Moment model of an uniformly magnetized permanent magnet based on point sources of the field

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Abstract. Permanent magnets are widely used in various technical devices. Mathematical models of magnetic fields of permanent magnets are considered to solve the problems of analysis and synthesis of such devices. The paper proposes a third model, based on the use of point sources of a magnetic field in addition to the two known models of uniformly magnetized permanent magnets (current and charge models). Each point source is characterized by a vector magnetic moment. The features of the model for permanent magnets of various shapes are considered. The advantage of the new model is to reduce the computation time of the magnetic field parameters by reducing the dimensionality of the modeling problem.

1. Introduction
Permanent magnets (PM) are increasingly used in electric motors, electromagnetic drives, electromechanical devices, due to their properties [1,2]. A more universal approach to solve inverse problems is based on the multiple solution of the direct problem of calculating magnetic fields when designing devices with PM and identifying their parameters [3-6]. As a rule, stationary magnetic field modeling is performed for various configurations of PM [7]. At present, two mathematical models of uniformly magnetized PM are used these are current and charge models [8,9]. The third model of PM is proposed in this work. Its basis is the magnetic moments of point sources of the magnetic field, constructed by analogy with the moments of similar sources of the electric field [10]. Let's call this model «the moment model».

2. Mathematical models
Consider the mathematical models of magnetic fields PM. The equation of the magnetic field created by the PM has the following form

\[ \text{rot} \vec{H} = 0 ; \text{div} \vec{B} = 0 \]  \hspace{1cm} (1)

in volume of PM

\[ \vec{B} = \mu_{\text{m弘}} \left( \vec{H} + \vec{M} \right) \]  \hspace{1cm} (2)

out of PM

\[ \vec{B} = \mu \vec{H} \]  \hspace{1cm} (3)

Further, the magnetization of the PM is assumed \( \vec{M} = \text{const} \).
Based on the first equation of system (1) it is assumed

$$\vec{H} = -\nabla \varphi$$  \hspace{1cm} (4)

where $\varphi$ – scalar magnetic potential.

The system (1) taking into account (4) is converted to

$$\text{div}(\mu \nabla \varphi) = 0$$

At the interface of media with different magnetic permeabilities $\mu^+$ and $\mu^-$ are obtained

$$\varphi^+ = \varphi^-; \quad \mu^+ \frac{\partial \varphi^+}{\partial n} - \mu^- \frac{\partial \varphi^-}{\partial n} = \mu^+ \left( M \cdot \hat{n} \right)$$  \hspace{1cm} (5)

Point sources are introduced to describe the magnetic field of PM [11]. The scalar potential of the magnetic field of a point source, located in the PM, with a moment $\vec{m}'$ in the space $V^-$, surrounding the PM in a spherical coordinate system is determined by the formula [10]

$$\varphi^-(r, \alpha) = \frac{(\vec{m}', \hat{e}_r)}{4\pi \mu_0 r^2} = \frac{m \cos \alpha}{r^2}$$  \hspace{1cm} (6)

where $m = m'/(4\pi \mu_0)$, $r$ – module of the radius-vector connecting the location point of the field source in the volume of the PM $V^+$ and point in $V^-$ with coordinates $r$ and $\alpha$, $\alpha$ – angle between radius vector and vector $\vec{m}'$, $e'_r$ – unit vector (Fig. 1).

![Figure 1. Point source of field with moment $\vec{m}'$.](image)

The scalar potential of the magnetic field in $V^+$ of the source $\vec{m}'_s$, located in $V^-$ is determined by the formula

$$\varphi^+(r, \alpha) = -\frac{(\vec{m}'_s, \hat{e}_r)}{4\pi \mu_0} r \cos \alpha = -m_s r \cos \alpha$$  \hspace{1cm} (7)

where $m_s = m'_s/(4\pi \mu_0)$.

