Research Article

On the General Dedekind Sums and Two-Term Exponential Sums

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We use the analytic methods and the properties of Gauss sums to study the computational problem of one kind hybrid mean value involving the general Dedekind sums and the two-term exponential sums, and give an interesting computational formula for it.

1. Introduction

Let \( q \) be a natural number and \( h \) an integer prime to \( q \). The classical Dedekind sums

\[ S(h, q) = \sum_{a=1}^{q} (\left(\frac{a}{q}\right))(\left(\frac{ah}{q}\right)), \]

(1)

where

\[ (x) = \begin{cases} x - \lfloor x \rfloor - \frac{1}{2}, & \text{if } x \text{ is not an integer;} \\ 0, & \text{if } x \text{ is an integer}, \end{cases} \]

(2)

describes the behaviour of the logarithm of the eta-function (see \cite{1,2}) under modular transformations. About the various arithmetical properties of \( S(h, q) \), many people had studied it and obtained a series of interesting results; see \cite{3–9}.

For example, Wang and Zhang \cite{6} and Wang and Pan \cite{7} had studied the hybrid mean value involving Dedekind sums and two-term exponential sums and proved the computational formulae

\[ \sum_{m=1}^{p-1} \sum_{n=1}^{p-1} |C(m, n, 3, 1; p)|^2 \cdot S(m\bar{n}, p) = \begin{cases} p \cdot h_p^2, & \text{if } p = 12k + 7, \\ 3p \cdot h_p^2, & \text{if } p = 12k + 11, \\ 0, & \text{if } p = 4k + 1, \end{cases} \]

(3)

\[ \sum_{m=1}^{p-1} \sum_{n=1}^{p-1} |C(m, n, 4, 2; p)|^2 \cdot S(m\bar{n}, p) = \begin{cases} 2p \cdot h_p^2, & \text{if } p = 8k + 7, \\ 0, & \text{if } p = 4k + 1 \text{ or } 8k + 3, \end{cases} \]

where \( h_p \) denotes the class number of the quadratic field \( \mathbb{Q}(\sqrt{-p}) \), \( \bar{n} \) denotes the solution of the congruence equation \( nx \equiv 1 \mod p \), and the two-term exponential sums \( C(m, n, h, k; q) \) are defined as

\[ C(m, n, h, k; q) = \sum_{a=1}^{q} e\left(\frac{ma^h + na^k}{q}\right), \]

(4)

\( e(y) = e^{2\pi iy} \). Some results related to \( C(m, n, h, k; q) \) can be found in \cite{10,11}.
On the other hand, Zhang [12] introduced a generalized Dedekind sums as follows:

$$S(h, m, q) = \sum_{a=1}^{q} B_n\left(\frac{a}{q}\right) \overline{B}_n\left(\frac{ah}{q}\right),$$  \hspace{1cm} (5)

where

$$B_n(x) = \begin{cases} B_n(x - [x]), & \text{if } x \text{ is not an integer;} \\ 0, & \text{if } x \text{ is an integer}, \end{cases}$$  \hspace{1cm} (6)

$$B_n(x)$$ denotes the nth Bernoulli polynomial and $$\overline{B}_n(x)$$ defined for all real $$0 < x \leq 1$$ is called the nth Bernoulli periodic function.

If $$n = 1$$, then $$S(h, 1; q) = S(h, q)$$, the classical Dedekind sums. About the arithmetical properties of $$S(h, n, q)$$ and $$B_n(x)$$, one can find them in [3, 12]. In this paper as a note of [6, 7], we consider the following hybrid mean value:

$$\sum_{u=1}^{p-1} \sum_{v=1}^{p-1} C(u, m, h, k; p) \cdot C(v, m, h, k; p) \cdot S(u \cdot \overline{v}, n; p)$$  \hspace{1cm} (7)

and use the analytic methods and the properties of Gauss sums to give an exact computational formula for (7). That is, we will prove the following conclusion.

