Orderable groups with Engel-like conditions

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Abstract. Let $x$ be an element of a group $G$. For a positive integer $n$ let $E_n(x)$ be the subgroup generated by all commutators $\ldots[[y,x],x],\ldots,x]$ over $y \in G$, where $x$ is repeated $n$ times. There are several recent results showing that certain properties of groups with small subgroups $E_n(x)$ are close to those of Engel groups. The present article deals with orderable groups in which, for some $n \geq 1$, the subgroups $E_n(x)$ are polycyclic. Let $h,n$ be positive integers and $G$ an orderable group in which $E_n(x)$ is polycyclic with Hirsch length at most $h$ for every $x \in G$. It is proved that there are $(h,n)$-bounded numbers $h^*$ and $c^*$ such that $G$ has a finitely generated normal nilpotent subgroup $N$ with $h(N) \leq h^*$ and $G/N$ nilpotent of class at most $c^*$.

1. Introduction

A group $G$ is called an Engel group if for every $x,y \in G$ the equation $[y,x,x,\ldots,x] = 1$ holds, where $x$ is repeated in the commutator sufficiently many times depending on $x$ and $y$. Throughout the paper, we use the left-normed simple commutator notation $[a_1,a_2,a_3,\ldots,a_r] = [...[[a_1,a_2],a_3],\ldots,a_r]$. The long commutators $[y,x,\ldots,x]$, where $x$ occurs $i$ times, are denoted by $[y,i,x]$. An element $x \in G$ is $n$-Engel if $[y,n,x] = 1$ for all $y \in G$. A group $G$ is $n$-Engel if $[y,n,x] = 1$ for all $x,y \in G$. Given $x \in G$, the subgroup $E_n(x)$ is the one generated by all elements of the form $[y,n,x]$ where $y$ ranges over $G$. Note that $E_n(x)$ is not the same as the more familiar subnormal subgroup $[G,n,x] = [[G,n-1,x],x]$. There are several recent results showing that certain properties of groups with small subgroups $E_n(x)$ are close to those of Engel groups (see for instance [3, 4, 10]). The present article

2010 Mathematics Subject Classification. 20F60, 20F45.

Key words and phrases. Orderable groups, polycyclic groups, Engel groups.

This research was supported by FAPDF and CNPq-Brazil.
deals with orderable groups. A group $G$ is called orderable if there exists a full order relation $\leq$ on the set $G$ such that $x \leq y$ implies $axb \leq ayb$ for all $a, b, x, y \in G$, i.e. the order on $G$ is compatible with the product of $G$. Kim and Rhemtulla proved that any orderable $n$-Engel group is nilpotent ([5], see also [7]). In the present article we consider orderable groups $G$ such that the subgroup $E_n(x)$ is polycyclic for each $x \in G$. Recall that a group is polycyclic if and only if it admits a finite subnormal series all of whose factors are cyclic. The Hirsch length $h(K)$ of a polycyclic group $K$ is the number of infinite factors in the subnormal series.

Our aim here is to prove the following theorem.

**Theorem 1.1.** Let $h, n$ be positive integers and $G$ an orderable group in which $E_n(x)$ is polycyclic with $h(E_n(x)) \leq h$ for every $x \in G$. There are $(h, n)$-bounded numbers $h^*$ and $c^*$ such that $G$ has a finitely generated normal nilpotent subgroup $N$ with $h(N) \leq h^*$ and $G/N$ nilpotent of class at most $c^*$.

Note that if $h = 0$, the proof shows that $N = 1$ and our result becomes the theorem of Kim and Rhemtulla.

One tool used in the proof of Theorem 1.1 deserves a special mention. A well-known theorem of Malcev states that a soluble group of automorphisms of a polycyclic-by-finite group is polycyclic [8]. We require the following quantitative variation of Malcev’s theorem.

