DISCRETE SPECTRUM ASYMPTOTICS FOR THE THREE-PARTICLE HAMILTONIANS ON LATTICES

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ABSTRACT. We consider the Hamiltonian of a system of three quantum mechanical particles on the three-dimensional lattice $\mathbb{Z}^3$ interacting via short-range pair potentials.

We prove for the two-particle energy operator $h(k)$, $k \in \mathbb{T}^3$ the two-particle quasi-momentum, the existence of a unique positive eigenvalue $\lambda(k)$ lying below the essential spectrum under assumption that the operator $h(0)$ corresponding to the zero value of $k$ has a zero energy resonance.

We describe the location of the essential spectrum of the three-particle discrete Schrödinger operators $H(K)$, $K$ the three-particle quasi-momentum by the spectra of $h(k)$, $k \in \mathbb{T}^3$.

We prove the existence of infinitely many eigenvalues of $H(0)$ and establish for the number of eigenvalues $N(0, z)$ lying below $z < 0$ the asymptotics

$$
\lim_{z \to -0} \frac{N(0, z)}{|\log |z||} = \frac{\lambda_0}{2\pi},
$$

where $\lambda_0$ a unique positive solution of the equation

$$
\lambda = \frac{8 \sinh \pi \lambda/6}{\sqrt{3} \cosh \pi \lambda/2}.
$$

We prove that for all $K \in U_0^\delta(0)$, where $U_0^\delta$ a $\delta > 0$ neighborhood of the origin, the number $N(K, 0)$ of eigenvalues the operator $H(K)$ below zero is finite and satisfy the asymptotics

$$
\lim_{|K| \to 0} \frac{N(K, 0)}{|\log |K||} = \frac{\lambda_0}{\pi}.
$$

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Key words: Schrödinger operators, quantum mechanical three-particle systems, short-range potentials, quasi-particles, eigenvalues, Efimov effect, essential spectrum, asymptotics, quasi-momentum lattices, zero energy resonances, excess mass phenomenon, solid state physics, zero energy resonances, Birman-Schwinger principle.

1. INTRODUCTION

We consider a system of three identical particles (bosons) on the three-dimensional lattice $\mathbb{Z}^3$ interacting by means of short-range pair attractive potentials.

The main goal of the present paper is to prove the finiteness or infiniteness (Efimov’s effect) of the number of eigenvalues lying below zero (the bottom of the essential spectrum of $H(0)$) of the three-particle discrete Schrödinger operator $H(K)$ depending on the total quasi-momentum $K \in U_0^\delta(0)$, where $U_0^\delta$ a $\delta > 0$ neighborhood of the origin.

Efimov’s effect is one of the remarkable results in the spectral analysis for three-particle Schrödinger operators associated to a system of three particles moving on Euclid space $\mathbb{R}^3$ or integer lattice $\mathbb{Z}^3$: if none of the three two-particle Schrödinger operators (corresponding to the two-particle subsystems) has negative eigenvalues, but at least two of them have a

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zero energy resonance, then this three-particle Schrödinger operator has an infinite number of discrete eigenvalues, accumulating at zero.

Since its discovery by Efimov in 1970 [11] much research have been devoted to this subject. See, for example [11, 6, 8, 13, 23, 26, 27, 28, 29].

The main result obtained by Sobolev [26] (see also [28]) is an asymptotics of the form 

$$U_0 |\log |\lambda||$$

for the number of eigenvalues below \(\lambda, \lambda < 0\), where the coefficient \(U_0\) does not depend on the two-particle potentials \(v_\alpha\) and is a positive function of the ratios \(m_1/m_2, m_2/m_3\) of the masses of the three-particles.

Recently the existence of Efimov’s effect for \(N\)-body quantum systems with \(N \geq 4\) has been proved by X.P. Wang in [30].

In fact in [30] for the total (reduced) Hamiltonian a lower bound on the number of eigenvalues of the form 

$$C_0 |\log (E_0 - \lambda)|$$

is given, when \(\lambda\) tends to \(E_0\), where \(C_0\) is a positive constant and \(E_0\) is the bottom of the essential spectrum.

