Dedicated to Prof. Emeritus Mihail Megan on the occasion of his 75th anniversary

Fixed points and the stability of the linear functional equations in a single variable

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ABSTRACT. In this paper we prove that an interesting result concerning the generalized Hyers-Ulam stability of the linear functional equation \( g(\varphi(x)) = a(x) \cdot g(x) \) on a complete metric group, given in 2014 by S.M. Jung, D. Popa and M.T. Rassias, can be obtained using the fixed point technique. Moreover, we give a characterization of the functions that can be approximated with a given error, by the solution of the linear equation mentioned above. Our results are also related to a recent result of G.H. Kim and Th.M. Rassias concerning the stability of Psi functional equation.

1. INTRODUCTION

"When a solution of an equation differing slightly from a given one must be somehow near to the solution of the given equation?" is the question formulated in 1940 by S.M. Ulam [33] while giving a lecture at the University of Wisconsin, on the stability of group homomorphisms. In a more precise formulation, its problem of stability reads as follows:

Let \((G_1, \circ)\) be a group, \((G_2, \ast)\) be a metric group with the metric \(d(\cdot, \cdot)\) and \(\varepsilon > 0\). Does there exists a \(\delta > 0\) such that if \(f : G_1 \to G_2\) satisfies

\[
d(f(x \circ y), f(x) \ast f(y)) \leq \delta, \quad \text{for all } x, y \in G_1
\]

there exists a homomorphism \(h : G_1 \to G_2\) with

\[
d(f(x), h(x)) \leq \varepsilon, \quad \text{for all } x \in G_1?
\]

A first answer to Ulam’s question was given by D. H. Hyers [22] in 1941 concerning the Cauchy functional equation. Afterwards different generalizations of that initial answer of Hyers were obtained. Hyers’ theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [31] for approximately linear mappings, by considering an unbounded Cauchy difference. See also [17], [30] and [32]. Nowadays we speak about the concept of Hyers-Ulam stability.

A further generalization was obtained by P. Gavruta [18] in 1994. See also [19] and [21] for more generalizations. The papers mentioned above use the direct method (of Hyers), i.e., the exact solution of the functional equation is explicitly constructed as a limit of a sequence, starting from the given approximate solution.

For other results and generalizations, see the books [5],[15], [23],[25] and their references.

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Among applications of the functional equations, we mention modeling in science and engineering (see [13]). An interesting application of the stability in the sense of Hyers-Ulam pointed out by D.H. Hyers, G. Isac and Th. M. Rassias is in the study of complementarity problems (see the book [23]). For other complementarity problems, see also the article of G. Isac [24].

On the other hand, J.A. Baker [2] used in 1991 the Banach fixed point theorem to give Hyers-Ulam stability results for a nonlinear functional equation. In 2003, V. Radu [29] proposed a new method, successively developed in [6], to obtain the existence of the exact solutions and the error estimations, based on the fixed point alternative. For some other applications of the fixed point theorem in the generalized Hyers-Ulam stability see the papers [7], [8], [9], [11], [12], [14], [16], [20], [28].

Recently, J. Brzdek, J. Chudziak & Z. Páles proved in [3] a general fixed point theorem for (not necessarily) linear operators and they used it to obtain Hyers-Ulam stability results for a class of functional equations in a single variable. A fixed point result of the same type was proved by J. Brzdek & K. Ciepliński [4] in complete non-Archimedean metric spaces as well as in complete metric spaces. Also, they formulated an open problem concerning the uniqueness of the fixed point.

In the paper [10] we obtained a fixed point theorem for a class of operators with suitable properties, in very general conditions. Also, we showed that some recent results in [3] and [4] can be obtained as particular cases of our theorem. Moreover, by using our outcome, we gave affirmative answer to the open problem of J. Brzdek & K. Ciepliński, formulated at the end of the paper [4]. We also showed that our main Theorem is an efficient tool for proving generalized Hyers-Ulam stability results of several functional equations in a single variable. To this end, we prove in this paper that an interesting result concerning generalized Hyers-Ulam-Rassias stability of a linear functional equation obtained in 2014 by S.M. Jung, D. Popa and M.T. Rassias in [26] is a particular case of a fixed point theorem given by us in [10]. Moreover, we give a characterization of the functions that can be approximated with a given error, by the solution of the previously mention linear equation.

We consider a nonempty set $X$, a complete metric space $(Y, d)$ and the mappings

$$
\Lambda: \mathbb{R}^+_X \to \mathbb{R}^+_X \quad \text{and} \quad T: Y^X \to Y^X.
$$

We recall that, for two sets $M$ and $N$, $N^M$ is the space of all mappings from $M$ to $N$ and if $(\delta_n)_{n \in \mathbb{N}}$ is a sequence of elements of $\mathbb{R}^+_X$, we write

$$\lim_{n \to \infty} \delta_n = 0 \quad \text{pointwise if} \quad \lim_{n \to \infty} \delta_n(x) = 0 \quad \forall x \in X.$$

$\mathbb{R}_+$ stands for the set of all nonnegative numbers, i.e., $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}^+_* = (0, \infty)$.

