Distributed Nash Equilibrium Seeking Algorithm Design for Multi-Cluster Games with High-Order Players

Zhenhua Deng, Yangyang Liu

Abstract—In this paper, a multi-cluster game with high-order players is investigated. Different from the well-known multi-cluster games, the dynamics of players are taken into account in our problem. Due to the high-order dynamics of players, existing algorithms for multi-cluster games cannot solve the problem. For purpose of seeking the Nash equilibrium of the game, we design a distributed algorithm based on gradient descent and state feedback, where a distributed estimator is embedded for the players to estimate the decisions of other players. Furthermore, we analyze the exponential convergence of the algorithm via variational analysis and Lyapunov stability theory. Finally, a numerical simulation verifies the effectiveness of our method.

Index Terms—Multi-cluster games, distributed algorithms, high-order multi-agent systems, Nash equilibrium.

I. INTRODUCTION

Distributed optimization and noncooperative games describe cooperative and competitive behaviors among multiple agents, respectively, which have widely applications in a variety of fields, such as smart grids, social networks, parameter estimation and radio networks, and have attracted considerable attention (see [1], [2], [3], [4], [5], [6]). In distributed optimization problems, all participants cooperate with their neighbors to search the optimal solution of the networks (see [7], [8], [9], [10], [11], [12]), while in noncooperative games, every player competes with other players to selfishly minimize its own cost function (see [13], [14], [15], [16]). Nevertheless, it is noteworthy that in numerous engineering practices, cooperation and competition among agents always coexist, such as healthcare networks and transportation networks (see [17], [18]). Multi-cluster games can simultaneously characterize cooperation relationship within clusters and competition relationship between clusters, which extends the aforementioned distributed optimization problems and noncooperative game problems, and have aroused the interest of many scholars (see [19],[20], [21], [22], [23]).

Multi-cluster games are conducted by multiple interacting clusters, and each cluster consists of a group of players. Every cluster wants to minimize its own cost function that is a summation of the cost functions of all players in the cluster. Consequently, the objective of these clusters are to seek the Nash equilibrium of the multi-cluster games. To this end, some Nash equilibrium seeking algorithms have been proposed for multi-cluster games recently. For example, for unconstrained multi-cluster games, [19] designed a Nash equilibrium seeking algorithm based on dynamic average consensus, and for constrained multi-cluster games, [20] presented a distributed Nash equilibrium seeking algorithm via projected operators. In order to reduce the communication and computation costs, [21] exploited a Nash equilibrium seeking algorithm for multi-cluster games with interference graphs. For multi-cluster games with partial-decision information, [22] proposed a distributed Nash equilibrium seeking algorithm based on the intra- and inter-communication of clusters. For multi-cluster games with non-smooth cost functions, [23] developed a Nash equilibrium seeking algorithm by Gaussian smoothing techniques.

Cyber-physical systems (CPSs) integrate computation, communication and physical processes, and commonly appear in multifarious engineering applications, such as power networks and transportation systems (see [24], [25]). With the development of CPSs, more and more distributed algorithms involved with the dynamics of systems have been exploited to study how physical systems autonomously accomplish distributed tasks. For example, [13], [14] investigated aggregate game problems of disturbed systems and Euler-Lagrange systems, respectively, and [12], [26] studied distributed optimization problems of second-order systems. On the other hand, many physical systems, such as generators, robots and satellites, can be depicted by high-order systems, and first- and second-order systems can be viewed as the special cases of high-order systems. Nevertheless, to the best of our knowledge, there are no results about multi-cluster games with high-order multi-agent systems. Moreover, without further integrating the control of high-order dynamics, existing Nash seeking algorithms for multi-cluster games, such as [19], [20], [21], [22], [23], are ineffective for the problem. These observations motivate us to study multi-cluster games of high-order multi-agent systems.

The objective of this paper is to investigate multi-cluster games of high-order players and design a distributed algorithm to seek the Nash equilibrium of the game. The contributions of this paper are summarized as follows:

1) We study the multi-cluster games of multi-agent systems, where the players have high-order dynamics. The formulation extends non-cooperative games discussed in [13], [14], [15], [16] by containing the distributed optimization of players within clusters, the distributed...
optimization problems studied in [7], [8], [9], [10], [11], [12] by considering the noncooperative games between clusters, and the multi-cluster games investigated in [19], [20], [21], [22], [23] by adding the high-order dynamics of players. Because the players have high-order dynamics, existing algorithms for multi-cluster games, such as [19], [20], [21], [22], [23], cannot be applied to our problem.

2) We design a distributed algorithm based on state feedback and gradient descent to seek the Nash equilibrium of multi-cluster games. Most of existing multi-cluster game algorithms need full decision information of all players (see [19], [21], [23]), and in contrast, the players with our algorithm only exchange information with their neighbors. Furthermore, we analyze the convergence of the algorithm via variational analysis and Lyapunov stability theory. Compared with the algorithms in [13], [14], [15], [16], [20], our algorithm exponentially instead of asymptotically converges to the Nash equilibrium of multi-cluster games.

The paper is organized as follows. Section II introduces some basic knowledge and describes our problem. Section III presents a distributed Nash equilibrium seeking algorithm and analyzes its convergence. Section IV provides a numerical example to illustrate the algorithm. Finally, Section V summarizes the conclusion.

Notations: \( \mathbb{R} \) and \( \mathbb{R}^n \) represent the set of real numbers and the \( n \)-dimensional Euclidean space, respectively, \( \otimes \) is the Kronecker product, \( \times \) is the Cartesian product, \( 0_n \) and \( 1_n \) denote the column vectors of \( n \) zeros and ones, respectively, \( \|x\| \) is the standard Euclidean norm of vector \( x \), \( \|A\| \) is the spectral norm of matrix \( A \). Define \( \text{col}(x_1, \ldots, x_n) = [x_1^T, \ldots, x_n^T]^T \), where \( x_i \) is a vector. \( I_n \) denotes the identity matrix. Let \( \lambda_{\min}(A) \) and \( \lambda_{\max}(A) \) be the smallest and the largest eigenvalues of matrix \( A \), respectively.