The components of the magnetic field are determined by the following formulas

$$H^+_r = -\frac{\partial \varphi^+}{\partial r} = m_s \cos \alpha; \quad H^+_\alpha = -\frac{1}{r} \frac{\partial \varphi^+}{\partial \alpha} = -m_s \sin \alpha$$  \hspace{1cm} (8)

$$H^-_r = -\frac{\partial \varphi^-}{\partial r} = \frac{2m \cos \alpha}{r^3}; \quad H^-_\alpha = -\frac{1}{r} \frac{\partial \varphi^-}{\partial \alpha} = \frac{m \sin \alpha}{r^3}$$  \hspace{1cm} (9)

In rectangular coordinate system
The equation of the PM field in $V^+$ and $V^-$ with cylindrical symmetry is performed in the spherical coordinate system, using the definition of potentials by the formulas (6) and (7)

$$\text{divgrad} \varphi = \frac{\partial}{\partial r} \left( r^2 \frac{\partial \varphi}{\partial r} \right) + \frac{1}{\sin \alpha} \frac{\partial}{\partial \alpha} \left( \sin \alpha \frac{\partial \varphi}{\partial \alpha} \right) = 0$$

Consider an example: the magnetic fields of a sphere uniformly magnetized along the 0z axis with parameters – radius $R$, magnetic permeability $\mu_{pm} = \mu_0$, magnetization $M = M\hat{e}_z$.

The analytical representation of the fields in $V^+$ and $V^-$ is determined using the approach described in [12]. Potentials are represented as

$$\varphi^+(r, \alpha) = K^+ r \cos \alpha; \quad \varphi^-(r, \alpha) = K^- \frac{\cos \alpha}{r^2}$$ (10)

The ratio based on the boundary conditions (5) at the point $N(R, 0, 0)$ are taken the form

$$K^+ R = \frac{K^-}{R^2}; \quad \mu_{pm} K^+ + \mu^- \frac{2K^-}{R^3} = \mu^+ M$$

where is supposed $\mu_{pm} = \mu^+ = \mu^- = \mu_0$, whence it follows that

$$K^+ = \frac{M}{3}; \quad K^- = \frac{MR^3}{3}$$ (11)

The relations (10) taking into account (11) are taken the following form

$$\varphi^+(r, \alpha) = \frac{M}{3} r \cos \alpha; \quad \varphi^-(r, \alpha) = \frac{MR^3 \cos \alpha}{3 r^2}$$ (12)

Consider the task of building a moment model of a ball uniformly magnetized along the 0z axis and having the parameters specified above. Three point sources are arranged as shown in figure 2.

**Figure 2.** Uniformly magnetized ball with three point sources of magnetic field, described by moments $\vec{m}'$ and $\vec{m}''$. 
Moments $m$ and $m_s$ are determined from the system of equations, based on the boundary conditions (5) for the point $N(R,0,0)$, lying on the surface of the ball
\[
\begin{cases}
\varphi^+ (R,0,0) = \varphi^- (R,0,0) \\
\mu_{pm} \left( 2H^+_r(R,0,0) + M \right) = \mu_0 H^-_r(R,0,0)
\end{cases}
\] (13)

The system of equations (13) taking into account (6) - (9) is converted to the form
\[
\begin{cases}
-2m_s R = \frac{m}{R^2} \\
2m_s + M = \frac{2m}{R^3}
\end{cases}
\] (14)

Solving the system (14), is obtained
\[
m = \frac{MR^3}{3}; \quad m_s = -\frac{M}{6}
\] (15)

Substitution (15) into (6) and (7) gives formulas for $\varphi^+$ and $\varphi^-$, which coincide with the analytical description of the fields (12), which proves the correctness of the chosen relations (6) and (7).

