ERRATA AND NOTES ON THE PAPER “A GENERALIZATION OF SPRINGER THEORY USING NEARBY CYCLES”

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ABSTRACT. We provide some corrections and clarifications to the paper [Gr3] of the title. In particular, we clarify the “left/right” conventions on complex reflection groups and their braid groups. Most importantly, we fill in a gap related to the treatment of cuts in the Picard-Lefschetz theory part of the argument. The statements of the main results are not affected.

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1. Introduction

The paper [Gr3] of the title contains a gap in the proof of its main result, [Gr3, Theorem 3.1], related to the treatment of cuts in the Picard-Lefschetz theory part of the argument. Namely, the proofs of [Gr3, Lemmas 4.2 & 4.3] are not satisfactory as written, and to the author’s knowledge, can not be fixed without substantial further argument. The main goal of this document is to fill in this gap. This is done by slightly modifying the statements of [Gr3, Lemmas 4.2 & 4.3], and by providing proofs of the modified lemmas. In addition, we take this opportunity to provide some minor corrections, clarifications, and additional details for the rest of [Gr3].

In more detail, the contents of this document are as follows. In Section 2, we clarify our “left/right” conventions on complex reflection groups and their braid groups. The material here is mostly notational, and some readers may find the notation to be overkill. However, the author has found it easy to make mistakes or to create ambiguities related to such “left/right” conventions (starting with the convention for multiplying loops in a fundamental group), and hopes that the notation presented here is helpful in avoiding such mistakes and ambiguities. In Section 3, we describe in detail and fix the gap related to [Gr3, Lemmas 4.2 & 4.3]. The problem is described in Section 3.1. Statements of the modified lemmas are given in Section 3.2, and the proofs are given in Sections 3.3-3.4. Sections 3.5-3.6 indicate how to adapt the rest of the proof of [Gr3, Theorem 3.1] to the modified versions of the lemmas. In addition, Section 3.5 corrects a couple of unrelated minor issues, and Section 3.6 substantially expands on the corresponding [Gr3, Section 4.3] by providing significant further detail. Finally, in Section 4, we make several corrections to [Gr3, Sections 5-6], and remark on the relation of the material in [Gr3, Section 6] to the recent paper [GVX].

We number the bibliography to extend the numbering in [Gr3]. We also provide the publication data for the references [Gr1] and [Gr2].
2. Left/Right Conventions

We begin by clarifying some basic conventions, pertaining to complex reflection groups and their braid groups, that are not made sufficiently clear in [Gr3].

2.1. The Centralizer of a Complex Reflection Group. Let $V$ be a finite dimensional complex vector space, and let $W \subset GL(V)$ be a finite complex reflection group acting on $V$. See [Bro] for an introduction to such groups, also known as Shephard-Todd groups. We say that $v \in V$ is regular if the stabilizer of $v$ in $W$ is trivial. We denote by $V_{\text{reg}}$ the set of all regular $v \in V$. Pick a $W$-orbit $\mathcal{O} \subset V_{\text{reg}}$. Let $S(\mathcal{O})$ be the group of permutations of the finite set $\mathcal{O}$. Note that $W$ is naturally a subgroup of $S(\mathcal{O})$. We denote by $W^\#(\mathcal{O}) \subset S(\mathcal{O})$ the centralizer of $W$ in $S(\mathcal{O})$.

The group $W^\#(\mathcal{O})$ is isomorphic to $W$ as an abstract group; but we prefer to think of $W^\#(\mathcal{O})$ as isomorphic to the opposite group $W^{\text{op}}$. More precisely, for every $v \in \mathcal{O}$, we can define a bijection of sets:

$$\varphi_v : W^\#(\mathcal{O}) \to W,$$

by the following condition:

$$\varphi_v(a)(v) = a(v) \text{ for every } a \in W^\#(\mathcal{O}).$$

It is not hard to check that $\varphi_v : W^\#(\mathcal{O}) \to W$ is an anti-homomorphism of groups. Indeed, for every pair $a, b \in W^\#(\mathcal{O})$, we have:

$$\varphi_v(ab)(v) = (ab)(v) = a(\varphi_v(b)(v)) = \varphi_v(b)(a(v)) = (\varphi_v(b)\varphi_v(a))(v),$$

where the third equality holds because $a \in W^\#(\mathcal{O})$ and $\varphi_v(b) \in W$ commute as elements of $S(\mathcal{O})$, by the definition of $W^\#(\mathcal{O})$. Thus, we obtain a group isomorphism:

$$\varphi_v : W^\#(\mathcal{O}) \to W^{\text{op}}.$$

Let $v_1, v_2 \in \mathcal{O}$ be a pair of points, and let $a \in W^\#(\mathcal{O})$ be the unique element such that $v_2 = a(v_1)$. One can check that the composition:

$$\varphi_{v_2}^{-1} \circ \varphi_{v_1} : W^\#(\mathcal{O}) \to W^\#(\mathcal{O}),$$

is given by:

$$\varphi_{v_2}^{-1} \circ \varphi_{v_1} : b \mapsto aba^{-1}.$$

Thus, we see that the group $W^\#(\mathcal{O})$ is non-canonically isomorphic to $W^{\text{op}}$.

Remark 2.1. The author is not aware of any source in the literature that highlights the group $W^\#(\mathcal{O})$, as distinct from $W$ or $W^{\text{op}}$. Of course, any statement about $W^\#(\mathcal{O})$ can be expressed in terms of $W$ and $W^{\text{op}}$. However, we feel that some things are made clearer by highlighting the group $W^\#(\mathcal{O})$. 
Note that the action of $W^\sharp(\mathcal{O})$ on $\mathcal{O}$ does not, in general, extend to a linear action on $V$. In fact, it does not even extend to a continuous action on $V^{\text{reg}}$. If we pick a point $v \in \mathcal{O}$, to make the identifications $\mathcal{O} \cong W$, $W^\sharp(\mathcal{O}) \cong W^{\text{op}}$, the action of $W^\sharp(\mathcal{O})$ on $\mathcal{O}$ becomes the right action of $W$ on itself. However, this point of view obscures the geometric distinction between the non-linear action of $W^\sharp(\mathcal{O})$ and the linear action of $W$ on $\mathcal{O}$.

2.2. The Braid Group of a Complex Reflection Group. Let $Q = W \setminus V$. Recall that, as an algebraic variety, the quotient $Q$ is isomorphic to $V$ (see [Bro, Theorem 4.1]). We will not be using any explicit linearization of $Q$; but we note that it naturally possesses an origin, given by the orbit of $0 \in V$. Let $g : V \to Q$ be the quotient map, and let $Q^{\text{reg}} = g(V^{\text{reg}}) \subset Q$. Pick a point $q \in Q^{\text{reg}}$. The braid group $B_W(q)$ of $W$ at $q$ is defined as a fundamental group:
$$B_W(q) = \pi_1(Q^{\text{reg}}, q).$$
At this point, it is essential to state the following convention, which ensures that monodromy representations of fundamental groups are, in fact, left representations (see [Sz, Remarks 2.3.1, 2.6.3]).

Con convention 2.2. We multiply loops in a fundamental group $\pi_1(X, x)$ by tracing the second loop first.

Note that $g$ restricts to a covering map:
$$g^{\text{reg}} : V^{\text{reg}} \to Q^{\text{reg}},$$
and that $W$ acts on $V^{\text{reg}}$ by deck transformations. Therefore, writing $\mathcal{O}_q = g^{-1}(q)$, we obtain a group homomorphism:
$$\eta_q : B_W(q) \to W^\sharp(\mathcal{O}_q),$$
given by the monodromy of the covering map $g^{\text{reg}}$. It is not hard to check that the homomorphism $\eta_q$ is surjective. Thus, we can think of $W^\sharp(\mathcal{O}_q)$ as a quotient of $B_W(q)$. If we further pick a point $v \in \mathcal{O}_q$, we can define a group homomorphism:
$$\eta^v = I_W \circ \varphi_v \circ \eta_q : B_W(q) \to W,$$
where $I_W : W^{\text{op}} \to W$ is the inverse map (we make no distinction between the group homomorphism $I_W$ and its inverse, as the underlying sets of $W$ and $W^{\text{op}}$ are the same).

Remark 2.3. The groups $W^\sharp(\mathcal{O}_q)$, for all $q \in Q^{\text{reg}}$, naturally form a local system over $Q^{\text{reg}}$. The same is true of the groups $B_W(q)$. The maps $\eta_q$, for all $q \in Q^{\text{reg}}$, give a map between these local systems.

