Stationary Josephson effect in a weak-link between nonunitary triplet superconductors

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A stationary Josephson effect in a weak-link between misoriented nonunitary triplet superconductors is investigated theoretically. The non-self-consistent quasiclassical Eilenberger equation for this system has been solved analytically. As an application of this analytical calculation, the current-phase diagrams are plotted for the junction between two nonunitary bipolar $f$-wave superconducting banks. A spontaneous current parallel to the interface between superconductors has been observed. Also, the effect of misorientation between crystals on the Josephson and spontaneous currents is investigated. Such experimental investigations of the current-phase diagrams can be used to test the pairing symmetry in the above-mentioned superconductors.

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I. INTRODUCTION

In recent years, the triplet superconductivity has become one of the modern subjects for researchers in the field of superconductivity \cite{1,2,3}. Particularly, the nonunitary spin triplet state in which Cooper pairs may carry a finite averaged intrinsic spin momentum has attracted much attention in the last decade \cite{1,3,4,5}. A triplet state in the momentum space $k$ can be described by the order parameter $\hat{\Delta}(k) = i(d(k) \cdot \hat{\sigma})d^\dagger(k)$ in a 2X2 matrix form in which $\hat{\sigma} = 2X2$ Pauli matrices $(\hat{\sigma} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z))$. The three dimensional complex vector $(d(k) \text{ (gap vector)})$ describes the triplet pairing state. In the nonunitary state, the product $\Delta(k)\Delta(k)^\dagger = d(k) \cdot d^\dagger(k) + i(d(k) \times d^\dagger(k)) \cdot \hat{\sigma}$ is not a multiple of the unit matrix. Thus in a non-unitary state the time reversal symmetry is necessarily broken spontaneously and a spontaneous moment $m(k) = i(d(k) \times d^\dagger(k))$ appears at each point $k$ of the momentum space. In this case the macroscopically averaged moment $<m(k)>$ integrated on the Fermi surface does not vanish. The value $m(k)$ is related to the net spin average by $Tr[\hat{\Delta}(k)^\dagger \hat{\sigma}_j \hat{\Delta}(k)]$. It is clear that the total spin average over the Fermi surface can be nonzero. As an application, the nonunitary bipolar state of $f$-wave superconductivity has been considered for the $B$-phase of superconductivity in the $UPt_3$ compound which has been created at low temperatures $T$ and small values of the magnetic field $H$ \cite{3,4,5,6}.

In the present paper, the ballistic Josephson weak link via an interface between two superconducting bulks with different orientations of the crystallographic axes is investigated. This type of weak link structure can be used for the demonstration of the pairing symmetry in the superconducting phase \cite{2}. Consequently, we generalize the formalism of paper \cite{6} for the weak link between triplet superconducting bulks with a nonunitary order parameter. In the paper \cite{6}, the Josephson effect in the point contact between unitary $f$-wave triplet superconductors has been studied. Also, the effect of misorientation on the charge transport has been investigated and a spontaneous current tangential to the interface between the $f$-wave superconductors, has been observed.

In this paper the nonunitary bipolar $f$-wave model of the order parameter is considered. It is shown that the current-phase diagrams are totally different from the current-phase diagrams of the junction between the unitary triplet (axial and planar) $f$-wave superconductors \cite{3}. Roughly speaking, these different characters can be used to distinguish between nonunitary bipolar $f$-wave superconductivity and the other types of superconductivity. In the weak-link structure between the nonunitary $f$-wave superconductors, the spontaneous current parallel to the interface has been observed as a fingerprint for unconventional superconductivity and spontaneous time reversal symmetry breaking. The effect of misorientation on the spontaneous and Josephson currents is investigated. It is possible to find the value of the phase difference in which the Josephson current is zero but the spontaneous current, which is produced by the interface and is tangential to the interface, is present. In some configurations and at the zero phase difference, the Josephson current is not generally zero but has a finite value. This finite value corresponds to a spontaneous phase difference which is related to the misorientation between the gap vectors $d$.

