Modular knots obey the Chebotarev law

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ABSTRACT

We study knots which behave like prime numbers. We discuss the planetary link raised from a hyperbolic fibered link in $S^3$ with an emphasis on surgeries, point out certain subtleness, and refine the construction. In addition, we point out a version of McMullen’s theorem for the cases over cusped orbifolds and deduce that the family of modular knots around any torus knot $K_{a,b}$ in $S^3$ together with the missing knot $K_{a,b}$ obey the Chebotarev law. Furthermore, we attach several remarks in the view of arithmetic topology.

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1. Introduction

The analogy between knots and primes has played important roles from the era of Gauss. In modern times, the analogy was initially pointed out by Mazur [Maz64] with an emphasis on Iwasawa theory and Alexander–Fox theory, and has been systematically and vastly developed by Kapranov, Reznikov, Morishita, Kim, and others [Kap95, Rez97, Rez00, Mor02, Mor12, Kim20]. This research field is called arithmetic topology. Mazur Maz12 further asked if there exists a sequence of knots in $S^3$ that behave like the set of all prime numbers in $\mathbb{Z}$, namely, that obeys the Chebotrov law. Here is the definition:

Definition 1.1 (The Chebotarev law). Let $(K_i) = (K_i)_{i \in \mathbb{N}_{>0}}$ be a sequence of disjoint knots in a 3-manifold $M$. For each $n \in \mathbb{N}_{>0}$ and $j > n$, put $L_n = \bigcup_{i \leq n} K_i$ and let $[K_j]$ denote the conjugacy class of $K_j$ in $\pi_1(M - L_n)$. We say that $(K_i)$ obeys the Chebotarev law if the density equality

$$\lim_{\nu \to \infty} \frac{\# \{ n < j \leq \nu \ | \ \rho([K_j]) = C \}}{\nu} = \frac{\# C}{\# G}$$

holds for any $n \in \mathbb{N}_{>0}$, any surjective homomorphism $\rho : \pi_1(M - L_n) \to G$ to any finite group, and any conjugacy class $C \subset G$.

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In this view, McMullen proved the following highly interesting theorem, generalizing Adachi-Sunada’s result [Sun84, Proposition II-2-12].

**Theorem 1.2 [McM13, Theorem 1.2].** Let \((K_i)\) be the closed orbits of any topologically mixing pseudo-Anosov flow on a closed 3-manifold \(M\), ordered by length in a generic metric. Then \((K_i)_i\) obeys the Chebotarev law.

We consult [Mos92, Fen08, Cal07] for the terminology of pseudo-Anosov flows. A key of the proof is the following: since such flows admit Markov partitions [FS79], the flow dynamics admit symbolic encodings, so the theory of symbolic dynamics (cf. [PP90]) applies.

By using this theorem, McMullen pointed out that the family \(L\) of closed orbits of the monodromy suspension flow of the figure-eight knot \(K = 4_1\) together with \(K\) in \(S^3\) obeys the Chebotarev law if ordered by length [McM13, Corollary 1.3], answering Mazur’s question. He also claimed that the same construction works for any hyperbolic fibered knot in \(S^3\).

However, there is something subtle to care about, even in the case of \(4_1\). In the former half of this article, we aim to verify his claim, as well as to improve the author’s argument in the previous paper [Uek21, Propositions 12, 15]. Namely, we re-prove the following assertion.

**Theorem 1.3.** (1) Let \(L\) be a fibered hyperbolic finite link in \(S^3\) and let \(\mathcal{L}\) denote the union of the closed orbits of the suspension flow of the monodromy map and \(L\) itself. Then the components of \(\mathcal{L}\) obey the Chebotarev law if ordered by length in a generic metric.

(2) For any finite branched cover \(h: M \to S^3\) branched along a finite sublink of \(\mathcal{L}\), the inverse image \(h^{-1}(L)\) obey the Chebotarev law, if ordered by length in a generic.

