A Frame Bundle Generalization of Multisymplectic Field Theories

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Abstract

This paper presents a generalization of symplectic geometry to a principal bundle over the configuration space of a classical field. This bundle, the vertically adapted linear frame bundle, is obtained by breaking the symmetry of the full linear frame bundle of the field configuration space, and it inherits a generalized symplectic structure from the full frame bundle. The geometric structure of the vertically adapted frame bundle admits vector-valued field observables and produces vector-valued Hamiltonian vector fields, from which we can define a Poisson bracket on the field observables. We show that the linear and affine multivelocity spaces and multiphase spaces for geometric field theories are associated to the vertically adapted frame bundle. In addition, the new geometry not only generalizes both the linear and the affine models of multisymplectic geometry but also resolves fundamental problems found in both multisymplectic models.

Keywords: symplectic geometry, multisymplectic geometry, frame bundle, Hamiltonian field theories, Poisson bracket.

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I. INTRODUCTION

Norris’s theory of n-symplectic geometry of the linear frame bundle has proven to be a powerful tool in the study of symplectic geometry and and the standard geometric model of Hamiltonian particle mechanics. In Norris’s theory, the canonical soldering form of the linear frame bundle of a manifold of particle configurations behaves as a vector-valued n-symplectic potential. The ensuing n-symplectic geometry not only generalizes the symplectic geometry of the cotangent bundle, but also provides information about “momentum frames” along the particle trajectories. The equations also give rise to Poisson algebras of tensorial classical observables.

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We find in the literature various linear and affine models of multisymplectic (multivariable symplectic) geometry on a covariant multiphase (multivariable phase) space of a classical field. There are two standard representations of the linear model of a multiphase space (Refs. 3 and 4), but Gotay has pointed out that both possess the same inherent flaw. Indeed, the linear model in either representation has a covariant multisymplectic form, but such a form is not naturally defined unless we arbitrarily specify an Ehresmann connection on the underlying field configuration space. The affine model suggested by Kijowski and refined by Gotay, et al., is an improvement because the affine multisymplectic structure can be defined intrinsically.

In the affine model, a variational principle applied to a given field Lagrangian would determine a unique Ehresmann connection. However, as the GIMMsy monograph demonstrates, the problem with the affine model is that the space of field momentum observables in the affine multiphase space is not closed under the naturally defined Poisson bracket. The generalized symplectic geometry developed in this paper addresses both the problem with the connection in the linear model and the problem with the bracket of momentum observables in the affine model.

In this paper we will show that a particular symmetry breaking of the linear frame bundle $LY$ of the field configuration space $Y$ produces a subbundle, the vertically adapted linear frame bundle $L_V Y$. The $n$-symplectic geometry on $LY$ pulled back by inclusion to $L_V Y$ generalizes both the linear and affine multisymplectic geometries. We will associate to $L_V Y$ the multivelocity spaces and the multiphase spaces for both the linear and the affine models. Then we can identify an Ehresmann connection on $Y$ with a symmetry breaking of $L_V Y$, thus addressing the shortcoming of the linear model. To solve the problem in the affine model, we construct a momentum observable from a projectable vector field on $Y$ and solve the generalized symplectic structure equation on $L_V Y$ to obtain a vector-valued Hamiltonian vector field on $L_V Y$. From the Hamiltonian vector fields we may define a Poisson bracket not only that generalizes the Poisson bracket on the momentum observables in the affine model but also that makes the vector space of momentum observables into an algebra under the bracket.

The format of this paper is as follows. In Sec. II, we summarize both the linear and affine multisymplectic geometries, and in Sec. III we summarize $n$-symplectic geometry. We construct a generalized symplectic geometry on $L_V Y$ in Sec. IV. Secs. V and VI show that the multivelocity spaces and multiphase spaces are associated to $L_V Y$. This work reveals the one-dimensional vector bundle structure of the affine multiphase space over the linear multiphase space. Sec. VII is devoted to the generation of the affine multisymplectic potential from the generalized symplectic potential of the vertically adapted frame bundle. This leads to the resolution of the problem with momentum observables.

II. MULTISYMPLECTIC GEOMETRY

This section summarizes the linear and the affine models of multisymplectic geometry. For an extensive bibliography see GIMMsy. Let $X$ be an oriented $n$-dimensional manifold and let $\pi_{XY} : Y \to X$ be a fiber bundle with a $k$-dimensional fiber. (Note: In general, $\pi_{BA}$ will denote a projection from $A$ onto $B$.) A classical field is a section of the field configuration space $Y$ over the parameter space $X$. From local coordinates $\{x^i\}$, $i = 1, \ldots, n$, on $X$ we
may construct local adapted coordinates \( \{x^i, y^A\}, i = 1, \ldots, n, A = 1, \ldots, k \) on \( Y \). Define the \textit{vertical subbundle} of \( TY \) to be

\[
V(TY) := \{w_y \mid y \in Y, w_y \in T_yY \text{ and } \pi_{XY^*}(w_y) = 0\},
\]

and define the \textit{linear multivelocity space} to be \( \text{Hom}_Y(TX, V(TY)) \), the vector bundle over \( Y \) whose fiber over \( y \in Y \) is the vector space \( \text{Lin}(T_{\pi_{XY}}(y)X, V(T_yY)) \). Define the \textit{linear multiphase space} \( J^*Y \) to be the vector bundle dual to the linear multivelocity space. We shall discuss two equivalent representations of \( J^*Y \), which we shall denote the Günther representation\(\text{[1]}\) and the Kijowski and Tulczyjew (KT) representation\(\text{[2]}\).

Each fiber of the vector bundle \( \text{Hom}_Y(TX, V(TY))^* \) is the linear dual to the corresponding fiber of \( \text{Hom}_Y(TX, V(TY)) \). Using the vector bundle isomorphism\(\text{[3]}\)

\[
\text{Hom}_Y(V(TY), TX) \to \text{Hom}_Y(TX, V(TY))^* : \alpha_y \mapsto \text{tr}(\alpha_y \circ \cdot),
\]

we obtain \( \text{Hom}_Y(V(TY), TX) \), the Günther representation of \( J^*Y \). From local adapted coordinates on \( Y \), define local canonical coordinates \( \{x^i, y^A, p_B^i\} \) on \( \text{Hom}_Y(V(TY), TX) \) where \( p_B^i(\alpha_y) := dx^i(\alpha_y(\frac{\partial}{\partial y_B^i})) \). There does not exist a natural multisymplectic potential on \( \text{Hom}_Y(V(TY), TX) \) that is analogous to the symplectic potential on the cotangent bundle. Indeed, \( \alpha_y \in \text{Hom}_Y(V(TY), TX) \) is defined only on vertical vectors in \( T_yY \). Choosing an Ehresmann connection on \( Y \) will remedy this problem\(\text{[4]}\). Let \( \gamma \) be an Ehresmann connection on \( Y \), represented as a projection \( \gamma : T_yY \to V(T_yY) \) fibered over \( Y \). Now define a \( TX \)-valued one-form \( \Theta_\gamma \) on \( J^*Y = \text{Hom}_Y(V(TY), TX) \) by

\[
\Theta_\gamma(\alpha_y) = \pi_{J^*Y}^*(\alpha_y \circ \gamma), \quad \alpha_y \in J^*Y.
\]

Günther\(\text{[1]}\) calls \( \Theta_\gamma \) a \textit{canonical polysymplectic form}, although its definition requires an object extrinsic to the natural geometry of \( J^*Y \). We shall call \( \Theta_\gamma \) a \textit{linear multisymplectic potential} on \( J^*Y \) in the Günther representation because \( \Theta_\gamma \) generates a linear multisymplectic geometry. In local coordinates, using the summation convention,

\[
\Theta_\gamma = \left(p_A^i dy^A + p_B^i \gamma_A^j dx^j\right) \otimes \frac{\partial}{\partial x^i}.
\]

The KT representation of \( J^*Y \) is \( \text{Hom}_Y(V(TY), \wedge^{n-1}X) \) where \( \wedge^{n-1}X \) is the bundle of \((n-1)\)-forms on \( X \), and the values of sections of \( \text{Hom}_Y(V(TY), \wedge^{n-1}X) \) are vector-densities\(\text{[2]}\). In local coordinates on \( X \) define

\[
\begin{align*}
d^n x &:= dx^1 \wedge \cdots \wedge dx^n, \\
d^{n-1} x_i &:= \frac{\partial}{\partial x^i} \bigwedge d^n x, \\
d^{n-2} x_{ij} &:= \frac{\partial}{\partial x^i} \bigwedge \left(\frac{\partial}{\partial x^j} \bigwedge d^n x\right), \quad \text{etc.,}
\end{align*}
\]

where \( \bigwedge \) denotes the inner product of a vector with a differential form. Local coordinates on \( \text{Hom}_Y(V(TY), \wedge^{n-1}X) \) are \( \{x^i, y^A, p_B^i\} \), where

\[
p_B^i(\alpha_y) = (-1)^{2-i} \frac{\partial}{\partial x^i} \bigwedge \frac{\partial}{\partial x^j} \bigwedge \cdots \bigwedge \frac{\partial}{\partial x^k} \bigwedge \left(\alpha_y(\frac{\partial}{\partial y_B^i})\right).
\]
and \( \overrightarrow{\partial} \) denotes omission of \( \frac{\partial}{\partial x_i} \). So the multisymplectic potential on \( J^*Y \) in the KT representation in local coordinates is

\[
\Theta_\gamma = \left( p_A^i dy^A + p_A^j \gamma_j^A dx^j \right) \otimes d^{n-1} x_i .
\]

Given a volume form \( \omega \) on \( X \), we may use the vector space isomorphism \( T_x X \rightarrow \Lambda_x^{n-1} X \) : \( v \mapsto v \downarrow \omega \) to define a vector bundle isomorphism from the Günther representation of \( J^*Y \) to the KT representation. In either representation \( d\Theta_\gamma \) is nondegenerate in the sense that for a vector field \( X \), \( X \downarrow d\Theta_\gamma = 0 \) if and only if \( X = 0 \). We conclude that \( (J^*Y, d\Theta_\gamma) \) is a connection-dependent linear multisymplectic manifold.

Ragionieri and Ricci use a hybrid representation of \( J^*Y \) using \( TX \otimes_Y V^*Y \), which is isomorphic to \( \text{Hom}_Y(\mathcal{V}(TY), TX) \) as a vector bundle, but they assume a volume form on \( X \) and obtain a “generalized Liouville n-form” on \( TX \otimes_Y V^*Y \). This form is also dependent upon the choice of a connection.

To avoid requiring a connection, we must reconsider the multivelocity space. Ragionieri and Ricci claim that the appropriate multivelocity space is not \( \text{Hom}_Y(\mathcal{V}(TY), TX) \) but rather, the (first-order) jet bundle \( JY \), the affine bundle over \( Y \) whose fiber over \( y \in Y \) consists of linear maps \( \gamma_y : T_{\pi Y}(y)X \rightarrow T_y Y \) satisfying \( \pi_{XY*} \circ \gamma_y = \text{Id}_{T_{\pi Y}(y)X} \). The affine structure of \( JY \) comes from a difference function \( \delta : JY \times JY \rightarrow \text{Hom}_Y(TX, \mathcal{V}(TY)) \).

Recall that a section of \( JY \) over \( Y \) can be identified with an Ehresmann connection on \( \pi_{XY} : Y \rightarrow X \).

The bundle of affine cojets is the vector bundle \( J^*Y \) over \( Y \) whose fiber over \( y \in Y \) is the set of affine maps from \( J_y Y \) to \( \wedge^n \pi_{XY*}(y)X \). It follows that \( \dim J^*Y = \dim JY + 1 \). We find an equivalent description of \( J^*Y \) to be useful. The affine multiphase space \( Z \) is the bundle of \( n \)-forms on \( Y \) whose fiber \( Z_y \) over \( y \) is

\[
Z_y := \{ z \in \wedge^n_y Y | v \downarrow w \downarrow z = 0 \ \forall v, w \in \mathcal{V}(T_y Y) \} .
\]

The affine multiphase space, originally defined by Kijowski, admits an \( n \)-form, \( \Theta_\gamma = \pi_{YZ}(z) \), which is the pullback via inclusion of the canonical \( n \)-form on \( \wedge^n Y \). (Kijowski attributes the notion of a multiphase space to an unpublished result of Tulczyjew.) GIMMSy proves that \( Z \) is “canonically” isomorphic to \( J^*Y \). On \( Z \), we can define local coordinates \( \{ x^i, y^A, p_B, p \} \) where

\[
p(z) = \frac{\partial}{\partial x^n} \downarrow \cdots \downarrow \frac{\partial}{\partial x^1} \downarrow z \quad \text{and} \quad p(z) = (-1)^{j-1} \frac{\partial}{\partial x^n} \downarrow \frac{\partial}{\partial x^{n-1}} \downarrow \cdots \downarrow \frac{\partial}{\partial x^j} \downarrow \cdots \downarrow \frac{\partial}{\partial x^1} \downarrow \frac{\partial}{\partial y_B} \downarrow z .
\]

So in local coordinates

\[
\Theta = p_A^i dy^A \wedge d^{n-1} x_i + pd^n x
\]

where \( d^n x \) and \( d^{n-1} x_i \) given in (2.1) are pulled up from \( X \) to \( Z \). The \((n+1)\)-form \( d\Theta \) is nondegenerate, so the pair \((Z, d\Theta)\) is an affine multisymplectic manifold. A connection is not required to define the affine multisymplectic structure.

