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Matrix Field Theory

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Abstract

This thesis studies matrix field theories, which are a special type of matrix models. First, the different types of applications are pointed out, from (noncommutative) quantum field theory over 2-dimensional quantum gravity up to algebraic geometry with explicit computation of intersection numbers on the moduli space of complex curves.

The Kontsevich model, which has proved the Witten conjecture, is the simplest example of a matrix field theory. Generalisations of this model will be studied, where different potentials and the spectral dimension are introduced. Because they are naturally embedded into a Riemann surface, the correlation functions are graded by the genus and the number of boundary components. The renormalisation procedure of quantum field theory leads to finite UV-limit.

We provide a method to determine closed Schwinger-Dyson equations with the usage of Ward-Takahashi identities in the continuum limit. The cubic (Kontsevich model) and the quartic (Grosse-Wulkenhaar model) potentials are studied separately.

The cubic model is solved completely for any spectral dimension $< 8$, i.e. all correlation functions are derived explicitly. Inspired by topological recursion, we propose creation and annihilation operators by differential and residue operators. The exact results are confirmed by perturbative computations with Feynman graphs renormalised by Zimmermann’s forest formula. The number and the amplitudes of the graphs grow factorially, which is known as renormalon problem. However, these series are convergent since the exact results are provided. A further differential operator is derived to determine all free energies. Additionally, by the theorem of Kontsevich, the intersection numbers of the moduli space of complex curves $\overline{M}_{g,b}$ are found.

For the quartic model, the 2-point function is derived for any spectral dimension $< 6$ explicitly. The first step is to derive an angle function which is, after analytic continuation, interpreted as an effective measure. On the 4-dimensional noncommutative Moyal space, the effective measure is given by a hypergeometric function. Its asymptotic behaviour changes the spectral dimension effectively to $4 - 2\frac{\arcsin(\lambda\pi)}{\pi}$ for $|\lambda| < \frac{1}{\pi}$. This dimension drop prevents the quantum field theoretical 4-dimensional $\Phi^4$-model on the Moyal space from the triviality problem. After combinatorial analysis, an explicit (not recursive) formula for any planar $N$-point function is provided.

The evident difference between the cubic and the quartic model is of algebraic-geometric nature. Computing correlation functions via topological recursion needs the spectral curve as initial data. This algebraic curve has for the cubic model only one branch point which coincides with the pole of the stable correlation functions. However, the quartic model has a spectral curve which admits infinitely many branch points in the continuum limit.
Zusammenfassung

Diese Dissertation beschäftigt sich mit Matrix-Feldtheorien, einer speziellen Form der Matrixmodelle. Zunächst werden unterschiedliche Anwendungsmöglichkeiten hervorgehoben, die von (nichtkommutativer) Quantenfeldtheorie über 2-dimensionale Quantengravitation bis hin zur algebraischen Geometrie mit expliziter Berechnung von Schnittzahlen auf dem Modulraum komplexer Kurven reichen.

Das Kontsevich-Modell ist ein paradigmatisches Beispiel einer Matrix Feldtheorie, mit der Wittens Vermutung bewiesen wurde. Es werden Verallgemeinerungen dieses Modells betrachtet, die durch die Wahl eines anderen Potentials und durch Einführung der spektralen Dimension. Die Korrelationsfunktionen werden durch das Geschlecht und die Zahl der Randkomponenten unterschieden, da diese eine natürliche Darstellung auf Riemannschen Flächen besitzen. Um dem UV Limes Bedeutung zu verleihen, werden Renormierungsmethoden aus der Quantenfeldtheorie verwendet.

Wir zeigen, wie geschlossene Schwinger-Dyson-Gleichungen für die Korrelationsfunktionen mit Hilfe von Ward-Takahashi-Identitäten im Kontinuumslimes bestimmt werden können. Die spezielle Wahl eines kubischen (Kontsevich-Modell) und eines quartischen (Grosse-Wulkenhaar-Modell) Potentials wird separat betrachtet und untersucht.

Das kubische Modell wird vollständig für eine spektrale Dimension $< 8$ gelöst, d. h. es werden alle Korrelationsfunktionen explizit berechnet. Die Erzeuger- und Vernichteroperatoren werden als Differential- und Residuumsoperator angegeben, wobei die Konstruktion durch topologische Rekursion inspiriert wurde. Die Resultate werden durch störungs-theoretische Rechnungen bestätigt, in denen in Feynman-Graphen entwickelt wird, die durch Zimmersmanns Waldformel renormiert werden. Die Anzahl der Graphen und die Amplitude der Graphen steigen mit $O(n!)$, welches als Renormalon-Problem bezeichnet wird; dennoch konvergieren diese Reihen, da wir sie konkret angeben. Es wird ein Differentialoperator angegeben, der die freien Energien berechnet und somit nach dem Theorem von Kontsevich die Schnittzahlen auf dem Modulraum der komplexen Kurven $\mathcal{M}_{g,b}$ erzeugt.

Für das quartische Modell geben wir die 2-Punkt-Funktion für die spektrale Dimension $< 6$ explizit an. Hierzu wird zunächst eine Winkelfunktion berechnet, die nach komplexer Fortsetzung als effektives Maß interpretiert werden kann. Auf dem 4-dimensionalen nichtkommutativen Moyalraum ist das effektive Maß durch eine hypergeometrische Funktion gegeben, die die spektrale Dimension effektiv zu $4 - 2\arcsin(\lambda/\pi)$ für $|\lambda| < \frac{1}{\pi}$ ändert.

Durch diese effektive Änderung wird das Trivialitätsproblem des quantenfeldtheoretischen $\Phi^4$-Modells auf dem 4-dimensionalen Moyalraum verhindert. Schließlich zeigen wir nach kombinatorischer Analyse, wie jede planare $N$-Punkt-Funktion im quartischen Modell explizit (nicht rekursiv) berechnet werden kann.

Die starken Unterschiede zwischen dem kubischen und quartischen Modell sind algebraisch-geometrischer Natur. Die Bestimmung der Korrelationsfunktionen durch topologische Rekursion bedient sich einer algebraischen Kurve, die im kubischen Modell nur einen Verzweigungspunkt hat, der mit der Polstelle der stabilen Korrelationsfunktionen übereinstimmt. Das quartische Modell hingegen hat im Kontinuumslimes unendlich viele Verzweigungspunkte, die zu unendlich vielen Zweigen führen.
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to my wife Caroline
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Chapter 1

Introduction

This thesis studies a special type of matrix models, namely matrix field theory models. These models have implications in modern areas of mathematics and mathematical physics which seem to be different but connected via matrix field theory. The various types of implications and applications of matrix field theory to quantum field theory, quantum field theory on noncommutative geometry, 2D quantum gravity and algebraic geometry will be introduced, respectively.

1.1 Quantum Field Theory

Nature is, on fundamental level, governed by four different interactions of two separated theories in physics. Elementary particles are described in the Standard Model with three interactions, the weak interaction, the strong interaction and the electromagnetic interaction. Quantum field theory (QFT) describes the dynamics of these elementary particles by fundamental principles. The fourth interaction is gravity and described by the theory of general relativity.

General relativity is, from a mathematical point of view, rigorously understood. The achievement of Einstein was to recognise that the 4-dimensional spacetime is curved by energy densities, and the motion occurs along geodesics. This theory is confirmed experimentally with astonishing precision, e.g. recently by the measurement of gravitational waves [Abb16].

Also the predictions of the Standard Model are verified day-by-day in huge particle colliders. The theoretical prediction, for instance, for the anomalous magnetic moment of an electron agrees with the experimental data up to eleven decimal digits [OHDG06]. However, the mathematical construction of QFT is, independent of the particle content, hard to formulate rigorously.

Wightman formulated these fundamental principles for a QFT on Minkowski space with natural axioms for operator-valued tempered distributions, smeared over the support of a test function, on a separable Hilbert space [Wig56, SW89]. The first application was to show that the 4-dimensional free scalar field satisfies these axioms, which indeed holds. Furthermore, Wightman’s powerful reconstruction theorem implies that if the full set of correlation functions is known, then under certain conditions the Hilbert space and the entire quantum field theory can be reconstructed. Unfortunately, the axiomatic formulation of Wightman has one problem: no interacting QFT model satisfying these axioms could be constructed in 4D, yet. An equivalent formulation to Wightman’s ax-
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ions on the Euclidean space, instead of Minkowski space, was found by Osterwalder and Schrader [OS73, OS75].

A different approach to QFT makes use of the path integral formalism. The idea behind is that a particle propagates between two points not along the path with extremal/minimal action, but along any path weighted by some probability. For a QFT, the particle is described by a field, a scalar field \( \phi \) can be for instance a Schwartz function \( \phi \in \mathcal{S}(\mathbb{R}^D) \). Therefore, the path integral translates into a sum (or even an integral) over all field configurations of the field content of the model [Pop84]. This expression is on Minkowski space not well-defined and has, even on Euclidean space, a lot of technical issues. Nevertheless, the path (or better: functional) integral formalism can be used to approximate correlation functions around the free theory, which is called perturbative expansion. To make the perturbative expansion well-defined, certain parameters of the model need to be adjusted (renormalised) appropriately. These approximated and renormalised correlation function can then be compared via LSZ reduction formula [LSZ55] to the experiment. This comparison of theory and experiment fits remarkably well.

Up to now, it is not clear whether the approximation of a correlation function by perturbation theory converges in any sense. The number of terms for the perturbative expansion grows factorially from order to order. Furthermore, the values of the different terms themselves increase after renormalisation (renormalon problem) such that even Borel summablility seems to be a hopeless concept [Dys52].

It will be proved in this thesis that matrix field theory provides non-trivial examples for models which have the same issues as QFT models, but the perturbative expansion is indeed convergent, in fact we will determine the function it converges to. We will define the dimension of the matrix field theory model in the natural sense given by Weyl’s law [Wey11]. The entire machinery of renormalisation will be necessary, as in QFT, to generate finite results for the perturbative expansion. We will see for selected examples that the number and the value of the terms grow for the perturbative expansion factorially, just like in QFT. The exact results of the correlation function will be computed directly and coincide with perturbative expansion after applying Zimmermann’s forest formula for renormalising all divergences and subdivergences.

From these examples, the following question arises: What are the mathematical conditions that the perturbative expansion (in the sense of QFT) converges?

1.2 Quantum Field Theory on Noncommutative Geometry

As mentioned before, QFT is described on a flat spacetime (Minkowski or Euclidean space). Since the theory of general relativity implies a curved spacetime, a natural question is whether both theories can be combined, which is first of all not the case. For instance, Heisenberg’s uncertainty relation of quantum mechanics implies for a spherical symmetric black hole (solution of Einstein’s field equation in general relativity) an uncertainty of the Schwarzschild radius. Applying this to a quantum field yields that the support of the quantum field cannot be localised better than the Planck scale \( l_P = \sqrt{\frac{G}{\hbar c}} \), where \( G \) is Newton’s constant, \( \hbar \) Planck’s constant and \( c \) the speed of light [MTW73].

Noncommutative geometry can avoid this gravitational collapse caused by localising events with extreme precision [DFR95]. The coordinate uncertainties have to satisfy cer-
tain inequalities which are induced by noncommutative coordinate operators $\hat{x}^\mu$ satisfying $[\hat{x}^\mu, \hat{x}^\nu] = i\Theta^{\mu\nu}$, where $\Theta^{\mu\nu}$ are the components of a 2-form with the properties $\langle \hat{\Theta},\hat{\Theta} \rangle = 0$ and $\langle \hat{\Theta},\ast\hat{\Theta} \rangle = 8l_p^4$ in 4D.

This suggests that, if QFT and gravity (in the classical sense of general relativity) are combined, spacetime itself should be noncommutative. First examples of scalar QFTs on noncommutative spaces face in the perturbative expansion the problem of mixing ultraviolet and infrared divergences [MVRS00]. This mixing problem was solved by adding a harmonic oscillator term depending on $\hat{\Theta}$ to the action [GW05b]. The most natural example of a scalar QFT is the quartic interacting model, the Grosse-Wulkenhaar model, which was proved to be renormalisable to all order in perturbation theory, a necessary condition for a QFT [GW05b].

The representation of a scalar QFT model on a noncommutative space (especially on the Moyal space) is approximated in momentum space by large matrices [GBV88]. At the self-dual point [LS02], this type of model becomes a matrix field theory model with a special choice for the external matrix (or better the Laplacian) defining the dynamics. The QFT model itself is reconstructed in the limit of infinitely large matrices.

1.3 2D Quantum Gravity

Quantum gravity designs a different approach to combine QFT and gravity. Spacetime, and therefore gravity itself, is quantised in the sense of a quantum field.

Remarkable results were achieved for quantum gravity in 2 dimensions, since orientable manifolds of dimension 2 are Riemann surfaces which are simpler than higher dimensional manifolds. The quantisation of gravity implies (in the sense of the path integral formalism) an average of special weights (corresponding to the physical theory) over all geometries of Riemann surfaces.

One way of doing so was by discretising the Riemann surfaces into polygons which are glued together. The dual picture of a discretisation of a Riemann surface is a ribbon graph such that a sum over discretised Riemann surfaces can be performed as a sum over the dual ribbon graphs [DFGZJ95]. In analogy to the perturbative expansion of QFT, ribbon graphs are generated by the Hermitian 1-matrix models. To end up in finite volumes for the Riemann surfaces in the continuum limit, the size of the polygons has to tend to zero, whereby the number of the polygons tends to infinite (double-scaling limit). Conjecturally, matrix models should provide 2-dimensional quantum gravity in this double-scaling limit, which was for a long time not understood rigorously.

A second approach to 2D quantum gravity was formulated by Polyakov [Pol81] under the name of Liouville quantum gravity. His idea was to sum over all metrics on a surface instead of summing over all surfaces. In 2 dimensions, any metric can be transformed in a conformal form, i.e. it is after transformation diagonal and characterised by a scalar, the Liouville field which can be coupled to gravity. The Jacobian to achieve the conformal form of the metric is called Liouville action which is by itself conformally invariant. This conformal invariance gives strong conditions on the correlation functions given by representations of the Virasoro algebra (due to the conformal group). Finite representations of the conformal group are classified by Kac’s table into $(p, q)$-minimal models, which implies that the partition function of a conformal field theory coupled to gravity is a $\tau$-function of KdV hierarchy (nonlinear partial differential equation of Painlevé type) [DFMS97].

Heuristic asymptotics yield the guess that the partition function of matrix models
is in the double-scaling limit a $\tau$-function of a $(p, q)$-minimal model. In other words, the partition function of a matrix model satisfies a partial differential equation in the double-scaling limit. This conjecture was later proved rigorously (see e.g. [Eyn16, Ch. 5]). Consequently, 2D quantum gravity was proved to be approximated by a particular discretisation of the underlying space.

The interest in matrix models increased due to the relation to Liouville quantum gravity. Further examples of matrix models were investigated. The Kontsevich model [Kon92] had even higher impact which is the first non-trivial example for a matrix field theory, where the attention of this thesis lies on. The ribbon graph expansion consists of weighted graphs with only trivalent vertices. Unexpectedly, the Kontsevich model was proved to be in the limit of infinite matrix size equivalent to the Hermitian 1-matrix model by a certain choice of the parameters, the so-called Miwa-transformation (or Kontsevich times) [AK93]. Hence, the Kontsevich model, as first non-trivial example for a matrix field theory, agrees with the $\tau$-function of KdV hierarchy and is therefore also an counterintuitive approximation for 2D quantum gravity.

1.4 Algebraic Geometry

A third approach to 2D quantum gravity goes back to concepts of algebraic geometry. This approach (so-called topological gravity) tries to take the sum over all Riemann surfaces up to holomorphic reparametrisations. The set of Riemann surfaces for given topology modulo holomorphic reparametrisation is called moduli space which is a finite dimensional complex variety. For the interest of quantum gravity, an integral over the moduli space (or better its compactification) should be performed. A volume form on the moduli space is constructed from wedging the Chern classes of the line bundles which are naturally constructed by the cotangent spaces at the marked points of the Riemann surface. If these forms are of top dimension, then the integral over the compactified moduli space provides a nonvanishing rational number, which is called the intersection number. These numbers are topological invariants characterising the corresponding moduli space.

The original motivation of integrating over the moduli spaces coming from 2D quantum gravity inspired Witten to his famous conjecture [Wit91b] that the generating function of the intersection numbers of stable Riemann surfaces (= stable complex curves) is a $\tau$-function of KdV hierarchy. Liouville quantum gravity is related to the KdV hierarchy. Otherwise stated, the approach of Liouville quantum gravity and the approach of topological gravity are equivalent.

This conjecture was proved by Kontsevich [Kon92] by relating the generating function for a special choice of the formal parameters (Kontsevich times) to the weighted ribbon graphs generated by the Kontsevich model. As mentioned before, the Kontsevich model is the easiest example for a matrix field theory and satisfies via the connection to Hermitian 1-matrix models the PDE’s of the KdV hierarchy.

Intensive studies on matrix models have shown that also the correlation functions (and not only the partition function) of the Hermitian 1-matrix model and the Kontsevich model are related in some sense. The correlation functions obey the same type of recursive relations, the so-called topological recursion. The beauty of topological recursion is that for a given initial data (the spectral curve) topological recursion universally produces symmetric meromorphic functions [EO07]. These are, in the case of matrix models, the correlation functions of the corresponding model.
1.5 Outline of the Thesis

The thesis starts in Ch. 2 with an introduction to matrix field theory in general. The basic definitions are given for the action of a matrix field theory, the partition function and the expectation values. To get an intuition for these models, Sec. 2.1 is included which explains the perturbative expansion in detail. The general setting of obtaining equations and identities between expectation values (Schwinger-Dyson equation and Ward-Takahashi identity) is described in Sec. 2.2. In Sec. 2.3, a scaling limit is performed which provides matrix field theory models of spectral dimension greater than 0 in the sense of QFT. For this limit, renormalisation (Sec. 2.4) is necessary which is a technique developed by physicists. The perturbative expansion needs for a renormalised matrix field theory a careful treatment by Zimmermann’s forest formula (Sec. 2.4.1) to avoid all divergences in the scaling limit. The chapter is finished by Sec. 2.5 which shows the explicit construction of QFT on the noncommutative Moyal space from a matrix field theory model.

Ch. 3 is dedicated to the simplest matrix field theory model with cubic interaction, the Kontsevich model. This model is solved completely in Sec. 3.2 which means that an algorithm is given to compute exactly any correlation function for any spectral dimension $D < 8$. The Kontsevich model is for higher spectral dimension $D \geq 8$ nonrenormalisable. The main theorems for the algorithm are Theorem 3.1 and Theorem 3.2. The free energies (and therefore the intersection numbers on the moduli space of stable complex curves) are determined in Sec. 3.3 via a Laplacian.

The case of quartic interaction (known as Grosse-Wulkenhaar model) is developed in Ch. 4. The total set of Schwinger-Dyson equations is derived in Sec. 4.1. The initial step in computing all correlation function starts for the quartic model with the 2-point correlation function described in Sec. 4.2. The exact solution of this function is given in Theorem 4.3 for spectral dimension $D < 6$, where the two important special cases of finite matrices and on the 4-dimensional Moyal space are explained in Sec. 4.2.3 and Sec. 4.2.4, respectively. We give in Sec. 4.3 an outline for the correlation function with higher topology. In the planar case with one boundary (of arbitrary length), the entire combinatorial structure is analysed in Sec. 4.4.

To make the thesis fluently readable, a lot of technical details are outsourced to the appendix. Basic properties of the Moyal space and the description of Schwinger functions on it are found in App. A and App. B, respectively. The proof of Theorem 3.1 is split
in several lemmata in App. C. An important cross-check for the validity of the results is derived in App. D by perturbative calculations with Feynman graphs and Zimmermann’s forest formula. Additionally, the perturbative analysis of the quartic model on the 4-dimensional Moyal space is discussed in much more detail in App. E. Examples for the combinatorial constructions used in Sec. 4.4 are given in App. F. The last appendix App. G provides a multi-matrix field theory model which interestingly shares properties of both models, the cubic model of Ch. 3 and the quartic model of Ch. 4.
Chapter 2
Matrix Field Theory

Let $\hat{\Phi} = \hat{\Phi}^\dagger$ be a self-adjoint compact linear operator on an infinite-dimensional Hilbert space $\mathcal{H}$. The operator $\hat{\Phi}$ will be called the field. Let $E$ be an unbounded self-adjoint positive operator on $\mathcal{H}$ with compact resolvent $(E - z1)^{-1}$. The following class of so-called action functionals will be considered

$$S[\hat{\Phi}] = V \text{Tr} \left( E \hat{\Phi}^2 + \mathcal{V}(\hat{\Phi}) \right),$$

(2.0.1)

defined on some subspace of compact operators depending on $E$. The trace and the products are defined for operators on Hilbert space $\mathcal{H}$. The potential $\mathcal{V}$ is of the form

$$\mathcal{V}(\phi) = \sum_{k=3}^d \lambda_k \hat{\Phi}^k$$

with coupling constants $\lambda_k \in \mathbb{R}$, and the parameter $V \in \mathbb{R}$ will be specified later.

The action implies that $E$ is interpreted as a Laplacian, where a canonical dimension is induced according to Weyl’s theorem [Wey11] by the asymptotic behaviour of spectrum of $E$:

**Definition 2.1.** The operator $E$ encodes the dimension $D := [\mathfrak{D}]$ via the spectral dimension

$$\mathfrak{D} := \inf \{ p \in \mathbb{R}_+ : \frac{1}{V} \text{Tr} [(1 + E)^{-p/2}] < \infty \}.$$  

Equivalently, the spectral dimension can be expressed by the spectral measure $dg(t)$ which is uniquely determined by $E$ by the spectral theorem of unbounded self-adjoint operators. The spectral dimension can therefore be computed additionally by

$$\mathfrak{D} = \inf \{ p \in \mathbb{R}_+ : \int \frac{dg(t)}{(1 + t)^{p/2}} < \infty \},$$

(2.0.2)

where it is not necessarily integer-valued. We will give an explicit example (see Sec. 4.2.4) where the Laplacian can effectively be changed $E \to E\lambda$, or $g \to g\lambda$, through the potential $\mathcal{V}$ and therefore the spectral dimension $\mathfrak{D} \to \mathfrak{D}_\lambda$ as well.

The field operator $\hat{\Phi}$ can be approximated by a matrix $\Phi$ of finite rank $(\mathcal{N} + 1)$, where the taken topology depends on the specific problem. The spectral theorem of finite-dimensional spaces implies the existence of a $\star$-homomorphism to Hermitian matrices $\Phi \mapsto (\Phi_{nm})$ in which $E$ is projected by the projection $P$ to a Hermitian $(\mathcal{N} + 1) \times (\mathcal{N} + 1)$-matrix. Let $(E_n)_{n=0}^\mathcal{N}$ be the eigenvalues of the projection $PEP$. We can choose without loss of generality that $E = (E_n \delta_{n,m})$ is diagonal with ordered eigenvalues $E_n \leq E_{n+1}$, since the later defined partition function (2.0.4) is invariant under a global unitary transformation.
\( \Phi \mapsto U^I \Phi U \) which can diagonalise \( E \). The action (2.0.1) is under the \( \ast \)-homomorphism after symmetrisation of the kinetic term of the form

\[
S[\Phi] = V \left( \sum_{n,m=0}^{N} \frac{H_{nm}}{2} \Phi_{nm} \Phi_{mn} + \sum_{k=3}^{N} \frac{\lambda_k}{k} \sum_{n_1,\ldots,n_k=0}^{N} \Phi_{n_1 n_2} \cdots \Phi_{n_k n_1} \right), \tag{2.0.3}
\]

\( H_{nm} := E_n + E_m \).

Notice that \( H_{nm} \) is not a \((N + 1) \times (N + 1)\)-matrix. It can be understood as the \((N + 1)^2 \times (N + 1)^2\)-matrix \((E \otimes \mathbb{I} + \mathbb{I} \otimes E)_{nm} = H_{nm} \), where its inverse takes the role of a free propagator.

Field theories in general have got a dynamical construction such that a propagating field carries an energy or momentum dependence. Matrix models are known to provide exact results which are established by the existence of a high symmetry. A matrix field theory combines both approaches, where the dynamics is considered by the external matrix \( E \). For constant \( E \), the usual Hermitian 1-matrix model is recovered.

**Remark 2.1.** Staying in the subspace of finite matrices would define only \( D = 0 \) matrix field theory models, since

\[
\frac{1}{V} \text{Tr}((1 + E)^{-p/2}) = \frac{1}{V} \sum_{n=0}^{N} \frac{1}{(1 + E_n)^{p/2}} = \int \frac{dt}{V} \sum_{n=0}^{N} \delta(t - E_n) = \int \frac{\delta(t) dt}{(1 + t)^{p/2}}
\]

is finite for any \( p > 0 \). The \( N \to \infty \) limit is of greater interest which is conveniently combined with a \( V \to \infty \) limit to achieve \( \mathcal{O} > 0 \).

The action (2.0.3) gives rise to a well-defined definition of the partition function \( \mathcal{Z}[J] \) depending on the Hermitian \((N + 1) \times (N + 1)\)-matrix \( (J_{nm}) \) (called the source) by

\[
\mathcal{Z}[J] = \int_{\text{formal}} D\Phi \exp \left( -S[\Phi] + V \text{Tr}(J\Phi) \right). \tag{2.0.4}
\]

The subscript formal means that \( e^{-V\text{Tr}(V(\phi))} \) is expanded as a formal series and the order of the integral and the series is exchanged. We will skip this subscript from now on. The integration is over all Hermitian \((N + 1) \times (N + 1)\)-matrices \( (\Phi_{nm}) \) with Lebesgue measure

\[
D\Phi := \prod_{n<m} d\Phi_{nm} \prod_{n\leq m} d\Phi_{nn}.
\]

Each variable is separated in the real and imaginary part

\[
\Phi_{nm} = \Phi_{nm}^R + i\Phi_{nm}^I \quad \text{with} \quad \Phi_{nm}^R = \Phi_{nm}^R \quad \text{and} \quad \Phi_{nm}^I = -\Phi_{nm}^I.
\]

Such that the partition function is defined over a \( \mathcal{N}(N + 1)^2 \) \((N + 1)^2\)-dimensional space. A partial derivative with respect to the source \( J_{nm} \) produces a factor \( \Phi_{nm} \) in the integrand. This means we have a correspondence \( \Phi_{nm} \leftrightarrow \frac{1}{V} \frac{\partial}{\partial J_{mn}} \), or more explicitly

\[
\frac{1}{V} \frac{\partial}{\partial J_{nm}} \int D\Phi f(\Phi) \exp \left( -S[\Phi] + V \text{Tr}(J\Phi) \right)
= \int D\Phi \left( \frac{f(\Phi)}{V} \frac{\partial}{\partial J_{nm}} \exp \left( -S[\Phi] + V \text{Tr}(J\Phi) \right) \right)
= \int D\Phi f(\Phi) \Phi_{nm} \exp \left( -S[\Phi] + V \text{Tr}(J\Phi) \right).
\]

Two partial derivatives commute \( \frac{\partial^2}{\partial J_{nm} \partial J_{kl}} = \frac{\partial^2}{\partial J_{kl} \partial J_{nm}} \). We employ this correspondence to rewrite the interaction term (or the potential) \( S_{\text{int}}[\Phi] := V \text{Tr}(V(\Phi)) \) as

\[
S_{\text{int}}[\Phi] \leftrightarrow S_{\text{int}} \left[ \frac{1}{V} \frac{\partial}{\partial J} \right].
\]
We will further combine the kinetic and the source term to
\[
\left( \frac{H_{nm}}{2} \Phi_{nm} \Phi_{mn} - J_{nm} \Phi_{mn} \right)
= \frac{H_{nm}}{2} \left( \Phi_{nm} - \frac{J_{nm}}{H_{nm}} \right) \left( \Phi_{mn} - \frac{J_{mn}}{H_{mn}} \right) - \frac{J_{nm} J_{mn}}{2H_{nm}}
\]
for any \( n, m \). Transforming the variables \( \Phi_{nm} \mapsto \Phi'_{nm} = \Phi_{nm} + J_{nm} H_{nm} \) with obviously invariant measure \( D\Phi = D\Phi' \) leads to a very useful form of the partition function
\[
Z[J] = K \exp \left( -S_{\text{int}} \left[ \frac{1}{V} \frac{\partial}{\partial J} \right] \right) Z_{\text{free}}[J],
\]
with \( Z_{\text{free}}[J] := \exp \left( V \sum_{n,m=0}^{N} H_{nm}^{2} \Phi_{nm} \Phi_{mn} \right) \).

The partition function gives rise to a definition of expectation values. We are mainly interested in the connected expectation values of a theory
\[
\langle \Phi_{p_{1}q_{1}} \Phi_{p_{2}q_{2}} \cdots \Phi_{p_{N}q_{N}} \rangle_{c},
\]
which is the connected part of the full expectation value defined by
\[
\langle \Phi_{p_{1}q_{1}} \Phi_{p_{2}q_{2}} \cdots \Phi_{p_{N}q_{N}} \rangle := \frac{\int D\Phi \Phi_{p_{1}q_{1}} \Phi_{p_{2}q_{2}} \cdots \Phi_{p_{N}q_{N}} e^{-S[\Phi]}}{\int D\Phi e^{-S[\Phi]}}.
\]

The full expectation value is given in terms of the connected ones by
\[
\langle \Phi_{p_{1}q_{1}} \Phi_{p_{2}q_{2}} \cdots \Phi_{p_{N}q_{N}} \rangle = \sum_{\text{Partitions}} \langle \Phi_{p_{1}q_{1}} \cdots \Phi_{p_{N}q_{N}} \rangle_{c},
\]
where the sum over partitions is understood as a sum over all possible decompositions. An equivalent definition of the connected expectation value is obtained by the correspondence between \( \Phi \) and the derivatives wrt to \( J \)
\[
\frac{1}{V^{N}} \frac{\partial^{N}}{\partial J_{p_{1}q_{1}} \partial J_{p_{2}q_{2}} \cdots \partial J_{p_{N}q_{N}}} \log \frac{Z[J]}{Z[0]} \bigg|_{J=0} = \langle \Phi_{p_{1}q_{1}} \Phi_{p_{2}q_{2}} \cdots \Phi_{p_{N}q_{N}} \rangle_{c}.
\]

The numbers \( p_{i}, q_{j} \in \{0, \ldots, N\} \) give different types of restrictions to the expectation values. To get an understanding which \( p_{i}, q_{j} \)'s produce a non vanishing expectation value, and how an expectation value might look one would first look at the perturbative expansion.

### 2.1 Perturbation Theory

Perturbation theory is a mathematical method to approximate a result or a solution which can possibly not expressed exactly. Quantum field theory has a perturbative expansion which is graphically described by Feynman graphs and its corresponding Feynman rules. The approximation via the perturbative expansion fits tremendously well with experimental data. An example how far perturbation theory can run is shown in the electron
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g − 2 anomaly calculation. The complete 4th order computation was recently finished \cite{Lap17} in a long-term project. Incredible 891 4-loop QED diagrams contributed to the calculation.

Nonperturbative results are quite rare in QFT. One possibility are constructive QFT models in lower dimensions, which were successful in the past for D < 4 \cite{Riv14}. Another example is numerical lattice calculations on computer clusters \cite{MM97}, which gave in the last years great insights to nonperturbative QFT. However, constructing or determining exact results in 4 dimensions was not yet accomplished.

One natural question is whether the perturbative expansion is mathematically rigorous, and whether we can extract nonperturbative information from it. Take the following example as an analogue to the quartic interaction

\[ \int_{-\infty}^{\infty} dx e^{-ax^2 - \frac{\lambda}{4}x^4}, \]  

which is finite for any \( \lambda > 0 \). A closed result of (2.1.1) exists in terms of the modified Bessel function \( K_\alpha \)

\[ e^\frac{x^2}{2\lambda} \sqrt{\frac{\lambda}{x}} K_{\frac{1}{4}} \left( \frac{a^2}{2\lambda} \right) \]

for \( a \geq 0 \). The result is holomorphic in \( \lambda \) in a certain domain, where \( \lambda = 0 \) lies on the boundary of the holomorphicity domain. For the perturbative approach we expand \( e^{-\frac{\lambda}{4}x^4} = \sum_{n=0}^{\infty} \frac{(-\frac{\lambda}{4})^n}{n!} x^{4n} \) at \( \lambda = 0 \) and naively exchange the order of the series and the integral

\[ \sum_{n=0}^{\infty} \frac{(-\frac{\lambda}{4})^n}{n!} \int_{-\infty}^{\infty} dx x^{4n} e^{-ax^2} = \frac{1}{\sqrt{a}} \sum_{n=0}^{\infty} \left( -\frac{\lambda}{4a^2} \right)^n \Gamma\left(\frac{4n+1}{2}\right) \frac{n!}{n!}. \]

By ratio test, the series has a vanishing convergence radius in \( \lambda \). Thus, the naive expansion does not reconstruct the exact result without further effort.

Borel summability addresses exactly this kind of problem. Let \( f(z) \) be a holomorphic function with formal power series \( \sum_{k=0}^{\infty} a_k z^k \) about \( z = 0 \). Define the Borel transform by \( B(t) = \sum_{k=0}^{\infty} \frac{a_k t^k}{k!} \) with nonvanishing convergence radius. Furthermore, suppose \( B(t) \) is well-defined for a neighbourhood of \( t \geq 0 \), then the integral

\[ \frac{1}{z} \int_{0}^{\infty} dt e^{-tz} B(t) \]

converges to \( f(z) \).

The assumption that the Borel transform can be continued analytically to the positive real line does not always hold and has therefore to be shown problem-specifically.

The hope is that certain QFT models are Borel summable. However, the perturbative expansion does not indicate Borel summability because the number of graphs (introduced later) grows at least with \( O(n!) \) and the amplitude of a graph can grow with \( O(n!) \) due to the renormalon problem as renormalisation artifact. Both properties appear also in the later considered matrix field theory models.

Nevertheless, an elegant way to classify the appearing integrals in the perturbative expansion was discovered by Richard Feynman: The usage of Feynman graphs and their associated Feynman rules.
2.1. PERTURBATION THEORY

2.1.1 Ribbon Graphs

A ribbon graph (not necessarily planar) consists of edges, vertices and faces. The end of an edge is either open or a vertex of degree \( d \geq 3 \). If it is open the edge is called open. A ribbon graph is called connected if any edges are connected. Ribbon graphs without open edges are called vacuum ribbon graphs. Each edge has two faces which can possibly be the same. A face is open if it is attached to an open edge, otherwise it is called closed. We identify two ribbon graphs if the edge-vertex connectivity and the vertex orientation is the same.

A ribbon graph can naturally be drawn on a Riemann surface, which is in particular a 2-dimensional orientable manifold. A Riemann surface is characterised by the genus \( g \), i.e. the number of handles, and the number \( b \) of boundary components (cycles). Cutting a Riemann surface in a finite number of polygons (triangles, quadrangles,..) is dual to the picture of ribbon graphs. A face of a polygon corresponds to a vertex of the ribbon graph, an edge of a polygon corresponds to an edge of the ribbon graph, and a vertex of the polygon corresponds to a face of the ribbon graph. For example, a triangulation corresponds to a ribbon graph with vertices only of degree 3.

A topological invariant of a Riemann surface is the Euler characteristic

\[
\chi = 2 - 2g - b. \tag{2.1.2}
\]

An open edge of a graph ends in a boundary component. Let \( e \) be the number of edges, \( f \) the number of faces and \( v \) the number of vertices, then the Euler characteristic is also given by (Euler’s formula)

\[
\chi = v + f - e. \tag{2.1.3}
\]

This formula originally holds for cutting a Riemann surface into polygons and counting the vertices, faces and edges after gluing. Due to the duality, this formula holds also for ribbon graphs since vertices and faces are exchanged \( v \leftrightarrow f \) under which the formula is invariant.

We will consider only the embedding of connected ribbon graphs with distinguished faces! Two different faces of the ribbon graph should also be distinguished after the embedding. By this assumption, it follows that an open face is attached to only one boundary component. If two open faces are attached to the same open edge, they correspond to the same boundary component. If two open faces \( F, F' \) are attached to the open edge \( E \), and \( F', F'' \) open are attached to the open edge \( E' \), then \( F \) and \( F'' \) correspond to the same boundary component. We define for that the infinite set of ribbon graphs with distinguished open faces which can be embedded into a Riemann surface

**Definition 2.2.** Let \( p_1^\beta, \ldots, p_{N_\beta}^\beta \) be for all \( \beta \in \{1, \ldots, b\} \) pairwise different. Let \( N_\beta \) be the number of open faces of the \( \beta \)th boundary component of a connected ribbon graph. Label the
a) The ribbon graph has two different faces (red and blue). It cannot be embedded into a Riemann surface of \( g = 1 \) and \( b = 1 \) since the embedding would identify both faces. This ribbon graph is embedded into a connected Riemann surface with \( g = 0 \) and \( b = 2 \).

b) The ribbon graph has two different faces (red and blue) which are clearly distinguished after the embedding. The Euler characteristic is \( \chi = -2 \) where two 3-valent vertices and two 4-valent vertices \( (v = 4) \), two faces (red and blue) \( (f = 2) \) and 9 edges \( (e = 9) \) are embedded into a Riemann surface with \( g = 1 \) and \( b = 2 \).

open faces of the \( \beta \)-th boundary component by the \( p^\beta_i \)'s, where two adjacent open faces are \( p^\beta_i \) and \( p^\beta_{i+1} \) in positive orientation, \( N^\beta + 1 \equiv 1 \). Then we define the set of connected ribbon graphs with labelled open faces which are naturally embedded in a Riemann surface with \( b \) boundary components and genus \( g \) by \( \Phi_{p_1^\beta p_2^\beta \ldots p_{N^\beta}^\beta}^{(g,b)} \).

2.1.2 Perturbative Expansion

Let \( \mathcal{N} + 1 \) be the size of the matrix. Consider for simplicity an interaction (potential) by \( \mathcal{V}(\Phi) = \frac{\lambda}{d} \Phi^d \) of degree \( d \) with \( d > 2 \). The perturbation theory follows from the naive expansion of \( e^{-V \operatorname{Tr}(\mathcal{V}(\Phi))} \). By definition (2.0.7), the numerator of the full expectation value is therefore given by

\[
\sum_{k=0}^{\infty} \frac{(-\lambda V)^k}{k!} \int D\Phi \Phi_{p_1 q_1 i} \cdots \Phi_{p_{N^\beta} q_{N^\beta} i} \left( \sum_{n_1 \ldots n_{p^\beta}} \Phi_{n_1 n_2} \cdots \Phi_{n_{d^\beta} n_{d^\beta}} \right)^k e^{-V \sum_{m,n} \frac{H_{nm}}{2} \Phi_{nm} \Phi_{mn}} \tag{2.1.4}
\]

The integral factorises into independent integrals of \( \Phi_{i j}^R \) and \( \Phi_{i j}^I \). Due to the Gaussian integral \( \int dx f(x)e^{-x^2} = 0 \) for \( f(x) \) odd and \( \int dx \, dy (x + iy)^2 e^{-x^2-y^2} = 0 \), the only nonvanishing contributions for (2.1.4) occur if the factor in front of the exponential consists of complex conjugated pairs of \( \Phi_{i j} \)'s. Considering again the denominator of the full expectation value (2.0.7), all nonvanishing contributions for any \( k \) in (2.1.4) are after factorisation
in $\Phi_{ij}^R$ and $\Phi_{ij}^I$ of the form
\[
\int_{-\infty}^{\infty} d\Phi_{pq}^R d\Phi_{pq}^I (\Phi_{pq} \Phi_{qp})^n e^{-V(H_{pq} \Phi_{pq} \Phi_{qp})} = \frac{n!}{(V H_{pq})^n}. \tag{2.1.5}
\]

We will call from now on $\frac{1}{H_{pq}} = \frac{1}{E_p + E_q}$ the free propagator.

The factorisation of (2.1.4) into a free propagator with the pair $\Phi_{ij}$ and $\bar{\Phi}_{ij} = \Phi_{ji}$ gives restrictions on the $n_4$ as well as on the $p_j$ and $q_j$. Integrating out a pair of complex conjugated $\Phi$’s is called Wick contraction. Since the interaction term has a cyclic order due to the trace, the $\Phi_{pq}$ are also forced to be of cyclic orders considering that nonvanishing integrals are necessarily of the form (2.1.5). Therefore, a nonvanishing expectation value of $b$ cycles, each of length $N_\beta$ with $\beta \in \{1, \ldots, b\}$, has necessarily the form
\[
\langle \Phi_{p_1 p_2} \Phi_{p_3 p_4} \cdots \Phi_{p_{N_1} p_{N_2}} \Phi_{p_1 p_2} \cdots \Phi_{p_{N_1} p_{N_2}} \rangle
\]
and can be completely separated into "Wick-contractible" integrals order by order in $\lambda$.

**Example 2.1.** Assume a quartic interaction $d = 4$ and the expectation value $\langle \Phi_{pq} \Phi_{qp} \rangle$ at order $\lambda^4$ with $q \neq p$. The following four contractions are possible
\[
\int D\Phi \Phi_{pq} \Phi_{qp} \sum_{n_1, n_2, n_3, n_4} \Phi_{n_1 n_2} \Phi_{n_2 n_3} \Phi_{n_3 n_4} \Phi_{n_4 n_1} e^{-V \sum_{n, m} \frac{H_{nm}}{2} \Phi_{nm} \Phi_{mn}} \sim \frac{1}{H_{pq} \sum_{n_1, n_2, n_4} 1 \overline{H_{n_1 n_2}}} \nonumber
\]
\[
\int D\Phi \Phi_{pq} \Phi_{qp} \sum_{n_1, n_2, n_3, n_4} \Phi_{n_1 n_2} \Phi_{n_2 n_3} \Phi_{n_3 n_4} \Phi_{n_4 n_1} e^{-V \sum_{n, m} \frac{H_{nm}}{2} \Phi_{nm} \Phi_{mn}} \sim \frac{1}{H_{pq} \sum_{n_1, n_2, n_4} 1 \overline{H_{n_1 n_2}}} \nonumber
\]
\[
\int D\Phi \Phi_{pq} \Phi_{qp} \sum_{n_1, n_2, n_3, n_4} \Phi_{n_1 n_2} \Phi_{n_2 n_3} \Phi_{n_3 n_4} \Phi_{n_4 n_1} e^{-V \sum_{n, m} \frac{H_{nm}}{2} \Phi_{nm} \Phi_{mn}} \sim \frac{1}{H_{pq} \sum_{n_1, n_2, n_4} 1 \overline{H_{n_1 n_2}}} \nonumber
\]
\[
\int D\Phi \Phi_{pq} \Phi_{qp} \sum_{n_1, n_2, n_3, n_4} \Phi_{n_1 n_2} \Phi_{n_2 n_3} \Phi_{n_3 n_4} \Phi_{n_4 n_1} e^{-V \sum_{n, m} \frac{H_{nm}}{2} \Phi_{nm} \Phi_{mn}} \sim \frac{1}{H_{pq} \sum_{n_1, n_2, n_4} 1 \overline{H_{n_1 n_2}}} \nonumber
\]
where the linked $\Phi$’s are Wick-contracted. It means for example that in the fourth line $n_1 = n_3 = p$ and $n_2 = n_4 = q$. Wick’s Theorem says that all possible contractions give a contribution to the expectation value of a certain order. Note that not all contractions give nonvanishing results
\[
\int D\Phi \Phi_{pq} \Phi_{qp} \sum_{n_1, n_2, n_3, n_4} \Phi_{n_1 n_2} \Phi_{n_2 n_3} \Phi_{n_3 n_4} \Phi_{n_4 n_1} e^{-V \sum_{n, m} \frac{H_{nm}}{2} \Phi_{nm} \Phi_{mn}}
\]
where $n_1 = n_4 = q$ and $n_2 = n_3 = p$. The last contraction is for $q \neq p$ not possible such that this integral vanishes.

**Example 2.2.** Assume a quartic interaction $d = 4$ and the expectation value $\langle \Phi_{pp} \Phi_{qq} \rangle$ at order $\lambda^4$ with $q \neq p$. We have one possible contraction
\[
\int D\Phi \Phi_{pp} \Phi_{qq} \sum_{n_1, n_2, n_3, n_4} \Phi_{n_1 n_2} \Phi_{n_2 n_3} \Phi_{n_3 n_4} \Phi_{n_4 n_1} e^{-V \sum_{n, m} \frac{H_{nm}}{2} \Phi_{nm} \Phi_{mn}} \sim \frac{1}{H_{pq} H_{pp} H_{qq}}.
\]
Both examples together show that if two or more $p_i$’s coincide, we will have a degenerate case:
\[
\langle \Phi_{pp} \Phi_{pp} \rangle = \langle \Phi_{pq} \Phi_{qp} \rangle \Big|_{q=p} + \langle \Phi_{pp} \Phi_{qq} \rangle \Big|_{q=p}.
\]
Assuming pairwise different $p_i$’s avoids this problem.
2.1.3 Feynman Rules

Associate to each $\Phi_{nm}$ of the integrand a half edge (ribbon) either open or connected. The left face is labelled by the first index and the right face with the second one. Associate a Wick contraction to a full edge (ribbon) by connecting two half edges (ribbons) with coinciding faces. Associate to any interaction term a vertex of degree $d$ with half edges labelled by $\Phi_{n_1n_2}\cdots\Phi_{n_dn_1}$.

Example 2.1 is associated to the following four graphs

The first two graphs are disconnected graphs and include vacuum graphs, respectively. The second vacuum graph is of genus $g = 1$.

Example 2.2 is associated to the following graph

The aforementioned consideration gives a map $h$ called Feynman rules from the set $\mathcal{G}^{(g,b)}_{p_1,p_2,\ldots,p_N}$ (set of Feynman graphs) to all terms appearing in the perturbative expansion.
of the connected expectation value \( \langle \Phi_{p_1 p_2} \Phi_{p_2 p_3} \cdots \Phi_{p_{N_1} p_1} \cdots \Phi_{p_{N_b} p_1} \rangle_c \) with pairwise different \( p_i \)'s by:

- An edge labelled by \( p, q \) corresponds to the factor \( \frac{1}{V_{pq}} \)
- A vertex of degree \( d \) corresponds to the factor \( -\lambda_d V \)
- Take a sum \( \sum_n \) over all closed faces \( n \)

An example appearing in the expectation value \( \langle \Phi_{p_1 p_2} \Phi_{p_2 p_3} \cdots \Phi_{p_{N_1} p_1} \cdots \Phi_{p_{N_b} p_1} \rangle_c \) of genus \( g = 1 \) and two boundary components with three open faces and one closed face, therefore an element of \( \mathfrak{S}_{p_1, p_2, \ldots, p^2} \), is

\[
\begin{pmatrix}
\begin{array}{c}
 p_1 \\
p_2
\end{array}
\end{pmatrix}
= \frac{-V^7 \lambda_3^5 \lambda_4^2}{V^{13} H_{p_1 p_2} H_{p_2 p_3} H_{p_3 p_4} H_{p_4 p_5} \sum_n \frac{1}{H_{p_1 n}}}
\]

Summing over all possible contractions (Wick’s Theorem) corresponds to summing over all Feynman graphs. We conclude:

**Proposition 2.1.** The formal expansion of a connected expectation value with pairwise different \( p_i \)'s is

\[
\langle \Phi_{p_1 p_2} \Phi_{p_2 p_3} \cdots \Phi_{p_{N_1} p_1} \cdots \Phi_{p_{N_b} p_1} \rangle_c = \sum_{g=0}^{\infty} \sum_{g \in \mathfrak{S}_{p_1, p_2, \ldots, p^2}} \mathfrak{h}(g).
\]

The full expectation value is given by the same formula, where the sum \( \sum_g \) is taken over all not necessarily connected ribbon graphs, since the map \( \mathfrak{h} \) has the property

\[
\mathfrak{h}(g \cup g') = \mathfrak{h}(g) \mathfrak{h}(g').
\]

Due to this property, all vacuum graphs are removed from the perturbative expansion of the expectation values \([2.0.7]\). The vacuum graphs factor out in the numerator in the formal expansion, whereas the denominator produces only vacuum graphs.

**Remark 2.2.** The Hermitian 1-matrix model \( (E = \text{const.}) \) has a much bigger set of graphs in its perturbative expansion. The expectation values are defined differently by \( \langle \text{Tr} \Phi^{k_1} \text{Tr} \Phi^{k_2} \cdots \text{Tr} \Phi^{k_n} \rangle \), where the trace gives also contributions if two or more \( p_i \)'s are equal. The degenerate case is therefore automatically included. This means from the graphical point of view that an open face can correspond to different boundary components. Furthermore,
two faces can have the same boundary component even if they are not connected along the edges.

The expectation value $\langle \text{Tr}\Phi^2 \rangle = \langle \sum_{n,m} \Phi_{nm} \Phi_{mn} \rangle$ has for genus 1 for all $m = n$ a contribution from the graph (different in comparison to Fig. 2.1 a) coming from the degenerated quadrangulation.

\[ 2.2 \text{ Schwinger-Dyson Equation and Ward-Takahashi Identity} \]

To have a convergent $N \to \infty$ limit an appropriate scaling in the parameter $V$ is necessary. We assume that the sum $\sum^N$ and the parameter $V$ are of the same order $N \sum \sim V$.

Let the interaction be $V(\Phi) = \sum_{k=3}^d \frac{\lambda_k}{k} \Phi^k$. We conclude from the perturbative expansion of $\langle \Phi_{p_1} \Phi_{p_2} \Phi_{p_3} q_1 \Phi_{p_4} \Phi_{p_5} \Phi_{p_6} \Phi_{p_7} \Phi_{p_8} \rangle_c$ that a graph of order $k_i$ in $\lambda_i$ has the factor

\[ \frac{V^{k_3+k_4+\ldots+k_d}}{V^{N+3k_3+4k_4+\ldots+dk_d}}, \tag{2.2.1} \]

where $N = N_1 + N_2 + \ldots + N_b$. On the other hand, Euler’s formula gives

\[ 2 - 2g - b = k_3 + \ldots + k_d + N + \sum - \frac{N + k_33 + \ldots + k_dd}{2}, \]

\[ \Rightarrow \quad \Sigma = 2 - 2g - b - (k_3 + \ldots + k_d + N) + \frac{N + 3k_3 + \ldots + dk_d}{2}, \]

where $\Sigma$ is the number of closed faces. Since the sum is taken over all closed faces and any sum will be of order $V$, we have to multiply (2.2.1) by $V^\Sigma$ and achieve the scaling factor

\[ V^{2-2g-b-N} \]

of $\langle \Phi_{p_1} \Phi_{p_2} \Phi_{p_3} q_1 \Phi_{p_4} \Phi_{p_5} \Phi_{p_6} \Phi_{p_7} \Phi_{p_8} \rangle_c$ being independent of the perturbative expansion. This gives rise to the definition of the correlation function which is finite in the limit $\lim_{V \to \infty} \sum^N \to \text{finite}$

\[ G_{p_1 \ldots p_{N_1} ; q_1 \ldots p_{N_b} } := V^{b+N_1+\ldots+N_b-2} \langle \Phi_{p_1} \Phi_{p_2} \Phi_{p_3} \Phi_{p_4} \Phi_{p_5} \Phi_{p_6} \Phi_{p_7} \Phi_{p_8} \rangle_c. \tag{2.2.2} \]
where $p_i^j$’s are pairwise different. We will call $G_{[p_1^1, \ldots, p_{N_1}^1], \ldots, [p_1^b, \ldots, p_{N_b}^b]}$ a $(N_1 + \ldots + N_b)$-point function which have a formal genus $g$-expansion

$$G_{[p_1^1, \ldots, p_{N_1}^1], \ldots, [p_1^b, \ldots, p_{N_b}^b]} := \sum_{g=0}^{\infty} V^{-2g} G^{(g)}_{[p_1^1, \ldots, p_{N_1}^1], \ldots, [p_1^b, \ldots, p_{N_b}^b]}$$  \hspace{1cm} (2.2.3)$$

defining the $(N_1 + \ldots + N_b)$-point function of genus $g$. The $b$ different boundary components are separated by vertical lines in the index of $G$. The definition (2.2.2) has the benefit that if two or more $p_i^j$’s coincide, the degeneracy is separated by the correlation functions. Take the earlier discussed example

$$G_{[p]} = V \langle \Phi_{pq} \Phi_{qp} \rangle |_{q=p}$$
$$G_{[p]} = V^2 \langle \Phi_{pp} \Phi_{qq} \rangle |_{q=p}$$
$$\langle \Phi_{pp}^2 \rangle_c = \frac{1}{V} G_{[p]} + \frac{1}{V^2} G_{[p]}.$$  

A more convenient form of the definition uses the correspondence between $\Phi$ and $J$-derivatives which shows that the partition function generates all correlation functions in powers of $J$

$$\log \frac{Z[J]}{Z[0]} = \sum_{b=1}^{\infty} \sum_{N_b=0}^{\infty} \sum_{\beta=1}^{\prod_{i=1}^{N_b} J_{p_i^b}} \sum_{g=0}^{\infty} V^{2-b-2g} G^{(g)}_{[p_1^1, \ldots, p_{N_1}^1], \ldots, [p_1^b, \ldots, p_{N_b}^b]} \prod_{\beta=1}^{b} \frac{1}{N_{\beta}}.$$  \hspace{1cm} (2.2.4)$$

The shorthand notation $\prod_{i=1}^{N_b} J_{p_i^b}$ with $N_b + 1 \equiv 1$ is used. The symmetry factor $\frac{1}{N_b}$ arises since a correlation function is by definition (2.2.2) invariant under a cyclic permutation in each boundary component labelled by $\beta$. The symmetry factor $\frac{1}{N_b}$ arises since a correlation function is by definition (2.2.2) symmetric under changing two boundary components $\beta \leftrightarrow \beta'$ and therefore under changing any boundary component.

Assuming pairwise different $p_i^j$’s a correlation function is directly extracted by partial derivatives from equation (2.2.4) at $J = 0$:

$$V^{b-2} \frac{\partial^{N_1 + \ldots + N_b}}{\partial J_{p_1^1} \ldots \partial J_{p_{N_b}^b}} \log \frac{Z[J]}{Z[0]} |_{J=0} = G_{[p_1^1, \ldots, p_{N_1}^1], \ldots, [p_1^b, \ldots, p_{N_b}^b]},$$

However, the interesting operation appears if the $J$-derivatives act on $\frac{Z[J]}{Z[0]}$ or have coinciding $p_i^j$’s. The first non-trivial example follows easily from (2.2.4):

$$\frac{\partial^2}{\partial J_{pp} \partial J_{pp}} \left( \frac{Z[J]}{Z[0]} \right) |_{J=0} = V G_{[pp]} + G_{[pp]} + V^2 G_{[p]} G_{[p]}$$

$$= \sum_{g=0}^{\infty} V^{-2g} \left( V G_{[pp]}^{(g)} + G_{[pp]}^{(g)} + V^2 \sum_{h+h'=g} G_{[p]}^{(h)} G_{[p]}^{(h')} \right)$$

$$= V^2 \langle \Phi_{pp} \Phi_{pp} \rangle_c + V^2 \langle \Phi_{pp}^2 \rangle_c = V^2 \langle \Phi_{pp} \Phi_{pp} \rangle_c.$$
Lemma 2.1. Let \( f(x) \) be a polynomial in \( x \), \( g(x) \) smooth and \( \partial_x = \frac{\partial}{\partial x} \). Define the operator 
\[
\exp(f(\partial_x)) := \sum_{k=0}^{\infty} \frac{f(\partial_x)^k}{k!}
\]
then we have 
\[
e^{f(\partial_x)}(x \cdot g(x)) = xe^{f(\partial_x)}g(x) + f'(\partial_x)e^{f(\partial_x)}g(x).
\]

Proof. Expanding \( e^{f(\partial_x)} \) by the definition and using the Leibniz rule to have 
\[
[f(\partial_x)^k, x] = kf'(\partial_x)f(\partial_x)^{k-1}
\]
gives the rhs after shifting the index \( k \rightarrow k + 1 \). \( \square \)

Applying Lemma 2.1 with \( f = S_{\text{int}} \) creates the derivative of \( S_{\text{int}} \) by 
\[
\left( \frac{\partial S_{\text{int}}}{\partial \Phi_{pq}} \right) \left[ \frac{1}{V} \frac{\partial}{\partial J} \right] = \sum_{k=3}^{d} \frac{\lambda_k}{V^{k-2}} \sum_{n_1, \ldots, n_k-2=0}^{N} \frac{\partial^{k-1}}{\partial J_{p_{n_1}} \cdots \partial J_{n_{k-2}q}}. \tag{2.2.5}
\]

The SDE is established for pairwise different \( p_i \)’s by 
\[
G_{[p_1, p_1, \ldots, p_1, p_h]} = V^{b-2}2 \sum_{k=3}^{d} \frac{\lambda_k}{V^{k-2}} \sum_{n_1, \ldots, n_k-2=0}^{N} \frac{\partial^{k-1}}{\partial J_{p_{n_1}} \cdots \partial J_{n_{k-2}q}} \log \frac{Z[J]}{Z[0]} \bigg|_{J=0} \bigg| = \frac{\delta_{k,1} \delta_{N_1,2}}{H_{p_1 | p_1}} - \frac{\partial S_{\text{int}}}{\partial \Phi_{p_1 | p_1}} \frac{1}{V} \frac{\partial}{\partial J} \frac{Z[J]}{\prod_{j=1}^{N_b}} \bigg|_{J=0} \bigg| = \delta_{k,1} \delta_{N_1,2} \prod_{j=1}^{N_b} \frac{\partial^{N_1+..+N_b}}{\partial J_{p_1} \cdots \partial J_{p_1} \cdots \partial J_{N_b} \cdots \partial J_{N_b}} \quad \text{and} \quad N_\beta + 1 = 1 \tag{2.2.6}
\]

The second line is achieved by deriving \( Z_{\text{free}} \) of (2.0.5) with respect to \( J_{p_1 | p_1} \) and third line by Lemma 2.1 as well as considering \( J = 0 \). Recall that \( \frac{\partial S_{\text{int}}}{\partial \Phi_{pq}} \) is of degree \( d \) such that the SDE (2.2.6) has for instance on the rhs \( ((N_1 + d - 2) + N_2 + \ldots + N_b) \)-point functions with \( d - 1 \)-summations, or different types of nonlinear terms if one or more summation indices coincide with other indices (degenerate case). The SDEs form a tower of equations. Writing precisely the first equations shows that the 1-point function depends on the 2-point, the 2-point on the 3-point and so on. The tower of equations does not decouple and is therefore inefficient to determine correlation functions explicitly.

Notice that equation (2.2.6) has a certain base point \( J_{p_1 | p_1} \). The two variables \( p_1 \) and \( p_2 \) play a special rôle. The resulting correlation function, however, obeys the cyclic symmetry in each boundary and an additional symmetry between the boundaries. In other words, a highly symmetric function fulfils an equation which is non-symmetric in one of its variables.

Nevertheless, a decoupling of the tower is possible and achieved in the large \( V \)-limit and by the Ward-Takahashi identity arising from a symmetry transformation of the partition function \( Z[J] \), first derived in [DGMR07]. Transform the integration variables \( (\Phi_{nm}) \) of the partition function (2.0.4) by a unitary transformation \( \Phi \mapsto \Phi' = U \Phi U^\dagger \) with \( U \in U(N + 1) \). The partition function is invariant under this transformation since the property of Hermitian matrices is preserved. From the invariant measure 
\[
\det \frac{\partial U \Phi U^\dagger}{\partial \Phi} = (\det U \det U^\dagger)^{N+1} = 1,
\]

it follows 
\[
0 = \int D\Phi \exp(-S[\Phi] + V \text{Tr}(J\Phi')) - \exp(-S[\Phi'] + V \text{Tr}(J\Phi')). \tag{2.2.7}
\]
2.2. SCHWINGER-DYSON EQUATION AND WARD-TAKAHASHI IDENTITY

Expanding the unitary transformation about the identity \( U = I + i\varepsilon A + \mathcal{O}(\varepsilon^2) \) gives the transformed matrix \( \Phi' = \Phi + i\varepsilon(A\Phi - \Phi A) + \mathcal{O}(\varepsilon^2) \) and finally, at order \( \varepsilon^1 \),

\[
0 = \int D\Phi \left( E\Phi^2 - \Phi^2 E - J\Phi + \Phi J \right)_{pq} \exp(-S[\Phi] + V \text{Tr}(J\Phi)). \tag{2.2.8}
\]

Note that the interaction is invariant under unitary transformation \( S_{\text{int}}[\Phi] = S_{\text{int}}[\Phi'] \), whereas the kinetic term and the source term are not. Since (2.2.8) vanishes for any interaction \( S_{\text{int}} \), the factor in front of the exponential has to vanish. We obtain after applying the correspondence between \( \Phi \) and \( J \)-derivatives

**Proposition 2.2.** ([DGMR07]) The partition function \( Z[J] \) obeys the Ward-Takahashi identity

\[
0 = \sum_{n=0}^{N} \left( \frac{E_p - E_q}{V} \frac{\partial^2}{\partial J_{qn} \partial J_{np}} + J_{nq} \frac{\partial}{\partial J_{np}} - J_{pn} \frac{\partial}{\partial J_{qn}} \right) Z[J].
\]

The Ward-Takahashi identity reduces a second-order derivative together with a sum over the intermediate index to a first-order derivative with an additional factor in \( J \). Precisely this reduction can decouple the tower of SDEs in the limit \( N, V \to \infty \). We emphasise that the decoupling is possible for an arbitrary interaction term \( S_{\text{int}} \). The Ward-Takahashi identity has to be applied therefore possibly several times.

We can further generalise Proposition 2.2 by a direct calculation instead of unitary transformation (observed in [HW18])

**Proposition 2.3.** The partition function \( Z[J] \) obeys the generalised Ward-Takahashi identity

\[
0 = \sum_{n=0}^{N} \left( \frac{\partial S_{\text{int}}}{\partial \Phi_{pq}} \left[ \frac{1}{V} \frac{\partial}{\partial J_{pq}} \right] \frac{\partial}{\partial J_{np}} - \left( \frac{\partial S_{\text{int}}}{\partial \Phi_{pn}} \left[ \frac{1}{V} \frac{\partial}{\partial J_{pq}} \right] \frac{\partial}{\partial J_{qn}} \right) \right) Z[J].
\]

**Proof.** Direct computation leads to

\[
\frac{E_p - E_q}{V} \frac{\partial^2}{\partial J_{qn} \partial J_{np}} Z[J]
\]

\[
= \frac{1}{V} \frac{\partial^2}{\partial J_{qn} \partial J_{np}} \left( (E_p + E_n) - (E_n + E_q) \right) Z[J]
\]

\[
= \frac{1}{V} \frac{\partial^2}{\partial J_{qn} \partial J_{np}} \left( -S_{\text{int}} \left[ \frac{1}{V} \frac{\partial}{\partial J_{pq}} \right] J_{np} - \frac{\partial}{\partial J_{np}} \exp \left( -S_{\text{int}} \left[ \frac{1}{V} \frac{\partial}{\partial J_{pq}} \right] \right) J_{nq} \right) Z_{\text{free}}[J]
\]

\[
= \left( J_{pn} \frac{\partial}{\partial J_{qn}} - J_{nq} \frac{\partial}{\partial J_{np}} \right) Z[J] - \left( \frac{\partial S_{\text{int}}}{\partial \Phi_{pq}} \left[ \frac{1}{V} \frac{\partial}{\partial J_{pq}} \right] \frac{\partial}{\partial J_{qn}} + \frac{\partial S_{\text{int}}}{\partial \Phi_{qn}} \left[ \frac{1}{V} \frac{\partial}{\partial J_{pq}} \right] \frac{\partial}{\partial J_{np}} \right) Z[J],
\]

where we used the partition function of the form of (2.0.5). The third line is achieved by acting with \( \frac{\partial}{\partial J_{np}} \) on \( Z_{\text{free}} \) for the first term and with \( \frac{\partial}{\partial J_{qn}} \) for the second term. For the last line, Lemma 2.1 was applied.

The Ward-Takahashi identity of Proposition 2.2 is recovered by taking the sum of the generalised identity of Proposition 2.3 over \( n \).
Both propositions have a problem, the prefactor \((E_p - E_q)\) cannot be divided out without any more ado. The decomposition of \(\mathcal{Z}\) into correlation functions is degenerate for coinciding indices.

Let us assume that two eigenvalues are distinct \(E_p \neq E_q\) for \(p \neq q\), which seems to be a strong assumption. However, all correlation functions depend on the \(E_p\)'s continuously according to the perturbative expansion. A sum over all eigenvalues can therefore be partitioned into a sum over distinct eigenvalues associated with a discrete measure governed their multiplicities. Including an appropriate measure therefore covers the assumption of distinct eigenvalues (see [Sec. 2.3]).

The distinct eigenvalues \(E_p\) with \(E_p < E_{p+1}\) can be understood as a discretisation of a continuously differentiable function. Furthermore, correlation functions are at least perturbatively rational functions of \(\mathcal{Z}\) with action \(S[\Phi] = \sum \text{Tr}(EB\Phi + V(\Phi))\), where \(E_{nm} = E_n \delta_{nm}\) is diagonal and \(V(\Phi)\) a polynomial, satisfies the Ward-Takahashi identity

\[
\sum_{n=0}^{N} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{qn} \partial J_{np}} = \delta_{p,q} (W_p^1[J] + W_p^2[J]) \mathcal{Z}[J] + \frac{V}{E_p - E_q} \sum_{n=0}^{N} \left( J_{pn} \frac{\partial}{\partial J_{qn}} - J_{nq} \frac{\partial}{\partial J_{np}} \right) \mathcal{Z}[J],
\]

\[
W_p^2[J] := \sum_{b=1}^{\infty} \sum_{N_1,..,N_b=1}^{\infty} \sum_{\beta=1}^{N_1} \sum_{g=0}^{\infty} \frac{V^{2-b-2g}}{b!} \prod_{\beta=1}^{N_1} \frac{1}{N_\beta} \times \left( \frac{1}{V^2} G_{[p\beta]\cdot}^{(g)} \right) + \frac{1}{V} \sum_{n=0}^{N} G_{[p\beta]\cdot}^{(g)} \mathcal{Z}[J],
\]

\[
W_p^1[J] := \sum_{b',b=1}^{\infty} \sum_{N_1,..,N_b=1}^{\infty} \sum_{\beta=1}^{N_1} \sum_{g'=0}^{\infty} \frac{V^{4-2b'-2(g'+g')}}{b'b!} \prod_{\beta=1}^{N_1} \frac{1}{N_\beta} \prod_{\beta'=1}^{N_1} \frac{1}{M_{\beta'}} \times \left( \frac{1}{V^2} G_{[\beta\beta]\cdot}^{(g')} \right) + \frac{1}{V} \sum_{n=0}^{N} G_{[p\beta]\cdot}^{(g')} \mathcal{Z}[J].
\]

The idea of the proof consists of writing

\[
\frac{\partial^2 \mathcal{Z}[J]}{\partial J_{qn} \partial J_{np}} = \mathcal{Z}[J] \left( \frac{\partial^2 \log \mathcal{Z}[J]}{\partial J_{qn} \partial J_{np}} \right) + \frac{\partial \log \mathcal{Z}[J]}{\partial J_{qn}} \left( \frac{\partial \log \mathcal{Z}[J]}{\partial J_{np}} \right).
\]

Then each part is separated in

\[
\frac{\partial^2 \log \mathcal{Z}[J]}{\partial J_{qn} \partial J_{np}} = \delta_{pq} W_p^2[J] + W_{pq}^{2,reg}[J],
\]

\[
\left( \frac{\partial \log \mathcal{Z}[J]}{\partial J_{qn}} \right) \left( \frac{\partial \log \mathcal{Z}[J]}{\partial J_{np}} \right) = \delta_{pq} W_p^1[J] + W_{pq}^{1,reg}[J].
\]
where $\delta_{p,q} W^i_p$ contains the degenerate terms coming from $q = p$. The $W^{i,\text{reg}}$ is the remaining part regular in the limit $q \to p$. The function $\log Z[J]$ generates the correlation functions through (2.0.5) such that the first term in $W^2_p$ is generated by $n = p$. The second in $W^2_p$ is the only 2-cycle and the third consists of all possible cycles of higher length. The quadratic term $W^1_p$ is generated if $q = n = p$. The regular terms obey by definition for $q \neq p$

$$\sum_{n=0}^{N'} \left( W^{2,\text{reg}}_{pq}[J] + W^{1,\text{reg}}_{pq}[J] \right) Z[J] = \frac{V}{E_p - E_q} \sum_{n=0}^{N'} \left( J_{pn} \frac{\partial}{\partial J_{qn}} - J_{np} \frac{\partial}{\partial J_{qn}} \right) Z[J]$$

with a unique limit $q \to p$. This limit is valid by extending $E_n$ to a differentiable function such that all correlation functions are differentiable in its indices and L'Hôpital's rule is applicable.

The SDE (2.2.6) consists of $k-1$ sums with $k \in \{3, ..., d\}$. Applying Theorem 2.1 $\left\lfloor \frac{k-1}{2} \right\rfloor$ times for each term in $\left( \frac{\partial S_{\text{int}}}{\partial \Phi |_{r_1]} \right)$ leads from a $((N_1 + k - 2) + N_2 + ... + N_b)$-point function to a $(N_1 + N_2 + ... + N_b)$-point function plus additional terms of different topologies. In the limit $N', V \to \infty$ and an additional topological expansion $G = \sum_g V^{-2g} G^{(g)}$, all equations decouple in a way that a correlation function of Euler characteristic $\chi$ obeys a linear equation, where the inhomogeneity does only depend on correlation functions of Euler characteristic $\chi' > \chi$.

