CP violation from pure gauge in extra dimensions

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Abstract

One of Sakharov’s conditions for baryogenesis is the violation of both C and CP. In the Standard Model, gauge interactions break maximally C, but CP is only broken through the Yukawa couplings in the poorly understood scalar sector. In extra-dimensional models, extra components of gauge fields behave as scalars in 4D and can acquire effective vev’s through (finite) quantum effects (Hosotani mechanism). This mechanism is used to build a toy model with 2 extra-dimensions compactified on a flat torus $T^2$, where a SU(2) gauge symmetry is broken to U(1) and CP violation (in 4D) is expected. This is verified by computing a non-vanishing electric dipole moment.

1 Introduction

In comparison with “pure gauge” theories, scalar interactions are badly understood — our ignorance being parametrized through a bunch of arbitrary (Yukawa) couplings. Moreover, while the gauge interactions are CP conserving (at least in 4D\textsuperscript{1}), the scalars break this symmetry, but still in an arbitrary manner (through the phases of the Yukawa coefficients). The situation is well-known in the Standard Model (SM) where CP-violating freedom is only empirically constrained. It is then a sensible belief that a better insight in the scalar sector could clarify the nature of CP violation, and vice versa.

Possibly more central than CP symmetry itself is the issue of matter-antimatter asymmetry. As was pointed out by Sakharov, the emergence of a matter-antimatter asymmetry from an initially symmetrical early universe requires in 4D both C and CP violation. A more general statement would be that C and any symmetry involving C must be broken, CP being just one particular case. This is pretty much the situation we will be discussing in the present note: how C or CP invariance can be broken in theories containing only fermions and their gauge interactions. More specifically, we will discuss how C or CP conservation behave in the dimensional reduction (in the present case from 6D to 4D).

In an attractive, though quite old idea, scalar fields are thought as spatial components of gauge fields in extra dimensions (ED)\textsuperscript{2}. When extra-dimensional space is not simply connected, non trivial holonomies (or Wilson lines (WL)) can appear dynamically for non contractible cycle\textsuperscript{3} and lead to dynamical symmetry breaking. At the level of our (3+1)-dimensional space, effective scalar fields acquire a vev, which could cause CP violation if scalar and pseudo-scalar contributions coexist. At the classical level, the WL are determined by the topology of ED and label degenerate classical vacua. The degeneracy disappears when quantum effects are taken into account, which select the physical solution. These are encoded into the effective potential for WL which depend on topology, matter content and Scherk-Schwarz (SchSch) phases (see below).

In a previous work, this idea was already used and revealed to be promising\textsuperscript{4,5}. One extra dimension was introduced, and the 5th components of gauge fields can yield the equivalent of pseudoscalar terms

\textsuperscript{1}We ignore mass terms which are, at least in chiral theories, a counterpart of scalar interactions.

\textsuperscript{2}This can be seen (at least for abelian cases) as finite magnetic fluxes through holes in the manifold. However, these holes being outside the physical space, a flux is always ill-defined, hence the use of holonomies.
in the 4D-reduced Lagrangian, leading to a complex mass matrix and possible CP violation. Of course, this is not enough, since we can always use a chiral rotation to make them real. Therefore real masses (or in other words, half of the scalar sector) were put in by hand. An appealing extension would be to add a second ED which will provide for this. This is in some way the situation we will be dealing for.

Before turning to 6D however, we should stress that this previous work viewed the Hosotani loops purely as external boundary conditions rather than dynamical variables (in the way of the Bohm-Aharanov effect). Here we will follow Hosotani’s view, which sees these loops as dynamical variables, and requires the evaluation of the effective Lagrangian, beyond the tree level. The problem proves difficult, and the present note deals with ”proof of concept”, namely the possibility of CP violation in 4D from pure gauge theory in 6D, but does not propose a realistic model. This is notably due to the difficulty of generating a “low mass scale”, providing non-zero mass to the zero modes of the compactified theory: in the present note, we will deal either with a massless low-energy sector separated from the Kaluza-Klein scale, or accept small masses controlled by arbitrary phases in the boundary conditions.

The paper is organized as follows. In section 2 we review the notions of P, C and CP symmetries in 4D and in 6D and link them through compactification schemes. Section 3 is devoted to Hosotani’s mechanism which takes place when compactification implies non simply connected ED. We summarize it in the special case of the flat torus $T^2$ and try to include Hosotani’s approach in the more modern one. In section 4 we use explicitly Hosotani mechanism to break CP through compactification and give simple examples in section 5. Finally, in section 6 we come back on anomaly issues which appear in chiral theories that we have neglected before. Conclusions and perspectives can be found in section 7.

### 2 P, C and CP in 4 and 6 dimensions

We use the notation $\gamma^\mu$ (resp. $\Gamma^A$) for 4D (resp. 6D) gamma matrices. The parity transformation is given respectively by $P^{-1}\psi(t, x)\mathcal{P} = \gamma^0\psi(t, -x)$ in 4D and $P^{-1}\Psi(t, x)\mathcal{P} = \Gamma^0\Psi(t, -x)$ in 6D, where $\psi$ and $\Psi$ are 4- and 6-dimensional Dirac spinors. Charge conjugation is given by $C^{-1}\psi(x)C^{-1} = \pm \gamma^0\gamma^\mu\gamma^5\psi(x)$ and $C^{-1}\Psi(x)C^{-1} = \pm \Gamma^0\Gamma^3\Gamma^5\Psi(x)$ where $C^{(4)}$ (resp. $C^{(6)}$) is a matrix which satisfies $C^{(4)}\gamma^\mu C^{(4)} = \pm \gamma^\mu \gamma^5$ (resp. $C^{(6)}\gamma^\mu C^{(6)} = \pm \Gamma^3\Gamma^5\Gamma^3\Gamma^5$). The $\pm$ sign in these relations can be used only in the absence of mass term (which is our case) and there is then an ambiguity in the definition, but we will see that this is unimportant for our purpose.

In 4 dimensions the two solutions are $\Psi^{(4)}_1 = \gamma_0\gamma_2$ and $\Psi^{(4)}_2 = \gamma_1\gamma_3$ (up to phase factors), while in 6 dimensions we find $\Psi^{(6)}_1 = \Gamma^0\Gamma^2\Gamma^4$ and $\Psi^{(6)}_2 = \Gamma^1\Gamma^3\Gamma^5$. In even dimensions, the spinors can be decomposed in two semi-spinors (or Weyl spinor) with the help of the chirality projectors $P_{L,R} = \mp \frac{i+\gamma^5}{2}$. Since $\gamma^5$ anticommute with all $\gamma$’s (as well as does $i\gamma^5$ with all $\Gamma$’s) it is obvious that charge conjugation in 4D links $\psi_L$ and $\psi_R$ (and vice versa), while in 6D it links $\Psi_+\psi_L$ with $\Psi_-\psi_R$. On the contrary, the parity connects + and − spinors in all cases ($L$ and $R$ in 4D). Then the CP operation which is the combination of these two connects $L$ and $L$ spinors in 4D, but + and − in 6D. As announced this is completely independent of the choice for $C^{(4)}$ (resp. $C^{(6)}$).