The kinematics of quantum particles on lattices, even in the two and three-particle sector, is rather exotic. For instance, due to the fact that the discrete analogue of the Laplacian or its generalizations are not translationally invariant, the Hamiltonian of a system does not separate into two parts, one relating to the center-of-mass motion and the other one relating to the internal degrees of freedom.

As a consequence any local substitute of the effective mass-tensor (of a ground state) depends on the quasi-momentum of the system and, in addition, it is only semi-additive (with respect to the partial order on the set of positive definite matrices). This is the so-called excess mass phenomenon for lattice systems (see, e.g., [20] and [22]): the effective mass of the bound state of an \(N\)-particle system is greater than and in general, not equal to the sum of the effective masses of the constituent quasi-particles.

The three-body problem on lattices can be reduced to the effective three-particle Schrödinger operators by using the Gelfand transform. The underlying Hilbert space \(\ell^2((\mathbb{Z}^3)^3)\) is decomposed as a direct von Neumann integral associated with the representation of the discrete group \(\mathbb{Z}^3\) by shift operators on the lattice and the total three-body Hamiltonian appears to be decomposable. In contrast to the continuous case, the corresponding fiber Hamiltonians \(H(K)\) associated with the direct decomposition depend parametrically the quasi-momentum, \(K \in \mathbb{T}^3 = (-\pi, \pi]^3\), which ranges over a cell of the dual lattice. Due to the loss of the spherical symmetry of the problem, the spectra of the family \(H(K)\) turn out to be rather sensitive to the quasi-momentum \(K \in \mathbb{T}^3\).

In particular, Efimov’s effect exists only for the zero value of the three-particle quasi-momentum \(K\), which is proven for Hamiltonians of a system of three particles interacting via pair zero-range attractive potentials on \(\mathbb{Z}^3\) (see, e.g., [2, 4, 16, 18, 19, 20] for relevant discussions and [3, 9, 10, 15, 20, 21, 22, 24] for the general study of the low-lying excitation spectrum for quantum systems on lattices).

Denote by \(\tau(K)\) the bottom of the essential spectrum of the three-particle discrete Schrödinger operator \(H(K), K \in \mathbb{T}^3\) and by \(N(K, z)\) the number of eigenvalues lying below \(z \leq \tau(K)\).

The main results of the present paper are as follows:

(i) the operator \(H(0)\) has infinitely many eigenvalues below the bottom of the essential spectrum and for the number of eigenvalues \(N(0, z)\) lying below \(z < 0\) the asymptotics

$$\lim_{z \to -0} \frac{N(0, z)}{|\log |z||} = \frac{\lambda_0}{2\pi},$$
holds, where $\lambda_0$ a unique positive solution of the equation

$$
\lambda = \frac{8 \sinh \pi \lambda/6}{\sqrt{3} \cosh \pi \lambda/2}.
$$

This result is similar to the asymptotics founded in the continuous case by Sobolev [26].

(ii) for some punctured $\delta > 0$ neighborhood $U^\delta_0(0)$ of the origin and for all $K \in U^\delta_0(0)$ the number $N(K, 0)$ is a finite and satisfy the following asymptotics

$$
\lim_{|K| \to 0} \frac{N(K, 0)}{\log |K|} = \frac{\lambda_0}{\pi}.
$$

This result is characteristic for the lattice system and does not have any analogue in the continuous case.

We underline that these results are in contrast to similar results for the continuous three particle Schrödinger operators, where the number of eigenvalues does not depend on the three-particle total momentum $K \in \mathbb{R}^3$.

Moreover these results are also in contrast with the results for two-particle operators, in which discrete Schrödinger operators have finitely many eigenvalues for all $k \in U_\delta(0)$, where $U_\delta(0) = \{ K \in T^3 : |K| < \delta \}$ is a $\delta-$neighborhood of the origin.

Note that to prove these results in the present paper we derive an asymptotics of the Birman-Schwinger operator $G(k, 0)$ resp. $G(0, z)$ as $k \to 0$ resp. $z \to 0$. In particular, we prove that the operator valued function $G(k, 0)$ resp. $G(0, z)$ is differentiable in $|k|$ at $k = 0 \in T^3$ resp. in $z$ at $z = 0$.

