Locally convex structures on higher local fields

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Abstract

We establish how a higher local field can be described as a locally convex vector space once an embedding of a local field into it has been fixed. This extends previous results that had been obtained in the two-dimensional case. In particular, we study bounded and compactoid submodules of these fields and establish a self-duality result once a suitable topology on the dual space has been introduced.

Introduction

In [1] we explained how characteristic zero two-dimensional local fields may be regarded as locally convex vector spaces once an embedding of a local field into them has been fixed.

This note, designed as a natural continuation of that work, explains how the locally convex approach to higher topologies works for a higher local field of arbitrary dimension.

It is perhaps necessary to explain the need to treat the arbitrary dimensional case separately. The two-dimensional case often supplies the first step of induction and therefore it is a good idea to treat it first. At the same time, the cases for dimension greater than two quickly turn into a rather involved exercise in notation and the application of arguments which are familiar from the case $n = 2$. As such, the proof of many results in this note often refers to [1] for the case $n = 2$ and then indicates how to proceed by induction.

Furthermore, there are many relevant functional analytic properties which may be shown to hold in the two-dimensional case and which fail in greater dimension or which one could only expect to hold in few particular cases; being bornological, reflexive or nuclear is an example of such properties.

However, it is possible to give explicit bases for the bornologies of bounded and compactoid $O$-submodules of $F$. We also show an explicit self-duality result in Theorem 6.3 which generalises [1, Theorem 6.2].

We not only often refer to concepts and results explained in [1], but also assume the reader to be familiar with it. In particular, we reviewed the definitions and results of the theory of locally convex vector spaces over a local field which are needed for this work in [1, §1]. Besides that, [8] and [9] contain suitable introductions to the topic.

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We outline the contents of this work. §1 is shaped very much after [1, §2] and summarizes certain results from the structure theory of higher local fields. At the end of this section we focus our attention on a standard higher local field, which is one of the form
\[ F = K \{ \{ t_1 \} \} \cdots \{ \{ t_r \} \} \{ (t_{r+1}) \} \cdots \{ (t_{n-1}) \} \]
and deduce our results first in this case.

Section §2 explains how the higher topology on \( F \) is locally convex. §3 exposes facts which are either immediate consequences of the results in the previous section or facts which were already known about higher topologies.

Sections §4 and §5 deal with the study of bounded submodules and compactoid submodules of \( F \), respectively.

We study duality issues in §6. In Theorem 6.3 we prove self-duality after topologizing the dual space adequately, generalising the work that was done for the two-dimensional case in [1, Theorem 6.2]. We also describe polars and pseudo-polars of relevant submodules of \( F \).

We explain how the results obtained in the previous sections may be extended from \( F \) to an arbitrary \( n \)-dimensional local field in §7 and we dedicate a few words to the positive characteristic and archimedean cases in section §8.

Finally, we discuss some interesting questions and directions of work specifically related to this note in §9.

Notation. Whenever \( F \) is a complete discrete valuation field, we will denote by \( \mathcal{O}_F, p_F, \pi_F, \mathfrak{F} \) its ring of integers, the unique nonzero prime ideal in the ring of integers, an element of valuation one and the residue field, respectively.

Throughout the text, \( K \) will denote a characteristic zero local field, that is, a finite extension of \( \mathbb{Q}_p \) for some prime \( p \). The cardinality of the finite field \( \mathbb{K} \) will be denoted by \( q \). The absolute value of \( K \) will be denoted by \( | \cdot | \), normalised so that \( | \pi_K | = q^{-1} \). Due to far too frequent apparitions in the text, we will ease notation by letting \( \mathcal{O} := \mathcal{O}_K \) and \( p := p_K \).

The conventions \( p^{-\infty} = K \), \( p^{\infty} = 0 \) and \( q^{-\infty} = 0 \) will be used.

The main object of study of this work is a field inclusion \( K \subset F \) where \( F \) is an \( n \)-dimensional local field. See §1 for details.

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1 Higher local fields arising from arithmetic contexts

A zero-dimensional local field is a finite field. An \( n \)-dimensional local field, for \( n \geq 1 \), is a complete discrete valuation field \( F \) such that \( \mathcal{F} \) is an \((n - 1)\)-dimensional local field. Thus, a local field in the usual sense is a one-dimensional local field.

An \( n \)-dimensional local field \( F \) determines then a collection of fields \( F_i \), \( i \in \{0, \ldots, n\} \), by letting \( F_n = F, \mathcal{F}_i = F_{i-1} \) for \( 1 \leq i \leq n \); being \( n \)-dimensional is then determined by the finiteness of \( F_0 \).
An excellent introduction to this topic may be found in [6], we also often refer to results explained in [3].

In [1] §2, we explained how, as was first introduced in [7], it is a good idea to regard two-dimensional local fields arising from an arithmetic context as vector spaces over a local field.

The construction may be generalised to an arithmetic scheme of any dimension as follows. Let $S$ be the spectrum of the ring of integers of a number field and $f : X \rightarrow S$ be an arithmetic scheme of dimension $n$ (for our purposes, it is enough to suppose that $X$ is an $n$-dimensional regular scheme and that $f$ is projective and flat). Given a complete flag of irreducible subschemes $\eta_0 \subset \cdots \subset \eta_n = X$, and assuming for simplicity that $\eta_n$ is regular in $\eta_i$ for each $0 \leq i \leq n-1$, define $F^n = \widehat{O}_{X,\eta_n}$ and

$$A^i = \widehat{A}^{i+1}, \quad i \in \{0, \ldots, n-1\}.$$

It can be shown [3] Remark 6.12] that $F = A^0$ is an $n$-dimensional local field. The ring homomorphism $\mathcal{O}_{S,f(x)} \rightarrow \mathcal{O}_{X,x}$ induces a field embedding $K \hookrightarrow F$, where $K = \text{Frac} \left( \mathcal{O}_{S,f(x)} \right)$. Our conclusion is that whenever $n$-dimensional local fields arise from an arithmetic-geometric setting they always come equipped with a prefixed embedding of a local field into them.

This justifies our decision to fix a characteristic zero local field $K$ and to study $n$-dimensional local fields not as fields $F$, but as pairs of a field $F$ and a field embedding $K \hookrightarrow F$. We shall refer to such a pair as an $n$-dimensional local field over $K$. A morphism of higher local fields over $K$ is therefore a commutative diagram of field embeddings

$$\begin{array}{ccc}
F_1 & \longrightarrow & F_2 \\
\uparrow & & \uparrow \\
K & \hookrightarrow & F_2
\end{array}$$

where $F_1$ and $F_2$ are higher local fields and $F_1 \rightarrow F_2$ is an extension of complete discrete valuation rings.

The structure of higher local fields is explained in [5], [3] §1] or [6] Theorem 2.18]. They may be classified using Cohen structure theory for complete rings. In particular, we have the following possibilities:

(i) If $\text{char } F$ is positive, then it is possible to choose $t_1, \ldots, t_n \in F$ such that

$$F \cong F_0((t_1)) \cdot \cdots \cdot ((t_n)).$$

In this work we assume $\text{char } F = 0$, and only treat the positive characteristic case in [8].