For this purpose, in Section 2 we first recall his construction with an emphasis on Dehn surgeries and observe how subtle things may occur. Afterward, in Section 3 we introduce the notion of a *generalized pseudo-Anosov flows*, which allows 1-pronged singular orbits, invoke *rational Fried surgeries*, and establish a version of Theorem 1.2 for such flows (Theorem 3.1). Finally, we prove Theorem 1.3 in Section 4.

We remark that the Chebotarev link \(L\) in Theorem 1.3 is also called *the planetary link* raised from \(L\) and sometimes admits another amazing property due to Ghrist and others [Ghr97, GHS97, GK04]: this link \(L\) contains every type of links. We also attach a fused question.

The modular knots around the trefoil \(K_{2,3} = 3_1\) in \(S^3\) is another famous example of infinite links of deep features, with a close connection to Lorenz knots (cf. [BWS83, Tue99, Ghy07], see Section 5 for more details). Since the planetary link raised from \(4_1\) and the family of modular knots are like siblings, it is desirable to have a similar result for modular knots. However, each of them has its specific nature and requires slightly different arguments. In this view, we would like to point out the following partial extension of Theorem 1.2.

**Theorem 1.4.** (1) Let \(\Gamma \leq \text{SL}_2 \mathbb{R}\) is any Fuchsian group, so that the orbifold \(\Sigma = \Gamma \backslash \mathbb{H}\) may have cusps. Let \(L\) be a finite link in a closed 3-manifold \(M\) and suppose that the exterior \(M - L\) is the unit tangent bundle \(T_1\Sigma\). Then, the closed orbits of (a lift of) the geodesic flow and the missing link \(L\) obey the Chebotarev law, if ordered by length in a generic metric.

(2) If instead \(M - L\) is a finite cover of \(T_1\Sigma\), then a similar result holds.

Now let \((a, b)\) be any pair of coprime integers and let \(\Gamma_{a,b} \leq \text{SL}_2 \mathbb{R}\) denote the \((a, b, \infty)\)-triangle group. Then a modular knot is defined to be a closed orbit of a standard flow called *the geodesic
flow on (i) the unit tangent bundle of the orbifold \( \Gamma_{a,b} \setminus \mathbb{H} \), which is homeomorphic to the exterior \( L(a,b) - \overline{K}_{a,b} \) of a certain knot in the lens space, or (ii) its certain finite cover homeomorphic to \( S^3 - K_{a,b} \), where \( K_{a,b} \) is the \((a,b)\)-torus knot. More precisely, we have \( \pi_1 L(a,b) = \mathbb{Z}/r\mathbb{Z} \) with \( r = ab - a - b \) and the corresponding \( \mathbb{Z}/r\mathbb{Z} \)-cover is \( S^3 - K_{a,b} \to L(a,b) - \overline{K}_{a,b} \) (cf. [Tsa13]). The classical case is \( \Gamma_{2,3} = SL_2\mathbb{Z}, K_{2,3} = \text{trefoil} \), and \( r = 1 \).

Theorem 1.4 particularly yields the following.

**Theorem 1.5.** For any pair \((a,b)\) of coprime integers, (i) the family of modular knots in \( L(a,b) - \overline{K}_{a,b} \) and the missing knot \( \overline{K}_{a,b} \) and (ii) the family of modular knots in \( S^3 - K_{a,b} \) and the missing torus knot \( K_{a,b} \) obey the Chebotarev law, if ordered by length in a generic metric.

As a corollary, we obtain an equidistribution formula for a certain ubiquitous function called the Rademacher symbol \( \Psi : \Gamma_{2,3} = SL_2\mathbb{Z} \to \mathbb{Z} \) (Corollary 6.1). This may be seen in contrast with Sarnak–Mozochi’s another distribution theorem. See Section 6 for the details.

