About Chow groups of certain hyperkähler varieties with non–symplectic automorphisms

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Abstract Let $X$ be a hyperkähler variety, and let $G$ be a group of finite order non–symplectic automorphisms of $X$. Beauville’s conjectural splitting property predicts that each Chow group of $X$ should split in a finite number of pieces. The Bloch–Beilinson conjectures predict how $G$ should act on these pieces of the Chow groups: certain pieces should be invariant under $G$, while certain other pieces should not contain any non–trivial $G$–invariant cycle. We can prove this for two pieces of the Chow groups when $X$ is the Hilbert scheme of a $K3$ surface and $G$ consists of natural automorphisms. This has consequences for the Chow ring of the quotient $X/G$.

Keywords Algebraic cycles · Chow groups · motives · hyperkähler varieties · non–symplectic automorphisms · $K3$ surfaces · Calabi–Yau varieties · Bloch–Beilinson conjectures · (weak) splitting property · multiplicative Chow–Künneth decomposition

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1 Introduction

Let $X$ be a hyperkähler variety of dimension $2m$ (i.e., a projective irreducible holomorphic symplectic manifold, cf. [4], [5]). Let $G \subset \text{Aut}(X)$ be a finite cyclic group of order $k$ consisting of non–symplectic automorphisms. We will be interested in the action of $G$ on the Chow groups $A^*(X)$. (Here, $A^i(X) := CH^i(X)_{\mathbb{Q}}$ denotes the Chow group of codimension $i$ algebraic cycles modulo rational equivalence with $\mathbb{Q}$–coefficients.) Let us suppose $X$ has a multiplicative Chow–Künneth decomposition, in the sense of [36]. This implies the Chow ring of $X$ is a bigraded ring $A^*(X)$, where each Chow group splits as

$$A^i(X) = \bigoplus_j A^i_{(j)}(X),$$

and the piece $A^i_{(j)}(X)$ is expected to be isomorphic to the graded $Gr_F^j A^i(X)$ for the conjectural Bloch–Beilinson filtration $F^*$ on Chow groups. (Conjecturally, all hyperkähler varieties have a multiplicative Chow–Künneth decomposition; this is related to Beauville’s conjectural splitting property [7]. The existence of a multiplicative Chow–Künneth decomposition has been established for Hilbert schemes of $K3$ surfaces [36], [41], and for generalized Kummer varieties [16].)

Since $H^{2k,0}(X) = H^{2,0}(X)^{\otimes k}$, the group $G$ acts as the identity on $H^{2k,0}(X)$. For $i < 2k$, we have that $\sum_{g \in G} g^*$ acts as 0 on $H^{i,0}(X)$. The Bloch–Beilinson conjectures [20], combined with the expected isomorphism $A^i_{(j)}(X) \cong Gr_F^j A^i(X)$, thus imply the following conjecture:

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Conjecture 1. Let $X$ be a hyperkähler variety of dimension $2m$, and let $G \subset \text{Aut}(X)$ be a finite cyclic group of order $k$ of non–symplectic automorphisms. Then

$$A^j_{(i)}(X) \cap A^2(X)^G = \begin{cases} 0 & \text{if } j < 2k ; \\ A^j_{(i)}(X) & \text{if } j = 2k ; \end{cases}$$

$$A^i_{(j)}(X) \cap A^i(X)^G = \begin{cases} 0 & \text{if } i < 2k ; \\ A^i_{(j)}(X) & \text{if } i = 2k ; \end{cases}$$

(Here $A^i(X)^G \subset A^i(X)$ denotes the $G$–invariant part of the Chow group $A^i(X)$.)

The main result in this note is a partial verification of conjecture 1 for a certain class of hyperkähler varieties and a certain class of automorphisms.

**Theorem (=theorem 25).** Let $S$ be a projective $K3$ surface, and let $X = \tilde{S}^{(m)}$ be the Hilbert scheme of length $m$ subschemes. Let $G \subset \text{Aut}(X)$ be a subgroup of order $k$ of natural non–symplectic automorphisms. Then

$$A^i_{(2)}(X) \cap A^i(X)^G = 0 \quad \text{for } i \in \{2, 2m\}.$$

A natural automorphism of $X$ is an automorphism induced by an automorphism of $S$. Theorem 25 applies to Hilbert schemes of any $K3$ surface $S$ having a finite order non–symplectic automorphism. Such $K3$ surfaces have been intensively studied, and there are lots of examples known [11], [12], [22], [24], [27], [33], [38], [11], [2], [3], [18]. It would be interesting to prove theorem 25 also for non–symplectic automorphisms that are non–natural; this seems considerably more difficult (cf. 25 for one special case where theorem 25 is proven for a non–natural involution).

Theorem 25 has interesting consequences for the Chow ring of the quotient:

**Corollary (=corollaries 28 and 29).** Let $X$ and $G$ be as in theorem 25, and let $Y := X/G$.

(i) Let $a \in A^{2m−1}(Y)$ be a 1–cycle which is in the image of the intersection product map

$$A^1(Y) \otimes A^2(Y) \otimes \cdots \otimes A^r(Y) \to A^{2m−1}(Y),$$

where all $i_j$ are $\leq 2$. Then $a$ is rationally trivial if and only if $a$ is homologically trivial.

(ii) Let $a \in A^{2m}(Y)$ be a 0–cycle which is in the image of the intersection product map

$$A^0(Y) \otimes A^{i_1}(Y) \otimes \cdots \otimes A^{i_r}(Y) \to A^{2m}(Y),$$

where all $i_j$ are $\leq 2$. Then $a$ is rationally trivial if and only if $a$ is homologically trivial.

These corollaries illustrate the following expectation: for certain special varieties with a multiplicative Chow–Künneth decomposition, the subring $A_{(0)} \subset A$ on which the cycle class map is injective should be larger than for hyperkähler varieties. Indeed, for a quotient $Y = X/G$ where $X$ is hyperkähler and $G \subset \text{Aut}(X)$ is a finite order group of non–symplectic automorphisms, one expects that codimension 2 cycles lie in $A^0_{(0)}(Y)$.

Results similar in spirit have been obtained for certain other hyperkähler varieties and their Calabi–Yau quotients in [24], [26], [25].

**Conventions.** In this article, the word variety will refer to a reduced irreducible scheme of finite type over $\mathbb{C}$. A subvariety is a (possibly reducible) reduced subscheme which is equidimensional.

All Chow groups will be with rational coefficients: we will denote by $A_1(X)$ the Chow group of $1$–dimensional cycles on $X$ with $\mathbb{Q}$–coefficients; for $X$ smooth of dimension $n$ the notations $A_j(X)$ and $A^{n−j}(X)$ are used interchangeably.

The notations $A_{(0)}(X)$, $A_{(1)}(X)$ will be used to indicate the subgroups of homologically trivial, resp. Abel–Jacobi trivial cycles. For a morphism $f : X \to Y$, we will write $\Gamma_f \in A_0(X \times Y)$ for the graph of $f$. The contravariant category of Chow motives (i.e., pure motives with respect to rational equivalence as in [14], [29]) will be denoted $\text{Mot}_{rat}$.

We will write $H^1(X)$ to indicate singular cohomology $H^1(X, \mathbb{Q})$.

