ON THE CENTRAL GEOMETRY OF NONNOETHERIAN DIMER ALGEBRAS

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Abstract. Let $Z$ be the center of a nonnoetherian dimer algebra on a torus. Although $Z$ itself is also nonnoetherian, we show that it has Krull dimension 3, and is locally noetherian on an open dense set of Max $Z$. Furthermore, we show that the reduced center $Z/\text{nil} Z$ is depicted by a Gorenstein singularity, and contains precisely one closed point of positive geometric dimension.

1. Introduction

In this article, all dimer quivers are nondegenerate and embed in a two-torus. A dimer algebra $A$ is noetherian if and only if its center $Z$ is noetherian, if and only if $A$ is a noncommutative crepant resolution of its 3-dimensional toric Gorenstein center \[ \text{[Br] D [B4]} \]. We show that the center $Z$ of a nonnoetherian dimer algebra is also 3-dimensional, and may be viewed as the coordinate ring for a toric Gorenstein singularity that has precisely one ‘smeared-out’ point of positive geometric dimension. This is made precise using the notion of a depiction, which is a finitely generated overring that is as close as possible to $Z$, in a suitable geometric sense (Definition 2.5).

Denote by $\text{nil} Z$ the nilradical of $Z$, and by $\hat{Z} := Z/\text{nil} Z$ the reduced ring of $Z$. Our main theorem is the following.

Theorem 1.1. (Theorems 3.19, 3.20.) Let $A$ be a nonnoetherian dimer algebra with center $Z$, and let $\Lambda := A/\langle p - q \mid p, q \text{ a non-cancellative pair} \rangle$ be its ghore algebra with center $R$.

1) The nonnoetherian rings $Z$, $\hat{Z}$, and $R$ each have Krull dimension 3, and the integral domains $\hat{Z}$ and $R$ are depicted by the cycle algebras of $A$ and $\Lambda$.

2) The reduced scheme of Spec $Z$ and the scheme Spec $R$ are birational to a noetherian affine scheme, and each contain precisely one closed point of positive geometric dimension.

Dimer models were introduced in string theory in 2005 in the context of brane tilings \[ \text{[HK] FHVWK} \]. The dimer algebra description of the combinatorial data of a
brane tiling arose from the notion of a superpotential algebra (or quiver with potential), which was introduced a few years earlier in [BD]. Stable (i.e., ‘superconformal’) brane tilings quickly made their way to the mathematics side, but the more difficult study of unstable brane tilings was largely left open, in regards to both their mathematical and physical properties. There were two main difficulties: in contrast to the stable case, the ‘mesonic chiral ring’ (closely related to what we call the cycle algebra\(^1\)) did not coincide with the center of the dimer algebra, and (ii) although the mesonic chiral ring still appeared to be a nice ring, the center certainly was not.

The center is supposed to be the coordinate ring for an affine patch on the extra six compact dimensions of spacetime, the so-called (classical) vacuum geometry. But examples quickly showed that the center of an unstable brane tiling could be infinitely generated. To say that the vacuum geometry was a nonnoetherian scheme – something believed to have no visual representation or concrete geometric interpretation – was not quite satisfactory from a physics perspective. However, unstable brane tilings are physically allowable theories. To make matters worse, almost all brane tilings are unstable, and it is only in the case of a certain uniform symmetry (an ‘isoradial embedding’) that they become stable. Moreover, in the context of 11-dimensional M-theory, stable and unstable brane tilings are equally ‘good’. The question thus remained:

*What does the vacuum geometry of an unstable brane tiling look like?*

The aim of this article is to provide an answer. In short, the vacuum geometry of an unstable brane tiling looks just like the vacuum geometry of a stable brane tiling, namely a 3-dimensional complex cone, except that there is precisely one curve or surface passing through the apex of the cone that is identified as a single ‘smeared-out’ point.

### 2. Preliminary definitions

Throughout, \( k \) is an uncountable algebraically closed field. Given a quiver \( Q \), we denote by \( kQ \) the path algebra of \( Q \); and by \( Q_0 \) and \( Q_1 \) the sets of vertices and arrows of \( Q \) respectively. The vertex idempotent at vertex \( i \in Q_0 \) is denoted \( e_i \), and the head and tail maps are denoted \( h, t : Q_1 \rightarrow Q_0 \). By monomial, we mean a nonconstant monomial.

#### 2.1. Dimer algebras, ghor algebras, and cyclic contractions.

**Definition 2.1.**

- Let \( Q \) be a finite quiver whose underlying graph \( \overline{Q} \) embeds into a real two-torus \( T^2 \), such that each connected component of \( T^2 \setminus \overline{Q} \) is simply connected and bounded

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\(^1\)The mesonic chiral ring is the ring of gauge invariant operators; these are elements of the dimer algebra that are invariant under isomorphic representations, and thus are cycles in the quiver.
by an oriented cycle of length at least 2, called a \textit{unit cycle}². The \textit{dimer algebra} of \( Q \) is the quiver algebra \( A := kQ/I \) with relations
\[
I := \langle p - q \mid \exists a \in Q_1 \text{ such that } pa \text{ and } qa \text{ are unit cycles} \rangle \subset kQ,
\]
where \( p \) and \( q \) are paths.

Since \( I \) is generated by certain differences of paths, we may refer to a path modulo \( I \) as a \textit{path} in the dimer algebra \( A \).

- Two paths \( p, q \in A \) form a \textit{non-cancellative pair} if \( p \neq q \), and there is a path \( r \in kQ/I \) such that
  \[
  rp = rq \neq 0 \quad \text{or} \quad pr = qr \neq 0.
  \]

\( A \) and \( Q \) are called \textit{non-cancellative} if there is a non-cancellative pair; otherwise they are called \textit{cancellative}.

- The \textit{ghor algebra} of \( Q \) is the quotient
  \[
  \Lambda := A/ \langle p - q \mid p, q \text{ is a non-cancellative pair} \rangle.
  \]

A dimer algebra \( A \) coincides with its ghor algebra if and only if \( A \) is cancellative, if and only if \( A \) is noetherian, if and only if \( \Lambda \) is noetherian [B4, Theorem 1.1].

- Let \( A \) be a dimer algebra with quiver \( Q \).
  - A \textit{perfect matching} \( D \subset Q_1 \) is a set of arrows such that each unit cycle contains precisely one arrow in \( D \).
  - A \textit{simple matching} \( D \subset Q_1 \) is a perfect matching such that \( Q \setminus D \) supports a simple \( A \)-module of dimension \( 1^{Q_0} \) (that is, \( Q \setminus D \) contains a cycle that passes through each vertex of \( Q \)). Denote by \( S \) the set of simple matchings of \( A \).
  - \( A \) is said to be \textit{nondegenerate} if each arrow of \( Q \) belongs to a perfect matching.³

Each perfect matching \( D \) defines a map
\[
n_D : Q_{\geq 0} \to \mathbb{Z}_{\geq 0}
\]
that sends path \( p \) to the number of arrow subpaths of \( p \) that are contained in \( D \). \( n_D \) is additive on concatenated paths, and if \( p, p' \in Q_{\geq 0} \) are paths satisfying \( p + I = p' + I \), then \( n_D(p) = n_D(p') \). In particular, \( n_D \) induces a well-defined map on the paths of \( A \).

Now consider dimer algebras \( A = kQ/I \) and \( A' = kQ'/I' \), and suppose \( Q' \) is obtained from \( Q \) by contracting a set of arrows \( Q_1^* \subset Q_1 \) to vertices. This contraction defines a \( k \)-linear map of path algebras
\[
\psi : kQ \to kQ'.
\]

²In forthcoming work, we consider the nonnoetherian central geometry of ghor algebras on higher genus surfaces. Dimer quivers on other surfaces arise in contexts such as Belyi maps, cluster categories, and bipartite field theories; see e.g., [BGH,BKM,FGU].

