Incomplete Continuous-time Securities Markets with Stochastic Income Volatility

Peter O. Christensen
Department of Economics and Business,
Aarhus University,
DK-8210 Aarhus V, Denmark
email: pochristensen@econ.au.dk

Kasper Larsen
Department of Mathematical Sciences,
Carnegie Mellon University,
Pittsburgh, PA 15213, USA
email: kasper1@andrew.cmu.edu

April 26, 2013

ABSTRACT: We derive closed-form solutions for the equilibrium interest rate and market price of risk processes in an incomplete continuous-time market with uncertainty generated by Brownian motions. The economy has a finite number of heterogeneous exponential utility investors, who receive partially unspanned income and can trade continuously. Countercyclical stochastic income volatility generates a countercyclical equilibrium market price of risk process and a procyclical equilibrium interest rate process, and we show that when the investors’ unspanned income volatility is countercyclical, the resulting equilibrium displays both lower interest rates and higher risk premia compared to the equilibrium in an otherwise identical complete market.

KEYWORDS: Incomplete markets · non-Pareto efficiency · stochastic volatility · stochastic interest rates · stochastic risk premia

---

We wish to thank Torben G. Andersen, Jerome Detemple, Darrell Duffie, Semyon Malamud, Claus Munk, Frank Riedel, Mark Schroder, Steve Shreve, Costis Skiadas, and Gordan Žitković for constructive comments.
1 Introduction

We consider an incomplete continuous-time securities market with uncertainty generated by Brownian motions, which allows us to derive closed-form solutions for the equilibrium interest rate and market price of risk processes. The economy has a finite number of heterogeneous exponential utility investors, who receive partially unspanned income. The closed-form solutions allow us to quantify the impact of stochastic income volatility and preference heterogeneity on equilibrium interest rates and risk premia in a setting with unspanned income risk both at the individual and aggregate level.

Recent empirical work (see, e.g., Bloom 2009, and Bloom, Floetotto, Jaimovich, Saporta, and Terry 2012) suggests that income volatility is strongly countercyclical, i.e., investors view their future income prospects as more uncertain in economic downturns. Our closed-form solutions for the equilibrium interest rate and market price of risk processes allow an explicit study of how the variation of income volatility over the business cycle affects the security-spot market equilibrium. We show that countercyclical income volatility generates a countercyclical market price of risk process and a procyclical interest rate process. The underlying economic mechanism is the following: In economic downturns with high income volatility, the demand for precautionary savings is high which drives down the equilibrium interest rate. High aggregate income volatility further increases the aggregate amount of risk to be shared among the finite number of investors implying that the equilibrium price of absorbing an additional unit of risk increases.

Labor income and other income from non-traded assets are important sources of wealth which influence consumption and portfolio decisions and, consequently, influence the equilibrium prices. Because these sources of income typically have large unhedgeable components, it is difficult to solve for the optimal decision rules in order to determine the equilibrium prices. We demonstrate that, in a setting with Brownian uncertainty and exponential utility investors, optimal individual consumption rules can be aggregated to obtain closed-form solutions for the security-spot market equilibrium, even when the investors’ income streams have partially unspanned components and stochastic volatility.

It is well-known that individual unspanned income risks lower the equilibrium interest rate compared to the Pareto efficient equilibrium in an otherwise identical complete market setting (see, e.g., Wang 2003, Krueger and Lustig 2010, and Christensen, Larsen, and Munk 2012). This is due to the inefficient sharing of these risks and the resulting increased demand for precautionary savings. We show that countercyclical unspanned income volatility produces higher equilibrium risk premia (measured over intervals) compared to an otherwise identical complete market. Our closed-form solution for the equilibrium market price of risk process allows us to determine the underlying economic mechanism: An increase in income volatility
increases the investors’ conditional expected marginal utility of optimal end-of-period consumption, and this effect is amplified if that risk cannot be efficiently shared. Consequently, if negative (positive) shocks to spanned risk are followed by an increase (a reduction) in unspanned income risk, the risk premium for spanned risk increases. The increase in equilibrium risk premia due to countercyclical unspanned income volatility is highest when the more risk-averse investors face the largest unspanned income risk. We show in numerical examples that this impact can be substantial.

Our analysis has important policy and empirical implications. We show that variations in income volatility over the business cycle affect both equilibrium interest rates and risk premia. If the investors’ perceived income volatility can be reduced in economic downturns by economic policy measures, the incentive for precaution savings is reduced and interest rates would increase while risk premia would decrease, and more so if investors have large components of unhedgeable income risk. In this regard, extending our analysis to a production economy would be an interesting topic for future research. Macro-finance and empirical asset pricing studies should recognize that interest rates and risk premia vary over the business cycle, and that reasonable levels of interest rates and risk premia (cf., the risk-free rate and risk premium puzzles) can only be assessed quantitatively if the impact of individual unspanned stochastic income volatility is accounted for. Our analysis relies on exponential utility functions and Brownian uncertainty. An important topic for future research is if the underlying economic mechanisms derived from our model carry over to more general specifications of preferences and uncertainty.

Christensen et al. (2012) show that unspanned income risk has no impact on risk premia for aggregate spanned income risk in settings with deterministic income volatility and exponential utility investors. We show, in a model-free manner, that the introduction of stochastic income volatility does not change this result as long as risk premia are measured instantaneously. This result follows since the instantaneous market price of risk process is determined by the cross-variation process between the investors’ marginal utility of optimal consumption and spanned risk. On the other hand, when risk premia are measured over finite time-intervals (as in empirical asset pricing studies), the market price of risk process measured over intervals is determined by the cross-variation process between the investors’ conditional expected marginal utility of end-of-period optimal consumption and spanned risk. Stochastic income volatility affects optimal consumption distributions over finite time-intervals, and we show that the market price of risk process measured over intervals is impacted when unspanned income streams have stochastic volatility.²

²Similarly, in the last section of the paper we show that when consumption and income are restricted to discrete time-points (but trading is still continuous), the equilibrium instantaneous market price of risk process can be increased by unspanned income risk whenever the income volatility is countercyclical.
The questions of existence and characterization of complete market equilibria in continuous time and state models are well-studied.$^3$ The most common technique applied is based on the martingale method from Karatzas, Lehoczky, and Shreve (1987) and Cox and Huang (1989), which in complete market settings provides an explicit characterization of the investor's optimizer. By using the so-called representative agent method, the search for a complete market equilibrium can be reduced to a finite-dimensional fixed-point problem. To the best of our knowledge, only Basak and Cuoco (1988), Cuoco and He (1994), Žitkovic (2012), Hugonnier (2012), and Christensen et al. (2012) consider the existence and characterization of non-Pareto efficient equilibria in continuous-time trading settings.

Our setting is similar to that of Christensen et al. (2012), who derive closed-form solutions for all the equilibrium quantities in an economy with a finite number of heterogeneous exponential utility investors, and dividends and unspanned income governed by arithmetic Brownian motions. The crucial difference between the model in Christensen et al. (2012) and our model is that we allow for stochastic income volatility and, still, we provide a tractable incomplete markets model for which the equilibrium price processes can be computed explicitly. Consequently, we can quantify the impact of market incompleteness in the more realistic setting of stochastic income volatility supported by empirical evidence (see op. cit. Bloom 2009 and Bloom et al. 2012). The stochastic income volatility is a sufficient ingredient in order to obtain an impact of unspanned income on the market price of risk process measured over intervals. We incorporate a stochastic volatility à la Heston’s model into the income and equilibrium risky security price dynamics. We derive explicit expressions for the equilibrium interest rate as well as for the various market price of risk processes in terms of the individual income dynamics and the absolute risk aversion coefficients. The resulting type of the equilibrium market price of risk processes has been widely used in various optimal consumption-portfolio models (see, e.g., Chacko and Viceira 2005 and Kraft 2005), and the resulting equilibrium interest rate process is similar to that in the celebrated CIR term structure model.

Translation invariant utility models (such as the exponential utility model we consider) allow consumption to be negative (see, e.g., the discussion in the textbook Skiadas 2009). Schroder and Skiadas (2005) show that this class of models is fairly tractable even when income is unspanned. Our method of proof is based on re-writing the individual investors’ consumption-portfolio problems as problems with spanned income but heterogeneous beliefs. In certain affine settings with a deterministic interest rate, the exponential investor’s value function is available in closed-form (see, e.g., Henderson 2005, Wang 2004, Wang 2006, and

$^3$See, e.g., Chapter 4 in Karatzas and Shreve (1998) and Chapter 10 in Duffie (2001) for an overview of this literature. More recent references on complete market equilibria include Žitkovic (2006), Cvitanić, Jourini, Malamud, and Napp (2012), Anderson and Raimondo (2008), and Hugonnier, Malamud, and Trubowitz (2012).
Christensen et al. 2012). The incorporation of stochastic income volatility necessarily produces a stochastic equilibrium interest rate preventing the corresponding HJB-equation from having the usual exponential affine form. Therefore, the individual investor’s value function is not available in closed-form in our setting. By using martingale methods, we obtain tractable expressions for the individually optimal consumption policies, which in turn are sufficient to produce the incomplete market equilibrium price processes.

In a discrete infinite time horizon model with a continuum of identical exponential utility investors, Wang (2003) illustrates the negative impact unspanned income risk can have on the equilibrium interest rate. Similarly, in a discrete-time setting, Krueger and Lustig (2010) provide sufficient conditions in a setting with a continuum of identical power utility investors under which unspanned idiosyncratic income risk will lower the equilibrium interest rate, but not affect the risk premium. Christensen et al. (2012) present a continuous-time model with a finite number of exponential utility investors exhibiting the same interest rate phenomena, but also with no impact of unspanned income on the instantaneous risk premium. We extend these results by showing that as long as the consumption and income dynamics are continuous over time and the uncertainty is governed by Brownian motions, any equilibrium based on exponential preferences produces the same instantaneous risk premium as the standard Pareto efficient analogue. On the other hand, as noted above, we also prove that the market price of risk process measured over intervals is increased due to unspanned income risk if there is stochastic countercyclical income volatility.

Constantinides and Duffie (1996), and various extensions including Storesletten, Telmer, and Yaron (2007), produce similar equilibrium implications for the impact of unspanned income risk on interest rates and risk premia. They rely on a discrete-time analysis and a continuum of identical power utility investors with idiosyncratic income risks which wash-out at the aggregate level using a law of large numbers. Given virtually any pattern of risky securities and bond prices, Constantinides and Duffie (1996) show that individual income processes can be derived so that the (no-trade) equilibrium is consistent with these prices. In particular, if the cross-sectional volatility of the individual investors’ income growth is countercyclical and sufficiently large, the model can produce equilibrium prices consistent with the high observed equity premium. Cochrane (2005, Chapter 21) argues that cross-sectional income data do not show such large dispersion. In contrast to Constantinides and Duffie (1996), we consider a finite number of heterogeneous investors such that there is unspanned income risk both at the individual and at the aggregate level. Importantly, while the countercyclical income volatility in Constantinides and Duffie (1996) pertains to the cross-sectional income distribution, the countercyclical income volatility in our model pertains to the individual investors’ unspanned income risk.

Models based on a continuum of agents, such as Constantinides and Duffie (1996) and
Krueger and Lustig (2010), rely on market clearing conditions defined by reference to a law of large numbers. Judd (1985) and Uhlig (1996) discuss both technical and interpretation issues related to using such averaging market clearing conditions. Our model uses a finite number of investors and our market clearing conditions are required to hold pointwise, i.e., the realized aggregate demands are required to equal the aggregate supplies in equilibrium.

The paper is organized as follows. The next section introduces the structure of the economy in terms of the exogenously given quantities, and in terms of conjectures for the equilibrium price processes. Section 3 presents the investors’ consumption-portfolio problems in which the investors take the conjectured price processes as given. Our main Section 4 first defines and then shows the existence of an equilibrium consistent with the conjectured price processes and, secondly, it examines the impact of market incompleteness on the equilibrium interest rates and risk premia. Section 5 discusses the discrete consumption and income version of our model, and Section 6 concludes. All proofs are in the appendix.

2 Endowment and price processes

We consider an endowment economy with a single non-storable consumption good which also serves as the numéraire, i.e., prices are quoted in terms of this good. The economy is populated by $I < \infty$ consumer-investors all living on the time interval $[0, T], T < \infty$. $(\Omega, \mathcal{F}, \mathbb{P})$ denotes the probability space on which all stochastic quantities are defined. $(W, Z)$ denotes an $1 + I$ dimensional Brownian motion, where $W$ is scalar valued and $Z = (Z_i)_{i=1}^I$ is a vector of investor-specific Brownian motions. All Brownian motions $(W, Z_1, ..., Z_I)$ are independent and the corresponding standard augmented Brownian filtration is denoted by $\mathcal{F}_t, t \in [0, T]$. We consider $\mathcal{F} := \mathcal{F}_T$ and we will often write $E_t[\cdot]$ instead of $E(\mathbb{P})[\cdot|\mathcal{F}_t]$. $L^p$ denotes the space of measurable and adapted processes $f$ such that

$$\int_0^T |f_u|^p du < \infty, \quad \mathbb{P}\text{-almost surely, } \quad p \in \{1, 2\}.$$

2.1 Exogenously specified quantities

The investors have time-additive negative exponential utility of consumption with possible different degrees of absolute risk tolerance $\tau_i > 0, i = 1, ..., I$. For simplicity, we assume that their time-preference rates are all equal to zero. Investor $i$’s utility function over consumption is therefore

$$U_i(x) := -e^{-x/\tau_i}, \quad x \in \mathbb{R}, \quad i = 1, ..., I.$$

The following process $v$ will be used to model stochastic income volatility. We define $v$
as the Feller process

\( dv_t := (\mu_v + \kappa_v v_t)dt + \sigma_v \sqrt{v_t}dW_t, \quad t \in [0, T], \quad v_0 > 0, \)

where \( \kappa_v, \mu_v, \sigma_v \) are constants such that \( v \) remains strictly positive on \([0, T]\). The strict positivity of \( v \) is ensured by the first part of following assumption (Feller’s condition).

