Conditional Distribution of Independent Brownian
Motions to Event of Coalescing Paths

Vitalii Konarovskyi* †  Victor Marx‡

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Abstract
We consider a family of independent real-valued Brownian motions starting at distinct points and are interested in the description of the conditional distribution of this family to the event that trajectories coalesce, which is of probability zero. We develop a general approach to construct a conditional probability to events of measure zero and apply it first when the above-mentioned family is finite and then when the family is infinite. In both cases, it leads us to the distribution of a modified massive Arratia flow [Kon17b] as the conditional distribution.

Keywords: Regular conditional probability, modified massive Arratia flow, cylindrical Wiener process, coalescing Brownian motions

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1 Introduction

1.1 A starting problem

The aim of this paper is to answer the following question. What is the conditional distribution of a finite or infinite family of independent Brown-
ian motions in $\mathbb{R}$ to the event that their paths coalesce? To introduce the problem, we start our discussion from the simplest case of two independent Brownian motions $X_1(t)$, $X_2(t)$, $t \geq 0$, with diffusion rates 2 which start at points $x_0^1 < x_0^2$ respectively. Define the set consisting of coalescing paths

$$\text{Coal} = \left\{ (x_1, x_2) \in C([0, \infty), \mathbb{R})^2 : \forall s \geq 0, \ x_1(s) = x_2(s) \right. \left. \implies x_1(t) = x_2(t) \text{ for all } t \geq s \right\}.$$ 

For $X := (X_1, X_2)$ we first wonder what the conditional distribution

$$\mathbb{P}[X \in \cdot | X \in \text{Coal}]$$

is. Trivially, $\mathbb{P}[X \in \text{Coal}] = 0$ and consequently, conditional distribution (1) cannot be well-defined considering $\{X \in \text{Coal}\}$ as an isolated event (see e.g. Borel-Kolmogorov paradox). However, one can make the definition rigorous considering the set $\text{Coal}$ as one fiber of some appropriate map $T$ and disintegrating the law of the Brownian motion $X$ with respect to the random element $T(X)$.

It turns out that under some choice of $T$, the conditional distribution $\mathbb{P}[X \in \cdot | X \in \text{Coal}]$ coincides with the distribution of the process $Y(t) = (Y_1(t), Y_2(t))$, $t \geq 0$, that is a strong solution to the following equation

$$\begin{aligned}
&dY_1(t) = \mathbb{I}_{\{t<\tau\}} dW_1(t) + \mathbb{I}_{\{t\geq\tau\}} \frac{dW_1(t)+W_2(t)}{2}, \\
&dY_2(t) = \mathbb{I}_{\{t<\tau\}} dW_2(t) + \mathbb{I}_{\{t\geq\tau\}} \frac{dW_1(t)+W_2(t)}{2}, \\
&Y_1(0) = x_0^1, \quad Y_2(0) = x_0^2,
\end{aligned}$$

where $\tau = \inf \{t \geq 0 : Y_1(t) = Y_2(t)\}$, and $W_1(t)$, $W_2(t)$, $t \geq 0$, are independent Brownian motions in $\mathbb{R}$ with diffusion rate 2. In words, $Y_1$ and $Y_2$ describe the evolution of two Brownian particles starting at $x_0^1$ and $x_0^2$, respectively, and moving independently with diffusion rate 2 until they meet at time $\tau$. After collision, both particles coalesce into a single Brownian particle with diffusion rate 1.

In order to justify this, we define the map

$$T : C[0, \infty)^2 \to C_0[0, \infty)$$

as follows

$$T(x)(t) = \begin{cases} x_1(\tau + t) - x_2(\tau + t), & \text{if } \tau < \infty, \\
0, & \text{if } \tau = \infty, \end{cases}$$

where $\tau = \inf \{t \geq 0 : x_1(t) = x_2(t)\}$ and $C[0, \infty)$ (resp. $C_0[0, \infty)$) denotes the space of continuous real-valued functions on $[0, \infty)$ (resp. taking zero value at 0). It is easily seen that $T^{-1} (\{0\}) = \text{Coal}$ and $T(X)$ is a standard Brownian motion.
1.2 Description of the main results of the paper

There exists a regular conditional probability\footnote{We will recall the definition of this notion in Definition 2.1 below.} of $X$ given $T(X)$:

$$p(A, z) = \mathbb{P}[X \in A | T(X) = z], \quad A \in \mathcal{B}(C[0, \infty)^2), \quad z \in C_0[0, \infty).$$

Consequently, we could define

$$\mathbb{P}[X \in A | X \in \text{Coal}] = \mathbb{P}[X \in A | T(X) = 0] = p(A, 0). \quad (4)$$

However, the problem is that the regular conditional probability $p$ is uniquely defined for almost all $z$ with respect to the distribution of $T(X)$. So, the value of $p$ at a fixed point $z_0$ depends on the choice of a version of $p$. Hence, to define $p$ at $z_0$ we either need to choose a special version of $p$ or make a choice of $\nu$ independently of the version of $p$. It can be easily done for $z_0$ belonging to the support of $T(X)$ if the map $z \mapsto p(\cdot, z)$ has a continuous version at point $z_0$, as a map from $C_0[0, \infty)$ to the space of probability measures on $C[0, \infty)^2$ furnished with the topology of weak convergence, whose value at $z_0$ can be taken as $p(\cdot, z_0)$.

To construct a continuous version of $p$, one can define $\Psi : \text{Coal} \times C_0[0, \infty) \to C[0, \infty)^2$

$$\Psi(y, z)(t) = \begin{cases} y_1(t) - z(t - \tau') \mathbb{1}_{\{t \geq \tau\}} & t \geq 0, \\ y_2(t) + z(t - \tau') \mathbb{1}_{\{t > \tau\}} & \end{cases}$$

where $\tau' := \inf\{t \geq 0 : y_1(t) = y_2(t)\}$ and then prove that $p(A, z) = \mathbb{P}[\Psi(Y, z) \in A]$ defines a continuous regular conditional probability of $X$ given $T(X)$. Moreover,

$$p(A, 0) = \mathbb{P}[\Psi(Y, 0) \in A] = \mathbb{P}[Y \in A],$$

which together with (4) shows that conditional distribution (1) coincides with the law of $Y$.

1.2 Description of the main results of the paper

The above example with two particles will be rigorously proved and extended, first to the case of finitely many initial particles and second to the case of infinitely many initial particles. All notions mentioned in this paragraph will be precisely defined in Sections 2 and 3.

Our goal is to show that the conditional distribution of a family of independent real-valued Brownian motions to the event of coalescing trajectories leads to the law of a modified massive Arratia flow (MMAF), investigated
in [Kon10, Kon14, Kon17b, Kon17a, Mar18, Kvr19] and which can be intuitively described as follows. Every particle has a mass and moves as a Brownian particle with diffusion rate inversely proportional to its mass. When two particles meet, they coalesce and form a new particle with the mass equal to the sum of masses of the colliding particles. The process \((Y_1, Y_2)\) given by equation (2) is an example of a MAAF with two particles.

Let us now informally describe the main two results of this paper. The first one deals with a finite family of independent real-valued Brownian motions, whereas for the second result we consider a cylindrical Wiener process in \(L_2[0,1]\) as an infinite family of independent Brownian motions. We will carefully explain in the introduction of Section 5 why the infinite-dimensional case leads to additional difficulties. In a nutshell, the major reason is related to the fact that in the finite-dimensional case, strong well-posedness of the MAAF is well known (e.g. strong well-posedness of equation (2) holds), whereas in the infinite-dimensional case, the question of uniqueness, in any form, remains an open problem.

**First main result** (we refer to Theorem 4.1 for the precise statement). Let \(W = (W_k(t))_{k=1}^n, t \geq 0,\) be a family of \(n\) independent Brownian motions starting at \(x^0 = (x^0_k)_{k=1}^n\) with diffusion rates \(\sigma^2 = (\sigma^2_k)_{k=1}^n.\) Let \(X = W\) and \(\text{Coal}\) describe the set of coalescing paths. Then there is a map \(T,\) similar to (3), such that

- \(\text{Coal}\) is the zero-fiber of \(T,\) i.e. \(\text{Coal} = T^{-1}(\{0\});\)
- \(T(X)\) is a Brownian motion of dimension \(n - 1;\)
- the regular conditional probability \(p\) of \(X\) given \(T(X)\) has a continuous version at 0;
- the conditional distribution \(\mathbb{P}[X \in \cdot | X \in \text{Coal}]\) is described by the value at 0 of \(p,\) which coincides with the distribution of a finite-dimensional MAAF starting at \(x^0\) with diffusion rate \(\sigma^2.\)

**Second main result** (we refer to Theorem 5.10 for the precise statement). We consider \(Y\) and \(W\) such that

- \(Y_t, t \geq 0,\) is a MAAF starting at a strictly increasing function \(g\) from \(L_p[0,1]\) for some \(p > 2;\)
- \(W_t, t \geq 0,\) is a cylindrical Wiener process in \(L_2[0,1]\) also starting at \(g;\)
- \(Y\) and \(W\) are related by an equation (we refer to (17)) which generalizes (2) in infinite dimension.

Now we define \(X = (Y, W)\) (and not \(X = W\) as in the finite-dimensional case), because due to the lack of uniqueness, it is not possible to associate to a given cylindrical Wiener process \(W\) a MAAF \(Y\) satisfying the infinite-dimensional form of (2).
We define Coal and T in the same spirit as in the finite-dimensional case. In particular, Coal is now seen as a subset of \( \mathcal{C}([0, \infty), L_2[0, 1]) \) and \( T(X) \) is a cylindrical Wiener process, identified with a random element in \( \mathcal{C}[0, \infty)^\mathbb{N} \). Also, we construct a regular conditional probability \( p \) of \( X \) given \( T(X) \), which in general is not continuous at 0. Thus, we define the value at 0 of \( z \mapsto p(\cdot, z) \) along a sequence \( \{\xi^n\} \) of processes such that

- \( \xi^n = (\xi^n_j)_{j \geq 1} \) and \( \xi^n_j, j \geq 1 \), are independent Ornstein-Uhlenbeck processes defined by (24);
- for all \( n \), the law of \( \xi^n \) is absolutely continuous with respect to the law of \( T(X) \);
- \( \xi^n \) converges in distribution to 0 in \( \mathcal{C}[0, \infty)^\mathbb{N} \).

The main result consists in showing that the value at 0 of \( z \mapsto p(\cdot, z) \) along \( \{\xi^n\} \) coincides with the law of the MMAF \( Y_t, t \geq 0 \).

**Content of the paper.** In order to state precisely the two main results described above, we need two preliminary sections. In Section 2, we give a definition of a regular conditional probability and introduce a notion of its value at a point along a sequence. We also explain how it is related to the intuitive idea of approximating a set \( C \) (like e.g. Coal) by some sequence \( \{C_\varepsilon\}_\varepsilon \). In Section 3, we recall a definition of a modified massive Arratia flow and list some its properties. Sections 4 and 5 are devoted to the proof of the main results which are stated in theorems 4.1 and 5.10, respectively.

## 2 Conditional distributions

In this section, we first recall the definition of a regular conditional probability and then explain how we can suitably choose the value of a regular conditional probability at a point, in order to describe the conditional distribution of a random variable to a fiber of some map.

### 2.1 One method of construction of a regular conditional probability

Let \( E \) be a Polish space and \( F \) be a metric space. We consider random elements \( X \) and \( \xi \) in \( E \) and \( F \), respectively. Let also \( \mathcal{B}(E) \) denote a Borel \( \sigma \)-algebra on \( E \) and \( \mathcal{P}(E) \) be the space of probability measures on \( (E, \mathcal{B}(E)) \) endowed with the topology of weak convergence.

**Definition 2.1.** A function \( p : \mathcal{B}(E) \times F \to [0, 1] \) is a regular conditional probability of \( X \) given \( \xi \) if

(R1) for every \( z \in F \), \( p(\cdot, z) \in \mathcal{P}(E) \);
2.1 One method of construction of a regular conditional probability

(R2) for every $A \in \mathcal{B}(E)$, $z \mapsto p(A, z)$ is measurable;

(R3) for every $A \in \mathcal{B}(E)$ and $B \in \mathcal{B}(F)$,

$$\mathbb{P}[X \in A, \xi \in B] = \int_B p(A, z) \mathbb{P}^\xi(dz),$$

where $\mathbb{P}^\xi := \mathbb{P} \circ \xi^{-1}$ denotes the law of $\xi$.

Recall the following existence and uniqueness result (see e.g. [Kal02, Thm. 6.3]):

**Proposition 2.2.** There exists a regular conditional probability of $X$ given $\xi$. Moreover, it is unique in the following sense: if $p$ and $p'$ are regular conditional probabilities of $X$ given $\xi$, then

$$\mathbb{P}^\xi [z \in F : p(\cdot, z) = p'(\cdot, z)] = 1.$$

Now we describe a simple approach to the construction of a regular conditional probability. Let $X$ be a random element in $E$ and $\xi = T(X)$, where $T : E \rightarrow F$ is a measurable map. Assume that there exists a quadruple $(G, \Psi, Y, Z)$ satisfying the following conditions

(P1) $G$ is a measurable space;

(P2) $Y$ and $Z$ are independent random elements in $G$ and $F$, respectively;

(P3) $\Psi : G \times F \rightarrow E$ is a measurable map such that $T(\Psi(Y, Z)) = Z$ a.s.;

(P4) $X$ and $\Psi(Y, Z)$ have the same distribution.

The following proposition holds.

**Proposition 2.3.** Let $\xi := T \circ X$ and $(G, \Psi, Y, Z)$ be a quadruple satisfying (P1)-(P4). Then $p(A, z) := \mathbb{P}[\Psi(Y, z) \in A], A \in \mathcal{B}(E), z \in F,$ is a regular conditional probability of $X$ given $\xi$.

**Proof.** Since $\Psi$ is measurable, $p$ satisfies properties (R1) and (R2) of Definition 2.1. Let us check Property (R3). For every $A \in \mathcal{B}(E)$ and $B \in \mathcal{B}(F)$

$$\mathbb{P}[X \in A, \xi \in B] = \mathbb{P}[X \in A, T(X) \in B]$$

$$\overset{(P4)}{=} \mathbb{P}[\Psi(Y, Z) \in A, T(\Psi(Y, Z)) \in B]$$

$$\overset{(P3)}{=} \mathbb{P}[\Psi(Y, Z) \in A, Z \in B].$$
By (P2), Y and Z are independent, so it follows from Fubini’s Theorem that
\[
P[\Psi(Y,Z) \in A, Z \in B] = \int_B P[\Psi(Y,z) \in A] P^Z(dz) \\
= \int_B p(A,z) P^Z(dz).
\]
Furthermore, since X and \(\Psi(Y,Z)\) have the same law, \(\xi = T(X)\) and \(Z = T(\Psi(Y,Z))\) have the same law too, so \(P^Z = P^\xi\). This concludes the proof of (R3).

Example 2.4. Let us illustrate the above statement on the well-known example of the Brownian bridge. In that case, \(E, F, G, T\) and \(\Psi\) may be defined as follows:
\[
E = \{f \in C[0,1] : f(0) = 0\}, \\
F = \mathbb{R}, \\
G = \{f \in E : f(1) = 0\}, \\
T(x) = x(1), \quad x \in E, \\
\Psi(y,z)(t) = y(t) + tz, \quad t \in [0,1], \quad (y,z) \in G \times F.
\]
Furthermore, let \(X = \{W_t, t \in [0,1]\}\) be a standard Brownian motion, seen as a random element in \(E\). Then the process \(Y = \{Y_t, t \in [0,1]\}\) defined by \(Y_t := W_t - tW_1\) and the random variable \(Z := W_1\) are independent. Moreover, \(X = \Psi(Y,Z)\). So properties (P1)-(P4) hold. Thus, by Proposition 2.3,
\[
p(A,z) = P[\Psi(Y,z) \in A], \quad A \in \mathcal{B}(E), \quad z \in F,
\]
is a regular conditional probability of the Brownian motion \(X\) given \(T(X)\).

2.2 Choice of a value of regular conditional probability at a given point

How can we define a conditional distribution of \(X\) to the event \(\{\xi = z_0\}\), where \(z_0\) is some given point in \(F\)? A naive candidate would be \(p(\cdot, z_0)\). But by Proposition 2.2, \(p(\cdot, z)\) is well-defined only for \(P^\xi\)-almost every \(z \in F\). Nevertheless, if \(z_0\) belongs to the support of \(P^\xi\) and if \(F \ni z \mapsto p(\cdot, z) \in \mathcal{P}(E)\) is continuous at \(z_0 \in F\), then \(p(\cdot, z_0)\) should be a good definition.

Example 2.5. In the case of Example 2.4, we note that \(z \mapsto \Psi(y,z)\) is continuous on \(\mathbb{R}\) for each fixed \(y \in G\). Hence, the map \(F \ni z \mapsto p(\cdot, z) \in \mathcal{P}(E)\) is continuous. Since, every point \(z_0 \in \mathbb{R}\) belongs to \(\text{supp } P^{W_1}\), we can define
\[
P[W \in A|W_1 = z_0] = p(A, z_0) = P[\{W_t - tW_1 + tz_0, \quad t \in [0,1]\} \in A]
\]
for all \(z_0 \in \mathbb{R}\). That distribution is known as being the law of the Brownian bridge between 0 and \(z_0\).
2.2 Choice of a value of regular conditional probability at a given point

Let us now explain our approach in the case where $p$ has no continuous version at $z_0$. In order to motivate it, consider again Example 2.4-2.5 and take a family of real-valued random variables $\{\xi^\varepsilon, \varepsilon > 0\}$ whose distributions $P_{\xi^\varepsilon}$ have densities

$$
\frac{dP_{\xi^\varepsilon}}{dP_{W_1}}(z) = \frac{1_{|z - z_0| < \varepsilon}}{P[|W_1 - z_0| < \varepsilon]}, \quad z \in \mathbb{R}.
$$

Then $\xi^\varepsilon \xrightarrow{d} z_0, \varepsilon \to 0$. By continuity of $p$ at $z_0$, for every bounded continuous function $f$ on $\mathbb{R}$

$$
E \left[ \int_{\mathbb{R}} f(x)p(dx, \xi^\varepsilon) \right] \to \int_{\mathbb{R}} f(x)p(dx, z_0), \quad \varepsilon \to 0. \quad (5)
$$

Remark that due to the absolute continuity $P_{\xi^\varepsilon} \ll P_{W_1}$, the composition $p'(\cdot, \xi^\varepsilon)$ is well-defined for every version $p'$ of $p$ and

$$
E \left[ \int_{\mathbb{R}} f(x)p'(dx, \xi^\varepsilon) \right] = E \left[ \int_{\mathbb{R}} f(x)p(dx, \xi^\varepsilon) \right].
$$

Consequently, (5) with $p'$ in the left hand side instead of $p$ leads to the value of the continuous version of the regular conditional probability at $z_0$ if it exists.

**Remark 2.6.** For that particular choice of $\xi^\varepsilon$, convergence (5) and the equality

$$
E \left[ \int_{\mathbb{R}} f(x)p(dx, \xi^\varepsilon) \right] = \int_{\mathbb{R}} f(x)p \left[ W \in dx \mid W(1) - z_0 < \varepsilon \right]
$$

yield

$$
P \left[ W \in \cdot \mid \left| W(1) - z_0 \right| < \varepsilon \right] \to P \left[ W \in \cdot \mid W(1) = z_0 \right] \text{ in } P(C[0,1]) \text{ as } \varepsilon \to 0.
$$

Taking into account the above idea, we will give the following definition of the value of a regular conditional probability at a point. Let $C_b(E)$ denote the space of bounded continuous functions on $E$ and $p : B(E) \times F \to [0,1]$ be a regular conditional probability of $X$ given $\xi$. We consider a sequence of random elements $\{\xi^n\}_{n \geq 1}$ in $F$ which satisfies the following two properties

- (B1) for each $n \geq 1$, the law of $\xi^n$ is absolutely continuous with respect to the law of $\xi$;
- (B2) $\{\xi^n\}_{n \geq 1}$ converges in distribution to $z_0$ in $F$.

**Definition 2.7.** A probability measure $\nu$ on $(E, B(E))$ is the value at $z_0$ of the regular conditional probability $p$ along $\{\xi^n\}$ if for every $f \in C_b(E)$

$$
E \left[ \int_E f(x)p(dx, \xi^n) \right] \to \int_E f(x)\nu(dx), \quad n \to \infty. \quad (6)
$$
2.2 Choice of a value of regular conditional probability at a given point

Remark 2.8. According to conditions (B1) and (B2), the left hand side of (6) is well-defined and is independent of the version of the regular conditional probability.

By the continuous mapping Theorem, if \( p \) has a continuous version at \( z_0 \), then \( \nu \) is the value of this version at \( z_0 \). It turns out that the inverse statement is also true.

Lemma 2.9. Let \( F \) be a Polish space and \( z_0 \) belongs to the support of \( P^{\xi} \). There exists a probability measure \( \nu \) which is the value at \( z_0 \) of \( p \) along any sequence \( \{\xi^n\}_{n \geq 1} \) satisfying (B1) and (B2) if and only if there exists a version of \( p \) which is continuous at \( z_0 \in F \). In this case, \( \nu \) is the value of the continuous version of \( p \) at \( z_0 \).

We postpone the proof of the lemma to Section A in the appendix.

Definition 2.10. Let \( X \) be a random element in \( E \), \( C \) be a measurable subset of \( E \), and \( T : E \to F \) satisfy \( C = T^{-1}(\{z_0\}) \) for some \( z_0 \in F \). A conditional distribution

\[
P[ X \in \cdot | X \in C]
\]

of \( X \) given \( X \in C \) is the value of \( p \) at \( z_0 \) along \( \{\xi^n\}_{n \geq 1} \), where \( p \) is a regular conditional probability of \( X \) given \( T(X) \) and \( \{\xi^n\}_{n \geq 1} \) is a sequence satisfying (B1) and (B2) with \( \xi = T(X) \).

Remark 2.11. It is important to mention that the conditional distribution \( P[ X \in \cdot | X \in C] \) depends on both \( T \) and the sequence \( \{\xi^n\}_{n \geq 1} \) satisfying (B1) and (B2).

The proposed definition can be also interpreted as a kind of generalization of the definition of the conditional distribution of \( X \) given \( X \in C \) as the limit

\[
P[ X \in \cdot | X \in C] = \lim_{\varepsilon \to 0} P[ X \in \cdot | X \in C_\varepsilon],
\]

where \( \{C_\varepsilon\}_\varepsilon \) is a sequence of subsets of \( E \) which somehow converges to \( C \), and \( P[ X \in C_\varepsilon] > 0 \) for all \( \varepsilon \). Here the limit usually also depends on the choice of the approximating sequence \( \{C_\varepsilon\}_\varepsilon \). Let us demonstrate that the conditional distribution defined via limit (7) can be obtained according to Definition 2.10, if e.g. \( C \) is a closed set and \( C_\varepsilon \) is the \( \varepsilon \)-extension of \( C \), i.e

\[
C_\varepsilon = \{ x \in E : d_E(C, x) < \varepsilon \}, \quad \varepsilon > 0,
\]

where \( d_E \) denotes the metric on \( E \) and \( d_E(C, x) = \inf \{d_E(y, x) : y \in C \} \).

We can take \( T(x) := d_E(C, x) \) and \( \xi := T(X) \).
Note that $T^{-1}(\{0\}) = C$. Assume that $0 \in \text{supp} \mathbb{P}^{\xi}$, that is, $\mathbb{P}[\xi < \varepsilon] = \mathbb{P}[X \in C_{\varepsilon}] > 0$ for every $\varepsilon > 0$. Let $\xi^{\varepsilon}$ be a random variable on $\mathbb{R}$ with distribution

$$\mathbb{P}[\xi^{\varepsilon} \in A] = \frac{1}{\mathbb{P}[\xi < \varepsilon]} \int_{A} \mathbb{1}_{\{x < \varepsilon\}} \mathbb{P}^{\xi}(dx) = \mathbb{P}[\xi \in A | X \in C_{\varepsilon}], \quad A \in \mathcal{B}(\mathbb{E}).$$

A simple computation shows that $\xi$ and the family $\xi^{\varepsilon}$, $\varepsilon > 0$, satisfy conditions (B1), (B2) with $z_0 = 0$ and

$$\int_{\mathbb{E}} f(x) \mathbb{P}[X \in dx | X \in C_{\varepsilon}] = \mathbb{E} \left[ \int_{\mathbb{E}} f(x) p(dx, \xi^{\varepsilon}) \right],$$

where $p$ is the regular conditional probability of $X$ given $\xi$. So (7) is equivalent to (6).