The idelic class field theory for number fields developed by Artin–Takagi and Chevalley sums up all local theories associated to primes to control the global abelian extensions with ramifications. Its topological analogue for a 3-manifold endowed with a nice infinite link has been developed in the series of works [Nii14, NU19, Mih19, KMNT21, NU23, Tas24, Tas25]. As proved in [Uek21] (see also the latest survey [Mor24, Section 7]), a link that obeys the Chebotarev law is stably generic in the sense of Mihara [Mih19] and guarantees a certain functionality. In Section 7 we remark further possible interests of Chebotarev links, such as the Lang–Trotter conjecture and the Neukirch–Uchida theorem, from the viewpoint of arithmetic topology.

**2. McMullen’s construction and Dehn surgeries**

Let us recall McMullen’s construction. The mapping torus of \( A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \) with \( \text{tr} \ A > 2 \) acting on \( T = \mathbb{R}^2 / \mathbb{Z}^2 \) is the result \( K(0) \) of the 0-surgery along the figure-eight knot \( K = 4_1 \) in \( S^3 \) [BZH13, Chapter 5], so that the monodromy suspension flow is a pseudo-Anosov flow on the torus bundle \( K(0) \) such that the exterior of the 0-orbit is homeomorphic to \( S^3 - K \). By [McM13, Corollary 2.2], the flow is topologically mixing if there are two orbits \( K_i \) and \( K_j \) whose length satisfy \( \ell(K_i)/\ell(K_j) \notin \mathbb{Q} \), hence a generic time change makes the pseudo-Anosov flow to be topologically mixing. Therefore by [McM13, Theorem 1.2], the closed orbits in \( K(0) \) obey the Chebotarev law. Since the Chebotarev law persists under Dehn surgeries along knots, the \( \infty \)-surgery along the 0-orbit yields a sequence of knots in \( S^3 \) containing \( K \) itself and obeying the Chebotarev law.

Here we attach a proof of a basic fact used in the construction:

**Proposition 2.1.** The Chebotarev law persists under Dehn surgeries along knots.

**Proof.** Let \( (K_i)_i \) be a sequence of disjoint knots in \( M \) obeying the Chebotarev law and let \( M' \) denote the result of Dehn surgeries along a link contained in \( L_N \) for \( N \in \mathbb{Z}_{>0} \). If \( N \leq n \), then we have \( M - L_n = M' - L_n \), hence the density equality persists. If \( N > n \), consider the composition

\[
\bar{\rho} : \pi_1(M - L_N) = \pi_1(M' - L_N) \to \pi_1(M' - L_n) \to G.
\]

Since we have \( \rho([K_j]) = \bar{\rho}([K_j]) \) for any \( j \geq N \), the density equality for \( \bar{\rho} \) yields that for \( \rho \). \( \square \)

If we try exactly the same construction for any other hyperbolic fibered knot or link in \( S^3 \), we
should notice that the 0-surgery is not always hyperbolic. If the result of 0-surgery is hyperbolic, then in general we do not necessarily obtain a pseudo-Anosov flow on a closed 3-manifold.

Indeed, Gabai [Gab97, p.27] initially pointed out by using Penner’s program that the 0-surgery along $K = 8_{20}$ results in a fibered manifold with reducible monodromy, so that the mapping torus is toroidal, hence not hyperbolic. This fact is known as the consequence of the fact that the complement of $8_{20}$ had an essential punctured torus with boundary slope 0 [HO89, p.479]. Here for the convenience of readers, we give a slightly generalized assertion (Proposition 6) suggested by Motegi. The crosscap number of a knot in the 3-sphere is defined as the minimal first Betti number of non-orientable subsurfaces bounded by the knot. Note that $K = 8_{20}$ is the pretzel knot $P(2,3,-3)$ and its crosscap number is two [IM10, Theorem 1.2].

**Proposition 2.2.** The 0-surgery along a hyperbolic knot $K$ with the crosscap number two in $S^3$ results in a toroidal 3-manifold.

