LIGHTLIKE HYPERSURFACES ON MANIFOLDS ENDOWED WITH A CONFORMAL STRUCTURE OF LORENTZIAN SIGNATURE

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Abstract. The authors study the geometry of lightlike hypersurfaces on manifolds \((M, c)\) endowed with a pseudoconformal structure \(c = CO(n - 1, 1)\) of Lorentzian signature. Such hypersurfaces are of interest in general relativity since they can be models of different types of physical horizons. On a lightlike hypersurface, the authors consider the fibration of isotropic geodesics and investigate their singular points and singular submanifolds. They construct a conformally invariant normalization of a lightlike hypersurface intrinsically connected with its geometry and investigate affine connections induced by this normalization. The authors also consider special classes of lightlike hypersurfaces. In particular, they investigate lightlike hypersurfaces for which the elements of the constructed normalization are integrable.

0 Introduction

The pseudo-Riemannian manifolds \((M, g)\) of Lorentzian signature play a special role in geometry and physics: they generate models of spacetime of general relativity. In the tangent space \(T_x\) at a point \(x\) of such a manifold, a real isotropic cone \(C_x\) is invariantly defined. From physical point of view, this cone is the light cone—the trajectories of light impulses emanating from the point \(x\) are tangent to the cone \(C_x\).

A hypersurface \(V^{n-1}\) on an \(n\)-dimensional manifold \(M\) of Lorentzian signature that is tangent to the cone \(C_x\) at each point \(x \in V\) is called lightlike. The lightlike hypersurfaces are also of interest for general relativity since they produce models of different type of horizons (event horizons, Cauchy’s horizons, Kruskal’s horizons— see, for example, [Ch 83] and [MTW 73]). Lightlike hypersurfaces are also studied in the theory of electromagnetism. This is the reason that there are many papers and two recent books [DB 96] and [Ku 96] in which lightlike hypersurfaces are investigated.

Many events and objects of general relativity are invariant under conformal transformations of a metric (see [AG 96], Ch. 4 and Ch. 5). In particular, a lightlike hypersurface is an example of the objects that are invariant under conformal transformations of a metric. Hence it is appropriate to study

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lightlike hypersurfaces not only on a pseudo-Riemannian manifold \((M, g)\) of Lorentzian signature but also on a manifold endowed with a conformal structure of Lorentzian signature.

In the present paper we study lightlike hypersurfaces on a differentiable manifold \(M^n\) endowed with a pseudoconformal structure \(CO(n-1, 1)\) of Lorentzian signature. We will denote such manifolds by \((M, c)\) where \(\dim M = n\) and \(c = CO(n-1, 1)\) is a conformal structure of signature \((n-1, 1)\).

Let us describe the contents of the paper. In Section 1 we write the basic equations of the manifold \((M, c)\) and consider pseudoconformal spaces \((C^n)_{x}\) of Lorentzian signature tangent to the manifold \((M, c)\) at its point \(x\). It appeared that it is very convenient to use the Darboux representation of a space \((C^n)_{x}\) as a hyperquadric \((Q^n)_{x}\) of a projective space \(P^{n+1}\).

As we proved in [AG 96] (see also [AG 97]), the isotropic geodesics of a pseudo-Riemannian manifold \((M, g)\) are invariant under conformal transformations of a metric. Thus they can be considered on a manifold \((M, c)\) endowed with a pseudoconformal structure. Under the development of the manifold \((M, c)\) onto a hyperquadric \((Q^n)_{x}\) along an isotropic geodesic, the latter is mapped into a rectilinear generator of the hyperquadric \((Q^n)_{x}\).

In Section 2 we consider differential geometry of lightlike hypersurfaces \(V^{n-1} \subset (M, c)\). In this section we construct a first-order frame bundle associated with \(V^{n-1}\); define a screen distribution \(S\) (see [DB 96]) and a field \(N\) of normalizing isotropic straight lines that is conjugate to \(S\); write the basic equations of lightlike hypersurfaces and prove the existence theorem for such hypersurfaces; prove that a lightlike hypersurface carries \((n-1)\)-parameter family of isotropic geodesics each of which possesses \(n-2\) real singular points if each of them is counted as many times as its multiplicity; and prove that under the development of the hypersurface \(V^{n-1}\) onto a hyperquadric \((Q^n)_{x}\) along an isotropic geodesic \(l\), to \(V^{n-1}\) there corresponds a ruled hypersurface which has the same tangent subspace at all regular points of its rectilinear generator.

In Section 3 we introduce the basic geometric objects and tensors defined in a second-order neighborhood of a point of a lightlike hypersurface \(V^{n-1}\) as well as geometric images associated with these objects and tensors. In particular, on each isotropic geodesic \(l\) of the hypersurface \(V^{n-1}\) we construct the harmonic pole \(H\) of the point \(x \in l \subset V^{n-1}\) with respect to singular points of the generator \(l\).

In studying submanifolds on a manifold \(M\) endowed with a differential geometric structure defined by a group \(G\), one of the most important problems is a construction of an invariant normalization and an affine connection intrinsically connected with the geometry of a submanifold in question (see [La 53], [No 76], [AG 93, 96]). In some simple cases such a normalization and a connection are defined in a first-order neighborhood. This is the case for submanifolds of the Riemannian manifold and for spacelike and timelike submanifolds of the pseudo-Riemannian manifold. However, for lightlike hypersurfaces of a pseudo-Riemannian manifold \((M, g)\) as well as of a manifold \((M, c)\) endowed with a
pseudoconformal structure, for constructing such a normalization and an affine connection elements of higher order differential neighborhoods are needed.

In Section 4 we give a geometric construction (called normalization) defining an affine connection on \( V^{n-1} \subset (M, c) \). However, the main purpose of this paper is a construction of an invariant normalization of \( V^{n-1} \) and an affine connection induced by this normalization. A solution of this problem is presented in Sections 5 and 6. In Section 5 we construct geometric objects defined in a third-order neighborhood of \( x \in V^{n-1} \) and apply them to construct a screen distribution \( S \), whose elements are subspaces \( S_x \subset T_x(V^{n-1}) \) passing through the point \( x \), and a complementary screen distribution \( \tilde{S} \), whose elements are subspaces \( \tilde{S}_H \) passing through the harmonic pole \( H \) of the point \( x \). The above construction can be done provided that two affinors are nondegenerate on \( V^{n-1} \). One of these affinors is defined in a second-order neighborhood of \( x \in V^{n-1} \), and the second one in a third-order neighborhood of \( x \in V^{n-1} \).

Under the same assumptions in Section 6 we construct a one-component geometric object which is defined in a fourth-order neighborhood of \( x \in V^{n-1} \) and determines a point \( C_n \) on a normalizing isotropic straight line. All these geometric objects are intrinsically defined by the geometry of \( V^{n-1} \). A geometric meaning of the geometric images associated with the constructed objects is also found. The invariant normalization of \( V^{n-1} \) we have constructed induces a torsion-free affine connection \( \gamma_1 \) whose fundamental group is the group \( G_1 = \text{GL}(n-1, \mathbb{R}) \). The curvature tensor of this connection is expressed in terms of elements of a fifth-order neighborhood of \( x \in V^{n-1} \).

In Section 7 we investigate the problem of integrability of the screen distributions \( S \) and \( \tilde{S} \). We prove that they are simultaneously integrable or simultaneously nonintegrable and find conditions of their integrability. If the distributions \( S \) and \( \tilde{S} \) are integrable, then the congruence of normalizing isotropic straight lines \( \tilde{l} \) is stratified into a one-parameter family of lightlike hypersurfaces.

Finally in Section 7, in addition to the torsion-free affine connection \( \gamma_1 \) induced by the constructed invariant normalization, we find another affine connection \( \gamma_2 \) whose fundamental group \( G_2 \) is isomorphic to the group \( \mathbb{R}^+ \times \text{GL}(n-2, \mathbb{R}) \). The connection \( \gamma_2 \) is not torsion-free and defined by elements a third-order neighborhood of \( x \in V^{n-1} \). The torsion and curvature tensors of this connection are expressed in terms of elements of a fourth-order neighborhood of \( x \in V^{n-1} \).

Note that the problem of construction of an invariant normalization and an invariant affine connection for a lightlike hypersurface \( V^{n-1} \) of a pseudo-Riemannian manifold \( (M, g) \) of Lorentzian signature (for definition see [ON 83]) was considered by many authors (see [DB 96], Ch. 4). However, as far as we know, an invariant normalization and an affine connection intrinsically connected with the geometry of \( V^{n-1} \) were not considered in these papers.

As to a construction of invariant normalization and an invariant affine connection for a lightlike hypersurface \( V^{n-1} \) on a manifold \( (M, c) \) endowed with a
pseudoconformal structure of Lorentzian signature, such a construction is given in the present paper for the first time.

The methods developed in the present paper and the results obtained in it can be used to study lightlike hypersurfaces in the pseudo-Riemannian space \((M, g)\) of Lorentzian signature, the pseudoconformal space \(C^n_1\), and the Minkowski space \(R^n_1\).

The present paper is related to our papers [AG 98a, b, c].

1 Basic equations of a manifold endowed with a conformal structure of Lorentzian signature

1. A pseudoconformal structure \(CO(n - 1, 1)\) on a manifold \(M\) of dimension \(n\) is a set of conformally equivalent pseudo-Riemannian metrics with the same signature \((n - 1, 1)\). Such a structure is called *conformally Lorentzian*.

A metric \(g\) can be given on \(M\) by means of a nondegenerate quadratic form

\[ g = g_{ij}du^i du^j, \]

where \(u^i, i = 1, \ldots, n\), are curvilinear coordinates on \(M\), and \(g_{ij}\) are the components of the metric tensor \(g\). On a conformal structure the quadratic form \(g\) is relatively invariant.

The equation \(g = 0\) defines in the tangent space \(T_x(M)\) a cone \(C_x\) of second order called the *isotropic cone*. Thus the conformal structure can be given on the manifold \(M\) by a field of cones of second order.

The cone \(C_x \subset T_x(M)\) remains invariant under transformations of the group

\[ G = \text{SO}(n - 1, 1) \times \text{H}, \]

where \(\text{SO}(n - 1, 1)\) is the special \(n\)-dimensional pseudoorthogonal group of signature \((n - 1, 1)\) (the connected component of the unity of the pseudoorthogonal group \(O(n - 1, 1)\)), and \(\text{H}\) is the group of homotheties. Note that the group \(G\) acts in the tangent space \(T_x(M)\). Thus the pseudoconformal structure \(CO(n - 1, 1)\) is a \(G\)-structure defined on the manifold \(M\) by the group \(G\) indicated above. For the pseudoconformal structure \(CO(n - 1, 1)\) the isotropic cone is real. Note that pseudo-Riemannian structures of arbitrary signature were studied in the book [ON 83] (they were called there "semi-Riemannian").

