The q-Euclidean algebra $U_q(e^N)$ and the corresponding q-Euclidean lattice

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Abstract

We present the Euclidean Hopf algebra $U_q(e^N)$ dual of $Fun(R^N_q \rtimes SO_{q-1}(N))$ and describe its fundamental Hilbert space representations [6], which turn out to be rather simple “lattice-regularized” versions of the classical ones, in the sense that the spectra of squared momentum components are discrete and the corresponding eigenfunctions normalizable.

A suitable notion of classical limit is introduced, so that we recover the classical continuous spectra and generalized (non-normalizable) eigenfunctions in that limit.

Introduction

Since their birth quantum groups [2] have found a number of different applications to physics and mathematics. In particular they can be used to generalize the ordinary notion of space(time) symmetry. This generalization is tightly coupled to a radical modification of the ordinary notion of space(time) itself. From this viewpoint inhomogenous group symmetries such as Poincaré’s and the Euclidean one yield physically relevant candidates for quantum group generalizations; Minkowski space $M^4$ and Euclidean $R^N$ one are then the corresponding space(time) manifolds. One can generalize the latter by the $N$-dimensional $(N \geq 3)$ Euclidean space $R^N_q$, its symmetry by the q-Euclidean one carried by the Hopf-algebra $E^N_q := R^N_q \rtimes SO_q(N)$ [9, 13, 10] or equivalently by its dual [4, 6], which here we will call $U_q(e^N)$. In Ref [6] we classified the fundamental Hilbert space representations of $U_q(e^N)$; here we represent the latter results in a more pedagogical and explicit way and add some new ones.

A major physical motivations for such generalizations is the desire to discretize space(time) (or momentum space) in a “wise” way for QFT regularization purposes. Nowadays such a discretization is usually performed by approximating the points of the space(time) (or
momentum space) continuum by the points of a lattice. In the case of the cubic Euclidean lattice, for instance, the coordinates $x^i$ ($i = 1, 2, ..., N$) can assume only the values $an^i$, where $a$ is the lattice spacing and $n^i \in \mathbb{Z}$; one chooses as a basis of the Hilbert space $\mathcal{H}$ of physical states the set $\{ |n^1, ..., n^N > \}_n \in \mathbb{Z}$ of eigenvectors of the $N$ commuting observables $x^i$ with eigenvalues $an^i$. On the other hand, it is known that standard lattices used in regularizing QFT do not carry representations of discretized versions (in the form of discrete subgroups) of the associated inhomogenous groups; actually, the notion of a group is too tight for this scope. For instance, the Euclidean cubic lattice is invariant only under a discretized version of the translation subgroup of the Euclidean group, but not of the rotation one; in other words, we are able only to represent the latter subgroup on $\mathcal{H}$. On the contrary, the notion of symmetry provided by quantum groups is broad enough to allow the existence of lattices whose points are mapped into each other under the action of the whole inhomogeneous q-groups. The main purpose of this paper is to describe how this occurs in the case of the q-Euclidean symmetry and how in the limit $q \to 1$ one recovers the ordinary representation spaces. One concludes that the q-Euclidean lattice introduced in ref. [6] seems very appealing in view of full covariant regularizations of Euclidean QFT; actually, a $q$-deformed version of the $\varepsilon$-tensor on $\mathbb{R}^N_q$ is also available [4, 6, 11], so allowing the construction of the pseudo-tensors which are needed for chiral field theories.

The main difference w.r.t the cubic lattice stems from the following fact. The $N$ configuration-space coordinates $x^i$ (as well as the momenta $p^i$) don’t commute with each other; therefore we can use a complete set of commuting observables consisting only partially (i.e. for about one half) of functions of the $p^i$ (or, alternatively, of the $x^i$) and, as for the rest, by angular momentum components. Their spectra are discrete. The lattice in the present situation has namely $\lfloor \frac{N+1}{2} \rfloor$ dimensions in $p$-space and $\lfloor \frac{N}{2} \rfloor$ dimensions ($\lfloor a \rfloor$ denotes the integer part of $a$) in angular momentum space; to each point of the lattice there corresponds a unique eigenvector belonging to a basis of $\mathcal{H}$ and labeled by $N$ integers. Notably, under the action of the generators of the q-Euclidean algebra each vector is mapped simply into a new one with labels differing at most by $\pm 1$. In the sequel we will consider as algebra of observables the one generated by $p^i$’s and the angular momentum components, since they generate the physically relevant q-deformed Euclidean algebra ($q$-translations + q-rotations); but both the commutation relations and the representation theory would be exactly the same (under the replacement $x^i \to p^i$) if we considered the $x^i$ instead.

In section 2 we briefly introduce the q-deformation $U_q(e^N)$ of the universal enveloping algebra of the Euclidean Lie algebra $e^N$ which we are going to adopt as quantum symmetry. We will be quite explicit in the case $N = 3, 4$, for which we also write down the analog of the Pauli-Lubanski casimirs. $U_q(e^N)$ is the Euclidean analogue of the q-deformed Poincare’ Hopf algebra (of u.e.a. type) Ref. [13, 3]. In both cases the inhomogeneous Hopf algebra contains the homogeneous one as a Hopf subalgebra which can be obtained from it by setting $p^i = 0, \Lambda = 1$ ($\Lambda$ is the “dilaton”), and all commutation relations are homogeneous in $p$, contrary to what happens for inhomogenous Hopf algebras obtained through contractions [1, 4, 8]. Representation theory is also developed in a similar way as in ref. [12].
Section 3 is devoted to a detailed description of fundamental (i.e. irreducible one-particle) Hilbert space *-representations of $U_q(e^N)$ (we will call them “irreps” in the sequel). The case $N = 3$ is analysed first, as an introduction to the general case. We choose a Cartan subalgebra (i.e. a complete set of commuting observables) consisting basically of two parts, $\left\lceil \frac{N+1}{2} \right\rceil$ squared momentum components and $\left\lceil \frac{N}{2} \right\rceil$ angular momentum components ([a] denotes the integer part of a). The points of the spectra make up a q-lattice. One important fact is that the irreps turn out to be of highest weight type. Moreover, they can be obtained from tensor products of the singlet one (i.e. the one describing a particle with zero $U_q(so(N))$-highest weight) and some representation of $U_q(so(N))$; for instance, the irreps with $N = 3$ are obtained from the tensor product of the q-boson (i.e. zero spin) representation of $U_q(e^3)$ with a representation of some spin $j \in N$ of $U_q(so(3)) \approx U_q(su(2))$, in analogy with the undeformed case. The spectra of all observables are discrete, in particular the spectra of squared momentum components, as expected. The corresponding eigenvectors are normalizable and make up an orthogonal basis of the Hilbert space of each irrep. A cumbersome “kinematical PT (parity + time-inversion) asymmetry” appears in the structure of the spectra of the angular momentum observables; it disappears in the limit $q \to 1^-$. 

In section 4 we clarify in which sense the Euclidean algebra/representations go to the classical ones in the limit $q \to 1$. In the classical representation we know that the eigenvectors of operators which are only functions of the momenta are distributions, typically they are delta-functions in momentum space. We show how to construct $q$-dependent integer labels $n_i(q)$ and coefficients $\alpha(q)$ such that $\alpha(q)|n_i(q) >_q$ (eigenvectors belonging to the $q$-representation) are delta-convergent functions in the limit $q \to 1$.

We can think of the irreps studied in section 3 as describing the (time-independent) dynamics of a free nonrelativistic particle with arbitrary “generalized” $U_q(so(N))$-spin on $\mathbb{R}_q^N$. The subalgebra $\hat{U}_q(e^N) := U_q(e^N)/(\Lambda - 1)$ can be considered as the quantum group symmetry of the hamiltonian

$$H := \frac{(p \cdot p)}{2M}, \quad (0.1)$$

of the system; therefore all states with a given energy should be obtained from each other by the action of $\hat{U}_q(e^N)$, as in the classical case, different eigenspaces of the energy should be obtained from each other by the action of the dilatation operators $\Lambda^{\pm 1}$.