### 2.3 Large $N', V$-Limit

As mentioned before, we assume that the sum and the parameter $V$ are of the same order

$$\sum_{n=0}^{N'} \sim V.$$ 

To be precise, let $(e_k)_{k=0}^{N'}$, $N' \leq N$ be the ordered distinct eigenvalues of the projection of $E$. Let $r_k$ be the multiplicity of $e_k$ of $E$. This means

$$E_0 = E_1 = ... = E_{r_0-1} = e_0$$
$$E_{r_0} = ... = E_{r_0+r_1-1} = e_1$$

$$\vdots$$

Let $f$ be a function depending on the eigenvalues $E_n$ and not on the multiplicities. We obtain for the sum

$$\frac{1}{V} \sum_{n=0}^{N} f(E_n) = \frac{1}{V} \sum_{k=0}^{N'} r_k f(e_k) = \int_0^\infty dt \ g(t) \ f(t) \quad (2.3.1)$$

where

$$g(t) = \frac{1}{V} \sum_{k=0}^{N'} r_k \delta(t - e_k). \quad (2.3.2)$$

The distinct eigenvalues $e_k$ are extended to a continuous differentiable function, where $e_k$ are discrete points. It is more convenient to perform the limit of $N'$ which indicates the limit of $N$. In the limit $N' \to \infty$ we assume the asymptotic behaviour

$$\lim_{k \to \infty} r_k \sim k^{\delta-1}$$
The limit $V \to \infty$ is adjusted such that the summation index $k$ converges to a continuous variable $x$ depending on the asymptotic behavior of $r_k$, namely

$$\lim_{N',V \to \infty} \frac{k}{V^{\frac{1}{2}}} \to x.$$  \hspace{1cm} (2.3.3)

From physical motivation, we want to denote the smallest eigenvalue by $e_0 = \frac{\mu^2}{2}$, where $\mu$ is called the mass. Take the monotonic, continuously differentiable functions $e(x)$ with $e(0) = 0$ and $r(x)$ from the multiplicities by

$$e \left( \frac{k}{V^{\frac{1}{2}}} \right) := e_k - \frac{\mu^2}{2}, \quad \lim_{N',V \to \infty} e \left( \frac{k}{V^{\frac{1}{2}}} \right) = e(x) \sim x \to \infty x$$  \hspace{1cm} (2.3.4)

$$r \left( \frac{k}{V^{\frac{1}{2}}} \right) := \frac{r_k}{V^{1-\frac{1}{2}}}, \quad \lim_{N',V \to \infty} r \left( \frac{k}{V^{\frac{1}{2}}} \right) = r(x) \sim x \to \infty x^{\delta-1},$$  \hspace{1cm} (2.3.5)

which are unique in the upper limit. The measure $g(t)$ converges to

$$g(t) = \lim_{N',V \to \infty} \frac{1}{V} \sum_{k=0}^{N'} r_k \delta(t - e_k) = \lim_{N',V \to \infty} \frac{1}{V^{\frac{1}{2}}} \sum_{k=0}^{N'} \frac{k^{\delta-1}}{V^{\frac{1}{2}}} \delta \left( t - \frac{\mu^2}{2} - e \left( \frac{k}{V^{\frac{1}{2}}} \right) \right)$$

$$= \int_{-\epsilon}^{\infty} dx \ r(x) \delta \left( t - \frac{\mu^2}{2} - e(x) \right) = \frac{r(-1)(t - \frac{\mu^2}{2})}{e(-1)(t - \frac{\mu^2}{2})},$$

which is equivalently to

$$g \left( \frac{\mu^2}{2} + e(x) \right) \ dx = r(x) \ dx.$$  

Finally, we find that the sum (2.3.1) converges in the limit (2.3.3) to

$$\lim_{N,V} \frac{1}{V} \sum_{n=0}^{N} f(E_n) = \int_{0}^{\infty} dx \ r(x) f \left( \frac{\mu^2}{2} + e(x) \right),$$

where the upper limit has its origin in the unbounded property of $E$. The last step is now to determine the spectral dimension $\mathcal{D}$ of Definition 2.1. Inserting the spectral measure $g(t)$ into (2.0.2) gives

$$\int_{0}^{\infty} dt \ \frac{g(t)}{(1 + t)^{p/2}} = \int_{0}^{\infty} dx \ \frac{r(x)}{(1 + \frac{\mu^2}{2} + e(x))^{p/2}}.$$

The asymptotic $r(x) \sim x^{\delta-1}$ and $e(x) \sim x$ provides the spectral dimension (Definition 2.1) if the integrand behaves asymptotically with $\frac{1}{x}$ such that

$$-1 = \delta - 1 - \frac{\mathcal{D}}{2} \Rightarrow \mathcal{D} = 2\delta.$$

**Remark 2.3.** The assumption that $e(x)$ behaves asymptotically linear is easily substantiated. Assume first the asymptotics $e(x) \sim x^k$ and $r(x) \sim x^{\delta-1}$ for large $x$ and transform the variable $x$ for the integral by $y = x^k$ and $k y^{\frac{1}{k}-1} \ dx = dy$. The asymptotic in $y$ is $e(x(y)) \sim y$ and $r(x(y)) \sim y^{\frac{1}{k}-1}$. Defining then $\tilde{e}(y) := e(x(y))$ and $\tilde{r}(y) := \frac{r(x(y))}{ky^{\frac{1}{k}-1}}$ with $\tilde{r}(y) \sim y^{\frac{\delta}{k}-1}$ satisfies the initial assumption by an adjusted measure $\tilde{r}$ with $\tilde{\delta} = \frac{\delta}{k}$. Note the modification of the spectral dimension to $\mathcal{D} = 2\frac{\delta}{k}$. 

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The next step is to understand the correlation function of genus \( g \) as a discretisation of a continuously differentiable function by

\[
G^g(x_1, x_2, ..., x_{N_1}, |x_1, ..., x_{N_2}|, |x_1|, ..., x_{N_b}) = G_{|p_1, ..., p_{k_1}| |p_{k_1}, ..., p_{k_b}| |p_j = x_j V^{2/D}}^g(2.3.6).
\]

We construct, in the same manner as the functions \( e(x) \) and \( r(x) \), the continuous function \( G^g \) which is unique in the upper limit.

For later purpose, we stronger assume that \( G^g \) is Hölder-continuous, i.e. \( \exists C > 0 \) such that

\[
|G^g(x, .., y, ..) - G^g(x, .., y, ..)| < C|x - y|^{\alpha}
\]

for all \( x, y \in U \subset \mathbb{R}_+ \) in every component of \( G^g \) with \( 0 < \alpha \leq 1 \).

### 2.4 Renormalisation

A sum converges in perturbative expansion in the large \( N, V \)-limit to an integral over all closed face variables. The measure \( \varrho(t) \) depends asymptotically on the spectral dimension which naturally leads in dimensions higher than zero to infinities related to the upper bound of the integral. For this purpose, let us modify the large \( N, V \)-limit with an additional definition of the cut-off \( \Lambda^2 \) as the ratio in the limit

\[
\lim_{N', V \to \infty} \frac{N'}{V^{2/D}} = \Lambda^2. \tag{2.4.1}
\]

Notice that \( N' + 1 \) was the number of distinct eigenvalues \( e_k \). Sending \( \Lambda^2 \) to infinity (also called UV-limit) produces divergences where the degree of the divergence depends on the spectral dimension.

**Example 2.3.** Take the third contraction of \( G_{[pq]}^{(0)} = V \langle \Phi_{pq} \Phi_{qp} \rangle \) at order \( \lambda^1 \). Let \( e_k \) be the distinct eigenvalues of multiplicity \( r_k = k \) which indicates \( D = d = 4 \). Then, the sum of the example converges in the limit discussed in Sec. 2.3 with the ratio \( (2.4.1) \) to

\[
\lim_{N', V \to \infty} \frac{1}{(e_q + e_p)^2} \sum_{k=0}^{N'} \frac{k}{e_k + e_q} = \frac{1}{(\mu^2 + e(x) + e(y))^2} \int_0^{\Lambda^2} dt \frac{t}{\mu^2 + e(t) + e(x)},
\]

where \( \frac{p}{\sqrt{V}} = x, \frac{q}{\sqrt{V}} = y, e(\frac{k}{\sqrt{V}}) = e_k - \frac{\mu^2}{2}, e_0 = \frac{\mu^2}{2} \) and \( e(t) \to \infty t \). The integral has therefore a linear divergence in \( \Lambda^2 \) for \( \Lambda^2 \to \infty \).

A systematic procedure, how the UV divergences can be compensated for QFTs, is called renormalisation and is described in its full generality by the BPHZ theorem [BP57, Hep66, Zim69]. Counterterms cancel all divergences and subdivergences by the Bogoliubov-Parasiuk R-operation such that the results are uniquely determined at any order through additional boundary conditions. A compact description of this procedure was given by Zimmermann and is also known as Zimmermann’s forest formula.

Adjusting the action with additional \( \Lambda^2 \)-depending constants, called renormalisation constants, produces exactly these counterterms coming from the R-operation by choosing...
the same boundary conditions. The renormalisation constants are usually divergent in the $\Lambda^2 \to \infty$ limit, typical examples are

- mass renormalisation: $\mu^2 \to \mu_{\text{bare}}^2(\Lambda^2)$
- field renormalisation: $\Phi \to Z^{1/2}(\Lambda^2)\Phi$
- coupling constant renormalisation: $\lambda_i \to \lambda_{i,\text{bare}}(\Lambda^2)$.

However, any finite number of further renormalisation constants is permitted. If a model is UV-finite (renormalised) after a finite number of renormalisation constants, it is called renormalisable, otherwise nonrenormalisable. How many renormalisation constants are necessary depends on the type of the model, the interactions and on the order of the divergences related to the dimension.

Let us recall Zimmermann’s forest formula for ribbon graphs. We admit that the complexity of the forest formula will get much easier for ribbon graphs and additionally on a Euclidean space. Faces of ribbon graphs are globally labelled whereas loops in ordinary QFT need locally assigned momenta. We follow the description of the forest formula for ribbon graphs given in [GSW18] with a slight generalisation for the dimension and interaction.

### 2.4.1 Zimmermann’s Forest Formula

Let $\Gamma$ be a connected ribbon graph. Let $B_\Gamma$ be the set of open faces and $F_\Gamma$ the set of closed faces of $\Gamma$. A ribbon subgraph $\gamma \subset \Gamma$ consists of a subset of closed faces $F_\gamma \subset F_\Gamma$ together with all bordering edges of $F_\gamma$ and all vertices at the end of the edges such that $\gamma$ is connected and after removing any vertex the subgraph is still connected. Each ribbon subgraph uniquely defines the set of adjacent faces of $\gamma$ by $E_\gamma \subset B_\Gamma \cup F_\Gamma \setminus F_\gamma$, i.e. any element of $E_\gamma$ is bordering an edge of $\gamma$. Let the extended subgraph $\bar{\gamma}$ of $\gamma$ be $\gamma$ together with all half-edges within $\Gamma$ which are connected to the vertices of $\gamma$, and its attached faces (which are now understood as open faces) of the half-edges which can possibly be outside $E_\gamma$.

Let $x_i$ with $i \in \{1, \ldots, k\}$ be the labellings of the faces in $E_\gamma$ and $f(\gamma) = \{x_1, \ldots, x_k\}$ the set of the face variables $x_i$. Further, let $r_\gamma(f(y), y_1, \ldots, y_m)$ be a rational function, where $y_i$ with $i \in \{1, \ldots, m\}$ are the labellings of the faces of $\gamma$. Let $(T^\omega_{f(\gamma)} r_\gamma)(f(\gamma), y_1, \ldots, y_m)$ be the $\omega$th order multivariate Taylor polynomial of $r_\gamma$ with respect to the variables $f(\gamma)$. The Taylor polynomial is $T^\omega_{f(\gamma)} r_\gamma \equiv 0$ for $\omega < 0$.

A forest $U_\Gamma$ in $\Gamma$ is a set of ribbon subgraphs $\{\gamma_1, \ldots, \gamma_l\}$ such that any pair of subgraphs $\gamma_i, \gamma_j$ obeys one of the three conditions

$$\gamma_i \subset \gamma_j, \quad \gamma_j \subset \gamma_i, \quad \gamma_i \cap \gamma_j = \emptyset,$$

where the empty set means that $\gamma_i, \gamma_j$ are disjoint also for edges, but not necessarily for vertices. The empty forest $U_\Gamma = \emptyset$ is included in the definition, whereas $\Gamma$ is not possible as forest since $\Gamma$ has open faces and any ribbon subgraph does not.

The three conditions for the elements of the forest equip the forest with a partial ordering by the following construction. Let a descendant $\gamma_{ij} \in U_\Gamma$ be the subset of $\gamma_i$, more precisely $\gamma_{ij} \subset \gamma_i \in U_\Gamma$, together with the conditions that any two different descendants are disjoint $\gamma_{ij} \cap \gamma_{il} = \emptyset$ and there exists no $\gamma' \in U_\Gamma$ such that $\gamma_{ij} \subset \gamma' \subset \gamma_i$. 


For any $\gamma_i \in \mathcal{U}_F$ there exists a unique set $o(\gamma_i) = \{\gamma_{i_1}, \ldots, \gamma_{i_k}\}$, $\gamma_{i_j} \in \mathcal{U}_F$ of descendants $\gamma_{i_j} \subset \gamma_i$ such that for any $\gamma_j \in \mathcal{U}_F \setminus \{\gamma_i, o(\gamma_i)\}$ one of the three conditions hold

$$\gamma_j \subset o(\gamma_i), \quad \gamma_i \subset \gamma_j, \quad \gamma_i \cap \gamma_j = \emptyset.$$ 

The first condition means that $\gamma_j$ is a subset of one descendant of $\gamma_i$.

Now let $I_F$ be the integrand of $h(\Gamma) V^{2g-2b-N}$ (see Sec. 2.1.3) of the ribbon graph $\Gamma$ with genus $g$, $b$ boundary components and $N$ open faces in the large $N$, $V$-limit, discussed in Sec. 2.3. Any forest $\mathcal{U}_F$ defines a unique partition due to the partial ordering of an integrand by

$$I_F = I_{\Gamma \setminus \mathcal{U}_F} \prod_{\gamma \in \mathcal{U}_F} I_{\setminus \gamma \setminus o(\gamma)}, \quad \prod_{\gamma \in \emptyset} I_{\emptyset} = 1.$$ 

We call $\omega(\gamma)$ the superficial degree of divergence of $\gamma$ which is defined as the degree of the numerator subtracted by the degree of the denominator of $I_\gamma$. Let $g_\gamma$, $b_\gamma$ the number of boundary components, $k_\gamma$ the number of $i$-valent vertices, $v_\gamma = k_3 + \ldots + k_d$ the number of all vertices, $N_\gamma$ the number of open faces and $e_\gamma$ the number of edges of the extended subgraph $\bar{\gamma}$ of the subgraph $\gamma$. Let $\Sigma_\gamma$ be the number of (closed) faces and $e_c$ the number of edges of $\gamma$ ($e_c$ can also be understood as number of edges attached to a closed face of $\bar{\gamma}$). The superficial degree of divergence is then

$$\omega(\gamma) = \frac{\Omega}{2} \Sigma_\gamma - e_c.$$ 

Euler’s formula gives the number of closed faces $\Sigma_\gamma = 2 - 2g_\gamma - b_\gamma + e_\gamma - N_\gamma - v_\gamma$. The 1PI construction of $\gamma$ as a subgraph leads to two further relations

$$e_\gamma = N_\gamma + 3k_3 + 4k_4 + \ldots + dk_d$$

$$e_\gamma = 3k_3 + 4k_4 + \ldots + dk_d - e_c,$$

so we can conclude:

$$\omega(\gamma) = \frac{\Omega}{2} (2 - 2g_\gamma - b_\gamma - v_\gamma) + \left(\frac{\Omega}{2} - 1\right) \frac{3k_3 + 4k_4 + \ldots + dk_d - N_\gamma}{2}. \quad (2.4.2)$$

Now we are ready to formulate the adapted BPHZ theorem for matrix field theory models in the large $N$, $V$-limit

**Theorem 2.2. ([Zim69])** A formal power series in the coupling constants $\lambda_i$ of appropriated chosen renormalisation constants $(\mu_{\text{bare}}^2, Z, \ldots)$ with the same number of normalisation conditions (with the reference point at zero momentum) results in the replacement of the integrand $I_\Gamma$ of any ribbon graph in the perturbation theory by

$$I_\Gamma \mapsto R(I_\Gamma) := \sum_{\mathcal{U}_F} I_{\Gamma \setminus \mathcal{U}_F} \prod_{\gamma \in \mathcal{U}_F} (-T|_{\omega(\gamma)} I_{\gamma \setminus o(\gamma)}),$$

where the sum over all forests includes the empty forest $\emptyset$. The product $\prod_{\gamma \in \mathcal{U}_F}$ over all elements of a forest takes the partial ordering into consideration

$$\prod_{\gamma \in \mathcal{U}_F} (-T|_{\omega(\gamma)} I_{\gamma \setminus o(\gamma)}) = \ldots (-T|_{\omega(\gamma)} I_{\gamma \setminus o(\gamma)} \prod_{\gamma' \in o(\gamma)} (-T|_{\omega(\gamma')} I_{\gamma' \setminus o(\gamma')}) \ldots$$

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The forest formula subtracts divergences in a minimal way with the reference point at zero momentum. This means that only divergences and subdivergences are removed. Finite graphs receive no subtraction. It is not clear from the beginning how the boundary conditions are fixed by the forest formula, e.g. \( G(0,0) = 1 \) or \( G(0,0) = \frac{1}{\mu^2} \).

In the forthcoming work, we will discuss for cubic interaction the dimensions \( D = 2, 4 \) which are super-renormalisable. Because of this and the later chosen normalisation conditions by \( G(0) = 0 \) in \( D = 2 \) and additionally \( \partial_x G(0)(x) |_{x=0} = G(0)(0,0) - 1 = 0 \) in \( D = 4 \), the forest formula has to be adapted since these conditions additionally renormalise finite graphs. For \( D = 6 \) is the model just-renormalisable and we will choose the conditions in the same way as the forest formula does.

Looking at the quartic interaction, the \( D = 2 \) case is super-renormalisable and the \( D = 4 \) case just-renormalisable. We are in particular interested in \( D = 4 \) and will later chose the renormalisation differently since the exact solution provides another natural choice for the boundary conditions.

All perturbative computations described by Zimmermann’s forest formula are collected in App. D.

**Remark 2.4.** To avoid redundant factors of the renormalised mass \( \mu \) (after renormalisation), we passed in the cubic model due to \( G(0)(0,0) = 1 \) to mass-dimensionless quantities without mentioning. More concretely, this can either be achieved by multiplying each quantity \( Q_i \) of mass-dimension \( k_i \) with \( \mu^{-k_i} \) or, equivalently, to choose \( \mu^2 = 1 \) from the beginning. A detailed treatment of appropriate factors in \( \mu \) can be found in [GSW18] for the cubic model.

### 2.5 QFT on the Moyal Space and Matrix Field Theory

The underlying \( D \)-dimensional space of a QFT is the \( \mathbb{R}^D \), we will restrict in this subsection to even \( D \in \{2, 4, 6, \ldots\} \) dimensions. In a rigorous formulation, a field operator of a QFT has to be smeared out by a test function \( g \in \mathcal{S}(\mathbb{R}^D) \). The space of complex-valued Schwartz functions consists of rapidly decaying functions

\[
\mathcal{S}(\mathbb{R}^D) = \{ g \in C^\infty(\mathbb{R}^D) : \sup_{x \in \mathbb{R}^D} |x^\alpha D^\beta g(x)| < \infty \quad \forall \alpha, \beta \in \mathbb{N}^D \},
\]

where \( \alpha, \beta \) are multi-indices, i.e. \( \alpha = (\alpha_1, \ldots, \alpha_D) \) and \( \beta = (\beta_1, \ldots, \beta_D) \). The expectation value of these operator-valued distributions defines the correlation functions of a QFT model. The underlying space \( \mathbb{R}^D \) of an ordinary QFT is equipped with local multiplications which are commutative \( f(x)g(x) = g(x)f(x) \) for \( f, g \in \mathcal{S}(\mathbb{R}^D) \).

As an alternative, we deform the space [Rie89] by the algebra \( \mathcal{A}_\star = (\mathcal{S}(\mathbb{R}^D), \star) \) to achieve the Moyal space, where the noncommutative Moyal \( \star \)-product is defined by [GBV88]

\[
(g \star h)(x) = \int \frac{d^D k}{(2 \pi)^D} \int d^D y \ g(x + \frac{1}{2} \Theta k) h(x + y) e^{i k \cdot y},
\]

where

\[
\Theta = 1_{D/2} \otimes \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}, \quad \theta \in \mathbb{R}, \quad x \in \mathbb{R}^D, \quad g, h \in \mathcal{S}(\mathbb{R}^D).
\]

The originally investigated Moyal space is 2-dimensional, whereas we take \( \frac{D}{2} \) copies of the 2-dimensional version multiplied by a Cartesian product. The structure of the \( D \)-dimensional Moyal space is covered by tensorial structure of \( \Theta \).
The $\star$-product generalises the ordinary Euclidean QFT construction since for $\theta = 0$ the $k$-integral generates the delta distribution $\delta(y)$ which gives after $y$-integration a pointwise multiplication $(g \star h)(x)|_{\theta=0} = g(x)h(x)$.

Due to the noncommutative nature and the $\frac{D}{2}$ copies, the Moyal product possesses a matrix base

$$b_{nm}(x) = f_{n_1m_2}(x_1, x_2)f_{n_2m_3}(x_3, x_4)\ldots f_{n_{D/2}m_{D/2}}(x_{D-1}x_D),$$

$$n = (n_1, \ldots, n_{D/2}) \in \mathbb{N}^{D/2}, \quad m = (m_1, \ldots, m_{D/2}) \in \mathbb{N}^{D/2}, \quad x = (x_1, \ldots, x_D) \in \mathbb{R}^D$$

which is separated into $\frac{D}{2}$ bases $f_{ij}(x, y)$ of the 2-dimensional Moyal spaces through the tensorial structure of $\Theta$. We refer to App. A for more details. The base has the matrix multiplication property and a trace by

$$\int d^Dx \ b_{nm}(x) = (2\pi)^{D/2}\delta_{n,m}$$

and even further a trace-class operator.

Using (2.5.5) as the action and writing out the first orders of the perturbative expansion has got the right properties of a field theory, where $\Phi \in \mathcal{S}(\mathbb{R}^D)$, $\Delta = \sum_{i=1}^{D} \frac{\partial^2}{\partial x_i^2}$ denotes the Laplacian and $\mathcal{V}^\star(\Phi) = \sum_{k=3}^{d} \frac{\lambda_k}{k} \Phi^{*k}$ as a potential with $\Phi^{*k} = \Phi \ast \ldots \ast \Phi$ exactly $k$ times. Using (2.5.5) as the action and writing out the first orders of the perturbative expansion was shown to have mixing of UV and IR divergence [MVRS00]. The origin of the mixing is the non-local definition of the product.

The works [GW03, GW05b] give a way to handle the UV/IR-mixing problem by adding a harmonic oscillator term to the action. The mixing problem is therefore a property of the Laplacian and not of chosen interaction. The idea was to add a term to the action such that Laplacian breaks in the matrix base (2.5.2) down to a matrix (only for $\Omega = 1$) instead of a tensor of rank 4. Therefore, let the action be

$$S[\Phi] = \int \frac{d^Dx}{(8\pi)^{D/2}} \left( \frac{1}{2} \Phi \star (-\Delta + \mu^2)\Phi + \mathcal{V}^\star(\Phi) \right)(x),$$

where $\Omega \in \mathbb{R}$ regulates the harmonic oscillator independently of $\theta$. The harmonic oscillator term breaks translational invariance, however invariance can be recovered in the $\theta \to \infty$ limit or $\Omega = 0$.

The expansion of $\Phi(x) = \sum_{n,m} \Phi_{nm} b_{nm}(x)$ and further properties of the Moyal $\star$-product listed in App. A give the action (2.5.6) after integration

$$S[\Phi] = \sum_{n,m \in \mathbb{N}^{D/2}} \Phi_{nm} G_{nm} \Phi_{kl} + \sum_{i=3}^{d} \frac{\lambda_i}{i} \sum_{n_1 \ldots, n_i \in \mathbb{N}^{D/2}} \Phi_{n_1n_2} \Phi_{n_2n_3} \ldots \Phi_{n_in_i}$$

for $\Omega = 1$.
with

\[ G_{nmkl} = \left( \frac{\theta}{4} \right)^{D/2} \left( \frac{\mu_0^2}{2} + \frac{1 + \Omega^2}{\theta} \left( \frac{D}{2} + |n| + |m| \right) \right) \delta_{m,k} \delta_{n,l} \]

\[ - \left( \frac{\theta}{4} \right)^{D/2} \frac{1 - \Omega^2}{\theta} \sum_{i=1}^{D/2} \left( \sqrt{n_m \delta_{n_i - 1, k_i} \delta_{m_i - 1, l_i}} + \sqrt{k_i l_i \delta_{n_i + 1, k_i} \delta_{m_i + 1, l_i}} \right) \delta_{m, l}^i \delta_{n, l}^i, \]

where \(|n| = n_1 + n_2 + \ldots + n_{D/2}\) and \(\delta_{nm}^i\) is \(\delta_{nm}\) with omitted \(\delta_{nm_i}\).

The Moyal \(\star\)-product is known to have a duality between position space and momentum space \(x \leftrightarrow p\) [LS02]. The action transforms under this duality by

\[ S[\Phi; \mu_0, \lambda, \Omega] \mapsto \Omega^2 S \left[ \Phi; \frac{\mu_0}{\Omega}, \frac{\lambda}{\Omega^2}, \frac{1}{\Omega} \right]. \]

The special case \(\Omega = 1\) is called self-dual since it leaves the action invariant under the position-momentum duality and breaks \(G_{nmkl}\), down to a matrix because all next-to-diagonal terms are cancelled (second line of (2.5.8)).

We consider from now on the self-dual noncommutative QFT model (\(\Omega = 1\)). The noncommutative QFT action (2.5.6) is then exactly a matrix field action (2.0.3) by truncating of the sum to \(|n| \in \mathbb{N}^{D/2}\) with \(\mathbb{N}^{D/2} := \{ |n| \in \mathbb{N}^D : |n| \leq N \} \) and identifying

\[ V = \left( \frac{\theta}{4} \right)^{D/2} \]

\[ \mu^2 = \mu_0^2 + \frac{D}{4V^2/\theta} \]

\[ e_{|n|} = \frac{\mu^2}{2} + \frac{|n|}{V^2/\theta} \]

\[ H_{|n|, |n|} = -2G_{nmkl} \delta_{ml} \delta_{nl} \]

\[ r_{|n|+1} = \left( \frac{|n| + D}{2} - 1 \right) \]

The model depends only on the norm \(|n|\), i.e. the explicit dependence on \(n\) drops out. The multiplicity of \(e_{|n|}\) is \(r_{|n|}\), which is the number of all \(n \in \mathbb{N}^{D/2}\) with norm \(|n|\). The correlation functions are naturally labelled by the tuples \(p_i^j \in \mathbb{N}^{D/2}\)

\[ G_{p_i^1, p_i^2, \ldots, p_i^N}, \]

where two correlation functions are the same if the norms of their indices are the same, i.e.

\[ G_{p_i^1, p_i^2, \ldots, p_i^N} = G_{q_i^1, q_i^2, \ldots, q_i^N}, \quad \text{if} \quad |p_i^j| = |q_i^j| \quad \forall i, j. \]

The deformation parameter \(\theta\) of the Moyal space is directly related to the parameter \(V\).

The discussed limit \(\mathcal{N}, V \to \infty\) (with \(\mathcal{N} = \sum_{i=1}^{\mathcal{N}} r_i\)) guarantees translational invariance of the action (2.5.6) and recovers the infinitely large base of the Moyal space. The monotonic, continuously differentiable functions \(e(x)\) (see (2.3.4)) and \(r(x)\) (see (2.3.5)) converge to

\[ e(x) = x \]
\[ r(x) = \frac{x^{D/2-1}}{(\frac{D}{2} - 1)!} \]

The model can now be treated as matrix field theory model. All correlation functions defined in Section 2.2 have to be determined in the limit of Section 2.3 with the renormalisation from Section 2.4.

However, we have to determine the correlation functions for the matrix field theory and insert them into the expansion in the base \( b_{\nu\mu}(x) \) to give statements for the expectation values on the Moyal space. Hence, we define the connected Schwinger function by

\[
S_c(\xi_1, \ldots, \xi_N) := \lim_{\Lambda^2 \to \infty} \lim_{V,N' \to \infty} \sum_{N_1+N_2+\ldots+N_b=N} \sum_{b=1}^{N_b} G[\mathcal{p}_1 \cdots \mathcal{p}_{N_1}] / (8\pi)^{D/2} b! \]

\[
\times \sum_{\beta=1}^{N} b_{\xi_1, \xi_2}(\xi_\sigma(\beta-1)) \ldots b_{\xi_{N_{\beta}}, \xi_1}(\xi_\sigma(\beta)) / V N_{\beta},
\]

where \( s_\beta = N_1 + \ldots + N_{\beta-1} \) and \( S_N \) is the symmetric group, consisting of all permutation of a set with \( N \) elements. The Schwinger function is, by definition, symmetric \( S_c(\xi_1, \ldots, \xi_j, \ldots, \xi_i, \ldots, \xi_N) = S_c(\xi_1, \ldots, \xi_j, \ldots, \xi_i, \ldots, \xi_N) \) for all \( i, j \).

The definition (2.5.10) is motivated by the partition function of the Moyal space in position space before applying the expansion. This partition function should be treated with caution and is formally defined as the functional integral

\[
\mathcal{Z}[J] = \int D[\Phi] \exp \left( -S[\Phi] + V \int d^D x \ (J * \Phi)(x) \right),
\]

where \( J \in \mathcal{S}(\mathbb{R}^D) \) is the source and the measure \( D[\Phi] \) should be understood formally by the function \( \Phi \in \mathcal{S}(\mathbb{R}^D) \). The measure as well as the entire expression have no well-defined limit such that the functional integral has to be understood more or less symbolically as the limit of (2.4.8). The source and the field are expanded by

\[
J(x) = \frac{1}{V (8\pi)^{D/2}} \sum_{\nu,\mu \in \mathbb{N}^{D/2}} J_{\nu\mu} b_{\nu\mu}(x) \quad \text{and} \quad \Phi(x) = \sum_{\nu,\mu \in \mathbb{N}^{D/2}} \Phi_{\nu\mu} b_{\nu\mu}(x).
\]

Staying at the formal level, the Schwinger function has originally the formal definition

\[
S_c(\xi_1, \ldots, \xi_N) = \frac{1}{(8\pi)^{D/2}} \log \left( \frac{\mathcal{Z}[J]}{\mathcal{Z}[0]} \right)_{J=0}.
\]

The expansion of \( J(x) \) leads with \( J_{\nu\mu} = \int d^D x \ b_{\nu\mu}(x) \) to \( \frac{\delta_{\nu\mu}}{S(\mathbb{R}^D)} = b_{\nu\mu}(x) \). This recovers the definition (2.5.10) from the chain rule together with (2.2.4).

The invariance of the Schwinger function (2.5.10) under the full Euclidean group is shown by another representation

\[
S_c(\xi_1, \ldots, \xi_N) = \sum_{\substack{N_1+\ldots+N_b=N \ \sigma \in S_N \ \ \text{even}}} \sum_{\substack{\beta=1 \ \text{even} \ \ N_{\beta}}} \left( \prod_{\beta=1}^{D/2} \frac{2^{D/2} N_{\beta}}{N_{\beta}} \right) \int \frac{d^D p^\beta}{(2\pi)^{D/2}} e^{i p^\beta (\xi_{\sigma(\beta+1)} - \xi_{\sigma(\beta+2)} + \ldots - \xi_{\sigma(\beta+N_{\beta})})} \]

\[
\times \left( \frac{\mu^2}{2} \right)^{N_1} \ldots \left( \frac{\mu^2}{2} \right)^{N_b},
\]

(2.5.11)
where \( G^g(x_1^1, ..., x_{N_1}^1|..|x_1^b, ..., x_{N_b}^b) = \lim_{V,N\to\infty} G^{(g)}_{|p_1^1,..,p_{N_1}^1|..|p_1^b,..,p_{N_b}^b||p_j^i=x_j^i V^{2/D}}. \)

This representation can be found for \( D = 4 \) in [GW13]. The derivation of the general case is outsourced to App. B.

We have completed the procedure to determine Schwinger functions of a self-dual real scalar QFT model on the Moyal space by applying first the matrix base expansion. To solve then the corresponding matrix field theory model in the combined limit \( V,N \to \infty \), we have to take the planar sector such that the Schwinger function is given by (2.5.11). This procedure obviously holds only in the discussed \( V,N \)-limit. For any finite \( V \), translational invariance is broken and for any finite \( N \), the Moyal space cannot be recovered. Going away from the self-dual point \( \Omega = 1 \) the limit of \( V \) could be relaxed. The off-diagonal terms of the covariance \( G_{mnkl} \) for \( \Omega \neq 1 \) survive, however it is possible to diagonalise \( G_{mnkl} \) in \( D = 2 \) [GW03] and in \( D = 4 \) by Meixner polynomials [GW05b]. Nevertheless, the next step of deriving correlation functions in this situation is an outstanding challenge.

**Remark 2.5.** From the representation (2.5.11) together with the correlation functions \( G^0 \), the reflection positivity property of Schwinger functions can be checked. If it holds, for instance in \( D = 4 \), the first exactly solvable non-trivial QFT model in 4 dimensions can be derived. All further Osterwalder-Schrader axioms [OS73, OS75] (except clustering) are fulfilled by the representation (2.5.11).
Chapter 3

Cubic Interaction:
The Renormalised Kontsevich Model

In this chapter, we consider a matrix field theory with cubic interaction $V(\Phi) = \frac{1}{3}\Phi^3$. Historically, this model has great importance. It was designed by Maxim Kontsevich [Kon92] to prove Edward Witten’s conjecture [Wit91b] about the equivalence between two different 2-dimensional quantum gravity approaches. In particular, Witten conjectured that the generating function of intersection numbers on the moduli space $\overline{M}_{g,b}$ of stable complex curves of genus $g$ and $b$ distinct marked points satisfies the string equation and an entire hierarchy of KdV equations. These equations are nonlinear partial differential equations which are recursively constructed by Gelfand-Dikii polynomials [GD77].

It was later discovered that the 0-dimensional Kontsevich model (finite matrices) has an underlying structure known as topological recursion, which was developed by Bertrand Eynard and Nicolas Orantin [EO07].

In Sec. 3.1 we recall the derivation of the SDEs. Sec. 3.2 is split into several subsections which provide the continuum limit together with the renormalisation depending on the spectral dimension $\mathcal{D}$. We will prove also in this section that the renormalised Kontsevich model obeys topological recursion by inverting a linear integral operator with combinatorial methods using Bell polynomials. Furthermore, we will construct a boundary creation operator independent of the dimension. The free energy will then be given in Sec. 3.3. As by-product by combining the boundary insertion operator with the inverse of an integral operator, we will derive a second-order differential operator to compute intersection numbers of $\psi$-classes. By doing so, we will prove that the stable partition function of the renormalised Kontsevich model is annihilated by the generators of a deformed Virasoro algebra due to the change of an implicitly defined constant $c$ depending on the dimension. In Sec. 3.4 we take up the question, whether over-subtraction or rather over-renormalisation will create problems for the cubic model.

3.1 Schwinger-Dyson Equations

All SDEs will be derived with the complete set of renormalisation constants for $\mathcal{D} < 8$. Due to the tadpole renormalisation, the action is equipped with an additional linear term

*The SDEs of this section were already derived in [GSWT18]
such that the renormalised Kontsevich model becomes
\[
S[\Phi] = V \left( \sum_{n,m=0}^{N} Z H_{nm} \Phi_{nm} \Phi_{mn} + \sum_{n=0}^{N} (\kappa + \nu E_n + \zeta E_n^2) \Phi_{nn} + \frac{\lambda_{bare} Z^{3/2}}{3} \sum_{n,m,k=0}^{N} \Phi_{nm} \Phi_{mk} \Phi_{kn} \right),
\]

(3.1.1)

\[H_{nm} = E_n + E_m,\]

where two eigenvalues \(E_i, E_j\) are not necessarily different. The partition function is also slightly affected by the renormalisation constants and is similarly to (2.0.5) given by
\[
Z[J] = \int \mathcal{D}\Phi \exp \left( -S[\Phi] + V \text{Tr}(J\Phi) \right)
= K \exp \left( -\frac{\lambda_{bare} Z^{3/2}}{3V^2} \sum_{n,m,k=0}^{N} \partial J_{nm} \partial J_{mk} \partial J_{kn} \right) Z_{\text{free}}[J],
\]

(3.1.2)

where \(K := \int \mathcal{D}\Phi \exp \left( -VZ \sum_{n,m=0}^{N} H_{nm} \Phi_{nm} \Phi_{mn} \right) = \prod_{n,m=0}^{N} \sqrt{\frac{2\pi}{g H_{nm}}} \). Finally, we recall the impact of the renormalisation constants on the Ward-Takahashi identity of Theorem 2.1
\[
\sum_{n=0}^{N} \partial^2 Z[J] = -V \sum_{n=0}^{N} \left( J_{pn} \partial J_{qn} - J_{nq} \partial J_{np} \right) Z[J] - \frac{V}{Z} (\nu + \zeta H_{pq}) \frac{\partial Z[J]}{\partial J_{pq}} + \delta_{E_p E_q} (W_p^1 [J] + W_p^2 [J]) Z[J].
\]

(3.1.3)

Notice that (3.1.3) holds even for \(E_p = E_q\) by regularity assumption even if \(p \mapsto E_p\) is not injective. The later computed correlation functions depend only on the distinct eigenvalues \(\epsilon_k\) with \(k \mapsto \epsilon_k\) injective. The sum can then be written as \(\sum_{n=0}^{N} \rightarrow \sum_{k=0}^{N'} r_k\), where \(N'\) are the numbers of distinct eigenvalues and \(r_k\) its multiplicities.

The renormalisation constants have singular behaviour in the later taken limit depending on the spectral dimension \(D\). Nevertheless, an appropriate choice of the constants leaves the correlation functions finite after removing the cut-off. Notice that there is still a freedom in the choice of the renormalisation constants by boundary conditions which we will fix for \(D < 6\) differently than in the perturbative expansion of Zimmermann’s forest formula:
\[
D \geq 2 : \quad G^{(0)}_{[0]} = 0,
\]

(3.1.4)
\[
D \geq 4 : \quad \left. \frac{\partial}{\partial p} G^{(0)}_{[p]} \right|_{p=0} = 0, \quad G^{(0)}_{[0]} = 1,
\]

(3.1.5)
\[
D \geq 6 : \quad \left. \frac{\partial^2}{\partial p^2} G^{(0)}_{[p]} \right|_{p=0} = 0, \quad \left. \frac{\partial}{\partial p} G^{(0)}_{[pq]} \right|_{p=q=0} = \left. \frac{\partial}{\partial q} G^{(0)}_{[pq]} \right|_{p=q=0} = 0.
\]

(3.1.6)

Let us first determine the 2-point function \(G_{[pq]}\), where \(E_p \neq E_q\), recursively from the 1-point function \(G_{[p]}\) by
\[
G_{[pq]} = \frac{1}{ZH_{pq}} - \frac{\lambda_{bare} Z^{1/2}}{V^2 H_{pq} Z[0]} \sum_{n=0}^{N} \partial J_{pq} \partial J_{qn} \partial J_{np} Z[J] \bigg|_{J=0}
\]

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where equation (2.2.6) together with (3.1.2) and (3.1.3) is applied. This expression is now equivalent to

\[
G_{|pq|} = \frac{1}{Z(1 - Z^{-1/2}\lambda_{bare}\zeta)} \left( E_p + E_q - \frac{Z^{-1/2}\lambda_{bare}\nu}{1 - Z^{-1/2}\lambda_{bare}\zeta} \right) + \frac{\lambda_{bare} Z^{1/2}}{Z(1 - Z^{-1/2}\lambda_{bare}\zeta)} \left( E_p - E_q \right) \left( E_p + E_q - \frac{Z^{-1/2}\lambda_{bare}\nu}{1 - Z^{-1/2}\lambda_{bare}\zeta} \right).
\]

(3.1.7)

Notice that the limit \( E_q \to E_p \) is uncrical since the 1-point function can be continued to differentiable function, where the derivative \( \frac{\partial}{\partial E_p} G_{|p|} \) has a meaningful expression. Furthermore, the perturbative expansion of \( G_{|pq|} \) is unique.

The two conditions (3.1.5) and (3.1.6) together with the expression (3.1.7) yield for \( g = 0 \) after genus expansion the relations

\[
\mu_{bare}^2 - Z^{1/2}\lambda_{bare}\nu = 1
\]

\[
Z(1 - Z^{-1/2}\lambda_{bare}\zeta) = 1,
\]

(3.1.8)

(3.1.9)

where \( \mu_{bare}^2 \) is the smallest eigenvalue of \( E \). It is also more convenient to define the renormalised coupling constant \( \lambda \) and the shifted eigenvalues \( F_p \) (which are UV-finite in the later taken limit) by

\[
\lambda := Z^{1/2}\lambda_{bare}
\]

\[
F_p := E_p - \frac{\lambda\nu}{2}.
\]

(3.1.10)

(3.1.11)

Inserting (3.1.8), (3.1.9) as well as (3.1.10), (3.1.11) into (3.1.7) gives the compact formula

\[
G_{|pq|} = \frac{1}{F_p + F_q} + \lambda \frac{G_{|p|} - G_{|q|}}{F_p^2 - F_q^2}.
\]

(3.1.12)

The setup is now established, and we are able to derive all SDEs starting with the 1-point function:

**Proposition 3.1.** The shifted 1-point function \( W_{|p|} := 2\lambda G_{|p|} + 2F_p \) satisfies

\[
(W_{|p|})^2 + 2\lambda\nu W_{|p|} + 2\lambda^2 \sum_{n=0}^{N} \frac{W_{|p|} - W_{|n|}}{F_p^2 - F_n^2} + 4\lambda^2 \frac{G_{|p|}}{V^2} = \frac{4F_p^2}{Z} + C,
\]

where \( C := -\frac{\lambda^2\nu^2(1+Z)+4\kappa\lambda}{Z} \). In particular, the genus expansion \( W_{|p|} =: \sum_{g=0}^{\infty} V^{-2g} W_{|p|}^{(g)} = 2F_p + 2\lambda \sum_{g=0}^{\infty} V^{-2g} G_{|p|}^{(g)} \) gives a linear equation for \( g > 0 \) and a nonlinear one for \( g = 0 \):

\[
\sum_{h+h'=g} W_{|p|}^{(h)} W_{|p|}^{(h')} + 2\lambda\nu W_{|p|}^{(g)} + 2\lambda^2 \sum_{n=0}^{N} \frac{W_{|p|}^{(g)} - W_{|n|}^{(g)}}{F_p^2 - F_n^2} + 4\lambda^2 G_{|p|}^{(g-1)} = \delta_{0,g} \left( \frac{4F_p^2}{Z} + C \right).
\]

(3.1.13)
Proof. By definition, the 1-point function is given by
\[
G_{[p]} = \frac{1}{V} \frac{\partial}{\partial J_{pp}} \log Z[J] |_{J=0}
\]
\[
= \frac{1}{ZH_{pp}} \left( -\kappa - \nu E_p - \zeta E_p^2 - \frac{\lambda_{bare} Z^{3/2}}{V^2 Z[0]} \sum_{n=0}^{N} \frac{\partial^2}{\partial J_{nn} \partial J_{np}} Z[J] \right) |_{J=0}
\]
\[
= \frac{1}{ZH_{pp}} \left( -\kappa - \nu E_p - \zeta E_p^2 - \lambda_{bare} Z^{3/2} \left( (G_{[p]})^2 + \frac{G_{[p]\bar{p}]}{V^2} + \frac{1}{V} \sum_{n=0}^{N} G_{[n\bar{n}]} \right) \right),
\]
where the second line is achieved by using (2.2.6) and considering the additional terms via (3.1.2). The definition of the correlation function gives the last line, where \( n = p \) produces the quadratic term and the \((1 + 1)\)-point function.

Inserting the conditions (3.1.8) and (3.1.9) together with the definitions (3.1.10), (3.1.11) and \( W_{[p]} := 2\lambda G_{[p]} + 2F_p \) leads to the first equation of the proposition since \( G_{[np]} \) (see (3.1.12) becomes
\[
G_{[np]} = \frac{1}{2} \frac{W_{[p]} - W_{[q]}}{F_p - F_q}. \tag{3.1.14}
\]
Expanding the shifted 1-point and the \((1 + 1)\)-point function in the genus expansion, gives at order \( V^{-2g} \) the second equation of the proposition. \( \square \)

Remark 3.1. The shift from \( G_{[p]} \) to \( W_{[p]} \) is understood as transformation \( \Phi \mapsto \Phi' = \Phi - \frac{E}{\lambda} \) of the integration variable of the partition function, which cancels the quadratic term of the action such that the action only consists of a linear and a cubic term in \( \Phi' \), and a constant.

The generalisation of the recursive equation (3.1.12) to all correlation functions is given by:

Proposition 3.2. For any \( j \in \{1, \ldots, b\} \) and \( i \in \{1, \ldots, N_j\} \), the \((N_1 + N_2 + \ldots + N_b)\)-point function is given recursively in terms of the \((N_1 + \ldots + N_{j-1} + (N_j - 1) + N_{j+1} + \ldots + N_b)\)-point function
\[
G_{[p_1| \ldots |p_i| \ldots |p_{N_j}|]} = -\lambda \frac{G_{[p_1| \ldots |p_i| |p_{N_j+1}| \ldots |p_N|]} - G_{[p_1| \ldots |p_{i+1}| |p_{N_j+2}| \ldots |p_N|]} \right). \]
3.1. SCHWINGER-DYSON EQUATIONS

\[ + \frac{\lambda_{\text{bare}} (\nu + \zeta H_{p_i^j})}{Z^{1/2} H_{p_i^j}} G[p_i^j; p_{N_j}^j], \]

where the second line is achieved by applying (2.2.6) and in the last two lines by (3.1.3).

Inserting the conditions (3.1.8), (3.1.9) together with the definitions (3.1.10), (3.1.11) leads to the result.

Since the lhs is regular if two or more \(E_{p_i^j}\)'s coincide, the rhs has a well-defined limit which can be understood by continuing the correlation functions to differentiable functions.

We emphasise that Proposition 3.2 implies recursively that any \((N_1 + \ldots + N_b)\)-point function depends linearly on \((1 + \ldots + 1)\)-point functions with \(b\) boundary components. The explicit result is formulated in:

**Proposition 3.3.** The explicit formula for the \((N_1 + \ldots + N_b)\)-point function is

\[ G[p_1^1 \ldots p_{N_1}^1 \ldots p_1^b \ldots p_{N_b}^b] = \lambda^{N-b} \sum_{k_1=1}^{N_1} \ldots \sum_{k_b=1}^{N_b} G[p_1^1 \ldots p_{N_1}^1 \ldots p_2^1 \ldots p_{N_2}^1 \ldots p_b^1 \ldots p_{N_b}^1] \prod_{\beta=1}^{b} \prod_{l_{ij} \neq k_{ij}}^{N_{ij}} \frac{1}{F_{p_k}^2 - F_{p_l}^2}, \]

where \(N := N_1 + \ldots + N_b\), and for \(b = 1\)

\[ G[p_1 \ldots p_N] = \lambda^{N-1} \sum_{k=1}^{N} W[p_k] \prod_{l=1}^{N} \frac{1}{F_{p_k}^2 - F_{p_l}^2}. \]

**Proof.** The proof is provided by induction. For \(N_1 = 2\) and \(N_\beta = 1\) for all \(\beta \in \{2, \ldots, b\}\) it holds. Assume it holds for the \((N_1 + \ldots + N_{b-1})\)-point function, then we have for the \(((N_1 + 1) + N_2 + \ldots + N_b)\)-point function with Proposition 3.2

\[ G[p_1 \ldots p_{N_1+1} \ldots p_1^1 \ldots p_{N_b}^b] = -\lambda \left( G[p_1^2 \ldots p_{N_1+1}^2 \ldots p_1^1 \ldots p_{N_b}^b] - G[p_1^1 \ldots p_{N_1+1}^1 \ldots p_1^2 \ldots p_{N_b}^b] \right) \]

\[ = -\lambda^{N+1-b} \sum_{k_1=2}^{N_1+1} \ldots \sum_{k_b=1}^{N_b} \frac{1}{F_{p_k}^2 - F_{p_l}^2} \times \left( \sum_{k_1=1}^{N_1+1} \frac{1}{F_{p_k}^2 - F_{p_l}^2} \right) \]

\[ \times \prod_{\beta=2}^{b} \prod_{l_{ij} \neq k_{ij}}^{N_{ij}} \frac{1}{F_{p_k}^2 - F_{p_l}^2} \]

\[ = -\lambda^{N+1-b} \sum_{k_1=1}^{N_1+1} \ldots \sum_{k_b=1}^{N_b} \left[ G[p_1^1 \ldots p_{N_1}^1 \ldots p_1^2 \ldots p_{N_b}^b] \prod_{l_{ij} \neq k_{ij}}^{N_{ij}} \frac{1}{F_{p_k}^2 - F_{p_l}^2} + G[p_1^2 \ldots p_{N_1}^2 \ldots p_1^1 \ldots p_{N_b}^b] \prod_{l_{ij} \neq k_{ij}}^{N_{ij}} \frac{1}{F_{p_k}^2 - F_{p_l}^2} \right] \]

\[ + \sum_{k_1=3}^{N_1+1} G[p_1^1 \ldots p_{N_1}^1 \ldots p_2^2 \ldots p_{N_b}^b] \prod_{l_{ij} \neq k_{ij}}^{N_{ij}} \frac{1}{F_{p_k}^2 - F_{p_l}^2} \]
In particular, the genus expansion, we observe that any \((N_1 + \ldots + N_b)\)-point function of the cubic model is not only cyclic symmetric within a boundary (by definition), it is fully symmetric. This is even surprising from the perspective of perturbative expansion. For example:

\[ G_{[p_1p_2p_3p_4]} = G_{[p_2p_1p_3p_4]}. \]

After genus expansion, we observe that any \((N_1 + \ldots + N_b)\)-point function of genus \(g\) is expressed by the \((1 + \ldots + 1)\)-point function of genus \(g\) with \(b\) boundary components. The final SDE to study is therefore the equation of the \((1 + \ldots + 1)\)-point function.

We impose the shorthand notation \(G_{[I]} := G_{[i^1, \ldots, i^b]}\) for the set \(I := \{i^1, \ldots, i^b\}\) with \(|I| = b\) and \(i^j \in \{0, \ldots, N\}\).

**Proposition 3.4.** Let \(J = \{p^1, \ldots, p^b\}\), then the \((1 + \ldots + 1)\)-point function with \(b\) boundary components satisfies the linear equation

\[
(W_{[p^1]} + \nu \lambda)G_{[p^1|J]} + \frac{\lambda^2}{V} \sum_{n=0}^{N} \frac{G_{[p^1|J]} - G_{[n|J]}}{F_{p^1}^2 - F_{n}^2} = -\lambda \sum_{\beta=2}^{b} G_{[p^1p^\beta|J\setminus\{\beta\}]} - \frac{\lambda}{V^2} \sum_{\beta=2}^{b} G_{[p^1p^\beta|J]} - \lambda \sum_{\beta=2}^{b} G_{[p^1|J]} G_{[p^1|J\setminus\{\beta\}]}.
\]

In particular, the genus \(g\) correlation function satisfies

\[
\sum_{h+h'=g} (W_{[p^1]}^{(h \downarrow)} + \delta_{h,0} \nu \lambda)G_{[p^1|J]}^{(h')} + \frac{\lambda^2}{V} \sum_{n=0}^{N} \frac{G_{[p^1|J]}^{(g)} - G_{[n|J]}^{(g)}}{F_{p^1}^2 - F_{n}^2} = -\lambda \sum_{\beta=2}^{b} G_{[p^1p^\beta|p^\beta|J\setminus\{p^\beta\}]} - \lambda \sum_{h+h'=g} \sum_{\beta=2}^{b} G_{[p^1|J]}^{(h')} G_{[p^1|J\setminus\{\beta\}]}.
\]
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Proof. Let the \( E_p \)'s be pairwise different. It follows from the definition

\[
G[p^1|J] = V^{b-2} \frac{\partial^b}{\partial J_{p^1p_1} \partial J_{p^2p_2} \cdots \partial J_{p^b p_b}} \log \left( \frac{Z[J]}{Z[\{0\}]} \right)_{J=0} \\
= - \frac{\lambda_{\text{bare}} Z^{1/2} V^{b-3}}{H_{p^1 p_1}} \frac{\partial^{b-1}}{\partial J_{p^2 p_2} \cdots \partial J_{p^b p_b}} \left[ \sum_{n=0}^{N} \frac{\partial^2}{\partial J_{n^1 n} \partial J_{n p^1}} Z[J] \right]_{J=0} \\
= - \frac{\lambda_{\text{bare}} Z^{1/2} V^{b-3}}{H_{p^1 p_1}} \frac{\partial^{b-1}}{\partial J_{p^2 p_2} \cdots \partial J_{p^b p_b}} \left[ \sum_{n=0}^{N} \left( \frac{\partial^2 \log Z[J]}{\partial J_{n^1 n} \partial J_{n p^1}} + \frac{\partial \log Z[J]}{\partial J_{n p^1}} \frac{\partial \log Z[J]}{\partial J_{n^1 n}} \right) \right]_{J=0} \\
= - \frac{\lambda_{\text{bare}} Z^{1/2}}{H_{p^1 p_1}} \left( \frac{1}{V} \sum_{n=0}^{N} G[p^1|n|J] + \frac{1}{V^2} G[p^1|p^1|J] + G[p^1|p^2|p^3|J\{p^3\}] + \sum_{I \neq J} G[p^1|I] G[p^1|p^1|I] \right),
\]

where in the last line the \((2 + 1 + \ldots + 1)\)-point function appears if \( n = p^1 \), and the \((3 + 1 + \ldots + 1)\)-point function if \( n = p^\beta \) with \( \beta \in \{2, \ldots, b\} \). Shifting the quadratic term with \(|I| = 0\) and \(|I| = b\) to the l.h.s and using \( H_{p^1 p_1} \lambda_{\text{bare}} Z^{1/2} G[p^1] = W[p^1] \lambda_{\text{bare}} Z^{1/2} \nu \) leads to the provided equation with \( \lambda_{\text{bare}} Z^{1/2} = \lambda \).

If two or more \( E_p \)'s coincide, both sides of the equation are uncritical. \( \square \)

Looking at all SDE of Proposition 3.1 and Proposition 3.4, we notice that any correlation function \( f(p) \) of Euler characteristic \( \chi = 2 - 2g - b \) can be computed by inverting the linear equation

\[
(W[p]^{(0)} + \nu \lambda)f(p) + \frac{\lambda^2}{V} \sum_{n=0}^{N} \frac{f(p) - f(n)}{F^p_p - F^p_n} = g_{\text{inh}}(p)
\]

(3.1.15)

where \( g_{\text{inh}}(p) \) is a inhomogeneity depending on correlation functions of Euler characteristic \( \chi' > \chi \). Only the 1-point function of genus \( g = 0 \) plays a special rôle and satisfies a nonlinear equation. All solutions are known in case of \( \delta = 0 \) and obey topological recursion. For higher dimensions, the limit \( N, V \rightarrow \infty \) needs to be performed before the correlation functions are computed.

3.2 Solution of the Schwinger-Dyson Equations

The solution of all planar correlation function was already found in [GSW18]. We have generalised these results in our paper [GHW19a] which will be presented in this section.

3.2.1 Large \( N, V \)-Limit

Following the limit discussed in Section 2.3 the distinct eigenvalues \( (e_0, e_1, \ldots, e_{N'}) \) with \( N' \leq N \) are given by the shifted eigenvalues \( F_p = E_p - \frac{\nu}{2} \) of multiplicities \( r_k \)

\[
F_0 = F_1 = \ldots = F_{r_0 - 1} = e_0 = \frac{1}{2} \\
F_{r_0} = \ldots = F_{r_0 + r_1 - 1} = e_1 \\
\vdots
\]

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The asymptotic behaviour of the multiplicities defines the spectral dimension of the model by \( \lim_{k \to \infty} r_k \sim k^{\frac{2}{D} - 1} \) since \( e_k \) can be assumed asymptotically linear in \( k \) (see Remark 2.3). The monotonic, continuously differentiable function \( e(x) \) with \( e(0) = 0 \) and \( r(x) \) are defined by (2.3.4) and (2.3.5) which are in the \( N, V \)-limit unique

\[
\lim_{N', V \to \infty} e_k - \frac{\mu^2}{2} = \lim_{N', V \to \infty} e \left( \frac{k}{\sqrt{\sigma}} \right) = e(x) \xrightarrow{x \to \infty} x
\]

\[
\lim_{N', V \to \infty} r_k \sigma = \lim_{N', V \to \infty} r \left( \frac{k}{\sqrt{\sigma}} \right) = r(x) \xrightarrow{x \to \infty} \frac{2}{x^{\frac{2}{D} - 1}}.
\]

The ratio \( N'/V^2 \) will be fixed in such a way that the sum converges to the integral with cut-off \( \Lambda^2 \)

\[
\lim \sum_{n=0}^{N'} f \left( \frac{n}{V^{2/3}} \right) = \lim \sum_{k=0}^{N'} r_k f \left( \frac{k}{V^{2/3}} \right) = \int_0^{\Lambda^2} dx \, r(x) f(x).
\]

The SDE of the previous section becomes together with the continuation of the correlation function (unique for \( N', V \to \infty \)) defined in (2.3.6):

\[
\sum_{h+h'=g} W^h(x)W^{h'}(x) + 2\lambda \nu W^g(x) + 2\lambda^2 \int_0^{\Lambda^2} dt \, r(t) \frac{W^g(x) - W^g(t)}{\left( \frac{1}{2} + e(x) \right)^2 - \left( \frac{1}{2} + e(t) \right)^2} = -4\lambda^2 G^{g-1}(x|x) + \delta_{0,0} \left( \frac{1 + 2e(x)^2}{Z} \right) + C,
\]

\[
\sum_{h+h'=g} (W^h(x^1) + \delta_{h,0} \lambda) G^{h'}(x^1|J) + \lambda^2 \int_0^{\Lambda^2} dt \, r(t) \frac{G^g(x^1|J) - G^g(t|J)}{\left( \frac{1}{2} + e(x^1) \right)^2 - \left( \frac{1}{2} + e(t) \right)^2} = -\lambda \sum_{\beta=0}^{b} G^g(x^1, x^\beta, x^{\beta'|J\setminus\{x^\beta\}}) - \lambda G^{g-1}(x^1|x^1|J) - \lambda \sum_{\substack{h+h'=g \\nu h'=f \\nu I=J \\nu |I|=b}} G^h(x^1|I)G^{h'}(x^1|I'),
\]

where \( W^g(x) := 2\lambda G^g(x) + \delta_{g,0}(1 + 2e(x)) \), \( C = \lim_{N', V} \left( -\frac{\lambda^2 \nu^2 (1+Z)^2 + 4\nu \lambda}{Z} \right) \) and \( J = \{ x^2, \ldots , x^b \} \). We used the notation \( G^g(I) := G^g(y^1|\ldots |y^b) \) for the set of variables \( I := \{ y^1, \ldots , y^b \} \) with \( |I| = b \) and \( y^l \in [0, \Lambda^2] \).

3.2.2 The Planar 1-Point Function

Computing the solution of the 1-point function is the hardest part in solving the model since it obeys the nonlinear integral equation. The 1-point function was originally solved in the appendix of [MS91] by transforming the problem to a boundary value problem, also known as Riemann-Hilbert problem. This solution can be extended from \( D = 0 \) to \( D < 2 \). The solution for higher spectral dimensions was then generalised by a similar ansatz in [GSW18] up to \( D < 8 \).

To stay self-contained, we recall these results. A more convenient form of the nonlinear integral equation (3.2.1) for \( g = 0 \) is achieved by the variable transformation

\[
X := (1 + 2e(x))^2, \quad \bar{W}(X(x)) := W^0(x)
\]
such that the nonlinear integral equation for the 1-point function takes the form
\[
\tilde{W}(X)^2 + 2\lambda\nu\tilde{W}(X) + \int_1^{(1+2e(\Lambda^2))^2} dY \varrho(Y) \frac{\tilde{W}(X) - \tilde{W}(Y)}{X - Y} + \frac{X}{Z} = C, \tag{3.2.4}
\]
where \( \varrho(Y) := \frac{2\lambda^2 \cdot r \left( e^{-1} \left( \frac{\sqrt{Y} - 1}{2} \right) \right)}{\sqrt{Y} \cdot e^{\left( e^{-1} \left( \frac{\sqrt{Y} - 1}{2} \right) \right)}}. \)

Its solution is:

**Proposition 3.5.** \cite{GSW18} For any measure \( \varrho(Y) \), the nonlinear integral equation (3.2.4) is solved by
\[
\tilde{W}(X) = \frac{\sqrt{X + c}}{\sqrt{Z}} - \lambda\nu + \frac{1}{2} \int_1^{(1+2e(\Lambda^2))^2} dY \varrho(Y) \left( \sqrt{X + c} + \sqrt{Y + c} \right) \sqrt{X + c},
\]
where \( c, Z \) and \( \nu \) are fixed by the renormalisation conditions.

**Proof.** Inserting the solution into the integral equation leads with \( \frac{\sqrt{X + c} - \sqrt{Y + c}}{X - Y} = \frac{1}{\sqrt{X + c} + \sqrt{Y + c}} \) to
\[
\int_1^{(1+2e(\Lambda^2))^2} dY \varrho(Y) \frac{\tilde{W}(X) - \tilde{W}(Y)}{X - Y} = \frac{1}{\sqrt{Z}} \int_1^{(1+2e(\Lambda^2))^2} dY \varrho(Y) \frac{\sqrt{X + c}}{\sqrt{Y + c}} + \frac{1}{\sqrt{Z}} \int_1^{(1+2e(\Lambda^2))^2} dY \varrho(Y) \frac{\sqrt{Y + c}}{\sqrt{X + c}} + \frac{1}{\sqrt{Z}} \int_1^{(1+2e(\Lambda^2))^2} dY \varrho(Y) \frac{\sqrt{Y + c}}{\sqrt{X + c}} - \frac{1}{2} \int_1^{(1+2e(\Lambda^2))^2} dY \varrho(Y) \frac{\sqrt{X + c}}{\sqrt{Y + c}} \right)^2.
\]

Using in the last line the symmetrisation \( \int_I \int_I dt \, dy \, f(t, y) = \int_I \int_I dt \, dy \, (f(y, t) + f(t, y)) \) factorises the integrals
\[
\int_1^{(1+2e(\Lambda^2))^2} dY \varrho(Y) \frac{\tilde{W}(X) - \tilde{W}(Y)}{X - Y} = \frac{1}{\sqrt{Z}} \int_1^{(1+2e(\Lambda^2))^2} dY \varrho(Y) \frac{\sqrt{X + c}}{\sqrt{Y + c}} + \frac{1}{\sqrt{Z}} \int_1^{(1+2e(\Lambda^2))^2} dY \varrho(Y) \frac{\sqrt{Y + c}}{\sqrt{X + c}} + \frac{1}{\sqrt{Z}} \int_1^{(1+2e(\Lambda^2))^2} dY \varrho(Y) \frac{\sqrt{Y + c}}{\sqrt{X + c}} - \frac{1}{2} \int_1^{(1+2e(\Lambda^2))^2} dY \varrho(Y) \frac{\sqrt{X + c}}{\sqrt{Y + c}} \right)^2.
\]
Comparing this with (3.2.4) identifies the constant \( C \) through \( c \). We therefore fix the renormalisation constants by (3.1.4), (3.1.5) and (3.1.6) and the definition of \( \tilde{W}(X) \) depending on the spectral dimension. \( \square \)

The proposition provides the asymptotic behaviour for the solution \( \tilde{W}(X) = \sqrt{X} + \mathcal{O}((\sqrt{X} - 1)^{3/2}) \). The conditions (3.1.4), (3.1.5) and (3.1.6) to fix the remaining renormalisation constants are translated to
\[
\tilde{W}(1) = 1, \quad \frac{\partial}{\partial X} \tilde{W}(X) \bigg|_{X=1} = \frac{1}{2}, \quad \frac{\partial^2}{\partial X^2} \tilde{W}(X) \bigg|_{X=1} = -\frac{1}{4}. \tag{3.2.5}
\]
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Remember that for $D < 2$ non of the conditions are necessary which implies the following implicit equation for $c$ (from the last line of the proof of Proposition 3.5 with $\nu = Z - 1 = C = 0$):

$$-c = \int_1^{(1+2e(\Lambda^2))^2} dY \frac{\varrho(Y)}{\sqrt{Y + c}}.$$  

Finite Matrices

To recover the solution for finite $(N' + 1) \times (N' + 1)$ matrices with distinct eigenvalues $\frac{1}{2} = e_0 < e_1 < e_{N'}$ of the external matrix $E$ with multiplicities $r_0, ..., r_{N'}$, the discrete Dirac measure can be used $\varrho(Y) = \frac{8\lambda^2}{V} \sum_{k=0}^{N'} r_k \delta(Y - 4e_k^2)$. This provides the solution

$$W^0(e_n) = \tilde{W}(4e_n^2) = \sqrt{4e_n^2 + c} + \frac{4\lambda^2}{V} \sum_{k=0}^{N'} \frac{r_k}{\sqrt{4e_n^2 + c} + \sqrt{4e_k^2 + c}} \sqrt{4e_k^2 + c},$$

with $-c = \int_1^{(1+2e(\Lambda^2))^2} dY \frac{\varrho(Y)}{\sqrt{Y + c}}.$

$$0 \leq D < 2$$

All renormalisation constants are independent of $\Lambda^2$ and can be set to $Z = 1$, $\nu = 0$, $C = 0$ and $\mu^2_{bare} = 1$. The solution becomes

$$\tilde{W}(X) = \sqrt{X + c} + \frac{1}{2} \int_1^{\infty} \frac{dY \varrho(Y)}{\sqrt{X + c + \sqrt{Y + c}} \sqrt{Y + c}}.$$ 

with $-c = \int_1^{\infty} dY \frac{\varrho(Y)}{\sqrt{Y + c}}.$

The integral of the implicit equation of $c$ converges since $r(e^{-1}(\sqrt{Y - 1})) \sim \sqrt{Y^{\frac{2}{2}} - 1} = Y^{\frac{2}{2} - \frac{1}{2}}$ and therefore $\varrho(Y) \sim \frac{1}{Y^{1 - \frac{1}{2}} \frac{1}{\sqrt{Y}}}$. The spectral dimension $D = 2$ is the critical dimension where the implicit equation diverges logarithmically.

Notice that the constant $c$ depends on $r(x), e(x)$ and $\lambda$ with $c = 0$ for $\lambda = 0$. From the implicit function theorem we know that $c(\lambda)$ is unique and differentiable in an open neighbourhood about $\lambda = 0$.

$$2 \leq D < 4$$

The first condition of (3.2.5) implies the following implicit equation for $c$:

$$1 = \sqrt{1 + c} \frac{1}{\sqrt{Z}} - \lambda \nu + \frac{1}{2} \int_1^{(1+2e(\Lambda^2))^2} dY \frac{\varrho(Y)}{\sqrt{1 + c + \sqrt{Y + c}} \sqrt{Y + c}}.$$ (3.2.6)

In this case the limit of the cut-off $\Lambda^2 \to \infty$ is safe since $\varrho(Y) \sim \frac{1}{Y^{1 - \frac{1}{2}} \frac{1}{\sqrt{Y}}}$. The integral representation of $\tilde{W}(X)$ of Proposition 3.5 converges for $\Lambda^2 \to \infty$. For $D = 4$, the expression...
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diverges logarithmically. The remaining renormalisation constants are \( Z = 1 \) and \( \nu = 0 \). Inserting the renormalisation constants yields

\[
\tilde{W}(X) = \sqrt{X + c} + \frac{1}{2} \int_1^\infty \frac{dY}{(\sqrt{X + c} + \sqrt{Y + c})\sqrt{Y + c}}
\]

\[
1 - \sqrt{1 + c} = \frac{1}{2} \int_1^\infty \frac{dY}{(1 + c + \sqrt{Y + c})\sqrt{Y + c}}.
\]

Since the rhs of the implicit function is positive for real \( \lambda \), the lhs indicates \( c \in \mathbb{C} \setminus -1, 0 \). The argumentation extends for complex \( \lambda \) to \( c \in \mathbb{C} \setminus -\infty, -1 \).

4 \leq D < 6

Subtracting the implicit function (3.2.6) from the solution of \( \tilde{W}(X) \) of Proposition 3.5 cancels \( \lambda \nu \) and leads to

\[
\frac{\tilde{W}(X) - 1}{\sqrt{X + c} - \sqrt{1 + c}} = \frac{1}{2} \int_1^{(1+2e(\Lambda^2))^2} \frac{dY}{(\sqrt{X + c} + \sqrt{Y + c})(\sqrt{1 + c} + \sqrt{Y + c})\sqrt{Y + c}}.
\]

(3.2.7)

Since \( \varrho(Y) \sim Y^{\frac{D}{2} - 1} \), the limit \( \Lambda^2 \to \infty \) implies logarithmic divergence for \( D = 6 \). The additional condition \( \frac{\varrho}{\partial X} \tilde{W}(X) \bigg|_{X=1} = \frac{1}{2} \) provides

\[
\sqrt{1 + c} = \frac{1}{2} \int_1^{(1+2e(\Lambda^2))^2} \frac{dY}{(1 + c + \sqrt{Y + c})^2\sqrt{Y + c}},
\]

(3.2.8)

where \( Z = 1 \) is safe. Sending \( \Lambda^2 \to \infty \) in (3.2.7) and (3.2.8) gives the renormalised solution for 4 \leq D < 6.

6 \leq D < 8

Since (3.2.7) is for \( D \geq 6 \) not divergent for \( Z = 1 \), we eliminate \( Z \) through (3.2.8) and get

\[
\tilde{W}(X) = \sqrt{X + c} \sqrt{1 + c} - c
\]

\[
+ \frac{1}{2} \int_1^{(1+2e(\Lambda^2))^2} \frac{dY}{(\sqrt{X + c} + \sqrt{Y + c})(\sqrt{1 + c} + \sqrt{Y + c})^2\sqrt{Y + c}}.
\]

where now \( \Lambda \to \infty \) is finite for all \( D < 8 \) since \( \varrho(Y) \sim Y^{\frac{D}{2} - 1} \). The last condition \( \frac{\varrho^2}{\partial X^2} \tilde{W}(X) \bigg|_{X=1} = -\frac{1}{4} \) gives

\[
-c = \int_1^\infty \frac{dY}{(\sqrt{1 + c} + \sqrt{Y + c})^2\sqrt{Y + c}}.
\]

Collecting all three cases gives the general result:
Corollary 3.1. Let \( D = 2 \left\lfloor \frac{D}{2} \right\rfloor \). The UV-finite, shifted planar 1-point function \( 3.2.3 \) is for the cubic matrix field of spectral dimension \( D < 8 \) with the renormalisation conditions \( 3.2.5 \) given by

\[
\tilde{W}(X) = \sqrt{X + c} \left( \sqrt{1 + c} \right)^{\delta_{D,6} + \delta_{D,4}} (1 - \sqrt{1 + c}) - \delta_{D,6} c + \frac{1}{2} \int_{1}^{\infty} \frac{dY \varrho(Y)}{(\sqrt{X + c} + \sqrt{Y + c})(\sqrt{1 + c} + \sqrt{Y + c})^{\frac{D}{2} - 1} \sqrt{Y + c}},
\]

where \( (1 - \sqrt{1 + c}) \left( \frac{1 + \sqrt{1 + c}}{2} \right)^{\delta_{D,6} + \delta_{D,4}} = \frac{1}{2} \int_{1}^{\infty} \frac{dY \varrho(Y)}{(\sqrt{1 + c} + \sqrt{Y + c})^{\frac{D}{2} / 2} \sqrt{Y + c}} \)

In all cases, \( c(\lambda) \) is a differentiable function in a small neighbourhood about \( \lambda = 0 \). The expansion of \( c \) in \( \lambda \) is expressed by the Lagrange inverse theorem

\[
c = \sum_{n=1}^{\infty} \frac{1}{n!} \frac{d^{n-1}}{dw^{n-1}} \Bigg|_{w=0} \left( \frac{w}{2} \int_{1}^{\infty} \frac{dY \varrho(Y)}{(1 + w)(1 + \sqrt{1 + w})^{\delta_{D,6} + \delta_{D,4}}} \right)^{n}, \tag{3.2.9}
\]

where \( D = 2 \left\lfloor \frac{D}{2} \right\rfloor \).

Remark 3.3. (\cite{GSW18}) The \( \beta \)-function \( \beta_\lambda \) of the running coupling constant is \( \lambda_{\text{bare}}(\Lambda^2) = \frac{\lambda}{\sqrt{2}} \). It is in dimension \( 6 \leq D < 8 \) and for real \( \lambda \) positive

\[
\beta_\lambda = \Lambda^2 \frac{d\lambda_{\text{bare}}(\Lambda^2)}{d\Lambda^2} = \frac{2\lambda^2 \Lambda^6}{(\sqrt{1 + c} + \sqrt{(1 + 2e(\Lambda^2)^2 + c)^2})\sqrt{(1 + 2e(\Lambda^2)^2 + c)}} > 0.
\]

This calculation is easily checked by \( 3.2.8 \) and assuming \( c \) is independent of \( \Lambda^2 \) which is achieved by choosing slightly different renormalisation conditions which converges in the limit \( \Lambda^2 \to \infty \) to the previous one. The \( \beta \)-function is finite for finite \( \Lambda^2 \) and therefore has no Landau pole.

Recall that the 1-point function is in any dimension given from the shifted 1-point function by

\[
G^0(x) = \frac{\tilde{W}((1 + 2e(x))^2) - (1 + 2e(x))}{2\lambda} \tag{3.2.10}
\]

Let’s look at the examples of the \( D \)-dimensional Moyal space with \( D \in \{2, 4, 6\} \).

