On $p$-schemes of order $p^3$

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Abstract

Let $(X,S)$ be a $p$-scheme of order $p^3$ and $T$ the thin residue of $S$. Now we assume that $T$ has valency $p^2$. It is easy to see that one of the following holds: (i) $|T| = p^2$ and $T \cong C_{p^2}$; (ii) $|T| = p^2$ and $T \cong C_p \times C_p$; (iii) $|T| < p^2$.

It is known that $(X,S)$ is Schurian if (i) holds. If (ii) holds, we will show that $(X,S)$ induces a partial linear space on $X/T$. Moreover, the character degrees of $(X,S)$ coincide with the sizes of the lines of the partial linear space. Under the assumption (iii) we will show a construction of non-Schurian $p$-schemes which are algebraically isomorphic to a Schurian $p$-scheme of order $p^3$.

Keywords: association scheme, $p$-scheme, partial linear space.

1. Introduction

An association scheme is a combinatorial object which is defined by some algebraic properties derived from a transitive permutation group (see Section 2 for definitions). Actually, any transitive permutation group $G$ on a finite set $\Omega$ induces an association scheme $(\Omega,R_G)$, where $R_G$ is the set of orbits of the induced action of $G$ on $\Omega \times \Omega$ (see [2, Example 2.1]). On the other hand, we can find many association schemes which are not induced by groups (see [4]).

We say that an association scheme $(X,S)$ is Schurian if $S = R_G$ for a transitive permutation group $G$ of $X$. In order to determine whether a given association scheme $(X,S)$ is Schurian or...
not, it is quite useful to consider its thin residue, i.e., the smallest subset \( T \) of \( S \) such that \( \bigcup_{t \in T} t \) is an equivalence relation on \( X \) and the factor scheme of \((X, S)\) over \( T \) is induced by a regular permutation group (see Section 2 for definitions). For example, if \( T \) is a group under the relational product and the set of normal subgroups of \( T \) is totally ordered with respect to the set-theoretic inclusion, then \((X, S)\) is Schurian (see [1]). In this paper we consider \( p \)-schemes which generalize \((\Omega, \mathcal{R}_G)\) for \( p \)-groups \( G \) and characterize them by their thin residues.

Let \((X, S)\) be a \( p \)-scheme, where \( p \) is a prime. Then \(|X|\) is a power of \( p \). It is known that \((X, S)\) is Schurian if \(|X| \in \{1, p, p^2\}\), and there are exactly three \( p \)-schemes of order \( p^2 \). Recall that the number of isomorphism classes of \( p \)-groups of order \( p^2 \) is five. In contrast to this situation the number of isomorphism classes of \( p \)-schemes of order \( p^3 \) depends on the choice of \( p \) (see [7]), and there is a method to construct non-Schurian \( p \)-schemes of order \( p^3 \) for each \( p > 5 \) (see [1]). By the discussion in the second paragraph, \((X, S)\) is Schurian if the thin residue \( T \) of \( S \) is a cyclic group under the relational product. Thus, the following cases are left when we enumerate \( p \)-schemes of order \( p^3 \) with thin residue \( T \) of valency \( p^2 \):

(i) \( T \) is the elementary abelian group of order \( p^2 \);
(ii) \( T \) is not a group.

In the first case, we will obtain a partial linear space \((P, \mathcal{L})\) (see Section 2 for definitions). According to [1] it is expected that quite many non-Schurian \( p \)-schemes of order \( p^3 \) can be obtained. However, it seems far from reach to classify all \( p \)-schemes of order \( p^3 \). We will consider \( \{ |L| \mid L \in \mathcal{L} \} \) and show that it coincides with \( \{ \chi(I) \mid \chi \in \text{Irr}(S) \} \), where \( I \) is the identity matrix and \( \text{Irr}(S) \) is the set of irreducible characters of the adjacency algebra of \((X, S)\) (see Section 2 for definitions). This result is obtained as Theorem 3.5 of this paper.

In the second case, it is known that there are three pairs of algebraically isomorphic 3-schemes of order 27, each consisting of one Schurian scheme and one non-Schurian scheme. In this paper we will generalize this fact to obtain the following:

**Theorem 1.1.** Let \((X, S)\) be an association scheme, \( T \) a closed subset of \( S \), and \( X_1, X_2, \ldots, X_m \) the cosets of \( T \) in \( X \). Suppose that \( \iota_1, \iota_2, \ldots, \iota_m \in \text{Aut}(T) \) extend to \( \tilde{\iota}_1, \tilde{\iota}_2, \ldots, \tilde{\iota}_m \in \text{Aut}(S) \) by \( s^{\tilde{\iota}_j} = s \) if \( s \in S \setminus T \) and \( s^{\tilde{\iota}_j} = s^j \) if \( s \in T \) for all \( j \). Let \( S' = \{ s' \mid s \in S \} \), where \( s' = s \) if \( s \in S \setminus T \) and \( s' = \bigcup_{j=1}^m (s^{\tilde{\iota}_j} \cap (X_j \times X_j)) \) if \( s \in T \). Then \((X, S')\) is an association scheme which is algebraically isomorphic to \((X, S)\).

**Theorem 1.2.** Let \( H \) and \( T \) be closed subsets of \( S \) such that \( H \subseteq T \) and \( T \) is strongly normal in \( S \). Assume that the following conditions are satisfied:

(i) \( sH = sT \) for each \( s \in S \setminus T \);
(ii) \( tH = tT \) for each \( t \in T \setminus H \).

Then, for each \( \iota \in \text{Aut}(T) \) fixing each element of \( H \), we have \( \tilde{\iota} \in \text{Aut}(S) \), where \( s^{\tilde{\iota}} = s \) if \( s \in S \setminus T \) and \( s^{\tilde{\iota}} = s' \) if \( s \in T \).

As a corollary of these theorems we can construct quite a many \( p \)-schemes of order \( p^3 \) by taking arbitrary sets of automorphisms of the cyclic group of order \( p \).

This paper is organized as follows. In Section 2 we review notations and terminology on association schemes and partial linear spaces. In Section 3 we show that the irreducible character degrees coincide with the sizes of the lines of the partial linear space. In Section 4 we give proofs of Theorem 1.1 and 1.2 and apply these results to a class of \( p \)-schemes obtaining association schemes with the same intersection numbers.
2. Preliminaries

In this section, we review some notations and known facts about association schemes and partial linear spaces. Throughout this paper, we use the notations given in [5, 13].

2.1. Association schemes

Let $X$ be a non-empty finite set. Let $S$ denote a partition of $X \times X$. Then we say that $(X, S)$ is an association scheme (or shortly scheme) if it satisfies the following conditions:

(i) $1_X := \{(x, x) \mid x \in X\} \in S$;
(ii) For each $s \in S$, $s^* := \{(x, y) \mid (y, x) \in s\} \in S$;
(iii) For all $s, t, u \in S$ and $x, y \in X$, $c_{st}^u := |xs \cap yt^*|$ is constant whenever $(x, y) \in u$, where
\[
xs := \{y \in X \mid (x, y) \in s\}.
\]

For each $s \in S$, we abbreviate $c_{s^*}^1$ as $n_s$, which is called the valency of $s$. For a subset $U$ of $S$, put $n_U = \sum_{u \in U} n_u$. We call $n_s$ the order of $(X, S)$. In particular, $(X, S)$ is called a $p$-scheme if $|X| \prod_{s \in S} n_s$ is a power of $p$, where $p$ is a prime.