2.3. The Case of a Coxeter Group. Suppose now that $V_\mathbb{R}$ is a finite dimensional real vector space, that $W$ is a finite Coxeter group acting on $V_\mathbb{R}$, and that $V$ is the complexification of $V_\mathbb{R}$. All of the above constructions can be applied to the pair $(V, W)$. We let $V_\mathbb{R}^{\text{reg}} = V^{\text{reg}} \cap V_\mathbb{R}$, $Q_\mathbb{R} = g(V_\mathbb{R})$, and $Q_\mathbb{R}^{\text{reg}} = g(V_\mathbb{R}^{\text{reg}})$. 
Remark 2.4. The space $Q^r_{\mathbb{R}}$ is contractible. Therefore, by Remark 2.3, the groups $W^r(\mathbb{O}_q)$, for all $q \in Q^r_{\mathbb{R}}$, are naturally isomorphic to each other. The same is true of the groups $B_W(q)$, for all $q \in Q^r_{\mathbb{R}}$. Based on this, we can unambiguously write $W^r = W^r(\mathbb{O}_q)$, $B_W = B_W(q)$, and $\eta = \eta_q : B_W \to W$, where $q$ is understood to be some point of $Q^r_{\mathbb{R}}$.

Remark 2.5. Let $V^+ \subset V_{\mathbb{R}}$ be a connected component of $V^r_{\mathbb{R}}$. Then, for every $v_1, v_2 \in V^+$, we have:

$$\varphi_{v_1} = \varphi_{v_2} : W^r \to W.$$ 

Based on this, we can unambiguously write:

$$\varphi_{V^+} = \varphi_v : W^r \to W \text{ and } \eta^\flat_{V^+} = \eta^\flat_v : B_W \to W,$$

where $v$ is understood to be some point of $V^+$.

Of special interest is the case of the Weyl group of a complex semisimple Lie algebra $g$. Every Cartan subalgebra $t \subset g$ comes equipped with a canonical real form $t_{\mathbb{R}} \subset t$, such that $i \cdot t_{\mathbb{R}}$ generates a compact torus (where $i = \sqrt{-1}$). Associated to each $t \subset g$, we have a Weyl group $W[t]$, which is a finite Coxeter group acting on $t_{\mathbb{R}}$. By Remark 2.4, we also have the groups $W^r[t]$ and $B_W[t]$, associated to $W[t]$.

Remark 2.6. The groups $W^r[t]$, for different choices of $t$, are naturally isomorphic to each other. The same is true of the braid groups $B_{W[t]}$, but not of the Weyl groups $W[t]$. Based on this, we can unambiguously write $W^r[g] = W^r[t]$ and $B_{W[g]} = B_{W[t]}$, where $t$ is understood to be some Cartan subalgebra of $g$. Furthermore, the homomorphism $\eta : B_{W[g]} \to W^r[g]$ of Remark 2.4 is independent of the Cartan subalgebra used to define it.

2.4. Comments on [Gr3, Section 2.1]. In [Gr3, Section 2.1], we gave a brief summary of the approach to constructing the Springer representations due to Lusztig and Borho-MacPherson. Here, we connect the exposition in that section to the terminology and notation introduced in Sections 2.1-2.3 above. We hope that the use of the centralizer group $W^r = W^r[g]$ in this context will add some clarity, as well as facilitate the connection to the nearby cycles approach to Springer theory (see Section 2.5 below).

At the top of [Gr3, p. 414], we consider the finite covering map:

$$q^{rs} : \tilde{g}^{rs} \to g^{rs}.$$ 

After making a choice of a point $x \in g^{rs}$ and a positive Weyl chamber for the Cartan subalgebra $g_x \subset g$ containing $x$, we claim that the Weyl group $W = W[g_x]$ of $g$ acts as the deck transformations of the covering $q^{rs}$.

The same idea can be expressed somewhat more precisely, by first making the following observation.
Remark 2.7. The group of deck transformations of the covering map $q^\ast$ is canonically isomorphic to the group $W^z = W[z]$ of Remark 2.6. This can be seen by identifying the fiber $q^{-1}(x)$ with the set of all Weyl chambers in $g$, i.e., of all connected components of $(g_x)^{reg}$. Let $g_x^+ \subset g_x$ be the chosen positive Weyl chamber. In order to obtain an action of the Weyl group $W = W[g_x]$ as the deck transformations of $q^\ast$, we must use Remark 2.7 plus the homomorphism:

$$ \varphi^{-1}_{g_x^+} \circ I_W : W \rightarrow W^z, $$

where $I_W : W \rightarrow W^{op}$ is the inverse map, and $\varphi_{g_x^+} : W^z \rightarrow W^{op}$ is the homomorphism of Remark 2.5.

Thus, one can say that Lusztig’s construction, described in the paragraph following [Gr3, Proposition 2.1], naturally produces an action of the centralizer group $W^z$ on the push-forward sheaf $P$; and that we need to make some auxiliary choices, and to apply the inverse map, in order to convert this action into an action of the Weyl group $W$. In particular, we can restate [Gr3, Theorem 2.5 (ii)] as follows.

Remark 2.8. The action of $W^z$ on $P$, arising from Remark 2.7 via Lusztig’s construction, gives an isomorphism $C[W^z] \cong \text{End}(P)$.

2.5. Comments on [Gr3, Section 2.2.1]. At the top of [Gr3, p. 416], we consider the adjoint quotient map $f : g \rightarrow W\backslash t$. Here, we clarify the definition of the nearby cycles sheaf $P_f$ of $f$ and the statement of [Gr3, Theorem 2.5 (ii)]. Let us write $Q = W\backslash t$ for the target of $f$. In order to fully specify the sheaf $P_f$, we must fix a test-arc $\gamma$ in $Q$. By this we mean an embedded complex analytic arc $\gamma : U \rightarrow Q$, where $U$ is a neighborhood of zero in $C$, such that $\gamma(0) = 0$, and $\gamma(z) \in Q^{reg}$ for $z \neq 0$; see the discussion at [Gr3, p. 415]. We will say that a test-arc $\gamma : U \rightarrow Q$ is real if $\gamma(z) \in Q_R (= f(t_R))$ for every $z \in U \cap R$.

Remark 2.9. Let $B_z \subset Q$ be a small ball around the origin, defined using some complex analytic local coordinates on $Q$. The local fundamental group $\pi_1(Q^{reg} \cap B_z)$ is trivial. Therefore, the nearby cycles sheaves $P_{f,\gamma}$, for all real test-arcs $\gamma$ in $Q$, are naturally isomorphic to each other.

Using Remark 2.9, we can resolve the “up-to-isomorphism” ambiguity in the definition of $P_f$ by assuming that $P_f = P_{f,\gamma}$ for some real test-arc $\gamma$ in $Q$. With this convention, the monodromy group acting on $P_f$ is the braid group $B_{W[g]}$ of Remark 2.6. To keep the notation close to [Gr3, Section 2.2.1], we will write $B_W = B_{W[g]}$. We can now restate [Gr3, Theorem 2.5 (ii)] as follows.

Remark 2.10. The monodromy action $\mu : B_W \rightarrow \text{Aut}(P_f)$ factors through the homomorphism $\eta : B_W \rightarrow W^z$ of Remark 2.6, producing an action of $W^z$ on $P_f$. The isomorphism of [Gr3, Theorem 2.5 (i)] agrees with the actions of $W^z$ on both sides (see Remark 2.8 for the action of $W^z$ on the LHS).
Remark 2.10 seems like the best way to clarify the meaning of [Gr3, Theorem 2.5 (ii)]. However, in order to read this theorem as stated (i.e., in terms of the Weyl group $W$ rather than the centralizer group $W^\#$), we must clarify what is meant by the “natural homomorphism $B_W \rightarrow W$”. For this, we need to invoke the same auxiliary choices that were used in Section 2.4 above. Namely, a Cartan subalgebra $\mathfrak{g}_x \subset \mathfrak{g}$, and a positive Weyl chamber $\mathfrak{g}_x^+ \subset \mathfrak{g}_x$. With these choices, we assume that $W = W[\mathfrak{g}_x]$, and we use the homomorphism:

$$\eta_{\mathfrak{g}_x^+} : B_W \rightarrow W,$$

of Remark 2.5 as the “natural homomorphism $B_W \rightarrow W$” of [Gr3, Theorem 2.5 (ii)].

Thus, we see that both from the point of view of Lusztig’s construction, and from the point of view of the nearby cycles construction, the centralizer group $W^\# = W^\#[\mathfrak{g}]$ emerges as the natural group of symmetries of the perverse sheaf $P$.

3. Existence of a Cyclic Picard-Lefschetz Class

Our comments in Sections 2.4 and 2.5 above amounted to clarifications rather than corrections. In this section, we address an actual gap in the proof of [Gr3, Theorem 3.1]. We being by explaining the problem.

3.1. The Problem. The proof of [Gr3] Theorem 3.1] contains a gap in the part of the argument dealing with Picard-Lefschetz cuts. More precisely, the proofs of [Gr3] Lemmas 4.2 & 4.3] are not satisfactory as written. To explain the problem, we focus on the proof of [Gr3, Lemma 4.2].

