CONVERGENCE OF A VECTOR-BGK APPROXIMATION FOR THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

ROBERTA BIANCHINI
Ecole Normale Supérieure de Lyon
UMPA, ENS-Lyon
46, allée d’Italie, 69364-Lyon Cedex 07, France

ROBERTO NATALINI*
Istituto per le Applicazioni del Calcolo “Mauro Picone”
Consiglio Nazionale delle Ricerche
via dei Taurini 19, I-00185 Rome, Italy

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Abstract. We present a rigorous convergence result for smooth solutions to a singular semilinear hyperbolic approximation, called vector-BGK model, to the solutions to the incompressible Navier-Stokes equations in Sobolev spaces. Our proof deeply relies on the dissipative properties of the system and on the use of an energy which is provided by a symmetrizer, whose entries are weighted in a suitable way with respect to the singular perturbation parameter. This strategy allows us to perform uniform energy estimates and to prove the convergence by compactness.

1. Introduction. We want to study the convergence of a singular perturbation approximation to the Cauchy problem for the incompressible Navier-Stokes equations on the $d$ dimensional torus $\mathbb{T}^d$:

\begin{align*}
\partial_t u^{NS} + \nabla \cdot (u^{NS} \otimes u^{NS}) + \nabla P^{NS} &= \nu \Delta u^{NS}, \\
\nabla \cdot u^{NS} &= 0,
\end{align*}

(1)

with $(t,x) \in [0, +\infty) \times \mathbb{T}^d$, and initial data

\begin{align*}
u^{NS}(0, x) &= u_0(x), \\
\nabla \cdot u_0 &= 0.
\end{align*}

Here $u^{NS}$ and $\nabla P^{NS}$ are respectively the velocity field and the gradient of the pressure term, and $\nu > 0$ is the viscosity coefficient.

We consider a semilinear hyperbolic approximation, called vector-BGK model, [13, 10], to the incompressible Navier-Stokes equations (1). The general form of this approximation is as follows:

\begin{align*}
\partial_t f_l^\varepsilon + \frac{\lambda_l}{\varepsilon} \cdot \nabla_x f_l^\varepsilon &= \frac{1}{\tau_0^2} (M_l(\rho^\varepsilon, \varepsilon \rho^\varepsilon u^\varepsilon) - f_l^\varepsilon),
\end{align*}

(3)
with initial data
\[ f_1^l(0, x) = N_l^\varepsilon (\rho, \varepsilon \rho u_0) = \tilde{M}_l^\varepsilon (\tilde{\rho}, \varepsilon \tilde{\rho} u_0) + \varepsilon g(\nabla u_0), \quad (4) \]
for \( l = 1, \cdots, L \), where the number of discrete velocities \( L \geq d + 1 \), and \( u_0 \) is the initial condition in (2). In order to give a more precise description of the model, we list here the main points:

- \( f_1^l(t, x), M_1^l(\rho, \varepsilon \rho u) \) take values in \( \mathbb{R}^{d+1} \);
- the Maxwellians \( M_1^l \) are Lipschitz continuous functions;
- \( \lambda_l = (\lambda_{l1}, \cdots, \lambda_{ld}) \) are vectors of constant velocities;
- \( \bar{\rho} > 0 \) is a strictly positive constant value;
- \( \varepsilon \) and \( \tau \) are strictly positive parameters.

In (4), \( M_1^l(\rho, \varepsilon \rho u) \) are the reference solutions, i.e. suitable perturbations of the Maxwellian functions, whose explicit expressions will be provided later on. In particular, in (4) \( g(\nabla u_0) \) is the first order correction of the Maxwellians in the Chapman-Enskog expansion of our singular perturbation system, see [29, 16] and references therein. Denoting by \( f_1^l, M_1^j \) for \( j = 0, \cdots, d \), the \( d+1 \) components of the vectors \( f_1^l, M_1^l \) for \( l = 1, \cdots, L \), let us set

\[ \rho^\varepsilon = \sum_{l=1}^{L} f_1^l(0, x) \quad \text{and} \quad q_j^\varepsilon = \varepsilon \rho^\varepsilon u_j^\varepsilon = \sum_{l=1}^{L} f_1^j(0, x), \quad (5) \]

where \( \rho^\varepsilon, q_j^\varepsilon \) are the macroscopic variables approximating the Navier-Stokes equations. In [13, 10], the convergence of the solutions to the vector-BGK model introduced above to the solutions to the incompressible Navier-Stokes equations is studied numerically. More precisely, assuming that, in a suitable functional space, \( \rho^\varepsilon \to \bar{\rho}, \quad u^\varepsilon \to \bar{u}, \quad \text{and} \quad \rho^\varepsilon - \bar{\rho} \to \bar{P}, \)

under some consistency conditions of the BGK approximation with respect to the Navier-Stokes equations, see [13], it can be shown that the couple \((\bar{u}, \bar{P})\) is a solution to the incompressible Navier-Stokes equations. The aim of the present paper is to provide a rigorous proof of this convergence in the Sobolev spaces.

Vector-BGK models come from the ideas of kinetic approximations for compressible flows. They are inspired by the hydrodynamic limits of the Boltzmann equation: see [3, 4, 14] for the limit to the compressible Euler equations, and see [15, 17] for the incompressible Navier-Stokes equations. In this regard, one of the main directions has been the approximation of hyperbolic systems with discrete velocities BGK models, as in [11, 21, 26, 8, 28]. Similar results have been obtained for convection-diffusion systems under the diffusive scaling [24, 9, 23, 2]. In the framework of the BGK approximations, one of the first important contributions was given in computational physics by the so called Lattice-Boltzmann methods, see for instance [30, 31]. Under some assumptions on the physical parameters, LBMs approximate the incompressible Navier-Stokes equations by scalar velocities models of kinetic equations, and a rigorous mathematical result on the validity of these kinds of approximations was proved in [22]. Other partially hyperbolic approximations of the Navier-Stokes equations were developed in [12, 27, 19, 18].

The vector-BGK systems studied in the present paper are a combination of the ideas of discrete velocities BGK approximations and LBMs. They are called **vector-BGK models** since, unlike the LBMs [30, 31], they associate every discrete velocity
with one vector of unknowns. Another fruitful property of vector-BGK approximations is their natural compatibility with a mathematical entropy, [8], which provides a nice analytical structure and stability properties. The present paper takes its roots in previous works [13, 10], where vector-BGK models for the incompressible Navier-Stokes equations were introduced. Here we prove a rigorous local in time convergence result for the smooth solutions to the vector-BGK system to the smooth solutions to the Navier-Stokes equations. In this paper, we focus on the two-dimensional case in space. Following [13], let us set $d = 2$, $L = 5$, and

$$w^\varepsilon = (\rho^\varepsilon, q^\varepsilon) = (\rho^\varepsilon, q^\varepsilon_1, q^\varepsilon_2) = (\rho^\varepsilon, \varepsilon \rho^\varepsilon u^\varepsilon_1, \varepsilon \rho^\varepsilon u^\varepsilon_2) = \sum_{l=1}^{5} f_l^\varepsilon \in \mathbb{R}^3. \quad (6)$$

Fix $\lambda, \tau > 0$ and let $\varepsilon > 0$ be a small parameter, which is going to zero in the singular perturbation limit. Thus, we get a five velocities model (15 scalar equations):

$$\begin{align*}
\partial_t f_1^\varepsilon + \frac{1}{\tau} \partial_y f_1^\varepsilon &= \frac{1}{\tau^2} (M_1(w^\varepsilon) - f_1^\varepsilon), \\
\partial_t f_2^\varepsilon + \frac{1}{\tau} \partial_y f_2^\varepsilon &= \frac{1}{\tau^2} (M_2(w^\varepsilon) - f_2^\varepsilon), \\
\partial_t f_3^\varepsilon - \frac{1}{\tau} \partial_y f_3^\varepsilon &= \frac{1}{\tau^2} (M_3(w^\varepsilon) - f_3^\varepsilon), \\
\partial_t f_4^\varepsilon - \frac{1}{\tau} \partial_y f_4^\varepsilon &= \frac{1}{\tau^2} (M_4(w^\varepsilon) - f_4^\varepsilon), \\
\partial_t f_5^\varepsilon &= \frac{1}{\tau^2} (M_5(w^\varepsilon) - f_5^\varepsilon).
\end{align*} \quad (7)$$

Here, the Maxwellian functions $M_j \in \mathbb{R}^3$ have the following expressions:

$$M_{1,3}(w^\varepsilon) = a w^\varepsilon \pm \frac{A_1(w^\varepsilon)}{2\lambda}, \quad M_{2,4}(w^\varepsilon) = a w^\varepsilon \pm \frac{A_2(w^\varepsilon)}{2\lambda}, \quad M_5(w^\varepsilon) = (1 - 4a)w^\varepsilon, \quad (8)$$

where

$$A_1(w^\varepsilon) = \left(\frac{(q^\varepsilon_1)^2 + P(\rho^\varepsilon)}{\rho^\varepsilon} \right), \quad A_2(w^\varepsilon) = \left(\frac{(q^\varepsilon_2)^2 + P(\rho^\varepsilon)}{\rho^\varepsilon} \right), \quad \rho^\varepsilon = \rho^\varepsilon - \bar{\rho}, \quad (9)$$

and

$$a = \frac{\nu}{2\lambda^2\tau}, \quad (10)$$

where $\nu$ is the viscosity coefficient in (1). In the following, our main goal is to obtain uniform energy estimates for the solutions to the vector-BGK model (7) in the Sobolev spaces and to get the convergence by compactness. We point out that in (10) our pressure is linear in order to simplify the computations below. However, the case of a general pressure term

$$P(\rho^\varepsilon) = \begin{cases} 
\frac{k}{\gamma - 1} [(\rho^\varepsilon)^\gamma - \bar{\rho}^\gamma], & \gamma > 1, \\
k [\rho^\varepsilon \log(\rho^\varepsilon) - \bar{\rho} \log(\bar{\rho})], & \gamma = 1,
\end{cases}$$

where $k$ is a positive constant value, can be handled in the same way, just by writing it as the sum of a linear part and a nonlinear one, the latter to be estimated together with all the other nonlinear terms.

In [13, 10], an $L^2$-stability estimate was obtained by using the entropy function associated with the vector-BGK model, whose existence is proved in [8], under the assumption of boundedness of the solutions to our approximating system. However, the kinetic entropy is obtained by inverting the Maxwellians, so generally its explicit dependency on the singular parameter is not known. Of course this implies that we do not know the weights, in terms of the diffusive parameter, of the
classical symmetrizer derived by the entropy, see [20] for the one dimensional case and [7, 22] for the general case. For this reason, the existence of an entropy is not enough to control the higher order estimates, besides that the kinetic entropy exists only under some assumptions. To deal with these problems, we use a constant right symmetrizer, which highlights the dissipative property of the linearized system. Precisely, the right symmetrization provides the starting setting to get the conservative-dissipative form introduced in [7]. Moreover, the entries of that symmetrizer are weighted in terms of the singular parameter in such a way that we are able to get uniform bounds on the nonlinear term, which is of course singular in the diffusive parameter.

We remark that in the following we adopt the terminology conservative-dissipative form for our symmetrized and dissipative system. Although it is not exactly the conservative-dissipative form introduced in [7], but the step immediately before, we choose the same name, being it reminiscent of the main idea under this approach: to use the dissipation of the linearized system, namely the one that we are somehow able to quantify in a proper way.

1.1. Plan of the paper. In Section 2 we introduce the vector-BGK approximation and the general setting of the problem. Section 3 is dedicated to the discussion on the symmetrizer and the conservative-dissipative form. In Section 4 we get uniform energy estimates to prove the convergence, in Section 5, of the solutions to the vector-BGK approximation to the solutions to the incompressible Navier-Stokes equations. Finally, Section 6 is devoted to our conclusions and perspectives.

2. General framework. Let us set

\[ U^\varepsilon = (f^\varepsilon_1, f^\varepsilon_2, f^\varepsilon_3, f^\varepsilon_4, f^\varepsilon_5) \in \mathbb{R}^{3 \times 5}, \]

and let us write the compact formulation of equations (7)-(4), which reads

\[ \partial_t U^\varepsilon + \Lambda_1 \partial_x U^\varepsilon + \Lambda_2 \partial_y U^\varepsilon = \frac{1}{\tau \varepsilon^2} (M(U^\varepsilon) - U^\varepsilon), \tag{12} \]

with initial data

\[ U_0^\varepsilon = f^\varepsilon_l(0, x) = M^\varepsilon_l(\bar{\rho}, \varepsilon \bar{\rho} u_0) = M^\varepsilon_l(\bar{\rho}, \varepsilon \bar{\rho} u_0) + \varepsilon g(\nabla u_0), \quad l = 1, \ldots, 5, \tag{13} \]

where \( M^\varepsilon_l \) are the perturbed Maxwellian functions, with \( M^\varepsilon_l \) the Maxwellians in (8), and

\[ g(\nabla u_0) = \begin{pmatrix} -a\lambda \tau \partial_x w_0 \\ -a\lambda \tau \partial_y w_0 \\ a\lambda \tau \partial_x w_0 \\ a\lambda \tau \partial_y w_0 \\ 0 \end{pmatrix}, \quad w_0 = (\bar{\rho}, \varepsilon \bar{\rho} u_0), \tag{14} \]

\[ \Lambda_1 = \begin{pmatrix} \frac{1}{\varepsilon} \text{Id} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\varepsilon} \text{Id} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\varepsilon} \text{Id} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\varepsilon} \text{Id} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \]

\( \text{Id} \) is the \( 3 \times 3 \) identity matrix, and

\[ M(U^\varepsilon) = (M^\varepsilon_1(w^\varepsilon), M^\varepsilon_2(w^\varepsilon), M^\varepsilon_3(w^\varepsilon), M^\varepsilon_4(w^\varepsilon), M^\varepsilon_5(w^\varepsilon)). \tag{15} \]
Assumption 1 (Dissipation condition). We assume the following condition:

\[ 0 < a < \frac{1}{4}, \]

where the parameter \( a \) is in (8).

