Negative Probabilities and Contextuality

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Abstract

There has been a growing interest, both in physics and psychology, in understanding contextuality in experimentally observed quantities. Different approaches have been proposed to deal with contextual systems, and a promising one is contextuality-by-default, put forth by Dzhafarov and Kujala. The goal of this paper is to present a tutorial on a different approach: negative probabilities. We do so by presenting the overall theory of negative probabilities in a way that is consistent with contextuality-by-default and by examining with this theory some simple examples where contextuality appears, both in physics and psychology.

Keywords: contextuality, extended probabilities, negative probabilities, quantum cognition

1. Introduction

In recent years there has been an increased interest in modeling psychological experiments with the mathematical tools of Quantum Mechanics (QM) [9]. The argument, already put forth by Bohr, is that the principle of complementarity in QM is not unique to physical events, but is also present in cognitive and social phenomena [33]. Since complementarity is ubiquitous, it should also be true that the Hilbert space formalism created by physicists at the beginning of the 20th Century can be applied to describe mathematical situations outside of physics. This line of thinking gave rise in recent times to a thriving line of research known as Quantum Interaction and, more specifically in the context of psychology, Quantum Cognition.
At the core of complementarity is the idea that it is not possible, in principle, to observe simultaneously certain characteristics of a system. In physics, this is the case for the well-known wave/particle duality: each experimental context determines which characteristic, undulatory or corpuscular, is observed. Similarly, it was proposed that complementarity in psychology appears when a subject has to deal with situations that have different and incompatible contexts. The key aspect of complementarity, for our purpose, is that of a dependency on the context. Therefore, quantum mathematical models, and also quantum cognition, are essentially descriptions of contextual observables.

That the mathematical apparatus of QM is well suited to describe those context-dependent observables found in physics is clear by the tremendous success of this theory to not only describe the microscopic world but also to predict surprising results. This success comes from the fact that complementarity, with its prohibition of simultaneous observation of certain quantities, implies an orthomodular lattice of propositions pertaining to the observable events, instead of a classic Boolean algebra of compatible observables. In a famous paper, Piron [49] proved that the orthomodular lattice of propositions have a representation in terms of Hilbert spaces. Therefore, it stands to reason that Hilbert spaces are a good candidate for modeling the probabilities of quantities that may not be simultaneously observable. In other words, the mathematics of QM is a well-suited extension of probability theory that offer a way to model the probabilistic outcomes of contextual observables [8].

However, the mathematical structure of QM does not come without a price. First, it is not the most universal generalization of probabilities for context-dependent systems. It is possible to imagine certain context-dependent situations of interest to researchers outside of physics which the Hilbert space formalism of QM fails to describe (see [16] for an example). Second, the quantum formalism predicts some results that are not reasonable in, say, psychology. For instance, one important result is the impossibility to clone an unknown quantum system, which is related to the impossibility of superluminal signaling. There is no analogue to this in psychology, and one should not expect the cloning of “cognitive states” to be impossible in principle (for instance, in principle, albeit not in practice, we could conceive of duplicating all the neural states of a given brain, with their corresponding firings and configurations).

It is thus reasonable to ask what other ways of describing contextual systems exist. This has been a matter of intense research in the past few years, and in this paper we provide one possible tool: Negative Probabilities (NP). Our purpose here is to lay out the main ideas necessary to describe certain contextual systems with NP. To do so, we organize this paper as follows. In Section 2, we start with a definition of contextuality, in line with the recent work of Dzhafarov and Kujala [28]. In Section 3, we go into the mathematical details of negative probabilities, and discuss possible interpretations. Finally, in Section 4, we present some examples and applications of NP.
Figure 1: A firefly inside the box, whose position is represented on the horizontal plane by the white dot on top of the box, shines its light at a certain instant of time. The box is designed such that if the firefly is on the left hand side of A, then only this side lights up, but the exact position of the insect on this plane cannot be inferred (similarly for B). Due to experimental constraints (also by design, we cannot look at both sides A and B simultaneously), even though the actual position of the firefly is given by, in the figure, left on A and right on B, we only know one at a time, but do not know them jointly (i.e., knowing A is left does not tell us what the value of B is, which in this case is either right or left).

2. Contextuality

To understand what we mean by contextuality, we need to lay down some notation to describe it. Let us start with a formal definition of probabilities, which will be useful later on, when we modify it to allow for contextual systems. We follow Kolmogorov’s axiomatic approach based on set theory in general[42], but for the present paper, will only need to use finite probability spaces.

A discrete probability space is determined by the triple \((\Omega, \mathcal{F}, p)\), where \(\Omega\) is the set of elementary events, \(\mathcal{F}\) is the algebra of events (which can be taken as the powerset \(2^{\omega}\) for our purposes), and \(p : \mathcal{F} \to [0, 1]\) is a function that yields the probability of each event \(S \in \mathcal{F}\). The elementary events define the most atomic outcomes of an experiment, and so the probability of a general event \(S\) is determined by the probabilities of elementary events: \(p(S) = \sum_{\omega \in S} p(\{\omega\})\).

It is important to note that one generally cannot observe the elementary events directly. Let us explain what we mean with the well-known firefly box[31], which will be useful later on when we introduce the concept of contextuality. Imagine we have a box with a firefly inside it emitting light at random times. The box is constructed such that its walls are translucent, but an observer can only see one side of the box at a time (see Figure 1). A possible \(\Omega\) may be the set of all possible joint values of A and B, namely \(\{RR, RL, LR, LL\}\), where \(RR\) corresponds to the firefly lighting up the right side of A and right side of B, \(RL\) to right side of A and left of B, and so on. For this \(\Omega\), the elementary event \(RR\) is never actually observed, since to observe it means seeing both sides of the box simultaneously, which is forbidden by experimental design.

In the firefly box, the set of elementary events could be even more fine grained. For instance, it could be \(\{T_RT_R, T_RT_L, \ldots, B_LB_R, B_LB_L\}\), where,

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1Here, for our purposes, we discard the possible state where the firefly is not blinking.
e.g., $T_RB_L$ corresponds to the firefly being on the top of the right side of $A$ and on the bottom of the left side on $B$. Then, if one observed the firefly on the left of $A$, any of the following elementary events might be result in this: $T_LT_R$, $T_LT_L T_RB_R T_LT_L B_LT_RB_L B_LT_RB_L B_LT_RB_L$. If we were to compute the probability of the event “left of $A$” happens, we would have to take the conjunction of all those elementary events, which once again are not directly observable.

The previous discussion motivates the idea of a random variable, a very important tool in modeling experimental outcomes. Intuitively, random variables (r.v.) are mathematical representations of outcomes of an experiment which may be stochastic, such as the outcomes of the firefly box, which are only “left on $A$,” “right on $A$,” “left on $B$,” or “right on $B$,” abbreviated by $L_A$, $R_A$, $L_B$, and $R_B$, respectively. Random variables model this experiment in the following way.

