A matrix decomposition and its applications

Fuzhen Zhang

Farquhar College of Arts and Sciences, Nova Southeastern University, Fort Lauderdale, FL, USA

Communicated by T.-Y. Tam

(Received 28 October 2013; accepted 27 May 2014)

We show the uniqueness and construction (of the $Z$ matrix in Theorem 2.1, to be exact) of a matrix decomposition and give an affirmative answer to a question proposed in [J. Math. Anal. Appl. 407 (2013) 436-442].

**Keywords:** accretive–dissipative matrix; Cartesian decomposition; matrix decomposition; numerical range; sectoral decomposition; unitarily invariant norm

**AMS Subject Classifications:** 15A45; 15A60; 47A30

1. Introduction

Several recent papers [1–5] are devoted to the study of matrices with numerical range in a sector of the complex plane. In particular, this includes the study of accretive–dissipative matrices and positive definite matrices as special cases. A matrix decomposition plays a fundamental role in these works. The aim of this paper is twofold: show the uniqueness along with other properties of the key matrix in the decomposition and give an affirmative answer to a question raised in [6].

As usual, the set of $n \times n$ complex matrices is denoted by $M_n$. For $A \in M_n$, the singular values and eigenvalues of $A$ are denoted by $\sigma_i(A)$ and $\lambda_i(A)$, respectively, $i = 1, \ldots, n$. The singular values are always arranged in nonincreasing order: $\sigma_1(A) \geq \cdots \geq \sigma_n(A)$. If $A$ is Hermitian, then all eigenvalues of $A$ are real and ordered as $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$. Note that $\sigma_j(A) = \lambda_j(|A|)$, where $|A|$ is the modulus of $A$, i.e. $|A| = (A^* A)^{1/2}$ with $A^*$ for the conjugate transpose of $A$. We denote $\sigma(A) = (\sigma_1(A), \ldots, \sigma_n(A))$ and $\lambda(A) = (\lambda_1(A), \ldots, \lambda_n(A))$.

Let $A \in M_n$. We write $A \succeq 0$ if $A$ is positive semidefinite (i.e. $x^* Ax \geq 0$ for all $x \in \mathbb{C}^n$) and $A \succ 0$ if $A$ is positive definite (i.e. $x^* Ax > 0$ for all nonzero $x \in \mathbb{C}^n$). For two Hermitian matrices $A$ and $B$ of the same size, we denote $A \succeq B$ if $A - B \succeq 0$. Note that $A \succeq B$ implies $\lambda_j(A) \geq \lambda_j(B)$ for all $j$.

For a square complex matrix $A$, recall the Cartesian (or Toeplitz) decomposition (see, e.g. [7, p.6] and [8, p.7]) $A = \Re A + i \Im A$, where

$$\Re A = \frac{1}{2} (A + A^*) , \quad \Im A = \frac{1}{2i} (A - A^*).$$

*Email: zhang@nova.edu

© 2014 Taylor & Francis
The Cartesian decomposition of a matrix is unique. There are many interesting properties for such a decomposition. For instance, \( \Re(A^*AR) = R^*(\Im A)R \) for any \( A \in \mathbb{M}_n \) and any \( n \times m \) matrix \( R \). A celebrated result due to Fan and Hoffman (see, e.g. [7, p.73]) states that

\[
\lambda_j(\Im A) \leq \sigma_j(A), \quad j = 1, \ldots, n. \tag{1}
\]

For \( A \in \mathbb{M}_n \), the numerical range of \( A \) is the set in the complex plane

\[
W(A) = \{ x^*Ax \mid x \in \mathbb{C}^n, \|x\| = 1 \}.
\]

The classic Toeplitz–Hausdorff theorem asserts that the numerical range of a matrix is a compact and convex subset of the complex plane (see, e.g. [9, p.108]).

If \( \Re A \) and \( \Im A \) are positive semidefinite. We call such a matrix \( A \) accretive–dissipative. Note that if \( A \) is accretive–dissipative and nonsingular, then \( W(A) \subseteq e^{i\pi/4}S_{\pi/4} \), i.e. \( W(e^{-i\pi/4}A) \subseteq S_{\pi/4} \). With a continuity argument, we assume that the accretive–dissipative matrices to be considered in this paper are nonsingular.

