Ill-posedness for the higher dimensional Camassa-Holm equations in Besov spaces

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Abstract: In the paper, by constructing a initial data $u_0 \in B_\infty^{\sigma}$ with $\sigma - 2 > \max\{1 + \frac{1}{p}, \frac{3}{2}\}$, we prove that the corresponding solution to the higher dimensional Camassa-Holm equations starting from $u_0$ is discontinuous at $t = 0$ in the norm of $B_\infty^{\sigma}$, which implies that the ill-posedness for the higher dimensional Camassa-Holm equations in $B_\infty^{\sigma}$.

Keywords: Higher dimensional Camassa-Holm equations; Ill-posedness; Besov spaces.

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1 Introduction

In this paper, we consider the following initial value problem for the higher dimensional Camassa-Holm equations:

$$
\begin{align*}
\partial_t m + u \cdot \nabla m + \nabla u^T \cdot m + (\text{div} u) m &= 0, & (t, x) &\in \mathbb{R}^+ \times \mathbb{R}^d, \\
m &= (1 - \Delta) u, & (t, x) &\in \mathbb{R}^+ \times \mathbb{R}^d, \\
u(0, x) &= u_0, & x &\in \mathbb{R}^d.
\end{align*}
$$

(1.1)

According to [44], we can transform Eq. (1.1) into the following form of transport equations:

$$
\partial_t u + u \cdot \nabla u = Q(u, u) + R(u, u),
$$

(1.2)

where

$$
Q(u, u) = -(I - \Delta)^{-1} \text{div} \left( \nabla u \nabla u + \nabla u \nabla u^T - \nabla u^T \nabla u - \nabla u (\text{div} u) + \frac{1}{2} 1 |\nabla u|^2 \right),
$$

$$
R(u, u) = -(I - \Delta)^{-1} \left( u (\text{div} u) + u \cdot (\nabla u)^T \right).
$$

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Eq. (1.1) was investigated as Euler-Poincaré equations describes geodesic motion on the diffeomorphism group with respect to the kinetic energy norm in [29], which can also be viewed as higher dimensional generalization of the following classical one dimensional Camassa-holm equation (CH):

$$m_t + um_x + 2u_xm = 0, \ m = u - u_{xx}.$$ 

The CH equation is completely integrable [8,17] and has a bi-Hamiltonian structure [7,24]. It also has the solitary waves and peak solitons [18,21]. It is worth mentioning that the peakons show the characteristic for the traveling waves of greatest height and arise as solutions to the free-boundary problem for the incompressible Euler equations over a flat bed, see [10,11,15,42]. The local well-posedness, global strong solutions, blow-up strong solutions of the CH equations were studied in [9,12–14,22,35,41]. The global weak solutions, global conservative solutions and dissipative solutions also have been investigated in [2,3,20,31,43]. For the continuity of the solutions map of the CH equations with respect to the initial data, it was only proved in the spaces $C([0,T];B_{p,r}^s(\mathbb{R}))$ for any $s' < s$ with $s > \max\{\frac{3}{2}, 1 + \frac{1}{p}\}$ by many authors. Moreover, Li and Yin in [40] proved that the index of the continuous dependence for the solutions to the Camassa-Holm type equations in $B_{p,r}^s(\mathbb{R}) (s > \max\{\frac{3}{2}, 1 + \frac{1}{p}\})$ can up to $s$, which improved many authors’ results, especially the Danchin’s results in [22,23]. Recently, Guo et al. [25] obtained the local ill-posedness for a class of shallow water wave equations (such as, the CH, DP, Novikov equations and etc.) in critical Sobolev space $H^{\frac{3}{2}}(\mathbb{R})$ and even in Besov space $B_{p,r}^{1+\frac{1}{p}}(\mathbb{R})$ with $p \in [1, +\infty]$, $r \in (1, +\infty]$. More recently, by use of the compactness argument and Lagrangian coordinate transformation rather than the usual techniques used in [40], Ye et al. [45] proved the CH equation is locally well-posed and continuous dependence in Besov spaces $B_{p,r}^{1+\frac{1}{p}}(\mathbb{R})$ with $p \in [1, +\infty)$, which implied $B_{p,1}^{1+\frac{1}{p}}(\mathbb{R})$ is the critical Besov spaces and the index $\frac{3}{2}$ is not necessary for the Camassa-Holm type equations. Further, the non-uniform continuity of the CH equation has been investigated in many papers, see [26–28,32,33].