### 3 Modified massive Arratia flow

In this section, we are going to recall the notion of the modified massive Arratia flow and state some properties needed to show that its law appears as the distribution of a family of Brownian motions conditioning to the event of coalescing paths.

Let $D((0,1), \mathbb{C}[0, \infty))$ be the space of càdlàg functions from $(0,1)$ to $\mathbb{C}[0, \infty)$, where $\mathbb{C}[0, \infty) = C([0, \infty), \mathbb{R})$ denotes the space of continuous functions from $[0, \infty)$ to $\mathbb{R}$ with topology of uniform convergence on compact sets. Let also $g$ belong to the set $L^{2+}_{2+}$ that consists of all non-decreasing càdlàg functions $g : (0,1) \to \mathbb{R}$ satisfying $\int_{0}^{1} |g(u)|^{2+\varepsilon} du < \infty$ for some $\varepsilon > 0$.

**Definition 3.1.** A random element $\mathcal{Y} = \{\mathcal{Y}(u,t), \ u \in (0,1), \ t \in [0, \infty)\}$ in the space $D((0,1), \mathbb{C}[0, \infty))$ is called a modified massive Arratia flow (or shortly MMAF) starting at $g$ if it satisfies the following properties:

1. **(E1)** for all $u \in (0,1)$ the process $\mathcal{Y}(u, \cdot)$ is a continuous square-integrable martingale with respect to the filtration
   $$\mathcal{F}_{t}^{\mathcal{Y}} = \sigma(\mathcal{Y}(u,s), \ u \in (0,1), \ s \leq t), \ t \geq 0; \quad (8)$$

2. **(E2)** for all $u \in (0,1)$, $\mathcal{Y}(u,0) = g(u)$;

3. **(E3)** for all $u < v$ from $(0,1)$ and $t \geq 0$, $\mathcal{Y}(u,t) \leq \mathcal{Y}(v,t)$;

4. **(E4)** for all $u, v \in (0,1)$, the joint quadratic variation of $\mathcal{Y}(u, \cdot)$ and $\mathcal{Y}(v, \cdot)$ is
   $$\langle \mathcal{Y}(u, \cdot), \mathcal{Y}(v, \cdot) \rangle_{t} = \int_{0}^{t} \frac{\mathbb{1}_{\{\tau_{u,v} \leq s\}}}{m(u,s)} ds, \quad t \geq 0,$$

   where $m(u,t) = \text{Leb} \{v : \exists s \leq t, \mathcal{Y}(v,s) = \mathcal{Y}(u,s)\}$ and $\tau_{u,v} = \inf\{t : \mathcal{Y}(u,t) = \mathcal{Y}(v,t)\}$. 

Remark 3.2. As it was already noted in Section 1.2, the massive modified Arratia flow describes the evolution of coalescing particles on the real line that carry a mass. The initial distribution of particle masses is defined as the push forward of the Lebesgue measure Leb on $[0, 1]$ under the map $g$. In particular, if $g$ is a step function with a finite number of jumps, i.e. it takes a finite number of pairwise distinct values $y_k^0$, $k \in [n] := \{1, \ldots, n\}$, with $\text{Leb}\{g^{-1}(\{y_k^0\})\} =: m_k > 0$, then the system consists of $n$ diffusion particles starting at points $y_k^0$ with masses $m_k$, $k \in [n]$. In that case, $\mathcal{Y}$ can be represented by a continuous process $y(t)$, $t \geq 0$, taking values in $\mathbb{R}^n$. This process will be precisely defined in Section 4.1 and we call it a MMAF starting at $y^0 = (y_k^0)_{k=1}^n$ with masses $\{m_k, k \in [n]\}$.

Remark 3.3. It is known that a MMAF exists if $g \in L^1_{2+, }$, see [Kon17a, Theorem 1.1]. However it is not known if properties (E1) - (E4) uniquely determine the distribution, except the case where $g$ is a step (piecewise constant) function with a finite number of jumps. However, all results obtained in the present paper deal just with some random element satisfying properties (E1) - (E4).

Let $L_p$ denote the space of $p$-integrable functions from $[0, 1]$ to $\mathbb{R}$ and $\| \cdot \|_{L_p}$ be the usual norm on $L_p$. We will consider the MMAF as a process in time, namely, we set $\mathcal{Y}_t = \mathcal{Y}(\cdot, t)$, $t \geq 0$.

Remark 3.4. The process $\mathcal{Y}_t$, $t \geq 0$, takes values in the subspace $L^+_2$ of $L_2[0, 1]$ consisting of functions which have non-decreasing versions. The fact that $\mathbb{E} \left[ \| \mathcal{Y}_t \|_{L_2}^2 \right] < \infty$ for every $t \geq 0$ follows from Lemma B.1 in appendix. Computations similar to ones in the proof of [KvR19, Lemma 3.1] show that the process $\mathcal{Y}_t, t \geq 0$, is a continuous $L^+_2$-valued martingale with quadratic variation

$$\langle \mathcal{Y} \rangle_t = \int_0^t \text{pr}_{\mathcal{Y}_s} \, ds, \quad t \geq 0,$$

that is,

(M1) $\mathcal{Y}_t, t \geq 0$, is a continuous $L^+_2$-valued process with $\mathbb{E} \left[ \| \mathcal{Y}_t \|_{L_2}^2 \right] < \infty$, $t \geq 0$;

(M2) for every $h \in L_2$ the $L_2$-inner product $(\mathcal{Y}_t, h)_{L_2}, t \geq 0$, is a continuous square integrable martingale with respect to the filtration generated by $\mathcal{Y}_t, t \geq 0$, that trivially coincides with $(\mathcal{F}^\mathcal{Y}_t)_{t \geq 0}$;

(M3) the joint quadratic variation of $(\mathcal{Y}_t, h_1)_{L_2}, t \geq 0$, and $(\mathcal{Y}_t, h_2)_{L_2}, t \geq 0$, equals

$$((\mathcal{Y}, h_1)_{L_2}, (\mathcal{Y}, h_2)_{L_2})_t = \int_0^t (\text{pr}_{\mathcal{Y}_s} h_1, h_2)_{L_2} \, ds, \quad t \geq 0,$$
where \( \text{pr}_f \) is the orthogonal projection operator in \( L_2 \) onto the subspace of \( \sigma(f) \)-measurable functions. In particular,

\[
\langle (\mathcal{Y}, h)_{L_2} \rangle_t = \int_0^t \| \text{pr}_{\mathcal{Y}_s} h \|_{L_2}^2 \, ds, \quad t \geq 0.
\]

We also remark that the inverse statement is true. To formulate it, we need the following definition.

**Definition 3.5.** Let \( \text{St} \) denote the set of non-decreasing step functions \( f : [0,1) \to \mathbb{R} \) of the form

\[
f = \sum_{j=1}^n f_j \mathbb{1}_{\pi_j},
\]

where \( n \geq 1, f_1 < \cdots < f_n \) and \( \{\pi_1, \ldots, \pi_n\} \) is an ordered partition of \([0,1)\) in half-open intervals of the form \( \pi_j = [a_j, b_j) \). The natural number \( n \) is denoted by \( N(f) \) and is by definition finite for every \( f \in \text{St} \).

**Proposition 3.6.** Let \( g \in L_2^{1+} \), and let \( \mathcal{Y}_t, t \geq 0, \) be a process starting at \( g \) and satisfying (M1) – (M3). Then there exists a MMAF \( \{\mathcal{Y}(u,t), u \in (0,1), t \in [0,\infty)\} \) such that \( \mathcal{Y}_t = \mathcal{Y}(\cdot,t) \) in \( L_2 \) a.s. for all \( t \geq 0 \). Moreover,

1) trajectories of \( \mathcal{Y}(u, \cdot), u \in (0,1), \) coalesce, i.e.

\[
P \left[ \forall u, v \in (0,1), \forall s \geq 0, \mathcal{Y}(u,s) = \mathcal{Y}(v,s) \implies \mathcal{Y}(u,t) = \mathcal{Y}(v,t), \forall t \geq s \right] = 1;
\]

2) for every \( t > 0 \) the system of particles, described by \( \mathcal{Y} \), contains a finite number of elements, that is,

\[
P \left[ \forall t > 0, \mathcal{Y}_t \in \text{St} \right] = 1.
\]

**Proof.** The statement directly follows from [Kon17a, Theorem 6.4] and propositions 2.3 and 6.2 ibid. \( \square \)

According to Remark 3.4 and Proposition 3.6, we may identify the modified massive Arratia flow \( \{\mathcal{Y}(u,t), u \in (0,1), t \in [0,\infty)\} \) and the \( L_2^+ \)-valued martingale \( \mathcal{Y}_t, t \geq 0 \), using both notations for the same object. We will also always assume that \( \{\mathcal{Y}(u,t), u \in (0,1), t \in [0,\infty)\} \) satisfies properties 1) and 2) of Proposition 3.6.

Let us recall (see e.g. [GM11, Definition 2.5]) that \( W_t, t \geq 0, \) defined on \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \) is an \( (\mathcal{F}_t) \)-cylindrical Wiener process (or shortly, cylindrical Wiener process) in a Hilbert space \( H \) if

1) for each \( t \geq 0 \), \( W_t : H \to L_2(\Omega, \mathcal{F}, \mathbb{P}) \) is a linear map;
ii) for any $h \in H$, $W_t(h)$, $t \geq 0$, is an $(\mathcal{F}_t)$-Brownian motion starting at 0;

iii) for any $h_1, h_2 \in H$ and $t \geq 0$, $\mathbb{E}[W_t(h_1)W_t(h_2)] = t(h_1, h_2)_H$.

We will also say that a cylindrical Wiener process $W$ starts at $g \in L_2$, if $W - g$ is a cylindrical Wiener process according to the definition above.

We finish this section by the following lemma which will play a key role in the disintegration of the law of a family of Brownian motions. But before we introduce the following notation. Let $h_t$, $t \geq 0$, be a progressively measurable process in $L_2$ and $B$ be a cylindrical Wiener process. We will denote

$$
\int_0^t h_s \cdot dB_s := \int_0^t L_s dB_s.
$$

for $L_t = (h_t, \cdot)_{L_2}$, $t \geq 0$. Let $\text{pr}^\perp_f$ denote the orthogonal projection in $L_2$ onto the orthogonal complement to the subspace of $\sigma(f)$-measurable functions.

**Lemma 3.7.** Let $\mathcal{Y}_t$, $t \geq 0$, be a MMAF starting at $g$. Let $B_t$, $t \geq 0$, be a cylindrical Wiener process in $L_2$ defined on the same probability space and independent of $\mathcal{Y}$. Then the process $W_t$, $t \geq 0$, defined by

$$
W_t := \mathcal{Y}_t + \int_0^t \text{pr}^\perp_{\mathcal{Y}} h_s \cdot dB_s, \quad t \geq 0, \tag{10}
$$

is a cylindrical Wiener process in $L_2$ starting at $g$.

In particular, equality (10) can be rewritten as

$$
W_t(h) := (\mathcal{Y}_t, h)_{L_2} + \int_0^t \text{pr}^\perp_{\mathcal{Y}} h_s \cdot dB_s, \quad t \geq 0, \quad h \in L_2.
$$

*Proof.* Let $(\mathcal{F}_t)_{t \geq 0}$ be a natural filtration generated by $\mathcal{Y}$ and $B$. It is trivial that the map $W_t : L_2 \to L_2(\Omega, \mathcal{F}_t, \mathbb{P})$ defined by (10) is linear. Let us check that $(W_t - g)(h)$, $t \geq 0$, is a Brownian motion with diffusion rate $\|h\|^2_{L_2}$ for any $h \in L_2$. Using (M2) – (M3) and the independence of $\mathcal{Y}$ and $B$, we have that $(W_t - g)(h)$, $t \geq 0$, is a continuous $(\mathcal{F}_t)$-martingale with the quadratic variation

$$
\langle (W - g)(h) \rangle_t = \int_0^t \|\text{pr}^\perp_{\mathcal{Y}} h_s \|_{L_2}^2 \, ds + \int_0^t \| \text{pr}^\perp_{\mathcal{Y}} h_s \|^2_{L_2} \, ds
$$

$$
= \int_0^t \|h\|^2_{L_2} \, ds = t\|h\|^2_{L_2}.
$$

\footnote{The process $\text{pr}^\perp_{\mathcal{Y}}$, $t \geq 0$, does not take values in the space of Hilbert-Schmidt operators in $L_2$. Therefore, the integral $\int_0^t \text{pr}^\perp_{\mathcal{Y}} h_s \cdot dB_s$ is not well-defined. Thus, we will define it as a map from $L_2$ to $L_2(\Omega)$ determined by $h \mapsto \int_0^t \text{pr}^\perp_{\mathcal{Y}} h_s \cdot dB_s$. Considering $f \in L_2$ as a map from $L_2$ to $L_2(\Omega)$, we always mean the map $h \mapsto (f, h)_{L_2}$.}
Lévy’s characterization of Brownian motion implies that \((W_t - g)(h), t \geq 0\), is a Brownian motion with diffusion rate \(\|h\|^2_{L_2}\). Next, the linearity of \(W_t - g\) and the polarization equality imply that the covariation between \((W_t - g)(h_1)\) and \((W_t - g)(h_2)\) equals \((h_1, h_2)_{L_2} t\) for every \(t \geq 0\) and \(h_1, h_2 \in L_2\). It achieves the proof of the lemma.

\[ \square \]

4 Conditional distribution for finite number of particles

In this section, we will consider a finite family of independent Brownian motions \(X_k(t), t \geq 0, k \in [n] = \{1, \ldots, n\}\), with diffusion rates \(\sigma^2_k, k \in [n]\), respectively, where \(n \geq 2\) is a fixed integer number. We set \(X(t) = (X_k(t))_{k=1}^n, t \geq 0, m_k = \frac{1}{\sigma_k^2}, k \in [n]\). Later \(m_k\) will play a role of particle masses in the modified massive Arratia flow. Without loss of generality, we may assume that \(m_1 + \cdots + m_n = 1\) and that the starting positions \(x_0^k\) of \(X_k\) are ordered \(x_0^1 \leq \ldots \leq x_0^n\). We will consider \(X = (X_k)_{k=1}^n\) as a random element of the Polish space

\[
E = C_1[0, \infty)^n = \{x = (x_k)_{k=1}^n \in C[0, \infty) : x_1(0) \leq \ldots \leq x_n(0)\}.
\]

Let us also define the set of coalescing functions from \(E\) as follows

\[
\text{Coal} = \left\{ x = (x_k)_{k=1}^n \in E : \forall k, l \in [n], \forall s \geq 0, x_k(s) = x_l(s) \implies x_k(t) = x_l(t), \forall t \geq s \right\}.
\]

The goal of this section is the computation of the conditional distribution \(\mathbb{P}[X \in \cdot | X \in \text{Coal}]\). For this purpose we will build a map \(T\) from \(E\) to a functional Polish space \(F\) such that \(T^{-1}(\{0\}) = \text{Coal}\), and find the continuous version of regular conditional probability of \(X\) given \(T(X)\). Then we will show that its value at 0 gives the distribution of the MMAF \(y(t), t \geq 0, starting at x^0 = (x_0^k)_{k=1}^n\) with masses \(\{m_k, k \in [n]\}\), which will be constructed in the next section (see also Remark 3.2). The following theorem is the main result of this section.

**Theorem 4.1.** Let \(y(t), t \geq 0, be the MMAF starting at \(x^0 = (x_0^k)_{k=1}^n\) with masses \(\{m_k, k \in [n]\}\). Then for every \(A \in \mathcal{B}(E)\)

\[
\mathbb{P}[X \in A | X \in \text{Coal}] = \mathbb{P}[y \in A].
\]

4.1 MMAF as a finite particle system

Let \(x^0 = (x_k)_{k=1}^n\) and \(\{m_k, k \in [n]\}\) be as in the previous section. The space \(\mathbb{R}^n\) equipped with inner product \((a, b)_m = \sum_{k=1}^n a_k b_k m_k\) will be denoted by
4.1 MMAF as a finite particle system

\( \mathbb{R}_m^n \). We also define a map \( \Xi_m : \mathbb{R}_m^n \to L_2 \) as follows

\[
\Xi_m(a) = \sum_{k=1}^n a_k 1_{x_k^0}, \quad a = (a_k)_{k=1}^n \in \mathbb{R}_m^n,
\]

where \( \pi_k^0 = [\sum_{j=1}^{k-1} m_j, \sum_{j=1}^k m_j], k \in [n] \). It is easy to see that \( \Xi_m \) is a bijection between \( \mathbb{R}_m^n \) and \( \Xi_m(\mathbb{R}_m^n) = \left\{ \sum_{k=1}^n a_k 1_{x_k^0}, a \in \mathbb{R}_m^n \right\} \). Moreover, \( (\Xi_m(a), \Xi_m(b))_{L_2} = (a, b)_m \) for all \( a, b \in \mathbb{R}_m^n \).

Consider a MMAF \( \{ \gamma(u, t), u \in (0, 1), t \in [0, \infty) \} \) starting at \( g = \Xi_m(x^0) \). By Proposition 3.6 and the definition of MMAF, there exists a unique process \( y(t) = (y_k(t))_{k=1}^n, t \geq 0 \), such that

\[
\gamma_t = \gamma(\cdot, t) = \Xi_m(y(t)), \quad t \geq 0.
\]

Moreover, it satisfies the following properties

(F1) for all \( k \in [n] \), the process \( y_k(t), t \geq 0 \), is a continuous real-valued square-integrable martingale with respect to the filtration

\[
\mathcal{F}_t = \sigma(y(s), s \leq t, l \in [n]);
\]

(F2) for all \( k \in [n] \), \( y_k(0) = x_k^0 \);

(F3) for all \( k < l \) from \( [n] \), \( y_k(t) \leq y_l(t) \);

(F4) for all \( k, l \in [n] \), the joint quadratic variation of \( y_k \) and \( y_l \) is

\[
\langle y_k, y_l \rangle_t = \int_0^t \frac{1_{\left\{ y_k(s) = y_l(s) \right\}}}{m_k(s)} ds, \quad t \geq 0,
\]

where \( m_k(t) = \sum_{l \in \pi_k(t)} m_l \) and \( \pi_k(t) = \{ l \in [n] : y_k(t) = y_l(t) \} \);

(F5) the coordinate processes coalesce, i.e.

\[
P \left[ \forall k, l \in [n], \forall s \geq 0, \ y_k(s) = y_l(s) \implies y_k(t) = y_l(t), \forall t \geq s \right] = 1.
\]

We also call the process \( y(t), t \geq 0 \), a modified massive Arratia flow. Moreover, this process satisfies the property that almost surely, there is a time for which the \( n \) coordinate processes coalesce to form a single particle.

**Lemma 4.2.** The process \( y(t), t \geq 0 \), satisfies the following property:

(F6) the number of distinct particles tend to be equal to one, i.e.

\[
P \left[ \exists \tau > 0, \forall t \geq \tau, \ y_1(t) = y_2(t) = \cdots = y_n(t) \right] = 1.
\]
Remark 4.3. By properties (F3) and (F5), property (F6) is equivalent to
\[ \mathbb{P} \left[ \exists \tau > 0, \ y_1(\tau) = y_n(\tau) \right] = 1. \]

Proof. According to Remark 4.3, we have to prove that almost surely there exists a time \( t > 0 \) such that \( y_1(t) = y_n(t) \). Let us denote \( \eta(t) := y_n(t) - y_1(t), \ t \geq 0, \) and \( \tau := \inf\{ t \geq 0 : \eta(t) = 0 \} \). By (F1) and (F2), \( \eta(t), \ t \geq 0, \) is a continuous \( \mathcal{F}^B_t \)-martingale starting at the non-negative point \( x_n^0 - x_1^0 \). Moreover, by (F4), its quadratic variation is equal to
\[
\langle \eta \rangle_t = \langle y_1 \rangle_t + \langle y_n \rangle_t - 2\langle y_1, y_n \rangle_t = \int_0^t \frac{1}{m_1(s)} \, ds + \int_0^t \frac{1}{m_n(s)} \, ds - 2 \int_0^t \mathbb{I}_{\tau \leq s} \frac{\| \eta \|}{m_1(s)} \, ds.
\]

because \( m_1(s) = m_n(s) \) for each \( s \geq \tau \). Therefore, \( \frac{d\langle \eta \rangle_t}{dt} = \frac{1}{m_1(t)} + \frac{1}{m_n(t)} \geq 2 \) for all \( t < \tau \) due to the boundedness of \( m_1(t) \) and \( m_n(t) \) from above by 1. Each continuous martingale whose quadratic variation has a derivative bounded from below by a positive constant reaches zero almost surely, see e.g. [IW89, Theorem 2.7.2]. Thus, \( \mathbb{P}[\tau < \infty] = 1 \), that concludes the proof. \( \square \)

For each \( a = (a_k)_{k=1}^n \in \mathbb{R}_m^n \), let \( \text{pr}_m^n \) denote the orthogonal projection operator in \( \mathbb{R}_m^n \) onto the subspace
\[
\mathbb{R}_m^n(a) := \{ b = (b_k)_{k=1}^n : b_k = b_l \text{ if } a_k = a_l, \ k, l \in [n] \}.
\]

Remark that \( \dim \mathbb{R}_m^n(a) \in [n] \) counts the number of distinct coordinates of the vector \( a \). The isometry between \( \Xi_m(\mathbb{R}_m^n) \) and \( \mathbb{R}_m^n \), equality (11) and properties (M2), (M3) imply that the process \( y(t), \ t \geq 0 \), is a continuous martingale in \( \mathbb{R}_m^n \) with the quadratic variation
\[
\langle y \rangle_t = \int_0^t \text{pr}_m^n(y(s)) \, ds, \ t \geq 0,
\]

that is, for any \( a \in \mathbb{R}_m^n \) the process \( (y(t), a)_m, \ t \geq 0 \), is a continuous martingale with quadratic variation
\[
\langle (y, a)_m \rangle_t = \int_0^t \| \text{pr}_m^n(y(s)) a \|_m^2 \, ds, \ t \geq 0,
\]

where \( \| \cdot \|_m \) denotes the norm in \( \mathbb{R}_m^n \).

Let \( \text{pr}_m^n(a)^\perp \) be the orthogonal projection operator in \( \mathbb{R}_m^n \) onto the orthogonal complement \( \mathbb{R}_m^n(a)^\perp \). The following lemma can be proved similarly as Lemma 3.7.
4.2 Construction of the map $T$

We next build a metric space $\mathcal{F}$ and a map $T : \mathcal{E} \rightarrow \mathcal{F}$ such that $T^{-1}(\{0\}) = \text{Coal}$. We fix $y \in \text{Coal}$ and set $\tau^y_0 = +\infty$,

$$\tau^y_k := \inf \{ t \geq 0 : \dim \mathbb{R}_m^n(y(t)) \leq k \}, \quad k \in [n].$$

Remark that we construct $\tau^y_k$, $k \in \{0, \ldots, n\}$ in nonstandard decreasing order, i.e. $0 = \tau^y_n \leq \tau^y_{n-1} \leq \cdots \leq \tau^y_1 \leq \tau^y_0 = +\infty$. We do this to be consistent with the infinite-dimensional case studied later in Section 5. Note that for each $k \in [n]$, $\tau^y_k$ represents the first time where the number of particles is less than or equal to $k$.