If \( W(A) \) is contained in the first quadrant of the complex plane, then \( \Re A \) and \( \Im A \) are positive semidefinite. We call such a matrix \( A \) accretive–dissipative. Note that if \( A \) is accretive–dissipative and nonsingular, then \( W(A) \subseteq e^{i\pi/4}S_{\pi/4} \), i.e. \( W(e^{-i\pi/4}A) \subseteq S_{\pi/4} \). With a continuity argument, we assume that the accretive–dissipative matrices to be considered in this paper are nonsingular.

More can be said about sectors and numerical ranges. Observe that a sector \( S_\alpha, \alpha \in [0, \frac{\pi}{2}) \), is a positive convex cone (i.e. \( ax + by \in S_\alpha \) for all positive \( a, b \in \mathbb{R} \) and all \( x, y \in S_\alpha \), it has the addition-closure property, that is, \( S_\alpha + S_\alpha \subseteq S_\alpha \). It follows immediately that if \( A, B \in \mathbb{M}_n \) satisfy \( W(A), W(B) \subseteq S_\alpha \) for some \( \alpha \in [0, \frac{\pi}{2}) \), then \( W(A + B) \subseteq W(A) + W(B) \subseteq S_\alpha \). A direct proof of this using the Cartesian decomposition goes as follows: let \( A = R_1 + iS_1, B = R_2 + iS_2 \). Since \( W(A) \) and \( W(B) \) are contained in \( S_\alpha \), we have \( R_1 + R_2 > 0 \). Note that for \( a, b, c, d > 0, (a + b)/(c + d) \leq \max\{a/c, b/d\} \). We compute, for any nonzero \( x \in \mathbb{C}^n \),

\[
\frac{|x^*(S_1 + S_2)x|}{x^*(R_1 + R_2)x} \leq \frac{|x^*S_1x| + |x^*S_2x|}{x^*(R_1 + R_2)x} \leq \frac{x^*|S_1|x + x^*|S_2|x}{x^*R_1x + x^*R_2x} \leq \max\left\{ \frac{x^*|S_1|x}{x^*R_1x}, \frac{x^*|S_2|x}{x^*R_2x} \right\} \leq \tan \alpha.
\]

This says \( |x^*(A + B)x| \leq x^*\Im(A + B)x \tan \alpha \). Thus, \( W(A + B) \subseteq S_\alpha \).

We note here that, by means of numerical range sectors, Li, Rodman and Spitkovsky studied fractional roots (powers) of elements in Banach algebras.[10]

In Section 2, we provide a detailed analysis of the so-called sectoral decomposition and show some important properties of it. In Section 3, we use the decomposition and majorization as a tool to obtain some norm inequalities; a question raised in [6] is answered.
2. A matrix decomposition with a sector

We begin with discussions on a matrix decomposition which we refer to as the sectoral decomposition. The existence of the matrix decomposition with numerical range contained in a sector has appeared in [1, Lemma 2.1]. A similar observation was made by London three decades ago (or even earlier by A. Ostrowski and O. Taussky) to prove a number of existing results by the factorization. This decomposition theorem, though simple as it looks, has been heavily used in recent papers.[1–5] In light of its importance and for completeness and convenience, we restate it here; we then show the uniqueness and give a way of constructing the key matrix Z in the decomposition.

Theorem 2.1 (Sectoral decomposition) Let A be an n × n complex matrix such that W(A) ⊆ Sα for some α ∈ [0, 2π]. Then there exist an invertible matrix X and a unitary and diagonal matrix Z = diag(eiθ1, . . . , eiθn) with all |θj| ≤ α such that A = XZX*. Moreover, such a matrix Z is unique up to permutation.

Proof Existence. Write A = M + Ni, where M = ℜA and N = ℑA are Hermitian. Since W(A) ⊆ Sα, A is invertible and M is positive definite. By [8, Theorem 7.6.4] or [9, Theorem 7.6], M and N are simultaneously *-congruent and diagonalizable, that is, P*MP and P*NP are diagonal for some invertible matrix P. It follows that we can write A = QDQ* for some diagonal invertible matrix D and invertible matrix Q. Since W(A) ⊆ Sα, we have W(D) ⊆ Sα. Thus we can write D = diag(d1eiθ1, . . . , dnieiθn), where d j > 0 and |θj| ≤ α, j = 1, . . . , n. Set X = Q diag(√d1, . . . , √dn) and Z = diag(eiθ1, . . . , eiθn). Then A = XZX*.

