Cyclic $n$-gonal surfaces and their automorphism groups - UNED Geometry Seminar

S. Allen Broughton, Rose-Hulman Institute of Technology
Aaron Wootton, University of Portland
March 18, 2010

Abstract

This paper is based upon two lectures on the authors’ joint work, presented by the first author at the UNED Geometry Seminar in February-March, 2009. As the detailed statements and proofs of results presented in the talks will be published elsewhere, this paper will only give an overview of cyclic $n$-gonal surfaces, their automorphism groups, and some examples illustrating the computational methods.

Contents

1 Cyclic $n$-gonal surfaces
  1.1 Plane models .......................................................... 2
  1.2 Overview of computing Aut($S$) ................................... 3

2 Translation to Fuchsian group triples
  2.1 Basics and canonical generators ...................................... 5
  2.2 Induced generators ...................................................... 6
  2.3 The spherical group $K$ ................................................ 7
  2.4 Fuchsian group invariants ............................................ 8

3 The sequence $\Gamma_C \hookrightarrow \Gamma_N \twoheadrightarrow K$
  3.1 Finding $\Pi \hookrightarrow \Gamma_C \twoheadrightarrow C$ ........................ 10
  3.2 Finding epimorphisms $\Gamma_N \twoheadrightarrow K$ ..................... 15

4 The Fuchsian group pair $\Gamma_N < \Gamma_A$
  4.1 Monodromy and word maps ........................................... 18
  4.2 Constrained and tight pairs ......................................... 19
  4.3 Classification steps for pairs ....................................... 21
  4.4 Maximal actions and signatures .................................... 22

5 Constraints on signatures
  5.1 Strong branching and weak normality $C < A$ ...................... 24
  5.2 Orbits and induced generators .................................... 26
1 Cyclic $n$-gonal surfaces

This paper is based upon two lectures [2], [3] on the author’s joint work, presented by the first author at the UNED Geometry Seminar in February and March of 2009. The full results for those talks will be given in the forthcoming papers [4] and [7], so in this paper we content ourselves with an overview of cyclic $n$-gonal surfaces, their automorphism groups, and a small number of examples illustrating the computational methods.

A closed Riemann surface $S$ of genus $\sigma \geq 2$ is called cyclic $n$-gonal if there exists a cyclic group of automorphisms $C$ of order $n$ such that the quotient space $S/C$ has genus 0. Such surfaces are of great interest since they have a simple plane model given in equation 1 below. The map $\pi_C : S \to S/C = \mathbb{P}^1$, where $\mathbb{P}^1$ denotes the Riemann sphere, is called a cyclic $n$-gonal morphism. When $n = 2$, $S$ is a hyperelliptic surface and so any generator of $C$ should be thought of as a generalization of a hyperelliptic involution.

We consider the general problem of determining $A = \text{Aut}(S)$, the full conformal automorphism group of $S$. When $C$ is normal in $A$, such as the hyperelliptic case or the prime order and large genus case, there are well-known methods for determining $A$ (see Section 3). The normal case has been investigated by several authors [1], [13], [14], [17]. The non-normal case has been considered in [18], and in some detail in the forthcoming work [7]. In general, a classification of surfaces and automorphism groups is hopeless. See the paper [4] for some examples of why the problem is complex when there are no restrictions on the $C$-action. When some restrictions are placed on the $C$-action or, the automorphism group $A$, then some progress can be made. Some possible restrictions are the following.

1. In [17] the cyclic group $C$ is assumed to be of prime order.

2. In [13] the $C$-action is fully ramified, i.e., $m_i = n$, or $\gcd(n, p_i) = 1$. See the next section for notation.

3. In [7] the cyclic group $C$ is assumed to be a weakly malnormal subgroup of $A$, in which case $C$ is normal if the genus is large enough. The definition is given in Section 5. This case includes the first two cases above.

4. Another case of great interest is quasi-platonic surfaces, namely $\pi_A : S \to S/A = \mathbb{P}^1$ is branched over three points. These cases are interesting since the surfaces are defined over number fields and give many computable examples of dessins d’enfants. Details will be in the forthcoming paper [8].
1.1 Plane models

One reason for looking at cyclic $n$-gonal surfaces is that there is some hope in determining the equation of the surface. There is a plane model of the form

$$y^n = f(x) = \prod_{i=1}^{r} (x - a_i)^{p_i},$$

where the $p_i$ and $p = p_1 + \cdots + p_r = \deg(f)$ satisfy

1. $0 < p_i < n$ (2)
2. $n$ divides $p$ (3)
3. $\gcd(n, p_1, \ldots, p_r) = 1$ (4)

The closed curve $\mathcal{S}$ in $\mathbb{P}^2$ defined by equation (1) is smooth except possibly where $y = 0, \infty$. The normalization map $\nu : S \to \mathcal{S}$ yields a smooth curve of genus

$$\sigma = \frac{1}{2} \left( 2 + (r - 2)n - \sum_{i=1}^{r} d_i \right),$$

where $d_i = \gcd(n, p_i)$. The group $C$ acts on the smooth part of $\mathcal{S}$ by

$$(x, y) \to (x, e^{2\pi ki/n} y)$$

and this action extends to $S$ via $\nu$. The $n$-gonal morphism is the $\nu$-lift of the map

$$\mathcal{S} \to \mathbb{P}^1, (x, y) \to x.$$ (7)

The $n$-gonal morphism $\pi_C : S \to \mathbb{P}^1$ is ramified only over the finite points $\{a_1, \ldots, a_r\}$, and the degree of ramification over $a_i$ is

$$m_i = \frac{n}{d_i} = \frac{n}{\gcd(n, p_i)}.$$ (8)

1.2 Overview of computing $\text{Aut}(S)$

For generically chosen $a_i$ we usually have $C = A = \text{Aut}(S)$. For special values of the $a_i$ and selections of the $p_i$ the automorphism group may be larger. See, for instance [13], [14], [17], and [18]. We shall not directly work with the defining equation or plane models in this paper, but use group theoretic methods instead.

Let $N = \text{Nor}_A(C)$ so that $C \preceq N \preceq A$. Our method is to lift this triple to a triple of covering Fuchsian groups $\Gamma_C \preceq \Gamma_N \preceq \Gamma_A$ and then to employ the group theory to implement classification. In Section 2 we describe the lifted triples, especially canonical generators and signatures. In Sections 3 and 4 we describe the inclusions $\Gamma_C \preceq \Gamma_N$ and $\Gamma_N < \Gamma_A$, respectively, using quite different methods in the two cases. In Section 3 the group $K = \Gamma_N/\Gamma_C \simeq N/C$ and its action on the generating system of $C$ will be fundamental to the discussion $\Gamma_C \preceq \Gamma_N$. 

3
In Section 4 we use permutation group methods on the coset space $\Gamma_A/\Gamma_N$ to describe the inclusions $\Gamma_N < \Gamma_A$. In particular we describe the notion of families of triples $\Gamma_C \leq \Gamma_N \leq \Gamma_A$ where $\Gamma_N < \Gamma_A$ has a fixed coset structure but $n = |C|$ varies over a family.

The general computational procedure for classification is

1. Specify a restriction on the triples $\Gamma_C \leq \Gamma_N \leq \Gamma_A$ by imposing a group theoretic, geometric, or arithmetic constraint as noted in the introductory paragraphs. This is described in Section 5. This step limits the complexity of calculations.

2. Compute all possible signature pairs $(S(\Gamma_N), S(\Gamma_A))$. See Section 2 for the definition of signature.

3. Determine the inclusions $\Gamma_N < \Gamma_A$ determining both exceptional cases and parametric families. This is discussed in Section 4.

4. Determine the exact sequences $\Gamma_C \rightarrow \Gamma_N \rightarrow K$. This is discussed in Section 3.

5. Fuse the pairs $\Gamma_N < \Gamma_A$ and $\Gamma_C < \Gamma_N$ together to compute $C \subseteq N \leq A$, or demonstrate that no extension exists. We discuss the computationally intensive methods very briefly in subsection 4.3. However, because of space limitations we use ad-hoc methods in the examples in Section 6.

6. The surface $S$ automatically exists as a quotient $\mathbb{H}/\Pi$ where $\Pi < \Gamma_C$. Constructing a model as in equation 1 takes much more work, see for instance [13], [14], [17], and [18]. We do not address construction of the plane models in this paper.

**Remark 1** The order of $A$ is given by

$$|A| = \frac{|A|}{|N|} \cdot |N| = mn|K|$$

where

$$n = |C|, m = |\Gamma_A/\Gamma_N|.$$ 

If the signature of $C$ is known – say the signatures of $N$ and $K$ are known – then

$$|C| = \gcd(\text{periods of } C)$$

Given $\Gamma_N < \Gamma_A$ and $\Gamma_C < \Gamma_N$ then $C \leq N \leq A$ is determined by finding a torsion free $\Pi < \Gamma_C$ such that $\Pi < \Gamma_A$. There are infinitely many such $\Pi$ but very few result in a cyclic $C$. Imposing an exact sequence $\Gamma_C \rightarrow \Gamma_N \rightarrow K$ with the additional constraints given in equations 9 and 10 eliminates many of the non-cyclic possibilities.
2 Translation to Fuchsian group triples

2.1 Basics and canonical generators

We take much of our notation from [7] and [18]. Recall that a compact Riemann surface $S$ of genus $\sigma \geq 2$ can be realized as a quotient of the upper half plane $\mathbb{H}/\Pi$ where $\Pi$ is a torsion free Fuchsian group called a surface group for $S$. Under such a realization, a group $G$ acts as a group of conformal automorphisms on $S$ if and only if there exists an epimorphism $\eta: \Gamma \to G$ with $\ker(\eta) = \Pi$ for some Fuchsian group $\Gamma$. We call $\eta$ a surface kernel epimorphism, and $\Gamma$ the covering Fuchsian group of $G$, usually denoting it by $\Gamma_G$. We identify the orbit spaces $\mathbb{H}/\Gamma$ and $S/G$. The quotient map $\pi_G: S \to S/G$ is branched over the same points as $\pi_{\Gamma}: \mathbb{H} \to \mathbb{H}/\Gamma$ with the same ramification indices. We define the signature of $\Gamma$ to be the tuple $S(\Gamma) = (\sigma_{\Gamma}; m_1, m_2, \ldots, m_r)$, where the quotient space $\mathbb{H}/\Gamma$ has genus $\sigma_{\Gamma}$ and the quotient map, $\pi_{\Gamma}$ (and also $\pi_G$) branches over $r$ points with ramification indices $m_i$ for $1 \leq i \leq r$. We call $\sigma_{\Gamma}$ the orbit genus of $\Gamma$ and the numbers $m_1, \ldots, m_r$ the periods of $\Gamma$. If $\sigma_{\Gamma} = 0$ the signature may be abbreviated to $(m_1, m_2, \ldots, m_r)$, which we may also write as $(m_1^{e_1}, \ldots, m_s^{e_s})$ to indicate repeated periods. The signature of $\Gamma$ provides information regarding a presentation for $\Gamma$, and in the special case that $\sigma_{\Gamma} = 0$, we have the following.

Theorem 2 If $\Gamma$ is a Fuchsian group with signature $(m_1, \ldots, m_r)$ then there exist an ordered set of elliptic (finite order) group elements $\mathcal{G} = \{\gamma_1, \ldots, \gamma_r\} \subseteq PSL(2, \mathbb{R})$, such that:

1. $\Gamma = \langle \gamma_1, \ldots, \gamma_r \rangle$.
2. Defining relations for $\Gamma$ are

$$\gamma_1^{m_1} = \gamma_2^{m_2} = \cdots = \gamma_r^{m_r} = \prod_{i=1}^{r} \gamma_i = 1.$$  

(11)

3. Each non-identity elliptic element (element of finite order) lies in a unique conjugate of $\langle \gamma_i \rangle$ for suitable $i$.

Definition 3 We call a set of elements of $\Gamma$ satisfying 1 and 2 of Proposition 2 canonical generators of $\Gamma$ for the signature $(m_1, \ldots, m_r)$.

Remark 4 The canonical generators are not unique and the periods of the signature may be permuted. The permutations can be built up from simple transpositions as follows. Set

$$m'_i = m_{i+1}, m'_{i+1} = m_i, m'_j = m_j \text{ otherwise}$$

and

$$\gamma'_i = \gamma_{i+1}, \gamma'_{i+1} = \gamma_i^{-1}\gamma_{i+1}\gamma_i, \gamma'_j = \gamma_j \text{ otherwise}.$$ 

Then the $\gamma'_i$ constitute a canonical generating set for the periods $m'_i$. 

5
Remark 5 Let \( \eta : \Gamma \rightarrow G \) be a surface kernel epimorphism with signature \((m_1, m_2, \ldots, m_r)\). Set \( g_i = \eta(\gamma_i) \). Then the vector \((g_1, \ldots, g_r)\) of elements satisfies

\[
o(g_i) = o(\gamma_i) = m_i, \quad 1 \leq i \leq r	ag{12}\]

\[
\prod_{i=1}^{r} g_i = 1	ag{13}\]

\[G = \langle g_1, \ldots, g_r \rangle.\tag{14}\]

Any such vector is called a generating \((m_1, \ldots, m_r)\)-vector of \(G\). We call the tuple \(S(\Gamma)\) the branching data or the signature of the \(G\)-action on \(S\). The definition can be extended to the case where \(\sigma_\Gamma > 0\), but we do not need it.

Remark 6 Using areas of fundamental regions one can show that \(A(\Gamma) = \pi(2\sigma - 2)/|G|\). Letting \(\tau\) be the orbit genus of \(\sigma(S/G) = \sigma(\mathbb{H}/\Gamma)\) then we get the Riemann-Hurwitz formula or

\[
\frac{2\sigma - 2}{|G|} = 2\tau - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right).
\]

Example 7 Suppose that \(C\) is the cyclic \(n\)-gonal group of the surface given by equation 6. Then the signature of the \(C\)-action is \((m_1, m_2, \ldots, m_r)\) with the \(m_i\) given by equation 8. The Riemann-Hurwitz equation applied to the \(C\)-action is then

\[
\frac{2\sigma - 2}{n} = \frac{2\sigma - 2}{|C|} = -2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right)
\]

or

\[
\sigma = \frac{1}{2} \left(2 + (r - 2) n - \sum_{i=1}^{r} d_i\right),
\]

confirming equation 5.

2.2 Induced generators

Note that Proposition 2 implies that, if \(\Gamma \leq \Delta\), then any elliptic element of \(\Gamma\) must be conjugate to an elliptic element of \(\Delta\). This motivates the following definition.

Definition 8 Suppose \(\Gamma \leq \Delta\) are Fuchsian groups, \(\theta \in \Gamma\) is an elliptic element and \(\theta\) is \(\Delta\)-conjugate to a power of \(\zeta \in \Delta\), i.e., \(\theta = x\zeta^k x^{-1}\). Then we say \(\theta\) is induced by \(\zeta\).