Eqs. (1.1) has numerous remarkable features and has been studies by many authors. Chae and Liu [6] established the local existence of weak solution in $W^{2,p}(\mathbb{R}^d)$, $p > d$ and local existence of unique classical solutions in $H^s(\mathbb{R}^d)$, $s > \frac{d}{2} + 3$ for (1.1). Then, Li, Yu and Zhai [39] proved that the solutions to (1.1) with a large class of smooth initial data blows up in finite time or exists globally in time, which settled an open problem raised by Chae and Liu [6]. Taking advantage of the Littlewood-Paley decomposition theory, Yan and Yin [44] further discussed the local existence and uniqueness of the solution to (1.1) in Besov spaces $B_{p,r}^s(\mathbb{R}^d)$ with $s > \max\{1 + \frac{d}{p}, \frac{3}{2}\}$ and $s = 1 + \frac{d}{p}, 1 \leq p \leq 2d, r = 1$. Recently, Li, Dai and Zhu [34] shown that the corresponding solution to (1.1) is not uniformly continuous dependence for that the initial data in $H^s(\mathbb{R}^d), s > 1 + \frac{d}{2}$. Also, Li, Dai and Li in [38] have shown that the data-to-solution map for (1.1) is not uniformly continuous dependence in Besov spaces $B_{p,r}^s(\mathbb{R}^d), s > \max\{1 + \frac{d}{2}, \frac{3}{2}\}$. For more results of higher dimensional Camassa-Holm equations, see [36,46].

However, the continuity of the data-to-solution map for the higher dimensional Camassa-Holm equations in Besov spaces $B_{p,\infty}^s(\mathbb{R}^d)$, $s > \max\{1 + \frac{d}{p}, \frac{3}{2}\}, 1 \leq p \leq +\infty$ has not been solved yet.
In this paper, we will pay our attention to studying the ill-posedness for the higher dimensional Camassa-Holm equations in Besov spaces. The key skill is to construct an initial data.

Now let us state our main result of this paper.

**Theorem 1.1.** Let $d \geq 2$ and $\sigma > 2 + \max \left\{ 1 + \frac{d}{p}, \frac{3}{2} \right\}$ with $1 \leq p \leq \infty$. There exists a $u_0 \in B^\sigma_{p,\infty}(\mathbb{R}^d)$ and a positive constant $\varepsilon_0$ such that the data-to-solution map $u_0 \mapsto u(t)$ of the Cauchy problem (1.1) satisfies

$$\limsup_{t \to 0^+} \| u(t) - u_0 \|_{B^\sigma_{p,\infty}} \geq \varepsilon_0.$$  

**Remark 1.1.** Theorem 1.1 demonstrates the ill-posedness of the higher dimensional CH equation in $B^\sigma_{p,\infty}$. More precisely, there exists a $u_0 \in B^\sigma_{p,\infty}$ such that the corresponding solution to the higher dimensional CH equation that starts from $u_0$ does not converge back to $u_0$ in the sense of $B^\sigma_{p,\infty}$-norm as time goes to zero. Our key argument is to construct an initial data $u_0$.

The remainder of this paper is organized as follows. In Section 2, we list some notations and recall known results. In Section 3, we present the proof of Theorem 1.1 by establishing some technical lemmas and propositions.

## 2 Preliminaries

### 2.1 General Notation

In the following, we denote by $\ast$ the convolution. Given a Banach space $X$, we denote its norm by $\| \cdot \|_X$. For $I \subset \mathbb{R}$, we denote by $C(I; X)$ the set of continuous functions on $I$ with values in $X$. Sometimes we will denote $L^p(0, T; X)$ by $L^p_T X$. For all $f \in \mathcal{S}'$, the Fourier transform $\mathcal{F}f$ (also denoted by $\hat{f}$) is defined by

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) dx \quad \text{for any} \ \xi \in \mathbb{R}^d.$$  

### 2.2 Littlewood-Paley Analysis

Next, we recall some useful properties about the Littlewood-Paley decomposition and the Besov spaces.

**Proposition 2.1** (Littlewood-Paley decomposition, See [1]). Let $\mathcal{B} := \{ \xi \in \mathbb{R}^d : |\xi| \leq \frac{4}{3} \}$ and $\mathcal{C} := \{ \xi \in \mathbb{R}^d : \frac{4}{3} \leq |\xi| \leq \frac{8}{3} \}$. There exist two radial functions $\chi \in C^\infty_c(\mathcal{B})$ and $\varphi \in C^\infty_c(\mathcal{C})$ both taking values in $[0, 1]$ such that

$$\chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1, \quad \forall \ \xi \in \mathbb{R}^d,$$