Some notational remarks are necessary before the beginning. For representation purposes we will assume in section 3 that $q \in \mathbb{R}^+$, and we will limit ourselves to the case $0 < q \leq 1$; the case $q > 1$ can be treated in an analogous way. We set $h = h(N) = \begin{cases} 0 & \text{if } N = 2n + 1 \\ 1 & \text{if } N = 2n \end{cases}$ to allow a compact way of writing relations valid both for even and odd $N$. Unless stated differently, in our notation a space index $i$ can take all the integer values between $-n$ and $n$ including/excluding $i = 0$ if $N = 2n + 1, 2n$ respectively. When $N = 2n$ there is a complete invariance of the validity of all the results under the exchange of indices $i = -1 \leftrightarrow i = 1$, so that we will normally omit writing down explicitly the results that can be obtained by such an exchange. We will often use the shorthand notation $[A, B]_a := AB - aBA$ ($\Rightarrow [\cdot, \cdot]_1 = [\cdot, \cdot]$). Indices are raised and
lowered through the q-deformed metric matrix $C := ||C_{ij}||$, for instance

$$a_i = C_{ij}a^j, \quad a^i = C^{ij}a_j, \quad C_{ij} := q^{-\rho_i}\delta_{i,j}, \quad (0.2)$$

where

$$\rho_i := \begin{cases} (n - \frac{1}{2}, n - \frac{3}{2}, \ldots, \frac{1}{2}, 0, -\frac{1}{2}, \ldots, \frac{1}{2} - n) & \text{if } N = 2n + 1 \\ (n - \frac{1}{2}, n - 2, \ldots, 0, 0, \ldots, 1 - n) & \text{if } N = 2n. \quad (0.3) \end{cases}$$

$C$ is not symmetric and coincides with its inverse: $C^{-1} = C$.

1 The Euclidean ∗-algebra $U_q(e^N)$

The Hopf algebra which we are going to use, $U_q(e^N)$, was constructed in Ref. [4] and in equivalent form in ref. [5] by an inhomogeneous extension of the Hopf algebra $U_q(so(N))$ of “infinitesimal q-rotations” (in analogy with the undeformed construction). $U_q(e^N)$ is the Hopf dual of $Fun(R_q^N \rtimes SO_q(N))$ [4, 5]. In ref. [4] work, we added to the Drinfeld-Jimbo generators of the latter first the q-derivatives on $R_q^N$ as infinitesimal generators $p^i$ of q-translations and then one more generator Λ, generating dilatations; the coalgebra and antipode for $U_q(e^N)$ were derived from the Leibnitz-rule of q-differential operators (the role of the coalgebra in representation theory is to allow the construction of many-particle representations starting from one-particle ones). On the algebra $U_q(e^N)$ there exists a notion of complex conjugation ∗, (which will play the role of hermitean conjugation of operators). However, since the coalgebra is incompatible, at least in the usual sense, with the ∗-structure, here we focus the attention on the algebra structure of $U_q(e^N)$ which we need to develop the theory of one-particle representations.

1.1 A Chevalley basis of $U_q(so(N))$

A Cartan-Weyl basis of $U_q(so(N))$ is the set $\{L^{ij}, (k^i)^{\pm \frac{1}{2}}\}$ ($i < j, \neq -j; n \geq l \geq 1$) with commutation relations given below. Its elements were realized in Ref. [4] as q-differential operators on $R_q^N$; this is the q-deformed analogue of realizing the generators of $so(N)$) as “angular momentum components”. To help the reader in the identification of the corresponding classical angular momentum components, we give here their classical limits

$$L^{ij} \xrightarrow{q \rightarrow 1} x^i \partial^j - x^j \partial^i, \quad \frac{k^i - 1}{q^2 - 1} \xrightarrow{q \rightarrow 1} x^i \partial^{-i} - x^{-i} \partial^i, \quad (1.1)$$

where $x^i, \partial^j$ denote the classical coordinates/derivatives, $\partial^i x^j = \delta^{i,j} + x^j \partial^i$; the latter are chosen not to be real, but complex combinations such that $(x^i)^* = x^{-i}, (\partial^i)^* = -\partial^{-i}$. According to this construction, $U_q(so(N))$ is realized as a subalgebra of the differential algebra on $R_q^N$.

The $k^i$’s generate a Cartan subalgebra of $U_q(so(N))$. The elements $L^{-i,i+1}, L^{-i-1,i}, k^i(k^i+1)^{-1}$ (together with $L^{12}, L^{-2,-1}, k^1k^2$ in the case $N = 2n$) for $i = h, h + 1, \ldots, n$ are “ Chevalley generators ” (i.e. algebraically independent generators) of $U_q(so(N))$ coinciding [4] with the Drinfeld-Jimbo ones, up to some rescaling of the roots $L$ by suitable functions of $k^i$. 
The correspondence between the Chevalley generators $L^{-i,j}$ corresponding to positive roots and the spots of the Dynkin diagram of $so(N)$ is shown in fig. 1. All the other generators $L^j_i$ can be constructed starting from them as follows:

\[
[L^{-j,i}, L^{-l,k}]_q = q^{n+1}L^{-i,j} \quad n \geq k > l > j \geq -h(N) \]

\[
[L^{l,k}, L^{-l,k}]_q = q^{n-1}L^{l,k} \quad 2 \leq l < k \leq n \]

\[
[L^{0k}, L^{01}] = q^{-1}L^{1k} \quad [L^{-10}, L^{-k0}] = L^{-k,1} \quad 1 < k \leq n \quad \text{if } N = 2n+1; \]

\[
\quad \text{these relations can be easily verified by the reader in the limit } q = 1 \text{ using the limits (1.1).} \]

Once introduced the basis $\{L^{ij}, k^l\} \quad (i < j, \neq -j; \quad n \geq l \geq 1)$, then the commutation relations satisfied by the Chevalley generators can be summarized in the following way.

- Commutation relations between the generators of the Cartan subalgebra and the simple roots:

\[
[k^i, L^{\pm(1-k),\pm k}]_q = 0 \quad a = \begin{cases} 
q^{\pm 2} & \text{if } i = k \leq n \\
q^{\pm 2} & \text{if } i = k - 1 \\
1 & \text{otherwise} 
\end{cases} \quad [k^i, k^j] = 0; \quad (1.5)
\]

- Commutation relations between positive simple roots (the ones appearing on the left of the $q$-commutators) and negative ones (the ones appearing on the right):

\[
[L^{-1-m,m}, L^{-k,k-1}] = 0 \quad a = \begin{cases} 
q^{-1} & \text{if } k > m + 1 = k \\
m + 1 & \text{if } k \neq m, m + 1 
\end{cases} \quad m, k \geq h(N)+1, \quad (1.6)
\]

\[
[L^{12}, L^{-2,1}] = 0 \quad [L^{-1,2}, L^{-2,-1}] = 0 \quad \text{if } N = 2n, \quad (1.7)
\]

\[
\begin{cases} 
[L^{-1-m,m}, L^{-m,m-1}]_q = q^{1+2\rho_m} \frac{1-k^{m-1}(k^{m})^{-1}}{q-q^{-1}} & \quad 2 \leq m \leq n \\
[L^{01}, L^{-1,0}]_q = q^{-1}\frac{1-(k^{1})^{-1}}{q-q^{-1}} & \quad \text{if } N = 2n + 1; 
\end{cases} \quad (1.8)
\]

- Serre relations:

\[
[L^{-1-m,m}, L^{-1-k,k}] = 0 \quad [L^{-m,m-1}, L^{-k,k-1}] = 0 \quad m, k > 0, \quad \vert m - k \vert > 1 \quad (1.9)
\]

\[
[L^{1-k,k}, L^{2-m,m}]_q = 0 \quad [L^{-m,m-2}, L^{-k,k-1}]_q = 0 \quad a = \begin{cases} 
q^{-1} & \text{if } k = m \\
q^{-1} & \text{if } k = m - 1 
\end{cases} \quad m \geq 3 \quad (1.10)
\]

\[
\begin{cases} 
[L^{01}, L^{12}]_q = 0 & \quad [L^{2,-1}, L^{-1,0}]_q = 0 \\
[L^{-1,2}, L^{02}]_q = 0 & \quad [L^{-2,0}, L^{-2,1}]_q = 0 
\end{cases} \quad \text{if } N = 2n+1. \quad (1.11)
\]

In the case $N = 3$, the relations among the generators $L^{01}, L^{-10}, k^1$ are simply

\[
\begin{cases} 
[k^1, L^{01}]_q = 0 \\
[k^1, L^{-10}]_q = 0 \\
[L^{01}, L^{-10}]_q = q^{-1}\frac{1-(k^{1})^{-1}}{q-q^{-1}}. 
\end{cases} \quad (1.12)
\]
1.2 Extending $U_q(so(N))$ to $U_q(e^N)$

The “infinitesimal” generators $p^i$ of q-translations and the generator $\Lambda$ of dilatations satisfy the commutation relations reported below.

\[ [k^h, p^i]_{a_{h,i}} = 0, \quad h = 1, 2, \ldots, n; \quad (1.13) \]

\[ [L^{01}, p^0] = -q^{-1}p^1 \quad [L^{-1,0}, p^0] = p^{-1} \quad i f \quad N = 2n + 1, \quad (1.14) \]

and in all the remaining cases

\[ [L^{1-m,m}, p^i]_{b_{m,i}} = q^{\rho_m}(\delta_m^i - \delta_{m-1}^i)p^{i+1} \quad [L^{-m,-m-1}, p^i]_{b_{m,i}} = q^{\rho_m}(\delta_{1-m}^i - \delta^i_m)p^{i-1}, \quad (1.15) \]

where

\[ a_{m,i} := q^{2(\delta_m^i - \delta_{m-1}^i)}, \quad b_{m,i} := (a_{m-1,i})^{\frac{1}{2}} (a_{m,i})^{-\frac{1}{2}}. \quad (1.16) \]