There exists a unique permutation $p^y = \{p^y_1, \ldots, p^y_{n-1}\} =: \{p_1, \ldots, p_{n-1}\}$ of $[n-1]$ such that

- for every $k \in [n-1]$ and small enough $\varepsilon > 0$, $y_{p_k}(\tau^y_k - \varepsilon) = y_{p_{k+1}}(\tau^y_k - \varepsilon)$ if $0 < \tau^y_k < +\infty$;
- for every $k \in [n-1]$ and $t \geq 0$, $y_{p_k}(t) < y_{p_{k+1}}(t)$ if $\tau^y_k = +\infty$;
- for every $k \in [n-1]$, $y_{p_k}(0) = y_{p_{k+1}}(0)$ if $\tau^y_k = 0$;
- for every $k \in [n-2]$ and $l \in [n-1-k]$, $p_k < \cdots < p_{k+l}$ if $\tau^y_k = \cdots = \tau^y_{k+l}$.

We remark that the partition $p^y$ describes the order of coalescing of paths in $y$, namely $p_k$ shows that the coordinate functions $y_{p_k}$ and $y_{p_k+1}$ coalesce at time $\tau^y_k$. If a couple of coalescences appears at the same time then we choose the corresponding $p_k$ in the increasing order.

Define the linear subspaces $\{0\} = H^y_0 \subseteq H^y_{n-1} \subseteq \cdots \subseteq H^y_1$ as the following orthogonal complements

$$H^y_k = \{ a = (a_l)_{l=1}^{n} \in \mathbb{R}_m^n : a_{p_l} = a_{p_{l+1}}, \quad \forall l \in \{k, \ldots, n-1\} \}^\perp, \quad k \in [n-1].$$

\(^3\)Equivalently, the coordinate processes $B_k$ are independent Brownian motions with diffusion rates $\sigma^2_k = \frac{\sigma^2_k}{p_{p_k}}$, respectively. If any assumption are not made on the starting point of a standard Brownian motion in $\mathbb{R}$ or a standard Wiener process in $\mathbb{R}_m^n$, we will assume the process starts at 0.
It is easy to see that \( \dim H_k^y = n - k \) and
\[
H_k^y = \mathbb{R}^n_k(y(\tau_k^y))^\perp \quad \text{if} \quad \tau_k^y < \tau_{k-1}^y. \tag{12}
\]

For every \( k \in [n-1] \), we take the unique normalized vector \( e_k^y \in H_k^y \otimes H_{k+1}^y \) whose first non-zero coordinate is positive, namely
\[
e_k^y = \frac{1}{\sqrt{m_{p_k} + m_{p_{k+1}}}} \left( \frac{m_{p_{k+1}}}{m_{p_k}} \varepsilon_{p_k} - \frac{m_{p_k}}{m_{p_{k+1}}} \varepsilon_{p_{k+1}} \right),
\]
where \( \varepsilon_1 = (1, 0, \ldots, 0), \ldots, \varepsilon_n = (0, \ldots, 0, 1) \) is the canonical basis of \( \mathbb{R}^n \).

Then \( H_k^y = \text{span} \{ e_l^y, \ l \in \{ k, \ldots, n-1 \} \} \) for all \( k \in [n-1] \).

**Remark 4.5.** It is not hard to see that such a choice of \( e_k^y \) is measurable, namely \( y \mapsto e_k^y \) is measurable as a map from \( \text{Coal} \) to \( \mathbb{R}^n_m \).

**Lemma 4.6.** For every \( k \in [n-1] \) and \( t \geq \tau_k^y \) one has \( (y(t), e_l^y)_m = 0 \) for all \( l \in \{ k, \ldots, n-1 \} \).

**Proof.** There exists \( j \in [k] \) such that \( \tau_k^y = \cdots = \tau_j^y < \tau_{j-1}^y \). Then \( H_j^y = \mathbb{R}^n_m(y(\tau_j^y))^\perp \). Moreover, by the coalescence property, \( y(t) \in \mathbb{R}^n_m(y(\tau_j^y)) \) for all \( t \geq \tau_j^y = \tau_k^y \). Consequently, for every \( l \in \{ k, \ldots, n-1 \} \subseteq \{ j, \ldots, n-1 \} \) we have \( (y(t), e_l^y)_m = 0 \).

**Lemma 4.7.** For every \( t \geq 0 \) and \( a \in \mathbb{R}^n_m \),
\[
\left( \text{pr}_{y(t)}^m \right)^\perp a = \sum_{l=1}^{n-1} \mathbf{1}_{\{ \cdot \geq \tau_l^y \}} (a, e_l^y)_m e_l^y.
\]

**Proof.** Take \( k \in [n] \) such that \( \tau_k^y < \tau_{k-1}^y \). By equality (12) and the fact that \( \mathbb{R}^n_m(y(\tau_k^y)) = \mathbb{R}^n_m(y(t)) \) for \( \tau_k^y \leq t < \tau_{k-1}^y \), we deduce that \( \left( \text{pr}_{y(t)}^m \right)^\perp \) is the orthogonal projection operator in \( \mathbb{R}^n_m \) onto \( H_k^y \). Thus for each \( t \in [\tau_k^y, \tau_{k-1}^y] \),
\[
\left( \text{pr}_{y(t)}^m \right)^\perp a = \sum_{l=k}^{n-1} (a, e_l^y)_m e_l^y.
\]
This implies the statement of the lemma.

For a function \( x \in \mathbb{E} = C_1[0, \infty)^n \) and \( y \in \text{Coal} \) we introduce the integral
\[
\int_0^t \text{pr}_{y(s)}^m \, dx(s) = \sum_{k=1}^n \left( \text{pr}_{y(\tau_k^y \wedge t)} x(\tau_{k-1}^y \wedge t) - \text{pr}_{y(\tau_k^y \wedge t)} x(\tau_k^y \wedge t) \right). \tag{13}
\]

**Lemma 4.8.** For every \( x \in C_1[0, \infty)^n \), there exists a unique \( y \in \text{Coal} \) such that
\[
y(t) = x(0) + \int_0^t \text{pr}_{y(s)}^m \, dx(s), \quad t \geq 0. \tag{14}
\]
4.2 Construction of the map $T$

Proof. The function $y \in \textbf{Coal}$ can be constructed step by step. First take $\sigma_0 = 0$ and $\tilde{y}^0(t) = x(0)\ t \geq 0$. Then set

$$\tilde{y}^k(t) := \tilde{y}^{k-1}(\sigma_{k-1}) + \text{pr}_m^{m} x(t) - \text{pr}_m^{m} x(\sigma_{k-1}), \ t \geq \sigma_{k-1},$$

and

$$\sigma_k := \inf \left\{ t > \sigma_{k-1} : \dim R_m^n(\tilde{y}^k(t)) < \dim R_m^n(\tilde{y}^{k-1}(\sigma_{k-1})) \right\}$$

for all $k \in [n-1]$. Remark that $\dim R_m^n(\tilde{y}^k(\sigma_k)) \in [n-k]$ for each $0 \leq k \leq n - 1$. We set $\sigma_n = +\infty$. The function $y$ can be defined as

$$y(t) = \tilde{y}^k(t) \quad \text{for} \ t \in [\sigma_{k-1}, \sigma_k), \ k \in [n].$$

By construction, $y$ belongs to $\textbf{Coal}$, satisfies (14) and is uniquely determined.

$\blacksquare$

Remark 4.9. If $x$ belongs to $\textbf{Coal}$, then $y = x$ is the unique element of $\textbf{Coal}$ satisfying equation (14).

Now, we take $F := C_0[0, \infty)^{n-1} = \{ h \in C[0, \infty) : h(0) = 0 \}^{n-1}$ and define the map $T : E \to F$ as follows. For a given $x \in E = C_+[0, \infty)^{n}$, let $y \in \textbf{Coal}$ be the unique solution to equation (14) associated with $x$. Let also $\tau_k^y, k \in [n-1]$, be constructed as above. Then define $T(x) := (T_k(x))_{k=1}^{n-1}$:

$$T_k(x)(t) = \begin{cases} (x(t + \tau_k^y), c_k^y)_m, & \text{if } \tau_k^y < \infty, \\ 0, & \text{if } \tau_k^y = \infty, \quad t \geq 0. \end{cases} \quad (15)$$

Lemma 4.10. The map $T : E \to F$ is well-defined.

Proof. We only need to show that $T(x)(0) = 0$, i.e. that for each $k \in [n-1]$, $(x(\tau_k^y), c_k^y)_m = 0$. Let us define $J_0^t \left( \text{pr}_m^{m} y(s) \right) \frac{1}{\text{dx}}(s)$ similarly as in (13) replacing $\text{pr}_m^{m} y(s)$ with $\left( \text{pr}_m^{m} y(s) \right)^\perp$. Since for every $a, b \in R_m^n$, $a = \text{pr}_m^{m} a + (\text{pr}_m^{m})^\perp a$, we have for each $k \in [n]$

$$x(\tau_k^y) = x(0) + \int_0^{\tau_k^y} \text{pr}_m^{m} y(s) \text{dx}(s) + \int_0^{\tau_k^y} \left( \text{pr}_m^{m} y(s) \right)^\perp \text{dx}(s)$$

$$= y(\tau_k^y) + \int_0^{\tau_k^y} \left( \text{pr}_m^{m} y(s) \right)^\perp \text{dx}(s).$$

We note that the integral of the right hand side of the latter equality consists of $\left( \text{pr}_m^{m} y(t) \right)^\perp$, $l \in \{k+1, \ldots, n-1 \}$, hence this integral is orthogonal to $c_k^y$ by Lemma 4.7. By Lemma 4.6, $y(\tau_k^y)$ is also orthogonal to $c_k^y$. Hence $(x(\tau_k^y), c_k^y)_m = 0$. $\blacksquare$
Lemma 4.11. The pre-image of zero function from $F$ under the map $T$ coincides with $\text{Coal}$, that is, 

$$T^{-1}(\{0\}) = \{x \in E : T(x) = 0\} = \text{Coal}. \quad (13)$$

Proof. Let $x \in \text{Coal}$. Then $y(t) = x(t)$, $t \geq 0$, by Remark 4.9. By Lemma 4.1, if $\tau_k^y < \infty$, $(x(t + \tau_k^y), e_k^y)_m = (y(t + \tau_k^y), e_k^y)_m = 0$ for all $k \in [n-1]$. Hence, $T(x) = 0$.

We next assume that $T(x) = 0$. Let $y \in \text{Coal}$ be the unique solution to equation (14) associated with $x$. We take $\tau_k^y \leq t < \tau_{k-1}^y$ for $k \in [n-1]$ and note that $(x(t), e_l^y)_m = 0$ for all $l \in \{k, \ldots, n-1\}$. Consequently, $x(t)$ is orthogonal to $H_k^y$. Thus, $x(t) \in R_m^y(y(\tau_k^y))$, by (12). The continuity of $x$ also implies that $x(\tau_{k-1}^y) \in R_m^y(y(\tau_k^y))$, $k \in [n-1]$. So, for $t \in [\tau_k^y, \tau_{k-1}^y)$ and $k \in [n-1]$

$$y(t) = x(0) + \int_0^t \text{pr}_{y(s)}^m dx(s) = x(0) + \sum_{l=k+1}^n \left( \text{pr}_{y(\tau_l^y)}^m x(\tau_l^y) - \text{pr}_{y(\tau_{l-1}^y)}^m x(\tau_{l-1}^y) \right) + \text{pr}_{y(\tau_k^y)}^m x(t) - x(\tau_k^y) = x(0) + \sum_{l=k+1}^n \left( x(\tau_{l-1}^y) - x(\tau_l^y) \right) + x(t) - x(\tau_k^y) = x(t).$$

The equality $x(t) = y(t)$ for $t \in [0, \tau_{n-1}^y)$ is trivial. This implies that $x \in \text{Coal}$. \hfill \Box

4.3 Proof of Theorem 4.1

In order to prove Theorem 4.1, we are going to use Proposition 2.3, namely we will construct a quadruple $(G, \Psi, Y, Z)$ satisfying (P1)-(P4). We take $G = \text{Coal}$ equipped with the induced topology of $E$. Define $\Psi : G \times F \rightarrow E$ by

$$\Psi(y, z)(t) = y(t) + \sum_{k=1}^{n-1} \mathbb{1}_{\{t > \tau_k^y\}} e_k^y z_k(t - \tau_k^y), \quad t \geq 0, \ y \in G, \ z \in F.$$

Remark 4.12. Obviously, for each fixed $y \in G$, the map $z \mapsto \Psi(y, z)$ is continuous.

Lemma 4.13. For any $z \in F$ and $y \in G$ with $\tau_k^y < \infty$ one has $T(\Psi(y, z)) = z$. 

4.3 Proof of Theorem 4.1

Proof. We take $x = \Psi(y, z)$ and first show that $y$ is the unique solution in CoaI to equation (14) associated with $x$. Note that $x(0) = \Psi(y, z)(0) = y(0)$. Then, for $t \in [\tau^y_k, \tau^y_{k-1}]$,

$$x(0) + \int_0^t \text{pr}^m_{y(s)} \, dx(s) = y(0) + \sum_{l=k+1}^{n} \left( \text{pr}^m_{y(t^y_l)} x(t^y_{l-1}) - \text{pr}^m_{y(t^y_l)} x(t^y_{l}) \right) + \text{pr}^m_{y(t^y_k)} x(t) - \text{pr}^m_{y(t^y_k)} x(t^y_{k}).$$

Remark that for every $s \in [\tau^y_I, \tau^y_{I-1})$,

$$\text{pr}^m_{y(t^y_I)} x(s) = \text{pr}^m_{y(t^y_I)} \left( y(s) + \sum_{i=1}^{n-1} \mathbb{1}_{\{s \geq \tau^y_i\}} e^y_i z_i(y, z - \tau^y_i) \right) = y(s) + \sum_{i=1}^{n-1} \text{pr}^m_{y(t^y_I)} e^y_i z_i(s, t^y_i).$$

By Lemma 4.6, $\text{pr}^m_{y(t^y_I)} e^y_i = 0$ for each $i \in \{I, \ldots, n-1\}$. Thus, $\text{pr}^m_{y(t^y_I)} x(s) = y(s)$ for every $s \in [\tau^y_I, \tau^y_{I-1})$ and due to the continuity of $\text{pr}^m_{y(t^y_I)}$, the latter equality remains true for $s = \tau^y_{I-1}$. This implies that

$$x(0) + \int_0^t \text{pr}^m_{y(s)} \, dx(s) = y(0) + \sum_{l=k+1}^{n} \left( y(t^y_{l-1}) - y(t^y_{l}) \right) + y(t) - y(t^y_{k}) = y(t).$$

Next, since $\tau^y_k < \infty$ for all $k \in [n-1]$, we have for $t \geq 0, k \in [n-1]$

$$\left( \Psi(y, z)(t + \tau^y_k), e^y_k \right)_m = \left( y(t + \tau^y_k) + \sum_{l=1}^{n-1} \mathbb{1}_{\{t + \tau^y_l \geq \tau^y_k\}} e^y_l z_l(t + \tau^y_k - \tau^y_l), e^y_k \right)_m = \left( y(t + \tau^y_k), e^y_k \right)_m + z_k(t) = z_k(t),$$

by Lemma 4.6 and by the orthogonality of $(e^y_l)_{l=1}^{n-1}$. Thus, $T(\Psi(y, z)) = z$. \hfill \Box

Lemma 4.14. Let $\theta(t), t \geq 0$, be a standard Wiener process in $\mathbb{R}_m^n$, and $y(t), t \geq 0$, be a modified massive Arratia flow that satisfies $(F1) - (F6)$ and is independent of $\theta$. Let also $\eta_k(t), k \in [n-1]$, be independent standard Brownian motions in $\mathbb{R}$ independent of $y$. Then

$$\text{Law} \left\{ (y(t), \int_0^t \left( \text{pr}^m_{y(s)} \right) \, d\theta(s)) , \ t \geq 0 \right\} = \text{Law} \left\{ (y(t), \sum_{k=1}^{n-1} \mathbb{1}_{\{t \geq \tau^y_k\}} e^y_k \eta_k(t - \tau^y_k)) , \ t \geq 0 \right\}.$$  \hfill (16)

In particular, $\Psi(y, \eta)$ is a standard Wiener process in $\mathbb{R}_m^n$ starting at $x^0$. 
4.3 Proof of Theorem 4.1

Proof. We first note that the last statement of Lemma 4.14 follows from (16) and Lemma 4.4. So, we need only to prove equality (16).

We denote \( \theta^y_k(t) = (\theta(t), e^y_k)_m, t \geq 0, k \in [n - 1], y \in \text{Coal}. \) Then using Lemma 4.7, we have

\[
\xi^y(t) := \int_0^t \left( \text{pr}^m_y(s) \right) \frac{1}{2} d\theta(s) = \int_0^t \sum_{k=1}^{n-1} \mathbb{1}_{\{s \geq \tau^y_k \}} e^y_k \, d\theta^y_k(s)
\]

\[
= \sum_{k=1}^{n-1} e^y_k (\theta^y_k(t) - \theta^y_k(\tau^y_k)) = \sum_{k=1}^{n-1} \mathbb{1}_{\{t \geq \tau^y_k \}} e^y_k (\theta^y_k(t) - \theta^y_k(\tau^y_k))
\]

for all \( t \geq 0. \)

Take a continuous bounded function \( G : (C[0, \infty)^n)^2 \to \mathbb{R} \) and compute

\[
\mathbb{E} \left[ G(y, \xi^y) \right] = \mathbb{E} \left[ \mathbb{E} \left[ G(y, \xi^y) | y \right] \right] = \mathbb{E} \left[ F(y) \right],
\]

where \( F(y) = \mathbb{E} \left[ G(y, \xi^y) | y = y \right] = \mathbb{E} \left[ G(y, \xi^y) \right] \) and

\[
\xi^y(t) = \sum_{k=1}^{n-1} \mathbb{1}_{\{t \geq \tau^y_k \}} e^y_k (\theta^y_k(t) - \theta^y_k(\tau^y_k)), \quad t \geq 0,
\]

for \( y \in \text{Coal}. \) This follows from the independence of \( y \) and \( \theta. \) Let us remark that \( e^y_k \) and \( \tau^y_k, k \in [n - 1], \) are deterministic. So, the family

\[
\left\{ \mathbb{1}_{\{t \geq \tau^y_k \}} e^y_k (\theta^y_k(t) - \theta^y_k(\tau^y_k)), \quad t \geq 0, \quad k \in [n - 1] \right\}
\]

has the same distribution as

\[
\left\{ \mathbb{1}_{\{t \geq \tau^y_k \}} \eta_k(t - \tau^y_k), \quad t \geq 0, \quad k \in [n - 1] \right\}.
\]

Hence,

\[
F(y) = \mathbb{E} \left[ G(y, \zeta^y) \right],
\]

where

\[
\zeta^y(t) = \sum_{k=1}^{n-1} \mathbb{1}_{\{t \geq \tau^y_k \}} e^y_k \eta_k(t - \tau^y_k), \quad t \geq 0.
\]

By independence of \( y \) and \( \eta_k, k \in [n - 1], \) we deduce as above that \( \mathbb{E} \left[ F(y) \right] = \mathbb{E} \left[ G(y, \xi^y) \right]. \) This implies \( \mathbb{E} \left[ G(y, \xi^y) \right] = \mathbb{E} \left[ G(y, \xi^y) \right]. \quad \square
\]

Proof of Theorem 4.1. By lemmas 4.2 and 4.13, \( \Psi \) satisfies (P3). Let \( Y : \Omega \to G, \ Z : \Omega \to F \) independent random elements such that \( Y = y \) is a MMAF satisfying \((F1) - (F6)\) and \( Z = (\eta_k)^{n-1}_{k=1} \) is a standard Brownian motion in \( \mathbb{R}^{n-1}. \) By Lemma 4.14, \( \Psi(Y, Z) \) is a standard Wiener process
starting at \( x^0 \), i.e. has the same law as \( X \). Thus the quadruple \((G,\Psi,Y,Z)\) satisfies (P1)-(P4). Therefore, by Proposition 2.3,

\[
p(A, z) = \mathbb{P} [\Psi(Y, z) \in A], \quad A \in \mathcal{B}(\mathbb{E}), \quad z \in \mathbb{F},
\]

is a regular conditional probability of \( X \) given \( \xi \), with \( \xi = T(X) \). Furthermore, \( T^{-1}(\{0\}) = \text{Coal} \) and the law of \( \xi \) is equal to the law of \( Z \), thus the element \( z_0 = 0 \) of \( F \) belongs to the support of \( \mathbb{P}^\xi \), which is the standard Wiener measure. By Remark 4.12, \( \Psi \) is also continuous with respect to \( z \). Hence, \( z \mapsto p(\cdot, z) \) is continuous and, according to Lemma 2.9, \( p(\cdot, 0) \) is the value of the regular conditional probability along any direction \( \{\xi^n\}_{n \geq 1} \) satisfying (B1), (B2). Therefore, for every \( A \in \mathcal{B}(\mathbb{E}) \)

\[
\mathbb{P} [X \in A | X \in \text{Coal}] = p(A, 0) = \mathbb{P} [\Psi(y, 0) \in A] = \mathbb{P} [y \in A],
\]

which concludes the proof of the theorem.

\[\square\]

5 Conditional distribution for infinite number of particles

The aim of the present section is the extension of the result of Theorem 4.1 to the infinite dimensional case. We are going to work with independent Brownian particles \( X_t(u), t \geq 0 \), labeled by \( u \in [0, 1] \), and starting at \( g(u) \), where \( g \in \mathcal{L}^2 \) is a strictly increasing function. As before, we want to show that the conditional distribution of \( X \) to the event that paths of \( X \) coalesce after meeting coincides with the distribution of the MMAF starting at \( g \). Since the coalescing of particles leads to the summation of their masses and slowing their diffusion rates, we must assume that all Brownian particles have zero masses and hence infinite diffusion rates. Therefore, a good candidate for such an “infinite family of independent Brownian particles \( X \) with infinite diffusion rates” would be a cylindrical Wiener process in \( L^2 \). Consequently, the main goal of this section is to find the conditional distribution of a cylindrical Wiener process in \( L^2 \) starting at \( g \) to the event that “its paths coalesce”.

Studying the conditional distribution of a cylindrical Wiener process, one must deal with three additional difficulties which did not appear in the case of a finite number of particles. First of all, the cylindrical Wiener process does not take values in the space \( L^2 \). Secondly, the analog to equation (14), which was needed to the construction of the map \( T \), cannot be solved for infinite dimensional case. And finally, we do not know if the regular conditional probability which we will obtain has a continuous version. The first and the third difficulties will be overcome by extending the state space and choosing the value of the regular conditional probability along some direction, respectively. However, the problem related with the solvability of an
analog to equation (14) cannot be overcome directly. We even do not know whether the equation
\[ Y_t = g + \int_0^t \text{pr}_{\gamma_s} dW_s, \quad t \geq 0, \] (17)
where \( W \) is a cylindrical Wiener process in \( L_2 \), admits a unique weak solution that takes values in \( L_2^\uparrow \). Note that (17) is a stochastic infinite-dimensional analog of (14).

To be able to find a conditional distribution of a cylindrical Wiener process to the set of “coalescing paths”, we will consider \( X \) as a pair \( X = (Y, W) \) of a cylindrical Wiener process \( W \) starting at \( g \) and a MMAF \( Y \) coupled⁴ by equality (17), and ask about the conditional distribution for the second component of \( X \). Let the pair \( X = (Y, W) \) be fixed during the whole section. The following statement claims that the distribution of \( X \) is independent of the choice of its second component.

**Proposition 5.1.** Let \( W_t, t \geq 0, \) be an another cylindrical Wiener process in \( L_2 \) starting at \( g \) such that the pair \( (Y, W) \) satisfies equation (17). Then \( \text{Law}(Y, W) = \text{Law}(Y, W) \). Moreover, there exists a cylindrical Wiener process \( B_t, t \geq 0, \) in \( L_2 \) independent of \( (Y, W) \) such that for every \( h \in L_2 \) almost surely
\[ W_t(h) = (\gamma_t, h)_{L_2} + \int_0^t \text{pr}_{\gamma_s} h \cdot dB_s, \quad t \geq 0. \] (18)

Since the proof of the proposition is technical, we will postpone it to the end of this section (see Section 5.4).