Example 3.1. (\( D = \mathcal{O} = 2 \) Moyal space)

The Moyal space admits linear eigenvalues \( e(x) = x \) and for \( D = 2 \) with multiplicity one, i.e. \( r(x) = 1 \), such that \( \varrho(Y) = \frac{2X}{\sqrt{Y}} \). Then Corollary 3.1 gives for the shifted 1-point function after integration and simplification

\[
\tilde{W}(X) = \sqrt{X + c} + \frac{2\lambda^2}{\sqrt{X}} \log \left( \frac{(\sqrt{X + c} + \sqrt{X})(\sqrt{X + 1})}{\sqrt{X} \sqrt{1 + c + \sqrt{X + c}}} \right)
\]
function after integration and simplification which is confirmed in App. D.1 by Feynman graph calculations.

Example 3.2. \((D = \mathcal{D} = 4\) Moyal space) Again, linear eigenvalues \(e(x) = x\) and for \(D = 4\) with growing multiplicity of the form \(r(x) = x\) such that \(\vartheta(Y) = \frac{\lambda^2(\sqrt{Y} - 1)^2}{\sqrt{Y}}\). Then Corollary 3.1 gives for the shifted 1-point function after integration and simplification

\[
\tilde{W}(X) = 1 + \sqrt{X} + c - \sqrt{1 + c} - \lambda^2 \left\{ \log \left( \frac{\sqrt{1 + c} + \sqrt{X} + c}{2(1 + \sqrt{1 + c})} \right) \right. \\
+ \frac{1}{\sqrt{X}} \log \left( \frac{\sqrt{X} + \sqrt{X} + c)(\sqrt{X} + 1)}{\sqrt{X} \sqrt{1 + c} + \sqrt{X} + c} \right) \\
\left. - 4c \right\}.
\]

The function \(c(\lambda)\) is uniquely invertible for \(|\lambda| < \lambda_c \approx 1.1203\). Expanding \(c\) in \(\lambda\) by \((3.2.9)\) gives \(c(\lambda) = -\lambda^2 2(1 - \log 2) + \lambda^4(2 - 5 \log 2 + 3(\log 2)^2) - \lambda^6(\frac{7}{4} - 7 \log 2 + \frac{27}{4}(\log 2)^2 - 4(\log 2)^3) + \mathcal{O}(\lambda^8)\). Inserting in \((3.2.10)\) gives the first orders of the 1-point function

\[
G^0(x) = \lambda \frac{x - (1 + x) \log(1 + x)}{1 + 2x} - \lambda^3 \frac{(1 - \log 2)^2(4x + 3)x^2}{(1 + 2x)^3} + \mathcal{O}(\lambda^5)
\]

which is confirmed in App. D.1 by Feynman graph calculations.

Example 3.3. \((D = \mathcal{D} = 6\) Moyal space) Again, linear eigenvalues \(e(x) = x\) and for \(D = 6\) with growing multiplicity of the form \(r(x) = \frac{x^2}{2}\) such that \(\vartheta(Y) = \frac{\lambda^2(\sqrt{Y} - 1)^2}{\sqrt{Y}}\). Then Corollary 3.1 gives for the shifted 1-point function after integration and simplification

\[
\tilde{W}(X) = \sqrt{X} + c\sqrt{1 + c} - c + \lambda^2 \left\{ \sqrt{1 + c} - \sqrt{X} + c + \log \left( \frac{\sqrt{X} + c + \sqrt{1 + c}}{2(1 + \sqrt{1 + c})} \right) \right. \\
+ \frac{(1 + X)}{2\sqrt{X}} \log \left( \frac{\sqrt{X} + \sqrt{X} + c)(1 + \sqrt{X})}{\sqrt{X} \sqrt{1 + c} + \sqrt{X} + c} \right) \\
\left. - 4c = \lambda^2 \left\{ 1 - 2\sqrt{1 + c} + 2(1 + c) \log \left( 1 + \frac{1}{\sqrt{1 + c}} \right) \right\} \right.
\]

The function \(c(\lambda)\) is uniquely invertible for \(|\lambda| < \lambda_c \approx 2.3647\). Expanding \(c\) in \(\lambda\) by \((3.2.9)\) gives \(c(\lambda) = -\lambda^2 2\log 2 - 1 + \lambda^4 8(\log 2)^2 - 10 \log 2 + 3 - \lambda^6 128(\log 2)^4 - 252(\log 2)^2 + 164 \log 2 - 35 + \mathcal{O}(\lambda^8)\). Inserting in \((3.2.10)\) gives the first orders of the 1-point function

\[
G^0(x) = \lambda \frac{2(1 + x)^2 \log(1 + x) - x(2 + 3x)}{4(1 + 2x)^2} + \lambda^3 \frac{x^3(2 + 3x)(2 \log 2 - 1)^2}{16(1 + 2x)^3} + \mathcal{O}(\lambda^5)
\]

which is confirmed in App. D.1 by Feynman graph calculations.
Remark 3.4. (GSW18) The cubic model on the $D = 6$ Moyal space admits the renormalon problem which provides no problem for the exact formula. Determining the amplitude of the Feynman graph below according to the Feynman rules together with Zimmermann’s forest formula gives

$$\frac{(-\lambda)^{4+2n}}{(1 + 2x)^4} \int_0^\infty y^2 \, dy \, \frac{1}{2 \cdot 4(1 + y)^{4+n}} \left( \frac{2(1 + y)^2 \log(1 + y) - y(2 + 3y)}{4(1 + 2y)} \right)^n \sim \frac{(-\lambda)^{4+2n}}{4^n(2(1 + 2x)^4)} \cdot n! .$$

... 

Remark 3.5. To show that the cubic matrix field theory model is in fact a QFT model, one has to check that the connected Schwinger functions are reflection positive. For the connected 2-point Schwinger function reflection positivity is by the work [Wid38] and formula (2.5.11) equivalent to say that the diagonal 2-point function $G^{(0)}(x, x) = 2\tilde{W}((1 + 2x)^2)$ is a Stieltjes function. It was proved in [GSW18] that the 2-point function on the $D = 2$ Moyal space is not a Stieltjes function and therefore not reflection positive. However, the function $G^{(0)}(x, x)$ is for $D = 4$ and $D = 6$ with $\lambda \in \mathbb{R}$ a Stieljes function [CSW18]. The 2-point function is special since reflection positivity is equivalent to the Stieltjes property which does not hold in general. Checking whether higher Schwinger functions are reflection positive is much harder and work in progress.

3.2.3 Analytic Continuation

The crucial step in deriving all solutions is to continue analytically the correlation functions with the “right” variable transformation. The result of [Proposition 3.5] provides a natural choice by

$$z(x) =: \sqrt{X + c} = \sqrt{(1 + 2e(x))^2 + c} .$$

Defining next the $(1 + .. + 1)$-point function of genus $g$ with $b$ boundary components from definition (2.3.6) by

$$G_g(z_1(x^1), .., z_b(x^b)) := G^g(x^1|..|x^b) \quad \text{for} \quad (g, b) \neq (0, 1)$$

and the planar 1-point function, which is already known by [Proposition 3.5] by

$$G_0(z(x)) := \tilde{W}(X(x)) = \tilde{W}^0(x) = G^0(x) + \frac{1 + 2e(x)}{2\lambda} \frac{z}{\sqrt{Z}} - \lambda' + \frac{1}{2} \int_{\sqrt{1+c}}^{\sqrt{(1+2e(A^2))^2+c}} dt \rho(t) \frac{\tilde{\rho}(t)}{(z + t)t} , \quad (3.2.11)$$
with \( \tilde{q}(t) := 2t \varrho(t^2 - c) = \frac{4\lambda^2 t \cdot \tilde{r}\left(e^{-1\left(\frac{\sqrt{t^2 - c} - 1}{2}\right)}\right)}{\sqrt{t^2 - c} \cdot e\left(e^{-1\left(\frac{\sqrt{t^2 - c} - 1}{2}\right)}\right)} \).

With this definition, the 1-point function extends uniquely to a sectional holomorphic function with branch cut along \(-\sqrt{(1 + 2e(\Lambda^2))^2 + c} - \sqrt{1 + c}\). Since we have \(-1 < c \leq 0\) for \(\lambda < \lambda_c\) for all \(\mathcal{D} < 8\), the 1-point function is holomorphic in a small region around \(z = 0\). Analogously, all SDEs are recursively complexified. The analyticity domains of the corresponding correlation functions will be discussed later.

For a more convenient reading, we define:

**Definition 3.1.** Let \(\hat{K}_z\) be the integral operator,

\[
\hat{K}_z f(z) := 2\lambda G_0(z) f(z) + \lambda \nu f(z) + \frac{1}{2} \int \frac{\sqrt{(1 + 2e(\Lambda^2))^2 + c} dt \cdot \tilde{q}(t)}{\sqrt{t^2 - c}} \frac{f(z) - f(t)}{z^2 - t^2}.
\]

Expressing (3.2.1) and (3.2.2) for all \((g, b) \neq (0, 1)\) with the variable \(z\) through the integral operator \(\hat{K}_z\) leads to

\[
\hat{K}_z g_g(z, J) + \lambda g_g-1(z_1, z_1, J) + \lambda \sum_{h = 0}^{J} \sum_{h' = J}^{J} G_h(z_1, J) G_h'(z_1, J') = (3.2.12)
\]

where \(J = \{z_2, ..., z_0\}\) and \(g_g(I) := g_g(\zeta_1, ..., \zeta_0)\) for the set \(I = \{\zeta_1, ..., \zeta_0\}\). The sum \(\sum'\) excludes \((h, I) = (0, 0)\) and \((h, I) = (g, J)\). The second line is achieved from the \((3 + 1 + .. + 1)\)-point function through the recursive equation of Proposition 3.2 which produces a derivative.

We mention that the dependence on the spectral dimension contributes in \(\hat{K}_z\) by the renormalisation constant \(Z\) for \(\mathcal{D} \geq 6\) and in the constant \(c\). The entire structure of the SDEs is independent of the spectral dimension.

For later purpose, we introduce the renormalised Kontsevich times which play a distinguished rôle:

**Definition 3.2.** Let the Kontsevich times \(\{\varrho_k\}_{k \in \mathbb{N}}\) be defined by

\[
\varrho_k := \frac{\delta_{k, 0}}{\sqrt{Z}} - \frac{1}{2} \int \frac{\sqrt{(1 + 2e(\Lambda^2))^2 + c} dt \cdot \tilde{q}(t)}{\sqrt{t^2 - c}} \frac{1}{t^{3 + 2k}}.
\]

with converging limit \(\Lambda^2 \rightarrow \infty\).

The Kontsevich times are related to the integral operator by:

**Lemma 3.1.** The operator \(\hat{K}_z\) defined in **Definition 3.1** satisfies

\[
\hat{K}_z \left(\frac{1}{z}\right) = \frac{1}{\sqrt{Z}}, \quad \hat{K}_z \left(\frac{1}{z^{2n + 2 - 2k}}\right) = \sum_{k=0}^{n} \frac{\varrho_k}{z^{2n + 2 - 2k}}.
\]
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Proof. Let \( a = \sqrt{1 + c} \) and \( b = \sqrt{(1 + 2e(\Lambda^2))^2 + c} \). The definition of the linear operator implies

\[
\hat{K} f(z) = \frac{z f(z)}{\sqrt{Z}} + \frac{1}{2} \int_a^b dt \tilde{g}(t) \frac{zf(z) - tf(t)}{t(z^2 - t^2)},
\]

where the case \( f(z) = \frac{1}{z} \) is directly clear. The identity

\[
\frac{1}{z} - \frac{1}{z} = -\sum_{i=0}^{k-1} \frac{z^i t^{k-i}}{z^k t^k}
\]
gives for \( f(z) = \frac{1}{z^{3+2n}} \)

\[
\hat{K} \left( \frac{1}{z^{3+2n}} \right) = \frac{1}{z^{2+2n} \sqrt{Z}} + \frac{1}{2} \int_a^b dt \tilde{g}(t) \frac{1}{t(z + t) (z - t)}
\]

\[
= \frac{1}{z^{2+2n} \sqrt{Z}} - \frac{1}{2} \int_a^b dt \tilde{g}(t) \sum_{l=1}^{2+2n} \frac{z^l t^{2+2n-l}}{z^{3+2n} t^{3+2l+n} (z + t)}
\]

\[
= \frac{1}{z^{2+2n} \sqrt{Z}} - \sum_{l=0}^{n} \frac{1}{z^{2+2n-2l} 2} \int_a^b dt \tilde{g}(t) \frac{1}{t^{3+2l}},
\]

which is the definition of the Kontsevich times. \( \square \)

3.2.4 Solution for \( \chi = 2 - 2g - b \geq -1 \)

As mentioned before, the 1-point function plays a special rôle since it obeys a nonlinear equation. Also the \((1 + 1)\)-point function is different from the others. The correlation functions are embedded into Riemann surfaces which are topologically clearly distinct into the cases \( \chi = 1, \chi = 0 \) and \( \chi < 0 \). The automorphism group for \( \chi \geq 0 \) (unstable) Riemann surfaces is infinite, whereas stable surfaces \( \chi < 0 \) have a finite group of automorphisms. These distinctions are inherited to the pole structure of the correlation functions.

Recall from (3.2.12) that the \((1 + 1)\)-point function obeys the integral equation

\[
\hat{K}_{z_1} \mathcal{G}_0(z_1, z_2) = -(2\lambda)^3 \frac{\partial}{\partial z_2} \mathcal{G}_0(z_1) - \mathcal{G}_0(z_2)
\]

(3.2.13)

Proposition 3.6. For \( \chi = 0 \), the \((1 + 1)\)-point function \((b = 2)\) of genus \( g = 0 \), which solves (3.2.13), is given by

\[
\mathcal{G}_0(z_1, z_2) = \frac{(2\lambda)^2}{z_1 z_2 (z_1 + z_2)^2}.
\]

Proof. Let \( a = \sqrt{1 + c} \) and \( b = \sqrt{(1 + 2e(\Lambda^2))^2 + c} \). The l.h.s. of (3.2.13) gives

\[
\hat{K}_{z_1} \mathcal{G}_0(z_1, z_2) = \frac{4\lambda^2}{z_2 (z_1 + z_2)^2 \sqrt{Z}} + \frac{4\lambda^2}{2} \int_a^b dt \tilde{g}(t) \frac{1}{z_2 (z_1 + z_2)^2} \frac{1}{t(z_1 + t)(z_1 - t)}
\]

\[
= \frac{4\lambda^2}{z_2 (z_1 + z_2)^2 \sqrt{Z}} - \frac{4\lambda^2}{2z_2} \int_a^b dt \tilde{g}(t) \frac{1}{t(z_1 + t)(z_1 + z_2)(t + z_2)(z_1 + z_2)^2(t + z_2)^2}
\]

\[
= - \frac{4\lambda^2}{z_2} \frac{1}{\partial z_2} \left( \frac{1}{(z_1 + z_2)^2} \sqrt{Z} - \frac{1}{2} \int_a^b dt \tilde{g}(t) \frac{1}{t(z_1 + t)(z_1 + z_2)(t + z_2)} \right)
\]

which coincides with the r.h.s. \( \square \)
Proposition 3.7. For $\chi = 2 - 2g - b = -1$, the $(1 + 1 + 1)$-point function ($b = 3$) of genus $g = 0$ and the 1-point function ($b = 1$) of genus $g = 1$ are given by

$$G_0(z_1, z_2, z_3) = -\frac{32\lambda^5}{\gamma_0 z_1^3 z_2^2 z_3^3}, \quad G_1(z) = \frac{\lambda^3 \gamma_1}{\gamma_0^2 z^3} - \frac{\lambda^3}{\gamma_0 z^3}.$$  

Proof. The $(1 + 1 + 1)$-point function obeys (3.2.12)

$$\tilde{K}_z G_0(z_1, z_2, z_3) + \lambda^2 G_0(z_1, z_2) G_0(z_1, z_3)$$

$$= -(2\lambda)^3 \left( \frac{\partial}{z_2 \partial z_2} G_0(z_1, z_3) - G_0(z_2, z_3) \right) + \frac{\partial}{z_3 \partial z_3} G_0(z_1, z_2) - G_0(z, z_2) \right).$$

The ansatz $G_0(z_1, z_2, z_3) = -\frac{\gamma}{z_1^3 z_2^3}$ leads with Lemma 3.1 to

$$\tilde{K}_z G_0(z_1, z_2, z_3) = -\frac{\gamma \gamma_0}{z_1^3 z_2^3 z_3^3}.$$  

Inserting Proposition 3.6 gives, after performing the derivatives and simplifying,

$$(2\lambda)^3 \left( \frac{\partial}{z_2 \partial z_2} G_0(z_1, z_3) - G_0(z_2, z_3) \right) + \frac{\partial}{z_3 \partial z_3} G_0(z_1, z_2) - G_0(z_3, z_2) \right)$$

$$= \frac{32\lambda^5}{z_1^3 z_2^3 z_3^3}$$

which gives by comparing the coefficient $\gamma = -\frac{32\lambda^5}{\gamma_0}$.

The 1-point function of genus $g = 1$ obeys (3.2.12)

$$\tilde{K}_z G_1(z) + \lambda G_0(z, z) = 0.$$  

The ansatz $G_1(z) = \frac{\alpha}{z^2} + \frac{\beta}{z}$ gives with Lemma 3.1 $\tilde{K}_z G_1(z) = \frac{\alpha \alpha_0}{\gamma_0^2} + \frac{\beta \beta_0}{\gamma_0^4}$. Comparing the coefficients with $G_0(z, z) = \frac{\lambda^2}{z^3}$ yields $\beta = -\frac{\lambda^2}{\gamma_0}$ and $\alpha = -\frac{\beta \gamma_1}{\gamma_0^2} = \frac{\lambda^3 \gamma_1}{\gamma_0^2}$. □

Notice that $G_0(z_1, z_2)$ is a meromorphic function with poles at $z_1 = 0 = z_2$ and at the diagonal $z_1 = -z_2$, whereas the 1-point function has even a branch cut at $z \in [-\sqrt{(1 + 2\alpha(\Lambda^2))^2 + c}, -\sqrt{1 + c}]$. For $\chi = -1$, the solutions of Proposition 3.7 are meromorphic functions with a pole only at $z_i = 0$. By the recursive hypothesis of (3.2.12), all further correlation functions with $\chi < -1$ may have poles at $z_i = 0, z_i \pm z_j$, and possibly a branch cut at $z_i \in [-\sqrt{(1 + 2\alpha(\Lambda^2))^2 + c}, -\sqrt{1 + c}]$. However, we will prove that any correlation function $G_0(z_1, ..., z_b)$ with $\chi < 0$ is a meromorphic function with poles of odd order at $z_i = 0$ for $i \in \{1, ..., b\}$.

3.2.5 Solution for $b > 1$ via Boundary Creation Operator

The goal is to construct an operator that increases the number of boundaries. This operator is already known for spectral dimensions $\varnothing < 2$ and used in [MS91, Wit91a]. Assume for the following consideration that $E_k$ are distinct eigenvalues of multiplicity

$^1$Parts of this subsection are taken from our paper [GHW19b]
one. Then, the boundary creation operator is defined for $\mathcal{D} = 0$ by $T_n := \frac{\lambda}{\pi E_n} \frac{\partial}{\partial E_n}$. The
definition makes sense by the following formal considerations, where the renormalisation
constants are taken trivially $Z - 1 = \kappa = \nu = \zeta = 0$, and the matrix size $N$ as well as $V$
are finite,

$$
T_n \log \left( \int D\Phi e^{-V \text{Tr}(E \Phi^2 + \frac{c}{2} \Phi)} \right)
= -\frac{V \lambda}{2Z[0]} \int D\Phi \sum_n \frac{\Phi_{nn} \Phi_{nn}}{E_n} e^{-V \text{Tr}(E \Phi^2 + \frac{c}{2} \Phi)}
= \frac{1}{2Z[0]} \int D\Phi \frac{1}{E_n} \left( \frac{\partial}{\partial \Phi_{nn}} + 2E_n \Phi_{nn} \right) e^{-V \text{Tr}(E \Phi^2 + \frac{c}{2} \Phi)}
= \frac{1}{Z[0]} \int D\Phi \Phi_{nn} e^{-V \text{Tr}(E \Phi^2 + \frac{c}{2} \Phi)} = G_n.
$$

We go back to the Dirac measure $r(x) = \frac{1}{\sqrt{2}} \sum_{n=0}^{\lambda'} r_n \delta(x - e_n)$, where $r_n$ are the multiplicities of the distinct eigenvalues $e_k$ of $E$. Take also the differen-
tiable function $e(x)$ discussed in Sec. 2.3 into account, then the boundary creation operator is expressed formally by the functional derivative

$$
T_n \rightarrow T(x) := \frac{2\lambda}{1 + 2e(x)} \frac{\delta}{\delta \left(1 + 2e(x)\right)} \frac{\delta}{\delta \left(r(x)dx\right)},
$$

where the derivative with respect to the measure is formally defined by

$$
\frac{\delta}{\delta r(t)} f(t) = f(x).
$$

The choice of the complex variable $z = \sqrt{(1 + 2e(x))^2 + c}$ and the property of Lemma 3.1 implies a dependence of the solutions on the parameters $g_l$ and $c$, where

$$
c = -8\lambda^2 \frac{dt r(t)}{\sqrt{(1 + 2e(x))^2 + c}} \text{ for } \mathcal{D} < 2.
$$

The formally defined boundary creation operator $T(x)$ gives on $c$ with the Leibniz rule

$$
T(x)c = \frac{8\lambda^3}{\sqrt{(1 + 2e(x))^2 + c}} + 4\lambda^2 \frac{dt r(t)}{\sqrt{(1 + 2e(x))^2 + c}} T(x)c
\Leftrightarrow T(x)c = \frac{8\lambda^3}{\varrho_0 \sqrt{(1 + 2e(x))^2 + c}} \quad (3.2.16)
$$

with Definition 3.2 for $\varrho_0$, and on $g_l$

$$
T(x)g_l = \frac{8\lambda^3(3 + 2l)}{\sqrt{(1 + 2e(x))^2 + c}} g_l^{5+2l} + (3 + 2l) \int \frac{4\lambda^2 dt r(t)}{\sqrt{(1 + 2e(x))^2 + c}} g_l^{5+2l} T(x)c
= \frac{8\lambda^3(3 + 2l)}{\sqrt{(1 + 2e(x))^2 + c}} g_l^{5+2l} - \frac{8\lambda^3(3 + 2l)g_{l+1}}{\varrho_0 \sqrt{(1 + 2e(x))^2 + c}},
$$

where (3.2.16) was used.

To avoid the formally defined functional derivative, the chain rule is applied to achieve partial derivatives with respect to $c$ and $g_l$. The next step is to switch to the complex
variable $z = \sqrt{(1 + 2e(x))^2 + c}$. We are then able to show that the boundary creation admits a rigorous and universal replacement for any $\mathcal{D} < 8$ by:
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**Definition 3.3.** Let be \( J = \{ z_1, \ldots, z_p \} \) with \( |J| := p \). Then, the boundary creation is

\[
\hat{A}_{J,z}^g := \sum_{l=0}^{3|J|} \left( -\frac{(3+2l)\varrho_{l+1}}{\varrho_0 z^3} + \frac{3+2l}{z^{5+2l}} \right) \partial_\varrho + \sum_{\zeta \in J} \frac{1}{\varrho_0 z^3 \zeta} \partial_\zeta. \tag{3.2.17}
\]

We emphasise that the last variable \( z \) in \( \hat{A}_{J,z}^g \) plays a very different rôle compared to all the \( z_i \in J \).

**Lemma 3.2.** The differential operators \( \hat{A}_{J,z}^g \) are commutative,

\[
\hat{A}_{J,z}^g \hat{A}_{J,z}^p = \hat{A}_{J,z}^p \hat{A}_{J,z}^g.
\]

**Proof.** Being a derivative, it is enough to verify \( \hat{A}_{J,z}^g \hat{A}_{J,z}^p (q_k) = \hat{A}_{J,z}^p \hat{A}_{J,z}^g (q_k) \) for any \( k \) and \( \hat{A}_{J,z}^g \hat{A}_{J,z}^p (z_i) = \hat{A}_{J,z}^p \hat{A}_{J,z}^g (z_i) \) for any \( z_i \in J \). This is guaranteed by

\[
\hat{A}_{J,z}^g \hat{A}_{J,z}^p (q_k) = \left( \frac{3 + 2k}{\varrho_0 z^3} \right) \frac{(3 + 2k)(5 + 2k)}{\varrho_0 z^{5+2k}} - \frac{(3 + 2k)(5 + 2k) - 3(3 + 2k) \varrho_{k+1}}{\varrho_0 z^{7+2k}} - \frac{3(3 + 2k) \varrho_{k+1}}{\varrho_0 z^{9+2k}}.
\]

This shows that boundary components labelled by \( z_i \) behave like bosonic particles at position \( z_i \). The creation operator \((2\lambda)^3 \hat{A}_{J,z}^g \) adds to a \(|J|\)-particle state another particle at position \( z \) of \(|J|\)-particle state is precisely given by \( G_y(J) \):

**Theorem 3.1.** Assume that \( G_y(z) \) is, for \( g \geq 1 \), an odd function of \( z \neq 0 \) and a rational function of \( \varrho_0, \ldots, \varrho_{3g-2} \) (true for \( g = 1 \)). Then the \((1 + 1 + \ldots + 1)\)-point function of genus \( g \geq 1 \) and \( b \) boundary components of the renormalised \( \Phi_D^3 \) matrix field theory model for spectral dimension \( \mathcal{D} < 8 \) has the solution

\[
G_y(z_1, \ldots, z_b) = (2\lambda)^{3b-3} \hat{A}_{z_1, \ldots, z_b}^g \left( \hat{A}_{z_1, \ldots, z_b}^g \left( \cdots \hat{A}_{z_1, z_2}^g G_y(z_1) \right) \right), \quad z_i \neq 0, \tag{3.2.18}
\]

where \( G_y(z_1) \) is the 1-point function of genus \( g \geq 1 \) and the boundary creation operator \( \hat{A}_{J,z}^g \) is defined in **Definition 3.3**. For \( g = 0 \) the boundary creation operators act on the \((1 + 1)\)-point function

\[
G_0(z_1, \ldots, z_b) = (2\lambda)^{3b-6} \hat{A}_{z_1, \ldots, z_b}^{10} \left( \hat{A}_{z_1, \ldots, z_b}^{10} \left( \cdots \hat{A}_{z_1, z_2}^{10} G_0(z_1) \right) \right) \tag{3.2.12}
\]

**Proof.** We rely on several Lemmata proved in **App. C**. Regarding (3.2.18) as a definition, we prove in **Lemma C.5** an equivalent formula for the linear integral equation (3.2.12). This expression is satisfied because **Lemma C.2** and **Lemma C.4** add up to 0. Consequently, the family of functions (3.2.18) satisfies (3.2.12). This solution is unique because of uniqueness of the perturbative expansion.

**Corollary 3.2.** Let \( J = \{ z_2, \ldots, z_b \} \). Assume that \( z \mapsto G_y(z) \) is holomorphic in \( \mathbb{C} \setminus \{ 0 \} \) with \( G_y(z) = -G_y(-z) \) for all \( z \in \mathbb{C} \setminus \{ 0 \} \) and \( g \geq 1 \). Then all \( G_y(z_1, J) \) with \( 2 - 2g - b < 0 \).
1. are holomorphic in every \( z_i \in \mathbb{C} \setminus \{0\} \)
2. are odd functions in every \( z_i \), i.e. \( G_g(-z, J) = -G_g(z, J) \) for all \( z, z_i \in \mathbb{C} \setminus \{0\} \).

**Proof.** The boundary creation operator \( \hat{A}_{jz}^{bg} \) of Definition 3.3 preserves holomorphicity in \( \mathbb{C} \setminus \{0\} \) and maps odd functions into odd functions. Thus, only the initial conditions need to be checked. They are fulfilled for \( G_0(z_1, z_2, z_3) \) and \( G_1(z_1) \) according to Proposition 3.7 for \( g \geq 2 \) by assumption.

The assumption will be verified later in Proposition 3.9.

**Corollary 3.3.** The boundary creation operator \( \hat{A}_{jz}^{bg} \) acting on a \((N_1 + \ldots + N_b)\)-point function of genus \( g \) and complex variables \( J = \{z_1, \ldots, z_N\} \) gives the \((1 + N_1 + \ldots + N_b)\)-point function of genus \( g \).

**Proof.** This follows from the change to complex variables in the equation of Proposition 3.3 and \( \hat{A}_{jz}^{bg} \left( \frac{1}{(z_1^2 - (z_2^2)^2)} \right) = 0 \) for \( j_1 \neq j_2 \) and \( i_1 \neq i_2 \).

**Remark 3.6.** The motivation of defining the boundary creation operator is due to the \( D < 2 \) case by the implicit equation (3.2.15) of c. For \( D \geq 2 \), we have formally a different result for \( T(x)c \) since a different implicit equation holds (see Corollary 3.1) due to the renormalisation constants. Nevertheless, the Definition 3.3 is universal for \( D < 8 \).

### 3.2.6 Solution for \( b = 1 \) and \( g > 1 \)

It remains to check that the 1-point function \( G_g(z) \) at genus \( g \geq 1 \) satisfies the assumptions of Theorem 3.1 and Corollary 3.2, namely:

1. \( G_g(z) \) depends only on the moments \( g_0, \ldots, g_{3g-2} \) of the measure,
2. \( z \mapsto G_g(z) \) is holomorphic on \( \mathbb{C} \setminus \{0\} \) and an odd function of \( z \).

We establish these properties by solving (3.2.12) for \( b = 1 \) via a formula for the inverse of \( K_z \). This formula is inspired by topological recursion.

**Definition 3.4.** The Bell polynomials are defined by

\[
B_{n,k}(x_1, \ldots, x_{n-k+1}) = \sum_{j_1, j_2, \ldots, j_{n-k+1}} \frac{n!}{j_1! j_2! \ldots j_{n-k+1}!} \left( \frac{x_1}{1!} \right)^{j_1} \left( \frac{x_2}{2!} \right)^{j_2} \ldots \left( \frac{x_{n-k+1}}{(n-k+1)!} \right)^{j_{n-k+1}}
\]

for \( n \geq 1 \), where the sum is over non-negative integers \( j_1, \ldots, j_{n-k+1} \) with \( j_1 + \ldots + j_{n-k+1} = k \) and \( 1j_1 + \ldots + (n-k+1)j_{n-k+1} = n \). Moreover, one defines \( B_{0,0} = 1 \) and \( B_{n,0} = B_{0,k} = 0 \) for \( n, k > 0 \).

An important application is Faà di Bruno’s formula, the \( n \)-th order chain rule:

\[
\frac{d^n}{dx^n} f(g(x)) = \sum_{k=1}^{n} f^{(k)}(g(x)) B_{n,k}(g'(x), g''(x), \ldots, g^{(n-k+1)}(x)).
\]  

---

1Parts of this subsection are taken from our paper [GHWI19a].
3.2. SOLUTION OF THE SCHWINGER-DYSON EQUATIONS

**Proposition 3.8.** Let \( f(z) = \sum_{k=0}^{\infty} \frac{a_k}{z^k} \) be an even Laurent series about \( z = 0 \) bounded at \( \infty \). Then the inverse of the integral operator \( \hat{K}_z \) of Definition 3.1 is given by the residue formula

\[
\left[ z^2 \hat{K}_z \right]^{-1} f(z) = - \text{Res}_{z'\to 0} [K(z, z') f(z') dz'],
\]

where \( K(z, z') := \frac{1}{\lambda(\mathcal{G}_0(z') - \mathcal{G}_0(-z'))(z'^2 - z^2)}. \)

**Proof.** The formulae (3.2.11) give rise to the series expansion

\[
\lambda(\mathcal{G}_0(z') - \mathcal{G}_0(-z')) = \sum_{l=0}^{\infty} \varrho_l \cdot (z')^{2l+1}, \quad (3.2.20)
\]

where the Kontsevich times \( \varrho_l \) are given by Definition 3.2. The series of its reciprocal is found using (3.2.19):

\[
\frac{1}{\lambda(\mathcal{G}_0(z') - \mathcal{G}_0(-z'))} = \sum_{m=0}^{\infty} \frac{(z')^{2m}}{m!} S_m, \quad (3.2.21)
\]

where \( S_m := \frac{d^m}{dz^m} \bigg|_{z=0} \left( \sum_{l=0}^{\infty} \frac{\varrho_l}{\varrho_0} z^l \right)^{-1} \). Then the series expansion is then

\[
K(z, z') = -\frac{1}{z^2 z' \varrho_0} \sum_{n,m=0}^{\infty} \frac{(z')^{2m+2n}}{m! z^{2n}} S_m. \quad (3.2.22)
\]

The residue of a monomial in \( f(z') = \sum_{k=0}^{\infty} \frac{a_k}{(z')^k} \) is then

\[
\text{Res}_{z'\to 0} \left[ K(z, z') \frac{dz'}{(z')^{2k}} \right] = -\frac{1}{\varrho_0} \sum_{j=0}^{k} \frac{S_j}{j! z^{2k-2j+2}}. \quad (3.2.23)
\]

In the next step we apply the operator \( z^2 \hat{K}_z^{-1} \) to (3.2.23), where Lemma 3.1 is used:

\[
z^2 \hat{K}_z \left( \frac{1}{z} \frac{(-1)^k}{\varrho_0} \sum_{j=0}^{k} \frac{S_j}{j! z^{2k-2j+2}} \right) = z^2 \sum_{j=0}^{k} \sum_{i=0}^{k-j} \frac{S_j \varrho_i}{j! z^{2k-2j-2i+2}} - \sum_{j=0}^{k} \frac{S_{k-j} \varrho_{j-i}}{(k-j)! z^{2j}}. \quad (3.2.24)
\]

The last sum over \( j \) is treated as follows, where the Bell polynomials are inserted for \( S_m \):

\[
\sum_{j=0}^{k} \frac{S_{k-j} \varrho_{j-i}}{(k-j)!} = \sum_{j=1}^{k-i} \frac{S_{k-j-i} \varrho_j}{(k-j-i)!}
\]

\[
= \sum_{j=1}^{k-i} \sum_{l=0}^{k-j-i} \frac{(-1)^l j!}{(k-j-i)! \varrho_0} \varrho_j B_{k-j-i,l}(1! \varrho_1, \ldots, (k-j-i-l+1)! \varrho_{k-j-i-l+1})
\]

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Thus, equation (3.2.26) holds for genus of sums, and used the following identity for the Bell polynomials \([\text{GSW17}, \text{Lemma 5.9}]\):

\[
\sum_{l=0}^{k-i} \frac{(-1)^l}{l!} \sum_{j=1}^{k-i-l} \binom{k-i}{j} j! \, g_j \, B_{k-j-i,l}(1!) \, \rho_1, \ldots, (k-j-i-l+1)! \, \rho_{k-j-i-l+1})
\]

\[
= \sum_{l=0}^{k-i} \frac{(-1)^l(l+1)!}{l!} B_{k-i,l+1}(1! \, \rho_1, \ldots, (k-i-l)! \, \rho_{k-i-l})
\]

\[
= -\rho_0 \, S_{k-i} \, (k-i)!
\]

We have used \(B_{n,0} = 0\) and \(B_{0,n} = 0\) for \(n > 0\) to eliminate some terms, changed the order of sums, and used the following identity for the Bell polynomials \([\text{GSW17}, \text{Lemma 5.9}]\):

\[
\sum_{j=1}^{n-k} \binom{n}{j} x_j \, B_{n-j,k}(x_1, \ldots, x_{n-j-k+1}) = (k+1) \, B_{n,k+1}(x_1, \ldots, x_{n-k}). \tag{3.2.25}
\]

Inserting back, we find that (3.2.24) reduces to the \((j = k)\)-term of the first sum in the last line of (3.2.24), i.e.

\[
z^2 \hat{K}_z \left( \frac{1}{z} \, \text{Res}_{z'} \left[ K(z, z') \frac{dz'}{(z')^{2k}} \right] \right) = -\frac{1}{z^{2k}}.
\]

This finishes the proof. \(\square\)

**Theorem 3.2.** For any \(g \geq 1\) and \(z \in \mathbb{C} \setminus \{0\}\) one has

\[
\mathcal{G}_g(z) = \frac{\lambda}{z} \, \text{Res}_{z'} \left[ K(z, z') \left\{ \sum_{h=1}^{g-1} \mathcal{G}_h(z') \mathcal{G}_{g-h}(z') + \mathcal{G}_{g-1}(z', z') \right\} (z')^2 dz' \right]. \tag{3.2.26}
\]

*Proof.* The formula arises when applying Proposition 3.8 to (3.2.11) with \(b = 1\) and holds if the function in \(\{\}\) is an even Laurent polynomial in \(z'\) bounded in \(\infty\). This is the case for \(g = 1\) where only \(\mathcal{G}_0(z', z') = \frac{\lambda^2}{(z')^4}\) contributes. Evaluation of the residue reconfirms the 1-point function of Proposition 3.7. We proceed by induction in \(g \geq 2\), assuming that all \(\mathcal{G}_h(z')\) with \(1 \leq h < g\) on the rhs of (3.2.26) are odd Laurent polynomials bounded in \(\infty\); their product is even. The induction hypothesis also verifies the assumption of Theorem 3.1 so that \(\mathcal{G}_{g-1}(-z', -z') = -\mathcal{G}_g(z', z') = \mathcal{G}_g(z', z')\) is even and, because of \(\mathcal{G}_{g-1}(z', z'') = (2\lambda)^3 \hat{A}_{g-1}^{\text{reb}} \mathcal{G}_{g-1}(z'')\), inductively a Laurent polynomial bounded in \(\infty\). Thus, equation (3.2.26) holds for genus \(g \geq 2\) and, as consequence of (3.2.23), \(\mathcal{G}_g(z)\) is again an odd Laurent polynomial bounded in \(\infty\). Equation (3.2.26) is thus proved for all \(g \geq 1\), and the assumption of Theorem 3.1 is verified. \(\square\)

A more precise characterisation can be given. It relies on

**Definition 3.5.** A polynomial \(P(x_1, x_2, \ldots)\) is called \(n\)-weighted if

\[
\sum_{k=1}^{\infty} k^x_k \frac{\partial}{\partial x_k} P(x_1, x_2, \ldots) = n \cdot P(x_1, x_2, \ldots).
\]

The Bell polynomials \(B_{n,k}(x_1, \ldots, x_{n-k+1})\) are \(n\)-weighted. The number of monomials in an \(n\)-weighted polynomial is \(p(n)\), the number of partitions of \(n\). The product of an \(n\)-weighted by an \(m\)-weighted polynomial is \((m + n)\)-weighted.
Proposition 3.9. For \( g \geq 1 \) one has
\[
G_g(z) = (2\lambda)^{4g-1} \sum_{k=0}^{3g-2} \frac{P_{3g-2-k}(q)}{\varrho_0^{2g-1-2k+3}},
\]
where \( P_0 \in \mathbb{Q} \) and the \( P_j(q) \) with \( j \geq 1 \) are \( j \)-weighted polynomials in \( \{ \varrho_1, \ldots, \varrho_k \} \) with rational coefficients.

**Proof.** The case \( g = 1 \) is directly checked. We proceed by induction in \( g \) for both terms in \( \{ \} \) in (3.2.26). The hypothesis gives \( G_g(z')G_{g-h}(z') = (2\lambda)^{4g-2} \sum_{k=0}^{3g-4} \frac{P_{3g-4-k}(q)}{\varrho_0^{2g-2}(z')^{2k+6}}. \) In the second term in \( \{ \} \), \( G_{g-1}(z, z) = (2\lambda)^3 \hat{A}_{g-1} \) which has the same structure as \( G_g(z')G_{g-h}(z') \). Application of (3.2.23) yields
\[
(2\lambda)^3 \hat{A}_{g-1} \hat{A}_{g-1}(z') = (2\lambda)^{4g-2} \sum_{k=0}^{3g-5} \sum_{l=0}^{3g-5-k} \frac{P_{3g-5-k-l}(q)}{(z')^{2k+4}} = (2\lambda)^{4g-2} \sum_{k=0}^{3g-5} \left( \frac{P_{3g-4-k}(q)}{(z')^{2k+6}} + \frac{P_{3g-5-k}(q)}{(z')^{2k+8}} \right),
\]
which has the same structure as \( G_h(z')G_{g-h}(z') \). Application of (3.2.23) yields
\[
\lambda \frac{\text{Res}}{z \to z'} [K(z, z')dz']^2 (2\lambda)^{4g-2} \sum_{k=0}^{3g-4} \frac{P_{3g-4-k}(q)}{\varrho_0^{2g-2}(z')^{2k+6}}
\]
\[
= (2\lambda)^{4g-1} \sum_{k=0}^{3g-4} \sum_{j=0}^{k+2} \frac{P_{3g-4-k}(q)}{\varrho_0^{2g-1-2k+3}}
\]
\[
= (2\lambda)^{4g-1} \sum_{k=0}^{3g-2} \frac{P_{3g-2-k}(q)}{\varrho_0^{2g-1-2k+3}},
\]
because \( S_j(q) \) is also a \( j \)-weighted polynomial by (3.2.21). \( \square \)

In particular, this proves the assumption of Theorem 3.1 namely that \( G_g(z) \) depends only on \( \{ \varrho_0, \ldots, \varrho_{3g-2} \} \). To be precise, we reciprocally increase the genus in Theorem 3.1 and Proposition 3.9.

### 3.2.7 Link to Topological Recursion

A \((1+1+\ldots+1)\)-point function of genus \(g\) with \(b\) boundary components fulfills a universal structure called *topological recursion*. To introduce it, we have to define the following functions:

**Definition 3.6.** The function \( \omega_{g,b} \) is defined by
\[
\omega_{g,b}(z_1, \ldots, z_b) := \left( \prod_{i=1}^{b} z_i \right) \left( G_g(z_1, \ldots, z_b) + 16\lambda^2 \frac{\delta_{g,0}\delta_{2,b}}{(z_1^2 - z_2^2)^2} \right) dz_1 \ldots dz_n
\]
as symmetric differential forms on \( \hat{\mathbb{C}}^n \) with \( \hat{\mathbb{C}} = \mathbb{C} \cup \{ \infty \} \), and the spectral curve \( y(z(x)) \) by \( x(z) = z^2 - c \) and

\[
y(z) = -G_0(z) = \frac{z}{2\lambda\sqrt{z}} + \frac{\nu}{2} - \frac{1}{4\lambda} \int \frac{\sqrt{(1+2\epsilon(\Lambda^2))^2+c}}{t(t-z)} \, dt.
\]

(3.2.27)

It can be checked that with these definitions, up to trivial redefinitions by powers of \( 2\lambda \), the theorems proved in [Eyn16] apply for topological recursion. These determine all \( \omega_{g,B} \) with \( 2-2g-B < 0 \) out of the initial data \( y(z), x(z) \) and \( \omega_{0,2} \):

**Theorem 3.3** (Eyn16 Thm. 6.4.4). For \( 2-2g-(1+b) < 0 \) and \( J = \{ z_1, ..., z_b \} \) the function \( \omega_{g,b+1}(z_0, ..., z_B) \) is given by topological recursion

\[
\omega_{g,b+1}(z_0, ..., z_B) = \text{Res}_{z \to 0} \left[ K(z_0, z) \, dz \left( \omega_{g-1,b+2}(z, -z, J) + \sum_{h+h'=g \atop I \cup I' = J} \omega_{h,|I|+1}(z, I) \omega_{h',|I'|+1}(-z, I') \right) \right],
\]

where \( K(z_0, z) = \frac{\lambda}{(z^2 - z_0^2)(y(z) - y(-z))} \) and the sum \( \sum' \) excludes \((h, I) = (0, \emptyset) \) and \((h, I) = (g, J) \).

This theorem motivated our ansatz for an inverse of \( \hat{K} \) as the residue involving \( K(z, z') \). The case \( J = \emptyset \) of Theorem 3.3 is essentially the same as Theorem 3.2. For us there is no need to prove the general case because higher \( \omega(J) \) can be obtained from Theorem 3.1.

Notice that our proof differs completely and is more combinatorial in comparison to the proof of Theorem 3.3, since Theorem 3.3 considers a deformation of a complex contour for which \( y(z) \) has to be a meromorphic function (which can be achieved for finite \( N', V \)) instead of a sectional holomorphic function. The contour is then moved through all possible poles and picks up the residues. However, moving the contour through a cut (even of infinite length for \( \Lambda^2 \to \infty \)) is much harder and cannot be done without further effort.

In other words, we have shown that the proofs of Theorem 3.1 and Theorem 3.2 are also valid if the large \( N', V \)-limit is performed first, which implies that the topological recursion and the large \( N', V \)-limit commutes in that special case.

### 3.3 Free Energy \( F_g \)

The free energy \( F_g \) generates the connected vacuum graphs of genus \( g \). Since the partition function \( Z[0] \) generates (not necessarily connected) graphs, the free energy is understood as the logarithm of the partition function or equivalently after genus expansion

\[
Z[0] = \exp \left( \sum_{g=0}^{\infty} V^{2-2g} F_g \right).
\]

To derive the free energies, the inverse of the boundary creation operator has to be defined in such a way that it produces the free energy \( F_g \) from the 1-point function \( G_g(z) \) of genus \( g \) uniquely.

\[\text{§ This section is taken from our paper GHW19a} \]
Definition 3.7. We introduce the operators

\[
\hat{A}_z^\dagger := \sum_{l=0}^{\infty} \left( -\frac{(3 + 2l) \varrho_l + 3 + 2l}{g_0 z^3} \right) \frac{\partial}{\partial \varrho_l}, \quad \hat{N} := -\sum_{l=0}^{\infty} \frac{\varrho_l}{g_l} \frac{\partial}{\partial \varrho_l},
\]

\[
\hat{A}_z f(\bullet) := -\sum_{l=0}^{\infty} \text{Res}_{z \to 0} \left[ \frac{z^{4+2l}}{3+2l} \varrho_l f(z) dz \right].
\]

We call \( \hat{A}_z \) a boundary annihilation operator acting on Laurent polynomials \( f \).

Proposition 3.10. There is a unique function \( F_g \) of \( \{ \varrho_l \} \) satisfying \( G_g(z) = (2\lambda)^3 \hat{A}_z^\dagger F_g \),

\[
F_1 = -\frac{1}{24} \log \varrho_0, \quad F_g = \frac{1}{(2g-2)(2\lambda)^3} \hat{A}_z G_g(\bullet) \quad \text{for } g \geq 1.
\]

The \( F_g \) have for \( g > 1 \) a presentation as

\[
F_g = (2\lambda)^{4g-4} \frac{P_{3g-3}(\varrho)}{\varrho_0^{2g-2}},
\]

where \( P_{3g-3}(\varrho) \) is a \((3g-3)\)-weighted polynomial in \( \{ \frac{\varrho_1}{\varrho_0}, \ldots, \frac{\varrho_{3g-3}}{\varrho_0} \} \).

Proof. The case \( g = 1 \) is checked by direct comparison with \((3.7)\). From Proposition 3.9 we conclude

\[
\frac{1}{(2\lambda)^3} \hat{A}_z G_g(\bullet) = -(2\lambda)^{4g-4} \text{Res}_{z \to 0} \left[ \sum_{l=0}^{\infty} \frac{\varrho_l z^{4+2l}}{3+2l} \sum_{k=0}^{3g-2} \frac{P_{3g-2-k}(\varrho)}{\varrho_0^{2g-1-2k} z^{2k+3}} dz \right]
= (2\lambda)^{4g-4} \sum_{k=1}^{3g-2} \frac{\varrho_{k-1}}{\varrho_0} \frac{P_{3g-2-k}(\varrho)}{\varrho_0^{2g-2}} = (2\lambda)^{4g-4} \frac{P_{3g-3}(\varrho)}{\varrho_0^{2g-2}},
\]

which confirms \((3.3.2)\). Observe that the total \( \varrho \)-counting operator \( \hat{N} \) applied to any polynomial in \( \{ \frac{\varrho_1}{\varrho_0}, \frac{\varrho_2}{\varrho_0}, \ldots \} \) is zero. Therefore, for \( g > 1 \),

\[
\hat{N} \left( \frac{1}{(2\lambda)^3} \hat{A}_z G_g(\bullet) \right) = (2g-2) \cdot \left( \frac{1}{(2\lambda)^3} \hat{A}_z G_g(\bullet) \right).
\]

The boundary annihilation operator is designed to satisfy \( \hat{A}_z \circ \hat{A}_z^\dagger = \hat{N} \). Dividing the previous equation by \( (2g-2) \) and inserting the ansatz for \( F_g \) given in the proposition, we have

\[
0 = \hat{N} F_g - \frac{1}{(2\lambda)^3} \hat{A}_z G_g(\bullet) = \hat{A}_z \left( \hat{A}_z^\dagger F_g - \frac{1}{(2\lambda)^3} G_g(\bullet) \right).
\]

Since \( f(z) := \hat{A}_z^\dagger F_g - \frac{1}{(2\lambda)^3} G_g(z) \) is by \((3.3.2)\) and Proposition 3.9 a Laurent polynomial bounded at \( \infty \), application of \( \hat{A}_z \) can only vanish if \( f(z) \equiv 0 \). This finishes the proof. \( \square \)

Remark 3.7. Proposition 3.10 shows that the \( F_g \) provide the most condensed way to describe the non-planar sector of the \( \Phi^3 \)-matricial QFT model. All information about the genus-\( g \) sector is encoded in the \( p(3g-3) \) rational numbers which form the coefficients in the \((3g-3)\)-weighted polynomial in \( \{ \frac{\varrho_1}{\varrho_0}, \frac{\varrho_2}{\varrho_0}, \ldots \} \). From these polynomials we obtain the \((1+\cdots+1)\)-point function with \( b \) boundary components via \( G_g(z) = (2\lambda)^3 \hat{A}_z^\dagger F_g \) followed by Theorem 3.1.
Lemma 3.3. For \((2g + b - 2) > 0\), the operator \(\tilde{N}\) measures the Euler characteristics,
\[
\tilde{N} \mathcal{G}_g(z_1, ..., z_b) = (2g + b - 2) \mathcal{G}_g(z_1, ..., z_b).
\]

Proof. Both cases with \((2g + b - 2) = 1\) are directly checked. The general case follows by induction from \([\tilde{N}, \tilde{A}_g] = \tilde{A}_g\) in combination with Theorem 3.1 and \(\tilde{N} F_g = (2g - 2) F_g\) for \(g \geq 2\).

Corollary 3.4.
\[
\tilde{A}_g \mathcal{G}_g(\bullet, z_2, ..., z_b) = (2\lambda)^3 (2g + b - 3) \mathcal{G}_g(z_2, ..., z_b)
\]
whenever \((2g + b - 3) > 0\).

Hence, up to a rescaling, \(\tilde{A}_g\) indeed removes the boundary component previously located at \(z\). We also have \(\tilde{A}_g F_g = 0\) for all \(g \geq 1\), so the \(F_g\) play the rôle of a vacuum or the free energy. Note that \(\mathcal{G}_0(z)\) cannot be produced with the operator \(\tilde{A}_g\) by whatever \(F_0\).

Remark 3.8. In the case of \(\mathcal{D} = 0\) the free energy of genus \(g = 0\) exists and is given by (adapted from [MS91])
\[
F_0 = \frac{1}{12\lambda^2} \int_0^{\Lambda^2} dt \frac{r(t)\sqrt{(1 + 2e(t))^2 + c^2}}{12\lambda^2} \int_0^{\Lambda^2} dt r(t) (1 + 2e(t))^3
\]
\[
- \frac{c}{8\lambda^2} \int_0^{\Lambda^2} dt \frac{r(t)\sqrt{(1 + 2e(t))^2 + c}}{12 \cdot 4^3 \lambda^4}
\]
\[
- \frac{1}{2} \int_0^{\Lambda^2} dx r(x) \int_0^{\Lambda^2} dt \frac{r(t)\log(\sqrt{(1 + 2e(t))^2 + c} + \sqrt{(1 + 2e(x))^2 + c})}{\sqrt{2\lambda}}.
\]

Acting with the operator \(T(x)\) (defined in (3.2.14)) on \(F_0\) gives precisely \(G^0(x) = \frac{W((1 + 2e(x))^2) - (1 + 2e(x))}{2\lambda}\). All parts proportional to \(T(x)c\) vanish which makes \(F_0\) stationary with respect to \(c\). By performing the computation \(T(x)F_0\), the implicit function of \(c\) (3.2.15) for \(\mathcal{D} < 2\) is of tremendous importance. Note that the planar free energy diverges for \(\Lambda^2 \to \infty\) such that finite \(\mathcal{N}, V\) or finite \(\Lambda^2\) are necessary which implies \(\mathcal{D} = 0\). Any renormalisation constant prevents to write down \(F_0\) in a closed form even for finite \(\Lambda^2\) since the implicit equation of \(c\) differs.

3.3.1 A Laplacian to Compute Intersection Numbers

The naive picture of intersection numbers is that they are counting the number of intersections of curves, which should give positive integers. However, if one assumes complex curves up to some equivalence class this will change tremendously and can give rational intersection numbers instead of integers.

Let \(\mathcal{M}_{g,b}\) be the moduli space of equivalence classes of complex curves of genus \(g\) with \(b\) distinct marked points, modulo biholomorphic reparametrisation. For a negative Euler characteristic \(\chi = 2 - 2g - b\), the moduli space \(\mathcal{M}_{g,b}\) is locally parameterised by \(d_{g,b} := (3g - 3 + b)\) complex parameters called moduli. The fact that \(\mathcal{M}_{g,b}\) has rational intersection numbers is due to the orbifold structure, which looks locally like the quotient space of \(\mathbb{C}^n\) under a linear action of a finite group. An orbifold is a generalisation of a manifold, where the finite group is trivial.
The moduli space \( \mathcal{M}_{g,b} \) is, in general, not compact and can be compactified by adding degenerate curves. The Deligne-Mumford compactification [DM69] provides an analytic structure and is the usual compactification used in the literature denoted by \( \overline{\mathcal{M}}_{g,b} \).

The \( b \) distinct marked points lead naturally to a family \( \{ \mathcal{L}_1, \ldots, \mathcal{L}_b \} \) of complex line bundles over \( \overline{\mathcal{M}}_{g,b} \). A classification of the line bundles is given by the first Chern class \( c_1(\mathcal{L}_i) \) which is an element of the second cohomology group \( H^2(\overline{\mathcal{M}}_{g,b}, \mathbb{Q}) \). The first Chern class \( c_1(\mathcal{L}_i) \) is independent of the choice of the connection on \( \mathcal{L}_i \) and has a curvature form as representative.

The intersection numbers of \( \overline{\mathcal{M}}_{g,b} \) are then defined by

\[
\langle \tau_{d_1} \cdots \tau_{d_s} \rangle := \int_{\overline{\mathcal{M}}_{g,b}} \bigwedge_{i=1}^{s} (c_1(\mathcal{L}_i))^{d_i}, \quad \sum_{i=1}^{s} d_i = d_{g,b} \tag{3.3.3}
\]

where the 2-forms \( c_1(\mathcal{L}_i) \) are multiplied with a commutative wedge product. These numbers are topological invariants of \( \overline{\mathcal{M}}_{g,b} \).

It was then conjectured by Witten [Wit91b] that the generating function of these intersection numbers satisfies a partial differential equation called the Korteweg-de Vries equation. Additionally, it was known that the integrability of matrix models is due to partial differential equation of the free energy which was also of Korteweg-de Vries type. This observation was subsequently proved to be not a coincidence

**Theorem 3.4. (Kon92)** The generating function of the intersection numbers of \( \psi \)-classes on the moduli space \( \overline{\mathcal{M}}_{g,b} \) defined in (3.3.3) is equal to the free energy \( F_g \) of the Kontsevich model

\[
F_g = \sum_{(k)} \langle \tau_{k_1} \cdots \tau_{k_s} \rangle \prod_{i=2}^{3g-2} \frac{k_i}{k_1!}, \quad \sum_{i=2}^{3g-2} (i-1)k_i = 3g - 3,
\]

where \( t_{i+1} := -(2i+1)!!t_i \) of Definition 3.2

Since we have now a prescription to derive the free energy \( F_g \) for any \( g > 0 \), also the intersection numbers can be derived simultaneously due to Theorem 3.4.

To achieve this, we express for \( b = 1 \) in (3.2.12) \( G_g(z) = (2\lambda)^3 \hat{A}_z^1 F_g \) and \( G_g(z,v) = (2\lambda)^3 (\hat{A}_z^1 + \frac{1}{4\vartheta} \frac{\partial}{\partial z}) (G_g(z)) \) and multiply by \( \frac{2V^2}{(2\lambda)^4} Z^np \). Summation over \( g \geq 1 \) gives

\[
0 = \left( \frac{2V^2}{(2\lambda)^4} \hat{K}_z \hat{A}_z^1 + \left( \hat{A}_z^1 + \frac{1}{4\vartheta} \frac{\partial}{\partial z} \right) \hat{A}_z^1 + \frac{V^2}{4(2\lambda)^4} \right) Z^np, \tag{3.3.4}
\]

where \( Z^np := \exp \left( \sum_{g=1}^{\infty} V^{2-2g} F_g \right) \).

We invert \( \hat{K}_z \) via Proposition 3.8 and apply \( \hat{A}_z \) given by the residue in Proposition 3.10

\[
\frac{2V^2}{(2\lambda)^4} \hat{N} Z^np = -\sum_{\ell=0}^{\infty} \text{Res}_{z=0} \left[ dz \left( z^{3+2\ell} \right) \text{Res}_{z'=0} \left[ d\alpha(z')^2 K(z, z') \left( \left( \hat{A}_z^1 + \frac{1}{4\vartheta} \frac{\partial}{\partial z} \right) \hat{A}_z^1 + \frac{V^2}{4(2\lambda)^4} \right) \right] Z^np \right].
\]

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We insert $K(z, z')$ from Proposition 3.8, expand only the geometric series about $z' = 0$ while keeping (3.2.20). Then the outer residue in $z$ is immediate

$$2V^2 \left( \frac{N}{z'} \right)^{np} = \text{Res}_{z' \to 0} \left[ (z')^3 dz' \sum_{l=0}^{\infty} \sum_{j=0}^{\infty} \frac{z_j^{2l} \rho_j (z')^{j'}}{\rho_0 (z')^{j'}} \left( \hat{A}_{z'}^1 + \frac{1}{\rho_0 (z')^4} \frac{\partial}{\partial z'} \hat{A}_{z'}^1 + \frac{V^2}{4(2\lambda)^4} \right) \right] Z^np.$$

We rename $z'$ to $z$ and introduce the function

$$\mathcal{R}(z) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{z_j^{2l} \rho_j z_j^{2j}}{\rho_0 z_j^{2j}} = \sum_{m=0}^{\infty} \mathcal{R}_m(z) z^{2m}.$$

The denominator is given by (3.2.21), without the $\frac{1}{z'\rho_0}$ prefactor. It combines with the numerator to

$$\mathcal{R}_m(z) = \frac{S_m(z)}{3m!} - \frac{m}{(3+2k)\rho_0 (m-k)!} \sum_{k=1}^{m} k\rho_k S_{m-k}(z) - \frac{2}{3} \sum_{k=1}^{m} \frac{k\rho_k S_{m-k}(z)}{(3+2k)\rho_0 (m-k)!},$$

where we have used (3.2.25) for the first $S_m(z)$ to achieve better control of signs.

The residue of $\frac{1}{z'\rho_0}$ is immediate and can be moved to the lhs:

$$2V^2 \left( \frac{N}{z'} - \frac{1}{24} \right) Z^np = \sum_{m=0}^{\infty} \mathcal{R}_m(z) \text{Res}_{z' \to 0} \left[ z^{3+2m} dz' \left( \hat{A}_{z'}^1 + \frac{1}{\rho_0 z' z^4} \frac{\partial}{\partial z'} \hat{A}_{z'}^1 \right) \right] Z^np$$

$$= \left[ \sum_{k=0}^{\infty} \frac{3(3+2k)\rho_1 \rho_k \mathcal{R}_1(z)}{\rho_0 \rho_0^2} + \frac{3(3+2k)\rho_k \mathcal{R}_2(z)}{\rho_0} \frac{\partial}{\partial \rho_k} \right] Z^np$$

$$= \sum_{k,l=0}^{\infty} \frac{(3+2k)(3+2l)\mathcal{R}_l(z)}{\rho_0} \frac{\partial}{\partial \rho_l} \frac{\partial}{\partial \rho_k}$$

Next we separate the $\rho_0$-derivatives:

$$\frac{2V^2}{(2\lambda)^4} \left( \frac{N}{z'} - \frac{1}{24} \right) Z^np$$

$$= \left[ \frac{9\mathcal{R}_1(z) \rho_0^2}{\rho_0^2} - \frac{18\mathcal{R}_2(z) \rho_1}{\rho_0} + \frac{9\mathcal{R}_3(z)}{\rho_0^2} \right] \frac{\partial^2}{\partial \rho_0^2}$$

$$+ \left( - \frac{9\rho_0^2 \mathcal{R}_1(z)}{\rho_0^2} + \frac{18\rho_0 \mathcal{R}_2(z)}{\rho_0} + \frac{15\mathcal{R}_1(z) \rho_0^2}{\rho_0} - \frac{30\mathcal{R}_3(z)}{\rho_0} \right) \frac{\partial}{\partial \rho_0}$$

$$+ \sum_{k=1}^{\infty} \frac{6(3+2k) \left( \mathcal{R}_k(z) \rho_{k+1} - \mathcal{R}_k(z) \rho_k \rho_{k+1} + \mathcal{R}_k(z) \rho_{k+2} \rho_0 - \mathcal{R}_k(z) \rho_{k+1} \rho_0 \rho_{k+1} \rho_0 \right) \partial \rho_k \partial \rho_0}.$$
We isolate $F_1$, i.e. $Z_V^{np} = \theta_0^{-\frac{1}{2}} Z_V^{stable}$, where $Z_V^{stable} = \exp \left( \sum_{g=2}^{\infty} V^{-2g} F_g \right)$. We commute the factor $\theta_0^{-\frac{1}{2}}$ in front of $\left[ \right]$ and move it to the other side:

\[
\frac{2V^2}{(2\lambda)^4} \hat{N} Z_V^{stable} = \left[ \left( 49 \theta_1^2 \theta_1 + \frac{49 \theta^4_2 \theta_1^2}{32 \theta_0^3} - \frac{5 R_1 \theta_1^2}{8 \theta_0^3} + \frac{105 R_3 \theta_1^2}{64 \theta_0^3} \right) + \left( \frac{9 R_1 \theta_1^2}{\theta_0^3} - \frac{18 R_2 \theta_2}{\theta_0^2} + 9 R_3 \theta_1 \right) \frac{\partial^2}{\partial \theta_0^2} \right.
\]
\[
+ \left( \left( -\frac{39 \theta_1^2 \theta_1^2}{4 \theta_0^3} + \frac{39 \theta_1^2 \theta_2^2}{2 \theta_0^3} + \frac{15 R_1 \theta_1^2}{\theta_0^3} - \frac{123 R_3 \theta_1^2}{4 \theta_0^3} \right) \frac{\partial}{\partial \theta_0} \right.
\]
\[
\left. + \sum_{k=1}^{\infty} 6(3+2k) \left( \frac{R_{k+3}(\theta)}{\theta_0} + \frac{R_{g+2}(\theta)}{\theta_0} + \frac{R_{g+1}(\theta)}{\theta_0} + \frac{R_{g+1}(\theta)}{\theta_0} \right) \frac{\partial}{\partial \theta_0} \right] Z_V^{stable}.
\]

Next observe

\[
\hat{N} Z_V^{stable} = \sum_{g=2}^{\infty} (V^{-2})^{g-1}(2g-2) Z_g = 2V^{-2} \frac{d}{dV^{-2}} \sum_{g=2}^{\infty} (V^{-2})^{g-1} Z_g = 2V^{-2} \frac{d}{dV^{-2}} Z_V^{stable}.
\]

Consequently, we obtain a parabolic differential equation in $V^{-2}$ which is easily solved. Inserting

\[
R_1(\theta) = -\frac{2}{15} \theta_1 \frac{\theta_1}{\theta_0}, \quad R_2(\theta) = \frac{2}{15} \theta_1^2 \frac{\theta_2}{\theta_0} - \frac{4}{21} \theta_2, \quad R_3(\theta) = -\frac{2}{15} \theta_1^3 + 2 \theta_1 \theta_2 + \frac{34}{105} \theta_1 \theta_2 \frac{\theta_1}{\theta_0} - \frac{2}{9} \theta_3,
\]

we have:

**Theorem 3.5.** When expressed in terms of the moments of the measure $\theta$, the stable partition function is given by

\[
Z_V^{stable} := \exp \left( \sum_{g=2}^{\infty} V^{-2g} F_g(\theta) \right) = \exp \left( -\frac{(2\lambda)^4}{V^2} \Delta_{\theta} + F_2(\rho) \right) 1,
\]
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where

\[ F_2 = \frac{(2\lambda)^4}{V^2} \left( - \frac{21 g_1^3}{160 g_0^6} + \frac{29 \varrho_1 \varrho_2}{128 g_1^4} - \frac{35 \varrho_3}{384 g_0^3} \right) \]  

\[ \Delta_p := -\left( -\frac{6g_1^3}{5g_0^3} + \frac{111\varrho_1 \varrho_2}{70 g_0^6} - \frac{3g_3}{2g_0^2} \right) \frac{\partial^2}{\partial \varrho_1^2} - \left( \frac{2g_1^3}{g_0^3} - \frac{1097 \varrho_1 \varrho_2}{280 g_0^3} + \frac{41g_3}{24g_0^3} \right) \frac{\partial}{\partial \varrho_0} \]

\[ - \sum_{k=1}^{\infty} (3 + 2k) \left( \frac{-2g_1^2}{5g_0^2} + \frac{2g_2}{2g_0^2} \right) \varrho_{k+1} - \frac{3R_{k+2}(\varrho) \varrho_1}{2 \varrho_0} + \frac{3R_{k+3}(\varrho)}{2} \right) \frac{\partial^2}{\partial \varrho_k \partial \varrho_0} \]

\[ + \sum_{k,l=1}^{\infty} (3 + 2k)(3 + 2l) \left( \frac{\varrho_1 \varrho_{l+1} \varrho_{k+1}}{30 \varrho_0^2} + \frac{\varrho_{k+1} R_{l+2}(\varrho)}{4 \varrho_0} + \frac{\varrho_{l+1} R_{k+2}(\varrho)}{4 \varrho_0} - \frac{R_{k+l+3}(\varrho)}{4} \right) \frac{\partial^2}{\partial \varrho_k \partial \varrho_l} \]

\[ - \sum_{k=1}^{\infty} (3 + 2k) \left( \frac{19g_1^2}{60g_0^5} \right) \varrho_{k+1} + \frac{\varrho_1 R_{k+2}(\varrho)}{16g_0^5} - \frac{R_{k+3}(\varrho)}{16g_0^5} \right) \frac{\partial^2}{\partial \varrho_k} \]

\[ - \frac{(5 + 2k) \varrho_1 \varrho_{k+2}}{30g_0^5} - \frac{(5 + 2k) R_{k+3}(\varrho)}{2g_0} \frac{\partial}{\partial \varrho_k} \]

\[ \text{(3.3.7)} \]

and \( R_m(\varrho) \) given by (3.3.5).

The \( F_g(\varrho) \) are recursively extracted from \( Z_g(\varrho) := \frac{1}{(g-1)!}(-\Delta_p + F_2(\varrho))^{g-1} \) by

\[ F_g(\varrho) = Z_g(\varrho) - \frac{1}{(g-1)!} \sum_{k=2}^{g-1} B_{g-1,k} \left( \{ h!F_{h+1}(\varrho) \}_{h=1}^{g-k} \right) \]

\[ = Z_g(\varrho) - \frac{1}{(g-1)!} \sum_{k=2}^{g-1} (-1)^k (k - 1)! B_{g-1,k} \left( \{ h!Z_{h+1}(\varrho) \}_{h=1}^{g-k} \right). \]

These free energies are listed in different conventions in the literature. As mentioned before, for any spectral dimension \( D < 8 \) the planar sector is renormalised such that \[ \text{Theorem 3.4} \]
holds. Therefore, \[ \text{Theorem 3.5} \]
produces the intersection numbers via the differential operator \( \Delta_p \).

The formula can easily be implemented in computer algebra and quickly computes the free energies \( F_g(t) \) to moderately large \( g \). For convenience we list

\[ F_3 = \frac{1225}{144} \frac{t_2^6}{6!(t_1)^6} + \frac{193}{288} \frac{t_2^3 t_3}{4!(t_1)^5} + \frac{205}{3456} \frac{t_2^2 t_4}{2!(t_1)^4} + \frac{53}{1152} \frac{t_2^3 t_4}{3!(t_1)^6} \]

\[ + \frac{583}{96768} \frac{t_3}{3!(t_1)^7} + \frac{241920}{5760} \frac{t_2 t_3 t_4}{2!(t_1)^7} + \frac{17}{1451520} \frac{t_2 t_5}{2!(t_1)^6} \]

\[ + \frac{503}{1451520} \frac{t_3 t_5}{6!(t_1)^6} + \frac{77}{414720} \frac{t_2 t_6}{t_1^5} + \frac{1}{82944} \frac{t_7}{t_1^5} \]

with \( t_{i+1} = -(2i + 1)! \varrho_i \), which is already given in [IZ92 eq. (5.30)].

\[ \text{Theorem 3.5} \]
seems to be closely related with \( \exp(\sum_{g \geq 0} F_g) = \exp(\hat{W}) \) 1 proved by Alexandrov [Ale11], where \( \hat{W} := \frac{2}{3} \sum_{k=1}^{\infty} (k + \frac{1}{2}) \varrho_k \hat{L}_{k-1} \) involves the generators \( \hat{L}_n \) of the Virasoro algebra. Including \( V \) and moving \( \exp(V^2 F_0 + F_1) \) to the other side, our \( \Delta_p \) is in principle obtained via Baker-Campbell-Hausdorff formula from Alexandrov’s equation.

### 3.3.2 Deformed Virasoro Algebra

The Kontsevich model without any renormalisation constants has some global constraints \( L_n Z = 0 \), where \( L_n \) is a differential operator with \( n \in \mathbb{N} \). These operators form a Witt
\[ [L_n, L_m] = (n - m)L_{n+m}, \]

an infinite-dimensional Lie algebra. The differential operator can be found by reducing the partition function to an integral over eigenvalues \( x_i \) of the Hermitian matrices. The partition function is then unchanged under the diffeomorphism \( x_i \mapsto x_i^{n+1} \frac{d}{dx_i} \) which gives exactly the generators \( L_n \) [MS91].

Since the renormalisation constants change the partition function, and are in general divergent in the limit \( \Lambda^2 \to \infty \), the Virasoro constraints are affected. Furthermore, the differential operator depends explicitly on \( c(\lambda) \) which obeys different implicit equations for different spectral dimensions.

To find the right Virasoro constraints for any \( D \) we return to (3.3.4), but instead of applying the inverse of \( \hat{K}_z \) we directly take the residue
\[
\hat{L}_n := \text{Res}_{z=0} \left[ z^{3+2n} \left( \frac{2V^2}{(2\lambda)^4} \hat{K}_z \hat{A}^\dagger_z + (\hat{A}^\dagger_z)^2 + \frac{1}{\varrho_z z^4} \frac{\partial \hat{A}^\dagger_z}{\partial z} + \frac{V^2}{4(2\lambda)^4 z^4} \right) dz \right].
\]

By construction, \( \hat{L}_n \varrho_z^{np} = 0 \). Recall that in the Kontsevich model \( L_n \) annihilates the full partition function. However, for \( D > 0 \) the free energy \( F_0 \) does not exist such that the constraints can annihilate only the non-planar part of the partition function. As explained below, these \( \hat{L}_n \) do not satisfy the commutation relations of the Virasoro algebra exactly.