Uniqueness. Suppose that A = XZ1X* = YZ2Y* are two decompositions of A, where X and Y are nonsingular, Z1 and Z2 are unitary and diagonal. We may assume Y = I (otherwise replace X with Y−1X). We show that Z1 and Z2 have the same main diagonal entries (regardless of order). For this, we show that β ∈ C is a diagonal entry of Z1 with multiplicity k if and only if β is a diagonal entry of Z2 with the same multiplicity. First, consider the case β = 1. Let Z1 = C1 + iS1 and Z2 = C2 + iS2 be the Cartesian decompositions of Z1 and Z2, respectively. Then XC1X* = C2 and XS1X* = S2. Since β = 1 is a diagonal entry of Z1 with multiplicity k, 1 appears on the diagonal of C1 k times, so S1 has k zeros on its diagonal. Thus rank(XS1X*) = n − k. As XS1X* = S2, we have rank(S2) = n − k. This implies that C2, thus Z2, contains k 1’s on its diagonal. If β ̸= 1, multiplying by β we have X(βZ1)X* = βZ2. Repeating the above argument with βZ1 = C1 + iS1 and βZ2 = C2 + iS2, we see that βZ1 and βZ2 each have k 1’s on their diagonals; so each Z1 and Z2 has k β’s on the diagonal. We conclude that Z2 is similar to Z1 through permutation, i.e. Z2 = PZ1P′ for a permutation matrix P, where P′ denotes the transpose of P.

Note that in the above proof, the uniqueness of the Cartesian decomposition of A results in the uniqueness of Z. This may be singled out as an independent result.

Corollary 2.2 If G is a nonsingular matrix such that GSG* = T, where S and T are both unitary and diagonal, then S is similar to T through permutation.
Note that $G$ in Corollary 2.2 and $X$ in Theorem 2.1 are not unique in general. At this point, the best we can say about $G$ is $GSG^*T^* = I$, the identity matrix. Observe that $\cos \alpha$ is decreasing in $\alpha$ on $[0, \frac{\pi}{2})$, the following are immediate.

**Corollary 2.3** Let $A$ be an $n \times n$ complex matrix such that $W(A) \subseteq S_\alpha$ for some $\alpha \in [0, \frac{\pi}{2})$ and let $A = ZXZ^*$ be a sectorial decomposition of $A$, where $X$ is invertible and $Z$ is unitary and diagonal. Then

- (i) $I \leq \sec \alpha \left(\Re (Z)\right)$.
- (ii) $RR^* \leq \sec \alpha (R(\Re (Z)^*)R^*) = \sec \alpha (R(\Re (Z)R^*)$ for any matrix $R$.
- (iii) $\sigma_j^2(R) \leq \sec \alpha \lambda_j (R(\Re (Z)^*)R^*) \leq \sec \alpha \sigma_j (R(\Re (Z)R^*)$ for any $R$ and $j$.
- (iv) $\sigma_j^2(X) \leq \sec \alpha \lambda_j (\Re (A)) \leq \sec \alpha \sigma_j (A)$ for all $j = 1, \ldots, n$.

The following result gives a way of constructing the unique matrix $Z$.

**Theorem 2.4** Let $A$ be an $n \times n$ complex matrix with the Cartesian decomposition $A = M + Ni$, where $M$ is positive definite and $N$ is Hermitian. Then the matrix $Z$ in Theorem 2.1 is determined by the eigenvalues of $M^{-1}N$. Let $\mu_j$ be the eigenvalues of $I + iM^{-1}N$ and let $\mu_j = |\mu_j| e^{i \gamma_j}, |\gamma_j| < \frac{\pi}{2}$, $j = 1, \ldots, n$. Then $Z = \text{diag}(e^{i \gamma_1}, \ldots, e^{i \gamma_n})$. Let $\gamma(A) = \max_j |\gamma_j|$, we see that $W(Z), W(I + Di)$ and $W(A)$ are all contained in $S_{\gamma(A)}$.

