Hydrodynamic dispersion in porous media with macroscopic disorder of parameters

D S Goldobin\textsuperscript{1,2} and B S Maryshev\textsuperscript{1,2}

\textsuperscript{1}Institute of Continuous Media Mechanics UB RAS, Perm, Russia
\textsuperscript{2}Perm State University, Perm, Russia
E-mail: Denis.Goldobin@gmail.com, bmaryshev@mail.ru

Abstract. We present an analytical derivation of the macroscopic hydrodynamic dispersion for flows in porous media with frozen disorder of macroscopic parameters: porosity and permeability. The parameter inhomogeneities generate inhomogeneities of filtration flow which perform fluid mixing and, on the large spacial scale, act as an additional effective diffusion (eddy diffusivity or hydrodynamic dispersion). The derivation is performed for the general case, where the only restrictions are (i) the spatial autocorrelation functions of parameter inhomogeneities decay with the distance $r$ not slower than $1/r^n$ with $n > 1$, and (ii) the amplitudes of inhomogeneities are small compared to the mean value of parameters. Our analytical findings are confirmed with the results of direct numerical simulation for the transport of a passive scalar in inhomogeneous filtration flow.

1. Introduction

Real porous media are characterized by two scales of porosity: microscopic, the scale of small pore channels or granules in granular materials, and macroscopic, the scale of inhomogeneities in the porous medium structure. Macroscopic inhomogeneities are associated with various processes (washing out, crack formation, etc.) and with stones and other inclusions. These inhomogeneities are always present (see, for example, [1, 2]), even in the ideal case, where the porous medium is formed by identical spheres.

During the imposed displacement of the fluid through porous medium, these inhomogeneities lead to spatially irregular currents and are responsible for the dispersion of the fluid particles, which causes additional (to molecular diffusion) hydrodynamic diffusion of passive impurities (see, e.g., [1, 2, 3, 4, 5, 6, 7]). In contrast to the molecular diffusion, the convective dispersion is anisotropic and non-constant, proportional to the filtration flowrate, say, $U$. The isotropic medium is characterized by the convective dispersion with two main quantities: the longitudinal dispersivity, $D_\parallel = Ud_1$, and the lateral one, $D_\perp = Ud_2$. For typical displacement flowrates, convective dispersion is not less important than molecular diffusion (cf. [1, 2, 7]). In addition to anisotropy and dependence on the flowrate of displacement, an important feature of macroscopic convective dispersion is that it provides initial perturbations on the interface of immiscible displaced fluids when the displacement of this surface is unstable [8, 9]. Thus, any information on the fundamental characteristics of convective dispersion is important for research on various processes in nature (groundwater flow, chemical and biological impurities in soils, soil and rock erosion, formation of natural gas hydrate deposits, etc. [10, 11, 12, 13]) and industry (filtering devices, cooling of reactors, carbon capture and storage, etc. [14, 15]).
The development of the theory of microscopic dispersion was the subject of series of works ([4, 5], a detailed review can be found in [2, 7]); later, the researchers turned to theoretical study of macroscopic dispersion, which is the result of irregular currents caused by large-scale inhomogeneities [16, 17, 18, 19, 20, 21]. The diffusivity of the random process is proportional to the square of the amplitude of the fluctuations and the length of the correlations. As result, the macroscopic diffusion can be more important than the microscopic diffusion, even with very weak macroscopic inhomogeneities, if the correlation length of the dispersing flows is sufficiently long. The paper [18] (which provoked the comment [22] and the answer [23]) became one of the fundamental works on the theory of macroscopic dispersion. This work had a crucial impact on all the subsequent development of this theory [19, 20, 21, 24], although it is not free of some limitations, beyond which we advance in our paper, and flaws in interpretation of analytical results. The assumption of ergodicity was adopted in the work unconditionally; however, it is incompatible with strong topological restrictions on the mutual diffusion of fluid particles for cases of one- and two-dimensional inhomogeneities (many of these flaws were realized and corrected, for example, in [19, 20, 21]).