An explicit expression is obtained from
\[
\hat{K}_z \hat{A}^\dagger_z = \sum_{l=0}^{\infty} \sum_{j=0}^{l} \frac{(3 + 2l) \varrho_{l-j}}{z^{4+2j}} \frac{\partial}{\partial \varrho_l},
\]
\[
\frac{1}{\varrho_z z^4} \frac{\partial}{\partial z} \hat{A}^\dagger_z = \sum_{l=0}^{\infty} \frac{3(3 + 2l) \varrho_{l+1}}{\varrho_0 z^2} - \frac{(3 + 2l)(5 + 2l)}{\varrho_0 z^{10+2l}} \frac{\partial}{\partial \varrho_l},
\]
\[
\hat{A}^\dagger_z \hat{A}^\dagger_z = \sum_{k=0}^{\infty} \frac{(5 + 2k) \varrho_{k+2}}{\varrho_0 z^3} - \frac{5 + 2k}{z^{7+2k}} \frac{(3 + 2k)}{\varrho_0 z^3} \frac{\partial}{\partial \varrho_k} + \sum_{k=0}^{\infty} \frac{(3 + 2l)(3 + 2k) \varrho_{k+1} \varrho_{l+1}}{\varrho_0 z^{2+2k}} \frac{\partial^2}{\partial \varrho_l \partial \varrho_k}
\]
\[
- \sum_{k=0}^{\infty} \frac{2(3 + 2l)(3 + 2k) \varrho_{l+1}}{\varrho_0 z^{8+2k}} \frac{\partial^2}{\partial \varrho_l \partial \varrho_k} + \sum_{k=0}^{\infty} (3 + 2l)(3 + 2k) \frac{\partial^2}{\partial \varrho_l \partial \varrho_k}.
\]

Evaluating the residues and defining \( A = \frac{(2\lambda)^4}{4V^2} \) and rescaling \( \hat{L}_n := A\hat{L}_n \) gives
\[
\hat{L}_0 = \frac{1}{16} + \frac{1}{2} \sum_{l=0}^{\infty} (3 + 2l) \varrho_l \frac{\partial}{\partial \varrho_l},
\]
\[
\hat{L}_1 = \frac{1}{2} \sum_{l=0}^{\infty} (5 + 2l) \varrho_l \frac{\partial}{\partial \varrho_{l+1}} + A \left( \sum_{k=0}^{\infty} \frac{(3 + 2k) \varrho_{k+1}}{\varrho_0} \frac{\partial}{\partial \varrho_k} - \frac{3 \varrho_l}{\varrho_0} \right) \sum_{l=0}^{\infty} (3 + 2l) \varrho_{l+1} \frac{\partial}{\partial \varrho_l},
\]
and for \( n \geq 2 \):
\[
\hat{L}_n = \frac{1}{2} \sum_{l=0}^{\infty} (3 + 2n+2l) \varrho_l \frac{\partial}{\partial \varrho_{n+l}} + A \delta_{n,2} \sum_{l=0}^{\infty} \frac{6(3 + 2l) \varrho_{l+1}}{\varrho_0} \frac{\partial}{\partial \varrho_l} - A \frac{2(2n-3)(2n-1)}{\varrho_0} \frac{\partial}{\partial \varrho_{n-3}}.
\]
\[ + A \sum_{l=0}^{n-3} (3+2l)(2n-2l-3) \frac{\partial^2}{\partial q_l \partial q_{n-3-l}} - A \sum_{l=0}^{\infty} \frac{2(3+2l)(2n-1)}{q_0} \frac{\partial^2}{\partial q_l \partial q_{n-1}} \frac{\partial}{\partial q_{n-2}} \frac{\partial}{\partial q_{l+1}}. \]

To write it in a more compact way, it is convenient to introduce the differential operator

\[ \hat{D} := \sum_{l=0}^{\infty} \frac{(3+2l)g_{l+1}}{q_0} \frac{\partial}{\partial q_l} \]  

(3.3.8)

Note that \( \frac{\partial}{\partial q_l} \hat{D} \neq \hat{D} \frac{\partial}{\partial q_l} \). The result is:

**Lemma 3.4.** The nonplanar partition function \( \mathcal{Z}_{np}^V := \exp \left( \sum_{g=1}^{\infty} V^{2-2g} F_g \right) \) satisfies the constraints \( \hat{L}_n \mathcal{Z}_{np} = 0 \) for all \( n \in \mathbb{N} \), where

\[ \hat{L}_0 = \frac{1}{16} + \frac{1}{2} \sum_{l=0}^{\infty} (3+2l) \frac{q_l}{g_l} \partial, \]

\[ \hat{L}_1 = \frac{1}{2} \sum_{l=0}^{\infty} (5+2l) \frac{q_l}{g_{l+1}} + A \hat{D}^2 \]

and for \( n \geq 2 \):

\[ \hat{L}_n = \frac{1}{2} \sum_{l=0}^{\infty} (3+2n+2l) \frac{q_l}{g_{n+1}} + A \sum_{l=0}^{n-3} \frac{2(3+2l)(2n-2l-3)}{q_0} \frac{\partial^2}{\partial q_l \partial q_{n-3-l}} \]

\[ - 2A(2n-1) \frac{\partial}{\partial q_{n-2}} \hat{D}, \]

where \( \hat{D} \) is the differential operator defined by (3.3.8) and \( A = \frac{(2\lambda)^4}{4V^2} \).

With the commutation rules

\[ \left[ \hat{D}, q_l \right] = \frac{3+2l}{q_0} q_{l+1}, \quad l \geq 0, \]

\[ \left[ \hat{D}, \frac{\partial}{\partial q_l} \right] = - \frac{1+2l}{q_0} \frac{\partial}{\partial q_{l+1}}, \quad l \geq 1, \]

\[ \left[ \hat{D}, \frac{\partial}{\partial g_0} \right] = \frac{1}{g_0} \hat{D}, \]

we end up after long but straightforward computation:

**Lemma 3.5.** The generators \( \hat{L}_n \) of Lemma 3.4 obey the commutation relation

\[ \left[ \hat{L}_0, \hat{L}_n \right] = -n \hat{L}_n \]

and for any \( m, n \geq 1 \),

\[ \left[ \hat{L}_m, \hat{L}_n \right] = (m-n) \hat{L}_{m+n} - 4A(m+1)B_{n-2} \hat{L}_{m-1} + 4A(n+1)B_{m-2} \hat{L}_{n-1} \]

\[ - 4A \delta_{m,1} \frac{n(n+1)}{g_0^2} \hat{L}_n \hat{L}_{n-2} + 4A \delta_{n,1} \frac{m(m+1)}{g_0^2} \hat{L}_m \hat{L}_{m-2} \]

where

\[ B_m := (2m+3) \frac{\partial}{\partial q_m} \frac{1}{g_0} \quad \text{for} \quad m \geq 0, \quad B_{-1} := - \frac{1}{2} \left\{ \hat{D}, \frac{1}{g_0} \right\}. \]
Remark 3.9. The differential operator $\hat{D}$ has its origin in the implicit definition of the constant $c$ (see Corollary 3.1) and the dependence of $\varrho_l$ on $c$. Since the expression

$$\varrho_{-1} := -8\lambda^2 \int_0^{\Lambda^2} dx \frac{r(x)}{\sqrt{(1 + 2e(x))^2 + c}}, \quad (\varrho_{-1} = c \text{ for } D < 2)$$

diverges for any $D \geq 2$ in the limit $\Lambda^2 \to \infty$, it was necessary to construct the analogue of the derivative $\frac{\partial}{\partial \varrho_{-1}}$ through the differential operator $\hat{D}$. Replacing the differential operator by $\hat{D} \mapsto \frac{\partial}{\partial \varrho_{-1}}$ and the generators by $\hat{L}_n \mapsto \hat{L}_n + \frac{1}{2} \varrho_{-1} \frac{\partial}{\partial \varrho_{-1}} = L_n,$

reverses the original undeformed Virasoro algebra. As explained above, $\varrho_{-1}$ and, consequently, the standard Virasoro generators do not exist in $D \geq 2$. The renormalisation necessary for $D \geq 2$ alters the definition of $c$ and prevents the construction of $\hat{L}_{-1}$ and $F_0$ which in $D = 0$ depend on $\varrho_{-1}$ (see Remark 3.8). Higher topologies ($\chi \leq 0$) are not affected because any explicit $c$-dependence drops out.

### 3.4 Over-Renormalisation

In this short section which has not yet appeared in the literature, we look upon the question whether over-renormalisation would produce any problems for the cubic model. We will understand over-renormalisation by renormalising a model of dimension $D$ as a model of dimension $\tilde{D} > D$ which involves more renormalisation than necessary. Of course, more conditions will fix these additional renormalisation constants. However, on the level of Feynman graphs over-renormalisation would create a big number of additional but finite counterterms due to Zimmermann’s forest formula [Zim69]. A very natural question is whether this over-subtraction of the additional counterterms converges or produces new problems. For each Feynman diagram, it is clear that the diagram is still finite after over-subtraction, but whether a global problem appears or not, is unclear. This question can only be answered if a model is resummable after renormalisation. Therefore, we can tackle this question perfectly for matrix field theories.

We saw that any correlation function is built recursively by the 1-point function with Theorem 3.1 and Theorem 3.2. It is therefore sufficient to show the convergence of the 1-point function. The general solution is given by Corollary 3.1. Taking the renormalisation with respect to a higher dimensional model, the integrals are clearly convergent in Corollary 3.1. This directly implies that the large number of over-subtracted terms can not destroy resummability.

To be more precise, we selected two examples:

**Example 3.4.** ($D = 2$ Moyal space with renormalisation of $D = 4$)

Linear eigenvalues $e(x) = x$ and for $D = 2$ with growing multiplicity of the form $r(x) = 1$ implies $\varrho(Y) = \frac{2\lambda^2}{\sqrt{Y}}$. Then Corollary 3.1 gives for $D = 4$ for the shifted 1-point function after integration and simplification

$$\hat{W}(X) = 1 + \sqrt{X + c} - \sqrt{1 + c} - 2\lambda^2 \left\{ \log \left( 1 + \frac{1}{\sqrt{1 + c}} \right) \right\}$$
The function $c(\lambda)$ is uniquely invertible for $|\lambda| < \lambda_c \approx 0.6886$. Expanding $c$ in $\lambda$ by (3.2.9) gives $c(\lambda) = -\lambda^2 2(2 \log 2 - 1) + \lambda^4 (5 - 16 \log 2 + 2(2 \log 2)^2 + 2(2 \log 2)^2 - 32(2 \log 2)^2) + O(\lambda^5)$. Inserting in (3.2.10) gives the first orders of the 1-point function

$$G^0(x) = \lambda \log (1 + x) - x - \lambda^2 \frac{x^2(3 + 4x)(2 \log 2 - 1)^2}{(1 + 2x)^3} + O(\lambda^5)$$

which is confirmed in App. D.1 by Feynman graph calculations.

**Example 3.5.** ($D = 2$ Moyal space with renormalisation of $D = 6$)

Again, linear eigenvalues $c(x) = x$ and for $D = 2$ with growing multiplicity of the form $r(x) = 1$ implies $g(Y) = \frac{2 \lambda^2}{\sqrt{Y}}$. Then Corollary 3.1 gives with $D = 6$ for the shifted 1-point function after integration and simplification

$$W(X) = \sqrt{X + c} \frac{1 + c}{c} - c + 2 \lambda^2 \left\{ \frac{\sqrt{X + c} - \sqrt{1 + c}}{\sqrt{1 + c}} \right\}$$

$$+ (\sqrt{1 + c} - \sqrt{X + c} - 1) \log \left( 1 + \frac{1}{\sqrt{1 + c}} \right)$$

$$- c \lambda^2 \left\{ \frac{1}{2(1 + c)} + 3 - 6\sqrt{1 + c} + 2(3c + 2) \log \left( 1 + \frac{1}{\sqrt{1 + c}} \right) \right\}.$$

The function $c(\lambda)$ is uniquely invertible for $|\lambda| < \lambda_c \approx 0.8891$. Expanding $c$ in $\lambda$ by (3.2.9) gives $c(\lambda) = -\lambda^2 (4 \log 2 - \frac{5}{3} + \lambda^4 \frac{960640000000}{3} \log 2^2 - \frac{1545}{4}) + O(\lambda^5)$. Inserting in (3.2.10) gives the first orders of the 1-point function

$$G^0(x) = \lambda \frac{2 \log (1 + x) - x(2 - x)}{2(1 + 2x)} + \lambda^2 \frac{x^3(2 + 3x)(2 \log 2 - 5)^2}{4(1 + 2x)^3} + O(\lambda^5)$$

which is confirmed in App. D.1 by Feynman graph calculations.

Unexpectedly, the convergence radius on the $D = 2$ Moyal space increases if the renormalisation corresponding to a higher dimensional model is taken. This counterintuitive behaviour is even more fascinating if we notice that the coupling constant is additionally renormalised by a finite factor for the 6-dimensional renormalisation. This finite factor (infinite for the original $D = 6$ Moyal space) shifts the convergence radius to a similar value in comparison to the $D = 2$ and $D = 4$ renormalisation.

### 3.5 Summary

The construction of the renormalised $\Phi^3$ matrix field theory model is complete. We established an algorithm to compute any correlation function $G^{(g)}(x_1^b, ... , x_N^b)$ for all genus $g$ with spectral dimension $\mathcal{D} < 8$.
1. Compute the free energy $F_g(\varrho)$ for $g \geq 2$ via Theorem 3.5. It encodes the $p(3g-3)$ intersection numbers of $\psi$-classes on the moduli space of complex curves of genus $g$. Take $F_1 = -\frac{1}{24} \log \varrho_0$ for $g = 1$.

2. Apply to $F_g(\varrho)$ according to Proposition 3.10 and Theorem 3.1 the boundary creation operators $\hat{A}_{z_1, \ldots, z_b}^{g_1} \circ \ldots \circ \hat{A}_{z_1, z_2}^{g} \circ \hat{A}_{z_1}^{g}$ defined in Definition 3.6. Multiply by $(2\lambda)^{4g+3b-4}$ to obtain $G_g(z_1, \ldots, z_b)$ for $g > 0$. Take Corollary 3.1 as well as the second part of Theorem 3.1 for $g = 0$.

3. Transform the variables by $z_\beta(x^\beta) = ((1 + 2e(x^\beta))^2 + c)^{1/2}$, where $c$ is given by the implicit equation of Corollary 3.1, to obtain $G^{(g)}(x^1|\ldots|x^b) = G_g(z_1(x^1), \ldots, z_b(x^b))$.

4. Pass to $G^{(g)}(x^1_1, \ldots, x^1_{N_1}|\ldots|x^b_1, \ldots, x^b_{N_b})$ via difference quotients by Proposition 3.2 which holds in the $N, V$-limit with $F_p \mapsto \frac{1}{2} + e(x)$.

We remark that, in spite of the relation to the integrable Kontsevich model, this $\Phi^3$-model provides a fascinating toy model for a possible QFT in higher dimensions which shows many facets of renormalisation. Our exact formulae can be expanded about $\lambda = 0$ via Corollary 3.1 and agree with the usual perturbative renormalisation which needs Zimmermann’s forest formula [Zim69]. Also note that at fixed genus $g$ one expects $O(n!)$ graphs with $n$ vertices so that a convergent summation at fixed $g$ cannot be expected a priori. To our knowledge, this is the first bosonic model which is just-renormalisable and still resummable for each genus. All correlation functions have a finite radius of convergent in $\lambda$. Taking the renormalisation procedure of a higher dimensional model for lower dimensions creates no problem, even though infinitely many finite counterterms are subtracted additionally. Moreover, for $6 \leq \mathcal{D} < 8$ the $\beta$-function of the coupling constant is positive for real $\lambda$ (see Remark 3.3), which in this particular case possesses not the slightest problem for summation. Furthermore, the renormalon problem appears (see Remark 3.4) also for $6 \leq \mathcal{D} < 8$. 

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Chapter 4

Quartic Interaction:
Grosse-Wulkenhaar Model

We consider in this chapter the matrix field theory model with quartic interaction. This type of interaction was also studied in the past with great interest, however from a completely different perspective than the cubic model. The motivation goes back to QFT.

In QFT, a scalar model with a quartic self-interaction is the first natural choice for a rigorous understanding of QFT in general. The potential is, without renormalisation, bounded from below such that the model has good convergence property. However, even in 2D the scalar $\Phi^4$ model was for a long time not understood rigorously. It was achieved by constructive QFT in a series of papers by Glimm and Jaffe [GJ68, GJ70a, GJ70b, GJ72].

Later, interest in QFT models on deformed spacetime arose which was motivated by a fundamental minimal scale in Nature also called Planck scale. An example of a deformation is the Moyal space described in Sec. 2.5. It was recognised that a model living on the $D$-dimensional Moyal space forces an UV/IR mixing problem [MVRS00]. This problem could be resolved by introducing a harmonic oscillator term [GW03, GW05b]. Assuming the self-dual point $\Omega = 1$ (see (2.5.9)) for the for the harmonic oscillator term, it was proved [DGMR07] that the $\beta$-function vanishes ($\beta = 0$) to all orders in perturbation theory. Furthermore, the model was proved to be renormalisable to all orders in perturbation theory in $D = 4$ [GW05a].

With the techniques described in Sec. 2.2, which were first used in [DGMR07], a nonlinear integral equation for the 2-point function was derived in [GW09] and proved to have a solution in [GW14a]. We will derive this solution as special case in Sec. 4.2.4 since the 4D Moyal space corresponds to a special case of the matrix field theory model.

We will start in Sec. 4.1 with computing the SDE in its full generality. The distinction between $(N_1 + \ldots + N_b)$-point function for even or odd $N_i$ will be discussed. This distinction does not exist in the cubic model. In Sec. 4.2 the solution of the 2-point function will be computed, where $\mathfrak{D} = 0$ (finite matrices) and the $D = 4$ Moyal space are discussed as particular examples. Applying the solution in Sec. 4.3 to the SDE creates a special form of the equation compared to the cubic model. The explicit structure of the recursive equation for the planar $N$-point function ($b = 1$) is analysed in Sec. 4.4 which is nonlinear in comparison to Proposition 3.3 for the cubic model.
4.1 Schwinger-Dyson Equations

We derive the SDEs in the way described in Sec. 2.2. In contrast to the cubic model, the Ward-Takahashi identity has to be applied in its full generality with the degenerate terms \( W_1[J], W_2[J] \) appearing in Theorem 2.1.

Recall the action with a quartic interaction

\[
S[\Phi] = V \left( \sum_{n,m=0}^{N} \frac{H_{nm} \Phi_{nm} \Phi_{mn}}{2} + \frac{\lambda}{4} \sum_{n,m,k,l=0}^{N} \Phi_{nm} \Phi_{mk} \Phi_{kl} \Phi_{ln} \right),
\]

\[H_{nm} = E_n + E_m,
\]

where two eigenvalues \( E_i, E_j \) are not necessarily distinct. The partition function is due to (2.0.5) given by

\[
Z[J] = \int D\Phi \exp \left( -S[\Phi] + V \text{Tr}(J\Phi) \right) = K \exp \left( -\frac{\lambda}{4V^3} \sum_{n,m,k,l=0}^{N} \frac{\partial^4}{\partial J_{nm}\partial J_{mk}\partial J_{kl}\partial J_{ln}} \right) Z_{\text{free}}[J],
\]

\[Z_{\text{free}}[J] := \exp \left( V \sum_{n,m=0}^{N} J_{nm} J_{mn} \right),
\]

and \( K := \int D\Phi \exp \left( -VZ \sum_{n,m=0}^{N} H_{nm} \Phi_{nm} \Phi_{mn} \right) = \prod_{n,m=0}^{N} \sqrt{\frac{2}{VZH_{nm}}}. \)

**Remark 4.1.** The action and the partition function are not renormalised in (4.1.1) and (4.1.2). The mass renormalisation changes \( \mu^2 \mapsto \mu_{\text{bare}}^2 \), the coupling constant renormalisation \( \lambda \mapsto \lambda_{\text{bare}} \) and the field renormalisation \( \Phi \mapsto \sqrt{Z} \Phi \). Inserting in the action gives the substitution \( E_n \mapsto Z E_n + \text{const} \) and \( \lambda \mapsto Z^2 \lambda_{\text{bare}} \) for all SDEs. We will see later that the coupling constant does not need renormalisation. Thus, we can take \( \lambda = \lambda_{\text{bare}} \). We will avoid the factors of \( Z \) for this section. Renormalisation of the quartic model will be discussed in greater detail in Sec. 4.2.

The quartic model has no 1-point function which becomes directly clear from perturbative expansion. Therefore, we start with the SDE of the 2-point function which was already computed before.

**Proposition 4.1.** ([GW14a eq. 3.4]) The 2-point function of the quartic matrix field theory model satisfies

\[
G_{[pq]} = \frac{1}{E_p + E_q} - \frac{\lambda}{E_p + E_q} \left\{ G_{[pq]} \left( \frac{1}{V^2} G_{[pp]} + \frac{1}{V} \sum_{n=0}^{N} G_{[pn]} \right) + \frac{1}{V^4} G_{[pppq]} \right\} + \frac{1}{V^3} \sum_{n=0}^{N} G_{[pn][pq]} + \frac{1}{V^2} \left( G_{[pppq]} + G_{[pppq]} \right) + \frac{1}{V} \sum_{n=0}^{N} \frac{G_{[pq]} - G_{[pq]}}{E_n - E_p} + \frac{1}{V^2} \frac{G_{[pq]} - G_{[pq]}}{E_q - E_p} \right\}.
\]

**Proof.** By definition, the 2-point function for \( p, q \) with \( E_p \neq E_q \) is

\[
G_{[pq]} = \frac{1}{V} \frac{\partial^2}{\partial J_{pq} \partial J_{qp}} \log Z[J] \bigg|_{J=0}
\]
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\[ \begin{align*}
&= \frac{1}{H_{pq}} - \frac{\lambda}{H_{pq} Z[0]} \sum_{n,m=0}^{N} \frac{\partial^2 Z[J]}{\partial J_{np} \partial J_{mn} \partial J_{mq}} \bigg|_{J=0} \\
&= \frac{1}{H_{pq}} - \frac{\lambda}{H_{pq} Z[0] V^3} \sum_{n,m=0}^{N} \frac{1}{E_{m} - E_{p}} \frac{\partial^2}{\partial J_{np} \partial J_{pq}} \left( J_{mn} \frac{\partial}{\partial J_{mn}} - J_{np} \frac{\partial}{\partial J_{np}} \right) Z[J] \bigg|_{J=0} \\
&= \frac{1}{H_{pq}} - \frac{\lambda}{H_{pq} Z[0] V^2} \sum_{n,m=0}^{N} \frac{1}{E_{n} - E_{p}} \left( \frac{1}{V^2} G_{[pq]} \left( \frac{1}{V^2} G_{[p|p]} + \frac{1}{V} \sum_{n=0}^{N} G_{[p|n]} \right) + \frac{1}{V^4} G_{[p|p|p]} + \frac{1}{V^3} \sum_{n=0}^{N} G_{[p|n|p]} \right)
\end{align*} \]

where we used in the second line of equation (2.2.6) and in the third and fourth line of Theorem 2.1. The last two lines are achieved by acting with the second order derivative \( \partial^2 \) on \( Z[J] \) or \( W^1_{p}[J] \), and fixing \( n \) or \( m \) by acting with \( \frac{\partial^2}{\partial J_{mn} \partial J_{pq}} \) on \( J_{mn} \) or \( J_{np} \), respectively.

Since the l.h.s is regular if \( E_p = E_q \), the r.h.s has a well-defined limit due to the continuation of the correlation functions to differentiable functions. \( \square \)

Expanding the SDE of Proposition 4.1 in the genus yields at order \( V^{-2g} \)

\[ G_{[pq]} = \frac{\delta_{0,q}}{E_p + E_q} - \frac{\lambda}{E_p + E_q} \left\{ \sum_{h=0}^{g-1} G_{[p|q]} \left( G_{[p|p]}^{(g-1-h)} \right) \right\} + \frac{g}{V} \sum_{h=0}^{g} G_{[p|p]}^{(g-h)} \frac{1}{V} \sum_{n=0}^{N} G_{[p|n]}^{(g-h)} + G_{[p|p|p]}^{(g-2)} \] (4.1.3)

Notice that the planar 2-point function satisfies a nonlinear equation, whereas the 2-point function of genus \( g > 0 \) of Euler characteristic \( \chi = 1 - 2g \) satisfies a linear equation with an inhomogeneity depending on correlation functions of Euler characteristic \( \chi' > \chi \).

The 2-point function of genus \( g \) depends on 4-point functions of genus \( g-1 \). In general, correlation functions with boundary lengths larger than 2 are recursively expressed by correlation functions only of lengths 1 or 2. This recursive behavior was found in [GW14a] for the \( N \)-point function and the \( (N+M) \)-point function of genus \( g \), where \( N+M \) is even. Let us define the shorthand notation \( G_{[\mathcal{I}]}^{(g)} := G_{[I^1,..,I^b]}^{(g)} \) for \( \mathcal{I} = \{I^1,..,I^b\} \) and \( I^\beta = \{p_{\beta}^1,..,p_{\beta}^{N_{\beta}}\} \) for \( \beta \in \{1,..,b\} \) and the cardinality \( |\mathcal{I}| = |I^1| + .. + |I^b| \).

The recursive equation, in full generality, reads for any number of boundary components:

**Proposition 4.2.** Let \( \mathcal{J} = \{J^1,..,J^b\} \), \( J^\beta = \{p_{\beta}^1,..,p_{\beta}^{N_{\beta}}\} \) and \( \beta \in \{2,..,b\} \). Then, the \( (N_1 + .. + N_b) \)-function with \( b \) boundary components satisfies for \( N_1 \geq 3 \) and \( N = \sum_{i=1}^{b} N_i \) even the recursive equation

\[ G_{[p_1^1..p_{N_1}^1]|\mathcal{J}|} = - \frac{\lambda}{E_{p_1^1} - E_{p_1^1}} \left\{ \frac{1}{V^2} \sum_{k=2}^{N_1} \frac{G_{[p_{k}^1..p_{k+1}^1..p_{N_1}^1]|\mathcal{J}|}}{E_{p_1^1} - E_{p_1^1}} - \frac{G_{[p_1^1..p_{k}^1..p_{k+1}^1..p_{N_1}^1]|\mathcal{J}|}}{E_{p_1^1} - E_{p_1^1}} \right\} \]

\[ + \sum_{\beta=2}^{b} \sum_{k=1}^{N_{\beta}} \frac{G_{[p_{k}^\beta..p_{k+1}^\beta..p_{N_{\beta}}^\beta]|\mathcal{J}|}}{E_{p_1^1} - E_{p_1^1}} \]
Bringing the global denominator of the proposition to the other side yields by definition to have clear distinction between these and the remaining $\lambda$. Assume $\lambda$. Proof. Assume $p_i^j$ such that all $E_{p_i^j}$ are pairwise different. Set $a = p_1^1$, $d = p_2^1$ and $c = p_{N_1}^1$ to have clear distinction between these and the remaining $p_i^j$. Define all derivatives besides the distinct ones by

$$\lambda = \frac{\partial^{N_1 + \ldots + N_{k-2}}}{\partial J_{d_i} \partial J_{a_i} \partial J_{d_i}} \log Z[J] |_{J=0}$$

$$= (E_d - E_c) V^{b-2} \frac{\partial^2}{\partial J_{c_i} \partial J_{d_i}} \log Z[J] |_{J=0}$$

$$= V^{b-2} \frac{\partial^2}{\partial J_{c_i} \partial J_{d_i}} (H_{ad} - H_{ac}) \log Z[J] |_{J=0}$$

$$= K \frac{\partial^3}{\partial J_{c_i} \partial J_{d_i} \partial J_{a_i}} \left( \frac{\partial}{\partial J_{c_i}} \frac{\exp(-V S_{int}(\frac{1}{2}) J_{da})}{Z[J]} - \frac{\partial}{\partial J_{a_i}} \frac{\exp(-V S_{int}(\frac{1}{2}) J_{ac})}{Z[J]} \right) Z_{free}[J] |_{J=0}$$

$$= - V^{b-4} \lambda \frac{\partial^3}{\partial J_{c_i} \partial J_{d_i} \partial J_{a_i}} \left( \frac{\partial}{\partial J_{c_i}} \frac{\exp(-V S_{int}(\frac{1}{2}) J_{da})}{Z[J]} - \frac{\partial}{\partial J_{a_i}} \frac{\exp(-V S_{int}(\frac{1}{2}) J_{ac})}{Z[J]} \right) Z[J] |_{J=0}.$$

where equation (2.2.6) was applied. For $E_m = E_a$, the bracket vanishes for regular and non-regular terms since $\frac{\partial}{\partial J_{c_i}}$ and $\frac{\partial}{\partial J_{a_i}}$ do not act on $\frac{1}{Z[J]}$ because it gives 0 after taking $J = 0$ (no cycle in $a$). Therefore, we can assume $E_m \neq E_a$ and apply the Ward-Takahashi identity of Proposition 2.2 to have

$$= - \lambda \frac{\partial^3}{\partial J_{c_i} \partial J_{d_i} \partial J_{a_i}} \left( \frac{\partial}{\partial J_{c_i}} \frac{\exp(-V S_{int}(\frac{1}{2}) J_{da})}{Z[J]} (E_m - E_a) (J_{mn} \frac{\partial}{\partial J_{a_i}} J_{m} - J_{mn} \frac{\partial}{\partial J_{a_i}} J_{m}) \right) Z[J] |_{J=0}.$$

In the first line, $\frac{\partial}{\partial J_{c_i}}$ has to act for the second term on $J_{na}$, where $n = c$. In the second line, $\frac{\partial}{\partial J_{a_i}}$ has to act for the first term on $J_{mn}$, where $n = d$. Both resulting terms cancel for any $m$. If $\frac{\partial}{\partial J_{c_i}}$ acts on $J_{mn}$ in the first line, and $\frac{\partial}{\partial J_{a_i}}$ on $J_{mn}$ in the second line, the resulting terms cancel as well. We end up with the surviving terms

$$= - \lambda \frac{\partial^3}{\partial J_{c_i} \partial J_{d_i} \partial J_{a_i}} \left( \frac{\partial}{\partial J_{c_i}} \frac{\exp(-V S_{int}(\frac{1}{2}) J_{da})}{Z[J]} (E_m - E_a) (J_{mn} \frac{\partial}{\partial J_{a_i}} J_{m} - J_{mn} \frac{\partial}{\partial J_{a_i}} J_{m}) \right) Z[J] |_{J=0}.$$

\[\text{69}\]
Applying next the identity \( \frac{\partial \partial f}{\partial x \partial y} = \partial_x \partial_y \log(f) + \partial_x \log(f) \partial_y \log(f) \) yield
\[
= -\lambda \hat{D}_{de} v^{b-3} \sum_{n,m} \frac{1}{E_m - E_a} \left\{ \frac{\partial}{\partial J_{ca}} J_{mn} \frac{\partial^2}{\partial J_{an} \partial J_{md}} \log Z[J] - \frac{\partial}{\partial J_{ad}} J_{nm} \frac{\partial^2}{\partial J_{cm} \partial J_{na}} \log Z[J] \right\} \\
+ \frac{\partial}{\partial J_{ca}} J_{mn} \left( \frac{\partial}{\partial J_{an}} \log Z[J] \right) - \frac{\partial}{\partial J_{ad}} J_{nm} \left( \frac{\partial}{\partial J_{cm}} \log Z[J] \right) \right|_{J=0}.
\]

The \( n, m \) are fixed by a derivative acting on \( J_{mn} \) (or \( J_{nm} \)). In the first line, there are two possibilities, either a derivative of the form \( \frac{\partial}{\partial J_{p_k l_{k+1}}} \) fixes the \( n, m \), which produces separated cycles, or a derivative of the form \( \frac{\partial}{\partial J_{p_k l_{k+1}}} \) with \( \beta > 1 \) the \( n, m \), which merges the first cycle with the \( \beta \)-th cycle. In the last two lines, it is only possible that a derivative of the form \( \frac{\partial}{\partial J_{p_k l_{k+1}}} \) fixes the \( n, m \), otherwise \( J = 0 \) vanishes. Acting with the remaining derivatives of \( \hat{D}_{de} \) on the product of the logarithms by considering the Leibniz rule leads to the assertion for pairwise different \( E_{p_j} \).

The expression stays true for coinciding \( E_{p_j} \) since the lhs is regular which induces a well-defined limit of the rhs by continuation to differentiable functions. From perturbative considerations it is clear that the number of external legs is necessarily even, which corresponds to even \( N_1 - k + 1 + |I| \).

After genus expansion, Proposition 4.2 gives at order \( V^{-2g} \)
\[
G^{(g)}_{[p_1 p_2 \ldots p_{N_1} | J]} = -\frac{\lambda}{E_{p_1} - E_{p_{N_1}}} \left\{ \sum_{k=2}^{N_1} \frac{G^{(g-1)}_{[p_1 p_2 \ldots p_{k-1} p_k l_k p_{k+1} | J]}}{E_{p_k} - E_{p_1}} - \frac{G^{(g-1)}_{[p_1 p_2 \ldots p_{k-1} p_k l_k p_{k+1} | J]}}{E_{p_k} - E_{p_1}} \right\} \\
+ \sum_{\beta=2}^{b} \sum_{k=1}^{N_\beta} \frac{G^{(g)}_{[p_1 p_2 \ldots p_{k-1} p_k l_k p_{k+1} \ldots p_{N_\beta} | J \setminus \{J_\beta\}]} - \frac{G^{(g)}_{[p_1 p_2 \ldots p_{k-1} p_k l_k p_{k+1} \ldots p_{N_\beta} | J \setminus \{J_\beta\}]}}{E_{p_k} - E_{p_1}} \right\}.
\]

Example 4.1. Take \( J = \emptyset \) and \( g = 0 \). Then, (4.1.4) gives
\[
G^{(0)}_{[p_1 p_2 \ldots p_{N_1} | J]} = -\lambda \sum_{k=1}^{N_2} \frac{G^{(0)}_{[p_1 p_2 \ldots p_{k+1} | J]} G^{(0)}_{[p_1 p_2 \ldots p_{k-1} | J]}}{(E_{p_{k+1}} - E_{p_1})(E_{p_2} - E_{p_{N_1}})},
\]
where \( p_i \equiv p_1 \). This result coincides with [GW14a, Prop. 3.4] since all correlation functions with boundary of odd length vanishes.

The recursive equation of Proposition 4.2 is nonlinear on the rhs (strictly different to the cubic model). A correlation function with boundary lengths larger than 3 is therefore built.
4.1. SCHWINGER-DYSON EQUATIONS

reversely by multiplications of correlation functions of shorter boundary lengths. This is a further big difference to the cubic model, since the recursive equation of Proposition 3.2 is linear.

Notice also that the first boundary labelled by \( p_1, ..., p_{N_1} \) as well as the labellings \( p_1^1, p_2^1 \) and \( p_1^N, p_1^{N_1} \) play a special rôle for the recursion. One could write the recursion for any other three different adjacent labellings \( p_k^\beta, p_{k+1}^\beta, p_{k-1}^\beta \), where \( p_k^\beta \) is then called the base point. Since all correlation functions have a unique solution, at least from the perturbative expansion in \( \lambda \), the recursive equation needs to be independent of the order of the base points chosen for each recursion until only correlation functions of boundary length 1 and 2 survive. The explicit structure is analysed and discussed extensively in Sec. 4.4 for Example 4.1.

**Remark 4.2.** The recursive equation of Proposition 3.2 is invariant under the renormalisation discussed in Remark 4.1 \( \lambda \to Z^2 \lambda \text{bare} \) and \( E_n \to Z E_n + \text{const} \). Since this holds in particular for the planar 4-point function by setting \( N = 4 \) in Example 4.1, there is no need for additional coupling constant renormalisation \( \lambda \text{bare} = \lambda \).

Next, we will consider an analogue to the boundary creation operator in the quartic interacting case. Assume for the following considerations that all \( E_k \) are distinct with multiplicity one. Assume further that the \( E_k \) can be varied in a small disjoint neighbourhood \( U_k \) such that we have

\[
\frac{\partial}{\partial E_i} \int D\Phi \ e^{-V \text{Tr} \left(E \Phi^2 + \frac{1}{4} \Phi^4 - J \Phi \right)} = -V \int D\Phi \sum_{n=0}^{N} \Phi_{in} \Phi_{ni} \ e^{-V \text{Tr} \left(E \Phi^2 + \frac{1}{4} \Phi^4 - J \Phi \right)}
\]

Having multiplicities \( r_k \) for the distinct eigenvalues \( e_k = E_q = .. = E_q+r_k-1 \) changes (4.1.5) to

\[
\frac{V}{r_k} \frac{\partial}{\partial E_q} \int D\Phi \ e^{-V \text{Tr} \left(E \Phi^2 + \frac{1}{4} \Phi^4 - J \Phi \right)} = -\sum_{n=0}^{N} \frac{\partial^2}{\partial J_{qn} \partial J_{nq}} \int D\Phi \ e^{-V \text{Tr} \left(E \Phi^2 + \frac{1}{4} \Phi^4 - J \Phi \right)}. \tag{4.1.6}
\]

We conclude with this idea that the derivative wrt \( E_q \) on a correlation function gives:

**Proposition 4.3.** Let \( J = \{J^1, .., J^b\} \), \( J^\beta = \{p_1^\beta, ..., p_{N_\beta}^\beta\} \) and \( \beta \in \{1, .., b\} \). Let \( e_k \) be the distinct eigenvalues of \( E \) of multiplicity \( r_k \) with \( e_k = E_q = .. = E_q+r_k-1 \). Then, we have for \( N = \sum_{i=1}^{b} N_i \) even, and \( E_q \neq E_{p_j^i} \)

\[
-\frac{V}{r_k} \frac{\partial}{\partial E_q} G_{[\mathcal{J}]} = \frac{1}{V} \sum_{n=0}^{N} G_{[qn],[\mathcal{J}]} + \frac{1}{2} V G_{[q\{q\}],[\mathcal{J}]} + \sum_{\beta=1}^{b} \sum_{k=1}^{N_\beta} G_{[p_k^\beta,p_{N_\beta+1}^\beta],[\mathcal{J}\setminus\mathcal{J}^\beta]} + \sum_{\mathcal{I} \subseteq \mathcal{J} \setminus \mathcal{J}^\beta} G_{[\mathcal{I}]} G_{[\mathcal{I}]} \tag{4.1.7}
\]

with \( p_{N_i+1}^\beta \equiv p_j^i \). The last sum over \( \mathcal{I} \) contains only terms with odd \( |\mathcal{I}| \).

**Proof.** Assume \( p_j^i \) such that all \( E_{p_j^i} \) are pairwise distinct. Let

\[
\partial^N_{n_1+..+n_b} D = \partial_{J_{p_1}^1 p_1^1} \partial_{J_{p_2}^2 p_2^2} \partial_{J_{p_3}^3 p_3^3} .. \partial_{J_{p_{N_1}}^1 p_1^1} \partial_{J_{p_{N_2}}^2 p_2^2} \partial_{J_{p_{N_3}}^3 p_3^3} .. \partial_{J_{p_{N_b}}^b p_b^b}.
\]

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Due to (4.1.6) the derivative wrt $E_q$ commutes with the derivative wrt $J_{nm}$. Using the definition of the correlation function yields

$$-\frac{V}{r_k} \frac{\partial}{\partial E_q} G_{[J]} = -\frac{V^{b-1}}{r_k} \frac{\partial}{\partial E_q} \log Z[J]_{|J=0}$$

$$= V^{b-2} \frac{\partial}{\partial J_{nm}} \frac{Z[J]}{Z[J]}_{|J=0}$$

$$= V^{b-2} \frac{\partial}{\partial J_{nm}} \left\{ \left( \frac{\partial \log Z[J]}{\partial J_{nm}} \right) \left( \frac{\partial \log Z[J]}{\partial J_{q}} \right) + \frac{\partial^2 \log Z[J]}{\partial J_{nm} \partial J_{q}} \right\}_{|J=0}.$$

Since $E_q \neq E_{p_j}$ the multiplicative term generates a cycle only in the case of $n = q$ and vanishes otherwise for $J = 0$. The last term survives and creates one more boundary for $n = q$ and merges two boundaries for $n = p_j^i$, which gives the expression of the Proposition.

For coinciding $E_{p_j}$ the rhs is regular which induces regularity on the lhs. $\square$

**Corollary 4.1.** Let $F := \frac{1}{V^2} \log Z[0]$. With the eigenvalue distribution assumed in Proposition 4.3, we have in particular

$$-\frac{V}{r_k} \frac{\partial}{\partial E_q} F = \frac{1}{V} \sum_{n=0}^{N} G_{[qn]} + \frac{1}{V^2} G_{[q|q]}.$$

Notice that Proposition 4.3 has a problem if $E_q = E_{p_j}$, therefore it is natural to define a derivative $T_p$ which ignores the fact that $G$ depends (possibly) explicitly on $E_q$:

**Definition 4.1.** Let $J = \{J^1, \ldots, J^b\}$, $J^\beta = \{p_1^\beta, \ldots, p_{N_\beta}^\beta\}$ and $\beta \in \{1, \ldots, b\}$. The operator $T_q$ is defined on the correlation function $G_{[J]}$ by

$$T_q G_{[J]} := \frac{1}{V} \sum_{n=0}^{N} G_{[qn],J} + \frac{1}{V^2} G_{[q|q],J} + \sum_{\beta=1}^{b} \sum_{k=1}^{N_\beta} G_{[q|\partial_{p_k} r_{\beta} J_{n+k}],J_{\{J^\beta\}}} + \sum_{\beta=1}^{b} G_{[q|J]} G_{[q|\bar{J}]}$$

where $q \in J$ is possible.

The operator $T_q$ can be understood due to the considerations of Proposition 4.3 as a derivative which does not act on any external energies. From the perturbative point of view, this operator acts only on the closed faces of the Feynman graphs, even if $E_q$ coincides with an external face.

The SDE of Proposition 4.1 can be rewritten with $T_q$ to

$$G_{[pq]} = \frac{1}{H_{pq}} - \frac{\lambda}{H_{pq}} \left\{ G_{[pq]} T_p F + \frac{1}{V} T_q G_{[pq]} + \frac{1}{V^2} \sum_{n=0}^{N'} G_{[pq]} - G_{[q|q]} + \frac{1}{V^2} \frac{G_{[p|q]} - G_{[q|q]}}{E_{n} - E_{p}} \right\}$$

with $F$ defined in Corollary 4.1

**Lemma 4.1.** Let $J = \{J^1, \ldots, J^b\}$, $J^\beta = \{p_1^\beta, \ldots, p_{N_\beta}^\beta\}$ and $\beta \in \{1, \ldots, b\}$. Let further be

$$\hat{D} = \frac{\partial^{N_1 + \ldots + N_\beta}}{\partial J_{p_1^1} \partial J_{p_2^1} \ldots \partial J_{p_1^1} \partial J_{p_2^1} \ldots \partial J_{p_1^1} \partial J_{p_2^1} \ldots \partial J_{p_1^1} \partial J_{p_2^1} \ldots \partial J_{p_1^1} \partial J_{p_2^1}}$$
with $G_{[\mathcal{J}]} = V^{b-2} \hat{D} \log \mathcal{Z}[J] |_{J=0}$. Then, the identity

$$\hat{D} (W_q^1 + W_q^2) = T_q G_{[\mathcal{J}]}$$

holds, where $W_q^1$ and $W_q^2$ are defined in Theorem 2.1.

**Proof.** Take the definition of $W_q^1$ and $W_q^2$ of Theorem 2.1 and act with $\hat{D}$ on it, which is exactly the Definition 4.1 of $T_q$ on $G_{[\mathcal{J}]}$. \hfill \Box

**Remark 4.3.** In the limit $\mathcal{N}, V \to \infty$, the operator $T_q$ converges with $e(x)$ and $r(x)$ described in Sec. 2.3 formally to the functional derivative

$$T_q \to D(x) := -\frac{\delta}{\delta e(t)} \frac{\delta}{\delta (r(t) dt)} |_{t=x}.$$ Setting $t = x$ after derivation, the operator $D(x)$ avoids the degenerate case, i.e. $D(x)$ ignores an explicit dependence on $e(x)$.

The next step is to derive the SDEs in its full generality. Proposition 4.2 shows that it is sufficient to assume boundaries of lengths 1 and 2. However, we have different SDEs if the base point is chosen from a boundary of length 1 or length 2. For the base point coming from boundary length 1, the SDE reads:

**Proposition 4.4.** Let $\mathcal{J} = \{J^1, J^{2b}, J^{2b+1}, \ldots, J^{2b+b'}\}$, $J^\beta = \{p^\beta\}$ for $\beta \in \{2, \ldots, 2b\}$ and $J^\beta = \{q_1^\beta, q_2^\beta\}$ for $\beta \in \{2b+1, \ldots, 2b+b\}$. Then, the $(1 + \ldots + 1 + 2 + \ldots + 2)$-point function with $2b$ boundaries of length 1 and $b'$ boundaries of length 2 satisfies

$$G_{[p], [\mathcal{J}]} = -\frac{\lambda}{H_{pp}} \left\{ \frac{1}{V} \sum_{m=0}^{N} G_{[p]; [\mathcal{J}]} - G_{[m]; [\mathcal{J}]} + \sum_{\beta=2}^{2b} G_{[p^\beta]; [\mathcal{J}\backslash \{J^\beta\}]} - G_{[p^\beta]; [\mathcal{J}\backslash \{J^\beta\}]} \right\} E_m - E_p$$

$$+ \sum_{i=1}^{b+b'} \sum_{\beta=2b+1}^{2b+b} \frac{G_{[q_1^\beta, q_2^\beta]; [\mathcal{J}\backslash \{J^\beta\}]} - G_{[q_1^\beta, q_2^\beta]; [\mathcal{J}\backslash \{J^\beta\}]} \right\} E_q - E_p$$

$$+ \frac{1}{\sqrt{2}} \mathcal{T}_p G_{[p]; [\mathcal{J}]} + \sum_{\mathcal{I} \cap \mathcal{J} = \emptyset} G_{[p]; [\mathcal{I}]} \mathcal{T}_p G_{[\mathcal{I}]} \right\},$$

where $q_1^\beta \equiv q_2^\beta$, $G_{[\emptyset]} = F$ of Corollary 4.1. $\mathcal{T}_p G$ is defined in Definition 4.1 and the sums over the sets are restricted to correlation functions where the boundary lengths sum to an even number.

**Proof.** Let us assume $p^i$, $q^i$ and $p$ such that $E_{p^i}, E_{q^i}$ and $E_p$ are pairwise different. Define further

$$\hat{D} = \frac{\partial^{2b+b'} - 1}{\partial J_{p^1} \partial J_{p^2} \ldots \partial J_{p^{2b+b}} \partial J_{q_1^{2b+1}} \partial J_{q_2^{2b+1}} \partial J_{q_1^{2b+1}} \partial J_{q_2^{2b+1}} \partial J_{q_1^{2b+1}} \partial J_{q_2^{2b+1}} \partial J_{q_2^{2b+1}} \partial J_{p^{2b+b'}}}.$$ By definition, the correlation function can be expressed by

$$G_{[p], [\mathcal{J}]} = \left. V^{2b+b'-2} \frac{\partial}{\partial J_{pp}} \log \mathcal{Z}[J] \right|_{J=0}$$

$$- \frac{V^{2b+b'-4}}{H_{pp}} \lambda \hat{D} \sum_{n, m} \frac{\partial^3}{\partial J_{pp} \partial J_{nm} \partial J_{mn}} \mathcal{Z}[J] \left|_{J=0} \right.$$


\[
\begin{align*}
\lambda D &= -\frac{V^{2b+b'}\beta}{H_{pp}} \mathbb{Z}[J]\left\{ \sum_{n,m} \lambda \frac{\partial}{\partial J_{mn}} \left( V E_m - E_p \frac{\partial}{\partial J_{nm}} - J_{nm} \right) \mathbb{Z}[J]
\right. \\
&\quad \left. + \lambda \frac{\partial}{\partial J_{pp}} (W_p^1[J] + W_p^2[J]) \mathbb{Z}[J] \right\}
\end{align*}
\]

where we used (2.2.6) and Theorem 2.1. The quotient difference term is treated as usual which gives the quotient difference terms of the proposition. The last line is rewritten to

\[
\frac{1}{\mathbb{Z}[J]} \frac{\partial}{\partial J_{pp}} (W_p^1[J] + W_p^2[J]) \mathbb{Z}[J] = \frac{\partial}{\partial J_{pp}} (W_p^1[J] + W_p^2[J]) + (W_p^1[J] + W_p^2[J]) \frac{\partial}{\partial J_{pp}} \log \mathbb{Z}[J].
\]

(4.1.7)

Apply Lemma 4.1 to the second and last term of (4.1.7). The first term produces a multiplication of two correlation functions, where each has a boundary of length 1 labelled by \(b\). The third term of (4.1.7) produces a cubic term since \(W_p^1[J]\) is already quadratic. Collecting all terms finishes the proof, after considering regularity on both sides of the equation for coinciding eigenvalues \(E_i\).

Example 4.2. Applying the genus expansion to Proposition 4.4 putting \(b = 1, b' = 0\) and \(\mathcal{J} = \mathcal{J}^2 = \{p^2\}\), we achieve the linear equation for the planar \((1+1)\)-point function

\[
G_{p[p^2]}^{0} = -\frac{\lambda}{H_{pp}} \left\{ \frac{1}{V} \sum_{m=0}^{N} G_{p[p^2]}^{0} \frac{C_{p[p^2]} - C_{m[p^2]}}{E_m - E_p} + \frac{C_{p[p^2]} - C_{p[p^2]}}{E_p^2 - E_p} + \frac{1}{V} \sum_{n=0}^{N} G_{n[p]}^{0} \right\}.
\]

For the base point coming from boundary length 2, the SDE reads:

Proposition 4.5. Let \(\mathcal{J} = \{J^1, J^{2b}, J^{2b+2}, \ldots, J^{2b+b'}\}\), \(J^\beta = \{p^\beta\}\) for \(\beta \in \{1, \ldots, 2b\}\) and \(J^\beta = \{q^\beta_1, q^\beta_2\}\) for \(\beta \in \{2b+2, \ldots, 2b+b'\}\). Then, the \((2+2+1+2+1+2\ldots+2)\)-point function with 2b boundaries of length 1 and \(b'\) boundaries of length 2 satisfies

\[
G_{\{q_1, q_2]\}[\mathcal{J}] = -\frac{\lambda}{H_{q_1 q_2}} \left\{ \frac{1}{V} \sum_{m=0}^{N} G_{\{q_1, q_2\}[\mathcal{J}]} - G_{\{m, q_2\}[\mathcal{J}]} \frac{1}{E_m - E_{q_1}} + \frac{1}{V^2} \left( G_{\{q_1, q_2\}[\mathcal{J}]} - G_{\{q_2, q_2\}[\mathcal{J}]} \right) \right. \\
&\left. + \sum_{\beta=1}^{2b} \frac{G_{\{q_2, q_2\}[\mathcal{J}] \setminus \{J^\beta\}} - G_{\{q_2, q_2\}[\mathcal{J}]} \frac{1}{E_{q_2} - E_{q_1}}}{E_{q_2} - E_{q_1}} + \sum_{i=1}^{2b+2} \frac{G_{\{q_2, q_2\}[\mathcal{J}] \setminus \{J^\beta\}} - G_{\{q_2, q_2\}[\mathcal{J}]} \frac{1}{E_{q_2} - E_{q_1}}}{E_{q_2} - E_{q_1}} ight. \\
&\left. + \sum_{\mathcal{J} \in \mathcal{J}^2} \frac{G_{\{q_2\}[\mathcal{J}]} - G_{\{q_2\}[\mathcal{J}]}}{E_{q_2} - E_{q_1}} + \frac{1}{V} \sum_{\mathcal{J} \in \mathcal{J}^2} T_{q_1} G_{\{q_1, q_2\}[\mathcal{J}]} \right\},
\]

where \(q^\beta_3 \equiv q^\beta_1\), \(G_{\{q\}} = F\) of Corollary 4.1, \(T_{q}G\) is defined in Definition 4.1, and the sums over the sets are restricted to correlation functions, where the boundary lengths sum to an even number.

Proof. Let us assume \(p^i, q^i\) and \(q_1, q_2\) such that \(E_{p^i}, E_{q^i}\) and \(E_p\) are pairwise different. Define further

\[
\begin{align*}
\hat{D} &= \frac{\partial^{2b+b'-2}}{\partial J_{p^1 p^1} \partial J_{p^2 p^2} \ldots \partial J_{q_1} \partial J_{q_2}^{2b+2} \partial J_{q_1}^{2b+2} \partial J_{q_2}^{2b+2} \ldots \partial J_{q_1}^{2b+b'} \partial J_{q_2}^{2b+b'} \partial J_{q_1}^{2b+b'} \partial J_{q_2}^{2b+b'}}.
\end{align*}
\]
By definition, the correlation function reads
\[ G_{[q_1q_2]} = V^{2b+b^\prime-2} \frac{\partial^2}{\partial J_{q_1q_2} \partial J_{q_2q_1}} \log Z[J] \bigg|_{J=0} \]
\[ = - \frac{V^{2b+b^\prime-4} \lambda \hat{D}}{H_{q_1q_2}} \frac{\partial}{\partial J_{q_2q_1}} \sum_{n,m} \frac{\partial J_{n,m} \partial J_{mq_2}}{Z[J]} Z[J] \bigg|_{J=0} \]
\[ = - \frac{V^{2b+b^\prime-4} \lambda \hat{D}}{H_{q_1q_2}} \frac{\partial}{\partial J_{q_2q_1}} \left( \sum_{n,m} \frac{\partial}{\partial J_{mq_2}} \frac{V}{E_m - E_{q_1}} \left( J_{mn} \frac{\partial}{\partial J_{q_n}} - J_{q_n} \frac{\partial}{\partial J_{mn}} \right) Z[J] \right. \]
\[ + \left. \frac{\partial}{\partial J_{q_1q_2}} (W_{q_1}^1[J] + W_{q_1}^2[J]) Z[J] \right) \bigg|_{J=0}, \]
where we have used (2.2.6) and Theorem 2.1. The quotient difference term is treated as usual which gives the quotient difference terms of the proposition, where more terms appear compared to Proposition 4.4. These terms are generated by acting with the derivative \( \frac{\partial}{\partial J_{q_2q_1}} \) on \( J_{mn} \) and \( J_{q_n} \) respectively and fixing \( m = q_2 \). The last line is rewritten to
\[ \frac{1}{Z[J]} \frac{\partial}{\partial J_{q_2q_1}} (W_{q_1}^1[J] + W_{q_1}^2[J]) Z[J] \]
\[ = \frac{\partial}{\partial J_{q_2q_1}} (W_{q_1}^1[J] + W_{q_1}^2[J]) + (W_{q_1}^1[J] + W_{q_1}^2[J]) \frac{\partial}{\partial J_{q_1q_2}} \log Z[J]. \quad (4.1.8) \]

Applying Lemma 4.1 and collecting all terms finishes the proof, where the regularity conditions manage coinciding eigenvalues \( E_i \).

**Example 4.3.** Applying the genus expansion to Proposition 4.5 setting \( b = 0, b^\prime = 2 \) with \( J = J^2 = \{ q_1^2, q_2^2 \} \) gives the linear equation for the planar \((2+2)\)-point function
\[ G_{[q_1q_2]q_2^2}^{(0)} = - \frac{\lambda}{H_{q_1q_2}} \left\{ \frac{1}{V} \sum_{m=0}^N C_{[q_1q_2]q_2^2}^{[m]} - G_{[mq_2]q_2^2}^{(0)} \right\} + \sum_{i=1}^2 \frac{C_{[q_1q_2]q_2^2}^{(0)}}{E_{q_2}^2 - E_{q_1}} \]
\[ + G_{[q_1q_2]q_2^2}^{(0)} \frac{1}{V} \sum_{n=0}^N C_{[nq_1]}^{(0)} + G_{[q_1q_2]}^{(0)} T_q G_{[q_1q_2]}^{(0)} \right\}. \]

We emphasise that the SDE for a correlation function with base point from a boundary of length 2 has originally a term of the form \( \frac{1}{V} \sum_{n=0}^N G_{[q_1n]J} \). Since we include this term inside the derivative \( T_q G_{[J]} \), the SDE has a much simpler recursive structure.

Performing the genus-expansion for Proposition 4.4 and Proposition 4.5 leads to linear recursive equations for a correlation function of Euler characteristic \( \chi \) where the inhomogeneous part is some \( g_{inh}^{a,J} \) depending on correlation functions of Euler characteristic \( \chi' > \chi \) by
\[ \hat{K}_p \dot{c}^{(g)}_{[p,J]} = g_{inh}^{a,J} \quad (4.1.9) \]
\[ \hat{K}_q \dot{c}^{(g)}_{[q,J]} = g_{inh}^{a,J}, \quad (4.1.10) \]
where
\[ \hat{K}_p f(p) := f(p) \left( H_{pp} + \frac{\lambda}{V} \sum_{n=0}^N \frac{1}{E_n - E_p} + G_{np}^{(0)} \right) \right) - \frac{\lambda}{V} \sum_{n=0}^N \frac{f(n)}{E_n - E_p}. \quad (4.1.11) \]
\[ K^2_{q_1} f(q_1, q_2) := f(q_1, q_2) - \sum_{n=0}^{N} \left( \frac{1}{E_n - E_{q_1}} + G^{(0)}_{nq_1} \right) - \sum_{n=0}^{N} \frac{f(n, q_2)}{E_n - E_{q_1}}. \] (4.1.12)

Comparing with the cubically interacting model the equations (4.1.9) and (4.1.10) share some similar structure with (3.1.15). The function \( H_{pp} + \frac{\lambda}{N} \sum_n G^{(0)}_{np} \) seems to take the analog rôle of \( W^{(0)}_p \). However, the first equation to solve is a nonlinear equation for \( G^{(0)}_{pq} \) which is a function depending on two variables instead of one. Solving this needs a completely different strategy than for cubic interaction. The result is changed tremendously and has a different structure which can also be seen in the perturbative expansion since hyperlogarithms survives at any order in \( \lambda \) (see App. D.2).

### 4.2 Solution of the Planar 2-Point Function

In this section we are analysing the structure of the planar 2-point function. The main idea for that came from an observation of the earlier known special case on the \( \mathcal{D} = D = 2 \) Moyal space [PW18]. The function \( I_D(w) \) defined in Definition 4.5 seems to have an “involutive” structure \( I_D(-I_D(w)) = w \) which holds only formally since the domains have to be specified. Nevertheless, this formal property gave the right ansatz for the general solution. Sec. 4.2.1-4.2.3 is taken from our joint paper [GHW19b] with H. Grosse and R. Wulkenhaar, where the major discovery is found by R. Wulkenhaar. The notation is adapted to the rest of the thesis, and later results will build up from this.

An important tool which plays an incredible rôle throughout this section is the Lagrange-Bürmann inversion formula:

**Theorem 4.1.** ([Lag70, Bür99]) Let \( \phi(w) \) be analytic at \( w = 0 \) with \( \phi(0) \neq 0 \) and \( f(w) := \frac{w}{\phi(w)} \). Then the inverse \( g(z) \) of \( f(w) \) with \( z = f(g(z)) \) is analytic at \( z = 0 \) and given by

\[ g(z) = \sum_{n=1}^{\infty} \frac{z^n}{n!} \frac{d^{n-1}w}{dw^{n-1}} \bigg|_{w=0} \left( \phi(w) \right)^n. \] (4.2.1)

More generally, if \( H(z) \) is an arbitrary analytic function with \( H(0) = 0 \), then

\[ H(g(z)) = \sum_{n=1}^{\infty} \frac{z^n}{n!} \frac{d^{n-1}w}{dw^{n-1}} \bigg|_{w=0} \left( H'(w) \left( \phi(w) \right)^n \right). \] (4.2.2)

Taking the renormalisation for \( \mathcal{D} < 6 \) for the quartic model into account (see Remark 4.1) gives the nonlinear equation for the planar 2-point function by Proposition 4.1

\[ ZG^{(0)}_{pq} = \frac{1}{E_p + E_q} - \frac{\lambda}{V(E_p + E_q)} \sum_{n=0}^{N} \left( ZG^{(0)}_{pn} ZG^{(0)}_{pq} - ZG^{(0)}_{pq} ZG^{(0)}_{np} \right). \] (4.2.3)

Taking the large \( N, V \)-limit discussed in Sec. 2.3 we write

\[ G^{(0)}_{pq} =: G(x, y) \bigg|_{x = E_a - \mu_{bare}/2, y = E_b - \mu_{bare}/2}, \] (4.2.4)
then $G(x,y)$ originally defined only on the (shifted) spectrum of $E$ extends to a sectionally holomorphic function which satisfies the integral equation

$$(\mu^2_{\text{bare}}+x+y)ZG(x,y) = 1 - \lambda \int_0^{\Lambda^2} dt \rho_0(t) \left( ZG(x,y) ZG(x,t) - \frac{ZG(t,y) - ZG(x,y)}{t-x} \right).$$

(4.2.5)

Here we have used $\rho_0(t) = r(t)$, where $e(x) = x$ and $r(x)$ are differentiable function defined in Sec. 2.3.

We assume that the measure $\rho_0(t)$ is a Hölder-continuous function. The final result will make perfect sense even for $\rho_0$ being a linear combination of Dirac measures. Intermediate steps become more transparent if $\rho_0 \in C^{0,\alpha}([0,\Lambda^2])$ is assumed. Using techniques for boundary values of sectionally holomorphic functions, explained in detail in [Tri85, GW14a, GW14b, PW18], one finds that a solution for $G(a,b)$ at $0 < a, b < \Lambda^2$ should be searched in the form

$$ZG(a,b) = \frac{e^{\mathcal{H}_a^\Lambda[\tau_a](\bullet)}}{\lambda \pi \rho_0(a)} - \frac{e^{\mathcal{H}_a^\Lambda[\tau_a](\bullet)}}{\lambda \pi \rho_0(b)},$$

(4.2.6)

where the angle function $\tau_a : (0, \Lambda^2) \to [0, \pi]$ for $\lambda > 0$ and $\tau_a : (0, \Lambda^2) \to [-\pi, 0]$ for $\lambda < 0$ remains to be determined. Here,

$$\mathcal{H}_a^\Lambda[f(\bullet)] := \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{[0,\Lambda^2) \setminus [a-c, a+c]} dt f(t) \frac{1}{t-a} = \lim_{\epsilon \to 0} \text{Re} \left( \frac{1}{\pi} \int_0^{\Lambda^2} dt \frac{f(t)}{t-(a+i\epsilon)} \right)$$

denotes the finite Hilbert transform. We go with the ansatz (4.2.6) into (4.2.5) at $x = a+i\epsilon$ and $y = b$:

$$(\mu^2_{\text{bare}} + a + b + \lambda \pi \mathcal{H}_a^\Lambda[\rho_0(\bullet)] + \frac{1}{\pi} \int_0^{\Lambda^2} dt e^{\mathcal{H}_a^\Lambda[\tau_a](\bullet)} \sin \tau_a(t) ) ZG(a,b)$$

$$= 1 + \mathcal{H}_a^\Lambda[e^{\mathcal{H}_a^\Lambda[\tau_a]} \sin \tau_a(\bullet)].$$

(4.2.7)

A Hölder-continuous function $\tau : (0, \Lambda^2) \to [0, \pi]$ or $\tau : (0, \Lambda^2) \to [-\pi, 0]$ satisfies

$$\mathcal{H}_a^\Lambda[e^{\mathcal{H}_a^\Lambda[\tau]} \sin \tau(\bullet)] = e^{\mathcal{H}_a^\Lambda[\tau]} \cos \tau - 1, \quad \int_0^{\Lambda^2} dt e^{\pm \mathcal{H}_a^\Lambda[\tau]} \sin \tau(t) = \int_0^{\Lambda^2} dt \tau(t).$$

The first identity appeared in [Tri85], the second one was proved in [PW18]. Inserting both identities into (4.2.7) gives with (4.2.6) a consistency relation for the angle function:

$$\tau_a(p) = \arctan \left( \frac{\lambda \pi \rho_0(p)}{\mu^2_{\text{bare}} + a + p + \lambda \pi \mathcal{H}_a^\Lambda[\rho_0(\bullet)] + \frac{1}{\pi} \int_0^{\Lambda^2} dt \tau_a(t)} \right),$$

(4.2.8)

where the arctan-branch in $[0, \pi]$ is selected for $\lambda > 0$ and the branch in $[-\pi, 0]$ for $\lambda < 0$.\(^*\)

\(^*\)The linear case $e(x) = x$ can be assumed without loss of generality. For arbitrary $e(x)$ the measure has to be changed to $\rho_0(t) = \frac{e^{-\tau(t)(x)}}{e^{-\tau(t)}}$. 

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4.2.1 Solution of the Angle Function

We succeed in solving (1.2.8) for any Hölder-continuous $\rho_0$ of spectral dimension $\mathcal{D} < 6$. The difficulty was to guess the solution; verifying it is a straightforward exercise in complex analysis. The main step is to deform the measure function. We first introduce structures for a fictitious measure $\rho_c$; later $\rho_c$ will be particularly chosen.

Definition 4.2. Let $\rho_c$ be a Hölder-continuous function on some interval $[\nu_D, \Lambda_2^D]$. For $\mu^2 = \mu^2_{\text{bare}}$ in $D = 2 \left( \frac{D}{2} \right)$, and $\mu^2 > \min(0, -\nu_D)$ a free parameter in $D = 2 \left( \frac{D}{2} \right) \in \{2, 4\}$, define functions $h_{00} \equiv h_0, h_{02}, h_4, h_2, h_4$ on $\mathbb{C} \setminus [\nu_D, \Lambda_2^D]$ by

$$h_{0D}(z) := \int_{\nu_D}^{\Lambda_2^D} \frac{dt \rho_c(t)}{t - z}, \quad h_0(z) := h_{00}(z), \quad (4.2.9)$$

$$h_2(z) := h_{02}(z) - h_{02}(-\mu^2) = (z + \mu^2) \int_{\nu_2}^{\Lambda_2^D} \frac{dt \rho_c(t)}{(t + \mu^2)(t - z)} ,$$

$$h_4(z) := h_{04}(z) - h_{04}(-\mu^2) - (z + \mu^2) h_{04}'(-\mu^2) = (z + \mu^2)^2 \int_{\nu_4}^{\Lambda_2^D} \frac{dt \rho_c(t)}{(t + \mu^2)^2(t - z)} .$$

Definition 4.3. For $\lambda \in \mathbb{C}$ and $h_D$ as given in Definition 4.2 we introduce functions $R_D$ on $\mathbb{C} \setminus [-\mu^2 - \Lambda_2^D, -\mu^2 - \nu_D]$ by

$$R_D(z) := z - \lambda h_D(-\mu^2 - z) \equiv z - \lambda(-z)^2 \int_{\nu_D}^{\Lambda_2^D} \frac{dt \rho_c(t)}{(t + \mu^2)(t + \mu^2 + z)} . \quad (4.2.10)$$

The limits $\lim_{\Lambda_2^D \to \infty} h_D(z)$ and $\lim_{\Lambda_2^D \to \infty} R_D(z)$ exist for $\rho_c$ of spectral dimension $\mathcal{D}$ according to Definition 2.1. We have

$$D \in \{0, 2\} \quad \Rightarrow \quad R'_D(z) = 1 + \lambda h_{0D}'(-\mu^2 - z) ,$$

which is uniformly positive on $\mathbb{R}_+$ for real $\lambda > -(h_{0D}'(-\mu^2))^{-1}$ in $D \in \{0, 2\}$. In contrast,

$$R'_4(z) = 1 - \lambda h_{04}'(-\mu^2) + \lambda h_{04}'(-\mu^2 - z) ,$$

which is uniformly positive in the opposite region of real $\lambda < (h_{04}'(-\mu^2))^{-1}$.

Lemma 4.2. Let $|\lambda| < \left( \int_{\nu_D}^{\Lambda_2^D} dt \frac{\rho_c(t)}{(t + \mu^2)^2} + \delta_{D,4} \int_{\nu_D}^{\Lambda_2^D} dt \frac{\rho_c(t)}{(t + \mu^2)^2} \right)^{-1}$. Then:

1. $R_D$ is a biholomorphic map from a right half plane $\mathcal{R}_\mu := \{ z \in \mathbb{C} : \text{Re}(z) > -\mu^2 \}$ onto a domain $U_D \subset \mathbb{C}$. For $\lambda$ real, $U_D$ contains $[R_D(\nu_D), \infty)$.

2. For $\lambda$ real, $\text{Im}(R_D(z))$ and $\text{Im}(z)$ have the same sign for every $z \in \mathcal{R}_\mu$.

Proof. 1. We show that $R_D$ is injective on $\mathcal{R}_\mu$. Any two points $z_0 \neq z_1 \in \mathcal{R}_\mu$ can be connected by a straight line $[0, 1] \ni s \mapsto c(s) = z_0 + (z_1 - z_0)s \in \mathcal{R}_\mu$. Then for $D \in \{0, 2\}$

$$|R_D(z_1) - R_D(z_0)| = |z_1 - z_0| \left( 1 + \lambda \int_0^1 ds \int_{\nu_D}^{\Lambda_2^D} \frac{dt \rho_c(t)}{(t + \mu^2 + c(s))^2} \right)$$

$$\geq |z_1 - z_0| \left( 1 - \sup_{s \in [0,1]} |\lambda| \int_{\nu_D}^{\Lambda_2^D} dt \frac{\rho_c(t)}{|t + \mu^2 + c(s)|^2} \right) > 0 .$$

\[^1\text{This subsection is taken from our paper [GHW19a].}\]
For \( D = 4 \) we have
\[
\left| R_4(z_1) - R_4(z_0) \right| = |z_1 - z_0| |1 - \lambda h_{04}(-\mu^2)| 1 + \frac{\lambda}{1 - \lambda h_{04}(-\mu^2)} \int_0^1 ds \int_{\mu_4}^{\Lambda^2_D} \frac{dt \rho_c(t)}{(t + \mu^2 + c(s))^2}
\]
which under the adapted condition leads to the same conclusion \( |R_4(z_1) - R_4(z_0)| > 0 \).

It follows from basic properties of holomorphic functions that \( R_D \) is, as holomorphic and injective function, even a biholomorphic map \( R_D : \mathcal{R}_\mu \to U_D := R_D(\mathcal{R}_\mu) \).

2. For \( \lambda \) real we have
\[
\text{Im}(R_D(x + iy)) = y \left\{ (1 - \lambda h_{04}(-\mu^2))^{s_D,4} + \lambda \int_{\nu_D}^{\Lambda^2_D} \frac{dt \rho_c(t)}{(t + \mu^2 + x)^2 + y^2} \right\}.
\]
The term in \( \{ \} \) is strictly positive by the same reasoning as above.

We can now define the ‘\( \lambda \)-deformed’ measure:

**Definition 4.4.** Given \( \lambda \in \mathbb{R}, \mu^2 > 0 \) and a Hölder-continuous function \( \rho_0 : [0, \Lambda^2] \to \mathbb{R}_+ \) of spectral dimension \( \mathcal{D} \) according to **Definition 2.1** Then a function \( \rho_\lambda \) on \([\nu_D, \Lambda^2_D]\) is implicitly defined by the equations
\[
\rho_0(t) =: \rho_\lambda(R_D^{-1}(t)) \quad \Leftrightarrow \quad \rho_\lambda(x) = \rho_0(R_D(x)) \\
\Lambda^2_D := R_D^{-1}(\Lambda^2), \quad \nu_D := R_D^{-1}(0),
\]
where \( h_D \) in \( R_D \) is defined via (4.2.10) and (4.2.9) by the same function \( \rho_c \mapsto \rho_\lambda \).

**Remark 4.4.** The deformation from \( \rho_0 \) to \( \rho_\lambda \) is the analogue of the deformation from \( E \) to \( E_c := \sqrt{E^2 + \frac{1}{2} c(\lambda)} \) in the cubic model of Ch. 3. There the deformation parameter \( c(\lambda) \) is implicitly defined in **Corollary 3.1**. Neither that equation nor (4.2.11) in the quartic model can in general be solved in terms of ‘known’ functions.