This result has been proved in the continuum case (see [26, 28]) using resolvent expansion established in [12].

The organization of the present paper is as follows.

Section 1 is an introduction.

In Section 2 we introduce the Hamiltonians of systems of two and three-particles in coordinate and momentum representations as bounded self-adjoint operators in the corresponding Hilbert spaces.

In Section 3 we introduce the total quasi-momentum and decompose the energy operators into von Neumann direct integrals, choosing relative coordinate systems.

In section 4 we introduce the concept of a zero energy resonance (threshold resonance).

In Section 5 we state the main results of the paper.

In Section 6 we study spectral properties of the two-particle discrete Schrödinger operator $h(k), k \in T^3$. We prove the existence of a positive eigenvalue below the bottom of the essential spectrum of $h(k), k \in T^3$ (Theorem 5.1) and obtain an asymptotics for the Birman-Schwinger operators associated to $h(k), k \in T^3$.

In Section 7 we introduce the channel operator and describe its spectrum by the spectrum of the two-particle discrete Schrödinger operators (Theorem 5.2).

In Section 8 we prove the Birman-Schwinger principle for the three identical particle Schrödinger operator on lattice $\mathbb{Z}^3$.

In Section 9 we derive the asymptotics for the number of eigenvalues $N(0, z)$ resp. $N(K, 0)$ of $H(0)$ as $z \to 0$ resp. $H(K)$ as $|K| \to 0$ (Theorem 5.3 resp. Theorem 5.4).

Throughout the present paper we adopt the following notations: We denote by $T^3$ the three-dimensional torus, i.e., the cube $(-\pi, \pi]^3$ with appropriately identified sides. The torus $T^3$ will always be considered as an abelian group with respect to the addition and multiplication by real numbers regarded as operations on the three-dimensional space $\mathbb{R}^3$ modulo $(2\pi \mathbb{Z})^3$. 


For each (sufficiently small) \( \delta > 0 \) the notation \( U_\delta(0) = \{ K \in \mathbb{T}^3 : |K| < \delta \} \) stands for a \( \delta \) neighborhood of the origin and \( U_\delta^c(0) = U_\delta(0) \setminus \{0\} \) for a punctured \( \delta \) neighborhood. The subscript \( \alpha \) (and also \( \beta \) and \( \gamma \)) always runs from 1 to 3 and we use the convention \( \alpha \neq \beta, \beta \neq \gamma, \gamma \neq \alpha \).

2. DESCRIPTION OF THE ENERGY OPERATORS OF TWO AND THREE ARBITRARY PARTICLES ON A LATTICE AND FORMULATIONS OF THE MAIN RESULTS

Let \( \mathbb{Z}^3 \) be the three-dimensional lattice and let \((\mathbb{Z}^3)^m, m \in \mathbb{N}\) be the Cartesian \( m \)-th power of \( \mathbb{Z}^3 \). Denote by \( \ell_2((\mathbb{Z}^3)^m) \) the Hilbert space of square-summable functions \( \varphi \) defined on \((\mathbb{Z}^3)^m \) and let \( \ell_2^s((\mathbb{Z}^3)^m) \subset \ell_2((\mathbb{Z}^3)^m) \) be the subspace of symmetric functions.

The free Hamiltonian \( \hat{h}_0 \) of a system of two identical quantum mechanical particles on the three dimensional lattice \( \mathbb{Z}^3 \) is defined by

\[
(\hat{h}_0\hat{\psi})(x_\beta, x_\gamma) = \frac{1}{2} \sum_{|s|=1} [2\hat{\psi}(x_\beta, x_\gamma) - \hat{\psi}(x_\beta + s, x_\gamma) - \hat{\psi}(x_\beta, x_\gamma + s)], \quad \hat{\psi} \in \ell_2^s((\mathbb{Z}^3)^2).
\]