$$\frac{1}{2} \leq \chi^2(\xi) + \sum_{j \geq 0} \varphi^2(2^{-j}\xi) \leq 1, \quad \forall \ \xi \in \mathbb{R}^d.$$
Proposition 2.2 (Bernstein’s inequalities, See [1]). Let $\mathcal{B}$ be a ball and $\mathcal{C}$ be an annulus. A constant $C > 0$ exists such that for all $k \in \mathbb{N}$, $1 \leq p \leq q \leq \infty$, and any function $f \in L^p(\mathbb{R})$, we have

$$ \text{Supp} \hat{f} \subset \lambda \mathcal{B} \Rightarrow \|D^k f\|_{L^q} = \sup_{|\alpha| = k} \|\partial^\alpha f\|_{L^q} \leq C^{k+1} \lambda^{k+d(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p}, $$

$$ \text{Supp} \hat{f} \subset \lambda \mathcal{C} \Rightarrow C^{-k} \lambda^k \|f\|_{L^q} \leq \|D^k f\|_{L^q} \leq C^{k+1} \lambda^k \|f\|_{L^p}. $$

Definition 2.1 (See [1]). For any $u \in S'(\mathbb{R}^d)$, the Littlewood-Paley dyadic blocks $\Delta_j$ are defined as follows

$$ \Delta_j u = \begin{cases} 0, & \text{if } j \leq -2; \\ \chi(D) u = \mathcal{F}^{-1}(\chi \mathcal{F} u), & \text{if } j = -1; \\ \varphi(2^{-j} D) u = \mathcal{F}^{-1}(\varphi(2^{-j}) \mathcal{F} u), & \text{if } j \geq 0. \end{cases} $$

In the nonhomogeneous case, the following Littlewood-Paley decomposition makes sense:

$$ u = \sum_{j \geq -1} \Delta_j u, \quad \forall \ u \in S'(\mathbb{R}^d). $$

Definition 2.2 (See [1]). Let $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$. The nonhomogeneous Besov space $B^s_{p,r}(\mathbb{R}^d)$ is defined by

$$ B^s_{p,r}(\mathbb{R}^d) := \left\{ f \in S'(\mathbb{R}^d) : \|f\|_{B^s_{p,r}(\mathbb{R}^d)} < \infty \right\}, $$

where

$$ \|f\|_{B^s_{p,r}(\mathbb{R}^d)} = \begin{cases} \left( \sum_{j \geq -1} 2^{sjr} \|\Delta_j f\|_{L^p(\mathbb{R}^d)}^r \right)^{\frac{1}{r}}, & \text{if } 1 \leq r < \infty, \\ \sup_{j \geq -1} 2^{js} \|\Delta_j f\|_{L^p(\mathbb{R}^d)}, & \text{if } r = \infty. \end{cases} $$

For simplicity, we always write $u \in B^s_{p,r}(\mathbb{R}^d)$ and $\nabla u \in B^s_{p,r}(\mathbb{R}^d)$ standing for $u \in (B^s_{p,r}(\mathbb{R}^d))^d$ and $\nabla u \in (B^s_{p,r}(\mathbb{R}^d))^d$, respectively.

Remark 2.1. It should be emphasized that the following embedding will be often used implicitly:

$$ B^{s_1}_{p,q}(\mathbb{R}^d) \hookrightarrow B^{s_2}_{p,r}(\mathbb{R}^d) \quad \text{for } s_1 > s_2 \text{ or } s_1 = s_2, \ 1 \leq q \leq r \leq \infty. $$

Finally, we give some important lemmas which will be also often used throughout the paper.

Lemma 2.1 (See [1, 44]). Let $(p, r) \in [1, \infty] \times [1, \infty] \times [1, \infty]$ and $s > \max \left\{ 1 + \frac{d}{p}, \frac{3}{2} \right\}$. Then

$$ \|uv\|_{B^{s-2}_{p,r}(\mathbb{R}^d)} \leq C \|u\|_{B^{s-2}_{p,r}(\mathbb{R}^d)} \|v\|_{B^{s-1}_{p,r}(\mathbb{R}^d)}. $$

