A UNIFORMLY CONTINUOUS LINEAR EXTENSION PRINCIPLE IN TOPOLOGICAL VECTOR SPACES WITH AN APPLICATION TO LEBESGUE INTEGRATION

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Abstract. We prove a uniformly continuous linear extension principle in topological vector spaces from which we derive a very short and canonical construction of the Lebesgue integral of Banach space valued maps on a finite measure space. The Vitali Convergence Theorem and the Riesz-Fischer Theorem are shown to follow as easy consequences from our construction.

1. Introduction and motivation

Since the birth of Lebesgue’s theory of integration, various introductions and modifications of the theory have been proposed in an attempt to
(1) make it more elementary ([9]),
(2) unify it with Riemann’s approach to integration ([5],[6],[8],[11]),
(3) extend it to a Banach space valued context ([1],[4],[10]),
(4) present it in the abstract setting of functional analysis ([3],[7]).

In this paper we shall present a very short and canonical construction of the Lebesgue integral via a uniformly continuous linear extension principle in topological vector spaces (Theorem 2.2) which addresses the above mentioned topics. The Vitali Convergence Theorem and the Riesz-Fischer Theorem are shown to follow as easy consequences from our construction.

2. Uniformly continuous linear extension in TVS

The following lemma is a well known result in the theory of uniform spaces, see e.g. [2].

Lemma 2.1. Let $X$ be a uniform space, $A \subset X$ a dense subset and $f$ a uniformly continuous map of $A$ into a complete Hausdorff uniform space. Then there exists a unique uniformly continuous extension of $f$ to $X$.

We say that a collection $\mathcal{K}$ of subsets of a complex vector space is closed under the formation of finite linear combinations iff the set

$$\alpha K_1 + \beta K_2 = \{ \alpha x + \beta y \mid x \in K_1, y \in K_2 \}$$

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belongs to $\mathcal{K}$ for all $K_1, K_2 \in \mathcal{K}$ and $\alpha, \beta \in \mathbb{C}$. We now apply Lemma 2.1 to obtain the following uniformly continuous linear extension principle in complex topological vector spaces (TVS).

**Theorem 2.2.** (Uniformly continuous linear extension principle in TVS) Let $E$ be a TVS, $F \subset E$ a vector subspace, $\mathcal{K}$ a collection of sets $K \subset F$ which covers $F$ and is closed under the formation of finite linear combinations. Let $\lambda$ be a linear map of $F$ into a complete Hausdorff TVS $E'$ which is uniformly continuous on each $K \in \mathcal{K}$. Then $\tilde{F} = \bigcup_{K \in \mathcal{K}} K$ is a vector space containing $F$ and there exists a unique linear extension of $\lambda$ to $\tilde{F}$ which is uniformly continuous on the closure of each $K \in \mathcal{K}$.

**Proof.** For $x, y \in \tilde{F}$ and $\alpha, \beta \in \mathbb{C}$, choose $K_1, K_2 \in \mathcal{K}$ such that $x \in \overline{K_1}$ and $y \in \overline{K_2}$. Then

$$\alpha x + \beta y \in \alpha \overline{K_1} + \beta \overline{K_2} \subset \overline{\alpha K_1 + \beta K_2} \subset \tilde{F}$$

the latter inclusion being a consequence of the fact that $\mathcal{K}$ is closed under the formation of finite linear combinations. We conclude that $\tilde{F}$ is a vector space, which contains $F$ because $\mathcal{K}$ covers $F$. Furthermore, for each $K \in \mathcal{K}$, Lemma 2.1 shows that $\lambda |_K$ extends uniquely to a uniformly continuous map $\lambda |_K$ of $K$ into $E'$. For $x \in \tilde{F}$, choose $K \in \mathcal{K}$ such that $x \in K$ and put $\tilde{\lambda}(x) = \overline{\lambda |_K(x)}$. The assignment $\tilde{\lambda}$ is well-defined. Indeed, suppose that $x \in \overline{K_1 \cap K_2}$ with $K_1, K_2 \in \mathcal{K}$ and let $V$ be the neighbourhood filter of 0 in $E$. Then, for each $V \in \mathcal{V}$, choose $y_V \in K_1 \cap (x + V)$ and $z_V \in K_2 \cap (x + V)$. Notice that $(y_V)_{V \in \mathcal{V}}$ (resp. $(z_V)_{V \in \mathcal{V}}$) is a net in $K_1$ (resp. $K_2$) converging to $x$. Now