Let $P$ and $Q$ be non-empty subsets of $S$. We define $PQ$ to be the set of all elements $s \in S$ such that there exist element $p \in P$ and $q \in Q$ with $c_{pq}^s \neq 0$. The set $PQ$ is called the complex product of $P$ and $Q$. If one of factors in a complex product consists of a single element $s$, then one usually writes $s$ for $\{s\}$.

A non-empty subset $T$ of $S$ is called closed if $TT \subseteq T$. Note that a subset $T$ of $S$ is closed if and only if $\bigcup_{t \in T} t$ is an equivalence relation on $X$. A closed subset $T$ is called thin if all elements of $T$ have valency 1. The set $\{s \mid n_s = 1\}$ is called the thin radical of $S$ and denoted by $O_\theta(S)$. Note that $T$ is thin if and only if $T$ is a group with respect to the relational product.

For a closed subset $T$ of $S$, $n_s / n_T$ is called the index of $T$ in $S$. A closed subset $T$ of $S$ is called strongly normal in $S$, denoted by $T \triangleleft S$, if $s^* Ts \subseteq T$ for every $s \in S$. We put $O_\theta^S := \bigcap_{T \triangleleft S} T$ and call it the thin residue of $S$. Note that $O_\theta^S(S)$ is the intersection of all closed subsets of $S$ which contain $\bigcup_{s \in S} s^*$s (see [13, Theorem 2.3.1]).

For each closed subset $T$ of $S$, we define $X/T := \{xT \mid x \in X\}$, called the set of cosets of $T$ in $X$, and $S//T := \{s^T \mid s \in S\}$, where $xT := \bigcup_{t \in T} xt$ and $s^T := \{(xT, yT) \mid y \in xTsT\}$. Then $(X/T, S//T)$ (or shortly $S//T$) is a scheme called the quotient (or factor) scheme of $(X, S)$ over $T$. Note that $T \triangleleft S$ if and only if $S//T$ is a group (see [13, Theorem 2.2.3]).

Let $(W, F)$ and $(Y, H)$ be association schemes. For each $f \in F$ we define
\[
\overline{f} := \{((w_1, y), (w_2, y)) \mid y \in Y, (w_1, w_2) \in f\}.
\]
For each $h \in H \setminus \{1_Y\}$ we define
\[
\overline{h} := \{((w_1, y_1), (w_2, y_2)) \mid w_1, w_2 \in W, (y_1, y_2) \in h\}.
\]

Let $F \cap H := \{f \mid f \in F\} \cup \{h \mid h \in H \setminus \{1_Y\}\}$. Then $(W \times Y, F \cap H)$ is an association scheme called the wreath product of $(W, F)$ and $(Y, H)$.

Let $(X, S)$ and $(X_1, S_1)$ be association schemes. A bijective map $\phi : X \cup S \to X_1 \cup S_1$ is called an isomorphism if it satisfies the following conditions:

(i) $X^\phi \subseteq X_1$ and $S^\phi \subseteq S_1$;
(ii) For all \(x, y \in X\) and \(s \in S\) with \((x, y) \in s\), \((x^s, y^s) \in s^\phi\).

An isomorphism \(\phi : X \cup S \to X \cup S\) is called an automorphism of \((X, S)\) if \(s^\phi = s\) for all \(s \in S\). We denote by \(\text{Aut}(X, S)\) the automorphism group of \((X, S)\).

On the other hand, we say that \((X, S)\) and \((X_1, S_1)\) are algebraically isomorphic or have the same intersection numbers if there exists a bijection \(\iota : S \to S_1\) such that \(c^t_{rs} = c^t_{\iota r \iota s}\) for all \(r, s, t \in S\). For a closed subset \(T\) of \(S\), we denote by \(\text{Aut}(T)\) the set of permutations of \(T\) which preserve the intersection numbers, namely

\[
\text{Aut}(T) = \{\iota \in \text{Sym}(T) \mid c^t_{rs} = c^t_{\iota r \iota s}, \text{ for all } r, s, t \in T\}.
\]

For each \(s \in S\), we denote the adjacency matrix of \(s\) by \(\sigma_s\). Namely \(\sigma_s\) is a matrix whose rows and columns are indexed by the elements of \(X\) and \((\sigma_s)_{xy} = 1\) if \((x, y) \in s\) and \((\sigma_s)_{xy} = 0\) otherwise. We denote by \(\mathbb{C}S\) the subspace of \(\text{Mat}_{|X|}(\mathbb{C})\) spanned by \(\{\sigma_s \mid s \in S\}\). It follows from the definition of a scheme that \(\mathbb{C}S = \bigoplus_{s \in S} \mathbb{C}\sigma_s\) is a subalgebra of \(\text{Mat}_{|X|}(\mathbb{C})\), and it is called the adjacency algebra of \((X, S)\). For a subset \(U\) of \(S\), we put \(\mathbb{C}U = \bigoplus_{u \in U} \mathbb{C}\sigma_u\) as a subset of \(\mathbb{C}S\). It is well-known that \(\mathbb{C}S\) is a semisimple algebra (see [13] Theorem 4.1.3). The set of irreducible characters of \(S\) is denoted by \(\text{Irr}(S)\). The map \(\sigma_s \mapsto n_s\) is called the principal character of \(S\) and denoted by \(1_s\). Now \(\mathbb{C}X = \bigoplus_{x \in X} \mathbb{C}x\) is a \(\mathbb{C}\)-module by \(x \sigma_s = \sum_{y \in s} y\) (see [13] p.97). Let \(\tau_s\) be the character afforded by \(\mathbb{C}X\). We call \(\tau_s\) the standard character of \(S\). For each irreducible character of \(S\), there exists a non-negative integer \(m_\chi\) such that \(\tau_s = \sum_{\chi \in \text{Irr}(S)} m_\chi \chi\). The number \(m_\chi\) is called the multiplicity of \(\chi\). For \(\chi \in \text{Irr}(S)\), the central primitive idempotent of \(\mathbb{C}S\) corresponding to \(\chi\) is denoted by \(e_\chi\). Note that \(\text{Irr}(S/\mathbb{O}^\theta(S))\) is embedded in \(\text{Irr}(S)\) (see [6]).

**Theorem 2.1** (see [3]). Let \(\chi \in \text{Irr}(S)\). Then \(\chi \in \text{Irr}(S/\mathbb{O}^\theta(S))\) if and only if \(m_\chi = \chi(1_\chi)\).

