Abstract

In usual dimensional counting, momentum has dimension one. But a function $f(x)$, when differentiated $n$ times, does not always behave like one with its power smaller by $n$. This inevitable uncertainty may be essential in general theory of renormalization, including quantum gravity. As an example, we classify possible singularities of a potential for the Schrödinger equation, assuming that the potential $V$ has at least one $C^2$ class eigen function. The result crucially depends on the analytic property of the eigen function near its 0 point.
1. INTRODUCTION

In usual dimensional counting, momentum has dimension one. But as a differential operator, momentum has no well-defined dimension unless the object it operates on is determined explicitly. For example, a function $f(x)$ with an essential singularity at $x = 0, (f(0) = 0)$, when differentiated $n$ times, does not always behave like one with its power smaller by $n$ in the neighborhood of $x = 0$. This fact is essential throughout this paper, and so, though I will investigate on the general behavior of possible singularities of a potential by a simple model below, the discussion can be applied to a wide class of differential equations of physical importance (See section 5.)

Quantum theory is accompanied with inevitable ambiguity beyond measurements. This work is motivated by an attempt to shed light on such ambiguity. For example, the analyticity of a wave function can not be determined by finite times of measurements. But if the wave function becomes very sharp at a point, momentum is dominated by the point. So, a theory should not be sensitive to such analyticity. Contrastingly, if a continuous wave function takes both plus and minus values, it must also takes 0, which is the qualitative fact not sensitive to the exact shape of a wave function. So in this paper I first try to distinguish these two kinds of singularities in section 2. Then I proceed to classify the most general types of possible singularities in section 3. The mathematical results derived there are physically explained in section 4. Applications to realistic cases are discussed in section 5 and 6. This paper has something to do with the theory of computability [1,2].

2. POSSIBILITY OF SINGULARITY AND DOMAIN OF DEFINITION

For simplicity, let us consider a one-dimensional Schrödinger equation,

\[ y'' = Vy - Ey. \]  \hspace{1cm} (2.1)

We will consider from now on if there exists a potential $V(x)$ for an arbitrary function $y(x)$ given as an eigen function.

For this purpose, it is necessary to define a ‘potential’. The potential between two particles with a large enough distance $r$ from each other is

\[
\begin{align*}
\rightarrow &1/r \text{ (for electroweak force and gravity)} \\
\rightarrow &\text{not exactly known but gets large (proportional to } r \text{ in a nuclei) (for strong force)}
\end{align*}
\]

More precisely, the coupling constants depend on the energy of the system, or the power of $r$ changes. For very short distance, the potential is completely unknown because current theory does not work beyond the Planck scale and it is also impossible to measure the size of an electron or a quark without any experimental error. So, in this paper I will say a function $y(x)$ has a potential $V(x)$ iff $y$ is a $C^2$ class function defined in $(0, a)$ and if there exists a continuous function $V(x)$ defined in $(0, a)$ and a constant $E$ which satisfies (2.1).

In fact, any $C^2$ class function $y$ satisfies (2.1) if we take
\[ V = y''/y, \quad E = 0. \]  
(2.2)

Here the replacement of the constant \( E \to E' \) is equivalent to \( V \to V - E' + E \), so from now on we take \( E = 0 \). There are 2 possible cases for (2.2) to have a singularity:

(I) there exists a \( x \) such that \( y(x) = 0, \quad 0 < x < a \),

(II) \( y'' \) does not converge (for \( x \to +0 \) or \( x \to a - 0 \)).

3. CLASSIFICATION OF THE POWER OF POSSIBLE SINGULARITIES

Now let us move the possible singularity to \( x = 0 \) by the redefinition of coordinates and consider the behavior of \( V \) as \( x \to +0 \). Let \( y(z) \) be an analytic continuation of \( y(x) \) to the complex plane.

(CASE 1) \( y(z) \) has no essential singularity at \( z = 0 \).

