AVERAGES OF THE MÖBIUS FUNCTION ON SHIFTED PRIMES

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Abstract. It is a folklore conjecture that the M"obius function exhibits cancellation on shifted primes; that is, \( \sum_{p \leq X} \mu(p + h) = o(\pi(X)) \) as \( X \to \infty \) for any fixed shift \( h > 0 \). We prove the conjecture on average for shifts \( h \leq H \), provided \( \log H / \log \log X \to \infty \). We also obtain results for shifts of prime \( k \)-tuples, and for higher correlations of M"obius with von Mangoldt and divisor functions. Our argument combines sieve methods with a refinement of Matomäki, Radziwiłł, and Tao’s work on an averaged form of Chowla’s conjecture.

1. Introduction

Let \( \mu : \mathbb{N} \to \{-1, 0, +1\} \) denote the M"obius function, defined multiplicatively on primes \( p \) by \( \mu(p) = -1 \) and \( \mu(p^k) = 0 \) for \( k \geq 2 \). Many central results in number theory may be formulated in terms of averages of the M"obius function. Notably, the prime number theorem is equivalent to the statement \( \sum_{n \leq X} \mu(n) = o(X) \), and \( \sum_{n \leq X} \mu(n) = O(X^\theta) \) for all \( \theta > \frac{1}{2} \) is equivalent to the Riemann hypothesis.

Clearly \( \mu(p) = -1 \) gives \( \sum_{p \leq X} \mu(p) = -\pi(X) \), but less is known about the M"obius function on shifted primes. It is a folklore conjecture that \( \sum_{p \leq X} \mu(p + h) = o(\pi(X)) \) for any fixed shift \( h > 0 \) (this appeared in print at least since Murty–Vatwani [13, (1.2)], and the case \( h = 1 \) is Problem 5.2 of [16]). We answer an averaged version of this conjecture with quantitative bounds.

Theorem 1.1. If \( \log H / \log_2 X \to \infty \) as \( X \to \infty \), then

\[
(1.1) \quad \sum_{h \leq H} \left| \sum_{p \leq X} \mu(p + h) \right| = o(H \pi(X)).
\]

Further if \( H = X^\theta \) for some \( \theta \in (0, 1) \), then for all \( \delta > 0 \)

\[
\sum_{h \leq H} \left| \sum_{p \leq X} \mu(p + h) \right| \ll \delta \frac{H \pi(X)}{(\log X)^{1/3 - \delta}}.
\]

An immediate consequence is that \( \sum_{p \leq X} \mu(p + h) \) exhibits cancellation for all but \( o(H) \) values of \( h \leq H = (\log X)^\psi(X) \) provided \( \psi(X) \to \infty \).

Remark 1.2. The weaker qualitative cancellation (1.1) in the longer regime \( H = X^\theta \) may be obtained more directly, using a recent Fourier uniformity result of Matomäki-Radziwiłł-Tao [12]. See Theorem 1.8 for details.

Theorem 1.1 is an illustrative example within a broader class of correlations that may be handled by the methods in this paper, see Theorem 6.2 for the full technical result. Below we highlight some further example correlations of general interest.
1.1. **Higher correlations.** The influential conjectures of Chowla [1] and Hardy–Littlewood [4] assert that for any fixed tuple $\mathcal{H} = \{h_1, \ldots, h_k\}$ of distinct integers,

$$\sum_{n \leq X} \mu(n + h_1) \cdots \mu(n + h_k) = o(X),$$

$$\sum_{n \leq X} \Lambda(n + h_1) \cdots \Lambda(n + h_k) = \mathcal{G}(\mathcal{H})X + o(X),$$

for the singular series $\mathcal{G}(\mathcal{H}) = \prod_p \frac{(1-\psi_p/p)}{(1-1/p)}$, where $\psi_p = \#\{h_1, \ldots, h_k(\text{mod } p)\}$. Both conjectures remain open for any $k \geq 2$.

We establish an average result for Chowla–Hardy–Littlewood correlations.

**Theorem 1.3.** Suppose $\log H/\log_2 X \to \infty$ as $X \to \infty$, and define $\psi_\delta$ by

$$(1.2) \quad \psi_\delta(X) = \min \left\{ \frac{\log_2 X}{\log H}, (\log X)^{1/3-\delta} \right\}.$$

Then for any $m, k \geq 1$, and fixed tuple $A = \{a_1, \ldots, a_k\}$ of distinct integers, we have

$$(1.3) \quad \sum_{h_1, \ldots, h_m \leq H} \left| \sum_{n \leq X} \prod_{j=1}^m \mu(n + h_j) \prod_{i=1}^k \Lambda(n + a_i) \right| \ll_{m, \delta, A} \frac{XH^m}{\psi_\delta(X)^m}.$$

It is worth emphasizing particular aspects of this result. First, (1.3) holds for an arbitrary fixed prime $k$-tuple. We must average over at least $m \geq 1$ copies of $\mu$ in order to obtain cancellation. Notably, the cancellation becomes quantitatively stronger for larger $m$, e.g. error savings $(\log X)^{m/3-\delta}$. For the case $m = 0, k = 2$, Matomäki-Radziwiłł-Tao [10] handled binary correlations $\sum_{n \leq X} \Lambda(n)\Lambda(n+h)$ on average with error savings $(\log X)^4$, though in the much larger regime $H \geq X^{8/33+\varepsilon}$.

In particular, the Chowla conjecture holds on average along the subsequence of primes.

**Corollary 1.4.** Suppose $\log H/\log_2 X \to \infty$ as $X \to \infty$. Then for any $m \geq 1$,

$$(1.4) \quad \sum_{h_1, \ldots, h_m \leq H} \left| \sum_{p \leq X} \mu(p + h_1) \cdots \mu(p + h_m) \right| = o_m(\pi(X)H^m).$$

Moreover, using Markov’s inequality we may obtain qualitative cancellation for almost all shifts, with arbitrary log factor savings in the exceptional set.

**Corollary 1.5.** Suppose $\log H/\log_2 X \to \infty$ as $X \to \infty$. Then for any $A > 0$,

$$\sum_{p \leq X} \mu(p + h_1) \cdots \mu(p + h_m) = o_m(\pi(X)),$$

for all except $O_A(H^m(\log X)^{-A})$ shifts $(h_1, \ldots, h_m) \in [1, H]^m$.

These results build on earlier work of Matomäki-Radziwiłł-Tao [9], who established an average form of Chowla’s conjecture,

$$(1.5) \quad \sum_{h_1, \ldots, h_m \leq H} \left| \sum_{n \leq X} \mu(n + h_1) \cdots \mu(n + h_m) \right| = o_m(XH^m),$$

for any $H = H(X) \to \infty$ arbitrarily slowly. Whereas, our results require the faster growth $H = (\log X)^{\psi(X)}$ with $\psi(X) \to \infty$ arbitrarily slowly.
1.2. **Correlations with divisor functions.** Consider fixed integers \(a \geq 1, k \geq l \geq 2\). The well studied correlation of two divisor functions \(d_k, d_l\) is predicted to satisfy
\[
\sum_{n \leq X} d_k(n + h) d_l(n) = C_{k,l,h} \cdot (X + o(X))(\log X)^{k-l-2},
\]
for a certain (explicit) constant \(C_{k,l,h} > 0\). Recently, Matomäki-Radziwiłł-Tao \([11]\) have shown the following averaged result, in the regime \(H \geq (\log X)^{10000k \log k}\),
\[
\sum_{h \leq H} \left| \sum_{n \leq X} d_k(n + h) d_l(n) - C_{k,l,h} \cdot X(\log X)^{k-l-2} \right| = o_k(HX(\log X)^{k+l-2}).
\]

For higher correlations of divisor functions with Möbius, we obtain

**Theorem 1.6.** For any \(j \geq 1, k_1, \ldots, k_j \geq 2\), let \(k = \sum_{i=1}^{j} k_i\) and take any fixed tuple \(A = \{a_1, \ldots, a_j\}\) of distinct integers. If \(\log H/\log_2 X \to \infty\), then
\[
\sum_{h \leq H} \left| \sum_{n \leq X} \mu(n + h) \prod_{i=1}^{j} d_{k_i}(n + a_i) \right| = o_{k,A}(HX(\log X)^{k-j}).
\]

Again, we emphasize the need to average over the shift \(h\) that inputs to Möbius \(\mu(n+h)\), while \(a_i\) may be fixed arbitrarily.

**Remark 1.7.** For simplicity, the results are stated for the Möbius function \(\mu\), but our results hold equally for its completely multiplicative counterpart, the Liouville function \(\lambda\). In fact, the proof strategy is to reduce from \(\mu\) to \(\lambda\).

The main number-theoretic input is the classical Vinogradov-Korobov zero-free region
\[
\left\{ \sigma + it : 1 - \sigma < \frac{c}{\max \{\log q, \log(|t| + 3)^{2/3} \log \log(|t| + 3)^{1/3}\}} \right\}
\]
for \(L(s, \chi)\), where \(\chi\) is a Dirichlet character of modulus \(q \leq (\log X)^{A}\) in the Siegel-Walfisz range, see \([6, \S 8]\).

1.3. **Beyond Möbius.** We also consider general multiplicative functions \(f : \mathbb{N} \to \mathbb{C}\), which do not pretend to be a character \(f(n) \approx n^it \chi(n)\) for some \(\chi (\mod q)\). More precisely, we follow Granville and Soundararajan \([3]\) and define the pretentious distance
\[
\mathbb{D}(f, g; X) = \left( \sum_{p \leq X} \frac{1 - \text{Re}(f(p)g(p))}{p} \right)^{1/2},
\]
and the related quantity
\[
M(f; X, Q) = \inf_{|t| \leq X} \mathbb{D}(f, n \mapsto n^it \chi(n); X)^2.
\]

We may apply recent work of Matomäki-Radziwiłł-Tao \([12]\) on Fourier uniformity, in order to more directly obtain (qualitative) cancellation for averages of non-pretentious multiplicative functions over shifted primes.
Theorem 1.8. Given \( \theta \in (0,1) \) let \( H = X^\theta \). Given a multiplicative function \( f : \mathbb{N} \to \mathbb{C} \) with \( |f| \leq 1 \). There exists \( \rho \in (0, \frac{1}{8}) \) such that, if \( M(f; X^2/H^2-\rho, Q) \to \infty \) as \( X \to \infty \) for each fixed \( Q > 1 \), then

\[
\sum_{h \leq H} \left| \sum_{p \leq X} f(p+h) \right| = o_{\theta, \rho}(H\pi(X)).
\]

In particular, \( f = \mu \) does not pretend to be a Dirichlet character, a fact equivalent to the prime number theorem in arithmetic progressions. Indeed,

\[
M(\mu, X, Q) \geq \inf_{|t| \leq X} \sum_{\chi(q), q \leq Q, e(\log X)^{2/3+\varepsilon} \leq p \leq X} \frac{1 + \text{Re} \chi(p)p^t}{p} \geq \left( \frac{1}{3} - \varepsilon \right) \log \log X + O(1),
\]

where the latter inequality is well-known to follow from the zero-free region (1.6).