Consider the problem of identifying a uniformly magnetized PM in the form of a ball. The radius $R$ of the ball, magnetic permeability $\mu_{pm} = \mu_0$ and the value of magnetic induction $B_z(z,0,0)$, measured at a point $(z,0,0)$, $z > R$ are known. It is required to determine the magnetization $M$ of PM. Based on the relations (8) and (9) is obtained
\[
B_z(z,0,0) = \mu_0 H^-_r(z,0,0) = \frac{2\mu_0 m}{z^3}
\]
whence it follows that
\[
m = \frac{z^3 B_z(z,0,0)}{2\mu_0}
\] (16)

consequently,
\[
M = \frac{3m}{R^3}
\] (17)

Thus, the moment model of PM in the form of a ball with known magnetization $M$, radius $R$ and $\mu_{pm} = \mu_0$ is a set of point sources with moments $\vec{m}_r^+$ and $\vec{m}_r^-$, relations (6) - (9), (15), as well as the coordinates of the location of point sources of fields $\vec{m}_r^+$ and $\vec{m}_r^-$. The identification of the PM in the form of a ball is carried out using relations (16), (17).

Note that the location on a plane of symmetry of point sources $\vec{m}_r^-$ can be arbitrary, since always $r \cos \alpha = R$.

Similarly, the moment models of PM of other forms are determined.

Consider the second example: plane-parallel stationary magnetic fields of a uniformly magnetized cylinder perpendicular to the axis $0z$ with parameters – radius $R$; magnetic permeability $\mu^+ = \mu_0$; magnetization $\vec{M} = M\hat{e}_y$.

The analytical representation of the fields in $D^+$ and $D^-$ in this case is [12]:
\( \varphi^+(r, \alpha) = K^+ r \cos \alpha ; \varphi^-(r, \alpha) = K^- \frac{\cos \alpha}{r} \) \tag{18}

At the point \( N(0, R) \) based on the boundary conditions (5), taking into account the equality \( \mu^+ = \mu^- = \mu_0 \) is received

\[
K^+ R = \frac{K^-}{R} ; \quad K^+ + \frac{2K^-}{R^2} = M
\]

whence it follows that

\[
K^+ = \frac{M}{2} ; \quad K^- = \frac{MR^2}{2}
\] \tag{19}

The relations (18) taking into account (19) are taken the following form

\[
\varphi^+(r, \alpha) = \frac{M}{2} r \cos \alpha ; \varphi^-(r, \alpha) = \frac{MR^2}{2} \frac{\cos \alpha}{r}
\] \tag{20}

Next, the construction of the moment model of a uniformly magnetized cylinder, having the above parameters, is performed.

Three point sources are located, as shown in Figure 3. The scalar potential of the magnetic field in \( D^+ \) the sources \( \vec{m}' \) located in \( D^- \) is determined by the formula

\[
\varphi^+(r, \alpha) = -2 \frac{(\vec{m}', \vec{e}_z)}{4\pi \mu_0} = -2m_s r \cos \alpha
\] \tag{21}

where \( m_s = m_s'/(4\pi \mu_0) \).

**Figure 3.** Cylinder, uniformly magnetized (perpendicular to the axis 0z) with three point sources with moments \( \vec{m}' \) and \( \vec{m}_s' \)

The scalar potential of the magnetic field in \( D^- \) a point source with a moment \( \vec{m}' \), located in \( D^+ \) is determined by the formula

\[
\varphi^-(r, \alpha) = \frac{(\vec{m}', \vec{e}_r)}{4\pi \mu_0 r} = \frac{m_s r \cos \alpha}{r}
\] \tag{22}

where \( m = m'/(4\pi \mu_0) \).
The components of the magnetic field strengths in $D^+$ and $D^-$ are determined by the following formulas

\begin{align}
H^+_r &= -\frac{\partial \varphi^+}{\partial r} = 2m_s \cos \alpha, \quad H^+_\alpha = -\frac{1}{r} \frac{\partial \varphi^+}{\partial r} = -2m_s \sin \alpha \tag{23} \\
H^-_r &= -\frac{\partial \varphi^-}{\partial r} = m \cos \alpha, \quad H^-_\alpha = -\frac{1}{r} \frac{\partial \varphi^-}{\partial r} = \frac{m \sin \alpha}{r^2} \tag{24}
\end{align}