Let \( \mathcal{X} Y \) be the Lie algebra of vector fields on \( Y \). Denote the space of vector fields of \( Y \) projectable to \( X \) by \( \mathcal{X}_{\pi_{YX}} Y \). Note that \( \mathcal{X}_{\pi_{YX}} Y \) is a Lie subalgebra of \( \mathcal{X} Y \), since \( [\pi_{XY*} v, \pi_{XY*} w] = \pi_{XY*} [v, w] \). (That is, the Lie brackets are \( \pi_{XY}\)-related.)
Definition Let $v \in \mathcal{X}_{\text{proj}} Y$. A momentum observable based on $v$ is an $(n-1)$-form $f_v$ on $Z$ defined by

$$f_v(z) := \pi^*_Y Z(v \lrcorner z).$$

Let $T^1(Z)$ denote the vector space of momentum observables.

If in local adapted coordinates on $Y$, $v = v^i(x^j) \frac{\partial}{\partial x^i} + v^A(x^j, y^B) \frac{\partial}{\partial y^A}$, then in local coordinates on $Z$,

$$f_v(z) = (p^i_A v^A + pv^i) d^{n-1} x_i - p^i_A v^i dy^A \wedge d^{n-2} x_{ij}. \quad (2.4)$$

The Hamiltonian vector field $X_{f_v}$ is obtained from $f_v$ via the multisymplectic structure equation

$$df_v = -X_{f_v} d\Theta. \quad (2.5)$$

The local coordinate expression for $X_{f_v}$ is

$$X_{f_v} = v^k \frac{\partial}{\partial x^k} + v^A \frac{\partial}{\partial y^A} + \left(\frac{p^i_A \partial v^i}{\partial x^j} - \frac{p^i_j \partial v^i}{\partial x^A} - \frac{p^i_B \partial v^i}{\partial y^A}\right) \frac{\partial}{\partial p^j_A} - \left(\frac{p^i \partial v^i}{\partial x^i} + \frac{p^i_A \partial v^i}{\partial x^A}\right) \frac{\partial}{\partial p}. \quad (2.6)$$

From equation (2.6) and Ref. 11, if $v, w \in \mathcal{X}_{\text{proj}} Y$ then $\pi^*_Y Z X_{f_v} X_{f_w} = [v, w]$. From (2.3), (2.4) and (2.6),

$$X_{f_v} \lrcorner \Theta = f_v. \quad (2.7)$$

Using (2.7),

$$[X_{f_v}, X_{f_w}] \lrcorner \Theta = \pi^*_Y Z ([v, w] \lrcorner z) = f_{[v, w]}(z). \quad (2.8)$$

Using equations (2.3) and (2.7) and the Lie derivative identity,

$$\mathcal{L}_X \alpha = X \lrcorner d\alpha + d(X \lrcorner \alpha) \quad (2.9)$$

it follows that

$$\mathcal{L}_{X_{f_v}} \Theta = 0. \quad (2.10)$$

Definition Let $f_v$ and $f_w \in T^1(Z)$ and let $X_{f_v}$ and $X_{f_w}$ be their corresponding Hamiltonian vector fields, respectively. Define the Poisson bracket of $f_v$ and $f_w$ to be

$$\{f_v, f_w\} := -X_{f_v} \lrcorner (X_{f_w} \lrcorner d\Theta).$$

The Poisson bracket is not a true Poisson bracket because there lacks an associative multiplication of $(n-1)$-forms on which the bracket acts as a derivation. Using $\mathcal{L}_{X_{f_v}} \Theta = 0$, equations (2.8), (2.9), (2.10), and the identity,

$$[X, Y] \lrcorner \alpha = \mathcal{L}_X (Y \lrcorner \alpha) - Y \lrcorner (\mathcal{L}_X \alpha),$$

we obtain

$$\{f_v, f_w\} = f_{[v, w]} - d(X_{f_v} \lrcorner (X_{f_w} \lrcorner \Theta)). \quad (2.11)$$

The exact form on the right side of equation (2.11) is not in $T^1(Z)$ because from equation (2.3) the Hamiltonian vector field of an exact form is the zero vector field on $Z$, but the momentum observable corresponding to the zero vector field is the zero $(n-1)$-form.
on $Z$. Thus, the space $T^1(Z)$ is not closed under the Poisson bracket. Equation (2.11) explains the remark in GIMMsy, that the Poisson bracket of two momentum observables “is up to the addition of exact terms, another momentum observable.”

Definitions

- The Lie algebra of locally Hamiltonian vector fields on $Z$ is
  
  $$LHV^1(Z) := \{ X \in \mathcal{X}Z \mid L_X d\Theta = 0 \}.$$

- The vector space of allowable Hamiltonian observables on $Z$ is
  
  $$HF^1(Z) := \{ f \in \wedge^{n-1}Z \mid df = -X \lrcorner d\Theta, X \in LHV^1(Z) \}.$$

- The vector space of Hamiltonian vector fields on $Z$ is
  
  $$HV^1(Z) := \{ X \in \mathcal{X}Z \mid X \lrcorner d\Theta = -df, f \in HF^1(Z) \}.$$

By equation (2.5), a momentum observable on $Z$ is an allowable Hamiltonian observable. Define the Euler vector field $E^{13}$ on $Z$ by $E \lrcorner d\Theta = \Theta$. Because $d\Theta$ is nondegenerate, $E$ is well-defined. If $f \in T^1(Z)$, then, using (2.5) and (2.7),

$$E \lrcorner df = E \lrcorner ( -X_f \lrcorner d\Theta ) = X_f \lrcorner \Theta = f.$$  

(2.12)

Define a linear operator $\tilde{E}$ on $\wedge^{n-1}Z$ by $\tilde{E}(f) := E \lrcorner df$. When restricted to $HF^1(Z)$, $\tilde{E}$ is a projection operator. If $f \in T^1(Z)$ then $\tilde{E}(f) = f$ by (2.12). If $n = 2$ and $k = 2$ or if $n \geq 3$ then $HF^1(Z)$ is an algebra under the Poisson bracket extended to $HF^1(Z)$ and the image of $HF^1(Z)$ under $\tilde{E}$ is $T^1(Z)$. Thus,

$$HF^1(Z) = T^1(Z) \oplus \text{Ker}(\tilde{E}|HF^1(Z)).$$

Furthermore, the nonzero exact $(n-1)$-forms on $Z$ are in $\text{Ker}(\tilde{E}|HF^1(Z))$, so $T^1(Z)$ is a vector subspace of $HF^1(Z)$, but by equation (2.11), $T^1(Z)$ is not a subalgebra of $HF^1(Z)$. Define a representation $HF^1(Z) \rightarrow HV^1(Z) : f \mapsto X_f$ and it follows that

$$HV^1(Z) \simeq HF^1(Z)/\{\text{closed (n-1)-forms}\}.$$  

(2.13)

If $n = 1$ then $Z = T^*Y$, $HF^1(Z) = C^\infty(T^*Y)$, the Poisson bracket is the usual Poisson bracket on the cotangent bundle $T^*Y$, and $HV^1(Z) = HF^1(Z)/\mathbb{R}$.

III. $n$-SYMPLECTIC GEOMETRY ON THE LINEAR FRAME BUNDLE

This section is a summary of Norris’s theory of $n$-symplectic geometry on the linear frame bundle. Let $M$ be an $n$-dimensional manifold and let $\tau : LM \rightarrow M$ be the bundle of linear frames of $M$. That is,

$$LM := \{(x,e_i) \mid x \in M, \{e_i\} \text{is a frame of } T_xM\}$$
The structure group of \( LM \) is the real general linear group \( GL(n) \), which acts freely on the right of \( LM \). Also, \( LM \) supports an \( R^n \)-valued soldering one-form \( \theta \), defined by

\[
\theta(Y) := u^{-1}(\tau_rY) \quad \forall \ Y \in T_uLM,
\]

where if \( \{ r_i \}, i = 1, 2, \ldots, n, \) denotes the standard basis of \( R^n \), then \( u = (x, e_i) : R^n \to T_{\tau(u)}M \) is the linear isomorphism \( \xi^i r_i \mapsto \xi^i e_j \). Compare \( \theta = \theta^i r_i \) to the real-valued canonical one-form \( \theta \) on \( T^*M \). The \( R^n \)-valued two-form \( d\theta \) on \( LM \) is closed and nondegenerate, so \( d\theta \) is considered to be an \( R^n \)-valued symplectic structure, or an \( n \)-symplectic structure on \( LM \).

The theory of \( n \)-symplectic geometry on \((LM, d\theta)\) is based on the generalized structure equation

\[
d\hat{f}^{i_1 i_2 \ldots i_p} = -p! X^{(i_1 i_2 \ldots i_{p-1}} \bigg] d\theta^{i_p),}
\tag{3.1}
\]

where the functions \( \hat{f}^{i_1 i_2 \ldots i_p} \) are the components of a \( \otimes^p R^n \)-valued function \( \hat{f} \), and the vector fields \( X^{i_1 i_2 \ldots i_{p-1}} \) are the components of a \( \otimes^{p-1} R^n \)-valued vector field \( X_f \). Because for \( g \in GL(n) \) the \( n \)-symplectic potential \( \theta \) transforms tensorially under right translations.

Hence \( n \)-symplectic geometry selects classes of allowable observables, in contrast with the fact that all smooth real-valued functions on \( T^*M \) are allowable observables.

The allowable observables in \( n \)-symplectic geometry divide naturally into the symmetric and antisymmetric Hamiltonian functions, which we denote by \( SHF \) and \( AHF \), respectively. The space \( SHF \) is the direct sum \( \bigoplus_{p=1}^{\infty} SHF^p \) where \( SHF^p \) is the space of \( \otimes^p R^n \)-valued functions defined on \( LM \) that are compatible with a symmetrized version of (3.1) and \( \otimes_s \) denotes the symmetric tensor product. The elements of \( SHF^p \) are degree \( p \) polynomials in the generalized local momentum coordinates on \( LM \) with coefficients that are constant on the fibers of \( LM \). In particular, the elements of \( SHF^p \) whose local coordinate representatives are homogeneous polynomials are in bijective correspondence with symmetric degree \( p \) contravariant tensor fields on \( M \) and are precisely the tensorial elements of \( SHF^p \). We call this subset \( ST^p \) (for \textbf{S}ymmetric and \textbf{T}ensorial) and define \( ST = \bigoplus_{p=1}^{\infty} ST^p \).

For \( ST^p, p \geq 1 \), equation (3.1) is replaced by

\[
d\hat{f}^{i_1 i_2 \ldots i_p} = -p! X^{(i_1 i_2 \ldots i_{p-1}} \bigg] d\theta^{i_p)} \tag{3.2}
\]

where the parentheses denote symmetrization over the enclosed indices. For each \( \hat{f} \in ST^p \) this equation determines an equivalence class of \( \binom{n+p-2}{p-1} \) vector fields. If \( p = 1 \) then the equivalence relation is equality. If \( p \geq 2 \) then the symmetrization on indices in (3.2) introduces a degeneracy which is not present for \( p = 1 \). More precisely, for each \( \hat{f} \in ST^p \) and each \( u \in LM \) there exists a local section \( X_u^{i_1 \ldots i_{p-1}} = X^{i_1 \ldots i_{p-1}}(r_{i_1} \otimes_s \cdots \otimes_s r_{i_{p-1}}) \) of the vector bundle \( T(LM) \otimes (\otimes_s)^{p-1} R^n \to LM \) defined on an open subset \( U_u \) of \( LM \) such that the vector fields \( X_u^{i_1 \ldots i_{p-1}} \) satisfy (3.2) for all \( (i_1, i_2, \ldots, i_p) \). In fact there exists more than one such local section since

\[
K_v = \{ Y_v | Y_v^{(i_1 \ldots i_{p-1}} \bigg] d\theta^{i_p)} = 0 \text{ for all } (i_1, i_2, \ldots, i_p) \}
\]
is a nontrivial subspace of $T_v(LM) \otimes \mathbb{R}^n$ for each $v \in LM$. Indeed, $K = \bigcup_{v \in LM} (K_v)$ is a vector subbundle of $T(LM) \otimes \mathbb{R}^n$ and for each $\hat{f} \in ST^p$ there exists a unique global section $\sigma$ of $(T(LM) \otimes \mathbb{R}^n)/K \rightarrow LM$ such that for each $u \in LM$ there exists a neighborhood $U$ of $u$ and a local section $X_\hat{f}$ of $T(LM) \otimes \mathbb{R}^n \rightarrow U$ having the property that $X_\hat{f}$ satisfies (3.2) and $\sigma(v) = X_\hat{f}(v) + K_v$ for each $v \in U$. Denote $\sigma$ by $[X_\hat{f}]$. If we fix $I = (i_1, \ldots, i_{p-1})$ then there is also a subbundle $K^I$ of $(T(LM))/K^I \rightarrow LM$ such that for $v \in LM$,

\[ K^I_v = \{ V_\nu^{i_1 \ldots i_{p-1}} \mid \text{d}\nu^{i_1 \ldots i_{p-1}} = 0 \text{ for all } i_p \}. \]

Moreover there is a unique section $\sigma^I$ of $(T(LM))/K^I \rightarrow LM$ such that for $u \in LM$, there is a neighborhood $U$ of $u$ and a local section $X^I_\hat{f}$ of $T(LM) \rightarrow U$ with the property that $X^I_\hat{f}$ satisfies equation (3.2) and $\sigma^I(v) = X^I_\hat{f}(v) + K^I_v$ for all $v \in U$. We denote this section $\sigma^I$ by $[X^I_\hat{f}] = [X^I_\hat{f}]_{i_1 \ldots i_{p-1}}$.