The arrangement of the rest of this paper is as follows. In Sec\textsuperscript{III} we describe the configuration that we have investigated. For a non-self-consistent model of the order parameter, the quasiclassical Eilenberger equations \cite{1} are solved and suitable Green functions have been obtained analytically. In Sec\textsuperscript{III} the formulas obtained for the Green functions have been used for the calculation of the current densities at the interface. An analysis of numerical results will be presented in Sec\textsuperscript{IV} together with some conclusions in Sec\textsuperscript{V}.
II. FORMALISM AND BASIC EQUATIONS

We consider a model of a flat interface $y = 0$ between two misoriented nonunitary $f$-wave superconducting half-spaces (Fig.1) as a ballistic Josephson junction. In the quasiclassical ballistic approach, in order to calculate the current, we use “transport-like” quasiclassical Eilenberger equations \( \mathbf{\tilde{g}}(\tilde{\mathbf{v}}_F, \mathbf{r}, \varepsilon_m) \) for the energy integrated Green functions $\mathbf{g}(\tilde{\mathbf{v}}_F, \mathbf{r}, \varepsilon_m)$

$$\mathbf{v}_F \cdot \nabla \mathbf{g} + \left[ \varepsilon_m \tilde{\sigma}_3 + i \Delta, \mathbf{g} \right] = 0,$$  \tag{1}

and the normalization condition $\mathbf{g} \cdot \mathbf{g} = 1$, where $\varepsilon_m = \pi T(2m + 1)$ are discrete Matsubara energies $m = 0, 1, 2, \ldots, T$ is the temperature, $\mathbf{v}_F$ is the Fermi velocity and $\tilde{\sigma}_3 = \sigma_3 \otimes \mathbf{1}$ in which $\sigma_j (j = 1, 2, 3)$ are Pauli matrices. The Matsubara propagator $\mathbf{g}$ can be written in the form:

$$\mathbf{g} = \left( \begin{array}{cc} g_{1} + \mathbf{g}_1 \cdot \tilde{\sigma} & (g_{2} + \mathbf{g}_2 \cdot \tilde{\sigma}) i \tilde{\sigma}_2 \\ i \tilde{\sigma}_2 (g_{3} + \mathbf{g}_3 \cdot \tilde{\sigma}) & g_{4} - \mathbf{g}_4 \cdot \tilde{\sigma}_2 \end{array} \right),$$ \tag{2}

where the matrix structure of the off-diagonal self energy $\Delta$ in the Nambu space is

$$\Delta = \left( \begin{array}{cc} 0 & \mathbf{d} \cdot \tilde{\sigma} i \tilde{\sigma}_2 \\ i \tilde{\sigma}_2 \mathbf{d}^* \cdot \tilde{\sigma} & 0 \end{array} \right).$$ \tag{3}

The nonunitary states, for which $\mathbf{d} \times \mathbf{d}^* \neq 0$, are investigated. Fundamentally, the gap vector (order parameter) $\mathbf{d}$ has to be determined numerically from the self-consistency equation \( \mathbf{d} \), while in the present paper, we use a non-self-consistent model for the gap vector which is much more suitable for analytical calculations \( \mathbf{g} \). Solutions to Eq. \( \mathbf{g} \) must satisfy the conditions for the Green functions and the gap vector $\mathbf{d}$ in the bulks of the superconductors far from the interface as follow:

$$\mathbf{g} = \frac{1}{\Omega_n} \left( \frac{\varepsilon_m (1 - \mathbf{A}_n \cdot \tilde{\sigma})}{i \tilde{\sigma}_2 [i \mathbf{d}_n^* \times \mathbf{A}_n] \cdot \tilde{\sigma}} - \frac{[i \mathbf{d}_n - \mathbf{d}_n \times \mathbf{A}_n] \cdot \tilde{\sigma} i \tilde{\sigma}_2}{-2 \mathbf{d}_n (1 + \mathbf{A}_n \cdot \tilde{\sigma}) \tilde{\sigma}_2} \right) \tag{4}
$$

where

$$\mathbf{A}_n = \frac{i \mathbf{d}_n \times \mathbf{d}_n^*}{\varepsilon_m^2 + \mathbf{d}_n \cdot \mathbf{d}_n^* + \sqrt{\varepsilon_m^2 + \mathbf{d}_n \cdot \mathbf{d}_n^*}^2 + (\mathbf{d}_n \times \mathbf{d}_n^*)^2}$$ \tag{5}

and

$$\Omega_n = \sqrt{\frac{2(\varepsilon_m^2 + \mathbf{d}_n \cdot \mathbf{d}_n^*)^2 + (\mathbf{d}_n \times \mathbf{d}_n^*)^2}{\varepsilon_m^2 + \mathbf{d}_n \cdot \mathbf{d}_n^* + \sqrt{\varepsilon_m^2 + \mathbf{d}_n \cdot \mathbf{d}_n^*}^2 + (\mathbf{d}_n \times \mathbf{d}_n^*)^2}}$$ \tag{6}