**Proof** Since $M > 0$ and $N$ is Hermitian, there is an invertible matrix $P$ such that $P^*MP = I$ and $P^*NP = D$ is diagonal (see, e.g. [9, p.213]). Recall that when $X$ and $Y$ are both $n \times n$ matrices, $XY$ and $YX$ have the same eigenvalues. We have $\lambda_j (P^*NP) = \lambda_j (PP^*N) = \lambda_j (M^{-1}N)$. It follows that $P^*AP = I + Di$ and $D$ is the diagonal matrix of the eigenvalues of $M^{-1}N$. Let $\mu_j$, $j = 1, \ldots, n$, be the eigenvalues of $I + Di$, that is, the eigenvalues of $I + iM^{-1}N$. Let $\mu_j = |\mu_j| e^{i \gamma_j}, |\gamma_j| < \frac{\pi}{2}$, $j = 1, \ldots, n$. Then $Z = \text{diag}(e^{i \gamma_1}, \ldots, e^{i \gamma_n})$. With $\gamma(A) = \max_j |\gamma_j|$, we see that $W(Z), W(I + Di)$ and $W(A)$ are all contained in $S_{\gamma(A)}$.

**Corollary 2.5** Let $A$ be an $n \times n$ complex matrix such that $W(A) \subseteq S_\alpha$ for some $\alpha \in [0, \frac{\pi}{2})$. Then there exist a normal matrix $\Lambda$ such that $A = (\Re (A))^{1/2} \Lambda (\Im (A))^{1/2}$ and the eigenvalues of $\Lambda$ all have real part 1 (i.e. in the form $1 + ir$, where $r \in \mathbb{R}$). Moreover, $\|\Lambda\|_2 \leq \sec \alpha$ for the spectral norm $\| \cdot \|_2$ on $M_n$.

**Proof** Let $A = M + Ni$ with $M = \Re A$ and $N = \Im A$. Take $\Lambda = M^{-1/2}AM^{-1/2} = I + M^{-1/2}NM^{-1/2}i$. Then $W(\Lambda) \subseteq S_\alpha$. One may check that $\Lambda$ is normal. For any unit vector $z$, $z^* \Lambda z$ is a point in the $xy$-plane with $x$-coordinate $x = 1$. It follows that the numerical radius of $\Lambda$, i.e. $w(\Lambda) = \max \{|z^* \Lambda z| \mid z \in \mathbb{C}^n, z^* z = 1\}$, is no more than $\sec \alpha$ (as the hypotenuse of the right triangle with the adjacent leg of length 1). Since $\Lambda$ is normal, all the singular values of $\Lambda$ are no more than $\sec \alpha$. In particular, for the spectral norm $\|\Lambda\|_2$, we have $\|\Lambda\|_2 \leq \sec \alpha$.

Let $\gamma_a$ and $\gamma_b$ be respectively the largest and smallest values of the $\gamma_j$’s in Theorem 2.4. For the $Z$ in the decomposition, $W(Z)$ is the polygonal region of the diagonal entries of $Z$ located on the unit circle from $e^{i \gamma_a}$ to $e^{i \gamma_b}$. For the $\Lambda$ in Corollary 2.5, $W(\Lambda)$ is the
vertical line segment from the point \(1 + i \tan \gamma_a\) to the point \(1 + i \tan \gamma_b\). All these figures are contained in \(S_{\gamma_c}\), where \(\gamma_c = \max(\gamma_a, |\gamma_b|)\), which is nothing but the \(\gamma(A)\) in Theorem 2.4. In practice, we find \(\Lambda_1\) first then \(Z\).

**Example 1** Let \(A \in \mathbb{M}_3\) have the Cartesian decomposition \(A = M + Ni\), where

\[
A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}, \quad N = i \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.
\]

Upon computation, the eigenvalues of \(M^{-1}N\) are \(0, \pm \sqrt{2}\). It follows that the eigenvalues of \(\Lambda = I + M^{-1/2}NM^{-1/2}i\) are \(1, 1 \pm \sqrt{2}i = \sqrt{3}e^{\pm i\rho}\), where \(\rho = \arctan \sqrt{2}\). Thus \(Z = \text{diag}(1, e^{i\rho}, e^{-i\rho})\). \(W(A)\) is the vertical line segment from \(1 + \sqrt{2}i\) to \(1 - \sqrt{2}i\); \(W(Z)\) is the triangle (with interior) with vertices \(1 + 0i, e^{i\rho}, e^{-i\rho}\); while \(W(A)\) is an oval disc (see [12, p.140]) contained in \(S_\rho = \{re^{i\theta} | r > 0, |\theta| \leq \rho\}\).