Now what does it mean? Since gauge interactions connect spinors of the same chirality, gauge symmetries give no reason to introduce both chiralities on an equal footing. Then, in all generality, P is not an automatic symmetry of gauge interactions in both 4 and 6 dimensions. However, while C symmetry is not automatic in 4D, this is always the case in 6D, and conversely for CP. For this reason we need scalar interactions in 4 dimensions to break CP (at perturbative level). In contrast if we write a theory in 6 dimensions with only (say) a + spinor then we break CP. Does it mean that the resulting effective 4D theory is not CP conserving? In other words, are the notions of CP in 4 and 6 dimensions directly related to each other? The answer is no.

To realize this we need to find a relation between 4D and 6D CP transformations. Let us focus on + spinor in 6D which is a Dirac spinor at the 4D level (with $L$ and $R$ components). We know that C transforms $\Psi^+_L(x)$ into $\Psi^+_R(x) \sim \gamma^5\gamma^3\Psi^+_L(x)$. On the other hand, + and − components being representations of the rotation group, we can use them to link $\Psi^+_L(x)$ with $\Psi^+_L(x) \sim \gamma^0\gamma^3\Psi^+_L(t, -x_1, -x_2, -x_3, x_4, x_5)$.

3We will return to this question later; in particular if complex mass terms are needed to generate CP à la Kobayashi-Maskawa, other sources of CP violation (through the Kaluza-Klein excitations for instance) remain in principle possible.

4Our choice of representation can be found in Appendix A.

5In 4D the + sign is identified with $L$ and the − sign with $R$.

6This is related to the fact that in 4D (resp. 6D) $\psi_L$ and $\psi_R$ (resp. $\Psi_+$ and $\Psi_-$) are equivalent representations of the Lorentz group.
where \((x'_4, x'_5)\) result from a rotation of \((x_4, x_5)\). Indeed, \(\Psi^{CP}_*(x)\) is then a CP transformation at the 4D level. One solution is to use a \(\pi\)-rotation in the 1 - 2 and 3 - 5 planes. Then \((x'_4, x'_5) = (x_4, -x_5)\).

But any additional rotation in the 4 - 5 plane leads to a valid definition. Since this combination of transformations is a symmetry of the 6D theory, the 4D effective theory will be CP violating only if the compactification is incompatible with all the symmetries:

\[
\begin{align*}
\Psi_+ &\rightarrow \Psi_+^* \\
X \equiv (x_4, x_5)^T &\rightarrow R \sigma^3 X = R(x_4, -x_5) \equiv \hat{X} = (\hat{x}_4, \hat{x}_5).
\end{align*}
\]

for any rotation \(R\). In other words, the 4D theory will be CP violating if we fail to chiral a rotation which reabsorbs the phases.

Let us take a simple example to illustrate this. Consider a flat torus \(T^2\) of radii \(R_4 = R_5 = R\) with the following Schrödinger boundary conditions (BC)\(^7\): \(\Psi(x_4 + 2\pi R, x_5) = e^{i\beta_1} \Psi(x_4, x_5)\) and \(\Psi(x_4, x_5 + 2\pi R) = e^{i\beta_2} \Psi(x_4, x_5)\). Under the prescribed transformation these BC become \(\Psi(\hat{x}_4 + 2\pi R \cos \theta, \hat{x}_5 + 2\pi R \sin \theta) = e^{-i\beta_1} \Psi(\hat{x}_4, \hat{x}_5)\) and \(\Psi(\hat{x}_4 + 2\pi R \sin \theta, \hat{x}_5 - 2\pi R \cos \theta) = e^{-i\beta_2} \Psi(\hat{x}_4, \hat{x}_5)\). The first relation is compatible only if \(\theta = \pi\) or if \(\theta = 0\) and \(\beta_1 \in \{0, \pi\}\), while the second one is compatible only if \(\theta = 0\) or \(\theta = \pi\) and \(\beta_2 \in \{0, \pi\}\). Then BC break effective 4D CP symmetry as soon as \(\beta_1\) and \(\beta_2\) are both different from \(0\) and \(\pi\). The result is of course independent of \(\theta\).

Note by the way that we can proceed in the same way for \(P\) and \(C\). It is straightforward to show that \(P\) invariance requires compatibility with the transformation \(X \rightarrow R \sigma^3 X\), while \(C\) requires compatibility with \(\Psi \rightarrow \Psi^*\) and \(X \rightarrow RX\). In our previous example, \(P\) is broken but not \(C\) (this leads then to CP violation).

As already mentioned in the introduction, the main point of breaking CP is to get a matter-antimatter asymmetry. Indeed even if \(C\) is broken, this is in general not enough to reach this goal. Indeed any other symmetries involving \(C\) (like CP, but \(CS\) in general) leads to matter-antimatter symmetry. In 6D the \(C\) symmetry is automatic for gauge interactions and the symmetry particle/antiparticle is respected. In 4D \(C\) is not automatic but \(CP\) leads to the same conclusion. Our idea to break this symmetry is precisely to introduce a compactification which breaks all these \(CS\) symmetries.

### 3 Hosotani mechanism with two ED

At the moment we work on flat space-time \(M^4 \times \mathbb{R}^2/G\) where the two ED are compactified by means of orbifolding through one of the 17 two-dimensional space groups \(G\)\(^8\). These groups correspond to isometries of \(\mathbb{R}^2\), which include translations, \(2\pi/n\)-rotations \((n = 2, 3, 4\) and \(6)\), reflections and glide reflections\(^9\). These isometries must obviously be symmetries of the 6D original lagrangian. For instance, only translations and rotations can be used with a chiral lagrangian, and the possible orbifolds in this case are \(T^2\), \(T^2/Z_2\), \(T^2/Z_3\), \(T^2/Z_4\) and \(T^2/Z_6\). We will see later that such lagrangians lead to highly non trivial issues which are due to chiral anomalies and to the interpretation of quantum corrections in ED models. Until then, we will nevertheless stick to them.

In any case, two kinds of compactification exist: the "non-magnetized" and the "magnetized one". In the first case, a non zero field strength is unstable and the only solutions are flat connections. In the second case, a non zero field strength can be stable and the solution corresponds to a physical flux orthogonal to the ED. The stability is ensured by the quantization of the flux for topological reasons\(^10\).

Let us focus on the flat torus \(T^2\) characterized by two radii \(R_4\) and \(R_5\) (we don’t consider here issues of gravitational stability, and they are seen as free parameters). Because of the translation symmetry on the torus, gauge fields on this manifold must be periodic up to a gauge transformation\(^11\):

\[
A_\alpha(y + 2\pi R_\alpha) = T_\alpha(y)A_\alpha(y)T_\alpha^{-1}(y) + \frac{i}{g} T_\alpha(y) \partial_\alpha T_\alpha^{-1}(y)
\]

The topology of the torus requires\(^12\): \(T_4(y + 2\pi R_4)T_5(y) = T_5(y + 2\pi R_4)T_4(y)\).

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\(^7\)Note that the effect of this rotation on the 4D fermion is obviously a chiral rotation.

\(^8\)For now on \(\Psi\) means \(\Psi_+\) unless otherwise stated.

\(^9\)In this note, "orbifold" refers to any quotient spaces regardless of the existence of fixed points.

\(^10\)Translations combined with mirror reflection.

\(^11\)The flat torus \(T^2\) has no fixed point and is generally not called orbifold.

\(^12\)This is true if we introduce fermions in a representation sensitive to the center of the group (e.g. the fundamental one). However, as long as we work with insensitive representations, the relation is valid up to an element of the center of the group\(^5\). We neglect this at the moment.

