CONSTRUCTING MEAN CURVATURE 1 SURFACES IN $H^3$ WITH IRREGULAR ENDS

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Abstract. With the developments of the last decade on complete constant mean curvature 1 (CMC-1) surfaces in the hyperbolic 3-space $H^3$, many examples of such surfaces are now known. However, most of the known examples have regular ends. (An end is irregular, resp. regular, if the hyperbolic Gauss map of the surface has an essential singularity, resp. at most a pole, there.) There are some known surfaces with irregular ends, but they are all either reducible or of infinite total curvature. (The surface is reducible if and only if the monodromy of the secondary Gauss map can be simultaneously diagonalized.) Up to now there have been no known complete irreducible CMC-1 surfaces in $H^3$ with finite total curvature and irregular ends.

The purpose of this paper is to construct countably many 1-parameter families of genus zero CMC-1 surfaces with irregular ends and finite total curvature, which have either dihedral or Platonic symmetries. For all the examples we produce, we show that they have finite total curvature and irregular ends. For the examples with dihedral symmetry and the simplest example with tetrahedral symmetry, we show irreducibility. Moreover, we construct a genus one CMC-1 surface with four irregular ends, which is the first known example with positive genus whose ends are all irregular.

Introduction

Let $H^3$ denote the unique simply connected complete 3-dimensional Riemannian manifold with constant sectional curvature $-1$, which we call the hyperbolic 3-space. Associated to a complete finite-total-curvature CMC-1 (constant mean curvature one) conformal immersion $f : M \to H^3$ of a Riemann surface $M$ are two meromorphic maps called the hyperbolic Gauss map and the secondary Gauss map, which we denote by $G$ and $g$ respectively (to be defined in the next section). Using these two Gauss maps, we can define two characteristics of the surface $f$:

1. It is known that $M$ is biholomorphic to a compact Riemann surface with a finite number of points removed, and hence each end is conformally a punctured disk. Therefore we may consider the order of the hyperbolic Gauss map $G$ at each end, and an end is called regular if $G$ has at most a pole singularity at this end. If $G$ has an essential singularity, the end is called irregular.

2. Although $G$ is single-valued on $M$, the secondary Gauss map $g$ might be multi-valued on $M$, so we can have a nontrivial monodromy representation defined on the first fundamental group of $M$. This monodromy group is a subgroup of SU(2), and if all members of this group can be diagonalized by the same conjugation, we say that the surface $f$ is reducible. Otherwise,
we say that \( f \) is irreducible. (Irreducibility depends on a global behavior of the surface but not on individual ends.)

If a CMC-1 immersion is reducible, the surface can be deformed preserving its hyperbolic Gauss map \( G \) and Hopf differential \(((2, 0)\)-part of the second fundamental form, see Section \([1]\). On the other hand, an irreducible surface is the only surface with given hyperbolic Gauss map and Hopf differential.

Recent progress in the theory of CMC-1 surfaces in \( H^3 \) has led to the discovery of many new examples of these surfaces. Many examples with regular ends are now known, and various properties of these surfaces are understood. Bryant \([Bry]\) found a local description for these surfaces in terms of holomorphic data that initiated this recent progress. The last two authors \([UY1]\–[UY7]\) developed the theory using Bryant’s description to find many examples and properties, and work in this direction has been continued by Small \([Sm]\), the authors \([RUY1]\–[RUY5]\), Costa-Sousa Neto \([CN]\), Earp-Toubiana \([ET1]\–[ET2]\), Yu \([Yu1]\–[Yu3]\), Levi-Rossman \([LR]\), Barbosa-Berard \([BB]\), do Carmo and Gomes and Lawson and Thorbergsson and Silveira \([CGT]\, \([CL]\, \([CS]\, \text{and others.}

Regarding properties of the ends of embedded examples, Collin, Hauswirth and Rosenberg \([CHR1]\) have recently shown that any embedded CMC-1 surface of finite total curvature is either a horosphere or all of its ends are asymptotic to catenoid cousin ends. In \([CHR1]\, \([Yu3]\) it is further shown that any irregular end cannot be embedded, and the limit points of such an end are dense at infinity. Recently, Pacard and Pimentel \([PP]\) established a method for attaching small handles between tangent horospheres and deforming to produce CMC-1 surfaces, and this construction produces many embedded CMC-1 surfaces of any genus. Also, Karcher \([Kar]\) has recently constructed periodic CMC-1 surfaces with fundamental domains in several different types of compact quotients of \( H^3 \).

A typical example of an irregular end is the end of the Enneper cousin, a surface first constructed by Bryant \([Bry]\). After that the last two authors \([UY2]\) constructed examples of genus zero and two irregular ends, and also many reducible CMC-1 surfaces of genus zero whose ends are all irregular, using deformations from minimal surfaces. (The conclusion of Remark 4.4 in \([UY2]\) contains an error. The number of ends should be \( ml + 2 \), and hence the genus of \( M_0^* \) is zero.) Recently, Daniel \([D]\) has investigated irregular ends from the viewpoint of Nevanlinna theory.

After \([UY2]\), no further surfaces with irregular ends and finite total curvature had been constructed. (However, such an example with infinite total curvature can be found in \([RUY3]\).) In particular, until now no irreducible CMC-1 surfaces with irregular ends and finite total curvature had been known.

The purpose of the paper is to construct countably many 1-parameter families of genus zero CMC-1 surfaces with irregular ends and finite total curvature, which have either dihedral or Platonic symmetries. We further show that examples with dihedral symmetries, and the simplest example with tetrahedral symmetry, must be irreducible. All of our examples have irregular ends of finite type in the sense of Daniel \([D]\).

To do the construction, we start with the meromorphic data for the genus zero irreducible CMC-1 surfaces with regular ends found in \([UY3]\) and \([RUY1]\) and modify this data to make surfaces with irregular ends. The spirit of the construction is similar to the construction of trinoids in \([UY3]\), where CMC-1 surfaces with prescribed Gauss maps are constructed by reflecting spherical triangles, and we use
monodromy killing arguments like in [RUY1] and [UY6], but the techniques are brought to bear more intricately here.

In Section 1 we give necessary preliminaries. As our construction is done by reflecting abstract spherical triangles, we discuss this in Section 2 and introduce a method to construct CMC-1 surfaces with irregular ends (Theorem 2.3), which is proved in Section 4. As an application of the theorem, we construct examples of genus zero with either dihedral or Platonic symmetries in Section 3. Finally, in Section 5 we construct a CMC-1 surface of genus 1 with four irregular ends, which is the first known example with positive genus whose ends are all irregular.

1. Preliminaries

Null meromorphic curves. Here we recall from [UY1, UY7] some fundamental properties of null meromorphic curves in SL(2, C).

Definition 1.1. Let $F : M \rightarrow \text{SL}(2, \mathbb{C})$ be a meromorphic map defined on a Riemann surface $M$ with a local complex coordinate $z$. Then $F$ is called null if $\det(F_z) \equiv 0$. (This condition does not depend on the choice of coordinate $z$.)

Let $F : M \rightarrow \text{SL}(2, \mathbb{C})$ be a null meromorphic map. We define a matrix $\alpha$ by

$$
\alpha = \begin{pmatrix}
\alpha_{11} & \alpha_{12} \\
\alpha_{21} & \alpha_{22}
\end{pmatrix} := \left(F^{-1} dF, \right)
$$

and set

$$(1.1) \quad g := \alpha_{11} / \alpha_{21}, \quad \omega := \alpha_{21}.$$

Then the pair $(g, \omega)$ is a meromorphic function $g$ and a meromorphic 1-form $\omega$ on $M$ satisfying

$$
F^{-1} dF = \begin{pmatrix}
g & -g^3 \\
1 & -g
\end{pmatrix} \omega.
$$

Conversely, let $g$ be a meromorphic function and $\omega$ a holomorphic 1-form on $M$. Then the ordinary differential equation (1.2) is integrable and the solution $F$ is a null map into $\text{SL}(2, \mathbb{C})$ (since we will always choose the initial condition to be in $\text{SL}(2, \mathbb{C})$) defined on the universal covering of $M \setminus \{\text{poles of } \alpha\}$. In general, $F$ might not be single-valued on $M$ itself, and $F$ may have essential singularities at poles of $\alpha$. We call the pair $(g, \omega)$ the Weierstrass data of $F$.

Definition 1.2. Let

$$
F = \begin{pmatrix}
F_{11} & F_{12} \\
F_{21} & F_{22}
\end{pmatrix}
$$

be a null meromorphic map of $M$ into $\text{SL}(2, \mathbb{C})$ with Weierstrass data $(g, \omega)$. We call

$$
G := \frac{dF_{11}}{dF_{21}} = \frac{dF_{12}}{dF_{22}}
$$

the hyperbolic Gauss map of $F$. Furthermore, we call $g$ in (1.1) the secondary Gauss map and $Q = \omega \cdot dg$ the Hopf differential of $F$.

