Phase separation and interface structure in two dimensions from field theory

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Abstract. We study phase separation in two dimensions in the scaling limit below criticality. The general form of the magnetization profile as the volume goes to infinity is determined exactly within the field theoretical framework which explicitly takes into account the topological nature of the elementary excitations. The result known for the Ising model from its lattice solution is recovered as a particular case. In the asymptotic infrared limit the interface behaves as a simple curve characterized by a Gaussian passage probability density. The leading deviation, due to branching, from this behavior is also derived and its coefficient is determined for the Potts model. As a byproduct, for random percolation we obtain the asymptotic density profile of a spanning cluster conditioned to touch only the left half of the boundary.

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1. Introduction

Boundary conditions notoriously play an important role in the theory of phase transitions. For a system of ferromagnetic spins taking discrete values, a pure phase of type \( a \) with translation invariant spontaneous magnetization below the critical temperature \( T_c \) can be selected by fixing all boundary spins to the value \( a \) and then sending the boundary to infinity. On the other hand, if the spins are fixed to a value \( a \) on the left half of the boundary and to a different value \( b \) on the right half, a pattern of phase separation between phases of type \( a \) and type \( b \) is expected in the large volume limit below \( T_c \), at least away from an interfacial region anchored to the points of the boundary where the boundary conditions change from \( a \) to \( b \). The properties of the phase separation and the notion of interface have been extensively studied in both two and three dimensions through rigorous [1], exact [2] and approximate [3] methods. The most advanced analytic results are available in two dimensions, where the exact asymptotic magnetization profile has been obtained for the Ising model [4, 5], exploiting its lattice solvability at any temperature. This result shows in particular that in two dimensions the Ising interface has middle point fluctuations which diverge as the square root of the volume, a property previously proved for low temperatures in [6]. The result for the Ising magnetization profile also admits a simple interpretation in terms of the passage probability of the interface through a point [7]. No exact result for the magnetization profile is available in three dimensions.

In this paper we use field theory as a general framework for the study of phase separation in the scaling limit below \( T_c \), for any two-dimensional model possessing a discrete set of ordered phases and undergoing a continuous phase transition. For a strip of width \( R \) we derive the large \( R \) asymptotics for the magnetization profiles along the longitudinal axis in the middle of the strip and show that a generalization of the Ising result holds whenever the surface tension between the phases \( a \) and \( b \) cannot be decomposed into a sum of smaller surface tensions. The formalism explicitly illustrates the role played by the topological nature of the elementary excitations (domain walls), which for a discrete set of ground states is peculiar to the two-dimensional case. The interpretation in terms of passage probability holds in general, with subsequent terms in the large \( R \) expansion accounting for the emergence of an interfacial region with finite width in a way that can be understood through renormalization group considerations.
The trajectories on the plane of the domain wall excitations of the field theory are naturally interpreted as the continuum limit of the boundaries of clusters made of nearest neighbors with the same value of the spin. In recent years the scaling properties of cluster boundaries have been extensively studied at criticality in the framework of Schramm–Loewner evolution (SLE, see, e.g., [8] for a review); the application of SLE methods to the off-critical case, on the other hand, is up to now much more limited (see [9, 10]). The renormalization group interpretation of our results below \( T_c \) is that the cluster boundary connecting the two boundary changing points on the edges and the interfacial curve between the two phases coincide as \( R \rightarrow \infty \), and then have the same Gaussian passage probability density; when \( R \) decreases, the interfacial region with finite width emerges via branching of the interface and formation of intermediate clusters. We write down the leading term associated with branching and determine its coefficient for the \( q \)-state Potts model. When the surface tension between the phases \( a \) and \( b \) is decomposable the formalism leads to a multiple interface description, the Ashkin–Teller model providing an interesting example of this type.

This paper is organized as follows. In section 2 we introduce the field theoretical formalism and derive the large \( R \) results for the magnetization profiles. Section 3 is then devoted to the interpretation of the results and the discussion of the interface structure. The specific cases of \( q \)-state Potts and Ashkin–Teller models as well as an application to percolation are finally discussed in section 4.