(a) If \( y(z) \) can be Laurent expanded around \( z = 0 \) as

\[ y = \sum_{n=k}^{\infty} a_n z^n, \quad a_k \neq 0, \]  
(3.3)

then

\[ \frac{y''}{y} = \frac{\sum_{n=k}^{\infty} a_n n(n-1)z^{n-2}}{\sum_{n=k}^{\infty} a_n z^n} \to \begin{cases} \frac{a_d d(d-1)z^{d-2-k}}{k(k-1)z^{-2}} & (0 \leq k) \\ \frac{a_d d(d-1)z^{d-2-k}}{k(k-1)z^{-2}} & (k < 0) \end{cases}, \]  
(3.4)

with \( d \) the lowest power such that \( a_d \neq 0 \) and \( 1 < d \) (if there is no such \( d, \ a_d = 0 \)).

(b) When we change the power of the finite number of terms in the type (a) expansion into an arbitrary real number, \( \nu \)

\[ \frac{y''}{y} \to \begin{cases} \frac{a_d d(d-1)z^{d-2-k}}{k(k-1)z^{-2}} & \text{if } y = a_0 + a_1 z + a_d z^d \cdots \text{ or } y = a_1 z + z_d z^d \cdots \\ k(k-1)z^{-2} & (k < 0) \end{cases}, \]  
(3.5)

where \( a_d \) is the coefficient of the lowest power except for 0, 1. Summarizing above, the powers \( \nu \) where the potential can behave like \( V \to x^\nu \) as \( x \to +0 \) are

for (I), \( \nu = -2 \); \( -1 \leq \nu \),

for (II), \( -2 \leq \nu < 0 \).

\[ ^* \text{From now on, the expansion coefficients are all real except if mentioned, and the branch is chosen so that the function takes unique real value at } z \to +0. \text{ More precisely, a branching point with the power of an irrational number is an essential singularity, but the difference is not important here.} \]
(CASE 2) $y(z)$ has an isolated essential singularity at $z = 0$. In complex analysis, a sequence of points can converge to any value depending on its approach to an essential singularity (with infinite order) \[3\]. But now that we deal with only the case along the real axis $z \to +0$, the limit is sometimes well defined. Let’s study the following cases.

(c) 
\[
y = \sum_{n=l}^{k} a_n (\log z)^n, \quad a_l \neq 0, \tag{3.6}
\]
if the above expansion is possible, then
\[
y'' \frac{y}{y} = \frac{\sum_{n=l}^{k} n a_n \{(n-1)(\log z)^{n-2} - (\log z)^{n-1}\}}{z^2 \sum_{n=l}^{k} a_n (\log z)^n} \to \frac{-k}{z^2 \log z}, \tag{3.7}
\]
where $\log z$ diverges as $z \to 0$, but for an arbitrary integer $n$, $z(\log z)^n$ tends to 0. So we can regard $\log z$ as ‘an infinitely small negative power’ $z^{-\epsilon}$ ($\epsilon > 0$). Then we can generalize type (b) expansion by the replacement of the finite number of terms
\[
a_n z^n \to z^n \sum_{m=t_n}^{k_n} a_m (\log z)^m \quad (m \in R). \tag{3.8}
\]
This has the effect of
\[
\begin{align*}
  z^{d-2-k} &\to z^{d-2-k}(\log z)^m \quad (m \in R) \\
  z^{-2} &\to z^{-2} / \log z
\end{align*} \tag{3.9}
\]
in \(3.3\), i.e.,
\[
\text{for (I), } \nu = -2(+\epsilon) ; \quad -1 \leq \nu. \tag{3.10}
\]

Let’s call this type of expansion type (c). We can define the index of power $k_y, \mu_y, \nu_y$ as $z \to 0$ for type (c) expansions as follows:
\[
y \to z^{k_y}, \quad \frac{y'}{y} \to z^{\mu_y}, \quad \frac{y''}{y} \to z^{\nu_y}. \tag{3.11}
\]
Type (c) property is invariant under finite times of summations, subtractions, and differentiations.

