Gauged N=4 Supergravity through Compactification and BPS Monopoles

Ali H. Chamseddine

Center for Advanced Mathematical Sciences
and
Physics Department
American University of Beirut
Beirut, Lebanon
E-mail chams@aub.edu.lb

Abstract

It is shown that N=4 gauged supergravity in four dimensions is obtained by compactifying N=1 supergravity in ten dimensions on the group manifold $S^3 \times S^3$. This could be further related to supergravity in eleven dimensions. Analysis of supersymmetry conditions of N=4 gauged supergravity in four dimensions reveals solutions which preserve 1/4 of the supersymmetries and are characterized by a BPS-monopole-type gauge field. These solutions are lifted to solutions of the ten and eleven dimensional theories.

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1 Introduction

Gauged supergravities in four dimensions have been known to be related to compactifications of supergravities in higher dimensions. Some of these results have been shown explicitly, but for others such as the conjectured relation between gauged N=8 supergravity in four dimensions and compactified eleven dimensional supergravity on $S^7$ a full proof is still missing [1]. For the simpler gauged N=4 supergravity, the no-go theorem of [2] stood as an obstacle to such a realisation. It turns out that the correct compactification of the ten dimensional theory is to take the internal six dimensionnal manifold to be the group manifold $S^3 \times S^3$. This is to be supplemented with a very specific relation between the components of the metric and the antisymmetric tensor along the internal directions. I will show how this is done explicitly, and further relate this to a compactification of eleven dimensional supergravity. I will also describe non-abelian monopole solutions preserving $\frac{1}{4}$ of supersymmetries. These are the only known non-abelian solutions which are not conformally flat or obtained by embedding the gravitational spin-connection in the gauge group [3]. Using our explicit relation between ten-dimensional supergravity and the Friedman-Schwarz model, the four dimensional solution is lifted to a ten-dimensional one. This could be further lifted to a solution of eleven-dimensional supergravity. The results presented here is based on work done in collaboration with M. Volkov [5].

2 Compactification of ten-dimensional supergravity on $S^3 \times S^3$

It is well known that torus compactification of ten-dimensional supergravity to four dimensions gives N=4 supergravity coupled to six vector multiplets [6]. The four dimensional vectors of supergravity and matter are linear combinations of the components of the metric and antisymmetric tensor along the internal directions. Truncating the matter multiplets is equivalent to identifying some components of the ten-dimensional metric with those of the antisymmetric tensor. Another known compactification of ten-dimensional supergravity is due to Scherk and Schwarz [7] where the internal space is taken as a group manifold. The maximal allowed group is $S^3 \times S^3$. Such a study was done in [8] for the dual formulation of supergravity. The resulting N=4 four dimensional supergravity theory has a non-compact gauge group containing $SU(2) \times SU(2)$. It is not possible to truncate the resulting theory to give the Friedman-Schwarz model. A careful analysis of this model suggests that the antisymmetric tensor must play a non-trivial role. We shall now show explicitly the correct compactification.

The bosonic part of N=1 supergravity action in ten dimensions is:

$$S_{10} = \int \left( -\frac{\hat{e}}{4} \hat{R} + \frac{\hat{e}}{2} \partial_M \hat{\phi} \partial^M \hat{\phi} + \frac{\hat{e}}{12} e^{-2\hat{\phi}} \hat{H}_{MNP} \hat{H}^{MNP} \right) d^4x d^6z \equiv S_G + S_{\hat{\phi}} + S_{\hat{H}}. \quad (1)$$

The notation is as follows: hatted symbols are used for 10-dimensional quantities. Late capital Latin letters stand for the base space indices, and early letters refer to tangent space indices. For four-dimensional space-time indices, late and early Greek letters denote base space and tangent space indices, respectively.
Similarly, the internal base space and tangent space indices are denoted by late and early Latin letters, respectively:

\[ \{ M \} = \{ \mu = 0, \ldots , 3; \ m = 1, \ldots , 6 \}, \ \ \ \{ \alpha \} = \{ \alpha = 0, \ldots , 3; \ a = 1, \ldots , 6 \}. \tag{2} \]