We remark that the secondary Gauss map $g$ satisfies

$$
g = \frac{dF_{22}}{dF_{21}} = \frac{dF_{12}}{dF_{11}}.
$$
Let $F : M \to \operatorname{SL}(2, \mathbb{C})$ be a null meromorphic map. Then for $a, b \in \operatorname{SL}(2, \mathbb{C})$, $F = aFb^{-1}$ is also a null meromorphic map. Then the associated two Gauss maps $\tilde{G}, \tilde{g}$, and the Hopf differential $\tilde{Q}$ of $\tilde{F}$ are

\begin{equation}
\tilde{G} = a \ast G, \quad \tilde{g} = b \ast g, \quad \text{and} \quad \tilde{Q} = Q,
\end{equation}

where, for any matrix $a = (a_{ij}) \in \operatorname{SL}(2, \mathbb{C})$ and any function $G$, $a \ast G$ is the Möbius transformation of $G$ by $a$:

\begin{equation}
a \ast G = \frac{a_{11}G + a_{12}}{a_{21}G + a_{22}}.
\end{equation}

Let $z$ be a complex coordinate on a neighborhood $U$ of $M$. Now we consider the Schwarzian derivatives $S(G)$ and $S(g)$ on $U$ of $G$ and $g$, where

\begin{equation}
S(G) = \left[ \frac{(G''G')'}{G'} - \frac{1}{2} \left( \frac{G''G'}{G'} \right)^2 \right] \frac{dz^2}{dz}, \quad \left( \frac{dz}{dz} = \frac{d}{dz} \right).
\end{equation}

The description of the Schwarzian derivative depends on the choice of complex coordinate. However, any difference of two Schwarzian derivatives, as a meromorphic 2-differential, is independent of the choice of complex coordinate. Note that the Schwarzian derivative is invariant under Möbius transformations:

\begin{equation}
S(G) = S(a \ast G) \quad (a \in \operatorname{SL}(2, \mathbb{C})).
\end{equation}

The following identity can be checked:

\begin{equation}
S(g) - S(G) = 2Q.
\end{equation}

Conversely, the following lemma holds:

**Lemma 1.3** ([Sm UY3 KUY]). Let $(G, g)$ be a pair of meromorphic functions on $M$ such that $S(g) - S(G)$ is not identically zero. Then there exists a unique (up to sign) null meromorphic map $F : M \to \operatorname{SL}(2, \mathbb{C})$ such that $G$ and $g$ are the hyperbolic Gauss map and the secondary Gauss map of $F$.

Now, for later use, we point out the following elementary fact from linear algebra:

**Lemma 1.4** ([RUY1]). A matrix $a \in \operatorname{SL}(2, \mathbb{C})$ satisfies $a \bar{a} = \text{id}$ if and only if it is of the form

\begin{equation}
a = \begin{pmatrix} p & i \gamma_1 \\ i \gamma_2 & \bar{p} \end{pmatrix}
\end{equation}

with $\gamma_1, \gamma_2 \in \mathbb{R}$, $p\bar{p} + \gamma_1\gamma_2 = 1$, $i = \sqrt{-1}$.

**CMC-1 surfaces in $H^3$.** We identify the Minkowski 4-space $L^4$, which has the canonical Lorentzian metric $(\cdot, \cdot)$ of signature $(-,+,+,+)$, with the space of $2 \times 2$ hermitian matrices $\text{Herm}(2)$. More explicitly, $(t, x_1, x_2, x_3) \in L^4$ is identified with the matrix

\begin{equation}
\begin{pmatrix}
t + x_3 & x_1 + ix_2 \\
x_1 - ix_2 & t - x_3
\end{pmatrix} \in \text{Herm}(2).
\end{equation}

The hyperbolic 3-space can be defined as the upper component

\begin{equation}
H^3 = \{ \xi = (t, x_1, x_2, x_3) \in L^4 \mid (\xi, \xi) = -1, \ t > 0 \}
\end{equation}

of the hyperboloid in $L^4$ with the induced metric. In $\text{Herm}(2)$ this is represented as

\begin{equation}
H^3 = \{ X \in \text{Herm}(2) ; \det X = 1, \ \text{trace} X > 0 \} = \{ aa^* ; a \in \operatorname{SL}(2, \mathbb{C}) \},
\end{equation}

where $aa^*$ is the conjugate transpose.
where \( a^* = \bar{\alpha} \). The complex Lie group \( \text{SL}(2, \mathbb{C}) \) acts isometrically on \( H^3 \) by 
\[
\rho(a)x = axa^*,
\]
where \( a \in \text{SL}(2, \mathbb{C}) \) and \( x \in H^3 \).

Let \( M \) be a Riemann surface and \( F : M \to \text{SL}(2, \mathbb{C}) \) a null holomorphic immersion. Then \( f = FF^* : M \to H^3 \) is a conformal CMC-1 immersion. Conversely, for any conformal CMC-1 immersion \( f : M \to H^3 \), there exists a null holomorphic immersion \( F : \tilde{M} \to \text{SL}(2, \mathbb{C}) \) such that \( f = FF^* \), where \( \tilde{M} \) denotes the universal covering of \( M \). We call \( F \) a lift of the conformal CMC-1 immersion \( f \). Let \( \tilde{F} \) be another lift of \( f \). Then, we have the expression \( \tilde{F} = Fb^{-1} \) for some matrix \( b \in SU(2) \). Let \((g, \omega)\) be the Weierstrass data of the null map of the lift \( F \). Then the first fundamental form \( ds^2 \) and the second fundamental form \( II \) are given by (see \([UY1]\), for example)
\[
ds^2 = (1 + |g|^2)^2 \omega \cdot \bar{\omega},
\]
\[
II = -Q - \overline{Q} + ds^2,
\]
where \( Q = \omega \cdot dg \) is the Hopf differential of \( F \).

Let \( f = FF^* : M \to H^3 \) be a complete conformal CMC-1 immersion whose total Gaussian curvature is finite. Since the Gaussian curvature \( K \) of CMC-1 surface is non-negative, finiteness of the total Gaussian curvature is equivalent to
\[
\int_M (-K) \, dA < \infty,
\]
where \( dA \) is the area element with respect to the first fundamental form. Then there is a compact Riemann surface \( \Sigma \) and a finite number of points \( \{p_1, \ldots, p_n\} \in \Sigma \) such that \( M = \Sigma \setminus \{p_1, \ldots, p_n\} \). We call each \( p_j \) an end of \( f \). The hyperbolic Gauss map \( G \) on \( M \) does not necessarily extend meromorphically on \( \Sigma \). The end \( p_j \) is called a regular end if \( p_j \) is at most a pole singularity of \( G \), and otherwise is called an irregular end. Namely, an irregular end is an end at which the hyperbolic Gauss map has an essential singularity.

On the other hand, the Hopf differential \( Q \) can always be extended as a meromorphic 2-differential on \( \Sigma \). We denote by \( \text{ord}_p Q \) the order of the first non-vanishing term of the Laurent expansion of the Hopf differential \( Q \) at \( p \in \Sigma \). (By definition, \( \text{ord}_p Q > 0 \) at zeros of \( Q \) and \( \text{ord}_p Q < 0 \) at poles of \( Q \).) The following lemma is well-known (cf. \([Bry]\), Lemma 2.3 of \([UY1]\)).

**Lemma 1.5.** An end \( p_j \) is regular if and only if \( \text{ord}_p Q \geq -2 \).

Now we set \( d\sigma_f = (-K) ds^2 \). Then \( d\sigma_f^2 \) is a pseudometric of constant curvature 1 with conical singularities (see the appendix of this paper, and also Proposition 4 of \([Bry]\)). It follows from (1.8) and the Gauss equation that
\[
d\sigma_f^2 = \frac{4 \, dg \cdot d\bar{g}}{(1 + |g|^2)^2}.
\]
Hence \( d\sigma_f^2 \) is the pull-back of the Fubini-Study metric \( d\sigma_0^2 \) on \( \mathbb{C}P^1 \) induced by the secondary Gauss map \( g : \tilde{M}^2 \to C \cup \{\infty\} = \mathbb{C}P^1 \). By (1.8) and (1.9) we have
\[
ds^2 \cdot d\sigma_f^2 = 4 \, Q \cdot \overline{Q}.
\]
In addition to having conical singularities at the ends \( p_j \), the pseudometric \( d\sigma_f^2 \) also has a conical singularity at each umbilic point \( q \in M \) of \( f \). The conical order of \( d\sigma_f^2 \) at each point is defined in the appendix of this paper. Since \( ds^2 \) is positive definite at \( q \), we have the following:
Lemma 1.6. At umbilic points, the conical order of $d\sigma^2$ equals the order of $Q$.

2. Reflections of an abstract spherical triangle

In this section, we introduce a method to construct CMC-1 surface with irregular ends. In [RUY1], examples with regular ends are constructed from the holomorphic data $G$ (the hyperbolic Gauss map) and $Q$ (the Hopf differential). However, since the hyperbolic Gauss map has essential singularities at irregular ends, it is hard to find an explicit expression of $G$ in our case. Thus, our construction is based on the secondary Gauss map $g$ and the Hopf differential $Q$. Though $g$ is not a well-defined meromorphic function on the surface, the pseudometric $d\sigma^2$ as in (1.9) is a spherical metric with conical singularities which is well-defined on the surface. So, we start by constructing a spherical metric $d\sigma^2$ with conical singularities, using reflections of spherical triangles (see [UY3]).

Abstract spherical triangles. In this section, we consider abstract spherical triangles and their extensions by reflection. First, we shall define abstract triangles. We set

$$\Delta := \{ x + iy \in \mathbb{C} \mid x \geq 0, y \geq 0, x + y \leq 1 \} ,$$

and label each vertex $V_1$, $V_2$, $V_3$ of this closed triangular region $\Delta$ as in Figure 1.

An abstract spherical triangle is a pair $(\Delta, d\sigma^2)$, where $d\sigma^2$ is a Riemannian metric defined on $\Delta$ with constant curvature 1 such that the three edges forming the boundary $\partial \Delta$ are geodesics. Let $A$, $B$ and $C$ be the interior angles of $\Delta$ at the vertices $V_1$, $V_2$, and $V_3$ with respect to the metric $d\sigma^2$ respectively. The angles $A$, $B$ and $C$ are positive, but may take any positive values, including those greater than or equal to $\pi$. Moreover, if $A, B, C \notin \pi \mathbb{Z}$, then we call the abstract spherical triangle $(\Delta, d\sigma^2)$ non-degenerate. The following fact is known:

Lemma 2.1 ([UY7]). Let $(\Delta, d\sigma^2)$ be a non-degenerate abstract spherical triangle, then the three angles $A$, $B$, $C$ satisfy the inequality

$$\cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C < 1 \, .$$

Figure 1. The triangle $\Delta$ and its reflections.
Conversely, if a triple of positive real numbers \((A, B, C)\) satisfies (2.1), then there exists a unique non-degenerate abstract spherical triangle such that the angles at \(V_1, V_2\) and \(V_3\) are \(A, B\) and \(C\) respectively.

