BOUNDS FOR SESHADRI CONSTANTS

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INTRODUCTION

In this paper we present an alternative approach to the boundedness of Seshadri constants of nef and big line bundles at a general point of a complex–projective variety.

Seshadri constants $\varepsilon(L, x)$, which have been introduced by Demailly [De92], measure the local positivity of a nef line bundle $L$ at a point $x \in X$ of a complex–projective variety $X$, and can be defined as

$$\varepsilon(L, x) := \inf_{C \ni x} \left\{ \frac{L \cdot C}{\text{mult}_x(C)} \right\},$$

where the infimum is taken over all reduced irreducible curves $C \subset X$ passing through $x$ (cf. also §1 below, [De92] or [EKL] for further characterizations and properties of Seshadri constants).

Over the last years there has been quite some activity in studying Seshadri constants, starting with the somehow surprising result by Ein and Lazarsfeld (cf. [EL]) that the Seshadri constant of an ample line bundle on a smooth surface is bounded below by 1 for all except perhaps countably many points.

On the other hand examples by Miranda (cf. [EKL, 1.5]) show that for any integral $n \geq 2$ and real $\delta > 0$ there is a smooth $n$–dimensional variety $X$, an ample line bundle $L$ on $X$ and a point $x \in X$ with $\varepsilon(L, x) < \delta$; in other words, there does not exist a universal lower bound for Seshadri constants valid for all $X$ and ample $L$ at every point $x \in X$.

Then it was proven by Ein-Küchle-Lazarsfeld [EKL] that, for a nef and big line bundle $L$ on an $n$–dimensional projective variety, the Seshadri constant at very general points

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(i.e. outside a countable union of proper subvarieties) is bounded below by $\frac{1}{n}$, and that this implies the existence of a lower bound at general points depending only on $n$.

Finally we want to mention the recent papers by Nakamaye [Na] and Steffens [St] dealing with the problem of "maximality" of Seshadri constants on abelian varieties, as well as variants due to Küchle [Kü] and Paoletti [Pa] concerning Seshadri constants along several points and higher dimensional subvarieties respectively.

Here we first reprove the existence of lower bounds for Seshadri constants of nef and big line bundles at general points of a projective variety. Although the bound obtained does, in general, not improve the one in [EKL], the method at hand may give better results in certain cases, since our bound can be expressed more flexibly in terms of the degrees of subvarieties with respect to the line bundle in question:

**Theorem 3.4.** Let $L$ be a nef line bundle on an $n$-dimensional irreducible projective variety $X$ and $\varepsilon > 0$ be a real number. Let $\alpha_1, \ldots, \alpha_n$ be positive rational numbers and $x \in X$ a general point. Put $\gamma = 1 + \sum_{i=1}^{n-1} \alpha_i$. Suppose that any $d$-dimensional $(1 \leq d \leq n)$ subvariety $V \subseteq X$ containing a very general point $y \in X$ satisfies

$$\deg_L V = L^d V \geq \varepsilon \cdot \frac{\gamma^n}{\alpha_n^{n-d}}.$$ 

Then $\varepsilon(L, x) \geq \varepsilon$.

One special feature of our proof is that there is no mention of "Seshadri–exceptional" curves; the objects of consideration are families of divisors with high multiplicity at prescribed points. The method itself is based upon ideas of Demailly's paper [De93] as explained and translated in an algebro–geometric language by Ein–Lazarsfeld–Nakamaye in [ELN]. The basic idea is, very roughly, to start with an effective divisor $E$ in a linear series $|kL|$ ($k \gg 0$) with large multiplicity at $x$, and to consider the locus $V$ of points where the singularities of $E$ are "concentrated" in a certain way. The possibility that $V$ is zero–dimensional imposes constraints on the local positivity of $L$ at $x$ in a sense. Otherwise one uses variational techniques to give a lower bound on the degree of $V$.

Instead of, as in [De93] and [ELN], making a positivity assumption on the tangent bundle of the manifold in question to be able to "differentiate", we apply the strategy of differentiation in parameter directions in the spirit of [EKL]. This, however, forces us to consider only very general points instead of arbitrary ones. The result of this method is the following Theorem, which might be of interest also in other contexts:

**Theorem 2.1.** Let $X$ be a smooth $n$-dimensional projective variety, $L$ a nef and big line bundle on $X$ and $\alpha > 0$ a rational number such that $L^n > \alpha^n$. Let

$$0 = \beta_1 < \beta_2 < \cdots < \beta_n < \beta_{n+1} = \alpha$$

be any sequence of rational numbers and $x \in X$ a very general point. Then either

(a) there exist $k \gg 0$ and a divisor $E \in |kL|$ having an isolated singularity with multiplicity at least $k(\beta_{n+1} - \beta_n)$ at $x$, or
(b) there exists a proper subvariety $V \subset X$ through $x$ of codimension $c \leq n - 1$ such that

$$\deg L V = L^{n-c} \cdot V \leq \frac{1}{(\beta_{c+1} - \beta_c)^c} \left(1 - n \sqrt{1 - \frac{\alpha^n}{L^n}}\right)^c L^n < \frac{\alpha^n}{(\beta_{c+1} - \beta_c)^c}.$$ 

To pass from Theorem 2.1 to actually bounding the Seshadri constant we use a rescaling–trick (cf. 3.3) in combination with the well known characterization of Seshadri constants via the generation of $s$–jets by certain adjoint linear systems (cf. 1.5).

After a first version of this paper was written, we realized that the rescaling argument mentioned above can also be applied to the results of [ELN], leading to bounds for Seshadri constants valid at arbitrary points; these bounds, however, depend on the line bundle $L$ and the manifold $X$, or, rather, its tangent bundle $T_X$:

**Corollary 4.4.** Let $X$ be a smooth $n$–dimensional projective variety, $x \in X$ any point, $A$ an ample line bundle and $\delta \geq 0$ a real number such that $T_X(\delta A)$ ist nef. Then

$$\varepsilon(A, x) \geq \min \left\{ \frac{1}{(n-1)^{n-1}(2n-1)}, \frac{1}{\delta} \right\}.$$ 

This gives in particular bounds valid at arbitrary points for the Seshadri constants of the canonical line bundle $K_X = (\Lambda^n T_X)^*$ or its inverse if these are ample, or for any ample line bundle in case $K_X$ is trivial (cf. 4.5).

Corollary 4.4 is in accordance with and should be compared to bounds following from recent work of Angehrn–Siu [AS] on the basepoint–freeness of adjoint linear series (cf. 4.6).

The paper is organized as follows. After fixing notations and establishing a general setup in §0 we recall some basic facts about Seshadri constants and collect some auxiliary statements in §1. Then, in §2, we prove the main technical result, Theorem 2.1. Finally we give the applications to bounding Seshadri constants at general points in §3, and at arbitrary points in §4.

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§0. **Notations and the General Setup**

(0.1). Throughout this paper we will work over the field $\mathbb{C}$ of complex numbers. Given a variety $Y$ a general point $y \in Y$ is a point in the complement of some proper subvariety
of $Y$, and a very general point is a point in the complement of some countable union of proper subvarieties.