Given a matrix \(A\), if the numerical range \(W(A)\) is contained in a half-plane, then we can rotate the numerical range so that it is relocated in a sector \(S_\alpha\) for some \(\alpha \in [0, \frac{\pi}{2})\). What would be the best possible (smallest) value of such an \(\alpha\)? Suppose that \(W(A)\) is contained in a region between two half-lines starting from the origin (or a wedge) and let \(\delta\) be the angle between the two half-lines, \(0 \leq \delta < \pi\). Then \(W(e^{i\theta}A) \subseteq S_{\delta/2}\) for some \(\theta \in \mathbb{R}\). Such a rotation has no impact on certain quantities of the matrix such as norm for \(\|e^{i\theta}A\| = \|A\|\). This observation suggests that some matrix problems (of stable matrices, say) may be studied through the matrices whose numerical ranges are contained in the right half-plane.

### 3. Norm inequalities for partitioned matrices

Recall that a norm \(\| \cdot \|\) on \(\mathbb{M}_n\) is unitarily invariant if \(\|UAV\| = \|A\|\) for any \(A \in \mathbb{M}_n\) and all unitary \(U, V \in \mathbb{M}_n\). The unitarily invariant norms of matrices are determined by nonzero singular values of the matrices via symmetric gauge functions (see, e.g. [9, Theorems 10.37 and 10.38]). If \(B\) is a submatrix of \(A \in \mathbb{M}_n\), then \(\|B\|\) is understood...
as the norm of the $n \times n$ augmented matrix with $B$ in the upper left corner and 0’s elsewhere, and conventionally $B$ has $n$ singular values with the trailing ones 0, that is, $\sigma(B) = (\sigma_1(B), \ldots, \sigma_r(B), 0, \ldots, 0) \in \mathbb{R}^n$, where $r$ is the rank of $B$. Thus $\sigma(A)$ and $\sigma(B)$ are both in $\mathbb{R}^n$.

Let $A$ be an $n$-square complex matrix partitioned in the form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where $A_{11}$ and $A_{22}$ are square. (2)

In [6], the following norm inequalities are proved (in Hilbert space).

(A) [6, Theorem 3.3]: Let $A \in M_n$ be accretive–dissipative and partitioned as in (2). Then for any unitarily invariant norm $\| \cdot \|$ on $M_n$,

$$\max\{\|A_{12}\|^2, \|A_{21}\|^2\} \leq 4\|A_{11}\|\|A_{22}\|. \quad (3)$$

(B) [6, Theorem 3.11]: Let $A \in M_n$ be accretive–dissipative and partitioned as in (2). Then for any unitarily invariant norm $\| \cdot \|$ on $M_n$,

$$\|A\| \leq \sqrt{2}(\|A_{11}\| + \|A_{22}\|). \quad (4)$$

It is asked in [6] as an open problem whether the factor 4 in (3) and the factor $\sqrt{2}$ in (4) can be improved. Indeed, the factor $\sqrt{2}$ in (4) is optimal. To construct such an accretive–dissipative matrix, we can first find a matrix whose numerical range is contained in the sector $S_{\pi/4}$, then rotate it by $+\pi/4$.

**Example 2** The normal matrix $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} i = \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix}$ has eigenvalues $1+i$ and $1-i$. So the matrix $A = e^{i\pi/4}B$ is accretive–dissipative. $A$ and $B$ have the same repeated singular value $\sqrt{2}$. Thus, for the trace norm (sum of all singular values),

$$2\sqrt{2} = \|A\| = \sqrt{2}(\|A_{11}\| + \|A_{22}\|) = \sqrt{2}(1+1).$$

However, the factor 4 in (3) can be improved to 2 (see Corollary 3.3). In this section, we extend (3) and (4) to some more general results.

Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. We denote $x \circ y = (x_1y_1, \ldots, x_ny_n)$ and write $x \leq y$ to mean $x_j \leq y_j$ for $j = 1, \ldots, n$. We rearrange the components of $x$ and $y$ in nonincreasing order: $x^*_1 \geq \cdots \geq x^*_n; \ y^*_1 \geq \cdots \geq y^*_n$. If $\sum_{i=1}^{k} x_i^* \leq \sum_{i=1}^{k} y_i^*$, $k = 1, \ldots, n$, we say that $x$ is weakly majorized by $y$, denoted by $x \prec_w y$. If, in addition, the last inequality is an equality, i.e. $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, we say that $x$ is majorized by $y$, written as $x \prec y$ (see, e.g. [13, p.12] or [9, p.326]).