**Proof**. We take a double (two identical copies) of \((\Delta, d\sigma^2)\) and glue them along their corresponding vertices and edges. Then we get a conformal pseudometric on \(S^2\) with three conical singularities with conical angles \(2A, 2B, 2C\). Consequently, the conical orders are

\[
\frac{A - \pi}{\pi}, \quad \frac{B - \pi}{\pi}, \quad \frac{C - \pi}{\pi}
\]

respectively. If \((\Delta, d\sigma^2)\) is non-degenerate, then \(A, B, C \notin \pi\mathbb{Z}\). By Corollary 3.2 of \([UY7]\), the metric on \(S^2\) is irreducible. Then (2.1) follows from (2.19) of \([UY7]\).

Conversely, suppose that (2.1) holds. By Theorem 2.4 of \([UY7]\), there exists a unique conformal pseudometric on \(S^2\) with three conical singularities with conical angles \(2A, 2B, 2C \notin 2\pi\mathbb{Z}\). The uniqueness of this pseudometric implies that it has a symmetry and can be considered as a gluing of two identical non-degenerate spherical triangles. \(\square\)

The above lemma implies that a non-degenerate abstract spherical triangle is uniquely determined by its angles \(A, B, C\). So we denote it by \(T(A, B, C) := (\Delta, d\sigma^2)\).

Now we fix a non-degenerate abstract spherical triangle \(T(A, B, C)\). Since \(\Delta\) is simply connected, there exists a meromorphic function \(g: \Delta \to \mathbb{C} \cup \{\infty\}\) such that the pull-back of the Fubini-study metric on \(\mathbb{C} \cup \{\infty\} = \mathbb{CP}^1\) by \(g\) is \(d\sigma^2\). However, such a choice of the developing map has an ambiguity up to an \(SU(2)\)-matrix action \(g \mapsto a \cdot g\) \((a \in SU(2))\). We shall now remove this ambiguity, using a normalization: There exists a unique (up to sign) developing map

\[
g = g_{A,B,C}: \Delta \longrightarrow \mathbb{C} \cup \{\infty\}
\]

of \(d\sigma^2\) satisfying (see Figure 1)

\[
e^{-iC} g(V_1) \in R \cup \{\infty\}, \quad g(V_2) \in R \cup \{\infty\} \quad \text{and} \quad g(V_3) = 0.
\]

We call the developing map \(g = g_{A,B,C}\) the normalized developing map of the triangle \(T(A, B, C)\).

Let \(\mu_j\) \((j = 1, 2, 3)\) be the reflections of the triangle \(T(A, B, C)\) across the three edges, as in Figure 1. Let \(\Delta_j\) be the closed domain obtained by reflecting \(\Delta\) with respect to \(\mu_j\) (see Figure 1). Then each reflection \(\mu_j\) can be regarded as an involution on the domain \(\Delta \cup \Delta_j\).

**Lemma 2.2** (Monodromy principle). Let \(T(A, B, C)\) be a non-degenerate abstract spherical triangle and \(g_{A,B,C}: \Delta \rightarrow \mathbb{C} \cup \{\infty\}\) \((j = 1, 2, 3)\) be the normalized developing map of \(T(A, B, C)\). Then the following identities hold:

\[
\begin{align*}
g_{A,B,C} \circ \mu_1 &= g_{A,B,C}, \\
g_{A,B,C} \circ \mu_2 &= e^{-2iC} g_{A,B,C}, \\
g_{A,B,C} \circ \mu_3 &= \left( \frac{q}{i\delta} \bar{q} \right)^* g_{A,B,C} \quad (\delta \in \mathbb{R}, \quad q\bar{q} + \delta^2 = 1),
\end{align*}
\]

\(^1\) Recently, an alternative proof and a geometric explanation of this lemma were given in \([FH]\) and \([F]\).
where

\[
q = \frac{i}{\sin C}(\cos A + e^{iC}\cos B).
\]

**Proof.** To simplify the notation, we set \( g = g_{A,B,C} \). Let \( \gamma_1, \gamma_2 \) and \( \gamma_3 \) be the three edges of \( \Delta \) which are stabilized by the reflections \( \mu_1, \mu_2 \) and \( \mu_3 \), respectively. Since the edge \( \gamma_1 \) is a geodesic, \( (2.2) \) implies that \( g(\gamma_1) \) lies on the real axis. Then by the reflection principle, \( g \circ \mu_1 = g \) holds. Similarly, by \( (2.2) \), \( e^{-iC}g(\gamma_2) \) is real. Hence

\[
e^{-iC}g \circ \mu_2 = e^{-iC}g
\]

holds. Then we have the second assertion.

There exists a rotation \( a \) centered at \( g(V_2) \) of the unit 2-sphere \( S^2(= C \cup \{ \infty \}) \) such that the image of \( g(V_1) \) is real. Such an isometry \( a \) of \( S^2 \) can be represented as a matrix \( a \in SU(2) \). Then we have \( a \ast g(\gamma_3) \) lies on the real axis. Hence by the reflection principle, \( a \ast g \circ \mu_3 = a \ast g \) holds, and then we have

\[
\bar{a} \ast g \circ \mu_3 = (\bar{a}^{-1}a) \ast g.
\]

In particular, we have

\[
(2.3) \quad g \circ \mu_1 = g, \quad g \circ \mu_2 = e^{-2iC}g, \quad g \circ \mu_3 = (\bar{a}^{-1}a) \ast g.
\]

Now we glue \( T(A,B,C) \) to a double of \( T(A,B,C) \) along corresponding edges and vertices, giving us a constant curvature one metric of three conical singularities on \( S^2 \) with conical angles \( 2A, 2B, \) and \( 2C \) just as in the proof of Lemma \( 2.1 \). The open domains \( \Delta_1, \Delta_2 \) and \( \Delta_3 \) can be regarded as the interior of the second triangle in the double of \( T(A,B,C) \). The monodromy of reflections for such metrics on \( S^2 \) are determined in \( [UY7] \). Then, as shown at the bottom of page 82 of \( [UY7] \), we have

\[
\bar{a}^{-1}a = \begin{pmatrix} q & i\delta \\ i\delta & \bar{q} \end{pmatrix}, \quad (\delta \in \mathbb{R}, q\bar{q} + \delta^2 = 1)
\]

with \( q \) as in \( (2.3) \). \( \square \)

**Closed Riemann surfaces generated by three reflections.** Let \( \overline{M} \) be a closed Riemann surface and \( D(\subset \overline{M}) \) a simply connected closed domain bounded by three real analytic curves \( \gamma_1, \gamma_2 \) and \( \gamma_3 \). We label the vertices \( V_1, V_2, V_3 \) of this closed triangular region \( D \) such that \( \gamma_1, \gamma_2, \) and \( \gamma_3 \) correspond to the three edges \( V_2V_3, V_3V_1 \) and \( V_1V_2 \), respectively. The Riemann surface \( \overline{M} \) is generated by \( D \) if there are three anti-holomorphic reflections \( \mu_1, \mu_2, \mu_3 \) of \( \overline{M} \) stabilizing the three edges \( \gamma_1, \gamma_2, \) and \( \gamma_3 \) of \( D \) such that any point of \( \overline{M} \) can be contained in the image of \( D \) by a suitable finite composition of these three reflections (see Figure 2). In this case, \( D \) is called a **fundamental domain** of \( \overline{M} \). A meromorphic 2-differential \( Q \) on \( \overline{M} \) is said to be symmetric with respect to \( D \) if

\[
Q \circ \mu_j = Q \quad (j = 1,2,3)
\]

holds.

We let \( \text{Met}_1(\overline{M}) \) denote the set of conformal pseudometrics with conical singularities on \( \overline{M} \). A metric \( d\sigma^2 \in \text{Met}_1(\overline{M}) \) is called symmetric with respect to the fundamental domain \( D \) if it is invariant under the three reflections \( \mu_1, \mu_2 \) and \( \mu_3 \). Moreover, if the restriction \( (D, d\sigma^2|_D) \) is isometric to \( T(A,B,C) \), we denote the metric by

\[
d\sigma^2 = d\sigma^2_{A,B,C}.
\]
A meromorphic function $g$ on $\mathcal{M}$ is called SU(2)-symmetric with respect to $D$ if
\begin{equation}
\text{d}\sigma^2_g := \frac{4\,dg \cdot d\bar{g}}{(1 + |g|^2)^2}
\end{equation}
is symmetric with respect to $D$.

The following is the main theorem in this paper:

**Theorem 2.3.** Let $\mathcal{M}$ be a closed Riemann surface which is generated by a triangular fundamental domain $D \subset \mathcal{M}$ by reflections, and label the vertices of $D$ as $V_1$, $V_2$ and $V_3$. Let $g$ be an SU(2)-symmetric meromorphic function on $\mathcal{M}$ with respect to $D$, and let $Q$ be a symmetric meromorphic 2-differential on $\mathcal{M}$ with respect to $D$. Suppose that:

1. There exist $A, B_0 \in \mathbb{R}^+ \setminus \pi\mathbb{Z}$ such that $\text{d}\sigma^2_g = \sigma^2_{A,B_0,\pi/2}$, with $\text{d}\sigma^2_g$ as in \text{(2.5)}.
2. $Q$ is holomorphic on $D \setminus \{V_2\}$ and has a pole at $V_2$ with $\text{ord}_{V_2}Q \leq -3$.
3. The branch point set of $g$ outside the poles of $Q$ equals to the zero set of $Q$, and at each zero of $Q$, the order of $Q$ is equal to the conical order of $\text{d}\sigma^2_g$.