**Assumption 2.1.** The following two conditions are satisfied:

\[ \mu_v \geq \frac{1}{2} \sigma_v^2, \quad \text{and} \quad \sigma_v \neq 0. \]

Investor \( i \)'s income is determined by the process

\( dY_{it} := (\mu_{Y_i} + \kappa_{Y_i} v_t)dt + \sqrt{v_t} \left( \sigma_{Y_i} dW_t + \beta_{Y_i} dZ_{it} \right), \quad Y_{i0} \in \mathbb{R}. \)

The parameters \((\mu_{Y_i}, \kappa_{Y_i}, \sigma_{Y_i}, \beta_{Y_i}), \ i = 1, ..., I, \) are constants. The Brownian motion \( W \) affects all investors’ income processes, whereas the Brownian motion \( Z_i \) models investor \( i \)'s idiosyncratic income risk. The income process \( Y_i \) consists of the dividends from the investor’s endowed portfolio of traded and non-traded assets with exogenous dividends plus the investor’s stream of labor income.

It is not immediate how to adjust our approach to cover the mean-reverting income models used in Wang (2004) and Wang (2006). The affine optimal investment models used in Wang (2004) and Wang (2006) are based on an exogenously specified deterministic interest rate. However, the corresponding equilibrium interest rate cannot be deterministic or even independent of the investors’ idiosyncratic income risk processes in these affine settings. The affine optimal investment problem tremendously. As we shall see, the income processes (2.2) produce a stochastic equilibrium interest rate, which is adapted to the filtration generated by \( W \), and for which the individual exponential utility investor’s optimal investment problem remains partially tractable.

The aggregate income process \( \mathcal{E}_t := \sum_{i=1}^{I} Y_{it} \) has the dynamics

\[ d\mathcal{E}_t = (\mu_{\mathcal{E}} + \kappa_{\mathcal{E}} v_t)dt + \sqrt{v_t} \left( \sigma_{\mathcal{E}} dW_t + \sum_{i=1}^{I} \beta_{Y_i} dZ_{it} \right), \quad t \in [0, T], \]

\[ ^{4}\text{Breeden (1986) show that in complete markets settings the equilibrium interest rate is an increasing function of expected aggregate consumption growth. In mean-reverting income models the expected aggregate consumption growth depends on the level of aggregate consumption and, hence, the equilibrium interest rate is likely to depend on both the } W \text{-risk and on the investors’ idiosyncratic risks } Z_i. \text{ Our income processes (2.2) ensure that individual and aggregate income shocks are fully persistent. Therefore, the expected aggregate consumption growth is independent of the idiosyncratic income risk processes.} \]
where we have defined the constants

\[
\tau_\Sigma := \sum_{i=1}^I \tau_i, \quad \sigma_\varepsilon := \sum_{i=1}^I \sigma_{Y_i}, \quad \kappa_\varepsilon := \sum_{i=1}^I \kappa_{Y_i}, \quad \mu_\varepsilon := \sum_{i=1}^I \mu_{Y_i}.
\]

In order to make the following discussions and interpretations unambiguous, we assume that \((\sigma_{Y_i}, \beta_{Y_i})\) are nonnegative, \(i = 1, \ldots, I\). The cross-variation process between the aggregate income process and the stochastic income volatility, i.e., \(d\langle \mathcal{E}, v \rangle_t = \sigma_v \sigma_{\varepsilon} v_t dt\), is controlled by the parameter \(\sigma_v\). In what follows \(\sigma_v\) plays an important role, and we allow for both countercyclical \((\sigma_v < 0)\) and procyclical \((\sigma_v > 0)\) stochastic income volatility.

As we noted in the Introduction, Bloom et al. (2012) demonstrate empirically that income uncertainty is strongly countercyclical both at the aggregate, the firm, and the individual level and, hence, \(\sigma_v < 0\). Moreover, Bloom (2009) and Bloom et al. (2012) demonstrate that income growth is negatively impacted by increases in the income volatility, for example, due to a “higher value of waiting to invest” with non-convex capital adjustment costs and, hence, \(\kappa_\varepsilon < 0\). In turn, this implies that the constant part of expected aggregate income growth must be positive, i.e., \(\mu_\varepsilon > 0\), in order to have positive expected aggregate income growth (on average). In addition, Bloom (2009) demonstrates that expected income growth rebounds following positive shocks to volatility. This is in our model captured by assuming that the volatility process is mean-reverting, i.e., \(\kappa_v < 0\). Therefore, in the following, the “empirically relevant setting” refers to the parameter configuration:

\[
\mu_\varepsilon > 0, \quad \kappa_\varepsilon < 0, \quad \sigma_v < 0, \quad \kappa_v < 0.
\]

In order to ensure finite zero-coupon bond prices for all maturities and to state our main equilibrium existence theorem, we need the following assumption on the exogenous model parameters.

**Assumption 2.2.** The parameters \((\kappa_v, \sigma_v, \kappa_\varepsilon, \sigma_\varepsilon, (\tau_i)_{i=1}^I, (\beta_{Y_i})_{i=1}^I)\) are such that the following two restrictions hold:

\[
(\kappa_v - \frac{\sigma_v}{\tau_\Sigma} \sigma_\varepsilon)^2 > 2\sigma_v^2 \frac{1}{\tau_\Sigma} \left( \sum_{i=1}^I \frac{\beta_{Y_i}^2}{2\tau_i} + \frac{\sigma_\varepsilon^2}{2\tau_\Sigma} - \kappa_\varepsilon \right), \quad \text{and} \quad \kappa_\varepsilon \neq \sum_{i=1}^I \frac{\beta_{Y_i}^2}{2\tau_i} + \frac{\sigma_\varepsilon^2}{2\tau_\Sigma}.
\]

Both restrictions in (2.6) trivially hold if

\[
\kappa_\varepsilon > \sum_{i=1}^I \frac{\beta_{Y_i}^2}{2\tau_i} + \frac{\sigma_\varepsilon^2}{2\tau_\Sigma}.
\]

In the empirically relevant setting (2.5), the constant \(\kappa_\varepsilon\) is negative implying that (2.7) fails.
In this case, we shall see that the equilibrium interest rate process is unbounded from below and, consequently, zero-coupon bond prices may explode in finite time. We demonstrate in the next subsection that the weaker condition (2.6) ensures finite zero-coupon bond prices for all maturities, and this is all we need to prove our main equilibrium existence theorem.

2.2 Endogenously determined quantities

The investors can trade continuously on the time interval \([0, T]\) in a money market account with price process \(S^{(0)}\) and a single risky security with price process \(S\). We begin with the money market account.

**Conjecture 2.3.** The equilibrium price of the money market account has the dynamics

\[
dS^{(0)}_t = S^{(0)}_t r_t dt, \quad t \in [0, T], \quad S^{(0)}_0 = 1,
\]

where the \(F^W_t := \sigma(W_u)_{u \in [0, t]}\)-adapted process \(r\) is defined by

\[
r_t := \frac{1}{\tau_\Sigma} \left\{ \mu_S + \left( \kappa_S - \sum_{i=1}^{I} \frac{\beta_{Y_i}^2}{2 \tau_i} - \frac{\sigma^2}{2 \tau_\Sigma} \right) v_t \right\}.
\]

For concreteness, we let the single risky security be an annuity paying out a unit dividend rate over \([0, T]\). We make the following conjecture.

**Conjecture 2.4.** There exists an \(F^W_t := \sigma(W_u)_{u \in [0, t]}\)-adapted process \(\sigma_S \in L^2\) with \(\sigma_{St} \neq 0\) for \(t \in [0, T]\) such that the equilibrium price of the risky security has the dynamics

\[
dS_t + dt = \left( r_t S_t + \sigma_{St} \mu_S \sqrt{v_t} \right) dt + \sigma_{St} dW_t, \quad S_0 > 0,
\]

where \(r\) is defined by (2.9), and the constant \(\mu_S\) is defined by

\[
\mu_S := \frac{\sigma_S}{\tau_\Sigma}.
\]

The idiosyncratic Brownian motions \((Z_i)_{i=1}^{I}\) do neither appear directly in the risky security price dynamics (2.10) nor in the interest rate dynamics (2.9). Nevertheless, a key point of this paper is to explicitly quantify the impact the presence of the idiosyncratic unspanned risks \((Z_i)_{i=1}^{I}\) can have on \((S, S^{(0)})\). This impact is due to spanned and unspanned income risks affecting the interest rate dynamics differently. The spot interest rate (2.9) is determined as the risk-adjusted expected aggregate consumption growth per capita, \(\frac{1}{\tau_\Sigma} \{ \mu_S + \kappa_S v_t \}\), minus the risk premium component for instantaneous aggregate consumption risk per capita \(\frac{1}{2 \tau_\Sigma} \{ \sum_{i=1}^{I} \beta_{Y_i}^2 / \tau_i + \sigma^2 / \tau_\Sigma \} v_t\). In the latter component, idiosyncratic unspanned income risks
are evaluated using the investors’ personal risk tolerances, whereas spanned income risk is
evaluated using the aggregate risk tolerance in the economy. This reflects that the latter can
be efficiently shared among the investors whereas the former cannot. We note that the risk
premium component is countercyclical if, and only if, the income volatility is countercyclical.

In order to state the third and final property regarding \((S, S^{(0)})\), we need the standard
concept of state-price densities (see, e.g., Section 6F in Duffie 2001). For clarity, we define
these processes explicitly.

**Definition 2.5.** A local state-price density \(\xi^\nu\) has the form

\[
\xi^\nu_t = \frac{M^\nu_t}{S_t^{(0)}}, \quad t \in [0, T], \quad \xi^\nu_0 = 1,
\]

where \(\nu \in \mathcal{L}^2\), \(W^\perp\) is a \(W\)-independent Brownian motion, and

\[
M^\nu_t := \exp \left(-\mu_S \int_0^t \sqrt{v_u} dW_u - \int_0^t \nu_u dW^\perp_u - \frac{1}{2} \int_0^t \left(\mu_S^2 v_u + \nu_u^2\right) du \right).
\]

If, in addition, \(\mathbb{E}[M^\nu_T] = 1\), we call \(\xi^\nu\) a state-price density.

The main property of local state-price densities is that both \(\xi^\nu_t S_t^{(0)}\) and \(\xi^\nu_t S_t\) are driftless under \(\mathbb{P}\). For \(\nu \in \mathcal{L}^2\), \(M^\nu\) is always a supermartingale with \(\mathbb{E}[M^\nu_T] \leq 1\). However, for \(\xi^\nu\) to be a state-price density, we require \(\nu \in \mathcal{L}^2\) to produce the martingale property of \(M^\nu\).

We will need the minimal state-price density \(\xi^{\min}\) for which \(\nu := 0\), i.e.,

\[
d\xi^{\min} := -\xi^{\min}_0 \left(r_t dt + \mu_S \sqrt{v_t} dW_t \right), \quad \xi^{\min}_0 := 1.
\]

The corresponding minimal martingale measure \(\mathbb{Q}^{\min}\) is defined via the Radon-Nikodym
derivative on \(\mathcal{F}_T\) as (see, e.g., the survey Föllmer and Schweizer 2010)

\[
dM^{\min}_t := -M^{\min}_t \mu_S \sqrt{v_t} dW_t, \quad M^{\min}_0 := 1, \quad \frac{d\mathbb{Q}^{\min}}{d\mathbb{P}} := M^{\min}_T > 0.
\]

Lemma A.1 (in the appendix) ensures that \(M^{\min}\) is a martingale. Therefore, \(\xi^{\min}\) is indeed a state-price density and not just a local state-price density. Consequently, Girsanov’s theorem ensures that

\[
dW^{\min}_t := dW_t + \mu_S \sqrt{t} dt, \quad W^{\min}_0 := 0,
\]

is a Brownian motion under \(\mathbb{Q}^{\min}\) which is independent of \((Z_1, ..., Z_I)\).

**Definition 2.6.** The instantaneous market price of risk process for the Brownian motion \(W\)
is defined to be \(\mu_S \sqrt{t}\) for \(t \in [0, T]\) with \(\mu_S\) defined by (2.11).
Since \( d(\mathcal{E}, \mu_S \sqrt{v})_t = \frac{1}{2} \sigma_v \sigma \mathcal{E} \mu_S \sqrt{v} t dt \), we see from Definition 2.6 that the instantaneous market price of risk process is countercyclical if, and only if, the income volatility is countercyclical, i.e., if, and only if, \( \sigma_v < 0 \).

The following conjecture explicitly identifies the price process of the risky security by identifying the volatility process \( \sigma_S \) appearing in the price dynamics (2.10).

**Conjecture 2.7.** The equilibrium price of the risky security (i.e., the unit annuity) has the representation (we note that \( S_T = 0 \))

\[
S_t = \mathbb{E}^Q_{\min} \left[ \int_t^T e^{-\int_{s}^{U} r_s ds} dU \right], \quad t \in [0, T].
\]

At a first glance, it may seem restrictive to take the single risky security to be an annuity. However, we can let the risky security be any security paying out dividends at rate \( \delta_t \) as long as the process \( \delta \) satisfies the following two properties:

1. \( \delta_t \) is an Itô-process adapted to the filtration \( \mathcal{F}_W^u := \sigma(W_u)_{u \in [0, t]} \).

2. The following process is well-defined

\[
\mathbb{E}^Q_{\min} \left[ \int_t^T e^{-\int_{s}^{U} r_s ds} \delta_U dU \right], \quad t \in [0, T],
\]

and the \( dW \)-coefficient in these dynamics is non-zero on \([0, T]\).

The second requirement is related to endogenous dynamic market completeness. Duffie and Huang (1985), Anderson and Raimondo (2008) and Hugonnier, Malamud, and Trubowitz (2012) provide conditions on the primitives of the economy under which an Arrow-Debreu equilibrium can be implemented by dynamic trading. In our setting, these conditions amount to ensuring that the \( dW \)-coefficient does not vanish in the above conditional expectation involving the \( \mathcal{F}_W^u \)-adapted dividends \( \delta_t \).

In the continuous-time securities market \((S^{(0)}, S)\) with \( \sigma_{St} \neq 0 \) for \( t \in [0, T) \) (cf. Conjecture 2.4), all European claims written on the risky security, i.e., claims paying out \( g(S_T) \) at time \( T \) for some bounded payoff function \( g \), are replicable.\(^6\) Hence, the assumption of only a single traded risky security with \( \mathcal{F}_W^u \)-adapted dividends is not restrictive. The key incompleteness property is that the individual investor’s income process \( Y_t \) cannot be fully hedged

---

\(^{5}\)One advantage of choosing the annuity as the risky security is that its stochastic return is only affected by changes in the stochastic volatility \( v \), and not by changes in aggregate income \( \mathcal{E} \) (recall that the aggregate income shocks are fully persistent). It is this property of the annuity which allows us to demonstrate that the equilibrium \( \sigma_{St} \) is non-zero on \([0, T]\).

