Stability of Exponential Tails in the Scattering on Wedges and Impurities

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We study a simple exactly solvable 2D model describing the interaction of a localized particle with an impurity. The localization potential $V(x) = -\alpha \delta(x)$ causes the particle to be trapped in the $y$-axis, and the ‘impurity’ is modeled by a straight impenetrable edge extending along the positive $x$-axis. We show that the problem described can be treated as the Sommerfeld diffraction from an infinite edge. We use the model to present qualitative arguments on the nature of interaction of polarized light travelling along an optical fiber with external impurities.

I. INTRODUCTION

There is a class of problems characterized by an example of polarized light propagating along an optical fiber (or a wave guide). It is interesting to see how the interaction with depolarizing impurities arises in such a problem and what effect the impurities have on the propagation of polarized light. The polarized light is described by the scalar Helmholtz equation. The optical fiber is a waveguide with appropriate boundary conditions imposed. The depolarizing interaction with impurities on the walls of the waveguide can be modeled by a photon spin-flip interaction. Since this problem is too complex to obtain an exact analytic solution, we will consider a much simplified model which, we believe, captures the essence of the problem. The Schrödinger equation for a particle of unit mass $m = 1/2$ is the same as the Helmholtz equation. We will model the waveguide by an attractive, localizing, potential $V(x, y)$ and the impurity by an impenetrable barrier which we will take to be a semi-infinite edge located along the positive $x$-axis. A simple argument based on the first order perturbation theory (the Born approximation) applied to a
bound state $\psi_0$ in a potential $V(x, y)$ suggests that an “impurity” located on the $x$-axis at $x = a$, and modeled by a potential $V_{\text{imp}}(x, 0) = \lambda \delta(x - a)\delta(y)$, modifies the bound state wave function by a correction which is quadratic in the value of this wave function at the point where the “impurity” is located. What this means is that in the case when the bound state wave function decays exponentially with $x$ as $|\psi_0(x, y)| \sim e^{-\alpha|x|}$, the correction to the wave function caused by an “impurity” is of the subleading order, $|\psi^{(1)}(x, y)| \sim e^{-2\alpha a} e^{-\alpha|x|}$. In this sense the bound state ‘exponential tails’ are rather rigid because the “impurity” changes the asymptotic behavior of the bound state by an exponentially negligible correction to an overall normalization factor.

II. THE MODEL

Consider a particle moving in the Euclidean plane. Along the positive $x$-axis we place an impenetrable barrier, a straight semi-infinite edge. A free quantum mechanical particle will be scattered by a straight edge in a manner first described by Sommerfeld [1, 2, 3, 4]. We are analyzing a situation where the particle is trapped in an attractive potential $V(x, y) = -\alpha \delta(x)$ that allows it to move freely only in the $y$ direction. The Hamiltonian for this particle is the sum of two one-dimensional Hamiltonians $H = H_1 + H_2$, $H_1 = -\partial_x^2 - \alpha \delta(x)$, $H_2 = -\partial_y^2$, where $\alpha > 0$. Therefore, the total Hamiltonian is

$$H = -\left(\partial_x^2 + \partial_y^2\right) - \alpha \delta(x).$$

(1)

In the simplest possible case that we deal with here, $H_1$ has only one bound state,

$$\psi_0(x) = e^{-\alpha|x|}.$$

(2)

In addition, there are scattering states,

$$\psi(x) = e^{ipx} + Ae^{-ipx}, x > 0, \psi(x) = Be^{ipx}, x < 0,$$

(3)

where the transmission and reflection coefficients are

$$A = \frac{i \alpha}{p - i \alpha}, B = \frac{p}{p - i \alpha}.$$ 

(4)
A simple pole at $p = i\alpha$ in the S-matrix corresponds to a bound state \( [2] \).