We start with a probability space, whose elementary events in $\Omega$ are sampled according to $p$. For this probability space, we choose functions $A : \Omega \rightarrow \{-1,1\}$ and $B : \Omega \rightarrow \{-1,1\}$ such that for a random sampling of elementary events $\omega \in \Omega$ following $p$, the probabilities of the outcomes $A(\omega) = -1$, $A(\omega) = 1$, $B(\omega) = -1$, and $B(\omega) = 1$, (which are given by respectively $p(A = -1)$, $p(A = 1)$, $p(B = -1)$, and $p(B = 1)$), are the same as the probabilities of observing “left on $A$”, “right on $A$”, “left on $B$”, “right on $B$”, respectively. In other words, what random variables do is set a partition on $\Omega$ such that each element of this partition (which is in $\mathcal{F}$) corresponds to an outcome of the experiment with the same probabilistic features. Thus, a discrete random variable is formally a function $\Omega \rightarrow E$ from the probability space to a certain set $E$ of possible values. For our example above, the random variables $A$ and $B$ are $\pm 1$-valued, with $E = \{-1,1\}$.

The **expected value** of a random variable $R$ or, for short, the **expectation of $R$**, on a probability space $(\Omega, \mathcal{F}, p)$, denoted $E(R)$, is defined as

$$E(R) = \sum_{\omega \in \Omega} p(\{\omega\}) R(\omega).$$

For two random variables $R$ and $S$, their **moment** is defined as the expectation of their product, $E(RS)$, and for three random variables $R$, $S$, and $T$, their **triple moment** is defined as the expectation of the triple product, $E(RST)$. Higher moments, are defined in the same way, as product expectations of four or more random variables.

So, the question is whether we can create random variables $A$ and $B$ that model the firefly box. What we mean here is whether there exists a probability space $(\Omega, \mathcal{F}, p)$ and discrete random variables on this space such that all statistical characteristics of the outcomes of observations of the box are the same as the statistical characteristics of the random variables. For example, if we observe “left on $A$” 50% of the time, then it must be the case that $E(A) = 0$ for a r.v. $A$ taking values $\pm 1$. Notice however that because we only observe $A$ or $B$, we cannot know what the value of the second moment $E(AB)$ is, and any probability spaces and r.v.’s on them satisfying the observed marginals $E(A)$
and $E(B)$ would be adequate. It is easy to prove that for this firefly box, we can always find a $(\Omega, \mathcal{F}, p)$ consistent with all observed marginals.\footnote{For instance, we can just choose a $(\Omega, \mathcal{F}, p)$ such that $A$ and $B$ are statistically independent, i.e. $E(AB) = 0$, since the moment is not observable by construction.}

However, a common probability space does not always exist for r.v’s representing a collection of properties that cannot all be observed simultaneously. To see this, let us consider a slightly more complicated firefly example. Imagine a box, shown in Figure 2 where we can observe not only $A$ and $B$, but also the top, $C$. The outcomes of an observation will modeled by ±1-valued random variables, $A$, $B$, and $C$, corresponding to which side of the cube’s face glows (marked in the figure with $+$ and $-$). It is clear that there is a one-one correspondence between the region inside the cube and what values the random variables take if the firefly blinks. For example, the faces of the cube divide it naturally into octants, and if the firefly is in one octant, the value of $A$, $B$, and $C$ will be determined. We can think of the firefly blinking in one octant as corresponding to an elementary event in the sample space of a probability space $(\Omega, \mathcal{F}, p)$, and we can label them according to the values of $A$, $B$, and $C$, i.e. $\Omega = \{\omega_{abc}, \omega_{abc}, \omega_{abc}, \omega_{abc}, \omega_{abc}, \omega_{abc}, \omega_{abc}, \omega_{abc}\}$, where we use the notation that the subscripts correspond to the outcome of the random variables, with the barred ones being $-1$ and the other $+1$ (e.g., $\omega_{abc}$ corresponds to the octant where $A = 1$, $B = -1$, and $C = 1$).

Let us further assume that, like the two-sided box of Figure 1, we cannot observe all three sides at the same time, but only two. This means that we do not only have access to the values of $E(A)$, $E(B)$, and $E(C)$, but also to their second moments, $E(AB)$, $E(BC)$, and $E(AC)$, (which together with the individual expectations fully determine the joint distribution of each pair of random variables). It is easy to see that if we start with the above sample space, we can impose constraints on the values of the moments. To see this, consider the following table:
Given that the table holds for individual values, the expected value for each of the columns\(^3\) are simply a convex combination of their values (which weights for this convex combination depends on the particular values of the observed expectations). An immediate consequence is that for the probability space given, the moments must be always such that

\[-1 \leq E(AB) + E(AC) + E(BC),\]

which is the right-hand-side of the Suppes-Zanotti inequalities \([53]\). In other words, if there is a probability space that describes all the moments for \(A\), \(B\), and \(C\), then inequality (1) must be satisfied.

Here we point out that violations of inequality \([53]\) correspond to violations of logical consistency, as indicated by Abramsky and Hardy \([3]\). To violate \([53]\), we need in the convex combination of elements at least some events that lead to values on the right column that are less than -1. One such element, for example, is \(E(AB) = E(AC) = E(BC) = -1\), which adds up to -3. For these moments, if \(A = 1\), then \(E(AB)\) implies \(B = -1\), and from \(E(BC)\) it follows that \(C = 1\), which finally leads, from \(E(AC)\), to \(A = -1\), a contradiction. The contradiction comes from the assumption that, say, the random variable \(A\) in the experiment that measures \(E(AB)\) is the same as the ones in the experiment \(E(AC)\). However, as we will see, this is not the case, and outcomes of experiments can depend on contexts.

To show this let us tweak the firefly example. As we mentioned, the box in Figure 2 is designed such that one can only observe two sides at a time. This could be done by having some mechanism attached to the box that prevents the observer to see what happens on one of the sides. Let us now connect the mechanism that selects which sides we can observe to a biasing mechanism inside the box. This biasing mechanism turns on (inside the box) little devices that release at random times\(^4\) pheromones that attract the firefly. If we place those pheromone-releasing devices in the right place, we can rig the box such that we have higher probabilities of finding the firefly only in certain octants. Furthermore, by a careful choice of octants, we can have it built such that the

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\(^3\)The quantity \(AB + AC + BC\) is itself a random variable.

\(^4\)But with expected time intervals that are of the same order of the expected period in between blinks for the firefly.
second moment of, say, \(A\) and \(B\), is close to \(-1\). Because the pheromone-releasing mechanism is connected to the side-selection mechanism, we can also make it change when we decide to observe \(B\) and \(C\) or \(A\) and \(C\), such that their second moment is also \(-1\). Of course, \(E(AB) = E(AC) = E(BC) \approx -1\) violates (1).

What is happening in the previous example is simple: inequalities (1) are violated because each observational setting, i.e. the decision of which two variables to observe, corresponds to a different experimental condition. This is because the choice of observing \(A\) and \(B\) instead of any other pair changes the places where the pheromones are being released. In other words, the probability space \((\Omega, \mathcal{F}, p)\) assumes that the values of the random variables \(A\) in the experiment with \(B\) are compatible with \(A\) in the experiment with \(C\). But such compatibility is impossible. To illustrate this in a different way, let us examine what happens to the octants as we impose the moments \(E(AB) = E(AC) = E(BC) = -1\).

