ALEXANDROV-FENCHEL INEQUALITY FOR CONVEX HYPERSURFACES WITH CAPILLARY BOUNDARY IN A BALL

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Abstract. In this paper, we first introduce the quermassintegrals for convex hypersurfaces with capillary boundary in the unit Euclidean ball $B^{n+1}$ and derive its first variational formula. Then by using a locally constrained nonlinear curvature flow, which preserves the $n$-th quermassintegral and non-decreases the $k$-th quermassintegral, we obtain the Alexandrov-Fenchel inequality for convex hypersurfaces with capillary boundary in $B^{n+1}$. This generalizes the result in [42] for convex hypersurfaces with free boundary in $B^{n+1}$.

1. INTRODUCTION

Isoperimetric inequality is one of the fundamental topics in differential geometry. The classical isoperimetric inequality in the Euclidean space says that among all bounded domains in $\mathbb{R}^{n+1}$ with fixed enclosed volume, the minimum of the area functional is achieved by round balls. The higher order generalizations of isoperimetric inequality in convex geometry are the Alexandrov-Fenchel inequalities for quermassintegrals, which say that for closed convex hypersurfaces $\Sigma \subset \mathbb{R}^{n+1}$, it holds that

$$\int_{\Sigma} H_k dA \geq \omega_n^{\frac{k-1}{n-1}} \left( \int_{\Sigma} H_l dA \right)^{\frac{n-k}{n-l}},$$

with equality holding on the round spheres. Here $H_k$ is the normalized $k$-th mean curvature of $\Sigma$ and $\omega_n$ denotes the surface area of $S^n$. See [8, 38, 39] for instance.

Fix a domain $B$ in $\mathbb{R}^{n+1}$, whose boundary $\partial B$ is named support hypersurface, the partitioning problem or the relative isoperimetric problem is to find hypersurfaces $\Sigma$ in $B$ with least area functional among all hypersurfaces with divide $B$ into two components with prescribed ratio. When $B$ is $\mathbb{B}^{n+1}$, the unit ball in $\mathbb{R}^{n+1}$, the solution to this problem is given by all spherical caps or totally geodesic balls with free boundary, cf. [6, 7, 49]. Here a hypersurface with free boundary means that it intersects with $\partial \mathbb{B}^{n+1} = S^n$ orthogonally. The higher order generalization of relative isoperimetric inequality – the relative Alexandrov-Fenchel inequalities in $\mathbb{B}^{n+1}$ have been considered by Scheuer-Wang-Xia in [42]. They gave the definition of quermassintegral in $\mathbb{B}^{n+1}$ from the viewpoint of the first variational formula, and they proved the highest order Alexandrov-Fenchel inequalities and the Gauss-Bonnet-Chern formula for convex hypersurfaces with free boundary in $\mathbb{B}^{n+1}$. Willmore type geometric inequalities for convex hypersurface with free boundary in a ball have been obtained in [30, 42].

Motivated by the study of the equilibrium shapes of a liquid confined in a given container, there is a more general relative isoperimetric problem in a fixed domain $B$, where the energy functional

$$|\partial \Omega \cap B| - \cos \theta |\partial B \cap \Omega|$$

is involved. Here $\Omega \subset B$ and $\theta \in (0, \pi)$, $|\partial B \cap \Omega|$ is the so-called wetting energy. For more physical background, we refer to Finn’s book [17]. It is known that among

Key words and phrases. Alexandrov-Fenchel inequalities, quermassintegral, capillary boundary, relative isoperimetric inequality.

2010 Mathematics Subject Classification. 53C21, 53C44, 35K96, 52A40.
all hypersurfaces which divide $\mathbb{B}^{n+1}$ into two components with prescribed energy, the ones with least energy functional are all spherical caps or totally geodesic balls which intersects at the constant angle $\theta$, cf. [49]. We shall call such boundary condition the $\theta$-capillary boundary.

In the same spirit of [42], we are interested in finding the quermassintegral for convex hypersurfaces with $\theta$-capillary boundary in $\mathbb{B}^{n+1}$, which should be the right higher order generalization of the energy functional. Moreover, in this paper, we shall study the Alexandrov-Fenchel inequalities and the Gauss-Bonnet-Chern formula for such hypersurfaces.

Let $n \geq 2$. Let $\mathbb{B}^n$ and $\mathbb{B}^{n+1}$ be the $n$-dimensional and $(n+1)$-dimensional open Euclidean unit ball centered at the origin, respectively. Let $S^n = \partial \mathbb{B}^{n+1}$ and $\hat{N}$ denote the unit outward normal of $S^n$. Let $\Sigma \subset \mathbb{B}^{n+1}$ be a smooth, orientable, compact hypersurface with boundary, given by an isometric embedding $x : \mathbb{B}^n \to \mathbb{B}^{n+1}$. $\Sigma$ is said to be properly embedded, if

$$\Sigma = x(\mathbb{B}^n) \subset \mathbb{B}^{n+1}, \quad \partial \Sigma = x(\partial \mathbb{B}^n) \subset S^n.$$ 

We are concerned with the case when $\Sigma$ is a convex hypersurface with capillary boundary. We choose $\nu$ to be the unit normal of $\Sigma$ such that $\Sigma$ is convex in the sense that the second fundamental form defined by $h(X,Y) = -\langle D_X Y, \nu \rangle$ is non-negative definite, where $D$ is the Euclidean connection.

**Definition 1.1.** For $\theta \in (0, \pi)$, $\Sigma$ is said to be with a $\theta$-capillary boundary, if $\Sigma$ intersects $S^n$ at the constant angle $\theta$, that is,

$$\langle \nu, \hat{N} \circ x \rangle = -\cos \theta, \text{ along } \partial \mathbb{B}^n.$$ 

In particular, if $\theta = \frac{\pi}{2}$, or equivalently say that $\Sigma$ intersects $\partial \mathbb{B}^{n+1}$ orthogonally, we say that $\Sigma$ is with free boundary.

The model examples of properly embedded hypersurfaces with a $\theta$-capillary boundary are the spherical cap of radius $r \in (0, \infty)$ around some constant unit vector field $e$ in $\mathbb{R}^{n+1}$ with $\theta$-capillary boundary, given by

$$C_{\theta,r}(e) := \{ x \in \mathbb{B}^{n+1} : |x - \sqrt{r^2 + 2r \cos \theta + 1}e| = r \},$$

and the flat ball around $e \in S^n$ with $\theta$-capillary boundary, given by

$$C_{\theta,\infty}(e) := \{ x \in \mathbb{B}^{n+1} : \langle x,e \rangle = \cos \theta \}. $$

We shall drop the argument $e$, in cases where it is not relevant.

For our purpose, we always assume $\Sigma$ is a convex hypersurface with $\theta$-capillary boundary for $\theta \in (0, \frac{\pi}{2}]$. Then $\partial \Sigma \subset S^n$ is a strictly convex hypersurface in $S^n$ (see Proposition 2.5) and it bounds a strictly convex body in $S^n$, which we denote by $\hat{\Sigma}$, cf. [13, 22]. Denote by $\hat{\Sigma}$ the domain enclosed by $\Sigma$ in $\mathbb{B}^{n+1}$ which contains $\partial \Sigma$. Let $\sigma_k$ denote the $k$-th elementary symmetric polynomial, evaluated at the principal curvatures of $\Sigma$, and

$$H_k = \frac{1}{(k)} \sigma_k, \quad 1 \leq k \leq n.$$ 

We make the convention that

$$\sigma_0 = H_0 = 1 \quad \sigma_{n+1} = H_{n+1} = 0.$$ 

Now we define the following geometric functionals for convex hypersurfaces with $\theta$-capillary boundary in $\mathbb{B}^{n+1}$ as follows, which we expect to be the correct counterparts to the quermassintegrals for closed convex hypersurfaces in $\mathbb{R}^{n+1}$:

$$W_{0,\theta}(\hat{\Sigma}) := |\hat{\Sigma}|,$$

$$W_{1,\theta}(\hat{\Sigma}) := \frac{1}{n+1} \left( |\Sigma| - \cos \theta W_0^{S^n}(\partial \Sigma) \right), \quad (1.4)$$
\[ W_{n+1,\theta}(\Sigma) := \frac{1}{n+1} \left[ \int_{\Sigma} H_n dA - \sum_{l=0}^{n-1} (-1)^{n+l} \binom{n}{l} \cos^{n-1-l} \theta \sin^{l} \theta \int_{\Sigma} W_{l}^{S^n}(\partial \Sigma) \right] \quad (1.5) \]

and for \( 1 \leq k \leq n - 1 \),

\[ W_{k+1,\theta}(\Sigma) := \frac{1}{n+1} \left\{ \int_{\Sigma} H_k dA - \cos \theta \sin^k \theta \int_{\Sigma} W_k^{S^n}(\partial \Sigma) \right. \]
\[ - \sum_{l=0}^{k-1} \left( -\frac{1}{n-l} \right)^{k+l} \left( \frac{k}{l} \right) \left[ (n-k) \cos^2 \theta + (k-l) \right] \cos^{k-1-l} \theta \sin^{l} \theta \int_{\Sigma} W_{l}^{S^n}(\partial \Sigma) \right\}. \quad (1.6) \]

Here for a \( k \)-dimensional submanifold \( M \subset \mathbb{R}^{n+1} \) (with or without boundary), \(|M|\) always denotes the \( k \)-dimensional Hausdorff measure of \( M \). \( W_k^{S^n} \) denotes the \( k \)-th quermassintegral of the closed convex hypersurface \( \partial \Sigma \subset \mathbb{S}^n \), see Section 2.4 for more discussion. We remark that \( W_{1,\theta}(\Sigma) \) is just a multiple of the energy functional (1.1) we mentioned before, or see e.g. \([37, 46, 49]\). When \( \theta = \frac{\pi}{2} \), \( W_{k,\theta}(\Sigma) \) is exactly the quermassintegrals defined in \([42]\).

The reason to define \( W_{k+1,\theta} \) to be the quermassintegral for convex hypersurfaces with \( \theta \)-capillary boundary in \( \mathbb{H}^{n+1} \) is the following first variational formula.

**Theorem 1.2.** Let \( \Sigma_t \subset \mathbb{H}^{n+1} \) be a family of smooth, properly embedded hypersurfaces with \( \theta \)-capillary boundary, given by \( x(t) : \mathbb{H}^n \to \mathbb{H}^{n+1} \), such that

\[(\partial_t x)^\perp = f \nu,\]

for some normal speed function \( f \). Then for \( 1 \leq k \leq n \),

\[ \frac{d}{dt} W_{k,\theta}(\Sigma_t) = \frac{n+1-k}{n+1} \int_{\Sigma_t} H_k f dA_t, \quad (1.7) \]

and

\[ \frac{d}{dt} W_{n+1,\theta}(\Sigma_t) = 0. \]

We obtain the following Alexandrov-Fenchel type inequalities and Gauss-Bonnet-Chern formula for convex hypersurfaces with \( \theta \)-capillary boundary in \( \mathbb{H}^{n+1} \).

**Theorem 1.3.** Let \( \theta \in (0, \frac{\pi}{2}) \) and \( \Sigma \subset \mathbb{H}^{n+1} \) be a convex hypersurface with \( \theta \)-capillary boundary. Then

\[ W_{n+1,\theta}(\Sigma) = \frac{\omega_n}{2(n+1)} I_{\sin^2 \theta} \left( \frac{n}{2} \right) \left( \frac{1}{2} \right), \quad (1.8) \]

where \( I_{\sin^2 \theta} \left( \frac{n}{2} \right) \) is the regularized incomplete beta function given by

\[ I_{\sin^2 \theta} \left( \frac{n}{2} \right) = \frac{\int_0^1 t^{\frac{n}{2}-1} (1-t)^{-\frac{n}{2}} dt}{\int_0^1 t^{\frac{n}{2}-1} (1-t)^{-\frac{n}{2}} dt}. \]

We also have that for \( 0 \leq k \leq n - 1 \),

\[ W_{n,\theta}(\Sigma) \geq (f_n \circ f_k^{-1})(W_{k,\theta}(\Sigma)), \quad (1.9) \]

where \( f_k := f_k(r) \) is the strictly increasing real function given by

\[ f_k(r) := W_{k,\theta}(C_{\theta, r}), \]

where \( C_{\theta, r} \) is the spherical cap given by (1.2). Moreover, equality holds if and only if \( \Sigma \) is a spherical cap or a flat ball with \( \theta \)-capillary boundary.

In particular, for \( n = 2 \), we obtain a Minkowski type inequality for convex surfaces in \( \mathbb{H}^3 \) with \( \theta \)-capillary boundary.
Corollary 1.4. Let \( \theta \in (0, \frac{\pi}{2}) \) and \( \Sigma \subset \mathbb{B}^3 \) be a convex surface with \( \theta \)-capillary boundary. Then
\[
3W_{3,\theta}(\hat{\Sigma}) = \int_{\Sigma} H_2 dA - \cos \theta |\partial \Sigma| + \sin \theta |\partial \Sigma| = 2\pi (1 - \cos \theta),
\]
and
\[
W_{2,\theta}(\hat{\Sigma}) = \frac{1}{6} \left( \int_{\Sigma} H_2 dA - \sin \theta \cos \theta |\partial \Sigma| + (1 + \cos^2 \theta) |\partial \Sigma| \right) \geq \left( f_2 \circ f_1^{-1} \right) \left( \frac{1}{3} |\Sigma| - \cos \theta |\partial \Sigma| \right). \tag{1.10}
\]
Moreover, equality holds if and only if \( \Sigma \) is a spherical cap or a flat disk with \( \theta \)-capillary boundary.

