Low-dimensional physics of ultracold gases with bound states and the sine-Gordon model

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Abstract. One-dimensional systems of interacting atoms are an ideal laboratory to study the Kosterlitz-Thouless phase transition. In the renormalization group picture there is essentially a two-parameter phase diagram to explore. We first present how detailed experiments have shown direct evidence for the theoretical treatment of this transition. Then generalization to the case of two-component systems with bound state formation is discussed. Trimer formation in the asymmetric attractive Hubbard model involve in a crucial this kind of physics.

1 Introduction

It is now routinely feasible to create one-dimensional ultracold quantum gases either Bose or Fermi \cite{1}. With a tube-shape potential the kinetic energy is totally frozen in the direction perpendicular to the tube. It should be noted that this by far the best practical way to achieve low-dimensionality. Another physical system that has been the focus of many studies is the two-dimensional electron gas. In this case the electronic motion perpendicular to the plane is quantized and one hopes that the characteristic energy of excitation of the perpendicular motion is higher than the other physically relevant energy scales. However even if this may be achieved, one should keep in mind that virtual transitions to the excited states renormalize the interactions in a way which is difficult to control. As such, the study of gases in confining tubes is cleaner and allows more detailed and precise comparisons with theory.

The low-dimension world has enhanced fluctuations that may forbid long-range order. This is well known in the context of magnetic systems where the Mermin-Wagner theorem gives restrictions of the allowed ordering and the corresponding phase transitions. One-dimensional systems at zero temperature are allowed to possess order corresponding to discrete symmetry breaking and there may be quantum phase transitions with appearance of Ising order. For example the XXZ antiferromagnetic spin-1/2 chain has a Néel ordered phase when the anisotropy is larger than one. This phase has a true discrete long-range order at T=0K which is allowed by the Mermin-Wagner theorem. When the anisotropy is less than one i.e. of XY type, the mean-field like XY long-range order is replaced by algebraic correlations decaying to zero at infinity. The appearance of phases with quasi long-range order is the striking feature of one dimensional quantum systems. The quantum phase transition that takes place at T=0K between Néel and algebraic phases is of the Kosterlitz-Thouless (KT) type, an infinite-order phase transition. In the realm of quantum gases we also expect to observe the KT transition in various guises. This brief review presents recent works that are directly related to this special phenomenon. After a brief presentation of theoretical underpinnings in section\textsuperscript{2} we discuss direct evidence for the control of the renormalization group phase diagram pf the KT transition in section\textsuperscript{3}. We next discuss the more involved system with...
two-component gases and the possibility of formation of many-particle bound states in section 2. Finally we present we give some conclusion about future prospects.

2 Field theory for low-dimensional gases

If we consider quantum systems in two or three space dimensions, then we classify excitations above the ground states in terms of particle-like excitations as well as collective excitations. In the case of Fermi systems there is a sharp Fermi surface at zero temperature and we can make low-energy particle-hole excitations above it. It is well known that there are also sound-like modes that can be excited. The particle or holes are in general dressed by interactions and have a complicated wavefunction. When there is still adiabatic continuity with respect to the non-interacting limit we have the Landau liquid description of the system which is relevant. Eventually the system may develop a superfluid/superconducting instability and the ensuing quasiparticles are gapped. But we still have a description in terms of Bogolyubov gapped quasiparticles and collectives with mutual coupling. This picture is successful in various condensed matter systems as well as in the description of atomic nuclei. Bose systems behave similarly: we can have individual particles excited above the condensates as well as various sound modes that are collective excitations of the system as a whole. This is a reasonably accurate description of the various liquid Heliums for example. If interactions are very strong the boundary between these entities particles and collective modes may become blurred and not so useful.

