Categories of two-colored pair partitions part I: categories indexed by cyclic groups

Alexander Mang\(^1\) · Moritz Weber\(^1\)

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Abstract
We classify certain categories of partitions of finite sets subject to specific rules on the coloring of points and the sizes of blocks. More precisely, we consider pair partitions such that each block contains exactly one white and one black point when rotated to one line; however, crossings are allowed. There are two families of such categories, the first of which is indexed by cyclic groups and is covered in the present article; the second family will be the content of a follow-up article. Via a Tannaka–Krein result, the categories in the two families correspond to easy quantum groups interpolating the classical unitary group \(U_n\) and Wang’s free unitary quantum group \(U_n^+\). In fact, they are all half-liberated in some sense and our results imply that there are many more half-liberation procedures than previously expected. However, we focus on a purely combinatorial approach leaving quantum group aspects aside.

Keywords
Quantum group · Unitary easy quantum group · Unitary group · Half-liberation · Tensor category · Two-colored partition · Partition of a set · Category of partitions

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Alexander Mang
s9almang@stud.uni-saarland.de

Moritz Weber
weber@math.uni-sb.de

\(^1\) Saarland University, Fachbereich Mathematik, 66041 Saarbrücken, Germany
Introduction

This article is part of the classification program begun in [11], having its roots in [3]. Our base objects are partitions of finite sets into disjoint subsets, the blocks. In addition, the points are colored either black or white. We represent partitions pictorially as diagrams using strings representing the blocks; see also [9], [10]. If a set of partitions is closed under certain natural operations like horizontal or vertical concatenation or reflection at some axes, we call it a category of partitions. Categories of partitions play a crucial role in Banica and Speicher’s approach ([3,14,15]) to compact quantum groups in Woronowicz’s sense ([16–19]). Although our investigations employ purely combinatorial means, let us briefly mention how they relate to the half-liberation procedures of Banica and Speicher.

The half-liberated orthogonal quantum group $O_n^*$, introduced by Banica and Speicher in [3], represents a midway point between the free orthogonal quantum group $O_n^+$, constructed by Wang in [13], and the classical orthogonal group $O_n$ over the complex numbers. It is defined by replacing the commutation relations $ab = ba$ by the half-commutation relations $abc = cba$. On the combinatorial side, these half-commutation relations are represented by the partition

![Partition diagram](image)

on non-colored points.

If attempting a similar procedure in the case of the free unitary quantum group $U_n^+$, also defined by Wang in [13], and the classical unitary group $U_n$, it is not clear what should be the analogue of the half-commutation relations. Here, the generators are no longer self-adjoint and hence their adjoints are involved. Equivalently, we now have to deal with partitions of two-colored points.

A natural starting point would be to consider the different possibilities for coloring the points of the above partition. And, indeed, this approach yields exactly two distinct categories. Each of the partitions

![Two partition diagrams](image)

and

generates one. These categories (or rather their associated quantum groups) appeared first in [6, Example 4.10] and [4, Definition 5.5]. Elaborating on the initial idea, one might next study the shapes
and apply various colorings to them. Thus, one arrives at a number of further categories, discovered by Banica and Bichon in [1]. See Sect. 9 for an overview of the previous work on half-liberations of $U_n$.

Here, however, we take a different approach. Even though both of the partitions embody the half-commutation relations $abc = cba$ in the orthogonal case, the simple, but crucial observation which unlocks the combinatorics of half-liberation in the unitary case is choosing the latter partition, which we call a bracket, rather than the former as our model. The power of this “bracket approach” becomes especially apparent in the follow-up article [8].

Combining the results of both articles, we eventually determine all possible half-liberations of $U_n$, i.e., all unitary easy quantum groups $G$ with $U_n \subseteq G \subseteq U_n^+$, or equivalently, all categories $\mathcal{C}$ with $\langle \mathcal{C} \rangle \supseteq \mathcal{C} \supseteq \langle \emptyset \rangle$. Easy quantum groups were first introduced by Banica and Speicher in the orthogonal setting in [3]; the extension to the unitary case is due to Tarrago and the second author [11,12]. More precisely, given any partition, we may associate a certain linear map to it; performing this for all partitions in a category of partitions, we obtain a tensor category in Woronowicz’s sense—and hence a compact matrix quantum group due to his Tannaka–Krein Theorem from [17]. This quantum group is called an “easy quantum group”. The investigation and classification of easy quantum groups is a young but quite active field. See for instance [15] for details and a survey on the latest research in this direction.

In the present article, we classify all categories above a certain category $\mathcal{S}_0$; the remainder will be found in [8]. Throughout, we employ combinatorial means exclusively, making no use of any quantum group notions or techniques.

1 Main results

We introduce and describe certain categories of partitions $(\mathcal{S}_w)_{w \in \mathbb{N}_0}$ which may be seen as categories indexed by the cyclic groups $(\mathbb{Z}/w\mathbb{Z})_{w \in \mathbb{N}_0}$.

Main Theorem 1 (a) For every $w \in \mathbb{N}_0$, a category of two-colored partitions is given by the set $\mathcal{S}_w$ containing all two-colored pair partitions with the following two properties satisfied when rotated to one line:
(1) Each block contains one point each of every color.
(2) Between the two legs of any block, the difference in the numbers of black and white points is a multiple of \( w \).

(b) The categories \((S_w)_{w \in \mathbb{N}_0}\) are pairwise distinct, and, for all \( w \in \mathbb{N} \) (i.e. \( w \neq 0 \)), the inclusions and equalities

\[
S_0 = \bigcap_{w' \in \mathbb{N}} S_{w'} \subseteq S_w \subseteq S_1 = \langle \bigodot \rangle
\]

are valid, and for all \( w, w' \in \mathbb{N} \) holds

\[
w' \mathbb{Z} \subseteq w \mathbb{Z} \implies S_{w'} \subseteq S_w.
\]

(c) For all \( w \in \mathbb{N} \), the category \( S_w \) is generated by the partition

while the category \( S_0 \) is cumulatively generated by the partitions \( \bigodot \) and

for all \( v \in \mathbb{N} \).

We classify all categories not contained in \( S_0 \).

**Main Theorem 2** For every category \( C \) of two-colored partitions with

\[
\langle \emptyset \rangle \subseteq C \subseteq \langle \bigodot \bigodot \rangle,
\]

either \( C \nsubseteq S_0 \) holds or there exists \( w \in \mathbb{N}_0 \) such that \( C = S_w \).

Our results in this article may be interpreted in two ways. First, we obtain a partial classification result for half-liberated unitary easy quantum groups, extending the work begun by Banica, Bichon and others. Second, we provide a new (combinatorial) picture...
for the previously known unitary half-liberations. More precisely, the quantum groups $U_n^{**}$ by Bichon and Dubois-Violette [6] and $U_{n,k}$ by Banica and Bichon [1] (with $U_{n,2} = U_n^{**}$) correspond to our categories $\mathcal{S}_k$ for $k \geq 1$, shedding a different light on their combinatorics. Banica and Bichon’s limiting case $U_{n,\infty}$, however, does not correspond to our limiting case $\mathcal{S}_0$; in the follow-up article [8] we will show that the category corresponding to $U_{n,\infty}$ is an honest subcategory of $\mathcal{S}_0$. See also Sect. 9 for more details on the interpretation of our results in the context of the previous research about half-liberations of $U_n$ and for a brief outlook on completing the classification of unitary easy half-liberations in [8].

2 Reminder about partitions and their categories

We refer to [11, Sect. 1] for an introduction to categories of two-colored partitions. However, let us briefly recall the basics.

2.1 Two-colored partitions

A (two-colored) partition, is a partition of a finite set into disjoint subsets, the blocks. Moreover, the points are distinguished into an upper and a lower row of points, each totally ordered. Finally, each point is colored either white (◦) or black (●). We say that these two colors are inverse to each other. If each block of a partition consists of exactly two points, we speak of a pair partition. The two points of such a block are called its legs. In this article we exclusively deal with pair partitions.

We illustrate two-colored partitions by two lines of black and white dots connected by strings representing the blocks.

2.2 Operations

On the set $\mathcal{P}^{\circ\bullet}$ of all two-colored partitions, one can introduce several operations: The tensor product $\otimes$ concatenates two partitions horizontally.

If we swap the roles of the upper and the lower row of $p \in \mathcal{P}^{\circ\bullet}$, we call this the involution $p^*$ of $p$.

A pair $(p, p')$ from $\mathcal{P}^{\circ\bullet}$ is composable if the upper row of $p$ and the lower row of $p'$ concur in size and coloration. In the composition $pp'$ of $(p, p')$, we then take the points and colors on the lower row from $p$ and on the upper row from $p'$. The blocks of $pp'$ are obtained by vertical concatenation of $p$ and $p'$. 
Reversing the order in each row of a partition is called *reflection*. If that is followed by inversion of colors, we speak of the *verticolor reflection* \( \tilde{p} \) of \( p \in \mathcal{P}^{\bullet} \).