\(^{6}\)The same also holds if \( g \) is a bounded path functional of \((W_t)_{t \in [0, T]}\).
due to the presence of $Z_i$ in the dynamics of $Y_i$. Therefore, $(S^{(0)}, S)$ constitutes an incomplete continuous-time securities market. Consequently, the standard method of describing the equilibrium by an representative agent cannot be applied.

We conclude this section by showing that our equilibrium conjecture produces exponential-affine zero-coupon bond prices. This property constitutes an important ingredient in the proof of our main equilibrium existence theorem stated in Section 4. We refer to the appendix in Kim and Omberg (1996) for a detailed description of Riccati equations.

**Lemma 2.8.** Under Assumptions 2.1-2.2, and under the assumption that Conjectures 2.3, 2.4 and 2.7 hold, the following coupled system of ODEs with $a(0) = b(0) = 0$ and for $s > 0$

\begin{align}
    b'(s) &= b(s)(\kappa_v - \frac{\sigma \varepsilon}{\tau \Sigma} \sigma_v) + \frac{1}{2} b(s)^2 \sigma_v^2 + \frac{1}{\tau \Sigma} \left( \sum_{i=1}^{I} \beta_{Y_i}^2 + \frac{\sigma \varepsilon^2}{2\tau \Sigma} - \kappa \varepsilon \right), \\
    a'(s) &= \frac{\mu \varepsilon}{\tau \Sigma} - b(s) \mu_v,
\end{align}

has unique non-explooding solutions satisfying $b(s) \neq 0$ for $s \in (0, \infty)$. Furthermore, for $\mu_S$ defined by (2.11), we have for $t \in [0, U]$ that the zero-coupon bond prices are given by

\begin{align}
    B(t, U) := \mathbb{E}_{t}^{Q^{\text{min}}} \left[ e^{-\int_{t}^{U} r_s ds} \right] = \exp \left( b(U - t) v_t - a(U - t) \right).
\end{align}

Depending on whether (2.7) holds, the second restriction in (2.6) ensures that (2.15) has a negative or positive solution $b(s)$ for $s \in [0, \infty)$. If (2.7) holds, i.e., if the impact of the volatility $v_t$ on expected aggregate income growth exceeds the impact on the risk premium component in the spot interest rate (2.9), the solution to (2.15) is negative. In the empirically relevant setting (2.5), in which (2.7) fails, the solution to (2.15) is positive. Therefore, when (2.7) fails, the zero-coupon bond prices are increasing in the volatility $v_t$. As we discuss at the end of the next section, this property is consistent with increasing incentives for precautionary savings when income risk increases.

### 3. The individual investor’s problem

Investor $i$ chooses trading strategies $(\theta^{(0)}, \theta)$ as well as some consumption rate process $c$ in excess of the income $Y_i$. $\theta_t$ denotes the number of units held of the risky security in addition to the endowed portfolio of this asset (the endowed portfolio has dividends included in $Y_i$). Since the money market account has endogenous dividends paid at time $T$, the dividends generated by the endowed portfolio $\theta^{(0)}_t$ of this asset are not included in the investor’s income process $Y_i$. Therefore, $\theta_t^{(0)}$ denotes the total number of units held of the money market account at time $t \in [0, T]$. Consequently, $X^{\theta, c}_{it} := \theta^{(0)}_t S^{(0)}_t + \theta_t S_t$ denotes the investor’s total financial wealth.
(in addition to income) with initial condition $X_{i0} := \theta_{i0-}^{(0)} S_{0}^{(0)} = \theta_{i0-}^{(0)}$. The self-financing condition becomes for $t \in [0, T]$

$$X_{it}^\theta,c = X_{i0} + \int_0^t \theta_u(u) dS_u + \int_0^t \theta_u dS_u(u) du - \int_0^t c_u du$$

$$= X_{i0} + \int_0^t r_u X_{iu}^\theta,c du + \int_0^t \theta_u \sigma S_S(u) \left( \mu_S \sqrt{v_u} du + dW_u \right) - \int_0^t c_u du,$$

since the risky security is an annuity paying a unit dividend stream.

In order to ensure well-posedness of the individual investor’s optimization problem, we need to impose conditions which ensure that the measurable and adapted processes $(\theta, c)$ are such that the wealth dynamics (3.1) are well-defined. Moreover, in order to rule out arbitrage, we need additional constraints on the possible choices. First, the investor is required to leave no obligations behind after the finite time horizon:

$$\mathbb{P}(X_{iT}^\theta,c \geq 0) = 1.$$  

(3.2)

Naturally, investor $i$ optimally chooses strategies $(\hat{\theta}_i, \hat{c}_i)$ such that $X_{iT}^{\hat{\theta}_i,\hat{c}_i} = 0$. We deem $(\theta, c)$ admissible if additionally the process

$$\xi_t^\nu X_{it}^\theta,c + \int_0^t \xi_u^\nu c_u du, \quad t \in [0, T],$$

(3.3)

is a supermartingale for all state-price densities $\xi^\nu$ (see Definition 2.5). In this case, we write $(c, \theta) \in A$. This supermartingale condition ensures that there are no arbitrage opportunities in the admissible set $A$. In order to verify this claim, we let $\tau$ be a stopping time valued in $[0, T]$. Doob’s optional sampling theorem produces

$$\mathbb{E} \left[ \xi_\tau^\nu X_{iT}^\theta,c + \int_0^\tau \xi_u^\nu c_u du \right] \leq \xi_{i0}^\nu X_{i0} = X_{i0}.$$ 

By using this inequality with $c := 0$, we see that there are no arbitrage opportunities on $[0, T]$ in the admissible set $A$.

Investor $i$ maximizes time-additive expected utility stemming from consumption in addition to the investor’s income, i.e., investor $i$ seeks $(\tilde{c}_i, \tilde{\theta}_i) \in A$ such that

$$\sup_{(c, \theta) \in A} \mathbb{E} \left[ \int_0^T U_i(c_u + Y_{iu}) du \right] = \mathbb{E} \left[ \int_0^T U_i(\tilde{c}_{iu} + Y_{iu}) du \right].$$

(3.4)

As detailed in the appendix, the following result follows from a variation of the martingale method for complete markets.
Theorem 3.1. Under Assumptions 2.1-2.2, and under the assumption that Conjectures 2.3, 2.4 and 2.7 hold, there exists a unique constant $\alpha_i > 0$ such that

$$
\mathbb{E} \left[ \int_0^T \xi_{u} \min c_{iu} du \right] = X_{i0},
$$

where

$$
\hat{c}_{i0} := -\tau_i \log (\tau_i \alpha_i) - Y_{i0},
$$

and the consumption process has the dynamics

$$
d\hat{c}_{it} := \left\{ \tau_i r_t + \left( \frac{1}{2} \tau_i \mu_S^2 + \frac{1}{2} \beta Y_i \right) v_t - \mu Y_i \right\} dt + \left( \tau_i \mu_S - \sigma Y_i \right) \sqrt{v_t} dW_t.
$$

Furthermore, there exists an investment strategy $\hat{\theta}_i$ such that the pair $(\hat{c}_i, \hat{\theta}_i) \in A$ is optimal for investor $i$, $i = 1, \ldots, I$.

The proof of Theorem 3.1 produces the optimal investment strategy $\hat{\theta}_i$ using the martingale representation theorem (see equation (A.4)) via the relation

$$
X_{t, \hat{\theta}_i, \hat{c}_i} = \mathbb{E}^{Q_{min}}_{t} \left[ \int_t^T e^{-\int_t^u r_s ds} \hat{c}_{iu} du \right], \quad t \in [0, T].
$$

However, a tractable expression for the optimal investment strategy $\hat{\theta}_i$ is not available because the interest rate $r_t$ is stochastic. Fortunately, our equilibrium approach only requires the abstract existence of $\hat{\theta}_i$. The proof of Theorem 3.1 shows that the optimal strategies $(\hat{\theta}_i, \hat{c}_i)$ are such that the process (3.3) is a martingale for all state-price densities $\xi^u$.

From (3.7) we see that the optimal consumption process including income has the dynamics

$$
d(\hat{c}_it + Y_{it}) = \left( \tau_i r_t + \left( \frac{1}{2} \tau_i \mu_S^2 + \frac{1}{2} \beta Y_i \right) v_t \right) dt + \sqrt{v_t} \left( \tau_i \mu_S dW_t + \beta Y_i dZ_{it} \right).
$$

Since $\mu_S = \sigma \varepsilon / \tau_\Sigma$, the investors can be seen as "selling off" their personal spanned income risk $\sqrt{\tau_i} \sigma Y_i dW_t$ while receiving their constant efficient risk sharing fraction $\tau_i / \tau_\Sigma$ of the aggregate spanned income risk $\sqrt{\tau_i} \varepsilon_d W_t$. On the other hand, due to the market incompleteness, the investors must retain their unspanned idiosyncratic income risk $\sqrt{\tau_i} \beta Y_i dZ_{it}$, although they are able to "smoothen" those income shocks over time through instantaneously riskless investments. Itô’s lemma produces the dynamics

$$
dU'_i(\hat{c}_it + Y_{it}) = -U'_i(\hat{c}_it + Y_{it}) \left( r_t dt + \sqrt{v_t} \mu_S dW_t + \sqrt{v_t} \beta Y_i dZ_{it} \right),$$

13
which implies that the cross-variation process for the marginal utility of optimal consumption and spanned income risk $W$ is given by

$$d\langle U'_i(\hat{c}_i + Y_i), W \rangle_t = -U'_i(\hat{c}_i + Y_i)\sqrt{\nu_i}\mu_S dt.$$  

Therefore, the instantaneous market price of risk process $\sqrt{\nu_i}\mu_S$ is indeed determined by the cross-variation process between the investors’ marginal utility of optimal consumption and spanned income risk.

The dynamics (3.9) and Itô’s lemma produce

$$\frac{U'_i(\hat{c}_iU + Y_i)}{U''_i(\hat{c}_i + Y_i)} = \exp\left(-\int_t^U \left(r_s + \frac{1}{2}v_s(\mu^2_S + \frac{\beta^2 Y_i}{r^2_\tau})\right) ds - \int_t^U \sqrt{\nu_s}(\mu_d W_s + \frac{\beta Y_i}{r_\tau}dZ_{is})\right),$$  

for $t \in [0, U]$. Since $r_t$ is measurable with respect to $\mathcal{F}_t^W := \sigma(W_u)_{u \in [0, t]}$ we have

$$E_t\left[\frac{U'_i(\hat{c}_iU + Y_iU)}{U''_i(\hat{c}_i + Y_i)}\right] = E_t^{Q^{\text{min}}}[\exp\left(-\int_t^U r_s ds\right)] = \exp\left(b(U - t)v_t - a(U - t)\right),$$  

where $Q^{\text{min}}$ is defined by (2.13), and the last equality follows from Lemma 2.8. In other words, in the empirically relevant setting in which the solution $b$ to (2.15) is positive, an increase in income volatility affects the investors’ expected marginal utility of optimal future consumption positively. This implies an increased demand for precautionary savings, which lowers the equilibrium zero-coupon interest rates.

### 4 Equilibrium

Before stating the following equilibrium definition (of the Radner-type), we recall that consumption $c_i$ is measured in excess of the income rates $Y_i$, and that the trading strategies $\theta_i$ denote the units held of the risky security in addition to the investors’ endowed portfolios of this asset. On the other hand, the trading strategies $\theta^{(0)}_i$ denote the total number of units held of the money market account. Since the money market account has endogenous dividends determined by the spot interest rates, this asset must be in zero net-supply in order to ensure that aggregate consumption is exogenous. Of course, this also implies that the endowments of the money market account must satisfy the clearing condition $\sum_{i=1}^I \theta^{(0)}_{i0} = \sum_{i=1}^I X_{i0} = 0$.

**Definition 4.1.** An equilibrium is a set of security price processes $(S^{(0)}, S)$, characterized by $(r, \mu_S, \sigma_S)$, and a set of investor strategies $(\hat{c}_i, \hat{\theta}_i) \in \mathcal{A}$ such that given $(r, \mu_S, \sigma_S)$, the...
processes \((\hat{c}_i, \hat{\theta}_i)\) are optimal for investor \(i\), \(i = 1, 2, \ldots, I\), and such that all markets clear, i.e.,

\[
\sum_{i=1}^{I} \hat{c}_{it} = 0, \quad \sum_{i=1}^{I} \hat{\theta}_{it} = 0, \quad \sum_{i=1}^{I} \hat{\theta}_{it}^{(0)} = 0, \quad \mathbb{P} \otimes \text{Leb-a.e.} \tag{4.1}
\]

The proof of the following main theorem shows that clearing in the good’s market, i.e., \(\sum_{i=1}^{I} \hat{c}_{it} = 0\), ensures market clearing for both the risky security and the money market.

**Theorem 4.2.** Under Assumptions 2.1 and 2.2, the security price processes \((S^{(0)}, S)\) defined by (2.8) and (2.14) with the resulting individually optimal strategies \((\hat{c}_i, \hat{\theta}_i)\) \(\in A\), \(i = 1, 2, \ldots, I\), constitute an equilibrium for which Conjectures 2.3, 2.4, and 2.7 hold.

The proof of Theorem 4.2 shows that Conjecture 2.7 holds with the volatility coefficient

\[
\sigma_{St} := \sigma_v \sqrt{v_t} \int_{t}^{T} B(t, U)b(U - t)dU, \quad t \in [0, T), \tag{4.2}
\]

which is non-zero on the interval \([0, T)\) under Assumption 2.2. The sign of \(\sigma_S\) is determined by the sign of \(\sigma_v\) and the sign of the function \(b\). In the empirically relevant setting (2.5) in which the income volatility is countercyclical \(\sigma_v < 0\), and in which increasing income volatility reduces the expected aggregate income growth \(\kappa E < 0\), implying that (2.7) fails), the function \(b\) is positive. This implies that the volatility process (4.2) is negative. Therefore, the instantaneous risk premium for the annuity, i.e., \(\sigma_{St} \mu_S \sqrt{v_t}/S_t\), is also negative. Of course, this result reflects that the annuity hedges adverse consequences of increased income volatility in this setting (see (3.10)).