2. Field theoretical results

Consider a ferromagnetic spin model of two-dimensional classical statistical mechanics in which each spin can take \( n \) discrete values that we label by an integer \( a = 1, 2, \ldots, n \). To be definite we refer to the case in which the energy of the system is invariant under global transformations of the spins according to a symmetry group; the spontaneous breaking of the symmetry below a critical temperature \( T_c \) is responsible for the presence on the infinite plane of \( n \) translation invariant pure phases; the phase of type \( a \) can be selected starting with the system on a finite domain with boundary spins fixed to the value \( a \), and then removing the boundary to infinity. We denote by \( Z_a \) and \( \langle \cdots \rangle_a \) the partition function and the statistical averages, respectively, in the phase \( a \).

We consider the scaling limit below \( T_c \), described by a Euclidean field theory defined on the plane with coordinates \((x, y)\); it corresponds to the analytic continuation to imaginary time of a \((1 + 1)\)-dimensional relativistic field theory with space coordinate \( x \) and time coordinate \( t = iy \). This theory possesses degenerate vacua \( |\Omega_a\rangle \) associated to the pure phases of the system. In \( 1 + 1 \) dimensions the elementary excitations will correspond to stable kink states \( |K_{ab}(\theta)\rangle \) which interpolate between different vacua\(^1\) \( |\Omega_a\rangle \) and \( |\Omega_b\rangle \); the rapidity \( \theta \) parameterizes the energy and momentum of the kinks as \((e, p) = (m_{ab} \cosh \theta, m_{ab} \sinh \theta)\), \( m_{ab} \) being the kink mass. In general, connection of \( |\Omega_a\rangle \) and \( |\Omega_b\rangle \) requires a multi-kink state \( |K_{aa_1}(\theta_1)K_{a_1a_2}(\theta_2) \cdots K_{a_{n-1}b}(\theta_n)\rangle \) passing through other vacua; we call adjacent vacua two vacua that can be connected through a single-kink excitation. There can be kinks with different masses connecting two adjacent vacua

\(^1\) In a kink state \( |K_{ab}(\theta)\rangle \) the spin field \( \sigma(x, y) \) takes the value \( \langle \sigma \rangle_a \) as \( x \rightarrow -\infty \) and the value \( \langle \sigma \rangle_b \) as \( x \rightarrow +\infty \). Propagation of the kink in Euclidean time \( y = it \) corresponds to domain wall configurations for the lattice model.
Consider now the scaling limit on a rectangle (figure 1) with horizontal sides of length $L$ and vertical sides of length $R$ (the origin of the coordinates is taken in the center of the rectangle), with the following choice of boundary conditions (boundary conditions of type $ab$): the boundary spins are fixed to a value $a$ for $x < 0$, and to a different value $b$ for $x > 0$. Let us denote by $Z_{ab}$ and $\langle \cdots \rangle_{ab}$ the partition function and the statistical averages for the system with this choice of boundary conditions. We consider the limit $L \to \infty$ and want to study the properties of the system as a function of the width $R$ of the resulting infinite strip, focusing on the asymptotic limit in which $R$ is much larger than the correlation length $\xi$.

Within the field theoretical formalism the boundary condition at time $t$ switching from $a$ to $b$ at a point $x_0$ is realized by a boundary state that we denote by $|B_{ab}(x_0; t)\rangle$. This state can be expanded over the basis of asymptotic particle states of the relativistic theory. The change of boundary conditions at the point $x_0$ requires that kink excitations interpolating between $|\Omega_a\rangle$ and $|\Omega_b\rangle$ are emitted/absorbed at that point. Then, if $|\Omega_a\rangle$ and $|\Omega_b\rangle$ are adjacent vacua, the boundary state has the form

$$|B_{ab}(x_0; t)\rangle = e^{-iHt+iPt} \left[ \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} f_{ab}(\theta)|K_{ab}(\theta)\rangle + \cdots \right] ,$$