1.4. Overview of the proof of Theorem 1.1. We now indicate the general form of the proof. We pursue a variation on the approach of Matomäki-Radziwiłł-Tao [9]. Namely, we first restrict (1.1) to ‘typical’ terms \( \mu(n) \) for \( n = p + h \in S \) that have prime factors lying in certain prescribed intervals \([P_1, Q_1], [P_2, Q_2] \). The terms with \( n \not\in S \) are sparse, and thus may be shown to contribute negligibly by standard sieve estimates. (For higher correlations, one may also use sieve estimates, along with work of Henriot [5] to handle a general class of functions with ‘moderate growth’ that are ‘amenable to sieves.’)

Once reduced to numbers with ‘typical factorization,’ we decouple the short interval correlation between Möbius and the indicator for the primes, using a Fourier identity and applying Cauchy-Schwarz (Lemma 2.1). This yields a bound of \( \pi(X) \ll X/\log X \) times a Fourier-type integral for \( \mu \),

\[
\sup_{\alpha} \int_0^X \left| \sum_{x \leq n \leq x+H} \mu(n)e(n\alpha) \right| dx.
\]

This decoupling step is a gambit. It has the advantage of only needing to consider \( \mu \) on its own, but loses a factor of \( \log X \) from the density of the primes. To make this gambit worthwhile, we must recover over a factor of \( \log X \) savings in the above Fourier integral for \( \mu \). However, Matomäki-Radziwiłł-Tao [9, Theorem 2.3] bound the above integral with roughly \( (\log X)^{1/250} \) savings (though their bound holds for any non-pretentious multiplicative function \( g \).) Therefore we must refine Matomäki-Radziwiłł-Tao’s argument in the special case of \( g = \mu \) to win back over a full factor of \( \log X \). We note this task is impossible unless \( H \) is larger than a power of \( \log \) (this already hints at why we must assume \( \log H/\log \log X \to \infty \)).

We accomplish this task in the ‘key Fourier estimate’ (Theorem 2.2), which bounds the above integral with \( (\log X)^A \) savings for any \( A > 0 \) (though \( S \) will implicitly depend on \( A \)). As with [9], this bound is proven by reducing to the analogous estimate with the completely multiplicative \( \lambda \), and splitting up \( \alpha \in [0,1] \) into major and minor arcs.

The main technical innovation here comes from the major arcs (Proposition 5.1), essentially saving a factor \( (\log X)^A \) in the mean values of ‘typical’ Dirichlet polynomials of the form

\[
\sum_{X \leq n \leq 2X \atop n \in S} \frac{\lambda(n)\chi(n)}{n^s}
\]
for a character $\chi$ of modulus $q \leq (\log X)^A$ in the Siegel-Walfisz range. This refines the seminal work of Matomäki-Radziwiłł [7], who obtained a fractional power of log savings for the corresponding mean values. However, Matomäki-Radziwiłł’s results apply to the general setting of (non-pretentious) multiplicative functions and appeal to Halász’s theorem, which offers small savings. By contrast, our specialization to the Möbius function affords us the full strength of Vinogradov-Korobov estimates (Lemma 4.5).

The Matomäki-Radziwiłł method saves roughly a fractional power of $P_1$ in the Dirichlet mean value when $Q_1 \approx H$. So in order to recover from our initial gambit, we are prompted to choose $P_1 = (\log X)^C$ for some large $C > 0$. Then by a standard sieve bound the size of $S$ is morally $O\left(\frac{\log P_1}{\log Q_1}\right) = O_C\left(\frac{\log \log X}{\log H}\right)$. This highlights the need for our assumption $\log H / \log \log X \to \infty$.

We remark that the Matomäki-Radziwiłł method requires two intervals $[P_1, Q_1], [P_2, Q_2]$ (that define $S$) in order to handle ‘typical’ Dirichlet polynomials in the regime $H = (\log X)^{\psi(X)}$ for $\psi(X) \to \infty$. Note in general [7] the slower $H \to \infty$ the more intervals we require (though by a neat short argument [8], only one interval is needed in the regime $H = X^\theta$ for $\theta > 0$).

\section*{Notation}

The Möbius function is defined multiplicatively from primes $p$ by $\mu(p) = -1$ and $\mu(p^k) = 0$ for $k \geq 2$. Similarly the Liouville function $\lambda$ is defined completely multiplicatively by $\lambda(p) = -1$.

We use standard asymptotic notation: $X \ll Y$ and $X = O(Y)$ both mean $|X| \leq CY$ for some absolute constant $C$, and $X \asymp Y$ means $X \ll Y \ll X$. If $x$ is a parameter tending to infinity, $X = o(Y)$ means that $|X| \leq c(x) Y$ for some quantity $c(x)$ that tends to zero as $x \to \infty$. Let $\log_k X = \log_{k-1} (\log X)$ denote the $k$th-iterated logarithm.

Unless otherwise specified, all sums range over the integers, except for sums over the variable $p$ (or $p_1, p_2, \ldots$) which are understood to be over the set of primes $\mathbb{P}$. Let $e(x) := e^{2\pi ix}$.

We use $1_S$ to denote the indicator of a predicate $S$, so $1_S = 1$ if $S$ is true and $1_S = 0$ if $S$ is false. When $S$ is a set, we write $1_S(n) = 1_{n \in S}$ as the indicator function of $S$. Also let $1_S f$ denote the function $n \mapsto 1_{S(n)} f(n)$.

\section*{2. Initial reductions}

We begin with a Fourier-type bound to decouple correlations of arbitrary functions.

\textbf{Lemma 2.1 (Fourier bound).} Given $f, g : \mathbb{N} \to \mathbb{C}$, let $F(X) := \sum_{n \leq X} |f(n)|^2$. Then

\begin{equation}
(2.1) \quad \sum_{|h| \leq H} \left| \sum_{n \leq X} f(n) g(n + h) \right|^2 \ll F(X + 2H) \cdot \sup_{\alpha} \int_0^X \left| \sum_{x \leq n \leq x + 2H} g(n) e(n\alpha) \right| \, dx.
\end{equation}

\textbf{Proof.} First, the lefthand side of (2.1) is

\begin{equation}
(2.2) \quad \sum_{|h| \leq H} \left| \sum_{n \leq X} f(n) g(n + h) \right|^2 \ll H^{-2} \sum_{|h| \leq 2H} (2H - |h|)^2 \sum_{n \leq X} f(n) g(n + h)^2 =: H^{-2} \Sigma.
\end{equation}
Expanding the square in $\Sigma$ and letting $h = m - n = m' - n'$, we have

$$\Sigma = \sum_{|h| \leq 2H} (2H - |h|)^2 \sum_{n, n' \leq X} f(n) \overline{f}(n') g(n + h) g(n' + h) = \sum_{n, n' \leq X} \sum_{m, m'} f(n) \overline{f}(n') g(m) \overline{g}(m') 1_{m-n=m'-n'} \left( \int_{0}^{X} 1_{x \leq n, m \leq x+2H} dx \right) \left( \int_{0}^{X} 1_{x' \leq n', m' \leq x'+2H} dx' \right).$$

Then orthogonality $1_{m-n=m'-n'} = \int_{0}^{1} e((m - n - m' + n')\alpha) d\alpha$ gives

$$\Sigma = \int_{0}^{1} \int_{0}^{X} \sum_{x \leq m \leq x+2H} g(m) e((m - n)\alpha) dx \cdot \int_{0}^{X} \sum_{x' \leq n', m' \leq x'+2H} \overline{f}(n') \overline{g}(m') e((n' - m')\alpha) dx' d\alpha = \int_{0}^{1} \left| \int_{0}^{X} \sum_{x \leq m \leq x+2H} f(n) g(m) e((m - n)\alpha) dx \right|^2 d\alpha.$$

Using Cauchy-Schwarz, we bound $\Sigma$ as

$$\Sigma \leq \int_{0}^{1} \int_{0}^{X} \left| \sum_{x \leq m \leq x+2H} g(m) e(m\alpha) \right|^2 dx \cdot \int_{0}^{X} \left| \sum_{y \leq n \leq y+2H} \overline{f}(n) e(n\alpha) \right|^2 dy d\alpha \equiv H \left( \sup_{\alpha} \int_{0}^{X} \left| \sum_{x \leq m \leq x+2H} g(m) e(m\alpha) \right| dx \right) \int_{0}^{1} \int_{0}^{X} \left| \sum_{y \leq n \leq y+2H} \overline{f}(n) e(n\alpha) \right|^2 dy d\alpha.$$

Using $\int_{0}^{1} e(n\alpha) d\alpha = 1_{n=0}$ again, the second integral in (2.3) is

$$\int_{0}^{1} \int_{0}^{X} \left| \sum_{y \leq n \leq y+2H} \overline{f}(n) e(n\alpha) \right|^2 dy d\alpha = \int_{0}^{X} \sum_{y \leq n, n' \leq y+2H} \overline{f}(n) \overline{f}(n') \int_{0}^{1} e((n - n')\alpha) d\alpha dy = \int_{0}^{X} \sum_{y \leq n \leq y+2H} |f(n)|^2 dy = \sum_{n \leq X+2H} |f(n)|^2 \int_{n-2H}^{n+2H} dy \ll HF(X + 2H).$$

Hence plugging the bound (2.3) for $H^{-2}\Sigma$ back into (2.2) gives the result. 

Next we consider numbers with ‘typical factorization.’