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However we must be careful, because the BC, $T_i$, do not fix the symmetry of the effective 4D theory. Indeed, the component of the gauge field in the ED, playing the role of scalar fields in 4D could very well acquire a "vev" through quantum effects. More precisely, the ED space being multiply-connected, some non-integrable phase factors become dynamical variables which can lead to effective symmetry breaking in 4D. Indeed, it's worth stressing that neither "vev" nor BC are gauge invariant concepts. The true gauge invariant quantities are the so called Wilson lines phases defined by Hosotani as the eigenvalues of $W_{C_i}(y)T_{C_i}$, with:

$$W_{C_i}(y) = \mathcal{P} \exp \left( ig \int_{C_i} dy^I \langle A_i(y') \rangle \right),$$

for all the non equivalent non-contractible cycles $C_i$, starting at $y$, and $T_{C_i}$ the associated BC.

In the following we will restrict ourselves to SU(N) gauge groups for which we have an important result \[8\][9]: because of the non existence of topological quantities on $T^2$, all stable configurations correspond to flat connexions $\langle F_{45} \rangle = 0$. In his approach \[3\], Hosotani takes this result as an hypothesis. Moreover, he restricts himself to homogeneous BC, i.e. $T_i(y) = T_i$. This is not mandatory, but it can help somehow to get a better insight of the physics. For this reason we first give a quick analysis of the simple case, followed by a more general, but also more technical one.

To elucidate the 4D symmetry, we are particularly interested in the zero modes ($y$ independent) of the gauge field. Obviously these correspond to directions in the gauge group which remain unbroken after the compactification. $F_{45} = 0$ makes them satisfy $\langle A_i \rangle$, $\langle A_5 \rangle = 0$. The homogeneous BC add the constraints $\langle A_i \rangle, T_i = 0$. In other words, $\langle A_4 \rangle$ and $\langle A_5 \rangle$ must be part of the Cartan subalgebra of the group. The selection of a particular solution is done at the quantum level through the so called Hosotani mechanism. Therefore, we need to compute the effective potential for $A_i$ to find the physical symmetry. The result is of course affected by the geometry and the matter content (see section \[4\]).

The "vev's" $\langle A_4 \rangle$ and $\langle A_5 \rangle$ can be gauged away by the transformation $\[13\]$:

$$\Omega(y) = \exp[-ig\langle A_4 \rangle y_4 + \langle A_5 \rangle y_5],$$

and the BC matrices $T_i$ then become $\[14\]$ $T_i^{sym} = \Omega(-2\pi R_i)T_i$. As previously mentioned, neither $T_i$, nor $\langle A_i \rangle$ are physical, but only an appropriate combination. Dynamics with different $T_i$ will give different $\langle A_i \rangle$, but the "symmetric" BC, $T_i^{sym}$, obtained when the "vev's" are gauged away, are all equal $\[15\]$. Therefore, in all generality, we can choose $T_i = 1$ at the beginning and compute the "vev" $\langle A \rangle^{phys}$ which contains all the physics.

Let us consider now the case where $T_i(y)$ can be $y$ dependent. The result $\langle F_{45} \rangle = 0$ is still valid $\[8\]$ and therefore the vacuum configuration for $\langle A \rangle$ must be pure gauge (this time we don't make any a priori assumption about $y$ dependence of it):

$$\langle A_a(y) \rangle = \frac{i}{g} U(y) \partial_a U^{-1}(y),$$

where $U$ must be compatible with the BC. If we use this expression for $\langle A_a \rangle$ into equation $\[2\]$ it is easy to show that $U$ must satisfy $U(y + 2\pi R_i) \partial_a U^{-1}(y + 2\pi R_i) = T_i(y)U(y)\partial_a (T_i(y)U(y))^{-1}$ what means:

$$U(y + 2\pi R_i) = T_i(y)U(y)V_i^{-1},$$

with $V_i$ a constant element of the gauge group such that $[V_4, V_5] = 0$ because of the topology. For some BC, all the classical vacua can be found by solving $\[6\]$ for all possible $V_i$. Since $\langle F_{45} \rangle = 0$, we know that solutions must exist, at least for some compatible $V_i$. Moreover, it can be shown $\[8\][9]$ that, for SU(N) groups on $T^2$, solutions exist for any compatible $V_i$. A particular vacuum is labelled by $T_i(y)$ and $U(y)$. Now let us perform a gauge transformation $U^{-1}$. Then $\langle A(y) \rangle = 0$ and $T_i(y) = V_i$. Therefore, all possible classical vacuum can be labelled by constant and commuting BC:

$$V_i = \exp(i\Theta_i),$$

where $\Theta_i$ are constant and commuting matrices of SU(N) algebra. Again quantum effects select the true vacuum which depends on geometry and matter content (see section \[4\]). Let us call it $\Theta_i^{phys}$. After the

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\[13\]: Since $\langle A_4 \rangle, \langle A_5 \rangle = 0$, $\Omega$ can be decomposed into $\Omega_\omega(y)\Omega_\zeta(y) = \Omega_\omega(y)\Omega_\zeta(y)$ with $\Omega_\omega(y) = \exp[-ig\langle A_4 \rangle y_4].$

\[14\]: $\Omega(-2\pi R_i)$ is a shorthand notation for $\Omega(-2\pi R_i, 0)$ or $\Omega(0, -2\pi R_i)$.

\[15\]: This is not true for all topologies. Indeed, it may be that some BC cannot be linked by any gauge transformation $\[4\]$ for topologically satisfactory $\langle A \rangle$. It follows that we could have more than one equivalence class for BC (see for example $\[10\]$). Here, any $T_i$ can be written as $\Omega(-2\pi R_i)$ thanks to the commutation properties and we have only one equivalence class.

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transformations (1) and (2) are good candidates for CP symmetry either if 
we can only be non zero for representations insensitive to the centre.

\[ \Theta = \exp \left( i \frac{\Theta_{\text{phys}}}{2\pi R_i} y_i \right) \]

we end up with trivial BC and a "vev" for the background that contains all the physics (as in the Hosotani approach):

\[ \langle A_i \rangle_{\text{phys}} = \frac{\Theta_i^{\text{phys}}}{2\pi g R_i} \]

This last identification is correlated by the computation of the WL phases \( \Omega \) in the two approaches. In the first one with trivial BC we find \( W_i = \exp(i\pi R_i \langle A_i \rangle_{\text{phys}}) \), while in the second with trivial "vev" we find \( W_i = \exp(i\Theta_i^{\text{phys}}) \). Note also that it shows that the natural scale for the effective "vev" are the dimensions of the ED. This is expected since they are the only dimensionfull parameters.

### 4 CP violation induced by BC

In the last section we saw that the BC alone are not meaningful by themselves. On the other hand, once we can always perform a gauge transformation that puts all the physics in the BC (in this gauge BC are identified with the WL). In all that follows we will work in this gauge. Therefore the fermionic fields \( \Psi \) have the BC (to simplify notation \( \Theta_i \) is identified with \( \Theta_i^{\text{phys}} \) defined in section 3):

\[ \Psi(y + 2\pi R_i) = \exp(i\beta_i) \exp(i\Theta_i) \Psi(y) \]

with additional phases \( \beta_i \) allowed because fermions appear always in bilinears.\(^\text{16}\) Note that \( \beta_i \) phases (or Scherk-Schwarz (SchSch) phases) are free external parameters and that we can choose them different for each fermionic field. They will enter the dynamics of fermion, possibly creating masses.