We note that, by Theorem 2 any elliptic generator of \(\Gamma\) in a set of canonical generators for \(\Gamma\) must be conjugate to a power of a unique elliptic generator of \(\Delta\) in a set of canonical generators of \(\Delta\). To determine exactly how elliptic generators of \(\Gamma\) and \(\Delta\) related, we can use the following important consequence of the main result of 13.
Theorem 9 Suppose that $\Gamma \leq \Delta$, $\zeta \in \Delta$ is a canonical generator of order $k$ and let $\rho: \Delta \to S[\Delta: \Gamma]$ denote the map induced by action of $\Delta$ on the left cosets of $\Gamma$. Then the number of canonical generators of $\Gamma$ induced by $\zeta$ is equal to the number of cycles of $\rho(\zeta)$ of orders less than $k$ and the order of these elements are given by $k/k_i$ where the $k_i$ run over the lengths of the cycles of $\rho(\zeta)$. Moreover, if $\zeta'$ is any other canonical generator of $\Delta$, then the canonical generators of $\Gamma$ induced by $\zeta'$ are distinct from the ones induced by $\zeta$.

In the special case where $\Gamma \vartriangleleft \Delta$, we have the following result.

Corollary 10 Suppose that $\Gamma \vartriangleleft \Delta$, $\zeta \in \Delta$ and $\theta$ is a canonical generator of $\Gamma$ induced by $\zeta$. Then the order $o(\theta)$ is the same for all canonical generators $\theta$ of $\Gamma$ induced by $\zeta$ and $\zeta$ induces $[\Delta: \Gamma]o(\zeta)/o(\theta)$ distinct canonical generators of $\Gamma$. Moreover, if $\zeta'$ is any other canonical generator of $\Delta$, then the canonical generators of $\Gamma$ induced by $\zeta'$ are distinct from the ones induced by $\zeta$.

2.3 The spherical group $K$

We fix some notation. Let $S$ denote a cyclic $n$-gonal surface of genus $\sigma$, $\Pi$ a surface group for $S$, $C$ an $n$-gonal group for $S$ and $\Gamma_C$ the covering Fuchsian group of $C$. Also, let $A$ denote the full automorphism group of $S$, $\Gamma_A$ its covering Fuchsian group, $N$ the normalizer of $C$ in $A$, $\Gamma_N$ its covering Fuchsian group. Next let $K = N/C = \Gamma_N/\Gamma_C$ and let $\eta: \Gamma_A \to A$ and $\chi: \Gamma_N \to K$ denote the canonical quotient maps. The relations are summarized in this diagram

$$
\begin{align*}
\Gamma_C & \hookrightarrow \Gamma_N \hookrightarrow \Gamma_A \\
\downarrow \eta & \quad \downarrow \eta \quad \downarrow \eta \\
C & \hookrightarrow N \hookrightarrow A
\end{align*}
$$

and the exact sequences

$$
\begin{align*}
\Pi & \hookrightarrow \Gamma_A \xrightarrow{\eta} A \\
\Gamma_C & \hookrightarrow \Gamma_N \xrightarrow{\chi} K
\end{align*}
$$

Notice that since the group $K = N/C$ acts on the surface $S/C = \mathbb{P}^1$, it follows that $K$ is a finite subgroup of $\text{PSL}(2, \mathbb{C})$, acting on $\mathbb{P}^1$ by linear fractional transformations. All such groups are well known as well as the properties of the corresponding quotient maps and can be thought of as a special case of Proposition 2. We summarize.

Theorem 11 Suppose that $K$ is a finite subgroup of $\text{PSL}(2, \mathbb{C})$. Then $K$ is conjugate to one of $C_k$, $D_k$, $A_4$, $S_4$ or $A_5$ (where $C_k$ denotes the cyclic group of order $k$ and $D_k$ the dihedral group of order $k$). The quotient map $\pi_K: \mathbb{P}^1 \to \mathbb{P}^1$ branches over $s$ points with ramification indices $m_i$ for $1 \leq i \leq s$. The signature of such a group is the tuple $(a_1, \ldots, a_e)$

- where $e = 2, 3$, $a_i \geq 2$
\[ \frac{1}{a_1} + \cdots + \frac{1}{a_e} > 1 \]
\[ a_1 = a_2 \text{ if } e = 2 \]

and any such tuple corresponds to a group. We tabulate all signatures in Table 1. Moreover, two groups in the table are isomorphic if and only if they have the same signature.

| Group | Signature |
|-------|-----------|
| \( C_k \) | \((k, k), k \geq 2\) |
| \( D_k \) | \((2, 2, k), k \geq 2\) |
| \( A_4 \) | \((2, 3, 3)\) |
| \( S_4 \) | \((2, 3, 4)\) |
| \( A_5 \) | \((2, 3, 5)\) |

As suggested by Theorem 2 the interplay among the signatures of \( \Gamma_C \), \( \Gamma_N \), \( \Gamma_A \) or the signatures of \( C \), \( N \), \( A \), and the epimorphisms \( \chi \) and \( \eta \) are closely related to the ramification properties of the quotient maps \( S/C \to S/N \to S/A \) or \( \mathbb{H}/\Gamma_C \to \mathbb{H}/\Gamma_N \to \mathbb{H}/\Gamma_A \) among the quotient surfaces. This relationship is explained in more detail in Sections 3 and 4. In Section 5 we discuss how the action of \( A \) on the ramification points of \( S \to S/A \) is related to induced generators and the ramification of \( \mathbb{H}/\Gamma \to \mathbb{H}/\Delta \).

### 2.4 Fuchsian group invariants

We fix some more notation. We denote the signatures of a pair \( \Gamma < \Delta \) (e.g., \( \Gamma_N < \Gamma_A \)) by \((m_1, m_2, \ldots, m_s)\) and \((n_1, n_2, \ldots, n_t)\) respectively. Let \( G_1 = \{\theta_1, \ldots, \theta_s\} \) and \( G_2 = \{\zeta_1, \ldots, \zeta_t\} \) be sets of canonical generators of \( \Gamma < \Delta \) respectively. Important Fuchsian group invariants and invariants of pairs may be read off from the signatures.

For single groups we have.

- **The area of a fundamental region:** \( A(\Gamma) = 2\pi \mu(\Gamma) \) where:

  \[
  \mu(\Gamma) = -2 + \sum_{j=1}^{s} \left( 1 - \frac{1}{m_j} \right) = (s - 2) - \sum_{j=1}^{s} \frac{1}{m_j}.
  \]

  For completeness, when the genus \( \sigma = \sigma(\Gamma) \) is greater than zero

  \[
  \mu(\Gamma) = 2(\sigma - 1) + \sum_{j=1}^{s} \left( 1 - \frac{1}{m_j} \right).
  \]

- **Teichmüller dimension** \( d(\Gamma) \) of \( \Gamma \): the dimension of the Teichmüller space of Fuchsian groups with signature \( S(\Gamma) \) given by

  \[
  d(\Gamma) = s - 3 = |G_1| - 3.
  \]
For completeness, when the genus $\sigma = \sigma(\Gamma)$ is greater than zero we have

$$d(\Gamma) = 3(\sigma - 1) + s.$$ 

For pairs we combine the invariants.

- For a finite index pair $\Gamma < \Delta$, we have
  $$|\Delta : \Gamma| = \mu(\Gamma)/\mu(\Delta)$$

- For finite index pair $\Gamma \leq \Delta$, we call the quantity
  $$d(\Gamma, \Delta) = d(\Gamma) - d(\Delta)$$
  the Teichmüller codimension of $\Gamma < \Delta$. If both groups have genus zero then $d(\Gamma, \Delta) = |G_1| - |G_2|$. 

3 The sequence $\Gamma_C \hookrightarrow \Gamma_N \twoheadrightarrow K$

First we consider any exact sequence $\Gamma_C \hookrightarrow \Gamma_N \twoheadrightarrow K$ where we only assume $\Gamma_C \trianglelefteq \Gamma_N$ is a pair of genus zero, finite area Fuchsian groups. We are not assuming any map $\Gamma_C \twoheadrightarrow C$. The induced map $\chi: \Gamma_N \twoheadrightarrow K$ is called a $K$-map.

**Definition 12** Given an exact sequence $\Gamma_C \hookrightarrow \Gamma_N \twoheadrightarrow K$ arising from a pair $\Gamma_C \trianglelefteq \Gamma_N$ of genus zero, finite area Fuchsian groups, we say that a canonical generator $\theta \in \Gamma_N$ is a $K$-generator if it has non-trivial image under the map $\chi: \Gamma_N \to \Gamma_N/\Gamma = K$.

**Proposition 13** Let $\Gamma_C \hookrightarrow \Gamma_N \twoheadrightarrow K$ be any exact sequence defined by a pair $\Gamma_C \trianglelefteq \Gamma_N$ of genus zero, finite area Fuchsian groups. Then, $K$ is a group acting on the sphere with signature given in Theorem 11. The images of the canonical generators of $\Gamma_N$ under the map $\chi: \Gamma_N \to K$ satisfy the relations of Theorem 2 for the signature of $K$. In particular, if $K$ is not trivial there are exactly $2$ ($K = C_k$) or $3$ ($K \neq C_k$) canonical generators for $\Gamma_N$ with non-trivial image under $\chi$.

By Remark 4 we may permute the periods of $\Gamma_N$ so that the $K$-generators occur first and the signature has the format $(m_1, \ldots, m_s) = (a_1b_1, a_2b_2, a_3b_3, m_4, \ldots, m_s)$ if $K$ has signature $(a_1, a_2, a_3)$ and $(m_1, \ldots, m_s) = (a_1b_1, a_2b_2, m_3, \ldots, m_s) = (kb_1, kb_2, m_3, \ldots, m_s)$ if $K$ has signature $(a_1, a_2) = (k, k)$. If $\Gamma_N$ has a signature of either form, after permutation, we say that $\Gamma_N$ has a $K$-compatible signature. We have the following converse to Proposition 13 which follows directly from Lemma 5.8 of 4.

**Proposition 14** Let $\{\theta_1, \ldots, \theta_s\}$ be a set of canonical generators corresponding to the $K$-compatible signature $(a_1b_1, a_2b_2, a_3b_3, m_4, \ldots, m_s)$. Then, there is an essentially unique epimorphism $\chi: \Gamma_N \to K$ such that $(x_1, x_2, x_3) = \chi(\theta_1, \ldots, \theta_s)$.
\[(\chi(\theta_1), \chi(\theta_2), \chi(\theta_3))\] is a generating \((a_1, a_2, a_3)\)-vector of \(K\). I.e., given two epimorphisms \(\chi_1: \Gamma_N \to K, \chi_2: \Gamma_N \to K\) such that \((\chi_s(\theta_1), \chi_s(\theta_2), \chi_s(\theta_3))\) are \((a_1, a_2, a_3)\)-vectors for \(s = 1, 2\) then \(\chi_2 = \omega \circ \chi_1\) for some \(\omega \in \text{Aut}(K)\). A similar statement holds for the cyclic case.

**Remark 15** Suppose we are given a generating \((a_1, \ldots, a_e)\)-vector \((x_1, \ldots, x_e)\) of \(K\). Then a \(K\) map \(\chi: \Gamma_N \to K\) may be defined by

\[
\chi(\theta_i) = x_i, \quad 1 \leq i \leq e, \quad \chi(\theta_i) = 1, \quad e + 1 \leq i \leq s. \quad (18)
\]

Once the factorization \((a_1b_1, \ldots, a_eb_e, m_{e+1}, \ldots, m_s)\) is fixed then the kernel \(\Gamma_C\) of the associated sequence \(\Gamma_C \hookrightarrow \Gamma_N \twoheadrightarrow K\) is unique.

### 3.1 Finding \(\Pi \hookrightarrow \Gamma_C \twoheadrightarrow C\)

We assume that our \(K\) map \(\chi: \Gamma_N \to K\) is given as in equation (18). We want to know when a \(K\) map arises from the normalizer of a cyclic \(n\)-gonal action. To this end let us denote by \(\Gamma_C\) the kernel of \(\chi\) so that we have an exact sequence of the form

\[
\Gamma_C \hookrightarrow \Gamma_N \twoheadrightarrow K.
\]

Let \(\{\xi_1, \ldots, \xi_r\}\) be an ordered set of canonical generators for \(\Gamma_C\). The canonical generators of \(\Gamma_C\) are in 1-1 correspondence to the branch points \(\mathbb{H} \to \mathbb{H}/\Gamma_C\) and \(K\) permutes these branch points. The canonical generators of \(\Gamma_N\) give rise to \(K\)-orbits of \(C\) branch points as follows. The \(K\)-generators correspond to orbits of size less than \(|K|\) and the other orbits are regular \(K\)-orbits. It follows that the branch points of \(\Gamma_C\) are: \(|K|/a_i\) branch points of period \(b_i\) (unless \(b_i = 1\)) for each \(i, 1 \leq i \leq e\), (singular \(K\)-orbits) and \(|K|\) branch points of period \(m_j\) for each \(j, e + 1 \leq j \leq s\) (regular \(K\)-orbits). Next we need a map \(\phi: \Gamma_C \to C\) where \(C\) is a cyclic group such that \(\Pi = \ker \phi\) is torsion free. Define \(z_i \in C\) by

\[
z_i = \phi(\xi_i) \quad (19)
\]

so that \((z_1, \ldots, z_r)\) is a generating \(S(\Gamma_C)\)-vector for the \(C\)-action. According to [12], in order that the vector exist and \(\Pi\) be torsion free, we must have:

- \(\prod_{i=1}^r z_i = 1,\)
- \(o(\xi_i) = o(z_i),\)
- \(|C| = \text{lcm}(o(\xi_1), \ldots, o(\xi_r)) = \text{lcm}(b_1, \ldots, b_e, m_{e+1}, \ldots, m_s),\)
- some additional constraints on the periods \(o(\xi_1), \ldots, o(\xi_r)\) given in Harvey’s work [12].

10
We now fix $C$ to have order $\text{lcm}(b_1, \ldots, b_e, m_{e+1}, \ldots, m_s)$, and assume the constraints in the fourth bullet above. Then, the set

$$X = \left\{ (z_1, \ldots, z_r) : o(z_i) = o(\xi_i), \prod_{i=1}^r z_i = 1 \right\} \quad (20)$$

of generating $S(\Gamma_C)$-vectors is non-empty. The set $X$ allows us to enumerate the epimorphisms $\phi : \Gamma_C \to C$ since $\phi \to (\phi(\xi_1), \ldots, \phi(\xi_r))$ is a 1-1 correspondence.

The group $\text{Aut}(C)$ acts without fixed points on the epimorphisms by $(\omega, \phi) \to \omega \circ \phi$, this action is transferred to $X$ by $(\omega, (z_1, \ldots, z_r)) \to (\omega(z_1), \ldots, \omega(z_r))$. The possible kernels $\Pi$ are in 1-1 correspondence with the $\text{Aut}(C)$ orbits on $X$, a finite computable set.