(ii) If $\text{char } F_1 = 0$, then there are $t_1, \ldots, t_{n-1} \in F$ such that

$$F \cong F_1((t_1)) \cdot \cdots \cdot ((t_{n-1})).$$

Moreover, if an embedding of fields $K \hookrightarrow F$ has been fixed, then we have a finite extension $K \hookrightarrow F_1$.

(iii) If none of the above holds, then there is a unique $r \in \{1, \ldots, n-1\}$ such that $\text{char } F_{r+1} \neq \text{char } F_r$. Then there is a characteristic zero local field $L$ and elements $t_1, \ldots, t_{n-1} \in F$ such that $F$ is a finite extension of

$$L\langle\{t_1\}\rangle \cdots \langle\{t_r\}\rangle \langle\{t_{r+1}\}\rangle \cdots \langle\{t_{n-1}\}\rangle.$$
Moreover, if char \( F_0 = p \), \( L \) may be chosen to be the unique unramified extension of \( \mathbb{Q}_p \) with residue field \( F_0 \). In this work we will not require this fact, but simply use the fact that given an embedding \( K \hookrightarrow F \), there is a finite subextension
\[
K \{ \{ t_1 \} \ldots \{ t_r \} \} (t_{r+1}) \cdots (t_{n-1}) \hookrightarrow F.
\]

Notation. We fix from now on, and until the beginning of \[7\]
\[
F = K \{ \{ t_1 \} \ldots \{ t_r \} \} (t_{r+1}) \cdots (t_{n-1})
\]
with \( 0 \leq r \leq n - 1 \). The extremal case \( r = 0 \) (resp. \( r = n - 1 \)) stands for \( F = K (t_1) \cdots (t_{n-1}) \) (resp. \( F = K \{ \{ t_1 \} \} \cdots \{ t_{n-1} \} \)). We also let
\[
L = K \{ \{ t_1 \} \} \cdots \{ t_r \} (t_{r+1}) \cdots (t_{n-2}),
\]
by which we simply mean that \( L \) is the subfield of \( F \) consisting of power series in \( t_1, \ldots, t_{n-2} \). It will be extremely convenient to use multi-index notation. For this purpose, let \( I = \mathbb{Z}^{n-1} \) and \( J = \mathbb{Z}^{n-2} \). For \( l \in \{ 1, \ldots, n-1 \} \), if we fix indices \( (i_1, \ldots, i_{n-1}) \in \mathbb{Z}^{n-l-1} \), we will denote
\[
I(i_1, \ldots, i_{n-1}) = \left\{ \alpha \in I; \ \alpha = (j_1, \ldots, j_{l-1}, i_1, \ldots, i_{n-1}) \text{ for some } (j_1, \ldots, j_{l-1}) \in \mathbb{Z}^{l-1} \right\}.
\]
Any element \( x \in F \) can be written uniquely as a power series
\[
x = \sum_{i_{n-1} \gg -\infty} \cdots \sum_{i_{r+1} \gg -\infty} \sum_{i_r \in \mathbb{Z}} \cdots \sum_{i_1 \in \mathbb{Z}} x_{i_1, \ldots, i_{n-1}} t_1^{i_1} \cdots t_{n-1}^{i_{n-1}},
\]
with \( x_{i_1, \ldots, i_{n-1}} \in K \). We will abbreviate such an expression to
\[
x = \sum_{\alpha} x(\alpha) t^\alpha,
\]
for \( \alpha \in I \) and \( x(\alpha) \in K \). Finally, for \( \alpha = (i_1, \ldots, i_{n-1}) \in I \), denote \( -\alpha = (-i_1, \ldots, -i_{n-1}) \).

Several proofs will use induction arguments. For such, it will be convenient to denote elements of \( L \) as \( \sum_\beta x(\beta) t^\beta \) for \( \beta \in J \) and \( x(\beta) \in K \), with this notation being analogous to that adopted for elements in \( F \). The statement \( \alpha = (\beta, i) \) for \( \alpha \in I \), \( \beta \in J \) and \( i \in \mathbb{Z} \) means that if \( \beta = (i_1, \ldots, i_{n-2}) \), then \( \alpha = (i_1, \ldots, i_{n-2}, i) \).

When necessary, \( I \) will be ordered with the inverse lexicographical order, that is: \( (i_1, \ldots, i_{n-1}) < (j_1, \ldots, j_{n-1}) \) if and only if for an index \( l \in \{ 1, \ldots, n-1 \} \) we have \( i_l < j_l \) and \( i_m = j_m \) for \( l < m \leq n-1 \).

By a net, we will refer to a set indexed by \( I \) or \( J \) (more generally, a net in a topological space is a subset indexed by a directed set, but we will only use this more general notion in Proposition 4.3). We will construct objects such as \( \mathcal{O} \)-submodules of and seminorms on \( F \) attached to a given net in \( \mathbb{Z} \cup \{ \pm \infty \} \). Instead of using notation \((n_\alpha)_{\alpha \in I}\), which is standard for sequences and was used thoroughly in [11], we will denote the elements of a net by \((n(\alpha))_{\alpha \in I}\), the net of coefficients \((x(\alpha))_{\alpha \in I} \subset K\) attached to an element \( x = \sum_\alpha x(\alpha) t^\alpha \in F \) being a first example. We will ease notation by letting, when given a net \((n(\alpha))_{\alpha \in I} \subset \mathbb{Z} \cup \{ \pm \infty \}\), \( n(\beta, i) := n((\beta, i)) \) for \( \beta \in J \), \( i \in \mathbb{Z} \).

4
2 Local convexity of higher topologies

The construction of the higher topology on $F$ is explained for example in [6 §4] and [5 §1]. It revolves around two basic constructions.

First suppose that a $L$ is a field on which a translation invariant and Hausdorff topology has been defined. Let $\{U_i\}_{i \in \mathbb{Z}}$ be a sequence of neighbourhoods of zero of $L$, with the property that there is an index $i_0 \in \mathbb{Z}$ such that $U_i = L$ for all $i \geq i_0$. The sets of the form

$$\sum_{i \in \mathbb{Z}} U_i t^i := \left\{ \sum_{i \gg -\infty} x_i t^i; \ x_i \in U_i \text{ for all } i \right\}$$

(1)

describe a basis of neighbourhoods of zero for a translation invariant Hausdorff topology on $L((t))$ [5 §1].

Second, suppose that $L$ is a complete discrete valuation field with $\text{char} L \neq \text{char} \mathbb{L}$, so that $L((t))$ is a complete discrete valuation field of mixed characteristic. Suppose that a translation invariant and Hausdorff topology has been defined on $L$. Let $\{V_i\}_{i \in \mathbb{Z}}$ be a sequence of neighbourhoods of zero of $L$ satisfying the following two conditions:

(i) There is $c \in \mathbb{Z}$ such that $p_c^i \subset V_i$ for every $i \in \mathbb{Z}$.

(ii) For every $l \in \mathbb{Z}$ there is $i_0 \in \mathbb{Z}$ such that for every $i \geq i_0$ we have $p_l^i \subset V_i$. This condition simply means that as $i \to \infty$ the neighbourhoods of zero $V_i$ become bigger and bigger. We will denote this condition by $V_i \to L$ as $i \to \infty$.

