STRING THEORY IN
COSMOLOGICAL SPACETIMES

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Abstract

Progress on string theory in curved spacetimes since 1992 are reviewed. After a short introduction on strings in Minkowski and curved spacetimes, we focus on strings in cosmological spacetimes.

The classical behaviour of strings in FRW and inflationary spacetimes is now understood in a large extent from various types of explicit string solutions. Three different types of behaviour appear in cosmological spacetimes: unstable, dual to unstable and stable. For the unstable strings, the energy and size grow for large scale factors $R \to \infty$, proportional to $R$. For the dual to unstable strings, the energy and size blow up for $R \to 0$ as $1/R$. For stable strings, the energy and proper size are bounded. (In Minkowski spacetime, all string solutions are of the stable type).

Recent progress on self-consistent solutions to the Einstein equations for string dominated universes is reviewed. The energy-momentum tensor for a gas of strings is then considered as source of the spacetime geometry and from the above string behaviours the string equation of state is determined. The self-consistent string solution exhibits the realistic matter dominated behaviour $R \sim (X^0)^{2/3}$ for large times and the radiation dominated behaviour $R \sim (X^0)^{1/2}$ for early times.

Finally, we report on the exact integrability of the string equations plus the constraints in de Sitter spacetime that allows to systematically find exact string solutions by soliton methods and the multistring solutions. Multistring solutions are a new feature in curved spacetimes. That is, a single world sheet simultaneously describes many different and independent strings. This phenomenon has no analogue in flat spacetime and appears as a consequence of the coupling of the strings with the spacetime geometry.
Since the previous Erice School on ‘String Quantum Gravity’ a host of impressive developments has taken place on strings in curved spacetime. For the state of the art in 1992 we refer to the 1992 Proceedings where the programme on string quantization in curved spacetimes initiated by us in 1987 is reviewed.

A consistent quantum theory of gravity is the strongest motivation for string theory and hence to study strings in curved spacetime. As stressed in the sec. 1 of the 1992 lectures, a quantum theory of gravity must be a theory able to describe all physics below the Planck scale $M_{\text{Planck}} = \hbar c/G = 1.22 \times 10^{16}\text{Tev}$. That means that a sensible theory of quantum gravity is necessarily part of a unified theory of all interactions. Pure gravity (a model containing only gravitons) cannot be a physical and realistic quantum theory. To give an example, a theoretical prediction for graviton-graviton scattering at energies of the order of $M_{\text{Planck}}$ must include all particles produced in a real experiment. That is, in practice, all existing particles in nature, since gravity couples to all matter.

String theory is therefore a serious candidate for a quantum description of gravity since it provides a unified model of all interactions overcoming at the same time the nonrenormalizable character of quantum fields theories of gravity.

The present lectures deal mainly with strings in cosmological spacetimes; substantial results were achieved in this field since 1992. The classical behaviour of strings in FRW and inflationary spacetimes is now understood in a large extent. This understanding followed the finding of various types of exact and numerical string solutions in FRW and inflationary spacetimes. For inflationary spacetimes, the exact integrability of the string propagation equations plus the string constraints in de Sitter spacetime is indeed an important help. This allowed to systematically find exact string solutions by soliton methods using the linear system associated to the problem (the so-called dressing method in soliton theory) and the multistring solutions.

In summary, three different types of behaviour are exhibited by the string solutions in cosmological spacetimes: unstable, dual to unstable and stable. For the unstable strings, the energy and size grow for large scale factors $R \to \infty$, proportional to $R$. For the dual to unstable strings, the energy and size blow up for $R \to 0$ as $1/R$. For stable strings, the energy and proper size are bounded. (In Minkowski spacetime, all string solutions are of the stable type). The equation of state for these string behaviours take the form

- (i) **unstable** for $R \to \infty$ $p_u = -E_u/(d-1) < 0$

- (ii) **dual to unstable** for $R \to 0$, $p_d = E_d/(d-1) > 0$.

- (iii) **stable** for $R \to \infty$, $p_s = 0$.

Here $E_u$ and $E_d$ stand for the corresponding string energies and $d-1$ for the number of spatial dimensions where the string solutions lives. For example, $d-1 = 1$ for a straight string, $d-1 = 2$ for a ring string, etc. This number $d$ is obviously less or equal than the number of spacetime dimensions $D$.

As we see above, the dual to unstable string behavior leads to the same equation of state than radiation (massless particles or hot matter). The stable string behavior leads to the equation
of state of massive particles (cold matter). The unstable string behavior is a purely ‘stringy’ phenomenon. The fact that it entails a negative pressure is however physically acceptable. For a gas of strings, the unstable string behaviour dominates in inflationary universes when $R \to \infty$ and the dual to unstable string behavior dominates for $R \to 0$.

The unstable strings correspond to the critical case of the so called {\it coasting universe} [15,29]. That is, classical strings provide a {\it concrete} realization of such cosmological models. The ‘unstable’ behaviour is called ‘string stretching’ in the cosmic string literature [16,17].

It must be stressed that while time evolves, a given string solution may exhibit two and even three of the above regimes one after the other (see sec. III). Intermediate behaviours are also observed in ring solutions [7,9]. That is,

$$P = (\gamma - 1)\ E \quad \text{with} \quad -\frac{1}{d-1} < \gamma - 1 < +\frac{1}{d-1}$$

Another new feature appeared in curved spacetimes: {\bf multistring solutions}. That is a single world-sheet simultaneously describes many different and independent strings. This phenomenon has no analogue in flat spacetime. This is a new feature appearing as a consequence of the interaction of the strings with the spacetime geometry.

The world-sheet time $\tau$ turns out to be an multi-valued function of the target string time $X^0$ (which can be the cosmic time $T$, the conformal time $\eta$ or for de Sitter universes it can be the hyperboloid time $q^0$). Each branch of $\tau$ as a function of $X^0$ corresponds to a different string. In flat spacetime, multiple string solutions are necessarily described by multiple world-sheets. Here, a single world-sheet describes one string, several strings or even an infinite number of different and independent strings as a consequence of the coupling with the spacetime geometry. These strings do not interact among themselves; all the interaction is with the curved spacetime. One can decide to study separately each of them (they are all different) or consider all the infinite strings together.

Of course, from our multistring solution, one could just choose only one interval in $\tau$ (or a subset of intervals in $\tau$) and describe just one string (or several). This will be just a {\it truncation} of the solution.

The really remarkably fact is that all these infinitely many strings come naturally together when solving the string equations in de Sitter spacetime as we did in refs. [5] - [7].

Here, interaction among the strings (like splitting and merging) is neglected, the only interaction is with the curved background.

The study of string propagation in curved spacetimes provide essential clues about the physics in this context but is clearly not the end of the story. The next step beyond the investigation of test strings, consist in finding {\bf self-consistently} the geometry from the strings as matter sources for the Einstein equations or better the string effective equations (beta functions). This goal is achieved in ref. [3] for cosmological spacetimes at the classical level. Namely, we used the energy-momentum tensor for a gas of strings as source for the Einstein equations and we solved them self-consistently.

To write the string equation of state we used the behaviour of the string solutions in cosmological spacetimes. Strings continuously evolve from one type of behaviour to another, as is explicitly shown by our solutions [5] - [6]. For intermediate values of $R$, the equation of state for gas of free strings is clearly complicated but a formula of the type:
\[ \rho = \left( u_R + \frac{d}{R} + s \right) \frac{1}{R^{D-1}} \tag{1.1} \]

where

\[ \lim_{R \to \infty} u_R = \begin{cases} 0 & \text{FRW} \\ u_\infty \neq 0 & \text{Inflationary} \end{cases} \tag{1.2} \]

This equation of state is qualitatively correct for all \( R \) and becomes exact for \( R \to 0 \) and \( R \to \infty \). The parameters \( u_R, d \) and \( s \) are positive constants and the \( u_R \) varies smoothly with \( R \).

The pressure associated to the energy density (1.1) takes then the form

\[ p = \frac{1}{D-1} \left( \frac{d}{R} - u_R R \right) \frac{1}{R^{D-1}} \tag{1.3} \]

Inserting this source into the Einstein-Friedman equations leads to a self-consistent solution for string dominated universes (see sec. IV) \[3\]. This solution exhibits the realistic matter dominated behaviour \( R \sim (X^0)^{2/(D-1)} \) for large times and the radiation dominated behaviour \( R \sim (X^0)^{2/D} \) for early times.

For the sake of completeness we analyze in sec. IV the effective string equations \[3\]. These equations have been extensively treated in the literature \[25\] and they are not our central aim.

It must be noticed that there is no satisfactory derivation of inflation in the context of the effective string equations. This does not mean that string theory is not compatible with inflation, but that the effective string action approach is not enough to describe inflation. The effective string equations are a low energy field theory approximation to string theory containing only the massless string modes. The vacuum energy scales to start inflation are typically of the order of the Planck mass where the effective string action approximation breaks down. One must also consider the massive string modes (which are absent from the effective string action) in order to properly get the cosmological condensate yielding inflation. De Sitter inflation does not emerge as a solution of the the effective string equations.

In conclusion, the effective string action (whatever be the dilaton, its potential and the central charge term) is not the appropriate framework in which to address the question of string driven inflation.

Early cosmology (at the Planck time) is probably the best place to test string theory. In one hand the quantum treatment of gravity is unavoidable at such scales and in the other hand, observable cosmological consequences are derivable from the inflationary stage. The natural gravitational background is an inflationary universe as de Sitter spacetime. Such geometries are not string vacua. This means that conformal and Weyl symmetries are broken at the quantum level. In order to quantize consistently strings in such case, one must enlarge the physical phase space including, in particular, the factor \( \exp \phi(\sigma, \tau) \) in the world-sheet metric [see eq. (2.6)]. This is a very interesting and open problem. Physically, the origin of such difficulties in quantum string cosmology comes from the fact that one is not dealing with an empty universe since a cosmological spacetime necessarily contains matter. In the other hand, conformal field theory techniques are only adapted to backgrounds for which the beta functions are identically zero, i.e. sourceless geometries.
The outline of these lectures is as follows. Section II presents an introduction to strings in curved spacetimes including basic notions on classical and quantum strings in Minkowski spacetime and introducing the main physical string magnitudes: energy-momentum and invariant string size. Section III deals with the string propagation and the string energy-momentum tensor in cosmological spacetimes. (In sections III.A, III.B and III.C we treat the straight strings, ring strings and generic strings respectively and derive the corresponding string equations of state). In section IV we treat self-consistent string cosmology including the string equations of state. (Section IV.A deals with general relativity, IV.B with the string thermodynamics). Section V discuss the effective (beta functions) string equations in the cosmological perspective and the search of inflationary solutions. Finally, in sec. VI, we briefly review the systematic construction of string solutions in de Sitter universe via soliton methods and the new feature of multistring solutions.

II. INTRODUCTION. STRINGS IN CURVED AND MINKOWSKI SPACETIMES.

Let us consider bosonic strings (open or closed) propagating in a curved D-dimensional spacetime defined by a metric $G_{AB}(X)$, $0 \leq A, B \leq D − 1$. The action can be written as

$$S = \frac{1}{2\pi\alpha'} \int d\sigma d\tau \sqrt{g} g_{\alpha\beta}(\sigma, \tau) G_{AB}(X) \partial^\alpha X^A(\sigma, \tau) \partial^\beta X^B(\sigma, \tau)$$

(2.1)

Here $g_{\alpha\beta}(\sigma, \tau)$ $(0 \leq \alpha, \beta \leq 1)$ is the metric in the worldsheet, $\alpha'$ stands for the string tension. As in flat spacetime, $\alpha' \sim (M_{\text{Planck}})^{-2} \sim (l_{\text{Planck}})^2$ fixes the scale in the theory. There are no other free parameters like coupling constants in string theory.

We will start considering given gravitational backgrounds $G_{AB}(X)$. That is, we start to investigate test strings propagating on a given spacetime. In section IV, the back reaction problem will be studied. That is, how the strings may act as source of the geometry.

String propagation in massless backgrounds other than gravitational (dilaton, antisymmetric tensor) can be investigated analogously.

The string action (2.1) classically enjoys Weyl invariance on the world sheet

$$g_{\alpha\beta}(\sigma, \tau) \rightarrow \lambda(\sigma, \tau) g_{\alpha\beta}(\sigma, \tau)$$

(2.2)

plus the reparametrization invariance

$$\sigma \rightarrow \sigma' = f(\sigma, \tau) \ , \quad \tau \rightarrow \tau' = g(\sigma, \tau)$$

(2.3)

Here $\lambda(\sigma, \tau)$, $f(\sigma, \tau)$ and $g(\sigma, \tau)$ are arbitrary functions.

The dynamical variables being here the string coordinates $X_A(\sigma, \tau)$, $(0 \leq A \leq D − 1)$ and the world-sheet metric $g_{\alpha\beta}(\sigma, \tau)$.