**Definition 1.1.** [10] We say that $T$ is $\Lambda$–contractive if for all $u, v \in Y^X$ and $\delta \in \mathbb{R}^+_X$ with

$$d(u(x), v(x)) \leq \delta(x), \quad \forall \ x \in X,$$

it follows

$$d((Tu)(x), (Tv)(x)) \leq (\Lambda \delta)(x), \quad \forall \ x \in X.$$

In the paper [10] we obtained the following fixed point theorem:

**Theorem 1.1.** We suppose that the operator $T$ is $\Lambda$–contractive, where $\Lambda$ satisfies the condition:

$$(C_1) \quad \text{for every sequence} \ (\delta_n)_{n \in \mathbb{N}} \ in \ \mathbb{R}^+_X \ \text{such that}$$

$$\lim_{n \to \infty} \delta_n = 0 \quad \text{pointwise, it follows that} \quad \lim_{n \to \infty} \Lambda \delta_n = 0 \quad \text{pointwise.}$$
We suppose that \( \varepsilon \in \mathbb{R}_+^X \) is a given function such that
\[
(C_2) \quad \varepsilon^*(x) := \sum_{k=0}^{\infty} (\Lambda^k \varepsilon)(x) < \infty, \quad \forall x \in X.
\]

We consider a mapping \( f \in Y^X \) such that
\[
(1.1) \quad d((Tf)(x), f(x)) \leq \varepsilon(x), \quad \forall x \in X.
\]
Then, for every \( x \in X \), the limit
\[
(1.2) \quad g(x) := \lim_{n \to \infty} (T^n f)(x),
\]
exists and the function \( g \) is the unique fixed point of \( T \) with the property
\[
(1.3) \quad d((T^m f)(x), g(x)) \leq \sum_{k=m}^{\infty} (\Lambda^k \varepsilon)(t), \quad x \in X, \ m \in \mathbb{N}.
\]
Moreover, if we have
\[
(C_3) \quad \lim_{n \to \infty} \Lambda^n \varepsilon^* = 0 \text{ pointwise},
\]
then \( g \) is the unique fixed point of \( T \) with the property
\[
(1.4) \quad d(f(x), g(x)) \leq \varepsilon^*(x), \forall x \in X.
\]

Theorem 1.1 generalizes a result of J. Brzdek and K. Cieplinski [4] concerning the existence of the fixed points. Moreover, our theorem provides a positive answer to the open question raised by these authors concerning the uniqueness of the fixed point.

2. Stability of the functional equation \( g(\varphi(x)) = a(x) \bullet g(x) \)

We take a nonempty set \( X \) and a complete metric group \((G, \bullet, d)\) with the metric \( d \) invariant to the left translation, i.e.,
\[
d(x \bullet y, x \bullet z) = d(y, z), \quad \text{for all } x, y, z \in G.
\]

We consider the given functions \( \varphi : X \to X \) and \( a : X \to G \).

We denote
\[
A_n(x) := a(\varphi^{n-1}(x)) \bullet \ldots \bullet a(\varphi(x)) \bullet a(x), \quad x \in X, \ n \geq 1.
\]

We have
\[
A_n(\varphi(x)) = A_{n+1}(x) \bullet (a(x))^{-1}, \quad x \in X, n \geq 1,
\]
and successive by
\[
A_n(\varphi^m(x)) = A_{n+m}(x) \bullet (A_m(x))^{-1}, \quad x \in X, \ m, n \geq 1.
\]

In this section we discuss the generalized Hyers-Ulam-Rassias stability of the functional equation
\[
(2.5) \quad g(\varphi(x)) = a(x) \bullet g(x), \ x \in X,
\]
where \( g : X \to G \) is the unknown function.

The equation (2.5) is equivalent to
\[
(2.6) \quad (a(x))^{-1} \bullet g(\varphi(x)) = g(x), \ x \in X.
\]
We remark also that
\[
(2.7) \quad g(\varphi^n(x)) = A_n(x) \bullet g(x), \quad x \in X, n \geq 1.
\]
In what follows we will show that the main result of the paper [26] concerning the generalized Hyers-Ulam-Rassias stability of the equation (2.5) is a consequence of our Theorem 1.1. To this end, we start with the presentation of the main result from [26]:

**Theorem 2.2.** [26] Let $\varepsilon : X \to \mathbb{R}_+$ be a given function with the property

$$
\varepsilon^*(x) := \sum_{k=0}^{\infty} \varepsilon (\varphi^k(x)) < \infty, \forall x \in X.
$$

Then, for every function $f : X \to G$ satisfying the inequality

$$
(2.8) \quad d(f(\varphi(x)), a(x) \cdot f(x)) \leq \varepsilon(x), \forall x \in X,
$$

there exists a unique solution $g$ of the equation (2.5) such that

$$
(2.9) \quad d(f(x), g(x)) \leq \varepsilon^*(x), \forall x \in X.
$$

This solution is given by the formula

$$
(2.10) \quad g(x) := \lim_{n \to \infty} (A_n(x))^{-1} \cdot f (\varphi^n(x)).
$$

We can easily see that the above theorem is a particular case of our fixed point result emphasized in the first section.