Let \(X\) be a representation of \(\mathbb{C}S\) affording the character \(\chi\). We let \(X(\sigma_s) = (x_{ij}(\sigma_s))\). Let \(\rho\) be a field automorphism of \(\mathbb{C}\). Then \(X^\rho : \sigma_s \mapsto (x_{ij}(\sigma_s)_{\rho}^\rho)\) is also a representation of \(\mathbb{C}S\), and \(X^\rho\) is irreducible if and only if so is \(X\). The character \(\chi^\rho\) afforded by \(X^\rho\) satisfies \(\chi^\rho(\sigma_s) = \chi(\sigma_s)^\rho\). We say that \(\chi^\rho\) is the algebraic conjugate of \(\chi\) by \(\rho\), or \(\chi\) and \(\chi^\rho\) are algebraically conjugate.

### 2.2. Strongly normal closed subsets of prime index

Let \((X, S)\) be an association scheme and \(T\) a strongly normal closed subset of \(S\). For \(x \in X\), the adjacency algebra of the subscheme \((X, S)_{xT}\) is isomorphic to \(\mathbb{C}T\) (see [13] for the definition). So we can regard \(\mathbb{C}T\) as the adjacency algebra. Put

\[
G := S/\mathbb{T}.
\]

Then the adjacency algebra \(\mathbb{C}S\) is a \(G\)-graded algebra. Based on the result given in [5], we build up this subsection.

Let \(\varphi\) be an irreducible character of \(T\) and \(L\) an irreducible right \(\mathbb{C}T\)-module affording \(\varphi\). The induction of \(L\) to \(S\) is defined by

\[
L^S = L \otimes_{\mathbb{C}T} \mathbb{C}S = \bigoplus_{s^T \in S/\mathbb{T}} L \otimes \mathbb{C}(TsT).
\]

Define the support \(\text{Supp}(L^S)\) of \(L^S\) by

\[
\text{Supp}(L^S) = \{s^T \in S/\mathbb{T} \mid L \otimes \mathbb{C}(TsT) \neq 0\}.
\]
and the stabilizer \( G\{L\} \) of \( L \) in \( G \) by

\[
G\{L\} = \{ s^T \in S//T \mid L \otimes \mathbb{C}(TsT) \cong L \text{ (as } \mathbb{C}T\text{-modules)} \}.
\]

Then \( G\{L\} \) is a subgroup of \( G \) and \( \text{Supp}(L^S) \) is a union of left cosets of \( G\{L\} \) in \( G \).

Under the assumption that \( T \) is a strongly normal closed subset of \( S \), we have the following result.

**Theorem 2.2** (see [4]). For any irreducible \( \mathbb{C}T\text{-module} \) \( L \) and \( s \in S \), \( L \otimes \mathbb{C}(TsT) \) is an irreducible \( \mathbb{C}T\text{-module} \) or 0.

For a character \( \varphi \) of \( T \) afforded by an irreducible \( \mathbb{C}T\text{-module} \) \( L \), we write \( \varphi^S \) for the induced character of \( \varphi \) to \( S \), and for a character \( \chi \) of \( S \) we write \( \chi_T \) for the restriction of \( \chi \) to \( T \). We put \( \text{Supp}(\varphi^S) := \text{Supp}(L^S) \) and \( G\{\varphi\} := G\{L\} \).

**Theorem 2.3** (see [3]). Let \((X,S)\) be an association scheme and \( T \) a strongly normal closed subset of \( S \). Suppose \( G = S//T \) is the cyclic group of prime order \( p \). Then for \( \chi \in \text{Irr}(S) \), one of the following statements holds:

1. \( \chi_T \in \text{Irr}(T) \) and \( (\chi_T)^S = \sum_{i=1}^{p} \chi_T^i \), where \( \text{Irr}(G) = \{ \chi_i \mid 1 \leq i \leq p \} \);
2. \( \chi(\sigma_s) = 0 \) for any \( s \in S \setminus T \) and \( \chi_T \) is a sum of at most \( p \) distinct irreducible characters. If \( \psi \) is an irreducible constituent of \( \chi_T \), then \( \psi^S = \chi \).

**Remark 2.4.** In Theorem 2.3(1), \( G\{\chi_T\} = G \) and \( \chi_T \in \text{Irr}(S) \). In Theorem 2.3(2), \( G\{\psi\} = \{1_G\} \) and \( \chi_T \) is a sum of \( k \) distinct irreducible characters of \( T \), where \( k = |\text{Supp}(\psi^S)| \).

**Theorem 2.5** (see [3]). Let \((X,S)\) be a scheme, \( T \) a closed subset of \( S \) and \( \psi \in \text{Irr}(T) \). Then

\[
\frac{n_s}{n_T} m_\psi = \sum_{\chi \in \text{Irr}(S)} (\psi^S,\chi)_S m_\chi,
\]

where \( (\psi^S,\chi)_S = a_\chi \) for \( \psi^S = \sum_{\varphi \in \text{Irr}(S)} a_\varphi \varphi \), and \( m_\psi \) and \( m_\chi \) are the multiplicities on \( T \) and \( S \), respectively.

### 2.3. Partial linear spaces and difference families

In [12, Section 4], it is shown that any reduced Klein configuration defines an incidence structure. By a parallel argument, we will obtain a partial linear space over a scheme under certain assumption.

In geometric terms, a **partial linear space** is a pair \((P,\mathcal{L})\), where \( P \) is a set of points and \( \mathcal{L} \) is a family of subsets of \( P \), called **lines**, such that every pair of points is contained in at most one line and every line contains at least two points. Moreover, \((P,\mathcal{L})\) is called a **linear space** if every pair of points is contained in exactly one line.

Let \((X,S)\) be an association scheme with \( T := O^p(S) \cong C_p \times C_p \) and let \( \{X_i \mid i = 1,2,\ldots,m\} \) denote the set of cosets of \( T \) in \( S \). Then we define the **left stabilizer** of \( s \)

\[
L(s) := \{ t \in T \mid ts = s \}.
\]

Note that \( L(s) = L(s') \) for \( s' \in TsT \). For given \( i, j \) and \( s \in S \) with \( s \cap (X_i \times X_j) \neq \emptyset \), we define \( L_{ij}(s) := L(s) \). Since \( L_{ij}(s) = L_{ij}(s') \) for \( s' \in TsT \), we write \( L_{ij} \) for \( L_{ij}(s) \). For \( \{1_X\} < M < T \), we put \( L_i(M) := \{i\} \cup \{j \mid L_{ij} = M\} \).

Define an incidence structure \((P,\mathcal{L})\) as follows:
(i) \( \mathcal{P} := \{i \mid i = 1, 2, \ldots, m\} \);
(ii) \( \mathcal{L} := \{L_i(M) \mid \{1_N\} < M < T, |L_i(M)| \neq 1, 1 \leq i \leq m\} \).

An automorphism \( \tau \) of \((\mathcal{P}, \mathcal{L})\) is a permutation of \(\mathcal{P}\) such that \(\mathcal{L}^\tau = \mathcal{L}\). We denote by \(\text{Aut}(\mathcal{P}, \mathcal{L})\) the set of automorphisms of \((\mathcal{P}, \mathcal{L})\).