The general coordinates \( \tilde{x}^M \) consist of spacetime coordinates \( x^\mu \) and internal coordinates \( z^m \). The flat Lorentz metric of the tangent space is chosen to be \((+,-,\ldots,-)\) with the internal dimensions all spacelike. One has \( \hat{\epsilon} = |\hat{e}^A_M| \), the metric is related to the vielbein by \( \hat{g}_{MN} = \hat{\eta}_{AB} \hat{e}^A_M \hat{e}^B_N = \eta_{\alpha\beta} \hat{e}^\alpha_M \hat{e}^\beta_N - \delta_{ab} \hat{e}^a_M \hat{e}^b_N \), and the antisymmetric tensor field strength is

\[ \hat{H}_{MNP} = \partial_M \hat{B}_{NP} + \partial_N \hat{B}_{PM} + \partial_P \hat{B}_{MN}. \tag{3} \]

The internal space spanned by \( z^m \) is assumed to form a compact group space. This means that there are functions \( U^a_m(z) \) subject to the condition

\[ (U^{-1})^m_\nu (U^{-1})^n_c \left( \partial_m U^a_n - \partial_n U^a_m \right) = \frac{f_{abc}}{\sqrt{2}}, \tag{4} \]

where \( f_{abc} \) are the group structure constants. The volume of the space is \( \Omega = \int |U^a_m| d^6z \). In the particular case when the internal space is the product manifold \( SU(2) \times SU(2) \) it is convenient to parametrize the 6 internal coordinates by a pair of indices: \( \{ m \} = \{ (s), i \} \), where \( s = 1, 2 \) and \( i = 1, 2, 3 \) and similarly for the tangent space coordinates: \( \{ a \} = \{ (s), a \} \), \( a = 1, 2, 3 \). Each of the two \( S^3 \)'s admits invariant 1-forms \( g(s)^a = \theta(s)^a_i dz(s)^i \) satisfying \( dg(s)^a + \frac{1}{2} \epsilon_{abc} g(s)^b \wedge g(s)^c = 0 \). If we choose \( U^a_m = U(s)^a_i = -\frac{\sqrt{2}}{g(s)^a_i} \), where \( g_a \) are the two gauge coupling constants, then the structure constants will be \( f_{abc} = \frac{1}{g_{abc}} = g_a \epsilon_{abc} \). Similarly, if one of the gauge coupling constants vanishes, say \( g_2 = 0 \), the internal space is \( SU(2) \times [U(1)]^3 \).

1. **The metric and the dilaton.**— Let us now return to the general parametrization of the internal space. Compactification of the action (1) starts by choosing the vielbein and the dilaton in the following form:

\[ \hat{e}^\alpha_\mu = e^{-\frac{\phi}{4}} e^\alpha_\mu, \quad \hat{e}^\alpha_m = \sqrt{2} e^{\frac{\phi}{4}} A^a_\mu, \]

\[ \hat{e}^\alpha_m = 0, \quad \hat{e}^\alpha_m = e^{\frac{\phi}{4}} U^a_m, \quad \hat{\phi} = -\frac{\phi}{2}, \tag{5} \]

where all quantities on the right, apart from \( U^a_m \), depend only on \( x^\mu \). One has \( \hat{\epsilon} = e^{-3\phi/2} |U^a_m| e^\phi \). The dual basis is given by

\[ e^\alpha_\mu = e^{\frac{\phi}{4}} e^\alpha_\mu, \quad e^\alpha_m = 0, \]

\[ e^\alpha_m = -\sqrt{2} e^{\frac{\phi}{4}} e^\alpha_\mu A^a_\mu (U^{-1})^m_\alpha, \quad e^\alpha_m = e^{\frac{\phi}{4}} (U^{-1})^m_\alpha. \tag{6} \]