Given a group $G \subset \text{Aut}(X)$ of automorphisms of $X$, we will write $A^i(X)^G$ (and $H^j(X)^G$) for the subgroup of $A^i(X)$ (resp. $H^j(X)$) invariant under $G$. 

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2 Preliminary

2.1 Quotient varieties

**Definition 2** A projective quotient variety is a variety

\[ Y = X/G, \]

where \( X \) is a smooth projective variety and \( G \subset \text{Aut}(X) \) is a finite group.

**Proposition 3 (Fulton [17])** Let \( Y \) be a projective quotient variety of dimension \( n \). Let \( A^*(Y) \) denote the operational Chow cohomology ring. The natural map

\[ A^i(Y) \to A^{n-i}(Y) \]

is an isomorphism for all \( i \).

**Proof** This is [17, Example 17.4.10].

**Remark 4** It follows from proposition 3 that the formalism of correspondences goes through unchanged for projective quotient varieties (this is also noted in [17, Example 16.1.13]). We can thus consider motives \([Y,p,0] \in \mathcal{M}_{\text{rat}}, \) where \( Y \) is a projective quotient variety and \( p \in A^n(Y \times Y) \) is a projector. For a projective quotient variety \( Y = X/G \), one readily proves (using Manin’s identity principle) that there is an isomorphism

\[ h(Y) \cong h(X)^G := ([X, \Delta^G_X, 0] \in \mathcal{M}_{\text{rat}}, \]

where \( \Delta^G_X \) denotes the idempotent

\[ \Delta^G_X := \frac{1}{|G|} \sum_{g \in G} \Gamma_g \in A^n(X \times X). \]

(NB: \( \Delta^G_X \) is a projector on the \( G \)-invariant part of the Chow groups \( A^*(X)^G \).)

2.2 MCK decomposition

**Definition 5 (Murre [28])** Let \( X \) be a projective quotient variety of dimension \( n \). We say that \( X \) has a CK decomposition if there exists a decomposition of the diagonal

\[ \Delta_X = \pi_0 + \pi_1 + \cdots + \pi_{2n} \text{ in } A^n(X \times X), \]

such that the \( \pi_i \) are mutually orthogonal idempotents and \( (\pi_i)_* H^*(X) = H^i(X), \) A CK decomposition is self–dual if \( \pi_i = t \pi_{2n-i} \) in \( A^n(X \times X) \) for all \( i \) (here \( t \pi \) denotes the transpose of \( \pi \)).

(NB: “CK decomposition” is shorthand for “Chow–Küneth decomposition”.)

**Remark 6** The existence of a CK decomposition for any smooth projective variety is part of Murre’s conjectures [28], [20].

**Definition 7 (Shen–Vial [36])** Let \( X \) be a projective quotient variety of dimension \( n \). Let \( \Delta^m_X \in A^{2n}(X \times X \times X) \) be the class of the small diagonal

\[ \Delta^m_X := \{(x,x,x) \mid x \in X\} \subset X \times X \times X. \]

An MCK decomposition is a CK decomposition \( \{\pi_i\} \) of \( X \) that is multiplicative, i.e. it satisfies

\[ \pi_k \circ \Delta^m_X \circ (\pi_i \times \pi_j) = 0 \text{ in } A^{2n}(X \times X \times X) \text{ for all } i + j \neq k. \]

(NB: “MCK decomposition” is shorthand for “multiplicative Chow–Küneth decomposition”.)
Remark 8 The small diagonal (seen as a correspondence from $X \times X$ to $X$) induces the multiplication morphism
\[ \Delta^{m}_{X} : h(X) \otimes h(X) \to h(X) \] in $\mathcal{M}_{\text{rat}}$.

Suppose $X$ has a CK decomposition
\[ h(X) = \bigoplus_{i=0}^{2n} h_{i}(X) \] in $\mathcal{M}_{\text{rat}}$.

By definition, this decomposition is multiplicative if for any $i$, $j$ the composition
\[ h_{i}(X) \otimes h_{j}(X) \to h(X) \otimes h(X) \xrightarrow{\Delta^{m}_{X}} h(X) \] factors through $h_{i+j}(X)$. It follows that if $X$ has an MCK decomposition, then setting
\[ A^{l}(X) := (\pi_{2j-l})_{*}A^{l}(X), \]
one obtains a bigraded ring structure on the Chow ring: that is, the intersection product sends $A^{i}(X) \otimes A^{j}(X)$ to $A^{i+j}(X)$.

It is expected (but not proven!) that for any $X$ with an MCK decomposition, one has
\[ A^{l}(X) \overset{??}{=} 0 \] for $j < 0$, $A^{1}(X) \cap A_{\text{hom}}^{l}(X) \overset{??}{=} 0$;
this is related to Murre’s conjectures B and D [28].

The property of having an MCK decomposition is severely restrictive, and is closely related to Beauville’s “(weak) splitting property” [7]. For more ample discussion, and examples of varieties with an MCK decomposition, we refer to [56] Section 8], as well as [41], [37], [16], [24].

Theorem 9 (Vial [41]) Let $S$ be an algebraic K3 surface, and let $X = S^{[m]}$ be the Hilbert scheme of length $m$ subschemes of $S$. Then $X$ has a self–dual MCK decomposition. One has
\[ A^{i}(X) = 0 \] for all $j$ odd and for all $j > i$.

Proof This is [41] Theorem 1. For later use, we briefly review the construction. First, one takes an MCK decomposition $\{\Pi^{S}_{r}\}$ for $S$ (this exists, thanks to [56]). Taking products, this induces an MCK decomposition $\{\Pi^{S^{[r]}}_{r}\}$ for $S^{[r]}$, $r \in \mathbb{N}$. This product MCK decomposition is invariant under the action of the symmetric group $\Sigma$, and hence it induces an MCK decomposition $\{\Pi^{S^{[(r)]}}_{r}\}$ for the symmetric products $S^{(r)}$, $r \in \mathbb{N}$.

There is the isomorphism of de Cataldo–Migliorini [13]
\[ \bigoplus_{\mu \in \mathfrak{B}(m)} (\tilde{\ell}_{\mu})_{*} : A^{i}(X) \xrightarrow{\approx} \bigoplus_{\mu \in \mathfrak{B}(m)} A^{i+l(\mu)−m}(S^{(\mu)}), \]
where $\mathfrak{B}(m)$ is the set of partitions of $m$, $l(\mu)$ is the length of the partition $\mu$, and $S^{(\mu)} = S^{(\mu)}/\Sigma_{l(\mu)}$, and $\tilde{\ell}_{\mu}$ is a correspondence in $A^{m+l(\mu)}(S^{[m]} \times S^{(\mu)})$. Using this isomorphism, Vial defines [41] Equation (4) a natural CK decomposition for $X$, by setting
\[ \Pi^{X}_{i} := \sum_{\mu \in \mathfrak{B}(m)} \frac{1}{m_{\mu}} \tilde{\ell}_{\mu} \circ \Pi^{S^{(\mu)}}_{i−2m+2l(\mu)} \circ \ell_{\mu}, \]
(1)
where the $m_{\mu}$ are rational numbers coming from the de Cataldo–Migliorini isomorphism. The $\{\Pi^{X}_{l}\}$ of definition (1) are proven to be an MCK decomposition.

