Is There Contextuality for a Single Qubit?

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Cabello and Nakamura have shown [A. Cabello, Phys. Rev. Lett. 90, 190401 (2003)] that the Kochen-Specker theorem can be applied to two-dimensional systems, if one uses the Positive Operator-Valued Measures. We show that the contextuality in their models is not of the Kochen-Specker type, but it is rather a result of not keeping track of the whole system on which the measurement is performed. This is connected to the fact that there is no one-to-one correspondence between the POVM elements and projectors on the extended Hilbert space and the same POVM element has to originate from two different projectors when used in Cabello’s and Nakamura’s models. Moreover, we propose a hidden-variable formulation of the above models.

For a long time there has been a debate whether a qubit is a truly quantum system\footnote{A. Cabello, Phys. Rev. Lett. 90, 190401 (2003)}. Although it may exist in a superposition of two orthogonal states it does not reveal the typical quantum oddities. The Kochen-Specker (KS)\footnote{A. Cabello, Phys. Rev. Lett. 90, 190401 (2003)} and Gleason\footnote{A. Cabello, Phys. Rev. Lett. 90, 190401 (2003)} theorems are valid only in at least three-dimensional Hilbert space and the Bell theorem\footnote{A. Cabello, Phys. Rev. Lett. 90, 190401 (2003)} applies to composite systems. Finally and most importantly there is a hidden-variable model describing every von Neumann measurement on a two-level quantum system\footnote{A. Cabello, Phys. Rev. Lett. 90, 190401 (2003)}. Only recently the special versions of KS and Gleason theorems have been presented for a single qubit. It was done by Cabello and Nakamura\footnote{A. Cabello, Phys. Rev. Lett. 90, 190401 (2003)} and by Busch\footnote{A. Cabello, Phys. Rev. Lett. 90, 190401 (2003)} respectively. The authors of both papers used the Positive Operator-Valued Measures (POVMs) to achieve their goals. The same year Aravind proposed how to generalize the CN method to obtain contextual POVMs for Hilbert spaces of arbitrary dimension\footnote{A. Cabello, Phys. Rev. Lett. 90, 190401 (2003)}.

The essence of the contextuality is that if an observable A is measured together with an observable B with which it commutes then it gives different outcome than if it is measured with an observable C with which it also commutes, and this fact has severe consequences when one wants to describe quantum mechanics within the framework of the hidden-variable models. If an outcome of a measurement of an observable was preassigned before the measurement, it would have to depend on the choice of the other observables co-measured with the observable of interest.

In the original KS theorem A, B and C are represented by projectors, i.e. by operators with eigenvalues 0 and 1, which makes them natural yes-no operators. According to quantum mechanics, from a set of mutually orthogonal projectors making up a measurement exactly one brings the value 1 and the others bring 0. The goal of KS theorem is to show that the pre-assignment of outcomes to a group of measurements inevitably leads to at least one measurement wrongly assigned.

There is no contextuality in von Neumann measurements for a two-dimensional system (a qubit). This is because the choice of the first projector automatically defines the second one and there is no freedom in choosing which one to measure together with the first projector. However, this freedom is restored if one uses POVM measurements instead of von Neumann ones. One can have as many POVM elements as one wants. The problem is that as for von Neumann measurements it is natural to ask whether an outcome of a measurement of an observable is somehow encoded in the state of the system prior to the measurement, for a POVM it is not that simple. To perform a POVM one has to measure an observable of a composite system — the qubit and an auxiliary quantum system (an ancilla) — thus fixing the outcome of a POVM prior to the measurement corresponds to fixing the outcome of the measurement of the observable on an extended Hilbert space. The important thing is that one should not assume that the outcome of a POVM is encoded in the qubit alone.

The projection postulate states that an act of a measurement defines the post measurement state of the system, therefore successive measurements of the same type should bring the same outcome. It is true in the case of the same von Neumann measurements because the second measurement of the system reveals the same outcome as the first one. On the other hand POVMs are not repeatable and the second measurement may bring a different outcome than the first one. Even worse, in order to perform the same POVM one has to reset the state of the ancilla, which results in preparation of the new state of the whole system. This is another example which shows that it is hard to speak of the KS theorem held for POVMs.

