PLUMBING IS A NATURAL OPERATION IN KOVANOV HOMOLOGY

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Abstract. Given a connect sum of link diagrams, there is an isomorphism which decomposes unnormalized Khovanov chain groups for the product in terms of normalized chain groups for the factors; this isomorphism is straightforward to see on the level of chains. Similarly, any plumbing $x * y$ of Kauffman states carries an isomorphism of the chain subgroups generated by the enhancements of $x * y$, $x$, $y$:

$$C_R(x * y) \rightarrow (C_{R,p=1}(x) \otimes C_{R,p=1}(y)) \oplus (C_{R,p=0}(x) \otimes C_{R,p=0}(y)).$$

We apply this plumbing of chains to prove that every homogeneously adequate state has enhancements $X^\pm$ in distinct $j$–gradings whose $A$–traces (cf §3.1) represent nonzero Khovanov homology classes over $F_2$, and that this is also true over $Z$ when all $A$–blocks' state surfaces are two–sided. We construct $X^\pm$ explicitly.

1. Introduction

Given a link diagram $D \subset S^2$, smooth each crossing in one of two ways, $\XLeft \X \XRight$. The resulting diagram $x$ is called a Kauffman state of $D$ and consists of state circles joined by $A$– and $B$–labeled arcs, one from each crossing. Enhance $x$ by assigning each state circle a binary label: $\bullet \Left \bullet \Right \bullet$, and let $R$ be a ring with 1. The enhanced states from $D$ form an $R$–basis for a bi-graded chain complex

$$C_R(D) = \bigoplus_{i,j \in Z} C_{i,j}^R(D),$$

which has a differential $d$ of degree $(1,0)$; the resulting homology groups are link–invariant. Khovanov homology categorifies the Jones polynomial in the sense that the latter is the graded euler characteristic of the former [4, 6, 11]. Section 2 reviews Khovanov homology in more detail.

What do (representatives of) nonzero Khovanov homology classes look like? The simplest examples come from adequate all–$A$ states $x_A$ and adequate all–$B$ states $y_B$: the all–1 enhancement of $x_A$ and the all–0 enhancement of $y_B$ are nonzero cycles with any coefficients. Further, any enhancement of $y_B$ with exactly one 1–label is a nonzero cycle over any $R$ in which 2 is not a unit; and the sum of all enhancements of $x_A$ with exactly one 0–label is a nonzero cycle over $R = F_2$.

Intriguingly, such states $x_A$, $y_B$ are essential in the sense that their state surfaces are incompressible and $\partial$–incompressible [9]. Does Khovanov homology detect essential surfaces in any more general sense? Letting $C_R(x)$ denote the submodule of $C_R(D)$ generated by the enhancements of any state $x$ of $D$, we ask:

**Main question.** For which essential states $x$ does $C_R(x)$ contain a nonzero homology class?

As this inquiry depends explicitly on the diagram, the chief motivation is not Khovanov homology in the abstract, but rather a geometric question: in what sense does Khovanov homology detect essential surfaces?

Which states $x$ are essential? A necessary condition is that $x$ must be adequate. For a sufficient condition, let $G_x$ denote the graph obtained from $x$ by collapsing each state circle to a vertex (each crossing arc is then
an edge). Cut \(G_x\) all at once along its cut vertices (ones whose deletion disconnects \(G_x\)), and consider the resulting connected components; the corresponding subsets of \(x\) are called blocks. The state \(x\) decomposes under plumbing (of states) into these blocks, and the state surface from \(x\) decomposes under plumbing (of surfaces) into the blocks’ state surfaces, each of which is a checkerboard surface for its block’s underlying link diagram. Section \(\square\) reviews state surfaces and plumbing in more detail.

If each block of \(x\) is essential, then \(x\) is essential too, as plumbing respects essentiality \([3, 9]\). In particular, if each block of \(x\) is adequate and either all–\(A\) or all–\(B\), then \(x\) is essential, and is called \textcolor{red}{homogeneously adequate} \(\square \). Our main result states that Khovanov homology over \(\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}\) detects all such states:

**Main theorem.** If \(x\) is a homogeneously adequate state, then \(C_{\mathbb{F}_2}^{i_x, j_x, \pm 1}(x)\) both contain (representatives of) nonzero homology classes. If also \(G_{x, \lambda}\) is bipartite, then \(C_{\mathbb{Z}}^{i_x, j_x, \pm 1}(x)\) contain such classes as well.

Here, \(i_x, j_x\) are integers that depend only on \(x\), and \(x, \lambda\) denotes the union of the \(A\)–blocks of \(x\). (We will define \(x, \lambda\) analogously; this is consistent with the earlier notation \(x, \lambda\).) The bipartite condition on \(G_{x, \lambda}\) is equivalent to the condition that the state surfaces from the \(A\)–blocks of \(x\) are all two–sided. In general, the condition of homogeneous adequacy is sensitive to changes in the link diagram, as are the homology classes from the main theorem, in the sense that Reidemeister moves generally do not preserve the fact that these classes have representatives in some \(C_H(x)\). In the adequate all–\(A\) case \(x = x, \lambda\), with \(G_x\) bipartite, the link \(L\) can be oriented so that the diagram \(D\) is \textit{positive}; if this \(D\) is a closed braid diagram, then the class from \(C_{\mathbb{Z}}^{i_x, j_x, \pm 1}(x)\) is Plamenevskaya’s \textit{distinguished element} \(\psi(L) \square\).

Section \(\square\) develops the operation \(\ast\) of plumbing on Khovanov chains in order to prove the main theorem by induction, extending the all–\(A\) and all–\(B\) cases to the homogeneously adequate case in general. The idea is simple: glue two enhanced states along a state circle where their labels match so as to produce a new enhanced state; then extend linearly. Unfortunately, even simplest case of plumbing—connect sum, \(\natural\)—reveals a technical wrinkle: the differential sometimes changes the labels on the state circle along which the two plumbing factors are glued together, upsetting the compatibility required for the plumbing. The workaround is to specify, by a rule of trumps, whether the labels on the first plumbing factor override those on the second or vice-versa. The upshot is a useful identity:

\[
d(X \ast Y) = dX \ast Y + (-1)^{|\circ|} X \ast dY.
\]

Roughly, this states that plumbing \(\ast\) behaves like an exterior product followed by interior multiplication. The effect of this workaround is that the inductive proof of the main theorem, although hopefully instructive, is somewhat complicated. Section \(\square\) offers an easier, direct proof. Section \(\square\) gives two easy examples of inessential states \(x\) with nonzero \(C_R(x)\), constructs a class of \textit{non-homogeneous} essential states \(y\), asks whether \(C_R(y)\) is nonzero for \(y\) in this class, and ends with further open questions.

**Notation:** For a diagram \(Z\) of any sort and any feature \(\circ\) which may appear in such diagram, \(|\circ|_Z\) denotes the number of \(\circ\)’s in \(Z\). For example, if \(D\) is a link diagram, then \(|X|_D\) counts the crossings in \(D\).

2. **Khovanov homology of a link diagram, after Viro**

2.1. Enhanced states. Index the crossings of a (connected) link diagram \(D\) as \(c^1, \ldots, c^{|X|_D}\), and make a binary choice at each crossing: \(X \leftarrow \leftarrow_{c^1} X \leftarrow_{c^2} X\). The resulting diagram \(x \subset S^2\) is called a Kauffman state of \(D\) and consists of \([\bigcup y \text{ state circles} \big]_{\bigcup y} \) state circles joined by \(A\)– and \(B\)–labeled arcs, one from each crossing. Index the state circles of \(x\) as \(x_1, \ldots, x_{|X|_D}\), and enhance \(x\) by making a binary choice at each state circle,
The Jones polynomial $V_K(q) = q + q^3 + q^5 - q^9$ of the RH trefoil via Khovanov chains.