The metric components are then given by:

\[ g_{\mu\nu} = e^{-\frac{\phi}{4}} g_{\mu\nu} - 2 e^{\frac{\phi}{4}} A^a_\mu A^a_\nu, \quad g_{\mu m} = \sqrt{2} e^{\frac{\phi}{4}} A^a_\mu U^a_m, \quad g_{m m} = -e^{\frac{\phi}{4}} U^a_m U^a_n, \tag{7} \]

similarly for \( \hat{g}^{\mu\nu} \). Using these expressions for the gravitational and dilaton terms in the action (1), one gets

\[ S_G + S_\phi = \Omega \int e \left( -\frac{1}{4} R + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{8} e^{2\phi} F^a_\mu F^{a\mu} + \frac{1}{32} e^{-2\phi} f_{abc}^2 \right) d^4x, \tag{8} \]
where
\[ F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f_{abc} A_\mu^b A_\nu^c. \]

2. The two-form. Now, the important role is played by the antisymmetric tensor field. The corresponding ansatz is
\[ \hat{B}_{\mu\nu} = B_{\mu\nu}, \quad \hat{B}_{\mu m} = -\frac{1}{\sqrt{2}} A_\mu^a U_m^a, \quad \hat{B}_{mn} = \tilde{B}_{mn}, \]
where \( B_{\mu\nu} = B_{\mu\nu}(x) \), while \( \tilde{B}_{mn} \) depend only on \( z \). Computation of the field strength gives
\[ \hat{H}_{\mu\nu\rho} = H_{\mu\nu\rho}, \quad \hat{H}_{\mu mn} = \frac{1}{2} f_{abc} A_\mu^a U_m^b U_n^c, \quad \hat{H}_{mnop} = \partial_m \hat{B}_{np} + \partial_n \hat{B}_{pm} + \partial_p \hat{B}_{mn}. \]

We also require that \( \hat{H}_{mnop} = \frac{1}{2\sqrt{2}} f_{abc} U_m^a U_n^b U_p^c U_\sigma^d \).

The next step is to compute the vielbein projections of \( H_{MNP} \). The result is
\[ \hat{H}_{\alpha\beta\gamma} = \epsilon^a \phi (H_{\alpha\beta\gamma} - \omega_{\alpha\beta\gamma}), \quad \hat{H}_{\alpha\beta\alpha} = -\frac{1}{\sqrt{2}} \epsilon^a \phi F_{\alpha\beta}^a, \]
\[ \hat{H}_{\alpha\beta\alpha} = 0, \quad \hat{H}_{\alpha\beta\epsilon} = \frac{1}{2\sqrt{2}} e^{-\frac{2}{3}} f_{abc}, \]
where \( F_{\alpha\beta}^a = e^\mu e^\nu F_{\mu\nu}^a \) are the tetrad projections of the gauge field tensor, and \( \omega_{\alpha\beta\gamma} \) are the tetrad projections of the gauge field Chern-Simons 3-form
\[ \omega_{\mu\nu\rho} = -6 \left( A_{\mu}^a \partial_\nu A_\rho^a + \frac{1}{3} f_{abc} A_\mu^b A_\nu^c A_\rho^c \right). \]