The sets of the form

$$\sum_{i \in \mathbb{Z}} V_i t^i := \left\{ \sum_{i \in \mathbb{Z}} x_i t^i \in L\{(t)\}; \ x_i \in V_i \text{ for all } i \right\}$$

(2)

constitute the basis of neighbourhoods of zero for a translation invariant and Hausdorff topology on $L\{(t)\}$ [5 §1].

The procedure for topologizing $F$ is an inductive application of the two constructions specified above. Namely, we endow $K$ with its $p$-adic topology, and for every $k \in \{1, \ldots, r\}$, we apply the second construction inductively on $E\{(t_k)\}$, with $E = K\{\{t_1\} \cdots \{t_{k-1}\}\}$. For $k \in \{r+1, \ldots, n-1\}$, we apply inductively the first construction on $E\{(t_k)\}$, with $E = K\{\{t_1\} \cdots \{t_r\} \{t_{r+1}\} \cdots \{t_{k-1}\}\}$.

The resulting topology on $F$ is called the higher topology.

If $r < n$ the higher topology on $F$ depends on the choice of a coefficient field, that is, a field inclusion $\overline{F} \subset F$ [5 §1.4]. This is due to the fact that, since in this case $\text{char} \overline{F} = 0$ and $\overline{F}$ is transcendental over $\mathbb{Q}$, there are infinitely many choices for such an embedding [4 II.5]. In our description of the higher topology, we are implicitly choosing an isomorphism $\overline{F} \cong K\{\{t_1\} \cdots \{t_r\} \{t_{r+1}\} \cdots \{t_{n-2}\}\}$. In the two-dimensional equal characteristic case, there is a unique coefficient field which factors the field embedding $K \hookrightarrow F$, namely the algebraic closure of $K$ in $F$ [1 §7.1]. Such a choice is not possible whenever $n \geq 3$.

**Proposition 2.1.** The higher topology on $F$ is locally convex. It may be described as follows. For any net $(n(\alpha))_{\alpha \in I} \subset \mathbb{Z} \cup \{-\infty\}$ subjected to the conditions:
(i) For any \( l \in \{ r + 1, \ldots, n - 1 \} \) and fixed indices \( i_{l+1}, \ldots, i_{n-1} \in \mathbb{Z} \), there is a \( k_0 \in \mathbb{Z} \) such that for every \( k \geq k_0 \) we have
\[
 n(\alpha) = -\infty \quad \text{for all } \alpha \in I(k, i_{l+1}, \ldots, i_{n-1}).
\]

(ii) For any \( l \in \{ 1, \ldots, r \} \) and fixed indices \( i_{l+1}, \ldots, i_{n-1} \), there is an integer \( c \in \mathbb{Z} \) such that
\[
 n(\alpha) \leq c \quad \text{for every } \alpha \in I(i_{l+1}, \ldots, i_{n-1}),
\]
and we have that
\[
 n(\alpha) \to -\infty, \quad \alpha \in I(k, i_{l+1}, \ldots, i_{n-1}), \quad \text{as } k \to \infty.
\]

Then, the open lattices of \( F \) are those of the form
\[
 \Lambda = \sum_\alpha p^{n(\alpha)} t^\alpha. \quad (3)
\]

**Remark 2.2.** Let us clarify what the second part in condition (ii) above stands for. The condition is that for any \( l \in \{ 1, \ldots, r \} \) and fixed indices \( i_{l+1}, \ldots, i_{n-1} \), given \( d \in \mathbb{Z} \) there is an integer \( k_0 \) such that for every \( k \geq k_0 \) and \( \alpha \in I(k, i_{l+1}, \ldots, i_{n-1}) \) we have \( n(\alpha) \leq d \).

**Proof.** We will prove the result by induction on \( n \). For \( n = 2 \), see [1, Propositions 3.1 and 3.7]. Suppose \( n > 2 \). Then write \( L = K\{\{t_1\}\} \cdots \{\{t_r\}\} \langle\langle(t_{r+1})\rangle\rangle \cdots \langle\langle(t_{n-2})\rangle\rangle \), with \( r \in \{0, n-2\} \). By induction hypothesis, the higher topology on \( L \) is locally convex and its open lattices are of the form
\[
 M = \sum_{\beta \in J} p^{n(\beta)} t^\beta, \quad (4)
\]
with \( (n(\beta))_{\beta \in J} \subset \mathbb{Z} \cup \{-\infty\} \) a net satisfying the conditions in the statement of the proposition.

Now we need to distinguish two cases. First, if \( r \leq n - 2 \), we must apply the construction in which neighbourhoods of zero are of the form \( (1) \), as \( F = L(\langle\langle t_{n-1} \rangle\rangle) \).

So we let
\[
 M_i = \sum_{\beta \in J} p^{n(\beta,i)} t^\beta, \quad i \in \mathbb{Z},
\]
with the property that there is an \( i_0 \in \mathbb{Z} \) such that for all \( i \geq i_0 \), \( M_i = L \). This last condition is equivalent to setting \( n(\beta,i) = -\infty \) for all \( \beta \in J \) and \( i \geq i_0 \). As the \( M_i \) describe a basis of neighbourhoods of zero for the higher topology on \( L \), the higher topology on \( F \) admits a basis of neighbourhoods of zero formed by sets of the form
\[
 \Lambda = \sum_{i \in \mathbb{Z}} M_i^i t^{i_{n-1}}.
\]

By induction hypothesis, the \( M_i \) are all \( \mathcal{O} \)-lattices, which is enough to show that \( \Lambda \) is an \( \mathcal{O} \)-lattice. So, in this case, we let \( \alpha = (\beta,i) \) so that a basis of neighbourhoods of zero for the higher topology is described by sets \( \Lambda = \sum_{\alpha \in I} p^{n(\alpha)} t^\alpha \). On top of the conditions which the indices \( n(\beta,i) \) satisfy by the induction hypothesis for \( \beta \in J \),
we must add the further condition that there is an integer \( i_0 \) such that for all \( i \geq i_0 \), \( n(\alpha) = -\infty \) for all \( \alpha \in I(i) \). This shows that our claim holds in this case.

The second case is the one in which \( r > n - 2 \) and we must apply the construction in which neighbourhoods of zero are given by sets of the form \( \{ t \} \), as \( F = L\{ t \} \).

So we set
\[
M_i = \sum_{\beta \in J} p^{n(\beta, i)} t^\beta, \quad i \in \mathbb{Z},
\]
subject to the properties:

(i) There is an integer \( c \) such that for every \( i \in \mathbb{Z} \), \( p^c \subset M_i \). By induction hypothesis, this means that \( n(\beta, i) \leq c \) for every \( \beta \in J \) and \( i \in \mathbb{Z} \).

(ii) \( M_i \rightarrow L \) as \( i \rightarrow \infty \). This is equivalent to \( n(\beta, i) \rightarrow -\infty \) for \( \beta \in J \) as \( i \rightarrow \infty \).