2. We consider the manifold \(M\), associate with any point \(x \in M\) its tangent space \(T_x(M)\), and define the frame bundle whose base is the manifold \(M\) and the fibers are the families of vectorial frames \(\{e_1, \ldots, e_n\}\) in \(T_x(M)\) defined up to a transformation of the general linear group \(\text{GL}(n)\). The frames indicated above are called the *frames of first order*. They form the first-order frame bundle which we will denote by \(\mathcal{R}^l(M)\). Let us denote by \(\{\omega^1, \ldots, \omega^n\}\) the co-frame dual to the frame \(\{e_1, \ldots, e_n\}\):

\[ \omega^j(e_j) = \delta_j^j. \]
Then an arbitrary vector $\xi \in T_x(M)$ can be written as

$$\xi = \omega^i(\xi)e_i.$$ 

The forms $\omega^i$ can be considered as differential forms on the manifold $M$ if we assume that $\xi = dx$ is the differential of the point $x \in M$. Thus the quadratic form $g$ can be written as

$$g = g_{ij}\omega^i\omega^j. \quad (1)$$

3. The structure equations of the $CO(n-1,1)$-structure can be reduced to the following form (see [AG 96], Section 4.1):

$$d\omega^i = \omega^0_0 \wedge \omega^i + \omega^k \wedge \omega^i_k, \quad (2)$$

$$d\omega^0_0 = \omega^0_0 \wedge \omega^0_0 + \omega^i \wedge \omega^0_i, \quad (3)$$

$$d\omega^i_j = \omega^0_0 \wedge \omega^i + \omega^k \wedge \omega^i_k + g_{jk}\omega^k \wedge g_{ij} + C_{ijkl}\omega^k \wedge \omega^j, \quad (4)$$

$$d\omega^0_i = \omega^0_i \wedge \omega^0_0 + \omega^j \wedge \omega^0_j + C_{ijk}\omega^j \wedge \omega^k, \quad (5)$$

and the metric tensor $g_{ij}$ satisfies the equations

$$dg_{ij} - g_{ik}\omega^k_j - g_{kj}\omega^k_i = 0. \quad (6)$$

Note that in equations (2)–(6) the forms $\omega^i$ are defined in a first-order frame bundle, the 1-forms $\omega^i_j$ and a scalar 1-form $\omega^0_0$ in a second-order frame bundle, and a covector form $\omega^0_0$ in the third-order frame bundle.

For $C^0_{ijkl} = C_{ijkl} = 0$, equations (2)–(6) coincide with the structure equations the pseudoconformal space $C^0_n$. For this reason the object $\{C^0_{ijkl}, C_{ijkl}\}$ is called the curvature object of the pseudoconformal structure $CO(n-1,1)$.

The quantities $C^i_{jkl}$ form a $(1,3)$-tensor which is called the Weyl tensor or the tensor of conformal curvature of the structure $CO(n-1,1)$.

Consider also the covariant tensor of conformal curvature

$$C_{ijkl} = g_{im}C^m_{jkl}. \quad (7)$$

This tensor allows us to write relations between the components of the tensor of conformal curvature in more convenient form:

$$\left\{\begin{array}{c}
C_{ijkl} = -C_{jikl} = -C_{ijlk} = C_{klij}, \\
C_{ijkl} + C_{iklj} + C_{iljk} = 0.
\end{array}\right. \quad (8)$$

In addition, the tensor $C^i_{jkl}$ is trace-free:

$$C^i_{jki} = 0. \quad (9)$$

(see, for example, [AG 96], Section 4.1).
The quantities $C_{ijk}$, that do not form a tensor, satisfy the conditions

$$C_{ijk} = -C_{ikj}. \quad (10)$$

Note also that the tensor of conformal curvature $C_{ijkl}$ is defined in a third-order neighborhood of the structure $CO(n-1,1)$, and the quantities $C_{ijk}$ are defined in its fourth-order neighborhood. The $CO(n-1,1)$-structure itself is a $G$-structure of finite type two (see [AG 96], Section 4.1). For $n \geq 4$, the condition $C_{ijkl} = 0$ is necessary and sufficient for a manifold $(M,c)$ to be conformally flat (see [AG 96], Section 4.1).

4. The 1-forms $\omega^i = \omega^0_0$, $\omega^i_0$, and $\omega^0_i$ defined by the $CO(n-1,1)$-structure on the manifold $M$ can be taken as components of infinitesimal displacement of a frame in the pseudoconformal space $C^n_{1}$. A conformal frame consists of two points $A_0$ and $A_{n+1}$ and $n$ hyperspheres $A_i$ passing through $A_0$ and $A_{n+1}$. The scalar products of the elements of this frame can be written as

$$\begin{align*}
(A_0, A_0) &= (A_{n+1}, A_{n+1}) = 0, \quad (A_0, A_{n+1}) = -1, \\
(A_0, A_i) &= (A_{n+1}, A_i) = 0, \quad (A_i, A_j) = g_{ij}
\end{align*} \quad (11)$$

The equations of infinitesimal displacement of this frame have the form

$$\begin{align*}
dA_0 &= \omega^0_0 A_0 + \omega^0_i A_i, \\
dA_i &= \omega^i_0 A_0 + \omega^i_j A_j + \omega^{n+1}_i A_{n+1}, \\
dA_{n+1} &= \omega^{i}_{n+1} A_i + \omega^{n+1}_{n+1} A_{n+1},
\end{align*} \quad (12)$$

where

$$\omega^{n+1}_i = g_{ij} \omega^0_j, \quad \omega^{i}_{n+1} = g^{ij} \omega^0_j, \quad \omega^{n+1}_{n+1} = -\omega^0_0,$$

and $g^{ij}$ is the inverse tensor of the tensor $g_{ij}$. In addition, the forms $\omega^i_j$ satisfy the system of equations (6). The family of frames in question forms a bundle of first-order conformal frames associated with the pseudoconformal structure $(M,c)$.

Equations (12) are completely integrable if and only if the tensor of conformal curvature of the $CO(n-1,1)$-structure vanishes. Then these equations define a fiber bundle in the whole space $C^n_{1}$. If the tensor of conformal curvature does not vanish, the system (12) can be integrated along any smooth curve $x = x(t)$ belonging to the manifold $M$. A solution of this system defines a development of this line and the frame bundle along it on the conformal space $C^n_{1}$. Moreover, if $x_1$ and $x_2$ are two points of the manifold $M$, and $l_1$ and $l_2$ are two smooth curves joining these points, then under integration of equations (12) along these curves, for the same initial conditions at the point $x_1$, we obtain different results at the point $x_2$. The difference of these two results is defined by the curvature of the pseudoconformal structure $CO(n-1,1)$ (see [Car 23] and also [AG 96]).

For study of conformal structures it is convenient to use Darboux mapping. Under the Darboux mapping to the conformal space $C^n_{1}$ there corresponds a
hyperquadric $Q^n_1$ of a projective space $P^{n+1}$; to the points $A_0$ and $A_{n+1}$ there correspond points of the hyperquadric $Q^n_1$ not belonging to a rectilinear generator of $Q^n_1$; and to the hyperspheres $A_i$ there correspond points of the space $P^{n+1}$ belonging to the intersection of the hyperplanes $T_x(Q^n_1)$ and $T_y(Q^n_1)$ tangent to the hyperquadric $Q^n_1$ at its points $x = A_0$ and $y = A_{n+1}$ (see Figure 1). We will denote the elements of a projective frame by the same letters which we used for the corresponding elements of a conformal frame. The equations of infinitesimal displacement of the projective frame in question have the same form (12) as the equations of infinitesimal displacement of the corresponding conformal frame.

The equation of the hyperquadric $Q^n_1$ with respect to the projective frames in question has the form:

$$ (x, x) = g_{ij}x^ix^j - 2x^0x^n = 0. \quad (13) $$

The quadratic form $g = g_{ij}x^ix^j$ is of signature $(n-1,1)$, and the equation $g_{ij}x^ix^j = 0$ defines the isotropic cone $C_x$ with the vertex at $x = A_0$ on the hyperquadric $Q^n_1$. This cone carries an $(n-2)$-parameter family of rectilinear generators corresponding to the isotropic lines of the space $C^n_1$.

For $\omega^i = 0$, equations (12) determine an admissible transformation of frames in the pseudoconformal space $(C^n_1)_x$ that is tangent to a manifold $(M, c)$ endowed a pseudoconformal structure $CO(n-1,1)$ at a point $x$.

## 2 Geometry of lightlike hypersurfaces of a manifold endowed with a conformal structure of Lorentzian signature

1. In this paper we consider a lightlike hypersurface $V^{n-1}$ on a manifold $M$ of dimension $n \geq 4$ endowed with a $CO(n-1,1)$-structure of Lorentzian signature
A lightlike hypersurface $V^{n-1}$ on such a manifold is a hypersurface which is tangent to the isotropic cone $C_x$ at each point $x \in V^{n-1}$.

Let $T_x(V^{n-1})$ be a tangent subspace to $V^{n-1}$ at a point $x$. In $T_x(M)$ we choose a projective frame such that $x = A_0$; the point $A_1$ belongs to the isotropic generator of the cone $C_x$ along which the subspace $T_x(V^{n-1})$ is tangent to $C_x$; the point $A_n$ also belongs to a rectilinear generator of the cone $C_x$ that does not belong to the subspace $T_x(V^{n-1})$; and we place the points $A_a$, $a = 2, \ldots, n-1$, into the $(n-2)$-dimensional intersection of the subspace $T_x(V^{n-1})$ and the subspace $T_{A_0A_n}(C_x)$ tangent to $C_x$ along $A_0A_n$. Then the scalar products of these points can be written as

\[
\begin{align*}
(A_1, A_1) &= (A_n, A_n) = 0, \\
(A_1, A_a) &= (A_n, A_a) = 0, \\
(A_a, A_b) &= g_{ab}, \\
(A_1, A_n) &= -1,
\end{align*}
\]

where $a, b = 2, \ldots, n-1$. The last relation in (14) is a result of an appropriate normalization of the points $A_0$ and $A_n$. The frames we have constructed are first-order frames associated with a lightlike hypersurface $V^{n-1}$ (see Figure 2).

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**Figure 2**

The isotropic straight lines $N_x = A_0 \wedge A_n$ are called the *normalizing straight lines* of a lightlike hypersurface $V^{n-1}$, and the subspaces $S_x = A_0 \wedge A_2 \wedge \ldots \wedge A_{n-1}$ belonging to $T_x(V^{n-1})$ are called the *screen subspaces* of $V^{n-1}$. There exists a bijective correspondence between the fields $N$ of normalizing straight lines $N_x$ of a lightlike hypersurface $V^{n-1}$ and its screen distributions $S$ of screen subspaces $S_x$.

A normalizing field $N$ on a lightlike hypersurface $V^{n-1}$ can be given with a big arbitrariness. One of the goals of the present paper is to find a method of
construction of a normal field $N$, and along with this field also a screen distribution $S$ both intrinsically connected with the geometry of a lightlike hypersurface $V^{n-1}$.

With respect to the projective moving frame chosen in the tangent space $T_x(M)$, the fundamental form $g$ of $M$ has the expression

$$g = g_{ab} \omega^a \omega^b - 2\omega^1 \omega^n, \quad a, b = 2, \ldots, n - 1,$$

and the isotropic cone $C_x$ is determined by the equation $g = 0$.

Now the components $g_{ij}$ of the tensor $g$ are the entries of the following matrix:

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & g_{ab} & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

(16)

where $(g_{ab})$ is a nondegenerate positive definite matrix. Equations (6) and (16) imply that

$$\begin{aligned}
\omega^n_1 &= \omega^n_0 = 0, \\
\omega^1_0 &= g_{a1} \omega^n_a, \\
dg_{ab} - g_{ac} \omega^c_b - g_{cb} \omega^c_a &= 0.
\end{aligned}$$

(17)

Since the points $A_1$ and $A_a$ of the frame $\{A_0, A_1, A_a, A_n\}$ of $T_x(V^{n-1})$ belong to the tangent subspace $T_x(V^{n-1})$, we have

$$dA_0 = \omega^0_0 A_0 + \omega^1_0 A_1 + \omega^n_a A_a.$$

This means that the hypersurface $V^{n-1}$ is defined by the following Pfaffian equation

$$\omega^n_0 = 0,$$

(18)

and the forms $\omega^1$ and $\omega^a$, $a = 2, \ldots, n - 1$, are basis forms of the hypersurface $V^{n-1}$.