**Proof.** Let $\text{Int}V_K$ denote the interior of a tubular neighborhood of $K$. Then in the exterior $S^3 - \text{Int}V_K$ of $K$, we have a once punctured Klein bottle $\Sigma$ whose boundary is a preferred longitude of $K$. A tubular neighborhood of $\Sigma$ in the exterior is a twisted $I$-bundle over $\Sigma$, and the $\partial I$-subbundle is a twice punctured torus with its boundary components again being preferred longitudes of $K$. Hence in the result $K(0)$ of the 0-surgery along $K$, we have a twisted $I$-bundle $X$ over a Klein bottle $\hat{\Sigma}$, in which the $\partial I$-subbundle is an incompressible torus $T$.

Consider the decomposition $K(0) = X \cup Y$ with $X \cap Y = T$. If $T$ is incompressible in $Y$ as well, then so it is in $K(0)$, and hence $K(0)$ is toroidal. Let us suppose that $T$ is compressible in $Y$ to deduce contradictions; If $Y$ is irreducible, then $Y$ is a solid torus, hence $K(0)$ is a prism manifold or $\mathbb{RP}^3 \# \mathbb{RP}^3$. In the former case, $\pi_1(K(0))$ is a finite group, contradicting $H_1(K(0)) \cong \mathbb{Z}$. In the latter case, $K(0)$ is reducible, contradicting Gabai’s great result [Gab87, Corollary 8.3] asserting that the result of the zero surgery along a knot in $S^3$ is a prime manifold. If $Y$ is reducible, then $Y$ is the connected sum $V \# Z$ of a solid torus $V$ and some closed 3-manifold $Z \neq S^3$. Hence we have $K(0) = W \# Y$ for a prism manifold $W$ or $K(0) = \mathbb{RP}^3 \# \mathbb{RP}^3 \# Z$, again contradicting Gabai’s result.

Examples of hyperbolic fibered knots such that the 0-surgeries yield Seifert fibered spaces with periodic monodromies were given in [MS05] (see also [IM09]).

3. Generalized pseudo-Anosov flows and rational Fried surgeries

Recall that a pseudo-Anosov flow admits the stable/unstable foliations with a finite number of $k$-pronged singular orbits with $k \geq 3$ and, away from them, it is just an Anosov flow. A generalized pseudo-Anosov flow, which is also called a 1-pronged pseudo-Anosov flow or a singular Anosov-flow, may admit a finite number of 1-pronged singular orbits and, away from them, it is a pseudo-Anosov flow. The nature of generalized pseudo-Anosov flows is less known, though there are several interesting examples, such as the one in $S^1 \times S^2$ and others described in [BF13].

One might expect the following: The whole argument for pseudo-Anosov flows in [FP79] persists, a generalized pseudo-Anosov flow admits a version of Markov partition, the proof of [McMI3, Theorem 1.2] (contained in Proof of [McMI3, Theorem 1.4]) works as well, and we have a version of [Theorem 1.2] for a generalized pseudo-Anosov flow. Instead of precisely verifying this argument, we invoke the notion of rational Fried surgeries to prove the assertion.

We first recollect the classical case. Let $K$ be a closed orbit of an Anosov flow and $\sigma$ a cross section of $K$, namely, a circle on the boundary of the tubular neighborhood of $K$, that
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transversely intersects each of the stable/unstable flow lines twice. Then a Fried surgery along \((K, \sigma)\) naturally induces an Anosov flow on the result of the Dehn surgery whose slope coincides with that of \(\sigma\). It is defined as the combination of the blow-up along \(K\) and the blow-down by parallel copies of \(\sigma\), so that the orbits of the flow and the stable/unstable foliations persist in the exterior \([Fri83]\). We remark that Fried surgeries are topologically equivalent to Goodman surgeries \([Goo83, Sha20]\).