$$\overline{\lambda |_{K_1}(x)} - \overline{\lambda |_{K_2}(x)}$$

$$(\lambda |_{K_1} \text{ and } \lambda |_{K_2} \text{ are continuous})$$

$$= \lim \lambda |_{K_1}(y_V) - \lim \lambda |_{K_2}(z_V)$$

$$= \lim \lambda(y_V) - \lim \lambda(z_V)$$

$$= \lim [\lambda(y_V) - \lambda(z_V)]$$

$$= \lim \lambda(y_V - z_V)$$

$$(K_1 - K_2 \in \mathcal{K})$$

$$= \lim \lambda |_{K_1 - K_2}(y_V - z_V)$$

$$(\lambda |_{K_1 - K_2} \text{ is continuous})$$

$$= 0$$

establishing the fact that $\tilde{\lambda}$ is well-defined. The linearity of $\tilde{\lambda}$ is proved analogously. We conclude that $\tilde{\lambda}$ has the desired properties and we are done. $\square$

**3. Construction of the Lebesgue integral**

In this section we give a brief outline of how a very short and canonical construction of the Lebesgue integral can be obtained via Theorem 2.2. Let $\Omega = (\Omega, \mathcal{A}, \mu)$ be a finite measure space, $E = (E, \| \cdot \|)$ a complex separable Banach space and $\mathbf{M}(\Omega, E)$ the TVS of Borel measurable maps $f$.
of $\Omega$ into $E$, equipped with the topology of convergence in measure. That is, the sets

$$V_\epsilon = \{ f \in M(\Omega, E) \mid \mu(\{ \| f \| \geq \epsilon \}) < \epsilon \}, \ \epsilon > 0,$$

consist of a base for the neighbourhood filter of 0. It is well known that the uniform structure of $M(\Omega, E)$ is complete and pseudometrisable, see e.g. [3]. Unless otherwise stated, all subsets of $M(\Omega, E)$ are equipped with the uniformity of convergence in measure.

A map $s \in M(\Omega, E)$ is called simple iff there exists a finite measurable partition $A_1, \ldots, A_n$ of $\Omega$ such that $s$ assumes a unique value $s_i \in E$ on each $A_i$. We denote the collection of simple maps as $S(\Omega, E)$. Notice that $S(\Omega, E)$ is a vector subspace of $M(\Omega, E)$. We define the \textit{integral} of $s \in S(\Omega, E)$ as

$$\int s = \sum_i \mu(A_i) s_i$$

with $A_1, \ldots, A_n$ the measurable partition and $s_1, \ldots, s_n$ the values associated with $s$.

\textbf{Proposition 3.1.} The mapping $\int$ of $S(\Omega, E)$ into $E$ is linear and, if $s \in S(\Omega, E)$, then $\|s\| \in S(\Omega, \mathbb{C})$ and $\int \|s\| \leq \int \|s\|$. In particular, if $E = \mathbb{C}$, then the mapping $\int$ of $S(\Omega, \mathbb{C})$ into $\mathbb{C}$ is positive in the sense that $\int s \geq 0$ if $s \geq 0$ and monotonic in the sense that $\int s \geq \int t$ if $s \geq t$.

\textit{Proof.} This is standard. \hfill \Box

We call a set $E \subset S(\Omega, E)$ elementary iff for each $\epsilon > 0$ there exists $\delta > 0$ such that $\int \|s\| 1_A < \epsilon$ whenever $s \in E$ and $A \in \mathcal{A}$ with $\mu(A) < \delta$. The collection of elementary sets in $S(\Omega, E)$ is denoted as $\mathcal{E}(\Omega, E)$.

\textbf{Proposition 3.2.} $\mathcal{E}(\Omega, E)$ contains all sets $E \subset S(\Omega, E)$ which are totally bounded under the weak uniformity for the mapping $\int \| \cdot \|$ of $S(\Omega, E)$ into $\mathbb{C}$. Furthermore, $\mathcal{E}(\Omega, E)$ is closed under the formation of finite linear combinations.

