General Relativity as an Attractor in Scalar–Tensor Stochastic Inflation

Juan García–Bellido and David Wands
Astronomy Centre, School of Mathematical and Physical Sciences, University of Sussex, Brighton BN1 9QH, UK

Abstract
Quantum fluctuations of scalar fields during inflation could determine the very large-scale structure of the universe. In the case of general scalar-tensor gravity theories these fluctuations lead to the diffusion of fundamental constants like the Planck mass and the effective Brans–Dicke parameter, ω. In the particular case of Brans–Dicke gravity, where ω is constant, this leads to runaway solutions with infinitely large values of the Planck mass. However, in a theory with variable ω we find stationary probability distributions with a finite value of the Planck mass peaked at exponentially large values of ω after inflation. We conclude that general relativity is an attractor during the quantum diffusion of the fields.

*PPARC postdoctoral research fellow. E-mail: j.bellido@sussex.ac.uk
†PPARC postdoctoral research fellow. E-mail: d.wands@sussex.ac.uk
I. INTRODUCTION

One of the most important problems for cosmology is the issue of initial conditions for the Big Bang [1]. The first models of inflation assumed that the universe started in a very hot state that supercooled in a metastable vacuum, which then decayed to the true vacuum through a first-order phase transition or just rolled down through a second-order phase transition [2]. Chaotic inflation [2] opened up the possibility of starting inflation from a wide range of initial conditions, including the Planck scale.

Quantum fluctuations in the inflaton field could produce the small perturbations observed in the background radiation [2,3], from which galaxies later evolved. Once we include quantum fluctuations of the scalar fields during inflation, we find that these can be large and dominate the classical evolution as we approach the Planck scale [1]. As a consequence, the scalar fields diffuse, very much like a particle in Brownian motion. The universe is then divided into causally independent inflationary domains, in which the fields acquire different values. One of the most fascinating features of inflation is the process of self-reproduction of the universe [1], by which the values of the fields in some inflationary domains diffuse towards larger rates of expansion, producing new domains and so on for ever. This process might still be occurring, at scales much larger than our present horizon. The global behavior of the universe can then be described with the formalism of stochastic inflation, using probability distributions for the values of the fields in physical space.

It has recently been shown that there are stationary solutions for the diffusion of the inflaton in general relativity [4]. A natural question to ask is whether this picture is still valid as we approach the Planck era, where quantum fluctuations of the metric become important [5]. Although it is generally assumed that the dynamics of the universe can be described by general relativity, the effective theory of gravity might be very different close to the Planck scale. So far the only consistent but by no means definite, since we lack the experimental observations needed to confirm it, theory of quantum gravity is string theory [6]. String theory contains in its massless gravitational sector a dilaton scalar field as well as the graviton. The low-energy effective theory from strings has the form a scalar-tensor theory of gravity [7], with non-trivial couplings of the dilaton to matter [8,9]. Therefore, it is expected that the description of gravitational phenomena, and in particular inflation, close to the Planck scale should also contain this scalar field [10,11].

The string dilaton field can be understood as a Brans–Dicke field [12], which acts like a dynamical gravitational ‘constant’. Jordan–Brans–Dicke theory is the simplest scalar-tensor theory, with a constant kinetic coupling $\omega$, which is bounded by primordial nucleosynthesis [13] and post-Newtonian experiments [14] to be $\omega > 500$. String theory predicts $\omega = -1$ in ten dimensions [15]. However, the low energy effective value of $\omega$ depends on the unknown details of the compactification mechanism and supersymmetry breaking [16]. In general, one would expect a functional dependence of $\omega$ on the dilaton field [17]. Such models were proposed in [14,15] for solving the graceful exit problem of extended inflation, and later suggested to be the generic asymptotic behavior of scalar-tensor theories’ approach to general relativity during the matter dominated era [16,17].