**Theorem 1.** Let $$p \geq 3$$ be a prime and $$n$$ any positive integer. Then for any positive integers $$h$$ and $$k$$ with $$(hk, p - 1) = 1$$ and integer $$m$$ with $$(m, p) = 1$$, one has the identity

$$\sum_{u=1}^{p-1} \sum_{v=1}^{p-1} C(u, m, h, k; p) \cdot C(v, m, h, k; p) \cdot S(u \cdot \overline{v}, n; p) = p^2 \cdot S(1, n; p).$$  \hspace{1cm} (8)

For $$n = 1, 2, 3$$, from our theorem we may immediately deduce the following.

**Corollary 2.** Let $$p \geq 3$$ be a prime. Then for any positive integers $$h$$ and $$k$$ with $$(hk, p - 1) = 1$$ and integer $$m$$ with $$(m, p) = 1$$, one has the identity

$$\sum_{u=1}^{p-1} \sum_{v=1}^{p-1} C(u, m, h, k; p) \cdot C(v, m, h, k; p) \cdot S(u \cdot \overline{v}, 1; p) = \frac{1}{12} p (p - 1) (p - 2).$$  \hspace{1cm} (9)

**Corollary 3.** Let $$p \geq 3$$ be a prime. Then for any positive integers $$h$$ and $$k$$ with $$(hk, p - 1) = 1$$ and integer $$m$$ with $$(m, p) = 1$$, one has the identity

$$\sum_{u=1}^{p-1} \sum_{v=1}^{p-1} C(u, m, h, k; p) \cdot C(v, m, h, k; p) \cdot S(u \cdot \overline{v}, 2; p) = \frac{1}{180} \left( p - 1 \right) \left( p^3 - 4 p^2 + 6 p + 6 \right).$$  \hspace{1cm} (10)

**Corollary 4.** Let $$p \geq 3$$ be a prime. Then for any positive integers $$h$$ and $$k$$ with $$(hk, p - 1) = 1$$ and integer $$m$$ with $$(m, p) = 1$$, one has the identity

$$\sum_{u=1}^{p-1} \sum_{v=1}^{p-1} C(u, m, h, k; p) \cdot C(v, m, h, k; p) \cdot S(u \cdot \overline{v}, 3; p) = \frac{1}{840} \left( p - 2 \right) \left( p - 1 \right) \left( p + 1 \right) \left( p + 2 \right) \left( p^2 + 5 \right).$$  \hspace{1cm} (11)

For general integer $$q > 3$$, whether there exists an exact computational formula for the hybrid mean value

$$\sum_{u=1}^{q} \sum_{v=1}^{q} C(u, m, h, k; q) \cdot C(v, m, h, k; q) \cdot S(u \cdot \overline{v}, n; q)$$

is an open problem, where $$h$$ and $$k$$ are positive integers with $$(hk, \phi(q)) = 1$$ and $$(m, q) = 1$$.  \hspace{1cm} (12)

**2. Several Lemmas**

In this section, we will give two lemmas, which are necessary in the proof of our theorem. Hereinafter, we will use many properties of character sums and Gauss sums; all of these can be found in [13], so they will not be repeated here. First we have the following.

**Lemma 1.** Let $$p$$ be an odd prime and $$\chi$$ any nonprincipal character mod $$p$$. Then for any positive integers $$h$$ and $$k \geq 1$$ with $$(hk, p - 1) = 1$$ and any integer $$m$$, one has the identity

$$\sum_{u=1}^{p-1} \chi(u) \cdot C(u, m, h, k; p) = \chi^h(m) \cdot \tau(\chi) \cdot \tau(\chi^k),$$

where $$\tau(\chi) = \sum_{a=1}^{p-1} \chi(a) e(a/p)$$ denotes the Gauss sums.