2.2. Grading. The writhe of an oriented diagram $D$ is $w_D = |\mathcal{X}_D| - |\mathcal{X}_D|$. For each state $x$ of $D$, let $\sigma_x := |\mathcal{X}_x| - |\mathcal{X}_x|$ and $i_x := \frac{1}{2}(w_D - \sigma_x)$. For any enhancement $X$ of $x$, define $\tau_X = |\bigcirc|X| - |\bigcirc|X| = -|\bigcirc|X| + 2 \sum a_r$ and $j_X := w_D + i_x - \tau_X$. The $R$–module $C_R(D)$ carries a bi-grading $C_R(D) = \bigoplus_{i,j} C^{i,j}_R(D)$, where each $C^{i,j}_R(D)$ is generated by the enhancements $Y$ of states $y$ of $D$ with $i = i_y =: i_Y$ and $j = j_Y$.

The Jones polynomial $V_K(q)$ of an oriented link $K$, unnormalized such that $V_{\text{unknot}}(q) = q + q^{-1}$, is given by Kauffman’s state sum formula [4, 5]. Enhancement foils this formula in order to express the Jones polynomial as the graded euler characteristic of $C_R(D)$ (cf Figure 1): 

$$V_K(q) = q^{3w_D} \sum_{\text{states } x} (-1)^{i_x} (q + q^{-1})^{\bigcirc|X|} = \sum_{\text{enhanced states } X} (-1)^{i_X} q^{i_X} = \sum_{i,j \in \mathbb{Z}} (-1)^{i_X} q^{i_X} \text{rk}(C^{i,j}(D)).$$
Blocks and zones. Associate to each state \( x \) a state graph \( G_x \) by collapsing each state circle of \( x \) to a point; maintain the \( A \)- and \( B \)-labels on the edges of \( G_x \), which come from the crossing arcs in \( x \). Cut

\[ \begin{align*}
\varepsilon \circ \tau & : C_R(D) \to C_R(D),
\end{align*} \]

Extend \( R \)-linearly to obtain the differential \( d : C_R(D) \to C_R(D) \), which has degree \((1,0)\) and obeys \( d \circ d = 0 \), giving \( C_R(D) \) the structure of a chain complex. A chain \( X \in C_R(D) \) is an \( R \)-linear combination of enhanced states from \( D \) — is called closed if \( dX = 0 \) and exact if \( X = dY \) for some \( Y \in C_R(D) \); closed chains are called cycles, exact chains boundaries. Take cycles mod boundaries to define Khovanov’s homology groups \( Kh_R(D) = \ker(d) / \text{image}(d) \), which are link–invariant.

The augmentation map \( \varepsilon : C_R(D) \to R \) is the \( R \)-linear map that sends each enhanced state \( X \) to 1. A subset \( B \subset C_R(D) \) is called primitive if, whenever \( r \in R \), \( X \in C_R(D) \), and \( rX \in B \), also \( uX \in B \) for some unit \( u \in R \). For example, a collection of enhanced states is primitive. If \( B \subset C_R(D) \) is primitive, then the projection map \( \pi_B : C_R(D) \to C_R(D) \) is the \( R \)-linear map that sends each chain \( X \) to itself when \( X \) is in the \( R \)-span of \( B \) and to 0 otherwise.

Normalization. Let \( D \) be a link diagram, \( R \) a ring with 1, and \( p \) a point on \( D \) away from crossings. For each state \( x \) of \( D \), define \( C_{R,p\to0}(x) \), \( C_{R,p\to1}(x) \) to be the subcomplexes of \( C_R(x) \) generated by those enhancements of \( x \) in which the state circle containing the point \( p \) has the indicated label. Note that \( C_R(x) = C_{R,p\to1}(x) \oplus C_{R,p\to0}(x) \) and thus

\[ C_R(D) = \bigoplus_{\text{states } x \text{ of } D} C_{R,p\to1}(x) \oplus C_{R,p\to0}(x). \]

Define the subcomplexes \( C_{R,p\to1}(D) := \bigoplus_x C_{R,p\to1}(x) \) and \( C_{R,p\to0}(D) := \bigoplus_x C_{R,p\to0}(x) \), with a shift of \( \pm 1 \) in the \( j \)-grading due to omitting the state circle containing the point \( p \) from the definitions of \( \tau \) and thus of \( j \), and with differentials obtained by restricting \( d \) as follows. If \( X \in C_{R,p\to1}(D) \), \( Y \in C_{R,p\to0}(D) \) are enhanced states, then their respective differentials in \( C_{R,p\to1}(D) \), \( C_{R,p\to0}(D) \) are

\[ \sum_{c^t} (-1)^{\mathcal{X}^t_{x}} \cdot \pi_{C_{R,p\to1}(D)} \circ d_{c^t} X, \quad \sum_{c^t} (-1)^{\mathcal{X}^t_{y}} \cdot \pi_{C_{R,p\to0}(D)} \circ d_{c^t} Y. \]

In other words, the differentials of \( X \) in \( C_{R,p\to1}(D) \), \( C_{R,p\to0}(D) \) are the same sums of enhanced states as \( dX \) in \( C_R(D) \), subject to the extra condition on the label at \( p \). The graded euler characteristics of the resulting homology groups \( Kh_{R,p\to1}(D) \), \( Kh_{R,p\to0}(D) \) both equal the normalized Jones polynomial, \( V_K(D) / (q^{-1} + q) \) .
G_x simultaneously along all its cut vertices. The subsets of x corresponding to the resulting (connected) components are called the blocks of x. If no block contains both A– and B–type crossing arcs, then x is called homogeneous [1][2]. A state x is called adequate if each crossing arc joins distinct state circles. If x is both adequate and homogeneous, it is called homogeneously adequate.

Given any state x, define x_A to be the union of all A–type crossing arcs and their incident state circles; define x_B analogously. If x is homogeneous, x_A and x_B are the respective unions of the A– and B–type blocks of x. In this case, define the A– and B–type homogeneous zones of x to be the components of x_A and x_B, respectively; call these A–zones and B–zones for short (cf Figure 3).

Define the equivalence relations ~_A, ~_B on enhanced states to be generated by X ~_A Y and X ~_B Y, respectively. Let [X]_A, [X]_B denote the associated equivalence classes. Note that [X]_A ∩ [X]_B ⊂ C^1(x,y) (x).

**Proposition 3.1.** If X enhances a homogeneous state x, then [X]_A ∩ [X]_B = {X}.

**Proof.** Suppose Y ∈ [X]_A ∩ [X]_B is an enhanced state. Deduce from X ~_A Y that X, Y are identical in x \ x_A, ie that each state circle of x_B \ x_A has the same label in X, Y. Likewise, X ~_B Y implies that X, Y are identical in x \ x_B. Hence, X and Y are identical in all of (x \ x_A) ∪ (x \ x_B) = x \ (x_A ∩ x_B). The fact that each zone of x has as many 0– (and 1–) labeled state circles in X and Y implies that innermost circles of x_A ∩ x_B are identical in X, Y; induction on height in x_A ∩ x_B completes the proof. □

Define the A–trace over R = F_2 of any enhanced state X to be tr_R X := ∑_{X ~_A Y} X Y. (The term is chosen in rough analogy with the field trace; we find no use for an analogous notion of B–trace.) To extend this notion to R = Z, suppose X ~_A Y are enhanced states; if every non-bipartite A–zone is all–1, define sgn(X → Y) to be 1 or –1 according to whether an even or odd number of X( ↔ ) Y moves take X to Y. Define the A–trace over Z of such an enhanced state X to be tr_Z X := ∑_{X ~_A Y} sgn(X → Y) Y. The notion of A–trace is generic to the main question in the following sense:

**Observation 3.2.** Over R = F_2 (resp. R = Z), every cycle X ∈ C_R (x) is a sum of traces, X = ∑_r tr_R X_r, and each component of x_A is adequate and either all–1 or (bipartite) with one 0–labeled circle.