Before proceeding to the substantive issue, we need to clarify the first sentence of this proof: “For each $w \in W$, pick a lift $b_w \in B_W$.” What is meant here is that, in the notation of Sections 2.1[2.2 above, we have a surjective group homomorphism:

$$\varphi_{\mathfrak{n}_w} \circ \eta \circ : B_W \rightarrow W^{op},$$

and the lift $b_w$ is taken with respect to this homomorphism.

With this understanding, in the proof of [Gr3, Lemma 4.2], we proceed by constructing a family of Picard-Lefschetz classes:

$$\{\mu(b_w) u_0 = u(e_w, \gamma_w, O_w)\}_{w \in W},$$

parametrized by the Weyl group $W$ of the polar representation $G|V$, associated to the Cartan subspace $c$. Here, each $e_w$ is a critical point of the algebraic function:

$$l|_F : F \rightarrow \mathbb{C},$$

each $\gamma_w$ is a smooth path in $\mathbb{C}$, connecting $l(e_w)$ to a fixed, large $\xi_0 > 0$, and each $O_w$ is an orientation of the “positive” real subspace $T_{e_w}[\gamma_w] \subset T_{e_w}F$, corresponding to the path $\gamma_w$. 

Next, we observe that the points \( \{e_w\}_{w \in W} \) are the only critical points of the function:

\[
\hat{l}|_\hat{F} : \hat{F} \to \mathbb{C},
\]

where \( \hat{F} \) is a certain compactification of \( F \) relative to \( l \), and \( \hat{l}|_\hat{F} \) is the extension of \( l|_F \) to \( \hat{F} \). We then conclude that the classes \( \{\mu(b_w)u_0\}_{w \in W} \) form a basis of the relative homology group:

\[
H^{-d}(\mathcal{F}P) \cong H_{d-r}(F, \{\xi(y) \geq \xi_0\}; \mathbb{C}),
\]

where \( \xi = \text{Re}(l) : F \to \mathbb{R} \). It is this conclusion that does not seem justified, because of insufficient control over the paths \( \{\gamma_w\}_{w \in W} \). More precisely, each of the \( \gamma_w \) is guaranteed to satisfy conditions (i)-(iv) of [Gr3, p. 424]. If, in addition, we knew that:

\( (v) \gamma_{w_1}(t_1) \neq \gamma_{w_2}(t_2) \), for all \( w_1 \neq w_2 \) and all \( (t_1, t_2) \in [0, 1] \times [0, 1] \setminus \{(1,1)\} \),

then the conclusion that \( \{\mu(b_w)u_0\}_{w \in W} \) is a basis would be justified and standard. However, our construction does not guarantee condition (v). It is easy to reduce [Gr3, Lemma 4.2] to the case where the critical values \( \{l(e_w)\}_{w \in W} \) are distinct. But nothing in our construction ensures that the interiors of the paths \( \{\gamma(e_w)\}_{w \in W} \) will not intersect. This is a puzzling oversight; but the author is unable to see how to complete the proof of [Gr3, Lemma 4.2] without some substantive further argument. In fact, it is not clear to the author whether the lemma is true as stated. The proof of [Gr3, Lemma 4.3] is analogous, and suffers from the same problem.

3.2. Modifying the Statements of [Gr3, Lemmas 4.2 & 4.3]. The statements of [Gr3, Lemmas 4.2 & 4.3] refer to a particular Picard-Lefschetz class \( u_0 \in H^{-d}(\mathcal{F}P) \). However, in defining this class (see [Gr3, p. 424]), we placed no restrictions on the critical point \( e_0 \in Z \), and no restrictions the path \( \gamma_0 : [0, 1] \to \mathbb{C} \), connecting \( l(e_0) \) to \( \xi_0 \), in addition to the general conditions (i)-(iv) of [Gr3, p. 424]. On the other hand, the proof of [Gr3, Theorem 3.1] made only some very mild use of the flexibility to make specific choices of \( e_0 \) and \( \gamma_0 \). For this reason, in order to fix the problem described in Section 3.1 above, it seems easiest to put the following two additional restrictions on the pair \( (e_0, \gamma_0) \). Recall that we write \( \xi = \text{Re}(l) \).

(A1) For every \( e \in Z \), we have \( \xi(e) \leq \xi(e_0) \).

(A2) For every \( t \in (0, 1) \), we have \( \xi(\gamma_0(t)) > \xi(e_0) \).

**Lemma 3.1.** The statement of [Gr3, Lemma 4.2] holds with the additional assumption that the pair \( (e_0, \gamma_0) \) satisfies conditions (A1) and (A2) above.

**Lemma 3.2.** The statement of [Gr3, Lemma 4.3] holds with the additional assumption that the pair \( (e_0, \gamma_0) \) satisfies conditions (A1) and (A2) above.
We will prove Lemmas 3.1 and 3.2 in Sections 3.3 and 3.4, respectively. In Section 3.5, we will indicate how to modify the proof of [Gr3, Theorem 3.1] to make use of Lemmas 3.1 and 3.2 in place of [Gr3, Lemmas 4.2 & 4.3].

3.3. Proof of Modified [Gr3, Lemma 4.2]. In this subsection, we provide a proof of Lemma 3.1.

Step 1. Recall the dual Cartan subspace $c^* \subset V^*$ of $c \subset V$ (see [Gr3, Proposition 2.13]). There exists a covector $l_1 \in c^*$, such that Re$(l_1(e)) < $ Re$(l_1(e_0))$ for every $e \in Z \setminus \{e_0\}$. This follows from the fact that the finite set $Z$ is the set of vertices of its convex hull.

Step 2. By considering the straight line path from $l$ to $l + l_1$, and replacing $l$ by $l + l_1$ in the statement of the lemma, if necessary, we can assume that the following stronger version of condition (A1) holds:

(A3) For every $e \in Z \setminus \{e_0\}$, we have $\xi(e) < \xi(e_0)$.

Step 3. By further perturbing the covector $l \in c^*$, we can ensure that the following two additional conditions hold. We write $\zeta = \text{Im}(l)$.

(A4) For every pair of distinct points $e_1, e_2 \in Z$, we have $\xi(e_1) \neq \xi(e_2)$.

(A5) For every pair of distinct points $e_1, e_2 \in Z$, we have $\zeta(e_1) \neq \zeta(e_2)$.

Step 4. Let $n$ be the order of $W$. We index the elements of $W$ by the set $I = \{0, \ldots, n-1\}$, so that the sequence $\{\xi(w_i e_0)\}_{i \in I}$ is strictly decreasing. This is possible by condition (A4) above. Note that, by condition (A3), we have $w_0 = 1 \in W$. To avoid cumbersome notation, we will write $e_i = e_{w_i}$ for every $i \in I$.

Step 5. In the proof of [Gr3, Theorem 3.1], we used Picard-Lefschetz classes defined using paths in $C$, connecting critical values of $l$ to the fixed large $\xi_0 > 0$. For the present argument, as a matter of convenience, we prefer to use paths connecting critical values of $l$ to the vertical line $\{z \in C \mid \text{Re}(z) = \xi_0\}$. For every $x \in C$, define a “horizontal” path $\gamma_h[x] : [0, 1] \to C$ by:

$$
\gamma_h[x] : t \mapsto (1 - t) \cdot \text{Re}(x) + t \cdot \xi_0 + i \cdot \text{Im}(x).
$$

All Picard-Lefschetz classes in the argument that follows will be defined using such horizontal paths. Note that the paths $\{\gamma_h[l(e_i)]\}_{i \in I}$ avoid each other by condition (A5) above.

Step 6. For each $i \in I$, consider the path:

$$
\gamma_h[i] = \gamma_h[l(e_i)] : [0, 1] \to \mathbb{C}.
$$

Recall the Hessian $\mathcal{H}_{e_i} : T_{e_i}F \to \mathbb{C}$ of $l|_F$ at $e_i$, and consider the positive eigenspace $T_h[i] \subset T_{e_i}F$ of the non-degenerate real quadratic form Re$(\mathcal{H}_{e_i}) : T_{e_i}F \to \mathbb{R}$ (here we use the Hermitian inner product $\langle \ , \ \rangle$ of [Gr3, Proposition 2.12]). Pick an orientation $\partial_h[i]$ of $T_h[i]$. 

The triple \((e_i, \gamma_h[i], \mathcal{O}_h[i])\) defines a Picard-Lefschetz class \(u[i] \in H^{-d}(\mathcal{F}P)\), as in [Gr3, p. 424].

**Step 7.** By assumptions (A1) and (A2) of the lemma, we have \(u_0 = \pm u[0]\). By reversing the orientation \(\mathcal{O}[0]\), if necessary, we can assume that the sign is a plus: \(u_0 = u[0]\).

**Step 8.** Since \(\{e_i\}_{i \in I}\) are the only critical points of the function \(\hat{l}|_{\hat{F}}\), and since the paths \(\{\gamma_h[i]\}_{i \in I}\) avoid each other (see Step 5), the elements \(\{u[i]\}_{i \in I}\) form a basis of \(H^{-d}(\mathcal{F}P)\). This proves the assertion of the lemma about the dimension of \(H^{-d}(\mathcal{F}P)\).