Extremizing $S$ with respect to them yields the classical equations of motion:

$$\partial^\alpha [\sqrt{g} G_{AB}(X) \partial_\alpha X^B(\sigma, \tau)] = \frac{1}{2} \sqrt{g} \partial_A G_{CD}(X) \partial_\alpha X^C(\sigma, \tau) \partial^\alpha X^D(\sigma, \tau)$$

$$0 \leq A \leq D − 1$$

(2.4)

$$T_{\alpha\beta} \equiv G_{AB}(X) [\partial_\alpha X^A(\sigma, \tau) \partial_\beta X^B(\sigma, \tau)$$

6
\[ -\frac{1}{2} g_{\alpha\beta}(\sigma, \tau) \partial_\gamma X^A(\sigma, \tau) \partial^\gamma X^B(\sigma, \tau) = 0 \quad 0 \leq \alpha, \beta \leq 1. \tag{2.5} \]

Eqs. (2.5) contain only first derivatives and are therefore a set of constraints. Classically, we can always use the reparametrization freedom \( (2.3) \) to recast the world-sheet metric on diagonal form

\[ g_{\alpha\beta}(\sigma, \tau) = \exp[\phi(\sigma, \tau)] \text{ diag}(-1, +1) \tag{2.6} \]

In this conformal gauge, eqs. (2.4) - (2.5) take the simpler form:

\[ \partial_+ X^A(\sigma, \tau) + \Gamma^A_{BC}(X) \partial_+ X^B(\sigma, \tau) \partial_- X^C(\sigma, \tau) = 0 \quad 0 \leq A \leq D - 1, \tag{2.7} \]

\[ T_{\pm+} \equiv G_{AB}(X) \partial_\pm X^A(\sigma, \tau) \partial_\mp X^B(\sigma, \tau) \equiv 0 \quad T_{+-} \equiv T_{-+} \equiv 0 \tag{2.8} \]

where we introduce light-cone variables \( x_\pm \equiv \sigma \pm \tau \) on the world-sheet and where \( \Gamma^A_{BC}(X) \) stand for the connections (Christoffel symbols) associated to the metric \( G_{AB}(X) \).

Notice that these equations in the conformal gauge are still invariant under the conformal reparametrizations:

\[ \sigma + \tau \rightarrow \sigma' + \tau' = f(\sigma + \tau) \quad , \quad \sigma - \tau \rightarrow \sigma' - \tau' = g(\sigma - \tau) \tag{2.9} \]

Here \( f(x) \) and \( g(x) \) are arbitrary functions.

The string boundary conditions in curved spacetimes are identical to those in Minkowski spacetime. That is,

\[ X^A(\sigma + 2\pi, \tau) = X^A(\sigma, \tau) \quad \text{closed strings} \]

\[ \partial_\sigma X^A(0, \tau) = \partial_\sigma X^A(\pi, \tau) = 0 \quad \text{open strings.} \tag{2.10} \]

**A. A brief review on strings in Minkowski spacetime**

In flat spacetime eqs.(2.7) become linear

\[ \partial_{-+} X^A(\sigma, \tau) = 0 \quad , \quad 0 \leq A \leq D - 1, \tag{2.11} \]

and one can solve them explicitly as well as the quadratic constraint \( (2.8) \) [see below] :

\[ \left[ \partial_{\pm} X^0(\sigma, \tau) \right]^2 - \sum_{j=1}^{D-1} \left[ \partial_{\pm} X^j(\sigma, \tau) \right]^2 = 0 \tag{2.12} \]

The solution of eqs.(2.11) is usually written for closed strings as

\[ X^A(\sigma, \tau) = q^A + 2p^A \alpha' \tau + \sqrt{\alpha'} \sum_{n \neq 0} \frac{1}{n} \{ \alpha_n^A \exp[in(\sigma - \tau)] + \tilde{\alpha}_n^A \exp[-in(\sigma + \tau)] \} \tag{2.13} \]
where $q^A$ and $p^A$ stand for the string center of mass position and momentum and $\alpha^A_n$ and $\tilde{\alpha}^A_n$ describe the right and left oscillator modes of the string, respectively. Since the string coordinates are real,

$$\bar{\alpha}^A_n = \alpha^A_n, \quad \bar{\tilde{\alpha}}^A_n = \tilde{\alpha}^A_n$$

This resolution is no more possible in general for curved spacetime where the equations of motion (2.7) are non-linear. In that case, right and left movers interact with themselves and with each other.

In Minkowski spacetime we can also write the solution of the string equations of motion (2.11) in the form

$$X^A(\sigma, \tau) = l^A(\sigma + \tau) + r^A(\sigma - \tau)$$

(2.14)

where $l^A(x)$ and $r^A(x)$ are arbitrary functions. Now, making an appropriate conformal transformation (2.9) we can turn any of the string coordinates $X^A(\sigma, \tau)$ (but only one of them) into a constant times $\tau$. The most convenient choice is the light-cone gauge where

$$U \equiv X^0 - X^1 = 2 p^U \alpha' \tau.$$  

(2.15)

That is, there are no string oscillations along the $U$ direction in the light-cone gauge. We have still to impose the constraints (2.12). In this gauge they take the form

$$\pm 2 \alpha' p^U \partial_\pm V(\sigma, \tau) = \sum_{j=2}^{D-1} \left[ \partial_\pm X^j(\sigma, \tau) \right]^2$$

(2.16)

where $V \equiv X^0 + X^1$. This shows that $V$ is not an independent dynamical variable since it expresses in terms of the transverse coordinates $X^2, \ldots, X^{D-1}$. Only $q^V$ is an independent quantity.

The physical picture of a string propagating in Minkowski spacetime clearly emerges in the light-cone gauge. The gauge condition (2.15) tells us that the string ‘time’ $\tau$ is just proportional to the physical null time $U$. Eqs.(2.13) shows that the string moves as a whole with constant speed while it oscillates around its center of mass. The oscillation frequencies are all integers multiples of the basic one. The string thus possess an infinite number of normal modes; $\alpha^A_n, \tilde{\alpha}^A_n$ classically describe their oscillation amplitudes. Only the modes in the direction of the transverse coordinates $X^2, \ldots, X^{D-1}$ are physical. This is intuitively right, since a longitudinal or a temporal oscillation of a string is meaningless. In summary, the string in Minkowski spacetime behaves as an extended and composite relativistic object formed by a $2(D-2)$-infinite set of harmonic oscillators.

Integrating eq.(2.14) on $\sigma$ from 0 to $2\pi$ and inserting eq. (2.13) yields the classical string mass formula:

$$m^2 \equiv p^U p^V - \sum_{j=2}^{D-1} (p^j)^2 = \frac{1}{\alpha} \sum_{j=2}^{D-1} \sum_{n=1}^{\infty} \left[ \alpha^j_n \alpha^j_{-n} + \tilde{\alpha}^j_n \tilde{\alpha}^j_{-n} \right]$$

(2.17)

We explicitly see how the mass of a string depends on its excitation state. The classical string spectrum is continuous as we read from eq.(2.17). It starts at $m^2 = 0$ for an unexcited string: $\alpha^j_n = \tilde{\alpha}^j_n = 0$ for all $n$ and $j$. 

8
The independent string variables are:
1. The transverse amplitudes \( \{ \alpha_j^n, \tilde{\alpha}_j^n, n \in \mathbb{Z}, n \neq 0, 2 \leq j \leq D - 1 \} \).
2. The transverse center of mass variables \( \{ q_j^i, p_j^i, 2 \leq j \leq D - 1 \} \), \( q^V \) and \( p^U \).

Up to now we have considered a classical string.

At the quantum level one imposes the canonical commutation relations (CCR)

\[
[\alpha_n^i, \alpha_m^j] = n \delta_{n,-m} \delta^{ij}, \\
[\tilde{\alpha}_n^i, \alpha_m^j] = n \delta_{n,-m} \delta^{ij}, \\
[\tilde{\alpha}_n^i, \alpha_m^j] = 0, \\
[q^i, p^j] = i \delta^{ij}, \quad [q^V, p^U] = i
\]

(2.18)

All other commutators being zero. An order prescription is needed to unambiguously express the different physical operators in terms of those obeying the CCR. The symmetric ordering is the simplest and more convenient.

The space of string physical states is the tensor product of the Hilbert space of the \( D - 1 \) center of mass variables \( q^V, p^U, \{ q^i, p^j, 2 \leq j \leq D - 1 \} \), times the Fock space of the harmonic transverse modes. The string wave function is then the product of a center of mass part times a harmonic oscillator part. The center of mass can be taken, for example, as a plane wave. The harmonic oscillator part can be written as the creation operators \( \alpha_j^n \), \( \tilde{\alpha}_j^n \), \( n \geq 1, 2 \leq j \leq D - 1 \) acting on the oscillator ground state \( |0> \). This state is defined as usual by

\[
\alpha_j^n |0> = \tilde{\alpha}_j^n |0> = 0, \quad \text{for all } n \geq 1, 2 \leq j \leq D - 1
\]

Notice that a string describes one particle. The kind of particle described depends on the oscillator wave function. The mass and spin can take an infinite number of different values. That is, there is an infinite number of different possibilities for the particle described by a string.

Let us consider the quantum mass spectrum. Upon symmetric ordering the mass operator becomes,

\[
m^2 = \frac{1}{2\alpha'} \sum_{j=2}^{D-1} \sum_{n=1}^{\infty} \left[ \alpha_n^j \alpha_{-n}^j + \alpha_n^j \alpha_{-n}^j + \tilde{\alpha}_n^j \tilde{\alpha}_{-n}^j + \tilde{\alpha}_n^j \tilde{\alpha}_{-n}^j \right].
\]

(2.19)

Using the commutation rules (2.18) yields

\[
m^2 = \frac{D - 2}{\alpha'} \sum_{n=1}^{\infty} n + \frac{1}{2\alpha'} \sum_{j=2}^{D-1} \sum_{n=1}^{\infty} \left[ \alpha_n^j \alpha_n^j + \tilde{\alpha}_n^j \tilde{\alpha}_n^j \right]
\]

(2.20)

The divergent sum in the first term can be defined through analytic continuation of the zeta function

\[
\zeta(z) \equiv \sum_{n=1}^{\infty} \frac{1}{n^z}
\]

(2.21)
One finds $\zeta(-1) = -1/12$ \cite{27}. Thus,

$$m^2 = -\frac{D - 2}{12\alpha'} + \frac{1}{2\alpha'} \sum_{j=2}^{D-1} \sum_{n=1}^{\infty} \left[ \alpha_j \dagger \alpha_j + \bar{\alpha}_j \dagger \bar{\alpha}_j \right]$$

(2.22)

Hence, the string ground state $|0\rangle$ has a negative mass squared

$$m_0^2 = -\frac{D - 2}{12\alpha'}$$

(2.23)

Such particles are called tachyons and exhibit unphysical behaviours. When fermionic degrees of freedom are associated to the string the ground state becomes massless (superstrings) \cite{26}.

Notice that the appearance of a negative mass square yields a dispersion relation $E^2 = p^2 - |m^2_0|$ similar to waves with Jeans unstabilities \cite{28}.

Let us consider now excited states.

The constraints (2.12) integrated on $\sigma$ from 0 to $2\pi$ impose

$$\sum_{j=2}^{D-1} \sum_{n=1}^{\infty} (\alpha_j \dagger) \alpha_j = \sum_{j=2}^{D-1} \sum_{n=1}^{\infty} (\bar{\alpha}_j \dagger) \bar{\alpha}_j$$

(2.24)

on the physical states. This means that the number of left and right modes coincide in all physical states.

The first excited state is then described by

$$|i, j \rangle = (\alpha_i \dagger) \alpha_j \dagger |0\rangle$$

(2.25)

times the center of mass wave function. We see that this wavefunction is a symmetric tensor in the space indices $i, j$. It describes therefore a spin two particle plus a spin zero particle (the trace part).

From eqs.(2.22-2.25) follows that

$$m^2 |i, j \rangle = -\frac{D - 26}{12\alpha'} |i, j \rangle$$

(2.26)

This state is then a massless particle only for $D = 26$. In such critical dimension we have then a graviton (massless spin 2 particle) and a dilaton (massless spin 0 particle) as string modes of excitation. For superstrings the critical dimension turns to be $D = 10$.

We shall consider, as usual, that only four space-time dimensions are uncompactified. That is, we shall consider the strings as living on the tensor product of a curved four dimensional space-time with lorentzian signature and a compact space which is there to cancel the anomalies. Therefore, from now on strings will propagate in the curved (physical) four dimensional space-time. However, we will find instructive to study the case where this curved space-time has dimensionality $D$, where $D$ may be 2, 3 or arbitrary.

B. The string energy-momentum tensor and the string invariant size

The spacetime string energy-momentum tensor follows (as usual) by taking the functional derivative of the action (2.1) with respect to the metric $G_{\alpha \beta}$ at the spacetime point $X$. This yields,
\[ \sqrt{-G} \ T^{AB}(X) = \frac{1}{2\pi\alpha'} \int d\sigma d\tau \left( \dot{X}^A \dot{X}^B - X'^A X'^B \right) \delta^{(D)}(X - X(\sigma, \tau)) \] (2.27)

where dot and prime stands for \( \frac{\partial}{\partial \tau} \) and \( \frac{\partial}{\partial \sigma} \), respectively.

Notice that \( X \) in eq. (2.27) is just a spacetime point whereas \( X(\sigma, \tau) \) stands for the string dynamical variables. One sees from the Dirac delta in eq. (2.27) that \( T^{AB}(X) \) vanishes unless \( X \) is exactly on the string world-sheet. We shall not be interested in the detailed structure of the classical strings. It is more useful to integrate the energy-momentum tensor (2.27) on a volume that completely encloses the string. It takes then the form

\[ \Theta^{AB}(X^0) = \frac{1}{2\pi\alpha'} \int d\sigma d\tau \left( \dot{X}^A \dot{X}^B - X'^A X'^B \right) \delta (X^0 - X^0(\tau, \sigma)) \] (2.28)

When \( X^0 \) depends only on \( \tau \), we can easily integrate over \( \tau \) with the result,

\[ \Theta^{AB}(X^0) = \frac{1}{2\pi\alpha'} \int_0^{2\pi} d\sigma \left[ \dot{X}^A \dot{X}^B - X'^A X'^B \right]_{\tau = \tau(X^0)} \] (2.29)

Another relevant physical magnitude for strings is the invariant size. We define the invariant string size \( ds^2 \) using the metric induced on the string world-sheet:

\[ ds^2 = G_{AB}(X) \, dX^A \, dX^B \] (2.30)

Inserting \( dX^A = \partial_+ X^A \, dx^+ + \partial_- X^A \, dx^- \), into eq. (2.30) and taking into account the constraints (2.8) yields

\[ ds^2 = 2 \ G_{AB}(X) \ \partial_+ X^A \partial_- X^B \left( d\tau^2 - d\sigma^2 \right) = G_{AB}(X) \ \dot{X}^A \dot{X}^B \left( d\tau^2 - d\sigma^2 \right) . \] (2.31)

Thus, we define the string size as the integral of \( \sqrt{G_{AB}(X) \dot{X}^A \dot{X}^B} \) over \( \sigma \) at fixed \( \tau \).