**Lemma 2.6.** Let \((X, S)\) be an association scheme such that \(T := \text{O}_\theta(S) = \text{O}^\theta(S) \cong C_p \times C_p\). For \(s_1, s_2 \in S \setminus T\), if \(L(s_1) = L(s_2) \cong C_p\), then either \(s_1^2 s_2 \subseteq T\) or \(s_2^2 s_1 = s_3\) for some \(s_3 \in S \setminus T\).

**Proof.** Note that \(n_{st} = n_{ts} = n_s\) and \(n_s > 1\) for each \(s \in S \setminus T\) and \(t \in T\). Since \(T\) acts transitively on \(TsT\) and \(n_p = p^2\), the length of \(T\)-orbit is 1 or \(p\). Thus, \(n_s\) is \(p\) or \(p^2\) for \(s \in S \setminus T\).

Suppose \(s_1^2 s_2 \not\subseteq T\). Then \(\sigma s_2 \sigma s_1\) is a linear combination of at most \(p\) adjacency matrices. Since \(s_1^2 s_2 = s_1^2 s_1\) for each \(t \in L(s_1)\), \(\sigma s_2 \sigma s_1 = p \sigma s_3\) for some \(s_3 \in S \setminus T\). \(\square\)

Using the key idea in the proof of [12, Lemma 4.3], we show the following theorem.

**Theorem 2.7.** Let \((X, S)\) be an association scheme such that \(T := \text{O}_\theta(S) = \text{O}^\theta(S) \cong C_p \times C_p\). Then \((\mathcal{P}, \mathcal{L})\) is a partial linear space such that any point belongs to at most \(p + 1\) lines, and \(\text{Aut}(X/T, S//T)\) is a subgroup of \(\text{Aut}(\mathcal{P}, \mathcal{L})\).

**Proof.** If \(\{j \in \mathcal{P} \mid \{1_N\} < L_{ij} < T\} = \emptyset\) for each \(i \in \mathcal{P}\), then \((\mathcal{P}, \mathcal{L})\) is a partial linear space without lines. Suppose that \(\{j \in \mathcal{P} \mid \{1_N\} < L_{ij} < T\} \neq \emptyset\) for some \(i \in \mathcal{P}\). By definition, any line contains at least two points.

First, we claim that for \(i \neq j\),

\[
j \in L_i(M) \Rightarrow L_i(M) = L_j(L_{ji}). \tag{1}\]

To show \(L_i(M) \supseteq L_j(L_{ji})\), let \(k \in L_j(L_{ji})\). If \(k = i\) or \(j\), then clearly \(k \in L_i(M)\). Assume \(k \neq i, j\). Then \(L_{jk} = L_{ji}\) so that \(L_{jk} = L_{jk}(s_1)\) and \(L_{ji} = L_{ji}(s_2)\) for some \(s_1, s_2 \in S \setminus T\). Since \(k \neq i\), we have \(s_2 \not\subseteq TsT\). By Lemma 2.6, \(s_2^2 s_1 = s_3\) for some \(s_3 \in S \setminus T\). Since \(L(s_2^2) = L(s_3) \cong C_p\), we have \(L_{ij} = L_{ik}\). Thus, by definition \(k \in L_i(L_{ij}) = L_i(M)\).

Since \(i \in L_i(L_{ij})\), by the same argument as above \(L_i(L_{ij}) \subseteq L_j(L_{ji})\). Thus, \(L_i(M) = L_j(L_{ji})\).

Now we prove that for any \(k, k' \in \mathcal{P}\) and for \(\{1_N\} < H, H' < T\),

\[
|L_k(H) \cap L_{k'}(H')| \geq 2 \Rightarrow L_k(H) = L_{k'}(H').
\]

Suppose that \(i, j \in L_k(H) \cap L_{k'}(H')\) and \(i \neq j\). We divide our consideration into two cases.

(i) \(i \neq k\).

By the definition, we have \(L_{ki} = H\). Also, by (11) \(L_k(H) = L_k(L_{ki}) = L_i(L_{ik})\). Since \(j \in L_k(H) = L_i(L_{ki})\), we have \(L_{ik} = L_{ij}\). It follows from (11) that \(L_i(L_{ik}) = L_i(L_{ij}) = L_j(L_{ji})\).

Thus, \(L_k(H) = L_i(L_{ij}) = L_j(L_{ji})\).

(ii) \(i = k\).

Since \(i \neq j\), we have \(j \neq k\). By argument of case(i), we get \(L_k(H) = L_i(L_{ij}) = L_j(L_{ji})\).

Similarly, we consider the following cases: \(i \neq k'\) and \(i = k'\). We can get \(L_{k'}(H') = L_i(L_{ij}) = L_j(L_{ji})\). Therefore, \(L_k(H) = L_{k'}(H')\), i.e., any two distinct points belong to at most one line. Since \(T\) has exactly \(p + 1\) proper subgroups, any point belongs to at most \(p + 1\) lines.

Finally, we show that

\[
\text{Aut}(X/T, S//T) \leq \text{Aut}(\mathcal{P}, \mathcal{L}).
\]
It suffices to prove that $\phi(L_i(M))$ belongs to $\mathcal{L}$ for any $\phi$ and $L_i(M) \in \mathcal{L}$. For convenience, we identify $X/T$ with $\mathcal{P}$. Note that $L_i(M)^\phi = \{i^\phi\} \cup \{j^\phi \mid L_{ij} = M\}$. Since $L_{ij} = L_{i\phi,j\phi}$, $L_i(M)^\phi = L_{i\phi}(M) \in \mathcal{L}$.

\textbf{Corollary 2.8.} Under the assumption of Theorem 2.7 if $\{n_s \mid s \in \mathcal{S}\} = \{1, p\}$, then $(\mathcal{P}, \mathcal{L})$ is a linear space. Moreover, $\frac{n_s}{n_p} \leq p^2 + p + 1$.

\textbf{Proof.} Clearly, $(\mathcal{P}, \mathcal{L})$ is a partial linear space. By the assumption, for given $i \neq j$ $L_{ij}$ is always a proper subgroup. Thus, every pair of points is contained in exactly one line.

We claim that each line contains at most $p + 1$ points. If not, any point which is not in the line containing at least $p + 2$ points would belong to at least $p + 2$ lines, a contradiction by Theorem 2.7. Therefore, for a fixed point, that point belongs to at most $p + 1$ lines, each of which containing at most $p + 1$ points. This implies that $|\mathcal{P}| \leq (p + 1)^2 - p = p^2 + p + 1$.