We remark that, when \( \theta = \frac{\pi}{2} \), Theorems 1.2 and 1.3 have been proved in [42]. The method of our proof follows the classical method for proving geometric inequalities by employing monotonicity properties along and convergence of a suitable curvature flow. We will use the locally constrained inverse type curvature flow with capillary boundary in this paper to achieve the inequality (1.9) in Theorem 1.3. Such locally constrained flow has been first considered by Guan-Li [24], and used in [5, 11, 24, 25, 26, 28] for closed hypersurfaces in space forms, and in [42, 50] for hypersurfaces with free boundary in balls to prove various geometric inequalities. The Minkowski formulas play an essential role to design these flows. We refer to [5, 42, 43] for more description to such locally constrained flows. We also refer to [2, 3, 12, 14, 19, 27, 31, 33, 48] and references therein for the studying of the Alexandrov-Fenchel type inequalities for closed hypersurfaces in space forms which used various types of curvature flow. In addition to the curvature flow approach, there are also many other interesting methods to study the Alexandrov-Fenchel inequalities for closed hypersurfaces, see e.g. [9, 10, 34, 35, 44]. Besides, we could establish the Alexandrov-Fenchel inequalities for capillary hypersurfaces in the half-space in a forthcoming paper [47].

For our purpose, we first generalize the Minkowski formula for hypersurfaces with capillary boundary in [49, 46] to higher order mean curvatures,
\[
\int_{\Sigma} H_{k-1} (\langle x, e \rangle + \cos \theta \langle \nu, e \rangle) dA = \int_{\Sigma} H_k \langle X_e, \nu \rangle dA, \tag{1.11}
\]
where \( e \) is a constant unit vector field in \( \mathbb{R}^{n+1} \) and \( X_e \) is defined in (2.3), which is a conformal Killing vector field parallel to \( S^n \). Then we consider the flow
\[
\begin{align*}
\partial_t x(\cdot, t) &= f(\cdot, t) \nu(\cdot, t) + T(\cdot, t), \quad &\text{in } \mathbb{B}^n \times [0, T),
\langle \nu(\cdot, t), N \circ x(\cdot, t) \rangle &= -\cos \theta \quad &\text{on } \partial \mathbb{B}^n \times [0, T),
\end{align*} \tag{1.12}
\]
where \( T(\cdot, t) \) is the tangential component of \( \partial_t x \) and
\[
f = \frac{\langle x + \cos \theta \nu, e \rangle}{m\sigma_n/\sigma_{n-1}} - \langle X_e, \nu \rangle.
\]
By the first variational formula (1.7) and (1.11), it is easy to see that under flow (1.12), \( W_{n,\theta}(\hat{\Sigma}) \) is preserved and \( W_{k,\theta}(\hat{\Sigma}) \), \( 1 \leq k \leq n - 1 \) is non-decreasing.

The remaining task is to prove the convergence of the flow to a spherical cap with capillary boundary. For this aim, we need to make a choice of \( e \), which depends only on the initial hypersurface \( x(\mathbb{B}^n, 0) = \Sigma_0 \), so that \( \langle x + \cos \theta \nu, e \rangle > 0 \) for \( \Sigma_0 \), due to the strict convexity and its \( \theta \)-capillary boundary of \( \Sigma_0 \). This will be done in Proposition 2.16. Under this choice, since we have barriers for the flow hypersurfaces, given by spherical caps around \( e \) with \( \theta \)-capillary boundary, we are able to prove that under the flow, \( \langle x + \cos \theta \nu, e \rangle \) is always bounded below by a uniform constant. On the other hand, we can prove the quantity \( \langle X_e, \nu \rangle \) is bounded below by a uniform constant, which enable us to write the flow equation to be a
This article is structured as follows. In Section 2, we give some preliminaries for \( \theta \)-capillary hypersurfaces, and prove a new Minkowski type formula (2.7). Besides, we introduce the definition of quermassintegral for hypersurfaces with \( \theta \)-capillary boundary and derive its first variational formula, i.e. Theorem 1.2. In Section 3, we introduce a locally constrained type curvature flow (3.1), derive the evolution equations for various geometric quantities, and obtain the uniform curvature estimates for convex hypersurfaces with \( \theta \)-capillary boundary in a ball along this flow, which follows the long-time existence and convergence of our flow (3.1). The last part is devoted to prove the Alexandrov-Fenchel inequalities for convex hypersurfaces with \( \theta \)-capillary boundary, i.e. Theorem 1.3.

2. Convex hypersurfaces with \( \theta \)-capillary boundary

2.1. Notation and preliminaries. Let \( \Sigma \subset \mathbb{R}^{n+1} \) be a smooth embedded, orientable hypersurface. Denote \( D \) be the Levi-Civita connection of \( \mathbb{R}^{n+1} \) with respect to the Euclidean metric \( \delta \), and \( \nabla \) be the Levi-Civita connection on \( \Sigma \) with respect to induced metric \( g \) from the embedding \( x \). We denote \( \text{div}, \Delta, \nabla^2 \) be the divergence, Laplacian, Hessian operator on \( \Sigma \) respectively. The second fundamental form of \( x \) is given by the Gaussian formula

\[
D_X Y = \nabla_X Y - h(X, Y)\nu.
\]

The Weingarten operator is defined via

\[
g(W(X), Y) = h(X, Y),
\]

and the Weingarten equation says that

\[
\nabla_X \nu = W(X).
\]

The Gauss-Codazzi equation says that

\[
Rm(X, Y, Z, W) = h(Y, Z)h(X, W) - h(Y, W)h(X, Z),
\]

\[
(\nabla_Z h)(X, Y) = (\nabla_Y h)(X, Z).
\]

where our convention for the Riemannian curvature tensor \( Rm \) is

\[
Rm(X, Y, Z, W) = g(Rm(X, Y)Z, W) = g(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z, W).
\]

Remark 2.1. We will simplify the notation by using the following shortcuts occasionally:

1. When dealing with complicated evolution equations of tensors, we will use a local frame to express tensors with the help of their components, i.e. for a tensor field \( T \in T^{k,l}(\Sigma) \), the expression \( T^{i_1...i_k}_{j_1...j_l} \) denotes

\[
T^{i_1...i_k}_{j_1...j_l} = T(e_{j_1}, \ldots, e_{j_l}, e^{i_1}, \ldots, e^{i_k}),
\]

where \( (e_i) \) is a local frame and \( (e^i) \) its dual coframe.

2. The coordinate expression for the \( m \)-th covariant derivative of a \( (k,l) \)-tensor field \( T \) is

\[
\nabla^m T = \left( T^{i_1...i_k}_{j_1...j_l;j_{l+1}...j_{l+m}} \right),
\]

where indices appearing after the semi-scolon denote the covariant derivatives.
(3) We shall use the convention of Einstein summation. For convenience the components of the Weingarten map $W$ are denoted by $(h^i_j) = (g^{ik}h_{kj})$, and $|h|^2$ be the norm square of the second fundamental form, that is $|h|^2 = g^{ik}h_{kj}g^{jl}$, where $(g^{ij})$ is the inverse of $(g_{ij})$. We use the metric tensor $(g_{ij})$ and its inverse $(g^{ij})$ to lower down and raise up the indices of tensor fields on $\Sigma$.

Let $\sigma_k(\kappa), 1 \leq k \leq n$, be the $k$-th elementary symmetric polynomial for $\kappa \in \mathbb{R}^n$ and $H_k(\kappa)$ be its normalization $H_k(\kappa) = \frac{1}{\kappa_k!} \sigma_k(\kappa)$. Denote by $\sigma_k(\kappa|i)$ the symmetric function $\sigma_k(\kappa)$ with $\kappa_i = 0$. We shall use the following basic properties in this paper.

**Proposition 2.2.**

1. $\sigma_k(\kappa) = \sigma_k(\kappa|i) + \kappa_i \sigma_{k-1}(\kappa|i)$, $\forall 1 \leq i \leq n$.
2. $\sum_{i=1}^{n} \sigma_k(\kappa|i) = (n-k)\sigma_k(\kappa)$.
3. $\sum_{i=1}^{n} \kappa_i \sigma_{k-1}(\kappa|i) = k\sigma_k(\kappa)$.
4. $\sum_{i=1}^{n} \kappa_i^2 \sigma_{k-1}(\kappa|i) = \sigma_1(\kappa)\sigma_k(\kappa) - (k+1)\sigma_{k+1}(\kappa)$.

**Proposition 2.3.** For $1 \leq k < l \leq n$, for $\kappa \in \Gamma_+ := \{ \kappa \in \mathbb{R}^n : \kappa_i > 0, 1 \leq i \leq n \}$, we have

$$H_kH_l \geq H_{k-1}H_{l+1},$$

with equality holding if and only if $\kappa = c(1, \ldots, 1)$ for some $c > 0$.

**Proposition 2.4.** For $1 \leq k \leq n$, $F(\kappa) = \frac{\sigma_k(\kappa)}{\sigma_{k-1}(\kappa)}$ is concave in $\Gamma_+$.

For a proof of Propositions 2.2 - 2.4, one can refer to [32, Chapter XV, Section 4] and [40, Lemma 2.10, Theorem 2.11] respectively.

2.2. Basic properties for $\theta$-capillary hypersurfaces.

Let $\Sigma \subset \mathbb{B}^{n+1}$ be a smooth, properly embedded, orientable hypersurfaces with $\theta$-capillary boundary, given by the embedding $x : \mathbb{S}^n \to \mathbb{B}^{n+1}$. Let $\mu$ be the unit outward co-normal of $\partial \Sigma$ in $\Sigma$ and $\nu$ be the unit normal to $\partial \Sigma$ in $\mathbb{S}^n$ such that $\{\nu, \mu\}$ and $\{\overline{\nu}, \overline{\mu}\}$ have the same orientation in the normal bundle of $\partial \Sigma \subset \mathbb{B}^{n+1}$. From Definition 1.1, it follows that

$$\begin{aligned}
\overline{\nu} = \cos \theta \mu + \sin \theta \nu,
\nu = \sin \theta \mu - \cos \theta \nu.
\end{aligned}$$

$\partial \Sigma$ can be viewed as a smooth closed hypersurface in $\mathbb{S}^n$, which bounds $\overline{\partial \Sigma}$ inside $\mathbb{S}^n$. By our convention, $\overline{\nu}$ is the unit outward normal of $\partial \Sigma$ in $\overline{\partial \Sigma} \subset \mathbb{S}^n$. The second fundamental form of $\partial \Sigma$ in $\mathbb{S}^n$ is given by

$$\overline{h}(X,Y) := -\langle \nabla_X^\mathbb{S} Y, \overline{\nu} \rangle = -\langle D_X Y, \nu \rangle, \quad X,Y \in T(\partial \Sigma).$$

The second equality holds since $\langle \nu, \overline{\nu} \rangle = 0$. The second fundamental form of $\partial \Sigma$ in $\Sigma$ is given by

$$h(X,Y) := -\langle \nabla_X Y, \mu \rangle = -\langle D_X Y, \mu \rangle, \quad X,Y \in T(\partial \Sigma).$$

The second equality holds since $\langle \nu, \mu \rangle = 0$.

**Proposition 2.5.** Let $\Sigma \subset \mathbb{B}^{n+1}$ be with $\theta$-capillary boundary. Let $\{e_\alpha\}_{\alpha=1}^{n+1}$ be an orthonormal frame of $\partial \Sigma$. Then along $\partial \Sigma$,

1. $\mu$ is a principal direction of $\Sigma$, that is, $h(\mu, e_\alpha) = 0$.
2. $\overline{h}_{\alpha\beta} = \sin \theta h_{\alpha\beta} - \cos \theta \delta_{\alpha\beta}$. 
(3) \( \tilde{h}_{\alpha \beta} = \cos \theta \hat{h}_{\alpha \beta} + \sin \theta \delta_{\alpha \beta} = \frac{1}{\sin \theta} \delta_{\alpha \beta} + \cot \theta h_{\alpha \beta} \).

(4) \( h_{\alpha \beta ; \mu} = \hat{h}_{\beta \gamma} (h_{\mu \mu} \delta_{\alpha \gamma} - h_{\alpha \gamma}) \).

Proof. The first assertion is well-known, cf. [49, Proposition 2.1]. For (2), we see
\[
h_{\alpha \beta} = -\langle D e_{\alpha} e_{\beta}, \nu \rangle = \langle \hat{h}_{\alpha \beta} \nu + \delta_{\alpha \beta} N, \nu \rangle = \sin \theta \hat{h}_{\alpha \beta} - \cos \theta \delta_{\alpha \beta}.
\]

For (3), we see
\[
\tilde{h}_{\alpha \beta} = -\langle D e_{\alpha} e_{\beta}, \mu \rangle = \langle \hat{h}_{\alpha \beta} \nu + \delta_{\alpha \beta} N, \mu \rangle = \cos \theta \hat{h}_{\alpha \beta} + \sin \theta \delta_{\alpha \beta}.
\]

For (4), by taking derivative of \( h(\mu, e_{\alpha}) = 0 \) with respect to \( e_{\beta} \), using the Codazzi equation and the fact (1), we get
\[
0 = e_{\beta}(h(\mu, e_{\alpha})) = h_{\alpha \mu ; \beta} + h(\nabla_{e_{\beta}} e_{\alpha}, \mu) + h(\nabla_{e_{\beta}} \mu, e_{\alpha}) = h_{\alpha \beta ; \mu} + \langle \nabla_{e_{\beta}} e_{\alpha}, \mu \rangle h_{\mu \mu} + \langle \nabla_{e_{\beta}} \mu, e_{\gamma} \rangle h_{\alpha \gamma} = h_{\alpha \beta ; \mu} - \hat{h}_{\beta \gamma} (h_{\mu \mu} \delta_{\alpha \gamma} - h_{\alpha \gamma}).
\]

Corollary 2.6. Let \( \Sigma \subset \mathbb{E}^{n+1} \) be with \( \theta \)-capillary boundary for \( \theta \in (0, \frac{\pi}{2}] \). If \( \Sigma \) is convex (strictly convex resp.), in the sense \( h \) is nonnegative definite (positive definite resp.), then \( \partial \Sigma \) is convex (strictly convex resp.) in both \( S^n \) and \( \Sigma \).

Proof. It follows directly from Proposition 2.5 (2) and (3) that \( \hat{h} \) and \( \tilde{h} \) are non-negative definite (positive definite resp.) provided \( h \) is so.