We point out that the previous assumption is necessary in order to guarantee the positivity of the spectrum of the Jacobian matrices associated with the Maxwellians, namely the stability of the scheme, see [8]. Further adjustments on the parameters of the model will be discussed later on.

2.1. Conservative variables. We define the following change of variables:

\[ w^\varepsilon = \sum_{l=1}^{5} f^\varepsilon_l, \quad m^\varepsilon = \frac{2}{3}(f^\varepsilon_1 - f^\varepsilon_5), \quad \xi^\varepsilon = \frac{2}{3}(f^\varepsilon_2 - f^\varepsilon_4), \quad k^\varepsilon = f^\varepsilon_1 + f^\varepsilon_3, \quad h^\varepsilon = f^\varepsilon_2 + f^\varepsilon_4. \] (16)

This way, the vector-BGK model (7) reads:

\[
\begin{align*}
\partial_t w^\varepsilon + \partial_x m^\varepsilon + \partial_y \xi^\varepsilon &= 0; \\
\partial_t m^\varepsilon + \frac{\lambda^2}{\tau} \partial_x k^\varepsilon &= \frac{1}{\tau^\varepsilon}(A_1(w^\varepsilon) - m^\varepsilon), \\
\partial_t \xi^\varepsilon + \frac{\lambda^2}{\tau} \partial_y h^\varepsilon &= \frac{1}{\tau^\varepsilon}(A_2(w^\varepsilon) - \xi^\varepsilon), \\
\partial_t k^\varepsilon + \partial_x m^\varepsilon &= \frac{1}{\tau^\varepsilon}(2aw^\varepsilon - k^\varepsilon), \\
\partial_t h^\varepsilon + \partial_y \xi^\varepsilon &= \frac{1}{\tau^\varepsilon}(2aw^\varepsilon - h^\varepsilon).
\end{align*}
\] (17)

We make a slight modification of system (17). Set \( \bar{w} = (\bar{\rho}, 0, 0) \) and

\[ w^{\varepsilon^*} := w^\varepsilon - \bar{w} = (w^\varepsilon_1 - \bar{\rho}, w^\varepsilon_2, w^\varepsilon_3), \quad k^{\varepsilon^*} = k^\varepsilon - 2aw^\varepsilon, \quad h^{\varepsilon^*} = h^\varepsilon - 2aw^\varepsilon. \] (18)

In the following, we are going to work with the modified variables. System (17) reads:

\[
\begin{align*}
\partial_t w^{\varepsilon^*} + \partial_x m^{\varepsilon^*} + \partial_y \xi^{\varepsilon^*} &= 0; \\
\partial_t m^{\varepsilon^*} + \frac{\lambda^2}{\tau^\varepsilon} \partial_x k^{\varepsilon^*} &= \frac{1}{\tau^\varepsilon}(A_1(w^{\varepsilon^*} + \bar{w}) - m^{\varepsilon^*}), \\
\partial_t \xi^{\varepsilon^*} + \frac{\lambda^2}{\tau^\varepsilon} \partial_y h^{\varepsilon^*} &= \frac{1}{\tau^\varepsilon}(A_2(w^{\varepsilon^*} + \bar{w}) - \xi^{\varepsilon^*}), \\
\partial_t k^{\varepsilon^*} + \partial_x m^{\varepsilon^*} &= \frac{1}{\tau^\varepsilon}(2aw^{\varepsilon^*} - k^{\varepsilon^*}), \\
\partial_t h^{\varepsilon^*} + \partial_y \xi^{\varepsilon^*} &= \frac{1}{\tau^\varepsilon}(2aw^{\varepsilon^*} - h^{\varepsilon^*}).
\end{align*}
\] (19)

Notice from (9) that

\[ A_1(w^\varepsilon) = \left( \begin{array}{c} (q_1^2)^2 \rho^2 + \rho^\varepsilon - \bar{\rho} \\ q_1^2 q_3^2 \rho^\varepsilon \\ \end{array} \right) = \left( \begin{array}{c} w_{2^*}^{\varepsilon^*} \\ w_{1^*}^{\varepsilon^*} \end{array} \right), \]

and, similarly,

\[ A_2(w^{\varepsilon^*}) = \left( \begin{array}{c} q_3^2 \rho^2 \\ \rho^\varepsilon - \bar{\rho} \\ \end{array} \right) = \left( \begin{array}{c} w_{3^*}^{\varepsilon^*} \\ \frac{w_{3^*}^{\varepsilon^*}}{w_{1^*}^{\varepsilon^*} + \rho} \end{array} \right) = A_2(w^{\varepsilon^*} + \bar{w}). \]

Hereafter, we will omit the apexes \( \varepsilon^* \) for \( w^{\varepsilon^*}, k^{\varepsilon^*}, h^{\varepsilon^*} \), and the apex \( \varepsilon \) for \( m^\varepsilon, \xi^\varepsilon \), when there is no ambiguity.
Let us define the $15 \times 15$ matrix
\[
C = \begin{pmatrix}
  \Id & \Id & \Id & \Id & \Id \\
  \varepsilon \lambda \Id & 0 & -\varepsilon \lambda \Id & 0 & 0 \\
  0 & \varepsilon \lambda \Id & 0 & -\varepsilon \lambda \Id & 0 \\
  \varepsilon^2 \Id & 0 & \varepsilon^2 \Id & 0 & 0 \\
  0 & \varepsilon^2 \Id & 0 & \varepsilon^2 \Id & 0 
\end{pmatrix},
\]
(20)
and set
\[
W = (w, \varepsilon^2 m, \varepsilon^2 \xi, \varepsilon^2 k, \varepsilon^2 h) := CU - (\bar{w}, 0, 0, 0, 0).
\]
(21)
Thus, we can write the translated system (19) in the compact form
\[
\partial_t W + B_1 \partial_x W + B_2 \partial_y W = \frac{1}{\tau \varepsilon^2} (\tilde{M}(W) - W),
\]
(22)
with initial conditions
\[
W_0 = CU_0 - (\bar{w}, 0, 0, 0, 0),
\]
(23)
where $U_0$ is given by (13),
\[
B_1 = C \Lambda_1 C^{-1}, \quad B_2 = C \Lambda_2 C^{-1},
\]
\[
B_1 = \begin{pmatrix}
  0 & \frac{1}{\varepsilon} \Id & 0 & 0 & 0 \\
  0 & 0 & 0 & \frac{\lambda}{\varepsilon} \Id & 0 \\
  0 & 0 & 0 & 0 & 0 \\
  0 & \Id & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad B_2 = \begin{pmatrix}
  0 & 0 & \frac{1}{\varepsilon} \Id & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 \\
  0 & 0 & \Id & 0 & 0 \\
  0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]
(24)
and
\[
\tilde{M}(W) = CM(C^{-1}W) = CM(U).
\]
Here, setting
\[
\sigma_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \sigma_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},
\]
(25)
we have the following expression of the source term,
\[
\frac{1}{\tau \varepsilon^2} (\tilde{M}(W) - W) = \frac{1}{\tau} \begin{pmatrix}
  \frac{A_1 (w + \bar{w})}{\varepsilon} - \frac{\varepsilon^2 m}{\varepsilon^2 x^2} \\
  \frac{A_2 (w + \bar{w})}{\varepsilon} - \frac{\varepsilon^2 \xi}{\varepsilon^2} \\
  2aw - \frac{\varepsilon^2 k}{\varepsilon^2 x^2} \\
  2aw - \frac{\varepsilon^2 h}{\varepsilon^2}
\end{pmatrix} = \frac{1}{\tau} \begin{pmatrix}
  \frac{w_2}{\varepsilon_{1,\beta}^2} + \frac{w_1}{\varepsilon_{2,\gamma}^2} \\
  \frac{w_3}{\varepsilon_{3,\delta}^2} + \frac{w_1}{\varepsilon_{4,\gamma}^2} \\
  2aw - \frac{\varepsilon^2 k}{\varepsilon^2 x^2} \\
  2aw - \frac{\varepsilon^2 h}{\varepsilon^2}
\end{pmatrix},
\]
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\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
\frac{\varepsilon_1}{\varepsilon_2} & -\frac{1}{\varepsilon} \operatorname{Id} & 0 & 0 & 0 \\
2\lambda \operatorname{Id} & 0 & 0 & -\frac{1}{\varepsilon} \operatorname{Id} & 0 \\
2\lambda \operatorname{Id} & 0 & 0 & 0 & -\frac{1}{\varepsilon} \operatorname{Id}
\end{pmatrix}
\begin{pmatrix}
0 \\
\frac{w_2}{w_1 + \rho} \\
\frac{w_3}{w_1 + \rho} \\
0 \\
0
\end{pmatrix}
\begin{pmatrix}
0 \\
\frac{w_2}{w_1 + \rho} \\
\frac{w_3}{w_1 + \rho} \\
0 \\
0
\end{pmatrix}
\begin{pmatrix}
\frac{1}{\varepsilon} \\
\frac{w_2}{w_1 + \rho} \\
\frac{w_3}{w_1 + \rho} \\
0 \\
0
\end{pmatrix}
\begin{pmatrix}
0 \\
\frac{w_2}{w_1 + \rho} \\
\frac{w_3}{w_1 + \rho} \\
0 \\
0
\end{pmatrix} \varepsilon \sigma_1
\]  

\[= -LW + N(w + \bar{w}), \quad (26)\]

where \(-L\) is the linear part of the source term of (22), while \(N\) is the remaining nonlinear one. Thus, we can rewrite system (22) as follows:

\[
\partial_t W + B_1 \partial_x W + B_2 \partial_y W = -LW + N(w + \bar{w}). \quad (27)
\]

3. The right symmetrizer and the conservative-dissipative form. According to the theory of semilinear hyperbolic systems, see for instance [25, 5], we need a symmetric formulation of system (27) in order to get energy estimates. However, we are dealing with a singular perturbation system, so any symmetrizer for system (27) is not enough. In other words, we look for a symmetrizer which provides a suitable dissipative structure for system (27). In this context, notice that the first equation of system (27) reads

\[
\partial_t w + \partial_x m + \partial_y \xi = 0,
\]

i.e. the first term of the source vanishes, and \(w\) is a conservative variable. To take advantage of this conservative property, rather than a classical Friedrichs left symmetrizer, see again [25, 5], we look for a right symmetrizer for (27), which allows us to get the step just before the conservative-dissipative form introduced in [7], here called again conservative-dissipative form. More precisely, the right multiplication easily provides the conservative structure in [7], while the dissipation is obtained by adjusting the parameters of the model. Besides, the symmetrizer \(\Sigma\) presents constant \(\varepsilon\)-weighted entries and this allows us to control the nonlinear part \(N\) of the source term (26) of system (27). To be complete, we point out that the inverse matrix \(\Sigma^{-1}\) is a left symmetrizer for system (27), according to the definitions given in [25, 5]. However, the product \(-\Sigma^{-1}L\) is a full matrix, so the symmetrized version of system (27), obtained by the left multiplication by \(\Sigma^{-1}\), does not provide the conservative-dissipative form. This is the reason why it is convenient to look for a right symmetrizer from the beginning. Let us explicitly write the symmetrizer

\[
\Sigma = \begin{pmatrix}
Id & \varepsilon \sigma_1 & \varepsilon \sigma_2 & 2a \varepsilon^2 \operatorname{Id} & 2a \varepsilon^2 \operatorname{Id} \\
\varepsilon \sigma_1 & 2\lambda^2 a \varepsilon^2 \operatorname{Id} & 0 & \varepsilon^3 \sigma_1 & 0 \\
\varepsilon \sigma_2 & 0 & 2\lambda^2 a \varepsilon^2 \operatorname{Id} & 0 & \varepsilon^3 \sigma_2 \\
2a \varepsilon^2 \operatorname{Id} & \varepsilon^3 \sigma_1 & 0 & 2a \varepsilon^4 \operatorname{Id} & 0 \\
2a \varepsilon^2 \operatorname{Id} & \varepsilon^3 \sigma_2 & 0 & 2a \varepsilon^4 \operatorname{Id}
\end{pmatrix}.
\]  