Multiple kinds of *rotations* can be defined: Given \( p \in \mathcal{P}^{\bullet} \), we remove the leftmost point \( \alpha \) on the upper row and add left to the points on the lower row a point \( \beta \) of the inverse color of \( \alpha \). The new point \( \beta \) belongs to the same block \( \alpha \) did before. The resulting partition is denoted by \( p^\ell \). Likewise, we can move the rightmost point on the upper row to the right end of the lower row, yielding \( p^\ell \). Rotating the outermost points from the lower to the upper row produces the partitions \( p^\ell \) and \( p^\ell \). *Cyclic rotations* are defined by \( p^\ell := (p^\ell)^\ell \) and \( p^\ell := (p^\ell)^\ell \).

Lastly, if we *erase* a set \( S \) of points in \( p \in \mathcal{P}^{\bullet} \), we obtain the partition \( E(p, S) \), which is constructed from \( p \) by removing \( S \) and combining all the remnants of the blocks which had a non-empty intersection with \( S \) into one block.

See [11] for examples of these operations.

### 2.3 Categories

If a subset of \( \mathcal{P}^{\bullet} \) is closed under tensor products, involution and composition and if it contains \( \downarrow \), \( \downarrow \), \( \updownarrow \), and \( \updownarrow \), it is called a *category of partitions*. Categories are invariant with respect to all rotations and verticolor reflection. Note that they might not be closed under mere reflection. For any set \( G \subseteq \mathcal{P}^{\bullet} \), we denote the smallest category containing \( G \) by \( \langle G \rangle \) and say that \( G \) *generates* \( G \). See [11]. Categories of partitions (with uncolored points) and the above operations were introduced by Banica and Speicher in [3].

### 3 Basic concepts for pair partitions with neutral blocks

#### 3.1 Cyclic order

In a partition \( p \in \mathcal{P}^{\bullet} \), the lower row \( L \) and the upper row \( U \) are each natively equipped with a total order, \( \leq_L \) and \( \leq_U \), respectively. We additionally endow the entirety \( L \cup U \) of the points of \( p \) with a *cyclic order*. It is uniquely determined by the following four conditions: It induces \( \leq_L \) on \( L \), but the *reverse* of \( \leq_U \) on \( U \), the minimum of \( \leq_L \) succeeds the minimum of \( \leq_U \), and the maximum of \( \leq_L \) precedes the maximum of \( \leq_U \).
In our convention of depicting the native ordering left to right on both rows, the cyclic order amounts to the counter-clockwise orientation.

With respect to this cyclic order, it makes sense to speak of intervals like \([\alpha, \beta]_p, \, [\alpha, \beta]_p\), etc. for points \(\alpha, \beta\) in \(p\), even if \(\alpha\) and \(\beta\) are not both contained in the same row of \(p\).

### 3.2 Connectedness

Blocks \(B\) and \(B'\) in \(p \in P^{\circ}\) are said to cross if \(B \neq B'\) and if there are pairwise distinct points \(\alpha, \beta \in B\) and \(\gamma, \delta \in B'\) occurring in the sequence \((\alpha, \gamma, \beta, \delta)\) with respect to the cyclic order. If no two blocks cross in \(p\), we say that \(p\) is non-crossing, in short: \(p \in NC^{\circ}\). (See [11] for all subcategories of \(NC^{\circ}\).)

We call the blocks \(B\) and \(B'\) connected if \(B = B'\), if \(B\) and \(B'\) cross, or if there exist blocks \(B_1, \ldots, B_m\) in \(p\) such that, writing \(B_0 := B\) and \(B_{m+1} := B'\), for every \(i \in \mathbb{N}_0\) with \(i \leq m\), block \(B_i\) crosses block \(B_{i+1}\).

The classes of this equivalence relation are the connected components of \(p\). And we say that \(p\) is connected if it has only a single connected component. Erasing the complement of any connected component \(S\) of \(p\) yields the factor partition of \(S\).
3.3 Color sum and neutral sets

The normalized color of a point is simply its color in the case of a lower point, but the inverse of its color in the case of an upper point. We think of it as the color the point would have if rotated to the lower line.

For any set $S$ of points in $p \in \mathcal{P}^\circ\bullet$, the color sum $\sigma_p(S)$ is given by the difference between the numbers of points in $S$ of normalized color $\bullet$ and of normalized color $\circ$.

$$\sigma_p(A) = \sigma_p(B) = 0$$

A neutral set $S$ of points of $p$ is a null set of the signed measure $\sigma_p$, i.e. one with $\sigma_p(S) = 0$. Categories of partitions are closed under erasing neutral intervals: Simply use compositions with the pair partitions $\updownarrow$ and $\downarrow\uparrow$ and their involutions.

**Definition 3.1** We say that $p \in \mathcal{P}^\circ\bullet$ is a pair partition with neutral blocks, $p \in \mathcal{P}^\circ\bullet_{2,\text{nb}}$, in short, if $p$ is a pair partition and all its blocks are neutral.

For example, the partition depicted in Sect. 2.1 is a pair partition with neutral blocks. The set $\mathcal{P}^\circ\bullet_{2,\text{nb}} = \langle \rangle$ is in fact a category of partitions, generated by the crossing partition. (See $\mathcal{O}_{\text{grp.loc}}$ in [11, Proposition 3.3, Lemma 8.2, Theorem 8.3].)

3.4 Sectors

If a proper subset $S$ of points in $p \in \mathcal{P}^\circ\bullet$ is an interval with respect to the cyclic order, we refer to the set comprising exactly the first point of $S$ and the last point of $S$ as the boundary $\partial S$ of $S$. In addition, define $\text{int}(S) := S \setminus \partial S$, the interior of the set $S$.

We call $S$ a sector in $p$ if $\partial S$ is a block of $p$. A sector $S'$ in $p$ with $S' \subseteq S$ is called a subsector of $S$. 
4 Definition of $S_w$, distinctness and set relationships [main Theorem 1 (b)]

The central result of this article concerns the following sets of partitions:

**Definition 4.1** For every $w \in \mathbb{N}_0$, denote by $S_w$ the set of all partitions $p \in \mathcal{P}^\circ_{2,\text{nb}}$, such that $\sigma_p(S) \in w\mathbb{Z}$ for all sectors $S$ in $p$.

Part (b) of Main Theorem 1 follows immediately from this definition. Moreover, we observe that the choices $w\mathbb{Z}$ or $\mathbb{Z}/w\mathbb{Z}$ as an index for $S_w$ would be natural alternatives. This fact, together with observations made in Sect. 9 and [8], justify the title of the present article.

**Proposition 4.2** The following identities and inclusions are valid for all $w \in \mathbb{N}$:

$$S_0 = \bigcap_{w' \in \mathbb{N}} S_{w'} \subseteq S_w \subseteq S_1 = \mathcal{P}^\circ_{2,\text{nb}} = \langle \mathcal{S}_0 \rangle$$

Furthermore, for all $w, w' \in \mathbb{N}$ holds

$$w'\mathbb{Z} \subseteq w\mathbb{Z} \implies S_{w'} \subseteq S_w.$$

Moreover, $S_w \neq S_{w'}$ for all $w, w' \in \mathbb{N}_0$ with $w \neq w'$.

**Proof** (1) For every $p \in \mathcal{P}^\circ_{2,\text{nb}}$, all its sectors $S$ necessarily satisfy $\sigma_p(S) \in \mathbb{Z}$ by definition. Hence, $S_1 = \mathcal{P}^\circ_{2,\text{nb}}$ follows. In [11, Proposition 3.3, Lemma 8.2, Theorem 8.3], it was proven that the category $\mathcal{P}^\circ_{2,\text{nb}}$ is generated by $\mathcal{S}_0$.

(2) For $w, w' \in \mathbb{N}$, a partition $p \in \mathcal{P}^\circ_{2,\text{nb}}$ and a sector $S$ in $p$, the relation $\sigma_p(S) \in w'\mathbb{Z}$ implies $\sigma_p(S) \in w\mathbb{Z}$ if $w'\mathbb{Z} \subseteq w\mathbb{Z}$. Hence, $S_{w'} \subseteq S_w$.

(3) For every integer $w \in \mathbb{Z}$ holds $w = 0$ if and only if $w\mathbb{Z} \subseteq w'\mathbb{Z}$ is true for all $w' \in \mathbb{N}$. We conclude $\bigcap_{w' \in \mathbb{N}} S_{w'} = S_0$.