Next we need to determine when the homomorphism $\phi$ extends to a homomorphism $\psi : \Gamma_N \to \tilde{K}$ such that

- $\tilde{K}$ is an overgroup of $C$ such that $C \triangleleft \tilde{K}$ and $\tilde{K}/C \simeq K$
- $\psi$ restricted to $\Gamma_C$ is $\phi : \Gamma_C \to C$

The group $\tilde{K}$ will equal $N$ when identified with a subgroup of $A$. To show that the two bullets hold, we shall employ the methods in [5]. For any $x \in \Gamma_N$ define $\phi_x : \Gamma_C \to C$ by $\phi_x(\gamma) = \phi(x\gamma x^{-1})$. The kernel of $\phi_x$ is $x^{-1}\Pi x$ and hence $\Pi$ is normal in $\Gamma_N$ if and only if $\ker(\phi_x) = \Pi$ for all $x \in \Gamma_N$. But $\phi_x$ and $\phi$ have the same kernel if and only if there is an $\omega_x \in \text{Aut}(C)$ such that $\phi_x = \omega_x \circ \phi$ or

$$\phi(x\gamma x^{-1}) = \phi_x(\gamma) = \omega_x(\phi(\gamma)), \gamma \in \Gamma_C. \quad (21)$$

We then have for $x, y \in \Gamma_N$ and $\gamma \in \Gamma_C$

$$\omega_{xy}(\phi(\gamma)) = \phi_{xy}(\gamma)$$
$$= \phi(xy\gamma y^{-1}x^{-1})$$
$$= \omega_x(\phi(y\gamma y^{-1}))$$
$$= \omega_x(\omega_y(\phi(\gamma)))$$

and so $\omega_{xy} = \omega_x \circ \omega_y$, and thus $x \to \omega_x$ is a homomorphism $\Gamma_N \to \text{Aut}(C)$. Since $C$ is abelian then $\omega_x = \text{id}$ for $x \in \Gamma_C$ and $x \to \omega_x$ factors through $K$, $g \to \omega_g$, $g \in K$.

**Remark 16** Observe that the homomorphisms $K \to \text{Aut}(C)$ are quite limited, since $\text{Aut}(C)$ is abelian. Thus $\omega : K \to \text{Aut}(C)$ factors through the abelianiza-
tion \( \omega : K_{ab} \to \text{Aut}(C) \). The abelianizations are given in Table 2.

**Table 2**

| Group | Signature       | Abelianization |
|-------|-----------------|----------------|
| \( C_k \)               | (\( k, k \)), \( k \geq 2 \) | \( \mathbb{Z}_k \) |
| \( D_k \)               | (\( 2, 2, k \)), \( k \geq 2, k \) even | \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) |
| \( D_k \)               | (\( 2, 2, k \)), \( k \geq 3, k \) odd | \( \mathbb{Z}_2 \) |
| \( A_4 \)               | (2, 3, 3)          | \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) |
| \( S_4 \)               | (2, 3, 4)          | \( \mathbb{Z}_2 \) |
| \( A_5 \)               | (2, 3, 5)          | (id)            |

**Remark 17** From Table 2 we see that for the non-cyclic case we only need to consider automorphisms of order 2. Let us write these down. For a cyclic group \( \mathbb{Z}_n \) the automorphism group is the group of units \( \mathbb{Z}_n^* \) which in turn is given by \( \mathbb{Z}_n^* = \prod_j \mathbb{Z}_{p_j^{e_j}} \) where \( n = \prod_j p_j^{e_j} \), since the Sylow subgroups are cyclic and invariant. The automorphisms of order dividing 2, and their fixed point subgroups are important to our analysis in Section 6. These automorphisms are given by \( x \to ax \) where \( a^2 = 1 \mod n \). According to the above decompositions we just need to determine the automorphisms for \( n = p^e \) a prime power. The automorphisms and their fixed points for the various prime power cases are given in the Table 3.

**Table 3**

| \( p^e \) | \( a \) | fixed point subgroup |
|----------|--------|---------------------|
| odd \( p \) | 1     | \( \mathbb{Z}_{p^e} \) |
| odd \( p \) | -1    | (0)                 |
| \( 2^e, e \geq 2 \) | 1     | \( \mathbb{Z}_{2^e} \) |
| \( 2^e, e \geq 2 \) | -1    | \( 2^{e-1}\mathbb{Z}_{2^e} \) |
| \( 2^e, e \geq 3 \) | \( 2^{e-1}+1 \) | \( 2^{e-2}\mathbb{Z}_{2^e} \) |
| \( 2^e, e \geq 3 \) | \( 2^{e-1}-1 \) | \( 2^{e-2}\mathbb{Z}_{2^e} \) |

The results in the table are derived by considering \( p^e | (a-1)(a+1) \).

Now let us compute the corresponding action of \( K \) on \( X \). For any canonical generator \( \xi_i, x_\xi x^{-1} \) is an elliptic element of \( \Gamma_C \) and hence belongs to \( y(\xi_j)y^{-1} \) for some canonical generator \( \xi_j \) and \( y \in \Gamma_C \), by 3 of Theorem 2. Since both \( x_\xi x^{-1} \) and \( y_\xi y^{-1} \) generate the stabilizer of the same point then \( x_\xi x^{-1} = y_\xi y^{-1} \) where \( a \) is relatively prime to the order of \( \xi_j \). By using covering space methods to construct the generating set \( \{ \xi_1, \ldots, \xi_r \} \), it can be shown that we may in fact take \( a = 1 \) and that the permutation representation \( q : i \to j \) is...
defined by the action of $K$ on the branch points of $\mathbb{H} \to \mathbb{H}/\Gamma$. We then have

$$\omega_x(z_i) = \phi_x(\xi_i)$$

$$= \phi(x\xi_ix^{-1})$$

$$= \phi(y\xi_jy^{-1})$$

$$= \phi(\xi_j)$$

$$= z_j$$

as $y \in \Gamma_C$. We piece together the data above to construct an action of $K$ on $X$ by

$$g \cdot (z_1, \ldots, z_r) = (\omega_g^{-1}(z_{q(1)}), \ldots, \omega_g^{-1}(z_{q(r)}))$$  \hspace{1cm} (22)

The vector $(z_1, \ldots, z_r)$ is fixed by $g$ if and only if

$$\omega_g(z_i) = z_{q(i)}, \ 1 \leq i \leq r.$$  \hspace{1cm} (23)

The following theorem allows us to identify normalizers of cyclic $n$-gonal actions, by finding the $K$-fixed points of the actions in equation 22 as we vary over all homomorphisms $K \to \text{Aut}(C)$. The proof of the theorem follows from the previous discussion.

**Theorem 18** Let the sequence $\Gamma_C \hookrightarrow \Gamma_N \twoheadrightarrow K$, the cyclic group $C$, the set of generating vectors $X$, and the permutation representation $q : K \to \Sigma_r$ be as defined above. Then we have the following.

- Let $(z_1, \ldots, z_r) \in X$ be a generating $S(\Gamma_C)$-vector of $C$ and $\Pi \twoheadrightarrow \Gamma_C \phi \rightarrow C$ the epimorphism sequence defined by $\phi(\xi_i) = z_i$. Assume that $\Pi$ is normal in $\Gamma_N$, that $\omega : K \to \text{Aut}(C)$ is the resulting homomorphism defined by equation 21, and let $K$ act on $X$ by equation 22. Then $(z_1, \ldots, z_r)$ is fixed by all $g$ in $K$.

- Let $\omega : K \to \text{Aut}(C)$ be any homomorphism, and let $K$ act on $X$ by equation 22. Assume that $(z_1, \ldots, z_r) \in X$ is fixed by all $y$ in $K$ and let $\Pi \twoheadrightarrow \Gamma_C \phi \rightarrow C$ be the epimorphism sequence defined by $\phi(\xi_i) = z_i$. Then $\Pi$ is normal in $\Gamma_N$.

**Example 19** Let $\Gamma_N$ have signature $(4, 4, 9, 11)$ written in factored form as $(2 \cdot 2, 2 \cdot 2, 3 \cdot 3, 11)$ where $K = D_3$ has signature $(2, 2, 3)$. Then $\Gamma_C$ has signature $(2^3 \cdot 3^2, 11^6)$. We determine all possible sequences

$$\Gamma_C \hookrightarrow \Gamma_N \twoheadrightarrow K$$

with $C$ cyclic. As noted in Table 3 $K_{ab} = \mathbb{Z}_2$ and we really only need to carefully consider the action of the reflections in $K$.

Let us first discuss the action of $K$ on the indices $\{1, \ldots, 14\}$. This action is derived from the $K$-action on the sphere, so we just need to describe it one orbit at a time. The indices $\{1, 2, 3\}$ correspond to one of the orbits of size three and
the $K$-action is just the standard $D_3$-action. Likewise for the indices $\{4, 5, 6\}$. The indices $\{7, 8\}$ correspond to the orbit of size two and so the reflections in $K$ interchange $7$ and $8$. Finally $\{9, 10, 11, 12, 13, 14\}$ constitutes a regular orbit and so we may arrange the indices so that the reflections in $K$ interchange $\{9, 10, 11\}$ and $\{12, 13, 14\}$ as sets.

Now let $C = C_{66} = C_2 \times C_3 \times C_{11}$ and from Remark $17$ $\text{Aut}(C) \cong \mathbb{Z}_2 \times \mathbb{Z}_{10}$. Define $g_2, g_3, g_{11}$, so that $C_2 = \langle g_2 \rangle$, $C_3 = \langle g_3 \rangle$, $C_{11} = \langle g_{11} \rangle$. Since the abelianization $K_{ab} \cong \mathbb{Z}_2$, the image $\omega_i$ of the non-trivial element of $K_{ab}$ must be in the subgroup of $\text{Aut}(C)$ generated by $\omega_1 : (x, y, z) \to (x, y^{-1}, z)$ and $\omega_2 : (x, y, z) \to (x, y, z^{-1})$, for $(x, y, z) \in C_2 \times C_3 \times C_{11}$.

Now let us consider a specific map $\omega_r \in \text{Aut}(C)$ and a specific vector. Set $\omega_1 = \omega_3, \omega_2$ and consider the following vector

$$(z_1, \ldots, z_{14}) = (g_2, g_2, g_2, g_2, g_2, g_3, g_3^{-1}, g_{11}, g_{11}, g_{11}, g_{11}^{-1}, g_{11}^{-1}, g_{11}^{-1}). \quad (24)$$

By construction $(z_1, \ldots, z_{14})$ satisfies equations 12, 13, 14, and it is also fixed by $K$ under the action given by equation 22 or equation 23. Thus the action of $C$ may be extended by $K$. The given vector is essentially unique. First we can only have $\omega_r = \omega_1 \omega_2$. If $\omega_3$ acts trivially on $C_3 \subset C$ then we must have $z_8 = \omega_r(z_7) = z_7$ by equation 22 but then

$$1 = z_1 \cdots z_{14} = z_1 \cdots z_6 z_7^2 z_9 \cdots z_{14}$$

Since $z_7^2$ has order $3$, it is not possible for this product to be trivial no matter what the values of other $z_i$ are. Likewise $\omega_1$ cannot act trivially on $C_3$ otherwise $z_9 = \cdots = z_{14}$, by equation 23, yielding another contradiction. Now that $\omega_r$ is determined it follows that we can only have a vector of the form in equation 24, where $g_2, g_3, g_{11}$ are suitably chosen generators. Any two such vectors are equivalent under $\text{Aut}(C)$.

**Remark 20** The above example demonstrates the following easily proved properties of $K$-fixed-vectors. Using the properties allows us to easily construct and enumerate the $K$-fixed vectors.

1. The element $z_i$ must be invariant under $\{\omega_g : q_g(i) = i\}$.
2. The collection of $z_i$ corresponding to a $K$-orbit $O \subseteq \{1, \ldots, r\}$ is determined by a single $z_{i_0}$ for any $i_0 \in O$. Just use the $K$-action.
3. If $O_1, \ldots, O_s$ are the $K$-orbits constituting $\{1, \ldots, r\}$ Then

$$\prod_{j=1}^s \left( \prod_{i \in O_j} z_i \right) = 1.$$

Now one simply finds an element in each orbit of the correct order satisfying statement 1. By statement 2 the orbit products $\prod_{i \in O_j} z_i$ are easily calculated and we just have to verify statement 3.
3.2 Finding epimorphisms $\Gamma_N \to K$

By proposition [14] there are epimorphisms $\Gamma_N \to K$ if and only if $\Gamma_N$ has a $K$-compatible signature. Finding maps $\Gamma_N \to K$ is fairly simple when all the periods are known constants. However, as we shall see in the next section, we want to consider that case when the periods are parameters, such as in Example 30. There needs to be some care to get an efficient enumeration of all the cases. We first consider an example.

**Example 21** Suppose that $\Gamma_N = T(2, d, 2d)$ with signature $(2, d, 2d), d \geq 4$ We want to permute and factor the signature $(2, d, 2d)$ so that it is $K$-compatible, i.e., in the form $(a_1b_1, a_2b_2, a_3b_3)$ or $(kb_1, kb_2, m_1)$ We put the results in Table 4 below. In the factorizations the variable $e$ may be any integer such that the signature $(2, d, 2d)$ is hyperbolic. Some factorizations are equivalent by permutations that leave the signature of $K$ fixed, they are listed contiguously. The signature of $C$ and the $n = |C|$ can be computed as at the beginning of subsection 3.2.

| $K$  | $S(K)$          | $S(\Gamma_N)$ factored | conditions on $d, e, k, n$ |
|------|-----------------|-------------------------|-----------------------------|
| $C_2$ | $(2, 2)$        | $(2, d, 2d) = (2 \cdot 1, 2 \cdot e, 4e)$ | $d = 2e, n = 4e$            |
|      | $(2, 2)$        | $(2, 2, 2d) = (2 \cdot e, 2 \cdot 1, 4e)$ | $d = 2e, n = 4e$            |
|      | $(2, 2, 2d) = (2 \cdot 1, 2 \cdot e, e)$ | $n = e = d, e$ odd, $e \geq 5$ |
| $C_k$ | $(k, k)$        | $(2d, 2, d) = (2 \cdot e, 2 \cdot 1, e)$ | $n = e = d, e$ odd, $e \geq 5$ |
|      | $(2d, 2, d) = (2 \cdot e, 2 \cdot 1, e)$ | $n = e = d, e$ odd, $e \geq 5$ |
| $D_2$ | $(2, 2, 2)$     | $(d, 2d, 2) = (2 \cdot e, 2 \cdot e, 2 \cdot 1)$ | $d = 2e, n = 2e$            |
|      | $(d, 2d, 2) = (2 \cdot e, 2 \cdot e, 2 \cdot 1)$ | $d = 2e, n = 2e$            |
| $D_k$ | $(2, 2, k)$     | $(2d, 2, d) = (2 \cdot e, ke, k \cdot e)$ | $d = n = e$                 |
|      | $(2d, 2, d) = (2 \cdot e, ke, k \cdot e)$ | $d = n = e$                 |
|      | $(2, 2, k) = (2d, 2, 2d) = (2 \cdot ke, 2 \cdot 1, k \cdot e)$ | $d = n = ke$                |
|      | $(2d, 2, d) = (2 \cdot ke, 2 \cdot 1, k \cdot e)$ | $d = n = ke$                |
|      | $(2, 2, k) = (2d, 2, d) = (2 \cdot e, 2 \cdot 1, k \cdot e)$ | $2d = ke, d$ even, $n = \text{lcm}(e, \frac{2d}{e})$ |
|      | $(2d, 2, d) = (2 \cdot e, 2 \cdot 1, k \cdot e)$ | $2d = ke, d$ even, $n = \text{lcm}(e, \frac{2d}{e})$ |

| $K$  | $S(K)$          | $S(\Gamma_N)$ factored | conditions on $d, e, n$ |
|------|-----------------|-------------------------|-----------------------------|
| $A_4$ | $(2, 3, 3)$     | $(2d, 2, d) = (2 \cdot 1, 3 \cdot e, 3 \cdot 2e)$ | $d = 3e, n = 2e$            |
|      | $(2d, 2, d) = (2 \cdot 1, 2e, 3 \cdot 2e)$ | $d = 3e, n = 2e$            |
| $S_4$ | $(2, 3, 4)$     | $(2d, 2, d) = (2 \cdot 1, 3 \cdot 2e, 4 \cdot 3e)$ | $d = n = 6e$               |
|      | $(2d, 2, d) = (2 \cdot 1, 3 \cdot 2e, 4 \cdot 3e)$ | $d = n = 6e$               |
| $A_5$ | $(2, 3, 5)$     | $(2d, 2, d) = (2 \cdot 1, 3 \cdot 5e, 5 \cdot 6e)$ | $d = 15e, n = 30e$          |
|      | $(2d, 2, d) = (2 \cdot 1, 3 \cdot 10e, 5 \cdot 3e)$ | $d = 15e, n = 30e$          |
We prove a few of the lines.

