On the boundary-value problems and the validity of the Post constraint in modern electromagnetism

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Abstract

We recall that the theory of electromagnetism consists of three building blocks: (a) the inhomogeneous Maxwell equations for the electric and magnetic excitations \( \mathbf{D}, \mathbf{H} \) (which reflects charge conservation), (b) the homogeneous Maxwell equations for the electric and magnetic field strengths \( \mathbf{E}, \mathbf{B} \) (which reflects flux conservation), and (c) the constitutive relation between \( \mathbf{D}, \mathbf{H} \) and \( \mathbf{E}, \mathbf{B} \). In the recent paper [1], Lakhtakia proposed to change the standard boundary conditions in electrodynamics in order to exclude certain constitutive parameters. We show that this is inadmissible both from the macroscopic and the microscopic points of view.

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I. INTRODUCTION

Let us consider magnetoelectric matter with the constitutive relation

\[ D = \varepsilon \varepsilon_0 E + \alpha B, \]
\[ H = \frac{1}{\mu \mu_0} B - \alpha E. \]  

Here \( \varepsilon_0 \) and \( \mu_0 \) are the electric and magnetic constants of the vacuum, \( \varepsilon \) and \( \mu \) are the permittivity and permeability of the matter, and \( \alpha \) is the axion (Tellegen) parameter.

In a recent discussion of the electrodynamics in magnetoelectric media of such a special type, Lakhtakia [1] proposed to replace the standard boundary (jump) conditions on an interface between two media characterized by different values of the constitutive parameters \( \varepsilon, \mu, \alpha \) by new boundary conditions which do not contain the axion parameter \( \alpha \).

The motivation for such a replacement is as follows. When the medium is completely homogeneous, it is easy to verify that Maxwell’s differential equations do not contain the parameter \( \alpha \) when (1)-(2) is substituted into these equations. However, for the inhomogeneous situation, when there are two spatial domains with different values of \( \alpha \), for example, one should implement the boundary conditions on the surface that separates the two domains.

The use of the standard boundary conditions allows for the influence of the nontrivial axion field \( \alpha \) on the physical processes (in particular, on waves propagating through the interface). Lakhtakia [1] modified the jump conditions such that this effect of \( \alpha \) is removed.

The authors of a comment [2] gave arguments against such an arbitrary modification of the boundary conditions. However, Lakhtakia [3] disagreed, referring to the difference between the microscopic and macroscopic approaches to electrodynamics. Here we show that the claim of Lakhtakia is incorrect, both macroscopically and microscopically.

II. MICROSCOPIC AND MACROSCOPIC ELECTRODYNAMICS

We start from the microscopic Maxwell equations (see eqs. (1)-(4) of [1]) that, after the spatial averaging, yield the equations of macroscopic electrodynamics:

\[ \varepsilon_0 \nabla \cdot E = \rho, \]
\[ \mu_0^{-1} \nabla \times B - \varepsilon_0 \dot{E} = J, \]  
\[ \nabla \cdot B = 0, \]
\[ \nabla \times E + \dot{B} = 0. \]
Here we agree completely with Lakhtakia, namely that all four dynamical equations contain only two fields, $E$ and $B$, and that all four equations hold in vacuum as well as in matter. There is however, an important difference: whereas in vacuum the electric charge and current densities $\rho$ and $J$ are defined by the *free* charges only, in matter the sources $\rho$ and $J$ are sums of both *free* and *bound* charges:

$$\rho = \rho^f + \rho^b, \quad J = J^f + J^b. \quad (5)$$

The bound sources are related to the polarization and magnetization which provide an averaged description of the physical response of the microscopic constituents of matter exposed to the action of the electric and magnetic fields:

$$\rho^b = -\nabla \cdot P, \quad J^b = \nabla \times M + \dot{P}. \quad (6)$$

Substituting (5) into (3), and making use of (6), we recast the inhomogeneous Maxwell equations into

$$\nabla \cdot D = \rho^f, \quad \nabla \times H - \dot{D} = J^f, \quad (7)$$

with the electric and magnetic excitation fields defined by

$$D = \varepsilon_0 E + P, \quad H = \mu_0^{-1} B - M. \quad (8)$$

### III. BOUNDARY CONDITIONS

Let us consider the case when space is divided into two domains by a plane boundary surface $S$. When matter in each of the domains is homogeneous, it is convenient technically to study the electrodynamical processes (waves, in particular) in the two half-spaces separately. However, if we want the physical picture in the whole space, we will need boundary (jump) conditions for the electric and magnetic fields across the surface $S$.