Let $G$ be a finite group of order $v$ acting on itself by right multiplication, and let $\mathcal{S} = \{B_1, \ldots, B_s\}$ be a family of subsets of $G$ satisfying \[\{|g \in G \mid gB_j = B_j\}| = 1 \text{ for } 1 \leq j \leq s,\]

and put $K := \{|B_j| \mid 1 \leq j \leq s\}$. If every non-identity element of $G$ occurs $\lambda$ times in $\bigcup_{1 \leq j \leq s} \Delta B_j$, where $\Delta B_j := \{ab^{-1} \mid a, b \in B_j, a \neq b\}$, then $\mathcal{S}$ is called a $(v, K, \lambda)$-difference family in $G$.

\textbf{Proposition 2.9} \textbf{[3].} Let $G$ be a finite group of order $v$, and let $\mathcal{S} = \{B_1, \ldots, B_s\}$ be a family of subsets of $G$ satisfying $\{|g \in G \mid gB_j = B_j\}| = 1$ for $1 \leq j \leq s$. Then $\mathcal{S}$ is a $(v, K, 1)$-difference family if and only if $(\mathcal{P}, \mathcal{L})$ is a linear space, where $\mathcal{P} := G$ and $\mathcal{L} := \{B_j g \mid g \in G, 1 \leq j \leq s\}$.

\textbf{Remark 2.10.} By Corollary 2.8 and Proposition 2.9, an association scheme $(X, \mathcal{S})$ such that $\mathcal{O}_\theta(S) = \mathcal{O}_\theta(S) \cong C_p \times C_p$ and and $\{n_s \mid s \in \mathcal{S}\} = \{1, p\}$ induces a difference family with $\lambda = 1$.

3. Thin residues as the elementary abelian $p$-group of rank two

Throughout this section we assume that $(X, \mathcal{S})$ is a scheme such that

(i) $\mathcal{O}_\theta(S)$ is the elementary abelian group of order $p^2$,
(ii) $S/\mathcal{O}_\theta(S)$ is the cyclic group of order $q$, where $p$ and $q$ are primes.

For convenience, we denote $S/\mathcal{T}$ by $G$ and $\mathcal{O}_\theta(S)$ by $T$.

\textbf{Lemma 3.1.} For all $\varphi \in \text{Irr}(T) \setminus \{1_T\}$ and $s \in S \setminus T$, the following are equivalent:

(i) $L(s) = \text{Ker}\varphi$;
(ii) $e_\varphi \sigma_s \neq 0$;
(iii) $s^T \in \text{Supp}(\mathbb{C}e_\varphi^S)$, i.e., $\mathbb{C}e_\varphi \otimes \mathbb{C}(TS\mathcal{T}) \neq 0$.

\textbf{Proof.} Note that $T = \langle t \rangle \text{Ker}\varphi$ for some $t \in T \setminus \text{Ker}\varphi$ and $e_\varphi = (\frac{1}{p} \sum_{i \in \mathbb{Z}_p} \varepsilon^i \sigma^i_t)(\frac{1}{p} \sum_{t' \in \text{Ker}\varphi} \sigma_{t'})$ for some primitive $p$-th root of unity $\varepsilon$.

First, we show that $L(s) = \text{Ker}\varphi$ if and only if $e_\varphi \sigma_s \neq 0$. Assume $L(s) = \text{Ker}\varphi$. Since $\langle t \rangle$ acts regularly on $TS\mathcal{T}$, $e_\varphi \sigma_s = (\frac{1}{p} \sum_{i \in \mathbb{Z}_p} \varepsilon^i \sigma^i_t)\sigma_s = (\frac{1}{p} \sum_{i \in \mathbb{Z}_p} \varepsilon^i \sigma_{1s}) \neq 0$. 

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Assume $e_\varphi \sigma_s \neq 0$. If $L(s) \neq \text{Ker}\varphi$, then $\langle t \rangle$ and $\text{Ker}\varphi$ act regularly on $TsT$,

$$e_\varphi \sigma_s = \left( \frac{1}{p} \sum_{i \in \mathbb{Z}_p} \varepsilon^i \sigma_i^1 \right) \left( \frac{1}{p} \sum_{t' \in \text{Ker}\varphi} \sigma_{t'} \right) \sigma_s = \left( \frac{1}{p} \sum_{i \in \mathbb{Z}_p} \varepsilon^i \sigma_i^1 \right) \left( \frac{1}{p} \sum_{s' \in TsT} \sigma_{s'} \right) = \left( \frac{1}{p^2} \sum_{i \in \mathbb{Z}_p} \varepsilon^i \right) \left( \sum_{s' \in TsT} \sigma_{s'} \right) = 0,$$

a contradiction.

Next, we show that $\mathbb{C}e_\varphi \otimes \mathbb{C}(TsT) = 0$ if and only if $e_\varphi \sigma_s = 0$. By Theorem 2.2, the dimension of $\mathbb{C}e_\varphi \otimes \mathbb{C}(TsT)$ is at most 1. Thus, $\mathbb{C}e_\varphi \otimes \mathbb{C}(TsT) = 0$ if and only if $e_\varphi \otimes \sigma_s = 0$.

Since $e_\varphi \otimes \sigma_s = e_\varphi \otimes e_\varphi \sigma_s$ by definition of tensor product, $e_\varphi \otimes \sigma_s = 0$ if and only if $e_\varphi \sigma_s = 0$. This completes the proof.

**Lemma 3.2.** For all $\varphi \in \text{Irr}(T) \setminus \{1_T\}$ and $s \in S \setminus T$, $s^T \in G\{\mathbb{C}e_\varphi\}$ if and only if $\text{Ker}\varphi = L(s) = L(s^*)$.

**Proof.** Suppose that $1^T_X \neq s^T \in G\{\mathbb{C}e_\varphi\}$. This implies that $e_\varphi \sigma_s$ is left invariant by the right multiplication of $e_\varphi$, i.e.,

$$(e_\varphi \sigma_s)e_\varphi = e_\varphi \sigma_s.$$ 

Since $e_\varphi \sigma_s$ is nonzero, so $\sigma_s e_\varphi$ is. Taking the adjoint operator on $\sigma_s e_\varphi 
eq 0$ we obtain that $(e_\varphi)^*(\sigma_s) = 0$. Note that $(\sigma_s)^* = \sigma_{s^*}$ and $(e_\varphi)^* = e_\varphi$ since $e_\varphi$ is a primitive idempotent of $\mathbb{C}T$ with $e_\varphi (e_\varphi)^* = 0$. Thus, we have $e_\varphi \sigma_{s^*} = 0$. Applying Lemma 3.1 for both $e_\varphi \sigma_s 
eq 0$ and $e_\varphi \sigma_{s^*} = 0$, we obtain that $\text{Ker}\varphi = L(s) = L(s^*)$.

Suppose $\text{Ker}\varphi = L(s) = L(s^*)$. By Lemma 3.1 we have $e_\varphi \sigma_s \neq 0$ and $e_\varphi \sigma_{s^*} \neq 0$.