The self–duality of the $\{\Pi^{X}_{l}\}$ is apparent from definition (1). The fact that $A_{l}^{j}(X)$ vanishes for $j$ odd is because $\Pi^{X}_{j} = 0$ for $j$ odd. The vanishing for $j > i$ follows from the fact that by construction, the projector $\Pi^{X}_{i}$ is supported on $V \times X$ with $\dim V = \ell$; this implies (for reasons of dimension) that
\[ (\Pi^{X}_{\ell}), A^{i}(X) = 0 \] for all $\ell < i$.
Remark 10 It follows from definition (1) that the de Cataldo–Migliorini isomorphism is compatible with the bigrading of the Chow ring, in the sense that there are induced isomorphisms

$$\bigoplus_{\mu \in \mathcal{B}(m)} (t^! \tau_{\mu})_* : A^i_{(j)}(X) \xrightarrow{\cong} \bigoplus_{\mu \in \mathcal{B}(m)} A^{i + \ell(\mu) - m}(S^\mu).$$

In particular, there are split injections

$$\bigoplus_{\mu \in \mathcal{B}(m)} (t^! \tau_{\mu})_* : A^i_{(j)}(X) \twoheadrightarrow \bigoplus_{\mu \in \mathcal{B}(m)} A^{i + \ell(\mu) - m}(S^\mu).$$

(Here, the right-hand side refers to the product MCK decomposition of $S^\mu$.)

Lemma 11 (Shen–Vial) Let $X$ be a projective quotient variety of dimension $n$, and suppose $X$ has a self-dual MCK decomposition. Then

$$\Delta_X \in A^n_{(0)}(X \times X),$$

$$\Delta_X^{sm} \in A^{2n}_{(0)}(X \times X \times X).$$

Proof The first statement follows from [37, Lemma 1.4] when $X$ is smooth. The same argument works for projective quotient varieties; the point is just that

$$\Delta_X = \sum_{i=0}^{2n} \Pi_i^X \circ \Pi_i^X$$

$$= \sum_{i=0}^{2n} (t^! \Pi_i^X \times \Pi_i^X)_* \Delta_X$$

$$= \sum_{i=0}^{2n} (\Pi_{2n-i}^X \times \Pi_i^X)_* \Delta_X$$

$$= (\Pi_{2n}^X \times X)_* \Delta_X \in A^n_{(0)}(X \times X).$$

(Here, the second line follows from Lieberman’s lemma [39, Lemma 3.3], and the last line is the fact that the product of 2 MCK decompositions is MCK [36, Theorem 8.6].)

The second statement is proven for smooth $X$ in [36, Proposition 8.4]; the same argument works for projective quotient varieties.

2.3 MCK for products

Proposition 12 Let $S$ be a $K3$ surface. There exist correspondences

$$\Theta_1, \ldots, \Theta_m \in A^{2m}(S^m \times S), \quad \Xi_1, \ldots, \Xi_m \in A^2(S \times S^m)$$

such that the composition

$$A^{2m}_{(2)}(S^m) \xrightarrow{((\Theta_1), \ldots, (\Theta_m)_*)} A^2(S) \oplus \cdots \oplus A^2(S) \xrightarrow{(\Xi_1)_* + \cdots + (\Xi_m)_*} A^{2m}(S^m)$$

is the identity.
Proof. By construction, the MCK decomposition for $S$ is given by

$$
\Pi_0^S = o S \times S, \quad \Pi_1^S = S \times o S, \quad \Pi_2^S = \Delta_S - \pi_0^S - \pi_1^S.
$$

Here $o S \in A^2(S)$ denotes the “distinguished point” of $S$ (any point lying on a rational curve in $S$ equals $o S$ in $A^2(S)$). Let

$$
p_{i,j} : S^{2m} \to S^2 \quad (1 \leq i < j \leq 2m)
$$

denote projection to the $i$-th and $j$-th factor, and let

$$
p_i : S^m \to S \quad (1 \leq i \leq m)
$$

denote projection to the $i$-th factor.

We now claim that there is equality

$$
\Pi_{4m-2} = (\Gamma_{p_1} \circ \Pi_2^S \circ \Gamma_{p_1} \circ ((p_{1,m+1})^*(\Delta_S) \cdot \prod_{2 \leq j \leq 2m, j \neq m+1} (p_j)^*(o S)) + \cdots + \Gamma_{p_m} \circ \Pi_2^S \circ \Gamma_{p_m} \circ ((p_{m,2m})^*(\Delta_S) \cdot \prod_{1 \leq j \leq 2m-1, j \neq m} (p_j)^*(o S))
$$

in $A^{2m}(S^m \times S^m)$.

Indeed, using Lieberman’s lemma [39 Lemma 3.3], we find that

$$
\Gamma_{p_1} \circ \Pi_2^S \circ \Gamma_{p_1} = (\Gamma_{p_{1,m+1}}, \Pi_2^S) = (p_{1,m+1})^*(\Pi_2^S),
$$

...\]

Let us now (by way of example) consider the first summand of the right-hand-side of (2). For brevity, let

$$
P : S^{3m} \to S^{2m}
$$

denote the projection on the first $m$ and last $m$ factors. Writing out the definition of composition of correspondences, we find that

$$
\Gamma_{p_1} \circ \Pi_2^S \circ \Gamma_{p_1} \circ ((p_{1,m+1})^*(\Delta_S) \cdot \prod_{2 \leq j \leq 2m, j \neq m+1} (p_j)^*(o S)) =
$$

$$
((p_{1,m+1})^*(\Pi_2^S)) \circ ((p_{1,m+1})^*(\Delta_S) \cdot \prod_{2 \leq j \leq 2m, j \neq m+1} (p_j)^*(o S)) =
$$

$$
P \left( ((\Delta_S)_{(1,m+1)} \times o S \times \cdots \times o S \times S \times \cdots \times S) \right.
$$

$$
\left. \times \left( S \times \cdots \times S \times (\Pi_2^S)_{(m+1,2m+1)} \times S \times \cdots \times S \right) \right)
$$

$$
\left. \times \left( S \times \cdots \times S \times (\Pi_2^S)_{(m+1,2m+1)} \times S \times \cdots \times S \right) \right)
$$

$$
= P \left( (\Delta_S \times S) \cdot (S \times \Pi_2^S)_{(1,m+1,2m+1)} \times o S \times \cdots \times o S \times S \times \cdots \times S \right) =
$$

$$
\Pi_2^S \times \Pi_4^S \times \cdots \times \Pi_4^S \text{ in } A^{2m}(S^m \times S^m).
$$

(Here, we use the notation $(C)_{(i,j)}$ to indicate that the cycle $C$ lies in the $i$th and $j$th factor, and likewise for $(D)_{(i,j,k)}$.)