We derive here the boundary conditions in a slightly different way than it is done in textbooks. Namely, we will do it directly for the fundamental fields $E$ and $B$ and not for the excitations. Since each of the fields is a vector, we expect three boundary conditions for each field (one for the normal projection and two for the tangential components). Indeed, with the help of the usual technique by integrating the Maxwell equations (3)-(4) in a thin region in a small vicinity of the surface $S$, we derive the conditions for the fields at the
boundary:

\[ \mathbf{E}_n^{(2)} - \mathbf{E}_n^{(1)} = \varepsilon_0^{-1} \rho_s, \quad \mathbf{E}_\tau^{(2)} - \mathbf{E}_\tau^{(1)} = 0, \quad (9) \]
\[ \mathbf{B}_\tau^{(2)} - \mathbf{B}_\tau^{(1)} = \mu_0 J_s, \quad \mathbf{B}_n^{(2)} - \mathbf{B}_n^{(1)} = 0. \quad (10) \]

The notation is obvious: the subscripts \( n \) and \( \tau \) denote the normal and tangential projections, whereas the superscripts \((1)\) and \((2)\) label the half-space domains.

The crucial feature of the boundary conditions across \( S \) is the presence of the surface charge and current densities \( \rho_s \) and \( J_s \) in (9) and (10). These surface densities have the form

\[ \rho_s = \rho_s^f + \rho_s^b, \quad J_s = J_s^f + J_s^b, \quad (11) \]

thus including on an equal footing both the surface density of the free sources \( \rho_s^f \) and \( J_s^f \) and the surface density of the bound sources \( \rho_s^b \) and \( J_s^b \). The physical origins of the two types of the surface charges and currents on \( S \) are somewhat different. Whereas \( \rho_s^f \) and \( J_s^f \) describe the possible presence of the free sources right at the boundary (prepared under the conditions of an experiment, for example), the surface sources \( \rho_s^b \) and \( J_s^b \) arise from the fact that matter (from the microscopic point of view) has different electromagnetic properties in the two half-spaces. As a result, the polarization and magnetization, while being continuous in each separate homogeneous domain, are discontinuous across the boundary \( S \).

We can easily derive the relation between the bound surface sources on \( S \) and the jump of the polarization and magnetization across \( S \). Integrating the equations (6) in a infinitely thin neighbourhood including \( S \) (in the same way like for the Maxwell equations), we derive

\[ \mathbf{P}_n^{(2)} - \mathbf{P}_n^{(1)} = -\rho_s^b, \quad \mathbf{M}_\tau^{(2)} - \mathbf{M}_\tau^{(1)} = J_s^b. \quad (12) \]

Substituting (11) and (12) into (9)-(10), and taking into account (8), we recast the boundary conditions in a more familiar form:

\[ \mathbf{D}_n^{(2)} - \mathbf{D}_n^{(1)} = \rho_s^f, \quad \mathbf{H}_\tau^{(2)} - \mathbf{H}_\tau^{(1)} = J_s^f. \quad (13) \]

IV. AXION (TELLEGEN) MAGNETOELECTRIC MEDIUM AND THE POST CONSTRAINT

Up to this point, we were quite general. Now, let us specialize to the case of the magnetoelectric matter with the constitutive relation \((1)-(2)\). The permittivity, permeability,
and the axion (Tellegen) field in the two half-space domains are \( \varepsilon_1, \mu_1, \alpha_1 \) and \( \varepsilon_2, \mu_2, \alpha_2 \), respectively. Using (11)-(2) in (13) and (9)-(10), we find the boundary conditions

\[
\varepsilon_0 \left( \varepsilon_2 \mathbf{E}^{(2)}_n - \varepsilon_1 \mathbf{E}^{(1)}_n \right) + (\alpha_2 - \alpha_1) \mathbf{B}^{(1)}_n = \rho^f_s, \quad (14)
\]

\[
\mu_0^{-1} \left( \mu_2^{-1} \mathbf{B}^{(2)}_v - \mu_1^{-1} \mathbf{B}^{(1)}_v \right) - (\alpha_2 - \alpha_1) \mathbf{E}^{(1)}_v = \mathbf{J}^f_s. \quad (15)
\]

As we see, the axion field shows up explicitly. The analysis of the wave propagation in such a magnetoelectric medium reveals the influence of the axion field on the reflected and transmitted wave [2, 4] (see also [5, 6]). One can use this effect to measure the axion (Tellegen) field [7].