**Definition 4.5.** Given \( \lambda \in \mathbb{R}, \mu^2 > 0 \) and a Hölder-continuous function \( \rho_0 : [0, \Lambda^2] \to \mathbb{R}_+ \) of spectral dimension \( \mathcal{D} \) according to **Definition 2.1** Let \( \rho_\lambda \) be its associated deformed measure according to **Definition 4.4** and let \( \lambda \) satisfy the requirements of **Lemma 4.2** so that \( R_D : \mathcal{R}_\mu \to U_D \) is biholomorphic. Then a holomorphic function \( I_D : U_D \setminus [0, \Lambda^2] \ni w \mapsto I_D(w) \in \mathbb{C} \) is defined by
\[
I_D(w) := -R_D(-\mu^2 - R_D^{-1}(w)) = \mu^2 + R_D^{-1}(w) + \lambda h_D(R_D^{-1}(w))
\]
where \( \mu \) in (4.2.12) and in **Definition 4.3** are the same and \( R_D, h_D \) are defined with the deformed measure \( \rho_c \mapsto \rho_\lambda \).

**Theorem 4.2.** Let \( \rho_0 : [0, \Lambda^2] \to \mathbb{R}_+ \) be a Hölder-continuous measure of spectral dimension \( \mathcal{D} \) and \( \rho_\lambda \) its deformation according to **Definition 4.4** for a real coupling constant \( \lambda \) with \( |\lambda| < (\int_{\nu_D}^{\Lambda^2_D} dt \frac{\rho_0(t)}{t + \mu^2} - \delta_{D,4} \int_{\nu_D}^{\Lambda^2_D} dt \frac{\rho_0(t)}{(t + \mu^2)^2})^{-1} \). Then the consistency equation (4.2.8) for the angle function is solved by
\[
\tau_a(p) = \lim_{\epsilon \to 0} \text{Im} \left( \log(a + I_D(p + i\epsilon)) \right),
\]
with \( I_D \) given by **Definition 4.5** provided that the following relations between \( \mu_{\text{bare}} \) and \( \mu \) are arranged: \( \mu_{\text{bare}}^2 = \mu^2 \) for \( \mathcal{D} < 2 \) and
\[
2 \leq \mathcal{D} \leq 4 : \quad \mu_{\text{bare}}^2 = \mu^2 - 2\lambda h_{02}(-\mu^2),
\]
\[
4 \leq \mathcal{D} < 6 : \quad \mu_{\text{bare}}^2 = \mu^2 (1 - \lambda h_{04}(-\mu^2)) - 2\lambda h_{04}(-\mu^2).
\]

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Proof. Assume \([4.2.13]\). Then for the given range of \(\lambda\) we have for \(0 < p < \Lambda^2\)

\[
\tau_a(p) = \lim_{\epsilon \to 0} \arctan \left( \frac{\text{Im}(a + I_D(p + i\epsilon))}{\text{Re}(a + I_D(p + i\epsilon))} \right) = \arctan \left( \frac{\lim_{\epsilon \to 0} \lambda \text{Im}(h_D(R_D^{-1}(p + i\epsilon)))}{\lim_{\epsilon \to 0} \lambda \text{Re}(a + I_D(p + i\epsilon))} \right) \\
= \arctan \left( \frac{\lambda \pi \rho_\lambda(R_D^{-1}(p))}{\lim_{\epsilon \to 0} \lambda \text{Re}(a + I_D(p + i\epsilon))} \right) \\
= \arctan \left( \frac{\lambda \pi \rho_\lambda(p)}{\lim_{\epsilon \to 0} \lambda \text{Re}(a + I_D(p + i\epsilon))} \right), \tag{4.2.15}
\]

where 2. of Lemma 4.2, the definition of \(h_D\) and the defining relation \([4.2.11]\) between \(\rho_0\) and \(\rho_\lambda\) have been used. The \(\arctan\) ranges in \([0, \pi]\) for \(\lambda > 0\) and in \([–\pi, 0]\) for \(\lambda < 0\). Comparison with \([4.2.8]\) shows (after renaming variables) that we have to prove

\[
\lim_{\epsilon \to 0} \text{Re}(I_D(a + i\epsilon)) = \mu_{\text{bare}}^2 + a + \lambda \pi \mathcal{H}_a[\rho_0(\bullet)] + \frac{1}{\pi} \int_0^{\Lambda^2} dp \, \tau_a(p). \tag{4.2.16}
\]

We evaluate the integral over \(\tau_a\). For \(p > \Lambda^2\) we have \(R_D^{-1}(p) > \Lambda_D^2\) and consequently \(\text{Im}(h_D(R_D^{-1}(p + i\epsilon))) = 0\). This implies

\[
\frac{1}{\pi} \int_0^{\Lambda^2} dp \, \tau_a(p) = \lim_{\epsilon \to 0} \frac{1}{\pi} \int_0^\infty dp \ \text{Im} \log(a + I_D(p + i\epsilon)) \\
= \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{R_D(\gamma_\epsilon)} dw \ \log(a + I_D(w)) = \lim_{\epsilon \to 0} T_D^\epsilon(a),
\]

where \(T_D^\epsilon(a) := \frac{1}{2\pi i} \int_{R_D(\gamma_\epsilon)} dw \ \log \left( \frac{a + I_D(w)}{a + \kappa_D + (R_D^{-1}(w) + \mu^2)/c_D} \right).\)

In the second line, the contour \(\gamma_\epsilon\) encircles \([\nu_D, \infty)\) clockwise at distance \(\epsilon\), i.e. it goes straight from \(+\infty - i\epsilon\) to \(\nu_D - i\epsilon\), in a left half circle to \(\nu_D + i\epsilon\) and straight again to \(+\infty + i\epsilon\). The denominator included in \(T_D^\epsilon(a)\) is holomorphic in \(U_D\) and does not contribute for \(\epsilon \to 0\). The constants \(c_D\) and \(\kappa_D\) are chosen as \(c_2 = c_3 = 1\), \(\kappa_0 = 0\) and

\[
c_4 = \frac{1}{1 - \lambda h_{04}(-\mu^2)}, \quad \kappa_D = -\lambda h_{0D}(-\mu^2) \quad \text{for} \quad D \in \{2, 4\}. \tag{4.2.17}
\]

We insert \([4.2.12]\) and transform to \(w = R(z)\):

\[
T_D^\epsilon(a) = \frac{1}{2\pi i} \int_{\gamma} dz \ R_D'(z) \log \left( \frac{a + \mu^2 + z + \lambda h_D(z)}{a + \kappa_D + (z + \mu^2)/c_D} \right) \\
= \frac{1}{2\pi i c_D} \int_{\gamma} dz \ (1 + \lambda c_D h_D'(z)(-\mu^2 - z)) \log \left( 1 + \frac{\lambda c_D h_D(z)}{c_D(a + \kappa_D) + z + \mu^2} \right). \tag{4.2.18}
\]

The function \(h_D\) in \([4.2.18]\) is defined with the \(\lambda\)-deformed measure \(\rho_\lambda\). We will now

- rename \(\rho_\lambda\) to \(\rho_c\) and the given coupling constant \(\lambda\) to \(\lambda_c \in \mathbb{R}\),
- consider a general complex \(\lambda \in \mathbb{C}\) (i.e. \(h_{0D}\) will be taken as in \([4.2.9]\) without any relation between \(\rho_c\) and \(\lambda\)),
- take \(\epsilon\) a fixed positive number.
In this setting, \( z \) in (4.2.18) keeps distance \( \epsilon \) from \([\nu_D, \infty)\) so that (4.2.18) becomes a holomorphic function of \( \lambda \) in a sufficiently small open ball around the origin. We choose its radius so small that the logarithm admits a uniformly convergent power series expansion on \( \gamma_\epsilon \). Hence, integral and series commute:

\[
T'_D(a) = -\sum_{n=1}^\infty \frac{(-\lambda c_D)^n}{nc_D} \frac{1}{2\pi i} \int_{\gamma_\epsilon} dz \left( 1 + \lambda c_D h'_0D(-\mu^2 - z) \right) \frac{(h_0D(z))^n}{(c_D(a + \kappa_D) + z + \mu^2)^n}.
\]

(4.2.19)

Since \( h_0D(z) \propto z^{-1} \) for \( |z| \to \infty \), we can close \( \gamma_\epsilon \) by a large circle to a closed contour \( \bar{\gamma}_\epsilon \) which avoids \([\nu_D, \infty)\).

We first evaluate the part without \( h'_0D \) (and the global factor \( c_D^{-1} \)) by the residue theorem. Since \( h_0D(z) \) is holomorphic in \( \mathbb{C} \setminus [\nu_D, \infty) \), only the pole of order \( n \) at \( z = -c_D(a + \kappa_D) - \mu^2 \) contributes:

\[
K'_D(a) := -\sum_{n=1}^\infty \frac{(-\lambda c_D)^n}{n} \frac{1}{2\pi i} \int_{\gamma_\epsilon} dz \frac{(h_0D(z))^n}{(c_D(a + \kappa_D) + z + \mu^2)^n}.
\]

\[
= -\sum_{n=1}^\infty \frac{(-\lambda c_D)^n}{n!} \frac{d^{n-1}}{dw^{n-1}} \bigg|_{w=0} \left( h_0D(w - c_D(a + \kappa_D) - \mu^2) \right)^n.
\]

Setting \( \phi(w) = h_0D(w - c_D(a + \kappa_D) - \mu^2) \), the Lagrange inversion formula (4.2.1) shows that \( w = -K'_D(a) \) is the inverse solution of the equation \( -\lambda c_D = f(-K'_D(a)) \), where \( f(w) = \frac{w}{\phi(w)} \). This means

\[
\lambda c_D h_0D\left( -K'_D(a) - c_D(a + \kappa_D) - \mu^2 \right) = K'_D(a) .
\]

(4.2.20)

Introducing \( z(a) := K'_D(a) + c_D(a + \kappa_D) \), equation (4.2.20) becomes

\[
z(a) - \lambda c_D h_0D(-\mu^2 - z(a)) = c_D(a + \kappa_D) .
\]

(4.2.21)

Comparing with Definition 4.3 equation (4.2.21) boils down for any \( D \in \{0, 2, 4\} \) to \( a = R_D(z(a)) \). But \( a \in [0, \Lambda^2] \subset U_D \) so that we can invert to \( z(a) = R_D^{-1}(a) \). In summary, we have proved a useful perturbative formula for \( R_D^{-1} \): 

**Lemma 4.3.** For any \( a \in [0, \Lambda^2] \) and \( \lambda \) satisfying the assumptions of Lemma 4.2 the inverse function of \( R_D \) defined in (4.2.10) admits a convergent representation

\[
R_D^{-1}(a) = c_D(a + \kappa_D) - \sum_{n=1}^\infty \frac{(-\lambda c_D)^n}{n!} \frac{d^{n-1}}{dw^{n-1}} \bigg|_{w=0} \left( h_0D(w - c_D(a + \kappa_D) - \mu^2) \right)^n .
\]

We continue with (4.2.19). We insert (4.2.9) for \( h'_0D \) and change the integration order:

\[
T'_D(a) = -\lambda c_D(a + \kappa_D) + \frac{1}{c_D} R_D^{-1}(a)
\]

\[
+ \lambda \int_{\nu_D}^{\Lambda^2_D} dt \rho_\epsilon(t) \int_{\gamma_\epsilon} dz \left( \frac{1}{2\pi i} \int_{\gamma_\epsilon} dz \left( 1 + \lambda c_D h'_0D(-\mu^2 - z) \right) \frac{(h_0D(z))^n}{(c_D(a + \kappa_D) + z + \mu^2)^n} \right).
\]

We first look at generic points \( t \neq c_D(a + \kappa_D) \). This is no restriction because for Hölder-continuous \( \rho_\epsilon \), ordinary and improper integral (the point \( t = c_D(a + \kappa_D) \) removed) agree.
The residue theorem picks up the simple pole at \( z = -\mu^2 - t \), for which we resum the series to the logarithm, and the pole of order \( n \) at \( z = -c_D(a + \kappa_D) - \mu^2 \):

\[
T_D^*(a) = -a - \kappa_D + \frac{R_D^{-1}(a)}{c_D} - \lambda \int_{\nu_D}^{\Lambda_2} dt \rho_c(t) \frac{d}{dt} \log \left( 1 + \frac{\lambda c_D h_{0D}(-t - \mu^2)}{c_D(a + \kappa_D) - t} \right) + \lambda \int_{\nu_D}^{\Lambda_2} dt \rho_c(t) \frac{d}{dt} \log \left( \sum_{n=1}^{\infty} \frac{(\lambda c_D)^n}{n} \frac{d^{n-1}}{dw^{n-1}} \left|_{w=0} \left( H_{t,a}^\prime(w) \left( h_{0D}(w-\mu^2-c_D(a+\kappa_D)) \right)^n \right) \right) ,
\]

(4.2.22)

where \( H_{t,a}(w) := \log \frac{w+t-c_D(a+\kappa_D)}{t-c_D(a+\kappa_D)} \). The original dependence on \( \epsilon \) dropped out. The Bürmann formula (4.2.2) identifies the term in \{ \} of the last line of (4.2.22) as \( H_{t,a}(-K_D^\epsilon(a)) \):

\[
T_D^*(a) = -a - \kappa_D + \frac{R_D^{-1}(a)}{c_D} + \lambda \int_{\nu_D}^{\Lambda_2} dt \rho_c(t) \frac{d}{dt} \log \left( \frac{t - K_D^\epsilon(a) - c_D(a + \kappa_D)}{t - c_D(a + \kappa_D) - \lambda c_D h_{0D}(-t-\mu^2)} \right) - a - \kappa_D + \frac{R_D^{-1}(a)}{c_D} + \lambda \int_{\nu_D}^{\Lambda_2} dt \rho_c(t) \frac{d}{dt} \log \left( \frac{t - R_D^{-1}(a)}{R_D(t) - a} \right) .
\]

(4.2.23)

We have used \( K_D^\epsilon(a) + c_D(a + \kappa_D) = R_D^{-1}(a) \) and rearranged the denominator with (4.2.10) to \( R_D(t) - a \).

We stress that (4.2.23) is proved for complex \( \lambda \) in a ball about the origin of small radius determined by \( \epsilon \). The identity theorem for holomorphic functions allows us to enlarge the domain of \( \lambda \) on both sides back to the original domain of the theorem. This includes the original real value \( \lambda = \lambda_c \) we started with, where \( \rho_c = \rho_\lambda \) on the rhs and \( \lim_{\epsilon \to 0} T_D^*(a) = \frac{1}{\pi} \int_0^{\Lambda_2} dp \tau_a(p) \) on the lhs. Therefore, for the original real \( \lambda \),

\[
\frac{1}{\pi} \int_0^{\Lambda_2} dp \tau_a(p) = -a - \kappa_D + \frac{R_D^{-1}(a)}{c_D} + \lambda \int_{\nu_D}^{\Lambda_2} dt \rho_\lambda(t) \frac{d}{dt} \log \left( \frac{t - R_D^{-1}(a)}{R_D(t) - a} \right) ,
\]

(4.2.24)

where also \( R_D \) is built from \( \rho_\lambda \).

The \( t \)-integral in (4.2.24) does not need any exception point. But for the next step it is useful to remove an \( \epsilon \)-interval about \( t = R_D^{-1}(a) \) to take the logarithms apart. These principal value integrals can equivalently be written as limit of the real part when shifting \( a \) to \( a + i \epsilon \):

\[
\frac{1}{\pi} \int_0^{\Lambda_2} dp \tau_a(p) = \lim_{\epsilon \to 0} \text{Re} \left( -a - \kappa_D + \frac{R_D^{-1}(a)}{c_D} + \lambda \int_{\nu_D}^{\Lambda_2} dt \rho_\lambda(t) \frac{d}{dt} \log \left( \frac{t - R_D^{-1}(a)}{R_D(t) - (a + i \epsilon)} \right) - \lambda \int_{\nu_D}^{\Lambda_2} dt \rho_\lambda(t) \frac{d}{dt} \log(R_D(t) - (a + i \epsilon)) \right) = \lim_{\epsilon \to 0} \text{Re} \left( -a - 2\kappa_D + R_D^{-1}(a) + \mu^2(1 - c_D逆) + \lambda h_D(R_D^{-1}(a + i \epsilon)) \right) - \lambda \int_0^{\Lambda_2} dx \rho_\lambda(R_D^{-1}(x)) \frac{x - (a + i \epsilon)}{x - (a + i \epsilon)} .
\]

Here we have completed the first \( t \)-integral \( h_{0D}(R_D^{-1}(a+i \epsilon)) \) with (4.2.9) to \( h_D(R_D^{-1}(a+i \epsilon)) \) and transformed in the second integral to \( x = R_D(t) \). Taking the relation (4.2.11) to the
original measure into account and recalling the definition \((4.2.12)\) of \(I_D(a)\), we precisely confirm our aim \((4.2.16)\) provided that

\[
\mu_b = 2\mu_D + c_D^{-1} \mu^2.
\]

This finishes the proof. \(\square\)

**Example 4.4.** On the \(D = D = 2\) Moyal space the eigenvalues increases linear \(e(x) = x\) with the measure \(r(x) = 1\) which induces for the undeformed measure \(\rho_0(x) = r(x) = 1\). The deformed measure \(\rho_{\lambda}(x)\) coincides with the undeformed measure due to Definition 4.4 \(\rho_{\lambda}(x) = \rho_0(R_2(x)) = 1\). The function \(R_2(x)\) was defined in Definition 4.3 such that we have

\[
R_2(x) = x + \lambda x \int_0^\infty \frac{dt}{(t + \mu^2)(t + \mu^2 + x)} = x + \lambda \log \left(1 + \frac{x}{\mu^2}\right). \tag{4.2.25}
\]

Setting \(\mu^2 = 1\) which corresponds to \(\mu_b = 1 - 2\lambda \log(1 + \Lambda^2)\) provides the inverses by the branches of Lambert-W [CGH+96], in particular

\[
R_2^{-1}(z) = \lambda W_0 \left(\frac{1}{\lambda} e^{\frac{1+i z}{\lambda}}\right) - 1. \tag{4.2.26}
\]

where \(W_0\) is the principal branch of the Lambert function. Inserting into Definition 4.2.12 leads to

\[
\tau_a(p) = \text{Im} \log \left(a + I(p + i\epsilon)\right), \quad I(z) := \lambda W_0 \left(\frac{1}{\lambda} e^{\frac{1+i z}{\lambda}}\right) - \lambda \log \left(1 - W_0 \left(\frac{1}{\lambda} e^{\frac{1+i z}{\lambda}}\right)\right), \tag{4.2.27}
\]

which was already found in [PW18].

### 4.2. Solution of the 2-Point Function

With \(\tau_a(p)\) determined, it remains to evaluate the Hilbert transform in the equation \((4.2.6)\) for the planar 2-point function \(G(a, b)\). We first establish a general integral representation. In the next subsection this integral will be evaluated for the case of finite matrices.

**Theorem 4.3.** The renormalised 2-point function of the quartic matrix field theory model with spectral dimension \(D\) and \(D = 2[\frac{D}{2}]\) is given by

\[
G(a, b) := \frac{\mu^{2\delta_D, 4} \exp(N_D(a, b))}{(\mu^2 + a + b)}, \tag{4.2.28}
\]

where

\[
N_D(a, b) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dt \left\{ \log \left(a - R_D(-\mu^2 - it)\right) \frac{d}{dt} \log \left(b - R_D(-\mu^2 + it)\right)
- \log \left(a - (-\mu^2 - it)\right) \frac{d}{dt} \log \left(b - (-\mu^2 + it)\right)
- \delta_D, 4 \log \left(-R_D(-\mu^2 - it)\right) \frac{d}{dt} \log \left(-R_D(-\mu^2 + it)\right)
+ \delta_D, 4 \log \left(-(-\mu^2 - it)\right) \frac{d}{dt} \log \left(-(-\mu^2 + it)\right) \right\}. \tag{4.2.29}
\]

\(^{\text{This subsection is taken from our paper [GHW19b]}}\)
and \( R_D \) is built via (4.2.10) and (4.2.9) with the deformed measure \( \rho_\lambda \) defined in (4.2.11). For \( 4 \leq \mathcal{D} < 6 \), \( G(a, b) \) is only determined up to a multiplicative constant which here is normalised to \( G(0, 0) = 1 \) independently of \( \mu \). For \( \mathcal{D} < 4 \) there is an alternative representation

\[
G(a, b) := \frac{(\mu^2 + a + b) \exp \left\{ \frac{1}{2\pi i} \int_{-\infty}^{\infty} dt \log \left( \frac{a - R_D(-\mu^2/2 - it)}{a - (-\mu^2/2 - it)} \right) \frac{d}{dt} \log \left( \frac{b - R_D(-\mu^2/2 + it)}{b - (-\mu^2/2 + it)} \right) \right\}}{(\mu^2 + b + R_D^{-1}(a))(\mu^2 + a + R_D^{-1}(b))}. \tag{4.2.30}
\]

**Proof.** We rely on structures developed during the proof of Theorem 4.2. The Hilbert transform of \( \tau_a \) given by (4.2.13) can be written as

\[
\mathcal{H}_b^\Lambda [\tau_a(\bullet)] = \lim_{\epsilon' \to 0} \lim_{\epsilon \to 0} \text{Re} \left( \frac{1}{\pi} \int_0^{\infty} dp \, \text{Im} \log \left( \frac{a + I_D(p + i\epsilon')}{\mu^2 + a + R_D^{-1}(p)} \right) \frac{1}{p - (b + i\epsilon')} \right).
\]

In the second line, \( \epsilon' \) must be chosen much larger than \( \epsilon \) so that \( R_D(\gamma_c) \) separates \( b + i\epsilon' \) from \( \mathbb{R}_+ \). As before, we are allowed to include a holomorphic denominator \( \mu^2 + a + R_D^{-1}(w) \).

In contrast to the procedure in Theorem 4.2 we choose it such that it has individually a limit for \( \Lambda \to \infty \). This leads to the large-\( w \) behaviour

\[
\frac{a + I_D(w)}{\mu^2 + a + R_D^{-1}(w)} \propto \frac{1}{c_D} + \mathcal{O}(1/R_D^{-1}(w)).
\]

Thus, for \( \mathcal{D} < 4 \) where \( c_D = 1 \), the integrand decays sufficiently fast to deform \( \gamma_c \) near \( \infty \). For \( \mathcal{D} \geq 4 \), however, \( c_4 = 0 \) prevents the deformation. This forces us to subtract the Hilbert transform \( \mathcal{H}_b^\Lambda [\tau_r(\bullet)] \) at some reference point \( a = b = r > 0 \). We first move \( R_D(\gamma_c) \) past the pole \( w = b + i\epsilon' \) at expense of its residue. In the remaining integral (which is automatically real) we transform to \( w = R_D(z) \):

\[
\mathcal{H}_b^\Lambda [\tau_a(\bullet)] - \delta_{D,4} \mathcal{H}_b^\Lambda [\tau_r(\bullet)] = \lim_{\epsilon' \to 0} \text{Re} \left( \log \left( \frac{a + I_D(b + i\epsilon')}{\mu^2 + a + R_D^{-1}(b + i\epsilon')} \right) - \delta_{D,4} \log \left( \frac{r + I_D(r + i\epsilon')}{\mu^2 + r + R_D^{-1}(r + i\epsilon')} \right) \right) \tag{4.2.31a}
\]

\[
+ \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{\gamma_c} dz \, R_D'(z) \left( \frac{\log \left( \frac{a + \mu^2 \tau_c + \lambda \rho_D(z)}{\mu^2 + a + \mu^2 \tau_c + \lambda \rho_D(z)} \right)}{R_D(z) - b} - \delta_{D,4} \log \left( \frac{r + \mu^2 + z + \lambda \rho_D(z)}{\mu^2 + r + z} \right) \right). \tag{4.2.31b}
\]

The line (4.2.31a) evaluates to

\[
(4.2.31a) = \log \left[ \frac{\lambda \pi \rho_D(b)}{\sin \tau_a(b)} \cdot \frac{1}{\mu^2 + a + R_D^{-1}(b)} \right] - \delta_{D,4} \log \left( \frac{\lambda \pi \rho_D(r)}{\sin \tau_r(r)} \cdot \frac{1}{\mu^2 + r + R_D^{-1}(r)} \right), \tag{4.2.32}
\]

where real and imaginary part of \( a + I_D(b + i\epsilon') \) are rearranged to \( \tau_a(b) \) as in (4.2.15).

In the last line (4.2.31b), we write

\[
\frac{R_D'(z)}{R_D(z) - b} = \frac{d}{dz} \log(R_D(z) - b) = \frac{1}{z - b} + \frac{d}{dz} \log \left( \frac{z - b - \lambda h_D(-\mu^2 - z)}{z - b} \right).
\]
Inserted back into (4.2.31b) we deform in the parts with products of logarithms the contour $\gamma_e$ into the straight line $-\frac{\mu^2}{2} + i\mathbb{R}$. No poles or branch cuts are hit during this deformation because $R_D(z)$ and $R_D(-\mu^2 - z)$ are holomorphic on the slit half plane $\{\text{Re}(z) > -\frac{\mu^2}{2}\} \setminus [\nu_D, \infty)$. In this way we produce integrals which are manifestly symmetric in both variables:

$$
\tilde{N}_D(a, b) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dt \left\{ \log \left( 1 + \frac{h_D(-\frac{\mu^2}{2} + it)}{\frac{\mu^2}{2} + a + it} \right) \frac{d}{dt} \log \left( 1 + \frac{\lambda h_D(-\frac{\mu^2}{2} - it)}{b + \frac{\mu^2}{2} - it} \right) - \delta_{D,4} \log \left( 1 + \frac{\lambda h_D(-\frac{\mu^2}{2} + it)}{r + \frac{\mu^2}{2} + r + it} \right) \right\}.
$$

(4.2.33)

The counterterm for $D \geq 4$ is indispensable for convergence. Now the line (4.2.31b)

$$
(4.2.31b) = \tilde{N}_D(a, b) + \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{\gamma_e} dz \left( \frac{\log \left( \frac{1}{\epsilon_D} + \frac{\epsilon D - \mu^2 + \lambda h_D(z)}{\frac{\mu^2}{2} + a + \epsilon z} \right)}{z - b} - \delta_{D,4} \frac{\log \left( \frac{1}{\epsilon_D} + \frac{\epsilon_D - \mu^2 + \lambda h_D(z)}{\frac{\mu^2}{2} + \epsilon r + z} \right)}{z - r} \right).
$$

(4.2.34)

For any $D \in \{0, 2, 4\}$ we can add the convergent integral $\frac{1}{2\pi i} \int_{\gamma_e} dz \left( \frac{\log \epsilon_D}{z - b} - \delta_{D,4} \frac{\log \epsilon_D}{z - r} \right) = 0$ (in $D \in \{0, 2\}$ we have $\epsilon_D = 1$ whereas for $D = 4$ we close $\gamma_e$ and use the residue theorem).

We follow the same strategy as in Theorem 4.2. $\rho_\lambda$ is renamed to $\rho_\epsilon$ and held fixed, $h_D$ and $R_D$ are built with $\rho_\epsilon$ and an independent complex $\lambda$ in a sufficiently small ball about the origin. Its radius is determined by $\epsilon$ which is also kept fixed. Also $\Lambda_2^D$ is still finite, and $a - ac_D^{-1}$ and $\kappa_D$ have according to (4.2.17) a factor $\lambda$ in front of them. After all, the logarithm in (4.2.34) admits a uniformly convergent power series expansion for any $z$ on $\gamma_e$. Every term of the expansion decays sufficiently fast for $z \to \infty$ to admit a closure of $\gamma_e$ to the contour $\gamma_e$ that avoids $[\nu_D, \infty)$. We proceed by the residue theorem. This is simpler than in Theorem 4.2 because $\frac{1}{z - b}$, $\frac{1}{z - r}$ and $h_0D(z)$ are holomorphic in the interior of $\gamma_e$ and on $\gamma_e$ itself:

$$
\frac{1}{2\pi i} \int_{\gamma_e} dz \left( \frac{\log \left( \frac{1}{\epsilon_D} + \frac{a - ac_D^{-1} + \kappa_D + \lambda h_D(z)}{\frac{\mu^2}{2} + a + \epsilon z} \right)}{z - b} - \delta_{D,4} \frac{\log \left( \frac{1}{\epsilon_D} + \frac{r - rc_D^{-1} + \kappa_D + \lambda h_D(z)}{\frac{\mu^2}{2} + \epsilon r + z} \right)}{z - r} \right)
= -\sum_{n=1}^{\infty} \frac{(-\lambda \epsilon_D)^n}{n!} \frac{d^{n-1}}{dw^{n-1}} \bigg|_{w=0} \left( H_{a,b}'(w)\left( \frac{a - ac_D^{-1} + \kappa_D}{\lambda} + h_0D(w - \mu^2 - a) \right)^n - \delta_{D,4} H_{a,b}'(w)\left( \frac{r - rc_D^{-1} + \kappa_D}{\lambda} + h_0D(w - \mu^2 - r) \right)^n \right),
$$

where $H_{a,b}(w) = \log \left( \frac{w - \mu^2 - a - b}{w - \mu^2 - a + b} \right)$. We apply the Bürmann formula (4.2.2). For that we need the auxiliary series

$$
-L(a) := \sum_{n=1}^{\infty} \frac{(-\lambda \epsilon_D)^n}{n!} \frac{d^{n-1}}{dw^{n-1}} \bigg|_{w=0} \left( a - ac_D^{-1} + \kappa_D + h_0D(w - \mu^2 - a) \right)^n.
$$

In the same way as in the proof of (4.2.20), the Lagrange inversion formula (4.2.1) yields

$$
-L(a) = -\lambda \epsilon_D \left( \frac{a - ac_D^{-1} + \kappa_D}{\lambda} + h_0D(-L(a) - \mu^2 - a) \right),
$$

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which by (4.2.9) and (4.2.10) rearranges into \( a = R_D(a + L_D(a)) \) for any \( D \in \{0, 2, 4\} \). We invert it to \( a + L_D(a) = R_D^{-1}(a) \), but question this step for \( D = 4 \) in Remark 4.5. The Bürmann formula (4.2.2) now gives

\[
(4.2.31b) = \tilde{N}_D(a, b) - H_{a,b}(-L_D(a)) + \delta_{D,A} H_{r,r}(-L_D(r))
= \tilde{N}_D(a, b) + \log \left( \frac{\mu^2 + a + b}{\mu^2 + b + R_D^{-1}(a)} \right) - \delta_{D,A} \log \left( \frac{\mu^2 + 2r}{\mu^2 + r + R_D^{-1}(r)} \right).
\]

By the identity theorem for holomorphic functions, this equation holds in the larger common \( \lambda \)-holomorphicity domain of both sides. It contains the original real coupling constant so that \( R_D \) in (4.2.35) extends to the situation formulated in the proposition.

It remains to collect the pieces: We want to evaluate (4.2.30). We set \( Z = 1 \) in \( D \in \{0, 2\} \) and \( Z = C_r e^{iL_D(\tau_r(\bullet))} \) in \( D = 4 \), where \( C_r \) is a finite number. We thus need the exponential of (4.2.31), which is the exponential of (4.2.32) times the exponential of (4.2.35). This is to be multiplied by \( \frac{\sin(\gamma_c(a))}{2\pi \rho_0(a)} \) which cancels with the corresponding term in (4.2.32):

\[
G(a, b) := \frac{(\mu^2 + a + b) \exp(\tilde{N}_D(a, b))}{(\mu^2 + b + R_D^{-1}(a))(\mu^2 + a + R_D^{-1}(b))} \left( C_r \frac{\lambda \pi \rho_0(r)}{\sin \tau_r(r)} \cdot \frac{\mu^2 + 2r}{(\mu^2 + r + R_D^{-1}(r))^2} \right)^{-\delta_{D,A}}.
\]

For \( D \in \{0, 2\} \) this already gives (4.2.30) after reconstructing \( R_D \) from \( H_D \).

As we will discuss in Remark 4.5 after the proof, this equation is not appropriate for all cases of \( D = 4 \). We can already in (4.2.31b) deform the contour \( \gamma_c \) to the straight line \(-\frac{\mu^2}{2} + iR\). After trading \( h_D \) in (4.2.33) for \( R_D \) via (4.2.10), equation (4.2.35) can be written as

\[
\log \left( \frac{\mu^2 + a + b}{\mu^2 + b + R_D^{-1}(a)} \right) \left( \frac{\mu^2 + 2r}{\mu^2 + r + R_D^{-1}(r)} \right)^{-\delta_{D,A}}
= \frac{1}{2\pi i} \int_{-\infty}^{\infty} dt \left\{ \log \left( \frac{a - R_D(-\frac{\mu^2}{2} - it)}{a - (-\frac{\mu^2}{2} - it)} \right) \frac{d}{dt} \log \left( b - (-\frac{\mu^2}{2} + it) \right)
- \delta_{D,A} \log \left( \frac{r - R_D(-\frac{\mu^2}{2} - it)}{r - (-\frac{\mu^2}{2} - it)} \right) \frac{d}{dt} \log \left( r - (-\frac{\mu^2}{2} + it) \right) \right\}.
\]

Inserting this and its flip \( a \leftrightarrow b \) back into (4.2.36) gives rise to a representation where \( R_D^{-1} \) is avoided completely:

\[
G(a, b) := \exp(\tilde{N}_D(a, b)) \left( \frac{C_r \lambda \pi \rho_0(r)}{\mu^2 + 2r \sin \tau_r(r)} \right)^{-\delta_{D,A}}
\tilde{N}_D(a, b) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dt \left\{ \log \left( a - R_D(-\frac{\mu^2}{2} - it) \right) \frac{d}{dt} \log \left( b - R_D(-\frac{\mu^2}{2} + it) \right)
- \log \left( a - (-\frac{\mu^2}{2} - it) \right) \frac{d}{dt} \log \left( b - (-\frac{\mu^2}{2} + it) \right)
- \delta_{D,A} \log \left( r - R_D(-\frac{\mu^2}{2} - it) \right) \frac{d}{dt} \log \left( r - R_D(-\frac{\mu^2}{2} + it) \right)
+ \delta_{D,A} \log \left( r - (-\frac{\mu^2}{2} - it) \right) \frac{d}{dt} \log \left( r - (-\frac{\mu^2}{2} + it) \right) \right\}.
\]
We can absorb the $r$-dependent factors arising for $D = 4$ by an appropriate choice of $C_r$ and then adjust $C_r$ further to have $G(0, 0) = 1$. This amounts to replace $\tilde{N}_4(a, b)$ by $N_4(a, b) := \tilde{N}_4(a, b) - \tilde{N}_4(0, 0)$. \hfill $\square$

**Remark 4.5.** The representation $\text{(4.2.30)}$, renormalised to $\frac{G(a, b)}{G(0, 0)}$, might fail for $4 \leq D < 6$. For finite $\Lambda$, as seen in the proof above, the representations $\text{(4.2.30)}$ and $\text{(4.2.28)+(4.2.29)}$ are equivalent for $|\lambda|$ small enough. But in the limit $\Lambda \to \infty$ it can happen that $R_4(\mathbb{R}^+)$ develops an upper bound for any $\lambda > 0$, independently of whether $\rho_4$ is discrete or continuous. In such a case $R_4^{-1}(a)$ does not exist for all $a \in \mathbb{R}$ and \textbf{(4.2.30)} becomes meaningless for $\Lambda \to \infty$, whereas \textbf{(4.2.28)+(4.2.29)} do not show any problem.

In Sec. 4.2.4 we prove that for the measure function $\rho_0(t) = t$, of spectral dimension exactly $D = D = 4$, there is no such problem. But other cases with $4 < D < 6$ are very likely affected. It is the identification $a + L_4(a) = R_4^{-1}(a)$ made before \textbf{(4.2.35)} which might fail for $\Lambda \to \infty$. For the same reasons, also $\tau_n(p)$ given in \textbf{(4.2.13)} with \textbf{(4.2.12)} does not have a limit $\Lambda \to \infty$ for $4 \leq D < 6$ and $\lambda > 0$. Such problems have been noticed in [GW14b]. They concern only auxiliary functions; the final result \textbf{(4.2.28)+(4.2.29)} is consistent for all $\lambda > 0$.

### 4.2. Solution for Finite Matrices ($D = 0$)\textsuperscript{4}

**Theorem 4.4.** Consider the quartic matrix field theory model with the self-adjoint $N \times N$-matrix $E$ having distinct eigenvalues $0 < e_1 < e_2 \leq \cdots < e_{N'}$ of multiplicities $r_1, r_2, \ldots, r_{N'}$. These data encode a meromorphic function

$$R(z) := z - \frac{\lambda}{V} \sum_{k=1}^{N'} \frac{\vartheta_k}{\varepsilon_k + z},$$

where $\{\varepsilon_k, \vartheta_k\}_{k=1, \ldots, N'}$ are the unique solutions in an open neighbourhood of $\lambda = 0$ of

$$e_l = \varepsilon_l - \frac{\lambda}{V} \sum_{k=1}^{N'} \frac{\vartheta_k}{\varepsilon_k + \varepsilon_l}, \quad 1 = \frac{r_l}{\vartheta_l} - \frac{\lambda}{V} \sum_{k=1}^{N'} \frac{\vartheta_k}{(\varepsilon_k + \varepsilon_l)^2}, \quad \text{for } l = 1, \ldots, N', \quad (4.2.37)$$

with $\lim_{\lambda \to 0} \varepsilon_k = e_k$ and $\lim_{\lambda \to 0} \vartheta_k = r_k$. For any $u \neq -\varepsilon_k$, let $z \in \{u, u^1, \ldots, u^{N'}\}$ be the list of roots of $R(z) = R(u)$. Then the planar 2-point function $G_{pq}^{(0)}$ satisfying the equation \textbf{(4.1.3)} for $g = 0$ that extends into the complex plane, in an open neighbourhood of $\lambda = 0$, is solved by the rational function

$$G_{(0)}^{(0)}(z, w) = \frac{1 - \frac{\lambda}{V} \sum_{k=1}^{N'} \frac{r_k}{(R(\varepsilon_k) - R(-w))(R(z) - R(\varepsilon_k))} \prod_{j=1}^{N'} \frac{R(w) - R(-\varepsilon_j)}{R(w) - R(\varepsilon_j)} \right)}{R(w) - R(-z)}$$

(4.2.38)

with $G_{pq}^{(0)} = G_{(0)}^{(0)}(\varepsilon_p, \varepsilon_q)$. This function $G_{(0)}^{(0)}(z, w)$ is symmetric in $z, w$ and defined outside poles located at $z + w = 0$, at $z = \varepsilon_l^m$ and at $w = \varepsilon_l^n$, for $k, l, m, n = 1, \ldots, N'$.

**Remark 4.6.** A further proof for Theorem 4.4 more intuitive, was found later and uses the right ansatz coming from the argumentation below [SW19]. Here, an important tool is applied

\textsuperscript{4}This subsection is taken from our paper [GHW19b].
which derives the inverse of a Cauchy matrix, which is a matrix of the form $(\frac{1}{x_i-y_j})_{i,j}$. The inverse of a Cauchy matrix is given by an explicit algebraic expression, where the sum of its rows and columns have a particular form [Sch59].

Proof. For the original problem of (finite) $N \times N$-matrices, the construction of the deformed measure is particularly transparent. It gives rise to a rational function $R$ for which the remaining integral of Theorem 4.3 can be evaluated.

In dimension $D = 0$ the special treatment of the lowest eigenvalue $E_1 = \frac{\nu_1^2}{2} = \frac{\mu^2}{2}$ is no longer necessary. The notation simplifies considerably when redefining $R(z) := \frac{\mu^2}{2} + R_0(z - \frac{\mu^2}{2})$. Let $0 < \epsilon_1 < \epsilon_2 < \cdots < \epsilon_{N'}$ be the eigenvalues of $E$ and $r_1, \ldots, r_{N'}$ their multiplicities, with $\sum_{k=1}^{N'} r_k = N$. We shift the measure to $\rho(t) := \rho_0(t - \frac{\mu^2}{2})$:

$$\rho(t) = \frac{1}{V} \sum_{k=1}^{N'} r_k \delta(t - e_k).$$

The deformed measure is according to (4.2.11) given by

$$\rho_\lambda(x - \frac{\mu^2}{2}) = \rho_0(R_0(x - \frac{\mu^2}{2})) = \rho(R(x)) = \frac{1}{V} \sum_{k=1}^{N'} \frac{r_k}{R'(R^{-1}(e_k))} \delta(x - R^{-1}(e_k)),$$

where $R_0$, and thus $R$, arises via (4.2.10) and (4.2.9) from the same measure $\rho_\lambda$:

$$R(z) = z - \lambda \frac{\sum_{k=1}^{N'} g_k}{\varepsilon_k + z}, \quad g_k := \frac{r_k}{R'(R^{-1}(e_k))}, \quad \varepsilon_k := R^{-1}(e_k). \quad (4.2.39)$$

This equation and its derivative evaluated at $z_l = R^{-1}(\epsilon_l) = \epsilon_l$ for $l = 1, \ldots, N'$ provide a system of $2N'$ equations for the $2N'$ parameters $\{\varepsilon_k, g_k\}$:

$$\epsilon_l = \epsilon_l - \lambda \frac{\sum_{k=1}^{N'} g_k}{\varepsilon_k + \epsilon_l}, \quad 1 = \frac{r_l}{\varepsilon_l} - \lambda \frac{\sum_{k=1}^{N'} g_k}{(\varepsilon_k + \epsilon_l)^2}. \quad (4.2.40)$$

The implicit function theorem guarantees a solution in an open $\lambda$-interval, and one explicitly constructs a sequence converging to the solution $\{\varepsilon_k, g_k\}$. Alternatively, (4.2.40) can be interpreted as a system of $2N'$ polynomial equations ($N'$ of them of degree $N'^2 + 1$, the other $N'$ of degree $2N' + 1$). Such system have many solutions, and they will indeed be needed in intermediate steps. The right solution is the one which for $\lambda \to 0$ converges to $\{e_k, r_k\}$.

Recall from (4.2.4) that $G^{(0)}_{pq} = G(x, y)$ where $x + \frac{\mu^2}{2} = R(\varepsilon_p)$ and $y + \frac{\mu^2}{2} = R(\varepsilon_q)$.

The ansatz (4.2.6) for $G(x, y)$ is turned with (4.2.31) and (4.2.32) into the representation

$$G^{(0)}_{pq} = G^{(0)}(\varepsilon_p, \varepsilon_q), \quad G^{(0)}(u, v) = \frac{1}{R(u) + v} \exp \left( \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{\gamma_\epsilon'} dz \frac{R'(z) \log(1 + \frac{\lambda \theta_0(z - \mu^2/2)}{z + R(u)})}{R(z) - R(v)} \right). \quad (4.2.41)$$

Here, the integration variable $z$ in (4.2.31b) is shifted into $z + \frac{\mu^2}{2} \mapsto z$, and $\gamma'_\epsilon$ is the shifted contour which encircles $[\nu_0 + \frac{\mu^2}{2}, \infty)$. We have $\lambda \theta_0(z - \mu^2/2) = -z - R(-z)$ from (4.2.10).
Lemma 4.4. For any \( v, z \notin \{-\varepsilon_1, \ldots, -\varepsilon_{N'}\} \), we can expand the rational function \( R(z) - R(v) \) according to (4.2.39) into

\[
R(z) - R(v) = (z - v) \prod_{k=1}^{N'} \frac{z - \hat{v}^k}{z + \varepsilon_k}.
\]  

(4.2.42)

Here, \( \hat{v}^1, \ldots, \hat{v}^{N'} \) are the other roots of the numerator polynomial; they are functions of \( v \) and the initial data \( E, \lambda \). For real \( v \) it follows from the intermediate value theorem that these roots are interlaced between the poles \( \{ -\varepsilon_k \} \) of \( R \). In particular, for \( v \geq 0 \) and \( \lambda > 0 \) all \( \hat{v}^k \) are real and located in \( -\varepsilon_{k+1} < \hat{v}^k < -\varepsilon_k \) for \( k = 1, \ldots N' - 1 \) and \( \hat{v}^{N'} < -\varepsilon_N' \).

Inserting (4.2.42) into (4.2.41) gives

\[
G^{(0)}(u, v) = \frac{1}{R(u) + v} \exp \left\{ \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{\gamma^*} dz \left( \frac{1}{z - v} + \sum_{k=1}^{N'} \frac{1}{z - \hat{v}^k} - \sum_{k=1}^{N'} \frac{1}{z + \varepsilon_k} \right) \times \log \left( 1 + \frac{\lambda}{z + R(u)} \cdot \frac{1}{V} \sum_{l=1}^{N'} \frac{q_l}{\varepsilon_l - z} \right) \right\}.
\]  

(4.2.43)

**Lemma 4.4.** For \( u, v > 0 \), a posteriori extended to a neighbourhood of \( [\nu_0 + \frac{u^2}{2}, \infty) \), one has

\[
G^{(0)}(u, v) = \frac{1}{R(v) - R(u)} \prod_{k=1}^{N'} \frac{R(u) - R(-\hat{v}^k)}{R(u) - R(\varepsilon_k)}.
\]  

(4.2.44)

**Proof.** As before, for finite \( \epsilon \) and for \( \lambda \) in a small open ball, the logarithm in (4.2.43) can be expanded. After closing the integration contour, the residue theorem picks up the simple poles at \( z = \hat{v}^k \) and \( z = -\varepsilon_k \) and the poles of \( n \)-th order at \( z = -R(v) \). The other candidates \( z = v \) and \( z = \varepsilon_k \) are outside the contour for real \( u, v \). The poles of \( n \)-th order combine (up to a global sign) to a Bührmann formula (4.2.2) for \( H_{u,v}(w) := \log \left( \frac{R(w - R(u)) - R(v)}{R(-R(u)) - R(v)} \right) \), where \( w \mapsto -L(u) \) solves the auxiliary integral

\[
-L(u) := -\lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{\gamma^*_L} dz \log \left( 1 + \frac{\lambda}{z + R(u)} \cdot \frac{1}{V} \sum_{l=1}^{N'} \frac{q_l}{\varepsilon_l - z} \right) \}
\]

\[
= \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n!} \left. \frac{d^{n-1}}{dt^{n-1}} \right|_{t=0} \left( \frac{1}{V} \sum_{l=1}^{N'} \frac{q_l}{\varepsilon_l + R(u) - t} \right)^n.
\]

The Lagrange inversion formula (4.2.1) gives

\[
-\lambda V \sum_{l=1}^{N'} \frac{q_l}{\varepsilon_l + R(u) + L(u)} = -L(u),
\]

which is solved by \( R(u) + L(u) = u \). Putting everything together, the integral (4.2.43) evaluates to

\[
G^{(0)}(u, v) = \frac{1}{R(u) + v} \cdot \frac{R(-R(u)) - R(v)}{R(-R(u)) - R(v)} \prod_{k=1}^{N'} \frac{R(u) - R(-\hat{v}^k)}{\hat{v}^k + R(u)} \prod_{k=1}^{N'} \frac{R(u) - \varepsilon_k}{R(u) - R(\varepsilon_k)}.
\]

The identity (4.2.42) applied for \( z = -R(u) \) simplifies this to (4.2.44). \( \square \)

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The representation \((4.2.44)\) is rational in the first variable. There are two ways to proceed. First, we can expand \((4.2.44)\) via \((4.2.42)\) to

\[
G^{(0)}(u, v) = \frac{\prod_{k=1}^{N'} (u - \varepsilon_k)}{(u + v) \prod_{k=1}^{N'} (u + \hat{\varepsilon}_k)} \frac{\prod_{k=1}^{N'} (u + \hat{\varepsilon}_k) \prod_{l=1}^{N'} (\hat{\varepsilon}_l)}{\prod_{k=1}^{N'} (\hat{\varepsilon}_k - \varepsilon_l) \prod_{l=1}^{N'} (\varepsilon_l - \varepsilon_l)}
\]

\[
= \frac{1}{u + v} \prod_{k,l=1}^{N'} (\varepsilon_k + \varepsilon_l)(\varepsilon_k - \varepsilon_l)(\hat{\varepsilon}_k - \hat{\varepsilon}_l)(\hat{\varepsilon}_k - \hat{\varepsilon}_l).
\]

(4.2.45)

This formula is manifestly symmetric in \(u, v\) — a crucial property below. But it needs all roots of \(R\), which exist only in a neighbourhood of \([\nu_0 + \frac{\mu_1}{2}, \infty)\), not globally.

The limit \(u \to \varepsilon_p\) of \((4.2.44)\) gives with \(r_p = g_p R'(\varepsilon_p)\):

**Corollary 4.2.** For any \(p = 1, \ldots, N'\) and \(v\) in a neighbourhood of \(\mathbb{R}_+\) one has

\[
-\frac{\lambda}{V} r_p G^{(0)}(\varepsilon_p, v) = \frac{\prod_{k=1}^{N'} (R(\varepsilon_p) - R(-\hat{\varepsilon}_k))}{\prod_{p \neq j=1}^{N'} (R(\varepsilon_p) - R(\varepsilon_j))}.
\]

(4.2.46)

In particular, for any \(p, q = 1, \ldots, N'\) one has

\[
G^{(0)}(\varepsilon_p, \varepsilon_q) = -\frac{V}{\lambda r_p} \frac{\prod_{k=1}^{N'} (R(\varepsilon_p) - R(-\varepsilon_q))}{\prod_{p \neq j=1}^{N'} (R(\varepsilon_p) - R(\varepsilon_j))} = -\frac{V}{\lambda r_q} \frac{\prod_{k=1}^{N'} (R(\varepsilon_q) - R(-\varepsilon_p))}{\prod_{q \neq j=1}^{N'} (R(\varepsilon_q) - R(\varepsilon_j))}.
\]

(4.2.47)

**Proof.** Next, we recall the basic lemma

\[
\sum_{j=0}^{N'} \frac{1}{\prod_{j \neq 0} (x_j - x_k)} = 1,
\]

(4.2.48)

valid for pairwise different \(x_0, \ldots, x_{N'}\) and any \(c_1, \ldots, c_{N'}\) (The rational function of \(x_0\) has potential simple poles at \(x_0 = x_k, k = 1, \ldots, N'\), but all residues cancel. Hence, it is an entire function of \(x_0\) by symmetry in all \(x_k\). The behaviour for \(x_0 \to \infty\) gives the assertion.). We use \((4.2.48)\) for \(x_0 = R(u), x_k = R(\varepsilon_k)\) and \(c_k = R(-\hat{\varepsilon}_k)\) to rewrite \((4.2.44)\) as

\[
G^{(0)}(u, v) = \frac{1}{R(v) - R(-u)} \left(1 + \sum_{k=1}^{N'} \frac{1}{R(u) - R(\varepsilon_k)} \frac{\prod_{l=1}^{N'} (R(\varepsilon_l) - R(-\hat{\varepsilon}_l))}{\prod_{k \neq l=1}^{N'} (R(\varepsilon_l) - R(\varepsilon_l))} \right)
\]

\[
= \frac{1}{R(v) - R(-u)} \left(1 + \frac{\lambda}{V} \sum_{k=1}^{N'} \frac{r_k G^{(0)}(\varepsilon_k, v)}{R(\varepsilon_k) - R(u)} \right).
\]

(4.2.49)

The second line results from \((4.2.46)\). Using the symmetry \(G^{(0)}(\varepsilon_k, v) = G^{(0)}(v, \varepsilon_k)\), the previous formulae give rise to a representation of \(G^{(0)}(u, v)\) which is rational in both variables. The assertion \((4.2.38)\) in **Theorem 4.4** follows from symmetry \(G(\varepsilon_k, w) = G(w, \varepsilon_k)\) and insertion of \(G(w, \varepsilon_k)\) given by \((4.2.44)\) into \((4.2.49)\). We could also insert the symmetrised version of \((4.2.49)\),

\[
G^{(0)}(\varepsilon_k, v) = \frac{1}{R(\varepsilon_k) - R(-v)} \left(1 + \frac{\lambda}{V} \sum_{l=1}^{N'} \frac{r_l G^{(0)}(\varepsilon_l, \varepsilon_k)}{R(\varepsilon_k) - R(v)} \right),
\]

(4.2.50)

back into \((4.2.49)\). The remaining assertion of **Theorem 4.4** about the poles of \(G^{(0)}(z, w)\) will be established in **Proposition 4.6** below.
Remark 4.7. For any $z \neq \varepsilon_k$ one has

$$R(z) + \frac{\lambda}{V} \sum_{k=1}^{N'} r_k G^{(0)}(z, \varepsilon_k) + \frac{\lambda}{V} \sum_{k=1}^{N'} \frac{r_k}{R(\varepsilon_k) - R(z)} = -R(-z) .$$  \hspace{1cm} (4.2.51)

We already know this identity. The original equation (4.2.3) for $G^{(0)}_{eq} = G^{(0)}(\varepsilon_p, \varepsilon_q)$ extends to complex variables $\varepsilon_p \mapsto z$ and $\varepsilon_q \mapsto w$, with $w + z \neq 0$ and $w, z \notin \{ \varepsilon_k \}_{k=1}^{N} \cup \{ \varepsilon_k \}_{k=1}^{N'}$:

$$\left\{ R(z) + R(w) + \frac{\lambda}{V} \sum_{k=1}^{N'} r_k G^{(0)}(z, \varepsilon_k) + \frac{\lambda}{V} \sum_{k=1}^{N'} \frac{r_k}{R(\varepsilon_k) - R(z)} \right\} G^{(0)}(z, w)$$

$$= 1 + \frac{\lambda}{V} \sum_{k=1}^{N'} r_k G^{(0)}(\varepsilon_k, w) .$$  \hspace{1cm} (4.2.52)

Now (4.2.51) follows by comparison with (4.2.49). Equation (4.2.51) has also been established for Hölder-continuous measure in Theorem 4.2. Namely, when expressed in terms of the angle function and variables $a + \frac{\mu^2}{2} = R(\varepsilon_p)$ and $b + \frac{\mu^2}{2} = R(\varepsilon_q)$, the terms \{ \} in the first line of (4.2.52) become $\lambda \pi \rho_0(a) \cot \tau_0(a)$. In (4.2.15) we had found $\lambda \pi \rho_0(a) \cot \tau_0(a) = \lim_{x \to 0} \text{Re}(b + \int_0^x (a + i \epsilon))$, which translates into $\lambda \pi \rho_0(a) \cot \tau_0(a) = R(a) - R(-a)$ for $v \neq \varepsilon_k$. From that starting point we had derived (4.2.44) so that finding (4.2.51) from (4.2.49) is no surprise.

But there is another line of arguments. We could have started with (4.2.44) as an ansatz, from which we only arrive at (4.2.49). If we could also prove (4.2.51) from (4.2.44), then (4.2.52) is a consequence of the ansatz (4.2.44), and we have proved that (4.2.51) solves (4.2.52). To directly verify (4.2.51) as identity for rational functions, note that both sides approach $z$ for $z \to \infty$. The rhs has poles only at $z = \varepsilon_j$ with residue $\frac{\lambda}{V^2} \theta_j$. The same poles with the same residues also arise on the lhs, taking $r_k/R'(\varepsilon_k) = \theta_k$ into account. But the lhs also has potential poles at $z = -\varepsilon_j$ and at $z = \varepsilon_k$. We have $\text{Res}_{\varepsilon_j \to \varepsilon_j} R(z) = -\frac{\lambda}{V} \theta_j$. Taking (4.2.44) for $G^{(0)}(z, \varepsilon_j)$ in which we have $\lim_{z \to \varepsilon_j} R(z) - R(\varepsilon_j) = 1$ for any $k, l$, one easily finds that $G^{(0)}(-\varepsilon_j, \varepsilon_l)$ is regular for $j \neq l$ and that $\text{Res}_{\varepsilon_j \to \varepsilon_j} \frac{\lambda}{V^2} r_j G^{(0)}(z, \varepsilon_j) = \frac{\lambda}{V^2} r_j$, which thus cancels $\text{Res}_{\varepsilon_j \to \varepsilon_j} R(z) = -\frac{\lambda}{V} \theta_j$.

Finally, from (4.2.49) we conclude

$$\text{Res}_{z \to \varepsilon_k} G^{(0)}(z, \varepsilon_k) = -\frac{\lambda r_j}{V R'(\varepsilon_k)} \frac{G^{(0)}(\varepsilon_k, \varepsilon_k)}{R(\varepsilon_k) - R(-\varepsilon_k)} = \frac{r_j}{R'(\varepsilon_k)} \frac{\prod_{n \neq j=1}^{N'} (R(\varepsilon_k) - R(-\varepsilon_k^n))}{\prod_{k \neq m=1}^{N'} (R(\varepsilon_k) - R(\varepsilon_m))} ,$$

where (4.2.47) together with $\theta_k R'(\varepsilon_k) = r_k$ has been used in the second equality. The basic lemma (4.2.48) in $N'$ variables $x_k = R(\varepsilon_k)$ gives $\text{Res}_{z \to \varepsilon_k} \sum_{k=1}^{N'} r_k G^{(0)}(z, \varepsilon_k) = \frac{r_k}{R'(\varepsilon_k)}$, which precisely cancels $\text{Res}_{z \to \varepsilon_k} \sum_{k=1}^{N'} \frac{r_k}{R(\varepsilon_k) - R(z)} = -\frac{r_j}{R'(\varepsilon_k)}$. In summary, (4.2.51) is a corollary of (4.2.44). \hfill \square

Proposition 4.6. The planar 2-point function has the (manifestly symmetric) rational fraction expansion

$$G^{(0)}(z, w) = \frac{1}{z + w} \left( 1 + \frac{\lambda^2}{V^2} \sum_{k,l,m=1}^{N'} \frac{C_{k,l,m}^{m,n}}{(z - \varepsilon_k^m)(w - \varepsilon_l^n)} \right) ;$$

$$C_{k,l,m}^{m,n} := \frac{(\varepsilon_k^m + \varepsilon_l^n) r_k r_l G^{(0)}(\varepsilon_k, \varepsilon_l)}{R'(\varepsilon_k^m)R'(\varepsilon_l^n)(R(\varepsilon_l) - R(-\varepsilon_k^m))(R(\varepsilon_k) - R(-\varepsilon_l^n))} .$$
CHAPTER 4. QUARTIC INTERACTION

Proof. Expanding the first denominator in (4.2.49) via (4.2.42), $G^{(0)}(u, v)$ has potential poles at $u = -\hat{\nu}^n$ for every $n = 1, \ldots, N$. However, for $u = -\hat{\nu}$ the sum in the first line of (4.2.49) becomes $\sum_{k=1}^{N}(\frac{1}{(R(-\hat{\nu}^n) - R(\hat{\nu}^n))}) = -1$ when using the basic lemma (4.2.48). Consequently, $G^{(0)}(z, w)$ is regular at $z = -\hat{\nu}$ and by symmetry at $w = -\hat{\nu}$.

This leaves the diagonal $z + w = 0$ and the complex lines $(z = \hat{\nu}_k^m, w = \hat{\nu}_l^n)$ and $(w = \hat{\nu}_k^m, w = \hat{\nu}_l^n)$ as the only possible poles of $G^{(0)}(z, w)$. The function $(z + w)G^{(0)}(z, w)$ approaches 1 for $z, w \to \infty$. Its residues at $z = \hat{\nu}_k^m, w = \hat{\nu}_l^n$ are obtained from (4.2.49):

$$\text{Res}_{z \to \hat{\nu}_k^m, w \to \hat{\nu}_l^n} (z + w)G^{(0)}(z, w) = -\frac{(\hat{\nu}_k^m + \hat{\nu}_l^n)}{(R(\hat{\nu}_k) - R(-\hat{\nu}_k^m))} V R((\hat{\nu}_k^m)_{w \to \hat{\nu}_l^n})$$

$$= \left(\frac{\lambda}{V}\right)^2 R((\hat{\nu}_k^m)_{w \to \hat{\nu}_l^n})(R(\hat{\nu}_k) - R(-\hat{\nu}_k^m))(R(\hat{\nu}_k) - R(-\hat{\nu}_l^n)).$$

The second line follows from (4.2.50). □

Example 4.5. The extreme case of a single $r_1 = N$-fold degenerate eigenvalue $E = \frac{\mu^2}{2} \cdot \text{id}$ corresponds to a standard Hermitian 1-matrix model with action $S[\Phi] = V \text{Tr}((\frac{\mu^2}{2} \cdot \text{id})^2 + \lambda(\Phi^4)$.

This purely quartic case was studied in [BIPZ78]. Transforming $\hat{M} \mapsto \sqrt{V} \mu \Phi$ and $g = \frac{\lambda}{\sqrt{V}}$ and $V = N$ brings [BIPZ78] eq. (3]) into our conventions. The equations (4.2.37) reduce for $E_1 = \frac{\mu^2}{2}$ and $N' = 1$ to

$$\frac{\mu^2}{2} = \varepsilon_1 - \frac{\lambda \rho_1}{N(2\varepsilon_1)}, \quad \rho_1 = \frac{\sqrt{\mu^4 + 12\lambda} - \mu^4 + 12\lambda}{18\lambda}. \quad (4.2.53)$$

with principal solution (i.e. $\lim_{\lambda \to 0} \varepsilon_1 = \frac{\mu^2}{2}$)

$$\varepsilon_1 = \frac{1}{6}(2\mu^2 + \sqrt{\mu^4 + 12\lambda}) \quad \rho_1 = \frac{\sqrt{\mu^4 + 12\lambda} - \mu^4 + 12\lambda}{18\lambda}. \quad (4.2.54)$$

The other root $\hat{\varepsilon}_1$ with $R(\hat{\varepsilon}_1) = \hat{\varepsilon}_1 - \frac{\lambda \rho_1}{N(\varepsilon_1 + \hat{\varepsilon}_1)} = \hat{R}(\varepsilon_1) = \frac{\mu^2}{2}$ is found to be

$$\hat{\varepsilon}_1 = -\frac{1}{6}(\mu^2 + 2\sqrt{\mu^4 + 12\lambda}) = \frac{\mu^2}{2} - 2\varepsilon_1. \quad (4.2.55)$$

The planar 2-point function $G^{(0)}_{11}$ can be evaluated via (4.2.47) or (4.2.45) to

$$G^{(0)}_{11} = -\frac{1}{\lambda} \left(\frac{\mu^2}{2} - R(-\hat{\varepsilon}_1)\right) = \frac{4}{3} \left(\frac{\mu^2 + 2\sqrt{\mu^4 + 12\lambda}}{(\mu^2 + \sqrt{\mu^4 + 12\lambda})^2}\right) \frac{2\hat{\varepsilon}_1}{(\varepsilon_1 - \hat{\varepsilon}_1)^2}. \quad (4.2.56)$$

The result can be put into $G^{(0)}_{11} = \frac{1}{4\sqrt{\mu^2}} a^2 (4 - a^2)$ for $a^2 = \frac{2\mu^2}{\mu^2 + \sqrt{\mu^4 + 12\lambda}}$ and thus agrees with the literature: This value for $a^2$, which corresponds to $\frac{2\mu^2}{\mu^2 + \sqrt{\mu^4 + 12\lambda}} = \varepsilon_1 - \frac{\mu^2}{2},$ solves [BIPZ78] eq. (17a)] for $g := \frac{\lambda}{\mu^2},$ so that (4.2.56) reproduces [BIPZ78] eq. (27)] for $p = 1$ (and the convention $G^{(0)}_{11} = \frac{1}{\mu^2} \text{for} \lambda = 0)$. 

\*In [BIPZ78] expectation values of traces $\langle \text{Tr}(M^{2p}) \rangle$ are studied, whereas we consider $\langle M_{11}M_{11} \rangle$, see also Remark 2.2.
4.2. SOLUTION OF THE PLANAR 2-POINT FUNCTION

The meromorphic extension $G^{(0)}(z, w)$ is most conveniently derived from Proposition 4.6 after cancelling the two representations (4.2.56) for $G^{(0)}_{11} = G^{(0)}(\varepsilon_1, \varepsilon_1)$:

$$G^{(0)}(z, w) = \frac{1}{z + w} \left(1 - \frac{(\varepsilon_1 + \varepsilon_1^*)^2}{(z - \varepsilon_1^*)(w - \varepsilon_1^*)}\right) \cdot \frac{1}{z + w} \left(1 - \frac{\mu^4(1 - a^2)^2}{(3a^2z + \mu^2)(3a^2w + \mu^2)}\right).$$ (4.2.57)

We have used $R'(\varepsilon_1^*) = \varepsilon_1^* - \varepsilon_1^*$.

**Corollary 4.3.** Let the planar free energy be $F^{(0)}$ with $F =: \sum_{q=0}^{\infty} V^{-2q} F^{(q)}$ and $F := V^{-2} \log Z[0]$. With the distinct eigenvalues $e_q$ of multiplicity $r_q$ of $E$, we have

$$\frac{V}{r_q} \frac{\partial}{\partial e_q} F^{(0)} = \frac{e_q - e_q}{\lambda} + \frac{1}{V} \sum_{k=0}^{N} r_k \left(\frac{1}{e_k - e_q} - \frac{1}{R'(\varepsilon_k)(\varepsilon_k - e_q)}\right).$$

**Proof.** Combing Corollary 4.1 after genus expansion with (4.2.51) with $z = e_q$ gives the assertion. \(\square\)

Integrating the rhs of the equation of Corollary 4.3 wrt $e_q$, multiplying with $\frac{r_q}{V}$ and summing over $q$ gives in principle the planar free energy $F^{(0)}$. This integration is very hard to perform, since all $\varepsilon_k$ and $R'(\varepsilon_k)$ depend intrinsically on $e_q$.

### 4.2.4 Solution on the $D = 4$ Moyal Space

On the $\mathcal{O} = D = 4$ Moyal space the eigenvalues increase linearly, $e(x) = x$, with the measure $r(x) = x$ which induces for the undeformed measure $\rho_0(x) = r(x) = x$. The deformed measure $\rho_\lambda$ is then, due to Definition 4.4, given by

$$\rho_\lambda(x) = \rho_0(R_\lambda(x)) = R_\lambda(x).$$

On the other hand, the function $R_\lambda(x)$ was defined in Definition 4.3 such that the deformed measure obeys a further integral equation

$$\rho_\lambda(x) = x - \lambda x^2 \int_0^{\infty} \frac{dt \rho_\lambda(t)}{t + \mu^2(t + \mu^2 + x)}.$$ (4.2.58)

This integral equation is a linear integral equation and of Fredholm type. Finding the solution of (4.2.58) solves together with Theorem 4.3 the noncommutative $\Phi^4_4$-QFT model exactly.

**Proposition 4.7.** Equation (4.2.58) is solved by

$$\rho_\lambda(x) = x \cos\left(\frac{\lambda \pi}{2}\right) - \frac{x}{\mu^2}, \quad \text{where} \quad \alpha_\lambda = \begin{cases} \frac{\arcsin(\lambda \pi)}{\lambda \pi} & \text{for } |\lambda| \leq \frac{1}{\pi}, \\ \frac{1}{\lambda} + \frac{\arccosh(\lambda \pi)}{\lambda \pi} & \text{for } \lambda \geq \frac{1}{\pi}. \end{cases}$$ (4.2.59)

Moreover, the particular choice $\mu^2 = \frac{\alpha_\lambda(1 - \alpha_\lambda)}{\lambda}$ provides a natural choice for the boundary condition differently from the condition $(\frac{\partial}{\partial a} G^{(0)}(a, 0)|_{a=0} = -1)$ induced by Zimmermann’s forest formula.

\[\text{This subsubsection is taken from our paper [GHW20].}\]
Proof. It is convenient to symmetrise the Fredholm equation (4.2.58). Dividing by \( \frac{x}{x^2+x} \) and defining \( \tilde{g}_\lambda(x) := \frac{R_\lambda(x)}{x^2+x} \), we have

\[
\tilde{g}_\lambda(x) = \frac{1}{\mu^2+x} - \lambda \int_0^\infty dt \frac{\tilde{g}_\lambda(t) tx}{(\mu^2+t)(\mu^2+x)(\mu^2+x+t)} = \frac{c_\lambda}{\mu^2+x} - \lambda \int_0^\infty dt \frac{\tilde{g}_\lambda(t)}{\mu^2+x+t},
\]

(4.2.60)

where \( c_\lambda = 1 + \lambda \mu^2 \int_0^\infty dt \frac{\tilde{g}_\lambda(t)}{\mu^2+t} = 1 + \lambda \mu^2 \int_0^\infty dt \frac{R_\lambda(t)}{\mu^2+t^2} \). The second line results by (not so obvious) rational fraction expansion. As proved in App. E.3, there exists for \( \lambda > \frac{1}{2} \) a solution \( \tilde{g}_\lambda \in L^2(\mathbb{R}_+) \), which means \( \lim_{t \to \infty} t \tilde{g}_\lambda(t) = 0 \). Another transformation \( \phi(x) = \mu^2 \tilde{g}_\lambda(x^2) \) simplifies the problem to

\[
\phi(x) = \frac{c_\lambda}{1+x} - \lambda \int_0^\infty dt \frac{\phi(t)}{1+t+x}, \quad \phi(0) = 1.
\]

(4.2.61)

The aim is to find the differential operator \( D_x \) acting on (4.2.61) which is reproduced under the integral on \( \phi(t) \) such that all appearing inhomogeneous parts vanish, i.e.

\[
D_x \phi(x) = -\lambda \int_0^\infty dt \frac{D_t \phi(t)}{1+t+x}.
\]

We compute derivatives and integrate by parts, taking the boundary values at 0 and \( \infty \) into account:

\[
\phi'(x) = -\frac{c_\lambda}{(1+x)^2} + \lambda \int_0^\infty dt \frac{\phi(t)}{1+t+x} + \frac{\lambda}{1+x}.
\]

(4.2.62)

Also the product with \( 1+x \) simplifies by integration by parts:

\[
(1+x)\phi'(x) = -\frac{c_\lambda}{(1+x)} - \lambda \int_0^\infty dt \frac{t \phi(t)}{1+t+x}.
\]

(4.2.63)

We differentiate once more:

\[
(1+x)\phi''(x) + \phi'(x) = \frac{c_\lambda}{(1+x)^2} + \lambda \int_0^\infty dt \frac{dt}{(1+t+x)} \frac{d}{dt} (t\phi'(t)),
\]

\[
(1+x)\phi''(x) = \frac{2c_\lambda}{(1+x)^2} + \lambda \int_0^\infty dt \frac{t \phi''(t)}{1+t+x} - \frac{\lambda}{1+x}.
\]

We multiply by \( x \) and integrate by parts:

\[
x(1+x)\phi''(x) = \frac{2c_\lambda}{(1+x)} - \frac{2c_\lambda}{(1+x)^2} - \lambda \int_0^\infty dt \frac{t(1+t)\phi''(t)}{1+t+x} + \frac{\lambda}{1+x}.
\]

We subtract twice (4.2.62) and add four times (4.2.63):

\[
x(1+x)\phi''(x) + (2+4x)\phi'(x) = -\frac{2c_\lambda + \lambda}{(1+x)} - \lambda \int_0^\infty dt \frac{t(1+t)\phi''(t) + (2+4t)\phi'(t)}{1+t+x}.
\]
Finally, we add \( \frac{2\lambda}{c\lambda} \) times (4.2.61) to get \( D_x = x(1 + x) \frac{d^2}{dx^2} + (2 + 4x) \frac{d}{dx} + \frac{2\lambda}{c\lambda} \), or equivalently
\[
0 = (\text{id} + \lambda \hat{A}_1)g, \quad \text{where}
\]
\[
g(x) = x(1 + x)\phi''(x) + (2 + 4x)\phi'(x) + \frac{2c\lambda + \lambda}{c\lambda} \phi(x),
\]
and \( \hat{A}_\mu \) is the integral operator with kernel \( \hat{A}_\mu(t, u) = \frac{1}{u+t+\mu^2} \). The arguments given in App. E.3 show that \( \hat{A}_\mu \) has spectrum \([0, \pi]\) for any \( \mu \geq 0 \). Therefore, equation (4.2.64) has for \( \lambda > -\frac{1}{\pi} \) only the trivial solution \( g(x) = 0 \), which is a standard hypergeometric differential equation. The normalisation \( \phi(0) = 1 \) uniquely fixes the solution to
\[
\phi(x) = 2F_1\left(1+\alpha\lambda, \frac{2-\alpha\lambda}{2} \left| -x \right. \right)
\]
\[
= \frac{1}{1 + x} 2F_1\left(\frac{\alpha\lambda}{2}, \frac{1-\alpha\lambda}{2} \left| -x \right. \right), \quad c\lambda = \frac{\lambda}{\alpha\lambda(1-\alpha\lambda)}. \quad (4.2.65)
\]

It remains to satisfy the boundary condition \( c\lambda = 1 + \lambda \int_0^\infty dt \frac{\phi(t)}{1+t} \) given after (4.2.60). The integral can be evaluated via the Euler integral [GR07 §9.111],
\[
\int_0^\infty dt \frac{\phi(t)}{1+t} = \frac{\Gamma(2)}{\Gamma(1-\alpha\lambda)\Gamma(1+\alpha\lambda)} \int_0^1 ds \int_0^1 du \frac{u^{-\alpha\lambda}(1-u)^{\alpha\lambda}}{(1+ut)^{\alpha\lambda}(1+t)^2}
\]
\[
= \frac{1}{\Gamma(1-\alpha\lambda)\Gamma(1+\alpha\lambda)} \int_0^1 du \int_0^1 ds \frac{u^{-\alpha\lambda}(1-u)^{\alpha\lambda}}{(1-(1-u)s)^{\alpha\lambda}}
\]
\[
= \frac{1}{\Gamma(1-\alpha\lambda)\Gamma(2+\alpha\lambda)} \int_0^1 du \frac{u^{\alpha\lambda}(1-u)^{-\alpha\lambda}}{\alpha\lambda} \left\{ \frac{(1+\alpha\lambda)}{\alpha\lambda} 2F_1\left(\frac{\alpha\lambda}{2+\alpha\lambda} \left| u \right. \right) \right. 
\]
\[
- \frac{1}{\alpha\lambda} 2F_1\left(\frac{\alpha\lambda}{2}, \frac{2+\alpha\lambda}{2} \left| u \right. \right) \left\} 
\]
\[
= \frac{1}{\alpha\lambda(1-\alpha\lambda)} \frac{\Gamma(1+\alpha\lambda)\Gamma(1-\alpha\lambda)\Gamma(1-\alpha\lambda)}{\Gamma(2+\alpha\lambda)\Gamma(1-\alpha\lambda)\Gamma(1-\alpha\lambda)} 
\]
\[
= \frac{\Gamma(1+\alpha\lambda)\Gamma(1-\alpha\lambda)\Gamma(1-\alpha\lambda)}{\Gamma(2)\Gamma(2)\Gamma(1)}
\]
\[
= \frac{\Gamma(1+\alpha\lambda)\Gamma(1-\alpha\lambda)}{\alpha\lambda(1-\alpha\lambda)} \Gamma(1-\alpha\lambda).
\]

Here we have transformed \( t = \frac{s}{1-x} \), evaluated first the s-integral [GR07 §9.111] to a hypergeometric function, used its contiguous relation [GR07 §9.137.17] so that the remaining integrals are known from [GR07 §7.512.4] and [GR07 §7.512.3]. We thus conclude
\[
c\lambda = 1 + \frac{\lambda}{\alpha\lambda(1-\alpha\lambda)} - \frac{\lambda\pi}{\sin(\alpha\lambda\pi)} = \frac{1}{\alpha\lambda(1-\alpha\lambda)}
\]
with solution
\[
\sin(\alpha\lambda\pi) = \lambda\pi, \quad \alpha\lambda = \begin{cases} \arcsin(\lambda\pi) & \text{for } |\lambda| \leq \frac{1}{\pi}, \\ \frac{\pi}{2} + \frac{\pi}{2} \arccosh(\lambda\pi) & \text{for } \lambda \geq \frac{1}{\pi}. \end{cases} \quad (4.2.66)
\]
The branch is uniquely selected by the requirement \( \lim_{\lambda \to 0} c_\lambda = 1 \). For \( \lambda < -\frac{1}{\pi} \) there is no solution for which \( c_\lambda \) and \( \phi \) are real. Transforming back to \( \hat{\rho}_\lambda \) and \( R_4 \) gives the result announced in Proposition 4.7, which provides the 2-point function \( G(x, y) \) via Theorem 4.3.