The high-dimension picture above is drastically modified in one space dimension. Here indeed all excitations are collective in character. The reason for this striking phenomenon is the change of available phase space. As a consequence all excitations can be expressed in terms of a single continuous Bose quantum field. This is the basis of the bosonization technique. It is important to note that the Bose field does not have the same physical significance as in quantum field theory in usual 3+1 dimensional Minkovsky space. Indeed the creation operator in second-quantized formalism is written as:

\[
\Psi^\dagger(x) \sim \left(\rho - \frac{1}{\pi} \partial_x \phi\right)^{1/2} \sum_n e^{in(kx - \phi)} e^{-i\theta},
\]

where \(k = \pi\rho, \rho\) being the average density, \(\phi\) is a Bose field and \(\theta\) its dual, \(n\) an odd integer for fermions and even for bosons. As a consequence the physical observables involve only gradients of \(\phi\) or exponential operators of the field and its dual. It is well known that a massless free field in 1+1 dimensions has a propagator which is logarithmically growing with distance but here we see that this pathological behavior does not arises in the observables. All correlation functions are well-behaved and decay with distance. If the effective field theory for the field \(\phi\) is free and massless, then the correlation functions needed to evaluate physical observables of the microscopic theory can be computed easily. Now in general the effective theory is never free. In fact the good way to think about this limiting case is in terms of fixed points in the renormalization group (RG) language. The appearance of exponentials of the field \(\phi\) means that the scale of the field is physically meaningful. In fact correlations have an algebraic decay with exponents that depend upon this scale. This is strikingly different from the 3+1 QFT behavior where the scale of \(\phi\) disappear in the LSZ reduction formula and hence the S-matrix elements do not depend upon it. If we discard interactions then the massless Bose field in 1+1 dimensions is a line of fixed points and the location along the line is given by the overall \(\phi\) scale. It is often written as the Luttinger parameter \(K\) in the condensed matter community and is also known as the radius of the boson in field theory circles. In general the value of the Luttinger parameter cannot be computed easily. If we do perturbation theory one can get a weak-coupling estimate. For integrable models it may be known. This is the case of the all-important Lieb-Liniger gas with delta function interactions which happens to be realized in ultracold atom systems. Finally several numerical techniques may be used to extract the value of the Luttinger parameter.

If we now translate the microscopic interactions in terms of the effective field theory the nature of the expansion Eq. (1) severely constraints the possible operators that are generated. There are powers of the gradient of the field and also exponentials of \(\phi\) or its dual \(\theta\). Strictly speaking there are certainly infinitely
many operators in the effective theory but it is very simple to classify allowed operators according to their relevance/irrelevance in the RG sense even if no exact solution is available. If all allowed operators are irrelevant for example then the effective theory in the long-distance limit will be the massless Bose field and accordingly there is algebraic long-range order instead of the true long-range order of higher-dimensional physics. If there is a relevant operator then strictly speaking the infrared behavior is out of reach of perturbation theory. However semiclassical reasoning may then give some clue about the physics of the system. In the simplest sine-Gordon case, there is also an exact solution which can guide us. So we focus on the case of a single relevant operator. The Euclidean Lagrangian formulation can be written as:

$$\mathcal{L} = \frac{1}{2} (\nabla \phi)^2 - \frac{\alpha}{\beta^2 a^2} \cos(\beta \phi).$$  \hspace{1cm} (2)

Here we have introduced the two dimensionless couplings $\alpha$ and $\beta$ defining the model as well as a length scale $a$ which is a short-distance cut-off. The RG flow of this model has been first studied by Kosterlitz and Thouless [2] but it is only later that a fully consistent calculation of the flow appeared [3]. The cosine operator is relevant if $\beta^2 < 8\pi$, marginal right at $\beta^2 = 8\pi$. We thus introduce for convenience the variable $\delta = \beta^2 / 8\pi - 1$ measuring the distance to marginality. The flow is then given by the coupled equations:

$$a \frac{\partial \alpha}{\partial a} = 2\alpha \delta + \frac{5}{64} \alpha^3,$$

$$a \frac{\partial \delta}{\partial a} = \frac{1}{32} a^2 - \frac{1}{16} a^2 \delta. \hspace{1cm} (4)$$

It is important to note that these equations are valid in the sense of a double perturbative expansion: both $\alpha$ and $\delta$ should be small. This fact is often overlooked in the condensed-matter literature where people extrapolate boldly to arbitrary values of $\beta$ far away from the KT point. This set of flow equations leads to a portrait which is given in Fig.1. The most salient fact is that there is a separatrix (in blue) that divides the diagram in two parts. In the figure the long distance limit is found by following the arrows.