(28)
where \( \sigma_1, \sigma_2 \) are in (25). It is easy to check that \( \Sigma \) is a constant right symmetrizer for system (27) since, taking \( B_1, B_2 \) and \( L \) in (24) and (26) respectively,

\[
B_1 \Sigma = \Sigma B_1^T, \quad B_2 \Sigma = \Sigma B_2^T,
\]

\[
-L \Sigma = \begin{pmatrix} 0 & 0 \\ 0 & -L \end{pmatrix}
\]

Now, we define the following change of variables:

\[
W, \tilde{\tau}
\]

with \( \Sigma \) in (21). System (27) reads:

\[
\Sigma \partial_t \tilde{W} + B_1 \Sigma \partial_x \tilde{W} + B_2 \Sigma \partial_y \tilde{W} = -L \tilde{W} \Sigma + N((\Sigma \tilde{W})_1 + \tilde{w}),
\]

where \((\Sigma \tilde{W})_1\) is the first component of the unknown vector \(\Sigma \tilde{W}\). We want to show that \(\Sigma\) in (28) is strictly positive definite. Thus,

\[
(\Sigma \tilde{W}, \tilde{W})_0 = ||\tilde{w}||^2_0 + 2\lambda^2 \sigma^2 ||\tilde{m}||^2_0 + ||\tilde{\xi}||^2_0 + 2a^2 \sigma^2 ||\tilde{h}||^2_0 + 2(\varepsilon^3 \sigma_1 \tilde{m}, \tilde{w})_0
\]

\[+ 2(\varepsilon^3 \sigma_2 \tilde{\xi}, \tilde{w})_0 + 4a^2 \sigma^2 (\tilde{k} + \tilde{h}, \tilde{w})_0 + 2\varepsilon^7 (\sigma_1 \tilde{k}, \tilde{m})_0 + 2\varepsilon^7 (\sigma_2 \tilde{h}, \tilde{\xi})_0
\]

\[= ||\tilde{w}||^2_0 + 2\lambda^2 \sigma^2 ||\tilde{m}||^2_0 + ||\tilde{\xi}||^2_0 + 2a^2 \sigma^2 ||\tilde{h}||^2_0 + I_1 + I_2 + I_3 + I_4 + I_5.
\]

Now, taking two positive constants \( \delta, \mu \) and by using the Cauchy inequality, we have:

\[
I_1 = 2\varepsilon^3 ||\tilde{m}_2, \tilde{w}_1||_0 + (\tilde{m}_1, \tilde{w}_2)_0 \geq -\delta \varepsilon^6 ||\tilde{m}_2||^2_0 - \frac{||\tilde{w}_1||^2_0}{\delta} - \delta \varepsilon^6 ||\tilde{m}_1||^2_0 - \frac{||\tilde{w}_2||^2_0}{\delta};
\]

\[
I_2 = 2\varepsilon^3 ||\tilde{\xi}_3, \tilde{w}_1||_0 + (\tilde{\xi}_1, \tilde{w}_3)_0 \geq -\delta \varepsilon^6 ||\tilde{\xi}_3||^2_0 - \frac{||\tilde{w}_1||^2_0}{\delta} - \delta \varepsilon^6 ||\tilde{\xi}_1||^2_0 - \frac{||\tilde{w}_3||^2_0}{\delta};
\]

\[
I_3 = 4a^2 \sigma^2 ||\tilde{k}, \tilde{w}||_0 + (\tilde{h}, \tilde{w})_0 \geq -2a\mu ||\tilde{w}||^2_0 - \frac{2a^8 \mu}{\varepsilon^6} ||\tilde{k}||^2_0 - 2a\mu ||\tilde{w}||^2_0 - \frac{2a^8 \mu}{\varepsilon^6} ||\tilde{h}||^2_0;
\]

\[
I_4 = 2\varepsilon^7 ||\tilde{k}, \tilde{m}_1||_0 + (\tilde{k}, \tilde{m}_2)_0 \geq -\frac{\varepsilon^7}{\delta} ||\tilde{k}||^2_0 - \delta \varepsilon^6 ||\tilde{m}_1||^2_0 - \frac{\varepsilon^7}{\delta} ||\tilde{k}||^2_0 - \delta \varepsilon^6 ||\tilde{m}_2||^2_0;
\]

\[
I_5 = 2\varepsilon^7 ||\tilde{h}, \tilde{\xi}_1||_0 + (\tilde{h}, \tilde{\xi}_3)_0 \geq -\frac{\varepsilon^7}{\delta} ||\tilde{h}||^2_0 - \delta \varepsilon^6 ||\tilde{\xi}_1||^2_0 - \frac{\varepsilon^7}{\delta} ||\tilde{h}||^2_0 - \delta \varepsilon^6 ||\tilde{\xi}_3||^2_0.
\]
Thus, putting them all together,
\[
(\Sigma \tilde{W}, \tilde{W})_0 \geq ||\tilde{w}_1||_0^2 \left[ 1 - \frac{2}{\delta} - 4a\mu \right] + ||\tilde{w}_2||_0^2 \left[ 1 - \frac{1}{\delta} - 4a\mu \right] + ||\tilde{w}_3||_0^2 \left[ 1 - \frac{1}{\delta} - 4a\mu \right]
\]
\[+ \varepsilon^6 ||\tilde{m}_1||_0^2 [2\lambda^2a - 2\delta] + \varepsilon^6 ||\tilde{m}_2||_0^2 [2\lambda^2a - 2\delta] + \varepsilon^6 ||\tilde{m}_3||_0^2 [2\lambda^2a]
\]
\[+ \varepsilon^6 ||\tilde{\xi}_1||_0^2 [2\lambda^2a - 2\delta] + \varepsilon^6 ||\tilde{\xi}_2||_0^2 [2\lambda^2a] + \varepsilon^6 ||\tilde{\xi}_3||_0^2 [2\lambda^2a - 2\delta]
\]
\[+ \varepsilon^8 ||\tilde{k}_1||_0^2 \left[ 2a - \frac{2a}{\mu} - \frac{1}{\delta} \right] + \varepsilon^8 ||\tilde{k}_2||_0^2 \left[ 2a - \frac{2a}{\mu} - \frac{1}{\delta} \right]
\]
\[+ \varepsilon^8 ||\tilde{k}_3||_0^2 \left[ 2a - \frac{2a}{\mu} \right] + \varepsilon^8 ||\tilde{h}_1||_0^2 \left[ 2a - \frac{2a}{\mu} - \frac{1}{\delta} \right]
\]
\[+ \varepsilon^8 ||\tilde{h}_2||_0^2 \left[ 2a - \frac{2a}{\mu} \right] + \varepsilon^8 ||\tilde{h}_3||_0^2 \left[ 2a - \frac{2a}{\mu} - \frac{1}{\delta} \right].
\] (32)

We can prove the following lemma.

**Lemma 3.1.** If Assumption 1 is satisfied and \( \lambda \) is big enough, then \( \Sigma \) is strictly positive definite.

**Proof.** From (32), we take
\[
\begin{aligned}
1 < \mu < \frac{1}{4a}; \\
\delta > \max\{\frac{2}{1-4a\mu}, \frac{1}{2a(1-\mu)}\}; \\
\lambda > \sqrt{\frac{2}{\mu}}.
\end{aligned}
\] (33)

Notice that we can choose the constant velocity \( \lambda \) as big as we need, therefore the third inequality is automatically verified. \( \square \)

At this point, we consider the linear part \(-L\Sigma\) of the source term of (31).

Thus,
\[
\tau(-L\Sigma \tilde{W}, \tilde{W})_0 = -2\lambda^2 a\varepsilon^4 \left( ||\tilde{m}_1||_0^2 + ||\tilde{\xi}_1||_0^2 \right) + 2a(2a - 1)\varepsilon^6 \left( ||\tilde{k}_1||_0^2 + ||\tilde{h}_1||_0^2 \right)
\]
\[+ \varepsilon^4 ||\tilde{m}_1||_0^2 + \varepsilon^4 ||\tilde{m}_2||_0^2 + \varepsilon^4 ||\tilde{m}_3||_0^2 + \varepsilon^4 ||\tilde{\xi}_1||_0^2 + \varepsilon^4 ||\tilde{\xi}_2||_0^2 + \varepsilon^4 ||\tilde{\xi}_3||_0^2 + 2\varepsilon^4 (\sigma_1\sigma_2\tilde{\xi}, \tilde{m})_0
\]
\[+ 2(2a - 1)\varepsilon^5 (\sigma_1\tilde{k}, \tilde{m})_0 + 4a\varepsilon^5 (\sigma_1\tilde{h}, \tilde{m})_0 + 4a\varepsilon^5 (\sigma_2\tilde{k}, \tilde{\xi})_0
\]
\[+ 2(2a - 1)\varepsilon^5 (\sigma_2\tilde{h}, \tilde{\xi})_0 + 8a^2\varepsilon^6 (\tilde{h}, \tilde{k})_0
\]
\[= (-2\lambda^2 a + 1)\varepsilon^4 \left( ||\tilde{m}_1||_0^2 + ||\tilde{m}_2||_0^2 + ||\tilde{\xi}_1||_0^2 + ||\tilde{\xi}_3||_0^2 \right)
\]
Now, taking a positive constant $\omega$ and by using the Cauchy inequality, we have

\[
J_1 = 2\varepsilon^4(\xi_3, \tilde{m}_2)_0 \leq \varepsilon^4(||\tilde{m}_1||^2_0 + ||\tilde{m}_2||^2_0);
\]

\[
J_2 = (4a - 2)\varepsilon^5[(\tilde{h}_2, \tilde{m}_1)_0 + (\tilde{k}_1, \tilde{m}_2)_0]
\leq (1 - 2a) \left\{\frac{\varepsilon^6}{\omega}(||\tilde{h}_2||^2_0 + ||\tilde{k}_1||^2_0) + \varepsilon^4\omega(||\tilde{m}_1||^2_0 + ||\tilde{m}_2||^2_0)\right\};
\]

\[
J_3 = 4a\varepsilon^5[(\tilde{h}_2, \tilde{m}_1)_0 + (\tilde{k}_1, \xi_3)_0]
\leq 2a \left\{\frac{\varepsilon^6}{\omega}||\tilde{h}_2||^2_0 + \varepsilon^4\omega||\tilde{m}_1||^2_0 + \frac{\varepsilon^6}{\omega}||\tilde{k}_1||^2_0 + \varepsilon^4\omega||\xi_3||^2_0\right\};
\]

\[
J_4 = 4a\varepsilon^5[(\tilde{h}_3, \xi_3)_0 + (\tilde{k}_1, \xi_3)_0]
\leq (1 - 2a) \left\{\frac{\varepsilon^6}{\omega}||\tilde{h}_3||^2_0 + \varepsilon^4\omega||\xi_3||^2_0 + \frac{\varepsilon^6}{\omega}||\tilde{k}_1||^2_0 + \varepsilon^4\omega||\xi_3||^2_0\right\};
\]

\[
J_5 = 2(2a - 1)\varepsilon^5[(\tilde{h}_3, \tilde{\xi}_3)_0 + (\tilde{h}_1, \tilde{\xi}_3)_0]
\leq (1 - 2a) \left\{\frac{\varepsilon^6}{\omega}||\tilde{h}_3||^2_0 + \varepsilon^4\omega||\tilde{\xi}_3||^2_0 + \frac{\varepsilon^6}{\omega}||\tilde{h}_1||^2_0 + \varepsilon^4\omega||\tilde{\xi}_3||^2_0\right\};
\]

\[
J_6 = 8a^2\varepsilon^6(\tilde{h}, \tilde{k})_0 \leq 4a^2\varepsilon^6(||\tilde{h}||^2_0 + ||\tilde{k}||^2_0).
\]

Putting them all together, we have

\[
\tau(-L\Sigma \dot{W}, \dot{W})_0 \leq \varepsilon^4(||\tilde{m}_1||^2_0[-2\lambda^2a + 1 + \omega] + \varepsilon^4||\tilde{m}_2||^2_0[-2\lambda^2a + 2 + \omega]
- 2\lambda^2 a^2 ||\tilde{m}_1||^2_0 + \varepsilon^4||\tilde{\xi}_1||^2_0[-2\lambda^2a + 1 + \omega]
- 2\lambda^2 a^2 ||\tilde{m}_2||^2_0 + \varepsilon^4||\tilde{\xi}_3||^2_0[-2\lambda^2a + 2 + \omega] + \varepsilon^6||\tilde{k}_1||^2_0 \left[2a(4a - 1) + \frac{1}{\omega}\right]
+ \varepsilon^6||\tilde{k}_2||^2_0 \left[2a(4a - 1) + \frac{(1 - 2a)}{\omega}\right] + \varepsilon^6||\tilde{k}_3||^2_0 \left[2a(4a - 1) + \frac{2a}{\omega}\right]
\]
\[ + \varepsilon^6 |\tilde{h}_1|^2_0 \left[ 2a(4a - 1) + \frac{1}{\omega} \right] + \varepsilon^6 |\tilde{h}_2|^2_0 \left[ 2a(4a - 1) + \frac{2a}{\omega} \right] \]

\[ + \varepsilon^6 |\tilde{h}_3|^2_0 \left[ 2a(4a - 1) + \frac{1 - 2a}{\omega} \right]. \]

This way, we obtain the following Lemma.