As \( M_i \) describe a basis of neighbourhoods of zero of the topology of \( L \), the sets of the form
\[
\Lambda = \sum_{i \in \mathbb{Z}} M_i t^n_{i-1}
\]
describe a basis of neighbourhoods of zero for the higher topology on \( F \). Since the \( M_i \) are \( \mathcal{O} \)-lattices, we get that \( \Lambda \) is an \( \mathcal{O} \)-lattice. Again, we let \( \alpha = (\beta, i) \), so that the \( \mathcal{O} \)-lattice \( \Lambda \) may be described as \( \Lambda = \sum_{\alpha \in I} p \cdot t^\alpha \cdot n(\alpha) \cdot t^\alpha \). On top of the conditions satisfied by the \( n(\alpha) \) which are inherited by induction, there are the two new conditions:

(i) There is an integer \( c \) such that \( n(\alpha) \geq c \) for all \( \alpha \in I \).

(ii) \( n(\alpha) \rightarrow -\infty \) for \( \alpha \in I(i) \), as \( i \rightarrow \infty \).

This shows that the result also holds in this case. \( \square \)

After showing that the higher topology on \( K \hookrightarrow F \) is locally convex, it is natural to describe it in terms of seminorms.

**Proposition 2.3.** The higher topology on \( F \) is the locally convex \( K \)-vector space topology defined by the seminorms of the form
\[
\| \cdot \| : F \rightarrow \mathbb{R}, \quad \sum_{\alpha} x(\alpha) t^\alpha \mapsto \sup_{\alpha} |x(\alpha)| q^{n(\alpha)}
\]
as \( (n(\alpha))_{\alpha \in I} \subset \mathbb{Z} \cup \{ -\infty \} \) varies over the nets described in the statement of Proposition 2.1.

**Proof.** It is necessary to show that the gauge seminorm associated to the open lattice \( \Lambda \) as in (3) is the one given by (5).

The gauge seminorm defined by \( \Lambda \) is by definition
\[
\| x \| = \inf_{x \in a \Lambda} |a|, \quad \text{for } x \in F.
\]

Let \( x = \sum_{\alpha} x(\alpha) t^\alpha \). We have that \( x \in a \Lambda \) if and only if \( x(\alpha) \in a p^{n(\alpha)} \) for every \( \alpha \in I \). That is, if and only if
\[
|x(\alpha)| q^{n(\alpha)} \leq |a| \quad \text{for all } \alpha \in I.
\]

The infimum of the values \( |a| \) for which (6) holds is the supremum of the values \( |x(\alpha)| q^{n(\alpha)} \) as \( \alpha \in I \). \( \square \)
Definition 2.4. The seminorms on $F$ defined in the previous proposition will be referred to as admissible seminorms.

An admissible seminorm $\| \cdot \|$ is attached to a net $(n(\alpha))_{\alpha \in I} \subset \mathbb{Z} \cup \{-\infty\}$. If we have chosen notation not to reflect this fact it is in pursuit of a lighter reading and understanding that the net $(n(\alpha))_{\alpha \in I}$, when needed, will be clear from the context.

3 First properties

We summarise some properties which were known already for higher topologies, or which are deduced immediately from the fact that these topologies are locally convex. We also state some properties which do not hold in general because they are known not to hold already for $n=2$.

The field $F$, equipped with a higher topology, is a locally convex $K$-vector space, as shown in Proposition 2.1. As such, it is a topological vector space. It is a previously known fact that higher topologies are Hausdorff. In order to show that this property holds in our setting it is enough to show that, given $x \in F \times$, there is an admissible seminorm $\| \cdot \|$ for which $\|x\| \neq 0$. If the $\alpha$-coefficient of $x$ is nonzero, any admissible seminorm for which $n(\alpha) > -\infty$ suffices.

Moreover, the reduction map $O_F \to F$ is open when $O_F$ is given the subspace topology and $F$ a higher topology compatible with the choice of coefficient field if $\text{char } F \neq \text{char } F$ [2, Proposition 3.6.(v)].

Multiplication $F \to F$ by a fixed nonzero element induces a homeomorphism of $F$, but multiplication $\mu : F \times F \to F$ is not continuous [3, §1.3.2]; the immediate reason why this is the case being that for any open lattice $\Lambda$ we have $\mu(\Lambda, \Lambda) = F$.

Higher topologies are not first-countable and therefore not metrizable [3, §1.3.2]. Moreover, in general, $F$ is not bornological, barrelled, reflexive nor nuclear and its rings of integers are not c-compact nor compactoid, the first counterexample being the field $K\{\{t\}\}$ [1].

Remark 3.1. Power series in the system of parameters $t_1, \ldots, t_{n-1}$ are convergent in the higher topology. If $x = \sum_{\alpha \in I} x(\alpha)t^\alpha \in F$, we define a net $(s(\alpha))_{\alpha \in I} \subset F$ by taking $s(\alpha) = \sum_{\alpha' \leq \alpha} x(\alpha')t^{\alpha'}$. If $\| \cdot \|$ is an admissible seminorm on $F$, then as $\alpha \in I$ grows, the value $\|x - s(\alpha)\|$ becomes arbitrarily small.

4 Bounded $O$-submodules

Let us study the bounded $O$-submodules of $F$. We start by describing a basis for the Von-Neumann bornology on $K \hookrightarrow F$.

Proposition 4.1. Let $(k(\alpha))_{\alpha \in I} \subset \mathbb{Z} \cup \{\infty\}$ be a net subjected to the conditions:

(i) For every $l \in \{r+1, \ldots, n-1\}$ and indices $i_{l+1}, \ldots, i_{n-1} \in \mathbb{Z}$ there is an index $j_0 \in \mathbb{Z}$ such that for every $j < j_0$ we have $k(\alpha) = \infty$ for all $\alpha \in I(j, i_{l+1}, \ldots, i_{n-1})$.

(ii) For every $l \in \{1, \ldots, r\}$ and $i_{l+1}, \ldots, i_{n-1} \in \mathbb{Z}$, there is an integer $d$ such that $k(\alpha) \geq d$ for all $\alpha \in I(i_{l+1}, \ldots, i_{n-1})$. 

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The Von-Neumann bornology of $F$ admits as a basis the collection of $O$-submodules
\[ B = \sum_{\alpha \in I} p^{k(\alpha)} t^\alpha \] (7)

as $(k(\alpha))_{\alpha \in I}$ varies over the nets specified above.

Proof. First, let us show that the sets $B$ are bounded. As we will use induction on $n$, the case $n = 2$ is thoroughly explained in [1, Propositions 4.2 and 4.4].

Let $\| \cdot \|$ be an admissible seminorm attached to the net $(n(\alpha))_{\alpha \in I} \subset \mathbb{Z} \cup \{-\infty\}$.

Let $x = \sum_\alpha x(\alpha) t^\alpha$. We have
\[ \|x\| = \sup_\alpha |x(\alpha)| q^{n(\alpha)} \leq \sup_\alpha q^{n(\alpha)-k(\alpha)} \]

and therefore it is enough to prove that the set \{$(n(\alpha) - k(\alpha))_{\alpha \in I}$ \} is bounded above.