³For our purposes, it suffices to assume that each cycle contains an arrow that belongs to a perfect matching; see [B1].
If $\psi(I) \subseteq I'$, then $\psi$ induces a $k$-linear map of dimer algebras, called a contraction,

$$\psi : A \to A'.$$

Denote by

$$B := k \left[ x_D : D \in S' \right]$$

the polynomial ring generated by the simple matchings $S'$ of $A'$. To each path $p \in A'$, associate the monomial

$$\bar{\tau}(p) := \prod_{D \in S'} x_D^{n_D(p)} \in B.$$

For each $i, j \in Q_0'$, this association extends to a $k$-linear map

$$\bar{\tau} : e_j A' e_i \to B.$$

This map is an algebra homomorphism if $i = j$, and injective if $A'$ is cancellative [B2, Proposition 4.29]. For each $i, j \in Q_0$, we also consider the $k$-linear map given by the composition of the $k$-linear maps $\psi$ and $\bar{\tau}$,

$$\bar{\tau}_\psi : e_j A e_i \xrightarrow{\psi} e_{\psi(j)} A' e_{\psi(i)} \xrightarrow{\bar{\tau}} B.$$

Given $p \in e_j A e_i$ and $q \in e_i A' e_k$, we will write

$$\bar{p} := \bar{\tau}_\psi(p) := \bar{\tau}(\psi(p)) \quad \text{and} \quad \bar{q} := \bar{\tau}(q).$$

$\psi$ is called a cyclic contraction if $A'$ is cancellative and

$$S := k \left[ \cup_{i \in Q_0} \bar{\tau}_\psi(e_i A e_i) \right] = k \left[ \cup_{i \in Q_0} \bar{\tau}(e_i A' e_i) \right] =: S'.$$

In this case, we call $S$ the cycle algebra of $A$. The cycle algebra is independent of the choice of cyclic contraction $\psi$ [B3, Theorem 3.14], and is isomorphic to the center of $A'$ [B2, Theorem 1.1.3]. Moreover, every nondegenerate dimer algebra admits a cyclic contraction [B1, Theorem 1.1].

In addition to the cycle algebra, the ghor center of $A$,

$$R := k \left[ \cap_{i \in Q_0} \bar{\tau}_\psi(e_i A e_i) \right],$$

also plays an important role. This algebra is isomorphic to the center of the ghor algebra $\Lambda$ of $Q$ [B2, Theorem 1.1.3].

**Notation 2.2.** Let $\pi : \mathbb{R}^2 \to T^2$ be a covering map such that for some $i \in Q_0$,

$$\pi(\mathbb{Z}^2) = i.$$

Denote by $Q^+ := \pi^{-1}(Q) \subset \mathbb{R}^2$ the covering quiver of $Q$. For each path $p$ in $Q$, denote by $p^+$ the unique path in $Q^+$ with tail in the unit square $[0, 1) \times [0, 1) \subset \mathbb{R}^2$ satisfying $\pi(p^+) = p$. For $u \in \mathbb{Z}^2$, denote by $C^u$ the set of cycles $p$ in $A$ such that

$$h(p^+) = t(p^+) + u \in Q_0^+.$$
Notation/Lemma 2.3. We denote by $\sigma_i \in A$ the unique unit cycle (modulo $I$) at $i \in Q_0$, and by $\sigma$ the monomial

$$\sigma := \sigma = \prod_{D \in S'} x_D.$$ 

The sum $\sum_{i \in Q_0} \sigma_i$ is a central element of $A$.

Lemma 2.4.

1. If $p \in C(0,0)$ is nontrivial, then $p = \sigma^n$ for some $n \geq 1$.
2. Let $u \in \mathbb{Z}^2$ and $p, q \in C_u$. Then $p = q\sigma^n$ for some $n \in \mathbb{Z}$.

Proof. (1) is [B2, Lemma 5.2.1], and (2) is [B2, Lemma 4.18]. □

2.2. Nonnoetherian geometry: depictions and geometric dimension. Let $S$ be an integral domain and a finitely generated $k$-algebra, and let $R$ be a (possibly nonnoetherian) subalgebra of $S$. Denote by $\text{Max} S$, $\text{Spec} S$, and $\text{dim} S$ the maximal spectrum (or variety), prime spectrum (or affine scheme), and Krull dimension of $S$ respectively; similarly for $R$. For a subset $I \subset S$, set $\mathcal{Z}_S(I) := \{ n \in \text{Max} S \mid n \supseteq I \}$.

Definition 2.5. [B5, Definition 3.1]

• We say $S$ is a depiction of $R$ if the morphism

$$\iota_{S/R} : \text{Spec} S \to \text{Spec} R, \quad q \mapsto q \cap R,$$

is surjective, and

1. $\{ n \in \text{Max} S \mid R_{n \cap R} = S_n \} = \{ n \in \text{Max} S \mid R_{n \cap R} \text{ is noetherian} \} \neq \emptyset$.

• The geometric height of $p \in \text{Spec} R$ is the minimum

$$\text{ght}(p) := \min \left\{ \text{ht}_S(q) \mid q \in \iota_{S/R}^{-1}(p), \ S \text{ a depiction of } R \right\}.$$ 

The geometric dimension of $p$ is

$$\text{gdim} p := \text{dim} R - \text{ght}(p).$$

We will denote the subsets of the algebraic variety $\text{Max} S$ by

1. $U_{S/R} := \{ n \in \text{Max} S \mid R_{n \cap R} = S_n \}$,
2. $U^*_{S/R} := \{ n \in \text{Max} S \mid R_{n \cap R} \text{ is noetherian} \}.$

The subset $U_{S/R}$ is open in $\text{Max} S$ [B5, Proposition 2.4.2].

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The morphism $\iota_{S/R}$ is well-defined: Suppose $q \in \text{Spec} S$. Then $S/q$ is an integral domain. Whence the subalgebra $R/(q \cap R) \subseteq S/q$ is an integral domain. Thus $q \cap R$ is a prime ideal of $R$. 

Example 2.6. Let $S = k[x, y]$, and consider its nonnoetherian subalgebra
\[ R = k[x, xy, xy^2, \ldots] = k + xS. \]

$R$ is then depicted by $S$, and the closed point $xS \in \text{Max } R$ has geometric dimension 1 \cite[Proposition 2.8]{B5}. Furthermore, $U_{S/R}$ is the complement of the line $\mathcal{Z}(x) = \{x = 0\} \subset \text{Max } S$.

In particular, $\text{Max } R$ may be viewed as 2-dimensional affine space $A^2_k = \text{Max } S$ with the line $\mathcal{Z}(x)$ identified as a single ‘smeared-out’ point. From this perspective, $xS$ is a positive dimensional point of $\text{Max } R$.

In the next section, we will show that the reduced center and ghor center of a nonnoetherian dimer algebra are both depicted by its cycle algebra, and both contain precisely one point of positive geometric dimension.

3. Proof of main theorem

Throughout, $A$ is a nonnoetherian dimer algebra with center $Z$ and reduced center $\hat{Z} := Z/\text{nil } Z$. By assumption $A$ is nondegenerate, and thus there is a cyclic contraction $\psi : A \to A'$ to a noetherian dimer algebra $A'$ \cite[Theorem 1.1]{B1}.

The center $Z'$ of $A'$ is isomorphic to the cycle algebra $S$ \cite[Theorem 1.1.3]{B2}, and the reduced center $\hat{Z}$ of $A$ is isomorphic to a subalgebra of $R$ \cite[Theorem 4.1]{B6}.

We may therefore write
\[ \hat{Z} \subseteq R. \]

The following structural results will be useful.

Lemma 3.1. Let $g \in B$ be a monomial.
\begin{itemize}
  \item[(1)] If $g \in R$, $h \in S$ are monomials and $g \neq \sigma^n$ for each $n \geq 0$, then $gh \in R$.
  \item[(2)] If $g \in R$ and $\sigma \nmid g$, then $g \in \hat{Z}$.
  \item[(3)] If $g \in R$, then there is some $m \geq 1$ such that $g^m \in \hat{Z}$.
  \item[(4)] If $g \in S$, then there is some $m \geq 0$ such that for each $n \geq 1$, $g^n \sigma^m \in \hat{Z}$.
  \item[(5)] If $\sigma g \in S$, then $g \in S$.
  \item[(6)] If a monomial $h \in S \setminus R$ satisfies $\sigma \nmid h$, then $h^n \notin R$ for each $n \geq 1$.
\end{itemize}

Furthermore, a monomial $h \in S \setminus R$ exists for which $\sigma \nmid h$.