Next, we demonstrate a version of Theorem 1.2 for a generalized pseudo-Anosov flow:

**Theorem 3.1.** Let \((K_i)_i\) be the closed orbits of any topologically mixing generalized pseudo-Anosov flow on a closed 3-manifold \(M\), ordered by length in a generic metric. Then \((K_i)_i\) obeys the Chebotarev law.

**Proof of Theorem 3.1.** If we instead apply a similar procedure for a closed orbit \(K\) of a generalized pseudo-Anosov flow and a cross section \(\sigma\) of \(K\) such that \(\sigma\) transversely intersects each of the stable/unstable flow lines more than once, then we mostly obtain a \(k\)-pronged orbit with \(k \geq 2\) and rarely a 1-pronged orbit in a new flow on the result of the corresponding rational Dehn surgery. Therefore, by applying the 0-fillings, rational Fried surgeries, a generic time change, \([McM13, Theorem 1.2]\), the \(\infty\)-Dehn surgeries, and Proposition 2.1 in this order, then we obtain the assertion.

\[\square\]

4. The planetary links

We prove Theorem 1.3 by using Theorem 3.1 and Proposition 2.1.

**Proof of Theorem 1.3.** Let \(L\) in \(S^3\) be any fibered hyperbolic link other than \(4_1\). Then Nielsen–Thurston’s classification together with Thurston’s theorem \([Thu86, Proposition 2.6, Theorem 0.1]\) yield that the exterior \(S^3 - L\) is the mapping torus of a pseudo-Anosov map on a punctured surface and that the monodromy suspension flow is pseudo-Anosov.

The pseudo-Anosov map on a punctured surface \(\Sigma\) extends to a map on a closed surface \(\hat{\Sigma}\) such that the stable/unstable foliations may admit 1-pronged singular closed orbits \([Thu88, CB88, FM12]\). The pseudo-Anosov suspension flow on the mapping torus of \(\Sigma\) extends to a generalized pseudo-Anosov flow on the surface \(\hat{\Sigma}\) such that the components of \(L\) are contained in the set of the closed orbits. By \([McM13, Corollary 2.2]\) based on \([PP90, Theorem 8.5]\), a generic time change makes the flow topologically mixing. Hence Theorem 3.1 yields a sequence of knots obeying the Chebotarev law. Applying Proposition 2.1 to the \(\infty\)-Dehn surgeries along \(L\) in the mapping torus of the closed surface \(\hat{\Sigma}\), we obtain the assertion.

\[\square\]

If the meridians of the link \(L\) are not parallel to the stable/unstable flow lines, then the \(\infty\)-Fried surgery is defined. So, we may directly apply \([McM13, Theorem 1.2]\) to the result of the \(\infty\)-Fried surgery on the result of the 0-filling, to obtain the result. However, for instance, the figure-eight knot \(4_1\) is not the case. Indeed, the \(\infty\)-blow down yields four singular points on the figure-eight knot; two of them are attracting and the other two are repelling.

We also remark that even if the \(\infty\)-Fried surgeries are not defined, we may still play an analogue of number theory on the foliated dynamical systems by adding Reeb components, as described in \([KMNT21]\).

The Chebotarev link \(L = \bigcup_i K_i\) obtained by Theorem 13 is called the planetary link rising from \(L\), after Birman–Williams \([BW83]\). Ghrist and others proved that if \(L\) belongs to a certain
large class of links (e.g., the figure-eight knot, the Borromean ring, and every fibered non-torus 2-bridge knot), then the planetary link $L$ contains all types of links \cite{Ghr97, GHS97, GK04}. Ghrist \cite[Section 5]{Ghr97} claims that the Whitehead link also belongs to this class, and it is suggested as a reader’s exercise in \cite[Remark 3.2.20]{GHS97}.

McMullen’s construction \cite{McM13} answers Mazur’s question \cite{Maz12} on the existence of a sequence of knots in $S^3$ obeying the Chebotarev law. Morishita further asks the following:

**Question 4.1.** Does there exist a sequence of disjoint knots $(K_i)$ in $S^3$ obeying the Chebotarev law such that every type of knot appears in the sequence exactly once?