(d) When we apply finite times of summations, subtractions, multiplications, divisions (by $\neq 0$), differentiations, and compositions (with the shape of $f(g(z))$, $0 \leq k_g$, $g(+0) = +0$ where $f, g$ are type (c) expansions), $k_y, \mu_y, \nu_y$ can also be defined. As an arbitrary type (d)

\[\text{†For a } C^2 \text{ class function } y, \quad -1 - \epsilon \text{ is impossible. And for (II), the region of } \nu \text{ is invariant.}\]
expansion $f(z)$ has a countable number of terms and a nonzero ‘radius of convergence' $r$ where the expansion converges for $0 < |z| < r$, it can be written as

$$f(z) = \sum_{n=0}^{\infty} f_n.$$  \hspace{1cm} (3.12)

As the ‘principal part’ which satisfies $k_f < 0$ consists of finite number of terms, a type (d) expansion diverges or converges monotonically as $z \to +0$, so enables the expansion of (3.12) in the order of ascending powers. As the expansion is almost the same as that of type (c) (the only differences are the multiplications by $(\log z)^n$ for an infinite number of terms and the appearance of the terms like $\log(z \log z)$), the region of $\nu_y$ is invariant.

(e) When the following expansion is possible (type (e)): $y = \pm e^{f(z)}$, where $f$ is a type (d) expansion. We can define the finite values $\mu_y, \nu_y$ by

$$\left\{\begin{array}{l}
\frac{y'}{y} = f' \to z^{\mu_y}, \quad \mu_y = k_f + \mu_f, \\
\frac{y''}{y} = f'^2 + f'' \to z^{\nu_y}, \quad \nu_y \geq \min(2k_f + 2\mu_f, k_f + \nu_f).
\end{array}\right.$$  

Let us consider the region of $\nu_y$. For $k_f \geq 0$ it is the same as for the type (d). For

$$y = e^{az^k}, \quad a, k \in R, \quad k \leq 0$$  \hspace{1cm} (3.13)

satisfies

$$\frac{y''}{y} = a^2 k^2 z^{2k-2} + ak(k-1)z^{k-2} \to \left\{\begin{array}{l}
z^{-2+\epsilon} \quad (k = -\epsilon) \\
a^2 k^2 z^{2k-2} \quad (k < 0)
\end{array}\right.$$  

combination with type (c) case leads to the region of $\nu_y$ being:

- for (I), $\nu_y \leq -2 + \epsilon$; $-1 \leq \nu_y$,
- for (II), an arbitrary negative number.

Let us then consider if we can fill the remaining ‘window’ of the region of $\nu_y$ for (I), $-2 + \epsilon < \nu_y < -1$.

(f) When we can write $y = f_0 + \sum_{n=1}^{m} (\pm) e^{f_n}$, where $f_n$ is of type (c), $k_{f_n} < 0$, and $(\pm)$ takes each of the signatures $+-$. We can assume that each terms in $\sum$ are ordered in the increasing absolute values for $z \to +0$. Then

$$e^{az^k} \to \left\{\begin{array}{l}
z^0 \quad (k \geq 0, \quad a = 0) \\
0 \quad (k < 0, \quad a < 0) \\
\infty \quad (k < 0, \quad a > 0)
\end{array}\right.$$  

and as $y \to 0$ for (I),

\footnote{The meaning of this term is different from the usual one because $z = 0$ can be a singularity point.}
\[ y = \left( \sum_{n=0}^{\infty} \sum_{m=l_n}^{m_n} a_{nm} z^n (\log z)^m \right) + \sum_{n=1}^{l} (\pm) e^{\sum_{i=k_n}^{\infty} \sum_{j=m_n}^{k_{ni}} a_{mij} z^i (\log z)^j}. \]  \hspace{1cm} (3.14)

If the second term sum at the R.H.S. is not 0, we can write

\[ k_1 < \cdots < k_l < 0, \quad a_{nk_n k_{ni}} < 0. \]  \hspace{1cm} (3.15)

As \( y \) is of \( C^2 \) class, the first term can be written as

\[ ( ) = a_{10} z + \sum_{n=2}^{\infty} \sum_{m=l_n}^{m_n} \cdots, \quad m_2 = 0. \]  \hspace{1cm} (3.16)