In this paper, we consider the problem of hydrodynamic dispersion on macroscopic inhomogeneities in the imposed displacement of a fluid that saturates a porous medium. The heterogeneity of both permeability and porosity is considered (the latter was ignored in many previous studies), the most general variants of the statistical properties of inhomogeneities are allowed. The only restrictions are (i) the spatial autocorrelation functions of parameter inhomogeneities decay with the distance \( r \) not slower than \( 1/r^n \) with \( n > 1 \), and (ii) the amplitudes of inhomogeneities are small compared to the mean value of parameters. These assumptions are reasonable for real geophysical and technological systems. We do not impose any additional assumptions on the statistical properties of inhomogeneities (such as Gaussian distribution, etc.). In Sec. 1 we provide the mathematical formulation of the problem and derive the equation for fluid flow through an inhomogeneous porous medium. In Sec. 2, the expressions for the dispersivity tensor for isotropic case are obtained and discussed. For the verification of analytical results, the direct numerical simulation (DNS) is performed (Sec. 4). In Conclusion (Sec. 5), the main results are discussed.

2. Flow through porous medium with macroscopic inhomogeneity of parameters

It is known that particles transported through a system of channels (pores) are subject to additional dispersion due to the inhomogeneity of the velocity field across the channel. This effect can lead, among other things, to non-Fickian dispersion (anomalous diffusion, e.g., see [25]) due to immobilization of solute particles. However, for solutes with significant molecular diffusion, the concentration distribution tends to be uniform across the pores (as, e.g., in appendix of [26]), and the solute transport occurs with a velocity equal to (at microscopic level) the average fluid velocity in the pores \( \mathbf{v} \), which is related to the filtration rate \( \mathbf{u} = \varphi \mathbf{v} \), where \( \varphi \) is the porosity.

Flows in a porous medium where the fluid is displaced by an imposed pressure gradient are determined by the Darcy law and by the law of conservation of matter:

\[
\mathbf{u} = -\frac{1}{\eta}K(\mathbf{r})\nabla p, \quad \nabla \cdot \mathbf{u} = 0, \tag{1}
\]

where \( K(\mathbf{r}) \) is the permeability of medium, \( \eta \) is the fluid viscosity and \( \mathbf{r} \) is the spatial coordinate vector. The medium parameters are inhomogeneous in space; permeability

\[
K(\mathbf{r}) = K_0(1 + \varepsilon \kappa(\mathbf{r}))
\]

and porosity

\[
\varphi(\mathbf{r}) = \frac{\varphi_0}{1 + \varepsilon \mu(\mathbf{r})},
\]
where $K_0$ and $\varphi_0$ are the reference values of permeability and porosity, respectively; $\varepsilon(r)$ and $\mu(r)$ are random functions of coordinates with zero mean values, $\langle \varepsilon(r) \rangle = \langle \mu(r) \rangle = 0$; and $\varepsilon \ll 1$ is a small dimensionless parameter which quantifies the characteristic relative variation of medium parameters.

We assume that the externally imposed pressure drop creates the mean gradient $\nabla P_0 = -\eta K_0^{-1} U$, where $U$ is the mean enforced filtration flowrate. Further, we decompose pressure $p$ into the mean-trend part and the fluctuating part $P$;

$$\nabla p = -\eta K_0^{-1} U + \nabla P.$$  \hfill (2)

Substitution of (2) into the law of matter conservation yields the equation for the pressure fluctuation; to the leading order in $\varepsilon$,

$$\Delta P = \varepsilon \eta K_0^{-1} U \cdot \nabla \varepsilon(r).$$  \hfill (3)

Equation (3) is a standard Poisson-type equation and its solution can be found by the Green’s function method

$$P(r) = \varepsilon \eta K_0^{-1} U \cdot \int \nabla \varphi(s) P_g(r - s) d^n s = -\varepsilon \eta K_0^{-1} U \cdot \int \nabla \varphi \nabla s P_g(r - s) d^n s,$$

where $n$ is the dimensionality of the problem (2- or 3-d) and the subscript of the nabla-operator indicates the coordinates for differentiation. Function $P_g(r)$ is the Green’s function for pressure fluctuations; it is $P_g(r) = (2\pi)^{-1} \ln r$ for two-dimensional flow and $P_g(r) = -1/(4\pi r)$ for three-dimensional flow (here $r = |r|$). The expression for filtration flowrate is

$$u(r) = -\eta^{-1} K(r) \nabla p \simeq (1 + \varepsilon \varphi(r)) U + \varepsilon \nabla \left( U \cdot \int \nabla \varphi \nabla s P_g(r - s) d^n s \right)$$