The quadratic form $\tilde{g}$ defining the conformal structure on $V^{n-1}$ has the form

$$\tilde{g} = g_{ab} \omega^a \omega^b$$

and it is of signature $(n - 2, 0)$, that is, the form $\tilde{g}$ is positive semidefinite on $V^{n-1}$.

Taking exterior derivative of equation (18) by means of (2), we obtain the exterior quadratic equation

$$\omega^a \wedge \omega^n_a = 0.$$

(19)

Applying Cartan’s lemma to this equation, we find that

$$\omega^n_a = \lambda_{ab} \omega^b, \quad \lambda_{ab} = \lambda_{ba}.$$

(20)
The quantities $\lambda_{ab}$ are defined in a second-order neighborhood of a point $x \in V^{n-1}$. It follows from equations (17) and (20) that

$$\omega^a_i = \lambda^b_i \omega^b,$$

(21)

where $\lambda^a_i = g^{ac} \lambda_{cb}$, and $g^{ab}$ is the inverse tensor of the tensor $g_{ab}$.

Let us prove the existence theorem for lightlike hypersurfaces.

**Theorem 1** Lightlike hypersurfaces on a manifold $(M, c)$ exist, and the solution of a system defining such hypersurfaces depends on one function of $n - 2$ variables.

**Proof.** The hypersurfaces in question are defined by equation (18) whose exterior differentiation leads to exterior quadratic equation (19). Equation (19) contains only the basis forms $\omega^a$ and does not contain the form $\omega^1$. This implies that for proving the existence we must consider only the forms $\omega^a$ as basis forms of an integral manifold. The number of these forms is $n - 2$. If we apply to equation (19) the Cartan test (see [BCGGG 91] or [AG 93], pp. 12–13), we find that the characters of this equation are $s_1 = s_2 = \ldots = s_{n-2} = 1$, and the Cartan number is $Q = (n-1)(n-2)/2$. A general integral element of equation (19) depends on $N$ arbitrary parameters, where $N$ is the number of independent coefficients $\lambda_{ab}$ in equations (20). Since $\lambda_{ab} = \lambda_{ba}$, we have $N = (n-1)(n-2)/2$. Thus we have $N = Q$. This proves Theorem 1. \(\blacksquare\)

An *isotropic geodesic* on the manifold $(M, g)$ is a geodesic that is tangent to the isotropic cone $C_x$ at each of its points $x$.

As was proved in [AG 96] (see also [AG 97]), isotropic geodesics are invariant with respect to a conformal transformation of the metric $g$. We will prove now the following theorem:

**Theorem 2** A lightlike hypersurface $V^{n-1} \subset M(c)$ carries a foliation formed by isotropic geodesics.

**Proof.** Since the straight lines $A_0A_1$ are tangent to isotropic lines on a hypersurface $V^{n-1}$, the equations of the isotropic foliation on $V^{n-1}$ have the form

$$\omega^a = 0,$$

(22)

Let us prove that the curves belonging to this foliation are isotropic geodesics. It is known (see [AG 96], Section 4.2) that the equations of geodesics in any of pseudo-Riemannian metrics compatible with the $CO(n-1, 1)$-structure can be written as

$$d\omega^i + \omega^j \omega^i_j = \alpha \omega^i, \quad i, j = 1, \ldots, n,$$

(23)
where $\alpha$ is a 1-form, and $d$ is the symbol of ordinary (not exterior) differentiation. In our moving frame equations (23) take the form
\[
\begin{cases}
  d\omega^1 + \omega^1 \omega^1_1 + \omega^a \omega^1_a + \omega^n \omega^n_1 = \alpha \omega^1, \\
  d\omega^a + \omega^1 \omega^a_1 + \omega^b \omega^a_b + \omega^n \omega^n_a = \alpha \omega^a, & a, b = 2, \ldots, n - 1, \\
  d\omega^n + \omega^1 \omega^n_1 + \omega^b \omega^n_b + \omega^n \omega^n_n = \alpha \omega^n.
\end{cases}
\]
(24)

By means of equations (18) and (21), which are valid on $V^{n-1}$, and equation (22) defining the isotropic foliation on $V^{n-1}$, the last two equations (24) are identically satisfied, and the remaining first equation determines $d\omega^1$ on a geodesic.

Note that for lightlike hypersurfaces of a pseudo-Riemannian space, a similar result in a slightly different terminology is given in [DB 96], p. 86.

2. Consider the development of isotropic geodesics of a hypersurface $V^{n-1}$ defined by equation (22) on the hyperquadric $Q^n$. By means of (18) and (22), it follows from equations (12) that
\[
dA_0 = \omega^0_0 A_0 + \omega^1_1 A_1, \quad dA_1 = \omega^0_1 A_0 + \omega^1_1 A_1.
\]
These equations prove that under the development, to the isotropic geodesics defined by equation (22) there corresponds an open part of the rectilinear generator $A_0 A_1$ of the hyperquadric $Q^n$. We assume that the isotropic geodesics of the hypersurface $V^{n-1}$ are prolonged in such a way that they are mapped onto the entire rectilinear generator $A_0 A_1$ which is a projective straight line $l$. From equation (22) it follows also that the family of isotropic geodesics on a hypersurface $V^{n-1}$ depends on $n - 2$ parameters, and the forms $\omega^a$ are independent linear combinations of differentials of these parameters. This implies the following theorem:

**Theorem 3** A lightlike hypersurface $V^{n-1}$ of a differential manifold $M$, $\dim M = n$, endowed with a pseudoconformal structure $\text{CO}(n-1,1)$ is the image of the product $M^{n-2} \times l$, where $l$ is a projective straight line, under a mapping $f$ of this product onto the manifold $M$, $V^{n-1} = f(M^{n-2} \times l)$.

Note also that since the isotropic geodesics of a lightlike hypersurface $V^{n-1}$ are the images of projective straight lines, then one can introduce projective coordinates both homogeneous and nonhomogeneous. In what follows we will use this remark.

Consider a displacement of the isotropic geodesic $l = A_0 A_1$ on a lightlike hypersurface $V^{n-1}$. From equations (12), (17), and (18) it follows that
\[
\begin{cases}
  dA_0 = \omega^0_0 A_0 + \omega^1_0 A_1 + \omega^0_a A_a, \\
  dA_1 = \omega^0_1 A_0 + \omega^1_1 A_1 + \omega^0_a A_a.
\end{cases}
\]
(25)
where the forms $\omega^a_i$ have expressions (21). Consider a point $Z = A_1 - sA_0$ on the straight line $l$. From equations (25) and (21) it follows that

$$dZ \equiv (\lambda^a_b - s\delta^a_b)\omega^b A_a \pmod{A_0, A_1}.$$  \hspace{1cm} (26)

The matrix $(J^a_b) = (\lambda^a_b - s\delta^a_b)$ is the Jacobi matrix of the mapping $f$, and its determinant,

$$J = \det(\lambda^a_b - s\delta^a_b)$$

is the Jacobian of this mapping.

Since the quasiaffinor $\lambda^a_b = g^{ac}\lambda_{cb}$ is symmetric, its characteristic equation

$$\det(\lambda^a_b - s\delta^a_b) = 0$$  \hspace{1cm} (27)

has $n - 2$ real roots if each of them is counted as many times as its multiplicity. This implies the following theorem.

**Theorem 4** *Any isotropic geodesic $l$ of a lightlike hypersurface $V^{n-1}$ of a manifold $M$ endowed with a pseudoconformal structure of Lorentzian signature carries $n - 2$ real singular points if each of them is counted as many times as its multiplicity.*

**Proof.** The tangent subspace to a lightlike hypersurface $V^{n-1}$ at a point $Z$ is a subspace of the space $T_x(M)$. By (25) and (26), this subspace is determined by the point $Z, A_1, \text{ and } C_b = (\lambda^a_b - s\delta^a_b)A_a$. If the Jacobian $J$ is different from 0, then these points are linearly independent and determine the $(n - 1)$-dimensional tangent subspace $T_Z(V^{n-1})$. In this case a point $Z$ is a regular point of the hypersurface $V^{n-1}$. If at a point $Z \in A_0A_1$ the Jacobian $J$ is equal to 0, then at this point $\dim T_Z(V^{n-1}) < n - 1$, and this point is a singular point of $V^{n-1}$. The coordinates $s$ of these singular points can be found from equation (27) which has $n - 2$ real roots. ■

It is obvious that the point $x = A_0$ is a regular point of the straight line $l$.

Denote by $s_a$ the roots of equation (27). Then the singular points of the isotropic geodesic $l$ have the expressions

$$F_a = A_1 - s_aA_0.$$  \hspace{1cm} (28)

In the paper [AG 98b], for a lightlike hypersurface of a pseudo-Riemannian de Sitter space we investigated the structure of these singular points and the structure of $V^{n-1}$ itself taking into account multiplicities of singular points. Many of the results of [AG 98b] are still valid for a lightlike hypersurface $V^{n-1}$ on a manifold endowed with a pseudoconformal structure.

One more important property of a lightlike hypersurface $V^{n-1}$ of a space with a pseudoconformal structure follows from our considerations. This property is described in the following theorem.
Theorem 5 Under the development of a lightlike hypersurface $V^{n-1}$ of a manifold $M$ endowed with a pseudoconformal structure of Lorentzian signature onto a hyperquadric $Q^0_1 \subset P^{n+1}$ along its isotropic geodesic $l$, to the tangent hyperplanes $T_Z(V^{n-1})$ at regular points $Z$ of the line $l$, there corresponds a unique subspace $T_l$ of dimension $n-1$ that is tangent to the hyperquadric $Q^0_1$ at all points of the line $l$.

Proof. In fact, from (25) and (26) it follows that at regular points $Z$ of the line $l$, i.e., for $J \neq 0$, the tangent subspace $T_Z$ is determined by the same points $A_0, A_1, A_2, \ldots, A_{n-1}$. Therefore these subspaces are not changed when a point $x$ moves along the line $l$. \[\blacksquare\]

3 The fundamental geometric objects and tensors of a lightlike hypersurface defined in a second-order neighborhood

1. Singular points $F_a$ are defined invariantly on an isotropic geodesic $l$ of a lightlike hypersurface $V^{n-1}$. But the coordinates $s_a$ of these points depend on the choice of the points $A_0$ and $A_1$ on this isotropic geodesic and on normalization of these points. The coefficients of characteristic equation (27) also depend on the choice and normalization of these two points. The point $A_0$ can move freely along the straight line $l$, since $A_0$ is an arbitrary point of a lightlike hypersurface $V^{n-1}$. A displacement of this point is determined by a parameter $u^1$ whose differential $du^1$ is contained in the basis form $\omega^1$. As to the point $A_1$, its freedom of motion can be restricted. For example, we can suppose that the point $A_1$ is the harmonic pole (introduced in [Cas 50]) of the point $A_0$ with respect to the foci $F_a$ of the isotropic geodesic $l$. Then the displacement of the point $A_1$ is determined by the same parameter $u^1$ which determines the displacement of $A_0$.

The coordinate $\lambda$ of the harmonic pole of the point $A_0$ with respect to the foci $F_a$ is equal to the arithmetic mean of coordinates of the foci $F_a$: 

$$\lambda = \frac{1}{n-2} \sum s_a.$$ 

But the sum of the roots of algebraic equation (27) is the negative of the coefficient in $s^{n-3}$ of this equation, that is, this sum is the trace of the quasiaffinor $\lambda_a^0$. Thus 

$$\lambda = \frac{1}{n-2} \lambda_a^0 = \frac{1}{n-2} \lambda ag^{ab},$$ 

and we have the following expression of the harmonic pole $H$: 

$$H = A_1 - \lambda A_0$$
If we superpose the point $A_1$ with the harmonic pole $H$, we obtain $\lambda = 0$. After such a normalization, all remaining coefficients of characteristic equation (27) become relative invariants of the hypersurface $V^{n-1}$. The weights of these invariants are equal to the degrees of a component of quasiaffinor $\lambda^a_b$ that occurs in the expressions of these components.