Finally, we mention that Theorem 4.2 does not make any uniqueness statement regarding the equilibrium. In other words, we are not claiming that \(S^{(0)}\) defined by (2.8) and \(S\) defined by (2.10) is the only equilibrium possible in our pure exchange economy.

### 4.1 Equilibrium impacts due to incompleteness

In this section we analytically show how the incomplete market equilibrium established in Theorem 4.2 can be used to simultaneously explain the risk-free interest rate puzzle and the equity premium puzzle. We compare the equilibrium characterized in Theorem 4.2 to the equilibrium in an otherwise identical complete market economy in which all risks are spanned.

In the complete market economy, there exists a representative agent, and the equilibrium is characterized by the representative agent’s first-order condition. The representative agent is modeled by the utility function

\[
U_{\text{rep}}(x; \gamma) := \sup_{\sum_{i=1}^{I} x_i = x} \sum_{i=1}^{I} \gamma_i U_i(x_i), \quad \gamma \in \mathbb{R}_+^I, \quad x \in \mathbb{R},
\]
where \( \gamma \) is a Negishi-weight vector. Since each investor is modeled by a negative exponential utility function, the representative agent’s utility function becomes (see, e.g., Section 5.26 in Huang and Litzenberger 1988)

\[
U_{\text{rep}}(x; \gamma) = -e^{-\frac{1}{\tau_x}x} \prod_{i=1}^{I} \left( \frac{\gamma_i}{\tau_i} \right)^{\frac{x_i}{\tau_i}}, \quad x \in \mathbb{R}.
\]

This expression shows that the weight \( \gamma \) does not matter for the representative agent’s preferences (Gorman aggregation). The first-order condition for the representative agent produces the proportionality requirement

\[
e^{-\frac{1}{\tau_x}E_t} \propto \xi_{\text{rep}}^t, \quad t \in [0, T],
\]

where the aggregate income process \( E_t \) is defined by (2.3), and \( \xi_{\text{rep}}^t \) is the unique state-price density in the representative agent setting. By computing the dynamics of both sides of (4.3) and matching the coefficients we find the spot interest rate based on the representative agent economy to be

\[
r_{\text{rep}}^t := \frac{1}{\tau_x} \mu_x + \frac{1}{\tau_x} \left( \kappa_x - \frac{1}{2\tau_x} \left( \sum_{i=1}^{I} \beta_i^2 + \sigma_x^2 \right) \right) v_t, \quad t \in [0, T].
\]

Since \( \tau_x := \sum_{i=1}^{I} \tau_i \), we have that \( \tau_x \geq \tau_i \) for all \( i \), which produces the key inequality

\[
\sum_{i=1}^{I} \frac{\beta_i^2}{\tau_i} \geq \frac{1}{\tau_x} \sum_{i=1}^{I} \beta_i^2.
\]

In an economy with unspanned idiosyncratic risks \( Z_i \), this inequality combined with Theorem 4.2 produces the interest rate reduction

\[
r_{t}^{\text{rep}} - r_t = \frac{1}{2\tau_x} \left( \sum_{i=1}^{I} \frac{\beta_i^2}{\tau_i} - \frac{1}{\tau_x} \sum_{i=1}^{I} \beta_i^2 \right) v_t \geq 0,
\]

which is an analogue of the result presented in Christensen et al. (2012) (compare to their equation (30)) although the interest rate reduction in our model is stochastic due to the common stochastic income volatility \( v \).

Similarly, from the dynamics of (4.3) we find that the instantaneous market price of risk process based on the representative agent is identical to the market price of risk process derived in Theorem 4.2, namely \( \mu_S \sqrt{v_t} = \frac{\sigma_x}{\tau_x} \sqrt{v_t} \). This is also an analogue of the result presented in Christensen et al. (2012) (compare to their equation (27)). This equilibrium
implication is not limited to our particular income model (2.1)-(2.2). Theorem 4.4 below shows that this result holds true in any model based on exponential investors and continuous income rates based on Itô-processes driven by Brownian motions.

We next establish that unspanned income risk with stochastic volatility can affect the risk premium measured over finite time-intervals, even though there is no impact on the instantaneous market price of risk process as demonstrated above. Our motivation is that empirical studies of asset pricing properties, such as the risk-free rate and the equity premium puzzles, necessarily must measure returns, spot interest rates, and risk premia over finite time-intervals, where the length $U > 0$ of the time-intervals is determined by the sampling frequency. In order to quantify risk premia measured over $[0, U]$, we introduce the minimal forward measure $Q^U$. Since the equilibrium spot interest rate derived in Theorem 4.2 is stochastic, the minimal martingale measure $Q^\text{min}$ and the minimal forward measure $Q^U$ differ. The probability measure $Q^U$ is defined by the Radon-Nikodym derivative on $\mathcal{F}_U$ as

$$
\frac{dQ^U}{dQ^\text{min}} := \exp\left(-\int_0^U r_u du\right) \frac{B(0, U)}{B(U, T)}, \quad U \in (0, T].
$$

Lemma 2.8 provides an explicit representation for equilibrium zero-coupon bond prices $B(t, U)$. Based on this lemma, Girsanov’s theorem ensures that

$$
dW^Q_t := dW^\text{min}_t - b(U - t)\sigma \sqrt{v_t} dt = dW_t + \left(\mu - b(U - t)\sigma \sqrt{v_t}\right) \sqrt{v_t} dt,
$$
is a $Q^U$-Brownian motion, where the deterministic function $b$ is defined by the Riccati equation (2.15). We can then make the following definition.

**Definition 4.3.** Under Assumption 2.2: The market price of risk process measured over $[0, U]$ is defined by $\mu^{[0,U]}(t) = \frac{\sigma}{\tau_\Sigma} - b(U - t)\sigma_v$, $t \in [0, U], \quad U \in (0, T]$.

Our reasoning behind Definition 4.3 is the following. Let $\sigma_X \in \mathcal{L}^2$ and consider a traded security with price process (use the wealth dynamics (3.1) with $c := 0$)

$$
dX_t := r_t X_t dt + \sigma_X dW^Q_t, \quad t \in [0, T], \quad X_0 \in \mathbb{R}.
$$

The main characterizing property of $Q^U$ is that all prices of traded securities denominated in terms of the price of the zero-coupon bond, $X_t/B(t, U)$, have no drift under $Q^U$. If $\sigma_X$ is
sufficiently integrable (such that \( X_t/B(t, U) \) is a martingale) we find
\[
\frac{X_0}{B(0, U)} = E^{Q^U}[X_U] = \frac{X_U}{B(U, U)},
\]
where the last equality follows from \( B(U, U) = 1 \). This identity implies that the expected return over the interval \([0, U]\) under the minimal forward measure \( Q^U \) is equal to the zero-coupon rate for this interval, i.e.,
\[
E^{Q^U}[X_U - X_0] = 1 - B(0, U)/B(0, U).
\]
In other words, the process \( \mu_S^{[0,U]}(t)\sqrt{v_t} \) is the drift-correction in the \( W \)-dynamics needed to produce the riskless return as the expected return under \( Q^U \) of \( X \) over the interval \([0, U]\).

It follows from (3.9), (3.10), and Itô’s lemma that
\[
dE_t[U'_i(\hat{c}_{iU} + Y_{iU})] = -E_t[U'_i(\hat{c}_{iU} + Y_{iU})] \left( (\mu_S - b(U - t)\sigma_v) dW_t + \frac{\beta Y_i}{\tau_i} dZ_{it} \right) \sqrt{v_t}.
\]
This implies that the cross-variation process for the conditional expected marginal utility of optimal consumption at time \( U \in [0, T] \) and the spanned income risk \( W \) is given by
\[
d\langle E_t[U'_i(\hat{c}_{iU} + Y_{iU})], W \rangle_t = -E_t[U'_i(\hat{c}_{iU} + Y_{iU})] \sqrt{v_t} (\mu_S - b(U - t)\sigma_v) dt.
\]
Consequently, the market price of risk process measured over \([0, U]\) is determined by the cross-variation process between the investors’ conditional expected marginal utility of optimal consumption at time \( U \) and spanned income risk. The key difference between the instantaneous market price of risk process and the market price of risk process measured over \([0, U]\) is that the former is determined by the instantaneous change in the marginal utility of concurrent optimal consumption and is, thus, not affected by stochastic income volatility, while the latter is determined by the instantaneous change in the conditional expected marginal utility of optimal consumption at the end of the time-interval \([0, U]\). Contrary to the instantaneous market price of risk process, the market price of risk process measured over \([0, U]\) depends both on the cyclicality of income volatility (\( \sigma_v \)) and the sensitivity of the investors’ conditional expected marginal utility of end-of-period optimal consumption to income volatility (the function \( b \) defined by (3.10)).

We focus on the market price of risk process measured over \([0, U]\), since the \( W \)-drift correction \( \mu_S^{[0,U]}(t)\sqrt{v_t} \) is universal across all traded securities. Alternatively, we could consider the risk premium measured over the interval \([0, U]\) for a security with the price process \( X_t \).
This premium is defined by the difference
\[
\mathbb{E} \left[ \frac{X_U - X_0}{X_0} \right] - \frac{1 - B(0, U)}{B(0, U)} = -\frac{1}{X_0} \text{Cov}_P \left( \frac{dQ^U}{dP}, X_U \right), \quad U \in (0, T],
\]
where the equality follows from (4.7). The Radon-Nikodym derivative \( \frac{dQ^U}{dP} \) is completely determined by the market price of risk process measured over \([0, U]\) via
\[
\frac{dQ^U}{dP} := M^Q_U, \quad dM^Q_U := -M^Q_U \mu_S^U(t) \sqrt{v_t} dW_t, \quad t \in [0, U], \quad M^Q_U(0) := 1.
\]
From this we see that the impact on the risk premium over the interval \([0, U]\) due to market incompleteness depends on the security. In other words, unlike the market price of risk process measured over \([0, U]\), the significance of the impact on the risk premium over \([0, U]\) depends on the security’s volatility process \(\sigma_X\). Moreover, contrary to when returns and risk premia are measured instantaneously, normalizing the risk premium over the interval \([0, U]\) by the standard deviation of the security’s return \(\frac{1}{X_0} \sqrt{\text{Var}_P[X_U]}\) to produce the “Sharpe ratio” measured over \([0, U]\) does not remove the dependence on the security’s volatility process \(\sigma_X\).

Similarly to the probability measures \(Q^{\text{min}}\) and \(Q^U\), we can introduce \(Q^{\text{rep}}\) and \(Q_{\text{rep}}^U\) corresponding to the representative agent based on the spot interest rate \(r_{\text{rep}}\) defined by (4.4). This interest rate \(r_{\text{rep}}\) produces the zero-coupon bond prices for \(0 \leq t \leq U \leq T\):
\[
B_{\text{rep}}(t, U) := \mathbb{E}_t^{Q^{\text{min}}} \left[ e^{-\int_t^U r_{\text{rep}} \, ds} \right] = \exp \left( b_{\text{rep}}(U - t) v_t - a_{\text{rep}}(U - t) \right),
\]
where \(a_{\text{rep}}\) and \(b_{\text{rep}}\) are defined by \(a_{\text{rep}}(0) = b_{\text{rep}}(0) = 0\) and for \(t \in [0, T]\)
\[
b_{\text{rep}}'(t) = b_{\text{rep}}(t)(\kappa_v - \mu_S \sigma_v) + \frac{1}{2} b_{\text{rep}}(t)^2 \sigma_v^2 + \frac{1}{\tau_\Sigma} \left( \frac{1}{2} \sum_{i=1}^I \beta_i^2 + \frac{\sigma_x^2}{2\tau_\Sigma} - \kappa \varepsilon \right),
\]
\[
a_{\text{rep}}'(s) = \frac{\mu_v}{\tau_\Sigma} - b_{\text{rep}}(s) \mu_v.
\]
By using the inequality (4.5), we see that Assumption 2.2 ensures that the Riccati equation describing \(b_{\text{rep}}\) has a unique non-exploding solution on \([0, \infty)\). Therefore, for \(U \in (0, T]\), the process
\[
dW_{\text{rep}}^Q := dW_{\text{rep}}^{Q^{\text{min}}} - b_{\text{rep}}(U - t) \sigma_v \sqrt{v_t} dt = dW_t + \left( \mu_S - b_{\text{rep}}(U - t) \sigma_v \right) \sqrt{v_t} dt,
\]
is a Brownian motion under the representative agent’s minimal forward measure \(Q_{\text{rep}}^U\). The market price of risk process measured over \([0, U]\) corresponding to the representative agent
is defined similarly to Definition 4.3 as $\mu_{0,U}^{S,\text{rep}}(t) \sqrt{v_t}$ where
\[
\mu_{0,U}^{S,\text{rep}}(t) := \mu_S - b_{\text{rep}}(U - t)\sigma_v = \frac{\sigma \xi}{\gamma_S} - b_{\text{rep}}(U - t)\sigma_v, \quad t \in [0,U].
\]

By comparing the coefficients for the two Riccati equations describing $b$ and $b_{\text{rep}}$ and using the inequality (4.5), we see that $b_{\text{rep}}(t) \leq b(t)$ for all $t \in [0,T)$. In other words, the sensitivity of the investors’ conditional expected marginal utility of end-of-period optimal consumption to income volatility is larger than that of the representative agent due to the latter evaluating all income risk with the aggregate risk tolerance in the economy, whereas investors in the incomplete market use their personal risk tolerances to evaluate unspanned income risk (see (4.5)). Consequently, provided that $\sigma_v \neq 0$ as in the second part of Assumption 2.1, we obtain an impact of unspanned income risk on the equilibrium market price of risk process measured over $[0,U]$. In particular, if the stochastic income volatility is countercyclical ($\sigma_v < 0$), the equilibrium market price of risk process measured over $[0,U]$ is higher than in an otherwise identical complete market setting.

