RIGIDITY OF RIEMANNIAN MANIFOLDS CONTAINING AN EQUATOR

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Abstract. In this paper, we prove that a Riemannian $n$-manifold $M$ with sectional curvature bounded above by 1 that contains a minimal 2-sphere of area $4\pi$ which has index at least $n-2$ has constant sectional curvature 1. The proof uses the construction of ancient mean curvature flows that flow out of a minimal submanifold. As a consequence we also prove a rigidity result for the Simon-Smith minimal spheres.

1. Introduction

Let $g$ be a complete Riemannian metric on the 2-sphere $S^2$. If its sectional curvature is between 0 and 1, it is known that any closed geodesic on $(S^2, g)$ has length at least $2\pi$ [16]. Moreover if such a closed geodesic has length $2\pi$, $(S^2, g)$ is isometric to the unit 2-sphere $S^2_1 = \{ p \in \mathbb{R}^3 \mid \|p\| = 1 \}$ with the induce metric. The proof of this result is given in [2] where the authors attribute the theorem to E. Calabi.

So a question is what happens in higher dimension. In dimension 3, one can replace geodesics by minimal 2-sphere. Actually one can prove that, if the sectional curvature is bounded above by 1, any minimal 2-sphere has area at least $4\pi$. In [12], H. Rosenberg and the author study the equality case. If $(M, g)$ is a Riemannian 3-manifold with sectional curvature $0 \leq K \leq 1$ that contains a minimal 2-sphere of area $4\pi$, they prove that the universal cover of $M$ is isometric to the unit 3-sphere $S^3_1$ or the product $S^2_1 \times \mathbb{R}$.

One purpose of this paper is to investigate generalizations of this result to higher dimensions. Actually if $(M, g)$ is a Riemannian $n$-manifold with sectional curvature $K \leq 1$, we still have that the area of a minimal 2-sphere is at least $4\pi$. So what can be said in the equality case?

A model of this situation is an equatorial 2-sphere in the unit $n$-sphere $S^n_1$. So one could expect that under some extra hypotheses this is the only example.

If $\Sigma$ is a minimal $m$-submanifold in $M$, $\Sigma$ is critical for the volume functional. The stability of this critical point is given by the Jacobi operator which is a self-adjoint second order elliptic operator that acts on sections of the normal bundle to $\Sigma$. As a critical point, the index of $\Sigma$ is given by the number of negative eigenvalues of this operator. In the case of an equatorial 2-sphere $S$ in $S^n_1$, the index of $S$ is $n-2$.

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The first main result of the paper is a rigidity result under such an instability hypothesis.

**Theorem.** Let $M$ be a Riemannian $n \geq 3$-manifold whose sectional curvature is bounded above by 1. Let us assume that $M$ contains an immersed minimal 2-sphere of area $4\pi$ which has index at least $n - 2$. Then the universal cover of $M$ is isometric to the unit sphere $S^n_1$.

Let us notice that the instability hypothesis can be replaced by another version.

**Definition 1.** Let $\Sigma$ be a minimal submanifold in $(M, g)$. We say that $\Sigma$ is unstable in any parallel directions if the restriction of the Jacobi operator to any parallel sub-bundle of the normal bundle to $\Sigma$ has index at least 1.

The above theorem gives an answer to a question that arises from a result in [2]. In [2, Corollary 5.11], L. Anderson and R. Howard prove that a Riemannian $n$-manifold $M$ with sectional curvature below 1 containing isometrically a neighborhood of the equator $S^{n-1}_1$ in $S^n_1$ is isometric to $S^n_1$. The hypothesis that a whole neighborhood of the equator belongs to $M$ seems strong and the question is to find a weaker hypothesis. Actually our main result gives an infinitesimal version of Anderson-Howard result. If $M$, with $K \leq 1$, contains a totally geodesic hypersurface isometric to $S^{n-1}_1$ that is unstable as a minimal hypersurface, then $M$ is isometric to $S^n_1$. The idea is that the totally geodesic $S^{n-1}_1$ contains a minimal 2-sphere of area $4\pi$ and index at least $n - 2$. Actually in the same spirit as Anderson-Howard theorem, there is a result by D. Panov and A. Petrunin [15, Theorem 1.4] with a weaker hypothesis: if $S$ is an equatorial 2-sphere in $S^n_1$ and $S^+$ denotes an hemisphere of $S$, Panov and Petrunin need just that $M$ contains isometrically a neighborhood of $S^+$ in $S^n_1$.

The proof of the main theorem uses ideas that already appear in [12]: if $S$ is an immersed 2-sphere we define the $F$ functional by $F(S) = A(S) + \int_S ||\vec{H}||^2 - 4\pi$ where $A(S)$ is the area of $S$ and $\vec{H}$ is the mean curvature vector of $S$. Actually if $F(S)$ vanishes, $S$ is totally umbilical and we obtain some information on the sectional curvature of $M$ along $S$. So if $S_0$ is the minimal 2-sphere given by the statement of the theorem $F(S_0) = 0$. The idea is to explore the geometry of $M$ by computing $F(S_t)$ along a deformation $\{S_t\}_t$ of $S_0$. One of the novelties is the construction of the family $\{S_t\}_t$. Actually we produce $\{S_t\}$ as a mean curvature flow that flows out of $S_0$. More precisely, we construct non trivial ancient solutions $\{S_t\}_{t \in (-\infty, b)}$ of the mean curvature flow such that, as $t \to -\infty$, $S_t$ converges to $S_0$.

The idea is that the eigen-sections of the Jacobi operator associated to the first eigenvalue give directions in which such an ancient mean curvature flow can be initiated. A similar idea appear in the work of K. Choi and C. Mantoulidis [4] where they construct ancient mean curvature flows "tangent" to the eigenspaces with negative eigenvalues. Then they prove several uniqueness results for ancient mean curvature flow in $S^n_1$. An other example
is where A. Mramor and A. Payne produce an eternal solution of the mean curvature flow that flows out of the catenoid. Let us notice that good introductions to the study of high codimension mean curvature flow can be found in the paper of K. Smoczyk and the PhD thesis of C. Baker.

The index hypothesis in the above theorem can appear very particular. Actually there are situations where it is quite natural. As critical points of the area functional, minimal hypersurfaces can be produced by a minimization process. However one have to consider a non-trivial class of hypersurfaces to produce a non-trivial critical point. So in order to solve this difficulty, a Morse theoretical approach has been developed. In F. Smith is able to contruct minimal 2-spheres in any riemannian $(S^3, g)$. It's proof is based on the following ideas. Let $\Lambda$ be the set of paths $\{\sigma_t\}_{t \in [-1,1]}$ in the space of 2-spheres in $S^3$ that sweeps out $S^3$ (see precise definitions and statements in Section 5). Then he considers the quantity

$$W(S^3, g) = \inf_{\{\sigma_t\} \in \Lambda} \max_{t \in [-1,1]} A(\sigma_t)$$

called the Simon-Smith width of $(S^3, g)$.

First this quantity is positive and Smith proves that it is realized by the area of a finite collection of minimal spheres. Besides it is reasonable to think that the index of these collection of minimal spheres is 1. F. C. Marques and A. Neves proved the upper-bound by 1. So the second main result of this paper is

**Theorem.** Let $(S^3, g)$ be a Riemannian 3-sphere whose sectional curvature is bounded above by 1. Then $W(S^3, g) \geq 4\pi$ and, if $W(S^3, g) = 4\pi$, then $(S^3, g)$ is isometric to $S^3_1$.

If one knows that $W(S^3, g)$ is realized by an index 1 minimal 2-sphere the above result is a direct consequence of our first rigidity result. So the difficulty is to deal with the case where $W(S^3, g)$ is realized by a 2-sphere of index 0. Actually one can think about the following example: the cylinder $S^2_1 \times [-1,1]$ capped by two hemispheres $S^3_1^+$ (see Figure 1). This defines a $C^{1,1}$ Riemannian metric $\bar{g}$ on $S^3$ whose sectional curvature is bounded above.
by 1 in any reasonable weak sense. Its Simon-Smith width is $4\pi$ so this implies that the above result is false for a weak sense of sectional curvature. Actually the above example is exactly the type of situation we have to consider in the proof of the above theorem: we prove that the Simon-Smith width is realized by a minimal 2-sphere which is not almost stable. Moreover $\bar{g}$ can be smoothed to produce sequences $(g_n)$ of smooth Riemannian metrics with sectional curvature bounded above by 1, $W(S^3, g_n) \to 4\pi$ and $g_n \to \bar{g}$. So the $g_n$ are far of the round metric of $S^3_1$. This implies that the above rigidity result is not stable.

Let us also notice studying the Simon-Smith width for reversed curvature inequalities has been done by F. C. Marques and A. Neves. This time just a control on the Ricci and scalar curvature is assumed.

**Theorem** (Marques, Neves [10]). Let $(S^3, g)$ be a Riemannian 3-sphere with positive Ricci curvature and scalar curvature $R \geq 6$. Then $W(S^3, g) \leq 4\pi$ and, if $W(S^3, g) = 4\pi$, then $(S^3, g)$ is isometric to $S^3_1$.

The paper is organized as follows. In Section 2 we recall some basic formulas and definitions of submanifold geometry. Section 3 is devoted to the construction of ancient solutions of the mean curvature flow (Theorem 1). In Section 4 we prove our first rigidity result (Theorem 6) and its Corollary concerning manifolds containing an equator of $S^n_1$. Section 5 is devoted to the study of the Simon-Smith width and the proof of the second rigidity result (Theorem 8). In Appendix A we prove a Schauder type estimate used in the proof of Theorem 1.

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2. **Geometry of submanifolds**

In this section we recall some classical notations and formulas concerning the geometry of submanifolds.

Let $(M, g)$ be a Riemannian manifold of dimension $n$ and $\Sigma$ a manifold of dimension $m$. If $F_0 : \Sigma \to M$ is an immersion, we can consider the induced Riemannian metric $g_0 = F_0^* g$ on $\Sigma$ making $F_0$ a local isometry. In the paper, we often identify $\Sigma$ with its image $\Sigma_0 = F_0(\Sigma)$ at least locally where $F_0$ is an embedding: for example, we often identify $T_p \Sigma$ with $(F_0)_*(T_p \Sigma) \subset T_{F_0(p)}M$.

If $\nabla$ and $\nabla^0$ are respectively the Levi-Civita connections on $M$ and $\Sigma$, we can define the second fundamental form on $\Sigma$ by

$$B_p(v, w) = \nabla_v w - \nabla^0_v w \in N_p \Sigma$$

where $v, w \in T_p \Sigma$ and $N_p \Sigma$ is the normal subspace to $\Sigma$ at $p$.

The mean curvature vector of $\Sigma$ is then

$$\vec{H}(p) = \frac{1}{m} \operatorname{tr}_{T_p \Sigma} B_p \in N_p \Sigma$$
where $\text{tr}_P$ denotes the trace operator on the subspace $P$. We define $\tilde{B}_p = B_p - \bar{H}(p)g_0$ the traceless part of the second fundamental form. We recall that the normal bundle $N\Sigma$ inherits from $g$ and $\nabla$ a normal connection $\nabla^\perp$.

Let $(F_t)_t$ be a smooth family of immersion of $\Sigma$ and define the vector field $X = \frac{d}{dt} \varphi_t|_{t=0}$ along $F_0(\Sigma)$ and let $\Sigma_t = F_t(\Sigma)$. We have a family of metrics $g_t = F_t^* g$ defined on $\Sigma$ with associated volume measure $d\sigma_t$. If, $X$ is orthogonal to $\Sigma$, it is well known that, for any function $f$ on $\Sigma$:

$$
\frac{d}{dt}|_{t=0} \int_\Sigma f d\sigma_t = - \int_\Sigma (X, m\bar{H}) f d\sigma_0
$$

So if $\Sigma$ is critical with respect to the $m$-volume functional $A$, we have $\bar{H} = 0$ along $\Sigma$: $\Sigma$ is minimal.

We are interested in understanding how the mean curvature vector $\bar{H}$ is deformed along the family $F_t$. So let us denote by $\bar{H}_t(p)$ the mean curvature vector of $\Sigma_t$ at $F_t(p)$.

**Lemma 1.** If $X$ is normal to $\Sigma$, we have

$$
\frac{D}{dt}|_{t=0} m\bar{H}_t = \Delta^{\perp} X + (R(e_i, X)e_i)^{\perp} + (X, B(e_i, e_j))B(e_i, e_j) - (m\bar{H}_0, \nabla e_i) X e_i
$$

with the convention that summations are made over repeated indices, $(e_1, \ldots, e_m)$ is an orthonormal frame of $\Sigma_0$, $R$ is the Riemann curvature tensor associated to $g$ and $\Delta^{\perp}$ denotes the Laplacian operator acting on normal sections: $\Delta^{\perp} X = \text{tr} \nabla^{\perp 2} X$.

**Proof.** Let $E_1, \ldots, E_m$ be an orthonormal frame on $(\Sigma, g_0)$ and consider along $F_t(p)$ the frame $e_i = (F_t)_* (E_i)$. We assume that $\nabla_{e_i} e_j = 0$ for any $i, j$ at $p$ where the computation is made. Let us denote $g_{ij} = (e_i, e_j)$ and $(g^{ij})$ the inverse matrix. We have

$$
m\bar{H}_t = g^{ij} (\nabla e_i, e_j)^{\perp}
$$

where $Y^{\perp}$ denotes the orthogonal projection to $N\Sigma_t$. At $t = 0$, we have $\frac{D}{dt} e_i = \nabla_{e_i} X$.

Notice that $g_{ij}|_{t=0} = \delta^j_i$, so at $t = 0$:

$$
\frac{d}{dt} g^{ij} = - \frac{d}{dt} g_{ij} = -(\frac{D}{dt} e_i, e_j) - (e_i, \frac{D}{dt} e_j) = - (\nabla_{e_i} X, e_j) - (e_i, \nabla_{e_j} X) = 2(X, B(e_i, e_j))
$$
If $Y$ is a vector field along $F_t(p)$. We have $Y^\perp = Y - g^{ij}(Y,e_j)e_i$, so at $t=0$:

$$
\frac{D}{dt}Y^\perp = \left(\frac{D}{dt}Y\right)^\perp + ((\nabla_{e_i}X,e_j) + (e_i,\nabla_{e_j}X))(Y,e_j)e_i
- (Y,\frac{D}{dt}e_i)e_i - (Y,e_i)\frac{D}{dt}e_i
$$

$$
= \left(\frac{D}{dt}Y\right)^\perp + (\nabla_{e_i}X,Y^\top)e_i + (Y,e_j)(\nabla_{e_j}X)^\top
- (Y,\nabla_{e_i}X)e_i - (Y,e_i)\nabla_{e_i}X
$$

$$
= \left(\frac{D}{dt}Y\right)^\perp - \sum_i (Y^\perp,\nabla_{e_i}X)e_i - (Y,e_i)(\nabla_{e_i}X)^\perp
$$

where $Z^\top$ denotes the tangential part of $Z$.