(4) The partitions listed in Main Theorem 1 provide examples of elements of the sets $S_w$ and prove $S_{w'} \neq S_w$ for $w, w' \in \mathbb{N}_0$ with $w' \neq w$. □

5 Category property of $S_w$ [main Theorem 1 (a)]

To show that $S_w$ is a category of partitions for every $w \in \mathbb{N}_0$, we need to check that the category operations preserve the defining property of $S_w$. The proof is facilitated by two simple facts:

**Lemma 5.1** If $S$ and $S'$ are the two sectors of a block in $p \in \mathcal{P}^\circ_{2,\text{nb}}$, then

$$\sigma_p(S') = -\sigma_p(S).$$

**Proof** The entirety $P_p$ of the points of $p$ is partitioned by the blocks of $p$. Since those are all neutral due to $p \in \mathcal{P}^\circ_{2,\text{nb}}$, so is their union: $\sigma_p(P_p) = 0$. The assumptions on $S$ and $S'$ imply that $P_p$ is also partitioned by $\text{int}(S)$, $\partial S = \partial S'$, and $\text{int}(S')$. Now, $\sigma_p(\partial S) = \sigma_p(\partial S') = 0$ yields the assertion. □
In conclusion, it suffices to check $\sigma_p(S) \in w\mathbb{Z}$ for only a single and arbitrary one of the two sectors $S$ of each block in a given $p \in \mathcal{P}_{2,\text{nb}}^{\circ\bullet}$ to prove $p \in S_w$. This can be done comfortably:

**Lemma 5.2** If $\alpha, \alpha'$ and $\beta$ are arbitrary points in $p \in \mathcal{P}_{2,\text{nb}}^{\circ\bullet}$, then

$$\sigma_p\left(\left[\alpha, \alpha'\right]_p\right) = \sigma_p\left(\left[\alpha, \beta\right]_p\right) + \sigma_p\left(\left[\beta, \alpha'\right]_p\right),$$

regardless of the order in which $\alpha, \alpha'$ and $\beta$ appear in $p$.

Similar results hold for other kinds of intervals and other decompositions, e.g.,

$$\sigma_p\left(\left[\alpha, \alpha'\right]_p\right) = \sigma_p\left(\left[\alpha, \beta\right]_p\right) + \sigma_p\left(\left[\beta, \alpha'\right]_p\right)$$

for points $\alpha, \alpha', \beta$ in $p \in \mathcal{P}_{2,\text{nb}}^{\circ\bullet}$.

**Proof** This lemma is clear if the order is $(\alpha, \beta, \alpha')$. If it is $(\alpha, \alpha', \beta)$ instead, we promptly employ the evident case of the lemma to find $\sigma_p\left(\left[\alpha, \beta\right]_p\right) + \sigma_p\left(\left[\beta, \alpha'\right]_p\right) = \sigma_p\left(\left[\alpha, \alpha'\right]_p\right) + \sigma_p\left(\left[\alpha', \beta\right]_p\right)$. The claim then follows from $\sigma_p\left(\left[\alpha', \alpha'\right]_p\right) = 0$, i.e. the fact that $p$ as a whole is neutral. 

**Proposition 5.3** For all $w \in \mathbb{N}$, the set $S_w$ is a category of partitions.

**Proof** Involution merely changes the sign of the color sum of a given sector. Moreover, since each partition in $\mathcal{P}_{2,\text{nb}}^{\circ\bullet}$, seen as the set of all of its points, is neutral, the color sums of sectors remain unchanged even if an entire such partition is injected via tensor product into a sector which is spread across both lines. To prove that $S_w$ is a category, we thus only need to show that $S_w$ is invariant under composition.

Let $(p, p')$ be a composable pairing from $S_w$. When dealing with compositions, it is convenient to use the following notation: We can reference points in partitions by their rank $x \in \mathbb{N}$ in the ordering of the row (not the cyclic order) if we write $\bullet x$ for a lower point and $\circ x$ for an upper one.

![Diagram](attachment:image.png)

We only treat the case of a through block $B$ in $pp'$, i.e. a block comprising both a lower point $\bullet a$ and an upper point $\circ b$. We now prove $\sigma_{pp'}\left(\left[\bullet a, \circ b\right]_{pp'}\right) \in w\mathbb{Z}$.
Because \( p \) and \( p' \) are pair partitions, the points of \( B \) must be linked through a sequence of blocks, alternately of \( p \) and of \( p' \). The blocks successively connect \( a \), oddly many points of indices \( x_1, \ldots, x_{2n+1} \) on the common row, and, finally, \( b \). Suppose that \( p' \) has \( k \) upper and \( l \) lower points and that \( p \) has \( m \) lower points.

We decompose, using Lemma 5.2,

\[
\begin{align*}
\sigma_{pp'}(\square_a, \bullet b_{pp'}) &= \sigma_{pp'}(\square_a, \bullet m_{pp'}) + \sigma_{pp'}(\bullet k, \bullet b_{pp'}) \\
&= \sigma_p(\square_a, \bullet m_p) + \sigma_{p'}(\bullet k, \bullet b_{p'}) \\
&= \sigma_p(\square_a, \bullet m_p) + \sigma_{p'}(\square x_{2n+1}, \bullet b_{p'}) - \sigma_{p'}(\square x_{2n+1}, \bullet l_{p'}) .
\end{align*}
\]

(5.1)

The order induced by the cyclic order of \( p' \) on its lower row is the exact opposite of the one induced by the cyclic order of \( p \) on its upper row. Whereas the colors of those two rows match, for a rank \( z \in \mathbb{N} \), \( z \leq l \), the normalized colors of the point \( a \) in \( p \) and the point \( b \) in \( p' \) are inverse to each other. Applying this to the set \( \square x_{2n+1}, \bullet l_{p'} \), we find

\[
\sigma_{p'}(\square x_{2n+1}, \bullet l_{p'}) = -\sigma_p(\square l, \bullet x_{2n+1}[p]) .
\]

Inserting this equality into Eq. (5.1) yields

\[
\sigma_{pp'}(\square a, \bullet b_{pp'}) = \sigma_p(\square a, \bullet m_p) + \sigma_{p'}(\square x_{2n+1}, \bullet b_{p'}) + \sigma_p(\bullet l, \bullet x_{2n+1}[p]) \\
= \sigma_p(\square a, \bullet x_{2n+1}[p]) + \sigma_{p'}(\square x_{2n+1}, \bullet b_{p'}) .
\]

(5.2)

If \( n = 0 \), the sets \( \square_a, \bullet x_{2n+1}[p] \) and \( \square x_{2n+1}, \bullet b_{p'} \) are the interiors of sectors in \( p \) and \( p' \), respectively. Hence, in this case, Eq. (5.2) already proves the claim since \( \sigma_p(\square a, \bullet x_{2n+1}[p]) \), \( \sigma_{p'}(\square x_{2n+1}, \bullet b_{p'}) \) \( \in w \mathbb{Z} \) by assumption.

If \( n > 0 \), we can employ again Lemma 5.2 iteratively to decompose

\[
\begin{align*}
\sigma_p(\square a, \bullet x_{2n+1}[p]) &= \sigma_p(\square a, \bullet x_1[p]) + \sum_{j=1}^n \sigma_p(\bullet x_{2j-1}, \bullet x_{2j}) \\
&+ \sum_{j=1}^n \sigma_p(\bullet x_{2j}, \bullet x_{2j+1}[p]) .
\end{align*}
\]

Using again the relationship between the orientations and the normalized colors of \( p \) and \( p' \) and combining this equality with Eq. (5.2) gives an expression for \( \sigma_{pp'}(\square a, \bullet b_{pp'}) \) as a sum of the color sums of sectors in \( p \) and \( p' \), proving that it must be a multiple of \( w \). That is what we needed to show. □
6 Proof technique: brackets

In this section, we develop general tools to classify subcategories of $\mathcal{P}_{2,\text{nb}}$ and find their generators. They will be employed in Sects. 7 and 8 to prove the remaining parts of the Main Theorems and in the follow-up article [8] to determine all subcategories of $\mathcal{S}_0$.

6.1 Brackets

To identify the generators of subcategories of $\mathcal{P}_{2,\text{nb}}$, we find a set $\mathcal{B}_{\text{res}} \subseteq \mathcal{P}_{2,\text{nb}}$ of “universal generators”, namely with the property $\mathcal{C} = \langle \mathcal{C} \cap \mathcal{B}_{\text{res}} \rangle$ for all categories $\mathcal{C} \subseteq \mathcal{P}_{2,\text{nb}}$ (see Proposition 6.13).

The set $\mathcal{B}_{\text{res}}$ will be defined in Sect. 6.2 as a special subset of the set of all brackets. These latter partitions we introduce and study in the following.

Definition 6.1  
(a) We call $p \in \mathcal{P}_{2,\text{nb}}$ a bracket if $p$ is projective, i.e. $p = p^*$ and $p_2 = p$, and if the lower row of $p$ is a sector in $p$.
(b) The set of all brackets is denoted by $\mathcal{B}$.

Categories in $\mathcal{P}_{2,\text{nb}}$ are closed under a certain kind of projection operation which produces brackets as described in the sequel.