Line 1: Since $S(\Gamma_N) = (2, d, 2d)$, $K = C_2$, the signature of $C$ is $(1, \frac{d}{k}, (2d)^k)$, setting $d = 2e$, $k = 2$ we get $S(C) = (e, 4e, 4e)$. According to [12], $n = 4e$ and a $C$-action exists on a surface of genus $2e - 2$ provided $e \geq 2$.

Line 3: The signature of $C$ is $(1, e, e^k)$ or $(e, e, e^k)$. A cyclic action with $n = e$ on a surface of genus $\sigma = (e - 1)/2$ exists if $e \geq 3$ and $e$ is odd.

Line 5: The signature of $C$ is $(e, 2e, 2^k)$. If either $e$ or $k$ is even, then the number of periods divisible by the highest power of 2 is odd, violating one the conditions in [12]. Thus $d$ is odd and $e \geq 3$. Harvey’s conditions now hold and $n = 2e$, $\sigma = \frac{d-1}{2}$.

Line 7: The signature of $C$ is $(e^2, d^2, 1^2)$ or $(e, e, 2e)$. A cyclic action with $n = 2e$ exists on a surface of genus $2e - 2$ if $e \geq 2$.

Line 9: The signature of $C$ is $(1^k, d^k, (\frac{d}{k})^2)$ or $(e, e, (ek)^k)$. We must have $n = d$ and $\sigma = \frac{k(d-3)+2}{2}$. If $d$ is odd or if $k$ is even then the signature meets the parity conditions in [12], and an action exists.

Lines 13: The signature of $C$ is $(1^k, \frac{d}{k}, (\frac{2d}{k})^2)$ or $(e, e, (\frac{ek}{k})^k)$, upon setting $2d = ek$. We must have $n = \text{lcm}(e, \frac{ek}{k})$ and $n = ek, \frac{ek}{k},$ or $\frac{4ek}{k}$ are all possible. The genus is and $\sigma = 1 + \frac{ek}{k} - \frac{4ek}{k}$.

Now we describe an algorithm for generating all possible maps $\Gamma_N \rightarrow K$, or equivalently the compatible, permuted signatures. See Example [22] for various steps of the process. First we build a list of all possibilities and then prune the list to remove redundancies.

1. Enumerate all distinct ordered pairs $(l_1, l_2)$ or triples of periods $(l_1, l_2, l_3)$ from the periods of $\Gamma_N$, depending on whether $K$ has 2 or 3 canonical generators.

2. Rewrite the periods in the form $(l_1, l_2, m_1, \ldots, m_u)$ or $(l_1, l_2, l_3, m_1, \ldots, m_u)$ so that the ordered pair or triple occurs first and the remaining periods are ordered lexicographically with respect to parameter variables, using increasing order on the coefficients.

3. For permuted $S(\Gamma_N)$ found in step 2 we solve $(l_1, l_2) = (kb_1, kb_2)$, $(l_1, l_2, l_3) = (a_1b_1, a_2b_2, a_3b_3)$. We split this into two cases depending on whether the signature of $K$ has parameters or not. Initially the parametric signatures are $(k, k)$ or $(2, 2, k)$ but these may be changed later on.

4. If the signature of $K$ consist only of constants we proceed as follows.
• We examine each $a_i$ in order, modifying $S(\Gamma_N)$ as needed.
• If $l_i$ is a constant not divisible by $a_i$ we reject the permuted $S(\Gamma_N)$.
• Otherwise write $l_i = c_i w_i$ where $c_i$ is a constant and $w_i$ is a parameter. Set $e_i = a_i / \gcd(a_i, c_i)$ and make the substitution $w_i \to e_i w_i$ throughout the signature. See Example 22 item 1.

5. If the signature of $K$ has a parameter $(k, k)$ or $(2, 2, k)$ we proceed as follows.

• We examine each $a_i$ in order, modifying $S(\Gamma_N)$ as needed.
• If $a_i$ is a constant then we proceed as in step 4.
• If $a_i$ is a parameter and $l_i$ is a constant then for each divisor $d$ of $l_i, d > 1$, solve the problem with $S(K) = (d, d)$ or $(2, 2, d)$ and $S(\Gamma_N)$. See Example 22 item 2.
• If $a_i$ has a parameter and $l_i$ has a parameter then we modify with a separate case for the dihedral and cyclic cases.
• Cyclic case $l_1 = c_1 w_1, l_2 = c_2 w_2$: Let $d$ be any divisor of $\gcd(c_1, c_2)$ then set $S(K) = (dk, dk)$ and make the substitution $w_i \to kw_i$ for each distinct $w_i$.
• Dihedral Case $l_3 = c_3 w_3$: First modify $S(\Gamma_N)$ as in the first bullet, possibly getting a new equation $l_3 = c_3 w_3$. Let $d$ be any divisor of $c_3$ then set $S(K) = (2, 2, dk)$ and make the substitution $w_3 \to kw_3$.

Example 22 Here are some examples of steps in the algorithm above. We denote the desired map $\chi : \Gamma_N \to K$ by $S(\Gamma_N)/S(K)$. Steps in the process corresponding to period $a_i$ of $K$ are denoted by the numbered arrow $\rightsquigarrow$.

1. First let $S(\Gamma_N) = (2, 2, x_1, 5x_1), S(K) = (2, 3, 5)$. The 12 permutations of $S(\Gamma_N)$ to be considered are

\[
(2, 2, x_1, 5x_1), (2, x_1, 2, 5x_1), (x_1, 2, 2, 5x_1), (2, 2, 5x_1, x_1),
(2, 5x_1, 2, x_1), (5x_1, 2, 2, x_1), (2, x_1, 5x_1, 2), (2, 5x_1, x_1, 2),
(x_1, 2, 5x_1, 2), (5x_1, 2, x_1, 2), (x_1, 5x_1, 2, 2), (5x_1, x_1, 2, 2).
\]

If we consider the case $(2, x_1, 5x_1, 2)$, then the sequence of substitutions required is:

\[
(2, x_1, 5x_1, 2)/(2, 3, 5) \stackrel{1}{\rightarrow} (2, x_1, 5x_1, 2)/(2, 3, 5) \stackrel{2}{\rightarrow} \]

\[
(2, 3x_1, 15x_1, 2)/(2, 3, 5) \stackrel{3}{\rightarrow} (2, 3x_1, 15x_1, 2)/(2, 3, 5)
\]

2. Let $S(\Gamma_N) = (6, x_1, 5x_1, 6), S(K) = (k, k)$ From $(k, k) = (2, 2)$ we get the sequence of substitutions required

\[
(6, x_1, 5x_1, 6)/(2, 2) \stackrel{1}{\rightarrow} (6, x_1, 5x_1, 6)/(2, 2) \stackrel{2}{\rightarrow} (6, 2x_1, 10x_1, 6)/(2, 2)
\]
and from $(k,k) = (3,3)$ we get

$$(6, x_1, 5x_1, 6)/(3,3) \xrightarrow{1} (6, x_1, 5x_1, 6)/(3,3) \xrightarrow{2} (6, 3x_1, 15x_1, 6)/(3,3)$$

3. Let $S(\Gamma_N) = (6x_1, 10x_1, 2, 2)$, $S(K) = (k,k)$. Then we get

$$(6x_1, 10x_1, 2, 2)/(k,k) \to (6kx_1, 10kx_1, 2, 2)/(k,k)$$

and

$$(6x_1, 10x_1, 2, 2)/(k,k) \to (6kx_1, 10kx_1, 2, 2)/(2k, 2k)$$

4. Let $S(\Gamma_N) = (2, x_1, 5x_1, 2)$, $S(K) = (2, 2, k)$. Then we get

$$(2, x_1, 5x_1, 2)/(2, 2, k) \xrightarrow{1} (2, x_1, 5x_1, 2)/(2, 2, k) \xrightarrow{2}$$

or

$$(2, x_1, 5x_1, 2)/(2, 2, k) \xrightarrow{1} (2, x_1, 5x_1, 2)/(2, 2, k) \xrightarrow{2}$$

or

$$(2, x_1, 5x_1, 2)/(2, 2, k) \xrightarrow{1} (2, x_1, 5x_1, 2)/(2, 2, k) \xrightarrow{2}$$

or

$$(2, x_1, 5x_1, 2)/(2, 2, k) \xrightarrow{1} (2, x_1, 5x_1, 2)/(2, 2, k) \xrightarrow{2}$$

or

$$(2, x_1, 5x_1, 2)/(2, 2, k) \xrightarrow{1} (2, x_1, 5x_1, 2)/(2, 2, k) \xrightarrow{2}$$

4 The Fuchsian group pair $\Gamma_N < \Gamma_A$

Since it is unlikely that $\Gamma_N$ is normal in $\Gamma_A$ we need to find ways to work with the structure of the inclusion of the pair $\Gamma_N < \Gamma_A$. We shall describe two different approaches: monodromy and word maps. Since these concepts require significant computational power to fully implement we only discuss them very briefly and refer the reader to [4] for full details. In our examples in Section 6 we shall use ad hoc methods to directly construct a candidate for the full automorphism group. Then we will use ad hoc applications of the monodromy group and a maximality result, discussed at the end of this section, to demonstrate that the candidate is the full automorphism group. On the other hand, the machinery of monodromy groups and word maps is necessary for full classification and computing the harder examples. Thus, we include an overview of those ideas to give a complete overview of the classification process.
In Singerman’s paper \[16\] on finite maximality, the inclusions $\Gamma_N < \Gamma_A$ where both $\Gamma_N$ and $\Gamma_A$ are triangle groups were determined. These pairs constitute the main part of what is known as “Singerman’s list”. Later, the authors of \[9\] presented methods useful in finding $A$, if it exists, given $N$ and $\Gamma_N < \Gamma_A$. However, their methods were restricted to pairs on Singerman’s list. As described in the signature theorem, Theorem \[12\] there may be pairs $\Gamma_N < \Gamma_A$ which do not appear in Singerman’s list. Hence, we need the more general discussion of pairs $\Gamma_N < \Gamma_A$ given in this section.

4.1 Monodromy and word maps

Let $\Gamma < \Delta$ be a finite index pair of genus zero Fuchsian groups and let $m = [\Gamma : \Delta]$. Any labeling of the cosets of $\Gamma$ in $\Delta$ gives rise to a permutation representation $\rho : \Delta \rightarrow \Sigma_m$. If another labeling is chosen then the two representations are related by $\rho_2 = \pi \rho_1 \pi^{-1}$ for some $\pi \in \Sigma_m$. Thus all the images $\rho(\Delta)$ are conjugate and are isomorphic to $\Delta/\text{Core}_\Delta(\Gamma)$. We call any of the images or $\Delta/\text{Core}_\Delta(\Gamma)$ itself the monodromy group $M(\Delta, \Gamma)$. The monodromy group $M(\Delta, \Gamma)$ is isomorphic to the monodromy of the branched cover $\mathbb{H}/\Delta \rightarrow \mathbb{H}/\Gamma$ away from the branch points. Since the groups are genus zero $\mathbb{H}/\Delta \rightarrow \mathbb{H}/\Gamma$ is just a branched covering of the sphere by itself.

If $G_2 = \{\zeta_1, \ldots, \zeta_t\}$ is the chosen set of canonical generators of $\Delta$, then the permutations

$$\pi_j = \rho(\zeta_j)$$

satisfy

$$\prod_{j=1}^{t} \pi_j = 1$$

because of equation \[11\]. The monodromy group $M(\Delta, \Gamma) = \langle \pi_1, \pi_2, \ldots, \pi_t \rangle$ is a transitive subgroup of $\Sigma_m$.

**Remark 23** If $\Gamma = \Gamma_N$ and $\Delta = \Gamma_A$ then $\Delta/\text{Core}_\Delta(\Gamma) \cong A/\text{Core}_A(N)$ and $\mathbb{H}/\Delta \rightarrow \mathbb{H}/\Gamma$ is the projection $S/N \rightarrow S/A$.

**Definition 24** Let notation be as above and set $P = (\pi_1, \ldots, \pi_t)$. The cycle type of $\pi_j$ determines a partition $p_j$ of $m$, set $P = (p_1, \ldots, p_t)$. The tuple of permutations $P$ is called the monodromy vector of $\Gamma < \Delta$ or $\mathbb{H}/\Delta \rightarrow \mathbb{H}/\Gamma$. The tuple of partitions $P$ is called the cycle vector of $\Gamma < \Delta$ or $\mathbb{H}/\Delta \rightarrow \mathbb{H}/\Gamma$. More generally, let $P = (p_1, \ldots, p_t)$ be a $t$-tuple of partitions and let $P = (\pi_1, \ldots, \pi_t)$ be $t$-tuple of permutations. Then $P$ is called a transitive $P$-monodromy vector if

$$\pi_j \text{ has cycle type } p_j$$

(25)

$$\prod_{j=1}^{t} \pi_j = 1$$

(26)

$\langle \pi_1, \pi_2, \ldots, \pi_t \rangle$ is a transitive subgroup of $\Sigma_m$. (27)
Remark 25 The signatures $S(\Gamma)$, $S(\Delta)$ determine the cycle types occurring in the cycle vector for $\Gamma < \Delta$. Indeed, let $p_j = (p_{j,1}, \ldots, p_{j,n_j})$ be the partition of $n$ determined by $\pi_j$. Then for each $p_{j,i}$ there is a distinct generator $\theta_{j,i}$ of $\Gamma$ of order $m_{j,i}$ such that
\[ o(\zeta_j) = p_{j,i} o(\theta_{j,i}) \tag{28} \]
or
\[ n_j = p_{j,i} m_{j,i} \tag{29} \]
where $S(\Delta) = (n_1, \ldots, n_t)$. We say that the pair of signatures $S(\Gamma) < S(\Delta)$ of signatures are $P$-compatible, and symbolize this by
\[ P : S(\Gamma) \to S(\Delta) \]
We call the sequence a numerical projection even though there may not be a projection of surfaces $\pi : \mathbb{H}/\Gamma \to \mathbb{H}/\Delta$.

The following variant of the Riemann existence theorem is important for our work.

Theorem 26 Let $\Gamma, \Delta$ be a two Fuchsian groups, and $P$ a cycle vector, and suppose that the signatures $S(\Gamma)$ and $S(\Delta)$ are $P$-compatible. Let $P$ be a transitive $P$-monodromy vector. Then there is a subgroup $\Gamma' < \Delta$, with the same signature as $\Gamma$, such that $P$ is the monodromy vector of the pair $\Gamma' < \Delta$.

Now we turn our attention to word maps.