In this Letter we show that the CN model is contextual because it does not give the complete information about the measurement. More precisely the same POVM element acting on the original qubit can correspond to different projectors on the extended Hilbert space. We also give an example of CN POVM, which can be described by non-contextual hidden-variable model of the system and the ancilla.

Let us now briefly describe the contextual set of mea-
These sets have to be 0. Now, the fifth set is made of elements which have already been assigned 0, since each element occurs in exactly two sets. Thus one cannot assign a value 1 to any POVM element in the fifth set. This ends the proof.

Nakamura proposed a more economic proof using the symmetry of a regular hexagon (see Fig.1 bottom). Now there are only three \( \pm \) pairs grouped in three POVM measurements

\[
\{ \varepsilon_{A_+}, \varepsilon_{A_-}, \varepsilon_{B_+}, \varepsilon_{B_-} \}, \quad \{ \varepsilon_{A_+}, \varepsilon_{A_-}, \varepsilon_{C_+}, \varepsilon_{C_-} \},
\{ \varepsilon_{B_+}, \varepsilon_{B_-}, \varepsilon_{C_+}, \varepsilon_{C_-} \},
\]

and all elements are given by

\[
\varepsilon_i = \frac{1}{2} |\psi_i\rangle\langle\psi_i|.
\]

By assigning 1 to any element we assign values to two sets, and therefore we assign 0 to all other elements. Since each element occurs exactly twice, there is always one set in which all elements are assigned 0.

We now show that the contextuality in both models comes from non-unique extension of POVM elements. More precisely, POVM measurement is performed as von Neumann measurement on the extended Hilbert space. The restriction of von Neumann projectors leads to POVM elements. As shown above, assignment of a physical reality to a POVM element corresponds to the assignment of a physical reality to a projector on the extended Hilbert space. The main point in our argument is that two different projectors can lead to the same POVM element and thus the contextuality comes from not keeping track of the whole system.

First, we briefly describe the relation between the POVM and von Neumann measurements. In order to perform POVM measurements one extends the Hilbert space by adding an ancilla and performs von Neumann measurement on a higher dimensional Hilbert space (see [9] and references therein). Then POVM elements \( \varepsilon_i \) are given by

\[
\varepsilon_i = \text{Tr}_A((\rho_A \otimes I)P_i).
\]

where \( \rho_A \) is the state of the ancilla, \( I \) is the identity on a qubit Hilbert space, \( P_i \) is a von Neumann projector on the whole Hilbert space and the right hand side is traced over the ancilla. One can always assume that the qubit and the ancilla are unentangled, because of the identity

\[
\text{Tr}((U\rho U^\dagger)P_i) = \text{Tr}(\rho(U^\dagger P_i U)).
\]

which corresponds to different choice of projectors. Our task is to examine the relation between projectors \( P_i \)'s that generate the CN POVM measurements. More precisely, we want to know if a particular POVM element which is in two different sets corresponds to the same projector. For example, if the ancilla is a qubit in the

FIG. 1: The structure of CN POVMs. Twenty vertices of Cabello’s dodecahedron (top) and six vertices of Nakamura’s hexagon (bottom).
We have introduced additional projectors orthogonal. From the first two sets (7) we see that both

$$\{P_{A_+}, P_{A_-}, P_{B_+}, P_{B_-}, P_{i}\},$$

$$\{P_{A_+}, P_{A_+}, P_{C_+}, P_{C_-}, P_{2}\},$$

$$\{P_{B_+}, P_{B_-}, P_{C_+}, P_{C_-}, P_{3}\}.$$  

We have introduced additional projectors $P_1$, $P_2$ and $P_3$ in order to have the resolution of the identity. Because they should not contribute to POVM they should satisfy:

$$\text{Tr}_A((\rho_A \otimes I) P_i) = 0$$  

for $i = 1, 2, 3$. All projectors in each set are mutually orthogonal. From the first two sets (7) we see that both $P_{A_+}$ and $P_{A_-}$ are orthogonal to $P_{B_+}$, $P_{B_-}$, $P_{C_+}$, $P_{C_-}$. Since the projectors in a set span the whole Hilbert space, from the third set we have that $P_{A_+}$ and $P_{A_-}$ have to be confined in $P_3$, but $P_3$ does not contribute to POVM (as we defined in (8)), thus $P_{A_+}$ and $P_{A_-}$ cannot contribute to POVM too. However, $\text{Tr}_A((\rho_A \otimes I) P_{A_+}) = \epsilon_{A_+}$ which is a required contradiction.