We also have

$$
\frac{D}{dt}\nabla_{e_i}e_i = R(e_i,X)e_i + \nabla_{e_i}\frac{D}{dt}e_i
= R(e_i,X)e_i + \nabla_{e_i}\nabla_{e_i}X
$$

So combining all the above computations at $\bar{p}$, we obtain

$$
\frac{D}{dt}mH_t = 2(X,B(e_i,e_j))B(e_i,e_j) + (R(e_i,X)e_i)^\perp + (\nabla_{e_i}\nabla_{e_i}X)^\perp - (mH_0,\nabla_{e_i}X)e_i
$$

Since $(\nabla_{e_i}\nabla_{e_i}X)^\perp = \nabla_{e_i}^\perp\nabla_{e_i}^\perp X - (X,B(e_i,e_j))B(e_i,e_j)$ we finally have

$$
\frac{D}{dt}mH_t = \Delta^\perp X + (X,B(e_i,e_j))B(e_i,e_j) + (R(e_i,X)e_i)^\perp - (mH_0,\nabla_{e_i}X)e_i
$$

As a consequence, if $\Sigma_0$ is minimal, the second derivative of the $m$-volume of $\Sigma_t = F_t(\Sigma)$ is given by

$$
\frac{d^2}{dt^2}A(\Sigma_t)|_{t=0} = - \int_{\Sigma}(X,\Delta^\perp X + (X,B(e_i,e_j))B(e_i,e_j) + (R(e_i,X)e_i))d\sigma_0
$$

$$
= \int_{\Sigma} \|\nabla^\perp X\|^2 - (R(e_i,X)e_i,e_i) - (X,B(e_i,e_j)^2)d\sigma_0^2
$$

$$
= Q_\Sigma(X,X)
$$

So $Q_\Sigma$ is a quadratic form acting on sections of the normal bundle $N\Sigma$. It is attached to the Jacobi operator acting on normal sections:

$$
LX = \Delta^\perp X + (R(e_i,X)e_i)^\perp + (X,B(e_i,e_j))B(e_i,e_j)
$$

This operator is elliptic and self-adjoint. It has a spectrum $\lambda_0 \leq \lambda_1 \leq \cdots$. If $\lambda_0 < 0$, $\Sigma$ is called unstable. The index of $L$ (the number of negative eigenvalues) is called the index of $\Sigma$. 
3. Ancient solutions of the mean curvature flow

3.1. The mean curvature flow. First let us recall some basics of the mean curvature flow and state our main result. For a good introduction to the high co-dimension case, one can have a look to Smoczyk’s paper [19].

Let \((M,g)\) be a Riemannian manifold and \(\Sigma\) a \(m\)-manifold. Let \(F: \Sigma \times I \to M\) (\(I\) an interval) be a smooth map such that \(F_t = F(\cdot, t)\) is an immersion for any \(t\). We say that \(F_t(\Sigma)\) evolve by mean curvature flow if for any \(p \in \Sigma\) and \(t \in I\)

\[
\frac{dF}{dt}(p, t) = m\vec{H}(p, t)
\]

where \(\vec{H} \in T_{F_t(p)}M\) is the mean curvature vector of \(F_t(\Sigma)\) at \(F_t(p)\).

For example, if \(F_0(\Sigma)\) is a minimal submanifold, then \(F_t = F_0\) for \(t \in I\) is a solution of the mean curvature flow: minimal submanifolds are fixed points of the mean curvature flow.

Our aim is to produce solutions that flow out of a minimal surface. More precisely, we construct non constant ancient solutions of the mean curvature flow (i.e. defined on a time interval \((-\infty, b)\)) such that, as \(t \to -\infty\), \(F_t(\Sigma)\) converges to a minimal surface.

It is well known that one difficulty in the solvability of \([\text{MCF}]\) is the invariance under the diffeomorphism group which causes a lack of parabolicity of the system. One solution to this difficulty consists in adding a tangential component to the time derivative of \(F\) which has no impact on the geometric evolution.

Let us explain such a solution. Let \(\Sigma\) be an immersed closed submanifold in \(M\). Let \(N\Sigma\) denote the normal bundle to \(\Sigma\). Then we can consider the map

\[
\Phi: \quad N\Sigma \quad \longrightarrow \quad M \\
(p, v) \quad \longmapsto \quad \exp_p(v)
\]

For \(\varepsilon > 0\), let us denote \(N\Sigma^c = \{(p, v) \in N\Sigma \mid \|v\| < \varepsilon\}\). If \(\varepsilon\) is small enough, the restriction of \(\Phi\) to \(N\Sigma^c\) is an immersion so the metric \(g\) can be lifted to \(h = \Phi^*g\) on \(N\Sigma^c\). Now studying immersed submanifolds close to \(\Sigma\) consists in looking at sections of \(N\Sigma^c\) close to 0. Actually one can extend the Riemannian metric \(h\) to the whole \(N\Sigma\) and just look at sections close to 0.

So the general setting we have to consider is the following. Let \(E\) be a vector bundle over a closed manifold \(\Sigma\) and consider \(g\) a Riemannian metric on the manifold \(E\). We say that \(E\) is a normal bundle if the fibers are orthogonal to \(\Sigma_0\) the image of the 0 section. If \(E\) is a normal bundle there is a natural identification between \(E\) and the normal bundle to \(\Sigma_0\). So as a normal bundle, \(E\) inherits a bundle metric \(g^\perp\) and a connection \(\nabla^\perp\).

If \(U\) is a section of \(E\), \(U(\Sigma)\) is a submanifold in \(E\). Then sections are a particular way to parametrize submanifolds in \(E\). Let \(U\) be a section of \(E\) and \(p \in \Sigma\). Since \(U\) is a section, the tangent space \(T_{U(p)}E\) splits as
where $E_p$ is the fiber of $E$ over $p$. Moreover, there is a natural identification of $T_{U(p)}E_p$ with $E_p$. So for any $Y \in T_{U(p)}E$, one can define $Y^\sharp$ the projection of $Y$ to $E_p$ parallel to $T_pU$.

With this type of notation, we can define the bundle mean curvature flow in the following way: let $U : \Sigma \times I \to E$ a smooth map such that $U_t = U(\cdot, t)$ is a section of $E$, we say that $(U_t)_{t \in I}$ evolves by bundle mean curvature flow if for any $p \in \Sigma$ and $t \in I$

\begin{equation}
\frac{dU}{dt}(p,t) = (m\vec{H}(U_t,p))^\sharp
\end{equation}

where $\vec{H} \in T_{U_t(p)}E$ is the mean curvature vector of the graph of $U_t$ at $U_t(p)$.

$(\vec{H}(U_t,p))^\sharp$ is equal to $\vec{H}(U_t,p)$ plus a tangent vector to $U_t(\Sigma)$. So solutions to (1) give rise to solutions to the mean curvature flow (MCF) after a reparametrization.

Let us define the operator $H : \Gamma(E) \to \Gamma(E)$ by $H(U)(p) = (m\vec{H}(U_t,p))^\sharp$. $H$ is a smooth quasilinear elliptic differential operator of order 2.

Let us assume $E$ is a normal bundle and $\Sigma_0$ is minimal, i.e. $H(0) = 0$. We can compute the differential of $H$ with respect to $U$ at 0, this gives

\[ DH(0)(V) = L(V) = \Delta^\perp V + (R(e_i, V)e_i)^\perp + (V, B(e_i, e_j))B(e_i, e_j) \]

which is an elliptic self-adjoint operator on $\Gamma(E)$. So $L$ has a discrete spectrum $\lambda_0 \leq \lambda_1 \leq \cdots$. Let us notice that $\Sigma_0$ is unstable if $\lambda_0 < 0$. So the main theorem of the section is the following

**Theorem 1.** Let $E \to \Sigma$ be as above. Assume that the first eigenvalue $\lambda_0$ of $L$ is negative. Then for any section $V$ in the first eigenspace, there is $U$ an ancient solution of (1) defined on $(-\infty, b)$ such that

\[ \lim_{t \to -\infty} e^{\lambda_0 t} U_t = V \]

One can compare this result with [4, Theorem 1.6 and Theorem 3.3] by Choi and Mantoulidis. The main difference is that here the ancient solution is parametrized by its asymptotic as $t \to -\infty$ while Choi and Mantoulidis parametrized it by its value at time $t = 0$.

### 3.2. The functional spaces.

In order to prove the above result we need to introduce some functional spaces. Following Solonnikov [20], we recall the definition of the Hölder spaces.

Let $\Omega \subset \mathbb{R}^m$ be a smooth domain and $P = \Omega \times [a, b]$. Then for $u : P \to \mathbb{R}^N$ and $\beta \in (0, 1)$, we define the Hölder semi-norms

\[ [u]_{\beta,P,x} = \sup_{(x,t) \neq (y,t) \in P} \frac{|u(x,t) - u(y,t)|}{|x - y|^{\beta}} \]

\[ [u]_{\beta,P,t} = \sup_{(x,t) \neq (x,s) \in P} \frac{|u(x,t) - u(x,s)|}{|t - s|^{\beta}} \]
and the uniform norm
\[ \|u\|_{0,P} = \sup_{X \in P} |u(X)| \]
For \( \alpha \in (0, 1) \) we define the combined H"older semi-norms
\[
[u]_{2,\alpha,P,x} = [\partial_x^2 u]_{\alpha,P} + [\partial_t u]_{\alpha,P}
\]
\[
[u]_{2,\alpha,P,t} = [\partial_x u](1+\alpha)/2,P,t + [\partial_t^2 u]_{\alpha/2,P,t} + [\partial_t u]_{\alpha/2,P,t}
\]
Finally we have the H"older norms
\[
\|u\|_{0,\alpha,P} = \|u\|_{0,P} + [u]_{\alpha,P,x} + [u]_{\alpha/2,P,t}
\]
\[
\|u\|_{2,\alpha,P} = \sum_{i=0}^2 \|\partial_x^i u\|_{0,P} + \|\partial_t u\|_{0,P} + [u]_{2,\alpha,P,x} + [u]_{2,\alpha,P,t}
\]
When \( u \) is defined on \( \Omega \), \( u \) does not depend on \( t \) so all the terms corresponding to the \( t \) variable disappear and we have the specific notations:
\[
|u|_{0,\alpha,\Omega} = \|u\|_{0,\Omega} + [u]_{\alpha,\Omega,x}
\]
\[
|u|_{2,\alpha,\Omega} = \sum_{i=0}^2 \|\partial_x^i u\|_{0,\Omega} + [u]_{2,\alpha,\Omega,x}
\]
We then have the associated H"older spaces \( C^{0,\alpha}(P) \), \( C^{2,\alpha}(P) \), \( C^{0,\alpha}(\Omega) \), \( C^{2,\alpha}(\Omega) \) made of applications \( u \) such that the above norms are well defined and finite.

This H"older spaces can be analogously defined on a closed Riemannian manifold \( (\Sigma, g) \) and for sections of a vector bundle \( E \) over \( \Sigma \) where \( E \) is equipped with a bundle metric \( h \) and a metric connection \( \nabla \). The vector bundle \( E \) can be extended as a vector bundle (still denoted by \( E \)) over \( \Sigma \times \mathbb{R} \). So if \( P = \Sigma \times [a,b] \) and \( U : P \to E \) is a section, we can define the H"older semi-norms
\[
[U]_{\beta,E_{[a,b]},x} = \sup_{(x,t),(y,t) \in P \atop d_g(x,y) < i_g} \frac{|U(x,t) - P_y,x U(y,t)|}{|x-y|^{\beta}}
\]
\[
[U]_{\beta,E_{[a,b]},t} = \sup_{(x,t),(x,s) \in P \atop x \neq y} \frac{|U(x,t) - U(x,s)|}{|t-s|^{\beta}}
\]
where \( i_g \) denotes the injectivity radius of \( \Sigma \) and \( P_{y,x} \) is the parallel transport operator from \( y \) to \( x \). Once this is defined we can construct the H"older norms similarly to the Euclidean case. The uniform norm:
\[
\|U\|_{0,E_{[a,b]}} = \sup_{X \in P} |U(X)|
\]
For \( \alpha \in (0, 1) \) we define the combined H"older semi-norms
\[
[U]_{2,\alpha,E_{[a,b]},x} = [\nabla^2 U]_{\alpha,E_{[a,b]},x} + [\partial_t U]_{\alpha,E_{[a,b]},x}
\]
\[
[U]_{2,\alpha,E_{[a,b]},t} = [\nabla U](1+\alpha)/2,E_{[a,b]},t + [\nabla^2 U]_{\alpha/2,E_{[a,b]},t} + [\partial_t U]_{\alpha/2,E_{[a,b]},t}
\]
Finally we have the Hölder norms
\[ \| U \|_{0, \alpha, E_{[a,b]}} = \| U \|_{0, E_{[a,b]}} + [U]_{\alpha, E_{[a,b]}} + [U]_{\alpha/2, E_{[a,b]}}, t} \]
\[ \| U \|_{2, \alpha, E_{[a,b]}} = \sum_{i=0}^{2} \| \nabla_{x}^{i} U \|_{0, E_{[a,b]}} + \| \partial_{t} U \|_{0, E_{[a,b]}} + [U]_{2, \alpha, E_{[a,b]}}, x} + [U]_{2, \alpha, E_{[a,b]}}, t} \]

When \( U \) is defined on \( \Sigma \), we have the specific notations:
\[ |U|_{0, \alpha, E} = \| U \|_{0, \Sigma} + [U]_{\alpha, E, x} \]
\[ |U|_{2, \alpha, E} = \sum_{i=0}^{2} \| \nabla_{x}^{i} U \|_{0, E} + [U]_{2, \alpha, E, x} \]

We then have the associated Hölder spaces \( C^{0, \alpha}(E_{[a,b]}), C^{2, \alpha}(E_{[a,b]}), C^{0, \alpha}(E), \]
\( C^{2, \alpha}(E) \). In the sequel we will also use the \( L^{2} \) norms \( \| \cdot \|_{L^{2}(E_{[a,b]})} \) and \( \| \cdot \|_{L^{2}(E)} \).

For a section \( U \) defined over \( \Sigma \times \mathbb{R} \), we denote \( U_{t}(\cdot) = U(\cdot, t) \).