The commutation relations of $p^i$’s among themselves are those of a quantum space $R^N_q$, $P_{A}^{ij}h^k p^h p^k = 0$, where $P_A$ is the projector appearing with negative eigenvalue in the projector decomposition of the $\hat{R}$ matrix of $SO_q(N)$ (the $q$-antisymmetrizer); they amount respectively to

\[ p^i p^l = q p^l p^i, \quad -l \neq i < l, \quad (1.17) \]

and

\[ \sum_{l=-j}^{j} p^l p_l = (p \cdot p)_j (1 + q^{-2\rho_j}) \quad (1.18) \]

where

\[ (p \cdot p)_j := \sum_{l=1}^{j} p^{-l} p_{-l} + \begin{cases} p^{\rho_m}, & i f \quad N = 2n + 1 \\ 0, & i f \quad N = 2n, \end{cases} \quad (1.19) \]

consequently

\[ [(p \cdot p)_j, p^l] = 0 \quad |l| \leq j. \quad (1.20) \]

We see that the algebra $\hat{U}_q(e^N)$ generated by $L, k, p$ is closed.

Finally

\[ [\Lambda, p^i]_{q^{-1}} = 0 \quad [\Lambda, k] = 0 \quad [\Lambda, L] = 0. \quad (1.21) \]

Note that all the commutation relations are homogeneous in $p$.

**Remark** Note that there exists a natural embedding $\hat{U}_q(e^N) \hookrightarrow \hat{U}_q(e^{N+2})$ obtained by setting equal to zero all the generators of $p^i, L^{ij}, k^i$ of $\hat{U}_q(e^{N+2})$ where either $i$ or $j$ takes the values $\pm (n+1)$.

The q-deformed analogue of the complex conjugation of the algebra of real translations and rotations of the real Euclidean space $R^N$ can be introduced whenever $q \in \mathbb{R}^+$: for such values of $q$ there exists a complex conjugation * which is consistent with the algebra relations of $U_q(e^N)$, in other words $U_q(e^N)$ equipped with * is a *-algebra.
The complex conjugation * acts on the Chevalley generators of $U_q(e^N)$ in the following way:

\[
(k^i)^* = k^i, \quad (L^{1-k,k})^* = q^{-2}L^{-k,k-1} \quad k \geq 2, \quad (L^{01})^* = q^{-2}L^{-10} \quad \text{if } N = 2n+1,
\]

\[
(p^i)^* = p^i C_{ji}, \quad \Lambda^* = \Lambda^{-1}; \quad (1.23)
\]

* is extended as an algebra antihomomorphism to all of $U_q(e^N)$, i.e. $(AB)^* = B^* A^*$.

1.3 New $L$ generators of the Euclidean algebra $U_q(e^N)$

The generators $L$’s presented in the previous subsection do not commute with $(p \cdot p)_i$. For representation-theoretical purposes it is convenient to introduce new generators $L$’s instead of the $L$’s by shifting the latter by some functions of the $p$’s, in such a way that $[L, (p \cdot p)_i] = 0$. The $L$’s have no classical analogue.

In section 2 we construct the fundamental Hilbert space representations of $U_q(e^N)$. One can show (Proposition 2) that for such representations either $(p \cdot p)_i \equiv 0$ identically, $\forall i \geq h$, or all $(p \cdot p)_i$ are strictly positive definite. In the former case the algebra reduces to the homogeneous one $U_q(so(N))$, in the latter case, which we here consider, it follows that we can define the inverse of $(p \cdot p)_i$.

We define

\[
\begin{cases}
L^{-m,m+1} := L_{-m,m+1}^{m+1} + \frac{q^{2+m+1}p}{(1-q^2)(pp)_m} p^{-m} p^{m+1} \\
L^{-m+1,m} := L_{-m+1,m}^{m} + \frac{q^{2+m+1}p}{(1-q^2)(pp)_m} p^{-m-1} p^{m}.
\end{cases} \quad (1.24)
\]

(similarly for $L_{12}$). Note that this redefinition is possible only when $q \neq 1$. The basic property of the new generators is the fact that (compare with relations (1.14),(1.15))

\[
[L^{-m,m+1}, p^i]_{b_i,m} = 0 \quad (1.25)
\]

implying

\[
[L^{-m,m+1}, (p \cdot p)_i] = 0 \quad [L^{-m+1,m}, p^i]_{b_i,m} = 0, \quad \forall i, m; \quad (1.26)
\]

moreover, it is easy to see that the $L$’s satisfy the same *-conjugation relations as the $L$’s.

Let us list now the commutation relations satisfied by the $L$’s. We can define other roots $L$ starting from simple ones, just in the same way as we did with the $L$’s, using relations (1.2)-(1.4) (with the replacement $L \to L$). Simple roots $L$ can be classified into positive and negative ones according to the same convention used for the $L$’s.

- Let $k \geq h + 1$. The commutation relations between positive and negative simple roots are

\[
[L^{1-m,m}, L^{-k,k-1}]_a = 0 \quad a = \begin{cases} q^{-1} & m \pm 1 = k \\
1 & \text{if } k \neq m, m \pm 1,
\end{cases} \quad (1.27)
\]

\footnote{Any definition $\Lambda^* = \alpha \Lambda^{-1}$, $\alpha \in \mathbb{C}$, is compatible with the algebra relations (1.21); here we will take $\alpha = 1$.}
The commutations relations among the $L$’s are the same as those among the $L$’s, if we add some “central charges” $(C_m)$. Let us compare the two sets of generators of $U_q(e^N)$ \{\{p, L, k\} and \{p, L, k\}. The $p$’s close a subalgebra, the $L, k$’s too; the $L, k$’s alone do not, but they close a subalgebra together with the elements $(p \cdot p)_j$ (and $p^{\pm 1}$, in the case $N = 2n$). However, the action of the $L, k$’s on the $p$’s is essentially trivial (they $q$-commute with the $p$’s and commute with the $(p \cdot p)_j$’s), whereas the $M$’s act non-trivially on the $p$’s.

Summing up, the commutations relations among the $L$’s are the same as those among the $L$’s, if we add some “central charges” $(C_m)$. Let us compare the two sets of generators of $U_q(e^N)$ \{\{p, L, k\} and \{p, L, k\}. The $p$’s close a subalgebra, the $L, k$’s too; the $L, k$’s alone do not, but they close a subalgebra together with the elements $(p \cdot p)_j$ (and $p^{\pm 1}$, in the case $N = 2n$). However, the action of the $L, k$’s on the $p$’s is essentially trivial (they $q$-commute with the $p$’s and commute with the $(p \cdot p)_j$’s), whereas the $M$’s act non-trivially on the $p$’s.

We collect below the whole set of algebra relations characterizing $U_q(e^3)$:

$$p^{-1}p^0 - qp^0p^{-1} = 0 \quad p^0p^1 - qp^1p^0 = 0 \quad p^{-1}p^1p^{-1} - (q^{1 \over 2} - q^{-{1 \over 2}})p^0p^0 = 0. \quad (1.32)$$

$$[k^1, p^i]_a = 0, \quad [L^{01}, p^j]_{a = \frac{1}{2}} = 0, \quad [L^{01}, p^j]_{a = \frac{1}{2}} = 0 \quad a = \begin{cases} 2 & \text{if } i = 1 \\ 0 & \text{if } i = 0 \\ -2 & \text{if } i = -1 \end{cases} \quad (1.33)$$

$$[k^1, L^{01}]_{q^2} = 0 = [k^1, L^{-10}]_{q^{-2}} \quad \quad [L^{01}, L^{-1,0}]_q = q^{-{1 \over 2}}(k^1)^{-1} + \frac{(p \cdot p)_1}{(p \cdot p)_0} \quad (1.34)$$

$$[\Lambda, L^{ij}] = 0 = [\Lambda, k^i] \quad [p^i, \Lambda]_q = 0. \quad (1.35)$$

### 1.4 Casimirs of $\hat{U}_q(e^N)$

As in the classical case, $\hat{U}_q(e^N)$ (the subalgebra generated by $L, k, p$ only) has $n + 1 - h$ casimirs; their general form mimics the classical one when given in terms of the $q$-epsilon tensor and the covariant generators of $U_q(so(N))$ [F]. The irreps of $\hat{U}_q(e^N)$ are characterized by the values of the casimirs. The simplest casimir is the square momentum casimir

$$\Omega^0 \equiv (p \cdot p)_n. \quad (1.36)$$
Apart from this we write here explicitly only the remaining “Pauli-Lubanski” casimir $\Omega^1$ for when $N = 3, 4$. In the limit $q = 1$ it is given respectively by

$$\Omega^1 = \begin{cases} \varepsilon_{ijk}l^jp^k & \text{if } N = 3 \\ w_l w_h & \text{if } N = 4 \end{cases}$$

where we have denoted here by $l^j$ the $\text{so}(N)$ generator of rotations in the plane $ij$. As in the classical case, $\Omega^1$ will vanish on the singlet representation.