To study whether or not these BC lead to CP violation at the 4D level, we need to check their compatibility with the transformations \( \Psi \to \Psi^* \) and \( Y \to Y \). Remember that \( Y \) can be any rotation of \( (y_4, -y_5) \) (see the transformation \( \Psi \) which makes explicit the link between C in 6D and CP in 4D) and that CP is conserved in 4D as long as we can find compatibility for one rotation. Here the gauge symmetry adds an additional freedom. Indeed, the transformations can be \( \Psi \to U^\dagger \Psi^* \), where \( U \) is any global symmetry matrix (since it keeps \( \langle A \rangle = 0 \).

Under the prescribed symmetries, the two BC become:\(^\text{17}\)

\[
\Psi_{\text{CP}}(y_4 + 2\pi R_4 \cos \theta, y_5 + 2\pi R_4 \sin \theta) = \exp \left[ -i \frac{\Theta_{\text{phys}}}{2\pi R_4} y_4 \right] \Psi_{\text{CP}}(y_4, y_5)
\]

\[
\Psi_{\text{CP}}(y_4 + 2\pi R_5 \sin \theta, y_5 - 2\pi R_5 \cos \theta) = \exp \left[ -i \frac{\Theta_{\text{phys}}}{2\pi R_5} y_5 \right] \Psi_{\text{CP}}(y_4, y_5)
\]

The Table \( \Gamma \) shows the different symmetries which might be compatible with BC. The angle \( \theta \) refers to the rotation \( \mathcal{R} \). The columns marked \( \beta_4 \) and \( \beta_5 \) indicate a possible constraint for these phases. The next two columns show the constraints on the \( U \) matrix introduced above.\(^\text{18}\) Note that for adjoint fermions, insensitive to the centre of the group, we have a little bit more freedom. The \( k \) and \( k' \) factors take this into account for SU(N) groups \( (T = \text{diag}(1,...,1,1-N)) \). \( k \) and \( k' \) can take all integer values for representations which are insensitive to the centre, but must be zero in the other case.

We may expect a large variety of situations depending of the gauge group. Let us look here to some simple examples in SU(2), which we are particularly interested in (see section 5). We always have \( \Theta_4 = a \theta_4 \) and \( \Theta_3 = b \theta_3 \) \( (\theta_3 = \sigma_3/2) \). Therefore, if the constraints on \( \beta \)’s and radii are fulfilled: the transformations (1) and (2) are good candidates for CP symmetry either if \( a = b = (j + k/2)\pi \), while the transformations (3) and (4) are good candidates either if \( a + b = (j + k/2)\pi \). \( k \) and \( j \) are integer numbers. \( j \) can always be non zero because it stands for the periodicity in the exponential factor, but \( k \) can only be non zero for representations insensitive to the centre.

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\(^\text{16}\)The notation refers explicitly to the fundamental representation, but it can be easily extended to the adjoint or others.

\(^\text{17}\)We stress again that matter content plays a crucial role in the dynamics that selects the physical vacuum at quantum level. At this point, we suppose this vacuum known and encoded in the BC.

\(^\text{18}\)It may be surprising that \( \langle A \rangle \) doesn’t change under the rotations. One can understand that if one remembers that the physical quantities are WL which are of course rotationally invariant.

\(^\text{19}\)We should write \( (U^{\text{phys}})_{\pm} \sim \pm \Theta^* \), but remember that \( \Theta_4^* = \Theta_4 \), then we can use \( \Theta_4^* \) instead of \( \Theta_4^* \). However, \( \Theta_4 \)’s are diagonal (or can be diagonalized because of the topology), and therefore we can use \( \Theta_4 \). Note also that these relations are not so strict. Indeed the periodicity of the exponential factor must be taken into account.
Table 1: Hypothetical transformations that could be identified with an effective CP symmetry in 4D if compatible with boundary conditions (BC).

| $R_4 \neq R_5$ | $\theta$ | $\beta_4$ | $\beta_5$ | $U\Theta_4U^{-1}$ | $U\Theta_5U^{-1}$ |
|----------------|----------|-----------|-----------|-------------------|-------------------|
|                | 0        | $[0, \pi]$ | $[0, 2\pi]$ | $-\Theta_4 + \frac{2\pi k}{N}T$ | $\Theta_5 + \frac{2\pi k}{N}T$ | (1) |
|          | $\pi$    | $[0, 2\pi]$ | $[0, \pi]$ | $\Theta_4 + \frac{2\pi k}{N}T$ | $-\Theta_5 + \frac{2\pi k}{N}T$ | (2) |
|          | $\pi/2$  | $-\beta_5$ | $-\beta_4$ | $-\Theta_4 + \frac{2\pi k}{N}T$ | $-\Theta_5 + \frac{2\pi k}{N}T$ | (3) |

Before closing the section, we would like to come back shortly on the 5D case studied in previous works \[1\] \[5\]. We will not return here to the already mentioned issue of WL dynamics. Here we would like to compare the approaches of investigating CP violation in compactification. In these pioneer works, the attention was entirely drawn on low scale sector (KK modes were neglected) and real 5D mass terms were introduced to complement the "pseudoscalar" terms coming from the 5th component of the vectors, leading to CP violation through the phases in the low-energy mass matrices. However, if we extend the analysis of section 2 to 5D, we can show that any compactification from 5D to 4D leads to a CP breaking effective theory. Therefore it seems that we don’t need to include 5D masses to generate CP violation in 4D: the KK modes should account for the breaking. However, the EDM remains the simplest observable, and since an EDM is also a P-violating, we have no hope to see any if compactification is P preserving. To break P, an "orientation" of the 4D world must be selected, e.g. with a kink \[4\] or with orbifolding \[5\].

5 Examples with SU(2)

For the next examples, we will work with one of the simplest groups, i.e. SU(2). In the two first examples the matter content consists in a fermion in the fundamental (resp. the adjoint) representation. In SU(2) there are two independent dynamical variables called $\theta_4$ and $\theta_5$ such that $\Theta_a = \begin{pmatrix} \theta_a & 0 \\ 0 & -\theta_a \end{pmatrix}$.

The effective potential can be decomposed into \[11\]:

$$V_{\text{eff}} = V \left( -V^{g+h}_{\text{eff}} + \sum_i 2V^i_{\text{eff}} + \sum_i 2V^{\text{adj}}_{\text{eff}} \right),$$

where $V$ is a positive constant, $V^{g+h}_{\text{eff}}$ the contribution from gauge and ghost fields, $V^i_{\text{eff}}$ the contribution from fundamental fermions and $V^{\text{adj}}_{\text{eff}}$ the one from adjoint fermions. Each contribution can be written as an infinite sum over fields modes. It worth noting that this expression is only valid for Dirac spinors, and

\[20\] We study the lightest mode since we look for an understanding of CP violation at low energy. However a zero EDM for this state doesn’t mean that CP is conserved (and that our previous analysis fails), as it may manifest itself at higher energy. Remember also that an EDM violates both P and CP. It is however easy to check that, with this mechanism, the 4D P symmetry is broken as soon as the CP one is.
not Weyl spinors. From now on, we will use it nonetheless, and postpone the justification to the next section.