Definition 27 Select canonical generating sets $G_1 = \{\theta_1, \ldots, \theta_s\}$ and $G_2 = \{\zeta_1, \ldots, \zeta_t\}$ of $\Gamma$ and $\Delta$ respectively. The word map of the pair $\Gamma \leq \Delta$ is a set of words $\{w_1, \ldots, w_s\}$ in the generators in $G_2$ such that
\[ \theta_i = \omega_i(\zeta_1, \ldots, \zeta_t), i = 1, \ldots, s. \]

Remark 28 If both groups have genus zero there is an easily implemented algorithm to calculate the word map, see [4]. The word maps for the inclusions in Singerman’s list have been calculated in [9].

Example 29 Suppose we have the signatures $S_1 = (2, 2, 2, 5), S_2 = (2, 4, 5)$. We want to show there is a pair $\Gamma < \Delta$ with $S(\Gamma) = S_1, S(\Delta) = S_2$. First find a compatible monodromy vector $P = (\pi_1, \pi_2, \pi_3)$ in $\Sigma_b$. We select
\[ \pi_1 = (1, 3)(4, 6), \pi_2 = (1, 2)(3, 5, 4, 6), \pi_3 = (1, 2, 3, 4, 5) \]
from which we get $M(\Delta, \pi) = A_6$. Define as before $\rho : \Delta \to \Sigma_b$ by $\rho : \zeta_i \mapsto \pi_i, i = 1 \ldots 3$. Then $\Gamma$ may be taken as the stabilizer of a point for the permutation action of $\Delta$ on $\{1, \ldots, 6\}$. From the algorithm, a generating set for $\Gamma$ is
\[ \theta_1 = (\zeta_1 \zeta_2) \zeta_1 (\zeta_1 \zeta_2)^{-1} \]
\[ \theta_2 = \zeta_2 \zeta_1 \zeta_2^{-1} \]
\[ \theta_3 = \zeta_2^2 \]
\[ \theta_4 = (\zeta_2^{-1} \zeta_1^{-1} \zeta_2^{-1} \zeta_1 \zeta_3 \zeta_1) \zeta_3 (\zeta_2^{-1} \zeta_1^{-1} \zeta_2^{-1} \zeta_1 \zeta_3 \zeta_1)^{-1} \]
Here is how the word maps may be used in conjunction with the monodromy vectors to expand an extension $C \triangleleft N$ to $C \triangleleft N < A$.

- Assume that we have pairs $\Gamma_N < \Gamma_A$ and $\Gamma_C \triangleleft \Gamma_N$ determined by monodromy groups $M(\Gamma_A, \Gamma_N)$ and $M(\Gamma_N, \Gamma_C) \simeq K$.
- According to Remark 28 there are word maps for the inclusions $\Gamma_C < \Gamma_N$ and $\Gamma_N < \Gamma_A$.
- The word maps may be composed to provide a word map for $\Gamma_C < \Gamma_A$.
- The word map may be used with the Todd-Coxeter algorithm to provide the monodromy group $M(\Gamma_A, \Gamma_C) = M(A, C)$.
- The stabilizer of a point in $M(A, C)$ is $C/\text{Core}_A(C)$. If $\text{Core}_A(C)$ is trivial then $C/\text{Core}_A(C)$ can be tested to see if it is cyclic. The trivial core condition is satisfied in the weakly malnormal case discussed in the next section.

### 4.2 Constrained and tight pairs

We need a mechanism to deal with families of inclusions. First we consider an example arising from Singerman’s list.

**Example 30** Let $T(l, m, n)$ denote the triangle Fuchsian group with signature $(l, m, n)$. Consider the possible case $\Gamma_N = T(2, d, 2d)$ and $\Gamma_A = T(2, 3, 2d)$ with $d \geq 4$. The index is

$$[\Gamma_A : \Gamma_N] = \frac{1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{d}}{1 - \frac{2}{3} - \frac{1}{2d}} = \frac{(d - 3)/2d}{(d - 3)/6d} = 3.$$  

With little more work a monodromy vector can be found $((1, 2), (1, 2, 3), (1, 3))$, and $M(\Gamma_A, \Gamma_N) = \Sigma_3$. Notice that in this case

$$o(\zeta_1) = o(\pi_1), \ o(\zeta_2) = o(\pi_2), \ o(\zeta_3) > o(\pi_3).$$

and that $o(\zeta_3)$ has a parameter $d \geq 4$.

To handle the notion of families we extend our consideration of Fuchsian groups to include parabolic elements $\delta_1, \ldots, \delta_q$. Thus we have $\Delta = \langle \gamma_1, \ldots, \gamma_t, \delta_1, \ldots, \delta_q \rangle$, with relations

$$\gamma_1^{n_1} = \gamma_2^{n_2} = \cdots = \gamma_t^{n_t} = \prod_{i=1}^{t} \gamma_i \prod_{j=1}^{q} \delta_j = 1.$$  

(30)

The Teichmüller dimension of the modified $\Delta$ is $d(\Delta) = t + q - 3$.

**Definition 31** Let $\rho : \Delta \to \Sigma_n$ as previously defined.
• A pair $\Gamma < \Delta$ is called constrained if $\Delta$ has no parabolic generators and $o(\zeta) = o(\rho(\zeta))$ for each elliptic generator.

• A pair $\Gamma < \Delta$ is called tight if $\Delta$ has at least one parabolic generator and $o(\zeta) = o(\rho(\zeta))$ for each elliptic generator.

**Remark 32** The definition depends only on the cycle types and not the permutations themselves. Hence, the definition depends only on the signature pair and may be applied to a numerical projection $P: S(\Gamma) \to S(\Delta)$.

**Proposition 33** Let $\Gamma < \Delta$ be a tight pair where $\Delta$ has $q$ parabolic elements. Then there is a $q$ parameter family $\Gamma(\ell_1, \ldots, \ell_q) < \Delta(\ell_1, \ldots, \ell_q)$ such that each member of the family has

- the same codimension $d(\Gamma, \Delta)$
- the same index $[\Delta : \Gamma]$
- the same monodromy $M(\Delta, \Gamma)$
- the same word map

**Remark 34** Every Fuchsian group pair is constrained or belongs to a unique family as above. The tight pair defining the family is called the parent tight pair.

**Example 35** The triangle group family $T(2, d, 2d) < T(2, 3, 2d)$ comes from the tight pair $T(2, \infty, \infty) < T(2, 3, \infty)$. The monodromy vector is $((1, 2), (1, 2, 3), (1, 3))$.

### 4.3 Classification steps for pairs

Here are steps for classification of the pairs $\Gamma_N < \Gamma_A$.

*Classify numerical projections by codimension.* For each codimension there are a finite number of constrained pairs and a finite number of tight pairs of numerical projections of signatures. The list of codimensions will depend on how the signature pairs have been limited.

*Compute monodromy vectors.* For each candidate signature pair, compute all the compatible monodromy vectors up to conjugacy equivalence. Each constrained numerical projection gives rise to a finite number (possibly none) of pairs $\Gamma_N < \Gamma_A$. Each tight numerical projection gives rise to a finite number (possibly none) of parametric family of pairs $\Gamma(\ell_1, \ldots, \ell_q) < \Delta(\ell_1, \ldots, \ell_q)$ all with the same monodromy. First one considers primitive pairs where $M(\Delta, \Gamma)$ is a primitive permutation group. This can be done by computer calculation and classification of primitive permutation groups (use Magma or GAP). In the general case there is a tower $\Gamma_N = \Gamma_1 < \cdots < \Gamma_e = \Gamma_A$ such that each inclusion $\Gamma_i < \Gamma_{i+1}$ is a primitive pair, already classified. A tower may be fused together by using word maps and the Todd Coxeter algorithm. An example of a tower is $T_{7,7,7} < T_{3,3,7} < T_{2,3,7}$, which occurs for the 7-gonal Klein quartic.
4.4 Maximal actions and signatures

Given a known group \( G \) of automorphisms of a surface \( S \), we want to know if \( G = A \), i.e., \( G \) has a maximal action. To demonstrate that \( G \) already has a maximal action in our examples in Section 6, we will use a simple test on the signatures. Our test rests on the concept of finite maximality developed in [16]. A Fuchsian group \( \Gamma \) is called finitely maximal if \( \Gamma \) is not contained in any other Fuchsian group with finite index. In [16] Singerman determines which Fuchsian groups are finitely maximal.

Now suppose that \( G \) acts on \( S \), then we have

\[
\begin{align*}
\Pi & \hookrightarrow \Gamma_G \hookrightarrow \Gamma_A \\
\downarrow \eta & \downarrow \eta \downarrow \eta \\
\langle 1 \rangle & \hookrightarrow G \hookrightarrow A
\end{align*}
\]

(31)

If \( \Gamma_G \) is finitely maximal then \( \Gamma_G = \Gamma_A \). If \( \Gamma_G \) is not finitely maximal we have

\[
\frac{|A|}{|G|} = \frac{|\Gamma_A/\Gamma_G|}{|\Gamma_G|} = \frac{A(\Gamma_G)}{A(\Gamma_A)} = \frac{\mu(\Gamma_G)}{\mu(\Gamma_A)}
\]

where \( |A| / |G| \) is an integer \( k \geq 2 \). If the signature of \( G \) is \((m_1, m_2, \ldots, m_r)\) and the signature of \( A \) is \((n_1, n_2, \ldots, n_t)\) then this may be rewritten.

\[
k = -2 + \sum_{i=1}^{r} \left( 1 - \frac{1}{m_i} \right) = r - 2 - \sum_{i=1}^{r} \frac{1}{m_i}
\]

(32)

\[
-2 + \sum_{j=1}^{t} \left( 1 - \frac{1}{n_j} \right) = t - 2 + \sum_{j=1}^{t} \frac{1}{n_j}
\]

where \( 3 \leq t \leq r \) and \( m_i \) divides some \( n_j \) for every \( i \). Equation (32) provides a restriction which may be enough to prove finite maximality since the quotient on the right hand side must be an integer. Rather than state and prove a general result we give an example sufficient for our needs. The example also follows from examining Singerman’s list in [16].

**Example 36** The Fuchsian group with signature \((2, 3, m)\), \( m \geq 7 \) is finitely maximal. To prove this let \( h = t - 2 + \sum_{j=1}^{t} \frac{1}{n_j} \) (note that \( t = 3 \)). Then

\[
1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{m} = kh, \quad \text{or} \quad h = \frac{1}{6k} \left( m - \frac{6}{m} \right) \quad \text{or} \quad m = \frac{6}{6kh - 1}
\]

Since \( k \geq 2 \), then \( h < \frac{1}{12} \), and there are only a finite number of signatures for which \( h < \frac{1}{12} \), namely, \((2, 3, 7)\), \((2, 3, 8)\), \((2, 3, 9)\), \((2, 3, 10)\), \((2, 3, 11)\), and \((2, 4, 5)\). None of these yield an integer value for \( m \) for any integer value of \( k \).
5 Constraints on signatures

As alluded to in Section 1 the full classification problem of cyclic \( n \)-gonal surfaces and their automorphism groups is too complex to be completed in its entirety. In this section we discuss some methods to limit the possible signature pairs \( S(\Gamma_A) \) so that the problem is more tractable. The limitations are chosen because of links to interesting group theoretic, geometrical or arithmetic properties of the restrictions. As discussed in the introduction there are two constraints we can consider.

- The action of \( C \) on \( S \) is weakly malnormal.
- \( S \) is a quasi-platonic surface.

The constraints in force because of weak normality have been completely described and proven in [7]. Later we recall the main theorem on signatures in that work, Theorem [42] below, and indicate how the theorem may be proven by consideration of the action of \( A \) on the singular \( A \)-orbits on \( S \). The constraints due to \( S \) being a quasi-platonic surface are not well known at this time other than to mention that the signature of \( A \) is quite restricted, and the potential for application to dessins.

To understand the simplification offered by weakly malnormal actions we first have to understand strong branching. Following that, we analyze the action of \( A \) on the points lying over the branch points of \( S \to S/A \). This analysis can be used to prove the signature theorem for weakly malnormal actions.

5.1 Strong branching and weak normality \( C < A \)

Previous work [1], [13], [14], [17], and [18] has shown that if the \( n \)-gonal morphism \( \pi_C : S \to S/C \) is highly ramified then we often have \( C \subseteq A \). This greatly simplifies the calculation of \( A \) since the calculations in Section 4 may be skipped. Some papers are restricted to the normal case [1], [13], [14], [17]. The non-normal case has been considered in [14] and [18]. In [1] Accola introduced a precise measure of “highly ramified” called strong branching. Strong branching is a condition that guarantees normality in many cases, in particular the prime cyclic case, and was used in [13], and [18]. Strong branching may be used to conclude that is the genus of a cyclic \( n \)-gonal surface is sufficiently large then \( C \subseteq A \).

An unramified covering \( \pi : S_1 \to S_2 \) of degree \( n \) satisfies \( 2\sigma_1 - 2 = n (2\sigma_2 - 2) \). If the covering is ramified then the formula is modified to:

\[
2\sigma_1 - 2 = n (2\sigma_2 - 2) + R_{\pi}
\]

where \( R_{\pi} \) may be determined from the Riemann-Hurwitz formula. Accola [11] calls \( \pi \) a strongly branched cover if

\[
R_{\pi} > 2n(n - 1)(\sigma_2 + 1)
\]
or
\[ \sigma_1 > n^2 \sigma_2 + (n - 1)^2. \]
If \( S_2 \) has genus 0 then the formulas are
\[ R_\pi > 2n(n - 1) \]
or
\[ \sigma_1 > (n - 1)^2. \]
In the case at hand, \( \pi : S \to S/C \) given by equation (1), if we define
\[ d_i = (n, p_i), \quad n_i = \frac{n}{d_i} \]
then
\[ R_\pi = n \sum_{i=1}^{t} \left( 1 - \frac{1}{n_i} \right) = \sum_{i=1}^{t} (n - d_i). \]

We see, that \( S \to S/C \) is strongly branched if, roughly, the right hand side of equation (1) has many factors. The main fact we need about strong branching is the following.

**Proposition 37** Let \( H \) be a group of automorphisms acting on a surface \( S \) such that \( S \to S/H \) is strongly branched. Then there is a unique minimal, normal, nontrivial subgroup \( L \) of \( \text{Aut}(S) \) such that \( L \leq H \), and \( S \to S/L \) is strongly branched.

In [5] the concept of weak normality was introduced, to take advantage of strong branching. It appears to be the weakest group theoretic constraint such that we can take advantage of strong branching.

**Definition 38** Let \( H \leq G \) be a pair of groups and let \( N = N_G(H) \). Then \( H \) is weakly malnormal in \( G \) if for each \( g \in G - N \) we have a trivial intersection \( H \cap H^g = \langle 1 \rangle \). A group action of \( H \) on a surface \( S \) is called weakly malnormal if \( H \) is a weakly malnormal subgroup of \( A = \text{Aut}(S) \).

**Remark 39** We can make some immediate remarks.

- Normal subgroups are trivially weakly malnormal.
- If \( H \leq G \) is a cyclic subgroup of prime order then \( H \) is weakly malnormal in \( G \).
- If \( C \leq A \) is a cyclic subgroup of \( A = \text{Aut}(S) \) and the map \( S \to S/C \) is fully ramified, then \( C \) is weakly malnormal in \( A \).
- Let \( H \leq G \) be a pair of groups such that \( H \) is weakly malnormal in \( G \), but not normal. If \( K \) is a nontrivial subgroup of \( H \), then \( N_G(H) = N_G(K) \).
• Assume the same hypotheses as above. Then the representation of $G$ on the left or right cosets of $H$ is faithful, for the kernel of the representation is $\bigcap_{g \in G} H^g$.