**Lemma 2.1.** Let $G = H \langle a \rangle$ where $H$ is a nilpotent of class $c$ normal subgroup and $a$ is an $n$-Engel element. Then $G$ is nilpotent with class at most $cn$.
We observe that $K_i = 0$ for $i \in \mathbb{N}$ and it follows that $K_i \leq Z_i(G)$ for $i = 1, 2, \ldots, n$. Therefore $K \leq Z_n(G)$. Passing to the quotient $G/Z_n(G)$ and using induction on $c$ we deduce that $G$ is nilpotent with class at most $cn$. \hfill \Box

We write $\gamma_i(G)$ for the $i$th term of the lower central series of a group $G$.

**Lemma 2.2.** For any positive integers $c, n$ there exists an integer $f = f(c, n)$ with the following property. Let $G = H \langle a \rangle$ where $H$ is a nilpotent of class $c$ normal subgroup. Then $\gamma_f(G) \leq E_n(a)$.

**Proof.** Fix $n \geq 1$ and use induction on $c$. If $H$ is abelian, we obviously have $\gamma_{n+1}(G) \leq E_n(a)$ and so it is enough to choose $f = n+1$. Assume that $c \geq 2$ and let $Z = Z(H)$. By induction there exists a bounded number $s$ such that $\gamma_s(G) \leq ZE_n(a)$. Let $E = E_n(a) \cap Z\gamma_s(G)$. So $ZE$ is normal in $G$ and $\gamma_s(G) \leq ZE$. Set $Z_0 = ZE$ and for $i = 0, 1, \ldots, s-1$ let $Z_i$ denote the full inverse image of $Z_i(G/Z_0)$. Further, for $i = 0, 1, \ldots, s-1$ we set $G_i = Z_i\langle a \rangle$. It is clear that $Z_{s-1} = G_{s-1} = G$.

Since $Z$ is abelian, $[Z, n] \leq E$. We observe that $Z$ and $E$ are commuting $a$-invariant subgroups and so $[Z_0, n] \leq E$. Let $T$ be the normal closure of $[Z_0, n]$ in $G_0$. It is clear that $T \leq E$. Since the image of $a$ in $G_0/T$ is $n$-Engel, Lemma 2.1 implies that there exists a bounded number $k$ such that $G_0/T$ is nilpotent with class at most $k-1$ and so $\gamma_k(G_0) \leq E$.

By induction on $i$ we will show that there exists a bounded number $k_i$ such that $\gamma_{k_i}(G_i) \leq E$. Once this is done, we will simply set $f = k_{s-1}$. Assume that for some $j \leq s - 1$ there exists $k_j$ with the property that $\gamma_{k_j}(G_j) \leq E$. If $j = s - 1$ we have nothing to prove so we suppose that $j \leq s - 2$. Since $G_{j+1}$ normalizes $G_j$, it follows that $\gamma_{k_j}(G_j)$ is normal in $G_{j+1}$. Recall that $\gamma_s(G) \leq G_0$. It follows that the image of $a$ in $G_{j+1}/\gamma_{k_j}(G_j)$ is $(s+k_j)$-Engel, whence by Lemma 2.1 the factor-group $G_{j+1}/\gamma_{k_j}(G_j)$ is nilpotent with bounded class, say $k_{j+1}$. We conclude that $\gamma_{k_{j+1}}(G_{j+1}) \leq E$. This completes the proof. \hfill \Box

The next lemma is rather obvious so the proof is omitted.

**Lemma 2.3.** Let $G$ be a finitely generated nilpotent group and $H$ a subgroup of finite index in $G$. Then $H'$ has finite index in $G'$.
Lemma 2.4. Let $G$ be a finitely generated nilpotent group and $\phi$ an automorphism of $G$ such that $[G, \phi]$ has finite index in $G$. Then $[G, \phi, \phi]$ has finite index in $G$ as well.

Proof. We use induction on the nilpotency class of $G$. Suppose first that $G$ is abelian and let $m$ be a positive integer such that $G^m \leq [G, \phi]$. Then $[G, \phi, \phi]$ contains $[G^m, \phi] = [G, \phi]^m$ which has finite index in $[G, \phi]$.

Now suppose that $G$ is non-abelian and both subgroups $[G, \phi, \phi]$ and $[G, \phi, \phi]'G'$ have finite index in $G'$. By Lemma 2.3, $[G, \phi, \phi]' = ([G, \phi, \phi]Z(G))'$ has finite index in $G'$. This implies that $[G, \phi, \phi]$ has finite index in $[G, \phi, \phi]G'$, whence the lemma follows. □

Corollary 2.5. Let $G = H(a)$ be a nilpotent group with a normal torsion-free subgroup $H$ of Hirsch length $h$. Then $G$ is nilpotent with $h$-bounded class.