Doing the same for the other summands in (2), one convinces oneself that both sides of (2) are equal to the product Chow–Künneth component

$$
\Pi_{4m-2} = \Pi_2^S \times \Pi_4^S \times \cdots \times \Pi_4^S + \cdots + \Pi_4^S \times \cdots \times \Pi_4^S \times \Pi_2^S \in A^{2m}(S^m \times S^m),
$$

thus proving the claim.
Let us now define

$$\Theta_i := \Gamma_{p_i} \circ (p_{i,m+1})^*(\Delta_S) \cdot \prod_{j \not\in \{i,m+1\}}^j (p_j)^*(\circ_S) \in A^{2m}(S^m \times S),$$

$$\Xi_i := t^\Gamma_{p_i} \circ \Pi_2^S \in A^2(S \times S^m),$$

where $1 \leq i \leq m$. It follows from equation (2) that there is equality

$$((\Xi_1 \circ \Theta_1 + \cdots + \Xi_m \circ \Theta_m)_* = (\Pi_{4m-2}^S)_* : A_{(i)}(S^m) \to A_{(i)}(S^m) \forall (i,j).$$

Taking $(i,j) = (2m, 2)$, this proves the proposition.

The following is a version of proposition 12 for the group $A^{2m}(S^m)$:

**Proposition 13** Let $S$ be a $K3$ surface. There exist correspondences

$$t_1 \Theta_1, \ldots, t_m \Theta_m \in A^{2m}(S \times S^m), \quad t_1 \Xi_1, \ldots, t_m \Xi_m \in A^2(S^m \times S)$$

such that the composition

$$A_{(2)}(S^m) \xrightarrow{((t_1 \circ \Xi_1 + \cdots + t_m \circ \Xi_m)_*)} A^2(S \oplus \cdots \oplus A^2(S) \xrightarrow{((t_1 + \cdots + t_m)_*)} A^2(S^m)$$

is the identity.

**Proof** By construction, the product MCK decomposition $\{\Pi_{i}^{S^m}\}$ satisfies

$$\Pi_2^{S^m} = t^\Gamma_1(\Pi_{4m-2}^S) \in A^{2m}(S^m \times S^m).$$

Hence, the transpose of equation (3) gives the equality

$$(\Pi_2^{S^m})_* = (t^\Gamma_1(\Pi_{4m-2}^S))_* = (t_1 \Theta_1 + \cdots + t_m \Theta_m)_* : A_{(i)}(S^m) \to A_{(i)}(S^m) \forall (i,j).$$

Taking $(i,j) = (2, 2)$, this proves the proposition.

2.4 Birational invariance

**Proposition 14 (Rieß[33], Vial [41])** Let $X$ and $X'$ be birational hyperkähler varieties. Assume $X$ has an MCK decomposition. Then also $X'$ has an MCK decomposition, and there are natural isomorphisms

$$A_{(i)}(X) \cong A_{(i)}(X') \text{ for all } i, j.$$

**Proof** As noted by Vial [41], this is a consequence of Rieß’s result that $X$ and $X'$ have isomorphic Chow motive (as algebras in the category of Chow motives). For more details, cf. [36, Section 6] or [25, Lemma 2.8].
2.5 A commutativity lemma

Lemma 15 Let $S$ be an algebraic K3 surface, and let $\{\Pi_i^S\}$ be the MCK decomposition as above. Let $h \in \text{Aut}(S)$. Then

$$
\Gamma_h \circ \Pi_i^S = \Pi_i^S \circ \Gamma_h \quad \text{in } A^2(S \times S) \quad \forall i .
$$

Proof It suffices to prove this for $i = 0$. Indeed, by definition of $\{\Pi_i^S\}$ we have

$$
\Pi_0^S := \Pi_0^S \quad \text{in } A^2(S \times S) ,
$$

$$
\Pi_0^S := \Delta_S - \Pi_0^S - \Pi_4^S .
$$

Supposing the lemma holds for $i = 0$, by taking transpose correspondences we get an equality

$$
\Gamma_h^{-1} \circ \Pi_4^S = \Pi_4^S \circ \Gamma_h^{-1} \quad \text{in } A^2(S \times S) .
$$

Composing on both sides with $\Gamma_h$, we get

$$
\Pi_4^S \circ \Gamma_h = \Gamma_h \circ \Pi_4^S \quad \text{in } A^2(S \times S) .
$$

Next, since obviously the diagonal $\Delta_S$ commutes with $\Gamma_h$, we also get

$$
\Gamma_h \circ \Pi_4^S = \Gamma_h \circ (\Delta_S - \Pi_0^S - \Pi_4^S) = (\Delta_S - \Pi_0^S - \Pi_4^S) \circ \Gamma_h = \Pi_4^S \circ \Gamma_h \quad \text{in } A^2(S \times S) .
$$

It remains to prove the lemma for $i = 0$. The projector $\Pi_0^S$ is defined as

$$
\Pi_0^S = \phi_S \times S \quad \text{in } A^2(S \times S) ,
$$

where $\phi_S \in A^2(S)$ is the “distinguished point” of [8]. Let $x \in S$ be a point lying on a rational curve. Then $h^*(\phi_S) = h^{-1}(x)$ is again a point lying on a rational curve, and so

$$
h^*(\phi_S) = \phi_S \quad \text{in } A^2(S) .
$$

Using Lieberman’s lemma [39] Lemma 3.3], we find that

$$
\Pi_0^S \circ \Gamma_h = (\Gamma_h \times \Delta_S)_*(\Pi_0^S)
$$

$$
= (\Gamma_h \times \Delta_S)_*(\phi_S \times S)
$$

$$
= h^*(\phi_S) \times S
$$

$$
= \phi_S \times S = \Pi_0^S \quad \text{in } A^2(S \times S) ,
$$

whereas obviously

$$
\Gamma_h \circ \Pi_0^S = (\Delta_S \times \Gamma_h)_*(\phi_S \times S) = \phi_S \times S = \Pi_0^S \quad \text{in } A^2(S \times S) .
$$

This proves the $i = 0$ case of the lemma.

The following lemmas establish some corollaries of lemma [15]

Lemma 16 Let $S$ be an algebraic K3 surface, and $G_S \subset \text{Aut}(S)$ a group of finite order $k$. For any $r \in \mathbb{N}$, let $\{\Pi_i^{S^r}\}$ denote the product MCK decomposition of $S^r$ induced by the MCK decomposition of $S$ as above. Let

$$
\Delta_{S^r}^G := \frac{1}{k} \sum_{g \in G_S} \Gamma_g \times \cdots \times \Gamma_g \quad \text{in } A^{2r}(S^r \times S^r) .
$$

Then

$$
\Delta_{S^r}^G \circ \Pi_i^{S^r} = \Pi_i^{S^r} \circ \Delta_{S^r}^G \quad \text{in } A^{2r}(S^r \times S^r)
$$

is an idempotent, for any $i$. 

Proof It suffices to prove the commutativity statement. (Indeed, since both $\Delta^G_S$ and $\Pi^S_r$ are idempotent, the idempotence of their composition follows immediately from the stated commutativity relation.) To prove the commutativity statement, we will prove more precisely that for any $h \in \text{Aut}(S)$ we have equality

$$\Gamma_{h \times r} \circ \Pi^S_r = \Pi^S_r \circ \Gamma_{h \times r} \in A^{2r}(S^r \times S^r). \tag{4}$$

This can be seen as follows: we have

$$\begin{align*}
\Gamma_{h \times r} \circ \Pi^S_r &= (\Gamma_h \times \cdots \times \Gamma_h) \circ (\sum_{i_1 + \cdots + i_r = 1} \pi^S_{i_1} \times \cdots \times \pi^S_{i_r}) \\
&= \sum_{i_1 + \cdots + i_r = 1} (\Gamma_h \circ \Pi^S_{i_1}) \times \cdots \times (\Gamma_h \circ \Pi^S_{i_r}) \\
&= \sum_{i_1 + \cdots + i_r = 1} (\Pi^S_{i_1} \circ \Gamma_h) \times \cdots \times (\Pi^S_{i_r} \circ \Gamma_h) \\
&= \sum_{i_1 + \cdots + i_r = 1} (\Pi^S_{i_1} \times \cdots \times \Pi^S_{i_r}) \circ (\Gamma_h \times \cdots \times \Gamma_h) \\
&= \Pi^S_r \circ \Gamma_{h \times r} \text{ in } A^{2r}(S^r \times S^r).
\end{align*}$$

Here, the first and last lines are the definition of the product MCK decomposition for $S^r$; the second and fourth line are just regrouping, and the third line is lemma $\text{[15]}$.