To answer his fused question in the affirmative, it suffices to show the equidistribution of each type of knot in the planetary link. Miller’s work \cite{Mil01} could give a cliff, which explicitly finds some types of geodesic knots.

## 5. Modular knots and Lorenz knots

We first recall the classical case. Let $\mathbb{H}^2 = \{ z \in \mathbb{C} \mid \text{Im} z > 0 \}$ denote the upper half plane. It is well-known that the unit tangent bundle $T_1(\PSL_2 \mathbb{Z})$ of the modular orbifold $\PSL_2 \mathbb{Z} \backslash \mathbb{H}^2$ is naturally isometric to the quotient space $\PSL_2 \mathbb{Z} \backslash \PSL_2 \mathbb{R} \cong SL_2 \mathbb{Z} \backslash SL_2 \mathbb{R}$ and is homeomorphic to the exterior of a trefoil $K_{2,3} = 3_1$ in $S^3$. The geodesic flow on $T_1(\PSL_2 \mathbb{Z} \backslash \mathbb{H}^2)$ may be seen as the flow on $SL_2 \mathbb{Z} \backslash SL_2 \mathbb{R}$ defined by multiplying $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ on the right, and its closed orbits are called modular knots around $K_{2,3}$.

An element $\gamma \in SL_2 \mathbb{Z}$ is said to be hyperbolic if $|\text{tr} \gamma | > 2$ holds. For each primitive hyperbolic element $\gamma$, we may define the corresponding modular knot $C_\gamma$ by $C_\gamma(t) = M_\gamma \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ ($0 \leq t \leq 0 \leq \log \xi_\gamma$), where $M_\gamma^{-1}\gamma M_\gamma = \begin{pmatrix} \xi_\gamma & 0 \\ 0 & \xi_\gamma^{-1} \end{pmatrix}$ with $\xi_\gamma > 1$. Every modular knot may be presented by some primitive hyperbolic $\gamma$, so that there is a natural surjection from the set of hyperbolic primitive elements to that of modular knots.

The geodesic flow on $\PSL_2 \mathbb{Z} \backslash \PSL_2 \mathbb{R}$ is classically known to be Anosov with a dense orbit and its structure is finely studied. We claim that modular knots and the missing trefoil obey the Chebotarev law, *if ordered by length in a generic metric*.

Unlike the cases of the suspension flows, the geodesic flow does not extend to a generalized pseudo-Anosov flow on the result of the $\infty$-filling resulting in $S^3$; in a natural extension of the flow, the missing trefoil consists of four heteroclinic orbits. Even worse, since any surgery along the trefoil in $S^3$ results in a non-hyperbolic manifold (cf. \cite{Sul13}), the procedure in the previous sections does not apply. Nevertheless, we may verify this assertion by proving Theorem 1.4 stated in Section 1, which is a partial extension of Theorem 1.2:

**Proof of Theorem 1.4** Let $\Gamma \leq SL_2 \mathbb{R}$ be an arbitrary Fuchsian group and let $T_1 \Sigma$ denote the unit tangent bundle of the orbifold $\Sigma = \Gamma \backslash \mathbb{H}$, which may have cusps. Note that a lift of the geodesic flow is pseudo-Anosov. Recently, Tsang gave an explicit construction of Markov partition that applies to cusped orbifolds also \cite[Remark 6.7]{Tsa23}. By substituting Tsang’s result (see also \cite{KU07} for the case of $SL_2 \mathbb{Z}$) to the place in the proof of \cite[Theorem 1.2]{McM13} (contained in the proof of \cite[Theorem 1.4]{McM13}) where the existence of a Markov partition (a Markov section) for a pseudo-Anosov flow on a closed 3-manifold \cite{FS79} is used, we obtain an extension of Theorem 1.2 to such $T_1 \Sigma$, that is, the closed orbits of the flow obey the Chebotarev law.
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(1) If $T_1 \Sigma = M - L$, then by the definition of the Chebotarev law, we may add the missing finite link $L$ to the family of closed orbits.