2. Consider the second prolongation of the basic differential equations (18) of a lightlike hypersurface $V^{n-1}$ in a manifold $M$ endowed with the pseudo-conformal $CO(n-1,1)$-structure of Lorentzian signature. To this end, using equations (3)-(5), we take exterior derivatives of equations (20) obtained in the first prolongation of equations (18). As a result, we arrive at the following exterior quadratic equations:

\[
[\nabla \lambda_{ab} - \lambda_{ab}(\omega^0_a + \omega^1_b) - g_{ab}\omega^0_1 + (2C^m_{ab1} + \lambda_{al}g^{lc}\lambda_{cb})\omega^1 + C^m_{abc}\omega^c] \land \omega^b = 0,
\]

where $\nabla \lambda_{ab} = d\lambda_{ab} - \lambda_{cb}(\omega^c_a - \delta^c_a\omega^b_0) - \lambda_{ac}(\omega^c_b - \delta^c_b\omega^a_0)$. By Cartan’s lemma, we find from (31) that

\[
\nabla \lambda_{ab} - \lambda_{ab}(\omega^0_a + \omega^1_b) - g_{ab}\omega^0_1 + (2C^m_{ab1} + \lambda_{al}g^{lc}\lambda_{cb})\omega^1 + C^m_{abc}\omega^c = \lambda_{abc}\omega^c,
\]

where $\lambda_{abc}$ are symmetric with respect to all lower indices.

The quantities $C^m_{ab1}$ are symmetric with respect to the indices $a$ and $b$ since by (7) and (8) we have

\[
C^m_{ab1} = -C^m_{1ab1} = -C^m_{b1a} = -C^m_{1ba} = C^m_{ba1}.
\]

If we alternate equations (32) with respect to the indices $a$ and $b$, then we find that $C^m_{ab1c} = 0$. This implies that $C^m_{abc} = C^m_{bac}$. By (7) and (8) we have

\[
C^m_{abc} = -C^m_{acb}.
\]

It follows that

\[
C^m_{abc} = -C^m_{acb} = -C^m_{cab} = C^m_{cba} = C^m_{cab} = -C^m_{bac} = -C^m_{abc}.
\]

Thus the components $C^m_{abc}$ of the curvature tensor satisfy the conditions

\[
C^m_{abc} = 0.
\]
As a result, equation (32) takes the form
\[ \nabla \lambda_{ab} - \lambda_{ab}(\omega_0^0 + \omega_1^1) - g_{ab}\omega_0^0 + (2C_{ab1}^0 + \lambda_{acl}\lambda_{bkl})\omega_1^1 = \lambda_{abc}\omega_c. \] (33)

Note also that using the operator \( \nabla \), we can write equations (6) and the corresponding equations for the tensor \( g^{ab} \) in the form
\[ \nabla g_{ab} = 2g_{ab}\omega_0^0, \quad \nabla g_{ab} = -2g_{ab}\omega_0^0. \] (34)

For a fixed point \( x \in V^{n-1} \) (i.e., for \( \omega_1^1 = \omega_a = 0 \)), we find from (32) that
\[ \nabla \delta \lambda_{ab} - \lambda_{ab}(\pi_0^0 + \pi_1^1) - g_{ab}\pi_0^0 = 0, \] (35)
where \( \delta = d|_{\omega_1^1 = \omega_a = 0} \), \( \pi_j^j = \omega_j^j|_{\omega_1^1 = \omega_a = 0} \), and \( \nabla \delta \lambda_{ab} = \delta \lambda_{ab} - \lambda_{cb}(\pi_0^c - \delta_0^c\pi_0^0) - \lambda_{ac}(\pi_0^c - \delta_0^c\pi_0^0). \)

Equations (35) prove that the quantities \( \lambda_{ab} \) do not form a tensor since they are changed under a displacement of the point \( A_1 \) along the isotropic geodesic \( l = A_0A_1 \). However, the quantities \( \{g_{ab}, \lambda_{ab}\} \) allow us to construct a tensor defined in a second-order differential neighborhood of a point \( x \in V^{n-1} \). To this end, we consider the geometric object \( \lambda \) defined by formula (29). We set \( \omega_1^1 = \omega_a = 0 \) and differentiate (29), using (35) and the relation \( \nabla \delta g_{ab} = -2g_{ab}\pi_0^0 \) (which follows from (34)). As a result, we find that \( \lambda \) satisfies the following differential equation:
\[ \delta \lambda + \lambda(\pi_0^0 - \pi_1^1) - \pi_0^0 = 0. \] (36)
Using the quantities \( \lambda_{ab}, \lambda, \) and \( g_{ab} \), we construct the quantities
\[ h_{ab} = \lambda_{ab} - \lambda g_{ab}. \] (37)
Differentiating (37) with respect to the fiber parameters and using (35), (36) and (34), we find that these new quantities satisfy the following differential equations:
\[ \nabla \delta h_{ab} = h_{ab}(\pi_0^0 + \pi_1^1), \] (38)
where \( \nabla \delta h_{ab} \) has the expression similar to that of \( \nabla \delta \lambda_{ab} \). It follows from (38) that the quantities \( h_{ab} \) form a symmetric relative \((0, 2)\)-tensor that is defined in a second-order differential neighborhood of a point \( x \in V^{n-1} \). Contracting equation (37) with the tensor \( g^{ab} \), we find that
\[ h_{ab}g^{ab} = 0, \] (39)
\[ \text{i.e., tensor } h_{ab} \text{ is apolar to the tensor } g_{ab}. \]

By means of the tensor \( h_{ab} \), we can construct the affinor
\[ h_a^b = g^{ac}h_{cb} = \lambda_a^b - \lambda\delta_a^b, \] (40)
that is also defined in a second-order differential neighborhood of a point \( x \in V^{n-1} \). This affinor is trace-free, since it is easy to check that \( h_a^a = 0 \).
3. Equations (37) prove that if we superpose the point $A_1$ with the harmonic pole $H$ of the point $A_0$ with respect to the foci $F_a$ of the isotropic geodesic $l$, then the quantities $\lambda_{ab}$ will be identically equal to the corresponding components of the tensor $h_{ab}$. It is naturally to call the tensor $h_{ab}$ the second fundamental tensor of the hypersurface $V^{n-1}$, and the affinor $h^a_b$ the Burali–Forti affinor of $V^{n-1}$ (cf. [Bu 12]). (Note that the authors of [DB 96] called $h^a_b$ the shape operator; see [DB 96], pp. 85, 154, and 160.)

The following two theorems clarify a geometric meaning of the affinor $h^a_b$.

**Theorem 6** The harmonic pole $H$ of the point $A_0 = x$ with respect to the foci $F_a$ of the isotropic geodesic $l = A_0A_1$ is its regular point if and only if $h^a_b \neq 0$.

**Proof.** Superpose the vertex $A_1$ of the frame associated with an isotropic geodesic $l = A_0A_1$ with the harmonic pole $H$ of its point $A_0$, $A_1 = H$. Then by (30), (37), (40), and (39), we find that

$$\lambda = 0, \quad \lambda_{ab} = h_{ab}, \quad \lambda^a_b = h^a_b, \quad h^a_a = 0,$$

and by (21), we obtain

$$\omega^a_1 = h^a_b \omega^b.$$  

Hence from (25) it follows that

$$dA_1 = \omega^0_1 A_0 + \omega^1_1 A_1 + h^0_b \omega^a A_b.$$  

The tangent subspace to the hypersurface $V^{n-1}$ at the point $A_1 = H$ is determined by the points $A_1, A_0$, and $C_a = h^0_a A_b$, and if $h \neq 0$, then $\dim T_H(V^{n-1}) = n - 1$. This implies Theorem 6. □

A point $x$ of a lightlike hypersurface $V^{n-1} \subset (M, c)$ is called *umbilical* if $h^a_b = 0$ at this point. A hypersurface $V^{n-1}$ is called *totally umbilical* if all its points are umbilical, i.e., if the tensor $h^a_b$ vanishes on $V^{n-1}$.

**Theorem 7** Every isotropic geodesic of a totally umbilical hypersurface $V^{n-1} \subset (M, c)$ carries a single $(n - 2)$-fold singular point coinciding with its harmonic pole $H$. Moreover,

(a) If a manifold $(M, c)$ is not conformally flat, then the point $H$ describes a singular curve on $V^{n-1}$.

(b) If a manifold $(M, c)$ is conformally flat, then the point $H$ is fixed. In this case the image of the hypersurface $V^{n-1}$ on a hyperquadric $Q^n_1$ is the isotropic cone $C_H = Q^n_1 \cap T_H(Q^n_1)$.

(c) Conversely, any isotropic cone $C_y$ of the conformal space $C^n_1$ is a totally umbilical lightlike hypersurface.
Proof. First note that for \( h^g_a = 0 \), it follows from (42) that
\[
\lambda^6_a = \lambda \delta^6_a.
\]
This allows us to write equation (27) defining the coordinates of singular points of the line \( l \) in the form
\[
(s - \lambda)^{n-2} = 0.
\]
It follows that the harmonic pole \( H = A_1 - \lambda A_0 \) coincides with a single \((n-2)\)-fold singular point of the line \( l \) and is not changed when the point \( A_0 \) moves along the line \( l \).

Next, on a totally umbilical lightlike hypersurface \( V^{n-1} \) equations (42) take the form
\[
\omega^a_1 = 0.
\]
Taking exterior derivatives of equations (44) and using (4), we find that
\[
\omega^a_1 \wedge \omega^b_0 + 2C^a_{11b} \omega^1 \wedge \omega^b + C^a_{1c} \omega^b \wedge \omega^c = 0.
\]
From (45) it follows that 1-form \( \omega^0_1 \) is a linear combination of the basis forms \( \omega^1 \) and \( \omega^a_1 \):
\[
\omega^0_1 = \mu \omega^1 + \mu_a \omega^a.
\]
Using equations (45), it is easy to prove that
\[
\mu = 0, \quad \mu_a = \frac{2}{n-3} C^b_{16a}.
\]
Of course, we should assume that \( n \geq 4 \).

By (44), we have
\[
dA_1 = \omega^0_1 A_0 + \omega^1_1 A_1.
\]
It follows that the point \( A_1 \) describes a curve tangent to the isotropic geodesic \( l = A_0 A_1 \). This proves part (a) of Theorem 7.

If a manifold \((M, c)\) is conformally flat, then it follows from (45) that \( \omega^0_1 = 0 \) and for \( n \geq 4 \), equation (44) takes the form
\[
dA_1 = \omega^1_1 A_1.
\]
This implies that the point \( A_1 = H \) is fixed. Under the Darboux mapping on a hyperquadric \( Q^n_y \) of the space \( P^{n+1} \), the isotropic geodesics of the hypersurface \( V^{n-1} \) are mapped into rectilinear generators of the isotropic cone \( C_y \) whose vertex is the image of the point \( A_1 \) under the Darboux mapping, \( y = A_1 \). This proves part (b) of Theorem 7.