3.3. Linear operators. If the fiber of \( E \) has dimension \( k \), sections of can locally be written has maps: \( u = (u^{a})_{1 \leq a \leq k} : \Omega \to \mathbb{R}^{k} \). In the sequel, we consider families of linear differential operators of order \( 2 \) acting on sections of \( E \) which in coordinates takes the form
\[ (L_{t}u)^{a} = \sum_{|I| \leq 2, b \leq k} A_{b}^{a I}(x, t) \partial_{I} u^{b} \]

where \( I \) denote a multi-index and \( \partial_{I} \) is the partial derivative associated to \( I \). \( L_{t} \) will be elliptic in the following sense: there is a constant \( \lambda > 0 \) such that for any \( \xi = (\xi_{1}, \ldots, \xi_{n}) \) and \( v = (v_{1}, \ldots, v_{k}) \) we have
\[ \sum_{i,j=1}^{2} A_{b}^{a ij} \xi_{j} v^{j} v_{i} \geq \lambda|\xi|^{2} |v|^{2} \]

Moreover we say that \( L_{t} \) has \( C^{\alpha} \) coefficients if the functions \( A_{b}^{a I} \) are in \( C^{0, \alpha} \). We denote by \( \Lambda \) the maximum of the \( C^{0, \alpha} \) norms of these coefficients.

An important result for us is the following Schauder estimate for solutions of parabolic systems associated to such operators \( L_{t} \)

**Theorem 2** ([20 Theorem 4.11]). Let \( \Omega' \subset \Omega \subset \mathbb{R}^{n} \) be smooth bounded domains with \( \overline{\Omega'} \subset \Omega \). Let \( P = \Omega \times [0, T] \) and \( P' = \Omega' \times [0, T] \). Let \( L_{t} \) be elliptic differential operators of order \( 2 \) as in (2) with \( C^{\alpha} \) coefficients in \( \Omega' \).
Then there is a constant \( C \) depending on \( \Omega, \Omega', \lambda, \Lambda, \alpha \) and \( T \) such that for any \( u \in C^{2, \alpha}(\overline{\Omega'}, \mathbb{R}^{k}) \) and \( f \in C^{0, \alpha}(\overline{P'}, \mathbb{R}^{k}) \) satisfying \( \partial_{t} u - L_{t} u = f \) we have
\[ \| u \|_{2, \alpha, P'} \leq C(\| f \|_{0, \alpha, P} + [u_{0}]_{2, \alpha, \Omega} + \| u \|_{L^{2}(P)}) \]

where \( u_{0}(\cdot) = u(\cdot, 0) \).

Similar estimates can also be found in Friedman’s paper [6].

Using finitely many local charts for a vector bundle \( E \to \Sigma \) (\( \Sigma \) is closed),
we can obtain an equivalent version for operators acting on sections of \( E \).
Proof. Let us consider the map

\[ U \rightarrow T \]

The unique solution actually it is uniformly bounded as \( \varepsilon > 0 \) there is \( \partial \mathcal{C} \) local charts with \( \mathcal{L} \) be a time independent elliptic differential operator of order \( 2 \) as in (2) in any local charts with \( C^\alpha \) coefficients. Then, for any \( U_0 \in C^{2,\alpha}(E) \) and \( F \in C^{0,\alpha}(E) \) satisfying \( \partial U - L_t U = F \), we have

\[
\|U\|_{2,\alpha,E[0,T]} \leq C(\|F\|_{0,\alpha,E[0,T]} + |U_0|_{2,\alpha,E} + \|U\|_{L^2(E[0,T])})
\]

A consequence is the following solution to the Cauchy problem

Theorem 4. Let \( E \) be a vector bundle over a closed manifold \( \Sigma \). Let \( L_1 \) be elliptic differential operators of order \( 2 \) as in (2) in any local charts with \( C^\alpha \) coefficients. Then, for any \( U_0 \in C^{2,\alpha}(E) \) and \( F \in C^{0,\alpha}(E) \), there is a unique \( U \in C^{2,\alpha}(E) \) such that

\[
\left\{
\begin{aligned}
\partial_t U - L_t U(\cdot,t) &= F(\cdot,t) \\
U(\cdot,0) &= U_0
\end{aligned}
\right.
\]

For a proof see [8, Theorem 2.4].

In Theorem 3, the constant \( C \) depends on the length \( T \) of the time interval: actually it is uniformly bounded as \( T \to 0 \) but not as \( T \to \infty \). However the proof can be adapted in order to obtain the following result where the constant is time independent. This is important for our following arguments.

Proposition 1. Let \( E \) be a vector bundle over a closed manifold \( \Sigma \). Let \( L \) be a time independent elliptic differential operator of order \( 2 \) as in (2) in any local charts with \( C^\alpha \) coefficients. Then there is a constant \( C \) (independent of \( T \)) such that for any \( U_0 \in C^{2,\alpha}(E) \) and \( F \in C^{0,\alpha}(E) \) satisfying \( \partial U - L U = F \) we have

\[
\|U\|_{2,\alpha,E[0,T]} \leq C(\|F\|_{0,\alpha,E[0,T]} + |U_0|_{2,\alpha,E} + \|U\|_{L^2(E[0,T])})
\]

See the proof in Appendix A

3.4. The ancient flow. In this section we prove Theorem 4. So \( E \to \Sigma \) is a vector bundle as in Theorem 1 and we use the notations introduced in the preceding sections. We start by giving a result that ensures the existence of solutions to (1).

Theorem 5. Let \( E \to \Sigma \) as above. There is \( \delta_0 \) such that for any \( \delta < \delta_0 \) there is \( \varepsilon > 0 \) such that, for any \( W \in C^{2,\alpha}(E) \) with \( |W|_{2,\alpha,E} \leq \varepsilon \), there is a unique solution \( U \in C^{2,\alpha}(E_{[0,1]}) \) of

\[
\left\{
\begin{aligned}
\partial_t U &= H(U) \\
U(\cdot,0) &= W
\end{aligned}
\right.
\]

with \( \|U\|_{2,\alpha,E_{[0,1]}} < \delta \).

Proof. Let us consider the map

\[
F : C^{2,\alpha}(E) \times C^{2,\alpha}(E_{[0,1]}) \to \begin{cases} C^{2,\alpha}(E) \times C^{0,\alpha}(E_{[0,1]}) \\ (W,U) \mapsto (\partial_t U - H(U), \partial_t U - H(U)) \end{cases}
\]
$F$ is a $C^1$ map and $F(0,0) = (0,0)$ since $\Sigma_0$ is minimal. If we compute the differential of $F$ with respect to $U$ at $(0,0)$ we have

$$D_U F(0,0) : \frac{C^{2,\alpha}(E_{[0,1]})}{Z} \to \frac{C^{2,\alpha}(E) \times C^{0,\alpha}(E_{[0,1]})}{(Z(\cdot,0), \partial_t Z - LZ)}$$

So the invertibility of this differential is given by the solution to the Cauchy problem (Theorem 4). Hence the implicit function theorem solves $F(W,U) = (0,0)$ for any $W$ with $|W|_{2,\alpha,E} < \delta$ small.

The above theorem produces solutions to the bundle MCF (4). Let $\varepsilon(\delta)$ be given by Theorem 5 for $\delta < \delta_0$. Actually it allows you to extend a solution $U$ as long as $|U_t|_{2,\alpha,\Sigma} < \varepsilon(\delta)$.

**Proposition 2.** Let $\delta < \delta_0$. Let $U$ be a solution of the bundle MCF defined on $\Sigma \times [a,b]$ with $\|X\|_{2,\alpha,E_{[a,b]}} \leq \delta$. Let $\tilde{t} \in (b-1,b)$ and assume that $|U_{\tilde{t}}|_{2,\alpha,\Sigma} \leq \varepsilon(\delta)$ then $U$ can be extended as a solution of the bundle MCF defined on $\Sigma \times [a,\tilde{t} + 1]$

**Proof.** Let $Z$ be the solution of (4) defined on $\Sigma \times [\tilde{t},\tilde{t} + 1]$ with $Z(\cdot,\tilde{t}) = U(\cdot,\tilde{t})$ given by Theorem 5. It suffices to prove that $Z = U$ on $\Sigma \times [\tilde{t},b]$ to conclude. This uniqueness is given by the following remark: we have

$$\partial_t (Z - U) = H(Z_t) - H(U_t) = \int_0^1 \frac{d}{ds} H(sZ_t + (1-s)U_t) ds$$

$$= \int_0^1 D(H)(sZ_t + (1-s)U_t)(Z_t - U_t) ds$$

$$= L_t(Z_t - U_t)$$

where $L_t$ are elliptic linear differential operators of order 2 acting on sections of $E$ with coefficient in $C^\alpha$. Then by the uniqueness part of Theorem 4 and since $(Z - U)|_{\tilde{t}} = 0$ we have $Z - U = 0$ on $\Sigma \times [\tilde{t},b]$. \qed

To prove Theorem 5 we consider $V$ an eigen-section associated to $\lambda_0$. We chose $\delta > 0$ as in Theorem 5. Let $a_\delta$ be such that $e^{-\lambda_0 a_\delta}|V|_{2,\alpha,E} = \varepsilon = \varepsilon(\delta)$. Then for any $a < a_\delta$, $e^{-\lambda_0 a}|V|_{2,\alpha,E} < \varepsilon$ so we can consider the section $U^{(a)}$ solution to the problem

$$\begin{cases}
\partial_t U = H(U) \\
U(\cdot,a) = e^{-\lambda_0 a}V
\end{cases}$$

on $\Sigma \times [a,b]$ where $b$ is chosen the largest possible such that $\|U^{(a)}\|_{2,\alpha,E_{[a,b]}} \leq \delta$, $|U^{(a)}|_{2,\alpha,\Sigma} \leq \varepsilon$ and $\|e^{-\lambda_0 t}V\|_{2,\alpha,E_{[a,b]}} \leq \delta$. So the proof consists in estimating the norm of $U^{(a)}$ in order to control $b$ and prove that, as $a \to -\infty$, $U^{(a)}$ converges to the desired solutions of (4).

Let us introduce $Z^{(a)} = U^{(a)} - e^{-\lambda_0 t}V$ for $t \in [a,b]$. We have the following result.
Lemma 2. There is \( \delta > 0 \) and \( b_0 \in \mathbb{R} \) such that for any \( a < \min(a_\delta, b_0) \), \( U^{(a)} \) is defined on \([a, b_0]\). Moreover for any \( a \leq b \leq b_0 \)

\[
\|Z^{(a)}\|_{L^2(E_{[a,b]})} \leq e^{-3\lambda_0 b/2}
\]

Proof. Let us choose \( \delta \in (0, \delta_0) \) as in Theorem 5 that will be fixed below. Let us write \( H = L + G \) where \( G \) satisfies \( |G(U)|_{\alpha,E} \leq C |U|_{2,\alpha,E}^2 \) for any section \( U \) of \( E \) with \( |U|_{2,\alpha,E} \leq \delta_0 \). Actually \( G \) satisfies \( \|G(U)\|_{\alpha,E_{[a,b]}} \leq C |U|_{2,\alpha,E_{[a,b]}}^2 \) for any section \( U \) of \( E_{[a,b]} \) with \( \|U\|_{2,\alpha,E_{[a,b]}} \leq \delta_0 \) and \( C \) independent of \( a, b \). In the computation below, the constant \( C \) will change line to line but independently of \( a \).

Let \( b_\delta \) be such that \( \|e^{-\lambda_0 tV}\|_{2,\alpha,E(-\infty, b_\delta]} \leq \delta \). Then for \( a < \min(a_\delta, b_\delta) \), we consider the solution \( U^{(a)} \) defined on \([a, b]\), then \( Z^{(a)} \) satisfies

\[
\partial_t Z^{(a)} = \partial_t U^{(a)} + \lambda_0 e^{-\lambda_0 tV} = L(Z^{(a)}) + G(U^{(a)}) = L(Z^{(a)}) + G(Z^{(a)} + e^{-\lambda_0 tV})
\]

Since \( \|Z^{(a)}\|_{2,\alpha,E_{[a,b]}} \leq 2\delta \), for \( c \in [a, b] \), Solonnikov’s estimate (Proposition [1]) gives:

\[
\|Z^{(a)}\|_{2,\alpha,E_{[a,c]}} \leq C(\|Z^{(a)}\|_{L^2(E_{[a,c]})} + \|G(Z^{(a)} + e^{-\lambda_0 tV})\|_{\alpha,E_{[a,c]}}) \\
\leq C(\|Z^{(a)}\|_{L^2(E_{[a,c]})} + C(\|Z^{(a)}\|_{2,\alpha,E_{[a,c]}}^2 + e^{-2\lambda_0 t})) \\
\leq C(\|Z^{(a)}\|_{L^2(E_{[a,c]})} + C(\delta \|Z^{(a)}\|_{2,\alpha,E_{[a,c]}} + e^{-2\lambda_0 t}))
\]

So we can choose and fix \( \delta \) small enough such that \( C\delta < 1 \) to obtain:

\[
\|Z^{(a)}\|_{2,\alpha,E_{[a,c]}} \leq C(\|Z^{(a)}\|_{L^2(E_{[a,c]})} + e^{-2\lambda_0 t})
\]

So if \( \|Z^{(a)}\|_{L^2(E_{[a,c]})} \leq e^{-3\lambda_0 c/2} \), we obtain \( \|Z^{(a)}\|_{2,\alpha,E_{[a,c]}} \leq C e^{-3\lambda_0 c/2} \) and \( \|U^{(a)}\|_{2,\alpha,E_{[a,c]}} \leq C e^{-\lambda_0 c} \leq \min(\delta, \epsilon) \) if \( c \) is less than some \( \bar{c} \) (we restrict the definition of \( U^{(a)} \) to \((-\infty, \bar{c}]\)). So as long as the estimate \( \|Z^{(a)}\|_{L^2(E_{[a,c]})} \leq e^{-3\lambda_0 c/2} \) is true the solution \( U^{(a)} \) is well defined. Let us now prove the estimate.

Since \( Z^{(a)}(\cdot, a) = 0 \) the estimate is true at \( c = a \). So let \( c_1 \) denote the first time where the estimate is not true. Because of (3), we have the expression

\[
Z^{(a)}_t = \int_a^t e^{(t-s)L} G(Z^{(a)}_s + e^{-\lambda_0 s}V) ds
\]
Since $\lambda_0$ is the first eigenvalue of $L$ we have
\[
|Z_t^{(a)}|_{L^2(E)} \leq \int_a^t e^{-\lambda_0(t-s)}|G(Z_s^{(a)} + e^{-\lambda_0 s} V)|_{L^2(E)} ds
\]
\[
\leq \int_a^t e^{-\lambda_0(t-s)}C|G(Z_s^{(a)} + e^{-\lambda_0 s} V)|_{0,E} ds
\]
\[
\leq \int_a^t e^{-\lambda_0(t-s)}C\left(\|Z_s^{(a)}\|^2_{L^2(E)} + e^{-2\lambda_0 s}\right) ds
\]
\[
\leq \int_a^t e^{-\lambda_0(t-s)}C(e^{-3\lambda_0 s} + e^{-2\lambda_0 s}) ds
\]
\[
\leq C e^{-\lambda_0 t} \int_a^t e^{-\lambda_0 s} ds \leq C e^{-2\lambda_0 t}
\]

Then $\|Z^{(a)}\|_{L^2(E_{[a,b_0]})} \leq C e^{-2\lambda_0 t}$. So we see that $c_1$ must satisfies $Ce^{-\lambda_0 c_1/2} \geq 1$; i.e. $c_1$ is bounded below by some universal constant $c_0$. So the Lemma is proved with $b_0 = \min(c_0, c, b_0)$.