Definition 6.2  
Given $p, p' \in \mathcal{P}_{2,\text{nb}}$ and sectors $S$ in $p$ and $S'$ in $p'$, we number the points in $\text{int}(S)$ and $\text{int}(S')$ with respect to the cyclic order. We say that $(p, S)$ and $(p', S')$ are equivalent if the following four conditions are met:

1. The sectors $S$ and $S'$ are of equal size.
2. The same normalized colors occur in the same order in $S$ and $S'$.
3. For all $i$, the point at position $i$ in $S$ belongs to a block crossing $\partial S$ in $p$ if and only if the block of $i$-th point in $S'$ crosses $\partial S'$ in $p'$.
4. For all $i, j$, the points at positions $i, j$ in $S$ form a block in $p$ if and only if the $i$-th and $j$-th points of $S'$ constitute one of the blocks of $p'$.

In other words: $p$ restricted to $S$ coincides with $p'$ restricted to $S'$ (up to rotation).

Definition 6.3  
Let $S$ be a sector in $p \in \mathcal{P}_{2,\text{nb}}$. We refer to the (uniquely determined) bracket $q$ with lower row $M$ which satisfies that $(p, S)$ and $(q, M)$ are equivalent as the bracket $B(p, S)$ associated with $(p, S)$.

We may construct such brackets as described in the proof of the following lemma.

Lemma 6.4  
For all sectors $S$ in $p \in \mathcal{P}_{2,\text{nb}}$ holds $B(p, S) \in \langle p \rangle$. 
Proof Rotations preserve equivalence. The category $\langle p \rangle$ contains as well the partition $p'$ which results from rotating $p$ in such a way that the image of $S$ comprises exactly the lower row of $p'$.

By a straightforward generalization of [7, Theorem 2.12] to two-colored partitions, the partition $p' (p')^* \in \langle p \rangle$ is projective. It is already the bracket $B(p, S)$.

Brackets can act via composition on suitable partitions to undo the erasing (see Sects. 2.2 and 3.3) of certain sets of points.

Definition 6.5 (1) A neutral set of two consecutive points (with respect to the cyclic order) is called a turn.

(2) A block $B$ in a partition $p \in P_{2, nb}$ is a turn block of a turn $T$ in $p$ if $B \neq T$ and $B \cap T \neq \emptyset$.

Lemma 6.6 Let $S$ be a sector in $p \in P_{2, nb}$ such that $\partial S$ is a turn block of a turn $T$ in $p$. Then,

$$\langle p \rangle = \langle E(p, T), B(p, S) \rangle .$$

Proof Since $\langle p \rangle$ is closed under erasing turns, Lemma 6.4 proves one direction of the claim. Conversely, start with the generators $E(p, T)$ and $B(p, S)$. As $T$ has a turn block, $p$ cannot consists of just two points. Moreover, we can assume that $p$ is rotated in such a way that it has no upper points and that $T$ consists of the first and the last point of the lower row of $p$. We only treat the case that the first point of the lower row of $p$ is black. The other case is handled in an analogous manner. The upper row of the partition $p' := (\bigotimes E(p, T))^{\odot}$, an element of $\langle E(p, T), B(p, S) \rangle$, has the same number of points and the same coloration as the lower row of $p$.

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Extend $B(p, S)$ to the right by a tensor product $u$ of suitable partitions from $\{1, \emptyset\}$ such that $(B(p, S) \otimes u, p')$ is composable and find that the composition $(B(p, S) \otimes u)p'$, which is an element of $\langle E(p, T), B(p, S) \rangle$, is equal to $p$. \hfill \Box

Besides by rotations, it is by this operation of “de-erasing” turns alone that the set $B$ generates all partitions of $P_{\cdot \cdot}^{\emptyset, n, b}$, as shown in the following.

**Lemma 6.7** For every category $C \subseteq P_{\cdot \cdot}^{\emptyset, n, b}$ holds $C = \langle C \cap B \rangle$. Moreover, for all $n \in \mathbb{N}$, the brackets in $C \cap B$ with at most $2n$ points are sufficient to generate all partitions of $C$ with at most $n + 1$ points.

**Proof** For all $n \in \mathbb{N}$, denote by $B_n$ the set of all brackets with at most $2n$ points. It was shown in [11, Proposition 3.3 (a)] that $P_{\cdot \cdot}^{\emptyset, \emptyset, n, b} \cap NC^{\emptyset, \emptyset} = \{\emptyset\}$, implying $C \cap NC^{\emptyset, \emptyset} \subseteq \langle C \cap B_n \rangle$ for all $n \in \mathbb{N}$. Especially, all partitions of $C$ with at most 2 points are contained in $(C \cap B_1)$ as they are all non-crossing. Let $n \in \mathbb{N}$ with $n \geq 2$ be arbitrary, suppose that $\langle C \cap B_{n-1} \rangle$ contains all partitions of $C$ with at most $n$ points and let $p \in C$ have $n + 1$ points. We can assume $p \notin NC^{\emptyset, \emptyset}$. Then, we can find a sector $S$ of a turn block $\partial S$ of a turn $T$ in $p$. The partition $B(p, S)$, an element of $C$ by Lemma 6.4, is a bracket with at most $2n$ points and thus an element of $C \cap B_n$ by definition of $B_n$. The partition $E(p, T)$ has $n - 1$, so no more than $n$, points and is hence contained in $\langle C \cap B_{n-1} \rangle$ by the induction hypothesis. As $C \cap B_{n-1} \subseteq C \cap B_n$, we infer $E(p, T) \in \langle C \cap B_n \rangle$. Having seen $E(p, T), B(p, S) \in \langle C \cap B_n \rangle$, Lemma 6.6 now proves $p \in \langle C \cap B_n \rangle$ and thus the claim. \hfill \Box

### 6.2 Residual brackets

While Lemma 6.7 provides a first tool for understanding subcategories of $P_{\cdot \cdot}^{\emptyset, \emptyset, n, b}$, the set $B$ of all brackets is still too large to be tractable. In this subsection, the subset $B_{\text{res}}$ of $B$ is defined and shown to also satisfy $C = \langle C \cap B_{\text{res}} \rangle$ for all categories $C \subseteq P_{\cdot \cdot}^{\emptyset, \emptyset, n, b}$. In several steps (leading up to Definition 6.12), we introduce the defining properties individually and note their respective significance. A first demand we can make on our universal generators is that they be connected (see Sect. 3.2). Connectedness is
a modest assumption in $\mathcal{P}_{2,\text{nb}}^{\circ\bullet}$ as the following result shows. Recall that the factor partition of a connected component $S$ of $p \in \mathcal{P}^{\circ\bullet}$ is $E(p, S^c)$, where $S^c$ denotes the complement of $S$.

**Lemma 6.8** A category contains a partition $p \in \mathcal{P}_{2,\text{nb}}^{\circ\bullet}$ if and only if it contains all the factor partitions of the connected components of $p$.

**Proof** If $p$ is not connected, there must exist a connected component $S$ of $p$ which is an interval with respect to the cyclic order. Since $p$ has neutral blocks, $S$ is a neutral set. Thus, we can erase it and conclude $E(p, S) \in \langle p \rangle$. Likewise, we can erase the complement $S^c$ of $S$ and find $E(p, S^c) \in \langle p \rangle$. The partition $E(p, S)$ is the factor partition of $S$, and $E(p, S^c)$ has one fewer connected component than $p$. We repeat the procedure until we are left with connected partitions only. These elements of $\langle p \rangle$ are rotations of the factor partitions of the connected components of the original partition $p$. That proves one direction of the claim. Because $p$ can be reassembled by appropriate tensor products and rotations from the factor partitions of its connected components, the reverse holds as well. \[\Box\]

By definition, brackets are already required to be fixed points of involution and composition with themselves. We can add further such symmetry conditions.

**Definition 6.9** A partition $p \in \mathcal{P}^{\circ\bullet}$ is called *verticolor-reflexive* if $\tilde{p} = p$.

Verticolor-reflexive partitions necessarily have an even number of points in every one of their rows.

**Definition 6.10** We refer to a bracket $p$ with lower row $S$ as *dualizable* if $p \in \mathcal{S}_0$, if $p$ is verticolor-reflexive, if $\text{int}(S)$ is non-empty, and if the two middle points of $\text{int}(S)$ form a turn in $p$ with turn blocks both of which cross $\partial S$.

Performing a quarter rotation on a dualizable bracket gives the same partition for both directions, and this partition is a bracket as well.

**Definition 6.11** For a dualizable bracket $p$ with $n$ points in its lower row, we call the bracket $p^\dagger := p \odot \frac{n}{2} = p \odot \frac{n}{2}$ the *dual bracket* of $p$.

The dual $p^\dagger$ of a dualizable bracket $p \in \mathcal{P}_{2,\text{nb}}^{\circ\bullet}$ is a dualizable bracket as well and it holds $(p^\dagger)^\dagger = p$ and $\langle p \rangle = \langle p^\dagger \rangle$. Combining connectedness and dualizability, we are now able to give the definition of residual brackets.
Definition 6.12 (a) Let $p$ be a bracket with lower row $S$.

