(X_t, U_t), \quad \sigma(U_t) \subset \sigma(W_1, \ldots, W_{t+1}),$

where $(W_t)_{t\in[1,T]}$ is a given sequence of independent random variables each with values in some
measurable set $(\bar{W}_t, \bar{W}_t)$. We refer to the random variable $W_{t+1}$ as a noise and throughout the
remainder of the article we assume the following on the sequence of noises.

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**Assumption 1.** Each random variable $W_t$ in Problem 1 has finite support and the sequence of random variable $(W_t)_{t \in [1,T]}$ is independent.

One approach to solving MSP problems is by dynamic programming, see for example [5, 8, 14][21]. For some integers $n, m \in \mathbb{N}$, denote by $X = \mathbb{R}^n$ the state space and $U = \mathbb{R}^m$ the control space. Both $X$ and $U$ are endowed with their euclidean structure and borelian structure. We define the pointwise Bellman operators $B^W_t$ and the average Bellman operators $B_t$ for every $t \in [0, T - 1]$. For each possible realization $w \in \mathbb{W}_{t+1}$ of the noise $W_{t+1}$, for every function $\phi : X \to \mathbb{R}$ taking extended real values in $\mathbb{R} = \mathbb{R} \cup \{\pm \infty\}$, the function $B^w_t(\phi)(\cdot) : X \to \mathbb{R}$ is defined by

$$\forall x \in X, \ B^w_t(\phi)(x) = \min_{u \in U} \left(c^w_t(x, u) + \phi(f^w_t(x, u))\right).$$

Now, the average Bellman operator $B_t$ is the mean of all the pointwise Bellman operators with respect to the probability law of $W_{t+1}$. That is, for every $\phi : X \to \mathbb{R}$, we have that

$$\forall x \in X, \ B_t(\phi)(x) = \mathbb{E}[B^w_{t+1}(\phi)(x)] = \mathbb{E}\left[\min_{u \in U} \left(c^w_{t+1}(x, u) + \phi(f^w_{t+1}(x, u))\right)\right].$$

The average Bellman operator can be seen as a one stage operator which computes the value of applying the best (average) control at a given state $x$. Note that in the hazard-decision framework assumed here, the control is taken after observing the noise. Now, the Dynamic Programming approach states that in order to solve MSP Problems 1, it suffices to solve the following system of Bellman equations

$$V_T = \psi \quad \text{and} \quad \forall t \in [0, T - 1], \ V_t = B_t(V_{t+1}). \quad (2)$$

Solving the Bellman equations means computing recursively backward in time the (Bellman) value functions $V_t$. Finally, the value $V_0(x_0)$ is the solution of the multistage Problem 1.

Grid-based approach to compute the value functions suffers from the so-called curse of dimensionality. Assuming that the value functions $\{V_t\}_{t \in [0,T]}$ are convex, one approach to bypass this difficulty is proposed by Pereira and Pinto [13] with the Stochastic Dual Dynamic Programming (SDDP) algorithm which computes piecewise affine approximations of each value function $V_t$. At a given iteration $k \in \mathbb{N}^*$ of SDDP, for every time step $t \in [0, T]$, the value function $V_t$ is approximated by $V^k_t = \max_{\phi \in F_t^k} \psi$ where $F_t^k$ is a finite set of affine functions. Then, given a realization of the noise process $(\tilde{W}_t)_{t \in [0,T]}$, the decision maker computes an optimal trajectory associated with the approximations $(\tilde{V}^k_t)_{t \in [0,T]}$ and add a new mapping, $\phi^{k+1}$ (named cut) to the current collection $F^k_t$ which define $V^k_t$, that is $E^{k+1} = F^k_t \cup \{\phi^{k+1}\}$. Although SDDP does not involve discretization of the state space, one of its computational bottleneck is the lack of efficient stopping criterion: SDDP easily builds lower approximations of the value function but upper approximations are usually computed through a costly Monte-Carlo scheme.

In order to build upper approximations of the value functions, Min-plus methods were studied (e.g. [12][18]) for optimal control problems in continuous time. When the value functions $\{V_t\}_{t \in [0,T]}$ are convex (or more generally, semiconcave), discrete time adaptations of Min-plus methods build for each $t \in [1, T]$ approximations of convex value function $V_t$ as finite infima of convex quadratic forms. That is, at given iteration $k \in \mathbb{N}$, we consider upper approximations defined as $V^k_t = \min_{\phi \in F^k_t} \psi$, where $F^k_t$ is a finite set of convex quadratic forms. Then, a sequence of trial points $(x^k_t)_{t \in [0,T]}$ are drawn (e.g. uniformly on the unit sphere as in [18]) and for every $t \in [0, T - 1]$ a new function $\phi^{k+1}$ is added, $F^{k+1} = F^k_t \cup \{\phi^{k+1}\}$. The function $\phi^{k+1}$ should be compatible with the Bellman equation, in particular it should be tight, i.e. the Bellman equations should be satisfied at the trial point,

$$B_t(\phi^{k+1})(x_t^k) = \phi^{k+1}(x_t^k).$$

In [2], the authors present a common framework for a deterministic version of SDDP and a discrete time version of Min-plus algorithms. Moreover, the authors give sufficient conditions on the way
the trial points have to be sampled in order to obtain asymptotic convergence of either upper or lower approximations of the value functions. Under these conditions, the main reason behind the convergence of these algorithm was shown to be that the Bellman equations are asymptotically satisfied on all cluster points of possible trial points. In this article, we would like to extend the work of [2] by introducing a new algorithm called Tropical Dynamic Programming (TDP).

In [4, 16], is studied approximation schemes where lower approximations are given as a suprema of affine functions and upper approximations are given as a polyhedral function. We aim in this article to extend, with TDP, the approach of [4, 16] considering more generally that lower approximations are max-plus linear combinations of some basic functions and upper approximations are min-plus linear combinations of other basic functions where basic functions are defined later.

TDP can be seen as a tropical variant of parametric approximations used in Adaptive Dynamic Programming (see [6, 17]) where the value functions are approximated by linear combinations of basis functions. In this article, we will:

1. Extend the deterministic framework of [2] to Lipschitz MSP defined in Equation (1) and introduce TDP, see Section 2.

2. Ensure that upper and lower approximations converge to the true value functions on a common set of points, see Section 3. The main result of Section 3 generalizes to any min-plus/max-plus approximation scheme the result of [4] which was stated for a variant of SDDP.

3. Explicitly give several numerically efficient ways to build upper and lower approximations of the value functions, as min-plus and max-plus linear combinations of some simple functions, see Section 4.

2 Tropical Dynamical Programming on Lipschitz MSP

2.1 Lipschitz MSP with independent finite noises

For every time step \( t \in [0, T] \), we denote by \( \text{supp}(W_t) \) the support of the discrete random variable \( W_t \) and for a given subset \( X \subseteq \mathbb{X} \), we denote by \( \pi_X \) the euclidean projector on \( X \). State and control constraints for each time \( t \) are modeled in the cost functions which may possibly take infinite values outside of some given sets. Now, we introduce a sequence of sets \( \{X_t\}_{t \in [0,T]} \) which only depend on the problem data and make the following compactness assumption:

**Assumption 2 (Compact state space).** For every time \( t \in [0, T] \), we assume that the set \( X_t \) is a nonempty compact set in \( \mathbb{X} \) where the sequence of sets \( \{X_t\}_{t \in [0,T]} \) is defined, for all \( t \in [0, T - 1] \), by

\[
X_t := \bigcap_{w \in \text{supp}(W_{t+1})} \pi_X(\text{dom} c^w_t),
\]

and for \( t = T \) by \( X_T = \text{dom} \psi \).

For each noise \( w \in \text{supp}(W_{t+1}), t \in [0, T - 1] \), we also introduce the constraint set-valued mapping \( \mathcal{U}^w_t : \mathbb{X} \rightrightarrows \mathbb{U} \) defined for every \( x \in \mathbb{X} \) by

\[
\mathcal{U}^w_t(x) := \left\{ u \in \mathbb{U} \mid c^w_t(x, u) < +\infty \text{ and } f^w_t(x, u) \in X_{t+1} \right\}.
\]

We will assume that the data of Problem (1) is Lipschitz in the sense defined below. Let us stress that we do not assume structure on the dynamics or costs like linearity or convexity, only that they are Lipschitz.

\[\text{The support of the discrete random variable } W_t \text{ is equal to the set } \{w \in W_t \mid \mathbb{P}(W_t = w) > 0\}.\]
Assumption 3 (Lipschitz MSP). For every time \( t \in [0, T-1] \), we assume that for each \( w \in \text{supp}(W_{t+1}) \), the dynamic \( f_t^w \), the cost \( c_t^w \) are Lipschitz continuous on \( \text{dom} c_t^w \) and the set-valued mapping constraint \( U_t^w \) is Lipschitz continuous on \( X_t \), i.e. for some constant \( L_{U_t^w} > 0 \), for every \( x_1, x_2 \in X_t \), we have
\[
d_H(U_t^w(x_1), U_t^w(x_2)) \leq L_{U_t^w} \|x_1 - x_2\|  \tag{5}
\]

Computing a (sharp) Lipschitz constant for the set-valued mapping \( U_t^w : X \rightrightarrows U \) is difficult. However, when the graph of the set-valued mapping \( U_t^w \) is polyhedral, as in the linear-polyhedral framework studied in Section 4, one can compute a Lipschitz constant for \( U_t^w \). We make the following assumption in order to ensure that the domains of the value functions \( V_t \) are chosen by the decision maker. It can be seen as a recourse assumption.

Assumption 4 (Recourse assumption). Given \( t \in [0, T-1] \), for every noise realization \( w \in \text{supp}(W_t) \) the set-valued mapping \( U_t^w : X \rightrightarrows U \) defined in (4) is nonempty compact valued.

\[\text{A priori, it might be difficult to compute the domain of each value function } V_t. \text{ However, under the recourse Assumption 3, we have that } \text{dom} V_t := X_t \text{ and thus the domain of each value function is known to the decision maker.}\]

Lemma 1 (Known domains of \( V_t \)). Under Assumptions 3 and 4 for every \( t \in [0, T] \), the domain of \( V_t \) is equal to \( X_t \).