Let us now consider a circular string as a simple example of a string solution in Minkowski spacetime.

\[ X^0(\sigma, \tau) = \alpha' E \tau \quad , \quad X^3(\sigma, \tau) = \alpha' \rho \tau \]
\[ X^1(\sigma, \tau) = \alpha' m \cos \tau \cos \sigma = \frac{\alpha' m}{2} \left[ \cos(\tau + \sigma) + \cos(\tau - \sigma) \right] \]
\[ X^2(\sigma, \tau) = \alpha' m \cos \tau \sin \sigma = \frac{\alpha' m}{2} \left[ \sin(\tau + \sigma) - \sin(\tau - \sigma) \right] \] (2.32)

This is obviously a solution of eqs. (2.11) where only the modes \( n = \pm 1, j = 1, 2 \) are excited. The constraints (2.12) yields

\[ E^2 = p^2 + m^2 \]

Eqs. (2.32) describe a circular string in the \( X^1, X^2 \) plane, centered in the origin and with an oscillating radius \( \rho(\tau) = \alpha' m \cos \tau \). In addition the string moves uniformly in the \( z \)-direction with speed \( p/E \). (That is, \( p \) is its momentum in the \( z \)-direction). The oscillation amplitude \( m \) can be identified with the string mass and \( E \) with the string energy. Notice that the string time \( \tau \) is here proportional to the physical time \( X^0 \) (this solution is not in the light-cone gauge).
It is instructive to compute the integrated energy-momentum tensor \( (2.29) \) for this string solution. We find in the rest frame \( (p = 0) \) that it takes the fluid form

\[
\Theta^B_A = \begin{pmatrix}
\rho & 0 & 0 & 0 \\
0 & -p & 0 & 0 \\
0 & 0 & -p & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\] (2.33)

where

\[
\rho = E = m , \quad p = -\frac{m}{2} \cos(2\tau)
\] (2.34)

We see that the total energy coincides with \( m \) as one could expect and that the (space averaged) pressure oscillates around zero. That is the string pressure goes through positive and negative values. The time average of \( p \) vanishes. The string behaves then as cold matter (massive particles).

The upper value of \( p \) equals \( E/2 \). This is precisely the relation between \( E \) and \( p \) for radiation (massless particles). (Notice that this circular strings lives on a two-dimensional plane). The lower value of \( p \) correspond to the limiting value allowed by the strong energy condition in General Relativity \(^\text{[11]}\). We shall see below that these two extreme values of \( p \) appear in the general context of cosmological spacetimes.

The invariant size of the string solution \( (2.32) \) follows by inserting eq.\((2.32)\) into eq.\( (2.31) \). We find

\[
ds^2 = (\alpha' m)^2 \left( d\tau^2 - ds^2 \right)
\] (2.35)

Therefore, this string solution has a constant size \( 2\pi\alpha' m \).

One obtains the invariant size for a generic solution in Minkowski spacetime inserting the general solution \( (2.13) \) into \( \partial_+ X^A \partial_- X^A \). This gives constant plus oscillatory terms. In any case the invariant string size is always \textbf{bounded} in Minkowski spacetimes. We shall see how differently behave strings in curved spacetimes.

**III. STRING PROPAGATION IN COSMOLOGICAL SPACETIMES**

We obtain in this section physical string properties from the exact string solutions in cosmological spacetimes.

We consider strings in spatially homogeneous and isotropic universes with metric

\[
ds^2 = (dT)^2 - R(T)^2 \sum_{i=1}^{D-1} (dX^i)^2
\] (3.1)

where \( T \) is the cosmic time and the function \( R(T) \) is called the scale factor. In terms of the conformal time

\[
\eta = \int T \frac{dT}{R(T)} ,
\] (3.2)
the metric (3.1) takes the form
\[ ds^2 = R(\eta)^2 \left[ (d\eta)^2 - \sum_{i=1}^{D-1} (dX^i)^2 \right] \] (3.3)

The classical string equations of motion can be written here as
\[ \partial^2 T - R(T) \frac{dR}{dT} \sum_{i=1}^{D-1} (\partial_{\mu} X^i)^2 = 0, \]
\[ \partial_{\mu} [R^2 \partial^\mu X^i] = 0, \quad 1 \leq i \leq D - 1, \] (3.4)
and the constraints are
\[ T_{\pm \pm} = (\partial_{\pm} T)^2 - R(T)^2 (\partial_{\pm} X^i)^2 = 0. \] (3.5)

The most relevant universes corresponds to power type scale factors. That is,
\[ R(T) = a T^\alpha = A \eta^{k/2} \] (3.6)
where \( \alpha = \frac{k}{k+2} \). For different values of the exponents we have either FRW or inflationary universes.

**FRW:** \( 0 < k \leq \infty, 0 < \alpha \leq 1 = \begin{cases} 
\alpha = 1/2, k = 2, & \text{radiation dominated} \\
\alpha = 2/3, k = 4, & \text{matter dominated} \\
\alpha = 1, k = \infty, & \text{stringy} 
\end{cases} \)

**Inflationary:** \( -\infty < k < 0, \alpha < 0 \) and \( \alpha > 1 = \begin{cases} 
\alpha = \infty, k = -2, & R(T) = e^{HT}, \text{ de Sitter}, \\
\alpha > 1, k < -2, & \text{power inflation} \\
\alpha < 0, -2 < k < 0, & \text{superinflationary} 
\end{cases} \) (3.7)

As we will see below, \( T^{AB}(X) \) takes the fluid form for string solutions in cosmological spacetimes, allowing us to define the string pressure \( p \) and energy density \( \rho \):
\[ T^B_A = \begin{pmatrix}
\rho & 0 & \cdots & 0 \\
0 & -p & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & -p
\end{pmatrix} \] (3.8)

Notice that the continuity equation
\[ D^A T^B_A = 0 \]
takes here the form
\[ \dot{\rho} + (D - 1) H (p + \rho) = 0 \] (3.9)
where \( H \equiv \frac{1}{R} \frac{dR}{dT} \).

For an equation of state of the type of a perfect fluid, that is
\[ p = (\gamma - 1) \rho, \quad \gamma = \text{constant}, \] (3.10)
eqs. (3.9) and (3.10) can be easily integrated with the result
\[ \rho = \rho_0 R^{\gamma(1-D)}. \] (3.11)

For \( \gamma = 1 \) this corresponds to cold matter \( (p = 0) \) and for \( \gamma = \frac{D}{D-1} \) this describes radiation with \( p = \frac{\rho}{D-1} \).

13
A. 1+1 Dimensional Universes. Straight Strings in D-Dimensional Universes

Let us start by considering strings in this simpler case. For a $D = 1+1$ dimensional universe or for a straight string parallel to the $X$-axis in a $D$-dimensional universe, the metric (3.1) takes the form

$$ds^2 = (dT)^2 - R(T)^2 (dX)^2$$

It is convenient to start by solving the constraints (3.5)

$$(\partial_\pm T)^2 = R(T)^2 (\partial_\pm X)^2.$$ \hspace{1cm} (3.12)

They reduce to

$$\partial_\pm T = \epsilon_\pm R(T) \partial_\pm X.$$ \hspace{1cm} (3.13)

where $\epsilon_\pm^2 = 1$. Using the conformal time (3.2), eq.(3.13) yields

$$\partial_\pm (\eta - \epsilon_\pm X) = 0.$$ \hspace{1cm} (3.14)

We find a first family of solutions choosing $\epsilon_\pm = \pm 1$. Then

$$\eta + X = \phi(\sigma + \tau), \quad \eta - X = \chi(\sigma - \tau)$$ \hspace{1cm} (3.14)

Where $\phi$ and $\chi$ are arbitrary functions of one variable. It is easy now to check that eq.(3.14) fulfills the string equations of motion (3.4).

The solution (3.14) is analyzed in detail for de Sitter spacetime ($R(T) = e^{HT}$) in ref. [4] where the global topology of the space is taken into account.

Since one can always perform conformal transformations

$$\sigma + \tau \to f(\sigma + \tau), \quad \sigma - \tau \to g(\sigma - \tau),$$

with arbitrary functions $f$ and $g$, the solution (3.14) has no degrees of freedom other than topological ones.

Let us compute the energy momentum tensor for the string solution (3.14). We find from eqs.(2.27) and (3.14),

$$\sqrt{-G} T^{00}(X) = \frac{1}{2\pi \alpha} \int d\sigma d\tau \left( \dot{\eta}^2 - \eta'^2 \right) \delta(\eta - \eta(\sigma, \tau)) \delta(X - X(\sigma, \tau))$$

$$= \frac{1}{2\pi \alpha} \frac{\dot{\eta}^2 - \eta'^2}{J}$$ \hspace{1cm} (3.15)

where $J = \frac{\partial(X, \eta)}{\partial(\sigma, \tau)}$ is the jacobian. From eq.(3.14) we find $J = -\chi' \phi'$ and $\dot{\eta}^2 - \eta'^2 = -\chi' \phi'$. Then,

$$\sqrt{-G} T^{00}(X) = \frac{1}{2\pi \alpha}.$$

We analogously find $\dot{X}^2 - X'^2 = -\chi' \phi'$. Then
\[
\sqrt{-G} \, T^{11}(X) = - \frac{1}{2\pi \alpha'} \, , \quad T^{01}(X) = 0 .
\]

That implies,
\[
\rho = \frac{1}{2\pi \alpha'} \, , \quad p = - \frac{1}{2\pi \alpha'} \, , \quad p + \rho = 0 . \tag{3.16}
\]

We find a constant energy density and a constant negative pressure. They exactly fulfill the continuity equation (3.9). These results hold for arbitrary cosmological spacetimes in \(1 + 1\) dimensions. That is, for arbitrary factors \(R(T)\). In particular they are valid for strings wound \(n\)-times around the de Sitter universe \([4]\).

A second family of string solutions follows from eq.(3.12) by choosing
\[
\eta = \pm X + C_\pm \tag{3.17}
\]
where \(C_\pm\) is a constant. Then, the string equations of motion (3.4) become
\[
\partial_\mu \left[ R(\eta)^2 \partial^\mu \eta \right] = 0 \tag{3.18}
\]

Using eq.(3.17), we find that the energy-momentum tensor (2.27) is traceless for this string solution:
\[T^{00} = T^{11} \, , \quad \text{that is} \quad p = \rho \tag{3.19}\]

Let us call
\[V(\eta) = \int^\eta R^2(x)dx\]

Then, eq.(3.18) implies that
\[
\left( \frac{\partial^2}{\partial \tau^2} - \frac{\partial^2}{\partial \sigma^2} \right) V(\eta) = 0 .
\]

The general solution \(\eta = \eta(\sigma, \tau)\) is implicitly defined by:
\[
\int^\eta R^2(x)dx = A(\sigma - \tau) + B(\sigma + \tau) .
\]

where \(A(x)\) and \(B(x)\) are arbitrary functions. Upon a conformal transformation, without loss of generality we can set
\[A(\sigma - \tau) + B(\sigma + \tau) \Rightarrow \tau .
\]

Hence, \(\eta\) is in general a function solely of \(\tau\) with
\[
\tau = \int^\eta R^2(x)dx \quad \text{or} \quad \frac{d\eta}{d\tau} = \frac{1}{R^2(\eta)} .
\]

The energy-momentum tensor (2.27) takes for this solution the form:
The δ(η ± X - C±) characterizes a localized object propagating on the characteristics at the speed of light. This solution describes a massless point particle since it has been possible to gauge out the σ dependence.

In summary, the two-dimensional string solutions in cosmological spacetimes generically obey perfect fluid equations of state with either

\[ p = -\rho \ (\gamma = 0) \quad \text{or} \quad p = +\rho \ (\gamma = 2) \]  

The respective energy densities being

\[ \rho = \rho_u \ (\rho_u = \text{constant}) \quad \text{for} \ \gamma = 0 \quad \text{or} \quad \rho = \frac{u}{R^2} \ (u = \text{constant}) \quad \text{for} \ \gamma = 2. \]  

These behaviors fulfill the continuity equation (3.9) for \( D = 2 \).

B. 2+1 Dimensional Universes. Ring Strings in D-Dimensional Universes

A large class of exact solutions describing one string and multistrings has been found in 2+1 dimensional de Sitter universe \([\frac{3}{2}] - \frac{5}{2}\). For power-like expansion factors \( R(\eta)^2 = A\eta^k \), \( k \neq -2 \) only ring solutions are known \([6]\). \( k = -2 \) corresponds to de Sitter spacetime. These string solutions are on a two-dimensional plane that can be considered embedded on a D-Dimensional universe.

Ring solutions correspond to the Ansatz \([3]\):

\[ T = T(\tau) \]
\[ X^1 = f(\tau) \cos \sigma \]
\[ X^2 = f(\tau) \sin \sigma \]  

(3.22)

The total energy of one string is then given by (recall \( G^{00} = 1 \))

\[ E(T) = \int d^{D-1}X \sqrt{-G} \ T^{00}(X) = \frac{1}{\alpha'} \frac{dT}{d\tau} \]

More generally, the energy-momentum tensor integrated on a volume that completely encloses the string, takes the form \([14]\)

\[ \Theta^{AB}(X) = \frac{1}{2\pi \alpha'} \int d\sigma d\tau \left( \dot{X}^A \dot{X}^B - X'^A X'^B \right) \delta(T - T(\tau)) \]
\[ = \frac{1}{2\pi \alpha' |X^0(\tau)|} \int_0^{2\pi} d\sigma \left[ \dot{X}^A \dot{X}^B - X'^A X'^B \right]_{\tau = \tau(T)} \]

For multistring solutions, one must sum over the different roots \( \tau_i \) of the equation \( T = T(\tau) \), for a given \( T \).