Hence, for the terms $Q(u, u)$, $Q(v, v)$, $R(u, u)$ and $R(v, v)$, we have

$$ \|Q(u, u) - Q(v, v)\|_{B^{s-1}_{p,r}(\mathbb{R}^d)} \leq C \|u - v\|_{B^{s-1}_{p,r}(\mathbb{R}^d)} \left( \|u\|_{B^{s}_{p,r}(\mathbb{R}^d)} + \|v\|_{B^{s}_{p,r}(\mathbb{R}^d)} \right), $$

$$ \|R(u, u) - R(v, v)\|_{B^{s-1}_{p,r}(\mathbb{R}^d)} \leq C \|u - v\|_{B^{s-1}_{p,r}(\mathbb{R}^d)} \left( \|u\|_{B^{s}_{p,r}(\mathbb{R}^d)} + \|v\|_{B^{s}_{p,r}(\mathbb{R}^d)} \right). $$
Lemma 2.2 (See [1]). For \((p, r) \in [1, \infty]^2\) and \(s > 0\), \(B^s_{p,r}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)\) is an algebra. Moreover, for any \(u, v \in B^s_{p,r}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)\), we have

\[\|uv\|_{B^s_{p,r}(\mathbb{R}^d)} \leq C\left(\|u\|_{B^s_{p,r}(\mathbb{R}^d)}\|v\|_{L^\infty(\mathbb{R}^d)} + \|v\|_{B^s_{p,r}(\mathbb{R}^d)}\|u\|_{L^\infty(\mathbb{R}^d)}\right).\]

In the paper, we also need some estimates for the following transport equation:

\[
\begin{align*}
\frac{\partial}{\partial t} f + v \cdot \nabla f &= g, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d, \\
f(0, x) &= f_0(x), \quad x \in \mathbb{R}^d. 
\end{align*}
\]

(2.1)

Lemma 2.3 (See [1]). Let \(d \in \mathbb{N}^+, 1 \leq p \leq \infty, 1 \leq r \leq \infty\) and \(\theta > -\min\left(\frac{d}{p}, 1 - \frac{d}{p}\right)\). Let \(f_0 \in B^\theta_{p,r}(\mathbb{R}^d), g \in L^1(0, T; B^\theta_{p,r}(\mathbb{R}^d)), v \in L^\rho(0, T; B^{\rho, M}_{\infty, \infty}(\mathbb{R}^d))\) for some \(\rho > 1\) and \(M > 0\), and

\[
\begin{align*}
\nabla v &\in L^1(0, T; B^{\frac{d}{p}}_{\infty, \infty}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)), \quad \text{if } \theta < 1 + \frac{d}{p}; \\
\nabla v &\in L^1(0, T; B^{\frac{d}{p} - 1}_{\infty, \infty}(\mathbb{R}^d)), \quad \text{if } \theta > 1 + \frac{d}{p} \text{ (or } \theta = 1 + \frac{d}{p}, \ r = 1). 
\end{align*}
\]

Then the problem (2.1) has a unique solution \(f\) in

- the space \(C([0, T]; B^\theta_{p,r}(\mathbb{R}^d))\), if \(r < \infty\),
- the space \((\bigcap_{\theta' < \theta} C([0, T]; B^{\theta'}_{p,r}(\mathbb{R}^d))) \cap C_w([0, T]; B^\theta_{p,\infty}(\mathbb{R}^d))\), if \(r = \infty\).

Lemma 2.4 (See [1, 40]). Let \(d \in \mathbb{N}^+, 1 \leq p, r \leq \infty, \theta > -\min\left(\frac{d}{p}, 1 - \frac{d}{p}\right)\). There exists a constant \(C\) such that for all solutions \(f \in L^\infty(0, T; B^\theta_{p,r}(\mathbb{R}^d))\) of (2.1) with initial data \(f_0\) in \(B^\theta_{p,r}(\mathbb{R}^d)\) and \(g\) in \(L^1(0, T; B^\theta_{p,r}(\mathbb{R}^d))\), we have, for a.e. \(t \in [0, T]\),

\[
\|f(t)\|_{B^\theta_{p,r}(\mathbb{R}^d)} \leq \|f_0\|_{B^\theta_{p,r}(\mathbb{R}^d)} + \int_0^t \|g(t')\|_{B^\theta_{p,r}(\mathbb{R}^d)} \, dt' + \int_0^t V'(t') \|f(t')\|_{B^\theta_{p,r}(\mathbb{R}^d)} \, dt'
\]

or

\[
\|f(t)\|_{B^\theta_{p,r}(\mathbb{R}^d)} \leq e^{CV(t)} \left(\|f_0\|_{B^\theta_{p,r}(\mathbb{R}^d)} + \int_0^t e^{-CV(t')} \|g(t')\|_{B^\theta_{p,r}(\mathbb{R}^d)} \, dt'\right)
\]

with

\[
V'(t) = \begin{cases} 
\|\nabla v(t)\|_{B^\theta_{p,\infty}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)} & \text{if } \theta < 1 + \frac{d}{p}; \\
\|\nabla v(t)\|_{B^{\frac{d}{p} - 1}_{p,\infty}(\mathbb{R}^d)} & \text{if } \theta > 1 + \frac{d}{p} \text{ (or } \theta = 1 + \frac{d}{p}, \ r = 1). 
\end{cases}
\]

If \(f = v\), then for all \(\theta > 0\), \(V'(t) = \|\nabla v(t)\|_{L^\infty(\mathbb{R}^d)}\).