**Lemma 3.2.** If Assumption 1 is satisfied and \( \lambda \) is big enough, then the symmetrized linear part of the source term \(-L\Sigma\) given by (29) is negative definite.

**Proof.** We need \( \omega \) and \( \lambda \) satisfying:

\[
\begin{align*}
\omega &> \frac{1}{2a(1 - 4a)}, \\
\lambda &> \sqrt{\frac{2 + \omega}{2a}}. 
\end{align*}
\]

(35)

Recalling (33), we assume

\[
\lambda > \max \left\{ \sqrt{\frac{\delta}{a}}, \sqrt{\frac{4a(1 - 4a) + 1}{4a^2(1 - 4a)}} \right\}. 
\]

(36)

Then, we take \( \omega > \frac{1}{2a(1 - 4a)} \), which ends the proof.

4. **Energy estimates.** Here we provide \( \varepsilon \)-weighted energy estimates for the solution \( W^\varepsilon \) to (27). Let us introduce \( T^\varepsilon \) the maximal time of existence of the unique solution \( \tilde{W}^\varepsilon \) for fixed \( \varepsilon \) to system (31), see [25]. In the following, we consider the time interval \([0, T^\varepsilon]\), with \( T \in [0, T^\varepsilon) \). Our setting is represented by the Sobolev spaces \( H^s(\mathbb{T}^2) \), with \( s > 3 \).

4.1. **Zero-order estimate.** We assume the following condition.

**Assumption 2.** Let \( \lambda \) satisfy (36) and

\[
\lambda > \sqrt{\frac{5 + \frac{1}{a(1 - 4a)}}{4a}}. 
\]

(37)

**Lemma 4.1.** If Assumptions 1 and 2 are satisfied, then the following zero-order energy estimate holds:

\[
\begin{align*}
|\tilde{w}(T)|^2_0 + \varepsilon^6 (||\tilde{m}(T)||^2_0 + ||\tilde{\xi}(T)||^2_0) + & \varepsilon^8 (||\tilde{k}(T)||^2_0 + ||\tilde{h}(T)||^2_0) \\
+ \int_0^T & \varepsilon^4 (||\tilde{m}(t)||^2_0 + ||\tilde{\xi}(t)||^2_0) + \varepsilon^6 (||\tilde{k}(t)||^2_0 + ||\tilde{h}(t)||^2_0) \ dt \leq c\varepsilon^2 (||u_0||^2_0 + ||\nabla u_0||^2_0) \\
+ c(||u||_{L^\infty([0,T]\times\mathbb{T}^2)}) & \int_0^T ||\tilde{w}(t)||^2_0 + \varepsilon^6 (||\tilde{m}(t)||^2_0 + ||\tilde{\xi}(t)||^2_0) + \varepsilon^8 (||\tilde{k}(t)||^2_0 + ||\tilde{h}(t)||^2_0) \ dt. 
\end{align*}
\]

(38)

**Proof.** We consider the symmetrized compact system (31) and we multiply \( \tilde{W} \) through the \( L^2 \)-scalar product. Thus, we have:

\[
\frac{1}{2} \frac{d}{dt} (\Sigma \tilde{W}, \tilde{W})_0 + (L\Sigma \tilde{W}, \tilde{W})_0 = (N((\Sigma \tilde{W})_1 + \tilde{w}), \tilde{W})_0.
\]
Integrating in time, we get:

\[
\frac{1}{2}(\Sigma \hat{W}(T), \hat{W}(T))_0 + \int_0^T (L \Sigma \hat{W}(t), \hat{W}(t))_0 \, dt \leq \frac{1}{2}(\Sigma \hat{W}(0), \hat{W}(0))_0
\]

\[
\quad + \int_0^T |(N((\Sigma \hat{W}(t))_1 + \tilde{w}), \hat{W}(t))_0| \, dt. \tag{39}
\]

Consider (32) and let us introduce the following positive constants:

\[
\Gamma_\Sigma := 1 - 4a\mu - \frac{2}{\beta}, \quad \Delta_\Sigma := 2(\lambda^2 a - \delta) \quad \Theta_\Sigma := 2a(1 - \frac{1}{\beta}) - \frac{1}{\beta}. \tag{40}
\]

Similarly, from (34), we define:

\[
\Delta_{L \Sigma} := 2(\lambda^2 a - 1) - \omega, \quad \Theta_{L \Sigma} := 2a(1 - 4a) - \frac{1}{\omega}.
\]

Thus, from (39), we get:

\[
\Gamma_\Sigma ||\tilde{w}(T)||_0^2 + \varepsilon^6 \Delta_\Sigma (||\tilde{m}(T)||_0^2 + ||\dot{\xi}(T)||_0^2) + \varepsilon^8 \Theta_\Sigma (||\dot{\tilde{m}}(T)||_0^2 + ||\tilde{\dot{m}}(T)||_0^2)
\]

\[
\quad + \frac{2}{\tau} \int_0^T \varepsilon^4 \Delta_{L \Sigma} (||\tilde{m}(t)||_0^2 + ||\dot{\xi}(t)||_0^2) + \varepsilon^6 \Theta_{L \Sigma} (||\dot{\tilde{m}}(t)||_0^2 + ||\tilde{\dot{m}}(t)||_0^2) \, dt
\]

\[
\leq (\Sigma \hat{W}_0, \hat{W}_0)_0 + 2 \int_0^T |(N((\Sigma \hat{W}(t))_1 + \tilde{w}), \hat{W}(t))_0| \, dt. \tag{42}
\]

Notice that, from (30),

\[
(\Sigma \hat{W}_0, \hat{W}_0)_0 = (\Sigma^{-1} W_0, \Sigma^{-1} W_0)_0 = (\Sigma^{-1} W_0, \Sigma^{-1} W_0)_0,
\]

where \( W_0 = W(0, x) = (w(0, x), \varepsilon^2 m(0, x), \varepsilon^2 \xi(0, x), \varepsilon^2 k(0, x), \varepsilon^2 h(0, x)) \), and, from (16), (18) and the initial conditions (4),

\[
w(0, x) = w_0 - \tilde{w} = (0, \varepsilon \rho_{v_01}, \varepsilon \rho_{v_02});
\]

\[
m(0, x) = \frac{\lambda}{\tau} (f_{10} - f_{13}) = \frac{\lambda}{\tau} (\rho_{v_01} - 2a \varepsilon \bar{\rho} \bar{\rho}_1 x_0 - \varepsilon \rho_{v_01} u_0 x_0 - 2a \varepsilon \bar{\rho}_2 x_0 u_0);
\]

\[
\xi(0, x) = \frac{\lambda}{\tau} (f_{20} - f_{40}) = \frac{\lambda}{\tau} (\rho_{u_01} - 2a \varepsilon \bar{\rho} \bar{\rho}_1 x_0 - \varepsilon \rho_{u_01} u_0 x_0 - 2a \varepsilon \bar{\rho}_2 x_0 u_0);
\]

\[
k(0, x) = f_{10} + f_{30} - 2a \tilde{w} = 2a \tilde{w}_0 - 2a \tilde{w} = 2a(0, \varepsilon \rho_{v_01}, \varepsilon \rho_{v_02});
\]

\[
h(0, x) = f_{20} + f_{40} - 2a \tilde{w} = 2a \tilde{w}_0 - 2a \tilde{w} = 2a(0, \varepsilon \rho_{u_01}, \varepsilon \rho_{u_02}).
\]

Besides, the explicit expression of the constant symmetric matrix \( \Sigma^{-1} \) is given by

\[
\Sigma^{-1} = \left(\begin{array}{ccccc}
\frac{I_d}{\varepsilon^2(1-4\lambda^2 a^2)} & 0 & 0 & -\frac{I_d}{\varepsilon^2(1-4\lambda^2 a^2)} & 0 \\
0 & H_1 & 0 & 0 & \frac{H_3}{\varepsilon^2(1-4\lambda^2 a^2)} \\
0 & 0 & H_2 & 0 & \frac{H_4}{\varepsilon^2(1-4\lambda^2 a^2)} \\
\frac{\sigma_1}{\varepsilon^2(1-4\lambda^2 a^2)} & 0 & \frac{\sigma_2}{\varepsilon^2(1-4\lambda^2 a^2)} & 0 & \frac{\sigma_3}{\varepsilon^2(1-4\lambda^2 a^2)} \\
\frac{\sigma_4}{\varepsilon^2(1-4\lambda^2 a^2)} & 0 & 0 & \frac{I_d}{\varepsilon^2(1-4\lambda^2 a^2)} & \frac{H_4}{\varepsilon^2(1-4\lambda^2 a^2)}
\end{array}\right), \tag{43}
\]

where

\[
H_1 = \begin{pmatrix}
\frac{2a}{\varepsilon^2(4\lambda^2 a^2 - 1)} & 0 & 0 \\
0 & \frac{2a}{\varepsilon^2(4\lambda^2 a^2 - 1)} & 0 \\
0 & 0 & \frac{1}{2\lambda^2 a^2}
\end{pmatrix};
\]

\[
H_2 = \begin{pmatrix}
\frac{2a}{\varepsilon^2(4\lambda^2 a^2 - 1)} & 0 & 0 \\
0 & \frac{1}{2\lambda^2 a^2} & 0 \\
0 & 0 & \frac{2a}{\varepsilon^2(4\lambda^2 a^2 - 1)}
\end{pmatrix};
\]
\[
H_3 = \begin{pmatrix}
\frac{4\lambda^2 a^2 - 2\lambda^2 a + 1}{\varepsilon^2 (4a-1)(4\lambda^2 a^2 - 1)} & 0 & 0 \\
0 & \frac{4\lambda^2 a^2 - 2\lambda^2 a + 1}{\varepsilon^2 (4a-1)(4\lambda^2 a^2 - 1)} & 0 \\
0 & 0 & \frac{2a-1}{2\varepsilon^2 (4a-1)}
\end{pmatrix};
\]
\[
H_4 = \begin{pmatrix}
\frac{4\lambda^2 a^2 - 2\lambda^2 a + 1}{\varepsilon^2 (4a-1)(4\lambda^2 a^2 - 1)} & 0 & 0 \\
0 & \frac{2a-1}{2\varepsilon^2 (4a-1)} & 0 \\
0 & 0 & \frac{4\lambda^2 a^2 - 2\lambda^2 a + 1}{\varepsilon^2 (4a-1)(4\lambda^2 a^2 - 1)}
\end{pmatrix}.
\]

It is easy to check that
\[
(\Sigma^{-1} W_0, W_0)_0 = \rho^2 \varepsilon^2 \|u_0\|^2 + \frac{2a\rho^2 \varepsilon^4}{4\lambda^2 a^2 - 1} (\|u_0^2 - 2a\lambda^2 \partial_x u_01\|^2 \\
+ \|u_0^2 - 2a\lambda^2 \partial_y u_02\|^2) \\
+ \frac{\rho^2 \varepsilon^4}{2\lambda^2 a} (\|u_01 u_02 - 2a\lambda^2 \partial_x u_02\|^2 + \|u_01 u_02 - 2a\lambda^2 \partial_y u_01\|^2) \\
\leq c \varepsilon^2 (\|u_0\|^2 + \|\nabla u_0\|^2),
\]

and so, from (42) we get the following inequality:
\[
\Gamma_{\Sigma} ||\hat{w}(T)||_0^2 + \varepsilon^6 \Delta_{\Sigma} (||\hat{m}(T)||_0^2 + ||\hat{\xi}(T)||_0^2) + \varepsilon^8 \Theta_{\Sigma} (||\hat{k}(T)||_0^2 + ||\hat{h}(T)||_0^2)
\]
\[
+ \frac{2}{\tau} \int_0^T \varepsilon^4 \Delta_{\Sigma} (||\hat{m}(t)||_0^2 + ||\hat{\xi}(t)||_0^2) + \varepsilon^6 \Theta_{\Sigma} (||\hat{k}(t)||_0^2 + ||\hat{h}(t)||_0^2) \, dt \tag{44}
\]
\[
\leq c \varepsilon^2 (\|u_0\|^2 + \|\nabla u_0\|^2) + 2 \int_0^T (|N((\Sigma \hat{w}(t))_1 + \hat{w}), \hat{W}(t)|_0 \, dt.
\]

It remains to deal with the last term of (44). Recall that \(w = (\rho - \tilde{\rho}, \varepsilon \rho u_1, \varepsilon \rho u_2)\).