(0.2). Given a smooth variety $Y$, an integer $m \geq 0$ and a subvariety $W \subset Y$ we denote by $\mathcal{I}_W^{(m)}$ the symbolic power sheaf of all functions vanishing to order at least $m$ along $W$. Then $\mathcal{I}_W^{(1)} = \mathcal{I}_W$ is the ideal sheaf of $W$, and $\mathcal{I}_Z^{(m)} = \mathcal{I}_Z^m$ for a smooth subvariety $Z \subset Y$.

(0.3). Let $M$ be a line bundle on a smooth variety $Y$. Given a divisor $E \in |kM|$ we call the normalized multiplicity

$$\text{ind}_y(E) = \frac{\text{mult}_y(E)}{k}$$

the index of $E$ at a point $y \in Y$.

We say that a divisor $E \in |kM|$ has an almost isolated singularity of index $\geq \alpha$ at a point $y \in Y$, if $\text{ind}_y(E) \geq \alpha$ and there is an open neighborhood $U \subset Y$ of $y$ such that $\text{ind}_x(E) < 1$ for all $x \in U \setminus \{y\}$.

(0.4). For a line bundle $L$ and a coherent sheaf of ideals $\mathcal{J}$ on $Y$ we denote by $|L \otimes \mathcal{J}|$ the linear subsystem of the complete linear system $|L|$ corresponding to sections in $L \otimes \mathcal{J}$. Given such a system $|L \otimes \mathcal{J}| \neq \emptyset$ on $Y$, by the base locus $\text{Bs}|L \otimes \mathcal{J}|$ we mean the support of the intersection of all members of $|L \otimes \mathcal{J}|$.

(0.5). We will be concerned with the following setup: Let be $X$ a smooth irreducible $n$–dimensional projective variety, $T$ a smooth irreducible affine variety and $g : T \rightarrow X$ a quasi-finite dominant morphism with graph $\Gamma$. Let $\text{pr}_X$ and $\text{pr}_T$ denote the projections from $Y = X \times T$ onto its factors. Note that the projections then map $\Gamma$ dominantly to $X$ resp. $T$. Given a Zariski-closed subset (or subscheme) $Z \subset X \times T$, we consider the fibre $Z_t$ of $\text{pr}_T$ as a subset (or subscheme) of $X$. Similarly, $Z_x \subset T$ is the fibre of $Z$ over $x \in X$. Given a sheaf $\mathcal{F}$ on $X \times T$ we write $\mathcal{F}_t$ for the induced sheaf on $X$.

Examples of this situation are provided by Zariski–open subsets $T \subset X$.

§1. Preliminaries

In this section we collect some preliminary results which will be needed in the sequel. We start with some remarks concerning multiplicity loci in a family.

Let $E$ be an effective divisor on a smooth variety $Y$. Then the function $y \rightarrow \text{mult}_y(E)$ is Zariski upper-semicontinuous on $Y$. For any given irreducible subvariety $Z \subset Y$ we refer with $\text{mult}_Z(E)$ to the value of $\text{mult}_z(E)$ at a general point $z \in Z$. The following lemma allows one to make fibrewise calculations of multiplicities.
Lemma 1.1. (cf. [EKL, 2.1]) Let $X$ and $T$ be smooth irreducible varieties, and suppose that $Z \subset X \times T$ is an irreducible subvariety which dominates $T$. Let $\mathcal{E} \subset X \times T$ be an effective divisor.

Then for a general point $t \in T$, and any irreducible component $W_t \subset Z_t$ of the fiber $Z_t$, we have

$$\text{mult}_{W_t}(\mathcal{E}_t) = \text{mult}_Z(\mathcal{E}).$$

□

The next elementary lemma gives a way to detect irreducible components of base loci.

Lemma 1.2. Let $m$ be a positive integer, $M$ a line bundle on a smooth variety $Y$ and $V \subseteq W \subseteq Y$ subvarieties such that $V$ is irreducible. Suppose that

1. $\mathcal{I}_W \otimes M$ is generated by global sections, and
2. $\mathcal{I}_W \subseteq \mathcal{I}_V^{(m)}$.

Then $V$ is an irreducible component of $\text{Bs}|\mathcal{I}_V^{(m)} \otimes M|$.

Proof. Consider the inclusions

$$V \subseteq \text{Bs}|\mathcal{I}_V^{(m)} \otimes M| \subseteq \text{Bs}|\mathcal{I}_W \otimes M| \subseteq W,$$

where the first inclusion is clear, the second follows from (2), and the third from the fact that $\text{Bs}|\mathcal{I}_W \otimes M| = W$ because $\mathcal{I}_W \otimes M$ is globally generated according to (1).

Now let $Z \subseteq \text{Bs}|\mathcal{I}_V^{(m)} \otimes M|$ be any irreducible component containing $V$. Then (*) and the assumption that $V \subseteq W$ is an irreducible component imply $V = Z$, hence the claim follows. □

Now we show that being an irreducible subvariety is well-behaved in families:

Lemma 1.3. Let $f : Y \to Z$ be a morphism between irreducible varieties, and $V, W \subseteq Y$ be subvarieties such that $V \subseteq W$ as an irreducible subvariety and $V$ maps dominant to $Z$.

Then over general $z \in Z$ every irreducible component $U_z$ of $V_z$ has dimension $\dim V - \dim Z$ and is an irreducible component of $W_z$.

Proof. For the first part we refer to [Ha, II.Ex 3.22]. The second assertion comes down to an easy dimension count as follows: Write

$$W = V \cup V'$$

with a subvariety $V' \subseteq Y$ not containing $V$, i.e. $\dim V > \dim (V \cap V')$. If $V \cap V'$ does not map dominantly to $Z$, then $V_z$ and $V'_z$ do not meet in a general fibre and the claim follows.
Otherwise we obtain over general \( z \in Z \):
\[
\dim V_z = \dim V - \dim Y \\
> \dim (V \cap V') - \dim Y \\
= \dim (V \cap V')_z,
\]
and therefore \( \dim U_z > \dim (V \cap V')_z \geq \dim (U_z \cap V'_z) \), where \( U_z \subseteq V_z \) is any irreducible component. Considering the decomposition \( V_z = U_z \cup U'_z \), where \( U_z \not\subseteq U'_z \), we conclude \( \dim U_z > \dim (U_z \cap (U'_z \cup V'_z)) \), which shows that \( U_z \) is an irreducible component of \( W_z = U_z \cup (V'_z \cup U'_z) \). □

For the reader’s convenience we recall some well–known facts concerning Seshadri constants. The next Lemma deals with the Seshadri constant at general versus the one at very general points (cf. [EKL, 1.4]).

**Lemma 1.4.** Let \( X \) be a smooth projective variety and \( L \) a nef and big line bundle on \( X \). Suppose \( \varepsilon(L, y) = \varepsilon \) for a point \( y \in X \). Then for any real \( \delta > 0 \) there exists a Zariski–open subset \( U(\delta) \subseteq X \) such that
\[
\varepsilon(L, x) \geq \varepsilon - \delta \text{ for all } x \in U(\delta). \quad □
\]

Finally we recall the relations between almost isolated singularities, generation of higher jets, and Seshadri constants (cf. [ELN, 1.1], [EKL, 1.3]).

**Theorem 1.5.** Let \( X \) be a smooth projective variety of dimension \( n \) and \( L \) a nef and big line bundle on \( X \).