**Proof.** We make the proof by backward induction on time. At time \( t = T \), we have \( V_T = \psi \) and thus \( \text{dom} V_T = \text{dom} \psi = X_T \). Now, for a given \( t \in [0, T-1] \), we assume that \( \text{dom} V_{t+1} = X_{t+1} \) and we prove that \( \text{dom} V_t = X_t \).

First, fix \( x \in X_t \). Then, for every \( w \in \text{supp}(W_{t+1}) \), using Assumption 4, \( U_t^w(x) \) is nonempty and thus \( V_t(x) < +\infty \). Moreover, by Assumptions 3 and Assumptions 4, the optimization problem
\[
\min_{u \in U} \left( c_t^w(x, u) + V_{t+1}(f_t^w(x, u)) \right) = \min_{u \in U_t^w(x)} \left( c_t^w(x, u) + V_{t+1}(f_t^w(x, u)) \right),
\]
consists in the minimization of a continuous function in \( u \) over a nonempty compact set. Denote by \( u^w \in U_t^w(x) \) a minimizer of this optimization problem. We have, denoting by \( \{p_w\}_{w \in \text{supp}(W_{t+1})} \) the discrete probability law of the random variable \( W_{t+1} \), that
\[
V_t(x) = \mathcal{B}_t(V_{t+1})(x) \\
= \mathbb{E}[\mathcal{B}_t^{W_{t+1}}(V_{t+1})(x)] \\
= \sum_{w \in \text{supp}(W_{t+1})} p_w \inf_{u \in U} \left( c_t^w(x, u) + V_{t+1}(f_t^w(x, u)) \right) \\
= \sum_{w \in \text{supp}(W_{t+1})} p_w \left( c_t^w(x, u^w) + V_{t+1}(f_t^w(x, u^w)) \right).
\]
As every term in the right hand side of the previous equation is finite, we have \( V_t(x) < +\infty \) and thus \( x \in \text{dom} V_t \).

Second, fix \( x \notin X_t \). Then, there exists an element \( w \in \text{supp}(W_{t+1}) \) such that \( c_t^w(x, u) = +\infty \) for every control \( u \in U \). We therefore have that \( V_t(x) = +\infty \) and \( x \notin \text{dom} V_t \).

We conclude that \( \text{dom} V_t = X_t \), which ends the proof. \( \square \)

In Section 4, it will be crucial for numerical efficiency to have a good estimation of the Lipschitz constant of the function \( \mathcal{B}_t(V_{t+1}^{X_t}) \).

We now prove that under Assumptions 3 and Assumptions 4, the operators \( \mathcal{B}_t \) preserve Lipschitz regularity. Given a \( L_{U_t^w} \)-Lipschitz function \( \phi \) and \( w \in \text{supp}(W_{t+1}) \), in order to compute
\[
\text{The Hausdorff distance } d_H \text{ between two nonempty compact sets } X_1, X_2 \text{ in } X \text{ is defined by}
\]
\[
d_H(X_1, X_2) = \max \left( \max_{x_1 \in X_1} d(x_1, X_2), \max_{x_2 \in X_2} d(X_1, x_2) \right) = \max \left( \min \left( \max_{x_1 \in X_1} d(x_1, x_2), \max_{x_2 \in X_2} d(x_1, x_2) \right) \right).
\]
a Lipschitz constant of the function $B^w_t(\phi)(\cdot)$ we exploit the fact that the set-valued constraint mapping $U^w_t$ and the data of problem (1) are Lipschitz in the sense of Assumptions [3]. This was mostly already done in [2], but for the sake of completeness, we will slightly adapt its statement and proof.

**Proposition 2** ($\mathcal{B}_t$ is Lipschitz regular). Let $\phi : \mathbb{X} \to \mathbb{R}$ be given. Under Assumptions [4] to [6] if for some $L_{t+1} > 0$, $\phi$ is $L_{t+1}$-Lipschitz on $X_{t+1}$, then the function $\mathcal{B}_t(\phi)$ is $L_t$-Lipschitz on $X_t$ for some constant $L_t > 0$ which only depends on the data of the problem [7] and $L_{t+1}$.

**Proof.** Let $\phi : \mathbb{X} \to \mathbb{R}$ be a $L_{t+1}$-Lipschitz function on $X_{t+1}$. We will show that for every $w \in \text{supp}(W_{t+1})$, the mapping $B^w_t(\phi)(\cdot)$ is $L_w$-Lipschitz for some constant $L_w$ which only depends on the data of problem [7]. Fix $w \in \text{supp}(\mathcal{W}_{t+1})$ and $x_1, x_2 \in X_t$. Denote by $u^*_w$ an optimal control at $x_2$ and $w$, that is $u^*_w \in \arg\min_{u \in U^w_t(x_2)} \{c_i^w(x_2, u) + \phi(f_i^w(x_2, u))\}$, or equivalently, $u^*_w$ satisfies

$$c_i^w(x_2, u^*_w) + \phi(f_i^w(x_2, u^*_w)) = B^w_t(\phi)(x_2). \quad (6)$$

Then, for every $u_1 \in U^w_t(x_1)$ we successively have

$$B^w_t(\phi)(x_1) \leq c_i^w(x_1, u_1) + \phi(f_i^w(x_1, u_1)) \leq B^w_t(\phi)(x_2) + c_i^w(x_1, u_1) + \phi(f_i^w(x_1, u_1)) - B^w_t(\phi)(x_2)$$

$$= B^w_t(\phi)(x_2) + (c_i^w(x_1, u_1) - c_i^w(x_2, u_2^*_w)) + \left(\phi(f_i^w(x_1, u_1)) - \phi(f_i^w(x_2, u_2^*_w))\right)$$

$$\leq B^w_t(\phi)(x_2) + \sum_{u \in U^w_t(x_2)} p_u \left|B^w_t(\phi)(x_1) - B^w_t(\phi)(x_2)\right|$$

where $L = \max(L_{c_i^w}, L_{t+1}L_{f_i^w})$. Now, as the set-valued mapping $U^w_t$ is $L_{u^w_t}$-Lipschitz, there exists $\bar{u}_1 \in U^w_t(x_1)$ such that

$$\|\bar{u}_1 - u^*_w\| \leq \sum_{w \in \text{supp}(W_{t+1})} \sum_{u \in U^w_t(x_2)} p_u \sum_{w \in \text{supp}(W_{t+1})} p_u \|x_1 - x_2\|.$$
2.2 Tight and valid selection functions

We formally define now what we call basic functions. In the sequel, the notation in bold \( F_t \) will stand for a set of basic functions and \( F_t \) will stand for a subset of \( F_t \).

**Definition 4** (Basic functions). Given \( t \in [0, T] \), a basic function \( \phi : X \rightarrow \mathbb{R} \) is a \( L_{V_t} \)-Lipschitz continuous function on \( X_t \), where the constant \( L_{V_t} > 0 \) is defined in Corollary 3.

In order to ensure the convergence of the scheme detailed in the introduction, at each iteration of TDP algorithm a basic functions which is be tight and valid in the sense below is added to the current sets of basic functions. The idea behind these assumptions is to ensure that the Bellman equations (2), however tightness and validity can be checked efficiently and this will be enough to ensure asymptotic convergence of our TDP algorithm.

There is a dissymmetry for the validity assumption which depends on whether the decision maker wants to build upper or lower approximations of the value functions. In (2) we will assume that the decision maker has, at his disposal, two sequences of selection functions \((S_t)_{t \in [0, T]}\) and \((\overline{S}_t)_{t \in [0, T]}\). The former to select basic functions for the upper approximations and the latter for the lower approximations of \( V_t \). We write \( S_t \) when designing either \( S_t \) or \( \overline{S}_t \) and denote by \( \overline{V}_t \) (resp. \( \underline{V}_t \)) the pointwise infimum (resp. pointwise supremum) of basic functions in \( F_t \) (resp. in \( \overline{F}_t \)) when approximating from above (resp. below) a mapping \( V_t \). The Figure 1 illustrates the formal definition of selection functions given below. Given a set \( Z \), we denote by \( \mathcal{P}(Z) \) its power set, i.e. the set of all subsets included in \( Z \).

**Definition 5** (Selection functions). Let a time step \( t \in [0, T - 1] \) be fixed. A selection function or simply selection function is a mapping \( S_t \) from \( \mathcal{P}(F_{t+1}) \times X_t \) to \( F_t \) satisfying the following properties

- **Tightness**: for every set of basic functions \( F_{t+1} \subset F_{t+1} \) and \( x \in X_t \), the mappings \( S_t(F_{t+1}, x) \) and \( \overline{B}_t(V_{F_{t+1}})(\cdot) \) coincide at point \( x \), that is
  \[
  S_t(F_{t+1}, x)(x) = \overline{B}_t(V_{F_{t+1}})(x).
  \]

- **Validity**: for every set of basic functions \( F_{t+1} \subset F_{t+1} \) and for every \( x \in X_t \) we have
  \[
  S_t(F_{t+1}, x) \geq \overline{B}_t(V_{F_{t+1}})(\cdot), \quad \text{ (when building upper approximations)}
  \]
  \[
  S_t(F_{t+1}, x) \leq \underline{B}_t(V_{F_{t+1}})(\cdot), \quad \text{ (when building lower approximations)}
  \]

For \( t = T \), we also say that \( S_T : X_T \rightarrow F_T \) is a selection function if the mapping \( S_T \) is tight and valid with a modified definition of tight and valid defined now. The mapping \( S_T \) is said to be valid if, for every \( x \in X_T \), the function \( S_T(x) \) remains above (resp. below) the value function at time \( T \) when building upper approximations (resp. lower approximations). The mapping \( S_T \) is said to be tight if it coincides with the value function at point \( x \), that is for every \( x \in X_T \) we have

\[
S_T(x)(x) = V_T(x).
\]

**Remark 6.** Note that the validity and tightness assumptions at time \( t = T \) is stronger than at times \( t < T \) as the final cost function is a known data, we are allowed to enforce conditions directly on the value function \( V_T \) and not just the on the image of the current approximations at time \( t + 1 \) as it is the case when \( t < T \).