The contradiction vanishes if for the same POVM element one uses different projectors in different sets. For example, if $\epsilon_{B_+}$ in the first POVM arose from $P_{B_+}$ and in the second POVM from $P_{B_+}$ there would be no contradiction in the Nakamura’s case. However, this change cancels the contextuality in the sets (7).

A similar contradiction can be obtained for the Cabello’s model. We introduce five projectors $P_1, P_2, \ldots, P_5$ which should have the property (9). The measurements on the extended Hilbert space give:

$$\{P_{A_+}, P_{A_-}, P_{C_+}, P_{C_-}, P_{i}, P_{j}, P_{k}, P_{l}, P_{m}, P_{n}, P_{i}\},$$

$$\{P_{A_+}, P_{A_+}, P_{D_+}, P_{D_-}, P_{j}, P_{k}, P_{l}, P_{m}, P_{n}, P_{i}\},$$

$$\{P_{B_+}, P_{B_-}, P_{D_+}, P_{D_-}, P_{F_+}, P_{F_-}, P_{j}, P_{k}, P_{l}, P_{m}, P_{n}, P_{i}\},$$

$$\{P_{C_+}, P_{C_-}, P_{E_+}, P_{E_-}, P_{F_+}, P_{F_-}, P_{G_+}, P_{G_-}, P_{i}\}.$$  

From the first two sets we see that $P_{A_+}$ and $P_{A_-}$ are orthogonal to $P_{C_+}, P_{C_-}, P_{i}, P_{j}, P_{k}, P_{l}, P_{m}, P_{n}, P_{D_+}. P_{D_-}, P_{G_+}, P_{G_-}.$ From the last three sets we can conclude that $P_{A_+}$ and $P_{A_-}$ have to be confined in each of the three subspaces spanned by

$$P_{B_+} + P_{B_-} + P_{E_+} + P_{E_-} + P_3,$$

$$P_{E_+} + P_{E_-} + P_{F_+} + P_{F_-} + P_3.$$  

From the last three sets we can also conclude that

$$P_{B_+} + P_{B_-} + P_{E_+} + P_{E_-},$$

$$P_{E_+} + P_{E_-} + P_{F_+} + P_{F_-},$$

thus $P_{A_+}$ and $P_{A_-}$ have to be confined in the subspace spanned by $P_3 + P_3 + P_3$, which brings us again to a contradiction.

The above reasoning shows that the contextuality in CN POVMs is a result of not keeping track of the ancilla, i.e. if one considers a qubit and an ancilla together, the sets of projectors are not contextual, but if one forgets what happens to the ancilla, the projectors become contextual POVMs. One may wonder if it is the same kind of contextuality as the one considered by KS (for different types of contextuality see the paper by Spekkens [10]). The problem is analogous to the one we present below. Consider two yes-no questions: 1) Is it raining in Poland? 2) Is it raining in the USA? It is obvious that the corresponding two answers may be different, but if we erase the ends of both questions we will have twice: Is it raining? If the answers to the questions are assigned before the erasure, we can have the same question with two different answers.

Let us now present an example of CN POVM measurement for which one can construct non-contextual hidden-variable model. Eqs. (2) and (4) define POVM elements as projectors multiplied by the same constant $(1/2$ for the Nakamura’s model and $1/4$ for the Cabello’s model). Since $\epsilon_{X_n}$ is orthogonal to $\epsilon_{X_n},$ without the constant the pair would give a von Neumann measurement on a qubit. Because of this symmetry CN POVM measurements may be implemented by performing von Neumann measurement on the ancilla and then, depending on the outcome, performing one of two von Neumann measurements on the qubit in the Nakamura’s case, or one of four in the Cabello’s case. The outcome of the first measurement defines which $\pm$ pair is going to be measured next. It is enough to prepare the ancilla in the superposition

$$|\phi\rangle = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} |i\rangle,$$  

