HELSON'S PROBLEM FOR SUMS OF A RANDOM MULTIPLICATIVE FUNCTION

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ABSTRACT. We consider the random functions $S_N(z) := \sum_{n=1}^N z(n)$, where $z(n)$ is the completely multiplicative random function generated by independent Steinhaus variables $z(p)$. It is shown that $(\mathbb{E}|S_N|^q)^{1/q} \gg q \sqrt{N} (\log N)^{-0.02152}$ for all $q > 0$.

1. INTRODUCTION

This paper deals with the following Question. Do there exist absolute constants $c > 0, 0 < \lambda < 1$ such that for every positive integer $N$ and every interval $I$ whose length exceeds some number depending on $N$, we have

$$\left| \sum_{n=1}^N n^{-it} \right| \geq c \sqrt{N}$$

on a subset of $I$ of measure larger than $\lambda |I|$?

We do not know the answer and can only conclude from our main result that we have, for every $\varepsilon > 0$ and suitable $c = c(\varepsilon)$,

$$\left| \sum_{n=1}^N n^{-it} \right| \geq c \sqrt{N} (\log N)^{-0.02152}$$

on a subset of measure $(\log N)^{-\varepsilon} |I|$ of every sufficiently large interval $I$.

Our question fits into the following general framework. We begin by associating with every prime $p$ a random variable $X(p)$ with mean 0 and variance 1, and we assume that these variables are independent and identically distributed. We then define $X(n)$ by requiring it to be a completely multiplicative function for every point in our probability space. Now suppose that $a(n)$ is an arithmetic function which is either 0 or 1 for every $n$. We refer to the sequence

$$C_N(X) := \sum_{n=1}^N a(n) X(n)$$

as the arithmetic chaos associated with $X$ and $a(n)$.

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Our question concerns the case when \( X(p) \) are independent Steinhaus variables \( z(p) \), i.e. the random variable \( z(p) \) is equidistributed on the unit circle. When \( a(n) \equiv 1 \), we refer to the resulting sequence

\[
S_N(z) := \sum_{n=1}^{N} z(n)
\]
as arithmetic Steinhaus chaos. The relation between our question and arithmetic Steinhaus chaos is given by the well-known norm identity

\[
\mathbb{E}(|S_N|^q) = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} \left| \sum_{n=1}^{N} n^{-it} \right|^q dt,
\]
valid for all \( q > 0 \) (see [7, Section 3]).

The point of departure for our research is a result from Helson’s last paper [6] saying that

\[
\mathbb{E}(|S_N|) \gg \sqrt{N}(\log N)^{-1/4};
\]
along with this result, Helson conjectured that \( \mathbb{E}(|S_N|) = o(\sqrt{N}) \) should hold when \( N \to \infty \). This means that Helson anticipated that our question has a negative answer. Our attempt to settle Helson’s conjecture has resulted in a reduction from \( 1/4 \) to \( 0.02152 \) in the exponent of the logarithmic factor in (3).

To get a picture of what our work is about, it is instructive to return for a moment to a general arithmetic chaos \( C := (C_N(X)) \). To this end, let us assume that \( X(p) \) is such that the moments

\[
\|C_N\|_q := \mathbb{E}(|C_N|^q)
\]
are well defined for all \( q > 0 \). We declare the number

\[
q(C) := \inf\left\{ q > 0 : \limsup_{N \to \infty} \|C_N\|_{q+\epsilon} / \|C_N\|_{q+\epsilon} = \infty \right\}
\]
to be the critical exponent of \( C \), setting \( q(C) = \infty \) should the set on the right-hand side be empty. A problem closely related to Helson’s conjecture is that of computing the critical exponent of a given arithmetic chaos. We observe that \( q(C) \geq 2 \) is equivalent to the statement that there exist absolute constants \( c > 0, 0 < \lambda < 1 \) such that

\[
\mathbb{P}\left(|C_N| \geq c \sqrt{N}\right) \geq \lambda
\]
holds for all \( N \), cf. our question. In our case, the critical exponent is strictly smaller than 4, and then a serious obstacle for saying much more is that only even moments are accessible by direct methods.

We will prove the following result about arithmetic Steinhaus chaos.

**Theorem 1.** We have

\[
\|S_N\|_q \gg_q \sqrt{N}(\log N)^{-0.02152}
\]
for all \( q > 0 \).
This estimate is of course of interest only for small $q$; our methods allow us to improve (4) when $q$ is smaller than and close to 2, but we consider this a less interesting point. In the range $q > 2$, we note that the $L^4$ norm has an interesting number theoretic interpretation and has been estimated with high precision [1]:

$$\|S_N\|_4^4 = \frac{12}{\pi^2} N^2 \log N + c N^2 + O\left(\frac{N^{19/13}}{\log N^{7/13}}\right)$$

with $c$ a certain number theoretic constant. This means in particular that the critical exponent of arithmetic Steinhaus chaos $S := (S_N)$ satisfies $q(S) < 4$. We mention without proof that, applying the Hardy–Littlewood inequality from [2] to $S_N^2$, we have been able to verify that in fact $q(S) \leq 8/3$. A further elaboration of our methods could probably lower this estimate slightly, but this would not alter the main conclusion that it remains unknown whether $q(S)$ is positive. Our research has led us to suspect, though, that $q(S) \geq 2$ and hence that Helson’s conjecture is false.