Let $b_0$ be given by Lemma 2. By (4), we have $\|Z^{(a)}\|_{2,\alpha,E_{[a,b_0]}} \leq C e^{-3\lambda_0 b_0/2}$. By Arzela-Ascoli theorem, there is $Z \in C^{2,\alpha}(E_{(-\infty,b_0)})$, such that $Z^{(a)}$ subconverge in $C^2$ to $Z$. Moreover $Z$ satisfies $\partial_t Z = L Z + G(Z + e^{-\lambda_0 t} V)$, i.e. $U = Z + e^{-\lambda_0 t} V$ is a solution of (1). Since $\|Z^{(a)}\|_{2,\alpha,E_{[a,t]}} \leq C e^{-3\lambda_0 t/2}$ for $t \leq b_0$, we have $\|Z\|_{2,\alpha,E_{(-\infty,t]}} \leq C e^{-3\lambda_0 t/2}$ and then $\lim_{t \to -\infty} e^{\lambda_0 t} X_t = V$ in $C^{2,\alpha}$.

## 4. The rigidity result
In this section we prove a rigidity result concerning $S^n_1 = \{p \in \mathbb{R}^{n+1} \mid \|p\| = 1\}$ endowed with the induced metric $g_{S^n_1}$. For $0 \leq k \leq n - 1$, let us consider the map:
\[
\Psi : S^k_1 \times \mathbb{R} \times S^{n-k-1}_1 \to S^n_1
\]
\[
(p, s, q) \mapsto ((\cos s)p, (\sin s)q)
\]
We notice that $\Psi(S^k_1 \times [0, \pi/2] \times S^{n-k-1}_1) = S^n_1$, $\Psi$ is injective on $S^k_1 \times (0, \pi/2) \times S^{n-k-1}_1$, $\Psi(p, 0, q) = (p, 0)$ and $\Psi(p, \pi/2, q) = (0, q)$. So $S^n_1$ can be seen as the joint of $S^k_1$ and $S^{n-k-1}_1$. Moreover $\Psi^*(g_{S^n_1}) = \cos^2 s g_{S^k_1} + ds^2 + \sin^2 s g_{S^{n-k-1}_1}$. The curves $s \mapsto \Psi(p, s, q)$ are geodesics of $S^n_1$.

For $k = 2$, we see that $\Psi(S^2_1, 0, q)$ is an immersed minimal sphere in $S^n_1$ which is isometric to $S^2$. Actually it is a totally geodesic equatorial 2-sphere in $S^n_1$. As a minimal surface its index is $n - 2$. Our rigidity result looks at such an immersed sphere in a Riemannian manifold.
Theorem 6. Let M be a Riemannian n \geq 3-manifold whose sectional curvature is bounded above by 1. Let us assume that M contains an immersed minimal 2-sphere of area 4\pi which is

- either of index at least n - 2
- or unstable in any parallel directions.

Then the universal cover of M is isometric to the sphere S^n_1.

Proof. Let S be an immersed 2-sphere in M, using Gauss and Gauss-Bonnet formulas we have

\[ 4\pi = \int_S K_S = \int_S K_{TS} + (B(e_1, e_1), B(e_2, e_2)) - \|B(e_1, e_2)\|^2 \]

\[ = \int_S K_{TS} + \|\vec{H}\|^2 - \|\hat{B}(e_1, e_1)\|^2 - \|B(e_1, e_2)\|^2 \]

\[ \leq \mathcal{A}(S) + \int_S \|\vec{H}\|^2 - \frac{1}{2} \|\hat{B}\|^2 \]

where (e_1, e_2) is an orthonormal basis of TS, K_S denotes the sectional curvature of S, K_{TS} the sectional curvature of M on the 2-plane TS and \(\hat{B}\) is the traceless part of B. As a consequence

\[ F(S) = \mathcal{A}(S) + \int_S \|\vec{H}\|^2 - 4\pi \geq \int_S \frac{1}{2} \|\hat{B}\|^2 \geq 0 \]

Hence \(F(S) = 0\) implies that S is totally umbilic, \(K_{TS} = 1\) and \(K_S = 1 + \|\vec{H}\|^2\).

The immersed 2-sphere in M lifts to its universal cover with the same instability property. So we assume that M is simply connected and X : S^2 \rightarrow M an immersed minimal 2-sphere as in the statement of the theorem. Let us notice that since \(\Sigma = X(S^2)\) is a minimal surface of area 4\pi, \(F(\Sigma) = 0\) and \(\Sigma\) is totally geodesic and \(K_\Sigma = 1\) so \(\Sigma\) is isometric to \(S^2_1\); we can choose X such that X is an isometry between \(S^2_1\) and \(\Sigma = X(S^2)\). Let us denote by NX the normal vector bundle \(\{(p, v) \in X^*TM \mid v \in T_pX^\perp\}\) and consider the map

\[ \Phi : NX \rightarrow M \quad (p, v) \mapsto \exp_{X(p)}(v) \]

We want to study the pull-back metric \(h = \Phi^*g\) on NX in order to control when \(\Phi\) is an immersion.

The first step of the proof is

Lemma 3. The normal bundle NX is parallelizable. Moreover for any \(p \in S^2_1\) and unit vectors \(e \in T_pX\) and \(v \in T_pX^\perp\), \((R(e, v)e, v) = 1\).

Proof of Lemma[3]. By hypothesis \(\Sigma\) is unstable so the Jacobi operator L has a negative eigenvalue \(\lambda\). Let us prove that \(\lambda = -2\) and the associated eigen-section V is a parallel section of NX.

We have \(LV = -\lambda V\). For small \(t\), we consider the immersed sphere \(\Sigma_t = \{\Phi(p, tV(p)); p \in S^2\}\). We then have \(F(\Sigma_t) \geq 0\) for any \(t\). \(F(\Sigma_0) = 0, \)
so the first derivative of \( t \mapsto F(\Sigma_t) \) must vanish at \( t = 0 \): it is confirmed by the computation

\[
\frac{d}{dt} F(\Sigma_t)_{t=0} = \int_{\Sigma} (-2\vec{H}, V) + \int_{\Sigma} \|\vec{H}\|^2 (-2\vec{H}, V) + \int_{\Sigma} (LV, \vec{H}) = 0
\]

Now the second derivative has to be non-negative and we have

\[
\frac{d^2}{dt^2} F(\Sigma_t)_{t=0} = \int_{\Sigma} -(LV, V) + \int_{\Sigma} \frac{1}{2}(LV, LV) = \int_{\Sigma} (\lambda + \frac{1}{2}\lambda^2)\|V\|^2
\]

So \( \lambda^2 + 2\lambda \geq 0 \): \( \lambda \leq -2 \). Since \( \Sigma \) is totally geodesic, we also have

\[
\lambda \int_{\Sigma} \|V\|^2 = \int_{\Sigma} -(LV, V) = \int_{\Sigma} \|\nabla^\perp V\|^2 - (R(e_i, V)e_i, V) - (V, B(e_i, e_j))^2 \\
= \int_{\Sigma} \|\nabla^\perp V\|^2 - (R(e_i, V)e_i, V) \geq \int_{\Sigma} -2\|V\|^2
\]

So \( \lambda \geq -2 \). This gives \( \lambda = -2 \).

The above computation shows also that \( V \) is a parallel normal vector field to \( \Sigma \) and \( (R(e, V)e, V) = 1 \) for any vector \( e \in T\Sigma \).

Let \( V_1, \ldots, V_d \) be a basis of the eigenspace associated to the eigen-value \(-2\). Let \( B \) be the sub-bundle of \( NX \) generated by \( V_1, \ldots, V_d \): \( B = \{(p, v) \in NX \mid v \in \text{span}(V_1(p), \ldots, V_d(p))\} \). \( B \) is parallelizable and, on \( B \), the stability operator is \( L = -\Delta^\perp - 2 \). So the index of \( L \) restricted to \( B \) is precisely \( d \). If \( d < n - 2 \), both hypotheses on \( \Sigma \) implies that the restriction of \( L \) to \( B^\perp \) must have a negative eigenvalue. Thus there is an eigensection of eigenvalue \(-2\) in \( B^\perp \) contradicting the definition of \( B \). So \( d = n - 2 \) and \( B = NX \) which ends the proof. \( \square \)

The sequel of the proof is a generalization of the above argument.

Let us fix \( V \) an eigen-section of \( L \) associated to the eigenvalue \(-2\). By Theorem [1], let \( (\Sigma_t)_{t \in (-\infty, b)} \) be the ancient solution of the mean curvature flow flowing out of \( \Sigma \) in the direction \( V \). \(( -\infty, b)\) is a maximal time interval of existence.
We look at the evolution of $F(\Sigma_t)$. We know that $\lim_{t\to-\infty} F(\Sigma_t) = 0$. Computing its derivative, we obtain

$$\frac{d}{dt} F(\Sigma_t) = \int_{\Sigma_t} -4\|\vec{H}\|^2 + \int_{\Sigma_t} -4\|\vec{H}\|^4 + \int_{\Sigma_t} -2\|\nabla^\perp \vec{H}\|^2$$

$$2(R(e_i, \vec{H})e_i, \vec{H}) + 2(\vec{H}, B(e_i, e_j))^2$$

$$\leq \int_{\Sigma_t} 2[(R(e_i, \vec{H})e_i, \vec{H}) - 2\|\vec{H}\|^2] + \int_{\Sigma_t} 2(\vec{H}, B(e_i, e_j))^2$$

$$\leq \int_{\Sigma_t} 2\|\vec{H}\|^2\|B\|^2 \leq 4 \sup_{\Sigma_t} \|\vec{H}\|^2 F(\Sigma_t)$$

By construction of $\Sigma_t$, we know that close to $-\infty$, $\sup_{\Sigma_t} \|\vec{H}\|^2 \leq Ce^{4t}$. Using this and $\lim_{t\to-\infty} F(\Sigma_t) = 0$, a Gronwall type argument gives $F(\Sigma_t) \leq 0$ for any $t$ and then $F(t) = 0$ for any $t$. So $\vec{B} = 0$ on $\Sigma_t$, $K_{\Sigma_t} = 1$.

This also implies that the derivative of $F(\Sigma_t)$ vanishes: $\vec{H}$ is a parallel normal vectorfield along $\Sigma_t$ and $(R(e, \vec{H})e, \vec{H}) = \|\vec{H}\|^2$ for any unit vector $e \in T\Sigma_t$. Since $\vec{H}$ is parallel, we have $\|\vec{H}\|$ is constant along $\Sigma_t$ (notice that $\|\vec{H}\| \neq 0$ by construction). So we can write $\vec{H} = H_1 \nu$ ($H_1 = \|\vec{H}\|(t)$) where $\nu$ is a unit normal vector field to $\Sigma$. Moreover $\nu$ is parallel along $\Sigma_t$.

Let us define a new time parameter $s = s(t) = \int_{-\infty}^{t} H_1 du$, so that $\frac{ds}{dt} = H_1$. Hence the derivative of $\Sigma_s$ with respect to $s$ is given by $\nu$.

If $q \in \Sigma_s$, the map $(a, b) \mapsto (R(a, b)a, b)$ defined for unit vectors $a, b \in T_qM$ is bounded above by $1$ and reaches its maximum at $(a, b) = (f, \nu)$ where $f \in T_q\Sigma_s$. So we have $(R(f, \nu)f, \nu) = 0$ for any $v \in \nu^\perp$ and $(R(f, \nu)v, \nu) = 0$ for any $v \in f^\perp$.

We can compute $\frac{\partial}{\partial s} 2\vec{H}_s$ in two ways:

$$\frac{\partial}{\partial s} 2\vec{H}_s = 2 \frac{d}{ds} H_s \nu + 2H_s \frac{\partial}{\partial s} \nu$$

and

$$\frac{\partial}{\partial s} 2\vec{H}_s = \Delta^\perp \nu + (R(e_i, \nu)e_i) + (\nu, B(e_i, e_j))B(e_i, e_j) - (\vec{H}, \nabla^\perp e_i \nu)e_i$$

$$= 2\nu + 2H_s^2 \nu$$

Hence $\frac{\partial}{\partial s} \nu = 0$; the evolution follows geodesics and $\Sigma_s = \{ \Phi(p, sV(p)) ; p \in \Sigma \}$. Besides $\frac{d}{ds} H_s = 1 + H_s^2$, so $H_s = \tan s$.

Let $\gamma_p$ be the geodesic $s \mapsto \Phi(p, sV(p))$. We are going to study some Jacobi fields along $\gamma_p$. Let $f_1, \ldots, f_{n-1}$ be parallel orthonormal fields along $\gamma_p$ such that $f_1, \ldots, f_{n-1}, \nu$ is orthonormal and, at $s = 0$, $f_1, f_2$ is a basis of $T_pX$. For $i \in \{1, 2\}$ we define $\partial_i$ the Jacobi field along $\gamma_p$ such that $\partial_i(0) = f_i$ and $\frac{\partial}{\partial s} \partial_i(0) = 0$. Actually if $e_i \in T_pS^2$ is such that $X_s(e_i) = f_i$, we have $\partial_i = D_{e_i}(\Phi(\cdot, sV(\cdot)))$ so $\partial_i(s)$ is tangent to $\Sigma_s$ as long as $\Sigma_s$ is well defined. $\partial_i$ is a Jacobi field so because of the above computations of the
Riemann tensor
\[ 0 = \frac{D^2}{ds^2} \partial_i + R(\nu, \partial_i)\nu = \frac{D^2}{ds^2} \partial_i + \partial_i \]
Decomposing this equation in \((f_1, \cdots, f_{n-1})\), we obtain that \(\partial_i = \cos s f_i\). We have \((\partial_1, \partial_2)\) is an orthogonal basis of \(T\Sigma_s\): \(\Sigma_s\) is an immersion for \(s \in [0, \pi/2]\).

As a consequence the orthogonal of \(T\Sigma_s\) is generated by \((\nu, f_3, \cdots, f_{n-1})\).

Let \(\partial_j (j \geq 3)\) be the Jacobi fields along \(\gamma_p\) with \(\partial_j(0) = 0\) and \(\frac{D}{ds} \partial_j(0) = f_j\). We have \((R(\nu, \partial_j)\nu, f_i) = (\partial_j, f_i)\) for \(i \in \{1, 2\}\) and \(s \in [0, \pi/2]\). So \((\partial_j, f_i)\) is solution of the ODE \(y'' + y = 0\) with vanishing initial value and derivative. Thus \((\partial_j, f_i) = 0\) and \(\partial_j\) belongs to \(\text{span}(f_3, \cdots, f_{n-1})\). Moreover Rauch comparison theorem implies that \(\partial_j, j \in \{3, \cdots, n-1\}\), are non vanishing on \((0, \pi)\).