Let $p_1, \ldots, p_n \in \mathcal{M}$ be the set of poles of $Q$. Then, for some $\varepsilon > 0$, there exist a smooth function $B(t) : (-\varepsilon, \varepsilon) \to \mathbb{R}$ satisfying $B(0) = B_0$ and a 1-parameter family of conformal CMC-1 immersions $f_t : \mathcal{M} \setminus \{p_1, \ldots, p_n\} \to H^3$ for $t \in (-\varepsilon, \varepsilon)$ with the following properties:

1. The Hopf differential of $f_t$ is $tQ$ and $\text{d}\sigma^2_{f_t} = \sigma_{A,B(t),\pi/2}$.
2. $f_t$ has irregular ends at $\{p_1, \ldots, p_n\}$.
3. $f_t$ is symmetric with respect to $D$. That is, the image of $f_t$ is generated by the reflections across the edges of $f_t(D)$.

**Remark 2.4.** The construction method for proving Theorem 2.3 will still work if $\text{ord}_{V_2}Q = -2$. The stronger assumption $\text{ord}_{V_2}Q \leq -3$ in \text{(2.5)} is required only to make the ends irregular (see Lemma 1.5).

The next theorem gives conditions which imply the surfaces $f_t$ in Theorem 2.3 are irreducible.

**Theorem 2.5.** Under the assumptions in Theorem 2.3 if $A \neq \pi/2 \pmod{\pi}$ and
\begin{equation}
\oint_{\gamma} \left( \frac{1}{g} \cdot -g^2 \right) \frac{Q}{dg} \neq 0
\end{equation}
for a local loop $\tau$ about $V_2$, then $f_t$ is irreducible for $t$ sufficiently close to zero.

The proofs of these theorems are given in Section 4.

3. CMC-1 surfaces with dihedral and Platonic symmetries

In this section, we construct examples of finite total curvature CMC-1 surfaces with irregular ends and either dihedral or Platonic symmetries. Examples with dihedral symmetries and the simplest example with tetrahedral symmetry (the case $m = 1$ in (3.1) and (3.2)) are irreducible. (Though we expect all the other examples to be irreducible, we have not checked them yet.)

CMC-1 surfaces with dihedral symmetries. Let $n \geq 3$ be an integer and

$$M_n := C \cup \{\infty\} \setminus \{1, \zeta, \ldots, \zeta^n-1\} \quad \left(\zeta = \exp \frac{2\pi i}{n}\right).$$

The Jorge-Meeks’ $n$-noid is the complete minimal immersion $f_{n,0}: M_n \to \mathbb{R}^3$ given by the Weierstrass representation as

$$f_{n,0} := \text{Re} \int ((1 - g_{n,0}^2), i(1 + g_{n,0}^2), 2g_{n,0}) \frac{Q_{n,0}}{dg_{n,0}},$$

where $g_{n,0} = z^{n-1}$, $Q_{n,0} = \frac{z^{n-2}}{(z^n - 1)^2} dz^2$.

The $\mathbb{Z}_2$ extension $D_n \times \mathbb{Z}_2$ of the dihedral group $D_n$ acts isometrically on the image of $f_{n,0}$.

There exists a one-parameter family of corresponding CMC-1 immersions of $M_n$ to $H^3$ such that the hyperbolic Gauss map is $g_{n,0}$ and the Hopf differential is proportional to $Q_{n,0}$ (see UY3, RUY1, and Figure 4 for the $n = 3$ case). Since $\text{ord}_{\zeta_j} Q_{n,0} = -2 (j = 0, \ldots, n-1)$, the ends of these corresponding surfaces are regular.

However, as we wish to produce CMC-1 surfaces in $H^3$ whose ends are not regular, we now modify our choices for $Q$ and $g$ to accomplish this: Let $m \geq 1$ be
SYMMETRIC CMC-1 SURFACES WITH IRREGULAR ENDS

Figure 4. Surfaces with dihedral symmetry: Figure (a) shows a CMC-1 surface with dihedral symmetry and three regular ends (the surface corresponding to a Jorge-Meeks surface) in the Poincaré model of $H^3$, and figure (b) is the fundamental region of the surface in figure (a). Figure (c) shows the fundamental piece of $f_{3,1,t}$, a surface of dihedral symmetry with three irregular ends. The central part of $f_{3,1,t}$ is similar to that of the hyperbolic correspondence of a Jorge-Meeks surface. The end of $f_{3,1,t}$ seen here is similar to the end of an Enneper cousin [Bry], which is shown in figure (d).

Let $\tau$ be a loop surrounding the end $V_2 = 1$. Since

$$\frac{Q_{n,m}}{g_{n,m}} = \left( \frac{1}{n(m+1)-1} \right) \frac{dz}{(z^n-1)^2(m+1)} ,$$

we have

$$\oint_{\tau} \frac{Q_{n,m}}{g_{n,m}} = \left( \frac{2\pi i}{n(m+1)-1} \right) \text{Res}_{z=1} \frac{1}{(z^n-1)^2(m+1)} \neq 0 .$$

Thus by Theorem 2.5 the surfaces are irreducible for sufficient small $t$. 

Let $\nu$ be a boundary loop on $\partial M$ as in Figure 3. Also, since $g_{n,m}$ and $Q_{n,m}$ are meromorphic functions on $M = \mathbb{C} \cup \{\infty\}$ respectively, which are symmetric with respect to the fundamental domain $\Omega_n$ as in Figure 3, we have

$$\oint_{\nu} \frac{g_{n,m}}{Q_{n,m}} = \frac{2\pi i}{n(m+1)-1} \text{Res}_{z=1} \frac{1}{(z^n-1)^2(m+1)} \neq 0 .$$

Then $g = g_{n,m}$ and $Q = Q_{n,m}$ are meromorphic functions and meromorphic 2-differentials on $M = \mathbb{C} \cup \{\infty\}$ respectively, which are symmetric with respect to the fundamental domain $\Omega_n$ as in Figure 3. Moreover, $(g, Q)$ satisfies the assumptions (1)–(3) of Theorem 2.3 for $A = \pi(m+1) - \pi/n$ and $B_0 = \pi/2$. Then, for each $n$ and $m$, there exists a one-parameter family of CMC-1 immersions $f_{n,m,t}: M_n \rightarrow H^3$ (0 < $|t|$ < $\varepsilon$) with symmetry group $D_n \times \mathbb{Z}_2$ whose Hopf differential is $tQ_{n,m}$ (see Figure 4). The total Gaussian curvature of these surfaces will be approximately $4\pi(n(m+1)-1)$.

Let $\tau$ be a loop surrounding the end $V_2 = 1$. Since

$$\oint_{\tau} \frac{Q_{n,m}}{g_{n,m}} = \frac{2\pi i}{n(m+1)-1} \text{Res}_{z=1} \frac{1}{(z^n-1)^2(m+1)} \neq 0 .$$

Thus by Theorem 2.5 the surfaces are irreducible for sufficient small $t$. 

Let $\nu$ be a boundary loop on $\partial M$ as in Figure 3. Also, since $g_{n,m}$ and $Q_{n,m}$ are meromorphic functions on $M = \mathbb{C} \cup \{\infty\}$ respectively, which are symmetric with respect to the fundamental domain $\Omega_n$ as in Figure 3, we have

$$\oint_{\nu} \frac{g_{n,m}}{Q_{n,m}} = \frac{2\pi i}{n(m+1)-1} \text{Res}_{z=1} \frac{1}{(z^n-1)^2(m+1)} \neq 0 .$$

Then $g = g_{n,m}$ and $Q = Q_{n,m}$ are meromorphic functions and meromorphic 2-differentials on $M = \mathbb{C} \cup \{\infty\}$ respectively, which are symmetric with respect to the fundamental domain $\Omega_n$ as in Figure 3. Moreover, $(g, Q)$ satisfies the assumptions (1)–(3) of Theorem 2.3 for $A = \pi(m+1) - \pi/n$ and $B_0 = \pi/2$. Then, for each $n$ and $m$, there exists a one-parameter family of CMC-1 immersions $f_{n,m,t}: M_n \rightarrow H^3$ (0 < $|t|$ < $\varepsilon$) with symmetry group $D_n \times \mathbb{Z}_2$ whose Hopf differential is $tQ_{n,m}$ (see Figure 4). The total Gaussian curvature of these surfaces will be approximately $4\pi(n(m+1)-1)$.
CMC-1 surfaces with tetrahedral symmetries. It is well-known that there exists a minimal immersion

\[ f_0 : M := C \cup \{ \infty \} \setminus \{ p_1, \ldots, p_4 \} \to \mathbb{R}^3 \]

with 4 catenoid ends and tetrahedral symmetry [Kat, Xn, BR, UY3] and corresponding CMC-1 surfaces in \( H^3 \) [UY3, RUY1] with regular ends.

We denote by \( Q_0 \) and \( g_0 \) the Hopf differential and the Gauss map of \( f_0 \) respectively. Since each end is asymptotic to a catenoid, \( \text{ord}_{p_j} Q_0 = -2 \) (\( j = 1, \ldots, 4 \)). Then there exists 4 umbilic points (zeros of \( Q_0 \)) \( q_1, \ldots, q_4 \) such that \( \text{ord}_{q_j} Q_0 = 1 \) (\( j = 1, \ldots, 4 \)). The Gauss map \( g_0 \) is a meromorphic function on \( C \cup \{ \infty \} \) whose branch points are \( \{ q_1, \ldots, q_4 \} \) each with branch order 1. Moreover, \( M \) is obtained by reflections across the edges of the fundamental domain \( D \), which is a triangle with vertices \( V_1 = q_1, V_2 = p_1, V_3 \). (See Figure 5. See also the construction in [UY3].) The Hopf differential \( Q_0 \) and the Gauss map \( g_0 \) are symmetric with respect to the fundamental domain \( D \).

Consider the Schwarzian derivative \( S(g_0) \) of \( g_0 \), as in (1.5), where \( z \) is the usual complex coordinate of \( C \cup \{ \infty \} \). Then \( S(g_0) \) is a meromorphic 2-differential on \( C \cup \{ \infty \} \). Moreover, since \( g_0 \) is symmetric, \( S(g_0) \) is invariant under reflections about the edges of \( D \).