2.3 The problem-child trajectory

From the previous section, given a set of basic functions and a point in \( X \), a selection function is used to computes a new basic function. We explain in this section the algorithm used to select the points which are used for searching new basic functions.
Figure 1: Given a time step $t \in [0, T-1]$, we illustrate the notions of tightness and validity of selection functions. A selection function takes as input a trial point $x$ in the domain $X_t$ of $V_t$ and a set of basic functions $F_t \subseteq F_{t+1}$ building the approximations at the future time step $t+1$ (right: pointwise suprema or infima of the basic functions). Then, the Bellman operator $\mathcal{B}_t$ translates one step backward in time the right picture to the picture on the left.

Tightness of the selection function enforces that the output is a function equal to the Bellman image of the future approximation of $V_{t+1}$ at $x$; it is a local property.

Validity enforces that the output of the selection function remains below, or above, the Bellman image the approximation of $V_{t+1}$ everywhere on the domain of $V_t$; it is a global property. More details on these examples of selection functions in Section 4.
In this section we present how to build a trajectory of states, without discretization of the whole state space. Selection functions for both upper and lower approximations of $V_t$ will be evaluated along it. This trajectory of states, coined *problem-child* trajectory, was introduced by Baucke, Downward and Zackeri in 2018 (see [4]) for a variant of SDDP first studied by Philpott, de Matos and Finardi in 2013 (see [16]).

We present in Algorithm 1 a generalized problem-child trajectory, it is the sequence of states on which we evaluate selection functions.

**Algorithm 1 Problem-child trajectory**

**Input:** Two sequences of functions from $X$ to $\mathbb{R}$, $\bar{\phi}_0, \ldots, \bar{\phi}_T$ and $\underline{\phi}_0, \ldots, \underline{\phi}_T$ with respective domains equal to $\text{dom}V_t$.

**Output:** A sequence of states $(x^*_0, \ldots, x^*_T)$.

Set $x^*_0 := x_0$.

for $t \in [0, T-1]$ do

for $w \in \text{supp}(W_{t+1})$ do

Compute an optimal control $u^w_t$ for $\bar{\phi}_{t+1}$ at $x^*_t$ for the given $w$

$$u^w_t \in \arg\min_{u \in U} \left( e^w_t(x^*_t, u) + \bar{\phi}_{t+1}(f^w_t(x^*_t, u)) \right).$$

end for

Compute “the worst” noise $w^* \in \text{supp}(W_{t+1})$, i.e. the one which maximizes the “future” gap

$$w^* \in \arg\max_{w \in \text{supp}(W_{t+1})} \left( \bar{\phi}_{t+1} - \underline{\phi}_{t+1} \right)(f^w_t(x^*_t, u^w_t)).$$

Compute the next state dynamics for noise $w^*$ and associated optimal control $u^w_t^*$:

$$x^*_{t+1} = f^w_t(x^*_t, u^w_t^*).$$

end for

One can interpret the problem child trajectory as the worst (for the noises) optimal trajectory (for the controls) of the lower approximations. It is worth mentioning that the problem-child trajectory is deterministic. The approximations of the value functions will be refined along the problem-child trajectory only, thus avoiding a discretization of the state space. The main computational drawback of such approach is the need to solve Problem (7) $|\text{supp}(W_{t+1})| \cdots |\text{supp}(W_T)|$ times. Except on special instances like the linear-quadratic case, one cannot expect to find a closed form expression for solutions of Equation (7). However, we will see in Section 4 examples where Problem (7) can be solved by Linear Programming or Quadratic Programming. Simply put, if one can solve efficiently the deterministic problem (7) and if at each time step the set $\text{supp}(W_t)$ remains of small cardinality, then using the problem-child trajectory and the Tropical Dynamical Algorithm presented below in Section 2.4 one can solve MSP problems with finite independent noises efficiently. This might be an interesting framework in practice if at each step the decision maker has a few different forecasts on which her inputs are significantly different.
2.4 Tropical Dynamic Programming

Algorithm 2 Tropical Dynamic Programming (TDP)

Input: For every \( t \in [0, T] \), two compatible selection functions \( \overline{S}_t \) and \( \underline{S}_t \). A sequence of independent random variables \( (W_t)_{t \in [0, T-1]} \), each with finite support.

Output: For every \( t \in [0, T] \), two sequence of sets \((\overline{V}^k_t)_{k \in \mathbb{N}}\), \((\underline{F}^k_t)_{k \in \mathbb{N}}\) and the associated functions \( \overline{V}^k_t = \inf_{\phi \in \overline{F}^k_t} \phi \) and \( \underline{V}^k_t = \sup_{\phi \in \underline{F}^k_t} \phi \).

Define for every \( t \in [0, T] \), \( \overline{F}^0_t = \emptyset \) and \( \underline{F}^0_t = \emptyset \).

for \( k \geq 0 \) do

Forward phase

Compute the problem-child trajectory \((x^k_t)_{t \in [0, T]}\) for the sequences \((\overline{V}^k_t)_{t \in [0, T]}\) and \((\underline{V}^k_t)_{t \in [0, T]}\) using Algorithm 1.

Backward phase

At \( t = T \), compute new basic functions \( \overline{\phi}_T := \overline{S}_T(x^k_T) \) and \( \underline{\phi}_T := \underline{S}_T(x^k_T) \).

Add them to current collections, \( \overline{F}^{k+1}_T := \overline{F}^k_T \cup \{ \overline{\phi}_T \} \) and \( \underline{F}^{k+1}_T := \underline{F}^k_T \cup \{ \underline{\phi}_T \} \).

for \( t \) from \( T-1 \) to 0 do

Compute new basic functions: \( \overline{\phi}_t := \overline{S}_t(F^{k+1}_{t+1}, x^k_t) \) and \( \underline{\phi}_t := \underline{S}_t(F^{k+1}_{t+1}, x^k_t) \).

Add them to the current collections: \( \overline{F}^{k+1}_t := \overline{F}^k_t \cup \{ \overline{\phi}_t \} \) and \( \underline{F}^{k+1}_t := \underline{F}^k_t \cup \{ \underline{\phi}_t \} \).

end for

end for

3 Asymptotic convergence of TDP along the problem-child trajectory

In this section, we will assume that Assumptions 1 to 4 are satisfied. We recall that, under Assumption 3, the sequence of sets \( \{X_t\}_{t \in [0, T]} \) defined in Equation 3 is known and for all \( t \in [0, T] \) the domain of \( V_t \) is equal to \( X_t \). We denote by \((x^k_t)_{k \in \mathbb{N}}\) the sequence of trial points generated by TDP algorithm at time \( t \) for every \( t \in [0, T] \), and by \((u^k_t)_{k \in \mathbb{N}}\) and \((w^k_t)_{k \in \mathbb{N}}\) the optimal control and worst noises sequences associated for each time \( t \) with \( x^k_t \) in the problem-child trajectory in Algorithm 1.

Now, observe that for every \( t \in [0, T] \), the approximations of \( V_t \) generated by TDP, \((\overline{V}^k_t)_{k \in \mathbb{N}}\) and \((\underline{V}^k_t)_{k \in \mathbb{N}}\), are respectively non increasing and non decreasing. Moreover, for every index \( k \in \mathbb{N} \) we have

\[ V^k_t \leq V_t \leq \overline{V}^k_t. \]

We refer to [2] Lemma 7 for a proof. Observing that the basic functions are all \( L_{V_t} \)-Lipschitz continuous on \( X_t \) one can prove using Arzelà-Ascoli Theorem the following proposition.

Proposition 7 (Existence of an approximating limit). Let \( t \in [0, T] \) be fixed, the sequences of functions \( (\overline{V}^k_t)_{k \in \mathbb{N}} \) and \( (\underline{V}^k_t)_{k \in \mathbb{N}} \) generated by Algorithm 2 converge uniformly on \( X_t \) to two functions \( \overline{V}^*_t \) and \( \underline{V}^*_t \). Moreover, \( \overline{V}^*_t \) and \( \underline{V}^*_t \) are \( L_{V_t} \)-Lipschitz continuous on \( X_t \) and satisfy \( \overline{V}^*_t \leq V_t \leq \underline{V}^*_t \).

Proof. Omitted as it is slight rewriting of [2] Proposition 9. \( \square \)

If we extract a converging subsequence of trial points, then using compactness, extracting a subsubsequence if needed, one can find a find a subsequence of trial points, and associated controls that jointly converge.
Lemma 8. Fix $t \in [0, T - 1]$ and denote by $(x^k_t)_{k \in \mathbb{N}}$ the sequence of trial points generated by Algorithm 4 and by $(u^k_t)_{k \in \mathbb{N}}$ the sequence of associated optimal controls. There exists an increasing function $\sigma : \mathbb{N} \to \mathbb{N}$ and a state-control ordered pair $(x^*_t, u^*_t) \in X_t \times U$ such that

$$
\begin{align*}
&x^\sigma(k)_{k \to +\infty} \to x^*_t, \\
u^\sigma(k)_{k \to +\infty} \to u^*_t.
\end{align*}
$$

(8)

Proof. Fix a time step $t \in [0, T-1]$. First, by construction of the problem-child trajectories, the sequence $(x^k_t)_{k \in \mathbb{N}}$ remains in the subset $X_t$ that is $x^k_t \in X_t$ for all $k \in \mathbb{N}$.

Second, we show that the sequence of controls $(u^k_t)_{k \in \mathbb{N}}$ is included in a compact subset of $U$. Under Assumption 3, $X_t$ is a nonempty compact subset of $\mathcal{X}$. For every $w \in \text{supp}(W_{t+1})$ the set-valued mapping $U^w_t$ is Lipschitz continuous on $X_t$ under Assumption 3 hence upper semicontinuous on $X_t$. Moreover, under recourse Assumption 4, $U^w_t$ is nonempty compact valued. Thus, by Proposition 11 p.112, its image $U^w_t(X_t)$ of the compact $X_t$ is a nonempty compact subset of $U$. Finally as the random variable $W_{t+1}$ has a finite support under Assumption 1, the set $U_t := \bigcup_{w \in \text{supp}(W_{t+1})} U^w_t(X_t)$ is a compact subset of $U$. The sequence $(u^k_t)_{k \in \mathbb{N}}$ remains in $U_t$ and therefore we conclude that it remains in a compact subset of $U$.