### 3 sine-Gordon theory for gases in a potential well

Up to now, our discussion has been quite general and we discuss the various physical systems related to the sine-Gordon flow. The first example is the finite-temperature transition in 2D superfluid systems like Helium films that was the subject of the original work of Kosterlitz and Thouless. This transition has been observed in ultracold gases [4,5]. It is then vortex-unbinding transition. The vortices have long-range interactions and can be regarded as a 2D Coulomb gas. It is precisely this problem of statistical mechanics that can be mapped onto a sine-Gordon model [6]. If we vary a single parameter, the temperature it means that the equivalent effective model will move along a line in the 2D figure of the KT flow. When this line crosses the separatrix, the KT transition takes place. The low-temperature phase being the one with algebraic order in the superfluid phase and only vortices bound in pairs while above the transition temperature the cosine operator is relevant and we have unbound vortices. Another physical system which belongs to the same universality class is the XY spin model where interacting spins are confined in a plane. It is also possible to map this system onto a 2D Coulomb gas and then onto the sine-Gordon model [6].

We now discuss the case of ultracold 1D atomic gases. As mentioned above it is the quantum problem at T=0K which can be mapped onto the sine-Gordon system. If we have a Bose gas in a weak lattice then its effective theory is given by the sine-Gordon model. Indeed without an optical lattice an ultracold gas of bosonic atomic is very precisely described by the Lieb-Liniger model of featureless bosons with delta function interactions that mimic the ultra-low-energy s-wave scattering. For such a system the Luttinger parameter, which is essentially the coefficient $\beta$ in Eq.(2), is related to the dimensionless strength of the interactions $\gamma = mg/n\hbar^2$ where $g$ is the 1D delta function strength and $n$ the density. By using a
Fig. 1. The Kosterlitz-Thouless RG flow: the vertical axis is the strength of the cosine operator while the horizontal axis is the Luttinger parameter. This last axis is a line of fixed point which are stable on the left part and unstable on the right.

Feshbach resonance it is possible to tune the scattering length between the atoms and thus to modify the Luttinger parameter in a controlled way. Now if we add a weak lattice potential which is commensurate with the density of atoms, this generates in the effective theory a cosine operator of the type described in the previous section and its strength is directly proportional to the strength of the lattice potential. Then the fate of the system depends upon the value of the interactions [7]. If they are weak enough the cosine operator is irrelevant and the system will remain superfluid for a weak lattice. If we increase the strength of the potential then at some point the system will cross the separatrix in phase diagram (1) and a KT transition will take place towards a Mott-Hubbard insulating phase with one boson pinned to each potential well. This is the standard Mott localization transition. However if we are in the right part of Fig.(1) then the atoms will be pinned immediately for any arbitrary weak coupling: the superfluid phase is immediately destroyed. This phenomenon is expected to be very general and will occur both for Bose and Fermi systems. It is has been recently demonstrated experimentally by ref. [8]. A gas of Cesium atoms was prepared in an array of 1D tubes and Cs Feshbach resonance allowed the tuning of the interactions. The gap of the system was measured by an amplitude modulation spectroscopy. The measurements revealing the immediate development of a Mott state as predicted by the RG treatment of the sine-Gordon model. In addition there is even quantitative estimates of the gap that are in agreement with the exact results for the gap of the sine-Gordon model, confirming the validity of the model as well as its phase diagram with a quantum phase transition.

4 Two-component systems and trimers

We now turn to the discussion of two-component fermionic systems. These involve mixtures of two ultracold atomic species. They may be different atoms like mixtures of $^6$Li and $^{40}$K isotopes or even different hyperfine states of a single species when transitions are blocked. If we put a mixture in an optical lattice it will be described by an asymmetric Hubbard model [14,16,17,18]:

$$
\mathcal{H} = -\sum_{i,\sigma=\uparrow,\downarrow} t_{\sigma} c_{i+1\sigma}^\dagger c_{i\sigma} + h.c. + U \sum_{i} n_{i\uparrow} n_{i\downarrow}.
$$

(5)
where the two hoppings are now different $t_\uparrow \neq t_\downarrow$. We consider the case of attractive interactions $U < 0$. Densities are in general different $\rho_\uparrow \neq \rho_\downarrow$ and thus the Fermi wavevectors are also different : $k_{F\uparrow} \neq k_{F\downarrow}$. So it is natural extension of the case of imbalanced superfluids [9,10,11,12,13].