2.3. Minkowski formula.

For a constant unit vector field \( e \in \mathbb{R}^{n+1} \), denote
\[
X_e := \langle x, e \rangle x - \frac{1}{2} (|x|^2 + 1) e.
\]

(2.3)

It has been observed in [49, Proposition 3.1] that \( X_e \) is a conformal Killing vector field such that
\[
\mathcal{L}_{X_e} \delta = \langle x, e \rangle \delta,
\]

and
\[
\langle X_e, N \rangle = 0 \text{ along } S^n.
\]

By using such property of \( X_e \), the following Minkowski type formula has been proved in [49, Proposition 3.2]: for a properly embedded hypersurface in \( \mathbb{E}^{n+1} \) with \( \theta \)-capillary boundary,
\[
n \int_{\Sigma} (\langle x, e \rangle + \cos \theta \langle \nu, e \rangle) dA = \int_{\Sigma} H(X_e, \nu) dA.
\]

(2.5)

Remark 2.7. From a direct calculation, we see that \( C_{\theta, r}(e) \) satisfies
\[
\langle x, e \rangle + \cos \theta \langle \nu, e \rangle = \frac{1}{r} \langle X_e, \nu \rangle.
\]

(2.6)

For our purpose, we generalize (2.5) to higher order Minkowski type formulas.

Proposition 2.8. Let \( \Sigma \subset \mathbb{E}^{n+1} \) be a smooth, properly embedded, orientable hypersurfaces with \( \theta \)-capillary boundary, given by the embedding \( x : \mathbb{E}^n \to \mathbb{E}^{n+1} \). For \( 1 \leq k \leq n \), it holds
\[
\int_{\Sigma} H_{k-1} (\langle x, e \rangle + \cos \theta \langle \nu, e \rangle) dA = \int_{\Sigma} H_k(X_e, \nu) dA.
\]

(2.7)
Proof. Let \( \{e_i\}_{i=1}^n \) be an orthonormal frame on \( \Sigma \). Define

\[
P_e := \langle \nu, e \rangle x - \langle x, \nu \rangle e.
\]

Let \( X_e^T \) and \( P_e^T \) be the tangential projection of \( X_e \) and \( P_e \) on \( \Sigma \) respectively. From (2.4), we have

\[
\frac{1}{2} \langle \nabla_i(X_e^T)_j + \nabla_j(X_e^T)_i \rangle = \langle x, e \rangle g_{ij} - h_{ij}(X_e, \nu).
\]

(2.8)

On the other hand, since \( \langle P_e, \nu \rangle = 0 \), by a direct computation, we know

\[
\nabla_i(P_e^T)_j = \langle \nu, e \rangle y_{ij} + h_u(\nu, e_i)\langle x, e_j \rangle - h_u(x, e_i)\langle x, e_j \rangle.
\]

(2.9)

By multiplying \( \sigma_{k-1}^{ij} \) into above equations (2.8) and (2.9), and combining with Proposition 2.2, it follows that

\[
\sigma_{k-1}^{ij} \nabla_i(X_e^T + \cos \theta P_e^T)_j = (n - k + 1)\sigma_{k-1}(\langle x, e \rangle + \cos \theta \langle \nu, e \rangle) - k\sigma_k(X_e, \nu).
\]

We need to show

\[
\int_{\Sigma} \sigma_{k-1}^{ij} \nabla_i(X_e^T + \cos \theta P_e^T)_j dA = 0,
\]

In fact, by integration by parts, and using the fact that \( \nabla_i\sigma_{k-1}^{ij} = 0 \) and \( \mu \) is a principal direction of \( \Sigma \), we get

\[
\int_{\Sigma} \sigma_{k-1}^{ij} \nabla_i(X_e^T + \cos \theta P_e^T)_j dA = \int_{\partial \Sigma} \sigma_{k-1}^{\mu \nu}(X_e + \cos \theta P_e, \nu) \mu ds.
\]

From (2.2), on \( \partial \Sigma \), we see that

\[
\langle X_e, \mu \rangle = \langle X_e, \sin \theta \nu + \cos \theta \nu \rangle = \cos \theta \langle X_e, \nu \rangle,
\]

\[
= \cos \theta (\langle x, e \rangle \langle x, \nu \rangle - \langle e, \nu \rangle) = -\cos \theta \langle e, \nu \rangle,
\]

and

\[
\langle P_e, \mu \rangle = \langle \nu, e \rangle \langle x, \mu \rangle - \langle x, \nu \rangle \langle e, \mu \rangle = (\sin \theta \nu + \cos \theta \mu, e) = \langle \nu, e \rangle,
\]

which follows that

\[
\langle X_e + \cos \theta P_e, \mu \rangle = 0.
\]

We complete the proof. \( \square \)

2.4. Quermassintegrals and first variational formula.

Recall that given a convex body \( K \subset \mathbb{S}^n \) with non-empty smooth boundary \( \partial K \), its quermassintegrals \( W_k^n \) are defined by

\[
W_k^n(K) := |\partial K|, \quad W_k^n(K) := \frac{|\partial K|}{n},
\]

and for \( 2 \leq k \leq n \),

\[
W_k^n(K) := \frac{1}{n} \int_{\partial K} H_k^n dA + \frac{k-1}{n-k+2} W_{k-2}^n(K),
\]

where \( H_k^n = \frac{1}{(n-k-1)!} \sigma_k^n \) and \( \sigma_k^n \) denote the \( k \)-th elementary symmetric polynomials, which evaluated at the \( (n - 1) \)-principal curvatures of the hypersurface \( \partial K \subset \mathbb{S}^n \), and we use the convention that \( H_0^n = 1, H_1^n = 0 \). In particular,

\[
W_0^n(K) = \frac{\omega_{n-1}}{n},
\]

due to the spherical Gauss-Bonnet-Chern’s Theorem, cf. [41]. The following first variational formula of \( W_k^n \) is proved by Reilly [36, Section 3], see also Barbosa-Colares [4, Section 4].

Proposition 2.9. Consider a family of convex bodies \( \{K_t\} \) whose boundary \( \partial K_t \) evolving by a normal speed function \( f \), then for \( 0 \leq k \leq n \),

\[
\frac{d}{dt} W_k^n(K_t) = \frac{1}{k} \int_{\partial K_t} \sigma_k^n f dA_t = \frac{n-k}{n} \int_{\partial K_t} H_k^n f dA_t.
\]

(2.10)
Next we define the geometric quantities $W_{k,\theta}$ for smooth, properly embedded, convex hypersurfaces in $\mathbb{B}^{n+1}$ with $\theta$-capillary boundary. Let $\Sigma \subset \mathbb{B}^{n+1}$ be such a hypersurface. Denote $\hat{\Sigma}$ be the enclosed bounded convex domain by $\Sigma$ inside $\mathbb{B}^{n+1}$ and $\partial \hat{\Sigma} \subset \mathbb{S}^n$ be the enclosed convex domain by $\partial \Sigma \subset \mathbb{S}^n$. We define the $W_{k,\theta}$ for $\hat{\Sigma} \subset \mathbb{B}^{n+1}$ as in (1.4), (1.5) and (1.6). In particular,

\[
W_{2,\theta}(\hat{\Sigma}) = \frac{1}{n(n+1)} \left\{ \int_{\Sigma} H dA - \sin \theta \cos \theta |\partial \Sigma| + [1 + (n - 1) \cos^2 \theta] |\partial \hat{\Sigma}| \right\}.
\]

We have the following first variational formula for $W_{k,\theta}$.

**Theorem 2.10.** Let $\Sigma_t \subset \mathbb{B}^{n+1}$ be a family of smooth, properly embedded hypersurfaces with $\theta$-capillary boundary, given by $x(\cdot, t) : \mathbb{B}^n \to \mathbb{B}^{n+1}$, such that

\[
(\partial_t x)^i = f \nu,
\]

for some normal speed function $f$. Then for $0 \leq k \leq n - 1$,

\[
dt \frac{W_{k+1,\theta}(\Sigma_t)}{W_{k,\theta}(\Sigma_t)} = \frac{n - k}{n + 1} \int_{\Sigma_t} H_{k+1} f dA_t,
\]

and

\[
dt \frac{W_{n+1,\theta}(\Sigma_t)}{W_{n,\theta}(\Sigma_t)} = 0.
\]

Before proving Theorem 2.10, we study the evolution equations for several geometric quantities under the flow

\[
\partial_t x = f \nu + T,
\]

where $T \in T\Sigma_t$.

**Proposition 2.11.** Along the general flow (2.14), it holds that

1. $\partial_t g_{ij} = 2 fh_{ij} + \nabla_i T_j + \nabla_j T_i$.
2. $\partial_t dA_t = (f H + \text{div}(T)) dA_t$.
3. $\partial_t \nu = -\nabla f + h(e_i, T) e_i$.
4. $\partial_t h_{ij} = -\nabla^2 f + fh_{ik} h^k_j + \nabla T h_{ij} + h^k_j \nabla_i T_k + h^k_i \nabla_j T_k$.
5. $\partial_t h^j_i = -\nabla^i \nabla_j f - fh^k_i h^j_k + \nabla T h^j_i$.
6. $\partial_t H = -\Delta f - |h|^2 f + \nabla T H$.
7. $\partial_t F = -f^i \nabla^i F - f F^i h^k_j h^j_k + \nabla T F$, for $F = F(h^j_i)$, where $F^i := \frac{\partial F}{\partial h^j_i}$.
8. $\partial_t \sigma_k = -\frac{\partial F}{\partial h^j_i} \nabla^i F - f(\sigma_1 \sigma_k - (k + 1) \sigma_{k+1}) + \nabla T \sigma_k$.

**Proof.** Due to the tangential part appearance in flow (2.14), we include a proof below for completeness. The assertion for $T = 0$ can be found for example in [20, Chapter 2, Section 2.3] or [16, Appendix B].

We choose an orthonormal frame $\{e_i\}_{i=1}^n$ around some point $p$. Recall the Gauss-Weingarten formula

\[
D_{e_i} x = e_i, \quad D_{e_i} e_j = \nabla_{e_i} e_j - h_{ij} \nu, \quad D_{e_i} \nu = h_{ik} e_k.
\]

We compute

\[
\partial_t g_{ij} = \partial_t (D_{e_i} x, D_{e_j} x) = (D_{e_i} (f \nu + T), e_j) + (e_i, D_{e_j} (f \nu + T)) = 2fh_{ij} + \nabla_i T_j + \nabla_j T_i.
\]

It follows that

\[
\partial_t dA_t = \frac{1}{2} \partial_t g_{ij} dA_t = g^{ij} (fh_{ij} + \nabla_i T_j) dA_t = (f H + \text{div}(T)) dA_t.
\]

Since $\langle \partial_t \nu, \nu \rangle = 0$, then

\[
\partial_t \nu = (\partial_t \nu, D_{e_i} x) e_i = -\langle \nu, \partial_t D_{e_i} x \rangle e_i = -\nabla f + h(e_i, T) e_i.
\]
We next compute
\[ \partial_t h_{ij} = \partial_t (D_{e_i} x, D_{e_j} \nu) = \langle D_{e_i} (f \nu + T), h_{jk} e_k \rangle + \langle e_i, D_{e_j} (-\nabla f + h(e_k, T)e_k) \rangle \]
\[ = -\nabla^2 f(e_i, e_j) + f h_{ik} h_{jk} + \nabla T h_{ij} + h_{jk} \nabla_i T_k + h_{ik} \nabla_j T_k, \]
where in the last equality we have used
\[ \langle e_i, D_{e_j} (h(e_k, T)e_k) \rangle = e_j (h(e_i, T)) + h(e_k, T) \langle e_i, \nabla e_j e_k \rangle = \nabla_T h(e_i, e_j) + h(e_i, \nabla e_j T) = \nabla_T h_{ij} + h_{ik} \nabla_j T_k. \]
It follows that
\[ \partial_t h_{ij} = \partial_t (g^{ik} h_{kj}) = -(2 f h_{ik} + \nabla_i T_k + \nabla_k T_i) h_{kj} \]
\[ - \nabla_{ij} f + f h_{ik} h_{kj} + \nabla_T h_{ij} + h_{jk} \nabla_i T_k + h_{ik} \nabla_j T_k \]
\[ = - \nabla_{ij} f - f h_{ik} h_{kj} + \nabla_T h_{ij} + h_{ik} \nabla_j T_k - h_{jk} \nabla_k T_i. \]
The last three assertions (6)-(8) follow directly from (5), just noticing that
\[ F^{ij} (h_{ik} \nabla_j T_k - h_{jk} \nabla_i T_k) = 0, \]
and the fact
\[ \frac{\partial \sigma_k}{\partial h_{ij}^k} h_{ij} = \sigma_1 \sigma_k - (k+1) \sigma_{k+1}, \]
which follows from Proposition 2.2 (4).