The Hamiltonian \( \hat{h} \) of a system of two quantum-mechanical identical particles interacting through a short-range pair potential \( \hat{\psi} \) is usually associated with the following bounded self-adjoint operator on the Hilbert space \( \ell_2^s((\mathbb{Z}^3)^2) \) and has form

\[
(\hat{\psi}\hat{\psi})(x_\beta, x_\gamma) = \hat{\psi}(x_\beta - x_\gamma)\hat{\psi}(x_\beta, x_\gamma), \quad \hat{\psi} \in \ell_2^s((\mathbb{Z}^3)^2).
\]

The free Hamiltonian \( \hat{H}_0 \) of a system of three identical quantum mechanical particles on the three-dimensional lattice \( \mathbb{Z}^3 \) is defined by

\[
(\hat{H}_0\hat{\psi})(x_1, x_2, x_3) = \frac{1}{2} \sum_{|s|=1} [3\hat{\psi}(x_1, x_2, x_3) - \hat{\psi}(x_1 + s, x_2, x_3) - \hat{\psi}(x_1, x_2 + s, x_3) - \hat{\psi}(x_1, x_2, x_3 + s)],
\]

\[
\hat{\psi} \in \ell_2^s((\mathbb{Z}^3)^3).
\]

The Hamiltonian \( \hat{H} \) of a system of three quantum-mechanical identical particles with the two-particle interaction \( \hat{\psi} = \hat{\psi}_\alpha = \hat{\psi}_\beta, \alpha, \beta, \gamma = 1, 2, 3 \) is a bounded perturbation of the free Hamiltonian \( \hat{H}_0 \)

\[
\hat{H} = \hat{H}_0 - \hat{V}_1 - \hat{V}_2 - \hat{V}_3,
\]

where \( \hat{V}_\alpha = \hat{V}, \alpha = 1, 2, 3 \) is multiplication operator on \( \ell_2^s((\mathbb{Z}^3)^3) \) defined by

\[
(\hat{V}\hat{\psi})(x_1, x_2, x_3) = \hat{v}(x_\beta - x_\gamma)\hat{\psi}(x_1, x_2, x_3), \quad \hat{\psi} \in \ell_2^s((\mathbb{Z}^3)^3).
\]

**Hypothesis 2.1.** The function \( \hat{v}(s) \) is a real, even, nonnegative function on \( \mathbb{Z}^3 \) and verify

\[
\lim_{|s| \to \infty} |s|^{3+\theta} \hat{v}(s) = 0, \quad \theta > \frac{1}{2}.
\]

**Remark 2.2.** It is clear that under of Hypothesis 2.1 the two- resp. three- particle Hamiltonian \( \hat{H}_2 \) resp. \( \hat{H}_3 \) is a bounded self-adjoint operator on the Hilbert space \( \ell_2^s((\mathbb{Z}^3)^2) \) resp. \( \ell_2^s((\mathbb{Z}^3)^3) \).

**Remark 2.3.** We note that Hypothesis 2.1 are far from being optimal, but we will not discuss further this point here.
2.1. The momentum representation. Let \((\mathbb{T}^3)^m, m \in \mathbb{N}\) be the Cartesian \(m\)-th power of the torus \(\mathbb{T}^3 = (-\pi, \pi]^3\) and let \(L_2^s((\mathbb{T}^3)^m) \subset L_2((\mathbb{T}^3)^m)\) be the subspace of symmetric functions.

Let \(\mathcal{F}_m : L_2((\mathbb{T}^3)^m) \to \ell^2((\mathbb{Z}^3)^m)\) be the standard Fourier transform. Since the subspace \(L_2^s((\mathbb{T}^3)^m)\) is invariant with respect to the group \(\mathcal{F}_m\), i.e., \(\mathcal{F}_m L_2^s((\mathbb{T}^3)^m) \subset \ell^2((\mathbb{Z}^3)^m)\), we denote by \(\mathcal{F}_m^s\) the restriction of \(\mathcal{F}_m\) on to the subspace \(L_2^s((\mathbb{T}^3)^m)\). We easily check that

\[
\mathcal{F}_m^s : L_2^s((\mathbb{T}^3)^m) \to \ell^2((\mathbb{Z}^3)^m).
\]