II. PRELIMINARIES AND FORMULATION

In this section, some preliminaries about graph theory and variational analysis are reviewed, and then our problem is formulated.

A. Preliminaries

Here some concepts about graph theory are presented (see [27]). Consider an undirected graph \( G = (\mathcal{V}, \mathcal{E}, A) \), where \( \mathcal{V} = \{1, \ldots, N\} \) is the vertex set, \( \mathcal{E} \subseteq \mathcal{V} \times \mathcal{V} \) is the edge set, and \( A = [a_{ij}]_{N \times N} \) is the adjacency matrix. An edge of \( G \) is denoted by \( \{i, j\} \in \mathcal{E} \) if vertices \( i \) and \( j \) can receive information from each other, i.e., they are neighbors. Besides, \( \{i, i\} \notin \mathcal{E}, \forall i \in \mathcal{V} \). A path of \( G \) is given by a sequence of distinct vertices connected by edges. The undirected graph \( G \) is connected if there exists a path between any pair of vertices. The element \( a_{ij} \) of the adjacency matrix \( A \) represents the weighting of \( \{i, j\} \), where \( a_{ij} = a_{ji} > 0 \) if \( \{i, j\} \in \mathcal{E} \), and \( a_{ij} = 0 \), otherwise. Let \( \deg_i = \sum_{j=1}^{N} a_{ij} \) be the degree of vertex \( i \). Define \( L = D - A \) as the Laplacian matrix of \( G \), where \( D = \text{diag} \{\deg_1, \ldots, \deg_N\} \). Obviously, \( L1_N = 0_N \). The eigenvalues of \( L \) are expressed as \( \lambda_1, \ldots, \lambda_N \), where \( \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N \) if \( i \leq j \). Moreover, \( G \) is connected if and only if \( \lambda_2 > 0 \).

Next, some definitions about variational analysis are introduced (see [28]).

A function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is convex if for any \( x, y \in \mathbb{R}^n \),
\[
f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y), \forall \alpha \in [0, 1].
\]

A function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is \( \omega \)-strongly monotone \( (\omega > 0) \) if for any \( x, y \in \mathbb{R}^n \),
\[
(x - y)^T (f(x) - f(y)) \geq \omega \|x - y\|^2.
\]

A function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is \( \theta \)-Lipschitz \( (\theta > 0) \) if for any \( x, y \in \mathbb{R}^n \),
\[
\|f(x) - f(y)\| \leq \theta \|x - y\|.
\]

B. Problem Formulation

Consider a multi-cluster game of \( N \) clusters over an undirected graph \( G_0 \). Cluster \( j \in \{1, \ldots, N\} \) consists of \( n_j \) players over an undirected graph \( G_j \), where \( G_j \) is a subgraph of \( G_0 \). Player \( i \) in cluster \( j \) has a continuously differentiable cost function \( f_i^j(x^i, x^{-j}) : \mathbb{R}^q \times \bigcup_{l=1}^{N} \mathbb{R}^{q_l} \rightarrow \mathbb{R} \), where \( x^i_j \in \mathbb{R}^q \) is the decision of player \( i \) in cluster \( j \), \( x^{-j} = \text{col}(x^1, \ldots, x^i - j, \ldots, x^N) \in \mathbb{R}^\sum_{l=1}^{N} q_l \), \( x^i \in \mathbb{R}^{q_i} \). Players in the same cluster are required to reach a common strategy. The objective of players in cluster \( j \) is to minimize \( f_i^j(x^i, x^{-j}) \) by competing with other clusters, where \( f_i^j(x^i, x^{-j}) = \sum_{l=1}^{N} f_l^j(x^i, x^{-j}) \) is the cost function of cluster \( j \). Specifically, cluster \( j \) faces the following multi-cluster game problem:
\[
\min_{x_i \in \mathbb{R}^{q_i}} f_i^j(x^i, x^{-j}) \\
\text{s.t. } x^i_j = x^i_k, \forall i, k \in \{1, \ldots, n_j\}.
\]

(1)

The Nash equilibrium of the multi-cluster game (1) is defined as follows (see [29], [30], [31]).

**Definition 1.** A strategy profile \( x^* = (x_1^*, \ldots, x_N^*) \) is said to be a Nash equilibrium of the multi-cluster game (1) if for all \( j \in \{1, \ldots, N\} \), we have
\[
f_i^j(x^i_1, x^{-i}_1) \leq f_i^j(x^i, x^{-j}), \forall x^j : (x^j, x^{-j}) \in \Omega_i
\]
where \( \Omega = \Omega_1 \cap \ldots \cap \Omega_N \) with \( \Omega_i = \{x^i \in \mathbb{R}^{q_i} : x^j_i = x^j_k, \forall i, k \in \{1, \ldots, n_j\} \} \).

Based on Definition 1, a Nash equilibrium is a strategy profile on which cluster \( j \) cannot reduce its cost \( f_i^j(x^i, x^{-j}) \) by unilaterally changing its own decision.

Some standard assumptions are given as follows.

**Assumption 1.** Undirected graphs \( G_0, \ldots, G_N \) are connected.

**Assumption 2.** The cost function \( f_i^j(x_i, x^{-i}) \) is convex in \( x_i \), and the map \( F(x) \) is \( \omega \)-strongly monotone and \( \theta \)-Lipschitz in \( x \), where \( F(\cdot) : \mathbb{R}^q \rightarrow \mathbb{R}^q \) is defined as
\[
F(x) = \text{col}(\nabla x_1 f_1^1(x_1^1, x^{-1}), \ldots, \nabla x_{q_1} f_1^1(x_{q_1}, x^{-1}), \ldots, \nabla x_i f_i^j(x_i^j, x^{-i}), \ldots, \nabla x_{q_n} f_n(x_{q_n}, x^{-N}))
\]
(2)
with \( x = \text{col}(x^1, \ldots, x^N) \in \mathbb{R}^\sum_q \) and \( q = \sum_{j=1}^{N} q_j \).
Under Assumption 2, we have the following lemma about the Nash equilibrium.