1. We call $p$ residual of the first kind if $p$ is connected and if $\text{int}(S)$ contains no turns of $p$.
2. We call $p$ residual of the second kind if $p$ is connected and dualizable and if $\text{int}(S)$ contains exactly one turn of $p$.
3. We call $p$ residual if $p$ is residual of the first or the second kind.

(b) The set of all residual brackets is denoted by $B_{\text{res}}$.

To reduce the set of brackets $B$ to its subset $B_{\text{res}}$, we decrease the number of turns occurring in a given bracket using the reversible category operation from Lemma 6.6. Residual brackets of the first kind represents those brackets where this reduction is possible until no turns remain. Residual brackets of the second kind, in contrast, arise as the set of brackets reproducing itself under this operation. How this works in detail is seen in the proof of the following enhancement of Lemma 6.7 and central result of this subsection.

Proposition 6.13 For every category $C \subseteq \mathcal{P}^{\bullet}_{2,\text{nb}}$, holds $C = \langle C \cap B_{\text{res}} \rangle$.

Proof A bracket has necessarily at least 4 points. We show inductively that, for all $n \in \mathbb{N}$ with $n \geq 2$, the set of brackets of $C$ with at most $2n$ points is contained in $\langle C \cap B_{\text{res}} \rangle$, which by Lemma 6.7 is sufficient to prove the claim. The set $C \cap B_{\text{res}}$ generates all brackets of size 4 as these are non-crossing and as $\langle \emptyset \rangle = \mathcal{P}^{\bullet}_{2,\text{nb}} \cap \mathcal{N}\mathcal{C}^{\bullet}$.

Let $n \in \mathbb{N}$ with $n \geq 3$ be arbitrary, let all brackets of $C$ with at most $2(n - 1)$ points be elements of $\langle C \cap B_{\text{res}} \rangle$, and let $p$ be a bracket of $C$ with $2n$ points. We distinguish three cases.

Case 1 First, assume that $p$ is not connected. If $p$ had exactly two connected components, then $p$ would have 4 points. Thus, $p$ has at least three connected components.

Then, the partition $p$ being a bracket, at most one connected component of $p$ can contain a through block and thus possibly have more than $n$ elements. (Here and in the following, such estimates for the number of points merely constitute convenient bounds and need not be optimal.) This component encompassing through blocks has no more than $2(n - 1)$ points and its factor partition is necessarily a bracket. The latter is hence contained in $\langle C \cap B_{\text{res}} \rangle$ by the induction hypothesis. All other connected components
of \( p \) have at most \( n \) points as they each comprise non-through blocks exclusively. Because, by the induction hypothesis, \( \langle C \cap B_{\text{res}} \rangle \) contains the brackets with \( 2(n - 1) \) points or less, by Lemma 6.7, all partitions with at most \( n \) points are elements of \( \langle C \cap B_{\text{res}} \rangle \). Consequently, all the factor partitions into which \( p \) decomposes, and thus by Lemma 6.8 the partition \( p \) itself, lie in \( \langle C \cap B_{\text{res}} \rangle \).

**Case 2** Now, suppose that \( p \) is connected, and let the lower row \( S \) of \( p \) contain a proper subsector \( S_0 \) such that \( \partial S_0 \) is a turn block of a turn \( T \) in \( \text{int}(S) \). Let \( T' \) and \( S_0' \) denote their respective counterparts on the upper row of \( p \).

The brackets \( B(p, S_0) \) and \( B(p, S_0') \), both elements of \( C \) by Lemma 6.4, have at most \( 2(n - 1) \) points and are, therefore, elements of \( \langle C \cap B_{\text{res}} \rangle \) by the induction hypothesis. The partition \( E(E(p, T), T') \) is a bracket of \( C \) with at most \( 2(n - 1) \) points as well. It, too, is hence contained in \( \langle C \cap B_{\text{res}} \rangle \) by the induction hypothesis. Because \( \langle C \cap B_{\text{res}} \rangle \) now contains both \( E(E(p, T), T') \) and \( B(p, S_0) = B(E(p, T), S_0') \), applying Lemma 6.6 yields \( E(p, T) \in \langle C \cap B_{\text{res}} \rangle \). A second application of this lemma hence shows \( p \in \langle C \cap B_{\text{res}} \rangle \) as \( E(p, T) \in \langle C \cap B_{\text{res}} \rangle \) and \( B(p, S_0) \in \langle C \cap B_{\text{res}} \rangle \).

**Case 3** Lastly, assume that \( p \) is connected but that only through blocks emanate from any turns \( \text{int}(S) \) might contain. If \( \text{int}(S) \) contains no turns, then \( p \) is residual of the first kind and hence an element of \( C \cap B_{\text{res}} \). So, suppose that \( \text{int}(S) \) contains at least one turn and let \( T \) be the rightmost turn inside \( \text{int}(S) \) and let \( S' \) be the sector to the right of the right turn block \( \partial S' \) of \( T \) (see next page for an illustration). Now, crucially, the bracket \( B(p, S')^\dagger \) is residual of the second kind and hence an element of \( C \cap B_{\text{res}} \). Indeed, since \( S' \) spreads symmetrically across both rows of \( p \) and \( p \) is projective, \( B(p, S') \) is verticolor-reflexive. Dualizable is \( B(p, S') \) because \( S' \) has at its center the boundaries of the sectors of \( p \) given by the lower and upper row of the projective \( p \) and because \( \partial S' \) is a through block of \( p \). With \( B(p, S') \) being a dualizable bracket, so is \( B(p, S')^\dagger \). Lastly, there is exactly one turn in the interior of the lower row of \( B(p, S')^\dagger \) since we chose \( T \) specifically to be the rightmost turn of \( \text{int}(S) \). Moreover, \( B(p, S')^\dagger \) is connected because \( p \) is. Especially, if \( p \) is dualizable itself, then \( p = B(p, S')^\dagger \).
Now, let $T'$ denote the counterpart of $T$ on the upper row again. The partition $E(E(p, T), T')$ is a bracket of $\mathcal{C}$ with $2(n - 1)$ elements at most and hence contained in $\langle \mathcal{C} \cap \mathcal{B}_{\text{res}} \rangle$ by the induction hypothesis. Because $T'$ is a connected component of $E(p, T)$ whose factor partition is a rotation of $\underbrace{\text{\_\_\_}}$ or $\underbrace{\text{\_\_\_}}$, we infer $E(p, T) \in \langle E(E(p, T), T') \rangle \subseteq \langle \mathcal{C} \cap \mathcal{B}_{\text{res}} \rangle$ with the help of Lemma 6.8. Lemma 6.6 now proves $p \in \langle \mathcal{C} \cap \mathcal{B}_{\text{res}} \rangle$ due to $E(p, T) \in \langle \mathcal{C} \cap \mathcal{B}_{\text{res}} \rangle$ and $B(p, S') \in \langle \mathcal{C} \cap \mathcal{B}_{\text{res}} \rangle$.

In conclusion, $p \in \langle \mathcal{C} \cap \mathcal{B}_{\text{res}} \rangle$ holds always, which is what we needed to show. □

### 6.3 Bracket arithmetics

Proposition 6.13 showed that the subcategories of $\mathcal{P}_{2, \text{nb}}$ are given precisely by the set $\{ \langle G \rangle \mid G \subseteq \mathcal{B}_{\text{res}} \}$. In this subsection, we begin investigating the map $\mathcal{P}(\mathcal{B}_{\text{res}}) \to \mathcal{P}(\mathcal{B}_{\text{res}})$, $G \mapsto \langle G \rangle \cap \mathcal{B}_{\text{res}}$. The fixed points of this map are in bijection with the subcategories of $\mathcal{P}_{2, \text{nb}}$. Especially, knowing which residual brackets generate which is key to proving the remaining parts of the Main Theorems in Sects. 7 and 8. And, in the follow-up article, we determine the full graph of the above map to find all the subcategories of $S_0$.

We require more precise notation to be able to address specific brackets and certain maps $\mathcal{B} \to \mathcal{B}$, particularly those mapping elements of $\mathcal{B}_{\text{res}}$ again to $\mathcal{B}_{\text{res}}$.

**Definition 6.14**

(a) If $p \in \mathcal{P}_{\text{nb}}$ is a bracket, the projective partition $\text{Arg}(p)$ which is obtained from $p$ by erasing in every row the left- and the rightmost point, is called the argument of $p$.

(b) Conversely, for each projective $a \in \mathcal{P}_{2, \text{nb}}$ and every color $c \in \{ \circ, \bullet \}$, denote by $\text{Br}(c \mid a \mid c)$ the bracket whose leftmost lower point is of color $c$ and which has the argument $a$.