It is well known (see, e.g. [13, p.368] or [9, p.375]) that for $A, B \in M_n$, $\|A\| \leq \|B\|$ for all unitarily invariant norms $\| \cdot \|$ on $M_n$ if and only if $\sigma(A) \prec_w \sigma(B)$. So, to some extent, the norm inequalities are essentially the same as the singular value majorization inequalities. As is known (see, e.g. [7, p.74]), $\sigma(\Re(A)) \prec_w \sigma(A)$, equivalently $\|\Re(A)\| \leq \|A\|$ for any $A \in M_n$ and any unitarily invariant norm $\| \cdot \|$ on $M_n$. The following is a reversal. Two useful facts are: the singular value majorization of product $\sigma(AB) \prec_w \sigma(A) \circ \sigma(B)$
(see, e.g. [9, p.363]) and its companion norm inequality \( \| AB \|_2^2 \leq \| AA^* \| \| B^* B \| \) (see, e.g. [14, p.212]).

**Lemma 3.1** Let \( A \in \mathbb{M}_n \) have \( W(A) \subseteq S_\alpha \) for some \( \alpha \in [0, \frac{\pi}{2}) \). Then

\[
\sigma(A) \prec_w \sec \alpha \lambda(\Re A).
\]

Equivalently, for all unitarily invariant norms \( \| \cdot \| \) on \( \mathbb{M}_n \),

\[
\| A \| \leq \sec \alpha \| \Re A \|.
\]

**Proof** Let \( A = XZX^* \) be a sectoral decomposition of \( A \), where \( X \) is invertible and \( Z \) is unitary and diagonal. Then

\[
\sigma(A) = \sigma(XZX^*) \prec_w \sigma(X) \circ \sigma(Z) \circ \sigma(X^*) = \sigma^2(X) \leq \sec \alpha \lambda(\Re A).
\]

The last inequality is by Corollary 2.3 (iv). The norm inequality (6) follows at once. \( \square \)

We point out that in [4] Drury and Lin presented a set of related inequalities:

\[
\sigma_j(A) \leq \sec^2 \alpha \lambda_j(\Re A).
\]

The inequality (7) gives a componentwise comparison of \( \sigma(A) \) and \( \lambda(\Re A) \), while (5) reveals a majorization. Note that the coefficient \( \sec^2 \alpha \) in (7) is bigger than \( \sec \alpha \) in (5) in general. For \( j = 1 \), (7) implies \( \sigma_1(A) \leq \sec^2 \alpha \lambda_1(\Re A) \), whereas (5) yields the stronger inequality \( \sigma_1(A) \leq \sec \alpha \lambda_1(\Re A) \). However, unlike (7), (5) gives no comparison of \( \sigma_j(A) \) and \( \lambda_j(\Re A) \) for \( j > 1 \).

**Theorem 3.2** Let \( A \in \mathbb{M}_n \) be partitioned as in (2) and assume \( W(A) \subseteq S_\alpha \) for some \( \alpha \in [0, \frac{\pi}{2}) \). Then for any unitarily invariant norm \( \| \cdot \| \) on \( \mathbb{M}_n \),

\[
\max \{ \| A_{12} \|^2, \| A_{21} \|^2 \} \leq \sec^2 \alpha \| A_{11} \| \| A_{22} \|.
\]

**Proof** Let \( A_{11} \) be \( p \times p \). By Theorem 2.1, we may assume that \( A = XZX^* \) is a sectoral decomposition of \( A \), where \( X \) is invertible and \( Z \) is unitary and diagonal. We partition \( X \) as \( X = \begin{bmatrix} X_1 & X_2 \end{bmatrix} \), \( X_1 \in \mathbb{M}_{p \times n} \). Then \( \Re A_{11} = X_1(\Re Z)X_1^*, \Re A_{22} = X_2(\Re Z)X_2^* \), and \( A_{12} = X_1ZX_2^* \). Using Corollary 2.3 (ii), we have

\[
\| A_{12} \|^2 = \| X_1ZX_2^* \|^2 \leq \| X_1X_1^* \| \| X_2Z^*ZX_2^* \|
\leq \sec^2 \alpha \| X_1(\Re Z)X_1^* \| \| X_2(\Re Z)X_2^* \|
= \sec^2 \alpha \| \Re A_{11} \| \| \Re A_{22} \|
\leq \sec^2 \alpha \| A_{11} \| \| A_{22} \|.
\]

