Stochastic Lorentz forces on a point charge moving near the conducting plate

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Abstract

The influence of quantized electromagnetic fields on a nonrelativistic charged particle moving near a conducting plate is studied. We give a field-theoretic derivation of the nonlinear, non-Markovian Langevin equation of the particle by the method of Feynman-Vernon influence functional. This stochastic approach incorporates not only the stochastic noise manifested from electromagnetic vacuum fluctuations, but also dissipation backreaction on a charge in the form of the retarded Lorentz forces. Since the imposition of the boundary is expected to anisotropically modify the effects of the fields on the evolution of the particle, we consider the motion of a charge undergoing small-amplitude oscillations in the direction either parallel or normal to the plane boundary. Under the dipole approximation for nonrelativistic motion, velocity fluctuations of the charge are found to grow linearly with time in the early stage of the evolution at the rather different rate, revealing strong anisotropic behavior. They are then asymptotically saturated as a result of the fluctuation-dissipation relation, and the same saturated value is found for the motion in both directions. The observational consequences are discussed.

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I. INTRODUCTION

The manifestation of vacuum fluctuations can be visualized through a mechanical effect on a materialized body. One of the most celebrated examples is the attractive Casimir force between two parallel conducting plates [1]. This phenomenon in general can be characterized by fluctuations under geometric constraints such that its spectrum on long wavelength modes is modified. Nevertheless, this induced-force effect can also be probed through the coupling to a test particle. For example, consider an atom in its ground state as a test particle located near a perfectly conducting plate. The atom then experiences a position-dependent energy shift due to the boundary effect on vacuum fluctuations from which to give rise to an attractive Casimir-Polder force toward the plate [2]. Thus, the presence of the boundary is expected to anisotropically change vacuum fluctuations. In this paper a charged test particle serves as a probe to understand the nature of electromagnetic vacuum fluctuations by observing their effects on the test-particle’s trajectory.

When a charged particle interacts with quantized electromagnetic fields, not only do the expectation values of fields determine the mean trajectory of the charge, but the accompanying quantum fluctuations also drive the charge into a zig-zag motion. The dynamics of the particle and field interaction has been studied quantum-mechanically in the system-plus-environment approach [3]. We treat the particle as the system of interest, and the degrees of freedom of fields as the environment. The influence of fields on the particle can be obtained with the method of Feynman-Vernon influence functional, by integrating out field variables within the context of the closed-time-path formalism [4, 5]. The Langevin equation can then be derived by ignoring the intrinsic quantum uncertainty of the particle, which is assumed to be much smaller than the resolution of the position measurements. This stochastic approach incorporates both dissipative backreaction arising from the interaction with fields, and a stochastic noise owing to the quantum fluctuations of fields. In particular, the non-uniformity of the charge’s motion will result in radiation that backreacts on itself through the electromagnetic self-force. The stochastic noise, which encodes the influence of quantum statistics of fields, drives the charge into a fluctuating motion [6, 7]. Furthermore, the noise-averaged result reduces to the known Abraham-Lorentz-Dirac equation with the self-force given by a third-order time derivative of the position as expected [6].

The anisotropy of electromagnetic vacuum fluctuations in the presence of the conducting
plate has been studied via an interference experiment of the electrons, and is manifested in
the form of the amplitude change and phase shift of the interference fringes \[8, 9, 10, 11\]. In
the previous article \[11\], we employ the method of influence functional, and obtain the evolu-
tion of the reduced density matrix of the electron with self-consistent backreactions from
quantized electromagnetic fields. Under the classical approximation with prescribed elec-
tron’s trajectories, it is shown that the modulus of the exponent in the influence functional
describes the change of the interference contrast, and its phase results in an overall shift of
the interference pattern. It is also found that the presence of the boundary anisotropically
modifies the contrast of the interference fringes. In Ref. \[12\], the Brownian motion of a
charged particle coupled to electromagnetic vacuum fluctuations near a perfectly conduct-
ing plate is studied for the case that the particle barely moves; thus its dissipation effects is
ignored. The behaviors of velocity fluctuations are shown different for the particle’s motion
in the directions perpendicular and parallel to the boundary plane. In this paper, we wish
to further explore the anisotropic nature of vacuum fluctuations due to the boundary by
the nontrivial motion of the charged particle where dissipative backreaction is incorporated
in a consistent manner. In the presence of the boundary, we expect that radiation emitted
by the charge in nonuniform motion should be bounced back and then impinge upon the
original charge at later times. This will give rise to an additional retardation effect, which
in turn results in a non-Markovian evolution of the particle. Thus, its mean trajectory will
be altered. Here we will apply red the approach of influence functional from which to derive
the Langevin equation beyond the mean-field approximation. The trajectory fluctuations off
the mean value driven by the stochastic noise will be studied with dissipation backreaction
taken into account in a way that a underlying fluctuation-dissipation relation is obeyed.

Our presentation is organized as follows. In Sec. \(\text{II}\) we introduce the closed-time-path for-
malism to describe the evolution of the density matrix of a nonrelativistic charge coupled to
quantized electromagnetic fields. We trace out field variables to obtain the coarse-grained ef-
fective action from which the Langevin equation is derived. The resulting stochastic Lorentz
force, which can be cast into a gauge invariant expression will be discussed in Sec. \(\text{III}\). The
solutions to the Langevin equation under a dipole approximation can be found through the
Laplace transform for the charge’s motion either parallel or perpendicular to the bound-
ary plane in Sec. \(\text{IV}\). Thus, velocity fluctuations are obtained in Sec. \(\text{V}\). The results are
summarized and discussed in Sec. \(\text{VI}\).
The Lorentz-Heaviside units with $\hbar = c = 1$ will be adopted unless otherwise noted. The metric is $\eta^{\mu\nu} = \text{diag}(+1, -1, -1, -1)$.

II. INFLUENCE FUNCTIONAL AND LANGEVIN EQUATION

We consider the dynamics of a nonrelativistic particle of charge $e$ interacting with quantized electromagnetic fields. In the Coulomb gauge, $\nabla \cdot \mathbf{A} = 0$, the Lagrangian is expressed as

$$L[q, A_T] = \frac{1}{2} m q^2 - V(q) - \frac{1}{2} \int d^3x d^3y \, \varrho(x; q) G(x, y) \varrho(y; q) + \int d^3x \left[ \frac{1}{2} (\partial_\mu A_T)^2 + j \cdot A_T \right],$$

(1)

in terms of the transverse components of the gauge potential $A_T$, and the position $q$ of the charged particle. The instantaneous Coulomb Green’s function $G(x, y)$ satisfies the Gauss’s law. The charge and current densities take the form, respectively,

$$\varrho(x; q(t)) = e \delta^{(3)}(x - q(t)), \quad j(x; q(t)) = e \dot{q}(t) \delta^{(3)}(x - q(t)).$$

(2)

Let $\hat{\rho}(t)$ be the density matrix of the particle-field system, and then it evolves unitarily according to

$$\hat{\rho}(t_f) = U(t_f, t_i) \hat{\rho}(t_i) U^{-1}(t_f, t_i)$$

(3)

with $U(t_f, t_i)$ the time evolution operator of the total system. The nonequilibrium partition function can be defined by taking the trace of the density matrix over the particle and field variables,

$$Z = \text{Tr} \left\{ U(t_f, t_i) \hat{\rho}(t_i) U^{-1}(t_f, t_i) \right\}.$$  

(4)

It is convenient to assume that the state of the particle-field at an initial time $t_i$ is factorizable as $\hat{\rho}(t_i) = \hat{\rho}_e(t_i) \otimes \hat{\rho}_{A_T}(t_i)$. The more sophisticated scheme of the density matrix involving initial correlations can be found in Ref. [13]. We also assume that the particle initially is in a localized state, and thus its density matrix can be expanded by the position eigenstate of the eigenvalue $q_i$,

$$\hat{\rho}_e(t_i) = |q_i, t_i \rangle \langle q_i, t_i|.$$  

(5)

The electromagnetic field at the time $t_i$ is assumed in thermal equilibrium at temperature $T = 1/\beta$, and thus its density operator takes the form

$$\hat{\rho}_{A_T}(t_i) = e^{-\beta H_{A_T}} / \text{Tr} \left\{ e^{-\beta H_{A_T}} \right\},$$

(6)
where $H_{\mathbf{A}_T}$ is the Hamiltonian of free vector potentials in the Coulomb gauge. Later, we will focus on the case that the initial state of fields is the vacuum state in the zero-temperature limit $\beta \to \infty$. The nonequilibrium partition functional can be computed with the help of the path integral along the contour in the complex time plane by taking the limits, $t_i \to -\infty$ and $t_f \to +\infty$, and it is given by [11],

$$
\mathcal{Z} = \int d^3q_f \int_{q_i}^{q_f} Dq^+ \int_{q_i}^{q_f} Dq^- \exp \left[ \frac{i}{\hbar} \int_{-\infty}^{\infty} dt \left( L_e[q^+] - L_e[q^-] \right) \right] \mathcal{F}[j^+, j^-],
$$

where the Lagrangian $L_e[q]$ is

$$
L_e[q] = \frac{1}{2} m \dot{q}^2 - V(q) - \frac{1}{2} \int d^3x d^3y \; g(x; q) G(x, y) g(y; q).
$$

The influence functional $\mathcal{F}[j^+, j^-]$, after tracing out field variables, can be written in terms of real-time Green’s functions of vector potentials,

$$
\mathcal{F}[j^+, j^-] = \exp \left\{ -\frac{1}{2\hbar^2} \int d^4x \int d^4x' \left[ j^+_i(x; q^+(t)) \langle A^+_i(x) A^+_j(x') \rangle j^+_j(x'; q^+(t')) 
- j^+_i(x; q^+(t)) \langle A^+_i(x) A^-_j(x') \rangle j^-_j(x'; q^-(t')) 
- j^-_i(x; q^-(t)) \langle A^-_i(x) A^-_j(x') \rangle j^+_j(x'; q^+(t')) 
+ j^-_i(x; q^-(t)) \langle A^-_i(x) A^-_j(x') \rangle j^-_j(x'; q^-(t')) \right] \right\},
$$

and contains full information about the influence of quantized electromagnetic fields. Here the explicit $\hbar$ dependence is restored in the expressions. The Green’s functions are defined by

$$
\langle A^+_i(x) A^+_j(x') \rangle = \langle A^+_i(x) A^+_j(x') \rangle \theta(t-t') + \langle A^+_i(x') A^+_j(x) \rangle \theta(t' - t),
$$

$$
\langle A^-_i(x) A^-_j(x') \rangle = \langle A^-_i(x') A^-_j(x) \rangle \theta(t-t') + \langle A^-_i(x) A^-_j(x') \rangle \theta(t' - t),
$$

$$
\langle A^+_i(x) A^-_j(x') \rangle = \langle A^+_i(x') A^-_j(x) \rangle \equiv \text{Tr} \left\{ \rho_{\mathbf{A}_T} A^+_i(x') A^-_j(x) \right\},
$$

$$
\langle A^-_i(x) A^+_j(x') \rangle = \langle A^-_i(x) A^+_j(x') \rangle \equiv \text{Tr} \left\{ \rho_{\mathbf{A}_T} A^-_i(x) A^+_j(x') \right\}.
$$