We distinguish two cases. Suppose first that $r \leq n - 2$. Then $F = L((t_{n-1}))$. On one hand, there is an index $j_0 \in \mathbb{Z}$ such that $k(\alpha) = \infty$ for every $\alpha \in I(j)$, $j < j_0$. On the other hand, there is an index $j_1 \in \mathbb{Z}$ such that $n(\alpha) = -\infty$ for every $\alpha \in I(j)$, $j > j_1$. It is therefore enough to show that each of the finitely many sets
\[ N(j) = \{n(\alpha) - k(\alpha); \alpha \in I(j)\}, \quad j_0 \leq j \leq j_1 \]

are bounded above. But for each $j \in \mathbb{Z}$, the net $(n(\alpha))_{\alpha \in I(j)}$ defines an admissible seminorm and $\sum_{\beta \in J} p^{k(\beta,j)} t^\beta \subset L$ is a bounded $O$-submodule; this implies the boundedness of $N(j)$.

The case in which $r = n - 1$, and therefore $F = L(\{t_{n-1}\})$, is simpler: we have that all the $n(\alpha)$ are bounded above and all the $k(\alpha)$ are bounded below; therefore the differences $n(\alpha) - k(\alpha)$ are bounded above.

Second, we have to show that any bounded subset of $F$ is contained in an $O$-submodule of $F$ of the form (7).

The elements of any bounded subset $D \subset F$ cannot have $t_{n-1}$-expansions with arbitrarily large coefficients in $L$ in a fixed degree: otherwise, suppose that this is not the case and that for $j \in \mathbb{Z}$ the $j$-th coefficients in the $t_{n-1}$-expansions of the elements in $D$ may be arbitrarily large. By choosing any admissible seminorm with $n(\alpha) > -\infty$ for some $\alpha \in I(j)$ we easily obtain that $D$ is not bounded.

Hence, $D$ is contained in an $O$-submodule of the form $C = \sum_{i \in \mathbb{Z}} C_i t_{n-1}$ with $C_i \subset L$ bounded $O$-submodules. By induction hypothesis, let us write
\[ C_i = \sum_{\beta \in J} p^{k(\beta,i)} t^\beta, \quad i \in \mathbb{Z}, \]

with $k(\beta,i) \in \mathbb{Z} \cup \{\infty\}$ satisfying the conditions exposed in the statement of the proposition.

By letting $\alpha = (\beta,i) \in I$, we may write $C = \sum_{\alpha \in I} p^{k(\alpha)} t^\alpha$ and we only have to show that the indices $k(\alpha)$ might be taken to satisfy the conditions exposed in the proposition. Suppose that this is not the case, and let us consider separate cases again.

First, if $r \leq n - 2$ and $F = L(\{t_{n-1}\})$, the indices $k(\alpha)$ may be taken to satisfy condition (ii) by induction hypothesis. Condition (i) is also satisfied by induction.
hypothesis for every \( l \in \{ r + 1, \ldots, n - 2 \} \). So we only have to show that if the \( k(\alpha) \) may not be taken to satisfy condition (i) in the case \( l = n - 1 \), then \( D \) cannot be bounded.

If the condition does not hold, then there is a decreasing sequence \((j_h)_{h \geq 0} \subset \mathbb{Z}_{< 0} \), an index \( \alpha_h \in I(j_h) \) and an element \( \xi_h \in D \) such that its \( \alpha_h \)-coefficient, which we label \( x(\alpha_h) \), is nonzero. Let

\[
n(\alpha) = \begin{cases} 
-j_h + v(x(\alpha_h)), & \text{if } \alpha = \alpha_h \in I(j_h), \\
-\infty & \text{otherwise}.
\end{cases}
\]

The net \( (n(\alpha))_{\alpha \in I} \) defines an admissible seminorm \( \| \cdot \| \). Since \( \| \xi_h \| \geq |x(\alpha_h)|q^{n(\alpha_h)} = q^{-j_h} \), \( D \) cannot be bounded.

Second, suppose that \( r = n - 1 \) and \( F = L\{t_{n-1}\} \). By induction hypothesis, we know that condition (ii) holds for every \( l \in \{1, \ldots, n - 2\} \), so we suppose that it does not hold in the case \( l = n - 1 \). In such case, at least one of the following must happen:

1. There is a decreasing sequence \((j_h)_{h \geq 0} \subset \mathbb{Z}_{< 0} \), an index \( \alpha_h \in I(j_h) \) and \( \xi_h \in D \) such that, if \( x(\alpha_h) \) denotes the \( \alpha_h \)-coefficient of \( \xi_h \), we have \( |x(\alpha_h)| \to \infty \) as \( h \to \infty \).

2. There is an increasing sequence \((j_h)_{h \geq 0} \subset \mathbb{Z}_{> 0} \), an index \( \alpha_h \in I(j_h) \) and \( \xi_h \in D \) such that, if \( x(\alpha_h) \) denotes the \( \alpha_h \)-coefficient of \( \xi_h \), we have \( |x(\alpha_h)| \to \infty \) as \( h \to \infty \).

Suppose that condition 1 holds. In this case, let

\[
n(\alpha) = \begin{cases} 
0, & \text{if } \alpha = \alpha_h \text{ for some } h \geq 0, \\
-\infty & \text{otherwise}.
\end{cases}
\]

The net \( (n(\alpha))_{\alpha \in I} \) defines an admissible seminorm \( \| \cdot \| \). Now, for \( h \geq 0 \), we have \( \| \xi_h \| \geq q^{-v(x(\alpha_h))} \) and hence \( D \) cannot be bounded.

Finally, if condition 1 does not hold, then condition 2 must happen. In such case, let

\[
n(\alpha) = \begin{cases} 
(v(x(\alpha_h)) - 1)/2, & \text{if } \alpha = \alpha_h \text{ for some } h \geq 0 \text{ and } v(x(\alpha_h)) \text{ odd}, \\
v(x(\alpha_h))/2, & \text{if } \alpha = \alpha_h \text{ for some } h \geq 0 \text{ and } v(x(\alpha_h)) \text{ even}, \\
-\infty & \text{otherwise}.
\end{cases}
\]

The net \( (n(\alpha))_{\alpha \in I} \) defines an admissible seminorm. Moreover, we have \( n(\alpha_h) - v(x(\alpha_h)) \to \infty \) as \( h \to \infty \). We have that \( \| \xi_h \| \geq |x(\alpha_h)|q^{n(\alpha_h)} = q^{n(\alpha_h) - v(x(\alpha_h))} \) and therefore \( D \) cannot be bounded. \( \Box \)

**Definition 4.2.** We will say that an \( \mathcal{O} \)-submodule of the form \((7)\) is a **basic bounded submodule** of \( F \).

The following result is necessary in order to compare compactoids and \( \varepsilon \)-compacts in the sequel.
Proposition 4.3. The submodules $B$ in Proposition 4.1 are complete.

Proof. Let $B = \sum_{\alpha} p^{k(\alpha)}t^\alpha$ with $(k(\alpha))_{\alpha \in I} \subset \mathbb{Z} \cup \{\infty\}$ satisfying conditions (i) and (ii) in the statement of Proposition 4.1.