As we saw above, the sample space \(\Omega\) is represented in terms of the octants, one for each elementary event in \(\Omega = \{\omega_{abc}, \omega_{a\overline{b}c}, \omega_{a\overline{b}\overline{c}}, \omega_{ab\overline{c}}, \omega_{a\overline{b}\overline{c}}, \omega_{ab\overline{c}}, \omega_{a\overline{b}c}, \omega_{a\overline{b}\overline{c}}\}\). To reproduce the \(E(AB) = -1\) observation (when \(C\) is not observable), the firefly would need to be in the two regions denoted by the two prisms on the left hand side of Figure 3. The further constraint that \(E(AC) = -1\) leads to a smaller region of the sample space, corresponding to only two cubes (center top on Figure 3). But that implies that \(E(BC) = 1\), and there are no points in the sample space that correspond to \(E(AB) = E(AC) = E(BC) = -1\) (this, by the way, is straightforward from our table above, and is also shown pictorially in Figure 3). Therefore, the regions where the firefly is depends on which sides
of the box you are observing. This characteristic is called *contextuality*.

Thus, contextuality, for us, can be stated in the following way. If a set of random variables, measured under different experimental conditions and never all at the same time, cannot be represented as partitions of a joint probability distribution, then they are contextual. From our example it should be clear that the nonexistence of a joint probability distribution for contextual random variables was not based on taking a coarse-grained probability space over the firefly’s path. But to make it explicit, we can notice that any probability space that reproduces the outcomes of A, B, and C must have as part of its algebra the elements $p_{abc}, \ldots, p_{abc'd'}$, and therefore it cannot have a proper probability distribution over it.

Following the Contextuality-by-Default (CbD) approach [28, 25, 44], one should assume that different experiments (e.g., observing (A, B), observing (B, C), and observing (A, C) in the above example) are stochastically unrelated and therefore modeled on distinct probability spaces. Indeed, only one experiment can be performed at a time so there is no pairing-scheme to justify defining the random variables of different experiments on the same probability space.

Thus, indexing properties by subscripts $i = 1, \ldots, M$ and different contexts by superscripts $j = 1, \ldots, N$, let us model the result of observing property $i$ in context $j$ by the random variable $R_{ij} : \Omega_j \rightarrow E_i$, where $R_{ij}$ with different $i$ but same $j$ are all jointly distributed but $R_{ij}$ and $R_{i'j'}$ for $j \neq j'$ are stochastically unrelated for all $i, i' \in \{1, \ldots, n\}$, equal or not. For keeping the notation uncluttered, we assume that when property $i$ does not appear in context $j$, the expression $R_{ij}$ is undefined and left out from any enumerations. Using this convention, we denote by $R^j = \{R_{ij} : i = 1, \ldots, m\}$ the jointly distributed set of random variables modeling the measurement of all properties appearing in context $j$.

**Definition 1.** A collection of random variables $R_{ij}$ is said to be *consistently connected* if $R_{ij} \sim R_{i'j'}$ for all $i \in \{1, \ldots, m\}$ and all contexts $j, j' \in \{1, \ldots, n\}$ in which the property $i$ appears (here we use the notation $A \sim B$ to signify that “$A$ has the same distribution as $B$”). If a system is not consistently connected, it is said to be *inconsistently connected*.

Intuitively, *consistently connected* means that one cannot find differences between a random variable in one context and another by solely observing this random variable. For example, if we observe A in the context of B, and we then observe A in the context of C, no scrutiny of the distribution or values of A can tell us which context it was observed in if the random variables are consistently connected.

Random variables that are not consistently connected are obviously context-dependent. As we will see in Section 4 below, many of the examples in physics and psychology are context-dependent because of being inconsistently connected.

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5This, by the way, is related to the no-signaling condition in physics.
However, it is still possible for a system of random variables to present contextuality even if they are consistently connected, and we need to distinguish those cases. This is the essence of the following definitions.

**Definition 2.** A *coupling* of random variables $X_1, \ldots, X_n$ (that may be defined on different probability spaces) is any jointly distributed set of random variables $Z_1, \ldots, Z_n$ such that $X_i \sim Z_i, \ldots, X_n \sim Z_n$.

Intuitively, a coupling imposes a joint distribution on a set of random variables and hence formalizes the concept of finding a common sample space for a set of random variables. Thus, we can define the traditional understanding of (non-)contextuality in a mathematically rigorous form as follows.

**Definition 3.** A collection of random variables $R^j_i$ is non-contextual if and only if there exists a coupling $Q^1, \ldots, Q^n$ of $R_1, \ldots, R_n$ such that $Q^j_i = Q^j_i$ for all $i \in \{1, \ldots, m\}$ and all contexts $j, j' \in \{1, \ldots, n\}$ in which the property $i$ appears.

Definition 3 only holds for consistently connected systems, as it requires, as its consequence, that $R^j_i \sim R^j_i$ for all $i \in \{1, \ldots, m\}$ and all contexts $j, j' \in \{1, \ldots, n\}$ in which property $i$ appears. If the coupling of Definition 3 exists, we can denote $Q_i = Q^j_i = Q^j_i = \ldots$ for all $i \in \{1, \ldots, m\}$ and all contexts $j, j', \ldots \in \{1, \ldots, n\}$ in which the property $i$ appears. We can think intuitively of the distributions of $R^j_i$'s as observable marginal distributions of the hypothetical larger system $Q_1, \ldots, Q_m$. Thus, the collection of r.v.'s is non-contextual if it is possible to “sew” the observed marginal probabilities together to produce a larger probability distribution over the whole set of properties [27, 24, 18, 22].

As mentioned, in Quantum Mechanics, and perhaps Psychology, it may not be possible to do that, but in many cases the marginal probabilities are compatible with a signed joint probability distribution of $Q_1, \ldots, Q_m$.

We are now left with the following three situations for collections of r.v.’s measured under different contexts: they are non-contextual (i.e., they can be imposed on a proper joint probability distribution in which r.v.’s representing the same property are always equal); the random variables are contextual and consistently connected (in the next section, we show that they can then be imposed on a signed joint distribution); and they are inconsistently connected.

For inconsistently connected systems, things are a little more subtle, and since NP cannot yet deal with them, we refer to the works of Dzhafarov and Kujala [44, 43].

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6Here we should add a comment on terminology. In the CbD approach, the definition of noncontextuality is extended [25, 23] to inconsistently connected systems by allowing $Q^j_i = Q^j_i = \ldots$ to not hold as long as the probability of it holding is in a certain well-defined sense maximal for each $i$. This allows one to detect contextuality on top of inconsistent connectedness and so in the most recent terminology of CbD, a system can be inconsistently connected and yet not contextual. For the present paper, since we are mostly focusing on the NP approach which does not apply to inconsistently connected systems, it suffices to use the traditional understanding of contextuality.
3. Negative Probabilities

As we mentioned in the previous section, contextuality appears not only in psychology but also in physical systems. In this Section we describe one possible approach to describing contextuality: Negative Probabilities (NP). NP are a generalization of standard Kolmogorovian probabilities to accommodate the observations performed in a system that behaves in a contextual way.