Lemma 17 Let $S$ be an algebraic K3 surface, and $G_S \subset \text{Aut}(S)$ a group of finite order $k$. For any $r \in \mathbb{N}$, let $X = S^{[r]}$ and let $G \subset \text{Aut}(X)$ be the group of natural automorphisms induced by $G_S$. Let $\{\Pi^X_i\}$ be the MCK decomposition of theorem $\text{[8]}$. Let $\Delta^G_X$ denote the correspondence

$$\Delta^G_X \coloneqq \frac{1}{k} \sum_{g \in G} \Gamma_g \in A^{2r}(X \times X).$$

Then

$$\Delta^G_X \circ \Pi^X_i = \Pi^X_i \circ \Delta^G_X \in A^{2r}(X \times X)$$

is an idempotent, for any $i$.

Proof Again, it suffices to prove the commutativity statement. This can be done as follows: for any $g \in G$, we can write $g = h^{[r]}$ where $h \in \text{Aut}(S)$. Then we have

$$\begin{align*}
\Gamma_g \circ \Pi^X_i &= \Gamma_g \circ \sum_{\mu \in \mathfrak{B}(k)} \frac{1}{m_{\mu}} \Gamma_\mu \circ \Pi^{S^{[r]}_{i-2k+2l(\mu)}} \circ \Gamma_\mu \\
&= \sum_{\mu \in \mathfrak{B}(k)} \frac{1}{m_{\mu}} \Gamma_g \circ \Gamma_\mu \circ \Pi^{S^{[r]}_{i-2k+2l(\mu)}} \circ \Gamma_\mu \\
&= \sum_{\mu \in \mathfrak{B}(k)} \frac{1}{m_{\mu}} \Gamma_\mu \circ \Gamma_h^{t(\mu)} \circ \Pi^{S^{[r]}_{i-2k+2l(\mu)}} \circ \Gamma_\mu \\
&= \sum_{\mu \in \mathfrak{B}(k)} \frac{1}{m_{\mu}} \Gamma_\mu \circ \Pi^{S^{[r]}_{i-2k+2l(\mu)}} \circ \Gamma_h^{t(\mu)} \circ \Gamma_\mu \\
&= \sum_{\mu \in \mathfrak{B}(k)} \frac{1}{m_{\mu}} \Gamma_\mu \circ \Pi^{S^{[r]}_{i-2k+2l(\mu)}} \circ \Gamma \circ \Gamma_g \\
&= \Pi^X_i \circ \Gamma_g \text{ in } A^{2r}(X \times X).
\end{align*}$$

Here, the first line follows from the definition of $\Pi^X_i$ (definition $\text{[1]}$). The second line is just regrouping, the third line is by construction of natural automorphisms of $X$, the fourth line is equality $\text{[4]}$ above, and the fifth line is again by construction of natural automorphisms.
Remark 18 In view of [37, Lemma 1.4] the commutativity property (5) is equivalent to the following: for any natural automorphism \( g \) of \( X \), the graph \( \Gamma_g \in \mathcal{A}^{0\mathcal{A}}(X \times X) \) is “of pure grade 0\( ^{t} \), i.e. \( \Gamma_g \in \mathcal{A}^{0\mathcal{A}}(X \times X) \).

Lemma 19 Let \( S \) be an algebraic K3 surface, and \( X = S^{[m]} \) the Hilbert scheme of length \( m \) subschemes. Let \( G \subset \text{Aut}(X) \) a group of finite order \( k \) of natural automorphisms. Then the quotient \( Y := X/G \) has a self−dual MCK decomposition.

Proof Let \( p : X \to Y \) denote the quotient morphism. One defines

\[
Y_j := \frac{1}{k} \Gamma_p \circ \Pi_j \times \Pi_j \in \mathcal{A}^{m}(Y \times Y),
\]

where \( \{ \Pi_j \} \) is the self−dual MCK decomposition of theorem 9. This defines a self−dual CK decomposition \( \{ Y_j \} \), since

\[
Y_j = \frac{1}{k^2} \Gamma_p \circ \Pi_j \times \Pi_j \circ \Gamma_p \circ \Pi_j \times \Pi_j \circ \Gamma_p,
\]

where

\[
Y_j = \begin{cases} 0 & \text{if } i \neq j; \\ \frac{1}{k} \Gamma_p \circ \Pi_j \times \Pi_j \circ \Gamma_p & \text{if } i = j. \\ \end{cases}
\]

(Here, in the third line we have used lemma 15.)

It remains to check this CK decomposition is multiplicative. To this end, let \( i, j, k \) be integers with \( k \neq i + j \). We note that

\[
Y_j \circ \Delta_{X}^{m} \circ (Y_j \times Y_j) = \frac{1}{k^2} \Gamma_p \circ \Pi_j \times \Pi_j \circ \Gamma_p \circ \Delta_{X}^{m} \circ \Gamma_p \circ \Pi_j \times \Pi_j \circ \Gamma_p \circ \Pi_j \times \Pi_j \circ \Gamma_p,
\]

but

\[
Y_j = \frac{1}{k} \Gamma_p \circ \Pi_j \times \Pi_j \circ \Gamma_p \circ \Pi_j \times \Pi_j \circ \Gamma_p \circ \Pi_j \times \Pi_j \circ \Gamma_p,
\]

This implies that

\[
\Gamma_p \circ \Delta_{X}^{m} \circ \Gamma_p \circ \Pi_j \times \Pi_j \circ \Gamma_p \circ \Pi_j \times \Pi_j \circ \Gamma_p = \frac{1}{k} \Gamma_p \circ \Pi_j \times \Pi_j \circ \Gamma_p \circ \Pi_j \times \Pi_j \circ \Gamma_p.
\]

But

\[
\Gamma_p \circ \Pi_j \times \Pi_j \circ \Gamma_p \circ \Pi_j \times \Pi_j \circ \Gamma_p = (\sum_{g \in G} \Gamma_g) \circ \Delta_{X}^{m} \circ \Gamma_p \circ \Pi_j \times \Pi_j \circ \Gamma_p.
\]

Hence

\[
\Delta_{X}^{m} = (p \times p \times p)_{\times} (\Delta_{X}^{m}) = \Gamma_p \circ \Delta_{X}^{m} \circ \Gamma_p \circ \Pi_j \times \Pi_j \circ \Gamma_p \circ \Pi_j \times \Pi_j \circ \Gamma_p \circ \Pi_j \times \Pi_j \circ \Gamma_p \circ \Pi_j \times \Pi_j \circ \Gamma_p.
\]