\textit{Proof.} Let $E \subset S(\Omega, E)$ be totally bounded under the weak uniformity for the mapping $\int \| \cdot \|$ of $S(\Omega, E)$ into $\mathbb{C}$. In order to show that $E \in \mathcal{E}(\Omega, E)$, fix $\epsilon > 0$. Then there exists a finite set $E_0 \subset S(\Omega, E)$ such that for all $s \in E$

$$\min_{t \in E_0} \int \|s - t\| \leq \epsilon. \quad (1)$$

Now, by (1), for $s \in E$ and $A \in \mathcal{A}$,

$$\min_{t \in E_0} \int \|s\| 1_A - \int \|t\| 1_A \leq \min_{t \in E_0} \int \|s\| - \|t\| 1_A \leq \min_{t \in E_0} \int \|s - t\| \leq \epsilon$$

whence

$$\int \|s\| 1_A \leq \max_{t \in E_0} \int \|t\| 1_A + \epsilon. \quad (2)$$

Choose a constant $C > 0$ such that

$$\max_{t \in F} \sup_{x \in \Omega} |t(x)| \leq C.$$
Then, by (2), for \( s \in E \) and \( A \in \mathcal{A} \),

\[ \int \|s\|_A1_A \leq \int \|s\|_A1_A \leq \max_{C \in \mathcal{F}} \int \|t\|_A1_A + \epsilon \leq C\mu(A) + \epsilon \]

entailing that \( E \in \mathcal{E}(\Omega, E) \). We now show that \( \mathcal{E}(\Omega, E) \) is closed under the formation of finite linear combinations. Fix \( E_1, E_2 \in \mathcal{E}(\Omega, E) \) and \( \alpha, \beta \in \mathbb{C} \). Then, for \( s_1 \in E_1 \), \( s_2 \in E_2 \) and \( A \in \mathcal{A} \),

\[ \int \|\alpha s_1 + \beta s_2\|_A1_A \leq \int (|\alpha| \|s_1\| + |\beta| \|s_2\|)1_A = |\alpha| \int \|s_1\|1_A + |\beta| \int \|s_2\|1_A \]

immediately yielding that \( \alpha E_1 + \beta E_2 \in \mathcal{E}(\Omega, E) \).

\[ \square \]

**Proposition 3.3.** The uniformity of convergence in measure on \( S(\Omega, E) \) is weaker than the weak uniformity for the mapping \( f \circ \cdot \| \) of \( S(\Omega, E) \) into \( \mathbb{C} \), and these uniformities coincide on elementary sets. In particular, the mapping \( f \) of \( S(\Omega, E) \) into \( E \) is uniformly continuous on elementary sets.\n
**Proof.** For \( s, t \in S(\Omega, E) \) and \( \epsilon > 0 \), the inequalities

\[ \mu(\{\|s - t\| \geq \epsilon\}) \leq \epsilon^{-1} \int \|s - t\|, \tag{3} \]

\[ \int \|s - t\| \leq \epsilon\mu(\Omega) + \int \|s - t\|1_{\{\|s - t\| \geq \epsilon\}}, \tag{4} \]

\[ \|s - t\| \leq \int \|s - t\| \tag{5} \]

are easily established. Now (3) implies that the uniformity of convergence in measure on \( S(\Omega, E) \) is weaker than the weak uniformity for the mapping \( f \circ \cdot \| \) of \( S(\Omega, E) \) into \( \mathbb{C} \), and (4) entails that these uniformities coincide on elementary sets. Finally, (5) shows that the mapping \( f \) of \( S(\Omega, E) \) into \( E \) is uniformly continuous on elementary sets. \[ \square \]

A map \( f \in M(\Omega, E) \) is called (Lebesgue) integrable iff it belongs to \( L(\Omega, E) = \cup_{E \in \mathcal{E}(\Omega, E)} E \). From Theorem 2.2 and the previous propositions we conclude that \( L(\Omega, E) \) is a vector space containing \( S(\Omega, E) \) and that there exists a unique linear extension of \( f \) to \( L(\Omega, E) \) which is uniformly continuous on the closure of each elementary set. We denote this extension again as \( f \) and we define the integral of \( f \in L(\Omega, E) \) as \( \int f \).

Let \( \overset{\mu}{\rightarrow} \) stand for convergence in measure.