Let \( \Sigma_t \subset \mathbb{B}^{n+1} \) be a family of smooth, properly embedded hypersurfaces with \( \theta \)-capillary boundary, evolving by (2.11). Then the tangential component \( (\partial_t x)^T \) of \( \partial_t x \), which we denote by \( T \in T \Sigma_t \), must satisfy
\[ T|_{\partial \Sigma_t} = f \cot \theta \mu + \tilde{T}, \] (2.15)
where \( \tilde{T} \in T(\partial \Sigma_t) \). In fact, the restriction of \( x(\cdot, t) \) on \( \partial \mathbb{B}^n \) is contained in \( \mathbb{S}^n \) and thus,
\[ f \nu + T|_{\partial \Sigma_t} = \partial_t x|_{\partial \mathbb{B}^n} \in T \mathbb{S}^n. \]
By virtue of (2.2), we have
\[ \nu = \frac{1}{\sin \theta} \tilde{\nu} - \cot \theta \mu. \]
Since \( \tilde{\nu} \in T \mathbb{S}^n \), we see \( (T - f \cot \theta \mu) \in T \mathbb{S}^n \cap T \Sigma_t \). Then (2.15) follows. Up to a diffeomorphism of \( \partial \mathbb{B}^n \), we can assume \( \tilde{T} = 0 \). For simplicity, in the following, we always assume that
\[ T|_{\partial \Sigma_t} = f \cot \theta \mu. \] (2.16)

**Proposition 2.12.** Let \( \Sigma_t \subset \mathbb{B}^{n+1} \) be a family of smooth, properly embedded hypersurfaces with \( \theta \)-capillary boundary. Then
\[ \nabla_\mu f = \left( \frac{1}{\sin \theta} + \cot \theta \mu \right) f \text{ along } \partial \Sigma_t. \] (2.17)

**Proof.** Let \( \{e_\alpha\}_{\alpha=1}^{n-1} \) be an orthonormal frame of \( T(\partial \Sigma_t) \subset T \mathbb{S}^n \). Then \( \{(e_\alpha)_{\alpha=1}^{n-1}, \mu\} \) forms an orthonormal frame for \( T \Sigma_t \). Since
\[ \langle \nu, \overline{N} \circ x \rangle = -\cos \theta, \text{ along } \partial \Sigma_t, \]
by taking the time derivative and using Proposition 2.11 (3), (2.2) and (2.16), we obtain along \( \partial \Sigma_t, \)
\[ 0 = \langle \partial_t \nu, \overline{N} \rangle + \langle \nu, D_f \nu + T \overline{N} \rangle \]
\[ = \langle -\nabla f + h(e_i, T)e_i, \sin \theta \mu - \cos \theta \nu \rangle + \langle \nu, f \nu + T \rangle \]
\[ = -\sin \theta \nabla_\mu f + \cos \theta h(\mu, \mu) f + f. \]
The assertion follows. \( \square \)
Proof of Theorem 2.10. By Proposition 2.11, using integration by parts and the fact $\mu$ is a principal direction, we obtain
\[
\frac{d}{dt} \int_{\Sigma_t} \sigma_k dA_t = \int_{\Sigma_t} \left[ -\frac{\partial \sigma_k}{\partial t} + \nabla^i \nabla_j f - f(\sigma_1 \sigma_k - (k+1)\sigma_{k+1}) + \nabla T \sigma_k \right] dA_t \\
+ \int_{\Sigma_t} \sigma_k (f \sigma_1 + \text{div}(T)) dA_t \\
= (k+1) \int_{\Sigma_t} f \sigma_{k+1} dA_t + \int_{\partial \Sigma_t} (\sigma_k(T, \mu) - \sigma_k^{\mu\nu} \nabla \mu f).
\]
Using (2.16), (2.17) and Proposition 2.2 (1), we see along $\partial \Sigma_t$,
\[
\sigma_k(T, \mu) - \sigma_k^{\mu\nu} \nabla \mu f = f \left[ \cot \theta (\sigma_k - \sigma_k^{\mu\nu} h_{\mu\nu}) - \frac{1}{\sin \theta} \sigma_k^\nu \right] \\
= f \left[ \cot \theta \sigma_k(h|h_{\mu\nu}) - \frac{1}{\sin \theta} \sigma_{k-1}(h|h_{\mu\nu}) \right].
\]
Recall in Proposition 2.5 (2), we have
\[
h_{\alpha\beta} = \sin \hat{\theta} h_{\alpha\beta} - \cos \theta \delta_{\alpha\beta},
\]
for an orthonormal frame $\{e_\alpha\}_{\alpha=1}^{n-1}$ of $T(\partial \Sigma_t)$, Thus
\[
\sigma_k(h|h_{\mu\nu}) = \sigma_k(\sin \hat{\theta} \hat{h} - \cos \theta I_{n-1}),
\]
where $I_{n-1}$ is the $(n-1) \times (n-1)$ identity matrix. In general, for a $(n-1) \times (n-1)$ symmetric matrix $B$ and $0 \leq k \leq n-1$, we know
\[
\sigma_k(I + B) = \sum_{l=0}^{k} \binom{k}{l} \sigma_k(I \cdot \ldots \cdot I, \underbrace{B, \ldots, B}_{k-l}) \\
= \sum_{l=0}^{k} \binom{k}{l} \binom{n-1}{n-l} \sigma_l(B) = \sum_{l=0}^{k} (n-l-1) \binom{n-l}{n-k-1} \sigma_l(B),
\]
Thus we have
\[
\sigma_k(h|h_{\mu\nu}) = \sigma_k(\sin \hat{\theta} \hat{h} - \cos \theta I_{n-1}) \\
= (-\cos \theta)^k \sum_{l=0}^{k} \binom{n-l-1}{n-k-1} (-\tan \theta)^l \sigma_l(\hat{h}). \quad (2.18)
\]
It follows that
\[
\cot \theta \sigma_k(h|h_{\mu\nu}) - \frac{1}{\sin \theta} \sigma_k-1(h|h_{\mu\nu}) \\
= \cos \theta \sin^{k-1} \theta \sigma_k(\hat{h}) \\
+ \frac{\cos^{k-1} \theta}{\sin \theta} \sum_{l=0}^{k-1} (-1)^{k+l} \left[ \cos^2 \theta \binom{n-l-1}{n-k-1} + \binom{n-l-1}{n-k} \right] \tan^l \theta \sigma_l(\hat{h}).
\]
Recall that $\hat{h}$ is the second fundamental form of $\partial \Sigma$ as a hypersurface in $\mathbb{S}^n$, we have $\sigma_k(\hat{h}) = \sigma_k^{\mu\nu}$. It follows that
\[
\frac{d}{dt} \int_{\Sigma_t} \sigma_k dA_t = (k+1) \int_{\Sigma_t} f \sigma_{k+1} dA_t + \cos \theta \sin^{k-1} \theta \int_{\partial \Sigma_t} f \sigma_k^{\mu\nu} \\
+ \frac{\cos^{k-1} \theta}{\sin \theta} \sum_{l=0}^{k-1} (-1)^{k+l} \left[ \cos^2 \theta \binom{n-l-1}{n-k-1} + \binom{n-l-1}{n-k} \right] \tan^l \theta \int_{\partial \Sigma_t} f \sigma_l^{\mu\nu}. 
\]
Recall that $T$ satisfies (2.16) and the flow (2.11) induces a normal hypersurface flow $\partial \Sigma t \subset S^n$ with normal speed $\frac{f}{\sin \theta}$; that is, 
$$\partial_t x |_{\partial \Sigma^n} = f \nu + f \cot \theta \mu = \frac{f}{\sin \theta} \nu.$$ 
From Proposition 2.9, we know 
$$\frac{d}{dt} W^S_k(\partial \Sigma t) = - \frac{1}{\sin \theta} \binom{n}{k}^{-1} \int_{\partial \Sigma t} f \sigma^S_{k-1}.$$ 
We conclude that for $0 \leq k \leq n-1$, 
$$\frac{d}{dt} \left\{ \int_{\Sigma t} \sigma_k dA_t - \binom{n}{k} \cos \theta \sin^k \theta W^S_k(\partial \Sigma t) \right\} 
- \cos^{k-1} \theta \sum_{l=0}^{k-1} (-1)^{k+l} \binom{n}{l} \left[ \cos^2 \theta \binom{n-l-1}{n-1} + \binom{n-l-1}{n-k} \right] \tan^l \theta W^S_l(\partial \Sigma t) \right\} 
= (k+1) \int_{\Sigma t} \sigma_{k+1} f dA_t.$$ 
By using 
$$\binom{n}{k} \binom{n-l-1}{n-k-1} = \binom{n}{k} \binom{k}{l} \frac{n-k}{n-l},$$ 
and 
$$\binom{n}{k} \binom{n-l-1}{n-k} = \binom{n}{k} \binom{k}{l} \frac{k-l}{n-l}$$ 
we get (2.12).

For $k = n$, the computation is similar, with replace (2.18) by 
$$\sigma_n(h \mu) = \det(\sin \theta h - \cos \theta I_{n-1}) = (-\cos \theta)^{n-1} \det(I_{n-1} - \tan \theta h),$$ 
then it follows 
$$\frac{d}{dt} \int_{\Sigma t} \sigma_n dA_t = - \int_{\partial \Sigma t} \frac{f}{\sin \theta} \sigma^\mu = \int_{\partial \Sigma t} (-1)^n \cos^{n-1} \theta \sin \theta \det(I_{n-1} - \tan \theta h)$$ 
$$= \frac{\cos^{n-1} \theta \sin \theta}{\sin \theta} \sum_{l=0}^{n-1} (-1)^{n+l} \tan^l \theta \int_{\partial \Sigma t} f \sigma^S_l.$$ 
We complete the proof. \hfill \Box

2.5. Estimates for convex hypersurfaces with $\theta$-capillary boundary.

To begin with, we state a theorem, due to Ghomi [21], about global convexity of a compact, connected, immersed $C^2$-hypersurface with positive curvature.

**Theorem 2.13** (Ghomi [21], Theorem 1.2.5). *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a compact, connected, immersed $C^2$-hypersurface with positive curvature. Then $\Sigma$ may be extended to a $C^2$-ovaloid if and only if for any boundary component $\Gamma$ and any $p \in \Gamma$, $\Gamma \cap T_p \Sigma = \{p\}$.)*

Recall that a closed immersed hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ is called an ovaloid if for any $p \in \Sigma$, $\Sigma$ lies on one side of $T_p \Sigma$. The classical Hadamard theorem says that a closed embedded hypersurface with positive curvature must be an ovaloid. Ghomi’s theorem can be viewed as a generalization of Hadamard’s theorem on hypersurfaces with boundary.

**Proposition 2.14.** *For any $p \in \mathbb{R}^n$ and $z \in \partial \Sigma \subset S^n$, it holds 
$$\langle z - x(p), \nu(p) \rangle \leq 0,$$
with equality holding if and only if $z = x(p)$.***
Proof. Consider \( \partial \Sigma \subset \mathbb{S}^n \). From Proposition 2.5, we see that
\[
(\hat{h}_{\alpha\beta}) > \cot \theta (\delta_{\alpha\beta}) \geq 0.
\]
From a comparison theorem (cf. [18, Theorem 0.1]), we have
\[
\max_{y \in \partial \Sigma} \text{dist}_{\mathbb{S}^n}(y, \partial \Sigma) < \theta.
\] (2.19)
For each \( p \in \mathbb{H}^n \), we denote by \( S_{p,\theta} \) the geodesic sphere in \( \mathbb{S}^n \) of radius \( \theta \) passing through \( x(p) \). It follows easily from (2.19) that
\[
\hat{\partial} \Sigma \subset \hat{S}_{p,\theta} \text{ with } \partial \Sigma \cap S_{p,\theta} = \{x(p)\}.
\] (2.20)
See Figure 1.

Figure 1

Since \( \Sigma \) is \( \theta \)-capillary, we see that \( T_p \Sigma \cap \mathbb{S}^n = S_{p,\theta} \) for each \( x(p) \in \mathbb{H}^n \). Therefore, (2.20) implies that \( \hat{\partial} \Sigma \setminus \{x(p)\} \) lies strictly on one side of \( T_p \Sigma \), which leads to the assertion. \( \square \)

Applying Ghomi’s theorem and Proposition 2.14, we have the following property for a strictly convex \( \theta \)-capillary hypersurface in the unit ball.

Proposition 2.15. Let \( \Sigma \subset \mathbb{H}^{n+1} \) be a strictly convex hypersurface with \( \theta \)-capillary boundary for \( \theta \in (0, \pi/2] \), which is given by the embedding \( x : \mathbb{H}^n \to \mathbb{H}^{n+1} \). Then for any \( p \in \Sigma \), \( \Sigma \) lies on one side of \( T_p \Sigma \).

Proof. Proposition 2.14 implies for \( p \in \partial \Sigma \), \( \partial \Sigma \cap T_p \Sigma = \{p\} \). By using Theorem 2.13, we see that \( \Sigma \) may be extended to a \( C^2 \)-ovaloid, in particular, our assertion follows. \( \square \)

The following facts about strictly convex hypersurface with \( \theta \)-capillary boundary will be used later.

Proposition 2.16. Let \( \Sigma \subset \mathbb{H}^{n+1} \) be a strictly convex hypersurface with \( \theta \)-capillary boundary for \( \theta \in (0, \pi/2] \), which is given by the embedding \( x : \mathbb{H}^n \to \mathbb{H}^{n+1} \). Then there exists \( e_0 \in \text{int}(\hat{\partial} \Sigma) \) and some \( \delta_0 > 0 \), depending on \( \Sigma \), such that the following properties hold:

1. \( \langle x, e_0 \rangle \geq \cos \theta + \delta_0 \).
2. \( \langle \nu, e_0 \rangle < -\cos^2 \theta \).
3. \( \langle \mu, e_0 \rangle > 0 \) on \( \partial \mathbb{H}^n \).
4. \( \langle x + \cos \theta \nu, e_0 \rangle \geq \delta_0 > 0 \).

Proof. The conclusion for \( \theta = \pi/2 \) has been proved by Lambert-Scheuer [29, Section 4]. Hence we prove the case for \( \theta \in (0, \pi/2) \).

(1) Step 1. Define \( \phi(p) := -\langle x(p), \nu(p) \rangle \), \( p \in \mathbb{H}^n \).
Note that $\phi|_{\partial B^n} \equiv \cos \theta > 0$. On the other hand, since $\mu$ is a principal direction and $\Sigma$ is strictly convex,

$$D_\mu \phi = -h_{\mu \mu}(x, \mu) = -h_{\mu \mu} \sin \theta < 0, \quad \text{along } \partial \mathbb{B}^n,$$

which implies that $\phi$ cannot attain its maximum on $\partial \mathbb{B}^n$. Thus, $\phi$ attains its maximum at some interior point, say $p_0 \in \mathbb{B}^n$. Therefore

$$\phi(p_0) > \phi|_{\partial B^n} = \cos \theta,$$

and at $p_0$, we have

$$0 = D_{e_i} \phi(p_0) = -h_{ij}(p_0) \langle x(p_0), e_j(p_0) \rangle,$$

for any $e_i \in T_{p_0} \Sigma$. Since $(h_{ij}(p_0))$ is positive definite, we get $\langle x(p_0), e_j(p_0) \rangle = 0$ for any $e_j \in T_{p_0} \Sigma$. This implies $x(p_0) \parallel \nu(p_0)$. Since $\phi(p_0) > \cos \theta > 0$, for $\theta \in (0, \frac{\pi}{2})$, we obtain

$$-\langle x(p_0), \nu(p_0) \rangle = |x(p_0)| > \cos \theta.$$

From Proposition 2.15, we see

$$\langle x(p) - x(p_0), \nu(p_0) \rangle \leq 0, \quad \text{for } p \in \mathbb{B}^n.$$

Define $e_0 := -\nu(p_0)$, it follows

$$\langle x(p), e_0 \rangle = \langle x(p), -\nu(p_0) \rangle \geq -\langle x(p), \nu(p_0) \rangle > \cos \theta,$$

for all $p \in \mathbb{B}^n$. See Figure 2.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{Figure 2}
\end{figure}

Next, we claim that $e_0$ must be contained in $\text{int}(\partial \Sigma)$. In fact, if not, there exists some $\tilde{p} \in \partial \mathbb{B}^n$ such that $x(\tilde{p}) \in \partial \Sigma$ and $\text{dist}_{\Sigma^n}(e_0, x(\tilde{p})) = \text{dist}_{\Sigma^n}(e_0, \partial \Sigma) > 0$. Using Proposition 2.15, we see that $\Sigma$ lies on one side of $T_{\tilde{p}} \Sigma$, say $T_{\tilde{p}} \Sigma^+$. By the fact $\theta \in (0, \frac{\pi}{2})$, we see that $T_{\tilde{p}} \Sigma^+$ does not contain the line segment $l$ through $e_0$ and the origin. However, $l$ goes through $x(p_0) \in \Sigma$ since $e_0 = -\nu(p_0) = \frac{x(p_0)}{|x(p_0)|}$. This leads to a contradiction and we get the claim.

