Two Reliable Iterative Methods for Solving Chaos synchronization

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Abstract. In this article we proposed two reliable iterative methods for solving Chaos synchronization. The iterative methods are the Adomain Decomposition method (ADM) and Variational Iteration Method (VIM). The ADM and VIM are solved several problems in different areas which accuracy and efficiency in the results. The solution which get is an approximate solution is accuracy as we well show that in figures and tables for the analysis of maximum error reminder various. The software which used in the study for the calculations was MATHEMATICA 11.

Keywords. Adomain Decomposition method (ADM), Variational Iteration Method (VIM), iterative methods, Chaos synchronization, maximum error reminder.

1. Introduction

The Nonlinear differential equations become from the most important matter in different sciences various of the problem that arise in engineering fields and physics can be depicted by non linear ordinary differential equation such as Duffing equations. Duffing equations was very important in engineering and physics because used in many problem like solid and structural mechanics as well as fluid mechanics [1, 2].

Synchronization phenomena have been important since the beginning physics at first the focus was devoted to the synchronization of periodic systems. The newly research for synchronization has convey to chaotic systems. The electrical circuit in fig (1) is a non-linear electric circuit paying by a sinusoidal voltage source depiction by the Duffing electrical oscillator, consists of the linear resistor in chain with a sinusoidal source. This two ingredients are linked in parallel with a capacitor and a non-linear inductor. Nonlinear inductor is an inductor with a ferromagnetic heart which can be description if an abstraction of the hysteresis phenomenon is made by using i-s nonlinear quality.

The quality of the nonlinear element is approximately described by the relation [3]:

\[ i = b_1 s + b_3 s^3 \]
Figure 1. (i) The electric circuit obeying to the Duffing equation and (ii) analog simulation circuit of the driven Duffing oscillator [3].

The fig (1) can be writing as following equation [3]:

\[ g u'' + \alpha u' + u + \epsilon u^3 = S \cos wx, \]

With initial condition:

\[ u(0) = A, \quad \text{and} \quad u'(0) = 0. \]

Where \( \epsilon = 1, \alpha = 0.18, S = 23.5, w = 0.8 \) and \( A = 1 \), then have

\[ u'' + 0.18 u' + u + u^3 = 23.5 \cos(0.8 x), \]  

With initial condition:

\[ u(0) = 1, \quad \text{and} \quad u'(0) = 0. \]

We take the taylor series of \( \cos(0.8 x) \) as follows

\[ \cos(0.8 x) = 1 - 0.320 x^2 + \cdots \]

In this article, two iterative methods will be used to solve the Chaos synchronization for obtain new approximate solutions. The first one is presented in 1980 by George Adomian [4]. The second method is established by Ji-Huan He in 1999 is now used to handle a wide variety of linear and nonlinear, homogeneous and inhomogeneous equations [5]. This methods solved many of various equations in various areas of science such as physics, chemistry, mechanics and other science.

2. The Basic Proceedings of the ADM

Let us consider the following ODE of ADM [6]:

\[ Lu + Ru + Nu = G, \]

Where, \( N \) is a nonlinear operator, \( L \) is highest-order derivative which is assumed to be invertible, \( R \) is a linear differential operator of order less than \( L \) and \( G \) is the nonhomogeneous term. The method starts by applying the operator \( L^{-1} \) formally to the expression

\[ Lu = G - Ru - Nu, \]

With initial condition:

\[ u(0) = a, \quad \text{and} \quad u'(0) = b. \]

So by using the given conditions, we have:

\[ u = b + L^{-1}G - L^{-1}Ru - L^{-1}Nu, \]  

Where \( L^{-1}(\cdot) = \int_{b}^{x} \cdot dx \), and \( u'(0) = b \), is the initial condition.
The ADM technique consists of approximating the solution of Eq. (2) as an infinite series:

\[ u = \sum_{n=0}^{\infty} u_n, \]

The nonlinear differential equation cannot be solved directly by ADM due to the nonlinear terms. In order to deal with the nonlinear term, we must calculate the so-called Adomian polynomial. Adomian’s polynomials are arranged to have the form:

\[ F(u) = Nu = \sum_{n=0}^{\infty} A_n, \]