The stationary solution of the Schrödinger equation describing a bound state of \( H \) and corresponding to the energy

\[
E = k^2 - \alpha^2,
\]

is

\[
\psi_0(x, y) = e^{iky-\alpha|x|}.
\]

The problem we want to solve is the problem of scattering on the straight edge $x > a$, $y = 0$. We assume that for $x > a$, $y = 0$ the wave function or its normal derivative must vanish, thus either $\psi(x, y) = 0$ or $\partial_y \psi(x, y) = 0$. These conditions are equivalent to the Dirichlet or Neumann boundary conditions, correspondingly. The idea that underlies the solution is to explore the Helmholtz equation

\[
\left( \Delta + k^2 \right) \psi = 0,
\]

in the complex $x$-$y$ space (complex $x$ and $y$ planes), where

\[
\Delta = \partial_x^2 + \partial_y^2,
\]

and $x$ and $y$ are independent complex variables. Elementary solutions that represent plane waves $\exp(ik_1 x + ik_2 y)$ are holomorphic everywhere in the $x$ and $y$ complex planes. Since the Helmholtz equation is linear, the superposition principle can be used to obtain more general solutions from the basic ones. In particular, the superposition principle can be applied to obtain bound state solutions.

Now,

\[
\psi_0 = A \exp(-ikr \cos(\varphi - \beta_0)),
\]

\[
\psi = \int_{C} A(\beta) \exp(-ikr \cos(\varphi - \beta)) d\beta.
\]

To get $\psi_0$ from the last formula we need to choose:
\[ A(\beta) = \exp(i\beta)/2\pi(\exp(i\beta) - \exp(i\beta_0)) \] (11)

and so

\[ \psi = \frac{1}{4\pi} \int_C \frac{\exp(i\gamma/2)\exp(ik\cos\gamma)}{\exp(i\gamma/2) - \exp(-i\chi/2)} d\gamma, \] (12)

where \( \chi = (\varphi - \beta_0)/2 \). For the scattering states arising in the scattering off the straight edge, \( \beta_0 = \pi/2 \). For the bound state propagating along the \( y \)-axis only, we have \( \beta_0 = \pi/2 \pm i\lambda \), where \( \lambda = \lambda(k, \alpha) \).

The cut starting at \( x = a \) and running all the way in the positive direction of \( x \)-axis suggests introducing a double covering of a complex \( z \)-plane, where \( z = y + i(x - a) \). To this end, we introduce a new complex variable \( w = \xi + i\eta \) defined by

\[ z = w^2. \] (13)

We can now write

\[ x - a = r\cos\varphi = -r\sin(\varphi - \pi/2), \] (14)

\[ y = r\sin\varphi = r\cos(\varphi - \pi/2), \] (15)

and identify \( \xi \) and \( \eta \) as

\[ \xi = \sqrt{r}\cos\frac{\chi}{2}, \eta = -\sqrt{r}\sin\frac{\chi}{2}, \] (16)

where \( \chi = \varphi - \pi/2 \). We observe that both \( \varphi = 0 \) and \( 2\pi \) correspond to \( y = 0 \) and \( x > a \) (as \( x = r + a \)), but the first case translates into \( \xi = \eta = \sqrt{r/2} \), whereas the second one into \( \xi = \eta = -\sqrt{r/2} \). For \( \varphi = \pi \) we obtain that \( \xi = -\eta = \sqrt{r/2} \). This corresponds to \( y = 0 \) but with \( x < a \). In these new variables the Laplacian reads

\[ \Delta = \frac{1}{4(\xi^2 + \eta^2)}(\partial^2_{\xi} + \partial^2_{\eta}). \] (17)

We will seek the solution to the scattering problem described by the Helmholtz equation (7) in the form of \( \psi = \psi_1 + \psi_2 \), where

\[ \psi_1 = \exp(-iky)V(\xi, \eta), \] (18)

\[ \psi_2 = \exp(iky)U(\xi, \eta). \] (19)
Plugging $\psi_1$ into (7) yields
\[ \left[ (\partial_\xi^2 + \partial_\eta^2) - 4ik(\xi \partial_\xi - \eta \partial_\eta) \right] V(\xi, \eta) = 0, \] (20)
and suggests that one should take $V(\xi, \eta) = V(\xi)$ and $U(\xi, \eta) = U(\eta)$. As a result, one obtains
\[ V(\xi) = \int_{-\infty}^{\xi} d\tau \exp(2ik\tau^2) = F(\xi) = C(\xi) + iS(\xi), \] (21)
where $C(\xi)$ and $S(\xi)$ are Fresnel integrals. The expression for $U(\eta)$ is identical to that of $V(\xi)$ except that it is in terms of $\eta$. Consequently,
\[ \psi(\xi, \eta) = \exp(-iky)F(\xi) - \exp(iky)F(\eta), \] (22)
where, as seen from (13),
\[ y = \xi^2 - \eta^2, \quad x = 2\xi \eta + a. \] (23)

For $y = 0$, we have
\[ \psi(x, 0) = F(\xi) - F(\eta) \] (24)
which equals 0 for $x > a$ and $(F(\xi) - F(-\xi))$ otherwise, as should be apparent from our earlier discussion of the relationship between $\xi$ and $\eta$ in the case $y = 0$.