Similarly to the derivation of $\sigma_S$ in (4.2) presented in the proof of Theorem 4.2, we can show that the annuity’s volatility coefficient in the representative agent setting is
\[
\sigma_{St}^{\text{rep}} := \sigma_v \sqrt{v_t} \int_t^T B_{\text{rep}}(t,U)b_{\text{rep}}(U - t)dU, \quad t \in [0,T].
\]

In the empirically relevant setting (2.5) in which $\sigma_v < 0$ and $\kappa_E < 0$, we have $0 \leq b_{\text{rep}}(t) \leq b(t)$ and, hence, also $a(t) \leq a_{\text{rep}}(t)$, for all $t \in [0,T)$. It therefore follows from (2.17) and (4.8) that $B_{\text{rep}}(t,U) \leq B(t,U)$ which produces the inequality
\[
\sigma_{St} \leq \sigma_{St}^{\text{rep}} < 0, \quad t \in [0,T).
\]

In other words, the drift-correction $\sigma_{St}^{\text{rep}}(0,U)^{S,\text{rep}}(t) \sqrt{v_t}$ in the $S$-dynamics needed to produce the riskless return as the expected return of the annuity over the interval $[0,U]$ in the complete market setting is larger (less negative) than in the incomplete market setting.

We summarize the equilibrium impacts due to market incompleteness as follows:

1. The equilibrium spot interest rate is impacted negatively.

2. The equilibrium instantaneous market price of risk process is unaffected. Theorem 4.4 below shows that this feature carries over to any model based on exponential utility investors and continuous income processes driven by Brownian motions.

3. The equilibrium market price of risk process measured over a finite time-interval $[0,U]$ is impacted, and the sign of the impact depends on the sign of $\sigma_v$. 

20
4. The equilibrium volatility coefficient of the risky security is impacted, and the sign of
the impact depends on the sign of $\sigma_v$.

In Section 5 we consider a setting in which investors only receive utility of terminal con-
sumption (but trading is still continuous). In this setting, we show that the equilibrium
instantaneous market price of risk process is equal to the market price of risk process mea-
sured over intervals and, thus, can be impacted due to market incompleteness. Therefore,
it is not the difference between the minimal martingale and forward measures per se, which
produces the impact of stochastic unspanned income volatility on the market price of risk
process measured over intervals. Instead, the key observation is that stochastic unspanned in-
come volatility has a non-trivial impact only when we consider risks over finite time-intervals,
such as returns measured over finite time-intervals or consumption only taking place at dis-
crete points in time. This ensures that the market price of risk process is not determined
by the cross-variation process between the investors’ marginal utility of concurrent optimal
consumption and spanned income risk, but rather determined by the cross-variation pro-
cess between the investors’ conditional expected marginal utility of end-of-period optimal
consumption and spanned income risk.

4.2 Numerical example

This section serves to illustrate that the impact on the equilibrium interest rate and the mar-
ket price of risk measured over $[0, U]$ stemming from investors receiving partially unspanned
income with stochastic volatility can be significant. The numerical values reported in this
section only serve to illustrate the potential impact.

The impact on the interest rate and on the market price of risk measured over $[0, U]$ is
determined by (4.5):

$$\Delta\beta := \frac{1}{\tau\Sigma} \sum_{i=1}^{I} \beta_i^2 \frac{\gamma_i}{\tau_i} - \frac{1}{\tau^2} \sum_{i=1}^{I} \beta_i^2 \gamma_i = \frac{1}{\tau^2} \sum_{i=1}^{I} \left( \frac{\Sigma}{\tau_i} - 1 \right) \beta_i^2 \gamma_i \geq 0.$$

We consider first a homogeneous investor setting in which all investors have the same risk
tolerance $\tau_i := \tau$ as well as the same unspanned income risk parameter $\beta_{Y_i} := \beta_Y$, $i = 1, ..., I$. In
this setting, $\tau\Sigma = I\tau$, and we find that

$$\Delta\beta = \frac{1}{\tau^2} \left( 1 - \frac{1}{I} \right) \beta_Y^2 \uparrow \frac{1}{\tau^2}\beta_Y^2,$$

as $I \to \infty$. For given parameter values, Table 1 shows the impact of unspanned income risk
with countercyclical stochastic volatility on the interest rate [column 2] and on the market
price of risk measured over $[0, U]$ with initial condition $v_0 := 1$ [column 3].

21
Table 1: Equilibrium effects of increasing the number of investors $I$ for the volatility parameters $v_0 := 1$, $\mu_v := 0.05$, $\kappa_v := -0.7$, and $\sigma_v := -0.3$. The investor parameters are $\tau_i := \frac{1}{2}$, $\beta_{Y_i} := 0.2$, $\kappa_{Y_i} := 0$, and $\sigma_{Y_i} := 0.3$ for all $i$. The horizon is $U := 1$.

| $I$ | $r_{01}^{\text{rep}} - r_0$ | $\mu_S^{[0,U]}(0) - \mu_S^{[0,U]}_{\text{rep}}(0)$ |
|-----|-----------------|-----------------|
| 2   | 0.0400          | 0.0094          |
| 5   | 0.0640          | 0.0151          |
| 10  | 0.0720          | 0.0169          |
| 100 | 0.0792          | 0.0186          |
| 1000| 0.0799          | 0.0188          |
| $\infty$ | 0.0800 | 0.0188 |

Secondly, we consider a heterogeneous investors setting in which we can split the population into two homogenous groups $A$ and $B$ with characteristics $(\tau_A, \beta_{Y_A})$ and $(\tau_B, \beta_{Y_B})$. The weight $w$ denotes group $A$’s proportion of the overall population. Table 2 reports the increase in the market price of risk measured over $[0, U]$, i.e., $\mu_S^{[0,U]}(0) - \mu_S^{[0,U]}_{\text{rep}}(0)$, for various combinations of risk tolerance parameters and population distributions in the limiting model ($I \to \infty$). We see that the impact on the market price of risk measured over $[0, U]$ is highest when the less risk tolerant investors face the largest unspanned income risk.

| $(\tau_A, \tau_B)$ | $w$ | $(\frac{1}{2}, \frac{1}{2})$ | $(\frac{1}{2}, \frac{1}{3})$ | $(\frac{1}{3}, \frac{1}{2})$ | $(\frac{1}{3}, \frac{1}{3})$ |
|-------------------|-----|-----------------|-----------------|-----------------|-----------------|
| 1.00              | 0.0047| 0.0047          | 0.0111          | 0.0111          |
| 0.75              | 0.0223| 0.0349          | 0.0332          | 0.0528          |
| 0.50              | 0.0400| 0.0720          | 0.0504          | 0.0946          |
| 0.25              | 0.0577| 0.1186          | 0.0642          | 0.1367          |
| 0.00              | 0.0755| 0.1789          | 0.0755          | 0.1789          |

Table 2: Increase in the market price of risk measured over $[0, U]$, i.e., $\mu_S^{[0,U]}(0) - \mu_S^{[0,U]}_{\text{rep}}(0)$, in the limiting case ($I \to \infty$) for various weights $w$ and various risk tolerance parameters $(\tau_A, \tau_B)$. The numbers are based on $\beta_{Y_A} := 0.1$, $\beta_{Y_B} := 0.4$, whereas the remaining exogenous parameters are as in Table 1.

4.3 No impact on the instantaneous market price of risk process

In this section we show that the instantaneous market price of risk process based on the representative agent is always identical to the equilibrium instantaneous market price of risk process in a setting based on exponential investors and continuous income processes governed
by Brownian motions. We consider the following model for \( t \in [0, T] \):

\begin{align*}
    dS_t^{(0)} &= r_t S_t^{(0)} dt, \quad S_0^{(0)} = 1, \\
    dS_t + \delta_t dt &= \left( r_t S_t + \lambda' \sigma'_S \right) dt + \sigma'_S dB_t, \quad S_0 \in \mathbb{R},
\end{align*}

(4.9)

for some \((\delta, r) \in L^1, (\sigma'_S, \lambda') \in L^2, \sigma'_S \neq 0\) and an arbitrary Brownian motion \( B \) with respect to \( F_t := \sigma(W_u, Z_{1u}, ..., Z_{Iu})_{u \in [0, t]} \). In the following theorem we refer to Definition 2.5 for the notion of a local state-price density \( \xi' \).

**Theorem 4.4.** For \( t \in [0, T] \) we consider the income dynamics

\( dY'_{it} = \mu'_{Yt} dt + \sigma'_{Yt} dB_t + \beta'_{Yt} dB^\perp_{it}, \quad Y'_{i0} \in \mathbb{R}. \)

Here \( B^\perp_1, ..., B^\perp_I \) denote possible dependent one-dimensional Brownian motions with respect \( F_t \) which are independent of \( B \), \( \mu'_{Yt} \in L^1 \), and \((\sigma'_{Yt}, \beta'_{Yt}) \in L^2 \). Assume that (4.9) constitutes an equilibrium in which each investor’s optimal consumption process \( \hat{c}_{it} \) satisfies the following first-order condition

\( (4.10) \quad U'_i(\hat{c}_{it} + Y'_{it}) \propto \hat{\xi}_{it}, \quad t \in [0, T], \quad i = 1, ..., I, \)

where \( \hat{\xi}_i \) is an investor-specific local state-price density. Then the equilibrium instantaneous market price of risk process \( \lambda' \) satisfies\( \frac{1}{\tau \Sigma^I_{i=1} \sigma'_{Yt}} \).

In the setting of this theorem, let \( \mathcal{E}'_t := \sum_{i=1}^I Y'_{it} \) denote the aggregate endowment. By computing the dynamics of the representative agent’s state-price density (proportional to \( e^{-\frac{1}{\tau \Sigma^I_{i=1} \sigma'_{Yt}}} \)), we see that the instantaneous market price of risk process based on the representative agent agrees with \( \lambda' \) stated in Theorem 4.4. In other words, Theorem 4.4 shows that any model based on exponential utility investors and continuous income processes governed by Brownian motions produces the same instantaneous market price of risk process as suggested by the standard representative agent model.

There is no loss of generality in assuming the above form for \( (Y'_{it})_{i=1}^I \) and \( S \). Indeed, by assuming that an equilibrium risky security price \( S \) exists, we can use Lévy’s characterization for Brownian motion as well as the martingale representation theorem for \( F_t := \sigma(W_u, Z_{1u}, ..., Z_{Iu})_{u \in [0, t]} \) to write the martingale component of \( dS \) as \( \sigma'_S dB_t \) for some Brownian motion \( B \) and some process \( \sigma'_S \in L^2 \). Subsequently, we can decompose the martingale part of \( Y'_{it} \) into its projection onto \( B \) and some residual orthogonal martingale component (possibly depending on \( i \)) which produces the above form for \( dY'_{it} \) for \( i = 1, ..., I \).

Let us finally discuss the first-order condition (4.10). In the case of utility functions defined on the positive semi-axis, Cvitanić, Schachermayer, and Wang (2001) show that the
introduction of unspanned endowments may require finitely additive measures in the dual space, in which case (4.10) makes no sense. However, Owen and Žitkovic (2009) show that for utility functions defined over \( \mathbb{R} \)—such as our setting—the dual optimizer is always a (countably additive) measure and (4.10) holds. Both papers Cvitanić et al. (2001) and Owen and Žitkovic (2009) consider the case of expected utility of terminal wealth only and instead of re-proving Owen and Žitkovic (2009) to fit our case of continuous consumption, we have opted for assuming (4.10) upfront.

5 Discrete consumption

Instead of running consumption, we consider in this section terminal consumption only. As we shall see in the next result, we need to allow \( \mu_S \) in (2.10) to be a continuous function on \([0, T] \). In this setting, the optimization problem (3.4) is replaced by

\[
\sup_{\theta \in \mathcal{A}^\text{term}} \mathbb{E} \left[ U_i(X_{iT}^\theta + Y_{iT}) \right] = \mathbb{E} \left[ U_i(X_{iT}^{\hat{\theta}} + Y_{iT}) \right].
\]

The wealth process \( X_i^\theta \) is defined by setting \( c := 0 \) in (3.1), i.e.,

\[
dX_i^\theta := r_t X_i^\theta dt + \theta_t \sigma_S(t) \sqrt{v_t} dt + dW_t, \quad X_i^0 = X_i^0 \in \mathbb{R}.
\]

We define the admissible strategies \( \mathcal{A}^\text{term} \) to be those measurable and adapted processes \( \theta \) for which \( X_i^\theta \) is well-defined and \( X^\theta \xi^\nu \) is a supermartingale for all state-price densities \( \xi^\nu \). The analogue of Theorem 4.2 is the following result.

**Theorem 5.1.** Under Assumptions 2.1 and 2.2, there exists an equilibrium for which Conjectures 2.3, 2.4, and 2.7 hold with \( r_t := 0 \) in (2.8) and \( \mu_S \) in (2.10) replaced by the deterministic function

\[
\mu_S(t) := \frac{\sigma^2}{\tau^2} - b(T - t) \sigma_v, \quad t \in [0, T],
\]

where \( b \) is defined by the Riccati equation (2.15).

In this setting of consumption at time \( T \) only, the interest rate cannot be determined in equilibrium, and we choose \( r_t := 0 \) for simplicity.\(^7\) Consequently, the minimum martingale measure \( Q^{\text{min}} \) and the minimum forward measure \( Q^U \) are identical, and the instantaneous market price of risk process is identical to the market price of risk process measured over finite

\(^7\)It is straightforward to extend this section’s setting to include consumption and income at various discrete-time points. This would produce an interest rate which is always negatively affected by market incompleteness.
time-intervals $[0, U], U \in [0, T]$. Therefore, in the following we will refer to these processes as the market price of risk process.

Contrary to Theorem 4.4, the setting of terminal consumption only produces an impact on the market price of risk process due to income incompleteness. In order to see this, we proceed as in Section 4.1 except that the first-order-condition (4.3) is only required to hold at $t = T$. To compute the market price of risk process corresponding to the representative agent, we need the dynamics of the martingale $\xi_t^{\text{rep}} := \mathbb{E}_t \left[ e^{-\frac{1}{2 \Sigma} \xi_T} \right] / \mathbb{E} \left[ e^{-\frac{1}{2 \Sigma} \xi_T} \right]$ for $t \in [0, T]$. Similarly to the proof of Lemma 2.8 we have

$$
(5.3) \quad d\xi_t^{\text{rep}} = -\xi_t^{\text{rep}} \sqrt{\nu_t} \left( \left\{ \frac{\sigma}{\tau_\Sigma} - b_{\text{rep}}(T - t) \sigma_v \right\} dW_t + \frac{1}{\tau_\Sigma} \sum_{i=1}^f \beta_i dZ_{it} \right),
$$

where $b_{\text{rep}}$ is defined in Section 4.1. By comparing (5.2) and the $dW$-coefficient in (5.3), we see from Section 4.1 that the incompleteness impact on the market price of risk process in the case of only terminal consumption at time $T$ is identical to the impact on the market price of risk process measured over the interval $[0, T]$ in the case of continuous consumption.