This implies that $\sigma_s e_\varphi = 0$ since $(\sigma_s)^* = \sigma_{s^*}$ and $(e_\varphi)^* = e_\varphi$. Therefore, the right multiplication of $e_\varphi$ to $\mathbb{C}e_\varphi \otimes \mathbb{C}(TsT)$ acts identically on it. It follows that $\mathbb{C}e_\varphi \otimes \mathbb{C}(TsT) \simeq \mathbb{C}e_\varphi$ as right $\mathbb{C}T$-modules, and hence, $s^T \in G\{\mathbb{C}e_\varphi\}$.

**Lemma 3.3.** The following are equivalent:

(i) $(X, S)$ is commutative;
(ii) For each $\varphi \in \text{Irr}(T) \setminus \{1_T\}$, $\text{Supp}(\mathbb{C}e_\varphi^S) = \{1^T_X\}$;
(iii) For each $s \in S \setminus T$ we have $|TsT| = 1$.

**Proof.** Suppose (i) and $\text{Supp}(\mathbb{C}e_\varphi^S) \neq \{1^T_X\}$ for some $\varphi \in \text{Irr}(T) \setminus \{1_T\}$. Then, by definition, $s^T \in \text{Supp}(\mathbb{C}e_\varphi^S)$ for some $s \in S \setminus T$. Since $(X, S)$ is commutative, we have $ss^* = s^*s$, or equivalently, $L(s) = L(s^*)$. It follows from Lemma 3.2 that $s^T \in G\{\mathbb{C}e_\varphi\}$. Note that $G(\mathbb{C}e_\varphi)$ is a subgroup of $G$ and $G$ is a cyclic group of prime order, it follows that $G\{\mathbb{C}e_\varphi\} = G$. By Lemma 3.2, $\text{Ker}\varphi = L(u) = L(u^*)$ for each $u \in S \setminus T$. Since $L(u) = uu^*$ for each $u \in S$, it follows that

$$\text{Ker}\varphi < T = O^\theta(S) = \langle uu^* \mid u \in S \rangle = \text{Ker}\varphi,$$

a contradiction.

Suppose (ii) and $|TsT| \neq 1$ for some $s \in S \setminus T$. Then $|TsT| > 1$ and $ss^*$ is a closed subset of $T$ with valency $p$, otherwise, $n_s = 1$, implying that $S$ is thin and $n_T = 1$, a contradiction. Take $\psi \in \text{Irr}(T) \setminus \{1_T\}$ such that $\text{Ker}\psi = L(s)$. Then, by Lemma 3.1, $s^T \in \text{Supp}(\mathbb{C}e_\psi^S)$, which contradicts (ii).
Suppose (iii). Let $s, u \in S \setminus T$. Since $S/T$ is abelian, $T(su)T = T(us)T$. If $us \subseteq T$, then $\sigma_u \sigma_s = p^2 \sigma_s = \sigma_s \sigma_u$. If not, then $us$ and $su$ are singletons by (iii), and hence $\sigma_u \sigma_s = \sigma_s \sigma_u$. Since $T$ acts trivially on $S \setminus T$ by the right and left multiplication respectively, it follows that $(X, S)$ is commutative. \hfill \Box

**Lemma 3.4.** For each $\varphi \in \text{Irr}(S) \setminus \{1_T\}$, if $\text{Supp}(\text{C}_\varphi S) \neq \{1_T\}$, then $\varphi S \in \text{Irr}(S)$.

**Proof.** Suppose $1_T \notin s^T \in \text{Supp}(\text{C}_\varphi S)$. Then we conclude from Lemma 3.3 that $(X, S)$ is not commutative, and from Lemma 3.1 that $L(s) = \text{Ker}\varphi$. Note that $\varphi_S(\sigma_{1_X}) = \dim \text{C}_\varphi \otimes \mathbb{C}$,

$$\text{C}_\varphi \otimes \mathbb{C} = \bigoplus_{s^T \in S \setminus T} \text{C}_\varphi \otimes \mathbb{C}(TsT) \text{ and dim} \text{C}_\varphi \otimes \mathbb{C}(TsT) \in \{0, 1\}. \quad (2)$$

We claim that $\text{Supp}(\text{C}_\varphi S) \neq G$. Otherwise, $L(u) = \text{Ker}\varphi$ for each $u \in S \setminus T$ by Lemma 3.1, which implies that $L(s) < T = \mathbb{O}_p^d(S) = \langle uu^* \mid u \in S \rangle = L(s)$, a contradiction.

Let $\chi \in \text{Irr}(S)$ be an irreducible constituent of $\varphi S$. Then $\varphi$ is an irreducible constituent of $\chi T$. So (1) or (2) of Theorem 2.3 holds. But, (1) never holds, since $\varphi_S(\sigma_{1_X}) < p$ by the claim and (2). This implies that $\varphi_S \notin \text{Irr}(S)$. \hfill \Box

**Theorem 3.5.** Suppose that $(X, S)$ is an association scheme which satisfies the following:

(i) $T := \mathbb{O}_p^d(S)$ is the elementary abelian group of order $p^2$;

(ii) $S/T$ is the cyclic group of order $q$;

Then $\{\chi(\sigma_{1_X}) \mid \chi \in \text{Irr}(S), \chi(\sigma_{1_X}) \neq 1\} = \{|l| \mid l \in \mathcal{L}\}$, where $\mathcal{L}$ is the set of lines on $X/T$ as in Section 2.

**Proof.** Assume $(X, S)$ is commutative. Then by Lemma 3.3 $|TsT| = 1$ for each $s \in S \setminus T$. This implies that $(\mathcal{P}, \mathcal{L})$ is a partial linear space without lines. Since every irreducible character of $S$ is linear, this case is done.

Assume $(X, S)$ is not commutative. Then, by Lemma 3.3, there exists $s \in S \setminus T$ such that $|TsT| = p$. Let $\chi \in \text{Irr}(S)$ such that $\chi(\sigma_{1_X}) \neq 1$. Since $\chi(\sigma_{1_X}) > 1$, we conclude from Theorem 2.3 that $\chi = \psi S$ for some $\psi \in \text{Irr}(T)$. Note that the equation obtained from (2) by replacement of $\varphi$ by $\psi$ holds. Since $\dim (\text{C}_\psi \varphi) \otimes \mathbb{C}(TsT) = 1$, it follows from Lemma 3.1 that

$$\psi^S(\sigma_{1_X}) = |\{s^T \in G \setminus \{1_X\} \mid L(s) = \text{Ker}\psi\}| + 1,$$

which is equal to the size of a line by definition of $L_s(L(s))$.

Conversely, let $L_i(M) \in \mathcal{L}$, where $M$ is a closed subset of $T$ with valency $p$. Then we can take $\psi \in \text{Irr}(T)$ such that $\text{Ker}\psi = M$. By Lemma 3.1, $\psi S \in \text{Irr}(S)$. Therefore, we conclude from Lemma 3.1 that $1 < |L_i(M)| = \psi^S(\sigma_{1_X})$ as desired. \hfill \Box

4. Proofs of the main theorems

For each $i \in \text{Aut}(T)$, we define $i : S \to S$ by $s \mapsto s$ if $s \in S \setminus T$ and $s \mapsto s^i$ if $s \in T$.