It is now straightforward to compute the last term in the action [12]:
\[ S_F = \Omega \int e \left( -\frac{1}{8} e^{2\phi} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{96} e^{-2\phi} f_{abc}^2 + \frac{1}{12} e^{4\phi} H_{\mu\nu\rho}^I H_{\mu\nu\rho}^I \right) d^4x, \]
where \( H_{\mu\nu\rho} = H_{\mu\nu\rho} - \omega_{\mu\nu\rho} \). Now, taking advantage of the identity \( \epsilon^a_{\sigma\mu\nu} \partial_\sigma H_{\mu\nu\rho} = 0 \), it is easy to see that the expression \( -\Omega \int \left( \frac{1}{8} e^{2\phi} F_{\mu\nu\rho}^a \partial_\nu a H_{\mu\nu\rho} \right) d^4x \) vanishes up to a surface term; here \( a \) is a Lagrange multiplier. Adding this to the action [9] it is possible to go to a first order formalism where both \( H_{\mu\nu\rho} \) and \( a \) are treated as independent fields. The equation of motion of \( a \) implies that \( H_{\mu\nu\rho} \) is a closed form and can be expressed locally as the curl of \( B_{\mu\nu} \). This is equivalent to varying \( H_{\mu\nu\rho} \) in the action with the result
\[ H_{\mu\nu\rho} = \omega_{\mu\nu\rho} + e^{-4\phi} \epsilon_{\sigma\mu\nu\rho} \partial_\sigma a. \]

Combining the above results gives
\[ S_{10} = \Omega \int e \left( -\frac{1}{4} R + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} e^{2\phi} \partial_\mu a - \frac{1}{4} e^{2\phi} F_{\mu\nu}^a F^{a\mu\nu} \right. \]
\[ - \left. \frac{1}{2} a F_{\mu\nu}^a \ast F^{a\mu\nu} + \frac{1}{48} e^{-2\phi} f_{abc}^2 \right) d^4x. \]

Finally, choosing \( U_m^a \) and \( f_{abc} \) for the \( SU(2) \times SU(2) \) case gives \( \langle f_{abc} \rangle^2 = 6 (g_1^2 + g_2^2) \), and thus the dimensionally reduced action exactly reproduces the bosonic part of the Friedman-Schwarz model of \( N=4 \) supergravity – up to an overall factor. Analysis of the fermionic sector is done in [9].
3 BPS Monopole solutions of N=4 gauged supergravity

For a purely bosonic configuration, the supersymmetry transformation laws of the action (16) are:

\[
\delta \bar{\chi} = -i \sqrt{2} \bar{\epsilon} \gamma^\mu \partial_\mu \phi + \frac{1}{2} e^\phi \bar{\epsilon} \alpha^a F^a_{\mu\nu} \sigma^{\mu\nu} + \frac{1}{4} e^{-\phi} \bar{\epsilon},
\]

\[
\delta \bar{\psi}_\rho = \bar{\epsilon} \left( \partial_\rho - \frac{1}{2} \omega_{\rho mn} \sigma^{mn} + \frac{1}{2} \alpha^a A^a_\rho \right) - \frac{1}{2\sqrt{2}} e^\phi \bar{\epsilon} \alpha^a F^a_{\mu\nu} \gamma_\mu \sigma^{\mu\nu} + \frac{i}{4\sqrt{2}} e^{-\phi} \bar{\epsilon} \gamma_\rho,
\]

the variations of the bosonic fields being zero. In these formulas, \( \epsilon \equiv \epsilon^I \) are four Majorana spinor supersymmetry parameters, \( \alpha^a \equiv \alpha^a_{IJ} \) are the SU(2) gauge group generators, whose explicit form is given in [4], and \( \omega_{\rho mn} \) is the tetrad connection.

We shall consider static, spherically symmetric, purely magnetic configurations of the bosonic fields, and for this we parameterize the fields as follows [9]:

\[
ds^2 = N \sigma^2 dt^2 - \frac{dr^2}{N} - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2),
\]

\[
\alpha^a A^a_\mu dx^\mu = w (-\alpha^2 \partial \theta + \alpha^1 \sin \theta \partial \varphi) + \alpha^3 \cos \theta \partial \varphi,
\]

where \( N, \sigma, w \), as well as the dilaton \( \phi \), are functions of the radial coordinate \( r \). Now, since we are unable to directly solve the equations of motion, we shall consider the supersymmetry conditions (17), which will give us a set of first integrals.