Let $H$ be a directed set and $(x(h))_{h \in H} \subset B$ a Cauchy net. Let us denote, for each $h \in H$, $x(h) = \sum_{\alpha} x(h)(\alpha)t^\alpha$ with $x(h)(\alpha) \in p^{k(\alpha)}$ for $\alpha \in I$.

We have that, for a fixed $\alpha \in I$, $(x(h)(\alpha))_{h \in H} \subset p^{k(\alpha)}$ is a Cauchy net. As $p^{k(\alpha)}$ is complete, the net converges to an element $x(\alpha) \in p^{k(\alpha)}$.

If the power series $\sum_{\alpha} x(\alpha)t^\alpha$ defines an element $x \in F$, then $x \in B$ and $x(h) \rightarrow x$. This is easy to check by induction on $n$ (the case $n = 2$ may be found in [1, §5]).

As we have explained, multiplication $\mu : F \times F \rightarrow F$ is not a continuous map. However, we may shown that it is bounded.

Proposition 4.4. Multiplication $\mu : F \times F \rightarrow F$ is a bounded map.

Proof. The argument for the proof is by induction on $n$. The case $n = 2$ is dealt with in [1, Proposition 4.8], and the same argument applies when looking at $F = L\{t_{n-1}\}$ or $L(\{t_{n-1}\})$.

5 Compactoid $O$-submodules

The result below outlines which basic bounded submodules of $F$ are compactoid, and thus describes a basis for the bornology on $F$ generated by compactoid $O$-submodules.

Proposition 5.1. Let $(k(\alpha))_{\alpha \in I} \subset \mathbb{Z} \cup \{\infty\}$ be a net satisfying the conditions:

(i) For every $l \in \{r + 1, \ldots, n - 1\}$ and indices $i_{l+1}, \ldots, i_{n-1} \in \mathbb{Z}$ there is an index $j_0 \in \mathbb{Z}$ such that for every $j < j_0$ we have $k(\alpha) = \infty$ for all $\alpha \in I(j, i_{l+1}, \ldots, i_{n-1})$.

(ii) For every $l \in \{1, \ldots, r\}$ and $i_{l+1}, \ldots, i_{n-1} \in \mathbb{Z}$, there is an integer $d \in \mathbb{Z}$ such that $k(\alpha) \geq d$ for all $\alpha \in I(i_{l+1}, \ldots, i_{n-1})$ and we have that $k(\alpha) \rightarrow \infty$ for $\alpha \in I(j, i_{l+1}, \ldots, i_{n-1})$, as $j \rightarrow -\infty$.

Then the $O$-submodule of $F$ given by

$$B = \sum_{\alpha} p^{k(\alpha)}t^\alpha$$

(8)

is compactoid.

The $O$-submodules of the form (8) are the only compactoid submodules amongst basic bounded submodules, and define a basis for the bornology on $F$ defined by compactoid submodules.

In the proof to this proposition we shall need to consider the projection maps to the coefficients of the $\alpha$-expansions of elements of $F$. For every $\alpha_0 \in I$, consider the continuous linear form:

$$\pi_{\alpha_0} : F \rightarrow K, \quad \sum_{\alpha} x(\alpha)t^\alpha \mapsto x(\alpha_0).$$
Proof. The result holds for \( n = 2 \) as shown in [1 §5].

First, let us show that the submodule \( B \) as in (3) is compactoid. Let \( \Lambda = \sum_{\alpha} p^{n(\alpha)} t^{\alpha} \) be an open lattice. We will show that there exist elements \( x_1, \ldots, x_m \in F \) such that \( B \subset \Lambda + \mathcal{O} x_1 + \cdots + \mathcal{O} x_m \).

Regardless of the value of \( r \in \{0, \ldots, n - 1\} \), there are two indices \( j_0, j_1 \in \mathbb{Z} \) such that

\[
k(\alpha) \geq n(\alpha), \quad \text{for all } \alpha \in I(j) \text{ with } j < j_0 \text{ or } j > j_1.
\]

(9)

if \( j_0 > j_1 \) then \( B \subset \Lambda \) and we are done. Henceforth, we assume \( j_0 \leq j_1 \).

Let us examine the situation for \( j_0 \leq j \leq j_1 \). For a fixed such \( j \), let \( \alpha = (\beta, j) \) with \( \beta \in J \). By induction hypothesis, the \( \mathcal{O} \)-submodule \( \sum_{\beta \in J} p^{k(\beta,j)} t^{\beta} \subset L \) is compactoid. Similarly, for a fixed such \( j \), \( \sum_{\beta} p^{n(\beta,j)} t^{\beta} \) is an open lattice in \( L \). Therefore, there exist a finite number of elements \( y_{j,1}, \ldots, y_{j,m_j} \in L \) for which we have, for \( j_0 \leq j \leq j_1 \),

\[
\sum_{\beta} p^{k(\beta,j)} t^{\beta} \subset \sum_{\beta} p^{n(\beta,j)} t^{\beta} + \mathcal{O} y_{j,1} + \cdots + \mathcal{O} y_{j,m_j}.
\]

(10)

Now, this implies that

\[
\sum_{j=j_0}^{j_1} \left( \sum_{\beta \in J} p^{k(\beta,j)} t^{\beta} \right) t_{n-1}^j \subset \sum_{j=j_0}^{j_1} \left( \sum_{\beta \in J} p^{n(\beta,j)} t^{\beta} + \sum_{s=1}^{m_j} \mathcal{O} y_{j,s} t_{n-1}^j \right).
\]

We rewrite this last equation as:

\[
\sum_{\alpha \in I(j)} p^{k(\alpha)} t^{\alpha} \subset \sum_{\alpha \in I(j)} p^{n(\alpha)} t^{\alpha} + \sum_{j=j_0}^{j_1} \sum_{s=1}^{m_j} \mathcal{O} y_{j,s} t_{n-1}^j \cdot
\]

(10)

The fact that

\[
B \subset \Lambda + \left( \sum_{j=j_0}^{j_1} \sum_{s=1}^{m_j} \mathcal{O} y_{j,s} t_{n-1}^j \right)
\]

follows from (9) and (10).

Second, let us show how any compactoid \( \mathcal{O} \)-submodule of \( F \) is contained in one of the form (3). Since compactoid \( \mathcal{O} \)-submodules are bounded, it is enough to show that any basic bounded submodule \( C = \sum_{\alpha} p^{k(\alpha)} t^{\alpha} \) is compactoid if and only if the indices \( k(\alpha) \) satisfy conditions (i) and (ii) in the statement of the Proposition. We proceed by induction, the result holds for \( n = 2 \) as mentioned above.

Now, suppose \( C \) is compactoid. Then, for every \( j \in \mathbb{Z} \), the \( \mathcal{O} \)-submodule of \( L \) given by

\[
C_j = \sum_{\beta} p^{k(\beta,j)} t^{\beta}
\]

is compactoid.
Next, we distinguish cases. Suppose that \( r \leq n - 2 \), so that \( F = L((t_{n-1})) \). In such case, by induction hypothesis, we only need to check that the indices \( k(\alpha) \) satisfy condition (i) for \( l = n - 1 \). But if this condition does not hold, then from the proof of Proposition 6.2, we deduce that \( C \) cannot be compactoid, as it is not bounded. So there is nothing more to say in this case.