For $A, \delta \geq 0$, define $\psi$ via $H = (\log X)^{\psi(X)}$ so that (1.2) becomes

$$\psi_\delta(X) = \min\{\psi(X), (\log X)^{1/3 - \delta}\}.$$ 

Consider the intervals

(2.4)  
\[ [P_1, Q_1] = [(\log X)^{3A}, (\log X)^{\psi(X) - 4A}], \]
\[ [P_2, Q_2] = [\exp((\log X)^{2/3 + \delta/2}), \exp((\log X)^{1 - \delta/2})], \]

and define the ‘typical factorization’ set

(2.5)  
\[ S = S(X, A, \delta) := \{n \leq X : \exists \text{ prime factors } p_1, p_2 \mid n \text{ with } p_j \in [P_j, Q_j] \}. \]

Using the Fourier bound, we shall reduce Theorem (1.1) to the following.
**Theorem 2.2** (Key Fourier estimate for $\mu$). Given any $A > 5$, $\delta > 0$, let $S = S(X, A, \delta)$ as in (2.5). Then if $\log H / \log_2 X \to \infty$,

$$\sup_{\alpha} \int_0^X \left| \sum_{x \leq n \leq x+H} \mu(n) e(n\alpha) \right| \, dx \ll_{A, \delta} \frac{HX}{(\log X)^{A/5}}.$$

**Proof of Theorem [7] from Theorem 2.2.** By a standard sieve upper bound [2, Theorem 7.1], for each $h \leq H$, $j = 1, 2$ we have

$$\# \{ p \leq X : q \nmid p + h \forall q \in [P_j, Q_j] \} \ll \pi(X) \frac{\log P_j}{\log Q_j} \cdot \frac{h}{\varphi(h)}.$$  

Thus, recalling the choice of $[P_j, Q_j]$ in (2.4), the terms $p + h \notin S$ trivially contribute to (1.1)

$$\sum_{h \leq H} \left| \sum_{p \leq X, p + h \notin S} \mu(p + h) \right| \ll \sum_{h \leq H} \sigma(h) \ll \pi(X) \frac{A}{\psi(X)} + (\log X)^{d - 1/3} \sum_{h \leq H} \frac{h}{\varphi(h)} \ll A \frac{\psi(X)}{\psi_\delta(X)} \sum_{h \leq H} \frac{h}{\varphi(h)} \ll A \frac{\psi(X)}{\psi_\delta(X)}.$$

On the other hand for $p + h \in S$, Lemma [21] with $f(n) = 1_{S}(n)$, $g(n) = 1_{S}(n)$ gives

$$\sum_{h \leq H} \left| \sum_{p \leq X, p + h \in S} \mu(p + h) \right|^2 \ll \pi(X + 2H) \cdot \sup_{\alpha} \int_0^X \left| \sum_{x \leq n \leq x+2H} \mu(n) e(n\alpha) \right| \, dx \ll_{A, \delta} \frac{HX^2}{(\log X)^{A/5+1}}.$$

Hence (2.7) and (2.8) with $A = 6$ give Theorem 1.1. \hfill \Box

Let $W = (\log X)^A$. Recall Theorem 2.2 asserts that $M_H(X) \ll XH/W^{1/5}$ for

$$M_H(X) := \sup_{\alpha} \int_0^X \left| \sum_{x \leq n \leq x+H} \mu(n) e(n\alpha) \right| \, dx.$$

We first note, that, for technical convenience, it suffices to establish $M_{H_0}(X) \ll XH_0/W^{1/5}$ with $H_0 := \min\{H, \exp((\log X)^{2/3})\}$. Indeed, if $H > H_0$ then by the triangle inequality

$$M_H(X) \leq \sum_{k \leq [H/H_0]} M_{H_0}(X + kH_0) \leq \sum_{k \leq [H/H_0]} \frac{(X + kH_0)H_0}{W^{1/5}} \ll \frac{H}{H_0} \cdot \frac{XH_0}{W^{1/5}} = \frac{XH}{W^{1/5}}$$

as desired. Hence we may assume $H \leq \exp((\log X)^{2/3})$ hereafter. This reduction is not strictly necessary, but will simplify the argument. For example, in this case $Q_1 < P_2$ so the intervals $[P_j, Q_j]$ are disjoint.

Consider the ‘refined typical factorization’ sets $S_d = \{n/d : d \mid n \in S\}$ for $d < P_1$, that is,

$$S_d = S_d(X, A, \delta) = \{m \leq X/d : \exists \text{ prime factors } p_1, p_2 \mid m \text{ with } p_j \in [P_j, Q_j]\}.$$

(2.9)
So far we have reduced Theorem 1.1 to Theorem 2.2 for \( \mu \). We now reduce further to the analogous estimate for its completely multiplicative counterpart \( \lambda \).

**Proposition 2.3** (Key Fourier estimate for \( \lambda \)). *Given any \( A > 5, \delta > 0 \), \( H = (\log X)^{\psi(X)} \) with \( \psi(X) \to \infty \) and \( \psi(X) \leq (\log X)^{2/3} \). For \( d \leq W = (\log X)^A \) and \( S_d = S_d(X, A, \delta) \) as in (2.9), we have*

\[
\sup_{\alpha} \int_0^X \left| \sum_{\substack{x \leq n \leq x+H \\
n \in S_d}} \lambda(n)e(\alpha n) \right| dx \ll_{A, \delta} \frac{HX}{d^{3/4}W^{1/5}}.
\]

**Proof of Theorem 2.2 from Proposition 2.3**. By Möbius inversion, we have \( \mu = \lambda \ast h \) for \( h = \mu \ast (\mu \lambda) \), where \( \ast \) denotes Dirichlet convolution. That is, \( h(d^2) = \mu(d) \) for squarefree \( d \), and zero otherwise. Thus we may write

\[
\sum_{x \leq n \leq x+H} 1_S(n)\mu(n)e(n\alpha) = \sum_{d \geq 1} h(d) \sum_{x \leq md \leq x+H} 1_S(md)\lambda(m)e(md\alpha),
\]

and so the triangle inequality gives

\[
(2.10) \quad \int_0^X \left| \sum_{x \leq n \leq x+H} 1_S(n)\mu(n)e(n\alpha) \right| dx \leq \sum_{d \geq 1} |h(d)| \int_0^X \left| \sum_{x \leq md \leq x+H} 1_S(md)\lambda(m)e(md\alpha) \right| dx.
\]

Note, using the trivial bound and swapping the order of summation and integration, the contribution of \( d > W \) to (2.10) is

\[
(2.11) \quad \ll \sum_{W < d \leq X} |h(d)| \sum_{md \leq X+H} H \ll \frac{HX}{W^{1/4}} \sum_{d \geq 1} |h(d)| \frac{d^{3/4}}{d^{3/4}} \ll \frac{HX}{W^{1/4}} \sum_{d \geq 1} d^{-3/2} \ll \frac{HX}{W^{1/4}}
\]

since \( |h| \leq 1 \) is supported on squares.

On the other hand the contribution of \( d \leq W \) to (2.10) is

\[
(2.12) \quad \ll \sum_{d \geq 1} |h(d)| \int_0^X \left| \sum_{x \leq md \leq x+H} 1_{S_d}(md)\lambda(m)e(md\alpha) \right| dx \ll \frac{HX}{W^{1/5}} \sum_{d \leq W} \frac{|h(d)|}{d^{3/4}} \ll \frac{HX}{W^{1/5}}
\]

assuming Proposition 2.3, and noting \( 1_S(md) = 1_{S_d}(m) \) since \( d \leq W < P_1 \). Together (2.11) and (2.12) give Theorem 2.2. \( \square \)

### 3. Key Fourier estimate

In this section, we establish Proposition 2.3 by the circle method, following the argument in [9], Proposition 2.4.

Take \( \alpha \in [0, 1] \). By Dirichlet’s approximation theorem there exists \( \frac{a}{q} \in \mathbb{Q} \) with \( (a, q) = 1 \) and \( 1 \leq q \leq Q_1 \) for which

\[
|\alpha - \frac{a}{q}| \leq \frac{1}{qQ_1}.
\]

So we may split \([0, 1]\) into major arcs \( \mathcal{M} \) and minor arcs \( \mathfrak{m} \), according to the size of denominator \( q \) compared to \( W \),

\[
\mathcal{M} = \bigcup_{q \leq W} \mathcal{M}(q) \quad \text{and} \quad \mathfrak{m} = [0, 1] \setminus \mathcal{M},
\]
Proposition 3.1 (Key minor arc estimate). Given any $A > 5$, $H = (\log X)^{\psi(X)}$ with $\psi(X) \to \infty$ and $\psi(X) \leq (\log X)^{2/3}$, let $d \leq W = (\log X)^A$ and $S_d = S_d(X, A, 0)$ as in (2.9). Then for any completely multiplicative $g : \mathbb{N} \to \mathbb{C}$ with $|g| \leq 1$, we have
\[
\sup_{\alpha \in \mathcal{M}} \int_0^X \left| \sum_{x \leq nd \leq x+H, n \in S_d} g(n) e(n\alpha) \right| \, dx \ll_A \frac{HX}{d^{3/4}W^{1/5}}.
\]

Proposition 3.2 (Key major arc estimate for $\lambda$). Given any $A > 5$, $\delta > 0$, $H = (\log X)^{\psi(X)}$ with $\psi(X) \to \infty$ and $\psi(X) \leq (\log X)^{2/3}$, let $d \leq W = (\log X)^A$ and $S_d = S_d(X, A, \delta)$ as in (2.9). Then we have
\[
\sup_{\alpha \in \mathcal{M}} \int_0^X \left| \sum_{x \leq nd \leq x+H, n \in S_d} \lambda(n) e(n\alpha) \right| \, dx \ll_{A, \delta} \frac{HX}{dW}.
\]

We remark that the bounds in the minor arc hold for any bounded multiplicative function, whereas in the major the specific choice of $\lambda$ is needed.

3.1. Minor arc. In this subsection, we prove Proposition 3.1. Recall for $\alpha \in \mathcal{M}$ in the minor arc, $|\alpha - a/q| < W^4/qH$ with $q \in [W, H/W^4]$. It suffices to show
\[
I_m := \int_{\mathbb{R}} \theta(x) \sum_{x \leq nd \leq x+H} 1_{S_d}(n) g(n) e(n\alpha) \, dx \ll HX (\frac{\log \log X}{dW})^{1/2} \psi(X),
\]
uniformly for any $\alpha \in \mathcal{M}$ and measurable $\theta : [0, X] \to \mathbb{C}$ with $|\theta(x)| \leq 1$. Letting $\mathcal{P} = \{p : P_1 \leq p \leq Q_1\}$, by definition each $n \in S_d$ has a prime factor in $\mathcal{P}$, so we use a variant of the Ramaré identity
\[
1_{S_d}(n) = \sum_{\substack{p \in \mathcal{P} \mid m \mid \mathcal{P} : q \mid m \}} \frac{1_{S_d^{(1)}}(mp)}{\# \{q \in \mathcal{P} : q \mid m \} + 1_{p|m}},
\]
where $S_d^{(1)} = \{m \leq X/d : \exists p \mid m, p \in [P_2, Q_2]\}$. As $g$ is completely multiplicative, we obtain
\[
I_m = \sum_{p \in \mathcal{P}} \sum_{m} \frac{1_{S_d^{(1)}}(mp) g(m) g(p) e(mp\alpha)}{\# \{q \in \mathcal{P} : q \mid m \} + 1_{p|m}} \int_{\mathbb{R}} \theta(x) 1_{x \leq mpd \leq x+H} \, dx.
\]
Next we split $\mathcal{P}$ into dyadic intervals $[P, 2P]$. It suffices to show for each $P \in [P_1, Q_1]$,
\[
\sum_{\substack{p \in \mathcal{P} \mid m \mid \mathcal{P} : q \mid m \}} \frac{1_{S_d^{(1)}}(mp) g(m) g(p) e(mp\alpha)}{\# \{q \in \mathcal{P} : q \mid m \} + 1_{p|m}} \int_{\mathbb{R}} \theta(x) 1_{x \leq mpd \leq x+H} \, dx \ll \frac{HX}{\log P} (\frac{\log \log X}{dW})^{1/2},
\]
since then (3.1) will follow by (2.4) and the triangle inequality, using
\[ \sum_{P \leq p < Q_1} \frac{1}{\log P} \ll \sum_{P \leq P, p} \frac{1}{j} \ll \log \frac{\log Q_1}{\log P} = \log \frac{\psi(X) - 4A}{33A} \ll \psi(X). \]