We find for the ring ansatz eq.(3.23):
\[ \Theta^{00}(X) = E(T) \]
\[ \Theta^{11}(X) = \Theta^{22}(X) = \frac{1}{2\alpha'[X^0(\tau)]} [f^2 - f'] \]
\[ \Theta^{01}(X) = \Theta^{02}(X) = \Theta^{12}(X) = 0. \]

That is,
\[ \Theta_B^A = \begin{pmatrix} E & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix} \]

where,
\[ E = \frac{1}{\alpha'} \dot{T}(\tau), \quad P = \frac{R(\tau)^2}{2\alpha'|X^0(\tau)|} [f^2 - f'] \tag{3.23} \]

Let us first consider ring strings in de Sitter universe \([5] - [7]\). We find there three different asymptotic behaviors: stable, unstable and its dual. The unstable regime appears for \( \eta \simeq \frac{\tau}{H} \to 0 \). That is, when the scale factor tends to infinity. From eqs.(3.23) and ref. [5], we find

\[ E(\tau) \xrightarrow{\tau \to 0} \frac{1}{\alpha'H} \left( \frac{1}{|\tau|} + 1 \right) \simeq \frac{1}{\alpha'H} [R(\tau) + 1] \to +\infty \]
\[ P(\tau) \xrightarrow{\tau \to 0} -\frac{1}{2\alpha'H|\tau|} \simeq \frac{R(\tau)}{2\alpha'H} \to -\infty \]

(Here \( H \) stands for the Hubble constant). The invariant string size grows as \( R/H \) in this unstable regime. Notice that the pressure is negative for the unstable strings and proportional to the expansion factor \( R \). In this regime we also see that

\[ P \stackrel{R \to \infty}{\rightarrow} -\frac{E}{2} + \frac{1}{2\alpha'H} + O(1/R) \to -\infty \tag{3.24} \]

with \( E = \frac{1}{\alpha'H} [R(\tau) + 1] \). The regime dual to the unstable regime appears when the conformal time \( \eta \) tends to infinity. For the solution \( q_-(\sigma, \tau) \) of ref. [5] in de Sitter universe, \( \eta \) diverges for finite \( \tau \to \tau_0 \) as

\[ \eta \xrightarrow{\tau \to \tau_0} 6 e^{-\tau_0} + O(1) \to +\infty \]

Here
\[ \tanh \frac{\tau_0}{\sqrt{2}} = \frac{1}{\sqrt{2}} \quad , \quad \tau_0 = \sqrt{2} \log(1 + \sqrt{2}) = 1.246450\ldots \]

Then,
$$R(\tau) \overset{\tau \to \tau_0}{=} \frac{e^{\tau_0}}{6} (\tau - \tau_0) \to 0^+$$

$$E(\tau) \overset{\tau \to \tau_0}{=} \frac{1}{\alpha' H (\tau - \tau_0)} = \frac{0.5796 \ldots}{\alpha' HR} \to +\infty$$

$$P(\tau) \overset{\tau \to \tau_0}{=} \frac{1}{2\alpha' H (\tau - \tau_0)} = E/2 \to +\infty$$

We call dual to this regime since it appears related to the unstable regime (3.24) through the exchange $R \leftrightarrow 1/R$. The invariant string size tends to $\frac{1}{\sqrt{2}H}$ in this regime.

In the stable regime, $\tau \to \infty$, (and the cosmic time $T \simeq \frac{\tau}{H} \to \infty$), from eq.(3.23) and ref. [5], we find

$$E(\tau) \overset{\tau \to \infty}{=} \frac{1}{\alpha' H}, \quad P(\tau) \overset{\tau \to \infty}{=} \frac{1 + \sqrt{2}}{2\alpha' H} e^{-\tau \sqrt{2}} \to 0. \quad (3.25)$$

For stable strings in de Sitter universe, the pressure is positive and vanishes asymptotically, and the invariant string size tends to $\frac{1}{\sqrt{2}H}$.

The solution $q_-(\sigma, \tau)$ in ref. [3] describes two ring strings:
a stable string for $q_0 \to +\infty (\tau \to \infty)$ ($q_0$ being the hyperboloid time-like coordinate) and
an unstable one for $q_0 \to -\infty (\tau \to \tau_0)$. The pressure $P$ depends on $\tau$; it is negative for $q_0 < l$ and positive for $q_0 > l$, where $l = -1.385145\ldots$.

It can be noticed that the behaviour eq.(3.24) for the energy can be interpreted as an unstable piece $R/(\alpha'H)$ plus a stable one $1/(\alpha'H)$. The constant term $1/(\alpha'H)$ is precisely the energy for the stable solution eq.(3.25).

Let us now study the energy and pressure for the ring string solutions in power-type inflationary universes considered in [3]. In terms of the conformal time $\eta$, we have as expansion factor $R^2(\eta) = A \eta^k$ with $k < 0$.

Near $\eta = 0$, two types of behavior were found [3]. The first one is a linear behavior

$$\eta \overset{\tau \to \tau_0}{=} \tau - \tau_0 + O((\tau - \tau_0)^2)$$

$$f(\tau) \overset{\tau \to \tau_0}{=} 1 - \frac{(\tau - \tau_0)^2}{2(k + 1)} \text{ for } k < -1$$

and

$$f(\tau) \overset{\tau \to \tau_0}{=} 1 + c (\tau - \tau_0)^{1-k} \text{ for } -1 < k < 0. \quad (3.26)$$

Eqs.(3.26) describe a expanding string for $k < 0$ with invariant size $\simeq (\tau - \tau_0)^{k/2}$. That is, in inflationary universes ($k < 0$), the string size grows indefinitely for $\eta \simeq \tau - \tau_0 \to 0$ as the universe radius $R \simeq (\tau - \tau_0)^{k/2} \to +\infty$. The growing of the proper string size for $\eta \simeq \tau - \tau_0 \to 0$ is a typical feature of string unstability.

Using eqs.(3.23) and (3.26), the energy and pressure take the form,

$$E(\tau) \overset{\tau \to \tau_0}{=} \frac{\sqrt{A}}{\alpha'} (\tau - \tau_0)^{k/2} = R/\alpha'$$

$$P(\tau) \overset{\tau \to \tau_0}{=} -\frac{\sqrt{A}}{2\alpha'} (\tau - \tau_0)^{k/2} = -R/(2\alpha'). \quad (3.27)$$

That yields
Eq. (3.28) is also valid in de Sitter universe for unstable strings [eq.(3.24)]. In all these cases strings exhibit negative pressure with an equation of state \( P = -E/2 \). This equation of state exactly saturates the strong energy condition in general relativity. This unstable string behavior dominates in all inflationary universes for \( R \to \infty \).

The second behavior present near \( \eta = 0 \) is \[ \eta \tau \to \tau_0 = \left( \tau - \tau_0 \right)^{1/(k+1)} \]
where \( f_0 \) must be set equal to zero for \(-1 < k < 0\). The invariant string size behaves for \( \tau \to \tau_0 \) as,
\[
S(\tau) \tau \to \tau_0 = \sqrt{A} \left( \tau - \tau_0 \right)^{k/2} A T \quad \text{for } k < 0
\]
This solution describes a string that collapses for \( k > -2 \) (power inflation) and blows up for \(-2 < k < 0 \) (super inflation). In both cases the string size is proportional to the horizon size which is of order \( T \) (cosmic time).

Here, the expansion factor tends to zero as
\[
R(\tau) \tau \to \tau_0 = \sqrt{A} \left( \tau - \tau_0 \right)^{k/2} \to 0.
\]
when \( k < -1 \) and blows up for \( 0 < k < -1 \).

From eq.(3.23) we find that
\[
P = \frac{A}{2(k+1)\alpha' R} \to +\infty , \quad E = \frac{A}{(k+1)\alpha' R} \to +\infty
\]
That is,
\[
P \tau \to \tau_0 = E/2 \quad \text{(3.30)}
\]
Notice that the pressure is here positive. This second behaviour is related by duality \( (R \leftrightarrow 1/R) \) to the first behaviour described by eqs.(3.27)-(3.28). This is the dual to unstable regime. In such regime the strings behave as massless particles (radiation).

Let us now consider ring string solutions in FRW universes considered in [9]. These solutions start ex-nihilo at the big bang \( (R = 0) \). In terms of the conformal time \( \eta \), we have as expansion factor \( R^2(\eta) = A \eta^k \) with \( k > 0 \).

Near \( \eta = 0 \), two types of behavior were found [9]. The first one is a linear behavior
\[
\eta \tau \to \tau_0 = \tau - \tau_0 + O(\tau - \tau_0)^2
\]
\[
f(\tau) \tau \to \tau_0 = 1 - \frac{(\tau - \tau_0)^2}{2(k+1)}.
\]
Eqs.(3.31) describe a collapsing string for \( k > 0 \) with invariant size \( \simeq (\tau - \tau_0)^{k/2} \). That is, in FRW universes \( (k > 0) \), the string size goes to zero for \( \eta \simeq \tau - \tau_0 \to 0 \) as the universe radius \( R \simeq (\tau - \tau_0)^{k/2} \to 0 \).
Using eqs. (3.23) and (3.31), the energy and pressure take the form,
\[ E(\tau) \stackrel{\tau \to \tau_0}{=} \frac{\sqrt{A}}{\alpha'} (\tau - \tau_0)^{k/2} = \frac{R}{\alpha'} , \]
\[ P(\tau) \stackrel{\tau \to \tau_0}{=} -\frac{\sqrt{A}}{2\alpha'} (\tau - \tau_0)^{k/2} = -\frac{R}{(2\alpha')} . \]  
(3.32)
That yields
\[ P \stackrel{\tau \to \tau_0}{=} -E/2 \]  
(3.33)
As we shall see below, this behaviour is subdominant in FRW universes. [Notice that eq.(3.33) is identical eq.(3.28) that holds for \( R \to \infty \) in power-like inflationary universes (\( k < 0 \)).]

The second behavior present near \( \eta = 0 \) is
\[ \eta \stackrel{\tau \to \tau_0}{=} (\tau - \tau_0)^{1/(k+1)} , \quad f(\tau) \stackrel{\tau \to \tau_0}{=} f_0 \pm (\tau - \tau_0)^{1/(k+1)} \]  
(3.34)
The invariant string size behaves for \( \tau \to \tau_0 \) as,
\[ S(\tau) \stackrel{\tau \to \tau_0}{=} \sqrt{A} (\tau - \tau_0)^{k/(2(k+1)} \to 0 \quad \text{for} \quad k > 0 \]
This solution describes a string that collapses for \( k > 0 \).

Here, the expansion factor tends to zero as
\[ R(\tau) \stackrel{\tau \to \tau_0}{=} \sqrt{A} (\tau - \tau_0)^{k/(2(k+1)} \to 0 . \]

since \( k > 0 \).

From eq.(3.23) we find that
\[ P = \frac{A}{2(k+1)\alpha'R} \to +\infty , \quad E = \frac{A}{(k+1)\alpha'R} \to +\infty \]
That is,
\[ P \stackrel{\tau \to \tau_0}{=} E/2 \]  
(3.35)
Notice that the pressure is here positive. This second behaviour is related by duality (\( R \leftrightarrow 1/R \)) to the first behaviour described by eqs.(3.32)-(3.33).

Hence, we see from eqs.(3.32,3.33) that near the big bang (\( R \to 0 \)) the dual behavior dominates over the unstable behavior.

It should also be noticed that for this dual behavior, the energy redshifts exactly as \( 1/R \) for \( R \to 0 \).

For large \( \tau \), the ring strings exhibit a stable behaviour,
\[ \eta(\tau) \stackrel{\tau \to \pm \infty}{=} \tau^{2/(k+2)} , \]
\[ T(\tau) \stackrel{\tau \to \pm \infty}{=} \frac{2\sqrt{A}}{k+2} \tau , \]
\[ f(\tau) \stackrel{\tau \to \pm \infty}{=} \frac{2}{k+2} \tau^{-k/(k+2)} \cos(\tau + \varphi) , \]  
(3.36)
where $\varphi$ is a constant phase and the oscillation amplitude has been normalized. For large $\tau$, the energy and pressure of the solution (3.36) behave as

$$E(\tau) \stackrel{\tau \to +\infty}{=} \frac{2\sqrt{A}}{\alpha'(k+2)} = \text{constant},$$

$$P(\tau) \stackrel{\tau \to +\infty}{=} -\frac{\sqrt{A}}{\alpha'(k+2)} \cos(2\tau + 2\varphi) \to 0.$$  (3.37)

This is the analog of the stable behaviour (3.25) in de Sitter universe for $\tau \to \infty$. Notice that eqs.(3.37) hold both for FRW ($k > 0$) and inflationary spacetimes ($k < 0$).

The behaviour described by eqs.(3.37) is similar to Minkowski spacetime. Notice that the factor $\tau^{-k/(k+2)}$ in the comoving radius [eq.(3.37)] is just the scale factor. That is, the physical string radius oscillates with constant amplitude as in Minkowski spacetime. The fact that the spacetime curvature tends to zero (except for de Sitter universe) when $T(\tau) \to \infty$ explains here the presence of a Minkowski-type behaviour.

For power-like inflationary universes with $k < -1$ a special exact ring solution exist [9] with

$$\eta = C \exp \frac{\tau}{\sqrt{-k - 1}}, \quad f(\tau) = C \exp \frac{\tau}{\sqrt{-k - 1}}.$$  (3.38)

where $C$ is an arbitrary constant. For $k = -2$ (de Sitter universe) this is the solution $q^{(o)}(\sigma, \tau)$ in ref. [5] which has constant string size.