where $H$ and $P$ are the energy and momentum operators of the (1+1)-dimensional theory, $f_{ab}(\theta)$ is the amplitude\(^2\) for the kink to be emitted at the boundary changing point and the dots correspond to states with total mass larger than $m_{ab}$. The partition function we are considering can be written as

$$Z_{ab}(R) = \langle B_{ab}(x_0; iR/2)|B_{ab}(x_0; -iR/2)\rangle = \langle B_{ab}(0; 0)|e^{-RH}|B_{ab}(0; 0)\rangle$$

and, as a consequence of (1), has the large $R$ asymptotics

$$Z_{ab}(R) \sim \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} e^{-m_{ab}R\cosh\theta} |f_{ab}(\theta)|^2 \sim \frac{|f_{ab}(0)|^2}{\sqrt{2\pi m_{ab}R}} e^{-m_{ab}R} .$$

\(^2\) We use the normalization $\langle K_{ab}(\theta)|K_{a'\theta'}(\theta')\rangle = 2\pi \delta_{aa'} \delta_{bb'} \delta(\theta - \theta')$. 

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4
The specific interfacial free energy, or surface tension, is given by

$$\Sigma_{ab} = - \lim_{R \to \infty} \frac{1}{R} \ln \frac{Z_{ab}(R)}{Z_a(R)},$$

where $Z_a(R)$ is the partition function for uniform boundary conditions of type $a$ on the strip. Since the lowest mass state entering the expansion of the boundary state $|B_a(t)\rangle$ for uniform boundary conditions is the vacuum $|\Omega_a\rangle$, $Z_a(R)$ tends to $\langle \Omega_a | \Omega_a \rangle = 1$ as $R \to \infty$, so that

$$\Sigma_{ab} = m_{ab}. \quad \text{(5)}$$

If $|\Omega_a\rangle$ and $|\Omega_b\rangle$ are not adjacent vacua the expansion of the boundary state $|B_{ab}(x_0; t)\rangle$ starts with a multi-kink state, and the corresponding surface tension is a sum of surface tensions between adjacent vacua; we defer to section 4 the illustration of this case.

Let us denote by $\sigma$ a generic component of the spin field, omitting for the time being the index which in general labels the different components. The magnetization profile along the horizontal axis in the middle of the strip for $ab$ boundary conditions and $a$ and $b$ adjacent phases is

$$\langle \sigma(x, 0) \rangle_{ab} = \frac{1}{Z_{ab}} \langle B_{ab}(0; 0) | e^{-HR/2+i\pi P} \sigma(0, 0) e^{-HR/2-i\pi P} | B_{ab}(0; 0) \rangle$$

$$\sim \frac{1}{Z_{ab}} \int \frac{d\theta}{2\pi} \int \frac{d\theta'}{2\pi} f^*(\theta) f(\theta') \langle K_{ab}(\theta) | \sigma(0, 0) | K_{ab}(\theta') \rangle$$

$$\times e^{m_{ab}[-(\cosh \theta + \cosh \theta')R/2 + i(\sinh \theta - \sinh \theta')x]}, \quad \text{(6)}$$

the last line being the large $R$ limit obtained from (1). The matrix element of the spin field it contains is related by the crossing relation$^3$

$$\langle K_{ab}(\theta) | \sigma(0, 0) | K_{ab}(\theta') \rangle = F^\sigma_{aba}(\theta - \theta' + i\pi) + 2\pi \delta(\theta - \theta') \langle \sigma \rangle_a \quad \text{(7)}$$

to the form factor

$$F^\sigma_{aba}(\theta_1 - \theta_2) \equiv \langle \Omega_a | \sigma(0, 0) | K_{ab}(\theta_1) K_{ba}(\theta_2) \rangle, \quad \text{(8)}$$

where the vacuum expectation value $\langle \sigma \rangle_a$ appearing in the disconnected part is the spontaneous magnetization in the phase $a$ on the infinite plane. When $\theta_1 - \theta_2 = i\pi$ the kink and the anti-kink in (8) have opposite energy and momentum and can annihilate each other. In 1+1 dimensions these annihilation configurations produce in general simple poles that have been characterized for general $k$-particle form factors in integrable field theories (see in particular [11]). For $k = 2$, however, integrability plays no role in the determination of the residue, which for the case of kink excitations reads [12]

$$\text{Res}_{\theta = i\pi} F^\sigma_{aba}(\theta) = i[\langle \sigma \rangle_a - \langle \sigma \rangle_b]. \quad \text{(9)}$$