$$= (1 + \varepsilon \varphi(r)) U - \varepsilon U \cdot \int \nabla(s + r) \overline{G}(s) d^n s,$$

where tensor $G_{ij}(r) = \nabla_i \nabla_j P_g(r)$. Finally, the equation for fluid particle motion reads

$$\dot{r} = v(r) = \frac{u(r)}{\varphi(r)} \simeq (1 + \varepsilon \{\mu(r) + \varphi(r)\}) V - \varepsilon V \cdot \int \nabla(s + r) \overline{G}(s) d^n s,$$  \hfill (4)

where $V \equiv \varphi_0^{-1} U$ is the average velocity of fluid particles. In the next section we calculate the diffusivity tensor from the equation of particle motion.

3. Diffusivity tensor

The diffusivity tensor for random process $r(t)$ is (cf. [27])

$$D_{ij} = \lim_{t \to \infty} \frac{1}{2t} \langle (r_i(t) - r_i(0) - \langle \dot{r}_i \rangle_t) (r_j(t) - r_j(0) - \langle \dot{r}_j \rangle_t) \rangle_t$$  \hfill (5)

where $\langle \cdot \rangle_t$ stands for the averaging in time and the subscripts $i$ and $j$ indicate spatial coordinates. Let the particle coordinate deviation from the average displacement be $q_i(t) \equiv r_i(t) - r_i(0) - V_i t$. The diffusivity tensor becomes [27]

$$D_{ij} = \lim_{t \to \infty} \frac{1}{2t} \langle q_i(t) q_j(t) \rangle_t = \frac{1}{2} \int_{-\infty}^{\infty} \langle \dot{q}_i(t) \dot{q}_j(t + \tau) \rangle_t d\tau.$$  \hfill (6)
Substituting (4) into (6) with \( r_i(t) - r_i(0) = Vt + q_i(t) \), one finds

\[
2D_{ij} = \varepsilon^2 V_i V_j \int_{-\infty}^{\infty} d\tau \left\langle \mu + \varepsilon^2 \left( Vt + q(t) \right) \right. \\
- \varepsilon^2 V_i V_k \int_{-\infty}^{\infty} d\tau \left[ \mu + \varepsilon^2 \left( V(t+\tau) + q(t+\tau) \right) + \mu + \varepsilon^2 \left( V(t) + q(t) + s \right) G_{kj}(s) \right] \\
- \varepsilon^2 V_k V_j \int_{-\infty}^{\infty} d\tau \left[ \mu + \varepsilon^2 \left( V(t+\tau) + q(t+\tau) \right) + \mu + \varepsilon^2 \left( V(t) + q(t) + s \right) G_{ik}(s) \right] \\
+ \varepsilon^2 V_k V_m \int_{-\infty}^{\infty} d\tau \int d^n s_1 \int d^n s_2 G_{ik}(s_1) G_{mj}(s_2) \left\langle \varepsilon^2 \left( V(t+\tau) + q(t+\tau) + s_2 \right) G_{mj}(s_2) \right\rangle_t,
\]

(7)

where \( [\mu + \varepsilon^2](r) \equiv \mu(r) + \varepsilon^2(r) \). One can introduce the correlation functions

\[
C_{f,g}(s_2 - s_1) = \left\langle f(s_1) g(s_2) \right\rangle, \quad C_f(s_2 - s_1) = \left\langle f(s_1) f(s_2) \right\rangle,
\]

the angle brackets without subscript indicate averaging over noise realizations) and rewrite expression (7) in a shorter form

\[
2D_{ij} = \varepsilon^2 V_i V_j \int_{-\infty}^{\infty} d\tau \left\langle \mu + \varepsilon^2 \left( Vt + q(t+\tau) - q(t) \right) \right\rangle_t
\]

(8)