It is found more convenient to change the variables $q^+$ and $q^-$ to the average and relative coordinates,

$$
q = \frac{1}{2} (q^+ + q^-), \quad r = q^+ - q^-.
$$

The nonequilibrium partition function in terms of the coarse-grained action becomes

$$
\mathcal{Z} = \int dq_f \int Dq Dr \exp \left\{ \frac{i}{\hbar} S_{CG} [q, r] \right\},
$$

\[ \text{(11)} \]
where the coarse-grained action $S_{CG}$ reads

$$S_{CG}[\mathbf{q}, r] = \int_{-\infty}^{\infty} dt \ r^i(t) \left\{-m \dddot{q}^i(t) - \nabla^i V(\mathbf{q}(t)) + e^2 \nabla^i G[\mathbf{q}(t), \mathbf{q}(t)] - e^2 \left( \xi^i(t) - \dot{q}^i(t) \nabla^i \right) \right\} \tag{15}$$

and thus the imaginary part of the coarse-grained action can be expressed as a functional integration over $\xi(t)$ weighted by the distribution function $P[\xi(t)]$. As a result, we end up with

$$\exp \left\{ \frac{i}{\hbar} S_{CG}[\mathbf{q}, r] \right\} = \int D\xi_i P[\xi_i(t)] \exp \left\{ \frac{i}{\hbar} \left[ \Re \{S_{CG} \} - \hbar e \int_{-\infty}^{\infty} dt \ r^i \left( \frac{d}{dt} - \dot{q}(t) \nabla^i \right) \xi^i \right] \right\}. \tag{14}$$

The expressions in the squared brackets on the right hand side is defined as the stochastic effective action, which consists of the real part of the coarse-grained effective action as well as the coupling term of the relative coordinate $r^i$ with the stochastic noise $\xi^i$.

The Langevin equation is obtained by extremizing the stochastic effective action and then setting $r^i$ to zero. By doing so, we have ignored intrinsic quantum fluctuations of the particle, and that holds as long as the resolution of the measurement on length scales is greater than its position uncertainty. The Langevin equation is then given by

$$m \dddot{q}^i + \nabla^i V(\mathbf{q}(t)) + e^2 \nabla^i G[\mathbf{q}(t), \mathbf{q}(t)] + e^2 \left( \delta^i \frac{d}{dt} - \dot{q}^i(t) \nabla^i \right) \times \int_{-\infty}^{\infty} dt' G^{ij}_{R}[\mathbf{q}(t), \mathbf{q}(t') \; t-t'] \dot{q}^j(t') = -\hbar e \left( \delta^i \frac{d}{dt} - \dot{q}^i(t) \nabla^i \right) \xi^i(t) \tag{15}$$
with the noise-noise correlation functions,
\begin{align*}
\langle \xi_i(t) \rangle &= 0, \\
\langle \xi_i(t) \xi_j(t') \rangle &= \frac{1}{\hbar} G^{ij}_H [\mathbf{q}(t), \mathbf{q}(t'); t - t'].
\end{align*}
(16)

This Langevin equation encompasses fluctuation and dissipation effects on the charge’s motion from quantized electromagnetic fields via the kernels $G^{ij}_H$ and $G^{ij}_R$ respectively, both of which are in turn linked by the fluctuation-dissipation relation \[14\]. The fluctuation-dissipation relation is known to play a pivotal role in balancing these two effects in order to dynamically stabilize the nonequilibrium evolution of the particle under a fluctuating environment. Mathematically, it relates the Fourier transform of the fluctuation kernel $G^{ij}_H$ to the imaginary part of the retarded kernel $G^{ij}_R$ as follows
\begin{align*}
G^{ij}_H [\mathbf{q}(t), \mathbf{q}(t'); \omega] &= \text{Im} \left\{ G^{ij}_R [\mathbf{q}(t), \mathbf{q}(t'); \omega] \right\} \coth \left[ \frac{\beta \omega}{2} \right].
\end{align*}
(17)

In the zero temperature limit, the relation reduces to
\begin{align*}
G^{ij}_H [\mathbf{q}(t), \mathbf{q}(t'); \omega] &= \text{Im} \left\{ G^{ij}_R [\mathbf{q}(t), \mathbf{q}(t'); \omega] \right\} \left[ \theta(\omega) - \theta(-\omega) \right].
\end{align*}
(18)

It is found that the backreaction kernel functions of electromagnetic fields in the Langevin equation \[15\] appear purely classical due to the fact that the coupling between the charge and electromagnetic potentials is linear. The noise-noise correlation functions can in principle be computed by taking an appropriate statistical average with the distribution functional $\mathcal{P}[\xi^i(t)]$. It is also seen from Eq. \[15\] that the influence of electromagnetic fields takes the form of an integral of the dissipation kernel over the past history of the charge’s trajectory, as well as a stochastic noise $\mathbf{\xi}$, which drives the charge into a fluctuating motion. As it stands, this is a nonlinear Langevin equation with non-Markovian backreaction, and the noise depends in a complicated way on the charge’s trajectory because the noise correlation function itself is a functional of the trajectory.

The general solution $q^i$ of the Langevin equation can be expressed as its mean value $q^i_h$ and a small deviation $\delta q^i$ from the mean. Expanding the equation with respect to $\delta q^i$ and then keeping its linear terms, we may decompose the stochastic equation into the equations of motion for $q^i_h$ and $\delta q^i$, respectively. The mean trajectory $q^i_h$ satisfies the homogeneous part of the Langevin equation, which describes the purely classical effects. On the other hand, the equation for the position fluctuations $\delta q^i$ involves the stochastic noise $\mathbf{\xi}$. The noise-driven position fluctuations $\delta q^i$ thus are entirely of quantum origin as seen from an
explicit $\hbar$ dependence in the noise term. The backreaction dissipation effect on the evolution of $\delta q^i$ is expected to balance with the effect from the accompanying stochastic noise via a fluctuation-dissipation relation where both effects are of quantum nature \[6\]. This issue will be further studied below.

III. STOCHASTIC LORENTZ FORCES WITH THE BOUNDARY

The integro-differential equation (15) can be cast into a form similar to the Lorentz equation. We consider a charged particle moving in the vicinity of a perfectly conducting plate. Let the plate be located at the $z = 0$ plane. Then, the tangential components of the electric field $E$ and the normal component of the magnetic field $B$ on the plate surface should vanish such that the boundary conditions of the vector potential $A$ are given by

$$A_0 = 0, \quad \text{and} \quad A_x = A_y = 0,$$

leading to

$$\frac{\partial A_z}{\partial z} = 0$$

as the result of the Coulomb gauge. The transverse vector potential $A_T$ in the $z > 0$ region is given by,

$$A_T(x) = \int \frac{d^2k_\parallel}{2\pi} \int_0^\infty \frac{dk_z}{(2\pi)^{1/2}} \frac{2}{\sqrt{2\omega}} \left\{ a_1(k) \hat{k}_\parallel \times \hat{z} \sin k_z z \\
+ a_2(k) \left[ i \hat{k}_\parallel \left( \frac{k_z}{\omega} \right) \sin k_z z - \hat{z} \left( \frac{k_\parallel}{\omega} \right) \cos k_z z \right] \right\} e^{ik_\parallel x_\parallel - i\omega t} + \text{H.C.} \right.$$  (21)

where the circumflex identifies unit vectors. The position vector $x$ is the shorthanded notation of $x = (x_\parallel, z)$, where $x_\parallel$ is the components parallel to the plate. Similarly, the wave vector is expressed by $k = (k_\parallel, k_z)$ with $\omega^2 = k_\parallel^2 + k_z^2$. The commutation relations of the creation and annihilation operators are satisfied by

$$[a_\lambda(k), a^{\dagger}_{\lambda'}(k')] = \delta_{\lambda\lambda'} \delta(k_\parallel - k'_\parallel) \delta(k_z - k'_z), \quad \text{with} \ \lambda, \lambda' = 1, 2,$$

and are zero otherwise. Then, the retarded Green’s function and Hadamard function can be explicitly expressed as the sum of the free-space and the boundary-induced contributions,

$$G_R^{ij}(q(t), q(t'); t - t') = i \theta(t - t') \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega} \left\{ \left( \delta^{ij} - \frac{k_i^i k_j^j}{\omega^2} \right) e^{ik_\parallel x_\parallel - i\omega t} \right\}$$  (23)
where the position \( \mathbf{q} \) denotes the location of the mirror image of the original charge at \( q \) with respect to the boundary at the \( z = 0 \) plane, and they are related by \( \overline{q} = (\delta^{ij} - 2\hat{z}^i \hat{z}^j)q^j \).

The integral expression in Eq. (15) can be realized in terms of the Liénard-Wiechert potential in the Coulomb gauge due to a moving charge,

\[
A^i_{\text{LW}}(q) = \int d^4x' G^{ij}_{\text{LW}}[q(t), x'] j^j(x'; q(t')).
\]

(25)

This potential \( A_{\text{LW}}(q) \) clearly depends on the past history of the charge’s motion [15].