Proof. (1) is \cite[Lemma 6.1]{B6}; (2) - (4) is \cite[Lemma 5.3]{B6}; and (5) is \cite[Lemma 4.18]{B2} for $u \neq (0, 0)$, and Lemma 2.4.1 for $u = (0, 0)$; and (6) is \cite[Proposition 3.14]{B4}. \hfill \Box

Lemma 3.2. The cycle algebra $S$ is a finite type integral domain.

\footnote{It is often the case that $\hat{Z}$ is isomorphic to $R$; an example where $\hat{Z} \neq R$ is given in \cite[Example 4.3]{B6}.}
Proof. The cycle algebra $S$ is generated by the $\tau_v$-images of cycles in $Q$ with no nontrivial cyclic proper subpaths. Since $Q$ is finite, there is only a finite number of such cycles. Therefore $S$ is a finitely generated $k$-algebra. $S$ is also an integral domain since it is a subalgebra of the polynomial ring $B$. □

It is well-known that the Krull dimension of the center of any cancellative dimer algebra (on a torus) is 3 (e.g. [Br]). The isomorphism $S \cong Z'$ therefore implies that the Krull dimension of the cycle algebra $S$ is 3. In the following we give a new and independent proof of this result.

Lemma 3.3. The cycle algebra $S$ has Krull dimension 3.

Proof. Fix $j \in Q'_0$ and cycles in $e_jA'e_j$,

\[(4)\quad s_1 \in C'^{(1,0)}, \quad t_1 \in C'^{(-1,0)}, \quad s_2 \in C'^{(0,1)}, \quad t_2 \in C'^{(0,-1)}.\]

Consider the algebra

$T := k[\sigma, s_1, s_2, t_1, t_2] \subseteq S' \subseteq S,$

where (i) holds since the contraction $\psi$ is cyclic.

Since $A'$ is cancellative, if

$p \in C'^u \quad \text{and} \quad q \in C'^v$

are cycles in $A'$ satisfying $p = q$, then $u = v$ [B4, Lemma 3.9]. Thus there are no relations among the monomials $s_1, s_2, t_1, t_2$, by our choice of cycles (4). However, by Lemma 2.4.1, there are integers $n_1, n_2 \geq 1$ such that

\[(5)\quad s_1 t_1 = \sigma^{n_1} \quad \text{and} \quad s_2 t_2 = \sigma^{n_2}.
\]

(i) We claim that dim $T = 3$. Since $T$ is a finite type integral domain and $k$ is algebraically closed, the variety Max $T$ is equidimensional [E, Ch. 13, Theorem A]. It thus suffices to show that the chain of ideals of $T$,

\[(6)\quad 0 \subset (\sigma, s_1, s_2) \subseteq (\sigma, s_1, s_2, t_1) \subseteq (\sigma, s_1, s_2, t_1, t_2),\]

is a maximal chain of distinct primes.

The inclusions in (6) are strict since the relations among the monomial generators are generated by the two relations (5).

Moreover, (6) is a maximal chain of primes of $T$: Suppose $s_i$ is in a prime $p$ of $T$. Then $\sigma$ is in $p$, by (5). Whence $s_{i+1}$ or $t_{i+1}$ is also in $p$, again by (5). Thus $(\sigma, s_1, s_2)$ is a minimal prime of $T$.

(ii) We now claim that dim $S = \dim T$. By Lemma 2.4.2, we have

$S[\sigma^{-1}] = T[\sigma^{-1}].$

Furthermore, $S$ and $T$ are finite type integral domains, by Lemma 3.2. Thus Max $S$ and Max $T$ are irreducible algebraic varieties that are isomorphic on their open dense sets $\{\sigma \neq 0\}$. Therefore dim $S = \dim T$. □
Corollary 3.4. The Krull dimension of the center of a noetherian dimer algebra is 3.

Proof. Follows from Lemma 3.3 and the isomorphism $Z' \cong S$. □

Lemma 3.5. The morphisms

\[(7) \quad \kappa_{S/\hat{Z}} : \text{Max } S \to \text{Max } \hat{Z}, \quad n \mapsto n \cap \hat{Z}, \]
\[\kappa_{S/R} : \text{Max } S \to \text{Max } R, \quad n \mapsto n \cap R, \]

and

\[\iota_{S/\hat{Z}} : \text{Spec } S \to \text{Spec } \hat{Z}, \quad q \mapsto q \cap \hat{Z}, \]
\[\iota_{S/R} : \text{Spec } S \to \text{Spec } R, \quad q \mapsto q \cap R, \]

are well-defined and surjective.

Proof. (i) We first claim that $\kappa_{S/\hat{Z}}$ and $\kappa_{S/R}$ are well-defined maps. Indeed, let $n$ be in $\text{Max } S$. By Lemma 3.2, $S$ is of finite type, and by assumption $k$ is algebraically closed. Therefore the intersections $n \cap \hat{Z}$ and $n \cap R$ are maximal ideals of $\hat{Z}$ and $R$ respectively (e.g., [B5, Lemma 2.1]).

(ii) We claim that $\kappa_{S/\hat{Z}}$ and $\kappa_{S/R}$ are surjective. Fix $m \in \text{max } \hat{Z}$. Then $Sm$ is a proper ideal of $S$ since $S$ is a subalgebra of the polynomial ring $B$. Thus, since $S$ is noetherian, there is a maximal ideal $n \in \text{Max } S$ containing $Sm$. Whence,

\[m \subseteq Sm \cap \hat{Z} \subseteq n \cap \hat{Z}.\]

But $n \cap \hat{Z}$ is a maximal ideal of $\hat{Z}$ by Claim (i). Therefore $m = n \cap \hat{Z}$. Similarly, $\kappa_{S/R}$ is surjective.

(iii) It is clear that $\iota_{S/\hat{Z}}$ and $\iota_{S/R}$ are well-defined maps (see footnote 4). Finally, we claim that $\iota_{S/\hat{Z}}$ and $\iota_{S/R}$ are surjective. By [B5, Lemma 3.6], if $D$ is a finitely generated algebra over an uncountable field $k$, and $C \subseteq D$ is a subalgebra, then $\iota_{D/C} : \text{Spec } D \to \text{Spec } C$ is surjective if and only if $\kappa_{D/C} : \text{Max } D \to \text{Max } C$ is surjective. Therefore, $\iota_{S/\hat{Z}}$ and $\iota_{S/R}$ are surjective by Claim (ii). □

Lemma 3.6. If $p \in \text{Spec } \hat{Z}$ contains a monomial, then $p$ contains $\sigma$.

Proof. Suppose $p$ contains a monomial $g$. Then there is a nontrivial cycle $p$ such that $p = g$.

Let $q^+$ be a path from $h(p^+)$ to $t(p^+)$. Then the concatenated path $(pq)^+$ is a cycle in $Q^+$. Thus, there is some $n \geq 1$ such that $pq = \sigma^n$, by Lemma 2.4.1.

By Lemma 3.5 there is a prime ideal $q \in \text{Max } S$ such that $q \cap R = p$. Furthermore, $pq = \sigma^n$ is in $q$ since $p \in p$ and $q \in S$. Hence $\sigma$ is also in $q$ since $q$ is prime. But $\sigma \in \hat{Z}$. Therefore $\sigma \in q \cap \hat{Z} = p$. □

Denote the origin of $\text{Max } S$ by

\[n_0 := (s \in S \mid s \text{ a nontrivial cycle}) \in \text{Max } S.\]
Consider the maximal ideals of \( \hat{Z} \) and \( R \) respectively,
\[
\mathfrak{j}_0 := n_0 \cap \hat{Z} \quad \text{and} \quad \mathfrak{m}_0 := n_0 \cap R.
\]

**Proposition 3.7.** The localizations \( \hat{Z}_{\mathfrak{j}_0} \) and \( R_{\mathfrak{m}_0} \) are nonnoetherian.

**Proof.** There is a monomial \( g \in S \setminus R \) such that \( g^n \notin R \) for each \( n \geq 1 \), by Lemma 3.1.6.