Finally, we will prove part (c) of Theorem 7. Let \( x \) be an arbitrary point of an isotropic cone \( C_y \) with vertex \( y \). With this cone we associate a first-order frame bundle in such a way that \( A_0 = x \) and \( A_1 = y \). Then since the point \( y \) is fixed, we have equations (46). They imply \( \omega^0_1 = 0, \lambda^6_a = 0, h^g_a = 0 \). This proves part (c).
4 An affine connection on a lightlike hypersurface

Let us find conditions under which on a lightlike hypersurface \( V^{n-1} \) of a manifold endowed with a pseudoconformal structure of Lorentzian signature there will be defined an affine connection. Such a hypersurface is defined by equations (17) and (18), and its basis forms are \( \omega^1, \omega^a, a = 2, \ldots, n - 1 \). Therefore, on such a hypersurface equations (2) take the form

\[
\begin{align*}
\text{d} \omega^1 &= \omega^1 \wedge (\omega^1 - \omega^0) + \omega^a \wedge \omega^1, \\
\text{d} \omega^a &= \omega^1 \wedge \omega^a + \omega^b \wedge (\omega^a - \delta^a_b \omega^0) \quad (a = 2, \ldots, n - 1).
\end{align*}
\]

Thus the matrix 1-form

\[
\omega = \begin{pmatrix}
\omega^1 - \omega^0 \\
\omega^0 \\
\omega^a - \delta^a_b \omega^0
\end{pmatrix}
\]

defines on \( V^{n-1} \) an affine structure. To define an affine connection, the form \( \omega \) must satisfy the structure equation

\[
\text{d} \omega + \omega \wedge \omega = \Omega,
\]

where \( \Omega \) is the curvature 2-form of this connection which is a linear combination of exterior products of the basis forms \( \omega^1 \) and \( \omega^a \) (see, for example, [KN 63], Ch. III).

We differentiate the form \( \omega \) componentwise and apply equations (3) and (4). As a result, we find that

\[
\begin{align*}
\text{d}(\omega^1 - \omega^0) - \omega^0 \wedge \omega^1 &= 2\omega^0 \wedge \omega^1 + \omega^a \wedge \omega^0 + C^1_{1kl} \omega^k \wedge \omega^1, \\
\text{d} \omega^a - \omega^1 \wedge (\omega^1 - \omega^0) - (\omega^b - \delta^b_a \omega^0) \wedge \omega^b &= \omega^0 \wedge \omega^1 - g_{ab} \omega^b \wedge \omega^a + C^a_{1kl} \omega^k \wedge \omega^1, \\
\text{d} \omega^a - (\omega^1 - \omega^0) \wedge \omega^a - \omega^b \wedge (\omega^a - \delta^a_b \omega^0) &= \omega^0 \wedge \omega^0 + C^a_{1kl} \omega^k \wedge \omega^1, \\
\text{d}(\omega^a - \delta^a_b \omega^0) - \omega^1 \wedge \omega^a - (\omega^c - \delta^c_b \omega^0) \wedge (\omega^a - \delta^a_b \omega^0) &= \omega^0 \wedge \omega^a + \omega^b \wedge \omega^0 + g_{bc} \omega^c \wedge \omega^0 - \delta^b_c (\omega^0 \wedge \omega^0 + \omega^c \wedge \omega^0) + C^a_{1kl} \omega^k \wedge \omega^1; \quad \text{for } a = 2, \ldots, n - 1. \quad (48)
\end{align*}
\]

in these formulas \( a, b = 2, \ldots, n - 1; k, l = 1, 2, \ldots, n - 1 \).

The right-hand sides of these equations are not expressed yet in terms of basis forms since the 1-forms \( \omega^1, \omega^0, \omega^a \), and \( \omega^a = g^{ab} \omega^b \) are fiber forms in the first-order frame bundle associated with a lightlike hypersurface \( V^{n-1} \). To make these forms principal, it is necessary to specialize our moving frames. The forms \( \omega^1 \) become principal if on \( V^{n-1} \) there is given a screen distribution \( S \), and along with distribution, also a field of normalizing isotropic lines \( A_0 A_1 \) is
given (see Section 2.1). As it was indicated in [DB 96], such a specialization is sufficient for defining an affine connection on a lightlike hypersurface $V^{n-1}$ of a pseudo-Riemannian manifold $(M, g)$ of Lorentzian signature.

However, in order to define an affine connection on a manifold $(M, c)$ endowed with a pseudoconformal structure of Lorentzian signature, it is not sufficient to have only a screen distribution. For the 1-forms $\omega^0_1$, $\omega^0_a$, and $\omega^0_n$ to be principal, it is necessary that in the tangent space $T_x(M)$ of a point $x \in V^{n-1}$, a hyperplane $L_x$ not passing through the point $x$ is given. Then a field $L$ of such hyperplanes $L_x$ together with a screen distribution define an affine connection on $M$. Thus the following theorem is valid.

**Theorem 8** To define an affine connection on a lightlike hypersurface $V^{n-1}$ of a manifold $(M, c)$, it is sufficient to assign on $V^{n-1}$ a screen distribution $S$ and a field $L$ of normalizing hyperplanes $L_x$ belonging to the tangent bundle $T(M)$.

Note that on a pseudo-Riemannian manifold $(M, g)$ the role of normalizing hyperplanes is played by the planes at infinity of the tangent spaces $T_x(M)$.

In what follows, on an isotropic hypersurface $V^{n-1}$, we will make an invariant construction of a screen distribution $S$ and a field $L$ of normalizing hyperplanes that are intrinsically connected with the geometry of the hypersurface $V^{n-1}$.

## 5 Construction of the main part of an invariant normalization of a lightlike hypersurface

1. We will derive now some formulas that will be used later. Differentiating equation (29) and applying equation (31), we obtain the following Pfaffian equation:

$$
\begin{align*}
\d \lambda + \lambda (\omega^0_0 - \omega^0_1) - \omega^0_1 &= -\frac{1}{n-2} g^{ab} (\lambda_{ae} g^{ec} \lambda_{cb} + 2 C^n_{ab1}) \wedge \omega^1 \\
&+ \frac{1}{n-2} g^{ab} \lambda_{abc} \wedge \omega^c.
\end{align*}
$$

(49)

From the last equation of (8) it follows that

$$
g^{ab} C^n_{ab1} = 0, \ C_{n11c} = 0.
$$

This allows us to write equation (49) in the form:

$$
\begin{align*}
\d \lambda + \lambda (\omega^0_0 - \omega^0_1) - \omega^0_1 &= -\frac{1}{n-2} g^{ab} \lambda_{ae} g^{ec} \lambda_{cb} \omega^1 + \frac{1}{n-2} g^{ab} \lambda_{abc} \omega^c.
\end{align*}
$$

(50)

Set

$$
\mu = \frac{1}{n-2} g^{ab} \lambda_{ae} g^{ec} \lambda_{cb}, \quad \mu_c = -\frac{1}{n-2} g^{ab} \lambda_{abc}.
$$

(51)
Then equation (50) can be written as
\[ d\lambda + \lambda (\omega^0_0 - \omega^1_1) - \omega^0_1 = -\mu \omega^1 - \mu_\alpha \omega^\alpha. \] (52)

Note that the quantities \( \mu \) and \( \mu_\alpha \) are defined in a second- and third-order neighborhood, respectively, of a point \( x \) of the hypersurface \( V^{n-1} \).

Note also that since the tensor \( g_{ab} \) is positive definite, the quantity \( \mu \) defined by (50) is equal to 0 at a point \( x \) if and only if the quasitensor \( \lambda_{ab} \) is equal to 0. But then equations (37) imply that the tensor \( h_{ab} \) is equal to 0 at the point \( x \), and as a result, the point \( x \) is umbilical. Thus the quantity \( \mu \) is equal to 0 at umbilical points of the hypersurface \( V^{n-1} \), and only at such points. In what follows we will assume that the hypersurface \( V^{n-1} \) does not have umbilical points and that \( \mu \neq 0 \) on \( V^{n-1} \). Moreover, from (51) it follows that \( \mu > 0 \).

If we take exterior derivative of equation (52), we obtain the following exterior quadratic equation:
\[
\begin{align*}
[d\mu + 2\mu(\omega^0_0 - \omega^1_1) - 2\lambda \omega^1_0] \wedge \omega^0_1 \\
+ \left[ \nabla_\mu a + \mu_\alpha (\omega^0_a - \omega^1_1) + h^b_b \omega^1_0 - (\mu \delta^b_b - \lambda \lambda^b_b) \omega^1_b \right] \\
+ \mu_b \lambda^b_\omega \omega^1_0 - 2(\lambda C^1_{ab} + C_{11a}) \omega^1_0 + (\lambda C^1_{ab} + C_{11b}) \omega^b_0 \wedge \omega^a = 0,
\end{align*}
\] (53)

where \( \nabla_\mu a = d\mu_a - \mu_b (\omega^b_a - \delta^b_a) \omega^0_0 \). Applying Cartan’s lemma to equation (53), we find that
\[
\begin{align*}
\begin{cases}
d\mu &+ 2\mu(\omega^0_0 - \omega^1_1) - 2\lambda \omega^1_0 = \nu \omega^1 + \nu_\alpha \omega^\alpha, \\
\nabla_\mu a &+ \mu_\alpha (\omega^0_a - \omega^1_1) + h^b_b \omega^1_0 - (\mu \delta^b_b - \lambda \lambda^b_b) \omega^1_b + \mu_b \lambda^b_\omega \omega^1_0 \\
&- 2(\lambda C^1_{ab} + C_{11a}) \omega^1_0 + (\lambda C^1_{ab} + C_{11b}) \omega^b_0 = \nu_\alpha \omega^1 + \nu_\alpha \omega^\alpha,
\end{cases}
\end{align*}
\] (54)

where \( \nu_{ab} = \nu_{ba} \). Here the quantities \( \nu \) and \( \nu_\alpha \) are defined in a third-order neighborhood of a point \( x \in V^{n-1} \), and the quantities \( \nu_{ab} \) are defined in a fourth-order neighborhood of \( x \in V^{n-1} \).

2. In what follows we assume that the point \( A_1 \) of an isotropic geodesic \( l = A_0 A_1 \) is superposed with the harmonic pole \( H \) of the point \( A_0 \) with respect to the singular points \( F_a \) of this straight line. Then equations (41) and (42) hold, equation (52) takes the form
\[ \omega^0_1 = \mu \omega^1 + \mu_\alpha \omega^\alpha, \] (55)

and equations (54) become
\[
\begin{align*}
\begin{cases}
d\mu &+ 2\mu(\omega^0_0 - \omega^1_1) = \nu \omega^1 + \nu_\alpha \omega^\alpha, \\
\nabla_\mu a &+ \mu_\alpha (\omega^0_a - \omega^1_1) + h^b_b \omega^1_0 - \mu_\alpha \omega^1_a - 2C_{11a} \omega^1 + C_{1ab} \omega^b = \nu_\alpha \omega^1 + \nu_\alpha \omega^\alpha.
\end{cases}
\end{align*}
\] (56)
Let us write the last equations for fixed principal parameters, i.e., for \( \omega^1 = \omega^a = 0 \):

\[
\begin{align*}
\delta \mu &= 2 \mu (\pi^1_1 - \pi^0_1), \\
\nabla \delta \mu + \mu_a (\pi^0_a - \pi^1_1) + h^b_a \pi^0_b - \mu \pi^1_1 &= 0.
\end{align*}
\] (57)

The first equation of (57) proves that after the specialization of moving frames described above, the quantity \( \mu \) becomes a relative invariant of weight 2. As we showed earlier, this relative invariant is different from 0 at non-umbilical points of the hypersurface \( V^{n-1} \).