Fix \( P \). We may replace \( 1_p \) with \( 1 \) in (3.3) at a cost of \( O(X/dP) \). Indeed, since the integral is \( \int X dx \ll \psi(X) \), and \( 1_{\psi} (mp) = 0 \) unless \( m \leq X/dP \), the cost of such substitution is
\[ \ll \sum_{P \leq p < 2P} \sum_{m \leq X/dP} H \ll P \frac{X}{dP} H = \frac{HX}{dP}. \]

Now the left hand side of (3.3) becomes
\[ \sum_{m \in \mathcal{S}^{(1)}_d} \frac{g(m)}{\# \{q \in \mathcal{P} : q \mid m \}} + 1 \sum_{P \leq p < 2P} g(p)e(m\alpha) \int \theta(x) \mathbf{1}_{mp \leq x} \mathbf{1}_{mp \leq x + H} \ dx \]
\[ \ll \sum_{P \leq p < 2P} \left| \sum_{P \leq p < 2P} g(p)e(m\alpha) \int \theta(x) \mathbf{1}_{mp \leq x} \mathbf{1}_{mp \leq x + H} \ dx \right| \]
\[ \ll \left( \frac{X}{dP} \right)^{1/2} \left( \sum_{P \leq p < 2P} \left| \sum_{P \leq p < 2P} g(p)e(m\alpha) \int \theta(x) \mathbf{1}_{mp \leq x} \mathbf{1}_{mp \leq x + H} \ dx \right|^2 \right)^{1/2}, \]
by the trivial bound and Cauchy-Schwarz. Hence for (3.3) it suffices to show
\[ (3.4) \sum_{m \leq X/dP} \left| \sum_{P \leq p < 2P} g(p)e(m\alpha) \int \theta(x) \mathbf{1}_{mp \leq x} \mathbf{1}_{mp \leq x + H} \ dx \right|^2 \ll \frac{H^2PX \log \log P}{W (\log P)^2}. \]

We expand the left hand side of (3.4) and sum the resulting geometric series on \( m \),
\[ \sum_{p_1, p_2 \in \mathcal{P} \cap [P, 2P]} \int g(p_1)g(p_2) \theta(x_1)\theta(x_2) \sum_{m \leq X/dP} e(m(p_1 - p_2)\alpha) \ dx_1 \ dx_2 \]
\[ \ll HX \sum_{p_1, p_2 \leq 2P} \min \left( \frac{H}{dP}, \frac{1}{\|(p_1 - p_2)\alpha\|} \right), \]
since for given \( d, p_1, p_2 \), there are \( O(X) \) choices for \( x_1 \) and \( O(H) \) subsequent choices for \( x_2 \) since \( x_2 = x_1(p_2/p_1) + O(H) \). Note \( \|z\| \) denotes the distance of \( z \in \mathbb{R} \) to the nearest integer.

Thus (3.4) reduces to showing
\[ (3.5) \sum_{p_1, p_2 \leq 2P} \min \left( \frac{H}{P}, \frac{1}{\|(p_1 - p_2)\alpha\|} \right) \ll \frac{HP \log \log P}{W (\log P)^2}. \]

The difference of primes is \( p_1 - p_2 \ll P \). Conversely, any integer \( n \ll P \) may be written as \( n = p_1 - p_2 \) for \( p_1, p_2 \leq 2P \) in \( \ll \frac{n}{\varphi(n)} P (\log P)^{-2} \ll \frac{P \log \log P}{(\log P)^2} \) ways by a standard upper
bound sieve, see [2, Proposition 6.22]. Hence for (3.5) it suffices to obtain
\[
\sum_{1 \leq n \leq P} \min \left( \frac{H}{n}, \frac{1}{\|n\alpha\|} \right) \ll \frac{H}{W} \quad (\alpha \in \mathbb{m}).
\]

But this follows by the standard ‘Vinogradov lemma’ [6, p.346].

**Lemma 3.3.** Given \( H, P > 1 \), take \( \alpha \in [0,1] \) with \( |\alpha - a/q| \leq 1/q^2 \) for some \( (a,q) = 1 \). Then
\[
\sum_{1 \leq n \leq P} \min \left( \frac{H}{n}, \frac{1}{\|n\alpha\|} \right) \ll \frac{H}{q} + \frac{H}{P} + (P + q) \log q.
\]

Observe \( H/q + H/P + (P + q) \log q \ll H/W \) since \( q \in [W,H/W^4], \ P \in [P_1,Q_1] = [W^{24},H/W^4] \). This completes the proof in the minor arc.

### 3.2. Major arc

In this subsection, we prove the key major arc estimate assuming the following mean value result for the (twisted) Liouville function.

**Proposition 3.4.** Given \( A > 5, \ \delta > 0 \), let \( q \leq W = (\log X)^A, \ d < W^{33}, \ \chi \mod q, \ h \in [H/W^5,H], \) and \( S_d = S_d(X,A,\delta) \) as in (2.9). Then for all \( Y \in [X/W^4,X] \), we have
\[
J_{d,h,q}(Y;\chi) := \int_Y^{2Y} \left| \frac{1}{h} \sum_{x \leq m \leq x+h} \lambda(m)\chi(m) \right|^2 \ dx \ll_{A,\delta} \frac{Y}{W^{10}}.
\]

**Proof of Proposition 3.2 from Proposition 3.4.** To obtain the key major arc estimate we shall prove the stronger bound,
\[
I_m := \sup_{\alpha \in \mathbb{R}} \int_0^X \left| \sum_{x \leq n \leq x+H} 1_{S_d}(n)\lambda(n)e(n\alpha) \right| \ dx \ll \frac{H X}{dW}.
\]

In the major arc recall \( \alpha = \frac{a}{q} + \theta \) with \( q \leq W \) and \( |\theta| \leq \frac{W^4}{qd} \). By partial summation with \( a_n = 1_{x/d}(n)1_{S_d}(n)\lambda(n)e(n\alpha/q) \), and \( A(t) = \sum_{n \leq t} a_n \), we have
\[
\sum_{x \leq nd \leq x+H} 1_{S_d}(n)\lambda(n)e(n\alpha) = e\left(\frac{x+H}{d}\theta\right)A\left(\frac{x+H}{d}\right) - e\left(\frac{x}{d}\theta\right)A\left(\frac{x}{d}\right) - 2\pi i \theta \int_{x/d}^{(x+H)/d} e(t\theta)A(t) \ dt
\]
\[
\ll \left| \sum_{x \leq nd \leq x+H} 1_{S_d}(n)\lambda(n)e(n\alpha/q) \right| + ||\theta| \int_0^{H/d} \left| \sum_{x/d \leq n \leq x/d+h} 1_{S_d}(n)\lambda(n)e(n\alpha/q) \right| \ dh.
\]

Thus taking the maximizing \( h \) and integrating over \( x \in [0,X] \), we obtain
\[
I_m \ll I_{H/d} + |\theta| \frac{H}{d} \sup_{h \leq H/d} I_h \ll I_{H/d} + \frac{W^4}{qd} \sup_{h \leq H/d} I_h,
\]
where
\[
I_h := \int_0^X \left| \sum_{x/d \leq n \leq x/d+h} 1_{S_d}(n)\lambda(n)e(n\alpha/q) \right| \ dx.
\]
Then splitting into residues $b \pmod{q}$ gives

$$I_h \leq \sum_{b(q)} |e(ab/q)| \int_0^X \left| \sum_{\substack{x/d \leq n \leq x/d+h \atop n \equiv b(q) \pmod{q}}} 1_{S_d}(n) \lambda(n) \right| dx = \sum_{b(q)} \int_0^X \left| \sum_{\substack{x/d \leq n \leq x/d+h \atop n \equiv b(q) \pmod{q}}} 1_{S_d}(n) \lambda(n) \right| dx.$$

Now suppose we have the bound

$$I_h \ll \frac{qhX}{W^5} \quad \text{for } h \in [qH/W^5, H/d]. \quad (3.10)$$

Then, combining with the trivial bound $I_h \leq hX$ when $h \leq qH/W^5$, (3.8) becomes

$$I_{2h} \ll I_{H/d} + \frac{W^4}{qd} \left( \sup_{qH/W^5 \leq h \leq H/d} I_h + \sup_{h \leq qH/W^5} hX \right) \ll \frac{qH}{dW^5} + \frac{W^4}{qd} \left( \frac{qH}{W^5} + \frac{qH}{W^5} \right) \ll \frac{HX}{dW},$$

for $q \leq W$ in the major arc. Hence it suffices to show (3.10).