$\text{Br}(\bullet \mid a \mid \circ)$
Definition 6.15 For any tuple \((c_1, \ldots, c_n)\) of colors \(c_1, \ldots, c_n \in \{\circ, \bullet\}, n \in \mathbb{N}\), denote by \(\text{Id}(c_1 \ldots c_n)\) the tensor product of \(n\) partitions from \(\{\frac{1}{2}, \frac{3}{2}\}\) such that the lower row (and thus also the upper row) has the coloration \((c_1, \ldots, c_n)\).

We define on the set of all brackets a partial associative operation applicable to all pairs of brackets whose lower rows start with the same color.

Definition 6.16 For every \(c \in \{\circ, \bullet\}\) and all projective \(a, b \in \mathcal{P}_{2}^{c, \bullet}\), we call
\[
\text{Br}(c \mid a \mid c) \otimes \text{Br}(c \mid b \mid c) := \text{Br}(c \mid a \otimes b \mid c)
\]
the \textit{bracket product} of \((\text{Br}(c \mid a \mid c), \text{Br}(c \mid b \mid c))\).

\[
\begin{array}{ccc}
\text{Br}(c \mid a \mid c) & \otimes & \text{Br}(c \mid b \mid c) \\
\end{array}
\]

For every bracket, we define two ways of changing the starting color and, in some sense, retaining the argument.

Definition 6.17 For every \(c \in \{\circ, \bullet\}\) and projective \(a \in \mathcal{P}_{2, \text{nb}}^{c, \bullet}\), we call
\[
\text{WIn} (\text{Br}(c \mid a \mid c)) := \text{Br}(c \mid \text{Br}(c \mid a \mid c) \mid a)
\]
the \textit{weak inversion} and
\[
\text{SIn} (\text{Br}(c \mid a \mid c)) := \text{Br}(c \mid \text{Id}(c) \otimes a \otimes \text{Id}(c) \mid c)
\]
the \textit{strong inversion} of \(\text{Br}(c \mid a \mid c)\).
Lemma 6.18 Let $p$ and $p'$ be two brackets starting with the same color.

(a) Categories are closed under the bracket product: $p \boxtimes p' \in \langle p, p' \rangle$.

(b) The brackets $p$ and $\tilde{p}$ start with the same color and $\text{Arg}(\tilde{p})$ is the verticolor-reflection of $\text{Arg}(p)$.

(c) Categories are closed under taking arguments of brackets: $\text{Arg}(p) \in \langle p \rangle$.

(d) It holds $\langle \overset{\downarrow}{\bullet\bullet\bullet} \rangle = \langle \overset{\downarrow}{\bullet\bullet\circ} \rangle = \langle \overset{\downarrow}{\circ\circ\bullet} \rangle = \langle \overset{\downarrow}{\circ\circ\circ} \rangle$.

(e) Weak inversion is a reversible category operation: $\langle p \rangle = \langle \text{WIn}(p) \rangle$.

(f) Strong inversion is reversible as well, but it is only available in certain categories: $\langle p, \overset{\downarrow}{\circ\circ\circ} \rangle = \langle \text{SIn}(p) \rangle$.

Proof Let $c \in \{\circ, \bullet\}$ be the starting color of $p$ and $p'$ and abbreviate $a := \text{Arg}(p)$.

(a) Erasing in $p \otimes p'$ the rightmost lower point formerly of $p$ and the leftmost lower point formerly of $p'$ and passing to the bracket associated with the lower row of the resulting partition produces the bracket $p \boxtimes p'$, which proves the claim by Lemma 6.4.

(b) Verticolor reflection turns $\text{Br}(c \mid a \mid c)$ into $\text{Br}(\tilde{c} \mid \tilde{a} \mid c) = \text{Br}(c \mid \tilde{a} \mid c)$.

(c) Erasing in $p$ the two turns formed, on the one hand, by the leftmost upper point and its successor and, on the other hand, by the rightmost lower point and its successor yields $\text{Arg}(p)$.

(d) We show the chain of inclusions from the left to the right. For the first, erase in $\overset{\downarrow}{\bullet\bullet\bullet}$ the two leftmost lower points and rotate the leftmost upper point to the lower row to obtain $\overset{\downarrow}{\circ\circ\circ}$. The second inclusion is seen to hold by recognizing that $\overset{\downarrow}{\bullet\bullet\circ}$ and $\overset{\downarrow}{\circ\circ\bullet}$ are cyclic rotations of each other. Using Lemma 6.4 to project to the right side sector of the middle block of $\overset{\downarrow}{\circ\circ\circ}$ gives $\overset{\downarrow}{\bullet\bullet\bullet}$. Since, in this chain, the roles of $\bullet$ and $\circ$ can be exchanged, the claim follows.

(e) The bracket associated with the lower row of the partition $(p^\perp)^\perp$ is $\text{WIn}(p)$. So, Lemma 6.4 proves one inclusion. The converse is a consequence of Part (c).

(f) Starting with the generators on the left hand side, employ Parts (a) and (d) to construct $\text{Br}(c \mid \text{Id}(\tilde{c}c) \mid \tilde{c}) \boxtimes p \boxtimes \text{Br}(c \mid \text{Id}(\tilde{c}c) \mid \tilde{c}) \in \langle p, \overset{\downarrow}{\circ\circ\circ} \rangle$. Erasing there the two last upper and the two first lower points yields $\text{SIn}(p)^\perp$. Conversely, applying the last step of the previous procedure to $\text{SIn}(p)$ returns $p^\perp$. And to obtain $\overset{\downarrow}{\circ\circ\circ}$, project with Lemma 6.4 to the left side sector of the block $\text{Id}(c)$ in the partition $\text{SIn}(p) = \text{Br}(\tilde{c} \mid \text{Id}(c) \otimes a \otimes \text{Id}(\tilde{c}) \mid c)$ and use Part (d).

□

7 Generators of $S_w$ [main Theorem 1 (c)]

We give a three-step proof of Part (c) of Main Theorem 1, using the results of Section 6. First, we apply the general results of Lemma 6.18 to the alleged generators of $S_w$ for $w \in \mathbb{N}$, i.e. $w > 0$.

Lemma 7.1 Let $w \in \mathbb{N}$ be arbitrary.

(a) It holds $\langle \overset{\downarrow}{\bullet\bullet\bullet} \rangle = \langle \overset{\downarrow}{\bullet\bullet\circ} \rangle = \langle \overset{\downarrow}{\circ\circ\bullet} \rangle = \langle \overset{\downarrow}{\circ\circ\circ} \rangle$. 

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(b) It holds \( \otimes \in (\mathcal{H}^w_{\otimes \otimes}) \).

(c) For all \( v \in \mathbb{N} \) hold \( \mathcal{H}^w_{\otimes \otimes \otimes} \in (\mathcal{H}^w_{\otimes \otimes}) \) and \( \mathcal{H}^w_{\otimes \otimes \otimes} \in (\mathcal{H}^w_{\otimes \otimes}) \).

**Proof** (a) Let \( c \in \{\circ, \bullet\} \) be arbitrary. To verify Part (a), by Lemma 6.18 (b), it suffices to show that \( \text{Br} (\overline{c} | \text{Id} (c^w) | c) \) is contained in \( \mathcal{C} := \langle \text{Br} (c | \text{Id} (c^w) | \overline{c}) \rangle \). By Lemma 6.18 (a) and (b), the category \( \mathcal{C} \) comprises \( \text{Br} (c | \text{Id} (c^w \otimes c^w) | \overline{c}) \). Erasing symmetrically the \( 2(w-1) \) middle points on the lower row of the latter and passing to the associated bracket of the result yields \( \text{Br} (c | \text{Id} (\overline{c} c) | \overline{c}) \in \mathcal{C} \). Moreover, Lemma 6.18 (a) now allows us to conclude \( \text{Br} (c | \text{Id} (\overline{c} c^w+1) | \overline{c}) \in \mathcal{C} \). Then, 6.18 (f) implies \( \text{Br} (\overline{c} | \text{Id} (c^w) | c) \in \mathcal{C} \) as \( \text{Br} (c | \text{Id} (\overline{c} c^w+1) | \overline{c}) = \text{Sln}(\text{Br} (\overline{c} | \text{Id} (c^w) | c)) \).

(b) In the proof of Part (a), we saw especially that \( \langle \mathcal{H}^w_{\otimes \otimes} \rangle \) contains \( \mathcal{H}^w_{\otimes} \). With 6.18 (d), we have thus already proven Part (b).

(c) Let \( c \in \{\circ, \bullet\} \) be arbitrary, and let \( k \in \mathbb{N} \) be large enough such that \( v \leq kw \). Then, using Part (a) and Lemma 6.18 (a), the category \( \mathcal{C} := \langle \mathcal{H}^w_{\otimes \otimes} \rangle \) contains the bracket \( \text{Br} (c | \text{Id}(c_{\otimes k}^w \otimes \overline{c} c_{\otimes k}^w) | \overline{c}) \). Erasing the \( 2(kw-v) \) middle points on the lower row and passing to the associated bracket of the result proves Claim (c).

\[ \square \]

We also need the following relationships between the supposed generators of \( \mathcal{S}_0 \).