Proof. We assume that $h \geq 1$. It is clear that $H$ has nilpotency class at most $h - 1$. In view of Lemma 2.1 we need to show that $a$ is $n$-Engel for some $h$-bounded number $n$. Lemma 2.4 implies that whenever $[H, i \cdot a]$ is infinite the subgroup $[H, i+1 \cdot a]$ has infinite index in $[H, i \cdot a]$. Therefore whenever $[H, i \cdot a]$ is infinite, $h([H, i+1 \cdot a]) < h([H, i \cdot a])$. Hence, $a$ is $n$-Engel with $n \leq h$. □

Given subgroups $X$ and $Y$ of a group $G$, we denote by $XY$ the smallest subgroup of $G$ containing $X$ and normalized by $Y$. We say that a group $G$ satisfies $\text{max}$ if $G$ satisfies the maximal condition on subgroups.

Lemma 2.6. Let $x$ and $y$ be elements of a group $G$ and suppose that for some $n \geq 1$ the subgroup $E_n(y)$ satisfies $\text{max}$. Then $\langle x \rangle^{(y)}$ is finitely generated.

Proof. Observe that $\langle x \rangle^{(y)}$ is generated by all commutators $[x, iy]$ with $i = 0, 1, \ldots$. Set $X = \langle x \rangle^{(y)} \cap E_n(y)$. We have

$$\langle x \rangle^{(y)} = \langle x, [x, y], \ldots, [x, n-1y], X \rangle.$$ 

Since $E_n(y)$ satisfies $\text{max}$, $X$ is finitely generated and so the lemma follows. □

Corollary 2.7. Let $y$ be an element of a group $G$ and $H$ a finitely generated subgroup. Suppose that for some $n \geq 1$ the subgroup $E_n(y)$ satisfies $\text{max}$. Then $H^{(y)}$ is finitely generated.

The following lemma is well-known. We supply the proof for the reader’s convenience.
LEMMA 2.8. If $G$ is a group generated by two elements $x$ and $y$, then $G' = \langle [x, y]^{x^r y^s} \mid r, s \in \mathbb{Z} \rangle$.

PROOF. Let $N = \langle [x, y]^{x^r y^s} \mid r, s \in \mathbb{Z} \rangle$. Of course, $N^y$ and $N^{y^{-1}}$ are both contained in $N$. Moreover,
\[
[x, y]^{x^r y^s} = [x, y]^{x^{r+1} y^{s} [y^s, x]} = [y^s, x]^{-1} [x, y]^{x^{r+1} y^s} [y^s, x].
\]
We have $[y^s, x] = [y, x]^{y^{-1} [y, x] y^{s-2} \cdots [y, x]}$, for all $s \geq 1$. This implies that $N^x \leq N$. Similarly we get $N^{x^{-1}} \leq N$ and so $N$ is normal in $G$. It follows that $G' = N$, as desired. 

□

LEMMA 2.9. Let $n \geq 1$ and $G$ be a group generated by a finite set $Y$ such that $E_n(y)$ satisfies max for all $y \in Y$. Then $G'$ is finitely generated.

PROOF. First assume that $Y = \{x, y\}$. Then $G' = \langle [x, y]^{x^r y^s} \mid r, s \in \mathbb{Z} \rangle$ by Lemma 2.8 and we are done since $\langle [x, y]^{(x)} \langle y \rangle \rangle$ is finitely generated by Corollary 2.7. Now suppose that $Y = \{y_1, \ldots, y_d\}$ with $d \geq 3$, and assume that the result is true for subgroups which can be generated by at most $d-1$ elements from $Y$. For $i = 1, \ldots, d$ set $G_i = \langle y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_d \rangle$. The induction hypothesis yields that $G_i'$ is finitely generated and, by Corollary 2.7, the same is true for $(G_i')^{(y_i)}$. It is easy to see that $K = \langle (G_i')^{(y_i)} \mid i = 1, \ldots, d \rangle$ is a normal subgroup of $G$ and hence $G'' = K$. In particular, $G'$ is finitely generated.