The field configuration (18) is supersymmetric, provided that there are non-trivial supersymmetry Killing spinors \( \epsilon \) for which the variations of the fermion fields defined by Eqs. (17) vanish. Inserting configuration (18) into Eqs. (17) and putting \( \delta \bar{\chi} = \delta \bar{\psi}_\mu = 0 \), the supersymmetry constraints become a system of equations for the four spinors \( \epsilon^I \).

The consistency of the algebraic constraints requires that the determinants of the corresponding coefficient matrices vanish and that the matrices commute with each other. These consistency conditions can be expressed by the following relations for the background:

\[
N \sigma^2 = e^{2(\phi - \phi_0)},
\]

\[
N = \frac{1 + w^2}{2} + e^{2\phi} \left( \frac{w^2 - 1}{2r^2} \right)^2 + \frac{r^2}{8} e^{-2\phi},
\]

\[
r\phi' = \frac{r^2}{8N} e^{-2\phi} \left( 1 - 4e^{4\phi} \right),
\]

\[
rw' = -2we^{-2\phi} \left( 1 + 2e^{2\phi} \right),
\]

with constant \( \phi_0 \). Under these conditions, the solution of the algebraic constraints yields \( \epsilon \) in terms of only two independent functions of \( r \). The remaining differential constraint then uniquely specify these two functions up to two
integration constants, which finally corresponds to two complex unbroken supersymmetries. We therefore conclude that the supersymmetry conditions for the bosonic background (13) are given in terms of Eqs. (19)–(22). Solutions of these Bogomolny equations describe the BPS states in N=4 gauge supergravity with 1/4 of the supersymmetries preserved.

Suppose that \( w(r) \) is not constant. Introducing the new variables \( x = w^2 \) and \( R^2 = \frac{1}{2} r^2 e^{-2\phi} \), Eqs. (19)–(22) become equivalent to one first order differential equation

\[
2xR(R^2 + x - 1) \frac{dR}{dx} + (x + 1) R^2 + (x - 1)^2 = 0. \tag{23}
\]

If \( R(x) \) is known, the radial dependence of the functions, \( x(r) \) and \( R(r) \), can be determined from (21) or (22). Eq. (23) is solved by the following substitution:

\[
x = \rho^2 e^{\xi(\rho)}, \quad R^2 = -\rho \frac{d\xi(\rho)}{d\rho} - \rho^2 e^{\xi(\rho)} - 1, \tag{24}
\]

where \( \xi(\rho) \) is obtained from

\[
\frac{d^2\xi(\rho)}{d\rho^2} = 2 e^{\xi(\rho)}. \tag{25}
\]

The most general (up to reparametrizations) solution of this equation which ensures that \( R^2 > 0 \) is \( \xi(\rho) = -2 \ln \sinh(\rho - \rho_0) \). This gives us the general solution of Eqs. (19)–(22). The metric is non-singular at the origin if only \( \rho_0 = 0 \), in which case

\[
R^2(\rho) = 2\rho \coth \rho - \frac{\rho^2}{\sinh^2 \rho} - 1, \tag{26}
\]

one has \( R^2(\rho) = \rho^2 + O(\rho^4) \) as \( \rho \to 0 \), and \( R^2(\rho) = 2\rho + O(1) \) as \( \rho \to \infty \).

The last step is to obtain \( r(s) \) from Eq. (22), which finally gives us a family of completely regular solutions of the Bogomolny equations:

\[
ds^2 = a^2 \sinh \rho \left\{ dt^2 - dr^2 - R^2(\rho)(d\theta^2 + \sin^2 \theta d\varphi^2) \right\}, \tag{27}
\]

\[
w = \pm \frac{\rho}{\sinh \rho}, \quad e^{2\phi} = a^2 \frac{\sinh \rho}{2 R(\rho)}, \tag{28}
\]

where \( 0 \leq \rho < \infty \), \( R(\rho) \) is given by Eq. (24), and we have chosen in Eq. (19) \( 2\phi_0 = -\ln 2 \). The geometry described by the line element (27) is everywhere regular, the coordinates covering the whole space whose topology is \( \mathbb{R}^4 \). The geometry becomes flat at the origin, but asymptotically it is not flat. The shape of the gauge field amplitude \( w(\rho) \), given by Eq. (28), corresponds to the gauge field of the regular magnetic monopole type. In fact, replacing \( \rho \) by \( r \), the amplitude exactly coincides with that for the flat space BPS solution.