(1.5.1) Suppose there exists a divisor \( E \in |kL| \) having an almost isolated singularity of index \( \geq n + s \) at \( x \in X \). Then
\[
H^1(X, \mathcal{O}_X(K_X + L) \otimes I_x^{s+1}) = 0.
\]
In particular, \( |K_X + L| \) generates \( s \)-jets at \( x \), i.e. the evaluation map
\[
H^0(X, \mathcal{O}_X(K_X + L)) \longrightarrow H^0(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{O}_X/I_x^{s+1})
\]
is surjective.

(1.5.2) Let \( \varepsilon(L, x) \) be the Seshadri constant of \( L \) at \( x \). If
\[
r > \frac{s}{\varepsilon(L, x)} + \frac{n}{\varepsilon(L, x)},
\]
then \( |K_X + rL| \) generates \( s \)-jets at \( x \in X \). The same statement holds if \( r = \frac{s + n}{\varepsilon(L, x)} \) and \( L^n > \varepsilon(L, x)^n \).
Conversely, suppose there is a real number $\varepsilon > 0$ plus a real constant $c$ such that $|K_X + rL|$ generates $s$-jets at $x \in X$ for all $s \gg 0$ whenever

$$r > \frac{s}{\varepsilon} + c.$$  

Then $\varepsilon(L, x) \geq \varepsilon$.

In other words: If $\varepsilon(L, x) < \alpha$, then for all $s_0 > 0$ and all real $c$ there exist an $s \geq s_0$ and an $r > \frac{s}{\alpha} + c$ such that $|K_X + rL|$ does not generate $s$-jets at $x$. \hfill \Box

§2. Relative moving part estimates

The aim of the section is to prove the following Theorem.

**Theorem 2.1.** Let $X$ be a smooth $n$-dimensional projective variety, $L$ a nef and big line bundle on $X$ and $\alpha > 0$ a rational number such that $L^n > \alpha^n$. Let

$$0 = \beta_1 < \beta_2 < \cdots < \beta_n < \beta_{n+1} = \alpha$$

be any sequence of rational numbers and $x \in X$ a very general point. Then either

(a) there exist $k \gg 0$ and a divisor $E \in |kL|$ having an isolated singularity of index $\text{ind}_x(E) \geq \beta_{n+1} - \beta_n$ at $x$; or

(b) there exists a proper subvariety $V \subset X$ through $x$ of codimension $c \leq n-1$ such that

$$\deg_L V = L^{n-c} \cdot V \leq \frac{1}{(\beta_{c+1} - \beta_c)^c} \left(1 - \sqrt[n]{\left(1 - \frac{\alpha^n}{L^n}\right)^c}\right) L^n.$$

(2.2) Families of divisors. Pick an arbitrary point $y \in X$ and a smooth affine neighbourhood $T \subset X$ of $y$ in $X$. Then the embedding $g : T \hookrightarrow X$ satisfies the properties of (0.5). Note that (very) general points of $T$ correspond to (very) general points of $X$. We will use the notations introduced in (0.5) henceforth.

Argueing as in [EKL, (3.8)], for $k \gg 0$ with $\alpha k \in \mathbb{Z}$ we obtain divisors $\mathcal{E}_k \in |pr_X^*(kL)|$ in $X \times T$ satisfying

$$\text{ind}_T(\mathcal{E}_k) > \alpha.$$  

The argument is, in brief, that for any $x \in X$ using Riemann-Roch and a parameter count one finds a divisor $E \in |kL|$ with $\text{mult}_x(E) > \alpha k$.

Hence the torsion free $\mathcal{O}_T$-module

$$R = pr_T^* \left(pr_X^*(kL) \otimes T^{(\alpha k)}_T\right)$$

has positive rank, and is globally generated since $T$ is affine. Therefore a non-zero section $\Phi$ of $R$ gives via the evaluation map $pr_T^* R \to pr_X^*(kL) \otimes T^{(\alpha k)}_T$ the desired divisor $\mathcal{E}_k$. 

\hfill \Box
(2.3) The multiplicity schemes $Z_{\sigma}(\mathcal{E})$. For $k \geq 0$ with $\alpha k \in \mathbb{Z}$ put

$$A_k = \lvert I_{\tau}^{(\alpha k)} \otimes pr_X^*(kL) \rvert.$$ 

Then by (2.2) $A_k$ is non-empty for sufficiently large $k$. For nonzero $\mathcal{E}_k \in A_k$ and any rational number $\sigma > 0$ we define

$$Z_{\sigma}(\mathcal{E}_k) = \{ y = (x, t) \in \mathcal{E}_k \mid \text{ind}_y(\mathcal{E}_k) \geq \sigma \}.$$ 

Note that $Z_{\sigma}(\mathcal{E}_k)$ is a Zariski-closed subset of $X \times T$. Its natural scheme structure is given locally by the vanishing of all partial derivatives of order $< k\sigma$ of a local equation for $\mathcal{E}_k$. We will be only interested in $Z_{\sigma}(\mathcal{E}_k)$ as an algebraic set, and for a general choice of $\mathcal{E}_k$.

Recall the following generalized version of Bertini’s Theorem, due to Kollár (cf. [Ko]):

**Theorem 2.3.0.** Let $Y$ be a smooth variety and $|B_1, \ldots, B_k|$ be a linear system on $Y$. Then a general member $B \in |B_1, \ldots, B_k|$ satisfies

$$\text{mult}_y B \leq 1 + \inf_i \{ \text{mult}_y B_i \}$$

at every point $y \in Y$. □

The following lemma, which is an analog of [ELN,(3.8)], says that the multiplicity loci $Z_{\sigma}(\mathcal{E}_k)$ are independent of $k$ and the choice of a general $\mathcal{E}_k \in A_k$ as soon as $k$ is sufficiently large; here by general we mean general in the sense of Theorem 2.3.0.

**Lemma 2.3.1.** For fixed $\sigma$ there is a positive integer $k_0$ such that $Z_{\sigma}(\mathcal{E}_{k_1}) = Z_{\sigma}(\mathcal{E}_{k_2})$ for all $k_1, k_2 \geq k_0$.

**Proof (cf. [ELN]).** As $\sigma$ and $\alpha$ are fixed, we will for simplicity write $Z(k)$ for $Z_{\sigma}(\mathcal{E}_k)$. Choose an integer $m \geq 2$ such that $A_k \neq \emptyset$ for $k \geq m$. Fixing an integer $a \geq m$ we then claim that there exists a positive integer $k(a)$ such that

$$Z(c) \subseteq Z(a) \text{ whenever } c \geq k(a).$$

To prove the claim, suppose that $y \notin Z(a)$ so that there exists $\eta > 0$ satisfying

$$\text{mult}_y(\mathcal{E}_a) \leq a\sigma - \eta.$$ 

Note that since the index is a discrete invariant, $\eta$ is bounded below independently of $y$; in fact if $m\sigma \in \mathbb{Z}$ then $\eta \geq 1/m$. Suppose $b \geq m$ is an integer relatively prime to $a$. Then any integer $c \geq ab$ can be expressed as $c = \alpha a + \beta b$, $\alpha, \beta \in \mathbb{Z}$ and $0 \leq \beta \leq a$. 