Finally, as the sequence $(x^k_t, u^k_t)_{k \in \mathbb{N}}$ is included in the compact subset $X_t \times U_t$ of $\mathcal{X} \times U$, one can extract a converging subsequence, hence the result. 

Lastly, we will use the following elementary lemma, whose proof is omitted.

Lemma 9. Let $(g^k)_{k \in \mathbb{N}}$ be a sequence of functions that converges uniformly on a compact $K$ to a function $g^*$. If $(y^k)_{k \in \mathbb{N}}$ is a sequence of points in $K$ that converges to $y^* \in K$ then one has

$$
g^k(y^k) \to_{k \to +\infty} g^*(y^*).
$$

We now state the main result of this article. For a fixed $t \in [0, T]$, as the Bellman value function $V_t$ is always sandwiched between the sequences of upper and lower approximations, if the gap between upper and lower approximations vanishes at a given state value $x$, then upper and lower approximations will both converge to $V_t(x)$. Note that, even though a MSP is a stochastic optimization problem, the convergence result below is not. Indeed, we have assumed (see Assumption 1) that the noises have finite supports, thus under careful selection of scenario as done by the Problem-child trajectory, we get a “sure” convergence.

Theorem 10 (Vanishing gap along problem-child trajectories). Denote by $(V^k_t)_{k \in \mathbb{N}}$ and $(V^*_t)_{k \in \mathbb{N}}$ the approximations generated by the Tropical Dynamic Programming algorithm. For every $k \in \mathbb{N}$ denote by $(x^k_t)_{0 \leq k \leq T}$ the current Problem-child trajectory.

Then, under Assumptions 7 to 4 we have that

$$
\nabla_t^k(x^k_t) - V^k_t(x^k_t) \to 0 \text{ as } k \to +\infty \quad \text{and} \quad V^*_t(x^*_t) = V^*_t(x^*_t),
$$

for every accumulation point $x^*_t$ of the sequence $(x^k_t)_{k \in \mathbb{N}}$.

Proof. We prove by backward recursion that, for every $t \in [0, T]$, for every accumulation point $x^*_t$ of the sequence $(x^k_t)_{k \in \mathbb{N}}$, we have

$$
V^*_t(x^*_t) = V^*_t(x^*_t).
$$

(9)

By a direct consequence of the tightness of the selection functions one has that for every $k \in \mathbb{N}$, $V^k_T(x^k_T) = V^*_T(x^k_T) = V^k_T(x^k_T)$. Thus, the equality (9) holds for $t = T$ by Lemma 8.

Now assume that for some $t \in [0, T-1]$, for every accumulation point $x^*_t$ of $(x^k_t)_{k \in \mathbb{N}}$ we have

$$
V^*_t(x^*_t) = V^*_t(x^*_t).
$$

(10)

3The compact valued set-valued mapping $U^w_t : \mathcal{X} \rightrightarrows U$ is upper semicontinuous on $X_t$ if, for all $x_t \in X_t$, if an open set $U \subset U$ contains $U^w_t(x_t)$ then $(x \in \mathcal{X} | U^w_t x \subset U)$ contains a neighborhood of $x_t$. 

\section{Conclusion}

In this paper, we have considered the problem of finding optimal solutions to a stochastic optimization problem using a tropical dynamic programming approach. The main result is that, under certain assumptions, the approximations generated by the algorithm converge to the true value function.

\section{Future Work}

Future work includes extending the results presented in this paper to more general stochastic optimization problems, as well as exploring the practical implications of the convergence results.

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On the one hand, for every index $k \in \mathbb{N}$ one has
\begin{align*}
\nabla_{t+1}^{k+1}(x_t^k) &= \mathcal{B}_t \left( \nabla_{t+1}^{k+1}(x_t^k) \right), \quad \text{(Tightness)} \\
&\geq \mathcal{B}_t \left( \nabla_{t+1}^k(x_t^k) \right), \quad \text{(Monotonicity)} \\
&= \mathbb{E} \left[ \mathcal{B}_t W_{t+1}^k \nabla_{t+1}^k(x_t^k) \right] \\
&= \mathbb{E} \left[ \mathcal{B}_t W_{t+1}(x_t^k, u_t^k W_{t+1}) + \nabla_{t+1}^k \left( f_{t+1}^k(x_t^k, u_t^k W_{t+1}) \right) \right] \quad \text{(by definition of } \mathcal{B}_t \text{)} \\
&= \sum_{w \in \text{supp}(W_{t+1})} \mathbb{P} \left[ W_{t+1} = w \right] \left( f_t^w(x_t^k, u_t^w) + \nabla_{t+1}^k \left( f_{t+1}^w(x_t^k, u_t^w) \right) \right) \\
&= \sum_{w \in \text{supp}(W_{t+1})} \mathbb{P} \left[ W_{t+1} = w \right] \left( f_t^w(x_t^k, u_t^w) + \nabla_{t+1}^k \left( f_{t+1}^w(x_t^k, u_t^w) \right) \right). 
\end{align*}

On the other hand, for every index $k \in \mathbb{N}$ one has
\begin{align*}
\nabla_{t+1}^{k+1}(x_t^k) &= \mathcal{B}_t \left( \nabla_{t+1}^{k+1}(x_t^k) \right), \quad \text{(Tightness)} \\
&= \mathbb{E} \left[ \mathcal{B}_t W_{t+1}^k \nabla_{t+1}^{k+1}(x_t^k) \right] \\
&\leq \mathbb{E} \left[ \mathcal{B}_t W_{t+1}(x_t^k, u_t^k W_{t+1}) + \nabla_{t+1}^{k+1} \left( f_{t+1}^k(x_t^k, u_t^k W_{t+1}) \right) \right] \quad \text{(Def. of pointwise } \mathcal{B}_t^w) \\
&\leq \mathbb{E} \left[ \mathcal{B}_t W_{t+1}(x_t^k, u_t^k W_{t+1}) + \nabla_{t+1}^{k+1} \left( f_{t+1}^k(x_t^k, u_t^k W_{t+1}) \right) \right] \quad \text{(Monotonicity)} \\
&= \sum_{w \in \text{supp}(W_{t+1})} \mathbb{P} \left[ W_{t+1} = w \right] \left( f_t^w(x_t^k, u_t^w) + \nabla_{t+1}^{k+1} \left( f_{t+1}^w(x_t^k, u_t^w) \right) \right). 
\end{align*}

By definition of the problem-child trajectory, recall that $u_t^k := u_t^w$, thus we have $x_{t+1}^k := f_t^w(x_t^k, u_t^w)$ and for every $k \in \mathbb{N}$
\begin{align*}
0 &\leq \nabla_{t+1}^{k+1}(x_t^k) - \nabla_{t+1}^{k+1}(x_t^k) \\
&\leq \sum_{w \in \text{supp}(W_{t+1})} \mathbb{P} \left[ W_{t+1} = w \right] \left( \nabla_{t+1}^k \left( f_t^w(x_t^k, u_t^w) \right) \right) \\
&\leq \nabla_{t+1}^{k+1}(x_t^k) - \nabla_{t+1}^{k+1}(x_t^k). 
\end{align*}

Thus, we get that for every function $\sigma : \mathbb{N} \rightarrow \mathbb{N}$
\begin{equation}
0 \leq \nabla_{t+1}^{k+1}(x_t^{\sigma(k)}) - \nabla_{t+1}^{k+1}(x_t^{\sigma(k)}) \leq \nabla_{t+1}^k(x_t^{\sigma(k)}) - \nabla_{t+1}^k(x_t^{\sigma(k)}) . \tag{11}
\end{equation}

By Lemma 8 and continuity of the dynamics, there exists an increasing function $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ such that the sequence of future states $x_{t+1}^{\sigma(k)} = f_t^{w_t^{\sigma(k)}}(x_t^{\sigma(k)}, u_t^{\sigma(k)})$, $k \in \mathbb{N}$, converges to some future state $x_{t+1}^* \in X_{t+1}$. Thus, by Lemma 9 applied to the $2L_{V_{t+1}}^\ast$-Lipschitz functions $g^k := \nabla_{t+1}^{\sigma(k)} - \nabla_{t+1}^{\sigma(k)}$, $k \in \mathbb{N}$ and the sequence $y^k := x_t^{\sigma(k)}$, $k \in \mathbb{N}$ we have that
\begin{align*}
\nabla_{t+1}^{\sigma(k)}(x_t^{\sigma(k)}) - \nabla_{t+1}^{\sigma(k)}(x_t^{\sigma(k)}) \rightarrow_{k \rightarrow +\infty} \nabla_{t+1}^\ast(x_t^*) - \nabla_{t+1}^\ast(x_t^*). 
\end{align*}

Likewise, by Lemma 8 applied to the $2L_{V_{t}}^\ast$-Lipschitz functions $g^k := \nabla_{t}^{\sigma(k)+1} - \nabla_{t}^{\sigma(k)+1}$, $k \in \mathbb{N}$ and the sequence $y^k := x_t^{\sigma(k)}$, $k \in \mathbb{N}$ we have that
\begin{align*}
\nabla_{t}^{\sigma(k)+1}(x_t^{\sigma(k)}) - \nabla_{t}^{\sigma(k)+1}(x_t^{\sigma(k)}) \rightarrow_{k \rightarrow +\infty} \nabla_{t}^\ast(x_t^*) - \nabla_{t}^\ast(x_t^*). 
\end{align*}

Thus, taking the limit in $k$ in Equation 11, we have that
\begin{align*}
0 \leq \nabla_{t+1}^\ast(x_t^*) - \nabla_{t+1}^\ast(x_t^*) \leq \nabla_{t+1}^\ast(x_t^*) - \nabla_{t+1}^\ast(x_t^*). 
\end{align*}

By induction hypothesis 10 we have that $\nabla_{t+1}(x_t^*) - \nabla_{t+1}(x_t^*) = 0$. Thus, we have shown that
\begin{align*}
\nabla_{t+1}^\ast(x_t^*) = \nabla_{t+1}^\ast(x_t^*). 
\end{align*}

This concludes the proof.
4 Illustrations in the linear-polyhedral framework

In this section, we first present a class of Lipschitz MSP that we call linear-polyhedral MSP where dynamics are linear and costs are polyhedral, i.e., functions with convex polyhedral epigraph. Second, we give three selection functions, one which generates polyhedral lower approximations (see §4.2), and two which generates upper approximations, one as infima of $U$-shaped functions (see §4.3) and one as infima of $V$-shaped functions (see §4.4).