From (26),
\[
N((\Sigma \hat{w})_1 + \hat{w}) = N(w + \hat{w}) = \frac{1}{\tau}
\]
\[
\begin{pmatrix}
0 \\
0 \\
u_1 w_2 \\
u_1 w_3 \\
0 \\
u_2 w_2 \\
u_2 w_3 \\
0 \\
0
\end{pmatrix}.
\]
Thus,
\[
(N(w + \bar{w}), \bar{W})_0 = \frac{1}{\tau}\{(u_1 w_2, \varepsilon^2 \bar{m}_2)_0 + (u_1 w_3, \varepsilon^2 \bar{m}_3)_0 \\
+ (u_2 w_2, \varepsilon^2 \bar{\xi}_2)_0 + (u_2 w_3, \varepsilon^2 \bar{\xi}_3)_0\}
\]
\[
\leq \frac{1}{2\tau}\{(||u_1 w_2||_0^2 + \varepsilon^4 ||\bar{m}_2||_0^2 + ||u_1 w_3||_0^2 + \varepsilon^4 ||\bar{m}_3||_0^2 + ||u_2 w_2||_0^2 \\
+ \varepsilon^4 ||\bar{\xi}_2||_0^2 + ||u_2 w_3||_0^2 + \varepsilon^4 ||\bar{\xi}_3||_0^2\} \leq c(||u||_{\infty})||w||_0^2 \\
+ \frac{\varepsilon^4}{2\tau}(||\bar{m}||_0^2 + ||\bar{\xi}||_0^2).
\]

By definition (30), explicitly we have:
\[
w = (\Sigma \bar{W}^c)_1 = \bar{w} + \varepsilon^3 \sigma_1 \bar{m} + \varepsilon^3 \sigma_2 \bar{\xi} + 2 \varepsilon^4 (\bar{k} + \bar{h}),
\]
and so,
\[
|(N(w + \bar{w}), \bar{W})_0| \leq c(||u||_{\infty})\{||\bar{w}||_0^2 + \varepsilon^6 (||\bar{m}||_0^2 + ||\bar{\xi}||_0^2) \\
+ \varepsilon^8 (||\bar{k}||_0^2 + ||\bar{h}||_0^2)\} + \frac{\varepsilon^4}{2\tau}(||\bar{m}||_0^2 + ||\bar{\xi}||_0^2).
\]

Putting them all together, (44) yields:
\[
\Gamma_{\Sigma}||\bar{w}(T)||_0^2 + \varepsilon^6 \Delta_{\Sigma}||\bar{m}(T)||_0^2 + ||\bar{\xi}(T)||_0^2 + \varepsilon^8 \Theta_{\Sigma}(||\bar{k}(T)||_0^2 + ||\bar{h}(T)||_0^2) \\
+ \frac{2}{\tau} \int_0^T \varepsilon^4 \Delta_{\Sigma\Sigma}(||\bar{m}(t)||_0^2 + ||\bar{\xi}(t)||_0^2) + \varepsilon^6 \Theta_{\Sigma\Sigma}(||\bar{k}(t)||_0^2 + ||\bar{h}(t)||_0^2) dt \leq c\varepsilon^2 (||u_0||_0^2 + ||\nabla u_0||_0^2) + \int_0^T \frac{\varepsilon^4}{\tau} (||\bar{w}(t)||_0^2 + ||\bar{\xi}(t)||_0^2) dt \\
+c(||u||_{L^\infty([0,T]\times T^2)}) \int_0^T \left[||\bar{w}(t)||_0^2 + \varepsilon^6 (||\bar{m}(t)||_0^2 + ||\bar{\xi}(t)||_0^2) \\
+ \varepsilon^8 (||\bar{k}(t)||_0^2 + ||\bar{h}(t)||_0^2)\right] dt.
\]

This gives:
\[
\Gamma_{\Sigma}||\bar{w}(T)||_0^2 + \varepsilon^6 \Delta_{\Sigma}||\bar{m}(T)||_0^2 + ||\bar{\xi}(T)||_0^2 + \varepsilon^8 \Theta_{\Sigma}(||\bar{k}(T)||_0^2 + ||\bar{h}(T)||_0^2) \\
+ \frac{1}{\tau} \int_0^T \varepsilon^4 (2\Delta_{\Sigma\Sigma} - 1)(||\bar{m}(t)||_0^2 + ||\bar{\xi}(t)||_0^2) + 2\varepsilon^6 \Theta_{\Sigma\Sigma}(||\bar{k}(t)||_0^2 + ||\bar{h}(t)||_0^2) dt \leq c\varepsilon^2 (||u_0||_0^2 + ||\nabla u_0||_0^2) \\
+c(||u||_{L^\infty([0,T]\times T^2)}) \int_0^T \left[||\bar{w}(t)||_0^2 + \varepsilon^6 (||\bar{m}(t)||_0^2 + ||\bar{\xi}(t)||_0^2) \\
+ \varepsilon^8 (||\bar{k}(t)||_0^2 + ||\bar{h}(t)||_0^2)\right] dt,
\]
where, by definition (41), $2\Delta_{\Sigma\Sigma} - 1 = 4\lambda^2 a - 4 - 2\omega$ is positive thanks to condition (37). This provides estimate (38).
4.2. Higher-order estimates.

Lemma 4.2. If Assumptions 1 and 2 are satisfied, then the following $H^s$-energy estimate holds:

$$
\begin{align*}
&||\hat{w}(T)||_2^2 + \varepsilon^6(||\hat{m}(T)||_2^2 + ||\hat{\xi}(T)||^2_2) + \varepsilon^8(||\hat{k}(T)||_2^2 + ||\hat{h}(T)||_2^2) \\
&+ \int_0^T \varepsilon^4(||\hat{m}(t)||_2^2 + ||\hat{\xi}(t)||^2_2) + \varepsilon^6(||\hat{k}(t)||_2^2 + ||\hat{h}(t)||_2^2) \, dt \\
&+ c(||u||_{L^\infty H^s_x}) \int_0^T ||\hat{w}(t)||_2^2 + \varepsilon^6(||\hat{m}(t)||_2^2 + ||\hat{\xi}(t)||^2_2) + \varepsilon^8(||\hat{k}(t)||_2^2 + ||\hat{h}(t)||_2^2) \, dt,
\end{align*}
$$

where, hereafter,

$$L^\infty_1 H^s_x := L^\infty([0,T], H^s(T)),$$

for $s \in \mathbb{R}$.

Proof. We take the $|\alpha|$-derivative, $0 < |\alpha| \leq s$, of the semilinear system given by (27). As done previously, we get:

$$
\begin{align*}
\Gamma_{\Sigma} &\int |D^\alpha \hat{w}(T)||_2^2 + \varepsilon^6 \Delta_{\Sigma}(|D^\alpha \hat{m}(T)||_2^2 + |D^\alpha \hat{\xi}(T)||^2_2) \\
&+ \varepsilon^8 \Theta_{\Sigma}(|D^\alpha \hat{k}(T)||_2^2 + |D^\alpha \hat{h}(T)||_2^2) \\
&+ \frac{2}{\tau} \int_0^T \varepsilon^4 \Delta_{LSigma}(|D^\alpha \hat{m}(t)||_2^2 + |D^\alpha \hat{\xi}(t)||^2_2) + \varepsilon^6 \Theta_{LSigma}(|D^\alpha \hat{k}(t)||_2^2 + |D^\alpha \hat{h}(t)||_2^2) \, dt \\
&\leq c\varepsilon^2(||D^\alpha u_0||_2^2 + |D^{\alpha+1} u_0||_2^2) + 2 \int_0^T |D^\alpha(N((\Sigma \hat{W}(t))_1 + \hat{w}), D^\alpha \hat{W}(t))_0| \, dt. \tag{49}
\end{align*}
$$

Now, from (45),

$$
\begin{align*}
&|\langle |D^\alpha N(w + \hat{w}), D^\alpha \hat{W} \rangle_0 \rangle| \leq \frac{1}{\tau} \{ ||(D^\alpha(u_1 w_2), D^\alpha \varepsilon^2 \hat{m}_2) ||_0 \\
&+ |D^\alpha(u_1 w_3), D^\alpha \varepsilon^2 \hat{m}_3) ||_0 + |(D^\alpha(u_2 w_2), D^\alpha \varepsilon^2 \hat{\xi}_2) ||_0 \\
&+ |D^\alpha(u_2 w_3), D^\alpha \varepsilon^2 \hat{\xi}_3) ||_0 \} \\
&\leq \frac{1}{2\tau} \{ |||(D^\alpha(u_1 w_2)||_2^2 + ||D^\alpha(u_1 w_3)||_2^2 \\
&+ ||D^\alpha(u_2 w_2)||_2^2 + ||D^\alpha(u_2 w_3)||_2^2) \\
&+ \varepsilon^4(|D^\alpha \hat{m})_2^2 + ||D^\alpha \hat{\xi})_2^2) \} \\
&\leq c(\langle |u||_s||w||_2^2 + \frac{\varepsilon^4}{2\tau}(||\hat{m})_2^2 + ||\hat{\xi})_2^2).}
\end{align*}
$$

By using (46) we have:

$$
\begin{align*}
&|\langle |D^\alpha N(w + \hat{w}), D^\alpha \hat{W} \rangle_0 \rangle| \leq c(\langle |u||_s||\hat{w})_2^2 + \varepsilon^6(||\hat{m})_2^2 + ||\hat{\xi})_2^2) \\
&+ \varepsilon^8(||\hat{k})_2^2 + ||\hat{h})_2^2) + \frac{\varepsilon^4}{2\tau}(||\hat{m})_2^2 + ||\hat{\xi})_2^2).
\end{align*}
$$

Thus, from (49),

$$
\begin{align*}
\Gamma_{\Sigma} &\int \hat{w}(T)||_2^2 + \varepsilon^6 \Delta_{\Sigma}(||\hat{m}(T)||_2^2 + ||\hat{\xi}(T)||^2_2) + \varepsilon^8 \Theta_{\Sigma}(||\hat{k}(T)||_2^2 + ||\hat{h}(T)||_2^2) \\
&+ \frac{2}{\tau} \int_0^T \varepsilon^4(2(LSigma) - 1)(||\hat{m}(t)||_2^2 + ||\hat{\xi}(t)||^2_2) + 2\varepsilon^6 \Theta_{LSigma}||\hat{k}(t)||_2^2 + ||\hat{h}(t)||_2^2) \, dt \\
&\leq c\varepsilon^2(||u_0||_2^2 + ||\nabla u_0||_2^2)
\end{align*}
$$

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Remark 1. In the case $s > 3$ is not an integer, by using the pseudodifferential operator $\lambda^s(\xi) = (1 + |\xi|^2)^{s/2}$ in the Fourier space, we get the same estimates in a standard way.

Now, we need a bound in the $H^s$-norm for the original variable $w = (\rho - \bar{\rho}, \varepsilon \rho u)$, which is the first component of the unknown vector $W$ in (21). By using estimate (49) and definition (30), we can prove the following proposition.

**Proposition 1.** If Assumptions 1 and 2 are satisfied, then the following estimate holds:

$$
||w(t)||_s^2 + \epsilon^6(||\tilde{m}(t)||_s^2 + ||\tilde{\xi}(t)||_s^2) + \epsilon^8(||\tilde{k}(t)||_s^2 + ||\tilde{h}(t)||_s^2)
\leq c\epsilon^2(||u_0||_s^2 + ||\nabla u_0||_s^2)e^{c(||u||_{L^\infty H^s})t},
$$

and

$$
\frac{||\rho(t) - \bar{\rho}||_s^2}{\epsilon^2} + ||\rho u(t)||_s^2 \leq c(||u_0||_s^2 + ||\nabla u_0||_s^2)e^{c(||u||_{L^\infty H^s})t},
$$

for $t \in [0, T^\infty)$.