The branch points of \( g_0 \) are poles of \( S(g_0) \), and each pole of \( S(g_0) \) has order \(-2\). So \( S(g_0) \) has 4 poles of order 2 at the \( q_j \) (\( j = 1, \ldots, 4 \)) and is holomorphic on \( C \cup \{ \infty \} \setminus \{ q_1, \ldots, q_4 \} \). Since the total order of a meromorphic 2-differential on \( C \cup \{ \infty \} \) is \(-4\), \( S(g_0) \) has 4 zeros (counting multiplicity). If there exists a zero of \( S(g_0) \) on the interior of \( D \), \( S(g_0) \) has at least 24 zeros because \( C \cup \{ \infty \} \) consists of 24 copies of the fundamental region \( D \), which is impossible. Similarly, if a zero of \( S(g_0) \) lies on the interior of an edge of \( D \), \( S(g_0) \) has at least 12 zeros, which is also impossible. If the vertex \( V_3 \) of \( D \) is a zero of \( S(g_0) \), there exist 6 zeros, which is again impossible. Since \( V_1 = q_1 \) is a pole of \( S(g_0) \), the set of zeros of \( S(g_0) \) must be \( \{ p_1, \ldots, p_4 \} \), and \( \text{ord}_{p_j} S(g_0) = 1 \) for \( j = 1, \ldots, 4 \).

For an integer \( m \geq 1 \), we define a meromorphic 2-differential \( Q_m \) as

\[
Q_m := \frac{Q_0^{m+1}}{S(g_0)^m},
\]

Figure 5. The fundamental domain of surfaces with tetrahedral symmetry
where \( Q_0^{m+1} \) (resp. \( S(y_0)^m \)) is the symmetric product of \( m+1 \) copies of \( Q_0 \) (resp. \( m \)-copies of \( S(y_0) \)). Since the poles of \( Q_0 \) and the zeros of \( S(y_0) \) are \( \{p_1, \ldots, p_4\} \), \( Q_m \) has the poles \( \{p_1, \ldots, p_4\} \) and is holomorphic on \( M \). More precisely,

\[
\text{ord}_{p_j} Q_m = -3m - 2 \quad \text{and} \quad \text{ord}_{q_j} Q_m = 3m + 1 \quad (j = 1, \ldots, 4)
\]

hold. Since \( Q_0 \) and \( S(y_0) \) are invariant under the reflections, so is \( Q_m \).

Consider an abstract spherical triangle \( T(A, B_0, C) \) with

\[
A = \pi m + \frac{2}{3} \pi , \quad B_0 = \frac{\pi}{3} , \quad C = \frac{\pi}{2}
\]

and identify it with the fundamental domain \( D \). Then we have a pseudometric \( d\sigma^2_{\Lambda, B_0, C} \in \text{Met}_1(C \cup \{\infty\}) \). Since neighborhoods of \( V_1, V_2 \) and \( V_3 \) are generated by \( 6, 6 \) and \( 4 \) copies of the fundamental domain \( D \), respectively, \( d\sigma^2_{\Lambda, B_0, C} \) is a pseudometric whose conical orders are all integers. Hence by Proposition A.1 in Appendix, the developing map \( g_m \) of \( d\sigma^2_{\Lambda, B_0, C} \) is meromorphic on \( C \cup \{\infty\} \).

Then one can easily check that \( (g_m, Q_m) \) satisfies the assumptions (1) (3) of Theorem 2.2. Hence for each \( m \), there exists a one-parameter family of CMC-1 immersions \( \{f_{m, t} : 0 < |t| < \varepsilon\} \) of \( M \) into \( H^3 \) with irregular ends and tetrahedral symmetry.

Finally, we check irreducibility for \( m = 1 \). We may set

\[
M = C \cup \{\infty\} \setminus \{1, \zeta, \zeta^2, \infty\} , \quad \text{where} \quad \zeta = \exp \frac{2}{3} \pi i ,
\]

and

\[
g_0 = \frac{1}{3\sqrt{2}} \left( z - \frac{4}{\sqrt{2}} \right) , \quad Q_0 = \frac{z(z^3 + 8)}{(z^3 - 1)^2} \, dz^2
\]

(see page 221 of [UY3]). Hence the umbilic points are \( \{0, -2, -2\zeta, -2\zeta^2\} \). By direct calculation, we have

\[
Q_1 = \frac{Q_0^2}{S(y_0)} = \frac{1}{96} \frac{z^4(z^3 + 8)^4}{(z^3 - 1)^3} \, dz^2.
\]

On the other hand, \( g_1 \) is a meromorphic function which branches at the umbilic points with branch order 4. Then we have \( \text{deg} g_1 = 9 \) by the Riemann-Hurwitz formula. Choose a rotation \( a \in \text{SU}(2) \) such that \( a \ast g_1(g_1) = \infty \), where we set \( q_1 = 0 \). Then \( q_j \) (\( j = 2, 3, 4 \)) are not poles of \( g := a \ast g_1 \), because the multiplicity of \( g \) at \( q_j \) is 5 for each \( j \) and \( \text{deg} g = 9 \). Moreover, \( d\sigma_{g_1}^2 = d\sigma_g^2 \) has a conical singularity at 0 with conical order 4. Hence, by symmetry, we have \( g_1(\zeta z) = \zeta g_1(z) \), and we may write

\[
dg = \beta \frac{(z^3 + 8)^4}{z^2(z^3 - a^3)^2} \, dz
\]

for some nonzero constants \( a \) and \( \beta \). Such a function \( g \) exists if and only if all residues of the right-hand side vanish, which is equivalent to \( a^3 = 16 \). Then one can check irreducibility by direct calculation and Theorem 2.2.

**CMC-1 surfaces with Platonic symmetries.** There are genus zero minimal surfaces in \( \mathbb{R}^3 \) with catenoidal ends and the symmetry of any Platonic solid ([Kat XG BR [UY3]). By similar arguments to the tetrahedral case above, one can obtain CMC-1 immersions with irregular ends and any Platonic symmetry. Table 1 shows the data for such surfaces.
4. PROOF OF THE MAIN THEOREM

Proof of Theorem 2.3.

Step 1: Take a real number $B \not\in \pi \mathbb{Z}$ and let

$$\rho_1 = \text{id}, \quad \rho_2 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix},$$

$$\rho_3 = \rho_3(B) = \begin{pmatrix} q(B) & i\delta(B) \\ i\delta(B) & \bar{q}(B) \end{pmatrix} \quad (\delta \in \mathbb{R}, \ q\bar{q} + \delta^2 = 1),$$

where

$$q(B) = i \cos A - \cos B.$$

Then we have

$$\bar{\rho}_j \rho_j = \text{id} \quad (j = 1, 2, 3).$$

Since $\mathcal{T}(A, B_0, \pi/2)$ is non-degenerate (by the assumption (1)), $A, B_0$ and $C = \pi/2$ satisfy the relation (2.7). Then for each $B$ sufficiently close to $B_0$, there exists an abstract spherical triangle with angles $A, B$ and $\pi/2$. We identify the domain $D \subset \overline{M}$ with this triangle. Then by reflecting the metric, we have $d\sigma^{A,B,\pi/2} \in \text{Met}_1(M)$. Let $M := \overline{M} \setminus \{p_1, \ldots, p_n\}$ and $\pi: \widetilde{M} \to M$ the universal covering. By Proposition A.1 in the appendix of this paper, the developing map $\hat{g}_{A,B,\pi/2}$ of $d\sigma^{A,B,\pi/2}$ is defined on $\widetilde{M}$. To simplify the notation, we set

$$\hat{Q} := \hat{g}_{A,B,\pi/2}: \widetilde{M} \to \mathbb{C} \cup \{\infty\}.$$

Then by the monodromy principle (Lemma 2.2), we have

$$\hat{g}_B \circ \mu_j = \rho_j \ast \hat{g}_B \quad (j = 1, 2, 3).$$

Step 2: One may regard the triangle $D \setminus \{V_2\}$ as generating $\widetilde{M}$ by the three reflections across its edges. We denote these reflections by $\hat{\mu}_1, \hat{\mu}_2, \hat{\mu}_3$; that is, each $\hat{\mu}_j$ ($j = 1, 2, 3$) is an antiholomorphic transformation on $\widetilde{M}$ which preserves the $j$’th edge of the triangle $D \setminus \{V_2\}$. We set

$$\hat{Q} := Q \circ \pi.$$

Let $F = F_{t,B}$ be a solution of the following ordinary differential equation on $\widetilde{M}$:

$$F^{-1}dF = t \left( \begin{array}{cc} \hat{g}_B & -\hat{g}_B \hat{g}_B \\ 1 & \hat{g}_B \end{array} \right) \frac{\hat{Q}}{d\hat{g}_B}, \quad F(V_3) = \text{id}.$$

| Symmetry      | $\#\{p_j\}$ | $\#\{q_j\}$ | $\text{ord}_{p_j} Q_0$ | $\text{ord}_{p_j} Q_m$ | $\text{ord}_{q_j} Q_m$ | $A$ | $B_0$ |
|--------------|-------------|-------------|-------------------|-------------------|-------------------|----|----|
| Dihedral     | $n$         | $n-2$       | $2(m+1)$          | $n(m+1) - 2$      | $\pi \left( m + 1 - \frac{1}{2} \right)$ | $\pm$ | $\pm$ |
| Tetrahedral  | $n$         | $n-2$       | $2(m+1)$          | $n(m+1) - 2$      | $\pi \left( m + 1 - \frac{1}{2} \right)$ | $\pm$ | $\pm$ |
| Octahedral   | $n$         | $n-2$       | $2(m+1)$          | $n(m+1) - 2$      | $\pi \left( m + 1 - \frac{1}{2} \right)$ | $\pm$ | $\pm$ |
| Icosahedral  | $n$         | $n-2$       | $2(m+1)$          | $n(m+1) - 2$      | $\pi \left( m + 1 - \frac{1}{2} \right)$ | $\pm$ | $\pm$ |

Table 1. Data for CMC-1 surfaces with Platonic symmetries
Such a solution $F_{t, B}$ is uniquely determined on $\tilde{M}$. By (4.3), the right-hand side of the ordinary differential equation is traceless and so $F_{t, B}$ takes values in $\text{SL}(2, C)$.