(2) If $M - L$ is a finite cover of $T_1 \Sigma$, then by noting that a Markov partition lifts to the covering space, we obtain a similar result. 

As explained in Section 1, for any pair $(a, b)$ of coprime integers, the notion of a modular knot of the $(a, b, \infty)$-triangle group $\Gamma_{a,b}$ is defined.

Proof of Theorem 1.5. Theorem 1.4 (1) and (2) apply to modular knots in (i) $L(a, b) - K_{a,b}$ and (ii) $S^3 - K_{a,b}$ respectively, yielding the assertion.

Here we give remarks on Lorenz knots. The classical Lorenz flow in $\mathbb{R}^3$ is defined by

$$\frac{dx}{dt} = 10(y - x), \quad \frac{dy}{dt} = 28x - y - xz, \quad \frac{dz}{dt} = xy - \frac{8}{3}z$$

This flow is well-known for its robustness and the chaotic behavior of its orbits, so it may suggest the existence of “fate” and “butterfly effects” at the same time. We have a certain geometric model for Lorenz flow as well (cf.

Lor63]. This flow factors through the conjugacy classes of PSL$_2 \mathbb{Z}$. (We may find in many literatures various confusions about the convention of the Rademacher symbol. Our convention is based on Matsusaka’s quite thorough investigation; See [Mat24].)

The Rademacher symbol $\Psi$ is known to be a highly ubiquitous function. Indeed, Atiyah proved the equivalence of seven definitions rising from very distinct contexts [Ati87], whereas Ghys gave further characterizations (cf. [BG92]) especially by using modular knots [Ghy07, Sections 3.3–3.5] (see also [DIT17, Appendix]), proving that for each primitive hyperbolic $\gamma \in \text{SL}_2 \mathbb{Z}$, the linking
number between the modular knot $C_\gamma$ and the missing trefoil $K$ coincides with the Rademacher symbol, namely,

$$\text{lk}(C_\gamma, K) = \Psi(\gamma)$$

holds. Note that the length of the image of $C_\gamma$ on the modular orbifold is given by $\ell(\gamma) = 2 \log \xi_\gamma$, where $\xi_\gamma = \frac{|\text{tr} \gamma| + \sqrt{(\text{tr} \gamma)^2 - 4}}{2}$ denotes the larger eigenvalue of $\gamma$, hence the order defined by the length of $C_\gamma$ in some metric coincides with that defined by $|\text{tr} \gamma|$. This order persists up to finite permutation under the filling and surgeries.

Now let $(K_i)$ denote the sequence of modular knots and the missing trefoil ordered by length and suppose that $L_n = \bigcup_{i \leq n} K_i$ contains the missing trefoil $K$. Let $0 \neq m \in \mathbb{Z}$. Applying the Chebotarev law for the surjective homomorphism $\rho : \pi_1(S^3 - L_n) \to \mathbb{Z}/m\mathbb{Z}; [C_\gamma] \mapsto \text{lk}(K, C_\gamma) = \Psi(\gamma) \mod m$, we obtain the following equidistribution formula:

**Corollary 6.1.** Suppose that $\gamma$ runs through primitive hyperbolic elements of $\text{SL}_2\mathbb{Z}$. For any $m \in \mathbb{Z}_{>0}$ and $k \in \mathbb{Z}/m\mathbb{Z}$, we have the density equality

$$\lim_{\nu \to \infty} \frac{\# \{\gamma \mid |\text{tr} \gamma| < \nu, \Psi(\gamma) = k \text{ in } \mathbb{Z}/m\mathbb{Z}\}}{\# \{\gamma \mid |\text{tr} \gamma| < \nu\}} = \frac{1}{m}.$$  

We remark that Simon recently gave a formula on the linking number of two modular knots [Sim22a, Sim22b], answering Ghys’s question in [Ghy07]. So, we may obtain a similar formula for every fixed modular knot.