The fact that elements of $ST^p$ determine equivalence classes of vector fields does not affect the basic algebraic structures in $n$-symplectic geometry. For each $p \geq 1$ the set of equivalence classes of $\otimes^p \mathbb{R}^n$-valued vector fields on $LM$ forms an infinite-dimensional vector space. Denote by $HV(ST^p)$ the vector space of $\otimes^p \mathbb{R}^n$-valued equivalence classes of vector fields determined by elements of $ST^p$ by equation (3.2). For $f \in ST^p$ and $\hat{g} \in ST^q$ define the Poisson bracket $\{ , \} : ST^p \times ST^q \rightarrow ST^{p+q-1}$ by

\[ \{ X_f^{i_1 i_2 \ldots i_{p+q-1}} \} = p! X_f^{(i_1 i_2 \ldots i_{p-1})} \left( \hat{g}_{i_1 i_2 \ldots i_{p+q-1}} \right) \]

where $X_f^{i_1 i_2 \ldots i_{p-1}}$ is any representative of the equivalence class $[X_f]_{i_1 i_2 \ldots i_{p-1}}$. The bracket defined in (3.3) is easily shown to be independent of the choice of representatives and it has all the properties of a Poisson bracket. In fact when the bracket in (3.3) is re-expressed on the base manifold $M$, it gives the differential concomitant of Schouten and Nijenhuis of the symmetric tensor fields corresponding to $\hat{f}$ and $\hat{g}$.

**Theorem 3.1** (Norris) The space $ST$ of symmetric tensorial functions on $LM$ is a Poisson algebra with respect to the Poisson bracket defined in (3.3).

Denote the direct sum of the vector spaces $HV(ST^p)$ by $HV(ST)$. There is a well-defined Lie bracket on the vector space $HV(ST)$ that satisfies

\[ [[X_f], [X_g]] = [X_{\{f, g\}}] \]

A degree one tensorial observable $\hat{f} \in T^1(LM) = ST^1$ corresponds to a unique vector field $f$ on $M$. In local canonical coordinates \{x^i, \pi^i_k := \pi^j \frac{\partial}{\partial x^j}\} on $LM$ we may write $\hat{f}^i = f^i(x)\pi^i_j$ where $f^i(x)\pi^i_j = f^i(x) \frac{\partial}{\partial x^j}$, and $x \in M$. For $p = 1$ equation (3.2) now has a unique solution $X_f \in HV(ST^1)$, given in local coordinates by

\[ X_f = f^i(x) \frac{\partial}{\partial x^i} - \frac{\partial f^i}{\partial x^j} \pi^j_k \frac{\partial}{\partial \pi^k}. \]

The vector field $X_f$ is the natural lift of the vector field $f$ to $LM$. See Ref. 1 for discussions of the full Poisson algebra $SHF$ and of the graded Poisson algebra $AHF$. 

8
All of the basic features of symplectic geometry on $T^*M$ are induced from the $n$-
symplectic geometry on $LM$. Indeed, it is well known that $T^*M$ is isomorphic to
the associated bundle $LM \times_{GL(n)} \mathbb{R}^{n^*}$. Furthermore, the relationship between
the canonical one-form $\theta$ on $T^*M$ and the soldering one-form $\theta$ on $LM$ is

$$\theta_{[(u, \alpha)]}(\tilde{X}) = \langle \theta_u(X), \alpha \rangle$$

where $u \in LM$, $[(u, \alpha)] \in T^*M \cong LM \times_{GL(n)} \mathbb{R}^{n^*}$ and $\tilde{X}$ is a tangent vector at $[u, \alpha]$ that
projects to the same vector as the tangent vector $X$ at $u$. Thus the symplectic potential $\phi$
for symplectic geometry on $T^*M$ is induced from the soldering one-form $\theta$ on $LM$. Moreover,
if for some nonzero $\alpha \in \mathbb{R}^{n^*}$ we define a map

$$\phi_\alpha : LM \to T^*M : u \mapsto [u, \alpha],$$

then the range of $\phi_\alpha$ is $T^*M$ with the zero-section deleted, and thus (3.4) becomes

$$\phi_\alpha^* \theta = \langle \theta, \alpha \rangle.$$ (3.5)

Also the homogeneous degree $p$ polynomial observables on $T^*M$ are induced from elements
of $ST^p$. Indeed, for $\tilde{f} \in ST^p$ and $u \in LM$, define

$$\tilde{f} : T^*M \to \mathbb{R} : [u, \alpha] \mapsto \langle \tilde{f}(u), \widehat{(\alpha, \ldots, \alpha)} \rangle.$$ (3.6)

Because $\tilde{f}$ is tensorial, $\tilde{f}$ is well defined. In local coordinates, $\pi^i_j(x, e_k)\alpha_i = e^i(\frac{\partial}{\partial x^j})\alpha_i = p_j(e^i\alpha_i)$ where $\{p_j\}$ are the standard local momentum coordinates on $T^*M$. For example,
if $p = 2$ and $\tilde{f} = \tilde{f}^{kl}(x)\pi^1_k \pi^2_l \otimes s r_j \in ST^2$ then

$$\tilde{f}([u, \alpha]) = f^{kl}(x)p_k p_l.$$

**Theorem 3.2 (Norris)** Let $\tilde{f} \in ST^p$, let $\mathbb{F}_{X_f}$ be the associated equivalence class of Hamiltonian vector fields determined by (3.3), and let $\tilde{f}$ be the degree $p$ homogeneous polynomial observable on $T^*M$ determined by $\tilde{f}$ as in (3.6). Then

$$X = p!\phi_{\alpha^*}(X_f^{i_1 i_2 \cdots i_{p-1}} \alpha_i \alpha_i \cdots \alpha_{i_{p-1}})$$

where $X_f^{i_1 i_2 \cdots i_{p-1}}$ denotes any set of representatives of $[X_f]$, $X$ is a vector field on $T^*M$
with the zero-section of $T^*M$ deleted, and $X = X_f$.

**IV. THE VERTICALLY ADAPTED LINEAR FRAME BUNDLE**

We now present a new principal bundle generalization of multisymplectic geometry. From Norris’s theory applied to the $(n + k)$-dimensional field configuration space $\pi_{XY} : Y \to X$,
the linear frame bundle $LY$ over $Y$ has an $(n + k)$-symplectic geometry.

**Definition** The vertically adapted frame bundle $L_VY$ is defined as

$$L_VY := \{(y, \{\epsilon_i, e_A\}) \in LY \mid \{\epsilon_A\} \text{ is a frame of } V(T_y Y)\}.$$
The terminology is motivated by the definition of an adapted frame. Again, as in Sec. II, $i = 1, \ldots, n$ and $A = 1, \ldots, k$.

Let $M_k(R^{n+k})$ denote the $k$th Grassmann manifold of $R^{n+k}$. A point in $M_k(R^{n+k})$ is the span of a set of $k$ linearly independent vectors $\{w_A\}$ in $R^{n+k}$. Define a left action of $GL(n+k)$ on $M_k(R^{n+k})$ by

$$g \cdot \text{span}\{w_A\} := \text{span}\{g^B_A w_B\}. \quad (4.1)$$

Let $i_1 : R^n \rightarrow R^{n+k} : v \mapsto \hat{v}$ be the inclusion into the first $n$ slots of $R^{n+k}$ and let $i_2 : R^k \rightarrow R^{n+k} : w \mapsto \hat{w}$ be the inclusion into the last $k$ slots. Let $\{r_i\}$ be the standard basis of $R^n$ and let $\{s_A\}$ be the standard basis of $R^k$. Then the isotropy subgroup of span{$\hat{s}_A$} expressed in a matrix representation with respect to the standard basis of $R^{n+k}$ is the adapted linear group,

$$G_A := \left\{ \begin{pmatrix} N & 0 \\ A & K \end{pmatrix} \mid N \in GL(n), K \in GL(k), A \in R^{k \times n} \right\}.$$  

The group $G_A$ is a semidirect product of $GL(n) \times GL(k)$ and $R^{k \times n}$. For convenience we write $\begin{pmatrix} N & 0 \\ A & K \end{pmatrix} \in G_A$ as $(N, K, A)$. Since the action of $GL(n+k)$ on $M_k(R^{n+k})$ is transitive, we have proven that $M_k(R^{n+k})$ is diffeomorphic to $GL(n+k)/G_A$.

**Theorem 4.1** Define the map

$$\phi : LY \rightarrow M_k(R^{n+k}) : w \mapsto w^{-1}(V(T_{\pi_Y Y}(w)Y))$$

where $w \in LY$ is a linear isomorphism $w : R^{n+k} \rightarrow T_{\pi_Y Y}(w)Y$. Then $\phi$ is a symmetry-breaking map and $L_Y Y = \phi^{-1}(\text{span}\{\hat{s}_A\})$. Thus, $L_Y Y$ is a symmetry-broken subbundle of $LY$ with structure group $G_A$.

**Proof** The action in (4.1) is transitive and $\phi$ is tensorial. Indeed,

$$\phi(w \cdot g) = (w \cdot g)^{-1}(V(T_{\pi_Y Y}(w)Y)) = g^{-1}(w^{-1}(V(T_{\pi_Y Y}(w)Y))) = g^{-1}(\phi(w)).$$

So, $\phi$ is a symmetry-breaking map. Finally, if $w = (y, E_\mu) \in L_Y Y$, $\mu = 1, \ldots, n + k$, then $\phi(w) = \text{span}\{\hat{s}_A\}$ if and only if $V(T_y Y) = w(\text{span}\{\hat{s}_A\}) := \text{span}\{w(\hat{s}_A)\}$. But $w(\hat{s}_A) = (y, E_\mu)(\hat{s}_A) = E_{n+\mu}$. So $w = (y, E_\mu) \in \phi^{-1}(\text{span}\{\hat{s}_A\})$ if and only if $V(T_y Y) = \text{span}\{E_{n+\mu}\}^{A=1}_{A=1}$ if and only if $w \in L_Y Y$. Thus, the structure group of $L_Y Y$ is the isotropy subgroup of $\text{span}\{\hat{s}_A\}$. □

**Definition** The bundle of vertical frames of $Y$, denoted $VF(Y)$, is defined by

$$VF(Y) := \{(y, \epsilon_A) \mid y \in Y, \{\epsilon_A\} \text{ is a frame of } V(T_Y Y)\}.$$  

The bundle $VF(Y) \rightarrow Y$ is a principal fiber bundle with structure group $GL(k)$. Let $\tau : LX \rightarrow X$ be the linear frame bundle over $X$. Using $\tau$ we construct first the pullback
bundle $\tau^*Y$ over $X$ and then the pullback bundle $\tau^*VF(Y)$ over $Y$. See diagram (4.2).

The elements of $\tau^*VF(Y)$ are of the form $(\pi_{XY}(y), y, f_i, \epsilon_A)$ where $(\pi_{XY}(y), f_i) \in LX$ and $(y, \epsilon_A) \in VF(Y)$.

$$\tau^*VF(Y) \rightarrow VF(Y)$$

$$\downarrow \downarrow$$

$$\tau^*Y \rightarrow Y$$

$$\downarrow \downarrow \pi_{XY}$$

$$LX \xrightarrow{\tau} X$$

Represent $GL(n) \times GL(k)$ as a subgroup of $GL(n + k)$, identifying the ordered pair $(N, K)$ with the matrix $\left( \begin{array}{ll} N & 0 \\ 0 & K \end{array} \right)$. Observe that $GL(n) \times GL(k)$ is a subgroup of $G_A$.

Define the right action of $GL(n) \times GL(k)$ on $\tau^*VF(Y)$ by

$$(\pi_{XY}(y), y, f_i, \epsilon_A) \cdot (N, K) = \{(\pi_{XY}(y), y, N^j_i, \epsilon_B K_A^B)\}.$$  \hspace{1cm} (4.3)

The action in (4.3) is free and we see from diagram (4.2) that $\tau^*VF(Y)$ is locally trivial over $Y$. Thus $\pi_{XY} : \tau^*VF(Y) \rightarrow Y$ is a principal fiber bundle with structure group $GL(n) \times GL(k)$.

The bundle $L_V Y$ projects smoothly to $\tau^*VF(Y)$ by the map

$$\eta : L_V Y \rightarrow \tau^*VF(Y) : (y, \{e_i, \epsilon_A\}) \mapsto (\pi_{XY}(y), y, \pi_{XY*}(e_i), \epsilon_A).$$

**Lemma 4.2** The bundle $\eta : L_V Y \rightarrow \tau^*VF(Y)$ is a principal fiber bundle with structure group $(R^{k \times n}, +)$. Furthermore, the projection map $\eta$ is $GL(n) \times GL(k)$-equivariant.