Then $\overline{F_{t, B} \circ \hat{\mu}_j} \, (j = 1, 2, 3)$ has the Hopf differential $\hat{Q} = \hat{Q} \circ \hat{\mu}_j$ and the secondary Gauss map satisfies $\hat{g}_B \circ \hat{\mu}_j = \rho_j \ast \hat{g}_B$. However, $F_{t, B} \rho_j^{-1}$ also has the Hopf differential $\hat{Q}$ and secondary Gauss map $\rho_j \ast \hat{g}_B$, by (4.3). Thus, by (4.3), we have

$$(F_{t, B} \circ \hat{\mu}_j)^{-1} d (F_{t, B} \circ \hat{\mu}_j) = (F_{t, B} \rho_j^{-1})^{-1} d (F_{t, B} \rho_j^{-1}) = \rho_j \left( \begin{array}{cc} \hat{g}_B & -\hat{g}_B^2 \\ 1 & -\hat{g}_B \end{array} \right) \frac{\hat{Q}}{d \hat{g}_B} \rho_j^{-1}.$$ 

This implies that $\overline{F_{t, B} \circ \hat{\mu}_j}$ and $F_{t, B} \rho_j^{-1}$ are both solutions of the same ordinary differential equation, and thus they differ only by the choice of initial values at the base point $V_3$. So there exists a matrix $\sigma_j(t, B) \in \text{SL}(2, C)$ such that

$$(F_{t, B} \circ \hat{\mu}_j)(V_3) = F(V_3) = \text{id} \quad (j = 1, 2, 3).$$

Since $F_{t, B} \circ \hat{\mu}_j(V_3) = F(V_3) = \text{id}$ for $j = 1, 2$,

$$\text{id} = \sigma_j(t, B) \rho_j^{-1} \quad (j = 1, 2).$$

holds. Thus we have

$$\sigma_1(t, B) = \rho_1 = \text{id}, \quad \sigma_2(t, B) = \rho_2 = \left( \begin{array}{cc} -i & 0 \\ 0 & i \end{array} \right).$$

In particular, the matrices $\sigma_1(t, B)$ and $\sigma_2(t, B)$ do not depend on $t$ nor the angle $B$. Since $F_{0, B} = \text{id}$, (4.3) implies that

$$\sigma_3(0, B) = \rho_3(B) = \left( \begin{array}{cc} q(B) & i \delta(B) \\ i \delta(B) & \bar{q}(B) \end{array} \right).$$

**Step 3**: Next we shall describe the matrix $\sigma_3(t, B)$. We have

$$F_{t, B} = F_{t, B} \circ \hat{\mu}_3 \circ \hat{\mu}_3 = \overline{F_{t, B} \circ \hat{\mu}_3 \circ \hat{\mu}_3} = \sigma_3(t, B)(F_{t, B} \circ \hat{\mu}_3) \rho_3(B)^{-1} = \sigma_3(t, B) F_{t, B} \rho_3(B)^{-1} \rho_3(B)^{-1}$$

$$= \sigma_3(t, B) \sigma_3(t, B) F_{t, B},$$

where we used the fact $\rho_3(B) \rho_3(B)$ is the identity. Thus we have

$$\sigma_3(t, B) \sigma_3(t, B) = \text{id}.$$  

By Lemma (4.3), the matrix $\sigma_3(t, B)$ can be written in the following form

$$\sigma_3(t, B) = \left( \begin{array}{cc} \rho(t, B) & i \nu_2(t, B) \\ i \nu_1(t, B) & \rho(t, B) \end{array} \right) \quad \text{with } \nu_1, \nu_2 \in \mathbb{R}, \quad \rho \nu + \nu \nu_2 = 1.$$  

We also have

$$F_{t, B} \circ \hat{\mu}_2 \circ \hat{\mu}_3 = \overline{F_{t, B} \circ \hat{\mu}_2 \circ \hat{\mu}_3} = \sigma_3(t, B)(F_{t, B} \circ \hat{\mu}_2) \rho_3(B)^{-1} = \sigma_3(t, B) \sigma_2(t, B) F_{t, B} \rho_2(B)^{-1} \rho_3(B)^{-1}.$$  

We may assume that the small disk centered at $V_1$ consists of $2l$-copies of $D$. Let $b$ be the branching order of $g$ at $V_1$. By the condition (3), we have

$$A = \pi \frac{b + 1}{l}.$$
Since $\tilde{\mu}_2 \circ \tilde{\mu}_3$ is the rotation of angle $2A$ at $V_1$, we have $(\tilde{\mu}_2 \circ \tilde{\mu}_3)^l = \text{id}$:

$$F_{t,B} = F_{t,B} \circ (\tilde{\mu}_2 \circ \tilde{\mu}_3)^l = \left( \sigma_3(t,B) \sigma_2(t,B) \right)^l F_{t,B} \left( \rho_3(B) \rho_2(B) \right)^{-l}.$$  

On the other hand, one can easily check that the eigenvalues of $\rho_3(B) \rho_2(B)$ are $\{-e^{iA}, -e^{-iA}\}$. Then by (4.8), the eigenvalues of $\left( \rho_3(B) \rho_2(B) \right)^l$ are $\{\pm 1, \pm 1\}$, that is

$$\left( \rho_3(B) \rho_2(B) \right)^l = \pm \text{id}.$$  

So we have

$$F_{t,B} = \pm \left( \sigma_3(t,B) \sigma_2(t,B) \right)^l F_{t,B}$$  

which implies that

$$\left( \sigma_3(t,B) \sigma_2(t,B) \right)^l = \pm \text{id}.$$  

This implies that the eigenvalues of $\sigma_3(t,B) \sigma_2(t,B)$ are of the form $\{e^{\pi i N/l}, e^{-\pi i N/l}\}$ for some integer $N$. Since $\sigma_3(t,B) \sigma_2(t,B)$ is continuous with respect to the parameter $t$, we can conclude that the eigenvalues of $\sigma_3(t,B) \sigma_2(t,B)$ do not change by $t$.

In particular,

$$(4.9) \quad \text{trace} \sigma_3(t,B) \sigma_2(t,B) = \text{trace} \sigma_3(0,B) \sigma_2(0,B).$$

On the other hand, since $F_{t,B} \circ \tilde{\mu}_2 \circ \tilde{\mu}_3 = \sigma_3(t,B) \sigma_2(t,B) F_{t,B} (\rho_3(B) \rho_2(B))^{-1}$ and $F_{0,B} = \text{id}$, we have

$$(4.10) \quad \sigma_3(0,B) \sigma_2(0,B) = \rho_3(B) \rho_2(B).$$

By (4.9), (4.10) and (4.7), we have

$$2 \text{Im} \, p(t,B) = \text{trace} \sigma_3(t,B) \sigma_2(t,B) = \text{trace} \rho_3(B) \rho_2(B) = 2 \cos A.$$  

**Step 4:** Since $\sigma_3(0,B) = \rho_3(B)$, we have

$$\operatorname{Re} p(0,B_0) = - \cos B_0.$$  

Now we would like to find a real valued smooth function $B(t)$ such that

$$\operatorname{Re} p(t,B(t)) = - \cos B_0 \quad (B(0) = B_0).$$

By the implicit function theorem, a sufficient condition for the existence of such a $B(t)$ is

$$\frac{\partial \operatorname{Re} p(t,B)}{\partial B} \bigg|_{(t,B) = (0,B_0)} \neq 0,$$

and by (4.7), (4.6), (4.2) and the assumption (1) we have

$$\frac{\partial \operatorname{Re} p(t,B)}{\partial B} \bigg|_{(t,B) = (0,B_0)} = \frac{\partial \operatorname{Re} p(0,B)}{\partial B} \bigg|_{B = B_0} = \frac{\partial \operatorname{Re} q(B)}{\partial B} \bigg|_{B = B_0} = - \frac{\partial \cos B}{\partial B} \bigg|_{B = B_0} = \sin B_0 \neq 0.$$  

This proves the existence of such a $B(t)$ ($|t| < \varepsilon$) for a sufficiently small $\varepsilon > 0$. 
Step 5: When \( t = 0 \), it holds that \( \sigma_3(t, B) = \rho_3(B) \), so \( \nu_1 \nu_2 > 0 \) for sufficiently small \( t \) \((|t| < \varepsilon)\), by continuity. Now we set

\[
F_t := \begin{pmatrix} u(t) & 0 \\ 0 & u(t)^{-1} \end{pmatrix} F_{t, B(t)}, \quad u(t) = \sqrt[4]{\frac{\nu_2(t, B(t))}{\nu_1(t, B(t))}}.
\]

Obviously, \( F_t \) satisfies the ordinary differential equation \((4.3)\) for \( B = B(t) \). In particular, \( F_t \) has the Hopf differential \( t\hat{Q} \) and the secondary Gauss map \( \hat{g}_{B(t)} \). Then by \((4.5)\) and \((4.7)\), we get the following relations

\[
F_t \circ \mu_j = \sigma_j(t) F_t \rho_j(B(t)) \quad (j = 1, 2, 3),
\]

where

\[
\sigma_1(t) = \text{id}, \quad \sigma_2(t) = \rho_2 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}
\]

and

\[
\sigma_3(t) = \left( \frac{p}{i \sqrt{\nu_1 \nu_2}} \right) \left( \frac{i \sqrt{\nu_1 \nu_2}}{\hat{p}} \right) (p = p(t, B(t)), \ \nu_j = \nu_j(t, B(t)), \ j = 1, 2).
\]

Since \( \text{Im} \ p(t, B(t)) = \cos A \) and \( \text{Re} \ p(t, B(t)) = -\cos B_0 \), we have

\[
p(t, B(t)) = i \cos A - \cos B_0 = q(B_0).
\]

Thus we have \( \sigma_j(t) = \rho_j(B_0) \) for \( j = 1, 2, 3 \). We now set

\[
f_t := F_t F_t^* : \tilde{M} \rightarrow H^3.
\]

By \((1.3)\), the first fundamental form of \( f_t \) is given by

\[
ds^2 := \left( 1 + |\hat{g}_{B(t)}|^2 \right)^2 \left| \frac{\hat{Q}}{d\hat{g}_{B(t)}} \right|^2.
\]

By the condition \((3)\) of the theorem, it is positive definite, and thus \( f_t \) is a conformal CMC-1 immersion whose Hopf differential is \( t\hat{Q} \) and the secondary Gauss map \( \hat{g}_{B(t)} \).