**Lemma 1.** Suppose Assumption 2 holds. $x = (x^i, x^j)$ is a Nash equilibrium of the multi-cluster game (1) if and only if
\[
\sum_{i=1}^{n_i} \nabla x_i^j f_i^j(x_i^j, x^j) = 0
\]
where $x_i^j$ is the $n$th order derivative of $x_i^j$ with $n \geq 1$, and $u_i^j \in \mathbb{R}^q$ is the control input.

The objective of this paper is to design a distributed algorithm for the high-order player (4) such that the outputs of all players converge to the Nash equilibrium of the multi-cluster game (1), which means the high-order player (4) can carry out the multi-cluster game task (1) autonomously.

**Remark 1.** Our formulation can be viewed as extensions of distributed optimization problems and noncooperative game problems: when $N = 1$, the problem (1) is reduced to the distributed optimization problems studied in [7], [8], [9], [10], [11], [12]; when $n_j = 1$, we have investigated in [13], [14], [15], [16]. Therefore, the multi-cluster game (1) involves cooperative and competitive behaviors of the players simultaneously: players in the same cluster collectively optimize the cost function of the cluster, while players in different clusters selfishly minimize their own cost functions of the clusters that they belong to. Moreover, without involving the high-order dynamics, existing Nash equilibrium seeking algorithms for multi-cluster games [19, 20, 21, 22, 23] cannot control the high-order player (4) to accomplish multi-cluster game task (1) autonomously. Also, the high-order dynamics of players and the nonlinearity of cost functions make it difficult to design and analyze distributed game algorithms.

**III. MAIN RESULTS**

In this section, we propose a distributed Nash equilibrium seeking algorithm for the multi-cluster game (1) with high-order player (4) in Subsection III-A, and then analyze its convergence in Subsection III-B.

**A. Distributed Algorithm Design**

This subsection provides a distributed Nash equilibrium seeking algorithm for the multi-cluster game (1) with high-order player (4).

Before giving our algorithm, the following characteristic polynomial associated with real coefficients $(k_1, \ldots, k_{n-1})$ is defined such that its roots are in the open left half plane (LHP).
\[
p(s) := s^{n-1} + k_{n-1}s^{n-2} + \ldots + k_2s + k_1
\]
which implies that the following companion matrix $A$ is Hurwitz.
\[
A = \begin{bmatrix} 0 & I_{n-2} \\ -k_1 & -k_2 & \ldots & -k_{n-1} \end{bmatrix}
\]

The following lemma is about the companion matrix $A$, which is used later (see [32, Theorem 5.6]).

**Lemma 2.** There is a positive definite symmetric matrix $P_1$ such that $P_1A + A^TP_1 = -I_{n-1}$ is satisfied, where $P_1 := [P_{ij}^n]_{(n-1) \times (n-1)}$.

The distributed Nash equilibrium seeking algorithm for player $i$ in cluster $j$ is designed as follows.
\[
\begin{align*}
\dot{x}_{i}^{j(n)} &= -\sum_{l=1}^{n_j} \epsilon \sum_{k=1}^{n_j} a_{ik}^j (x_{i}^{j(n)} - x_{k}^{j(n)}) - \nabla x_i^j f_i^j(x_i^j, x^j) \\
\dot{y}_{i}^j &= \kappa \sum_{k=1}^{n_j} a_{k}^j (x_{i}^{j(n)} - x_{k}^{j(n)}) \\
\dot{\hat{x}}_{i}^{j} &= -\kappa \sum_{v=1}^{n_j} \sum_{u=1}^{n_j} a_{uv}^j (\hat{x}_{i}^{j} - \hat{x}_{u}^{j}) + \tilde{a}_{in}^0 (\hat{x}_{i}^{j} - x_{i}^{j(n)})
\end{align*}
\]

where $\hat{x}_{i}^{j} \in \mathbb{R}^q$ is the estimation of player $i$ in cluster $j$ on $x_{i}^{j(n)}$ of player in cluster $n$ with $m \in \{1, 2, \ldots, N\}$ and $n \in \{1, \ldots, n_j\}$, $\dot{x}^j = \text{col}(x_{i}^{j(1)}, x_{i}^{j(2)}, \ldots, x_{i}^{j(n)})$, $\hat{x}_{i}^{j} = \text{col}(\hat{x}_{i}^{j(1)}, \hat{x}_{i}^{j(2)}, \ldots, \hat{x}_{i}^{j(n)})$, $a_{uv}^j$ is the element of the adjacency matrix $A_0$ of $G_0$, $a_{ik}^j$ is the element of the adjacency matrix $A_j$ of $G_j$, $k_1, \ldots, k_{n-1}$ are the coefficients of the characteristic polynomial (5) with roots in the open LHP, $\epsilon > \sqrt{\frac{3}{2}}(\sqrt{\theta} + 1)$, $\kappa_2 > \frac{12 \epsilon \omega}{n_j (2 - \epsilon)}$, $P_2$ and $Q$ are symmetric positive-definite matrices satisfying $P_2(L \otimes I_N + M) + (L \otimes I_N + M)^T P_2 = Q$, $M = \text{diag}(\tilde{a}_{in}^0)$, $L$ is the Laplacian matrix of $G_0$, $\tilde{a}_{in}^0 = \max\{2p_1(n-1) + k_2, \ldots, 2p_2(n-2)(n-1) + k_2 + k_1\}$, $p_1(n-1)$ is the last element of the $i$th row vector of matrix $P_1$ defined in Lemma 2, $k_{max} = \max\{1, k_2, \ldots, k_{n-1}\}$, $L = \text{diag}\{L^1, \ldots, L^N\}$ with $L^j$ being the Laplacian matrix of $G_j$.
Remark 3. In contrast to many existing multi-cluster game algorithms that require every player to have access to the decisions of all players, such as [19], [20], [21], [23], players with the algorithm (7) only exchange necessary information with their neighbors. Besides, the cost functions and gradients of players are not shared with any other players, which signifies that the algorithm (7) is conducive to protect these information.