Together with the instantaneous Coulomb potential term, the backreaction can be expressed in a gauge invariant way to necessarily maintain underlying gauge symmetry, respected by the Lagrange we begin with. With the definition of the current density in Eq. (2), the straightforward algebraic manipulation further shows that the Langevin equation can be re-expressed as retarded Lorentz forces and the stochastic components,

\[
m\ddot{q}^i + \nabla^i V(q) = e \left[ E^i(q) + \epsilon_{ijk} \dot{q}^j(t) B^k(q) \right] - \hbar e \left( \delta^{ij} \frac{d}{dt} - \dot{q}^j(t) \nabla_q \right) \xi^j(t).
\]

(26)

The electromagnetic fields are defined by

\[
E = -\frac{\partial}{\partial t} A_{\text{LW}} - \nabla_q G,
\quad
B = \nabla_q \times A_{\text{LW}}.
\]

(27)

Next we write electromagnetic fields in reference to the retarded spacetime coordinates and explicitly have them divided into the free-space contribution and the correction out of the boundary. Let the free-space part \( G^{ij;\text{LW}}_{\text{R}}(q, q'; \tau) \) and the boundary correction \( G^{ij;\text{B}}_{\text{R}}(q, q'; \tau) \) of the retarded Green’s function be respectively given by

\[
G^{ij;\text{LW}}_{\text{R}}(q, q', \tau) = \frac{\delta^{ij}}{2(2\pi)^3} \int \frac{dk}{2\omega} \left\{ \delta^{ij} - \frac{k^i k^j}{\omega^2} \right\} e^{i\mathbf{k} \cdot (\mathbf{q}' - \mathbf{q}) - i\omega \tau} + \text{C.C.},
\]

\[
G^{ij;\text{B}}_{\text{R}}(q, q', \tau) = -i \theta(t - t') \int \frac{d^3k}{2(2\pi)^3} \frac{1}{2\omega} \left\{ \delta^{ij} - \frac{k^i k^j}{\omega^2} \right\} e^{i\mathbf{k} \cdot (\mathbf{q}' - \mathbf{q}) - i\omega \tau} + \text{C.C.},
\]

where \( q' = q(t') \) and \( \tau = t - t' \). The similar decomposition is applied to the instantaneous Coulomb potential. Therefore, we end up with

\[
E = E^{(0)}(q, R) + E^{(b)}(q, R),
\]

\[
B = [n \times E^{(0)}(q, R) + \overline{n} \times E^{(b)}(q, R)]_{\text{ret}},
\]

(28)
where

\[
E^{(0)}(q, R) = e \left[ \frac{n - \dot{q}}{\gamma^2 (1 - \dot{q} \cdot n)^3 R^2} \right]_{\text{ret}} + e \left[ \frac{n \times [(n - \dot{q}) \times \ddot{q}]}{(1 - \dot{q} \cdot n)^3 R} \right]_{\text{ret}},
\]

\[
E^{(b)}(q, R) = -e \left[ \frac{\overline{n} - \dot{\overline{q}}}{\gamma^2 (1 - \dot{\overline{q}} \cdot \overline{n})^3 \overline{R}^2} \right]_{\text{ret}} - e \left[ \frac{\overline{n} \times [(\overline{n} - \dot{\overline{q}}) \times \ddot{\overline{q}}]}{(1 - \dot{\overline{q}} \cdot \overline{n})^3 \overline{R}} \right]_{\text{ret}},
\]

in which \( R = q(t) - q(t_R) \), \( n = R/R \), and \( \gamma = (1 - \dot{q}^2)^{-1/2} \). The retarded time \( t_R \) is defined as \( t_R = t - R \). The variables with the overbar are given by replacing the source point \( q(t_R) \) with its image position \( \overline{q}(t_R) \) in their definition. The minus sign in the definition of the retarded electric field \((30)\) implies that this field can be interpreted as radiation out of the image charge due to the boundary. Thus Eqs. \((28)-(30)\) are consistent with what would be obtained for electromagnetic fields due to a moving charge and its image \([15]\).

The free-space part of the Liénard-Wiechert potentials involves its associated retarded Green’s function, which is nonvanishing only for lightlike spacetime intervals. Since the worldline of a massive particle is timelike, in the absence of the boundary the charge at the present time can not be affected by backreaction from previously emitted radiation due to the nonuniform motion of the charge itself \([15]\); however, radiation can backreact on the charge at the time when it is just emitted. Thus, the nonlocal form of the free-space contribution of the Liénard-Wiechert potential reduces to a purely local effect. Besides it may lead to short-distance divergence in the coincidence limit. This ultraviolet divergence arises from the assumption that the point-like particle interacts with fields, and must be regularized to have a finite and unambiguous result. Then the divergent part is absorbed by mass renormalization, while the finite backreaction takes the form of electromagnetic self-

forces, given by a third-order time derivative of the position. It may bring about issues such as the runaway solutions and acausality on the dynamical evolution of a point charge \([16]\).

Needless to say, the emitted radiation may backscatter off the boundary, and in turn affects the charge’s motion at a later time. Thus, this backreaction owing to the presence of the boundary depends on the past history of the charge’s trajectory, leading to the memory effect. This non-Markovian processes can also be understood as if radiation was emitted at a retarded time from the image charge at \( \overline{q} \), which is the location of the mirror reflection of the point charge at \( q \) due to the presence of the conducting plate. The lag corresponds to the time delay for radiation to travel from the image counterpart at an earlier time to the charge itself at the present time, roughly being equal to the round-trip traveling time of
radiation between the plane boundary and the charge.

In addition, from the interpretation of the stochastic noise and the noise-noise correlation, we may formally identify $-\hbar \xi^i$ as the stochastic vector potentials $A^i_s$ such that their correlation functions are

$$\langle A^i_s(t) \rangle = 0, \quad \langle A^i_s(t) A^j_s(t') \rangle = \hbar G^{ij}_H[q(t), q(t'); t - t']. \quad (31)$$

The noise term on the right hand side of Eq (26) can then be thought of as the stochastic Lorentz force $F^i_s$,

$$F^i_s = e \left[ E^i_s + \epsilon_{ijk} \dot{q}^j(t) B^k_s \right], \quad (32)$$

where

$$E^i_s = -\frac{\partial}{\partial t} A^i_s, \quad \text{and} \quad B^i_s = -\nabla^i q \times A^i_s. \quad (33)$$

The origin of the stochastic Lorentz force by construction comes from electromagnetic vacuum fluctuations. Here we note that the stochastic electromagnetic fields $(E^i_s, B^i_s)$ involve only the transverse components of gauge potentials. In the Coulomb gauge, the instantaneous Coulomb potential, which is determined by the Gauss law, is not a dynamical variable when the intrinsic quantum fluctuations of the charge are ignored, and hence it has no corresponding stochastic component. In the end, the Langevin equation (26) for the point charge interacting with quantized electromagnetic fields in the presence of the conducting plate can be nicely cast into a gauge-invariant form,

$$m \ddot{q}^i + \nabla^i V(q) = e \left[ E^i(q) + \epsilon_{ijk} \dot{q}^j(t) B^k(q) \right] + e \left[ E^i_s + \epsilon_{ijk} \dot{q}^j(t) B^k_s \right]. \quad (34)$$

Then the expressions in the first pair of squared brackets on the right hand side denote the retarded electromagnetic forces, while the terms in the second pair are its stochastic components, manifested from the quantum fluctuations of electromagnetic fields.

**IV. Langevin Equation Under Dipole Approximation and Its Solution**

The nonlinear, non-Markovian Langevin equation is far too complicated to study further without any approximation. We will consider that a charged particle undergoes the harmonic motion, and assume that the amplitude of oscillation is sufficiently small. The appropriate approximation for a non-relativistic motion will be the dipole approximation.
This approximation amounts to considering the backreaction solely from electric fields, and linearizing the Langevin equation in such a way that the equation of motion remains non-Markovian. Since the presence of the boundary will anisotropically modify the effects of electromagnetic fields on the evolution of the charge, we will consider the motion in two different directions, that is, either parallel or perpendicular to the plane boundary.

A. parallel motion

When the charged particle moves parallel to the boundary, say, in the $x$ direction, let the equilibrium point be located at the coordinates $(x, y, z) = (0, 0, z_0)$. The linearized Langevin equation for the motion displaced from its equilibrium position reduces to

$$m\ddot{q}_x(t) + \partial_x^2 V(z_0) q_x(t) + e^2 \int_0^t dt' \hat{g}_R^x[z_0, z_0; t - t'] \dot{q}_x(t') = -\hbar e \dot{\xi}_x(t), \quad (35)$$

from Eqs. (15), (23), and (24) under the dipole approximation. Here $q^i$ denotes the displacement from the equilibrium point. The $xx$ component of the retarded Green’s function in the dipole approximation is denoted by $g_R^x[z_0, z_0; t - t']$, and can be written in terms of the spectral density $\rho^x$ as

$$g_R^x[z_0, z_0; t - t'] = -\theta(t - t') \int_0^\infty \frac{dk}{\pi} \rho^x(z_0, z_0; k) \sin[k(t - t')], \quad (36)$$

where the spectral density is given by

$$\rho^x(z_0, z_0; k) = -\frac{k}{\pi} \left[ \frac{1}{3} - \frac{\sin(2kz_0)}{2(2kz_0)} - \frac{\cos(2kz_0)}{2(2kz_0)^2} + \frac{\sin(2kz_0)}{2(2kz_0)^3} \right]. \quad (37)$$

The instantaneous Coulomb potential in this case is found to have no effect on the motion in the $x$ direction, but will establish a static attraction force between the charge and its image in the $z$ direction. Hence, the applied potential $V$ is assumed to have an additional component, other than the harmonic potential, to counteract the static force so that the motion of the charge remains on the $z = z_0$ plane. In addition, the motion has been assumed to start at $t = 0$, and its initial conditions are chosen to be $q_x(t < 0) = q_x(0)$ and $\dot{q}_x(t < 0) = 0$. In general, it is insufficient for a non-Markovian equation to have a unique solution if a finite number of initial conditions are specified. Thus in this case, the initial conditions are given over half of the real axis $t < 0$, although later it will become clear that the requirement on specifying initial conditions can be less stringent in the current case. Physically, these
The finite backreaction effect is given by the third-order time derivative of the position. The boundary-induced backreaction in the squared brackets on the right hand side of Eq. (40) reveals the non-Markovian nature in the argument of each individual term. The divergence is then absorbed into mass renormalization as shown above.
assumption of small amplitude for the charged oscillator simplifies a general non-Markovian integro-differential equation into an ordinary differential equation with a fixed time delay. This is expected due to the fact that by small amplitude, we mean the variation of the position for the charge oscillator is much smaller than its distance to the plate \( z_0 \). It implies that the time variation owing to the oscillation amplitude is ignorable compared to the time \( 2z_0/c \) needed for the radiation to have a round trip between the charge and the boundary. Therefore, the dominant contribution of the time delay will be just given by a fixed constant \( T = 2z_0/c \). The term with \( \ddot{q}^x(t - T) \) is derived from the acceleration field of electric fields while the rest two terms come from the velocity field. The time difference \( t - T \) indicates that the backreaction effect due to the boundary on the charge at the time \( t \) depends on the charge’s dynamics at an earlier time \( t - T \). The time delay \( 2z_0/c \) may also be thought of as the traveling time for radiation emitted from the image charge at an earlier time to reach the charge at a later time. The memory effect is thus described by such a time delay differential equation. This equation can also be obtained from Eq. (26) by expanding the expression of the retarded Lorentz force to the linear terms in \( q^x, \dot{q}^x \) and \( \ddot{q}^x \) in the non-relativistic limit.