Step 6: We shall now prove that the conformal CMC-1 immersion \( f_t \) is single-valued on \( M = \tilde{M} \setminus \{p_1, \ldots, p_n\} \). Let \( s \) be a non-negative integer and \( \hat{\mu}_{i_1}, \hat{\mu}_{i_2}, \ldots, \hat{\mu}_{i_s} \) are sequences of three reflections \( \hat{\mu}_{i_1}, \hat{\mu}_{i_2}, \hat{\mu}_{i_3} \) on \( \tilde{M} \) such that

\[
\pi \circ \hat{\mu}_{i_1} \circ \hat{\mu}_{i_2} \circ \cdots \circ \hat{\mu}_{i_s} = \pi,
\]

where \( \pi : \tilde{M} \rightarrow M \) be the universal covering. To show the single-valued property of \( f_t \), it is sufficient to show that \( f_t = f_t \circ \hat{\mu}_{i_1} \circ \hat{\mu}_{i_2} \circ \cdots \circ \hat{\mu}_{i_s} \). In fact, we have

\[
F_t \circ \hat{\mu}_{i_1} \circ \hat{\mu}_{i_2} \circ \cdots \circ \hat{\mu}_{i_s} = \left( \frac{\rho_{i_1}(B_0) \rho_{i_2}(B_0) \cdots \rho_{i_{s-1}}(B_0) \rho_{i_s}(B_0)}{\rho_{i_1}(B(t)) \rho_{i_2}(B(t)) \cdots \rho_{i_{s-1}}(B(t)) \rho_{i_s}(B(t))} \right) F_t.
\]

Since \( g \) is single-valued on \( M \), \( \hat{g} := g \circ \pi \) satisfies

\[
\hat{g} = \hat{g} \circ \hat{\mu}_{i_1} \circ \cdots \circ \hat{\mu}_{i_s}.
\]

On the other hand, since \( \hat{g} \) is the secondary Gauss map of \( F_{t, B_0} \), we have by \((4.3)\) that

\[
\hat{g} = \hat{g} \circ \hat{\mu}_{i_1} \circ \cdots \circ \hat{\mu}_{i_s} = \left( \frac{\rho_{i_1}(B_0) \rho_{i_2}(B_0) \cdots \rho_{i_{s-1}}(B_0) \rho_{i_s}(B_0)}{\rho_{i_1}(B(t)) \rho_{i_2}(B(t)) \cdots \rho_{i_{s-1}}(B(t)) \rho_{i_s}(B(t))} \right) * g.
\]
Thus we can conclude that
\[ \rho_1(B_0) \rho_2(B_0) \cdots \rho_{2s-1}(B_0) \rho_{2s}(B_0) = \pm \text{id} . \]
Since \( \rho_j(B(t)) \in \text{SU}(2) \) \((j = 1, \ldots, 2s)\), this implies that \( f_t = F_t F_t^* \) is single-valued on \( M \).

Moreover, by the assumption \[2\] and Lemma \[1.5\], each end is irregular. The Hopf differential of \( f_t \) is \( Q_t \), and the pseudometric \( d\sigma^2_t \) defined in \[1.9\] is \( d\sigma^2_{A,B(t),\pi/2} \). Since they are symmetric with respect to \( D \), \[1.10\] and \[1.8\] imply that the first and second fundamental forms of \( f_t \) are invariant under the reflections \( \mu_j \) \((j = 1, 2, 3)\). Then by the fundamental theorem of surfaces, \( f_t \) is symmetric with respect to \( D \).

**Proof of Theorem 2.3** Let \( \tau \) be a loop surrounding the point \( V_2 \) with the base point \( V_3 \) and \( T \) the covering transformation of \( \tilde{M} \) corresponding to \( \tau \). Suppose that a neighborhood of \( V_2 \) is generated by \( 2k \)-copies of \( D \). Then
\[ T := (\hat{\mu}_3 \circ \hat{\mu}_1)^k \]
holds. Thus we have
\[ F_{t,B} \circ \tau = F_{t,B} \circ (\hat{\mu}_3 \circ \hat{\mu}_1)^k \]
\[ = \left( \sigma_3(t,B) \sigma_1 \right)^k F_{t,B} \left( \rho_1^{-1} \rho_3(B)^{-1} \right)^k = \sigma_3(t,B)^k F_{t,B} \rho_3(B)^{-k} . \]
Here, by the argument of Step 6 of the proof of Theorem 2.3, we have \( \rho_3(B)^k = \pm \text{id} \). Hence at the base point \( V_3 \),
\[ (4.11) \quad \frac{\partial F_{t,B} \circ \tau}{\partial t} \bigg|_{(t,B)=(0,B_0)} = \frac{\partial}{\partial t} \sigma_3(t,B)^k F_{t,B} \rho_3(B)^{-k} \bigg|_{(t,B)=(0,B_0)} \]
\[ = \pm \frac{\partial}{\partial t} \sigma_3(t,B)^k \bigg|_{(t,B)=(0,B_0)} \]
since \( F_{t,B}(V_3) = \text{id} \).

Since \( F_{t,B} \) is a solution of \[1.8\], it holds that
\[ \frac{\partial}{\partial z} \left[ \frac{\partial F_{t,B}}{\partial t} \bigg|_{(t,B)=(0,B_0)} \right] d\bar{z} = d \left[ \frac{\partial F_{t,B}}{\partial t} \bigg|_{(t,B)=(0,B_0)} \right] \]
\[ = \frac{\partial}{\partial t} dF_{t,B} \bigg|_{(t,B)=(0,B_0)} \]
\[ = \frac{\partial}{\partial t} \left[ tF_{t,B} \left( \hat{g}_B \hat{\bar{g}} - \hat{\bar{g}} \hat{g}_B \right) \hat{Q} \right] \bigg|_{(t,B)=(0,B_0)} \]
\[ = \left( \hat{g} \begin{pmatrix} 1 & -\hat{g}^2 \\ \hat{g} & 1 \end{pmatrix} \hat{Q} \right) d\bar{g} , \]
because \( \hat{g}_B = \hat{g} = g \circ \pi \). Then by the assumption and \[4.11\], we have
\[ \frac{\partial}{\partial t} \sigma_3(t,B)^k \bigg|_{(t,B)=(0,B_0)} = \pm \frac{\partial}{\partial t} \left( g \begin{pmatrix} 1 & -g^2 \\ 1 & -g \end{pmatrix} \right) \hat{Q} \neq 0 . \]
This implies that \( \sigma_3(t) = \sigma_3(t,B(t)) \) is not constant on \( \{ t \; | \; |t| < \varepsilon \} \) for sufficiently small \( \varepsilon > 0 \), and hence we have
\[ (4.12) \quad B(t) \neq B_0 \quad \text{for} \quad 0 < |t| < \varepsilon . \]
Hence the eigenvalues of $\rho_3(t) := \rho_3(B(t))$ are different from those of $\rho_3(B_0)$.

The secondary Gauss map $\hat{g}_t$ of $F_t$ changes by the covering transformation $T$ as

$$\hat{g}_t \circ T = \hat{g}_t \circ (\hat{\mu}_3 \circ \hat{\mu}_1)^k = \rho_3(t)^k \cdots \hat{g}_t.$$ 

Now, let $\tilde{V}_2 = \hat{\mu}_2(V_2)$ (see Figure 6). We denote by $\hat{\mu}_3$ the reflection about the edge $V_1 \tilde{V}_2$. Then we have

$$\hat{\mu}_3 = \hat{\mu}_2 \circ \hat{\mu}_3 \circ \hat{\mu}_2.$$ 

Let $\hat{T}$ be a loop surrounding $\tilde{V}_2$ with base point $V_3$, and let $\tilde{T}$ be the covering transformation corresponding to $\hat{T}$. Then we have

$$\tilde{T} = (\hat{\mu}_1 \circ \hat{\mu}_3)^k = (\hat{\mu}_1 \circ \hat{\mu}_2 \circ \hat{\mu}_3 \circ \hat{\mu}_2)^k,$$

and

$$\hat{g}_t \circ \tilde{T} = \left(\rho_1 \rho_2 \rho_3(t) \rho_2\right)^k \cdots \hat{g}_t = (-1)^{k-1} \rho_2 \left(\rho_3(t)^k\right) \rho_2 \cdots \hat{g}_t.$$ 

So, to show irreducibility, it is sufficient to show that the matrices

$$a := \rho_3(t)^k \quad \text{and} \quad b := \rho_2 \rho_3(t)^k \cdots \rho_2 = \rho_2 \cdots \rho_2$$

do not commute. By Lemma 1.4, the off-diagonal components of $a$ are coincide and pure imaginary. Set

$$a = \rho_3(t)^k = \begin{pmatrix} r \cdot \beta & i \beta \\ i \beta & -i \beta \end{pmatrix} \quad (\beta \in \mathbb{R}, r \beta + \beta^2 = 1).$$ 

Then we have

$$b = \rho_2 \cdots \rho_2 = \begin{pmatrix} -r & i \beta \\ i \beta & -r \end{pmatrix},$$ 

and

$$[a, b] = ab - ba = \begin{pmatrix} 0 & -2 \beta \text{Im}r \\ 2 \beta \text{Im}r & 0 \end{pmatrix},$$

that is, $a$ and $b$ commute if and only if $\beta = 0$ or $r$ is a real number.