(i) Fix \( n \geq 1 \). We claim that \( g^n \) is not in the localization \( R_{\mathfrak{m}_0} \). Assume otherwise; then there is an \( a \in R \), \( b \in \mathfrak{m}_0 \), \( \beta \in k^\times \), such that
\[
g^n = \frac{a}{\beta + b} \in R_{\mathfrak{m}_0},
\]
since \( \mathfrak{m}_0 \) is a maximal ideal of \( R \). Whence,
\[
bg^n - a = -\beta g^n \notin R.
\]
Thus \( bg^n \notin R \) since \( a \in R \) and \( \beta g^n \notin R \).

Since \( \mathfrak{m}_0 \) is generated by monomials in \( R \), we may write the polynomial \( b \in \mathfrak{m}_0 \) as a sum of monomials in \( \mathfrak{m}_0 \),
\[
b = \sum_j b_j + \sum_k b'_k,
\]
where for each \( j \) and \( k \),
\[
\begin{align*}
    b_j g^n & \notin R \quad \text{and} \quad b'_k g^n \in R.
\end{align*}
\]

Consider the polynomial
\[
h := \sum_j b_j g^n + \beta g^n = a - \sum_k b'_k g^n.
\]
If \( h = 0 \), then \( \sum_j b_j g^n = -\beta g^n \). Whence \( \sum_j b_j = -\beta \) since \( B \) is an integral domain. But \( \sum_j b_j \) is in \( \mathfrak{m}_0 \) whereas \( \beta \in k^\times \) is not, a contradiction. Therefore \( h \neq 0 \).

View \( R \subset S \) as \( k \)-vector subspaces of the polynomial ring \( B \). These subspaces have bases given by all monomials with scalar coefficient 1 in \( R \) and \( S \) respectively. Define a symmetric bilinear form on \( S \): for monomials \( m_1, m_2 \in S \) in the basis, set
\[
(m_1, m_2) := \begin{cases} 
1 & \text{if } m_1 = m_2 \\
0 & \text{otherwise}
\end{cases},
\]
and extend \( k \)-bilinearly to \( S \).

Now \( h \) is nonzero and equals \( \sum_j b_j g^n + \beta g^n \), and thus may be written entirely in terms of basis vectors that lie in the orthogonal complement to \( R \) (with respect to \((\cdot, \cdot)) \). Thus \( h \) itself does not lie in \( R \). But \( h \) also equals \( a - \sum_k b'_k g^n \), and thus lies in \( R \), a contradiction. Therefore \( g^n \) is not in the localization \( R_{\mathfrak{m}_0} \).

(ii) We now claim that \( R_{\mathfrak{m}_0} \) is nonnoetherian.

By Lemma 3.1.4, there is an \( m \geq 1 \) such that for each \( n \geq 1 \),
\[
h_n := g^n \sigma^m \in R.
\]
Consider the chain of ideals of $R_m$:

$$0 \subset (h_1) \subseteq (h_1, h_2) \subseteq (h_1, h_2, h_3) \subseteq \ldots$$

Assume to the contrary that the chain stabilizes. Then there is an $N \geq 1$ such that

$$h_N = \sum_{n=1}^{N-1} c_n h_n,$$

with $c_n \in R_{m0}$. Thus, since $R$ is an integral domain,

$$g_N = \sum_{n=1}^{N-1} c_n g^n.$$

Furthermore, since $R$ is a subalgebra of the polynomial ring $B$ and $g$ is a monomial, there is some $1 \leq n \leq N - 1$ such that

$$c_n = g^{N-n} + b,$$

with $b \in R_{m0}$. But then $g^{N-n} = c_n - b \in R_{m0}$, contrary to Claim (i).

(iii) Similarly, $\hat{Z}_0$ is nonnoetherian.

Recall that by monomial, we mean a nonconstant monomial.

Lemma 3.8. Suppose that each monomial in $\hat{Z}$ is divisible (in $B$) by $\sigma$. If $p \in \text{Spec } \hat{Z}$ contains a monomial, then $p = z_0$.

Proof. Suppose $p \in \text{Spec } \hat{Z}$ contains a monomial. Then $\sigma$ is in $p$ by Lemma 3.6.

Furthermore, there is some $q \in \text{Spec } S$ such that $q \cap \hat{Z} = p$, by Lemma 3.5.

Suppose $g$ is a monomial in $\hat{Z}$. By assumption, there is a monomial $h$ in $B$ such that $g = \sigma h$. By Lemma 3.15, $h$ is also in $S$. Whence $g = \sigma h \in q$ since $\sigma \in p \subseteq q$.

But $g \in \hat{Z}$. Therefore $g \in q \cap \hat{Z} = p$. Since $g$ was arbitrary, $p$ contains all monomials in $\hat{Z}$.

Remark 3.9. In Lemma 3.8, we assumed that $\sigma$ divides all monomials in $\hat{Z}$. An example of a dimer algebra with this property is given in Figure 1, where the center is the nonnoetherian ring $\hat{Z} \cong \hat{Z} = R = k + \sigma S$, and the cycle algebra is the quadric cone, $S = k[xz, xw, yz, yw]$. The contraction $\psi : A \to A'$ is cyclic.

Lemma 3.10. Suppose that there is a monomial in $\hat{Z}$ which is not divisible (in $B$) by $\sigma$. Let $m \in \text{Max } \hat{Z} \setminus \{z_0\}$. Then there is a monomial $g \in \hat{Z} \setminus m$ such that $\sigma \nmid g$.

Proof. Let $m \in \text{Max } \hat{Z} \setminus \{z_0\}$.

(i) We first claim that there is a monomial in $\hat{Z} \setminus m$. Assume otherwise. Then

$$n_0 \cap \hat{Z} \subseteq m.$$

But $z_0 := n_0 \cap \hat{Z}$ is a maximal ideal by Lemma 3.5. Thus $z_0 = m$, contrary to assumption.
(ii) We now claim that there is a monomial in $\hat{Z} \setminus m$ which is not divisible by $\sigma$. Indeed, assume to the contrary that every monomial in $\hat{Z}$, which is not divisible by $\sigma$, is in $m$. By assumption, there is a monomial in $\hat{Z}$ that is not divisible by $\sigma$. Thus there is at least one monomial in $m$. Therefore $\sigma$ is in $m$, by Lemma 3.6.

There is an $n \in \text{Max } S$ such that $n \cap \hat{Z} = m$, by Lemma 3.5. Furthermore, $\sigma \in n$ since $\sigma \in m$. Suppose $g \in \hat{Z}$ is a monomial for which $\sigma \mid g$; say $g = \sigma h$ for some monomial $h \in B$. Then $h \in S$ by Lemma 3.1.5. Whence, $g = \sigma h \in n$. Thus

$$g \in n \cap \hat{Z} = m.$$ 

It follows that every monomial in $\hat{Z}$, which is divisible by $\sigma$, is also in $m$. Therefore every monomial in $\hat{Z}$ is in $m$. But this contradicts our choice of $m$ by Claim (i).
Definition 3.11. A column and pillar of a dimer quiver $Q$ are subquivers as shown in Figure 2.

Lemma 3.12. Let $z = \sum_{i \in Q_0} q_i$ be a central element of $A$ such that for each $i \in Q_0$, $q_i$ is a cycle in $e_iAe_i$. (In particular, $q_i = q_j$ for each $i, j \in Q_0$.) If $\sigma \nmid q_i$, then the set of representatives $\tilde{q}_i \in e_i kQe_i$ of the $q_i$ partition $Q$ into columns and pillars.