As

\[ y'' \rightarrow \begin{cases} (z^n (\log z)^m)' \rightarrow z^{n-m \epsilon - 2} \left( n - m \epsilon \right. & \text{The term such that} \\ & \text{is the smallest} \right) \quad (\exists a_{nm} \neq 0) \\
\left\{ \left( a_{nk_n k_{ni}} z^{k_n} (\log z)^{k_{ni}} \right)^2 + \left( a_{nk_n k_{ni}} z^{k_n} (\log z)^{k_{ni}} \right)^{\nu} \right\} \right. \\
\times e^{\sum_{i=k_n}^{\infty} \sum_{j=m_n}^{k_{ni}} a_{mij} z^i (\log z)^j} \quad (\forall a_{nm} = 0) \end{cases} \]

for \( z \rightarrow +0, \)

\[ \frac{y''}{y} \rightarrow \begin{cases} z^{n-m \epsilon - 3} \left( a_{10} \neq 0 \text{ and } \exists a_{nm} \neq 0 \right) \\
\left. z^{2k_n - 2k_{ni} \epsilon - 2} e^{a_{nk_n k_{ni}} z^{k_n} (\log z)^{k_{ni}}} \rightarrow 0 \right( a_{10} \neq 0 \text{ and } \forall a_{nm} = 0 \right) \\
z^{-2} \left( a_{10} = 0 \text{ and } \exists a_{nm} \neq 0 \right) \\
\left. z^{2k_n - 2k_{ni} \epsilon - 2} \right( a_{10} = 0 \text{ and } \forall a_{nm} = 0 \right) \end{cases} \]  \hspace{1cm} (3.17)

The possible values of \( \nu_y \) for (I) remain the same: \( \nu_y \leq -2 + \epsilon ; \quad -1 \leq \nu_y. \)

(g) Whole of the expansions obtained from type (f) expansions by finite times of summations, subtractions, multiplications, divisions (by \( \neq 0 \)), differentiations, and compositions (with the shape of \( f(g(z)), \quad 0 \leq k_g, \quad g(+0) = +0 \) where \( f, g \) are type (f) expansions).

This type of expansion is very complicated compared to an ordinary Laurent expansion, but in any case has a countable number of terms and a nonzero ‘radius of convergence’ \( r \) where \( y \) is analytic for \( 0 < |z| < r \). This can also be ordered partially in the ascending powers and we can write the first term explicitly, and so monotonically diverges or converges but never oscillates as \( z \rightarrow +0 \). Its general shape is the whole sum

\[ (1)_i + (2)_j + \cdots + (m)_k, \]  \hspace{1cm} (3.19)

where

\[ ^5 \text{Of course, the meaning is different from the usual one.} \]
$$\begin{align}
(1)_i & := \left( \sum_{n \in \{n_i\}, m_1, \cdots, m_{d_i} = -\infty}^\infty \sum_{m = m_{d_i}} a_{nm \cdots m_{d_i}} z^n (-\log z)^{m_1} (-\log(-z/\log z))^{m_2} \\
& \quad \cdots (-\log(-z/(-\log(-z/\log \cdots z))))^{m_{d_i}} \right)_i,
\end{align}$$

$$(2)_{\pm j} := \sum_{i \in \{i_j\}} (\pm) e^{\pm (1)_i},$$

$$(3)_{\pm k} := \sum_{j \in \{j_k\}} (\pm) e^{\pm (2)_j},$$

$$\vdots$$

(3.20)

Here the \((\pm)\) in front of \(e\) takes each of the signatures depending on each \(i\) (or \(j, k, \cdots\)), while the \(\pm\) on the shoulder of \(e\) and in front of \(j, k, \cdots\) takes the signature such that the coefficient of the first term in \(\sum\) is of the same signature as \(j\) after choosing the signatures. Each term is ordered in partially ascending powers with regards for any sums. The sum with index \(n\) is performed according to the monotonically non-decreasing sequence of real numbers \(\{n_i\}\) \((-\infty < n_i)\) depending on \(i\). In the same manner, the sum with index \(i, j, \cdots\) is performed according to the finite, monotonically non-decreasing sequence \(\{i_j\}, \{j_k\}\) \(\cdots\) of natural numbers. \(m_1, \cdots, m_{d_i}\) take finite values, but they increase in correspondence with \(n\) and grows \(\rightarrow \infty\) as \(n \rightarrow \infty\), and depend on \(i\). \(d_i\) is the maximal ‘depth’ of the composition of logs, or the number of logs, depending on \(i\) and of finite value.