Because the equilibrium interest rate is deterministic, the investors’ value functions is of the exponential-affine form. Consequently, the investors’ optimal trading strategies $\hat{\theta}_t$ can be computed explicitly using HJB-techniques.

### 6 Concluding remarks

In this paper, we have derived closed-form solutions for the equilibrium interest rate, the instantaneous market price of risk, and the risky asset’s volatility processes in an incomplete continuous-time securities market with uncertainty generated by Brownian motions and a finite number of heterogeneous exponential utility investors. Motivated by recent empirical evidence of strongly countercyclical income volatility, we showed that countercyclical stochastic income volatility of the CIR-type generates a countercyclical equilibrium instantaneous market price of risk process and a procyclical equilibrium interest rate process. We further showed that a fixed-income annuity is a hedge against increases in income volatility. These results have clear empirical implications for the study of the relationship between the business cycle of the real economy and equilibrium financial market outcomes.

We showed that the presence of unspanned idiosyncratic income risk affects the equilibrium interest rate process negatively compared to an otherwise identical complete markets economy. We also showed, in a model-free manner, that unspanned income risk can never affect the equilibrium instantaneous market price of spanned risk process in a setting with exponential utility investors and continuous income processes governed by Brownian motions. In any empirical analysis, returns, interest rates and risk premia can only be measured over
finite time-intervals (given by the sampling frequency). We showed that the equilibrium market price of spanned risk process measured over intervals is affected positively by unspanned income risk if the stochastic income volatility of unspanned income is countercyclical. This result is due to the fact that a negative shock to income does not only reduce contemporaneous optimal consumption but it also increases the volatility of future optimal consumption, and the fact that the impact of shocks to income volatility on conditional expectations of future marginal utilities of optimal consumption is larger when these risks cannot be efficiently shared than when these risks can be efficiently shared among investors.

This result suggests a new avenue of empirical studies of interest rates and risk premia based on disaggregate income and consumption data as opposed to the common use of aggregate consumption per capita data in empirical studies of the risk-free interest rate and equity premium puzzles. In a numerical example, we showed that the impact of unspanned income risk with stochastic countercyclical volatility on equilibrium interest rates and on the market price of spanned risk process measured over intervals can be substantial.

A Proofs

Lemma A.1. Let $v$ be defined by (2.1). For any continuous function $f : [0, T] \to \mathbb{R}$ the following process is a martingale:

$$dM_t := -f(t)\sqrt{v_t}M_t dW_t, \quad t \in [0, T], \quad M_0 := 1.$$

Proof. Since $v$ defined by (2.1) is a Feller process, $v_t$ is non-centrally $\chi^2$-distributed with a continuous finite variance function $\mathbb{V}[v_t] := \mathbb{E}[v_t^2] - \mathbb{E}[v_t]^2$. Furthermore, the function

$$[0, T] \ni t \to \mathbb{E}\left[e^{\Delta v_t}\right] \in [0, \infty],$$

is a finitely-valued continuous function provided that $\Delta > 0$ satisfies $\max_{t \in [0, T]} \mathbb{V}[v_t] < \frac{1}{2\Delta}$. By Corollary 5.14 in Karatzas and Shreve (1988) it suffices to verify Novikov’s condition locally, i.e., find $N \in \mathbb{N}$ such that $\Delta := \frac{T}{N}$ and $t_n := n\Delta$ for $n = 0, ..., N$ satisfy

$$\mathbb{E}\left[e^{\frac{1}{2} \int_{t_n}^{t_{n+1}} f(u)^2 v_u du}\right] < \infty.$$

To verify this property we define $\Delta > 0$ such that

$$\max_{t \in [0, T]} \mathbb{V}[v_t] \max_{u \in [0, T]} f(u)^2 < \frac{1}{\Delta}.$$

Such a constant $\Delta$ exists because $f$ is continuous on $[0, T]$ and $\max_{t \in [0, T]} \mathbb{V}[v_t] = \mathbb{V}[v_T] < \infty.$
Jensen’s inequality and Tonelli’s theorem produce
\[
\mathbb{E} \left[ e^{\frac{1}{2} \int_{t_n}^{t_{n+1}} f(u) \, du} \right] \leq \frac{1}{\Delta} \mathbb{E} \left[ \int_{t_n}^{t_{n+1}} e^{\frac{1}{2} \Delta f(u) \, du} \right] = \frac{1}{\Delta} \int_{t_n}^{t_{n+1}} \mathbb{E} \left[ e^{\frac{1}{2} \Delta f(u) \, du} \right] \, du,
\]
which is finite because continuous functions are always integrable on finite intervals.

\[ \diamond \]

**Proof of Lemma 2.8.** The discriminant corresponding to the Riccati equation (2.15) is defined as
\[
q := \left( \kappa_v - \frac{\sigma_v^2 \sigma_v}{\tau_\Sigma} \right)^2 - 2 \sigma_v^2 \left( \sum_{i=1}^{I} \frac{\beta_i^2}{2 \tau_i} + \frac{\sigma_v^2}{2 \tau_\Sigma} - \kappa_v \right).
\]
Under Assumption 2.2, \( q \) is positive. The appendix in Kim and Omberg (1996) on normal Riccati equations ensures that (2.15) has a non-exploding unique normal solution \( b \) with \( b(s) \neq 0 \) for \( s \in (0, \infty) \).

In order to calculate the zero-coupon bond prices, we need the dynamics of the volatility process \( v \) defined by (2.1) under the minimal martingale measure \( Q^{\text{min}} \):

\[
dv_t = \left( \mu_v + (\kappa_v - \mu_s \sigma_v) v_t \right) dt + \sigma_v \sqrt{v_t} dW^{Q^{\text{min}}}.
\]
Therefore, \( v_t \) is also a Feller process under the minimal measure \( Q^{\text{min}} \). By Itô’s lemma we see that the process \( N_t := \exp \left( (b(U - t)v_t - a(U - t))/S_t(0) \right) \) is a local martingale under \( Q^{\text{min}} \) which has the dynamics

\[
dN_t = N_t b(U - t) \sigma_v \sqrt{v_t} dW^{Q^{\text{min}}}, \quad N_U = 1/S_U(0).
\]
Lemma A.1 ensures that \( N \) is a martingale on \([0, U]\). This martingale property and the terminal condition \( N_U = 1/S_U(0) \) show that \( B(t, U) = N_t \) for \( t \in [0, U] \) and the claim follows.

\[ \diamond \]

In the later proofs we will need the following result, where the main complication is that \( \nu \) can depend on both \( W \) and \( W^\perp \) and, hence, the random variable \( \int_0^T \nu_u dW_u^\perp \) is not independent of \( \mathcal{F}^{W}_t := \sigma(W_u)_{u \in [0,t]} \).

**Lemma A.2.** Under the assumptions of Theorem 3.1: Let \( \xi_t^\nu = M_t^\nu / S_t(0) \) be a state-price
density as in Definition 2.5. Then for $0 \leq s \leq t \leq T$ we have

$$
\mathbb{E}[M_t^\nu | \mathcal{F}_s \vee \mathcal{F}_t^W] = M_s^\nu \frac{M_{\min}^t}{M_{\min}^s}, \quad \mathbb{P}\text{-a.s.,}
$$

where $\xi_{\min}^t = M_{\min}^t / S_t^{(0)}$ is the minimal state-price density.

**Proof.** By the definition of a state-price density $\xi^\nu$, we can find a $W$-independent Brownian motion $W^\perp$ as well as $\nu \in L^2$ such that

$$
dM_u^\nu = -M_u^\nu \left( \mu_S \sqrt{\nu_u} dW_u + \nu_u dW_u^\perp \right), \quad M_0^\nu = 1,
$$

is a martingale. We define the corresponding $\mathbb{P}$-equivalent probability measure $Q^\nu$ by $dQ^\nu = dP$.

For a set $A_t \in \mathcal{F}_t^W$ the martingale representation theorem produces a $\mathcal{F}_t^W$-adapted process $f \in L^2$ such that

$$
g_v := \mathbb{E}^{Q^\min}_{\mathcal{F}_v} [1_{A_t} | \mathcal{F}_v] = \mathbb{E}^{Q^\min}_{\mathcal{F}_s} [1_{A_t}] + \int_0^v f_u dW_u^{Q^\min}, \quad v \in [0, t].
$$

Since $1_{A_t}$ is bounded, the process $g_v$ is a bounded $Q^{\min}$-martingale. Furthermore, since $W^{Q^\min}$ is also a $Q^\nu$-Brownian motion, $g_v$ is a local $Q^\nu$-martingale. However, by $g_v$’s boundedness property, $g_v$ is actually a $Q^\nu$-martingale.

To conclude the proof, we let $A_s \in \mathcal{F}_s$ be arbitrary. Then we have

$$
\mathbb{E}[M_t^\nu 1_{A_t} 1_{A_s}] = \mathbb{E}[1_{A_s} \mathbb{E}[M_t^\nu | \mathcal{F}_s]]
= \mathbb{E}[1_{A_s} \mathbb{E}^{Q^\nu}_{\mathcal{F}_s} [g_t | \mathcal{F}_s] M_s^\nu]
= \mathbb{E}[1_{A_s} g_s M_s^\nu]
= \mathbb{E} \left[ 1_{A_s} \frac{\mathbb{E}[M_t^\nu 1_{A_t} | \mathcal{F}_s]}{M_{\min}^s} M_s^\nu \right] = \mathbb{E} \left[ 1_{A_s} 1_{A_t} M_{\min}^\nu M_s^\nu \right].
$$

The first equality follows from iterated expectations and the $\mathcal{F}_s$-measurability of $A_s$. The second equality is Bayes’ rule for conditional expectations. The third equality is $g$’s martingale property under $Q^\nu$. The fourth equality is again Bayes’ rule, whereas the last equality is produced by the $\mathcal{F}_s$-measurability of $A_s$, $M_t^\nu$, $M_{\min}^s$ and iterated expectations. The arbitrariness of $A_t \in \mathcal{F}_t^W$, $A_s \in \mathcal{F}_s$ and the $\mathcal{F}_s \vee \mathcal{F}_t^W$-measurability of $M_{\min}^t M_s^\nu$ conclude the proof. 

$\diamondsuit$
Next, we introduce a technique which allows us to partially solve the individual investor’s problem (3.4). Because the interest rate $r_t$ is stochastic, the PDE produced by the HJB-approach does not have the usual exponential affine solution that Henderson (2005) and Christensen et al. (2012) rely on. Instead we follow Section 5 in Christensen et al. (2012) and convert the optimization problem into an equivalent problem with spanned income but heterogeneous beliefs. We define the $\mathbb{P}$-equivalent probability measures $\mathbb{P}_i, i = 1, \ldots, I,$ via the Radon-Nikodym derivative

$$d\mathbb{P}_i d\mathbb{P} := \pi_i dt > 0$$

on $\mathcal{F}_T,$ where

$$\pi_i := \exp \left( -\frac{\beta Y_i}{\tau_i} \int_0^t \sqrt{v_u} dZ_{iu} - \frac{1}{2} \frac{\beta^2 Y_i}{\tau_i^2} \int_0^t v_u du \right), \quad t \in [0, T].$$

Lemma A.1 allows us to Girsanov’s theorem to see that under each $\mathbb{P}_i, i = 1, \ldots, I,$ the processes $W$ and $Z_{it} + \frac{\beta Y_i}{\tau_i} \int_0^t \sqrt{v_u} du$ are independent Brownian motions. We will need the processes $\tilde{Y}_i$ defined by

$$d\tilde{Y}_it := \mu Y_i dt + \left( \kappa Y_i - \frac{1}{2} \frac{\beta^2 Y_i}{\tau_i} \right) v_t dt + \sigma Y_i \sqrt{v_t} dW_t,$$

for $t \in [0, T].$ By using the processes $(\pi_i, \tilde{Y}_i),$ we can re-write the objective in (3.4) as

$$\mathbb{E} \left[ \int_0^T U_i (c_u + \tilde{Y}_{iu}) du \right] = \mathbb{E} \left[ \int_0^T \pi_i u U_i (c_u + \tilde{Y}_{iu}) du \right]$$

$$= \mathbb{E}^{\mathbb{P}_i} \left[ \int_0^T U_i (c_u + \tilde{Y}_{iu}) du \right],$$

where the last equality follows from the martingale property of $\pi_i$ and iterated conditional expectations. Problem (3.4) can then be re-stated as

$$(A.3) \quad \sup_{(c, \theta) \in A} \mathbb{E}^{\mathbb{P}_i} \left[ \int_0^T U_i (c_u + \tilde{Y}_{iu}) du \right],$$

which can be seen as a complete market consumption-portfolio optimization problem with the spanned income rate process $\tilde{Y}_i$ and heterogeneous beliefs $\mathbb{P}_i.$

From the following proof we see that the individual investors’ optimal state-price densities are defined as $\hat{\xi}_it := \pi_i t \xi_{it}^{\min}.$ Indeed, the below equation (A.9) produces

$$U_{i} (\hat{c}_{iu} + Y_{iu}) = \pi_i U_i (\hat{c}_{iu} + \tilde{Y}_{iu}) = \alpha_i \pi_i \xi_{it}^{\min},$$

where the Lagrange multiplier $\alpha_i$ is defined by (3.5).