It is important at this point to understand why changing the theory of probability is a desirable approach. When we have context-dependent r.v.’s, such as the A, B, and C in the firefly box, one could argue that the nonexistence of a joint probability distribution comes from a mistake: the identification of a random variable, say A, in two different contexts (e.g. (A, B) and (A, C) as being the same). This is clearly what is happening, and the solution to it is, following Dzhafarov and Kujala’s Contextuality-by-Default approach, to clearly label each r.v. according to its context. However, there are cases when such distinction may not highlight important non-trivial features of a system. One such case is the famous Bell-EPR experiment, which we describe below. For this experiment, because the experiments that measure, e.g., A and B should not interfere with each other, for physical reasons, it makes no sense to label them differently. However, the Bell-EPR system is contextual, and using the same label brings this contextuality to the surface in a very dramatic way. Therefore an extended probability theory may help shed light in some of those contextual cases.

Because contextuality is equivalent to the non-existence of a joint probability distribution (see Proposition 4 below) for a collection of random variables, some proposals for dealing with contextual systems are to simply change the theory of probability. This is what was done in quantum mechanics, where the complementarity principle, whereupon some variables were forbidden in principle to be observed simultaneously, opened up the need to describe such contextual systems with the formalism of measures over Hilbert spaces. However, a question in QM is what are the principles behind such a specific generalized probability theory? Why do we use Hilbert spaces? These questions form an important topic of research, and are yet unanswered. Similarly, these questions can also be asked for psychology. Why is the quantum formalism adequate to model psychological systems? Up to now, it seems that all arguments about using the quantum formalism are related to specific examples that form a subset of those in physics, since they all involve inconsistently connected systems.

So, instead of using quantum probabilities, as do researchers in quantum cognition, we propose a more general framework given by negative probabilities. Our definition of NP is a straightforward generalization of Kolmogorov’s probability. A discrete signed probability space is given by a triple $(Ω, F, p)$ with the same components as a proper probability space, except that the function $p : Ω → \mathbb{R}$ is allowed to attain negative values and values larger than 1, as long as it still satisfies $p(Ω) = 1$. The probability of an event $E \in F$ is still calculated as $p(S) = \sum_{\omega \in S} p(\{\omega\})$, like in proper probability spaces, and random variables and expectations are defined analogously to those of proper probability spaces.
Let us motivate the above definition. First, we notice that, in Definition\(^3\) contextuality was defined as the impossibility to impose a joint distribution on \(R_1, \ldots, R^n\) such that the stochastically unrelated random variables representing the same property are always equal in this joint. However, for consistently connected systems, it turns out it is always possible to find such a joint on a signed probability space:

**Proposition 4.** For a collection \(R^j_i\) of discrete r.v.’s on finite sample spaces, the following are equivalent

1. there exists on a signed probability space jointly distributed r.v.’s \(Q_1, \ldots, Q_m\) such that for all \(j = 1, \ldots, n\), it holds \(\{R^j_i, R^j_i\} \sim \{Q_i, Q_i\}\) where \(i, i', \ldots\) are the properties appearing in context \(j\). (Here “\(\sim\)” is taken to refer to the joint distributions of the two sets of r.v.’s).
2. the collection \(R^j_i\) is consistently connected.

**Proof.** See [1, 5, 46]. \(\square\)

**Definition 5.** Let \(R^j_i\) be a consistently connected collection of r.v.’s. Then, the minimum \(L_1\) probability norm, denoted \(M^*\), or simply minimum probability norm, is given by \(M^* = \min \sum_{\omega \in \Omega} |p(\{\omega\})|\), where the minimization is over all signed probability spaces \((\Omega, \mathcal{F}, p)\) and r.v.’s \(Q_1, \ldots, Q_n\) on it that satisfy condition 1 of Proposition 4.

From this definition it is easy to prove the following:

**Proposition 6.** A consistently connected collection of r.v.’s is non-contextual if and only if \(M^* = 1\).

**Proof.** See [22]. \(\square\)

If follows that since \(M^*\) can be greater than one for contextual systems, and that the greater the value of \(M^*\) the further away from a proper probability distribution it lies (due to the strong relationships imposed, e.g., by the moments of the random variables), it is natural to interpret \(M^*\) as a measure of contextuality: the larger the value of \(M^*\), the more contextual the system [18].

From Proposition 4 it follows that inconsistently connected systems of random variables cannot be described with negative probabilities. Here we are left with only one possibility. If, for a system of random variables, some of them are not consistently connected, then we need to face the fact that they are not the same random variable, and label them accordingly, following the prescription of Contextuality-by-Default [28, 26].

We end this Section with some comments about the meaning of NP. One of the main obstacles to the use of NP is the lack of an interpretation. After all, what meaning should we give to them? If probabilities are, as in some objective views, given by relative frequencies of actual realizable events, how can we even consider a probability to be negative?

First, we should point out that NP are not directly observable, but only inferrable. For instance, in the firefly box, negative probabilities appear exactly
because we cannot observe all three sides simultaneously (or know where the firefly is). Were we able to observe all three simultaneously, then a joint probability distribution would necessarily exist. With the observable moments, any attempt to create probabilities that have marginals consistent with the moments lead to NP.

Our second point is that NP may be useful in certain applications. For example, in physics an important question is what are the physical principles that define QM. NP may be an adequate tool to help us understand those principles. We will not explore this application of NP here, but the interested reader is referred to [46, 47].

That said, there are ways to interpret NP, even consistently with a frequentist interpretation. Here we will briefly sketch how some of those interpretations work, but the interested reader should refer to the cited references. We start with Andrei Khrennikov’s $p$-adic interpretation. Khrennikov [34, 35, 36, 37, 38, 39] showed that for the frequentist interpretation proposed by von Mises, where probabilities are defined as the convergent ratio of infinite sequences, NP appear in sequences where the usual Archimedian metric does not converge to a specific value (i.e., sequences not satisfying the principle of stabilization). When Archimedian metrics do not converge, $p$-adic metrics may do so, and in those cases NP appear as the $p$-adic limiting case. In other words, Khrennikov interprets infinite sequences that do not satisfy the principle of stabilization as arising from contextuality, and describable by negative probabilities. However, the relationship between NP and observations, in this interpretation, is not straightforward, as it depends on the particular $p$-adic metric chosen.

Another interpretation of NP, also frequentist, is the one proposed by Abramsky and Brandenburger [1, 2]. They use, in the context of sheaf theory, the concept that events may have two different types that may annihilate each other. In most circumstances, when quantities are observed, no events are annihilated; however, when there are context-dependent observables, they are context-dependent because each context determines a different interaction between the observables through their annihilation.

Finally there is Szekely’s “half-coin” interpretation of NP [50, 54]. The idea is that two probability distributions that are negative may give rise to a non-negative proper probability distribution. In this interpretation, negative probabilities $P$ are related to a proper probability $p$ via a convolution equation $P \ast p_- = p_+$, which is always possible to be found [50, 54]. This convolution means that for a random variable $X$ whose (negative) probability distribution is $P$, there exists two other random variables, $X_+$ and $X_-$ with proper probability distributions ($p_+$ and $p_-$, respectively) and such that $X = X_+ - X_-$. As one can see, this interpretation is closely related to that of Abramsky and Brandenburger.

In this paper we favor a more pragmatic “interpretation.” Negative probabilities are taken here to be simply an accounting tool, one that provides us the best subjective information about systems which do not have an objective probability distribution, as it is the closest distribution to a proper one (via normalization of the L1 norm). This is analogous to the use of negative num-
bers in mathematics, which was considered by many absurd. For example, the famous mathematician Augustus De Morgan wrote the following about negative numbers [45, pg. 72].

