Integrable discretizations for lattice systems:
local equations of motion
and their Hamiltonian properties

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Abstract. We develop the approach to the problem of integrable discretization based on the
notion of $r$–matrix hierarchies. One of its basic features is the coincidence of Lax matrices of
discretized systems with the Lax matrices of the underlying continuous time systems. A common
feature of the discretizations obtained in this approach is non–locality. We demonstrate how to
overcome this drawback. Namely, we introduce the notion of localizing changes of variables and
construct such changes of variables for a large number of examples, including the Toda and the
relativistic Toda lattices, the Volterra lattice and its integrable perturbation, the second flows
of the Toda and of the Volterra hierarchies, the modified Volterra lattice, the Belov–Chaltikian
lattice, the Bogoyavlensky lattices, the Bruschi–Ragnisco lattice. We also introduce a novel class
of constrained lattice KP systems, discretize all of them, and find the corresponding localizing
change of variables. Pulling back the differential equations of motion under the localizing changes
of variables, we find also (sometimes novel) integrable one–parameter deformations of integrable
lattice systems. Poisson properties of the localizing changes of variables are also studied: they
produce interesting one–parameter deformations of the known Poisson algebras.
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1 Introduction

This paper deals with some aspects of the following general problem: how to discretize one or several of independent variables in a given integrable system, maintaining the integrability property? We call this the problem of integrable discretization.

To assure the coincidence of the qualitative properties of the discretized models with that of the continuous ones becomes one of the central ideas of the modern numerical analysis, which therefore comes to a close interplay with different aspects of the theory of dynamical systems. One of the most advanced examples of this approach is the symplectic integration, on which recently the first monograph appeared [SSC].

The problem of integrable discretization constitutes another aspect of this general line of thinking. It arose in the course of development of the theory of solitons. This theory, born exactly 30 years ago, has grown so tremendously, that it is difficult to keep an overview of the whole variety of different notions and contexts of integrability, not to say about the concrete results. (Recall that recently there appeared a thick book under the title ”What is integrability?” [Zakh]).

Correspondingly, various approaches to the problem of integrable discretization are currently available. They began to be discussed sporadically in the soliton literature starting
from the mid–70s. Following ones should be mentioned:

1. An approach based on the representation of an integrable system as a compatibility condition of two auxiliary linear problems. A natural proposition is to discretize one or the both of them [AL1]. This, however, can be made in a great variety of ways, cf., for instance, different spatial discretizations of the nonlinear Schrödinger equation and of the sine–Gordon equation, found in [AL1] and in [IK]. One attempt of fixing discretization crystallized with the development of the Hamiltonian approach. Namely, Faddeev and Takhtajan, based on the experience of the Leningrad soliton school, formulated in [FT2] the following rule for a transition from models with one continuous space variable to lattice models: the $r$–matrix should be preserved, the linear Poisson bracket being replaced by the quadratic one. See [FT2] for a collection of examples showing productivity of this approach.

2. One of the most intriguing and universal approaches is the Hirota’s one [H], based on the notion of the $\tau$–function and on a bilinear representation of integrable systems. It seems to be able to produce discrete versions of the majority of soliton equations, but still remains somewhat mysterious, and the mechanism behind it is yet to be fully understood. One successful way to do it was proposed in [DJM], where also a large number of integrable discretizations was derived. Among the most interesting products of this approach is the so called Hirota–Miwa equation [H1], [Mi], which is sometimes claimed to contain ”everything”, i.e. the majority if not all soliton equations (continuous and discrete) are particular or limiting cases of this single equation, cf. [Za].

3. A fruitful method is based on the ”direct linearization” [NCW], [QNCV], [CWN], [WC], [NPCQ], [NC]. Its basic idea is to derive integrable nonlinear differential equations which are satisfied by the solutions of certain linear integral equations. A large variety of continuous and discrete soliton equations has been obtained on this way.

4. Approach based on the variational principle (discrete Lagrangian equations), combined with matrix factorizations [V], [MV], [DLT2]. Historically, it was the work of Veselov and Moser that consolidated the more or less isolated results to a separate branch of the theory of integrable systems.

5. Considering stationary and restricted flows of soliton hierarchies, closely related to the ”nonlinearization” of spectral problems, often leads to interesting discrete equations [QRT], [R], [RR].

6. Differential equations describing various geometric problem (surfaces of the constant mean curvature, motion of the curve in the space, etc.) turn out to be integrable [Syl].
Correspondingly, a discretization of geometric notions naturally leads to discrete integrable equations \[ B1 \], \[ B2 \], \[ BP1 \], \[ BP2 \], \[ DS \], \[ Do \].

7. There exist integrable discretizations which belong to the most beautiful examples, but were derived by guess, without any systematical approach \[ S1 \], \[ S5 \].

8. Last but not least we mention an approach to the temporal discretization in which the auxiliary spectral problem is not discretized at all. In other words, the basic feature of this approach is maintaining the Lax matrix of the continuous time system. The first example of this approach is the work by Ablowitz–Ladik \[ AL2 \], further developed towards the practical algorithms in \[ TA \]. This feature was also put in the basis of the work by Gibbons and Kupershmidt \[ GK \], \[ K2 \]. The discretizations found in all these papers were somewhat unsatisfactory from the esthetical point of view, namely they suffered from being nonlocal, as opposed to the underlying continuous time systems. Moreover, these authors did not recognize the connection with the factorization problem, which did not allow them to identify their discretizations as certain members of the corresponding hierarchies and to establish the Poisson properties of these discretizations. This led Gibbons and Kupershmidt to call this method “the method of the bizarre ansatz”.

Recently the author pushed forward the last mentioned approach to the problem of integrable discretization, putting it in a connection with the \( r \)–matrix theory of integrable hierarchies (see \[ RSTS \] for a review of this theory). In this context the method could be understood properly, and became rather natural and simple. It was applied to a number of integrable lattice systems \[ S6 \]–\[ S11 \], \[ S13 \], \[ S14 \]. Its clear advantage is universality. The method is in principle applicable to any system admitting an \( r \)–matrix interpretation, which is the common feature of the great majority of the known integrable systems. As for the drawback of nonlocality, there exist several ways to repair it. The first one, connected with the notion of discrete time Newtonian equations of motion, was followed in \[ S6 \], \[ S7 \], \[ S10 \], \[ S11 \]. A splitting of complicated flows into superpositions of more simple ones was used in \[ S13 \], \[ S14 \]. The present work is devoted to another way connected with the so called localizing changes of variables.

The paper has the following structure. In Sect. 2 we give an accurate formulation of the problem of integrable discretization. Sect. 3, 4 are devoted to a general framework of integrable \( r \)–matrix hierarchies of Lax equations on associative algebras. In Sect. 5 we formulate a general recipe of integrable discretization, and in Sect. 6 we introduce the notion of localizing changes of variables and discuss their general properties. Sect. 7 contains the description of algebras used for the analysis of all integrable lattice systems in this paper. The rest of the paper is devoted to a detailed elaboration of a number of examples, including the most prominent ones, such as the Toda lattice and the Volterra lattice, and less well known ones, such as the Belov-Chaltikian lattice.
2 The problem of integrable discretization

Let us formulate the problem of integrable discretization more precisely. Let $\mathcal{X}$ be a Poisson manifold with a Poisson bracket $\{\cdot,\cdot\}$. Let $H$ be a completely integrable Hamilton function on $\mathcal{X}$, i.e. the system

$$\dot{x} = \{H,x\}$$

possesses many enough functionally independent integrals $I_k(x)$ in involution.

The problem consists in finding a map $\mathcal{X} \mapsto \mathcal{X}$ described by a formula

$$\tilde{x} = \Phi(x;h)$$

depending on a small parameter $h > 0$, and satisfying the following requirements:

1. The map (2.2) is a discrete time approximation for the flow (2.1) in the following sense:

$$\Phi(x;h) = x + h\{H,x\} + O(h^2)$$

(of course, one might require also a higher order of approximation). In all our considerations and formulas we pay a special attention to a simple and transparent control of the continuous limit $h \to 0$.

2. The map (2.2) is Poisson with respect to the bracket $\{\cdot,\cdot\}$ on $\mathcal{X}$ or with respect to some its deformation $\{\cdot,\cdot\}_h$ such that $\{\cdot,\cdot\}_h = \{\cdot,\cdot\} + O(h)$.

3. The map (2.2) is integrable, i.e. possesses the necessary number of independent integrals in involution $I_k(x;h)$ approximating the integrals of the original system: $I_k(x;h) = I_k(x) + O(h)$.

3 Lax representations

Our approach to the problem of integrable discretization is applicable to any system allowing an $r$–matrix interpretation, but we formulate the basic recipe in a simplified form, applicable to systems with a Lax representation of one of the following types:

$$\dot{L} = \left[ L, \pi_+ (f(L)) \right] = - \left[ L, \pi_- (f(L)) \right]$$

or

$$\dot{L}_j = L_j \cdot \pi_+ (f(T_{j-1})) - \pi_+ (f(T_j)) \cdot L_j = - L_j \cdot \pi_- (f(T_{j-1})) + \pi_- (f(T_j)) \cdot L_j$$

Let us discuss the notations.
Let \( g \) be an associative algebra, supplied with a nondegenerate scalar product, which allows to identify \( g^* \) with \( g \). One can introduce in \( g \) the structure of Lie algebra in a standard way. Let \( g_+, g_- \) be two subalgebras such that as a vector space \( g = g_+ \oplus g_- \). Denote by \( \pi_{\pm} : g \mapsto g_{\pm} \) the corresponding projections. Finally, let \( f : g \mapsto g \) be an \( \text{Ad}^* \)-covariant function on \( g \), and let \( L \) stand for a generic element of \( g \). Then (3.1) is a certain differential equation on \( g \).

Further, let \( g = \bigotimes_{j=1}^{m} g \) be a direct product of \( m \) copies of the algebra \( g \). A generic element of \( g \) is denoted by \( L = (L_1, \ldots, L_m) \). We use also the notation
\[
T_j = T_j(L) = L_j \cdot \ldots \cdot L_1 \cdot L_m \cdot \ldots \cdot L_{j+1}
\] (3.3)
Then (3.2) is a certain differential equation on \( g \). Such equations are sometimes called Lax triads.

One says that (3.1), resp. (3.2), is a Lax representation of the flow (2.1), if there exists a map \( L : X \mapsto g \), resp. \( L : X \mapsto g \), such that the former equations of motion are equivalent to the latter ones. Let us stress that when considering equations (3.1), resp. (3.2) in the role of Lax representation, the letter \( L \) (resp. \( L \)) does not stand for a generic element of the corresponding algebra any more; rather, it represents the elements of the images of the maps \( L : X \mapsto g \) and \( L : X \mapsto g \), correspondingly. The elements \( L(x) \), resp. \( L(x) \) (and the map \( L \), itself) are called Lax matrices.

Equations (3.1) and (3.2) have several remarkable features. First of all, they are Hamiltonian with respect to a certain linear \(-\)-matrix Poisson bracket on \( g \), resp. on \( g \) \( \text{[STS]} \). Moreover, under some natural conditions (for example, if \( g_+ \) and \( g_- \) serve as orthogonal complements to each other) one can define quadratic and cubic \(-\)-matrix brackets on \( g \), compatible with the linear one, such that the equations above are Hamiltonian with respect to all three brackets \( \text{[STS]}, \text{[LP]}, \text{[OR]}, \text{[S6]}, \text{[S12]} \).

The Lax representations are especially useful, if the maps \( L \), resp. \( L \), corresponding to the Lax matrices, are Poisson with respect to one of the above mentioned \(-\)-matrix brackets; then the manifold consisting of the Lax matrices is a Poisson submanifold. In such cases one says that the Lax representation admits an \(-\)-matrix interpretation.

4 Factorization theorems

As a further remarkable feature of the equations (3.1) and (3.2) we consider the possibility to solve them explicitly in terms of a certain factorization problem in the Lie group \( G \) corresponding to \( g \) \( \text{[TTY]}, \text{[STS]}, \text{[RSTS]} \). (Actually, this can be done even in a more general situation of hierarchies governed by \textit{R}-operators satisfying the so-called modified Yang–Baxter equation, see \( \text{[RSTS]} \)). The factorization problem is described by the equation
\[
U = \Pi_+(U)\Pi_-(U), \quad U \in G, \quad \Pi_\pm(U) \in G_\pm
\] (4.1)
where \( G_\pm \) are two subgroups of \( G \) with the Lie algebras \( g_\pm \), respectively. This problem has a unique solution in a certain neighbourhood of the group unit. In what follows we suppose that \( G \) is a matrix group, and write the coadjoint action of the group elements on \( g^* \cong g \) as a conjugation by the corresponding matrices. [In this context we write \( \Pi_\pm^{-1}(U) \) for \( (\Pi_\pm(U))^{-1} \). Correspondingly, we call \( \text{Ad}^*-\text{covariant} \) functions \( g \mapsto g \) also ”conjugation covariant”. This notation has an additional advantage of being applicable also to functions \( g \mapsto G \).

For the history of the following fundamental theorem and its different proofs the reader is referred to [RSTS].

**Theorem 4.1** Let \( f : g \mapsto g \) be a conjugation covariant function. Then the solution of the differential equation (3.1) with the initial condition \( L(0) = L_0 \) is given, at least for \( t \) small enough, by

\[
L(t) = \Pi_+^{-1}(e^{tf(L_0)}) L_0 \Pi_+ \left( e^{tf(L_0)} \right) = \Pi_- \left( e^{tf(L_0)} \right) L_0 \Pi_-^{-1}(e^{tf(L_0)})
\]  

(4.2)

**Proof.** We give a proof based on direct and simple verification. Denote

\[
\mathcal{L}(t) = \Pi_+ \left( e^{tf(L_0)} \right), \quad \mathcal{R}(t) = \Pi_- \left( e^{tf(L_0)} \right)
\]

so that

\[
e^{tf(L_0)} = \mathcal{L}(t) \mathcal{R}(t), \quad \mathcal{L}(t) \in G_+, \quad \mathcal{R}(t) \in G_-
\]  

(4.3)

Now we set

\[
L(t) = \mathcal{L}^{-1}(t) L_0 \mathcal{L}(t) = \mathcal{R}(t) L_0 \mathcal{R}^{-1}(t)
\]  

(4.4)

(these two expressions for \( L(t) \) are equal due to \( \text{Ad}^* \)-covariance of \( f(L) \)), and check by direct calculation that this \( L(t) \) satisfies the differential equation (3.1). The theorem will follow by the uniqueness of solution. We see immediately that \( L(t) \) satisfies the following Lax type equation:

\[
\dot{L} = [L, \mathcal{L}^{-1} \dot{\mathcal{L}}] = -[L, \mathcal{R} \mathcal{R}^{-1}]
\]

and it remains to show that

\[
\mathcal{L}^{-1} \dot{\mathcal{L}} = \pi_+(f(L)), \quad \mathcal{R} \mathcal{R}^{-1} = \pi_- (f(L))
\]

Since, obviously, \( \mathcal{L}^{-1} \dot{\mathcal{L}} \in g_+, \mathcal{R} \mathcal{R}^{-1} \in g_- \), we need to demonstrate only that

\[
\mathcal{L}^{-1} \dot{\mathcal{L}} + \mathcal{R} \mathcal{R}^{-1} = f(L)
\]  

(4.5)

To do this, we differentiate (4.3) and derive, using \( \text{Ad}^*-\text{covariance} \) of \( f \) and the definition (4.4):

\[
\dot{\mathcal{L}} \mathcal{R} + \mathcal{L} \dot{\mathcal{R}} = e^{tf(L_0)} f(L_0) = \mathcal{L} \mathcal{R} f(L_0) = \mathcal{L} f(L) \mathcal{R}
\]
This is equivalent to (4.3). □

For an arbitrary conjugation covariant function $F : g \mapsto G$ one can define the map $B_F : g \mapsto g$ according to the formula

$$\tilde{L} = B_F(L) = \Pi^{-1}_+ (F(L)) \cdot L \cdot \Pi_+ (F(L)) = \Pi_+ (F(L)) \cdot L \cdot \Pi^{-1}_+ (F(L)) \quad (4.6)$$

("B" is for "Bäcklund"). Theorem 4.1 shows that the flows defined by the differential equations (3.1) consist of maps having such a form with $F(L) = e^{tf(L)}$. The very remarkable feature of such maps is their commutativity for different $F$’s.

**Theorem 4.2** For two arbitrary conjugation covariant functions $F_1, F_2 : g \mapsto G$

$$B_{F_2} \circ B_{F_1} = B_{F_2F_1} \quad (4.7)$$

and therefore the maps $B_{F_1}, B_{F_2}$ commute.

**Proof.** Denote:

$$L_1 = B_{F_1}(L), \quad L_2 = B_{F_2} \circ B_{F_1}(L) = B_{F_2}(L_1)$$

So, by definition we have:

$$L_1 = \mathcal{L}^{-1}_1 \mathcal{L} = \mathcal{R}_1 \mathcal{L} \mathcal{R}^{-1}_1, \quad L_2 = \mathcal{L}^{-1}_2 \mathcal{L}_1 \mathcal{L}_2 = \mathcal{R}_2 \mathcal{L}_1 \mathcal{R}^{-1}_2 \quad (4.8)$$

where the matrices $\mathcal{L}_i \in \mathbf{G}_+, \mathcal{R}_i \in \mathbf{G}_- (i = 1, 2)$ come from the following factorizations:

$$F_1(L) = \mathcal{L}_1 \mathcal{R}_1, \quad F_2(L_1) = \mathcal{L}_2 \mathcal{R}_2$$

From (4.8) we have:

$$L_2 = \mathcal{L}^{-1} \mathcal{L} \mathcal{R}_2 \mathcal{R}_1, \quad \text{where} \quad \mathcal{L} = \mathcal{L}_1 \mathcal{L}_2 \in \mathbf{G}_+, \quad \mathcal{R} = \mathcal{R}_2 \mathcal{R}_1 \in \mathbf{G}_- \quad (4.9)$$

Now the following chain of equalities holds:

$$\mathcal{L} \mathcal{R} = \mathcal{L}_1 \mathcal{L}_2 \mathcal{R}_2 \mathcal{R}_1 = \mathcal{L}_1 F_2(L_1) \mathcal{R}_1 = \mathcal{L}_1 F_2(\mathcal{L}_1^{-1} \mathcal{L} \mathcal{L}_1) \mathcal{R}_1 = F_2(L) \mathcal{L}_1 \mathcal{R}_1 = F_2(L) F_1(L)$$

In view of (4.9) we get:

$$\mathcal{L} = \Pi_+ \left( F_2(L) F_1(L) \right), \quad \mathcal{R} = \Pi_- \left( F_2(L) F_1(L) \right)$$

and the theorem is proved. □

Theorem 4.2 implies that the flows of two arbitrary differential equations of the form (3.1) commute. Another important consequence of Theorem 4.2 is the following discrete-time counterpart of Theorem 4.1, going back to [Sy].

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Theorem 4.3 Let $F : g \mapsto G$ be a conjugation covariant function. Then the solution of the difference equation

$$\tilde{L} = \Pi_+^{-1}(F(L)) \cdot L \cdot \Pi_+(F(L)) = \Pi_-(F(L)) \cdot L \cdot \Pi_-^{-1}(F(L))$$  \hspace{1cm} (4.10)

where $L = L(n), \tilde{L} = L(n+1)$, with the initial condition $L(0) = L_0$, is given by

$$L(n) = \Pi_+^{-1}(F^n(L_0)) \cdot L_0 \cdot \Pi_+(F^n(L_0)) = \Pi_-(F^n(L_0)) \cdot L_0 \cdot \Pi_-^{-1}(F^n(L_0))$$  \hspace{1cm} (4.11)

Proof. From Theorem 4.2 there follows by induction that $(B_F)^n = B_{F^n}$.

Comparing the formulas (4.11), (4.2), we see that the map (4.10) is the time shift along the trajectories of the flow (3.1) with

$$f(L) = h^{-1} \log(F(L))$$

The above results are purely kinematic, in the sense that no additional Hamiltonian structure is necessary neither to formulate nor to prove them. However, as mentioned above, the equations (3.1) often admit a Hamiltonian or even a multi-Hamiltonian interpretation. If this is the case, then we get some useful additional information. In particular, all maps (4.10) are Poisson with respect to the invariant Poisson bracket of the hierarchy (3.1), being shifts along the trajectories of Hamiltonian flows. Further, if the set of Lax matrices $L(\mathcal{X})$ for the system at hand forms a Poisson submanifold for one of the $r$-matrix brackets on $g$, then this manifold is left invariant by the flows (3.1) and by the maps (4.10). The functions on $\mathcal{X}$ of the form $I \circ L$, where $I$ are conjugation invariants of $g$, are integrals of motion of the corresponding systems and in involution with respect to $\{,\}$. We close this section by giving analogous results for the Lax equations on the direct products $g = \otimes_{j=1}^m g_j$.

Theorem 4.4 For a conjugation covariant function $f : g \mapsto g$ the solution of the equation (3.2) with the initial value $L(0)$ is given, at least for $t$ small enough, by the formula

$$L_j(t) = \Pi_+^{-1}\left( e^{tf(T_j(0))} \right) \cdot L_j(0) \cdot \Pi_+\left( e^{tf(T_j-1(0))} \right) = \Pi_-\left( e^{tf(T_j(0))} \right) \cdot L_j(0) \cdot \Pi_-^{-1}\left( e^{tf(T_j-1(0))} \right)$$  \hspace{1cm} (4.12)

Theorem 4.5 For a conjugation covariant function $F : g \mapsto G$ consider the following system of difference equations on $g$:

$$\tilde{L}_j = \Pi_+^{-1}(F(T_j)) \cdot L_j \cdot \Pi_+(F(T_j-1)) = \Pi_-(F(T_j)) \cdot L_j \cdot \Pi_-^{-1}(F(T_j-1))$$  \hspace{1cm} (4.13)

Its solution with the initial value $L(0)$ is given by the formula

$$L_j(n) = \Pi_+^{-1}(F^n(T_j(0))) \cdot L_j(0) \cdot \Pi_+(F^n(T_j-1(0))) = \Pi_-(F^n(T_j(0))) \cdot L_j(0) \cdot \Pi_-^{-1}(F^n(T_j-1(0)))$$  \hspace{1cm} (4.14)

The proofs are kinematic and absolutely parallel to the case of the "small" algebras $g$.  

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5 Recipe for integrable discretization

The results of the previous section inspire the following recipe for integrable discretization, clearly formulated for the first time in [S6, S7].

**Recipe.** Suppose you are looking for an integrable discretization of an integrable system (2.1) allowing a Lax representation of the form (3.1). Then as a solution of your task you may take the difference equation (4.10) with the same Lax matrix $L$ and some conjugation covariant function $F : g \mapsto G$ such that

$$F(L) = I + hf(L) + O(h^2)$$

Analogously, if your system has a Lax representation of the form (3.2) on the algebra $g$, then you may take as its integrable discretization the difference Lax equation (4.13) with $F$ as above.

Of course, this prescription makes sense only if the corresponding factors $\Pi_{\pm} (F(L))$ [resp. $\Pi_{\pm} (F(T_j))$] admit more or less explicit expressions, allowing to write down the corresponding difference equations in a more or less closed form. The choice of $F$ is a transcendent problem, which however turns out to be solvable for many of (hopefully, for the majority of or even for all) the known integrable systems. The simplest possible choice $F(L) = I + hf(L)$ works perfectly well for a vast set of examples considered below.

Let us stress the advantages of this approach to the problem of integrable discretization.

- Although we formulate our recipe only for systems with Lax representations of the particular form, it is in fact more universal. Almost without changes it may be applied to any system whose Lax representation is governed by an $R$–operator satisfying the modified Yang–Baxter equation.

- The discretizations obtained in this way share the Lax matrix and therefore the integrals of motion with their underlying continuous time systems.

- If the Lax representation (3.3) [resp. (3.2)] allows an $r$–matrix interpretation, then our discretizations share also the invariant Poisson bracket with the underlying continuous time systems. In particular, if the Lax matrices $L$ [resp. $L$] form a Poisson submanifold for some $r$–matrix bracket, then this submanifold is left invariant by the corresponding Poisson map (4.10) [resp. (4.13)].

- The initial value problem for our discrete time equations can be solved in terms of the same factorization in a Lie group as the initial value problem for the continuous time system.

- Interpolating Hamiltonian flows also belong to the set of granted by–products of this approach.
6 Localizing changes of variables

Along with these advantageous properties our recipe has also an important drawback: it produces, as a rule, nonlocal difference equations, when applied to lattice systems with local interactions. Under \textit{locality} we understand the following property: in some coordinates \((x_1, \ldots, x_N)\) on \(\mathcal{X}\) the equations of motion \((6.1)\) have the form
\[
\dot{x}_k = \phi_k(x_k, x_{k\pm 1}, \ldots, x_{k\pm s})
\]
with a fixed \(s \in \mathbb{N}\). Nonlocal difference equations produced by our scheme have the form
\[
\bar{x}_k = x_k + h\Phi_k(x_k), \quad \Phi_k(x, 0) = \phi_k(x)
\]
where \(\Phi_k\) depends explicitly on all \(x_j\), not only on \(2s\) nearest neighbours of \(x_k\). The aim of the present paper is to demonstrate on a large number of examples how this drawback can be overcome, i.e. how to bring the latter difference equations into a local form; the price we have to pay is that they become implicit.

The general strategy will be to find \textit{localizing changes of variables} \(\mathcal{X}(x) \mapsto \mathcal{X}(x)\) such that in the variables \(x\) the map \((6.2)\) may be written as
\[
\bar{x}_k = x_k + h\Psi_k(x, \bar{x}_j; h), \quad \Psi_k(x, x; 0) = \phi_k(x)
\]
where \(\Psi_k\) depends only on the \(x_j\)'s and \(\bar{x}_j\)'s with correct indices \(|j - k| \leq s\). Such implicit local equations of motion are much better suited for the purposes of numerical simulation and are much more satisfactory from the esthetical point of view. Moreover, in all our examples the functions \(\Psi_k\) actually depend only on \(x_j\)'s with \(k \leq j \leq k + s\) and on \(\bar{x}_j\)'s with \(k - s \leq j \leq k\), which makes the practical implementation of the corresponding difference equations even more effective (if, for instance, one uses the Newton’s iterative method to solve \((6.3)\) for \(\bar{x}\), then one has to solve only linear systems whose matrices are \textit{triangular} and have a band structure, i.e. only \(s\) nonzero diagonals). Further, it has to be remarked that, when considered as equations on the lattice \((t, k)\), the equations \((6.3)\) often allow transformations of independent variables (mixing \(t\) and \(k\)) bringing these equations into local \textit{explicit} form (cf. \[PNC\]). This last remark will be the subject of a separate publication.

It is by no means evident that such localizing changes of variables exist, but we give in this paper a large number of examples which, hopefully, will convince the reader that this is indeed the case. The probably first examples appeared in the context of the Bogoyavlensky lattices in \[SS\], and will be reproduced here in a clarified form for the sake of completeness.

The localizing changes of variables turn out to have many additional remarkable properties. They are always given by the formulas
\[
x_k = x_k + h\Xi_k(x; h)
\]
with local functions $\Xi_k$. However, the inverse change of variables is always nonlocal. Therefore nothing guarantees a priori that the pull–back of the differential equations of motion (6.1) under the change of variables (6.4) will be given by local formulas. Nevertheless, this turns out to be the case. This gives a way of producing (sometimes novel) one–parameter families of integrable local deformations of lattice systems (see [K1] for a general concept and some examples of integrable deformations).