**Proof** The right action of $G_A$ on $L_V Y$ is given by

$$(y, \{e_i, \epsilon_A\}) \cdot (N, K, A) := (y, \{e_j N^j_i + \epsilon_B A^B_K, \epsilon_B K_A^B\})$$ \hspace{1cm} (4.4)

and this action is clearly free. Any two points in $\eta^{-1}(x, y, f_i, \epsilon_A)$ are related by a unique translation by an element of $(R^{k \times n}, +)$ expressed as the subgroup $\{I_n, I_k, W\}, W \in R^{k \times n}$, of $G_A$. Indeed, if $\eta(y, \{e_i, \epsilon_A\}) = \eta(y', \{e_i', \epsilon'_{A}\})$ then $y = y'$, $\epsilon_A = \epsilon'_{A}$, and $\pi_{XY*}e_i = \pi_{XY*}e_i$. Thus the unique element of $G_A$ that maps $(y, \{e_i, \epsilon_A\})$ to $(y, \{e_i', \epsilon'_{A}\})$ is $(I_n, I_k, A)$, so $e_i' = e_i + A^A e_A$. Also, a local trivialization $\eta^{-1}(\pi_{XY}^{-1}(U)) \simeq \pi_{XY}^{-1}(U) \times R^{k \times n}$ can be obtained in some open neighborhood $U$ of $y \in Y$. Finally, using (4.3) and (4.4),

$$\eta((y, \{e_i, \epsilon_A\}) \cdot (N, K, 0)) = \eta(y, \{e_i, \epsilon_A\}) \cdot (N, K). \hspace{1cm} \Box$$

(4.5)

Let $(y, \{e^i, e^A\})$ be the coframe dual to the vertically adapted frame $(y, \{e_i, \epsilon_A\})$. We may define the right action of $G_A$ on $(y, \{e^i, \epsilon^A\})$ by

$$(y, \{e^i, \epsilon^A\}) \cdot (N, K, A) = (y, \{(N^{-1})^j_i e^i, -(K^{-1} A N^{-1})^j_i e^j + (K^{-1})^A_B e^B\}).$$

(4.6)
The coframe in (4.6) is dual to the frame \((y, \{e, \epsilon_A\}) \cdot (N, K, A)\).
Define a left action of \(G_A\) on \(R^{k \times n}\) by
\[
(N, K, A) \cdot W := KWN^{-1} - AN^{-1}.
\] (4.7)
The action in (4.7) transitive and the isotropy subgroup of \(\{0\}\) is \(GL(n) \times GL(k)\).

\textbf{Lemma 4.3} \(\tau^*VF(Y) \times R^{k \times n} \to Y\) is a principal fiber bundle with structure group \(G_A\).

\textbf{Proof} First, define a right action of \(G_A\) on \(\tau^*VF(Y) \times R^{k \times n}\) by
\[
(u, W) \cdot (N, K, A) = \left( u \cdot (N, K), (N, K, A)^{-1} \cdot W \right),
\] (4.8)
where the action on \(\tau^*VF(Y)\) is given by (4.3) and the \(G_A\)-action on \(R^{k \times n}\) is given by (4.7).
If \((u, W) \cdot (N, K, A) = (u, W)\) then \(u \cdot (N, K) = u\) and \((N, K, A)^{-1} \cdot W = W\). The action on \(\tau^*VF(Y)\) is free, so \(N = I_n\) and \(K = I_k\), and thus \((I_n, I_k, A)^{-1} \cdot W = W\), so \(A = 0\). Thus the \(G_A\)-action is free. The projection from \(\tau^*VF(Y) \times R^{k \times n}\) onto \(Y\) is \(\pi \circ \text{Pr}_1\), where \(\text{Pr}_1\) is projection in the first slot of the Cartesian product. So for \(y \in Y\) there exists an open neighborhood \(U\) of \(y\) such that \(\text{Pr}_1^{-1}(\pi^{-1}(U)) \simeq U \times G_A\). \(\square\)

\textbf{Theorem 4.4} The following are in pairwise bijective correspondence:

\(i\) An Ehresmann connection on \(\pi_{XY} : Y \to X\)

\(ii\) A \(GL(n) \times GL(k)\)-equivariant global section of \(\eta : LVY \to \tau^*VF(Y)\)

\(iii\) A global trivialization of \(LVY\) over \(\tau^*VF(Y)\),
\[
\Lambda : LVY \to \tau^*VF(Y) \times R^{k \times n}, w \mapsto (\eta(w), \lambda(w))
\]
where \(\lambda : LVY \to R^{k \times n}\) is a \(G_A\)-equivariant function

\(iv\) A symmetry breaking \(\lambda : LVY \to R^{k \times n}\) where \(\lambda^{-1}(0) \simeq \tau^*VF(Y)\)

\(v\) A global section of \(JY \to Y\)

\textbf{Proof} \((i) \iff (v)\): See GIMMsys.

\((ii) \Rightarrow (iv)\): The proof is motivated by Ref. 18. Let \(\sigma : \tau^*VF(Y) \to LVY\) be a \(GL(n) \times GL(k)\)-equivariant section. Since any element of \(\eta^{-1}(u)\) can be expressed uniquely as \(\sigma(u) \cdot A\) for some \(A \in R^{k \times n}\), we may define \(\lambda : LVY \to R^{k \times n}\) by \(\lambda(\sigma(u) \cdot A) = A\). Let \(B \in R^{k \times n}\). Then
\[
\lambda((\sigma(u) \cdot B) \cdot (N, K, A)) = \lambda(\sigma(u) \cdot (N, K, BN + A))
\]
\[
= \lambda(\sigma(u) \cdot (N, K)) \cdot (K^{-1}BN + K^{-1}A)
\]
\[
= K^{-1}BN + K^{-1}A
\]
\[
= (N, K, A)^{-1} \cdot \lambda(\sigma(u) \cdot B).
\]
By definition, \(\lambda(\sigma(u) \cdot A) = 0 \iff A = 0\), so \(\lambda^{-1}(0) \simeq \tau^*VF(Y)\).
(iii) ⇒ (ii): This proof also is motivated by Ref. [18]. Define $\sigma: \tau^*VF(Y) \to L_Y Y$ by $\sigma(u) = \Lambda^{-1}(u, 0)$. Using (4.8),
\[
\Lambda(\sigma(u) \cdot (N, K), 0) = (u, 0) \cdot (N, K, 0) = (u \cdot (N, K), 0) = \Lambda(\sigma(u) \cdot (N, K)).
\]
Since $\Lambda$ is bijective, $\sigma(u \cdot (N, K)) = \sigma(u) \cdot (N, K, 0)$.

(i) ⇒ (iv): Let $\gamma_y: T_Y Y \to V(T_Y Y)$ be an Ehresmann connection. Define a map
\[
\lambda_y: L_Y Y \to \mathbf{R}^{k \times n} : (y, \{e_i, e_A\}) \mapsto \epsilon^A |_{V(T_Y Y)}(\gamma(e_i))E^A_y
\]
where $\{E^A_y\}$ is the standard basis of $\mathbf{R}^{k \times n}$. Now, if $v \in V(T_Y Y)$ then $e'(v) = 0$. So,
\[
\lambda_y((y, \{e_i, e_A\}) \cdot (N, K, A))
\]
\[
= \left((K^{-1})_A^B \epsilon^A - (K^{-1} AN^{-1})_k^B \epsilon^k \right) \mid_{V(T_Y Y)} \left(\gamma(e_j N^j + \epsilon_A A^j_i)\right) E^B_y
\]
\[
= \left((K^{-1})_A^B \epsilon^A |_{V(T_Y Y)}(\gamma(e_j))N^j + (K^{-1})B_i^A \epsilon^B \right) E^B_y
\]
\[
= (N^{-1}, K^{-1}, -K^{-1} AN^{-1}) \cdot (\epsilon_B |_{V_T (\gamma(e_i))} E^B_y)
\]
\[
= (N, K, A)^{-1} \cdot \lambda_y((y, \{e_i, e_A\})).
\]

(iv) ⇒ (i): Let $\lambda(y, \{e_i, e_A\}) := \lambda_y(y, \{e_i, e_A\})E^B_y$. Define a linear operator $\gamma_y$ on $T_Y Y$ by $\gamma_y(\epsilon_A) := \epsilon_A$ and $\gamma_y(\epsilon_j) := \lambda_y(y, \{e_i, e_A\})\epsilon_B$. Then $\gamma_y$ is the projection onto $V(T_Y Y)$ along span{$\epsilon_j - \lambda_B^A(y, \{e_i, e_A\})\epsilon_B$}. The definition of $\gamma$ is independent of choice of a point in $\pi_Y^{-1}(y)$. Indeed, if $(y, \{e'_i, e'_A\}) \in \pi_Y^{-1}(y)$ then $(y, \{e'_i, e'_A\}) = (y, \{e_i, e_A\}) \cdot (N, K, A) = (y, \{e_j N^j + \epsilon_B A^j_i, \epsilon_B K^B_A\})$ for some $\gamma_y$ by $\gamma_y(\epsilon_A') := \epsilon_A'$ and $\gamma_y(\epsilon_j') := \lambda_y(y, \{e'_i, e'_A\})\epsilon_B$. Hence $\epsilon_B K^B_A = \gamma_y(\epsilon_B K^B_A) = \gamma_y(\epsilon_B K^B_A)$ and thus $\epsilon_A = \gamma_y(\epsilon_A')$. Also, since $\lambda$ is $G_A$-equivariant,
\[
\gamma_y(e_k) = \lambda_y((y, \{e_i, e_A\}) \cdot (N, K, A))\epsilon_B
\]
\[
= \left((K^{-1})_A^B \lambda_y^{cA}(y, \{e_i, e_A\})N^j + (K^{-1})A_i^c A^j_A \right) \epsilon_B K^B_A
\]
\[
= \gamma_y(e_j) N^j_k + \epsilon_B A^j_i
\]
\[
= \gamma_y(e_j) N^j_k + \gamma_y(e'_j) - \gamma_y(e_j) N^j_k,
\]
so $\gamma_y(e_i) = \gamma_y(e_i)$. Therefore $\gamma_y$ and $\gamma_y$ coincide on the frame $(y, \{e_i, e_A\})$.

(iv) ⇒ (iii): Define $\Lambda(w) := (\eta(w), \lambda(w))$. Using (4.5) and (4.8),
\[
\Lambda(w \cdot (N, K, A)) = \left(\eta((w \cdot AN^{-1}) \cdot (N, K, 0)), \lambda(w \cdot (N, K, A))\right)
\]
\[
= \left(\eta(w) \cdot (N, K), (N, K, A)^{-1} \cdot \lambda(w)\right)
\]
\[
= \Lambda(w) \cdot (N, K, A).
\]
The transitive $G_A$-action on $\mathbf{R}^{k \times n}$ defined in (4.7) ensures that $\lambda$ is surjective. To show that $\Lambda$ is surjective, let $\eta(w, B) \in \tau^*VF(Y) \times \mathbf{R}^{k \times n}$. If $C \in \mathbf{R}^{k \times n}$ then $\eta(w \cdot C) = \eta(w)$ and $\lambda(w \cdot C) = (I_n, I_k, C)^{-1} \cdot \lambda(w) = \lambda(w) \cdot C$. If $C = B - \lambda(w)$ then $\Lambda(w \cdot C) = (\eta(w), B)$. To show that $\Lambda$ is injective, let $\Lambda(w) = \Lambda(w')$ for some $w, w' \in L_Y Y$. Then $w' \in \eta^{-1}(\eta(w))$, so $w' = w \cdot A$ and $\lambda(w) = \lambda(w \cdot A) = (I_n, I_k, A)^{-1} \cdot \lambda(w)$. Hence, $A = 0$. Finally, note that $\Lambda$ is $\mathbf{R}^{k \times n}$-equivariant since $\mathbf{R}^{k \times n} \subset G_A$. □
Corollary 4.5  An Ehresmann connection on $\pi_{XY}: Y \to X$ induces a flat connection on $\eta: LVY \to \tau^*VF(Y)$.

Proof  If $\lambda_\gamma$ denotes the $G_A$-equivariant function obtained from an Ehresmann connection $\gamma$ in the proof of Theorem 4.4 ((i) $\Rightarrow$ (iv)), then $R_A^* d\lambda_\gamma = d(\lambda_\gamma + A) = d\lambda_\gamma = Ad(A^{-1}) \cdot d\lambda_\gamma$ for $A \in \mathbb{R}^{k \times n}$. Let $A = A_i^B E_B \in \mathbb{R}^{k \times n}$ and $w = (y, \{e_i, \epsilon_A\}) \in LVY$. Let $A_w^*$ be the fundamental vertical vector for $A$. Then,

$$d\lambda_\gamma(A_w^*) = d\lambda_\gamma \left( \frac{d}{dt} (y, \{e_i, \epsilon_A\}) \cdot \exp tA \Big|_{t=0} \right)$$

$$= \frac{d}{dt} \lambda_\gamma(y, \{e_i + tA_i^B \epsilon_B, \epsilon_A\}) \Big|_{t=0}$$

$$= \frac{d}{dt} \left( \epsilon^C - tA_i^C e^j \gamma(e_i + tA_i^B \epsilon_B) \right) \Big|_{t=0} E^i_C$$

$$= \frac{d}{dt} \epsilon^B \gamma(e_i) + tA_i^B \Big|_{t=0} E^i_B$$

$$= A.$$  

The connection $d\lambda_\gamma$ is flat because the Lie algebra $(\mathbb{R}^{k \times n}, +)$ is Abelian. $\Box$

The pullback of the $(n+k)$-symplectic structure $d\theta$ via inclusion $i: LVY \hookrightarrow LY$ is closed and nondegenerate on $LVY$, just as $d\theta$ is on $LY$. The $(n+k)$-symplectic structure equation for $LVY$ is

$$d\hat{f} = -X_{\hat{f}} \cdot i^*d\theta,$$

(4.9)

where $\hat{f} \in C^\infty(LVY, \mathbb{R}^{n+k})$ and $X_{\hat{f}} \in \mathcal{X}(LVY)$, the vector space of vector fields of $LVY$. Like the $p = 1$ case of equation (3.1), equation (4.9) admits neither all vector fields nor all $\mathbb{R}^{n+k}$-valued functions.

Definitions

- $T^1(LVY)$ is the vector space of tensorial $\mathbb{R}^{n+k}$-valued functions on $LVY$.
- The Lie algebra of locally Hamiltonian vector fields on $LVY$ is $LHV^1(LVY) := \{X \in \mathcal{X}(LVY) \mid \mathcal{L}_X i^*d\theta = 0\}$.
- The vector space of allowable Hamiltonian observables on $LVY$ is $HF^1(LVY) := \{\hat{f} \in C^\infty(LVY, \mathbb{R}^{n+k}) \mid d\hat{f} = -X_{\hat{f}} \cdot i^*d\theta, \ X \in LHV^1(LVY)\}$.
- The vector space of Hamiltonian vector fields on $LVY$ is $HV^1(LVY) := \{X \in \mathcal{X}(LVY) \mid X \cdot i^*d\theta = -d\hat{f}, \ \hat{f} \in HF^1(LVY)\}$. 