In order to find the solutions, it apparently requires the knowledge of \( q^x \) at earlier times. Since the Langevin equation under consideration reduces to an ordinary differential equation with a fixed time delay \( T \), the specification of the initial conditions can be relaxed to the conditions

\[
q^x(-T < t < 0) = q^x(0), \quad \dot{q}^x(-T < t < 0) = 0, \quad (41)
\]

in the interval \(-T < t < 0\), instead of the whole negative real axis of \( t \). Then the equation can be solved by means of the Laplace transform. From Eq. (35), the Laplace transformed Langevin equation reads

\[
m_r \left[ s^2 \ddot{q}^x(s) - s q^x(0) + \omega_0^2 q^x(s) \right] + e^{2\gamma_c} g_{gR}[z_0, z_0; s] \left[ s^2 \ddot{q}^x(s) - s q^x(0) \right] = -\hbar e \tilde{\xi}^x(s), \quad (42)
\]

where the initial conditions (11) have been applied. The Laplace transformation of a function \( f(t) \) is defined by

\[
\tilde{f}(s) = \int_0^{\infty} dt \ e^{-st} f(t), \quad t > 0, \]

while the inverse Laplace transform of \( \tilde{f}(s) \) is

\[
f(t) = \frac{1}{2\pi i} \int_C ds \ \tilde{f}(s) e^{st},
\]

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where the kernel $\tilde{K}(z_0, z_0; s)$ yields

\[
\tilde{g}_R[z_0, z_0; s] = -\int_{-\infty}^{\infty} \frac{dk}{\pi} \rho^x(z_0, z_0; k) \frac{k}{k^2 + s^2} - \frac{\Lambda}{3\pi^2},
\]

(43)

The solution to the Laplace transformed equation is readily obtained as

\[
\tilde{q}^x(s) = \frac{s \left\{ 1 + \frac{e^2}{m} \tilde{g}_R[z_0, z_0; s] \right\} q^x(0) - \frac{he}{m} \tilde{\xi}^x(s)}{s^2 + \omega_0^2 + \tilde{\Sigma}_x(s)},
\]

(44)

where the $\tilde{\Sigma}_x(s)$ kernel is

\[
\tilde{\Sigma}_x(s) = \frac{e^2}{m} s^2 \tilde{g}_R[z_0, z_0; s].
\]

(45)

Thus, the solution in the time domain can be given by the inverse Laplace transformation of $\tilde{q}^x(s)$, and it reads in terms of the mean trajectory and its deviation from the mean,

\[
q^x(t) = \left[ K_x(t) + \frac{e^2}{m} \int_0^t dt' g_R[z_0, z_0; t - t'] K_x(t') \right] q^x(0) - \frac{eh}{m} \int_0^t dt' K_x(t - t') \tilde{\xi}^x(t'),
\]

(46)

where the kernel $K_x(t)$ is the inverse Laplace transform of $\tilde{K}_x(s) = [s^2 + \omega_0^2 + \tilde{\Sigma}_x(s)]^{-1}$,

\[
K_x(t) = \int_C \frac{ds}{s^2 + \omega_0^2 + \tilde{\Sigma}_x(s)} e^{st},
\]

(47)

with $K_x(0) = 0$ and $K_x(0) = 1$. It is seen from Eqs. (43) and (45) that the kernel $\tilde{\Sigma}_x(s)$ has a branch-cut along the imaginary $s$ axis, so the discontinuity of the kernel $\tilde{\Sigma}_x(s)$ over the branch-cut can be made explicitly by letting $s = i\omega + 0^\pm$. In terms of $\omega$, the kernel $\tilde{\Sigma}_x$ becomes

\[
\tilde{\Sigma}_x(s = i\omega + 0^\pm) = \text{Re} \Sigma_x(\omega) \pm i \text{Im} \Sigma_x(\omega),
\]

(48)

where

\[
\text{Re} \Sigma_x(\omega) = \frac{e^2}{m} \omega^2 \left[ \int_{-\infty}^{\infty} \frac{dk}{\pi} \rho^x(z_0, z_0; k) P \left( \frac{1}{k^2 - \omega^2} \right) + \frac{\Lambda}{3\pi^2} \right],
\]

(49)

\[
\text{Im} \Sigma_x(\omega) = -\frac{e^2}{m} \text{sgn}(\omega) \omega^2 \int_0^\infty dk \rho^x(z_0, z_0; k) \delta(k^2 - \omega^2).
\]

(50)

Carrying out the integrals yields

\[
\text{Re} \Sigma_x(\omega) = \frac{e^2}{4\pi m} \omega^3 \left[ \frac{1}{(2\omega z_0)^3} + \frac{\cos(2\omega z_0)}{(2\omega z_0)} - \frac{\sin(2\omega z_0)}{(2\omega z_0)^2} - \frac{\cos(2\omega z_0)}{(2\omega z_0)^3} \right],
\]

(51)
FIG. 1: (a) On the first Riemann sheet, the branch-cut lies along the real axis of $\omega$ and a runaway pole sits at the negative imaginary axis. All the other poles stay on the second Riemann sheet. Thus, the Bromwich contour reduces to two straight lines parallel but infinitesimally close to the cut from both sides if the contribution of the runaway pole is discarded. (b) The resonance modes are found asymptotically in the complex $\omega$ plane along the curve $C_p$.

\[
\text{Im} \Sigma_x(\omega) = \frac{e^2}{4\pi m} \text{sgn}(\omega) \omega^3 \left[ \frac{2}{3} - \frac{\sin(2\omega z_0)}{(2\omega z_0)} - \frac{\cos(2\omega z_0)}{(2\omega z_0)^2} + \frac{\sin(2\omega z_0)}{(2\omega z_0)^3} \right]. \tag{52}
\]

In general, the branch-cut of the $\tilde{\Sigma}_x(s)$ kernel lies within the intervals $(\omega_{\text{th}}, \Lambda)$ and $(-\Lambda, -\omega_{\text{th}})$, where $\Lambda$ is the energy cutoff and $\omega_{\text{th}}$ is the threshold energy above which the process of particle creation is possible \[17\]. However, for quantized electromagnetic fields, its threshold energy vanishes due to masslessness of the photon, i.e. $\omega_{\text{th}} = 0$. Thus the branch-cut will extend along the imaginary $s$ axis from $-i\Lambda$ to $i\Lambda$.

The locations of the poles on $1/(s^2 + \omega_0^2 + \tilde{\Sigma}_x(s))$ can be found by solving the following equation,

\[
\frac{s^2}{3} + \omega_0^2 + \tilde{\Sigma}_x(s) = 0. \tag{53}
\]

This pole equation turns out to be a transcendental function of $s$ due to the presence of the boundary. It may have an infinite number of solutions. This in turn implies that the integro-differential equation can be viewed as an infinite order of the ordinary differential equation. Thus an infinite number of initial conditions are needed \[18\]. For the weak coupling constant $e^2$, perturbatively, we may let $s = i(\omega_0 + \Delta) + \delta\gamma$, where $\Delta = i \delta\gamma$ is the shift of the pole.
owing to interaction with electromagnetic fields. It is then plugged into the equation (53) to find $\delta \omega$ and $\delta \gamma$. We assume $\delta \gamma > 0$ for the moment. To the order $\epsilon^2$, the perturbative solution is found to be $\delta \omega = \text{Re} \Sigma_x(\omega_0)/(2\omega_0)$, and $\delta \gamma = -\text{Im} \Sigma_x(\omega_0)/(2\omega_0)$. However, since from Eq. (52), $\text{Im} \Sigma_x(\omega) > 0$ for all positive values of $\omega$, it leads to $\delta \gamma < 0$, inconsistent with the assumption we made earlier. A similar contradictory result will be found through the same argument by assuming $\delta \gamma < 0$. As a result, the poles do not exist in the first Riemann sheet, and may appear in the second sheet. This is the known fact for the resonance [20]. In fact, there are still an infinite number of poles lying on the upper half of the second Riemann sheet. However, one exception is the runaway pole, which resides on the positive real axis of the complex $s$ plane. Since the kernel $\tilde{\Sigma}_x(s)$ is real when evaluated at the runaway pole, this pole stays on the first Riemann sheet. It corresponds to either a runaway motion of the charge or the acausal evolution due to preacceleration. Since it is viewed as unphysical if the asymptotically bounded motion is considered [15], we then discard the contributions of the runaway-type poles to the inverse Laplace transformation [17]. Accordingly, the contour on the first Riemann sheet can be deformed to be parallel and infinitesimally close to the branch-cut as shown in Fig. 1. The real-time solution of $K_x(t)$ can be written as

$$K_x(t) = \int_{\omega_{th}=0}^{\infty} \frac{d\omega}{2\pi} \frac{4 \text{Im} \Sigma_x(\omega)}{[\omega^2 - \omega_0^2 - \text{Re} \Sigma_x(\omega)]^2 + [\text{Im} \Sigma_x(\omega)]^2} \sin(\omega t), \quad t > 0.$$  \hspace{1cm} (54)

Similarly, the time derivative of the $K_x$ kernel is given by

$$\dot{K}_x(t) = \int_{\omega_{th}=0}^{\infty} \frac{d\omega}{2\pi} \frac{4\omega \text{Im} \Sigma_x(\omega)}{[\omega^2 - \omega_0^2 - \text{Re} \Sigma_x(\omega)]^2 + [\text{Im} \Sigma_x(\omega)]^2} \cos(\omega t), \quad t > 0.$$  \hspace{1cm} (55)

The integrand in a weak coupling limit shows a Breit-Wigner feature. The width of the resonance is related to the imaginary part of the kernel $\Sigma_x$, and its peak is located at the resonance frequency $\Omega_x$ to be determined later. For the sufficiently late times, $t \gg T$, the time derivative of $K_x(t)$ can be approximated by taking into account only the contribution of the resonance mode, and is given by

$$\dot{K}_x(t) \sim Z_x \cos(\Omega_x t + \alpha_x) e^{-\Gamma_x t}, \quad t > 0.$$  \hspace{1cm} (56)

where the resonance frequency $\Omega_x$ and the decay constant $\Gamma_x$ are given by

$$\Omega_x \sim \omega_0 + \frac{\text{Re} \Sigma_x(\omega_0)}{2\omega_0}, \quad \Gamma_x \sim Z_x \frac{\text{Im} \Sigma_x(\Omega_x)}{2\Omega_x},$$  \hspace{1cm} (57)
respectively. In addition, the phase shift $\alpha_x$ and $Z_x$ are

$$Z_x \sim \left[1 - \frac{\partial \text{Re} \Sigma_x(\Omega_x)}{\partial \Omega_x^2}\right]^{-1}, \quad \alpha_x \sim Z_x \frac{\partial \text{Im} \Sigma_x(\Omega_x)}{\partial \Omega_x^2}.$$  \hspace{1cm} (58)