The system (6.1) often admits one or several invariant local Poisson brackets. Nothing guarantees a priori that the pull–backs of these brackets under the change of variables (6.4) are also given by local formulas. Indeed, as a rule these pull–backs are non–local. However, in the multi–Hamiltonian case it often turns out that pull–backs of certain linear combinations of invariant Poisson brackets are local again!

These facts still wait to be completely understood. It seems that the remarkable properties of the localizing maps have the same nature as that of the Miura maps (“miraculous cancellations”). Moreover, actually our localizing changes of variables are Miura maps, and in this image some of them already appeared in [K1]. However, the observation that they bring integrable discretizations into a local form, seems to be completely new. The scope of the present work is restricted to elaborating a large number of examples of localizing changes of variables, along with their Poisson properties, in a hope to attract the attention of the soliton community to these fascinating and beautiful objects.

7 Basic algebras and operators

Two concrete algebras play the basic role in our presentation. They are well suited to describe various lattice systems with the so called open–end and periodic boundary conditions, respectively. Here are the relevant definitions.

For the open–end case we always set $g = gl(N)$, the algebra of $N \times N$ matrices with the usual matrix product, the Lie bracket $[u, v] = uv – vu$, and the nondegenerate bi–invariant scalar product $\langle u, v \rangle = \text{tr}(u \cdot v)$. As a linear space, $g$ may be presented as a direct sum

$$g = g_+ \oplus g_-$$

where the subalgebras $g_+$ and $g_-$ consist of lower triangular and of strictly upper triangular matrices, respectively.

The Lie group $G$ corresponding to the Lie algebra $g$ is $GL(N)$, the group of $N \times N$ nondegenerate matrices. The subgroups $G_+, G_-$ corresponding to the Lie algebras $g_+, g_-$ consist of nondegenerate lower triangular matrices and of upper triangular matrices with unit diagonal, respectively. The $\Pi_+\Pi_-$ factorization is well known in the linear algebra under the name of the LU factorization.

In the periodic case we always choose as $g$ a certain twisted loop algebra over $gl(N)$. A loop algebra over $gl(N)$ is an algebra of Laurent polynomials with coefficients from $gl(N)$.
and a natural commutator $[u \lambda^j, v \lambda^k] = [u, v] \lambda^{j+k}$. Our twisted algebra $g$ is a subalgebra singled out by the additional condition

$$g = \left\{ u(\lambda) \in gl(N)[\lambda, \lambda^{-1}] : \Omega u(\lambda)\Omega^{-1} = u(\omega \lambda) \right\},$$

where $\Omega = \text{diag}(1, \omega, \ldots, \omega^{N-1})$, $\omega = \exp(2\pi i/N)$. In other words, elements of $g$ satisfy

$$u(\lambda) = \sum_p \sum_{j \equiv k \pmod{N}} \lambda^p u_{jk} E_{jk}$$  \hspace{1cm} (7.1)

(Here and below $E_{jk}$ stands for the matrix whose only nonzero entry is on the intersection of the $j$th row and the $k$th column and is equal to 1). The nondegenerate bi–invariant scalar product is chosen as

$$\langle u(\lambda), v(\lambda) \rangle = \text{tr}(u(\lambda) \cdot v(\lambda))_0$$  \hspace{1cm} (7.2)

the subscript 0 denoting the free term of the formal Laurent series. This scalar product allows to identify $g^*$ with $g$.

As a linear space, $g$ is again a direct sum

$$g = g_+ \oplus g_-$$

with the subalgebras

$$g_+ = \bigoplus_{k \geq 0} \lambda^k g_k, \hspace{1cm} g_- = \bigoplus_{k < 0} \lambda^k g_k$$  \hspace{1cm} (7.3)

The group $G$ corresponding to the Lie algebra $g$ is a twisted loop group, consisting of $GL(N)$–valued functions $U(\lambda)$ of the complex parameter $\lambda$, regular in $\mathbb{C}P^1 \setminus \{0, \infty\}$ and satisfying $\Omega U(\lambda)\Omega^{-1} = U(\omega \lambda)$. Its subgroups $G_+$ and $G_-$ corresponding to the Lie algebras $g_+$ and $g_-$, are singled out by the following conditions:

- $U(\lambda) \in G_+$ are regular in the neighbourhood of $\lambda = 0$;
- $U(\lambda) \in G_-$ are regular in the neighbourhood of $\lambda = \infty$ and $U(\infty) = I$.

We call the corresponding $\Pi_+ \Pi_-$ factorization the generalized $LU$ factorization. It is uniquely defined in a certain neighbourhood of the unit element of $G$. As opposed to the open–end case, finding the generalized $LU$ factorization is a problem of the Riemann–Hilbert type which is solved in terms of algebraic geometry rather than in terms of linear algebra.
8 Toda lattice

8.1 Equations of motion and tri–Hamiltonian structure

The equations of motion of the Toda lattice (hereafter TL) read:

\[ \dot{a}_k = a_k(b_{k+1} - b_k), \quad \dot{b}_k = a_k - a_{k-1}, \quad 1 \leq k \leq N, \] (8.1)

with one of the two types of boundary conditions: open–end \((a_0 = a_N = 0)\), or periodic (all subscripts are taken \((\text{mod} \ N)\), so that \(a_0 \equiv a_N, b_{N+1} \equiv b_1\)).

The phase space of the TL in the case of the periodic boundary conditions is

\[ \mathcal{T} = \mathbb{R}^{2N}(b_1, a_1, \ldots, b_N, a_N) \] (8.2)

There exist three compatible local Poisson brackets on \(\mathcal{T}\) such that the system TL is Hamiltonian with respect to each one of them \([A], [K1]\), see also \([D]\). We adopt once and forever the following conventions: the Poisson brackets will be defined by writing down all nonvanishing brackets between the coordinate functions; the indices in the corresponding formulas are taken \((\text{mod} \ N)\).

The ”linear” Poisson structure on \(\mathcal{T}\) is defined by the brackets

\[ \{b_k, a_k\}_1 = -a_k, \quad \{a_k, b_{k+1}\}_1 = -a_k \] (8.3)

the corresponding Hamilton function for the flow TL is given by:

\[ H_2(a, b) = \frac{1}{2} \sum_{k=1}^{N} b_k^2 + \sum_{k=1}^{N} a_k \] (8.4)

The ”quadratic” Poisson structure has the following definition:

\[ \{b_k, a_k\}_2 = -a_k b_k, \quad \{a_k, b_{k+1}\}_2 = -a_k b_{k+1} \] \[ \{a_k, a_{k+1}\}_2 = -a_{k+1} a_k, \quad \{b_k, b_{k+1}\}_2 = -a_k \] (8.5)

The Hamilton function generating TL in this bracket is:

\[ H_1(a, b) = \sum_{k=1}^{N} b_k \] (8.6)

Finally, the ”cubic” bracket on \(\mathcal{T}\) is given by the relations

\[ \{b_k, a_k\}_3 = -a_k (b_k^2 + a_k), \quad \{a_k, b_{k+1}\}_3 = -a_k (b_{k+1}^2 + a_k), \] \[ \{a_k, a_{k+1}\}_3 = -2a_k a_{k+1} b_{k+1}, \quad \{b_k, b_{k+1}\}_3 = -a_k (b_k + b_{k+1}), \] \[ \{a_k, b_{k+2}\}_3 = -a_k a_{k+1}, \quad \{b_k, a_{k+1}\}_3 = -a_k a_{k+1} \] (8.7)
The expression for the corresponding Hamilton function, suitable in both the periodic and the open–end case, is nonlocal in the coordinates \((a, b)\). However, in the periodic case one has an alternative Hamilton function:

\[
H_0(a, b) = \frac{1}{2} \sum_{k=1}^{N} \log(a_k)
\]

(8.8)

**8.2 Lax representation**

The Lax representation of the Toda lattice [F], [M] lives in the algebra \(g\) introduced in Sect. 7. Actually, there exist different versions of the Lax representation connected with different ways to represent the algebra \(g\) as a direct sum of its two subalgebras [DLT1]. We discuss here only the \(LU\) version which corresponds in the open–end case to the decomposition of an arbitrary matrix into the sum of a lower triangular and a strictly upper triangular matrices.

The Lax matrix \(T : T \mapsto g\) of the TL corresponding to the generalized \(LU\) decomposition is:

\[
T(a, b, \lambda) = \lambda^{-1} \sum_{k=1}^{N} a_k E_{k,k+1} + \sum_{k=1}^{N} b_k E_{k,k} + \lambda \sum_{k=1}^{N} E_{k+1,k}
\]

(8.9)

We use here and below a convention according to which in the periodic case \(E_{N+1,N} = E_{1,N}, E_{N,N+1} = E_{N,1}\); in the open–end case \(E_{N+1,N} = E_{N,N+1} = 0\) and we may set \(\lambda = 1\).

The equations of motion (8.1) are equivalent to the Lax equations

\[
\dot{T} = [T, B_+] = -[T, B_-]
\]

(8.10)

with

\[
B_+(a, b, \lambda) = \pi_+(T(a, b, \lambda)) = \sum_{k=1}^{N} b_k E_{k,k} + \lambda \sum_{k=1}^{N} E_{k+1,k}
\]

(8.11)

\[
B_-(a, b, \lambda) = \pi_-(T(a, b, \lambda)) = \lambda^{-1} \sum_{k=1}^{N} a_k E_{k,k+1}
\]

(8.12)

where \(\pi_\pm : g \mapsto g_\pm\) are the projections to the subalgebras \(g_\pm\) defined as in Sect. 7 (the generalized \(LU\) decomposition).

Spectral invariants of the Lax matrix \(T(a, b, \lambda)\) serve as integrals of motion of this system. Note that all Hamilton functions in different Hamiltonian formulations belong to these spectral invariants. For instance,

\[
H_2(a, b) = \frac{1}{2} \left( \text{tr} T^2(a, b, \lambda) \right)_0, \quad H_1(a, b) = \left( \text{tr} T(a, b, \lambda) \right)_0
\]

where the subscript "0" is used to denote the free term of the corresponding Laurent series.
All spectral invariants turn out to be in involution with respect to each of the Poisson brackets (8.3), (8.5), (8.7). Most directly it follows from the $r$–matrix interpretation of the Lax equation (8.10), which can be given for all three brackets [AM], [DLT1], [OR], [S5], [MP].

8.3 Discretization

In order to find an integrable time discretization for the flow TL, we apply the recipe of Sect. 5 with $F(T) = I + hT$, i.e. we take as a solution of this problem the map described by the discrete time Lax equation

$$\tilde{T} = B_+^{-1}TB_+ = B_-TB_-^{-1} \quad \text{with} \quad B_\pm = \Pi_\pm(I + hT) \quad (8.13)$$

**Theorem 8.1** [S6] (see also [GR]). The discrete time Lax equation (8.13) is equivalent to the map $(a,b) \mapsto (\tilde{a}, \tilde{b})$ described by the following equations:

$$\tilde{a}_k = a_k \frac{\beta_{k+1}}{\beta_k}, \quad \tilde{b}_k = b_k + h \left( \frac{a_k}{\beta_k} - \frac{a_{k-1}}{\beta_{k-1}} \right) \quad (8.14)$$

where the functions $\beta_k = \beta_k(a,b) = 1 + O(h)$ are uniquely defined by the recurrent relation

$$\beta_k = 1 + h\beta_k - \frac{h^2a_{k-1}}{\beta_{k-1}} \quad (8.15)$$

and have the asymptotics

$$\beta_k = 1 + hb_k + O(h^2) \quad (8.16)$$

**Remark.** The matrices $B_\pm$ have the following expressions:

$$B_+(a,b,\lambda) = \Pi_+(I + hT) = \sum_{k=1}^{N} \beta_k E_{k,k} + h\lambda \sum_{k=1}^{N} E_{k+1,k} \quad (8.17)$$

$$B_-(a,b,\lambda) = \Pi_-(I + hT) = I + h\lambda^{-1} \sum_{k=1}^{N} \frac{a_k}{\beta_k} E_{k,k+1} \quad (8.18)$$

**Proof.** The bi–diagonal structure of the factors $B_\pm$, as well as the expressions for the entries of $B_-$, follow from the tri–diagonal structure of the matrix $T$. The recurrent relation (8.13) for the entries of $B_+$ is equivalent to $B_+B_- = I + hT$. The equations of motion (8.14) are now nothing but the componentwise form of the matrix equation $B_+\tilde{T} = TB_+$. 

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The map (8.14) will be denoted dTL. Due to the asymptotics (8.16) it is easy to see that the equations of motion (8.14) of the dTL serve as a difference approximation to the Toda flow TL (8.1). The construction assures numerous positive properties of this discretization: the map dTL is Poisson with respect to each one of the Poisson brackets (8.3), (8.5), (8.7), it has the same integrals of motion as the flow TL, the Lax representation for the dTL lies in its very definition, etc. The only unpleasant property of the equations of motion (8.14), when compared with their continuous time counterparts (8.1), is the nonlocality. The source of nonlocality are the functions $\beta_k$. In the open–end case they have explicit expressions in terms of finite continued fractions:

$$
\beta_k = 1 + h b_k - \frac{h^2 a_{k-1}}{1 + h b_{k-1} - \ldots - \frac{h^2 a_1}{1 + h b_1}}
$$

In the periodic case for small $h$ the $\beta_k$’s may be expressed as analogous infinite (periodic) continued fractions.

### 8.4 Local equations of motion for dTL

Fortunately, there exist different ways to bring these equations of motion into a local form connected with a localizing change of variables in the sense of Sect. 6. Consider another copy of the phase space $T$. The coordinates in this another copy will be denoted by $a_k$, $b_k$. The localizing change of variables for dTL is the map $T(a, b) \mapsto T(a, b)$ defined by the following formulas:

$$
a_k = a_k (1 + h b_k), \quad b_k = b_k + h a_{k-1} \quad (8.19)
$$

The implicit functions theorem assures that this is a local diffeomorphism between the two copies of $T$ by small enough values of $h$. This change of variables appeared in [K1] as a Miura map in connection with the problem of deformation of integrable systems, but without any relation to integrable discretizations.

**Theorem 8.2** The change of variables (8.19) conjugates the map dTL with the map $(a, b) \mapsto (\tilde{a}, \tilde{b})$ described by the following equations of motion:

$$
\tilde{a}_k (1 + h \tilde{b}_k) = a_k (1 + h b_{k+1}), \quad \tilde{b}_k = b_k + h (a_k - \tilde{a}_{k-1}) \quad (8.20)
$$

**Proof.** The key point is the following observation: the auxiliary functions $\beta_k$ acquire in the new coordinates local expressions, namely,

$$
\beta_k = 1 + h b_k \quad (8.21)
$$
To demonstrate this, it is enough to notice that from (8.19) there follows that the quantities
\[ 1 + h b_k \]
satisfy the same recurrent relation as the quantities \( \beta_k \), namely
\[
1 + h b_k = 1 + hb_k - \frac{h^2 a_{k-1}}{1 + h b_{k-1}}
\]
Due to the uniqueness of the solution with the asymptotics \( 1 + O(h) \) the formula (8.21) is proved. The statement of the theorem follows now immediately from (8.14), (8.21).

Of course, the map (8.20) is Poisson with respect to pull–backs of the three invariant Poisson structures of the Toda lattice. These pull–backs are described by highly nonlocal and non–polynomial formulas. However, there exist certain linear combinations of the basic Poisson structures in coordinates \((a, b)\) whose pull–backs to the coordinates \((a, b)\) are local.

**Theorem 8.3**

a) The pull–back of the bracket

\[
\{ \cdot, \cdot \}_1 + h \{ \cdot, \cdot \}_2
\]

on \( T(a, b) \) under the change of variables (8.19) is the following bracket on \( T(a, b) \):

\[
\{ b_k, a_k \} = -a_k(1 + h b_k), \quad \{ a_k, b_{k+1} \} = -a_k(1 + h b_{k+1})
\]

b) The pull–back of the bracket

\[
\{ \cdot, \cdot \}_2 + h \{ \cdot, \cdot \}_3
\]

on \( T(a, b) \) under the change of variables (8.19) is the following bracket on \( T(a, b) \):

\[
\begin{align*}
\{ b_k, a_k \} &= -a_k(b_k + h a_k)(1 + h b_k), & \{ a_k, b_{k+1} \} &= -a_k(b_{k+1} + h a_k)(1 + h b_{k+1}) \\
\{ a_k, a_{k+1} \} &= -a_k a_{k+1}(1 + h b_{k+1}), & \{ b_k, b_{k+1} \} &= -a_k(1 + h b_k)(1 + h b_{k+1})
\end{align*}
\]

c) The brackets (8.23), (8.25) are compatible. The map (8.20) is Poisson with respect to both of them.

**Proof.** To prove the theorem, one has, for example, in the (less laborious) case a) to verify the following statement: the formulas (8.23) imply that the nonvanishing pairwise Poisson brackets of the functions (8.19) are

\[
\begin{align*}
\{ b_k, a_k \} &= a_k(1 + h b_k), & \{ a_k, b_{k+1} \} &= -a_k(1 + h b_{k+1}) \\
\{ a_k, a_{k+1} \} &= -h a_k a_{k+1}, & \{ b_k, b_{k+1} \} &= -h a_k
\end{align*}
\]

This verification consists of straightforward calculations. In what follows we do not repeat analogous arguments in the similar situations. ■
The map (8.20) was first found in [HTI], along with the Lax representation. In [S0] it was stressed that this map is nothing other than the so-called $qd$ algorithm well known in the numerical analysis. Its Poisson structure and its place in the continuous time Toda hierarchy were not discussed in [HTI]. The previous theorem provides a bi–Hamiltonian structure of the $qd$ algorithm. This result in a slightly different form was found in [S6].

**Theorem 8.4** [K1]. The pull–back of the flow TL under the change of variables (8.19) is described by the following differential equations:

$$\dot{a}_k = a_k(b_{k+1} - b_k), \quad \dot{b}_k = (a_k - a_{k-1})(1 + h b_k)$$

(8.26)

**Proof.** To determine the pull–back of the flow TL, we can use the Hamiltonian formalism. An opportunity to apply it is given by the Theorem 8.3. We shall use the statement a) only. Consider the function $h^{-1} H_1(a, b) = h^{-1} \sum_{k=1}^{N} b_k$. It is a Casimir of the bracket $\{ \cdot, \cdot \}_1$, and generates exactly the flow TL in the bracket $h\{ \cdot, \cdot \}_2$. Hence it generates the flow TL also in the bracket (8.22). The pull–back of this Hamilton function is equal to $h^{-1} \sum_{k=1}^{N} (b_k + ha_{k-1})$. It remains only to calculate the flow generated by this function in the Poisson brackets (8.23). This results in the equations of motion (8.26). ■

**9 Second flow of the Toda hierarchy**

Now we want to demonstrate that our method for finding integrable discretizations and local equations of motion for them works not only for the flow TL, but equally well for the higher flows of the Toda hierarchy. We consider here the second flow (called hereafter TL2).

**9.1 Equations of motion and tri–Hamiltonian structure**

This is the flow on $\mathcal{T}$ governed by the differential equations

$$\dot{a}_k = a_k(b_{k+1}^2 - b_k^2 + a_{k+1} - a_{k-1}), \quad \dot{b}_k = a_k(b_{k+1} + b_k) - a_{k-1}(b_k + b_{k-1})$$

(9.1)

This flow is Hamiltonian with respect to all three brackets (8.3), (8.5), (8.7). The corresponding Hamilton functions are:

$$H_3(a, b) = \frac{1}{3} \sum_{k=1}^{N} b_k^3 + \sum_{k=1}^{N} b_k(a_k + a_{k-1}) = \frac{1}{3} \left( \text{tr} \ T^3(a, b, \lambda) \right)_0$$

(9.2)

for the bracket $\{ \cdot, \cdot \}_1$, $H_2(a, b)$ for the bracket $\{ \cdot, \cdot \}_2$, and $H_1(a, b)$ for the bracket $\{ \cdot, \cdot \}_3$. 

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9.2 Lax representation

Naturally, the Lax matrix for the flow TL2 is the same as for the flow TL. The difference lies in the auxiliary matrices $B_\pm$ taking part in the Lax representation (since in this section we are dealing only with the flow TL2, using the same notations $B_\pm$ as in Sect. 8 will not lead to confusions; the same holds also for some other notations in this section).

The equations of motion (9.1) are equivalent to the Lax equations in $g$

$$\dot{T} = [T, B_+] = -[T, B_-] \quad \text{with} \quad B_\pm = \pi_\pm(T^2)$$

so that

$$B_+(a, b, \lambda) = \sum_{k=1}^N (b_k^2 + a_k + a_{k-1}) E_{k,k} + \lambda \sum_{k=1}^N (b_{k+1} + b_k) E_{k+1,k} + \lambda^2 \sum_{k=1}^N E_{k+2,k}$$

$$B_-(a, b, \lambda) = \lambda^{-1} \sum_{k=1}^N (b_{k+1} + b_k) a_k E_{k,k+1} + \lambda^{-2} \sum_{k=1}^N a_{k+1} a_k E_{k,k+2}$$

9.3 Discretization

In order to obtain an integrable discretization of the flow TL2 we can apply the recipe of Sect. 8 with $F(T) = I + hT^2$, i.e. consider the map described by the discrete time Lax equation

$$\tilde{T} = B_+^{-1} T B_+ = B_- T B_-^{-1} \quad \text{with} \quad B_\pm = \Pi_\pm(I + hT^2)$$

**Theorem 9.1** The discrete time Lax equation (9.6) is equivalent to the map $(a, b) \mapsto (\tilde{a}, \tilde{b})$ described by the following equations:

$$\tilde{a}_k = a_k \frac{\beta_{k+1}}{\beta_k}, \quad \tilde{b}_k = b_k + h \left( a_k \delta_k \frac{a_k \delta_{k-1}}{\beta_k} - \frac{a_{k-1} \delta_{k-1}}{\beta_{k-1}} \right)$$

where the auxiliary functions $\delta_k = \delta_k(a, b) = O(1)$ and $\beta_k = \beta_k(a, b) = 1 + O(h)$ are uniquely defined for $h$ small enough by the recurrent relations

$$\delta_k = b_{k+1} + b_k - \frac{h a_{k-1} \delta_{k-1}}{\beta_{k-1}}$$

$$\beta_k = 1 + h(b_k^2 + a_k + a_{k-1}) - \frac{h^2 a_{k-1} \delta_{k-1}^2}{\beta_{k-1}} - \frac{h^2 a_{k-1} a_{k-2}}{\beta_{k-2}}$$

and have the asymptotics

$$\delta_k = b_{k+1} + b_k + O(h)$$

$$\beta_k = 1 + h(b_k^2 + a_k + a_{k-1}) + O(h^2)$$
Remark. The matrices $B_\pm$ from this theorem have the following expressions:

$$B_+(a, b, \lambda) = \sum_{k=1}^{N} \beta_k E_{k,k} + h\lambda \sum_{k=1}^{N} \delta_k E_{k+1,k} + h\lambda^2 \sum_{k=1}^{N} E_{k+2,k} \quad (9.12)$$

$$B_-(a, b, \lambda) = I + h\lambda^{-1} \sum_{k=1}^{N} a_k \delta_k E_{k,k+1} + h\lambda^{-2} \sum_{k=1}^{N} \frac{a_{k+1}a_k}{\beta_k} E_{k,k+2} \quad (9.13)$$

Proof. The tri-diagonal structure of the matrix $T$ assures that the factors $B_\pm$ have the structure as in (9.12), (9.13). The expressions for the entries of $B_-$ and the recurrent relations for the entries of $B_+$ follow easily from the equality $B_+ B_- = I + hT^2$. After that the equations of motion follow from the equation $B_+ T = T B_+$. ■

9.4 Local equations of motion for dTL2

For the map (9.7), called hereafter dTL2, there exists a localizing change of variables, however different from the one used for the map dTL. It is given by the formulas

$$a_k = a_k (1 + ha_{k-1})(1 + h b_k^2), \quad b_k = b_k (1 + ha_{k-1}) + h b_{k-1} a_{k-1} \quad (9.14)$$

(Actually, discretizations of all higher flows of the Toda hierarchy should possess their own localizing changes of variables).

The name ”localizing change of variables” is justified by the following theorems.

**Theorem 9.2** The change of variables (9.14) conjugates the map dTL2 with the map $(a, b) \mapsto (\tilde{a}, \tilde{b})$ governed by the following local equations of motion:

$$\tilde{a}_k (1 + h\tilde{a}_{k-1})(1 + h\tilde{b}_k^2) = a_k (1 + ha_{k+1})(1 + h b_{k+1}^2) \quad (9.15)$$

$$\tilde{b}_k - b_k = ha_k (b_k + b_{k+1}) - h\tilde{a}_{k-1}(\tilde{b}_{k-1} + \tilde{b}_k)$$

**Proof.** This time the key point of the proof is obtaining the following local expressions for the coefficients of the factor $\Pi_+ (I + hT^2)$:

$$\beta_k = (1 + ha_k)(1 + ha_{k-1})(1 + h b_k^2) \quad (9.16)$$

$$\delta_k = (1 + ha_k)(b_k + b_{k+1}) \quad (9.17)$$

Indeed, the equations of motion (9.15) follow directly from (9.7) and the latter formulas. To prove the latter formulas, define the quantities $\beta_k$, $\delta_k$ by the equations (9.16), (9.17). A straightforward calculation convinces that then the recurrent relations (9.9), (9.8) are
satisfied. The uniqueness of solution to these equations proves the desired expressions for $\beta_k, \delta_k$. ■

There still exists a linear combination of invariant Poisson brackets of the Toda hierarchy, whose pull–back with respect to the above change of variables is local.

**Theorem 9.3** The pull–back of the bracket

$$\{\cdot, \cdot\}_1 + h\{\cdot, \cdot\}_3$$

on $T(a, b)$ under the change of variables (9.14) is the following bracket on $T(a, b)$:

$$\{b_k, a_k\} = -a_k(1 + ha_k)(1 + hb_k^2)$$

(9.19)

$$\{a_k, b_{k+1}\} = -a_k(1 + ha_k)(1 + h b_{k+1}^2)$$

The map (9.17) is Poisson with respect to (9.19).

**Proof** – by straightforward verification. ■

**Theorem 9.4** The pull–back of the flow $TL$ under the map (9.14) is described by the following differential equations:

$$\dot{a}_k = a_k(b_{k+1} - b_k)(1 + ha_k), \quad \dot{b}_k = (a_k - a_{k-1})(1 + h b_k^2)$$

(9.20)

**Proof.** The equations of motion we are looking for describe the flow $TL$ as a Hamiltonian system in the Poisson bracket (9.18). The corresponding Hamilton function is, obviously, given by

$$(I + hT^2)\nabla H(T) = T$$

Hence we have $H(T) = (2h)^{-1}\left(\log\det(I + hT^2)\right)_0$. From the expressions for the factors of the generalized $LU$ factorization of the matrix $I + hT^2$ given in the formulas (9.12), (9.13) we conclude that $H(T) = (2h)^{-1}\sum_{k=1}^{N}\log(\beta_k)$. Taking into account the expressions (9.16), we find finally:

$$H = \frac{1}{2h}\sum_{k=1}^{N}\log(1 + h b_k^2) + \frac{1}{h}\sum_{k=1}^{N}\log(1 + h a_k)$$

(9.21)

The Hamiltonian equations of motion generated by this Hamilton function and the Poisson brackets (9.19), coincide with (9.20). ■

The differential equations (9.20) are a particular case of a two–parameter deformation of $TL$ found in [K1].