In this paper we study the very large scale structure of the universe assuming that the gravitational interaction is described during inflation, close to Planck scale, by a general scalar-tensor theory of gravity, in the context of the stochastic inflation formalism. In Brans–Dicke stochastic inflation [14,15], we found runaway solutions to the diffusion of the dilaton and inflaton fields due to the fact that the Planck boundary, for generic chaotic potentials, is a line and the probability distribution does not become stationary, as occurs in general relativity [1], but slides along this boundary. As a consequence, the value of Planck mass at the end of inflation is not well defined (it would depend on new dynamics at large values of the fields, e.g. quantum loop corrections to the potential [20]). We will study a particular case in which the Brans–Dicke parameter has a simple pole with respect to the Brans–Dicke field, which will actually give stationary probability distributions for the diffusion of the inflaton and dilaton fields in physical space. We find a probability distribution peaked about domains that produce general relativity as the effective theory of gravity at late times and therefore we conclude that
it is most probable to live one of those domains. This prediction seems to be in good agreement with observations.

II. RUNAWAY SOLUTIONS IN BRANS–DICKE COSMOLOGY

In this section we review the problem of runaway solutions in Brans–Dicke models of stochastic inflation. For a detailed analysis see Refs. [18,19]. Let us consider the evolution of the inflaton field \( \sigma \) with a generic chaotic potential in a JBD theory of gravity with dilaton field \( \phi \),

\[
S = \int d^4x \sqrt{-g} \left[ \frac{1}{8\omega} \phi^2 R - \frac{1}{2} (\partial \phi)^2 - \frac{1}{2} (\partial \sigma)^2 - V(\sigma) \right],
\]

where Planck mass is written in terms of the dilaton field as

\[
M^2_P(\phi) = \frac{2\pi}{\omega} \phi^2.
\]

For generic inflaton potentials of the type \( V(\sigma) = \lambda \sigma^{2n}/2n \), the equations of motion for the homogeneous fields in the slow-roll approximation are

\[
\begin{align*}
\dot{\phi} &= \frac{H}{\omega}, \\
\dot{\sigma} &= -\frac{n}{2} \frac{H \phi^2}{\sigma^2}, \\
H &= \left( \frac{2\omega \lambda}{3n} \right)^{1/2} \frac{\sigma^n}{\phi}.
\end{align*}
\]

They correspond to circular motion along the lines \( \phi^2 + \frac{2n}{\sigma^2} = \text{constant} \), as shown in Fig. 1, with the angular variable \( z = \sqrt{n/2} (\phi/\sigma) \) evolving as

\[
\dot{z} = \frac{Hz}{\omega} (1 + z^2).
\]

The end of inflation occurs for \( |\dot{H}| = H^2 \), corresponding to the line \( z^2 \simeq \omega/2 \), where the slow-roll approximation for the inflaton field breaks down, while the slow-roll approximation for the dilaton simply requires \( \omega \gg 1 \). The Planck boundary is the curve \( V(\sigma) \simeq M^2_P(\phi) \), \( \phi^2 \simeq (\omega \sqrt{2} / 2\pi \sqrt{2n}) \sigma^n \). In the simplest case \( n = 2 \), it is the line \( z^2_p = \omega \sqrt{2} / 4\pi \).

Apart from classical motion there are quantum fluctuations that act on the background fields as stochastic forces and produce a random motion of those fields, very much like Brownian motion. The amplitude of quantum fluctuations of \( \delta \phi \) and \( \delta \sigma \), whose wavelengths are stretched beyond the horizon, can be computed in the slow-roll limit, as in [19,23], by solving the linearized perturbation equations in a de Sitter background \( (R = 12H^2 = \text{const}) \). They turn out to be

\[
\begin{align*}
\delta \phi &= \langle 4\pi k^3 |u_k|^2 \rangle^{1/2} \simeq H \frac{2}{2\pi}, \\
\delta \sigma &= \langle 4\pi k^3 |v_k|^2 \rangle^{1/2} \simeq H \frac{2}{2\pi}.
\end{align*}
\]