**Proof.** The Gronwall inequality applied to (49) yields:

$$
\Gamma_\Sigma||\tilde{w}(t)||^2_+ + \epsilon^6(\Sigma_1||\tilde{m}(t)||^2_+ + ||\tilde{\xi}(t)||^2_+) + \epsilon^8(\Sigma_2(||\tilde{k}(t)||_s^2 + ||\tilde{h}(t)||_s^2)
\leq c\epsilon^2(||u_0||_s^2 + ||\nabla u_0||_s^2)e^{c(||u||_{L^\infty H^s})t}.
$$

Recalling (46),

$$
\tilde{w} = w - \epsilon^3\sigma_1 \tilde{m} - \epsilon^3\sigma_2 \tilde{\xi} - 2a\epsilon^4(\tilde{k} + \tilde{h}).
$$

Thus,

$$
||\tilde{w}||^2_+ = ||w||^2_+ + \epsilon^6(||\tilde{m_1}||^2_+ + ||\tilde{m_2}||^2_+ + ||\tilde{\xi_1}||^2_+ + ||\tilde{\xi_2}||^2_+) + 4a^2\epsilon^8||\tilde{k} + \tilde{h}||^2_s
\leq 2\epsilon^3(w, \sigma_1 \tilde{m})_s - 2\epsilon^3(w, \sigma_2 \tilde{\xi})_s - 4a\epsilon^4(w, \tilde{k} + \tilde{h})_s + 2\epsilon^6(\sigma_1 \tilde{m}, \sigma_2 \tilde{\xi})_s
+ 4a\epsilon^7(\sigma_1 \tilde{m}, \tilde{k} + \tilde{h})_s + 4a\epsilon^7(\sigma_2 \tilde{\xi}, \tilde{k} + \tilde{h})_s
$$

$$
= ||w||^2_+ + \epsilon^6(||\tilde{m_1}||^2_+ + ||\tilde{m_2}||^2_+ + ||\tilde{\xi_1}||^2_+ + ||\tilde{\xi_2}||^2_+) + 4a^2\epsilon^8||\tilde{k} + \tilde{h}||^2_s
+ Y_1 + Y_2 + Y_3 + Y_4 + Y_5 + Y_6.
$$

Now, taking two positive constants $\eta, \zeta$ and using the Cauchy inequality, from (52) we have:

$$
Y_1 = -2\epsilon^3(w, \sigma_1 \tilde{m})_s \geq -\frac{||w_1||^2_+}{\eta} - \epsilon^6\eta||\tilde{m}_2||^2_s - \frac{||w_2||^2_+}{\eta} - \epsilon^6\eta||\tilde{m}_1||^2_s.
$$
\[ Y_2 = -2\varepsilon^3(w, \sigma_2 \xi) \geq -\frac{|w_2|^2}{\eta} - \varepsilon^6 \eta |\xi_3|^2 - \frac{|w_3|^2}{\eta} - \varepsilon^6 \eta |\xi_1|^2; \]

\[ Y_3 = -4\varepsilon^4(w, \tilde{k} + \tilde{h}) \geq \frac{2\varepsilon^2}{\xi} |w|^2 - 2\alpha \varepsilon^8 |\tilde{k} + \tilde{h}|^2; \]

\[ Y_4 = 2\varepsilon^6(\tilde{m}_2, \xi_3) \geq -\varepsilon^6(|\tilde{m}_2|^2 + |\xi_3|^2); \]

\[ Y_5 = 4\varepsilon^7[(\tilde{m}_2, \tilde{k} + \tilde{h}) + (\tilde{m}_1, \tilde{k} + \tilde{h})] \geq -2\alpha \varepsilon^6 \eta |\tilde{m}_2|^2 - \frac{2\alpha^2}{\eta} |\tilde{k} + \tilde{h}|^2 \]

\[ -2\alpha \varepsilon^6 \eta |\tilde{m}_1|^2 - \frac{2\alpha^2}{\eta} |\tilde{k} + \tilde{h}|^2; \]

\[ Y_6 = 4\varepsilon^7[(\xi_3, \tilde{k} + \tilde{h}) + (\xi_1, \tilde{k} + \tilde{h})] \geq -2\alpha \varepsilon^6 \eta |\tilde{m}_1|^2 - \frac{2\alpha^2}{\eta} |\tilde{k} + \tilde{h}|^2 \]

\[ -2\alpha \varepsilon^6 \eta |\tilde{m}_3|^2 - \frac{2\alpha^2}{\eta} |\tilde{k} + \tilde{h}|^2. \]

The left hand side of (51) and the previous calculations provide the following inequality:

\[
\Gamma_\Sigma |\tilde{w}|^2 + \varepsilon^6 \Delta_\Sigma (||\tilde{m}||^2 + ||\xi||^2) + \varepsilon^6 \Theta_\Sigma (||\tilde{k}||^2 + ||\tilde{h}||^2) \geq \Gamma_\Sigma \left[ 1 - \frac{2}{\eta} - \frac{2\alpha}{\zeta} \right] |w_1|^2 \]

\[
+ \Gamma_\Sigma \left[ 1 - \frac{1}{\eta} - \frac{2\alpha}{\zeta} \right] (||w_2|^2 + ||w_3|^2) + \varepsilon^6 \Theta_\Sigma (||\tilde{k}||^2 + ||\tilde{h}||^2) \]

\[
+ \varepsilon^6 (||\tilde{m}_1||^2 + ||\tilde{k}||^2)[\Delta_\Sigma + \Gamma_\Sigma(1 - \eta - 2\alpha\eta)] \]

\[
+ \varepsilon^6 (||\tilde{m}_2||^2 + ||\tilde{k}||^2)[\Delta_\Sigma + \Gamma_\Sigma(-\eta - 2\alpha\eta)] \]

\[
+ \varepsilon^6 (||\tilde{m}_3||^2 + ||\tilde{k}||^2)[\Delta_\Sigma + \varepsilon^6||\tilde{k}_1 + \tilde{h}_1||^2 \Gamma_\Sigma \left[ 4\alpha^2 - 2\alpha \zeta - \frac{4\alpha}{\eta} \right] \]

\[
+ \varepsilon^8 \Gamma_\Sigma (||\tilde{k}_2 + \tilde{h}_2||^2 + ||\tilde{k}_3 + \tilde{h}_3||^2) \left[ 4\alpha^2 - 2\alpha \zeta - \frac{2\alpha}{\eta} \right]. \]

Fixed \( \beta > 1 \), the Cauchy inequality yields \( ||\tilde{k} + \tilde{h}||^2 \geq (1 - \frac{1}{\beta})||\tilde{k}||^2 + (1 - \beta)||\tilde{h}||^2 \), then the last term of (53) is bounded from below by the following expression:

\[
\varepsilon^8 (||\tilde{k}_1||^2 + ||\tilde{h}_1||^2) \left[ \Theta_\Sigma + (1 - 1/\beta) \Gamma_\Sigma \left[ 4\alpha^2 - 2\alpha \zeta - \frac{4\alpha}{\eta} \right] \right] \]

\[
+ \varepsilon^8 (||\tilde{k}_2||^2 + ||\tilde{h}_2||^2 + ||\tilde{k}_3||^2 + ||\tilde{h}_3||^2) \left[ \Theta_\Sigma + (1 - \beta) \Gamma_\Sigma \left[ 4\alpha^2 - 2\alpha \zeta - \frac{2\alpha}{\eta} \right] \right]. \]
Thus, in order to get estimate (50), we require:
\[
\begin{align*}
1 - \frac{2}{\eta} - \frac{4a}{\zeta} > 0; \\
\Delta_{\Sigma} - \eta \Gamma_{\Sigma} (1 + 2a) > 0; \\
\Theta_{\Sigma} + (1 - 1/\beta) \Gamma_{\Sigma} \left[ 4a^2 - 2a \zeta - \frac{4a}{\eta} \right] > 0; \\
\Theta_{\Sigma} + (1 - \beta) \Gamma_{\Sigma} \left[ 4a^2 - 2a \zeta - \frac{4a}{\eta} \right] > 0.
\end{align*}
\] (55)

Recalling definition (40), \( \Delta_{\Sigma} = 2(\lambda^2 a - \delta) \), and so the second inequality is satisfied for \( \lambda \) big enough. Precisely, we take \( \lambda \) as in Assumption 2 and
\[
\lambda > \sqrt{\frac{\delta}{a} + \frac{\eta \Gamma_{\Sigma} (1 + 2a)}{2a}}.
\]
Moreover, the first condition of (55) is verified if
\[
\eta > \frac{2 \zeta}{\zeta - 4a}, \quad \zeta > 4a.
\] (56)

Since \( \Theta_{\Sigma} \) and \( \Gamma_{\Sigma} \) are positive, taking \( 1 - \beta < 0 \), i.e. \( \beta > 1 \), the last inequality is verified if
\[
2a \zeta + \frac{4a}{\eta} - 4a^2 > 0.
\]
From (56),
\[
2a \zeta + \frac{4a}{\eta} - 4a^2 > 8a^2 + \frac{4a}{\eta} - 4a^2 = 4a^2 + \frac{4a}{\eta} > 0,
\]
then the last inequality in (55) holds under (56). Now, the third condition in (55) is satisfied if
\[
\zeta < \frac{\Theta_{\Sigma}}{2a \Gamma_{\Sigma} (1 - 1/\beta)} + 2(a - 1/\eta).
\]
Thus, if \( \eta > \frac{1}{a} \) we can take
\[
4a < \zeta < \frac{\Theta_{\Sigma}}{2a \Gamma_{\Sigma} (1 - 1/\beta)},
\] (57)
with \( \eta \) and \( \zeta \) satisfying (56). In particular, we show that there exists \( \beta > 1 \) such that:
\[
4a < \frac{\Theta_{\Sigma}}{2a \Gamma_{\Sigma} (1 - 1/\beta)}, \quad \text{i.e.} \quad 8a^2 \Gamma_{\Sigma} (1 - 1/\beta) < \Theta_{\Sigma}.
\] (58)

From (40), \( \Gamma_{\Sigma} = 1 - 4a \mu - \frac{2}{\beta} \) and, from Lemma 3.1, \( 0 < \Gamma_{\Sigma} < 1 \). Thus, in order to verify (58), we require:
\[
8a^2 (1 - 1/\beta) < \Theta_{\Sigma},
\]
which is automatically verified if \( 8a^2 \leq \Theta_{\Sigma} \). Otherwise, it yields \( \beta < \frac{8a^2}{8a^2 - \Theta_{\Sigma}} \).

Finally, since \( \beta > 1 \), we need
\[
1 < \frac{8a^2}{8a^2 - \Theta_{\Sigma}}, \quad \text{i.e.} \quad \Theta_{\Sigma} > 0,
\]
which is already satisfied thanks to Lemma 3.1.
This way, from (54), (51) and (55), we get some positive constants \( \Gamma_{\Sigma}, \Delta_{\Sigma}, \Theta_{\Sigma} \) such that
\[
\Gamma_{\Sigma}^1||u(t)||_s^2 + \varepsilon^6 \Delta_{\Sigma}^1(||\tilde{m}(t)||_s^2 + ||\tilde{\xi}(t)||_s^2) + \varepsilon^8 \Theta_{\Sigma}^1(||\tilde{h}(t)||_s^2 + ||\tilde{h}(t)||_s^2) \\
\leq c\varepsilon^2(||u_0||_s^2 + ||\nabla u_0||_s^2)e^{c(||u||_{L^\infty H^s})t},
\]
(59)
and, in particular,
\[
||w(t)||_s^2 \leq c\varepsilon^2(||u_0||_s^2 + ||\nabla u_0||_s^2)e^{c(||u||_{L^\infty H^s})t},
\]
i.e.
\[
\frac{||\rho(t) - \bar{\rho}||_s^2}{\varepsilon^2} + ||\rho u(t)||_s^2 \leq c(||u_0||_s^2 + ||\nabla u_0||_s^2)e^{c(||u||_{L^\infty H^s})t}.
\]
(60)

Thus, we are able to prove that the maximal time \( T^\varepsilon \) of existence of the solutions to the vector-BGK scheme is bounded form below by a positive time \( T^\star \), which is independent of \( \varepsilon \).

**Proposition 2.** There exist \( \varepsilon_0 \) and a fixed time \( T^\star \), independent of \( \varepsilon \), \( T^\star < T^\varepsilon \), such that for all \( \varepsilon \leq \varepsilon_0 \) the following uniform bounds hold:
\[
||\rho u(t)||_s \leq M, \quad t \in [0, T^\star],
\]
(61)
\[
||\rho(t) - \bar{\rho}||_s \leq \varepsilon M, \quad i.e. \quad ||\rho(t)||_s \leq \bar{\rho}|T^2| + \varepsilon M, \quad t \in [0, T^\star],
\]
(62)
and
\[
||u(t)||_s \leq M(\bar{\rho}|T^2| + \varepsilon M), \quad t \in [0, T^\star].
\]
(63)

**Proof.** Let \( u_0 \in H^{s+1}(T^2) \) and, from (4), recall that \( \rho_0 = \bar{\rho} \). Then, there exists a positive constant \( M_0 \) such that
\[
||\rho_0 u_0||_{s+1} = \bar{\rho}||u_0||_{s+1} \leq M_0
\]
(64)
Let \( M > M_0 \) be any fixed constant, and
\[
T_0^\varepsilon := \sup \left\{ t \in [0, T^\varepsilon] \mid \frac{||\rho(t) - \bar{\rho}||_s^2}{\varepsilon^2} + ||\rho u(t)||_s^2 \leq M^2, \quad \forall \varepsilon \leq \varepsilon_0 \right\}.
\]
(65)
Notice that, from (65),
\[
||\rho - \bar{\rho}||_s \leq c_s ||\rho - \bar{\rho}||_s \leq c_s M \varepsilon, \quad t \in [0, T_0^\varepsilon],
\]
where \( c_s \) is the Sobolev embedding constant, i.e.
\[
\bar{\rho} - c_s M \varepsilon \leq \rho \leq \bar{\rho} + c_s M \varepsilon, \quad t \in [0, T_0^\varepsilon],
\]
Taking \( \varepsilon_0 \) such that \( \rho - c_s M \varepsilon_0 > \frac{\bar{\rho}}{2} \), i.e. \( \bar{\rho} > 2c_s M \varepsilon_0 \), we have
\[
\rho > \frac{\bar{\rho}}{2}, \quad t \in [0, T_0^\varepsilon], \quad \varepsilon \leq \varepsilon_0.
\]
(66)
Now, since \( s > 3 = \frac{d}{2} + 2 \),
\[
||u||_s \leq ||\rho u||_s ||1/\rho||_s.
\]
Moreover,
\[
||1/\rho||_s \leq c \left( \frac{|T^2|}{\bar{\rho}} + ||\rho||_s \right) \leq c_1 + c_2 ||\rho||_s.
\]
From (65),
\[
||\rho||_s \leq c(|T^2|/\bar{\rho} + M \varepsilon),
\]
so

\[ \|1/\rho\|_s \leq c_1 + c_2 M \varepsilon, \]

and

\[ \|u\|_s \leq cM(c_1 + c_2 M \varepsilon). \]

From (60),

\[ \frac{\|\rho(t) - \bar{\rho}\|^2}{\varepsilon^2} + \|\rho u(t)\|^2_s \leq cM^2 e^{c(M(c_1 + c_2 M \varepsilon)t)}, \quad t \in [0, T_0^\varepsilon]. \]

We take \( T^* \leq T_0^\varepsilon \) such that

\[ cM^2 e^{c(M(c_1 + c_2 M \varepsilon))T^*} \leq M^2, \]

i.e.

\[ T^* \leq \frac{1}{c(M(c_1 + c_2 M \varepsilon))} \log(M^2/(cM^2_0)) \quad \forall \varepsilon \leq \varepsilon_0. \]

This way,

\[ \|u(t)\|_s \leq cM(c_1 + c_2 M \varepsilon), \quad t \in [0, T^*] \quad \text{and} \quad \|\rho u\|_s \leq M \quad \forall \varepsilon \leq \varepsilon_0. \]

4.3. Time-derivative estimate. In order to use the compactness tools, we need a uniform bound for the time derivative of the unknown vector field.