In Table 1 we illustrate the flexibility made available by TDP to the decision maker to approximate value functions. Implementations were done in the programming language Julia 1.4.2 using the optimization interface JuMP 0.21.3, [9]. The code is available online (https://github.com/BenoitTran/TDP) as a collection of Julia Notebooks.

| Selection mapping | Tight | Valid | Averaged | Computational difficulty |
|-------------------|-------|-------|----------|--------------------------|
| SDDP              | ✓     | ✗     | ✓        | Card($\mathbf{W}_{t+1}$) LPs |
| $\mathbf{U}$      | ✓     | ✗     |          | Card($\mathbf{W}_{t+1}$) · Card($\mathbf{F}$) QPs |
| $\mathbf{V}$      | ✓     | ✓     | ✗        | one LP |

Table 1: Summary of the three selection functions presented in Section 4

4.1 Linear-polyhedral MSP

We want to solve MSPs where the dynamics are linear and the costs are polyhedral. That is, we want to solve optimization problems of the form (1) where for each time step $t$, we want to solve optimization problems of the form (1) where for each time step $t$

$$f_t^w(x, u) = A_t^w x + B_t^w u$$

for some matrices $A_t^w$ and $B_t^w$ of coherent dimensions and the cost is polyhedral:

$$c_t^w(x, u) = \max_{i \in I_t} \langle c_t^w, (x; u) \rangle + d_t^w + \delta_{P_t^w}(x, u).$$

(12)

where $I_t$ is a finite set, $c_t^w \in \mathbb{X} \times \mathbb{U}$, $d_t^w$ is a scalar and $P_t^w$ is a convex polyhedron. The final cost function $\psi$ is of the form $\psi(x) = \max_{i \in I_T} \langle c_T^w, x \rangle + d_T^w + \delta_{\mathbb{X}_T}$ where $\mathbb{X}_T$ is a nonempty convex polytope. We assume that Assumption [1][2] and 4 are satisfied.

**Proposition 11 (Linear-polyhedral MSP are Lipschitz MSP).** Linear-polyhedral MSP are Lipschitz MSP in the sense of Assumption 3.

**Proof.** By construction, the costs $c_t^w$ and the dynamics $f_t^w$ are Lipschitz continuous with explicit constants. We show that for every $t \in [0, T-1]$ and each $w \in \text{supp}(\mathbf{W}_{t+1})$, the constraint set-valued mapping $\mathcal{U}_t^w$ is Lipschitz continuous. From [19] Example 9.35, it is enough to show that the graph of $\mathcal{U}_t^w$ is a convex polyhedron. By assumption dom $c_t^w$ is a convex polyhedron and by recourse Graph $\mathcal{U}_t^w$ is nonempty. As a nonempty intersection of convex polyhedron is a convex polyhedron, we only have to show that $\{(x, u) \in \mathbb{X} \times \mathbb{U} \mid f_t^w(x, u) \in \mathbb{X}_{t+1}\}$ is a convex polyhedron as well.

Using Equation [3] we have that $X_{t+1}$ is given by $X_{t+1} = \cap_{w \in \text{supp}(\mathbf{W}_{t+2})} \pi_{\mathbb{X}}(\text{dom} c_{t+1}^w)$, which is the nonempty intersection of convex polyhedron. Thus, $X_{t+1}$ is a convex polyhedron which implies that there exist a matrix $Q_{t+1}$ and a vector $b_{t+1}$ such that $X_{t+1} = \{x \in \mathbb{X} \mid Q_{t+1} x \leq b_{t+1}\}$. Therefore, we obtain that the two following sets coincide

$$\{(x, u) \in \mathbb{X} \times \mathbb{U} \mid f_t^w(x, u) \in \mathbb{X}_{t+1}\} = \{(x, u) \in \mathbb{X} \times \mathbb{U} \mid Q_{t+1} A_t^w x + Q_{t+1} B_t^w u \leq b_{t+1}\}.$$

The latter being convex polyhedral we obtain that the former is convex polyhedral. This ends the proof.

Now, observe that as linear-polyhedral MSP are Lipschitz MSP, by Corollary 3, the value function $V_t$ is $L_{V_t}$-Lipschitz continuous on $X_t$ for all $t \in [0, T]$. Moreover, under the recourse assumption 4 we can show that the Bellman operators $\mathcal{B}_{ht \in [0, T-1]}$ preserves polyhedrality in the sense defined below.
Lemma 12 (Bₜ preserves polyhedrality). For every \( t \in [0, T-1] \), if \( \phi : X \to \mathbb{R} \) is a polyhedral function, i.e. its epigraph is a convex polyhedron, then \( \mathcal{B}_t(\phi) \) is a polyhedral function as well.

Proof. For every \( w \in \text{supp}(W_{t+1}) \), we have shown in the proof of Proposition \([1]\) that the graph of \( \mathcal{U}_t^w \) is a convex polyhedron. Thus, \( (x, u) \mapsto c_t^w(x, u) + \phi(f_t^w(x, u)) + \delta_{\text{Graph}_t^w}(x, u) \) is convex polyhedral and by \([2]\) Proposition 5.1.8.e, \( \mathcal{B}_t^w(\phi) \) is polyhedral as well. Finally, under Assumption \([\text{I}]\), we deduce that \( \mathcal{B}_t(\phi) := \sum_{w \in \text{supp}(W_{t+1})} \mathcal{B}_t^w(\phi) \) is polyhedral as a finite sum of polyhedral functions. This ends the proof. \( \square \)

4.2 SDDP lower approximations

Stochastic Dual Dynamic Programming is a popular algorithm which was introduced by Perreira and Pinto in 1991 (see \([13]\)) and studied extensively since then, e.g. \([14, 15, 16, 20, 22]\).

Lemma 12 is the main intuitive justification of using SDDP in linear-polyhedral MSPs: if the final cost function is polyhedral, as the operators \( \{\mathcal{B}_t\}_{t \in [0, T-1]} \) preserve polyhedrality, by backward induction on time, we obtain that the value function \( V_t \) is polyhedral for every \( t \in [0, T] \). Hence, the decision maker might be tempted to construct polyhedral approximations of \( V_t \).

We now present a way to generate polyhedral lower approximations of value functions, as done in the literature of SDDP, by defining a proper selection mapping. When the value functions are convex, it builds lower approximations as suprema of affine cuts. We put SDDP in TDP’s framework by constructing a lower selection function.

First, for every time step \( t \in [0, T] \), define the set of basic functions,

\[
\mathcal{F}^\text{SDDP}_t := \{ (a, \cdot) + b + \delta_{X_t} \mid (a, b) \in X \times \mathbb{R} \text{ s.t. } \|a\| \leq L_{V_t} \}.
\]

At time \( t = T \), given a trial point \( x \in X_T \), we define \( \mathcal{S}^\text{SDDP}_T(x) = \langle a_x, \cdot \rangle - x + b_x \), where \( a_x \) is a subgradient of the convex polyhedral function \( \psi \) at \( x \) and \( b_x = \psi(x) \). Tightness and validity of \( \mathcal{S}^\text{SDDP}_t \) follows from the given expression. Now, for \( t \in [0, T-1] \), we compute a tight and valid cut for \( \mathcal{B}_t^w \) for each possible value of the noise \( w \) then average it to get a tight and valid cut for \( \mathcal{B}_t \). The details are given in Algorithm 3.

Algorithm 3 SDDP Selection function \( \mathcal{S}^\text{SDDP}_t \) for \( t < T \)

**Input:** A set of basic functions \( \mathcal{F}^\text{SDDP}_{t+1} \subset \mathcal{F}^\text{SDDP}_t \) and a trial point \( x_t \in X_t \).

**Output:** A tight and valid basic function \( \phi \in \mathcal{F}^\text{SDDP}_t \).

for \( w \in \text{supp}(W_{t+1}) \) do

Solve by linear programming \( b^w := \mathcal{B}_t^w(\mathcal{U}_t^w)(x) \) and compute a subgradient \( a^w \) of \( \mathcal{B}_t^w(\mathcal{U}_t^w) \) at \( x \).

end for

Set \( \phi := \langle a, \cdot \rangle + b + \delta_{X_t} \), where \( a := \sum_{w \in \text{supp}(W_{t+1})} p_w a^w \) and \( b = \sum_{w \in \text{supp}(W_{t+1})} p_w b^w \).

We say that \( S_t^w \) is a selection function for \( \mathcal{B}_t^w \), for a given noise value \( w \in \text{supp}(W_{t+1}) \) if Definition 5 is satisfied when replacing \( \mathcal{B}_t \) by \( \mathcal{B}_t^w \). We now prove that \( \mathcal{S}^\text{SDDP}_t \) is a selection function, i.e. it is tight and valid in the sense of Definition 5. It follows from the general fact that by averaging functions which are tight and valid for the pointwise Bellman operators \( \mathcal{B}_t^w \), \( w \in \text{supp}(W_{t+1}) \), then one get a tight and valid function for the average Bellman operator \( \mathcal{B}_t \). Note that the average of affine functions is still an affine function, the set of basic functions \( \mathcal{F}^\text{SDDP}_t \) is stable by averaging.