**Step 2.** Denote the tangent hyperplane $T_{p_0} \Sigma := \mathcal{H}$, where $p_0$ is determined in Step 1. It is known that $\mathcal{H}$ intersects $S^n$ at the angle $\theta_1 < \theta$, where $\theta_1$ is given by

$$\cos \theta_1 = |x(p_0)| > \cos \theta.$$

Now we parallel transport $\mathcal{H}$ downwards along the direction $-e_0$ until it achieves a position $\mathcal{H}'$ which satisfies that $\mathcal{H}'$ intersects with $S^n$ at the angle $\theta$, see Figure 3.\[\text{Figure 3}\]
We construct the foliation $C_{\theta,s}(e_0), s \geq 0$ so that $C_{\theta,\infty}(e_0) := \mathcal{H}'$. Recall $C_{\theta,s}(e_0), s > 0$ is the spherical cap of radius $s$ around $e_0$ with $\theta$-capillary boundary, see (1.2).

Since $C_{\theta,0}(e_0) = \mathcal{H}'$ does not touch $\Sigma$ and lies below $\Sigma$, there exists some $s_0 > 0$, depending on $\theta_1$, such that $C_{\theta,s_0}(e_0)$ touches $\Sigma$ at the first time. Hence

$$\langle x, e_0 \rangle \geq \min_{x \in C_{0,0}(e_0)} \langle x, e_0 \rangle := \cos \theta + \delta_0,$$

for some $\delta_0 > 0$. From above, it is clear that $\delta_0$ depends on $s_0$ and in turn on $\Sigma$.

Hence we complete the proof of statement (1).

(2) From (1), we know that $\Sigma \subset H_{e_0} := \{\langle x, e_0 \rangle > 0\}$.

Since $e_0 \in \text{int}(\bar{\Sigma})$, we see that

$$\bar{\Sigma} \subset H_{e_0}.$$

On the other hand, we see from Corollary 2.6 that $\partial \Sigma$ is a strictly convex hypersurface in $\mathbb{S}^n$. It follows from [13, 22] that, there exist two disjoint open connected components $A$ and $B$, such that

$$\mathbb{S}^n \setminus \nu(\partial \mathbb{B}^n) = A \cup B,$$

where $A$ is the interior of the strictly convex body in $\mathbb{S}^n$ which $\nu(\partial \mathbb{B}^n)$ bounds. Since $\bar{\Sigma} \subset H_{e_0}$, we have

$$\partial A \subset H_{-e_0},$$

which implies

$$\langle \nu(p), e_0 \rangle \leq 0, \quad p \in \partial \mathbb{B}^n.$$

Using (2.2), we get

$$\langle \nu(p) + \cos \theta x(p), e_0 \rangle \leq 0, \quad p \in \partial \mathbb{B}^n.$$

Combining with $\langle x(p), e_0 \rangle > \cos \theta$ from (1), it yields that

$$\langle \nu(p), e_0 \rangle \leq - \cos \theta \langle x(p), e_0 \rangle < - \cos^2 \theta, \quad p \in \partial \mathbb{B}^n. \quad (2.21)$$

On the other hand, assume

$$\mathbb{S}^n \setminus \nu(\partial \mathbb{B}^n) = \hat{A} \cup \hat{B},$$

where $\hat{A}$ is the component which contains $e_0$. In view of (2.21), it holds that

$$\langle y, e_0 \rangle < - \cos^2 \theta, \quad y \in \hat{B}.$$

Since $\nu(\mathbb{B}^n)$ is simply connected, $\nu(\mathbb{B}^n)$ is either $\hat{A}$ or $\hat{B}$. However, $\nu(\mathbb{B}^n)$ cannot contain $e_0$. Thus $\nu(\mathbb{B}^n) = \hat{B}$. Therefore

$$\langle \nu(p), e_0 \rangle < - \cos^2 \theta, \quad p \in \mathbb{B}^n.$$
Remark 2.18. In the case \( \theta = \pi/2 \), Lambert-Scheuer [29] showed that for strictly convex hypersurfaces with free boundary and \( e_0 \) given by the \( \partial \Sigma \), let \( \Sigma \) be a family of smooth, properly embedded, strictly convex hypersurfaces with \( e_0 \) such that \( \partial \Sigma \subset H_{e_0} \), there exists some constant \( \epsilon > 0 \) depending on \( \Sigma \), such that \( \langle X_{e_0}, \nu \rangle > \epsilon \cos^2 \theta \).

Recall that \( \theta \) is given in Proposition 2.16. Then there exists \( \epsilon > 0 \) depending on \( \Sigma \), such that \( \langle X_{e_0}, \nu \rangle > \epsilon \cos^2 \theta \).

Proof. Since \( \Sigma \) is strictly convex and \( e_0 \in \partial \Sigma \), then
\[
\langle x - e_0, \nu \rangle \geq 0. \tag{2.22}
\]

Since \( \Sigma \) is compact and \( e_0 \notin \Sigma \), there exists \( \epsilon > 0 \) such that \( \frac{1}{2} |x - e_0|^2 \geq \epsilon \) on \( \Sigma \). Recall that \( X_{e_0} = \langle x, e_0 \rangle x - \frac{|x|^2 + 1}{2} e_0 \). Thus
\[
\langle X_{e_0}, \nu \rangle = \langle x, e_0 \rangle \langle x, \nu \rangle - \frac{|x|^2 + 1}{2} \langle \nu, e_0 \rangle
= \langle x, e_0 \rangle \langle x - e_0, \nu \rangle - \frac{|x - e_0|^2}{2} \langle \nu, e_0 \rangle \geq \epsilon \cos^2 \theta.
\]

where the last inequality follows from (2.22) and Proposition 2.16 (2). \( \square \)

3. Locally constrained Curvature Flow

In this section, we study the flow. Let \( \Sigma_0 := \Sigma \subset \mathbb{B}^{n+1} \) be a smooth, properly embedded, strictly convex hypersurface with \( \theta \)-capillary boundary, given by the embedding \( x_0 : \mathbb{B}^n \to \mathbb{B}^{n+1} \). Choose \( e_0 \) as in Proposition 2.16, which depends only on \( \Sigma_0 \). For simplicity of notation, we omit the subscription and write \( e = e_0 \). Let \( \Sigma_t \subset \mathbb{B}^{n+1} \) be a family of smooth, properly embedded, strictly convex hypersurfaces with \( \theta \)-capillary boundary, given by \( x(\cdot, t) : \mathbb{B}^n \to \mathbb{B}^{n+1} \), evolving by

\[
\begin{aligned}
\begin{cases}
\partial_t x(\cdot, t) = f(\cdot, t) \nu(\cdot, t) + T(\cdot, t), & \text{in } \mathbb{B}^n \times [0, T), \\
\nu(\cdot, t), \nabla x(\cdot, t) = -\cos \theta & \text{on } \partial \mathbb{B}^n \times [0, T), \\
x(\cdot, 0) = x_0(\cdot) & \text{in } \mathbb{B}^n.
\end{cases}
\end{aligned}
\tag{3.1}
\]

where
\[
f = \frac{\langle x + \cos \theta \nu, e \rangle}{F} - \langle X_e, \nu \rangle, \tag{3.2}
\]

with
\[
F = \frac{H_n}{H_{n-1}} = \frac{n \sigma_n}{\sigma_{n-1}},
\]

and \( T \in T \Sigma_t \) such that (2.16) holds. The following property of the flow (3.1) is crucial in this paper.

Proposition 3.1. As long as \( \Sigma_t \) is strictly convex and the flow (3.1) exists, \( W_{n, \theta} \) is preserved and \( W_{k, \theta} \) is non-decreasing for \( 1 \leq k \leq n-1 \).
Proposition 3.3. The family of

In particular, there exists some

Thus there exists some

from Proposition 2.16 there exists some

Also us to transform the flow equation to be a scalar equation with oblique boundary

deformation of (2.6) implies that

Thus the scalar equation is a strictly parabolic equation. Then the short time

existence follows from the standard parabolic theory.

The aim of this section is the following existence and convergence result for the

flow (3.1).

Theorem 3.2. Let \( \Sigma_0 \subset \mathbb{B}^{n+1} \) be a smooth, properly embedded, strictly convex hypersurface with \( \theta \)-capillary boundary for some \( \theta \in (0, \frac{\pi}{2}) \), given by the embedding

\( x_0 : \mathbb{B}^n \to \mathbb{B}^{n+1} \). Choose \( \epsilon_0 \) as in Proposition 2.16. Then \( x(\cdot, t) \) to the flow (3.1) exists for \( t \in [0, \infty) \). Moreover, \( x(\cdot, t) \) converges smoothly to the spherical cap around a with \( \theta \)-capillary boundary, which has the same \( W_{n,\theta} \) as \( \Sigma_0 \).

3.1. Barriers. First of all, we have the short time existence for the flow (3.1). This is because the strict convexity and the \( \theta \)-capillary boundary of \( \Sigma_0 \) imply that \( \Sigma_0 \) satisfies \( \langle X_e, \nu \rangle \geq \epsilon \cos^2 \theta \) for some \( \epsilon > 0 \) by Proposition 2.17. This allows us to transform the flow equation to be a scalar equation with oblique boundary condition, by using the Möbius coordinate transformation see [29] or [42]. Also from Proposition 2.16 there exists some \( \delta_0 > 0 \) depending on \( \Sigma_0 \) such that

Thus the scalar equation is a strictly parabolic equation. Then the short time

existence follows from the standard parabolic theory.

Let \( T^* \) be the maximal time of smooth existence of a solution to (3.1). The positivity of \( F \) implies that \( \Sigma_t \) is strictly convex up to \( T^* \) for flow (3.1). From Proposition 2.16 (1), It follows that

Thus there exists some \( 0 < R_1 < R_2 < \infty \), such that

The family of \( C_{\theta,r}(e) \) forms natural barriers of (3.1).

Proposition 3.3. \( \Sigma_t, t \in [0, T^*) \) satisfies

In particular, there exists some \( \epsilon_0 > 0 \), depending only only on \( R_1 \) and \( R_2 \), such that

Proof. Recall that \( C_{\theta,r}(e) \) satisfies (2.6). Thus for each \( r > 0 \), it is a static solution to the flow (3.1). The assertion follows from the avoidance principle for strictly parabolic equation with capillary boundary condition (see [1, Section 2.6] or [46, Proposition 4.2]). The height estimate holds since \( \langle x, e \rangle \geq \cos \theta + \epsilon_0 \) in \( C_{\theta,R_2}(e) \), and \( \langle x, e \rangle \leq 1 - \epsilon_0 \) in \( C_{\theta,R_1}(e) \), for some \( \epsilon_0 > 0 \).

We have the following direct consequence.
Corollary 3.4. $\Sigma_t, t \in [0, T^*)$ satisfies

$$\langle x + \cos \theta \nu, e_0 \rangle \geq \epsilon_0.$$ (3.3)

3.2. Evolution equations.
In order to avoid confusion with tensor indices, we abbreviate $X = X_e$, keeping in mind that $X$ depends on $e$. We introduce the operator

$$\mathcal{L} := \partial_t - \frac{\langle x + \cos \theta \nu, e \rangle}{F^2} F^{ij} \nabla_{ij}^2 - \langle x - \frac{\cos \theta}{F} e + T, \nabla \rangle,$$

Denote $F := \sum_{i=1}^n F_i = \sum_{i=1}^n \frac{\partial F}{\partial h_i}$.

For $F = \frac{n \sigma_n}{\sigma_{n-1}}$, use Proposition 2.2, we have

$$F - F^{ij} h_{ij} = F - 1 \geq 0,$$ (3.4)

$$\frac{F^{ij} h^k_i h_{kj}}{F^2} = 1.$$ (3.5)

Proposition 3.5. Along the flow (3.1), we have

$$\mathcal{L}(\langle x, e \rangle) = 2F^{-1} \langle x + \cos \theta \nu, e \rangle \langle \nu, e \rangle + \cos \theta F^{-1} |e^T|^2 - \langle X, e \rangle,$$

where $e^T$ is the tangential projection of $e$ on $\Sigma_t$.