□

Now, an easy induction gives us the following corollary.

COROLLARY 2.10. Let $G$ be a finitely generated group such that for each $g \in G$ there exists $n \geq 1$ with the property that $E_n(g)$ satisfies max. Then each term of the derived series of $G$ is finitely generated.

We will also require the following lemma.

LEMMA 2.11. Let $G$ be a group such that $G'$ is nilpotent and let $N$ be a normal subgroup of $G$. Suppose that the elements $x, y \in G$ are both Engel in the subgroups $N \langle x \rangle$ and $N \langle y \rangle$, respectively. Then their product $xy$ is Engel in the subgroup $N \langle xy \rangle$.

PROOF. Set $C_0 = 1$ and $C_{i+1}/C_i = C_{N/C_i}(G'C_i/C_i)$ for $i = 0, 1, \ldots$. Thus, $C_0 \leq C_1 \leq \ldots$ is a series in $N$ all of whose factors centralize $G'$. Since $G'$ is nilpotent, there is a number $s$ such that $C_s = N$. If $s = 0$, then $N$ is trivial and there is nothing to prove. So we assume that $s \geq 1$. Arguing by induction on $s$ assume that the lemma holds for the group $G_i/C_i$. Thus, there is a number $j$ such that $[g, j, xy] \leq C_1$ for each $g \in N$. Therefore it is sufficient to prove the lemma with
\[C_1 \text{ in place of } N\]. Hence, without loss of generality we assume that \([N, G'] = 1\). Let \(\bar{G} = G/C_G(N)\). The group \(\bar{G}\) naturally acts on \(N\) and we can view \(N\) as a subgroup in the semidirect product of \(N\) by \(\bar{G}\). We note that \(N\langle xy \rangle\) is nilpotent if and only if so is \(N\langle \bar{x}\bar{y}\rangle\). Since \(\bar{G}\) is abelian, both subgroups \(N\langle \bar{x}\rangle\) and \(N\langle \bar{y}\rangle\) are normal in \(N\bar{G}\). Moreover both are nilpotent. It follows that their product is nilpotent, too. The lemma follows. \(\Box\)

3. On soluble groups of automorphisms of polycyclic groups

Malcev proved that if \(\Gamma\) is a soluble group of automorphisms of a polycyclic-by-finite group, then \(\Gamma\) is polycyclic [8]. In fact a more specific information about \(\Gamma\) can be deduced. The aim of this short section is to prove the following proposition. The proof given here was suggested by Dan Segal.

**Proposition 3.1.** Let \(N\) be a polycyclic-by-finite group with \(h(N) = h\) and let \(\Gamma\) be a soluble group of automorphisms of \(N\). Then \(h(\Gamma) < h^2 + 2h\).

First, we require a lemma.

**Lemma 3.2.** Let \(\alpha\) be an automorphism of a group \(G\) and suppose that \(\alpha\) centralizes a normal subgroup \(N \leq G\) of finite index \(m\). Then \(\alpha^m\) is an inner automorphism.

**Proof.** Since the group of automorphisms of \(G/N\) has order dividing \((m-1)!\), it follows that \(\beta = \alpha^{(m-1)!}\) stabilizes the series \(1 \leq N \leq G\). Let \(\beta^G = \{\beta_1, \ldots, \beta_s\}\). Note that the automorphisms \(\beta_1, \ldots, \beta_s\) commute. Therefore \(\prod_{i=1}^s \beta_i\) centralizes \(G\). Further, for every \(i\) the element \(\beta \beta_i^{-1}\) belongs to \(N\). Let \(K = G\langle \alpha \rangle\). Write

\[
\beta^s = \prod_{i=1}^s \beta_i \cdot \prod_{i=1}^s \beta \beta_i^{-1} \in C_K(G)N.
\]

We see that \(\beta^s\) is an inner automorphism of \(G\) (induced by an element of \(N\)). Since \(s\) is a divisor of \(m\), we conclude that \(\beta^m\) is an inner automorphism of \(G\). It remains to note that \(\beta^m = \alpha^m\). \(\Box\)