We know that \(NX\) is parallelizable so we can fix an isometric parametrization by \(NX \simeq S^2_1 \times \mathbb{R}^{n-2}\) and we have a map
\[ \Phi : S^2_1 \times \mathbb{R}^{n-2} \rightarrow M \]
Using a polar decomposition of \(\mathbb{R}^{n-2}\) as \(\mathbb{R}_+ \times S^{n-3}\) and coordinates \((p, s, q) \in S^2 \times \mathbb{R}_+ \times S^{n-3}\), this gives a map \(\Psi : S^2 \times \mathbb{R}_+ \times S^{n-3} \rightarrow M\) defined by \(\Psi(p, s, q) = \Phi(p, s, q)\). The above study of the Jacobi fields along the geodesic gives that the lift of the metric is given by
\[ \Psi^* g = \cos^2 s g_{S^2_1} + ds^2 + g_{p, s} \]
for \(s \in [0, \pi/2]\) and \(g_{p, s}\) is a smooth family of metrics on \(S^{n-3}\) depending on \((p, s) \in S^2 \times (0, \pi/2)\). Let us notice that \(g_{p, s} = s^2 g_{S^2_1} + o_0(s^2)\) and \(g_{p, s}\) is a well defined metric on \(S^{n-3}\).

As a consequence \(\Psi(p, \frac{\pi}{2}, q)\) is a point \(Q \in M\) that does not depend on \(p\). If \(\bar{p}\) is fixed \(\Sigma' = \Psi(\bar{p}, \frac{\pi}{2}, S^{n-3})\) is then an immersed submanifold of \(M\). Let us study the geometry near \(\Sigma'\). The geodesics \(s \mapsto \Psi(p, s, q)\) arrive orthogonally to \(\Sigma'\) when \(s = \frac{\pi}{2}\). Let us fix \(\bar{q} \in S^{n-3}\) and define a map
\[ G : S^2 \rightarrow U_{\bar{q}}X' \]
\[ p \mapsto \frac{d}{ds} \Psi(p, s, \bar{q})|_{s=\frac{\pi}{2}} \]
where \(U_{\bar{q}}X'\) is the unit sphere in the normal bundle \(TX'\) at \(\bar{q}\).

For \(r \in [0, \pi/2]\) let us define \(F_r : S^2 \rightarrow M, p \mapsto \exp_{\bar{q}}(-r G(p))\). We have \(F_r(p) = \Psi(\bar{p}, \frac{\pi}{2} - r, \bar{q})\) so \(F_r^* g = \sin^2 r g_{S^2}\). So \(G^* g_{\bar{q}} = \lim_{r \rightarrow 0} \frac{1}{r} F_r^* g = g_{S^2}\). So \(G\) is a linear isometry between \(S^2\) and \(U_{\bar{q}}X'\). As a consequence \(G(-p) = -G(p)\). Thus \(\Psi(p, s + \frac{\pi}{2}, \bar{q}) = \exp_{\bar{q}}(s G(p)) = \exp_{\bar{q}}(-s G(-p)) = \Psi(-p, \frac{\pi}{2} - s, \bar{q})\) for \(s \in [0, \frac{\pi}{2}]\). So \(\Psi(p, \pi, q) = \Psi(-p, 0, q) = X(-p)\). This implies that \(\pi\) is a conjugate time for the Jacobi fields \(\partial_j (3 \leq j \leq n-1)\). Then by Rauch comparison theorem, \(|\partial_j(s)| = |\sin s|\) and \((R(\partial_j, \nu)\partial_j, \nu) = \sin^2 s\). Thus \(R(f_j, \nu)f_j, \nu) = 1\) for any \(j\). So \(\Psi^* g = \cos^2 s g_{S^2_1} + ds^2 + \sin^2 s g_{S^2_1}^{-1}\). As a consequence, \(\Psi\) generates a local isometry \(\Psi'\) from the joint of \(S^2\) with
$S^{n-3}$ endowed with the metric $\cos^2 s g_{S^1} + ds^2 + \sin^2 s g_{S^{n-3}}$ to $M$. So $\Psi'$ is a local isometry and thus a covering map from $S^n$ to $M$. Since $M$ is simply connected $\Psi'$ is a global isometry.

The above result has a corollary concerning manifold containing an "equator". This an infinitesimal version of [2, Corollary 5.11].

**Corollary 1.** Let $M$ be a Riemannian $n \geq 3$-manifold whose sectional curvature is bounded above by 1. Let us assume that there is a totally geodesic isometric immersion $f : S^{n-1} \to M$. Moreover, assume that $f$ is unstable as a minimal hypersurface. Then the universal cover of $M$ is isometric to $S^n$.

**Proof.** Let $G(2, n-1)$ be the set of totally geodesic 2 spheres in $S^{n-1}$: the intersections of $S^{n-1}$ with any 3-plane in $\mathbb{R}^n$. For any $S$ in $G(2, n-1)$, $f(S)$ is totally geodesic in $M$. Moreover $f(S)$ has index at least $n-3$. So it is enough to prove that one of these $S$ has index at least $n-2$ to conclude by Theorem 6. For any $S \in G(2, n-1)$, let $Q_S$ be the quadratic form associated to the Jacobi operator on $f(S)$.

Let $\nu$ be the unit normal to $f(S)$. For $S \in G(2, n-1)$, span($\nu$) is a parallel normal bundle along $f(S)$. Let $\lambda_0 < 0$ be the first eigenvalue of the Jacobi operator on $f(S)$ and $u$ a first eigenfunction. On $S^{n-1}$, we can define a quadratic form $q = du \otimes du - (R(\nu, \cdot)\nu, \cdot)$. Then the quadratic form $Q_{S^{n-1}}$ associated to the Jacobi operator on $f(S)$ satisfies to

$$Q_{S^{n-1}}(u, u) = \int_{S^{n-1}} tr_{S^{n-1}} q = \lambda_0 \int_{S^{n-1}} u^2 < 0$$

Now there is a dimensional constant $c_n$ such that

$$c_n \int_{S^{n-1}} tr_{S^{n-1}} q = \int_{G(2, n-1)} dS \int_S tr_S q$$

$$= \int_{G(2, n-1)} Q_S(\nu, \nu) dS$$

where $tr_P$ denotes the trace operator on the subspace $P$ and $dS$ is the Haar measure on $G(2, n-1)$ coming from the Haar measure on the Grassmannian of 3-planes in $\mathbb{R}^{n+1}$. Thus there is $S$ such that $Q_S(\nu, \nu) < 0$. So the restriction to span($\nu$) of the stability operator for $S$ is unstable: $S$ has index at least $n-2$. 

5. RIGIDITY OF SIMON-SMITH WIDTH

Here we state Smith theorem concerning the existence of a minimal sphere in any Riemannian 3-sphere ($S^3$, $g$) and our second rigidity result.

We start with the standard sweep-out of the sphere $S^3$ given by horizontal spheres

$$S_t = \{(x_1, x_2, x_3, x_4) \in S^3 \mid x_4 = t\} \text{ for } t \in [-1, 1]$$
If \( F_t : \mathbb{S}^3 \to \mathbb{S}^3 \) \( (t \in [-1, 1]) \) is a smooth family of diffeomorphisms isotopic to the identity map, we can define a general sweep-out of \( \mathbb{S}^3 \) by spheres as the family \( \sigma_t = F_t(S_t) \) \( (t \in [-1, 1]) \). Let \( \Lambda \) denote the set of all general sweep-outs of \( \mathbb{S}^3 \) by spheres. The Simon-Smith width of a Riemannian 3-sphere \( (\mathbb{S}^3, g) \) is then defined by
\[
W(\mathbb{S}^3, g) = \inf_{\{\sigma_t\} \in \Lambda} \left( \max_{t \in [-1, 1]} A(\sigma_t) \right)
\]

For \( \sigma \in \Lambda \), we denote \( L(\sigma) = \max_{t \in [-1, 1]} A(\sigma_t) \). So a minimizing sequence for the Simon-Smith width \( W \) is a sequence \( (\sigma^n) \) in \( \Lambda \) such that \( \lim L(\sigma^n) = W \). For such a minimizing sequence, one can consider a sequence \( (t_n) \in [-1, 1] \) such that \( \lim A(\sigma_{t_n}^n) = W \). Such a sequence is called a min-max sequence. The main result in [18] is

**Theorem 7** (Smith). There is a min-max sequence that converges in the sense of varifolds to a disjoint union of embedded minimal spheres (possibly with multiplicity) whose area is \( W(\mathbb{S}^3, g) \).

Moreover the quantity \( W(\mathbb{S}^3, g) \) is positive so the collection of minimal 2-spheres is not empty.

If the sectional curvature of \( (\mathbb{S}^3, g) \) is bounded above by 1 the area of each sphere in the collection is at least \( 4\pi \) so \( W(\mathbb{S}^3, g) \geq 4\pi \). We notice that for the Euclidean sphere \( \mathbb{S}^3_1 \) we have \( W(\mathbb{S}^3_1) = 4\pi \). The main result of this section is a rigidity result for the equality case.

**Theorem 8.** Let \( (\mathbb{S}^3, g) \) be a Riemannian 3-sphere whose sectional curvature is bounded above by 1. Then \( W(\mathbb{S}^3, g) \geq 4\pi \) and, if \( W(\mathbb{S}^3, g) = 4\pi \), then \( (\mathbb{S}^3, g) \) is isometric to \( \mathbb{S}^3_1 \).

In order to give the proof of this result we need to understand the index property of the minimal spheres that realize \( W(\mathbb{S}^3, g) \). In the sequel we introduce de notion of almost stable minimal hypersurfaces. We will use concept from geometric measure theory, for the notations we refer to [5] and [11].

### 5.1. The Simon-Smith min-max surface

In this section we recall some aspect of the min-max construction by Smith in [18]. Actually we mainly refer to the statements contained in the paper by Colding and De Lellis [5].

If the varifold \( V \) is a varifold limit of some min-max-sequence we say that \( V \) is a min-max varifold. So the proof of Theorem 7 consists in finding a min-max varifold which is smooth and has the right topological type.

The first step in this proof is that there is a minimizing sequence \( (\sigma^n) \) such that any min-max varifold coming from \( (\sigma^n) \) is stationary (see [5, Proposition 4.1]). The second step toward regularity use the notion of almost-minimizing surface.
Definition 2. Given $\varepsilon > 0$, an open set $U \in S^3$ and a surface $\Sigma$. $\Sigma$ is $\varepsilon$-almost minimizing in $U$ if there is no isotopy $\psi$ supported in $U$ such that

$$A(\psi_t(\Sigma)) \leq A(\Sigma) + \varepsilon/8 \text{ for all } t \quad A(\psi_1(\Sigma)) \leq A(\Sigma) - \varepsilon$$

A sequence $(\Sigma_n)$ is said to be almost minimizing in $U$ if $\Sigma_n$ is $\varepsilon_n$-almost minimizing in $U$ for some sequence $\varepsilon_n \to 0$

For $p$ and $r > 0$, we define the set of annuli centered at $p$ of outer radius less that $r$

$$AN_r(p) = \{ B_t(p) \setminus \overline{B_s(p)} : 0 < s < t < r \}$$

Then one can select a positive function on $S^3$ and a min-max sequence $(\sigma_n^0)$ which is almost minimizing in any small annuli: i.e. in any $A \in AN_r(p)$ for all $p \in S^3$ (see [5, Proposition 5.1]). Then the author proves that if $\sigma_n^0 \to V^*$ as varifold, $V^*$ has the expected properties. Concerning smoothness, the author introduce the notion of replacement.

Definition 3. Let $V$ be a stationary varifold and $U$ is an open subset of $S^3$. A stationary varifold $V'$ is said to be a replacement of $V$ in $U$ if

1. $V' = V$ on $G(M \setminus \overline{U})$ and $\|V'\|(M) = \|V\|(M)$
2. $V'|U$ is supported by a stable minimal surface $\Sigma$ with $\Sigma \setminus \Sigma \subset \partial U$.

The proof of the smoothness $V^*$ consist basically in proving that $V^*$ has replacements in any small annuli and that they coincide with $V^*$ (see [3, Theorem 7.1]).

Let $(\Sigma_n) = (\sigma_n^0)$ we know that $\Sigma_n \to V^*$ as varifold. Moreover, viewing $\Sigma_n$ as a flat cycle, we can assume that $\Sigma_n \to T \in \mathbb{Z}_2(M, \mathbb{Z}_2)$ as current. So one can ask the relation between $T$ and $V^*$. Since $V^*$ is made of smooth surface with integer multiplicities, there is a corresponding element $[V^*] \in \mathbb{Z}_2(M, \mathbb{Z}_2)$ by reducing each multiplicities mod 2. So one can suspect that $T = [V^*]$. This is confirmed by the result below (we are inspired by a similar result for the Almgrenn-Pitts min-max theory by Marques ans Neves [9, Proposition 4.10]). Actually the support of $T$ has to be contained in spt $V^*$. So if $\{S_i\}_{1 \leq i \leq N}$ is the collection of connected components of spt $V^*$, the Constancy Theorem implies that $T = n_1S_1 + \cdots + n_NS_N$ for some $n_i \in \{0, 1\}$.

Proposition 3. Let $(\Sigma_n)$ be a min-max sequence which is almost-minimizing in any small annuli and such that $\Sigma_n \to V$ as varifold and

$$V = m_1S_1 + \cdots + m_NS_N$$

where each $S_i$ is a smooth embedded minimal surface. Let us assume that $\Sigma_n \to T \in \mathbb{Z}_2(M, \mathbb{Z}_2)$ as current and

$$T = n_1S_1 + \cdots + n_NS_N$$

with $n_i \in \{0, 1\}$. Then $m_i = n_i \mod 2$.

One tool in the proof of the smoothness of $V^*$ is the following result of Meeks, Simon and Yau.
Theorem 9 (Meeks-Simon-Yau [13]). Let \( \Sigma \) be a surface in \( M \) and \( U \) an open subset of \( M \). Let \( (\Sigma_k) \) be a minimizing sequence for the Problem \((\Sigma, \mathcal{I}\mathcal{S}(U))\), i.e. \( \Sigma_k = \psi(\Sigma) \) for some \( \psi \in \mathcal{I}\mathcal{S}(U) \) and
\[
\mathcal{A}(\Sigma_k) \to \inf_{\psi \in \mathcal{I}\mathcal{S}(U)} \mathcal{A}(\psi(\Sigma)),
\]
which converges to a varifold \( V \). Then \( V \cap U \) is an integer rectifiable varifold whose support is a stable minimal surface \( M \) open subset of \( M \). Let \( U \) be an annulus centered at \( p \) such that \( spt V \) implies \( \Delta = \Delta \).