3.2. State surfaces. Given a link diagram D on S^2 ⊂ S^3, embed the underlying link L in S^3 by inserting tiny, disjoint balls ∥C^t = C at the crossing points c^t and pushing the two arcs of D ∩ C^t to the hemispheres of ∂C^t \ S^2 indicated by the over–under information at c^t. In this setup, the states of D are precisely the closed 1–manifolds L ∩ S^2 ⊂ x ∩ (L ∪ ∂C) ∩ S^2. Given a state x in this setup, x ∩ L intersects each ∂C^t in a circle. Cap each such circle with a disk in C^t, called a crossing band, and cap the state circles of x with
The plumbed state depends explicitly on a gluing map \( g \) of two states \( x \) and spanning surfaces. Plumbing two states \( x \) and \( y \) simply involves gluing these states along a single state circle in such a way that the resulting diagram is also a state. Such a plumbing \( x \circledast y = z \) is external in the sense that it depends on a gluing map \( g : (S^2, x) \sqcup (S^2, y) \to (S^2, z) \) (cf Figure 4); the notation \( x \circledast y \) makes this dependence explicit.

The plumbed state \( z = x \circledast y \) de-plumbs as a gluing of the states \( g(x), g(y) \) along the state circle \( g(x) \cap g(y) \). Viewing the plumbing factors \( g(x), g(y) \) as subsets of \( z \) and identifying \( g(x) \) with \( x \), \( g(y) \) with \( y \) in the obvious way, denote this de-plumbing by \( z = x \ast y \). This (de–)plumbing \( z = x \ast y \) is internal in the sense that \( x \) and \( y \) are subsets of \( z \), and so no extra gluing information is needed. The distinction between internal and external plumbing, taken in analogy with internal and external free products of groups, will help with labeling; usually the distinction is immaterial and we make no comment.

If \( x \ast y = z \) is a plumbing of states, then there is an associated plumbing of link diagrams, \( D_x \ast D_y = D_z \), and an associated plumbing of the underlying links. There is also an associated plumbing of state surfaces, \( F_x \ast F_y = F_z \); here is how this works. Viewing \( x \ast y = z \) as an internal plumbing, let \( z_0 \) be the state circle comprising \( x \cap y \), and let \( U \) be the disk that \( z_0 \) bounds in the state surface \( F_z \). There is an embedded sphere \( Q \subset S^3 \) transverse to the projection sphere \( S^2 \) with \( Q \cap F_z = U \) and \( Q \cap S^3 = z_0 \); let \( B_x, B_y \) denote the (closed) balls into which \( Q \) cuts \( S^3 \), such that \( x \subset B_x, y \subset B_y \). The surfaces \( F_x := F_z \cap B_x, F_y := F_z \cap B_y \) are the state surfaces for \( x, y \), respectively, and plumbing these surfaces along \( Q \) produces \( F_x \ast F_y = F_z \).

For general interest, we briefly describe two more general notions of (de-)plumbing of spanning surfaces. First, suppose \( F \) spans a link \( K \subset S^3 \) and \( Q \subset S^3 \) is a sphere which intersects \( F \) (non–transversally) in a disk \( U = Q \cap F \). If \( B_0, B_1 \) are the (closed) balls into which \( Q \) cuts \( S^3 \) and \( F_0 = B_0 \cap F, F_1 = B_1 \cap F \), so that \( F_0 \cap F_1 = F \cap B_0 \cap B_1 = F \cap Q = U \), then the sphere \( Q \) is said to de-plumb \( F \) as \( F = F_0 \ast F_1 \).

A second, more general notion of plumbing, better suited for iteration, views a regular neighborhood of \( \text{int}(F) \) in the link complement \( S^3 \setminus K \) in terms of a line bundle \( \rho : N \to \text{int}(F) \) and allows de-plumbing

![Figure 4](image-url)

**Figure 4.** Use crossing balls \( C = \bigcup C^t \) to embed a link and its states in \((S^2 \setminus C) \cup \partial C\).
4. Plumbing Khovanov chains

The all–A state of a link diagram $D$ is always homogeneous; if this state is adequate, then $D$ is called A–adequate; B–adequacy is defined analogously. It is easy to see, recalling Figure 5, that Khovanov homology over any coefficient ring $R$ with 1 detects any A– or B–adequate state $x$, in the sense that $C_R(x)$ contains a non–exact cycle. The main theorem extends this fact in case $R = F_2$ to all homogeneously adequate states, and in case $R = Z$ (or any other ring with 1 in which 2 is not a unit) to such states whose A–zones are bipartite. The inductive key for extending in this way is the operation of plumbing on Khovanov chains.

Suppose that $z = x * y$ is a plumbing of states along a state circle $z_0 = g(x_{r_0}) = g(y_{s_0})$, and that $p_z = g(p_x) = g(p_y)$ is a point away from crossing arcs, so that $p_x \in x_{r_0}$, $p_y \in y_{s_0}$, $p_z \in z_0$. If $X$, $Y$ enhance $x$, $y$, respectively, in such a way that $p_x$ has the same label in $X$ that $p_y$ has in $Y$, then there is an enhancement $Z$ of $z = x * y$ which assigns each state circle $g(x_r) \subset g(x)$ the same label that $X$ assigns $x_r$, and which assigns each state circle $g(y_s) \subset g(y)$ the same label that $Y$ assigns $y_s$; denote this enhancement by $Z = X * Y$ and call it the (external) plumbing of the chains $X$, $Y$ by $g$. Extend $R$–linearly to view plumbing as an $R$–module isomorphism:

$$g : (C_{R,p_x \to 0}(x) \otimes C_{R,p_y \to 0}(y)) \oplus (C_{R,p_x \to 0}(x) \otimes C_{R,p_y \to 0}(y)) \to C_R(x * y),$$

$$g : X \otimes Y \to X * Y.$$
There is also an internal notion of plumbing on chains, in which $x$ and $y$ are viewed as subsets of $z$. For convenience, we take a mixed approach, viewing $x$ and $y$ as subsets of $z$ for simplicity, and using the gluing map $g : (S^2, x) \sqcup (S^2, y) \to (S^2, z)$ for relabeling. This should cause no confusion.

How does plumbing of chains, $*$, interact with the differential, $d$? Consider the simplest case, connect sum $\natural$.