To conclude this part, we have demonstrated that the solution to the pure scattering problem under study in the entire $x$-$y$ plane is given by
\[ \psi(x, y) = C_0 \left[ \exp(-iky)\int_{-\infty}^{\xi} d\tau \exp(2ik\tau^2) - \exp(iky)\int_{-\infty}^{\eta} d\tau \exp(2ik\tau^2) \right], \] (25)
where $\xi$ and $\eta$ are related to $x$ and $y$ via (23) and $C_0$ is a constant.

Let us now consider the case of the bound state and assume for simplicity that $a = 0$. Writing $\psi_0(x, y)$ given by (24) as $\psi_0(x, y) = \exp(IS)$, we obtain
\[ IS = -iky - \alpha|x| = ikr \sin \varphi - \alpha \varepsilon r \cos \varphi \]
\[ = r \left( (k + \alpha \varepsilon) \exp(i\varphi) - (k - \alpha \varepsilon) \exp(-i\varphi) \right)/2 \]
\[ = ikr \cos(\varphi - \pi/2 - i\lambda) = ikr \cos(\varphi - \chi_0) = ikr \cos \chi, \] (26)
where we used the substitutions $k + \alpha \varepsilon = \kappa \exp \lambda$ and $k - \alpha \varepsilon = \kappa \exp(-\lambda)$. Moreover, $\varepsilon = \varepsilon(x) = x/|x|$ is the signum function and $\chi_0 = \pi/2 + i\lambda$. The last sequence of formulas is valid for the positive values of energy. The expression for $IS$ that corresponds to negative energies can be easily
obtained by analytical continuation. By analogy to the free scattering case, one can introduce new complex variables $\xi$ and $\eta$ related to the radial and angular coordinates $r$ and $\chi$ in the same manner as in (16) except that now

$$
\xi = \sqrt{\frac{r}{2}} \left( \cos \frac{\varphi - i\lambda}{2} + \sin \frac{\varphi - i\lambda}{2} \right),
$$

(27)

$$
\eta = \sqrt{\frac{r}{2}} \left( \cos \frac{\varphi - i\lambda}{2} - \sin \frac{\varphi - i\lambda}{2} \right).
$$

(28)

One can show that for $\varphi$ equal 0 and $2\pi$, $\xi(\varphi) = \eta^*(\varphi)$, where the asterisk denotes the complex conjugation. Pursuing further the analogy to the free scattering, it is straightforward to write the solution for the case under consideration,

$$
\psi = \exp(-\alpha|x|) \left( \exp(-iky)F(\xi) - \exp(iky)F(\eta^*) \right).
$$

(29)

This is an exact solution to the scattering problem on the half-infinite edge impurity which is valid both for real and imaginary $\kappa$. The case of imaginary $\kappa$, or negative energy, is of interest because it corresponds to the case of a single bound state of the attractive Dirac-delta potential in the transverse direction $x$. In this case we find an agreement with the simple perturbation theory argument presented in the introduction. Our investigation can be extended in many directions. In particular, it seems possible to extend the method presented here to localized wave packets.

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[1] A. Sommerfeld, “Matematische Theorie der Diffraction”, Math. Ann. 47, 317-374, 1896.
[2] A. Sommerfeld, Optics, Academic Press, New York, 1954, pp.247-272.
[3] Pauli, W., “On Asymptotic Series for Functions in the Theory of Diffraction of Light”, Phys. Rev. 54, 924-931, 1938.
[4] De-Witt-Morette, C., Low, S. G., Schulman, L. S. and Shiekh, A. Y., “Wedges I”, Foundations of Physics 16, 311-349, 1986.
[5] More generally, we will assume that it extends from $x = \alpha \geq 0$ to infinity.