For this solution we find

$$E = \frac{1}{\alpha'} \sqrt{-\frac{A}{k - 1}} \exp \left[ \frac{\tau(k+2)}{2\sqrt{-k - 1}} \right] = K R^{1+2/k},$$

$$P = -\left( \frac{1}{2} + \frac{1}{k} \right) E$$

where $K$ is a constant.

This is a fluid-like equation of state with $\gamma = \frac{1}{2} - \frac{1}{k}$. Notice that $\frac{1}{2} < \gamma < \frac{3}{2}$. For this solution, the energy grows with $R$ as $R$ to a power $1 + 2/k$ where, since $k < -1$,

$$-1 < 1 + \frac{2}{k} < 1.$$  

($E$ and $P$ of this solution are constants in de Sitter spacetime [$k = -2$]). This means that these special strings are subdominant both for $R \to \infty$ and for $R \to 0$ where the unstable strings ($E \approx R$) and their dual ($E \approx R^{-1}$) dominate respectively.

In ref. [4] the exact general evolution of circular strings in $2 + 1$ dimensional de Sitter spacetime is described closely and completely in terms of elliptic functions. Such solutions follow the form of eq.(3.22). The evolution depends on a constant parameter $b$, related to the string energy, and falls into three classes depending on whether $b < 1/4$ (oscillatory motion), $b = 1/4$ (degenerated, hyperbolic motion) or $b > 1/4$ (unbounded motion). The novel feature here is that one single world-sheet generically describes infinitely many (different and independent)
strings. The world-sheet time $\tau$ is an infinite-valued function of the string physical time, each branch yields a different string. This has no analogue in flat spacetime.

We computed in ref. [7] the string energy $E$ as a function of the string proper size $S$, and analyze it for the expanding and oscillating strings. For expanding strings ($\dot{S} > 0$): $E \neq 0$ even at $S = 0$, $E$ decreases for small $S$ and increases $\propto S$ for large $S$. For an oscillating string ($0 \leq S \leq S_{\text{max}}$), the average energy $<E>$ over one oscillation period is expressed as a function of $S_{\text{max}}$ as a complete elliptic integral of the third kind [7].

Let us briefly describe here the periodic solutions found in ref. [7]. The scale factor takes the form

$$R(T) = e^{HT}(\tau) = e^{\tau \left[ \frac{k}{1+k^2} + \Omega \frac{\vartheta_4'(\Omega y)}{\pi \vartheta_1(\Omega y)} \right] + \left| \frac{\vartheta_4(\Omega \tau - y)}{\vartheta_4(\Omega \tau)} \right|}$$

where $0 \leq k \leq 1$ stands for the elliptic modulus,

$$\Omega \equiv \frac{\pi}{2 \sqrt{k^2 + 1} K(k)}, \quad y \equiv \sqrt{k^2 + 1} F\left( \frac{1}{\sqrt{k^2 + 1}}, k \right)$$

and $F(\phi, k)$ stands for a 1st. class elliptic integral. Notice that $1 \geq \Omega \geq 0$ for $0 \leq k \leq 1$. $b$ relates with $k$ through $\sqrt{b} = \frac{k}{1+k^2}$. We see that $R(T)$ is an oscillatory function of $\tau$.

The cosmic time can be explicitly written as a Fourier series:

$$HT(\tau) = \tau \left[ \frac{k}{1+k^2} + \Omega \frac{\vartheta_4'(\Omega y)}{\pi \vartheta_1(\Omega y)} \right] + \log \left| \frac{\vartheta_4(\Omega \tau - y)}{\vartheta_4(\Omega \tau)} \right| - 4 \sum_{n=1}^{\infty} \frac{q^n}{n(1-q^{2n})} \sin(ny\Omega) \sin[n\pi \Omega(2\tau - y)] .$$

The comoving string radius $f(\tau)$ [see eq.(3.22)] takes here the form

$$f(\tau) = e^{-HT(\tau)} \frac{k}{\sqrt{1+k^2}} \text{sn} \left( \frac{\tau}{\sqrt{1+k^2}}, k \right) = \frac{2\pi}{K(k)\sqrt{1+k^2}} \sum_{n=1}^{\infty} q^{n-1/2} \frac{q^{n-1} \sin[(2n-1)\Omega \tau]}{1-q^{2n-1}} .$$

The cosmic time and the comoving string radius are therefore completely regular functions of $\tau$, and it follows that this string solution which is oscillating regularly as a function of world-sheet time $\tau$, is also oscillating regularly when expressed in terms of hyperboloid time or cosmic time. This solution represents one stable string.

The invariant string size results:

$$S(\tau) = \frac{k}{H\sqrt{1+k^2}} \text{sn} \left( \frac{\tau}{\sqrt{1+k^2}}, k \right)$$

We see that the string oscillates inside the horizon:

$$0 < S(\tau) < \frac{1}{H}$$

We see from eqs. (3.40)-(3.40) that the string oscillations do not follow a pure harmonic motion as in flat Minkowski spacetime, but they are precise superpositions of all frequencies.
\[(2n - 1)\Omega \ (n = 1, 2, ..., \infty)\] with uniquely defined coefficients. The non-linearity of the string equations in de Sitter spacetime fixes the relation between the mode coefficients. In the present case the basic frequency \(\Omega\) depends on the string energy, while in Minkowski spacetime the frequencies are merely \(n\).

The energy (3.23) of this solution is given by [7]:

\[
E = \frac{1}{H\alpha'} \left[ \frac{k}{1 + k^2} + \frac{1}{\sqrt{1 + k^2}} \operatorname{zn}(\frac{y}{\sqrt{1 + k^2}}, k) \right] + \text{oscillating terms},
\]

(3.42)

where \(\operatorname{zn}(x, k)\) stands for the Jacobi zeta-function [27]. Averaging over a period \(2K\sqrt{1 + k^2}\), the average energy \(\langle E \rangle\) is just the square bracket term.

The pressure (3.23) averaged over a period can be expressed here as a combination of complete elliptic integrals of the third kind. Numerical evaluation gives zero with high accuracy. We have checked in addition that the first orders in the expansion in the \(k\) identically vanish. We conclude that the average pressure is zero as in Minkowski spacetime. Thus, in average the oscillating strings describe cold matter.

In the \(k \to 0\) limit the amplitude of the string size oscillation and the energy goes to zero [see eq.(3.41-3.42)]. Actually the solution disappears in such limit.

The other interesting limit is \(k \to 1\) where the elliptic solutions turn into hyperbolic solutions. Then \(\langle E \rangle \to 1/(H\alpha')\).

More generally a numerical analysis shows that \(\langle E \rangle\) is a monotonically increasing function of \(b\) for \(b \in [0, 1/4]\).

In summary, three asymptotic behaviors are exhibited by ring solutions.

i) Unstable for \(R \to \infty\) and \(R \to 0\):
\[
E = (\text{constant}) \ R \to +\infty, \ P = -E/2 \to -\infty, \ S \simeq R \to \infty.
\]

ii) Dual to unstable for \(R \to 0\) and \(R \to \infty\):
\[
E = (\text{constant}) \ R^{-1} \to +\infty, \ P = +E/2 \to +\infty, \ S \simeq R \to 0 \ (\text{except for de Sitter where } S \to \frac{1}{H}).
\]

iii) Stable for \(R \to \infty\):
\[
E = \text{constant}, \ P = 0, \ S = \text{constant}.
\]

Recall that the unstable behaviour dominates for \(R \to \infty\) in inflationary universes whereas the dual behaviour dominates in FRW universes for \(R \to 0\).

In addition we have the special behavior (3.39) for \(k < -1\). Notice that the three behaviors i)-iii) appear for all expansion factors \(R(\eta)\). The behaviours i) and ii) are related by the duality transformation \(R \leftrightarrow 1/R\), the case iii) being invariant under duality. In the three cases we find perfect fluid equations [see eq.(3.10)] with different \(\gamma\):

\[
\gamma_u = 1/2, \ \gamma_d = 3/2, \ \gamma_s = 1,
\]

where the indices \(u, d\) and \(s\) stand for ‘unstable’, ‘dual’ and ‘stable’, respectively. Assuming a perfect gas of strings on a volume \(R^2\), the energy density \(\rho\) will be proportional to \(E/R^2\). This yields the following scaling with the expansion factor using the energies from i)-iii):

\[
\rho_u = \text{constant} \ R^{-1}, \ \rho_d = \text{constant} \ R^{-3}, \ \rho_s = \text{constant} \ R^{-2}.
\]

(3.43)
All three densities and pressures obey the continuity equation (3.9), as it must be.

The factor \(1/2\) in the relation between \(P\) and \(E\) for the cases i) and ii) is purely geometric. Notice that this factor was one in \(1+1\) dimensions [eq.(3.20)].

C. D-Dimensional Universes

The solutions investigated in secs. II.A and II.B can exist for any dimensionality of the spacetime. Embedded in D-dimensional universes, the \((1+1)\) solutions of section II.A describe straight strings, the \((2+1)\) solutions of section II.B are circular rings. In D-dimensional spacetime, one expects string solutions spread in \(D-1\) spatial dimensions. Their treatment has been done asymptotically in ref. \([13]\). One finds for \(\eta \approx \tau - \tau_0 \to 0\), \(R \to \infty\),

\[
P_u = -\frac{1}{D-1} \rho_u \quad , \quad \gamma_u = \frac{D-2}{D-1} \quad \text{for unstable strings.} \tag{3.44}
\]

This relation coincides with eq.(3.16) for \(D = 2\) and with eqs.(3.24) and (3.28) for \(D = 3\).

The energy density scales with \(R\) as

\[
\rho_u = u R^{2-D} \quad , \quad R \to \infty \tag{3.45}
\]

(where \(u\) is a constant) in accordance with eq.(3.21) for \(D = 2\) and with eq.(3.43) for \(D = 3\).

For the dual regime, \(R \to 0\), we have:

\[
P_d = +\frac{1}{D-1} \rho_d \quad , \quad \gamma_d = \frac{D}{D-1} \quad \text{with} \quad \rho_d = d R^{-D} \quad , \quad R \to 0 \quad , \tag{3.46}
\]

(where \(d\) is a constant). Eq.(3.46) reduces to eq.(3.19) for \(D = 2\) and to eqs.(3.25) and (3.30) for \(D = 3\). In this dual regime strings have the same equation of state as massless radiation.

Finally, for the stable regime we have

\[
P_s = 0 \quad , \quad \gamma_s = 1 \quad \text{with} \quad \rho_s = s R^{1-D} \tag{3.47}
\]

(where \(s\) is a constant). This regime is absent in \(D = 2\) and appears for \(D = 3\) solutions in eqs.(3.25) and (3.31). The lack of string transverse modes in \(D = 2\) explains the absence of the stable regime there. The equation of state for stable strings coincides with the one for cold matter.

In conclusion, an ideal gas of classical strings in cosmological universes exhibit three different thermodynamical behaviours, all of perfect fluid type:

1) Unstable strings : negative pressure gas with \(\gamma_u = \frac{D-2}{D-1}\)
2) Dual behaviour : positive pressure gas similar to radiation , \(\gamma_d = \frac{D}{D-1}\)
3) Stable strings : positive pressure gas similar to cold matter, \(\gamma_s = 1\).

Tables I and II summarize the main string properties for any scale factor \(R(X^0)\).
TABLE 1. String energy and pressure as obtained from exact string solutions for various expansion factors $R(X^0)$.

**STRING PROPERTIES FOR ARBITRARY $R(X^0)$**

| $D = 1 + 1$: two families of solutions | Energy | Pressure | Equation of State: $p = (\gamma - 1)\rho$ |
|----------------------------------------|--------|----------|------------------------------------------|
| (i) $\eta \pm X = f_\pm(\sigma \pm \tau)$ | $E = u R$ | $P = -E$ | $\gamma = 0$ |
| (ii) $\eta \pm X = \text{constant}$ | $E = d/R$ | $P = +E$ | $\gamma = 2$ |

$D = 2 + 1$: Ring Solutions, three asymptotic behaviours (u, d, s)

| $E = \frac{1}{\alpha'} \dot{X}^0(\tau)$ | $P = \frac{R(\tau)^2}{2\alpha'|\dot{X}^0(\tau)|}[\dot{f}^2 - f^2]$ |

(i) unstable for $R \to \infty$
(ii) dual to (i) for $R \to 0$
(iii) stable for $R \to \infty$

| $E_u R \to \infty$ | $u R \to \infty$ | $P_u = -E_u/2 \to -\infty$ | $\gamma_u = 1/2$ |
| $E_d R \to \infty$ | $d/R \to \infty$ | $P_d = +E_d/2 \to \infty$ | $\gamma_d = 3/2$ |
| $E_s = \text{constant}$ | $P_s = 0$ | $\gamma_s = 1$ |

D-Dimensional spacetimes: general asymptotic behaviour

| $E_u R \to \infty$ | $u R \to \infty$ | $P_u = -\frac{E_u}{D-1} \to -\infty$ | $\gamma_u = \frac{D-2}{D-1}$ |
| $E_d R \to \infty$ | $d/R \to \infty$ | $P_d = +\frac{E_d}{D-1} \to \infty$ | $\gamma_d = \frac{D}{D-1}$ |
| $E_s = \text{constant}$ | $P_s = 0$ | $\gamma_s = 1$ |
TABLE 2. The string energy density and pressure for a gas of strings can be summarized by the formulas below which become exact for \( R \to 0 \) and for \( R \to \infty \).