With different atoms and attractive interactions one may have few-body bound states. It is possible to study this phenomenon by looking at the zero-density limit of Eq. (5). It is convenient to use the formalism introduced by D. Mattis [22] that leads to an integral equation whose solution allows to determine the bound state spectrum [21]. As soon as $t_\uparrow \neq t_\downarrow$, there are three-body bound states for all negative values of $U$. The trimer binding energy is displayed in fig. (2). At equal hoppings it is known from the Bethe-Ansatz solution [15] of the Hubbard model that there are no bound states with more than two particles so we find vanishing binding in the symmetric limit.

The situation becomes more interesting when we go to finite density. It is not clear that the bound states will survive. We thus define a many-body trimer gap by:

$$\Delta_{tr} = - \lim_{L \to \infty} \left[ E_L(N_{\uparrow} + 1, N_{\downarrow} + 2) + E_L(N_{\uparrow}, N_{\downarrow}) - E_L(N_{\uparrow} + 1, N_{\downarrow} + 1) - E_L(N_{\uparrow}, N_{\downarrow} + 1) \right],$$

(6)

where $E_L(N_{\uparrow}, N_{\downarrow})$ is the ground state energy of a gas with spin populations $N_{\uparrow}, N_{\downarrow}$ in a chain of size $L$. This quantity is computed by DMRG for large system sizes. It is plotted in Fig. (3) where we see that trimers do survive as long as the density is not too high. It remains to understand the nature of the interacting trimer liquid. We use the bosonization technique applied to the asymmetric Hubbard model [20]. There is then a Bose field for each fermionic species $\phi_\uparrow$ and $\phi_\downarrow$. Each of these fields will have a velocity $v_\sigma$, $\sigma = \uparrow, \downarrow$ as well as a Luttinger parameter $K_\sigma$. The effective low-energy long-wavelength theory will contain a free part which is the most general quadratic form in the fields and there are also cosine operators generated by the appearance of higher harmonics in the expression Eq. (1). They are typically given by:

$$\mathcal{O}_{p,q} = \cos \left[ 2(p k_F^{\uparrow} - q k_F^{\downarrow})x - 2(p \phi_\uparrow - q \phi_\downarrow) \right],$$

(7)

where $p$ and $q$ are integers. Such operators are rapidly oscillating in space as long as $p n_{\uparrow} - q n_{\downarrow} \neq 0$ and can be neglected in the continuum limit. But for special commensurabilities $p n_{\uparrow} - q n_{\downarrow} \equiv 0$ we have to consider an operator which survives the continuum limit. Note that this effect has nothing to do with the underlying lattice periodicity, it exists even in the continuum. Due to the appearance of the integer factor $p$ and $q$ it is not clear that this operator can ever be relevant since its scaling dimension near the free boson fixed points involves $p^2$ and $q^2$ factors. In the absence of relevant cosine operators the
Fig. 3. The trimer gap as a function of $n_↓$ generalizing the binding energy at finite density ($n_↓ = 2n_↑$). The bottom curve is for $U = -2$ and next curve is for $U = -4$. The zero-density limit is the binding energy of fig.2 as indicated by arrows.). Note that high density is detrimental to trimerization.

Fig. 4. Modulus of superconducting correlations as a function of distance. We use $U = -4$, $t_↓ = 0.3$ The upper curve is for $n_↓ = 0.7$, $n_↑ = 0.3$ off-commensurate. There is algebraic decay. The commensurate case is the lower curve with now $n_↓ = 0.6 = 2n_↑$; there is exponential decay in the trimer phase.