**Proposition 1** \[6\] When $N = 3, 4$, the Casimirs $\Omega_1$ in terms of $p, L, k$ generators take respectively the form

$$\Omega_1 = p^0 (k^1)^{-\frac{1}{2}} - q(q + 1) \frac{(p \cdot p)_1}{p^0} (k^1)^{\frac{1}{2}} + q^2 (1 - q) (1 - q^2) L^{-1,0} L^{0,1} (k^1)^{\frac{1}{2}} p^0$$

and

$$\Omega_1 = (L^{-1,2} L^{-2,1}) (L^{-2,1} L^{1,2}) k^2 (p \cdot p)_1 + \frac{q^2}{(q^2 - 1)^2} (p \cdot p)_1 \{k^1 (L^{-2,1} L^{1,2}) + (k^1)^{-1} (L^{-2,1} L^{1,2}) \}$$

$$+ \frac{q^{-4} (p \cdot p)_1 (k^2)^{-1}}{(q^2 - 1)^4} \left[ 1 - q^2 k^2 \frac{(p \cdot p)_2}{(p \cdot p)_1} \right]^2 - \frac{q^{-2} (p \cdot p)_2}{(1 - q^2)^2 (p \cdot p)_1} [p^{-1} p L^{-2,1} L^{1,2} + p^1 p L^{-2,1} L^{1,2}] k^2.$$ (1.39)

### 2 The fundamental Hilbert space representations of $U_q(e^N)$

A $\ast$-representation $\Gamma$ \[14\] of a $\ast$-algebra $A$ on a Hilbert $\mathcal{H}$ space is essentially a representation of $A$ such that $\Gamma(a^\ast) = \Gamma(a)^\dagger$ ($T^\dagger$ is the adjoint of $T$) at least on a dense subset of the Hilbert space $\mathcal{H}$. In this section we describe the main features of the fundamental Hilbert space $\ast$-representations of $U_q(e^N)$ (denoted by “irreps” in the sequel). In particular we focus on the singlet representation, which is the one describing a free zero-spin boson on $\mathbb{R}^N$ in the limit $q = 1$. For further details and proofs see Ref. \[6\]. We find a basis of $\mathcal{H}$ and show how the generators of $U_q(e^N)$ are to be represented as operators on the elements of the basis. We don’t deal with premature questions regarding domains of definition of the operators. The positivity of the scalar product

$$\{ \begin{array}{l} <u|u> \geq 0, \\
<u|u> = 0 \iff |u> = 0, \end{array} \} \quad \forall |u> \in \mathcal{H}$$

will be imposed apriori at each step of our construction, and of course will be essential in determining the structure of the representations.
2.1 Choice of the observables

Contrary to the classical case, the momenta $p^i$ don’t commute with each-other, therefore cannot be all chosen as elements a set of commuting observables in order to study Hilbert spaces of the irreps of $U_q(\mathbf{e}^N)$. On the contrary, among the commuting observables of a complete set characterizing an irrep we can always take $p_0, (p \cdot p)_1, (p \cdot p)_{n-1}, (p \cdot p)_n, k^1, ..., k^n$ (in fact we can check that they actually make up a complete set for the “singlet” irrep).

\begin{equation}
\begin{cases}
(p \cdot p)_i \equiv 0 & \text{if } N = 2n \quad (2.2)
\end{cases}
\end{equation}

(in fact we can check that they actually make up a complete set for the “singlet” irrep).

It is easy to realize from the commutation relations of $U_q(\mathbf{e}^N)$ in the case $N = 2n + 1$ that the sign of the eigenvalues of $p_0$ will be the same within each irrep. This will mean that to obtain the q-deformed analog of a classical irrep we have to sum two irreps of $U_q(\mathbf{e}^N)$ differing only by the sign of $p_0$.

**Proposition 2** \textsuperscript{[6]} There are only the following two alternatives in $\mathcal{H}$:

\begin{equation}
\begin{cases}
1) & (p \cdot p)_i \equiv 0 \quad \text{identically} \quad \forall i = h, h + 1, ..., n; \\
2) & (p \cdot p)_i > 0 \quad \text{strictly} \quad \forall i = h, h + 1, ..., n.
\end{cases} \quad (2.3)
\end{equation}

The case 1) corresponds to a “trivial” irrep of $U_q(\mathbf{e}^N)$, i.e. to setting $p_i \equiv 0$; the irrep reduces to an irrep of $U_q(\mathbf{so}(N))$, and therefore won’t be considered in the sequel. In the case 2), as a consequence of the proposition, $[(p \cdot p)_i]^{-1}$ and the operators $L$ of section 2.3 will be well-defined.

As an introduction to the results of representation theory \textsuperscript{[6]} for general $N$, we derive them in the case $N = 3$.

2.2 The case $N=3$

The three observables $(p \cdot p)_1, p_0, \log_q(k^1)$ are respectively the q-deformed analogues of 1) the square angular momentum; 2) the momentum component along the $x^0$ direction; 3) the total angular momentum component along the same direction; of a one-particle system in $\mathbf{R}^3$. In the case $q=1$ this is a convenient set of observables for instance if the particle is free or subject to no other force than the one coming from a magnetic field in the $x^0$ direction.

For our derivation we use the algebra relations in formulae (1.32)-(1.35). Let $\mathcal{H}$ denote the Hilbert space of an irrep of $U_q(\mathbf{e}^3)$.

As a first step, we study the representation of the $p$-subalgebra. We make an ansatz, assuming existence of an eigenspace $\hat{\mathcal{H}} \subset \mathcal{H}$ of $p_0, (p \cdot p)_1$

\begin{equation}
(p \cdot p)_1 \hat{\mathcal{H}} = M^2 q^2 \hat{\mathcal{H}} \quad p_0 \hat{\mathcal{H}} = q^2 m \hat{\mathcal{H}} \quad (2.4)
\end{equation}

consisting only of normalizable eigenvectors; $M^2$ is a nonnegative constant with dimensions of a squared mass which we assume to be positive (see proposition 2). Then we find that $\mathcal{H}$ entirely consists of eigenspaces of normalizable eigenvectors, too. The remaining
Proposition 3 \(L^{01}, L^{-10}, k^1\) of \(U_q(e^3)\) will map each of these eigenspace into itself, since they commute with \(p_0, (p \cdot p)_1\).

Given any vector \(|\psi >\in \mathcal{H}\), according to eq. (1.20),(1.32) \(|\psi_{\pm r} >:= (p^{\pm 1})^r|\psi > (r \in \mathbb{N}, l \leq n)\) will also be an eigenvector of \(p_0, (p \cdot p)_1\). The eigenvalues of \(|\psi >, |\psi_{\pm r} >\) will differ by an integer power of \(q\); the norm of \(|\psi_{-r-1} >\) will be given by

\[
< \psi_{-r-1} | \psi_{-r-1} > = < \psi_{-r} | \psi_{-r} > (M^2 - q^{-2}r^2 m^2)q^2
\]

(2.5)

If \(0 < q < 1\), there must exist a \(r\) such that, \(\forall|\psi >\in \mathcal{H}\) \((p^{-1})^{r+1}|\psi > = 0\), otherwise the above norm would get negative for large \(r\). In other words there must exist an eigenspace of \(p_0, (p \cdot p)_1\), which is annihilated by \(p^{-1}\), we call it \(\mathcal{H}_0\); this also fixes the eigenvalue of \((p_0)^2\) up to a sign. Consequently

\[
(p \cdot p)_1 \mathcal{H}_0 = M^2 q^2 \mathcal{H}_0 \quad p_0 \mathcal{H}_0 = \pm M [1 + q^{-1}] \frac{1}{2} q^2 \mathcal{H}_0
\]

(2.6)