The main use of weak normality is given in the following proposition, which shows that the non-normal cases occur only for small genus. For instance the hyperelliptic involution is always normal and any non-normal cyclic trigonal case must occur in genus 2, 3, or 4.

**Proposition 40** Let $H$ be a group of automorphisms acting on a surface $S$ such that $S \to S/H$ is strongly branched and $H$ is weakly malnormal in $A = \text{Aut}(S)$. Then $H$ is normal in $A$. If the action of a group $C$ of order $n$ on a surface of genus $\sigma > (n-1)^2$ is weakly malnormal and $S/C$ has genus zero then $C$ is normal in $A$.

**Example 41** There are examples of cyclic 4-gonal actions on surfaces of arbitrarily high genus, but where $C$ is not normal in $A$. See [7].

The main restriction imposed for weakly malnormal actions is the signature theorem below. The theorem is proved in [7] by directly working with canonical generators, though it may also be proven from the analysis in the next subsection.

**Theorem 42** If the action of $C$ on $S$ is weakly malnormal, then $\Gamma_N$ has at most 3 additional periods to $\Gamma_A$. The signatures for $\Gamma_A$ and $\Gamma_N$ appear as a pair in Table 5, where $(a_1, a_2, a_3)$ or $(k, k)$ is the signature of $K = \Gamma_N/\Gamma_C$. The column labeled Codim is the Teichmüller codimension $d(\Gamma_A, \Gamma_N)$.

| Case | Codim | Signature of $\Gamma_N$ | Signature of $\Gamma_A$ |
|------|-------|-------------------------|-------------------------|
| 0A   | 0     | $(a_1 m_1, a_2 m_2, a_3 m_3, n_1, \ldots, n_r)$ | $(b_1, b_2, n_1, \ldots, n_r)$ |
| 0B   | 0     | $(km_1, km_2, n_1, \ldots, n_r)$ | $(b_1, b_2, n_1, \ldots, n_r)$ |
| 1A   | 1     | $(a_1 m_1, a_2 m_2, a_3 m_3, n_1, \ldots, n_r)$ | $(b_1, b_2, n_1, \ldots, n_r)$ |
| 1B   | 1     | $(km_1, km_2, n_1, \ldots, n_r)$ | $(b_1, n_1, \ldots, n_r)$ |
| 2A   | 2     | $(a_1 m_1, a_2 m_2, a_3 m_3, n_1, \ldots, n_r)$ | $(b_1, n_1, \ldots, n_r)$ |
| 2B   | 2     | $(km_1, km_2, n_1, \ldots, n_r)$ | $(n_1, \ldots, n_r)$ |
| 3A   | 3     | $(a_1 m_1, a_2 m_2, a_3 m_3, n_1, \ldots, n_r)$ | $(n_1, \ldots, n_r)$ |

### 5.2 Orbits and induced generators

We now return to the general situation. We want to closely link the signatures of $\Gamma_A$ and $\Gamma_N$, by studying the singular $A$-orbits on $S$. 

26
Definition 43 Let $H \subseteq A$ be any subgroup. The orbit $Hx$ is called $H$-regular if $|Hx| = |H|$, and is called $H$-singular if $|Hx| < |H|$.

The facts in the following lemma are easily shown, we leave most details to the reader.

Lemma 44 Let $H \subseteq M \subseteq A$ be subgroups of $A$. Then

1. If $Hx$ is singular then the order of the $H$-stabilizer $H_y$ of any point $y \in Hx$ is $|H| / |Hx|$.

2. The orbit $Mx$ is a union of $H$-orbits, and $Mx$ is $M$-singular if any of the $H$-orbits is $H$-singular.

3. Let $H \trianglelefteq M$, $L = M/H$, and $y$ be any point of $S$. Then $My$ is a union of $H$-orbits of the same size. The number of $H$-orbits in an $M$-orbit $My$ is less than $|L|$ if and only if $y$ is fixed by an element of $M - H$.

Proof. Only Statement 3 requires any work. Let $My$ be any $M$-orbit, it is a disjoint union of $H$-orbits. Since $M$ normalizes $H$ then $M$ permutes the $H$-orbits so they must all be the same size. Again by normality, $L$ permutes the $H$-orbits comprising $My$ transitively. If there are less than $|L|$ $H$-orbits, then for some $g, g_1 \in M$, $g \in M - H$, $gHg_1y = Hg_1y$. It follows that $gh_1g_1y = g_1y$ for some $h_1 \in H$ and so $g_1^{-1}gh_1g_1y = y$. If $g_1^{-1}gh_1g_1 = g_1^{-1}gg_1g_1^{-1}h_1g_1 \in H$ then so must $g_1^{-1}gg_1 \in H$ and hence $g \in H$. This is a contradiction and so $g_1^{-1}gh_1g_1 \in M - H$. On the other hand if $My$ is not contained in $H$ then $L$ has a nontrivial fixed point when acting on the set of $H$-orbits. It follows that there are fewer than $|L|$ $H$-orbits. ■

Now suppose that $\zeta \in \Gamma_A$ is a canonical generator. The elliptic element $\zeta$ has a unique fixed point $z \in \mathbb{H}$, let $x = \pi_H(z)$ be the image on $S$. The map $\eta : \Gamma_A \to A$ maps $\langle \zeta \rangle$ isomorphically onto the stabilizer $Ax$. If a conjugate of $\zeta$ is chosen we simply get another point of $Ax$. Thus, there is a 1-1 correspondence between the canonical generators of $\Gamma_A$ and the singular orbits of $A$

$$\zeta \leftrightarrow \langle \zeta \rangle^{\Gamma_A} \leftrightarrow Ax.$$  

Moreover,

$$|\langle \zeta \rangle| = |Ax| = |A| / |Ax|.$$  

(33)

A similar statement applies to any subgroup $H \subseteq A$ and the elliptic canonical generators of the corresponding group $\Gamma_H$. The following proposition details the relationship between induced generators and singular orbits.

Proposition 45 Let $H \subseteq M \subseteq A$ be a tower of groups and $\Gamma_H \subseteq \Gamma_M \subseteq \Gamma_A$ the covering Fuchsian groups. Assume that the genus of $S/H$ is zero. Then, we have the following.
1. The canonical generators $\zeta \in \Gamma_H$ are in 1-1 correspondence with the $H$-singular orbits $Hx$ via
   \[ \zeta \leftrightarrow \langle \zeta \rangle^{\Gamma_H} \leftrightarrow Hx, \]
   where $x = \pi_H(z)$ for the fixed point $z$ of $\zeta$, $\langle \zeta \rangle^{\Gamma_H}$ is a conjugacy class of stabilizers in $\Gamma_H$. The order of $\zeta$ is $|Hx| = |H|/|Hx|$. 

2. Suppose that $\zeta \in \Gamma_A$ is a canonical generator and $z, x$ are as item 1. Then the canonical generators of $H$ induced by $\zeta$ are in 1-1 correspondence to the singular $H$-orbits contained in $Ax$. Moreover if $Hy \subseteq Ax$ is a singular $H$-orbit then the order of the corresponding induced canonical generator of $\Gamma_H$ is $|H|/|Hy|$. 

3. Let $\text{gens}(\Gamma)$ be a set of canonical generators of $\Gamma$. Then the signatures satisfy
   \[ |\text{gens}(\Gamma_A)| \leq |\text{gens}(\Gamma_M)| \leq |\text{gens}(\Gamma_H)|. \]

Proof. Statement 1 was demonstrated in the discussion preceding the statement of the Proposition. Statement 2 is a reformulation of the theorem from Singerman. To prove statement 3 observe that
   \[ |\text{gens}(\Gamma_M)| = 3 + \text{Teichmüller dimension } \Gamma_M \]
   \[ \leq 3 + \text{Teichmüller dimension } \Gamma_H \]
   \[ = |\text{gens}(\Gamma_H)| \]
   The identical argument works for the other inequality. 

We are now going to focus on the relation between the singular $N$-orbits and the singular $A$-orbits when $S/C$ has genus zero. To this end we identify exactly three ways in which an $N$-orbit can be singular.

Remark 46 Let the notation for the groups $C \subseteq N \subseteq A$ be as above and assume that $K = N/C$ is non-trivial. Then the singular $N$-orbits are of three types:

1. Type 1: The orbit $Nx$ consists of $|K|$ singular $C$-orbits. For each $y \in Nx$ the stabilizer $N_y \subseteq C$ and so $N_y = C_y$. This is according to Statement 3 of Lemma 44.

2. Type 2: The orbit $Nx$ consists of fewer than $|K|$ regular $C$-orbits. For each $y \in Nx$ $N_y \cap C$ is trivial. There is an element $g \in N$ of order $|N|/|Ny|$ in $N$ such that each stabilizer in $Nx$ is conjugate to $\langle g \rangle$. The order of $g$ is one of the periods of $K$.

3. Type 3: The orbit $Nx$ consists of fewer than $|K|$ singular $C$-orbits. Let $\mathfrak{N}$ be the orbit $Nx$, so that $|K\mathfrak{N}| < |K|$. The value $a = |K|/|K\mathfrak{N}|$ is one of the periods of $K$ acting on $S/H$, let $m = |N_x \cap C|$. Then there is an element $g \in N$ of order am such that $\langle g \rangle = N_x$, and $\langle g^a \rangle = N_x \cap C$. 

28
Remark 47 (Continuation of above Remark) Suppose that \( K \) is non-trivial. Then there are two possible signatures of \( \Gamma_N \), namely \((km_1, km_2, n_1, \ldots, n_r)\) or \((a_1m_1, a_2m_2, a_3m_3, n_1, \ldots, n_r)\), depending on the signature of \( K \). The orbits of Type 1 produce the canonical generators of orders \( n_1, \ldots, n_r \). The orbits of Type 2 and 3 produce canonical generators of orders \( a_1m_1, a_2m_2, a_3m_3 \) or \( km_1, km_2 \) depending on signature of \( K \). For the orbits of Type 2, \( m_1 = 1 \). There are either two or three orbits of Type 2 or 3 if \( K \) is non-trivial. If \( K \) is trivial then the only singular orbits are of Type 1. The canonical generators corresponding to the orbits of type 2 or 3 are \( K \)-generators.

Remark 48 In the previous situation we do not need \( C \) to be cyclic.

Now let us assume that \( C \) is weakly malnormal in \( A \) and determine the consequences for the orbits and the signatures.

Lemma 49 Suppose that \( C \) is weakly malnormal in \( A \). Then we have the following:

1. If \( Ny \subseteq Ax \) is a singular orbit of Type 1 or Type 3 we have equality of stabilizers \( Ny = Ay \).

2. Each singular orbit \( Ax \) contains at most one \( N \)-orbit of Type 1.

Proof. Assume that \( Ny \subseteq Ax \) is a singular orbit of Type 1 or Type 3. By definition the stabilizer \( C_y \) is nontrivial and the cyclic subgroup \( Ay \supseteq Cy \) and so \( Ay \) normalizes a non-trivial subgroup of \( C \). It follows that \( Ay \subseteq N \) and hence \( Ny = Ay \).

Suppose that \( Ax \) is a singular \( A \)-orbit and that \( Ax \) contains an \( N \)-orbit \( Ny \) of Type 1. Then, by Remark 48 and the first statement above, \( Cy = Ny = Ay \).

Now suppose that \( Ng'y \) is a Type 1 orbit distinct from \( Ny \) for some \( g \in A - N \). We must also have that

\[ C_{gy} = A_{gy} = gAyg^{-1} = gCyg^{-1}. \]

But then

\[ C_{gy} \subseteq C \cap gCyg^{-1} \subseteq C \cap gCg^{-1} = \langle 1 \rangle. \]

Thus we have a contradiction if there are two distinct \( N \)-orbits of Type 1. □

Remark 50 The proof techniques just used automatically show the following. If the \( n \)-gonal morphism \( S \to S/C \) is fully ramified, i.e., has signature \( (n, \ldots, n) \), then the action is weakly malnormal. To see this let \( x \) be any point of \( S \) and observe that the stabilizer \( C_x = C \) or \( C_x = \langle 1 \rangle \). There is some point \( x \in S \) where \( C = C_x \subseteq Ax \). Let \( g \in A - N \) and consider

\[ C_{gx} = \{ c \in C : cgx = gx \} = \{ c \in C : (g^{-1}cg) x = x \} = C \cap gA\cap g^{-1} \supseteq C \cap gCg^{-1}. \]

29
If $|C_{gx}| = 1$ then $C \cap gCg^{-1} = \langle 1 \rangle$. On the other hand, seeking a contradiction, assume that $|C_{gx}| > 1$. By the fully ramified hypothesis we must have $|C_{gx}| = |C|$, but then, we must have $C_{gx} = C$, which implies $gA_xg^{-1} \supseteq C$. As $A_x$ and $gA_xg^{-1}$ are cyclic they have unique subgroups of order $|C|$ and we conclude that $C = gCg^{-1}$, a contradiction.

Remark 51 Lemma 49 still holds if $C$ is not cyclic.

It is useful to classify the decomposition of $A$-orbits into $N$-orbits.

Lemma 52 Assume that $C$ is weakly malnormal in $A$, and assume that $\zeta$ is a canonical generator of $\Gamma_A$ which corresponds to the orbit $A_x$. Then we have the following possibilities.

1. The orbit $A_x$ contains no singular $N$-orbits and $\zeta$ does not induce any canonical generator of $\Gamma_N$ or $\Gamma_C$.

2. The orbit $A_x$ contains a singular $N$-orbit of Type 1 and no other singular orbits. Then $\zeta$ induces a canonical generator $\theta$ of $\Gamma_N$ of the same order as $\zeta$, and exactly $|K|$ canonical generators of $\Gamma_C$ with same order as $\zeta$.

3. The orbit $A_x$ is as in 1, 2 above except that it additionally contains up to three $N$-orbits of Type 2 or type 3 subject to the constraint that the total number of orbits of Type 2 and Type 3 is 2 or 3. Let $\theta \in \Gamma_N$ be the corresponding generator induced by $\zeta$ for an orbit of Type 2 or 3. In the case of Type 2 we have $o(\theta) < o(\zeta)$ and in the type 3 case we have $o(\theta) = o(\zeta)$.

Proof. In Case 1 all the $N$-orbits and $C$-orbits are regular and hence no canonical generators are induced. The number and order of induced canonical generators in the remaining cases follow from Lemma 44 and Lemma 49.

The discussion in the section may be used to prove the signature theorem 42. Here is a proof sketch. Consider any canonical generator $\zeta$ of $\Gamma_A$, the corresponding singular $A$-orbit $A_x$ and its decomposition into $N$-orbits. If $A_x$ contains an $N$-orbit of Type 1, then $\zeta$ induces one canonical generator of $\Gamma_N$ of the same order as $\zeta$ and possibly others. Since two Type 1 $N$-orbits cannot occupy the same $A$-orbit then all the canonical generators of $\Gamma_A$ inducing Type 1 generators of $\Gamma_N$ are distinct. This leads to the sequence $n_1, \ldots, n_r$ in both signatures. It follows that

$$|\text{gens}(\Gamma_N)| \geq |\text{gens}(\Gamma_A)| \geq |\text{gens}(\Gamma_N)| - 3.$$ 

The remaining generators come from Type 2 and Type 3 orbits. Thus the signature $\Gamma_N$ is known and the periods $b_i$ of $\Gamma_A$ are simply fill-ins, except that each period of $\Gamma_N$ must divide some period of $\Gamma_A$.