Quantum fluctuations of the fields act as stochastic forces on the classical background fields. We will describe the quantum diffusion of the coarse-grained fields in terms of the probability distribution \( P_p(\sigma, \phi; t) \) to find, at a given time \( t \) in a domain with a given physical volume, the fields with mean values \( \phi \) and \( \sigma \) [18],

\[
\begin{align*}
\frac{\partial P_p}{\partial t} &= \frac{\partial}{\partial \sigma} \left[ \frac{M^2_P(\phi)}{4\pi} \frac{\partial H}{\partial \sigma} P_p + \frac{H^{3/2}}{8\pi^2} \frac{\partial}{\partial \sigma} \left( H^{3/2} P_p \right) \right] \\
&\quad + \frac{\partial}{\partial \phi} \left[ \frac{M^2_P(\phi)}{2\pi} \frac{\partial H}{\partial \phi} P_p + \frac{H^{3/2}}{8\pi^2} \frac{\partial}{\partial \phi} \left( H^{3/2} P_p \right) \right] + 3H P_p.
\end{align*}
\]

\(^4\text{Note that Coleman’s mechanism for the vanishing of the cosmological constant in the context of Brans–Dicke theory also predicts general relativity as a low energy effective theory of gravity [2].}\)
where the first and second terms in each bracket correspond to the classical drift and quantum diffusion, respectively. The last term takes into account the different quasi-exponential growth of the proper physical volume in different parts of the universe. Those few domains that jump in the opposite direction to the classical trajectory contribute with a larger physical space and therefore dominate the proper physical volume of the universe. Such domains will split into smaller domains, some with lower values of the scalar fields, some with higher. As a consequence of the diffusion process, there will always be domains which are still inflating, and this corresponds to what is known as the self-reproduction of the inflationary universe [1].

Beyond a certain point, the quantum fluctuations $\delta z$ of the fields dominate their classical motion in a time interval $\Delta t = H^{-1}$ and the universe enters the self-reproduction regime. In Brans–Dicke inflation this corresponds to the line $z = z_s$, such that

$$\delta z = \dot{z} H^{-1}.$$  

(2.6)

For $n = 2$, this is a straight line with $z_s^4 \simeq \omega^3 \lambda/12\pi^2 \ll 1$. For $z_s < z < z_p$, new domains create more inflationary domains and so on for ever. (The requirement that $z_s < z_p$ is guaranteed by the slow-roll condition, $\omega \gg 1$.) In this picture, most of the volume of the universe today is occupied by regions that are still inflating close to Planck boundary, while our own causal domain is thought to have evolved from one of those inflationary domains, down the inflaton potential, through reheating and into the radiation and matter dominated eras.

The probability of finding given values of the scalar fields in a given domain in physical space can be computed, in a first approximation, with ordinary diffusion equations. In general relativity, where the Planck boundary corresponds to a certain value of the inflaton field, it is possible to find stationary solutions to the quantum diffusion of the inflaton [4]. However, in Brans–Dicke gravity, diffusion occurs in the two-dimensional space $(\sigma, \phi)$, where the Planck boundary is a line. For the usual chaotic inflation potentials, the diffusion of the probability distribution along the Planck boundary is unbounded and the fields evolve indefinitely along this line [18,19] towards larger and larger values producing what we called runaway solutions. The diffusion equation for the probability distribution $P_p(\phi, \sigma, t)$ along the Planck boundary, where $H_p^2 = (8\pi/3)M_p^2(\phi) = (8\pi/3)V(\sigma)^{1/2}$ and diffusion dominates classical drift, can be written as

$$\frac{\partial P_p}{\partial t} = \frac{1}{8\pi^2} \frac{\partial}{\partial \sigma} \left( H_p^{3/2} \frac{\partial}{\partial \sigma} \left( H_p^{3/2} P_p \right) \right) + 3H_p P_p.$$  

(2.7)

There are no stationary solutions for generic chaotic potentials of the type $V(\sigma) \sim \lambda \sigma^{2n}$. It was argued in Ref. [18] that a possible way to stop this runaway behavior of the fields is to introduce a cutoff in the potential at $\sigma = \sigma_b$ due, for instance, to quantum corrections. It is also possible to find stationary solutions within Brans–Dicke stochastic inflation by considering a non minimal coupling of the inflaton to the curvature scalar, see Ref. [20].