Consider the divisor $E'_c = \alpha E_a + \beta E_b \in A_c$. Then
\[
\text{mult}_y(E_c) \leq 1 + \text{mult}_y(E'_c) = 1 + \alpha \cdot \text{mult}_y(E_a) + \beta \cdot \text{mult}_y(E_b)
\leq 1 + \alpha a \sigma - \alpha \eta + \beta \cdot \text{mult}_y(E_b)
= c \left( \sigma - \frac{\eta \left(1 - \frac{\beta b}{c}\right)}{a} + \frac{1 + \beta \cdot \text{mult}_y(E_b)}{c} \right),
\]
where the first inequality is a consequence of Theorem 2.3.0. Since $\eta, \beta,$ and $b$ are bounded independently of $c$, it follows that $\text{mult}_y(E_c) < c \sigma$ for $c \gg 0$. Hence $y \notin Z(c)$ for all sufficiently large $c$ as claimed.

If $Z(c) = Z(a)$ for all $c \gg 0$ then we are finished. If not, then by $(*)$ there exists $a' > 0$ such that $Z(a') \subsetneq Z(a)$. The argument can then be repeated with $a'$ instead of $a$. This process cannot go on indefinitely and this establishes the Lemma. \[\square\]

The next Lemma is an adaptation of [ELN, (1.5), (1.6)] to our situation. We present a sketch of its proof for the reader’s convenience.

**Gap-Lemma 2.3.2.** Let $E \subset X \times T$ be a family of effective divisors on $X$ with $\text{ind}_\Gamma(E) > \alpha$ along the graph $\Gamma \subset X \times T$ of $g : T \to X$. Let
\[0 = \beta_1 < \beta_2 < \cdots < \beta_n < \beta_{n+1} = \alpha\]
be any sequence of rational numbers. Define
\[Z_0 = X \times T \text{ and } Z_j = Z_{\beta_j}(E) = \{ y \in E \mid \text{ind}_y(E) \geq \beta_j \} \text{ for } 1 \leq j \leq n + 1.\]
Then there exists an index $c$, $1 \leq c \leq n$, and an irreducible subvariety $V \subset X \times T$ such that:

1. $\text{codim}(V) = c$,
2. $\Gamma \subset V$, and
3. $V$ is an irreducible component of both $Z_c$ and $Z_{c+1}$.

This means that the index of $E$ “jumps” by at least $\beta_{c+1} - \beta_c$ along $V$, i.e. $\text{ind}_y(E) \geq \beta_{c+1}$ for every $y \in V$ and there is an open set $U \subset X \times T$ meeting $V$ such that $\text{ind}_v(E) < \beta_c$ for every $v \in U \setminus V$.

**Sketch of Proof.** The sets $Z_i$ lie in a chain
\[\Gamma \subset Z_{n+1} \subset \cdots \subset Z_1 = E \subset Z_0 = X \times T.\]
Starting with $Z_{n+1}$ and working up in dimension, we can choose irreducible components $V_j$ of $Z_j$ containing $\Gamma$ such that $V_{j+1} \subset V_j$. So we arrive at a chain of irreducible varieties
\[\Gamma \subset V_{n+1} \subset V_n \subset \cdots \subset V_1 \subset V_0 = X \times T,\]
and since $X \times T$ is irreducible of dimension $2n = \text{dim}(\Gamma) + n$, at least two consecutive links in the chain must coincide, say $V_c = V_{c+1}$, and we take $V = V_c$. Using elementary combinatorial arguments one shows that also the condition $\text{codim}(V) = c$ can be achieved. For details we refer to the proof of Lemma (1.6) in [ELN]. \[\square\]
(2.4). From now on we fix the set of rational numbers

\[ 0 = \beta_1 < \beta_2 < \cdots < \beta_n < \beta_{n+1} = \alpha. \]

Then by Lemma 2.3.1 there exists an integer \( k_0 \) such that the multiplicity loci \( \mathcal{Z}_{\beta_i}(\mathcal{E}_k) \), \( 1 \leq i \leq n+1 \), are independent of \( k \) as soon as \( k \geq k_0 \) and \( \mathcal{E}_k \in \mathcal{A}_k \) are general. Therefore also the multiplicity ”jumping” loci \( \mathcal{V} \) obtained by the Gap-Lemma 2.3.2 can be chosen independently of \( \mathcal{E} = \mathcal{E}_k \) and \( k \) up to the above restrictions. Fix such a \( \mathcal{V} \) and put \( \beta = \beta_{c+1} - \beta_c \).

**Proposition 2.4.1.** For all sufficiently divisible \( k \gg 0 \) the jumping locus \( \mathcal{V} \) is an irreducible component of the base locus of the linear system

\[ |\mathcal{I}(\beta k) \otimes \text{pr}_X^* (kL)|. \]

Here by sufficiently divisible we mean that \( \beta_i k \in \mathbb{Z} \) for all \( i = 1, \ldots, n+1 \).

We start by recalling some general facts concerning the differentiation of sections of line bundles \( \text{pr}_X^* (kL) \) in parameter directions and its connection to certain multiplicity loci (cf. also [EKL, §2] and [ELN, §2]).

Let \( \mathcal{D}^\ell_{X \times T}(\text{pr}_X^* (kL)) \) be the sheaf of differential operators of order \( \leq \ell \) on \( \text{pr}_X^* (kL) \) and let \( \mathcal{D}^\ell_T \) be the sheaf of differential operators of order \( \leq \ell \) on \( T \). Since there is a canonical inclusion of vector bundles

\[ \text{pr}_T^* (\mathcal{D}_T^\ell) \hookrightarrow \mathcal{D}^\ell_{X \times T}(\text{pr}_X^* (kL)), \]

the sections of \( \mathcal{D}_T^\ell \) act naturally on the space of sections of \( \text{pr}_X^* (kL) \). A section \( \psi \in \Gamma(X \times T, \text{pr}_X^* (kL)) \) determines a homomorphism of sheaves

\[ \mathfrak{d}_\ell(\psi) : \text{pr}_T^* (\mathcal{D}_T^\ell) \to \text{pr}_X^* (kL). \]

Locally, represent \( \psi \) by a function \( f \), then \( \mathfrak{d}_\ell(\psi) \) is just taking a differential operator \( D \) to the function \( D(f) \). Since \( \text{pr}_X^* (kL) \) is a line bundle there exists a sheaf of ideals \( \mathcal{I}_{\Sigma_\ell(\psi)} \) such that

\[ \text{Im}(\mathfrak{d}_\ell(\psi) : \text{pr}_T^* (\mathcal{D}_T^\ell) \to \text{pr}_X^* (kL)) = \mathcal{I}_{\Sigma_\ell(\psi)} \otimes \text{pr}_X^* (kL). \]

Let \( \psi \) be a defining section for a divisor \( \mathcal{E} \in |\text{pr}_X^* (kL)| \). Then we claim that

\[ \Sigma_\ell(\psi) = \{ (x,t) \in X \times T \mid \text{mult}_t(\mathcal{E}_x) > \ell \}. \]  

Indeed, the scheme structure on the right hand side is given locally by the vanishing of all partial derivatives in \( T \) direction of order \( \leq \ell \) of a local equation for \( \mathcal{E} \).
Note also that the $I_{\Sigma_\ell(\psi)} \otimes pr_\ast^* (kL)$ are as quotients of the globally generated sheaf $pr_\ast^* (D_\ell^T)$ generated by global sections.

**Proof of the Proposition.** The plan is to apply Lemma 1.2. By assumption

$$\alpha k, \ p = \beta_c k \text{ and } q = \beta k$$

are integers.