## 3 Proof of Theorem 1.1

### 3.1 Construction of Initial Data

We need to introduce smooth, radial cut-off functions to localize the frequency region. Precisely, let \(\hat{\phi} \in C_0^\infty(\mathbb{R})\) be an even, real-valued and non-negative function on \(\mathbb{R}\) and satisfy

\[
\hat{\phi}(\xi) = \begin{cases} 
1, & \text{if } |\xi| \leq \frac{1}{4}, \\
0, & \text{if } |\xi| \geq \frac{1}{2}.
\end{cases}
\]
In [37], it has been verified that for \( f_n := \phi(x_1) \cos(\frac{17}{12} \ell x_1) \phi(x_2) \cdot \phi(x_n) \) and \( n \geq 2 \),
\[
\Delta_j(f_n) = \begin{cases} 
  f_n, & \text{if } j = n, \\
  0, & \text{if } j \neq n.
\end{cases}
\] (3.1)

We can obtain the similar result:

**Lemma 3.1.** Let \( 6 \leq k, n \in \mathbb{N}^+ \). Define the function \( g_{m,n}^k(x) \) by
\[
g_{m,n}^k(x) := \phi(x_1) \cos \left( \frac{17}{12} (2^{kn} \pm 2^m) x_1 \right) \phi(x_2) \cdots \phi(x_d) \quad \text{with} \quad 0 \leq m \leq n - 1.
\]
Then we have
\[
\Delta_j(g_{m,n}^k) = \begin{cases} 
  g_{m,n}^k, & \text{if } j = kn, \\
  0, & \text{if } j \neq kn.
\end{cases}
\]

**Proof.** The proof is similar to that of in [37], and here we omit it. \( \square \)

**Lemma 3.2.** Define the initial data \( u_0(x) \) as
\[
u_0(x) = (u_0^1(x), \ldots, u_0^d(x)) := \left( \sum_{n=0}^{\infty} 2^{-kn \sigma} f_n^k(x), 0, \cdots, 0 \right),
\]
ce4 where
\[
f_n^k(x) := \phi(x_1) \cos \left( \frac{17}{12} 2^{kn} x_1 \right) \phi(x_2) \cdots \phi(x_d), \quad n \geq 0.
\]
Then for any \( \sigma \in (2 + \max\{\frac{3}{2}, 1 + \frac{d}{p}\}, +\infty) \) and for some \( k \) large enough, we have
\[
|u_0|^2(x) = \left( u_0^1(x) \right)^2,
\]
\[
u_0 \cdot \nabla u_0 = \left( \frac{1}{2} \partial_1 \left( (|u_0|^2) \right), 0, \cdots, 0 \right) = \left( \frac{1}{2} \partial_1 (|u_0|^2), 0, \cdots, 0 \right),
\]
\[
\|u_0\|_{B_{p,\infty}^\sigma} \leq C,
\]
\[
\|\Delta_{kn}(|u_0|^2)\|_{L^p} \geq c 2^{-kn \sigma}.
\]

**Proof.** Thanks to the definition of Besov spaces, the support of \( \varphi(2^{-j} \cdot) \) and (3.1), we see
\[
\|u_0\|_{B_{p,\infty}^\sigma} = \sup_{j \geq -1} 2^{j \sigma} \|\Delta_j u_0\|_{L^p}
\]
\[
= \|\phi\|_{L^p}^{d-1} \sup_{j \geq 0} \left\| \phi(x_1) \cos \left( \frac{17}{12} 2^j x_1 \right) \right\|_{L^p}
\]
\[
\leq C.
\]