**Proposition 3.4.** The mapping \( \int \) of \( L(\Omega, E) \) into \( E \) is linear and, if \( f \in L(\Omega, E) \), then \( \|f\| \in L(\Omega, \mathbb{C}) \) and \( \int \|f\| \leq \int \|f\| \). In particular, if \( E = \mathbb{C} \), then the mapping \( \int \) of \( L(\Omega, \mathbb{C}) \) into \( \mathbb{C} \) is positive in the sense that \( \int f \geq 0 \) if \( f \geq 0 \) and monotonic in the sense that \( \int f \geq \int g \) if \( f \geq g \). That is, Proposition 2.7 continues to hold if we replace \( S(\Omega, E) \) by \( L(\Omega, E) \) and \( S(\Omega, \mathbb{C}) \) by \( L(\Omega, \mathbb{C}) \).

**Proof.** The linearity of \( \int f \) was already established. Fix \( f \in L(\Omega, E) \). We prove that \( \|f\| \in L(\Omega, \mathbb{C}) \) and \( \int \|f\| \leq \int \|f\| \). By definition of \( L(\Omega, E) \), there exists a sequence \((s_n)_n\) such that \( \{s_n \mid n\} \in \mathcal{E}(\Omega, E) \) and \( s_n \overset{\mu}{\rightarrow} f \). One easily sees that \( \{\|s_n\| \mid n\} \in \mathcal{E}(\Omega, \mathbb{C}) \) and that \( \|s_n\| \overset{\mu}{\rightarrow} \|f\| \). Hence
\[ \| f \| \in L(\Omega, \mathbb{C}) \text{ and, } \int \text{ being continuous on the closure of each elementary set, Proposition 3.1 gives} \]
\[ \left\| \int f \right\| = \lim_{n} \left\| \int s_n \right\| \leq \lim_{n} \int \| s_n \| = \int \| f \|. \]

The other properties are now easily established. \qed

We call a set \( F \subset L(\Omega, E) \) uniformly integrable iff for each \( \epsilon > 0 \) there exists \( \delta > 0 \) such that \( \int \| f \|_A < \epsilon \) whenever \( f \in F \) and \( A \in \mathcal{A} \) with \( \mu(A) < \delta \).

Notice that the closure of an elementary set is uniformly integrable because \( \int \) is continuous on such a set. The collection of uniformly integrable sets in \( L(\Omega, E) \) is denoted as \( I(\Omega, E) \).

**Proposition 3.5.** \( I(\Omega, E) \) contains all sets \( F \subset S(\Omega, E) \) which are totally bounded under the weak uniformity for the mapping \( \int \| \cdot \| \) of \( L(\Omega, E) \) into \( \mathbb{C} \). Furthermore, \( I(\Omega, E) \) is closed under the formation of finite linear combinations. That is, Proposition 3.2 continues to hold if we replace \( \mathcal{E}(\Omega, E) \) by \( I(\Omega, E) \) and \( S(\Omega, E) \) by \( L(\Omega, E) \).

**Proof.** Let \( F \subset L(\Omega, E) \) be totally bounded under the weak uniformity for the mapping \( \int \| \cdot \| \) of \( L(\Omega, E) \) into \( \mathbb{C} \). In order to show that \( F \in I(\Omega, E) \), fix \( \epsilon > 0 \). Copying the first part of the proof of Proposition 3.2, we find a finite set \( F_0 \subset L(\Omega, E) \) such that for each \( f \in F \) and each \( A \in \mathcal{A} \)
\[ \int \| f \|_A \leq \max_{g \in F_0} \int \| g \|_A + \epsilon. \] (6)

By definition of \( L(\Omega, E) \), there exists, for each \( g \in F_0 \), a set \( E_g \in \mathcal{E}(\Omega, E) \) such that \( g \in F_g \). Now (6), in combination with the fact that \( \bigcup_{g \in F_0} E_g \in \mathcal{E}(\Omega, E) \) and the continuity of \( \int \) on the closure of each elementary set, easily implies that \( F \in I(\Omega, E) \). As in the proof of Proposition 3.2, the fact that \( I(\Omega, E) \) is closed under the formation of finite linear combinations is a straightforward application of the linearity and the monotonicity of \( \int \). \qed

**Proposition 3.6.** The uniformity of convergence in measure on \( L(\Omega, E) \) is weaker than the weak uniformity for the mapping \( \int \| \cdot \| \) of \( L(\Omega, E) \) into \( \mathbb{C} \), and these uniformities coincide on uniformly integrable sets. In particular, the mapping \( \int \) of \( L(\Omega, E) \) into \( E \) is uniformly continuous on uniformly integrable sets. That is, Proposition 3.3 continues to hold if we replace \( S(\Omega, E) \) by \( L(\Omega, E) \) and ‘elementary set’ by ‘uniformly integrable set’.