B. Convergence Analysis

The convergence of algorithm (7) is analyzed in this subsection.

Let

\[ x = \text{col}(x^1, \ldots, x^N) \in \mathbb{R}^q \]

\[ x^i = \text{col}(x^i_1, \ldots, x^i_{n_j}) \in \mathbb{R}^{q_{n_j}} \]

\[ y = \text{col}(y^1, \ldots, y^N) \in \mathbb{R}^q \]

\[ y^i = \text{col}(y^i_1, \ldots, y^i_{n_j}) \in \mathbb{R}^{q_{n_j}} \]

\[ x^{(l)} = \text{col}(x^{(l)}_1, \ldots, x^{(l)}_{N(l)}) \in \mathbb{R}^q \]

\[ x^{i(l)} = \text{col}(x^{i(l)}_1, \ldots, x^{i(l)}_{n_j}) \in \mathbb{R}^{q_{n_j}} \]

\[ \hat{x}^{ij} = \text{col}(\hat{x}^{ij}_1, \ldots, \hat{x}^{ij}_{n_k}) \in \mathbb{R}^q \]

where \( l \in \{1, \ldots, n\} \) and \( N = \sum_{j=1}^N n_j. \)

Combining (4) with (7), we have

\[ \dot{x} = x^{(1)} \quad (8a) \]

\[ x^{(n)} = -\sum_{l=1}^{n-1} e^{n-l} k_l x^{(l)} - y - F(\hat{x}) \quad (8b) \]

\[ \dot{y} = \kappa_1 ((L \otimes I_q) x), \quad y(0) = 0_q \quad (8c) \]

\[ \dot{\hat{x}} = -\kappa_2 ((L \otimes \hat{N} \otimes I_q) \hat{x} + (M \otimes I_q) (\hat{x} - 1_N \otimes x)) \quad (8d) \]

where \( F(\hat{x}) = \text{col}(\nabla x_1 f_1(x^1, \hat{x}^{-1}), \ldots, \nabla x_{n_1} f_1(x^1, \hat{x}^{-1}), \ldots, \nabla x_{n_N} f_{n_N}(x_{n_N}, \hat{x}^{-N})). \)

The following lemma is about \( F(\hat{x}) \), which is used later.

Lemma 3. Under Assumption 2, \( F(\hat{x}) \) is \( \theta \)-Lipschitz.

Proof: Based on the previous definitions, we have \( F(\hat{x}) = \text{col}(\nabla x_1 f_1(x^1, \hat{x}^{-1}), \ldots, \nabla x_{n_1} f_1(x^1, \hat{x}^{-1}), \ldots, \nabla x_{n_N} f_{n_N}(x_{n_N}, \hat{x}^{-N})). \) Define \( \hat{y}^{-j} \), \( \hat{y}^{-j} \) and \( y \) in the same way as \( \hat{x}^{-j}, \hat{x}^{-j} \) and \( x \). Then, by Assumption 2, it is obvious that \( \|\nabla x_{j} f_{j}(x^{j}, \hat{x}^{-j}) - \nabla x_{j} f_{j}(x^{j}, \hat{y}^{-j})\| \leq \|F(\hat{x}) - F(\hat{y})\| \leq \theta \|\hat{x}^{-j} - \hat{y}^{-j}\|, \)

\[ \forall j \in \{1, \ldots, N\}, \forall i \in \{1, \ldots, n_j\}. \]

Regarding the following inequalities are obtained: \( \|F(\hat{x}) - F(\hat{y})\|^2 = \sum_{j=1}^N \sum_{k=1}^{n_j} \|\nabla x_{j} f_{j}(x^{j}, \hat{x}^{-j}) - \nabla x_{j} f_{j}(x^{j}, \hat{y}^{-j})\|^2 \leq \theta^2 \sum_{j=1}^N \sum_{k=1}^{n_j} \|\hat{x}^{-j} - \hat{y}^{-j}\|^2 \), which implies that \( F(\hat{x}) \) is \( \theta \)-Lipschitz. \[ \square \]

Next, the relationship between the equilibrium point of (8) and the Nash equilibrium of the multi-cluster game (1) is analyzed, which yields the following result.

Theorem 1. Under Assumptions 1 and 2, \( x^* \) is a Nash equilibrium of the multi-cluster game (1) if and only if there exist \( y^* \in \mathbb{R}^q \) and \( \hat{x}^* \in \mathbb{R}^q \) such that \( (x^*, y^*, \hat{x}^*) \) is an equilibrium point of (8).

Proof: (i) The equilibrium point of (8) satisfies the following equations:

\[ x^{(1)} = 0_q \quad (9a) \]

\[ \vdots \]

\[ -\sum_{l=1}^{n-1} e^{-n-l} k_l x^{(l)} - y^* - F(\hat{x}) = 0_q \quad (9b) \]

\[ (L \otimes I_q) x^* = 0_q \quad (9c) \]

\[ -((\mathcal{L} \otimes I_N) \otimes I_q) x^* + (M \otimes I_q)(\hat{x}^* - 1_N \otimes x^*) = 0_{Nq}. \quad (9d) \]