Proof of Theorem 3.1. Let $\hat{c}_i$ be defined as in the theorem’s statement. Our first task is to
show that the following process is well-defined (we note that \( \dot{X}_{it} = 0 \))

\[
\dot{X}_t := \mathbb{E}_{t}^{Q_{\min}} \left[ \int_{t}^{T} e^{-\int_{t}^{u} r_s ds} \hat{c}_{iu} du \right], \quad t \in [0, T].
\]

(A.4)

Under Assumption 2.2, we can find a constant \( p > 1 \) such that

\[
(k_v - \frac{\sigma \varepsilon}{\tau \Sigma} \sigma_v)^2 > 2p \sigma_v^2 \left( \sum_{i=1}^{I} \frac{\beta_{i}^2}{2 \tau_i} + \frac{\sigma^2}{2 \tau \Sigma} - \kappa \varepsilon \right).
\]

(A.5)

We then consider the coupled system of ODEs for \( s \in (0, \infty) \)

\[
\begin{aligned}
\tilde{b}'(s) &= \tilde{b}(s)(k_v - \frac{\sigma \varepsilon}{\tau \Sigma} \sigma_v) + \frac{1}{2} \tilde{b}(s)^2 \sigma_v^2 + \frac{p}{\tau \Sigma} \left( \sum_{i=1}^{I} \frac{\beta_{i}^2}{2 \tau_i} + \frac{\sigma^2}{2 \tau \Sigma} - \kappa \varepsilon \right), \quad \tilde{b}(0) = 0,
\end{aligned}
\]

\[
\begin{aligned}
\tilde{a}'(s) &= p \frac{\mu \varepsilon}{\tau \Sigma} - \tilde{b}(s) \mu_v, \quad \tilde{a}(0) = 0.
\end{aligned}
\]

The restriction (A.5) ensures a positive discriminant corresponding to \( \tilde{b} \)'s ODE. Therefore, the appendix in Kim and Omberg (1996) on normal solutions ensures that \( \tilde{b} \) and, hence, also \( \tilde{a} \), is a continuous function on \([0, \infty)\). Arguing as in the proof of Lemma 2.8, we find

\[
\mathbb{E}_{t}^{Q_{\min}} \left[ e^{-\int_{t}^{u} r_s ds} \right] = e^{\tilde{b}(u-t)\mu_v - \tilde{a}(u-t)}, \quad 0 \leq t \leq u \leq T.
\]

To verify that (A.4) indeed is well-defined, we can use Tonelli’s theorem to write

\[
\mathbb{E}_{t}^{Q_{\min}} \left[ \int_{t}^{T} e^{-\int_{t}^{u} r_s ds} \hat{c}_{iu} du \right] = \int_{t}^{T} \mathbb{E}_{u}^{Q_{\min}} \left[ e^{-\int_{u}^{u} r_s ds} \hat{c}_{iu} \right] du.
\]

By Hölder’s inequality it therefore suffices to show that the expectations

\[
\mathbb{E}_{t}^{Q_{\min}} \left[ e^{-p \int_{u}^{T} r_s ds} \right] \quad \text{and} \quad \mathbb{E}_{t}^{Q_{\min}} \left[ \hat{c}_{iu} \right]^{\frac{p}{p-1}},
\]

(A.6)

are bounded uniformly in \( u \in [t, T] \). We start with the first term. If (2.7) holds, \( r_t \) defined by (2.9) is bounded from below, and the claim follows since \( p > 1 \). On the other hand, if
(2.7) fails, we get the inequality for \( u \in [t, T] \)

\[
\hat{e}^\tilde{b}(u) v_0 - \hat{a}(u) = E^{Q_{\min}} \left[ e^{-p \int_t^u r_s ds} \right]
\]

\[
= E^{Q_{\min}} \left[ e^{-p \int_t^u \left( \sum_{t=1}^T \beta_i^2 \frac{\sigma_i^2}{\tau_i^2} - \kappa \epsilon \right) ds} \right]
\]

\[
\geq e^{-p \int_t^u \left( \sum_{t=1}^T \beta_i^2 \frac{\sigma_i^2}{\tau_i^2} - \kappa \epsilon \right) ds}
\]

\[
= E^{Q_{\min}} \left[ e^{-p \int_t^u r_s ds} \right].
\]

Since both \( \hat{a} \) and \( \hat{b} \) are continuous functions on \([0, T]\) and, hence, bounded, we obtain

\[
E^{Q_{\min}} \left[ e^{-p \int_t^u r_s ds} \right] \leq e^{p \int_t^u \max_{s \in [0, T]} \hat{b}(s) v_0 - \hat{a}(s)} < \infty, \quad u \in [t, T].
\]

We will next provide a uniform bound (in \( u \in [0, T] \)) of the second term in (A.6). In the following argument \( C_1, C_2, \ldots \) denote various irrelevant positive constants. Since \( v_1 > 0 \), we have the following chain of inequalities

\[
E^{Q_{\min}} \left[ |\hat{c}_1u|^{\frac{p}{p-1}} \right]
\]

\[
\leq C_1 + C_2 E^{Q_{\min}} \left[ \left( \int_0^u v_s ds \right)^{\frac{p}{p-1}} \right] + C_3 E^{Q_{\min}} \left[ \left( \int_0^u \sqrt{v_s dW_s^{Q_{\min}}} \right)^{\frac{p}{p-1}} \right]
\]

\[
\leq C_1 + C_2 u^{\frac{1}{p-1}} \int_0^u E^{Q_{\min}} \left[ v_s^{\frac{p}{p-1}} \right] ds + C_4 E^{Q_{\min}} \left[ \left( \int_0^u v_s ds \right)^{\frac{1}{2(p-1)}} \right]
\]

\[
\leq C_1 + C_2 T^{\frac{1}{p-1}} \int_0^T E^{Q_{\min}} \left[ v_s^{\frac{p}{p-1}} \right] ds + C_4 \left( T^{\frac{1}{p-1}} \int_0^T E^{Q_{\min}} \left[ v_s^{\frac{p}{p-1}} \right] ds \right)^{\frac{1}{2}}
\]

The first inequality follows from the definition of \( \hat{c}_1 \). The second inequality uses Jensen’s inequality (recall \( p > 1 \)) and Tonelli’s theorem on the \( ds \)-integral, whereas the estimate of the \( dW^{Q_{\min}} \)-integral follows from the Burkholder-Davis-Gundy inequality (see, e.g., Theorem 3.28 on p.166 in Karatzas and Shreve (1988)). The third inequality first uses \( u \leq T \) and Jensen’s inequality on the second \( ds \)-integral. The last estimate is similar. The dynamics (A.1) for \( v \) ensure that \( v_s \) is non-central \( \chi^2 \)-distributed under \( Q_{\min} \) and, hence, the \( ds \)-integrals are finite. All in all, we have shown that when Assumption 2.2 holds, the process (A.4) is well-defined.
and finite.

We next establish the existence of \( \alpha_i > 0 \) satisfying (3.5). The requirement (3.5) becomes

\[
X_{i0} = S_0 \hat{c}_{i0} + \mathbb{E}^{\mathbb{Q}_{\min}} \left[ \int_0^T e^{-\int_0^u r_s ds} \left( \int_0^u \hat{c}_{is} ds \right) du \right].
\]

Inserting the definition of \( \hat{c}_{i0} \) from (3.6) into (A.7) produces an equation uniquely characterizing the Lagrange multiplier \( \alpha_i \in (0, \infty) \).

We now turn to the admissibility requirement. Using the relation between \( \mathbb{Q}_{\min} \) and \( \xi_{\min} \) and \( \hat{X}_{it} \)'s definition produce for \( t \in [0, T] \)

\[
\frac{\hat{X}_{it}}{S_t^{(0)}} + \int_0^t \frac{\hat{c}_{iu}}{S_u^{(0)}} du = \mathbb{E}^{\mathbb{Q}_{\min}} \left[ \int_0^T \frac{\hat{c}_{iu}}{S_u^{(0)}} du \right] = \mathbb{E}^{\mathbb{Q}_{\min}} \left[ \int_0^T \frac{\hat{c}_{iu}}{S_u^{(0)}} du \right] + \int_0^t f_{iudW}^{\mathbb{Q}_{\min}} u = X_{i0} + \int_0^t \frac{1}{S_u^{(0)}} \hat{\theta}_{iu} \sigma_{Su} dW_u^{\mathbb{Q}_{\min}}.
\]

The integrand \( f_i \in L^2 \) appearing in the second equality comes from the martingale representation theorem for \( \mathcal{F}_W^t := \sigma(W_u)_{u \in [0,t]} \) after noticing that all involved quantities are \( \mathcal{F}_W^t \)-adapted. The last equality follows from (3.5) and by defining

\[
\hat{\theta}_{iu} := \frac{S_u^{(0)} f_{iu}}{\sigma_{Su}}, \quad u \in [0, T),
\]

which is possible since we are assuming that Conjecture 2.4 holds. All in all, this shows that \( \hat{X}_{it} \) has the form (3.1) and that

\[
\xi_{\min} \hat{X}_{it} + \int_0^t \xi_{iu} \hat{c}_{iu} du, \quad t \in [0, T].
\]

is a \( \mathbb{P} \)-martingale. We will next show the supermartingale requirement (3.3) by proving the stronger martingale property. By the definition of a state-price density \( \xi_{i}' \), we can find a
The first equality follows from the established martingale property, whereas the last equality gives us

\[ \mathbb{E} \left[ \xi_t^\nu \hat{X}_{it} + \int_0^t \xi_u^\nu \hat{c}_{iu} \, du \bigg| \mathcal{F}_s \right] = \mathbb{E} \left[ M_t^\nu \left( \hat{X}_{it}/S_t^0 + \int_0^t \hat{c}_{iu}/S_u^0 \, du \right) \bigg| \mathcal{F}_s \right] 
\]

\[ = \mathbb{E} \left[ \mathbb{E} \left[ M_t^\nu \big| \mathcal{F}_s \right] \left( \hat{X}_{it}/S_t^0 + \int_0^t \hat{c}_{iu}/S_u^0 \, du \right) \bigg| \mathcal{F}_s \right] 
\]

\[ = \frac{M_t^\nu}{M_s^{\min}} \mathbb{E} \left[ M_t^{\min} \left( \hat{X}_{it}/S_t^0 + \int_0^t \hat{c}_{iu}/S_u^0 \, du \right) \bigg| \mathcal{F}_s \right] 
\]

\[ = \xi_t^\nu \hat{X}_{is} + \int_0^s \xi_u^\nu \hat{c}_{iu} \, du. \]

The first equality follows from \( \xi_t^\nu = M_t^\nu/S_t^0 \) and the martingale property of \( M_t^\nu \). The second equality follows from iterated expectations and the \( \mathcal{F}_t^W \)-measurability of \( \hat{X}_{it}, (\hat{c}_{iu})_{u \in [0,t]} \) and \( (S_u^0)_{u \in [0,t]} \). The third equality uses Lemma A.2, whereas the last equality is \( \xi_t^{\min} = M_t^{\min}/S_t^0 \) combined with the already established martingale property of (A.8). This shows \( (\hat{\theta}_i, \hat{c}_i) \in \mathcal{A} \).

Finally, we will verify the optimality of \( (\hat{\theta}_i, \hat{c}_i) \) for problem (A.3) and, hence, also for problem (3.4). For the case of positive wealth processes the standard argument can be found in Section 3.6 in Karatzas and Shreve (1998). In the following, \( V \) denotes the convex conjugate of \( U \) (see Section 3.4 in Karatzas and Shreve 1998). By Fenchel’s inequality, we have

\[ U(c_u + \hat{Y}_{iu}) \leq V(\alpha_i \xi_u^{\min}) + \alpha_i \xi_u^{\min} \left( c_u + \hat{Y}_{iu} \right), \quad u \in [0,T]. \]

Integrating with respect to \( du \) and adding the positive random variable \( \alpha_i \xi_T^{\min} X_{i,T}^{c,\theta} \), see (3.2), give us

\[ \int_0^T U(c_u + \hat{Y}_{iu}) \, du \leq \int_0^T \left\{ V(\alpha_i \xi_u^{\min}) + \alpha_i \xi_u^{\min} \left( c_u + \hat{Y}_{iu} \right) \right\} \, du + \alpha_i \xi_T^{\min} X_{i,T}^{c,\theta}. \]

Since \( \pi_u \xi_T^{\min} \) is a state-price density and the subjective probability measure \( \mathbb{P}_i \) is defined by

\[ \frac{d\mathbb{P}_i}{d\mathbb{P}} := \pi_{iT}, \]

we can use the supermartingale property (3.3) to obtain the inequality

\[ \mathbb{E}^{\mathbb{P}_i} \left[ \int_0^T U(c_u + \hat{Y}_{iu}) \, du \right] \leq \mathbb{E}^{\mathbb{P}_i} \left[ \int_0^T \left\{ V(\alpha_i \xi_u^{\min}) + \alpha_i \xi_u^{\min} \hat{Y}_{iu} \right\} \, du \right] + \alpha_i X_{i0} 
\]

\[ = \mathbb{E}^{\mathbb{P}_i} \left[ \int_0^T \left\{ V(\alpha_i \xi_u^{\min}) + \alpha_i \xi_u^{\min} \hat{Y}_{iu} \right\} \, du \right] + \alpha_i \mathbb{E}^{\mathbb{P}_i} \left[ \int_0^T \xi_u^{\min} \hat{c}_{iu} \, du \right] 
\]

\[ = \mathbb{E}^{\mathbb{P}_i} \left[ \int_0^T U(\hat{c}_{iu} + \hat{Y}_{iu}) \, du \right]. \]

The first equality follows from the established martingale property, whereas the last equality
follows from the first-order condition
\begin{equation}
U_i'(c_{iu} + \hat{Y}_{iu}) = \alpha_i \xi_u^{\text{min}}, \quad u \in [0, T],
\end{equation}

and the relation between $U$ and $V$ stated in Lemma 4.3(i) in Karatzas and Shreve (1998). In order to verify that (A.9) holds, we use (3.6) to see that (A.9) holds for $u = 0$. Furthermore, by using (3.7) we see that the dynamics of both sides of (A.9) are identical and, hence, (A.9) holds for all $u \in [0, T]$.

\begin{proof}[Proof of Theorem 4.2] We define $S(0)$ by (2.8) and $S$ by (2.14). The already proven Lemma 2.8 produces the zero-coupon bond dynamics
\begin{equation}
 dB(t, U) = B(t, U) \left( r_t dt + b(U - t) \sigma_v \sqrt{v_t} dW^\text{Qmin}_t \right).
\end{equation}

We can then use Tonelli’s theorem to re-write (2.14) as follows
\begin{equation}
 S_t = \int_t^T E^\text{Qmin}_t \left[ e^{-\int_t^U r_s ds} \right] dU = \int_t^T B(t, U) dU, \quad t \in [0, T].
\end{equation}

Leibnitz’ rule for stochastic integrals produces the dynamics
\begin{equation}
 dS_t = -B(t, t) dt + r_t \int_t^T B(t, U) dU dt + \sigma_v \sqrt{v_t} \int_t^T B(t, U) b(U - t) dU dW^\text{Qmin}_t
\end{equation}
\begin{equation}
 = -dt + r_t S_t dt + \sigma_v \sqrt{v_t} \int_t^T B(t, U) b(U - t) dU dW^\text{Qmin}_t.
\end{equation}

Therefore, Conjecture 2.4 holds with the volatility coefficient (4.2).