4.1. **Connect sum of chains.** A state $z$ is a connect sum if there is a simple closed curve $\gamma \subset S^2$ which intersects $z$ transversally in two points, $p, b$, neither of them on crossing arcs (this implies that $p, b$ lie on the same state circle, $z_t$), such that both components of $S^2 \setminus \gamma$ contain crossing arcs. In this case, $z$ decomposes as the connect sum $x \natural y = z$, where $x \subset z$ consists $z_t$ together with all state circles and crossing arcs to one side of $\gamma$ in $S^2$, and $y \subset z$ consists of $z_t$ together with all state circles and crossing arcs to the opposite side of $\gamma$. If $D_x, D_y, D_z$ are the underlying link diagrams for $x, y, z$ then there is also a connect sum of link diagrams, $D_x \natural D_y = D_z$. Moreover, every state $z'$ of $D_z$ decomposes as a connect sum $x' \natural y' = z'$, where $x'$ is a state of $D_x$ and $y'$ is a state of $D_y$. If $Z'$ enhances $x' \natural y' = z'$, then $Z'$ restricts on $x', y'$ to enhancements $X', Y'$, whose labels match at $p$: either both are 1 or both are 0. This supplies the $R$–module isomorphism:

$$
C_R(D_x \natural D_y) = C_{R, p \to 1}(D_x \natural D_y) \oplus C_{R, p \to 0}(D_x \natural D_y) \\
\to (C_{R, p \to 1}(D_x) \otimes C_{R, p \to 1}(D_y)) \oplus (C_{R, p \to 0}(D_x) \otimes C_{R, p \to 0}(D_y)).
$$

Use this isomorphism to write each enhanced state $Z$ from $D_z = D_x \natural D_y$ as $Z = X \natural Y$. If $D_z = D_x \natural D_y$ respects orientation, then $t_Z = i_X + i_Y$, and $j_Z = j_X + j_Y + 1$ in case $X \in C_{R, p \to 1}(D_x), Y \in C_{R, p \to 1}(D_y)$ or $j_Z = j_X + j_Y - 1$ in case $X \in C_{R, p \to 0}(D_x), Y \in C_{R, p \to 0}(D_y)$.

If $X \in C_{F_2, p \to 1}(D_x), Y \in C_{F_2, p \to 1}(D_y)$, then $d(X \natural Y) = dX \natural Y + X \natural dY$. This follows straight from the definition of $d$ (cf Figure 2). The case $X \in C_{F_2, p \to 1}(D_x), Y \in C_{F_2, p \to 0}(D_y)$ is more awkward, requiring variants of the operation $\natural$ in left– and right–trumps, $\triangledown, \triangledown : C_{F_2}(D_x) \otimes C_{F_2}(D_y) \to C_{F_2}(D_x \natural D_y)$.

Suppose that $x \natural y = z$ is a connect sum of states along a state circle $z_{t_0} = x \cap y$, with $z_{t_0} \ni p$, and suppose that $X, Y$ enhance $x, y$, not necessarily with matching labels at $p$. Define $X \triangledown Y$ to be the enhancement of $x \natural y$ which assigns each state circle of $x$ the same label that $X$ assigns it, and which assigns each state circle of $y$ except possibly $z_{t_0}$, the same label that $Y$ does. Likewise, define $Y \triangledown X$ to be the enhancement of $z$ which assigns each state circle of $x$ the same label that $X$ does, and which assigns each state circle of $y$, except possibly $z_{t_0}$, the same label that $Y$ does. Thus, $X \triangledown Y$ and $Y \triangledown X$ are the enhancements of $z$ which match $X$ and $Y$ away from $z_{t_0}$; at $z_{t_0}$, the label on $z_{t_0}$ from $X$ trumps the label from $Y$ in $X \triangledown Y$, and vice-versa in $Y \triangledown X$. Whenever $X \triangledown Y$ is defined, it equals both $X \triangledown Y$ and $Y \triangledown X$. The most immediate payoff is the following identity, which holds over any $R$ with 1:

$$
d(X \triangledown Y) = dX \triangledown Y + (-1)^{|X|} X \triangledown dY,
$$

and in particular over $R = F_2$:

$$
d(X \triangledown Y) = dX \triangledown Y + X \triangledown dY.
$$

4.2. **General construction.** Let $x \ast y = z$ be a plumbing of states by a gluing map $g : x \sqcup y \to z$, and let $D_x \ast D_y = D_z$ be the associated plumbing of link diagrams. Index the crossings $c_t$ of $D_z$ so that those from $D_x$ precede those from $D_y$: $c_t = c_t^x$ for $1 \leq t \leq |X|_x$, $c_t^y = c_t^y - |X|_x$ for $1 + |X|_x \leq t \leq |X|_x + |X|_y$.

Likewise, index the state circles $z_t$ of $z$ so that those from $x$ precede those from $y$: $z_r = x_r$ for $1 \leq r \leq |X|_x$, $z_r = y_{r+1-|X|_x}$ for $|X|_x \leq r \leq |X|_x + |X|_y - 1$. Note that $z_{|X|_x} = x_{|X|_x} = y_1$. 
Let $X$, $Y$ enhance $x$, $y$, and write $X = q^{a_1} \otimes \cdots \otimes q^{a_{|\mathcal{I}|x}}$, $Y = q^{b_1} \otimes \cdots \otimes q^{b_{|\mathcal{I}|y}}$, with each $a_r, b_r \in \{0, 1\}$ according to whether the associated state circle is labeled 0 or 1, as in [2]. The plumbing of $X$ and $Y$ by $g$ is the enhancement $X \ast Y = Z$ of the state $x \ast y = z$ which matches $X$ on the state circles from $x$ and which matches $Y$ on those from $y$, if such an enhancement exists:

$$X \ast Y = \begin{cases} q^{a_1} \otimes \cdots \otimes q^{a_{|\mathcal{I}|x} - 1} \otimes q^{b_1} \otimes q^{b_2} \otimes \cdots \otimes q^{b_{|\mathcal{I}|y}} & \text{if } a_{|\mathcal{I}|x} = b_1, \\ \text{undefined} & \text{if } a_{|\mathcal{I}|x} \neq b_1. \end{cases}$$

Extend $R$–linearly to obtain the following isomorphism of $R$–modules:

$$g^\#: \left( \mathcal{C}_{R,p-1}(x) \otimes \mathcal{C}_{R,p-1}(y) \right) \oplus \left( \mathcal{C}_{R,p-0}(x) \otimes \mathcal{C}_{R,p-0}(y) \right) \rightarrow \mathcal{C}_R(x \ast y),$$

$$X \otimes Y \mapsto X \ast Y.$$ 

Recall from [1] that Khovanov homology over $\mathbb{F}_2$ detects adequate all–$A$ and all–$B$ states, in the sense that such a state has enhancements $X^\pm$ with $j_{X^+} = j_{X^-} + 2$, such that $\text{tr}_{\mathbb{F}_2} X^+$, $\text{tr}_{\mathbb{F}_2} X^−$ represent nonzero homology classes. Moreover, every homogeneously adequate state is a plumbing of adequate all–$A$ and all–$B$ states. The main theorem will follow inductively from this setup, using the interaction between plumbing and the differential, which we describe next.

### 4.3. Trumps

Let $x \ast y = z$ be a plumbing by a gluing map $g$, and let $x'$, $y'$ be arbitrary states of $D_x$, $D_y$. In an abuse of terminology and notation, define the plumbing $x' \ast y$ to be the state of $D_x \ast D_y = D_z$ whose smoothings match those of $x'$ and $y$; likewise, define the plumbing $x \ast y'$ to be the state of $D_z$ whose smoothings match those of $x$ and $y'$. In terms of the crossing ball setup from [3,2],

$$x' \ast y = \left( (x' \cup y) \setminus (x \cap y) \right) \cup (x' \cap y), \quad x \ast y' = \left( (x \cup y') \setminus (x \cap y) \right) \cup (x \cap y').$$