where $\{i\}$ is an orthonormal basis and $N = 2$ for the Nakamura’s model ($N = 4$ for the Cabello’s model), and then to measure it in the basis $\{i\}$. There is always a hidden-variable model of a measurement in one basis (in our case $\{i\}$). For example, let $A \in \{0, 1\}$
\((\Lambda \in \{0,1,2,3\})\) be a discrete hidden-variable for the Nakamura’s model (the Cabello’s model). If the total description of the ancilla is given by \(\Lambda\), the state after measurement is \(|\Lambda\rangle\) with the probability \(1/2\) \((1/4)\) if \(\Lambda\) is uniformly distributed. For the next measurement which is made on the qubit, we can describe it within the framework of the Bell model \([3]\). The corresponding projectors on the qubit Hilbert space are:

\[ V = \frac{I + \vec{v} \cdot \vec{\sigma}}{2}, \quad (13) \]

where \(\vec{v}\) is the real unit vector pointing at the \(V\) direction on the Bloch sphere and \(\vec{\sigma}\) is the vector of Pauli matrices. Similarly, we define the state of a qubit as

\[ |n\rangle\langle n| = \frac{I + \vec{n} \cdot \vec{\sigma}}{2}, \quad (14) \]

with \(\vec{n}\) being a unit vector. The hidden-variable is another unit vector \(\vec{m}\) which is uniformly distributed over all directions on the Bloch sphere. The outcome of the measurement of a projector \((13)\) is given by

\[ v_n(V) = 1, \quad \text{if} \quad (\vec{m} + \vec{n}) \cdot \vec{v} > 0, \]

\[ v_n(V) = 0, \quad \text{if} \quad (\vec{m} + \vec{n}) \cdot \vec{v} < 0. \quad (15) \]

Hidden-variables allow a pre-assignment of outcomes to both von Neumann measurements. In both cases the value 1 is assigned to exactly one element, whereas the rest of elements are assigned 0. The two successive measurements on the ancilla and the qubit are equivalent to one measurement on the whole system using von Neumann projectors of the form \(|i\rangle\langle i|_A \otimes V_{\Lambda,Q}\), thus we can multiply the values assigned to the outcomes of the first measurement by the values assigned to the outcomes of the second measurement. As a result we obtain a group of measurements which is definitely non-contextual, because in every measurement there is exactly one element assigned to the value 1.

It is instructive to show how different extensions of POVM elements appear in the Nakamura’s model. First we consider the first set of POVM elements. The outcome \(|0\rangle\) of a measurement performed on the ancilla’s state (which was prepared in equally weighted superposition of \(|0\rangle\) and \(|1\rangle\)) tells us that we should perform a von Neumann measurement on a qubit along the \(A_x\) direction, while the outcome \(|1\rangle\) tells us that we should perform a von Neumann measurement on the \(A_z\) direction. Now for the second set if we want to have the same extension of POVM elements the outcome \(|0\rangle\) has to lead to a von Neumann measurement along the \(A_x\) direction and then the outcome \(|1\rangle\) leads to a von Neumann measurement along the \(C_x\). However in the third set the outcome \(|1\rangle\) cannot lead to a von Neumann measurements in both \(B_x\) and \(C_x\) directions. One of them has to be conditioned on the outcome \(|0\rangle\). Thus for example POVM elements \(\varepsilon_{B \perp}\) have different extensions.

We have shown that the contextuality in the Cabello-Nakamura’s model is not of the Kochen-Specker type. It is rather a result of not keeping track of the whole system on which the measurement is performed. We proved that in the CN model there is no one-to-one correspondence between POVM elements and projectors on the extended Hilbert space. It means that some POVM elements appearing in two different POVMs have to originate from two distinct von Neumann projectors. Therefore, there is nothing surprising in the fact that they bring different outcomes when measured in two different POVMs. Moreover, POVMs are probabilistic measurements in the sense that subsequent measurements of the same type may bring different outcomes. The basic assumption of all hidden-variable models is that an outcome of every measurement is pre-encoded in the state of the system, and since in our case the whole system is not only the qubit, it is incorrect to assume that the outcome of the POVM is encoded only in the qubit’s state. One should rather speak of preassigning an outcome of a von Neumann measurement to the whole system — a qubit and an ancilla (which is not considered in the CN model).

It would be interesting to find contextual POVMs for a qubit with a one-to-one correspondence between their elements and von Neumann projectors or to show that every contextual POVM on a qubit cannot be obtained unless we give up one-to-one correspondence. We conjecture that if the corresponding von Neumann projectors form a contextual group of measurements one may obtain a contextual one-to-one POVMs.

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