Where \( A_n \) are the Adomain polynomial of \( u_0, u_1, \ldots, u_n \) that are the terms of the analytical expansion of \( Nu \), where

\[ u = \sum_{i=0}^{\infty} \lambda^i u_i, \]

around \( \lambda=0 \) [11]. That is:

\[ A_n = \frac{1}{n!} \left[ \frac{\partial^n}{\partial \lambda^n} F \left( \sum_{i=0}^{n} \lambda^i u_i \right) \right]_{\lambda=0}, \quad n = 0, 1, 2, \ldots \]

Adomian polynomials are arranged to have the form [6]:

\[ A_0 = F(u_0), \]
\[ A_1 = u_1 F'(u_0), \]
\[ A_2 = u_2 F(u_0) + \frac{\mu^2}{2!} F''(u_0), \]
\[ A_3 = u_3 F(u_0) + u_1 u_2 F''(u_0) + \frac{\mu^3}{3!} F'''(u_0), \]

Now, we parameterize Eq. (2) in the form:

\[ u = b + L^{-1}(G) - \lambda L^{-1} R u - \lambda L^{-1} N u, \]

where, \( \lambda \) is just an identifier of collecting terms in a suitable way such that \( u_n \) depends on \( u_0, u_1, \ldots, u_n \) and we will later set \( \lambda=1 \).

\[ \sum \lambda^n u^n = b + L^{-1} G - \lambda L^{-1} R \sum \lambda^n u^n - \lambda L^{-1} \lambda^n A_n, \]

Equating the coefficients of equal powers of \( \lambda \), we obtain:

\[ u_0 = b + L^{-1} G, \]
\[ u_1 = -L^{-1} R(u_0) - L^{-1}(A_0), \]
\[ u_2 = -L^{-1} R(u_1) - L^{-1}(A_1), \]

and in general

\[ u_n = -L^{-1} R(u_{n-1}) - L^{-1}(A_{n-1}). \]

In order to illustrate the application of ADM for Duffing equation. Also, we calculate the error remainder with the maximal error remainder parameters for the approximate solution, the convenience function of the error remainder will be [7, 5]:

\[ ER_n(x) = u''(x) + k_1 u''(x) + k_2 u_n + k_3 u_n^2 - f(x), \]

The maximal error remainder parameters are:

\[ MER_n = \max_x |ER_n(x)|. \quad (3) \]

3. The Basic Proceedings of the VIM

To illustrate the basic concepts of the VIM, we consider the following nonlinear equation [8]:

\[ Lu(t) + Nu(t) = g(t), \quad t > 0. \quad (4) \]
Where, \( L \) is a linear operator, \( N \) a nonlinear operator and \( g(x) \) is the source the inhomogeneous term. The VIM introduces functional for Eq. (4) in the form:

\[
\int_0^t \left[ A(t)(L_0 u(t) - N \tilde{u}(t) - g(t)) \right] dt, \tag{5}
\]

Where, \( \lambda \) is a general Lagrangian multiplier calculated in the following form \([9, 10]\):

\[
\lambda = \frac{(-1)^n}{(n-1)!} (t-x)^{n-1}, \quad n \geq 1. \tag{6}
\]

Consequently, the solution is given by:

\[
u(x) = \lim_{n \to \infty} u_n(x).
\]

4. **ADM for solving Chaos synchronization**

We write Eq. (1) in an operator form then have:

\[
L_{xx} u = 11.75x^2 - 0.6266x^4 - 0.18 L_x u(x) - u - u^3(x),
\]

where the differential operator \( L \) is given by \( L_x = \frac{d}{dx} \) and \( L_{xx} = \frac{d^2}{dx^2} \), and there for the inverse operator is:

\[
L_x^{-1}(\cdot) = \int_0^x (\cdot) dx, \quad \text{and} \quad L_{xx}^{-1}(\cdot) = \int_0^x (\cdot) dx dx.
\]

Applying \( L_{xx}^{-1} \) to both sides and using the initial condition then obtain:

\[
u(x) = 1 + 11.75x^2 - 0.6266x^4 - L_x^{-1}(0.18 u(x)) - L_{xx}^{-1}(u(x) + u^3(x)).
\]

Now using the Adomain method to find the approximate solution of Eq. (1):

\[
u(x) = \sum_{n=0}^{\infty} u_n(x),
\]

and the nonlinear term \( u^3 \) is:

\[
u^3 = \sum_{n=0}^{\infty} A_n.
\]