In this paper we explore another possibility which arises in more general scalar-tensor gravity theories where the Brans–Dicke parameter $\omega$ becomes a function of the dilaton field. Such theories possess the observationally desirable feature of including the general relativistic behavior in the limit $\omega \to \infty$. As we shall see, these models can also yield stationary probability distributions.

### III. VARIABLE OMEGA PARAMETER

Let us now consider the classical evolution of the inflaton field with a generic chaotic potential, in the context of an arbitrary scalar-tensor theory of gravity,

$$\mathcal{S} = \int d^4x \sqrt{-g} \left[ f(\phi) R - \frac{1}{2}(\partial \phi)^2 - \frac{1}{2}(\partial \sigma)^2 - V(\sigma) \right].$$  

(3.1)

Here the parameter $\omega$, a constant in Brans–Dicke theory, becomes a function of the effective Planck mass.
Friedmann–Robertson–Walker metric as

\[ M_0^2(\phi) \equiv 16 \pi f(\phi) , \]

\[ \omega(f) \equiv \frac{f(\phi)}{2[f'(\phi)]^2} , \]  

where \( 16 \pi f(\phi) \) acts like the Brans–Dicke scalar \[12\]. The equations of motion of theory \[3.1\] can be written as

\[ \nabla^2 \phi = f'(\phi) R , \]

\[ \nabla^2 \sigma = -V'(\sigma) , \]  

\[ 2f(\phi) \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = g_{\mu\nu} V(\sigma) + 2 (\nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla^2) f(\phi) \]

\[ + \left( \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} (\partial \phi)^2 \right) + \left( \partial_\mu \sigma \partial_\nu \sigma - \frac{1}{2} g_{\mu\nu} (\partial \sigma)^2 \right) . \]  

We can then write the exact equations for the homogeneous fields in a spatially flat \((k = 0)\) Friedmann–Robertson–Walker metric as

\[ (2\omega + 3) \left( \ddot{f}(\phi) + 3H \dot{f}(\phi) \right) + \omega'(f) \dot{f}(\phi)^2 = 2V(\sigma) - \frac{1}{2} \dot{\sigma}^2 , \]

\[ \ddot{\sigma} + 3H \dot{\sigma} = -V'(\sigma) , \]  

\[ f(\phi) \left( 6\ddot{H} + 12H^2 \right) = 2V(\sigma) - 3\ddot{f}(\phi) - 9H \dot{f}(\phi) - \frac{1}{2} \dot{\sigma}^2 - \frac{1}{2} \dot{\phi}^2 , \]

\[ f(\phi) 6H^2 = V(\sigma) - 6H \dot{f}(\phi) + \frac{1}{2} \dot{\sigma}^2 + \frac{1}{2} \dot{\phi}^2 , \]  

where \( H \equiv \dot{a}/a, \) and \( a \) is the scale factor. Note that for \( f(\phi) = \phi^2/8\omega \) we recover the usual BD equations.

During inflation, we can write the equations of motion of the homogeneous fields \( \phi \) and \( \sigma \), in the slow-roll approximation, as

\[ \dot{\phi} = 4f'(\phi) H = -\frac{M_0^2(\phi)}{2\pi} \frac{\partial H}{\partial \phi} , \]

\[ \dot{\sigma} = -2f(\phi) \frac{V'(\sigma)}{V(\sigma)} H = -\frac{M_0^2(\phi)}{4\pi} \frac{\partial H}{\partial \sigma} , \]

\[ H^2 = \frac{V(\sigma)}{6f(\phi)} . \]  

For the slow-roll solution to be an attractor, \( f(\phi) \) must satisfy the following conditions,

\[ f'(\phi)^2 \ll f(\phi) \implies \omega(f) \gg 1 \]

\[ |f''(\phi)| \ll 1 \implies \frac{f(\phi)\omega'(f)}{\omega^2(f)} \ll 1 . \]  

In addition we have the straightforward generalization to scalar-tensor gravity of the familiar slow-roll conditions for the inflaton potential, \( f(\phi)(V'/V)^2 \ll 1 \) and \( f(\phi)|V''/V| \ll 1 \).

The end of inflation in our theory occurs when \( |\dot{H}| = H^2 \), or \( V = \dot{\sigma}^2 + \dot{\phi}^2 + 3\ddot{f}(\phi) + 3H \dot{f}(\phi) \).