Lemma 13. Let a time step \( t \in [0, T-1] \) be fixed and let be given for every noise value \( w \in \text{supp}(W_{t+1}) \) a selection function \( S_t^w \) for \( \mathcal{B}_t^w \). Then, the mapping \( S_t \) defined by \( S_t = \mathbb{E}[S_t^W] \) is a selection mapping for \( \mathcal{B}_t \).
Proof. Fix $t \in [0, T-1]$. Given a trial point $x \in X_t$ and a set of basic functions $F$, the pointwise tightness (resp. validity) equality (resp. inequality) is satisfied for every realization $w$ of the noise $W_{t+1}$, that is

$$S^w_t(F, x)(x) = B^w_t(V_F)(x),$$  \hspace{1cm} \text{(Pointwise tightness)}

$$S^w_t(F, x) \geq B^w_t(\bar{V}_F),$$  \hspace{1cm} \text{(Pointwise validity when building upper approximations)}

$$S^w_t(F, x) \leq B^w_t(\underline{V}_F).$$  \hspace{1cm} \text{(Pointwise validity when building lower approximations)}

Recall that $B_t(V_F)(x) = E[B_t^{W_{t+1}}(V_F)(x)]$, thus taking the expectation in the above equality and inequalities, one gets the lemma. \qed

Proposition 14 (SDDP Selection function). For every $t \in [0, T]$, the mapping $S^{SDDP}_t$ is a selection function in the sense of Definition 3.

Proof. For $t = T$, for every $x_T \in X_T$, by construction we have

$$S^{SDDP}_T(x_T) = \psi(x_T) = V_T(x_T).$$

Thus, $S^{SDDP}_T$ is tight and it is valid as $S^{SDDP}_T(x_T) = \langle a, \cdot - x_T \rangle + \psi(x_T)$ is an affine minorant of the convex function $\psi$ which is exact at $x_T$. Now, fix $t \in [0, T-1]$, a set of basic functions $E_t \subset F^{SDDP}_t$ and a trial point $x_t \in X_t$. By construction, $S^{SDDP}_t$ is tight as we have

$$S^{SDDP}_t(E_t, x_t) = \langle a, x_t - x_t \rangle + E[B_t^{W_{t+1}}(V_E)(x_t)] = B_t(V_E)(x_t).$$

Moreover, for every $w \in \text{supp}(W_{t+1})$, $a^w$ (see Algorithm 3) is a subgradient of $B_t^w(V_E)$ at $x_t$. Thus as $a$ is equal to $E[a^{W_{t+1}}]$ it is a subgradient of $B_t(V_E)$ at $x_t$. Hence, the mapping $S^{SDDP}_t$ is valid. \qed

4.3 $U$-upper approximations

We have seen in Lemma 13 that in order to construct a selection function for $B_t$, it suffices to construct a selection function for each pointwise Bellman operator $B^w_t$. In order to do so, for upper approximations we exploit the min-additivity of the pointwise Bellman operators $B^w_t$. That is, given a set of functions $F$, we use the following decomposition

$$\forall t \in [0, T-1], \forall x \in X, \forall w \in \text{supp}(W_{t+1}), B^w_t \left( \inf_{\phi \in F} \phi \right)(x) = \inf_{\phi \in F} B^w_t(\phi)(x).$$

This is a decomposition of the computation of $B^w_t(\bar{V}_F)$ which is possible for upper approximations but not for lower approximations as for minimization problems, the Bellman operators (average or pointwise) are min-plus linear but generally not max-plus linear.

However, in linear-polyhedral MSP, the value functions are polyhedral. Approximating from above value function $V_t$ by infima of convex quadratics is not suited: in particular, one cannot ensure validity of a quadratic at a kink of the polyhedral function $V_t$. Still, we present a selection function which is tight but not valid. In the numerical experiment of Figure 2 we illustrate that the selection function defined below might not be valid, but the error is still reasonable. Yet, this will motivate the use of other basic functions more suited to the linear-polyhedral framework, as done in §4.4.

We consider basic functions that are $U$-shaped, i.e. of the form $\frac{c}{2} \|x - a\|^2 + b$ for some constant $c > 0$, vector $a$ and scalar $b$. We call such function a $c$-function. We now fix a sequence of constants $(c_t)_{t \in [0, T]}$ such that $c_t > L_{V_t}$, for every time $t \in [0, T]$, define the set of basic functions

$$F^U_t = \left\{ \frac{c_t}{2} \|x - a\|^2 + b + \delta_x, \quad (a, b) \in X \times R \right\}.$$
Figure 2: U-SDDP approximations of the value functions. In the bottom right we see that the $U$-shaped basic functions might not be valid when the trial point is associated with a kink of value function. Still, we observe that the gap between upper and lower approximations vanishes along the problem-child trajectory (in dashed lines).

At time $t = T$, we select the $c_T$-quadratic mapping which is equal to $\psi$ at point $x \in X_T$ and has same (sub)gradient at $x$, i.e. $S^U_T(x) = \frac{c_T}{2} \| \cdot - a \|^2 + b$ where $a = x - \frac{1}{c_T} \lambda$ and $b = \psi(x) - \frac{1}{2c_T} \| \lambda \|^2$ with $\lambda$ being a subgradient of $\psi$ at $x$.

The mapping $S^U_t$, defined in Algorithm 4, is tight but not necessarily valid, see an illustration in Figure 2. As with SDDP, in order to build a tight selection function at $t < T$ for $B_t$, we first compute a tight selection function for each $B^w_t$, $w \in \text{supp}(W_{t+1})$, which can be done numerically by quadratic programming.

**Algorithm 4 U Selection function $S^U_t$ for $t < T$**

**Input:** A set of basic functions $F_{t+1} \subset F^U_{t+1}$ and a trial point $x_t \in X_t$.

**Output:** A tight basic function $\overline{\phi}_t \in F^U_t$.

for $w \in \text{supp}(W_{t+1})$ do

Solve by quadratic programming $v^w := B^w_t \left( \nabla F_{t+1} \right)(x) = \inf_{\phi \in F_{t+1}} B^w_t \left( \phi \right)(x)$ and compute $a^w = x - \frac{1}{c} \lambda$ and $b^w = v^w - \frac{1}{2c} \| \lambda \|^2$ with $\lambda$ being a subgradient of $B^w_t \left( \nabla F_{t+1} \right)$ at $x$.

end for

Set $\overline{\phi} := v^w \| \cdot - a \|^2 + b + \delta X_t$, where $a := \mathbb{E}[a^w_t + 1]$ and $b = \mathbb{E}[\| \cdot - a \|^2 + b^w_{t+1}]$.

### 4.4 V-upper approximations

We have seen in §4.3 that $U$-shaped basic functions may not be suited to approximate polyhedral functions. In [16], upper approximations which were polyhedral as well were introduced. In this
section we propose upper approximations of $V_t$ as infima of $V$-shaped functions. Even though when $V_t$ is polyhedral the approach of [16] seems the most natural, their approximations cannot be easily expressed as a pointwise infima of basic functions.

In future works we will add a max-plus/min-plus projection step to TDP in order to broaden the possibilities of converging approximations available to the decision maker. In particular, polyhedral upper approximations as in [16] will be covered.

In this section, by introducing a new tight and valid selection function, we would like to emphasize on the flexibility already available to the decision maker by adopting the framework of TDP.

We consider $V$-shaped functions, i.e., functions of the form $L\|x-a\|_1 + b$ with $a \in \mathbb{X} = \mathbb{R}^n$ and $b \in \mathbb{R}$ and a constant $L > 0$. We define for every time step $t \in [0, T[$, the set of basic functions

$$\overline{\mathcal{F}}_t^V := \left\{ \frac{L\psi_t}{\sqrt{n}} \cdot -a\|_1 + b \mid (a, b) \in \mathbb{X} \times \mathbb{R} \right\}.$$  

At time $t = T$, we compute a $V$-shaped function at $\psi(x)$, i.e., given a trial point $x \in X_T$, using the expression $\overline{\mathcal{S}}_T^V(x) = \frac{L\psi_T}{\sqrt{n}} \cdot -x\|_1 + \psi(x)$. For time $t \in [0, T-1]$, the selection function is given in Algorithm 5. The main difference with the previous cases treated in §4.2 and in §4.3 is that $V$-shaped function are not stable by averaging as the average of several $V$-shaped function is a polyhedral function.

---

**Algorithm 5 V Selection function $\overline{\mathcal{S}}_t^V$ for $t < T$**

**Input:** A set of basic functions $\overline{\mathcal{F}}_{t+1} \subset \overline{\mathcal{F}}_{t+1}^V$ and a trial point $x_t \in X_t$.

**Output:** A tight and valid basic function $\overline{\phi}_t \in \overline{\mathcal{F}}_t^V$.

Solve by linear programming $b := \mathcal{B}_t \left( \overline{\mathcal{V}}_{t+1} \right) (x_t)$.

Set $\overline{\phi}_t := \frac{L\psi_t}{\sqrt{n}} \cdot -x_t\|_1 + b$.

---

**Proposition 15 (V Selection function).** For every $t \in [0, T[$, the mapping $\overline{\mathcal{S}}_t^V$ described in Algorithm 5 is a selection function in the sense of Definition 3.

**Proof.** At time $t = T$, for every $x_T \in X_T$, we have $\overline{\mathcal{S}}_T^V(x_T) = \frac{L\psi_T}{\sqrt{n}} \cdot -x_T\|_1 + \psi(x_T)$. Thus, $\overline{\mathcal{S}}_T^V(x_T)(x_T) = \psi(x_T)$ and $\overline{\mathcal{S}}_T^V$ is a tight mapping. As the polyhedral function $\psi(x) = \max_{i \in I_T} (c_i^T, x) + d_i^T + \delta_{X_T}$ is $L_{\psi_T}$-Lipschitz continuous, by Cauchy-Schwarz inequality, for every $x \in X_T$ and $i \in I_T$, we have

$$\langle c_i^T, x - x_T \rangle \leq \|c_i^T\|_2 \|x - x_T\|_2 \leq L_{\psi_T} \frac{1}{\sqrt{n}} \|x - x_T\|_1.$$  

Adding $\langle c_i^T, x_T \rangle + d_i^T$ on both sides of the last inequality and taking the maximum over $i \in I_T$ we have that

$$\psi(x) = \max_{i \in I_T} (c_i^T, x) + d_i^T \leq L_{\psi_T} \frac{1}{\sqrt{n}} \|x - x_T\|_1 + \psi(x_T) = \overline{\mathcal{S}}_T^V(x_T)(x),$$  

which gives that $\overline{\mathcal{S}}_T^V$ is a valid mapping.