The proof of Proposition 3.1 will use the concept of plinth. A plinth of a group \(G\) is a non-trivial finitely generated free abelian normal subgroup \(A\) containing no non-trivial subgroup of lower rank which is normal in any subgroup of finite index in \(G\). Thus \(G\) and all its subgroups of finite index act rationally irreducibly on \(A\).
Proof of Proposition 3.1. If \( h(N) = 0 \), then \( h(\Gamma) = 0 \) so we will assume that both groups \( N \) and \( \Gamma \) are infinite. We write \( G \) for the product of \( N \) by \( \Gamma \). By [9 Exercise 1C9] \( N \) contains a characteristic infinite free abelian subgroup. Therefore \( N \) also contains a plinth \( N_1 \) of some normal subgroup \( G_1 \) of finite index in \( G \) (see [6 Theorem 7.1.10]). If \((N \cap G_1)/N_1 \) is infinite, we repeat the above and find a plinth \( N_2/N_1 \) contained in \((N \cap G_1)/N_1 \) of some normal subgroup \( G_2/N_1 \) of finite index in \( G_1/N_1 \). Continuing the process we find a series

\[
1 < N_1 < \cdots < N_s
\]

and a subgroup \( G_s \) of finite index in \( G \) such that \( N_s \) has finite index in \( N \cap G_s \), the subgroups \( N_i \) are normal in \( G_s \) and each factor \( N_{i+1}/N_i \) is a plinth for \( G_s/N_i \).

Set \( K = G_s \cap \Gamma \). It is clear that \( h(N_s) = h \) and \( h(K) = h(\Gamma) \) so it is sufficient to show that \( h(K) \) is at most \( h^2 + 2h \). Set \( \Delta = C_K(N_s) \), 

\[
\Delta_1 = C_K(N_1), \quad \Delta_2 = C_K(N_s/N_1), \quad \text{and } \Delta_0 = \Delta_1 \cap \Delta_2.
\]

Let \( h_1 = h(N_1) \). The group \( K/\Delta_1 \) faithfully acts on \( N_1 \) and so \( K/\Delta_1 \) embeds in \( GL(h_1, \mathbb{Z}) \). By [6 Theorem 3.1.8] \( K/\Delta_1 \) is abelian-by-finite. Now we use the fact that rationally irreducible abelian subgroups of \( GL(n, \mathbb{Z}) \) have torsion-free rank at most \( n-1 \) by the Dirichlet Unit Theorem and conclude that \( h(K/\Delta_1) \leq h_1 - 1 \).

Set \( h_2 = h - h_1 \). The group \( K/\Delta_2 \) naturally acts on \( N_s/N_1 \). Arguing by induction on \( h \) we can assume that \( h(K/\Delta_2) \leq h_2^2 + 2h_2 \). Further, the group \( \Delta_0/\Delta \) acts faithfully on \( N_s \) and stabilizes the series \( 1 \leq N_1 \leq N_s \). By [9 Proposition 1B11] \( \Delta_0/\Delta \) is isomorphic to a subgroup of \( Der(N_1, N_s/N_1) \) which is of rank \( h_1h_2 \).

Finally, note that if \( [N : N_s] = m \), by Lemma 3.2 \( \Delta^m \) embeds in the group of inner automorphisms of \( N \) and therefore \( h(\Delta) \leq h \). Thus,

\[
h(K) \leq h(K/\Delta_1) + h(K/\Delta_2) + h(\Delta_0/\Delta) + h(\Delta) \\
\leq h_1 - 1 + h_2^2 + 2h_2 + h_1h_2 + h < h^2 + 2h.
\]

\[ \square \]