STRING ENERGY DENSITY AND PRESSURE FOR ARBITRARY \( R(X^0) \)

| Qualitatively correct formulas for all \( R \) and \( D \) | Energy density: \( \rho \equiv \frac{E}{R^{D-1}} \) | Pressure |
|--------------------------------------------------------|-------------------------------------------------|----------|
| \( \rho = \left(u \frac{R}{R} + \frac{d}{R} + s\right) \frac{1}{R^{D-1}} \) | \( p = \frac{1}{D-1} \left(\frac{d}{R} - u \frac{R}{R}\right) \frac{1}{R^{D-1}} \) |

TABLE 3. The self-consistent cosmological solution of the Einstein equations in General Relativity with the string gas as source.

STRING COSMOLOGY IN GENERAL RELATIVITY

| Einstein equations (no dilaton field) | Expansion factor \( R(X^0) \) | Temperature \( T(R) \) |
|----------------------------------------|-------------------------------|-----------------------|
| \( X^0 \to 0 \) \( D \) \( \left[ \frac{2d}{(D-1)(D-2)} \right]^{\frac{1}{D-2}} (X^0) \frac{1}{D} \) | \( \frac{dD}{S(D-1)} \frac{1}{R} \) |

| \( X^0 \to \infty \) \( (\text{without string decay}) \) | \( \left[ \frac{(D-2)u}{2(D-1)} \right]^{\frac{1}{D-2}} (X^0) \frac{2}{D} \) | \( \frac{(D-2)u}{(D-1)S} R \) |

| \( X^0 \to \infty \) \( (\text{with string splitting}) \) | \( \left[ \frac{(D-1)s}{2(D-2)} \right]^{\frac{1}{D-1}} (X^0) \frac{2}{D-1} \) | usual matter dominated behaviour |
The unstable string behaviour corresponds to the critical case of the so-called coasting universe \[15,29\]. In other words, classical strings provide a concrete matter realization of such cosmological model. Till now, no form of matter was known to describe coasting universes \[15\].

Finally, notice that strings continuously evolve from one type of behaviour to the other two. This is explicitly seen from the string solutions in refs. \([1] - [9]\). For example the string described by \(q_{-}(\sigma, \tau)\) for \(\tau > 0\) shows unstable behaviour for \(\tau \to 0\), dual behaviour for \(\tau \to \tau_0 = 1.246450...\) and stable behaviour for \(\tau \to \infty\).

The equation of state for strings in four dimensional flat Minkowski spacetime is discussed in ref. \[21\]. One finds the values \(4/3\), \(2/3\) and \(1\) for \(\gamma\) by choosing appropriate values of the average string velocity in chap. 7 of ref. \[21\].

**IV. SELF-CONSISTENT STRING COSMOLOGY**

In the previous section we investigated the propagation of test strings in cosmological space-times. Let us now investigate how the Einstein equations in General Relativity and the effective equations of string theory (beta functions) can be verified self-consistently with our string solutions as sources.

We shall assume a gas of classical strings neglecting interactions as string splitting and coalescing. We will look for cosmological solutions described by metrics of the type \((3.1)\). It is natural to assume that the background will have the same symmetry as the sources. That is, we assume that the string gas is homogeneous, described by a density energy \(\rho = \rho(T)\) and a pressure \(p = p(T)\). In the effective equations of string theory we consider a space independent dilaton field. Antisymmetric tensor fields will be ignored.

**A. String Dominated Universes in General Relativity (no dilaton field)**

The Einstein equations for the geometry \((3.1)\) take the form

\[
\frac{1}{2} (D - 1)(D - 2) H^2 = \rho ,
\]

\[
(D - 2) \dot{H} + p + \rho = 0 .
\]

(4.1)

where \(H \equiv \frac{dR}{d\tau}/R\). We know \(p\) and \(\rho\) as functions of \(R\) in asymptotic cases. For large \(R\), the unstable strings dominate \([eq.(3.45)]\) and we have

\[
\rho = u R^{2-D} , \quad p = -\frac{\rho}{D-1} \quad \text{for} \ R \to \infty \quad (4.2)
\]

For small \(R\), the dual regime dominates with

\[
\rho = d R^{-D} , \quad p = \frac{\rho}{D-1} \quad \text{for} \ R \to 0 \quad (4.3)
\]

We also know that stable solutions may be present with a contribution \(\sim R^{1-D}\) to \(\rho\) and with zero pressure. For intermediate values of \(R\) the form of \(\rho\) is clearly more complicated but a formula of the type
\[ \rho = \left( u_R R + d \frac{R}{R} + s \right) \frac{1}{R^{D-1}} \]  

(4.4)

where

\[ \lim_{R \to \infty} u_R = \begin{cases} 
0 & \text{FRW} \\
\text{Inflationary} & u_\infty \neq 0 
\end{cases} \]  

(4.5)

This equation of state is qualitatively correct for all \( R \) and becomes exact for \( R \to 0 \) and \( R \to \infty \). The parameters \( u_R, d \) and \( s \) are positive constants and the \( u_R \) varies smoothly with \( R \).

The pressure associated to the energy density (4.4) takes then the form

\[ p = \frac{1}{D-1} \left( \frac{d}{R} - u_R R \right) \frac{1}{R^{D-1}} \]  

(4.6)

Inserting eq.(4.4) into the Einstein-Friedmann equations [eq.(4.1)] we find

\[ \frac{1}{2} (D-1)(D-2) \left( \frac{dR}{dT} \right)^2 = \left( u_R R + d \frac{R}{R} + s \right) \frac{1}{R^{D-3}} \]  

(4.7)

We see that \( R \) is a monotonic function of the cosmic time \( T \). Eq.(4.7) yields

\[ T = \sqrt{\frac{(D-1)(D-2)}{2}} \int_0^R dR \frac{R^{D/2-1}}{\sqrt{u_R R^2 + d + s}} \]  

(4.8)

where we set \( R(0) = 0 \).

It is easy to derive the behavior of \( R \) for \( T \to 0 \) and for \( T \to \infty \).

For \( T \to 0, \ R \to 0 \), the term \( d/R \) dominates in eq.(4.7) and

\[ R(T) \overset{T \to 0}{\sim} \frac{D}{2} \left[ \frac{2d}{(D-1)(D-2)} \right]^{\frac{1}{D}} (T)^{\frac{2}{D}} \]  

(4.9)

For \( T \to \infty, \ R \to \infty \) and the term \( u_R R \) dominates in eq.(4.7). Hence,

\[ R(T) \overset{T \to \infty}{\sim} \left[ \frac{(D-2)u_\infty}{2(D-1)} \right]^{\frac{1}{D-2}} (T)^{\frac{2}{D}} \]  

(4.10)

\([u_R \text{ tends to a constant } u_\infty \text{ for } R \to \infty]\). This expansion is faster than (cold) matter dominated universes where \( R \simeq [T]^{2/3} \). For example, for \( D = 4 \), \( R \) grows linearly with \( T \) whereas for matter dominated universes \( R \simeq [T]^{2/3} \). However, eq.(4.10) is not a self-consistent solution. Assuming that the term \( u_R R \) dominates for large \( R \) we find a scale factor \( R(T) \sim (T)^{2/3} \sim \eta^{2/3} \) for \( D \neq 4 \) and \( R(T) \sim T \sim e^\eta \) at \( D = 4 \). This is not an inflationary universe but a FRW universe. The term \( u_R R \) is absent for large \( R \) in FRW universes as explained before. Therefore, we must instead use for large \( R \)

\[ \rho = \left( \frac{d}{R} + s \right) \frac{1}{R^{D-1}} \]  

(4.11)
Now, for $T \to \infty, R \to \infty$ and we find a matter dominated regime:

$$R(T) \overset{T \to \infty}{\sim} \left[ \frac{(D-1)s}{2(D-2)} \right]^{\frac{1}{D-1}} (T)^{\frac{2}{D-1}}$$  \hspace{1cm} (4.12)

For intermediate values of $T$, $R(T)$ is a continuous and monotonically increasing function of $T$.

In summary, the universe starts at $T = 0$ with a singularity of the type dominated by radiation. (The string behaviour for $R \to 0$ is like usual radiation). Then, the universe expands monotonically, growing for large $T$ as $R \simeq [T]^{2/3}$. In particular, this gives $R \simeq [T]^{2/3}$ for $D = 4$.

It must be noticed that the qualitative form of the solution $R(T)$ does not depend on the particular positive values of $u_R, d$ and $s$.

We want to stress that we achieve a **self-consistent** solution of the Einstein equations with string sources since the behaviour of the string pressure and density given by eqs.(4.4)-(4.6) precisely holds in universes with power like $R(T)$.

In ref. [3] similar results were derived using arguments based on the splitting of long strings.

**B. Thermodynamics of strings in cosmological spacetimes**

Let us consider a comoving volume $R^{D-1}$ filled by a gas of strings. The entropy change for this system is given by:

$$TdS = d(\rho R^{D-1}) + p d(R^{D-1})$$  \hspace{1cm} (4.13)

The continuity equation (3.9) and (4.13) implies that $dS/dt$ vanishes. That is, the entropy per comoving volume stays constant in time. Using now the thermodynamic relation [20]

$$\frac{dp}{dT} = \frac{p + \rho}{T}$$  \hspace{1cm} (4.14)

it follows [21] that

$$S = \frac{R^{D-1}}{T} (\rho + \rho) + \text{constant}$$  \hspace{1cm} (4.15)

Let us first ignore the possibility of string decay. Then, eq.(4.13) together with eqs.(4.4) and (4.6) yields the temperature as a function of the expansion factor $R$. That is,

$$T = \frac{1}{S} \left[ s + \frac{1}{D-1} \left[ \frac{D}{R} + (D-2) u \right] R \right]$$  \hspace{1cm} (4.16)

where $S$ stands for the (constant) value of the entropy.

Eq.(4.16) shows that for small $R$, $T$ scales as $1/R$ whereas for large $R$ it scales as $R$. The small $R$ behaviour of $T$ is the usual exhibited by radiation.

For large $R$, in FRW universes $u_R \to 0$ and the constant term in $s$ dominates. We just find a cold matter behaviour for large $R$.

For large $R$ in inflationary universes, $u_R \to u_\infty$ and eq.(4.16) would indicate a temperature that **grows** proportionally to $R$. However, as stressed in ref. [3], the decay of long strings (through splitting) makes $u_R$ exponentially decreasing with $R$.
V. EFFECTIVE STRING EQUATIONS WITH THE STRING SOURCES INCLUDED

Let us consider now the cosmological equations obtained from the low energy string effective action including the string matter as a classical source. In D spacetime dimensions, this action can be written as

\[
S = S_1 + S_2
\]

\[
S_1 = \frac{1}{2} \int d^D x \sqrt{-G} e^{-\Phi} \left[ R + G_{AB} \partial^A \Phi \partial^B \Phi + 2 \ U(G, \Phi) - c \right]
\]

\[
S_2 = -\frac{1}{4 \pi \alpha'} \sum_{\text{strings}} \int d\sigma d\tau \ G_{AB}(X) \partial_\mu X^A \partial^\mu X^B , \tag{5.1}
\]

Here \( A, B = 0, \ldots, D - 1 \). This action is written in the so called ‘Brans-Dicke frame’ (BD) or ‘string frame’, in which matter couples to the metric tensor in the standard way. The BD frame metric coincides with the sigma model metric to which test strings are coupled.

Eq.(5.1) includes the dilaton field \( \Phi \) with a potential \( U(G, \Phi) \) depending on the dilaton and graviton backgrounds; \( c \) stands for the central charge deficit or cosmological constant term. The antisymmetric tensor field was not included, in fact it is irrelevant for the results obtained here. Extremizing the action (5.1) with respect to \( G_{AB} \) and \( \Phi \) yields the equations of motion

\[
R_{AB} + \nabla_{AB} \Phi + 2 \ \frac{\partial U}{\partial G_{AB}} - \frac{G_{AB}}{2} R = e^\Phi T_{AB}
\]

\[
R + 2 \nabla^2 \Phi - (\nabla \Phi)^2 - 2 U - \frac{\partial U}{\partial \Phi} = 0 , \tag{5.2}
\]

which can be more simply combined as

\[
R_{AB} + \nabla_{AB} \Phi + 2 \ \frac{\partial U}{\partial G_{AB}} - \frac{G_{AB}}{2} \partial^\mu G_{AB} = e^\Phi T_{AB}
\]

\[
R + 2 \nabla^2 \Phi - (\nabla \Phi)^2 - c + 2 U - \frac{\partial U}{\partial \Phi} = 0 \tag{5.3}
\]

Here \( T_{AB} \) stands for the energy momentum tensor of the strings as defined by eq.(2.27). It is also convenient to write these equations as

\[
R_{AB} - \frac{G_{AB}}{2} R = T_{AB} + \tau_{AB} \tag{5.4}
\]

where \( \tau_{AB} \) is the dilaton energy momentum tensor:

\[
\tau_{AB} = -\nabla_{AB} \Phi + \frac{G_{AB}}{2} \left[ 2 \ \frac{\partial U}{\partial \Phi} - R \right]
\]

The Bianchi identity

\[
\nabla^A \left( R_{AB} - \frac{G_{AB}}{2} R \right) = 0
\]

yields, as it must be, the conservation equation,
\[ \nabla^A (T_{AB} + \tau_{AB}) = 0 \]  

(5.5)

It must be noticed that eqs. (5.3) do not reduce to the Einstein equations of General Relativity even when \( \Phi = U = 0 \). Eqs. (5.3) yields in that case the Einstein equations plus the condition \( R = 0 \).

### A. Effective String Equations in Cosmological Universes

For the homogeneous isotropic spacetime geometries described by eq. (3.1) we have

\[
\begin{align*}
R^0_0 &= -(D-1)(\dot{H} + H^2) \\
R^k_i &= -\delta^k_i \left[ \dot{H} + (D-1)H^2 \right] \\
R &= -(D-1)(2\dot{H} + DH^2).
\end{align*}
\]

(5.6)

where \( H \equiv \frac{1}{R} \frac{dR}{dT} \).