By (51) the invariant \( \mu \) can be written now in the following form:

\[
\mu = \frac{1}{n-2} g^{ab} h_{ac} g^{ce} h_{eb} = \frac{1}{n-2} h^b_a h^b_b.
\] (58)

The second equation of (57) contains two groups of fiber forms, \( \omega^0_a \) and \( \omega^1_a \), and this is the reason that the reduction of the object \( \mu_a \) to 0 does not make these forms principal. Hence we should also consider the object \( \nu_a \) occurring in equations (56) which is also defined in a third-order neighborhood of a point \( x \in V^{n-1} \). In order to find differential equations which this object satisfies, we take the exterior derivative of the first equation of (56). As a result, we find that

\[
\begin{align*}
&[d \nu + 3 \nu (\omega^1_0 - \omega^1_1)] - \nu_i \omega^1_i + 4 \mu \omega^0_i \wedge \omega^1_i \\
&+ [\nabla \nu_a + 2 \nu_a (\omega^0_0 - \omega^1_1) + 2 \mu \omega^0_a - (2 \mu h^b_a + \nu \delta^b_a) \omega^1_b] \wedge \omega^a = 0
\end{align*}
\] (59)

Substituting the forms \( \omega^1_a \) and \( \omega^0_a \) in equation (59) by their values taken from equations (42) and (55), we obtain

\[
\begin{align*}
&[d \nu + 3 \nu (\omega^0_0 - \omega^1_1)] - \nu_i \omega^1_i + 4 \mu \omega^0_i \wedge \omega^1_i \\
&+ [\nabla \nu_a + 2 \nu_a (\omega^0_0 - \omega^1_1) + 2 \mu \omega^0_a - (2 \mu h^b_a + \nu \delta^b_a) \omega^1_b] \wedge \omega^a = 0
\end{align*}
\] (60)

where \( \nabla \nu_a = d \nu_a - \nu_b (\omega^b_a - \delta^b_a \omega^0) \). Applying Cartan’s lemma to equation (60), we find that

\[
\begin{align*}
&d \nu + 3 \nu (\omega^0_0 - \omega^1_1) = \rho \omega^1 + \rho_a \omega^a, \\
&\nabla \nu_a + 2 \nu_a (\omega^0_0 - \omega^1_1) + 2 \mu \omega^0_a - (2 \mu h^b_a + \nu \delta^b_a) \omega^1_b \\
&+ (\nu_b h^b_a - 4 \mu \omega^1_a) \omega^1 + 2 \mu (2 C^1_{11a} \omega^1 - C^1_{1ab} \omega^b) = \rho_a \omega^1 + \rho_a \omega^b.
\end{align*}
\] (61)

The coefficients \( \rho, \rho_a, \) and \( \rho_{ab} \) in equations (61) are connected with a fourth-order differential neighborhood of a point \( x \in V^{n-1} \) and \( \rho_{ab} = \rho_{ba} \).

For fixed principal parameters (i.e., for \( \omega^1 = \omega^a = 0 \)), equations (61) take the form

\[
\begin{align*}
&d \nu + 3 \nu (\pi^0_0 - \pi^1_1) = 0, \\
&\nabla \nu_a + 2 \nu_a (\pi^0_0 - \pi^1_1) + 2 \mu \pi^0_a - (2 \mu h^b_a + \nu \delta^b_a) \pi^1_b = 0.
\end{align*}
\] (62)
When we derived (62), we took into account that by (41) and (55), \( \pi^a_1 = \pi^0_1 = 0 \).

3. Using the geometric objects \( \mu_a \) and \( \nu_a \), we will construct now an invariant normalization of a lightlike hypersurface \( V^{n-1} \).

In the tangent space \( T_x(V^{n-1}) \), we consider the subspace \( R_x \) which is complementary to the straight line \( A_0A_1 \), and take the points

\[
C_a = A_a + y_a A_0 + z_a A_1
\]

as basis points of this space. The subspace \( R_x \) is invariant if and only if

\[
\delta C_a = \sigma^b_a C_b.
\]

Differentiating equation (63) with respect to fiber parameters, we find that

\[
\delta C_a = (\nabla_\delta y_a + \pi^0_a)A_0 + (\nabla_\delta z_a + z_a (\pi^1_1 - \pi^0_0) + \pi^1_a)A_1 + \pi^b_a C_b.
\]

Thus the conditions for the subspace \( R_x \) to be invariant is

\[
\begin{cases} 
\nabla_\delta y_a + \pi^0_a = 0, \\
\nabla_\delta z_a + z_a (\pi^1_1 - \pi^0_0) + \pi^1_a = 0.
\end{cases} \tag{64}
\]

Next, using the geometric objects \( \mu_a \) and \( \nu_a \) defined earlier, we should construct normalizing geometric objects satisfying equations (64). Let us write one more time the equations which the objects \( \mu_a \) and \( \nu_a \) satisfy:

\[
\begin{cases} 
\nabla_\delta \mu_a + \mu_a (\pi^0_0 - \pi^1_1) + h^b_a \pi^0_b - \mu \pi^1_a = 0, \\
\nabla_\delta \nu_a + 2\nu_a (\pi^0_0 - \pi^1_1) + 2\mu \pi^0_a - (2\mu h^b_a + \nu \delta^b_a) \pi^1_b = 0.
\end{cases} \tag{65}
\]

We will try to solve these equations for 1-forms \( \pi^0_a \) and \( \pi^1_a \). Construct the objects

\[
M_a = h^b_a \mu_b + \frac{\nu}{2\mu} \mu_a - \frac{1}{2} \nu_a, \quad N_a = \frac{1}{2} h^b_a \nu_b - \mu \mu_a. \tag{66}
\]

Differentiating these equations with respect to fiber parameters and applying formulas (65), (57), and (38), we find that

\[
\begin{cases} 
\nabla_\delta M_a + 2M_a (\pi^0_0 - \pi^1_1) + H^b_a \pi^0_b = 0, \\
\nabla_\delta N_a + 3N_a (\pi^0_0 - \pi^1_1) - \mu H^b_a \pi^1_b = 0,
\end{cases} \tag{67}
\]

where

\[
H^b_a = h^c_a h^b_c + \frac{\nu}{2\mu} h^b_a - \mu \delta^b_a. \tag{68}
\]

We will establish some properties of the tensor \( H^b_a \).
1) The tensor $H^b_a$ is of weight 2. In fact, differentiating equation (68) with respect to fiber parameters, we find that

$$\nabla_\delta H^b_a = 2H^b_a(\pi^1_1 - \pi^0_0).$$

(69)

Since equation (68) contains a relative invariant $\nu$ defined in a third-order neighborhood, the tensor $H^b_a$ is also connected with this neighborhood.

2) The tensor $H^b_a$ is trace-free. In fact, we have

$$H_a^a = h^c_a h^a_c - (n - 2)\mu,$$

since $h^b_a$ is a trace-free tensor. But by formula (58) defining the invariant $\mu$, it follows that $H_a^a = 0$.

3) The tensor $H^b_a$ can be reduced to a diagonal form simultaneously with the tensors $g_{ab}$ and $h^b_a$. In fact, since the tensor $g_{ab}$ is positive definite, and by (40) the tensor $h^b_a$ satisfies the condition $g_{ac}h^c_b = g_{bc}h^c_a$, it follows that the tensors $g_{ab}$ and $h^b_a$ can be reduced simultaneously to diagonal forms:

$$g_{ab} = \delta_{ab}, \quad h^b_a = \delta_{ab}, \quad a, b = 2, \ldots, n - 1.$$

But now it follows from (68) that the tensor $H^b_a$ is also reduced to a diagonal form and has the following eigenvalues:

$$H_a = h^2_a + \frac{\nu}{2\mu}h_a - \mu.$$  

(70)

4) If the tensor $H^b_a$ is degenerate, then the relative invariants $\mu$ and $\nu$ of a lightlike hypersurface $V^{n-1}$ are connected by an algebraic equation, and the hypersurface is of a special type. In fact, suppose that $\det(H^b_a) = 0$. Then at least one of the eigenvalues $H_a$ of the tensor $H^b_a$ is equal to 0. Suppose that $H_2 = 0$. This and (70) imply that

$$h^2_2 + \frac{\nu}{2\mu}h_2 - \mu = 0.$$

Moreover since $h_2 \neq 0$, we have $\mu \neq 0$. The above written relation is an algebraic equation indicated earlier.

5) If $n = 4$, then the tensors $H^b_a$ and $h^b_a$ are proportional:

$$H^b_a = \frac{\nu}{2\mu}h^b_a.$$  

(71)

In fact, if $n = 4$, we have $a, b = 2, 3$, and

$$\mu = -\frac{1}{2}(h^2_2 + h^2_3), \quad H_2 = \frac{1}{2}(h^2_2 - h^2_3) + \frac{\nu}{2\mu}h_2, \quad H_3 = \frac{1}{2}(h^2_3 - h^2_2) + \frac{\nu}{2\mu}h_3.$$
Since the tensor $h^{a}_{b}$ is trace-free, we have $h_{2} + h_{3} = 0$, and the first terms in the expressions for $H_{2}$ and $H_{3}$ vanish. This implies equation (71).

Formula (71) implies that for $n = 4$ the tensor $H^{b}_{a}$ is the zero-tensor either if the tensor $h^{b}_{a}$ is the zero-tensor (i.e., the hypersurface $V^{3}$ is umbilical) or if $\nu = 0$.

4. Suppose now that the tensor $h^{b}_{a}$ is nondegenerate. As we proved in Theorem 7, this means that the harmonic pole $H = A_{1}$ of the point $x = A_{0}$ is a nonsingular point. The invariant $\mu$ defined by the formula (51) is different from 0. Suppose further that the relative tensor $H^{b}_{a}$ defined by formula (68) is also nondegenerate and denote by $\tilde{H}^{b}_{a}$ the inverse tensor of the tensor $H^{b}_{a}$. By (69), the tensor $\tilde{H}^{b}_{a}$ satisfies the equations

$$\nabla_{\delta} \tilde{H}^{b}_{a} = 2\tilde{H}^{b}_{a}(\pi_{0}^{0} - \pi_{1}^{1}).$$

We construct also two more objects

$$P_{a} = \tilde{H}^{b}_{a}M_{b}, \quad Q_{a} = \frac{1}{\mu}\tilde{H}^{b}_{a}N_{b}.$$  (73)

By (67) and (69), these objects satisfy the equations

$$\nabla_{\delta}P_{a} + \pi_{0}^{a} = 0, \quad \nabla_{\delta}Q_{a} + Q_{a}(\pi_{1}^{1} - \pi_{0}^{0}) + \pi_{a}^{1} = 0.$$  (74)

Comparing (74) and (64), we see that equations (69) are satisfied if we set

$$y_{a} = P_{a}, \quad z_{a} = Q_{a}.$$  (75)

This implies that in the tangents subspace $T_{x}(V^{n-1})$ the points

$$C_{a} = A_{a} + P_{a}A_{0} + Q_{a}A_{1}$$

define an invariant $(n - 3)$-dimensional subspace $R_{x}$ that is intrinsically connected with the geometry of the hypersurface $V^{n-1}$. The formulas (66) and (73) show that the geometric objects $P_{a}$ and $Q_{a}$ are expressed algebraically in terms of the objects $\mu_{a}$ and $\nu_{a}$ defined in a third-order neighborhood of a point $x \in V^{n-1}$. Hence the invariant subspace $R_{x}$ is defined in a third-order neighborhood of a point $x \in V^{n-1}$ too.

The invariant subspace $R_{x}$ defines an invariant screen subspace $S_{x} = [x, R_{x}]$ and a complementary screen subspace $\tilde{S}_{x} = [H(x), R_{x}]$ which is the span of the harmonic pole $H(x)$ of a point $x \in V^{n-1}$ and the subspace $R_{x}$ (see Figure 4). On a lightlike hypersurface $V^{n-1}$ these subspaces define a screen distribution $S = \cup_{x \in V^{n-1}}S_{x}$ and a complementary screen distribution $\tilde{S} = \cup_{x \in V^{n-1}}\tilde{S}_{x}$ that are intrinsically connected with the geometry of the hypersurface $V^{n-1}$ and are defined in its third-order differential neighborhood. Note that in the above
formulas, \( x \in V^{n-1} \) means that all regular points \( x \) are taken for which the harmonic points are regular too.