So the inequality (8) is true for \( A_{12} \). The inequality for \( A_{21} \) is similarly proven. \( \square \)

If \( A \) is a positive definite matrix, then \( \alpha = 0 \) and \( \sec \alpha = 1 \) in (8).
Let $A \in \mathbb{M}_n$ be accretive–dissipative and partitioned as in (2). Then for any unitarily invariant norm $\| \cdot \|$ on $\mathbb{M}_n$,

$$\max\{\|A_{12}\|^2, \|A_{21}\|^2\} \leq 2 \|A_{11}\| \|A_{22}\|. \quad (9)$$

**Proof** Set $\alpha = \pi/4$ in Theorem 3.2. Then sec$^2 \alpha = 2$. □

The inequality (9) is stronger than (3). Moreover, the constant factor 2 is best possible for all accretive–dissipative matrices and unitarily invariant norms.

**Example 3** Let $B = \begin{bmatrix} 1 & 1-i \\ 0 & 1 \end{bmatrix}$. One checks that $\Re B > 0$ and $\Re B \geq \pm \Im B$, which yield $x^*(\Re B)x \geq |x^*(\Im B)x|$ for all $x \in \mathbb{C}^2$. (Note that $\Re B \not\geq |\Im B|$.) So $W(B) \subseteq S_{\pi/4}$ and $A = e^{i\pi/4}B$ is accretive–dissipative. For the trace norm, apparently, $\|A_{12}\|^2 = 2 = 2(1 \cdot 1) = 2\|A_{11}\| \|A_{22}\|$. This answers a question raised in [6, p. 442].

To present next theorem, we need a lemma which is interesting in its own right.

**Lemma 3.4** Let $H = \begin{bmatrix} H_{11} & * \\ * & H_{22} \end{bmatrix}$ be an $n \times n$ positive semidefinite matrix, where $H_{11}$ and $H_{22}$ are square submatrices (possibly of different sizes) and $*$ stands for irrelevant entries. With the eigenvalues arranged in nonincreasing order, we have

$$\lambda(H) \prec \lambda(H_{11}) + \lambda(H_{22}). \quad (10)$$

Consequently, for all unitarily invariant norms $\| \cdot \|$ on $\mathbb{M}_n$,

$$\|H\| \leq \|H_{11}\| + \|H_{22}\|. \quad (11)$$

**Proof** Note that a matrix $P$ is positive semidefinite if and only if $P = Q^*Q$ for some matrix $Q$. Let $H = \begin{bmatrix} S & S^* \\ T^* & * \end{bmatrix}$ with $H_{11} = S^*S$ and $H_{22} = T^*T$. Using the fact that matrices $XY$ and $YX$ have the same nonzero eigenvalues for any $(p \times q)$ matrix $X$ and any $(q \times p)$ matrix $Y$, we arrive at

$$\lambda(H) = \lambda\left(\begin{bmatrix} S^* \\ T^* \end{bmatrix} [S, T] \begin{bmatrix} S^* \\ T^* \end{bmatrix}\right) = \lambda\left(\begin{bmatrix} S^* \\ T^* \end{bmatrix} [S, T] \begin{bmatrix} S^* \\ T^* \end{bmatrix}\right)$$

$$= \lambda(\langle S\rangle S^* + T^*T^*) < \lambda(\langle S\rangle S^*) + \lambda(T^*T^*)$$

$$= \lambda(H_{11}) + \lambda(H_{22}).$$

Here we regard $\lambda(H_{11})$ and $\lambda(H_{22})$ as vectors in $\mathbb{R}^n$ (by adding 0’s if necessary) with components arranged in nonincreasing order. □

**Remark 1** It is known [13, p. 308] that if $H = \begin{bmatrix} H_{11} & * \\ * & H_{22} \end{bmatrix}$ is Hermitian, then

$$\langle \lambda(H_{11}), \lambda(H_{22}) \rangle = \lambda(H_{11} \oplus H_{22}) < \lambda(H).$$