The equations of motion (5.3) read

\[
\begin{align*}
\ddot{\Phi} - (D-1)\left( \dot{H} + H^2 \right) - \frac{\partial U}{\partial \Phi} &= e^\Phi \rho \\
\dot{H} + (D-1)H^2 - H\dot{\Phi} + \frac{\partial U}{\partial \Phi} + \frac{R}{D-1} \frac{\partial U}{\partial R} &= e^\Phi p \\
2\ddot{\Phi} + 2(D-1)H\dot{\Phi} - \dot{\Phi}^2 - (D-1)(2\dot{H} + DH^2) - 2\left( \frac{\partial U}{\partial \Phi} \right) &- c + 2U = 0
\end{align*}
\]

(5.7)

where dot \( \cdot \) stands for \( \frac{d}{dT} \), and

\[ \rho = T^0_0, \quad -\delta^k_i p = T^k_i. \]

(5.8)

The conservation equation takes the form of eq. (3.9)

\[
\dot{\rho} + (D-1)H(p + \rho) = 0.
\]

(5.9)

By defining,

\[
\Psi \equiv \Phi - \log \sqrt{-G} = \Phi - (D-1) \log R \\
\bar{\rho} = e^\Phi \rho, \quad \bar{p} = e^\Phi p,
\]

(5.10)

equations (5.7) can be expressed in a more compact form as

\[
\begin{align*}
\dddot{\Psi} - (D-1)H^2 - \frac{\partial U}{\partial \Psi} \bigg|_R &= \bar{\rho} \\
\dot{H} - H\dddot{\Psi} + \frac{R}{D-1} \frac{\partial U}{\partial R} \bigg|_\Psi &= \bar{p} \\
\dot{\Psi}^2 - (D-1)H^2 - 2\bar{\rho} - 2U + c &= 0
\end{align*}
\]

(5.11)

The conservation equation reads
\[ \dot{\rho} - \dot{\Psi} \bar{\rho} + (D - 1) H \bar{\rho} = 0 \]  

(5.12)

As is known, under the duality transformation \( R \rightarrow R^{-1} \), the dilaton transforms as \( \Phi \rightarrow \Phi + (D - 1) \log R \). The shifted dilaton \( \Psi \) defined by eq.(5.10) is invariant under duality. The transformation

\[ R' \equiv R^{-1} \]  

(5.13)

implies

\[ \Psi' = \Psi \quad , \quad H' = -H \quad , \quad \bar{p}' = -p \quad , \quad \bar{p}' = \bar{\rho} \]  

(5.14)

provided \( u = d \), that is, a duality invariant string source. This is the duality invariance transformation of eqs.(5.11).

Solutions to the effective string equations have been extensively treated in the literature \[25\] and they are not our main purpose. For the sake of completeness, we briefly analyze the limiting behaviour of these equations for \( R \rightarrow \infty \) and \( R \rightarrow 0 \).

It is difficult to make a complete analysis of the effective string equations (5.11) since the knowledge about the potential \( U \) is rather incomplete. For weak coupling (\( e^\Phi \) small ) the supersymmetry breaking produces an effective potential that decreases very fast (as the exponential of an exponential of \( \Phi \)) for \( \Phi \rightarrow -\infty \).

Let us analyze the asymptotic behavior of eqs.(5.11) for \( R \rightarrow \infty \) and \( R \rightarrow 0 \) assuming that the potential \( U \) can be ignored. It is easy to see that a power behaviour Ansatz both for \( R \) and for \( e^\Psi \) as functions of \( T \) is consistent with these equations. It turns out that the string sources do not contribute to the leading behaviour here, and we find for \( R \rightarrow 0 \)

\[ R_\mp = C_1 (T)^{\pm 1/\sqrt{D-1}} \rightarrow 0 \quad , \quad e^{\Psi_\mp} = C_2 (T)^{-1} \rightarrow \begin{cases} \infty \\ 0 \end{cases} \]  

(5.15)

Where \( C_1 \) and \( C_2 \) are constants. Here the branches \((-\) and \(+)\) correspond to \( T \rightarrow 0 \) and to \( T \rightarrow \infty \) respectively. In both regimes \( R_\mp \rightarrow 0 \) and \( e^{\Psi_\mp} \rightarrow 0 \).

The potential \( U(\Phi) \) is hence negligible in these regimes. In terms of the conformal time \( \eta \), the behaviours (5.15) result

\[ R_\mp = C'_1 (\eta)^{\pm 1/\sqrt{D-1}} \rightarrow 0 \quad , \quad e^{\Psi_\mp} = C'_2 (\eta)^{-1} \rightarrow \begin{cases} \infty \\ 0 \end{cases} \]  

(5.16)

Where \( C'_1 \) and \( C'_2 \) are constants. The branch \((-\) would describe an expanding non-inflationary behaviour near the initial singularity \( T = 0 \), while the branch \(+)\) describes a ‘big crunch’ situation and is rather unphysical.

Similarly, for \( R \rightarrow \infty \) and \( e^\Phi \rightarrow \infty \), we find

\[ R_\mp = D_1 (T)^{\mp 1/\sqrt{D-1}} \rightarrow \infty \quad , \quad e^{\Psi_\mp} = D_2 (T)^{-1} \rightarrow \begin{cases} \infty \\ 0 \end{cases} \]  

(5.17)
Where $D_1$ and $D_2$ are constants. Here again, the branches ($-$) and (+) correspond to $T \to 0$ and to $T \to \infty$ respectively, but now in both regimes $R_{\mp} \to \infty$ and $e^{\Phi_{\mp}} \to \infty$. (In this limit, one is not guaranteed that $U$ can be consistently neglected). In terms of the conformal time, eqs. (5.17) read

$$R_{\mp} = D_1' (\eta) \mp \sqrt{D-1} \pm \to \infty$$
$$e^{\Psi_{\mp}} = D_2' (\eta) - \sqrt{D-1} \to \left\{ \begin{array}{ll} \infty & \\
0 & \end{array} \right.$$  (5.18)

The branch (+) describes a noninflationary expanding behaviour for $T \to \infty$ faster than the standard matter dominated expansion, while the branch (−) describes a super-inflationary behaviour $\eta^{-\alpha}$, since $0 < \alpha < 1$, for all $D$.

The behaviours (5.15) for $R_{\mp} \to 0$ and (5.17) for $R_{\mp} \to \infty$ are related by duality $R \leftrightarrow 1/R$.

**B. String driven inflation?**

Let us consider now the question of whether de Sitter spacetime may be a self-consistent solution of the effective string equations (5.7) with the string sources included. The strings in cosmological universes like de Sitter spacetime have the equation of state (4.4)-(4.6). Since

$$e^{\Psi} = e^\rho R^{1-D} :$$

$$\dot{\rho} = e^{\Psi} \left( u \frac{d}{R} + s \right)$$
$$\ddot{\rho} = \frac{e^{\Psi}}{D-1} \left( \frac{d}{R} - u \frac{d}{R^2} \right)$$  (5.19)

In the absence of dilaton potential and cosmological constant term, the string sources do not generate de Sitter spacetime as discussed in sec. III.A. We see that for $U = c = 0$, and $R = e^{HT}$, eqs. (5.11) yields to a contradiction (unless $D = 0$) for the value of $\Psi$, required to be $-HT + \text{constant}$.

A self-consistent solution describing asymptotically de Sitter spacetime self-sustained by the string equation of state (5.19)-(5.20) is given by

$$R = e^{HT}, \quad H = \text{constant} > 0,$$
$$2U - c = \text{constant},$$
$$\Psi_{\pm} = \mp HT \pm i\pi + \log \left( \frac{(D-1)H^2}{\rho_{\pm}} \right),$$
$$\rho_{\pm} \equiv u, \quad \rho_{\pm} \equiv d$$  (5.21)

The branch $\Psi_{\pm}$ describes the solution for $R \to \infty$ ($T \to +\infty$), while the branch $\Psi_{\pm}$ corresponds to $R \to 0$ ($T \to -\infty$). De Sitter spacetime with lorentzian signature self-sustained by the strings necessarily requires a constant imaginary piece $\pm i\pi$ in the dilaton field. This makes $e^{\Psi} < 0$ telling us that the gravitational constant $G \sim e^{\Psi} < 0$ here describes antigravity.

It is interesting to notice that in the euclidean signature case, i.e. $(+++.+.+)$, the Ansatz $\dot{H} = 0, \ 2U - c = \text{constant}$, yields a constant curvature geometry with a real dilaton, but
which is of Anti-de Sitter type. This solution is obtained from eqs. (5.20)-(5.21) through the transformation

\[ \hat{X}^0 = iT, \quad \hat{H} = -iH, \quad X^i = X^i, \quad \Psi = \Psi \]  

which maps the Lorentzian de Sitter metric into the positive definite one

\[ ds^2 = (d\hat{X}^0)^2 + e^{\hat{H}\hat{X}^0} (d\hat{X})^2. \]  

The equations of motion (5.11) within the constant curvature Ansatz (\( \dot{\hat{H}} = \ddot{\Psi} = 0 \)) are mapped onto the equations

\[ (D - 1) \hat{H}^2 - \frac{\partial U}{\partial \Psi} \bigg|_R = \bar{\rho} \]  
\[ \hat{H} \frac{d\Psi}{d\hat{X}^0} + \frac{R}{D - 1} \frac{\partial U}{\partial R} \bigg|_\Psi = \bar{\rho} \]  
\[ -(\frac{d\Psi}{d\hat{X}^0})^2 + (D - 1) \hat{H}^2 - 2\bar{\rho} - 2U + c = 0, \]  

with the solution

\[ R = e^{\hat{H}\hat{X}^0}, \quad \hat{H} = \text{constant} > 0, \]  
\[ c - 2U = D \hat{H}^2 = \text{constant} \]  
\[ \Psi_\pm = \mp \hat{H}\hat{X}^0 + \log (D - 1) \hat{H}^2 \]  
\[ \rho_+ \equiv u, \quad \rho_- \equiv d \]  

Both solutions (5.25) and (5.21) are mapped one into another through the transformation (5.22).

It could be recalled that in the context of (point particle) field theory, de Sitter spacetime (as well as anti-de Sitter) emerges as an exact selfconsistent solution of the semiclassical Einstein equations with the back reaction included [31] - [32]. (Semiclassical in this context, means that matter fields including the graviton are quantized to the one-loop level and coupled to the (c-number) gravity background through the expectation value of the energy-momentum tensor \( T^A_B \). This expectation value is given by the trace anomaly: \( < T^A_A >= \bar{\gamma} R^2 > \). On the other hand, the \( \alpha' \) expansion of the effective string action admits anti-de Sitter spacetime (but not de Sitter) as a solution when the quadratic curvature corrections (in terms of the Gauss-Bonnet term) to the Einstein action are included [33]. It appears that the corrections to the anti-de Sitter constant curvature are qualitatively similar in the both cases, with \( \alpha' \) playing the rôle of the trace anomaly parameter \( \bar{\gamma} \) [32].

The fact that de Sitter inflation with true gravity \( G \sim e^\Psi > 0 \) does not emerge as a solution of the effective string equations does not mean that string theory excludes inflation. What means is that the effective string equations are not enough to get inflation. The effective string action is a low energy field theory approximation to string theory containing only the massless string modes (massless background fields).
The vacuum energy scales to start inflation (physical or true vacuum) are typically of the order of the Planck mass [21] - [22] where the effective string action approximation breaks down. One must consider the massive string modes (which are absent from the effective string action) in order to properly get the cosmological condensate yielding de Sitter inflation. We do not have at present the solution of such problem.

TABLE 4. Asymptotic solution of the string effective equations (including the dilaton).

**EFFECTIVE STRING EQUATIONS**

**SOLUTIONS IN COSMOLOGY**

| Effective String equations | $R(X^0) \to 0$ behaviour | $R(X^0) \to \infty$ behaviour |
|---------------------------|--------------------------|-------------------------------|
| $X^0 \to 0$               | $\sim (X^0)^{1/\sqrt{D-1}}$ | $\sim (X^0)^{-1/\sqrt{D-1}}$ |
| $X^0 \to \infty$         | $\sim (X^0)^{-1/\sqrt{D-1}}$ | $\sim (X^0)^{1/\sqrt{D-1}}$ |

**VI. MULTI-STRINGS AND SOLITON METHODS IN DE SITTER UNIVERSE**

Among the cosmological backgrounds, de Sitter spacetime occupies a special place. This is, in one hand relevant for inflation and on the other hand string propagation turns to be specially interesting there [2] - [8]. String unstability, in the sense that the string proper length grows indefinitely is particularly present in de Sitter. The string dynamics in de Sitter universe is described by a generalized sinh-Gordon model with a potential unbounded from below [4]. The sinh-Gordon function $\alpha(\sigma, \tau)$ having a clear physical meaning: $H^{-1}e^{\alpha(\sigma, \tau)/2}$ determines the string proper length. Moreover the classical string equations of motion (plus the string constraints) turn to be integrable in de Sitter universe [4,5]. More precisely, they are equivalent to a non-linear sigma model on the grassmannian $SO(D,1)/O(D)$ with periodic boundary conditions (for closed strings). This sigma model has an associated linear system [34] and using it, one can show the presence of an infinite number of conserved quantities [35]. In addition, the string constraints imply a zero energy-momentum tensor and these constraints are compatible with the integrability.

The so-called dressing method [34] in soliton theory allows to construct solutions of non-linear classically integrable models using the associated linear system. In ref. [3] we systematically construct string solutions in three dimensional de Sitter spacetime. We start from a given exactly known solution of the string equations of motion and constraints in de Sitter [5] and then we “dress” it. The string solutions reported there indeed apply to cosmic strings in de Sitter spacetime as well.