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Local coordinates on $L_Y L_Y$ are $\{x^i, y^A, \pi_j^i, \pi_B^A, \pi_i^A\}$, where $\pi_j^i := e^j_i(\frac{\partial}{\partial x^j})$, $\pi_B^A := e^A_i(\frac{\partial}{\partial y^B})$ and $\pi_i^A := e^A_i(\frac{\partial}{\partial y^B})$. In these coordinates,

$$i^*d\theta = d\pi_j^i \wedge dx^j \otimes \hat{r}_j + (d\pi_A^B \wedge dx^B + d\pi_i^A \wedge dy^B) \otimes \hat{s}_A .$$

An element of $T^1(L_Y L_Y)$ can be expressed in local coordinates as

$$\hat{f} = f^i(x^k, y^C)\pi_j^i \otimes \hat{r}_j + f^A(x^k, y^C)\pi_B^A \otimes \hat{s}_B + f^i(x^k, y^C)\pi_i^A \otimes \hat{s}_B .$$

For $\hat{f} \in T^1(L_Y L_Y)$ we solve equation (4.13) locally for $X_{\hat{f}}$. This yields

$$X_{\hat{f}} = f^i \frac{\partial}{\partial x^i} + f^A \frac{\partial}{\partial y^A} - \frac{\partial f^k}{\partial x^j} \pi_k^i \frac{\partial}{\partial \pi_j^i} - \frac{\partial f^C}{\partial y^B} \pi^A \frac{\partial}{\partial \pi_B^A} - \left( \frac{\partial f_j^i}{\partial x^i} \pi_j^A \frac{\partial}{\partial \pi_B^A} + \frac{\partial f^B}{\partial x^i} \pi_i^A \frac{\partial}{\partial \pi_B^A} \right) \frac{\partial}{\partial \pi_i^A} ,$$

subject to the constraints on $\hat{f}$,

$$\frac{\partial f_i^i}{\partial y^A} = 0 \quad \forall i = 1, \ldots, n \quad \text{and} \quad A = 1, \ldots, k.$$  \hspace{1cm} (4.11)

Thus, $T^1(L_Y L_Y) \not\subset HF^1(L_Y L_Y)$. Define

$$T^1_V(L_Y L_Y) := HF^1(L_Y L_Y) \cap T^1(L_Y L_Y) .$$

Since $L_Y L_Y \subset L_Y$, a point $w \in L_Y L_Y$ is a linear isomorphism $w : \mathbb{R}^{n+k} \rightarrow T_{\pi_Y L_Y Y(w)}Y$. Moreover, for $g \in G_A$, $(w \cdot g)V = w(gV)$ for each $V \in \mathbb{R}^{n+k}$. Hence $\hat{f} \in T^1(L_Y L_Y)$ corresponds bijectively to $f \in X Y$ by the relation $\hat{f}(w) = w^{-1}(f(\pi_Y L_Y Y(w)))$. Now, if $\hat{f} \in T^1_V(L_Y L_Y)$ then, by (4.11), $\hat{f}$ is induced from a projectable vector field. Conversely, $f \in X_{\pi_Y L_Y Y}$ gives a tensorial function $\hat{f}$ satisfying (4.11). Thus, $T^1_V(L_Y L_Y)$ is in bijective correspondence with $X_{\pi_Y L_Y Y}$.

For $n \geq 2$ and $k \geq 2$ a vector field $X$ on $L_Y L_Y$ satisfies $\mathcal{L}_X(i^*d\theta) = 0$ if and only if in local coordinates

$$X = g^i(x^k) \frac{\partial}{\partial x^i} + g^A(x^k, y^C) \frac{\partial}{\partial y^A} - \left( g^B, B(x^k, y^C) \pi_B^A + \xi^A, B(x^k, y^C) \right) \frac{\partial}{\partial \pi_B^A}$$

$$- \left( \xi^j, d(x^k) \pi_j^i \right) \frac{\partial}{\partial \pi_j^i}$$

$$- \left( g^B, i(x^k, y^C) \pi_B^A + \xi^B, i(x^k) \pi_B^A + \xi^A, i(x^k, y^C) \right) \frac{\partial}{\partial \pi_i^A} ,$$

where $g, i = \frac{\partial g}{\partial y^A}$ and $g, B = \frac{\partial g}{\partial y^B}$. Observe that $\pi_Y L_Y L_Y X \in X_{\pi_Y L_Y Y}$. From (4.13) and (4.12) we obtain the local expression for $\hat{g} \in HF^1(L_Y L_Y)$.

$$\hat{g} = \left( g^i(x^k) \pi_j^i + \xi^i(x^k) \right) \otimes \hat{r}_i + \left( g^A(x^k) \pi_B^A + g^A(x^k, y^C) \pi_i^A + \xi^B(x^k, y^C) \right) \otimes \hat{s}_B .$$

As a result of equation (4.11), if $n \geq 2$ and $k \geq 2$ then

$$HF^1(L_Y L_Y) \simeq T^1_V(L_Y L_Y) \oplus C^\infty(X, \mathbb{R}^n) \oplus C^\infty(Y, \mathbb{R}^k) .$$

Observe that the kernel of the representation $HF^1(L_Y L_Y) \rightarrow HV^1(L_Y L_Y) : \hat{f} \mapsto X_{\hat{f}}$ is the set of the constant $\mathbb{R}^{n+k}$-valued functions. Thus,

$$HV^1(L_Y L_Y) \simeq HF^1(L_Y L_Y)/\mathbb{R}^{n+k}$$

in analogy to the result on the full frame bundle $L_Y$ and to equation (2.13).
V. ASSOCIATING THE MULTIVELOCITY SPACES TO $L_VY$

Define a linear left action of $G_A$ on $\mathbb{R}^{k \times n}$ by

$$(N, K, A) \odot W := KW^{-1}.$$

(We’ll reserve the more conventional “"·"" for the nonlinear left action in (5.1).)

Use (4.4) and (5.1) to define an equivalence relation on $L_VY \times \mathbb{R}^{k \times n}$,

$$((y, \{e_i, e_A\}), W) \sim \left((y, \{e_i, e_A\}) \cdot (N, K, A), (N, K, A)^{-1} \odot W\right).$$

(5.2)

Equivalence classes $[(y, \{e_i, e_A\}), W]$ are points in the associated bundle $L_VY \times_{G_A} \mathbb{R}^{k \times n}$.

(We’ll reserve the more conventional "\([,]\)" for a different equivalence class.)

**Lemma 5.1** The map

$$\psi : L_VY \times_{G_A} \mathbb{R}^{k \times n} \to \text{Hom}_Y(TX, V(TY)) : \left(\{(y, \{e_i, e_A\}), W\}\right) \mapsto (y, -W_j^B(\pi_{XY}, e_j) \otimes \epsilon_B)$$

is a vector bundle isomorphism over $Y$, where $\left\{(\pi_{XY}, e_j)\right\}$ is the coframe of $X$ at $\pi_{XY}(y)$ dual to the frame $\left\{(\pi_{XY}, e_j)\right\} := \left\{\pi_{XY}, e_j\right\}$ and $W = W_j^B E^j_B \in \mathbb{R}^{k \times n}$.

**Proof** By (5.2), $\psi$ is well defined. Since $\text{Hom}_Y(TX, V(TY))$ is vector bundle isomorphic to $T^XY \otimes Y V(TY)$, we represent $-W_j^B(\pi_{XY}, e_j) \otimes \epsilon_B$ as an element of $\text{Lin}(T_{\pi_{XY}(y)}X, V(T_y Y))$. Let $\psi[[\{(y, \{e_i, e_A\}), W\}]] = \psi[[\{(y', \{e_i, e_A\}), W'\}], \text{ then } y = y' \text{ and there exists an } (N, K, A) \in G_A \text{ such that } (y', \{e_i, e_A\}) = (y, \{e_i, e_A\}) \cdot (N, K, A). \text{ Subsequently, } \psi[[\{(y', \{e_i, e_A\}), W'\}]] = \psi[[\{(y, \{e_i, e_A\}), KW^N \cdot W'\}]]. \text{ So } W = KW^N \cdot W', \text{ and thus } W' = (N, K, A)^{-1} \odot W. \text{ So } ((y', \{e_i, e_A\}), W') \sim ((y, \{e_i, e_A\}), W)$, and thus $\psi$ is injective. Now $\psi$ is linear on each fiber over $y$ and $\dim(\text{Lin}(T_{\pi_{XY}(y)}X, V(T_y Y))) = kn = \dim \left((L_VY)_y \times_{G_A} \mathbb{R}^{k \times n}\right)$. Thus $\psi$ is surjective on each fiber over $y$. \qed

Use (4.4) and (4.7) to define a different equivalence relation on $L_VY \times \mathbb{R}^{k \times n}$,

$$((y, \{e_i, e_A\}), W) \sim \left((y, \{e_i, e_A\}) \cdot (N, K, A), (N, K, A)^{-1} \odot W\right).$$

(5.3)

Equivalence classes $[(y, \{e_i, e_A\}), W]$ are points in the associated bundle $(L_VY \times_{G_A} \mathbb{R}^{k \times n})_{\text{Aff}}$.

Ragionieri and Ricci show that $JY$ is an affine bundle over $Y$ modeled on $\text{Hom}_Y(TX, V(TY))$.

We now can reproduce this result with bundles associated to $L_VY$.

**Theorem 5.2** The bundle $(L_VY \times_{G_A} \mathbb{R}^{k \times n})_{\text{Aff}}$ is an affine bundle over $Y$ with underlying vector bundle $L_VY \times_{G_A} \mathbb{R}^{k \times n}$, and the map

$$\hat{\psi} : (L_VY \times_{G_A} \mathbb{R}^{k \times n})_{\text{Aff}} \to JY : \left([y, \{e_i, e_A\}], W\right) \mapsto \left(y, -W_j^B(\pi_{XY}, e_j) \otimes \epsilon_B + (\pi_{XY}, e_j) \otimes e_j\right)$$

is an affine bundle isomorphism onto the jet bundle $JY$. 

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\textbf{Proof} Define a difference function $\hat{\delta}$ on $(L_Y Y \times_{G_A} R^{k \times n})_{Aff}$ on each fiber over $Y$ by

$$\hat{\delta}_y([w, W_1], [w, W_2]) = [w, W_1 - W_2]$$

where $y = \pi_{L_Y Y}(w)$. The range is $(L_Y Y)_y \times_{G_A} R^{k \times n}$. The function $\hat{\delta}_y$ is well defined. Indeed, if $w' \in \pi_{L_Y Y}(\pi_{L_Y Y}(w))$, then there exists $(N, K, A) \in G_A$ such that $w' = w \cdot (N, K, A)$. Let $[w', W'_1] = [w, W_1]$ and $[w', W'_2] = [w, W_2]$. Then $W_1 = (N, K, A) \cdot W'_1$ and $W_2 = (N, K, A) \cdot W'_2$ and thus

$$[w', W'_1 - W'_2] = [w \cdot (N, K, A), K^{-1} A - (K^{-1} W_1 N + K^{-1} A)]$$

$$= [w \cdot (N, K, A), (N, K, A)^{-1} \odot (W_1 - W_2)]$$

$$= [w, W_1 - W_2].$$

Similarly, we may verify the other properties of the difference function.

Let $(y, \gamma_y) = \hat{\psi}([y, \{e_i, \epsilon_A\}], W)$. Then $\gamma_y \in \text{Lin}(T_{\pi_{XY}(y)} X, T_y Y)$ and $\pi_{XY^*} \circ \gamma_y = \Id_{T_x X}$. Therefore, $(y, \gamma_y) \in JY$. Using (5.3), we can show that $\hat{\psi}$ is well defined. To show that $\hat{\psi}$ is surjective choose an arbitrary point $(y, \{e_i, \epsilon_A\}) \in L_Y Y$ and define $\gamma_y^B(y)$ by $\gamma_y(\pi_{XY^*} e_j) = e_j + \gamma_y^B(y) \epsilon_B$. Then $\gamma_y(\gamma_y^B(y)) = \gamma_y^B(y) E_B^j$. To show that $\hat{\psi}$ is injective let $\hat{\psi}([y, \{e_i, \epsilon_A\}], W) = \hat{\psi}([y', \{e'_i, \epsilon'_A\}], W')$. Then $y = y'$ and there is a unique $(N, K, A) \in G_A$ such that $(y', \{e'_i, \epsilon'_A\}) = (y, \{e_i, \epsilon_A\}) \cdot (N, K, A)$. So

$$\hat{\psi}((y, \{e_i, \epsilon_A\}), (N, K, A) \cdot W') = \hat{\psi}([y, \{e_i, \epsilon_A\}], (N, K, A) \cdot W')$$

$$= (y, -(K W' N^{-1} - A N^{-1}) B (\pi_{XY^*} e_j) \otimes \epsilon_B + (\pi_{XY^*} e_j) \otimes e_j),$$

giving us $K W' N^{-1} - A N^{-1} = W$ or $W' = (N, K, A)^{-1} \cdot W$. Thus $((y, \{e_i, \epsilon_A\}), W) \sim ((y', \{e'_i, \epsilon'_A\}), W')$. Finally, let $\delta$ be the difference function on $JY$. Let $w = (y, \{e_i, \epsilon_A\}) \in L_Y Y, \pi_{L_Y Y}(w) = y$ and let $W$ and $W' \in R^{k \times n}$. Then,

$$\psi(\delta([w, W], [w, W'])) = \delta (\hat{\psi}[w, W], \hat{\psi}[w, W'])$$

where $\psi$ is defined in Lemma 5.1. \hfill $\Box$

\section{ASSOCIATING THE MULTIPHASE SPACES TO $L_Y Y$}

Define a linear left action of $G_A$ on $R^{n \times k}$ by

$$(N, K, A) \circ B := N B K^{-1},$$

and define an equivalence relation on $L_Y Y \times R^{n \times k}$ similar to (5.2), using (4.4) and (6.1). The resulting equivalence classes, denoted by $[(y, \{e_i, \epsilon_A\}), B]_{Gu}$, are points in the associated bundle $(L_Y Y \times_{G_A} R^{n \times k})_{Gu}$. The following result comes from modifying the proof of Lemma 5.1.