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10 Volterra lattice

10.1 Equations of motion and bi–Hamiltonian structure

The second flow \((\mathcal{H}_2)\) of the Toda hierarchy, unlike the first one, allows an important reduction \(b_k = 0\), and it results in the famous Volterra lattice (hereafter VL), known also under the names of Lotka–Volterra system, discrete KdV equation, Langmuir lattice, Kac–van Moerbecke lattice, etc. \([\text{M}], [\text{KM}]\):

\[
\dot{a}_k = a_k(a_{k+1} - a_{k-1}) \tag{10.1}
\]

The phase space of VL in the case of periodic boundary conditions is

\[
\mathcal{V} = \mathbb{R}^N(a_1, \ldots, a_N) \tag{10.2}
\]

Clearly, it is the subspace \(b_k = 0\) of \(\mathcal{T}(a, b)\). Unfortunately, neither of three invariant Poisson brackets \(\{·, ·\}_1, \{·, ·\}_2, \text{ and } \{·, ·\}_3\) of the Toda hierarchy can be properly restricted to the set \(\mathcal{V}\). However, it is easy to see that the quadratic bracket \(\{·, ·\}_2\) allows a Dirac reduction to this set, the reduced bracket being defined by the relations

\[
\{a_k, a_{k+1}\}_2 = -a_k a_{k+1} \tag{10.3}
\]

The system VL is Hamiltonian with respect to this bracket with the Hamilton function

\[
H_1(a) = \sum_{k=1}^{N} a_k
\]

There exists one more local Poisson bracket on \(\mathcal{V}\) invariant with respect to VL and compatible with \((10.3)\) \([\text{K1}], [\text{FT1}]\). It is given by the relations

\[
\{a_k, a_{k+1}\}_3 = -a_k a_{k+1}(a_k + a_{k+1}), \quad \{a_k, a_{k+2}\}_3 = -a_k a_{k+1} a_{k+2} \tag{10.4}
\]

the corresponding Hamilton function being equal to

\[
H_0(a) = \frac{1}{2} \sum_{k=1}^{N} \log(a_k)
\]

10.2 Lax representation

The Lax representation of the previous section survives in the process of restriction to \(\mathcal{V}\). So, we obtain the Lax representation of the VL with the matrices from \(\mathcal{G} \ [\text{M}], \ [\text{KM}]\):

\[
\dot{T} = [T, B_+] = -[T, B_-] \tag{10.5}
\]
where
\[
T(a, \lambda) = \lambda^{-1} \sum_{k=1}^{N} a_k E_{k,k+1} + \lambda \sum_{k=1}^{N} E_{k+1,k} \quad (10.6)
\]
\[
B_+(a, \lambda) = \pi_+(T^2) = \sum_{k=1}^{N} (a_k + a_{k-1}) E_{k,k} + \lambda^2 \sum_{k=1}^{N} E_{k+2,k} \quad (10.7)
\]
\[
B_-(a, \lambda) = \pi_-(T^2) = \lambda^{-2} \sum_{k=1}^{N} a_{k+1} a_k E_{k,k+2} \quad (10.8)
\]
These equations allow an $r$–matrix interpretation in case of the quadratic bracket $\{\cdot, \cdot\}_2$, see [S4]. The Volterra hierarchy consists of the flows allowing Lax representations of the form (10.5) with $B_\pm = \pi_\pm(f(T^2))$, where $f : g \mapsto g$ is Ad$^*$-covariant.

10.3 Discretization

The discretization of the second Toda flow (9.7) also may be restricted to the set $\mathcal{V}$. It is easy to see from (9.8), (9.7) that in this situation we have with necessity $\delta_k = 0$, and the recurrent relations (9.9) turn into
\[
\beta_k = 1 + h(a_k + a_{k-1}) - \frac{h^2 a_{k-1} a_{k-2}}{\beta_{k-2}} \quad (10.9)
\]

**Theorem 10.1** [S8] (see also [K2]). The discrete time Lax equation
\[
\bar{T} = B_+^{-1} T B_+ = B_- T B_-^{-1} \quad \text{with} \quad B_\pm = \Pi_\pm (I + hT^2) \quad (10.10)
\]
is equivalent to the following map $a \mapsto \bar{a}$:
\[
\bar{a}_k = a_k \frac{\beta_{k+1}}{\beta_k} \quad (10.11)
\]
where the quantities $\beta_k = \beta_k(a) = 1 + O(h)$ are uniquely defined by the recurrent relations
\[
\beta_k - ha_k = \frac{\beta_{k-1}}{\beta_{k-1} - ha_{k-1}} = 1 + \frac{ha_{k-1}}{\beta_{k-1} - ha_{k-1}} \quad (10.12)
\]
and have the asymptotics
\[
\beta_k = 1 + h(a_k + a_{k-1}) + O(h^2) \quad (10.13)
\]
Remark. The matrices $B_±$ have the following expressions:

$$B_+(a, \lambda) = \Pi_+(I + hT^2) = \sum_{k=1}^{N} \beta_k E_{k,k} + h\lambda^2 \sum_{k=1}^{N} E_{k+2,k} \quad (10.14)$$

$$B_-(a, \lambda) = \Pi_- (I + hT^2) = I + h\lambda^{-2} \sum_{k=1}^{N} \frac{a_k a_{k+1}}{\beta_k} E_{k,k} + 2 \quad (10.15)$$

Proof. Referring to Theorem 9.1, we need only to prove that the auxiliary quantities $\beta_k$ defined by the recurrent relations (10.9), may be alternatively characterized as solutions to (10.12). To do this, consider the matrix equation $B_+ \tilde{T} = TB_+$. In coordinates it is equivalent to the set of two scalar equations: (14.17) and

$$\beta_{k+1} - ha_{k+1} = \frac{\beta_k}{\beta_{k-1}} (\beta_{k-1} - ha_{k-1})$$

The last equation means that the following quantity does not depend on $k$:

$$\frac{(\beta_k - ha_k)(\beta_{k-1} - ha_{k-1})}{\beta_{k-1}} = c = \text{const} \quad (10.16)$$

and we need only to prove that $c = 1$. But applying (10.16) twice, we have:

$$\beta_k - ha_k = c + \frac{ha_{k-1}}{\beta_{k-1} - ha_{k-1}} = c + \frac{ha_{k-1}(\beta_{k-2} - ha_{k-2})}{\beta_{k-2}} = c + ha_{k-1} - \frac{h^2a_{k-1}a_{k-2}}{\beta_{k-2}}$$

Comparing this with (10.9), we conclude that $c = 1$. \[\blacksquare\]

The map (10.11), (10.12) will be denoted dVL. It is Poisson with respect to the brackets (10.3), (10.4), but nonlocal, as opposed to the flow VL. In the open-end case we have terminating continued fractions for the nonlocal quantities $\beta_k$ (in the periodic case for small $h$ the $\beta_k$ may be expressed as analogous infinite periodic continued fractions). From (10.9) we derive, for example, for $\beta_k$ with odd indices:

$$\beta_{2k+1} = 1 + h(a_{2k+1} + a_{2k}) - \frac{h^2a_{2k}a_{2k-1}}{1 + h(a_{2k-1} + a_{2k-2}) - \cdots - \frac{h^2a_1a_0}{1 + ha_1}} \quad (10.17)$$

(with obvious changes for $\beta_{2k}$). Analogously, from (10.12) we derive alternative continued fractions:

$$\beta_k - ha_k = 1 + \frac{ha_{k-1}}{1 + \frac{ha_{k-2}}{1 + \cdots + \frac{ha_2}{1 + ha_1}}} \quad (10.18)$$
One says that the continued fractions (10.17) are obtained from (10.18) with the help of compression.

### 10.4 Local equations of motion for dVL

The localizing change of variables $\mathcal{V}(a) \mapsto \mathcal{V}(a)$ for dVL follows from (9.14) upon setting $b_k = 0$:

$$a_k = a_k(1 + ha_{k-1})$$ (10.19)

Again, this change of variables was found in [K1] in the context of integrable deformations, but with no relation to integrable discretizations.

As it follows from the proof of Theorem 9.2, in the variables $a_k$ the auxiliary quantities $\beta_k$ become local:

$$\beta_k = (1 + ha_k)(1 + ha_{k-1})$$ (10.20)

and we obtain the following result.

**Theorem 10.2** The change of variables (10.19) conjugates the map $dVL$ with the map $a \mapsto \tilde{a}$ governed by the following local equations of motion:

$$\tilde{a}_k(1 + h\tilde{a}_{k-1}) = a_k(1 + ha_{k+1})$$ (10.21)

The only known local invariant Poisson bracket (9.19) for the local form of $dTL_2$ does not admit a proper restriction or a reduction to the subset $\mathcal{V}(a)$ of $T(a, b)$ characterized by $b_k = 0$. Nevertheless, there appears to exist a local bracket on $\mathcal{V}(a)$ invariant under the local form of dVL.

**Theorem 10.3** The pull–back of the bracket

$$\{\cdot, \cdot\}_2 + h\{\cdot, \cdot\}_3$$

on $\mathcal{V}(a)$ under the change of variables (10.19) is the following bracket on $\mathcal{V}(a)$:

$$\{a_k, a_{k+1}\} = -a_k a_{k+1}(1 + ha_k)(1 + ha_{k+1})$$ (10.22)

This bracket is invariant under the map (10.21).

**Proof** – by direct calculation. ■

The local equations of motion (10.21) together with Lax representation were found by different methods in [THO], [PN], [S8]. The invariant Poisson structure and the relation to factorization problem were pointed out only in the latter of these references.
Theorem 10.4 [K1]. The pull–back of the flow VL under the map (10.19) is described by the following equations of motion:

\[ \dot{a}_k = a_k(1 + h a_k)(a_{k+1} - a_{k-1}) \]  

(10.23)

Proof. Since the flow VL has a Hamilton function \((2h)^{-1} \sum_{k=1}^{N} \log(a_k)\) in the Poisson bracket \(h\{\cdot, \cdot\}_3\), and this function is a Casimir of the bracket \(\{\cdot, \cdot\}_2\), we conclude that this function generates the flow VL also in the bracket \(\{\cdot, \cdot\}_2 + h\{\cdot, \cdot\}_3\). That means that the pull–back of the flow VL is a Hamiltonian flow in the bracket (10.22) with the Hamilton function

\[ (2h)^{-1} \sum_{k=1}^{N} \log \left( a_k(1 + h a_k) \right) \]

Calculating the corresponding equations of motion, we arrive at (10.23). ■

The system (10.23) is known under the name of the modified Volterra lattice.

Remark. An additional map

\[ a_k \mapsto \bar{a}_k = \frac{a_k}{1 + h a_k} \]

delivers an alternative version of the localizing change of variables

\[ a_k = \frac{a_k}{(1 - h a_k)(1 - h a_{k-1})} \]  

(10.24)

The corresponding local version of the dVL reads:

\[ \bar{a}_k \frac{a_k}{(1 - h \bar{a}_k)(1 - h \bar{a}_{k-1})} = \frac{a_k}{(1 - h a_k)(1 - h a_{k+1})} \]  

(10.25)

This version was introduced in [HT] under the name ”discrete Lotka–Volterra equation of type II” (while the type I was assigned to (10.21)). The modified Volterra lattice in the variables \(a_k\) takes the following form:

\[ \dot{a}_k = a_k \left( \frac{a_{k+1}}{1 - h a_{k+1}} - \frac{a_{k-1}}{1 - h a_{k-1}} \right) \]  

(10.26)

A remarkable feature of the variables \(a_k\) (not mentioned in [HT]) is the formal coincidence of the invariant Poisson bracket (10.22) with the quadratic invariant Poisson bracket of the continuous time flow VL. Indeed, in the variables \(a_k\) the bracket (10.22) takes the form

\[ \{a_k, a_{k+1}\} = -a_k a_{k+1} \]  

(10.27)
11 Second flow of the Volterra hierarchy

Now we apply the general procedure of integrable discretization to the second flow of the Volterra hierarchy. We use the notations $\beta_k, B_\pm$ etc. for objects analogous to those of the previous section without danger of confusion.

11.1 Equations of motion and bi–Hamiltonian structure

The second flow of the Volterra hierarchy (hereafter VL2) is described by the following differential equations on $\mathcal{V}$:

$$\dot{a}_k = a_k (a_{k+1} (a_{k+2} + a_{k+1} + a_k) - a_{k-1} (a_k + a_{k-1} + a_{k-2}))$$

(11.1)

This flow is Hamiltonian with respect to both Poisson brackets (10.3), (10.4). The corresponding Hamilton functions are:

$$H_2(a) = \frac{1}{2} \sum_{k=1}^{N} a_k^2 + \sum_{k=1}^{N} a_{k+1} a_k$$

for the quadratic bracket $\{\cdot, \cdot\}_2$, and $H_1(a)$ for the cubic bracket $\{\cdot, \cdot\}_3$.

11.2 Lax representation

The Lax representation for the flow VL2 is of the type (3.1) with $f(T) = T^4$.

**Theorem 11.1** The flow (11.1) admits the following Lax representation in $g$:

$$\dot{T} = [T, B_+] = - [T, B_-]$$

(11.2)

with the matrices

$$B_+(a, \lambda) = \pi_+(T^4) = \sum_{k=1}^{N} \left( a_{k+1} a_k + (a_k + a_{k-1})^2 + a_{k-1} a_{k-2} \right) E_{kk}$$

$$+ \lambda^2 \sum_{k=1}^{N} (a_{k+2} + a_{k+1} + a_k + a_{k-1}) E_{k+2,k} + \lambda^4 \sum_{k=1}^{N} E_{k+4,k}$$

$$B_-(a, \lambda) = \pi_-(T^4) = \lambda^2 \sum_{k=1}^{N} (a_{k+2} + a_{k+1} + a_k + a_{k-1}) a_{k+1} a_k E_{k,k+2}$$

$$+ \lambda^{-4} \sum_{k=1}^{N} a_{k+3} a_{k+2} a_{k+1} a_k E_{k,k+4}$$

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11.3 Discretization

Applying the recipe of Sect. 5 with \( F(T) = I + hT^4 \), we take as a discretization of the flow VL2 the map described by the discrete time Lax equation

\[
\tilde{T} = B_+^{-1}TB_+ = B_-TB_-^1 \quad \text{with} \quad B_{\pm} = \Pi_{\pm}(I + hT^4) \tag{11.3}
\]

Theorem 11.2 The discrete time Lax equations (11.3) are equivalent to the map \( a \mapsto \tilde{a} \) described by the following equations:

\[
\tilde{a}_k = a_k \frac{\beta_{k+1}}{\beta_k} \tag{11.4}
\]

where the functions \( \beta_k = \beta_k(a) = 1 + O(h) \) are uniquely defined for \( h \) small enough simultaneously with the functions \( \delta_k = O(1) \) by the system of recurrent relations

\[
\beta_k - h(\delta_k - a_{k+2})a_k = \frac{\beta_{k-1}}{\beta_{k-1} - h(\delta_{k-1} - a_{k+1})a_{k-1}} \tag{11.5}
\]

\[
\delta_k = a_{k+2} + a_{k+1} + a_k + a_{k-1} - \frac{ha_{k-1}a_{k-2}\delta_{k-2}}{\beta_{k-2}} \tag{11.6}
\]

The auxiliary functions \( \beta_k \) have the asymptotics

\[
\beta_k = 1 + h\left(a_{k+1}a_k + (a_k + a_{k-1})^2 + a_{k-1}a_{k-2}\right) + O(h^2) \tag{11.7}
\]

Remark. The matrices \( B_{\pm} \) have the following expressions:

\[
B_+(a, \lambda) = \sum_{k=1}^{N} \beta_k E_{k,k} + h\lambda^2 \sum_{k=1}^{N} \delta_k E_{k+2,k} + h\lambda^4 \sum_{k=1}^{N} E_{k+4,k}
\]

\[
B_-(a, \lambda) = I + h\lambda^2 \sum_{k=1}^{N} \frac{a_{k+1}a_k \delta_k}{\beta_k} E_{k+2,k} + h\lambda^4 \sum_{k=1}^{N} \frac{a_{k+3}a_{k+2}a_{k+1}a_k}{\beta_k} E_{k+4,k}
\]

Proof. The scheme of the proof is standard. First of all, the general structure of the factors \( B_{\pm} \) is clear from the bi–diagonal structure of the matrix \( T \). The expressions for the entries of \( B_- \) and the recurrent relations for the entries of \( B_+ \) follow from the equality \( B_+B_- = I + hT^4 \). However, in this way we come to the recurrent relation (11.6) for \( \delta_k \) and the following recurrent relation for \( \beta_k \):

\[
\beta_k = 1 + h\left(a_{k+1}a_k + (a_k + a_{k-1})^2 + a_{k-1}a_{k-2}\right) - \frac{h^2a_{k-1}a_{k-2}\delta_{k-2}^2}{\beta_{k-2}} \quad \frac{h^2a_{k-1}a_{k-2}a_{k-3}a_{k-4}}{\beta_{k-4}} \tag{11.8}
\]
different from (11.3). In order to "decompress" this recurrent relation for \( \beta_k \) and to bring it into the form (11.3), we proceed as follows. The matrix equation \( B_+ \tilde{T} = T B_+ \) is equivalent to the set of three scalar ones: the equation of motion (11.4) and

\[
\beta_{k+1} - h\delta_k a_{k+1} = \frac{\beta_k}{\beta_{k-1}} (\beta_{k-1} - h\delta_{k-1} a_{k-1}) \tag{11.9}
\]

\[
\delta_{k+1} - a_{k+3} = \delta_k - a_{k-1} \frac{\beta_k}{\beta_{k-1}} \tag{11.10}
\]

From the last two equations it is easy to derive:

\[
\beta_{k+1} - h(\delta_{k+1} - a_{k+3})a_{k+1} = \frac{\beta_k}{\beta_{k-1}} (\beta_{k-1} - h(\delta_{k-1} - a_{k+1})a_{k-1})
\]

which means that the following quantity does not depend on \( k \):

\[
\frac{(\beta_k - h(\delta_k - a_{k+2})a_k)(\beta_{k-1} - h(\delta_{k-1} - a_{k+1})a_{k-1})}{\beta_{k-1}} = c = \text{const} \tag{11.11}
\]

It remains to prove that this constant is equal to 1. To this end we apply (11.11) twice (as in the proof of Theorem 10.1) to derive

\[
\beta_k = c + h(\delta_k - a_{k+2})a_k + h(\delta_{k-1} - a_{k+1})a_{k-1} - \frac{h^2(\delta_{k-1} - a_{k+1})(\delta_{k-2} - a_k)a_{k-1}a_{k-2}}{\beta_{k-2}}
\]

Transforming the second and the third terms on right–hand side with the help of (11.6), and the fourth one with the help of (11.10), we see that the \( O(h) \)-terms of the last formula exactly coincide with the \( O(h) \)-terms on the right–hand side of (11.8). Equating the free terms, we get \( c = 1 \).

### 11.4 Local equations of motion for dVL2

The localizing change of variables for the map dVL2 is given by the formulas

\[
a_k = \frac{(1 + ha_{k-1}^2)}{(1 - ha_k a_{k-1})(1 - ha_{k-1} a_{k-2})} \tag{11.12}
\]

**Theorem 11.3** The change of variables (11.12) conjugates the map dVL2 with the map \( a \mapsto \tilde{a} \) described by the following local equations of motion:

\[
\tilde{a}_k = \frac{(1 + h\tilde{a}_{k-1}^2)}{(1 - h\tilde{a}_k \tilde{a}_{k-1})(1 - h\tilde{a}_{k-1} \tilde{a}_{k-2})} = a_k \frac{(1 + ha_{k+1}^2)}{(1 - ha_{k+2} a_{k+1})(1 - ha_{k+1} a_k)} \tag{11.13}
\]
Proof. The statement will follow immediately, if we prove the following expressions for the auxiliary quantities $\beta_k$:

$$\beta_k = \frac{(1 + ha_k^2)(1 + ha_{k-1}^2)}{(1 - ha_{k+1}a_k)(1 - ha_{k-1}^2a_{k-2})}$$  \hspace{1cm} (11.14)

To do this, it is sufficient to verify that the recurrent relations (11.5), (11.6) are satisfied by the quantities (11.14) and

$$\delta_k = \frac{a_{k+1} + a_{k+2}}{1 - ha_{k+1}a_{k+2}} + \frac{a_k + a_{k-1}}{1 - ha_ka_{k-1}}$$  \hspace{1cm} (11.15)

Such a verification is a matter of straightforward, though somewhat tedious algebra. We omit it, giving only an important intermediate formula:

$$\beta_k - h(\delta_k - a_{k+2})a_k = \frac{(1 + ha_{k-1}^2)}{(1 - ha_k(a_{k-1}a_{k-2}))}$$  \hspace{1cm} (11.16)

(it implies immediately (11.3)).

Unlike the situation with the previously encountered localizing changes of variables, we did not find a linear combination of invariant Poisson brackets of the Volterra hierarchy, which would be pulled back into a local bracket on $\mathcal{V}(a)$ (presumably, such a linear combination does not exist). Therefore, to find a pull back of the flow $\mathcal{VL}$ under the change of variables (11.12), we cannot use the Hamiltonian formalism and are forced to turn to a direct analysis of equations of motion.

**Theorem 11.4** The pull–back of the flow $\mathcal{VL}$ under the change of variables (11.12) is described by the following equations of motion:

$$\dot{a}_k = a_k(1 + ha_k^2) \left( \frac{a_{k+1}}{1 - ha_{k+1}a_k} - \frac{a_{k-1}}{1 - ha_ka_{k-1}} \right)$$  \hspace{1cm} (11.17)

**Proof.** It is a matter of straightforward calculations to verify that the equations (11.17) are sent into (10.1) by the change of variables (11.12).

### 11.5 Local discretization of the KdV

Probably the most interesting feature of the flow $\mathcal{VL}2$ is that a certain linear combination of this flow with the original $\mathcal{VL}$ gives a direct spatial discretization of the famous KdV [AL]. Indeed, the equations of motion of the linear combination $\alpha \cdot \mathcal{VL} + \mathcal{VL}2$ read:

$$\dot{a}_k = a_k \left( a_{k+1}(\alpha + a_{k+2} + a_{k+1} + a_k) - a_{k-1}(\alpha + a_k + a_{k-1} + a_{k-2}) \right)$$  \hspace{1cm} (11.18)
Setting $\alpha = -6$ and

$$a_k = 1 + \varepsilon^2 p_k$$

we get the equations of motion

$$\dot{p}_k = (1 + \varepsilon^2 p_k) \left(p_{k+2} - 2p_{k+1} + 2p_{k-1} - p_{k-2} + \varepsilon^2 p_{k+1} (p_{k+2} + p_{k+1} + p_k) - \varepsilon^2 p_{k-1} (p_k + p_{k-1} + p_{k-2})\right)$$

Assuming that $p_k(t) \approx p(t, k\varepsilon)$ with a smooth $p(t, x)$ and rescaling the time $t \mapsto t/(2\varepsilon^3)$, we see that the previous equation approximates

$$p_t = p_{xxx} + 3p_x p$$

which is the KdV. Therefore a local discretization of the linear combinations of the flows VL and VL2 will result (for $\alpha = -6$) in a local spatio–temporal discretization of the KdV (see [TA] for a highly nonlocal integrable discretization).

**Theorem 11.5** A localizing change of variables for the discretization of the above flow corresponding to $F(T) = I + h(\alpha T^2 + T^4)$ is given by the formulas

$$a_k = a_k \frac{(1 + h\alpha a_{k-1} + ha_{k-1}^2)}{(1 - h a_{k-1} a_{k-2})} (1 - h a_{k-1} a_{k-2})$$

(11.19)

This change of variables conjugates the above discretization with the map $a \mapsto \tilde{a}$ described by the local equations of motion

$$\tilde{a}_k = \frac{(1 + h\alpha \tilde{a}_{k-1} + h\tilde{a}_{k-1}^2)}{(1 - h \tilde{a}_{k-1} \tilde{a}_{k-2})} = a_k \frac{(1 + h\alpha a_{k+1} + ha_{k+1}^2)}{(1 - h a_{k+2} a_{k+1} + a_{k+1})} (1 - h a_{k+2} a_{k+1})$$

(11.20)

**Proof** is completely analogous to that of Theorem [11.3] and will not be repeated here.

## 12 Modified Volterra lattice

### 12.1 Equations of motion and Hamiltonian structure

As we have seen many times in the preceding exposition, the localizing changes of variables are interesting not only in their connection with the problem of integrable discretization, but already on the level of continuous time systems. Namely, they may be viewed as Miura transformations and used to find the so called modified equations of motion. These modified systems are often interesting in their own rights – and one can also wish to discretize them as well. We consider in the present section one example of modified systems, namely, the modified Volterra lattice (MVL).
To this end we use the change of variables (10.19) in slightly different notations. Namely, we define a Miura change of variables $\mathcal{M} : \mathcal{M}V(q) \mapsto V(a)$ by the formula

$$a_k = q_k(1 + \alpha q_{k-1})$$

(12.1)

We denote the parameter $\alpha$ instead of $h$ (it is not related to the time step of discretizations and is not supposed to be small). The MVL is the pull−back of the VL (10.1) under the change of variables (12.1). Its equations of motion were already determined in Sect. 10 to be

$$\dot{q}_k = q_k(1 + \alpha q_k)(q_{k+1} − q_k)$$

(12.2)

As usual, this system may be considered under periodic or open−end boundary conditions, the phase space in case of the periodic ones being

$$\mathcal{M}V = \mathbb{R}^N(q_1, \ldots, q_N)$$

As pointed out in Sect. 10, this system is Hamiltonian with respect to the following Poisson bracket on $\mathcal{M}V$:

$$\{q_k, q_{k+1}\} = −q_k q_{k+1}(1 + \alpha q_k)(1 + \alpha q_{k+1})$$

(12.3)

and with one of the Hamilton functions

$$H_0(q) = \sum_{k=1}^N \log(q_k) \quad \text{or} \quad H_0(q) = \alpha^{-1} \sum_{k=1}^N \log(1 + \alpha q_k)$$

(12.4)

(Their difference is a Casimir of the bracket (12.3)). The bracket (12.3) is the pull−back under the Miura change of variables (12.1) of the invariant Poisson bracket $\{\cdot, \cdot\}_2 + \alpha \{\cdot, \cdot\}_3$ of the VL (see (10.3), (10.4) for the relevant definitions).

### 12.2 Discretization

We define the discretization $d\mathcal{M}VL$ of $\mathcal{M}VL$ (12.2) as a pull−back of the $dVL$ under the Miura change of variables (12.1).