As the sum of the shape of \((m)_i\) can always be represented as the exp of the infinite sum of the same shape,

$$(m)_i = (\pm) e^{(m)_0}, \quad (m)_0 := \log \left( \text{sum of the finite number of } e^{(m-1)_i} s \right)$$

$$= (m-1)_1 + \log \left( 1 \pm e^{(m-1)_2} + \cdots \right),$$

(3.21)
type \((g)\) expansion can in fact be written in only ‘one term’ \(\exp(m)_{i+1}\).

Now, for the part of \(i \leq 0\) in \((m)_i\), satisfying \(0 \leq k_{(m)_i}\), \(\exp(m)_i\) can be written within the shape of \((m)_i\) as the composition of \(e^z\) and \((m)_i\); Then we can write for (I)

$$y = bz + \sum_{n=2}^\infty a_n z^n \sim + \cdots + \sum_{i<0} (\pm) e^{-biz^i} \sim \cdots + \sum_{j<0} (\pm) e^{-ez^j} \sim \cdots \cdots$$

$$+ \sum_{k<0} (\pm) e^{-dz^k} \sim \cdots \cdots,$$

(3.22)

where \(b, c_j, d_k, \cdots > 0\), \(\sim\) represents the abbreviation of \(\log z \sim\), and \(\cdots\) the higher order terms. The power of \(y''/y\) can be classified by whether \(b = 0\) or not, and what is the first of \(b, c_j, d_k, \cdots\) such that the corresponding term is not 0:

\[**\text{The power is smaller when } m_1 + m_2 + \cdots + m_{i_1} \text{ is greater for the same } n, \text{ and when it is also the same and } m_1 \text{ is smaller, and when it is also the same and } m_2 \text{ is smaller, } \cdots, \text{ and so on.}\]
\[
\frac{y''}{y} \rightarrow \begin{cases} 
(\pm) z^{n-m\epsilon-3} & (b \neq 0 \text{ and } \exists a_n \neq 0, \ n-m\epsilon \geq 2) \\
(\pm) 0 & (b \neq 0 \text{ and } \exists a_n = 0 \text{ and } \exists b_i > 0) \\
+z^{-2} & (b = 0 \text{ and } \exists a_n \neq 0) \\
+z^{2i+2k-2} & (b = \exists a_n = 0 \text{ and } \exists b_i > 0) \\
+\infty & (b = \exists a_n = \exists b_i = 0 \text{ and } \exists c_j \text{ or } d_k \cdots > 0)
\end{cases}, \quad (3.23)
\]
where \(\exists b_i = 0\) means that there is no term in \(\sum_{i<0}\).

After all, \(\nu_y \leq -2 + \epsilon, \ -1 \leq \nu_y\) for (I), where \(\epsilon\) represents the power like \(\log z \sim\).

(h) It is unclear to me whether there are other cases.

(CASE 3) \(y(z)\) has a non-isolated essential singularity at \(z = 0\).

(i) When we allow complex coefficients in (g). The discussion above is almost valid in this case, except that when \(a\) is complex \(e^{az}\) shows oscillatory behavior, and so \(y\) is not monotonic as \(z \to 0\) and generally has an accumulation point of poles or essential singularities, keeping us away from defining \(k_y, \mu_y,\) or \(\nu_y\). For example,
\[
y = z^5 \sin(z^{-1}) \quad (3.24)
\]

satisfies the condition of (I) and the term with the smallest power in \(y\) cancels that of \(y''\), yet higher order oscillation remains.

(j) It is unclear to me whether there are other cases. In such a case \(\nu_y\) would not be clearly physical, even if defined.

4. PHYSICAL EXPLANATION OF THE RESULT

The above result is not mathematically perfect, but shows that very wide types of functions, only by satisfying the second order differential equation, can restrict the behavior of the potential. Or physically, if there exists a wave function that can be applied to every point of the world, the point of nonzero charge should also be included in the domain, which determines the shape of a force.