**Remark 5.2.** For the definition of the map \( T \) in the finite dimensional case we also worked with a pair \( (y, W) \) coupled by equation (14). However in that case, the modified massive Arratia flow \( y \) was uniquely determined by the standard Wiener process \( W \) in \( \mathbb{R}_n^m \), by Lemma 4.8. As we already noted, the uniqueness of MMAF remains an open problem for general initial condition \( g \). Therefore, the MMAF \( Y \) must be fixed and the distribution of \( X \) can only depend on \( Y \).

### 5.1 Set of coalescing trajectories

In this section, we will define a set of coalescing paths in the space \( C([0, \infty), L_2^\uparrow) \), denoted by \( \text{Coal} \), and build a basis \( \{e_k^\gamma, k \geq 0\} \) in \( L_2 \) generated by the

---

⁴For every MMAF \( Y \) there exists (probably on an extended probability space) a cylindrical Wiener process \( W \) such that (17) holds, according to properties (M1)-(M3) from Remark 3.4 and [GM11, Corollary 2.2]. However, it remains an open problem whether equation (17) has a strong solution.
5.1 Set of coalescing trajectories

MMAF $\mathcal{G}$ similarly as it was done in Section 4. The basis will be needed for
definition of the random element $\xi = T(X)$ in Section 5.2.

We will define $\textbf{Coal}$ as a set of functions $y$ from $C([0, \infty), L^2)$ which have
a version from $D((0,1), C[0, \infty))$, denoted also by $y$, such that

(G1) $y_0 = g$;

(G2) for each $t > 0$, $y_t \in \text{St}$;

(G3) for each $u, v \in (0, 1)$ and $s \geq 0$, $y_s(u) = y_s(v)$ implies $y_t(u) = y_t(v)$ for
every $t \geq s$;

(G4) the number of steps of the function $t \mapsto N(y_t)$, $t \geq 0$, is a càdlàg non-
increasing integer-valued function with jumps of height on $e$ and which
is constant equal to 1 for sufficiently large time.

The function $N(y_t)$ denotes a number of distinct values of $y_t$ for every $t$. Informally speaking, Condition (G4) requires that at every time $t$ at most
one coalescing of two particles can occur, and after some time only one
particle remains. This condition is not necessary but it will simplify our
computations.

We note that, according to Lemma C.2, the set $\textbf{Coal}$ is measurable in
$C([0, \infty), L^2)$. We will also consider $\textbf{Coal}$ as a metric subspace of $C([0, \infty), L^2)$. Moreover, at the end of this section we will prove that the MMAF belongs
to $\textbf{Coal}$ almost surely.

Let us introduce for every $y \in \textbf{Coal}$ the corresponding coalescence times:

$$\tau^y_k := \inf \{ t \geq 0 : N(y_t) \leq k \}, \quad k \geq 0. \quad (19)$$

Since $g$ is a strictly increasing function, one has that $N(g) = +\infty$, and
therefore, the family $\{ \tau^y_k, k \geq 0 \}$ is strictly decreasing for all $y \in \textbf{Coal}$, i.e.

$$+\infty = \tau^y_0 > \tau^y_1 > \tau^y_2 > \cdots > 0,$$

by Condition (G4).

Now we are going to define the basis $\{ e^y_k, k \geq 0 \}$. For each $h \in L^2$ denote
by $L_2(h)$ the subspace of $L_2$ consisting of all $\sigma(h)$-measurable functions.
Since $y_t$, $t \geq 0$, is an $L_2$-valued continuous function and $L_2(g) = L_2$ due to
the strong increase of $g$, it is easily seen that the closure of $\bigcup_{k=1}^{\infty} L_2(\tau^y_k)$
coincides with $L_2$. Let $H^y_k$ be the orthogonal complement of $L_2(\tau^y_k)$ in $L_2$,
k $\geq 1$.

**Lemma 5.3.** For every $y \in \textbf{Coal}$ there exists a unique basis $\{ e^y_k, k \geq 1 \}$ in
$L_2$ which satisfies the following properties
5.1 Set of coalescing trajectories

1) the family \( \{e^y_k, k < n\} \) is a basis of \( L_2(y_{\tau_k^y}) \) for each \( n \geq 1 \);

2) \( (e^y_k, \mathbb{I}_{(0,u)})_{L_2} \geq 0 \) for every \( u \in (0,1) \).

Moreover, the family \( \{e^y_k, k \geq n\} \) is a basis of \( H_n^y \) for each \( n \geq 1 \).

Proof. Let us construct the family \( \{e^y_k, k \geq 0\} \) explicitly by induction. The function \( y_{\tau_k^y} \) is constant on \([0,1]\), thus \( L_2(y_{\tau_k^y}) = \text{span}\{\mathbb{I}_{(0,1)}\} \). The only possible choice for a vector of norm 1 satisfying Condition 1) is \( e^y_0 = \mathbb{I}_{(0,1)} \).

Note that \( e^y_0 \) is a basis of \( L_2(y_t) \) for each \( t \geq \tau_k^y \), since \( L_2(y_t) = L_2(y_{\tau_k^y}) \) by the coalescence Property (G3).

Then fix \( k \geq 1 \) and assume that \( \{e^y_0, \ldots, e^y_{k-1}\} \) is already constructed basis of \( L_2(y_{\tau_k^y}) \). Since \( y_{\tau_k^y} \) is a non-decreasing step function with \( k+1 \) steps, there exists a partition of \([0,1]\) in ordered intervals \( I_1, \ldots, I_{k+1} \) such that

\[
L_2(y_t) = \left\{ \sum_{i=1}^{k+1} x_i \mathbb{I}_{I_i}, (x_1, \ldots, x_{k+1}) \in \mathbb{R}^{k+1} \right\}, \quad t \in [\tau_{k+1}^y, \tau_k^y). 
\]

At time \( \tau_k^y \) a coalescence occurs. So, there is \( i \in [k+1] \) such that \( I_i = [a,b) \) and \( I_{i+1} = [b,c) \), with \( a < b < c \), merge to form \( J_i = I_i \cup I_{i+1} = [a,c) \). Thus,

\[
L_2(Y_{\tau_k^y}) = \left\{ \sum_{\substack{i \in [k+1] \\cap J_i \neq \emptyset \\cap i \neq i+1}} x_i \mathbb{I}_{I_i} + x_i \mathbb{I}_{J_i}, (x_1, \ldots, x_i, x_{i+2}, \ldots, x_{k+1}) \in \mathbb{R}^k \right\}
\]

The vector \( e^y_k \) has to be normalized, to belong to \( L_2(y_{\tau_{k+1}^y}) \) and to be orthogonal to every element of \( L_2(y_{\tau_k^y}) \). There are exactly two possibilities:

\[
f = \frac{1}{\sqrt{c-a}} \left( \sqrt{\frac{c-b}{b-a}} \mathbb{I}_{(a,b)} - \sqrt{\frac{b-a}{c-b}} \mathbb{I}_{(b,c)} \right) \tag{20}
\]

and \(-f\). Only \( f \) satisfies Condition 1). So the only possible choice for \( e^y_k \) is \( f \). Hence the family \( \{e^y_0, \ldots, e^y_k\} \) is an orthonormal basis of \( L_2(y_{\tau_{k+1}^y}) \) and is unique. Since \( \bigcup_{k=1}^{\infty} L_2(y_{\tau_k^y}) = L_2 \), we get that \( \{e^y_k, k \geq 0\} \) form a basis in \( L_2 \).

Since for each \( n \geq 1 \), \( H_n^y = L_2(y_{\tau_n^y})_{\perp} \), it follows from Property 1) that \( \{e^y_n, k \geq n\} \) is an orthonormal basis of \( H_n^y \).

Remark 5.4. The construction of the basis \( \{e^y_k, k \geq 0\} \) in the proof of Lemma 5.3 easily implies that the map \( \text{Coal} : y \mapsto e^y_k \in L_2 \) is measurable for any \( k \geq 0 \), where \( \text{Coal} \) is endowed with the induced topology of \( C([0,\infty), L_2^1) \). Moreover, for every \( k \geq 1 \), \( e^y_k \) can be constructed using only \( y_t, t \leq \tau_k^y \).
Remark 5.5. All results of this section will remain true if $g$ is a general (not necessarily strictly) increasing function with $N(g) = +\infty$. Then one needs to work with $L_2(g)$ instead of $L_2$ as the state space. The case $N(g) < \infty$ corresponds to a finite number of particles, which was studied in Section 4.

Lemma 5.6. The process $\gamma_t$, $t \geq 0$, belongs almost surely to $\text{Coal}$.

Proof of Lemma 5.6. According to Proposition 3.6, properties (G1)-(G3) are satisfied. It remains to check that $\gamma$ satisfies (G4) almost surely. The monotonicity of $N(\gamma_t)$, $t \geq 0$, is a consequence of the coalescence property 1) in Proposition 3.6. Moreover, the fact that $N(\gamma_t) = 1$ for $t$ large enough is equivalent to $\tau^\gamma_1 < +\infty$ and the fact that every jump of $t \mapsto N(\gamma_t)$ is of height 1 is equivalent to say that the sequence $\{\tau^\gamma_k\}_{k \geq 1}$ is strictly decreasing. Those two facts are proved in Lemma B.2. 

5.2 Statement of the main result

The aim of this section is the construction of a random element $\xi = T(\gamma, \mathcal{W})$ and to state that the regular conditional probability of $X$ given $\xi$ leads to the MMAF. The random element $\xi$ will be constructed similarly as in Section 4, using the basis $e_k^\gamma$ and times $\tau^\gamma_k$, $k \geq 1$. According to the definition of $\tau^\gamma_k$, $\tau^\gamma_k$ are $(\mathcal{F}_t^\gamma)$-stopping times for all $k \geq 0$, where $(\mathcal{F}_t^\gamma)_{t \geq 0}$ is the complete right-continuous filtration generated by the MMAF $\gamma$. Furthermore, Remark 5.4 yields that $e^\gamma_k$ is an $\mathcal{F}_{\tau^\gamma_k}$-measurable random element in $L_2$. To simplify the notation, we will write $e_k$ and $\tau_k$ instead of $e^\gamma_k$ and $\tau^\gamma_k$, respectively.

Formally, $\xi_t$, $t \geq 0$, will be defined as follows

$$\xi_t = T_t(\gamma, \mathcal{W}) = \sum_{k=1}^\infty e_k \mathcal{W}_{t+\tau_k}(e_k), \quad t \geq 0.$$ 

Similarly as the cylindrical Wiener process $\mathcal{W}$, $\xi$ can not be defined as a random process taking values in $L_2$. Thus, we define $\xi_t$ as a map from an Hilbert space $L^0_2 := L_2 \ominus \text{span}\{1_{[0,1]}\}$ to $L_2(\Omega)$. We set

$$\xi_t(h) = T_t(\gamma, \mathcal{W})(h) := \sum_{k=1}^\infty (h, e_k) L_2 \mathcal{W}_{t+\tau_k}(e_k), \quad t \geq 0, \quad h \in L^0_2. \quad (21)$$

Let us first explain why $\mathcal{W}_{t+\tau_k}(e_k)$, $k \geq 1$, $t \geq 0$ is well-defined. It is not clear at first glance because $\mathcal{W}_t$ is a map from $L_2$ into $L_2(\Omega)$ for each $t \geq 0$. But recall that, by Proposition 5.1, there exists a cylindrical Wiener process $B_t$, $t \geq 0$, in $L_2$ independent of $\gamma$ such that

$$\mathcal{W}_t = \gamma_t + \int_0^t \mathcal{P}^{\gamma_t}_s dB_s, \quad t \geq 0.$$
Since the process \( \Pr_{Y_t \leq \tau_k} e_k, t \geq 0 \), is independent of \( B \), we can define for every \( k \geq 1 \)
\[
W_{t+\tau_k}(e_k) = (Y_{t+\tau_k}, e_k)_{L_2} + \int_0^{t+\tau_k} \mathbb{1}_{s \geq \tau_k} e_k \cdot dB_s
\]
\[
= \int_0^t e_k \cdot d(B_{s+\tau_k} - B_{\tau_k}), \quad t \geq 0.
\]
(22)

Here we have used Lemma 5.3 to conclude that \( \Pr_{Y_t \leq \tau_k} e_k = \mathbb{1}_{s \geq \tau_k} e_k \) and \((Y_{t+\tau_k}, e_k) = 0\).

**Proposition 5.7.** For every \( h \in L^0_2 \) the sum (21) converges almost surely in \( C[0, \infty) \). Moreover, \( \xi \) is a cylindrical Wiener process in \( L^0_2 \) that is independent of the MMAF \( \gamma \).

The proposition will be proved in the Section 5.3.1. Now, we formulate the main result about the conditional distribution of \( X = (Y, W) \). As we already noted, the cylindrical Wiener process \( W \) is not a random element in \( C([0, \infty), L_2) \). Therefore, we need to define an appropriate space \( E \) for \( X \). Let \( \{h_j, j \geq 0\} \) be a fixed orthonormal basis of \( L_2 \) such that \( h_0 = 1_{[0,1]} \). In particular, \( \{h_j, j \geq 1\} \) is an orthonormal basis of \( L^0_2 \). Then we can identify the cylindrical Wiener process \( W \) with the following sequence of independent Brownian motions
\[
\hat{W}_l = \left( \hat{W}_j(t) \right)_{j \geq 0} := (W_l(h_j))_{j \geq 0}, \quad t \geq 0,
\]
by the relation
\[
W_l(h) = \sum_{j=0}^{\infty} \hat{W}_j(t)(h, h_j)_{L_2}, \quad t \geq 0, \quad h \in L_2,
\]
where the series converges in \( C([0, \infty)) \) almost surely for every \( h \in L_2 \). Similarly, we identify \( \xi \) with
\[
\hat{\xi}_l = \left( \hat{\xi}_j(t) \right)_{j \geq 1} := (\xi_l(h_j))_{j \geq 1}, \quad t \geq 0,
\]
by the relation
\[
\xi_l(h) = \sum_{j=1}^{\infty} \hat{\xi}_j(t)(h, h_j)_{L_2}, \quad t \geq 0, \quad h \in L^0_2.
\]
Note that \( \hat{\xi} \) and \( \hat{W} \) are related by
\[
\hat{\xi}_j(t) = \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} (e_k, h_j)_{L_2} (e_k, h_i)_{L_2} \hat{W}_i(t + \tau_k), \quad t \geq 0, \quad j \geq 1.
\]
(23)
Trivially, \( \hat{W} \) and \( \hat{\xi} \) are random elements in \( C[0, \infty)^{\mathbb{N}_0} \) and \( C[0, \infty)^{\mathbb{N}} \), respectively, where \( C[0, \infty)^{\mathbb{N}_0} \) and \( C[0, \infty)^{\mathbb{N}} \) are Polish spaces (with the metric generated by the product topology) and \( \mathbb{N}_0 = \{0, 1, 2, \ldots \} \). Equality (23) implies that there exists a measurable map
\[
\hat{T} : C([0, \infty), L^2_{\mathbb{J}}) \times C[0, \infty)^{\mathbb{N}_0} \to C[0, \infty)^{\mathbb{N}}
\]
such that \( \hat{\xi} = \hat{T}(\gamma', \hat{W}) \) almost surely.

Remark 5.8. A simple computation shows that almost surely
\[
\hat{T}(\gamma', \hat{W}) = T(\gamma', \hat{W}).
\]

Let also
\[
\hat{\gamma}_t := \left( \hat{\gamma}^j(t) \right)_{j \geq 0} := (\gamma_t, h_j)_{L_2}, \quad t \geq 0.
\]

It is easily seen that
\[
\hat{\gamma}_t := \sum_{j=0}^{\infty} \hat{\gamma}^j(t) h_j, \quad t \geq 0,
\]
where the series converges almost surely in \( C([0, \infty), L_2) \).

We set \( \hat{X} = (\gamma', \hat{W}) \), \( E := C([0, \infty), L^2_{\mathbb{J}}) \times C[0, \infty)^{\mathbb{N}_0} \) and \( F := C[0, \infty)^{\mathbb{N}} \).

Let for each \( n \geq 1 \), \( \xi^n := (\xi^n_j)_{j \geq 1} \) be a sequence of the Ornstein-Uhlenbeck processes that are strong solutions to the equations\(^5\)
\[
\begin{align*}
\text{d}\xi^n_j(t) &= -\alpha^n_j \mathbf{1}_{\{t \leq n\}} \xi^n_j(t) \text{d}t + d\xi_j(t), \\
\xi^n_j(0) &= 0,
\end{align*}
\]
where \( \{\alpha^n_j, n, j \geq 1\} \) is a family of non-negative real numbers such that

\begin{itemize}
  \item[(O1)] for every \( n \geq 1 \) the series \( \sum_{j=1}^{\infty} (\alpha^n_j)^2 < +\infty \);
  \item[(O2)] for every \( j \geq 1 \), \( \alpha^n_j \to +\infty \) as \( n \to \infty \).
\end{itemize}

Remark 5.9. (i) Using Kakutani’s theorem [Kak48, p. 218] and Jensen inequality, it is easily seen that Condition (O1) guaranties the absolute continuity of \( F^{\xi^n} \) with respect to \( F^{\hat{\xi}} \) on \( C[0, \infty)^{\mathbb{N}} \). Hence, Assumption (B1) of Definition 2.7 is satisfied by the sequence \( \{\xi^n\}_{n \geq 1} \).

(ii) Condition (O2) yields the convergence in distribution of \( \{\xi^n\}_{n \geq 1} \) to 0 in \( C[0, \infty)^{\mathbb{N}} \) (see Lemma 5.17). Thus Assumption (B2) is also satisfied.

The following theorem is the main result of the present section.

**Theorem 5.10.** The law of \( (\gamma', \hat{\gamma}) \) is the value at 0 of the regular conditional probability of \( \hat{X} \) given \( \hat{\xi} \) along the sequence \( \{\xi^n\}_{n \geq 1} \).

\(^5\)We must write the indicator function in the drift of equation (24) to ensure the absolute continuity of the law of \( \xi^n_j \) with respect to the law of the Brownian motion \( \hat{\xi}_j \) on \( C[0, \infty) \). Otherwise, the laws are singular. For instance, without \( \mathbf{1}_{\{t \leq n\}} \) the process \( \xi^n_j \) would not satisfy the laws of iterated logarithms which is satisfied by \( \hat{\xi}_j \).
5.3 Proof of the main result

The main goal of this section is the proof of Theorem 5.10. We start from the proof of the fact that $\xi$ is a cylindrical Wiener process in $L^0_2$ stated in Proposition 5.7.

5.3.1 Proof of Proposition 5.7

Let us denote

$$\eta_k(t) = \eta_k(e_k) := W_t(e_k) + \tau_k(e_k) = \int_0^t e_k \cdot d(B_{s+\tau_k} - B_{\tau_k}), \quad t \geq 0, \quad k \geq 1.$$  

**Lemma 5.11.** The processes $\eta_k, k \geq 1$, are independent standard Brownian motions that do not depend on the MMAF $\gamma$.

**Proof.** We fix $n \geq 1$ and show that the processes $\gamma, \eta_k, k \in [n]$, are independent. Let

$$F_0 : C([0, \infty), L^1_0) \to \mathbb{R}, \quad F_k : C[0, \infty) \to \mathbb{R}, \quad k \in [n],$$

be bounded measurable functions. Using the strong Markov property of $B$ and the independence of $B$ and $\gamma$, we have that $B_{s+\tau_k} - B_{\tau_k}$ is also independent of $\gamma$. Hence for every $y \in \text{Coal},$

$$\eta_k(y) := \int_0^t e_k(y) \cdot d(B_{s+\tau_k} - B_{\tau_k}), \quad t \geq 0, \quad k \in [n],$$

are independent standard Brownian motions which do not depend on $\gamma$. Therefore, we can compute

$$\mathbb{E} \left[ F_0 (\gamma) \prod_{k=1}^n F_k (\eta(e_k)) \right] = \mathbb{E} \left[ \mathbb{E} \left[ F_0 (\gamma) \prod_{k=1}^n F_k (\eta(e_k)) \bigg| \gamma \right] \right]$$

$$= \mathbb{E} \left[ \mathbb{E} \left[ F_0 (y) \prod_{k=1}^n F_k (\eta(y)) \bigg| \gamma = \gamma \right] \right]$$

$$= \mathbb{E} \left[ \mathbb{E} \left[ F_0 (y) \prod_{k=1}^n F_k (w_k) \bigg| \gamma = \gamma \right] \right]$$

$$= \mathbb{E} \left[ F_0 (\gamma) \prod_{k=1}^n \mathbb{E} [F_k(w_k)] \right] = \mathbb{E} \left[ F_0 (\gamma) \prod_{k=1}^n \mathbb{E} [\eta(e_k)] \right],$$

where $w_k, k \in [n]$, are independent standard Brownian motions that do not depend on $\gamma$. This completes the proof of the lemma. \qed
5.3 Proof of the main result

Let \( h \in L^0_2 \) be fixed. For every \( y \in \text{Coal} \) and \( n \in \mathbb{N} \) we define

\[
M'^{y,n}_{t}(h) := \sum_{k=1}^{n} (e'_k, h)_{L_2} \eta_k(t), \quad t \geq 0.
\]

By Lemma 5.11, \( \eta_k, k \geq 1 \), are independent standard Brownian motions, hence \( M'^{y,n}_{t}(h), t \geq 0 \), is a continuous square-integrable martingale with respect to the filtration \((\mathcal{F}'_t)_{t \geq 0} \) generated by \( \eta_k, k \geq 1 \), with quadratic variation

\[
\langle M'^{y,n}(h) \rangle_t = \sum_{k=1}^{n} (e'_k, h)_{L_2}^2 t, \quad t \geq 0.
\]

Moreover, for each \( T > 0 \) the sequence of processes \( \{ M'^{y,n}(h) \}_{n \geq 1} \) restricted to the interval \([0,T] \) converges in \( L_2(\Omega, \mathcal{C}[0,T]) \). Indeed, for each \( m < n \), by Doob’s inequality

\[
\mathbb{E} \left[ \max_{t \in [0,T]} |M'^{y,n}_{t}(h) - M'^{y,m}_{t}(h)|^2 \right] = \mathbb{E} \left[ \max_{t \in [0,T]} \left| \sum_{k=m+1}^{n} (e'_k, h)_{L_2} \eta_k \right|^2 \right] \\
\leq 4 \sum_{k=m+1}^{n} (e'_k, h)_{L_2}^2 T,
\]

The sum \( \sum_{k=1}^{n} (e'_k, h)_{L_2}^2 \) converges to \( ||h||_{L_2}^2 \) because \( \{ e'_k, k \geq 1 \} \) is an orthonormal basis of \( L^0_2 \). Thus, \( \{ M'^{y,n}(h) \}_{n \geq 1} \) is a Cauchy sequence in \( L_2(\Omega, \mathcal{C}[0,T]) \), and hence, it converges to a limit denoted by \( M^y(h) = \sum_{k=1}^{\infty} (e'_k, h)_{L_2} \eta_k \). Trivially, \( M^y(h) \) can be well-defined for all \( t \geq 0 \), and, by [CE05, Lemma B.11], \( M^y(h), t \geq 0 \), is a continuous square-integrable \((\mathcal{F}'_t)\)-martingale with quadratic variation \( \langle M^y(h) \rangle_t = \lim_{n \to \infty} \langle M'^{y,n}(h) \rangle_t = ||h||_{L_2}^2 t, t \geq 0 \).