4. The main theorem

It is also easy to see that any orderable group is torsion-free. Moreover, if \( x, y \) are elements of an orderable group such that \([x, y^m] = 1\) for some \( m \geq 1 \), then \( x \) and \( y \) commute [1 Lemma 2.5.1 (i)]. The class of orderable groups is closed under taking subgroups but a quotient of an orderable group is not necessarily orderable [1 Section 2.1]. A subgroup \( C \) of an ordered group \((G, \leq)\) is called \textit{convex} if \( x \in C \) whenever \( 1 \leq x \leq c \) for some \( c \in C \). Obviously \( \{1\} \) and \( G \) are convex subgroups of \( G \); and, if \( C \) is a convex subgroup, then every conjugate
of $C$ is convex. It is also clear that all convex subgroups of an ordered group form, by inclusion, a totally ordered set, which is closed under intersection and union. If $C$ and $D$ are convex subgroups of an ordered group $G$, with $C < D$, and there is not a convex subgroup $H$ of $G$ such that $C < H < D$, we say that the pair $(C, D)$ is a convex jump in $G$. Orders on a group $G$ in which $\{1\}$ and $G$ are the only convex subgroups are very well known. By a result of Hölder [1, Theorem 1.3.4], a group $G$ with such an order is order-isomorphic to a subgroup of the additive group of the real numbers under the natural order. This implies that, if $(C, D)$ is a convex jump of an ordered group, then $C$ is normal in $D$ and $D/C$ is abelian [1, Lemma 1.3.6].

If $x, y$ are elements of an orderable group such that $[x, y^m] = 1$ for some $m \geq 1$, then $x$ and $y$ commute [1, Lemma 2.5.1 (i)]. The following lemma can be easily deduced from the fact that in an orderable group $[x, y^m] = 1$ for some $m \geq 1$.}

**Lemma 4.1.** Let $G$ be an orderable group having a nilpotent subgroup $N$ of finite index. Then $G$ is nilpotent of the same class as $N$.

**Lemma 4.2.** Let $G$ be an orderable group in which for each $x$ there exists $n$ such that $E_n(x)$ satisfies max. Then each convex subgroup in $G$ is normal.

**Proof.** Suppose that $C$ is convex and not normal in $G$. Since convex subgroups form a chain, we have either $C^x < C$ or $C < C^x$ for some $x \in G$. Without loss of generality assume that $C < C^x$ and let $c^x \in C^x \setminus C$ for a suitable $c \in C$. Then $C^{x^i} < C^{x^{i+1}}$ for any integer $i$. Moreover, by Lemma 2.6, the subgroup $\langle c \rangle^{(x)}$ is finitely generated so that $\langle c \rangle^{(x)} = \langle c^{x_{i_1}}, \ldots, c^{x_{i_k}} \rangle$ where $i_1, \ldots, i_k$ are integers. We may assume $i_1 < i_2 < \ldots < i_k$. It follows that $\langle c \rangle^{(x)} \leq C^{x_{i_k}}$. Hence $c^{x_{i_k+1}} \in C^{x_{i_k}}$ and therefore $c^x \in C$, a contradiction.

**Lemma 4.3.** Let $n, h \geq 1$. Let $G$ be an orderable group in which $E_n(x)$ is polycyclic with $h(E_n(x)) \leq h$. Then $G'$ is nilpotent with $(h, n)$-bounded class.

**Proof.** It is sufficient to establish the result under the additional hypothesis that $G$ is finitely generated. Thus, assume that $G$ is finitely generated. We know that the convex subgroups in $G$ are normal and let $C$ be a convex subgroup such that $G/C$ is soluble. Since by Corollary 2.11 all terms of the derived series of $G$ are finitely generated, it follows that $G/C$ has finite rank and therefore, by [1, Theorem 3.3.1], the derived group $(G/C)'$ is nilpotent. We conclude that the image of $E_n(x)$ in $G/C$ is nilpotent for each $x \in G$. Hence, each element of $(G/C)'$ is
Engel and so, by Corollary 2.3 there is an $h$-bounded number $n_0$ such that each element of $\langle G/C \rangle'$ is $n_0$-Engel. A result of Zelmanov states that a nilpotent torsion-free $n_0$-Engel group is nilpotent with bounded nilpotency class (see also [2]). In particular, we deduce that $G/C$ has $n$-bounded derived length, say $d$.