We now establish clearing in the goods market. By summing up the expressions for $d\hat{c}_{it}$, we find $d \sum_{i=1}^I \hat{c}_{it} = 0$, see (2.9)-(2.11) for the definitions of $r_t$ and $\mu_S$. Since $\sum_{i=1}^I X_{i0} = 0$, we see from (A.7) that $\sum_{i=1}^I \hat{c}_{i0} = 0$ and, hence, the goods market clears.

To see that the risky security market also clears, we sum over $i = 1, ..., I$ in (3.8) to see
\begin{equation}
 \sum_{i=1}^I X_t^{\hat{\theta}_{i}, \hat{c}_i} = 0. \quad \text{By dividing both sides in this relation by } S_t(0), \text{ we find the } Q^{\text{min}}\text{-dynamics}
\end{equation}
\begin{equation}
 0 = d \sum_{i=1}^I \frac{X_t^{\hat{\theta}_{i}, \hat{c}_i}}{S_t(0)} = \sum_{i=1}^I \frac{1}{S_t(0)} \left( \hat{\theta}_{it} \sigma_S dW^\text{Qmin}_t - \hat{c}_{it} dt \right)
\end{equation}
\begin{equation}
 = \frac{1}{S_t(0)} \left( \sum_{i=1}^I \hat{\theta}_{it} \right) \sigma_S dW^\text{Qmin}_t.
\end{equation}

The second equality follows from the definition of $W^\text{Qmin}$, and the last equality is due to
clearing in the goods market. By matching the $dW^{Q_{\min}}$-coefficients and using $\frac{1}{\sqrt{\sigma_{S}}}\sigma_{S} \neq 0$, we obtain the clearing condition.

Finally, to show clearing in the money market, we use

$$0 = \sum_{i=1}^{I} X_{t}^{i, \hat{\theta}_{i}} = \sum_{i=1}^{I} \left( \hat{\theta}_{it} S_{t} + \hat{\theta}_{it}^{(0)} S_{t}^{(0)} \right) = S_{t}^{(0)} \sum_{i=1}^{I} \hat{\theta}_{it}^{(0)}.$$

The first equality was established above, whereas the last equality follows from the already established clearing in the risky security market. Since $S_{t}^{(0)} > 0$, the clearing condition in the money market follows.

\textbf{Proof of Theorem 4.4.} In the first-order condition for the individual investor (4.10), the investor-specific state-price density $\hat{\xi}_{i}$ has the dynamics

$$d \hat{\xi}_{it} = -\hat{\xi}_{it} \left( r_{t} dt + \lambda'_{t} dB_{t} + dM_{\perp}^{i} \right), \quad i = 1, \ldots, I,$$

for some local martingale $M_{\perp}^{i}$ orthogonal to $B$, i.e., $\langle B, M_{\perp}^{i} \rangle_{t} = 0$ for all $t \in [0, T]$. Computing the dynamics of both sides of (4.10) gives us the relation

$$d \hat{c}_{it} = \ldots dt + \left( \tau_{i} \lambda'_{t} - \sigma_{Y_{i, t}}' \right) dB_{t} + \ldots dB_{\perp}^{i} + \ldots dM_{\perp}^{i}.$$

By summing over investors and matching the $dB$-integrals, we see that the equilibrium instantaneous market price of risk process satisfies

$$\lambda'_{t} = \frac{1}{\tau_{\Sigma}} \sum_{i=1}^{I} \sigma_{Y_{i, t}}, \quad t \in [0, T].$$

\textbf{Proof of Theorem 5.1.} In this setting the minimal martingale measure $Q_{\min}$ on $\mathcal{F}_{T}$ is defined by $\frac{dQ_{\min}}{dP} := \xi_{T}^{\min}$, where $\xi_{t}^{\min} > 0$ is the martingale

$$d \xi_{t}^{\min} := -\xi_{t}^{\min} \mu_{S}(t) \sqrt{v_{t}} dW_{t}, \quad t \in [0, T], \quad \xi_{0}^{\min} := 1.$$

Let $\bar{Y}_{t}$ be defined by (A.2). Similarly to the proof of Theorem 3.1, we can use the martingale representation theorem to produce $\hat{\theta}_{i} \in \mathcal{A}^{\text{term}}$ such that the corresponding wealth process $(X_{t}^{\hat{\theta}_{i}})_{t \in [0, T]}$ satisfies the first-order condition

$$U_{i}'(X_{T}^{\hat{\theta}_{i}} + \bar{Y}_{iT}) = \alpha_{i} \xi_{T}^{\min}.$$
Here $\alpha_i$ is the Lagrange multiplier corresponding to the budget constraint, i.e., $\alpha_i > 0$ satisfies the analogue of (A.7):

\[ (A.10) \quad X_{i0} = E^{Q_{\min}} \left[ X_{iT}^i \right] = E^{Q_{\min}} \left[ -\tau_i \log(\tau_i \alpha_i \xi_T^{\min}) - \tilde{Y}_iT \right]. \]

In order to see that all markets clear we introduce the martingale for $t \in [0, T]$

\[ N_t := \tau \Sigma \sigma v \int_0^t b(T - u) \sqrt{\nu_t} dW_u - E_t \left[ \int_0^T \left( \kappa_{\nu} - \sum_{i=1}^I \frac{\beta_{Y_i}^2}{2\tau_i} - \frac{\tau \Sigma \mu_S(u)^2}{2} \right) v_2 du \right]. \]

By using Fubini’s theorem for conditional expectations we find the dynamics

\[ dN_t = \sigma v \left( \tau \Sigma b(T - t) - \int_t^T \left( \kappa_{\nu} - \sum_{i=1}^I \frac{\beta_{Y_i}^2}{2\tau_i} - \frac{\tau \Sigma \mu_S(u)^2}{2} \right) e^{-\kappa_{\nu}(t - u)} du \right) \sqrt{\nu_t} dW_t \]

\[ = \sigma v \tau \left( b(T - t) + \int_t^T \left( b'(T - u) - b(T - u) \kappa_{\nu} \right) e^{-\kappa_{\nu}(t - u)} du \right) \sqrt{\nu_t} dW_t, \]

where the second equality follows from (2.15). However, by using integration by parts together with $b(0) = 0$ we obtain $dN_t = 0$ for $t \in [0, T]$.

We can then finish the proof and as in the proof of Theorem 4.2 it suffices to show

\[ \sum_{i=1}^I X_{iT}^i = 0, \quad P\text{-a.s.}, \]

to ensure clearing in all markets. By the definition of $\xi^{\min}_t$ the requirement $\sum_{i=1}^I X_{iT}^i = 0, \quad P\text{-a.s.}$, is equivalent to

\[ \sum_{i=1}^I \tilde{Y}_iT = -\sum_{i=1}^I \tau_i \log(\alpha_i \tau_i \xi_T^{\min}) \]

\[ = -\sum_{i=1}^I \tau_i \log(\alpha_i \tau_i) + \tau \left( \int_0^T \mu_S(t) \sqrt{\nu_t} dW_t + \frac{1}{2} \int_0^T \mu_S(t)^2 \nu_t dt \right). \]

By the definitions of $\mu_S$ and $\tilde{Y}_t$ this requirement is equivalent to

\[ \sum_{i=1}^I \left( \tilde{Y}_i0 + \mu Y_i T + \int_0^T (\kappa Y_i - \frac{\beta_{Y_i}^2}{2\tau_i}) v_t dt \right) \]

\[ = -\sum_{i=1}^I \tau_i \log(\alpha_i \tau_i) + \tau \left( \sigma v \int_0^T b(T - t) \sqrt{\nu_t} dW_t + \frac{1}{2} \int_0^T \mu_S(t)^2 \nu_t dt \right). \]

By the definition of the martingale $N$, this requirement can be re-written as

\[ (A.11) \quad \sum_{i=1}^I \tilde{Y}_i0 + \mu \epsilon T + \sum_{i=1}^I \tau_i \log(\alpha_i \tau_i) = N_T = N_0. \]
Since $\sum_{i=1}^I X_{i0} = 0$, the requirement (A.11) holds by (A.10).

References

Anderson, R. M. and R. C. Raimondo (2008). Equilibrium in continuous-time financial markets: endogenously dynamically complete markets. *Econometrica* 76, 841–907.

Basak, S. and D. Cuoco (1988). An equilibrium model with restricted stock market participation. *Review of Financial Studies* 11, 309–341.

Bloom, N. (2009). The impact of uncertainty shocks. *Econometrica* 77, 623–685.

Bloom, N., M. Floetotto, N. Jaimovich, I. Saporta, and S. Terry (2012). Really uncertain business cycles. Working Paper, Stanford University, http://www.stanford.edu/~nbloom/RUBC_DRAFT.pdf.

Breeden, D. T. (1986). Consumption, production, inflation and interest rates. *Journal of Financial Economics* 16, 3–39.

Chacko, G. and L. M. Viceira (2005). Dynamic consumption and portfolio choice with stochastic volatility in incomplete markets. *Review of Financial Studies* 18, 1369–1402.

Christensen, P. O., K. Larsen, and C. Munk (2012). Equilibrium in securities markets with heterogeneous investors and unspanned income risk. *Journal of Economic Theory*. 147, 1035–1063.

Cochrane, J. H. (2005). *Asset Pricing*. Princeton University Press.

Constantinides, G. M. and D. Duffie (1996). Asset pricing with heterogeneous consumers. *Journal of Political Economy* 104, 219–240.

Cox, J. C. and C. F. Huang (1989). Optimal consumption and portfolio policies when asset prices follow a diffusion process. *Journal of Economic Theory* 49, 33–83.

Cuoco, D. and H. He (1994). Dynamic equilibrium in infinite-dimensional economies with incomplete financial markets. Working Paper.

Cvitanić, J., E. Jouini, S. Malamud, and C. Napp (2012). Financial markets equilibrium with heterogeneous agents. *Review of Finance* 16, 285–321.

Cvitanić, J., W. Schachermayer, and H. Wang (2001). Utility maximization in incomplete markets with random endowment. *Finance and Stochastics* 5, 259–272.

Duffie, D. (2001). *Dynamic Asset Pricing Theory* (3 ed.). Princeton University Press.

Duffie, D. and C.-F. Huang (1985). Implementing Arrow-Debreu equilibria by continuous trading of few long-lived securities. *Econometrica* 53, 1337–1356.
Föllmer, H. and M. Schweizer (2010). The minimal martingale measure. *Encyclopedia of Quantitative Finance*, 1200–1204.

Henderson, V. (2005). Explicit solutions to an optimal portfolio choice problem with stochastic income. *Journal of Economic Dynamics and Control* 29, 1237–1266.

Huang, C. and R. Litzenberger (1988). *Foundations for Financial Economics*. Prentice-Hall.

Hugonnier, J. (2012). Rational asset pricing bubbles and portfolio constraints. *Journal of Economic Theory*, to appear.

Hugonnier, J., S. Malamud, and E. Trubowitz (2012). Endogenous completeness of diffusion driven equilibrium markets. *Econometrica* 80, 1249–1270.

Judd, K. L. (1985). The law of large numbers with a continuum of iid random variables. *Journal of Economic Theory* 35, 19–25.

Karatzas, I., J. P. Lehoczky, and S. E. Shreve (1987). Optimal portfolio and consumption decisions for a “small investor” on a finite horizon. *SIAM Journal on Control and Optimization* 25, 1557–1586.

Karatzas, I. and S. E. Shreve (1988). *Brownian Motion and Stochastic Calculus* (2 ed.). Springer.

Karatzas, I. and S. E. Shreve (1998). *Methods of Mathematical Finance*. Springer.

Kim, T. S. and E. Omberg (1996). Dynamic non-myopic portfolio behavior. *Review of Financial Studies* 9, 141–161.

Kraft, H. (2005). Optimal portfolios and Heston’s stochastic volatility model. *Quantitative Finance* 5, 303–313.

Krueger, D. and H. Lustig (2010). When is market incompleteness irrelevant for the price of aggregate risk (and when is it not)? *Journal of Economic Theory* 145, 1–41.

Owen, M. P. and G. Žitkovic (2009). Optimal investment with an unbounded random endowment and utility-based pricing. *Mathematical Finance* 19, 129–159.

Schroder, M. and C. Skiadas (2005). Lifetime consumption-portfolio choice under trading constraints, recursive preferences and nontradeable income. *Stochastic Processes and their Applications* 115, 1–30.

Skiadas, C. (2009). *Asset Pricing Theory*. Princeton University Press.

Storesletten, K., C. Telmer, and A. Yaron (2007). Asset pricing with idiosyncratic risk and overlapping generations. *Review of Economic Dynamics* 10, 519–548.

Uhlig, H. (1996). A law of large numbers for large economies. *Economic Theory* 8, 41–50.
Wang, N. (2003). Caballero meets Bewley: The permanent-income hypothesis in general equilibrium. *American Economic Review* 51, 927–936.

Wang, N. (2004). Precautionary saving and partially observed income. *Journal of Monetary Economics* 51, 1645–1681.

Wang, N. (2006). Generalizing the permanent-income hypothesis: Revisiting Friedman’s conjecture on consumption. *Journal of Monetary Economics* 53, 737–752.

Žitkovic, G. (2006). Financial equilibria in the semimartingale setting: Complete markets and markets with draw-down constraints. *Finance and Stochastics* 10, 99–119.

Žitkovic, G. (2012). An example of a stochastic equilibrium with incomplete markets. *Finance and Stochastics* 16, 177–206.