**Step 9.** Let \(M^0 \subset H^{-d}(\mathcal{F}P)\) be the linear subspace generated by the image of \(u[0]\) under the monodromy action:

\[\mu_* : B_W \to \text{End}(H^{-d}(\mathcal{F}P))\]

To complete the proof of the lemma, it suffices to show that \(u[i] \in M^0\) for every \(i \in I\). We will do so by induction on \(i\). For \(i = 0\), the statement is obvious. Let \(I_+ = I \setminus \{0\}\). Fix a \(k \in I_+\), and assume that \(u[i] \in M^0\) for every \(i \in I\) with \(i < k\). We need to show that \(u[k] \in M^0\).

**Step 10.** Recall the regular part \(c^{reg} = c \cap V^{rs}\) of \(c\). We have the following claim.

**Claim:** There exists a smooth path:

\[\beta : [0, 1] \to c^{reg},\]

with \(\beta(0) = e_0\) and \(\beta(1) = e_k\), satisfying conditions (B1) and (B2) below. For each \(i \in I\), let \(\beta[i] : [0, 1] \to c^{reg}\) be the translate of \(\beta\) by the element \(w_i \in W\). More precisely, we let:

\[\beta[i] : t \mapsto w_i \beta[0](t).\]

(B1) For each \(i \in I\), the derivative \((\zeta \circ \beta[i])'\) does not vanish at any point of \([0, 1]\).

(B2) For each pair \(i, j \in I\), there is at most one \(t \in [0, 1]\), such that:

\[(\zeta \circ \beta[i])(t) = (\zeta \circ \beta[j])(t).\]

Moreover, the root \(t\) satisfies \(t \in (0, 1)\) and:

\[(\zeta \circ \beta[i])'(t) \neq (\zeta \circ \beta[j])'(t).\]

**Proof:** The path \(\beta\) can be obtained as a \(C^1\)-small perturbation of the straight line the path \(\beta_{st} : [0, 1] \to c\), given by:

\[\beta_{st} : t \mapsto (1 - t) \cdot e_0 + t \cdot e_k.\]

Conditions (B1) and (B2) for \(\beta_{st}\) in place of \(\beta\) follow from conditions (A4) and (A5) of Step 3. It remains to note that (B1) and (B2) are \(C^1\)-open conditions, and that the complement \(c \setminus c^{reg}\) is a complex hyperplane arrangement in \(c\) (see [Bro, Theorem 4.7]).
Step 11. Recall the quotient map \( f : V \to Q \). Consider the composition:
\[
\vec{\beta} = f \circ \beta : [0, 1] \to Q_{reg}.
\]
Note that \( \vec{\beta}(0) = f(e_0) = \lambda \) and \( \vec{\beta}(1) = f(e_k) = \lambda \). Thus, \( \vec{\beta} \) represents an element of \( B_{W} = \pi(Q_{reg}, \lambda) \). We will show that \( u[k] \subset M^0 \) (see Step 9 above) by examining the monodromy operator:
\[
\mu_*(\vec{\beta}) : H_{l}^{-d}(\mathcal{F}P) \to H_{l}^{-d}(\mathcal{F}P).
\]

Step 12. To analyze the monodromy operator \( \mu_*(\vec{\beta}) \), we need to consider a parametrized version of the stalk cohomology group \( H_{l}^{-d}(\mathcal{F}P) \). Let us write \( T = [0, 1] \) (not to be confused with the tangent spaces that appear in Step 6). For every \( t \in T \), let \( F_t = f^{-1}(\vec{\beta}(t)) \), and let \( P_t \in \mathcal{P}_c(E) \) be the nearby cycles sheaf (with constant coefficients) given by the specialization of \( F_t \) to \( \text{As}(F_t) = E \) (see the discussion at the top of [Gr3, p. 421]). To streamline the notation, we write:
\[
M_t = H_{l}^{-d}(\mathcal{F}P_t).
\]
Note that the vector spaces \( \{M_t\}_{t \in T} \) form a local system over the segment \( T \), and that we have \( M_0 = M_1 = H_{l}^{-d}(\mathcal{F}P) \). For each pair \( t_1, t_2 \in T \), we write:
\[
\nu_{t_1, t_2} : M_{t_1} \to M_{t_2},
\]
for the parallel transport of the local system \( \{M_t\}_{t \in T} \). Note that, with this notation, we have:
\[
\nu_{0, 1} = \mu_*(\vec{\beta}) : M_0 \to M_1.
\]

Step 13. Next, we consider parametrized versions of the constructions of Step 6. For every \( t \in T \), let \( Z_t \) be the critical locus of \( l|_{F_t} \). Note that we have:
\[
Z_t = \{\beta[i](t)\}_{i \in I}.
\]
For each \( i \in I \) and each \( t \in T \), consider the horizontal path:
\[
\gamma_h[i, t] = \gamma_h[l \circ \beta[i]](t) : [0, 1] \to \mathbb{C},
\]
as defined in Step 5. Also, consider the positive eigenspace:
\[
T_h[i, t] \subset T_{\beta[i](t)} F_t,
\]
defined by analogy with the positive eigenspace \( T_h[i] \) of Step 6. The real vector spaces \( \{T_h[i, t]\}_{t \in T} \) form a local system over \( T \). Let \( \Theta_h[i, t] \) be the orientation of \( T_h[i, t] \) obtained from the orientation \( \theta_h[i] \) of \( T_h[i] = T_h[i, 0] \) of Step 6 via the parallel transport of the local system \( \{T_h[i, t]\}_{t \in T} \).

Step 14. The triple \( (\beta[i](t), \gamma_h[i, t], \Theta_h[i, t]) \) does not define a Picard-Lefschetz class in \( M_t \) for every pair \( (i, t) \). This is because the path \( \gamma_h[i, t] \) can collide with the set of critical values \( l(Z_t) \) for some \( t > 0 \). However, by condition (B2) of Step 10, such collisions can only occur for finitely many \( t \in T \). More precisely, for each pair \( i, j \in I \), let \( \Theta_{i, j} \subset T \) be the set of roots
of the equation of condition (B2) of Step 10. The set \( \Theta_{i,j} \) is either empty or consists of one element. Consider the union:

\[
\Theta = \bigcup_{i,j \in I} \Theta_{i,j};
\]

it is a finite subset of \( T \). Define \( T^o = T \setminus \Theta \). Note that, by condition (B2) of Step 10, we have \( \{0,1\} \subset T^o \). Now, for each \( i \in I \) and each \( t \in T^o \), we can use the triple \((\beta[i](t), \gamma_h[i,t], \Omega_h[i,t])\) to define a Picard-Lefschetz class \( u[i,t] \in M_t \) by analogy with the class \( u[i] \) of Step 6. Moreover, just as in Step 8, for every \( t \in T^o \), the elements \( \{u[i,t]\}_{i \in I} \) form a basis of \( M_t \).

**Step 15.** For each \( i \in I \) and each pair \( t_1, t_2 \in T^o \), such that \([t_1, t_2] \subset T^o\), we have:

\[
\nu_{t_1, t_2}(u[i, t_1]) = u[i, t_2].
\]

Now, let \( \theta \in \Theta \). To describe what happens to the basis element \( u[i, t] \), as \( t \) passes through \( \theta \), we will need the following notation. For each \( i \in I \), let \( J[i, \theta] \subset I \) be the set of all \( j \in I \), such that:

\[
(\zeta \circ \beta[i])(\theta) = (\zeta \circ \beta[j])(\theta) \quad \text{and} \quad (\xi \circ \beta[i])(\theta) < (\xi \circ \beta[j])(\theta).
\]

Let \( t_1, t_2 \in T^o \) be a pair, such that: \( t_1 < \theta < t_2 \), \([t_1, \theta] \subset T^o \), and \((\theta, t_2] \subset T^o \). Then, by a standard Picard-Lefschetz theory argument, we have:

\[
\nu_{t_1, t_2}(u[i, t_1]) = u[i, t_2] + \sum_{j \in J[i, \theta]} a_{i,j} \cdot u[j, t_2],
\]

for some integers \( \{a_{i,j}\}_{j \in J[i, \theta]} \).

**Step 16.** Recall the subspace \( M^0 \subset M_0 \), defined in Step 9. For every \( t \in T \), let \( M^0_t = \nu_{0,t}(M^0) \subset M_t \). Note that, by the definition of \( M^0 \) and the last equation of Step 12, we have:

\[
M^0_1 = M^0_0 = M^0 \subset M_0 = M_1.
\]

The requisite containment \( u[k] \in M^0 \) will be implied by the following claim.

**Claim:** Let \( i \in I \) and \( t \in T^o \). Assume that:

\[
(\xi \circ \beta[i])(t) > \xi(e_k).
\]

Then we have \( u[i, t] \in M^0_t \).