\textbf{Lemma 6.1} Let $\{E_j^C\}$ be the standard basis of $R^{n \times k}$ and $B = B_j^C E_i^C \in R^{n \times k}$. Then

$$\rho_{Gu} : (L_Y Y \times_{G_A} R^{n \times k})_{Gu} \to \text{Hom}_Y (V(T Y), T X)$$

$$[(y, \{e_i, \epsilon_A\}), B]_{Gu} \mapsto (y, B_j^C E_i^C |_{V(T_y Y)} \otimes f_j)$$

is a vector bundle isomorphism over $Y$.
If, instead of the action defined in (6.1), we use a $GL(n)$-normalized action
\[(N, K, A) \cdot B = \det(N^{-1})NBK^{-1},\] (6.2)
then the bundle with fiber $\mathbb{R}^{n \times k}$ associated to $L_V Y$ is $\text{Hom}_Y(V(TY), \wedge^{n-1}X)$. Using (4.4) and (6.2), the new equivalence classes denoted by $[(y, \{e_i, \epsilon_A\}), B]$ are points in the associated bundle $L_V Y \times_{G_A} \mathbb{R}^{n \times k}$. Define
\[
\rho_{KT}: L_V Y \times_{G_A} \mathbb{R}^{n \times k} \rightarrow \text{Hom}_Y(V(TY), \wedge^{n-1}X)
\]
\[
[(y, \{e_i, \epsilon_A\}), B] \mapsto (y, B^j_i \epsilon^B_i \wedge \omega(e)_j + \lambda \omega(e)),
\]
where if $f$ is a set of $n$ linearly independent vectors $\{f_i\}$ then
\[
\omega(f) := f^1 \wedge f^2 \wedge \ldots \wedge f^n \quad \text{and} \quad \omega(f)_j := f_j \downarrow \omega(f).
\] (6.3)
Like $\rho_{Gu}$ in Lemma 6.1, $\rho_{KT}$ is a vector bundle isomorphism over $Y$.

Define a linear left action of $G_A$ on the vector space $\mathbb{R}^{n \times k} \times \mathbb{R}$ as follows.
\[
(N, K, A) \cdot (B, \lambda) := \det(N^{-1}) \left( NBK^{-1}, \lambda - \text{Tr} (BK^{-1}A) \right)
\] (6.4)
The associated vector bundle $L_V Y \times_{G_A} (\mathbb{R}^{n \times k} \times \mathbb{R})$ is constructed using (4.4) and (6.4). Equivalence classes (points in the vector bundle) are denoted by $[(y, \{e_i, \epsilon_A\}), (B, \lambda)]$.

**Theorem 6.2** The map
\[
\rho_Z: L_V Y \times_{G_A} (\mathbb{R}^{n \times k} \times \mathbb{R}) \rightarrow \wedge^n Y
\]
\[
[(y, \{e_i, \epsilon_A\}), (B, \lambda)] \mapsto (y, B^j_i \epsilon^B_i \wedge \omega(e)_j + \lambda \omega(e))
\]
is a vector bundle monomorphism over $Y$, and the range of $\rho_Z$ is $Z$.

**Proof** We may use (4.4), (6.1), (6.3) and (6.4) to show that $\rho_Z$ is well defined. Observe that $\omega(e)$ kills vertical vectors, and if $n \geq 2$ then $e^A \wedge \omega(e)_i$ kills two vertical vectors. Thus, the range of $\rho_Z$ is in $Z$. For any $z \in Z$, in local coordinates,
\[
z = \rho_Z \left[ \left( \pi_{YZ}(z), \left\{ \frac{\partial}{\partial x^A}, \frac{\partial}{\partial y^A} \right\} \right), \left( (p^i_A(z)), p(z) \right) \right]
\]
where $p^i_A(z)$ and $p(z)$ are the coordinates defined in (2.2). Also, $\rho_Z$ is linear on fibers over $Y$, $\rho_Z$ preserves fibers, and the fibers have the same dimension. $\square$

Use $\text{Pr}_1: \mathbb{R}^{n \times k} \times \mathbb{R}$ to $\mathbb{R}^{n \times k}$ and the maps $\rho_{KT}$ and $\rho_{Z}^{-1}|_Z$ to define the projection $\pi_{KTZ}: Z \rightarrow \text{Hom}_Y(V(TY), \wedge^{n-1}X)$. See diagram (6.5).
\[
\begin{array}{ccc}
L_V Y \times_{G_A} (\mathbb{R}^{n \times k} \times \mathbb{R}) & \xrightarrow{\rho_Z^{-1}|_Z} & Z \\
\downarrow & & \downarrow \pi_{KTZ} \\
L_V Y \times_{G_A} \mathbb{R}^{n \times k} & \xrightarrow{\rho_{KT}} & \text{Hom}_Y(V(TY), \wedge^{n-1}X)
\end{array}
\] (6.5)
Using $\rho_Z$, the relationships between local canonical coordinates \( \{x^i, y^A, \pi^B_k, \pi^D_i\} \) on $Z$ and local coordinates \( \{x^i, y^A, \pi^B_k, \pi^D_i\} \) on $L_Y$ are

\[
p^J_B = \det(\pi^J_m) B^i_A \pi^A_B (\pi^{-1})^J_i \quad \text{and} \quad p = \det(\pi^J_m) (B^i_A \pi^A_B (\pi^{-1})^J_i + \lambda). \tag{6.6}
\]

The first equation in (6.6) also relates coordinates in $\text{Hom}_Y(V(TY), \wedge^{n-1} X)$ to those in $L_Y$.

The identification of the linear and affine multiphase spaces with bundles associated to $L_Y$ allows us to recast certain known relationships between these spaces in our more general context.

**Theorem 6.3** The bundle $Z$ is an affine bundle over $\text{Hom}_Y(V(TY), \wedge^{n-1} X)$. (See Ref. [5].)

The vector bundle underlying $Z$ is $\text{Hom}_Y(V(TY), \wedge^{n-1} X) \times_Y \pi^{YZ} \wedge^n X$.

**Proof** Define a difference function $\delta : Z \times Z \to \text{Hom}_Y(V(TY), \wedge^{n-1} X) \times_Y \pi^{YZ} \wedge^n X$ by $\delta([u, (B, \lambda_1)], [u, (B, \lambda_2)]) = ([u, B], (\lambda_1 - \lambda_2) \omega(u))$, where if $u = (y, \{e_i, e_A\})$ then $\omega(u) = \omega(e)$. The map $\delta$ is well defined and preserves fibers over $\text{Hom}_Y(V(TY), \wedge^{n-1} X)$. To check that $\delta$ is well defined, let $(u', (B', \lambda')) \sim (u, (B, \lambda))$. Then $u' = u \cdot g$ and $(B', \lambda') = g^{-1} (B, \lambda)$ for some $g = (N, K, A) \in G_A$ and $i = 1, 2$. Now, $\omega(u \cdot g) = \det(N^{-1}) \omega(u)$, so,

\[
\delta([u', (B', \lambda')], [u', (B', \lambda_2)]) = \delta([u \cdot g, \lambda_1], [u \cdot g, \lambda_2]) = \delta([u \cdot g, \det(N^{-1}BK, \lambda_1 + \text{Tr} (BAN^{-1})]), [u \cdot g, \det(N^{-1}BK, \lambda_2 + \text{Tr} (BAN^{-1})])
\]

\[
= \left(\delta([u \cdot g, \lambda_1 \cdot B], \det(N\lambda_1 - \lambda_2)\omega(u \cdot g)) = \right. \left. \delta([u, (B, \lambda_1)], [u, (B, \lambda_2)])
\right)
\]

It is straightforward to verify the other properties of the difference function $\delta$. □

**Lemma 6.4** Any section of the bundle $\pi_{KTZ} : Z \to \text{Hom}_Y(V(TY), \wedge^{n-1} X)$ may be expressed as a map $[u, B] \mapsto [u, (B, \eta(u, B))]$, where $\eta : L_Y \times R^{n \times k} \to R$ satisfies the equation

\[
(\det N) \left( \eta(u, B) + \text{Tr} (BAN^{-1}) \right) = \eta(u \cdot (N, K, A), (\det N)N^{-1} BK) \tag{6.7}
\]

for all $(N, K, A) \in G_A$. Conversely, any map $\eta : L_Y \times R^{n \times k} \to R$ that satisfies (6.7) induces a section $[u, B] \mapsto [u, (B, \eta(u, B))]$ of the bundle $\pi_{KTZ} : Z \to \text{Hom}_Y(V(TY), \wedge^{n-1} X)$.

**Proof** By definition, a section $s : \text{Hom}_Y(V(TY), \wedge^{n-1} X) \to Z$ must satisfy $s[u, B] = [u, (B, \lambda)]$ where $u \in L_Y$, $B \in R^{n \times k}$ and $\lambda \in R$. For each $u \in L_Y$, define $\sigma_u : Z \to R^{n \times k} \times R : [u, (B, \lambda)] \mapsto (B, \lambda)$. Because $Z \simeq L_Y \times (R^{n \times k} \times R)$, $\sigma_u$ is well defined. Define $\eta : L_Y \times R^{n \times k} \to R$ by $\eta(u, B) := \text{Pr}_2(\sigma_u(s[u, B]))$, where $\text{Pr}_2$ is projection in the second slot. Then $k[u, B] = [u, (B, \eta(u, B))]$. Now $s[u, B] = s[u \cdot g, \lambda_1 \cdot B]$ for $g \in G_A$, so $[u \cdot g, (g^{-1} \cdot B, \eta(u \cdot g, g^{-1} \cdot B))] = [u, (B, \eta(u, B))] = [u \cdot g, \lambda_1 \cdot (B, \eta(u, B))]$. So $g^{-1} \cdot (B, \eta(u, B)) = (g^{-1} \cdot B, \eta(u \cdot g, g^{-1} \cdot B))$ and thus we obtain equation (6.7). Conversely, if $s$ satisfies (6.7) then $s$ is well defined. □
Theorem 6.5 (Sardanashvily) An Ehresmann connection on \( Y \) is equivalent to a splitting of the short exact sequence of vector bundle homomorphisms over \( \text{Id}_Y \),

\[
0 \to \pi_{XY}^*(\wedge^n X) \to Z \to \text{Hom}_Y(V(TY), \wedge^{n-1} X) \to 0.
\]

**Proof** Assume an Ehresmann connection on \( Y \). By Theorem 4.4, a connection is equivalent to a \( G_A \)-equivariant map \( \lambda : L_V Y \to \mathbb{R}^{k \times n} \). Define a map

\[
\eta : L_V Y \times \mathbb{R}^{n \times k} \to \mathbb{R} : (u, B) \mapsto \text{Tr}(B \lambda(u))
\]

Let \( g = (N, K, A) \in G_A \). Using (1.7),

\[
(\det N) \left( \eta(u, B) + \text{Tr}(BAN^{-1}) \right) = \text{Tr} \left( N^{-1} BK(K^{-1}(u)N + K^{-1}A) \right)
\]

so \( \eta \) satisfies (5.4). By Lemma 5.4, we may define a section \( s : \text{Hom}_Y(V(TY), \wedge^{n-1} X) \to Z \) and it is routine to check that \( s \) is linear on fibers over \( Y \).

Conversely, let the section \( s : \text{Hom}_Y(V(TY), \wedge^{n-1} X) \to Z \) be linear on fibers over \( Y \). By Lemma 5.4, we write \( s[u, B] = [u, (B, \xi(u, B))] \), and \( \xi \) satisfies \( \xi(u, B_1 + cB_2) = \xi(u, B_1) + c\xi(u, B_2) \). For \( u \in L_V Y \) we define \( \xi_u : \mathbb{R}^{n \times k} \to \mathbb{R} \) by \( \xi_u(B) := \xi(u, B) \). Since \( \xi_u \) is linear we may uniquely represent \( \xi_u \) as \( \xi_u(B) = \text{Tr}(BW(\xi_u)) \) where \( W(\xi_u) \in \mathbb{R}^{k \times n} \).