The potential must be studied numerically. The results for a theory with only fundamental fermions are simple and given in Table 22:

| $\beta_4 \in [0, \pi/2]$ | $\beta_5 \in [\pi/2, \pi]$ | $\beta_6 \in [\pi, 3\pi/2]$ | $\beta_7 \in [3\pi/2, 2\pi]$ |
|--------------------------|--------------------------|--------------------------|--------------------------|
| $R_4 \beta_5$           | $R_5 \beta_6$           | $R_6 \beta_7$           | $R_7 \beta_8$           |
| $\theta = 0$            | $\theta = 0$            | $\theta = 0$            | $\theta = 0$            |

Table 2: Wilson line (WL) phases for a SU(2) theory with a 6D spinor in the fundamental representation.

According to [11], this result is valid for $R_4 = R_5$, but our study shows that this remains exact even for $R_4 \neq R_5$. To be more precise, the potential shape depends only on $r = R_5/R_4$. When $r > 1$, the potential flattens in the $y_3$ direction, but the global minimum stays unchanged at least for $r \lesssim 5$. Beyond, an other local minimum becomes very close to the global one and it is hard to select the right one with numerical calculations. Nevertheless, the two candidates lead to the same phenomenological issues that we will describe here. First, it’s worth noting that $\theta = 0$ and $\theta = \pi$ are particular values since then $-\theta = \theta$, $\exp[i\theta_{a}] = \pm 1$ and the gauge symmetry remains unbroken because all SU(2) generators commute with transition functions $T_{i}$. However CP symmetry can still be broken because of the SchSch phases or $R_4 \neq R_5$ (see Table 1), but another big issue is the absence of a light fermion, even with $\beta$’s tuned to be small. Indeed, when $\beta$’s are small, the WL are large and vice versa. More precisely one can show that the smallest “distance” between $(\beta + \theta)/2\pi$ and an integer is 0.25. Then the fermion masses are bounded from below (with $R_4 = R_5 = R$) $m_f > \sqrt{2/4R} \sim 0.35/R$ and there is a poor gap between the lightest mode and the KK tower.

Let us focus now on a more interesting example. Richer phenomenology can be reached if we replace the fundamental fermion by an adjoint. We will not try to give an exhaustive study of the effective potential in this case. Refs and personal analysis show that, at least in the interesting regime $\beta_4, \beta_5 \in [0, 0.1]$ and $0.9 < r = R_5/R_4 < 1$, $(\theta_4, \theta_5) = (\pi/2, \pi/2)$. This is interesting because this time the SU(2) symmetry is spontaneously broken into U(1), and after this breaking we have a neutral fermion with mass $\sim \beta/\sqrt{2\pi R}$ which can be choose to be small. Moreover Table 1 tell us that CP can be broken with non zero $\beta$’s. If $r = 1$, $\beta_4$ must be different from $\beta_5$, but if $r \neq 1$ this is not even necessary. We will verify these affirmations with the EDM of our light fermion. Details about particle content and effective interactions can be found in the Appendix B. The EDM is given by [22]:

$$
\frac{dE}{e} = \sum_{nm} \left\{ F_{nm}^+ \sin(\varphi_{3,0} - \varphi_{+nm}) + J_{nm}^+ \sin \varphi_{3,0} + K_{nm}^+ \cos \varphi_{3,0} \right\} 
+ \sum_{nm} \left\{ F_{nm}^- \sin(\varphi_{3,0} - \varphi_{-nm}) + J_{nm}^- \sin \varphi_{3,0} + K_{nm}^- \cos \varphi_{3,0} \right\}
$$

(6)

where the coefficients $F$, $J$ and $K$ and the phases $\varphi_{3,0}$ and $\varphi_{nm}$ are functions of the $\vartheta$’s, the $\beta$’s and $r$. Their explicit form can be found in the Appendix C. When $r = 1$ and $\beta_4 = \beta_5$, it is easy to check that (see Appendix C):

$$
F_{nm}^\pm = F_{nm}^\pm; \quad J_{nm}^\pm = -K_{mn}^\pm; \quad \varphi_{\pm nm} = \frac{\pi}{2} - \varphi_{\pm nm}; \quad \varphi_{3,0} = \frac{\pi}{4}(3 - 2 \text{sign}(\beta)).
$$

(7)

Therefore:

$$
\left| \frac{dE}{e} \right| \sim (\sin \varphi_{3,0} - \cos \varphi_{3,0}) = 0.
$$

This is no more true when $r \neq 1$ or $\beta_4 \neq \beta_5$. We will illustrate this with numerical evaluations. Our results can be found in Table 23. We use the notation $\beta = \beta_4, \Delta \beta = \beta_4 - \beta_5, \Delta r = 1 - r$.

The behaviour of the lightest mass is easily predicted. Indeed (see Appendix C) we have $m_{\text{light}}R \simeq \beta(1 + \Delta \beta/\beta + \Delta r)$, and its order of magnitude is directly related to $\beta$. On the other hand, the behaviour

21The case $r < 1$ is completely symmetric.

22We normalize to the scale R and the coupling constant $c$ of the SU(2) gauge interaction.
of the EDM is less intuitive from the analytic solutions, because of the summations and integrations in its expression. Nevertheless, we could expect a behaviour of the type:

\[ \left| \frac{d\mathcal{E}R}{\epsilon^3} \right| \approx C \cdot \left( \Delta r + \kappa \frac{\Delta \beta}{\beta} \right), \]

where \( C \) and \( \kappa \) are (almost) constant factors. Numerical evaluations show this is the case (with a pretty good accuracy) with \( C \sim 10^{-2} \) and \( \kappa \sim 4.5 \). Obviously, the dominant CP source (\( \Delta r \) or \( \Delta \beta/\beta \)) dictates the order of magnitude for the EDM.

## 6 Chiral anomaly in 6D

Gauge theories in more than 4 space-time dimensions are not renormalizable and it could then seem dangerous to consider quantum corrections (in the effective potential for instance) in this context. However, the WL-dependent part of \( V_{\text{eff}} \) turns out finite, at least at the one loop level, and can then be evaluated unambiguously \[12\]. For chiral theory, however, things become worse, because of the presence of anomalies. In 6D, they come from the square diagram \( \square \) (equivalent to triangle diagram in 4D) which certainly plays a role in the effective potential.

Concretely, anomalies originate from UV divergences, but they are finite and calculable IR effects (even in more than 4D) which then do not depend on the UV completion of the theory \[13\] \[14\]. For non renormalizable theories, which are only valid under a certain energy scale, anomalies can cancel among themselves (like in 4D), but they can also cancel with effects originating from an unknown UV sector. It is not our point to discuss these issues here, and we will avoid them with the introduction of both 6D chiralities (+ and −) in the same representation. However, we will use the BC to differentiate the masses of the light excitations of these fields. As announced, this will lead to small modifications in the effective potential, but our previous results will remain intact.

The only modification of \( V_{\text{eff}} \) appears in the fermion contributions. We must do the replacements:

\[
2V^l_{\text{eff}}(\beta_4, \beta_5) \quad \rightarrow \quad V^l_{\text{eff}}(\beta_{4+}, \beta_{5+}) + V^l_{\text{eff}}(\beta_{4-}, \beta_{5-})
\]

\[
2V^{\text{ad}}_{\text{eff}}(\beta_4, \beta_5) \quad \rightarrow \quad V^{\text{ad}}_{\text{eff}}(\beta_{4+}, \beta_{5+}) + V^{\text{ad}}_{\text{eff}}(\beta_{4-}, \beta_{5-}).
\]

We have checked numerically that, in the range of SchSch phases we work with, this keeps the minimum of the total effective potential at \( (\theta_4, \theta_5) = (\pi/2, \pi/2) \). This is why we used it in section \[15\] even if at that

\[23\]In non abelian theories, there exists other pathological diagrams, but they can be related to this one through gauge invariance.
time, we had only introduced one chirality. It is easy to verify that in the case of degenerate SchSch phases for $+\,$ and $−\,$ chiralities, the EDMs are exactly opposite for the two sectors, thus restoring CP ("CP doubling"). To be more precise, the lightest modes could still be distinguished through their different couplings, but we prefer to provide a case where CP violation is explicit in terms of low energy parametrization. This is easily obtained if we choose different SchSch phases for $+$ and $−\,$ chiralities. As a simple class of examples, let us take $\Delta r = 0$, $\Delta \beta_− = 0$ and $\Delta \beta_+ \neq 0$. In this way, in the $−\,$ sector $d_E = 0$ while in the $+$ sector $d_E \neq 0$.