Proof.

$$\partial_t \langle x, e \rangle = \left( \frac{\langle x + \cos \theta \nu, e \rangle}{F} - \langle x, \nu \rangle \right) \langle \nu, e \rangle + \langle T, e \rangle.$$

Thus

$$F^{ij} \nabla_{ij}^2 \langle x, e \rangle = -F^{ij} h_{ij} \langle \nu, e \rangle = -F \langle \nu, e \rangle.$$

Thus

$$\mathcal{L}(\langle x, e \rangle) = \left( \partial_t \langle x, e \rangle - \frac{\langle x + \cos \theta \nu, e \rangle}{F^2} F^{ij} \nabla_{ij}^2 \langle x, e \rangle - \left( \frac{X - \frac{\cos \theta}{F} e + T, \nabla \langle x, e \rangle \right) \right)$$

$$= 2F^{-1} \langle x + \cos \theta \nu, e \rangle \langle \nu, e \rangle + \cos \theta F^{-1} |e^T|^2 - \langle X, e \rangle.$$

Proposition 3.6. Along the flow (3.1), we have

$$\mathcal{L} F = \frac{2F^{ij}}{F^2} \langle x + \cos \theta \nu, e \rangle_i F_{ij}$$

$$- \frac{2\langle x + \cos \theta \nu, e \rangle}{F^3} F^{ij} F_i F_j + (1 - F) \langle \nu, e \rangle.$$ (3.6)

and

$$\nabla_{\mu} F = 0 \text{ along } \partial \Sigma_t.$$ (3.7)

Proof. From Proposition 2.11, we have

$$\partial_t F = -F^{ij} f_{ij} - F F^{ij} h_i^k h_{kj} + \nabla_T F.$$ (3.8)
Next we compute the first term in the right hand side above.
\[
-F^{ij} f_{ij} = -F^{ij} \left( \frac{\langle x + \cos \theta \nu, e \rangle}{F} - \langle X, \nu \rangle \right)
\]
\[
= \frac{\langle x, e \rangle}{F^2} F^{ij} f_{ij} + \frac{2 F^{ij} \langle x, e \rangle_i F_{ij}}{F^2} - 2 \langle x, e \rangle \frac{F^{ij} F_{i}^{\cdot}}{F^3} F_{ij} + \frac{F^{ij} h_{kij}^{\cdot} \langle \nu, e \rangle}{F^3} + \frac{\cos \theta F^{ij} \langle \nu, e \rangle}{F} F^{ij} f_{ij} \\
+ \cos \theta \frac{2 F^{ij} \langle \nu, e \rangle_i F_{ij}}{F^2} \langle \nu, e \rangle \\
- \cos \theta \frac{2 F^{ij} \langle \nu, e \rangle_i}{F^3} + \cos \theta \frac{F^{ij} h_{kij}^{\cdot} \langle \nu, e \rangle}{F} - \cos \theta \frac{F^{ij} \langle e_k, e \rangle}{F}
\]
\[
+ F^{ij} \langle X, \nu \rangle_i f_{ij}.
\]
Similarly, by conducting a direct computation, (one can also refer to [42, Lemma 3.3]), we know that
\[
\langle X, \nu \rangle_i f_{ij} = h_k^{i} \langle X, e_k \rangle + \langle x, e \rangle h_{ij} - h_k^{i} h_{kij} \langle X, \nu \rangle - \langle e, e \rangle g_{ij}
\]
\[
+ h_k^{i} \frac{\langle e_i, e \rangle \langle x, e_k \rangle - \langle e_i, x \rangle \langle e, e_k \rangle}{F^3} + h_k^{i} \frac{\langle e_j, x \rangle \langle e, e_k \rangle - \langle e_j, x, e \rangle}{F^3}.
\]
which yields that
\[
F^{ij}(X, \nu)_{ij} = F^{ij} \left( h_k^{i} \langle X, e_k \rangle + h_{ij} \langle x, e \rangle - g_{ij} \langle \nu, e \rangle - h_k^{i} h_{kij} \langle X, \nu \rangle \right)
\]
\[
= F_{k}(X, e_k) + (F \langle x, e \rangle - F \langle \nu, e \rangle) - F^{ij} h_k^{i} h_{kij} \langle X, \nu \rangle.
\]
And we have
\[
-f F^{ij} h_k^{i} h_{kij} = -\left( \frac{\langle x + \cos \theta \nu, e \rangle}{F} - \langle X, \nu \rangle \right) F^{ij} h_k^{i} h_{kij}.
\]
Plugging those terms into equation (3.8), after simplifications, we conclude that
\[
\partial_t F = \left[ \frac{\langle x + \cos \theta \nu, e \rangle}{F^2} F^{ij} f_{ij} + F_{k}(X, e_k) \right] + \frac{2 F^{ij} F_{ij}}{F^2} \langle \langle x, e \rangle_i \rangle + \cos \theta \langle \nu, e \rangle_i \\
- \frac{2 F^{ij} \langle x, e \rangle}{F^3} F^{ij} f_{ij} - \cos \theta \frac{F^{ij} \langle e_k, e \rangle}{F} + \left( \frac{F^{ij} h_{ij}}{F} - F \langle \nu, e \rangle \right) F_i^{\cdot} F_{ij} \\
+ F \langle x, e \rangle \left( 1 - \frac{F^{ij} h_k^{i} h_{kij}}{F^2} \right) + \nabla_T F,
\]
therefore, we obtain
\[
\mathcal{L} F = \frac{2 F^{ij} \langle x + \cos \theta \nu, e \rangle_i F_{ij}}{F^2} - \frac{2 \langle x + \cos \theta \nu, e \rangle}{F^3} F^{ij} F_{ij} \\
+ \left( \frac{F^{ij} h_{ij}}{F} - F \langle \nu, e \rangle \right) F_i^{\cdot} F_{ij} + F \langle x, e \rangle \left( 1 - \frac{F^{ij} h_k^{i} h_{kij}}{F^2} \right).
\]
In view of (3.4) and (3.5), we complete the proof of the evolution equation.

Next we prove $\nabla_\mu F = 0$. From Proposition 2.12, we have along $\partial \Sigma_t$,
\[
\nabla_\mu f = \left( \frac{1}{\sin \theta} + \cot \theta h_{\mu \mu} \right) f.
\]
From [49, Proposition 3.3], we know along $\partial \Sigma_t$,
\[
\nabla_\mu \langle X, \nu \rangle = \left( \frac{1}{\sin \theta} + \cot \theta h_{\mu \mu} \right) \langle X, \nu \rangle,
\]
and
\[
\nabla_\mu \langle x + \cos \theta \nu, e \rangle = \left( \frac{1}{\sin \theta} + \cot \theta h_{\mu \mu} \right) \langle x + \cos \theta \nu, e \rangle.
\]
From the definition (3.2) of $f$, we conclude that $\nabla_\mu F = 0$. □
Proposition 3.7. Along the flow (3.1), we have
\[
\mathcal{L} H = H(x, e) + (HF^{-1} - n + \cos \theta |h|^2 F^{-1}) \langle \nu, e \rangle \\
+ (x + \cos \theta \nu, e) (H - 2|h|^2 F^{-1} - 2F^{-3} |\nabla F|^2) \\
+ 2F^{-2} \langle \nabla F, e \rangle + 2 \cos \theta F^{-1} g^{ij} F_j i^k (e_k, e) \\
+ F^{-2} \langle x + \cos \theta \nu, e \rangle g^{ij} h_{kl} h_{stij} \frac{\partial^2 F}{\partial h_{kl} \partial h_{st}}.
\]
and
\[
\nabla \mu H \leq 0 \text{ along } \partial \Sigma_t.
\]

Proof. By Proposition 2.11, we have
\[
\partial_t H = -\Delta f - |h|^2 f + \nabla_T H = -\Delta \left[ \frac{\langle x + \cos \theta \nu, e \rangle}{F} - \langle X, \nu \rangle \right] \\
- |h|^2 \left( \frac{\langle x + \cos \theta \nu, e \rangle}{F} - \langle X, \nu \rangle \right) + \nabla_T H.
\]
Note that, from equation (3.9), it holds
\[
\Delta \langle X, \nu \rangle = H_{,k} \langle X, e_k \rangle + \langle x, e \rangle H - |h|^2 \langle X, \nu \rangle - n\langle \nu, e \rangle,
\]
and
\[
-\Delta \left( \frac{\langle x + \cos \theta \nu, e \rangle}{F} \right) = \frac{H}{F} \langle \nu, e \rangle - \cos \theta \left( \langle \nabla H, e \rangle - |h|^2 \langle \nu, e \rangle \right) \\
+ 2F^{-2} \left( \langle \nabla F, e \rangle + \cos \theta g^{ij} F_j i^k (e_k, e) \right) \\
+ \Delta F \cdot F^{-2} \langle x + \cos \theta \nu, e \rangle - 2F^{-3} \langle x + \cos \theta \nu, e \rangle |\nabla F|^2.
\]
Next we compute the term containing $\Delta F$. Due to the Codazzi equations, the Ricci identities and the Gauss equation, there holds (see for example [42])
\[
h_{kl}^a = h_{ij}^a + (h_{ai} h_{jk} - h_{ik} h_{ja}) h_{i}^a + (h_{ia} h_{ji} - h_{ii} h_{ja}) h_{i}^a,
\]
and thus
\[
g^{ij} F^{kl} h_{kl}^a = F^{kl} h_{kl}^a - F |h|^2.
\]
It follows that
\[
\Delta F = g^{ij} \left( \frac{\partial^2 F}{\partial h_{kl} \partial h_{st}} h_{kl} h_{stij} + \frac{\partial F}{\partial h_{kl}} h_{kl} h_{ij} \right) \\
= F^{kl} H_{,kl} + HF^{kl} h_{kl}^a - F |h|^2 + g^{ij} h_{kl} h_{stij} \frac{\partial^2 F}{\partial h_{kl} \partial h_{st}}.
\]
By adding above terms together into equation (3.12), we get the evolution equation for $H$.

Next we prove $\nabla \mu H \leq 0$ on $\partial \Sigma_t$. Let $\{e_\alpha\}_{\alpha=1}^{n-1}$ be an orthonormal frame of $T(\partial \Sigma_t)$, such that the second fundamental form of $\Sigma_t$ is diagonal with respect to $\{e_\alpha, \mu\}_{\alpha=1}^{n-1}$. From (3.7), we have
\[
0 = \nabla \mu F = F^{\mu \nu} h_{\mu \nu} + \sum_{\alpha=1}^{n-1} F^{\alpha \alpha} h_{\alpha \alpha}.
\]
It follows that
\[
\nabla \mu H = h_{\mu \nu} + \sum_{\alpha=1}^{n-1} h_{\alpha \alpha} = -\sum_{\alpha=1}^{n-1} F^{\alpha \alpha} h_{\alpha \alpha} + \sum_{\alpha=1}^{n-1} h_{\alpha \alpha} \\
= \sum_{\alpha=1}^{n-1} \frac{1}{F^{\mu \mu}} (F^{\mu \mu} - F^{\alpha \alpha}) (h_{\mu \mu} - h_{\alpha \alpha}) h_{\alpha \alpha}.
\]
In the last equality we have used Proposition 2.5 (4). Since $F$ is a concave function,
\[(F^{\mu\mu} - F^{\alpha\alpha})(h_{\mu\mu} - h_{\alpha\alpha}) \leq 0, \quad \forall \alpha.\]
Also, $\tilde{h}_{\alpha\alpha} \geq 0$ due to Corollary 2.6. It follows that
\[
\nabla_{\mu} H = \sum_{\alpha=1}^{n-1} \frac{1}{F_{\mu\mu}} (F^{\mu\mu} - F^{\alpha\alpha})(h_{\mu\mu} - h_{\alpha\alpha}) \tilde{h}_{\alpha\alpha} \leq 0.
\]

\[\square\]

### 3.3. Curvature estimates.

First we have the uniform lower bound of $F$.

**Proposition 3.8.** Along the flow (3.1), it holds
\[F(p, t) \geq \min_{\mathbb{B}^n} F(\cdot, 0), \quad \forall (p, t) \in \mathbb{B}^n \times [0, T^*).\]

*Proof.* From Proposition 3.6, (3.4) and Proposition 2.16 (2), we know
\[\mathcal{L} F \geq 0, \quad \text{mod} \quad \nabla F,
\]
Taking into account (3.7), the assertion follows from the maximum principle.  \[\square\]

In particular, since $F = \frac{\alpha n}{\sigma_{n-1}}$, from the uniform lower bound of $F$, we get uniform curvature positive lower bound.

**Corollary 3.9.** $\Sigma_t, t \in [0, T^*)$ is uniformly convex, that is, there exists $c > 0$ depending only on $\Sigma_0$, such that the principal curvatures of $\Sigma_t$,
\[\min_i \kappa_i(p, t) \geq c, \quad \forall (p, t) \in \mathbb{B}^n \times [0, T^*).\]

Then by Proposition 2.17, we get the following

**Corollary 3.10.** Along the flow (3.1), there exists $c > 0$ depending only on $\Sigma_0$, such that
\[\langle X_e, \nu \rangle \geq c, \quad \forall (p, t) \in \mathbb{B}^n \times [0, T^*).\]

Next we obtain the uniform upper bound of $F$.