The long-time dynamics of the kernel $\dot{K}(t)$ can also be investigated by examining the low-frequency behavior of the integrand in Eq. (55) along the branch-cut in the neighborhood of threshold energy. When the resonance peak is far away from threshold energy, $\omega_{th} \neq \Omega$, the imaginary part of the kernel $\Sigma$ generally behaves like $\text{Im} \Sigma \propto (\omega - \omega_{th})^n$ as $\omega$ approaches $\omega_{th}$. For nonzero threshold energy, $\omega_{th} \neq 0$, the integration over $\omega$ in Eq. (55) leads to

$$\dot{K}(t) \propto \int_{\omega_{th}}^{\infty} d\omega \ (\omega - \omega_{th})^n \cos(\omega t) = \frac{1}{t^{n+1}} \cos(\omega_{th} t + \frac{n+1}{2} \pi),$$  \hspace{1cm} (59)

which results in a power-law decay for its late-time dynamics. However, for electromagnetic fields with vanishing threshold energy, infrared photons can be generated even with an infinitesimally small amount of energy. The backreaction is expected to damp out the motion of the particle more effectively than that of the massive field. Thus as shown in Eq. (52), because $\text{Im} \Sigma$ is proportional to $\omega^3$ for small $\omega$, we find

$$\dot{K}(t) \propto \int_{0}^{\infty} d\omega \ \omega^4 \cos(\omega t) = 0,$$  \hspace{1cm} (60)

after the proper regularization. This result can also be obtained from Eq. (59). The conclusion holds true for all higher-order time derivatives of $K(t)$. The kernel $K(t)$ thus decay faster than the power law, and its time derivative $\dot{K}(t)$ at asymptotical times is described by an exponential decay given by Eq. (56).

To account for the dynamics of $K(t)$ at all times, we need the knowledge of all solutions to the relevant transcendental equation $\omega^2 - \omega_0^2 - \text{Re} \Sigma_x(\omega) - i \text{Im} \Sigma_x(\omega) = 0$. It is highly improbable to analytically locate all of them; nonetheless, when $|\omega|$ is much larger than unity, with the help of Eqs. (51) and (52), this equation may be transformed into an exponential polynomial such that the asymptotic solutions can be found. We decompose the solution $\omega$ into its real part $u$ and imaginary part $v$, i.e. $\omega = u + iv$. The coupled equations of $u$ and $v$ can be given by

$$\ln \sqrt{u^2 + v^2} - 2z_0v = \ln \frac{3}{4z_0},$$  

$$\tan^{-1} \frac{u}{u + 2z_0u} = (2n + 1/2)\pi, \quad n \in \mathbb{Z}. $$
Thus asymptotically, the solutions are evenly distributed in the $u$ sense along the curve $\ln \sqrt{u^2 + v^2} - 2z_0v = \ln(3/4z_0)$ on the $u-v$ plane as shown in Fig. 1. With $|u| \gg |v|$, both $u$ and $v$ are given by

$$u = \frac{1}{T}(2n + 1/2)\pi, \quad |n| \gg 1,$$

$$v = \frac{1}{T} \left(-\ln \frac{3}{4z_0} + \ln |u|\right) > 0.$$  

It can be seen that $v$ is always positive so that all the asymptotical solutions are on the upper half side of the complex $\omega$ plane. These modes will contribute to the motion via the time evolution factor

$$\exp(i\omega t) \propto n^{-\frac{1}{2}} \exp(i 2n\pi \frac{t}{T}), \quad |n| \gg 1.$$  

Their contributions are thus negligible for the dynamics of $\dot{K}_x(t)$ at times $t \gg T$. These relatively high frequency modes are significant only in the very early stage of the evolution. Therefore, the resonance mode, which peaks at $\Omega$ with width $\Gamma$, have the dominant effect for the late-time dynamics of the non-Markovian Langevin equation in a weak coupling limit. The memory effect or the effect of the time-delay will register in the parameters such as the resonance frequency $\Omega$, the decay constant $\Gamma$, and so on.

The Laplace transform method seems too complicated to properly take all asymptotical resonance modes into consideration in order to deal with the very early-time behavior of the charged oscillator. It would be more straightforward to apply the iteration techniques. The idea goes as follows. From the initial conditions in the interval $-T < t < 0$, we may find the solution to Eq. (40) for the times $0 < t < T$. Because in this time interval, the retardation effect does not settle in yet, the equation of motion essentially describes the Markovian motion due to the damped oscillator with a driving source. Next, this solution is then plugged into the retardation terms on the right hand side of Eq. (40) to iteratively find the solution for the time interval $T < t < 2T$. We may continue with these procedures, but the secular terms are expected to appear after several iterations. The method of dynamical renormalization group can then be implemented by resumming these secular terms to address more on the non-Markovian nature of the full-time dynamics of the solution.
B. perpendicular motion

We now consider a charged particle moving in the \( z \) direction perpendicular to the plane boundary. Contrary to the previous case, the modified instantaneous Coulomb term due to the boundary will make a shift to the oscillation frequency of the motion. Besides, it causes a static attraction between the charge and its image, so an applied potential is introduced to counteract the attractive Coulomb force. The corresponding linearized Langevin equation for the displacement from the equilibrium position is obtained from Eqs. (15), (23), and (24) as

\[
\ddot{q}_z(t) + \left[ \frac{\partial^2 V(z_0)}{\partial z^2} \frac{e^2}{4\pi (2z_0)^3} \right] q_z(t) + e^2 \frac{1}{4\pi (2z_0)^2} + e^2 \int_0^t dt' \, \dot{g}_R^z[z_0, z_0; t - t'] \dot{q}(t') = -\hbar e \dot{\xi}_z(t),
\]

where the \( zz \) component retarded Green’s function \( g_R^z \) can be expressed in terms of the spectral density,

\[
g_R^z[z_0, z_0; t - t'] = -\theta(t - t') \int_0^\infty \frac{dk}{\pi} \rho^z(z_0, z_0; k) \sin[k(t - t')],
\]

with

\[
\rho^z(z_0, z_0; \omega) = -\frac{k}{\pi} \left[ \frac{1}{3} - \frac{\cos(2kz_0)}{(2kz_0)^2} + \frac{\sin(2kz_0)}{(2kz_0)^3} \right].
\]

The noise-noise correlation functions are then obtained as

\[
\langle \xi_z(t) \rangle = 0, \quad \langle \xi_z(t) \xi_z(t') \rangle = \frac{1}{\hbar} \, g_R^z[z_0, z_0; t - t'],
\]

\[
g_H^z[z_0, z_0; t - t'] = -\int_0^\infty \frac{dk}{2\pi} \rho^z(z_0, z_0; k) \cos[k(t - t')].
\]

As discussed before, a fluctuation-dissipation relation is obeyed.

After mass renormalization, a similar time-delay differential equation for the charged oscillator moving in the \( z \) direction in the presence of the boundary is given by

\[
m_r \ddot{q}_z(t) + \left[ m_r \omega_0^2 - \frac{e^2}{4\pi (2z_0)^3} \right] q_z(t) = -\hbar e \dot{\xi}_z(t) + \frac{e^2}{6\pi} \dot{\xi}_z(t)
+ \frac{2e^2}{4\pi} \left[ \dot{q}_z(t - T) \frac{1}{(2z_0)^2} + \dot{q}_z(t - T) \frac{1}{(2z_0)^3} \right].
\]

The applied potential \( V(q) \) has been turned on to drive the charged particle into a harmonic motion of frequency \( \omega_0 \) in the \( z \) direction. In addition, the retarded Green’s function also
has the contribution to the frequency shift. This equation explicitly shows the memory effect due to the boundary. In particular, all backreaction terms arising from the boundary come from the velocity field. The contribution of the acceleration field vanishes under the dipole approximation since the relative direction from the image charge to the charge itself is perpendicular to its motion. Then, the real-time solution \( q^z \) to the inhomogeneous stochastic equation (62) takes a similar form to Eq. (46). The time evolution of the kernel \( K_z(t) \) from the inverse Laplace transform is obtained as

\[
\dot{K}_z(t) \sim Z_z \cos(\Omega_z t + \alpha_z) e^{-\Gamma_z t}, \quad t > 0,
\]

with the resonance frequency \( \Omega_z \) and the decay constant \( \Gamma_z \) given by

\[
\Omega_z \sim \omega_0^2 = \frac{e^2}{4\pi m_r (2z_0)^3} + \frac{\text{Re} \Sigma_z(\omega_0)}{2\omega_0}, \quad \Gamma_z \sim Z_z \frac{\text{Im} \Sigma_z(\Omega_z)}{2\Omega_z},
\]

respectively where the frequency shift due to the boundary corrections of the instantaneous Coulomb potential has been taken into account. Moreover, the phase shift \( \alpha_z \) and \( Z_z \) are found to be

\[
Z_z \sim \left[ 1 - \frac{\partial \text{Re} \Sigma_z(\Omega_z)}{\partial \Omega_z^2} \right]^{-1}, \quad \alpha_z \sim Z_z \frac{\partial \text{Im} \Sigma_z(\Omega_z)}{\partial \Omega_z^2}.
\]

Both the mass and the real part of the \( \Sigma_z(\omega) \) kernel have been renormalized. The corresponding \( \tilde{\Sigma}_z(s) \) kernel is

\[
\tilde{\Sigma}_z(s) = \frac{e^2}{m} s^2 \tilde{g}_R[z_0, z_0; s] = -\frac{e^2}{m} \left[ \int_{-\Lambda}^{\Lambda} \frac{dk}{\pi} \rho^*(z_0, z_0; k) \frac{s^2}{s^2 - k^2} - \frac{\Lambda}{3\pi^2} \right],
\]

and the real and imaginary parts of \( \tilde{\Sigma}_z(s) \) in the vicinity of the branch-cut are given, respectively, by

\[
\text{Re} \Sigma_z(\omega) = \frac{e^2}{4\pi m} \omega^3 \left[ \frac{2}{(2\omega z_0)^3} - \frac{2 \sin(2\omega z_0)}{(2\omega z_0)^2} - \frac{2 \cos(2\omega z_0)}{(2\omega z_0)^3} \right],
\]

\[
\text{Im} \Sigma_z(\omega) = \frac{e^2}{4\pi m} \text{sgn}(\omega) \omega^3 \left[ \frac{2}{3} \frac{2 \cos(2\omega z_0)}{(2\omega z_0)^2} + \frac{2 \sin(2\omega z_0)}{(2\omega z_0)^3} \right].
\]

The presence of the boundary apparently modifies the behavior of the charged oscillator in an anisotropic way. This characteristic is especially noticeable near the boundary where the electric fields parallel to the plate vanish, but their normal components become doubled, compared with their counterparts without the boundary. This anisotropic feature is encoded
in the spectral density. Thus, the quantities that can be expressed with the spectral density should share the same feature. As a result, the decay constant $\Gamma$ for motion parallel to the boundary turns out to be smaller than in the perpendicular case with a similar configuration. Additionally, the effect of the stochastic noise on the oscillator ought to be much weaker in the parallel case than the perpendicular one.