If \(q > 1\) one would find similarly that \(\mathcal{H}_0\) exists and which is is annihilated by \(p^1\). Let \(\mathcal{H}_\pi := \mathfrak{H}_\pi = \Lambda^{\pi_1}(p^1)^{\pi_0} \mathcal{H}_0\) are eigenspaces of \(p_0, (p \cdot p)_1\). Clearly the maps \(p^{\pm 1} : \mathcal{H}_\pi \rightarrow \mathcal{H}_{\pi \pm \epsilon_0}\) \((\epsilon_0 \equiv (1,0))\) are invertible \((p^{-1}\) is invertible only in the orthogonal complement of its kernel \(\mathcal{H}_0\)), the inverse being \(p^{1-1} = \frac{q^{-\frac{1}{2}}}{(p \cdot p)_1 - (p \cdot p)_0} p^{-1}\) and \(p^{1-1} = \frac{q^{-\frac{1}{2}}}{(p \cdot p)_1 - q^{-2}(p \cdot p)_0} p^1\) respectively, as one can easily check using equations (1.32). Using the irreducibility of \(\mathcal{H}\) and relations (1.32),(2.6) we arrive at the proposition

**Proposition 3** \(\mathcal{H}\) can be decomposed into the direct sum

\[
\mathcal{H} = \bigoplus_{\pi \in \mathbb{N} \times \mathbb{Z}} \mathcal{H}_\pi, \quad \mathcal{H}_\pi := \Lambda^{\pi_1}(p^1)^{\pi_0} \mathcal{H}_0
\]

(2.7)

of orthogonal eigenspaces \(\mathcal{H}_\pi\) of the observables \(p_0, (p \cdot p)_1\),

\[
(p \cdot p)_1 \mathcal{H}_\pi = M^2 q^{2(1+\pi_1)} \mathcal{H}_\pi \quad p_0 \mathcal{H}_\pi = \pm M [1 + q^{-1}] \frac{1}{2} q^{2m} \mathcal{H}_\pi
\]

(2.8)

We can visualize the preceding results by thinking of the vectors of \(\mathcal{H}_\pi\) as functions of \(\vec{p}\) with support concentrated on the circles \(\mathcal{T}^1 \subset \mathbb{R}^3_{\vec{p}}\) drawn in fig. 2. The arrows in the figure show the action of the generators \(\Lambda, p^{\pm 1}\).

As already anticipated, the generators \(L^{01}, L^{-10}, k^1\) map each \(\mathcal{H}_\pi\) into itself; as a second step, we study these maps. We can stick to \(\mathcal{H}_0\) since, due to equations (1.33),(2.7), the maps on the other subspaces can be obtained from these ones through application to \(\mathcal{H}_\pi\) of powers in the momenta.

The projections of the casimir \(\Omega_1 (1.38)\)and of relation (1.34)_3 onto \(\mathcal{H}_0\) read

\[
\Omega = L^{-10} L^{01} (k^1)^{\frac{1}{2}} + q^{\frac{1}{2}} (k^1)^{-\frac{1}{2}} - (k)^{\frac{1}{2}} - \frac{(k^1)^{\frac{1}{2}}}{(q^2 - 1)(q - 1)}.
\]

(2.9)
\[ [L^{0,1}, L^{-1,0}]_q = q^{\frac{1}{2}} \frac{q^{-1} + (k^1)^{-1}}{1 - q^2} \]  

Let \( \Omega H_{\tilde{q}} = \omega H_{\tilde{q}}, \omega \in \mathbb{R}, \) let \( \psi \in H_{\tilde{q}} \) be an eigenvector of \( k^1, k^1|\psi| = \mu^2|\psi|, \) and define \( |\psi_{\pm m}\rangle := (L^{0,\pm 1})^m |\psi\rangle; |\psi_{\pm m}\rangle := (L^{0,\pm 1})|\psi\rangle \) are new eigenvectors with eigenvalues \( \mu^2 q \pm 2m \). The norm of \( |\psi_{m+1}\rangle \) reads

\[
< \psi_{m+1} |\psi_{m+1} >= q^{-\frac{3}{2}} < \psi_m |L^{-10}L^{01}|\psi_m >= < \psi_m |\psi_m > \left\{ \omega q^{-m} \mu^{-1} + q^\frac{1}{2} \frac{1 - \mu^2 q^{-2m}}{(q^2 - 1)(q - 1)} \right\}
\]

Since \( 0 < q < 1 \), there must exist a \( m \) such that \( |\psi_{m+1}\rangle = 0 \) otherwise the above norm would get negative for large \( r \). In other words there must exist a \( w \in \mathbb{R} \) and a highest weight vector \( |\tilde{0}, 0 >^w \in H_{\tilde{q}} \) such that

\[
L^{0,1}|\tilde{0}, 0 >^w = 0, \quad k^1|\tilde{0}, 0 >^w = q^w |\tilde{0}, 0 >^w, \quad \omega = q^\frac{1}{2} \frac{q^{-w} - q^w}{(1 - q^2)(1 - q)}. \quad (2.12)
\]

If we repeat the same argument with \( |\psi_{-m}\rangle := (L^{-10})^m |\psi\rangle \), we see that its norm keeps positive for large \( m \), hence there exists no lowest weight vector. On the contrary, if it were \( q > 1 \) there would exist a lowest weight vector and no highest weight one.

Defining normalized vectors \( |\tilde{0}, j >^w := N_j(L^{01})|\tilde{0}, 0 >^w \), we find from irreducibility that they form a basis of \( H_{\tilde{q}} \) and that \( k^1|\tilde{0}, j >^w = q^{2j+w}|\tilde{0}, j >^w, L^{0,1}|\tilde{0}, j >^w \in |\tilde{0}, j \pm 1 >. \) The coefficient of proportionality in the latter relation can be found using formula (2.11) after replacing \( |\psi_m\rangle \) by \( |\tilde{0}, j >^w \). This completes the study of the structure of \( H_{\tilde{q}} \).

Now we can easily extend this knowledge to the other spaces \( H_{\tilde{q}} \). By defining \( |\tilde{\pi}, j >^w \) as the normalized vector proportional to \( \Lambda^{\pi} p^1 \tilde{\pi}, j - \pi_0 >^w \), one determines an orthonormal basis of \( H \) consisting of eigenvectors of the commuting observables \( p_0, (p^1 p^1), k^1 \).

We collect the results in the

**Theorem 1** A basis \( B^w_{\tilde{q}} \) of \( H^w \) in the case \( N = 3 \) is the set \( \{|\tilde{\pi}, j >^w \} (\tilde{\pi} \in \mathbb{N} \times \mathbb{Z}, j \in J(\tilde{\pi}) := \{ j \in \mathbb{Z} \mid j \leq \pi_0 \} \), with the following properties.

\[
\begin{align*}
\langle \pi; j >^w &= \pm M[1 + q^{-1}] \frac{1}{2} q^{j=0} (1 + \pi_h) |\pi; j >^w, \\
(p \cdot p)_1 |\pi; j >^w &= M^2 q^{2(1+\pi_1)} |\pi; j >^w \\
k^1 |\pi; j >^w &= q^{2j_1+w} |\pi; j >^w,
\end{align*}
\]

Moreover

\[
\begin{align*}
\Lambda^{\pm 1} |\tilde{\pi}; j >^w &= |\tilde{\pi} \pm e_1; j >^w, \\
p^1 |\tilde{\pi}; j >^w &= M[1 - q^{2(\pi_0+1)}] \frac{1}{2} q^{1+\pi_1} |\tilde{\pi} + e_0; j + 1 >^w \\
p^{-1} |\tilde{\pi}; j >^w &= M[1 - q^{2\pi_0}] \frac{1}{2} q^{-\pi_0+1+\pi_1} |\tilde{\pi} - e_0; j - 1 >^w,
\end{align*}
\]

(we have set all the arbitrary phase factors equal to 1). Here \( e_0 \equiv (1, 0), e_1 \equiv (0, 1), M \) is a constant with dimensions of a mass, defined modulo integer \( q \)-powers and characterizing the irrep together with \( w \). Finally

\[
\begin{align*}
L^{0,1} |\tilde{\pi}; j >^w &= q^{-\frac{1}{2} - \pi_0} \left[ \frac{(1+q^{-w+j+\pi_0})(1-q^{-j+\pi_0})}{(q^{-1} - 1)(1-q)} \right]^{\frac{1}{2}} |\tilde{\pi}; j + 1 >^w \\
L^{-1,0} |\tilde{\pi}; j >^w &= q^{-\pi_0} \left[ \frac{(1+q^{1-w+j+\pi_0})(1-q^{1-j+\pi_0})}{(q^{-1} - 1)(1-q)} \right]^{\frac{1}{2}} |\tilde{\pi}; j - 1 >^w.
\end{align*}
\]
We have appended suffix \( w \) to specify that the value of the casimir \( \Omega \) given by relation (2.12).

### 2.3 The general case

We introduce a sort of Borel decomposition of \( U_q(e^N) \). As a consequence of its existence, Hilbert space representations will be of highest weight type.

**Definition** We denote by \( U^+_{q,N} \) the subalgebra of \( U_q(e^N) \) generated by the positive roots \( L's \) and by \( p^{-l}, \ l > h(N) \); by \( U^-_{q,N} \) the subalgebra generated by the negative roots \( L's \), by \( \Lambda^{\pm 1} \), by \( p^l, \ l \geq h(N) \) and, in the case \( N = 2n \) only, by \( p^{-1} \). Clearly \( U_q(e^N) = U^-_{q,N} \otimes U^+_{q,N} \).