For $R \to \infty$ the integral in (6) is dominated by small rapidities and the leading contribution can be written as

$$\langle \sigma(x, 0) \rangle_{ab} \sim \langle \sigma \rangle_a + \frac{i}{2\pi} [\langle \sigma \rangle_a - \langle \sigma \rangle_b] \int_{-\infty}^{\infty} d\theta - \frac{1}{\theta} e^{-m_{ab}R\theta^2/8 + im_{ab}xt_{\theta}}, \quad \text{(10)}$$

$^3$ In field theory ‘crossing’ a particle from the initial to the final state (or vice versa) involves reversing the sign of its energy and momentum. Within the parameterization introduced above this amounts to an $i\pi$ rapidity shift.
where we used (3), (7) and (9), $\theta_- = \theta - \theta'$, and we integrated over $\theta_+ \equiv \theta + \theta'$. The last integral is regularized by moving the pole slightly above the real axis, so that the usual relation $(x - i0)^{-1} = i\pi \delta(x) + \text{p.v.} x^{-1}$ finally gives

$$
\langle \sigma(x, 0) \rangle_{ab} \sim \frac{1}{2} \left[ \langle \sigma \rangle_a + \langle \sigma \rangle_b \right] - \frac{1}{2} \left[ \langle \sigma \rangle_a - \langle \sigma \rangle_b \right] \text{erf} \left( \sqrt{\frac{2m_{ab}}{R}} x \right),
$$

where the principal value of the integral in (10) has been expressed in terms of the error function $\text{erf}(x) \equiv (2\sqrt{\pi}) \int_0^x d\eta e^{-\eta^2}$ (see, e.g., [13]). The same result can be obtained by differentiating (10) with respect to $x$ in order to get rid of the pole, and then integrating the result of the integral over $\theta_-$ with the condition $\langle \sigma(\pm, 0) \rangle_{ab} = \langle \sigma \rangle_b$.

For the Ising model ($\langle \sigma \rangle_a = -\langle \sigma \rangle_b$) the result (11) coincides with the scaling limit of that obtained from the lattice in [4, 5]. Even in our more general setting, it shows that for $R \to \infty$ $\langle \sigma(\alpha(m_{ab}R)^{\beta}/m_{ab}, 0) \rangle_{ab}$ tends to the pure values $\langle \sigma \rangle_a$ or $\langle \sigma \rangle_b$ for $\beta > 1/2$ and $\alpha$ negative or positive, respectively, and to the average value $(\langle \sigma \rangle_a + \langle \sigma \rangle_b)/2$ for $\beta < 1/2$.

The result (11) is produced by the leading term in the small rapidity expansion of the emission amplitude in (1) and the matrix element (7). More generally, for the latter we write

$$
F_{aba}(\theta + i\pi) = \sum_{k=-1}^{\infty} c_{ab}^{(k)} \theta^k,
$$

with $c_{ab}^{(-1)}$ given by (9). As for the emission amplitude, it satisfies $f_{ab}(\theta) = f_{ba}(-\theta)$ as a consequence of reflection symmetry about the vertical axis. In any model in which $a$ and $b$ play a symmetric role we will have $f_{ab}(\theta) = f_{ba}(\theta)$ and $f_{ab}(\theta) = f_{ab}(0) + O(\theta^2)$. Then it is easy to check that the next contribution to $\langle \sigma(x, 0) \rangle_{ab}$ produced by the small rapidity expansion is

$$
c_{ab}^{(0)} \sqrt{\frac{2}{\pi m_{ab}R}} e^{-2m_{ab}x^2/R}.
$$

Notice that the error function in (11) is leading with respect to (13) as $R \to \infty$ for $x \sim (m_{ab}R)^{\beta}/m_{ab}$ with $\beta > 0$; the two terms are of the same order for $\beta = 0$.