Proof. See [B2, Lemmas 4.8.3 and 4.12]. □

Lemma 3.13. Let $p \in C^u$, $q \in C^v$ be cycles in $Q$ such that $u, v \in \mathbb{Z}^2$ are linearly independent over $\mathbb{R}$. If $\sigma \nmid q$, then

$$p \in S \setminus \hat{Z}, \quad q \in \hat{Z},$$

and $\sigma \nmid \tilde{q}$, then

$$pq \in \hat{Z}.$$

Proof. Since $\tilde{q}$ is a monomial in $\hat{Z}$, for each $j \in Q_0$ there is a cycle $q_j \in e_jAe_j$ such that $q_j = \tilde{q}$, $q_{t(q)} = q$, and the sum

$$z := \sum_{j \in Q_0} q_j$$

is in the center $Z$ of $A$. Furthermore, since $\sigma \nmid \tilde{q}$, the set of representatives of the $q_j$ partition $Q$ into columns and pillars, by Lemma 3.12.

Fix a representative $\tilde{p}$ of $p$ (that is, $\tilde{p} + I = p$), and for each $j \in Q_0$, choose a representative $\tilde{q}_j$ of $q_j$. By assumption, $u$ and $v$ are linearly independent over $\mathbb{R}$. Thus, for each $j \in Q_0$, the cycles $\tilde{q}_j$ and $\tilde{p}$ intersect at some vertex $i(j) \in Q_0$. Factor $\tilde{q}_j$ and $\tilde{p}$ into paths

$$\tilde{q}_j = q_2 e_{i(j)} q_1,$$

$$\tilde{p} = p_2 e_{i(j)} p_1,$$

and consider the cycles

$$r_j := q_2 p_1 q_2 + I \in e_j A e_j.$$  

Note that

$$r_j = \overline{pq} = \overline{pq}.$$

Thus, to prove the lemma, it suffices to show that the element

$$r := \sum_{j \in Q_0} r_j$$

is in the center $Z$ of $A$.

(a) We first claim that $r$ is independent of the choice of representatives $\tilde{q}_j$ of $q_j$. Fix $j \in Q_0$, and suppose $\tilde{q}_j$ and $\tilde{q}_j'$ are two representatives of $q_j$. By Lemma 3.12, it suffices to suppose that $\tilde{q}_j$ and $\tilde{q}_j'$ bound a pillar, and $\tilde{p}$ intersects this pillar as shown in Figure 3. Factor $\tilde{p}$, $\tilde{q}_j$, $\tilde{q}_j'$ into paths

$$\tilde{p} = p_2 b p_1, \quad \tilde{q}_j = q_5 q_3 q_2 q_1, \quad \tilde{q}_j' = q_5 q_4 q_3 q_2 q_1.$$
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Then

\[ q_5q_3bp_1p_2q_2q_1 \equiv (i) q_5q'_4q'_3cbp_1p_2q_2q_1 \]
\[ \equiv q_5q'_4\sigma_{h(p_1)}p_1p_2q_2q_1 \]
\[ \equiv (ii) q_5q'_4p_1p_2\sigma_{t(p_2)}q_2q_1 \]
\[ \equiv q_5q'_4p_1p_2bq'_3cq_2q_1 \]
\[ \equiv (iii) q_5q'_4p_1p_2bq'_3q_1. \]

where (i) and (iii) hold by the dimer relations since each arrow in the interior of the column is an arrow in the quiver, and (ii) holds by Lemma 2.3. Therefore, the two representatives \( \tilde{q}_j, \tilde{q}'_j \) of \( q_j \) define the same cycle \( r_j \in e_jAe_j \) in (9).

(b) We now claim that \( r \) is in \( Z \). Since \( r \) is a sum of cycles, \( r \) trivially commutes with the vertex idempotents. Thus, to show that \( r \) is in \( Z \), it suffices to show that \( ra = ar \) for each arrow \( a \in Q_1 \).

Fix an arrow \( a \). Set \( j := t(a) \) and \( k := h(a) \).

If \( a \) is a leftmost arrow subpath of a representative \( \tilde{r}_k \) of \( r_k \), then there is a representative \( \tilde{r}_j \) of \( r_j \) that is a cyclic permutation of \( \tilde{r}_k \), by Claim (a). Whence

\[ ra = r_k a = ar_j = ar. \]

So suppose \( a \) is not a leftmost arrow subpath of any representative of \( r_k \). There are three cases to consider, shown in Figure [4]

Case (i): Suppose \( \tilde{q}_j \) and \( \tilde{q}_k \) bound a column, and \( \tilde{p} \) intersects this column as shown in Figure [ii]. Factor \( \tilde{p}, \tilde{q}_j, \tilde{q}_k \) into paths

\[ \tilde{p} = p_2bp_1, \quad \tilde{q}_j = q_3q_2q_1, \quad \tilde{q}_k = q'_2q'_1. \]

Then

\[ (q'_2(bp_1p_2)q_1)a \equiv (i) q'_2bp_1p_2bq_2q_1 \]
\[ \equiv (i) a(q_3(p_1p_2b)q_2q_1), \]

where (i) and (ii) hold by the dimer relations since each arrow in the interior of the column is an arrow in the quiver.

Case (ii): Suppose \( \tilde{q}_j \) and \( \tilde{q}_k \) bound a column, and \( \tilde{p} \) intersects this column as shown in Figure [ii]. Factor \( \tilde{p}, \tilde{q}_j, \tilde{q}_k \) into paths

\[ \tilde{p} = p_2cp_1, \quad \tilde{q}_j = q_3q_2q_1, \quad \tilde{q}_k = q'_2q'_1. \]
Thus, \[(q_3'(p_1p_2c)q_1')a \equiv (ii) q_3'p_1p_2cbq_1\]
\[\equiv q_3'p_1p_2\sigma_t(q_2)q_1\]
\[(i) \equiv q_2'\sigma_h(p_1)p_1p_2q_1\]
\[\equiv q_2'bq_2cp_1p_2q_1\]
\[(ii) \equiv a(q_3q_2(cp_1p_2)q_1),\]

where (i) and (iii) hold by the dimer relations since each arrow in the interior of the column is an arrow in the quiver, and (ii) holds by Lemma 2.3.

Case (iii): Suppose \(q_j'\) and \(q_k'\) bound a pillar, and \(p\) intersects \(q_j\) as shown in Figure 4.iii. Factor \(p, q_j, q_k\) into paths
\[p = p_2p_1, \quad q_j = q_4q_3q_2q_1, \quad q_k = q_4'q_3q_2q_1'.\]

Whence,
\[(q_4'q_3(p_1p_2)q_2q_1')a \equiv (i) q_4'q_3p_1p_2q_2\sigma_t(q_2)q_1\]
\[\equiv q_4'\sigma_h(q_3)q_3p_1p_2q_2q_1\]
\[(ii) \equiv a(q_4q_3(p_1p_2)q_2q_1),\]

where (i) and (iii) hold by the dimer relations since each arrow in the interior of the pillar is an arrow in the quiver, and (ii) holds by Lemma 2.3.

Therefore \(ra = ar\) holds in each case.
Figure 4. The three cases for Claim (b) in the proof of Lemma 3.13.

Lemma 3.14. Let $p \in C^u$, $q \in C^v$ be cycles in $Q$ such that $u, v \in \mathbb{Z}^2$ are linearly dependent over $\mathbb{R}$. If $
olinebreak[4]\bar{p} \in S \setminus \hat{Z}$, \ $
olinebreak[4]\bar{q} \in \hat{Z}$, and $\sigma$ does not divide $\bar{p}$ or $\bar{q}$, then there is a cycle $r$ and integers $m, n \geq 1$ such that $r^n = q$, $\bar{r} \in \hat{Z}$, and $\bar{p} \bar{r}^m \in \hat{Z}$.
Proof. (i) First suppose $u = (0, 0)$. Since $p \not\in \hat{Z}$, $p$ is not a vertex. Thus, there is some $\ell \geq 1$ such that $p = \sigma^\ell$, by Lemma 2.4.1. Whence $\sigma \nmid p$, contrary to assumption.

(ii) Assume to the contrary that there are positive integers $n_1, n_2 \geq 1$ such that $n_1 u = n_2 v$. By assumption, $\sigma \nmid p$. Thus there is a simple matching $D$ such that $x_D \nmid p$. Whence $\sigma \nmid p^{n_1}$. Similarly, $\sigma \nmid q^{n_2}$. Consequently,

$$(10) \quad p^{n_1} (i) = q^{n_2} \in \hat{Z} (ii) \subseteq R,$$

where (i) holds by Lemma 2.4.2, and (ii) holds by [B6, Theorem 1.1.3].