7 Conclusion and perspectives

We made use of the Hosotani mechanism to generate both gauge and CP symmetry breaking through compactification from a 6-dimensional model. Though we found examples where it works, our solutions is far from being realistic, and they must be seen more as "proof of concept". One of the major difficulty of the work is the high level of entanglement in the approach. Indeed, the final result depends both on matter content (representations), BC (SchSch phases) and WL phases, while the latter depend in turn on the formers and are dynamically determined through a potential which must be numerically evaluated.

The next steps in this program should be the resolution of the two main drawbacks of the present solutions. First new compactification mechanism (like orbifold or flux compactification) might be employed to reach a chiral theory in 4D (at this point the only difference between left and right couplings in the gauge sector comes through a phase). Moreover, we’d like to avoid the presence of two (nearly) identical fermionic sectors without introducing anomalies in the theory. Secondly (but this maybe even more ambitious), a mechanism which produces a low energy sector naturally separated from the Kaluza-Klein scale would be very welcome. For instance, in more complex situations, one can hope for an effective low energy potential between the remaining scalars, what would provide the lower mass scale, but this goes beyond this "proof of concept" paper.

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A Dirac matrices in 6D

We use the following representation for the Dirac matrices in 6D:

\[ \Gamma^A = \begin{pmatrix} 0 & \Sigma^A \\ \Sigma^A & 0 \end{pmatrix} \quad \text{and} \quad \Gamma^7 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]

where $\Sigma^\mu = \gamma^0 \gamma^\mu$, $\Sigma^4 = i \gamma^0 \gamma^5$, $\Sigma^5 = \gamma^0$ and $\Sigma^0 = \gamma^0$, $\Sigma^A \neq 0 = -\Sigma^A \neq 0$. The $\gamma$’s are 4D Dirac matrices:

\[ \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix} \quad \text{and} \quad \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]

where $\sigma^\mu = (1, \sigma_i)$ and $\bar{\sigma}^\mu = (1, -\sigma_i)$.

B Effective 4D theory for an SU(2) adjoint fermion

Not considering here the anomalies, we work only with a 6D Weyl fermion $\Psi$ in the adjoint representation of SU(2) and a gauge field $A_A$. These fields can be decomposed in the Cartan basis $\{T_+, T_-, T_3\}$ which satisfies $[T_+, T_-] = T_3$ and $[T_2, T_3] = \pm T_3$. The 6D lagrangian can be written

\[ \mathcal{L} = -\frac{1}{2} \text{Tr}[F_{AB} F^{AB}] + 2 \text{Tr}[i \bar{\Psi} \Sigma^A D_A \Psi], \]

\[ \text{Note by the way that the same conclusion holds for fundamental representation and our previous results stay qualitatively identical. There is no way to force } \theta \text{ values to be different from 0 or } \pi \text{ which lead to unbroken gauge symmetry. Moreover, the minimum always arranges to prevent small masses in the spectrum.} \]

\[ \text{See for instance interesting application for gauge symmetry breaking in} \]
with $F_{AB} = \partial_A A_B - \partial_B A_A - i e [A_A, A_B]$, $D_A = \partial_A - i e [A_A, \cdot]$ the covariant derivative and $\Sigma^A = \gamma^0 \cdot \{ \gamma^0, -\gamma^i, -i \gamma^5, -1 \}$. If we define $\psi = \gamma^0 \Psi$ we can write the fermionic part of the lagrangian in the following form:

$$
\mathcal{L} \supset i \bar{\psi} \gamma^\mu \partial_\mu \psi + i \bar{\psi} \gamma^\mu \partial_\mu \psi + i \bar{\psi} \gamma^\mu \partial_\mu \psi + i \bar{\psi} \gamma^\mu \partial_\mu \psi
$$

To get real (and positive) masses, we perform a chiral rotation by:

$$
A
$$

The real masses are then:

$$
\psi
$$

To find the 4D effective lagrangian we need to decompose $\psi$ and $A_A$ into fundamental modes which satisfy BC. For an adjoint fermion these are given by:

$$
\psi(y + 2\pi R_i) = e^{i \beta_i} e^{i \theta_i T_3} \psi(y) e^{-i \theta_i T_3},
$$

or, in the Cartan basis:

$$
\begin{align*}
\psi_3(y + 2\pi R_i) &= e^{i \beta_3} \psi_3(y) \\
\psi_\pm(y + 2\pi R_i) &= e^{i (\beta_\pm + \theta_\pm)} \psi_\pm(y).
\end{align*}
$$

Therefore the (normalized) mode decompositions are:

$$
\begin{align*}
\psi_3(y) &= \frac{1}{2\pi \sqrt{R_4 R_5}} \sum_{nm} e^{i \left( \frac{m}{2\pi} \right)} e^{i \left( \frac{n \pm \theta_3}{2\pi} \right)} \psi_3_{nm} \\
\psi_\pm(y) &= \frac{1}{2\pi \sqrt{R_4 R_5}} \sum_{nm} e^{i \left( \frac{m + \beta_\pm + \theta_\pm}{2\pi} \right)} e^{i \left( \frac{n \pm \theta_3}{2\pi} \right)} \psi_\pm_{nm}.
\end{align*}
$$

The decompositions for $A_A$ are obtained with $\beta_3 = \beta_5 = 0$.

Let us introduce these decompositions in (8). The first line gives the kinetic energy for each mode. The second line gives the effective 4D masses:

$$
\begin{align*}
m_{3,nm} &= -\frac{1}{R} \left[ \left( m + \frac{\beta_3}{2\pi} \right) - i \gamma^5 \right] \left( n + \frac{\beta_4}{2\pi} \right) \\
m_{\pm,nm} &= -\frac{1}{R} \left[ \left( m + \frac{\beta_\pm + \theta_\pm}{2\pi} \right) - i \gamma^5 \right] \left( n + \frac{\beta_4}{2\pi} \right).
\end{align*}
$$

To get real (and positive) masses, we perform a chiral rotation $\psi \rightarrow e^{i \frac{\theta_3}{2} \gamma^5} \psi$, where the phases are given by:

$$
\begin{align*}
\exp[i \varphi_{3,nm}] &= -\frac{(m + \frac{\beta_3}{2\pi}) + ir \left( n + \frac{\beta_4}{2\pi} \right)}{\sqrt{(m + \frac{\beta_3}{2\pi})^2 + r^2 \left( n + \frac{\beta_4}{2\pi} \right)^2}} \\
\exp[i \varphi_{\pm,nm}] &= -\frac{(m + \frac{\beta_\pm + \theta_\pm}{2\pi}) + ir \left( n + \frac{\beta_4}{2\pi} \right)}{\sqrt{(m + \frac{\beta_\pm + \theta_\pm}{2\pi})^2 + r^2 \left( n + \frac{\beta_4}{2\pi} \right)^2}}.
\end{align*}
$$