It results from (9d) that \( -((\mathcal{L} \otimes I_N) \otimes I_q + (M \otimes I_q)) x^* = -((\mathcal{L} \otimes I_N) \otimes I_q + (M \otimes I_q))(1_N \otimes x^*). \) Then, because \( G^0 \) is a connected undirected graph and \( M \) is a diagonal matrix with at least one diagonal element being positive, \( (\mathcal{L} \otimes I_N) \otimes I_q + (M \otimes I_q) \) is positive define, which implies that \( \hat{x}^* = 1_N \otimes x^* \), i.e., \( F(\hat{x}^*) = F(1_N \otimes x^*). \) Besides, since \( G^1, \ldots, G^N \) are undirected and connected graphs, i.e., \( L_1 = 0_N, L_1^N L = 0_N, \)

(9c) yields that \( x_{i}^* = x_{k}^*, \forall i, k \in \{1, \ldots, n_j\}, \) and \((1^T_{n_j} \otimes I_q) y^j = 0_q, \forall j \in \{1, \ldots, N\}. \) Further, we have \( \sum_{j=1}^{n_j} \nabla x_j f_j(x_{j}^*, x^{-j*}) = 0_q, \forall j \in \{1, \ldots, N\}. \) Take \( \hat{x}^* = x_{m}^* \) and \( y^* = F(\hat{x}^*). \) Thus (9) holds.

Theorem 1 shows that if (8) converges to its equilibrium points, high-order player (4) approaches the Nash equilibrium of the multi-cluster game (1). Consequently, we can obtain the following result by analyzing the convergence of (8).

Theorem 2. Under Assumptions 1 and 2, the high-order player (4) with the algorithm (7) globally exponentially converges to the Nash equilibrium of the multi-cluster game (1).

Proof: We complete the proof in two steps.

Step 1: Coordinate transformations for (8). Make the following coordinate transformation.

\[ \tilde{x} = x - x^* \]

\[ \tilde{y} = y - y^* \]

\[ \tilde{x}^{(l)} = x^{(l)} - x^* \]

(10a) where \( l \in \{1, \ldots, n-1\}. \) and (8) and (9) yield that

\[ \tilde{x} = x^{(1)} \]
\[ \dot{x}^{(n)} = -\sum_{l=1}^{n-1} \varepsilon^{n-l} k_l \dot{x}^{(l)} - \bar{y} - h \quad (10b) \]
\[ \dot{y} = k_1 ((L \otimes I_q) \dot{x}) \quad (10c) \]
\[ \dot{\tilde{x}} = -k_2 ((L \otimes I_N) \otimes I_q + (M \otimes I_q)) \tilde{x} - 1_N \otimes \dot{\tilde{x}} \quad (10d) \]
where \( h = F(\tilde{x}) - F(\bar{x}). \)

With the above transformation, the equilibrium point of (10) is the origin. Let

\[ \tilde{x} = \text{col}(x^{(1)}, \ldots, x^{(n-1)}) \]
\[ \bar{x}^T = \text{col}(\bar{x}^{(1)}, \ldots, \bar{x}^{(n)}) \]
\[ \tilde{x}^T = \text{col}(\tilde{x}^{(1)}, \ldots, \tilde{x}^{(1)}) \]

where \( \tilde{x}^{(l)} = \frac{x^{(l)}}{\pi} \) with \( l \in \{1, \ldots, n-1\} \).

Then (10) can be rewritten as

\[ \dot{\tilde{x}} = -\varepsilon (A \otimes I_{Nq}) \tilde{x} - b \otimes I_{Nq} (\bar{y} + h) \quad (11a) \]
\[ \dot{\tilde{y}} = k_1 ((L \otimes I_q) \dot{x}) \quad (11b) \]
\[ \dot{\tilde{x}} = -k_2 ((L \otimes I_N) \otimes I_q + (M \otimes I_q)) \tilde{x} - 1_N \otimes \dot{\tilde{x}} \quad (11c) \]

where \( b = [0_{n-2} \ 1]^T \) and \( A \) is defined in (6).

It is obvious that (10) and (11) are equivalent.

Utilize the following orthogonal transformation,

\[ x = \text{col}(x_1, x_2) = ([r \ R]^T \otimes I_q) \tilde{x} \quad (12a) \]
\[ x^{(l)} = \text{col}(x^{(l)}, \chi_2^{(l)}) = ([r \ R]^T \otimes I_q) \tilde{x}^{(l)} \quad (12b) \]
\[ \eta = \text{col}(\eta_1, \eta_2) = ([r \ R]^T \otimes I_q) \tilde{y} \quad (12c) \]

where \( x_1, x_2 \in \mathbb{R}^q, \chi_1^{(l)}, \chi_2^{(l)}, \eta_2 \in \mathbb{R}^{(N-1)q}, \ l \in \{1, \ldots, n-1\}, \ r = \frac{1}{\sqrt{2}} 1_N, r^T R = 0_{N, N}, r^T R = I_{N-1} \) and \( RR^T = I_N - \frac{1}{N-1} 1_N 1_N^T. \)

Without loss of generality, let \( q = 1 \) for simplicity, and let

\[ \tilde{x}_1 = \text{col}(x_1^{(1)}, \ldots, x_1^{(n-1)}) \quad \tilde{x}_2 = \text{col}(x_2^{(1)}, \ldots, x_2^{(n-1)}). \]

Thus (11) can be described as

\[ \dot{\tilde{x}}_1 = \varepsilon \tilde{x}_1^{(1)} \quad (13a) \]
\[ \dot{\tilde{x}}_1 = \varepsilon A \tilde{x}_1 - \frac{1}{\varepsilon^{n-1}} b(\eta_1 + r^T h) \quad (13b) \]
\[ \dot{\eta}_1 = 0 \quad (13c) \]
\[ \dot{\tilde{x}}_2 = \varepsilon \tilde{x}_2 \quad (13d) \]
\[ \dot{\tilde{x}}_2 = \varepsilon (A \otimes I_{N-1}) \tilde{x}_2 - \frac{1}{\varepsilon^{n-1}} (b \otimes I_{N-1})(\eta_2 + R^T h) \quad (13e) \]
\[ \dot{\eta}_2 = \kappa_1 R^T \bar{L} \bar{R} \chi_2 \quad (13f) \]
\[ \dot{\chi}_2 = = -\kappa_2 ((L \otimes I_N) + M) \tilde{x} - 1_N \otimes \dot{\tilde{x}}. \quad (13g) \]

Obviously, \( \chi_2 \) approaches the Nash equilibrium of the multi-cluster game (1) if (13) tends to the origin. Consequently, our next task is to analyze the convergence of (13).