For the generic chaotic potential \( V(\sigma) = \frac{1}{4} M_P^2(\phi) \phi^4 \), the end of inflation corresponds, in the slow-roll approximation, to \( \sigma_e \approx M_P \phi_e / \sqrt{\pi} \). Note that in our theory, inflation ends when the inflaton field starts oscillating around the minimum of its potential, while the dilaton field
becomes essentially constant. On the other hand, the Planck boundary is approximately given by \( V(\sigma) = M_P^4(\phi) \) \[^4\] or \( \sigma_p \approx \sqrt{2} \lambda^{-1/4} M_P(\phi_p) \).

The quantum fluctuations in a variable \( \omega(f) \) model remain those of Eqs. (2.4) since they were calculated in the slow-roll limit, which should still hold here. The variable \( \omega(f) \) then only affects how the diffusion of \( \phi \) is reflected in \( f(\phi) \). Along the Planck boundary the rate of expansion is given by \( H_P = (8\pi/3) M_P^2(\phi) \) and thus will still diffuse towards larger values of \( M_P \).

A simple form to consider is \( \omega(f) = \omega_0 + \omega_m f^m \) \[^4\], which satisfies the slow-roll conditions for \( \omega_0 \gg 1 \) and \( \omega_m > 0 \). For \( m > 0 \) we expect that quantum fluctuations of the fields will drive the effective value of the Brans–Dicke parameter to infinity, recovering general relativity in this limit. However, in general, we will not recover a finite value of \( M_P \), i.e. there will still be runaway solutions for the quantum diffusion. A divergent \( \omega \) is not sufficient to recover a stationary probability distribution. However, in the next section we analyze a scalar-tensor theory that has an upper bound on the Planck mass which does produce both a stationary distribution and a divergent \( \omega \).

**IV. THE MODEL**

In this section we study the classical evolution of the theory (3.1), for a particular non minimal coupling \( f(\phi) \). We are interested in functions that acquire a maximum (\( f'(\phi) = 0 \)) at some value of \( \phi \), which will lead to a singular value of \( \omega(f) \), see Eq. (3.2). We choose a simple case for which there are analytical solutions, \( f(\phi) = f_0 \sin^2 a\phi \). (4.1)

In this case, the Brans–Dicke parameter behaves as,

\[
\omega(f) = \omega_0 \frac{1}{1 - f/f_0} = \omega_0 \frac{\cos^2 a\phi}{\cos^2 a\phi}, \tag{4.2}
\]

where \( a^2 = (8\omega_0 f_0)^{-1} \equiv 2\pi G_0/\omega_0 \). Equation (4.2) has a simple pole at \( f = f_0 \), where we recover general relativity with a gravitational constant \( G_0 \). This type of pole behavior for the \( \omega \) parameter was used in Ref. \[^4\] to allow for a graceful exit of extended inflation. Note that it behaves as ordinary Brans–Dicke theory with \( \omega = \omega_0 \) for \( a\phi \ll 1 \).

The slow-roll equations of motion (3.7) for the theory \( V(\sigma) = \lambda \sigma^4/4 \) become,

\[
\dot{\phi} = \left( \frac{\lambda}{3\omega_0} \right)^{1/2} \sigma^2 \cos a\phi, \\
\dot{\sigma} = -\left( \frac{\lambda}{3\omega_0} \right)^{1/2} \frac{\sigma}{a} \sin a\phi, \\
H = \left( \frac{\lambda \omega_0}{3} \right)^{1/2} \frac{\omega_0}{a} \sin a\phi. \tag{4.3}
\]

We can redefine \( u = (\lambda/3\omega_0)^{1/2} t/a \), \( x = a\sigma \), \( y = \sin a\phi \), under which the equations of motion (4.3) become

\[
y' = x^2(1 - y^2), \\
x' = -xy, \tag{4.4}
\]

where a prime here denotes a derivative with respect to \( u \). The slow-roll conditions for the dilaton (3.8) require \( \omega_0 \gg 1 - y^2 \) and \( \omega_0 \gg y^2 \), which are both satisfied by \( \omega_0 \gg 1 \).