$$\mathbf{d}(\pm \infty) = \mathbf{d}_{2,1}(\mathbf{r}, \mathbf{v}_F) \exp \left( \mp \frac{i \phi}{2} \right)$$ \tag{7}

where $\phi$ is the external phase difference between the order parameters of the bulks and $n = 1, 2$ label the left and right half spaces respectively. It is clear that poles of the Green function in the energy space are in

$$\Omega_n = 0.$$ \tag{8}

Consequently,

$$-E^2 + \mathbf{d}_n \cdot \mathbf{d}_n^* + (\mathbf{d}_n \times \mathbf{d}_n^*)^2 = 0$$ \tag{9}

and

$$E = \pm \sqrt{\mathbf{d}_n \cdot \mathbf{d}_n^* \pm i \mathbf{d}_n \times \mathbf{d}_n^*}$$ \tag{10}

in which $E$ is the energy value of the poles. The Eq. \( \mathbf{g} \) has to be supplemented by the continuity conditions at the interface between superconductors. For all quasiparticle trajectories, the Green functions satisfy the boundary conditions both in the right and left bulks as well as at the interface. The system of equations \( \mathbf{g} \) and the self-consistency equation for the gap vector $\mathbf{d}$ \( \mathbf{d} \) can be solved only numerically. For unconventional superconductors such solution requires the information about the interaction between the electrons in the Cooper pairs and the nature of unconventional superconductivity in novel compounds which in most cases are unknown. Also, it has been shown that the absolute value of a self-consistent order parameter is suppressed near the interface and at the distances of the order of the coherence length, while its dependence on the direction in the momentum space almost remains unaltered \( \mathbf{d} \). This suppression of the order parameter changes the amplitude value of the current, but does not influence the current-phase dependence drastically. For example, it has been verified in Refs. \( \mathbf{d} \) for the junction between unconventional $d$-wave superconductors, in Ref. \( \mathbf{d} \) for the case of unitary “$f$-wave” superconductors and in Ref. \( \mathbf{d} \) for pinholes in $3He$, that there is good qualitative agreement between self-consistent and non-self-consistent results for not very large angles of misorientation. It has also been observed that the results of the non-self-consistent model in \( \mathbf{d} \) are similar to experiment \( \mathbf{d} \). Consequently, despite the fact that this solution cannot be applied directly to a quantitative analysis of a real experiment, only
a qualitative comparison of calculated and experimental current-phase relations is possible. In our calculations, a simple model of the constant order parameter up to the interface is considered and the pair breaking and scattering on the interface are ignored. We believe that under these strong assumptions our results describe the real situation qualitatively. In the framework of such a model, the analytical expressions for the current can be obtained for a certain form of the order parameter.

III. ANALYTICAL RESULTS

The solution of Eq. (11) allows us to calculate the current densities. The expression for the current is:

$$\mathbf{j}(\mathbf{r}) = 2i\pi e TN(0) \sum_m \langle \mathbf{v}_F g_1 (\mathbf{v}_F, r, \varepsilon_m) \rangle$$ (11)

where $\langle ... \rangle$ stands for averaging over the directions of an electron momentum on the Fermi surface $\mathbf{v}_F$, and $N(0)$ is the electron density of states at the Fermi level of energy. We assume that the order parameter is constant in space and in each half-space it equals its value far from the interface in the left or right bulk. For such a model, the current-phase dependence of a Josephson junction can be calculated analytically. It enables us to analyze the main features of current-phase dependence for any model of the nonunitary order parameter. The Eilenberger equations (11) for Green functions $\tilde{g}_i$, which are supplemented by the condition of continuity of solutions across the interface, $y = 0$, and the boundary conditions at the bulk, are solved for a non-self-consistent model of the order parameter analytically. In the ballistic case the system of equations for functions $g_i$ and $\tilde{g}_i$ can be decomposed into independent blocks of equations. The set of equations which enables us to find the Green function $g_1$ is:

$$v_F k \mathbf{v} \cdot \mathbf{g}_1 = i(\mathbf{d} \cdot \mathbf{g}_3 - \mathbf{d}^* \cdot \mathbf{g}_2);$$ (12)

$$v_F k \mathbf{v} \cdot \mathbf{g}_- = -2(\mathbf{d} \times \mathbf{g}_3 + \mathbf{d}^* \times \mathbf{g}_2);$$ (13)