**Lemma 7.2** Let \( v \in \mathbb{N} \) be arbitrary.

(a) For every \( v' \in \mathbb{N} \) with \( v' \leq v \) holds \( \mathcal{H}^v_{\otimes v'} \in (\mathcal{H}^v_{\otimes v'}) \) and \( \mathcal{H}^v_{\otimes v'} \in (\mathcal{H}^v_{\otimes v'}) \).

(b) It holds \( \mathcal{H}^v_{\otimes v'} = (\mathcal{H}^v_{\otimes v'}) \).

(c) The category \( \mathcal{H}^v_{\otimes v'} \) comprises all connected dualizable brackets whose lower rows contain at most \( 2v + 2 \) elements and exactly one turn.

**Proof** (a) For \( c \in \{\circ, \bullet\} \) and \( v' \in \mathbb{N} \), erase in \( \text{Br} (c | \text{Id}(c_{\otimes v'} \otimes \overline{c}_{\otimes v'}) | \overline{c}) \) the neutral set of the middle \( 2(v-v') \) points on the lower row and pass to the associated bracket of the result to obtain \( \text{Br} (c | \text{Id}(c_{\otimes v'} \otimes \overline{c}_{\otimes v'}) | \overline{c}) \in (\text{Br} (c | \text{Id}(c_{\otimes v'} \otimes \overline{c}_{\otimes v'}) | \overline{c})) \).

(b) For all \( c \in \{\circ, \bullet\} \) and \( v' \in \mathbb{N} \), write \( p_{c,v'} := \text{Br} (c | \text{Id}(c_{\otimes v'} \otimes \overline{c}_{\otimes v'}) | \overline{c}) \). Note that \( p_{c,1} = p_{\overline{c},1} \) proves the claim for \( v = 1 \).

Thus, suppose \( v > 1 \). By Part (a) holds \( p_{c,v'} \in (p_{c,v}) \) and thus

\[ q_{c,v'} := \text{Id}(c_{\otimes (v-v')}) \otimes p_{c,v'} \otimes \text{Id}(\overline{c}_{\otimes (v-v')}) \in (p_{c,v}). \]

\[ \square \]
for all $v' \in \mathbb{N}$ with $v' < v$. The identity

$$p_{c,v} = p_{c,v_1}q_{c,v-1} \cdots q_{c,2}q_{c,1}$$

then proves Claim (b).

(c) To prove Part (c), let $v' \in \mathbb{N}$ satisfy $v' \leq v$, and let $p$ be a connected and dualizable bracket with a lower row $S$ starting with color $c \in \{\circ, \bullet\}$, with $2v' + 2$ points and exactly one turn in $S$. The partition $p_{c,v}$ as above is an element of $\langle \otimes v \otimes v \rangle$ by Parts (a) and (b). If $p \neq p_{c,v'}$, the partitions $p_{c,v'}$ and $p$ differ by the fact that $S$ has $m \in \mathbb{N}$ proper subsectors in $p$, whereas such do not exist in $p_{c,v'}$. Denote by $v_1, \ldots, v_m \in \mathbb{N}$ with $v_i < v'$ for every $i \in \{1, \ldots, m\}$ the numbers such that $S$ has a proper subsector with $2v_i + 2$ elements for every $i \in \{1, \ldots, m\}$. Then, again $q_{c,v_i}$ is contained in $\langle \otimes v \otimes v \rangle$ by Parts (a) and (b) for every $i \in \{1, \ldots, m\}$. And from $p = p_{c,v'}q_{c,v_m} \cdots q_{c,v_2}q_{c,v_1}$ follows Claim (c).

By combining Proposition 6.13 and the preceding two lemmata, we show Part (c) of Main Theorem 1.

**Proposition 7.3**

(a) It holds

$$S_0 = \left\langle \begin{array}{c} \overline{0} \psi \overline{0} \psi \\ \bigcirc \end{array} \right\rangle \big| v \in \mathbb{N} \right\rangle.$$

(b) For every $w \in \mathbb{N}$ holds

$$S_w = \left\langle \begin{array}{c} \otimes w \\ \bigcirc \end{array} \right\rangle.$$

**Proof**

(a) Writing $C_0 := \left\langle \begin{array}{c} \overline{0} \psi \overline{0} \psi \\ \bigcirc \end{array} \right\rangle \big| v \in \mathbb{N} \right\rangle$, it suffices to prove $S_0 \cap B_{\text{res}} \subseteq C_0$ by Proposition 6.13. Thus, let $p$ be a residual bracket of $S_0$ and $S$ its lower row. If $p$ was residual of the first kind, then, in particular, int$(S)$ would be non-empty and free of turns. It would follow $\sigma_p(S) \neq 0$, violating the assumption $p \in S_0$. Hence, we infer that $p$ must be residual of the second kind. Especially, int$(S)$ contains precisely one turn. If this one turn in int$(S)$ is not the only turn in all of $S$, then all further turns of $S$ must intersect $\partial S$. If so, then each point of $\partial S$ must be element of a turn as $p$ is verticolor-reflexive. It follows that in all of $S$ there exist either just one turn or three turns. Equivalently,

$$p = \begin{array}{c} \overline{0} \psi \overline{0} \\ \bigcirc \end{array}, \quad p = \begin{array}{c} \overline{0} \psi \overline{0} \\ \bigcirc \end{array}, \quad p = p', \quad p = \text{WIn}(p') \quad \text{or} \quad p = \text{SIn}(p')$$

for a residual bracket of the second kind $p'$ whose lower row contains precisely one turn. In the first two cases, 6.18 (d) shows $p \in C_0$. In the remaining three, Lemma 7.2 (a) and Lemma 7.2 (c) prove $p' \in C_0$, from which then $p \in C_0$ follows by 6.18 (e) and Lemma 6.18 (f).

(b) Again, by Proposition 6.13, it suffices to show $S_w \cap B_{\text{res}} \subseteq C_w := \left\langle \begin{array}{c} \otimes w \\ \bigcirc \end{array} \right\rangle$. Thus, let $p$ be a residual bracket of $S_w$ and $S$ its lower row. First, suppose that $p$ is residual of the first kind. Because then int$(S)$ is non-empty and free of turns, the defining condition $\sigma_p(S) \in w\mathbb{Z}$ of $S_w$ forces $p$ to be of the form

$$p = \text{Br} \left( c_1 \left| \text{Id}(c_2^\otimes w) \big| \overline{c_1} \right. \right)$$

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for some colors \( c_1, c_2 \in \{ \circ, \bullet \} \) and some \( k \in \mathbb{N} \). Lemma 7.1 (a) shows that \( C_w \) contains the bracket \( q := \text{Br}(c_1 | \text{Id}(c_2^w) | c_1^w) \). Applying Lemma 6.18 (a) hence proves \( p = q \boxtimes k \in C_w \).

Now, suppose that \( p \) is residual of the second kind. Recognize that this already implies \( p \in S_0 \). Then, \( p \in \langle \otimes \cdot \cdot \cdot \cdot \rangle \) by Part (a). Hence, we can employ all three assertions of Lemma 7.1 to infer \( p \in C_w \).

\[ \square \]

In the follow-up [8] to this article, it is shown that the category \( S_0 \) is not finitely generated.

**8 Classification above \( S_0 \) [main Theorem 2]**

For now, it only remains to prove Main Theorem 2.

**Proposition 8.1** If \( \mathcal{C} \subseteq \mathcal{P}^{\circ, \bullet}_{2,nb} \) is a category, then either \( \mathcal{C} \subseteq S_0 \) or there exists \( w \in \mathbb{N}_0 \) such that \( \mathcal{C} = S_w \).

**Proof** Writing \( B_c \) for the set of all brackets whose lower row starts with a \( c \)-colored point, \( (\mathcal{C} \cap B_c, \boxtimes) \) is a monoid for every \( c \in \{ \circ, \bullet \} \) by Lemma 6.18. Define the mapping

\[ H: \mathcal{C} \cap B \rightarrow \mathbb{Z}, \ p \mapsto \sigma_p(S_p) = \sigma_p(\text{int}(S_p)) , \]

where \( S_p \) denotes the lower row of \( p \). For each \( c \in \{ \circ, \bullet \} \), the map \( H \) is a monoid homomorphism from \( (\mathcal{C} \cap B_c, \boxtimes) \) to \( (\mathbb{Z}, +) \). Moreover, \( H(p) = -H(p) \) for all \( p \in C \cap B_c \). Third, \( H(\text{WIn}(w)) = H(p) \) for all \( p \in C \cap B_c \), proving \( H(\mathcal{C} \cap B_\bullet) = H(\mathcal{C} \cap B_\circ) \) according to Lemma 6.18 (e). Therefore, \( H(\mathcal{C} \cap B) \) is an additive subgroup of \( \mathbb{Z} \). Hence, \( H(\mathcal{C} \cap B) = w \mathbb{Z} \) for some \( w \in \mathbb{N}_0 \).