Now to bound $I_h$, we extract the gcd. Let $c := (b, q)$ so that $c \mid n$, and we let $b' = b/c$, $q' = q/c, h' = h/c, m = n/c$. Thus since $\lambda$ is completely multiplicative, we have

$$I_h \leq \sum_{c \mid q} |\lambda(c)| \sum_{b'(q')} \int_0^X \left| \sum_{\substack{x/cd \leq m \leq x/cd+h/c \atop m \equiv b'(q') \pmod{q}}} 1_{S_{cd}}(cm) \lambda(m) \right| dx$$

$$\leq \sum_{c \mid q} cd \sum_{b'(q')} \int_0^{X/cd} \left| \sum_{\substack{y \leq m \leq y+h' \atop m \equiv b'(q') \pmod{q}}} 1_{S_{cd}}(m) \lambda(m) \right| dy,$$

using the substitution $y = x/cd$, and noting $1_{S_{cd}}(cm) = 1_{S_{cd}}(m)$ since $c \leq q < P_1$. Then recalling orthogonality of characters $\varphi(q')1_{m \equiv b'(q')} = \sum_{\chi(q')} \chi(b') \chi(m)$, we obtain

$$I_h \leq \sum_{c \mid q} cd \sum_{b'(q')} \int_0^{X/cd} \left| \sum_{\chi(q')} \left| \varphi(q') \sum_{\chi(q')} \chi(b') \right| \chi(m) \lambda(m) \right| dy \int_0^{X/cd} \left| \sum_{\chi(q')} \left| \varphi(q') \sum_{\chi(q')} \chi(b') \right| \chi(m) \lambda(m) \right| dy$$

$$\leq \sum_{c \mid q} cd \sum_{\chi(q')} \int_0^{X/cd} \left| \sum_{y \leq m \leq y+h'} 1_{S_{cd}}(m) \lambda(m) \chi(m) \right| dy \quad (3.11)$$

We may discard the contribution to (3.11) of the integral over $y \leq X/dW^5$, since $h' = h/c$ and $q \leq W$ imply an admissible cost

$$\ll \sum_{c \mid q} cd \varphi(q') \frac{X}{dW^5} h' \leq \frac{hX}{W^5} \sum_{q' \mid q} \varphi(q') = \frac{qH}{W^5}.$$

For the remaining $y \in [\frac{X}{dW^5}, \frac{X}{cd}]$ in (3.11), we split into dyadic intervals so that

$$I_h \leq \sum_{c \mid q} cd \sum_{\chi(q')} \int_{2^J}^{2^{J+1}} \left| \sum_{y \leq m \leq y+h'} 1_{S_{cd}}(m) \lambda(m) \chi(m) \right| dy + O\left( \frac{qH}{W^5} \right). \quad (3.12)$$
By assumption, Proposition 3.4 implies $J_{c,d,h',q'}(Y;\chi) \ll Y/W^10$, so Cauchy-Schwarz gives
\[
\int_Y^{2Y} \left| \sum_{y \leq m \leq y+h'} 1_{S_{c,d}}(m)\lambda(m)\chi(m) \right| dy \ll h' \sqrt{Y} \cdot J_{c,d,h',q'}(Y;\chi) \ll A \frac{Yh'}{W^5}.
\]
So plugging back into (3.12), we obtain
\[
I_h \ll \sum_{c|q} cd\varphi(q') \sum_{\frac{Y}{2W^7} \leq Y < \frac{2X}{cd}} \frac{Yh'}{W^5} = \frac{hd}{W^5} \sum_{q'|q} \varphi(q') \sum_{\frac{X}{2W^7} < Y < \frac{2X}{cd}} Y \ll \frac{qhX}{W^5}.
\]
This gives (3.10) as desired. □

4. Preparatory lemmas

We collect some standard lemmas on Dirichlet polynomials.

The first is the integral mean value theorem [6, Theorem 9.1].

Lemma 4.1 (mean value). For $D(s) = \sum_{n \leq N} a_n n^{-s}$, we have
\[
\int_{-T}^{T} |D(it)|^2 dt = (T + O(N)) \sum_{n \leq N} |a_n|^2.
\]

One may discretize the mean value theorem by replacing the integral over $[-T,T]$ with a sum over a well-spaced set $W \subset [-T,T]$.

Definition 4.2. A set $W \subset \mathbb{R}$ is well-spaced if $|w-w'| \geq 1$ for all $w, w' \in W$.

Next is the Halász-Montgomery inequality [6, Theorem 9.6], which offers an improvement to the (discretized) mean value theorem when the well-spaced set is ‘sparse.’

Lemma 4.3 (Halász-Montgomery). Given $D(s) = \sum_{n \leq N} a_n n^{-s}$ and a well-spaced set $W \subset [-T,T]$. Then
\[
\sum_{t \in W} |D(it)|^2 \ll (N + |W|\sqrt{T}) \log 2T \sum_{n \leq N} |a_n|^2.
\]

We also need a bound on the size of well-spaced sets $W$ in terms of the values of prime Dirichlet polynomials on $1 + iW$ [7, Lemma 8].

Lemma 4.4. Let $a_p \in \mathbb{C}$ be indexed by primes, with $|a_p| \leq 1$, and define the prime polynomial
\[
P(s) = \sum_{L \leq p \leq 2L} \frac{a_p}{P^s}.
\]
Suppose a well-spaced set $W \subset [-T,T]$ satisfies $|P(1+it)| \geq 1/U$ for all $t \in W$. Then
\[
|W| \ll U^2 T^{2(\log U + \log \log T)/\log L}.
\]

Lemma 4.5. Given $A, K > 0, \theta > \frac{2}{3}$, and a Dirichlet character $\chi \mod q \leq (\log X)^A$. Assume $\exp((\log X)^\theta) \leq P \leq Q \leq X$, and let $P(s,\chi) = \sum_{P \leq p \leq Q} \chi(p)p^{-s}$. Then for any $|t| \leq X$,
\[
|P(1+it,\chi)| \ll A,K,\theta \frac{\log X}{1+|t|} + (\log X)^{-K}.
\]
Proof. This follows as with [8, Lemma 2], except that the Vinogradov–Korobov zero-free region for \( \zeta(s) \) is replaced by that of \( L(s, \chi) \). \qed

We also use a Parseval-type bound. This shows that the average of a multiplicative function in almost all short intervals can be approximated by its average on a long interval, provided the mean square of the corresponding Dirichlet polynomial is small.

**Lemma 4.6** (Parseval bound). Given \( T_0 \in [(\log X)^{1/15}, X^{1/4}] \), and take a sequence \((a_m)_{m=1}^{\infty}\) with \( |a_m| \leq 1 \). Assume \( 1 \leq h_1 \leq h_2 \leq X/T_0^3 \). For \( x \in [X, 2X] \), define

\[
S_j(x) = \sum_{x \leq m \leq x+h_j} a_m, \quad \text{and} \quad A(s) = \sum_{X \leq m \leq 4X} \frac{a_m}{m^s}.
\]

Then

\[
\frac{1}{X} \int_X^{2X} \left| \frac{1}{h_1} S_1(x) - \frac{1}{h_2} S_2(x) \right|^2 \, dx \ll \frac{1}{T_0} + \int_{T_0}^{X/h_1} |A(1 + it)|^2 \, dt + \max_{T \geq X/h_1} \frac{X/h_1}{T} \int_{T}^{2T} |A(1 + it)|^2 \, dt.
\]

Proof. This follows as in [7, Lemma 14] with \((\log X)^{1/15}\) replaced by general \( T_0 \). \qed

We have a general mean value of products, via the Ramaré identity [7, Lemma 12].

**Lemma 4.7.** For \( V, P, Q \geq 1 \), denote \( \mathcal{P} = \{P, Q\} \cap \mathbb{P} \). Let \( a_m, b_m, c_p \) be bounded sequences for which \( a_{mq} = b_m c_q \) when \( q \nmid m \) and \( q \in \mathcal{P} \). Let

\[
Q_{v,V}(s) = \sum_{q \in \mathcal{P}} \frac{c_q}{q^s},
\]

\[
R_{v,V}(s) = \sum_{Xe^{-v/V} \leq m \leq 2Xe^{-v/V}} \frac{b_m}{m^s} \cdot \frac{1}{\# \{p \mid m : p \in \mathcal{P}\} + 1},
\]

and take a measurable set \( T \subset [-T, T] \). Then for \( \mathcal{I} = [[V \log P], V \log Q] \cap \mathbb{Z} \), we have

\[
\int_T \left| \sum_{X \leq n \leq 2X} \frac{a_n}{n^1+it} \right|^2 \, dt \ll V \log \left( \frac{Q}{V} \right) \sum_{v \in \mathcal{I}} \int_T |Q_{v,V}(1 + it) R_{v,V}(1 + it)|^2 \, dt
\]

\[+ \left( \frac{T}{X} + 1 \right) \left( \frac{1}{V} + \frac{1}{P} + \sum_{X \leq n \leq 2X} |a_n|^2 / n \right).\]

In the next result we employ the Fundamental Lemma of the sieve, along with the Siegel-Walfisz theorem.

**Lemma 4.8.** Given \( A, K > 0 \), \( q \leq (\log x)^A \), Dirichlet character \( \chi \) (mod \( q \)), and let \( \mathcal{D} = \prod_{p \in \mathcal{P}} p \) for any set of primes \( \mathcal{P} \subset (q, x^{1/\log \log x}) \). Then

\[
\sum_{\substack{m \leq x \\ (m, \mathcal{D}) = 1}} \lambda(m) \chi(m) \ll_{A,K} \frac{x}{(\log x)^K}.
\]
Lemma 6.11], the main term is uniformly in (4.2) above error is \(\ll s\).

Hence (4.2) follows as claimed, noting \(\nu\) by pairing up terms (4.3).

Now it suffices to prove

\[
(4.2) \quad \sum_{m \in A^{(b,\nu)}} 1_{(m,D)=1} = \frac{x}{2q} \prod_{p \mid D} \left(1 - \frac{1}{p}\right) + O_{A,K}(x(\log x)^{-K})
\]

uniformly in \(b, \nu\), from which it will follow

\[
S_0 = \frac{x}{2qd} \prod_{p \mid D} \left(1 - \frac{1}{p}\right) \sum_{\nu \in \{\pm 1\}} \nu \chi(b) + O(x(\log x)^{-K}) \ll \frac{x}{(\log x)^K},
\]

by pairing up terms \(\nu = \pm 1\). This will give the lemma.

Now to show (4.2), write \(A = A^{(b,\nu)}\). For \(d \mid D\) the set of multiples \(A_d = \{m \in A : d \mid m\}\) has size

\[
|A_d| = \frac{x}{2qd} + R_d,
\]

where \(|R_d| \ll \frac{x}{d} \log(x/d)^{-2K} + q\).

Now for any \(D > 1\) the indicator \(1_{m(D)=1}\) is bounded in between \(\sum_{d \leq D} \lambda_d^\pm\), for the standard linear sieve weights \(\{\lambda_d^\pm\}_{d \leq D}\), see [2, Lemma 6.11]. Thus the desired sum in (4.2) is bounded in between

\[
\sum_{d \mid D, \ d < D} \lambda_d^- |A_d| \leq \sum_{m \in A} 1_{(m,D)=1} \leq \sum_{d \mid D, \ d < D} \lambda_d^+ |A_d|.
\]

Note by (4.3), the upper and lower bounds are given by

\[
\sum_{d \mid D, \ d < D} \lambda_d^\pm |A_d| = \frac{x}{2q} \sum_{d \mid D, \ d < D} \frac{\lambda_d^\pm}{d} + \sum_{d \mid D, \ d < D} \lambda_d^\pm R_d.
\]

Let \(z = x^{1/\log_2 x}\) so that \(P \subset (q,z)\). Then choosing \(D = z^s\) for \(s = 2K \log_2 x/\log_3 x\), the above error is \(\ll \sum_{d < D} |R_d| \ll x(\log x)^{-K}\) by (4.3). And by the Fundamental Lemma [2, Lemma 6.11], the main term is

\[
\sum_{d \mid D, \ d < D} \frac{\lambda_d^\pm}{d} = (1 + O(s^{-s})) \prod_{p \mid D} \left(1 - \frac{1}{p}\right).
\]

Hence (4.2) follows as claimed, noting \(s^{-s} \ll (\log x)^{-K}\). \(\square\)
5. Mean value of multiplicative functions

In this section we prove Proposition 3.4 based on the following mean value theorem for Dirichlet polynomials with typical factorization. This refines Matomäki-Radziwill [7, Proposition 12] in the case of $g = \lambda \chi$, by leveraging Vinogradov-Korobov type bounds.