Next section will be devoted to studying the time evolution of velocity fluctuations of the charged oscillator due to both anisotropically modified vacuum fluctuations by the boundary and associated dissipative backreaction from electromagnetic fields.

V. VELOCITY FLUCTUATIONS

When the charged particle couples to quantized electromagnetic fields, the stochastic Lorentz force, manifested from vacuum fluctuation of fields, drives the particle into a fluctuating trajectory in analogy to the Brownian motion. Thus, it is of interest to study velocity fluctuations of the charged oscillator to see how they are affected by the boundary and are asymptotically saturated as a result of the fluctuation-dissipation relation. We will compute them in the interval $1/\Omega \ll t \ll 1/\Gamma$ for the linearly growing regime in which backreaction dissipation can be ignored, as well as in the interval of $1/\Gamma \ll t$ for the saturation regime where the effects of fluctuations and dissipation come into balance.

As the particle starts to move at $t = 0$, its velocity fluctuations at time $t$ in the direction $i$ driven by electromagnetic vacuum fluctuations are given by

$$\langle \Delta v_i^2(t) \rangle = \frac{e^2}{m^2} \int_0^t dt' \int_0^t dt'' K_i(t') \frac{d^2}{dt' dt''} \left[ g_H^i(z_0, z_0; t' - t'') \right] K_i(t''),$$

where $g_H^i$ is the Hadamard function of vector potentials in the dipole approximation. We have implicitly assumed that $\langle \Delta v_i^2 \rangle$ vanishes initially. The charged oscillator is allowed to move either parallel or perpendicular to the boundary, and thus the relevant Hadamard function takes the form of Eq. (39) or (65). The function $K_i(t)$ is the kernel of the equation of motion, and its time derivative is approximated by Eq. (56) or (67). This approximate solution holds for $t > 1/\Omega_i \geq 2z_0$ by ignoring the contributions from high-frequency modes. Then, it may give rise to error in computing the integral (73) for the time regime $t \leq 2z_0$. In fact, one can argue that the error for a large distance $z_0$ can be trivially neglected because the boundary correction is negligible. Moreover, although the presence of the boundary will
result in significant contributions for small \(z_0\), the time interval \(0 \leq t \leq 2z_0\) over which integrations in Eq. (73) are performed, are also small. Only a minor error is introduced.

In writing so, the integrand of the integrations in Eq. (73) are performed, are also small. Only a minor error is introduced. By introducing a cutoff.

Two competing scales \(1/\pi t\) and \(1/z\), the boundary effect becomes insignificant. Thus, the result of the integration (74) relies on after doing integration over \(z\) space, which will lead to heavy cancelation as the distance \(z_0\) and time \(t\), respectively. The full expression of the function \(I\) is too intricate to be of any help, but in the time intervals \(1/\Omega_i \ll t \ll 1/\Gamma_i\) and \(t \gg 1/\Gamma_i\), it can be reduced greatly,

1. \(1/\Omega_i \ll t \ll 1/\Gamma_i\): linearly growing regime, with \(\Gamma_i t\) set to zero,
   \[
   I_0(t, k) \equiv I \left( \frac{1}{\Omega_i} \ll t \ll \frac{1}{\Gamma_i} ; k \right) = \frac{Z_i^2}{2(k^2 - \Omega_i^2)^2} \left\{ 2(k^2 + \Omega_i^2) + (k^2 - \Omega_i^2) \cos(2\alpha_i) + (k^2 - \Omega_i^2) \cos(2\Omega_i t + 2\alpha_i) \right. \\
   - (k + \Omega_i)^2 \cos[(k - \Omega_i)t] - (k - \Omega_i)^2 \cos[(k + \Omega_i)t] \\
   - (k^2 - \Omega_i^2) \cos(\Omega_i t - kt + 2\alpha_i) - (k^2 - \Omega_i^2) \cos(\Omega_i t + kt + 2\alpha_i) \right\}.
   \]

2. \(\Gamma_i t \gg 1\): saturation regime, with \(\Gamma_i t\) set to infinity,
   \[
   I_{\infty}(k) \equiv I(t \to \infty ; k) = Z_i^2 \frac{(k^2 + \Gamma_i^2) \cos^2 \alpha_i - 2\Omega_i \Gamma_i \cos \alpha_i \sin \alpha_i + \Omega_i^2 \sin^2 \alpha_i}{[(k - \Omega_i)^2 + \Gamma_i^2] \left[(k + \Omega_i)^2 + \Gamma_i^2\right]}.
   \]

Notice that the function \(I\) has a Breit-Wigner feature in the \(k\) space with the resonance peak at about \(\Omega_i\) and its width being approximately of order \(\pi/t\) at early times or \(\Gamma_i\) for the late-time regime. The narrow width may result in a prominent peak. On the other hand, the spectral density \(\rho^i\) reveals the oscillatory behavior on the scales \(1/z_0\) in the \(k\) space, which will lead to heavy cancelation as the distance \(z_0\) is sufficiently large so that the boundary effect becomes insignificant. Thus, the result of the integration (74) relies on two competing scales \(1/z_0\), and \(\pi/t\) or \(\Gamma_i\). Additionally, the behavior of the integrand in Eq. (74) increases linear in \(k\) when \(k\) is sufficiently large. Therefore, velocity fluctuations after doing integration over \(k\) will have a quadratic divergence, which has to be regularized by introducing a cutoff. \(Z_i\) is approximately equal to unity in the weak coupling limit.
A. Growing regime

Here we study the motion of the charged oscillator at the early stage for $1/\Omega_i \ll t \ll 1/\Gamma_i$ when dissipation backreaction can be ignored. Velocity fluctuations thus mainly result from the stochastic noise. For $t \gg 2z_0$ when the retardation effect out of the boundary becomes effective, we may expect that the spectral density varies relatively slowly with $k$ in comparison with the function $I$ around the resonance peak.

Therefore, the integration (74) can be approximated by pulling the spectral density $\rho^i$ out of the integral and evaluating it with $k \sim \Omega_i$ in the neighborhood of the peak. With this in mind, we then rewrite Eq. (74) in terms of the dimensionless parameters denoted as $\gamma_i = \Gamma_i/\Omega_i$, $\bar{z}_{0i} = \Omega_i z_0$, $y = k/\Omega_i$, and $\tau_i = \Omega_i t$, and obtain

$$\langle \Delta v^2_i(t) \rangle = -\frac{e^2\Omega_i^2}{m^2} \int_0^\infty \frac{dy}{2\pi} y^2 \bar{\rho}^i(\bar{z}_{0i}, \bar{z}_{0i}; y) I(\tau_i; y), \quad (76)$$

where

$$\bar{\rho}^i(\bar{z}_{0i}, \bar{z}_{0i}; y) = \frac{1}{\Omega_i} \rho^i(z_0, z_0; k), \quad I(\tau_i; y) = \Omega_i^2 I_0(t; k).$$

Since the function $I$ demonstrates a sharp peak about $y = 1$ with the width of order $\pi/\tau_i$, as long as $\bar{z}_{0i}/\tau_i \ll 1$, the velocity fluctuations can be approximately given by

$$\langle \Delta v^2_i(t) \rangle \sim \langle \Delta v^2_i(t) \rangle_{\text{div.}} - \frac{e^2\Omega_i^2}{m^2} \bar{\rho}^i(\bar{z}_{0i}, \bar{z}_{0i}; 1) \int_0^\infty \frac{dy}{2\pi} I(\tau_i; y), \quad (77)$$

where

$$\int_0^\infty \frac{dy}{2\pi} I(\tau_i; y) \approx \frac{\tau_i}{4} + \text{terms oscillatory with time} \quad (78)$$

leads to a contribution linearly growing in time. The high-frequency contributions $\langle \Delta v^2_i(t) \rangle_{\text{div.}}$ need regularization by inserting a convergent factor $e^{-y\epsilon_i}$ into the integrand. Here $\epsilon_i$ is the dimensionless short-distance cutoff. Then, it ends up with

$$\langle \Delta v^2_i(t) \rangle \sim \Omega_i \Gamma_i t + \frac{e^2\Omega_i^2}{16\pi^2 m^2} \left[ 2 + \cos(2\alpha_i) + \cos(2\Omega_i t + 2\alpha_i) \right] \left\{ \begin{array}{ll} \frac{4}{3\lambda^2}, & x \text{ direction} \\ \frac{4}{3\lambda^2} + \frac{1}{\bar{z}_{0i}}, & z \text{ direction} \end{array} \right\},$$

$$\sim \frac{\Omega_0}{m} \Gamma_i t + \frac{e^2}{16\pi^2 m^2} \left[ 3 + \cos(2\Omega_0 t) \right] \left\{ \begin{array}{ll} \frac{4}{3\lambda_{\text{dB}}^2}, & x \text{ direction} \\ \frac{4}{3\lambda_{\text{dB}}^2} + \frac{1}{\bar{z}_{0i}}, & z \text{ direction} \end{array} \right\}. \quad (79)$$

Notice that to obtain the last expression, the dimensionless short-distance cutoff $\epsilon_i$ is chosen as a product of the resonance frequency $\Omega_i$ and the de Broglie wavelength of the charge $\lambda_{\text{dB}},$
that is, \( \epsilon_i = \Omega_i \lambda_{dB} \). The cutoff dependence terms are known to arise from the unbounded Minkowski vacuum fluctuations, experienced by the charged oscillator. They weakly depend on the modified oscillatory motion that results from the coupling to vacuum fluctuations in the presence of boundary. We will make further approximations by \( \Omega_i \sim \Omega_0 \), \( \alpha_i \sim 0 \), and \( Z_i \sim 1 \), ignoring higher order effect on \( z_0 \). For motion perpendicular to the boundary, the dominant \( z_0 \) dependence of high frequency contributions is thus the \( 1/z_0^2 \) term, and should be small as compared with the cut-off dependence term in accordance with the classical assumption on the particle. However, the corresponding term is missing for motion parallel to the boundary. This can be understood as the consequence that there exists no corresponding boundary for the motion in the \( x \) direction; thus no length scale is introduced in this direction.