**Proof of Theorem 1.1**
Clearly, \(1'_x = 1_x\) and \((s')^* \in S'\) for each \(s' \in S'\). It is enough to show that, for all \(u', v', w' \in S'\), 
\(|xu' \cap y(v')^*|\) is constant whenever \((x, y) \in w'\).

We divide our consideration into four cases depending on \(|\{u, v, w\} \cap T|\).

(i) \(|\{u, v, w\} \cap T| = 3\). For each \((x, y) \in w'\), if \(x, y \in X_i\), then 
\(|xu' \cap y(v')^*| = |xu'^i \cap y(v^i)^*| = c_{u'^i, v^i}^w\).

Since \(\iota_i \in \text{Aut}(T)\), we have \(c_{u'^i, v^i}^w = c_{u'^i, v^i}^w\). This implies that \(c_{u'^i, v^i}^w\) is well-defined.

(ii) \(|\{u, v, w\} \cap T| = 2\). First, we assume \(w \in S \setminus T\). Then \(u, v \in T\) and \(w = w\). For each 
\((x, y) \in w'\), if \((x, y) \in X_i \times X_j\) for some \(i \neq j\), then 
\(|xu' \cap y(v')^*| = |xu'^i \cap y(v^i)^*| = c_{u'^i, v^i}^w\).

Since \(T\) is a closed subset of \(S\), \(c_{u'^i, v^i}^w = 0\) and \(c_{u'^i, v^i}^w\) is well-defined. Now we assume \(w \in T\). Then \(v \in S \setminus T\), or \(u \in S \setminus T\). Since \(n_wc_{u^w}^w = n_v c_{w^w}^w = n_u c_{u^w}^w\), we obtain the same conclusion as in the case when \(w\) belongs to \(S \setminus T\).

(iii) \(|\{u, v, w\} \cap T| = 1\). First, we assume that \(u \in T\) and \(v, w \in S \setminus T\). For each \((x, y) \in w'\), if \(x \in X_i\), then 
\(|xu' \cap y(v')^*| = |xu'^i \cap yv^i| = c_{u'^i, v^i}^w\).

Since \(v, w \in S \setminus T\) and \(\iota_i \in \text{Aut}(S)\), we have \(c_{u'^i, v^i}^w = c_{u'^i, v^i}^w = c_{u^w}^w\). Thus, \(c_{u^w}^w = c_{u^w}^w\) and \(c_{u^w}^w\) is well-defined. Similarly, we obtain the same conclusion as in the case when \(v\) or \(w\) belongs to \(T\).

(iv) \(|\{u, v, w\} \cap T| = 0\). It is clear that \(c_{u'^i, v^i}^w\) is well-defined and \(c_{u^w}^w = c_{u^w}^w\).

This completes the proof. \(\square\)

**Proof of Theorem 1.2**

Let \(\iota\) be an element of \(\text{Aut}(T)\) fixing each element of \(H\). For convenience, we denote \(s^\iota\) by \(s^{\tilde{\iota}}\). We will show that \(c_{u^w}^w = c_{\tilde{u}^w}^w\) for all \(u, v, w \in S\).

We divide our consideration into four cases depending on \(|\{u, v, w\} \cap T|\).

(i) \(|\{u, v, w\} \cap T| = 3\). Since \(\iota\) is an element of \(\text{Aut}(T)\), we have \(c_{\tilde{u}^w}^w = c_{u^w}^w = c_{u^w}^w\).

(ii) \(|\{u, v, w\} \cap T| = 2\). Since \(T\) is a closed subset of \(S\), \(c_{u^w}^w = 0 = c_{\tilde{u}^w}^w\). Thus we have \(c_{\tilde{u}^w}^w = c_{u^w}^w\).

(iii) \(|\{u, v, w\} \cap T| = 1\). First, we assume that \(u \in T \setminus H\). If \(uv^* \not\subseteq T\), then clearly \(c_{u^w}^w = 0 = c_{\tilde{u}^w}^w\). Now assume \(uv^* \subseteq T\). Let \((x, y) \in \tilde{w} = \tilde{w}\). Note that \(xu\) is a union of cosets of \(H\) in \(xT\) by the second condition, and

\[yv^* = yv^* \cap yv^* T = yv^* \cap yv^* T = \bigcup_{i=1}^{m} (xu^i \cap yv^* H),\]

where \(\{x_1, x_2, \ldots, x_m\}\) is a transversal of \(H\) in \(yv^* T\) such that \((y, x_i) \in v^*\) for each \(i\).

Since \(|yv^* \cap x_i H| = c_{v^* H}^w\) does not depend on the choice of \(i\), it follows that

\[|xu \cap yv^*| = \left| \bigcup_{i=1}^{m} (xu \cap (yv^* \cap x_i H)) \right| = (\text{the number of cosets of } H \text{ in } xu) \times c_{v^* H}^w = n_{u} n_{H} c_{v^* H}^w.\]

Since \(n_u = n_{\tilde{u}}, c_{\tilde{u}^w}^w\) is equal to \(c_{u^w}^w\). Thus, since \(\tilde{w} = \tilde{w}\) and \(\tilde{v} = v\), we have \(c_{\tilde{u}^w}^w = c_{\tilde{u}^w}^w\). Next, we assume that \(u \in H\). Then it is clear that \(c_{u^w}^w = c_{\tilde{u}^w}^w\). Since \(n_u c_{\tilde{u}^w}^w = n_v c_{\tilde{u}^w}^w = n_u c_{\tilde{u}^w}^w\), we obtain the same conclusion as in the case when \(v\) or \(w\) belongs to \(T\).
(iv) \(|\{u,v,w\} \cap T| = 0. By the definition of \(i\), clearly \(c^w_{uv} = c^w_{uv}\).

This completes the proof. □

Remark 4.1. Suppose that \((X,S)\) is a \(p\)-scheme of order \(p^2\) and \(T := O^p(S)\) is not a group. Put \(H := \{t \in T \mid n_t = 1\}\). Then \(T\) and \(H\) satisfy the conditions in Theorem 1.2.

The following is an example about Remark 4.1.

Example 4.2. Let \(V := \mathbb{F}_p^2\) and \(G\) be a group generated by \(\{\tau_y, \iota \mid y \in V\}\), where \(\tau_y\) and \(\iota\) are functions from \(V\) to \(V\) defined by \(\tau_y(x) = x + y\) and \(\iota((x_1, x_2, x_3)) = ((x_1 + x_2, x_2 + x_3, x_3))\), respectively. Then the orbitals of \(G\) on \(V\) are

\[
(0,0)^G, (0,e_1)^G, (0,2e_1)^G, \ldots, (0,(p-1)e_1)^G,
\]

\[
(0,e_2)^G, (0,2e_2)^G, \ldots, (0,(p-1)e_2)^G,
\]

\[
(0,e_3)^G, (0,e_3 + e_1)^G, \ldots, (0,e_3 + (p-1)e_1)^G,
\]

\[
(0,2e_3)^G, (0,2e_3 + e_1)^G, \ldots, (0,2e_3 + (p-1)e_1)^G,
\]

\[
\vdots
\]

\[
(0,(p-1)e_3)^G, (0,(p-1)e_3 + e_1)^G, \ldots, (0,(p-1)e_3 + (p-1)e_1)^G,
\]

where \(\{e_1, e_2, e_3\}\) is the standard basis of \(V\).