**Proposition 5.1.** Given any $A > 5$, $\delta > 0$, denote $B = 11A$. Write $H = (\log X)^{\psi(X)}$ with $
abla(\log X) \to \infty$ and $\psi(X) \le (\log X)^{2/3}$. Take $q \le (\log X)^{A}$, $d < (\log X)^{(3-A)}$, a Dirichlet character $\chi \pmod q$, and let $S_d = S_d(X, A, \delta)$ as in (2.9). For any $Y \in [X^{1/2}, X^2]$, define

$$G(s) = \sum_{\substack{d \le Y/2 \ni \in \mathcal{S} \\text{n} \in S_d}} \frac{\lambda(n)\chi(n)}{n^s}.$$ 

Then for any $T \in [Y^{1/2}, Y]$, we have

$$\int \limits_{(\log X)^{2B}}^T |G(1+it)|^2 \, dt \ll_{A, \delta} \left( \frac{Q \sqrt{T}}{Y} + 1 \right) (\log X)^{-B}. \tag{5.1}$$

**Proof of Proposition 5.1 from Proposition 5.1.** We shall prove

$$J := \int \limits_Y^{2Y} \left| \frac{1}{h_1} S_{h_1}(x) \right|^2 \, dx \ll_{A, \delta} \frac{Y}{(\log X)^{10A}} \tag{5.2}$$

for $Y \in [X/W^4, X]$, $h_1 = h \in [qH/W^5, H]$ and the sum

$$S_l(x) := \sum_{x \le m < x+l, m \in \mathcal{D}_l} \lambda(m)\chi(m).$$

First we claim $S_x(0) \ll x(\log x)^{-K}$ for all $K > 0$. To this, recall from (2.9) that each $m \in \mathcal{D}_l$ has prime factors $p_1, p_2 \mid m$ with $p_j \in [P_j, Q_j]$. So by inclusion-exclusion, the indicator is $1_{\mathcal{D}_l}(m) = \sum_{j=0}^3 (-1)^j 1_{(m, p_j)=1}$ where $\mathcal{D}_j = \prod_{p \in P_j} p$ for the sets of primes $P_0 = \emptyset$, $P_1 = [P_1, Q_1]$, $P_3 = [P_2, Q_2]$, $P_2 = P_1 \cup P_3$. Hence applying Lemma 4.8 to each $\mathcal{D}_j$ gives

$$S_x(0) = \sum_{j=0}^3 (-1)^j \sum_{m \le x} 1_{(m, P_j)=1} \lambda(m)\chi(m) \ll_{A, K} x(\log x)^{-K}. \tag{5.3}$$

In particular, letting $B = 11A$ we have

$$S_{h_2}(x) = S_{x+h_2}(0) - S_x(0) \ll_A \frac{x}{(\log x)^{7B}} \ll_A \frac{h_2}{(\log x)^B},$$

where $h_2 \asymp x(\log x)^{-6B}$, and so

$$J = \frac{1}{Y} \int \limits_Y^{2Y} \left| \frac{1}{h_1} S_{h_1}(x) \right|^2 \, dx \ll (\log X)^{-B} + \frac{1}{Y} \int \limits_Y^{2Y} \left| \frac{1}{h_1} S_{h_1}(x) - \frac{1}{h_2} S_{h_2}(x) \right|^2 \, dx. \tag{5.4}$$

Then Lemma 4.5 (Parseval) with $T_0 = (\log X)^{2B}$ and $h_2 = Y/T_0^3 = Y/(\log X)^{6B}$ gives

$$J \ll (\log X)^{-B} + \int \limits_{T_0}^{Y/h_1} |G(1+it)|^2 \, dt + \max \limits_{T \ge Y/h_1} \frac{Y/h_1}{T} \int \limits_T^{2T} |G(1+it)|^2 \, dt. \tag{5.5}$$
Now for the latter integral over \([T, 2T]\), we apply the Lemma \[4.1\] (mean value) if \(T \geq X/2\), and apply Proposition \[5.1\] if \(T \in [Y/h_1, X/2]\). Doing so, \((5.5)\) becomes

\[
J \ll (\frac{Y}{h_1} + 1)(\log X)^{-B} + \max_{T \geq X/2} \frac{Y}{h_1}(T/X + 1)
\]

\[
+ \max_{Y/h_1 \leq T \leq X/2} \frac{Y}{h_1} \left( \frac{Q_t T}{Y} + 1 \right)(\log X)^{-B}
\]

\[(5.6)\] \[
\ll \left( \frac{Q_1}{h_1} + 1 \right)(\log X)^{-B} + \frac{Y}{h_1 X} \ll W(\log X)^{-B} = (\log X)^{A-B}.
\]

Here we used \(h_1 \geq H/W^5\), \(Q_1 = H/W^4\), and \((Y \in [X/W^4, X])\). Hence recalling \(B = 11A\) gives Proposition \[3.3\] as claimed. \(\square\)

5.1. Mean value of Dirichlet polynomials. In this subsection, we prove Proposition \[5.1\]. Recall the definitions \[2.3\], \[2.9\],

\[
[P_1, Q_1] = [(\log X)^{3/4}, (\log X)^{\psi(X)-4A}],
\]

\[
[P_2, Q_2] = [\exp ((\log X)^{2/3+\delta/2}), \exp ((\log X)^{1-\delta/2})],
\]

\[
S_d(X, A, \delta) = \{m \leq X/d : \exists p_1, p_2 \mid m \text{ with } p_j \in [P_j, Q_j]\}.
\]

Let \(B = 11A\), and \(\alpha = 1/5\). Let \(V = P_1^{1/3} = (\log X)^B\) and define the prime polynomial

\[(5.7)\] \[
Q_{v,j}(s) := \sum_{P_j \leq p \leq Q_j \atop e^{v/V} \leq p \leq e^{(v+1)/V}} \frac{\lambda(p)\chi(p)}{p^s}.
\]

Note \(Q_{v,j}(s) \neq 0\) only if \(v \in \mathcal{T}_j := \{v \in \mathbb{Z} : P_j \leq e^{v/V} \leq Q_j\} = [\lceil V \log P_j \rceil, V \log Q_j]\).

We decompose \([T_0, T] = \mathcal{T}_1 \cup \mathcal{T}_2\) as a disjoint union, where \(\mathcal{T}_2 = [0, 1] \setminus \mathcal{T}_1\) and

\[(5.8)\]

\[
\mathcal{T}_1 = \{t : \lvert Q_{v,1}(1 + it) \rvert \leq e^{-av/V} \ \forall v \in \mathcal{T}_1\}.
\]

For \(j = 1, 2\) denote by \(S_{d,j}^{(j)}\) the integers containing a prime factor in the interval \([P_i, Q_i]\) with \(i \neq j\) and possibly, but not necessarily, with \(i = j\). That is,

\[
S_{d,j}^{(j)} = \{m \leq X/d : \exists p \mid m \text{ with } p \in [P_i, Q_i] \text{ for } i \neq j\}.
\]

Also define the polynomial

\[(5.9)\] \[
R_{v,j}(s) = \sum_{Y^{-v/V} \leq m \leq 2Ye^{-v/V} \atop m \in S_{d,j}^{(j)}} \frac{\lambda(m)\chi(m)}{m^s} \cdot \frac{1}{\# \{m \mid P_j \leq p \leq Q_j\} + 1}.
\]

Now Lemma \[4.7\] (Ramaré) applies with \(V = V, P = P_j, Q = Q_j\) and \(a_m = \lambda \chi 1_{S_j}(m)\), \(c_p = \lambda \chi(p), b_m = \lambda \chi 1_{S_j}(m)\), giving

\[
\int_{T_j} |G(1 + it)|^2 \ dt \ll V \log Q_j \sum_{v \in \mathcal{T}_j} \int_{T_j} |Q_{v,j}(1 + it) R_{v,j}(1 + it)|^2 \ dt \]

\[
+ \frac{1}{V} + \frac{1}{P_j} + \sum_{Y \leq n \leq 2Y \atop \text{prime } p \in [P_j, Q_j]} \frac{1_{S_d}(n)}{n}.
\]

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We crucially note the latter sum vanishes since each \( n \in S_d \) has a prime factor \( p \in [P_j, Q_j] \). Summing over \( j = 1, 2 \), the second and third terms above contribute

\[
\ll \sum_{1 \leq j \leq 2} \left( \frac{1}{T} + \frac{1}{P_j} \right) \ll \frac{1}{T} = (\log X)^{-B}.
\]

Hence the desired integral is

\[
\int_{T_0}^{T} |G(1 + it)|^2 \, dt = \int_{T_1 \cup T_2} |G(1 + it)|^2 \, dt \ll E_1 + E_2 + (\log X)^{-B},
\]

where

\[
E_j = V \log Q_j \sum_{v \in \mathcal{I}_j} \int_{T_j} |Q_{v,j}(1 + it) R_{v,j}(1 + it)|^2 \, dt.
\]

Hence it suffices to bound \( E_1, E_2 \ll (Q_1 T/Y + 1)(\log X)^{-B} \).

**Bound for \( E_1 \):** By definition of \( t \in T_1 \), we have \( |Q_{v,1}(1 + it)| \leq e^{-\alpha v/V} \) for all \( v \in \mathcal{I}_1 \), so

\[
E_1 \ll V \log Q_1 \sum_{v \in \mathcal{I}_1} e^{-2\alpha v/V} \int_{T_1} |R_{v,1}(1 + it)|^2 \, dt \ll V \log Q_1 \sum_{v \in \mathcal{I}_1} e^{-2\alpha v/V} \left( \frac{T}{Y/e^\alpha v} + 1 \right)
\]

by Lemma 4.1 (mean value). Summing the resulting geometric series gives

\[
E_1 \ll V \log Q_1 \frac{P_1^{-2\alpha}}{1 - e^{-2\alpha v/V}} \left( \frac{Q_1 T}{Y} + 1 \right) \ll (\log X)^{1-6\alpha} \left( \frac{Q_1 T}{Y} + 1 \right) \ll (\log X)^{-B} \left( \frac{Q_1 T}{Y} + 1 \right),
\]

noting \( V/(1 - e^{-2\alpha v/V}) = O(1) \) and \( 1 - 6B/5 < -B \).