**Theorem 12.1** The equations of motion of the map $d\mathcal{M}VL$ read:

$$\tilde{q}_k = q_k \frac{\gamma_{k+1}}{\gamma_k}$$

(12.5)

where the quantities $\gamma_k = \gamma_k(q) = 1 + O(h)$ are uniquely defined by $h$ small enough by the system of recurrent relations

$$\gamma_k − h q_k = \frac{\gamma_{k-1} + h \alpha q_k q_{k-1}}{\gamma_{k-1} − h q_{k-1}} \quad \Leftrightarrow \quad \gamma_k − h q_k + \alpha q_k = (1 + \alpha q_k) \frac{\gamma_{k-1}}{\gamma_{k-1} − h q_{k-1}}$$

(12.6)

and have the asymptotics

$$\gamma_k = 1 + h(q_k + q_{k-1} + \alpha q_k q_{k-1}) + O(h^2)$$

(12.7)
Proof. From the equations of motion (12.5) and the definition (12.6) we derive:

\begin{equation}
1 + \alpha \tilde{q}_k = \frac{1}{\gamma_k} (\gamma_k + h\alpha q_{k+1}q_k) \left( 1 + \frac{\alpha q_k}{\gamma_k - hq_k} \right) = (1 + \alpha q_k) \frac{\gamma_{k-1}(\gamma_{k-1} - hq_{k-1})}{\gamma_k (\gamma_{k-1} - hq_{k-1})} \tag{12.8}
\end{equation}

Hence the variables $a_k$ defined as in (12.1) satisfy the equations of motion (10.11) with

\begin{equation}
\beta_k = \frac{\gamma_k}{\gamma_{k-2}} (\gamma_{k-1} - hq_{k-1})(\gamma_{k-2} - hq_{k-2}) \tag{12.9}
\end{equation}

It remains to prove that these quantities $\beta_k$ satisfy the recurrent relations (10.12). But this follows immediately from the formula

\[
\beta_k - ha_k = \frac{\gamma_{k-1}}{\gamma_{k-2}} (\gamma_{k-2} - hq_{k-2})
\]

which is an easy consequence of (12.9), (12.1), and (12.6). □

Clearly, by definition the map $dMVL$ is Poisson with respect to the bracket (12.3), but the above theorem renders $dMVL$ nonlocal. In particular, in the open–end case we have terminating continued fractions

\begin{equation}
\gamma_k - hq_k = 1 + \frac{hq_{k-1}(1 + \alpha q_k)}{1 + \cdots + hq_1(1 + \alpha q_1)} \tag{12.10}
\end{equation}

In the periodic case these continued fractions are also periodic.

### 12.3 Local equations of motion for dMVL

It is possible to find a localizing change of variables $\mathcal{M}(q) \mapsto \tilde{\mathcal{M}}(\tilde{q})$ for the map $dMVL$:

\begin{equation}
q_k = \frac{q_k(1 + hq_{k-1})}{1 - h\alpha q_k q_{k-1}} \tag{12.11}
\end{equation}

As usual, this map is a local diffeomorphism for $h$ small enough.

**Theorem 12.2** The change of variables (12.11) conjugates the map $dMVL$ with the map $q \mapsto \tilde{q}$ described by the following local equations of motion:

\begin{equation}
\frac{\tilde{q}_k}{1 + \alpha \tilde{q}_k} (1 + h\tilde{q}_{k-1}) = \frac{q_k}{1 + \alpha q_k} (1 + hq_{k+1}) \tag{12.12}
\end{equation}

or, equivalently,

\begin{equation}
\frac{1 + \alpha \tilde{q}_k}{1 - h\alpha \tilde{q}_k q_{k-1}} = \frac{1 + \alpha q_k}{1 - h\alpha q_{k+1}q_k} \tag{12.13}
\end{equation}
Proof. The first step is to prove the following local expressions for the quantities $\gamma_k$ in the variables $q_k$:

$$\gamma_k = \frac{(1 + hq_k)(1 + hq_{k-1})}{1 - h\alpha q_k q_{k-1}}$$ \hspace{1cm} (12.14)

Indeed, from (12.5), (12.11) and (12.14) we derive immediately the equations of motion in the form

$$\dot{a}_k(1 + h\tilde{q}_k) \frac{q_{k+1}}{1 - h\alpha \tilde{q}_k q_{k+1}} = q_k(1 + hq_{k+1}) \frac{q_{k-1}}{1 - h\alpha q_k q_{k-1}}$$

which is equivalent to either of (12.12) and (12.13). To prove (12.14), it is as usual enough to verify that the quantities defined by this formula satisfy the recurrent relations (12.6). This verification is the matter of a simple algebra. ■

The equations of motion (12.12) were introduced in [HT].

In general, the only expression for a Poisson bracket invariant with respect to the map (12.12) is non–local and non–polynomial (it may be characterized as the pull–back of the bracket (12.3) under the localizing change of variables (12.11)). However, a direct analysis of equations of motion allows to prove the following statement.

**Theorem 12.3** The pull–back of the flow (12.2) under the change of variables (12.11) is described by the following equations of motion:

$$\dot{q}_k = q_k(1 + \alpha q_k)(1 + hq_k) \left( \frac{q_{k+1}}{1 - h\alpha q_k q_{k+1} q_k} - \frac{q_{k-1}}{1 - h\alpha q_k q_{k-1} q_k} \right)$$ \hspace{1cm} (12.15)

This system may be called the second modification of the Volterra lattice. Notice its remarkable symmetry with respect to the interchange $\alpha \leftrightarrow h$. Also the expressions for $a_k$ in terms of $q_k$ enjoy this symmetry. Indeed, the composition of two changes of variables (12.1) and (12.11) results in

$$a_k = \frac{q_k(1 + hq_{k-1})(1 + \alpha q_{k-1})}{(1 - h\alpha q_k q_{k-1})(1 - h\alpha q_k q_{k-2})}$$ \hspace{1cm} (12.16)

However, in the discrete equations of motion (12.12) this symmetry goes lost.

The formula (12.16) allows also to translate the Miura map $M : MV(q) \mapsto V(a)$ into the language of localizing variables. Namely, if we define the map $\tilde{M} : MV(q) \mapsto V(a)$ by the formula

$$a_k = \frac{q_k(1 + \alpha q_{k-1})}{1 - h\alpha q_k q_{k-1}}$$ \hspace{1cm} (12.17)

then an easy calculation shows the commutativity of the following diagram:
Remark. Sometimes another form of MVL is more convenient:

\[ \dot{c}_k = (c_k^2 - \varepsilon^2)(c_{k+1} - c_{k-1}) \]  \hspace{1cm} (12.18)

It is related to (12.2) with \( \alpha = (2\varepsilon)^{-1} \) by means of a linear change of variables:

\[ q_k = c_k - \varepsilon \]  \hspace{1cm} (12.19)

and time rescaling \( t \mapsto t/\alpha \). Analogous linear change of localizing variables \( q_k = c_k - \varepsilon \) accompanied by the rescaling of the time step \( h \mapsto h/\alpha \) allows to derive from (12.12) the following local integrable discretization of the flow (12.18):

\[ \tilde{c}_k - \varepsilon \frac{\tilde{c}_k - \varepsilon}{\tilde{c}_k + \varepsilon} (1 + 2h\varepsilon \tilde{c}_{k+1}) = \frac{c_k - \varepsilon}{c_k + \varepsilon} (1 + 2h\varepsilon c_{k+1}) \]  \hspace{1cm} (12.20)

12.4 Particular case \( \alpha \to \infty \)

There exists a particular case of MVL, for which a more detailed information is available. It corresponds to the \( \alpha \to \infty \) limit of the previous constructions, which has to be accompanied by the time rescaling \( t \mapsto t/\alpha \). (Alternatively, one could consider the \( \varepsilon \to 0 \) limit of the system (12.18)). Let us list the corresponding results. The equations of motion read:

\[ \dot{q}_k = q_k^2(q_{k+1} - q_{k-1}) \]  \hspace{1cm} (12.21)

and the Miura map \( \mathcal{M} : \mathcal{MV}(q) \mapsto \mathcal{V}(a) \) relating this system to the VL, takes the form

\[ a_k = q_k q_{k-1} \]  \hspace{1cm} (12.22)
An invariant Poisson bracket \((12.3)\) in the limit \(\alpha \to \infty\) turns (being suitably rescaled) into
\[
\{q_k, q_{k+1}\}_3 = -q_k^2 q_{k+1}^2
\] (12.23)
(see below for the reason for assuming the index "3" to this bracket). The Hamilton function of the flow \((12.21)\) in this bracket is equal to
\[
H_0(q) = \sum_{k=1}^{N} \log(q_k)
\]
The localizing change of variables \((12.11)\) after rescaling \(h \mapsto h/\alpha\) and going to the limit \(\alpha \to \infty\) becomes
\[
q_k = \frac{q_k}{1 - h q_k q_{k-1}}
\] (12.24)
and the discrete time equations of motion \((12.13)\) become
\[
\frac{\dot{q}_k}{1 - h q_k q_{k-1}} = \frac{q_k}{1 - h q_k q_{k+1}}
\] (12.25)
or, equivalently,
\[
\bar{q}_k - q_k = h \bar{q}_k q_k (q_{k+1} - \bar{q}_{k-1})
\] (12.26)
Finally, the pull–back of the system \((12.21)\) under the localizing change of variables \((12.24)\) reads:
\[
q_k = q_k^2 \left( \frac{q_{k+1}}{1 - h q_{k+1} q_k} - \frac{q_{k-1}}{1 - h q_k q_{k-1}} \right)
\] (12.27)
Now notice that the bracket \((12.23)\) is the pull–back with respect to the Miura map \(\mathcal{M}\) \((12.22)\) of the cubic invariant bracket \(\{\cdot, \cdot\}_3\) of the Volterra hierarchy. It turns out that in this particular case the pull–back of the quadratic bracket \(\{\cdot, \cdot\}_2\) is given by more elegant formulas than in the general case. Namely, the corresponding bracket on \(\mathcal{M}V(q)\) is quadratic albeit still nonlocal:
\[
\{q_k, q_j\}_2 = \pi_{kj} q_k q_j
\] (12.28)
with the coefficients
\[
\pi_{kj} = \begin{cases} 0 & k - j = 0 \\ -1 & k - j > 0 \text{ and odd or } k - j < 0 \text{ and even} \\ 1 & k - j > 0 \text{ and even or } k - j < 0 \text{ and odd} \end{cases}
\] (12.29)
The corresponding Hamilton function of the flow \((12.21)\) is given by
\[
H(q) = \sum_{k=1}^{N} q_k q_{k+1}
\]
So, in this case we know a simple formula for an additional invariant Poisson bracket. This allows to find a nice formula for an invariant Poisson structure of the map (12.26) and of the flow (12.27) – the result lacking in the general case.

**Theorem 12.4** The pull–back of the bracket

\[ \{\cdot, \cdot\}_2 + h\{\cdot, \cdot\}_3 \]  

on \( \mathcal{M}(q) \) under the map (12.24) is the following bracket on \( \mathcal{M}(q) \):

\[ \{q_k, q_j\} = \pi_{kj} q_k q_j \]  

with the coefficients (12.29). The map (12.26) and the flow (12.27) are Poisson with respect to this bracket.

**Proof.** A direct calculation shows that the brackets (12.31) for the variables \( q_k \) are sent by the map (12.24) into the brackets (12.30) for the variables \( q_k \).

We close this section by noticing that the translation of the Miura map \( \mathcal{M} \) (12.22) into the localizing variables is the map \( M : \mathcal{M}(q) \mapsto V(a) \) given by the formula

\[ a_k = \frac{q_k q_{k-1}}{1 - \frac{1}{h} q_k q_{k-1}} \]  

(12.32)

For the localizing variables \( a_k \) of VL introduced by (10.24), we have:

\[ a_k = q_k q_{k-1} \]  

(12.33)

### 13 Bogoyavlensky lattices

#### 13.1 Equations of motion and Hamiltonian structure

There are three basic families of integrable lattice systems carrying the name of Bogoyavlensky [B] (although some of them were found earlier in [N], [K1], [I]). These systems are enumerated by integer parameters \( m, p \geq 1 \) (\( p > 1 \) for the third one) and read:

\[ \dot{a}_k = a_k \left( \sum_{j=1}^{m} a_{k+j} - \sum_{j=1}^{m} a_{k-j} \right) \]  

(13.1)

\[ \dot{a}_k = a_k \left( \prod_{j=1}^{p} a_{k+j} - \prod_{j=1}^{p} a_{k-j} \right) \]  

(13.2)
\begin{align}
\dot{a}_k &= p^{-1} \prod_{j=1}^{k-1} a_{k+j} - p^{-1} \prod_{j=1}^{k-1} a_{k-j} \\
\text{(13.3)}
\end{align}

We shall call these systems \(\text{BL1}(m)\), \(\text{BL2}(p)\), and \(\text{BL3}(p)\), respectively.

The lattices \(\text{BL1}(m)\) and \(\text{BL2}(p)\) serve as generalizations of the Volterra lattice, which arises from them by \(m = 1\) and \(p = 1\), respectively. The lattice \(\text{BL3}(p)\) after the change of variables \(a_k \mapsto q_k = a_k^{-1}\) and \(t \mapsto -t\) turns into

\begin{align}
\dot{q}_k &= q_k^2 \left( p^{-1} \prod_{j=1}^{k-1} q_{k+j} - p^{-1} \prod_{j=1}^{k-1} q_{k-j} \right) \\
\text{(13.4)}
\end{align}

We call the latter system \(\text{modified BL2}(p)\). It serves as a generalization of the modified Volterra lattice \((12.21)\), which is the \(p = 2\) particular case of \((13.4)\).

These systems may be considered on an infinite lattice (all the subscripts belong to \(\mathbb{Z}\)), and admit also periodic finite–dimensional reductions (all indices belong to \(\mathbb{Z}/N\mathbb{Z}\), where \(N\) is the number of particles). The lattices BL1 and BL2 admit also finite–dimensional versions with the open–end boundary conditions. The phase space of the periodic BL’s is

\begin{align}
\mathcal{B} &= \mathbb{R}^N(a_1, \ldots, a_N) \\
\text{(13.5)}
\end{align}

The Hamiltonian structure of the BL1 was determined in \([K1]\) and later in \([B]\); for the BL2 and BL3 in the infinite setting this was done in \([ZTOF]\); in the open–end and the periodic setting (where some subtleties come out) the Hamiltonian structures were determined in \([S4]\). We reproduce here the corresponding result for the periodic boundary conditions.

The invariant quadratic Poisson brackets for Bogoyavlensky lattices are given by the formula (we set \(p = 1\) for BL1\((m)\), \(m = 1\) for BL2\((p)\), and \(m = -1\) for BL3\((p)\)):

\begin{align}
\{a_k, a_j\}_2 &= \pi_{kj} a_k a_j \\
\text{(13.6)}
\end{align}

with the coefficients

\begin{align}
\pi_{kj} &= \frac{1}{2} \left( \delta_{k,j+m} - \delta_{k+m,j} + w_{k+m,j}^{(p)} - w_{k,j}^{(p)} - w_{k,m+j}^{(p)} + w_{k,j}^{(p)} \right) \\
\text{(13.7)}
\end{align}

Here, in turn, the coefficients \(w_{kj}^{(p)}\) are given in the \(N\)–periodic case with g.c.d.\((N, p) = 1\) by the formula

\begin{align}
w_{kj}^{(p)} &= \begin{cases} 
\text{sgn}(k - j) & k \equiv j \pmod{N} \\
2n/p - 1 & k - j \equiv nN \pmod{p}, \ 1 \leq n \leq p - 1
\end{cases} \\
\text{(13.8)}
\end{align}

The Poisson structure \((13.6)\) is nonlocal unless \(p = 1\) (the case of BL1\((m)\)), when it is given by the following brackets:

\begin{align}
\{a_k, a_{k+1}\} &= -a_k a_{k+1}, \ldots \{a_k, a_{k+m}\} = -a_k a_{k+m} \\
\text{(13.9)}
\end{align}
In particular, for \( m = 1 \) we obtain the quadratic Poisson bracket for the Volterra lattice.

The corresponding Hamiltonians are:

\[
H(a) = \sum_{k=1}^{N} a_k \quad \text{for BL1}(m) \tag{13.10}
\]

\[
H(a) = \sum_{k=1}^{N} a_k a_{k+1} \ldots a_{k+p-1} \quad \text{for BL2}(p) \tag{13.11}
\]

\[
H(a) = \sum_{k=1}^{N} a_k^{-1} a_{k+1}^{-1} \ldots a_{k+p-1}^{-1} \quad \text{for BL3}(p) \tag{13.12}
\]

### 13.2 Lax representations

The Lax representations of the Bogoyavlensky lattices fall into the class considered in [K1], and were also specified in [B]. They have the form

\[
\dot{T} = [T, B] \tag{13.13}
\]

with the spectral-dependent (in the periodic case) matrices \( T = T(a, \lambda) \), \( B = B(a, \lambda) \). Actually, the natural phase space of all these Lax representations is the same algebra \( g \) as before. For BL1\((m)\) the matrices \( T, B \) are given by

\[
T(a, \lambda) = \lambda \sum_{k=1}^{N} E_{k+1,k} + \lambda^{-m} \sum_{k=1}^{N} a_k E_{k,k+m} \tag{13.14}
\]

\[
B(a, \lambda) = \pi_+ \left( T^{m+1}(a, \lambda) \right) = \sum_{k=1}^{N} (a_k + a_{k-1} + \ldots + a_{k-m}) E_{k,k} + \lambda^{m+1} \sum_{k=1}^{N} E_{k+m+1,k} \tag{13.15}
\]

for BL2\((p)\) they are given by

\[
T(a, \lambda) = \lambda^p \sum_{k=1}^{N} E_{k+p,k} + \lambda^{-1} \sum_{k=1}^{N} a_k E_{k,k+1} \tag{13.16}
\]

\[
B(a, \lambda) = -\pi_- \left( T^{p+1}(a, \lambda) \right) = -\lambda^{-p-1} \sum_{k=1}^{N} a_k a_{k+1} \ldots a_{k+p} E_{k,k+p+1} \tag{13.17}
\]

and for BL3\((p)\) they are given by

\[
T(a, \lambda) = \lambda^p \sum_{k=1}^{N} E_{k+p,k} + \lambda \sum_{k=1}^{N} a_{k+1} E_{k+1,k} \tag{13.18}
\]
\[ B(a, \lambda) = -\pi \left( T^{-p+1}(a, \lambda) \right) = -\lambda^{-p+1} \sum_{k=1}^{N} a_{k+1}^{-1} a_{k+2}^{-1} \ldots a_{k+p-1}^{-1} E_{k,k+p-1} \] (13.19)

All three Lax representations above may be seen as having the form
\[ \dot{T} = \left[ T, \pm \pi \left( d\varphi(T) \right) \right] \]
with appropriate \( Ad \)-invariant functions \( \varphi \) on \( g \), namely,
\[ \varphi(T) = \frac{1}{m+1} \left( \text{tr}(T^{m+1}) \right)_{0} \] for \( BL1(m) \) (13.20)
\[ \varphi(T) = \frac{1}{p+1} \left( \text{tr}(T^{p+1}) \right)_{0} \] for \( BL2(p) \) (13.21)
\[ \varphi(T) = -\frac{1}{p-1} \left( \text{tr}(T^{-p+1}) \right)_{0} \] for \( BL3(p) \) (13.22)

(it is easy to see that the values of these functions in coordinates \( a_{k} \) coincide with (13.10)–(13.12), respectively).

These Lax equations allow an \( r \)-matrix interpretation \([\mathbb{S}]\), in which a certain quadratic bracket on \( g \) and its Dirac reduction to the manifold of Lax matrices are involved. It may be demonstrated that in general the Dirac correction to the Lax equations of motion vanishes for the Hamilton functions of the form \( \varphi(T) = \left( \text{tr}(T^{n}) \right)_{0} \), where \( n = (p_{1} + m_{1})n_{1} \) with some \( n_{1} \in \mathbb{Z} \) and \( m = m_{1}d, p = p_{1}d, d = \text{g.c.d.}(m, p) \). In particular, this explains the Lax equations for the Hamilton functions (13.20)–(13.22).

### 13.3 Discretization of BL1

We are now in a position to apply the recipe of Sect. \([\mathbb{E}]\) to the problem of discretizing the Bogoyavlensky lattices. This was done for the first time in \([\mathbb{S}]\). Closely related results were obtained in \([\mathbb{THO}], [\mathbb{PN}]\) by different methods. We reproduce here the results of \([\mathbb{S}]\) without proofs. As usual, the construction gives automatically for all discrete time systems (called hereafter dBL1, dBL2, dBL3, respectively) the invariant Poisson structure, the Lax representation, the integrals of motion, the interpolating Hamiltonian flows etc. The maps of all three families are nonlocal, but we demonstrate how to bring them to a local form by means of a suitable change of variables. Of course, the local forms of the maps dBL1–dBL3 are Poisson with respect to the Poisson brackets on \( \mathcal{B}(a) \) which are the pull–backs of the corresponding brackets on \( \mathcal{B}(a) \). However, we did not succeed in finding more or less nice formulas for such pull–backs (the only exceptions – the Volterra and the modified Volterra lattices).

As dBL1\((m)\) we take the map described by the discrete time Lax equation
\[ \bar{T} = B_{-1}^{-1}TB_{+}, \quad B_{+} = \Pi_{+}(I + hT^{m+1}) \] (13.23)
Theorem 13.1. The discrete time Lax equation (13.23) is equivalent to the map \( a \mapsto \tilde{a} \) described by the equations

\[
\tilde{a}_k = \frac{\beta_{k+m}}{\beta_k} a_k
\]  

(13.24)

where the functions \( \beta_k = \beta_k(a) = 1 + O(h) \) are uniquely defined for \( h \) small enough by the recurrent relations

\[
\beta_k - ha_k = \prod_{j=1}^{m} \left( 1 + \frac{ha_{k-j}}{\beta_{k-j} - ha_{k-j}} \right)
\]  

(13.25)

and have the following asymptotics:

\[
\beta_k = 1 + h \sum_{j=0}^{m} a_{k-j} + O(h^2)
\]  

(13.26)

Remark. The factor \( B_+ \) is of the form

\[
B_+(a, \lambda) = \Pi_+ \left( I + hT^{m+1} \right) = \sum_{k=1}^{N} \beta_k E_{k,k} + h\lambda^{m+1} \sum_{k=1}^{N} E_{k+m+1,k}
\]  

(13.27)

The map (13.24) is nonlocal due to the functions \( \beta_k \). In the simplest case \( m = 1 \) one has the continued fractions, terminating for the open–end boundary conditions:

\[
\beta_k - ha_k = 1 + \frac{ha_{k-1}}{1 + \frac{ha_2}{1 + ha_1}}
\]  

(13.28)

and periodic for the periodic boundary conditions. For \( m > 1 \) there lacks even such an expressive mean as continued fractions to represent these nonlocal functions.

The localizing change of variables \( B(a) \mapsto \mathcal{B}(a) \) for \( \text{dBL1}(m) \) is:

\[
a_k = a_k \prod_{j=1}^{m} (1 + ha_{k-j})
\]  

(13.29)

Obviously, the map (13.29) for \( h \) small enough is a local diffeomorphism.

Theorem 13.2. The change of variables (13.29) conjugates the map \( \text{dBL1}(m) \) with the following one:

\[
\tilde{a}_k \prod_{j=1}^{m} (1 + h\tilde{a}_{k-j}) = a_k \prod_{j=1}^{m} (1 + ha_{k+j})
\]  

(13.30)
Proof. Let us define the quantities $\beta_k$ by the relation
\[
\beta_k = \prod_{j=0}^{m} (1 + h a_{k-j}) \tag{13.31}
\]
and prove that they satisfy the recurrent relations (13.25). Indeed, from (13.29), (13.31) it follows:
\[
\beta_k - h a_k = \prod_{j=1}^{m} (1 + h a_{k-j}) = \frac{a_k}{a_k}
\]
Hence $a_k = a_k / (\beta_k - h a_k)$, which, being substituted in the previous formula, implies (13.25). Now the uniqueness of solution of this latter recurrent system yields (13.31). Plugging (13.31), (13.29) into the equations of motion (13.24) results in (13.30). ■

Theorem 13.3 The pull-back of equations of motion (13.1) under the change of variables (13.29) is given by the following formula:
\[
\ddot{a}_k = a_k (1 + h a_k) \frac{\prod_{j=1}^{m} (1 + h a_{k+j}) - \prod_{j=1}^{m} (1 + h a_{k-j})}{h} \tag{13.32}
\]
Proof. By a direct calculation one checks that the equations (13.32) under the change of variables (13.29) are mapped to the equations of motion (13.1). ■

13.4 Discretization of BL2

As dBL2($p$) we take the map described by the discrete time Lax equation
\[
\bar{T} = C_- T C_-^{-1}, \quad C_- = \prod_{j=1}^{m} (1 + h a_{k+j}) - \prod_{j=1}^{m} (1 + h a_{k-j}) \tag{13.33}
\]

Theorem 13.4 [8]. The discrete time Lax equation (13.33) is equivalent to the map $a \mapsto \bar{a}$ described by the equations
\[
\bar{a}_k = \frac{a_k - h \gamma_{k-p}}{a_{k+p+1} - h \gamma_{k+1}} a_{k+p+1} \tag{13.34}
\]
where the functions $\gamma_k = \gamma_k(a) = O(1)$ are uniquely defined for $h$ small enough by the recurrent relations
\[
a_k - h \gamma_{k-p} = \frac{a_k}{1 + h \prod_{j=1}^{p} (a_{k-j} - h \gamma_{k-p-j})} \tag{13.35}
\]
and have the asymptotics
\[
\gamma_k = \prod_{j=0}^{p} a_{k+j} (1 + O(h)) \tag{13.36}
\]
Remark. The factor $C_{-}$ is of the form

$$C_{-}(a, \lambda) = \Pi_{-}\left(I + hT^{p+1}\right) = I + h\lambda^{-(p+1)} \sum_{k=1}^{N} \gamma_{k} E_{k,k+p+1}$$

(13.37)

The quantities $\gamma_{k}$ render the equations of motion (13.34) nonlocal. The localizing change of variables for dBL2($p$) reads:

$$a_{k} = a_{k} \left(1 + h \prod_{j=1}^{p} a_{k-j}\right)$$

(13.38)

As usual, its bijectivity follows from the implicit functions theorem.

**Theorem 13.5** The change of variables (13.38) conjugates the map dBL2($p$) with the following one:

$$\tilde{a}_{k} \left(1 + h \prod_{j=1}^{p} \tilde{a}_{k-j}\right) = a_{k} \left(1 + h \prod_{j=1}^{p} a_{k+j}\right)$$

(13.39)

**Proof.** We proceed according to the by now already usual scheme. Define the quantities $\gamma_{k}$ by the formula

$$\gamma_{k} = \prod_{j=0}^{p} a_{k+j}$$

(13.40)

Then we immediately derive:

$$a_{k} - h\gamma_{k-p} = a_{k}$$

(13.41)

and plugging this expression for $a_{k}$ into (13.38) shows that the recurrent relations (13.35) are satisfied. The uniqueness of the system of functions $a_{k} - h\gamma_{k-p}$ satisfying these relations justifies the expressions (13.40). Now putting (13.38), (13.41) into the equations of motion (13.34) allows to rewrite them as (13.39).