Now if a monomial $g \in S$ is not in $R$ and $\sigma \nmid g$, then $g^\ell$ is also not in $R$ for each $\ell \geq 1$, by Lemma 3.1.6. Thus (10) implies that $p$ is in $R$. Therefore $p$ is in $\hat{Z}$ since $\sigma \nmid p$, by Lemma 3.1.2. But this contradicts our choice of $p$.

(iii) Finally, suppose there are positive integers $n_1, n_2 \geq 1$ such that $n_1 u = -n_2 v$. Let $\hat{v} \in \mathbb{Z}^2$ be the vector of minimal length (with respect to the standard $\mathbb{R}^2$ metric) such that $v = n_3 \hat{v}$ for some $n_3 \geq 1$. Then there is a cycle $r \in \mathcal{Q}^\hat{v}$ such that $\sigma \nmid r$, by [B1, Corollary 3.9]. Thus $r \in \hat{Z}$ since $q \in \hat{Z}$, by Lemma 3.1.6. Furthermore, setting $m := n_2 n_3 n_1$ we have

$$u = -m \hat{v}.$$

Therefore there is some $\ell \geq 1$ such that

$$p^m (i) = \sigma^\ell \in \hat{Z},$$

where (i) holds by Lemma 2.4.1. \hfill \square

Recall the subsets (2) of Max $S$ and the morphisms (7).

**Proposition 3.15.** Let $n \in \text{Max } S$. Then

$$n \cap \hat{Z} \neq \emptyset_0 \quad \text{if and only if} \quad \hat{Z}_{n \cap \hat{Z}} = S_n,$$

and

$$n \cap R \neq \emptyset_0 \quad \text{if and only if} \quad R_{n \cap R} = S_n.$$ 

Consequently,

$$\kappa_{S/\hat{Z}}(U_{S/\hat{Z}}) = \text{Max } \hat{Z} \setminus \emptyset_0 \quad \text{and} \quad \kappa_{S/R}(U_{S/R}) = \text{Max } R \setminus \emptyset_0.$$ 

**Proof.**

(i) Set $m := n \cap \hat{Z}$, and suppose $m \neq \emptyset_0$. We first claim that

$$(11) \quad S \subset \hat{Z}_m.$$

Consider $g \in S \setminus \hat{Z}$. By [B2, Proposition 5.14], $S$ is generated by $\sigma$ and a set of monomials in $B$ not divisible by $\sigma$. Furthermore, $\sigma$ is in $\hat{Z}$. It thus suffices to suppose that $g$ is a monomial which is not divisible by $\sigma$; let $p$ be a cycle for which $\tilde{p} = g$.

(i.a) First suppose $\sigma$ does not divide all monomials in $\hat{Z}$. By Lemma 3.10, there is a nontrivial cycle $q \in A$ such that

$$\tilde{q} \in \hat{Z} \setminus m \quad \text{and} \quad \sigma \nmid \tilde{q}.$$
Let \( u, v \in \mathbb{Z}^2 \) be such that
\[
p \in C^u \quad \text{and} \quad q \in C^v.
\]
If \( u, v \) are linearly independent over \( \mathbb{R} \), then \( p\bar{q} \) is in \( \hat{Z} \), by Lemma 3.13. Whence
\[
g = \bar{p} = (p\bar{q})q^{-1} \in \hat{Z}_m.
\]
If instead \( u, v \) are linearly dependent over \( \mathbb{R} \), then there is a cycle \( r \) and integers \( m, n \geq 1 \) such that
\[
r^n = q, \quad r \in \hat{Z}, \quad \bar{p}r^m \in \hat{Z},
\]
by Lemma 3.14. Furthermore, \( r \not\in \mathfrak{m} \) since \( r^n = q \) and \( q \not\in \mathfrak{m} \). Thus,
\[
g = \bar{p} = (\bar{p}r^m)r^{-m} \in \hat{Z}_m.
\]
Therefore, in either case, \( g \) is in the localization \( \hat{Z}_m \). Consequently, (11) holds if \( \sigma \) does not divide all monomials in \( \hat{Z} \).

(i.b) Now suppose \( \sigma \) divides all monomials in \( \hat{Z} \). Then \( \mathfrak{m} \) does not contain any monomials since \( \mathfrak{m} \neq \mathfrak{z}_0 \), by Lemma 3.8. In particular, \( \sigma \not\in \mathfrak{m} \). By Lemma 3.14, there is an \( n \geq 0 \) such that \( g\sigma^n \in \hat{Z} \). Thus
\[
g = (g\sigma^n)\sigma^{-n} \in \hat{Z}_m.
\]
Therefore (11) also holds if \( \sigma \) divides all monomials in \( \hat{Z} \).

(ii) Denote by \( \hat{\mathfrak{m}} := \mathfrak{m}\hat{Z}_m \) the maximal ideal of \( \hat{Z}_m \). Then, since \( \hat{Z} \subset S \), we have
\[
\hat{Z}_m = \hat{Z}_{\hat{\mathfrak{m}} \cap \hat{Z}} \subseteq S_{\hat{\mathfrak{m}} \cap S} \subseteq (\hat{Z}_m)_{\hat{\mathfrak{m}} \cap \hat{Z}_m} = (\hat{Z}_m)_{\hat{\mathfrak{m}}\hat{Z}_m} = \hat{Z}_m,
\]
where (i) holds by (11). Therefore
\[
S_n = \hat{Z}_m.
\]

(iii) Now suppose \( n \cap R \neq \mathfrak{m}_0 \). We claim that \( R_{n \cap R} = S_n \).

Since \( n \cap R \neq \mathfrak{m}_0 \), there is a monomial \( g \in R \setminus n \). Thus there is some \( n \geq 1 \) such that \( g^n \in \hat{Z} \), by Lemma 3.13. Furthermore, \( g^n \not\in n \) since \( n \) is a prime ideal. Consequently,
\[
g^n \in \hat{Z} \setminus (n \cap \hat{Z}).
\]
Whence
\[
(12) \quad n \cap \hat{Z} \neq \mathfrak{z}_0.
\]
Therefore
\[
S_n \overset{(1)}{=} \hat{Z}_{n \cap \hat{Z}} \subseteq R_{n \cap R} \subseteq S_n,
\]
where (i) holds by (12) and Claim (i), and (ii) holds by (3). It follows that \( R_{n \cap R} = S_n \).

(iv) Finally, we claim that
\[
\hat{Z}_{\mathfrak{z}_0} \neq S_{\mathfrak{z}_0} \quad \text{and} \quad R_{\mathfrak{m}_0} \neq S_{\mathfrak{m}_0}.
\]
These inequalities hold since the local algebras \( \hat{\mathbb{Z}}_{m_0} \) and \( R_{m_0} \) are nonnoetherian by Proposition 3.7, whereas \( S_n \) is noetherian by Lemma 3.2. □

**Lemma 3.16.** Let \( q \) and \( q' \) be prime ideals of \( S \). Then
\[
qu \cap \hat{\mathbb{Z}} = q' \cap \hat{\mathbb{Z}} \quad \text{if and only if} \quad q \cap R = q' \cap R.
\]

**Proof.** (i) Suppose \( q \cap \hat{\mathbb{Z}} = q' \cap \hat{\mathbb{Z}} \), and let \( s \in q \cap R \). Then \( s \in R \). Whence there is some \( n \geq 1 \) such that \( s^n \in \hat{\mathbb{Z}} \), by Lemma 3.13. Thus
\[
s^n = q \cap \hat{\mathbb{Z}} = q' \cap \hat{\mathbb{Z}}.
\]
Therefore \( s^n \in q' \). Thus \( s \in q' \) since \( q' \) is prime. Consequently, \( s \in q' \cap R \). Therefore \( q \cap R \subseteq q' \cap R \). Similarly, \( q \cap R \supseteq q' \cap R \).