Remark that \( \sum_{k=1}^{\infty} (e'_k, h)_{L_2} \eta_k \) is a sum of independent random elements in \( \mathcal{C}[0,T] \). Hence, by Itô-Nisio’s Theorem 3.1 [IN68], the sequence \( \{ M'^{y,n}(h) \}_{n \geq 1} \) converges almost surely to \( M^y(h) \) in \( \mathcal{C}[0,T] \), and therefore, in \( \mathcal{C}[0,\infty) \). Recall that by Lemma 5.11, the sequence \( \{ \eta_k \}_{k \geq 1} \) is independent of \( \gamma' \), and by Lemma 5.6, \( \gamma' \) belongs to \( \text{Coal} \) almost surely. Then \( \sum_{k=1}^{\infty} (e_k, h)_{L_2} \eta_k \) also converges almost surely in \( \mathcal{C}[0,\infty) \) to a limit that we have called \( \xi(h) \). Indeed,

\[
\mathbb{P} \left[ \sum_{k=1}^{\infty} (e_k, h)_{L_2} \eta_k \text{ converges in } \mathcal{C}[0,\infty) \right] \\
= \mathbb{E} \left[ \mathbb{P} \left[ \sum_{k=1}^{\infty} (e_k, h)_{L_2} \eta_k \text{ converges in } \mathcal{C}[0,\infty) \mid \gamma' \right] \right] \\
= \mathbb{E} \left[ \mathbb{P} \left[ \sum_{k=1}^{\infty} (e'_k, h)_{L_2} \eta_k \text{ converges in } \mathcal{C}[0,\infty) \mid y = \gamma' \right] = 1. \right.
\]
5.3 Proof of the main result

We next are going to show that ξ is independent of ξ'. It is enough to check the independence of the processes ξ and \{ξ(h_i), \ i \in [n]\} for every h_i ∈ L^0_2, i ∈ [n], n ⩾ 1. But this can be proved similarly as Lemma 5.11.

Let us show that ξ is a cylindrical Wiener process. Obviously, h ↦ ξ(h) is a linear map. We denote h̃ for every h ∈ L^0_2, ξ(h) is an (F^η_t)-Brownian motion. According to Lévy’s characterization of Brownian motion [IW89, Theorem II.6.1], it is enough to show that ξ(h) is a continuous square-integrable \(\hat{F}^η_t\)-martingale with quadratic variation \(\|h\|^2_{L^2} t\). So, we take n ⩾ 1 and a boundend measurable function

\[ F : \mathcal{C}(0, \infty)^n \times \mathcal{C}([0, \infty), L^2) \to \mathbb{R}. \]

Then using Lemma 5.11 and the fact that \(M^η(h)\) is an \(F^η_t\)-martingale, we have for every \(s < t\)

\[
\mathbb{E}[\xi_t(h)F((\eta_k(\cdot \wedge s))_{k=1}^n \mid \mathcal{F}^η_s)] = \mathbb{E}\left[\mathbb{E}[\xi_t(h)F((\eta_k(\cdot \wedge s))_{k=1}^n \mid \mathcal{F}^η_s)] \mid \mathcal{F}^η_s\right] \\
= \mathbb{E}\left[\mathbb{E}\left[M^η_s(h)F((\eta_k(\cdot \wedge s))_{k=1}^n, y) \bigg| \mathcal{F}^η_s\right) \bigg| \mathcal{F}^η_s\right] \\
= \mathbb{E}\left[\mathbb{E}\left[M^η_s(h)F((\eta_k(\cdot \wedge s))_{k=1}^n, y) \bigg| \mathcal{F}^η_s\right) \bigg| \mathcal{F}^η_s\right] \\
= \mathbb{E}\left[\xi_t(h)F((\eta_k(\cdot \wedge s))_{k=1}^n \mid \mathcal{F}^η_s)\right].
\]

Hence, \(ξ(h)\) is an \(\hat{F}^η_t\)-martingale. Similarly, one can prove that \(\xi_t(h)^2 - \|h\|^2_{L^2} t, \ t ⩾ 0\), is also \(\hat{F}^η_t\)-martingale. This proves that ξ(h) is a continuous square-integrable \(\hat{F}^η_t\)-martingale with quadratic variation \(\|h\|^2_{L^2} t, \ t ⩾ 0\). The equality \(\mathbb{E}[\xi_t(h_1)\xi_t(h_2)] = t(h_1, h_2)_{L^2}, \ t ⩾ 0\), trivially follows from the polarization equality and the fact that \(ξ(h_1)\) and \(ξ(h_2)\) are martingales with respect to the same filtration \(\hat{F}^η_t\). Thus, \(ξ\) is a \(\hat{F}^η_t\)-cylindrical Wiener process in \(L^0_{L^2}\). This finishes the proof of the proposition.

5.3.2 Construction of the regular conditional probability

We will now apply the general method described in Section 2 in order to construct the conditional probability of \(\hat{X}\) given \(\xi\). For this we need to construct a quadruple \((G, Ψ, Y, Z)\) satisfying (P1)-(P4).

Let \(Z\) be a cylindrical Wiener process in \(L^0_2\) that is independent of \(Y\). We first define

\[
ψ(\mathcal{F}^η, Z) := (\mathcal{F}^η, ϕ(\mathcal{F}^η, Z)),
\]

where

\[
ϕ_t(\mathcal{F}^η, Z) := ϕ_t + \sum_{k=1}^{∞} e_k \mathbf{1}_{\{t ≥ τ_k\}} Z_{t-τ_k}(e_k), \ t ⩾ 0.
\]
As for $\xi$, the series in the right hand side of (25) do not converge in $L_2$. Consequently, we define $\varphi_t(\mathcal{Y}', \mathcal{Z})$ as a map from $L_2$ to $L_2(\Omega)$ determined by

$$\varphi_t(\mathcal{Y}', \mathcal{Z})(h) = (\mathcal{Y}_t, h)_{L_2} + \sum_{k=1}^{\infty} (e_k, h)_{L_2} \mathbb{1}_{\{t \geq \tau_k\}} Z_{t-\tau_k}(e_k)$$

(26)

for all $t \geq 0$ and $h \in L_2$. Since $\mathcal{Z}$ is independent of $e_k, k \geq 1$, $Z_t(e_k)$ is well-defined e.g. by $Z_t(e_k) = \int_0^t e_k \cdot \mathrm{d}Z_s, t \geq 0$. Moreover, as in the proof of Lemma 5.11, one can show that $\mathcal{Z}(e_k), k \geq 1,$ are independent standard Brownian motions that do not depend on $\mathcal{Y}'$.

**Proposition 5.12.** For each $h \in L_2$, the sum in (26) converges almost surely in $C[0, \infty)$. Furthermore, $\varphi(\mathcal{Y}', \mathcal{Z})$ is a cylindrical Wiener process in $L_2$ starting at $g$ and the law of $\psi(\mathcal{Y}', \mathcal{Z})$ is equal to the law of $X = (\mathcal{Y}, \mathcal{W})$.

**Proof.** Let us first show that the sum in (26) converges almost surely in $C[0, \infty)$. Fixing $y \in \text{Coal}$ and $h \in L_2$, we define for every $n \geq 1$

$$R^{y,n}_t(h) := \sum_{k=1}^n (e^y_k, h)_{L_2} \mathbb{1}_{\{t \geq \tau^y_k\}} Z_{t-\tau^y_k}(e_k), \quad t \geq 0.$$

Since $\mathcal{Z}(e_k), k \geq 1,$ are independent standard Brownian motions, one can easily check that $R^{y,n}_t(h), t \geq 0,$ is a continuous square-integrable martingale with respect to the filtration generated by $Z_{t-\tau^y_k}(e_k), k \geq 1$. As in the proof of Proposition 5.7, one can show that the sequence of partial sums $\{R^{y,n}_t(h)\}_{n \geq 1}$ converges in $C[0, \infty)$ almost surely for each $y \in \text{Coal}$. By the independence of $\mathcal{Z}(e_k), k \geq 1,$ and $\mathcal{Y}'$, one can see that the series

$$R^{y}_t(h) := \sum_{k=1}^{\infty} (e_k, h)_{L_2} \mathbb{1}_{\{t \geq \tau_k\}} Z_{t-\tau_k}(e_k), \quad t \geq 0,$$

also converges almost surely in $C[0, \infty)$.

Next, we claim that there exists a cylindrical Wiener process $\theta_t, t \geq 0,$ in $L^0_2$ independent of $\mathcal{Y}'$ such that

$$\mathcal{W}_t = \mathcal{Y}_t + \int_0^t \frac{\mathrm{pr}}{\mathcal{Y}_s} \mathrm{d}\theta_s, \quad t \geq 0.$$

(27)

Indeed, by Proposition 5.1, there is a cylindrical Wiener process $B_t, t \geq 0,$ in $L_2$ independent of $\mathcal{Y}'$ and satisfying equation (18). Taking $\theta$ equal to the restriction of $B$ to the sub-Hilbert space $L^0_2$, we easily check that
\[
\int_0^t \text{pr}_{\tilde{\mathcal{Y}}_s} \, d\theta_s = \int_0^t \text{pr}_{\tilde{\mathcal{Y}}_s} \, dB_s, \quad t \geq 0,
\]
for all \(s \geq 0\), \(\text{pr}_{\tilde{\mathcal{Y}}_s} = \text{pr}_{L^2} \circ \text{pr}_{\tilde{\mathcal{Y}}_s}\) almost surely. Furthermore, almost surely
\[
\int_0^t \text{pr}_{\tilde{\mathcal{Y}}_s} \, d\theta_s = \sum_{k=1}^{\infty} (e_k, h)_{L^2} \mathbb{1}_{\{t \geq \tau_k\}} (\theta_t(e_k) - \theta_{\tau_k}(e_k)), \quad t \geq 0.
\]

Thus, using the same argument as in the proof of (16), one can check that
\[
\text{Law} \left\{ (\mathcal{Y}_t, \int_0^t \text{pr}_{\tilde{\mathcal{Y}}_s} \, d\theta_s), t \geq 0 \right\} = \text{Law} \left\{ (\mathcal{Y}_t, \sum_{k=1}^{\infty} e_k \mathbb{1}_{\{t \geq \tau_k\}} Z_{t-\tau_k}(e_k)), t \geq 0 \right\}.
\]
This relation and equality (27) yield that the law of \(X = (\mathcal{Y}, W)\) is equal to
\[
\psi(\mathcal{Y}, Z) = (\mathcal{Y}, \varphi(\mathcal{Y}, Z)).
\]
In particular, \(\varphi(\mathcal{Y}, Z)\) is a cylindrical Wiener process in \(L^2\) starting at \(g\).

Proposition 5.13. Let \(T(\psi(\mathcal{Y}, Z))\) be defined by (21) with \((\mathcal{Y}, \mathcal{W})\) replaced by \(\psi(\mathcal{Y}, Z)\). Then almost surely \(T(\psi(\mathcal{Y}, Z)) = Z\), that is, for every \(h \in L^0_2\) almost surely \(T(\psi(\mathcal{Y}, Z))(h) = Z(h)\).

Proof. By the continuity of \(T(\psi(\mathcal{Y}, Z))(h), t \geq 0\), and \(Z(h), t \geq 0\), (in \(t\)), it is enough to show that for each \(t \geq 0\) almost surely
\[
T_t(\mathcal{Y}, \varphi(\mathcal{Y}, Z))(h) = Z_t(h).
\]
By definition (21) of \(T_t\),
\[
T_t(\mathcal{Y}, \varphi(\mathcal{Y}, Z))(h) = \sum_{k=1}^{\infty} (e_k, h)_{L^2} \varphi_{t+\tau_k}(\mathcal{Y}, Z)(e_k).
\]
Using definition (26) of \(\varphi_t\) and Lemma 5.3, we have
\[
\varphi_{t+\tau_k}(\mathcal{Y}, Z)(e_k) = (\mathcal{Y}_{t+\tau_k}, e_k)_{L^2} + \sum_{k=1}^{\infty} (e_l, e_k)_{L^2} \mathbb{1}_{\{t+\tau_k \geq \tau_l\}} Z_{t+\tau_k-\tau_l}(e_l) = \mathbb{1}_{\{t+\tau_k \geq \tau_l\}} Z_{t+\tau_k-\tau_l}(e_k) = Z_t(e_k).
\]
Hence, almost surely
\[
T_t(\mathcal{Y}, \varphi(\mathcal{Y}, Z))(h) = \sum_{k=1}^{\infty} (e_k, h)_{L^2} Z_t(e_k) = Z_t(h),
\]
where the last equality follows from the fact that \(h \in L^0_2\) and \(\{e_k, \quad k \geq 1\}\) is an orthonormal basis of \(L^0_2\).
5.3 Proof of the main result

Let \( \{h_j, j \geq 0\} \) be the basis of \( L_2 \) fixed in Section 5.2. Since for every \( k \geq 1 \)

\[
\mathbb{Z}_{-\tau_k}(e_k) = \sum_{i=1}^{\infty} (h_i, e_k)_{L_2} \mathbb{Z}_{-\tau_k}(h_i),
\]

where the series converges in \( C[0, \infty) \) almost surely, we get almost surely

\[
\varphi_t(Y, Z)(h) = \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} (e_k, h)_{L_2} (h_i, e_k)_{L_2} \mathbb{I}_{\{t \geq \tau_k\}} \mathbb{Z}_{t-\tau_k}(h_i), \quad t \geq 0.
\]

Hence, there exists a measurable map

\[
\hat{\varphi} : E \to C[0, \infty)^{\mathbb{N}_0}
\]

such that

\[
\hat{\varphi}(Y, Z) = \overline{\varphi(Y, Z)}.
\]

**Corollary 5.14.** Almost surely \( \hat{T}(Y, \hat{\varphi}(Y, Z)) = \hat{Z} \).

**Proof.** Using Remark 5.8 and Proposition 5.13, we obtain

\[
\hat{T}(Y, \hat{\varphi}(Y, Z)) = \hat{T}(\varphi(Y, Z)) = T(\psi(Y, Z)) = T(\psi(Y, Z)) = \hat{Z}.
\]

We now define \( G := \text{Coal}, Y := \gamma, Z := \hat{Z} \) and the map \( \Psi : G \times F \to E \) by

\[
\Psi(y, z) := (y, \hat{\varphi}(y, z)).
\]

Then, trivially, conditions (P1) and (P2) are satisfied. By Corollary 5.14, we have \( \hat{T}(\Psi(Y, Z)) = Z \) almost surely, that implies (P3). The fact that \( \hat{X} \) and \( \Psi(Y, Z) \) have the same distribution follows from Proposition 5.12. So, (P4) also holds. Hence, by Proposition 2.3, the probability kernel \( p \) defined by

\[
p(A, z) := \mathbb{P}[\Psi(Y, z) \in A] = \mathbb{P}[\gamma, \hat{\varphi}(\gamma, z) \in A] \quad (28)
\]

for all \( A \in B(E) \) and \( z \in F \), is a regular conditional probability of \( \hat{X} \) given \( T(\hat{X}) \).

### 5.3.3 Value of \( p \) along a sequence of Ornstein-Uhlenbeck processes

The last part of the proof of Theorem 5.10 consists in showing that convergence (6) holds for the regular conditional probability \( p \) and the sequence \( \{\xi^n\}_{n \geq 1} \) defined by (28) and equation (24), respectively. Let \( \{h_j, j \geq 0\} \) be
the orthonormal basis of $L_2$ fixed in Section 5.2. Recall that $\{h_j, j \geq 1\}$ is an orthonormal basis of $L_0^2$. For $y \in \textbf{Coal}$ we consider

$$
\Psi(y, \xi^n) = (y, \hat{\varphi}(y, \xi^n)),
$$

where the map $\hat{\varphi} : E \to C[0, \infty)^{\mathbb{N}_0}$ was defined in Section 5.3.2. Since for every $n \geq 1$ the law of $\xi^n$ is absolutely continuous with respect to the law of $\hat{\xi}$ (or $\hat{Z}$), we have that for almost all $y \in \textbf{Coal}$ with respect to the law of $Y$

$$
\hat{\varphi}_j (y, \xi^n) = (y, h_j) + \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} (e^n_k, h_j)_{L_2} (e^n_l, e^n_k)_{L_2} \mathbb{1}_{\{t \geq t^n_j\}} \xi^n(t - t^n_j),
$$

for each $j \geq 0$, where the series converges in $C[0, \infty)$ almost surely. Without loss of generality, we may assume that equality (29) holds for all $y \in \textbf{Coal}$. Otherwise, we can work with a measurable subset of $\textbf{Coal}$ for which equality (29) is valid almost surely.

**Proposition 5.15.** Let $\varepsilon \in (0, 1)$ and $y \in \textbf{Coal}$ be such that the series $\sum_{k=1}^{\infty} (t^n_k)^{1-\varepsilon}$ converges. Then the sequence of processes $\Psi(y, \xi^n)$, $n \geq 1$, converges in distribution to $(y, \hat{y})$ in $E = C([0, \infty), L^1_j) \times C([0, \infty) \mathbb{N}_0$, where $\hat{y} = ((y, h_j)_{L_2})_{j \geq 0}$.

Let us fix $y \in \textbf{Coal}$ satisfying the assumption of Proposition 5.15. Before starting the proof, we define for all $j \geq 1$

$$
R^n_j (t) := \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} (e^n_k, h_j)_{L_2} (e^n_l, e^n_k)_{L_2} \mathbb{1}_{\{t \geq t^n_j\}} \xi^n(t - t^n_j), \quad t \geq 0,
$$

and $R^n_t := (R^n_j (t))_{j \geq 0}, t \geq 0$. Note that it is sufficient to prove that

$$
R^n \overset{d}{\to} 0 \quad \text{in} \quad C[0, \infty)^{\mathbb{N}_0}, \quad n \to \infty.
$$

Indeed, this will imply that

$$
\Psi(y, \xi^n) = (y, \hat{\varphi}(y, \xi^n)) = \left( y, \hat{y} + \hat{R}^n \right) \overset{d}{\to} (y, \hat{y}) \quad \text{in} \quad E,
$$

where $\hat{R}^n = (0, R^n_1, R^n_2, \ldots)$.

Let us first prove some auxiliary lemmas.

**Lemma 5.16.** For every $j, n \geq 1$ and $0 \leq s \leq t \leq n$,

$$
\mathbb{E} \left[ (\xi^n_j(t) - \xi^n_j(s))^2 \right] \leq \frac{1}{\alpha_j^n} \wedge (t - s),
$$

where $\frac{1}{\alpha_j} := +\infty$. 

**Proof.**
5.3 Proof of the main result

Proof. The covariation of the Ornstein-Uhlenbeck processes $\xi^n_j$, $j, n \geq 1$, is equal to

$$\text{cov} (\xi^n_j(t), \xi^n_j(s)) = \frac{e^{-\alpha_j^n(t-s)} - e^{-\alpha_j^n(t+s)}}{2\alpha_j^n}, \quad 0 \leq s \leq t \leq n.$$ 

Denoting by $x = 1 - e^{-\alpha_j^n(t-s)}$, a simple computation shows that

$$\mathbb{E} \left[ (\xi^n_j(t) - \xi^n_j(s))^2 \right] = \frac{2x - x^2 e^{-2\alpha_j^n s}}{2\alpha_j^n} \leq \frac{x}{\alpha_j^n} = 1 - e^{-\alpha_j^n(t-s)}.$$ 

This yields inequality (31). \qed

**Lemma 5.17.** The sequence $\{\xi^n\}_{n \geq 1}$ converges in distribution to 0 in $C[0, \infty)^N$.

Proof. In order to prove the lemma, we first show that the sequence $\{\xi^n\}_{n \geq 1}$ is tight in $C[0, \infty)^N$. This will imply that the sequence $\{\xi^n\}_{n \geq 1}$ is relatively compact, by Prohorov’s theorem. Then we will show that every (weakly) convergent subsequence of $\{\xi^n\}_{n \geq 1}$ converges to 0. This will immediately yield that $\xi^n \xrightarrow{d} 0$ in $C[0, \infty)^N$.

According to [EK86, Proposition 3.2.4], the tightness of $\{\xi^n\}_{n \geq 1}$ will follow from the tightness of $\{\xi^n_j\}_{n \geq 1}$ in $C[0, \infty)$ for every $j \geq 1$. So, let $j \geq 1$ and $T > 0$ be fixed. We take $0 \leq s \leq t \leq T$ and, using Lemma 5.16 and the fact the $\xi^n_j$ is a Gaussian process, one can estimate for $n \geq T$

$$\mathbb{E} \left[ (\xi^n_j(t) - \xi^n_j(s))^4 \right] \leq 3 \mathbb{E} \left[ (\xi^n_j(t) - \xi^n_j(s))^2 \right]^2 \leq 3(t - s)^2.$$ 

Moreover, $\xi^n_j(0) = 0$. Hence, by the Kolmogorov-Chentsov tightness criterion (see e.g. [Kal02, Corollary 16.9]), the sequence of processes $\{\xi^n_j\}_{n \geq 1}$ restricted to $[0, T]$ is tight in $C[0, T]$. Since $T > 0$ was arbitrary, we get that $\{\xi^n_j\}_{n \geq 1}$ is tight in $C[0, \infty)$. Hence, $\{\xi^n\}_{n \geq 1}$ is tight in $C[0, \infty)^N$.

Next, let $\{\xi^n\}_{n \geq 1}$ converges in distribution to $\xi^0$ in $C[0, \infty)^N$ along a subsequence $N \subseteq \mathbb{N}$. Then for every $t \geq 0$ and $j \geq 1$ $\{\xi^n_j(t)\}_{n \geq 1}$ converges in distribution to $\xi^0_j(t)$ in $\mathbb{R}$ along $N$. But on the other hand,

$$\mathbb{E} \left[ (\xi^n_j(t))^2 \right] \leq \frac{t}{\alpha^n_j} \xrightarrow{n \to \infty} 0,$$

by Lemma 5.16 and Assumption (O2) in Section 5.2. Hence, $\xi^0_j(t) = 0$ almost surely for all $t \geq 0$ and $j \geq 1$. Thus, we have obtained that $\xi^0 = 0$, and therefore, $\xi^n \xrightarrow{d} 0$ in $C[0, \infty)^N$ as $n \to \infty$. \qed

To prove that $\{R^n\}_{n \geq 1}$ converges to 0, we will use the same argument as in the proof of Lemma 5.17. So, we start from the tightness of $\{R^n\}$. 

Lemma 5.18. Under the assumption of Proposition 5.15, the sequence \( \{R^n_j\}_{n \geq 0} \) is tight in \( C[0, \infty) \).

Proof. Again, according to [EK86, Proposition 3.2.4], it is enough to check that the sequence \( \{R^n_j\}_{n \geq 1} \) is tight in \( C[0, \infty) \) for every \( j \geq 1 \). So, let \( j \geq 1 \) be fixed. We set

\[
R_j^{n,1}(t) := \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} (e_k^n, h_j)L_2(h_l, e_k^n) L_2 \xi^n_i(t), \quad t \geq 0,
\]

and

\[
R_j^{n,2}(t) := \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} (e_k^n, h_j)L_2(h_l, e_k^n) L_2 \left( \mathbb{1}_{\{t \geq \tau_k^n\}} \xi^n_i(t - \tau_k^n) - \xi^n_i(t) \right), \quad t \geq 0.
\]

Then \( R_j^n = R_j^{n,1} + R_j^{n,2} \). We will prove the tightness separately for \( \{R_j^{n,1}\}_{n \geq 1} \) and \( \{R_j^{n,2}\}_{n \geq 1} \).

**Tightness of \( \{R_j^{n,1}\}_{n \geq 1} \).**

Using the fact that \( \{e_k^n, k \geq 1\} \) and \( \{h_j, j \geq 1\} \) are bases of \( L^0_2 \), a simple computation shows that almost surely

\[
\Gamma(\hat{\xi}) := \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} (e_k^n, h_j)L_2(h_l, e_k^n) L_2 \hat{\xi}_l = \hat{\xi}_j.
\]

Due to the absolute continuity of the law of \( \xi^n \) with respect to the law of \( \hat{\xi} \) and the equality \( \Gamma(\xi^n) = R_j^{n,1} \), we get that \( R_j^{n,1} = \xi^n_j \). Hence

\[
R_j^{n,1} \overset{d}{\to} 0 \quad \text{in } C[0, \infty), \quad n \to \infty,
\]

by Lemma 5.17. In particular, \( \{R_j^{n,1}\}_{n \geq 1} \) is tight in \( C[0, \infty) \), according to Prohorov’s theorem.