3. Passage probability and interface structure

The results of the section 2 allow an interpretation based on renormalization group and probabilistic considerations. It is clear that the problem has two length scales: the correlation length $\xi$, proportional to the inverse of the kink masses, which is the scale of the fluctuations within the pure phases, and the width $R$ of the strip, which sets the scale at which we observe the system with $ab$ boundary conditions. In an expansion around $R/\xi = \infty$ the leading term corresponds to the crudest description of phase separation in which all short distance features are washed out and one is left with two pure phases sharply separated by a simple curve connecting the two boundary changing points (figure 2(a)). Hence, the notion of a curvilinear interface naturally arises in this limit and can be formulated directly in the continuum. It is clear, however, that this picture cannot hold true for finite values of $R/\xi$, and that one needs to switch from the notion of sharp separation and a curvilinear interface to that of an interfacial region (or thick interface).
with a width which diverges as the correlation length when the critical point is approached; such a divergence simply reflects the fact that phase separation disappears at criticality. Within the large $R/\xi$ expansion the leading deviations from the simple curvilinear picture are expected from effects such as branching and recombination as well as self-intersection of the curve $^4$ (figures 2(b)–(d)). These effects appear in the expansion through insertions (delta functions) localized on the curve separating the two pure phases.

According to this discussion the large $R/\xi$ expansion for the magnetization at a point $x$ on the axis $y = 0$ when the interface passes through a point $u$ on this axis can be expected to start as

$$\sigma_{ab}(x|u) = \theta(u - x)\langle \sigma \rangle_a + \theta(x - u)\langle \sigma \rangle_b + A_{ab}^{(0)} \delta(x - u) + A_{ab}^{(1)} \delta'(x - u) + \cdots,$$

where $\theta(x)$ is the step function equal to 1 for $x > 0$ and zero for $x < 0$, and the prime denotes differentiation. If $p_{ab}(u) \, du$ is the probability that the curve intersects the axis $y = 0$ in the interval $[u, u + du]$, with $p_{ab}(u) = p_{ab}(-u)$ and $\int_{-\infty}^{\infty} du \, p_{ab}(u) = 1$, then the average magnetization is

$$\langle \sigma(x, 0) \rangle_{ab} = \int_{-\infty}^{\infty} du \, p_{ab}(u) \sigma_{ab}(x|u) + \cdots = \langle \sigma \rangle_a \int_{-\infty}^{x} du \, p_{ab}(u) + \langle \sigma \rangle_b \int_{-\infty}^{x} du \, p_{ab}(u) + A_{ab}^{(0)} p_{ab}(x) - A_{ab}^{(1)} p'_{ab}(x) + \cdots,$$

where the dots in the first line stay for the contribution of multi-kink states. Comparison with (11) and (13), as is easily seen, shows correspondence between the two expansions and determines

$$p_{ab}(u) = \sqrt{\frac{2m_{ab}}{\pi R}} e^{-2m_{ab}u^2/R},$$

$$A_{ab}^{(0)} = \frac{c_{ab}^{(0)}}{m_{ab}}.$$  

The last as well as additional terms in (15) should be compared with those produced in (6) by further expansion around $\theta = \theta' = 0$. The Gaussian form of $p_{ab}(u)$ is model independent

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$^4$ At a later stage in the expansion thickness is generated also by multi-kink terms in the boundary state (1), which will produce a bundle of thin interfaces rather than just one (figure 2(e)). Multi-kink contributions to $Z_{ab}$ are suppressed at large $R$ as $e^{-MR}$, $M$ being the total mass.
and of course coincides with the scaling limit of that deduced in [7] for the Ising model\(^5\). As shown in the previous section, the integral, non-local terms in \((15)\) are entirely due to the pole term in \((12)\), which in turn is produced by the non-locality of the kinks with respect to the spin field\(^6\).