It is also known (see [15] or [16]) that if $H = \begin{bmatrix} H_{11} & K \\ K^* & H_{22} \end{bmatrix}$ is positive semidefinite, where $K$ is Hermitian or skew-Hermitian, then

$$\lambda(H) < \lambda(H_{11} + H_{22}).$$
We must also point out that (11) has appeared in [14, p.217] and a more general result is available in [17, Theorem 2.1]. We include our proof here as it is short and elementary; and it is the most elegant one in author’s opinion.

**Theorem 3.5** Let \( A \in \mathbb{M}_n \) be partitioned as in (2) and let \( W(A) \subseteq S_\alpha \) for some \( \alpha \in [0, \frac{\pi}{2}) \). Then for any unitarily invariant norm \( \| \cdot \| \) on \( \mathbb{M}_n \),

\[
\| A \| \leq \sec \alpha (\| A_{11} \| + \| A_{22} \|).
\]  

(12)

**Proof** By Lemma 3.1 and noticing that \( \Re A = \begin{bmatrix} \Re A_{11} & * \\ * & \Re A_{22} \end{bmatrix} > 0 \), we have

\[
\| A \| \leq \sec \alpha \| \Re A \| \leq \sec \alpha (\| \Re A_{11} \| + \| \Re A_{22} \|).
\]

The desired inequality follows at once since \( \| \Re X \| \leq \| X \| \) for any \( X \). \( \square \)

If \( A \) is positive definite, then \( \alpha = 0 \) and Theorem 3.5 reduces to (11). If \( A \) is accretive–dissipative, then (4) is immediate by setting \( \alpha = \pi/4 \) in (12).

**Acknowledgements**

The author is thankful to S.W. Drury and M. Lin for reference [4] which initiated this work; he is also indebted to M. Lin for his valuable input and discussions.

**References**

[1] Drury SW. Fischer determinantal inequalities and Higham’s Conjecture. Linear Algebra Appl. 2013;439:3129–3133.
[2] Drury SW. A Fischer type determinantal inequality. Linear Multilinear Algebra. 2013; http://dx.doi.org/10.1080/03081087.2013.832244.
[3] Drury SW. Principal powers of matrices with positive definite real part. Forthcoming.
[4] Drury SW, Lin M. Singular value inequalities for matrices with numerical ranges in a sector. Forthcoming; http://dx.doi.org/10.1080/03081087.2013.865732.
[5] Li CK, Sze N. Determinantal and eigenvalue inequalities for matrices with numerical ranges in a sector. J. Math. Anal. Appl. 2014;410:487–491.
[6] Lin M, Zhou D. Norm inequalities for accretive-dissipative operator matrices. J. Math. Anal. Appl. 2013;407:487–491.
[7] Bhatia R. Matrix analysis. GTM 169. New York (NY): Springer-Verlag; 1997.
[8] Horn RA, Johnson CR. Matrix analysis. 2nd ed. New York (NY): Cambridge University Press; 2013.
[9] Zhang F. Matrix theory: basic results and techniques. 2nd ed. New York (NY): Springer; 2011.
[10] Li CK, Rodman L, Spitkovsky I. On numerical ranges and roots. J. Math. Anal. Appl. 2003;282:329–340.
[11] London D. A note on matrices with positive definite real part. Proc. Am. Math. Soc. 1981;82:322–324.
[12] Gustafson KE, Rao DKM. Numerical range. New York (NY): Springer; 1997.
[13] Marshall AW, Olkin I, Arnold B. Inequalities: theory of majorization and its applications. 2nd ed. New York (NY): Springer; 2011.
[14] Horn RA, Johnson CR. Topics in matrix analysis. New York (NY): Cambridge University Press; 1991.
[15] Turkmen R, Paksoy V, Zhang F. Some inequalities of majorization type. Linear Algebra Appl. 2012;437:1305–1316.

[16] Lin M, Wolkowicz H. An eigenvalue majorization inequality for positive semidefinite block matrices. Linear Multilinear Algebra. 2012;60:1365–1368.

[17] Lee EY. Extension of Rotfel’d theorem. Linear Algebra Appl. 2011;435:735–741.