The choice of \( \mu^2 = \frac{\alpha_4(1-\alpha_4)}{\lambda} \) which is a natural choice is discussed in great details by perturbative analysis in App. E.1 by comparison to the angle function.

The proof presented here was not the way how we found the solution, however it is for a reader the most transparent one. The perturbative analysis outsourced to the App. E.1 provided the first more natural idea. We compared the perturbative result of (4.2.58) with the perturbative result of (4.2.58). The details came by using the Maple package HYPERINT [Pan15], which computed iterated integral in a symbolical way. Up to the 10th order in \( \lambda \), two coupled differential equations were found which gave a conjectural solution. This conjectural solution was then proved with the Meijer G-function which can be found in [App. E.2]. The easy proof above was realised after these discoveries.

### Effective Spectral Dimension

Let \( g_0(x)dx \) be the spectral measure of the operator \( E \) in the initial action of the non-commutative \( \Phi_4^2 \) model. The main discovery of Sec. 4.2.1 was that the interaction \( \frac{1}{4} \text{Tr}(\Phi^4) \) effectively modifies the spectral measure to \( g_\lambda(x)dx \). What before, when expressed in terms of \( g_0(x)dx \), was intractable became suddenly exactly solvable in terms of the deformation \( g_\lambda(x)dx \). For 4-dimensional Moyal space, one has \( g_0(x) = x \) and \( g_\lambda(x) = R_\lambda(x) \). The explicit solution (4.2.59) shows that the deformation is drastic: it changes the spectral dimension \( D \) defined in Definition 2.1 to an effective spectral dimension \( D_\lambda := \inf \{ p : \int_0^\infty dt \frac{\phi_\lambda(t)}{(1+t)^{p+2}} < \infty \} \).

**Lemma 4.5.** For any \( |\alpha_\lambda| < \frac{1}{2} \) one has

\[
\frac{1}{(1 + x)^{\alpha_\lambda}} \leq 2 F_1(\alpha_\lambda, 1 - \alpha_\lambda | - x) \leq \frac{\Gamma(1 - 2\alpha_\lambda)}{\Gamma(2 - \alpha_\lambda)\Gamma(1 - \alpha_\lambda)} \frac{1}{(1 + x)^{\alpha_\lambda}}.
\]

**Proof.** We transform with [GR07, §9.131.1] to

\[
2 F_1(\alpha_\lambda, 1 - \alpha_\lambda | - x) = \left( \frac{1}{1 + x} \right)^{\alpha_\lambda} 2 F_1\left(\frac{2 - \alpha_\lambda, 1 - \alpha_\lambda}{2} | \frac{x}{1 + x} \right) \frac{x}{1 + x}.
\]

By [PV77, Thm. 1.10], the fraction on the rhs is strictly increasing from 1 at \( x = 0 \) to its limit \( \frac{\Gamma(1-2\alpha_\lambda)}{\Gamma(2-\alpha_\lambda)\Gamma(1-\alpha_\lambda)} \) for \( x \to \infty \). \( \square \)

**Corollary 4.4.** For \( |\lambda| < \frac{1}{\pi} \), the deformed measure \( g_\lambda = R_4 \) of 4-dimensional Moyal space has spectral dimension \( D_\lambda = 4 - 2 \frac{\arcsin(\lambda \pi)}{\pi} \).

**Proof.** Lemma 4.5 together with \( g_\lambda(x) = R_4(x) \) and (4.2.59) gives the assertion. \( \square \)

The change of spectral dimension is important. If instead of (4.2.58) the function \( R_4 \) was given by \( \check{R}(x) = x - \lambda x^2 \int_0^\infty dt \frac{\phi_\lambda(t)}{(1+t)^{\alpha_\lambda}} \), then for \( g_0(x) = x \) this function \( \check{R} \) is bounded above. Hence, \( \check{R}^{-1} \) needed in higher topological sectors could not exist globally on \( \mathbb{R}_+ \), which would render the model inconsistent for any \( \lambda > 0 \). The dimension drop down to \( D_\lambda = 4 - 2 \frac{\arcsin(\lambda \pi)}{\pi} \) avoids this (triviality) problem.
Perturbative Expansion of the 2-Point Function

For a perturbative expansion, the exact solution of the planar 2-point function via [Theorem 4.3] and [Proposition 4.7] is not practical to expand in small \( \lambda \). The expression \[(4.2.6)\] is much more convenient for a perturbative analysis, where the measure is \( \varrho_0(a) = a \) and the field renormalisation constant \( Z \) is chosen to satisfy \( G(0, 0) = 1 \). The angle function \( \tau_b \) is derived perturbatively from \[(4.2.8)\], where the mass renormalisation is taken by

\[
\mu^2_{\text{bare}} = 1 - \lambda \Lambda^2 - \frac{1}{\pi} \int_0^{\Lambda^2} dt \tau_0(t).
\]

The computations are first done for finite \( \Lambda^2 \) where the limit is obviously convergent. All details about the perturbative expansion to higher order is discussed very detailed in [App. E.1]. We present here only the first two orders which are

\[
p\lambda \pi \cot(\tau_a(p)) = 1 + a + p + \lambda ((1 + p) \log(1 + p) - p \log(p)) \\
+ \lambda^2 (-p \zeta_2 + (1 + p) \log(1 + p)^2 + (1 + 2p) \text{Li}_2(-p)) + O(\lambda^3)
\]

and after inserting in \[(4.2.6)\]

\[
G(a, b) = \frac{1}{1 + a + b} - \frac{\lambda}{(1 + a + b)^2} \left\{ (1 + a) \log(1 + a) + (1 + b) \log(1 + b) \\
+ \frac{\lambda^2}{(1 + a + b)^3} \left\{ \zeta_2 ab + (1 + a)(1 + b) \log(1 + a) \log(1 + b) \\
- a(1 + b) \log(1 + b)^2 - b(1 + a) \log(1 + a)^2 \\
- (1 + b + 2a + 2ab + a^2) \text{Li}_2(-a) - (1 + a + 2b + 2ab + b^2) \text{Li}_2(-b) \right\}
\]

\[+ O(\lambda^3).\]

This result coincides only with the perturbative expansion through Feynman graphs for hyperlogarithms with two letters (see [App. D] for the definition of hyperlogarithms) according to the computations in [App. D.2]. Since the boundary conditions for Zimmermann’s forest formula are different both expansion cannot coincide completely. It is hard to adjust the forest formula to obey \( \frac{\partial}{\partial a} G(a, 0)|_{a=0} = -1 - \lambda + \lambda^2 + O(\lambda^3) \) which is the right boundary condition for \[(4.2.67)\].

On the other hand, it is also possible to change \( \mu^2 \neq \frac{\alpha \lambda (1 - \alpha \lambda)}{\lambda} \) in Theorem 4.3 and Proposition 4.7 such that the condition \( \frac{\partial}{\partial a} G(a, 0)|_{a=0} = -1 \) which includes recursively some work. One would determine for arbitrary \( \mu^2(\lambda) \) the functions \( R_4, R_4^{-1} \) and \( I(w) \) in an expansion in \( \lambda \) to get the angle function \( \tau_b(a) \) as an expansion (depending on \( \mu^2 \)). Inserting then the expanded angle function in \[(4.2.6)\] such that \( G(0, 0) = 1 \) and \( \frac{\partial}{\partial a} G(a, 0)|_{a=0} = -1 \) holds order by order, fixes \( \mu^2 \) at the first orders in \( \lambda \). This is done more explicitly in [App. D.2] where it is shown that for the same boundary conditions both approaches coincide perfectly!

**Remark 4.8.** The quartic model on the \( D = 4 \) Moyal space admits the renormalon problem which generates no problem for the exact formula. Determining the amplitude of the Feynman graph below according to the Feynman rules together with Zimmermann’s forest formula gives

\[
\left( -\lambda \right)^{3+2n} \left( 1 + 2a \right)^6 \int_0^\infty dy \frac{(1 + y) \log(1 + y) - y)^n}{(1 + a + y)^{3+n}} \sim \left( -\lambda \right)^{3+2n} \left( 1 + 2a \right)^6 \int_0^\infty dy \frac{y^2 \log(y)^n}{\sim n!}.
\]
4.3 Higher Order Correlation Functions

The 2-point function is the starting point in solving the entire hierarchy of all correlation functions for the quartic model. We have seen that the SDEs of Proposition 4.4 and Proposition 4.5 have when using \( T_q \) defined in Definition 4.1 a universal structure of the form

\[
\hat{K}_1^{(g)} |_{p,J} = g_{inh}^{q,J}
\]

and

\[
\hat{K}_2^{(g)} |_{q_1,q_2,J} = g_{inh}^{q,J},
\]

where the operators \( \hat{K}^i \) are

\[
\hat{K}_1^i f(p) := f(p) \left\{ H_{pp} + \frac{\lambda}{V} \sum_{n=0}^{N} \left( \frac{1}{E_n - E_p} + G^{(0)}_{\{np\}} \right) \right\} - \frac{\lambda}{V} \sum_{n=0}^{N} \frac{f(n)}{E_n - E_p}
\]

and \( g_{inh}^{q,J} \) is a inhomogeneity of less topology, i.e. a larger Euler characteristic than \( G^{(g)} \).

On the other hand, we have used the important identity (4.2.51) for the proof of Theorem 4.4 mentioned in Remark 4.7, which implies with the definition of \( R(z) \) at \( z = \varepsilon_p \)

\[
E_p + \frac{\lambda}{V} \sum_{n=0}^{N} \left( \frac{1}{E_n - E_p} + G^{(0)}_{\{np\}} \right) = -R(-R^{-1}(E_p)),
\]

where \( R^{-1}(z) \) is the principal branch with \( R(\varepsilon_p) = E_p \) due to Lemma 4.2. The expression makes sense for \( n = p \) after inserting it into \( \hat{K}^1 \) and \( \hat{K}^2 \) and assuming a differentiable interpolation between the discrete point \( E_p \). It is natural to pass from \( G^{(0)} \to G^{(0)}(\ldots) \) by

\[
G^{(g)}_{\{p_1,\ldots,p_N\}} := G^{(g)}(\varepsilon_{p_1}, \varepsilon_{p_2}, \ldots, \varepsilon_{p_N}),
\]

where \( \varepsilon_p = R^{-1}(E_p) \) due to Theorem 4.4.

Assuming that \( G^{(g)} \) can be analytically continued except of for some particular points (poles) yields for the SDEs (4.3.1) and (4.3.2)
4.3. HIGHER ORDER CORRELATION FUNCTIONS

\[(R(z) − R(−w))G^{(g)}(z, w | \tilde{J}) − \frac{\lambda}{V} \sum_{n=0}^{N'} r_n \frac{G^{(g)}(\varepsilon_n, w | \tilde{J})}{R(\varepsilon_n) − R(z)} = g^{\alpha}_{\text{inh}}, \quad (4.3.5)\]

where \(\tilde{J}\) is a set of some complex numbers.

Comparing with [Definition 3.1] and (3.2.12) of the cubic model implies exactly the same structure for the SDE (4.3.4), where the base point is taken from the boundary of length one. The SDE (4.3.5), where the base point is taken from the boundary of length two, shows a new structure of more complexity.

Formally, all correlation functions can be derived recursively by inverting the equations (4.3.4) and (4.3.5). Take the planar \((1 + 1)\)-point function as an example. It obeys with \(g = 0\) and \(J = \{w\}\) for (4.3.4) the equation

\[(R(z) − R(−z))G^{(0)}(z | w) − \frac{\lambda}{V} \sum_{k=0}^{N'} r_k G^{(0)}(\varepsilon_k | w) = \frac{\lambda G^{(0)}(z, w) − G^{(0)}(w, w)}{R(z) − R(w)}. \quad (4.3.6)\]

Let \(z \in \{0, \pm \alpha_0, \ldots, \pm \alpha_{N'}\}\) be the solutions of \(R(z) − R(−z) = 0\), with all \(\alpha_k > 0\). When treating (4.3.6) as a Carleman-type singular integral equation as in [GW14a], it is clear that \(G^{(0)}(\alpha_k | w)\) is regular for all \(w > 0\). Therefore, setting \(z \mapsto \alpha_k\) gives a system of \(k\) affine equations

\[\frac{1}{V} \sum_{l=0}^{N'} r_l G^{(0)}(\varepsilon_l | w) = \frac{G^{(0)}(\alpha_k, w) − G^{(0)}(w, w)}{R(\alpha_k) − R(w)}, \quad k = 0, \ldots, N'. \quad (4.3.7)\]

They are easily solved by the inverse Cauchy matrix [SW19] for \(\frac{1}{V} G^{(0)}(\varepsilon_l | w), l = 0, \ldots, N'\), in terms of the planar 2-point function \(G^{(0)}(z, w)\), which are already known. Moreover, since \(G^{(0)}(z, w)\) depends on \(z\) only via \(R(\pm z)\), setting instead \(z \mapsto −\alpha_k\) in (4.3.6) gives the same \(G^{(0)}(\varepsilon_l | w)\). With these \(G^{(0)}(\varepsilon_l | w)\) determined, (4.3.6) gives the explicit formula for \(G^{(0)}(z | w)\):

\[G^{(0)}(z | w) = \frac{\lambda}{V} \sum_{k=0}^{N'} r_k G^{(0)}(\varepsilon_k | w) + \frac{\lambda G^{(0)}(z, w) − G^{(0)}(w, w)}{R(z) − R(w)}. \quad (4.3.8)\]

The global denominator \(R(z) − R(−z)\) introduces a pole only at \(z = 0\), but not at \(z = ±\alpha_k\).

This procedure can be applied recursively for (4.3.4) and (4.3.5) (see [SW19] for more information). However, the procedure does not capture the analytic structure of the correlation function. The poles which characterise the analytic structure are not revealed. It is known for instance that the \((1 + 1)\)-point function is symmetric \(G^{(0)}(z | w) = G^{(0)}(w | z)\) which is hardly to check through the representation (4.3.8).

Comparing to the cubic model and its link to topological recursion (see Sec. 3.2.7) it is natural to conjecture also here a connection to topological recursion through the identity (1.2.51) and all SDEs. The spectral curve is conjecturally a rational plane algebraic curve given by

\[\mathcal{E}(x(z), y(z)) = 0 \quad \text{with} \quad x(z) = R(z), \quad y(z) = −R(−z) \quad (4.3.9)\]

with the corresponding Riemann surface \(\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}\). The rational function \(R(z)\) was defined in [Theorem 4.4] implicitly by the eigenvalues \(e_k\) of the external matrix \(E\).
together with its corresponding multiplicities $r_k$. However, it is not yet clear how the meromorphic $n$-forms $\omega_{g,n}$ coming from the spectral curve $\mathcal{E}(x(z), y(z))$, $z \in \mathbb{C}$, the $1$-form $\omega_{0,1}(z) = y(z) \, dx(z)$ and the $2$-form $\omega_{0,2}(z_1, z_2) = \frac{dz_1 \cdot dz_2}{(z_1 - z_2)^2}$ through topological recursion [EO07, Eyn16], are linked to the correlation functions of the quartic matrix field theory model. A detailed analysis of the pole structure of $\omega_{g,n}$ can bring more light into the dark, but the state of not knowing the branch point $dx(z) = 0$ explicitly makes the approach of topological recursion more difficult. On the other hand, applying the procedure of inverse Cauchy matrices yields also inconvenient results. Studying the connection between the quartic matrix field theory model and topological recursion is work in progress.

4.4 Explicit Form of the Planar Recursive Equation**

This section studies the recursive equation of the planar $N$-point function with one boundary component in more detail

$$G_{p_0 \ldots p_{N-1}}^{(0)} = -\lambda \sum_{l=1}^{N-2} \frac{G_{p_0 \ldots p_{2l-1}}^{(0)} \cdot G_{p_{2l} \ldots p_{N-1}}^{(0)} - G_{p_1 \ldots p_{2l}}^{(0)} \cdot G_{p_{2l+1} \ldots p_{N-1}}^{(0)}}{(E_{p_0} - E_{p_{2l}})(E_{p_1} - E_{p_{2l+1}})}.$$  \hspace{1cm} (4.4.1)

which was given by Example 4.1. We will omit the superscript of $G^{(0)}$, which indicates the genus, and let $\lambda = -1$, which is for the analysis of the recursion irrelevant, for the rest of the section.

Interestingly, the same relation (4.4.1) appears in the planar sector of the $2$-matrix model for mixed correlation functions [EO05]. The distinction between even $b_{2i}$ and odd $b_{2i+1}$ matrix indices in (4.4.1) corresponds to the different matrices of the $2$-matrix model.

The complete expression for the $(N = 2k + 2)$-point function $G_{p_0 p_1 \ldots p_{2k+1}}$ according to (4.4.1) yields $2^k c_k$ terms of the form

$$\pm \frac{G_{p_i p_j} \cdots G_{p_m p_n}}{(E_{p_i} - E_{p_m}) \cdots (E_{p_j} - E_{p_n})}.$$  \hspace{1cm} (4.4.2)

with $i < j$, $l < u$, $n < m$ and $v < w$, where $c_k = \frac{1}{k+1}(\frac{2k}{k})$ is the $k$th Catalan number. However, some of the terms cancel. We will answer the questions: Which terms survive the cancellations? Can they be explicitly characterised, without going into the recursion? The answer will be encoded in Catalan tables.

First we discuss the symmetries of $G_{p_0 \ldots p_{N-1}}$ induced by (4.4.1) and not by its definition as an expectation value (which actually are the same). Then, we will introduce Catalan tuples and Catalan tables, certain trees and operations on them. The Catalan numbers $c_k = \frac{1}{k+1}(\frac{2k}{k})$ will count various parts of our results and will be related to the number $d_k = \frac{1}{k+1}(\frac{3k+1}{k})$ of Catalan tables of length $k + 1$, see Proposition 4.8.

The main part is Theorem 4.3 that Catalan tables precisely encode the surviving terms in the expansion of $G_{p_0 \ldots p_{N-1}}$ with specified designated node.

Both the Catalan tables and the $G_{p_0 \ldots p_{N-1}}$ can be depicted conveniently as chord diagrams with threads, which is discussed in App. F.2. Through these diagrams it will become clear that the recursion relation (4.4.1) is related to well-known combinatorial problems [DN02, Noy98].

**This section is taken from our paper [dJHW19]
4.4. Explicit Form of the Planar Recursive Equation

4.4.1 Symmetries

Theorem 4.3 proves the symmetry of the 2-point function, \( G_{p_0p_j} = G_{p_jp_0} \). Because there is an even number of antisymmetric factors in the denominator of each term, it follows immediately that

\[
G_{p_0p_1...p_{N-1}} = G_{p_{N-1}...p_1p_0} .
\] (4.4.3)

Our aim is to prove cyclic invariance \( G_{p_0p_1...p_{N-1}} = G_{p_{N-1}...p_1p_0} \). We proceed by induction. Assuming that all \( n \)-point functions with \( n \leq N - 2 \) are cyclically invariant, it is not difficult to check that

\[
G_{p_0p_1...p_{N-1}} = \sum_{l=1}^{N-2} \frac{G_{p_0...p_{2l-1}} \cdot G_{p_{2l}...p_{N-1}} - G_{p_1...p_{2l}} \cdot G_{p_{0p_{2l+1}}...p_{N-1}}}{(E_{p_0} - E_{p_{2l}})(E_{p_1} - E_{p_{N-1}})}
\]

\[
= -\sum_{l=1}^{N-2} \frac{G_{p_0p_{N-1}...p_{2l+1}} \cdot G_{p_{2l+1}...p_{N-1}} - G_{p_1p_{N-1}...p_{2l}} \cdot G_{p_{2l+1}...p_{1}p_0}}{(E_{p_0} - E_{p_{2l}})(E_{p_1} - E_{p_{N-1}})}
\]

\[
= \sum_{k=1}^{N-2} \frac{G_{p_0p_{N-1}...p_{N-2k+1}} \cdot G_{p_{N-2k+1}...p_{N-1}} - G_{p_1p_{N-1}...p_{N-2k}} \cdot G_{p_{N-2k+1}...p_1}}{(E_{p_0} - E_{p_{N-2k}})(E_{p_{N-1}} - E_{p_1})}
\]

\[
= G_{p_0p_{N-1}...p_1} = G_{p_{N-1}...p_1p_0} .
\] (4.4.4)

The transformation \( 2l = N - 2k \) and the symmetry (4.4.3) are applied here to rewrite the sum. This shows cyclic invariance.

Although the \( N \)-point functions are invariant under a cyclic permutation of its indices, the preferred expansion into surviving terms (4.4.2) will depend on the choice of a designated node \( p_0 \), the root. Our preferred expansion will have a clear combinatorial significance, but it cannot be unique because of

\[
\frac{1}{E_{p_i} - E_{p_j}} \cdot \frac{1}{E_{b_j} - E_{b_n}} + \frac{1}{E_{b_n} - E_{b_i}} \cdot \frac{1}{E_{b_i} - E_{b_j}} + \frac{1}{E_{b_j} - E_{b_n}} \cdot \frac{1}{E_{b_n} - E_{b_i}} = 0 .
\] (4.4.5)

These identities must be employed several times to establish cyclic invariance of our preferred expansion.

4.4.2 Catalan Tuples

Definition 4.6 (Catalan tuple). A Catalan tuple \( \tilde{e} = (e_0, \ldots, e_k) \) of length \( k \in \mathbb{N}_0 \) is a tuple of integers \( e_j \geq 0 \) for \( j = 0, \ldots, k \), such that

\[
\sum_{j=0}^{k} e_j = k \quad \text{and} \quad \sum_{j=0}^{l} e_j > l \quad \text{for} \quad l = 0, \ldots, k - 1 .
\]

The set of Catalan tuples of length \( |\tilde{e}| := k \) is denoted by \( C_k \).

For \( \tilde{e} = (e_0, \ldots, e_k) \) it follows immediately that \( e_k = 0 \) and \( e_0 > 0 \), if \( k > 0 \).

Example 4.6. We have \( C_0 = \{(0)\} \), \( C_1 = \{(1, 0)\} \) and \( C_2 = \{(2, 0, 0), (1, 1, 0)\} \). All Catalan tuples of length 3 are given in the first column of Table 4.1.

We now define two particular compositions of these objects. App. F.1 provides a few examples.
Definition 4.7 ($\circ$-composition). The composition $\circ : C_k \times C_l \rightarrow C_{k+l+1}$ is given by

$$(e_0, \ldots, e_k) \circ (f_0, \ldots, f_l) := (e_0 + 1, e_1, \ldots, e_{k-1}, e_k, f_0, f_1, \ldots, f_l).$$

No information is lost in this composition, i.e. it is possible to uniquely retrieve both terms. In particular, $\circ$ cannot be associative or commutative. Consider for a Catalan tuple $\tilde{e} = (e_0, \ldots, e_k)$ partial sums $p_l : C_k \rightarrow \{0, \ldots, k\}$ and maps $\sigma_l : C_k \rightarrow \{0, \ldots, k\}$ defined by

$$p_l(\tilde{e}) := -l + \sum_{j=0}^{l} e_j, \quad \text{for } l = 0, \ldots, k - 1,$$

$$\sigma_l(\tilde{e}) := \min\{l \mid p_l(\tilde{e}) = a\}.$$  \hfill (4.4.6)

Then

$$\tilde{e} = (e_0, \ldots, e_k) = (e_0 - 1, e_1, \ldots, e_{\sigma_l(\tilde{e})}) \circ (e_{\sigma_l(\tilde{e})+1}, \ldots, e_k).$$  \hfill (4.4.7)

Because $\sigma_1(\tilde{e})$ exists for any $\tilde{e} \in C_k$ with $k \geq 1$, every Catalan tuple has unique $\circ$-factors. Only these two Catalan tuples, composed by $\circ$, yield $(e_0, \ldots, e_k)$. This implies that the number $c_k$ of Catalan tuples in $C_k$ satisfies Segner’s recurrence relation

$$c_k = \sum_{m=0}^{k-1} c_m c_{k-1-m}$$

together with $c_0 = 1$, which is solved by the Catalan numbers $c_k = \frac{1}{k+1} \binom{2k}{k}$.

The other composition of Catalan tuples is a variant of the $\circ$-product.

Definition 4.8 ($\bullet$-composition). The composition $\bullet : C_k \times C_l \rightarrow C_{k+l+1}$ is given by

$$(e_0, \ldots, e_k) \bullet (f_0, \ldots, f_l) = (e_0 + 1, f_0, \ldots, f_l, e_1, \ldots, e_k).$$

As in the case of the composition $\circ$, [Definition 4.7] no information is lost in the product $\bullet$. It is reverted by

$$\tilde{e} = (e_0, \ldots, e_k) = (e_0 - 1, e_{1+\sigma_{e_0-1}(\tilde{e})}, \ldots, e_k) \bullet (e_1, \ldots, e_{\sigma_{e_0-1}(\tilde{e})}).$$  \hfill (4.4.8)

Because $\sigma_{e_0-1}(\tilde{e})$ exists for any $\tilde{e} \in C_k$ with $k \geq 1$ (also for $e_0 = 1$ where $\sigma_{e_0-1}(\tilde{e}) = k$), every Catalan tuple has a unique pair of $\bullet$-factors.

Out of these Catalan tuples we will construct three sorts of trees: pocket tree, direct tree, opposite tree. They are all planted plane trees, which means they are embedded into the plane and planted into a monovalent phantom root which connects to a unique vertex that we consider as the (real) root. We adopt the convention that the phantom root is not shown; its implicit presence manifests in a different counting of the valencies the real root. Pocket tree and direct tree are the same, but their rôle will be different. Their drawing algorithms are given by the next definitions.

Definition 4.9 (direct tree, pocket tree). For a Catalan tuple $(e_0, \ldots, e_k) \in C_k$, draw $k + 1$ vertices on a line. Starting at the root $l = 0$:

- unless $l = 0$, connect this vertex to the last vertex ($m < l$) with an open half-edge;
- if $e_l > 0$: $e_l$ half-edges must be attached to vertex $l$;

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- move to the next vertex.

For direct trees, vertices will be called nodes and edges will be called threads; they are oriented from left to right. For pocket trees, vertices are called pockets.

**Definition 4.10** (opposite tree). For a Catalan tuple \((e_0, \ldots, e_k) \in \mathcal{C}_k\), draw \(k + 1\) vertices on a line. Starting at the root \(l = 0\):

- if \(e_l > 0\): \(e_l\) half-edges must be attached to vertex \(l\);
- if \(e_l = 0\):
  - connect this vertex to the last vertex \((m < l)\) with an open half-edge;
  - if this vertex is now not connected to the last vertex \((n \leq m < l)\) with an open half-edge, repeat this until it is;
- move to the next vertex.

For opposite trees, vertices will be called nodes and edges will be called threads; they are oriented from left to right.

Examples of these trees can be seen in Figure 4.1 and Table 4.1. It will be explained in Sec. 4.4.4 how these trees relate to the recurrence relation (4.4.1) and how to label the nodes. The pocket trees will often be represented with a top-down orientation, instead of a left-right one.

![Figure 4.1: Direct tree (upper) and the opposite tree (lower) for the Catalan tuple (6, 0, 0, 1, 3, 0, 0, 0, 2, 2, 0, 0, 0, 0) = (5, 0, 0, 1, 3, 0, 0, 0, 2, 2, 0, 0, 0, 0) • (0).

4.4.3 Catalan Tables

A Catalan table is a ‘Catalan tuple of Catalan tuples’:

**Definition 4.11** (Catalan table). A Catalan table of length \(k\) is a tuple \(T_k = \langle \tilde{e}^{(0)}, \tilde{e}^{(1)}, \ldots, \tilde{e}^{(k)} \rangle\) of Catalan tuples \(\tilde{e}^{(j)}\), such that \((1 + |\tilde{e}^{(0)}|, |\tilde{e}^{(1)}|, \ldots, |\tilde{e}^{(k)}|)\), the length tuple of \(T_k\), is itself a Catalan tuple of length \(k\). We let \(\mathcal{T}_k\) be the set of all Catalan tables of length \(k\). The constituent \(\tilde{e}^{(j)}\) in a Catalan table is called the \(j\)-th pocket.

We will show in Sec. 4.4.4 that a Catalan table contains all information about individual terms in the expansion (4.4.2) of the \(N\)-point function \(G_{p_0 \ldots p_{N-1}}\).

Recall the composition \(\circ\) from Definition 4.7 and the fact that any Catalan tuple of length \(\geq 1\) has a unique pair of \(\circ\)-factors. We extend \(\circ\) as follows to Catalan tables:
Table 4.1: The Catalan tuples and the corresponding planted plane trees for \( k = 3 \). The phantom roots are not shown. The real root is on top for the pocket tree and on the left for direct and opposite trees.

| Catalan tuple | pocket tree | direct tree | opposite tree |
|---------------|-------------|-------------|--------------|
| \((3,0,0,0)\) | ![Diagram](image1) | ![Diagram](image2) | ![Diagram](image3) |
| \((2,1,0,0)\) | ![Diagram](image4) | ![Diagram](image5) | ![Diagram](image6) |
| \((2,0,1,0)\) | ![Diagram](image7) | ![Diagram](image8) | ![Diagram](image9) |
| \((1,2,0,0)\) | ![Diagram](image10) | ![Diagram](image11) | ![Diagram](image12) |
| \((1,1,1,0)\) | ![Diagram](image13) | ![Diagram](image14) | ![Diagram](image15) |

**Definition 4.12** (\(\circ\)-operation). The operation \( \circ: \mathcal{T}_k \times \mathcal{T}_l \to \mathcal{T}_{k+l} \) is given by

\[
\langle \tilde{e}^{(0)}, \ldots, \tilde{e}^{(k)} \rangle \circ \langle \tilde{f}^{(0)}, \ldots, \tilde{f}^{(l)} \rangle := \langle \tilde{e}^{(0)} \circ \tilde{f}^{(0)}, \tilde{e}^{(1)}, \ldots, \tilde{e}^{(k)}, \tilde{f}^{(1)}, \ldots, \tilde{f}^{(l)} \rangle.
\]

Now suppose the Catalan table on the right-hand side is given. If the 0th pocket has length \( \geq 1 \), then it uniquely factors into \( \tilde{e}^{(0)} \circ \tilde{f}^{(0)} \). Consider

\[
\hat{k} = \sigma_{1+|\tilde{f}^{(0)}|}\left((1 + |\tilde{e}^{(0)} \circ \tilde{f}^{(0)}|, |\tilde{e}^{(1)}|, \ldots, |\tilde{e}^{(k)}|, |\tilde{f}^{(1)}|, \ldots, |\tilde{f}^{(l)}|)\right). \tag{4.4.9}
\]

By construction, \( \hat{k} = k \) so that \( \circ \) can be uniquely reverted. Note also that Catalan tables \( \langle (0), \tilde{e}_1, \ldots, \tilde{e}_k \rangle \) do not have a \(\circ\)-decomposition.

The composition \(\bullet\) of Catalan tuples is extended as follows to Catalan tables:

**Definition 4.13** (\(\bullet\)-operation). The operation \( \bullet: \mathcal{T}_k \times \mathcal{T}_l \to \mathcal{T}_{k+l} \) is given by

\[
\langle \tilde{e}^{(0)}, \ldots, \tilde{e}^{(k)} \rangle \bullet \langle \tilde{f}^{(0)}, \ldots, \tilde{f}^{(l)} \rangle := \langle \tilde{e}^{(0)}, \tilde{e}^{(1)} \bullet \tilde{f}^{(0)}, \tilde{f}^{(1)}, \ldots, \tilde{f}^{(l)}, \tilde{e}^{(2)}, \ldots, \tilde{e}^{(k)} \rangle.
\]

If the 1st pocket has length \( \geq 1 \), it uniquely factors as \( \tilde{e}^{(1)} \bullet \tilde{f}^{(0)} \), and we extract

\[
\hat{l} := \sigma_{|\tilde{e}^{(0)}|+|\tilde{e}^{(1)}|+1}\left((1 + |\tilde{e}^{(0)}|, |\tilde{e}^{(1)} \bullet \tilde{f}^{(0)}|, |\tilde{f}^{(1)}|, \ldots, |\tilde{f}^{(l)}|, |\tilde{e}^{(2)}|, \ldots, |\tilde{e}^{(k)}|)\right). \tag{4.4.10}
\]

By construction \( \hat{l} = l \), and \(\bullet\) is uniquely reverted.

We let \( \mathcal{S}_k = \{ \langle \tilde{e}_0, (0), \tilde{e}_2, \ldots, \tilde{e}_k \rangle \in \mathcal{T}_k \} \) be the subset of length-\( k \) Catalan tables having \((0)\) as their first pocket. The Catalan tables \( S \in \mathcal{S}_k \) are precisely those which do not have a \(\bullet\)-decomposition. The distinction between \( \mathcal{S}_l \) and its complement in \( \mathcal{T}_l \) is the key to determine the number of Catalan tables:
Proof. Let

\[ D(x) := \sum_{k=1}^{\infty} x^k \sum_{T \in \mathcal{T}_k} T \quad \text{and} \quad H(x) := \sum_{k=1}^{\infty} x^k \sum_{S \in \mathcal{S}_k} S \]

be the generating function of the set of all Catalan tables and of those having (0) as their first pocket, respectively. Then

\[ D(x) = D(x) \cdot D(x) + H(x) \quad (4.4.12) \]

because precisely the complements \( \mathcal{T}_k \setminus \mathcal{S}_k \) have a unique \( \bullet \)-decomposition. With the exception of \((0), (0)\) \( \in \mathcal{S}_1 = \mathcal{T}_1 \), all \( S = (\tilde{e}^1, (0), \tilde{e}^2, \ldots, \tilde{e}^k) \in \mathcal{S}_k \) with \( k \geq 2 \) have \( |\tilde{e}^0| \geq 1 \). Therefore, they have a unique \( \circ \)-decomposition, where the left factor necessarily belongs to \( \mathcal{S}_l \) for some \( l \):

\[ H(x) = H(x) \circ D(x) + x((0), (0)) \quad (4.4.13) \]

Introducing the generating functions

\[ D(x) = \sum_{k=0}^{\infty} x^{k+1} d_k \quad \text{and} \quad H(x) = \sum_{k=0}^{\infty} x^{k+1} h_k \]

of the cardinalities \( d_k = |\mathcal{T}_{k+1}| \) and \( h_k = |\mathcal{S}_{k+1}| \), eqs. (4.4.12) and (4.4.13) project to quadratic relations

\[ D(x) = D(x) \cdot D(x) + H(x) \quad \text{and} \quad H(x) = H(x) \cdot D(x) + x \quad (4.4.14) \]

Multiplying the first equation by \( H(x) \) and the second one by \( D(x) \) gives \( x \cdot D(x) = H^2(x) \), which separates (4.4.14) into cubic relations

\[ D(x)(1 - D(x))^2 = x \quad \text{and} \quad \frac{H(x)}{\sqrt{x}} \left( 1 - \left( \frac{H(x)}{\sqrt{x}} \right)^2 \right) = \sqrt{x} \quad (4.4.15) \]

The coefficients (4.4.11) can now be obtained by the Lagrange inversion formula. The second equation of (4.4.15) results by taking \( f(w) = w(1-w)^2 \) in Theorem 4.1 i.e. \( \phi(w) = \frac{1}{(1-w)^2} \) and \( G(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!} \left. \frac{d^{n-1}}{dw^{n-1}} \right|_{w=0} \frac{1}{(1-w)^2} \). For the first equation of (4.4.15), set \( \frac{H(x)}{\sqrt{x}} = w, z = \sqrt{x} \) and \( \phi(w) = \frac{1}{1-w} \) in Theorem 4.1 Then \( H(x) = \sqrt{x} \sum_{n=1}^{\infty} \frac{x^n}{n!} \left. \frac{d^{n-1}}{dw^{n-1}} \right|_{w=0} \frac{1}{1-w^2} \).

\[ \square \]

Remark 4.9. Equations (4.4.15) are higher-order variants of the equation \( C(x)(1-C(x)) = x \) for the generating function \( C(x) = \sum_{n=0}^{\infty} c_n x^{n+1} \) of Catalan numbers.

Corollary 4.5. The number \( d_k \) of Catalan tables satisfies

\[ d_k = \sum_{(e_0, \ldots, e_{k+1}) \in \mathcal{C}_{k+1}} c_{e_0-1} c_{e_1} \cdots c_{e_k} c_{e_{k+1}} \quad (4.4.16) \]

Proof. There are \( c_{|\tilde{e}_0|} \cdots c_{|\tilde{e}_{k+1}|} \) Catalan tables \( (\tilde{e}_0, \ldots, \tilde{e}_{k+1}) \) of the same length tuple \( (|\tilde{e}_0|+1, |\tilde{e}_1|, \ldots, |\tilde{e}_{k+1}|) \in \mathcal{C}_{k+1} \). Set \( e_0 = |\tilde{e}_0|+1 \) and \( e_j = |\tilde{e}_j| \) for \( j = 1, \ldots, k+1 \). \[ \square \]
4.4.4 The Bijection between Catalan Tables and Contributions to $G_{p_0 \ldots p_{N-1}}$

Definition 4.14. To a Catalan table $T_{k+1} = (\tilde{e}^{(0)}, \tilde{e}^{(1)}, \ldots, \tilde{e}^{(k+1)}) \in \mathcal{T}_{k+1}$ with $N/2 = k + 1$ we associate a monomial $[T]_{p_0, \ldots, p_{N-1}}$ in $G_{p_0 p_m}$ and $E_{p_m^{-1} E_{p_m'}}$ as follows:

1. Build the pocket tree for the length tuple $(1 + |\tilde{e}^{(0)}|, |\tilde{e}^{(1)}|, \ldots, |\tilde{e}^{(k+1)}|) \in C_{k+1}$. It has $k + 1$ edges and every edge has two sides. Starting from the root and turning counterclockwise, label the edge sides in consecutive order from $p_0$ to $p_{N-1}$. An edge labelled $p_i p_m$ encodes a factor $G_{p_i p_m}$ in $G_{p_0 \ldots p_{N-1}}$.

2. Label the $k + 2$ vertices of the pocket tree by $P_0, \ldots, P_{k+1}$ in consecutive order when turning counterclockwise around the tree. Let $v(P_m)$ be the valency of vertex $P_m$ (number of edges attached to $P_m$) and $L_m$ the distance between $P_m$ and the root $P_0$ (number of edges in shortest path between $P_m$ and $P_0$).

3. For every vertex $P_m$ that is not a leaf, read off the $2v(P_m)$ side labels of edges connected to $P_m$. Draw two rows of $v(P_m)$ nodes each. Label the nodes of the first row by the even edge side labels in natural order, i.e. starting at the edge closest to the root and proceed in the counterclockwise direction. Label the nodes of the other row by the odd edge side labels using the same edge order. Take the $m$-th Catalan tuple $\tilde{e}^{(m)}$ of the Catalan table. If $L_m$ is even, draw the direct tree encoded by $\tilde{e}^{(m)}$ between the row of even nodes. If $L_m$ is odd, draw the opposite tree encoded by $\tilde{e}^{(m)}$ between the row of odd nodes. Encode a thread from $b_l$ to $b_m$ in the direct or opposite tree by a factor $E_{p_l^{-1} E_{p_m}}$.

Remark 4.10. In the proof below we sometimes have to insist that one side label of a pocket edge is a particular $p_k$, whereas the label of the other side does not matter. Is such a situation we will label the other side by $p_T$. Note that if $p_k$ is $\{\text{even}\}$ then $p_T$ is $\{\text{odd}\}$.

Remark 4.11. For the purpose of this article it is sufficient to mention that an explicit construction for the level function $L_m : C_{k+1} \to \{0, \ldots, k\}$ exists.

Example 4.7. Let $T = ((2, 0, 0), (1, 1, 0), (0), (0), (0), (1, 0), (0)) \in \mathcal{T}_6$. Its length tuple is $(3, 2, 0, 0, 1, 0) \in C_6$, which defines the pocket tree:

![Diagram of the pocket tree](image)

The edge side labels encode

$$G_{p_0 p_0} G_{p_0 p_2} G_{p_1 p_1} G_{p_2 p_3} G_{p_3 p_3} G_{p_3 p_5} G_{p_5 p_5} G_{p_5 p_{11}} G_{p_{11} p_{11}}.$$

For vertex $P_0$, at even distance, we draw direct and opposite tree encoded in $\tilde{e}^{(0)} = (2, 0, 0)$:

![Diagram of the encoded trees](image)

††This is the same order as in [Sta99, Fig. 5.14].
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For vertex $P_1$, at odd distance, we draw opposite and direct tree encoded in $\tilde{e}^{(1)} = (1, 1, 0)$:

![Tree diagram for $P_1$]

For vertex $P_5$, at odd distance, we draw opposite and rooted tree encoded in $\tilde{e}^{(5)} = (1, 0)$:

![Tree diagram for $P_5$]

They give rise to a factor

$$
\frac{1}{(E_{p_0} - E_{p_6})(E_{p_3} - E_{p_4})(E_{p_0} - E_{p_4})(E_{p_1} - E_{p_4})(E_{p_4} - E_{p_{10}})}
\times \frac{1}{(E_{p_5} - E_{p_7})(E_{p_5} - E_{p_{11}})(E_{p_5} - E_{p_3})(E_{p_1} - E_{p_3})(E_{p_{11}} - E_{p_3})}.
$$

Later in Figure F.3 we give a diagrammatic representation of this Catalan table.

The following theorem shows that the Catalan tables correspond bijectively to the terms in the expansion of the recurrence relation (4.4.1).

**Theorem 4.5.** The recurrence (4.4.1) of $N$-point functions in the quartic matrix field theory model has the explicit solution

$$
G_{p_0 \ldots p_{N-1}} = \sum_{T \in \mathcal{T}_{k+1}} [T]_{p_0 \ldots p_{N-1}},
$$

where the sum is over all Catalan tables of length $N/2 = k + 1$ and the monomials $[T]_{p_0 \ldots p_{N-1}}$ are described in Definition 4.14.

**Proof.** We proceed by induction in $N$. For $N = 2$ the only term in the 2-point function corresponds to the Catalan table $\langle (0), (0) \rangle \in \mathcal{T}_1$. Its associated length tuple $(1, 0)$ encodes the pocket tree

![Pocket tree diagram]

whose single edge corresponds to a factor $G_{p_0 p_1}$. The Catalan tuples of both pockets have length 0, so that there is no denominator.

For any contribution to $G$ with $N \geq 4$, encoded by a length-$N/2$ Catalan table $T_{N/2}$, it must be shown that $T_{N/2}$ splits in one or two ways into smaller Catalan tables whose corresponding monomials produce $T_{N/2}$ via (4.4.1). There are three cases to consider.

[1] Let $T_{k+1} = \langle (0), \tilde{e}^{(1)}, \ldots, \tilde{e}^{(k+1)} \rangle \in \mathcal{T}_{k+1}$ with $N/2 = k + 1$. It follows from Definition 4.13 that there are uniquely defined Catalan tables $T_i = \langle \tilde{f}, \tilde{e}^{(2)}, \ldots, \tilde{e}^{(l+1)} \rangle \in \mathcal{T}_l$ and $T_{k-l+1} = \langle (0), \tilde{e}, \tilde{e}^{(l+2)}, \ldots, \tilde{e}^{(k+1)} \rangle \in \mathcal{T}_{k-l+1}$ with $\tilde{e}^{(1)} = \tilde{e} \cdot \tilde{f}$ and consequently $T_{k-l+1} \cdot T_i = T_{k+1}$. The length $l = \tilde{i}$ is obtained via (4.4.10). Recall that $T_{k+1}$ cannot be obtained by the $\circ$-composition because the zeroth pocket has length $|\langle 0 \rangle| = 0$. By induction, $T_i$ encodes a unique contribution $[T_i]_{p_1 \ldots p_2}$ to $G_{p_1 \ldots p_2}$, and $T_{k-l+1}$ encodes a unique contribution $[T_{k-l+1}]_{p_{2l+1} \ldots p_{N-1}}$ to $G_{p_{2l+1} \ldots p_{N-1}}$. We have to show that

$$
\frac{\sum_{T_i} [T_i]_{p_1 \ldots p_2} [T_{k-l+1}]_{p_{2l+1} \ldots p_{N-1}}}{(E_{p_0} - E_{p_2})(E_{p_1} - E_{p_{N-1}})}
$$

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agrees with $[T_{k+1}]_{p_0 \ldots p_{N-1}}$ encoded by $T_{k+1}$. A detail of the pocket tree of $T_{k+1}$ sketching $P_0, P_1$ and their attached edges is

$$\text{(4.4.17)}$$

Only the gluing of the direct and opposite tree encoded by $\tilde{e} = (e_0, \ldots, e_p)$ with the direct and opposite tree encoded by $\tilde{f} = (f_0, \ldots, f_q)$ via a thread from $p_0$ to $p_{2l}$ and a thread from $p_{N-1}$ to $p_1$ remains to be shown; edge sides encoding a 2-point function and all other pockets are automatic. A symbolic notation is used now to sketch the trees. Horizontal dots are used to indicate a general direct tree and horizontal dots with vertical dots above them indicate an opposite tree. Unspecified threads are indicated by dotted half-edges. The four trees mentioned above are depicted as

$$\text{OT}_{\tilde{e}} = \quad \text{OT}_{\tilde{f}} =$$

$$\text{DT}_{\tilde{e}} = \quad \text{DT}_{\tilde{f}} =$$

Here $\tilde{e}$ describes $P_1$, at odd distance, so that even-labelled nodes are connected by the opposite tree. Every edge in the pocket tree has two sides labelled $p_r$ and $p_s$, where the convention of Remark 4.10 is used when the other side label does not matter.

The first edge in the pocket tree has side labels $p_0p_{N-1}$ and descends from the root pocket. The following edge is $p_{2l+1}p_{2l+1}$ where $2l + 2 \leq 2l + 1 \leq N - 2$ is an even number. The final edge is $p_{N-2}p_{N-2}$ where $2l + 1 \leq N - 2 \leq N - 3$ is an odd number.

Next, $\tilde{f}$ encodes $P_0$ in the pocket tree belonging to $[T_k]_{p_1 \ldots p_2}$. It lies at even distance, but, because the labels at $G_{p_1 \ldots p_{2l}}$ start with an odd one, the odd nodes of $\tilde{f}$ are connected by the direct tree and the even nodes by the opposite tree. Again, $2 \leq l \leq 2l$ denotes an even number and $1 \leq 2l \leq 2l - 1$ an odd number. When pasting $\tilde{f}$ into $\tilde{e}$, the first edge remains $p_0p_{N-1}$, which descends from the root. Then all edges from $\tilde{f}$ follow and, finally, the remaining edges of $\tilde{e}$. Thus, before taking the denominators into account, the four trees are arranged as:

$$\text{OT}_{\tilde{e}} \cup \text{OT}_{\tilde{f}}: \quad \text{DT}_{\tilde{e}} \cup \text{DT}_{\tilde{f}}:$$

$$\text{(4.4.18)}$$

The denominator of $1/(E_{p_0} - E_{p_{2l}})(E_{p_{N-1}} - E_{p_1})$ (with rearranged sign) corresponds to a thread
between the nodes $p_0$ and $p_{2l}$ and one between the nodes $p_{N-1}$ and $p_1$:

\[ \text{OT}_{\hat{e} \circ \hat{f}}: \]
\[ \text{DT}_{\hat{e} \circ \hat{f}}: \] (4.4.19)

The result is precisely described by $\hat{e} \circ \hat{f} = (e_0 + 1, f_0, \ldots, f_q, e_1, \ldots, e_p)$ with Definitions 4.9 and 4.10. Indeed, the increased zeroth entry corresponds to one additional half-thread attached to the first node $p_{N-1}$ and one additional half-thread to $p_0$. For the direct tree the rules imply that the next node, $p_1$, is connected to $p_{N-1}$. This is the new thread from the denominators. The next operations are done within $\hat{f}$, labelled $p_1, \ldots, p_{2l}$, without any change. Arriving at its final node $p_{2l}$ all half-threads of $\hat{f}$ are connected. The next node, labelled $p_{2l+1}$, connects to the previous open half-thread, which is the very first node $p_{N-1}$. These and all the following connections arise within $\hat{e}$ and remain unchanged. Similarly, in the opposite tree, we first open $e_0 + 1$ half-threads at the zeroth node $p_0$. Since $f_0 > 0$, we subsequently open $f_0$ half-threads at the first node $p_{T}$. The next operations remain unchanged, until we arrive at the final node $p_{2l}$ of $\hat{f}$. It corresponds to $f_0 = 0$, so that we connect it to all previous open half-threads, first within $\hat{f}$. However, because $e_0 + 1 > 0$, it is connected by an additional thread to $p_0$ and encodes the denominator of $\frac{1}{E_{p_0} - E_{p_{2l}}}$. This consumes the additional half-thread attached to $p_0$. All further connections are the same as within $\hat{e}$. In conclusion, we obtain precisely the Catalan table $T_{k+1} = \langle (0), \hat{e}(1) \ldots \hat{e}(N/2) \rangle$ we started with.

[II] Let $T_{k+1} = \langle \hat{e}(0), (0), \hat{e}(2), \ldots, \hat{e}(k+1) \rangle \in \mathcal{T}_{k+1}$ and $N/2 = k + 1$. There are uniquely defined Catalan tables $T_l = \langle \hat{e}, (0), \hat{e}(2), \ldots, \hat{e}(l) \rangle \in \mathcal{T}_l$ and $T_{k-l+1} = \langle \hat{f}, \hat{e}(l+1), \ldots, \hat{e}(k+1) \rangle \in \mathcal{T}_{k-l+1}$ with $\hat{e}(0) = \hat{e} \circ \hat{f}$ and, consequently, $T_l \circ T_{k-l+1} = T_{k+1}$. The length $l = k$ is obtained via (4.4.9). Recall that $T_{k+1}$ cannot be obtained by the $\circ$-composition, because the first entry has length $|((0))| = 0$. By the induction hypothesis, $T_l$ encodes a unique contribution $[T_l]_{p_0 \ldots p_{2l}}$ to $G_{p_0 \ldots p_{2l}}$ and $T_{k-l+1}$ encodes a unique contribution $[T_{k-l+1}]_{p_{2l} \ldots p_{N-1}}$ to $G_{2l \ldots p_{N-1}}$. It remains to be shown that

\[ \frac{[T_l]_{p_0 \ldots p_{2l}} [T_{k-l+1}]_{p_{2l} \ldots p_{N-1}}}{(E_{p_0} - E_{p_{2l}})(E_{p_1} - E_{p_{N-1}})} \]

agrees with $[T_{k+1}]_{p_0 \ldots p_{N-1}}$ encoded by $T_{k+1}$. A detail of the pocket tree of $T_{k+1}$ sketching $P_0, P_1$ and their attached edges is

\[ 0 \]

As in case [I] only the gluing of the direct and opposite tree encoded by $\hat{e} = (e_0, \ldots, e_p)$ with the direct and opposite tree encoded by $\hat{f} = (f_0, \ldots, f_q)$ via a thread from $p_0$ to $p_{2l}$ and a thread from $p_1$ to $p_{N-1}$ must be demonstrated. Everything else is automatic. These
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trees are

\[
\begin{align*}
\text{DT}_{\tilde{e}} &= \begin{array}{c}
p_0 \\
p_1 \\
p_2 \\
\vdots \\
p_{N-1}
\end{array} \quad \text{DT}_f = \begin{array}{c}
p_{2l} \\
p_{N-1}
\end{array} \\
\text{OT}_{\tilde{e}} &= \begin{array}{c}
p_1 \\
p_\pi \\
p_{2l-1}
\end{array} \quad \text{OT}_f = \begin{array}{c}
p_{2l} \\
p_{N-1}
\end{array}
\end{align*}
\tag{4.4.21}
\]

The notation is the same as in case [I]. The first pocket \(P_1\), described by the Catalan tuple \((0),\) is only 1-valent so that the first edge is labelled \(p_0p_1\). The direct trees in \((4.4.21)\) are put next to each other and a thread between \(p_0\) and \(p_{2l}\) is drawn for the denominator of \(\frac{1}{E_{p_0} - E_{p_{2l}}}\). Similarly, the opposite trees in \((4.4.21)\) are put next to each other and a thread between \(p_1\) and \(p_{N-1}\) is drawn for the denominator of \(\frac{1}{E_{p_1} - E_{p_{N-1}}}\):

\[
\begin{align*}
\text{DT}_{\tilde{e} \circ f} &= \begin{array}{c}
p_0 \\
p_2 \\
p_{2l-1} \\
p_{2l} \\
p_{N-1}
\end{array} \\
\text{OT}_{\tilde{e} \circ f} &= \begin{array}{c}
p_1 \\
p_\pi \\
p_{2l-1} \\
p_{2l} \\
p_{N-1}
\end{array}
\end{align*}
\]

The result are precisely the direct and opposite trees of the composition \(\tilde{e} \circ \tilde{f} = (e_0 + 1, e_1, \ldots, e_p, f_0, \ldots, f_q)\). The increase \(e_0 \rightarrow e_0 + 1\) opens an additional half-thread at \(b_0\) and an additional half-thread at \(p_1\). In the direct tree, this new half-thread is not used by \(e_1, \ldots, e_p\). Only when we are moving to \(f_0\), labelled \(p_{2l}\), we have to connect it with the last open half-thread, i.e. with \(p_0\). After that the remaining operations are unchanged compared with \(\tilde{f}\). In the opposite tree, the additional half-thread at \(p_1\) is not used in \(e_1, \ldots, e_p\). Because \(f_0\), labelled \(p_{2l}\), opens enough half-threads, it is not consumed by \(f_0, \ldots, f_{q-1}\) either. Then, the last node \(f_q\), labelled \(p_{N-1}\), successively connects to all nodes with open half-threads, including \(p_1\). In conclusion, we obtain precisely the Catalan table \(T_{k+1} = (\tilde{e}(0), \tilde{e}(1), \tilde{e}(2), \ldots, \tilde{e}(k+1))\) we started with.

[III] Finally, we consider a general \(T_{k+1} = (\tilde{e}(0), \tilde{e}(1), \tilde{e}(2), \ldots, \tilde{e}(k+1)) \in T_{k+1}\) with \(k + 1 = N/2\), \(|\tilde{e}(0)| \geq 1\) and \(|\tilde{e}(1)| \geq 1\). There are uniquely defined Catalan tables \(T_l = (\tilde{e}, \tilde{e}(1), \tilde{e}(2), \ldots, \tilde{e}(l)) \in T_l\) and \(T_{k-l+1} = (\tilde{f}, \tilde{e}(l+1), \ldots, \tilde{e}(k+1)) \in T_{k-l+1}\) with \(\tilde{e}(0) = \tilde{e} \circ \tilde{f}\) and consequently \(T_l \circ T_{k-l+1} = T_{k+1}\). Moreover, uniquely defined Catalan tables \(T_{l'} = (\tilde{f}', \tilde{e}(2), \ldots, \tilde{e}(l'+1)) \in T_{l'}\) and \(T_{k-l'+1} = (\tilde{e}(0), \tilde{e}', \tilde{e}(l'+2), \ldots, \tilde{e}(k+1)) \in T_{k-l'+1}\) exist, such that \(\tilde{e}(1) = \tilde{e}' \circ \tilde{f}'\) and consequently \(T_{k-l'+1} \circ T_{l'} = T_{k+1}\). We necessarily have \(l' \leq k - 1\) and \(l \geq 2\), because \(l' = k\) corresponds to case [I] and \(l = 1\) to case [II]. By the induction hypothesis, these Catalan subtables encode unique contributions \([T_l]_{p_0 \ldots p_{2l-1}}\) to \(G_{p_0 \ldots p_{2l-1}}\), \([T_{k-l+1}]_{p_{2l} \ldots p_{N-1}}\) to \(G_{p_{2l} \ldots p_{N-1}}\), \([T_{l'}]_{p_{l'} \ldots p_{2l'}}\) to \(G_{p_{l'} \ldots p_{2l'}}\) and \([T_{k-l'+1}]_{p_{2l'+1} \ldots p_{N-1}}\) to \(G_{p_{2l'+1} \ldots p_{N-1}}\). We have to show that

\[
\frac{[T_l]_{p_0 \ldots p_{2l-1}}[T_{N/2-l} p_{2l} \ldots p_{N-1}]}{(E_{p_0} - E_{p_{2l}})(E_{p_{2l}} - E_{p_{N-1}})} - \frac{[T_{l'}]_{p_{l'} \ldots p_{2l'}}[T_{N/2-l'} p_{2l'} \ldots p_{N-1}]}{(E_{p_0} - E_{p_{2l'}})(E_{p_{2l'}} - E_{p_{N-1}})} = \frac{1}{(E_{p_0} - E_{p_{2l}})(E_{p_{2l}} - E_{p_{N-1}})} \tag{4.4.22}
\]

agrees with \([T_{k+1}]_{b_0 \ldots b_{N+1}}\).

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In the pocket tree of $T_{k+1}$, there must be an edge with side labels $p_0p_h$, where $3 \leq h \leq N - 3$ and $h$ is odd. Here is a detail of the pocket tree of $T_{k+1}$ showing $P_0, P_1$:

\[ (4.4.23) \]

The direct and opposite trees for $\tilde{e}, \tilde{f}$ and $\tilde{e}^{(1)}$ can be sketched as

\[ \text{DT}_\tilde{e} \cup \text{DT}_\tilde{f} = \quad \text{OT}_\tilde{e} \cup \text{OT}_\tilde{f} = \]

\[ \text{DT}_\tilde{e}^{(1)} = \quad \text{OT}_\tilde{e}^{(1)} = \]

The denominators of $\frac{1}{(E_{p_0} - E_{p_1})(E_{p_1} - E_{p_{N-1}})}$ in (4.4.22) add threads from $p_0$ to $p_{2l}$ and from $p_1$ to $p_{N-1}$. The first one connects the direct trees for $\tilde{e} \cup \tilde{f}$ to the direct tree encoded by $\tilde{e}^{(0)} = \tilde{e} \circ \tilde{f}$. The second thread does not give a valid composition of the opposite trees for $\tilde{e} \cup \tilde{f}$.

This is a problem. The solution is to split this contribution. Half of the contribution is sacrificed to bring the other half in the desired form. Afterwards, the same procedure is repeated for the other term in (4.4.22) with a minus-sign. The remainders are the same and cancel each other, whereas the other halves add up to yield the sought for monomial.

Returning to trees, we note that in the direct tree for the pocket $\tilde{e}^{(1)}$, there is always a thread from $p_h$ to $p_{l_1}$, encoding a factor $\frac{1}{E_{p_h} - E_{p_{l_1}}}$. With the factor $\frac{1}{E_{p_{l_1}} - E_{p_{N-1}}}$ it fulfils

\[ \frac{1}{E_{p_h} - E_{p_{l_1}}} \cdot \frac{1}{E_{p_{l_1}} - E_{p_{N-1}}} = \frac{1}{E_{p_h} - E_{p_{l_1}}} \cdot \frac{1}{E_{p_{l_1}} - E_{p_{N-1}}} + \frac{1}{E_{p_h} - E_{p_{N-1}}} \cdot \frac{1}{E_{p_{l_1}} - E_{p_{N-1}}}. \]

(4.4.25)

The first term on the right-hand side of (4.4.25) leaves the direct tree $\text{DT}_{\tilde{e}^{(1)}}$ as it is and connects the parts of $\text{OT}_{\tilde{e}} \cup \text{OT}_f$ via the thread from $p_h$ to $p_{N-1}$ to form $\text{OT}_{\tilde{e}^{(0)}}$, where $\tilde{e}^{(0)} = \tilde{e} \circ \tilde{f}$.

The final term in (4.4.25) also unites $\text{OT}_{\tilde{e}} \cup \text{OT}_f$ and forms $\text{OT}_{\tilde{e}^{(0)}}$, but it removes in $\text{DT}_{\tilde{e}^{(1)}}$ the thread between $p_h$ and $p_{l_1}$. It follows from $\tilde{e}^{(1)} = \tilde{e}' \cdot \tilde{f}'$ that this tree falls apart into the subtrees $\text{DT}_{\tilde{e}'}$, containing $p_h$, and $\text{DT}_{\tilde{f}'}$, which contains $p_{l_1}$. These are multiplied by a factor $\frac{1}{E_{p_{l_1}} - E_{p_{N-1}}}$. The second term in (4.4.22) will remove them.

Indeed, direct and opposite trees for $\tilde{e}^{(0)}, \tilde{e}'$ and $\tilde{f}'$ can be sketched as

\[ \text{DT}_{\tilde{e}^{(0)}} = \quad \text{OT}_{\tilde{e}^{(0)}} = \quad \text{DT}_{\tilde{e}' \cup \tilde{f}'} = \]

\[ \text{DT}_{\tilde{e}^{(1)}} = \quad \text{OT}_{\tilde{e}^{(1)}} = \quad \text{DT}_{\tilde{e}' \cup \tilde{f}'} = \]

(4.4.26)
CHAPTER 4. QUARTIC INTERACTION

The direct tree $\mathcal{DT}_{\tilde{c}(0)}$ remains intact and the thread from $p_0$ to $p_{2l}'$ encoded in the factor $\frac{1}{(E_{p_0} - E_{p_{2l}'})}$ in (4.4.22) connects the opposite trees for $\tilde{c}' \cup f'$ to form the opposite tree for $\tilde{c}(1) = \tilde{c}' \bullet f'$. The direct trees $\mathcal{DT}_{\tilde{c}} \mathcal{DT}_{\tilde{f}}$ remain disconnected and are multiplied by $\frac{1}{(E_{p_1} - E_{p_{N-1}})}$ from (4.4.22). With the minus-sign from (4.4.22) they cancel the terms described in [*]. The other trees combined yield precisely the direct and opposite trees for both $\tilde{c}(0)$ and $\tilde{c}(1)$, so that the single Catalan table we started with is retrieved.

This completes the proof. Bijectivity between Catalan tables and contributing terms to $(N' < N)$-point functions is essential: Assuming the above construction [I]–[III] missed Catalan subtables $T_{l}, T_{N/2 - l}$, then their composition $T_{l} \bullet T_{N/2 - l}$ would be a new Catalan table of length $N/2$. However, all Catalan tables of length $N/2$ are considered. Similarly for $T_{l}' \bullet T_{N/2 - l}'$.

This theorem shows that there is a one-to-one correspondence between Catalan tables and the diagrams/terms in $G_{p_0, \ldots, p_{N-1}}$ with designated node $p_0$. The choice of designated node does not influence $G_{p_0, \ldots, p_{N-1}}$, but it does alter its expansion.

4.5 Summary

We derived all SDEs for the quartic matrix field theory model and showed that they have one of the following particular forms (Proposition 4.4 and Proposition 4.5)

$$\hat{K}_1 G_{[p_1, \ldots, p_{N-1}]}^{(g)} = g_{inh}^{a_j}$$

(4.5.1)

$$\hat{K}_2 G_{[q_1, q_2, \ldots, q_n]}^{(g)} = g_{inh}^{a_j}$$

(4.5.2)

if the base point is taken from a boundary component of length 1 or 2. The operators $\hat{K}_i$ are linear operators defined in (4.1.11) and (4.1.12), and $g_{inh}$ the inhomogeneous part of the equation depending on correlation functions of less topology than $G^{(g)}$.

Especially, the form (4.5.2) is achieved by the operator $\mathcal{T}_q$ defined in Definition 4.1 which rewrites $\frac{1}{N} \sum_n G_{[q_1, n, \ldots, q_n]}^{(g)}$ as a derivative such that $\hat{K}_2$ gets really the form (4.1.12).

If a correlation function has boundary components of lengths $\geq 3$, it can be recursively written by applying Proposition 4.2 through correlation functions of boundary components of length one and/or two. The nonlinear rhs of Proposition 4.2 is strictly different to the linear rhs of the recursive equation Proposition 3.2 of the cubic model.

The main achievement for the quartic model was to find the exact solution of the planar 2-point function ($g = 0$) for any spectral dimension $\mathcal{D} < 6$ with an appropriate renormalisation. For $4 \leq \mathcal{D} < 6$, the quartic matrix field theory model is just-renormalisable. It was known that the angle function on the $D = 2$ and $D = 4$ Moyal space plays an important rôle, which gave us the right ansatz for the general case with the measure $\varrho_0(x)$, and the functions $R_D(z)$ and $I(w)$. The angle function is then computed by Theorem 4.2, where the important observation was to deform the measure to $\varrho_\lambda(x) = \varrho_0(R_D(x))$. The solution of the planar 2-point function is summarised in Theorem 4.3, which is more or less straightforward to compute from the solution of the angle function. The Example 3.1 on the $D = 2$ Moyal space recovers the result of [PW18].

Two more examples are of particular interest, the solution for finite matrices ($\mathcal{D} = 0$) discussed in Sec. 4.2.3 with Theorem 4.4 and the solution on the $D = 4$ Moyal space discussed in Sec. 4.2.4.
For finite matrices, the planar 2-point function is given as a rational function. Furthermore, the rational function $R(z)$ (defined in Theorem 4.4) induces a preferred variable transformation $x \mapsto z = R^{-1}(x)$ which brings the SDEs (4.5.1) and (4.5.2) after analytic continuation to the form

$$(R(z) - R(-z))G^{(g)}(z|\tilde{J}) - \frac{\lambda}{V} \sum_{n=0}^{N'} r_n \frac{G^{(g)}(\varepsilon_n|\tilde{J})}{R(\varepsilon_n) - R(z)} = g\varepsilon,$$

$$(R(z) - R(-w))G^{(g)}(z, w|\tilde{J}) - \frac{\lambda}{V} \sum_{n=0}^{N'} r_n \frac{G^{(g)}(\varepsilon_n, w|\tilde{J})}{R(\varepsilon_n) - R(z)} = g\varepsilon,$$

where $G^{(g)}(\varepsilon_p, \varepsilon_{p_1}, \ldots, \varepsilon_{p_{N_b}}) = G^{(g)}_{|p_{N_b}, p_{N_b-1}, \ldots, p_1, p_2, \ldots, p_{N_b-1}}$ with $\varepsilon_p = R^{-1}(E_p)$. The new SDEs (after variable transformation) indicate a possible relation to topological recursion with the cubic model (no hyperlogarithms survive). Fixing the boundary condition by adjusting $\mu^2$ in Proposition 4.7 to obey $\frac{\partial}{\partial a} G(a, 0)|_{a=0} = -1$, the perturbative expansion of the exact solution and the perturbative expansion through Zimmermann’s forest formula coincide perfectly, which is shown in App. D.2. The surviving hyperlogarithms of the perturbative expansion are due to the integral representation of the exact solution given in Theorem 4.3 after inserting $R_4$. Notice also that the Feynman diagrammatic expansion admits on the $D = 4$ Moyal space the renormalon problem (see Remark 4.8) and the number of planar graphs with one boundary grows for $n$ vertices with $\mathcal{O}(n!)$. However, we provide here a second example, where the expansion with Zimmermann’s forest formula for a just-renormalisable model is resummable, and has additionally a potential which is bounded from below.

Any planar $N$-point function is given recursively through the 2-point function via (4.4.1). Writing out the recursion, cancellations appear which are related to non-crossing chord diagrams (see App. F.2). The combinatorial structure is captured by Catalan tables defined in Definition 4.11 which can be understood as Catalan tuples of Catalan tuples. The bijection between the recursive equation after cancellations and Catalan tables is given in Theorem 4.5 which probably has a higher topological generalisation due to the general recursion (4.1.4) for a correlation function of genus $g$ and $b$ boundary components.
Chapter 5

Conclusion and Outlook

One of the main achievements of this thesis was the generalisation of the solution of the Kontsevich model to the spectral dimensions $D < 8$. We found a method to derive intersection numbers on the moduli space $\mathcal{M}_{g,b}$ with partial differential equations after having applied the theorem of Kontsevich [Kon92]. The definition of $c$ is changed by the renormalisation procedure, but the structure of all correlation functions is not.

Furthermore, the exact solution of the 2-point function and therefore of any planar $N$-point function was computed for the Grosse-Wulkenhaar model for $D < 6$. For the special case on the 4-dimensional Moyal space, it was proved earlier that a solution exists [GW14a]. However, Theorem 4.3 gives the solution for any eigenvalue distribution of the external matrix $E$ of spectral dimension $D < 6$.

The computed results coincide with perturbative calculations via Zimmermann’s forest formula after having taken the same boundary conditions. To our knowledge, these two models provide the first bosonic examples for exactly solvable models which are just-renormalisable. Perturbatively, the number of graphs grows factorially and the renormalon appears (see Remark 3.4 and 4.8). Even in the cubic case, the $\beta$-function is positive (see Remark 3.3) which is also the case in quantum electrodynamics. Nevertheless, the resummability implies that cancellations prevent the renormalon problem. The factorial growth of the Feynman graphs, the renormalon problem and a positive $\beta$-function do not imply that a just-renormalisable model is not resummable.