For any lattice configuration there is a cluster (let us call it the left cluster) formed by nearest neighboring spins with value (color) \(a\) and whose external perimeter is formed by the left half of the boundary of the strip together with a path connecting the two boundary changing points. This path, whose identification may be ambiguous and require some lattice dependent prescription, becomes a simple curve in the continuum limit. A second curve connecting the two boundary changing points completes the perimeter of the cluster of color \(b\) anchored to the right half of the boundary (the right cluster). In general the two curves, which can touch but not intersect, enclose other clusters in between them. Among these intermediate clusters, those adjacent to the left (right) cluster have color different from \(a\) (\(b\)). The first few terms in \((14)\) and \((15)\) are compatible with a picture in which a uniform magnetization \(\langle \sigma \rangle_a (\langle \sigma \rangle_b)\) is assigned to the region enclosed by the perimeter of the left (right) cluster: the two curves become coincident as \(R/\xi \to \infty\), with asymptotic passage probability density given by \((16)\); the first deviation from this situation as \(R/\xi\) decreases is expected to happen via bifurcation and recombination around a cluster of color \(c \neq a, b\) (figure 2b), and to be associated with the term containing \(A^{(0)}_{ab}\). We will see in the section 4 that this term is indeed absent in the Ising model, where bifurcation is not allowed\(^7\).

4. Specific models

Potts model

The lattice Hamiltonian \([16]\)

\[
H_{\text{Potts}} = -J \sum_{\langle \mathbf{x}_1, \mathbf{x}_2 \rangle} \delta_{s(\mathbf{x}_1), s(\mathbf{x}_2)}, \quad s(\mathbf{x}) = 1, \ldots, q, \tag{18}
\]

is invariant under global permutations of the values of the spins \(s(\mathbf{x})\). For \(J > 0\) in two dimensions the phase transition is continuous for \(q \leq 4\) and above \(J_c\) there are \(q\) degenerate vacua located at the vertices of a hypertetrahedron in the \((q - 1)\)-dimensional order parameter space. Kinks with equal masses run along the edges of the hypertetrahedron and all the vacua are adjacent according to the definition given in section 2. The results we obtained for the magnetization profile apply to each component \(\sigma_c(\mathbf{x}) \equiv \delta_{s(\mathbf{x}), c} - 1/q, \ c = 1, \ldots, q,\) of the spin field; taking into account that \(\sum_{c=1}^q \sigma_c = 0\)

\(^5\) The Gaussian probability density \((16)\) indicates an effective Brownian behavior of the interface. The convergence of the interface to a Brownian bridge for all subcritical temperatures has been proved in [14] for the Ising model and in [15] for the \(q\)-state Potts model. We thank Velenik for these references.

\(^6\) If we consider the form factor of the energy density \(\varepsilon\), the residue at \(i\pi\) is given by \((9)\) with \(\sigma\) replaced by \(\varepsilon\), and vanishes because the expectation value of \(\varepsilon\) is the same in all stable phases. This reflects the fact that even for a domain wall excitation the energy density is spatially localized (on the wall).

\(^7\) Splitting into an odd number of paths (figure 2(c)), however, is allowed and encoded by subsequent terms in the expansion.