The invariant interval in $D$-dimensional de Sitter space-time is given by
\[ ds^2 = dT^2 - \exp[2HT] \sum_{i=1}^{D-1} (dX^i)^2. \] (6.1)

Here \( T \) is the so-called cosmic time. In terms of the conformal time \( \eta \),
\[
\eta \equiv -\frac{\exp[-HT]}{H}, -\infty < \eta \leq 0,
\]
the line element becomes
\[
ds^2 = \frac{1}{H^2\eta^2}[d\eta^2 - \sum_{i=1}^{D-1} (dX^i)^2].
\]

The de Sitter spacetime can be considered as a \( D \)-dimensional hyperboloid embedded in a \( D+1 \) dimensional flat Minkowski spacetime with coordinates \((q^0, ..., q^D)\):
\[
ds^2 = \frac{1}{H^2}[-(dq^0)^2 + \sum_{i=1}^{D} (dq^i)^2] \tag{6.2}
\]
where
\[
q^0 = \sinh HT + \frac{H^2}{2} \exp[HT] \sum_{i=1}^{D-1} (X^i)^2,
\]
\[
q^1 = \cosh HT - \frac{H^2}{2} \exp[HT] \sum_{i=1}^{D-1} (X^i)^2,
\]
\[
q^{i+1} = H \exp[HT] X^i, \quad 1 \leq i \leq D-1, \quad -\infty < T, \quad X^i < +\infty. \tag{6.3}
\]

The complete de Sitter manifold is the hyperboloid
\[-(q^0)^2 + \sum_{i=1}^{D} (q^i)^2 = 1.
\]

The coordinates \((T, X^i)\) and \((\eta, X^i)\) cover only the half of the de Sitter manifold \(q^0 + q^1 > 0\).

We will consider a string propagating in this \( D \)-dimensional space-time. The string equations of motion (2.7) in the metric (6.2) take the form:
\[
\partial_+ q + (\partial_+ q, \partial_+ q) q = 0 \quad \text{with} \quad q.q = 1, \tag{6.4}
\]
where \(\cdot\) stands for the Lorentzian scalar product \(a.b \equiv -a_0b_0 + \sum_{i=1}^{D} a_i b_i\), \(x_\pm \equiv \frac{1}{2}(\tau \pm \sigma)\) and \(\partial_\pm q = \frac{\partial q}{\partial x_\pm}\). The string constraints (2.8) become for de Sitter universe
\[
T_{\pm \pm} = \frac{\partial q}{\partial x_\pm} \cdot \frac{\partial q}{\partial x_\pm} = 0. \tag{6.5}
\]

Eqs. (6.4) describe a non compact O(D,1) non-linear sigma model in two dimensions. In addition, the (two dimensional) energy-momentum tensor is required to vanish by the constraints eqs. (6.5). This system of non-linear partial differential equations can be reduced by choosing an appropriate basis for the string coordinates in the \((D+1)\)-dimensional Minkowski space-time \((q^0, ..., q^D)\) to a noncompact Toda model [4].
These equations can be rewritten in the form of a chiral field model on the Grassmanian $G_D = SO(D,1)/O(D)$. Indeed, any element $g \in G_D$ can be parametrized with a real vector $q$ of the unit pseudolength

$$g = 1 - 2|q\rangle\langle q|J, \quad \langle q|J|q\rangle = 1. \quad (6.6)$$

In terms of $g$, the string equations (6.4) have the following form

$$2g_{\xi\eta} - g_{\xi} g g_{\eta} - g_{\eta} g g_{\xi} = 0, \quad (6.7)$$

and the conformal constraints (6.5) become

$$\text{tr} g_{\xi}^2 = 0, \quad \text{tr} g_{\eta}^2 = 0. \quad (6.8)$$

The fact that $g \in G_D$ implies that $g$ is a real matrix with the following properties:

$$g = Jg^\dagger J, \quad g^2 = I, \quad \text{tr} g = D - 1, \quad g \in SL(D+1, R). \quad (6.9)$$

These conditions are equivalent to the existence of the representation (6.6). Equation (6.7) is the compatibility condition for the following overdetermined linear system:

$$\Psi_{\xi} = \frac{U}{1-\lambda} \Psi, \quad \Psi_{\eta} = \frac{V}{1+\lambda} \Psi, \quad (6.10)$$

where

$$U = g_{\xi} g, \quad V = g_{\eta} g. \quad (6.11)$$

Or, in terms of the vector $q$:

$$U = 2 q_{\xi} \langle q J - 2 q \rangle \langle q\xi J, \quad V = 2 q_{\eta} \langle q J - 2 q \rangle \langle q\eta J.$$

Eq. (6.11) can be easily inverted yielding $q$ in terms of the matrix $g$:

$$q_0 = \sqrt{\frac{g_{00} - 1}{2}}, \quad q_i = \sqrt{\frac{1 - g_{ii}}{2}} \quad 1 \leq i \leq D \quad (\text{no sum over } i) \quad (6.12)$$

The use of overdetermined linear systems to solve non-linear partial differential equations associated to them goes back to refs. [36]. (See refs. [37] - [38] for further references).

In order to fix the freedom in the definition of $\Psi$ we shall identify

$$\Psi(\lambda = 0) = g. \quad (6.13)$$

This condition is compatible with the above equations since the matrix function $\Psi$ at the point $\lambda = 0$ satisfies the same equations as $g$. Thus the problem of constructing exact solutions of the string equations is reduced to finding compatible solutions of the linear equations (6.10) such that $g = \Psi(\lambda = 0)$ satisfies the constraints eqs. (6.8) and (6.9).

We concentrate below on the linear system (6.10) since this is the main tool to derive new string solutions in de Sitter spacetime.
In ref. [6] the dressing method was applied as follows. We started from the exact ring-shaped string solution $q(0)$ and we find the explicit solution $\Psi(0)(\lambda)$ of the associated linear system, where $\lambda$ stands for the spectral parameter. Then, we propose a new solution $\Psi(\lambda)$ that differs from $\Psi(0)(\lambda)$ by a matrix rational in $\lambda$. Notice that $\Psi(\lambda = 0)$ provides in general a new string solution.

We then show that this rational matrix must have at least four poles, $\lambda_0, 1/\lambda_0, \lambda_0^*, 1/\lambda_0^*$, as a consequence of the symmetries of the problem. The residues of these poles are shown to be one-dimensional projectors. We then prove that these projectors are formed by vectors which can all be expressed in terms of an arbitrary complex constant vector $|x_0\rangle$ and the complex parameter $\lambda_0$. This result holds for arbitrary starting solutions $q(0)$.

Since we consider closed strings, we impose a $2\pi$-periodicity on the string variable $\sigma$. This restricts $\lambda_0$ to take discrete values that we succeed to express in terms of Pythagorean numbers.

In summary, our solutions depend on two arbitrary complex numbers contained in $|x_0\rangle$ and two integers $n$ and $m$. The counting of degrees of freedom is analogous to 2+1 Minkowski spacetime except that left and right modes are here mixed up in a non-linear and precise way.

The vector $|x_0\rangle$ somehow indicates the polarization of the string. The integers $(n, m)$ determine the string winding. They fix the way in which the string winds around the origin in the spatial dimensions (here $S^2$). Our starting solution $q(0)(\sigma, \tau)$ is a stable string winded $n^2 + m^2$ times around the origin in de Sitter space.

The matrix multiplications involved in the computation of the final solution were done with the help of the computer program of symbolic calculation “Mathematica”. The resulting solution $q(\sigma, \tau) = (q^0, q^1, q^2, q^3)$ is a complicated combination of trigonometric functions of $\sigma$ and hyperbolic functions of $\tau$. That is, these string solitonic solutions do not oscillate in time. This is a typical feature of string instability [3] - [13] - [39]. The new feature here is that strings (even stable solutions) do not oscillate neither for $\tau \to 0$, nor for $\tau \to \pm \infty$.

We plot in figs. 1-7 the solutions for significative values of $|x_0\rangle$ and $(m, n)$ in terms of the comoving coordinates $(T, X^1, X^2)$

$$T = \frac{1}{H} \log(q^0 + q^1), \quad X^1 = \frac{1}{H} \frac{q^2}{q^0 + q^1}, \quad X^2 = \frac{1}{H} \frac{q^3}{q^0 + q^1}$$

The first feature to point out is that our solitonic solutions describe multiple (here five or three) strings, as it can be seen from the fact that for a given time $T$ we find several different values for $\tau$. That is, $\tau$ is a multivalued function of $T$ for any fixed $\sigma$ (fig.1-2). Each branch of $\tau$ as a function of $T$ corresponds to a different string. This is an entirely new feature for strings in curved spacetime, with no analogue in flat spacetime where the time coordinate can always be chosen proportional to $\tau$. In flat spacetime, multiple string solutions are described by multiple world-sheets. Here, we have a single world-sheet describing several independent and simultaneous strings as a consequence of the coupling with the spacetime geometry. Notice that we consider free strings. (Interactions among the strings as splitting or merging are not considered). Five is the generic number of strings in our dressed solutions. The value five can be related to the fact that we are dressing a one-string solution ($q(0)$) with four poles. Each pole adds here an unstable string.

In order to describe the real physical evolution, we eliminated numerically $\tau = \tau(\sigma, T)$ from the solution and expressed the spatial comoving coordinates $X^1$ and $X^2$ in terms of $T$ and $\sigma$.
We plot $\tau(\sigma, T)$ as a function of $\sigma$ for different fixed values of $T$ in fig.3-4. It is a sinusoidal-type function. Besides the customary closed string period $2\pi$, another period appears which varies on $\tau$. For small $\tau$, $\tau = \tau(\sigma, T)$ has a convoluted shape while for larger $\tau$ (here $\tau \leq 5$), it becomes a regular sinusoid. These behaviours reflect very clearly in the evolution of the spatial coordinates and shape of the string.

The evolution of the five (and three) strings simultaneously described by our solution as a function of $T$, for positive $T$ is shown in figs. 5-7. One string is stable (the 5th one). The other four are unstable. For the stable string, $(X^1, X^2)$ contracts in time precisely as $e^{-HT}$, thus keeping the proper amplitude ($e^{HT}X^1, e^{HT}X^2$) and proper size constant. For this stable string $(X^1, X^2) \leq \frac{1}{H}$. (1/H = the horizon radius). For the other (unstable) strings, $(X^1, X^2)$ become very fast constant in time, the proper size expanding as the universe itself like $e^{HT}$. For these strings $(X^1, X^2) \geq \frac{1}{H}$. These exact solutions display remarkably the asymptotic string behaviour found in refs. [4,13].

In terms of the sinh-Gordon description, this means that for the strings outside the horizon the sinh-Gordon function $\alpha(\sigma, \tau)$ is the same as the cosmic time $T$ up to a function of $\sigma$. More precisely,

$$\alpha(\sigma, \tau) \overset{T \gg \frac{1}{H}}{=} 2HT(\sigma, \tau) + \log \left\{2H^2 \left[ (A^1(\sigma))^2 + (A^2(\sigma))^2 \right] \right\} + O(e^{-2HT}).$$  \hfill (6.15)

Here $A^1(\sigma)$ and $A^2(\sigma)$ are the $X^1$ and $X^2$ coordinates outside the horizon. For $T \rightarrow +\infty$ these strings are at the absolute minimum $\alpha = +\infty$ of the sinh-Gordon potential with infinite size. The string inside the horizon (stable string) corresponds to the maximum of the potential, $\alpha = 0$. $\alpha = 0$ is the only value in which the string can stay without being pushed down by the potential to $\alpha = \pm \infty$ and this also explains why only one stable string appears (is not possible to put more than one string at the maximum of the potential without falling down). These features are generically exhibited by our one-soliton multistring solutions, independently of the particular initial state of the string (fixed by $|x^0>$ and $(n, m)$). For particular values of $|x^0>$, the solution describes three strings, with symmetric shapes from $T = 0$, for instance like a rosette or a circle with festoons (fig. 5-7).

The string solutions presented here trivially embed on $D$-dimensional de Sitter spacetime ($D \geq 3$). It must be noticed that they exhibit the essential physics of strings in D-dimensional de Sitter universe. Moreover, the construction method used here works in any number of dimensions.
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Figure Captions:

Figure 1: Plot of the function $HT(\tau)$, for two values of $\sigma$, for $n = 4, |x^0 > = (1 + i, .6 + .4i, .3 + .5i, .77 + .79i)$. The function $\tau(T)$ is multivalued, revealing the presence of five strings.

Figure 2: Same as fig.1, for $n = 4, |x^0 > = (1, -1, i, 1)$. Because of a degeneracy, there are now only three strings.

Figure 3: $\tau = \tau(\sigma, T)$ for fixed $T$ for $n = 4, |x^0 > = (1, -1, i, 1)$. Three values of HT are displayed, corresponding to HT=0 (full line), 1 (dots), and 2 (dashed line). For each HT, three curves are plotted, which correspond to the three strings. They are ordered with $\tau$ increasing.

Figure 4: Same as fig. 3 for $n = 4, x^0 > = (1 + i, .6 + .4i, .3 + .5i, .77 + .79i)$. a) The five curves corresponding to the five strings at HT=2. b) The five curves for three values of HT: HT=0 (full line), 1 (dots), and 2 (dashed line).

Figure 5: Evolution as a function of cosmic time $HT$ of the three strings, in the comoving coordinates $(X^1, X^2)$, for $n = 4, |x^0 > = (1, -1, i, 1)$. The comoving size of string (1) stays constant for $HT < -3$, then decreases around $HT = 0$, and stays constant again after $HT = 1$. The invariant size of string (2) is constant for negative $HT$, then grows as the expansion factor for $HT > 1$, and becomes identical to string (1). The string (3) has a constant comoving size for $HT < -3$, then collapses as $e^{-HT}$ for positive $HT$.

Figure 6: Evolution of three of the five strings for $n = 4, |x^0 > = (1 + i, .6 + .4i, .3 + .5i, .77 + .79i)$.

Figure 7: Evolution of the three strings for the degenerate case $n = 6, |x^0 > = (1, -1, i, 1)$.
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