Proof. From the definitions of $$C(u, m, h, k; p)$$ and Gauss sums we have

$$\sum_{u=1}^{p-1} \chi(u) \cdot C(u, m, h, k; p) = \sum_{u=1}^{p-1} \chi(u) \sum_{a=1}^{p} e\left(\frac{ua^h + ma^k}{p}\right) = \sum_{a=1}^{p} \frac{\sum_{u=1}^{p-1} \chi(u) \cdot e\left(\frac{ua^h + ma^k}{p}\right)}{p} = \tau(\chi) \sum_{a=1}^{p} \chi(a) \cdot e\left(\frac{ma^k}{p}\right).$$  \hspace{1cm} (14)

Since $$(hk, p - 1) = 1$$, then there exits one integer $$\overline{k}$$ such that $$\overline{k} \cdot k \equiv 1 \mod (p - 1)$$ and $$(k, p - 1) = 1$$. From the properties of reduced residue system mod $$p$$ we know that if $$a$$ pass through a reduced residue system mod $$p$$, then $$a^k$$ also...
pass through a reduced residue system mod $p$. So from (14) and Fermat little theorem we have

$$\sum_{u=1}^{p-1} \chi(u) \cdot C(u, m, h, k; p)$$

$$= \left( \frac{\chi}{\chi(\chi)} \right)^{p-1} \sum_{u=1}^{p-1} \left( a^k \right) \cdot e\left( \frac{ma}{p} \right)$$

$$= \left( \frac{\chi}{\chi(\chi)} \right)^{p-1} \sum_{u=1}^{p-1} \left( a^h \right) \cdot e\left( \frac{ma}{p} \right)$$

$$= \chi^h(m) \cdot \chi(\chi) \cdot \tau\left( \chi^h \right).$$

This proves Lemma 1.

**Lemma 2.** Let $q \geq 3$ be an integer and $h$ any integer with $(h, q) = 1$. Then for any positive integer $n$, one has the following identities.

(i) If $n$ is an odd number, then

$$S(h, n; q) = \frac{(n!)^2}{4^{n-1} q^{2n-1} \pi^{2n}} \sum_{d \mid (q)} \sum_{\chi \mod d} \chi(h) |L(n, \chi)|^2.$$

(ii) If $n$ is an even number, then

$$S(h, n; q) = \frac{(n!)^2}{4^{n-1} q^{2n-1} \pi^{2n}} \sum_{d \mid (q)} \sum_{\chi \mod d} \chi(h) |L(n, \chi)|^2 - \frac{(n!)^2}{4^{n-1} \pi^{2n}} \cdot \zeta^2(n).$$

where $L(1, \chi)$ denotes the Dirichlet $L$-function corresponding to character $\chi \mod d$ and $\zeta(s)$ is the famous Riemann zeta-function.

**Proof.** See [12].

### 3. Proof of the Theorem

In this section, we will complete the proof of our theorem. If $n$ is an odd number, then note that for any nonprincipal character $\chi \mod p$, $|\tau(\chi)| = \sqrt{p}$ and $\tau(\chi) \cdot \tau(\chi) = \chi(-1) \cdot \tau(\chi) \cdot \tau(\chi) = \chi(-1) \cdot p$. So for any integer $u$ with $(u, p) = 1$, from (i) of Lemma 2 we have

$$\sum_{u=1}^{p-1} C(u, m, h, k; p) \cdot C(v, m, h, k; p)$$

$$= \frac{(n!)^2 \cdot p}{4^{n-1} (p - 1) \pi^{2n}}$$

$$\cdot C(v, m, h, k; p) \cdot C(u, m, h, k; p)$$

$$= \frac{(n!)^2 \cdot p}{4^{n-1} (p - 1) \pi^{2n}} \chi(u \cdot \overline{v}) |L(n, \chi)|^2$$

$$= \frac{(n!)^2 \cdot p^2}{4^{n-1} (p - 1) \pi^{2n}} \cdot |L(n, \chi)|^2 - \frac{(n!)^2 \cdot p^2}{4^{n-1} (p - 1) \pi^{2n}} \frac{\zeta^2(n)}{\pi^{2n}}.$$
From (18) and (20) we have the identity
\[
\sum_{u=1}^{p-1} C(u, m, h, k; p) \cdot C(v, m, h, k; p) \cdot S(u \cdot \bar{v}, 3; p) = \frac{1}{840} \cdot \frac{(p-2)(p-1)(p+1)(p+2)(p^2+5)}{p^3}.
\] (26)

This completes the proof of Corollary 4.

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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