Let $E \in A_k$ be a divisor determining $V$ and $\psi$ a section defining $\mathcal{E}$. For integers $\ell$ put $\Sigma_\ell = \Sigma_\ell(\psi)$. We claim that

$$I_{\Sigma_{p-1}} \subset I^{(q)}_V.$$  \hfill (2.4.3)

To prove this let $f$ be a local equation for $E$ over some open set $U$. Then $I_{\Sigma_{p-1}}$ is locally generated by all functions

$$\{ D(f) \mid D \in (pr_\ast^* D_\ell^T(U)) \}.$$

On the other hand we have that $\tilde{D}(f) \in I_V$ for every $\tilde{D} \in D_\ell^{p+q-1}(pr_\ast^* (kL))(U)$, since $V$ is an irreducible component of $Z_{p-1}(E)$. And in particular $\tilde{D}(f) \in I_V$ for every $\tilde{D} \in (pr_\ast^* D_\ell^{p+q-1}(U))$. Hence for all $D \in pr_\ast^* D_\ell^{p-1}$ the function $D(f)$ vanishes to the order $\geq q$ on $V$ which shows (2.4.3).

Since $I_{\Sigma_{p-1}} \otimes pr_\ast^* (kL)$ is globally generated, application of Lemma 1.2 to our situation will give the desired result once we show that $V$ is also an irreducible component of $\Sigma_{p-1}$. This is the content of Lemma 2.4.4 below. \hfill $\square$

**Lemma 2.4.4.** Let $\sigma k$ be a positive integer, $E \in |pr_\ast^* (kL)|$ be an effective divisor on $X \times T$ and $V \subset Z_\sigma (E)$ an irreducible component dominating $X$. Then $V$ is also an irreducible component of $\Sigma_{k\sigma-1}(E)$.

**Proof.** By definition $Z_\sigma (E) \subset \Sigma_{k\sigma-1}(E)$. Let $W \subset \Sigma_{k\sigma-1}(E)$ be an irreducible component containing $V$. If we can show that $W \subset Z_\sigma (E)$, then we are done, because that implies $V = W$. Lemma 1.1 shows $\text{ind}_W(E) = \text{ind}_{W_x}(E)$ for general $x \in X$ and any irreducible component $W'_x$ of $W_x$. Hence the assertion follows from $\text{ind}_{W_x}(E_x) \geq \sigma$, where we used (2.4.2). \hfill $\square$

**Corollary 2.4.5.** For all sufficiently divisible $k \gg 0$, we have

$$|I^{(\alpha k)}_T \otimes pr_\ast^* (kL)| \subset |I^{(\beta k)}_V \otimes pr_\ast^* (kL)|.$$  

**Proof.** By the above any sufficiently general $E \in A_k$ determines the same $V$, in particular such an $E$ satisfies $\text{ind}_V(E) \geq \beta_c$, and this implies $E \in |I_{\Sigma_{p-1}} \otimes pr_\ast^* (kL)|$ by (2.4.2), where again we assume that $\alpha k, \ p = \beta_c k$ and $q = \beta k$ are integral. Then the claim follows from (2.4.3). \hfill $\square$
Proposition 2.4.6. For all sufficiently divisible $k \gg 0$ and very general $t \in T$ the following hold:

(1) There exists an irreducible subvariety $V \subseteq X$ of codimension $c$ containing $g(t)$ which is an irreducible component of the base locus of the linear system

$$|J_k| := \left| \left( I_V^{(\beta k)} \right)_t \otimes kL \right|$$

on $X$ such that $\text{mult}_V D \geq k\beta$ for all $D \in |J_k|$. In particular, if $c = n$, i.e. $V = \Gamma$, then by Bertini’s Theorem there exists a divisor $D \in |kL|$ having an isolated singularity of index $\geq \beta$ at the point $x = g(t)$.

(2) $\dim H^0(X, I_{g(t)}^{\alpha k} \otimes kL) \leq \dim H^0(X, J_k)$.

Proof. To begin with we study the situation for a fixed $k$. First we note that after possibly shrinking $T$ we can assume that the coherent sheaf $F = I_V^{(\beta k)} \otimes pr_X^* (kL)$ is flat over $T$. In fact $pr_T: X \times T \to T$ is projective and $T$ is affine and integral, hence the assertion follows from consideration of the Hilbertpolynomials of the $F_t$ (cf. also [Ha, III.9.9]): These do not depend on $t$ for $t$ in an open dense subset of $T$.

After possibly shrinking $T$ more it follows from semicontinuity that there is a natural isomorphism

$$(*) \quad H^0(X \times T, F) \otimes k(t) \simeq H^0(X, F_t).$$

(cf. [Ha, III.12.9]). In other words, taking global sections of $F$ commutes with restricting to fibres over general $t \in T$, and therefore $(\text{Bs}|F|)_t = \text{Bs}|J_k|$ for such $t$.

Now we can prove (1). By Proposition 2.4.1 we have $\Gamma \subseteq V \subseteq \text{Bs}|F|$ with $V$ an irreducible component of $\text{Bs}|F|$, hence Lemma 1.3 shows that any irreducible component $V$ of $V_t$ is an $(n-c)-$dimensional irreducible component of $(\text{Bs}|F|)_t = \text{Bs}|J_k|$. It remains to show $\text{mult}_V D \geq k\beta$ for all $D \in |J_k| = |F_t|$. But this follows from $(*)$ and Lemma 1.2.

Assertion (2) follows in the same way from Corollary 2.4.5 and the fact that

$$\left( I_{\Gamma}^{(\alpha k)} \right)_t = \left( I_{T}^{(\alpha k)} \right)_t = I_{g(t)}^{\alpha k}.$$ 

To complete the proof of the Proposition we only have to remark that, since $V$ does not depend on $k$, the above arguments work simultaneously for all divisible $k \gg 0$ if we replace the general $t \in T$ by a very general $t \in T$. \hfill $\square$

(2.5) Bounding the degree of irreducible components of base loci. In this subsection we complete the proof of Theorem 2.1 by bounding the degree of the irreducible component $V$ from 2.4.6 using a strategy essentially due to Fujita (cf. [Fu82], [Fu94]). Alternatively one could carry out an approach via graded linear series as in [ELN] leading to slightly weaker bounds.
Let \( k \gg 0 \) be sufficiently large and divisible, and fix a very general \( t \in T \). Let \( J_k = (Z_{\nu}^{(3k)})_t \otimes kL \) and \( V \subset X \) be as in (2.4.6); recall that \( V \) depends on \( t \) but not on \( k \), and that \( V \) is an \((n-c)\)-dimensional irreducible component of \( Bs|J_k| \). We may assume that \( \dim V > 0 \), since otherwise the assertion of Theorem 2.1 follows from (2.4.6.1).

(2.5.1). Resolving the base locus of \( \Lambda := |J_k| \) we can find a sequence

\[
X' = X_s \rightarrow \cdots \rightarrow X_r \rightarrow X_{r-1} \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 = X
\]

of birational morphisms \( \tau_i : X_i \rightarrow X_{i-1} \) together with linear systems \( \Lambda_i \) on \( X_i \) such that

1. \( \Lambda_0 = \Lambda \).
2. \( \tau_i : X_i \rightarrow X_{i-1} \) is the blow-up of a smooth subvariety \( C_i \) of \( X_{i-1} \).
3. \( \tau_i^* \Lambda_{i-1} = \Lambda_i + m_i E_i \) for some nonnegative integers \( m_i \), where \( E_i \) is the exceptional divisor on \( X_i \) lying over \( C_i \) and \( E_i \not\subseteq Bs\Lambda_i \).
4. \( Bs\Lambda_s = \emptyset \).