**Proof.** This is identical to the proof of Proposition 3.3 \qed

**Corollary 3.7.** The space \( L(\Omega, E) \), equipped with the weak uniformity for the mapping \( \int \| \cdot \| \) of \( L(\Omega, E) \) into \( \mathbb{C} \), contains \( S(\Omega, E) \) as a dense subspace.

**Proof.** Fix \( f \in L(\Omega, E) \). Then, by definition of \( L(\Omega, E) \), there exists a sequence \( (s_n)_n \) such that \( \{ s_n \mid n \} \in \mathcal{E}(\Omega, E) \) and \( s_n \overset{\mu}{\to} f \). We know that \( \{ s_n \mid n \} \cup \{ f \} \in I(\Omega, E) \) and thus Proposition 3.1 implies that, on the set \( \{ s_n \mid n \} \cup \{ f \} \), the uniformity of convergence in measure coincides with the weak uniformity for the mapping \( \int \| \cdot \| \) of \( L(\Omega, E) \) into \( E \). Hence \( \int \| f - s_n \| \to 0 \), finishing the proof of the corollary. \qed

**Corollary 3.8.** (Vitali) Fix \( f \in M(\Omega, E) \) and \( (f_n)_n \) in \( L(\Omega, E) \). Then the following are equivalent.
(1) $f \in L(\Omega, E)$ and $\int \| f - f_n \| \to 0$.
(2) $\{ f_n \mid n \} \in I(\Omega, E)$ and $f_n \overset{\mu}{\to} f$.

Proof. (1) $\Rightarrow$ (2) Suppose that $f \in L(\Omega, E)$ and $\int \| f - f_n \| \to 0$. Then $\{ f_n \mid n \}$ is totally bounded under the weak uniformity for the mapping $f \circ \cdot \| \cdot \| : L(\Omega, E) \to C$ and thus, by Proposition 3.5, $\{ f_n \mid n \} \in I(\Omega, E)$. Furthermore, by Proposition 3.6, the uniformity of convergence in measure on $L(\Omega, E)$ is weaker than the weak uniformity for the mapping $f \circ \cdot \| \cdot \|$ of $L(\Omega, E)$ into $C$. Hence we conclude that $f_n \overset{\mu}{\to} f$. (2) $\Rightarrow$ (1) Suppose that $\{ f_n \mid n \} \in I(\Omega, E)$ and $f_n \overset{\mu}{\to} f$. Corollary 3.7 allows us to choose, for $n \in \mathbb{N}_0$, $s_n \in S(\Omega, E)$ such that $\int \| f_n - s_n \| \leq 1/n$. Now one easily establishes that $\{ s_n \mid n \} \in E(\Omega, E)$, whence $f \in \{ s_n \mid n \} \subset L(\Omega, E)$. Furthermore, Proposition 3.6 reveals that, on the set $\{ f_n \mid n \} \cup \{ f \}$, the uniformity of convergence in measure coincides with the weak uniformity for the mapping $f \circ \cdot \| \cdot \|$ of $L(\Omega, E)$ into $C$. As a consequence, $\int \| f - f_n \| \to 0$. \qed

Corollary 3.9. (Riesz-Fischer) The space $L(\Omega, E)$, equipped with the weak uniformity for the mapping $f \circ \cdot \| \cdot \|$ of $L(\Omega, E)$ into $C$, is complete.

Proof. Let $(f_n)_n$ be Cauchy in $L(\Omega, E)$. Proposition 3.6 entails that the uniformity of convergence in measure is weaker than the weak uniformity for the mapping $f \circ \cdot \| \cdot \| : L(\Omega, E) \to C$. In particular, $(f_n)_n$ is also Cauchy for the uniformity of convergence in measure. The completeness of $M(\Omega, E)$ allows us to find $f \in M(\Omega, E)$ such that $f_n \overset{\mu}{\to} f$. Now, $(f_n)_n$ being Cauchy in $L(\Omega, E)$, the set $\{ f_n \mid n \}$ is totally bounded under the weak uniformity for the mapping $f \circ \cdot \| \cdot \| : L(\Omega, E) \to C$ and thus, by Proposition 3.5, $\{ f_n \mid n \} \in I(\Omega, E)$. Finally, an application of Corollary 3.8 reveals that $f \in L(\Omega, E)$ and $\int \| f - f_n \| \to 0$. \qed

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