$$v_F k \mathbf{v} \cdot \mathbf{g}_2 = -2\varepsilon_m \mathbf{g}_3 + 2i\mathbf{d} + \mathbf{d} \times \mathbf{g}_-;$$ (14)

$$v_F k \mathbf{v} \cdot \mathbf{g}_3 = 2\varepsilon_m \mathbf{g}_3 - 2i\mathbf{d}^* + \mathbf{d}^* \times \mathbf{g}_-;$$ (15)

where $g_\pm = g_1 - g_4$. The Eqs. (12)-(15) can be solved by integrating over the ballistic trajectories of electrons in the right and left half-spaces. The general solution satisfying the boundary conditions (11) at infinity is

$$g_1^{(n)} = \frac{\varepsilon_m}{\Omega_n} + a_n e^{-2s\Omega_n t};$$ (16)

$$g_-^{(n)} = -2\frac{\varepsilon_m}{\Omega_n} \mathbf{A}_n + C_n e^{-2s\Omega_n t};$$ (17)

$$g_2^{(n)} = -\frac{i\mathbf{d}_n - \mathbf{d}_n \times \mathbf{A}_n}{\Omega_n} - 2i\alpha_n \mathbf{d}_n + \frac{\mathbf{d}_n \times \mathbf{C}_n e^{-2s\Omega_n t}}{2s\Omega_n - 2\varepsilon_m};$$ (18)

$$g_3^{(n)} = -\frac{i\mathbf{d}_n^* + \mathbf{d}_n^* \times \mathbf{A}_n}{\Omega_n} + 2i\alpha_n \mathbf{d}_n^* - \frac{\mathbf{d}_n^* \times \mathbf{C}_n e^{-2s\Omega_n t}}{2s\Omega_n + 2\varepsilon_m};$$ (19)

where $t$ is the time of flight along the trajectory, $sgn (t) = sgn (y) = s$ and $\eta = sgn (v_y)$. By matching the solutions (11),(14) at the interface ($y = 0, t = 0$), we find constants $a_n$ and $C_n$. Indices $n = 1,2$ label the left and right half-spaces respectively. The function $g_1 (0) = g_1^{(1)} (-0) = \frac{\varepsilon_m}{\Omega_n}$.
\( g_1^{(2)} (+0) \) which is a diagonal term of the Green matrix and determines the current density at the interface, \( y = 0 \), as follows:

\[
g_1 (0) = \frac{\eta (d_2 \cdot d_1 (\eta \Omega_1 + \varepsilon)^2 - d_1 \cdot d_1 (\eta \Omega_2 - \varepsilon)^2 + B)}{(d_2 (\eta \Omega_1 + \varepsilon) + d_1 (\eta \Omega_2 - \varepsilon))^2} \tag{20}
\]

where \( B = i d_1 \times d_2 \cdot (A_1 + A_2) / (\eta \Omega_2 - \varepsilon) (\eta \Omega_1 + \varepsilon) \). We consider a rotation \( \tilde{R} \) only in the right superconductor (see Fig.1), i.e., \( d_2 (k) = \tilde{R} d_1 (\tilde{R}^{-1} k) \); \( k \) is the unit vector in the momentum space. The crystallographic \( c \)-axis in the left half-space is selected parallel to the partition between the superconductors (along the \( z \)-axis in Fig.1). To illustrate the results obtained by computing the formula \( \tag{20} \), we plot the current-phase diagrams for two different geometries. These geometries correspond to the different orientations of the crystals in the right and left sides of the interface (Fig.1):

(i) The basal \( ab \)-plane in the right side has been rotated around the \( c \)-axis by \( \alpha \); \( \hat{c}_1 \parallel \hat{c}_2 \).

(ii) The \( c \)-axis in the right side has been rotated around the \( b \)-axis by \( \alpha \); \( \hat{b}_1 \parallel \hat{b}_2 \).

Further calculations require a certain model of the gap vector (order parameter) \( d \).