There are exact solutions to the system of equations (4.4), \( x^2 - \ln(1 - y^2) = \) constant, see Fig. 1. For small values of \( y \), we recover the circular solutions found in the Brans–Dicke case \[^4\]. while in general we have

\[^4\]
\[ \omega(y) = \omega(y_i) \exp \left( x_i^2 - x^2 \right) . \]  

(4.5)

Note that during inflation \( \omega \) increases exponentially. This behavior will be important for the approach to general relativity.

In the variable \( z = y/x \), the end of inflation and the Planck boundary correspond to

\[ z_e^2 \simeq \frac{\omega_0}{2} , \]
\[ z_p^2 \simeq \frac{\omega_0 \sqrt{\lambda}}{4\pi} . \]

(4.6)

The simple analytic results for slow-roll inflation in this model allows us to describe very simply the evolution of various quantities such as the number of e-foldings from the end of inflation,

\[ N = \omega_0 \ln \left[ \frac{y^2}{y^2} \left( 1 - y^2 \right) \right] , \]

(4.7)

which approaches \( N \simeq \omega_0 x^2 \) for \( x \gg x_e \).

The new feature in this particular model is the existence of a boundary in the motion of the dilaton. As it grows towards \( y = 1 \) it slows down and eventually stops while the inflaton field \( x \) ends inflation. This effect will be critical for the behavior of the probability distribution, as we will see shortly. It is simply a consequence of the local maximum in the function \( f(\phi) \), and will occur in a wide range of functions.

V. SELF-REPRODUCTION AND STATIONARY DISTRIBUTIONS

We will now study the onset of the self-reproduction of the inflationary universe in our model. The bifurcation line is defined by the values of the scalar fields for which the maximum of the probability distribution \( P_p \) starts increasing, or equivalently, where the quantum diffusion of the fields dominates its classical motion in the time interval \( H^{-1} \). The bifurcation line can be written, in the \( z \) variable, as

\[ z_s^2 \simeq \left( \frac{\lambda \omega_0^3}{12 \pi^2} \right)^{1/2} . \]

(5.1)

There will be self-reproduction of the universe for \( z_s > z > z_p \). The probability distribution \( P_p(\phi, \sigma, t) \) follows the diffusion equation (2.5), although, as in the Brans–Dicke case discussed in section I, it will very rapidly diffuse along the angular direction \( z \), towards the Planck boundary, which is an absorbing boundary \[18\]. It is possible to estimate the relative dispersion in \( z \) due to quantum fluctuations,

\[ \frac{\delta z}{z} = z_s^2 \omega_0 z^2 \left[ (1 - y^2) + z^2 \right]^{1/2} . \]

(5.2)

Note that the prefactor \( z_s^2/\omega_0 z^2 < 2/\sqrt{3 \omega_0} \) for \( z > z_p \) so the dispersion must be small for small values of \( z \) and decrease as we approach \( y = 1 \), which strongly suggests that we can approximate the diffusion as being essentially one dimensional, along the Planck boundary. The distribution then evolves along this boundary following Eq. (2.7), towards large values of \( \sigma \) until it reaches the line \( y = 1 \), which acts like an effective cutoff and the distribution acquires a peak. Diffusion in \( \phi \) across the boundary \( y = 1 \) is automatically identified with values of \( y < 1 \), see Eq. (4.1). The upper limit on \( \sigma \) is then \( \sigma = (2/G_0 \sqrt{\lambda})^{1/2} \equiv \sigma_b \). At this point, the parameter \( \omega(f) \) becomes

\[ \delta \left( \frac{\omega(\sigma)}{\sqrt{3 \omega_0}} \right) \left( 1 - \frac{y^2}{y^2} \right) \left( 1 - \frac{y^2}{y^2} \right) \]

5Note that these are the same values as in Brans–Dicke theory \[18\].
infinite, while \( f \omega'(f)/w^2 \to 0 \), corresponding to the general relativistic post-Newtonian limit \( \frac{23}{24} \).