Let \( \tau = \tau_s \circ \cdots \circ \tau_1 \) be the composition, \( E_i^* \) resp. \( E_i' \) be the total resp. proper transforms of \( E_i \) on \( X' \), and \( Y_i = \tau(E_i^*) = \tau(E_i') \). The \( Y_i \) coincide with the image of \( C_i \) in \( X \).

Let \( F_i \) denote the pull back to \( X' \) of the general member of the linear system \( \Lambda_i \) on \( X_i \). Finally, let \( H \) be a general member of \( \Lambda_s \), \( F = \tau^*(kL) \), and \( E \) the fixed part of \( \tau^* \Lambda \), so that \( H = F - E \) where \( E = \sum_{i=1}^s m_i E_i^* \).

By assumption there exists an index \( r \) with \( Y_r = V \), and

\[
m_r \geq k \beta
\]

since \( \text{mult}_V(D) \geq k \beta \) for all \( D \in |J_k| \). We also may and will assume that the resolution \( \tau \) is chosen in such a way that \( \dim(Y_i) < \dim(V) = n - c \) for all \( i < r \).

Lemma 2.5.3. \( F^{n-c}.E.H^{c-1} \geq k^{n-c}m_r^c \deg_L V \).

Proof. First of all note that, since \( F \) and \( H \) are nef and \( E - m_r E_r^* \) is an effective divisor, we have

\[
F^{n-c}.E.H^{c-1} \geq m_r F^{n-c}.E_r^*.H^{c-1}.
\]

Next we use that

\[
F^{n-c}.E_r^*.H^{c-1} = F^{n-c}.E_r^*.F_r^{c-1},
\]

which follows from the standard Segre class computations because of

\[
\dim \left( \text{supp} \left( \tau^* (E_r^*.F_r^{c-1} - E_r^*.H^{c-1}) \right) \right) < n - c.
\]
Finally we compute $F^{n-c}.E_r^* F_r^{c-1}$. Since $\dim(Y_i) < n - c$ for all $i < r$, we have

$$F^{n-c}.F_r^c = F^{n-c} (F - \sum_{i \leq r} m_i E_i^*)^c = F^{n-c} (F - m_r E_r^*)^c,$$

hence $F^{n-c}.F_r^c = F^{n-m_c} \deg_{kL}(V)$ by the birationality of the morphism $C_r \to Y_r = V$. A similar argument shows $F^{n-c+1}.F_r^{c-1} = F^n$, and therefore

$$F^{n-c}.E_r^* F_r^{c-1} = \frac{1}{m_r} (F^{n-c} F_r^{c-1} - \sum_{i < r} m_i E_i^*)$$

$$= \frac{1}{m_r} (F^{n-c+1} F_r^{c-1} - F^{n-c} F_r^c)$$

$$= m_r^{-1} k^{n-c} \deg_{L} V.$$

Combining this with (*) and (**) proves the Lemma. □

**Lemma 2.5.4.** For any $\varepsilon > 0$ there exist a sufficiently large and divisible $k$ and a resolution $\tau : X' \to X$ of the rational map given by $|J_k|$ satisfying the properties in (2.5.1) and $H^n = (F - E)^n \geq k^n (L^n - \alpha^n - \varepsilon)$.

**Proof.** The proof follows closely the proof of the Theorem in [Fu94]. Therefore we only give an outline and indicate the necessary modifications. For varying $k$ consider $|J_k|$ and denote by $(X'_k, H_k)$ the pair $(X', H)$ obtained as in (2.5.1). We will derive a contradiction assuming that $H_k^n < k^n (L^n - \alpha^n - \varepsilon)$ on $X'_k$ for all large and divisible $k$.

Letting $\varepsilon$ grow if necessary, we may assume that $H_k^n \geq \ell^n \left( L^n - \alpha^n - \varepsilon - \frac{\varepsilon}{(2n)!} \right)$ on $X'_k$ for one fixed large and divisible $\ell$. Now, for any integer $s > 0$, we claim that

$$h^0(X, J_s) \leq h^0(X'_s, sH_s) + \frac{n(s\ell)^n \varepsilon}{(2n)!},$$

which is proven exactly as in [Fu94] by using the lower bound on $H_k^n$ and considering an appropriate resolution of $\Lambda = |J_s|$. From (2.4.6.2) and asymptotic Riemann–Roch we then obtain

$$\frac{(s\ell)^n}{n!} (L^n - \alpha^n) + \varphi(s) \leq h^0(X, I_x^{as\ell} \otimes s\ell L)$$

$$\leq h^0(X, J_{s\ell})$$

$$\leq h^0(X'_s, sH_s) + \frac{n(s\ell)^n \varepsilon}{(2n)!}$$

$$\leq \frac{s^n}{n!} \ell^n (L^n - \alpha^n - \varepsilon) + \frac{n(s\ell)^n \varepsilon}{(2n)!} + \psi(s),$$

where $\varphi$ and $\psi$ are functions with $\lim_{s \to \infty} \frac{\varphi(s)}{s^n} = \lim_{s \to \infty} \frac{\psi(s)}{s^n} = 0$. This gives the desired contradiction. □

Before stating the main result of this section which will complete the proof of Theorem 2.1 we need to recall some well known facts (cf. [De93, 5.2], [Fu82, 1.2]).
Lemma 2.5.5. Let $F$, $H$ be nef divisors on an $n$-dimensional smooth projective variety $Y$. Then:

1. $F^dH^{n-d} \geq \sqrt[n]{F^n}^d \sqrt[n]{H^n}^{n-d}$ holds for all $0 \leq d \leq n$.

2. If $E = F - H$ is effective, then $F^aH^{n-a} \geq F^bH^{n-b}$ for any $a \geq b$. □

Proposition 2.5.6. With the above notations the degree of $V$ satisfies

$$\deg L V \leq \frac{1}{\beta^c} \left( 1 - \sqrt[n]{1 - \alpha^n L^n} \right) L^n.$$  

Proof. By Lemma 2.5.3 we have

$$\deg L V \leq \frac{1}{k^{n-c}m^c_r} F^{n-c}E.H^{c-1}.$$  

Note that $F^{n-c}E.H^{c-1} = (F^{n-c+1}H^{c-1} - F^{n-c}H^c)$. So if we bound the first term using (2.5.2) and bound the second term using (2.5.1), we find

$$\deg L V \leq \frac{1}{k^{n-c}m^c_r} (F^{n-c+1}H^{c-1} - F^{n-c}H^c) \leq \frac{1}{k^{n-c}m^c_r} (F^n - (F^n)^n H^n) \leq \frac{1}{k^{n-c}m^c_r} (L^n - (L^n)^n L^n) \leq \frac{1}{\beta^c} \left( 1 - \sqrt[n]{1 - \alpha^n L^n} \right) L^n,$$  

where the last steps are Lemma 2.5.4 plus the fact that $\deg L V$ is integral, and (2.5.2). □

§3. Applications

Theorem 3.1. Let $X$ be an $n$-dimensional smooth projective variety, $x \in X$ be a very general point and $L$ be a nef and big line bundle. Let $r$ and $s$ be positive integers, $\gamma > 1$ a rational number satisfying

$$(rL)^n > (\gamma(n+s))^n,$$

and $\alpha_1, \ldots, \alpha_{n-1}$ be positive rational numbers with $\sum_{i=1}^{n-1} \alpha_i = \gamma - 1$.