(ii) Now suppose \( q \cap R = q' \cap R \), and let \( s \in q \cap \hat{\mathbb{Z}} \). Then \( s \in \hat{\mathbb{Z}} \subseteq R \). Thus
\[
s \in q \cap R = q' \cap R.
\]
Whence \( s \in q' \cap \hat{\mathbb{Z}} \). Therefore \( q \cap \hat{\mathbb{Z}} \subseteq q' \cap \hat{\mathbb{Z}} \). Similarly, \( q \cap \hat{\mathbb{Z}} \supseteq q' \cap \hat{\mathbb{Z}} \). □

**Proposition 3.17.** The subsets \( U_{S/\hat{\mathbb{Z}}} \) and \( U_{S/R} \) of \( \text{Max} \ S \) are equal.

**Proof.** (i) We first claim that
\[
U_{S/\hat{\mathbb{Z}}} \subseteq U_{S/R}.
\]
Indeed, suppose \( n \in U_{S/\hat{\mathbb{Z}}} \). Then since \( \hat{\mathbb{Z}} \subseteq R \subseteq S \), we have
\[
S_n = \hat{\mathbb{Z}}_{w \cap \hat{\mathbb{Z}}} \subseteq R_{w \cap R} \subseteq S_n.
\]
Thus
\[
R_{w \cap R} = S_n.
\]
Therefore \( n \in U_{S/R} \), proving our claim.

(ii) We now claim that
\[
U_{S/R} \subseteq U_{S/\hat{\mathbb{Z}}}.
\]
Let \( n \in U_{S/R} \). Then \( R_{w \cap R} = S_n \). Thus by Proposition 3.15
\[
n \cap R \neq n_0 \cap R.
\]
Therefore by Lemma 3.16
\[
n \cap \hat{\mathbb{Z}} \neq n_0 \cap \hat{\mathbb{Z}}.
\]
But then again by Proposition 3.15,
\[
\hat{\mathbb{Z}}_{w \cap \hat{\mathbb{Z}}} = S_n.
\]
Whence \( n \in U_{S/\hat{\mathbb{Z}}} \), proving our claim. □

We denote the complement of a set \( W \subseteq \text{Max} \ S \) by \( W^c \).
Theorem 3.18. The following subsets of $\text{Max } S$ are open, dense, and coincide:

\[
U_{S/\hat{Z}}^* = U_{S/\hat{Z}} = U_{S/R}^* = U_{S/R} = \kappa_{S/\hat{Z}}^{-1}(\text{Max } \hat{Z} \setminus \{z_0\}) = \kappa_{S/R}^{-1}(\text{Max } R \setminus \{m_0\})
\]

\[
= Z_S(z_0S)^c = Z_S(m_0S)^c.
\]

In particular, $\hat{Z}$ and $R$ are locally noetherian at all points of $\text{Max } \hat{Z}$ and $\text{Max } R$ except at $z_0$ and $m_0$.

Proof. For brevity, set $Z(I) := Z_S(I)$.

(i) We first show the equalities of the top two lines of (13). By Proposition 3.17, $U_{S/R} = U_{S/\hat{Z}}$.

By Lemma 3.2, $S$ is noetherian. Thus for each $n \in \text{Max } S$, the localization $S_n$ is noetherian. Therefore, by Proposition 3.15,

\[
U_{S/\hat{Z}}^* = U_{S/\hat{Z}} \quad \text{and} \quad U_{S/R}^* = U_{S/R}.
\]

Moreover, again by Proposition 3.15,

\[
U_{S/\hat{Z}} = \kappa_{S/\hat{Z}}^{-1}(\text{Max } \hat{Z} \setminus \{z_0\}) \quad \text{and} \quad U_{S/R} = \kappa_{S/R}^{-1}(\text{Max } R \setminus \{m_0\}).
\]

(ii) We now claim that the complement of $U_{S/R} \subset \text{Max } S$ is the zero locus $Z(m_0S)$. Suppose $n \in Z(m_0S)$; then $m_0S \subseteq n$. Whence,

\[
m_0 \subseteq m_0S \cap R \subseteq n \cap R.
\]

Thus $n \cap R = m_0$ since $m_0$ is a maximal ideal of $R$. But then $n \not\in U_{S/R}$ by Claim (i). Therefore $U_{S/R}^c \supseteq Z(m_0S)$.

Conversely, suppose $n \in U_{S/R}^c$. Then $\kappa_{S/R}(n) \not\in \kappa_{S/R}(U_{S/R})$, by the definition of $U_{S/R}$. Thus $n \cap R = \kappa_{S/R}(n) = m_0$, by Claim (i). Whence,

\[
m_0S = (n \cap R)S \subseteq n,
\]

and so $n \in Z(m_0S)$. Therefore $U_{S/R}^c \subseteq Z(m_0S)$.

(iii) Finally, we claim that the subsets (13) are open dense. Indeed, there is a maximal ideal of $R$ distinct from $m_0$, and $\kappa_{S/R}$ is surjective by Lemma 3.5. Thus the set $\kappa_{S/R}^{-1}(\text{Max } R \setminus \{m_0\})$ is nonempty. Moreover, this set equals the open set $Z(m_0S)^c$, by Claim (ii). But $S$ is an integral domain. Therefore $Z(m_0S)^c$ is dense since it is nonempty and open.

Theorem 3.19. The center $Z$, reduced center $\hat{Z}$, and ghor center $R$ of $A$ each have Krull dimension 3,

\[
\dim Z = \dim \hat{Z} = \dim R = \dim S = 3.
\]

Furthermore, the fraction fields of $\hat{Z}$, $R$, and $S$ coincide,

\[
\text{Frac } \hat{Z} = \text{Frac } R = \text{Frac } S.
\]
Proof. Recall that \( S \) is of finite type by Lemma \[3.2\] and \( \hat{Z} \subseteq R \subseteq S \) are integral domains since they are subalgebras of the polynomial ring \( B \).

The sets \( U_{S/\hat{Z}} \) and \( U_{S/R} \) are nonempty, by Theorem \[3.18\]. Thus \( \hat{Z}, R, \) and \( S \) have equal fraction fields \[B5, \text{Lemma 2.4}\]; and equal Krull dimensions \[B5, \text{Theorem 2.5.4}\]. In particular, \( \dim \hat{Z} = \dim R = \dim S = 3 \), by Lemma \[3.3\]. Finally, each prime \( p \in \text{Spec} \ Z \) contains the nilradical \( \text{nil} \ Z \), and thus \( \dim Z = \dim \hat{Z} \). \( \Box \)

Recall that the reduction \( X_{\text{red}} \) of a scheme \( X \), that is, its reduced induced scheme structure, is the closed subspace of \( X \) associated to the sheaf of ideals \( \mathcal{I} \), where for each open set \( U \subseteq X \),

\[ \mathcal{I}(U) := \{ f \in \mathcal{O}_X(U) \mid f(p) = 0 \text{ for all } p \in U \}. \]

\( X_{\text{red}} \) is the unique reduced scheme whose underlying topological space equals that of \( X \). If \( R := \mathcal{O}_X(X) \), then \( \mathcal{O}_{X_{\text{red}}}(X_{\text{red}}) = R/\text{nil} R \).

Theorem 3.20. Let \( A \) be a nonnoetherian dimer algebra, and let \( \psi : A \to A' \) a cyclic contraction.

(1) The reduced center \( \hat{Z} \) and ghor center \( R \) of \( A \) are both depicted by the center \( Z' \cong S \) of \( A' \).

(2) The affine scheme \( \text{Spec} \ R \) and the reduced scheme of \( \text{Spec} \ Z \) are birational to the noetherian scheme \( \text{Spec} \ S \), and each contain precisely one closed point of positive geometric dimension, namely \( \mathfrak{m}_0 \) and \( \mathfrak{z}_0 \).