Thus we have proved the following result.

**Theorem 9** On a lightlike hypersurface \( V^{n-1} \) of a manifold endowed with a pseudoconformal structure of Lorentzian signature, an invariant screen distribution \( S \) and an invariant complementary screen distribution \( \tilde{S} \) that are intrinsically connected with the geometry of \( V^{n-1} \) are defined by elements of a third-order differential neighborhood of a point \( x \in V^{n-1} \) and can be constructed in the way indicated above.

Note that the problem of construction of an invariant normalization as well as of an affine connection for a lightlike hypersurface \( V^{n-1} \) of a pseudo-Riemannian manifold \((M, g)\) of Lorentzian signature (for definition see [ON 83]) was considered by many authors (see [DB 96], Ch. 4). However, as far as we know, an invariant normalization and an affine connection intrinsically connected with the geometry of \( V^{n-1} \) were not considered in these papers. In [DB 96] (see pp. 115–117) the authors consider a canonical normalization (canonical screen distribution) that is not invariant with respect to the Lorentzian transformations of the tangent space \( T_x(M) \). A similar normalization was considered in [Bo 72].

**6 A congruence of normalizing straight lines**

1. In order to complete the construction of an invariant normalization of a lightlike hypersurface \( V^{n-1} \), we need only to construct an invariant point on the
isotropic normalizing straight line $xC_n$ that is conjugate to the screen subspace $S_x$. To simplify our construction, we superpose the vertices $A_a$ of our moving frame with the basis points $C_a$ of the invariant subspace $R_x$ that are defined by formulas (75). As a result, we find that $P_a = Q_a = 0$. Then equations (73) imply that $M_b = N_b = 0$. Since the tensor $H^b_a$ is nondegenerate, equations (66) imply that $\mu_a = \nu_a = 0$.

As a result of the specialization of moving frames we have made, the second equations of (56) and (61) take the form:

\begin{align*}
  h^b_a \omega^0_a - \mu \omega^1_a - 2C_{11a} \omega^3_a + C_{1ab} \omega^b = \nu_{ab} \omega^b, \\
  2\mu \omega^0_a - (2\mu h^b_a + \nu \delta^b_a) \omega^1_a + 2\mu (2C_{11a} \omega^1_a - C_{1ab} \omega^b) = \rho_a \omega^1_a + \rho_{ab} \omega^b,
\end{align*}

The coefficients in the basis forms $\omega^1$ and $\omega^a$ in the right-hand sides of equations (76) are defined by elements of a fourth-order neighborhood of a point $x \in V^{n-1}$.

Since det$(H^b_a) \neq 0$, one can solve equations (76) with respect to the 1-forms $\omega^0_a$ and $\omega^1_a$. We will write these solutions in the form

\begin{align*}
  \omega^0_a &= \sigma_a \omega^1 + \sigma_{ab} \omega^b, \\
  \omega^1_a &= \tau_a \omega^1 + \tau_{ab} \omega^b.
\end{align*}

The coefficients of these decompositions are expressed algebraically in terms of the coefficients of equation (76). Thus they are also defined by elements of a fourth-order neighborhood of a point $x \in V^{n-1}$.

Taking exterior derivative of the second equation of (77), applying Cartan’s lemma to exterior quadratic equation obtained, and fixing the principal parameters, we find that

\begin{align*}
  \nabla_\delta \tau_a &= 0, \\
  \nabla_\delta \tau_{ab} + \tau_{ab}(\pi^1_a - \pi^0_a) - g_{ab} \pi^0_n &= 0.
\end{align*}

The first equation of (78) proves that the quantities $\tau_a$ form a covector. The second equation of (78) allows us to construct a geometric object that fixes a point on a rectilinear generator $A_0A_1$. This can be done as follows.

Consider the quantity

\[ \tau = \frac{1}{n - 2} \tau_{ab} g^{ab}. \]

By means of the second equation of (78) and equation (17), this quantity satisfies the equation

\[ \delta \tau + \tau (\pi^0_a + \pi^1_a) - \pi^0_n = 0. \]

Consider a point $Z = A_n + z A_0$ on the normalizing straight line $A_0A_1$. Differentiating this point with respect to fiber parameters, we find that

\[ \delta Z = [\delta z + z (\pi^0_n + \pi^1_n) + \pi^0_n] A_0 - \pi^1_n Z. \]
It follows that the point \( Z \) is invariant if and only if its coordinate \( z \) satisfies the equation

\[
\delta z + z(\pi_0^0 + \pi_1^1) + \pi_n^0 = 0. \tag{81}
\]

Comparing equations (80) and (81), we see that \( z = -\tau \) is a solution of equation (81). Thus the point

\[
C_n = A_n - \tau A_0 \tag{82}
\]
is not only an invariant point but also this point is intrinsically connected with the geometry of a lightlike hypersurface \( V^{n-1} \) and is defined by elements of a fourth-order neighborhood of a point \( x \in V^{n-1} \).

We proved the following result.

**Theorem 10** On a lightlike hypersurface \( V^{n-1} \) of a manifold \((M, c)\), a complete invariant normalization intrinsically connected with the geometry of \( V^{n-1} \) is defined by elements of a fourth-order differential neighborhood of a point \( x \in V^{n-1} \). This complete normalization induces an affine connection that is also intrinsically connected with the geometry of \( V^{n-1} \).

The last statement of Theorem 10 follows from Theorem 8. The normalizing hyperplane \( L_x \) discussed in Theorem 8 is defined by the harmonic pole \( H \) of the point \( x \) with respect to the singular points of the isotropic geodesic \( \tilde{l} \) of the hypersurface \( V^{n-1} \); the normalizing \((n-3)\)-plane \( R_x \) which is the intersection of the screen subspace and the complementary screen subspace (both belong to the hyperplane \( T_x(V^{n-1}) \)); and finally the invariant point \( C_n \) (defined by formula (82) on the normalizing straight line \( A_0A_n \)) which is conjugate to the screen subspace \( S_x \) with respect to the isotropic cone \( C_x \).

2. Let us clarify a geometric meaning of the normalizing point \( C_n \) on the straight line \( A_0A_n \). We consider this straight line as the line belonging to a local hyperquadric \((Q^n)_x \) that is tangent to a manifold \((M, c)\) at a point \( x \). On the manifold \((M, c)\), to this line \( A_0A_n \) there corresponds an isotropic geodesic \( \tilde{l} \) which we will also denote by \( A_0A_n \), \( \tilde{l} = A_0A_n \).

When a point \( A_0 \) describes a lightlike hypersurface \( V^{n-1} \subset (M, c) \), the isotropic geodesic \( \tilde{l} \) describes an isotropic congruence \( U \). As a point manifold, this congruence is an \( n \)-dimensional domain on the manifold \((M, c)\). This is the reason that we will denote it by \( U^n \). Moreover, \( U^n = \tilde{f}(V^{n-1} \times \tilde{l}) \), where \( \tilde{f} \) is a differentiable mapping of the direct product \( V^{n-1} \times \tilde{l} \) onto \((M, c)\).

Let us find the Jacobian of the mapping \( \tilde{f} \). To this end we consider an arbitrary point \( Z = A_n + zA_0 \) on the isotropic geodesic \( \tilde{l} \). The differential of this point has the form

\[
dZ = -\omega^1 Z + [dz + z(\omega^0_0 + \omega^1_1) + \omega^0_n] A_0 \\
+ (\tau^a_0 + z\delta^a_0)\omega^h A_n + (zA_1 + \tau^a A_n - A_{n+1})\omega^1,
\]
where \( \tau^a = g^{ab} \tau_b \) and \( \tau^a_a = g^{ac} \tau_{cb} \). At regular points \( Z \) of the mapping \( \tilde{f} \), the tangent space to the domain \( U^n \) coincides with the \( n \)-dimensional tangent subspace of the manifold \( (M, c) \) which is defined by the points \( Z, A_0, A_n, \) and \( A_{n+1} - z A_1 - \tau^a A_a \). At singular points \( Z \), the dimension of this tangent space is reduced. This happened at the points at which the rank of the Jacobian matrix of the mapping \( \tilde{f} \),

\[
\tilde{J} = (\tau^a_b + z \delta^a_b), \quad a, b = 2, \ldots, n-1,
\]

is reduced, and only at such points. At such points the Jacobian vanishes:

\[
\det(\tau^a_b + z \delta^a_b) = 0. \tag{83}
\]

Denote the eigenvalues of the matrix \( (\tau^a_b) \) by \( \tilde{\tau}_a \). In general, the matrix \( (\tau^a_b) \) is not symmetric, since the matrix \( (\tau_{ab}) \) occurring in equation (77) is not symmetric. Thus the eigenvalues \( \tilde{\tau}_a \) of the matrix \( (\tau^a_b) \) can be either real or complex conjugate. Hence the solutions of equation (81), that are defined by the formula \( z_a = -\tilde{\tau}_a \), as well as the singular points

\[
Z_a = A_n - \tilde{\tau}_a A_0 \tag{84}
\]

can be also complex conjugate.

But even in the case of complex conjugate roots of equation (83), by Vieta’s theorem, the sum of the roots of equation (83) is the negative trace of the matrix \( (\tau^a_b) \),

\[
\sum_{a=2}^{n-1} z_a = -\sum_{a=2}^{n-1} \tau_a = -\sum_{a=2}^{n-1} \tau^b_b = -\tau_{ab} g^{ab}. \tag{85}
\]

This relation allows us to find a geometric meaning of the invariant point \( C_n \) on the isotropic geodesic \( \tilde{l} = A_0 A_n \). Since the coefficient \( \tau \) occurring in the formula (82) defining the point \( C_n \) is determined by equation (79), the point \( C_n \) is the harmonic pole of the point \( A_0 \) with respect to the singular points \( Z_a \) of the mapping \( \tilde{f} \).

7 Integrability of screen distributions

Consider the screen distribution \( S \) and the complementary screen distribution \( \tilde{S} \) of a lightlike hypersurface \( V^{n-1} \) which we have constructed in Section 5.4. The distribution \( S \) is formed by the subspaces \( S_x = A_0 \wedge C_2 \wedge \ldots \wedge C_{n-1} \), where the points \( C_a \) are defined by equations (75), and the distribution \( \tilde{S} \) is formed by the subspaces \( \tilde{S}_x = H \wedge C_2 \wedge \ldots \wedge C_{n-1} \), where \( H \) is the harmonic pole of the point \( A_0 \) with respect to the singular points of the isotropic geodesic \( \tilde{l} = A_0 A_1 \) of the hypersurface \( V^{n-1} \).

As we indicated in Section 6.1, assuming that the tensors \( h^a_b \) and \( H^a_b \) are nondegenerate, we can reduce a frame bundle associated with the hypersurface.
$V^{n-1}$ in such a way that the points $A_1$ and $H$ as well as the points $A_a$ and $C_a$ will be superposed. This gives

$$S_x = A_0 \wedge A_2 \wedge \ldots \wedge A_{n-1}, \quad \tilde{S}_x = A_1 \wedge A_2 \wedge \ldots \wedge A_{n-1}.$$ 

As a result, the screen distribution $S$ of the hypersurface $V^{n-1}$ is defined by the differential equation

$$\omega_1^0 = 0,$$ 

and the complementary screen distribution $\tilde{S}$ is defined by the equation

$$\omega_1^0 = 0.$$ 

Let us find the conditions of integrability of the screen distributions $S$ and $\tilde{S}$. Since in the reduced frame bundle the geometric object $\mu_a$ vanishes, equation (55) connecting the 1-forms $\omega_1^0$ and $\omega_1^1$ takes the form

$$\omega_1^0 = \mu \omega_1^1,$$ 

where the invariant $\mu$ is different from 0. Thus if one of the distributions $S$ and $\tilde{S}$ defined by equations (86) and (87), respectively, is integrable, then another one is also integrable.