**IV. ANALYSIS OF NUMERICAL RESULTS**

In the present paper, the nonunitary \( f \)-wave gap vector in the \( B \)-phase (low temperature \( T \) and low field \( H \)) of superconductivity in \( UPt_3 \) compound has been considered. This nonunitary bipolar state which explains the weak spin-orbit coupling in \( UPt_3 \) is \( \vec{k} \):

\[
d(T, v_F) = \Delta_0 (T) k_z \vec{x} \left( k_x^2 - k_y^2 \right) + \gamma 2ik_z k_y. \tag{21}
\]

The coordinate axes \( \hat{x}, \hat{y}, \hat{z} \) are selected along the crystallographic axes \( \hat{a}, \hat{b}, \hat{c} \) in the left side of Fig.1. The function \( \Delta_0 = \Delta_0 (T) \) describes the dependence of the gap vector on the temperature \( T \) (our numerical calculations are done at the low value of temperature \( T/T_c = 0.1 \)). Using this model of the order parameter \( \tag{20} \) and solution to the Eilenberger equations \( \tag{20} \), we have calculated the current density at the interface numerically. These numerical results are listed below:

1) The nonunitary property of Green’s matrix diagonal term consists of two parts. The explicit part which is in the \( B \) mathematical expression in Eq.\( \tag{20} \) and the implicit part in the \( \Omega_{1,2} \) and \( d_{1,2} \) terms. These \( \Omega_{1,2} \) and \( d_{1,2} \) terms are different from their unitary counterparts. In the mathematical expression for \( \Omega_{1,2} \) the nonunitary mathematical terms \( A_{1,2} \) are presented. The explicit part will be present only in the presence of misorientation between gap vectors, \( B = i d_1 \times d_2 \cdot (A_1 + A_2) / (\eta \Omega_2 - \varepsilon) (\eta \Omega_1 + \varepsilon) \), but the implicit part will be present always. So, in the absence of misorientation \( (d_1 \parallel d_2) \), although the implicit part of nonunitary exists the explicit part is absent. This means that in the absence of misorientation, current-phase diagrams for planar unitary and nonunitary bipolar systems are the same but the maximum values are slightly different.

2) A component of current parallel to the interface \( j_z \) for geometry (i) is zero similar to the unitary case \( \vec{k} \) while the other parallel component \( j_z \) has a finite value (see Fig.4). This latter case is a difference between unitary and nonunitary cases. Because in the junction between unitary \( f \)-wave superconducting bulks all parallel components of the current \( (j_x \text{ and } j_z) \) for geometry (i) are absent \( \vec{k} \).
3) In Figs.2,3 the Josephson current $j_y$ is plotted for a certain nonunitary model of $f$–wave and different geometries. Figs.2,3 are plotted for the geometry (i) and geometry (ii) respectively. They are completely unusual and totally different from their unitary counterparts which have been obtained in [8].

4) In Fig.2 for geometry (i), it is observed that by increasing the misorientation, some small oscillations appear in the current–phase diagrams as a result of the non-unitary property of the order parameter. Also, the Josephson current at the zero external phase difference $\phi = 0$ is not zero but has a finite value. The Josephson current will be zero at the some finite values of the phase difference.

5) In Fig.3 for geometry (ii), it is observed that by increasing the misorientation the new zeros in current–phase diagrams appear and the maximum value of the current will be changed non-monotonically. In spite of the Fig.2 for geometry (i), the Josephson currents at the phase differences $\phi = 0$, $\phi = \pi$, and $\phi = 2\pi$ are zero exactly.

6) The current–phase diagram for geometry (i) and $x$–component (Fig.4) is totally unusual. By increasing the misorientation, the maximum value of the current increases. The components of current parallel to the interface for geometry (ii) are plotted in Fig.4 All the terms at the phase differences $\phi = 0$, $\phi = \pi$, and $\phi = 2\pi$ are zero. The maximum value of the current–phase diagrams is not a monotonic function of the misorientation.

V. CONCLUSIONS

Thus, we have theoretically studied the supercurrents in the ballistic Josephson junction in the model of an ideal transparent interface between two misoriented $UPT_3$ crystals with nonunitary bipolar $f$–wave superconducting bulks which are subject to a phase difference $\phi$. Our analysis has shown that misorientation between the gap vectors creates a current parallel to the interface and different misorientations between gap vectors influence the spontaneous parallel and normal Josephson currents. These have been shown for the currents in the point contact between two bulks of unitary axial and planar $f$-wave superconductor in [8] separately. Also, it is shown that the misorientation of the superconductors leads to a spontaneous phase difference that corresponds to the zero Josephson current and to the minimum of the weak link energy in the presence of the finite spontaneous current. This phase difference depends on the misorientation angle. The tangential spontaneous current is not generally equal to zero in the absence of the Josephson current. The difference between unitary planar and nonunitary bipolar states can be used to distinguish between them. This experiment can be used to test the pairing symmetry and recognize the different phases of $UPT_3$.