**Proposition 3.11.** Along the flow (3.1), there exists $C > 0$ depending only on $\Sigma_0$, such that
\[F(p, t) \leq C, \quad \forall (p, t) \in \mathbb{B}^n \times [0, T^*).\]

*Proof.* Consider the function
\[\Phi := \log F - \alpha \langle x, e \rangle\]
where $\alpha > 0$ will be determined later. Using (3.7) and Proposition 2.16 (3), we have on $\partial \Sigma_t$,
\[\nabla_{\mu} \Phi = -\alpha \nabla_{\mu} \langle x, e \rangle = -\alpha \langle \mu, e \rangle < 0.\]
Thus $\Phi$ attains its maximum value at an interior point, say $p_0 \in \mathbb{B}^n$, then we have
\[\nabla \Phi(p_0) = 0, \quad \nabla^2 \Phi(p_0) \leq 0, \quad \partial_t \Phi(p_0) \geq 0.
\]
Now all the computation below are conducted at the point $p_0$. We have
\[(\log F)_{;i} = \alpha \langle x, e \rangle_{;i}, \quad \text{and} \quad \mathcal{L} \Phi \geq 0.\quad (3.13)\]
From (3.6) and (3.13),
\[
\mathcal{L} \log F = F^{-1} \left[ \frac{2F_{ij}}{F^2} (x + \cos \theta \nu, e)_i F_{ij} - \frac{2(1 + \cos \theta \nu, e)}{F^3} \right] F_{ij} F_{ij} + (1 - F) \langle \nu, e \rangle \\
+ \frac{(x + \cos \theta \nu, e)}{F^2} F_{ij} (\log F)_i (\log F)_j
\]
\[
= F^{-2} (-\alpha^2 (x + \cos \theta \nu, e) + 2\alpha) \cdot F_{ij} (x, e)_i (x, e)_j \\
+ F^{-1} (1 - F) \langle \nu, e \rangle + 2 \cos \theta \alpha F^{-2} F_{ij} (x, e)_j \langle \nu, e \rangle_i.
\]
By choosing orthonormal frame with principal directions, we may assume that \( h_{ij} \) is diagonal at \( p_0 \in \mathbb{B}^n \). By \( (h_{ij}) > 0 \), we have
\[
2 \cos \theta \alpha F^{-2} F_{ij} (x, e)_j \langle \nu, e \rangle_i = 2 \alpha \cos \theta F^{-2} F_{ij} (e_j, e) h_{ik} (e_k, e) \\
\leq 2 \alpha \cos \theta F^{-2} \| F_{ij} h_{ik} \| = 2 \alpha \cos \theta F^{-1}.
\]
Also by \( (h_{ij}) > 0 \) we have \( F_{ii} \leq 1 \) for each \( i \). Thus
\[
F^{-1} (1 - F) \langle \nu, e \rangle \leq CF^{-1}.
\]
By choosing \( \alpha \) large, we have
\[
\mathcal{L} \log F \leq CF^{-1}.
\]
Combining with Proposition 3.5, we get
\[
0 \leq \mathcal{L} \Phi (x_0) = \mathcal{L} (\log F - \alpha (x, e)) \\
\leq CF^{-1} - \alpha \left[ 2F^{-1} (x + \cos \theta \nu, e) \langle \nu, e \rangle + \cos \theta F^{-1} | e^T |^2 - \langle X, e \rangle \right] \\
\leq CF^{-1} + C \alpha F^{-1} + \alpha \langle X, e \rangle.
\]
By using Proposition 3.3, we have
\[
\langle X, e \rangle = \langle x, e \rangle^2 - \frac{1}{2} \langle x^2 \rangle = - \frac{1}{2} (1 - \langle x, e \rangle^2) - \frac{1}{2} (1 - \langle x, e \rangle^2) \leq -c_0,
\]
for some \( c_0 > 0 \). The upper bound for \( F \) follows from (3.14). \( \square \)

Next we obtain the uniform bound of the mean curvature.

**Proposition 3.12.** Along the flow (3.1), there exists \( C > 0 \) depending only on \( \Sigma_0 \), such that
\[
H(p, t) \leq C, \quad \forall (p, t) \in \mathbb{B}^n \times [0, T^*].
\]

**Proof.** Firstly, from (3.11), we know that \( \nabla H \leq 0 \) on \( \partial \Sigma_t \). Thus \( H \) attains its maximum value at some interior point, say \( p_0 \in \mathbb{B}^n \). We conduct all the computation below at \( p_0 \).

Recall the evolution equation (3.10) of \( H \). From the concavity of \( F = \frac{n \sigma_n}{\sigma_{n-1}} \) and Corollary 3.4, we see
\[
F^{-2} \langle x + \cos \theta \nu, e \rangle g^{ij} h_{kl;ij} \frac{\partial^2 F}{\partial h_{kl} \partial h_{st}} \leq 0.
\]
Thus
\[
0 \leq \mathcal{L} H (p_0) \leq J_1 + J_2 + J_3,
\]
where
\[
J_1 := H \left( (x, e) + \langle x + \cos \theta \nu, e \rangle \right) + (HF^{-1} - n) \langle \nu, e \rangle,
\]
\[
J_2 := 2F^{-2} \langle \nabla F, e \rangle - 2x F^{-3} (x + \cos \theta \nu, e) | \nabla F |^2,
\]
\[
J_3 := \cos \theta (h_{ij}^2 F^{-1} \langle \nu, e \rangle - 2| h_{ij} |^2 F^{-1} \langle x + \cos \theta \nu, e \rangle \\
- 2(1 - \varepsilon) F^{-3} (x + \cos \theta \nu, e) | \nabla F |^2 + 2 \cos \theta F^{-2} g^{ij} F_{ij} h_{jk} (e_k, e).
\]
Here \( \varepsilon > 0 \) will be chosen later.
By Cauchy-Schwarz inequality and the bound for $F$, we see easily that
\[ J_1 \leq CH, \quad J_2 \leq C \varepsilon. \]

Next we tackle $J_3$. By choosing orthonormal frame with principal directions, we may assume that $g_{ij} = \delta_{ij}$ and $h_{ij}$ is diagonal at $p_0 \in \mathbb{H}^n$. Then
\[
F \cdot J_3 = - \left[ \langle x, e \rangle + \langle x + \cos \theta \nu, e \rangle \right] |h|^2 - 2(1 - \varepsilon) F^{-2} \langle x + \cos \theta \nu, e \rangle |\nabla F|^2 + 2 \cos \theta F^{-1} \sum_{i=1}^n h_{ii} F_i (e_i, e).
\]
\[
= -W \sum_{i=1}^n \left( F_i - \frac{V_i}{2W} h_{ii} \right)^2 + \frac{1}{W} \sum_{i=1}^n \left( \frac{V_i^2}{4} - UW \right) h_{ii}^2,
\]
where for notation simplicity we used
\[ U := \langle x, e \rangle + \langle x + \cos \theta \nu, e \rangle, \quad W := 2(1 - \varepsilon) F^{-2} \langle x + \cos \theta \nu, e \rangle, \]
and
\[ V_i := 2F^{-1} \cos \theta \langle e_i, e \rangle. \]

Next, we analyze the term $V_i^2 - 4UUW$. For each $i$,
\[
V_i^2 - 4UUW = 4F^{-2} \left[ \cos^2 \theta \langle e_i, e \rangle^2 - 2(1 - \varepsilon) \langle x + \cos \theta \nu, e \rangle \langle \langle x, e \rangle + \langle x + \cos \theta \nu, e \rangle e \rangle \right] \leq 4F^{-2} \left[ \cos^2 \theta |e|^2 - 2(1 - \varepsilon) \langle x + \cos \theta \nu, e \rangle \langle \langle x, e \rangle + \langle x + \cos \theta \nu, e \rangle e \rangle \right].
\]
Recall that we have
\[ \langle x, e \rangle \geq \cos \theta + \varepsilon_0 = \frac{1}{\sqrt{1 - \varepsilon_1}} \cos \theta \]
for some $\varepsilon_1 > 0$, and
\[ \langle x + \cos \theta \nu, e \rangle \geq \varepsilon_0 > 0, \]
due to Proposition 3.3, and Corollary 3.4. Then
\[
2(1 - \varepsilon) \langle x + \cos \theta \nu, e \rangle \langle \langle x, e \rangle + \langle x + \cos \theta \nu, e \rangle e \rangle + \cos^2 \theta \langle \nu, e \rangle^2 = 3(1 - \varepsilon) \langle x + \cos \theta \nu, e \rangle \langle \langle x, e \rangle + \langle x + \cos \theta \nu, e \rangle e \rangle^2 + \varepsilon \cos^2 \theta \langle \nu, e \rangle^2 \geq 3(1 - \varepsilon) \varepsilon_0^2 + \frac{1 - \varepsilon}{1 - \varepsilon_1} \cos^2 \theta.
\]
By choosing $\varepsilon = \varepsilon_1$ in the above inequality, we obtain
\[
\cos^2 \theta |e|^2 - 2(1 - \varepsilon) \langle x + \cos \theta \nu, e \rangle \langle \langle x, e \rangle + \langle x + \cos \theta \nu, e \rangle e \rangle \leq -3(1 - \varepsilon_1) \varepsilon_0^2.
\]
It follows that
\[
V_i^2 - 4UUW \leq -12(1 - \varepsilon_1) \varepsilon_0^2 F^{-2},
\]
and hence
\[ J_3 \leq -C|h|^2. \]
Therefore,
\[ 0 \leq LH \leq J_1 + J_2 + J_3 \leq CH + C - C|h|^2. \]
We conclude that $H$ is bounded above. \qed

It follows from Proposition 3.12 and Corollary 3.9 that all the principal curvatures are bounded.

**Corollary 3.13.** $\Sigma_t, t \in [0, T^*)$, has uniform curvature bound, that is, there exists $C > 0$ depending only on $\Sigma_0$, such that the principal curvatures of $\Sigma_t$,
\[ \max_i \kappa_i(p, t) \leq C, \quad \forall(p, t) \in \mathbb{H}^n \times [0, T^*). \]
3.4. Convergence of the flow.

Proposition 3.14. The flow (3.1) exists for all time with uniform $C^\infty$-estimates.

Proof. Because of Corollary 3.10, we can reduce the flow (3.1) to a scalar parabolic equation with oblique boundary condition, by introducing a conformal transformation map as in [42, 46, 50].

Without loss of generality, we may assume $e = E_{n+1} = (0, \ldots, 0, 1)$. Define the transformation map

$$
\varphi : \mathbb{F}^{n+1} \to \mathbb{R}^{n+1}_+ := \{(y', y_{n+1}) \in \mathbb{R}^{n+1} : y_{n+1} > 0\}
$$

$$(x', x_{n+1}) \mapsto \frac{2x' + (1 - |x'|^2 - x_{n+1}^2)E_{n+1}}{|x'|^2 + (x_{n+1} - 1)^2} := (y', y_{n+1}) = y,
$$

where $x = (x', x_{n+1})$ with $x' = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Note that $\varphi$ maps $\mathbb{S}^n \setminus \{e\}$ to $\partial \mathbb{R}^{n+1}_+$, and

$$
\varphi^*(\delta_{\mathbb{R}^{n+1}_+}) = \frac{4}{(|x'|^2 + (x_{n+1} - 1)^2)^2} \delta_{\mathbb{R}^{n+1}_+} := e^{-2u} \delta_{\mathbb{R}^{n+1}_+},
$$

which means that $\varphi$ is a conformal transformation from $\left(\mathbb{F}^{n+1}_+, \delta_{\mathbb{R}^{n+1}_+}\right)$ to $\left(\mathbb{R}^{n+1}_+, \delta_{\mathbb{R}^{n+1}_+}\right)$. Since $\left(\mathbb{F}^{n+1}_+, \delta_{\mathbb{R}^{n+1}_+}\right)$ and $\left(\mathbb{F}^{n+1}_+, (\varphi^{-1})^*(\delta_{\mathbb{R}^{n+1}_+})\right)$ are isometric, a proper embedded hypersurface $\Sigma_t = x(\mathbb{F}^n, \tau)$ in $\left(\mathbb{F}^{n+1}_+, \delta_{\mathbb{R}^{n+1}_+}\right)$ can be identified as $\widehat{\Sigma}_t := y(\mathbb{F}^n, \tau)$ in $\left(\mathbb{R}^{n+1}_+, (\varphi^{-1})^*(\delta_{\mathbb{R}^{n+1}_+})\right)$, where $y := \varphi \circ x$.

As shown in [50], $\varphi_* (X_e) = y$. Therefore, since $\langle X_e, \nu \rangle > 0$ on $\Sigma_t$, we see $\widehat{\Sigma}_t \subset \left(\mathbb{R}^{n+1}_+, (\varphi^{-1})^*(\delta_{\mathbb{R}^{n+1}_+})\right)$ can be written as a radial graph over $\mathbb{S}^n_+$, that is, there exists some positive function $\rho(z', \tau)$ defined on $\mathbb{S}^n_+ \times [0,T^*)$ such that

$$
y = \rho(z'(p, \tau), \tau) z'(p, \tau),
$$

where $(p, \tau) \in \mathbb{S}^n_+ \times [0,T^*)$. Denote $z' := (\beta, \xi) \in [0, \frac{\pi}{2}] \times \mathbb{S}^{n-1}$ and $u := \log \rho$, then the flow (3.1) is equivalent to the following scalar parabolic equation on $\mathbb{S}^n_+$, see [50] for a detailed computation.

$$
\begin{aligned}
\begin{cases}
\partial_t u = \frac{\nu}{\rho e^n} \tilde{f} & \text{in } \mathbb{S}^n_+ \times [0,T^*), \\
\nabla_{\partial_\beta} u = \cos \theta v & \text{in } \partial \mathbb{S}^n_+ \times [0,T^*), \\
u(\cdot, 0) = u_0(\cdot) & \text{on } \mathbb{S}^n_+.
\end{cases}
\end{aligned}
$$

(3.15)

Here $v := \sqrt{1 + |\nabla u|^2}$, $u_0 = \log \rho_0$, $\rho_0$ is the corresponding quantity for $\Sigma_0$ under the transformation $\varphi$, $\tilde{f}$ is the corresponding quantity of $f$ under the transformation $\varphi$, and $\partial_\beta$ is the unit outward normal of $\partial \mathbb{S}^n_+$ on $\mathbb{S}^n_+$.

From the a priori estimates in Corollaries 3.9, 3.10 and 3.13, we see that $u$ is uniformly bounded in $C^2(\mathbb{S}^n_+ \times [0,T^*))$ and the scalar equation in (3.15) is uniformly parabolic. Note that $|\cos \theta| < 1$, the boundary value condition in (3.15) satisfies the uniformly oblique property, from the standard parabolic theory (see e.g. [15, Theorem 6.1, Theorem 6.4 and Theorem 6.5], also [45, Theorem 5] and [32, Theorem 14.23]), we conclude the uniform $C^\infty$-estimates and the long-time existence of solution to (3.15). \hfill \Box

Proposition 3.15. $x(\cdot, \tau)$ smoothly converges to a uniquely determined spherical cap around a with $\theta$-capillary boundary, as $\tau \to \infty$.

Proof. Recall $W_{1,\theta}(\widehat{\Sigma}_t)$ is non-decreasing. Precisely,

$$
\partial_t W_{1,\theta}(\widehat{\Sigma}_t) = \frac{1}{n+1} \int_{\Sigma_t} \left( \frac{\sigma_1 \sigma_{n-1}}{n \sigma_n} - n \right) (x + \cos \theta \nu, e) \geq 0.
$$
Since we have long time existence and uniform $C^\infty$-estimates, we obtain
\[
\int_{\Sigma_t} \left( \frac{\sigma_1 \sigma_{n-1}}{n \sigma_n} - n \right) \langle x + \cos \theta \nu, e \rangle \to 0, \quad \text{as } t \to +\infty.
\]
Hence from the equality characterization of the Newton-MacLaurin inequality \( \frac{\sigma_1 \sigma_{n-1}}{n \sigma_n} \geq n \), we see that any convergent subsequence must converge to a spherical cap. Next we show that any limit of a convergent subsequence is uniquely determined, which implies the flow smoothly converges to a unique spherical cap. We shall use the argument in [42].