Now we consider the ratio of the cut-off dependence term to the term linear in time, obtained form the above expression. With the help of the explicit expressions of the relaxation constants \( \Gamma_i \), given by Eqs. (52), (57), (68), and (72),

\[
\begin{align*}
\Gamma_x(\Omega_0) &= \frac{e^2}{8\pi m} \Omega_0^2 \left[ \frac{2}{3} - \frac{\sin(2\Omega_0 z_0)}{(2\Omega_0 z_0)^2} - \frac{\cos(2\Omega_0 z_0)}{(2\Omega_0 z_0)^3} + \frac{\sin(2\Omega_0 z_0)}{(2\Omega_0 z_0)^3} \right], \\
\Gamma_z(\Omega_0) &= \frac{e^2}{8\pi m} \Omega_0^2 \left[ \frac{2}{3} - \frac{2\cos(2\Omega_0 z_0)}{(2\Omega_0 z_0)^2} + \frac{2\sin(2\Omega_0 z_0)}{(2\Omega_0 z_0)^3} \right],
\end{align*}
\]

(80) (81)

it can be shown that the ratio is of the order

\[
\mathcal{O}\left( \frac{1}{\Omega_0 t} \frac{1}{\epsilon^2} \right) \sim \mathcal{O}\left( \frac{1}{\Omega_0 t} \frac{1}{\Omega_0^2 \lambda_{dB}^2} \right).
\]

Apparently, the high-frequency contributions can be dominant for the sufficiently low oscillation frequency, or at very early stage of motion \([12]\). At later times, \( \epsilon^{-2} \Omega_0^{-1} \ll t \), they then become insignificant as it can be further seen from the numerical result to be discussed later.

Therefore, the velocity fluctuations are found to grow linearly with time. The growing rate is related to the relaxation constant \( \Gamma_i \), evaluated at the resonance frequency \( \Omega_0 \). Back-reaction dissipation will set in at later times, and will asymptotically counteract the effects of fluctuations.
B. Saturation regime

Next we will investigate the behavior of velocity fluctuations at much later times $t \gg 1/\Gamma_i$ by incorporating backreaction dissipation. In this regime, they are described by

$$\langle \Delta v_i^2 \rangle = -\frac{e^2 \Omega_i^2}{m^2} \int_0^\infty \frac{dy}{2\pi} y^2 \tilde{\rho}^i(\bar{z}_{0i}, \bar{z}_{0i}; y) \mathcal{I}(y),$$

where

$$\mathcal{I}(y) = \Omega_i^2 I_\infty(k).$$

In general, since Eq. (82) has a quadratic divergence, a regulator of the form $e^{-y\epsilon_i}$ will be introduced to regularize the integral. The influence from the boundary can be significant when the dimensionless distance of the oscillator from the boundary $\bar{z}_{0i}$ is much smaller than unity.

Thus, we will expand the result of velocity fluctuations in terms of small $\bar{z}_{0i}$, after we plug in the $\bar{z}_{0i}$ dependence of the parameters $\gamma_i$, $\alpha_i$, and $\tilde{\rho}^i$ for the motion in the direction $i$, which are given by,

$$\tilde{\rho}^x(\bar{z}_{0x}, \bar{z}_{0x}; y) = -\frac{y}{\pi} \left[ 1 - \frac{\sin(2\bar{z}_{0x})}{2(2\bar{z}_{0x})} - \frac{\cos(2\bar{z}_{0x})}{2(2\bar{z}_{0x})^2} + \frac{\sin(2\bar{z}_{0x})}{2(2\bar{z}_{0x})^3} \right],$$

$$\tilde{\rho}^z(\bar{z}_{0z}, \bar{z}_{0z}; y) = -\frac{y}{\pi} \left[ 1 - \frac{\cos(2\bar{z}_{0z})}{2(2\bar{z}_{0z})^2} + \frac{\sin(2\bar{z}_{0z})}{(2\bar{z}_{0z})^3} \right],$$

$$\alpha_x = \bar{r}_{cx} \left[ 1 - \frac{\cos(2\bar{z}_{0x})}{2} + \frac{\sin(2\bar{z}_{0x})}{2(2\bar{z}_{0x})} \right], \quad \alpha_z = \bar{r}_{cz} \left[ 1 + \frac{\sin(2\bar{z}_{0z})}{(2\bar{z}_{0z})} \right],$$

and the dimensionless relaxation constant can be obtained from Eqs. (80) and (81) as $\gamma_i = -\pi \bar{r}_{ci} \tilde{\rho}^i$ with $\bar{r}_{ci} = \Omega_i r_c$. The parameter $r_c = e^2/(4\pi m)$ is the particle’s classical radius. Here, another small parameter $\bar{r}_{ci}$, characterizing the distance $r_c$ over the oscillation time scales due to a nonrelativistic motion, will also be used to extract the dominant contributions.

The saturated value of velocity fluctuations can be found analytically. For the motion parallel to the boundary, it is given by

$$\langle \Delta v_x^2 \rangle = \frac{Z_x^2}{16\pi^2 m^2} \Omega_x^2 \left\{ \frac{8}{3 \epsilon_x^2} + \frac{2\pi}{\bar{r}_{cx}} + \frac{16}{3} \ln \frac{2\bar{z}_{0x}}{\epsilon_x} - \frac{40}{3} \right\} + \frac{776}{73} - \frac{32}{3} \gamma_c - \frac{32}{5} \ln 2\bar{z}_{0x} \bar{z}_{0x}^2 + \mathcal{O}(\bar{z}_{0x}^4 \ln \bar{z}_{0x}, \bar{r}_{cx})$$

$$\sim \frac{e^2}{16\pi^2 m^2} \Omega_x^2 \left\{ \frac{8}{3 \epsilon_x^2} + \frac{2\pi}{\bar{r}_{c}} \right\} = \frac{e^2}{8\pi^2 m^2} \left\{ \frac{4}{3 \lambda_{db}^2} + \frac{\pi \Omega_0}{r_c} \right\}.$$
For the perpendicular case, we find
\[
\langle \Delta v_z^2 \rangle = Z^2 \frac{e^2}{16\pi^2 m^2} \Omega^2 \left\{ \frac{8}{3\epsilon^2} + \frac{2}{z_0^2} + \left[ \frac{2\pi}{r_{cz}} - \frac{16}{3} \left( \ln 2z_0 + \ln \epsilon \right) + \frac{40}{3} - \frac{32}{3} \gamma_i \right] + \right. \\
\left. + \left\{ -\frac{448}{75} + \frac{16}{5} \gamma_i + \frac{16}{5} \ln 2z_0 \right\} \frac{z_0^2}{z_0^2} + \mathcal{O}(z_0^4 \ln z_0, \bar{r}_{cz}) \right\} \\
\sim \frac{e^2}{16\pi^2 m^2} \Omega^2 \left\{ \frac{8}{3\epsilon^2} + \frac{2}{z_0^2} + \frac{2\pi}{\bar{r}_c} \right\} = \frac{e^2}{8\pi^2 m^2} \left\{ \frac{4}{3\lambda_{dB}^2} + \frac{1}{z_0^2} + \frac{\pi \Omega_0}{r_c} \right\},
\]
where \( \gamma_i \) is Euler’s constant, with numerical value \( \sim 0.577216 \). The last lines of Eqs (83) and (84) are further approximated by \( Z_i \sim 1, \Omega_i \sim \Omega_0 \), ignoring higher-order \( \bar{z}_0 \)-dependent terms where \( \bar{z}_0 = \Omega_0 z_0, \bar{r}_c = \Omega_0 r_c \), and \( \epsilon = \Omega_0 \lambda_{dB} \); hence typically \( \bar{z}_0 > \epsilon \gg \bar{r}_c \). The parameter \( \Omega_0 \) is the renormalized oscillation frequency with the frequency shift due to the interaction with fields. Here for simplicity, the same \( \Omega_0 \) is chosen for the motion in both directions because the anisotropy it introduces is of next order in \( e^2 \). In general, the results (83) and (84) should show strong dependence on the relative orientation between the boundary and the direction of the motion for small \( z_0 \), and thus are anisotropic. It can be understood as a result of vacuum fluctuations of electromagnetic fields under the influence of the boundary \[ 11 \]. The enhancement in velocity fluctuations due to the presence of the boundary in the direction normal to the conducting plate arises from large induced \( \mathbf{E} \)-field fluctuations near the boundary, in comparison with the fluctuations parallel to the plate \[ 12 \]. On the other hand, the term depending on \( \Omega_0 \) results from particle’s motion, and is found to be boundary-independent. The physics behind this features can be explored in a more general context as follows. Starting from Eq. (82), for small \( \bar{z}_0i \), and \( \bar{r}_{ci} \), the contribution from the resonance peak is given by approximating the integral over \( y \) with the spectral function evaluated at the dimensionless resonance frequency, \( y \sim 1 \),
\[
\int_{1-\gamma}^{1+\gamma} \frac{dy}{2\pi} y^2 \bar{\rho}(\bar{z}_0i, \bar{z}_0i; y) \mathcal{I}(y) \sim \bar{\rho}(\bar{z}_0i, \bar{z}_0i; 1) \int_{1-\gamma}^{1+\gamma} \frac{dy}{2\pi} \mathcal{I}(y) \\
\sim \bar{\rho}(\bar{z}_0i, \bar{z}_0i; 1) \int_0^\infty \frac{dy}{2\pi} \mathcal{I}(y) \sim \frac{\bar{\rho}(\bar{z}_0i, \bar{z}_0i; 1)}{8\gamma_i} \sim -\frac{1}{8\pi} \frac{1}{r_c},
\]
where again the approximation \( \Omega_i \sim \Omega_0 \) has been applied to obtain the last expression. Apparently, the result superficially depends on the spectral function \( \bar{\rho} \); however, the integral to the right of the spectral function yields an expression proportional to \( 1/\gamma_i \). Since the dimensionless relaxation constant \( \gamma_i \) can be related to the spectral density by \( -\pi \bar{r}_{ci} \bar{\rho}(\bar{z}_0i, \bar{z}_0i; 1) \), the boundary dependence due to the spectral density \( \bar{\rho} \) is canceled. Thus, the result does
not depend on the relaxation constant, hence independent of the orientation. Recall that the function $\bar{\rho}^i$ is proportional to the rate at which velocity fluctuations grow, while the time scales to reach the saturated regime are given by $1/\gamma_i$. Thus it implies that although the velocity fluctuations increases at different rates at early times, they will take different amount of time to reach saturation. In the end, the saturated values are independent of the direction of the motion, and are only determined by the oscillation frequency of the charged particle. This cancelation of the boundary dependence, in fact, can be understood as reminiscence of the fluctuations-dissipation relation since the function $\bar{\rho}^i$ comes from the noise kernel $g^i_H$ while the dimensionless relaxation constant $\gamma_i$ is inherited from the dissipation kernel $g^i_R$.