The proof of the regularity of \( V \cap U \) is local so is contained in [13]. As above this regularity implies the existence of flat chain mod 2, \( [V] \). The proof of the convergence as current is a byproduct of the regularity proof as noticed by Almgren and Simon in [11, Remark 5.20].

As mentioned above the smoothness proof uses the fact that replacement of \( V^* \) coincide with \( V^* \). This is a natural property of stationary varifold with smooth support.

Lemma 4. Let \( V \) be a integer stationary varifold whose support is a smooth minimal surface \( S \) and \( p \in S \). Then there is \( \rho > 0 \) such the following is true: if \( V' \) is a stationary varifold such that \( V' \cap B_p(\rho)^c = V \cap B_p(\rho)^c \), then \( V' = V \).

Proof. Let \( r > 0 \) be chosen such that \( \Delta = S \cap B_r \) is connected, stable and with non-empty smooth boundary. Let \( u \) be a non negative eigenfunction for the first eigenvalue of the Jacobi operator on \( \Delta \) with \( u = 0 \) on \( \partial \Delta \). Let \( \Delta_t = \{ \exp_p(tu(p)\nu(p)), p \in \Delta \} \) where \( \nu \) is the unit normal to \( \Delta \). There is \( \varepsilon > 0 \) such that, for \( |t| \leq \varepsilon \), \( \Delta_t \) is a smooth surface with mean curvature pointing to \( \Delta = \Delta_0 \). Let \( \rho \) such that \( B_p(\rho) \subset \cup_{|t| < \varepsilon} \Delta_t \).

Let \( V' \) be as in the Lemma, we notice that \( spt V' \subset S \cup B_p(\rho) \). Using the surfaces \( \Delta_t \) as barriers and the maximum principle [22], we have that \( spt V' \subset S \). Then the Constancy theorem and \( V' \cap B_p(\rho)^c = V \cap B_p(\rho)^c \) implies \( V' = V \).

We then have

Proof of Proposition [3]. Let \( p \in S_1 \) and \( \rho \) given by Lemma [4]. Let \( A \subset B_p(\rho) \) be an annulus centered at \( p \) such that \( \Sigma_n \) is \( \varepsilon_n \)-almost minimizing in \( A \).

Let
\[
\mathcal{I}\mathcal{S}_n(A) = \{ \psi \in \mathcal{I}\mathcal{S}(A) \mid \mathcal{A}(\psi(\Sigma_n)) \leq \mathcal{A}(\Sigma_n) + \varepsilon_n/8 \}
\]
and \( \psi^k \in \mathcal{I}\mathcal{S}_n(A) \) such that
\[
\mathcal{A}(\psi^k(\Sigma_n)) \to \inf_{\psi \in \mathcal{I}\mathcal{S}_n(A)} \mathcal{A}(\psi_1(\Sigma_n))
\]

Let \( q \in S_1 \cap A \) and \( \varepsilon > 0 \) such that \( B_{\varepsilon}(q) \subset A \). We first fix \( n \) and \( k \). Let \( \varphi^j \in \mathcal{I}\mathcal{S}(B_{\varepsilon}(q)) \) be a sequence such that
\[
\mathcal{A}(\varphi^j(\Sigma^k_n)) \to \inf_{\varphi \in \mathcal{I}\mathcal{S}(B_{\varepsilon}(q))} \mathcal{A}(\varphi_1(\Sigma^k_n))
\]
Let $W_k^n$ be the varifold limit of $\varphi_1^f(\Sigma_n^k)$ and $R_k^n$ be the limit of $\varphi_1^f(\Sigma_n^k)$ for the flat convergence. By Theorem 0, $R_k^n \subseteq B_\varepsilon(q) \supseteq W_k^n \subseteq B_\varepsilon(q)$. Besides we have $R_k^n = \varphi_1^f(\Sigma_n^k)$. Letting $k \to \infty$ we consider $W_n$ a varifold limit of $W_k^n$ and $R_n$ a limit of $R_k^n$ for the flat convergence. Since $\operatorname{spt}(W_k^n) \cap B_\varepsilon(q)$ is a stable minimal surface, we have curvature estimates and the convergence of $\operatorname{spt}(W_k^n) \cap B_\varepsilon(q)$ to $\operatorname{spt}(W_n)$ is locally smoothly graphical. So taking multiplicities of these graphical leaves into account, varifold and flat convergences give $R_n \subseteq B_\varepsilon(q) = [W_n]$. 

There is a subsequence $(\varphi_1^f(\Sigma_n^k))$ which converges to $W_n$ as varifold and $R_n$ as current. By [5, Lemma 7.1] there is $\Phi^f(\Sigma_n^k)$ such that $\varphi_1^f(\Sigma_n^k) = \varphi_1^f(\Sigma_n^k)$ and $A(\Phi^f(\Sigma_n^k)) \leq A(\Sigma_n^k) + \varepsilon_n/8$. This implies that $\varphi_1^f(\Sigma_n^k)$ can be constructed from $\Sigma_n$ by an isotopy in $\mathcal{J}_A$. Then $(\varphi_1^f(\Sigma_n^k))$ is a minimizing sequence in $\mathcal{J}_A$. So $\operatorname{spt}(W_n) \cap A$ is a stable minimal surface [5, Lemma 7.4]. Moreover a varifold limit $W$ of $W_n$ is a replacement for $V$ in $A$ [5, Proposition 7.5] and then $V = W$ because of Lemma 4 and $A \subseteq B_\rho(p)$. Let $R$ be a current limit of $R_n$.

As above since $\operatorname{spt}(W_n) \cap A$ is a stable minimal surface, the varifold and flat convergence give $R_n A = [W_n A] = [V A]$. Actually $\operatorname{spt} R \subseteq \operatorname{spt} W = \operatorname{spt} V$ and $R_n A^c = T_n A^c$. So by the Constancy Theorem, $R = T$. So $T^c A = [V A]$, this implies that the multiplicity of $T$ on $S_1$ is equal to the one of $V$ mod 2. Then $T = [V]$ by considering $p$ on other connected components. 

5.2. Almost stable minimal hypersurfaces. If $\Sigma$ is a stable minimal hypersurface in a Riemannian manifold $M$, then $\Sigma$ is a local minimum for the area functional (see [21]). In this section we introduce almost stable minimal hypersurfaces as hypersurfaces which have this local minimum property: one can give a sense to stability of a minimal hypersurface $\Sigma$ whose first Jacobi eigenvalue vanishes, i.e., degenerate stable minimal hypersurface.

Actually we focus only on 2-sided minimal hypersurfaces, the 1-sided case can be considered similarly. So let $\Sigma$ be a 2-sided minimal hypersurface and parameterize the $\varepsilon$-neighborhood of $\Sigma$ by $\Phi : \Sigma \times (-\varepsilon, \varepsilon) \to M; (p, t) = \exp_p(t\nu(p))$. This defines a vectorfield $\partial_t$ on the neighborhood. If $u$ is a smooth function on $\Sigma$, we define $\Phi_u : \Sigma \to M$ by $\Phi_u(p) = \Phi(p, u(p))$ and $\Sigma_u = \Phi_u(\Sigma)$. If $u$ is small, $\Sigma_u$ is immersed and we define $\nu_u(p)$ the unit normal to $\Sigma_u$ at $\Phi_u(p)$ such that $(\nu_u, \partial_t) \geq 0$ and $H_u(p) \in \mathbb{R}$ such that the mean curvature vector of $\Sigma_u$ is $H_u(p)\nu_u(p)$ at $\Phi_u(p)$. Let $W_u$ be the jacobian of the map $\Phi_u$. Then the area of $\Sigma_u$ can be computed as $A(\Sigma_u) = A(u) = \int_\Sigma W_u$. For $u$ and $v$ two functions we then have

$$DA(u)(v) = \int_\Sigma H_u(\nu_u, \partial_t)vW_u = \int_\Sigma h_u v$$
where \( h_u = H_u(\nu_u, \partial_t)W_u \). The map \( u \mapsto h_u \) is a smooth operator of order 2.

**Lemma 5.** Let be a degenerate stable minimal 2-sided hypersurface \( \Sigma \) with first Jacobi eigen-function \( u_0 \). There is \( \varepsilon > 0 \) and a smooth map \( v : (-\varepsilon, \varepsilon) \times \Sigma \to \mathbb{R}; (t, p) \mapsto v_t(p) \) with the following properties.

- \( v_0 = 0, \partial_t v_{|t=0} = 0 \) and \( \int_{\Sigma} u_0v_t = 0 \).
- for each \( t \in (-\varepsilon, \varepsilon) \), \( h_{t+u_0+v_t} \) is a multiple of \( u_0 \).

**Proof.** Let \( V \) be the \( L^2 \) orthogonal of \( u_0 \) and \( \pi_V \) the orthogonal projection on \( V \). Let \( V^{k,\alpha} = V \cap C^{k,\alpha}(\Sigma) \). Let us define the map: \( G : \mathbb{R} \times V^{2,\alpha} \to V^{0,\alpha}; (t, v) \mapsto \pi_V(h_{t+u_0+v}) \). We notice that \( G(0,0) = 0 \) and \( D_vG(0,0)(w) = \pi_V(Lw) \) where \( L \) is the Jacobi operator. Since \( L \) is invertible from \( V^{2,\alpha} \) to \( V^{0,\alpha} \), the implicit function theorem gives the map \( v \). \( \square \)

Once the family \( \{v_t\} \) is constructed we then define a foliation of a neighborhood of \( \Sigma \) by \( S_t = \Sigma_{u_0+v_t} \); this foliation is called the canonical foliation given by \( \Sigma \). Let \( a(t) = A(S_t) \). We notice that by construction \( a'(t) = 0 \) if and only if \( S_t \) is a minimal surface. Moreover, for small \( u, \Sigma_u \) is a minimal surface iff \( \Sigma_u = S_t \) for some small \( t \). It is possible that \( a(t) = A(\Sigma) \) for \( t \in (-\varepsilon, \varepsilon) \), \( t \in [0, \varepsilon) \) or \( t \in (-\varepsilon, 0] \). In the first case, a whole neighborhood of \( \Sigma \) is foliated by minimal hypersurfaces, we say that \( \Sigma \) is **minimally foliating on both side**. In both remaining cases, only one side of the neighborhood is foliated by minimal hypersurfaces, we say that \( \Sigma \) is **minimally foliating only on one side**.

If \( \{S_t\}_t \) is such a smooth family of minimal surfaces with \( \Sigma = S_0 \), we notice that the derivative at \( t \) of the family is given by a Jacobi field on \( S_t \). So up to a change of parametrization \( \{S_s\}_s \), we can assume that this Jacobi field as unit \( L^2 \)-norm.

One can try to extend such a family. Let \( \{S_s\}_{s \in (s^-, s^+)]} \) be such a family of minimal hypersurfaces with \( S_0 = \Sigma \) (possibly \( s^- = 0 \) and \( s \in (0, s^+] \)). First we notice that the hypersurfaces \( S_s \) are disjoint for \( s \) close to 0. So if \( S_s \cap S_{s'} \neq \emptyset \) for some \( 0 \leq s' < s \), there is a larger \( \sigma \) such that all the hypersurfaces in \( \{S_s\}_{0 \leq s < \sigma} \) are disjoint and \( S_\sigma \) intersect \( \Sigma \). Then the maximum principle implies that \( \Sigma = S_\sigma \) and then \( S_s = S_{s+\sigma} \) and the family is periodic and defined on \( \mathbb{R} \).

Assume now that the family is made of disjoint minimal hypersurfaces. Each \( S_s \) is an index 0 minimal hypersurface of area \( \mathcal{A}(\Sigma) \). So by Sharp compactness result \([17]\), as \( s \to s^+ \), a subsequence converges in the varifold sense to a minimal hypersurface \( \Sigma^+ \) : if \( \Sigma^+ \) is two-sided the convergence is smooth, if \( \Sigma^+ \) is one sided the convergence is with multiplicity 2 and smooth in the double cover of the tubular neighborhood of \( \Sigma^+ \). Actually this implies that as \( s \to s^+ \) the whole family \( \{S_s\} \) converges to \( \Sigma^+ \). The same can be done as \( s \to s^- \) to produce \( \Sigma^- \). If \( \Sigma^+ \) is one sided, then the whole tubular neighborhood of \( \Sigma^+ \) is foliated by \( \{S_s\}_{s \in (s^-, s^+]} \) where \( S_{s^+} = \Sigma^+ \). If \( \Sigma^+ \) is two sided, the family extends for \( s = s^+ \) and \( S_{s^+} = \Sigma^+ \) is either minimally
foliating only on one side and the family \( \{S_t\}_{t \in [s^-, s^+]} \) can’t be extended across \( s^+ \) or \( \Sigma^+ \) is minimally foliating on both sides and the family extends to \( t \in (s^-, s') \) with \( s^+ < s' \). As a consequence we are in one of the following cases

1. the family extends to \( \{S_s\}_{s \in [s^-, s^+]} \) and \( S_{s^\pm} \) are minimally foliating only on one side or, in the case \( s^- = 0 = s^+ \), \( \Sigma \) is not minimally foliating on any side
2. the family is periodic and gives a global foliation of \( M \)
3. the family extends to \( \{S_s\}_{s \in [s^-, s^+]} \) such that \( S_s \to \Sigma^- \) a 1-sided minimal hypersurface as \( s \to s^- \) and \( S_{s^+} \) is minimally foliating only on one side.
4. the family extends to \( \{S_s\}_{s \in [s^-, s^+]} \) such that \( S_s \to \Sigma^\pm \) two 1-sided minimal hypersurfaces as \( s \to s^\pm \). In that case \( \{S_s\}_{s \in [s^-, s^+]} \) with \( S_{s^\pm} = \Sigma^\pm \) gives a global foliation of \( M \).

We notice that the family can’t be defined on an unbounded interval without being periodic because of Sharp’s compactness result and the fact that the family is parameterized at unit speed.

We then say that \( \Sigma \) is almost stable if, in the above cases (1) and (3), the area function \( a^\pm \) associated to the canonical foliation given by \( S_{s^\pm} \) also satisfies locally \( a^\pm \geq A(S_{s^\pm}) = A(\Sigma) \). We notice that, in case (1) and (3), \( S_{s^\pm} \) is minimally foliating only on one side so there are value of \( t \) close to 0 such that \( a^\pm(t) > A(\Sigma) \).

5.3. Almost stable minimal hypersurfaces are local minima. In the above classification we are interested in the situation of type 1 almost stable minimal hypersurface.