Lakhtakia’s proposal to change the standard boundary conditions in order to eliminate the contribution of axion (thus proving the so called Post constraint [4]) is physically unsubstantiated. One cannot treat the boundary conditions as a kind of supplementary conditions that one can choose arbitrarily (like the the fixing of the gauge, for example). The boundary conditions (9)-(10) and (13) are the Maxwell equations written in a different (integral and not differential) form for a specific region of space (i.e., for the infinitely thin neighbourhood of the boundary surface \( S \)). By changing a boundary condition, Lakhtakia actually changes the physical laws, namely, the Maxwell equations at the interface between the two media.

An alternative (equivalent) explanation of the presence of the axionic terms in (14)-(15) is as follows. Let us look at the right-hand sides of these equations. The free surface density sources \( \rho^f_s \) and \( \mathbf{J}^f_s \) arise because, originally, in the Maxwell equations (3) there is a \( \delta \)-function distribution of the free charge and current densities of the type \( \rho^f \cong \rho^f_s \delta(\mathbf{x} - \mathbf{x}(S)) \) (and similarly for the free current density). The same applies also to the bound surface density sources. Indeed, the axion field, considered on the whole space, is a step function

\[
\alpha(x) = \begin{cases} 
\alpha_1, & \text{for } x \in 1st \ domain \\
\alpha_2, & \text{for } x \in 2nd \ domain 
\end{cases} \quad (16)
\]

Using this function in (6), we find a \( \delta \)-function contribution from the magnetoelectric piece in the polarization and magnetization, namely, \( \rho^b \cong (\alpha_1 - \alpha_2) B_n \delta(\mathbf{x} - \mathbf{x}(S)) \) (and a similar expression for the bound current density).

When these two delta-functions (one for the free and another for the bound sources) are substituted into the Maxwell equations (3), integration in the infinitesimally thin region around the boundary \( S \) yields the two contributions to the right-hand side of (9)-(10). The
free source delta-function is responsible for the surface density terms $\rho^f_s$ and $\mathbf{J}^f_s$, whereas the bound source delta-function gives rise exactly to the axion terms in (14)-(15). A similar argument was used in the previous comment [2]. In his response [3], Lakhtakia claimed that a microscopic approach and a “homogenization” of the fields might support his proposal. However, here we have analysed the problem starting from a microscopic viewpoint. It is unclear how any kind of “homogenization” can eliminate a delta-function at the boundary between the two domains filled with different matter.

If we take any point at the boundary $S$ and perform averaging and “homogenization” in an arbitrarily small neighbourhood of this point, we will necessarily find two portions of space to the left and to the right of $S$, in which the electric and magnetic properties are homogeneous within the respective portions of the neighbourhood, but are not homogeneous and even not continuous across $S$. There just cannot be any “homogenization” across the boundary since $S$ divides the two materials with essentially different physical properties. For example, we can have vacuum in the first half-space and a magnetoelectric medium in the second half-space. The vacuum is not polarized and magnetized. In contrast, the magnetoelectric medium becomes electrically polarized in a magnetic field and/or becomes magnetized in electric field, with the parameter $\alpha$ determining such polarization and magnetization. As a result, $\mathbf{P}$ and $\mathbf{M}$ are both discontinuous across $S$, and “homogenization” cannot change this fact.

V. CONCLUSION

Classical macroscopic electrodynamics (which can be consistently derived with the help of the spatial averaging from the microscopic electrodynamics) consists of three building blocks [8]: (a) the inhomogeneous Maxwell equations (3) (which reflects charge conservation), (b) the homogeneous Maxwell equations (4) (which reflects flux conservation), and (c) the constitutive relation between the electromagnetic field excitations ($\mathbf{D}, \mathbf{H}$) and the electromagnetic field strength ($\mathbf{E}, \mathbf{B}$). The latter encodes the response of the medium to the action of the electric and magnetic fields in terms of the polarization and magnetization fields that are related to the bound charge and current source densities. Although some constitutive parameter (like the axion field $\alpha$) may drop out of the differential equations [3], it still enters the constitutive law, reflecting the state of polarization of matter induced by
the magnetic field and/or the state of magnetization induced by the electric field.

A modification of the boundary (jump) conditions across the surface $S$ between the two different media, proposed in [1], is physically inadmissible because such a change of the boundary conditions amounts to a change of the Maxwell equations. Moreover, the averaging and “homogenization” arguments cannot eliminate the discontinuous behavior of the polarization and magnetization across the boundary $S$ which is manifest in the delta-function like contributions both to the sources of the free charge and the bound charge. The Post constraint is unphysical and invalid.

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