**Bound for \( E_2 \):** We choose the maximizing \( v \in \mathcal{I}_2 \) for \( E_2 \). Thus since \( |\mathcal{I}_2| < V \log Q_2 \),

\[
E_2 = V \log Q_2 \sum_{v \in \mathcal{I}_2} \int_{T_2} |Q_{v,2} \cdot R_{v,2}(1 + it)|^2 \, dt \ll (V \log Q_2)^2 \int_{T_2} |Q_{v,2} \cdot R_{v,2}(1 + it)|^2 \, dt
\]

\[
\leq (V \log Q_2)^2 \sum_{n} \sup_{t_n \in [n,n+1] \cap T_2} |Q_{v,2} \cdot R_{v,2}(1 + it_n)|^2
\]

\[
\leq 2(V \log Q_2)^2 \sum_{t \in \mathcal{W}} |Q_{v,2} \cdot R_{v,2}(1 + it)|^2;
\]

for a well-spaced set \( \mathcal{W} \subset \mathcal{T}_2 \). For instance, one may take \( \mathcal{W} \) as the even or odd integers in \( \mathcal{T}_2 \) (choose the parity that gives a larger contribution). We shall see \( \mathcal{W} \) is easier to analyze than \( \mathcal{T}_2 \) itself.

Now is the critical step for the choice \( g = \lambda_X \) and \( \log P_2 = (\log X)^{2/3+\delta} \): by Lemma 4.5 (Vinogradov-Korobov), we have for all \( t \in [T_0, T] \)

\[
|Q_{v,2}(1 + it)| \leq \frac{\log X}{1 + T_0} (\log X)^{-B} \ll (\log X)^{-B},
\]

\[
(5.13)\]
for $T_0 = (\log X)^{2B}$ and $B = 11A$. So by Lemma 4.3 (Halász-Montgomery), we have

\[
E_2 \ll (V \log Q_2)^2(\log X)^{2-4B} \sum_{t \in \mathcal{W}} |R_{t,2}(1 + it)|^2
\]

\[
\ll (V \log Q_2)^2(\log X)^{2-4B}(Y e^{-v/V} + |\mathcal{W}| \sqrt{T}) \frac{e^{v/V}}{Y}
\]

\[
\ll (\log X)^{-B}(1 + |\mathcal{W}| \sqrt{T} Q_2)
\]

recalling $\log T \asymp \log X$, $V = (\log X)^B$, and $e^{v/V} \leq Q_2$.

Thus it suffices to bound $|\mathcal{W}|$. We shall obtain

\[
(5.14) \quad E_2 \ll (\log X)^{-B} \left(1 + \frac{T}{Y}\right) \ll_{A,\delta} (\log X)^{-B}
\]

provided we show $|\mathcal{W}| \ll T^{1/2}/Q_2$. To prove this, by definition of $\mathcal{T}_2 \supset \mathcal{W}$, we first partition $\mathcal{W} = \bigcup_{u \in \mathcal{T}_1} \mathcal{W}^{(u)}$ where

\[
|Q_{u,1}(1 + it)| > e^{-u\alpha/V} \quad \text{for all} \quad t \in \mathcal{W}^{(u)}.
\]

Hence for each $u \in \mathcal{T}_1$, we may apply Lemma 4.4 to the prime polynomial $Q_{u,1}$ with $U = e^{u\alpha/V}$ and $L = e^{u/V}$, so that

\[
(5.15) \quad |\mathcal{W}| = \sum_{u \in \mathcal{T}_1} |\mathcal{W}^{(u)}| \ll \sum_{u \in \mathcal{T}_1} U^2 T^2 \frac{\log U \log T}{\log L} \ll |\mathcal{Z}_1| U^2 T^{2\alpha + \frac{2 \log \log T}{\log L}} \ll T^{2/5+2/5A+o(1)},
\]

since $|\mathcal{Z}_1| \ll V \log Q_1 \ll T^{o(1)}$, $U^2 \ll Q_2^{2\alpha} \ll T^{o(1)}$, $\log L \gg \log P_1 \geq 5A \log \log T$, by recalling $[P_1, Q_1] = [(\log X)^{33A}, (\log X)^{\psi(X)-44A}]$ and $T \in [X^{1/4}, X^2]$. Hence $A > 5$ gives $|\mathcal{W}| \ll T^{1/2}/Q_2$, and completes the proof of Proposition 5.1.

6. Average Chowla-type correlations

In this section, we establish the results for higher correlations stated in the introduction.

We first exhibit quantitative cancellation among a broad class of correlations with a typical factor $1_S\mu$. We use a standard ‘van der Corput’ argument and then apply the key Fourier estimate.

**Lemma 6.1.** Given any $A > 5$, $\delta > 0$, let $S = S(X, A, \delta)$ as in (2.5). Assume $H < X$ satisfies $\log H/\log_2 X \to \infty$, and $G : \mathbb{N} \to \mathbb{C}$ satisfies $\sum_{n \leq X} |G(n)|^2 \ll X (\log X)^{A/20}$. Then

\[
\sum_{h \leq H} \left| \sum_{n \leq X} 1_S \mu(n + h)G(n) \right|^2 \ll_{A,\delta} \frac{HX}{(\log X)^{A/40}}.
\]

**Proof.** Let $g = 1_S\mu$. By Cauchy-Schwarz it suffices to show

\[
\frac{H X^2}{(\log X)^{A/20}} \gg \sum_{h \leq H} \left| \sum_{n \leq X} g(n + h)G(n) \right|^2 = \sum_{n, n' \leq X} G(n) \overline{G(n')} \sum_{h \leq H} g(n + h)g(n' + h).
\]

Using Cauchy-Schwarz again, the right hand side above is bounded by

\[
\sum_{n \leq X} |G(n)|^2 \cdot \left( \sum_{n, n' \leq X} \left| \sum_{h \leq H} g(n + h)g(n' + h) \right|^2 \right)^{1/2}.
\]
recalling $g$ is supported on $[1, X]$. By assumption $\sum_{n \leq X} |G(n)|^2 \ll X \log X^{A/20}$, so it suffices to prove

$$\frac{H^2 X^2}{(\log X)^{A/5}} \gg \sum_{n, n'} \left| \sum_{h \leq H} g(n + h) g(n' + h) \right|^2 = \sum_{|h| < H} \left| \sum_{n} g(n) g(n + h) \right|^2.$$ 

But this indeed holds: since $g = 1_{S\mu}$, we apply Lemma 2.1 (Fourier bound) with $f = g = 1_{S\mu}$. Thus the trivial bound $F(X) \ll X$ and Theorem 2.2 (Key Fourier estimate) give

$$\sum_{|h| \leq H} \left| \sum_{n \leq X} 1_{S\mu}(n) 1_{S\mu}(n + h) \right|^2 \ll F(X + 2H) \sup_{\alpha} \int_0^X \left| \sum_{x \leq n \leq x + 2H} 1_{S\mu}(n) e(n\alpha) \right| dx \ll \frac{HX^2}{(\log X)^{A/5}}.$$ 



We now prove the main technical result of the article, which handles averaged Chowla-type correlations with $m \geq 1$ copies of the Möbius function $\mu$ and with any function $G : \mathbb{N} \to \mathbb{C}$ of ‘moderate growth’ which is ‘amenable to sieves.’

**Theorem 6.2.** Given any $A > 5$, $\delta > 0$, let $S = S(X, A, \delta)$ as in (2.5). Assume $H < X$ satisfies $\log H / \log_2 X \to \infty$, and $G : \mathbb{N} \to \mathbb{C}$ satisfies $\sum_{n \leq X} |G(n)|^2 \ll X \log X^{A/20}$. Then

$$\sum_{h_1, \ldots, h_m \leq H} \left| \sum_{n \leq X} G(n) \prod_{j=1}^m \mu(n + h_j) \right|^2 \ll_{A, \delta} \sum_{h_1, \ldots, h_m \leq H} \sum_{n \leq X} |G(n)| \prod_{j=1}^m 1_{\mathcal{S}}(n + h_j) + \frac{mX H^m}{(\log X)^{A/40}}.$$ 

**Proof.** We observe from Lemma 6.1 that any correlation with a factor $1_{S\mu}$ exhibits strong cancellation. So we split up $\mu = 1_{S^m \mu} + 1_{S\mu}$ until each term has a factor $1_{S\mu}$, except for one term with $m$ factors of $1_{S\mu}$. Thus the product in (6.1) becomes

$$\prod_{j=1}^m \mu(n + h_j) = \prod_{j=1}^m 1_{S\mu}(n + h_j) + \sum_{i=1}^m 1_{S\mu}(n + h_i) \prod_{1 \leq j < i} 1_{S\mu}(n + h_j) \prod_{i < j \leq m} \mu(n + h_j).$$ 

Hence we bound the left hand side of (6.1) by $\Sigma_1 + \Sigma_2$, where

$$\Sigma_1 = \sum_{h_1, \ldots, h_m \leq H} \left| \sum_{n \leq X} G(n) \prod_{j=1}^m 1_{\mathcal{S}}(n + h_j) \right|,$$

$$\Sigma_2 = \sum_{i=1}^m \sum_{h_1, \ldots, h_m \leq H} \left| \sum_{n \leq X} 1_{S\mu}(n + h_i) G_i(n) \right|,$$

where $G_i(n) = G(n) \prod_{1 \leq j < i} 1_{S\mu}(n + h_j) \prod_{i < j \leq m} \mu(n + h_j).$ In particular $|G_i(n)| \leq |G(n)|$. Thus Lemma 6.1 applies to each $1_{S(n + h_i) G_i(n)}$, so that $\Sigma_2$ is bounded by

$$\Sigma_2 \ll_{A} \frac{mH^m X}{W^{1/40}}.$$ 

6.1. Deduction of results.

Proof of Theorem 6.3. Let \( G(n) = \prod_{j=1}^{k} \Lambda(n + a_j) \) for the tuple \( A = \{a_1, \ldots, a_k\} \). Then \( \sum_{n \leq X} |G(n)|^2 \ll X(\log X)^{k} \), and by a standard sieve upper bound

\[
\sum_{n \leq X} G(n) \prod_{j=1}^{m} 1_{\mathcal{S}}(n + h_j) \ll_{m,\mathcal{A}} X \left( \prod_{p \in [P_1, Q_1]} + \prod_{p \in [P_2, Q_2]} \right) \left( 1 - \frac{1}{p} \right)^m \ll_{m,\mathcal{A}} \frac{X}{\psi_{\delta}(X)^m},
\]

using Mertens’ product theorem. Hence Theorem 6.2 with \( A = 20(m + k) \) gives

\[
(6.4) \quad \sum_{h_1, \ldots, h_m \leq H} \left| \sum_{n \leq X} \prod_{j=1}^{k} \Lambda(n + a_j) \prod_{j=1}^{m} \mu(n + h_j) \right| \ll_{m,\delta,\mathcal{A}} \frac{XH^m}{\psi_{\delta}(X)^m}.
\]

\[ \square \]