Proof. (1) We first claim that \( \hat{Z} \) and \( R \) are depicted by \( S \). By Theorem \[3.18\],

\[ U_{S/\hat{Z}}^* = U_{S/\hat{Z}} \neq \emptyset \quad \text{and} \quad U_{S/R}^* = U_{S/R} \neq \emptyset. \]

Furthermore, by Lemma \[3.5\], the morphisms \( t_{S/\hat{Z}} \) and \( t_{S/R} \) are surjective.\(^6\)

(2.i) As schemes, \( \text{Spec} \ S \) is isomorphic to \( \text{Spec} \hat{Z} \) and \( \text{Spec} R \) on the open dense subset \( U_{S/\hat{Z}} = U_{S/R} \), by Theorem \[3.18\]. Thus all three schemes are birationally equivalent. Furthermore, \( \hat{Z} \) and \( \text{Spec} R \) each contain precisely one closed point where \( \hat{Z} \) and \( R \) are locally nonnoetherian, namely \( z_0 \) and \( m_0 \), again by Theorem \[3.18\].

(2.ii.a) We claim that the closed point \( m_0 \in \text{Spec} R \) has positive geometric dimension.

Indeed, since \( A \) is nonnoetherian, there is a cycle \( p \) such that \( \sigma \nmid \bar{p} \) and \( \bar{p}^n \in S \setminus R \) for each \( n \geq 1 \), by Lemma \[3.1.6\].

If \( \bar{p} \) is in \( m_0 S \), then there are monomials \( g \in R, h \in S \), such that \( gh = \bar{p} \). Furthermore, \( g \neq \sigma^n \) for all \( n \geq 1 \) since \( \sigma \nmid \bar{p} \). But then \( \bar{p} = gh \) is in \( R \) by Lemma \[3.1.1\], a contradiction. Therefore \( \bar{p} \not\in m_0 S \).

\(^6\)The fact that \( S \) is a depiction of \( R \) also follows from \[B5, \text{Theorem D.1}\], since the algebra homomorphism \( \tau_\psi : \Lambda \to M_{|Q_\psi|}(B) \) is an impression \[B2, \text{Theorem 5.9.1}\].
Consequently, for each \( c \in k \), there is a maximal ideal \( n_c \in \text{Max} S \) such that
\[
(p - c, m_0)S \subseteq n_c.
\]
Thus,
\[
m_0 \subseteq (p - c, m_0)S \cap R \subseteq n_c \cap R.
\]
Whence \( n_c \cap R = m_0 \) since \( m_0 \) is maximal. Therefore by Theorem 3.18,
\[n_c \in U_{S/R}^c\]
Set
\[
q := \bigcap_{c \in k} n_c.
\]
The intersection of radical ideals is radical, and so \( q \) is a radical ideal. Thus, since \( S \) is noetherian, the Lasker-Noether theorem implies that there are minimal primes \( q_1, \ldots, q_\ell \in \text{Spec} S \) over \( q \) such that
\[
q = q_1 \cap \cdots \cap q_\ell.
\]
We claim that at least one \( q_i \) is non-maximal in \( S \). Setting \( \mathcal{Z}(I) := \mathcal{Z}_S(I) \), we have
\[
\bigcup_{i=1}^\ell \mathcal{Z}(q_i) = \mathcal{Z}(\bigcap_{i=1}^\ell q_i) \overset{(1)}{=} \mathcal{Z}(q) \overset{(ii)}{=} \mathcal{Z}(\cap_{c \in k} n_c) = \bigcup_{c \in k} \mathcal{Z}(n_c),
\]
where (1) holds by (16), and (ii) holds by (15). Thus, if each \( q_i \) were maximal, then
\[
\{\mathcal{Z}(q_1), \ldots, \mathcal{Z}(q_\ell)\} = \{\mathcal{Z}(n_c) : c \in k\}.
\]
In particular, there would be an infinite set of distinct zero-dimensional points that is equal to a finite set of zero-dimensional points, which is not possible.

Therefore at least one \( q_i \) is a non-maximal prime, say \( q_1 \). Whence,
\[
m_0 = \bigcap_{c \in k} (n_c \cap R) = \bigcap_{c \in k} n_c \cap R = q \cap R \subseteq q_1 \cap R.
\]
Consequently, \( q_1 \cap R = m_0 \) since \( m_0 \) is maximal in \( R \).

Since \( q_1 \) is a non-maximal prime ideal of \( S \),
\[
\text{ht}(q_1) < \dim S.
\]
Furthermore, \( S \) is a depiction of \( R \) by Claim (1). Thus
\[
\text{ght}(m_0) \leq \text{ht}(q_1) < \dim S \overset{(i)}{=} \dim R,
\]
where (i) holds by Theorem 3.19. Therefore
\[
gdim m_0 = \dim R - \text{ght}(m_0) \geq 1,
\]
proving our claim.

(2.ii.b) Finally, we claim that the closed point \( z_0 \in \text{Spec} \hat{Z} \) has positive geometric dimension.

Again, since \( A \) is nonnoetherian, there is a cycle \( p \) such that \( \sigma \nmid \bar{p} \) and \( \bar{p} \in S \setminus R \), by Lemma 3.16.
If \( p \) is in \( \mathfrak{z}_0S \), then there are monomials \( g \in \hat{Z}, \ h \in S \), such that \( gh = p \). Furthermore, \( g \in R \) since \( \hat{Z} \subseteq R \) by (3); and \( g \neq \sigma^n \) for all \( n \geq 1 \) since \( \sigma \nmid \bar{p} \). But then \( \bar{p} = gh \) is in \( R \) by Lemma 3.1.1, a contradiction. Therefore
\[
\bar{p} \notin \mathfrak{z}_0S.
\]
The proof then follows as in Claim (2.ii.a).

**Remark 3.21.** Although \( \hat{Z} \) and \( R \) determine the same variety using depictions, their associated affine schemes
\[
(\text{Spec } \hat{Z}, \mathcal{O}_{\hat{Z}}) \quad \text{and} \quad (\text{Spec } R, \mathcal{O}_R)
\]
will not be isomorphic if their rings of global sections, \( \hat{Z} \) and \( R \), are not isomorphic.

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**REFERENCES**

[BKM] K. Baur, A. King, B. Marsh, Dimer models and cluster categories of Grassmannians, *Proc. London Math. Soc.*, 113(2):213-260, 2016.
[B1] C. Beil, Cyclic contractions of dimer algebras always exist, *Algebr. Represent. Theor.*, 22:1083-1100, 2019.
[B2] ______, Dimer algebras, ghor algebras, and cyclic contractions, [arXiv:1711.09771](https://arxiv.org/abs/1711.09771).
[B3] ______, Morita equivalences and Azumaya loci from Higgsing dimer algebras, *J. Algebra*, 453:429-455, 2016.
[B4] ______, Noetherian criteria for dimer algebras, *J. Algebra*, 585:294-315, 2021.
[B5] ______, Nonnoetherian geometry, *J. Algebra Appl.*, 15(09), 2016.
[B6] ______, The central nilradical of nonnoetherian dimer algebras, [arXiv:1902.11299](https://arxiv.org/abs/1902.11299).
[BD] D. Berenstein, M. Douglas, Seiberg Duality for Quiver Gauge Theories, [arXiv:hep-th/0207027](https://arxiv.org/abs/hep-th/0207027).
[BGH] S. Bose, J. Gundry, Y. He, Gauge Theories and Dessins d’Enfants: Beyond the Torus, *J. High Energy Phys.*, 01:135, 2015.
[Br] N. Broomhead, Dimer models and Calabi-Yau algebras, *Memoirs AMS*, 1011, 2012.
[D] B. Davison, Consistency conditions for brane tilings, *J. Algebra*, 338:1-23, 2011.
[E] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Springer-Verlag, 1995.
[FGU] S. Franco, E. García-Valdecasas, A. Uranga, Bipartite field theories and D-brane instantons, *J. High Energy Phys.*, 11:98, 2018.
[FHVKW] S. Franco, A. Hanany, D. Vegh, B. Wecht, K. Kennaway, Brane dimers and quiver gauge theories, *J. High Energy Phys.*, 01:096, 2006.
[HK] A. Hanany, K. D. Kennaway, Dimer models and toric diagrams, [arXiv:0503149](https://arxiv.org/abs/0503149).

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