If $X'$ is an enhancement of $x'$ and $Y$ is an enhancement of $y$, define the left–trump plumbing $X' \overset{\diamond}{\ast} Y$ by $g$ to be the enhancement of $x' \ast y$ which assigns each state circle $x'_r \subset x'$ the same label that $X'$ assigns $x'_r$, and which assigns each state circle $y_r \subset y$, except possibly $y_1$, which need not be a state circle in $x' \ast y$, the same label that $Y$ assigns $y_r$. Likewise, if $Y'$ is an enhancement of a state $y'$ of $D_y$ and $X$ is an enhancement of $x$, the right–trump plumbing $X \overset{\diamond}{\ast} Y'$ is the enhancement of $x \ast y'$ which assigns each state circle $x_r \subset x$, except possibly $x_1$, the same label that $X$ does, and which assigns each state circle $y'_r \subset y'$ the same label that $Y'$ does. That is, $X' \overset{\diamond}{\ast} Y$ and $X \overset{\diamond}{\ast} Y'$ are the respective enhancements of $x' \ast y$ and $x \ast y'$ which match $X'$, $Y$ and $X$, $Y'$ away from $g(x) \cap g(y) = z_{|\mathcal{I}|y}$; at $z_{|\mathcal{I}|y}$, the labels from $X'$ trump the labels from $Y$ in $X' \overset{\diamond}{\ast} Y$, and the labels from $Y'$ trump those from $X$ in $X \overset{\diamond}{\ast} Y'$:

$$\overset{\diamond}{\ast} : \mathcal{C}_R(D_x) \otimes \mathcal{C}_R(D_y) \rightarrow \mathcal{C}_R(D_x \overset{\diamond}{\ast} D_y),$$

$$\overset{\diamond}{\ast} : \mathcal{C}_R(D_x \overset{\diamond}{\ast} D_y) \rightarrow \mathcal{C}_R(D_x \overset{\diamond}{\ast} D_y),$$

$$X' \otimes Y \mapsto X' \overset{\diamond}{\ast} Y,$$

$$X \otimes Y' \mapsto X \overset{\diamond}{\ast} Y'.$$

**Proposition 4.1.** If $X \ast Y$ enhances $x \ast y = z$, then $d(X \ast Y) = dX \overset{\diamond}{\ast} Y + (-1)^{|\mathcal{I}|x} X \overset{\diamond}{\ast} dY$. 


Proposition 4.3. If gluing map $g : X \to Y$, then $X * Y$ is a cycle (cf Proposition 4.1).

Observation 4.2. If $X, X' \in C_R(x)$ identical away from $x$ and $X * Y + X' * Y' \in C_R(x * y)$, then $X * Y + X' * Y'$ is also a cycle (cf Proposition 4.3).

Proof. Let $D_x, D_y, D_z$ be the link diagrams for $x, y, z$. Index the crossings as in §4.2, and again let $|X|^i_x$ denote the number of crossings $c_i^x$ in $D_x$ with $r < t$ at which $x$ has an $A$-smoothing. Now:

$$d(X * Y) = \sum_{i=1}^{|D_x|} (-1)^{|c^x_i|} d_{c^x_i} (X * Y) + \sum_{i=1+|D_x|} (-1)^{|c^y_i|} d_{c^y_i} (X * Y)$$

$$= \sum_{i=1}^{|D_x|} (-1)^{|c^x_i|} d_{c^x_i} X * Y + (-1)^{|c^y_i|} \sum_{i=1}^{|D_y|} (-1)^{|c^y_i|} X * d_{c^y_i} Y$$

$$= dX * Y + (-1)^{|D_y|} X * dY. \quad \Box$$

4.4. Cycles. Let $\kappa : (S^2, x) \sqcup (S^2, \bigcirc) \to (S^2, x)$ be a gluing map of a state $x$ with the state $\bigcirc$ of the trivial diagram, so that $x = x * \bigcirc$ and $\kappa|_x = \mathbb{1}_x$. Suppose $p$ is a point on the state circle $\kappa^{-1}(\bigcirc) \subset x$, and $X \in C_{R,p^{-1}}(x)$, $X' \in C_{R,p^{-1}}(x)$ are chains. Say that $X$ and $X'$ are identical away from $p$ if $X * \bigcirc \bigcirc = X'$, or equivalently if $X' * \bigcirc \bigcirc = X$.

Observation 4.2. If $X, X' \in C_R(x)$ are identical away from $p$, and if $x * y = z$ is a plumbing of states by a gluing map $g : x \sqcup y \to z$ with $x * g^{-1}(y) = x * \kappa^{-1}(\bigcirc) \ni p$, then $X * \bigcirc \bigcirc = X' * \bigcirc \bigcirc$ for any $Y \in C_R(y)$.

Proposition 4.3. If $x * y$ is a plumbing of states and $X, X', Y + Y'$ are cycles, with $X \in C_{R,p^{-1}}(x)$, $X' \in C_{R,p^{-1}}(x)$ identical away from $p \in x \sqcup y$ and $Y \in C_{R,p^{-1}}(y)$, $Y' \in C_{R,p^{-1}}(y)$, then $X * Y + X' * Y'$ is also a cycle.

Proof. Since $X$ and $X'$ are identical away from $p$, Observation 4.2 implies that $X * \bigcirc \bigcirc dY' = X' * \bigcirc \bigcirc dY'$. This and Proposition 4.3 now yield:

$$d(X * Y + X' * Y') = dX * Y + dX' * Y' + (-1)^{|D_y|} (X * \bigcirc \bigcirc dY + X * \bigcirc \bigcirc dY')$$

$$= (-1)^{|D_y|} (X * dY + X * dY')$$

$$= (-1)^{|D_y|} (X * d(Y + Y')) = 0. \quad \Box$$
Observation 4.4. If \( x \ast \{ \} = x \) is a plumbing by \( \kappa \) as above, with \( \kappa^{-1}(\{ \}) \cap x_A = \emptyset \) and \( X \in \mathcal{C}_R(x) \), then:
\[
d(X \ast \{ \}) = dX \ast \{ \} \quad \text{and} \quad d(X \ast \{ \}) = dX \ast \{ \}.
\]
In particular, if \( X \) is a cycle with \( \kappa^{-1}(\{ \}) \cap x_A = \emptyset \), then \( X \ast \{ \}, X \ast \{ \} \) are also cycles.

The point is that, because the state circle \( \kappa^{-1}(\{ \}) \) is incident to no \( A \)-type crossing arcs, every enhanced state \( Y \) with \( \pi_{RY} \circ dX \neq 0 \) contains \( \kappa^{-1}(\{ \}) \) and assigns it the same label that \( X \) does.

Proposition 4.5. Let \( z = x \ast y \) be a plumbing of states such that \( x \cap y \subset y \setminus y_A \). If \( X, Y \) enhance \( x, y \) such that both \( \text{tr}_R X \) and \( \text{tr}_R Y \) are cycles and \( X \ast Y \) is defined, then \( \text{tr}_R (X \ast Y) \) is also a cycle.

Proof. Use the indexing from 4.3 so that the state circles in \( z \) from \( x \) precede those from \( y \), with \( x \cap y = x|\{ \} | y = y_1 \). The cycle condition on \( \text{tr}_R X \), \( \text{tr}_R Y \) implies that each component of \( x_A, y_A \) is adequate and contains at most one 0–labeled circle, by Observation 3.2. Write \( X = \bigotimes_{r=1}^{l} q^r \) and \( Y = \bigotimes_{r=1}^{l} q^{b_r} \). Let \( Y' := 1 \otimes \bigotimes_{r=2}^{l} q^{r} \) and \( Y'' := q \otimes \bigotimes_{r=2}^{l} q^{b_r} \), so that \( Y', Y'' \) are identical away from \( y_1 \), and one of \( Y', Y'' \) equals \( Y \). Thus, one of one of \( \text{tr}_R Y', \text{tr}_R Y'' \) equals \( \text{tr}_R Y \), which is a cycle; the assumption that \( y_1 \subset y \setminus y_A \) implies that the other is also a cycle, by Observation 4.4. Write \( \text{tr}_R X = X' \ast X'' \), where \( X' \in \mathcal{C}_{R,p \to 1}(x) \) and \( X'' \in \mathcal{C}_{R,p \to 0}(x) \) with \( p \in x|\{ \} | y \). Proposition 4.3 now implies that \( \text{tr}_R (X \ast Y) = X' \ast \text{tr}_R Y' + X'' \ast \text{tr}_R Y'' \) is a cycle. \( \square \)