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and \( \langle \sigma_a \rangle_b = (q\delta_{ab} - 1)/(q - 1)\langle \sigma_a \rangle_a \), one obtains
\[
\langle \sigma_c(x, 0) \rangle_{ab} = \frac{\langle \sigma_a \rangle_a}{2} \left[ \frac{q(\delta_{ca} + \delta_{cb}) - 2}{q - 1} - \frac{2q(\delta_{ca} - \delta_{cb})}{q - 1} \int_0^x du p(u) \right] + \cdots,
\]
where \( p(u) \) is (16) with \( m_{ab} = m \). Potts field theory is integrable [17] and from the known form factors [12] we obtain
\[
B = \frac{1}{2\sqrt{3}}, \frac{2}{3\sqrt{3}}
\]
for \( q = 3, 4 \), respectively. For \( c \neq a, b \) the integral term in (19) is absent and the \( x \)-dependence of the magnetization profile is entirely due to the structure of the interface. The Gaussian term in (19) is produced by the leading deviation from the picture of the interface as a simple curve separating the phases \( a \) and \( b \), which was argued in the previous section to correspond to the appearance of an island of the phase \( c \) via bifurcation and recombination of the curve. Bifurcation is not allowed in the Ising model, and indeed the coefficient of \( p(x) \) vanishes at \( q = 2 \), where \( c \) necessarily coincides with \( a \) or \( b \). Directly at \( q = 2 \), the same conclusion is obtained from the explicit form \( F_{aba}(\theta) = i(\sigma)_a \tanh(\theta/2) \) of the spin form factor, implying that (12) contains only the terms with \( k \) odd.

**Percolation**

It is well known [18, 16] that the partition function of the Potts model admits an expansion over Fortuin–Kasteleyn (FK) clusters\(^8\) of spins with the same color which as \( q \to 1 \) become the clusters of random percolation. If we consider the Potts model with boundary conditions of type \( ab \) on the strip, the probability \( \langle \delta_{s(x,0),ab} \rangle \) that the spin \( s(x,0) \) has color \( a \) is given by the probability \( P(x,0) \) that it belongs to a cluster touching the part of the boundary with \( x < 0 \) (which has color \( a \)), plus \( 1/q \) times the probability \( 1 - P(x,0) - P(-x,0) \) that it belongs to a bulk cluster. This can be rewritten as
\[
\langle \sigma_a(x, 0) \rangle_{ab} = \frac{q - 1}{q} P(x, 0) - \frac{1}{q} P(-x, 0).
\]

The FK expansion of \( Z_{ab} \) does not contain configurations with clusters connecting the boundary regions with \( x < 0 \) and \( x > 0 \); this restriction is inherited by the percolation problem we consider. When the occupation probability \( p \) for the sites is above the percolation threshold \( p_c \) (this corresponds to our case \( J > J_c \) in the Potts model), even for \( R \to \infty \) there is a positive probability of having a spanning cluster which connects the upper and lower parts of the boundary with \( x < 0 \) (figure 3). Then the probability \( P(x,0)|_{q=1} \) that the site \( (x,0) \) is connected to the left part of the boundary is given by the probability \( P_s(x,0) \) that it belongs to a cluster touching the upper or lower edge. Since the clusters of the latter type have an average linear extension of order \( 1/m \), \( P_{ns}(x,0) \) vanishes exponentially when \( mR \to \infty \). Hence in this limit we have \( P_s(x,0) \sim P(x,0)|_{q=1} \), and from (19), (21).

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\(^8\) The FK clusters differ from the ‘geometrical’ spin clusters we referred to in the section 3 due to the fact that nearest neighbors with the same color do not necessarily belong to the same cluster.
Figure 3. For percolation on the strip of width $R$, consider only the configurations without clusters connecting the parts of the edges with $x < 0$ to the parts with $x > 0$. Equation (22) gives the probability that a point on the $x$ axis belongs to a cluster $s$ spanning between the negative parts of the edges, in the scaling limit above $p_c$ and for $R \gg \xi = 1/2m$.

we obtain

$$P_s(x, 0) = \frac{P}{2} \left[ 1 - \text{erf}\left(\sqrt{\frac{2m}{R}} x\right) - \gamma \sqrt{\frac{2}{\pi mR}} e^{-2mx^2/R} + \cdots \right], \quad (22)$$

where $P = \lim_{q \to 1} \frac{(q)}{(q-1)} \langle \sigma_a \rangle$ is the probability that a site belongs to the infinite cluster on the infinite plane, and the mass $m$ is related to the exponential correlation length above $p_c$ as $\xi = 1/2m$ (see [19, 20]); $\gamma \equiv \text{Res}_{q=1} B(q)$ is not known because for $q \to 1$ the form factor (8) is available only for real rapidities [19].