Finally, suppose that \( r = n - 1 \), so that \( F = L(\{t_{n-1}\}) \). By hypothesis induction, the indices \( k(\alpha) \) satisfy condition (ii) in the statement of the Proposition for \( 1 \leq l \leq n - 2 \). If condition (ii) for \( l = n - 1 \) does not hold, then there is a decreasing sequence \( (j_h)_{h \geq 0} \subset \mathbb{Z}_{<0} \) and an index \( \alpha_h \in I(j_h) \) for each \( h \geq 0 \) such that the sequence \( (k(\alpha_h))_{h \geq 0} \) is bounded above. Let \( M \in \mathbb{Z} \) be such that \( k(\alpha_h) < M \) for every \( h \geq 0 \). Let

\[
n(\alpha) = \begin{cases} M, & \text{if } \alpha = \alpha_h \text{ for some } h \geq 0, \\ -\infty & \text{otherwise}. \end{cases}
\]

The net \( (n(\alpha))_{\alpha \in I} \) defines an open lattice \( \Lambda = \sum_{\alpha} p^{n(\alpha)}t^\alpha \). Now, suppose that \( x_1, \ldots, x_m \in F \) satisfy that \( C \subset \Lambda + O x_1 + \cdots + O x_m \). Let us write, for \( 1 \leq l \leq m \),

\[
x_l = \sum_{\alpha} x_l(\alpha)t^\alpha.
\]

Since \( x_l(\alpha) \to 0 \) for \( \alpha \in I(j) \) as \( j \to -\infty \), we have that there is an index \( k \in \mathbb{Z} \) such that for every \( j \leq k \), we have \( v(x_l(\alpha)) > M \) for every \( \alpha \in I(j) \) and \( 1 \leq l \leq m \). Fix an \( h \geq 0 \) such that \( j_h \leq k \). Now, for such an \( h \), we have

\[
\pi_{\alpha_h}(C) \subset \pi_{\alpha_h}(\Lambda + O x_1 + \cdots + O x_m),
\]

which implies

\[
p^{k(\alpha_h)} \subset p^M + p^{v(x_1(\alpha_h))} + \cdots + p^{v(x_m(\alpha_h))} = p^M.
\]

This last inclusion implies that \( M \leq k(\alpha_h) \), a contradiction. Hence, we must have \( k(\alpha) \to \infty \) for \( \alpha \in I(j) \) as \( j \to -\infty \).

**Definition 5.2.** We will refer to the \( O \)-submodules of \( F \) of the form (8) as basic compactoid submodules of \( F \).

**Corollary 5.3.** The basic compactoid \( O \)-submodules of \( F \) are c-compact.

**Proof.** An \( O \)-submodule of a locally convex \( K \)-vector space is c-compact and bounded if and only if it is compactoid and complete [1, Prop. 12.7]. So the result follows from the fact that these \( O \)-submodules are bounded, compactoid and complete. \( \square \)

## 6 Duality

Let us discuss some issues regarding the dual space of \( F \). We showed in [1, Theorem 6.2] how in the two-dimensional case \( F \) is isomorphic in the category of locally convex \( K \)-vector spaces to \( F'_c \), its continuous dual space topologized using the c-topology, that is: the topology of uniform convergence on compactoid submodules.

The c-topology is defined on the continuous dual of any \( n \)-dimensional \( F \) by the collection of seminorms

\[
| \cdot |_B : F' \to \mathbb{R}, \quad l \mapsto \sup_{x \in B} |l(x)|,
\]

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for any basic compactoid submodule $B \subset F$.

Our first goal in this section is to construct an isomorphism of locally convex vector spaces $F \cong F'_c$.

We have already come across some continuous nonzero linear forms on $F$ in the previous section, we recall that these were the projections, for every $\alpha_0 \in I$:

$$\pi_{\alpha_0} : F \to K, \quad \sum_{\alpha \in I} x(\alpha)t^\alpha \mapsto x(\alpha_0).$$

In particular, denote by $\pi_0$ the continuous linear form on $F$ constructed as in the previous example for $\alpha_0 = (0, \ldots, 0) \in I$.

We relate $F$ and its continuous dual space. Define

$$\gamma : F \to F', \quad x \mapsto \pi_x,$$

with

$$\pi_x : F \to K, \quad y \mapsto \pi_0(xy).$$

Lemma 6.1. If $x = \sum_{\alpha \in I} x(\alpha)t^\alpha$ and $y = \sum_{\alpha \in I} y(\alpha)t^\alpha$ are elements in $F$, we have that

$$\pi_x(y) = \sum_{\alpha \in I} x(-\alpha)y(\alpha) = \sum_{i_{n-1} \in \mathbb{Z}} \cdots \sum_{i_1 \in \mathbb{Z}} x(-i_{1}, \ldots, -i_{n-1})y(i_1, \ldots, i_{n-1}).$$

Proof. The result becomes clear once notation is unwound and the 0-th coefficient of $xy$ is taken for each parameter $t_i$ separately, for $l$ descending from $n - 1$ to 1.

Lemma 6.2. Let $w \in F'$. Define, for each $\alpha \in I$, $x(\alpha) = w(t^{-\alpha}) \in K$. Then, the expression $\sum_{\alpha \in I} x(\alpha)t^\alpha$ defines an element in $F$.

Proof. The result may be shown by induction; the argument is the same as in [1, Lemma 6.3].

Theorem 6.3. The map $\gamma : F \to F'_c$ is an isomorphism of locally convex vector spaces.

Proof. The map $\gamma$ is linear and injective. Surjectivity follows from Lemma 6.2 as if $w \in F'$, we apply the lemma to obtain $x = \sum_{\alpha \in I} x(\alpha)t^\alpha \in F$. Then, for any $y = \sum_{\alpha \in I} y(\alpha)t^\alpha \in F$, we have

$$w(y) = w \left( \sum_{\alpha} y(\alpha)t^\alpha \right) = \sum_{\alpha} y(\alpha)w(t^\alpha) = \sum_{\alpha} y(\alpha)x(-\alpha) = \pi_x(y).$$

The second equality follows from Remark 6.1.

In order to show bicontinuity of $\gamma$, the argument is very similar to the one given in the proof of [1, Theorem 6.2]; given a basic compactoid $O$-submodule $B = \sum_{\alpha \in I} p^{k(\alpha)}t^\alpha$ of $F$ as in Proposition 5.1, the net $(-k(-\alpha))_{\alpha \in I}$ defines an admissible seminorm $\| \cdot \|$ on $F$. We have that, for $x \in F$,

$$\|x\| \leq q^n$$

if and only if $|\pi_x|_B \leq q^n$, which concludes the proof.
If \( A \subset F \) is an \( \mathcal{O} \)-submodule, we denote \( A^\gamma = \gamma^{-1}(A^p) \subset F \), with \( A^p \subset F' \) being the pseudo-polar of \( A \).