4 Lifting the solution to ten and eleven dimensions

The results of the previous section imply that any solution of the gauged supergravity model in four dimensions given in terms of the metric \( g_{\mu\nu} \), gauge
fields $A_\mu^a$, the axion $a$ and the dilaton $\phi$, can be lifted to ten dimensions as a solution of the N=1 supergravity. The ten-dimensional metric, the vielbein and the dilaton $\hat{\phi}$ are then given by Eqs. (5) – (7).

Choosing $A_\mu^{(2)a} = g_2 = 0$ and $g_1 = 1$, the lifted solutions can be represented as follows. The metric and the dilaton are

$$g_{MN} = 2e^{-\hat{\phi}} \hat{g}_{MN}, \quad \hat{\phi} = -\frac{\phi(\rho)}{2},$$

(29)

where the metric in the string frame, $\hat{g}_{MN}$, is specified by the line element

$$ds^2 = dt^2 - d\rho^2 - R^2(\rho) d\Omega^2_2 - \Theta^a \Theta^a - (dz^4)^2 - (dz^5)^2 - (dz^6)^2.$$

(30)

Here $d\Omega^2_2$ is the standard metric on the unit 2-sphere, $\Theta^a \equiv A^a - \theta^a = A^a_\mu dx^\mu - \theta^a_i dz^i$,

(31)

where $\theta^a_i$ are the Maurer-Cartan forms on $S^3$ parametrized by \{z^i\} = \{z^1, z^2, z^3\}. If $T_a$ are the SU(2) group generators, $[T_a, T_b] = i\epsilon^{abc} T_c$, then the gauge field is given by

$$A \equiv T_a A^a \equiv T_a A^a_\mu dx^\mu = w(\rho) \left(-T_2 d\theta + T_1 \sin \theta d\varphi\right) + T_3 \cos \theta d\varphi.$$

(32)

The non-vanishing vielbein projections of the antisymmetric tensor field are

$$\hat{H}_{\alpha\beta a} = -\frac{1}{2\sqrt{2}} e^{-\frac{2}{3} \phi} F_{\alpha\beta}^a, \quad \hat{H}_{abc} = \frac{1}{2\sqrt{2}} e^{-\frac{4}{3} \phi} \epsilon_{abc},$$

(33)

where $F_{\alpha\beta}^a$ are the tertad projections of the gauge field tensor corresponding to the gauge field (32) for the tetrad $e_a$ specified by the four-dimensional part of the string metric (28).

To lift the solution further to eleven dimensions we first note that N=1 supergravity in ten-dimensions is obtained from eleven dimensional supergravity by dimensional reduction and then compactification. One has the following identifications: $e_M^A = e^{-\frac{4}{3} \phi} e_M^A$ and $e_{11}^1 = e^{\frac{4}{3} \phi}$. For the antisymmetric tensor we have $A_{MN11} = B_{MN}$ and $A_{MNP} = 0$. This implies that we can write the different components of the eleven dimensional metric as follows:

$$g_{\mu\nu}^{(11)} = e^{-\frac{4}{3} \phi} (g_{\mu\nu} - 2e^{\phi} A_\mu^a A_\nu^a), \quad g_{\mu m}^{(11)} = \sqrt{2} e^{\frac{2}{3} \phi} A_\mu^a U^a_m,$$

$$g_{mn}^{(11)} = -e^{\frac{2}{3} \phi} U^a_m U^a_n, \quad g_{1111}^{(11)} = e^{-\frac{4}{3} \phi}.$$

(34)

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