Remark 53 If $K$ is trivial then by the argument in the proof of the proposition of the proposition $\Gamma_A$ and $\Gamma_N$ have the signature $(n_1, \ldots, n_r)$. It follows that $A = N$ and $C$ is normal in $A$. 

30
Example 54 Consider Klein’s quartic. $C = \mathbb{Z}_7$, $C = \mathbb{Z}_3 \times \mathbb{Z}_7$, $A = PSL_2(7)$, $K = \mathbb{Z}_3$. Then we have

| $\text{Signature of } \Gamma_A$ | $\text{Signature of } \Gamma_N$ | $\text{Signature of } \Gamma_C$ |
|-------------------------------|-------------------------------|-------------------------------|
| $(2,3,7)$                     | $(3,3,7)$                     | $(7,7,7)$                     |

The $A$-orbits split into $N$-orbits as follows

| order of canonical generator | $2$ | $3$ | $7$ | Size of $N$-orbits |
|-------------------------------|-----|-----|-----|--------------------|
| Size of $A$-orbit             | 84  | 56  | 24  |                    |
| Regular $N$-orbits            | 4   | 2   | 1   | 21                 |
| Type 1 $N$-orbits             | 0   | 0   | 1   | 3                  |
| Type 2 $N$-orbits             | 0   | 2   | 0   | 7                  |
| Type 3 $N$-orbits             | 0   | 0   | 0   | 1                  |

The table entries are interpreted as follows. The $A$-orbit corresponding to a canonical generator of order 2 consists of 84 points which breaks up into 4 regular $N$-orbits of size 21. The $A$-orbit corresponding to a canonical generator of order 3 has 56 points and breaks up into 2 regular $N$-orbits of size 21 and 2 Type 2 orbits of size 7. The $A$-orbit corresponding to a canonical generator of order 7 has 246 points and breaks up into a regular $N$-orbit of size 21 and one Type 1 orbit of size 3. There cannot be any Type 3 orbits since $N$ is not cyclic.

6 Examples

6.1 Constrained examples

Only two constrained examples have been found as of the writing of this paper. Both are discussed in [18] and are well known curves.

| Name               | genus | $C$   | $N$         | $A$        | $K$ |
|--------------------|-------|-------|-------------|------------|-----|
| Klein’s quartic    | 3     | $C_7$ | $C_3 \ltimes C_7$ | $PSL_2(7)$ | $C_5$ |
| Bring’s curve      | 4     | $C_5$ | $C_4 \ltimes C_5$ | $\Sigma_5$ | $C_4$ |

6.2 Examples with parametric families.

We conclude with some examples of parametric families suggested Table 4. First we consider Fermat curves. We will show that they give a parametric family of curves with weakly malnormal cyclic $n$-gonal actions where the automorphism group strictly contains the normalizer of the $n$-gonal action.

Example 55 Consider the variant of the Fermat curve $F_n$ given by

$$x^n + y^n = -1,$$  \hspace{1cm} (34)
or better, by its homogeneous equation,

\[ X^n + Y^n + Z^n = 0. \]  

(35)

From the affine equation \( X^3 \) we see that the curve is cyclic \( n \)-gonal. From the projective form \( \Sigma_3 \) we determine that the linear group \( \Sigma_3 \times \mathbb{Z}_n^3 \) acts on \( F_n \) with \( \Sigma_3 \) acting as permutations of the coordinates, and \( \mathbb{Z}_n^3 \) acting by

\[
(a, b, c) \cdot (X : Y : Z) = \left(e^{2\pi i a/n} X : e^{2\pi i b/n} Y : e^{2\pi i c/n} Z\right),
\]

in homogeneous coordinates. The diagonal subgroup \( D = \{(a, a, a) : a \in \mathbb{Z}_n\} \) acts trivially. In fact, \( G = \Sigma_3 \times \mathbb{Z}_n^3 / D \) is the automorphism group of \( F_n \) as we shall see shortly. The affine model of \( F_n \) can be obtained via \( x = X/Z \) and \( y = Y/Z \). In the affine setting \((a, b, c)\) acts via

\[
(a, b, c) \cdot (x, y) = \left(\exp\left(\frac{2\pi i a - c}{n}\right) x, \exp\left(\frac{2\pi i b - c}{n}\right) y\right)
\]

and the coordinate transpositions correspond to birational maps as in the following table. The projections

| permutation | projective automorphism | birational affine map |
|-------------|-------------------------|-----------------------|
| (1, 2)     | \((X : Y : Z) \leftrightarrow (Y : X : Z)\) | \((x, y) \rightarrow (y, x)\) |
| (1, 3)     | \((X : Y : Z) \leftrightarrow (Z : Y : X)\) | \((x, y) \rightarrow (1/x, y/x)\) |
| (2, 3)     | \((X : Y : Z) \leftrightarrow (X : Z : Y)\) | \((x, y) \rightarrow (x/y, 1/y)\) |

Using the projection \( \mathbb{Z}_3^3 \rightarrow \mathbb{Z}_2^3 \), \((a, b, c) \rightarrow (a - c, b - c)\) with kernel \( D \), we may more conveniently denote the automorphism group of the affine model by writing \( G \cong \Sigma_3 \times \mathbb{Z}_2^3 \) with \((1, 2), (1, 3), \) and \((2, 3)\) acting on \( \mathbb{Z}_n^2 \) by \((1, 2) : (a, b) \rightarrow (b, a)\), \((1, 3) : (a, b) \rightarrow (-a, -a + b)\), and \((2, 3) : (a, b) \rightarrow (a - b, -b)\).

Let \( C = \mathbb{Z}_n \) be the cyclic group \((1) \times \{(0, a, 0) : a \in \mathbb{Z}_n\}\) of \( \Sigma_3 \times \mathbb{Z}_n^3 \) corresponding to the subgroup \((1) \times \{(0, a) : a \in \mathbb{Z}_n\}\) of \( G \). The subgroup \( C \) corresponds to the standard \( n \)-gonal action on \( F_n \) when we write the affine equation in the form \( y^n = -1 - x^n \), with projection \((x, y) \rightarrow x\), or \((X : Y : Z) \rightarrow (X : Z)\) in projective coordinates. The normalizer \( N \) of \( C \) in \( G \) is \( N = \langle (1, 3) \rangle \times \mathbb{Z}_n^2 \). The group \( N \) acts by multiplying \( X \) and \( Z \) by \( n \)th roots of unity and switching the \( X \) and \( Z \) coordinates. There are two other \( n \)-gonal projections, the information for all three projections is summarized in the following table. The projections are equivalent under the full automorphism group.

| affine map | projective map | \( C \) | \( N \) |
|------------|---------------|--------|--------|
| \((x, y) \rightarrow x\) | \((X : Y : Z) \rightarrow (X : Z)\) | \{\(0, a\) : \(a \in \mathbb{Z}_n\)\} | \langle (2, 3) \rangle \times \mathbb{Z}_n^2 \|
| \((x, y) \rightarrow y\) | \((X : Y : Z) \rightarrow (Y : Z)\) | \{\(a, 0\) : \(a \in \mathbb{Z}_n\)\} | \langle (1, 3) \rangle \times \mathbb{Z}_n^2 \|
| \((x, y) \rightarrow x/y\) | \((X : Y : Z) \rightarrow (X : Y)\) | \{\(a, a\) : \(a \in \mathbb{Z}_n\)\} | \langle (1, 2) \rangle \times \mathbb{Z}_n^2 \|

By Example 7 the signature of \( C \) is \((n, \ldots, n)\) \((n \text{ times})\) since \( x^n + 1 \) has\n distinct linear factors. The dihedral action of \( N/C \cong \mathbb{Z}_2 \times \mathbb{Z}_n \) on \( S/C \) is the
standard dihedral action of \( D_n \) on the sphere with fixed points as follows. The branch points of \( C \) form a single \( N/C \) orbit consisting of points of ramification order 2, there is another \( N/C \) orbit of points of ramification order 2 consisting of \( C \)-regular points, and finally an orbit consisting two \( C \)-regular points with \( N/C \) ramification order \( n \). Therefore, the signature of \( N \) is \((2,2,n)\) when written compatibly with the \((2,2,n)\) signature of \( K = D_n \). Alternatively, it is easily directly verified that \( N/C \cong D_n \). For completeness let us construct a generating \((2,2n)\)-vector for \( N = \langle (1,2) \rangle \cong \mathbb{Z}_2 \). Let \( g = ((1,2), (0,0)), h = (1, (1,0)) \), and \( k = (1, (0,0)) \). Then \( g^2 = 1, h^n = k^n = 1 \), and \( ghk = kh \). Hence \( gh \) has order \( 2n \) and \((g, gh, h^{-1})\) is a generating \((2,2n)\)-vector for \( N \).

Now we turn to the full automorphism group. It is easily verified that \( C \) is weakly malnormal in \( G \). If we can show \( G \) is the full automorphism group then our example is complete. The monodromy group \( M(G,N) \) is easily calculated to be \( \Sigma_3 \), and a little work shows that the signature of \( G \) is \((2,3,2n)\). By Example \( \text{26} \) \( \Gamma_G \) is finitely maximal and so the full automorphism group \( A \) of \( F_n \) equals \( G \). This case is line 9 from Table 4 with \( k = d = n \) and \( e = 1 \). Again for completeness we construct a generating \((2,3,2n)\)-vector for \( G \). The generating vector projects to a \((1 \cdot 2 \cdot 3, 1 \cdot 2)\)-monodromy vector of \( M(G,N) \) and so we start with the elements \( g_1 = ((1,3), (0,0)) \) and \( g_2 = ((1,2), (0,0)) \) of \( \Sigma_3 \) to construct \((g_1, g_1g_2, g_2)\) a \((1 \cdot 2 \cdot 3, 1 \cdot 2)\)-monodromy vector. Setting \( h = (1, (1,0)) \) as before we see that \((g_1g_2h^{-1})^3 = 1 \) and \( o(hg_2) = 2n \). Thus \((g_1, g_1g_2h^{-1}, hg_2)\) is a generating \((2,3,2n)\)-vector.

**General \( K = D_k \), with trivial action on \( C \)** Before going on to our remaining examples we shall examine the general case where \( K = D_k \) and the \( D_k \)-action on \( C \) is trivial and see what we can conclude about the structure of \( N \). From Table 2 we see that \( D_k \) acts on \( C \) by factoring through a group of order 2 or 4. Because of space considerations we are going to restrict our attention to the case where all of \( D_k \) acts trivially on \( C \). In the sequence \( C \to N \to D_k \), let \( E \) be the inverse image of \( C_k \subset D_k \), so that \( E \cong N \) and we have an exact sequence \( C \to E \to C_k \), with \( C_k \) acting trivially on \( C \). In this case, it may be verified simply that \( E \) is an abelian group, which we shall write in additive notation. For \( x \in E \) we denote the image in \( C_k \) by \( \overline{x} \).

**Remark 56** In case \( k \) is odd then \( C_k \subset D_k \) automatically acts trivially on \( C \) and \( E \) is automatically abelian.

The structure of \( E \) is strongly influenced by the action of \( D_k \) on \( E \). If \( g \in N - E \) then conjugation by \( g \) induces an automorphism \( \phi \) of \( E \). Since \( g^2 \in C_k \), then \( \phi^2 = 1 \) and \( \phi \) does not depend on the \( g \) chosen. The subgroup \( C \) is invariant under \( \phi \), and by assumption \( \phi(x) = x, x \in C \). On the quotient group \( C_k = E/C \) the induced map acts by \( \phi(\overline{x}) = -\overline{x} \). To utilize the action of \( \phi \) to determine the structure \( E \) we need the even/odd Sylow decomposition of \( E \). Decompose \( E = S_2 \times S_3 \) where \( S_2 \) is the 2-Sylow subgroup of \( E \) and \( S_3 \) is the subgroup of elements of odd order, a direct sum of odd order Sylow subgroups.
We will determine the structure of the two subgroups separately, considering the odd piece first.

Since \( S_o \) has odd order, division by 2 is well-defined. We will use the “eigenspace” decomposition of \( E \) induced by \( \phi \). For any \( x \) we may write

\[
x = x^+ + x^-
\]

where

\[
x^+ = \frac{x + \phi(x)}{2}, \quad x^- = \frac{x - \phi(x)}{2}.
\]

We observe from equation 37 and \( \phi^2 = 1 \) that \( \phi(x^+) = x^+ \) and \( \phi(x^-) = -x^- \).

Let \( S_o^+ = \{ x \in S_o : \phi(x) = x \} \) and \( S_o^- = \{ x \in S_o : \phi(x) = -x \} \). By equation 36 \( S_o = S_o^+ + S_o^- \). We also have \( S_o^+ \cap S_o^- \) is trivial since any \( x \in S_o^+ \cap S_o^- \) satisfies \( x = \phi(x) = -x \), forcing \( x = 0 \), by the odd order condition. It follows that \( S_o = S_o^+ + S_o^- \). Now \( C \subseteq S_o^+ \). If \( C \) is properly contained in \( S_o^+ \) then some element \( \pi \in S_o^+ / C \subseteq E / C \) satisfies \( \phi(\pi) = \pi \), a contradiction. It follows that \( C = S_o^+ \) and \( S_o^- \) maps injectively to \( C_k \). Both \( S_o^+ \) and \( S_o^- \) are cyclic.

The analysis of \( S_2 \) is a bit more fussy. Let \( S_C = S_2 \cap C \) and \( \overline{S} = S_2 / S_C \) be the corresponding 2-Sylow subgroups of \( C \) and \( C_k = E / C \). Note that \( S_C \to S_2 \to \overline{S} \) is exact, and \( S_C \) and \( \overline{S} \) are cyclic and we may suppose they are non-trivial. As in our previous analysis set \( S_2^+ = \{ x \in S_2 : \phi(x) = x \} \) and \( S_2^- = \{ x \in S_2 : \phi(x) = -x \} \). Now, however, \( S_2^+ \cap S_2^- \subseteq \{ x \in S_2 : x = -x \} \) the subgroup of elements of order 2. Also it is not true that \( S_2 = S_2^+ + S_2^- \). Let \( h \in S_2 \) be an element such that \( \overline{S} = \langle h \rangle \), set \( H = \langle h \rangle \). Then the map \( S_C \times H \to S_2 \), \((x,y) \to xy\) is surjective and the kernel is \( Z = S_C \cap H = \{ (x,-x) : x \in S_C \cap \langle h \rangle \} \) isomorphic to a cyclic subgroup of \( H \). We know that \( \phi(h) = -h + c \) for some \( c \in C \). If we can choose \( c = 0 \) then the \( S_C \cap H \) is contained in \( S_2^+ \cap S_2^- \) and so \( Z \) has order 2 or 1. This leads to two cases.

\[
S_2 \cong C \times \langle h \rangle
\]

and

\[
S_2 \cong C \times \langle h \rangle / Z, \quad |Z| = 2.
\]

As an example of the first take \( N = S_C \times D_k \) where \( k = |\overline{S}| \). For the second let \( E = S_C \times C_{2k} \) and let \( z_1, z_2 \) be the unique elements of order 2 in \( S_C \) and \( C_{2k} \), and let \( Z = \langle (z_1, z_2) \rangle \). Then \( Z \) is a normal subgroup of \( S_C \times D_k \), and \( S_C \times D_k / Z \) is an example satisfying equation 39. In the first case \( S_2 \) is a product of two cyclic groups. In the second case if \( |S_C| = 2 \) or \( |H| = 2 \) then \( S_2 \) is cyclic, otherwise \( S_2 \) is the product of two cyclic 2-groups.