Let $R$ be the intersection of all (normal) convex subgroups $N$ of $G$ such that $G/N$ is soluble. The above argument shows that $G^{(d+1)} \leq R$. Since all terms of the derived series of $G$ are finitely generated, we conclude that $R$ is finitely generated, too. If $R \neq 1$, among the convex subgroups properly contained in $R$ we can choose a maximal one, say $D$. It follows that $R/D$ is abelian and so $G/D$ is soluble. This is a contradiction since $R$ is the intersection of all convex subgroups $N$ of $G$ such that $G/N$ is soluble. The conclusion is that $R = 1$ and $G$ is soluble with derived length at most $d$. Again we observe that $G$ has finite rank whence $G'$ is nilpotent with $(h, n)$-bounded class. 

We are now ready to complete the proof of our main theorem which we restate here for the reader’s convenience.

**Theorem.** Let $h, n$ be positive integers and $G$ an orderable group in which $E_n(x)$ is polycyclic with $h(E_n(x)) \leq h$ for every $x \in G$. There are $(h, n)$-bounded numbers $h^*$ and $c^*$ such that $G$ has a finitely generated normal nilpotent subgroup $N$ with $h(N) \leq h^*$ and $G/N$ nilpotent of class at most $c^*$.

**Proof.** Choose an arbitrary element $x \in G$. Since the Hirsch length of a subgroup of infinite index of a polycyclic group is strictly smaller than that of the group, it follows that in the series

$$E_n(x) \geq [E_n(x), x] \geq [E_n(x), x, x] \geq \ldots$$

at most $h$ terms $[E_n(x), i, x]$ have infinite index in $[E_n(x), i-1, x]$. In view of Lemma 1.3, $E_n(x)$ is nilpotent. The element $x$ naturally acts on $E_n(x)$ by conjugation and Lemma 2.4 shows that if, for some $i \geq 1$, the subgroup $[E_n(x), i, x]$ has finite index in $[E_n(x), i-1, x]$, then $[E_n(x), i+s, x]$ has finite index in $[E_n(x), i-1, x]$ for any $s \geq 1$. It follows that $[E_n(x), h+s, x]$ has finite index in $[E_n(x), h, x]$ for any $s \geq 1$. For all $x \in G$ set $U(x) = [E_n(x), h, x]$. It follows that an element $x$ is an Engel element if and only if $U(x) = 1$ and each Engel element in $G$ is $(n + h)$-Engel.

We know from Lemma 1.3 that $G'$ is nilpotent with $(h, n)$-bounded class. By Lemma 2.2 there exists a bounded number $f$ such that for each element $x \in G$ we have $\gamma_f((x, G')) \leq E_{n+h}(x) \leq U(x)$. Observe that for each $i$ the subgroup $[U(x), i, x]$ is contained in $\gamma_{n+h+i+1}((x, G'))$. 


Since by our assumptions for each positive \( i \) the subgroup \([U(x), x]\) has finite index in \( U(x) \), we conclude that also \( \gamma_{f+i}(\langle x, G' \rangle) \) has finite index in \( U(x) \). In particular, \( h(\gamma_{f+i}(\langle x, G' \rangle)) = h(U(x)) \). In the sequel we will use without mentioning explicitly that all subgroups of the form \( \gamma_j(\langle x, G' \rangle) \) are normal in \( G \).