Let us compute the shape of the distribution close to the peak, since that will give us some idea of possible deviations from general relativity. The diffusion equation (2.5) for the probability distribution in the physical frame, along the Planck boundary (2.7), for \( z_p \ll 1 \), can be written in the form of a Schrödinger equation,

\[
-\frac{\partial^2 \Psi}{\partial s^2} + W(s)\Psi(s) = -E\Psi(s),
\]

where \( \Psi(s) \) is the wave function, \( s^2 = 1/\sigma \), and \( W(s) = -72\pi/s^2\sqrt{\lambda} \) is the singular potential. It is well known from quantum mechanics that there are no regular solutions of (5.3) at \( s = 0 \) for a singular potential like \(-1/s^2\). However, in our case, the boundary at \( \sigma = \sigma_b \) permits the existence of stationary solutions, with \( \Psi(\sigma_b) = 0 \). A numerical solution is presented in Fig. 2.

Unfortunately, we do not have an analytic expression for it, but one can easily write down WKB approximate solutions as

\[
\Psi(\sigma) \sim \left(1 - \frac{\sigma}{\sigma_*}\right)^{-1/4} \exp\left\{-\frac{72\pi}{\sqrt{\lambda}} \left(\frac{\sigma_*}{\sigma} - 1 - \arcsin\left(\frac{\sigma_*}{\sigma}\right)\right)\right\}, \quad \sigma < \sigma_*,
\]

\[
\Psi(\sigma) \sim 2 \left(\frac{\sigma}{\sigma_*} - 1\right)^{-1/4} \cos\left\{\frac{72\pi}{\sqrt{\lambda}} \left[1 - \frac{\sigma_*}{\sigma} - \ln\left(\frac{\sigma}{\sigma_*} + \frac{\sigma_*}{\sigma} - 1\right)\right] + \pi\right\}, \quad \sigma > \sigma_*,
\]

where \( \sigma_* \equiv E\sqrt{\lambda}/72\pi = \sigma_b(1 - (9\pi\sqrt{\lambda}/128)^{1/3}) \) is the WKB turning point. This solution has a very sharp maximum close to \( \sigma_* \) and an exponential decay for small \( \sigma \).

It is now possible to study the way general relativity is approached as the probability distribution moves towards the critical point \( \sigma_b \). The value of the variable Brans–Dicke parameter \( \omega_P(\sigma) \) along the Planck boundary can be written as

\[
\omega_P(\sigma) = \frac{\omega_0}{1 - (\sigma/\sigma_b)^2},
\]

which shows a pole singularity at \( \sigma_b \). Thus near the peak of the probability distribution function we find

\[
\omega_P(\sigma_*) = \left(\frac{16}{9\pi}\right)^{1/3} \omega_0 \lambda^{-1/6},
\]

which may be quite large for the values of \( \lambda \simeq 10^{-12} \) required by density perturbations \( [1] \). However, the main increase of \( \omega \) comes from the classical evolution of those domains that started close to the peak of the distribution and later evolved towards the end of inflation[**] It is possible to compute this increase using the slow-roll approximate solution (4.5). By the end of inflation, the effective value of \( \omega \) coming from domains at the peak of the distribution takes the expression

\[
\omega^*_{\text{end}} \simeq \omega_P(\sigma_*) \exp\left[\frac{2}{\omega_0} \left(\frac{2\pi}{\sqrt{\lambda}} - 1\right)\right],
\]

**Note that diffusion is important between the Planck boundary and the self-reproduction boundary, however one expects that, as in the case of Brans–Dicke [12], the peak of the distribution will closely follow the classical trajectory.
which is exponentially large, compared to $\omega_0$. We conclude that stochastic inflation drives the effective value of $\omega$ at the end of inflation to be exponentially large, with a sharply peaked probability distribution. It is important to emphasize here that once the distribution on the Planck boundary is peaked at $\sigma_*$, it is the classical motion towards the end of inflation which is responsible for the exponential increase of the effective Brans–Dicke parameter.