This implies that

\[
\Gamma_p \circ \Delta_{X}^{m} \circ \Gamma_p \circ \Pi_j \times \Pi_j \circ \Gamma_p \circ \Pi_j \times \Pi_j \circ \Gamma_p = \frac{1}{k} \Gamma_p \circ \Pi_j \times \Pi_j \circ \Gamma_p \circ \Pi_j \times \Pi_j \circ \Gamma_p.
\]

and thus

\[
\Gamma_p \circ \Delta_{X}^{m} \circ \Gamma_p \circ \Pi_j \times \Pi_j \circ \Gamma_p \circ \Pi_j \times \Pi_j \circ \Gamma_p = (\sum_{g \in G} \Gamma_g) \circ \Delta_{X}^{m} \circ \Gamma_p \circ \Pi_j \times \Pi_j \circ \Gamma_p \circ \Pi_j \times \Pi_j \circ \Gamma_p.
\]

as claimed.
There is also the following commutativity relation:

Lemma 21 Let

$$\Xi_1, \ldots, \Xi_m \in A^2(S \times S^m)$$

be as in propositions 12 and 13. Let $$h \in \text{Aut}(S)$$. The diagrams

$$A^2_2(S^m) \xrightarrow{((t \Xi_1|_{S^m+1})^*, \ldots, (t \Xi_m|_{S^m+1})^*)} A^2_2(S) \oplus \cdots \oplus A^2_2(S)$$

$$\downarrow (h \times m)^* \quad \downarrow (h_*, \ldots, h_*)$$

$$A^2_2(S^m) \xrightarrow{((t \Xi_1|_{S^m+1})^*, \ldots, (t \Xi_m|_{S^m+1})^*)} A^2_2(S) \oplus \cdots \oplus A^2_2(S)$$

and

$$A^2_2(S) \oplus \cdots \oplus A^2_2(S) \xrightarrow{(\Xi_1^*, \ldots, \Xi_m^*)^*} A^2_2(S^m)$$

$$\downarrow (h_*, \ldots, h_*) \quad \downarrow (h \times m)^*$$

$$A^2_2(S) \oplus \cdots \oplus A^2_2(S) \xrightarrow{(\Xi_1^*, \ldots, \Xi_m^*)^*} A^2_2(S^m)$$

are commutative.

Proof First, we observe that $$h \times m$$ and $$h$$ preserve the bigrading in view of (4), so the diagrams make sense. Next, we recall (proposition 12) that $$t \Xi_i$$ is defined on $$A^2_2(S^m)$$ as projection on the $$i$$th factor (which also preserves the bigrading, cf. [37, Corollary 1.6]). The commutativity of the first diagram now follows from the commutativity of

$$S^m \xrightarrow{p_i} S \xrightarrow{h} S$$

As for the second diagram: $$\Xi_1$$ acts on $$A^2_2(S)$$ as $$(p_i)^*$$. Since we can write $$h_* = (h^{-1})^*$$, the second diagram is also commutative.

2.6 Natural automorphisms of Hilbert schemes

Definition 22 (Boissière [11]) Let $$S$$ be a surface, and let $$X = S^m$$ denote the Hilbert scheme of length $$m$$ subschemes. An automorphism $$\psi \in \text{Aut}(S)$$ induces an automorphism $$\psi^m$$ of $$X$$. This determines a homomorphism

$$\text{Aut}(S) \to \text{Aut}(X), \quad \psi \mapsto \psi^m,$$

which is injective [11]. The image of this homomorphism is called the group of natural automorphisms of $$X$$.

Remark 23 It is known [12, Theorem 1] that an automorphism of a Hilbert scheme is natural if and only if it fixes the exceptional divisor of the Hilbert–Chow morphism. To find examples of non–natural automorphisms of a Hilbert scheme $$X$$, Boissière and Sarti introduce the notion of index of an automorphism of $$X$$. For Hilbert schemes of a generic algebraic $$K'$$ surface, the index of an automorphism is 1 if and only if the automorphism is natural [12, section 4].
3 Main result

This section contains the proof of the main result of this note, theorem 25.

Definition 24 Let $S$ be a $K3$ surface, and let $h \in \text{Aut}(S)$ be an automorphism of order $k$. We say that $h$ is non–symplectic if

$$h^* = \nu \cdot \text{id}: H^{2,0}(S) \to H^{2,0}(S),$$

where $\nu$ is a primitive $k$–th root of unity.

(NB: this is sometimes referred to as a “purely non–symplectic automorphism”.)

Theorem 25 Let $S$ be a projective $K3$ surface, and let $X = S^{[m]}$ be the Hilbert scheme of length $m$ subschemes. Let $G \subset \text{Aut}(X)$ be a subgroup of order $k$ of natural non–symplectic automorphisms. Then

$$A^i_{(2)}(X) \cap A^i(X)^G = 0 \quad \text{for} \quad i \in \{2, 2m\}.$$ 

Proof Let us start with the case $i = 2$, i.e. codimension 2 cycles. To prove the required vanishing

$$A^2_{(2)}(X) \cap A^2(X)^G = 0$$

is equivalent to showing that

$$(\Delta^G \circ \Pi^X_2)_* = 0 : A^2(X) \to A^2(X)$$

where $\Pi^X_2$ is part of an MCK decomposition for $X$.

As we have seen (remark 10, plus the obvious fact that $A^1_{(2)}(S^{m-1}) = 0$), there is a commutative diagram

$$
\begin{array}{ccc}
A^2_{(2)}(X) & \hookrightarrow & A^2_{(2)}(S^m) \\
\downarrow (\Delta^G_*) & & \downarrow (\Delta^G_{2m})_* \\
A^2_{(2)}(X) & \hookrightarrow & A^2_{(2)}(S^m)
\end{array}
$$

where horizontal arrows are split injective. Here, the correspondence $\Delta^G_{2m}$ is defined as

$$\Delta^G_{2m} := \sum_{h \in G_S} \Gamma_h \times \Gamma_h \times \cdots \times \Gamma_h \in A^2(S^m \times S^m),$$

and the diagram commutes because of the construction of natural automorphisms of $X$.

To prove (6), we are thus reduced to proving that

$$(\Delta^G_{2m} \circ \Pi^S_2)_* = 0 : A^2(S^m) \to A^2(S^m),$$

where $\Pi^S_2$ is part of an MCK decomposition $\{\Pi^S_i\}$ for $S^m$. We will suppose $\{\Pi^S_i\}$ is the product MCK decomposition used in the proof of theorem 9.