**Theorem 2** [6] The subspace \( \mathcal{H}_G \) of “highest weight vectors”, i.e.

\[
\mathcal{H}_G := \{ | \phi > \in \mathcal{H} \mid u|\phi >= 0, \ \forall u \in U^+_{q,N} \} \tag{2.18}
\]

is infinite-dimensional. A basis of \( \mathcal{H}_G \) is provided by the vectors \( \{(\Lambda^s)|\phi >, \ s \in \mathbb{Z} \} \) and \( \{(\Lambda^s(p^{\pm 1})^r)|\phi >, \ s \in \mathbb{Z}, \ r \in \mathbb{N} \} \) in the cases \( N = 2n + 1 \) and \( N = 2n \) respectively; \(|\phi >\) is any nontrivial vector of \( \mathcal{H}_G \). \(|\phi >\) is cyclic in \( \mathcal{H} \) w.r.t. the subalgebra \( U^-_{q,N} \). (In the sequel by “the highest weight vector” we will mean a particular one of these vectors). The eigenvalues \( k^j \) of the operators \( k^j \) are of the type \( k^j = q^{2j} \lambda_1, \ j \in \mathbb{Z}, \) and the constant \( \lambda_1, 1 \geq \lambda_1 > q^2 \), is a function of the casimirs characterizing the irrep.

The existence of highest weight vectors follows when \( 0 < q < 1 \) from the requirement of nonnegativity of the scalar product and from the Borel decomposition given at the beginning of section 2.3. The theorem is proved considering first the Hilbert space representations of the \( p \)-subalgebra, then the Hilbert space representations of the subalgebra of the \( L,k \)'s within each eigenspace of the observables \( (p \cdot p)^{j} \)'s; this is possible because of formula (1.26). Contrary to the case of representation theory of \( U_q(so(N)) \), in each such eigenspace there is no lowest weight vector in \( \mathcal{H}_G \), due to the presence of non-vanishing \( C_m \)'s in the commutation relations (1.29); therefore each such eigenspace is infinite-dimensional.

In the sequel we stick to irreps characterized by \( \lambda_1 = 1,q \) (among which we can find those having classical analogue). For this class of irreps we can introduce a vector \( \vec{w} \in \mathbb{Z}^n \) such that \( k^j|\phi >= q^{w^j}|\phi >. \) The vector \( \vec{w} \) depends on the casimirs and together with mass-scale \( M \) (defined modulo \( q^2 \)) completely characterizes an irrep. We will therefore attach it as a superscript to the symbol \( \mathcal{H} \) and write \( \mathcal{H}^{\vec{w}} \). Now we can formulate the main proposition of this section.

**Theorem 3** A basis \( \mathcal{B}_q^{\vec{w}} \) of \( \mathcal{H}^{\vec{w}} \) is the set \( \{|\vec{\pi}; \vec{j}, \alpha >\} \) \( (\vec{\pi} \in \mathbb{N}^{n-h} \times \mathbb{Z}, \ \vec{j} \in \mathcal{J}, \ \alpha \in A) \) with the following properties.

\[
\begin{align*}
|p_0|\vec{\pi}; \vec{j}, \alpha > &= \pm M[1 + q^{-1}] \sum_{k=0}^{n} (1+\pi_k) |\vec{\pi}; \vec{j}, \alpha >, \quad \text{if } N = 2n+1; \\
(p \cdot p)^{\frac{1}{2}}|\vec{\pi}; \vec{j}, \alpha > &= M^2 q^{\pi_k} \sum_{k=1}^{n} 2(1+\pi_k) |\vec{\pi}; \vec{j}, \alpha > \quad i \geq 1 \\
k^j|\vec{\pi}; \vec{j}, \alpha > &= q^{2j+w^j}|\vec{\pi}; \vec{j}, \alpha >
\end{align*}
\tag{2.19}
\]
Moreover

\begin{equation}
(A)^\pm 1 |\vec{\pi};\vec{j},\alpha \rangle = |\vec{\pi} \pm \vec{e}_n;\vec{j},\alpha \rangle, \quad (2.20)
\end{equation}

\begin{equation}
p^l |\vec{\pi};\vec{j},\alpha \rangle = M[1 - q^{2(\pi l - 1 + 1)}] \frac{1}{2} \sum_{k=1}^{n} (1 + \pi_k) |\vec{\pi} + \vec{e}_{l-1};\vec{j} + \vec{y}_l,\alpha' \rangle \quad (2.21)
\end{equation}

\begin{equation}
p^{-l} |\vec{\pi};\vec{j},\alpha \rangle = M[1 - q^{-2(\pi l - 1)}] \frac{1}{2} \sum_{k=1}^{n} (1 + \pi_k) |\vec{\pi} - \vec{e}_{l-1};\vec{j} - \vec{y}_l,\alpha' \rangle \quad (2.22)
\end{equation}

\begin{equation}
p^{\pm 1} |\vec{\pi};\vec{j},\alpha \rangle = M q^{k-1} |\vec{\pi};\vec{j} \pm y_1,\alpha' \rangle, \quad \text{if} \quad N = 2n \quad (2.23)
\end{equation}

(we have set all the arbitrary phase factors equal to 1). Here \(l > h\) and we have set all the arbitrary phase factors equal to 1). Here \(l > h\) and set all the arbitrary phase factors equal to 1). Hence \(l > h\) and set all the arbitrary phase factors equal to 1).

The domain \(\mathcal{J}\) of \(\vec{j}\) is

\begin{equation}
\mathcal{J} := \{ \vec{j} \in \mathbb{Z}^n \mid j_i \leq \pi_{i-1}, \quad i = h + 1, h + 2, \ldots, n; \quad j_i \in \mathbb{Z} \quad \text{if} \quad N = 2n \}. \quad (2.25)
\end{equation}

The coefficients \(D_m, D'_m\) and the values of \(\alpha'\) depend on the particular irrep under consideration. \(D_m = 0\) if \(j_i = \pi_{i-1}\). \(\mathcal{A} = \mathcal{A}(\vec{w}, \vec{\pi}, \vec{j})\) is a finite set and \(\alpha \in \mathcal{A}\) are additional labels identifying the eigenvalues of the observables, if any, which have to be added to the ones of formula (100) to get a complete set. \(\mathcal{A}\) is trivial (i.e. it has only one element, therefore label \(\alpha\) can be omitted) 1) for any value of \(\vec{\pi}, \vec{j}\) if \(\vec{w} = 0; 2)\) whenever \(j_i = \pi_{i-1}\).

Remarks.

- A basis of the subspace \(\mathcal{H}_G\) (see theorem 4) is \(\{ |\vec{\pi} = s\vec{e}_n;\vec{j} = \vec{0};\alpha_0 \rangle \mid s \in \mathbb{Z} \) if \(N = 2n + 1, \{ |\vec{\pi} = s\vec{e}_n;\vec{j} = r\vec{y}_l;\alpha_0 \rangle \mid s, r \in \mathbb{Z} \) if \(N = 2n\).

- As expected the spectra of \((p \cdot p)_l\) are discrete; they are particularly simple, since they consist only of \(q\)-powers. Note that none of them contains the zero eigenvalue (but the latter is an accumulation point of the spectra); in particular \((p \cdot p)_n > 0\) always, i.e. “there is no state in which the nonrelativistic quantum particle is at rest”.

- Theorem 5 summarizes the essential features of the promised “\(q\)-latticization” in momentum space \(\mathbb{R}^N_p\). For each vector \(\vec{\pi}\), the equations \((p \cdot p)_l = M^2 q_{2l}^{k-1}\), \(l = h, h + 1, \ldots, n\) single out a \(n\)-Torus subgroup \(\mathcal{T}^n\) within \(\mathbb{R}^N_p\), on which the vectors with fixed \(\vec{\pi}\) have support; let us call \(\mathcal{H}_\pi \in \mathcal{H}\) the linear span of such vectors. The additional specification of a vector \(\vec{j}\), however, selects in \(\mathcal{H}_\pi\) vector(s) having well-defined angular momentum components \(\vec{k}\), but no well-defined \(p\)-angles; in other words the support of each state is not concentrated on a point of \(\mathcal{T}^n \subset \mathbb{R}^N_p\). For no choice of a complete set of commuting observables the corresponding eigenvectors
would have a point-like support in $\mathbf{R}^N_{\vec{p}}$, since no such set can include $N$ functions of the (non-commuting variables) $p$’s. The q-lattice $\{ (\vec{\pi}, \vec{j}), \alpha \}$ has to be understood in a space where $n+1-h$ dimensions (corresponding to the first $n+1-h$ observables (2.2)) are of “momentum” type, and the remaining are of “angular momentum” type. The action of the generators $p, L, \Lambda^{\pm}$ on a vector $|\vec{\pi}; \vec{j}, \alpha \rangle$ can be visualized as a mapping of the point $(\vec{\pi}; \vec{j})$ of the q-lattice into one of its nearest neighbour points.