4.5. Boundaries. Consider the following chains from Figure 4.

All four are closed; are they exact? The first three cannot be exact since their \( B \)-type crossing arcs, if there are any, join distinct 0–labeled circles; this holds over both \( R = \mathbb{F}_2, \mathbb{Z} \). To see that \( X := \bigotimes_{r=1}^{l} q^r \) is not exact over \( \mathbb{F}_2, \mathbb{Z} \), apply Proposition 3.1 and the homogeneity of \( x := \bigotimes_{r=1}^{l} q^r \) to see that \( [X]_A \cap [X]_B = \{ X \} \). Since each \( B \)-type crossing arc in \( X \) (and in its two \( B \)-equivalent enhanced states) joins distinct state circles, at most one of them labeled 1, the image of the map \( \varepsilon \circ \pi_{[X]_B} \circ d \) is in \( 2R \). This implies that \( X \) cannot be exact:

Proposition 4.6. If \( X \) enhances a state \( x \) of a diagram \( D \) such that \( [X]_A \cap [X]_B = \{ X \} \) (eg if \( x \) is homogeneous), and if \( \varepsilon \circ \pi_{[X]_B} \circ d : \mathcal{C}_R(D) \to 2R \) with \( 2R \subseteq R \), then \( \text{tr}_R X \) is not exact.

Proof. If \( \text{tr}_R X \) were exact, say \( \text{tr}_R X = dY, Y \in \mathcal{C}_R(D) \), then \( 2 \) would be a unit in \( R \), contrary to assumption:
\[
1 = \varepsilon(X) = \varepsilon \circ \pi_{[X]_B} \circ \text{tr}_R X = \varepsilon \circ \pi_{[X]_B} \circ d(Y) \in 2R. \quad \square
\]

Thus, \( \text{tr}_R X = X := \bigotimes_{r=1}^{l} q^r \) is not exact because \( [X]_A \cap [X]_B = \{ X \} \) and \( \varepsilon \circ \pi_{[X]_B} \circ d : \mathcal{C}_R(D) \to 2R \). Plumbing preserves homogeneity, which implies the former property. Plumbing also preserves the latter property:

Proposition 4.7. If \( X \ast Y \) is a plumbing of chains, where \( X, Y \) enhance \( x, y \) with \( \varepsilon \circ \pi_{[X]_B} \circ d : \mathcal{C}_R(D_x) \to 2R, \varepsilon \circ \pi_{[Y]_B} \circ d : \mathcal{C}_R(D_y) \to 2R \), and if \( x \ast y = z \) is a plumbing of states, then there are \( X' \in [X]_B, Y' \in [Y]_B \) such that \( X' \ast Y' := Z \) satisfies \( \varepsilon \circ \pi_{[Z]_B} \circ d : \mathcal{C}_R(D_Z) \to 2R \).

Proof. Deduce from \( \varepsilon \circ \pi_{[X]_B} \circ d : \mathcal{C}_R(D_x) \to 2R \) that \( X \) has no more than one 1–labeled circle in any \( B \)-zone of \( x \) (cf Figure 2); likewise for \( Y, y \). Let \( z_0 \) denote the state circle \( x \cap y \), and let \( p \) be a point on \( z_0 \) away from crossings. Let \( X' = X \) and \( Y' = Y \), unless \( z_0 \) is in \( B \)-zones of both \( x, y \) such that both zones
contain a 1–labeled circle. In that case, choose \( X' \in [X]_B \cap C_{R,p \to 1}(x) \) and \( Y' \in [Y]_B \cap C_{R,p \to 1}(y) \), so that \( X' \star Y' := Z \in C_{R,b \to 1}(z) \). In all cases, we have chosen \( X', Y' \) so that, if \( z_0 \) is in a \( B \)-zone of \( z \), then this zone contains at most one 1–labeled circle in \( Z \).

We claim that these choices for \( X' \star Y' =: Z \) always satisfy \( \varepsilon \circ \pi_{[Z]} \circ d : C_R(D_z) \to 2R \). Suppose \( W \) enhances a state \( w \) of \( D_z \) such that \( \pi_{[Z]} \circ d(W) \neq 0 \). The state \( w \) must differ from \( z \) at a single crossing. Either this crossing is from \( y \), and \( w = x \star y' \) for a state \( y' \) of \( D_y \); or the crossing is from \( x \), and \( w = x' \star y \) for a state \( x' \) of \( D_z \); wlog assume the latter. Then, since the \( B \)-zone of \( y \) containing \( z_0 \), if there is one, has no 1–labeled circles other than \( z_0 \), \( W \) assigns each circle of \( y \setminus z_0 \) the same label that \( Y' \) does; ie \( W = X'' \circ Y' \) for some enhancement \( X'' \) of \( x' \). Thus, \( \pi_{[Z]} \circ d(X'' \circ Y') = \pi_{[Z]} \circ (dX'' \circ Y') \), giving:

\[
\varepsilon \circ \pi_{[Z]} \circ d(W) = \varepsilon \circ \pi_{[Z]} \circ (dX'' \circ Y') = \varepsilon \circ \pi_{[Z]} \circ d(X'') \in 2R.
\]

\[\square\]

4.6. Inductive proof of the main theorem. Two examples will show how plumbing is used to build up the main theorem’s nonzero cycles. First, with either \( R = F_2 \) or \( R = \mathbb{Z} \), consider:

\[
X_1 = \begin{array}{c}
\bullet \bullet \\
\end{array}, \quad X_2 = \begin{array}{c}
\bullet \\
\end{array}, \quad X_3 = \begin{array}{c}
\bullet \\
\end{array}.
\]

Each of \( \text{tr}_R X_1 = X_1 \), \( \text{tr}_R X_2 = X_2 \), \( \text{tr}_R X_3 = X_3 \) is a cycle; also each \( [X_r]_A \cap [X_r]_B = \{ X_r \} \), and \( \varepsilon \circ \pi_{[X_r]} \circ d \) maps to \( 2R \); Proposition 4.6 implies that \( \text{tr}_R X_1 \), \( \text{tr}_R X_2 \), \( \text{tr}_R X_3 \) represent nonzero homology classes. Propositions 4.5, 4.7 further imply that

\[
\text{tr}_R(X_1 \star X_2) = X_1 \star X_2 = \begin{array}{c}
\bullet \\
\end{array}.
\]

also represents a nonzero homology class, as does

\[
\text{tr}_R(X_1 \star X_2 \star X_3) = \begin{array}{c}
\bullet \\
\end{array}.
\]

While the previous example holds over both \( \mathbb{Z} \), \( F_2 \), the next example works over \( F_2 \) only. Let

\[
Y_1 = \begin{array}{c}
\bullet \\
\end{array}, \quad Y_2 = \begin{array}{c}
\circ \\
\end{array}, \quad Y_3 = \begin{array}{c}
\circ \\
\end{array}.
\]

By the same reasoning as the last example, \( \text{tr}_{F_2} Y_1 = Y_1 \), \( \text{tr}_{F_2} Y_2 = Y_2 + \begin{array}{c}
\circ \\
\end{array} \), and \( \text{tr}_{F_2} Y_3 = Y_3 + \begin{array}{c}
\circ \\
\end{array} \) represent nonzero homology classes, as do

\[
\text{tr}_{F_2}(Y_1 \star Y_2) = \begin{array}{c}
\bullet \\
\end{array} + \begin{array}{c}
\circ \\
\end{array} + \begin{array}{c}
\circ \\
\end{array}, \quad \text{and}
\]

\[
\text{tr}_{F_2}(Y_1 \star Y_2 \star Y_3) = \begin{array}{c}
\bullet \\
\end{array} + \begin{array}{c}
\circ \\
\end{array} + \begin{array}{c}
\circ \\
\end{array} + \begin{array}{c}
\circ \\
\end{array} + \begin{array}{c}
\circ \\
\end{array}.
\]

Both proofs of the main theorem will establish the following, which is stronger than the version from \[\text{[]}\]
Main theorem. If \( z \) is a homogeneously adequate state, then for any point \( p \) away from crossing arcs, \( z \) has enhancements \( Z^- \in C_{R,p \to 0}(x) \), \( Z^+ \in C_{R,p \to 1}(z) \), identical away from \( p \), such that both \( tr_R Z^\pm \) represent nonzero homology classes. If also \( G_{x_A} \) is bipartite, then both \( tr_Z Z^\pm \) represent nonzero homology classes.