The real masses are then:

$$
\begin{align*}
m_{3,nm} &= \frac{1}{R} \sqrt{(m + \frac{\beta_3}{2\pi})^2 + r^2 \left( n + \frac{\beta_4}{2\pi} \right)^2} \\
m_{\pm,nm} &= \frac{1}{R} \sqrt{(m + \frac{\beta_\pm + \theta_\pm}{2\pi})^2 + r^2 \left( n + \frac{\beta_4}{2\pi} \right)^2}.
\end{align*}
$$

Note that the two last relations, valid for $\beta_4 = \beta_5$ and $r = 1$ can be easily proven here with the definitions. If we remind that the effective potential imposes $\theta_4 = \theta_5$, we see that the exchange of $n$ and $m$ in these relations is equivalent to the exchange of real and imaginary part of the phases, what
means $\varphi_{k,n}=\pi/2-\varphi_{k,n}$. Finally $\varphi_{3,0}=\pm\pi/2$, since in this case it has its real and imaginary parts equal. The sign is determined by the sign of $\beta$, and the solution can be written synthetically as $\varphi_{3,0}=\frac{\pi}{2}(3-2\text{sign}(\beta))$.

The third line in [4] gives the effective interactions with 4D vector bosons, while the fourth and fifth ones give the interactions with 4D scalars bosons. To get an interesting form we need to perform the chiral rotation, but also to go in the mass eigenbasis for the bosons. To study this let us have a look the the quadratic part of the gauge lagrangian:

$$\mathcal{L} \supset -\frac{1}{4}\left(\partial_{\mu}A_{3,\nu} - \partial_{\nu}A_{3,\mu}\right)^2 + \frac{1}{2}\left(\partial_{\mu}A_{3;4}\right)^2 + \frac{1}{2}\left(\partial_{\mu}A_{3;5}\right)^2 + \frac{1}{2}\left(\partial_{\mu}A_{3;6}\right)^2 + \frac{1}{2}\left(\partial_{\mu}A_{3;7}\right)^2$$

From BC we can convert $\partial_4, \partial_5$ into mass matrices for the vector and scalar bosons. The vector bosons $A_{3,\mu,nm}$ (resp. $A_{+;\mu,nm}$) have masses $M_{3,\mu}$ (resp. $M_{+;\mu}$) given by:

$$M_{3,\mu} = \frac{1}{R} \sqrt{m^2 + r^2 n^2}$$

$$M_{+;\mu} = \frac{1}{R} \sqrt{\left(m + \frac{\theta_4}{2\pi}\right)^2 + r^2 \left(n + \frac{\theta_4}{2\pi}\right)^2}$$

One of the bosons in the spectrum ($A_{3,\mu,00}$) remains massless as expected by the symmetry breaking pattern. On the other hand $A_{+;\mu,00}$ acquires a mass through the Hosotani mechanism. It is worth noting that, except for $A_{3,\mu,00}$, all the 4D vector bosons can be expressed in terms of complex fields.

In the scalar sector, there is a mixing between $A_4$ and $A_5$. The mass matrices are given by:

$$\begin{pmatrix} A_{3;4,nm} & A_{3;5,nm} \\ A_{3;6,nm} & A_{3;7,nm} \end{pmatrix} \left[ \begin{array}{c} A_{3;4,nm} \\ A_{3;5,nm} \end{array} \right] = \frac{1}{R^2} \begin{pmatrix} m^2 & -r nm \\ -r nm & r^2 n^2 \end{pmatrix} \begin{pmatrix} A_{3;4,nm} \\ A_{3;5,nm} \end{pmatrix}$$

$$\begin{pmatrix} A_{+;4,nm} & A_{+;5,nm} \\ A_{+;6,nm} & A_{+;7,nm} \end{pmatrix} \left[ \begin{array}{c} A_{+;4,nm} \\ A_{+;5,nm} \end{array} \right] = \frac{1}{R^2} \begin{pmatrix} m + \frac{\theta_4}{2\pi} & -r \left(n + \frac{\theta_4}{2\pi}\right) \\ -r \left(n + \frac{\theta_4}{2\pi}\right) & m + \frac{\theta_4}{2\pi} \end{pmatrix} \begin{pmatrix} A_{+;4,nm} \\ A_{+;5,nm} \end{pmatrix}$$

The mass eigenstates $g_{3,\mu,nm}$ and $g_{+;\mu,nm}$ are massless, while $h_{3,\mu,nm}$ and $h_{+;\mu,nm}$ have masses $M_{3,\mu,nm}$ and $M_{+;\mu,nm}$. They are given by:

$$g_{3,\mu,nm} = \frac{m A_{3,\mu,nn} + r n A_{3;4,nn}}{\sqrt{m^2 + r^2 n^2}}$$

$$h_{3,\mu,nm} = \frac{m A_{3,\mu,nn} - r n A_{3;5,nn}}{\sqrt{m^2 + r^2 n^2}}$$

$$g_{+;\mu,nm} = \frac{m + \frac{\theta_4}{2\pi}}{\sqrt{m^2 + r^2 n^2}} A_{+;5,nn} + r \left(n + \frac{\theta_4}{2\pi}\right) A_{+;4,nn}$$

$$h_{+;\mu,nm} = \frac{m + \frac{\theta_4}{2\pi}}{\sqrt{m^2 + r^2 n^2}} A_{+;4,nn} - r \left(n + \frac{\theta_4}{2\pi}\right) A_{+;5,nn}$$

If we perform a rotation toward the mass eigenbasis in [10], we find that $g$ scalar bosons play the role of goldstone bosons. They are eaten by the vector boson which acquire masses. The only physical goldstone boson is $g_{3,00}$. Actually, $h_{3,00}$ is massless too and the effective theory contains two massless scalar degrees of freedom, what could be a drawback.

We can now find all the interaction term in the right basis. In addition to the fermion-fermion-vector and fermion-fermion-scalar interactions, we have still a bunch of vector-scalar interactions implying 3 or 4 particles. We will not write all of them but focus ourselves on the one participating in the one loop diagrams for the EDM. These are the 3 particles interactions with at least one $A_{3,\mu,00}$ boson (the external "photon"). They come from the following part of the 6D lagrangian:

$$\mathcal{L} \supset ie A_{3;4,\mu} A_{+;5,\mu} A_{+;4,\mu} A_{+;5,\mu} A_{+;4,\mu} A_{+;5,\mu} A_{+;4,\mu} A_{+;5,\mu} A_{+;4,\mu} + h.c.$$
Let us now introduce the mode decompositions (with only the mode (00) for $A_{3;\mu}$) to yield:

$$
\mathcal{L} \supset i e A_{\nu;\mu}^\mu A_{\nu;\mu}^\nu \partial_\nu A_{3;\mu} + i e (\partial_\nu A_{\nu;\mu} - \partial_\mu A_{\nu;\mu}) A_{\nu;\mu}^\mu + e M_{\nu;\mu} g_{\nu;\mu} A_{3;\mu}^\mu + i e g_{\nu;\mu} \partial_\nu g_{\nu;\mu} A_{3;\mu}^\mu + i e h_{\nu;\mu} \partial_\nu h_{\nu;\mu} A_{3;\mu}^\mu + h.c.
$$

We give all the corresponding diagrams below (Figures 1 to 7). The additional diagrams are for the fermion-fermion-vector or fermion-fermion-scalar interactions implying at least one $A_{3;\mu,00}$ boson or a $\psi_{3;00}$ fermion. We use simplified notations: $\psi_3 = \psi_{3;00}$, $A_+ = A_{+;\mu,00}$, $g_+ = g_{+;\mu,00}$, $h_+ = h_{+;\mu,00}$, $M_+ = M_{+;\mu,00}$, $\varphi = \varphi_{3;00}$, $\varphi_\pm = \varphi_{\pm;\mu,00}$, and $\varphi_0 = \varphi_{0;\pm;\mu,00} (\beta_4 = \beta_5 = 0)$. Finally, all the wiggled lines without label are $A_{3;\mu,00}$ "photons". Note that charge conservation combined with momentum conservation in ED imposes $A_{+;\mu,00}$ interacts with $\psi_{3;00}$ and either $\psi_{3;\mu,00}$ or $\psi_{-;\mu,00}$.