**Proposition 3.** If Assumptions 1 and 2 hold, for \( M_0 \) in (64) and \( M \) in (61), we have:

\[ \|\partial_t w\|_{s-1}^2 + \varepsilon^2 ||\partial_t m\|_{s-1}^2 + ||\partial_t \xi\|_{s-1}^2 + \varepsilon^2 (||\partial_t \tilde{\xi}\|_{s-1}^2 + ||\partial_t \tilde{\xi}\|_{s-1}^2) \leq \varepsilon^2 c(||u_0||_{s+1}) e^{c(M^3)} \leq \varepsilon^2 c(M_0, M) \quad \text{in} \ [0, T^*], \]

with \( T^* \) in (67). This also yields the uniform bound:

\[ \|\partial_t (\rho - \bar{\rho})\|_{s-1}^2 + \|\partial_t (\rho u)\|_{s-1}^2 \leq c(||u_0||_{s+1}) \leq M^2 \quad \text{in} \ [0, T^*]. \]

**Proof.** Let us take the time derivative of system (31). Defining \( \tilde{V} = \partial_t \tilde{W} \), from (26) we get:

\[ \partial_t \Sigma \tilde{V} + \tilde{\Lambda}_1 \Sigma \partial_y \tilde{V} + \tilde{\Lambda}_2 \Sigma \partial_y \tilde{V} = -L \Sigma \tilde{V} + \partial_t N((\Sigma \tilde{W})_1 + \bar{w}) = -L \Sigma \tilde{V} + \partial_t N(w + \bar{w}), \]

where

\[ \partial_t N(w + \bar{w}) = \frac{1}{\tau} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2u_1 \partial_t w_2 - \varepsilon u_1^2 \partial_t w_1 & u_2 \partial_t w_2 + u_1 \partial_t w_3 - \varepsilon u_1 u_2 \partial_t w_1 \\ 0 & u_2 \partial_t w_2 + u_1 \partial_t w_3 - \varepsilon u_1 u_2 \partial_t w_1 & 2u_2 \partial_t w_3 - \varepsilon u_1^2 \partial_t w_1 \end{pmatrix}. \]

Taking the scalar product with \( \tilde{V} \), we have:

\[ \frac{1}{2} \frac{d}{dt}(\Sigma \tilde{V}, \tilde{V}) + (L \Sigma \tilde{V}, \tilde{V}) \leq |(\partial_t N(w + \bar{w}), V)|. \]
Here,
\[
|\langle \partial_t N(w + \tilde{w}), \tilde{V} \rangle_0| = \frac{1}{\tau} |\langle 2u_1 \partial_t w - \varepsilon u_1^2 \partial_t w_1, \varepsilon^2 \partial_t \tilde{m} \rangle_0 + (u_2 \partial_t w_2 + u_1 \partial_t w_3 - \varepsilon u_1 u_2 \partial_t w_1, \varepsilon^2 \partial_t \tilde{\xi} + \varepsilon^2 \partial_t \tilde{m} \rangle_0 + (2u_2 \partial_t w_3 - \varepsilon u_2^2 \partial_t w_1, \varepsilon^2 \partial_t \tilde{\xi} \rangle_0| \leq c(||u||_\infty) ||\partial_t w||_0^2 + \frac{\varepsilon^4}{2\tau} (||\partial_t \tilde{m}||_0^2 + ||\partial_t \tilde{\xi}||_0^2).
\]

Similarly to (48), we get:
\[
\Gamma_{\Sigma} ||\partial_t \tilde{w}||_0^2 + \varepsilon^0 \Delta_{\Sigma} (||\partial_t \tilde{m}||_0^2 + ||\partial_t \tilde{\xi}||_0^2) + \varepsilon^8 (||\partial_t \tilde{k}||_0^2 + ||\partial_t \tilde{h}||_0^2)
\]
\[
+ \frac{1}{\tau} \int_0^T (2\Delta_{L\Sigma} - 1) \varepsilon^4 (||\partial_t \tilde{m}||_0^2 + ||\partial_t \tilde{\xi}||_0^2) + 2\varepsilon^6 (||\partial_t \tilde{k}||_0^2 + ||\partial_t \tilde{h}||_0^2) \, dt
\]
\[
\leq \varepsilon^2 ||\partial_t w||_{t=0}^2
\]
\[
+ c(||u||_{L^\infty(0,T\times T^2)} \int_0^T \left[ ||\partial_t \tilde{w}||_0^2 + \varepsilon^6 (||\partial_t \tilde{m}||_0^2 + ||\partial_t \tilde{\xi}||_0^2) + \varepsilon^8 (||\partial_t \tilde{k}||_0^2 + ||\partial_t \tilde{h}||_0^2) \right] \, dt.
\]

Now, from the first equation given by (19),
\[
\partial_t w|_{t=0} = -\partial_x m|_{t=0} - \partial_y \xi|_{t=0},
\]
where, from (16), (4), and (9),
\[
m|_{t=0} = \frac{A_1(w_0)}{\varepsilon} = 2a\lambda^2 \tau \partial_x w_0 = \bar{\rho} \left( \begin{array}{c} u_{01} \\ \varepsilon u_{01} - 2a\lambda^2 \varepsilon \partial_x u_{01} \end{array} \right),
\]
\[
\xi|_{t=0} = \frac{A_2(w_0)}{\varepsilon} = 2a\lambda^2 \tau \partial_y w_0 = \bar{\rho} \left( \begin{array}{c} u_{02} \\ \varepsilon u_{01} u_{02} - 2a\lambda^2 \varepsilon \partial_y u_{01} \\ u_{02} - 2a\lambda^2 \varepsilon \partial_y u_{02} \end{array} \right).
\]

By definition of \( w \) in (6), \( \partial_t w|_{t=0} = (\partial_t \rho)|_{t=0}, \varepsilon \partial_t (\rho u)|_{t=0} \). This implies that
\[
\partial_t \rho|_{t=0} = -\bar{\rho} (\nabla \cdot u_0) = 0,
\]
since \( u_0 \) is divergence free. This way,
\[
\partial_t u|_{t=0} = -\partial_x \left( \begin{array}{c} u_{01}^2 - 2a\lambda^2 \tau \partial_x u_{01} \\ \varepsilon u_{01} u_{02} - 2a\lambda^2 \varepsilon \partial_x u_{02} \end{array} \right) - \partial_y \left( \begin{array}{c} u_{01} u_{02} - 2a\lambda^2 \tau \partial_y u_{01} \\ u_{02} - 2a\lambda^2 \tau \partial_y u_{02} \end{array} \right).
\]
Thus,
\[ \Gamma \Sigma \| \partial_t \bar{w} \|^{2}_{\epsilon} + \varepsilon^6 \Delta \Sigma (\| \partial_t \bar{m} \|^{2}_{\epsilon} + \| \partial_t \bar{\xi} \|^{2}_{\epsilon}) + \varepsilon^8 \Theta \Sigma (\| \partial_t \bar{k} \|^{2}_{\epsilon} + \| \partial_t \bar{h} \|^{2}_{\epsilon}) \]
\[ + \frac{1}{\tau} \int_{0}^{T} (2 \Delta \Sigma - 1) \varepsilon^4 (\| \partial_t \bar{m} \|^{2}_{\epsilon} + \| \partial_t \bar{\xi} \|^{2}_{\epsilon}) + 2 \varepsilon^6 \Theta \Sigma (\| \partial_t \bar{k} \|^{2}_{\epsilon} + \| \partial_t \bar{h} \|^{2}_{\epsilon}) \ dt\]
\[ \leq \varepsilon^2 (\| u_{0} \|^{2}_{\epsilon} + \| \nabla u_{0} \|^{2}_{\epsilon} + \| \nabla^{2} u_{0} \|^{2}_{\epsilon}) \]
\[ + c(M) \int_{0}^{T} \| \partial_t \bar{w} \|^{2}_{\epsilon} + \varepsilon^6 (\| \partial_t \bar{m} \|^{2}_{\epsilon} + \| \partial_t \bar{\xi} \|^{2}_{\epsilon}) + \varepsilon^8 (\| \partial_t \bar{k} \|^{2}_{\epsilon} + \| \partial_t \bar{h} \|^{2}_{\epsilon}) \ dt, \]
where the last inequality follows form the Sobolev embedding theorem and from (61).

Similarly, taking the \(|\alpha|-\text{derivative}, for |\alpha| \leq s - 1, of (71) and multiplying by \( D^\alpha \bar{V} \) through the scalar product, we get:
\[ \frac{1}{2} \frac{d}{dt} (\Sigma D^\alpha \bar{V}, D^\alpha \bar{V})_{0} + (L \Sigma D^\alpha \bar{V}, \bar{D}^\alpha \bar{V})_{0} \leq \| (D^\alpha \partial_t N(w + \bar{w}), D^\alpha \bar{V})_{0}, \]
where
\[ \| (D^\alpha \partial_t N(w + \bar{w}), D^\alpha \bar{V})_{0} = \frac{1}{\tau} [(D^\alpha (2u_{1} \partial_t w_{2} - \varepsilon u_{2}^{2} \partial_t w_{1}), \varepsilon^2 \partial_t D^\alpha \bar{m}_{2})_{0} \]
\[ + (D^\alpha (u_{2} \partial_t w_{2} + u_{1} \partial_t w_{3} - \varepsilon u_{1} u_{2} \partial_t w_{1}), \varepsilon^2 \partial_t D^\alpha \bar{m}_{3} + \varepsilon^2 \partial_t D^\alpha \bar{\xi}_{2})_{0} \]
\[ + (D^\alpha (2u_{2} \partial_t w_{3} - \varepsilon u_{2}^{2} \partial_t w_{1}), \varepsilon^2 \partial_t D^\alpha \bar{\xi}_{3})_{0} | \]
\[ \leq c (\| u_{0} \| s_{-1}) \| \partial_t \bar{w} \|^{2}_{s_{-1}} + \frac{\epsilon^4}{2\tau} (\| \partial_t \bar{m} \|^{2}_{s_{-1}} + \| \partial_t \bar{\xi} \|^{2}_{s_{-1}}) \]
\[ \leq c(M) \| \partial_t \bar{w} \|^{2}_{s_{-1}} + \frac{\epsilon^4}{2\tau} (\| \partial_t \bar{m} \|^{2}_{s_{-1}} + \| \partial_t \bar{\xi} \|^{2}_{s_{-1}}), \]
where the last inequality follows from (61). Finally, we obtain:
\[ \Gamma \Sigma \| \partial_t \bar{w} \|^{2}_{s_{-1}} + \varepsilon^6 \Delta \Sigma (\| \partial_t \bar{m} \|^{2}_{s_{-1}} + \| \partial_t \bar{\xi} \|^{2}_{s_{-1}}) + \varepsilon^8 \Theta \Sigma (\| \partial_t \bar{k} \|^{2}_{s_{-1}} + \| \partial_t \bar{h} \|^{2}_{s_{-1}}) \]
\[ + \frac{1}{\tau} \int_{0}^{T} (2 \Delta \Sigma - 1) \varepsilon^4 (\| \partial_t \bar{m} \|^{2}_{s_{-1}} + \| \partial_t \bar{\xi} \|^{2}_{s_{-1}}) + 2 \varepsilon^6 \Theta \Sigma (\| \partial_t \bar{k} \|^{2}_{s_{-1}} + \| \partial_t \bar{h} \|^{2}_{s_{-1}}) \ dt\]
\[ \leq c \varepsilon^2 (\| u_{0} \|^{2}_{s_{-1}} + \| \nabla u_{0} \|^{2}_{s_{-1}} + \| \nabla^{2} u_{0} \|^{2}_{s_{-1}}) \]
\[ + c(M) \int_{0}^{T} \| \partial_t \bar{w} \|^{2}_{s_{-1}} + \varepsilon^6 (\| \partial_t \bar{m} \|^{2}_{s_{-1}} + \| \partial_t \bar{\xi} \|^{2}_{s_{-1}}) + \varepsilon^8 (\| \partial_t \bar{k} \|^{2}_{s_{-1}} + \| \partial_t \bar{h} \|^{2}_{s_{-1}}) \ dt. \]

(74)

**Lemma 4.3.** If Assumption 1 and 2 hold, then there exists a positive constant \( c \) such that:
\[ \| \partial_t \bar{w} \|^{2}_{s_{-1}} \leq c (\| \partial_t \bar{w} \|^{2}_{s_{-1}} + \varepsilon^6 (\| \partial_t \bar{m} \|^{2}_{s_{-1}} + \| \partial_t \bar{\xi} \|^{2}_{s_{-1}}) + \varepsilon^8 (\| \partial_t \bar{k} \|^{2}_{s_{-1}} + \| \partial_t \bar{h} \|^{2}_{s_{-1}})). \]

**Proof.** The proof of Proposition 1 can be adapted here with slight modifications. \( \square \)
We end the proof by applying the Gronwall inequality to (74) and using Lemma 4.3.