Definition We define the singlet Irrep as the one characterized by the highest weight $\vec{w} = 0$.

With straightforward computations one can verify that the in the singlet representation coefficients $D_m, D'_m$ $(m \geq 2)$ appearing in formula (2.24) read

$$D_m(\vec{\pi}, \vec{j}) = q^{-1-\rho_m-\pi_m-1+\pi_m-2} \left[ \frac{(1-q^{j_m-1-\pi_m-2-j_m+\pi_m-1})(1-q^{j_m-1-\pi_m-2-j_m+\pi_m-1-2})}{(1-q^2)^2} \right]^{\frac{1}{2}}$$

$$D_m(\vec{\pi}, \vec{j}) = q^{1-\rho_m-\pi_m-1+\pi_m-2} \left[ \frac{(1-q^{j_m-1-\pi_m-2-j_m+\pi_m-1+2})(1-q^{j_m-1-\pi_m-2-j_m+\pi_m-1})}{(1-q^2)^2} \right]^{\frac{1}{2}}$$

(we have set all arbitrary phase factors equal to 1).

It is easy to verify that by making the tensor product $(\tilde{\Gamma}^u, \tilde{\mathcal{H}}^u)$ of the singlet Irrep $(\Gamma^0, \mathcal{H}^0)$ of $\hat{U}_q(e^N)$ and an Irrep $(\Gamma^u_{hom}, \mathcal{H}^u_{hom})$ of $U^N_q \equiv U_q(\mathfrak{so}(N))$ with highest weight $\vec{u}$ we find a reducible Hilbert space representation of $\hat{U}_q(e^N)$ characterized by the same momentum scale $M$, as it occurs in the classical case. To find the Irreps contained in $\tilde{\Gamma}^u$ one proceeds as in the classical Lie algebra representations; namely, using orthogonality, one determines all the highest weight vectors contained in $\mathcal{H}^u$.

Proposition 4 Possible highest weights are of the form $\vec{w} = \vec{u} - l \vec{y}_1$, $0 \leq l \leq 2u_1$, if $N = 2n + 1$, $\vec{w} = \vec{w}(l, l') := \vec{u} - l \cdot \text{sign}(u_2 - u_1)(\vec{y}_2 - \vec{y}_1) - l'(\vec{y}_2 + \vec{y}_1)$, $0 \leq l \leq |u_2 - u_1|$, $0 \leq l' \leq u_1 + u_2$, if $N = 2n$; $\vec{u}$ denote weights of $U^N_q$. In particular, when $N = 3, 4$ the sets $\{ \vec{w} \}$ of weight satisfy the relations $\{ w_1 \} = \mathbf{Z}$, $\{ \vec{w} \} \subset \mathbf{Z} \otimes \mathbf{Z}$ respectively. We have the following tensor product decomposition

$$\tilde{\Gamma}^u = \bigoplus_{l=0}^{2u_1} \Gamma^{u-l\vec{y}_1} \bigoplus_{0 \leq l \leq |u_2 - u_1|; 0 \leq l' \leq u_1 + u_2} \Gamma^{\vec{u}(l, l')} \quad \text{if} \quad N = 2n + 1$$

$$\text{if} \quad N = 2n + 1$$

(2.27)

Highest weight vectors can be easily determined from the above described tensor product construction procedure.

Note that only the irreps with $w = 0, 1, 2, \ldots$ have classical analogue.

We give an intuitive picture of the physical content of the spectra of the observables (2.2) in the singlet representation. The subspace $\mathcal{H}^0_{\vec{p}} := \bigoplus_{\{ \vec{p}, | \vec{\pi}, \lambda \}} \mathcal{H}^0_{\vec{p}}$ is the eigenspace of the observable $p^{-i}p^{-i} = (p \cdot p)_i - (p \cdot p)_{i-1}$ with the minimum eigenvalue compatible
with a given eigenvalue of \((p \cdot p)_i\), namely 
\[ p^{-i}p_{-i} = M^2 q_{k=i} \sum_{k=i}^{n} 2(1+\pi_k) (q_2 - 1); \] the latter quantity never vanishes when \(q \neq 1\). This means that the there is always a “point zero” momentum component available in the plane of the coordinates \(i, -i\). Now let us ask in which “directions” of this plane this point zero momentum component can be pointed.

For the above choice of the momenta, the admitted eigenvalues of \(\ln_q (k^i)\), i.e. of the angular momentum component in the plane, are \(j_i \leq 0\) (see formula (2.25)) and show that (except when \(N = 2n, i = 1\)) only a “clockwise” or “radial” orientation are possible. The anticlockwise is excluded! If \(N = 3\), for instance, minimum \(p^1 p_1\) means that the momentum is “almost pointed” in the \(p^0\) direction; \(j_1\) represents the \(p^0\)-direction component of the (orbital) angular momentum and cannot take positive values. This amounts to sort of a purely “kinematical” PT (parity+time-inversion) asymmetry of the allowed momentum space (under a PT transformation \(\vec{p}\) would remain unchanged, whereas the angular momentum component \(h\) would change sign). This is a surprising feature for a lattice theory; in fact, at least usual equispaitated lattice theories, which are commonly used nowadays for regularization purposes, cannot have a parity asymmetry by a well-known no-go-theorem [12]. In next section we will see in which sense in the classical limit \(q \to 1\), however, parity symmetry is recovered.

### 2.4 Configuration space realization

One can show [3] that the singlet irrep can be realized in “\(Fun(\mathbb{R}^N_q)\)-configuration space”. By this we mean that the vectors of \(\mathcal{H}_\delta^0\) can be realized as elements (“wave-functions”) of \(Fun(\mathbb{R}^N_q)\) and the elements of \(U_q(e^N)\) as \(q\)-differential operators acting on them. Actually in Ref. [3] we give two equivalent “\(Fun(\mathbb{R}^N_q)\)-configuration space” realizations, which we call the unbarred and the barred. The scalar product of two vectors of \(\mathcal{H}_\delta^0\) is realized as a \(Fun(\mathbb{R}^N_q)\)-integral involving both the barred and the unbarred wavefunctions. These “\(Fun(\mathbb{R}^N_q)\)-configuration space” realizations should be useful for future functional analysis studies, such as an intrinsic characterization of \(\mathcal{H}_\delta^0\), questions regarding in concrete cases the domains of definition of operators representing some elements of \(U_q(e^N)\), etc. For further details we refer the reader to Ref. [3].

### 3 Classical limit of the singlet irrep

In order that the representations of \(U_q(e^N)\) presented in this work can be considered as physically realistic, they should describe a system of one free particle on \(\mathbb{R}^N\) and \(U(so(N))\)-spin \(\vec{w}\) in the limit (understood in some reasonable sense) \(q \to 1\). For simplicity, let us stick to the case of the singlet irrep \(\vec{w} = 0\), drop the superscript \(\delta\) and introduce a subscript \(q\) on the ket symbols: we will write \(|ket >_q\) instead of \(|ket >^\delta\).

The commuting observables

\[ p_0, (p \cdot p)_1, ..., (p \cdot p)_{n-1}, (p \cdot p)_n; h_1, ..., h_n \]  

\((p_0 \equiv 0 \quad if \quad N = 2n + 1)\), (3.1)
(where \( h_i := \log_q^2(k^i) \)) make up a complete set both when \( q \neq 1 \) and \( q = 1 \). We have chosen \( h_i \) instead of \( k^i \) because it is the set of generators \( \{L^{i,j}, h_{i,j}, p^i\} \) which has classical commutation relations in the limit \( q \to 1 \). The eigenvectors \(|\tilde{\pi}; \tilde{j} >_q \) of the observables (2.2) form the orthonormal basis \( B_q \) of theorem 3 for all \( q \in \mathbb{R}^+ - \{1\} \); when \( q = 1 \) the vectors of \( B_{q=1} \) are (orthogonal) distributions, i.e. elements of the space of functionals on some space of smooth functions on \( \mathbb{R}^n \), e.g. \( S(\mathbb{R}^N) \). We ask whether they can be obtained by some sort of limiting procedure when \( q \to 1 \).