We now argue that \( u \mapsto W(\xi_u) \) is a symmetry-breaking map and thus is equivalent to an Ehresmann connection on \( Y \). Indeed, the action of \( G_A \) on \( \mathbb{R}^{k \times n} \) given in (4.7) is transitive, so it remains only to show equivariance. By equation (5.4),

\[
\begin{align*}
\det N(\xi_u(B)) + \text{Tr}(BAN^{-1}) &= \xi_{u-g}((\det N)N^{-1}BK) \forall B \\
\text{Tr}(BW(\xi_u)) + \text{Tr}(BAN^{-1}) &= \text{Tr}(BKW(\xi_{u-g}))N^{-1} \forall B \\
\text{Tr}(B(W(\xi_u) + AN^{-1} - KW(\xi_{u-g}))N^{-1}) &= 0 \forall B \\
KW(\xi_{u-g})N^{-1} &= W(\xi_u) + AN^{-1} \\
W(\xi_{u-g}) &= K^{-1}W(\xi_u)N + K^{-1}A \\
W(\xi_{u-g}) &= g^{-1} \cdot W(\xi_u) \quad \square
\end{align*}
\]

If \( f : Z \to \pi_{XY}^*(\wedge^n X) \) splits the sequence in Theorem 6.5, then we may write \( f[u, (B, \lambda)] = [u, (0, \lambda - \xi(u, B))] \), where \( \xi \) is the map in the proof of Theorem 6.5. Let \( \gamma \) be the connection equivalent to \( f \). In local coordinates,

\[
f : p^i_A dy^A \wedge d^n x_i + p d^n x_i \mapsto (p + p^i_A \gamma^A_i) d^n x_i.
\]

**VII. AFFINE MULTISYMPLECTIC GEOMETRY FROM (n+k)-SYMPLECTIC GEOMETRY**

We will construct the multisymplectic potential \( \Theta \) on \( Z \) from the \( (n+k) \)-symplectic potential \( i^* \theta \) on \( L_V Y \). First, some preliminary remarks about tensor-valued differential forms on \( LY \) are necessary.
Let \( \{ R_\mu \}, \mu = 1, \ldots, n + k \), be the standard basis of \( \mathbf{R}^{n+k} \) and let \( \{ R^\mu \} \) be the corresponding dual basis. For convenience, define \( R_{\mu_1 \cdots \mu_m} := R_{\mu_1} \wedge \cdots \wedge R_{\mu_m} \in \wedge^m \mathbf{R}^{n+k} \) and \( R^{\mu_1 \cdots \mu_m} := R^{\mu_1} \wedge \cdots \wedge R^{\mu_m} \in \wedge^m \mathbf{R}^{n+k} \). Let \( \alpha \) be a \( \wedge^p \mathbf{R}^{n+k} \)-valued \( p \)-form and let \( \beta \) be a \( \wedge^q \mathbf{R}^{n+k} \)-valued \( q \)-form on a manifold. Then \( \alpha = \alpha^{\mu_1 \cdots \mu_p} \otimes R_{\mu_1 \cdots \mu_p} \) and \( \beta = \beta^{\mu_1 \cdots \mu_q} \otimes R_{\mu_1 \cdots \mu_q} \). Define
\[
\alpha \wedge \beta := (\alpha^{\mu_1 \cdots \mu_p} \wedge \beta^{\mu_1 \cdots \mu_q}) \otimes R_{\mu_1 \cdots \mu_p \mu_1 \cdots \mu_q}.
\]
Observe that \( \alpha \wedge \beta = (-1)^{pq+r}s \beta \wedge \alpha \). Let \( \theta \) be the canonical soldering form on \( LY \). Define the \( \wedge^m \mathbf{R}^{n+k} \)-valued \( m \)-form \( \wedge^m \theta \) on \( LY \) by \( \wedge^0 \theta := 1 \) and
\[
\wedge^m \theta := \underbrace{\theta \wedge \cdots \wedge \theta}_{m \text{ factors}} \text{ if } m \geq 1.
\]

Note that
\[
d(\wedge^m \theta) = m d \theta \wedge (\wedge^{m-1} \theta).
\] (7.1)

**Lemma 7.1** If \( 1 \leq m \leq n + k - 1 \) then \( d(\wedge^m \theta) \) is closed and nondegenerate.

**Proof** If \( m = 1 \) then the above form is \( d \theta \). Let \( 2 \leq m \leq n + k - 1 \). The form is closed because \( d^2(\wedge^m \theta) = 0 \). Let \( X \in \mathcal{X}(LY) \) satisfy \( 0 = X \lrcorner d(\wedge^m \theta) \). By equation (7.1), \( 0 = X \lrcorner (d \theta \wedge (\wedge^{m-1} \theta)) \). We may introduce a torsionless connection \( \omega \) on \( LY \) without loss of generality. Thus \( 0 = D \theta = d \theta + \omega \wedge \theta \), or
\[
d \theta^\mu = -\omega^\mu \wedge \theta^\nu.
\]

At \( w \in LV Y \), we can write \( X_w = X^\mu B_\mu(w) + X_\nu^\mu E_\nu^\mu(w) \), where \( \{ E_\nu^\mu(w) \} \) is the frame of fundamental vertical vector fields and \( \{ B_\mu(w) \} \) is the horizontal frame complementary to \( \{ E_\nu^\mu(w) \} \). So,
\[
0 = X_w \lrcorner (\theta^\nu \wedge \omega^\mu,_{[\mu_1]} \wedge \theta^\nu,_{\mu_2} \wedge \cdots \wedge \theta^\nu,_{\mu_m}])
\]
\[
0 = X^\nu \omega^\mu,_{[\mu_1]} \wedge \theta^\nu,_{\mu_2} \wedge \cdots \wedge \theta^\nu,_{\mu_m} - \theta^\nu \wedge X^\nu \omega^\mu,_{[\mu_1]} \theta^\nu,_{\mu_2} \wedge \cdots \wedge \theta^\nu,_{\mu_m} + \theta^\nu \wedge \omega^\mu,_{[\mu_1]} X^\nu \theta^\nu,_{\mu_2} \wedge \cdots \wedge \theta^\nu,_{\mu_m}
\]
\[
+ \sum_{i=3}^m (-1)^i \theta^\nu \wedge \omega^\mu,_{[\mu_1]} \theta^\nu,_{\mu_2} \wedge \cdots \wedge X^\nu \theta^\nu,_{\mu_i} \wedge \cdots \wedge \theta^\nu,_{\mu_m}
\]
\[
0 = X^\nu \omega^\mu,_{[\mu_1]} \theta^\nu,_{\mu_2} \wedge \cdots \wedge \theta^\nu,_{\mu_m} - \theta^\nu \wedge X^\nu \omega^\mu,_{[\mu_1]} \theta^\nu,_{\mu_2} \wedge \cdots \wedge \theta^\nu,_{\mu_m}
\]
\[
+ \sum_{i=2}^m (-1)^i \theta^\nu \wedge X^\nu \omega^\mu,_{[\mu_1]} \theta^\nu,_{\mu_2} \wedge \cdots \wedge \theta^\nu,_{\mu_i} \wedge \cdots \wedge \theta^\nu,_{\mu_m}
\]

where \( \hat{\theta}^\mu,_{\mu_i} \) denotes the omission of \( \theta^\mu,_{\mu_i} \). Now use \( [\mu_i, \mu_1, \ldots, \hat{\mu}_i, \ldots, \mu_m] = (-1)^{i-1} [\mu_1, \ldots, \mu_m] \) to combine terms, obtaining
\[
0 = X^\nu \omega^\mu,_{[\mu_1]} \theta^\nu,_{\mu_2} \wedge \cdots \wedge \theta^\nu,_{\mu_m} - \theta^\nu \wedge X^\nu \omega^\mu,_{[\mu_1]} \theta^\nu,_{\mu_2} \wedge \cdots \wedge \theta^\nu,_{\mu_m}
\]
\[
- (m - 1) \theta^\nu \wedge X^\nu \omega^\mu,_{[\mu_1]} \theta^\nu,_{\mu_2} \wedge \theta^\nu,_{\mu_3} \wedge \cdots \wedge \theta^\nu,_{\mu_m}. \] (7.2)
Now evaluate both sides of equation (7.2) at \((B_\alpha, \ldots, B_\alpha)(w)\) to obtain
\[
0 = -\theta^\nu \wedge X^\mu_1 \theta^\mu_2 \wedge \cdots \wedge \theta^\mu_m (B_\alpha, \ldots, B_\alpha)(w)
\]
\[
0 = (X^\mu_1 \theta^\mu_2 \wedge \cdots \wedge \theta^\mu_m (B_\alpha, \ldots, B_\alpha)(w)
\]
\[
0 = \sum_{\sigma \in \text{Perm}(m)} \text{sgn}(\sigma) \theta^\nu X^\mu_{\sigma(1)} \theta^\mu_{\sigma(2)} \otimes \cdots \otimes \theta^\mu_{\sigma(m)} (B_\alpha, \ldots, B_\alpha)(w)
\]
\[
0 = \sum_{\sigma \in \text{Perm}(m)} \text{sgn}(\sigma) X^\mu_{\sigma(1)} \delta^\mu_{\sigma(2)} \delta^\mu_{\sigma(3)} \cdots \delta^\mu_{\sigma(m)}
\]
\[
0 = X^\nu I_{\alpha_1} \delta^\nu_{\alpha_2} \cdots \delta^\nu_{\alpha_m}.
\]

Choose a sequence \(\{\mu_1\}\) such that the \(\mu_i\) are distinct for \(2 \leq I \leq m\) and choose a sequence \(\{\alpha_1\}\) such that \(\alpha_I = \mu_I\) for \(2 \leq I \leq m\). Thus \(X^\mu_{\alpha_1} = 0\), for any choice of \(\mu_1\) and \(\alpha_1\). Return to (7.2) and evaluate both sides of the equation at \((E_\beta^\alpha, B_\alpha, \ldots, B_\alpha)\). Thus, by a calculation similar to that above,
\[
0 = X^\alpha \delta^\nu_{\beta \alpha_2} \cdots \delta^\nu_{\alpha_m} + (m - 1) \delta^\alpha_{\alpha_2} X^\mu_1 \delta^\mu_{\alpha_3} \cdots \delta^\mu_{\alpha_m}.
\]

Again, choose a sequence \(\{\mu_1\}\) such that the \(\mu_i\) are distinct if \(2 \leq I \leq m\). Now let \(\alpha_I = \mu_I\) for \(2 \leq I \leq m\) and let \(\alpha = \alpha_2 = \mu_2 = \beta = \mu_1\). Then the second term in equation (7.3) vanishes, else \(\alpha = \alpha_2 = \beta\) which contradicts the hypothesis that the \(\mu_i\) are distinct. \(\square\)

Pulling back \(d(\wedge^m \theta)\) via inclusion \(i : L_Y V \to L_Y Y\), it follows that \(d(\wedge^m i^* \theta)\) is a closed, nondegenerate, \(\wedge^m \mathbb{R}^{n+k}\)-valued form on \(L_Y Y\) if \(1 \leq m \leq n + k - 1\). When expressing the standard basis of \(\mathbb{R}^{n+k}\) as \(\{\hat{r}_i, \hat{s}_A\}, i = 1, \ldots, n, A = 1, \ldots, k\) instead of \(\{R_\mu\}, \mu = 1, \ldots, n + k\), but use upper case and lower case Latin indices in place of lower case Greek indices. Then \(i^* \theta = \theta^i \otimes \hat{r}_i + \theta^A \otimes \hat{s}_A\), and, so by induction on \(m\),
\[
\wedge^m i^* \theta = \sum_{l=0}^{m} \binom{m}{l} \theta^{A_1} \wedge \cdots \wedge \theta^{A_l} \wedge \theta^{i_{l+1}} \wedge \cdots \wedge \theta^{i_{m-l}} R_{A_1 \cdots A_{i_{l+1}} \cdots i_{m-l}}.
\]

Let \(m = n\). If \(\sigma\) is a permutation of \(n\) elements expressed by \(\sigma(1), \ldots, \sigma(n) = (i_1, \ldots, i_n)\) then let \(e_{i_1 \cdots i_n}\) denote the sign of \(\sigma\). Let \((y, \{e_i, e_A\}) \in L_Y Y\). Using (1.3),
\[
\omega(e) = \frac{1}{n!} e_{i_1 \cdots i_n} e^{i_1} \wedge \cdots \wedge e^{i_n}
\]
and
\[
\omega(e)_j = \frac{1}{(n - 1)!} e_{j_{i_1 \cdots i_{n-1}}} e^{i_1} \wedge \cdots \wedge e^{i_{n-1}}.
\]

Define
\[
\phi_{(B, \lambda)} : L_Y Y \to Z : w \mapsto \rho_Z[w, (B, \lambda)]
\]
The map \(\phi_{(B, \lambda)}\) is fiber preserving over \(Y\).

**Theorem 7.2** Let \(n \geq 2\). The \(\wedge^n \mathbb{R}^{n+k}\)-valued \(n\)-form \(\wedge^n i^* \theta\) on \(L_Y Y\) can be related to the canonical \(n\)-form \(\Theta\) on \(Z\) by
\[
\langle \wedge^n i^* \theta, V(B, \lambda) \rangle = \phi_{(B, \lambda)}^* \Theta
\]
(7.7)
where the map $V : \mathbb{R}^{n \times k} \times \mathbb{R} \to \Lambda^n \mathbb{R}^{n \times k^*}$ has components

$$V_{i_1 \ldots i_n}(B, \lambda) = \frac{1}{n!} \lambda e_{i_1 \ldots i_n},$$

$$V_{A i_1 \ldots i_{n-1}}(B, \lambda) = \frac{1}{n!} B^j A e_{j i_1 \ldots i_{n-1}}$$

and

$$V_{A_1 \ldots A_i i_1 \ldots i_{n-1}}(B, \lambda) = 0 \ \forall \ l \geq 2.$$

**Proof** Using equations (7.4), (7.5) and (7.6), for $w = (y, \{e_i, \epsilon_A\}) \in L_Y Y$,

$$\langle \wedge^n \iota \theta, V(B, \lambda) \rangle (w) = \frac{1}{n!} \lambda e_{i_1 \ldots i_n} \theta^{i_1} \wedge \cdots \wedge \theta^{i_n} (w) + \frac{n}{n!} B^j A e_{j i_1 \ldots i_{n-1}} \epsilon^A \wedge \theta^{i_1} \wedge \cdots \wedge \theta^{i_{n-1}} (w)$$

$$= \left( \frac{1}{n!} \lambda e_{i_1 \ldots i_n} \epsilon^{i_1} \wedge \cdots \wedge \epsilon^{i_n} + \frac{1}{(n-1)!} B^j A e_{j i_1 \ldots i_{n-1}} \epsilon^A \wedge \epsilon^{i_1} \wedge \cdots \wedge \epsilon^{i_{n-1}} \right) (w)$$

$$= (\lambda \omega(e) + B^j A \epsilon^A \wedge \omega(e)_j)(w)$$

$$= \phi_{(B, \lambda)}^* \Theta(w). \quad \square$$

Equation (7.7) holds for $n = 1$ if $V_i(B, \lambda) = \lambda$ and $V_A(B, \lambda) = B_A$, producing a parametrized version of equation (3.5) for Hamiltonian particle mechanics. Note that we cannot adapt Theorem 7.2 to construct a connection-independent “canonical” $n$-form on $J^*Y$, because a connection is needed to define an inclusion map from $J^*Y$ into $\Lambda^n Y$.