So we have a family of minimal hypersurface \( \{S_s\}_{s \in [s^-, s^+]} \) that we can view as a subset \( \mathcal{M} \) of currents in \( \mathcal{Z}_n(M, \mathbb{Z}_2) \). We notice that \( \mathcal{M} \) is compact for the flat topology. In \( \{S_s\} \), there are two particular element \( S^- = S_{s^-} \) and \( S^+ = S_{s^+} \) that are minimally foliating only on one side; if \( s^- = 0 = s^+ \), \( S^- = \Sigma = S^+ \) and \( \Sigma \) is not minimally foliating on any side. In the the non minimally foliated side, the leaves of the canonical foliation given by \( S^\pm \) have area at least \( A(\Sigma) \). However, it may contain minimal hypersurfaces of area \( A(\Sigma) \), we add to \( \mathcal{M} \) all these minimal hypersurfaces that are sufficiently close to \( S^\pm \) to define \( \mathcal{M}' \). \( \mathcal{M}' \) is still compact for the flat topology.

We have the following result

**Proposition 4.** Let \( \Sigma \) be a type 1 almost stable minimal hypersurface and \( \mathcal{M}' \) be the set of minimal hypersurface of area \( A(\Sigma) \) in the canonical foliation as defined above. Then there is an open set \( U \) containing \( \Sigma \) such that \( A(\Sigma) \leq M(T) \) for any \( T \in \mathcal{Z}_n(M, \mathbb{Z}_2) \) homologous to \( \Sigma \) with support in \( U \) and, if \( A(\Sigma) = M(T) \), then \( T \in \mathcal{M}' \).

**Proof.** The proof is similar to the one of [21, Theorem 2]. So we consider the neighborhoods \( K_r, 0 \leq r \leq \varepsilon \) of \( \Sigma \) given by [21, Theorem 1]. Let \( T \),
minimize the mass $\mathcal{M}$ among all currents homologous to $\Sigma$ in $K_r$. As $r \to 0$, the currents $T_r$ must converge to $\Sigma$.

Since $T_r$ are uniformly almost minimizing, the convergence $T_r \to \Sigma$ and regularity theory implies that $T_r$ can be described as the normal graph of a function $u_r$ on $\bar{V}$ with $u_r \to 0$ in $C^{1,\alpha}$. As $u_r$ is close to 0, on can write $u_r = w_t + f_r$ where $w_t = tu_0 + v_t$ is given by Lemma 5 and $f_r \in V^{1,\alpha}$. So one can estimate the mass of $T_r$ by

$$\mathcal{M}(T_r) = A(u_r)$$

$$= A(w_t) + \int_0^1 DA(w_t)(f_r)(f_r)dt$$

$$= A(w_t) + DA(w_t)(f_r) + \int_0^1 \int_0^1 tD^2A(w_t + tf_r)(f_r, f_r)dsdt$$

By Lemma 5 and 6, $DA(w_t)(f_r) = 0$ and, for $r$ close to 0, there is $c > 0$ such that $D^2A(w_t)(f_r, f_r) \geq c\|f_r\|^2$ since $D^2A(0)$ is given by the Jacobi operator. So $\mathcal{M}(T_r) = A(w_t) + \frac{c}{2}\|f_r\|^2 \geq A(\Sigma) + \frac{c}{2}\|f_r\|^2$. Since $T_r$ is minimizing $\mathcal{M}(T_r) \leq A(\Sigma)$. So $f_r = 0$ and $A(w_t) = A(\Sigma) : T_r = \Sigma_{w_t}$ which is minimal. So $T_r \in \mathcal{M}'$ for $r$ close to 0.

**Proposition 5.** Let $\Sigma$, $\mathcal{M}$ and $\mathcal{M}'$ be as above. There is $\varepsilon > 0$ so that for every $T \in \mathcal{Z}_n(M, \mathbb{Z}_2)$ with $\mathcal{F}(T, \mathcal{M}) < \varepsilon$, we have $\mathcal{M}(T) > A(\Sigma)$ unless $T \in \mathcal{M}'$ where $\mathcal{M}(T) = A(\Sigma)$.

**Proof.** The proof follows the lines of the proof of [9, Proposition 6.2].

First let $(T_i) \in \mathcal{Z}_n(M, \mathbb{Z}_2)$ be a sequence converging to some $S$ in $\mathcal{M}$ in the flat topology with $\mathcal{M}(T_i) \leq A(\Sigma)$. From Proposition 4, we obtain a neighborhood $U$ of $S$ such that elements in $\mathcal{M}'$ are the only minimizers of $\mathcal{M}$ among cycles contained in $U$ homologous to $S$. Let us consider a smaller neighborhood $V$ with $V \subset U$. In [9], Marques and Neves then construct a sequence of cycles $R_i$ satisfying the following properties

- $\mathcal{M}(R_i) \leq \mathcal{M}(T_i) \leq A(\Sigma)$
- $R_i \to S$ in the flat topology
- the support of $R_i \setminus T_i$ is contained in $M \setminus V$
- the support of $R_i$ is in $U$ for large $i$

Since $R_i \to S$, $R_i$ is homologous to $S$ for large $i$. So because of Proposition 4, $R_i = S_i$ for some $S_i \in \mathcal{M}$. Moreover, since $R_i \to \Sigma$, $S_i \subset V$ for large $i$ then $\mathcal{M}(T_i) = \mathcal{M}(S_i + T_i \cap (M \setminus V)) = A(S_i) + \mathcal{M}(T_i \cap (M \setminus V))$ so $T_i \cap (M \setminus V) = 0$ and $T_i = R_i = S_i$ i.e. $T_i \in \mathcal{M}'$.

So the consequence of this is the following result.

**Proposition 6.** Let $\Sigma$ and $\mathcal{M}$ be as above. For any $\varepsilon > 0$, there is $\delta > 0$ and a neighborhood $N \subset \mathcal{Z}_n(M, \mathbb{Z}_2)$ of $\mathcal{M}$, such that $N' \subset N(\mathcal{M}) = \{T \in \mathcal{Z}_n(M, \mathbb{Z}_2) \mid \mathcal{F}(T, \mathcal{M}) < \varepsilon\}$ and, for any $T \in \partial N$, we have $\mathcal{M}(T) \geq A(\Sigma) + \delta$. 
Proof. First we assume that \( \varepsilon \) is such that Proposition \(^5\) applies. Let \( S_{\pm} \) denote both extremal hypersurfaces of the family (possibly \( S^{+} = \Sigma = S^{-} \)). Let \( \{ w_{t}^{\pm} \} = \{ tu_{0}^{\pm} + v_{i}^{\pm} \} \) be the families of functions given by Lemma \(^5\) associated to \( S^{\pm} \). Since \( \Sigma \) is almost minimizing and \( S^{\pm} \) are minimally foliating only on one side, there are \( \delta > 0 \) and \( t^{+} > 0 \) (resp. \( t^{-} < 0 \)) such that \( S_{w_{t}^{\pm}}^{\pm} \in \mathcal{N}_{\varepsilon}(\mathcal{M}) \) for \( 0 < s \leq t^{+} \) (resp. \( t^{-} \leq s \leq 0 \)) and \( A(w_{t}^{\pm}) > A(\Sigma) \).

Let \( \mathcal{M} = \mathcal{M} \cup \{ S_{w_{t}^{\pm}}, 0 \leq \pm s \leq t^{\pm} \} \). We can notice that any element in \( \mathcal{M}' \setminus \tilde{\mathcal{M}} \) is at a positive \( \mathcal{F} \) distance from \( \tilde{\mathcal{M}} \). Let \( \eta > 0 \) such that \( \mathcal{N}_{\eta}(\tilde{\mathcal{M}}) \subset \mathcal{N}_{\varepsilon}(\mathcal{M}) \) and \( \mathcal{N}_{\eta}(\mathcal{M}) \cap \mathcal{M}' = \tilde{\mathcal{M}} \cap \mathcal{M}' \). If \( T_{i} \) is a sequence in \( \partial \mathcal{N}_{\eta}(\mathcal{M}) \) such that \( \mathcal{M}(T_{i}) \to \inf \{ \mathcal{M}(T), T \in \partial \mathcal{N}_{\eta}(\tilde{\mathcal{M}}) \} \). By compactness we can assume that \( T_{i} \to T' \) and, by lower-semicontinuity of the mass, \( \mathcal{M}(T) = \inf \{ \mathcal{M}(T), T \in \partial \mathcal{N}_{\eta}(\tilde{\mathcal{M}}) \} \). Moreover \( \mathcal{F}(T, \tilde{\mathcal{M}}) = \eta \) and \( \mathcal{F}(T, \mathcal{M}) \leq \varepsilon \). So by Proposition \(^5\) \( \mathcal{M}(T) \geq A(\Sigma) \) with equality iff \( T \in \mathcal{M}' \) which would imply \( T \in \tilde{\mathcal{M}} \) a contradiction. So \( \mathcal{M}(T) > A(\Sigma) \) and the result is proved. \( \square \)

5.4. The rigidity result. In this section we finally prove Theorem \(^8\). So let us fix a Riemannian 3-sphere \((\mathbb{S}^{3}, g)\) with sectional curvature bounded above by 1. Its Simon-Smith width is realized by a collection of minimal spheres whose areas are at least \( 4\pi \) so the width is at least \( 4\pi \). If the width is \( 4\pi \), the width is then realized by a minimal 2-sphere \( \Sigma \) with multiplicity 1.

If \( \Sigma \) has index at least 1, the rigidity comes from Theorem \(^6\). So we need to prove that index 0 cannot occur.

If \( \Sigma \) has vanishing first Jacobi eigenvalue then \( \Sigma \) may belong to a family of minimal spheres \( \{ s_{s} \} \). Since we are in \( \mathbb{S}^{3} \), this family is of type 1. Let \( \mathcal{M} = \{ S_{s}, s \in [s_{-}, s_{+}] \} \) be the associated compact subset in \( \mathcal{Z}_{2}(M, \mathbb{Z}_{2}) \) (\( \mathcal{M} = \{ \Sigma \} \) if \( \Sigma \) is stable).

If \( \Sigma \) is almost-stable (or \( \Sigma \) stable), let \( \varepsilon \) be less than the \( \mathcal{F} \) distance from \( \mathcal{M} \) to the 0 cycle and let \( \mathcal{N} \) be the neighborhood of \( \mathcal{M} \) given by Proposition \(^6\).

Since \( \Sigma \) realizes the width, there are sequences \( \{ \sigma^{n} \} \) in \( \Lambda \) and \( \{ t_{n} \} \) in \([-1, 1]\) such that \( \sigma^{n}_{t_{n}} \to \Sigma \) in the varifold sense and \( (\sigma^{n}_{t_{n}}) \) is a min-max sequence which is almost minimizing in small annuli. Because of Proposition \(^3\) we have \( \sigma^{n}_{t_{n}} \to \Sigma \in \mathcal{Z}_{2}(M, \mathbb{Z}_{2}) \) for the \( \mathcal{F} \) metric. So for \( n \) large enough, the continuous path \( t \mapsto \sigma^{n}_{t} \in \mathcal{Z}_{2}(M, \mathbb{Z}_{2}) \) enters into \( \mathcal{N} \). As this path starts and ends at the 0 cycle, it must cross \( \partial \mathcal{N} \) so

\[
\max_{t \in [-1, 1]} A(\sigma^{n}_{t}) \geq 4\pi + \delta
\]

where \( \delta \) is given by Proposition \(^6\). So \( \{ \sigma^{n} \} \) can not be a minimizing sequence.

Thus the width is realized by a minimal sphere \( \Sigma \) in \( M \) of area \( 4\pi \) with vanishing first Jacobi eigenvalue and which is not almost-stable. \( \Sigma \) belongs to a local foliation by minimal surfaces (it may contain only one leaf) which contain one minimal sphere \( \mathcal{S} \) of area \( 4\pi \) whose canonical foliation \( \{ S_{t} \} \) contains leaves such that \( A(S_{t}) < A(\Sigma) \) for \( t > 0 \) arbitrarily close to 0.
Notice that \( \overline{S} \) has also vanishing first Jacobi eigenvalue. Let \( a(t) = A(S_t) \) be the associated area function. We notice that, if \( a'(t) = 0 \), \( S_t \) is a minimal sphere and then \( a(t) \geq 4\pi \). This implies that

- either \( a(t) \geq 4\pi \) for \( t \in [0, \varepsilon) \),
- or \( a(t) \) is decreasing on \([0, \varepsilon)\).

Since \( \overline{S} \) is not almost-stable, we are in the second situation. In order to exploit this situation we need to introduce a slightly different local foliation near \( \overline{S} \).

**Lemma 6.** Let \( \Sigma \) be a minimal 2-sided hypersurface with vanishing first Jacobi eigenvalue. There is \( \varepsilon > 0 \) and a smooth map \( \tilde{v} : (-\varepsilon, \varepsilon) \times \Sigma \to \mathbb{R} \) with the following properties.

- \( \tilde{v}_0 = 0 \), \( \partial_t \tilde{v}_t|_{t=0} = 0 \) and \( \int_\Sigma u_0 \tilde{v}_t = 0 \).
- for each \( t \in (-\varepsilon, \varepsilon) \), \( H_{tu_0+\tilde{v}_t} \) is a multiple of \( u_0 \).

The proof is the same as Lemma 5 proof and gives the same consequence: if \( \Sigma_w \) is minimal hypersurface with \( w \) small \( w = tu_0 + \tilde{v}_t \) for some \( t \).

So let \( \{ \tilde{S}_t \} = \{ \overline{S}_{tu_0+\tilde{v}_t} \} \) be the family given by applying the above lemma to \( \overline{S} \). Let \( \tilde{a}(t) = A(\tilde{S}_t) \) be the associated area function. If \( \tilde{a}'(t) = 0 \), \( \tilde{S}_t \) is a minimal sphere and then \( \tilde{a}(t) \geq 4\pi \). So we also have the alternative

- either \( \tilde{a}(t) \geq 4\pi \) for \( t \in [0, \varepsilon) \),
- or \( \tilde{a}(t) \) is decreasing on \([0, \varepsilon)\).

Let us see that we are in the second case. If we are in the first one, there is \( t_0 > 0 \) such that \( \tilde{a}'(t_0) \geq 0 \), this implies that the mean curvature vector of \( \tilde{S}_{t_0} \) points to \( \overline{S} \). For \( t > 0 \) small, \( S_t \) is between \( \overline{S} \) and \( \tilde{S}_{t_0} \), let \( t_1 \) be the first \( t \) such that \( \tilde{S}_t \) intersect \( \tilde{S}_{t_0} \), then the mean curvature vector of \( \tilde{S}_t \) must points to \( \overline{S} \) which contradict \( a'(t_1) < 0 \). So we are in the second case.