Another distribution theorem is due to Sarnak–Mozzochi, proved by using the trace formula:

**Theorem 6.2** [Sar08, Sar10, Moz13]. Suppose that $\gamma$ runs through conjugacy classes of primitive hyperbolic elements in $\text{SL}_2\mathbb{Z}$. Then for any $-\infty \leq a \leq b \leq \infty$, we have

$$\lim_{\nu \to \infty} \frac{\# \{\gamma \mid \ell(\gamma) < \nu, a \leq \Psi(\gamma) \leq b\}}{\# \{\gamma \mid \ell(\gamma) < \nu\}} = \frac{\tan^{-1}(\frac{\pi b}{3}) - \tan^{-1}(\frac{\pi a}{3})}{\pi}.$$  

Changing the order of the orbits is an important issue in the context of dynamical systems. The study of Chebyshev bias [RS94] of prime numbers and prime geodesics is of a similar interest. We wonder whether these two equidistribution formulas above may be interpreted from a unified viewpoint.

Recently, Matsusaka and the author gave a thorough study of modular knots around any torus knot in $S^3$, establishing a generalization of Ghys’s formula $\text{lk}(C_\gamma, K) = \Psi(\gamma)$, as well as pointing out a version of Theorem 6.2 ([MU23, Theorem A, Corollary 5.3], see also [vE14, Bur22, BvE22]). This should be compared with Dehornoy–Pinsky’s works on so-called templates and codings [Deh13, DP18], as is done by Matsusaka–Shin [MS24]. So, every argument and question for classical modular knots above extends to the case around any torus knot.

### 7. Further remarks

**Stably Chebotarev**

By comparing Theorem 1.3 (2) and Theorem 1.4 (2), one might wonder if every Chebotarev link is stably Chebotarev, namely, if an infinite link $L$ in a 3-manifold $M$ obeys the Chebotarev law if ordered by length in a generic metric and $h : N \to M$ is a finite branched cover along a finite sublink $L$ of $L$, then the inverse image $h^{-1}(L)$ has the same property. To discuss this assertion,
the Hilbert ramification theory for 3-manifolds ([Uek14], see also [Mor12 Chapter 5]) would play a role, but it turns out to be troublesome, since the covering degrees of the orbits affect the order of the orbits.

**The $p$-adic torsions of knots and the Lang–Trotter conjecture**

Let $p$ be a prime number. Let $K$ be a knot in $S^3$ and let $(M_n \to S^3)_n$ denote the system of branched $\mathbb{Z}/n\mathbb{Z}$-covers. We proved in [UY22] that the sequence $(|H_1(M_p^n)_{\text{tor}}|)_n$ of the homology torsion sizes converges in the ring $\mathbb{Z}_p$ of $p$-adic integers. In addition, we pointed out that a knot $K$ with the $p$-adic limit value being 1 is an analog of a so-called supersingular prime of a fixed elliptic curve. It is known to be of high interest that the set of supersingular primes is an infinite set but its density in the set of all prime numbers is zero [Elk87]. This is a part of so-called the Lang–Trotter conjecture. To examine their analogues for Chebotarev links would be highly interesting also.

**The Neukirch–Uchida theorem in anabelian geometry**

In the context of anabelian geometry, the reconstruction problem is one of the main concerns (cf. [Moc15]). An important cornerstone is known as Neukirch–Uchida theorem [NSW08 Chapter XII] for number fields. In the spirit of arithmetic topology, we may formulate its analogue for a 3-manifold endowed with a Chebotarev link. Analogues of the Hilbert ramification theory and the Chebotarev density theorem play essential roles there [GUW24].

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