Before turning to the proof of Theorem 1, we mention the following simple fact: There exists a constant $c < 1$ such that $\|S_N\|_1 \leq c\|S_N\|_2$ when $N \geq 2$. To see this, we apply the Cauchy–Schwarz inequality to the product of $(1 - \varepsilon z(2))S_N$ and $(1 - \varepsilon z(2))^{-1}$ to obtain

$$\|S_N\|_1^2 \leq \frac{1}{(1 - \varepsilon^2)} \cdot \left( (1 - \varepsilon)^2 [N/2] + \left\lfloor (N + 1)/2 \right\rfloor \right) \leq \frac{N - (\varepsilon + \varepsilon^2/2)(N - 1)}{1 - \varepsilon^2}$$

for every $0 < \varepsilon < 1$. Choosing a suitable small $\varepsilon$, we obtain the desired constant $c < 1$.

2. PROOF OF THEOREM [1]

Our proof starts from a decomposition of $S_N$ into a sum of homogeneous polynomials. To this end, we set

$$E_{N,m} := \{n \leq N : \Omega(n) = m\},$$

where $\Omega(n)$ is the number of prime factors of $n$, counting multiplicities. Correspondingly, we introduce the homogeneous polynomials

$$S_{N,m}(z) := \sum_{n \in E_{N,m}} z(n)$$

so that we may write

$$S_N(z) = \sum_{m=0}^{(\log N)/\log 2} S_{N,m}(z).$$

(5)

We need two lemmas. The first is a well-known estimate of Sathe; the standard reference for this result is Selberg’s paper [8]. To formulate this lemma, we introduce the function

$$\Phi(z) := \frac{1}{\Gamma(z+1)} \prod_p \frac{(1 - 1/p)^z}{(1 - z/p)^{-1}},$$

where the product runs over all prime numbers $p$. This function is meromorphic in $\mathbb{C}$ with simple poles at the primes and zeros at the negative integers.
Lemma 2. When $N \geq 3$ and $1 \leq m \leq (2 - \varepsilon) \log \log N$ for $0 < \varepsilon < 1$, we have
\[
|E_{N,m}| = \frac{N}{\log N} \Phi \left( \frac{m}{\log \log N} \right) \frac{(\log \log N)^{m-1}}{(m-1)!} \left( 1 + O \left( \frac{1}{\log \log N} \right) \right),
\]
where the implied constant in the error term only depends on $\varepsilon$.

We will use the following consequence of this estimate. Suppose that $m = \beta \log \log N$ with $\beta \leq 2 - \varepsilon$. Then by Lemma 2 and Stirling’s formula, we have
\[
|E_{N,m}| \asymp N(\log N)^{\beta - \beta \log \beta - 1/2},
\]
where the implied constants only depend on $\varepsilon$.

The second lemma is a general statement about the decomposition of a holomorphic function into a sum of homogeneous polynomials. For simplicity, we consider only an arbitrary holomorphic polynomial $P(z)$ in $d$ complex variables $z = (z_1, \ldots, z_d)$. Such a polynomial has a unique decomposition
\[
P(z) = \sum_{m=0}^{k} P_m(z),
\]
where $k$ is the degree of $P$ and
\[
P_m(z) = \sum_{|\alpha| = m} a_\alpha z^\alpha
\]
is a homogeneous polynomial of degree $m$. Here we use standard multi-index notation, which means that $\alpha = (\alpha_1, \ldots, \alpha_d)$, where $\alpha_1, \ldots, \alpha_d$ are nonnegative integers,
\[
z^\alpha = z_1^{\alpha_1} \cdots z_d^{\alpha_d},
\]
and $|\alpha| = \alpha_1 + \cdots + \alpha_d$. At this point, the reader should recognize that if we represent an arbitrary integer $n \leq N$ by its prime factorization $p_1^{\alpha_1} \cdots p_d^{\alpha_d}$ (here $d = \pi(N)$) and set $\alpha(n) = (\alpha_1, \ldots, \alpha_d)$, then we may write
\[
S_N(z) = \sum_{n=1}^{N} z^{\alpha(n)}.
\]
Hence, as already pointed out, (5) is the decomposition of $S_N$ into a sum of homogeneous polynomials, and we also see that $|\alpha(n)| = \Omega(n)$.