**Theorem 13.6** The pull–back of the equations of motion (13.2) under the change of variables (13.38) is given by the following formula:

$$\dot{a}_{k} = a_{k} \left(\prod_{j=1}^{p} a_{k+j} - \prod_{j=1}^{p} a_{k-j}\right) \prod_{n=1}^{p} \left(1 + h \prod_{i=1}^{p} a_{k-n+i}\right)$$

(13.42)

**Proof.** A direct, though tedious calculation shows that the equations of motion (13.42) are mapped on (13.2) by means of the map (13.38).
13.5 Discretization of BL3

As dBL3($p$) we define the map with the discrete time Lax representation

$$\tilde{T} = \mathbf{A}_- T \mathbf{A}_-^{-1}, \quad \mathbf{A}_- = \Pi_-(I + hT^{-p+1}) \quad (13.43)$$

**Theorem 13.7** [SS]. The discrete time Lax equation (13.43) is equivalent to the map $a \mapsto \tilde{a}$ described by the equations

$$\tilde{a}_k = \frac{a_k - h\alpha_{k-p}}{a_{k+p-1} - h\alpha_{k-1}} a_{k+p-1} \quad (13.44)$$

where the functions $\alpha_k = \alpha_k(a) = O(1)$ are uniquely defined for $h$ small enough by the recurrent relations

$$\alpha_k = \prod_{j=1}^{p-1} \frac{1}{a_{k+j} - h\alpha_{k+j-p}} \quad (13.45)$$

and have the asymptotics

$$\alpha_k = \prod_{j=1}^{p-1} a_{k+j}^{-1} (1 + O(h)) \quad (13.46)$$

**Remark.** The factor $\mathbf{A}_-$ is of the form

$$\mathbf{A}_-(a, \lambda) = \Pi_-(I + hT^{-p+1}) = I + h\lambda^{-p+1} \sum_{k=1}^{N} \alpha_k E_{k,k+p-1} \quad (13.47)$$

The nonlocality of the equations of motion (13.44) is due to the functions $\alpha_k(a)$. For example, for $p = 2$ they can be expressed as periodic continued fractions of the following structure:

$$h\alpha_k = \underbrace{h}_{a_{k+1}} - \underbrace{h}_{a_k} - \underbrace{h}_{a_{k-1}} - \ldots$$

The localizing change of variables for dBL3($p$) is given by:

$$a_k = a_k \left(1 + h \prod_{j=0}^{p-1} a_{k-j}^{-1}\right) \quad (13.48)$$

The bijectivity of this map is assured by the implicit functions theorem.
Theorem 13.8 The change of variables \( (13.48) \) conjugates the map \( dBL3(p) \) with the following one:

\[
\tilde{a}_k \left( 1 + h \prod_{j=0}^{p-1} \tilde{a}_{k-j} \right) = a_k \left( 1 + h \prod_{j=0}^{p-1} a_{k+j} \right)
\]

(13.49)

Proof. Defining the quantities \( \alpha_k \) by the formula

\[
\alpha_k = \prod_{j=1}^{p-1} a_{k+j}^{-1}
\]

(13.50)

we obtain with the help of \( (13.48) \):

\[
a_k - h\alpha_{k-p} = a_k
\]

(13.51)

Substituting this expression into \( (13.50) \), we see that the recurrent relations \( (13.45) \) are satisfied, which proves \( (13.50) \). Substituting \( (13.48) \), \( (13.51) \) into \( (13.44) \), we immediately arrive at the equations of motion \( (13.49) \).]

Theorem 13.9 The pull–back of the equations of motion \( (13.3) \) under the change of variables \( (13.48) \) is given by the following formula:

\[
\hat{a}_k = \left( \prod_{j=1}^{p-1} a_{k+j}^{-1} - \prod_{j=1}^{p-1} a_{k-j}^{-1} \right) \prod_{n=0}^{p-1} \left( 1 + h \prod_{i=0}^{p-1} a_{k-n+i}^{-1} \right) ^{-1}
\]

(13.52)

Proof – by direct (but tiresome) calculations.

We close the discussion of the local equations of motion for the \( dBL3(p) \) by noticing that under the change of variables \( a_k \mapsto q_k = a_k^{-1} \), \( h \mapsto -h \) the map \( (13.49) \) turns into

\[
\tilde{q}_k \left( 1 - h \prod_{j=0}^{p-1} \tilde{q}_{k-j} \right)^{-1} = q_k \left( 1 - h \prod_{j=0}^{p-1} q_{k+j} \right)^{-1}
\]

(13.53)

which is a local integrable discretization of the system \( (13.4) \), while the differential equations \( (13.52) \) under the change of variables \( a_k \mapsto q_k = q_k^{-1} \), \( h \mapsto -h \), \( t \mapsto -t \) go into

\[
\dot{q}_k = q_k^2 \left( \prod_{j=1}^{p-1} q_{k+j} - \prod_{j=1}^{p-1} q_{k-j} \right) \prod_{n=0}^{p-1} \left( 1 - h \prod_{i=0}^{p-1} q_{k-n+i} \right)^{-1}
\]

(13.54)

which is an integrable one–parameter deformation of \( (13.4) \).
13.6 Particular case \( p = 2 \)

The case \( p = 2 \) of the Bogoyavlensky lattice BL3 is remarkable in several respects. The equations of motion for this case read:

\[
\dot{a}_k = \frac{1}{a_{k+1}} - \frac{1}{a_{k-1}} \tag{13.55}
\]

The localizing change of variables for the discretization of the system (13.55) obtained in the previous subsection, is:

\[
a_k = a_k \left( 1 + \frac{h}{a_k a_{k-1}} \right) = a_k + \frac{h}{a_k a_{k-1}} \tag{13.56}
\]

In the variables \( a_k \) we have the following discretization:

\[
\tilde{a}_k - a_k = h \left( \frac{1}{a_{k+1}} - \frac{1}{a_{k-1}} \right) \tag{13.57}
\]

(When considered on the lattice \((t,k)\), the equation (13.57) is equivalent to the so called lattice KdV, which is a very popular object nowadays, cf. [NC], [NS], and references therein). The localizing change of variables (13.56) brings the system (13.55) itself into the form

\[
\dot{a}_k = \left( \frac{1}{a_{k+1}} - \frac{1}{a_{k-1}} \right) \left( 1 + \frac{h}{a_k a_{k+1}} \right)^{-1} \left( 1 + \frac{h}{a_k a_{k-1}} \right)^{-1}
\]

\[
= a_k \left( \frac{1}{a_{k+1} a_k + h} - \frac{1}{a_k a_{k-1} + h} \right) \tag{13.58}
\]

The special properties of this case begin with the following observation. The subset of \( g \) consisting of the Lax matrices

\[
T(a, \lambda) = \lambda^2 \sum_{k=1}^{N} E_{k+2,k} + \lambda \sum_{k=1}^{N} a_{k+1} E_{k+1,k} \tag{13.59}
\]

is a Poisson submanifold for the corresponding quadratic \( r \)-matrix bracket, so that the Dirac reduction is not needed in giving an \( r \)-matrix interpretation to the bracket

\[
\{a_k, a_j\}_2 = \pi_{kj} a_k a_j \tag{13.60}
\]

The coefficients \( \pi_{kj} \) are given (in the case of odd \( N \)) by the formula (12.23).
Moreover, not only the quadratic \( r \)-matrix bracket, but also the linear one may be properly restricted to the set of the matrices (13.59). The coordinate representation of the induced bracket on this set is given by:

\[
\{a_k, a_{k+1}\}_1 = -1
\]  

(13.61)

The two Poisson brackets (13.60) and (13.61) are compatible, hence the system (13.55) and its discretization given by Theorem 13.7 with \( p = 2 \) are bi–Hamiltonian. Obviously, the Hamilton function of the system (13.55) in the Poisson bracket (13.61) may be taken as

\[
H_0(a) = \sum_{k=1}^{N} \log(a_k)
\]

(13.62)

It is easy to check that this function is a Casimir of the quadratic bracket.

Being bi–Hamiltonian, the system (13.55) and its discretization admit also an arbitrary linear combination of the brackets (13.60) and (13.61) as an invariant Poisson structure. A further remarkable feature is the following: there exists a linear combination of these two brackets whose pull–back under the map (13.56) allows a nice representation in the variables \( a_k \).

**Theorem 13.10** The pull–back of the bracket

\[
\{\cdot, \cdot\}_2 + h\{\cdot, \cdot\}_1
\]

(13.63)

on \( B(a) \) under the map (13.56) is the following bracket on \( B(a) \):

\[
\{a_k, a_j\} = \pi_{kj} a_k a_j
\]

(13.64)

with the coefficients (12.29). The map (13.57) and the flow (13.58) are Poisson with respect to this bracket.

**Proof.** A direct calculation shows that the brackets (13.64) for the variables \( a_k \) are sent by the map (13.56) into the brackets (13.63) for the variables \( a_k \). ■

This theorem allows also to derive the equations of motion (13.58) in a Hamiltonian manner. Indeed, these equations describe the Hamiltonian flow with the Hamilton function

\[
h^{-1} \sum_{k=1}^{N} \log \left( a_k(1 + h a_k^{-1} a_{k-1}^{-1}) \right)
\]

in the Poisson brackets (13.64). Indeed, this function is a pull–back of \( h^{-1} \sum_{k=1}^{N} \log(a_k) \), which generates (13.55) in the bracket (13.63). A direct calculation shows that this Hamiltonian system is governed by the differential equations (13.58).

**Remark.** The results of this subsection agree with the results of the subsection (12.4) after the change of variables \( a_k \mapsto q_k = a_k^{-1}, \; a_k \mapsto q_k = a_k^{-1} \).
14 Alternative approach to Volterra lattice

Starting from this point, we consider the systems with Lax representations in the direct products $g = g^\otimes m$ rather than in $g$ itself. We start with an alternative approach to the Volterra lattice based on a Lax representation in $g \otimes g$.

14.1 Equations of motion and bi–Hamiltonian structure

The version of VL we consider here is:

$$\dot{u}_k = u_k(v_k - v_{k-1}), \quad \dot{v}_k = v_k(u_{k+1} - u_k) \tag{14.1}$$

Usually we let the subscript $k$ change in the interval $1 \leq k \leq N$ and consider either open–end boundary conditions ($v_0 = u_{N+1} = 0$) or periodic ones (all indices are taken (mod $N$)). We consider mainly the periodic case, because the open–end one is similar and more simple. The relation to the form (10.1) is achieved by re-naming the variables according to

$$u_k \mapsto a_{2k-1}, \quad v_k \mapsto a_{2k} \tag{14.2}$$

So, in the present setting the $N$–periodic VL consists of $2N$ particles. The phase space of VL in the case of periodic boundary conditions is the space

$$\mathcal{W} = \mathbb{R}^{2N}(u_1, v_1, \ldots, u_N, v_N) \tag{14.3}$$

Two compatible local Poisson brackets on $\mathcal{W}$ invariant under the flow VL are given by the relations

$$\{u_k, v_k\}_2 = -u_kv_k, \quad \{v_k, u_{k+1}\}_2 = -v_ku_{k+1} \tag{14.4}$$

and

$$\begin{align*}
\{u_k, v_k\}_3 &= -u_kv_k(u_k + v_k), \\
\{v_k, u_{k+1}\}_3 &= -v_ku_{k+1}(v_k + u_{k+1}) \\
\{u_k, u_{k+1}\}_3 &= -u_kv_ku_{k+1}, \\
\{v_k, v_{k+1}\}_3 &= -v_ku_{k+1}v_{k+1}
\end{align*} \tag{14.5}$$

respectively. The corresponding Hamilton functions for the flow VL are equal to

$$H_1(u, v) = \sum_{k=1}^{N} u_k + \sum_{k=1}^{N} v_k \tag{14.6}$$

and

$$H_0(u, v) = \sum_{k=1}^{N} \log(u_k) \quad \text{or} \quad H_0(u, v) = \sum_{k=1}^{N} \log(v_k) \tag{14.7}$$

(the second function makes sense only in the periodic case; the difference of these two functions is a Casimir of $\{\cdot, \cdot\}_3$ whose value is fixed to $\infty$ in the open–end case).
14.2 Lax representation

Consider the following two matrices:

\[ U(u,v,\lambda) = \sum_{k=1}^{N} u_k E_{k,k} + \lambda \sum_{k=1}^{N} E_{k+1,k}, \quad V(u,v,\lambda) = I + \lambda^{-1} \sum_{k=1}^{N} v_k E_{k,k+1} \tag{14.8} \]

These formulas define the "Lax matrix" \((U, V) : W \rightarrow g = g \otimes g\).

**Theorem 14.1** [K1] (see also [S12]). The flow (14.1) admits the following Lax representation in \(g \otimes g\):

\[ \dot{U} = UC_+ - B_+ U = B_- U - UC_- \]

\[ \dot{V} = VB_+ - C_+ V = C_- V - VB_- \tag{14.9} \]

with the matrices

\[ B_+(u,v,\lambda) = \sum_{k=1}^{N} (u_k + v_{k-1}) E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k} \tag{14.10} \]

\[ C_+(u,v,\lambda) = \sum_{k=1}^{N} (u_k + v_k) E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k} \tag{14.11} \]

\[ B_-(u,v,\lambda) = \lambda^{-1} \sum_{k=1}^{N} u_k v_k E_{k,k+1} \tag{14.12} \]

\[ C_-(u,v,\lambda) = \lambda^{-1} \sum_{k=1}^{N} u_{k+1} v_k E_{k,k+1} \tag{14.13} \]

**Corollary.** The matrices

\[ T_+(u,v,\lambda) = U(u,v,\lambda)V(u,v,\lambda), \quad T_-(u,v,\lambda) = V(u,v,\lambda)U(u,v,\lambda) \tag{14.14} \]

satisfy the usual Lax equations in \(g\):

\[ \dot{T}_+ = [T_+, B_+] = -[T_-, B_-], \quad \dot{T}_- = [T_-, C_+] = -[T_-, C_-] \tag{14.15} \]

The matrices \(T_\pm\) are easy to calculate explicitly. From the corresponding formulas one sees that the matrices \(B_\pm, C_\pm\) allow the following representations:

\[ B_+ = \pi_+(T_+) = \pi_+(UV), \quad B_- = \pi_+(T_-) = \pi_+(VU) \]

\[ C_+ = \pi_-(T_+) = \pi_-(UV), \quad C_- = \pi_-(T_-) = \pi_-(VU) \]

The Lax equations (14.9) may be given an \(r\)-matrix interpretation [S12] in terms of a certain quadratic bracket on \(g = g \otimes g\), the induced bracket on \(W\) being \(\{\cdot, \cdot\}_2\).
14.3 Discretization

To find an integrable time discretization for the flow VL, we apply the recipe of Sect. 5 with $F(T) = I + hT$, i.e. we consider the map described by the discrete time "Lax triads"

$$
\tilde{U} = B_+^{-1}UC_+ = B_-UC_-^{-1}, \quad \tilde{V} = C_+^{-1}UB_+ = C_-VB_-^{-1}
$$

(14.16)

with

$$
B_\pm = \Pi_\pm(I + hUV), \quad C_\pm = \Pi_\pm(I + hVU)
$$

**Theorem 14.2** The discrete time Lax equations (14.16) are equivalent to the map $(u, v) \mapsto (\tilde{u}, \tilde{v})$ described by the following equations:

$$
\tilde{u}_k = u_k \frac{\gamma_k}{\beta_k}, \quad \tilde{v}_k = v_k \frac{\beta_{k+1}}{\gamma_k}
$$

(14.17)

where the functions $\beta_k = \beta_k(u, v) = 1 + O(h)$, $\gamma_k = \gamma_k(u, v) = 1 + O(h)$ are uniquely defined for $h$ small enough by the recurrent relations

$$
\beta_k - hu_k = \frac{\gamma_{k-1}}{\gamma_{k-1} - hv_{k-1}} = 1 + \frac{hv_{k-1}}{\gamma_{k-1} - hv_{k-1}}
$$

(14.18)

$$
\gamma_k - hv_k = \frac{\beta_k}{\beta_k - hu_k} = 1 + \frac{hu_k}{\beta_k - hu_k}
$$

(14.19)

and have the asymptotics

$$
\beta_k = 1 + h(u_k + v_{k-1}) + O(h^2)
$$

(14.20)

$$
\gamma_k = 1 + h(u_k + v_k) + O(h^2)
$$

(14.21)

**Remark.** The matrices $B_\pm, C_\pm$ have the following expressions:

$$
B_\pm(u, v, \lambda) = \Pi_\pm(I + hUV) = \sum_{k=1}^{N} \beta_k E_{k,k} + h\lambda \sum_{k=1}^{N} E_{k+1,k}
$$

(14.22)

$$
C_\pm(u, v, \lambda) = \Pi_\pm(I + hVU) = \sum_{k=1}^{N} \gamma_k E_{k,k} + h\lambda \sum_{k=1}^{N} E_{k+1,k}
$$

(14.23)

$$
B_\pm(u, v, \lambda) = \Pi_\pm(I + hUV) = I + h\lambda^{-1} \sum_{k=1}^{N} \frac{u_kv_k}{\beta_k} E_{k,k+1}
$$

(14.24)

$$
C_\pm(u, v, \lambda) = \Pi_\pm(I + hVU) = I + h\lambda^{-1} \sum_{k=1}^{N} \frac{u_{k+1}v_k}{\gamma_k} E_{k,k+1}
$$

(14.25)
Proof. The general structure of the factors $B_{\pm}, C_{\pm}$, as given in (14.22), as well as the expressions for the entries of $B_{-}, C_{-}$, follow directly from the defining equalities $B_{+}B_{-} = I + hUV, C_{+}C_{-} = I + hVU$. For the entries $\beta_k, \gamma_k$ of $B_{+}, C_{+}$ one obtains the following recurrent relations:

$$\beta_k = 1 + h(u_k + v_{k-1}) - \frac{h^2 u_{k-1}v_{k-1}}{\beta_{k-1}}, \quad \gamma_k = 1 + h(u_k + v_k) - \frac{h^2 u_k v_{k-1}}{\gamma_{k-1}}$$  \hspace{1cm} (14.26)

Now notice that these relations coincide with (10.9) after re-naming (14.2) and $\beta_k \mapsto \beta_{2k-1}, \gamma_k \mapsto \beta_{2k}$.

Hence we may use the proof of Theorem 10.1 to establish the alternative recurrent relations (14.18), (14.19). The equations of motion (14.17) follow directly from $B_+\tilde{U} = UC_+, C_+\tilde{V} = V B_+$. It is important to notice that (14.17) become identical with (10.11) after the above mentioned re-namings of variables. This allows to denote consistently the map constructed in this theorem by dVL. \hfill \blacksquare

14.4 Local equations of motion for dVL

Now we can simply reformulate the results of Sect. 10 in our new notations.

The localizing change of variables $W(u, v) \mapsto W(u, v)$ for dVL is given by the following formulas:

$$u_k = u_k(1 + hv_{k-1}), \quad v_k = v_k(1 + hu_k)$$  \hspace{1cm} (14.27)

Due to the implicit function theorem, these formulas define a local diffeomorphism for $h$ small enough.

Theorem 14.3 The change of variables (14.27) conjugates the map dVL with the map $(u, v) \mapsto (\tilde{u}, \tilde{v})$ governed by the following local equations of motion:

$$\tilde{u}_k(1 + h\tilde{v}_{k-1}) = u_k(1 + hv_k), \quad \tilde{v}_k(1 + h\tilde{u}_k) = v_k(1 + hu_{k+1})$$  \hspace{1cm} (14.28)

Let us mention that the functions $\beta_k(u, v), \gamma_k(u, v)$ in the localizing variables are given by the formulas:

$$\beta_k = (1 + hu_k)(1 + hv_{k-1}), \quad \gamma_k = (1 + hv_k)(1 + hu_k)$$  \hspace{1cm} (14.29)

so that

$$\beta_k - hu_k = 1 + hv_{k-1}, \quad \gamma_k - hv_k = 1 + hu_k$$  \hspace{1cm} (14.30)
**Remark.** Considering the equations (14.28) as lattice equations on the lattice \((t, k)\), and performing a linear change of independent variables, one arrives at the *explicit* version of \(dVL\), cf. [V1, FV].

**Theorem 14.4** The pull–back of the bracket
\[
\{\cdot, \cdot\}_2 + h\{\cdot, \cdot\}_3
\]
on \(W(u, v)\) under the change of variables (14.27) is the following bracket on \(W(u, v)\):
\[
\{u_k, v_k\} = -u_k v_k (1 + hu_k)(1 + hv_k), \quad \{v_k, u_{k+1}\} = -v_k u_{k+1} (1 + hv_k)(1 + hu_{k+1})
\]
The map (14.28) is Poisson with respect to the bracket (14.32).

**Theorem 14.5** The pull–back of the flow \(VL\) under the map (14.27) is described by the following equations of motion:
\[
\dot{u}_k = u_k (1 + hu_k)(v_k - v_{k-1}), \quad \dot{v}_k = v_k (1 + hv_k)(u_{k+1} - v_k)
\]

### 14.5 Lax representation for \(VL_2\)

We consider now the \(g \otimes g\) formulation of the second flow \(VL_2\) of the Volterra hierarchy. We use the notations \(\beta_k, \gamma_k, B_\pm, C_\pm\) etc. for objects analogous to those relevant for \(VL\) without danger of confusion.

The flow \(VL_2\) is described by the following differential equations on \(W\):
\[
\dot{u}_k = u_k (v_k(u_{k+1} + v_k + u_k) - v_{k-1}(u_k + v_{k-1} + u_{k-1})) \\
\dot{v}_k = v_k (u_{k+1}(v_{k+1} + u_{k+1} + v_k) - u_k(v_k + u_k + v_k))
\]
The Hamilton functions of this flow are:
\[
H_2(u, v) = \frac{1}{2} \sum_{k=1}^{N} (u_k^2 + v_k^2) + \sum_{k=1}^{N} (u_{k+1}v_k + v_ku_k)
\]
in the quadratic bracket \(\{\cdot, \cdot\}_2\), and \(H_1(u, v)\) in the cubic bracket \(\{\cdot, \cdot\}_3\).

The Lax representation for the flow \(VL_2\) is of the type (3.2) with \(m = 2\) and \(f(T) = T^2\).
Theorem 14.6 [K1]. The flow (14.34) admits the following Lax representation in \( g \otimes g \):

\[
\dot{U} = UC_+ - B_+ U = B_- U - UC_-
\]

\[\text{(14.35)}\]

\[
\dot{V} = VB_+ - C_+ V = C_- V - VB_-
\]

with the matrices

\[
B_+(u, v, \lambda) = \sum_{k=1}^{N} \left((u_k + v_{k-1})^2 + u_kv_k + u_{k-1}v_{k-1}\right)E_{kk} + \lambda \sum_{k=1}^{N} (u_{k+1} + v_k + u_k + v_{k-1})E_{k+1,k} + \lambda^2 \sum_{k=1}^{N} E_{k+2,k}
\]

\[
C_+(u, v, \lambda) = \sum_{k=1}^{N} \left((u_k + v_k)^2 + u_{k+1}v_k + u_kv_{k-1}\right)E_{kk} + \lambda \sum_{k=1}^{N} (u_{k+1} + v_{k+1} + u_k + v_k)E_{k+1,k} + \lambda^2 \sum_{k=1}^{N} E_{k+2,k}
\]

\[
B_-(u, v, \lambda) = \lambda^{-1} \sum_{k=1}^{N} (u_{k+1} + v_k + u_k + v_{k-1})u_kv_kE_{k,k+1} + \lambda^{-2} \sum_{k=1}^{N} u_{k+1}v_{k+1}u_kv_kE_{k+1,k+2}
\]

\[
C_-(u, v, \lambda) = \lambda^{-1} \sum_{k=1}^{N} (u_{k+1} + v_{k+1} + u_k + v_k)u_{k+1}v_kE_{k,k+1} + \lambda^{-2} \sum_{k=1}^{N} u_{k+2}v_{k+1}u_{k+1}v_kE_{k+2,k+2}
\]

Obviously, the expressions for \(B_+\), \(C_+\) may be obtained from (9.4) with the help of the substitutions (14.47), (14.48), respectively, and \(B_-\), \(C_-\) follow analogously from (9.3).

14.6 Discretization of VL2

Applying the recipe of Sect. B with \(F(T) = I + hT^2\), we take as a discretization of the flow VL2 the map described by the discrete time Lax triads

\[
\tilde{U} = B_+^{-1}UC_+ = B_- UC_-, \quad \tilde{V} = C_+^{-1}UB_+ = C_- VB_-
\]

\[\text{(14.36)}\]

with

\[
B_\pm = \Pi_\pm \left(I + h(UV)^2\right), \quad C_\pm = \Pi_\pm \left(I + h(VU)^2\right)
\]
Theorem 14.7 The discrete time Lax equations (14.36) are equivalent to the map \((u, v) \mapsto (\tilde{u}, \tilde{v})\) described by the following equations:

\[
\begin{align*}
\tilde{u}_k &= u_k \frac{\gamma_k}{\beta_k}, \\
\tilde{v}_k &= v_k \frac{\beta_{k+1}}{\gamma_k}
\end{align*}
\] (14.37)

where the functions \(\beta_k = \beta_k(u, v) = 1 + O(h)\), \(\gamma_k = \gamma_k(u, v) = 1 + O(h)\) are uniquely defined for \(h\) small enough simultaneously with the functions \(\delta_k = O(1), \epsilon_k = O(1)\) by the system of recurrent relations

\[
\begin{align*}
\beta_k - h(\delta_k - u_{k+1})u_k &= \frac{\gamma_{k-1}}{\gamma_k - 1 - h(\epsilon_{k-1} - v_k)v_{k-1}} \\
\gamma_k - h(\epsilon_k - v_{k+1})v_k &= \frac{\beta_k}{\beta_k - h(\delta_k - u_{k+1})u_k} \\
\delta_k &= u_{k+1} + v_k + u_k + v_{k-1} - \frac{huv_{k-1}\delta_{k-1}}{\beta_{k-1}} \\
\epsilon_k &= u_{k+1} + v_{k+1} + u_k + v_{k-1} - \frac{huv_{k-1}\epsilon_{k-1}}{\gamma_{k-1}}
\end{align*}
\] (14.38) (14.39) (14.40) (14.41)

The auxiliary functions \(\beta_k, \gamma_k\) have the asymptotics

\[
\begin{align*}
\beta_k &= 1 + h \left((u_k + v_{k-1})^2 + u_k v_k + u_{k-1} v_{k-1}\right) + O(h^2) \\
\gamma_k &= 1 + h \left((u_k + v_k)^2 + u_{k+1} v_k + u_k v_{k-1}\right) + O(h^2)
\end{align*}
\] (14.42) (14.43)

Remark. The matrices \(B_+, C_+\) have the following expressions:

\[
\begin{align*}
B_+(u, v, \lambda) &= \Pi_+ \left(I + h(UV)^2\right) = \sum_{k=1}^{N} \beta_k E_{k,k} + h\lambda \sum_{k=1}^{N} \delta_k E_{k+1,k} + h^2 \sum_{k=1}^{N} E_{k+2,k} \\
C_+(u, v, \lambda) &= \Pi_+ \left(I + h(VU)^2\right) = \sum_{k=1}^{N} \gamma_k E_{k,k} + h\lambda \sum_{k=1}^{N} \epsilon_k E_{k+1,k} + h^2 \sum_{k=1}^{N} E_{k+2,k}
\end{align*}
\]

Proof. The scheme of the proof is standard. First of all, the general structure of the factors \(B_\pm, C_\pm\) is clear from the bi–diagonal structure of the matrices \(U, V\). The recurrent relations for the entries of the matrices \(B_+, C_+\) follow, in principle, from (9.8), (9.9) with the help of substitutions (14.47), (14.48). It is easy to see that these relations coincide with (11.8), (11.6) after re–naming (14.2) and

\[
\begin{align*}
\beta_k \mapsto \beta_{2k-1}, \quad \gamma_k \mapsto \beta_{2k}, \quad \delta_k \mapsto \delta_{2k-1}, \quad \epsilon_k \mapsto \delta_{2k}
\end{align*}
\]

The equations of motion (14.37), following from \(B_+ \bar{U} = UC_+, C_+ \bar{V} = VB_+,\) also coincide with (11.4) after the above re–naminings. Thus the discretization introduced in this theorem agrees with the one from Sect. [11] and may be consistently denoted by dVL2. ■

14.7 Local equations of motion for dVL2

Here we translate the corresponding results from Sect. 11 into our present notations. The localizing change of variables for the map dVL2 is given by the formulas

\[ u_k = u_k \frac{(1 + h v_k^2)}{(1 - h u_k v_{k-1})(1 - h v_{k-1}u_k)}, \quad v_k = v_k \frac{(1 + h u_k^2)}{(1 - h v_k u_k)(1 - h u_k v_{k-1})} \] (14.44)

**Theorem 14.8** The change of variables (14.44) conjugates the map dVL2 with the map \((u, v) \mapsto (\tilde{u}, \tilde{v})\) described by the following local equations of motion:

\[ \tilde{u}_k = \tilde{u}_k \frac{(1 + h \tilde{v}_{k-1}^2)}{(1 - h \tilde{u}_k \tilde{v}_{k-1})(1 - h \tilde{v}_{k-1} \tilde{u}_k)} = u_k \frac{(1 + h v_k^2)}{(1 - h u_k v_{k-1})(1 - h v_{k-1} u_k)} \] (14.45)

\[ \tilde{v}_k = \frac{(1 + h \tilde{u}_{k+1}^2)}{(1 - h \tilde{v}_{k+1} u_k)(1 - h \tilde{u}_k \tilde{v}_{k+1})} = v_k \frac{(1 + h u_k^2)}{(1 - h v_k u_{k+1})(1 - h u_{k+1} v_k)} \]

**Theorem 14.9** The pull–back of the flow VL under the change of variables (14.44) is described by the following equations of motion:

\[ \dot{u}_k = u_k (1 + h u_k^2) \left( \frac{v_k}{1 - h v_k u_k} - \frac{v_{k-1}}{1 - h u_k v_{k-1}} \right) \] (14.46)

\[ \dot{v}_k = v_k (1 + h v_k^2) \left( \frac{u_{k+1}}{1 - h u_{k+1} v_k} - \frac{u_k}{1 - h v_k u_k} \right) \]

14.8 Miura relations to the Toda hierarchy

We have seen in Sect. 10 that the flow VL may be considered as a restriction of the second flow TL2 of the Toda hierarchy. There exists a relation of a completely different nature with the flow TL. Namely, the flow VL is Miura related to the first flow TL of the Toda hierarchy, while the flow VL2 is Miura related to TL2. To see this notice that the matrices \(T_{\pm}(u, v, \lambda)\) from (14.14) have the same tri–diagonal structure as the Toda lattice Lax matrix (8.3) with

\[ a_k = u_k v_k, \quad b_k = u_k + v_{k-1} \] (14.47) 

or

\[ a_k = u_{k+1} v_k, \quad b_k = u_k + v_k \] (14.48) 

respectively. These two pair of formulas may be considered as two Miura maps \(M_{\pm} : \mathcal{W}(u, v) \mapsto \mathcal{T}(a, b)\). The following holds (see [K1]):

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1. The both maps $M_{\pm}$ are Poisson, if $W$ is equipped with the bracket (14.4), and $T$ is equipped with the bracket (8.7).