**Tightness of \( \{R_j^{n,2}\}_{n \geq 1} \).**

**Step I.** For any \( t \geq 0 \) and \( n \geq t \) the vector

\[
V^n_t := \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} e_k^n(h_l) L_2 \left( \mathbb{1}_{\{t \geq \tau_k^n\}} \xi^n_i(t - \tau_k^n) - \xi^n_i(t) \right)
\]

belongs almost surely to \( L^0_2 \) and \( \mathbb{E} \left[ \|V^n_t\|_{L^2_2}^2 \right] \leq \sum_{k=1}^{\infty} (t \wedge \tau_k^n) < \infty \).

Indeed, by Parseval’s equality (with respect to the orthonormal family \( \{e_k^n, k \geq 1\} \)),

\[
\|V^n_t\|^2_{L^2_2} = \sum_{k=1}^{\infty} \left( \sum_{l=1}^{\infty} (e_k^n, h_l)L_2 \left( \mathbb{1}_{\{t \geq \tau_k^n\}} \xi^n_i(t - \tau_k^n) - \xi^n_i(t) \right) \right)^2.
\]
Recall that the sequence \( \{\xi^n_t\}_{t \geq 1} \) is independent. Thus, it follows that

\[
E \left[ \left\| V^n_t \right\|_{L^2}^2 \right] \leq \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} (e^y_k, h_l)_{L^2}^2 E_{k,l}^n(t),
\]

where

\[
E_{k,l}^n(t) = E \left[ \left( \mathbb{1}_{\{t \geq \tau^n_k\}} \xi^n_k(t - \tau^n_k) - \xi^n_l(t) \right)^2 \right].
\]

Since \( \xi^n_t(0) = 0 \), we have

\[
E_{k,l}^n(t) = \mathbb{1}_{\{t \geq \tau^n_k\}} E \left[ \left( \xi^n_k(t - \tau^n_k) - \xi^n_l(t) \right)^2 \right] + \mathbb{1}_{\{t < \tau^n_k\}} E \left[ \left( \xi^n_l(0) - \xi^n_l(t) \right)^2 \right].
\]

By inequality (31), we can deduce that

\[
E_{k,l}^n(t) \leq \mathbb{1}_{\{t \geq \tau^n_k\}} \tau^n_k + \mathbb{1}_{\{t < \tau^n_k\}} t = t \wedge \tau^n_k.
\]

Therefore,

\[
E \left[ \left\| V^n_t \right\|_{L^2}^2 \right] \leq \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} (e^y_k, h_l)_{L^2}^2 (t \wedge \tau^n_k)
\]

\[
= \sum_{k=1}^{\infty} \left\| e^y_k \right\|_{L^2}^2 (t \wedge \tau^n_k) = \sum_{k=1}^{\infty} (t \wedge \tau^n_k),
\]

by Parseval’s identity (with respect to the orthonormal family \( \{h_l, l \geq 1\} \)). Moreover, \( \sum_{k=1}^{\infty} (t \wedge \tau^n_k) \leq t^2 \sum_{k=1}^{\infty} (\tau^n_k)^{1-\varepsilon} < \infty \). Therefore, for any \( t > 0 \), \( V^n_t \) belongs to \( L^2 \) almost surely. In particular, for every \( t \geq 0 \) and \( n \geq t \) the inner product \( (V^n_t, h_j)_{L^2} \) is well-defined, and almost surely \( R^n_{\varepsilon,2}(t) = (V^n_t, h_j)_{L^2} \).

Step II. Let \( T > 0 \). There exists \( C_{y,\varepsilon} \) depending on \( y \) and \( \varepsilon \) such that for all \( 0 \leq s \leq t \leq T \) and \( n \geq T \),

\[
E \left[ \left( R^n_{\varepsilon,2}(t) - R^n_{\varepsilon,2}(s) \right)^2 \right] \leq C_{y,\varepsilon} (t - s)^{\varepsilon}.
\]

Indeed, proceeding as in Step I, we get

\[
E \left[ \left\| V^n_t - V^n_s \right\|_{L^2}^2 \right] = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} (e^y_k, h_l)_{L^2}^2 E \left[ \left( \mathbb{1}_{\{t \geq \tau^n_k\}} \xi^n_k(t - \tau^n_k) - \xi^n_l(t) - \mathbb{1}_{\{s \geq \tau^n_k\}} \xi^n_l(s - \tau^n_k) + \xi^n_l(s) \right)^2 \right] \]

\[
\leq \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} (e^y_k, h_l)_{L^2}^2 4 ((t - s) \wedge \tau^n_k) = 4 \sum_{k=1}^{\infty} ((t - s) \wedge \tau^n_k),
\]
where the inequality follows as previously from inequality (31). It follows that
\[
E \left[ \left( R_{j}^{n,2}(t) - R_{j}^{n,2}(s) \right)^{2} \right] \leq E \left[ \| V_{t}^{n} - V_{s}^{n} \|_{L_2}^{2} \right] \\
\leq 4 \sum_{k=1}^{\infty} \left( (t - s) \wedge \tau_{k}^{y} \right) \\
\leq 4 (t - s)^{\xi} \sum_{k=1}^{\infty} \left( \tau_{k}^{y} \right)^{1 - \varepsilon}.
\]

By assumption on \( y \), the series \( \sum_{k=1}^{\infty} \left( \tau_{k}^{y} \right)^{1 - \varepsilon} \) converges, so the proof of Step II is achieved.

**Step III.** There exists \( \alpha > 0, \beta > 0 \) and \( C_{y,\varepsilon} \) depending on \( y \) and \( \varepsilon \) such that for all \( 0 \leq s \leq t \leq T \) and \( n \geq T \),
\[
E \left[ \left( R_{j}^{n,2}(t) - R_{j}^{n,2}(s) \right)^{\alpha} \right] \leq C_{y,\varepsilon} (t - s)^{1 + \beta}.
\]

Indeed, for any \( s \leq t \) from \( [0, T] \), \( R_{j}^{n,2}(t) - R_{j}^{n,2}(s) \) is a random variable with normal distribution \( N(0, \sigma_{j}^{2}) \). By Step II, \( \sigma_{j}^{2} \leq C_{y,\varepsilon} (t - s)^{\varepsilon} \). Therefore, for any \( p \geq 1 \),
\[
E \left[ \left( R_{j}^{n,2}(t) - R_{j}^{n,2}(s) \right)^{2p} \right] \leq (2p - 1)!! \left( \sigma_{j}^{2} \right)^{p} \leq C_{p,y,\varepsilon} (t - s)^{\varepsilon p}.
\]
The proof of Step III follows by choosing \( p \) larger than \( \frac{1}{\varepsilon} \).

**Step IV.** By Kolmogorov-Chentsov tightness criterion (see e.g. [Kal02, Corollary 16.9]), it follows from Step III and the equality \( R_{j}^{n,2}(0) = 0, n \geq 1 \), that the sequence of processes \( \{ R_{j}^{n,2} \}_{n \geq 1} \) restricted to \( [0, T] \) is tight in \( C[0, T] \) for every \( T > 0 \). Hence, \( \{ R_{j}^{n,2} \}_{n \geq 1} \) is tight in \( C[0, \infty) \).

**Conclusion.** As the sum of two tight sequences, the sequence \( \{ R_{j}^{n} \}_{n \geq 1} \) is tight in \( C[0, \infty) \) for any \( j \geq 1 \). Since \( C[0, \infty)^{N} \) is equipped with the product topology, it follows from [EK86, Proposition 3.2.4] that the sequence \( \{ R^{n} \}_{n \geq 1} \) is tight in \( C[0, \infty)^{N} \). \( \square \)

**Lemma 5.19.** For every \( j \geq 1 \) and \( t \geq 0 \), \( E \left[ (R_{j}^{n}(t))^{2} \right] \to 0 \) as \( n \to \infty \).

**Proof.** Let \( j \geq 1 \) and \( t \geq 0 \) be fixed. We recall that \( R_{j}^{n} = R_{j}^{n,1} + R_{j}^{n,2} \). Remark that \( R_{j}^{n,1} = \xi_{j}^{n} \) almost surely. Thus, using inequality (31), we have
\[
E \left[ \left( R_{j}^{n,1}(t) \right)^{2} \right] = E \left[ (\xi_{j}^{n}(t))^{2} \right] \leq \frac{t}{\alpha_{j}^{n}} \to 0, \quad n \to \infty.
\]
Due to the equality $R_n^{n,2}(t) = (V_t^n, h_j)_{L^2}$, we can estimate for $n \geq t$

$$
\mathbb{E} \left[ \left( R_n^{n,2}(t) \right)^2 \right] \leq \mathbb{E} \left[ \| V_t^n \|_{L^2}^2 \right] = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} (e_k^n, h_l)_L^2 E_{k,l}(t).
$$

Recall that estimates (34) and (35) give that

$$
E_{k,l}(t) \leq t \wedge \tau_k^l, \quad k, l \geq 1,
$$

and $\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} (e_k^n, h_l)_L^2 (\tau_k \wedge t) < \infty$. Since (33) and (31) imply for every $k, l \geq 1$

$$
0 \leq E_{k,l}^n(t) \leq \frac{1}{\alpha_l^n} \to 0, \quad n \to \infty,
$$

we have

$$
\mathbb{E} \left[ \| V_t^n \|_{L^2}^2 \right] \to 0, \quad n \to \infty,
$$

by the dominated convergence theorem.

Consequently,

$$
\mathbb{E} \left[ \left( R_n^{n}(t) \right)^2 \right] \leq 2 \mathbb{E} \left[ \left( R_n^{n,1}(t) \right)^2 \right] + 2 \mathbb{E} \left[ \left( R_n^{n,2}(t) \right)^2 \right] \to 0, \quad n \to \infty,
$$

that concludes the proof of the lemma. \hfill \square

**Proof of Proposition 5.15.** Lemma 5.18 and Prohorov’s theorem yield that the sequence $\{R_n\}_{n \geq 1}$ is relatively compact in $C[0, \infty)^{\mathbb{N}}$. To show that the sequence $\{R_n\}_{n \geq 1}$ converges in distribution to 0, it is enough to show that every its weakly convergence subsequence has the same limit equals 0. Let $R_n \overset{d}{\to} R^0$ in $C[0, \infty)^{\mathbb{N}}$ along a subsequence $N \subseteq \mathbb{N}$. Then for every $t \geq 0$ and $j \geq 1$, $R_n(t) \overset{d}{\to} R_j^0(t)$ in $\mathbb{R}$ along $N$. But by Lemma 5.19, $\mathbb{E} \left[ (R_n^j(t))^2 \right] \to 0$ as $n \to \infty$. Hence, $R_j^0(t) = 0$ almost surely for every $t \geq 0$ and $j \geq 1$. This implies that $R^0 = 0$ and consequently, $R_n \overset{d}{\to} 0$ in $C[0, \infty)^{\mathbb{N}}$ as $n \to \infty$. The proof of the proposition is finished. \hfill \square

**Proof of Theorem 5.10.** Let $\{\xi^n\}_{n \geq 1}$ be as before and $p$ be defined by (28). For the proof of Theorem 5.10, we need to show that for every $f \in C_b(\mathbb{E})$,

$$
\mathbb{E} \left[ \int_{\mathbb{E}} f(x) p(dx, \xi^n) \right] \to \int_{\mathbb{E}} f(x) \nu(dx), \quad n \to \infty,
$$

where $\nu = p(\cdot, 0) = \text{Law}(\mathbb{Y}, \mathbb{\hat{Y}})$.

For any $f \in C_b(\mathbb{E})$ and $z \in \mathbb{F}$, it follows from the definition of $p$ that

$$
\int_{\mathbb{E}} f(x) p(dx, z) = \mathbb{E} [f(\Psi(\mathbb{Y}, z))].
$$
Therefore, by Fubini’s Theorem, we have
\[
\mathbb{E} \left[ \int \mathbb{E} \left[ f(x)p(dx, \xi^n) \right] \right] = \mathbb{E} \left[ \mathbb{E} \left[ f(\Psi(y, z)) \right] \right] = \mathbb{E} \left[ \mathbb{E} \left[ f(\Psi(y, \xi^n)) \right] \right].
\]

Next, using lemmas 5.6 and B.5, we obtain that \( Y \) belongs almost surely to \( \text{Coal} \) and the series \( \sum_{k=1}^{\infty} (\tau_k^y)^{1-\varepsilon} \) converges almost surely for each \( \varepsilon \in (0, \frac{1}{2}) \). Therefore, Proposition 5.15 implies that \( \mathbb{E} \left[ \mathbb{E} \left[ f(\Psi(y, \xi^n)) \right] \right] \) converges almost surely for each \( \varepsilon \in (0, \frac{1}{2}) \).

Therefore, Proposition 5.15 implies that \( \mathbb{E} \left[ \mathbb{E} \left[ f(\Psi(y, \xi^n)) \right] \right] \) almost surely to \( f(Y, \hat{X}) \). By the dominated convergence theorem, we finally deduce that
\[
\mathbb{E} \left[ \mathbb{E} \left[ f(\Psi(y, \xi^n)) \right] \right] \rightarrow \mathbb{E} \left[ f(Y, \hat{X}) \right] = \int \mathbb{E} f(x)p(dx), \quad n \rightarrow \infty.
\]

This concludes the proof of the theorem. \( \square \)

We remark that the map \( \hat{T} \) was defined up to the set of measure zero with respect to the law of \( \hat{X} \). Therefore, we may not talk about the preimage \( T^{-1}(\{0\}) \) because it is not well-defined. However, it turns out that the zero-fiber of the function \( z \mapsto \varphi(y, z) \) coincides almost surely with \( \text{Coal} \). We state this result precisely in the following lemma.

**Lemma 5.20.** For every \( y \in \text{Coal} \) and \( z = (z_k)_{k \leq 1} \in C_0(0, \infty)^\mathbb{N} \) define similarly to (25)
\[
\varphi(y, z) = y_1 + \sum_{k=1}^{\infty} e_k^y \mathbb{1}_{\{t > \tau_k^y\}} z_k(t - \tau_k^y), \quad t \geq 0,
\]
if the series converges in \( C([0, \infty), L_2) \). Then for each \( y \in \text{Coal} \), \( \varphi(y, z) \) belongs to \( \text{Coal} \) if and only if \( z = 0 \).

**Proof.** It is obvious that \( \varphi(y, 0) = y \in \text{Coal} \).

We assume now that \( \varphi(y, z) \) belongs to \( \text{Coal} \) and prove that \( z = 0 \). Set
\[
\gamma(y, z) = \sum_{k=1}^{\infty} e_k^y \mathbb{1}_{\{t > \tau_k^y\}} z_k(t - \tau_k^y), \quad t \geq 0,
\]
and show that \( \gamma(y, z) = 0 \). This will immediately imply \( z = 0 \).

**Step I.** Let \( k \geq 1 \) be fixed. By (20), there exist \( a < b < c \) such that
\[
e_k^y = \frac{1}{\sqrt{c-a}} \left( \sqrt{\frac{c-b}{b-a}} \mathbb{1}_{(a,b)} - \sqrt{\frac{b-a}{c-b}} \mathbb{1}_{(b,c)} \right).
\]
The goal of this step is to show that $\varphi_{\tau_k^y}(y, z)(u) = y_{\tau_k^y}(u)$ for every $u \in [a, c)$, in other words, that $\gamma_{\tau_k^y}(y, z)$ is equal to zero on the interval $[a, c)$.

By the construction of $\tau_k^y$ and $e_k^y$, $y_{\tau_k^y}$ is constant on the interval $[a, c)$. Furthermore, since $\varphi(y, z) \in \text{Coal}$, $\varphi_{\tau_k^y}(y, z)$ belongs to $L_2^+$. Hence, we can deduce that $\gamma_{\tau_k^y}(y, z) = \varphi_{\tau_k^y}(y, z) - y_{\tau_k^y}(y, z)$ is non-decreasing on $[a, c)$, as a difference of a non-decreasing function and a constant function. Furthermore,

$$\gamma_{\tau_k^y}(y, z) = \sum_{l=k}^{\infty} e_l^y z_l(\tau_k^y - \tau_l^y) = 0,$$

since $z_k(0) = 0$. Hence, $\gamma_{\tau_k^y}(y, z)$ belongs to span $\{e_l^y, l \geq k + 1\}$, whereas $\mathbf{1}_{[a, b]}$ and $\mathbf{1}_{[a, c)}$ both belong to span $\{e_l^y, l \leq k\}$. Indeed, $\mathbf{1}_{[a, c)} \in L_2(y_{\tau_k^y}) = \text{span} \{e_l^y, l < k\}$, by Lemma 5.3, and $\mathbf{1}_{[a, b]} \in \text{span} \{\mathbf{1}_{[a, c)}, e_k^y\}$. Recall that $\{e_l^y, l \geq 0\}$ is an orthonormal basis of $L_2$. Thus,

$$\left(\gamma_{\tau_k^y}(y, z), \mathbf{1}_{[a, b]} \right)_{L_2} = \left(\gamma_{\tau_k^y}(y, z), \mathbf{1}_{[a, c)} \right)_{L_2} = 0.$$

So, we can deduce that $u \mapsto (\gamma_{\tau_k^y}(y, z), \mathbf{1}_{[a, u]})_{L_2}$ is a convex function on $[a, c]$ which vanishes at $a, b$ and $c$. Thus, it is zero everywhere on $[a, c]$. In particular, $\gamma_{\tau_k^y}(y, z)(u) = 0$ for every $u \in (a, c)$. Consequently, $\varphi_{\tau_k^y}(y, z)(u) = y_{\tau_k^y}(u)$ for every $u \in (a, c)$. The equality also holds for $u = a$, by the right-continuity of $\varphi_{\tau_k^y}(y, z)$ and $y_{\tau_k^y}$.

Step II. Now let $t > 0$ be fixed. By Property (G2) of the definition of Coal in Section 5.1, $y_t$ belongs to $S_t$, and thus,

$$y_t(u) = \sum_{j=1}^n y_j \mathbf{1}_{[a_j, c_j]}(u),$$

for pairwise distinct $y_j, j \in [n]$. Fix $j \in [n]$. By coalescence Property (G3), there exists $k \geq 1$ such that $u \mapsto y_t(u)$ is constant on $[a_j, c_j)$ for every $s \geq \tau_k^y$ and non-constant on $[a_j, c_j)$ for every $s < \tau_k^y$. By Step I, $y_{\tau_k^y} = \varphi_{\tau_k^y}(y, z)$ on $[a_j, c_j)$. Thus, $\varphi_{\tau_k^y}(y, z)$ is constant on $[a_j, c_j)$. By Property (G3) again, now applied to $\varphi(y, z)$, $\varphi_t(y, z)$ is constant on $[a_j, c_j)$ due to $t \geq \tau_k^y$. As the difference of two constant functions, $\gamma_t(y, z)$ is also constant on $[a_j, c_j)$. Moreover, by the construction of $\gamma$ and Lemma 5.3, $\gamma_t(y, z)$ is orthogonal to $L_2(y_t)$. Hence $\gamma_t(y, z)$ is also orthogonal to $\mathbf{1}_{[a_j, c_j)}$. Therefore, we can conclude that $\gamma_t(y, z) = 0$ on $[a_j, c_j)$. Since $j \in [n]$ and $t > 0$ were arbitrary, we deduce that $\gamma_t(y, z) = 0$ on $[0, 1)$ for every $t > 0$. This finishes the proof of the lemma.

$\square$
5.4 Proof of Proposition 5.1

We first remark that by the construction of the basis \( \{ e_k, \, k \geq 0 \} \) in the proof of Lemma 5.3, \( e_k^y \) is uniquely defined by \( y(t \wedge \tau_k) \), \( t \geq 0 \), for every \( k \geq 1 \). This implies that for every \( k \geq 1 \), the random element \( e_k \) is \( \mathcal{F}_{t-}^B \)-measurable. Hence, \( \text{pr}^\gamma_t e_k = \mathbb{1}_{\{t \geq \tau_k\}} e_k, \, t \geq 0 \), is a right-continuous \( (\mathcal{F}_t^\gamma) \)-adapted process in \( L_2 \). We take a cylindrical Wiener process \( W_t, \, t \geq 0 \), in \( L_2 \) independent of \( (\gamma, \mathcal{W}) \), fix an orthonormal basis \( \{ h_j, \, j \geq 0 \} \) in \( L_2 \) and define for every \( k \geq 0 \)

\[
B_k(t) := \int_0^t \mathbb{1}_{\{s < \tau_k\}} h_k \cdot dW_s + \int_0^t \mathbb{1}_{\{s \geq \tau_k\}} e_k \cdot dW_s, \quad t \geq 0. \tag{36}
\]

Recall that \( \tau_0 = +\infty \), so \( B_0(t) = W_t(h_0) \).

**Lemma 5.21.** The processes \( B_k, \, k \geq 0 \), defined by (36), are independent standard Brownian motions.

**Proof.** Let \( (\mathcal{F}_t^\gamma)_{t \geq 0} \) be the complete right-continuous filtration generated by \( X = (\gamma, \mathcal{W}) \). Then trivially, \( \mathcal{W} \) is an \( (\mathcal{F}_t^\gamma) \)-cylindrical Wiener process. Since the cylindrical Wiener process \( W \) is independent of \( X \), the filtration \( (\mathcal{F}_t^\gamma) \) can be extended to the complete right-continuous filtration \( (\mathcal{F}_t^{\gamma, W})_{t \geq 0} \) generated by \( X \) and \( W \). In this case, \( \mathcal{W} \) and \( W \) will be \( (\mathcal{F}_t^{\gamma, W}) \)-cylindrical Wiener processes. Since the \( L_2 \)-valued right-continuous processes \( \mathbb{1}_{\{t < \tau_k\}} h_k, \, t \geq 0 \), and \( \mathbb{1}_{\{t \geq \tau_k\}} e_k, \, t \geq 0 \), are \( (\mathcal{F}_t^{\gamma, W}) \)-adapted, they are \( (\mathcal{F}_t^{\gamma, W}) \)-progressively measurable, and hence, the processes \( B_k(t), \, t \geq 0, \, k \geq 0 \), are \( (\mathcal{F}_t^{\gamma, W}) \)-continuous martingales. We next compute their joint quadratic variations

\[
(B_k, B_l)_t = \int_0^t \mathbb{1}_{\{s < \tau_k \wedge \tau_l\}} (h_k, h_l)_{L_2} ds + \int_0^t \mathbb{1}_{\{s \geq \tau_k \vee \tau_l\}} (e_k, e_l)_{L_2} ds = \mathbb{1}_{\{k = l\}} t,
\]

for every \( k, l \geq 0 \) and \( t \geq 0 \). By Lévy’s characterization of Brownian motion [IW89, Theorem II.6.1], \( B_k, \, k \geq 0 \), are independent \( (\mathcal{F}_t^{\gamma, W}) \)-Brownian motions. \( \square \)

Our next goal is to show that the family of independent Brownian motions \( B_k, \, k \geq 0 \), is independent of \( \gamma \). Before we prove some auxiliary statements.

**Lemma 5.22.** Let \( g \in \text{St} \) and \( W \) be a cylindrical Wiener process. Then there exists a unique continuous \( L_2^g \)-valued process \( Y \) such that almost surely

\[
Y_t = g + \int_0^t \text{pr}_Y dW_t^g, \quad t \geq 0, \tag{37}
\]

where \( W_t^g = \int_0^t \text{pr}_g dW_s, \, t \geq 0 \).
5.4 Proof of Proposition 5.1

Proof. Using [Kon17a, Proposition 2.2], we can conclude that almost surely $\text{pr}_y \text{pr}_{Y_s} = \text{pr}_{Y_0} \text{pr}_Y$, $s \geq 0$. Hence, we get that equation (37) is equivalent to the equation

$$Y_t = g + \int_0^t \text{pr}_{Y_s} dW_s, \quad t \geq 0.$$ In particular, every strong solution to (37) satisfies properties of (M1)-(M3). By Proposition 3.6, $Y_t$, $t \geq 0$, is a MMAF starting at $g \in \text{St}$. Following Section 4.1, there exists a MMAF $y(t)$, $t \geq 0$, such that $Y_t = \Xi_m(y(t))$, $t \geq 0$, for some masses $m = \{m_k, k \in [n]\}$ and $x^0$ satisfying $g = \Xi_m(x^0)$. A simple computation shows that $y$ solves the following equation

$$y(t) = x^0 + \int_0^t \text{pr}_{y(s)} d\eta(s), \quad t \geq 0,$$ (38) with the standard Brownian motion $\eta_t$, $t \geq 0$, in $\mathbb{R}^n_m$, satisfying $W^g(t) = \Xi_m(\eta(t))$, $t \geq 0$. Inversely, if $y$ is a strong solution to equation (38), then $Y_t = \Xi_m(y(t))$, $t \geq 0$, solves (37) with $W^g(t) = \Xi_m(\eta(t))$ and $g = \Xi_m(x^0)$. Hence the statement of the lemma directly follows from Lemma 4.8.