Before proving the Claim, we note that it readily implies that \( u[k] \in M^0 \). Indeed, pick a \( t \in T^o \), such that \([t, 1] \subset T^o \). Consider the element \( u[0, t] \in M_t \). By condition (B1) of Step 10 for \( i = 0 \), we know that the composition \( \xi \circ \beta[0] : [0, 1] \to \mathbb{R} \) is monotone decreasing. Therefore, we have:

\[
(\xi \circ \beta[0])(t) > (\xi \circ \beta[0])(1) = \xi(e_k).
\]
Thus, by the Claim, we have \( u[0, t] \in M_t^0 \). Now, by the first equation of Step 15, applied to \( t_1 = t \) and \( t_2 = 1 \), we have:

\[
u_{t, 1}(u[0, t]) \in \nu_{t, 1}(M_t^0) = M_t^0 = M_0^0 \subset M_0.
\]

It remains to note that, by construction, we have:

\[
u_t = \nu_t(M_0^0) = M_0^1 = M_0^0 \subset M_0.
\]

Indeed, the Picard-Lefschetz classes \( u[k, 0] \) and \( u[0, 1] \) are defined using the same critical point \( e_k \in Z \) and the same path \( \gamma_h[k] : [0, 1] \to \mathbb{C} \). The only difference between these two classes comes from the difference between the orientations \( O_h[k, 0] \) and \( O_h[0, 1] \) of the real vector space \( T_h[0, k] = T_h[1, 0] \). Therefore, these classes are the same up to sign.

Steps 17-22 below comprise our proof of the Claim.

**Step 17.** We proceed to prove the Claim of Step 16, arguing by contradiction. Assume that the claim is false. Consider the set:

\[ S = \{(i, t) \in I \times T^\circ \mid u[i, t] \notin M^0_t\}. \]

Define:

\[ \xi_{\text{max}}(i, t) = \sup_{(i, t) \in S} (\xi \circ \beta[i])(t). \]

By assumption, we have \( \xi_{\text{max}} > \xi(e_k) \).

Let \( C(T^\circ) = \pi_0(T^\circ) \) be the set parametrizing the connected components of \( T^\circ \). For each \( \tau \in C(T^\circ) \), let \( T^\circ[\tau] \subset T^\circ \) be the corresponding connected component. Thus, we have:

\[ T^\circ = \bigcup_{\tau \in C(T^\circ)} T^\circ[\tau]. \]

The first equation of Step 15 implies that, for every pair \( (i, \tau) \in I \times C(T^\circ) \), we have either \( \{(i) \times T^\circ[\tau]\} \subset S \) or \( \{(i) \times T^\circ[\tau]\} \cap S = \emptyset \). Define:

\[ C(S) = \{(i, \tau) \in I \times C(T^\circ) \mid \{(i) \times T^\circ[\tau]\} \subset S\}. \]

Thus, we have:

\[ S = \bigcup_{(i, \tau) \in C(S)} \{i\} \times T^\circ[\tau]. \]

For every \( (i, \tau) \in C(S) \), let:

\[ \xi_{\text{max}}(i, \tau) = \sup_{t \in T^\circ[\tau]} (\xi \circ \beta[i])(t). \]

Since \( C(S) \) is a finite set, we have:

\[ \xi_{\text{max}} = \max_{(i, \tau) \in C(S)} \xi_{\text{max}}(i, \tau). \]
Therefore, there exists a pair \((i, \tau) \in C(S)\), such that \(\xi_{\text{max}}[i, \tau] = \xi_{\text{max}} > \xi(e_k)\). Let us fix such a pair \((i, \tau)\). In subsequent steps, we analyze the possibilities for this pair.

**Step 18.** We consider four cases for the pair \((i, \tau) \in C(S)\) fixed at the end of Step 17. (Recall condition (B1) of Step 10.)

*Case 1:* We have \((\xi \circ \beta[i])'(t) > 0\) for all \(t \in T\), and \(1 \in T^\circ[\tau]\).

*Case 2:* We have \((\xi \circ \beta[i])'(t) > 0\) for all \(t \in T\), and \(1 \notin T^\circ[\tau]\).

*Case 3:* We have \((\xi \circ \beta[i])'(t) < 0\) for all \(t \in T\), and \(0 \in T^\circ[\tau]\).

*Case 4:* We have \((\xi \circ \beta[i])'(t) < 0\) for all \(t \in T\), and \(0 \notin T^\circ[\tau]\).

In Steps 19-22 below, we take up these four cases one-by-one, reaching a contradiction in each case.

**Step 19.** Suppose Case 1 of Step 18 obtains. Then we have:

\[
\xi_{\text{max}} = \xi_{\text{max}}[i, \tau] = (\xi \circ \beta[i])(1).
\]

By the assumption of Step 17, we have \(\xi_{\text{max}} > \xi(e_k)\). Therefore, by Step 4, we have \(\beta[i](1) = e_j\), for some \(j < k\). By the induction hypothesis of Step 9, we have \(u[j] \in M^0\). But, by an argument as in Step 16, we have:

\[
u[j] = u[j, 0] = \pm u[i, 1] \in M_0 = M_1.\]

Thus, we conclude that \(u[i, 1] \in M^0 = M^0_1\) (see the first displayed equation of Step 16), which contradicts the assumption that \((i, \tau) \in C(S)\).

**Step 20.** Suppose Case 2 of Step 18 obtains. Let \(\theta = \sup(T^\circ[\tau])\). Since \(1 \notin T^\circ[\tau]\), we have \(\theta \in \Theta\). We also have:

\[
\xi_{\text{max}} = \xi_{\text{max}}[i, \tau] = (\xi \circ \beta[i])(\theta).
\]

Recall the index set \(J[i, \theta] \subset I\) of Step 15. Pick a \(t_1 \in T^\circ[\tau]\). Next, pick a \(t_2 \in (\theta, 1]\), such that \((\theta, t_2) \subset T^\circ\), and we have:

\[
(\xi \circ \beta[j])(t_2) > \xi_{\text{max}},
\]

for every \(j \in J[i, \theta]\). This is possible by the definition of \(J[i, \theta]\).

Next, consider the final equation of Step 15 for the pair \(t_1, t_2 \in T^\circ\). We have:

\[
\nu_{t_1, t_2}(u[i, t_1]) = u[i, t_2] + \sum_{j \in J[i, \theta]} a_{i,j} \cdot u[j, t_2].
\]

By assumption, we have \(u[i, t_1] \notin M^0_{t_1}\). By the definition of the subspaces \(\{M^0_t \subset M_t\}_{t \in T}\), we have \(\nu_{t_1, t_2}(M^0_{t_1}) = M^0_{t_2}\). Therefore, we have \(\nu_{t_1, t_2}(u[i, t_1]) \notin M^0_{t_2}\). On the other hand, every term in the RHS of the above equation is contained in \(M^0_{t_2}\), since by construction, all of them correspond to critical points of \(l|_{F_{t_2}}\) with \(\xi > \xi_{\text{max}}\). Thus, we obtain a contradiction.
**Step 21.** Suppose Case 3 of Step 18 obtains. Then we have:

\[ \xi_{max} = \xi_{max}[i, \tau] = (\xi \circ \beta[i])(0) = \xi(e_i). \]

By the assumption of Step 17, we have \( \xi_{max} > \xi(e_k) \). Therefore, by Step 4, we have \( i < k \). But, by the induction hypothesis of Step 9, we have \( u[i, 0] = u[i] \in M^0 \), which contradicts the assumption that \((i, \tau) \in C(S)\).

**Step 22.** Suppose Case 4 of Step 18 obtains. Our argument is then parallel to Step 20. Let \( \theta = \inf(T^o[\tau]) \). Since \( 0 \notin T^o[\tau] \), we have \( \theta \in \Theta \). We also have:

\[ \xi_{max} = \xi_{max}[i, \tau] = (\xi \circ \beta[i])(\theta). \]

Recall the index set \( J[i, \theta] \subset I \) of Step 15. Pick a \( t_2 \in (0, \theta) \), such that \( [t_1, \theta) \subset T^o \). Next, pick a \( t_2 \in T^o[\tau] \), such that we have:

\[ (\xi \circ \beta[j])(t_2) > \xi_{max}, \]

for every \( j \in J[i, \theta] \). This is possible by the definition of \( J[i, \theta] \).

Next, we can rewrite the final equation of Step 15 for the pair \( t_1, t_2 \in T^o \) as follows:

\[ u[i, t_2] = \nu_{t_1, t_2}(u[i, t_1]) - \sum_{j \in J[i, \theta]} a_{i,j} \cdot u[j, t_2]. \]

By assumption, we have \( u[i, t_2] \notin M^0_{t_2} \). On the other hand, every term in the RHS of the above equation is contained in \( M^0_{t_2} \). Indeed, we have:

\[ (\xi \circ \beta[i])(t_1) > (\xi \circ \beta[i])(\theta) = \xi_{max}. \]

Therefore, we have \( u[i, t_1] \in M_{t_1} \) and \( \nu_{t_1, t_2}(u[i, t_1]) \in M_{t_2} \). Finally, by the choice of \( t_2 \) above, we have \( u[j, t_2] \in M^0_{t_2} \) for every \( j \in J[i, \theta] \). Thus, we obtain a contradiction, which completes our proof of Lemma 3.1.