**Step 2: The convergence analysis of (13) to the origin.**

Take the following candidate Lyapunov function

\[ V = \tilde{x}_1^T P_1 \tilde{x}_1 + \tilde{x}_2^T (P_1 \otimes I_{N-1}) \tilde{x}_2 + \tilde{x}_2^T R \tilde{x}_2 \]
\[ + \frac{1}{2} k_1 \chi_1 + \sum_{i=1}^{n-2} k_{i+1} \chi_1^{(i)} + \chi_1^{(n-1)} \| h \|^2 \]
\[ + \frac{1}{2} k_1 \chi_2 + \sum_{i=1}^{n-2} k_{i+1} \chi_2^{(i)} + \chi_2^{(n-1)} + \mu \eta_2 \| \|^2 \quad (14) \]

where \( P_1 \) and \( P_2 \) are symmetric positive-definite matrices such that \( P_1 A + A^T P_1 = -I_{n-1} \) and \( P_2 (L \otimes I_N + M) + (L \otimes I_N + M)^T P_2 = Q \) hold, respectively (see Lemma 2).

The derivative of \( V \) along (13) is

\[ \dot{V} = -\kappa_2 \tilde{x}^T Q \tilde{x} - 2 \tilde{x}^T P_2 (1_N \otimes \tilde{x}) - \frac{k_1}{\varepsilon^{n-1}} (\chi_1^T r^T + \chi_2^T r^T) h \]
\[ - \varepsilon \| \tilde{x} \|^2 - \frac{\mu}{\varepsilon^{n-1}} \| \eta_2 \|^2 - \frac{\mu}{\varepsilon^{n-1}} \| \eta_1 \|^2 r^T h \]
\[ - \frac{1}{\varepsilon^{n-1}} \left( \sum_{l=1}^{n-2} (2p_l(n-1) + k_{l+1}) \chi_1^{(l)} + \frac{2}{\varepsilon^{n-1}} \| \eta_1 \|^2 r^T h \right) \]
\[ - \frac{1}{\varepsilon^{n-1}} \left( \sum_{l=1}^{n-2} (2p_l(n-1) + k_{l+1}) \chi_2^{(l)} + \frac{2}{\varepsilon^{n-1}} \| \eta_2 \|^2 r^T h \right) \]
\[ - \frac{k_1}{\varepsilon^{n-1}} \chi_1 \chi_2 + \kappa_1 \kappa_2 \chi_1 \chi_2 + \kappa_1 \kappa_2 \chi_2 \chi_2^T + \kappa_1 \kappa_2 \chi_2 \chi_2^T \quad (15) \]

where \( \tilde{x} = \text{col}(\tilde{x}_1, \tilde{x}_2). \)

Because \( Q \) is a symmetric positive-definite matrix, we have

\[ -\kappa_2 \tilde{x}^T Q \tilde{x} \leq -\kappa_2 \lambda_{\min}(Q) \| \tilde{x} \|^2. \quad (16) \]

According to the \( \omega \)-strongly monotonicity of \( F(x) \) (see Assumption 2), the Lipschitz continuity of \( F(\tilde{x}) \) (see Lemma 3) and (12), we have

\[ -\chi_1^T r^T h + \chi_2^T r^T h \leq -\tilde{x}^T (F(\tilde{x}) - F(1_N \otimes x) + F(x) - F(\bar{x}^T)) \leq -\omega \| \tilde{x} \|^2 + \theta \| \tilde{x} \| \| \tilde{x} - 1_N \otimes x \| \]
\[ \leq -\omega \| \tilde{x} \|^2 + \theta 2 \omega \| \tilde{x} \|^2. \quad (17) \]

It results from the orthogonal transformation (12) and the Lipschitz continuity of \( F(\tilde{x}) \) that

\[ -\frac{\mu}{\varepsilon^{n-1}} (\eta_1^T r^T + \eta_2^T r^T) h \leq -\frac{\mu \theta}{\varepsilon^{n-1}} \| \tilde{y} \| \| \tilde{x} \| \]
\[ \leq \frac{\mu \theta}{\varepsilon^{n-1}} \| \eta_1 \|^2 + \frac{\theta^2 \mu^2}{\varepsilon^{n-1}} \| \tilde{x} \|^2 \quad (18) \]

and

\[ -\frac{1}{\varepsilon^{n-1}} \left( \sum_{l=1}^{n-2} (2p_l(n-1) + k_{l+1}) \chi_1^{(l)} + (2p_l(n-1) + 1) \chi_1^{(n-1)} \right) r^T h \]
\[
\begin{align*}
&\leq \frac{\theta}{\varepsilon^{n-1}} \left( \sum_{i=1}^{n-2} \left( 2p_i(n-1) + k_i + 1 \right) \chi_2^{(i)} \right)^T \mathbf{R} \tilde{h} \\
&\leq \frac{\theta}{\varepsilon^{n-1}} \left( \sum_{i=1}^{n-2} \left( 2p_i(n-1) + k_i + 1 \right) \| \chi_1^{(i)} \| \right) \\
&\leq \frac{\theta}{\varepsilon^{n-1}} \left( \sum_{i=1}^{n-2} \left( 2p_i(n-1) + k_i + 1 \right) \| \chi_2^{(i)} \| \right) \\
&\leq \frac{\theta}{\varepsilon^{n-1}} \left( \sum_{i=1}^{n-2} \left( 2p_i(n-1) + k_i + 1 \right) \| \chi_1^{(i)} \| \right) \\
&\leq \frac{\theta}{\varepsilon^{n-1}} \left( \sum_{i=1}^{n-2} \left( 2p_i(n-1) + k_i + 1 \right) \| \chi_2^{(i)} \| \right)
\end{align*}
\]