We let $\mu_d$ denote normalized Lebesgue measure on $\mathbb{T}^d$ and define
\[
\|P\|_q^d := \int_{\mathbb{T}^d} |P(z)|^q d\mu_d(z)
\]
for every $q > 0$. The variables $z_1, \ldots, z_d$ can be viewed as independent Steinhaus variables so that $\|S_N\|_q$ has the same meaning as before.

Lemma 3. There exists an absolute constant $C$, independent of $d$, such that
\[
\|P_m\|_q \leq \begin{cases} \|P\|_q, & q \geq 1 \\ C m^{1/q-1} \|P\|_q, & 0 < q < 1 \end{cases}
\]
holds for every holomorphic polynomial $P$ of $d$ complex variables.
Proof. We introduce the transformation $z_w = (wz_1, ..., wz_d)$, where $w$ is a point on the unit circle $\mathbb{T}$. We may then write

$$P(z_w) = \sum_{m=0}^{k} P_m(z) w^m.$$  

It follows that we may consider the polynomials $P_m(z)$ as the coefficients of a polynomial in one complex variable. Then a classical coefficient estimate (see [3, p. 98]) shows that

$$|P_m(z)|^q \leq \begin{cases} \int_{\mathbb{T}} |P_m(z_w)|^q d\mu_1(w), & q \geq 1 \\ C m^{1-q} \int_{\mathbb{T}} |P_m(z_w)|^q d\mu_1(w), & 0 < q < 1. \end{cases}$$  

Integrating this inequality over $\mathbb{T}^d$ with respect to $d\mu_d(z)$ and using Fubini’s theorem, we obtain the desired estimate. \hfill \Box

We now turn to the proof of Theorem [1] The idea of the proof can be related to an interesting study of Harper [5] from which it can be deduced that, asymptotically, the square-free part of the homogeneous polynomial $S_{N,m}/\sqrt{N}$ has a Gaussian distribution when $m = o(\log \log N)$. When $m = 1/2 \log \log N$ for $\beta$ bounded away from 0, this is no longer so, but what we will use, is a much weaker statement: When $\beta$ is small enough, the $L^2$ and $L^4$ norms are comparable. The proof will consist in identifying for which $\beta$ this holds.

To this end, we assume that $m = \beta \log \log N$ and observe first that then (6) implies that

$$\|S_{N,m}\|_2^2 = |E_{N,m}| = N(\log N)^{\beta - \beta \log \beta - 1} m^{-1/2}. \quad (7)$$

To estimate $\|S_{N,m}\|_4^4$, we begin by noting that

$$|S_{N,m}|^2 = |E_{N,m}| + \sum_{k=0}^{m} \sum_{a,b \in E_{N,k}, (a,b)=1} |E_{N/\max(a,b),m-k}| z(a) \overline{z(b)}. \quad (8)$$

Squaring this expression and taking expectation, we obtain

$$\|S_{N,m}\|_4^4 = |E_{N,m}|^2 + 2 \sum_{k=0}^{m} \sum_{a,b \in E_{N,k}, (a,b)=1, a<b} |E_{N/b,m-k}|^2 \leq |E_{N,m}|^2 + 2 \sum_{k=0}^{m} \sum_{b \in E_{N,k}} |E_{b,k}| \cdot |E_{N/b,m-k}|^2 \leq |E_{N,m}|^2 + 2(m+1) \max_{0 \leq k \leq m} \sum_{b \in E_{N,k}} |E_{b,k}| \cdot |E_{N/b,m-k}|^2. \quad (8)$$

To estimate the maximum in (8), we fix $k$ and let $\gamma \leq \beta$ be the number such that $k = \gamma \log \log N$. To simply the writing, we also set $\kappa := \beta - \gamma$. Then using (6), we obtain

$$\sum_{b \in E_{N,k}} |E_{b,k}| \cdot |E_{N/b,m-k}|^2 \ll \frac{N^2}{\sqrt{k(m-k)}} \sum_{b \in E_{N,k}} b^{-1}(\log b)^{\gamma - \gamma \log \gamma - 1} \left(\log \frac{N}{b}\right)^{2(\kappa - \kappa \log \kappa - 1)} \ll \frac{N^2}{k(m-k)}(\log N)^{2(\gamma - \gamma \log \gamma + \kappa \log \kappa - 3)},$$
where we in the last step used Abel’s summation formula when summing with respect $b$. A calculus argument shows that the maximum occurs when $\gamma = \beta/2$ so that, returning to (8) and also using (7), we obtain

$$\|S_{N,m}\|_4^4 \ll \|S_N\|^2 \left(1 + (\log N)^{2\beta \log 2 - 1}\right).$$

It follows that the two norms are comparable whenever $\beta \leq (2\log 2)^{-1}$, in which case Hölder’s inequality yields

$$\|S_{N,m}\|_2 \ll \|S_{N,m}\|_q$$

for $0 < q < 2$. We notice that

$$\delta := - \left(\frac{(2\log 2)^{-1} + (2\log 2)^{-1}\log(2\log 2) - 1}{2} - 0.02152\right)$$

and see that if $m \leq (2\log 2)^{-1}\log\log N$, then (9) along with (7) and Lemma 3 gives

$$\sqrt{N}(\log N)^{-\delta} (\log\log N)^{-1/2} \ll \|S_{N,m}\|_q \ll (\log\log N)^{\max(1/q - 1, 0)} \|S_N\|_q$$

when $0 < q < 2$.