### 3.4. Proof of Modified [Gr3, Lemma 4.3]

Our proof of Lemma 3.2 is closely parallel to the proof of Lemma 3.1 described in the previous subsection. In this subsection, we only indicate the adaptations that are needed for Lemma 3.2. Note that [Gr3, Lemma 4.3] is stated in terms of the local system \( \mathcal{L} \), and the stalk \( \mathcal{L}_l \) of \( \mathcal{L} \) at \( l \). By definition, we have:

\[ \mathcal{L}_l = H^{-d}_l(\mathcal{F} P). \]

In the argument below, we will mostly be referring to the stalk \( \mathcal{L}_l \) as the RHS of the above equation, in order to keep the notation parallel with the proof of Lemma 3.1 above.

Steps 1-7 of the proof of Lemma 3.1 carry over to Lemma 3.2 without any changes. In Step 8, we need to omit the last sentence. In Step 9, the first sentence needs to be modified as follows.
Step 9. Let $M^0 \subset H^{-d}(\mathcal{F} P)$ be the linear subspace generated by the image of $u[0]$ under the holonomy action:

$$
h : \pi_1((V^*)^{rs}, l) \to \text{End}(H^{-d}(\mathcal{F} P)).$$

The rest of Step 9 is unchanged. Steps 10 through 14 need to be modified as follows.

Step 10. Recall that we have $l \in c^*$, where $c^* \subset V^*$ is the Cartan subspace of $\text{Gr}_3$, Proposition 2.13. Consider the regular part $(c^*)^{reg} = c^* \cap (V^*)^{rs}$. For each $w \in W$, we will write $w^* : c^* \to c^*$ for the adjoint operator of $w : c \to c$. We have the following claim.

Claim: There exists a smooth path:

$$\beta : [0, 1] \to (c^*)^{reg},$$

with $\beta(0) = l$ and $\beta(1) = (w_k)^* l$, satisfying conditions (B1) and (B2) below.

(B1) For each $i \in I$, the real part $\text{Re}(\beta'(t)(e_i))$ does not vanish at any point $t \in [0, 1]$.

(B2) For each pair $i, j \in I$, there is at most one $t \in [0, 1]$, such that:

$$\text{Im}(\beta(t)(e_i)) = \text{Im}(\beta(t)(e_j)).$$

Moreover, the root $t$ satisfies $t \in (0, 1)$ and:

$$\text{Im}(\beta'(t)(e_i)) \neq \text{Im}(\beta'(t)(e_j)).$$

Proof: The path $\beta$ can be obtained as a $C^1$-small perturbation of the straight line path $\beta_{st} : [0, 1] \to c^*$, given by:

$$\beta_{st} : t \mapsto (1 - t) \cdot l + t \cdot (w_k)^* l,$$

as in Step 10 of the proof of Lemma 3.1. □

Step 11. Let $G \cdot l \subset (V^*)^{rs}$ be the $G$-orbit of $l$. Recall that $G$ is connected. Therefore, so is $G \cdot l$. Note that $\beta(1) = (w_k)^* l \in G \cdot l$. Pick a smooth path:

$$\beta_1 : [0, 1] \to G \cdot l,$$

with $\beta_1(0) = (w_k)^* l$ and $\beta_1(1) = l$. Define a loop:

$$\bar{\beta} : [0, 1] \to (V^*)^{rs},$$

by composing the paths $\beta$ and $\beta_1$. More precisely, let:

$$\bar{\beta}(t) = \begin{cases} 
\beta(2t) & \text{for } t \in [0, 1/2] \\
\beta_1(2t - 1) & \text{for } t \in [1/2, 1].
\end{cases}$$
Note that $\bar{\beta}$ represents an element of $\pi_1((V^*)^s, l)$. We will show that $u[k] \subset M^0$ (see Step 9 above) by examining the holonomy operator:

$$h(\bar{\beta}) : H^{-d}(\mathcal{F} P) \to H^{-d}(\mathcal{F} P).$$

**Step 12.** To analyze the holonomy operator $h(\bar{\beta})$, we need to consider a parametrized version of the basis $\{u[i]\}_{i \in I}$ of Step 8. We begin by defining a parametrized version of the stalk cohomology group $H^{-d}(\mathcal{F} P)$. Let us write $T = [0, 1]$. For every $t \in T$, let:

$$M_t = H^{-d}_{\beta(t)}(\mathcal{F} P) = \mathcal{L}_{\beta(t)}.$$

Note that the vector spaces $\{M_t\}_{t \in T}$ form a local system over the segment $T$. For each pair $t_1, t_2 \in T$, let:

$$\nu_{t_1, t_2} : M_{t_1} \to M_{t_2},$$

be the parallel transport of the local system $\{M_t\}_{t \in T}$. Also, let:

$$\nu_{1,2} : M_1 \to M_0,$$

be the holonomy of the local system $\mathcal{L}$ along the path $\beta_1$. Note that, with this notation, we have:

$$\nu_{1,2} \circ \nu_{0,1} = h(\bar{\beta}) : M_0 \to M_0.$$

**Step 13.** Next, we consider parametrized versions of the constructions of Step 6. For every $t \in T$, let $Z_t$ be the critical locus of $\beta(t)|_F$. Note that the set $Z_t$ is independent of $t \in T$. More precisely, for every $t \in T$, we have:

$$Z_t = Z = \{e_i\}_{i \in I}.$$

For each $i \in I$ and each $t \in T$, consider the path:

$$\gamma_h[i, t] = \gamma_h[\beta(t)(e_i)] : [0, 1] \to \mathbb{C},$$

as defined in Step 5. Also, consider the positive eigenspace:

$$T_{h}[i, t] \subset T_{e_i} F,$$

defined by analogy with the positive eigenspace $T_{h}[i]$ of Step 6, and using $\beta(t) \in (c^*)^\text{reg}$ in place of $l$. The real vector spaces $\{T_{h}[i, t]\}_{t \in T}$ form a local system over $T$. Let $\mathcal{O}_h[i, t]$ be the orientation of $T_{h}[i, t]$ obtained from the orientation $\mathcal{O}_h[i]$ of $T_{h}[i] = T_{h}[i, 0]$ of Step 6 via the parallel transport of the local system $\{T_{h}[i, t]\}_{t \in T}$.

**Step 14.** Just as in Step 14 of the proof of Lemma 3.1, we define a finite subset $\Theta \subset T$, and its complement $T^c = T \setminus \Theta \supset \{0, 1\}$. Now, for each $i \in I$ and each $t \in T^c$, let $u[i, t] \in M_t$ be the Picard-Lefschets class defined by the triple $(e_i, \gamma_h[i, t], \mathcal{O}_h[i, t])$. Just as in Step 8, for every $t \in T^c$, the elements $\{u[i, t]\}_{i \in I}$ form a basis of $M_t$. 
In Step 15, we only need to modify the definition of the index set \( J[i, \theta] \). Namely, for each \( i \in I \) and each \( \theta \in \Theta \), let \( J[i, \theta] \subset I \) be the set of all \( j \in I \), such that:

\[
\text{Im}(\beta(\theta)(e_i)) = \text{Im}(\beta(\theta)(e_j)) \quad \text{and} \quad \text{Re}(\beta(\theta)(e_i)) < \text{Re}(\beta(\theta)(e_j)).
\]

The rest of Step 15 is unchanged. Step 16 needs to be modified as follows.

\textbf{Step 16.} Recall the subspace \( M^0 \subset M_0 \), defined in Step 9. For every \( t \in T \), let \( M^0_t = \nu_{0,t}(M^0) \subset M_t \). Note that, by the definition of \( M^0 \) and the last equation of Step 12, we have:

\[
M^0_0 = \nu_{1,2}(M^0_1) = M^0 \subset M_0.
\]

The requisite containment \( u[k] \in M^0 \) will be implied by the following claim.

\textbf{Claim:} Let \( i \in I \) and \( t \in T^\circ \). Assume that:

\[
\text{Re}(\beta(t)(e_i)) > \xi(e_k).
\]

Then we have \( u[i, t] \in M^0_t \).