Further, we employ the assumption of smallness of macroscopic inhomogeneities of porosity and permeability, which allows one neglecting \( |q(t+\tau) - q(t)| \) in equation (8). Indeed, \( |q(t+\tau) - q(t)| \) is related to distortion of the homogeneous macroscopic flow and \( \propto \varepsilon \); therefore, for vanishing \( \varepsilon \) the perturbation of the arguments of correlation functions in (8) should be either small for nonlarge \( \tau \) or small compared to \( V\tau \) for large \( \tau \). Hence, equation (8) reads

\[
D_{ij} = \frac{\varepsilon^2}{2V} V_i V_j \int_{-\infty}^{\infty} dl C_{\mu + \varepsilon^2}(le_U) - \frac{\varepsilon^2}{2V} V_i V_k \int d^n s G_{kj}(s) \int_{-\infty}^{\infty} dl C_{\mu + \varepsilon^2}(le_U + s)
\]

\[
- \frac{\varepsilon^2}{2V} V_k V_j \int d^n s G_{ik}(s) \int_{-\infty}^{\infty} dl C_{\mu + \varepsilon^2}(le_U + s)
\]

\[
+ \frac{\varepsilon^2}{2V} V_k V_m \int d^n s_1 G_{ik}(s_1) \int d^n s_2 G_{mj}(s_2) \int_{-\infty}^{\infty} dl C_{\varepsilon^2}(le_U + s_2 - s_1) + O(\varepsilon^4),
\]

(9)

where \( l = V\tau \) and \( e_U \) is the unit vector along the direction of the average imposed displacement flux, \( e_U = U/U = V/V \).

With expression (9), one can calculate the diffusivity for any small random inhomogeneity of permeability and porosity. However, the employed definition (5) is the definition for the diffusion in time. The spacial dispersivity of fluid particles equals to the temporal diffusivity only for an ergodic process. In the next section, the case of the inhomogeneities with isotropic correlation functions is considered and expression is validated by comparison to the results of direct numerical simulation. The issue of the system ergodicity is also discussed.
4. Isotropic statistical properties of inhomogeneities

For the case of isotropic correlation functions, $C_{f,g}(s) = C_{f,g}(|s|)$ and equation (9) yields

$$D_{ij} = \frac{\varepsilon^2}{2V} V_i V_j \int_{-\infty}^{\infty} dl \ C_{\mu+\kappa}(l) + \mathcal{O}(\varepsilon^4).$$

The tensor $D_{ij}$ is symmetric and can be rewritten into the diagonal form with components $D'_{11} = D_{\parallel}$, $D'_{22} = D_{\perp}$, and $D'_{12} = D'_{21} = 0$. In these terms,

$$D_{\parallel} = \frac{\varepsilon^2 V}{2} \int_{-\infty}^{\infty} C_{\mu+\kappa}(l) dl, \quad D_{\perp} = \mathcal{O}(\varepsilon^4). \quad (10)$$

In order to validate expressions (10), we performed direct numerical simulations for dimensionless equations (in which $\eta = 1$, $K_0 = 1$). Firstly, we generated numerically random functions $-0.5 < \kappa(r) < 0.5$ and $-0.5 < \mu(r) < 0.5$ with isotropic statistical properties (see Figure 1). Secondly, equations (1) for pressure were solved by means of a standard finite difference method of second order accuracy for periodic boundary conditions (a sample field of pressure fluctuations is plotted in Figure 1). Thirdly, the velocity of fluid particles was calculated as

$$v = (1 + \varepsilon \mu) \ (1 + \varepsilon \kappa) \ (V - \varphi^{-1} \nabla P) .$$

Finally, the advection-diffusion equation for passive scalar ($C$) was solved with prescribed molecular diffusivity $D_m$ and calculated filtration velocity field $u$:

$$\frac{\partial C}{\partial t} + v \cdot \nabla C = \frac{D_m}{\varphi(r)} \ \nabla \cdot (\varphi(r) \nabla C) . \quad (11)$$

The solution for equation (11) was obtained by the direct numerical simulation with the explicit finite difference scheme of second order accuracy in space and first order accuracy in time. This solution was further compared with the fundamental solution of diffusion equation with symmetric diffusivity tensor $D_{ij}$, which reads

$$C_s(x, y, t) = \frac{1}{4 \pi t \sqrt{D_{11} D_{22} - D_{12}^2}} \exp \left[ - \frac{D_{22} x^2 - 2 D_{12} xy + D_{11} y^2}{4 t (D_{11} D_{22} - D_{12}^2)} \right] , \quad (12)$$

for the estimation of diffusivity tensor components.
The isolines of the concentration field. The field $C$ (numerical solution of (11)) is plotted in the right panel, the Gaussian approximation $C_*$ (12) is plotted in the left panel. The direction of average filtration flow is indicated with the arrow, $\phi_0^{-1} U = V = (0.86, 0.52)$. The calculation of $C(x, y, t)$ was performed with the pressure field presented in Figure 1 and molecular diffusivity $D_m = 10^{-3}$. The components of diffusivity tensor $D_{ij}$ were estimated by BFGS method ($D_{22} = 6.22 \cdot 10^{-3}, D_{11} = 1.43 \cdot 10^{-2}, D_{12} = 8.08 \cdot 10^{-3}$).