First, we consider the case $\beta = 0$. Then $a = \rho_3(t)^k$ is a diagonal matrix whose eigenvalues are different from $\pm 1$ for sufficiently small $t$, because $B(t)$ is not constant. In particular, the two eigenvalues of $a$ are distinct. This implies that the eigenspaces of $a$ coincide of those of $\rho_3(t)$. Since $a$ is diagonal, this implies that $\rho_3(t)$ is also a diagonal matrix, a contradiction.
Next, assume \( r \) is real. Then there exists a real number \( \theta \) such that

\[
a = \rho_3(t)^k = \begin{pmatrix} \cos \theta & i \sin \theta \\ i \sin \theta & \cos \theta \end{pmatrix} = P \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} P^{-1},
\]

where \( P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \) and \( \theta \in R \setminus \pi Z \).

In this case, \((P^{-1}\rho_3(t)P)^k\) is a diagonal matrix whose eigenvalues differ from \( \pm 1 \), for sufficiently small \( t \neq 0 \). Then, by a similar argument to the previous case, we have \( P^{-1}\rho_3(t)P \) is diagonal. If \( A \neq \pi/2 \) (mod \( \pi \)), this contradicts to (4.1) and (4.2). Hence \( a \) and \( b \) do not commute. \( \Box \)

5. An example of genus one

In the final section, we construct an example of a CMC-1 surface of genus one with four irregular ends.

Let \( \Gamma = Z \oplus iZ \) be the lattice of Gaussian integers of \( C \) and let

\[
\mathcal{M} := C/2\Gamma.
\]

We consider \( \mathcal{M} \) as the square \([-\frac{1}{2}, \frac{3}{2}] \times [-\frac{1}{2}, \frac{3}{2}] \) in \( C = R^2 \), with opposite edges identified. Take a triangle \( D \) on \( \mathcal{M} \) as in Figure 7. Then \( \mathcal{M} \) is obtained from \( D \) by reflecting across the edges of \( D \).

Using the Weierstrass \( \wp \) function with respect to \( \Gamma \) (not with respect to \( 2\Gamma \)), we set

\[
Q = (\wp'(z))^2 \, dz^2.
\]
Then \( Q \) has poles at \( \{p_1, p_2, p_3, p_4\} = \{0, 1, 1 + i, i\} \), each with order 6. The \( \varphi \)-function with respect to the square lattice has the following properties

\[
\varphi(z) = \varphi(z), \quad \varphi(-z) = \varphi(z), \quad \varphi(i\bar{z}) = -\varphi(z).
\]

Hence \( Q \) is symmetric with respect to \( D \).

Consider an abstract spherical triangle \( T(A, B_0, C) \) with

\[
A = \frac{3}{4}\pi, \quad B_0 = \frac{\pi}{2}, \quad C = \frac{3}{2}\pi,
\]

and identify the triangle with the fundamental region \( D \). Then the metric of \( T(A, B_0, C) \) can be extended to \( d\sigma^2_{A, B_0, C} \in \text{Met}_1(\overline{M}) \) by reflections. Since \( A, B_0 \) and \( C \) satisfy (2.1), \( d\sigma^2_{A, B_0, C} \) is non-degenerate. Let \( g \) be the developing map of \( d\sigma^2_{A, B_0, C} \). Since the conical orders of \( d\sigma^2_{A, B_0, C} \) are integers, \( g \) is well-defined on \( C \).

Now, we prove that \( g \) is well-defined on \( \overline{M} \). By the monodromy principle (Lemma 2.2), one can choose \( g \) such that \( g \circ \hat{\mu}_j = \rho_j \ast g \ (j = 1, 2, 3) \), where

\[
\rho_1 := \text{id}, \quad \rho_2 := \left( \begin{array}{cc} i & 0 \\ 0 & -i \end{array} \right), \quad \rho_3 := \frac{1}{\sqrt{2}} \left( \begin{array}{cc} i & \pm i \\ \pm i & -i \end{array} \right)
\]

and \( \hat{\mu}_1, \hat{\mu}_2 \) and \( \hat{\mu}_3 \) are reflections along the edges \( V_3V_1, V_1V_2 \) and \( V_2V_3 \), respectively. We denote by \( \tau_1 \) and \( \tau_2 \) the translations \( z \mapsto z + 1 \) and \( z \mapsto z + i \) respectively. Then

\[
\tau_1 = \hat{\mu}_2 \circ \hat{\mu}_3 \circ \hat{\mu}_1 \circ \hat{\mu}_3, \quad \tau_2 = \hat{\mu}_3 \circ \hat{\mu}_2 \circ \hat{\mu}_3 \circ \hat{\mu}_1
\]

holds. So we have

\[
g \circ \tau_1 = \overline{\rho_2} \rho_3 \overline{\rho_1} \rho_2 \ast g = \rho_2 \ast g,
\]

\[
g \circ \tau_2 = \overline{\rho_3} \rho_2 \overline{\rho_1} \rho_1 \ast g = \pm \left( \begin{array}{cc} 0 & i \\ i & 0 \end{array} \right) \ast g.
\]

Thus

\[
g(z + 2) = g \circ \tau_1 \circ \tau_1(z) = g(z), \quad g(z + 2i) = g \circ \tau_2 \circ \tau_2(z) = g(z)
\]

hold. This shows that \( g \) is invariant under the action of the lattice \( 2\Gamma \). Hence \( g \) is a meromorphic function on \( \overline{M} \).

One can easily see that the same result as Theorem 2.3 holds when \( C = 3\pi/2 \), instead of \( \pi/2 \). Hence we have a one-parameter family \( \{f_t\} \) of CMC-1 immersions of \( \overline{M} \setminus \{p_1, p_2, p_3, p_4\} \) into \( H^3 \) with irregular ends.

**Appendix A.**

For a compact Riemann surface \( \overline{M} \) and points \( p_1, \ldots, p_n \in \overline{M} \), a conformal metric \( d\sigma^2 \) of constant curvature 1 on \( M := \overline{M} \setminus \{p_1, \ldots, p_n\} \) is an element of \( \text{Met}_1(\overline{M}) \) if there exist real numbers \( \beta_1, \ldots, \beta_n > -1 \) so that each \( p_j \) is a conical singularity of conical order \( \beta_j \), that is, if \( d\sigma^2 \) is asymptotic to \( c_j |z - p_j|^{2\beta_j} dz \cdot d\bar{z} \) at \( p_j \), for \( c_j \neq 0 \) and \( z \) a local complex coordinate around \( p_j \). We call the formal sum

\[
(A.1) \quad D := \sum_{j=1}^{n} \beta_j p_j
\]

the *divisor* corresponding to \( d\sigma^2 \). For a pseudometric \( d\sigma^2 \in \text{Met}_1(\overline{M}) \) with divisor \( D \), there is a holomorphic map \( g: \overline{M} \to \mathbb{CP}^1 \) such that \( d\sigma^2 \) is the pull-back of the Fubini-Study metric of \( \mathbb{CP}^1 \). This map, called the *developing map* of \( d\sigma^2 \), is
uniquely determined up to Möbius transformations $g \mapsto a \ast g$ for $a \in \text{SU}(2)$. We have the following expression

$$\pi^* d\sigma^2 = \frac{4 \, dg \cdot d\bar{g}}{(1 + |g|^2)^2},$$

where $\pi : \tilde{M} \to M$ is a covering projection.

Consider $d\sigma^2 \in \text{Met}_1(M)$ with divisor $D$ as in (A.1) and with the developing map $g$. Since the pull-back of the Fubini-Study metric of $\mathbb{C}P^1$ by $g$ is invariant under the deck transformation group $\pi_1(M) := M \setminus \{p_1, \ldots, p_n\}$, there is a representation

$$\rho_g : \pi_1(M) \to \text{SU}(2)$$

such that

$$g \circ T^{-1} = \rho_g(T) \ast g \quad (T \in \pi_1(M)).$$

The metric $d\sigma^2$ is called reducible if the image of $\rho_g$ is a commutative subgroup in $\text{SU}(2)$, and is called irreducible otherwise. Since the maximal abelian subgroup of $\text{SU}(2)$ is $\text{U}(1)$, the image of $\rho_g$ for a reducible $d\sigma^2$ lies in a subgroup conjugate to $\text{U}(1)$, and this image might be simply the identity. We call a reducible metric $d\sigma^2$ $H^3$-reducible if the image of $\rho_g$ is the identity, and $H^1$-reducible otherwise (for more on this, see [RUY1, Section 3]).

The following assertion was needed in Section 2:

**Proposition A.1.** Let $d\sigma^2$ be a metric of constant curvature 1 defined on $M = \overline{M \setminus \{p_1, \ldots, p_n\}}$ whose conical order at each $p_j$ is an integer. Then the developing map $g$ of $d\sigma^2$ is single-valued on the universal covering of $M$.

Let $p_1, \ldots, p_{n-1}$ be distinct points in $\mathbb{C}$ and $p_n = \infty$. We set

$$M_{p_1, \ldots, p_n} := \overline{\mathbb{C} \setminus \{\infty\} \setminus \{p_1, p_2, \ldots, p_n\}} \quad (p_n = \infty)$$

and $\tilde{M}_{p_1, \ldots, p_n}$ its universal covering.

**Corollary A.2.** Let $d\sigma^2$ be a metric of constant curvature 1 defined on $M_{p_1, \ldots, p_n}$ ($p_n = \infty$) whose conical order at each $p_j$ is an integer. Then the developing map $g$ of $d\sigma^2$ is single-valued on $M_{p_1, \ldots, p_n}$ and extends as a meromorphic function on $\mathbb{C} \cup \{\infty\}$. Moreover, the divisor of $d\sigma^2$ coincides with the ramification divisor of the meromorphic function $g$.

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