Before proving the Claim, we note that it readily implies that \( u[k] \in M^0 \). Indeed, pick a \( t \in T^\circ \), such that \( [t, 1] \subset T^\circ \). Consider the element \( u[0, t] \in M_t \). By condition (B1) of Step 10 for \( i = 0 \), we know that the real part:

\[
\text{Re}(\beta(t)(e_0)) : [0, 1] \to \mathbb{R},
\]

is monotone decreasing. Therefore, we have:

\[
\text{Re}(\beta(t)(e_0)) > \text{Re}(\beta(1)(e_0)) = \xi(e_k),
\]

where the last equality follows from the condition \( \beta(1) = (w_k)^* l \) of Step 10. Thus, by the Claim, we have \( u[0, t] \in M^0_t \). Now, by the first equation of Step 15, applied to \( t_1 = t \) and \( t_2 = 1 \), we have:

\[
u_{1,1}(u[0, t]) \in \nu_{1,1}(M^0_t) = M^0_t.
\]

It remains to note that, by construction, and by the first equation of this step, we have:

\[
u_{1,2}(u[0, 1]) \in M^0_0.
\]

More precisely, for the second equality above, observe that the set of critical values of \( \beta_1(t)|_F \) is independent of \( t \) for \( t \in [0, 1] \). The Picard-Lefschetz classes \( u[k, 0] \) and \( u[0, 1] \) correspond to critical points of \( \beta_1(0)|_F \) and \( \beta_1(1)|_F \), respectively, with the same critical value \( l(e_k) \). Furthermore, the classes \( u[k, 0] \) and \( u[0, 1] \) are defined using the same path:

\[
\gamma_h[k, 0] = \gamma_h[0, 1] = \gamma_h[l(e_k)] : [0, 1] \to \mathbb{C}.
\]

It follows that the classes \( u[k, 0] \) and \( \nu_{1,2}(u[0, 1]) \) correspond to the same critical point \( e_k \) of \( l|_F \), and the same path \( \gamma_h[l(e_k)] : [0, 1] \to \mathbb{C} \). Therefore, these classes are the same up to sign.
Our proof of the Claim of Step 16 is a straightforward adaptation of Steps 17-22 of the proof of Lemma 3.1. We omit the remaining details. This completes our proof of Lemma 3.2.

3.5. **Further Comments on the Proof of [Gr3, Theorem 3.1]: the Stable Case.** The proof of [Gr3, Theorem 3.1] in the stable case is readily fixed by replacing [Gr3, Lemmas 4.2 & 4.3] by Lemmas 3.1 and 3.2 above, and by further making the following four changes. Note that two out of these changes (the first and the third) are unrelated to the main problem described in Section 3.1, as we take this opportunity to correct some unrelated inaccuracies.

First, in stating [Gr3 Lemma 2.9], we neglected to require that \( X \subset \hat{X} \) be a stratum of the stratification \( \hat{X} \) of \( \hat{X} \) provided by that lemma. The construction of [Gr1, Section 3.4] provides a stratification \( \hat{X} \) satisfying this requirement. Moreover, the lemma with this requirement is trivially implied by the lemma as stated. Thus, in applying [Gr3 Lemma 2.9] in the second paragraph of [Gr3, Section 4.2], we must assume that \( F \) is a stratum of the stratification of \( \hat{F} \). This justifies the conclusion that \( Z \subset c \subset V \), where \( Z \) is the stratified critical locus of the restriction \( \hat{l}|_{\hat{F}} \).

Second, at [Gr3, p. 424], we must choose \( e_0 \) and \( \gamma_0 \) so that conditions (A1) and (A2) are satisfied.

Third, in the second paragraph of the proof of claim (iii) of the theorem at [Gr3, p. 425], we must use the homomorphism \( \eta_\lambda : B_W(\lambda) \to W^Z(Z) \) of Section 2.2 above, to replace the sentence:

“Each cluster consists of \( n \) points surrounding a point in \( W \cdot v_1 \), and the action of \( \sigma \in W \) cyclically permutes the points in each cluster.”

by:

“Each cluster consists of \( n \) points surrounding a point in \( W \cdot v_1 \), and the action of \( \eta_\lambda(\sigma) \in W^Z(Z) \) cyclically permutes the points in each cluster.”

And fourth, in last paragraph of the proof of claim (iii) of the theorem, we must modify the construction of the Picard-Lefschetz class \( u_0 \), so that Lemma 3.2 would apply. Namely, we must choose a point \( e_0 \in Z \), satisfying condition (A1). Next, we must choose a smooth path \( \gamma_0 : [0,1] \to \mathbb{C} \), satisfying conditions (i)-(iv) of [Gr3, p. 424] for \( e = e_0 \), as well as condition (A2). Finally, we must pick an orientation \( \mathcal{O}_0 \) of the real subspace \( T_{e_0}[\gamma_0] \subset T_{e_0}F \), and define \( u_0 = u(e_0, \gamma_0, \mathcal{O}_0) \). Figure 1 at [Gr3, p. 426] must be modified accordingly, to show the path \( \gamma_0 \) originating from \( l(e_0) \), one of the right-most points of the image \( l(Z) \). The rest of the argument in the stable case remains unchanged.
3.6. Further Comments on the Proof of [Gr3, Theorem 3.1]: the Nonstable Case. As indicated in [Gr3, Section 4.3], the proof of [Gr3, Theorem 3.1] in the case when \( G|V \) is not stable is analogous to the proof in the stable case, with Morse critical points of the restriction \( l|_F \) replaced by Morse-Bott critical manifolds (see [Gr3, Corollary 2.16 (ii)]). In this subsection, we indicate the changes to the proof that have to be made in the nonstable case. Compared to [Gr3, Section 4.3], we reflect the changes described in Section 3.5, and supply some more detail.

As in the stable case, we pick a basepoint \( l \in (V^*)^{rs} \), lying in the Cartan subspace \( c^* \) of [Gr3, Proposition 2.13], consider the compactification \( \hat{F} \) of \( F \) relative to \( l \), fix a stratification of \( \hat{F} \) as in [Gr3, Lemma 2.9], such that \( F \subset \hat{F} \) is a stratum, and write \( Z \) for the stratified critical locus of the restriction \( \hat{l}|_F \). Let \( Z_1 = Z \cap c \), it is a single \( W \)-orbit in \( c \). Note that, by [Gr3, Corollary 2.16 (ii)], we have:

\[
(3.1) \quad l(Z \cap F) = l(Z_1),
\]

but the image \( \hat{l}(Z) \) is potentially larger than \( l(Z_1) \).

The construction of Picard-Lefschetz classes described at [Gr3, p. 424] can be adapted to this new setting as follows. Let \( e \in Z_1 \) and let \( \gamma : [0, 1] \to \mathbb{C} \) be a smooth path satisfying conditions (i)-(iv) of [Gr3, p. 424]. Unlike in the stable case, the Hessian \( \mathcal{H}_e : T_eF \to \mathbb{C} \) of \( l|_F \) at \( e \) is now degenerate. However, we may still consider the positive eigenspace \( T_e[\gamma] \subset T_eF \) of the real quadratic form \( \text{Re}(\mathcal{H}_e/\gamma'(0)) \). By [Gr3, Corollary 2.16 (ii)], we have \( \dim_{\mathbb{R}} T_e[\gamma] = d_0 - r \). Fix an orientation \( \mathcal{O} \) of \( T_e[\gamma] \). The triple \( (e, \gamma, \mathcal{O}) \) defines a Picard-Lefschetz class:

\[
u = \nu(e, \gamma, \mathcal{O}) \in H_{d_0 - r}(F, \{\xi(y) \geq \xi_0\}; \mathbb{C}),
\]

where \( \xi = \text{Re}(l) \) and \( \xi_0 \) is large, exactly as in the stable case. Moreover, as before, we can use [Gr3, Lemma 2.8] and duality with supports to regard \( u \) as an element of \( H^{d_0 - d}(\mathcal{F}P) \).

In order to analyze such Picard-Lefschetz classes, we introduce the following notation. For each \( x \in \hat{l}(Z) \), let \( Z_x = Z \cap \hat{l}^{-1}(x) \) and let:

\[
\hat{l}[x] = \hat{l}|_{\hat{F}} - x : \hat{F} \to \mathbb{C}.
\]

Consider the vanishing cycles sheaf:

\[
Q_x = \phi_{\hat{l}[x]}(j_{!}(\mathbb{C}_F[d - r]));
\]

it is a a perverse sheaf on \( \hat{F} \), satisfying:

\[
(3.2) \quad \text{supp}(Q_x) \subset Z_x.
\]

By [Gr3, Lemma 2.8], we have:

\[
(3.3) \quad H^{d_0 - r}(\mathcal{F}P) \cong \bigoplus_{x \in \hat{l}(Z)} \mathbb{H}^{d_0 - d}(\hat{F}; Q_x),
\]

where the isomorphism depends on a choice of a system of cuts, as usual.
By the inequalities of [Gr3, Lemmas 2.9 & 4.1] and equations (3.1)-(3.2) above, we have:

\[(3.4) \quad H^{d-d_0}(\hat{F}; Q_x) = 0 \quad \text{for every} \quad x \in \hat{l}(Z) \setminus l(Z_1).\]

In other words, the critical values \(x \in \hat{l}(Z) \setminus l(Z_1)\) are “invisible” from the point of view of the direct sum decomposition (3.3). This observation enables us to extend the notation for Picard-Lefschetz classes as follows. Let \(e \in Z_1\) and let \(\gamma : [0, 1] \to \mathbb{C}\) be a smooth path satisfying conditions (i), (iii), and (iv) of [Gr3, p. 424], plus the following condition:

\[(i') \quad \gamma(t) \notin l(Z_1), \quad \text{for} \quad t > 0.\]

Define the real subspace \(T_e[\gamma] \subset T_eF\) as usual, and fix an orientation \(O\) of \(T_e[\gamma]\). Then we can define:

\[u(e, \gamma, O) = u(e, \gamma_1, O) \in H^{-d_0}_i(F P),\]

where \(\gamma_1 : [0, 1] \to \mathbb{C}\) is a smooth \(C^1\)-perturbation of the path \(\gamma\), satisfying conditions (i)-(iv) of [Gr3, p. 424], and such that \(\gamma_1'(0) = \gamma'(0)\). Equation (3.4) ensures that the class \(u(e, \gamma, O)\) is well defined. This enhances the analogy with the stable case, as we can basically ignore the critical values \(x \in \hat{l}(Z) \setminus l(Z_1)\) for the purposes of discussing the paths used to define Picard-Lefschetz classes.