As stated above, the integrand of Eq. (82) grows linearly for large values of $y$, so the contribution of this portion is given by

$$\int_{\sigma}^{\infty} \frac{dy}{2\pi} y^2 \bar{\rho}^i(z_0, \bar{z}_0; y) \mathcal{I}(y) e^{-y\epsilon} \sim -\frac{1}{16\pi^2} \left\{ \begin{array}{ll} \frac{8}{3\epsilon^2}, & \text{in the } x \text{ direction}, \\ \frac{8}{3\epsilon^2} + \frac{2}{\bar{z}_0^2}, & \text{in the } z \text{ direction}, \end{array} \right.$$  \hspace{1cm} (85)

where $\epsilon = \Omega_0 \lambda dB$, and the lower limit $\sigma$ of the integral is chosen to be greater than unity but otherwise arbitrary in order to exclude the integration region around the resonance peak. As long as $\bar{z}_0 \sigma \ll 1$, the integral is pretty much independent of the choice of $\sigma$. Clearly, the integral (85) is boundary-dependent and thus anisotropic. The quadratic dependence on the cutoff can be characterized by the width of the particle’s wave wavefunction. When the spatial extension of the wave function is smaller, it tends to probe the finer structure of fluctuations, which in turn causes larger variation in particle’s velocity.

Some comments are in order. In the limit $\Omega_0 \rightarrow 0$, the motion of the charged particle becomes insignificant and the dissipation effect is ignored. Hence, it is of interest to compare our results with the earlier study [12]. They are consistent as $t \rightarrow \infty$, apart from a factor 1/2 due to an average over a period in the oscillatory motion. In this limit, the backreaction dissipation can be safely ignored because the relaxation constant $\Gamma_i$ with $\Omega_0^2$ dependence, is vanishingly small for such a slow motion. Nonetheless, for a finite value of $\Omega_0$, the nonuniform motion results in a dissipative effect, which damps out the particle’s motion.
C. Numerical results

The typical behavior of time variation of velocity fluctuations is shown in Fig. 2, where the results are generated numerically. The value of the cutoff is set by the de Broglie wavelength, \( \lambda_{dB} \sim \frac{h}{(mv)} \), so the ratio of the charge’s classical radius to its de Broglie wavelength is \( r_c / \lambda_{dB} \sim \left( \frac{e^2}{hc} \right) (v/c) \sim 10^{-4} \) with \( v/c \sim 10^{-2} \) for nonrelativistic motion. In Fig. 2 we choose \( \bar{z}_0 = z_0 \Omega_0 / c \sim 1 \), for example, when the distance to the plate \( z_0 \) is of the order \( \mu \)m, the oscillation frequency is given by \( \Omega_0 \sim 10^{14} \text{s}^{-1} \) below the plasma frequency of the plate \( 10^{16} \text{s}^{-1} \) [15]. Now for an arbitrary charged particle, without loss of generality, we may take appropriate values of the parameters to numerically compute full evolution of velocity fluctuations. Here we let \( m \sim 10^3 m_e \) where \( m_e \) is the mass of an electron, leading to the value of \( \epsilon = \lambda_{dB} \Omega_0 / c \sim 0.1 \) and \( \bar{r}_c = r_c \Omega_0 / c = 5 \times 10^{-4} \). The horizontal axis is the dimensionless time \( \tau = \Omega_0 t \), and the vertical axis is velocity fluctuations, normalized by the parameter \( \hbar \Omega_0 / 2m \). It is seen that at early times, the velocity fluctuations increase linearly at different rates for motions parallel and perpendicular to the conducting plate, and then saturate to the same value at late times. In this case the cutoff-dependent terms is negligible except for very early times \( t \leq \epsilon^{-2} \Omega_0^{-1} \sim 10^2 \Omega_0^{-1} \), which is vanishingly short when compared with the relaxation time scales for motion in either direction, about the order of \( 1/\Gamma \sim 10^4 \Omega_0^{-1} \). Thus, their contribution can not possibly be seen in Fig. 2.

VI. DISCUSSIONS AND CONCLUDING REMARKS

The anisotropic behavior of velocity fluctuations can be observed at the early stage when the velocity fluctuations grow linearly in time. It becomes more significant for the small value of \( z_0 \Omega_0 / c \), and can be estimated analytically from the ratio obtained from Eq. (81) as

\[
\frac{\Delta v_x^2}{\Delta v_z^2} e^{-2 \Omega_0^{-1} \ll t \ll \Gamma^{-1}_z \sim 0.4 (z_0 \Omega_0/c)^2, \quad z_0 \Omega_0 / c \ll 1.}
\] (86)

However, the values of \( z_0 \Omega_0 / c \) cannot be arbitrarily small, and are constrained by the underlying assumptions. To be consistent with the dipole approximation, the amplitude of the charged oscillator should be much smaller than its distance to the plate, and can be set to the order of \( 10^{-2} z_0 \), for example. Thus, the corresponding velocity \( v \) of the charged oscillator is about \( v/c \sim 10^{-2} (z_0 \Omega_0/c) \). It can be argued that the charge’s motion cannot be too slow.
FIG. 2: Velocity fluctuations of the charge oscillator moving in the direction either parallel or normal to the plane boundary grow linearly at early times, and then saturate to a constant at late times. They start off at different rates, but approach the same saturated value $\hbar \Omega_0/2m$. The values of parameters are chosen as $\bar{z}_0 = z_0 \Omega_0/c = 1.0$ and $\bar{r}_c = r_c \Omega_0/c = 5 \times 10^{-4}$, and $\epsilon = 0.1$; thus $\gamma_z^{-1} = 0.363 \times 10^4$ and $\gamma_x^{-1} = 0.931 \times 10^4$.

since it may give rise to a large position uncertainty to jeopardize the assumption of a point-like particle in the stochastic approach. It is of interest to take the electron as an example. Because the de Broglie wavelength, characterizing the size of the electron wavefunction, is

$$\lambda_{\text{DB}} \sim \frac{h}{(m_e v)} \sim 10^{-12} (v/c)^{-1} \text{m},$$

when the velocity is overly small, the spatial extension of the electron wavefunction can be the same as or even larger than the distance to the plate $z_0$. Then it contradicts to the earlier assumptions. Nevertheless, the velocity of the electron may still be chosen as small as $v/c \sim 10^{-4}$, consistent with $\lambda_{\text{DB}} \ll z_0 \sim \mu\text{m}$, ending up with $z_0 \Omega_0/c \sim 10^{-2}$. Thus, the oscillation frequency for such a slow motion is reduced to the value of $\Omega_0 \sim 10^{12} \text{s}^{-1}$. It will lead to the relaxation constant $\Gamma \sim (e^2\Omega_0/m_e c)\Omega_0 \sim 10 \text{s}^{-1}$ from Eq. (81). In addition, the dimensionless cutoff is then given by $\epsilon = \lambda_{\text{DB}} \Omega_0/c \sim 10^{-4}$. Thus, even for an electron in rather slow motion, appreciable anisotropy $\Delta v_x^2/\Delta v_z^2 \sim 10^{-4}$ can be found during the time regime $10^{-4} \text{s} < t < 0.1 \text{s}$ in which the cutoff dependence effect can be safely ignored while the saturation has not been reached yet in both directions.
At later times, the velocity fluctuations of the charged oscillator near the boundary reach saturation, independent of the orientation of the boundary, as shown in Fig. 2. The saturated value is approximately given by the $\Delta v_i^2(\infty) \sim \hbar \Omega_0 / 2m$ as consequence of its nonuniform motion. The contribution of the velocity fluctuations, resulting from the imposition of the boundary conditions on electromagnetic fields, are at most the same order of magnitude as the cutoff-dependent terms. Both of them can be argued to be ignored at late times. Finally, the effective temperature corresponding to saturated velocity fluctuations for such Brownian motion can be estimated by

$$T_{\text{eff}} \sim \frac{\hbar \Omega_0}{k_B} \sim 10 \left( \frac{\Omega_0}{10^{12} \text{s}^{-1}} \right) \text{K},$$

where $k_B$ is the Boltzmann constant.

To conclude, we study the influence due to quantized electromagnetic fields on the motion of a nonrelativistic charged particle near the conducting plate. The nonlinear, non-Markovian Langevin equation of the particle is derived with the method of Feynman-Vernon influence functional, and it incorporates both dissipation backreaction on the charge in the form of the retarded self-force as well as the stochastic noise, manifested from vacuum fluctuations of quantized electromagnetic fields. The dipole approximation, an appropriate approximation for a non-relativistic motion, is implemented to find the solution to the Langevin equation. We consider that the charged particle undergoes a small-amplitude oscillation in the direction either parallel or perpendicular to the boundary plane. The noise-averaged trajectory of this charged oscillator is governed by the classical dynamics. Furthermore, the evolution of the kernel $K_i(t)$, obtained by solving the Langevin equation, is found to be dominated by the narrow resonance in the weak coupling limit where the ratio of the decay width $\Gamma_i$ over the oscillation frequency $\Omega_i$ is much less than unity, $\Gamma_i / \Omega_i \ll 1$. Thus, the memory effects or the effects of the time-delay on backreaction terms due to the presence of the boundary are just to modify the quantities such as $\Omega_i$, $\Gamma_i$ and so on. Then, velocity fluctuations of the charged oscillator driven by stochastic forces are found to grow linearly with time in the early stage of the evolution at the rate given by the relaxation constant. It turns out to be smaller in the parallel case than in the perpendicular one with a similar configuration, and reveals strong anisotropic behavior. They are then asymptotically saturated as a result of the fluctuation-dissipation relation at rather different relaxation time scales. However, we find that the same saturated value is obtained for motion in both direc-
tions at late times, resulting from delicate balancing effects between dissipation backreaction and accompanying fluctuations, and thus the value is mainly determined by its oscillatory motion. So, at late times the effects from boundary-modified vacuum fluctuations on the velocity dispersion of the charged particle can be hardly seen.

The dipole approximation amounts to linearizing the Langevin equation obtained above. However, beyond the dipole approximation, one expects to introduce the additional drift effects on the dynamics of the particle from the trajectory-dependent Green’s functions. In particular, it may give rise to the noise-induced-drift forces owing to the correlations of stochastic forces. This effect has been considered in the context of the fast moving particle [21], and may have observational consequences.

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