**Proof.** We take in Theorem 1.1

$$(Tu)(x) = (a(x))^{-1} \cdot u(\varphi(x)) \quad \text{and} \quad (\Lambda \delta)(x) = \delta(\varphi(x)).$$

So, it follows

$$
d ((Tu)(x), (Tv)(x)) = d(u(\varphi(x)), v(\varphi(x))) \leq (\Lambda \delta)(x)
$$

if

$$
d(u(x), v(x)) \leq \delta(x),
$$

hence the operator $T$ is $\Lambda-$ contractive in the sense of the Definition 1.1.

On the other hand, by using the invariance property to the left translation of the metric $d$ and the assumption (2.8), we obtain that (1.1) holds.

Uniqueness of $g$ results also from Theorem 1.1. In fact, we prove that $\Lambda$ satisfies the hypothesis $(C_3)$:

$$
\Lambda^n(\varepsilon^*(x)) = \Lambda^n \left( \sum_{k=0}^{\infty} \varepsilon (\varphi^k(x)) \right) = \sum_{k=0}^{\infty} \varepsilon (\varphi^{n+k}(x)) = \sum_{n=n}^{\infty} \varepsilon (\varphi^n(x)).
$$

Thus

$$
\lim_{n \to \infty} \Lambda^n(\varepsilon^*(x)) = 0, \; x \in X.
$$

□

In the second Theorem of this section we will give a characterization of the functions $f : X \to G$ that can be approximated with a given error, by a solution of the equation (2.5).

We denote by

$$
{\mathcal{E}}_\varphi = \left\{ \varepsilon \in \mathbb{R}_+^X, \lim_{n \to \infty} \varepsilon (\varphi^n(x)) = 0, \forall x \in X \right\}.
$$
Theorem 2.3. The following statements are equivalent:

(i) There exists a unique solution \( g \) of (2.5) such that
\[
d(f(x), g(x)) \leq \varepsilon(x), \forall x \in X.
\]

(ii) \( d(f(\varphi^n(x)), A_n(x) \ast f(x)) \leq \varepsilon(x) + \varepsilon(\varphi^n(x)), x \in X, n \geq 1. \)

(iii) there exists \( \delta \in \mathcal{E}_\varphi \) such that
\[
d(f(\varphi^n(x)), A_n(x) \ast f(x)) \leq \varepsilon(x) + \delta(\varphi^n(x)), x \in X, n \geq 1.
\]

Proof. (i) \( \Rightarrow \) (ii). We have, by using (2.7)
\[
d(f(\varphi^n(x)), A_n(x) \ast f(x)) \leq d(f(\varphi^n(x)), g(\varphi^n(x))) + d(g(\varphi^n(x)), A_n(x) \ast f(x))
\]
\[
\leq \varepsilon(\varphi^n(x)) + d(A_n(x) \ast g(x), A_n(x) \ast f(x))
\]
\[
= \varepsilon(\varphi^n(x)) + \varepsilon(x).
\]

(ii) \( \Rightarrow \) (iii). We take in (ii) \( \delta = \varepsilon. \)