The parameter estimation was performed by the least squares method with residuals’ sum

$$E = \sum_{x,y,t} \left( C(x, y, t) - C_*(x, y, t) \right)^2,$$

where summation was performed for all nodes of space mesh and for 30 time steps (from number 500 to number 530, after the transient process). The procedure of the error minimization with respect of components of $D_{ij}$ was based on conventional Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm [28]. The example of calculated fields $C$ and $C_*$ is presented in Figure 2.

In Table 1 the values of components of the diffusivity tensor estimated by BFGS method and compared to the diffusivity tensor calculated with expression (10), read as $D_{\|} = \varepsilon^2 V I$ with the integral $I = (1/2) \int_0^\infty C_{\mu+x}(l) \, dl$. Equation (11) is solved for the case of a small but non-zero molecular diffusivity. The latter is included into consideration for two reasons: firstly, the molecular diffusion is present for transport processes in the real porous media, and, secondly

Table 1. The analytical calculations with equation (10) are compared to the results of DNS for isotropic disorder in permeability and porosity for two-dimensional system with non-negligible molecular diffusivity $D_m$. Results are present for the average interstitial fluid velocity $V = (0.86, 0.52)$; the integral $I = \frac{1}{2} \int_{-\infty}^{\infty} C_{\mu+x}(l) \, dl$.

| parameters | $D_m$ | $I$ | $D_{\|}$ | $D_{\|} + D_m$ | $D_{\|,net}$ | $D_{\perp,net}$ |
|------------|-------|-----|--------|-------------|-------------|-------------|
| $10^{-1}$  | $10^{-3}$ | 1.82 | 1.82 $\cdot 10^{-2}$ | 1.92 $\cdot 10^{-2}$ | 1.93 $\cdot 10^{-2}$ | 1.23 $\cdot 10^{-3}$ |
| $10^{-1}$  | $5 \cdot 10^{-4}$ | 1.63 | 1.63 $\cdot 10^{-2}$ | 1.68 $\cdot 10^{-2}$ | 1.67 $\cdot 10^{-2}$ | 6.12 $\cdot 10^{-4}$ |
| $10^{-1}$  | $10^{-4}$ | 1.71 | 1.71 $\cdot 10^{-2}$ | 1.72 $\cdot 10^{-2}$ | 1.72 $\cdot 10^{-2}$ | 2.17 $\cdot 10^{-4}$ |
| $5 \cdot 10^{-2}$ | $10^{-3}$ | 1.58 | 3.95 $\cdot 10^{-3}$ | 4.95 $\cdot 10^{-3}$ | 4.98 $\cdot 10^{-3}$ | 1.08 $\cdot 10^{-3}$ |
| $5 \cdot 10^{-2}$ | $5 \cdot 10^{-4}$ | 1.72 | 4.31 $\cdot 10^{-3}$ | 4.81 $\cdot 10^{-3}$ | 4.85 $\cdot 10^{-3}$ | 5.52 $\cdot 10^{-4}$ |
| $5 \cdot 10^{-2}$ | $10^{-4}$ | 1.48 | 3.72 $\cdot 10^{-3}$ | 3.82 $\cdot 10^{-3}$ | 3.81 $\cdot 10^{-3}$ | 1.68 $\cdot 10^{-4}$ |
| $10^{-2}$  | $10^{-3}$ | 1.92 | 1.92 $\cdot 10^{-4}$ | 1.19 $\cdot 10^{-3}$ | 1.19 $\cdot 10^{-3}$ | 1.00 $\cdot 10^{-3}$ |
| $10^{-2}$  | $5 \cdot 10^{-4}$ | 1.73 | 1.73 $\cdot 10^{-4}$ | 6.73 $\cdot 10^{-4}$ | 6.73 $\cdot 10^{-4}$ | 5.00 $\cdot 10^{-4}$ |
| $10^{-2}$  | $10^{-4}$ | 1.67 | 1.67 $\cdot 10^{-4}$ | 2.67 $\cdot 10^{-4}$ | 2.67 $\cdot 10^{-4}$ | 1.01 $\cdot 10^{-4}$ |
(even more important mathematically), it brings the ergodicity into the system. The diffusivity components are presented in the coordinate system associated to the filtration velocity $U$. $D_{∥,\text{net}}$ and $D_{⊥,\text{net}}$ are the net longitudinal and lateral diffusion coefficients contributed by both the molecular diffusion and the hydrodynamic dispersion. Provided the transport in the system is ergodic, the Brownian fluctuations of the particle position and the fluctuations owned by the macroscopic flow inhomogeneities are mutually independent; in this case, $D_{∥/⊥,\text{net}} = D_{∥/⊥,\text{net}} + D_m$.