Now, fix $t < T$, we show that the mapping $\overline{\mathcal{S}}_t^V$ is tight and valid as well. By construction, for every set of basic functions $\overline{\mathcal{F}}_{t+1} \subset \overline{\mathcal{F}}_{t+1}$ and trial point $x_t \in X_t$, we have

$$\overline{\mathcal{S}}_t^V (\overline{\mathcal{F}}_{t+1}, x_t)(x_t) = b = \mathcal{B}_t \left( \overline{\mathcal{V}}_{F_{t+1}} \right) (x_t).$$  

Hence, $\overline{\mathcal{S}}_t^V$ is a tight mapping.
Figure 3: V-SDDP approximations of the value functions. As the selection function $S^V_t$ does not average other basic functions to compute a new one (compare with $S^U_t$ or $S^{SDDP}_t$), we lose the regularizing effect of averaging: the upper basic functions added are very sharp. We still observe that the gap between upper and lower approximations vanishes along the problem-child trajectory (in dashed lines).

We check that $S^V_t$ is a valid mapping. First, as each basic function $\phi \in F^{t+1}_t$ is $L^{V_t}$-Lipschitz continuous on $X_t$, we show that $\overline{V}_{t+1}$ is $L_{V_{t+1}}$-Lipschitz continuous on $X_t$ as well. Given $x_1, x_2 \in X_t$, we have

$$\begin{align*}
|\overline{V}_{t+1}(x_1) - \overline{V}_{t+1}(x_2)| &= \left| \inf_{\phi \in F^{t+1}_t} \phi(x_1) - \inf_{\phi \in F^{t+1}_t} \phi(x_2) \right| \\
&\leq \sup_{\phi \in F^{t+1}_t} |\phi(x_1) - \phi(x_2)| \\
&\leq L_{V_t} \|x_1 - x_2\|.
\end{align*}$$

As the Bellman operator $B_t$ is Lipschitz regular in the sense of Proposition 2, $B_t \left( \overline{V}_{t+1} \right)$ is $L_{V_t}$-Lipschitz continuous.

Second, by min-additivity of the Bellman operator $B_t$, we have that

$$B_t \left( \overline{V}_{t+1} \right)(x) = B_t \left( \inf_{\phi \in F^{t+1}_t} \phi \right)(x) = \inf_{\phi \in F^{t+1}_t} B_t(\phi)(x).$$

Recall that by Lemma 12, the Bellman operator $B_t$ preserves polyhedrality. As $\phi \in F^{t+1}_t$ is polyhedral, $B_t(\phi)$ is polyhedral as well and as in the case $t = T$, mutatis mutandis we have that $S^V_t$ is valid. 

□
Conclusion

• TDP generates simultaneously monotonic approximations \((V^k_t)\) and \((V^k_t)\) of \(V_t\).
• Each approximation is either a min-plus or max-plus linear combinations of basic functions.
• Each basic function should be tight and valid.
• The approximations are refined iteratively along the Problem-child trajectory without discretizing the state space.
• The gap between upper and lower approximation vanishes along the Problem-child trajectory.
• TDP generalizes a similar approach done in [16] and proved by [4] for a variant of SDDP in convex MSPs.

Perspectives

• Consider an additional min-plus/max-plus projection step of suprema/infima of basic functions.
• Extensive numerical comparisons with existing methods, namely classical SDDP and the upper approximations obtained by Fenchel duality of [11].
• Extend the scope of TDP to encompass Partially Observed Markov Decision Processes. A first attempt to do so can be found in Appendix A.

A Tropical Dynamic Programming for POMDP

In this section, we present an on-going work to apply TDP on Partially Observed Markov Decision Processes (POMDP).

A.1 Recalls on POMDP

Formally, a POMDP is described (in the finite settings) by a finite set of states \(X = \{x_1, \ldots, x_{|X|}\}\), a finite set of actions \(U = \{u_1, \ldots, u_{|U|}\}\), a finite set of observations \(O = \{o_1, \ldots, o_{|O|}\}\), transition probabilities of the Markov chain

\[
P^u_t(x_i, x_j) = \mathbb{P}\{X_{t+1} = x_j \mid X_t = x_i, U_t = u\},
\]

and conditional law of the observations

\[
Q_{t+1}(o \mid x, u) = \mathbb{P}\{O_{t+1} = o \mid X_{t+1} = x, U_t = u\},
\]

a real-valued cost function \(L_t(x, u)\) for any \(t \in [0, T-1]\), a final cost \(K(x)\) and an initial probability law in the simplex of \(\mathbb{R}^{|X|}\) called the initial belief \(b_0\). We assume here that the state space the control space and the observation space dimensions do not vary with time but for the sake of clarity we will use the notation \(X_t\) to designate the state space at time \(t\) even if it is equal to \(X\) and the same for control and observation states.

Under Markov assumptions, we can use at time \(t\) a probability distribution \(b_t\), whose name is a reminder of belief, over current states as a sufficient statistic for the history of actions and observations up to time \(t\). The space of beliefs is the simplex of \(\mathbb{R}^{|X|}\), denoted \(\Delta_{|X|}\). The belief dynamics, at time \(t\), driven by action \(u_t\) and observation \(o_{t+1}\) is given by by the equation

\[
b_{t+1} = \tau_t(b_t, u_t, o_{t+1})
\]
with \( b_{t+1} \in \Delta_{|X|} \) given by
\[
    b_{t+1}(x_{t+1}) = \beta_{t+1} Q_{t+1}(o_{t+1} \mid x_{t+1}, u_t) \left( \sum_{x_t \in X_t} b(x_t) P_t^u(x_t, x_{t+1}) \right) \quad \forall x_{t+1} \in X_{t+1} ,
\]
where \( \beta_{t+1} \) is a normalization constant to ensure that \( b_{t+1} \in \Delta_{|X|} \), that is
\[
    \beta_{t+1}^{-1} = \sum_{x_{t+1} \in X_{t+1}} Q_{t+1}(o_{t+1} \mid x_{t+1}, u_t) \left( \sum_{x_t \in X_t} P_t^u(x_t, x_{t+1}) b(x_t) \right) .
\]
To simplify the notation we introduce the (sub-stochastic) matrix defined as follows
\[
    M_t^{u, o_{t+1}}(x_t, x_{t+1}) = Q_t(o_{t+1} \mid x_t, u_t) P_t^u(x_t, x_{t+1}) \quad \forall (x_t, x_{t+1}) \in X_t \times X_{t+1} ,
\]
where we have \( \sum_{o_{t+1}} \sum_{x_{t+1}} M_t^{u, o_{t+1}}(x_t, x_{t+1}) = 1 \). Using matrix notations, where beliefs are represented by row vector and \( \mathbf{1} \) is a column vector full of ones, we can rewrite the beliefs dynamics as
\[
    \tau_t(b_t, u_t, o_{t+1}) = \frac{b_t M_t^{u, o_{t+1}}}{b_t M_t^{u, o_{t+1}} \mathbf{1}} \in \Delta_{|X|} .
\]

In general the object of the optimization problem is to generate a policy that minimizes expected finite horizon cost for the controlled Markov chain \( \{X_t^u\}_{t \in \mathbb{N}} \) with transition matrix \( P^u \). That is consider the minimization problem given by Equation 17.

\[
    J(b_0) = \min_{U_1, \ldots, U_T} \mathbb{E} \left[ \sum_{t=0}^{T-1} L_t(X_t, U_t) + K(X_T) \mid b_0 \right] .
\]

It is classical to derive a Bellman equation for the beliefs given by the bellman operators for \( t \in [0, T - 1] \)
\[
    B^u_t(V) = \inf_{u \in U} B_t^u(V) ,
\]
where for each \( u \in U \) and \( t \in [0, T - 1] \), the Bellman operator \( B_t^u \) is defined by
\[
    B_t^u(V)(b) = b L_t^u + \sum_{o \in \Omega_{t+1}} (b M_t^{u, o} \mathbf{1}) V(b M_t^{u, o} \mathbf{1}) ,
\]
where \( L_t^u \) is the column vector \( (L_t^u(x_t))_{x_t \in X_t} \). Note that the mapping \( o_{t+1} \in \Omega_{t+1} \mapsto (b_t M_t^{u, o_{t+1}} \mathbf{1}) \) is a probability distribution on \( \Omega_{t+1} \) (\( \sum_{o \in \Omega_{t+1}} b_t M_t^{u, o_{t+1}} \mathbf{1} = 1 \)).

The Bellman operator can be also written as
\[
    B_t^u(V)(b) = b L_t^u + \sum_{b' \in \Delta_{|X|}} P_t^u(b, b') V(b') ,
\]
where, \( P_t^u \) is a controlled Markov chain transition matrix in the belief space. Indeed
\[
    P_t^u(b, b') = \begin{cases} 
        (b M_t^{u, o} \mathbf{1}) & \text{when } b' = \frac{b M_t^{u, o} \mathbf{1}}{b M_t^{u, o} \mathbf{1}} \text{ with } o \in \Omega_{t+1} , \\
        0 & \text{if not} ,
    \end{cases}
\]
which is a classical Bellman equation of a controlled Markov chain but with a state space in the belief space.