For every $k \geq 1$ we remark that $W^k_t := W_{t+\tau_k} - W_{\tau_k}$, $t \geq 0$, is a cylindrical Wiener process independent of $\mathcal{F}^X_{\tau_k}$. Since $\text{pr}_{\mathcal{F}^X_{\tau_k}}$ is $\mathcal{F}^X_{\tau_k}$-measurable, the process

$$\zeta^k_t := \int_0^t \text{pr}_{\mathcal{F}^X_{\tau_k}} dW^k_s, \quad t \geq 0,$$

is a well-defined continuous $L^2$-valued $(\mathcal{F}^X_{t+\tau_k})$-martingale.

Definition 5.23. Let $\mathcal{G}_k$ be the complete $\sigma$-algebra generated by $X(t \wedge \tau_k) = (\mathcal{Y}_{t \wedge \tau_k}, \mathcal{W}_{t \wedge \tau_k})$, $t \geq 0$, and by $\zeta^k_t$, $t \geq 0$.

Lemma 5.24. For every $k \geq 1$ the MMAF $\mathcal{Y}$ is $\mathcal{G}_k$-measurable as a map from $\Omega$ to $\mathcal{C}([0, \infty), L^2)$.

Proof. In order to show the measurability of $\mathcal{Y}$ with respect to $\mathcal{G}_k$, it is enough to show that $\mathcal{Y}_{\tau_k+t}$, $t \geq 0$, is uniquely determined by $\zeta^k$ and $\mathcal{Y}_{\tau_k}$.

We consider the equation

$$Z_t = \mathcal{Y}_{\tau_k} + \int_0^t \text{pr}_{Z_s} d\zeta^k_s, \quad t \geq 0.$$ (39)

Since $Y_{\tau_k}$ belongs to St almost surely and is independent of $W^k$, equation (39) has a unique solution taking values in $L^2$, by Lemma 5.22. On the other
This implies that \( Z_t = \mathcal{Y}_{t+\tau_k}, t \geq 0 \), and therefore, \( \mathcal{Y}_{t+\tau_k}, t \geq 0 \), is uniquely determined by \( \zeta^k \) and \( \mathcal{Y}_{\tau_k} \). This implies that \( \mathcal{Y} \) is \( \mathcal{G}_k \)-measurable.

**Lemma 5.25.** Let \( W_t, t \geq 0 \), be a cylindrical Wiener process, \( g_0 \in \text{St} \) and \( \{ g_l, l \in [n] \} \) be an orthonormal family belonging to \( L_2(g_0)^\perp \). Then

\[
W_t(g_l) = \int_0^t g_l \cdot dW_s, \quad t \geq 0, \quad l \in [n]
\]

are independent standard Brownian motions that do not depend on \( g_0 \).

**Proof.** Let \( f_j, j \in [k] \), be an orthonormal basis of \( L_2(g_0) \). Then the family \( \{ f_j, g_l, j \in [k], l \in [n] \} \) is orthonormal. Consequently, \( W(f_j), W(g_l), j \in [k], l \in [n] \), are independent Brownian motions. Since

\[
W_t^{g_0} = \sum_{j=1}^k f_j W_t(f_j), \quad t \geq 0,
\]

the statement of the lemma holds.

For \( l \geq k \geq 1 \) we recall that \( \tau_1 \leq \tau_k \) almost surely and the random element \( e_l \) is \( \mathcal{F}^X_{\tau_1} \)-measurable. Hence, \( e_l \) is \( \mathcal{F}^X_{\tau_k} \)-measurable due to \( \mathcal{F}^X_{\tau_1} \subseteq \mathcal{F}^X_{\tau_k} \). Since \( \mathcal{W}_t^k \) is a cylindrical Wiener process in \( L_2 \) independent of \( \mathcal{F}^X_{\tau_k} \), the process

\[
\mathcal{W}_t^k(e_l) := \int_0^t e_l \cdot d\mathcal{W}_s^k = \int_{\tau_k}^{t+\tau_k} e_l \cdot d\mathcal{W}_s^k, \quad t \geq 0,
\]

is well-defined. It is easily seen that \( \mathcal{W}_t^k(e_l) = B_l(\cdot + \tau_k) - B_l(\tau_k) \). In particular, we have almost surely

\[
\mathcal{W}_t^k(e_l) = \mathcal{W}_{t+\tau_k-\tau_1}^l(e_l) - \mathcal{W}_{\tau_k-\tau_1}^l(e_l), \quad t \geq 0, \quad (40)
\]

for every \( l \geq k \geq 1 \).

**Lemma 5.26.** For every \( k \geq 1 \) the processes \( \mathcal{W}_t^k(e_l), l \geq k \), are independent and do not depend on \( \mathcal{G}_k \). Furthermore, for each \( l > k \), \( \mathcal{W}_{l\wedge \tau_k}^k(e_l) \) is \( \mathcal{G}_k \)-measurable, where \( \tau_{k,l} := \tau_k - \tau_l \).
5.4 Proof of Proposition 5.1

Proof. We first show that $\mathcal{G}_k$, $\mathcal{W}^k(\epsilon_l)$, $l \geq k$, are independent. Let $n \geq k$ and $m \geq 1$ be fixed. Let $h_j$, $j \geq 0$, be an arbitrary orthonormal basis of $L_2$. We consider bounded measurable functions

$$F_0 : \mathcal{C}([0, \infty), L^2_\mathbb{R}) \times \mathcal{C}([0, \infty), L_2) \times \mathcal{C}([0, \infty)^m \to \mathbb{R}$$

and

$$F_1 : \mathcal{C}(0, \infty) \to \mathbb{R}, \quad l = k, \ldots, n,$$

Using the independence of $\mathcal{W}^k$ from $\mathcal{F}^\mathcal{X}$, we can compute the following expectation

$$E := \mathbb{E} \left[ F_0 \left( \mathcal{G} \cup \tau_k, \zeta^k, (\mathcal{W} \cup \tau_k(h_j))_{j=1}^m \right) \prod_{l=k}^n F_1 \left( \mathcal{W}^k(\epsilon_l) \right) \right]$$

$$= \mathbb{E} \left[ \mathbb{E} \left[ F_0 \left( \mathcal{G} \cup \tau_k, \zeta^k, (\mathcal{W} \cup \tau_k(h_j))_{j=1}^m \right) \prod_{l=k}^n F_1 \left( \mathcal{W}^k(\epsilon_l) \right) \right] \Big| \mathcal{F}^\mathcal{X} \right]$$

$$= \mathbb{E} \left[ \mathbb{E} \left[ F_0 \left( \mathcal{G} \cup \tau_k, \zeta^k, (\mathcal{W} \cup \tau_k(h_j))_{j=1}^m \right) \prod_{l=k}^n F_1 \left( \mathcal{W}^k(\epsilon_l) \right) \right] \right|_{y=\mathcal{G} \cup \tau_k, x_j=\mathcal{W} \cup \tau_k(h_j)}$$

where

$$\zeta^{y,k}_t = \int_0^t \text{pr}_{y,k} \, d\mathcal{W}^k_s, \quad t \geq 0.$$ 

By Lemma 5.25, $\mathcal{G}^{y,k}$, $\mathcal{W}^k(\epsilon_l)$, $l = k, \ldots, n$, are independent. Thus,

$$E = \mathbb{E} \left[ \mathbb{E} \left[ F_0 \left( y, \zeta^{y,k}_t, (x_j)_{j=1}^m \right) \right] \right|_{y=\mathcal{G} \cup \tau_k, x_j=\mathcal{W} \cup \tau_k(h_j)}$$

where $w_j$, $l = k, \ldots, n$, are standard independent Brownian motions that do not depend on $\mathcal{G}$ and $\mathcal{W}$. Hence,

$$E = \prod_{l=k}^n \mathbb{E} \left[ F_1 \left( \mathcal{W}^k(\epsilon_l) \right) \right] \mathbb{E} \left[ F_0 \left( y, \zeta^{y,k}_t, (x_j)_{j=1}^m \right) \right|_{y=\mathcal{G} \cup \tau_k, x_j=\mathcal{W} \cup \tau_k(h_j)}$$

This implies the independence of $\mathcal{W}^k(\epsilon_l)$, $l \geq k$, and $\mathcal{G}_k$.

We next show that for every $l > k$, the process $\mathcal{W}^k_{\mathcal{G} \cup \tau_k}(\epsilon_l)$ is $\mathcal{G}_k$-measurable. We remark that $\eta_l$ and $\tau_l$ are $\mathcal{G}_k$-measurable because they are $\mathcal{F}^\mathcal{X}_{\tau_l}$-measurable and $\mathcal{F}^\mathcal{X}_{\tau_l} \subseteq \mathcal{F}^\mathcal{X}_{\tau_k} \subseteq \mathcal{G}_k$. Then the process $\mathcal{W}^k_{\mathcal{G} \cup \tau_k,l} = \mathcal{W}_{(\mathcal{G} \cup \tau_k,l) + \tau_l} - \mathcal{W}_{\tau_l}$, $t \geq 0$, is $\mathcal{G}_k$-measurable, and consequently, $\mathcal{W}^k_{\mathcal{G} \cup \tau_k,l}(\epsilon_l)$ is also $\mathcal{G}_k$-measurable. This finishes the proof of the lemma. \qed
We define a map \( \text{Gl} : C_0[0, \infty)^2 \times [0, \infty) \rightarrow C_0[0, \infty) \) as follows
\[
\text{Gl}(x_1, x_2, r)(t) = x_1(t \wedge r) + x_2((t - r)^+), \quad t \geq 0,
\]
where \( a^+ := a \vee 0 \). It is easily seen that the map \( \text{Gl} \) is continuous and therefore measurable.

Using equality (40), a simple computation shows that for every \( l > k \geq 1 \) almost surely
\[
W^l(e_l) = \text{Gl} \left( W^l_{\tau_{k,l}}(e_l), W^k(e_l), \tau_{k,l} \right),
\]
where \( \tau_{k,l} := \tau_k - \tau_l \). This equality will be used for the proof of the following lemma.

**Lemma 5.27.** The processes \( \gamma \), \( W^k(e_k), k \geq 1 \), are independent.

**Proof.** To prove the lemma we will use the mathematical induction. Let \( F_0 : C([0, \infty), L^2_2) \rightarrow \mathbb{R} \) and \( F_l : C([0, \infty) \rightarrow \mathbb{R}, l \geq 1 \), be bounded measurable functions. Let \( n \geq 2 \) be fixed. Using lemmas 5.24 and 5.26, we first compute the following expectation
\[
\mathbb{E} \left[ F_n \left( W^m(e_n) \right) F_0 \left( \gamma \right) \right] = \mathbb{E} \left[ \mathbb{E} \left[ F_n \left( W^m(e_n) \right) F_0 \left( \gamma \right) \mid G_n \right] \right]
= \mathbb{E} \left[ F_0 \left( \gamma \right) \mathbb{E} \left[ F_n \left( W^m(e_n) \right) \right] \right]
= \mathbb{E} \left[ F_0 \left( \gamma \right) \mathbb{E} \left[ F_n \left( W^m(e_n) \right) \right] \right].
\]
This implies the independence of \( \gamma \) and \( W^m(e_n) \). We next assume that \( \gamma \), \( W^{k+1}(e_{k+1}), \ldots, W^m(e_n) \) are independent. Again, one has:
\[
\mathbb{E} \left[ F_0 \left( \gamma \right) \prod_{l=k}^{n} F_l \left( W^l(e_l) \right) \right] = \mathbb{E} \left[ \mathbb{E} \left[ F_0 \left( \gamma \right) \prod_{l=k}^{n} F_l \left( W^l(e_l) \right) \mid G_k \right] \right].
\]
Using lemmas 5.24, 5.26 and equality (41), we can compute
\[
\mathbb{E} \left[ F_0 \left( \gamma \right) \prod_{l=k}^{n} F_l \left( W^l(e_l) \right) \mid G_k \right]
= F_0 \left( \gamma \right) \mathbb{E} \left[ \prod_{l=k+1}^{n} F_l \left( \text{Gl} \left( W^l_{\tau_{k,l}}(e_l), W^k(e_l), \tau_{k,l} \right), F_k \left( W^k(e_k) \right) \mid G_k \right) \right]
= F_0 \left( \gamma \right) \mathbb{E} \left[ \prod_{l=k+1}^{n} F_l \left( \text{Gl} \left( x_l, W^k(e_l), \tau_{k,l} \right) \right) F_k \left( W^k(e_k) \right) \mid x_{l} = W^l_{\tau_{k,l}}(e_l), t = \tau_{k,l} \right]
= F_0 \left( \gamma \right) \mathbb{E} \left[ \prod_{l=k+1}^{n} F_l \left( W^l(e_l) \right) \mid G_k \right] \mathbb{E} \left[ F_k \left( W^k(e_k) \right) \right].
\]
Finally, by the induction assumption,
\[
\mathbb{E} \left[ F_0(\mathcal{Y}) \prod_{l=k}^{n} F_l(\mathcal{W}^l(e_l)) \right] = \mathbb{E} \left[ F_0(\mathcal{Y}) \prod_{l=k+1}^{n} F_l(\mathcal{W}^l(e_l)) \left. \right| F_k(\mathcal{W}^k(e_k)) \right] \\
= \mathbb{E} \left[ F_0(\mathcal{Y}) \prod_{l=k}^{n} \mathbb{E} \left[ F_l(\mathcal{W}^l(e_l)) \right] \right].
\]

Hence, for every \( n \geq 2 \), \( \mathcal{Y}, \mathcal{W}^1(e_1), \ldots, \mathcal{W}^n(e_n) \) are independent. This implies the statement of the lemma.

**Lemma 5.28.** Let \( w_i(t), t \geq 0, i = 1, 2, \) be independent standard Brownian motions and \( r > 0 \). Then the process
\[
w(t) = \text{Gl}(w_1, w_2, r)
\]
is a Brownian motion.

**Proof.** The statement of the lemma trivially follows from the fact that \( w \) is a continuous Gaussian process with covariance \( \mathbb{E}[w(s)w(t)] = s \wedge t \).

**Lemma 5.29.** The processes \( \mathcal{Y}, B_k, k \geq 0 \), are independent.

**Proof.** The lemma will be proved similarly to Lemma 5.26. Let \( w_k, k \geq 1 \), denote independent standard Brownian motions independent of \( \mathcal{Y}, \mathcal{W} \) and \( W \). We first remark that
\[
B_k = \text{Gl} \left( W(h_k), \mathcal{W}^k(e_k), \tau_k \right), \quad k \geq 1.
\]
Let \( G_0 : C([0, \infty), L^2_\mathbb{R}) \to \mathbb{R}, F_k : C[0, \infty) \to \mathbb{R}, k \geq 0 \), be bounded measurable functions. Using the independence of \( \mathcal{Y}, W(h_0), W(h_k), \mathcal{W}^k(e_k), k \geq 1 \), which follows from Lemma 5.27, and the measurability of \( e_k, \tau_k, k \geq 1 \), with respect to \( \mathcal{F}^\mathcal{Y} := \sigma(\mathcal{Y}) \), we obtain
\[
\mathbb{E} \left[ G_0(\mathcal{Y}) F_0(B_0) \prod_{k=1}^{n} F_k(B_k) \right] \\
= \mathbb{E} \left[ G_0(\mathcal{Y}) F_0(W(h_0)) \prod_{k=1}^{n} F_k \left( \text{Gl} \left( W(h_k), \mathcal{W}^k(e_k), \tau_k \right) \right) \right] \\
= \mathbb{E} \left[ G_0(\mathcal{Y}) \mathbb{E} \left[ F_0(W(h_0)) \prod_{k=1}^{n} F_k \left( \text{Gl} \left( W(h_k), \mathcal{W}^k(e_k), \tau_k \right) \right) \left| \mathcal{F}^\mathcal{Y} \right. \right] \right] \\
= \mathbb{E} \left[ G_0(\mathcal{Y}) \mathbb{E} \left[ F_0(W(h_0)) \prod_{k=1}^{n} F_k \left( \text{Gl} \left( W(h_k), \mathcal{W}^k(e_k), \tau_k^y \right) \right) \right. \left| y = \mathcal{Y} \right. \right].
\[ \begin{align*}
\mathbb{E} \left[ G_0 (\mathcal{G}) \mathbb{E} [F_0 (W(h_0))] \prod_{k=1}^{n} \mathbb{E} [F_k (w_k)] \right] \\
= \mathbb{E} [G_0 (\mathcal{G})] \mathbb{E} [F_0 (B_0)] \prod_{k=1}^{n} \mathbb{E} [F_k (B_k)],
\end{align*} \]

where we have used Lemma 5.28 to conclude that \( G_l (W(h_k), w^k(e_k), \tau^k_l) \), \( k \geq 1 \), is a family of independent standard Brownian motions for every \( y \in \text{Coal} \).

To finish the proof of Proposition 5.1, we take
\[ B_t(h) := \sum_{k=0}^{\infty} (h, e_k)_{L_2} B_k(t), \quad h \in L_2. \]

Since \( B_k, k \geq 0 \), are independent Brownian motions that do not depend on \( \mathcal{G} \) and hence on \( e_k, k \geq 1 \), one can show similarly to the proof of Lemma 5.11 that the series converges in \( C[0, \infty) \) almost surely for every \( h \in L_2 \), and \( B_t, t \geq 0 \), is a cylindrical Wiener process in \( L_2 \).

Moreover, \( B \) is independent of \( \mathcal{G} \). Indeed, it is sufficient to show that for each \( n \geq 1 \) and each \( h_1, \ldots, h_n \) in \( L_2 \), \( B(h_1), \ldots, B(h_n) \) are independent of \( \mathcal{G} \). Indeed, for any bounded and measurable functions \( F : C[0, \infty)^n \to \mathbb{R} \) and \( G : C([0, \infty), L_2^*) \to \mathbb{R} \),
\[ \mathbb{E} [F(B(h_1), \ldots, B(h_n)) G(\mathcal{G})] = \mathbb{E} \left[ \mathbb{E} \left[ F(B(h_1), \ldots, B(h_n)) \mid \mathcal{G} \right] G(\mathcal{G}) \right] = \mathbb{E} \left[ \mathbb{E} \left[ F(w_1, \ldots, w_n) \mid \mathcal{G} \right] G(\mathcal{G}) \right] = \mathbb{E} \left[ F(B(h_1), \ldots, B(h_n)) \right] \mathbb{E} \left[ G(\mathcal{G}) \right], \]

where \( w_l, l \in [n] \), denote independent standard Brownian motions independent of \( \mathcal{G} \) and \( B \).

Moreover, for each \( h \in L_2 \) almost surely
\[ \int_0^t pr_{\mathcal{G}^s}^+ h \cdot dB_s = \sum_{k=1}^{\infty} \int_0^t \left( pr_{\mathcal{G}^s}^+ h, e_k \right)_{L_2} dB_k(s) \]
\[ = \sum_{k=1}^{\infty} \int_0^t \mathbb{1}_{s \geq \tau_k} \left( pr_{\mathcal{G}^s}^+ h, e_k \right)_{L_2} dW_k(e_k) \]
\[ = \sum_{k=1}^{\infty} \int_0^t \mathbb{1}_{s \geq \tau_k} \left( pr_{\mathcal{G}^s}^+ h, e_k \right)_{L_2} dW_s(e_k) \]
\[ = \sum_{k=1}^{\infty} \int_0^t \left( pr_{\mathcal{G}^s}^+ h, e_k \right)_{L_2} dW_s(e_k) \]
\[
\begin{align*}
&= \int_0^t \varphi_\gamma^t h \cdot dW_s' \\
&= \int_0^t h \cdot dW_s' - \int_0^t \varphi_\gamma^s h \cdot dW_s \\
&= W_t(h) - (g, h)_{L^2} - (\gamma_t - g, h)_{L^2} = W_t(h) - (\gamma_t, h)_{L^2}
\end{align*}
\]
for all \( t \geq 0 \).

If \((\gamma', W')\) is another coupling satisfying (17), then there exists a cylindrical Wiener process \(B'\) starting at \(g\) and independent of \(\gamma\) such that
\[
W'_t = \gamma_t + \int_0^t \varphi_\gamma^s dB'_s, \quad t \geq 0.
\]
Since \((\gamma, B)\) and \((\gamma', B')\) have the same distribution, we can conclude that \((\gamma', W')\) and \((\gamma', W')\) have the same distribution too. This finishes the proof of Proposition 5.1.

A Appendix: Proof of Lemma 2.9

We first recall that the sufficiency of Lemma 2.9 immediately follows from the continuous mapping theorem.

We next prove the necessity. We first choose a family \(\{f_k, k \geq 1\} \subset C_b(E)\) which strongly separate points in \(E\). One can show that such a family exists since \(F\) is separable (see also [BK10, Lemma 2]). By [EK86, Theorem 4.5] (or [BK10, Theorem 6] for weaker assumptions on the space \(F\)), any sequence \(\{\mu_n\}_{n \geq 1}\) of probability measures on \(E\) converges weakly to a probability measure \(\mu\) if and only if
\[
\int_E f_k(x) \mu_n(dx) \to \int_E f_k(x) \mu(dx), \quad n \to \infty,
\]
for all \(k \geq 1\).

We define the following sets
\[
A_{m}^{k,+} = \left\{ z \in F : \int_E f_k(x)p(dx, z) - \int_E \nu(dx) \geq \frac{1}{m} \right\},
\]
\[
A_{m}^{k,-} = \left\{ z \in F : \int_E \nu(dx) - \int_E f_k(x)p(dx, z) \geq \frac{1}{m} \right\}
\]
for all \(k \geq 1\) and \(m \geq 1\). Let also \(A_{m}^{k} = A_{m}^{k,+} \cup A_{m}^{k,-}\).

Lemma A.1. If for every \(k \geq 1\) and \(m \geq 1\) there exists \(\delta_m^k > 0\) such that
\[
\mathbb{P}^z \left[ A_{m}^{k} \cap B_{m}^{k} \right] = 0,
\]
(42)
where $B_m^k$ is the ball in $F$ with center $z_0$ and radius $\delta_m^k$, then $p$ has a version continuous at $z_0$. Moreover, it can be taken as

$$p'(\cdot, z) = \begin{cases} p(\cdot, z), & \text{if } z \not\in \bigcup_{k,m=1}^{\infty} (A_m^k \cap B_m^k), \\ \nu, & \text{otherwise.} \end{cases}$$

**Proof.** We first remark that according to (42), $p' = p$ $P^\xi$-a.e. Next, let $z_n \to z_0$ in $F$ as $n \to \infty$. Without loss of generality, we may assume that $z_n \not\in \bigcup_{k,m=1}^{\infty} (A_m^k \cap B_m^k)$ for all $n \geq 1$. Let $m \geq 1$ and $k \geq 1$ be fixed. Then there exists a number $N$ such that $z_n \in B_m^k$ for all $n \geq N$. Consequently, $z_n \not\in A_m^k$, $\forall n \geq N$, that yields

$$\left| \int_E f_k(x)p(dx, z_n) - \int_E f_k(x)\nu(dx) \right| < \frac{1}{m}$$

for all $n \geq N$. This finishes the proof of the lemma. \qed

We come back to the proof of Lemma 2.9. Let us assume that $p$ has no version continuous at $z_0$. Then, according to Lemma A.1, there exists $k \geq 1$ and $m \geq 1$ such that for every $\delta > 0$

$$P^\xi \left[ A_m^k \cap B_\delta \right] > 0,$$

where $B_\delta$ denotes the ball with center $z_0$ and radius $\delta$. For every $n \geq 1$, let $\xi^n$ be a random variable on $E$ with distribution

$$P^\xi^n [A] = \int_E q_n(z)P^\xi [dz], \quad A \in \mathcal{B}(E),$$

where

$$q_n(z) = \frac{1}{P^\xi \left[ A_m^{k,\pm} \cap B_1^{\pm} \right]} \mathbb{1}_{A_m^{k,\pm} \cap B_1^{\pm}}(z), \quad z \in F.$$ 

Here $A_m^{k,\pm} = A_m^{k,+}$, if $P^\xi \left[ A_m^{k,+} \cap B_1^{+} \right] > 0$, and $A_m^{k,\pm} = A_m^{k,-}$, otherwise.