Proof. We argue by induction on the number of zones in \( z \) that \( z \) has enhancements \( Z^+ \in C_{R,p \to 0}(z) \), \( Z^- \in C_{R,p \to 1}(z) \), identical away from \( p \), such that both \( A \)-traces \( tr_R Z^\pm \) are cycles, and 2\( R \) contains the images of \( \varepsilon \circ \pi_{[Z^\pm]} B \circ d \). The last condition implies that neither \( Z^\pm \) is exact, by Proposition 4.6.

The base case checks out. For the inductive step, de-plumb \( z = x \ast y \) by a gluing map \( g : x \sqcup y \to z \) with \( g(x_{r_0}) = g(y_{s_0}) = z_{t_0} \), where either \( x_{r_0} \subset x \setminus x_A \) or \( y_{s_0} \subset y \setminus y_A \).

If \( p \in z_{t_0} \), then apply the inductive hypothesis to \( x \) and \( y \) to obtain \( X^+ \in C_{R,p \to 0}(x) \), \( X^- \in C_{R,p \to 1}(x) \), identical away from \( p \), and \( Y^+ \in C_{R,p \to 0}(y) \), \( Y^- \in C_{R,p \to 1}(y) \), identical away from \( p \), such that all four of \( tr_R X^\pm \), \( tr_R Y^\pm \) are cycles, and such that 2\( R \) contains the images of all four of \( \varepsilon \circ \pi_{[X^\pm]} B \circ d \), \( \varepsilon \circ \pi_{[Y^\pm]} B \circ d \). Let \( Z^- := X^- \ast Y^- \), \( Z^+ := X^+ \ast Y^+ \). Then \( Z^- \), \( Z^+ \) are identical away from \( z_{t_0} \); \( tr_R Z^- \), \( tr_R Z^+ \) are cycles; and 2\( R \) contains the images of \( \varepsilon \circ \pi_{[Z^-]} B \circ d \), \( \varepsilon \circ \pi_{[Z^+]} B \circ d \).

Assume instead \( p \notin z_{t_0} \); wlog \( p \in x \setminus y \). Apply the inductive hypothesis to obtain \( X^+ \in C_{R,p \to 0}(x) \), \( X^- \in C_{R,p \to 1}(x) \), identical away from \( p \), such that \( tr_R X^\pm \) are cycles and 2\( R \) contains the images of \( \varepsilon \circ \pi_{[X^\pm]} B \circ d \). Also let \( b \) be a point in \( z_{t_0} \), and apply the inductive hypothesis to obtain \( Y^+ \in C_{R,b \to 0}(y) \), \( Y^- \in C_{R,b \to 1}(y) \), identical away from \( p \), such that \( tr_R Y^\pm \) are cycles and 2\( R \) contains the images of \( \varepsilon \circ \pi_{[Y^\pm]} B \circ d \). Since \( X^\pm \) are identical away from \( p \), they assign the same label to \( z_{t_0} \). If this label is 1, then let \( Z^+ := X^+ \ast Y^+ \) and \( Z^- := X^- \ast Y^- \); if it is 0, then define \( Z^+ := X^+ \ast Y^- \) and \( Z^- := X^- \ast Y^+ \). Either way, \( Z^\pm \) are identical away from \( p \), \( tr_R Z^\pm \) are cycles, and 2\( R \) contains the images of \( \varepsilon \circ \pi_{[Z^\pm]} B \circ d \).

5. Direct proof of the main theorem.

Throughout this section, fix a homogeneously adequate state \( x \) of a link diagram \( D \). Here is the plan. Several propositions will establish two conditions on enhancements \( X \) of \( x \) which together guarantee that \( tr_Z X \) represents a nonzero homology class: each \( A \)-zone must contain at most one 0–labeled circle, and each \( B \)-zone must contain at most one 1–labeled circle. These conditions also suffice over \( R = Z \) when \( G_{x_A} \) is bipartite. An explicit construction will then fashion enhancements \( X^\pm \) of \( x \) which satisfy these conditions, with \( j(X^+) = j(X^-) + 2 \).

Proposition 5.1. If \( X \) enhances \( x \) with at most one 0–labeled circle in each \( A \)-zone, then \( d( tr_Z X ) = 0 \). Further, \( d( tr_Z X ) = 0 \) if every non-bipartite \( A \)-zone of \( X \) is all-1.

Proof. Let \( c \) be an arbitrary crossing of the link diagram \( D \); it will suffice to show that \( d_c( tr_R X ) = 0 \) for \( R = F_2 \), \( R = Z \). Assume that \( x \) has an \( A \)-type crossing arc at \( c \), or else we are done. Partition the enhanced states in \( [X]_A \) as follows. Let one equivalence class consist of all enhancements for which both state circles incident to \( c \) are labeled 1; \( d_c(X') = 0 \) for each \( X' \) in this class. Partition any remaining enhanced states in \( [X]_A \) into pairs \( \{ X_s, X_{s'} \} \) which are identical except with opposite labels on the two state circles incident to \( c \). For each such pair, \( d_c(X_s) = d_c(X_{s'}) \) over both \( R = F_2 \) and \( R = Z \); also, \( \text{sgn}(X \to X_s) = -\text{sgn}(X \to X_{s'}) \) in case \( R = Z \). Conclude in both cases:

\[
d(tr_Z X) = \sum_{X' \in [X]_A} \text{sgn}(X \to X') d_c X' = \sum_{\text{pairs } \{ X_s, X_{s'} \}} \text{sgn}(X \to X_s) (d_c X_s - d_c X_{s'}) = 0.
\]
Figure 9. Constructing $X^\pm$ from the state $x$, left.

Proposition 5.2. If $X$ enhances $x$ so that no $B$–zone contains more than one 1–labeled circle, then

$$\varepsilon \circ \pi_{[X]_B} \circ d : C_R(D) \to 2R.$$

Proof. Let $Y$ be any enhanced state. If $\pi_{[X]_B} \circ d(Y) \neq 0$, then the underlying state $y$ of $Y$ must differ from $x$ at precisely one crossing, $c$, at which $y$ must have an $A$–smoothing with one incident state circle, which must be labeled 0 in $Y$ because each $X' \in [X]_B$ has at most one 1–labeled circle in each $B$–zone. Thus, $\pi_{[X]_B} \circ d_c(Y) = X_s + X'_s$, where $X_s, X'_s$ are identical except with opposite labels on the two state circles of $x$ incident to $c$. In particular, $\varepsilon \circ \pi_{[X]_B} \circ d(Y) = 1 + 1 \in 2R$. \hfill \Box

Proposition 5.3. If $X$ enhances $x$ with at most one 0–labeled state circle in each $A$–zone and at most one 1–labeled state circle in each $B$–zone, then $tr_{F_2} X$ represents a nonzero homology class. Further, if every $A$–zone containing a 0–labeled circle in $X$ is bipartite, then $tr_Z X$ represents a nonzero homology class.