We conclude this section by the following lemma

**Proposition 16.** The value functions \( \{V_t\}_{t \in [0, T]} \) solutions of the Bellman Equation
\[
    \forall b \in \mathbb{R}^{|X|} \quad V_T(b) = b K \quad \text{and} \quad \forall t \in [0, T - 1] \quad V_t(b) = \inf_{u \in U} B_t^u(V_{t+1}(b)) ,
\]
where the operator \( B_t^u \) is given by Equation 19 are such that \( V_0(b_0) \) is the optimal value of the minimization problem given by Equation 17.
A.2 The Bellman operator defined in Equation (18) propagate Lipschitz mappings

Proposition 17. For $t \in [0, T-1]$, assume that the mappings $L_t(u, \cdot)$ satisfy $\|L_t(u, \cdot)\|_\infty \leq c^i$ for all $u \in U$ and assume that a mapping $K$ satisfy $\sup_{x \in X}\|K(x)\| = K < +\infty$. Then the solution of the Bellman Equation (22) are Lipschitz mappings.

Proof.

• We consider the operator $\tilde{B}_t^u$ defined for mappings $\tilde{V}: \mathbb{R}_{+}^{\mathcal{X}} \to \mathbb{R}$ by

$$\tilde{B}_t^u(\tilde{V})(c) = cL_t^u + \sum_{o \in \mathcal{O}_{t+1}} \tilde{V}(cM_t^{u,o}) \quad \forall c \in \mathbb{R}_{+}^{\mathcal{X}},$$

where $L_t^u$ stands for the column vector $(L_t(x, u))_{x \in \mathcal{X}}$, and we recall that beliefs are row vectors. We consider $\{\tilde{V}_t\}_{t \in [0, T]}$ solution of the Bellman Equation

$$\forall c \in \mathbb{R}_{+}^{\mathcal{X}} \quad \tilde{V}_t(c) = cK \quad \text{and} \quad \forall t \in [0, T-1] \quad \tilde{V}_t(c) = \inf_{u \in U} \tilde{B}_t^u(\tilde{V}_{t+1})(c). \quad (24)$$

First, we straightforwardly obtain by backward induction that the value functions $(\tilde{V}_t)_{t \in [0, T]}$ are homogeneous of degree 1. Second we prove that the operator $\tilde{B}_t^u$ preserves Lipschitz regularity. We proceed as follows. Consider $c$ and $c'$ in $\mathbb{R}_{+}^{\mathcal{X}}$ and suppose that $|\tilde{V}(c) - \tilde{V}(c')| \leq \mathcal{V}|c' - c|_1$. Then we have that

$$\tilde{B}_t^u(\tilde{V})(c') - \tilde{B}_t^u(\tilde{V})(c) = (c' - c)L_t^u + \sum_{o \in \mathcal{O}_{t+1}} \tilde{V}(c'M_t^{u,o}) - \tilde{V}(cM_t^{u,o})$$

$$\leq \mathcal{L}|c' - c|_1 + \sum_{o \in \mathcal{O}_{t+1}} \mathcal{V}|c'M_t^{u,o} - cM_t^{u,o}|_1$$

$$\leq \mathcal{L}|c' - c|_1 + \mathcal{V} \sum_{o \in \mathcal{O}_{t+1}} \left| \sum_{x \in \mathcal{X}} (c'(x) - c(x))M_t^{u,o}(x, x') \right|$$

$$\leq \mathcal{L}|c' - c|_1 + \mathcal{V} \sum_{x \in \mathcal{X}} \sum_{o \in \mathcal{O}_{t+1}} |c'(x) - c(x)| M_t^{u,o}(x, x')$$

$$\leq \mathcal{L}|c' - c|_1 + \mathcal{V} \sum_{x \in \mathcal{X}} |c'(x) - c(x)|$$

As a pointwise minimum of Lipschitz mappings having the same Lipschitz constant is Lipschitz, we obtain the same Lipschitz constant for the operators $\inf_{u \in U} \tilde{B}_t^u$. Then, using the fact that $\nabla_T = K$ we obtain by backward induction that the Bellman value function $\tilde{V}_t$ is $(\mathcal{L}(T-t) + \mathcal{K})$-Lipschitz for $t \in [0, T]$ where $\mathcal{K} = \|K(\cdot)\|_\infty$.

• We prove now an intermediate result to link the solutions of the Bellman Equation (22) to the Bellman Equation (22). Suppose that $\tilde{V}$ is 1-homogeneous and such that $\tilde{V}(b) = V(b)$ for all $b \in \Delta_{\mathcal{X}}$. Then, We prove that $\tilde{B}_t^u(\tilde{V})(b) = B_t^u(V)(b)$ for all $b \in \Delta_{\mathcal{X}}$. For $b \in \Delta_{\mathcal{X}}$, we successively

\footnote{Since the state space is finite we identify mappings $\phi: \mathcal{X} \to \mathbb{R}$ with vectors in $\mathbb{R}^{\mathcal{X}}$.}
have that
\[
\tilde{B}^u_t(\tilde{V})(b) = bL^u_t + \sum_{o \in \Omega_{t+1}} \tilde{V}(bM^u_{t,o})
\]
\[
= bL^u_t + \sum_{o \in \Omega_{t+1}} (bM^u_{t,o}1)\tilde{V}\left(\frac{bM^u_{t,o}}{bM^u_{t-1}1}\right)
\]
\[
= bL^u_t + \sum_{o \in \Omega_{t+1}} (bM^u_{t,o}1)\tilde{V}\left(\frac{bM^u_{t,o}}{bM^u_{t-1}1}\right)
\]
\[
= \tilde{B}^u_t(V)(b).
\] (26)

• Now we turn to solutions of Bellman Equation (22). Since $\tilde{V}_T(c) = cK$ for all $c \in \mathbb{R}_+^{|X|}$ and $V_T(b) = bK$ for all $b \in \Delta_{|X|}$, the two mappings $V_T$ and $\tilde{V}_T$ coincide on the simplex of dimension $|X|$. Then gathering the previous steps we obtain that $V_t$ and $\tilde{V}_t$ coincide also on the simplex of dimension $|X|$ for all $t \in [0,T]$. Finally, for all $t \in [0,T]$ $\tilde{V}_t$ being $(\mathcal{L}(T - t) + K)$-Lipschitz we obtain the same result for $V_t$. 
\[\square\]

A.3 Value of $B_t(V_{t+1})$ when $V_{t+1} = \min_{\alpha \in \Gamma_{t+1}} \langle \alpha, b \rangle$

Assume that $V_{t+1} : b \mapsto \min_{\alpha \in \Gamma_{t+1}} \langle \alpha, b \rangle$ where $\Gamma_{t+1} \subset \mathbb{R}^{|X|}$. Then we obtain that

\[
B_t(V_{t+1})(b) = \min_{u \in U_t} \left(bL^u_t + \sum_{o \in \Omega_{t+1}} (b_{t}\tilde{M}^u_{t,o}1)V_{t+1}\left(\frac{bM^u_{t,o}}{bM^u_{t-1}1}\right)\right)
\]
\[
= \min_{u \in U_t} \left(bL^u_t + \sum_{o \in \Omega_{t+1}} (bM^u_{t,o}1)\min_{\alpha \in \Gamma_{t+1}} \left(\frac{bM^u_{t,o}}{bM^u_{t-1}1}\right)\right)
\]
\[
= \min_{u \in U_t} \left(bL^u_t + \sum_{o \in \Omega_{t+1}} bM^u_{t,o}\alpha^t(u,o)\right) \quad \text{(with } \alpha^t(u,o) = \arg \min_{\alpha \in \Gamma_{t+1}} \frac{bM^u_{t,o}}{bM^u_{t-1}1}\text{)}
\]
\[
= \min_{u \in U_t} \left(L^u_t + \sum_{o \in \Omega_{t+1}} M^u_{t,o}\alpha^t(u,o)\right)
\]
\[
= \min_{\alpha \in \Gamma_t} \langle \alpha, b \rangle,
\] (30)

with $\Gamma_t = \{L^u_t + \sum_{o \in \Omega_{t+1}} M^u_{t,o}\alpha^t(u,o) \mid u \in U_t \text{ and } \alpha^t(u,o) = \arg \min_{\alpha \in \Gamma_{t+1}} \frac{bM^u_{t,o}}{bM^u_{t-1}1}\}$. We therefore obtain that the Bellman value function at time $t$ has the same form as the Bellman value function at time $t + 1$.

We are in a context where the Bellman function that is to be computed is polyhedral concave with a huge polyhedron. It is thus tempting to use our algorithm with polyhedral concave upper approximations and sup of quadratic or Lipschitz mappings as lower approximations.

The Problem-child trajectory technique is used in POMDP algorithms as an heuristic but without a convergence proof as far as we have investigated.
A.4 A lower bound of $B_t(V_{t+1})$

We consider a special case where $V_{t+1} : X \rightarrow \mathbb{R}$ is given by $V_{t+1}(b) = \langle b, \hat{V}_{t+1} \rangle$ and we compute $B_t(V_{t+1})$ as follows

$$B_t(V_{t+1})(b) = \min_{u \in U_t} \left( bL_t^u + \sum_{o \in \mathcal{O}_{t+1}} bM_{t,o}^u \hat{V}_{t+1} \right)$$

$$= \min_{u \in U_t} \left( bL_t^u + \sum_{o \in \mathcal{O}_{t+1}, x \in X_t, x' \in X_{t+1}} Q_{t+1}(o \mid x', u)P_t^u(x, x')b(x)\hat{V}_{t+1}(x') \right)$$

$$\geq \sum_{x \in X_t} b(x) \min_{u \in U_t} \left( L_t(u, x) + \sum_{x' \in X_{t+1}} P_t^u(x, x')\hat{V}_{t+1}(x') \right)$$

$$= \sum_{x \in X_t} b(x)\hat{V}_t(x) = b\hat{V}_t,$$

with

$$\hat{V}_t(x) = \min_{u \in U_t} \left( L_t(u, x) + \sum_{x' \in X_{t+1}} P_t^u(x, x')\hat{V}_{t+1}(x') \right). \quad (31)$$

Using the fact that at time $T$ we have that $V_T = \langle b, \hat{V}_T \rangle$ with $\hat{V}_T = K$ we obtain that for all $t \in [0, T]$ $V_t \geq \langle b, \hat{V}_t \rangle$ where $\hat{V}_t$ is the Value function of the fully observed Bellman equation associated to the POMDP.

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