We state a lemma:

Lemma 26 The surface $R := S/G_S$ has

$$A^2_{\text{hom}}(R) = 0.$$ 

Equivalently, for any MCK decomposition $\{\Pi^S_i\}$ one has

$$(\Delta^G_S \circ \Pi^S_2)_* = 0 : A^2(S) \to A^2(S).$$

Proof The quotient variety $R$ has geometric genus 0. Since quotient singularities are rational singularities, there exists a resolution $Y \to R$ with $p_g(Y) = 0$. Since $Y$ is not of general type, Bloch’s conjecture is known to hold for $Y$ [10], i.e. $A^2_{\text{hom}}(Y) = 0$. This implies that also $A^2_{\text{hom}}(R) = 0$. 
Armed with this lemma, we can prove the vanishing (7): There is a commutative diagram

\[
A^2_{(2)}(S^{m}) \xrightarrow{(t \Xi_{1}|_{g+1}), \ldots, (t \Xi_{m}|_{g+1})} A^2_{(2)}(S) \oplus \cdots \oplus A^2_{(2)}(S) \\
\downarrow (\Delta^G_{S^{m}}) \quad \downarrow \quad (\Delta^G_{S^{m}}) \quad \downarrow (\Delta^G_{S^{m}}) \quad \downarrow (\Delta^G_{S^{m}}) \\
A^2_{(2)}(S^{m}) \xrightarrow{(t \Xi_{1}|_{g+1}), \ldots, (t \Xi_{m}|_{g+1})} A^2_{(2)}(S) \oplus \cdots \oplus A^2_{(2)}(S)
\]

The commutativity of this diagram is lemma 21. Horizontal arrows are injections thanks to proposition 13. Since the right vertical arrow is the zero map (lemma 26), the left vertical arrow is also the zero map; this proves the vanishing (7).

The statement for \( i = 2m \) is proven similarly; in view of remark 10, there is a commutative diagram

\[
A^2_{(2)}(X) \xrightarrow{} A^2_{(2)}(S^{m}) \\
\downarrow (\Delta^G_{S^{m}}) \\
A^2_{(2)}(X) \xrightarrow{} A^2_{(2)}(S^{m})
\]

where horizontal arrows are split injective. It thus suffices to prove the right vertical arrow is the zero map. Thanks to proposition 12 and lemma 21, there is a commutative diagram

\[
A^2_{(2)}(S) \oplus \cdots \oplus A^2_{(2)}(S) \xrightarrow{(\Xi_{1})_{s}, \ldots, (\Xi_{m})_{s}} A^2_{(2)}(S^{m}) \\
\downarrow (\Delta^G_{S^{m}}) \\
A^2_{(2)}(S) \oplus \cdots \oplus A^2_{(2)}(S) \xrightarrow{(\Xi_{1})_{s}, \ldots, (\Xi_{m})_{s}} A^2_{(2)}(S^{m})
\]

where horizontal arrows are surjections. Combined with lemma 26, this settles the \( i = 2m \) case.

**Remark 27** Let \( X \) and \( G \) be as in theorem 25. Let \( X' \) be a hyperkähler variety birational to \( X \), and let \( G' \) be the group of birational self–maps of \( X' \) induced by \( G \). Applying proposition 14, it follows from theorem 25 that also

\[
A^i_{(2)}(X') \cap A^i(X')^{G'} = 0 \quad \text{for } i \in \{2, 2m\}.
\]

### 4 Some corollaries

**Corollary 28** Let \( X \) and \( G \) be as in theorem 25 and let \( Y := X/G \) be the quotient. For any \( r \in \mathbb{N} \), let

\[
E^r(Y^r) \subset A^r(Y^r)
\]

be the subalgebra generated by (pullbacks of) \( A^1(Y^r) \) and \( A^2(Y^r) \) and \( \Delta_Y, \Delta_Y^{sm} \). Then the cycle class map induces maps

\[
E^i(Y^r) \to H^{2i}(Y^r)
\]

that are injective for \( i \geq 2mr - 1 \).

**Proof** First, it follows from lemma 13 that \( Y \), and hence \( Y^r \), has a self–dual MCK decomposition. Consequently, the Chow ring \( A^*(Y^r) \) is a bigraded ring. Theorem 25 (plus the obvious fact that \( A_{hom}^1(Y^r) = 0 \)) implies that

\[
A^i(Y) = \bigoplus_{j \leq 0} A^j_{(j)}(Y) \quad \text{for } i \leq 2.
\]

Lemma 11 ensures that

\[
\Delta_Y \in A^{2m}_{(0)}(Y \times Y), \quad \Delta_Y^{sm} \in A^{4m}_{(0)}(Y^3).
\]
Since pullbacks for projections of type \( Y^r \to Y^s, s < r \), preserve the bigrading (this follows from [37, Corollary 1.6], or alternatively can be checked directly), this implies that
\[
E^*(Y^r) \subset \bigoplus_{j \leq 0} A^r_{(j)}(Y^r) .
\]

The corollary now follows from the fact that
\[
A^r_{(j)}(Y^r) \to A^r_{(j)}(X^r)
\]
is injective (this is true for any \( i \) and \( j \)), and the fact that
\[
A^r_{(i)}(X^r) \cap A^i_{hom}(X^r) = 0 \quad \text{for} \quad i \geq 2mr - 1
\]
(as noted in [41, Introduction]).

**Corollary 29** Let \( X \) and \( G \) be as in theorem 25, and let \( Y := X/G \) be the quotient. Let \( a \in A^{2m}(Y) \) be a 0–cycle which is in the image of the intersection product map
\[
A^{i_1}(Y) \otimes A^{i_2}(Y) \otimes \cdots \otimes A^{i_s}(Y) \to A^{2m}(Y) ,
\]
with all \( i_j \leq 2 \) (and \( i_1 + \cdots + i_s = 2m - 3 \)). Then \( a \) is rationally trivial if and only if \( \deg(a) = 0 \).

**Proof** The point is that
\[
A^3(Y) = \bigoplus_{r \leq 2} A^3_{(r)}(Y) ,
\]
\[
A^{i_1}(Y) = \bigoplus_{r \leq 0} A^{i_1}_{(r)}(Y) \quad \text{for} \quad i_1 \leq 2
\]
(theorem 25), and so
\[
a \in \bigoplus_{r \leq 2} A^{2m}_{(r)}(Y) .
\]
But we know that \( A^{2m}_{(r)}(Y) = 0 \) for \( r < 0 \) (this is a general fact for any variety with an MCK decomposition), and we have seen that \( A^{2m}_{(2)}(Y) = 0 \) (theorem 25), and so
\[
a \in A^{2m}_{(0)}(Y) \cong \mathbb{Q} .
\]

**Remark 30** Results similar to corollaries 28 and 29 have been obtained for 0–cycles on certain Calabi–Yau varieties. If \( Y \) is a Calabi–Yau variety (of dimension \( n \)) that is a generic complete intersection in projective space, it is known that the image of the intersection product
\[
\text{Im} \left( A^i(Y) \otimes A^{n-i}(Y) \to A^n(Y) \right) , \quad 0 < i < n ,
\]
is of dimension 1, and hence injects into cohomology [42], [14].

Going beyond the Calabi–Yau case, there is also a result of L. Fu for generic hypersurfaces \( Y \) of general type. Here, the image of the intersection product
\[
\text{Im} \left( A^{i_1}(Y) \otimes A^{i_2}(Y) \otimes \cdots \otimes A^{i_m}(Y) \to A^n(Y) \right) , \quad i_j > 0 ,
\]
is again of dimension 1, provided \( m \) is large enough relative to the degree of \( Y \) [14 Theorem 2.13]. This is very similar to the behaviour of the Chow ring exhibited in corollary 29.