Proof. Such $tr_{F_2} X$, $tr_Z X$ are cycles by Proposition 5.1. If $tr_R X$ were exact over $R = F_2$ or $R = \mathbb{Z}$, say $tr_R X = dY$, then Propositions 3.1 and 5.2 would imply that 2 is a unit in $R$: $1 = \varepsilon(X) = \varepsilon \left( \sum_{X' \in [X]_A \cap [X]_B} X' \right) = \varepsilon \circ \pi_{[X]_B} (tr_R X) = \varepsilon \circ \pi_{[X]_B} \circ dY \in 2R$. \hfill \Box

Putting all this together proves that Khovanov homology over $F_2$ detects every homogeneously adequate state $x$ in two distinct gradings, $(i_x, j_x \pm 1)$, where

$$j_x = w_D + i_x + |O|_{x_B} - |O|_{x_A} + \#(B$–zones of $x) - \#(A$–zones of $x)$.

Main theorem. If $x$ is a homogeneously adequate state, then for any point $p$ away from crossing arcs, $x$ has enhancements $X^- \in C_{R,p \to 1}(x)$, $X^+ \in C_{R,p \to 0}(x)$, identical away from $p$, such that both $tr_{F_2} X^\pm$ represent nonzero homology classes. If also $G_{x_A}$ is bipartite, then both $tr_Z X^\pm$ represent nonzero homology classes.
A state need not be essential in order for $C_{F_2}(x)$ to be nonzero; indeed, $x$ need not even be adequate. Consider two examples. First, for the trivial diagram of two components, $\bigcirc \bigcirc$, the homology groups are

$$Kh_{F_2}^{0,-2} = F_2 \cdot \bigcirc \bigcirc, \quad Kh_{F_2}^{0,0} = F_2 \cdot \bigcirc \bigcirc \oplus F_2 \cdot \bigcirc \bigcirc, \quad Kh_{F_2}^{0,2} = F_2 \cdot \bigcirc \bigcirc.$$ 

Now perform a Reidemeister–2 (R2) move to get the connected diagram $\bigcirc \bigcirc$. Each of the four homology generators can still be taken to be the $\lambda$-trace of an enhancement of a single state, namely $\bigcirc$ or $\bigcirc$. Yet, these states are not essential, since their state surfaces are connected and span a split link.

Second, consider the enhancement $X = \bigcirc \bigcirc$ of the state $x = \bigcirc \bigcirc$ of the diagram $D = \bigcirc \bigcirc$ (cf Figure 10); $X$ is a cycle with any coefficients. Moreover, $X$ is not exact unless 2 is a unit in $R$, as $\varepsilon \circ \pi_B \circ d \equiv 0$ over $F_2$, where $B$ is an $F_2$–basis for $C_{F_2}^{0,-2}(D)$ (cf Figure 10).

Here is an idea for extending the main theorem: establish a class of essential states which are nonzero in two distinct $j$–gradings in Khovanov homology, say over $F_2$—for simplicity, insist that the initial class must consist only of checkerboard states—and then aim to extend by plumbing. The easiest such class of checkerboard states consists of those which are alternating; plumbing these gives the adequate homogeneous states. To construct a new (non-alternating) essential checkerboard state, consider any (non-alternating) link diagram $D$ which admits no $n \to 0$ wave moves. This means that, whenever $\alpha \subset D$ is a smooth arc whose endpoints are away from crossings and whose interior contains $n$ underpasses and no overpasses, or vice-versa, every arc $\beta \subset S^2$ with the same endpoints as $\alpha$ intersects $K$ in its interior. Construct either checkerboard surface $F$ for $D$, and replace each of its half-twist crossing bands with a full-twist band in the same sense. (Any band with at least two half-twists in the same sense suffices.)

The resulting link diagram $D'$ has twice as many crossings as $D$, and the resulting surface $F'$ is an essential, two–sided checkerboard surface with the same euler characteristic as $F$. (To prove essentiality, use Menasco’s crossing ball structures, hypothesize a compression disk $\Delta$ for $F'$ which intersects the crossing ball structure minimally, characterize the outermost disks of $\Delta \setminus (S^2 \cup C)$, and observe that the only viable configuration
for a height one component of $\Delta \cap (S^2 \cup C)$ implies that $D$ admitted an $n \to 0$ wave move, contrary to assumption.) Does Khovanov homology detect the essential checkerboard states from this construction?

The simplest non-alternating diagram admitting no $n \to 0$ wave moves is a 4–crossing diagram of the trefoil. Following Figure 11, construct an 8–crossing diagram from this one in the manner just described, and consider $X = \sum_{n=0}^3 c_n \otimes \tau$, with $\tau_R X = \sum_{n=0}^3 c_n \otimes \tau$, and $Y = \sum_{n=0}^3 c_n \otimes \tau$. Both are cycles over $R = F_2$ and $R = \mathbb{Z}$, but exactness is not so easy. When considering the exactness of a cycle $\tau_R Z$ from an enhancement $Z$ of an adequate homogeneous state, it sufficed to consider $\pi[Z] \circ d$. In the case of $X$ from above, considering the map $\pi_{C_2(X)} \circ d$ proves that $\tau_{Z} X$ is not a cycle; yet this computation proves nothing regarding $\tau_{F_2} X$ or $Y$. We leave it as an open question whether this construction yields states $x$ with $C_{F_2}(x) \cap \ker(d) \not\subset \text{image}(d)$, even in this simplest example. We also ask:

**Question 6.1.** If $x$ is an essential state, does $C_{F_2}(x)$ always contain a nonzero homology class?

**Question 6.2.** Is there a general method for distinguishing those Khovanov homology classes that correspond to essential states from those that do not?

**Question 6.3.** Does every link have a diagram with an essential state? A homogeneously adequate state?

References

[1] P. Bartholomew, S. McQuarrie, J. Purcell, K. Weser, Volume and geometry of homogeneously adequate knots., J. Knot Theory Ramifications 24 (2015), no. 8, 1550044, 29pp.
[2] P.R. Cromwell, Homogeneous links, J. London Math. Soc. (2) 39 (1989), no. 3, 535-552.
[3] D. Gabai, The Murasugi sum is a natural geometric operation, Low-dimensional topology (San Francisco, Calif., 1981), 131-143, Contemp. Math., 20, Amer. Math. Soc., Providence, RI, 1983.
[4] V.F.R. Jones, A polynomial invariant for knots via von Neumann algebras, Bull. Amer. Math. Soc. (N.S.) 12 (1985), no. 1, 103-111.
[5] L.H. Kauffman, State models and the Jones polynomial, Topology 26 (1987), no. 3, 395-407.
[6] M. Khovanov, A categorification of the Jones polynomial, Duke Math. J. 101 (2000) 359-426.
[7] W. Menasco, Closed incompressible surfaces in alternating knot and link complements, Topology 23 (1984), no. 1, 37-44.
[8] K. Murasugi, On a certain subgroup of the group of an alternating link, Amer. J. Math. 85 1963 544-550.
[9] M. Ozawa, Essential state surfaces for knots and links, J. Aust. Math. Soc. 91 (2011), no. 3, 391-404.
[10] O. Plamenevskaya, Transverse knots and Khovanov homology, Math. Res. Lett. 13 (2006), no. 4, 571-586.
[11] O. Viro, Khovanov homology, its definitions and ramifications, Fund. Math. 184 (2004), 317-342.