![Figure 1](image1)

**Figure 1:** $e(p_\mu g_{\mu\rho} + p_\rho g_{\mu\nu} - 2p_\mu g_{\mu\rho} + k_\mu g_{\rho\nu} + k_\rho g_{\mu\nu} - 2k_\rho g_{\mu\rho})$.

![Figure 2](image2)

**Figure 2:** (a) $= e(2p_\mu - k_\mu)$; (b) and (c) $= eM_+$.

![Figure 3](image3)

**Figure 3:** (a) $= -e \exp \left( -i(\varphi - \varphi_+) \gamma^5/2 \right) \gamma_\mu$; (b) $= -e \exp \left[ i(\varphi - \varphi_+) \gamma^5/2 \right] \gamma_\mu$.

## C Electric dipole moment of $\psi_{3;00}$

Six kinds of diagram are involved in the one loop evaluation of the EDM for $\psi_{3;00}$. We show them for a $\psi_{3;\mu,00}$ in the loop in Figure 8. For the $\psi_{-;\mu,00}$, the fields $A_+, g_+$ and $h_+$ must be replaced by complex conjugate fields (or the arrows reversed). At the end we must sum up the $+$ and $-$ contributions and sum over all $nm$ modes.

The contributions to $F_{nm}$ come from diagrams $8d$ and $8f$. The diagrams $8a$ and $8b$ give no contributions. Finally the contributions to $J_{nm}$ and $K_{nm}$ come from diagrams $8c$ and $8d$.
The only difference between + and − that can be "contracted" with covariant derivative. For + these are $\bar{\Sigma}^B$, and for − $D^+$ difference appears because $\Psi^+$.

Thus, the only remaining difference, is a sign in the fifth component of the covariant derivative $m\pm$.

It is now easy to check the two first relations (7). To compute $F^\pm_{nm}$, we must exchange $n$ and $m$ in all the masses $m_{\pm;\pm\pm\pm\pm}$ and $M_{\pm;\pm\pm\pm\pm}$. Since $\theta_1 = \theta_5$ (imposed by the effective potential), and $\beta_4 = \beta_5$ (imposed by hand as an hypothesis), these stay unchanged and $F$ as well. In the same way $J^\pm_{nm}$ and $K^\pm_{nm}$ do not change through $\tilde{\Delta}_{\pm;\pm\pm\pm\pm}$ but only through the phase $\varphi^0_{\pm}$. The expressions (8) with $\theta_1 = \theta_5$ ($\beta_4 = \beta_5 = 0$ by definition) show that exchanging $n$ and $m$ is equivalent to exchanging real and imaginary part, or $\cos \varphi^\pm_{\pm}$ and $\sin \varphi^\pm_{\pm}$. We then conclude easily that $J^\pm_{nm} = -K^\pm_{nm}$.

\section*{D and − chirality sector}

The only difference between + and − chirality lagrangians is the matrices used to form a 6D vectors that can be "contracted" with covariant derivative. For + these are $\Sigma^A$ matrices defined in Appendix B and for − these are $\Sigma^{\tilde{A}}$ matrices, defined as $\Sigma^0 = \Sigma^0$ and $\Sigma^{A\neq0} = -\Sigma^{A\neq0}$. At the 4D level, an other difference appears because $\Psi_+ \sim \left(\bar{\psi}_R, \psi_L\right)$, while $\Psi_- \sim \left(\bar{\psi}_L, \psi_R\right)$. To form the usual Dirac kinetic terms, we have to rewrite the lagrangians in terms of the matrices $\Sigma^A{\gamma}^0 = (\gamma^\mu, i\gamma^5, -1)$ and $\gamma^0\Sigma^A = (\gamma^\mu, i\gamma^5, 1)$. Thus, the only remaining difference, is a sign in the fifth component of the covariant derivative $D_5$. This doesn’t change the mass spectrum, but only the chiral phases and the interactions with $A_5$ bosons (see Appendix B for more details). This is equivalent to change sign of $m$, $\beta_5$ and $\theta_5$ in (8), sign of $\varphi^0_{\pm}$ and $e$ in diagrams of Figure 6 (interaction with $g_5$ bosons) and sign of $\varphi^0_{\pm}$ in diagrams of Figure 7 (interaction with $h_5$ boson). Now we see that $F^\pm_{nm}$ and $J^\pm_{nm}$ don’t change, while $K^\pm_{nm}$ changes sign. But we must not forget that $J^\pm_{nm}$ and $K^\pm_{nm}$ come from interaction with one $g_5$ boson (see Appendix B), then they undergo
Figure 6: (a) and (b) $= -e \exp \left[ i \left( \frac{\varphi_+ + \varphi_- - 2\varphi_0^n}{2} \right) \gamma^5 \right]$; (c) and (d) $= -e \exp \left[ i \left( \frac{\varphi_+ - \varphi_- - 2\varphi_0^n}{2} \right) \gamma^5 \right]$.

Figure 7: (a) and (b) $= e \exp \left[ i \left( \frac{\varphi_+ + \varphi_- - 2\varphi_0^n + \pi}{2} \right) \gamma^5 \right]$; (c) and (d) $= e \exp \left[ i \left( \frac{\varphi_+ + \varphi_- + 2\varphi_0^n + \pi}{2} \right) \gamma^5 \right]$.

an additional change of sign. We have then the following transformations:

$$
P_{nm}^\pm \sin(\varphi_{3,00} - \varphi_{\pm;\pm n \pm m}) \rightarrow P_{nm}^\pm (-\sin(\varphi_{3,00} - \varphi_{\pm;\pm n \pm m}))
$$

$$
J_{nm}^\pm \sin \varphi_{3,00} \rightarrow (-J_{nm}^\pm) \sin \varphi_{3,00}
$$

$$
K_{nm}^\pm \cos \varphi_{3,00} \rightarrow K_{nm}^\pm (-\cos \varphi_{3,00})
$$

which lead to the conclusion that $d_E$ changes sign (see (6)).

Therefore, if we choose the same SchSch phases for $+$ and $-$ chiralities, we end up with two fermionic sectors (let us call them $P$- and $M$-sectors, which interact only through gauge and scalar interactions) with exactly the same mass spectra, and with an equal and opposite EDM for the two lightest modes.
Figure 8: One-loop contributions to EDM for $\psi_{3,00}$. Contributions with $\psi_{-n-m}$ in the loop must be included as well.
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