As for the non-commutative character of quantum mechanics, type (g) expansion is valid under the special rule that we must not decompose an exponential until the end of the calculation. Each expansion has several infinite series of different order. Having nonzero `radius of convergence’, it can be calculated as a usual function. Instead, near \(z = 0\), if we do not obey the rule and try to calculate by extracting all the terms below a certain order, the result, even if finite, may depend on the arrangement of terms. (It is known in mathematics that infinite series that do not converge absolutely do not always converge to a unique value.) This implies an interesting non-commutative property.

I notice also that the difficulties caused by point-like particles may be absent here. If we assume that the existence of an eigen function is more fundamental than that of a potential, there can be the region where the potential is not defined (where the eigen function is
0). Perhaps this possibility is too subtle to be distinguished by experiment, though. The analyticity of matter field is not a quantity distinguished by finite times of measurement. Conversely, this inevitable ambiguity may be the origin of gauge uncertainty \[5\].

### 5. APPLICATIONS

I will comment on possible physical applications of the result. The first application is to general relativity, where it shows directly that in quantum mechanics, an eigen function and a potential obey different transformation rules for a nonlinear coordinate transformation. In usual quantum mechanics, the special property of an eigen function, that its 0-points are of order one and near them it behaves like \(\sin(x)\), saves the potential from divergence.

Another application is to the general theory of renormalization. The above consideration explains why some theories nonrenormalizable in the usual sense are partially computable. The first example of such case is quantum gravity, where the one loop quantum corrections to the Newton potential are determined by assuming the Einstein-Hilbert action and the perturbation around the flat metric and calculating the effective action \[7\] \[8\] \[6\]. The result is

\[
V(r) = -\frac{Gm_1 m_2}{r} \left(1 - \frac{G(m_1 + m_2)}{rc^2} - \frac{127Gh}{30\pi^2 r^2 c^3}\right),
\]

where \(G, h, c, m_1, m_2\) are respectively the Newton constant, Planck constant, the speed of light in a vacuum, and the masses of the particles. This naturally contains all the corrections at the distance, including the classical relativistic correction. The first term in (5.25) is attractive force and others are repulsive. They correspond to the type (e) singularity of the eigen function. Of course, whether or not the assumption is valid for very high energy is another story \[12\], though.

The second example is perturbative QCD, where it is shown that in a confined theory the poles and branch points of the true Green functions are generated by the Physical hadron states in the unitarity relation, and no singularities related to the underlying quark and gluon degrees of freedom should appear \[9\]. Detailed discussions about these topics - Borel summation, renormalons, Landau singularities - are in \[10\], so I only mention here that the Callan-Symanzik equation

\[
\mu \frac{d}{d\mu} g_\mu = \beta(g_\mu)
\]

just means that the cutoff scale \(\mu(g)\) as the function of a coupling constant, when differentiated once, does not always behave like one with its power smaller by 1. (It seems peculiar that the cutoff scale depends on a coupling constant, but I think the idea of multi-valued coupling constant interesting.)

Other applications may include the spherical-symmetric part of the effective field equation of the Higgs potential, where we can extend the potential to the more general functional of scalar field \(\phi\) without breaking gauge symmetry.
6. EXAMPLE

We can extend the results to dimension \( N > 1 \) as follows. If we assume that the eigen function \( y \) is a \( N \)-dimensional spherical symmetric function \( R(r) \) (i.e. orbital angular momentum is 0), short distance limit behavior (3.23) is clearly replaced by

\[
\frac{\Delta R(r)}{R(r)} = \frac{R''(r)}{R(r)} + \frac{N-1}{r} \frac{R'(r)}{R(r)}
\]

\[
\rightarrow \begin{cases} 
+ (N-1)r^{-2} \ (b \neq 0) \\
+ n(n+N-2) r^{-2} \ (b = 0 \text{ and } \exists a_n \neq 0) \\
+ (-ib_i)^2 r^{2i+2x} \ (b = \gamma \ a_n = 0 \text{ and } \exists b_i > 0) \\
+ \infty \ (b = \gamma \ a_n = \gamma b_i = 0 \text{ and } \exists c_j \text{ or } d_k \text{ or } \cdots > 0)
\end{cases}.
\tag{6.27}
\]