2. The both maps $M_{\pm}$ are Poisson, if $W$ is equipped with the bracket (14.5), and $T$ is equipped with the bracket (8.7).

3. The flow $VL$ (14.1) is conjugated with the flow $TL$ (8.1) and the flow $V L_2$ (14.34) is conjugated with the flow $TL_2$ (9.1) by either of the maps $M_{\pm}$.

We now translate these statements into the language of localizing variables. Since we have two different localizing changes of variables (for $dVL$ and $dVL_2$), two different translations are necessary.

We start with the case of the localizing change of variables (14.27).

Theorem 14.10

a) Define two maps $M_{\pm}: W(u,v) \mapsto T(a,b)$ by the formulas

$$a_k = u_k v_k, \quad 1 + h b_k = (1 + hu_k)(1 + hv_{k-1})$$

and

$$a_k = u_{k+1} v_k, \quad 1 + h b_k = (1 + hu_k)(1 + hv_k)$$

respectively. Then the following diagram is commutative:

$$
\begin{array}{ccc}
W(u,v) & \xrightarrow{M_{\pm}} & T(a,b) \\
\downarrow \text{(14.27)} & & \downarrow \text{(8.19)} \\
W(u,v) & \xrightarrow{M_{\pm}} & T(a,b)
\end{array}
$$

b) The both maps $M_{\pm}$ are Poisson, if $W(u,v)$ is equipped with the bracket (14.32), and $T(a,b)$ is equipped with the bracket (8.23).
c) The local form of the dVL (14.28) is conjugated with the local form of the dTL (8.20) by either of the maps $M_{\pm}$.

**Proof.** The first statement is verified by a direct check, the second and the third ones are its consequences.

In [K1] this theorem was formulated without any relation to the problem of integrable discretization. In the context of discrete time systems the formulas (14.49), (14.50) were found also in [HT], however, without discussing Poisson properties of these maps.

Concerning the localizing change of variables (14.44) for dVL2, we come to the following results.

**Theorem 14.11** Define the maps $N_{\pm} : W(u, v) \mapsto T(a, b)$ by the formulas

$$a_k = \frac{u_k v_k}{1 - h u_k v_k}, \quad b_k = \frac{u_k + v_{k-1}}{1 - h u_k v_{k-1}} \quad (14.51)$$

and

$$a_k = \frac{u_{k+1} v_k}{1 - h u_{k+1} v_k}, \quad b_k = \frac{u_k + v_k}{1 - h u_k v_k} \quad (14.52)$$

respectively. Then the following diagram is commutative:

$$\begin{align*}
W(u, v) \xrightarrow{N_{\pm}} & \quad T(a, b) \\
\downarrow^{M_{\pm}} & \quad \downarrow^{\mathcal{M}_{\pm}} \\
W(u, v) \xrightarrow{T(a, b)}
\end{align*}$$

and the local form of the dVL2 (14.43) is conjugated with the local form of the dTL2 (9.13) by either of the maps $N_{\pm}$.

**Proof** – by a direct check. ■

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15 Relativistic Toda lattice

15.1 Equations of motion and tri–Hamiltonian structure

The relativistic Toda lattice was invented by Ruijsenaars [Ruij], and further studied in [BRI], [OFZR] and numerous other papers. In particular, the tri–Hamiltonian structure was elaborated in the latter reference.

We consider here two flows of the relativistic Toda hierarchy:

\[ \dot{d}_k = d_k(c_k - c_{k-1}), \quad \dot{c}_k = c_k(d_{k+1} + c_{k+1} - d_k - c_{k-1}) \] (15.1)

and

\[ \dot{d}_k = d_k \left( \frac{c_k}{d_k d_{k+1}} - \frac{c_{k-1}}{d_k d_{k-1}} \right), \quad \dot{c}_k = c_k \left( \frac{1}{d_k} - \frac{1}{d_{k+1}} \right) \] (15.2)

The both systems may be considered either under open–end boundary conditions \((c_0 = c_N = 0)\), or under periodic ones (all the subscripts are taken \((\mod N)\), so that \(d_{N+1} \equiv d_1, d_0 \equiv d_N, c_{N+1} \equiv c_1, c_0 \equiv c_N\)). We shall denote the first flow by RTL+, and the second one by RTL−. The phase space of the flows RTL± in the case of the periodic boundary conditions may be defined as

\[ \mathcal{R} = \mathbb{R}^{2N}(d_1, c_1, \ldots, d_N, c_N) \] (15.3)

This space carries three compatible local Poisson bracket, with respect to which the flows RTL± are Hamiltonian.

The linear Poisson structure on \(\mathcal{R}\) is defined as

\[ \{d_k, c_k\}_1 = -c_k, \quad \{c_k, d_{k+1}\}_1 = -c_k, \quad \{d_k, d_{k+1}\}_1 = c_k \] (15.4)

The quadratic invariant Poisson structure on \(\mathcal{R}\) is given by the brackets

\[ \{d_k, c_k\}_2 = -d_k c_k, \quad \{c_k, d_{k+1}\}_2 = -c_k d_{k+1}, \quad \{c_k, c_{k+1}\}_2 = -c_k c_{k+1} \] (15.5)

Finally, the cubic Poisson bracket on \(\mathcal{R}\) is given by the relations

\[ \{d_k, c_k\}_3 = -d_k c_k (d_k + c_k), \quad \{c_k, d_{k+1}\}_3 = -c_k d_{k+1} (c_k + d_{k+1}), \]

\[ \{d_k, d_{k+1}\}_3 = -d_k c_k d_{k+1}, \quad \{c_k, c_{k+1}\}_3 = -c_k c_{k+1} (c_k + 2d_{k+1} + c_{k+1}), \]

\[ \{d_k, c_{k+1}\}_3 = -d_k c_k c_{k+1}, \quad \{c_k, d_{k+2}\}_3 = -c_k c_{k+1} d_{k+2}, \]

\[ \{c_k, c_{k+2}\}_3 = -c_k c_{k+1} c_{k+2} \] (15.6)
The Hamilton functions for the flow RTL+ in the brackets (15.4), (15.5), (15.6) are equal to $H_2(c,d)$, $H_1(c,d)$, and $H_0(c,d)$, respectively, where

$$H_2(c,d) = \frac{1}{2} \sum_{k=1}^{N} (d_k + c_k)^2 + \sum_{k=1}^{N} (d_k + c_k)c_{k-1}$$  \hspace{1cm} (15.7)$$

$$H_1(c,d) = \sum_{k=1}^{N} (d_k + c_k)$$  \hspace{1cm} (15.8)$$

$$H_0(c,d) = \sum_{k=1}^{N} \log(d_k)$$  \hspace{1cm} (15.9)$$

Similarly, the Hamilton functions for the flow RTL− in these three brackets are equal to $-H_0(c,d)$, $H_{-1}(c,d)$, $H_{-2}(c,d)$, respectively, where

$$H_{-1}(c,d) = \sum_{k=1}^{N} \frac{d_k + c_k}{d_k d_{k+1}}$$  \hspace{1cm} (15.10)$$

$$H_{-2}(c,d) = \frac{1}{2} \sum_{k=1}^{N} \frac{(d_k + c_k)^2}{d_{k+1}^2 d_k^2} + \sum_{k=1}^{N} \frac{(d_{k-1} + c_{k-1})c_k}{d_{k+1} d_k^2 d_{k-1}}$$  \hspace{1cm} (15.11)$$

### 15.2 Lax representation

The most natural Lax representation for the relativistic Toda hierarchy is the one living in $g = g \otimes g$ [S3], which is in many respects analogous to the Lax representation for the Volterra hierarchy considered in the previous section.

Introduce the matrices

$$U(c,d,\lambda) = \sum_{k=1}^{N} d_k E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k}, \quad V(c,d,\lambda) = I - \lambda^{-1} \sum_{k=1}^{N} c_k E_{k,k+1}$$  \hspace{1cm} (15.12)$$

**Theorem 15.1** [S3]. The equations of motion (15.1) are equivalent to the following Lax equations in $g \otimes g$ (Lax triads):

$$\dot{U} = UB - AU, \quad \dot{V} = VB - AV$$  \hspace{1cm} (15.13)$$

where

$$A(c,d,\lambda) = \sum_{k=1}^{N} (d_k + c_{k-1}) E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k}$$  \hspace{1cm} (15.14)$$

$$B(c,d,\lambda) = \sum_{k=1}^{N} (d_k + c_k) E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k}$$  \hspace{1cm} (15.15)$$
The equations of motion (15.2) are equivalent to the following Lax triads:

\[ \dot{U} = UD - CU, \quad \dot{V} = VD - CV \]  

(15.16)

where

\[
C(c, d, \lambda) = -\lambda^{-1} \sum_{k=1}^{N} \frac{c_k}{d_{k+1}} E_{k, k+1} \tag{15.17}
\]

\[
D(c, d, \lambda) = -\lambda^{-1} \sum_{k=1}^{N} \frac{c_k}{d_k} E_{k, k+1} \tag{15.18}
\]

Corollary. The matrices

\[
T_+(c, d, \lambda) = U(c, d, \lambda)V^{-1}(c, d, \lambda), \quad T_-(c, d, \lambda) = V^{-1}(c, d, \lambda)U(c, d, \lambda) \tag{15.19}
\]

satisfy the usual Lax equations in \( g \). Namely, for the flow (15.1)

\[ \dot{T}_+ = [T_+, A], \quad \dot{T}_- = [T_-, B] \tag{15.20} \]

and for the flow (15.2)

\[ \dot{T}_+ = [T_+, C], \quad \dot{T}_- = [T_-, D] \tag{15.21} \]

The following formulas are easy to establish by a direct calculation:

\[
A = \pi_+(T_+), \quad B = \pi_+(T_-) \tag{15.22}
\]

\[
C = \pi_-(T_+^{-1}), \quad D = \pi_-(T_-^{-1}) \tag{15.23}
\]

Hence the Lax equations (15.20) and (15.21) may be presented as

\[ \dot{T} = [T, \pi_+(T)] \quad \text{and} \quad \dot{T} = [T, \pi_-(T^{-1})] \]

respectively. These Lax equations in \( g \) may be given an \( r \)-matrix interpretation in the cases of the linear and of the quadratic Poisson brackets; for the Lax triads in \( g \otimes g \) an \( r \)-matrix interpretation is known in the case of the quadratic bracket only [S3].

15.3 Discretization of the relativistic Toda hierarchy

We see that actually the pairs \((U, V^{-1})\) satisfy the Lax equations of the form (3.2) (with \( m = 2 \)). This allows to apply our general recipe to find integrable discretizations of the flows \( \text{RTL} \pm \). This results in considering the following discrete time Lax triads:

\[ \tilde{U} = \Pi_+^{-1}(F(T_+)) \cdot U \cdot \Pi_+(F(T_-)) = \Pi_-(F(T_+)) \cdot U \cdot \Pi_-^{-1}(F(T_-)) \]

\[ \tilde{V} = \Pi_+^{-1}(F(T_+)) \cdot V \cdot \Pi_+(F(T_-)) = \Pi_-(F(T_+)) \cdot V \cdot \Pi_-^{-1}(F(T_-)) \]

with

\[ F(T) = I + hT \quad \text{and} \quad F(T) = I - hT^{-1} \]

respectively. It turns out that for the flow \( \text{RTL}+ \) the version with the \( \Pi_+ \) factors is more suitable, while for the \( \text{RTL}− \) flow the version with the \( \Pi_- \) factors is preferable.
15.4 Discretization of the flow RTL+

Consider the discrete time Lax triad

\[ \tilde{U} = A^{-1} U, \quad \tilde{V} = A^{-1} V \]

with

\[ A = \Pi(I + hT_+), \quad B = \Pi(I + hT_-) \]

implying also either of the two equivalent forms of a convenient Lax equation:

\[ \tilde{T}_+ = A^{-1} T_+ A, \quad \tilde{T}_- = B^{-1} T_- B \]

**Theorem 15.2** \cite{S7}. The equations (15.24) are equivalent to the map \((c, d) \mapsto (\tilde{c}, \tilde{d})\) described by the following equations:

\[ \tilde{d}_k = d_k \frac{b_k}{a_k}, \quad \tilde{c}_k = c_k \frac{b_{k+1}}{a_k} \]

where the functions \(a_k = a_k(c, d) = 1 + O(h)\) are uniquely defined by the recurrent relation

\[ a_k = 1 + hd_k + \frac{hc_{k-1}}{a_{k-1}} \]

and the coefficients \(b_k = b_k(c, d) = 1 + O(h)\) are given by

\[ b_k = a_{k-1} \frac{a_k + hc_k}{a_{k-1} + hc_{k-1}} = a_k \frac{a_{k+1} - hd_{k+1}}{a_k - hd_k} \]

The following asymptotics hold:

\[ a_k = 1 + h(d_k + c_{k-1}) + O(h^2) \]

\[ b_k = 1 + h(d_k + c_k) + O(h^2) \]

**Remark.** The auxiliary matrices \(A, B\) are bi–diagonal:

\[ A(c, d, \lambda) = \Pi(I + hT_+) = \sum_{k=1}^{N} a_k E_{kk} + h\lambda \sum_{k=1}^{N} E_{k+1,k} \]

\[ B(c, d, \lambda) = \Pi(I + hT_-) = \sum_{k=1}^{N} b_k E_{kk} + h\lambda \sum_{k=1}^{N} E_{k+1,k} \]
Proof. The general bi–diagonal structure of the factors $A, B$ follows from their definition. The simplest way to derive the recurrent relations (15.27) for the entries of $A$ is to notice that
\[ A = \Pi_+ \left( I + hUV^{-1} \right) = \Pi_+ (V + hU), \quad \text{because} \quad V \in G_- \]

The equations of motion (15.26) and the relations (15.28) follow now from $A\tilde{U} = UB, A\tilde{V} = VB.$

The map (15.26) will be denoted dRTL+. Due to the asymptotic relations (15.29), (15.30) it clearly approximates the flow RTL+. As usual, it is tri–Poisson, allows the same integrals and the same Lax matrix as the flow RTL, but has a drawback of nonlocality. Its source are the functions $a_k$, which in the open–end case may be expressed as terminating continued fractions:
\[ a_k = 1 + h d_k + \frac{hc_{k-1}}{1 + h d_{k-1} + \cdots + hc_1} \]

In the periodic case the $a_k$’s may be expressed as infinite periodic continued fractions of an analogous structure.

15.5 Local equations of motion for dRTL+

Introduce another copy of the phase space $\mathcal{R}$ parametrized by the variables $c_k, d_k$ and consider the change of variables $\mathcal{R}(c, d) \mapsto \mathcal{R}(c, d)$ defined by the following formulas:
\[ d_k = d_k(1 + h c_{k-1}), \quad c_k = c_k(1 + h d_k)(1 + h c_{k-1}) \] (15.33)

Obviously, due to the implicit function theorem, for $h$ small enough this map is locally a diffeomorphism.

Theorem 15.3 The change of variables (15.33) conjugates dRTL+ with the map described by the following local equations of motion:
\[ \tilde{d}_k(1 + h\tilde{c}_{k-1}) = d_k(1 + h c_k) \]
\[ \tilde{c}_k(1 + h\tilde{d}_k)(1 + h\tilde{c}_{k-1}) = c_k(1 + h d_{k+1})(1 + h c_{k+1}) \] (15.34)

Proof. The crucial point in the proof of this theorem is the following observation: the parametrization of the variables $(c, d)$ according to (15.33) allows to find the coefficients $a_k$ (defined by the recurrent relations (15.27)) in the closed form, namely:
\[ a_k = (1 + h d_k)(1 + h c_{k-1}) \] (15.35)
Indeed, if we accept the last formula as the definition of the quantities $a_k$, then we obtain successively from (15.35) and (15.33):

$$a_k = 1 + h d_k (1 + h c_{k-1}) + h c_{k-1} = 1 + h d_k + \frac{h c_{k-1}}{a_{k-1}}$$

So, the quantities defined by (15.35) satisfy the recurrent relation (15.27), and due to the uniqueness of solution our assertion is demonstrated. Now (15.28) and (15.35) yield:

$$b_k = (1 + h d_k)(1 + h c_k)$$

and the equations of motion (15.34) follow directly from (15.26), (15.35), (15.36).

It turns out that the pull–back of either of the brackets (15.4), (15.5), (15.6) is highly nonlocal. Nevertheless, there exist certain linear combinations thereof, whose pull–backs are local.

**Theorem 15.4**  

a) The pull–back of the bracket

$$\{\cdot, \cdot\}_1 + h \{\cdot, \cdot\}_2$$

on $\mathcal{R}(c, d)$ under the change of variables (15.33) is the following Poisson bracket on $\mathcal{R}(c, d)$:

$$\begin{align*}
\{d_k, c_k\} &= -c_k (1 + h d_k) \\
\{c_k, d_{k+1}\} &= -c_k (1 + h d_{k+1}) \\
\{d_k, d_{k+1}\} &= c_k \frac{(1 + h d_k)(1 + h d_{k+1})}{(1 + h c_k)}
\end{align*}$$

b) The pull–back of the bracket

$$\{\cdot, \cdot\}_2 + h \{\cdot, \cdot\}_3$$

on $\mathcal{R}(c, d)$ under the change of variables (15.33) is the following Poisson bracket on $\mathcal{R}(c, d)$:

$$\begin{align*}
\{d_k, c_k\} &= -d_k c_k (1 + h d_k)(1 + h c_k) \\
\{c_k, d_{k+1}\} &= -c_k d_{k+1} (1 + h c_k)(1 + h d_{k+1}) \\
\{c_k, c_{k+1}\} &= -c_k c_{k+1} (1 + h c_k)(1 + h d_{k+1})(1 + h c_{k+1})
\end{align*}$$

c) The brackets (15.38), (15.40) are compatible. The map (15.34) is Poisson with respect to both of them.
Proof – by a direct check. Notice a curious feature of the bracket \((15.38)\): it is nonpolynomial in coordinates (though still local). ■

**Theorem 15.5** The pull–back of the flow RTL+ under the map \((15.33)\) is described by the following equations of motion:

\[
\dot{d}_k = d_k(1 + h d_k) (c_k - c_{k-1}) \\
\dot{c}_k = c_k(1 + h c_k) \left( d_{k+1} + c_{k+1} + h d_{k+1} c_{k+1} - d_k - c_{k-1} - h d_k c_{k-1} \right)
\]

**Proof.** We will use by the proof only the bracket of the part b) of the previous theorem. Obviously, the pull–back we are looking for is a Hamiltonian system with the Hamilton function, which is a pull–back of \(h^{-1} H_0(c, d)\) (indeed, this function is a Casimir function for \(\{·, ·\}_2\) and is the Hamilton function of the flow RTL+ in the bracket \(h\{·, ·\}_3\)). Calculating the equations of motion generated by the Hamilton function

\[
h^{-1} \sum_{k=1}^{N} \log \left( d_k(1 + h c_k) \right)
\]

in the Poisson brackets \((15.40)\), we obtain \((15.41)\). ■

### 15.6 Discretization of the flow RTL–

Consider the discrete time Lax triad

\[
\tilde{U} = C U D^{-1}, \quad \tilde{V} = C V D^{-1}
\]

with

\[
C = \Pi_-(I - h T_+^{-1}), \quad D = \Pi_-(I - h T_-^{-1})
\]

implying also the convenient Lax equations:

\[
\tilde{T}_+ = C T_+ C^{-1}, \quad \tilde{T}_- = D T_- D^{-1}
\]

**Theorem 15.6** \([S7]\). The equations \((15.42)\) are equivalent to the map \((c, d) \mapsto (\tilde{c}, \tilde{d})\) described by the following equations:

\[
\tilde{d}_k = d_{k+1} \frac{c_k}{\partial_k}, \quad \tilde{c}_k = c_{k+1} \frac{c_k}{\partial_k + 1}
\]
where the functions $\varphi_k = \varphi_k(c, d) = O(1)$ are uniquely defined by the recurrent relation

$$\varphi_k = \frac{c_k}{d_k - h - h\varphi_{k-1}} \quad (15.45)$$

and the coefficients $c_k = c_k(c, d) = O(1)$ are given by

$$c_k = \varphi_k \frac{d_k - h\varphi_{k-1}}{d_{k+1} - h\varphi_k} = \varphi_{k+1} \frac{c_k + h\varphi_k}{c_{k+1} + h\varphi_{k+1}} \quad (15.46)$$

The following asymptotics hold:

$$\varphi_k = \frac{c_k}{d_k} + O(h), \quad c_k = \frac{c_k}{d_{k+1}} + O(h) \quad (15.47)$$

**Remark.** The auxiliary matrices $C, D$ in the discrete time Lax equations admit the following expressions:

$$C(c, d, \lambda) = I + h\lambda^{-1} \sum_{k=1}^{N} c_k E_{k,k+1} \quad (15.48)$$

$$D(c, d, \lambda) = I + h\lambda^{-1} \sum_{k=1}^{N} \varphi_k E_{k,k+1} \quad (15.49)$$

**Proof** – analogous to that of Theorem 15.2.

The map (15.44) will be called hereafter $dRTL−$. Like $dRTL+$, it is tri–Poisson etc. It is nonlocal because of the presence of the functions $\varphi_k$, which in the open–end case have the following finite continued fractions expressions:

$$\varphi_k = \frac{c_k}{d_k - h - \cdots - h\varphi_1} \quad (15.44)$$

**15.7 Local equations of motion for dRTL−**

To bring the map $dRTL−$ to the local form, another change of variables $\mathcal{R}(c, d) \mapsto \mathcal{R}(c, d)$ is necessary:

$$d_k = d_k \left(1 + \frac{h c_{k-1}}{d_{k-1} d_k}\right), \quad c_k = c_k \left(1 - \frac{h}{d_k}\right) \quad (15.50)$$

Again, for $h$ small enough this map is locally a diffeomorphism, due to the implicit function theorem.
Theorem 15.7  The change of variables (15.50) conjugates dRTL— with the map described by the following local equations of motion:

\[
\tilde{d}_k \left(1 + \frac{h\tilde{c}_{k-1}}{\tilde{d}_{k-1}\tilde{d}_k}\right) = d_k \left(1 + \frac{hc_k}{d_k d_{k+1}}\right)
\]

\[
\tilde{c}_k \left(1 - \frac{h}{d_k}\right) = c_k \left(1 - \frac{h}{d_{k+1}}\right)
\]

(15.51)

Proof. This time the crucial component of the proof is the following remarkably simple local formula for the coefficients \(d_k\) (defined by the recurrent relations (15.45)) in the coordinates \((c_k, d_k)\):

\[
\vartheta_k = \frac{c_k}{d_k}
\]

(15.52)

Indeed, if we use both (15.50) and (15.52) as definitions, then we obtain:

\[
\frac{c_k}{\vartheta_k} = d_k - h = d_k - h - h\vartheta_{k-1}
\]

Hence, the quantities defined by (15.52) satisfy the recurrent relation (15.45), and due to the uniqueness of solution our assertion is proved. From (15.46) and (15.52) we obtain also:

\[
c_k = \frac{c_k}{d_{k+1}}
\]

(15.53)

Now the equations of motion (15.51) follow directly from (15.44), (15.52), (15.53). □

The Poisson properties of the change of variables (15.50) are similar to that of (15.33). Namely, the pull–backs of either of the brackets (15.4), (15.5), (15.6) are nonlocal, but there exist linear combinations thereof, whose pull–backs are local.