The main distinction from the stable case is that there is no a priori geometric reason for the class \(u(e, \gamma, O)\) to be non-zero. This is because the Morse-Bott manifold containing \(e\) is non-compact. Thus, we can not draw an immediate conclusion from equation (3.3) regarding the dimension \(\dim H^{-d_0}_i(F P)\). To analyze this dimension, let:

\[(c^*)^{rs, O}[\lambda] = \{l' \in (c^*)^{rs} \mid |l'(Z_1)| = |Z_1| = |W|\}.\]

Note that \((c^*)^{rs, O}[\lambda]\) is Zariski open in \((c^*)^{rs}\). We incorporate the regular value \(\lambda \in Q^{reg}\) in the notation to indicate the dependence on the fiber \(F = f^{-1}(\lambda)\).

Assume that \(l \in (c^*)^{rs, O}[\lambda]\). From the inequalities of [Gr3, Lemmas 2.9 & 4.1], [Gr3, Corollary 2.16 (ii)], and equation (3.2) above, we can conclude that:

\[\dim H^{d-d_0}(\hat{F}; Q_x) \in \{0, 1\} \quad \text{for every} \quad x \in l(Z_1).\]

Moreover, by construction, we have:

\[(3.5) \quad u(e, \gamma, O) = 0 \iff H^{d-d_0}(\hat{F}; Q_x) = 0,\]

where \(x = l(e)\). We can use the equivalence (3.5) and the monodromy action of \(B_W\) on \(H^{-d_0}_i(F P)\) to conclude that if the hypercohomology group \(H^{d-d_0}(\hat{F}; Q_x)\) vanishes for some \(x \in l(Z_1)\), then it vanishes for all \(x \in l(Z_1)\). However, in the latter case, by (3.3), it would follow that \(H^{-d_0}_i(F P) = 0\). Since the same conclusion would hold for every \(l' \in (c^*)^{rs, O}[\lambda]\), by [Gr3, Theorem 2.6] and [Gr3, Corollary 2.16 (i)], it would then follow that \(P = 0\). This contradiction shows that:

\[(3.6) \quad \dim H^{d-d_0}(\hat{F}; Q_x) = 1 \quad \text{for every} \quad x \in l(Z_1),\]
and we have:

\[(3.7) \dim H^{-d_0}(FP) = |W|.
\]

So far, we have only established equation (3.7) for \(l \in (c^*)^{rs,0}[\lambda]\). However, the LHS of this equation is independent of the choice of the regular value \(\lambda \in Q^{rs}\), and the sets \((c^*)^{rs,0}[\lambda]\), for different choices of \(\lambda\), cover the entirety of \(c^rs\). Therefore, equation (3.7) holds for all \(l \in (c^*)^{rs}\), and by \(G\)-equivariance, for all \(l \in (V^*)^{rs}\). Putting this together with [Gr3, Theorem 2.6] and [Gr3, Corollary 2.16 (i)], we can conclude that:

\[FP|_{(V^*)^{rs}} \cong L[d_0],\]

where \(L\) is a rank \(|W|\) local system on \((V^*)^{rs}\).

Note that equations (3.5)-(3.6) imply that \(u(e, \gamma, 0) \neq 0\) whenever \(l \in (c^*)^{rs,0}[\lambda]\). The same conclusion can be readily extended to all \(l \in (c^*)^{rs}\), for example, by considering the parallel transport of the local system \(L\) from a point \(l \in (c^*)^{rs} \setminus (c^*)^{rs,0}[\lambda]\) to a nearby point \(l_1 \in (c^*)^{rs,0}[\lambda]\). Thus, we see that the Picard-Lefschetz class \(u(e, \gamma, 0) \in H^{-d_0}(FP)\) is non-zero for every triple \((e, \gamma, 0)\), as in the stable case.

The rest of the proof of [Gr3, Theorem 3.1] in the nonstable case proceeds by working with such Picard-Lefschetz classes in complete analogy with the stable case. We omit the rest of the details.

4. Comments on [Gr3 Sections 5-6]

In this section, we correct several minor errors from [Gr3 Sections 5-6].

First, in the second paragraph of [Gr3 Section 5], the subgroup \(G_{\sigma} \subset G\) should be defined as the connected subgroup corresponding to the subalgebra \(\mathfrak{g}_{\sigma} \subset \mathfrak{g}\), not as the adjoint form of \(\mathfrak{g}_{\sigma}\).

Second, in the last paragraph of the proof of [Gr3 Theorem 5.2], the first sentence should be modified as follows:

"Let \(g_{\sigma} : Q_{\sigma} \to Q\) be the map \(g_{\sigma} : f_{\sigma}(v) \mapsto f(v + v_1), v \in V_{\sigma}^*\)."

And in the next line, the words "the point \(f_{\sigma}(v_1)\)" should be replaced by "the origin."

Third, in the first paragraph of [Gr3 Section 6], the sentence:

"Then \(\mathfrak{g}^+\) is a Lie algebra, and the adjoint form \(G^+\) of \(\mathfrak{g}^+\) acts on the symmetric space \(\mathfrak{g}^-\) by conjugation."

should be replaced by:
“Then $\mathfrak{g}^+$ is a Lie algebra, and if $G$ is the adjoint of $\mathfrak{g}$, and $G^+ \subset G$ is the connected subgroup corresponding to $\mathfrak{g}^+ \subset \mathfrak{g}$, then $G^+$ acts on the symmetric space $\mathfrak{g}^-$ by conjugation.”

And fourth, in the proof of [Gr3, Theorem 6.1 (i)], the sentence:

“It follows from the root space decomposition for $\mathfrak{g}_\mathbb{R}$ that, in terms of Proposition 5.1, we have $G_\sigma = SO(s(i) + 1)$, $V_\sigma = \mathbb{C}^{s(i) + r}$, and $G_\sigma$ acts by the standard representation on the first $s(i) + 1$ coordinates.”

should be replaced by:

“It follows from the root space decomposition for $\mathfrak{g}_\mathbb{R}$ that, in the notation of Section 5, there are isomorphisms $V_\sigma \cong \mathbb{C}^{s(i) + 1} \oplus c_\sigma$ and $Q_\sigma \cong \mathbb{C} \oplus c_\sigma$, such that $f_\sigma = (\tilde{f}_\sigma, \text{Id}_c)$, where $\tilde{f}_\sigma : \mathbb{C}^{s(i) + 1} \to \mathbb{C}$ is a non-degenerate homogenous quadric.”

We conclude this section with a remark regarding the proof of [Gr3, Theorem 6.1 (ii)].

**Remark 4.1.** The proof of [Gr3, Theorem 6.1 (ii)] is essentially left as an exercise to the reader. Arguably, more should have been said. It is indeed easy to check that the answer provided for the homomorphism $\rho$ is accurate up to sign. One way to pin down the signs is to trace the effect of both the monodromy action $\mu$ and the holonomy of the local system $L$ on the top homology group $H_{d-r}(F; \mathbb{C}) \cong \mathbb{C}$, regarded as a subgroup of the stalk $L_1$ (see the first displayed equation at [Gr3, p. 424]).

The recent paper [GVX] can be regarded as a sequel to [Gr3, Section 6], giving a generalization of [Gr3, Theorem 6.1 (ii)] to nearby cycles with coefficients in certain rank one local systems on the general fiber $F$ of $f$. In particular, [GVX, Section 6] treats the case of constant coefficients in detail, and [GVX, Proposition 6.14] provides the analysis of the top homology group $H_{d-r}(F; \mathbb{C})$ mentioned above. The relationship between the material in [Gr3, Section 6] and [GVX, Section 6] is explained in the introduction to the latter.

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