Notice that the simple fact
\[
\cos(A + B) + \cos(A - B) = 2 \cos A \cos B
\]
and
\[
\sum_{n=0}^{\infty} \sum_{m=0, m \neq n}^{\infty} A_n A_m = 2 \sum_{n=0}^{\infty} \sum_{m=0}^{n-1} A_n A_m,
\]
then direct computations give
\[
|u_0|^2(x) = (u_0^1(x))^2
= \frac{1}{2} \sum_{n=0}^{\infty} 2^{-2kn} \phi^2(x_1) \phi^2(x_2) \cdots \phi^2(x_d) + \frac{1}{2} \sum_{n=0}^{\infty} 2^{-2kn} \phi^2(x_1) \cos \left( \frac{17}{12} 2^{kn+1} x_1 \right) \phi^2(x_2) \cdots \phi^2(x_d)
+ \sum_{n=1}^{n-1} \sum_{m=0}^{n-1} 2^{-k(n+m) \sigma} \phi^2(x_1) \left[ \cos \left( \frac{17}{12} (2^{kn} - 2^{km}) x_1 \right) + \cos \left( \frac{17}{12} (2^{kn} + 2^{km}) x_1 \right) \right] \phi^2(x_2) \cdots \phi^2(x_d).
\]
Lemma 3.1 yields
\[
\Delta_{kn}(|u_0|^2) = \Delta_{kn}((u_0^1)^2)
= 2^{-kn} \phi^2(x_1) \left[ \cos \left( \frac{17}{12} (2^{kn} - 1) x_1 \right) + \cos \left( \frac{17}{12} (2^{kn} + 1) x_1 \right) \right] \phi^2(x_2) \cdots \phi^2(x_d)
+ \sum_{m=1}^{n-1} 2^{-k(n+m) \sigma} \phi^2(x_1) \left[ \cos \left( \frac{17}{12} (2^{kn} - 2^{km}) x_1 \right) + \cos \left( \frac{17}{12} (2^{kn} + 2^{km}) x_1 \right) \right] \phi^2(x_2) \cdots \phi^2(x_d)
=: I_1 + I_2,
\]
where we denote
\[
I_1 := 2 \cdot 2^{-kn} \phi^2(x_1) \cos \left( \frac{17}{12} 2^{kn} x_1 \right) \cos \left( \frac{17}{12} x_1 \right) \phi^2(x_2) \cdots \phi^2(x_d),
I_2 := 2 \sum_{m=1}^{n-1} 2^{-k(n+m) \sigma} \phi^2(x_1) \cos \left( \frac{17}{12} 2^{kn} x_1 \right) \cos \left( \frac{17}{12} 2^{km} x_1 \right) \phi^2(x_2) \cdots \phi^2(x_d).
\]
For the first term $I_1$, after a simple calculation, we have
\[
\|I_1\|_{L^p(\mathbb{R}^d)} \geq 2^{-kn} \left\| \phi^2(x_1) \cos \left( \frac{17}{12} 2^{kn} x_1 \right) \cos \left( \frac{17}{12} 2^{kn} x_1 \right) \right\|_{L^p(\mathbb{R})} \|\phi\|_{L^{2(d-1)}}^{2(\delta-1)}. \tag{3.2}
\]
From Lemma 3.2 in [37], we have for some $\delta > 0$
\[
\left\| \phi^2(x_1) \cos \left( \frac{17}{12} 2^{kn} x_1 \right) \cos \left( \frac{17}{12} 2^{kn} x_1 \right) \right\|_{L^p(\mathbb{R})} \geq c(p, \delta, \phi(0)). \tag{3.3}
\]
Then we obtain from (3.2) that
\[
\|I_1\|_{L^p} \geq c 2^{-kn \sigma}. \tag{3.4}
\]
For the second term $I_2$, from a straightforward calculation, we deduce
\[
\|I_2\|_{L^p} \leq C \|\phi\|_{L^{2p}}^{2d} \sum_{m=1}^{n-1} 2^{-k(n+m) \sigma} \leq C 2^{-k(n+1) \sigma}. \tag{3.5}
\]
(3.4) and (3.5) together yield that
\[
\|\Delta_{kn}(|u_0|^2)\|_{L^p} \geq (c - C 2^{-k \sigma}) 2^{-kn \sigma}.
\]
We choose $k \geq 6$ such that $c - C 2^{-k \sigma} > 0$ and then finish the proof of Proposition 3.2. \qed
3.2 Error Estimates

We first recall the following local-in-time existence of strong solutions to (1.1) in [44].