Then, Eq. (1) will be then:

\[
\sum_{n=0}^{\infty} u_n(x) = 1 + 11.75x^2 - 0.6266x^4 - L_x^{-1}(0.18 \sum_{n=0}^{\infty} u_n(x)) - L_{xx}^{-1}(u_n(x) + \sum_{n=0}^{\infty} A_n)).
\]

The following recursive relation:

\[
\begin{align*}
 u_0 &= 1 + 11.75x^2 - 0.6266x^4, \\
 u_{n+1} &= -L_x^{-1}(0.18 u_n) - L_{xx}^{-1}(u_n + A_n), \quad n = 0, 1, \ldots
\end{align*}
\]

Now, find the Adomain polynomial \( A_n \),

\[
\begin{align*}
 A_0 &= f(u_0) = u_0^3, \\
 A_1 &= u_1 f'(u_0) = 3u_0^2u_1, \\
 A_2 &= u_2 f''(u_0) + \frac{1}{2!} u_1^2 f'''(u_0) = 3u_0^2u_2 + 3u_0u_1^2,
\end{align*}
\]

Follows immediately. Consequently, we obtain:

\[
\begin{align*}
 u_0 &= 1 + 11.75x^2 - 0.6266x^4, \\
 u_1 &= -L_x^{-1}(0.18 u_0) - L_{xx}^{-1}(u_0 + A_0), \\
 u_1 &= -0.18x - x^2 - 0.70499x^3 - 3.91666x^4 + 0.022559x^5 - 13.72269x^6 + \cdots, \\
 u_2 &= -L_x^{-1}(0.18 u_1) - L_{xx}^{-1}(u_1 + A_1), \\
 u_2 &= 0.0162x^2 + 0.18x^3 + 0.365058x^4 + 0.9165x^5 + 2.8715454x^6 + 3.2930x^7 + 13.2400x^8 + \cdots,
\end{align*}
\]

and so on.

Continuing in this technique, the approximation solution is:
\[ u(x) = \sum_{n=0}^{\infty} u_n(x), \]

\[ u(x) = 1 - 0.18x + 10.7662x^2 - 0.52499x^3 - 4.1782749x^4 + 0.93906x^5 - 10.8511490x^6 + 3.293088x^7, \]

Eq. (1) has approximate solution and calculate the numerical result by MER. See table (1).

| n | MER       |
|---|-----------|
| 1 | 0.210519  |
| 2 | 0.00653478|
| 3 | 0.000154972|
| 4 | 0.000001385 \times 10^{-6} |

**Figure 2.** The MER of Eq. (1) where n= 1, ..., 4.

5. VIM for solving Chaos synchronization

The correction functional is:

\[ u_{n+1} = u_n(x) + \int_0^x \lambda(t)(u''_n + 0.18u'_n + u_n + u_n^3 - g)(t)dt, \]

By using the formula in Eq. (1. 21) given above then have:

\[ \lambda = 1 - x \text{ in this equation:} \]

\[ u_{n+1}(x) = u_n(x) + \int_0^x (t-x)(u''_n + 0.18u'_n + u_n + u_n^3 - g)(t)dt, \]

Now can select:

\[ u_0 = u(0) + xu'(0) = 1, \]

From the given conditions. Applying the algorithm of (VIM):

\[ u_0 = 1, \]

\[ u_1 = 1 + \int_0^x (x-t)(u''_0 + 0.18u'_0 + u_0 + u_0^3 - 23.5 + 7.52x^2)(t)dt, \]

\[ u_1 = 1 + 10.75x^2 - 0.6266x^4, \]
\[ u_2 = u_1 + \int_0^x (t-x)(u''_1 + 0.18u'_1 + u_1 + u_1^3 - 23.5 + 7.52x^2)(t)dt, \]

\[ u_2 = 1 + 10.75x^2 - 0.645x^3 - 4.209x^4 + 0.02255x^5 - 11.472694x^6 - 21.4620x^8 + \ldots \]

and so on. Then

\[ u(x) = 1 + 10.75x^2 - 0.645x^3 - 4.209x^4 + 0.0225599x^5 - 11.472694x^6 - 21.46208x^8 + 2.40088x^{10} + \ldots \]

| \( n \) | MER         |
|-------|------------|
| 1     | 0.85216    |
| 2     | 0.010933   |
| 3     | 0.000803926|
| 4     | 4.150 \times 10^{-7} |

Figure 3. MER of Eq. (1) where \( n = 1, \ldots, 4 \).

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