What remains to investigate is whether the genus expansion

$$G = \sum_{g=0}^{\infty} V^{-2g} G^{(g)}$$

is Borel summable. For a generic set of regular spectral curves, it was proved that the free energies $F^{(g)}$ obtained by topological recursion grow at most like $\mathcal{O}((kg)! r^{-g})$ for some $r > 0$ and $k \leq 5$ [Eyn19]. But the property of Borel summability is still unclear, since the Borel transform needs an analytic continuation along the positive real line.

From a quantum field theoretical perspective, the question of analytic continuation $V \to 0$ is of big interest because it recovers the commutative space. However, it should be treated with caution. The harmonic oscillator term breaks translational invariance for any $V < \infty$. Therefore, the first step would be to generalise the results of this thesis for QFTs on noncommutative spaces away from the self-dual point at $\Omega = 1$. With the help of Meixner polynomials, it was proved that the quartic model, for instance, is in
4 dimensions perturbatively renormalisable for any $0 < \Omega \leq 1$ to all orders \cite{GW05b}. However, the limit $\Omega \rightarrow 0$ can be performed in $D = 2$, but not in $D = 4$ because this generates again the UV/IR mixing problem.

Further analysis suggests that the 4-point connected Schwinger function on the 4-dimensional Moyal space is not reflection positive. Therefore, the naive construction of the Schwinger functions via equation (2.5.10) fails for a noncommutative space. A different way to construct the Schwinger function is proposed in \cite{Wul19}. On a noncommutative geometry, Connes’ distance formula \cite{Con94} provides a metric structure via states. This implies that it is more natural for noncommutative geometry to construct Schwinger functions via states (see \cite{Wul19} for a detailed construction).

From a more geometric perspective, it is a natural question whether the quartic matrix field theory model satisfies topological recursion \cite{EO07}. Comparing the conjectured spectral curve of the quartic model (4.3.9) with the one of the cubic model (3.2.27) leads to an interesting observation. For finite matrices (take $V = \mathcal{N}$ for simplicity), both spectral curves have the general description (up to trivial factors of $2\lambda$)

$$y(z) = z \pm \frac{\lambda}{\mathcal{N}} \sum_{k=1}^{\mathcal{N}} \frac{r_k}{x'(\varepsilon_k)(\varepsilon_k - z)}, \quad e_k = x(\varepsilon_k),$$

where $x(z) = z^2 - c$ holds with the upper sign for the cubic model and $x(z) = z - \frac{\lambda}{\mathcal{N}} \sum_{k=1}^{\mathcal{N}} \frac{r_k}{x'(\varepsilon_k)(z + \varepsilon_k)}$ with the lower sign for the quartic model.

At this point, it is important also to recall the spectral curve of the Hermitian 1-matrix model and the Hermitian 2-matrix model. Assuming a genus zero spectral curve for the Hermitian 2-matrix model yields the combined form for their meromorphic $y$-function with

$$y(z) = az + \sum_{k=1}^{\tilde{d}-1} \frac{\alpha_k}{z^k}, \quad \text{where} \quad a = 0 \quad \text{for the Hermitian 1-matrix model}$$

and $d$ is the degree of the potential. The meromorphic $x$-function is

$$x(z) = \gamma \left(z + \frac{1}{z}\right) + b \quad \text{for the Hermitian 1-matrix model}$$

$$x(z) = \frac{a}{z} + \sum_{k=1}^{\tilde{d}-1} \beta_k z^k \quad \text{for the Hermitian 2-matrix model},$$

where $\tilde{d}$ is the degree of the second potential of the Hermitian 2-matrix model. The constants $\gamma, a, b, \alpha_k, \beta_k$ are implicitly defined by the coupling constants of the potential(s) \cite{Eyn03} \cite{Eyn16}. If both potentials of the Hermitian 2-matrix model coincide the number of free parameters is reduced with the identities $\alpha_k = \beta_k$ and $\tilde{d} = d$.

It should be emphasised that the meromorphic $x$-function of the cubic model and of the Hermitian 1-matrix model has a global symmetry given by $x(z) = x(-z)$ and $x(z) = x(\frac{1}{z})$, respectively. By contrast, the $x$-function of the quartic model and the Hermitian 2-matrix model has no symmetry. However, another symmetry can be observed by

$$y(z) = -x(-z) \quad \text{for the quartic model}$$

$$y(z) = x\left(\frac{1}{z}\right) \quad \text{for the Hermitian 2-matrix model with coinciding potentials.}$$
From this observations and the fact that the double-scaling limit of the Hermitian 1-matrix model coincides with the large $\mathcal{N}$ limit of the Kontsevich model [AK93], a relation between the quartic matrix field theory model and the Hermitian 2-matrix model (with coinciding potentials) can be conjectured. The spectral curves of these models are natural generalisations of the Hermitian 1-matrix model and the cubic model, where the asymmetry between $x(z)$ and $y(z)$ is achieved with different global involutions.

This conjecture is supported by the observation that the planar correlation function with one mixed boundary in the Hermitian 2-matrix model [EO05] obeys exactly the same recursive equation as the planar $\mathcal{N}$-point function for the quartic matrix field theory (Example 4.1). It indicates that correlation functions with mixed boundaries are possibly related in general to correlation functions of even length.

In Sec. 4.4, the combinatorial structure of Example 4.1 was analysed and proved to consists after cancellations of $d_k = \frac{1}{1+k}(3k+1 \choose k)$ terms. This number can be understood due to Corollary 4.5

$$d_k = \sum_{(e_0, \ldots, e_{k+1}) \in C_{k+1}} c_{e_0-1}c_{e_1} \cdots c_{e_k}c_{e_{k+1}}$$

as Catalan numbers of Catalan numbers. Studying the generalisation of Example 4.1 given by equation (4.1.4) can give more insight into the conjecture.

Catalan numbers $c_n$ can be generalised in several forms. One of them is given by the numbers $C_{g,b}$ which is a topological generalisation graded by the genus $g$ and the number of boundaries $b$. The usual Catalan number $c_n$ corresponds in this picture to the genus $g = 0$ and $b = 1$ case. The Laplace transform of the generating functions of $C_{g,b}$ was proved to satisfy topological recursion [DMSS13], too. A further question is whether the number $d_k$ has a topological generalisation as well, which should be encoded in the recursive equation (4.1.4).

If a relation between the Hermitian 2-matrix model and the quartic model turns out to exist, the quartic model would not satisfy topological recursion in the sense of [EO07], because also the Hermitian 2-matrix model has a completely different topological recursive form [EO08]. Additionally, the nonlinearity of equation (4.1.4) suggests that the quartic model has a more complex topological recursive structure.

Characterising the poles and branch points for the correlation functions of Euler characteristic $\chi = 0$ and $\chi = -1$ will be the first step. The branch points are difficult to handle because they cannot be computed explicitly. In the large $V, \mathcal{N}$-limit, the number of poles and branch points tends to infinity which is (to our knowledge) different to any other model described by topological recursion, especially the cubic model with coinciding pole and branch point at $z = 0$. 

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Appendix A

Moyal Base

The following was developed in [GBV88]. In our $D$-dimensional notation, it is formulated as follows:

Let the continuous variable be $x = (x_1, ..., x_D) \in \mathbb{R}^D$ and the discrete $\mathbb{n} = (n_1, ..., n_{D/2}) \in \mathbb{N}^{D/2}$. The $\star$-product is defined by

$$ (g \star h)(x) = \int \frac{d^Dk}{(2\pi)^D} \int d^Dy \, g(x + \frac{1}{2}\Theta k) \, h(x + y) e^{ik \cdot y}, \quad (A.1) $$

where

$$ \Theta = 1_{D/2} \otimes \left( \begin{array}{cc} 0 & \theta \\ -\theta & 0 \end{array} \right), \quad \theta \in \mathbb{R}, \quad g, h \in S(\mathbb{R}^D). $$

Let $\theta > 0$. The starting point of finding the full base is the unit vector (or vacuum) given by the Gaussian

$$ b_0(x) = 2^{D/2} e^{-\frac{1}{2}(x_1^2 + x_2^2 + .. + x_D^2)}. $$

Inserting into (2.5.1) leads after factorising the integrals to

$$ (b_0 \star b_0)(x) = 2^D \int d^Dy \int \frac{d^Dk}{(2\pi)^D} e^{ik \cdot y} e^{-\frac{1}{2} \sum_{i=1}^{D/2} (x_{2i-1} + \frac{1}{2}k_{2i})^2 + (x_{2i} + \frac{1}{2}k_{2i-1})^2 + (x_{2i-1} + y_{2i-1})^2 + (x_{2i} + y_{2i})^2} = b_0(x). $$

The creation and annihilation operators are defined by

$$ a_i(x) = \frac{1}{\sqrt{2}} (x_{2i-1} + ix_{2i}), \quad a_i^\dagger(x) = \frac{1}{\sqrt{2}} (x_{2i-1} - ix_{2i}), $$

$$ \frac{\partial}{\partial a_i} = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial x_{2i-1}} - i \frac{\partial}{\partial x_{2i}} \right), \quad \frac{\partial}{\partial a_i^\dagger} = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial x_{2i-1}} + i \frac{\partial}{\partial x_{2i}} \right). $$

Direct calculation leads for $g \in S(\mathbb{R}^D)$ to

$$ (a_i \star g)(x) = a_i(x)g(x) + \frac{\theta}{2} \frac{\partial g}{\partial a_i^\dagger}(x), \quad (g \star a_i)(x) = a_i(x)g(x) - \frac{\theta}{2} \frac{\partial g}{\partial a_i}(x), \quad (A.2) $$

$$ (a_i^\dagger \star g)(x) = a_i^\dagger(x)g(x) - \frac{\theta}{2} \frac{\partial g}{\partial a_i}(x), \quad (g \star a_i^\dagger)(x) = a_i^\dagger(x)g(x) + \frac{\theta}{2} \frac{\partial g}{\partial a_i^\dagger}(x). \quad (A.3) $$
Since $a_i, a_j$ (and $a_i^\dagger, a_j^\dagger$) act for $i \neq j$ on different variables, they commute with respect to the star product
\[
([a_i, a_j] \star g)(x) = ([a_i, a_j]^m \star g)(x) = ([a_i^\dagger, a_j^\dagger] \star g)(x) = ([a_i, a_j]^m \star g)(x) = ([a_i^\dagger, a_j^\dagger] \star g)(x)
\]
\[
= (g \star [a_i, a_j])(x) = (g \star [a_i, a_j]^m)(x) = (g \star [a_i^\dagger, a_j^\dagger])(x) = (g \star [a_i, a_j]^m)(x) = 0,
\]
where $[g, h] = g \star h - h \star g$. Acting with $a_i$ from the left or with $a_j^\dagger$ from the right for some $i, j \in \{1, \ldots, D/2\}$ on the unit $b_0$ with respect to the $\star$-product vanishes exactly
\[
(a_i \star b_0)(x) = (b_0 \star a_j^\dagger) = 0,
\]
whereas $a_i^\dagger$ from the left and $a_j$ from the right take the rôle as creation operator with
\[
((a_i^\dagger)^m \star b_0)(x) = 2^m (a_i^\dagger)^m(x) b_0(x)
\]
\[
(b_0 \star (a_j^\dagger)^m)(x) = 2^n (a_j)^n(x) b_0(x),
\]
where $g^m = g \star g \ast \ldots \ast g$ exactly $m$ times. One further verifies for $m, n > 0$
\[
(a_i \star (a_j^\dagger)^m \star b_0)(x) = m \theta((a_i^\dagger)^{m-1} \star b_0)(x), \tag{A.4}
\]
\[
(b_0 \ast a_j^\dagger)^n \ast (a_j)(x) = n \theta(b_0 \ast a_j^{n-1})(x). \tag{A.5}
\]
Defining now the base with $i \in \{1, \ldots, \frac{D}{2}\}$ for the 2-dimensional space
\[
f_{n,m_i}(x_{2i-1}, x_{2i}) := \frac{1}{\sqrt{n!m!\theta m_i+m_i}}((a_i^\dagger)^{m_i} \ast 2e^{-\frac{\hbar}{2} (x_i^2 + x_{i+1}^2)} \ast a_i^{m_i})(x)
\]
leads to the base of the $D$-dimensional Moyal space with the correct normalisation factor by
\[
b_{n,m}(x) := \frac{1}{\sqrt{n!m!\theta |a|+|\theta|}}((a_i^\dagger)^{m_i} \ast b_0 \ast a_i^m)(x)
\]
\[
= f_{n_1,m_2}(x_1, x_2) f_{n_2,m_2}(x_3, x_4) \cdots f_{n_{D/2},m_{D/2}}(x_{D-1}, x_D),
\]
where $n! = n_1! n_2! \cdots n_{D/2}!$, $|\theta| = n_1 + n_2 + \cdots + n_{D/2}$, $(a_1^\dagger)^{n_1} = (a_1^\dagger)^{n_1} \ast (a_2^\dagger)^{n_2} \ast \cdots \ast (a_{D/2}^\dagger)^{n_{D/2}}$ and $a^m = (a_1)^{m_1} \ast (a_2)^{m_2} \ast \cdots \ast (a_{D/2})^{m_{D/2}}$.

The matrix multiplication rule follows by [A.4] and [A.5]
\[
(b_{n,m} \ast b_{k,l})(x) = \delta_{n,k} b_{n,m}(x),
\]
where $\delta_{n,m} = \delta_{n_1,m_1} \cdots \delta_{n_{D/2}, m_{D/2}}$.

For $g, h \in \mathcal{S}(\mathbb{R}^D)$ we have
\[
\int d^Dx \, (g \ast h)(x) = \int d^Dx \, g(x) h(x). \tag{A.6}
\]
This is proved by inserting the definition [A.1] and transforming $x \to x' = x + \frac{\hbar}{2} \Theta k$ to get $g$ independent of the $k$ and $y$ integral. The next transformation is $y \to y' = y - \frac{\hbar}{2} \Theta k$, where the exponential become $ik \cdot (y' + \frac{\hbar}{2} \Theta k) = ik \cdot y'$. Now, the integral over $k$ is performed and gives a $\delta$-distribution $\delta(y')$. The integral over $y'$ leads to $g(x) h(x)$ inside the integral over $x$. 

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The condition \([A.6]\) implies a cyclic property (as for traces)
\[
\int d^Dx \ (g \ast h)(x) = \int d^Dx \ g(x)h(x) = \int d^Dx \ (h \ast g)(x).
\]  \hspace{1cm} (A.7)

furthermore, the base \(b_{nm}\) obeys a trace property
\[
\int d^Dx \ b_{nm}(x) = \frac{1}{\sqrt{|n!m!\theta^{|n|+|m|}|}} \int d^Dx \ ((a^\dagger)^n \ast b_0 \ast a^m)(x)
\]
\[
= \frac{1}{\sqrt{|n!m!\theta^{|n|+|m|}|}} \int d^Dx \ (a^m \ast (a^\dagger)^n \ast b_0)(x)
\]
\[
= \delta_{nm} \int d^Dx \ b_0(x) = \delta_{nm} (2\pi\theta)^{D/2},
\]

where we have used the cyclic property and \([A.4], [A.5]\) in the third line.

Any Schwartz function \(g \in \mathcal{S}(\mathbb{R}^D)\) can be expanded into a convergent series
\[
g(x) = \sum_{n,m} g_{nm} b_{nm}(x),
\]
where the sum goes over all \(n_i, m_j\) from 0 to infinity and \(y_{2m} = g_{n_1 m_1} \ldots g_{n_{D/2} m_{D/2}}\). Furthermore, \(g\) lies in \(\mathcal{S}(\mathbb{R}^D)\) iff
\[
\sum_{n_i, m_i=0}^{\infty} \left(2n_i + 1\right)^{2k} \left(2n_i + 1\right)^{2k} |g_{n_i m_i}|^2 \right)^{1/2} < \infty
\]
for all \(i, k\).

Let \(\Phi \in \mathcal{S}(\mathbb{R}^D)\) be the scalar field. The kinetic part (with the harmonic oscillator) of the action for scalar fields on the Moyal space is
\[
\int \frac{d^Dx}{(8\pi)^{D/2}} \left(\frac{1}{2} \Phi \ast (-\Delta + \Omega^2 ||2\Theta^{-1} \cdot x||^2 + \mu^2_1^2)\Phi\right)(x)
\]  \hspace{1cm} (A.8)
\[
= \sum_{n,m,k,l} \Phi_{nm} G_{nm-kl} \Phi_{kl}
\]  \hspace{1cm} (A.9)

with the regulator \(\Omega \in \mathbb{R}\), the Laplacian \(\Delta = \sum_{i=1}^{D/2} \left(\frac{\partial^2}{\partial x_{2i-1}^2} + \frac{\partial^2}{\partial x_{2i}^2}\right) = \sum_{i=1}^{D/2} \frac{\partial^2}{\partial a_i \partial a_i} + \frac{\partial^2}{\partial a_i \partial a_i}\)

and expansion \(\Phi(x) = \sum_{n,m} \Phi_{nm} b_{nm}(x)\). The kinetic term reads in the matrix base with \(||x||^2 = \sum_{i=1}^{D/2} a_i a_i^\dagger + a_i^\dagger a_i\) and \([A.2], [A.3]\)
\[
G_{nm-kl} = \int \frac{d^Dx}{(8\pi)^{D/2}} \left(\frac{1}{2} b_{nm} \ast \left(- \sum_{i=1}^{D/2} \frac{\partial^2}{\partial x_i^2} + \frac{4\Omega^2}{\theta^2} ||x||^2 + \mu^2_1^2\right) b_{kl}\right)(x)
\]
\[
= \int \frac{d^Dx}{(8\pi)^{D/2}} \left(\frac{1}{2} b_{nm} \ast \left(- \sum_{i=1}^{D/2} \frac{\partial^2}{\partial a_i \partial a_i} + \frac{\partial^2}{\partial a_i \partial a_i} - \frac{4\Omega^2}{\theta^2} (a_i a_i^\dagger + a_i^\dagger a_i) + \mu^2_0\right) b_{kl}\right)(x)
\]
\[
= \int \frac{d^Dx}{(8\pi)^{D/2}} \left(\frac{1 + \Omega^2}{2\theta^2} \ b_{nm} \ast \sum_{i=1}^{D/2} (a_i \ast a_i^\dagger + a_i^\dagger \ast a_i) \ast b_{kl}\right)
\]
\[ + \frac{1 + \Omega^2}{2\theta^2} b_{2m} \ast b_{kl} \ast \sum_{i=1}^{D/2} (a_i \ast a_i^\dagger + a_i^\dagger \ast a_i^\dagger) \]
\[ - \frac{1 - \Omega^2}{\theta^2} b_{2m} \ast \sum_{i=1}^{D/2} (a_i^\dagger \ast b_{kl} \ast a_i + a_i \ast b_{kl} \ast a_i^\dagger) + \frac{\mu_0^2}{2} b_{2m} \ast b_{kl} \] (x)

\[ = \left( \frac{\theta}{4} \right)^{D/2} \left( \frac{\mu_0^2}{2} + \frac{1 + \Omega^2}{\theta} \sum_{i=1}^{D/2} (1 + n_i + m_i) \right) \delta_{\text{m},k} \delta_{\text{n},l} \]  
\[ - \left( \frac{\theta}{4} \right)^{D/2} \left( \frac{1 - \Omega^2}{\theta} \sum_{i=1}^{D/2} (\sqrt{n_i} m_i \delta_{n_i - 1, i} \delta_{m_i - 1, i} + \sqrt{k_i} l_i \delta_{n_i + 1, i} \delta_{m_i + 1, i}) \delta_{\text{m}+1, k} \delta_{\text{n}+1, l}, \right) \]

where \( \delta_{\text{m},k} \) is \( \delta_{2,m} \) with omitted \( \delta_{m,m} \).
Appendix B

Schwinger Function on the Moyal Space

We are following the derivation given in [GW13] and extend it to any dimension $D$. The connected Schwinger function is defined as

$$S_c(\xi_1, \ldots, \xi_N) := \lim_{\Lambda^2 \to \infty} \lim_{N' \to \infty} \sum_{N_1 + \ldots + N_b = N} \sum_{p_1, \ldots, p_{N_b}^N \in N_{N_b}^{D/2}} \frac{G_{|p_1 \cdots p_{N_b}^N|}}{(8\pi)^{D/2} b!} \sum_{\sigma \in S_N} \prod_{\beta = 1}^b \frac{b_{\sigma_1} \cdots b_{\sigma_{b\beta}} (\xi_{\sigma(s_{\beta} + 1)} \cdots \xi_{\sigma(s_{\beta} + N_{\beta})})}{V N_{\beta}},$$  \hspace{1cm} (B.1)$$

where $s_{\beta} = N_1 + \ldots + N_{\beta-1}$ and $S_N$ is the symmetric group.

Abbrivations are used for the correlation functions $G_{|p_1 \cdots p_{N_b}^N|} = G_{|p_1 \cdots p_{N_b}^N|}$ with $P^{N_b} = p_1^\beta \cdots p_{N_b}^\beta$. Additionally, we define the total norm $\|P^{N_b}\| = |p_1^\beta| + \ldots + |p_{N_b}^\beta|$. Furthermore, we assume that the correlation function $G_{|p_1 \cdots p_{N_b}^N|}$ has a Fourier-Laplace transform $\tilde{G}_{N,V}(t_1, \omega_{N_1}^N \cdots t_b, \omega_{N_b}^N)$ depending on $N$ and $V$, where the Laplace transform is used to distinguish the time $t_i \geq 0$ coordinate and $\omega_{N_b}^N = (\omega_1^\beta, \ldots, \omega_{N_b-1}^\beta) \in \mathbb{R}^{N_b-1}$. Define further the product $\langle \omega_{N_b}^N, P^{N_b} \rangle = \sum_{i=1}^{N_b-1} \omega_i (|p_i^\beta| - |p_{i+1}^\beta|)$ such that the Fourier-Laplace transform takes the form

$$G_{|p_1 \cdots p_{N_b}^N|} = \int_{\mathbb{R}_+^b} d(t_1, \ldots, t_b) \int_{\mathbb{R}^{N-6}} d(\omega_{N_1}^N, \ldots, \omega_{N_b}^N) \hat{G}_{N,V}(t_1, \omega_{N_1}^N \cdots t_b, \omega_{N_b}^N) \prod_{\beta = 1}^b e^{-\frac{t_{\beta}^2}{V^2} \|P^{N_b}\| + \frac{1}{V} \langle \omega_{N_b}^N, P^{N_b} \rangle}.$$  \hspace{1cm} (B.2)$$

Since the limit $N', V \to \infty$ is taken, the Fourier-Laplace transform converges in this limit to $\hat{G}(t_1, \omega_{N_1}^N \cdots t_b, \omega_{N_b}^N)$ such that the inverse Fourier-Laplace transformation gets independent of $N, V$.

Inserting (B.2) into (B.1) the index-sum and the product can be interchanged

$$\sum_{p_1^1 \cdots p_{N_b}^N \in N_{N_b}^{D/2}} \prod_{\beta = 1}^b = \prod_{\beta = 1}^b \sum_{p_1^1 \cdots p_{N_b}^N \in N_{N_b}^{D/2}} \prod_{\beta = 1}^b$$
since the Fourier-Laplace transform \( \tilde{G}_{N,V}(t_1, \omega^N | \ldots | t_b, \omega^N) \) is independent of the index-sum and the exponentials factorise. For each \( \beta \in \{1, \ldots, b\} \) the following series has to be determined

\[
\frac{1}{VN_\beta} \sum_{\xi_1^\beta \ldots \xi_N^\beta \in N^{D/2}_\beta} b_{\xi_1^\beta \xi_2^\beta} (\xi_{\sigma(s_\beta + 1)}) \ldots b_{\xi_{N^\beta}^\beta \xi_1^\beta} (\xi_{\sigma(s_\beta + N_\beta)}) e^{-\frac{i\beta}{\sqrt{2D}} \|\xi_N\| + \frac{1}{\sqrt{2D}} \langle \omega^N, \xi^N \rangle}
\]

\[
= \frac{1}{VN_\beta} \sum_{\xi_1^\beta \ldots \xi_N^\beta \in N^{D/2}_\beta} b_{\xi_1^\beta \xi_2^\beta} (\xi_{\sigma(s_\beta + 1)}) \ldots b_{\xi_{N^\beta}^\beta \xi_1^\beta} (\xi_{\sigma(s_\beta + N_\beta)}) (z_1^\beta \xi_1^\beta \ldots (z_N^\beta \xi_N^\beta)^{2N_\beta})
\]

(B.3)

where

\[
z_i^\beta := \begin{cases} 
  e^{-\frac{i\beta}{\sqrt{2D}} + i\frac{\omega_i^\beta}{\sqrt{2D}}} & i = 1 \\
  e^{-\frac{i\beta}{\sqrt{2D}} + i\frac{\omega_i^\beta - \omega_j^\beta}{\sqrt{2D}}} & i \in \{2, \ldots, N_\beta - 1\} \\
  e^{-\frac{i\beta}{\sqrt{2D}} - i\frac{\omega_{N_\beta - 1}^\beta}{\sqrt{2D}}} & i = N_\beta
\end{cases}
\]

Recall next a lemma and its corollary proved in \([GW13]\) about the 2D base \( f_{nm}(\xi), \xi \in \mathbb{R}^2 \) of the Moyal plane

**Lemma B.1.** (\([GW13]\) Lemma 4 + Corollary 5) Let for \( \xi, \eta \in \mathbb{R}^2 \) be the scalar product, the norm and the determinant of the matrix \( (\xi, \eta) \) given by \( \langle \xi, \eta \rangle, \|\xi\| \) and det(\( \xi, \eta \)), respectively. Then, for \( \xi_i \in \mathbb{R}^2, z_i \in \mathbb{C} \) with \( |z_i| < 1 \) and \( N \in \mathbb{N} \) (with the cyclic structure \( N + i \equiv i \))

\[
\sum_{p_1 \ldots p_N = 0}^{\infty} \prod_{i=1}^N f_{n_ip_i+1}(\xi_i) z_i^{p_i} = \frac{2^N}{1 - \prod_{i=1}^N (-z_i)} \exp \left(-\sum_{i=1}^N \frac{\|\xi_i\|^2}{4V_i^2} \left(1 + \prod_{l=i}^N (-z_l)\right)\right)
\]

\[
\times \exp \left(-\sum_{1 \leq k < l \leq N} \frac{\langle \xi_k, \xi_l \rangle - i \text{det}(\xi_k, \xi_l)}{2V_i^2} \left(1 - \prod_{l=i}^N (-z_l)\right)\right)
\]

Let \( N^{D/2} \ni p_i^\beta = ((p_i^\beta)_1, \ldots, (p_i^\beta)_D/2) \) and \( \mathbb{R}^{D/2} \ni \chi_i = (\chi_i)_1, \ldots, (\chi_i)_D/2 \), then (B.3) can be rearranged to

\[
\sum_{p_1^\beta \ldots p_{N_\beta}^\beta \in N^{D/2}_\beta} b_{p_1^\beta p_2^\beta} (\xi_{\sigma(s_\beta + 1)}) \ldots b_{p_{N_\beta}^\beta p_1^\beta} (\xi_{\sigma(s_\beta + N_\beta)}) (z_1^\beta \xi_1^\beta \ldots (z_N^\beta \xi_N^\beta)^{2N_\beta})
\]

\[
= \sum_{p_1^\beta \ldots p_{N_\beta}^\beta \in N^{D/2}_\beta \cap \mathbb{R}^{D/2}} \prod_{i=1}^{D/2} \prod_{j=1}^{N_\beta} f((p_i^\beta)_i, (p_{i+1}^\beta)_i, (\xi_{\sigma(s_\beta + j)})_i, (\xi_{\sigma(s_\beta + j)})_i) (z_j^\beta)^{(p_j^\beta)}
\]

\[
= \prod_{i=1}^{D/2} \sum_{(p_1^\beta), \ldots, (p_{N_\beta}^\beta)} \prod_{i=0}^{\infty} \prod_{j=1}^{N_\beta} f((p_i^\beta)_i, (p_{i+1}^\beta)_i, (\xi_{\sigma(s_\beta + j)})_i, (\xi_{\sigma(s_\beta + j)})_i) (z_j^\beta)^{(p_j^\beta)}
\]

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in the $\mathcal{N}' \to \infty$ limit such that Lemma B.1 can be used. The splitting of the double scaling limit $V, \mathcal{N}' \to \infty$ with $\frac{N'}{V^{1/2}} = \Lambda^2$ and interchanging it with $G_{\mathcal{N}',V}$ is uncritical. However, all $z_i^{\beta}$'s get critical $|z_i^{\beta}| = 1$. In an intermediate step by deriving Lemma B.1, the series
$$\sum_{\mathcal{N}} \frac{(p+k)!}{p!k!} ((-z_1)...(-z_N)) = \frac{1}{(1-((-z_1)...(-z_N)))^{1+\epsilon}}$$
has to be taken, where a restriction to $\mathcal{N}'$ summands would give an error term proportional to $(z_1...z_N)^{\mathcal{N}'} = e^{-\Lambda^2 N t}$. This error term vanishes in the following limit of $V \to \infty$ together with the final limit of $\Lambda^2 \to \infty$.

With this argumentation, Lemma B.1 can be used.

As $|z_i|$ converges to 1 in the $V \to \infty$ limit, the denominator $1 - \prod_{i=1}^{N} (-z_i) = 1 - (-1)^{N} e^{-\frac{N t}{V^{1/2}}}$ converges to 2 for $N$ odd. For even $N$, the denominator converges at leading order in $V$ to $\frac{N t}{V^{1/2}}$. Therefore, performing the limits in the discussed way leads to

$$\lim_{\Lambda^2 \to \infty} \lim_{V,\mathcal{N}' \to \infty} \frac{2^{DN_{\beta}/2}}{N_{\beta}(N_{\beta}t_{\beta})^{D/2}} \exp \left( - \frac{\|\sum_{j=1}^{N_{\beta}} (-1)^{j-1} \xi_{(s_{\beta}+j)} \|}{2N_{\beta}t_{\beta}} \right)$$

for $N_{\beta}$ even

and

$$0$$

for $N_{\beta}$ odd.

Rewrite this result with the Gaussian integral to

$$\frac{2^{DN_{\beta}/2}}{N_{\beta}(N_{\beta}t_{\beta})^{D/2}} \exp \left( - \frac{\|\sum_{j=1}^{N_{\beta}} (-1)^{j-1} \xi_{(s_{\beta}+j)} \|}{2N_{\beta}t_{\beta}} \right)$$

$$= \frac{2^{DN_{\beta}/2}}{(2\pi)^{D/2}N_{\beta}} \int_{\mathbb{R}^{D/2}} dp_{\beta} e^{-\frac{N_{\beta}}{2}\|p_{\beta}\|^{2}t_{\beta}} e^{i\sum_{j=1}^{N_{\beta}} (-1)^{j-1} \xi_{(s_{\beta}+j)}} \cdot \frac{\prod_{\beta=1}^{D} 2^{DN_{\beta}/2}}{N_{\beta}}$$

Inserting this back into (B.1) together with (B.2) give the inverse Laplace transform in the variable $t_{\beta}$. The integrals over $\xi_{(s_{\beta})}$ give the inverse Fourier transform if within each boundary $\beta$ the variables are taken to be the same. We end up in the following representation of the Schwinger function

$$S_{\beta}(\xi_{1},...\xi_{N}) = \sum_{N_{1}+...+N_{\beta}=N} \sum_{\sigma \in S_{N}} \left( \prod_{\beta=1}^{D} \frac{2^{DN_{\beta}/2}}{N_{\beta}} \int_{(2\pi)^{D/2}} dp_{\beta} \cdot \frac{G^{(g)}(\xi_{(s_{\beta}+1)}-\xi_{(s_{\beta}+2)}+...-\xi_{(s_{\beta}+N_{\beta})})}{2^N} \right)$$

$$\times \frac{1}{(8\pi)^{D/2}} G^{0} \left( \left| \frac{\|p_{\beta}\|^{2}}{2} \right|_{N_{1}} \left| \frac{\|p_{\beta}\|^{2}}{2} \right|_{N_{2}} \cdots \left| \frac{\|p_{\beta}\|^{2}}{2} \right|_{N_{N_{\beta}}} \right),$$

where $G^{0}(x_{1}^{1},...,x_{1}^{1}...x_{N_{1}}^{1},...,x_{N_{N_{\beta}}}^{1}) = \lim_{V,\mathcal{N}',\Lambda^2 \to \infty} G^{(g)}_{\mathcal{N}',V}[\xi_{1}^{1},...\xi_{1}^{1},...|\xi_{2}^{1},...\xi_{N_{N_{\beta}}}^{1}]=x_{V^{1/2}}^{1}$.
Appendix C

Relevant Lemmata for Theorem 3.1*

Assumption C.1. We assume that \( G_g(z) \) is, for \( g \geq 1 \), a function of \( z \) and of \( \varrho_0, \ldots, \varrho_{3g-2} \) (true for \( g = 1 \)). We take eq. (3.2.18) and in particular \( G_g(z, J) := (2\lambda)^3 \hat{A}_{j,z}^g G_g(J) \) as a definition of a family of functions \( \hat{G}_g(z_1, J) \) and derive equations for that family.

Lemma C.1. Let \( J = \{z_2, \ldots, z_b\} \). Then under Assumption C.1 and with Definition 3.1 of the operator \( \hat{K}_{z_1} \) one has

\[
\hat{K}_{z_1} G_g(z_1, J) = \frac{(2\lambda)^3}{z_1^2} \left( \sum_{l=0}^{3g-3+|J|} (3 + 2l) \frac{\partial G_g(J)}{\partial \varrho_l} \sum_{k=0}^{l} \frac{\varrho_k}{z_1^{2+2l-2k}} + \sum_{\xi \in J} \frac{1}{\partial \xi} \frac{\partial G_g(J)}{\partial \xi} \right).
\]

Proof. Take Definition 3.3 for \( \hat{A}_{j,z}^g G_g(J) \) and apply Lemma 3.1. \( \square \)

Lemma C.2. Let \( J = \{z_2, \ldots, z_b\} \). Then under Assumption C.1 one has

\[
\frac{(2\lambda)^3}{\zeta} \frac{\partial}{\partial \zeta} G_g(z_1, J\setminus\{\zeta\}) - G_g(\zeta, J\setminus\{\zeta\}) = 2\lambda G_0(z_1, \zeta) G_g(z_1, J\setminus\{\zeta\})
\]

\[
= (2\lambda)^6 \left[ \sum_{l=0}^{3g-4+|J|} \left( -\sum_{n=0}^{l} \frac{(3 + 2l)(1 + 2n) \varrho_{l+1}}{\varrho_0 z_1^{4-2n} \zeta^{3+2n}} + \sum_{n=0}^{l+2} \frac{(3 + 2l)(1 + 2n)}{z_1^{6+2l-2n} \zeta^{3+2n}} \right) \frac{\partial G_g(J\setminus\{\zeta\})}{\partial \varrho_l} \right]
\]

\[
+ \sum_{\xi \in J\setminus\{\zeta\}} \sum_{n=0}^{l} \frac{1 + 2n}{\varrho_0 \xi z_1^{4-2n} \zeta^{3+2n}} \frac{\partial G_g(J\setminus\{\zeta\})}{\partial \xi} \right].
\]

Proof. Definition 3.3 gives with \( \frac{z_1^{1+2g} - y^{1+2g}}{z_1^{2} - y^{2}} = -\sum_{l=0}^{2g+2} \frac{z_1^{1+2g-l} y^{2g+2-l}}{z_1^{2} - y^{2}} \) for the first term

\[
G_g(z_1, J\setminus\{\zeta\}) - G_g(\zeta, J\setminus\{\zeta\})
\]

\[
= \frac{(2\lambda)^3}{z_1^{2} - \zeta^{2}} \sum_{l=0}^{3g-4+|J|} \left( \frac{(3 + 2l) \varrho_{l+1}}{\varrho_0 z_1^{4-2n} \zeta^{3+2n}} \left( \frac{1}{z_1^{2} - \zeta^{2}} \right) \right) \frac{\partial G_g(J\setminus\{\zeta\})}{\partial \varrho_l}
\]

\[
+ \sum_{\xi \in J\setminus\{\zeta\}} \frac{1}{\varrho_0 \xi z_1^{2} - \zeta^{2}} \frac{\partial G_g(J\setminus\{\zeta\})}{\partial \xi}
\]

*This is taken from the appendix of our paper [GHW19a]
The remaining \( \zeta \)-derivative confirms the assertion. \( \square \)

**Lemma C.3.** Let \( J = \{ z_2, \ldots, z_b \} \). Then under Assumption C.1 one has

\[
-(2\lambda)^{3b-3} \hat{K}_{z_1, \ldots, z_b} \hat{A}^{lg}_{z_1, z_2} \hat{K}_{z_1} g_g(z_1) = \lambda \sum_{h=1}^{g-1} \sum_{I \subset J} g_h(z_1, I) g_{g-h}(z_1, J \setminus I) + \lambda g_{g-1}(z_1, z_1, J).
\]

**Proof.** Equation (3.2.12) can be rewritten for \( b = 1 \) as

\[
-\hat{K}_{z_1} g_g(z_1) = \lambda \sum_{h=1}^{g-1} g_h(z_1) g_{g-h}(z_1) + \lambda g_{g-1}(z_1, z_1).
\]

Operating with \(-(2\lambda)^{3b-3} \hat{A}^{lg}_{z_1, \ldots, z_b} \hat{A}^{lg}_{z_1, z_2} \) and taking the Leibniz rule into account, the assertion follows. \( \square \)

**Lemma C.4.** Let \( J = \{ z_2, \ldots, z_b \} \). Then under Assumption C.1 one has

\[
(2\lambda)^3 [\hat{K}_{z_1} \hat{A}^{lg}_{z_1, \ldots, z_b}] g_g(z_1, J \setminus \{ z_b \})
\]

\[
= (2\lambda)^6 \left[ 3g-4+|J| \sum_{l=0}^{3g-4+|J|} \left( \frac{3 + 2l}{z_1^{3+2l}} \left( \frac{q_{l+1}}{q_0 z_1^{3}} + \frac{3q_{l+1}}{z_1^{3+2l}} - \frac{1}{z_1^{4+2l}} \right) \frac{\partial}{\partial q_l} \right) - \sum_{l=0}^{3g-4+|J|} \sum_{k=0}^{l} \frac{(3 + 2l)(3 + 2k)}{z_1^{4+2l-2k} z_1^{3+2k}} \frac{\partial}{\partial q_l} - \sum_{\zeta \in J \setminus \{ z_b \}} \frac{1}{q_0 z_1^{3} z_1^{2}} \left( \frac{1}{z_1^{2}} + \frac{3}{z_1^{3}} \right) \frac{\partial}{\partial \zeta} \right] g_g(J \setminus \{ z_b \}).
\]
APPENDIX C. RELEVANT LEMMATA FOR THEOREM 3.1

Proof. The first term of the lhs, \( \hat{K}_z \hat{A}_{1g} \hat{G}_g(z_1, J \setminus \{ z_b \}) \), is given by Lemma C.1 and \( G_g(J) = (2\lambda)^3 \hat{A}_{1g} \hat{G}_g(J \setminus \{ z_b \}) \) to

\[
\begin{align*}
\hat{K}_z (G_g(z_1, J)) &= (2\lambda)^6 \left[ \sum_{l=0}^{3g-3+|J|} (3 + 2l) \sum_{k=0}^{l} \frac{\partial}{\partial q_l} \left( \frac{3g-3+|J|}{2} \frac{1}{\theta_0 z_{b}^3} \frac{1}{\theta_0 z_{b}^3} \frac{1}{\theta_0 z_{b}^3} \right) \left( G_g(J \setminus \{ z_b \}) \right) \right] \\
&= (2\lambda)^6 \left[ \sum_{l=0}^{3g-3+|J|} (3 + 2l) \sum_{k=0}^{l} \frac{1}{\theta_0 z_{b}^3} \frac{1}{\theta_0 z_{b}^3} \frac{1}{\theta_0 z_{b}^3} \right] \left( G_g(J \setminus \{ z_b \}) \right)
\end{align*}
\]

We have used that \( G_g(J \setminus \{ z_b \}) \) can only depend on \( q_l \) for \( l \leq 3g - 4 + |J| \). For the second term of the lhs, \( \hat{A}_{1g} \hat{K}_z \hat{G}_g(z_1, J \setminus \{ z_b \}) \), Lemma C.1 can also be used with \( b - 1 \) instead of \( b \):

\[
\begin{align*}
(2\lambda)^3 \hat{A}_{1g} \hat{K}_z \hat{G}_g(z_1, J \setminus \{ z_b \}) &= (2\lambda)^6 \left[ \sum_{l=0}^{3g-3+|J|} \frac{1}{\theta_0 z_{b}^3} \frac{1}{\theta_0 z_{b}^3} \frac{1}{\theta_0 z_{b}^3} \right] \left( G_g(J \setminus \{ z_b \}) \right)
\end{align*}
\]
\[ + \sum_{\xi \in J \setminus \{z_b\}} \left( \sum_{l=0}^{3g-4+|J|} \frac{1}{\theta_0 z_1^{3+2l-2k}} \left( 3 + 2l \right) g_k \frac{\partial^2}{\partial \xi \partial \xi} + \sum_{\zeta \in J \setminus \{z_b\}} \frac{1}{\theta_0 z_1^{2+2l-2k}} \left( 3 + 2l \right) g_k \frac{\partial^2}{\partial \zeta \partial \zeta} \right) \]

Subtracting the second from the first expression proves the Lemma. \( \square \)

**Lemma C.5.** Let \( J = \{z_2, \ldots, z_b\} \). The linear integral equation (3.2.12) is under Assumption C.1 and with Definition 3.3 equivalent to the expression

\[ 0 = (2\lambda)^3 [\hat{K}_{z_1, \hat{A}^g_{z_1, \ldots, z_b}, J}] G_g(z_1, J \setminus \{z_b\}) + 2\lambda G_0(z_1, z_b) G_g(z_1, J \setminus \{z_b\}) + (2\lambda)^3 \frac{1}{z_b} \frac{\partial}{\partial z_b} G_g(z_1, J \setminus \{z_b\}) \]

**Proof.** With Lemma C.3 we can rewrite the linear integral equation (3.2.12) in the form

\[ 0 = (2\lambda)^{3b-3} [\hat{K}_{z_1, \hat{A}^g_{z_1, \ldots, z_b}, \ldots, \hat{A}^g_{z_1, z_2}, J}] G_g(z_1) + (2\lambda)^{3b-3} \hat{A}^g_{z_1, \ldots, z_b} \hat{K}_{z_1, J} G_g(z_1) \]

\[ + 2\lambda G_g(z_1) G_0(z_1, J) + 2\lambda \sum_{1 \leq |I| < |J|} G_0(z_1, I) G_g(z_1, J \setminus I) \]

\[ + (2\lambda)^3 \frac{1}{z_b} \frac{\partial}{\partial z_b} G_g(z_1, J \setminus \{z_b\}) - G_g(z_1, J \setminus \{z_b\}) \]

(C.1)

By using this formula for \( \hat{A}^g_{z_1, \ldots, z_b, \ldots, \hat{A}^g_{z_1, z_2}, \hat{K}_{z_1, J} G_g(z_1) \) and inserting it back into (C.1) gives

\[ 0 = (2\lambda)^{3b-3} [\hat{K}_{z_1, \hat{A}^g_{z_1, \ldots, z_b}, \ldots, \hat{A}^g_{z_1, z_2}, J}] G_g(z_1) \]

\[ + 2\lambda G_g(z_1) G_0(z_1, J) \]

\[ + 2\lambda \sum_{1 \leq |I| < |J|} G_0(z_1, I) G_g(z_1, J \setminus I) \]

\[ + (2\lambda)^3 \frac{1}{z_b} \frac{\partial}{\partial z_b} G_g(z_1, J \setminus \{z_b\}) \]

\[ - (2\lambda)^6 \hat{A}^g_{z_1, \ldots, z_b, \ldots, \hat{A}^g_{z_1, z_2}, \hat{K}_{z_1, J} G_g(z_1) \]

The second and third line break down to \( 2\lambda G_0(z_1, z_b) G_g(z_1, J \setminus \{z_b\}) \). Therefore, the assertion follows if we can show that, in the fourth line, the part of the sum which excludes \( \zeta = z_b \) cancels with the fifth line. This is true because of

\[ \left[ \hat{A}^g_{z_1, \ldots, z_b}, \frac{1}{z_b} \frac{\partial}{\partial z_b} \right] = 0 \quad \text{and} \quad \hat{A}^g_{z_1, \ldots, z_b} \frac{1}{z_b^2 - \zeta^2} = 0. \]
Consequently, the linear integral equation can be written by operators of the form given in this Lemma.
Appendix D

Perturbative Computations on the Moyal Space

The following calculations confirm the results derived in the thesis. We will show the first non-trivial examples in different dimensions which are already complicated enough. For correlation functions with more boundary components or higher genus, the number of Feynman graphs is too large for an appropriate example. For a more convenient representation, the ribbon graphs will be drawn with lines instead of ribbons.

D.1 Cubic Interaction

We will focus on the planar 1-point function on the Moyal space in different dimensions. The lowest order Feynman graphs with only 3-valent vertices $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5 \in \mathcal{G}_2^{(6,1)}$ are:

\begin{align*}
\Gamma_1 &= \begin{array}{cc}
x & y \\
\end{array} \\
\Gamma_2 &= \begin{array}{cc}
x & y_1 \quad y_2 \\
\end{array} \\
\Gamma_3 &= \begin{array}{cc}
x & y_1 \quad y_2 \\
\end{array} \\
\Gamma_4 &= \begin{array}{cc}
x & y_1 \quad y_2 \\
\end{array} \\
\Gamma_5 &= \begin{array}{cc}
x & y_1 \quad y_2 \\
\end{array}
\end{align*}

where the $\Gamma_i$'s should be understood as ribbon graphs. Applying the Feynman rules $\hbar$ of Sec. 2.1.3, performing the $\Lambda', V$-limit of Sec. 2.3 with the ratio $\frac{\Lambda'^2}{V^{\frac{1}{2}}} = \Lambda^2$ and taking the eigenvalue distribution induced by the $D$-dimensional Moyal space of Sec. 2.5 leads
for the first diagram $\Gamma_1$ to

$$\lim_{N', V \to \infty} \beta(\Gamma_1) = \frac{-\lambda}{1 + 2x} \int_0^{\Lambda^2} \frac{dy \, r(y)}{1 + x + y}, \quad \text{where} \quad r(x) = \frac{x^{D/2-1}}{(\frac{D}{2} - 1)!}.$$ 

To renormalise this kind of graphs we apply the procedure described in Sec. 2.4.1 with Theorem 2.2. The superficial degree of divergence of the unique subgraph $\gamma \in \Gamma_1$ is by formula (2.4.2)

$$\omega(\gamma) = \frac{D}{2} (2 - 0 - 1 - 1 - 1) + \left( \frac{D}{2} - 1 \right) \frac{3 \cdot 1 - 1}{2} = \frac{D}{2} - 1.$$

The graph $\Gamma_1$ has two forests $U_{\Gamma_1} = \emptyset$ and $U_{\Gamma_1} = \gamma$, where $\gamma$ consists of the face labelled by $y$, the attached edge and the vertex. Then Theorem 2.2 leads to the renormalised diagram by

$$-\lambda \int_0^{\Lambda^2} dy \, r(y) \left\{ \frac{1}{(1 + x + y)(1 + 2x)} \right\}_{U_{\Gamma_1} = \emptyset} - \left\{ \frac{1}{1 + x + y} \right\}_{I_{\Gamma_1} \cup U_{\Gamma_1} = I_{\Gamma_1} \backslash \gamma}$$

$$= -\lambda \int_0^{\Lambda^2} dy \, r(y) \left\{ \frac{1}{1 + x + y} - \left( T_x^{D-1} \right) \left( \frac{1}{1 + x + y} \right) \right\}_{U_{\Gamma_1} = \gamma}$$

where $\Lambda^2 \to \infty$ is now finite. This formula matches with our renormalisation conditions (3.1.4), (3.1.5) and (3.1.6). Taking $D = 2, 4, 6$ gives specially

$$D = 2 : \quad -\lambda \int_0^{\infty} dy \left( \frac{1}{1 + x + y} - \frac{1}{1 + y} \right) = \lambda \frac{\log(1 + x)}{1 + 2x}$$

$$D = 4 : \quad -\lambda \int_0^{\infty} dy \left( \frac{1}{1 + x + y} - \frac{1}{1 + y} + \frac{x}{(1 + y)^2} \right) = \lambda \frac{x - (1 + x) \log(1 + x)}{1 + 2x}$$

$$D = 6 : \quad -\lambda \int_0^{\infty} dy \frac{y^2}{2} \left( \frac{1}{1 + x + y} - \frac{1}{1 + y} + \frac{x}{(1 + y)^2} - \frac{x^2}{(1 + y)^3} \right)$$

$$= \lambda \frac{2(1 + x)^2 \log(1 + x) - x(2 + 3x)}{4(1 + 2x)}$$

which coincides for the first order with Example 3.1, Example 3.2 and Example 3.3.

For the over-subtracted form in the $D = 2$ with the renormalisation of $D = 4$ and $D = 6$ respectively, we have

$$-\lambda \int_0^{\infty} dy \left( \frac{1}{1 + x + y} - \frac{1}{1 + y} + \frac{x}{(1 + y)^2} \right) = \lambda \frac{\log(1 + x) - x}{1 + 2x}$$

$$-\lambda \int_0^{\infty} dy \left( \frac{1}{1 + x + y} - \frac{1}{1 + y} + \frac{x}{(1 + y)^2} - \frac{x^2}{(1 + y)^3} \right)$$

$$= \lambda \frac{2 \log(1 + x) - x(2 - x)}{2(1 + 2x)}$$

which coincides for the first order with Example 3.4 and Example 3.5.
Iterated Integrals

The integrals appearing the first time for $\Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5$ are of new complexity. The integration theory of the appearing integrals is completely understood in form of iterated integrals [Bro09]. They form a shuffle algebra, which is symbolically implemented in the Maple package HYPERINT [Pan15].

The hyperlogarithms $H\log$ appearing in HYPERINT are defined by the iterated integrals via

$$H\log(a, [k_1, \ldots, k_n]) := \int_0^{a} \frac{dx_1}{x_1 - k_1} \int_0^{x_1} \frac{dx_2}{x_2 - k_2} \cdots \int_0^{x_{n-1}} \frac{dx_n}{x_n - k_n},$$

where the $k_i$ are called letters. An alternative notation is $H\log(a, [k_1, \ldots, k_n]) = L_{k_1,\ldots,k_n}(a)$. Important special cases are $H\log(a, [-k, \ldots, -k]) = \frac{\log((1 - \frac{a}{k})^n)}{n!}$ for $k \in \mathbb{N}^*$,

$$H\log(a, [0, \ldots, 0]) := \frac{\log(a)^n}{n!}$$

and $H\log(a, [0, \ldots, 0, -1]) = -Li_{1+n}(-a)$.

Renormalised $\Gamma_2$

The graph $\Gamma_2$ has four forests which are $U_{\Gamma_2} = \emptyset$, $U_{\Gamma_2} = \{\gamma_1\}$, $U_{\Gamma_2} = \{\gamma_2\}$ and $U_{\Gamma_2} = \{\gamma_1, \gamma_2\}$, where $\gamma_i$ is the ribbon subgraph with the face labelled by $y_i$ and its attached edges and vertices. Note there is no subgraph which includes both faces since this is not a 1PI graph. We have $g_{\gamma_1} = g_{\gamma_2} = 0$, $b_{\gamma_1} = b_{\gamma_2} = 1$, $v_{\gamma_1} = 2v_{\gamma_2} = 2$, $N_{\gamma_1} = 2N_{\gamma_2} = 2$, and $k_3 = 2$ vertices for $\gamma_1$ and $k_3 = 1$ vertex for $\gamma_2$. The superficial degree of divergence is therefore

$$\omega(\gamma_1) = \frac{D}{2} - 2, \quad \omega(\gamma_2) = \frac{D}{2} - 1.$$ 

The set of variables for the Taylor-subtraction is $f(\gamma_1) = f(\gamma_2) = \{x\}$ and the rational functions $r_{\gamma_1} = \frac{1}{1 + x + y_1}$ and $r_{\gamma_2} = \frac{1}{1 + x + y_2}$. Inserting in Theorem 2.2, writing out the sum over the forests and simplifying yields

$$\frac{(-\lambda)^3}{(1 + 2x)^2} \int_0^\infty dy_1 r(y_1) \left\{ \frac{1}{(1 + x + y_1)^2} - \left( \frac{T_x^{p-2}}{1 + x + y_1} \right) \left( \frac{1}{(1 + x + y_1)^2} \right) \right\} \times \int_0^\infty dy_2 r(y_2) \left\{ \frac{1}{1 + x + y_2} - \left( \frac{T_x^{p-1}}{1 + x + y_2} \right) \left( \frac{1}{1 + x + y_2} \right) \right\}.$$ 

This formula matches with our renormalisation conditions (3.1.4), (3.1.5) and (3.1.6). Plugging in the different dimensions gives

$$D = 2 : \quad \lambda^3 \frac{\log(1 + x)}{(1 + 2x)^2(1 + x)}$$

$$D = 4 : \quad \lambda^3 \frac{(1 + x)^3}{(1 + 2x)^2}$$

$$D = 6 : \quad \lambda^3 \frac{2(1 + x)^3 \log(1 + x)^2 - x(1 + x)(5x + 4) \log(1 + x) + x^2(3x + 2)}{4(1 + 2x)^2}$$
and for the over-subtracted from in $D = 2$ with

- renormalisation of $D = 4$:
  \[
  \lambda^3 \frac{x(x - \log(1 + x))}{(1 + 2x)^3(1 + x)}
  \]
- renormalisation of $D = 6$:
  \[
  \lambda^3 \frac{x^2(2 \log(1 + x) + x(x - 2))}{2(1 + x)(1 + 2x)^2}
  \]

**Renormalised $\Gamma_3$**

The graph $\Gamma_3$ is renormalised in the same way as $\Gamma_1$ for each subgraph, respectively. The general result is

\[
\left(-\lambda\right)^3 \int_0^\infty dy_1 r(y_1) \left\{ \frac{1}{1 + x + y_1} - \left(T_{x}^{D-1}\right) \left(\frac{1}{1 + x + y_1}\right) \right\} \times \int_0^\infty dy_2 r(y_2) \left\{ \frac{1}{1 + x + y_2} - \left(T_{x}^{D-1}\right) \left(\frac{1}{1 + x + y_2}\right) \right\}.
\]

This formula matches with our renormalisation conditions \((3.1.4), \ (3.1.5)\) and \((3.1.6)\). Plugging the different dimensions in gives

- $D = 2$:
  \[-\lambda^3 \frac{\log(1 + x)^2}{(1 + 2x)^3}\]
- $D = 4$:
  \[-\lambda^3 \frac{(x - (1 + x) \log(1 + x))^2}{(1 + 2x)^3}\]
- $D = 6$:
  \[-\lambda^3 \frac{(2(1 + x)^2 \log(1 + x) - x(2 + 3x))^2}{16(1 + 2x)^3}\]

and for the over-subtracted from in $D = 2$ with

- renormalisation of $D = 4$:
  \[-\lambda^3 \frac{(\log(1 + x) - x)^2}{(1 + 2x)^3}\]
- renormalisation of $D = 6$:
  \[-\lambda^3 \frac{(2 \log(1 + x) - x(2 - x))^2}{4(1 + 2x)^3}\]

**Renormalised $\Gamma_4$**

The graph $\Gamma_4$ has six forests which are $\mathcal{U}_{\Gamma_4} = \emptyset$, $\mathcal{U}_{\Gamma_4} = \{\gamma_1\}$, $\mathcal{U}_{\Gamma_4} = \{\gamma_2\}$, $\mathcal{U}_{\Gamma_4} = \{\gamma_{12}\}$, $\mathcal{U}_{\Gamma_4} = \{\gamma_{12}, \gamma_1\}$ and $\mathcal{U}_{\Gamma_4} = \{\gamma_{12}, \gamma_2\}$, where $\gamma_{12}$ is the ribbon subgraph with the face labelled by $y_1$ and $y_2$ together with its attached edges and vertices. The superficial degrees of divergence is therefore

\[
\omega(\gamma_1) = \frac{D}{2} - 4, \quad \omega(\gamma_2) = \frac{D}{2} - 1, \quad \omega(\gamma_{12}) = D - 4.
\]

The set of variables for the Taylor-subtraction is $f(\gamma_1) = f(\gamma_{12}) = \{x\}$ and $f(\gamma_2) = \{y_1\}$. The rational functions are $r_{\gamma_1} = r_{\gamma_{12}} = \frac{1}{1 + x + y_1} \frac{1}{(1 + 2y_1)(1 + y_1 + y_2)}$ and $r_{\gamma_2} = \frac{1}{1 + y_1 + y_2}$. Note that $\gamma_{12}$ has the subgraph $\gamma_2 = o(\gamma_{12})$ or $\gamma_1 = o(\gamma_{12})$, but then $\gamma_{12} \setminus \gamma_1$ is only a face which does not count. We will neglect the forest $\mathcal{U}_{\Gamma_4} = \{\gamma_1\}$ since the degree of divergence
matters for dimensions $D \geq 8$. Inserting in Theorem 2.2 writing out the sum over the forests and simplifying yields
\[
\frac{(-\lambda)^3}{1 + 2x} \int_0^\infty dy_1 \frac{T_x^{D-4}}{(1 + x + y_1)^2} \left\{ \frac{1}{(1 + x + y_1)^2} - \left( T_{y_1}^{D-1} \right) \left( \frac{1}{1 + y_1 + y_2} \right) \right\}.
\]

Since the normalisation conditions (3.1.4), (3.1.5) and (3.1.6) imply a different subtraction, we see here the first time an adaption of the forest formula in the first line by changing the degree of divergence
\[
\frac{(-\lambda)^3}{1 + 2x} \int_0^\infty dy_1 \frac{T_x^{D-4}}{(1 + x + y_1)^2} \left\{ \frac{1}{(1 + x + y_1)^2} - \left( T_{y_1}^{D-1} \right) \left( \frac{1}{1 + y_1 + y_2} \right) \right\} \times \frac{1}{1 + 2y_1} \int_0^\infty dy_2 \frac{T_x^{D-4}}{(1 + y_1 + y_2)^2} \left\{ \frac{1}{(1 + y_1 + y_2)^2} - \left( T_{y_2}^{D-1} \right) \left( \frac{1}{1 + y_1 + y_2} \right) \right\}.
\]

which corresponds to an appropriate subtraction of the $\gamma_{12}$ graph. Note that the $D = 6$ case is unchanged.

Plugging in the different dimensions (D.1) and using HYPERINT gives
\[
\begin{align*}
D = 2 : & \quad \lambda^3 \left( \frac{2\text{Hlog}(x, [0, -1]) + 4x(1 + x)(\log(2)^2 - \zeta_2 + 1) + 1}{(1 + 2x)^3} - \frac{2\log(1 + x)}{x(1 + 2x)^2} \right) \\
D = 4 : & \quad \lambda^3 \left( \frac{(1 + 2x + 2x^2)\text{Hlog}(x, [0, -1]) - x^2(4x + 3)(\log(2)^2 - 2\log 2 - \zeta_2 + 3)}{(1 + 2x)^3} \right) \\
& \quad + \left( \frac{1 + x}{1 + 2x} \right) \log(1 + x) - x(x + 2) \\
D = 6 : & \quad \lambda^3 \left( \frac{-2x(1 + x)(1 + 3x + 3x^2)\text{Hlog}(x, [0, -1]) + 2x^4 - 15x^3 - 16x^2 - 4x}{4(1 + 2x)^3} \\
& \quad + \left( \frac{1 + x}{1 + 2x} \right)^2(2 + 7x + 7x^2) \log(1 + x) + 2x^3(3x + 2)(\log(2)^2 - \log 2 - \zeta_2) \right) / 8(1 + 2x)^3
\end{align*}
\]

and for the over-subtracted from in $D = 2$ with
\[
\begin{align*}
\text{renormalisation of } D = 4 : & \quad \lambda^3 \left( \frac{2\text{Hlog}(x, [0, -1])}{(1 + 2x)^3} - \frac{(1 + x)\log(1 + x)}{x(1 + 2x)^3} \right) \\
& \quad - \frac{8x^2(4x + 3)(\log(2)^2 - \log 2 - \zeta_2) + 60x^3 + 44x^2 + 3x - 2}{2(1 + 2x)^3} \\
\text{renormalisation of } D = 6 : & \quad \lambda^3 \left( \frac{2\text{Hlog}(x, [0, -1])}{(1 + 2x)^3} - \frac{(1 + x + x^2 + x^3)\log(1 + x)}{x(1 + 2x)^3} \right) \\
& \quad + \frac{24x^3(3x + 2)(4\log(2)^2 - 5\log 2 - 4\zeta_2) + 584x^4 + 392x^3 + 8x^2 - 9x + 6}{6(1 + 2x)^3}
\end{align*}
\]

**Renormalised $\Gamma_5$**

The graph $\Gamma_5$ has six forests which are $U_{\Gamma_5} = \emptyset$, $U_{\Gamma_5} = \{\gamma_1\}$, $U_{\Gamma_5} = \{\gamma_2\}$, $U_{\Gamma_5} = \{\gamma_{12}\}$, $U_{\Gamma_5} = \{\gamma_{12}, \gamma_1\}$ and $U_{\Gamma_5} = \{\gamma_{12}, \gamma_2\}$ defined as above. For this graph the overlapping divergences need the forest formula in its full generality. The superficial degrees of divergence
The set of variables for the Taylor-subtraction is \( f = \{x, y\} \) and \( f(\gamma_2) = \{x\} \). The rational functions are \( r_1 = (1 + x + y^2) (1 + y_1 + y_2) \) and \( r_2 = (1 + y_1 + y_2) (1 + x + y_2) \). Inserting in Theorem 2.2 writing out the sum over the forests and simplifying yields

\[
\frac{(-\lambda)^3}{1 + 2x} \int_0^\infty dy_1 r(y_1) \int_0^\infty dy_2 r(y_2) \times \left\{ \frac{1}{(1 + y_1 + y_2)} \left( 1 - \left( T^{D-4}_x \right) \right) \frac{1}{(1 + x + y_1)^2 (1 + x + y_2)} \right\} \\
+ \left\{ \left( 1 - \left( T^{D-4}_x \right) \right) \frac{1}{(1 + x + y_1)^2} \left( - T^{D-3}_{x, y_2} \right) \frac{1}{(1 + y_1 + y_2)(1 + x + y_2)} \right\} \\
+ \left\{ \left( 1 - \left( T^{D-4}_x \right) \right) \frac{1}{1 + x + y_2} \left( - T^{D-3}_{x, y_2} \right) \frac{1}{(1 + y_1 + y_2)(1 + x + y_2)} \right\}.
\]

Notice that in the last and the second last line \( T^{D-4}_x \) acts also on the other Taylor polynomial recursively, which depends on \( x \). Since the normalisation conditions (3.1.4), (3.1.5) and (3.1.6) implies a stronger subtraction, we adapt the forest formula by changing the degree of divergence again for the \( \gamma_1 \) graph to

\[
\frac{(-\lambda)^3}{1 + 2x} \int_0^\infty dy_1 r(y_1) \int_0^\infty dy_2 r(y_2) \times \left\{ \frac{1}{(1 + y_1 + y_2)} \left( 1 - \left( T^{D-1}_x \right) \right) \frac{1}{(1 + x + y_1)^2 (1 + x + y_2)} \right\} \\
+ \left\{ \left( 1 - \left( T^{D-4}_x \right) \right) \frac{1}{(1 + x + y_1)^2} \left( - T^{D-3}_{x, y_2} \right) \frac{1}{(1 + y_1 + y_2)(1 + x + y_2)} \right\} \\
+ \left\{ \left( 1 - \left( T^{D-4}_x \right) \right) \frac{1}{1 + x + y_2} \left( - T^{D-3}_{x, y_2} \right) \frac{1}{(1 + y_1 + y_2)(1 + x + y_2)} \right\}.
\]

Plugging in the different dimensions and using HYPERINT gives

\[
D = 2 : \quad \lambda^3 \left( \log(1 + x)^2 - 2 H \log(x, [0, -1]) + 4 x (1 + x) (\zeta_2 - 1) - 1 \right) \frac{1}{(1 + 2x)^3} \\
+ \frac{\log(1 + x)}{x(1 + x)(1 + 2x)^2}
\]

\[
D = 4 : \quad \lambda^3 \left( - x (1 + x) \log(1 + x)^2 - (1 + 2x + 2x^2) H \log(x, [0, -1]) \right) \frac{1}{(1 + 2x)^3} \\
+ \frac{-x^2(4x + 3)\zeta_2 + 8x^3 + 8x^2 + 2x - \log(1 + x)}{(1 + 2x)^3}
\]

\[
D = 6 : \quad \lambda^3 \left( - (1 + 3x)(1 + x)^3 \log(1 + x)^2 + x2(1 + x)(1 + 3x + 3x^2) H \log(x, [0, -1]) \right) \frac{1}{(1 + 2x)^3} \\
+ \frac{x^3(3x + 2)\zeta_2 - 4x^4 + 4x^3 + 7x^2 + 2x - (1 + x)(3x + 2) \log(1 + x)}{4(1 + 2x)^3}
\]
and for the over-subtracted from in \( D = 2 \) with

\[
\text{renormalisation of } D = 4 : \quad \lambda^3 \left( \frac{\log(1+x)^2 - 2\text{Hlog}(x, [0, -1])}{(1+2x)^3} \right) + \frac{\log(1+x)}{x(1+2x)^2(1+x)} - \frac{8x^2\zeta_2(4x + 3)(1+x) + 52x^4 + 88x^3 + 41x^2 + x - 2}{2(1+2x)^3(1+x)}
\]

\[
\text{renormalisation of } D = 6 : \quad \lambda^3 \left( \frac{\log(1+x)^2 - 2\text{Hlog}(x, [0, -1])}{(1+2x)^3} \right) + \frac{\log(1+x)}{x(1+2x)^2(1+x)} + \frac{96x^3\zeta_2(3x + 2)(1+x) - 476x^5 - 784x^4 - 319x^3 + 7x^2 + 3x - 6}{6(1+2x)^3(1+x)}.
\]

**Sum of \( \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5 \)**

Taking the sum of the results of all diagrams of order \( \lambda^3 \) breaks down to

\[
D = 2 : \quad \lambda^3 \left( \frac{2 \log 2}{(1+2x)^3} (1+x) \right)
\]

\[
D = 4 : \quad - \lambda^3 \left( \frac{1 - \log 2}{(1+2x)^3} (4x + 3)x^2 \right)
\]

\[
D = 6 : \quad \lambda^3 \frac{x^3(2 + 3x)(2 \log 2 - 1)^2}{16(1+2x)^3}
\]

and for the over-subtracted from in \( D = 2 \) with

\[
\text{renormalisation of } D = 4 : \quad - \lambda^3 \frac{x^2(3 + 4x)(2 \log 2 - 1)^2}{(1+2x)^3}
\]

\[
\text{renormalisation of } D = 6 : \quad \lambda^3 \frac{x^3(2 + 3x)(8 \log 2 - 5)^2}{4(1+2x)^3}
\]

which confirms Example 3.1, Example 3.2, Example 3.3, Example 3.4 and Example 3.5.

We emphasise that for the final results all hyperlogarithms cancels perfectly.

### D.2 Quartic Interaction

For the \( D = 4 \) Moyal space, the 2-point function possesses at order \( \lambda \) two graphs and at order \( \lambda^2 \) nine graphs. The exact result for the 2-point function given in Sec. 4.2.4 has a natural choice for the boundary conditions \( G(0, 0) = 1 \) also holds for the exact solution. From (4.2.67), the first orders for the derivative are \( \frac{\partial}{\partial a} G(0, a) \mid_{a=0} = -1 - \lambda + \lambda^2 + O(\lambda^3) \). On the other hand, Zimmermann’s forest formula demands \( \frac{\partial}{\partial a} G(0, a) \mid_{a=0} = -1 \), which cannot coincide with the exact solution without further effort. Assuming a general \( \mu^2 \) in Theorem 4.3 and Proposition 4.7 and computing recursively order by order \( R_4, R_4^{-1} \) and \( I(w) \), this is done in App. E.1.2. The angle function \( \tau_b(a) \) has now a general boundary condition.
condition can then be fixed by inserting the angle function into (4.2.6), where \( Z \) and \( \mu^2 \) are chosen to satisfy \( G(0,0) = 1 \) and \( \frac{\partial}{\partial a} G^{(0)}(a,0)|_{a=0} = -1 \).

The first order is the same as usual
\[
\frac{1}{1 + a + b}
\]
With \( \mu^2 = 1 + \lambda \mu_1^2 + \lambda \mu_2^2 + \ldots \) and \( Z \) = \( C_\lambda \cdot e^{H_0[\tau]} \) with \( C_\lambda = 1 + \lambda k_1 + \lambda^2 k_2 + \ldots < \infty \), it follows for the next order
\[
\lambda \left( \frac{k_1}{1 + a + b} - \frac{\mu_1^2 + 1}{(1 + a + b)^2} \right) \left( 1 + a \right) \log(1 + a) + (1 + b) \log(1 + b) \right) \left( 1 + a + b \right)^2.
\]
The boundary conditions are achieved with \( k_1 = 1 \) and \( \mu_1^2 = 0 \), which leads to
\[
\lambda \left( \frac{a + b}{(1 + a + b)^2} - \frac{1 + a \log(1 + a) + (1 + b) \log(1 + b)}{(1 + a + b)^2} \right).
\]
(D.1)
The second order is also straightforward to compute, which is with \( k_1 = 1 \) and \( \mu_1^2 = 0 \)
\[
\frac{\lambda^2}{(1 + a + b)^2} \left[ \zeta_2 (1 + a + b + ab) - \mu_2^2 (1 + a + b) + (a + b)^2 - (1 + a + b) + k_2 (1 + a + b)^2 \right.
\]
\[
+ (1 + a) (1 + b) \log(1 + a) \log(1 + b) - a (1 + b) \log(1 + b)^2 - b (1 + a) \log(1 + a)^2
\]
\[
- (1 + b + 2 a + 2 a b + a^2) \text{Li}_2(-a) - (1 + a + 2 b + 2 a b + b^2) \text{Li}_2(-b)
\]
\[
+ ((1 + a - b) - (1 + a) (1 + a + b)) \log(1 + a)
\]
\[
+ ((1 + b - a) - (1 + b) (1 + a + b)) \log(1 + b) \left].
\]
(D.2)
The boundary conditions are fulfilled with \( \mu_2^2 = \zeta_2 - 2 \) and \( k_2 = -1 \). We see that it now coincides with the graph expansion below (D.10).

**Graph Expansion**

The graphs up to second order are the following:
We will determine the expressions for the graphs analogously to previous section. Only the last graph, the sunrise diagram, needs more discussions through Zimmermann’s forest formula.

For the two graphs at order $\lambda$, the forest formula gives

\[
-\frac{\lambda}{(1 + a + b)^2} \int_0^\infty y dy \left( \frac{1}{1 + a + y} + \frac{1}{1 + b + y} - \frac{2}{1 + y} + \frac{a}{(1 + y)^2} + \frac{b}{(1 + y)^2} \right)
\]

\[
= -\lambda \frac{(1 + a) \log(1 + a) + (1 + b) \log(1 + b) - (a + b)}{(1 + a + b)^2}
\]

which coincides with (D.1).