We can extend the results to \( r \rightarrow \infty \) case as follows. If we change the variable to \( z := \frac{1}{r} \) and assume that \( R(z) \) is \( C^2 \) class (expanded as below)

\[
R = a + bz + \sum_{n=2}^{\infty} a_n z^n + \cdots + \sum_{i<0} (\pm) e^{-b_i z^i} \cdots \\
+ \sum_{j<0} (\pm) e^{-c_j z^j} \cdots + \sum_{k<0} (\pm) e^{-d_k z^k} \cdots,
\tag{6.28}
\]

(3.27) is clearly replaced by

\[
\frac{\Delta R(r)}{R(r)} = \frac{1}{R(z)} \left\{ \frac{dz}{dr} \left( \frac{dz}{dr} \frac{dR(z)}{dz} \right) + (N-1) z \frac{dz}{dr} \frac{dR(z)}{dz} \right\}
\]

\[
= z^4 \frac{R''(z)}{R(z)} - z^3 (N-3) \frac{R'(z)}{R(z)}
\]

\[
\rightarrow \begin{cases} 
(3-N) \frac{b}{a} z^2 \ (a \neq 0 \text{ and } b \neq 0 \text{ and } N \neq 3) \\
(n-N+2) n \frac{a_n z^{n+2}}{a} \ (a \neq 0 \text{ and } b = 0 \text{ and } \exists a_n \neq 0 \text{ and } N \neq 3) \\
(n-1) n \frac{a_n z^{n+2}}{a} \ (a \neq 0 \text{ and } \exists a_n \neq 0 \text{ and } N = 3) \\
(\pm) 0 \ (a = 0 \text{ and } b = \gamma a_n = 0 \text{ and } \exists b_i \text{ or } c_j \text{ or } d_k \text{ or } \cdots > 0) \\
(3-N) z^2 \ (a = 0 \text{ and } b \neq 0 \text{ and } N \neq 3) \\
(n-1) n \frac{a_n z^{n+1}}{b} \ (a = b \neq 0 \text{ and } \exists a_n \neq 0 \text{ and } N = 3) \\
(\pm) 0 \ (a = 0 \text{ and } b \neq 0 \text{ and } \gamma a_n = 0 \text{ and } \exists b_i \text{ or } c_j \text{ or } d_k \text{ or } \cdots \rightline{> 0 \text{ and } N = 3} \\
(n-N+2) n z^2 \ (a = b = 0 \text{ and } \exists a_n \neq 0) \\
+ (-ib_i)^2 z^{2i+2x} \ (a = b = \gamma a_n = 0 \text{ and } \exists b_i > 0) \\
+ \infty \ (a = b = \gamma a_n = \gamma b_i = 0 \text{ and } \exists c_j \text{ or } d_k \text{ or } \cdots > 0)
\end{cases}.
\tag{6.29}
\]

Noting that \( 2 \leq n \) and \( i < 0 \), we conclude that the power of potential \( V \rightarrow r^\nu \) as \( r \rightarrow \infty \) is \( \nu \leq -3 \); \( -2 - \epsilon \leq \nu \). There is no reason to assume that \( R(z) \) is \( C^2 \) class, but more natural normalizability condition that \( R(r) \) is a \( L^2 \) function leads to small modifications \( a = b = 0 \) and \( N < 2n \) (instead of \( 2 \leq n \)) in (3.28) and (3.29). Notice also that (1.27) for the more general cases of \( N, a \) can be obtained from (6.29) by the trivial replacement \( N \rightarrow 4-N \) and \( z \rightarrow r \) with its power smaller by 4.
Comment on normalizability
If \( R(r) \) is not a \( L^2 \) function, that does not always mean contradiction. I think that \( \delta \) function like sharpness of \( R(r) \) is not realistic but for \( r \to \infty \) the ‘generalized expectation value’ of a physical operator \( A \) can be defined for \( R(r) \) as follows:

\[
< A > := \lim_{L \to \infty} \frac{\int_0^L r^{N-1} dr R^*(r) A R(r)}{\int_0^L r^{N-1} dr R^*(r) R(r)}.
\]

(6.30)

The above definition may have little physical meaning in case it only depends on the value of the field on the surface of a sphere, but sometimes, for example when \( N = 1 \) and \( R(r) = \sin r \), can be valid.