“Above all, he [the student] must reject the definition still sometimes given of the quantity $-a$, that it is less than nothing. It is astonishing that the human intellect should ever have tolerated such an absurdity as the idea of a quantity less than nothing; above all, that the notion should have outlived the belief in judicial astrology and the existence of witches, either of which is ten thousand times more possible.”

However, nowadays we understand that negative numbers can be a useful bookkeeping device. For example, when tracking a store inventory, one would not be overly concerned about something such as “$-30$ rolls of toilet paper” in our spreadsheet, and equate such a line to “the existence of witches.” We approach NP the same way, asking whether it can be a useful device that may not only help us in computations but also give us further insights in some situations, as mentioned above. But, as De Morgan, we consider a statement such as “event A has probability $-0.1$” on equal terms with judicial astrology.

4. Some examples and applications

Let us now examine some examples of contextual systems, and how they can be described (or not) with negative probabilities. We already gave an example of a contextual system above, with the $A, B, C$ random variables from the three-sided firefly box. Here we will look at examples from physics, in particular Quantum Mechanics, and then move to psychology.

Perhaps the most important example of contextuality in physics is the double-slit experiment, as it contains in its essence the complementarity principle. So, here we start this section with this experiment, but in a simplified version given by the Mach-Zehnder interferometer, which captures the main features of complementarity. We then re-examine the firefly box in more details, showing how negative probabilities can model some of its outcomes. Next, we review the first example where contextuality was recognized as playing a key role in Quantum Mechanics, the famous Kochen-Specker theorem [41]. Finally, as a last physics example, we investigate the Bell-EPR with negative probabilities. We then move to the contextual cases in quantum cognition, and we discuss how those are related to the different physics cases shown before. We end this section with a discussion of negative probabilities as a possible way to measure the contextuality of an observable system.

\[\text{For some of our physics examples, we assume that the reader is familiar with the mathematical formalism of QM. Readers not familiar with it may wish to skip the details, since they do not affect the overall understanding of this paper, or may refer to the many available texts on this subject (e.g. [12] or [45]).}\]
Figure 4: Mach Zehnder Interferometer (MZI). A source $S$ sends a particle beam that impinges on the first beam splitter $BS$. The beam is then divided by $BS$ into equal-intensity (i.e., particle numbers, on average) beams that travel to both arms (paths) $A$ and $B$ of the interferometer, reflecting on surfaces $M_A$ and $M_B$. The beams from arms $A$ and $B$ are then recombined in the second beam splitter. The outcomes are the two beams detected at $D_1$ and $D_2$.

4.1. Interference experiments

In the double-slit experiment, a particle impinges on a solid barrier that has on it two small and parallel slits close to each other. The particle has a probability of passing through the slits, later on being detected on a scintillating screen. Contextuality in this experiment appears as a manifestation of the wave/particle duality: the places where the screen scintillates depend on whether we know any which-path information for the particle, i.e., whether the particle went through one slit and not the other (see [30] for a detailed discussion of the double slit).

Since the detection rates of an observed event depend on the context, it is immediate that the double-slit experiment exhibits trivial contextuality. It thus follows that it cannot be described using negative probabilities. However, due to its importance in many applications of the quantum formalism to psychology, we present a brief discussion of it here in a simplified form. To do so, we use a conceptually similar setup where the slits and the screen are replaced by the Mach-Zehnder interferometer shown in Figure (see Figure 4; for a more detailed discussion of the MZI, see references [22] [21]). Beam splitters send the particles into two directions in a random way, such that if we place a particle detector after each of the outputs of the beam splitter, we will see that in the long run the number of particles going to one side approaches that of going to the other side. After the first beam splitter, some reflecting surfaces redirect the beams to another beam splitter, and the beams are recombined. From QM, the whole system can be described mathematically by a wavefunction.

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Except if one makes special counterfactual assumptions, as common in certain physics experiments [19] [22].
and the recombination of the two beams in the second beam splitter leads to an interference effect. Careful positioning of the beam splitters and reflecting surfaces allow for perfect interference, namely all particles reaching $D_1$ and none reaching $D_2$.

What makes the MZI interesting is that the placement of a detector on either path $A$ or path $B$ causes a collapse of the wavefunction, thus changing the outcomes of a measurement of $D_1$ and $D_2$: they now have the same probability of detecting a particle. However, let us recall that if no detectors are placed on $A$ or $B$, the particle has zero probability of reaching $D_2$. Furthermore, if we simply block one of the paths, say by putting a barrier in $A$, half of the particles going through $B$ will reach $D_2$. This is seemingly disturbing, for how can we increase the probability of detection of $D_2$ when we actually decrease the number of ways in which the particle can reach $D_2$? This is the main difficulty of the double-slit experiment.

To see that the observations of $D_1$ and $D_2$ are contextual, let $P$ and $D$ be two $\pm 1$-valued random variables representing which-path information and detection: $P = 1$ if the particle is detected on $A$ and $P = -1$ otherwise, $D = 1$ if the particle is detected in $D_1$ and $D = -1$ otherwise. The MZI has two contexts: there is a detector on $A$ or $B$, providing which-path information, or no detector. $D$ measured under the no-which-path context has expected value $E(D) = 1$, whereas a joint measurement of $D$ and $P$ gives as marginal expectation the result $E(D) = 0$. Thus, according to the above definition, $D$ is inconsistently connected.

So, since it is inconsistently connected, how would we model the MZI with NP? The fact that $D$ changes when measured with $P$ or not leads to the necessity of defining two different random variables, $D$ and $D_P$, where $D_P$ is simply the representation of the detectors under the which-path information context. Clearly we can always write down a joint probability distribution for $P$, $D$, and $D_P$, but then there is no contextuality in this system, and no need for negative probabilities.

4.2. Three-sided Firefly Box

For the three-sided firefly box of Figure 2 Suppes and Zanotti proved a necessary and sufficient condition for the existence of a joint probability distribution, namely that $A$, $B$, and $C$ need to satisfy the following inequalities:

$$-1 \leq E(AB) + E(BC) + E(AC) \leq 1 + 2 \min \{E(AB), E(BC), E(AC)\}.$$  \hfill (2)

This example actually shows up in physics, and a weaker form of inequalities \cite{3} are known in the physics literature as the Leggett-Garg inequalities. If we restrict our variables to consistently connected systems, it is straightforward to compute a (negative) joint probability distribution consistent with expectations.
violating (2). In fact, imagine we have the following moments

\[
E(AB) = \epsilon_1, \\
E(BC) = \epsilon_2, \\
E(AC) = \epsilon_3.
\]