If \( \tilde{w}_t = tu_0 + \tilde{v}_t \), we define on the neighborhood of \( \overline{S} \) the vectorfield \( X \) by \( X(\Phi_{\tilde{w}_t}(p)) = (\partial_2 \tilde{v}_t)\partial_t \) such that \( \tilde{S}_t \) is the image of \( \overline{S} \) by the flow given by \( X \). We also consider \( \tilde{v}_t \) the unit normal to \( \tilde{S}_t \) and \( Y = (X, \tilde{v}_t)\tilde{v}_t \). This vectorfield is normal to \( \tilde{S}_t \) and \( \tilde{S}_t \) is the image of \( \overline{S} \) by the flow given by \( Y \).

So looking at the \( F \) functional on \( \tilde{S}_t \) we have:

\[
\frac{d}{dt} F(\tilde{S}_t) = -\int_{\tilde{S}_t} 2\tilde{H}_t(Y, \tilde{v}_t)d\sigma - \int_{\tilde{S}_t} 2\tilde{H}^2_t(Y, \tilde{v}_t)d\sigma
\]

\[
+ \int_{\tilde{S}_t} (\Delta^\perp Y, H_t \tilde{v}_t) + (Y, B(e_i, e_j))(B(e_i, e_j), \tilde{H}_t \tilde{v}_t)d\sigma
\]

\[
+ \int_{\tilde{S}_t} \tilde{H}_t R(e_i, Y)e_i, \tilde{v}_td\sigma
\]

\[
= \int_{\tilde{S}_t} \tilde{H}_t(X, \tilde{v}_t)(\Ric(\tilde{v}_t) - 2)d\sigma + \int_{\tilde{S}_t} \tilde{H}_t(X, \tilde{v}_t)\|\tilde{B}\|^2d\sigma
\]

\[
- \int_{\tilde{S}_t} (\nabla(X, \tilde{v}_t), \nabla \tilde{H}_t)d\sigma
\]
where \( \Delta \) and \( \nabla \) are operators on \( S_t \) and \( \bar{H}_t \) is the mean curvature of \( S_t \).

Let \( \bar{u}_t \) be the positive function defined on \( S_t \) by \( \bar{u}_t(\Phi_{\bar{\omega}}(p)) = u_0(p) \). By construction there is \( \bar{c}_t \in \mathbb{R} \) such that \( \bar{H}_t = \bar{c}_t \bar{u}_t \). Moreover since we are in the second case, \( \bar{c}_t > 0 \) and \( \bar{H}_t > 0 \). This also implies \( \bar{H}_t(X, \bar{\nu}_t) \geq 0 \) and \( (\nabla(X, \bar{\nu}_t), \nabla \bar{H}_t) = c_t(\nabla(X, \bar{\nu}_t), \nabla \bar{u}_t) \). At \( t = 0 \), \( (\nabla(X, \bar{\nu}_t), \nabla \bar{u}_t) = \| \nabla u_0 \|^2 \geq 0 \). So if \( u_0 \) is not constant \( \int_{S_t}(\nabla(X, \bar{\nu}_t), \nabla \bar{H}_t)d\sigma \geq 0 \) for small \( t \) and, if \( u_0 \) is constant, \( \bar{H}_t = \bar{c}_t \bar{u}_t \) is constant and \( \int_{S_t}(\nabla(Y, \bar{\nu}_t), \nabla \bar{H}_t)d\sigma \) vanishes. In both cases

\[
\frac{d}{dt} F(S_t) \leq CF(S_t)
\]

for some constant \( C \). This implies that \( F(S_t) = 0 \) for any \( t \). So we are in the equality case: \( \text{Ric}(\nu) = 2 \) since \( c_t > 0 \). So \( \text{Ric}(\nu_0) = 2 \) and the Jacobi operator on \( S \) is \( \Delta + 2 \) which contradict that 0 is its first eigenvalue.

**Appendix A. A Schauder estimate**

In this appendix, we prove Proposition 1. To lighten notations, we denote \( E_T = E_{[0,T]} \) and \( P_T = \mathbb{R}^n \times [0,T] \). First, we give some complementary notations for H"older norms. For a map \( u : D \subset \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^k \), we define

\[
[u]_{2,\alpha,D} = [u]_{2,\alpha,D,x} + [u]_{2,\alpha,D,t} \\
[u]_{1,\alpha,D,x} = [\partial_x u]_{0,\alpha,D,x} \\
[u]_{1,\alpha,D,t} = [u]_{1+\alpha/2,D,t} + [\partial_x u]_{\alpha/2,D,t} \\
[u]_{1,\alpha,D} = [u]_{1,\alpha,D,x} + [u]_{1,\alpha,P,t} \\
\|u\|_{1,\alpha,D} = \|u\|_{0,D} + \|\partial_x u\|_{0,D} + [u]_{1,\alpha,D}
\]

Associated to these norms, we have the \( C^{l,\alpha} \) H"older space. We also define the space \( C^{l,\alpha}_0(P_T) \) as maps \( u \in C^{l,\alpha}(P_T) \) satisfying \( \partial^i_t u|_{t=0} = 0 \) for \( 0 \leq i \leq l/2 \).

We have some classical interpolation inequalities.

**Lemma 7.** Let \( l, m \in \{0,1,2\}, \alpha, \beta \in [0,1] \) such that \( l + \alpha > m + \beta \) and let \( T > 0 \). Then for any \( \varepsilon > 0 \) there is a constant \( C \) such that, for any \( u : P_T \to \mathbb{R}^k \)

\[
\|u\|_{m,\beta, P_T} \leq C\|u\|_{L^2(P_T)} + \varepsilon [u]_{l,\alpha,P_T}
\]

**Proof.** Let us explain the proof when \( l = m = 0 \) and \( \beta = 0 < \alpha \). Let us notice that its sufficient to look only at one component \( u^a \) of \( u \). Let \( X \in P_T \) and consider \( \delta > 0 \). For \( \delta \) small, we can consider a box \( B \) which is a translate of \( [0,\delta] \times [0,\delta^2] \) such that \( B \subset P_T \) and \( X \in B \) (\( \delta \) can be chosen independently of \( X \)). Then there is \( \bar{X} \in B \) such that \( \int_B u^a = \delta^{n+2} u^a(\bar{X}) \).
So $|u^a(X)| \leq |u^a(X)| + |u^a(X) - u^a(X)|$

\[
\leq \frac{1}{\delta^{n+2}} \int_B |u^a| + 2\delta^\alpha |u^a|_{0,\alpha,B}
\leq \frac{1}{\delta^{n/2+1}} \|u^a\|_{L^2(B)} + 2\delta^\alpha |u^a|_{0,\alpha,B}
\leq \frac{1}{\delta^{n/2+1}} \|u^a\|_{L^2(P_T)} + 2\delta^\alpha |u^a|_{0,\alpha,P_T}
\]

So choosing $\delta$ small enough we have the result.

Once this first estimate is established, the other ones can be obtained by similar arguments (for example, see Section 6.8 in [7]).

The second result that we shall need is a Schauder type estimate for solution of

\[
\partial_t u - Lu = f
\]

over $\mathbb{R}^n \times [0, T]$ where $L$ is an operator as in (2) with constant coefficients and only second order terms.

**Lemma 8.** Let $L$ be an operator as in (2) with constant coefficients and only second order terms. Then there is a constant $C$ such that the following statement is true. If $u \in C^{2,\alpha}_0(P_T)$ is a solution of (6) with $f \in C^{0,\alpha}_0(P_T)$ such that $u_t$ has compact support for any $t$ then

\[
[u]_{2,\alpha,P_T} \leq C[f]_{0,\alpha,P_T}
\]

This result is established in Section 15 of [20] (see Theorem 4.1 and Equation (4.43))

We now want a similar result when $L$ depends on the variable $x$ and have terms of any order.

**Lemma 9.** Let $L$ be an operator as in (2) with $C^\alpha$ coefficients independent of $t$. Then there is a constant $C$ such that the following statement is true. If $u \in C^{2,\alpha}_0(P_T)$ is a solution of (6) with $f \in C^{0,\alpha}_0(P_T)$ such that $u_t$ has compact support for any $t$ then

\[
[u]_{2,\alpha,P_T} \leq C[f]_{0,\alpha,P_T} + \|u\|_{L^2(P_T)}
\]

\[
\|u\|_{2,\alpha,P_T} \leq C[f]_{0,\alpha,P_T} + \|u\|_{L^2(P_T)}
\]

**Proof.** Let $p$ be a point in $\mathbb{R}^n$, we are going to prove the estimate near $p$. Let $\delta > 0$ and consider $\varphi_\delta$ be a non-negative $C^\infty$ function on $\mathbb{R}^n$ with support on the ball $B$ centered at $p$ and radius $2\delta$, equal to 1 on the ball $B'$ of radius $\delta$ and such that $[\varphi_\delta]_{i,E} \leq \frac{C}{\delta^r}$. We are going to estimate $\varphi_\delta u$.

Let $L_p$ denote the operator $L(p)$ and $\mathcal{L}_p$ the part of $L_p$ with only second order terms. Since $\partial_t u - Lu = f$ we have

\[
\partial_t (\varphi_\delta u) - L_p (\varphi_\delta u) = \varphi_\delta f + \varphi_\delta Lu - L(\varphi_\delta u) + L(\varphi_\delta u) - L_p (\varphi_\delta u) + L_p (\varphi_\delta u) - L_p (\varphi_\delta u)
\]
So the estimate (7) gives
\[
[\varphi \delta u]_{2, \alpha, B_T} \leq C([\varphi \delta f]_{0, \alpha, B_T} + [\varphi \delta L - L(\varphi \delta u)]_{0, \alpha, B_T} + [L(\varphi \delta u) - L_p(\varphi \delta u)]_{0, \alpha, B_T} + [L_p(\varphi \delta u)]_{0, \alpha, B_T})
\]
where \( B_T \) denotes \( B \times [0, T] \).

In the right hand side of the above estimate, the first term can be estimated by \([\varphi \delta f]_{0, \alpha, B_T} \leq C \| f \|_{0, \alpha, B_T} \) (in the sequel \( C \) will denote a constant that depends on \( \delta \)). In \( \varphi \delta L - L(\varphi \delta u) \), the terms where the second derivatives of \( u \) appear cancel, so \([\varphi \delta L - L(\varphi \delta u)]_{0, \alpha, B_T} \leq C \| u \|_{1, \alpha, B_T} \). Similarly for the last term, \([L_p(\varphi \delta u)]_{0, \alpha, B_T} \leq C \| u \|_{1, \alpha, B_T} \). For the third term, if \( \Lambda \) bounds the \( C^{0, \alpha} \) norm of the coefficients of \( L_p \), we have
\[
[L(\varphi \delta u) - L_p(\varphi \delta u)]_{0, \alpha, B_T} \leq \Lambda \delta^\alpha \| \varphi \delta u \|_{2, \alpha, B_T} + 2\Lambda \| \varphi \delta u \|_{2, B_T}
\]
So fixing \( \delta \) such that \( C\Lambda \delta^\alpha < 1 \) we obtain
\[
[\varphi \delta u]_{2, \alpha, B_T} \leq C_\delta ([f]_{0, \alpha, B_T} + \| u \|_{1, \alpha, B_T} + \| \varphi \delta u \|_{2, B_T})
\]
Since \( \varphi = 1 \) on \( B' \) we obtain
\[
[u]_{2, \alpha, B_T'} \leq C_\delta ([f]_{0, \alpha, B_T} + \| u \|_{1, \alpha, B_T} + \| u \|_{2, B_T})
\]
Since we can consider any point \( p \) in \( \mathbb{R}^n \), we have
\[
[u]_{2, \alpha, B_T} \leq C_\delta ([f]_{0, \alpha, B_T} + \| u \|_{1, \alpha, B_T} + \| u \|_{2, B_T})
\]
By interpolation inequalities, we have \( \| u \|_{1, \alpha, B_T} + \| u \|_{2, B_T} \leq \| u \|_{L^2(B_T)} \). So choosing \( \varepsilon \) such that \( C\delta \varepsilon < 1 \), we obtain
\[
[u]_{2, \alpha, B_T} \leq C_\delta ([f]_{0, \alpha, B_T} + \| u \|_{L^2(B_T)})
\]
Using interpolation inequality again we finally have
\[
\| u \|_{2, \alpha, B_T} \leq C_\delta ([f]_{0, \alpha, B_T} + \| u \|_{L^2(B_T)})
\]
\( \square \)

We can now give the proof of Proposition 4.

**Proof of Proposition 4.** Let \( U \) and \( F \) be as in the proposition. If \( T \leq 1 \) the result is given by Theorem 3 so let us assume that \( T \geq 1 \) and let \( \eta : \mathbb{R}_+ \to \mathbb{R}_+ \) such that \( \eta = 1 \) in a neighborhood of 0 and \( \eta = 0 \) on \([1/2, +\infty) \). So we can write \( U = \eta(t)U + (1 - \eta(t))U \). By Theorem 3 we have \( \| \eta(t)U \|_{2, \alpha, E_T} \leq C \| U \|_{E_T} \leq C(\| F \|_{0, \alpha, E_T} + \| U \|_{L^2(E_T)}) \).

The section \( W = (1 - \eta)U \in C^0 \alpha(E_T) \) is then a solution of
\[
\partial_t W - LW = (1 - \eta)F - \eta'U
\]
Let \( (\varphi_i)_{1 \leq i \leq m} \) be a partition of the unity such that the \( \varphi_i \) has support in local charts \( \Omega^j \) of \( E \). Then each \( W_i = \varphi_i W \) is a solution of \( \partial_t W_i - LW_i = G_i \).

In the chart, \( W_i \) can be written as a section \( w_i \) over \( \mathbb{R}^n \) with compact support which is a solution of \( \partial_t w_i - Lw_i = g_i \). So, by Lemma 9
\[
\| w_i \|_{2, \alpha, B_T} \leq C(\| g_i \|_{0, \alpha, B_T} + \| w_i \|_{L^2(B_T)})
\]
This gives
\[
\|W\|_{2,\alpha,\Omega^i_T} \leq C(\|g_i\|_{0,\alpha,\Omega^i_T} + \|w_i\|_{L^2(\Omega^i_T)})
\]
where \(\Omega^i_T\) denotes the bundle over \(\Omega^i \times [0,T]\). Summing these estimates and using that a finite number of \(W_i\) is sufficient we obtain
\[
\|W\|_{2,\alpha,1,E_T} \leq C(\|F\|_{0,\alpha,E_T} + \|U\|_{0,\alpha,E_1} + \|U\|_{L^2(E_T)})
\]
\[
\leq C(\|F\|_{0,\alpha,E_T} + \|U_0\|_{2,\alpha,E} + \|U\|_{L^2(ET)})
\]
where we have used \(\eta' = 0\) outside \([0,1]\) and Solonnikov’s estimate on \(E_1\).

So adding both estimates for \(\eta U\) and \((1 - \eta)U\) gives the desired result. \(\square\)

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