Combining (15)-(20), we have
\[
\dot{V} \leq - \left( \frac{2\varepsilon}{3} - \frac{1}{\varepsilon^{n-1}} \left( \frac{\theta d_1}{2} + \bar{a}_1 \right) \right) \| \tilde{x} \|^2
\]
\[
- \left( \frac{\omega k_1}{2\varepsilon^{n-1}} - \kappa_1 \left( \frac{3n-3}{2\varepsilon} + \frac{\mu^2}{2} + k_1 \mu \| \mathbf{L} \| \right) \right) \| \chi \|^2
\]
\[
- \left( \frac{1}{\varepsilon^{n-1}} \left( \mu - \frac{2n-1}{4} \omega \right) - \frac{1}{2} \mu^2 \kappa_1 \| \mathbf{L} \|^2 \right) \| \eta_2 \|^2
\]
\[
- \left( \kappa_2 \lambda_{\text{min}}(Q) - \frac{\mu^2}{2} + \frac{\omega \theta(n-1)}{2\varepsilon^{n-1}} \right) \| \tilde{x} \|^2.
\]

(21)

(21) shows that \( \dot{V} \) is negative definite. Besides, the Lyapunov function \( V \) and its derivative are quadratic, which indicates that (13) is globally exponentially stable, i.e., the high-order player (4) under the algorithm (7) globally exponentially converges to the Nash equilibrium of the multi-cluster game (1).

Remark 4. The algorithm (7) is exponentially convergent, which is different from the asymptotically convergent algorithms in [13], [14], [15], [16], [20]. Furthermore, compared with the algorithms in [21] and [23], the algorithm (7) converges to the exact Nash equilibrium rather than the neighborhood of the Nash equilibrium.

IV. Numerical Examples

In this section, a numerical example is presented to illustrate the algorithm (7).

Consider a multi-cluster game with \( N \) clusters, where cluster \( j \in \{1,\ldots,N\} \) is composed of \( n_j \) players. The clusters compete with each other for their own benefits, and the players in the same cluster cooperate with each other. Cluster \( j \in \{1,\ldots,N\} \) faces the following multi-cluster game:

\[
\min_{\mathbf{x} \in \mathbb{R}^{n_{\text{col}}}} f_j^T (\mathbf{x}^j, \mathbf{x}^{-j})
\]

\[
x^j_i = x^j_k, \quad \forall i, k \in \{1,\ldots,n_j\}
\]

where \( f_j^T (\mathbf{x}^j, \mathbf{x}^{-j}) = \sum_{i=1}^{n_j} f_i^j (x^j_i, \mathbf{x}^{-j}) \) is the cost function of cluster \( j \), \( f_i^j (x^j_i, \mathbf{x}^{-j}) \) is the cost function of player \( i \) in cluster \( j \), \( x^j_i \) is the decision of player \( i \) in cluster \( j \), \( \mathbf{x}^j = (x^j_1, \ldots, x^j_{n_j}) \), \( \mathbf{x}^{-j} = (x^1_1, \ldots, x^1_{n_1}, \ldots, x^j_{n_j}, \ldots, x^{N}_{n_N}) \). Here, we consider \( N = 3, n_j = 4, \forall j \in \{1,\ldots,N\} \). Particularly, the cost functions of all players are expressed as follows:

\[
f_i^j (x^1_i, \mathbf{x}^{-1}) = \frac{3.4(x^1_i)^2}{6.2 \sqrt{5.2(x^1_i)^2 + 27}} + 1.9(x^1_i)^2 - 60x^1_i
\]

Fig. 1: The communication topology.
The dynamics of player $i$ in cluster $j$ are $\dot{x}_i^{(4)} = u_j^i$. The communication topology among players is depicted as Fig. 1. The algorithm parameters are chosen as $k_1 = 1$, $k_2 = 2$, $k_3 = 1$, $\varepsilon = 3.71$, $\kappa_1 = 0.05$ and $\kappa_2 = 386$.

The simulation results are presented in Fig. 2, where the solid lines, the dotted lines and the dot-dash lines are the evolutions of outputs of clusters 1, 2 and 3, respectively. As shown in Fig. 2, the decisions of players in the same cluster reach a common strategy, and the decisions of all players converge to the Nash equilibrium under algorithm (7). These simulation results verify the effectiveness of our method.

V. CONCLUSIONS

This paper has investigated the multi-cluster games of high-order multi-agent systems. To seek the Nash equilibrium of the multi-cluster game, we have designed a distributed algorithm via gradient descent and state feedback. In the algorithm, a distributed estimator has been employed such that players can estimate the decisions of other players. In comparison with other results for multi-cluster games, players only need to share some information with their neighbors by our algorithm. Besides, we have analyzed the convergence of the algorithm. Under the algorithm, all high-order players exponentially converge to the exact Nash equilibrium of the multi-cluster game. Finally, a numerical example has illustrated the result.