Lemma 3.3 (See [44]). Let \( d \geq 2, 1 \leq p, r \leq \infty \) and \( s > \max\{1 + \frac{d}{p}, \frac{3}{2}\} \). Assume that \( u_0 \in B^s_{p,r}(\mathbb{R}^d) \),

then there exists a time \( T = T(s,p,r,\|u_0\|_{B^s_{p,r}(\mathbb{R}^d)}) > 0 \) such that (1.1) has a unique solution \( u \in C([0,T];B^s_{p,r}(\mathbb{R}^d)) \). Moreover, for all \( t \in [0,T] \), there holds

\[
\text{achm} \|u(t)\|_{B^s_{p,r}(\mathbb{R}^d)} \leq C\|u_0\|_{B^s_{p,r}(\mathbb{R}^d)}.
\]

Proof. Due to Lemma 3.3, we know that there exists a positive time \( T = T(s,p,r,\|u_0\|_{B^s_{p,r}(\mathbb{R}^d)}) \) such that

\[
\|u(t)\|_{L^\infty_T B^s_{p,r}} \leq C\|u_0\|_{B^s_{p,r}} \leq C. \tag{3.6}
\]

Moreover, for \( \gamma > \frac{1}{2} \), taking advantage of Lemma 2.4 and (3.6), we have

\[
\|u(t)\|_{L^\infty_T B^s_{p,r}} \leq C\|u_0\|_{B^s_{p,r}}. \tag{3.7}
\]

By the Mean Value Theorem, we obtain from (1.2) and (3.7) that

\[
\|u(t) - u_0\|_{B^s_{p,r}} \leq \int_0^t \|\partial_t u\|_{B^s_{p,r}} \, d\tau
\]

\[
\leq \int_0^t \|Q(u,u)\|_{B^s_{p,r}} \, d\tau + \int_0^t \|R(u,u)\|_{B^s_{p,r}} \, d\tau + \int_0^t \|u \cdot \nabla u\|_{B^s_{p,r}} \, d\tau
\]

\[
\leq Ct\|u\|_{L^\infty_T B^s_{p,r}}^2 + \|u\|_{L^\infty_T L^\infty} \|\nabla u\|_{L^\infty_T B^s_{p,r}}
\]

\[
\leq Ct\|u\|_{L^\infty_T B^s_{p,r}}^2 + \|u\|_{L^\infty_T B^{s-1}_{p,r}} \|u\|_{L^\infty_T B^{s+1}_{p,r}}
\]

\[
\leq Ct\|u_0\|_{B^s_{p,r}}^2 + \|u_0\|_{B^{s-1}_{p,r}} \|u_0\|_{B^{s+1}_{p,r}},
\]

where we have used that \( B^{s-1}_{p,r} \rightarrow L^\infty \) with \( s - 1 > \max\{\frac{d}{p}, \frac{1}{2}\} \).

Following the same procedure as above, according to Lemmas 2.1 and 2.2, we see

\[
\|u(t) - u_0\|_{B^{s-1}_{p,r}} \leq \int_0^t \|\partial_t u\|_{B^{s-1}_{p,r}} \, d\tau
\]

\[
\leq \int_0^t \|Q(u,u)\|_{B^{s-1}_{p,r}} \, d\tau + \int_0^t \|R(u,u)\|_{B^{s-1}_{p,r}} \, d\tau + \int_0^t \|u \cdot \nabla u\|_{B^{s-1}_{p,r}} \, d\tau
\]

\[
\leq Ct\|u\|_{L^\infty_T B^{s-1}_{p,r}} + \|u\|_{L^\infty_T B^{s+1}_{p,r}}
\]

\[
\leq Ct\|u_0\|_{B^{s-1}_{p,r}} \|u_0\|_{B^{s+1}_{p,r}}.
\]
and
\[
\|u(t) - u_0\|_{B^{p+1}_{p,\infty}} \leq \int_0^t \|\partial_\tau u\|_{B^{p+1}_{p,\infty}} d\tau \\
\leq \int_0^t \|Q(u, u)\|_{B^{p+1}_{p,\infty}} d\tau + \int_0^t \|R(u, u)\|_{B^{p+1}_{p,\infty}} d\tau + \int_0^t \|u \cdot \nabla u\|_{B^{p+1}_{p,\infty}} d\tau \\
\leq Ct(\|u\|_{L^\infty_t B^s_{p,\infty}} \|u\|_{L^\infty_t B^{p+1}_{p,\infty}} + \|u\|_{L^\infty_t B^{s-1}_{p,\infty}} \|u\|_{L^\infty_t B^{p+2}_{p,\infty}}) \\
\leq Ct(\|u_0\|_{B^s_{p,\infty}} \|u_0\|_{B^{p+1}_{p,\infty}} + \|u_0\|_{B^{s-1}_{p,\infty}} \|u_0\|_{B^{p+2}_{p,\infty}}).
\]
Thus, we finish the proof of Proposition 3.1.