The results of the analytical theory and direct numerical simulations were found to be in good agreement up to as large values of $\varepsilon$ as 0.1 (see Table 1).

5. Conclusion
The expression for macroscopic hydrodynamic dispersivity for inhomogeneous flows in porous medium has been derived analytically on the basis of the fact that the dispersion is produced by macroscopic inhomogeneities of porosity and permeability of the medium. The derivation has been performed for the most general case of statistical properties of inhomogeneities. The only restrictions are (i) relative smallness of inhomogeneities and (ii) finiteness of the autocorrelation length. The analytical results have been verified by comparison to the results of direct numerical simulation of the transport of a passive scalar in two-dimensional inhomogeneous filtration flow with a non-negligible molecular diffusivity.

Acknowledgments
The work was partially supported by the Grant of the President of Russian Federation (Grant No. MK-1447.2017.5).

References
[1] Collins R E 1961 Flow of Fluids through Porous Materials (New York: Reinhold)
[2] Sahimi M 1993 Rev. Mod. Phys. 65 1393
[3] von Rosenberg D U 1956 AIChE J. 2 55
[4] Saffman P G 1959 J. Fluid Mech. 6 321
[5] Saffman P G 1960 J. Fluid Mech. 7 194
[6] Dagan G 1986 Water Resour. Res. 22 1208
[7] Dagan G 1987 Annu. Rev. Fluid Mech. 19 183
[8] Saffman P G and Taylor G 1958 Proc. Roy. Soc. Lond. A 245 312
[9] Lyubimov V D, Shklyaev S, Lyubimova T P and Zikanov O 2009 Phys. Fluids 21 014105
[10] Davie M K and Buffett B A 2001 J. Geophys. Res. 106 497
[11] Goldobin D S and Brilliantov N V 2011 Phys. Rev. E 84 056328
[12] Hunter S J, Goldobin D S, Haywood A M, Ridgwell A and Rees J G 2013 Earth Planet. Sci. Lett. 367 105
[13] Goldobin D S and Krauzin P V 2015 Phys. Rev. E 92 063032
[14] Holloway S 1997 Energy Conversion and Management 38 S193
[15] Research Consortium for Methane Hydrate Resources in Japan 2008 Japan’s Methane Hydrate R&D Program, Phase 1 Comprehensive Report of Research Results
[16] Buyevich Y A, Leonov A I and Safrai V M 1969 J. Fluid Mech. 37 371
[17] Gelhar L W, Gutjahr A L and Naff R J 1979 Water Resour. Res. 15 1387
[18] Gelhar L W and Axness C L 1983 Water Resour. Res. 19 161
[19] Neuman S P, Winter C L and Newman C M 1987 Water Resour. Res. 23 453
[20] Neuman S P and Zhang Y 1990 Water Resour. Res. 23 887
[21] Dagan G 1991 J. Fluid Mech. 233 197
[22] Cushman J H 1983 Water Resour. Res. 19, 1641
[23] Gelhar L W and Axness C L 1983 Water Resour. Res. 19 1643
[24] Dagan G 1988 Water Resour. Res. 24 1491
[25] Maryshev B, Joelson M, Lyubimov D, Lyubimova T, Néel M-C N 2009 J. Phys. A: Math. Theor. 42 115001
[26] Goldobin D S and Shklyaeva E V 2009 J. Stat. Mech.: Theor. Exp. P01024
[27] Gardiner C W 1997 Handbook of Stochastic Methods (Berlin Germany: Springer-Verlag Heidelberg)
[28] Nocedal J and Wright S J 2006 Numerical Optimization (New York USA: Springer-Verlag)