A quadratic form coupling the two Bose fields can be diagonalized and this leads to two free eigenmodes: this is a two-component Luttinger liquid as happens in ordinary condensed-matter systems involving electrons with spin. Here in the case of an attractive mixture of fermions the physical picture is a 1D analog of the so-called FFLO phase. All correlations will have algebraic order the pair correlation function revealing the superconducting properties of the system will oscillate with the FFLO momentum given by the difference of Fermi wavevectors: $|k_{F↑} - k_{F↓}|$. 
If we are at one special commensurability $p n_\uparrow - q n_\downarrow = 0$ for some $p$ and $q$ values and if the cosine operator is relevant then a simple semiclassical picture suggests that the cosine will pin the special combination of fields $p \phi_\uparrow - q \phi_\downarrow$ to one of the minimum of the potential. This locked mode will have only gapped excitations corresponding to small vibration around the minimum energy position while there should exist another combination of the fields that will remain gapless. This is a one-component Luttinger liquid like the Luther-Emery system. The question of relevance/irrelevance of this cosine cannot be easily solved by perturbation technique. A decisive answer can be obtained by using the DMRG [20,25] technique applied to the microscopic model Eq.(5). The “easiest” commensurability is expected to be that of 2:1 ratio of densities (otherwise operators are even less likely to be relevant). Bosonization of correlation functions can be used as a guide to find a way to characterize the two phases. If we express all correlations in terms of the field which is pinned in the one-component phase as well as an unknown combination of $\phi_\uparrow$ and $\phi_\downarrow$ which is gapless then we find that all two-body pairing channels decay exponentially. We thus expect that the simplest microscopic pairing operator should have the behavior:

$$
\langle c_{x,\uparrow}^\dagger c_{x,\downarrow}^\dagger c_{0,\downarrow} c_{0,\uparrow} \rangle \propto \frac{\exp(-|x|/\xi)}{|x|^\alpha} \cos(|k_{F\uparrow} - k_{F\downarrow}|x) \tag{8}
$$

with a correlation length typically $\propto 1/\Delta t_\uparrow$. So the change from the two-component phase to the one-component phase can be seen by the change of the decay law of the pair propagator. This is seen in Fig.(4) where we see clearly that commensurability of type 2:1 leads to a new phase involving locking of the field $2\phi_\uparrow - \phi_\downarrow$. As is clear from the strong coupling limit, the physical interpretation is that one has a Luttinger liquid of trimers. This liquid is gapless and has some non-trivial Luttinger exponent. Excitations corresponding to trimer break-up on the contrary corresponds to gapped modes. An operator which has algebraic decay in the trimer phase is given simply by the combination $\Psi_\downarrow^\dagger \Psi_\uparrow^\dagger$. So there is existence of this trimerized phase in the asymmetric Hubbard model. To map out its domain of existence one can use the fact that the central charge $c$ of the effective theory is either $c = 2$ in the absence of trimers and $c = 1$ when they are present [25]. The measurement of the central charge can be done efficiently by measuring the entanglement entropy by the DMRG algorithm.

A slice at fixed low density is displayed in Fig.(5). It shows an extended trimer phase for all values of $U$ that exists as soon as the asymmetry is large enough. When the asymmetry becomes too large i.e. $t_\downarrow \rightarrow 0$ we are dealing with the so-called Falicov-Kimball limit. In this case one species does not move and one has to solve the classical problem of minimizing the energy of the other species. This leads to a complex staircase structure [19] which involves all kinds of commensurabilities. This limit however is beyond the reach of bosonization and in addition the DMRG algorithm no longer converges. The wide region of trimer stability shrinks when we rise the density and finally disappear for total density close to unity.

5 Conclusion

several ultracold gas systems that are under intense scrutiny can be described accurately by prototypical field theory models like the sine-Gordon model with one or more components. It is likely that precision measurements will be able to map out phase structure of these systems in detail as is already the case with the KT system. Notably the study of multicomponent systems is hampered theoretically by the fact that the velocities have no reason to be equal. The detailed comparison of theory and experiments should allow for a better understanding of this complex situation.

Concerning trimers, in principle their detection is accessible to experiments [24,23] and this would allow the experimental observation of these new commensurability effects which attract much attention [20,27].
Fig. 5. The trimer phase exists for densities not too high and enough hopping asymmetry. Here the conventional phase is the 2-component Luttinger liquid and the trimer phase appears for all $U$ by reducing $t_d$. For higher densities the trimer phase shrinks and disappears beyond some critical value $n_c$.

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