**Proposition 6.4.** Let \( A = \sum_{\alpha \in I} p^{k(\alpha)} t^\alpha \subset F \) be an \( \mathcal{O} \)-submodule, with \( k(\alpha) \in \mathbb{Z} \cup \{ \pm \infty \} \). We have that \( A^\gamma = \sum_{\alpha \in I} p^{1-k(-\alpha)} t^\alpha \).

**Proof.** The argument is the same as the one exposed in the proof of [1, Proposition 6.7].

The isomorphism \( \gamma \) is not unique as, for example, choosing \( \pi_{-\alpha} \) for any \( \alpha \in I \) instead of \( \pi_0 \) in the definition of \( \gamma \) would have given a different isomorphism. Thus, the shape of \( A^\gamma \) depends ultimately on our choice of \( \gamma \).

However, there are certain facts which are general for pseudo-polars of \( \mathcal{O} \)-submodules in any locally convex \( K \)-vector space. As such, we recall that taking the pseudo-polar exchanges open lattices and compactoid \( \mathcal{O} \)-submodules, and that the pseudo-bipolar of an \( \mathcal{O} \)-submodule is equal to its closure.

These facts are highlighted in the previous Proposition for the \( \mathcal{O} \)-submodules of the form \( \sum_{\alpha \in I} p^{k(\alpha)} t^\alpha \), which are closed. The facts that for an open lattice \( \Lambda \) we have that \( \Lambda^\gamma \) is compactoid and that for a basic compactoid set \( B \) we have that \( B^\gamma \) is an open lattice are evident by checking that for the nets \((n(\alpha))_{\alpha \in I}\) and \((1-n(-\alpha))_{\alpha \in I}\), one of them satisfies conditions (i) and (ii) in Proposition 2.1 if and only if the other one satisfies conditions (i) and (ii) in Proposition 5.1.

7 The general case

In this section, let \( K \hookrightarrow F \) be a general \( n \)-dimensional local field over \( K \). By structure theory, there is an \( r \in \{0, \ldots, n\} \) such that \( F \) is a finite extension of \( F_0 := K\{t_1\} \cdots \{t_r\}\{t_{r+1}\} \cdots \{t_{n-1}\} \) as explained in §1. Denote the degree of such extension by \( e \). The higher topology on \( F \) may be defined as the product topology on \( F \cong (F_0)^e \) [3, Remark after 1.3.2].

Since the product topology on a product of locally convex vector spaces is again locally convex, a higher topology on \( F \) is locally convex. Open lattices (resp. continuous seminorms) on \( F \) may be described using Proposition 2.1 (resp. Proposition 2.3) and [1, Proposition 1.1].

Finally, from Theorem 6.3 we recover the existence of an isomorphism \( F \cong F' \) via the chain

\[
F' \cong (F_0')' \cong (F_0')' \cong (F_0')' \cong F_0' \cong F.
\]

Explicit nonzero continuous linear forms on \( F \) may be obtained by composing the forms \( \pi_\alpha : F_0 \rightarrow K \) for \( \alpha \in I \) with \( \text{Tr} F | F_0 \).

8 Other types of higher local fields

Let us center our attention, for the sake of completeness, on the higher local fields which we have not treated in the previous.
First, suppose that char $F = p$. In such case, as explained in §1, there is a finite field $\mathbb{F}_q$ and elements $t_1, \ldots, t_n \in F$ such that

$$F \cong \mathbb{F}_q((t_1)) \cdot \cdots \cdot (t_n).$$

The field $\mathbb{F}_q((t_1))$ is a local field and the results in this work may be applied to $F$ if we let $K = \mathbb{F}_q((t_1))$. However, after §9, we are only stating that the higher topology on $F$ is a linear topology when we regard $F$ as a vector space over $\mathbb{F}_q$. The choice between linear-topological structures over $\mathbb{F}_q$ or locally convex structures over $\mathbb{F}_q((t_1))$ is merely a matter of language in our case.

Now let $K = \mathbb{R}$ or $\mathbb{C}$ and denote the usual absolute value by $| \cdot |$. The theory of higher local fields is also developed by looking at complete discrete valuation fields $F$ that have an $n$-dimensional structure on them and such that $F_1 = K$. For these, there are $t_1, \ldots, t_{n-1} \in F$ for which

$$F \cong K((t_1)) \cdot \cdots \cdot (t_{n-1}).$$

As hinted at in §8, we can apply the archimedean theory of locally convex spaces to study these fields.

The open disks

$$D_\rho = \{a \in K; |a| < \rho\}, \quad \rho \in \mathbb{Q}_{>0} \cup \{\infty\}$$

supply a basis of convex sets for the euclidean topology on $K$.

The higher topology on $F$ is constructed by iterating the construction in §8.

Proposition 8.1. Let $I = \mathbb{Z}^{n-1}$ and $(\rho(\alpha))_{\alpha \in I} \subset \mathbb{Q}_{>0} \cup \{\infty\}$ be a net restricted to the condition:

For any $l \in \{1, \ldots, n-1\}$ and fixed indices $i_{l+1}, \ldots, i_{n-1} \in \mathbb{Z}$ there is a $k_0 \in \mathbb{Z}$ such that for every $k \geq k_0$ we have $\rho(\alpha) = \infty$ for all $\alpha \in I(k, i_{l+1}, \ldots, i_{n-1})$.

The higher topology on $F$ is locally convex and it is defined by the seminorms of the form

$$\| \cdot \| : F \to \mathbb{R}, \quad \sum_{\alpha \in I} x(\alpha) t^\alpha \mapsto \sup_{\alpha \in I} \frac{|x(\alpha)|}{\rho(\alpha)},$$

with the convention that $a/\infty = 0$ for any $a \in \mathbb{R}_{\geq 0}$.

Proof. The result follows from [1, Proposition 8.3] and straightforward adaptation of the arguments used in Proposition 2.1 and Corollary 2.3.

9 Future work

Let us start this discussion by saying that §10 contains a list of topics worth studying, and that many of them are closely related to the topics dealt with in this note.

Among the directions outlined there, there is one which particularly has a direct impact on the study of functional analytic properties of higher local fields of arbitrary dimension.

Structure and topology on higher local fields may be studied successfully as an iteration of applications of inverse limits in the form of completions and direct limits.
in the form of localizations. In order to describe functional analytic structures that hold in any dimension and regardless of the characteristic type of \( F \), it seems that two initial ingredients are necessary: a theory of locally convex \( \mathcal{O} \)-modules and a study of which functional analytic properties of these modules are preserved after taking direct and inverse limits.

On a different direction, the dependence of higher topologies on choices of coefficient fields as soon as \( \text{char } F = 0 \) is a well-known handicap of the theory. For this reason, showing a class of subsets of \( F \) with an interesting topological or functional analytic property and which would remain stable under change of coefficient field would be extremely important.

Similarly, we have not dealt with the different rings of integers of a higher local field in this note, on purpose: although they are \( \mathcal{O} \)-submodules which are very relevant for arithmetic purposes, as highlighted already by comparing \( K((t)) \) and \( K\{\{t\}\} \) in \([1]\), the functional analytic properties of such rings change drastically according to the characteristic of the residue field. It would also be interesting to establish whether there is a relevant topological or analytic property which highlights the relevance of these arithmetically interesting \( \mathcal{O} \)-submodules.

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