Note that we have proved that \( x(\cdot, t) \) subconverges smoothly to a capillary boundary spherical cap \( C_{\theta, r}(e) \). Since \( W_{n, \theta} \) is preserved along the flow (3.1), the radius \( r_\infty \) is independent of the choice of the subsequence of \( t \). We next show in the following that \( e_\infty = e \). Denote \( r(\cdot, t) \) be the radius of the unique spherical cap centered at \( e \) with contact angle \( \theta \) passing through the point \( x(\cdot, t) \). Due to the spherical barrier estimate, i.e. Proposition 3.3, we know

\[
r_{\text{max}}(t) := \max_{\cdot} r(\cdot, t) = r(\xi_t, t),
\]
is non-increasing with respect to \( t \), hence the limit \( \lim_{t \to +\infty} r_{\text{max}}(t) \) exists. Next we claim that

\[
\lim_{t \to +\infty} r_{\text{max}}(t) = r_\infty.
\]  \hfill (3.16)

We prove this claim by contradiction. Suppose (3.16) is not true, then there exists \( \varepsilon > 0 \) such that

\[
r_{\text{max}}(t) > r_\infty + \varepsilon, \quad \text{for } t \text{ large enough.} \]

(3.17)

By definition, \( r(\cdot, t) \) satisfies

\[
2 \langle x, e \rangle \sqrt{r^2 + 2r \cos \theta + 1} = |x|^2 + 2r \cos \theta + 1.
\]  \hfill (3.18)

By taking the time derivative for (3.18), we get

\[
\left( \frac{(r + \cos \theta) \langle x, e \rangle}{\sqrt{r^2 + 2r \cos \theta + 1}} - \cos \theta \right) \partial_t r = \langle \partial_t x, x - \sqrt{r^2 + 2r \cos \theta + 1} e \rangle = \langle f \nu + T, x - \sqrt{r^2 + 2r \cos \theta + 1} e \rangle.
\]

We evaluate at \( (\xi_t, t) \). Since \( \Sigma_t \) is tangential to \( C_{\theta, r}(e) \) at \( x(\xi_t, t) \), we have

\[
\nu_{\Sigma_t}(\xi_t, t) = \nu_{\partial C_{\theta, r}(e)}(\xi_t, t) = \frac{1}{r} \left( x - \sqrt{r^2 + 2r \cos \theta + 1} e \right).
\]

Thus we deduce

\[
\left( \frac{(r_{\text{max}} + \cos \theta) \langle x, e \rangle}{\sqrt{r_{\text{max}}^2 + 2r_{\text{max}} \cos \theta + 1}} - \cos \theta \right) \frac{d}{dt} r_{\text{max}} = r \left( \frac{\langle x + \cos \theta \nu, e \rangle}{F} - \langle X_e, \nu \rangle \right).
\]  \hfill (3.19)

We first claim that there exists \( \delta_1 > 0 \) such that

\[
\frac{(r_{\text{max}} + \cos \theta) \langle x, e \rangle}{\sqrt{r_{\text{max}}^2 + 2r_{\text{max}} \cos \theta + 1}} - \cos \theta \geq \delta_1.
\]  \hfill (3.20)

This follows directly from (3.18). In fact, from (3.18), we see

\[
\langle x, e \rangle^2 (r^2 + 2r \cos \theta + 1) = \frac{(|x|^2 + 1)^2}{4} + r \cos \theta (|x|^2 + 1) + r^2 \cos^2 \theta.
\]  \hfill (3.21)
By using (3.18) again and (3.21), we have

\[
(r + \cos \theta)\langle x, e \rangle - \cos \theta \sqrt{r^2 + 2r \cos \theta + 1}
\]

\[
= (r + \cos \theta)\langle x, e \rangle - \cos \theta \left| x \right|^2 + 2r \cos \theta + 1
\]

\[
= \frac{1}{r\langle x, e \rangle} \left\{ r(r + \cos \theta)\langle x, e \rangle^2 - r^2 \cos^2 \theta - r \cos \theta \left| x \right|^2 + 1 \right\}
\]

\[
= \frac{1}{r\langle x, e \rangle} \left\{ (|x|^2 + 1)^2 + r \cos \theta(|x|^2 + 1) - r \cos \theta \langle x, e \rangle^2 - \langle x, e \rangle^2 - r \cos \theta \left| x \right|^2 + 1 \right\}
\]

\[
= \frac{1}{r\langle x, e \rangle} \left\{ (|x - e||x + e|)^2 + r \cos \theta \left| x - e \right|^2 \right\}.
\]

This yields (3.20).

Since the spherical caps \(C_{\theta, r_{\max}}(e)\) are the static solution to (3.1) and \(x(\cdot, t)\) is tangential to \(C_{\theta, r_{\max}}(e)\) at \(x(\xi_t, t)\), we see from (2.6)

\[
\frac{\langle x + \cos \theta \nu, e \rangle}{\langle X_e, \nu \rangle} \bigg|_{x(\xi_t, t)} = \frac{\langle x + \cos \theta \nu, e \rangle}{\langle X_e, \nu \rangle} \bigg|_{C_{\theta, r_{\max}}(e)} = \frac{1}{r_{\max}(t)}. \tag{3.22}
\]

Since \(x(\cdot, t)\) converges to \(C_{\theta, r_{\infty}}(e_{\infty})\) and \(r_{\infty}\) is uniquely determined, we have

\[
F = \frac{n \sigma_n}{\sigma_{n-1}} \to \frac{1}{r_{\infty}} \text{ uniformly,}
\]

as \(t \to +\infty\). Thus there exists \(T_0 > 0\) such that

\[
\frac{1}{F} - r_{\infty} < \frac{\epsilon}{2},
\]

and hence

\[
\frac{1}{F} - r_{\max}(t) < -\frac{\epsilon}{2},
\]

for all \(t > T_0\). Taking into account of (3.22), we see

\[
\left( \frac{1}{F} - \frac{\langle x + \cos \theta \nu, e \rangle}{\langle X_e, \nu \rangle} \bigg|_{x(t, \xi_t)} \right) < -\frac{\epsilon}{2},
\]

for all \(t > T_0\). Finally, we conclude from (3.19) that there exists some \(C > 0\) such that

\[
\frac{d}{dt} r_{\max} \leq -C \epsilon.
\]

This is a contradiction to the fact that \(\lim_{t \to +\infty} \frac{d}{dt} r_{\max} = 0\), hence the claim (3.16) is true. Similarly, we can obtain that

\[
\lim_{t \to +\infty} r_{\min}(t) = r_{\infty}. \tag{3.23}
\]

Combining this with claim (3.16), we know that \(\lim_{t \to +\infty} r(t, \cdot) = r_{\infty}\), which is a constant. This implies any limit of a convergent subsequence is the spherical cap around \(e\), the claimed uniqueness. We complete the proof of Proposition 3.15. \(\Box\)

Combining Propositions 3.14 and 3.15, we get the assertion in Theorem 3.2.

3.5. Proof of Theorem 1.3.

With the preparations above, using the same approach as in [42, Section 4], we can prove the main result, i.e. Theorem 1.3.

**Proof of Theorem 1.3.** Firstly, for given \(\theta \in (0, \frac{\pi}{2})\), we claim that

\[
f_k(r) := W_k(C_{\theta, r}(e)),
\]
is strictly increasing with respect to \( r > 0 \). To prove this claim, given \( r_0 > 0 \), consider the hypersurface flow \( y : \mathbb{R}^n \times [0, +\infty) \to \mathbb{R}^{n+1} \) given by

\[
\begin{cases}
\partial_t y = X_\varepsilon, & \text{on } \mathbb{R}^n \times [0, +\infty) \\
y(\cdot, 0) = C_{\theta, r_0}(e) & \text{on } \mathbb{R}^n.
\end{cases}
\]

The flow hypersurfaces of this flow are \( C_{\theta, r(t)}(e) \), where the radius \( r(t) \) is some strictly increasing function. In fact, \( r(t) \) satisfies

\[
r'(t) = r\sqrt{r^2 + 2r \cos \theta + 1},
\]

with \( r(0) = r_0 \). Hence from Theorem 2.10, we have

\[
\frac{\partial_t W_{k, \theta}(C_{\theta, r(t)}(e))}{\partial_t} = \frac{k}{n+1} \left( \frac{n}{k-1} \right)^{-1} \int_{C_{\theta, r(t)}(e)} \sigma_k(X_\varepsilon, \nu) dA > 0,
\]

where the last inequality follows from the strictly convex of the spherical caps, hence it yields that \( f_k(r) \) are strictly increasing functions, which follows the claim.

Assume that \( \Sigma \) is strictly convex. By evolving along the flow (3.1) with initial strictly hypersurface \( \Sigma \), we denote its convergence limit be the spherical cap given by \( C_{\theta, r, \infty}(e) \), then we have

\[
W_{n, \theta}(\Sigma) = W_{n, \theta}(C_{\theta, r, \infty}(e)) = f_n(r_{\infty}) = f_n \circ f_k^{-1} \circ f_k(r_{\infty}) \geq f_n \circ f_k^{-1}(W_{k, \theta}(\Sigma)),
\]

where equality holds iff \( \Sigma \) is a spherical cap. Also,

\[
W_{n+1, \theta}(\Sigma) = W_{n+1, \theta}(C_{\theta, r, \infty}(e)).
\]

Note that when \( r \to 0 \), \( C_{\theta, r}(e) \) is more and more close to the cap \( C_{\theta, r}^{\mathbb{R}^{n+1}} \) of radius \( r \) in the half space \( \mathbb{R}^{n+1}_+ \) with \( \theta \)-capillary boundary. Then

\[
(n + 1) \lim_{r \to 0} W_{n+1, \theta}(C_{\theta, r}(e)) = \lim_{r \to 0} \int_{C_{\theta, r}^{\mathbb{R}^{n+1}}} H_n = \frac{\omega_n}{2} I_{\sin^2 \theta} \left( \frac{n}{2}, \frac{1}{2} \right).
\]

The second equality follows from the simple fact that

\[
C_{\theta, r}^{\mathbb{R}^{n+1}} \mid_{r^n} = C_{\theta, 1}^{\mathbb{R}^{n+1}} \mid_{r^n} = \frac{\omega_n}{2} I_{\sin^2 \theta} \left( \frac{n}{2}, \frac{1}{2} \right) r^n.
\]

When \( \Sigma \) is convex but not strictly convex, the inequality (1.9) and the equality (1.8) follows by approximation. The equality characterization in (1.9) can be proved similar to [42] Section 4, by using an argument of [23]. We omit the details here. \( \square \)

In particular, we have following simplified expansion formulas for a convex hypersurface \( \Sigma \) with \( \theta \)-capillary boundary in \( \mathbb{R}^{n+1} \).

1. For \( n = 2 \),

\[
3W_{3, \theta}(\Sigma) = \int_{\Sigma} H_2 dA - \cos \theta |\partial \Sigma| + \sin \theta |\partial \Sigma| = 2\pi(1 - \cos \theta),
\]

due to \( I_{\sin^2 \theta} \left( 1, \frac{1}{2} \right) = 1 - \cos \theta \).

2. For \( n = 3 \),

\[
4W_{4, \theta}(\Sigma) = \int_{\Sigma} H_3 dA + \sin^2 \theta \int_{\partial \Sigma} H_1^3 + |\partial \Sigma| - \sin \theta \cos \theta |\partial \Sigma|
\]

\[
= 2\pi(\theta - \sin \theta \cos \theta),
\]

due to \( I_{\sin^2 \theta} \left( \frac{3}{2}, \frac{1}{2} \right) = \frac{20 - \sin(2\theta)}{\pi} \).
(3) For \( n = 4 \),
\[
5W_{5,\theta}(\hat{\Sigma}) = \int_{\Sigma} H_4 dA + \sin^3 \theta \int_{\partial \Sigma} H_2^3 - \frac{3}{2} \cos \theta \sin^2 \theta \int_{\partial \Sigma} H_1^3 \\
- \cos \theta \left( 1 + \frac{1}{2} \sin^2 \theta \right) |\partial \Sigma| \sin \theta \left( 1 - \frac{\sin^2 \theta}{3} \right) |\partial \Sigma| \\
= \frac{2\pi^2}{3} \left( 2 - 2 \cos \theta - \cos \theta \sin^2 \theta \right).
\]

Remark 3.16. For \( n = 2 \), we can use another viewpoint to show (3.24) directly. From Gauss-Bonnet formula, we see
\[
2\pi = \int_{\Sigma} K dA + \int_{\partial \Sigma} \kappa_g ds = \int_{\Sigma} K dA + \int_{\partial \Sigma} (\cos \theta \kappa + \sin \theta) ds \\
= \int_{\Sigma} K dA + \sin \theta |\partial \Sigma| + \cos \theta (2\pi - |\partial \Sigma|),
\]
where \( K = H_2 \) is the Gauss curvature, \( \kappa_g \) and \( \kappa \) denote the geodesic curvature of \( \partial \Sigma \) in \( \Sigma \) and \( \partial \Sigma \) respectively. From \( \mu = \cos \theta \nu + \sin \theta \mathbf{N} \) implies \( \kappa_g = \cos \theta \kappa + \sin \theta \). Hence it follows
\[
3W_{3,\theta}(\hat{\Sigma}) = \int_{\Sigma} K dA - \cos \theta |\partial \Sigma| + \sin \theta |\partial \Sigma| = 2\pi (1 - \cos \theta).
\]

Acknowledgment: LW is supported by project funded by China Postdoctoral Science Foundation (No. 2021M702143) and NSFC (Grant No. 12171260). CX is supported by NSFC (Grant No. 11871406). Parts of this work was done while LW was visiting the Tianyuan Mathematical Center in Southeast China and school of mathematical sciences at Xiamen University under the support of NSFC (Grant No. 12126102). He would like to express his deep gratitude to the center for its hospitality. We thank the anonymous referee for his/her careful reading and critical comments.

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