Comment on uniqueness
The solution of the following ‘2 dimensional weak exterior Dirichlet problem’ is not unique: the function \( u(x, y) \) is defined on and on the exterior of the circle \( x^2 + y^2 = a^2 \) (called \( C \)), satisfying the Laplace equation \( \Delta u(x, y) = 0 \) and being 0 on \( C \). Determine \( u(x, y) \).

The proof is as follows. Let \( u(x, y) \) be \( \text{Re} \ z - \frac{a^2}{z} \), where \( z = x + iy \). Because of the Cauchy-Riemann equation, the real part of an analytic function becomes automatically harmonic, which shows that \( u(x, y) \) is a nontrivial solution. It clearly is 0 on \( C \). In fact we can find infinite number of solutions to the problem by the replacement \( z \to \frac{z + 1}{a} \). Furthermore, if we include multi-valued function, another type of solutions can be found as follows.

1. Let’s take two multi-valued function \( f(z) \) and \( g(z) \) analytic on and on the exterior of \( C \), having a single-valued branch defined outside the cut \( z = 0, 0 \leq \text{Re} \ z \).

2. Combine them so as not to have a gap at the cut on \( C \). For example, define a new function \( h(z) := \Delta_g f(z) - \Delta_f g(z) \), where \( \Delta_f := f(\alpha e^{i\theta}) - f(\alpha e^{i(\theta - 2\pi)}) \), \( \Delta_g := g(\alpha e^{i\theta}) - g(\alpha e^{i(\theta - 2\pi)}) \) are the gap of \( f(z), g(z) \).

3. As the branch \( \text{Re} \ h(z), z = \alpha e^{i\theta} \), \( 0 \leq \theta \leq 2\pi \) is continuous on \( C \) and is 0 at \( z = a \), there is the unique sine Fourier expansion of it on \( C \), i.e., \( \text{Re} \ \hat{h}(\alpha e^{i\theta}) = \sum_{n=0}^{\infty} h_n \sin(\frac{\alpha n \theta}{2}) \).

4. There is the corresponding function \( \hat{h}(z) := \sum_{n=0}^{\infty} -h_n \frac{(2z)^n}{2} \) such that \( \text{Im} \ \hat{h}(z) = \sum_{n=0}^{\infty} h_n \sin(\frac{\alpha n \theta}{2}) \) on \( C \).

5. Define \( \tilde{H}(x, y) := \text{Re} \ h(z) - \text{Im} \ \hat{h}(z), \text{where} \ z := x + iy \). Then, the branch \( \tilde{H}(x, y) \) can be a nontrivial solution to the problem for, for example, \( f(z) = \frac{1}{\log z}, g(z) = \frac{1}{z \log z} \).

Notice that \( \tilde{H}(x, y) \) is generally multi-valued on the cut except for the point \( (a, 0) \).

Comment on relativistic effect and the sign of potential
For the time-independent and spherical-symmetric \( U(1) \) gauge field \( A^\mu = (\phi(r), 0, 0, 0) \), \( (6.29) \) becomes an exact relativistic Schrödinger equation by the replacement \( V(r) \to \frac{m^2 c^2 - (E - e\phi)^2}{\hbar^2} \), where \( E, e, m \) are respectively dimensionless the energy, the charge, and the mass of the spherical-symmetric scalar field \( R(r) \). Of course, fermion field equation is another story. Above results show that for a physical dimension \( N = 1, 2, 3 \), the sign of a potential \( V \) must be positive for \( \nu \leq -2 + \epsilon \ \text{(} r \to 0 \) ) and \( -2 - \epsilon \leq \nu \ \text{(} r \to \infty \) ), but can be negative for other cases.
7. CONCLUSION

In this paper we classified possible singularities of a potential for the one dimensional Schrödinger equation, assuming that a potential $V$ has at least one $C^2$ class eigen function. The result crucially depends on the analytic property of the eigen function near its 0 point. We also discussed the extension to dimension $N$ and the long distance limit. There are interesting applications to quantum gravity and to the general theory of renormalization.

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