VI. CONCLUSIONS

Why do the constants of nature take the values we observe? Couplings range fourteen orders of magnitude; masses are smaller than $10^{-17}M_P$ and range eleven orders of magnitude; the vacuum energy is smaller than $10^{-120}M_P^4$, and so on. A possible answer, very popular among particle physicists, is that there is a unique logically consistent theory of everything, where all fundamental constants are determined from its vacuum state. Unfortunately, this state is probably not unique, e.g. in superstrings it strongly depends on the compactification mechanism and supersymmetry breaking [6]. On the other hand, quantum cosmology proposes that the so-called wave function of the universe provides a probability distribution for all fundamental constants. It is usually studied in the canonical or Euclidean approach, which has problems of interpretation related to the choice of measure. In Ref. [5] it was suggested that the stochastic inflation formalism could provide a reasonable framework within which to answer these questions. Here an exponentially large, causally disconnected inflationary domain replaces a single nucleated universe of Euclidean quantum cosmology. This formalism proposes that the global measure should be given by the probability distribution in physical space [5,23], which takes into account the proper volume of the universe.

Stochastic inflation describes the quantum diffusion of fields close to the Planck boundary. It uses branching diffusion equations to derive the probability of finding a given value of the scalar fields that drive inflation in a given physical proper volume [4]. It can be analyzed in the context of general relativity or in other theories of gravity, like scalar-tensor theories, where the gravitational coupling (Newton’s constant in general relativity), becomes another dynamical field, the Brans–Dicke field. In the stochastic picture this leads to different values of the effective Planck mass in different exponentially large, causally disconnected, parts of the universe. This picture may be incorporated in a theory of evolution of the universe [23], where quantum fluctuations of Planck mass could act as a mechanism for mutation, while a selection mechanism establishes that its value should be as large as possible in order to increase the rate of expansion, and therefore the proper volume of the universe. Unfortunately, in the simplest scalar-tensor theory, Brans–Dicke theory, diffusion close to the Planck boundary leads to runaway solutions where the global volume of the universe becomes dominated by regions with infinitely large Planck mass, in conflict with observations unless new dynamics is introduced into the model [18,20].

In scalar-tensor theories with power-law behavior of the Brans–Dicke parameter, there are still runaway solutions, but by considering a scalar-tensor theory with an upper bound on the BD field (corresponding to a pole in the BD parameter), we have shown that not only do we recover a stationary probability distribution for the fields along the Planck boundary (peaked at the maximum allowed value of the Planck mass), but that the low-energy effective $\omega$ parameter becomes exponentially large, thus recovering the general relativistic behavior. It is important to emphasize that this result is expected to be generic in all models involving a maximum value of the Planck mass, not just the particular model considered here.

This purely quantum diffusion process towards large values of $\omega$ is then reinforced by the subsequent classical evolution of the inflationary universe during which the Brans–Dicke parameter exponentially approaches the general relativistic limit. The ability of infinite $\omega$ to act as an attractor in classical cosmology is well-known during inflation [15] and matter dominated era [14]. What we have presented in this paper is a quantum process which enables us to attribute a relative probability of finding a given value of $\omega$ in the post-inflationary universe. Along with the classical evolution, this quantum diffusion mechanism predicts an effective theory of gravity which at late-times is indistinguishable from general relativity.
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FIG. 1. The classical evolution of dilaton and inflaton fields during inflation in the \((x, y)\) plane, see Eq. (4.4), for the scalar-tensor theory defined by (4.1) is represented by the continuous curves, while the dashed curves correspond to the solutions in ordinary Brans–Dicke theory. Classical motion starts at Planck boundary \((z_p)\) and ends at the end of inflation boundary \((z_e)\), represented by the thick straight lines, while the dotted line corresponds to the self-reproduction boundary \((z_\ast)\). The dot-dashed horizontal line corresponds to the general relativistic limit \(y \to 1\). Note that the classical motion in our theory very quickly approaches that limit.

FIG. 2. The probability distribution \(\Psi(\sigma)\) along the Planck boundary, for \(\sigma_b = 1\) and \(\sigma_\ast = 0.812\). The WKB solutions (5.4) are good approximations to the numerical result away from the turning point \(\sigma_\ast\), marked here with a vertical line.
Fig. 2

$\psi(\sigma)$ vs $\sigma$