Proof of Theorem 6.6. Let \( G(n) = \prod_{i=1}^{j} d_{k_i}(n + a_i) \) for the tuple \( A = \{a_1, \ldots, a_j\} \) and recall \( k = \sum_{i=1}^{j} k_i \). Using work of Henriot [5, Theorem 3], we may obtain

\[
\sum_{h \leq H} \sum_{n \leq X} 1_{\mathcal{S}}(n + h) \prod_{i=1}^{j} d_{k_i}(n + a_i) \ll_{\mathcal{A}} \frac{HX}{(\log X)^{j+1}} \sum_{n \leq \sqrt{X}} \frac{1_{\mathcal{S}}(n)}{n} \prod_{i=1}^{j} \sum_{n \leq X} \frac{d_{k_i}(n)}{n}.
\]

By the divisor bound \( \sum_{n \leq X} d_{k}(n)/n \ll X(\log X)^{k} \), and by Mertens’ product theorem

\[
\sum_{n \leq \sqrt{X}} \frac{1_{\mathcal{S}}(n)}{n} \ll \log X \left( \prod_{p \in [P_1, Q_1]} + \prod_{p \in [P_2, Q_2]} \right) \left( 1 - \frac{1}{p} \right) \ll_{\delta} \frac{\log X}{\psi_{\delta}(X)}.
\]

Thus since \( \sum_{n \leq X} |G(n)|^2 \ll X(\log X)^{k} \), Theorem 6.2 with \( A = 20k \) gives

\[
\sum_{h \leq H} \sum_{n \leq X} \mu(n + h) \prod_{i=1}^{j} d_{k_i}(n + a_i) \ll_{\delta,\mathcal{A}} \frac{HX}{\psi_{\delta}(X)}(\log X)^{k-j}.
\]

\[ \square \]

6.2. Almost all shifts. Corollary 1.5 follows from the following result by the triangle inequality for \( g_j = \mu \).

Theorem 6.3. Suppose \( \log H/\log_2 X \to \infty \) as \( X \to \infty \). Let \( g_1 = \mu \) and take any \( g_j : \mathbb{N} \to \mathbb{C} \) with \( |g_j| \leq 1 \) for \( 1 < j \leq k \). Then for any fixed shifts \( h_2, \ldots, h_k \leq H, K > 0 \) we have

\[
(6.5) \quad \sum_{p \leq X} \prod_{j=1}^{k} g_j(p + h_j) = o(\pi(X)),
\]

for all except \( O_K(H(\log X)^{-K}) \) shifts \( h_1 \leq H \).

Proof. Given \( \varepsilon > 0 \) and fixed shifts \( h_2, \ldots, h_k \leq H \), we aim to show \( |\mathcal{E}| \ll_{\varepsilon} H(\log X)^{-K} \) for the exceptional set

\[
(6.6) \quad \mathcal{E} = \left\{ h \leq H : \left| \sum_{p \leq X} \mu(p + h) \prod_{j=2}^{k} g_j(p + h_j) \right| > 2\varepsilon \pi(X) \right\}.
\]
To this, by Markov’s inequality we have

\[ |\mathcal{E}|(\varepsilon \pi(X)) \ll \sum_{h \in \mathcal{E}} \left| \sum_{p \leq X} \mu(p + h) \prod_{j=2}^{k} g_j(p + h_j) \right| \]

\[ \ll \sum_{h \in \mathcal{E}} \left| \sum_{p \leq X} 1_{S}(p + h) \right| + \sum_{h \leq H} \left| \sum_{p \leq X} 1_{S} \mu(p + h) \prod_{j=2}^{k} g_j(p + h_j) \right| \]

\[ \ll A \frac{\pi(X)}{\psi(X)} \sum_{h \in \mathcal{E}, \, p|h, \, p>P_1} (1 + \frac{1}{p}) + \frac{H \pi(X)}{(\log X)^{A/40}}, \]

using Proposition 6.1 when \( p + h \in S \), and a standard sieve upper bound [2] Theorem 7.1 when \( p + h \notin S \). Here \( S = S(X, A, \delta) \) as in (2.5) with \( A = 80K \) and \( \delta = 1/10 \), say.

Observe for any \( h \leq H = (\log X)^{\psi(X)} \) the above product is at most \( \prod_{P_1 < p \leq z}(1 + \frac{1}{p}) \ll \frac{\log z}{\log P_1} \) where \( z = P_1 + \psi(X) \log 2X \). Recalling \( P_1 = (\log X)^{33} \) this gives

\[ \frac{\pi(X)}{\psi(X)} \sum_{h \in \mathcal{E}, \, p|h, \, p>P_1} (1 + \frac{1}{p}) = o(\mathcal{E} \pi(X)). \]

Hence we conclude \( |\mathcal{E}| \ll \frac{1}{\varepsilon} H (\log X)^{-K} \) as desired.

7. Non-pretentious multiplicative functions

In this section we prove Theorem 1.8 which we restate below.

**Theorem 1.8** Let \( H = X^\theta \) for \( \theta \in (0, 1) \), and take a multiplicative function \( f : \mathbb{N} \to \mathbb{C} \) with \( |f| \leq 1 \). There exists \( \rho \in (0, \frac{1}{8}) \) such that, if \( M(f; X^2/H^{2-\rho}, Q) \to \infty \) as \( X \to \infty \) for each fixed \( Q \), then

\[ \sum_{h \leq H} \left| \sum_{p \leq X} f(p + h) \right| = o_{\theta, \rho}(H \pi(X)). \]

**Proof.** Consider the exponential sum \( F_x(\alpha) = \sum_{x \leq m \leq x+H} f(m)e(m\alpha) \). The hypotheses of our theorem are made in order to satisfy [12] Theorem 1.4], which in this case gives

\[ \int_0^X \sup_{\alpha} |F_x(\alpha)| \, dx = o_{\theta, \rho}(HX). \]

We critically note the supremum is inside the integral.

Now on to the proof, it suffices to show \( S_f = o(HX) \) where

\[ S_f := \sum_{h \leq H} \left| \sum_{n \leq X} \Lambda(n)f(n + h) \right| \ll \sum_{h \leq 2H} (2H - h) \left| \sum_{n \leq X} \Lambda(n)f(n + h) \right|. \]
We introduce coefficients $c(h)$ to denote the phase of $\sum_{n \leq X} \Lambda(n) f(n + h)$, so that
\[
S_f \ll \frac{1}{H} \sum_{h \leq H} (H - h) c(h) \sum_{n \leq X} \Lambda(n) f(n + h)
\]
\[
= \frac{1}{H} \sum_{h \leq H} c(h) \sum_{n \leq X} \Lambda(n) \sum_{m \leq X + H} f(m) 1_{m = n + h} \cdot \int_0^X 1_{x \leq n, m \leq x + H} \, dx
\]
\[
= \frac{1}{H} \int_0^X \int_{0}^{1} \sum_{h \leq H} c(h) e(h\alpha) \sum_{x \leq n, m \leq x + H} \Lambda(n) f(m) e((n - m)\alpha) \, d\alpha \, dx,
\]
by orthogonality $\int_0^1 e(n\alpha) \, d\alpha = 1_{n=0}$. That is, we have the following triple convolution
\[
(7.2) \quad S_f \ll \frac{1}{H} \int_0^X \int_0^1 C_0(\alpha)L_x(-\alpha)F_x(\alpha) \, d\alpha \, dx,
\]
denoting the sums $C_0(\alpha) = \sum_{h \leq H} c(h)e(h\alpha)$ and $L_x(\alpha) = \sum_{x \leq h \leq x + H} \Lambda(n)e(n\alpha)$.

We shall split the inner integral on $\alpha$ according to the size of $L_x$. Specifically, for each $x$ let $T_x = \{\alpha \in [0, 1]: |L_x(\alpha)| \geq \delta H\}$. Then by Markov’s inequality, $T_x$ has measure
\[
(7.3) \quad \int_{T_x} d\alpha \leq \frac{1}{(\delta H)^4} \int_{T_x} |L_x(\alpha)|^4 d\alpha \ll \frac{1}{\delta^4 H},
\]
since the Fourier identity implies
\[
\int_0^1 |L_x(\alpha)|^4 d\alpha = \sum_{x \leq n_1, n_2, n_3, n_4 \leq x + H} \Lambda(n_1)\Lambda(n_2)\Lambda(n_3)\Lambda(n_4) 1_{n_1 + n_2 = n_3 + n_4}
\]
\[
\ll (\log X)^4 \sum_{x \leq p_1, p_2, p_3, p_4 \leq x + H} 1 \ll \theta H^3,
\]
by a standard sieve upper bound [2, Theorem 7.1]. Thus plugging (7.3) into (7.2), we obtain
\[
S_f \ll \frac{1}{H} \int_0^X \int_{[0,1] \setminus T_x} C_0(\alpha)L_x(-\alpha)F_x(\alpha) \, d\alpha \, dx + \frac{1}{\delta^4 H^2} \int_0^X \sup_{\alpha \in T_x} |C_0(\alpha)L_x(-\alpha)F_x(\alpha)| \, dx.
\]

Denote the two integrals above by $I_1$ and $I_2$. Observe $I_2 \ll \theta \delta^{-4} \int_0^X \sup_{\alpha} |F_x(\alpha)| \, dx$, using $|C_0(\alpha)| \ll H$ trivially and $|L_x(\alpha)| \ll \theta H$ by the Brun–Titchmarsh theorem. Then by definition of $T_x$, Cauchy-Schwarz implies
\[
I_1 \leq \delta \int_0^X \int_{[0,1] \setminus T_x} |C_0(\alpha)F_x(\alpha)| \, d\alpha \, dx \leq \delta \int_0^X \left( \int_0^1 |C_0(\alpha)|^2 \, d\alpha \cdot \int_0^1 |F_x(\alpha)|^2 \, d\alpha \right)^{1/2} \, dx \leq \delta H X,
\]
by Parseval’s identity applied to $C_0$ and $F_x$. Thus combining bounds for $I_1, I_2$ gives
\[
(7.4) \quad S_f \ll \theta \delta H X + \delta^{-4} \int_0^X \sup_{\alpha} |F_x(\alpha)| \, dx.
\]
Hence taking $\delta \to 0$, the Fourier uniformity bound (7.1) gives $S_f = o_{\theta, \rho}(HX)$ as claimed. \quad \Box
ACKNOWLEDGMENTS

The author is grateful to Joni Teräväinen for suggesting the problem and for many valuable discussions. The author thanks Joni Teräväinen and James Maynard for careful readings of the manuscript and for helpful feedback. The author is supported by a Clarendon Scholarship at the University of Oxford.

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