REFERENCES

[1] B. Gharesifard, T. Basar, and A. D. Dominguez-Garcia, “Price-based coordinated aggregation of networked distributed energy resources,” IEEE Transactions on Automatic Control, vol. 61, no. 10, pp. 2936–2946, Oct. 2016.
[2] J. Ghaderi and R. Srikant, “Opinion dynamics in social networks with stubborn agents: Equilibrium and convergence rate,” Automatica, vol. 50, no. 12, pp. 3209–3215, 2014.
[3] S. S. Ram, V. V. Veeravalli, and A. Nedík, “Distributed and recursive parameter estimation in parametrized linear state-space models,” IEEE Transactions on Automatic Control, vol. 55, no. 2, pp. 488–492, Feb. 2010.
[4] M. Cao, “Merging game theory and control theory in the era of AI and autonomy,” National Science Review, vol. 7, no. 7, pp. 1122–1124, 2020.
[5] D. Yuan, D. W. C. Ho, and S. Xu, “Stochastic strongly convex optimization via distributed epoch stochastic gradient algorithm,” IEEE Transactions on Neural Networks and Learning Systems, pp. 1–14, 2020, to be published.
[6] Y. Lou, Y. Hong, L. Xie, G. Shi, and K. H. Johansson, “Nash equilibrium computation in subnetwork zero-sum games with switching communications,” IEEE Transactions on Automatic Control, vol. 61, no. 10, pp. 2920–2935, Oct. 2016.
[7] S. Liang, L. Y. Wang, and G. Yin, “Exponential convergence of distributed primal–dual convex optimization algorithm without strong convexity,” Automatica, vol. 105, pp. 298–306, 2019.
[8] Y. Yang, Q. Liu, and J. Wang, “Distributed optimization based on a multigent system in the presence of communication delays,” IEEE Transactions on Systems, Man, and Cybernetics: Systems, vol. 47, no. 5, pp. 717–728, May 2017.
[9] R. Li and G.-H. Yang, “Consensus control of a class of uncertain nonlinear multiagent systems via gradient-based algorithms,” IEEE transactions on cybernetics, vol. 49, no. 6, pp. 2085–2094, Jun. 2019.
[10] X. He, T. Huang, J. Yu, C. Li, and Y. Zhang, “A continuous-time algorithm for distributed optimization based on multigent networks,” IEEE Transactions on Systems, Man, and Cybernetics: Systems, vol. 49, no. 12, pp. 2700–2709, Dec. 2019.
[11] S. S. Kia, J. Cortés, and S. Martínez, “Distributed convex optimization via continuous-time coordination algorithms with discrete-time communication,” Automatica, vol. 55, pp. 254–264, 2015.
[12] Y. Zhang, Z. Deng, and Y. Hong, “Distributed optimal coordination for multiple heterogeneous Euler-Lagrange systems,” Automatica, vol. 79, pp. 207–213, May 2017.
[13] Z. Deng, “Distributed algorithm design for aggregative games of Euler-Lagrange systems and its application to smart grids,” IEEE Transactions on Cybernetics, 2021, to be published.
[14] M. Bianchi and S. Grammatico, “Continuous-time fully distributed generalized Nash equilibrium seeking for multi-integrator agents,” Automatica, vol. 129, p. 109660, 2021.
[15] A. R. Romano and L. Pavel, “Dynamic NE seeking for multi-integrator networked agents with disturbance rejection,” IEEE Transactions on Control of Network Systems, vol. 7, no. 1, pp. 129–139, Mar. 2020.
[17] O. Shehory and S. Kraus, “Methods for task allocation via agent coalition formation,” *Artificial intelligence*, vol. 101, no. 1-2, pp. 165–200, 1998.
[18] T. J. Peng, A., and M. Bourne, “The coexistence of competition and cooperation between networks: implications from two taiwanese healthcare networks,” *British Journal of Management*, vol. 20, no. 3, pp. 377–400, 2009.
[19] M. Ye, G. Hu, and F. L. Lewis, “Nash equilibrium seeking for N-coalition noncooperative games,” *Automatica*, vol. 95, pp. 266–272, 2018.
[20] X. Zeng, J. Chen, S. Liang, and Y. Hong, “Generalized Nash equilibrium seeking strategy for distributed nonsmooth multi-cluster game,” *Automatica*, vol. 103, pp. 20–26, 2019.
[21] M. Ye, G. Hu, F. L. Lewis, and L. Xie, “A unified strategy for solution seeking in graphical N-coalition noncooperative games,” *IEEE Transactions on Automatic Control*, vol. 64, no. 11, pp. 4645–4652, Nov. 2019.
[22] M. Meng and X. Li, “On the linear convergence of distributed Nash equilibrium seeking for multi-cluster games under partial-decision information,” *arXiv preprint arXiv:2005.06923*, 2020.
[23] Y. Pang and G. Hu, “Gradient-free Nash equilibrium seeking in N-cluster games with uncoordinated constant step-sizes,” *arXiv preprint arXiv:2008.13088*, 2020.
[24] K. D. Kim and P. R. Kumar, “Cyber-physical systems: A perspective at the centennial,” *Proceedings of the IEEE*, vol. 100, pp. 1287–1308, May, 2012.
[25] X. Zhang, A. Papachristodoulou, and N. Li, “Distributed control for reaching optimal steady state in network systems: An optimization approach,” *IEEE Transactions on Automatic Control*, vol. 63, no. 3, pp. 864–871, Mar. 2018.
[26] Z. Deng, “Distributed algorithm design for resource allocation problems of second-order multiagent systems over weight-balanced digraphs,” *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 51, no. 6, pp. 3512–3521, June, 2021.
[27] C. D. Godsil and G. Royle, *Algebraic Graph Theory*. New York: Springer, 2001.
[28] F. Facchinei and J.-S. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems*. Springer Science & Business Media, 2003.
[29] C. A. Holt and A. E. Roth, “The Nash equilibrium: A perspective,” *Proceedings of the National Academy of Sciences*, vol. 101, no. 12, pp. 3999–4002, 2004.
[30] Z. Deng and X. Nian, “Distributed generalized Nash equilibrium seeking algorithm design for aggregative games over weight-balanced digraphs,” *IEEE Transactions on Neural Networks and Learning Systems*, vol. 30, no. 3, pp. 695–706, Mar. 2018.
[31] F. Facchinei and C. Kanzow, “Generalized Nash equilibrium problems,” *Annals of Operations Research*, vol. 175, no. 1, pp. 177–211, 2010.
[32] C.-T. Chen, *Linear System Theory and Design*, 3rd ed. New York: Oxford University Press, 1999.