Theorem 15.8  a) The pull–back of the Poisson bracket

\[
\{\cdot, \cdot\}_2 - h\{\cdot, \cdot\}_1
\]

(15.54)

on \(\mathcal{R}(c, d)\) under the change of variables (15.50) is the following bracket on \(\mathcal{R}(c, d)\):

\[
\{d_k, c_k\} = -d_k c_k \left(1 - \frac{h}{d_k}\right), \quad \{c_k, d_{k+1}\} = -c_k d_{k+1} \left(1 - \frac{h}{d_{k+1}}\right),
\]

\[
\{c_k, c_{k+1}\} = -c_k c_{k+1} \left(1 - \frac{h}{d_{k+1}}\right)
\]

(15.55)
b) The pull–back of the Poisson bracket

\[ \{ \cdot, \cdot \}_3 - h \{ \cdot, \cdot \}_2 \]  

(15.56)

on \( \mathcal{R}(c, d) \) under the change of variables (15.50) is the following bracket on \( \mathcal{R}(c, d) \):

\[
\begin{align*}
\{d_k, c_k\} &= -d_k c_k (d_k + c_k) \left( 1 - \frac{h}{d_k} \right) \\
\{c_k, d_{k+1}\} &= -c_k d_{k+1} (c_k + d_{k+1}) \left( 1 - \frac{h}{d_{k+1}} \right) \\
\{d_k, d_{k+1}\} &= -d_k c_k d_{k+1} \left( 1 - \frac{h}{d_k} \right) \left( 1 - \frac{h}{d_{k+1}} \right) \\
\{c_k, c_{k+1}\} &= -c_k c_{k+1} (c_k + 2d_{k+1} + c_{k+1}) \left( 1 - \frac{h}{d_{k+1}} \right) \\
\{d_k, c_{k+1}\} &= -d_k c_k c_{k+1} \left( 1 - \frac{h}{d_k} \right) \left( 1 - \frac{h}{d_{k+1}} \right) \\
\{c_k, d_{k+2}\} &= -c_k c_{k+1} d_{k+2} \left( 1 - \frac{h}{d_{k+1}} \right) \left( 1 - \frac{h}{d_{k+2}} \right) \\
\{c_k, c_{k+2}\} &= -c_k c_{k+1} c_{k+2} \left( 1 - \frac{h}{d_{k+1}} \right) \left( 1 - \frac{h}{d_{k+2}} \right)
\end{align*}
\]

(15.57)

c) The brackets (15.55), (15.57) are compatible. The map (15.51) is Poisson with respect to both of them.

**Proof** consists of straightforward calculations. ■

**Theorem 15.9** The pull–back of the flow RTL– under the map (15.50) is described by the following equations of motion:

\[
\begin{align*}
\dot{d}_k &= (d_k - h) \left( \frac{c_k}{d_k d_{k+1}} \left( 1 + \frac{hc_k}{d_k d_{k+1}} \right)^{-1} - \frac{c_{k-1}}{d_{k-1} d_k} \left( 1 + \frac{hc_{k-1}}{d_{k-1} d_k} \right)^{-1} \right) \\
\dot{c}_k &= c_k \left( \frac{1}{d_k} - \frac{1}{d_{k+1}} \right) \left( 1 + \frac{hc_k}{d_k d_{k+1}} \right)^{-1}
\end{align*}
\]

(15.58)

**Proof.** We use the part a) of the previous Theorem. The flow under consideration is a Hamiltonian system with the Hamilton function, which is a pull–back of \( h^{-1}H_0(c, d) \) (indeed, this function is a Casimir function for \( \{ \cdot, \cdot \}_2 \) and is a Hamilton function of the flow RTL–
in the bracket $-h\{\cdot, \cdot\}_1$). Calculating the equations of motion generated by the Hamilton function
\[
 h^{-1} \sum_{k=1}^{N} \log(d_k) + h^{-1} \sum_{k=1}^{N} \log \left(1 + \frac{hc_k}{d_k d_{k+1}}\right)
\]
in the Poisson brackets (15.55), we arrive at (15.58).

15.8 Third appearance of the Volterra lattice

It is interesting to remark that the flow RTL+ allows the reduction $d_k = 0$, in which it turns into the Volterra lattice
\[
 \dot{c}_k = c_k(c_{k+1} - c_{k-1})
\]
The Lax representation of the RTL+ flow survives this reduction, delivering a new (third) Lax representation for the VL. Also the quadratic and the cubic Poisson brackets (15.5) and (15.6) allow this reduction and turn into the corresponding objects for the VL.

It may be verified that the map $d\text{RTL} +$ in the reduction $d_k = 0$ turns into $d\text{VL}$, although this discretization is based on a completely different Lax representation. Naturally, the same holds for the local forms of these maps.

16 Belov–Chaltikian lattice

16.1 Equations of motion and bi–Hamiltonian structure

In their studies of the lattice analogs of $W$–algebras, Belov and Chaltikian [BC, ABC] found an interesting integrable lattice (hereafter BCL):
\[
 \dot{b}_k = b_k(b_{k+1} - b_{k-1}) - c_k + c_{k-1}, \quad \dot{c}_k = c_k(b_{k+2} - b_{k-1})
\]
This system may be viewed as an extension of the Volterra lattice (which appears as a $c_k = 0$ reduction of the above system). Belov and Chaltikian established also the bi–Hamiltonian structure of this system. Namely, its phase space, which in the periodic case is
\[
 \mathcal{BC} = \mathbb{R}^{2N}(b_1, c_1, \ldots, b_N, c_N)
\]
carries two compatible local Poisson brackets, with respect to which the system BCL is Hamiltonian. The first ("quadratic") Poisson bracket is given by
\[
\begin{align*}
\{b_k, b_{k+1}\}_2 &= -b_k b_{k+1} + c_k, & \{c_k, c_{k+1}\}_2 &= -c_k c_{k+1} \\
\{b_k, c_{k+1}\}_2 &= -b_k c_{k+1}, & \{c_k, b_{k+2}\}_2 &= -c_k b_{k+2} \\
\{c_k, c_{k+2}\}_2 &= -c_k c_{k+2}
\end{align*}
\]
the corresponding Hamilton function being

\[ H_1(b, c) = \sum_{k=1}^{N} b_k \]  

(16.4)

The second ("cubic") Poisson bracket on \( \mathcal{B} \mathcal{C} \) is given by

\[
\{b_k, c_k\}_3 = -c_k (b_k b_{k+1} - c_k) \\
\{c_k, b_{k+1}\}_3 = -c_k (b_k b_{k+1} - c_k) \\
\{b_k, b_{k+1}\}_3 = -(b_k + b_{k+1})(b_k b_{k+1} - c_k) \\
\{c_k, c_{k+1}\}_3 = -c_k c_{k+1} (b_k + b_{k+2}) \\
\{b_k, c_{k+1}\}_3 = -b_k c_{k+1} (b_k + b_{k+1}) + c_k c_{k+1} \\
\{c_k, b_{k+2}\}_3 = -c_k b_{k+2} (b_{k+1} + b_{k+2}) + c_k c_{k+1} \\
\{b_k, b_{k+2}\}_3 = -b_k b_{k+1} b_{k+2} + b_k c_{k+1} + c_k b_{k+2} \\
\{c_k, c_{k+2}\}_3 = -c_k c_{k+2} (b_{k+1} + b_{k+2}) \\
\{b_k, c_{k+2}\}_3 = -c_k b_{k+2} (b_{k+1} + b_{k+2}) - c_k \\
\{c_k, b_{k+3}\}_3 = -c_k (b_k b_{k+2} - c_{k+2}) \\
\{c_k, c_{k+3}\}_3 = -c_k b_{k+2} c_{k+3}
\]

(16.5)

and the corresponding Hamilton function is equal to

\[
H_0(b, c) = \frac{1}{3} \sum_{k=1}^{N} \log(c_k)
\]

(16.6)

**16.2 Lax representation**

The Lax matrix of the BCL found in [BC], [ABC], is given in terms of the matrices

\[
U(\lambda) = \lambda \sum_{k=1}^{N} E_{k+1,k}, \quad V(b, c, \lambda) = I - \lambda^{-1} \sum_{k=1}^{N} b_k E_{k,k+1} + \lambda^{-2} \sum_{k=1}^{N} c_k E_{k,k+2}
\]

(16.7)

**Theorem 16.1** The equations of motion (16.1) are equivalent to the following matrix differential equation:

\[
\dot{V} = VB - AV
\]

(16.8)

where

\[
A(b, c, \lambda) = \pi_+(UV^{-1}) = \sum_{k=1}^{N} b_{k-1} E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k} \\
B(b, c, \lambda) = \pi_+(V^{-1}U) = \sum_{k=1}^{N} b_k E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k}
\]

(16.9)
so that also the following equation holds identically:

\[ UB - AU = 0 \]  \hspace{1cm} (16.10)

**Corollary** \([BC],[ABC]\). The matrix

\[ T(b, c, \lambda) = U \cdot V^{-1}(b, c, \lambda) \]  \hspace{1cm} (16.11)

satisfies the usual Lax equation in \(g\):

\[ \dot{T} = [T, A] \]  \hspace{1cm} (16.12)

This Lax equation can be given an \(r\)-matrix interpretation in the case of quadratic Poisson bracket, which can be lifted to an \(r\)-matrix interpretation of the Lax triads for the pairs \((U, V) \in g \otimes g\).

### 16.3 Discretization

Since the Lax equation (16.12) has the form (3.1), and moreover, the pairs \((U, V^{-1})\) satisfy the Lax triads of the form (3.2), we can apply the recipe of Sect. 5. Taking, as usual, \(F(T) = I + hT\), we come to the discrete time matrix equation

\[ \bar{V} = A^{-1}V B \]  \hspace{1cm} (16.13)

with

\[ A = \Pi_+(I + hUV^{-1}), \quad B = \Pi_+(I + hV^{-1}U) \]

Moreover, since the equation

\[ U = A^{-1}UB \]  \hspace{1cm} (16.14)

holds, we have also the usual Lax equation

\[ \bar{T} = A^{-1}TA \]  \hspace{1cm} (16.15)

**Theorem 16.2** The equations (16.13), (16.14) are equivalent to the following equations:

\[ \bar{b}_k = b_k \frac{\alpha_{k+2}}{\alpha_k} - h \left( c_k \frac{1}{\alpha_k} - c_k - 1 \frac{\alpha_{k+2}}{\alpha_k \alpha_{k-1}} \right), \quad \bar{c}_k = c_k \frac{\alpha_{k+3}}{\alpha_k} \]  \hspace{1cm} (16.16)
where the coefficients \( \alpha_k = \alpha_k(b, c) = 1 + O(h) \) are uniquely defined by the recurrent relation

\[
\alpha_k = 1 + \frac{h b_{k-1}}{\alpha_{k-1}} + \frac{h^2 c_{k-2}}{\alpha_{k-1} \alpha_{k-2}} \tag{16.17}
\]

The following asymptotics hold:

\[
\alpha_k = 1 + h b_{k-1} + O(h^2) \tag{16.18}
\]

Remark. The auxiliary matrices \( A, B \) are bi–diagonal:

\[
A(b, c, \lambda) = \sum_{k=1}^{N} \alpha_k E_{kk} + h \lambda \sum_{k=1}^{N} E_{k+1,k} \tag{16.19}
\]

\[
B(b, c, \lambda) = \sum_{k=1}^{N} \alpha_{k+1} E_{kk} + h \lambda \sum_{k=1}^{N} E_{k+1,k} \tag{16.20}
\]

Proof is by now standard. The general bi–diagonal structure of the factors \( A, B \) follows from \( A = \Pi_+(I + hUV^{-1}) = \Pi_+(V + hU) \), the latter representation implies also the recurrent relation for the entries \( \alpha_k \) of the matrix \( A \). From (16.14) one derives immediately \( \beta_k = \alpha_{k+1} \). The equations of motion (16.16) follow then easily from \( A \tilde{V} = V B \). □

Hereafter we call the map (16.16) dBCL. By construction, it is bi–Poisson with respect to the brackets (16.3) and (16.5), approximates the flow BCL due to the asymptotics (16.18), but is nonlocal due to the nature of the auxiliary quantities \( \alpha_k \).

### 16.4 Local equations of motion for the dBCL

The localizing change of variables for dBCL is the map \( \mathcal{BC}(b, c) \mapsto \mathcal{BC}(b, c) \) given by the formulas:

\[
b_k = b_k(1 + h b_{k-1}) - h c_{k-1}, \quad c_k = c_k(1 + h b_{k-1}) \tag{16.21}
\]

As usual this is a local diffeomorphism for \( h \) small enough.

Theorem 16.3 The change of variables (16.21) conjugates the map dBCL with the following one:

\[
\tilde{b}_k(1 + \tilde{b}_{k-1}) - h \tilde{c}_{k-1} = b_k(1 + h b_{k+1}) - h c_k
\]

\[
\bar{c}_k(1 + \bar{c}_{k-1}) = c_k(1 + h b_{k+2}) \tag{16.22}
\]

Proof. Introducing the quantities

\[
\alpha_k = 1 + h b_{k-1} \tag{16.23}
\]
we immediately see via simple check that they satisfy the recurrent relations (16.17). Hence they represent the unique solution of these recurrences with the asymptotics $\alpha_k = 1 + O(h)$. Now the equations of motion follow directly from (16.16), (16.21), (16.23).

**Theorem 16.4** The pull back of the bracket

$$\{\cdot, \cdot\}_2 + h\{\cdot, \cdot\}_3$$

on $BC(b,c)$ under the change of variables (16.21) is the following local Poisson bracket on $BC(b,c)$:

\[
\begin{align*}
\{b_k, c_k\} &= -hc_k(b_k b_{k+1} - c_k)(1 + h b_k) \\
\{c_k, b_{k+1}\} &= -hc_k(b_k b_{k+1} - c_k)(1 + h b_{k+1}) \\
\{b_k, b_{k+1}\} &= -(b_k b_{k+1} - c_k)(1 + h b_k)(1 + h b_{k+1}) \\
\{c_k, c_{k+1}\} &= -c_k c_{k+1} (1 + h b_k + h b_{k+2} + h^2(b_k b_{k+1} - c_k) + h^2(b_{k+1} b_{k+2} - c_{k+1})) \\
\{b_k, c_{k+1}\} &= -c_{k+1} (b_k + h(b_k b_{k+1} - c_k))(1 + h b_k) \\
\{c_k, b_{k+2}\} &= -c_k (b_{k+2} + h(b_{k+1} b_{k+2} - c_{k+1}))(1 + h b_{k+2}) \\
\{c_k, c_{k+2}\} &= -c_k c_{k+2} (1 + h b_{k+1} + h b_{k+2} + h^2(b_{k+1} b_{k+2} - c_{k+1}))
\end{align*}
\]

The map (16.22) is Poisson with respect to this bracket.

**Proof** – by a straightforward but tiresome calculation.

**Theorem 16.5** The pull–back of the flow BCL under the change of variables (16.21) is described by the following equations of motion:

\[
\begin{align*}
\dot{b}_k &= (1 + h b_k) (b_k(b_{k+1} - b_{k-1}) - c_k + c_{k-1}) \\
\dot{c}_k &= c_k (b_{k+2}(1 + h b_{k+1}) - b_{k-1}(1 + h b_k) - h c_{k+1} + h c_{k-1})
\end{align*}
\]

**Proof.** We can use the Hamiltonian formalism. The pull–back we are looking for, is a Hamiltonian system on $BC(b,c)$ with the Poisson bracket (16.25) and the Hamilton function which is a pull–back of $(3h)^{-1} \sum_{k=1}^{N} \log(c_k)$. Indeed, this function is a Casimir function for $\{\cdot, \cdot\}_2$ and is the Hamilton function for BCL in the bracket $h\{\cdot, \cdot\}_3$. Calculating the equations of motion generated by the Hamilton function

$$\begin{align*}
(3h)^{-1} \sum_{k=1}^{N} \log(c_k) + (3h)^{-1} \sum_{k=1}^{N} \log(1 + h b_k)
\end{align*}$$

in the bracket (16.23), we arrive at (16.26).
17 A perturbation of the Volterra lattice

17.1 Equations of motion and bi–Hamiltonian structure

Consider the following lattice system:

\begin{align*}
\dot{u}_k &= u_k(w_k - w_{k-1} + u_kw_k - u_{k-1}w_{k-1}) \\
\dot{w}_k &= w_k(u_{k+1} - u_k + u_{k+1}w_{k+1} - u_kw_k)
\end{align*}

(17.1)

It may be considered as a perturbation of the Volterra lattice, therefore we adopt a name PVL for it. Its phase space in the periodic case is

\[ PV = \mathbb{R}^{2N}(u_1, w_1, \ldots, u_N, w_N) \]  

(17.2)

The system PVL is bi–Hamiltonian. First of all, it is Hamiltonian with respect to a quadratic Poisson bracket on \( PV \) which is identical with the invariant quadratic Poisson bracket of the Volterra lattice:

\[ \{u_k, w_k\}_2 = -u_kw_k, \quad \{w_k, u_{k+1}\}_2 = -w_kw_{k+1} \]  

(17.3)

The corresponding Hamilton function is equal to

\[ H_1(u, w) = \sum_{k=1}^{N}(u_k + w_k + u_kw_k) \] 

(17.4)

The second ("cubic") invariant Poisson bracket, compatible with the previous one, is different from the cubic bracket of the VL and is given by

\begin{align*}
\{u_k, w_k\}_3 &= -u_kw_k(u_k + w_k + u_kw_k), & \{w_k, u_{k+1}\}_3 &= -w_kw_{k+1}(w_k + u_{k+1}) \\
\{u_k, u_{k+1}\}_3 &= -u_kw_{k+1}(w_k + u_kw_k), & \{w_k, w_{k+1}\}_3 &= -w_kw_{k+1}(u_{k+1} + u_{k+1}w_{k+1}) \\
\{w_k, u_{k+2}\}_3 &= -w_kw_{k+1}w_{k+1}u_{k+2}
\end{align*}

(17.5)

The corresponding Hamilton function may be taken as

\[ H_0(u, w) = \sum_{k=1}^{N}\log(u_k) \quad \text{or} \quad H_0(u, w) = \sum_{k=1}^{N}\log(w_k) \]  

(17.6)

(the difference of these two functions is a Casimir of the bracket \( \{w_k, u_{k+1}\}_2 \)).
17.2 Lax representation

The Lax representation for PVL is given in terms of three matrices from $g$:

$$U(u, w, \lambda) = \sum_{k=1}^{N} u_k E_{k,k} + \lambda \sum_{k=1}^{N} E_{k+1,k}$$

$$V(u, w, \lambda) = I - \lambda^{-1} \sum_{k=1}^{N} u_kw_k E_{k,k+1}$$

$$W(u, w, \lambda) = I + \lambda^{-1} \sum_{k=1}^{N} w_k E_{k,k+1}$$

(17.7)

Theorem 17.1 The equations of motion (17.1) are equivalent to the following matrix differential equations:

$$\dot{U} = UC - AU, \quad \dot{W} = WB - CW$$

(17.8)

and imply also the matrix differential equation

$$\dot{V} = VB - AV$$

(17.9)

with the auxiliary matrices

$$A(u, w, \lambda) = \sum_{k=1}^{N} (u_k + u_{k-1}w_{k-1} + w_{k-1}) E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k}$$

$$B(u, w, \lambda) = \sum_{k=1}^{N} (u_k + u_k w_k + w_{k-1}) E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k}$$

(17.10)

$$C(u, w, \lambda) = \sum_{k=1}^{N} (u_k + u_k w_k + w_k) E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k}$$

Proof – an elementary check. ■

It is easy to establish the following fact:

$$A = \pi_{+}(UWV^{-1}), \quad B = \pi_{+}(V^{-1}UW), \quad C = \pi_{+}(WV^{-1}U)$$

so that the triples $(U, V^{-1}, W) \in g \otimes g \otimes g$ satisfy the Lax equations of the type (3.2) with $m = 3$ and $f(T) = T$. These equations may be given an $r$–matrix interpretation, at least in the case of the quadratic bracket $\{\cdot, \cdot\}_2$. The corresponding quadratic bracket on $g \otimes g \otimes g$ turns out to be identical with the one introduced in [S12].
17.3 Discretization

To discretize the PVL, we can apply the recipe of Sect. 5 with \( F(T) = I + hT \). So, we have to consider the following discrete time Lax representation:

\[
\tilde{U} = A^{-1} UC, \quad \tilde{V} = A^{-1} VB, \quad \tilde{W} = C^{-1} WB \tag{17.11}
\]

where

\[
A = \Pi_+ \left( I + hUWV^{-1} \right), \quad B = \Pi_+ \left( I + hV^{-1}UW \right), \quad C = \Pi_+ \left( I + hWV^{-1}U \right)
\]

**Theorem 17.2** The discrete time Lax equations (17.11) are equivalent to the following equations of motion:

\[
\tilde{u}_k = u_k \frac{c_k}{a_k}, \quad \tilde{w}_k = w_k \frac{b_{k+1}}{c_k} \tag{17.12}
\]

where the functions \( a_k = a_k(u, w) = 1 + O(h) \) are uniquely defined by the system of recurrent relations

\[
a_{k+1} = 1 + h(u_{k+1} + w_k) + \frac{h(1-h)u_kw_k}{a_k} \tag{17.13}
\]

and the coefficients \( b_k = b_k(u, w) = 1 + O(h) \), \( c_k = c_k(u, w) = 1 + O(h) \) are given by

\[
b_k = a_{k-1} \frac{a_k + hu_kw_k}{a_{k-1} + hu_{k-1}w_{k-1}}, \quad c_k = a_k \frac{a_{k+1} - hu_{k+1}}{a_k - hu_k} \tag{17.14}
\]

The following asymptotics hold:

\[
a_k = 1 + h(u_k + u_{k-1}w_{k-1} + w_{k-1}) + O(h^2) \tag{17.15}
\]

\[
b_k = 1 + h(u_k + u_kw_k + w_{k-1}) + O(h^2) \tag{17.16}
\]

\[
c_k = 1 + h(u_k + u_kw_k + w_k) + O(h^2) \tag{17.17}
\]

**Remark.** The auxiliary matrices \( A, B, C \) are bi–diagonal:

\[
A = \sum_{k=1}^{N} a_k E_{k,k} + h \lambda \sum_{k=1}^{N} E_{k+1,k} \tag{17.18}
\]

\[
B = \sum_{k=1}^{N} b_k E_{k,k} + h \lambda \sum_{k=1}^{N} E_{k+1,k} \tag{17.19}
\]

\[
C = \sum_{k=1}^{N} c_k E_{k,k} + h \lambda \sum_{k=1}^{N} E_{k+1,k} \tag{17.20}
\]

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Proof. We have: $A = \Pi_+(I + hUWV^{-1}) = \Pi_+(V + hUW)$, since $V \in G_-$. Here

$$V + hUW = \sum_{k=1}^{N} (1 + h u_k + h w_{k-1}) E_{k,k} - \lambda^{-1} \sum_{k=1}^{N} (1 - h) u_k w_k E_{k,k+1} + h \lambda \sum_{k=1}^{N} E_{k+1,k}$$

and the recurrent relations (17.13) for $a_k$, the entries of the $\Pi_+$ factor of this tri-diagonal matrix, follow immediately. The expressions for $b_k$, $c_k$ through $a_k$, as well as the equations of motion (17.12), follow directly from the Lax equations $A \tilde{U} = UC$, $A \tilde{V} = VB$, $CW = WB$.

We denote the map defined in this theorem by dPVL. As usual, it shares with the system PVL the bi-Hamiltonian structure, the integrals of motion, and so on, but is highly nonlocal.

17.4 Local equations of motion for dPVL

The localizing change of variables for the map dPLV is given by the formulas

$$u_k = u_k \frac{1 + h w_{k-1}}{1 - h u_{k-1} w_{k-1}}, \quad w_k = w_k \frac{1 + h u_k}{1 - h u_k w_k} \quad (17.21)$$

Indeed, the following statement holds.

Theorem 17.3 The change of variables (17.21) conjugates the map dPVL with the following one:

$$\tilde{u}_k \frac{1 + h \tilde{w}_{k-1}}{1 - h \tilde{u}_{k-1} \tilde{w}_{k-1}} = u_k \frac{1 + h w_k}{1 - h u_k w_k} \quad (17.22)$$

$$\tilde{w}_k \frac{1 + h \tilde{u}_k}{1 - h \tilde{u}_k \tilde{w}_k} = w_k \frac{1 + h u_{k+1}}{1 - h u_{k+1} w_{k+1}} \quad (17.23)$$

Proof. The statement will follow immediately, if we establish the local expressions for the quantities $a_k$:

$$a_k = \frac{(1 + h u_k)(1 + h w_{k-1})}{(1 - h u_{k-1} w_{k-1})} \quad (17.24)$$

Indeed, from (17.21), (17.22) and the formulas (17.14) we derive immediately:

$$b_k = \frac{(1 + h u_k)(1 + h w_{k-1})}{(1 - h u_k w_k)} \quad (17.25)$$

$$c_k = \frac{(1 + h u_k)(1 + h w_k)}{(1 - h u_k w_k)} \quad (17.26)$$
and then (17.12) imply (17.22). To prove (17.23), we take this formula as a definition of the quantities $a_k$ and by means of a simple algebra verify that then the recurrent relations (17.13) hold. The reference to the uniqueness of solution to these recurrent relations finishes the proof.

**Theorem 17.4** The pull-back of the bracket

$$\{\cdot, \cdot\}_2 + h\{\cdot, \cdot\}_3$$

on $\mathcal{PV}(u, w)$ under the change of variables (17.21) is the following bracket on $\mathcal{PV}(u, w)$:

$$\{u_k, w_k\} = -u_kw_k(1 + hu_k)(1 + hw_k), \quad \{w_k, u_{k+1}\} = -w_ku_{k+1}(1 + hw_k)(1 + hu_{k+1})$$

(17.27)

The map (17.22) is Poisson with respect to the bracket (17.27).

**Proof** – by a straightforward verification.

It is very interesting that the bracket (17.27) again turns out to be identical with the invariant local bracket (14.32) of the local version of dVL (so that the contributions of different cubic brackets for VL and PVL are somehow compensated by different localizing changes of variables).

**Theorem 17.5** The pull-back of the flow PVL under the change of variables (17.21) is described by the following differential equations:

$$\dot{u}_k = u_k(1 + hu_k) \left( \frac{w_k + u_kw_k}{1 - hu_kw_k} - \frac{w_{k-1} + u_{k-1}w_{k-1}}{1 - hu_{k-1}w_{k-1}} \right)$$

(17.28)

$$\dot{w}_k = w_k(1 + hw_k) \left( \frac{u_{k+1} + u_{k+1}w_{k+1}}{1 - hu_{k+1}w_{k+1}} - \frac{u_k + u_kw_k}{1 - hu_kw_k} \right)$$

**Proof.** To obtain these differential equations, one has to calculate the Hamiltonian equations of motion generated by the Hamiltonian function

$$h^{-1} \sum_{k=1}^{N} \left( \log(w_k) + \log(1 + hu_k) - \log(1 - hu_kw_k) \right)$$

with respect to the Poisson brackets (17.27).