Let \((X,S)\) be the Schurian scheme \((V,R_G)\) induced by \(G\) on \(V \times V\). Put

\[H := \{(0,0)^G, (0,e_1)^G, (0,2e_1)^G, \ldots, (0,(p-1)e_1)^G\}\]

and

\[T := H \cup \{(0,e_2)^G, (0,2e_2)^G, \ldots, (0,(p-1)e_2)^G\}.\]

Note that \(G \cong V \rtimes \langle \iota \rangle\), \(H \cong C_p\) and \(T \cong C_p \wr C_p\) as a wreath product of schemes. Then one can check that \(T\) and \(H\) satisfy the conditions in Theorem 1.2. Further, we can observe the following:

For \(s \in S \setminus T\) and \(t \in T \setminus H\),

(i) there exists a unique \(s_1 \in S \setminus T\) such that \(c^{s_1}_{ss} = 1\);
(ii) \(ss^* = s^*s = \{1_x\} \cup (T \setminus H)\);
(iii) \(|tt| = 1\).

Note that (ii) implies that \(c^{s_1}_{ss} = 1\) for each \(t \in T \setminus H\), and hence \(c^{s_1}_{ss} = 1\) since \(n_s = n_t = p\). We will denote by \(t'\) the element in \(tt\) which is unique by (iii).

Theorem 4.3. For each odd prime \(p\), there exists a non-Schurian scheme with the same intersection numbers as the Schurian scheme \((X,S)\) given in Example 4.2.

Proof. Fix \(s_1 \in S \setminus T\) and \(t_1 \in T \setminus H\). Let \(x \in X\) and \(y_1 \in xs_1\). Then there exists a unique \(y_2 \in y_1t_1 \cap xs_1\) since \(c^{s_1}_{s_1s_1} = 1\). Inductively, we can define \(y_i \in y_{i-1}t_1 \cap xs_1\) for \(i = 1, \ldots, p-1\). Note that \((y_i, y_1) \in t_1\). For each \(y_i \in xs_1\), there exists a unique \(z_i \in xs_2 \cap y_is_1\), where \(s_2\) is a unique element of \(S \setminus T\) such that \(c^{s_2}_{s_1s_1} = 1\). Note that \(\{z_i \mid 1 \leq i \leq p\}\) are distinct and \((z_i, z_{i+1}) \in t'_1\).

On the other hand, it is possible to take \(t \in \text{Aut}(T)\) such that \((t'_1)^t = t_1\) and \(t|_H\) is the identity map. By Theorem 1.2 \(t\) is extended to an element of \(\text{Aut}(S)\). Let \((X,S')\) be the scheme obtained
by applying Theorem 1.1 for the case where \( \iota_j = \iota \) for the coset \( X_j \) of \( T \) containing \( z_1 \) and \( \iota_i \) is the identity except for \( j \).

Next, we claim that \((X, S')\) is not isomorphic to \((X, S)\). Suppose to the contrary that \( \phi : X \cup S' \rightarrow X \cup S \) is an isomorphism such that for each \( x, y \in X \),

\[(x, y) \in s \iff (x^{\phi}, y^{\phi}) \in s^{\phi}.\]

Then the adjacency among elements in \( \{x, y, z_i \mid 1 \leq i \leq p\} \) must be invariant under \( \phi \). Note that both \((y_1, y_2)\) and \((z_1, z_2)\) lie in \( t'_1 \). Therefore, both \((y_1^{\phi}, y_2^{\phi})\) and \((z_1^{\phi}, z_2^{\phi})\) lie in \( t'_1^{\phi} \). However, such a structure does not occur in \((X, S)\) by the construction of \((X, S')\), a contradiction.

Finally, we claim that \((X, S')\) is not Schurian. Let \( x \in X, s_1 \in S \setminus T \) and \( y_1 \in xs_1 \). Then a pair \((x, y_1)\) induces unique \( y_2 \in xs_1 \cap y_1 t_1 \) and \( z_i \in xs_2 \cap y_is_1 \) for \( i = 1, 2 \). If \((X, S')\) is Schurian, then the relation containing \((z_1, z_2)\) is uniquely determined. On the other hand, by the construction of \((X, S')\), we can take \((x, y_1), (\hat{x}, \hat{y}_1) \in s\) such that the induced pairs \((z_1, z_2)\) and \((\hat{z}_1, \hat{z}_2)\) by \((x, y_1)\) and \((\hat{x}, \hat{y}_1)\) respectively, are not in the same relation. But, this is a contradiction. \( \square \)

References

[1] S. Bang, M. Hirasaka, Construction of association schemes from difference sets, Europ. J. Combin. 26 (2005) 59–74.

[2] E. Bannai, T. Ito, Algebraic combinatorics I: association schemes, Benjamin/Cummings, Menlo Park, 1984.

[3] T. Beth, D. Jungnickel, H. Lenz, Design theory, Cambridge University Press, 1999.

[4] A. Hanaki, Clifford theory for association schemes, J. Algebra 321 (2009) 1686–1695.

[5] A. Hanaki, Characters of association schemes containing a strongly normal closed subset of prime index. Proc. Am. Math. Soc. 135 (2007) 2683–2687.

[6] A. Hanaki, Representations of association schemes and their factor schemes, Graphs Combin. 19 (2003) 195–201.

[7] A. Hanaki, I. Miyamoto, Classification of association schemes of small order, Online catalogue. [http://kissme.shinshu-u.ac.jp/as](http://kissme.shinshu-u.ac.jp/as)

[8] M. Hirasaka, M. Muzychuk, Association schemes generated by a non-symmetric relation of valency 2, Discrete Math. 244 (2002) 109–135.

[9] M. Hirasaka, P.-H. Zieschang, Sufficient conditions for a scheme to originate from a group, J. Combin. Theory Ser. A 104 (2003), 17–27.

[10] K. Kim, Characterization of \( p \)-schemes of prime cube order, J. Algebra 331 (2011) 1–10.

[11] M. Klin, M. Muzychuk, C. Pech, A Woldar, P.-H. Zieschang, Association scheme on 28 points as mergings of a half-homogeneous coherent configuration, Europ. J. Combin. 28 (2007) 1994–2025.
[12] M. Muzychuk, I. Ponomarenko, On quasi-thin association schemes, J. Algebra 351 (2012) 467–489.

[13] P.-H. Zieschang, An Algebraic Approach to Association Schemes, Lecture Notes in Mathematics 1628, Springer, Berlin, 1996.

[14] P.-H. Zieschang, Theory of association schemes, Springer monographs in mathematics, Springer, Berlin, 2005.