(iii) \( \Rightarrow \) (i). In (iii) with \( \varphi^m(x) \) instead of \( x \), we have
\[
d(f(\varphi^{n+m}(x)), A_n(\varphi^m(x)) \ast f(\varphi^m(x))) \leq \varepsilon(\varphi^m(x)) + \delta(\varphi^{n+m}(x)),
\]
which means
\[
d(f(\varphi^{n+m}(x)), A_{n+m}(x) \ast (A_m(x))^{-1} \ast f(\varphi^m(x))) \leq \varepsilon(\varphi^m(x)) + \delta(\varphi^{n+m}(x)),
\]
hence
\[
d((A_{n+m}(x))^{-1} \ast f(\varphi^m(x)), (A_m(x))^{-1} \ast f(\varphi^m(x))) \leq \varepsilon(\varphi^m(x)) + \delta(\varphi^{n+m}(x)).
\]
It follows that the sequence
\[
\left\{(A_n(x))^{-1} \ast f(\varphi^n(x))\right\}_{n \geq 1}
\]
is a Cauchy sequence. Since \((G, \ast, d)\) is complete, it results that there exists
\[
g(x) := \lim_{n \to \infty} (A_n(x))^{-1} \ast f(\varphi^n(x)), x \in X.
\]
We have
\[
g(\varphi(x)) = a(x) \ast \lim_{n \to \infty} (A_{n+1}(x))^{-1} \ast f(\varphi^{n+1}(x)) = a(x) \ast g(x), x \in X,
\]
hence \( g \) is a solution of (2.5) and
\[
d(g(x), (A_m(x))^{-1} \ast f(\varphi^m(x))) \leq \varepsilon(\varphi^m(x)), x \in X, m \geq 1.
\]
By (iii) it follows that
\[
d((A_n(x))^{-1} \ast f(\varphi^n(x)), f(x)) \leq \varepsilon(x) + \delta(\varphi^n(x))
\]
and by letting \( n \) go to infinity, we obtain
\[
d(f(x), g(x)) \leq \varepsilon(x), \forall x \in X.
\]
We prove now the uniqueness of \( g \). To this end, let us consider a solution \( h : X \to G \)
of the equation (2.5), satisfying the relation
\[
d(h(x), f(x)) \leq \varepsilon(x), \forall x \in X.
\]
By replacing \( x \) by \( \varphi^m(x) \), we have
\[
d(h(\varphi^m(x)), f(\varphi^m(x))) \leq \varepsilon(\varphi^m(x)), \forall x \in X.
\]
Having in mind that \( h(\varphi^n(x)) = A_m(x) \ast h(x) \), it follows
\[
d(h(x), (A_m(x))^{-1} \ast f(\varphi^m(x))) \leq \varepsilon(\varphi^m(x)), x \in X.
\]
Letting $m$ go to infinity, we obtain
\[ d(h(x), g(x)) = 0, \forall x \in X. \]

As a direct application of the Theorem 2.3 we will obtain the following result concerning the characterization of the functions $f : \mathbb{R}_+^* \rightarrow \mathbb{R}$ that can be approximated with a given error, by the solutions of Digamma functional equation

(2.11)
\[ g(x + 1) = g(x) + \frac{1}{x}, \quad x \in \mathbb{R}_+^*. \]

**Corollary 2.1.** The following statements are equivalent:

(i) There exists a unique solution $g$ of (2.11) such that
\[ |f(x) - g(x)| \leq \varepsilon(x), \forall x \in \mathbb{R}_+^*. \]

(ii) \[ |f(x + n) - f(x) - \sum_{k=0}^{n-1} \frac{1}{x+k}| \leq \varepsilon(x) + \varepsilon(x+n), \quad x \in \mathbb{R}_+^*, n \geq 1. \]

(iii) There exists
\[ \delta \in \mathcal{E}_\varphi := \{ \varepsilon : X \rightarrow \mathbb{R}_+, \lim_{n \rightarrow \infty} \varepsilon(x + n) = 0, \forall x \in \mathbb{R}_+^* \} \]

so that
\[ |f(x + n) - f(x) - \sum_{k=0}^{n-1} \frac{1}{x+k}| \leq \varepsilon(x) + \delta(x+n), \quad x \in \mathbb{R}_+^*, n \geq 1. \]

**Proof.** The result follows immediately by taking in Theorem 2.3, $X = \mathbb{R}_+^*, (G, \cdot) = (\mathbb{R}, +)$, $d$ the Euclidean metric on $\mathbb{R}$, $\varphi(x) = x + 1, a(x) = \frac{1}{x}, x \in \mathbb{R}_+^*$. \(\square\)

We give below a more general result obtained in the same way, which is in connection with the recent paper of G.H. Kim and Th. M. Rassias [27].

**Corollary 2.2.** Let $p$ be a positive real number. The following statements are equivalent:

(i) There exists a unique solution $g$ of the functional equation
\[ g(x + p) = g(x) + a(x), \quad x \in \mathbb{R}_+^* \]

such that
\[ |f(x) - g(x)| \leq \varepsilon(x), \forall x \in \mathbb{R}_+^*. \]

(ii) \[ |f(x + np) - f(x) - \sum_{k=0}^{n-1} a(x + kp)| \leq \varepsilon(x) + \varepsilon(x + np), \quad x \in \mathbb{R}_+^*, n \geq 1. \]

(iii) There exists
\[ \delta \in \mathcal{E}_\varphi := \{ \varepsilon : X \rightarrow \mathbb{R}_+, \lim_{n \rightarrow \infty} \varepsilon(x + np) = 0, \forall x \in \mathbb{R}_+^* \} \]

so that
\[ |f(x + np) - f(x) - \sum_{k=0}^{n-1} a(x + kp)| \leq \varepsilon(x) + \delta(x + np), \quad x \in \mathbb{R}_+^*, n \geq 1. \]
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