For each eigenvector \(|\tilde{\pi}, \tilde{j} >_q \) the eigenvalues \( j_i \) of \( h_i := \log_q^2(k^i) \) don’t depend on \( q \) and are integers; whereas the eigenvalues of \((p \cdot p)_i \) (non-uniformly) “collapse” to \( M^2 \) (see formula (2.19)\(_3\)):

\[
\lim_{q \to 1} c_i(q, \tilde{\pi}) = M^2 \quad \text{where} \quad (p \cdot p)_i|\tilde{\pi}, \tilde{j} > =: c_i(q, \tilde{\pi})|\tilde{\pi}, \tilde{j} > \quad i = 1, 2, ..., n, \]

\[
\lim_{q \to 1} c_0(q, \tilde{\pi}) = M \quad \text{where} \quad p_0|\tilde{\pi}, \tilde{j} > =: c_0(q, \tilde{\pi})|\tilde{\pi}, \tilde{j} > \quad \text{if} \quad N = 2n + 1. \tag{3.2}
\]

If we kept \( \tilde{\pi}, \tilde{j} \) fixed and let \( q \to 1 \), \(|\tilde{\pi}, \tilde{j} >_q \) would remain a normalized eigenvector and its eigenvalues \( c_i \) would go to \( M^2 \) (independently of \( \tilde{\pi} \)). Consequently, the above limit cannot be given a literal sense \( B_{q=1} = \{\lim_{q \to 1} |\tilde{\pi}, \tilde{j} >_q, |\tilde{\pi}, \tilde{j} >_q \in B_q \} \).

However, we note that for each fixed \( 0 \leq q < 1 \) and each \( \mu_i \in \mathbb{R} \) there exist \( \tilde{\pi} \) large enough such that \( c_i(q, \tilde{\pi}) \) are close to \( \mu_i \), and the difference can be made smaller and smaller as \( q \) approaches 1, because in that limit the point density of the set \( \{q^n\}_{q \in \mathbb{Z}} \) gets higher and higher around each fixed point on the real axis. This suggests a more adequate notion of “representation limit”, as given below. First of all, for each distribution \(|\bar{\mu}; \bar{j} >_c \in B_1 \) of the classical representation defined by

\[
(p \cdot p)_i|\bar{\mu}, \bar{j} >_c = \mu_i|\bar{\mu}, \bar{j} >_c, \quad h_i|\bar{\mu}, \bar{j} >_c = j_i|\bar{\mu}, \bar{j} >_c, \quad < \bar{\mu}', \bar{j}'|\bar{\mu}, \bar{j} >_c = \delta_{\bar{j}'j} \delta(\bar{\mu} - \bar{\mu}')
\]

\[
(p_0|\bar{\mu}, \bar{j} > = \mu_0|\bar{\mu}, \bar{j} > \quad \text{if} \quad N = 2n + 1) \tag{3.3}
\]

\((i = 1, ..., n, \mu_{i+1} \geq \mu_i \geq 0, \bar{j} \in \mathbb{Z}^n \) and the second \( \delta \) is a Dirac’s \( \delta \), we can find a vector function \( \tilde{\pi}(\bar{\mu}, q) \) such that

\[
\mu_i = \lim_{q \to 1} c_i(q, \tilde{\pi}(\bar{\mu}, q)). \tag{3.4}
\]

It is easy to see that this condition is fulfilled e.g. by

\[
\tilde{\pi}_i := \left[ \log_q^2\left(\frac{\mu_i}{\mu_{i+1}}\right) \right] \quad \text{(and} \quad \tilde{\pi}_0 := \left[ \log_q^2\left(\frac{(\mu_0)^2}{\mu_1}\right) \right] \quad \text{if} \quad N = 2n + 1) \tag{3.5}
\]

([\( a \] \( \in \mathbb{Z} \) denotes the integral part of \( a \in \mathbb{R} \), and \( \mu_{n+1} \equiv M^2 \)).

Therefore we are led to define

\[
||\bar{\mu}; \bar{j} >_q := \alpha(q, \bar{\mu})|\tilde{\pi}(\bar{\mu}, q); \bar{j} >_q. \tag{3.6}
\]

and choose the function \( \alpha(q, \bar{\mu}) \) in such a way that

\[
\lim_{q \to 1} < \bar{\mu}': \bar{j}'||\bar{\mu}; \bar{j} >_q = \delta(\bar{\mu}' - \bar{\mu})\delta_{\bar{j}'j}; \tag{3.7}
\]
the latter limit is in the sense of convergence of \( \langle \vec{p}'; \vec{j} \| \vec{p}; \vec{j} \rangle \) in the space of functionals on smooth functions of the two variables \( \vec{p}', \vec{p} \). This is finally the adequate notion of the limit we were looking for. Symbolically

\[
\langle \vec{p}; \vec{j} \rangle = \lim_{q \to 1^-} \| \vec{p}; \vec{j} \|. \tag{3.8}
\]

It is easy to check that a choice of \( \alpha \) satisfying relation (3.7) is

\[
\alpha(q, \vec{p}) := \prod_{i=1}^{n} [\mu_i(q^{-2} - 1) + \frac{1}{2}] - \frac{1}{2} \cdot \left\{ \begin{array}{ll}
1 & \text{if } N = 2n + 1 \ni \mu_0(q^{-1} - 1) \frac{1}{2} \text{ otherwise.}
\end{array} \right. \tag{3.9}
\]

Let us verify that \( D(\vec{p}', \vec{p}) := \langle \vec{p}; \vec{j} \| \vec{p}; \vec{j} \rangle \) is really a \( \delta \)-convergent functional.

Suppose first that \( N = 2n \). We consider a smooth function \( f(\vec{p}') \) and the integral \( \int d\vec{p}' D \cdot f(\vec{p}') \); we want to show that its limit is \( f(\vec{p}) \). We note that

\[
\langle \tilde{\pi}(\vec{p}', q); \vec{j} \| \tilde{\pi}(\vec{p}, q); \vec{j} \rangle = \prod_{i=1}^{n} \chi_{[q^{2g_i(\vec{p}, q)} \cdot \cdot \cdot q^{2g_i(\vec{p}, q) - 2}]}(\frac{\mu_i}{\mu_i + 1}),
\]

\[
\chi_{[a, b]}(z) := \left\{ \begin{array}{ll}
1 & \text{if } z \in [a, b) \\
0 & \text{otherwise.}
\end{array} \right.
\tag{3.10}
\]

Therefore, setting \( z_i := \frac{\mu_i}{\mu_i + 1}, i = 1, \ldots, n \), we find

\[
\prod_{i=1}^{n} d\mu_i = \prod_{i=1}^{n} dz_i J(z) \quad J(z) := M^N \cdot z_{n-1}^{n-2} \ldots z_2
\tag{3.11}
\]

and

\[
\int d\vec{p}' D(\vec{p}', \vec{p}) \cdot f(\vec{p}') = \alpha(q, \vec{p}) M^N \int d\vec{z} J(\vec{z}) \alpha(q, \vec{p}(z)) f(\vec{p}(z)) \prod_{i=1}^{n} \chi_{[q^{2g_i(\vec{z}, q)} \cdot \cdot \cdot q^{2g_i(\vec{z}, q) - 2}]}(z_i)(z_0)
\]

\[
\approx \int \frac{|\alpha(q, \vec{p})|^2 M^N f(\vec{p})}{q^{-1}} d\vec{z} J(\vec{z}) \prod_{i=1}^{n} \chi_{[q^{2g_i(\vec{z}, q)} \cdot \cdot \cdot q^{2g_i(\vec{z}, q) - 2}]}(z_i)(z_0)
\]

\[
= \frac{(n)_{q-2}}{n!}(q^{-2} - 1)^n M^2 n \sum_{i=1}^{q-1} i \tilde{\chi}_i(\vec{p}, q) |\alpha(q, \vec{p})|^2 f(\vec{p}) \approx (q^{-2} - 1)^n (\prod_{i=1}^{n} \mu_i) |\alpha(q, \vec{p})|^2 f(\vec{p})
\tag{3.12}
\]

and the last expression goes to \( f(\vec{p}) \) in the limit \( q \to 1^- \), due to relations (3.9). Similarly one proves the result in the case \( N = 2n + 1 \).

In the limit \( q \to 1^- \) the " PT asymmetry " in the spectrum of the observables \( \mathbf{k}^i \) noticed at the end of section 2.3 disappears, " almost everywhere " in momentum space. (Actually, the only points where this does not occur are charaterized by the condition \( (p \cdot p)_{i-1} = (p \cdot p)_i \), namely they belong to a cylinder in the classical momentum space \( \mathbf{R}^N_p \); the latter is a subset of \( \mathbf{R}^N_p \) of zero measure). In fact, whenever \( \mu_{i-1} < \mu_i \), i.e. \( (p \cdot p)_{i-1} < (p \cdot p)_i \), the range of each \( j_i \) (as a function of the square momenta) becomes the whole set \( \mathbf{Z} \) in the limit \( q \to 1^- \), since then \( \lim_{q \to 1^-} \pi_{i-1}(\vec{p}, q) = \infty \). The same is true also in the irreps with highest weight \( \neq 0 \).
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