5. Convergence to the incompressible Navier-Stokes equations. Now we state our main result.

Theorem 5.1. Let $s > 3$. If Assumptions 1 and 2 hold, there exists a subsequence $W^s = (w^s, \varepsilon^2 m^s, \varepsilon^2 \xi^s, \varepsilon^2 k^s, \varepsilon^2 h^s)$, with $w^s = (\rho^s - \bar{\rho}, \varepsilon \rho^s u^s)$ and $\bar{\rho} > 0$, of the solutions to the vector-BGK model (22) with initial data (23) and $u_0 \in H^{s+1}(\mathbb{T}^2)$ in (2), such that

$$(\rho^s, u^s) \to (\bar{\rho}, u^{NS}) \text{ in } C([0,T^*], H^{s'}(\mathbb{T}^2)),$$

with $T^*$ in (67), $s - 1 < s' < s$, and where $u^{NS}$ is the unique solution to the incompressible Navier-Stokes equations in (1), with initial data $u_0$ above and $P^{NS}$ the hydrostatic pressure. Moreover,

$$\nabla \left( \frac{\rho^s - \bar{\rho}}{\varepsilon^2} \right) \to \nabla P^{NS} \text{ in } L^\infty H^{s-3}.$$

Proof. First of all, consider the previous bounds in (61), (62), (63) and (70):

$$\sup_{t \in [0,T^*]} \frac{||\rho^s - \bar{\rho}||}{\varepsilon} \leq M, \quad \sup_{t \in [0,T^*]} \frac{||\partial_t (\rho^s - \bar{\rho})||}{\varepsilon} \leq M_1, \quad (75)$$

$$\sup_{t \in [0,T^*]} ||\rho^s u^s||_s \leq N, \quad \sup_{t \in [0,T^*]} ||\partial_t (\rho^s u^s)||_{s-1} \leq N_1, \quad (76)$$

where $M, M_1, N, N_1$ are positive constants. The Lions-Aubin Lemma in [6] implies that, for $s - 1 < s' < s$,

$$\rho^s \to \bar{\rho} \text{ strongly in } C([0,T^*], H^{s'}(\mathbb{T}^2)),$$

and there exists $m^s$ such that

$$m^s = \frac{\rho^s u^s}{\rho^s} \to m^* \text{ strongly in } C([0,T^*], H^{s'}(\mathbb{T}^2)).$$

Notice also that $u^s = \frac{m^s}{\rho^s}$, where

$$\frac{1}{\rho^s} \to \frac{1}{\bar{\rho}} \text{ strongly in } C([0,T^*], H^{s'}(\mathbb{T}^2)),$$

since we can take $\bar{\rho}$ with $\rho^s > \frac{\bar{\rho}}{2}$ as in (66). Then

$$u^s = \frac{m^s}{\rho^s} \to m^* =: u^* \text{ strongly in } C([0,T^*], H^{s'}(\mathbb{T}^2)).$$

Now, consider system (19) in the following formulation:

$$\begin{cases}
\partial_t w^s + \partial_x m^s + \partial_y \xi^s = 0; \\
\varepsilon \partial_t m^s + \frac{1}{\varepsilon} \partial_x k^s = \frac{1}{\varepsilon} \left( \frac{A_1(w^s + \bar{w})}{\varepsilon} - \frac{m^s}{\varepsilon} \right), \\
\varepsilon \partial_t \xi^s + \frac{1}{\varepsilon} \partial_y h^s = \frac{1}{\varepsilon} \left( \frac{A_2(w^s + \bar{w})}{\varepsilon} - \frac{\xi^s}{\varepsilon} \right), \\
\varepsilon \partial_t k^s + \varepsilon \partial_x m^s = \frac{(2\omega w^s - K^s)}{\varepsilon^2}, \\
\varepsilon \partial_t h^s + \varepsilon \partial_y \xi^s = \frac{(2\omega w^s - H^s)}{\varepsilon^2},
\end{cases} \quad (77)$$

From (77) and $2a\lambda^2 \tau = \nu$ as in (11), it follows that

$$\begin{cases}
m^s = \frac{A_1(w^s + \bar{w})}{\varepsilon} - \nu \partial_x w^s + \varepsilon^2 \lambda^2 \tau^2 (\partial_x k^s + \partial_{xx} m^s) - \varepsilon^2 \tau \partial_t m^s; \\
\xi^s = \frac{A_2(w^s + \bar{w})}{\varepsilon} - \nu \partial_y w^s + \varepsilon^2 \lambda^2 \tau^2 (\partial_y h^s + \partial_{yy} \xi^s) - \varepsilon^2 \tau \partial_t \xi^s.
\end{cases}$$
Substituting the expansions above in the first equation of (77), we get the following expression:

\[
\frac{\partial_t w^\varepsilon + \frac{\partial_x A_1 (w^\varepsilon + \bar{w})}{\varepsilon} + \frac{\partial_y A_2 (w^\varepsilon + \bar{w})}{\varepsilon} - \nu \Delta w^\varepsilon}{\varepsilon} = \varepsilon^2 \tau \partial_{tx} m^\varepsilon + \varepsilon^2 \tau \partial_{ty} \xi^\varepsilon - \varepsilon^2 \lambda^2 \tau^2 (\partial_{txx} k^\varepsilon + \partial_{txxx} m^\varepsilon + \partial_{tuy} h^\varepsilon + \partial_{uyy} \xi^\varepsilon).
\]

We recall that \( W^\varepsilon = \Sigma \tilde{W}^\varepsilon \) by definition (30), with \( W^\varepsilon, \tilde{W}^\varepsilon \) in (21) and (30) respectively. This yields:

\[
\begin{align*}
\varepsilon^2 m^\varepsilon &= \varepsilon \sigma_1 \tilde{w}^\varepsilon + 2a \lambda^2 \varepsilon \tilde{m}^\varepsilon + \varepsilon^5 \sigma_1 k^\varepsilon; \\
\varepsilon^2 \xi^\varepsilon &= \varepsilon \sigma_2 \tilde{w}^\varepsilon + 2a \lambda^2 \varepsilon \tilde{m}^\varepsilon + \varepsilon^5 \sigma_2 \tilde{h}^\varepsilon; \\
\varepsilon^2 k^\varepsilon &= 2a \varepsilon \tilde{w}^\varepsilon + \varepsilon^5 \sigma_1 \tilde{m}^\varepsilon + 2a \varepsilon \tilde{h}^\varepsilon; \\
\varepsilon^2 h^\varepsilon &= 2a \varepsilon \tilde{w}^\varepsilon + \varepsilon^5 \sigma_2 \tilde{h}^\varepsilon + 2a \varepsilon \tilde{h}^\varepsilon.
\end{align*}
\]

From (69), (50)-(68) and (78) it follows that, for a fixed constant value \( c > 0 \),

\[
\tau \varepsilon^2 \| \partial_{tx} m^\varepsilon + \partial_{ty} \xi^\varepsilon - \lambda^2 \tau (\partial_{txx} k^\varepsilon + \partial_{txxx} m^\varepsilon + \partial_{tuy} h^\varepsilon + \partial_{uyy} \xi^\varepsilon) \|_{s=3} = O(\varepsilon^2),
\]

then

\[
\left\| \partial_t w^\varepsilon + \frac{\partial_x A_1 (w^\varepsilon + \bar{w})}{\varepsilon} + \frac{\partial_y A_2 (w^\varepsilon + \bar{w})}{\varepsilon} - \nu \Delta w^\varepsilon \right\|_{s=3} = O(\varepsilon^2).
\]

The last two equations and the previous bounds (75) and (76) yield:

\[
\left\| \partial_t (\rho^\varepsilon \mathbf{u}^\varepsilon) + \nabla \cdot (\rho^\varepsilon \mathbf{u}^\varepsilon \otimes \mathbf{u}^\varepsilon) + \frac{\nabla (\rho^\varepsilon - \bar{\rho})}{\varepsilon^2} - \nu \Delta (\rho \mathbf{u}^\varepsilon) \right\|_{s=3} = O(\varepsilon),
\]

and, in particular,

\[
\left\| \nabla (\rho^\varepsilon - \bar{\rho}) \right\|_{s=3} \leq c,
\]

i.e. there exists \( \nabla P^* \in L^\infty_t H^{s-3}_x \) such that

\[
\frac{\nabla (\rho^\varepsilon - \bar{\rho})}{\varepsilon^2} \rightharpoonup \nabla P^* \quad \text{in} \quad L^\infty_t H^{s-3}_x.
\]

Since \( \rho^\varepsilon \rightharpoonup \bar{\rho} \) and \( \mathbf{u}^\varepsilon \rightharpoonup \mathbf{u}^* \) in \( C([0, T^*], H^s(\mathbb{T}^2)) \), from \( \| \partial_t (\rho^\varepsilon \mathbf{u}^\varepsilon) \|_{s=1} \leq N_1 \) as in (76), it follows also that

\[
\partial_t (\rho^\varepsilon \mathbf{u}^\varepsilon) \rightharpoonup \bar{\rho} \mathbf{u}^* \quad \text{in} \quad L^\infty_t H^{s-3}_x,
\]

while

\[
\nabla \cdot (\rho^\varepsilon \mathbf{u}^\varepsilon \otimes \mathbf{u}^\varepsilon) \rightharpoonup \bar{\rho} \nabla \cdot (\mathbf{u}^* \otimes \mathbf{u}^*) \quad \text{in} \quad L^\infty_t H^{s-3}_x.
\]

Thus, from (80) we have the weak* convergence in \( L^\infty_t H^{s-3}_x \), i.e.

\[
\partial_t (\rho^\varepsilon \mathbf{u}^\varepsilon) + \nabla \cdot (\rho^\varepsilon \mathbf{u}^\varepsilon \otimes \mathbf{u}^\varepsilon) + \frac{\nabla (\rho^\varepsilon - \bar{\rho})}{\varepsilon^2} - \nu \Delta (\rho^\varepsilon \mathbf{u}^\varepsilon) \rightharpoonup \bar{\rho} \mathbf{u}^* + \nabla \cdot (\mathbf{u}^* \otimes \mathbf{u}^*) + \frac{\nabla P^*}{\bar{\rho}} - \nu \Delta \mathbf{u}^*.
\]

On the other hand, the first equation of (79) yields

\[
\partial_t (\rho^\varepsilon - \bar{\rho}) + \nabla \cdot (\rho^\varepsilon \mathbf{u}^\varepsilon) - \nu \Delta (\rho^\varepsilon - \bar{\rho}) = O(\varepsilon^2).
\]
Notice that $\|\partial_t (\rho^\varepsilon - \bar{\rho})\|_{s-1} = O(\varepsilon)$ and $\|\Delta (\rho^\varepsilon - \bar{\rho})\|_{s-2} = O(\varepsilon)$ thanks to (75), while

$$\rho^\varepsilon \to \bar{\rho} \quad \text{and} \quad u^\varepsilon \to u^* \quad \text{in} \quad C([0,T^*], H^s(T^2)),$$

This way, from (81) we finally recover the divergence free condition

$$\nabla \cdot u^* = 0.$$

6. Conclusions and perspectives. In this paper we proved the convergence of the solutions to the vector-BGK model to the solutions to the incompressible Navier-Stokes equations on the two dimensional torus $\mathbb{T}^2$. It is worth extending these results to the whole space and to a general bounded domain with suitable boundary conditions, but new ideas are needed to approach these cases. Rather than the more classical kinetic entropy approach, in this paper our main tool was the use of a constant right symmetrizer, which provides the conservative-dissipative structure introduced in [7], and allows us to get higher order energy estimates. Another interesting problem is to estimate the rate of convergence, in terms of the difference $\|u^* - u^{NS}\|_s$, with $u^*, u^{NS}$ the velocity fields associated with the BGK system in (5) and the Navier-Stokes equations in (1) respectively. This could be done by dealing with the non-explicit expression, in terms of the dependency with respect to the singular perturbation parameter, of the kinetic entropy associated with the model.

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E-mail address: roberta.bianchini@ens-lyon.fr
E-mail address: roberto.natalini@cnr.it