18 Some constrained lattice KP systems

In this section we introduce a large family of systems generalizing simultaneously the Volterra lattice, the relativistic Toda lattice, the Belov–Chaltikian lattice, and the perturbed Volterra lattice. For some reasons it is convenient to call these systems constrained lattice KP systems.
18.1 Equations of motion and Hamiltonian structure

Each system of this family may be treated as consisting of \( m \) sorts of particles. The phase space of such systems in the periodic case is described as:

\[
\mathcal{K}_m = \mathbb{R}^{mN}(v^{(1)}, \ldots, v^{(m)})
\]

where each vector

\[
v^{(j)} = (v_1^{(j)}, \ldots, v_N^{(j)}) \in \mathbb{R}^N
\]

represents the set of particles of the \( j \)th sort. We introduce the notion of the signature of the constrained lattice KP as an ordered \( m \)-tuple of numbers:

\[
\epsilon = (\epsilon_1, \ldots, \epsilon_m), \quad \epsilon_j \in \{0, 1\}, \quad \epsilon_1 = 0
\]

The constrained KP lattice (hereafter cKPL(\( m \))) with the signature \( \epsilon \) is the following system of differential equations:

\[
\dot{v}_k^{(j)} = v_k^{(j)} \left( \sum_{i=1}^{j-1} (v_i^{(j)} - v_i^{(j)}) + \epsilon_j (v_{k+1}^{(j)} - v_{k-1}^{(j)}) + \sum_{i=j+1}^{m} (v_i^{(j)} - v_i^{(j)}) \right)
\]

This statement can be easily checked and generalizes the quadratic brackets (14.4) and (15.5) for the VL and the RTL, respectively. It would be important to find out, when do the analogs of the linear bracket (for the RTL) and of the cubic bracket (for both the VL and the RTL) hold, and to find the corresponding expressions.
18.2 Lax representation

The natural Lax representation of the cKPL\((m)\) \([18.4]\) lives in \(g^\otimes m\) and is given in terms of the following matrices from \(g\):

\[
U_1(v^{(1)}, \lambda) = \lambda \sum_{k=1}^{N} E_{k+1,k} + \sum_{k=1}^{N} v^{(1)}_k E_{k,k}
\]

\[
V_j(v^{(j)}, \lambda) = I + \sigma_j \lambda^{-1} \sum_{k=1}^{N} v^{(j)}_k E_{k,k+1}, \quad j = 1, 2, \ldots, m
\]

where

\[
\sigma_j = \begin{cases} 
1 & \epsilon_j = 0 \\
-1 & \epsilon_j = 1 
\end{cases} = 1 - 2\epsilon_j
\]

**Theorem 18.2** The equations of motion \([18.4]\) are equivalent to the following matrix differential equations:

\[
\dot{U}_1 = U_1 B_m - B_1 U_1
\]

\[
\dot{V}_j = \begin{cases} 
V_j B_{j-1} - B_j V_j & \epsilon_j = 0 \\
V_j B_j - B_{j-1} V_j & \epsilon_j = 1 
\end{cases} \quad 2 \leq j \leq m
\]

where

\[
B_j(v, \lambda) = \sum_{k=1}^{N} \left( \sum_{i=1}^{j} v^{(i)}_k + \sum_{i=j+1}^{m} v^{(i)}_{k-1} \right) E_{k,k} + \lambda \sum_{k=1}^{N} E_{k,k+1}
\]

The evolution of the monodromy matrices

\[
T_j(v^{(1)}, \ldots, v^{(m)}, \lambda) = V_1^{\sigma_1}(\lambda) \cdot \ldots \cdot V_2^{\sigma_2}(\lambda) \cdot U_1(\lambda) \cdot V_m^{\sigma_m}(\lambda) \cdot \ldots \cdot V_{j+1}^{\sigma_{j+1}}(\lambda)
\]

is governed by usual Lax equations:

\[
\dot{T}_j = [T_j, B_j]
\]

The matrices \(B_j\) allow the representation

\[
B_j = \pi_+(T_j)
\]
Proof – an easy check. ■

It is easy to see that this Lax representation is exactly of the form (3.2), if one considers the $m$–tuple of matrices

$$(U_1, V_{2}^{\sigma_2}, \ldots, V_{m}^{\sigma_m}) \in \mathfrak{g} \otimes m$$

as the Lax matrix. This Lax representation allows an $r$–matrix interpretation for the quadratic bracket of the previous theorem. As a matter of fact, the corresponding quadratic bracket in $\mathfrak{g} \otimes m$ literally coincides with the bracket introduced for the BL1 in [S12]. The Lax representation (18.11), (18.12) serves as a starting point for applying the recipe of Sect. 5.

18.3 Discretization

Taking in this recipe $F(T) = I + hT$, we come to the discrete time Lax equations

$$\tilde{U}_1 = B_1^{-1} U_1 B_m$$

$$\tilde{V}_j = \begin{cases} B_j^{-1} V_j B_{j-1} & \epsilon_j = 0 \\ B_{j-1}^{-1} V_j B_{j} & \epsilon_j = 1 \end{cases} \quad 2 \leq j \leq m \quad (18.17)$$

with $B_j = \Pi_\epsilon (I + hT_j)$.

**Theorem 18.3** The discrete time Lax equations (18.17) are equivalent to the map $v \mapsto \tilde{v}$ described by the equations

$$\tilde{v}_k^{(1)} = v_k^{(1)} \frac{b_k^{(m)}}{b_k^{(1)}}, \quad \tilde{v}_k^{(j)} = \begin{cases} v_k^{(j)} \frac{b_k^{(j-1)}}{b_k^{(1)}} & \epsilon_j = 0 \\ v_k^{(j)} \frac{b_k^{(j+1)}}{b_k^{(j-1)}} & \epsilon_j = 1 \end{cases} \quad 2 \leq j \leq m \quad (18.18)$$

where the functions $b_k^{(j)} = b_k^{(j)}(v) = 1 + O(h)$ satisfy the following equations:

$$\frac{b_k^{(m)}}{b_k^{(1)}} = \frac{b_k^{(1)} - hv_k^{(1)}}{b_k^{(1)} - hv_k^{(1)}} \quad (18.19)$$

$$\frac{b_{k+1}^{(j-1)}}{b_k^{(j)}} = \frac{b_{k+1}^{(j)} - hv_k^{(j)}}{b_k^{(j)} - hv_k^{(j)}}, \quad \epsilon_j = 0, \quad 2 \leq j \leq m \quad (18.20)$$

$$\frac{b_{k+1}^{(j)}}{b_k^{(j-1)}} = \frac{b_k^{(j-1)} + hv_k^{(j)} + 1}{b_k^{(j-1)} + hv_k^{(j)} + 1}, \quad \epsilon_j = 1, \quad 2 \leq j \leq m \quad (18.21)$$
Proof. First of all notice that the matrices $B_j$ must have the following structure:

$$B_j(\lambda) = \sum_{k=1}^{N} b^{(j)}_k E_{k,k} + h\lambda \sum_{k=1}^{N} E_{k+1,k}$$

Now the equations of motion are derived straightforwardly. For example, the $\epsilon_j = 1$ variant of the last $m-1$ equations in (18.17), i.e. the matrix equation $B_{j-1} \dot{V}_j = V_j B_j$, is equivalent to the following system of scalar equations:

$$\begin{cases} v_k^{(j-1)} \tilde{v}_k^{(j)} = v_k^{(j)} b_k^{(j)} \\ v_k^{(j-1)} - h\tilde{v}_k^{(j)} = v_k^{(j)} - hv_k^{(j)} \end{cases}$$

This is equivalent to the corresponding variant of equations of motion (18.18) together with the relation (18.21).

Remark. It is important to notice that the statement of the last theorem deviates from the usual scheme in that it does not contain a system of equations which determine $b^{(j)}_k$ uniquely. In fact, in order to find such a system one has to deduce from the equations (18.19)–(18.21) $m$ formulas of the type

$$b^{(j)}_{k+q} = \frac{\Psi^{(j)}_k}{\Psi^{(j)}_{k+1}}$$

Here the number $q$ does not depend on $j$; $\Psi^{(j)}_k$ are certain expressions of the type $\prod_{i=1}^{m} \psi^{(i)}_{k+n_i}$, and all $\psi^{(i)}_k = b^{(i)}_k - hv^{(i)}_k$ or $\psi^{(i)}_k = \left(b^{(i-1)}_k + hv^{(i)}_k\right)^{-1}$. This, in turn, implies that there hold certain equations of the type

$$b_k^{(j)} \cdot \ldots \cdot b_{k+q-1}^{(j)} \Psi_k^{(j)} = \text{const} \quad (18.22)$$

(here we assumed for definiteness that $q > 0$). The value of the constant on the right-hand side is uniquely defined by the conditions $B_j = \Pi_+ (I + hT_j)$. (As a matter of fact, it is easy to see that this constant does not depend on $j$). The value of the constant being determined, the equations (18.22) give the desired system which defines $b_k^{(j)}$ uniquely. However, the outfit of this system depends heavily on the signature $\epsilon$ of the cKPL, and the general formulas would contain too many indices to be instructive enough. It is simpler to derive such formulas for each concrete signature separately. Nevertheless, the formulas (18.19)–(18.21) completely characterize the coefficients of the matrices which serve as the factors $\Pi_+ (\alpha I + hT_j)$ with some $\alpha$, so that every solution of this system leads to a discretization based on the factorization of $I + h'T$ with $h' = h/\alpha$, which enjoys all the positive properties of our general construction. We call the maps introduced in the previous theorem dcKPL($m$).
18.4 Local equations of motion for dcKPL

The dcKPL\((m)\) can be brought into the local form for an arbitrary signature \(\epsilon\).

**Theorem 18.4** The change of variables \(K_m(v) \mapsto \tilde{K}_m(v)\),

\[
v_k^{(j)} = v_k^{(j)} \prod_{i=1}^{j-1} (1 + h v_k^{(i)}) \cdot (1 + \epsilon_j h v_k^{(j)}) \cdot \prod_{i=j+1}^{m} (1 + h v_k^{(i)}) \tag{18.23}
\]

conjugates dcKPL\((m)\) with the following map:

\[
\tilde{v}_k^{(j)} \prod_{i=1}^{j-1} (1 + h \tilde{v}_k^{(i)}) \cdot (1 + \epsilon_j h \tilde{v}_k^{(j)}) \cdot \prod_{i=j+1}^{m} (1 + h \tilde{v}_k^{(i)})
\]

\[
= v_k^{(j)} \prod_{i=1}^{j-1} (1 + h v_{k+1}^{(i)}) \cdot (1 + \epsilon_j h v_{k+1}^{(j)}) \cdot \prod_{i=j+1}^{m} (1 + h v_k^{(i)}) \tag{18.24}
\]

**Proof.** It is easy to calculate that if (18.23) holds, and if the quantities \(b_k^{(j)}\) are defined by the formula

\[
b_k^{(j)} = \prod_{i=1}^{j} (1 + h v_k^{(i)}) \prod_{i=j+1}^{m} (1 + h v_k^{(i)}) \tag{18.25}
\]

then

\[
b_k^{(j)} - h v_k^{(j)} = \prod_{i=1}^{j-1} (1 + h v_k^{(i)}) \prod_{i=j+1}^{m} (1 + h v_k^{(i)}), \quad \epsilon_j = 0
\]

\[
b_k^{(j-1)} + h v_k^{(j)} = \prod_{i=1}^{j} (1 + h v_k^{(i)}) \prod_{i=j+1}^{m} (1 + h v_k^{(i)}), \quad \epsilon_j = 1
\]

and it is easy to check now that the equations (18.19)–(18.21) are satisfied. Indeed, for \(\epsilon_j = 0\) we find:

\[
\frac{b_k^{(j)}}{v_k^{(j)} - h v_k^{(j)}} = \frac{b_k^{(j-1)}}{b_k^{(j)} - h v_k^{(j)}} = 1 + h v_k^{(j)}
\]

which proves (18.20), while for \(\epsilon_j = 1\) we find

\[
\frac{b_k^{(j-1)} + h v_k^{(j)}}{b_k^{(j)}} = \frac{b_k^{(j-1)} + h v_{k+1}^{(j)}}{b_k^{(j-1)} + h v_{k+1}^{(j)}} = 1 + h v_k^{(j)}
\]

which proves (18.21). The verification of (18.19) is completely analogous. The pull–back of the equations of motion is calculated now straightforwardly. ■
Unfortunately, we do not now a general formula for the second invariant Poisson structure for cKPL’s. This prevents us from applying our general scheme for finding the local invariant Poisson brackets for the localized maps. However, by a direct analysis of equations of motion the following statement can be proved.

**Theorem 18.5** The pull–back of the system (18.4) under the change of variables (18.23) is given by the formula

\[ \dot{v}_k^{(j)} = v_k^{(j)}(1 + h v_k^{(j)}) \left( \prod_{i=1}^{j-1} (1 + h v_{k+1}^{(i)}) \cdot (1 + \epsilon_j h v_{k+1}^{(j)}) \cdot \prod_{i=j+1}^{m} (1 + h v_k^{(i)}) - \prod_{i=1}^{j-1} (1 + h v_k^{(i)}) \cdot (1 + \epsilon_j h v_{k-1}^{(j)}) \cdot \prod_{i=j+1}^{m} (1 + h v_{k-1}^{(i)}) \right) / h \]  

(18.26)

**18.5 Example 1**: \( \epsilon = (0, 0, 0) \)

As illustrations we consider the systems with \( m = 3 \) - the simplest possible ones after VL and RTL+. We denote for simplicity

\[ v_k^{(1)} = u_k, \quad v_k^{(2)} = v_k, \quad v_k^{(3)} = w_k \]

The Hamilton function is in all cases equal to

\[ H_1(u, v, w) = \sum_{k=1}^{N} (u_k + v_k + w_k) \]

The nonvanishing Poisson brackets consist of the signature independent part,

\[
\begin{align*}
\{ u_k, v_k \}_2 &= -u_k v_k, & \{ v_k, u_{k+1} \}_2 &= -u_{k+1} v_k \\
\{ u_k, w_k \}_2 &= -u_k w_k, & \{ w_k, u_{k+1} \}_2 &= -u_{k+1} w_k \\
\{ v_k, w_k \}_2 &= -v_k w_k, & \{ w_k, v_{k+1} \}_2 &= -v_{k+1} w_k
\end{align*}
\]

(18.27)

supplemented by

\[ \{ v_k^{(j)}, v_{k+1}^{(j)} \}_2 = -v_k^{(j)} v_{k+1}^{(j)} \]

for those \( j \) where \( \epsilon_j = 1 \).

In particular, if all \( \epsilon_j = 0 \), then all nonvanishing Poisson brackets of the coordinate functions are exhausted by (18.27), and we arrive at the system

\[
\begin{align*}
\dot{u}_k &= u_k (v_k + w_k - v_{k-1} - w_{k-1}) \\
\dot{v}_k &= v_k (u_{k+1} + w_k - u_k - w_{k-1}) \\
\dot{w}_k &= w_k (u_{k+1} + v_{k+1} - u_k - v_k)
\end{align*}
\]

(18.28)
which becomes the usual Bogoyavlensky lattice BL1(2) after the re-naming

\[ u_k \mapsto a_{3k-2}, \quad v_k \mapsto a_{3k-1}, \quad w_k \mapsto a_{3k} \]

The localizing change of variables for its discretization:

\[ u_k = u_k(1 + hv_{k-1})(1 + hw_{k-1}) \]
\[ v_k = v_k(1 + hu_k)(1 + hw_{k-1}) \]  \hspace{1cm} (18.29)
\[ w_k = w_k(1 + hu_k)(1 + hv_k) \]

The local discretization of the system (18.28):

\[ \tilde{u}_{k} = u_k(1 + h\tilde{v}_{k-1})(1 + h\tilde{w}_{k-1}) = u_k(1 + hv_k)(1 + hw_k) \]
\[ \tilde{v}_{k} = v_k(1 + h\tilde{u}_{k})(1 + h\tilde{w}_{k-1}) = v_k(1 + hu_{k+1})(1 + hw_k) \]  \hspace{1cm} (18.30)
\[ \tilde{w}_{k} = w_k(1 + h\tilde{u}_{k})(1 + h\tilde{v}_{k}) = w_k(1 + hu_{k+1})(1 + hv_{k+1}) \]

18.6 Example 2: \( \epsilon = (0, 1, 0) \)

In this example the signature dependent part of the Poisson brackets reads

\[ \{v_k, v_{k+1}\}_2 = -v_kv_{k+1} \]

and the equations of motion take the following form:

\[ \dot{u}_k = u_k(v_k + w_k - v_{k-1} - w_{k-1}) \]
\[ \dot{v}_k = v_k(u_{k+1} + v_{k+1} + w_k - u_k - v_{k-1} - w_{k-1}) \]  \hspace{1cm} (18.31)
\[ \dot{w}_k = w_k(u_{k+1} + v_{k+1} - u_k - v_k) \]

The localizing change of variables for the discretization of this system:

\[ u_k = u_k(1 + hv_{k-1})(1 + hw_{k-1}) \]
\[ v_k = v_k(1 + hu_k)(1 + hv_{k-1})(1 + hw_{k-1}) \]  \hspace{1cm} (18.32)
\[ w_k = w_k(1 + hu_k)(1 + hv_k) \]

The local form of equations of motion for the discretization of (18.31):

\[ \tilde{u}_{k}(1 + h\tilde{v}_{k-1})(1 + h\tilde{w}_{k-1}) = u_k(1 + hv_k)(1 + hw_k) \]
\[ \tilde{v}_{k}(1 + h\tilde{u}_{k})(1 + h\tilde{v}_{k-1})(1 + h\tilde{w}_{k-1}) = v_k(1 + hu_{k+1})(1 + hv_{k+1})(1 + hw_k) \]
\[ \tilde{w}_{k}(1 + h\tilde{u}_{k})(1 + h\tilde{v}_{k}) = w_k(1 + hu_{k+1})(1 + hv_{k+1}) \]  \hspace{1cm} (18.34)
Let us mention that the system (18.31) allows an interesting reduction

\[ v_k = u_k w_k \quad (18.35) \]

which is, moreover, compatible with the quadratic Poisson brackets. In this reduction we arrive at the system PVL. It is easy to check that in the variables \( u_k, v_k, w_k \) the reduction (18.35) takes the form

\[ v_k = \frac{u_k w_k}{1 - h u_k w_k}, \quad 1 + h v_k = \frac{1}{1 - h u_k w_k} \]

This makes a link with the results of Sect. 17.

18.7 Example 3: \( \epsilon = (0, 1, 1) \)

In this case the Poisson brackets (18.27) have to be supplemented by

\[ \{v_k, v_{k+1}\}_2 = -v_k v_{k+1}, \quad \{w_k, w_{k+1}\}_2 = -w_k w_{k+1} \]

and the equations of motion take the form:

\[ \begin{align*}
\dot{u}_k &= u_k(v_k + w_k - v_{k-1} - w_{k-1}) \\
\dot{v}_k &= v_k(u_{k+1} + v_{k+1} + w_k - u_k - v_{k-1} - w_{k-1}) \\
\dot{w}_k &= w_k(u_{k+1} + v_{k+1} + w_{k+1} - u_k - v_k - w_{k-1})
\end{align*} \quad (18.36) \]

The localizing change of variables for the discretization of this system:

\[ \begin{align*}
u_k &= u_k(1 + h v_{k-1})(1 + h w_{k-1}) \\
v_k &= v_k(1 + h u_k)(1 + h v_{k-1})(1 + h w_{k-1}) \\
w_k &= w_k(1 + h u_k)(1 + h v_k)(1 + h w_{k-1})
\end{align*} \quad (18.37) \]

The local form of equations of motion for the discretization of (18.36):

\[ \begin{align*}
u_k(1 + h v_{k-1})(1 + h w_{k-1}) &= u_k(1 + h v_k)(1 + h w_k) \\
v_k(1 + h u_k)(1 + h v_{k-1})(1 + h w_{k-1}) &= v_k(1 + h u_{k+1})(1 + h v_{k+1})(1 + h w_k) \\
w_k(1 + h u_k)(1 + h v_k)(1 + h w_{k-1}) &= w_k(1 + h u_{k+1})(1 + h v_{k+1})(1 + h w_{k+1})
\end{align*} \quad (18.38) \]

Let us discuss the following reduction of the system (18.36):

\[ u_k = 0 \quad (18.39) \]
It is compatible with the quadratic Poisson brackets, so that we arrive at the following reduced system:

\[
\dot{v}_k = v_k (v_{k+1} + w_k - v_{k-1} - w_{k-1}) \\
\dot{w}_k = w_k (v_{k+1} + w_{k+1} - v_k - w_k)
\]

Interestingly enough, this system is again nothing but the usual Bogoyavlensky lattice BL1(2), which becomes obvious after the renaming

\[ v_k \mapsto a_{2k-1}, \quad w_k \mapsto a_{2k} \]

So, we have found the third Lax representation for BL1(2).

It is easy to see that the maps \( M_\pm : K_3(0, v, w) \mapsto BC(b, c) \) defined as

\[
M_+: \quad b_k = v_k + w_k, \quad c_k = v_k w_{k+1}
\]

and

\[
M_-: \quad b_k = v_{k+1} + w_k, \quad c_k = v_{k+2} w_k
\]

conjugate the flow (18.40) with the Belov–Chaltikian flow BCL, and are Poisson, if the both spaces are equipped with the brackets \( \{\cdot, \cdot\}_2 \). So, the system BCL is Miura related to BL1(2) (this fact is similar to the Miura relation between the Toda and the Volterra hierarchy).

The localizing change of variables for the discretization of the reduced system (18.40) is given by

\[
v_k = v_k (1 + h v_{k-1})(1 + h w_{k-1}), \quad w_k = w_k (1 + h v_k)(1 + h w_{k-1})
\]

and the corresponding local equations of motion read:

\[
\tilde{v}_k (1 + h \tilde{v}_{k-1})(1 + h \tilde{w}_{k-1}) = v_k (1 + h v_{k+1})(1 + h w_k) \\
\tilde{w}_k (1 + h \tilde{v}_k)(1 + h \tilde{w}_{k-1}) = w_k (1 + h v_{k+1})(1 + h w_{k+1})
\]

So, the discretizations of BL1(2) based on different Lax representations, agree with one another.

It turns out that the Miura maps \( M_\pm \) are still given by nice local formulas, when translated to the localizing variables. Namely, the following diagram is commutative:
if the maps $M_\pm$ are defined by the formulas
\begin{align*}
M_+ : \quad 1 + h b_k &= (1 + h v_k)(1 + h w_k), \quad c_k = v_k w_{k+1}(1 + h w_k)(1 + h v_{k+1}) \quad (18.45) \\
M_- : \quad 1 + h b_k &= (1 + h v_{k+1})(1 + h w_k), \quad c_k = v_{k+2} w_k(1 + h v_{k+1})(1 + h w_{k+1}) \quad (18.46)
\end{align*}
This statement may be verified by a simple calculation.

19 Bruschi–Ragnisco lattice

The Bruschi–Ragnisco lattice (hereafter BRL) was introduced in [BR2]:
\begin{align*}
\dot{b}_k &= b_{k+1} c_k - b_k c_{k-1}, \quad \dot{c}_k = c_k (c_k - c_{k-1}) \quad (19.1)
\end{align*}
It may be considered either under open–end boundary conditions ($b_{N+1} = c_0 = c_N = 0$), or
under periodic ones (all the subscripts are taken (mod $N$), so that $c_0 \equiv c_N$, $b_{N+1} \equiv b_1$). The
phase space of the Bruschi–Ragnisco lattice:
\begin{equation*}
\mathcal{BR} = \mathbb{R}^{2N}(b_1, c_1, \ldots, b_N, c_N)
\end{equation*}
Two compatible brackets may be defined on $\mathcal{BR}$ such that the system (19.1) is Hamiltonian
with respect to each one of them. The linear Poisson bracket is given by
\begin{equation*}
\{b_k, c_k\}_0 = -\{b_{k+1}, c_k\}_0 = -c_k \quad (19.2)
\end{equation*}
while the quadratic one - by
\[
\{b_k, b_{k+1}\}_1 = -b_{k+1}c_k, \quad \{b_k, c_k\}_1 = c_k^2, \quad \{b_k, c_{k+1}\}_1 = -c_k c_{k+1}
\] (19.3)
The corresponding Hamilton functions are:
\[
H_1(b, c) = \sum_{k=1}^N b_{k+1}c_k \quad \text{and} \quad H_0(b, c) = \sum_{k=1}^N b_k
\] (19.4)
for the brackets \{·, ·\}_0 and \{·, ·\}_1, respectively.

It has been pointed out in [BR2] that this system allows a very complete study by different methods of the soliton theory. As was demonstrated in [S2], this is due to its extreme simplicity. Namely, in a certain gauge the Lax representation of this system is a linear matrix equation. Namely, if the entries of the Lax matrix \(T = T(b, c) \in g = gl(N)\) are defined as
\[
T_{kj} = \begin{cases} 
  b_j \prod_{i=k+1}^{j-1} c_i & k \leq j \\
  b_j \left(\prod_{i=j+1}^{k-1} c_i\right)^{-1} & k > j
\end{cases}
\] (19.5)
then the system (19.1) is equivalent to the equation
\[
\dot{T} = [T, M]
\] (19.6)
with the constant matrix \(M\)
\[
M = \sum_{k=1}^{N-1} E_{k,k+1} \quad \text{or} \quad \sum_{k=1}^{N-1} E_{k,k+1} + CE_{N,1}
\] (19.7)
for the open–end and periodic case, respectively (in the latter case it is supposed that the dynamics of the BRL is restricted to the set \(c_1\ldots c_N = C\)). The whole hierarchy of the BRL consists of equations
\[
\dot{T} = [T, M^m]
\] which are linear and may be immediately integrated:
\[
T(t) = \exp(-tM^m) \cdot T(0) \cdot \exp(tM^m)
\] (19.8)
The brackets (19.2), (19.3) were shown in [S2] to give a coordinate representation of certain Lie–Poisson brackets on \(g\), restricted to the subset of the Lax matrices \(T(b, c)\).

Obviously, the recipe of Sect. 3 cannot be literally applied to the BRL. However, the philosophy behind this recipe is, of course, applicable, and requires to seek for the discrete
time Bruschi–Ragnisco lattice in the same hierarchy. It should share the Lax matrix with the continuous time system, and its explicit solution should be given by

\[ T(nh) = (I + hM)^{-n}T(0)(I + hM)^n \]  

(cf. (19.8)). Hence the corresponding discrete Lax equation should have the form

\[ \tilde{T} = (I + hM)^{-1}T(I + hM) \]  

(19.10)

**Theorem 19.1** The discrete time Lax equation (19.10) is equivalent to the following map on the space \( \mathcal{B}R \):

\[ \tilde{b}_k(1 + h\tilde{c}_{k-1}) = b_k + hb_{k+1}c_k, \quad \tilde{c}_k = c_k \frac{1 + h\tilde{c}_k}{1 + h\tilde{c}_{k-1}} \]  

(19.11)

**Proof** – an easy calculation. 

By construction, this map is Poisson with respect to the both brackets (19.2), (19.3). We see that the extreme simplicity of the BRL allows to find its local discretization in the original variables. The localizing change of variables is not necessary for this system.

### 20 Conclusion

This paper contains a rich collection of examples illustrating the procedure of constructing local integrable discretizations for integrable lattice systems. The construction is based on the notion of the \( r \)-matrix hierarchy and consists of three steps of a rather different nature.

The first step is to find a Lax representation for a given lattice system, living in an associative algebra \( g \). This Lax representation has to be a member of a hierarchy governed by an \( R \)-operator on \( g \) satisfying the modified Yang–Baxter equation. In all examples treated here this operator is simply a difference of projections to two complementary subalgebras.

The second step is an application of a general recipe for integrable discretization. This step is almost algorithmic, the only non–formalized (and, probably, non–formalizable) point being the choice of the function \( F(T) \) approximating \( \exp(hT) \) for \( T \in g \) (cf. Sect. 5). In all examples treated here the simplest possible choice \( F(T) = I + hT \) works perfectly. The difference equations obtained on this step share the invariant Poisson structures, the integrals of motion, the Lax matrices, etc. with the underlying continuous time systems. However, as a rule, they are non–local. This feature is unpleasant from both the esthetical and the practical point of view, because it makes the equations ugly and not well suited for practical realization.
The third step is finding the localizing change of variables. This step is again absolutely non-algorithmic (at least, at our present level of knowledge). These changes of variables have remarkable properties: they often produce one-parameter local deformations of Poisson brackets algebras, and always produce one-parameter integrable deformations of the lattice systems themselves. At the moment we cannot provide a rational explanation neither for these properties nor for the mere existence of localizing changes of variables. However, our collection seems to be representative enough to convince that these phenomena are very general. We feel that they are connected with the Poisson geometry of certain $r$-matrix brackets on associative algebras and of monodromy maps, but we prefer to stop at this point.

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