Then either

(a) $|K_X + rL|$ generates $s$-jets at $x$, or
(b) there is a proper subvariety $V \subset X$ of positive dimension $d$ containing $x$ with

$$\deg_L V = L^d V \leq \frac{(rL)^n}{\alpha_d^{n-d} (n+s)^{n-d}} \left( 1 - \sqrt{\left( 1 - \frac{\gamma^n(n+s)^n}{(rL)^n} \right)^{n-d}} \right).$$

Proof. Put

$L' = rL,$

$\alpha = \beta_{n+1} = \gamma(n+s),$ 

$\beta_n = (\gamma - 1)(n+s),$ and downward recursively

$\beta_i = \beta_{i+1} - \alpha_{n-i}(n+s)$ for $i = n-1, \ldots, 1.$

Let $x \in X$ be a very general point and apply Theorem 2.1. Then either there exists $k \gg 0$ and a divisor $E \in |kL'|$ having an almost isolated singularity of index $\geq n+s$ at $x,$ in which case by (1.5.1) the linear series $|K_X + L'| = |K_X + rL|$ generates $s-$jets at $x,$ or there exists a subvariety $V \subset X$ with the wanted properties. \hfill $\Box$

Corollary 3.2. With the assumptions of Theorem 3.1 there exists $\kappa > 0$ such that either

(a) $|K_X + rL|$ generates $s-$jets at $x,$ or

(b) there is a proper subvariety $V \subset X$ of positive dimension $d$ containing $x$ with

$$\deg_L V = L^d V < \left( \frac{n+s}{r} \right)^d \frac{\gamma^n}{\alpha_d^{n-d}} \left( \frac{1}{1 - \kappa} \right).$$

Given constants $c_1 > c_2 > 0,$ such $\kappa$ can be chosen independently of $\frac{n+s}{r}, \gamma$ and $\alpha_d$ if the condition $c_1 \geq \frac{n+s}{r} \gamma \geq c_2$ holds.

Proof. This follows directly from Theorem 3.1 using the estimate

$$g \left( \frac{n+s}{r} \gamma; \kappa \right) := \left( 1 - \frac{(n+s)^n \gamma^n}{(rL)^n} \left( \frac{1}{1 - \kappa} \right) \right)^n - \left( 1 - \frac{(n+s)^n \gamma^n}{(rL)^n} \right)^{n-d} < 0$$

for integral $1 \leq d < n,$ fixed $\frac{n+s}{r} \gamma,$ and sufficiently small $\kappa.$ The independence of $\frac{n+s}{r} \gamma$ is a consequence of the fact that $g$ satisfies a Lipschitz condition with respect to $\frac{n+s}{r} \gamma.$ \hfill $\Box$

Remark 3.3. Up to the assumption on the positivity of $T_X$ and the genericity of $x \in X$ Theorem 3.1 looks similar to [ELN, Theorem 4.1]. Note however that in the estimate of the degree $\deg_L V$ we have $(n+s)^d$ compared to $(n+s)^n$ in [ELN], which turns out to be crucial when bounding the Seshadri constant. This improvement is achieved by "rescaling" the intervall $[0, \alpha]$ from 2.1.
Theorem 3.4. Let $L$ be a nef line bundle on an $n$-dimensional irreducible projective variety $X$ and $\varepsilon > 0$ be a real number. Let $\alpha_1, \ldots, \alpha_n$ be positive rational numbers and $x \in X$ be a general point. Put $\gamma = 1 + \sum_{i=1}^{n-1} \alpha_i$. Suppose that any $d$-dimensional $(1 \leq d \leq n)$ subvariety $V \subseteq X$ containing a very general point $y \in X$ satisfies

$$\deg_L V = L^d V \geq \frac{\varepsilon^d \cdot \gamma^n}{\alpha_d^{n-d}}.$$  

Then $\varepsilon(L, x) \geq \varepsilon$.

Proof. First we note that, since we are only considering general points, there is no loss of generality in supposing that $X$ is smooth (cf. [EKL, 3.2] for the precise argument).

Suppose that $\varepsilon(L, x) < \varepsilon$ at general points $x \in X$. Then by Lemma 1.4 for all $\delta > 0$ we have $\varepsilon(L, x) < \varepsilon + \delta$ at any point $y \in X$, and accordingly by 1.5.3 for any $y \in X$ there exist positive integers $s$ and $r$ with $r > \frac{s+n}{\varepsilon + \delta}$ such that $|K_X + rL|$ does not separate $s$-jets at $y$. Clearly we can assume $2\varepsilon \geq \delta + \varepsilon > \frac{s+n}{r} \geq \frac{\varepsilon}{2}$, and that $\delta$ is so small that

$$\gamma \geq \gamma' := \frac{\varepsilon \gamma}{\delta + \varepsilon} \geq \frac{\gamma - 1}{2} + 1.$$  

Since by assumption

$$(rL)^n \geq r^n (\varepsilon \gamma)^n > (s+n)^n \cdot (\gamma')^n,$$

we can apply Theorem 3.1 to $\gamma'$, and $\alpha'_i = \alpha_i \frac{\varepsilon (\gamma - 1)}{(\delta + \varepsilon) (\gamma - 1)}$. By construction

$$2\varepsilon \gamma \geq \gamma' \frac{s+n}{r} \geq \frac{\varepsilon}{2} \left( \frac{\gamma - 1}{2} + 1 \right),$$

so that from Corollary 3.2 we deduce the existence of a constant $\kappa > 0$ and a proper subvariety $V \subset X$ containing $y$ of degree

$$\deg_L V = L^d V < \left( \frac{n+s}{r} \right)^d \frac{(\gamma')^n}{(\alpha'_d)^{n-d}} \left( \frac{1}{1 - \kappa} \right)$$

independently of the choice of $\delta$. For $\delta > 0$ small enough this gives the existence of a subvariety $V$ containing $y$ such that

$$L^d V < \frac{\varepsilon^d \cdot \gamma^n}{\alpha_d^{n-d}},$$

leading to a contradiction. $\square$

From the above Theorem one can deduce easily various boundedness statements by specifying the $\alpha_i$. 
Example 3.5. Let $X$ be a smooth projective variety of dimension $n$, $L$ a nef and big line bundle on $X$ and $x \in X$ a general point.

a) Setting $\alpha_1 = \cdots = \alpha_n = 1$, one obtains that the Seshadri constant satisfies the universal bound

$$\varepsilon(L, x) \geq \frac{1}{n^n}.$$ 

b) With the following choice one comes closer to the bound obtained in [EKL]: put

$$\alpha_i = \frac{n - 1}{2^i(1 - 2^{1-n})},$$

and define

$$\mu(d) := \min \{ L^d \cdot V \},$$

where the minimum runs over all $d$-dimensional subvarieties $V \subseteq X$ containing very general points. Then

$$\varepsilon(L, x) \geq \min_{1 \leq d \leq n} \left\{ d \left( \frac{\mu(d)}{n^d 2^d (1 - 2^{1-n})^d} \right) \right\}.$$ 