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References

1. M. Artebani and A. Sarti, Non–symplectic automorphisms of order 3 on K3 surfaces, Math. Ann. 342 No. 4 (2008), 903—921.
2. M. Artebani, A. Sarti and S. Taki, K3 surfaces with non–symplectic automorphisms of prime order, Math. Z. 268 (2011), 507—533.
3. D. Al Tabbba, A. Sarti and S. Taki, Classification of order 16 non–symplectic automorphisms on K3 surfaces, J. Korean Math. Soc.,
4. A. Beauville, Some remarks on Kähler manifolds with c1 = 0, in: Classification of algebraic and analytic manifolds (Katata, 1982), Birkhäuser Boston, Boston 1983,
5. A. Beauville, Variétés Kählériennes dont la première classe de Chern est nulle, J. Differential Geom. 18 no. 4 (1983), 755—782.
6. A. Beauville, Sur l’anneau de Chow d’une variété abélienne, Math. Ann. 273 (1986), 647—651.
7. A. Beauville, On the splitting of the Bloch–Beilinson filtration, in: Algebraic cycles and motives (J. Nagel and C. Peters, editors), London Math. Soc. Lecture Notes 344, Cambridge University Press 2007,
8. A. Beauville and C. Voisin, On the Chow ring of a K3 surface, J. Alg. Geom. 13 (2004), 417—426.
9. S. Bloch, Lectures on algebraic cycles, Duke Univ. Press Durham 1980,
10. S. Bloch, Lectures on the Chow ring of the Hilbert cube $X^3$, Memoirs of the AMS 240 (2016), no.1139,
11. S. Bloch, Automorphismes naturels de l’espace de Douady de points sur une surface, Canad. J. Math. (2009).
12. S. Blois et alii, Motivic sheaves and filtrations on Chow groups, in: Motives (U. Jannsen et alii, editors), London Math. Soc. Lecture Notes 344, Cambridge University Press 2007,
13. M. de Cataldo and L. Migliorini, The Chow groups and the motive of the Hilbert scheme of points on a surface, Journal of Algebra 251 no. 2 (2002), 824—848.
14. L. Fu, Decomposition of small diagonals and Chow rings of hypersurfaces and Calabi–Yau complete intersections, Advances in Mathematics (2013), 894—924,
15. L. Fu, On the action of symplectic automorphisms on the CH0–groups of some hyper-Kähler fourfolds, Math. Z. 280 (2015), 307—334,
16. L. Fu, Z. Tian and C. Vial, Motivic hyperkähler resolution conjecture for generalized Kummer varieties, arXiv:1608.04968.
17. W. Fulton, Intersection theory, Springer–Verlag Ergebnisse der Mathematik, Berlin Heidelberg New York Tokyo 1984,
18. A. Garbagnati and M. Penegini, K3 surfaces with a non–symplectic automorphism and product–quotient surfaces, Rev. Mat. Iberoam. 31 vol. 4 (2015), 1277—1310,
19. D. Huybrechts, Symplectic automorphisms of K3 surfaces of arbitrary order, Math. Res. Letters 19 (2012), 947—951,
20. U. Jannsen, Motivic sheaves and filtrations on Chow groups, in: Motives (U. Jannsen et alii, editors), Proceedings of Symposia in Pure Mathematics Vol. 55 (1994), Part 1,
21. B. Kahn, J. Murre and C. Pedrini, On the transcendental part of the motive of a surface, in: Algebraic cycles and motives (J. Nagel and C. Peters, editors), Cambridge University Press, Cambridge 2007,
22. S. Kondo, Automorphisms of K3 surfaces which act trivially on Picard groups, J. Math. Soc. Japan, 44, No. 1 (1992), 75—98,
23. R. Laterveer, Algebraic varieties with small Chow groups, J. Math. Kyoto Univ. Vol. 38 No 4 (1998), 673—694,
24. R. Laterveer, Algebraic cycles on a very special EPW sextic, submitted,
25. R. Laterveer, Bloch’s conjecture for certain hyperkähler fourfolds, and EPW sextics, submitted,
26. R. Laterveer, Algebraic cycles on a very special EPW sextic, submitted,
27. R. Laterveer, Algebraic varieties with small Chow groups, J. Math. Kyoto Univ. Vol. 38 No 4 (1998), 673—694,
28. R. Laterveer, Bloch’s conjecture for certain hyperkähler fourfolds, and EPW sextics, submitted,
29. R. Laterveer, Algebraic cycles on a very special EPW sextic, submitted,
30. R. Laterveer, Algebraic cycles on some special hyperkähler varieties, to appear in Rendiconti di Matematica e delle sue applicazioni,
31. R. Livné, M. Schütt and N. Yui, The modularity of K3 surfaces with non–symplectic group actions, Math. Ann. 348 (2010), 333—355,
32. J. Murre, On a conjectural filtration on the Chow groups of an algebraic variety, Part I and II, Indag. Math. 4 (1993), 177—201,
33. J. Murre, J. Nagel and C. Peters, Lectures on the theory of pure motives, Amer. Math. Soc. University Lecture Series 61, Providence 2013,
34. V. Nikulin, Finite automorphism groups of Kählerian surfaces of type K3, Trans. Moscow Math. Soc. 38 No. 2 (1980), 71—135,
35. V. Nikulin, Factor groups of groups of automorphisms of hyperbolic forms with respect to subgroups generated by 2–reflections, Algebraic geometry applications, J. Sovi. Math. 22 (1983), 1401—1475,
36. P. O’Sullivan, Algebraic cycles on an abelian variety, J. Reine Angew. Math. 654 (2011), 1—81,
37. U. Rieß, On the Chow ring of birational irreducible symplectic varieties, Manuscripta Math. 145 (2014), 473—501,
38. T. Scholl, Classical motives, in: Motives (U. Jannsen et alii, editors), Proceedings of Symposia in Pure Mathematics Vol. 55 (1994), Part 1,
39. M. Schütt, K3 surfaces with non–symplectic automorphisms of 2–power order, Journal of Algebra 323 (2010), 206—223,
40. M. Shen and C. Vial, The Fourier transform for certain hyperkähler fourfolds, Memoirs of the AMS 240 (2016), no.1139,
41. M. Shen and C. Vial, The motive of the Hilbert cube $X^3$, Forum Math. Sigma 4 (2016),
42. S. Taki, Non–symplectic automorphisms of 3–power order on K3 surfaces, Proc. Japan Acad. 86, Ser. A (2010), 125—130,
43. C. Vial, Remarks on motives of abelian type, to appear in Tohoku Math. J.,
44. C. Vial, Niveau and coniveau filtrations on cohomology groups and Chow groups, Proceedings of the LMS 106(2) (2013), 410—444,
45. C. Vial, On the motive of some hyperkähler varieties, to appear in J. für Reine u. Angew. Math.,
46. C. Voisin, Chow rings and decomposition theorems for K3 surfaces and Calabi–Yau hypersurfaces, Geom. Topol. 16 (2012), 433—473,
43. C. Voisin, The generalized Hodge and Bloch conjectures are equivalent for general complete intersections, Ann. Sci. Ecole Norm. Sup. 46, fascicule 3 (2013), 449—475.
44. C. Voisin, Chow Rings, Decomposition of the Diagonal, and the Topology of Families, Princeton University Press, Princeton and Oxford, 2014.
45. S. Vorontsov, Automorphisms of even lattices arising in connection with automorphisms of algebraic $K^3$ surfaces (in Russian), Vestnik Moskov. Uni. Ser. I mat. Mekh. No. 2 (1983), 19—21.