Recall from Secs. II and IV that $X_{\text{proj}} Y$ is in bijective correspondence not only with $T^1(Z)$ but also with $T^1 V (L_Y Y)$. The following theorem is an exact analogue of Theorem 3.2

**Theorem 7.3** Let $v \in X_{\text{proj}} Y$. Let $\hat{f}_v \in T^1 V (L_Y Y)$ correspond to $v$ and let $X_{\hat{f}_v}$ be the Hamiltonian vector field on $L_Y Y$ obtained from $\hat{f}_v$ via (4.3). Let $f_v \in T^1 (Z)$ also correspond to $v$ and let $X_{f_v}$ be the Hamiltonian vector field on $Z$ obtained from $f_v$ via (4.3). Then

$$\phi_{(B, \lambda)}^* X_{f_v} = X_{\hat{f}_v}$$

**Proof** From the definition of $\phi_{(B, \lambda)}$ we express its derivative map $\phi_{(B, \lambda)}^*$ in local adapted coordinates. Let a vector field $X$ on $L_Y Y$ have local adapted coordinate expression

$$X = X^i \frac{\partial}{\partial x^i} + X^A \frac{\partial}{\partial y^A} + X^j_i \frac{\partial}{\partial \pi^j_i} + X^A_B \frac{\partial}{\partial \pi^A_B} + X^i_A \frac{\partial}{\partial \pi^A_i}.$$

Let $\gamma(t)$ be a curve in $L_Y Y$ such that $\gamma(0) = w$ and $\dot{\gamma}(0) = X_w$. Then

$$\phi_{(B, \lambda)}^* X_w = \frac{d}{dt} \phi_{(B, \lambda)} \circ \gamma(t)|_{t=0}.$$

Using (6.6) and the matrix identity

$$\frac{\partial}{\partial \pi^k_s} \det \pi^r_s = \det(\pi^r_s)(\pi^{-1})^r_k$$

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we can derive the local coordinate formula,

\[
\phi_{(B,\lambda)_w} X_w = X^i \frac{\partial}{\partial x^i} + X^A \frac{\partial}{\partial y^A} \\
+ \det(\pi^*_w) B^i_A \left( (\pi^{-1})^i_j X^B_A - \pi^*_A (\pi^{-1})_k^i \pi^B_k (\pi^{-1})^j_k X^k_i \right) \frac{\partial}{\partial p^A} \\
+ \det(\pi^*_w) \left( B^i_A (\pi^{-1})^i_k X^k_i + (\pi^{-1})^i_j X^k_i + (\pi^{-1})^i_j \pi^A_j X^i_k \right) \\
+ \lambda (\pi^{-1})_k^i X^k_i \frac{\partial}{\partial p^i}.
\]  
(7.8)

Let \( X = X_{f_v} \) and substitute the local coordinate expressions for the components of \( X_{f_v} \) from (4.10) into (7.8). After using the coordinate conversions (6.6), we obtain equation (2.6), making it a candidate for a higher degree generalization of the (induction).

Let \( \pi \in L_V^* \) for each \( 0 \leq m \leq n+k \) define a representation of \( T^1_V(L_V Y) \) into the space of \( \wedge^{m+1} \mathbb{R}^{n+k} \)-valued \( m \)-forms on \( L_V Y \) by

\[
\hat{f} \mapsto \hat{f} \wedge (\wedge^m i^* \theta).
\]  
(7.9)

The image of \( T^1_V(L_V Y) \) under the representation (7.9) is the set of degree \( m \) momentum observables on \( L_V Y \). Obtain \( X_f \) from \( \hat{f} \) via (4.9). By (4.10), \( X_f \cdot i^* \theta = \hat{f} \). Then by induction,

\[
X_f \cdot \wedge^m i^* \theta = m \hat{f} \wedge (\wedge^{m-1} i^* \theta). 
\]  
(7.10)

By Lemma 7.1, for each \( 0 \leq m \leq n+k-2 \) the form \( d(\wedge^m i^* \theta) \) is closed and nondegenerate, making it a candidate for a higher degree generalization of the \((n+k)\)-symplectic form \( i^* d\theta \). However, using equation (7.4), we may instead use \( i^* d\theta \wedge (\wedge^m i^* \theta) \) to eliminate a factor of \((m+1)\) in the following generalization of equation (4.9).

\[
d(\hat{f} \wedge (\wedge^m i^* \theta)) = -X_f \cdot \wedge^m (i^* d\theta \wedge (\wedge^m i^* \theta)).
\]  
(7.11)

Equation (7.11) was derived using (7.1) and (7.10).

Let \( \hat{f}, \hat{g} \in T^1_V(L_V Y) \). For \( 0 \leq m \leq n+k \) define a bracket on their images under representation (7.9) by

\[
\{ \hat{f} \wedge (\wedge^m i^* \theta), \hat{g} \wedge (\wedge^m i^* \theta) \} := -X_f \cdot \wedge^m (i^* d\theta \wedge (\wedge^m i^* \theta)).
\]  
(7.12)

If \( m = 0 \) then (7.12) defines the bracket on \( T^1_V(L_V Y) \) analogous to (3.3) where \( p = q = 1 \),

\[
\{ \hat{f}, \hat{g} \} := X_f(\hat{g}) = -X_f \cdot \wedge^m i^* d\theta.
\]  
(7.13)

The vector space \( T^1_V(L_V Y) \) is a Lie algebra under (7.13).
Theorem 7.4 Let \( \hat{f}, \hat{g} \in T^1_V(L_Y) \) and let \( 0 \leq m \leq n + k \). Then
\[
\{ \hat{f} \wedge (\wedge^m i^* \theta), \hat{g} \wedge (\wedge^m i^* \theta) \} = \{ \hat{f}, \hat{g} \} \wedge (\wedge^m i^* \theta) + m d(\hat{f} \wedge \hat{g} \wedge (\wedge^{m-1} i^* \theta)).
\]

Proof Assume \( m \geq 2 \). Then, using (7.10), (7.12) and (7.13),
\[
\{ \hat{f} \wedge (\wedge^m i^* \theta), \hat{g} \wedge (\wedge^m i^* \theta) \}
= -X_f \mathcal{J} ((X_g \mathcal{J} i^* d\theta) \wedge (\wedge^m i^* \theta) + i^* d\theta \wedge (X_g \mathcal{J} \wedge^m i^* \theta))
= X_f \mathcal{J} (d\hat{g} \wedge (\wedge^m i^* \theta) + m \hat{g} \wedge i^* \theta \wedge (\wedge^{m-1} i^* \theta))
= X_f (\hat{g}) \wedge (\wedge^m i^* \theta) - d\hat{g} \wedge (X_f \mathcal{J} \wedge^m i^* \theta)
\quad + m \hat{g} \wedge (X_f \mathcal{J} i^* d\theta) \wedge (\wedge^{m-1} i^* \theta) + m \hat{g} \wedge i^* d\theta \wedge (X_f \mathcal{J} \wedge^{m-1} i^* \theta)
= \{ \hat{f}, \hat{g} \} \wedge (\wedge^m i^* \theta) - m d\hat{g} \wedge \hat{f} \wedge (\wedge^{m-1} i^* \theta) - m \hat{g} \wedge d\hat{f} \wedge (\wedge^{m-1} i^* \theta)
\quad + m (m - 1) \hat{g} \wedge i^* d\theta \wedge \hat{f} \wedge (\wedge^{m-2} i^* \theta)
= \{ \hat{f}, \hat{g} \} \wedge (\wedge^m i^* \theta) + m (\hat{f} \wedge d\hat{g} \wedge (\wedge^{m-1} i^* \theta) + d\hat{f} \wedge \hat{g} \wedge (\wedge^{m-1} i^* \theta)
\quad + \hat{f} \wedge \hat{g} \wedge d(\wedge^{m-1} i^* \theta))
= \{ \hat{f}, \hat{g} \} \wedge (\wedge^m i^* \theta) + m d(\hat{f} \wedge \hat{g} \wedge (\wedge^{m-1} i^* \theta)).
\]

The proof of the \( m = 1 \) case is a minor modification of the above argument. The \( m = 0 \) case is obvious. \( \Box \)

Theorem 7.4 is of particular interest when \( m = n - 1 \). In this case, we have reproduced for \( L_Y \) the exact analogue of the problem in equation (2.11). Just like \( T^1(Z) \), the set of degree \( (n - 1) \) momentum observables on \( L_Y \) is obstructed from closing under their bracket, and the source of the obstruction is an exact \( (n - 1) \)-form.

Any extension of the set of degree \( (n - 1) \) momentum observables on \( L_Y \) to include closed \( \mathbb{R}^{n+k} \)-valued \( (n-1) \)-forms would have a representation to \( HV^1(L_Y) \) with a kernel larger than that of the representation from \( T^1_V(L_Y) \) to the same vector fields. The space \( T^1_V(L_Y) \) of observables that are both Hamiltonian and tensorial already forms a well-defined Lie algebra under the bracket defined in equation (7.13). \( T^1_V(L_Y) \) and \( T^1(Z) \) are in bijective correspondence, and Theorem 7.3 dictates a relationship between the corresponding spaces of Hamiltonian vector fields. Thus we may argue that the \( (n+k) \)-symplectic geometry of \( L_Y \) geometry of \( Z \) for classical fields, but also possesses a Lie algebra structure for field observables that \( Z \) lacks.

VIII. CONCLUSIONS

This investigation shows that the \((n+k)\)-symplectic geometry of the bundle of vertically adapted linear frames \( L_Y \) of a field configuration space \( Y \) provides a general description of classical Hamiltonian field theories analogous to the description of Hamiltonian particle mechanics using Norris’s \( n \)-symplectic geometry. A vector space of Hamiltonian degree one tensorial functions serves as the space of allowable classical field observables, and the allowable observables produce Hamiltonian vector fields via the \((n+k)\)-symplectic equation.
The \((n + k)\)-symplectic geometry of \(L_Y Y\) generalizes not only the affine multisymplectic geometry of the prototypic affine multiphase space \(Z\) but also the linear multisymplectic geometry of the linear multiphase space \(J^*Y\). This new geometry illustrates exactly why the linear multiphase space \(J^*Y\) requires a choice of a background connection in order to model classical field theories. Specifically, a particular symmetry breaking of \(L_Y Y\) is equivalent to a choice of a connection on \(Y\) which is necessary in order to define a linear multisymplectic potential on \(J^*Y\). This symmetry breaking of \(L_Y Y\) reinforces the notion that the connection-independent affine theory on \(Z\) is a more natural starting point for multisymplectic field theories. We also use the \((n + k)\)-symplectic geometry of \(L_Y Y\) to resolve a problem with the naturally defined Poisson bracket of two momentum observables on \(Z\) by the construction of an analogous bracket of two Hamiltonian degree one tensorial observables on \(L_Y Y\).

In addition to generalizing the existing multisymplectic field theories, our generalized symplectic geometry has an advantage over these theories. In order to witness events in a covariant field theory, we must choose a preferred reference frame. The principal bundle \(L_Y Y\) is useful because it describes the space of all reference frames, and sections of \(L_Y Y\) describe preferential frames of observers.

The results of current work will demonstrate the utility of this generalized symplectic geometry. In a paper with R. O. Fulp and L. K. Norris, the author shows that \((n + k)\)-symplectic geometry on the full frame bundle \(LY\) generalizes the natural geometry of any skew-symmetric tensor bundle over \(Y\). Additionally, in a forthcoming publication, the author will introduce momentum mappings on \(L_Y Y\) for field theories, setting the stage for geometric representations of frames of field momenta and field conservation laws.

As foreshadowed by the superscript “1” in the notation (such as in \(T^1_Y (L_Y Y)\)), the vector spaces of momentum observables may be extended to higher degree tensor fields, and the definitions of the Poisson bracket may be extended to these new spaces. (This has been accomplished for particle mechanics, resulting in full Poisson and graded Poisson Schouten-Nijenhuis algebras of tensorial observables on \(LM\) under the bracket in \(\{\cdot,\cdot\}\). See Refs. [1] and [21].) Mathematically, the extended Poisson bracket would be a true Poisson bracket, acting as a derivation on the tensor algebra. Physically, this will create new geometrical models of Hamiltonian tensor fields, such as electromagnetism and Hamiltonian gravity. Early successes in extending the algebras of observables on \(LM\) and on \(J^*Y\) (see Ref. [22]) motivate us to pursue this direction of inquiry.

Finally there may be applications of these results to a principal bundle formulation of geometric quantization of fields, similar to that done for particles. A prolongation of \(L_Y Y\) may be the appropriate setting in which to find a faithful representation of classical observables into a space of prequantum operators, leading, for example, to a geometric derivation of the Dirac equation.

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