3. CONCLUDING REMARKS

1. We will now deduce (1) from Theorem 1. Since $t \mapsto \sum_{n=1}^N n^{-it}$ is an almost periodic function, it suffices to consider the interval $I = [0, T]$ for some large $T$. Moreover, by (2), it amounts to the same to estimate the measure of the set

$$E := \{z : |S_N(z)| \geq c \sqrt{N}(\log N)^{-0.02152}\}$$

for a suitable $c$ depending on $\epsilon$. We find that

$$\|S_N\|_q^q \leq c^q N^{q/2} (\log N)^{-0.02152q} + \int_{\mathcal{E}} |S_N(z)|^q d\mu_{\mathcal{E}}(z)$$

$$\leq c^q N^{q/2} (\log N)^{-0.02152q} + \|S_N\|_2^q |\mathcal{E}|^{1-q/2},$$

where we in the last step used Hölder’s inequality. Given $\epsilon > 0$, we choose $q$ such that $\epsilon/2 = 0.02152q/(1 - q/2)$ and adjust $c$ accordingly so that (10) and Theorem 1 together yield

$$|\mathcal{E}| \geq (c^{q/2} (1-q/2)^{-1} (\log N)^{-\epsilon/2}.$$

2. Our proof shows that we essentially need $\beta \leq (2\log 2)^{-1}$ for the projection

$$P_{\beta}S_N := \sum_{m \leq \beta \log\log N} S_{N,m}$$

to have comparable $L^2$ and $L^4$ norms. To use our method of proof to show that Helson’s conjecture fails, it would suffice to know that the projection $P_1 S_N$ has comparable $L^2$ and $L^q$ norms for some $q > 2$, because in that case $\|P_1 S_N\|_2 \geq (1 + o(1)) \sqrt{N}$. It seems reasonable to conjecture that this holds and hence that Helson’s conjecture is false.

3. Finally, we would like to point out that Helson’s problem makes sense for other distributions. An interesting case is when $X(p)$ are independent Rademacher functions $\epsilon(p)$ taking values $+1$.
and −1 each with probability 1/2. If we set \( a(n) = |\mu(n)| \) (here \( \mu(n) \) is the Möbius function), then we obtain arithmetic Rademacher chaos:

\[
R_N(\varepsilon) := \sum_{n=1}^{N} |\mu(n)| \varepsilon(n).
\]

Rademacher chaos was first considered by Wintner [9] and has been studied by many authors, see e.g. [4, 5]. We suspect in this case as well, for the same reason as just indicated for \( S_N \), that the critical exponent is at least 2.

**REFERENCES**

[1] A. Ayyad, T. Cochrane, and Z. Zheng, *The congruence \( x_1 x_2 = x_3 x_4 \pmod{p} \), the equation \( x_1 x_2 = x_3 x_4 \), and mean values of character sums*, J. Number Theory 59 (1996), 398–413.

[2] A. Bondarenko, W. Heap, M. Radziwiłł, and K. Seip, *An inequality of Hardy–Littlewood type for Dirichlet polynomials*, arXiv:1405.6516, 2014.

[3] P. L. Duren, *Theory of \( H^p \) Spaces*, Academic Press, New York 1970; reprinted by Dover, Mineola NY, 2000.

[4] G. Halász, *On random multiplicative functions*, In Proceedings of the Hubert Delange colloquium, (Orsay, 1982), pp 74–96. Publ. Math. Orsay, Univ. Paris XI, 1983.

[5] A. Harper, *On the limit distributions of some sums of a random multiplicative function*, J. Reine Angew. Math. 678 (2013), 95–124.

[6] H. Helson, *Hankel forms*, Studia Math. 198 (2010), 79–84.

[7] E. Saksman and K. Seip, *Integral means and boundary limits of Dirichlet series*, Bull. London Math. Soc. 41 (2009), 411–422.

[8] A. Selberg, *Note on a theorem of L. G. Sathe*, J. Indian Math. Soc. 18 (1954), 83–87.

[9] A. Wintner, *Random factorizations and Riemann’s hypothesis*, Duke Math. J. 11 (1944), 267–275.