To make the computations simpler, let us also assume that \(E(A) = E(B) = E(C) = 0\). Then, we can construct a (negative) probability space \((\Omega, \mathcal{F}, P)\) with \(\Omega = \{\omega_{abc}, \omega_{a\overline{bc}}, \omega_{\overline{a}bc}, \omega_{ab\overline{c}}, \omega_{a\overline{b}c}, \omega_{\overline{a}b\overline{c}}, \omega_{ab\overline{c}}\}\), and a \(P\) satisfying the above marginals is given by

\[
\begin{align*}
p(\omega_{abc}) &= \frac{1}{4} (1 + \epsilon_1 + \epsilon_2 + \epsilon_3) - \alpha, \\
p(\omega_{a\overline{bc}}) &= \frac{1}{4} (\alpha - \epsilon_1 - \epsilon_2), \\
p(\omega_{\overline{a}bc}) &= \frac{1}{8} (1 + \epsilon_1 - \epsilon_2 + \epsilon_3), \\
p(\omega_{ab\overline{c}}) &= \frac{1}{8} (1 + \epsilon_1 - \epsilon_2 - \epsilon_3), \\
p(\omega_{a\overline{b}c}) &= \frac{1}{4} (1 + \epsilon_3) - \alpha, \\
p(\omega_{\overline{a}b\overline{c}}) &= \frac{1}{8} (1 - \epsilon_1 + \epsilon_2 - \epsilon_3), \\
p(\omega_{ab\overline{c}}) &= \frac{1}{8} (1 + \epsilon_1 - \epsilon_2 - \epsilon_3), \\
p(\omega_{a\overline{b}\overline{c}}) &= \alpha,
\end{align*}
\]

where \(\alpha\) is a free parameter that takes a range of values given by the moments \(\epsilon_1, \epsilon_2,\) and \(\epsilon_3\) and by the minimization of the L1 norm. Notice that if the moments violate (2), then some of the probabilities above will be negative, regardless of the values of \(\alpha\), as we should expect.

To see what further information negative probabilities may provide, we follow an example from [15, 20]. Imagine a decision-maker, Deana, who wants to invest in stocks. She considers three companies, A, B, and C, about which she knows nothing. In a wise move, Deana hires three “experts,” Alice, Bob, and Carlos, to provide her with information about the companies. However, each expert is specialized only in two of the companies, but not in all (e.g. Alice knows a lot about A and B, but nothing about C). Imagine now that the \(\pm 1\)-valued random variables, \(A, B,\) and \(C\), are supposed to model the experts’ beliefs of a stock value going up if \(+1\) and down if \(-1\) whenever asked about it. We assume that our experts’ opinions about each company A, B, or C are consistently connected, i.e. they all agree about the expectations of \(A, B,\) and \(C\). To make it simple for our toy example, we set

\[
E(A) = E(B) = E(C) = 0. \tag{3}
\]

Since Alice only knows about A and B, she can add to (3) information about
the second moment, and she claims

$$E_A(AB) = -1,$$  

(4)

where we use the subscript $A$ to remind us that (4) corresponds to Alice’s subjective belief.

Equation (4) has a simple interpretation: Alice believes that if the value of $A$ goes up, $B$ will certainly go down and vice versa. Bob’s and Carlos’s beliefs are that

$$E_B(AC) = -\frac{1}{2},$$  

(5)

and

$$E_C(BC) = 0.$$  

(6)

It is easy to see from (2) that (4)–(6) do not have a proper joint probability distribution. However, because the random variables are consistently connected, there exists a negative probability distribution consistent with (3)–(6).

What can Deana do with her inconsistent expert information? The only unknown to Deanna, in a certain sense, is the triple moment. The minimization of the L1 norm provides a range of possible values for the triple moments, namely, for the above expectations,

$$-\frac{1}{2} \leq E(XYZ) \leq \frac{1}{2}.$$ 

So, NP provide a range of possible values for the triple moment that could be thought as the most reasonable range, given that the minimization of L1 puts the negative measure as close to a proper probability distribution as possible.

4.3. Kochen-Specker Theorem

Very early on, a heated discussion in the foundations of quantum mechanics was whether the process of measuring a quantum system revealed the actual value of a property or created it. To answer this question, Kochen and Specker asked whether it was possible to assign values 0 or 1 to a set of quantum properties (corresponding to projection operators, the quantum equivalent to yes/no measurements). If measurements revealed a property, then this assignment of 0 and 1 values should be possible, but Kochen and Specker showed this was not the case. To do so, they used 117 projection operators (projectors). However, a simpler proof with only 18 projectors in a four dimensional Hilbert space exists, and that form is followed here. Let $P_i$ be a collection of projectors, and let $V_i$ be ±1-valued random variables taking values $-1$ or $+1$ depending on whether the property $P_i$ is false or true, respectively. Since $P_i$ is determined uniquely by a vector in the Hilbert space, we use this vector as the index $i$ for the projector and the random variable. Consider the following set

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9Our example is not easily translatable into objective probabilities, but one could devise a situation where certain biases on the experts’ sides could increase their assessment of second moments, thus recreating the moments we use.
of equations, guaranteed to be satisfied by the algebra of the chosen projection operators.

\[ V_{0,0,0,1}V_{0,0,1,0}V_{1,1,0,0}V_{1,-1,0,0} = -1, \quad (7) \]
\[ V_{0,0,0,1}V_{0,1,0,0}V_{1,0,1,0}V_{1,0,1,0} = -1, \quad (8) \]
\[ V_{1,-1,1,-1}V_{1,-1,0,1}V_{1,1,0,0}V_{0,0,1,1} = -1, \quad (9) \]
\[ V_{1,-1,1,-1}V_{1,1,1,1}V_{1,0,0,0}V_{0,1,0,0} = -1, \quad (10) \]
\[ V_{0,0,1,0}V_{0,1,0,0}V_{1,0,1,0}V_{1,1,0,1} = -1, \quad (11) \]
\[ V_{1,-1,1,1}V_{1,1,1,1}V_{1,0,0,1}V_{0,1,0,1} = -1, \quad (12) \]
\[ V_{1,1,-1,1}V_{1,1,1,1}V_{1,-1,0,0}V_{0,0,1,1} = -1, \quad (13) \]
\[ V_{1,1,-1,1}V_{-1,1,1,1}V_{1,0,1,0}V_{0,1,0,1} = -1, \quad (14) \]
\[ V_{1,1,1,1}V_{-1,1,1,1}V_{1,0,0,1}V_{0,1,-1,0} = -1, \quad (15) \]

A quick examination will reveal that the r.v.’s on each line correspond to a set of commuting projectors. Because the \( P_i \) in each line are orthogonal, only one of the \( V_i \)’s in each line can be true at a time, and therefore the product of them must be \(-1\). The commutation of observables for each line means that each corresponding random variable can be measured simultaneously, though this is not true for all random variables in different lines. We can think of each line as representing a particular context for the experiment.

We can multiply the left hand side of (7)-(15), and because each variable appears twice, their product must be one (since \( V_i^2 = 1 \), because it is a \( \pm 1 \)-valued random variable). However, if we multiply the right hand side of (7)-(15), their product is \(-1\), and we reach a contradiction. The contradiction comes from assuming that the random variable (say, \( V_{0,0,0,1} \)) in one experimental context (i.e., measured with \( V_{0,0,1,0}, V_{1,1,0,0}, V_{1,-1,0,0} \)) is the same as the random variable in a different context (i.e., \( V_{0,0,0,1} \) in the context \( V_{0,1,0,0}, V_{1,0,1,0}, V_{1,0,-1,0} \)). Since each of the value combinations for the \( V_i \)’s correspond to an \( \omega \) in a (course-grained) probability space, it follows that there is no joint probability distribution underlying it. Therefore the algebra of observables in Quantum Mechanics is contextual.