**Graphs of the Second Order in $\lambda$**

The first four graphs are computed as the graphs of order $\lambda^1$ which leads to the four results

\[
\lambda^2 \frac{(1 + a) \log(1 + a) - a)^2}{(1 + a + b)^3}
\] (D.3)

\[
\lambda^2 \frac{(1 + b) \log(1 + b) - b)^2}{(1 + a + b)^3}
\] (D.4)

\[
\lambda^2 \frac{(1 + b) \log(1 + b) - b)((1 + a) \log(1 + a) - a)}{(1 + a + b)^3}
\] (D.5)

\[
\lambda^2 \frac{(1 + a) \log(1 + a) - a)((1 + b) \log(1 + b) - b)}{(1 + a + b)^3}
\] (D.6)

The two graphs which have the second loop on top have the forests $\emptyset, \{\gamma_1\}, \{\gamma_2\}$ and $\{\gamma_1, \gamma_2\}$ according to the notation of the previous section. The subgraph $\{\gamma_{12}\}$ is not possible since it becomes disjoint after removing the vertex. The result of these two graphs is therefore easily computed to

\[
\lambda^2 a \log(1 + a) + b \log(1 + b) - (1 + a) \log(1 + a)^2 - (1 + b) \log(1 + b)^2
\] (D.7)

For the two graphs where the second loop is inside the first, Zimmermann’s forest formula leads analogously to the discussions of the previous section of the graph $\Gamma_4$ to

\[
-\lambda^2 (1 + 2a) \text{Li}_2(-a) + (1 + 2b) \text{Li}_2(-b) + (1 + a) \log(1 + a) + (1 + b) \log(1 + b)
\] (D.8)

The sunrise graph needs Zimmermann’s forest formula in its full beauty. Using the six forests $\emptyset, \{\gamma_1\}, \{\gamma_2\}, \{\gamma_{12}\}, \{\gamma_1, \gamma_2\}$ and $\{\gamma_{12}, \gamma_2\}$, inserting it in the forest formula and counting each degree of divergence leads after simplifying to

\[
\frac{\lambda^2}{(1 + a + b)^2} \int_0^\infty \int_0^\infty y_1 dy_1 y_2 dy_2
\]

\[
\times \left[ \left\{ \frac{\lambda_1}{(1 + y_1 + y_2) \left( 1 - \left( T_{a,b}^1 \right) \right) (1 + a + y_1)(1 + b + y_2) } \right\} + \left\{ 1 - \left( T_{a,b}^1 \right) \right\} \frac{1}{1 + b + y_2} \left( - T_{a,y_2}^0 \right) \frac{1}{(1 + y_1 + y_2)(1 + a + y_1)} \right]
\]

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\[
\frac{\lambda^2}{(1 + a + b)^3} \left\{ \zeta_2 ab - a - b + (1 + a)(1 + b) \log(1 + a) \log(1 + b) \\
- a(1 + b) \log(1 + b)^2 - b(1 + a) \log(1 + a)^2 \\
- (1 + b + 2a + 2ab + a^2) \text{Li}_2(-a) - (1 + a + 2b + 2ab + b^2) \text{Li}_2(-b) \\
- (a + 2b + a^2 + ab) \log(1 + a) - (b + 2a + b^2 + ab) \log(1 + b) \right\} \tag{D.11}
\]

Adjusting the boundary conditions for the exact solution in the right way, leads to the same results compared to (D.2). The natural choice for the \( \mu^2 \) in (4.2.67) coincides with graphs expansions for the hyperlogarithms with two letters. The hyperlogarithms with less letters, e.g. \( \log(1 + a) \) and \( \log(1 + b) \), are affected by the boundary conditions.
Appendix E

Analysis of the Fredholm Equation and a Second Proof of Proposition 4.7∗

E.1 Perturbative Analysis

E.1.1 Direct Expansion

Expanding equation (4.2.8) with the renormalisation motivated by Taylor-subtraction
\[ \mu_{\text{bare}}^2 = 1 - \lambda \Lambda^2 - \frac{1}{\pi} \int_0^{\Lambda^2} dt \tau_0(t). \]
and finite cut-off gives
\[ p\lambda\pi \cot(\tau_a(p)) = 1 + a + p + \lambda p \log \left( \frac{\Lambda^2 - p}{p} \right) + \frac{1}{\pi} \int_0^{\Lambda^2} dt (\tau_p(t) - \tau_0(t)). \] (E.1)

The first order is read out directly
\[ p\lambda\pi \cot(\tau_a(p)) = 1 + a + p + O(\lambda^1) \quad \Rightarrow \quad \tau_a(p) = \frac{p\lambda\pi}{1 + a + p} + O(\lambda^2), \]
which gives after inserting back at the next order
\[ p\lambda\pi \cot(\tau_a(p)) = 1 + a + p + \lambda \left( (1 + p) \log(1 + p) - p \log(p) + p \log \left( \frac{\Lambda^2 - p}{1 + p + \Lambda^2} \right) + \log \left( \frac{1 + \Lambda^2}{1 + p + \Lambda^2} \right) \right) + O(\lambda^2). \]

The limit \( \Lambda^2 \to \infty \) gives finite results for \( \cot(\tau_a(p)) \) as well as for \( \tau_a(p) \) order by order, however the limit has to be taken with caution. Integral and limit do not commute. Namely, for and expansion \( \tau_a(p) = \sum_{n=1}^{\infty} \lambda^n \tau_a^{(n)}(p) \) we have
\[ \lim_{\Lambda^2 \to \infty} \int_0^{\Lambda^2} dt \left( \tau_p^{(n)}(t) - \tau_0^{(n)}(t) \right) \neq \int_0^{\infty} dt \lim_{\Lambda^2 \to \infty} \left( \tau_p^{(n)}(t) - \tau_0^{(n)}(t) \right), \quad n > 1. \]

*This is taken from our paper [GHW20]
As an example we will look at the next order of both integrals. They give

\[
\lim_{\Lambda^2 \to \infty} \frac{1}{\pi} \int_0^{\Lambda^2} dt \left( \tau_p^{(2)}(t) - \tau_0^{(2)}(t) \right)
= (1 + p) \log(1 + p)^2 + (1 + 2p)\text{Li}_2(-p) - p\zeta_2,
\]

\[
\frac{1}{\pi} \int_0^{\infty} dt \lim_{\Lambda^2 \to \infty} \left( \tau_p^{(2)}(t) - \tau_0^{(2)}(t) \right)
= \int_0^{\infty} dt \left( \frac{t \log(t) - (1 + t) \log(1 + t)}{(1 + t)^2} - \frac{t \log(t) - (1 + t) \log(1 + t)}{(1 + t)^2} \right)
= (1 + p) \log(1 + p)^2 + (1 + 2p)\text{Li}_2(-p) + 2p\zeta_2,
\]

respectively, where Li_n(x) is the n-th polylogarithm and \( \zeta_n \equiv \zeta(n) \) is the Riemann zeta value at integer n. The last term makes the difference. Taking the "wrong" second result and plugging it back into \( (E.1) \) would lead to divergences at the next order. Consequently, we have to treat the perturbative expansion of \( (E.1) \) with a finite cut-off \( \Lambda^2 \) at all orders, where each order has a finite limit.

We computed the first 6 orders via HyperInt described in \[\text{App. D}\] for finite \( \Lambda^2 \). Sending \( \Lambda^2 \to \infty \) is well-defined at any order as expected. The first orders read explicitly

\[
\lim_{\Lambda^2 \to \infty} p\lambda \pi \cot(\tau_a(p)) = 1 + a + p + \lambda ((1 + p) \log(1 + p) - p \log(p))
+ \lambda^2 (-p\zeta_2 + (1 + p) \log(1 + p)^2 + (1 + 2p)\text{Li}_2(-p))
+ \lambda^3 (\zeta_2 \log(1 + p) - \frac{1 + p}{2p} \log(1 + p)^2 + (1 + p) \log(1 + p)^3
+ 2p\zeta_3 - 2p\text{Li}_3(-p) - (1 + 2p)\text{Hlog}(p, [-1, 0, -1])
- 2(2 + 3p)\text{Hlog}(p, [0, -1, -1]) + O(\lambda^4).
\]

The definition of the hyperlogarithms Hlog is given by iterated integrals and can be found in \[\text{App. D}\].

The perturbative expansion shows that the branch point at \( p = -1 \) plays an important role. Its boundary value is found to be \( \lim_{\Lambda^2 \to \infty} \cot(\tau_0(-1+i\varepsilon)) = -i + O(\lambda^7) \). It is natural to conjecture that it holds at any order,

\[
\lim_{\Lambda^2 \to \infty} \cot(\tau_0(-1+i\varepsilon)) = -i.
\]

The perturbative expansion with a finite cut-off \( \Lambda^2 \) is quite inefficient. The boundary value \( (E.3) \) admits a more efficient strategy. We take the derivative of \( (E.1) \) with respect to \( p \):

\[
1 + \lambda \log \left( \frac{\Lambda^2 - p}{p} \right) - \lambda \frac{\Lambda^2}{\Lambda^2 - p} + \frac{1}{\pi} \int_0^{\Lambda^2} dt \frac{d\tau_p(t)}{dp} = \lambda \pi \cot(\tau_a(p)) + p\lambda \pi \frac{\partial}{\partial p} \cot(\tau_a(p)).
\]

Multiplying this equation by \( p \) and subtracting it from \( (E.1) \) again leads to

\[
-p^2 \lambda \pi \frac{\partial}{\partial p} \cot(\tau_a(p)) = 1 + a + \lambda \frac{p\Lambda^2}{\Lambda^2 - p} + \frac{1}{\pi} \int_0^{\Lambda^2} dt \left( \tau_p(t) - \tau_0(t) - p \frac{d\tau_p(t)}{dp} \right),
\]
where the limit $\Lambda^2 \to \infty$ is now safe from the beginning and commutes with the integral. We divide (E.4) by $-p^2$ and integrate it for all orders higher than $\lambda^1$ over $p$ from $-1$ (here (E.3) is assumed) up to some $q$ to get $\lim_{\Lambda^2 \to \infty} \lambda \pi \cot(\tau_a(q))$ on the lhs. On the rhs the order of integrals $\int_{-1}^q dp \int_0^\infty dt$ can be exchanged. The integral over $p$ is

$$\int_{-1}^q dp \frac{1}{p^2} \left( \tau_p(t) - \tau_0(t) - \frac{d\tau_p(t)}{dp} \right),$$

(E.5)

assuming Hölder continuity of $\tau_p(t)$ so that the integral splits after taking principal values. The last term is computed for small $\epsilon$ and all $O(\lambda^{-1})$-contributions via integration by parts

$$\int_{[-1,q]([-\epsilon,\epsilon)}) \frac{d\tau_p(t)}{p} = \frac{\tau_p(t)}{p} \bigg|_{p=\epsilon} - \frac{\frac{\tau_p(t)}{p}}{p} \bigg|_{p=-1} + \int_{[-1,q][-\epsilon,\epsilon)} \frac{d\tau_p(t)}{p^2}$$

$$= \frac{\tau_0(t)}{q} + \tau_0(t) + \int_{[-1,q][-\epsilon,\epsilon)} \frac{d\tau_p(t)}{p^2} - \frac{\tau_0(t) + \tau_0(t)}{\epsilon}.$$  
(E.6)

The first term in (E.5) cancels. The second term in (E.6) integrates to a boundary term $+2\frac{\tau_0(t)}{\epsilon}$, which is also cancelled by the last term of (E.6). Multiplying by $q$ and including the special $O(\lambda)$-contribution we arrive in the limit $\Lambda^2 \to \infty$ where (E.2) is (conjecturally) available at

$$q \lambda \pi \cot(\tau_a(q)) = 1 + a + q - \lambda q \log(q) + \frac{1}{\pi} \int_0^\infty dt \left( \frac{\tau_0(t)}{q} - (1 + q)\tau_0(t) + q \tau_0(t) \right).$$

(E.7)

This equation is much more appropriate for perturbation theory because the number of terms is reduced tremendously order by order. Obviously, the first six order coincide with the earlier but much harder perturbative expansion of (E.1).

Using (E.7) the perturbative expansion is increased up to $\lambda^0$ with HYPERINT. As consistency check of assumption (E.3) we inserted the next orders $\tau_a^{(n)}(p)$ into (4.2.6) to get the expansion $G(a, b) = \sum_{n=0}^{\infty} \lambda^n G^{(n)}(a, b)$ which confirmed the symmetry $G^{(n)}(a, b) = G^{(n)}(b, a)$ which would easily be lost by wrong assumptions. We are thus convinced to have the correct expressions for $\tau_a^{(n)}(p)$ for $6 < n < 10$.

### E.1.2 Expansion of the Fredholm Equation

To access the angle function $\tau_a(p)$ we first have to determine the expansion of the deformed measure $g_\lambda(x) = R_\lambda(x)$ through the Fredholm equation (4.2.58). The constant $\mu^2(\lambda)$ is not yet fixed and needs a further expansion

$$\mu^2 = \sum_{n=0}^{\infty} \lambda^n \mu_n^2.$$  

First orders of the deformed measure are given iteratively through (4.2.58)

$$g_\lambda(x) = x - \lambda((x + \mu_0^2)Hlog(x, [-\mu_0^2]) - x)$$

$$- \frac{\lambda^2}{\mu_0^2}(-\mu_0^2 x Hlog(x, [0, -\mu_0^2]) + \mu_0^2(\mu_1^2 + \mu_0^2 + x)Hlog(x, [-\mu_0^2]) - x(\mu_1^2 + \mu_0^2))$$

$$+ O(\lambda^3).$$

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Recall that the inverse of $R_4(x) = g_4(x) = p$ exists for all $p \in \mathbb{R}^+$ in case $\lambda < 0$. If $g_4(x)$ had the same asymptotics as $g_0(x) = x$ then $R_4^{-1}$ could not be defined globally for $\lambda > 0$. We proved in Sec. 4.2.4 that the asymptotics of $g_4(x)$ is altered in such a way that $R_4^{-1}$ is defined. Anyway, in each order of perturbative expansion the inverse $R_4^{-1}$ is globally defined on $\mathbb{R}^+$. At this point it suffices to assume that $R_4^{-1}(p)$ is a formal power series in $\lambda$, which is achieved by (4.2.58)

$$
R_4^{-1}(p) = p + \lambda (R_4^{-1}(p))^2 \int_0^\infty \frac{dt}{(t + \mu^2)^2(t + \mu^2 + R_4^{-1}(p))}.
$$

Expanding $g_4(t)$ and $\mu^2$, the first orders are

$$
R_4^{-1}(p) = p - \lambda (p - (\mu_0^2 + p) \log(p, [-\mu_0^2]))
- \frac{\lambda^2}{\mu_0^2}(p \mu_0^2 \log(p, [0, -\mu_0^2]) - 2 \mu_0^2 (p + \mu_0^2) \log(p, [-\mu_0^2, -\mu_0^2])
- \mu_0^2 (p^2 + \mu_0^2) \log(p, [-\mu_0^2]) + p (\mu_0^2 + \mu_0^2)) + O(\lambda^3).
$$

The last step is to determine $\lim_{\epsilon \to 0} \text{Re} I(p + i \epsilon) = p \lambda \pi \cot(\tau_0(p))$ for $\Lambda^2 \to \infty$ via

$$
I(z) = \mu^2 + R_4^{-1}(z) + \lambda (\mu^2 + R_4^{-1}(z))^2 \int_0^\infty \frac{dt}{(t + \mu^2)^2(t - R_4^{-1}(z))},
$$

as a formal series. The first few orders are

$$
\lim_{\epsilon \to 0} I(p + i \epsilon)
= \mu_0^2 + p + \lambda (i \pi p + \mu_0^2 + \mu_1^2 + (\mu_0^2 + p) \log(p, [-\mu_0^2]) + p \log(\mu_0^2) - p \log(p))
+ \lambda^2 (\mu_0^2 (1 - \zeta_2) + \mu_1^2 + \mu_2 - \pi \zeta_2 + (\mu_0^2 + \mu_1^2) \log(p, [-\mu_0^2])
+ 2 (\mu_0^2 + p) \log(p, [-\mu_0^2, -\mu_0^2]) - (\mu_0^2 + 2 p) \log(p, [0, -\mu_0^2])) + O(\lambda^3).
$$

Comparing it with (E.2) through equation (4.2.13) fixes every $\mu_1^2$ uniquely and confirms

$$
\lim_{\epsilon \to 0} I(p + i \epsilon) = \lambda \pi p \cot(\tau_0(p)) + i \lambda \pi p.
$$

Furthermore, the first 10 orders are identical with the expansion of (E.7), provided that the $\mu_1^2$’s are fixed to

$$
\mu^2 = 1 - \lambda + \frac{1}{6} (\pi \lambda)^2 - \lambda \frac{1}{3} (\pi \lambda)^2 + \frac{3}{40} (\pi \lambda)^4 - \lambda \frac{8}{45} (\pi \lambda)^4 + \frac{5}{112} (\pi \lambda)^6 - \lambda \frac{4}{35} (\pi \lambda)^6
+ \frac{35}{1152} (\pi \lambda)^8 - \lambda \frac{128}{1575} (\pi \lambda)^8 + \frac{63}{2816} (\pi \lambda)^10 + O(\lambda^{11}). \tag{E.8}
$$

The conjectured behavior of $\cot(\tau_0(p))$ at $p = -1 + i \epsilon$ in the previous subsection (E.3) is now equivalent to

$$
\lim_{\epsilon \to 0} I(-1 + i \epsilon) = 0 \quad \Rightarrow \quad R_4^{-1}(-1) = -\mu^2.
$$

We find that the expansion (E.8) of $\mu^2$ obeys an unexpected boundary condition

$$
\int_0^\infty \frac{dt}{(\mu^2 + t)^3} = \frac{1}{2} + O(\lambda^{10}). \tag{E.9}
$$
For further study we pass as in Sec. 4.2.4 to the rescaled measure \( \phi(x) = \mu^2 \tilde{g}_\lambda(\mu^2 x) := \frac{\rho(x)}{\mu^2 x(1 + x)} \). The pattern of coefficients of the \( \mu^2 \)-expansion in (E.8) suggests to distinguish between even an odd powers in \( \lambda \). The even powers \( \lambda^{2n} \) are given by the formula

\[
\frac{(2n-1)!!}{(2n)!!(2n+1)} = \frac{(2n)!}{4^n n!(2n+1)},
\]

and the odd powers \( \lambda^{2n+1} \) by

\[
\frac{2(2n)!!}{(2n+1)!!(2n+2)} = \frac{2}{(2n+2)!}.
\]

Both series are convergent for \(|\lambda| < \frac{1}{\pi}\) with the result (up to order \( \lambda^{10} \))

\[
\mu^2 = \frac{\arcsin(\lambda\pi)}{\lambda\pi} - \lambda \left( \frac{\arcsin(\lambda\pi)}{\lambda\pi} \right)^2.
\]

This result suggests that \( \frac{\arcsin(\lambda\pi)}{\lambda\pi} \) is a better expansion parameter than \( \lambda \) itself. The factors \( \pi^{2n} \) are produced by \( \zeta_{2n} \) in the iterated integrals. We thus reorganise the perturbative solution of (4.2.61) into a series in \( \frac{\arcsin(\lambda\pi)}{\lambda\pi} \). The power of \( \frac{\arcsin(\lambda\pi)}{\lambda\pi} \) depends on the number of letters of the hyperlogarithm, which alternate between \(-1\) and \(0\). The expansion which holds up to order \( \lambda^{10} \) is given by

\[
\phi(x) = c_\lambda \frac{\arcsin(\lambda\pi)}{\lambda\pi(1 + x)} \sum_{n=0}^{\infty} Hlog(x, [0, -1, \ldots, 0, -1]) \left( \frac{\arcsin(\lambda\pi)}{\pi} \right)^{2n},
\]

where the underbrace with \( n \) means that we have \( n \) times the letters \( 0 \) and \( -1 \) in an alternating way.

In the limit \( x \to 0 \) only the terms with \( n = 0 \) in both sums survive,

\[
1 \equiv \phi(0) = c_\lambda \frac{\arcsin(\lambda\pi)}{\lambda\pi} \lim_{x \to 0} \frac{Hlog(x, [\ldots])}{1 + x} - \lambda c_\lambda \frac{\arcsin(\lambda\pi)^2}{(\lambda\pi)^2} \lim_{x \to 0} \frac{Hlog(x, [-1])}{x}.
\]

This value was found in Sec. 4.2.4 by another method. We also remark that \( c_\lambda = \frac{1}{\mu^2} \) for the special renormalisation.

Next define the functions

\[
f(x) := \sum_{n=0}^{\infty} Hlog(x, [0, -1, \ldots, 0, -1]) \alpha^{2n}_\lambda,
\]

\[
g(x) := \sum_{n=0}^{\infty} Hlog(x, [-1, 0, -1, \ldots, 0, -1]) \alpha^{2n}_\lambda,
\]
where \( \alpha_\lambda = \frac{\arcsin(\lambda \pi)}{\pi} \). Both together obey the differential equations

\[
f'(x) = \frac{\alpha_\lambda^2}{x} g(x) \quad g'(x) = \frac{1}{1 + x} f(x),
\]
or equivalently

\[
f''(x) + \frac{f'(x)}{x} - \alpha_\lambda^2 \frac{f(x)}{(1 + x)x} = 0, \quad g''(x) + \frac{g'(x)}{1 + x} - \alpha_\lambda^2 \frac{g(x)}{(1 + x)x} = 0,
\]
with the boundary conditions \( f(0) = 1, f'(0) = \alpha_\lambda^2, g(0) = 0 \) and \( g'(0) = 1 \). The solution is given by hypergeometric functions.

In summary, the solution of equation (4.2.61) is conjectured to be

\[
f(x) = \alpha_\lambda^2 \frac{f'(x)}{x} = x_2 \frac{1}{\alpha_\lambda^2} \left( \frac{1}{2} \right) \left( x_2 \frac{1}{2} \right) - \alpha_\lambda^2 \left[ \frac{1}{2} \left( x_2 \frac{1}{2} \right) - x \right].
\]

In summary, the solution of equation (4.2.61) is conjectured to be

\[
\phi(x) = \frac{\alpha_\lambda c_\lambda}{\lambda(1 + x)^2} \left( x_2 \frac{1}{2} \right),
\]
or equivalently for (4.2.58)

\[
R_4(x) = \phi(x) = \frac{x}{\mu^2} \left( x_2 \frac{1}{2} \right) = x_2 \left( x_2 \frac{1}{2} \right) - \frac{x}{\mu^2}.
\]

where we have used the Gauss recursion formula [GR07, § 9.137.7] for hypergeometric functions. Finally, we note that

\[
\int_0^\infty \frac{dt}{(1 + t)^{2\beta}} \frac{\phi(t)}{t} = \lim_{x \to 0} \frac{x - \phi(x)}{\lambda x^2} = \frac{\alpha_\lambda(1 - \alpha_\lambda)}{2\lambda \mu^2} = \frac{1}{2\lambda \mu^2}.
\]

Thus choosing \( \mu^2 = \frac{\alpha_\lambda(1 - \alpha_\lambda)}{\lambda} \) we confirm (E.9) exactly.

### E.2 Proof with Meijer G-Function

We find it interesting to directly check that the hypergeometric function \( \tilde{\phi}(x) = \frac{1}{\mu^2} \phi(\frac{x}{\mu^2}) \), see (4.2.65), solves the integral equation (4.2.60). The hypergeometric function can be expressed through the more general Meijer-G function. A Meijer G-function is defined by

\[
G_{p,q}^{m,n}(z \mid a_1, \ldots, a_p ; b_1, \ldots, b_q) = \frac{1}{2\pi i} \int_L \prod_{j=1}^m \Gamma(b_j - s) \prod_{j=1}^n \Gamma(1 - a_j + s) \prod_{j=m+1}^q \Gamma(1 - b_j + s) z^s ds,
\]

with \( m, n, p, q \in \mathbb{N} \), with \( m \leq q \) and \( n \leq p \), and poles of \( \Gamma(b_j - s) \) different from poles of \( \Gamma(1 - a_j + s) \). The infinite contour \( L \) separates between the poles of \( \Gamma(b_j - s) \) and \( \Gamma(1 - a_j + s) \), and its behavior to infinity depends on \( m, n, p, q \) (see [GR07, §9.3]).

The Meijer G-function has by definition the property

\[
G_{p,q}^{m,n}(z \mid a_1, \ldots, a_p ; b_1, \ldots, b_q) = \frac{1}{z} G_{q,p}^{m,n}(z \mid -b_1, \ldots, -b_q).
\]
It obeys the convolution formula [GR07, § 7.811.1]
\[
\int_0^\infty dx G^{m,n}_{p,q}(\alpha x|a_1,\ldots,a_p) G^{m',n'}_{p',q'}(\beta x|b_1,\ldots,b_q) = \frac{1}{\alpha} G^{m+m',n+n'}_{q+q',p+p'} \left( \frac{\beta}{\alpha} - b_1, \ldots, -b_m, a_1', \ldots, a_{p'}', -b_{m+1}, \ldots, -b_q \right),
\]
(E.3)
which is the source of numerous impressive integrals over \( \mathbb{R}_+ \) of products of special functions. If no two \( b_j \) differ by an integer, either \( p < q \) or \( p = q \) with \( |z| < 1 \), then a Meijer G-function can be expressed by hypergeometric functions
\[
G^{m,n}_{p,q}(z|a_1,\ldots,a_p) = \sum_{k=1}^m \frac{\prod_{j=m+1}^m \Gamma(b_j - b_k) \prod_{j=1}^m \Gamma(1 + b_k - a_j)}{\prod_{j=m+1}^p \Gamma(1 + b_k - b_j) \prod_{j=n+1}^q \Gamma(a_j - b_k)} z^{b_k} \times \mu^{p-1} \left( 1 + b_k - a_1, \ldots, 1 + b_k - a_p \right) (1)^{p-n-m} z),
\]
where primed sum and the \( \star \) means that the term with \( j = k \) is omitted.

We need another identity which is derived directly from the definition
\[
G^{2,2}_{3,3}(z|0,0,1) = \frac{1}{2\pi i} \int_L \frac{\Gamma(b_1 - s) \Gamma(b_2 - s) \Gamma(-s) \Gamma(1 + s)^2}{\Gamma(1 - s)} z^s ds
\]
\[
= \frac{1}{2\pi i} \int_L \Gamma(b_1 - s) \Gamma(b_2 - s) \Gamma(s) \Gamma(1 + s) z^s ds
\]
\[
= \Gamma(b_1) \Gamma(b_2) - \frac{1}{2\pi i} \int_L \Gamma(b_1 - s) \Gamma(b_2 - s) \Gamma(s) \Gamma(1 + s) z^s ds
\]
\[
= \Gamma(b_1) \Gamma(b_2) - G^{2,2}_{2,2}(z|0,1),
\]
where the contour is changed \( L \to L' \) such that it is moved through \( s = 0 \) and picked up the residue. The contour \( L' \) fulfils the definition (E.1) for \( G^{2,2}_{2,2}(z|0,1) \).

From (E.4) one can establish
\[
\tilde{\varphi}_\lambda(t) = \frac{1}{\mu^2} G^{2,2}_{2,2}(t|\alpha_\lambda, 0, -1),
\]
and \( \frac{1}{x+t+\mu^2} = \frac{1}{x+\mu^2} F_1 \left( | \begin{array}{c} \frac{t}{x+\mu^2} \\ 0 \end{array} \right) = \frac{1}{x+\mu^2} G^{1,1}_{1,1}(\frac{t}{x+\mu^2}|0,0,1). \) The convolution theorem (E.3) of Meijer G-functions thus allows to evaluate the integral
\[
\lambda \int_0^\infty \frac{dt \tilde{\varphi}_\lambda(t)}{x+t+\mu^2}
\]
\[
= \frac{\lambda}{\mu^2 \Gamma(2 - \alpha_\lambda) \Gamma(1 + \alpha_\lambda)} G^{2,3}_{3,3} \left( \frac{x + \mu^2}{\mu^2} \left| \alpha_\lambda - 1, -\alpha_\lambda, 0 \right. \right)
\]
\[
= \frac{\lambda}{(x + \mu^2) \Gamma(2 - \alpha_\lambda) \Gamma(1 + \alpha_\lambda)} G^{3,2}_{3,3} \left( \frac{\mu^2}{x + \mu^2} \left| 0, 0, 1 - \alpha_\lambda, 0, \alpha_\lambda \right. \right)
\]
\[
= \frac{\lambda}{(x + \mu^2) \left( \Gamma(1 - \alpha_\lambda) \Gamma(1 + \alpha_\lambda) - G^{2,2}_{2,2} \left( \frac{\mu^2}{x + \mu^2} \left| 0, 1 - \alpha_\lambda, \alpha_\lambda \right. \right) \right)
\]
\[
= \frac{1}{(x + \mu^2) \left( 1 - \alpha_\lambda \right)}
\]
\[ - \frac{\Gamma(2\alpha - 1)\Gamma(1 - \alpha)}{\Gamma(1 + \alpha)} \left( \frac{\mu^2}{x + \mu^2} \right)^{1-\alpha} \mathcal{F}_1 \left( \begin{array}{c} 2-\alpha, 1-\alpha \\ -2-2\alpha \\ \frac{\mu^2}{x + \mu^2} \end{array} \right) \\
\quad - \frac{\Gamma(2 - 2\alpha)\Gamma(\alpha)}{\Gamma(2 - \alpha)} \left( \frac{\mu^2}{x + \mu^2} \right) \alpha \mathcal{F}_1 \left( \begin{array}{c} 1+\alpha, \alpha \\ 2\alpha \\ \frac{\mu^2}{x + \mu^2} \end{array} \right) \right) \\
= \frac{1}{(x + \mu^2)\alpha(1 - \alpha)} - \frac{\lambda \pi}{\sin(\alpha \pi)} \hat{\varrho}(x). \quad (E.6) \]

We have used the expansion of a Meijer G-function into hypergeometric functions and applied in the last step \[ \text{GR07} \] §9.132.1. The result is precisely (4.2.60) provided that \( c_\lambda = \frac{\lambda}{\alpha(1 - \alpha)} \) (see (4.2.65)) and \( \sin(\alpha \pi) = \frac{\lambda \pi}{\sin(\alpha \pi)} \) (see (4.2.66)).

### E.3 On the Spectrum of the Fredholm Integral Operator

Abstractly, the integral equation (4.2.60) is of the form

\[ \psi = f_\mu - \lambda A_\mu \psi, \]

where \( \psi(t) = \tilde{\varrho}_\lambda(t) \), \( f_\mu(t) = (t + \mu^2)^{-1} \) and \( A_\mu \) is the operator with integral kernel

\[ A_\mu(t, u) = \frac{ut}{(u + \mu^2)(u + t + \mu^2)(t + \mu^2)}. \quad (E.1) \]

Note that \( A_\mu \) is symmetric and positive. The equation can thus be solved for \( \psi \) if \( \lambda > \lambda_c = -\|A_\mu\|^{-1} \).

By scaling, the spectrum of \( A_\mu \) is independent of \( \mu \) for \( \mu > 0 \). We claim that

\[ \|A_\mu\| = \|A_0\| = \pi. \quad (E.2) \]

In particular, \( \lambda_c = -1/\pi \).

Since \( A_\mu \) has a positive kernel which is monotone in \( \mu \), one readily obtains \( \|A_\mu\| \leq \|A_0\| \). On the other hand, \( A_0 \) is the weak limit of \( A_\mu \) as \( \mu \to 0 \), hence \( \|A_0\| \leq \liminf_{\mu \to 0} \|A_\mu\| \), which proves that \( \|A_\mu\| = \|A_0\| \). Rational fraction expansion of \( A_\mu \) gives \( f_\mu \) an additional factor and changes the integral kernel to \( A_\mu \to (u + t)^{-1} \). Introducing logarithmic coordinates, we have

\[ \int_0^\infty \int_0^\infty \frac{\phi(u)\phi(t)}{u + t} \phi^*(e^s)\phi^*(e^v)e^{v+s}dvds = \int_\mathbb{R} \int_\mathbb{R} \frac{\phi^*(e^s)e^{v/2}\phi^*(e^s)e^{s/2}}{2 \cosh(\frac{1}{2}(v-s))}dvds \]

which can be diagonalised via Fourier transforms. Since

\[ \int_\mathbb{R} \frac{1}{2 \cosh(v/2)}dv = \pi, \]

this shows that the spectrum of \( A_0 \) equals \([0, \pi]\), and indeed \( \|A_0\| = \|A_\mu\| = \pi \).

\[^1\text{Contributed by Robert Seiringer to GHW20}\]
Appendix F

Examples for Catalan Tuples, Catalan Tables and its Diagrammatic Representation

F.1 Examples for Catalan Tuples and Catalan Tables

Catalan tuples are introduced in Definition 4.6 and Catalan tables in Definition 4.11.

Example F.1. We have $(1, 0) = (0) \circ (0)$, $(2, 0, 0) = (1, 0) \circ (0)$, $(1, 1, 0) = (0) \circ (1, 0)$ and $(3, 1, 0, 0, 2, 0, 0) = (2, 1, 0, 0) \circ (2, 0, 0)$.

Example F.2. We have $(1, 0) = (0) \bullet (0)$, $(2, 0, 0) = (1, 0) \bullet (0)$, $(1, 1, 0) = (0) \bullet (1, 0)$ and $(3, 1, 0, 0, 2, 0, 0) = (2, 0, 2, 0, 0) \bullet (1, 0)$.

Example F.3. We have

$$T_1 = \{((0), (0))\},$$

$$T_2 = \{((1, 0), (0), (0)), ((0), (1, 0), (0))\},$$

$$T_3 = \{((2, 0, 0), (0), (0), (0)), ((1, 1, 0), (0), (0), (0)), ((1, 0), (1, 0), (0), (0)), ((1, 0), (0), (1, 0), (0)), ((0), (2, 0, 0), (0), (0)), ((0), (1, 1, 0), (0), (0)), ((0), (1, 0), (1, 0), (0))\}.$$

Later in Figure F.1 and F.2 we give a diagrammatic representation of the Catalan tables in $T_2$ and $T_3$, respectively.

Example F.4. We have $\langle(2, 0, 0), (0), (0), (0)\rangle = \langle(1, 0), (0), (0)\rangle \bullet \langle(0), (0)\rangle$ and $\langle(1, 1, 0), (0), (0), (0)\rangle = \langle(0), (0)\rangle \circ \langle(1, 0), (0), (0)\rangle$. In Example 4.7 and Figure F.3 we considered the Catalan table $\langle(2, 0, 0), (1, 1, 0), (0), (0), (0), (1, 0), (0)\rangle = \langle(1, 0), (1, 1, 0), (0), (0), (0)\rangle \circ \langle(0), (1, 0), (0)\rangle$. Another example will be given in Example F.6.

Example F.5. We have $\langle(0), (2, 0, 0), (0), (0)\rangle = \langle(0), (1, 0), (0)\rangle \bullet \langle(0), (0)\rangle$ and $\langle(0), (1, 1, 0), (0), (0)\rangle = \langle(0), (0)\rangle \circ \langle(1, 0), (0), (0)\rangle$. In Example 4.7 and Figure F.3 we considered the Catalan table $\langle(2, 0, 0), (1, 1, 0), (0), (0), (0), (1, 0), (0)\rangle = \langle(2, 0, 0), (0), (0), (1, 0), (0)\rangle \bullet \langle(1, 0), (0), (0)\rangle$. Another example will be given in Example F.7.

*This is taken from the appendix of our paper [JdHW19].
F.2 Chord Diagrams with Threads

For uncovering the combinatorial structure of (4.4.1), it was extremely helpful for us to have a graphical presentation as diagrams of chords and threads. To every term of the expansion (4.4.2) of an $N$-point function we associate a diagram as follows:

**Definition F.1** (diagrammatic presentation). Draw $N$ nodes on a circle, label them from $p_0$ to $p_{N-1}$. Draw a green chord between $p_r, p_s$ for every factor $G_{p_r p_s}$ in (4.4.2) and an (orange for $t, u$ even, blue for $t, u$ odd) thread between $p_t, p_u$ for every factor $E_{p_t} - E_{p_u}$. The convention $t < u$ is chosen so that the diagrams come with a sign.

It was already known in [GW14a] that the chords do not cross each other (using cyclic invariance (4.4.4)) and that the threads do not cross the chords (using (4.4.5)). But the combinatorial structure was not understood in [GW14a] and no algorithm for a canonical set of chord diagrams could be given. This work repairs this omission.

The $N/2 = k + 1$ chords in such a diagram divide the circle into $k + 2$ pockets. The pocket which contain the arc segment between the designated nodes $p_0$ and $p_{N-1}$ is by definition the root pocket $P_0$. Moving in the counterclockwise direction, every time a new pocket is entered it is given the next number as index, as in Definition 4.14. The tree of these $k + 2$ pockets, connecting vertices if the pockets border each other, is the pocket tree. A pocket is called even (resp. odd) if its index is even (resp. odd).

Inside every even pocket, the orange threads (between even nodes) form the direct tree, the blue threads (between odd nodes) form the opposite tree. Inside every odd pocket, the orange threads (between even nodes) form the opposite tree, the blue threads (between odd nodes) form the direct tree.

The sign $\tau$ of the diagram is given by

$$\tau(T) = (-1)^{\sum_{j=1}^{k+1} e_0^{(j)}},$$

where $e_0^{(j)}$ is the first entry of the Catalan tuple corresponding to a pocket $P_j$. Indeed, for every pocket that is not a leaf or the root pocket, the chain of odd nodes starts with the highest index, which implies that every thread emanating from this node contributes a factor $(-1)$ to the monomial (4.4.2) compared with the lexicographic order chosen there. In words: count for all pockets other than the root pocket the total number $K$ of threads which go from the smallest node into the pocket. The sign is even (odd) if $K$ is even (odd).

Figure F.1 and F.2 show Catalan tables and chord diagrams of the 4-point function and 6-point function, respectively. Figure F.3 shows the chord diagram discussed in Example 4.7.

![Figure F.1: The two chord diagrams and Catalan tables of $G_{p_0 p_1 p_2 p_3}^{(4)}$.](image)

Now that a visual way to study the recursion relation (4.4.1) has been introduced, it is much easier to demonstrate the concepts introduced in Secs. 4.4.2 and 4.4.3.
F.2. CHORD DIAGRAMS WITH THREADS

Figure F.2: The seven chord diagrams and Catalan tables of $G_{p_0 p_1 p_2 p_3 p_4 p_5}$.

Figure F.3: A chord diagram and Catalan table contributing to $G_{p_0 p_1 p_2}$... Pocket tree and all non-trivial direct and opposite trees have been given in Example 4.7.
Example F.6. The operation $\circ$ is best demonstrated by an example:

\[
\langle (1,0), (0), (0) \rangle \circ \langle (0), (1,0), (0) \rangle = \langle (2,0,0), (0), (0), (1,0), (0) \rangle .
\]

The corresponding chord diagrams are

\[
\begin{array}{ccc}
\langle (1,0), (0), (0) \rangle & \circ & \langle (0), (1,0), (0) \rangle \\
\hline
\langle (2,0,0), (0), (0), (1,0), (0) \rangle
\end{array}
\]

The diagrammatic recipe is to cut both diagrams on the right side of the designated node and paste the second into the first, where the counterclockwise order of the nodes must be preserved. Then both designated nodes (here $p_0$, $p_4$) are connected by a orange thread and nodes $p_1$ and $p_7 = p_{N-1}$ by a blue thread.

To $\circ$-decompose the Catalan table $\langle (2,0,0), (0), (0), (1,0), (0) \rangle$, we first $\circ$-factorise the zeroth pocket $(2,0,0)$ via (4.4.7). Here $\sigma_1((2,0,0)) = 1$ and, hence, $(2,0,0) = (1,0) \circ (0)$. Next, we evaluate the number $\hat{k}$ defined in (4.4.9). We have $1 + |\hat{f}(0)| = 1$ and $\sigma_1((3,0,0,1,0)) = 2$. Consequently, we get from Definition 4.12

\[
\langle (2,0,0), (0), (0), (1,0), (0) \rangle = \langle (1,0), (0), (0) \rangle \circ \langle (0), (1,0), (0) \rangle .
\]

Example F.7. We employ the same example (with diagrams switched) to demonstrate the operation $\bullet$. In terms of Catalan tables this becomes

\[
\langle (0), (1,0), (0) \rangle \bullet \langle (1,0), (0), (0) \rangle = \langle (0), (2,1,0,0), (0), (0), (0) \rangle ,
\]

for which the chord diagrams are

\[
\begin{array}{ccc}
\langle (0), (1,0), (0) \rangle & \bullet & \langle (1,0), (0), (0) \rangle \\
\hline
\langle (0), (2,1,0,0), (0), (0), (0) \rangle
\end{array}
\]

The diagrammatic recipe is to cut the first diagram on the left side of the designated node and the second diagram on the right side. Then paste the second into the first, where the counterclockwise order of the nodes must be preserved. The threads in the second diagram switch blue and orange by doing so. Then, the designated node of the first diagram is connected to the last node of the second by a orange thread, the designated node of the second diagram is connected to the last node of the first diagram by a blue thread.

Conversely, to $\bullet$-decompose the Catalan table $\langle (0), (2,1,0,0), (0), (0), (0) \rangle$, we first $\bullet$-factorise the first pocket $c^{(1)} = (2,1,0,0)$ via (4.4.8). We have $e_0^{(1)} = 1$, hence consider $\sigma_1((2,1,0,0)) = 2$ and conclude $(2,1,0,0) = (1,0) \bullet (1,0)$. Next, we evaluate the number $\hat{\bar{k}}$ in (4.4.10). With $|\hat{c}(0)| + |\hat{c}(1)| + 1 = 0 + 1 + 1 = 2$ the decomposition follows from $\sigma_2((3,0,0,1,0)) = 2$ and yields

\[
\langle (0), (2,1,0,0), (0), (0), (0) \rangle = \langle (0), (1,0), (0) \rangle \bullet \langle (1,0), (0), (0) \rangle .
\]
Appendix G

3-Coloured Model with Cubic Interaction

This appendix will consider a different type of matrix field theory which can be seen as multi-matrix field theory, studied in our paper [HW18]. We assume three different matrices $\Phi^a$ with the "colours" $a \in \{1, 2, 3\}$. The interaction is a cubic interaction such that the 1-point function does not exist. Accordingly, the action is given by

$$S[\Phi] = V \left( \sum_{a=1}^{3} \sum_{n,m=0}^{N} \frac{H_{nm}}{2} \Phi^a_{nm} \Phi^a_{mn} + \frac{\lambda}{3} \sum_{a,b,c=1}^{3} \sum_{n,m,l=0}^{N} \sigma_{abc} \Phi^a_{nm} \Phi^b_{ml} \Phi^c_{ln} \right)$$

$$H_{nm} := E_n + E_m,$$

where $(\Phi^a_{nm})$ are Hermitian matrices and $\sigma_{abc} = 1$ for $a \neq b \neq c \neq a$ and $\sigma_{abc} = 0$ else.

We demonstrate the techniques to determine correlation functions from the partition function for a coloured model. The partition function $Z[J]$ of the 3-colour model with external Hermitian matrices $(J^a_{nm})$ and $a \in \{1, 2, 3\}$ is formally defined by

$$Z[J] := \int \left( \prod_{a=1}^{3} D\Phi^a \right) \exp \left( -S[\Phi] + \sum_{a=1}^{3} \sum_{n,m=0}^{N} J^a_{nm} \Phi^a_{nm} \right)$$

$$= K \exp \left( -\frac{\lambda}{3V^2} \sum_{a,b,c=1}^{3} \sum_{n,m,l=0}^{N} \sigma_{abc} \frac{\partial^3}{\partial J^a_{nm} \partial J^b_{ml} \partial J^c_{ln}} \right) Z_{\text{free}}[J],$$

$$Z_{\text{free}}[J] := \exp \left( \sum_{a=1}^{3} \sum_{n,m=0}^{N} \frac{V}{2H_{nm}} J^a_{nm} J^a_{mn} \right),$$

$$K := \int \left( \prod_{a=1}^{3} D\Phi^a \right) \exp \left( -\sum_{a=1}^{3} \sum_{n,m=0}^{N} \frac{VH_{nm}}{2} \Phi^a_{nm} \Phi^a_{mn} \right).$$

The logarithm of $Z[J]$ will be expanded into a series of moments with different number $b$ of boundary components. The sources are cyclic within every boundary $\beta \in \{1, ..., b\}$. For simplification we use the notation $\mathcal{J}^{a_1 \ldots a_{N_\beta}}_{p_1 \ldots p_{N_\beta}} := \prod_{i=1}^{N_\beta} J^{a_i}_{p_i p_{i+1}}$ with $N_\beta + 1 \equiv 1$. The

*This is a summary of our paper [HW18]
correlation functions are then defined by

$$\log \frac{Z[J]}{Z[0]} = \sum_{b=1}^{\infty} \sum_{N_1, \ldots, N_b=1}^{\infty} \sum_{N} \prod_{b=1}^{3} \frac{G_{[p_1] \ldots [p_{N_b}]}^{a_1 \ldots a_{N_b}}}{b!} \sum_{N_\beta} \int_{p_1 \ldots p_{N_\beta}} \frac{a_1^{\beta} \ldots a_{N_\beta}^{\beta}}{N_\beta}. \quad (G.5)$$

Due to the vanishing 1-point function for the 3-colour model, the partition function can be expanded with (G.5) to

$$\frac{Z[J]}{Z[0]} = 1 + \sum_{a,b=1}^{\infty} \sum_{n,m=0}^{\infty} \sum_{N} \left( \frac{V}{2} G_{[a]n}^{[a]} n_m + \frac{1}{2} G_{[a][b]}^{[a][b]} n_m^{\alpha} m_\beta \right) \quad (G.6)$$

The calculation rule for later purpose is

$$\frac{\partial}{\partial J_{p_1 p_2}} J_{p_3 p_4} = \delta_{a b} \delta_{p_1 p_3} \delta_{p_2 p_4} + J_{p_3 p_4} \frac{\partial}{\partial J_{p_1 p_2}}.$$

### G.1 Ward-Takahashi Identity

The Ward-Takahashi identity is obtained by the requirement of invariance of $Z[J]$ under inner automorphisms (see Proposition 2.2). For a colour model, we choose a transformation as follows: $\phi^a \mapsto (\phi^a)' = U^\dagger \phi^a U$ for $U \in U(N)$ for one colour $a \in \{1, 2, 3\}$. The Ward-Takahashi identity following from this transformation for $E_{p_1} \neq E_{p_2}$ is given by

$$\sum_{m=0}^{N} \frac{\partial^2}{\partial J_{p_1 m}} \partial J_{m p_2} Z[J] + \frac{V}{E_{p_1} - E_{p_2}} \sum_{m=0}^{N} \left( J_{p_2 m} \frac{\partial}{\partial J_{p_1 m}} - J_{m p_1} \frac{\partial}{\partial J_{m p_2}} \right) Z[J] = \frac{\lambda}{V(E_{p_1} - E_{p_2})} \sum_{m,n=0}^{N} \sum_{b,c=1}^{3} \sigma_{abc} \left( \frac{\partial^3}{\partial J_{p_1 m} \partial J_{m n} \partial J_{c p_2}} - \frac{\partial^3}{\partial J_{p_1 m} \partial J_{m n} \partial J_{c p_2}} \right) Z[J]. \quad (G.1)$$

The interaction terms are not invariant under the transformation of only one colour. However, the sum over all colours in (G.1) gives

$$\sum_{a=1}^{3} \sum_{m=0}^{N} \frac{\partial^2}{\partial J_{p_1 m} \partial J_{m p_2}} Z[J] = \frac{V}{(E_{p_1} - E_{p_2})} \sum_{a=1}^{3} \sum_{m=0}^{N} \left( J_{m p_1} \frac{\partial}{\partial J_{m p_2}} - J_{p_2 m} \frac{\partial}{\partial J_{p_1 m}} \right) Z[J], \quad (G.2)$$
which has the usual form of a Ward-Takahashi identity (see Proposition 2.2). Equation (G.2) shows that the interaction term is invariant under the simultaneous transformation of all three colours.

A more general identity plays the crucial rôle (analog to Proposition 2.3):

**Proposition G.1.** Let $E_{p_1} \neq E_{p_2}$. The generalised Ward-Takahashi identity for the 3-colour matrix model with an external field $E$ is

$$\sum_{m=0}^{N} \frac{\partial^2}{\partial J_{p_1m} \partial J_{mp_2}} Z[J] + \frac{V}{E_{p_1} - E_{p_2}} \sum_{m=0}^{N} \left( J_{p_2m}^b \frac{\partial}{\partial J_{mp_1}} - J_{mp_1}^a \frac{\partial}{\partial J_{mp_2}} \right) Z[J]$$

$$= \frac{\lambda}{V(E_{p_1} - E_{p_2})} \sum_{m,n=0}^{N} \sum_{c,d=1}^{3} \left( \sigma_{bcd} \frac{\partial}{\partial J_{p_1m}^a} \frac{\partial}{\partial J_{mn}^c} \frac{\partial}{\partial J_{np_2}^d} - \sigma_{acd} \frac{\partial}{\partial J_{p_1m}^c} \frac{\partial}{\partial J_{mn}^d} \frac{\partial}{\partial J_{np_2}^e} \right) Z[J].$$

**Proof.** Let $S_{int}[\Phi] = V\frac{\lambda}{3} \sum_{a,b,c=1}^{3} \sum_{n,m,l=1}^{N} \sigma_{abc} \Phi_{nm}^a \Phi_{ml}^b \Phi_{ln}^c$ be the interaction term of the action. Direct computation gives then

$$\frac{E_{p_1} - E_{p_2}}{V} \sum_{m=0}^{N} \frac{\partial^2}{\partial J_{p_1m} \partial J_{mp_2}} Z[J]$$

$$= \frac{1}{V} \sum_{m=0}^{N} \frac{\partial^2}{\partial J_{p_1m} \partial J_{mp_2}} (E_{p_1} + E_m - (E_{p_2} + E_m)) Z[J]$$

$$= K \sum_{m=0}^{N} \left\{ \frac{\partial}{\partial J_{p_1m}^a} \exp \left( -S_{int} \left[ \frac{1}{V} \frac{\partial}{\partial J} \right] \right) J_{mp_1}^a \right.$$  \left. \right.$$ - \frac{\partial}{\partial J_{p_1m}^b} \exp \left( -S_{int} \left[ \frac{1}{V} \frac{\partial}{\partial J} \right] \right) J_{mp_2}^b \right\} Z_{free}[J]$$

$$= \sum_{m=0}^{N} \left( J_{mp_1}^a \frac{\partial}{\partial J_{mp_2}^b} - J_{mp_2}^b \frac{\partial}{\partial J_{mp_1}^a} \right) Z[J]$$

$$- \frac{\lambda}{V^2} \sum_{m,n=0}^{N} \sum_{c,d=1}^{3} \left( \sigma_{acd} \frac{\partial}{\partial J_{p_1m}^c} \frac{\partial}{\partial J_{mn}^d} \frac{\partial}{\partial J_{np_2}^e} - \sigma_{acd} \frac{\partial}{\partial J_{p_1m}^c} \frac{\partial}{\partial J_{mn}^d} \frac{\partial}{\partial J_{np_2}^e} \right) Z[J].$$

We have used the second form of $Z[J]$ in (G.3) and the Leibniz rule in the last step. Technically, one expands the exponential function and resums after using the Leibniz rule. Since $E_{p_1} \neq E_{p_2}$ the proof is finished.

Equation (G.1) is a special case of Proposition G.1 by setting $b = a$. The derivation of both identities is completely different. Proposition G.1 cannot be obtained by a symmetry transformation of only one colour due to the discrete mixing of the colours if $a \neq b$. Applying the procedure of the proof of Proposition G.1 it is also possible to derive the usual Ward-Takahashi identity even in other models.

For later purpose, we combine two identities to get a more useful expression:

**Lemma G.1.** Let $a$ be fixed and $E_{p_1} \neq E_{p_2}$, then it follows

$$\sum_{b,c=1}^{3} \sum_{m=0}^{N} \sigma_{abc} \frac{\partial^2}{\partial J_{p_1m} \partial J_{mp_2}} Z[J]$$
\[\frac{V}{E_{p_1} - E_{p_2}} \left[ \sum_{b,c=1}^{3} \sum_{m=0}^{N} \sigma_{abc} \sum_{n=0}^{N} \left( J_{mp_1}^{b} \frac{\partial}{\partial J_{mp_2}^{b}} - J_{mp_2}^{c} \frac{\partial}{\partial J_{mp_2}^{c}} \right) \right] + \frac{\lambda}{V^2} \sum_{b=1}^{3} \left\{ \sum_{m=0}^{N} \left( \frac{\partial^3}{\partial J_{p_1m}^{b} \partial J_{mp_1}^{b} \partial J_{mp_2}^{b}} - \frac{\partial^3}{\partial J_{p_2m}^{b} \partial J_{mp_2}^{b} \partial J_{mp_2}^{b}} \right) \right\} Z[J].\]

**Proof.** Inserting Proposition G.1 for the lhs yields

\[\frac{V}{E_{p_1} - E_{p_2}} \sum_{b,c=1}^{3} \sum_{m=0}^{N} \sigma_{abc} \sum_{n=0}^{N} \left( J_{mp_1}^{b} \frac{\partial}{\partial J_{mp_2}^{b}} - J_{mp_2}^{c} \frac{\partial}{\partial J_{mp_2}^{c}} \right) Z[J] \quad (G.3)\]

By the sum over the colours \(b, c, d, e,\) we obtain for the multiplication of two \(\sigma\)'s with one common index

\[\sigma_{abc} \sigma_{cde} = \sigma_{abc} (\delta_{ad}\delta_{be} + \delta_{ae}\delta_{bd})\]

\[\sigma_{abc} \sigma_{bde} = \sigma_{abc} (\delta_{ad}\delta_{ce} + \delta_{ae}\delta_{cd}).\]

Therefore, the last line in (G.3) gives

\[\frac{\lambda}{V(E_{p_1} - E_{p_2})} \sum_{m,n=0}^{N} \sum_{b,c=1}^{3} \sigma_{abc} \left( \frac{\partial^3}{\partial J_{p_1m}^{b} \partial J_{mn}^{d} \partial J_{np}^{b}} + \frac{\partial^3}{\partial J_{p_1m}^{b} \partial J_{mn}^{c} \partial J_{np}^{b}} \right) Z[J].\]  

The first and the last term in parentheses vanish because of the total symmetry of \(\sigma_{abc}.\)

Adding \(0 = \left( \frac{\partial^3}{\partial J_{p_1m}^{b} \partial J_{mn}^{d} \partial J_{np}^{b}} - \frac{\partial^3}{\partial J_{p_2m}^{b} \partial J_{mn}^{d} \partial J_{np}^{b}} \right) Z[J]\) and renaming the indices, (G.4) can be rewritten to

\[\frac{\lambda}{V(E_{p_1} - E_{p_2})} \sum_{m,n=0}^{N} \sum_{b=1}^{3} \left( \frac{\partial^3}{\partial J_{p_1m}^{b} \partial J_{mn}^{d} \partial J_{np}^{b}} - \frac{\partial^3}{\partial J_{p_1m}^{b} \partial J_{mn}^{c} \partial J_{np}^{b}} \right) Z[J].\]

Inserting (G.2) for \(E_n \neq E_{p_1}\) in the first and \(E_m \neq E_{p_2}\) in the second term finally gives
after renaming indices

$$\frac{\lambda}{V(E_{p_1} - E_{p_2})} \sum_{b=1}^{3} \left\{ \sum_{m=0}^{N} \left( \frac{\partial^3}{\partial J_{p_1 m} \partial J_{b m} \partial J_{a p_2}} - \frac{\partial^3}{\partial J_{b p_2} \partial J_{p_1 m} \partial J_{a m}} \right) \right. $$

$$+ \sum_{m,n=0}^{N} \frac{V}{E_{p_1} - E_{n}} \frac{\partial}{\partial J_{n p_2}} \left( J_{m p_1} \frac{\partial}{\partial J_{b m}} - J_{b m} \frac{\partial}{\partial J_{p_1 m}} \right) $$

$$- \sum_{m,n=0}^{N} \frac{V}{E_{p_2} - E_{n}} \frac{\partial}{\partial J_{p_1 n}} \left( J_{p_2 m} \frac{\partial}{\partial J_{b m}} - J_{b m} \frac{\partial}{\partial J_{p_2 m}} \right) \right\} \mathcal{Z}[J].$$

The identity follows by combining (G.3) and (G.5).

\[ \square \]

### G.2 Schwinger-Dyson Equations for \( b = 1 \)

In this section we derive the SDEs with the help of Ward-Takahashi identity.

**Proposition G.2.** The SDE for the 2-point function in the 3-colour matrix model with an external field \( E \) is for \( E_{p_1} \neq E_{p_2} \) given by

$$G_{[p_1 p_2]}^{aa} = \frac{1}{H_{p_1 p_2}} + \frac{\lambda}{E_{p_1} - E_{p_2}} \left[ \sum_{m=0}^{N} \left( G_{[p_1 p_2]}^{aa} \left( G_{[p_2 m]}^{bb} - G_{[p_1 m]}^{bb} \right) + \frac{1}{V} \left( G_{[p_2 p_1]}^{aab} - G_{[p_1 p_2]}^{aab} \right) \right) \right. $$

$$+ \sum_{b=1}^{3} \frac{1}{V^2} \left( \sum_{m=0}^{N} \left( G_{[p_2 p_1]}^{aab} - G_{[p_1 p_2]}^{aab} \right) + \frac{1}{V} \left( G_{[p_2 p_1]}^{a bb} - G_{[p_1 p_2]}^{a bb} \right) \right) \right. $$

$$+ \sum_{b=1}^{3} \left( \frac{1}{V^3} \left( G_{[p_2 p_1]}^{b ba} - G_{[p_1 p_2]}^{b ba} \right) + \frac{1}{V} G_{[p_2 p_1]}^{a ab} \left( G_{[p_2 p_1]}^{b bb} - G_{[p_1 p_2]}^{b bb} \right) \right) \right. $$

$$+ \sum_{m=0}^{N} \frac{G_{[p_1 p_2]}^{aa} - G_{[p_2 m]}^{aa}}{E_{p_2} - E_{m}} - \sum_{m=0}^{N} \frac{G_{[p_2 p_1]}^{aa} - G_{[p_1 m]}^{aa}}{E_{m} - E_{p_1}} + \frac{1}{V} \left[ \sum_{m \neq p_1}^{N} \frac{G_{[p_2 m]}^{aa}}{E_{p_2} - E_{m}} + \frac{G_{[p_2 p_1]}^{aa}}{E_{m} - E_{p_1}} \right].$$

**Proof.** Assuming \( E_{p_1} \neq E_{p_2} \) the 2-point function is given via definition (G.5) and expansion (G.6). Using (G.3) leads to

$$G_{[p_1 p_2]}^{aa} = \frac{1}{V} \frac{\partial^2}{\partial J_{p_1 p_2} \partial J_{a p_2}} \log \mathcal{Z}[J] \bigg|_{J=0} = \frac{1}{V} \frac{\partial^2}{\partial J_{a p_2} \partial J_{p_1 p_2}} \mathcal{Z}[J] \bigg|_{J=0} $$

$$= \frac{K}{H_{p_1 p_2} \mathcal{Z}[0]} \frac{\partial}{\partial J_{a p_2}} \exp \left( -\frac{1}{V} \frac{\partial}{\partial J} \right) \left. J_{p_2 p_1} \mathcal{Z}_{free}[J] \right|_{J=0} $$

$$= \frac{K}{H_{p_1 p_2} \mathcal{Z}[0]} \frac{\partial}{\partial J_{a p_2}} \left. \sum_{b,c=1}^{3} \sum_{m=0}^{N} \gamma_{abc} \frac{\partial^2}{\partial J_{p_1 m} \partial J_{c m}} \mathcal{Z}[J] \right|_{J=0}.$$

Inserting the expansion of (G.6) would give the SDE between the 2-point and 3-point function. At first sight, the application of Lemma G.1 seems to make the equation more...
complicated. However, it yields a better behaviour in the V-expansion. The first term on the rhs of the equation of Lemma G.1 vanishes by setting \( J \) to zero. Therefore, we obtain

\[
\frac{1}{H_{p_1p_2}} \frac{\lambda^2}{V^3} \left[ (E_{p_1}^2 - E_{p_2}^2) \mathcal{Z}[0] \mathcal{V}^3 \right]
\]

\[
\times \left\{ \sum_{b=1}^{3} \sum_{m=0}^{N} \left( \frac{\partial^4}{\partial J_{p_1p_1}^m \partial J_{p_2p_2}^m \partial J_{p_1p_2}^m \partial J_{p_2p_1}^m} - \frac{\partial^4}{\partial J_{p_1p_2}^m \partial J_{p_1p_1}^m \partial J_{p_2p_2}^m \partial J_{p_2p_1}^m} \right) \mathcal{Z}[J] \right|_{J=0} + \sum_{m,n=0}^{N} \frac{V}{E_{p_1} - E_{m}} \left( \frac{\partial^2}{\partial J_{p_2p_2}^m \partial J_{p_1p_1}^m} - \frac{\partial^2}{\partial J_{p_1p_1}^m \partial J_{p_2p_2}^m} \right) \mathcal{Z}[J] \right|_{J=0} \left\}.
\]

where \( H_{p_1p_2}(E_{p_1} - E_{p_2}) = (E_{p_1}^2 - E_{p_2}^2) \) has been used and the fact that in the last two lines only colour \( a \) survives. By taking \( E_{p_1} \neq E_{p_2} \) into account and \( J = 0 \) gives with the Leibniz rule

\[
\frac{1}{H_{p_1p_2}} \frac{\lambda^2}{V^3} \left[ (E_{p_1}^2 - E_{p_2}^2) \mathcal{Z}[0] \mathcal{V}^3 \right]
\]

\[
\times \left\{ \sum_{b=1}^{3} \sum_{m=0}^{N} \left( \frac{\partial^4}{\partial J_{p_1p_1}^m \partial J_{p_2p_2}^m \partial J_{p_1p_2}^m \partial J_{p_2p_1}^m} - \frac{\partial^4}{\partial J_{p_1p_2}^m \partial J_{p_1p_1}^m \partial J_{p_2p_2}^m \partial J_{p_2p_1}^m} \right) \mathcal{Z}[J] \right|_{J=0} + \sum_{m,n=0}^{N} \frac{V}{E_{p_1} - E_{m}} \left( \frac{\partial^2}{\partial J_{p_2p_2}^m \partial J_{p_1p_1}^m} - \frac{\partial^2}{\partial J_{p_1p_1}^m \partial J_{p_2p_2}^m} \right) \mathcal{Z}[J] \right|_{J=0} \left\}.
\]

The first line generates for \( E_m \neq E_{p_1} \) and \( E_m \neq E_{p_2} \) either a 4-point function with one boundary or two 2-point functions with one boundary, respectively. Functions with higher boundaries \( b \geq 2 \) appear in case of \( E_m = E_{p_1} \) or \( E_m = E_{p_2} \). All terms are found by comparing with the expansion \((G.6)\).

We remind that in \( \) Proposition G.2 \( \) correlation functions of genus \( g \geq 1 \) are also included. The SDE of the 2-point function depends on \( \lambda^2 \), since graphs exist only with an even number of vertices.

**Proposition G.3.** Let \( N \geq 3 \). The SDE for the \( N \)-point function in the 3-colour matrix model with an external field \( E \) is for pairwise different \( E_{p_1}, E_{p_2} \) given by

\[
G_{[p_1 \ldots p_N]}^{a_1 \ldots a_N} = -\frac{\lambda}{(E_{p_1}^2 - E_{p_2}^2)} \sum_{b=1}^{3} \left( \sigma_{a_1 \ldots a_N b} C_{[p_2 \ldots p_N - 1 p]}^{a_2 \ldots a_{N-1} b} - \sigma_{a_1 a_2 b} C_{[p_1 p_3 p_4 \ldots p_N]}^{a_2 a_3 a_4 \ldots a_N} \right)
\]
\[ G^{a_1 \ldots a_N}_{[p_1 \ldots p_N]} = \frac{1}{V} \left. \frac{\partial^N Z[J]}{\partial J_{a_1 p_1} \ldots J_{a_N p_N}} \right|_{J=0} = - \frac{\lambda}{H_{p_1 p_2} V^2 Z[0]} \sum_{b, c=1}^3 \sum_{n=0}^N \left( \partial^2_{J_{b p_n}} \right) \frac{\partial^N Z[J]}{\partial J_{a_1 p_3} \ldots J_{a_N p_N}} |_{J=0} \cdot 
abla^2 \left( E_{p_1}^2 - E_{p_2}^2 \right) \left( \sum_{m=0}^{N'} G^{a_1 a_2 \ldots a_N}_{[p_1 p_2 \ldots p_N, mn]} - C^{a_1 a_2 \ldots a_N}_{[p_1 p_2 \ldots p_N] m} \right) - \frac{\lambda}{V^2 (E_{p_1}^2 - E_{p_2}^2)} \sum_{k=2}^N \left( \sum_{m=0}^{N'} \frac{G^{a_1 a_2 \ldots a_N}_{[p_k p_2 \ldots p_k, p_{k+1} \ldots p_N, mn]} - C^{a_1 a_2 \ldots a_N}_{[p_k p_2 \ldots p_k, p_{k+1} \ldots p_N] m}}{E_{p_1} - E_{p_k}} \right) \right) \]
\[ = - \frac{\lambda}{(E_{p_1}^2 - E_{p_2}^2) V Z[0]} \partial^N Z[J] \partial_{p_1} \partial_{p_2} \]
The first term of (G.1a) contributes only for $b = a_N$ and $E_m = E_{pN}$ and the second term only for $c = a_2$ and $E_m = E_{p_3}$. This generates the term proportional to $\lambda$. Line (G.1b) produces three different types of terms for arbitrary $E_m$, the $(2 + N)$-point functions with one boundary, the multiplication of 2-point with $N$-point functions, and $(2 + N)$-point functions with two boundaries. If in (G.1b) $E_m = E_{p_k}$ for the first term with $2 \leq k \leq N$ (for the second term with $3 \leq k \leq N$ or $k = 1$), additionally $(k + (N + 2 - k))$-point functions with two boundaries and the multiplication of $k$-point with $(N + 2 - k)$-point functions with one boundary are generated. In case of $E_m = E_{p_1}$ for the left term ($E_m = E_{p_2}$ for the right term) (G.1b) produces either $(1 + 1 + N)$-point functions with three boundaries, $(1 + (1 + N))$-point functions with two boundaries or the multiplication of $(1 + 1)$-point with $N$-point functions.

Finally, we look at (G.1c) and (G.1d) together. The first terms again contribute only for $b = a_N$ and $E_m = E_{p_2}$ in (G.1c) or for $b = a_2$ and $E_m = E_{p_1}$ in (G.1d). Since the sum over $n$ survives, $N$-point functions arise. If $E_n = E_{p_k}$ for $k \neq 1$ in (G.1c) and for $k \neq 2$ in (G.1d) one gets either $(k + (N - k))$-point functions or the multiplication of $k$-point functions with $(N - k)$-point functions with one boundary. For the second term in (G.1c) and (G.1d), each derivative have to be taken into account. If the derivative in front of the brackets in (G.1c) and (G.1d) acts on $J_{nm}^b$ or $J_{mn}^b$, the sum over $n$ survives again and has a prefactor depending on $E_n$, but no $n$ appears in the $N$-point function. If any other derivative $\partial_{J_{pk+1}^{b+2}}$, for some $k \geq 1$, acts on the second term, $n, m, b$ will be fixed and it will produce $N$-point functions, $(k + (N - k))$-point functions with two boundaries and the multiplication of $k$-point with $(N - k)$-point functions. Collecting all and making use of (G.5) to get the correct prefactor in $V$, one finds all the terms appearing in Proposition G.3.

The first term shows that a $(N - 1)$-point function only contributes for different adjacent colours, because of $\sigma_{a_1a_2b}$ and $\sigma_{a_1a_2b}$. This fact is in perfect accordance with a loop expansion. Furthermore, the 2-point function is assigned with a special rôle, since the sum over $m$ only appears for the $N$-point and 2-point function even in the large $N, V$-limit.

It should be emphasised that not all combinations of the colours for the correlation functions are possible. The 2-point function is of the form $G_{[p_1p_2]}^{aa}$ and the 3-point function $\sigma_{abc}G_{[p_1p_2p_3]}^{abc}$. There exists no 4-point function equipped with all three colours simultaneously, and so on. These properties which are first recognized by loop expansion are intrinsically presented in the SDEs.
Correlation functions with more boundary components satisfy also SDE which can be computed. However, to determine them an analog of Theorem 2.1 for the coloured model is necessary.

G.3 Link to the Cubic and Quartic Model

The 3-coloured model with cubic interaction is of particular interest because its Feynman graphs are a subset of the Feynman graphs of the cubic model (see Ch. 3). However, the graphs are additionally decorated by colours which prohibits for instance the tadpole graph and therefore induces a vanishing 1-point function. On the other hand, some graphs need to be counted several times since a graph can have with different colourings which gives a symmetry factor. This symmetry is due to the external matrix $E$ which is taken equally for each colour.

A more detailed perturbative analysis is performed in our paper [HW18]. The perturbative calculation of the 2-point function through the SDE is compared to the Feynman graph calculation up to three loops on the two dimensional Moyal space, and of course both results coincide perfectly. No renormalisation was necessary since each graph is UV finite for spectral dimension $D < 4$.

The SDE of the 2-point function (Proposition G.2) takes a much easier form after $V$-expansion

$$G_{[p_1p_2]}^{aa} = \frac{1}{H_{p_1p_2}} + \frac{\lambda^2}{E_{p_1}^2 - E_{p_2}^2} \left[ G_{[p_1p_2]}^{aa} \frac{1}{V} \sum_{m=0}^{N} \sum_{b=1}^{3} \left( G_{[p_2m]}^{bb} - G_{[p_1m]}^{bb} \right) \right] \quad (G.1)$$

$$+ \frac{1}{V} \sum_{m=0}^{N} \frac{G_{[m]}^{aa} - G_{[p_2]}^{aa}}{E_{p_2} - E_m} - \frac{1}{V} \sum_{m=0}^{N} \frac{G_{[p_1]}^{aa} - G_{[p_2]}^{aa}}{E_m - E_{p_1}} + O(V^{-1}).$$

This equation is manifestly symmetric in $p_1, p_2$. Symmetrising the SDE (Proposition 4.1) of the 2-point function for the quartic model after $V$-expansion yields

$$G_{[pq]} = \frac{1}{E_p + E_q} - \frac{\lambda}{2(E_p + E_q)} \left[ G_{[pq]} \frac{1}{V} \sum_{n=0}^{N} \left( G_{[pn]} + G_{[qn]} \right) \right] \quad (G.2)$$

$$+ \frac{1}{V} \sum_{n=0}^{N} \frac{G_{[pq]} - G_{[nq]}}{E_n - E_q} + \frac{1}{V} \sum_{n=0}^{N} \frac{G_{[pq]} - G_{[np]}}{E_n - E_p} + O(V^{-1}).$$

The equations show an incredible similarity. Looking at the graphs of the 3-coloured model, all propagators of a chosen colour can be contracted. It means that all vertices which were connected by this colour are now coincident. The resulting graph has vertices of valence 4 each weighted with a factor $\lambda^2$. Furthermore, the vertices carry the dynamics of the contracted propagator. However, topologically the same graphs appears in the perturbative expansion as in the quartic model, but with some additional decoration and constraints.

This contraction of propagators of one chosen colour is understood as integrating out the corresponding field for the partition function. It is a straightforward calculation to integrate out for instance the field $\Phi^3$ to get

$$Z[0] = \int \prod_{a=1}^{3} D\Phi^a \exp \left\{ - \text{Tr} \left( \sum_{a=1}^{3} E\Phi^a\Phi^a + \lambda\Phi^1\Phi^2\Phi^3 + \lambda\Phi^1\Phi^3\Phi^2 \right) \right\}$$
Figure G.1: On the lhs, two Feynman graphs are at order $\lambda^2$ of the 3-coloured model. After contracting the green coloured propagator (rhs), two Feynman graphs appears with a quartic interaction.

$$=C(E) \int D\Phi^1 D\Phi^2 \exp \left\{ - \text{Tr}(E\Phi^1\Phi^1 + E\Phi^1\Phi^1) + \frac{\lambda^2}{2} \sum_{n,m=0}^N (\Phi^1\Phi^2)_{nm}(\Phi^2\Phi^1)_{mn} \right\},$$

where $C(E)$ is a constant depending on $E$. Now, we have a quartic interaction with two propagators of each colour attached to the vertex. The vertex has the weight $\lambda^2$ and an additional dynamics through the denominator of $\frac{(\Phi^1\Phi^2)_{nm}(\Phi^2\Phi^1)_{mn}}{E_n + E_m}$, which fits perfectly with the considerations at the Feynman graph level above.
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