Next suppose that we cannot choose \( c = 0 \). If \( b \in C \) is any other element and \( h' = h + b \) then \( \phi(h') = -h + c + b = -h' + c + 2b \). Thus if \( c \not\in 2C \) then we may choose \( c \) to be an explicit generator of \( C \) as \( C / 2C = C_2 \). Also observe that \( h \) and \( \phi(h) \) have the same order and so \( |C| \leq |H| \). As before, construct the exact sequence \( Z \to S_C \times \langle h \rangle \to S_2 \) so that the kernel \( Z = \{ (x, -x) : x \in S_C \cap \langle h \rangle \} = \langle r h \rangle \) for some \( r \), a power of 2. Since the sequence \( Z \to S_C \times \langle h \rangle \to S_2 \) is short exact

\[
|Z| = \frac{|S_C \times \langle h \rangle|}{|S_2|} = \frac{|S_C| o(h)}{|S_C| |\overline{S}|} = \frac{o(h)}{|\overline{S}|}.
\]
and \( r = o(h)/|Z| = |\overline{S}| > 1 \). As \( rh \in C \) then \( rh = sc \) for some \( s \). Noting that \( rh \) is \( \phi \)-invariant we get \( rh = \phi(rh) = -rh + rc \), or \( rc = 2rh \). Since both \( Sc \) and \( H \) are nontrivial it follows that \( o(h) = 2o(c) \). From \( rc = 2rh = 2sc \), there are two possible values for \( s \), namely \( s = \frac{r}{2} \) and \( s = \frac{r + o(c)}{2} \). Thus \( Z = \langle (\frac{r}{2}, -rh) \rangle \) or \( Z = \langle (\frac{r + o(c)}{2}, c, -rh) \rangle \). Correspondingly, assuming \( r > 1 \) is a power of 2, we may construct a model for the \( \phi \)-module \( S_2 \) as \( Sc \times \langle h \rangle / Z \), where \( Z = \langle (\frac{r}{2}, -rh) \rangle \) or \( Z = \langle (\frac{r + o(c)}{2}, c, -rh) \rangle \) and \( \phi : Sc \times \langle h \rangle \to \) is the map \((nc, mh) \to ((n + m)c, -mh)\). The subgroup \( Z \) is invariant since \( \phi(\frac{r}{2}, -rh) = ((\frac{r}{2} - r)c, rh) = (\frac{r}{2}, -rh) \), or \( \phi(\frac{r}{2}, -rh) = \left( \frac{o(c)}{2} - \frac{r}{2} \right)c, rh \) = \((\frac{r + o(c)}{2}, c, -rh) \). It follows that \( Sc \times \langle h \rangle / Z \) has all the correct properties.

The preceding discussion gives us a good representation of \( S_2 \) when it is not cyclic. It will be useful to write down an alternate, specific representation of \( S_2 \) when it is cyclic. Let \( S_2 = \langle h \rangle, q = o(h) \), and let \( \overline{h} \) be the image of in \( \overline{S} \), so that \( Sc = \langle rh \rangle, \) and \( \overline{S} = \langle \overline{h} \rangle \). Then

\[
\phi(h) = ah, \\
\phi(rh) = rh, \\
\phi(h) = -\overline{h}.
\]

Since \( \overline{h} + \overline{h} = 0 \) then \( \overline{h} = -\overline{h} \) for some \( l \) or \( e = -1 + lr \) mod \( q \). Next

\[
a^2h = \phi^2(h) = \phi((-1 + lr)h) \\
= \phi(-h)\phi(lrh) = -eh + lrh \\
= (1 - lr)h = h,
\]

so it follows that \( a^2 = 1 \) mod \( q \) or that \( q|(a - 1)(a + 1) \). From Table 3, for \( r \geq 8 \), there are four possibilities for \( a \), namely \( a = 1, q/2 - 1, q/2 + 1, q - 1 \) mod \( q \). The case \( a = 1 \) is eliminated unless \( |\overline{S}| = 1, 2 \). In the case \( a = q/2 - 1 \), \( Sc \) is contained in the subgroup \( \langle \frac{\overline{h}}{2} \rangle \) of order 2. If \( Sc = \langle \frac{\overline{h}}{2} \rangle \) then \( \phi(x) = -x + \frac{\overline{h}}{2}x \), where \( \frac{\overline{h}}{2}x \in Sc \) so that \( \phi(x) = -\overline{h} \). In the case \( e = q/2 + 1 \), \( Sc \) is contained in subgroup \( \langle 2h \rangle \) of order \( q/2 \). If \( Sc = \langle 2h \rangle \) then equation (42) holds trivially otherwise equation (42) fails. The case \( a = r - 1 = -1 \) mod \( r \) is eliminated unless \( |Sc| = 1, 2 \). Finally if \( q = 4 \) then the cases \( e = 1, 3 \) both lead to the solution \( Sc \cong \overline{S} \cong C_2 \).

Our remaining examples come from various lines of Table 4 where \( \Gamma_N = T(2, 2d) \) or a variant. The computer calculations in [7] show that the only possible overgroup \( \Gamma_A \) of \( \Gamma_N = T(2, d, 2d) \) for generic \( d \) is \( \Gamma_A = T(2, 3, 2d) \) with monodromy group \( M(\Gamma_A, \Gamma_N) = \Sigma_3 \).

**Example 57** Suppose that \( \Gamma_N = T(2, 2d, d) \) and that \( K = C_k, k > 2 \) as in Table 4. We first determine the possible \( N \) and then show that a full automorphism group \( A \) with \( M(A, N) = \Sigma_3 \) as described in Example 56 is not possible.
Table 4 line 5 we see that $d$ is odd $|C| = 2\frac{d}{k}$ that $N$ has an element order $2d = |K||C| = |N|$, thus $N$ is cyclic. By direct construction there are cyclic actions with signature $(d, 2d, 2)$ since $d$ is odd. Since $M(A, N) = \Sigma_3$ then $|\text{Core}_A(N)| = d$ and is a proper cyclic subgroup of $C$. But $\text{Core}_A(N)$ is a proper subgroup of $C$ of order $d$ and hence $\text{Core}_A(N) = \text{Core}_A(C)$ contradicting weak normality. Even dropping the assumption that $C$ is weakly malnormal, does not yield any larger groups at least in the case when we assume that $d$ is relatively prime to 6.

Example 58 Suppose that $\Gamma_N = T(2, 2d, d)$ and that $K = D_k$, $k > 2$ as given in line 9 of Table 4. We first determine the possible $N$ and then determine whether there can be a full automorphism group $A$ with $M(A, N) = \Sigma_3$ as described in Example 57. From Table 4 we see that $n = d = ek$ and $|N| = 2ek^2$. For simplicity’s sake let us assume that $k$ and $e$ are coprime odd numbers, coprime to $|\Sigma_3|$. Let us determine the various $K$-actions and $K$-fixed points on $X$ the set of generating vectors of $C$ defined in equation (26). The generating vector for $C$ has two elements of order $e$ in one $K$-orbit $O_1$ and $k$ elements of order $n$ in another $K$-orbit $O_2$. For the orbit $O_1$ the point stabilizers are $C_k$ so no condition is imposed and for points of $O_2$ the $z_i$ must be $\phi$-invariant. As the $z_i$ are generators $\phi$ must be trivial, and hence $D_k$ acts trivially on $C$. A typical $K$-invariant vector has the form $(g, g, h, \ldots, h)$, where $g$ has order $e$ and each of the $k$ repeats of $h$ has order $n$. Thus $h$ can be any of the $\phi(n)$ elements of order $n$. We must have $g^2h^k = 1$ or $g^2 = h^{-k}$. Since $h^{-k}$ has exact order $e$ then $g^2 = h^{-k}$ has a unique solution in $C_e$ since squaring is injective. Now let us find the possible groups $N$. According to our previous analysis the subgroup $E \subseteq N$ is isomorphic to $C \times C_k$. Since 2 divides $|N|$ there is an element $y \in N - E$ of order 2. Putting everything together $N \cong C_e \times C_2 \times (C_k \times C_k)$, with $y$ acting by $(z_1, z_2) \mapsto (z_1, -z_2)$ on $C_k \times C_k$. The subgroup $C_2 \times \{ (0, z) : z \in C_k \}$ is a subgroup of $N$ mapping onto $K$. If we let $C_e = \{ x \}$, $C_2 = \{ y \}$, $C_k \times C_k = \{ z_1 \times (z_2) \}$ then a generating $(2, 2d, d)$-vector is $(y, yxz_1z_2, (xz_1z_2)^{-1})$. For, $yxz_1z_2yxz_1z_2 = xz_1^2$ has order $n$, and $xz_1z_2$ and $x^2z_1^2$ generate $C_e \times (C_k \times C_k)$, so the vector generates all of $N$.

Now let us find $A$. Since $M(A, N) = \Sigma_3$, then $\text{Core}_A(N)$ is of index 2 in $N$, and so $\text{Core}_A(N) = E = C_e \times C_k \times C_k$. The subgroups $C_e$ and $C_k \times C_k$ are characteristic subgroups of $E$ and hence they are normal in $A$. The action of $\Sigma_3$ on $C_e$ is induced by the action of $\omega(g)$ and its conjugates, and $A$ acts trivially on $C_e$. Using Sylow theory and the fact that $k^2e$ and 6 are coprime we can show that $E \rightarrow A \rightarrow \Sigma_3$ is split and $A \cong C_e \times \Sigma_3 \times (C_k \times C_k)$. But the $A$-action cannot have a generating $(2, 3, 2d)$-vector because the image of the vector under the epimorphism $A \rightarrow C_e$ can only have signature $(1, 1, e)$, or $(1, 1, 1)$ a contradiction in both cases. Therefore, $N$ is the full automorphism group. In the Fermat case $e = 1$ and the contradiction is eliminated.

Example 59 Our last example will have a non-trivial action of $K$ on $C$. Suppose that $\Gamma_N = T(2, d, 2d)$ and that $K = D_k$, $k > 2$ as given in line 13 of Table
4. From Table 4 we see that \( d \) is even and \( 2d = ek \). There are many cases to consider depending on the parity of \( e \) and \( k \). We shall assume that \( k = 4l \), where \( e \) and \( l \) are odd coprime integers. The branching data of \( C \) collected into \( K \)-orbits is \((1^k,(el)^k,e^2)\). Upon permutation and dropping trivial generators we get a signature of \( (e,e,(el)^k) \), so \( n = el \), and \( C = C_e \times C_l \). Let \( (a,b,c) \) be a generating \( (2,2,k) \)-vector of \( K \), with \( a \) the stabilizer of a \( 1^k \) orbit, \( b \) the stabilizer of an \((el)^k \) orbit and \( c \) the stabilizer of an \( e^2 \) orbit. Then there is no restriction on \( a, b \) must fix an element of order \( n = el \) and \( c \) must fix an element of order \( e \).

A non-trivial action of \( K \) on \( C = C_e \times C_l \) satisfying the fixed point restriction is

\[
\begin{align*}
    a : (x,y) &\rightarrow (x, y^{-1}), \\
    b : (x,y) &\rightarrow (x, y), \\
    c : (x,y) &\rightarrow (x, y^{-1}).
\end{align*}
\]

Using Sylow subgroup analysis and previous techniques one can show that \( N \cong D_k \ltimes C \) with action defined above. Also as previously argued the \( N \)-action does not extend to a \((2,3,2d)\)-action.

References

[1] R.D.M. Accola, *Strongly Branched Coverings of Closed Riemann Surfaces*, Proc. Amer. Math. Soc. 26 (1970) 315–322.

[2] S.A. Broughton, *Full Automorphism Groups of Cyclic \( n \)-gonal Surfaces*, First of two talks in the UNED Geometry Seminar, February 2009, [http://www.rose-hulman.edu/˜brought/Epubs/UNED/UNED.html](http://www.rose-hulman.edu/˜brought/Epubs/UNED/UNED.html)

[3] S.A. Broughton, *Classification of Pairs of Fuchsian Groups*, Second of two talks in the UNED Geometry Seminar, March 2009, [http://www.rose-hulman.edu/˜brought/Epubs/UNED/UNED.html](http://www.rose-hulman.edu/˜brought/Epubs/UNED/UNED.html)

[4] S.A. Broughton, *Fuchsian Group Pairs I*, in preparation.

[5] S.A. Broughton, A. Wootton, *Finite Abelian Subgroups of the Mapping Class Group*, Algebraic & Geometric Topology 7 (2007) [http://msp.warwick.ac.uk/agt/2007/07/p066.xhtml](http://msp.warwick.ac.uk/agt/2007/07/p066.xhtml)

[6] S.A. Broughton, A. Wootton, *Topologically Unique Maximal Elementary Abelian Group Actions on Compact Oriented Surfaces*, Journal of Pure and Applied Algebra, 213 (2009) 557-572.

[7] S.A. Broughton, A. Wootton, *Full Automorphism Groups of Cyclic \( n \)-gonal Surfaces*, in preparation.

[8] S.A. Broughton, A. Wootton, *Quasiplatonic Cyclic \( n \)-gonal Surfaces*, in preparation.
[9] E. Bujalance, F.J. Cirre, M. Conder, *On extendability of Group Actions on Compact Riemann Surfaces*, Trans. Amer. Math. Soc. **355** (2003), 1537-1557.

[10] G. Gonzalez-Diez, *On Prime Galois Coverings of the Riemann Sphere*, Ann. Mat. Pura Appl. (4) **168** (1995), 1–15.

[11] G. Gonzalez-Diez, *Loci of Curves which are Prime Galois Coverings of $\mathbb{P}^1$*, Proc. London Math. Soc. (3) **62** (3)(1991) 469–489.

[12] W.J. Harvey, *Cyclic Groups of Automorphisms of a Compact Riemann Surface*, Quart. J. Math. Oxford Ser. (2) **17** (1966), 86-97.

[13] A. Kontogeorgis, *The Group of Automorphisms of Cyclic Extensions of Rational Function Fields*, J. Algebra **216** (1999), no. 2, 665–706.

[14] T. Shaska, *Determining the Automorphism Group of a Hyperelliptic Curve*, Proceedings of the 2003 international symposium on symbolic and algebraic computation.

[15] D. Singerman, *Subgroups of Fuchsian Groups and Finite Permutation Groups* Bull. London Math. Soc., **2** (1970), 319-323.

[16] D. Singerman, *Finitely Maximal Fuchsian Groups*, J. London Math. Society(2) **6**, (1972),17-32.

[17] A. Wootton, *Defining Equations for Cyclic Prime Covers of the Riemann Sphere*, Israel J.of Math,**157**,1 (2007).

[18] A. Wootton, *The Full Automorphism Group of a Cyclic $p$-gonal Surface*, Journal of Algebra, **312**, 1 (2007), 377-396.