Now choose \( a \in G \) such that the Hirsch length \( h_0 \) of \( \gamma_f(\langle a, G' \rangle) \) is as big as possible. If \( h_0 = 0 \), then \( U(x) = 1 \) for each \( x \in G \) and so all elements of \( G \) are \((n + h)\)-Engel. Hence, by Zelmanov’s result \([11]\), \( G \) is nilpotent with bounded class. Therefore we will assume that \( h_0 \geq 1 \). Let \( b \in C_G(\gamma_f(\langle a, G' \rangle)) \). Recall that in an orderable group \([x, y^n] = 1 \) implies that \([x, y] = 1 \). Since \( b \) centralizes \( \gamma_f(\langle a, G' \rangle) \), which is a subgroup of finite index in \( U(a) \), it follows that \( b \) centralizes \( U(a) \). Set \( S = \gamma_f(\langle b, G' \rangle) \). Define \( T = 1 \) if \( S \) is abelian and \( T = Z(S) \) (the center of \( S \)) otherwise. Note that if \( U(b) \neq 1 \), then \( S/T \) is infinite. Since \( b \) centralizes \( U(a) \), we deduce that \( U(b) \cap U(a) \leq T \). Further, observe that \( a \) acts on \( S/T \) as an \((n + h)\)-Engel element. Since \( b \in C_G(\gamma_f(\langle a, G' \rangle)) \), the action of \( ab \) on \( \gamma_f(\langle a, G' \rangle) \) is the same as that of \( a \). Therefore the subgroups of the form \( [\gamma_f(\langle a, G' \rangle), ab] \) have finite index in \( \gamma_f(\langle a, G' \rangle) \) for each \( i \). It follows that \( \gamma_f(\langle ab, G' \rangle) \) intersects \( \gamma_f(\langle a, G' \rangle) \) by a subgroup having finite index in both \( \gamma_f(\langle ab, G' \rangle) \) and \( \gamma_f(\langle a, G' \rangle) \). Therefore the Hirsch length of \( \gamma_f(\langle ab, G' \rangle) \) is precisely \( h_0 \). Since \( b \) centralizes a subgroup of finite index in \( \gamma_f(\langle ab, G' \rangle) \), we conclude that \( b \) centralizes \( \gamma_f(\langle ab, G' \rangle) \). Taking into account that \( S \cap \gamma_f(\langle a, G' \rangle) \leq T \) we further deduce that \( S \cap \gamma_f(\langle ab, G' \rangle) \leq T \). Hence, \( ab \) acts on \( S/T \) as an \((n + h)\)-Engel element. Thus, both \( a \) and \( ab \) act on \( S/T \) as Engel elements. Lemma \([2,11]\) now shows that also \( b \) acts on \( S/T \) as an Engel element. We know that \( [S, i] \) has finite index in \( S \) for every \( i \). Therefore we now deduce that \( S = 1 \), that is, \( b \) is an Engel element in \( G \). Recall that \( b \) was chosen in \( C_G(\gamma_f(\langle a, G' \rangle)) \) arbitrarily. Thus, each element of \( C_G(\gamma_f(\langle a, G' \rangle)) \) is \((n + h)\)-Engel in \( G \).

Let \( F \) be the Fitting subgroup of \( G \). We know that \( F \) consists of \((n + h)\)-Engel elements. Therefore, by Zelmanov’s result \([11]\), \( F \) is nilpotent with bounded class. Moreover both \( G' \) and \( C_G(\gamma_f(\langle a, G' \rangle)) \) are contained in \( F \). Further, using Lemma \([1,1]\) we note that \( G/F \) is torsion-free. The group \( G/C_G(\gamma_f(\langle a, G' \rangle)) \) faithfully acts on \( \gamma_f(\langle a, G' \rangle) \). By Proposition \([3,1]\) \( G/C_G(\gamma_f(\langle a, G' \rangle)) \) has \( h \)-bounded Hirsch length. Therefore \( G/F \) is abelian with \( h \)-bounded number of generators. Write \( G = \langle F, a_1, \ldots, a_k \rangle \), where \( a_1, \ldots, a_k \) are (boundedly many) generators of \( G \) modulo \( F \). For \( i = 1, \ldots, k \) set \( G_i = \langle F, a_i \rangle \). All the subgroups \( G_i \) are normal in \( G \) since \( F \) contains \( G' \). According to Lemma \([2,2]\) there is an \((h, n)\)-bounded number \( f_0 \) such that \( \gamma_{f_0}(G_i) \leq E_n(a_i) \). Our hypotheses imply that \( \gamma_{f_0}(G_i) \) is polycyclic with Hirsch length at most
h for each \( i = 1, \ldots, k \). Let \( N \) be the subgroup generated by all \( \gamma_{f_0}(G_i) \). It follows that \( N \) is polycyclic with Hirsch length at most \( kh \). Further, \( G/N \) is a product of \( k \) normal subgroups \( G_iN/N \), each of which is nilpotent of class at most \( f_0 - 1 \). It follows that \( G/N \) is nilpotent of class at most \( kf_0 - k \). Thus, we can take \( h^* = kh \) and \( c^* = kf_0 - k \). □

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