**Proposition 3.2.** Let \( s = \sigma - 2 \) and \( u_0 \in B^s_{p,\infty} \). Assume that \( u \in L^\infty_T B^\sigma_{p,\infty} \) be the solution of the Cauchy problem (1.1), we have
\[
\|w(t, u_0)\|_{B^s_{p,\infty}} \leq Ct^2(\|u_0\|^3_{B^s_{p,\infty}} + \|u_0\|^2_{B^{s-1}_{p,\infty}} \|u_0\|_{B^s_{p,\infty}} + \|u_0\|^2_{B^{s-1}_{p,\infty}} \|u_0\|_{B^{p+2}_{p,\infty}}),
\]
here and in what follows we denote
\[
w(t, u_0) := u(t) - u_0 - t v_0, \\
v_0 := -u_0 \cdot \nabla u_0 + Q(u_0, u_0) + R(u_0, u_0).
\]
In particular, we obtain
\[
\|w(t, u_0)\|_{B^{s-2}_{p,\infty}} \leq C(\|u_0\|_{B^s_{p,\infty}}) t^2.
\]
**Proof.** Taking advantage of the Mean Value Theorem and (1.2), and then using Lemma 2.1 and Lemma 2.2, we find
\[
\|w(t, u_0)\|_{B^s_{p,\infty}} \leq \int_0^t \|\partial_\tau u - v_0\|_{B^s_{p,\infty}} d\tau \\
\leq \int_0^t \|Q(u, u) - Q(u_0, u_0)\|_{B^s_{p,\infty}} + \|R(u, u) - R(u_0, u_0)\|_{B^s_{p,\infty}} d\tau \\
+ \int_0^t \|u \cdot \nabla u - u_0 \cdot \nabla u_0\|_{B^s_{p,\infty}} d\tau \\
\leq C \int_0^t \|u(\tau) - u_0\|_{B^s_{p,\infty}} \|u_0\|_{B^s_{p,\infty}} d\tau + C \int_0^t \|u(\tau) - u_0\|_{B^{s-1}_{p,\infty}} \|u(\tau)\|_{B^{p+1}_{p,\infty}} d\tau \\
+ C \int_0^t \|u(\tau) - u_0\|_{B^{s-1}_{p,\infty}} \|u_0\|_{B^{p-1}_{p,\infty}} d\tau \\
\leq Ct^2(\|u_0\|^3_{B^s_{p,\infty}} + \|u_0\|^2_{B^{s-1}_{p,\infty}} \|u_0\|_{B^s_{p,\infty}} + \|u_0\|^2_{B^{s-1}_{p,\infty}} \|u_0\|_{B^{p+2}_{p,\infty}}),
\]
where we have used Proposition 3.1 in the last step.

Thus, we complete the proof of Proposition 3.2.

Now we present the proof of Theorem 1.1.
Proof of Theorem 1.1: Using Proposition 2.2, Proposition 3.2 and Lemma 3.2, we get

\[ \|u(t) - u_0\|_{B^\sigma_{p,\infty}} \geq \frac{2^{kn\sigma}}{2^{kn}} \|\Delta_{kn}(u(t) - u_0)\|_{L^p} = 2^{kn\sigma} \|\Delta_{kn}(tv_0 + w(t, u_0))\|_{L^p} \]

\[ \geq t^{2kn\sigma} \|\Delta_{kn}(v_0)\|_{L^p} - 2^{kn}t^{2kn(\sigma - 2)} \|\Delta_{kn}(w(t, u_0))\|_{L^p} \]

\[ \geq t^{2kn(\sigma + 1)} \|\Delta_{kn}(|u_0|^2)\|_{L^p} - Ct \left( \|u_0\|_{B^\sigma_{p,\infty}}^2 + \|\nabla u_0\|_{B^\sigma_{p,\infty}^{-1}}^2 \right) \]

\[ - C't^{2kn} \|w(t, u_0)\|_{B^\sigma_{p,\infty}^{-2}} \]

\[ \geq t^{2kn(\sigma + 1)} \|\Delta_{kn}(|u_0|^2)\|_{L^p} - C't^{2kn}t^2 \]

\[ \geq ct^{2kn} - Ct - C't^{2kn}t^2. \]

Then, for \( k \geq 6 \), taking \( n > N \) large enough such that \( c2^{kn} \geq 2C \), we deduce that

\[ \|u(t) - u_0\|_{B^\sigma_{p,\infty}} \geq ct^{2kn} - C't^{2kn}t^2. \]

Thus, choosing \( t^{2kn} \approx \varepsilon \) with small \( \varepsilon \), we finally conclude that

\[ \|u(t) - u_0\|_{B^\sigma_{p,\infty}} \geq c\varepsilon - C\varepsilon^2 \geq c_1\varepsilon. \]

This proves Theorem 1.1.

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Conflict of interest The authors declare that they have no conflict of interest.

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