From the structure equations (2)—(5) of a manifold $(M, c)$ endowed with the pseudoconformal structure $CO(n - 1, 1)$ it follows that

$$d\omega_0^1 = \omega_0^0 \wedge \omega_1^0 + \omega_0^1 \wedge \omega_1^1 + \omega_0^a \wedge \omega_1^a,$$

$$d\omega_1^0 = \omega_1^0 \wedge \omega_0^0 + \omega_1^1 \wedge \omega_0^1 + \omega_1^a \wedge \omega_0^a + 2C_{1ab}\omega_1^a \wedge \omega_0^b + C_{1ab}\omega_0^a \wedge \omega_1^b.$$ 

Substituting the values of the 1-forms $\omega_1^a$ and $\omega_0^a$ taken from equations (77) into the last two equations, we find that

$$d\omega_0^1 \equiv \tau_{ab}\omega_a^a \wedge \omega_b^b \pmod{\omega_1^1},$$

$$d\omega_1^0 \equiv (h^a_c \sigma_{cb} + C_{1ab})\omega_a^a \wedge \omega_b^b \pmod{\omega_1^1}.$$ 

This and the Frobenius theorem (see, for example, [BCGGG 91]) imply that the condition of integrability of the screen distribution $S$ has the form

$$\tau_{ab} = \tau_{ba},$$ 

and the condition of integrability of the complementary screen distribution $\tilde{S}$ has the form

$$h^a_c \sigma_{cb} + C_{1ab} = h^a_c \sigma_{ca} + C_{1ba}.$$ 

Let us prove that conditions (89) and (90) are equivalent. In fact, in the reduced frame bundle we have equations (76) and (77). Substituting the values of $\omega_0^a$ and $\omega_1^a$ taken from (77) into the first equation of (76) and using the fact
that the forms $\omega^1$ and $\omega^a$, $a = 2, \ldots, n - 1$, are linearly independent, we find that
\[-h_a^b \sigma_b - \mu \tau_a + 2C_{1a} = 0; \quad -h_a^c \sigma_{cb} - \mu \tau_{ab} = C_{1ab} + \nu_{ab}.\]
Alternating the second equation with respect to the indices $a$ and $b$, we obtain that
\[h_a^c \sigma_{cb} - h_b^c \sigma_{ca} + C_{1ab} - C_{1ba} = -2\mu \tau_{[ab]} . \quad (91)\]
Since $\mu \neq 0$, it follows from (91) that conditions (89) and (90) are equivalent.

Suppose that the screen distribution $S$ of a lightlike hypersurface $V^{n-1}$ is integrable. Denote by $S(x)$ an integral manifold of this distribution passing through a point $x = A_0$ of the isotropic geodesic $l$ of the hypersurface $V^{n-1}$. Then Pfaffian equation (86) determines also a stratification of the normalizing congruence $U^n$ of the hypersurface $V^{n-1}$ into a one-parameter family of lightlike hypersurfaces $U^{n-1}(x)$ whose generators $\tilde{l} = A_0 A_n$ pass through the points of the manifold $S(x)$.

The following theorem combines the results obtained in this subsection.

**Theorem 11** If on a lightlike hypersurface $V^{n-1}$ the conditions (89) hold, then

(a) The screen distribution $S$ and the complementary screen distribution $\tilde{S}$, are integrable.

(b) The normalizing lightlike congruence $U^n$ of a hypersurface $V^{n-1}$ is stratified into a one-parameter family of lightlike hypersurfaces $U^{n-1}$.

(c) All singular points of the congruence $U^n$ are real and coincide with singular points of the hypersurfaces $U^{n-1}$.

Part (c) of Theorem 11 follows from the symmetry of the matrix $(\tau_{ab})$: all eigenvalues of such a matrix (roots of (83)) are real.

Note also that if a $CO(n - 1, 1)$-structure on the manifold $(M, c)$ is conformally flat, then equations (76) and (89) imply that the affinors $\tau^a_b$ and $\sigma^a_b = g^{ac} \sigma_{cb}$ of a lightlike hypersurface $V^{n-1}$ are diagonalized simultaneously with the affinors $h_a^b$ and $H_a^b$. Geometrically this means that the torso formed by isotropic geodesics on every lightlike hypersurface $U^{n-1}$ correspond one to another.

### 8 Construction of affine connections intrinsically connected with a lightlike hypersurface

1. In Section 4, we have already considered the question of finding an affine connection on a lightlike hypersurface $V^{n-1}$ of a manifold $(M, c)$. As we proved in Theorem 10, an affine connection on $V^{n-1}$—denote it by $\gamma_1$—is defined in a
fourth-order differential neighborhood. Equations (47) show that this connection is torsion-free. For finding the curvature tensor of the connection \( \gamma_1 \) we substitute the values (77) of the forms \( \omega_0^a \) and \( \omega_1^a \) into (48). In addition, since the vertex \( A_n \) coincides with the harmonic pole \( C_n \) of the point \( A_0 = x \) with respect to the singular points \( Z_n \) of the isotropic geodesic \( A_0A_n \), the 1-form \( \omega_0^0 \) becomes a principal form:

\[
\omega_0^0 = \varphi_1 \omega^1 + \varphi_a \omega^a.
\] (92)

The coefficients \( \varphi_1 \) and \( \varphi_a \) in (92) are defined in a fifth-order neighborhood of a point \( x \in V^{n-1} \). This implies that the curvature tensor of the affine connection \( \gamma_1 \) induced by the invariant normalization of \( V^{n-1} \) we have constructed is also defined in a fifth-order neighborhood of a point \( x \in V^{n-1} \).

However, there is another way to construct an affine connection on a hypersurface \( V^{n-1} \). To this end, we note that in a reduced frame bundle associated with a third-order neighborhood of a point \( x \in V^{n-1} \) the 1-forms \( \omega_1^a \) and \( \omega_1^b \) in equations (47) become principal: they are expressed by formulas (42) and (77). Substituting the expressions of these forms into equations (47), we find that

\[
\begin{align*}
\omega^1 &= \omega^1 + \omega^a \wedge (\tau_a \omega^1 + \tau_{ab} \omega^b), \\
\omega^a &= \omega^a + \delta_a^b \omega^b + h_a^b \omega^1 \wedge \omega^b.
\end{align*}
\] (93)

Thus the forms \( \omega^1 - \omega_0^0 \) and \( \omega^a - \delta_a^b \omega_0^b \) can be considered as the only connection forms of the affine connection \( \gamma_2 \), and the 2-forms

\[
\begin{align*}
\Theta^1 &= \omega^a \wedge (\tau_a \omega^1 + \tau_{ab} \omega^b), \\
\Theta^a &= h_a^b \omega^1 \wedge \omega^b.
\end{align*}
\] (94)

as the torsion 2-forms of this connection.

In the decompositions (48) of exterior differentials of the 1-forms \( \omega_0^a \) and \( \omega_1^a \) we have only the forms \( \omega_0^1 \), \( \omega_0^a \), and \( \omega_1^a = g^{ab} \omega_1^b \), and the form \( \omega_0^0 \) does not occur in these decompositions. Thus the torsion and curvature tensors of the affine connection \( \gamma_2 \) are defined in a fourth-order neighborhood of a point \( x \in V^{n-1} \), not the fifth-order as was the case for the connection \( \gamma_1 \).

If the principal parameters are fixed, then the first and the last subsystems of system (48) take the form

\[
\begin{align*}
\delta (\pi_1^1 - \pi_0^1) &= 0, \\
\delta (\pi_0^a - \delta_a^b \pi_0^b) &= (\pi_0^a - \delta_a^c \pi_0^c) \wedge (\pi_0^a - \delta_a^c \pi_0^c).
\end{align*}
\] (95)

Equations (95) are the structure equations of the group \( G_2 \) defining the connection \( \gamma_2 \) on the hypersurface \( V^{n-1} \). It follows from (95) that the group \( G_2 \) is the direct product of the group \( \mathbb{R}^+ \) of homotheties (\( \mathbb{R}^+ \) is the multiplicative group
of positive real numbers), and the general linear group $GL(n-2, \mathbb{R})$ over the field of real numbers:

$$G_2 = \mathbb{R}^+ \times GL(n-2, \mathbb{R}).$$

It follows from equations (48) that the curvature forms of the connection $\gamma_2$ can be written as

$$\begin{cases}
\Omega^1_a = \omega^a_0 \wedge \omega^a_1 + \omega^a_k \wedge \omega^a_l + C^a_{1kl} \omega^k \wedge \omega^l,
\Omega^a_b = \omega^a_0 \wedge \omega^a_1 + \omega^a_k \wedge \omega^a_b + \omega^a_l + g_{ab} g^{ae} \omega^e_0 \wedge \omega^e_1 - \delta^a_b \omega^0_0 \wedge \omega^0_1 + C^a_{bkl} \omega^k \wedge \omega^l,
\end{cases}$$

where $a, b = 2, \ldots, n-1$; $k, l = 1, 2, \ldots, n-1$. Substituting the values (77) of the forms $\omega^a_0, \omega^a_1$, and $\omega^a_n = g^{ab} \omega^1_b$ into these expressions and using (42), we find the following values of the curvature 2-forms $\Omega^1_1$ and $\Omega^a_b$:

$$\begin{cases}
\Omega^1_1 = (2C^1_{11a} - \sigma_a - h^a_c \tau_c) \omega^1 \wedge \omega^a + (\sigma_{ab} + h^a_c \tau_{cb} + C^1_{1ab}) \omega^a \wedge \omega^b,
\Omega^a_b = (\delta^a_c \sigma_c + \delta^a_b \sigma_a - g_c \sigma^a + h^a_c \tau_a - h_{bc} \tau^a + 2C^a_{b1c}) \omega^1 \wedge \omega^c,
+ (\delta^a_c \sigma_{bc} + g_{bc} \sigma^a + h^a_c \tau_{bc} + h_{bc} \tau^a - \delta^a_b \sigma_{ce} + C^a_{bce}) \omega^c \wedge \omega^e,
\end{cases}$$

where $\sigma^a = g^{ab} \sigma_b$, $\tau^a = g^{ab} \tau_b$, $\sigma^a_{bc} = g^{a[c} \sigma_{bc]}$, and $\tau^a_{bc} = g^{a[c} \tau_{bc]}$. It follows that the components of the curvature tensor of the connection $\gamma_2$ are determined by the following formulas:

$$\begin{cases}
R^1_{11a} = 2C^1_{11a} - \sigma_a - h^a_c \tau_c,
R^1_{1ab} = \sigma_{[ab]} + h^a_c \tau_{[c]} + C^1_{1ab},
R^a_{b1c} = \delta^a_c \sigma_c + \delta^a_b \sigma_a - g_c \sigma^a + h^a_c \tau_a - h_{bc} \tau^a + 2C^a_{b1c},
R^a_{bec} = \delta^a_c \sigma_{[c]} + g_{[b[c]} \sigma^a_{[c]} + h^a_c \tau_{[c]} + h_{[b[c]} \tau^a_{[c]} - \delta^a_b \sigma_{[ce]} + C^a_{bce},
\end{cases}$$

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