In rectangular coordinate system

$$
H^-_y = m \left( 2 \cos^2 \alpha - 1 \right)
$$

The scalar potentials represented by formulas (22) and (23) satisfy the following magnetic field equation in $D^+$ and $D^-$ in the cylindrical coordinate system, taking into account the relation $\frac{\partial \varphi}{\partial z} = 0$ for the plane-parallel field

$$
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \varphi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \varphi}{\partial \alpha^2} = 0
$$

This indicates the correctness of the selected relations (21) and (22).

The location on the symmetry plane of point sources $m_s'$ can be arbitrary, since, as in the problem of the ball model, always $r \cos \alpha = R$.

The system of equations for the collocation point $N(0,R)$ is compiled on the basis of the boundary conditions (5) to determine the moments $m$ and $m_s$.

$$
\begin{cases}
\varphi^+(R,0) = \varphi^-(R,0) \\
\mu^+(2H^+_r(0,R) + M) = \mu^- H^-_r(0,R)
\end{cases} \tag{25}
$$

The system of equations (25) taking into account (21) - (24) and $\mu^+ = \mu^- = \mu_0$ is converted to the equations

$$
-2m_sR = \frac{m}{R}; \quad 2m_s + M = \frac{m}{R} \tag{26}
$$

The solution of system (26) is

$$
2m_s = -\frac{M}{2}; \quad m = \frac{MR^2}{2} \tag{27}
$$

Formulas for the definition of $\varphi^+$ and $\varphi^-$, which coincide with the analytical description of the fields (20), are obtained by substituting (27) into (21) and (22).

Consider a more complex case: PM in the shape of a parallelepiped (figure 4).

The field sources in $V^-$ are located in $V^+$ on the middle surface of the $S_{ms}$, which has a potential $\varphi = 0$ due to the symmetry of the PM. Sources of the field in $V^+$ are located outside the volume of the PM. Collocation points $N_i$ are located on the upper surface $S_h$ of the PM. Figure 5 shows a view of the PM from the top and from the side with the location of sources on the plane $S_{ms}$.
Figure 4. PM in the shape of a parallelepiped with the location of sources and collocation points $N_i$

Figure 5. View of the PM from the top (a) and from the side (b) with the location of point sources on the plane of symmetry $S_{ms}$ and the collocation points $N_1$ and $N_2$ on the surface $S_h$. 
It is necessary to determine the values of magnetic moments \( m_1, m_2, m_3 \) and the coordinates of the source points \( y_2 = -y_4 = y \), the other coordinates are assumed to be known, making two equations for each collocation point of the form (13) to build the moment model considered PM.

The formula for determining the required number of collocation points \( N_{col} \) is

\[
N_{col} = \frac{X_m + X_c}{2}
\]

where \( X_m \) – number of unknown moments, \( X_c \) – number of unknown coordinates.

In the case of the total number of sources of the field \( X_n = 9 \) (Fig. 5) is received: \( X_m = 3, X_c = 1, N_{col} = 2 \). Thus, two points of collocation \( N_1 \) and \( N_2 \) are chosen in the example. Next, a system of four equations of the form (13) is composed.

3. Conclusion

The paper proposes a third model, based on the use of point sources of a magnetic field in addition to the two known models of uniformly magnetized PM (current and charge models).

Each point source is characterized by a vector magnetic moment and coordinates of the position of the corresponding point in the PM volume. It is shown that the position of point sources located outside the PM is arbitrary.

A formula for determining the number of points of collocation is proposed. The system of equations constructed on the basis of boundary conditions is used to determine the moments and their coordinates.

The advantage of the new model is to reduce the computation time of magnetic induction and magnetic field strength in the space, surrounding the PM. In the case of using the model in solving the problem of identifying the PM, the number of operations is reduced.

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