Ashkin–Teller model

The lattice model corresponds to two Ising spins $\sigma_1(x), \sigma_2(x) = \pm 1$ on each site interacting as specified by the Hamiltonian

$$H_{AT} = - \sum_{\{x_1, x_2\}} \left\{ J[\sigma_1(x_1)\sigma_1(x_2) + \sigma_2(x_1)\sigma_2(x_2)] + J_4 \sigma_1(x_1)\sigma_1(x_2)\sigma_2(x_1)\sigma_2(x_2) \right\}. \quad (23)$$

Each site can be in one of four states $(\sigma_1, \sigma_2)$ that we label $a = 1, 2, 3, 4$, corresponding to $(+, +), (++, -), (-, -), (--, +)$, respectively. The model that we consider for $J > 0$ is well known to possess a line of critical points parameterized by $J_4$ [21]. In the scaling limit close to this line it renormalizes into the sine–Gordon field theory (see [22, 23]), where a parameter $\beta^2$ plays the role of $J_4$; $\beta^2 = 4\pi$ describes a free fermionic theory and corresponds to the decoupling point $J_4 = 0$. Below the critical temperature the model possesses four degenerate vacua $|\Omega_a\rangle$ and for any value of $\beta^2$ there are kinks $|K_{a,a\pm1(mod\ 4)}\rangle$ with the same mass $m$.

For $\beta^2 < 4\pi$ the interaction among these kinks (which correspond to sine–Gordon solitons) is attractive and produces bound states, the lightest of which have mass $m' = 2m \sin(\pi \beta^2 / 2(8\pi - \beta^2))$ and are kinks $|K_{a,a\pm2(mod\ 4)}\rangle$. Hence, in this regime any pair of vacua is connected by a single-kink excitation, all the vacua are adjacent and the boundary state $|B_{ab}\rangle$ has in any case the form (1). The results of the previous sections apply with surface tensions $\Sigma_{a,a\pm1(mod\ 4)} = m$ and $\Sigma_{a,a\pm2(mod\ 4)} = m'$; the bifurcation coefficients (17)
can be obtained from the form factors computed in [22, 23]. For $\beta^2 = 2\pi$ the masses $m$ and $m'$ coincide and one recovers the $q = 4$ Potts model.

For $\beta^2 \geq 4\pi$ there are no bound states and the vacua with indices differing by two units are not adjacent, with surface tension $\Sigma_{a,a\pm 2(\text{mod} 4)} = 2m$. In this case (1) is replaced by

\begin{equation}
|B_{a,a\pm 2}(x_0; t)\rangle = e^{-itH + i\pi a^2} \left[ \sum_{c=\pm 1} \int \frac{d\theta_1}{2\pi} \int \frac{d\theta_2}{2\pi} f_{a,c,a\pm 2}(\theta_1, \theta_2) |K_{c,a}(\theta_1)K_{c,a\pm 2}(\theta_2)\rangle + \cdots \right] \tag{24}
\end{equation}

(indices are intended mod 4) and (14) has to be replaced by a description in terms of two interfaces (figure 2(e)).

Let us mention that studies of cluster boundaries at criticality can be found in particular in [24, 25] for the Potts model and in [26]–[28] for the Ashkin–Teller model; we refer the reader to [29] for results on cluster densities in critical percolation.

In summary, in this paper we have shown how field theory naturally accounts for phase separation in two dimensions in the scaling limit below criticality. The derivation is simple but requires us to take into account that the large distance properties are determined by kink (domain wall) excitations. In recent decades two-dimensional field theories with kinks have been studied within the non-Lagrangian framework of integrable field theory based on asymptotic states, form factors and spectral sums; the derivation of section 2 exploits this framework but is general and does not require integrability. We have shown how it naturally leads to a Gaussian passage probability density for the interface and accounts for the deviations from curvilinear behavior. The leading deviation is produced by branching of the interface and has been determined exactly for the Potts model. Specializations of the formalism to percolation and the Ashkin–Teller model have also been discussed.

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Phase separation and interface structure in two dimensions from field theory

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