By the construction, $P^\xi^n \ll P^\xi$, $n \geq 1$. Moreover, it is easy to see that $\xi^n \to z_0$ in distribution as $n \to \infty$. But

$$E \left[ \int_E f_k(x)p(dx, \xi^n) \right] \not\to \int_E f_k(x)\nu(dx), \quad n \to \infty.$$ 

Indeed, for every $n \geq 1$ the random variable takes values on $A_m^{k,\pm} \cap B_1^{\pm}$ that is contained in $A_m^{k,+}$ or $A_m^{k,-}$. This immediately implies that

$$\left| E \left[ \int_E f_k(x)p(dx, \xi^n) \right] - \int_E f_k(x)\nu(dx) \right| \geq \frac{1}{m}, \quad n \geq 1.$$ 

We have obtained the contradiction with assumption (6). This finishes the proof of Lemma 2.9.
B Some properties of MMAF

Let $\gamma_t, t \geq 0$, be a MMAF starting at $g \in L_{2+}^1$, and the family $\tau_k^\gamma, k \geq 1$, be constructed in Section 5.1 for every $y \in \text{Coal}$.

**Lemma B.1.** If $\|g\|_{L_{2+}} < \infty$ for some $\varepsilon > 0$, then for every $T > 0$ and $\delta \in \left(0, \frac{\varepsilon}{2T}\right)$ there exists $C_{T,\delta}$ such that

$$
E \left[ \max_{t \in [0,T]} \|\gamma_t - g\|_{L_{2+}}^{2+\delta} \right] \leq C_{T,\delta} \left( 1 + \|g\|_{L_{2+}} \right).
$$

**Proof.** In order to check the estimate, one need to repeat the proof of [Kon17a, Proposition 4.4] replacing the summation with the integration. □

Recall that for every step function $f \in \text{St}$, $N(f)$ denotes the number of steps of $f$ (see Definition 3.5). We write $N(f) = \infty$ for each non-decreasing càdlàg function $f$ which does not belong to $\text{St}$. Since $\gamma \in \text{Coal}$, we have that

$$
\tau_k^\gamma = \inf \{ t \geq 0 : N(\gamma_t) \leq k \}, \quad k \geq 0.
$$

The following statement shows that with probability one, each two different particles collide in finite time and two collisions cannot happen at the same time.

**Lemma B.2.** For any initial condition $g$ such that $N(g) \geq 2$

$$
P \left[ \forall k < N(g), \tau_k^\gamma < \tau_{k+1}^\gamma \right] = 1 \quad (43)
$$

and

$$
P \left[ \tau_1^\gamma < +\infty \right] = 1. \quad (44)
$$

Let us first prove the following statement which corresponds to a particular case of Lemma B.2. For any $y \in \text{Coal}$ with $y_0 \in \text{St}$ denote

$$
\sigma_1(y) := \inf \{ t \geq 0 : N(y_t) \leq N(y_0) - 1 \};
$$

$$
\sigma_2(y) := \inf \{ t \geq 0 : N(y_t) \leq N(y_0) - 2 \}.
$$

**Lemma B.3.** Let $g \in \text{St}$, $N(g) \geq 3$, and $\gamma$ be a MMAF starting at $g$. Then

$$
0 < \sigma_1(\gamma) < \sigma_2(\gamma).
$$

almost surely.
Proof. By continuity of \( \gamma_t, t \geq 0 \), \( \sigma_1(\gamma') > 0 \) almost surely. Moreover, denoting \( n = N(g) \), there exist positive numbers \( m_1, \ldots, m_n, m_1 + \cdots + m_n = 1 \), and a process \( y(t) = (y_k(t))_{k=1}^n \) satisfying \( \gamma'_t = \Xi_m(y(t)), t \geq 0 \), and properties (F1)-(F5). Therefore, on the time interval \([0, \sigma_1(\gamma')]\), the process \( y \) coincides with a Brownian motion \((w_k)_{k=1}^n \) in \( \mathbb{R}^n \) which starts at \( y_k(0) \) and whose coordinate processes have diffusion rates \( \frac{1}{m_k} \), \( k \in [n] \). Since the initial values \( y_k(0), k \in [n] \), are pairwise distinct, the stopping times

\[
\delta_k := \inf\{t \geq 0 : w_k(t) = w_{k+1}(t)\}, \quad k \in [n-1],
\]

are almost surely pairwise distinct. At time \( \sigma_1(\gamma') \), which coincides with \( \min(\delta_k, k \in [n-1]) \), exactly two paths are equal each other almost surely. Hence, \( N(\gamma_{\sigma_1(\gamma')}) = n - 1 \) almost surely. By continuity of \( \gamma' \), \( \sigma_2(\gamma') > \sigma_1(\gamma') \) almost surely. \( \square \)

Proof of Lemma B.2. Take \( r \) from the set \( Q_+ \) of (strictly) positive rational numbers. Note that \( \gamma'_r := \gamma_{r+t}, t \geq 0 \), is a MMAF starting at \( \gamma_t \) due to the martingale properties of \( \gamma_t, t \geq 0 \) (see Remark 3.4 and Proposition 3.6). Moreover, almost surely, \( \gamma'_r = \gamma_t \) belongs to \( \mathcal{S} \). For each \( f \in \mathcal{S} \) we denote by \( \mathbb{P}_f \) the law on \((\mathcal{C}([0, \infty), L^2), \mathcal{B}(\mathcal{C}([0, \infty), L^2)))\) of a MMAF starting at \( f \), which is uniquely determined, according to Remark 3.3. By [IW89, Theorem I.3.3] and [Kon17a, Proposition 3.4], for every \( A \in \mathcal{B}(\mathcal{C}([0, \infty), L^2)) \) and \( B \in \mathcal{B}(L^2) \),

\[
\mathbb{P}[\gamma'_r \in A, \gamma'_0 \in B] = \int_B \mathbb{P}_f(A) \mu_r(\mathrm{d}f), \tag{45}
\]

where \( \mu_r \) denotes the law of \( \gamma'_0 \).

In order to prove (43), we first remark that

\[
\mathbb{P}\left[ \exists k < N(g), \tau^\gamma_{k+1} = \tau^\gamma_k \right] \leq p_1 + p_2,
\]

where

\[
p_1 := \mathbb{P}\left[ \forall k < N(g), \tau^\gamma_{k+1} = \tau^\gamma_k \right],
\]

\[
p_2 := \mathbb{P}\left[ \exists k < N(g) - 1, \tau^\gamma_{k+2} < \tau^\gamma_{k+1} = \tau^\gamma_k \right].
\]

Let us check that \( p_1 = 0 \) by distinguishing two cases. In the case where \( n = N(g) \) is finite, \( \tau^\gamma_n = 0 \) and \( \tau^\gamma_{n-1} = \sigma_1(\gamma') > 0 \) almost surely, by Lemma B.3. Thus, \( p_1 = 0 \). In the case where \( N(g) = +\infty \), the continuity of \( \gamma' \) implies that \( \tau^\gamma_k > 0 \) almost surely for every \( k \geq 1 \). In that case, \( p_1 \leq \mathbb{P}\left[ \exists r \in Q_+, \forall k \geq 1, \tau^\gamma_k > r \right] \leq \mathbb{P}\left[ \exists r \in Q_+, N(\gamma'_r) = \infty \right] \). But almost surely, \( \gamma'_r \in \mathcal{S} \) for every \( r > 0 \), and thus, \( p_1 = 0 \).
Moreover,
\[ p_2 \leq \mathbb{P}\left[ \exists r \in \mathbb{Q}_+, \exists k < N(g) - 1, \tau_k^y < r < r_{k+1}^y = \tau_{k+1}^y \right] \]
\[ \leq \sum_{r \in \mathbb{Q}_+} \mathbb{P}\left[ \exists k < N(g) - 1, \tau_k^y < r < r_{k+1}^y = \tau_{k+1}^y \right]. \]

Remark that \( \tau_{k+2}^y < r < r_{k+1}^y = \tau_{k+1}^y \) implies that \( N(\gamma_0^y) = k + 2 \geq 3 \) and that \( \sigma_1(\gamma^r) = \tau_{k+1}^y - r, \sigma_2(\gamma^r) = \tau_{k+1}^y - r \) satisfy \( \sigma_1(\gamma^r) = \sigma_2(\gamma^r) \).

Thus,
\[ \mathbb{P}\left[ \exists k < N(g) - 1, \tau_k^y < r < r_{k+1}^y = \tau_{k+1}^y \right] \leq \mathbb{P}\left[ N(\gamma_0^y) \geq 3, \sigma_1(\gamma_0^y) = \sigma_2(\gamma_0^y) \right] \]
\[ = \int_{\{f: N(f) \geq 3\}} \mathbb{P}(\sigma_1(\gamma^r) = \sigma_2(\gamma^r)) \mu_r(df) = 0, \]
where we used (45) to get the equality, and Lemma B.3 to obtain the last inequality. Hence, we can deduce that \( p_2 = 0 \). This implies that \( \mathbb{P}\left[ \exists k < N(g), \tau_k^y = \tau_{k+1}^y \right] = 0 \), i.e. equality (43) holds true.

Furthermore, note that \( \mathbb{P}\left[ \tau_1^y = \infty \right] = \mathbb{P}\left[ \tau_1^{y_1} = \infty \right] \), where \( \tau_1^{y_1} \) is defined similarly as \( \tau_1^y \) with \( y \) replaced by \( y_1^y = y_{1+t}, t \geq 0 \). Using equality (45),
\[ \mathbb{P}\left[ \tau_1^{y_1} = \infty \right] = \int_{\text{St}} \mathbb{P}\left[ \tau_1^y = \infty \right] \mu_r(df). \]

By Lemma 4.2, for each \( f \in \text{St}, \mathbb{P}(\tau_1^y = \infty) = 0 \). This yields that \( \mathbb{P}\left[ \tau_1^y = \infty \right] = 0 \) and hence equality (44).

**Lemma B.4.** For every \( y \in \text{Coal}, \beta > 0 \) and \( n \geq 1 \) one has
\[ \sum_{k=n}^{\infty} (\tau_k^y)^\beta = \beta \int_0^{\tau_n^y} (N(y_t) - n)t^{\beta-1}dt. \]

**Proof.** For simplicity of notation we will omit the superscript \( y \) in \( \tau_k^y \). We write for \( m > n \)
\[ \sum_{k=n}^{m} \tau_k^\beta = \sum_{k=n+1}^{m-1} (k - n)(\tau_k^\beta - \tau_{k-1}^\beta) + (m + 1 - n)\tau_m^\beta \]
\[ = \sum_{k=n+1}^{m-1} (N(y_{\tau_k}) - n)(\tau_k^\beta - \tau_{k-1}^\beta) + (N(y_{\tau_{m+1}}) - n)\tau_m^\beta \]
\[ = \int_0^{\tau_n^y} (N(y_t^m) - n)dt^\beta = \beta \int_0^{\tau_n^y} (N(y_t^m) - n)t^{\beta-1}dt, \]
where \( y_t^m := y_{t \vee \tau_{m+1}} \). Hence, the statement of the lemma follows from the monotone convergence theorem. \( \square \)
Lemma B.5. Let \( \gamma \) be a MMAF starting at \( g \in L^1_{2+} \). Then for every \( \beta > \frac{1}{2} \),
\[
\sum_{k=1}^{\infty} (\tau_k^\gamma)^\beta < +\infty \text{ almost surely.}
\]

Proof. Let \( g \in L_{2+}^\varepsilon \) for some \( \varepsilon > 0 \). In order to prove the lemma, we will
use the estimate
\[
E \left[ N(\gamma_t) \right] \leq \frac{C_{\varepsilon,T}}{\sqrt{t}} (1 + \|g\|_{L_{2+}^\varepsilon}) , \quad t \in (0, T],
\]
from [Kon17a, Remark 4.6], where \( C_{\varepsilon,T} \) is a constant depending on \( \varepsilon \) and
\( T > 0 \). Take an arbitrary number \( T > 0 \) and estimate for \( \beta > \frac{1}{2} \)
\[
E \left[ \int_0^{\tau_{Y1} \wedge T} N(\gamma_t) t^{\beta-1} dt \right] \leq \int_0^T E \left[ N(\gamma_t) \right] t^{\beta-1} dt
\]
\[
\leq C_{\varepsilon,T} (1 + \|g\|_{L_{2+}^\varepsilon}) \int_0^T t^{\beta - \frac{3}{2}} dt < +\infty.
\]
This implies that \( \int_0^{\tau_{Y1} \wedge T} N(\gamma_t) t^{\beta-1} dt < \infty \) almost surely. Thus, the statement
of the lemma follows directly from Lemma B.4. \qed

C  Measurability of Coal

We recall that the set \( D((0, 1), C[0, \infty)) \) denotes the space of càdlàg functions
from \((0, 1)\) to \( C[0, \infty)\) equipped with the Skorokhod distance, which makes
it a Polish space. Set
\[
D^1 C := \{ y \in D((0, 1), C[0, \infty)) : \forall 0 < u < v < 1, \quad x_t(u) \leq x_t(v) \forall t \geq 0 \}.
\]
It is easily seen that \( D^1 C \) a closed subspace of \( D((0, 1), C[0, \infty)) \). So, we will
consider \( D^1 C \) as a Polish subspace of \( D((0, 1), C[0, \infty)) \). Let
\[
D^1_2 C := \left\{ y \in D^1 C : \forall T \in \mathbb{N}, \exists K \in \mathbb{N}, \exists \delta \in \mathbb{Q}_+, \max_{t \in [\frac{1}{T}, T]} \|y_t\|_{L_{2+\delta}} \leq K \right\}
\]
\[
\cap \left\{ y \in D^1 C : \|y_t - y_0\|_{L_2} \to 0, \quad t \to 0 \right\} =: D^1 \cap D^2.
\]

Lemma C.1. For every \( A \in B(D^1 C) \) the set \( A \cap D^1_2 C \) is a Borel measurable
subset of \( C L^1_{2+} := C([0, \infty), L_2^1) \).

Proof. First we are going to show that \( D^1_2 C \) is a subset of \( C L^1_{2+} \). So, we take
\( y \in D^1_2 C \) and check that \( y \) is a continuous \( L_2 \)-valued function. The continuity
of \( y \) at 0 follows from the definition of \( D^1_2 C \). Let \( t > 0 \) and \( t_n \to t \) as \( n \to \infty \).
Without loss of generality, we may assume that \( t_n \in [\frac{1}{T}, T] \) for some \( T \in \mathbb{N} \)
and all \( n \geq 1 \). We are going to show that \( y_n \to y_t \) in \( L_2 \), \( n \to \infty \). Let us note that the sequence \( \{y_n\}_{n \geq 1} \) is relatively compact, according to [Kon17a, Lemma 5.1] and the fact that \( y_n \in L_2^+ \), \( n \geq 1 \), are uniformly bounded in \( L_{2+\delta} \)-norm. This implies that there exists a subsequence \( N \subseteq \mathbb{N} \) and \( f \in L_2^+ \) such that \( y_n \to f \) in \( L_2 \) along \( N \). On the other hand, \( y_n \to y_t \) pointwise, that implies the equality \( f = y_t \). Moreover, it yields that every convergent subsequence of \( \{y_n\}_{n \geq 1} \) converges to \( y_t \) in \( L_2 \). Using the relatively compactness of \( \{y_n\}_{n \geq 1} \), we can conclude that \( y_n \to y_t \) in \( L_2 \) as \( n \to \infty \). Thus, \( y \in CL_2^+ \).

Next, we will check that the set \( D_2^+ \) is measurable in \( D^+ \). We fix \( t \geq 0 \) and make the following observation. For every \( y \in D^+ \) the real valued function \( y_t \) is non-decreasing on \((0, 1)\). This implies that it has at most countable number of discontinuous points. Hence, by [EK86, Proposition 3.5.3], the convergence \( y^n \to y \) in \( D^+ \) implies the convergence of \( y^n_t \to y_t \) a.e. (with respect to the Lebesgue measure on \([0, 1]\)). Using Fatou’s lemma, we get that the set

\[
\Lambda(t, f, K, p) := \left\{ y \in D^+ : \|y_t - f\|_{L_p} \leq K \right\}
\]

is closed in \( D^+ \) (46) for every \( K \geq 0 \), \( p \geq 2 \) and \( f \in L_2 \). Hence the set

\[
D^1 = \bigcap_{T=1}^{\infty} \bigcup_{K=1}^{\infty} \bigcup_{\delta \in \mathbb{Q}^+} \bigcap_{t \in [0, T]} \Lambda(t, 0, K, 2 + \delta)
\]

is Borel measurable in \( D^+ \). Using the standard argument and (46), one can check the measurability of \( D^2 \). So, the set \( D_2^+ = D^1 \cap D^2 \) is Borel measurable in \( D^+ \).

We claim that the identity map \( \Phi : D_2^+ \to CL_2^+ \) is Borel measurable. Indeed, let

\[
B_r^T(y) := \left\{ x \in CL_2^+ : \max_{t \in [0, T]} \|x_t - y_t\|_{L_2} \leq r \right\}.
\]

Then the preimage

\[
\Phi^{-1} \left( B_r^T(y) \right) = \bigcap_{t \in [0, T]} \Lambda(t, y_t, r, 2)
\]

is a closed set in \( D^+ \), by (46). Since the Borel \( \sigma \)-algebra on \( CL_2^+ \) is generated by the family \( \{B_r^T(y) \mid T, r > 0, \ y \in CL_2^+\} \), [Kal02, Lemma 1.4] implies that \( \Phi \) is a Borel measurable function. Moreover, it is an injective map. So, using the Kuratowski theorem (see [Par67, Theorem 3.9]) and the fact that \( A \cap D_2^+ \subseteq B(D^+) \), we obtain that the image \( \Phi(A \cap D_2^+) = A \cap D_2^+ \subseteq B(CL_2^+) \) for every \( A \subseteq B(D^+) \).
Lemma C.2. Let $\text{Coal}$ be defined in Section 5.1. Then $\text{Coal}$ is a Borel measurable subset of $CL^+_2$.

Proof. Let $\text{Coal}_D$ consists of all functions from $D^+C$ which satisfies conditions (G1)-(G4) of the definition of $\text{Coal}$ in Section 5.1. Since every function $f \in St$ has a finite $L_p$-norm for every $p \geq 2$, it is easily seen that

$$\text{Coal}_D \cap D^+_2C = \text{Coal}.$$ 

Hence, according to Lemma C.1, the statement of the lemma will immediately follow from the measurability of $\text{Coal}_D$ in $D^+C$.

We will write $\text{Coal}_D = \bigcap_{i=1}^4 G_i$, where for every $i \in [4]$ the set $G_i$ consists of all functions from $D^+C$ which satisfy Condition $(Gi)$. Then

$$G_1 := \left\{ y \in D^+C : y_0 = g \right\};$$

$$G_2 := \left\{ y \in D^+C : \forall t > 0, y_t \in St \right\};$$

$$G_3 := \left\{ y \in D^+C : \forall u, v \in (0, 1), \forall s \geq 0, y_s(u) = y_s(v) \right\};$$

$$G_4 := \left\{ y \in D^+C : \forall s > 0, N(y_s) = 1 \text{ or } \exists t > s, N(y_t) - N(y_s) = 1, \text{ and } N(y_t), t \geq 0, \text{ is càdlàg and non-increasing} \right\},$$

where $N(y_t)$ is a number of distinct values of $y_t$. It is not clear if the sets $G_i, i \in [4]$, are Borel measurable in $D^+C$. So, we are going to replace them by another ones for which the measurability will easily follow. We set

$$\tilde{G}_2 := \left\{ y \in D^+C : \forall r \in Q_+, \exists K \in N, N(y_r) \leq K, \forall t \geq r \right\},$$

$$\tilde{G}_3 := \left\{ y \in D^+C : \forall u, v \in (0, 1) \cap Q_+, \forall s > 0, y_s(u) = y_s(v) \right\};$$

Then, the fact that paths of functions $y$ from $G_3$ coalesce implies that $G_2 \cap G_3 = \tilde{G}_2 \cap \tilde{G}_3$. By the right-continuity of $u \mapsto y_t(u)$ for every $t \geq 0$, we have $\tilde{G}_2 \cap G_3 = \tilde{G}_2 \cap \tilde{G}_3$. Thus, $G_2 \cap G_3 = \tilde{G}_2 \cap \tilde{G}_3$. We remark that for every $y \in \tilde{G}_2 \cap \tilde{G}_3$, the function $N(y_t), t \geq 0$, is càdlàg and integer-valued. Consequently, $\tilde{G}_2 \cap \tilde{G}_3 \cap G_4 = \tilde{G}_2 \cap \tilde{G}_3 \cap \tilde{G}_4$, where

$$\tilde{G}_4 := \left\{ y \in D^+C : \forall s \in Q_+, N(y_s) = 1 \text{ or } \exists t > s, t \in Q_+, N(y_s) - N(y_t) = 1 \right\}.$$

So, we have obtained that $\text{Coal}_D = \bigcap_{i=1}^4 \tilde{G}_i$, where $\tilde{G}_1 := G_1$.

We next show that $\tilde{G}_i \in B(D^+C), i \in [4]$. The measurability of $\tilde{G}_1$ is trivial. In order to check that the set $\tilde{G}_2$ is Borel measurable, we will note that the set

$$\Gamma(t, K) := \left\{ y \in D^+C : N(y_t) \leq K \right\}$$

is closed in $D^+C$. (47)
Hence, the set
\[ \tilde{G}_2 = \bigcap_{r \in \mathbb{Q}^+} \bigcup_{K=1}^{\infty} \bigcap_{t \in [r, \infty)} \Gamma(t, K) \]
belongs to \( B(D^\uparrow \mathcal{C}) \). By [EK86, Proposition 3.7.1], the measurability of \( \tilde{G}_3 \)
will follow from the measurability of the set
\[ \left\{ x = (x_1, x_2) \in \mathcal{C}[0, \infty)^2 : \forall s \geq 0, \ x_1(s) = x_2(s) \right. \]
\[ \left. \text{implies } x_1(t) = x_2(t), \ \forall t \geq s \right\} \]
in \( \mathcal{C}[0, \infty)^2 \). But one can easily get it from the fact that the functions
\[ \mathcal{C}[0, \infty)^2 \ni x \mapsto \inf \{ t \geq 0; \ x_1(t) = x_2(t) \} \in [0, \infty), \]
and
\[ \mathcal{C}[0, \infty)^2 \times [0, \infty) \ni (x, s) \mapsto x(\cdot \wedge s) \in \mathcal{C}[0, \infty)^2 \]
are measurable. It only remains to show that \( \tilde{G}_4 \in B(D^\uparrow \mathcal{C}) \). We remark
that the map \( D^\uparrow \mathcal{C} \ni y \mapsto N(y_t) \in [1, \infty] \) is measurable for every \( t \geq 0 \),
according to (47). Since the set \( \tilde{G}_4 \) is defined via the countably family of \( \mathcal{B} \)-measurable functions \( y \mapsto N(y_t), \ t \in \mathbb{Q}^+ \), we get that \( \tilde{G}_4 \in B(D^\uparrow \mathcal{C}) \).
This finishes the proof of the lemma.

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