For each sector \( S \) in a partition \( p \in \mathcal{C} \), the bracket \( B(p, S) \) associated with \( (p, S) \) is an element of \( \mathcal{C} \cap B \) and \( \sigma_p(S) = H(B(p, S)) \). That proves \( \mathcal{C} \subseteq S_w \).

Now, assume \( \mathcal{C} \not\subseteq S_0 \). We then find a bracket \( p \in \mathcal{C} \cap B_\bullet \) with lower row \( S \) and \( \sigma_p(S) = w \neq 0 \). We may assume that \( \text{int}(S) \) contains no turns of \( p \) since erasing a turn would leave \( \sigma_p(S) \) unchanged. Hence, all points in \( \text{int}(S) \) must belong to through blocks, meaning \( p = \overbrace{\overbrace{\overbrace{\circ \circ} \circ \circ} \circ \circ} \circ \circ \circ \) or \( p = \overbrace{\overbrace{\overbrace{\circ \circ} \circ \circ} \circ \circ} \circ \circ \circ \), which, according to Proposition 7.3 and Lemma 7.1 (a), proves \( S_w \subseteq \mathcal{C} \).

\[ \square \]

In [8], we will determine all subcategories of \( S_0 \), thus completing the full classification of all categories of pair partitions with neutral blocks. But already the consequence of Propositions 8.1 and 4.2 that for every category \( \mathcal{C} \subseteq \mathcal{P}^{\circ, \bullet}_{2,nb} \) holds either \( \mathcal{C} \subseteq S_0 \) or \( S_0 \subseteq \mathcal{C} \) reveals a remarkable fact about the categories of pair partitions with neutral blocks.
9 Concluding remarks

9.1 Comparison with the previous research on half-liberations of $U_n$

Our results are consistent with the previous research on this topic:

(1) In [4, Definition 5.5] and [5, Definition 2.8], Bhowmick, D’Andrea, Das and Dabrowski used the relations represented by

\[
\begin{array}{c}
\text{\includegraphics[width=1cm]{relation}}
\end{array}
\]

to define an algebra $A^*_u(n)$. For the quantum group associated with $A^*_u(n)$, Banica and Bichon later used the symbol $U^*_n$ in [2, Definition 3.2 (3)]. Proposition 7.3 (a) shows that $\langle \rangle$ is a subcategory of $S_0$. We will put the category $\langle \rangle$ into a broader context in [8].

(2) Bichon and Dubois-Violette defined in [6, Example 4.10] a quantum group with algebra $A^{**}_u(n)$. The partition

\[
\begin{array}{c}
\text{\includegraphics[width=1cm]{partition}}
\end{array}
\]

generates its intertwiner spaces. Later, Banica and Bichon denoted the corresponding quantum group by $U^{**}_n$ in [1] and [2]. And in [1, Definition 7.1], Banica and Bichon introduced a series of quantum groups referred to as $(U_{n,k})_{k \in \mathbb{N}}$ in [1] and later in [2]. For all $k \in \mathbb{N}$, the “$k$-half-liberated unitary quantum group” $U_{n,k}$ has its intertwiner spaces generated by

\[
\begin{array}{c}
\text{\includegraphics[width=12cm]{intertwiner_spaces}}
\end{array}
\]

Especially, $U_{n,1}$ corresponds to $U_n \sim \langle \varnothing \rangle$ and they show that $U_{n,2}$ is identical to $U^{**}_n \sim \langle \varnothing \rangle$. In general, by composing the above generator of $U_{n,k}$ with elements of $\langle \varnothing \rangle$ in the following way

\[
\begin{array}{c}
\text{\includegraphics[width=12cm]{composition}}
\end{array}
\]
it is seen that the category of $U_{n,k}$ contains the partition

![Partition Diagram]

Conversely, by composing this partition $k$ times with itself with successively increasing offset, we can recover the original generator of $U_{n,k}$:

![Composed Diagram]

Thus, both partitions generate the category of $U_{n,k}$. Because
and

the category of \( U_{n,k} \) is precisely our category \( S_k \) by Proposition 7.3 (b).

(3) Lastly, the quantum group \( U_{n,\infty} \) is defined by Banica and Bichon in [1, Definition 8.1] as some limit case of their series \( (U_{n,k})_{k \in \mathbb{N}} \). The two partitions

\[
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{partition_1}
\end{array}
\quad \text{and} \quad
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{partition_2}
\end{array}
\]

generate its associated category. We can write these generators equivalently in the form

\[
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{equivalent_1}
\end{array}
\quad \text{and} \quad
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{equivalent_2}
\end{array}
\]
by rotating each generator cyclically in counter-clockwise direction. And Lemma 6.18 (d) showed that the second partition of the last pair and \( {\mathfrak{x}} \) generate the same category. So, \( U_{n,\infty} \) corresponds to the category \( \langle {\mathfrak{x}}\rangle \). As \( U_{n}^{\infty} \) and \( U_{n,\infty} \) are distinct, an immediate consequence is \( \langle {\mathfrak{x}}\rangle \subset \langle {\mathfrak{y}}\rangle \). Again, from Proposition 7.3 (b), we know \( \langle {\mathfrak{y}}\rangle \) and \( \langle {\mathfrak{y}}\rangle \) to be subcategories of \( S_{0} \), both to be treated in [8]. Especially, we will show that \( \langle {\mathfrak{y}}\rangle \) and \( S_{0} \) are indeed distinct.

### 9.2 Outlook on [8]

In the present article we established combinatorially that every category \( C \subseteq \langle {\mathfrak{x}}\rangle \) is either one of the categories \( (S_{w})_{w \in \mathbb{N}_{0}} \) or a proper subcategory of \( S_{0} \) (Proposition 8.1). In particular, \( S_{0} \) is revealed to be a halfway point for all categories \( C \subseteq \langle {\mathfrak{x}}\rangle \) as every such category necessarily satisfies either \( C \subseteq S_{0} \) or \( S_{0} \subseteq C \). The follow-up article [8] will continue the combinatorial analysis of categories in the realm \( S_{0} \). We will determine all subcategories of \( S_{0} \), their description in combinatorial terms and their generating partitions. Many more categories will be shown to exist in \( S_{0} \) besides the known examples. While the categories \( C \) with \( S_{0} \subseteq C \subseteq \langle {\mathfrak{x}}\rangle \) are equivalent to the cyclic groups (see also Section 4), we will prove that the subcategories of \( S_{0} \) are dual to the subsemigroups of \( (\mathbb{N}_{0},+) \). Specifically, we can contrast the present article and [8] as follows:

| Main theorem | Present article | [8] |
|--------------|-----------------|-----|
| 1            | Subsets \( (S_{w})_{w \in \mathbb{N}_{0}} \) of \( \langle {\mathfrak{x}}\rangle \) are categories, where \( \mathbb{N}_{0} \) corresponds to the set of cyclic groups. \( w'Z \subseteq wZ \implies S_{w'} \subseteq S_{w} \). Description of generators of \( S_{w} \). | Subsets \( (I_{D})_{D \in \mathbb{N}_{0}'} \) of \( \langle {\mathfrak{x}}\rangle \) are categories, where \( D \) denotes the set subsemigroups of \( (\mathbb{N}_{0},+) \). \( D' \subseteq D \implies I_{D'} \supseteq I_{D} \). Description of generators of \( I_{D} \). |
| 2            | Classification of all categories \( C \) with \( S_{0} \subseteq C \subseteq \langle {\mathfrak{x}}\rangle \). | Classification of all categories \( C \) with \( S_{0} \not\subseteq C \subseteq \langle {\mathfrak{x}}\rangle \). |

In [8], we will continue to employ the techniques of bracket partitions developed in Sect. 6. It was shown in Proposition 6.13 that every category \( C \subseteq \langle {\mathfrak{x}}\rangle \) is generated by its set of residual brackets (Definition 6.12). These come in two flavors: None of the residual brackets of the first kind, \( \otimes_{w} \), \( \oplus_{w} \), \( \mathfrak{y}_{w} \), and \( \otimes_{w} \) for arbitrary \( w \in \mathbb{N} \), are elements of \( S_{0} \). Rather, for fixed \( w \), they all generate the category \( S_{w} \) (Proposition 7.3 (b)). The residual brackets of the second kind collectively generate \( S_{0} \) (Proposition 7.3 (a)). So far, we have only explicitly encountered the few examples: \( \mathfrak{y} \) and \( \mathfrak{y} \) for arbitrary \( v \in \mathbb{N} \) as well as \( \mathfrak{y} \) and \( \mathfrak{y} \). In [8] we consider arbitrary sets of residual brackets of the second kind and determine which categories they generate.
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