It is worth mentioning that the lack of a joint probability for the above example is a consequence of the algebra of observables being state independent. What this means is that for any system describable by a four-dimensional Hilbert space we will reach the above contradiction, regardless of how this system was initially prepared. Assuming consistent connectedness (i.e. that the marginal expectations (\( V_i \)) match between contexts), it is possible to find a negative probability distribution that describes this system. However, such distributions are quite large, consisting of signed probabilities for \( 2^{18} = 262,144 \) elementary events.

4.4. Bell-EPR non-local contextuality

Perhaps the most celebrated example of contextuality in QM is the Bell-EPR thought experiment, which we present here in terms of random variables. In
Bell-EPR experiment. A source emits two photons, one toward Alice’s lab and another toward Bob’s. Each experimenter can make a decision on which direction of spin to measure, represented in the figure by the settings $A$ and $A'$ for Alice and $B$ and $B'$ for Bob. Outcomes of measurements are $\pm 1$, with equal probabilities.

This experiment, two spin-$1/2$ particles $A$ and $B$ are emitted by a source and go to opposite directions, where Alice and Bob measure them (see Figure 5). One of the possible states that can be prepared for such a source is

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|+\rangle - |-\rangle),$$

(16)

where $|+\rangle$ ($|-\rangle$) corresponds to $A$ having spin polarization “$+1$” (“$-1$”) and $B$ “$-1$” (“$+1$”) in the $z$ direction. It is clear that their spin $z$ is negatively correlated, as a “$+1$” or “$-1$” outcomes for particle $A$ will result in the same for particle $B$. Therefore, if Alice measures $A$’s spin in the $z$ direction, then Bob’s measurement in the same direction is moot: his experimental outcomes are already determined by Alice’s. Einstein, Podolsky, and Rosen used this example to argue that QM is incomplete: if we can know something (they called it “element of reality”) about particle $B$ without affecting it (since we measured $A$, whose measurement may be separated by spacelike interval from Bob’s own measurement), then the assumption in QM that the vector $|\psi\rangle$ is a complete description of its physical state is incorrect [29].

We can argue that there is still no mystery with QM up to now, but just an argument that QM should be incomplete. Einstein’s proposal was to search for a more complete theory (often called a hidden-variable theory), whereas Bohr defended that no such theory could be satisfactorily produced. However, things become more interesting when, following Bell, we use angles that are different from simply measuring vertical polarization (e.g. combinations of other directions). In the 1960’s, John Bell showed that (local) hidden-variable theories were incompatible with the predictions of QM. Stating in the formalism we put forth, Bell showed that if QM is correct, then some random-variables variables representing the outcomes of spacelike-separated experiments are contextual.

About a decade later, Aspect, Grangier, and Gérard [6] provided the first evidence that QM was correct, and recent (loophole-free) experiments seem to corroborate their conclusions [32].

Bell’s result comes out of the construction of a simple random variable $S$ defined as

$$S = AB + A'B + AB' - A'B',$$

(17)

10The choice of the $z$ direction is arbitrary. For simplicity we use units where $\hbar/2 = 1$. 
where $A$, $A'$, $B$, and $B'$ are $\pm 1$-valued random variables corresponding to outcomes of experiments for Alice and Bob, with the prime denoting different spin-measurement angles. It is straightforward to check that $S$ can take values $-2$ or $2$ (to verify this, one can make a table with all 16 possible values for $A$, $A'$, $B$, and $B'$ and compute $S$). Therefore, the expected value of $S$ must be between $-2$ and $2$, and we obtain the Clauser-Horne-Shimony-Holt (CHSH) inequalities (permutations of the minus sign gives you the other ones) \[11\]

$$-2 \leq \langle AB \rangle + \langle A'B' \rangle + \langle AB' \rangle - \langle A'B \rangle \leq 2,$$  

where here we introduce a shorter standard notation for expectation, i.e. $E(\cdot) = \langle \cdot \rangle$. As in the above example for the firefly box, if (18) is violated, there is no joint probability distribution, and the system of random variables is contextual.

The Bell-EPR setup differs significantly from the Kochen-Specker. The random variables in Bell-EPR are necessarily consistently connected. If they were not, it would be possible to used EPR-type correlated systems to communicate superluminally: a choice of measurement direction by Alice would instantly affect the mean value of Bob’s measurements, and she could use entangled particles to communicate with Bob. This would be incompatible with the causal structure of special relativity, and would require a complete rethinking of relativistic physics. Thus, the absence of a joint probability distribution comes from the (non-trivial) correlations imposed by the experimental outcomes (through the values of the moments). But, more importantly, the Bell-EPR case provides a situation where two parts of a system are correlated in ways that cannot be explained by the existence of a common cause (hidden-variable) because they are contextual. This is particularly disturbing to the physicist because those two parts may be arbitrarily far away from each other, and the events that are correlated may be spacelike separated. A striking way to see how this is difficult to understand is if we look at a firefly box-like construction for the Bell-EPR variables. We will not attempt to do this here, as it would be lengthy, but we refer the interested reader to an interesting paper by Blasiak \[7\].

Once again, a general solution may be obtained for the joint moments in (18), and we have the following (maybe negative, depending on whether (18) is
violated or not) joint probability distribution:

\[
p(\omega_{aa'bb'}) = \frac{1}{4} (\langle AB \rangle + \langle A'B \rangle + \langle AB' \rangle + \langle A'B' \rangle) + \alpha_3 + \alpha_4 - \alpha_7
\]

\[
p(\omega_{aa'bb'}) = \frac{1}{4} (\langle AB \rangle + \langle A'B \rangle) + \alpha_3 + \alpha_4 - \alpha_7,
\]

\[
p(\omega_{aa'tbb'}) = \alpha_7,
\]

\[
p(\omega_{aa'tbb'}) = -\frac{1}{4} (\langle AB \rangle + \langle A'B \rangle - \langle AB' \rangle - \langle A'B' \rangle) + \alpha_2 - \alpha_3 + \alpha_7,
\]

\[
p(\omega_{aa'tbb'}) = -\frac{1}{4} (\langle A'B' \rangle) + \alpha_1 + \alpha_3 - \alpha_7,
\]

\[
p(\omega_{aa'bb'}) = \frac{1}{4} (1 - \langle A'B \rangle) - \alpha_3 - \alpha_4 - \alpha_6,
\]

\[
p(\omega_{aa'bb'}) = \alpha_6,
\]

\[
p(\omega_{aa'bb'}) = \frac{1}{4} (\langle A'B' \rangle - \langle A'B' \rangle) - \alpha_2 + \alpha_3 + \alpha_6,
\]

\[
p(\omega_{aa'bb'}) = \frac{1}{4} (1 + \langle A'B' \rangle) - \alpha_1 - \alpha_3 - \alpha_6,
\]

\[
p(\omega_{aa'tbb'}) = -\frac{1}{4} (\langle AB \rangle + \langle AB' \rangle) + \alpha_1 - \alpha_4 + \alpha_5,
\]

\[
p(\omega_{aa'tbb'}) = \frac{1}{4} (1 + \langle AB' \rangle) - \alpha_1 - \alpha_3 - \alpha_5,
\]

\[
p(\omega_{aa'tbb'}) = \frac{1}{4} (1 + \langle AB \rangle) - \alpha_1 - \alpha_2 - \alpha_5,
\]

\[
p(\omega_{aa'tbb'}) = \alpha_5,
\]

\[
p(\omega_{aa'tbb'}) = \alpha_4,
\]

\[
p(\omega_{aa'tbb'}) = \alpha_3,
\]

\[
p(\omega_{aa'tbb'}) = \alpha_2,
\]

\[
p(\omega_{aa'tbb'}) = \alpha_1,
\]

where \(\alpha_i\) are free parameters. Once again, the ranges of \(\alpha_i\) depend on the values of the moments, but one point is relevant here. On the \(A, B, C\) example, we had only one free parameter, while here we have seven. The reason is that in the Bell-EPR setup, only the four individual expectations and four moments are given, and together with the requirement that \(\sum p(\omega_i) = 1\) this amounts to 9 equations for sixteen elementary events, thus it is a more underdetermined case.