§4 Bounds for Seshadri constants at arbitrary points

In this section, we show how to apply the strategy of §3 to obtain certain bounds for Seshadri constants at arbitrary points using the following result of [ELN]:

Theorem 4.1. (Ein–Lazarsfeld–Nakamaye) Let $X$ be a smooth $n$–dimensional variety with tangent bundle $T_X$, $x \in X$ any point, $A$ an ample line bundle on $X$ and $\delta \geq 0$ a real number such that $T_X(\delta A)$ is nef. Let

$$0 = \beta_1 < \cdots < \beta_{n+1} < n$$

be rational numbers. Then either

(a) there exists $E \in |kA|$ $(k \gg 0)$ with an almost isolated singularity at $x$ of index

$$\frac{\beta_{n+1} - \beta_n}{1 + \delta \beta_n},$$

or

(b) there exists an irreducible subvariety $V \ni x$ of codimension $c \neq n$ with

$$A^{n-c}V = \deg_A V \leq \left( \frac{1 + \delta \beta_c}{\beta_{c+1} - \beta_c} \right)^c \cdot \beta_c^n.$$

Theorem 4.1 follows from [ELN, Theorem 3.9] together with [ibid, 1.5, 1.6] and the remark (from the proof of Theorem 3.9) that in any event $V$ is an irreducible component of the base locus of the linear system $|k \varepsilon \cdot V^{(k\varepsilon)}(k(1 + \delta \sigma)A)|$ on $X$, which gives, in case $V$ is 0–dimensional, by Bertini’s Theorem the divisor in (b).

Then the argument proceeds as in §3:
Corollary 4.2. Let $X$, $A$ and $\delta$ be as in (4.1), moreover $r, s$ be positive integers, and $\gamma > 1$ rational such that $(rA)^n > (\gamma(n + s))^n$. Let $\alpha_1, \ldots, \alpha_{n-1}$ be positive rational numbers with
\[
1 + \left(1 + \frac{\delta}{r}(n + s)\right) \cdot \sum_{i=1}^{n-1} \alpha_i = \gamma.
\]
Then for any $x \in X$ either $K_X + rA$ separates $s$–jets at $x$, or there exists a subvariety $V \subseteq X$ through $x$ of dimension $d \geq 1$ and degree
\[
A^{n-c} \cdot V \leq \frac{\gamma^n(n+s)^d}{r^d} \left(1 + \frac{\delta}{r}(n + s)\right) \sum_{i=d+1}^{n-1} \alpha_i
\]
Proof. Put $A' = rA$, $\beta_{n+1} = \gamma(n+s)$, $\beta_n = (n+s) \cdot \sum_{i=1}^{n-1} \alpha_i$, and $\beta_i = \beta_{i+1} - \alpha_{n-i}(n+s)$. Then apply the Theorem 4.1 to $A'$ and $\delta' = \frac{\delta}{r}$, and use (1.5.1). □

Theorem 4.3. Let $X$, $A$ and $\delta$ be as in (4.1), moreover $\varepsilon > 0$ real and $\alpha_1, \ldots, \alpha_n$ be positive rational numbers. Let $x \in X$ be any point and suppose that any $d$–dimensional $(1 \leq d \leq n)$ subvariety $V \subseteq X$ containing $x$ satisfies
\[
\deg_A V = A^d \cdot V \geq \varepsilon^d \left(1 + \delta \varepsilon \sum_{i=d+1}^{n-1} \alpha_i \right) \sum_{i=1}^{n-1} \alpha_i.
\]
Then $\varepsilon(A, x) \geq \varepsilon$.

Proof. Fix $x \in X$ and suppose $\varepsilon(A, x) < \varepsilon$. Then there exist positive integers $r$ and $s$ with $r > \frac{s+n}{\delta}$ such that $|K_X + rA|$ does not generate $s$–jets at $x$. Put $\gamma = 1 + (1 + \frac{\delta}{r}(n+s)) \sum_{i=1}^{n-1} \alpha_i$. Then by assumption
\[
(rA)^n \geq r^n \varepsilon^n \left(1 + \left(1 + \frac{\delta}{r} \sum_{i=d+1}^{n-1} \alpha_i \right) \sum_{i=1}^{n-1} \alpha_i \right)^n.
\]
Therefore Corollary 4.2 gives the existence of $V \supseteq x$ of dimension $d \geq 1$ with
\[
A^d \cdot V \leq \frac{\gamma^n(n+s)^d}{r^d} \left(1 + \frac{\delta}{r}(n + s)\right) \sum_{i=d+1}^{n-1} \alpha_i
\]
\[
< \varepsilon^d \gamma^n \left(1 + \delta \varepsilon \sum_{i=d+1}^{n-1} \alpha_i \right) \sum_{i=1}^{n-1} \alpha_i
\]
\[
< \varepsilon^d \left(1 + \delta \varepsilon \sum_{i=d+1}^{n-1} \alpha_i \right) \sum_{i=1}^{n-1} \alpha_i \cdot \left(1 + (1 + \delta \varepsilon) \sum_{i=1}^{n-1} \alpha_i \right)^n.
\]
contradicting the assumptions. \( \square \)

Setting \( \alpha_1 = \cdots = \alpha_n = 1 \) one then obtains:

**Corollary 4.4.** Let \( X, A \) and \( \delta > 0 \) be as in (4.1), and \( x \in X \) be any point. Then

\[
\varepsilon(A, x) \geq \min \left\{ \frac{1}{(n-1)^{n-1}(2n-1)^n}, \frac{1}{\delta} \right\}.
\]

\( \square \)

**Remark 4.5.** It is well known that, for a very ample line bundle \( H \) on \( X \), the twisted tangent bundle \( T_X(K_X + nH) \) is globally generated, and in particular nef. In case \( A = K_X \) is ample on \( X \) one therefore can use one of the available effectivity statements for very ampleness of ample line bundles (e.g. [De93]) to determine explicit values for \( \delta \), making Theorem 4.3 or Corollary 4.4 effective in the way that the bounds for the Seshadri constant of \( K_X \) at any \( x \in X \) do only depend on the dimension \( n \). The same argument works in case \( A = -K_X \) is ample, or for any ample \( A \) in case \( K_X \) is trivial.

**Remark 4.6.** Let us finally compare Corollary 4.4 with the bounds that can be obtained using Angehrn–Siu’s basepoint–free Theorem. Namely, Angehrn and Siu prove that, for an ample line bundle \( A \) on \( X \), the adjoint line bundles \( mA + K_X \) are free for \( m \geq \frac{1}{2}n(n+1) + 1 \). An elementary argument (cf. e.g. [Kü, 3.3]) shows that \( \varepsilon(A, x) \geq 1 \) for all ample basepoint–free line bundles \( A \). Moreover, if \( T_X(\delta A) \) is nef, so is the \( \mathbb{Q} \)-line bundle \( M := \text{det}(T_X(\delta A)) = -K_X + n\delta A \).

By definition Seshadri constants have a sublinearity property saying that, for any rational \( \lambda, \mu \geq 0 \) and nef line bundles \( L \) and \( M \), the inequality \( \varepsilon(\lambda L + \mu M, x) \geq \lambda \cdot \varepsilon(L, x) + \mu \cdot \varepsilon(M, x) \) holds. This shows

\[
\varepsilon(A, x) \geq \frac{2}{n(n + 2\delta + 1) + 2}.
\]
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