### 4.5. Quantum contextuality in psychology

We did not try to give an exhaustive list of all contextual systems in QM, but mainly those which provide further conceptual understanding of the difficulties faced by physicists trying to understand quantum theory. We present those examples to provide a background for the discussion of contextuality in psychology, a theme that is at the core of current attempts to use the mathematics
of QM to model cognition. Once again, we will not try to give an exhaustive account of all different contextual cases, and the interested reader is referred to Busemeyer and Bruza’s book [9]. Here we briefly examine a few cases that exemplify quantum-like contexts in psychology: violations of the sure-thing-principle [4] [41] in decision making and order effects [56].

Savage’s Sure-Thing-Principle was stated the following way [51, pg. 21]:

“A businessman contemplates buying a certain piece of property. He considers the outcome of the next presidential election relevant to the attractiveness of the purchase. So, to clarify the matter for himself, he asks whether he should buy if he knew that the Republican candidate were going to win, and decides that he would do so. Similarly, he considers whether he would buy if he knew that the Democratic candidate were going to win, and again finds that he would do so. Seeing that he would buy in either event, he decides that he should buy, even though he does not know which event obtains, or will obtain, as we would ordinarily say. It is all too seldom that a decision can be arrived at on the basis of the principle used by this businessman, but, except possibly for the assumption of simple ordering, I know of no other extralogical principle governing decisions that finds such ready acceptance.”

For example, imagine you have \( B \) and \( P \) as a \( \pm 1 \)-valued random variables corresponding to “not buy” (\( B = -1 \)) or “buy” (\( B = +1 \)), and “Republican president” (\( P = -1 \)) or “Democrat president” (\( P = +1 \)). The STP corresponds to the probabilistic statement that

\[
P(B = 1) = P(B = 1|P = 1) P(P = 1) + P(B = 1|P = -1) P(P = -1) \\
\geq P(B = 1|P = 1) P(P = 1) + P(B = -1|P = -1) P(P = -1) \\
= P(B = -1),
\]

if

\[
P(B = 1|P = 1) \geq P(B = -1|P = 1)
\]

and

\[
P(B = 1|P = 1) \geq P(B = -1|P = 1).
\]

Tversky and Shafir showed that human decision makers often do not follow the STP [52, 55]. Since STP follows in a straightforward way from the axioms of probability theory, violations of STP by human decision makers imply they do not follow those axioms themselves, but perhaps some type of generalized probability theory. Such generalized probability theory, as some have proposed, is the one given by probabilities defined over an orthomodular lattice resulting from measures over a Hilbert space, i.e., quantum probabilities [8].

For example, the STP can be given by a quantum description of the MZI paradigm [21]. In the MZI, where which path information causes a collapse of the wave function, therefore changing the probability distributions of the outcomes.
of the experiment. So, if we use the analogy that in the MZI the responses “buy” or “not buy” correspond to detectors at the end of the interferometer, and the which-path information corresponding to “Republican president” or “Democrat president,” the collapse of the wave function would change the distributions of “buy” or “not buy” depending on the context of knowing which is president, similar to the Tversky and Shafir’s experiments. Therefore, violations of STP show a clear case of contextuality. However, it is obviously trivial contextuality, since the “measurement” of which-path creates a direct change in the expectation values of the “buy”/”not buy” random variable.

We now turn to order effects. Order effects are well-known in quantum systems, where successive measurements of incompatible quantities (e.g. spin in two orthogonal directions) give different results depending on the order. Recently, in a model similar to the quantum model for the MZI, Wang et al. [56], showed that not only can quantum models correctly reproduce the observed order effect of outcomes of many different experiments, but they can also predict a non-trivial relation for the order effect: the QQ equality. This equality, which holds exactly for the quantum formalism, seems to also hold with good fit for most order effect experiments investigated by Wang et al., a surprising finding, since it seems the QQ equality cannot be derived in any straightforward way from other approaches. However, as in the STP example, the random variables are inconsistently connected.

5. Final remarks

In this paper we described negative probabilities, and showed how they can be used to describe some contextual systems. We tried to show in the examples some of the cases where negative probabilities work well, but also those where no clear approach with negative probabilities exist (i.e. for inconsistently connected systems). Our goal was to provide a different approach to contextual systems than the formalism of Quantum Mechanics, one that may perhaps be useful in quantum cognition. The advantage of NP is that it can model not only those situations where QM is applied, but it is also more general.

As an example, let us think about the three-sided firefly box. In QM, if we have three observables that can be observed simultaneously in pairs, it follows that they can also be observed all together. This is a characteristic of the Hilbert space formalism, and can be easily demonstrated (see [13, 16]). However, it is also possible to show that, under certain reasonable assumptions, one should expect a neural stimulus-response model to be able to reproduce the types of correlations that we find in the three random variable case, where no joint probability distribution exists [14, 20]. Thus, we are left with the possibility of a plausible contextual system that is forbidden by the quantum formalism and that can easily be described by NP, as we saw in Section [1]. Additionally, as mentioned earlier, there are many surprising theorems in QM that seem to have no counterparts in psychology or social sciences, and a more general contextual theory of probabilities might be advantageous.
We should point out that despite all the discussions about contextuality in
social systems, recently Dzhafarov, Zhang, and Kujala analyzed many psychol-
ogy experiments, and found no evidence of non-trivial contextuality \[23\]. This
means that more subtle examples, such as the firefly box or systems equivalent
to the Bell-EPR where contextuality comes from the correlations and not from
inconsistently connected random variables, were not found. Their analysis was
made using the apparatus of Contextuality-by-Default, an approach that is more
general than the NP. This does not mean that NP are not necessarily useful in
the social sciences, but it seems that up to now attempts to find non-trivially
contextual systems have failed.

As we saw in the examples, as well as in the discussions that followed Propo-
sition \[6\], the minimum value of the L1 norm, $M^*$, can be interpreted as a mea-
sure of contextuality \[17\]. This is connected to standard views in QM, where
the values of $\langle S \rangle$ (equation \[17\]) are taken as a measure of departure from loc-
ality for Bell-EPR systems, with higher values of $\langle S \rangle$ corresponding to more
non-local systems (therefore more contextual). This is also true for the three
random variable system $A$, $B$, and $C$, where $M^*$ is associated to the expecta-
tion of $AB + BC + AC$ present in equation \[2\]. It would be interesting to see
how $M^*$ compares to other measures of contextuality, namely the one given by
Contextuality-by-Default, for more complex systems, and whether interesting
classifications can arise from different measures of contextuality.

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