The two-resp. three-particle Hamiltonians in the momentum representation are given by bounded self-adjoint operators on the Hilbert spaces \(L_2^s((\mathbb{T}^3)^2)\) resp. \(L_2^s((\mathbb{T}^3)^3)\) as follows

\[
h = (\mathcal{F}_2^s)^{-1} h \mathcal{F}_2^s,
\]

resp.

\[
H = (\mathcal{F}_3^s)^{-1} \hat{H} \mathcal{F}_3^s.
\]

The two-particle Hamiltonian \(h\) is of the form

\[
h = h^0 - v.
\]

The operator \(h^0\) is the multiplication operator by the function \(\varepsilon(k_1) + \varepsilon(k_2)\):

\[
(h^0 f)(k_1, k_2) = (\varepsilon(k_1) + \varepsilon(k_2)) f(k_1, k_2), \quad f \in L_2^s((\mathbb{T}^3)^2),
\]

where \(k_\alpha, \alpha = 1, 2\) is the quasi-momentum of the particle \(\alpha\).

The integral operator \(v\) is of convolution type

\[
(v f)(k_\beta, k_\gamma) = (2\pi)^{-\frac{3}{2}} \int_{(\mathbb{T}^3)^2} v(k_\beta - k'_\beta) \delta(k_\beta + k_\gamma - k'_\gamma) f(k'_\beta, k'_\gamma) dk'_\beta dk'_\gamma,
\]

\[
f \in L_2^s((\mathbb{T}^3)^2),
\]

where \(\delta(\cdot)\) denotes the Dirac delta-function at the origin.

The functions \(\varepsilon(k)\) and \(v(k)\), are given by the Fourier series

\[
\varepsilon(k) = \sum_{j=1}^{3} (1 - \cos k(j)), \quad v(k) = (2\pi)^{-3/2} \sum_{s \in \mathbb{Z}^3} \hat{v}(s) e^{i(k,s)},
\]

with

\[
(k, s) = \sum_{j=1}^{3} k(j)s(j), \quad k = (k^{(1)}, k^{(2)}, k^{(3)}) \in \mathbb{T}^3, \quad s = (s^{(1)}, s^{(2)}, s^{(3)}) \in \mathbb{Z}^3.
\]

The three-particle Hamiltonian \(H\) is of the form

\[
H = H_0 - V_1 - V_2 - V_3,
\]

where \(H_0\) is the multiplication operator by the function \(\sum_{\alpha=1}^{3} \varepsilon(k_\alpha)\)

\[
(H_0 f)(k_1, k_2, k_3) = \sum_{\alpha=1}^{3} \varepsilon(k_\alpha) f(k_1, k_2, k_3),
\]

\[
(V f)(k_1, k_2, k_3) = (2\pi)^{-\frac{3}{2}} \int_{(\mathbb{T}^3)^3} v(k_\beta - k'_\beta) \times
\]

\[
x \delta(k_\alpha - k'_\alpha) \delta(k_\beta + k_\gamma - k'_\beta - k'_\gamma) f(k'_1, k'_2, k'_3) dk'_1 dk'_2 dk'_3, \quad f \in L_2^s((\mathbb{T}^3)^3).
\]
3. Decomposition of Hamiltonians into von Neumann Direct Integrals. Quasimomentum and Coordinate Systems

Given $m \in \mathbb{N}$, denote by $\hat{U}_t^m$, $t \in \mathbb{Z}^3$ the unitary operators on the Hilbert space $\ell_2((\mathbb{Z}^3)^m)$ defined by:

$$(\hat{U}_t^m f)(n_1, n_2, ..., n_m) = f(n_1 + t, n_2 + t, ..., n_m + t), \quad f \in \ell_2((\mathbb{Z}^3)^m).$$

We easily see that

$$\hat{U}_{t+\tau}^m = \hat{U}_t^m \hat{U}_\tau^m, \quad t, \tau \in \mathbb{Z}^3,$$

i.e., $\hat{U}_t^m$, $t \in \mathbb{Z}^3$ is a unitary representation of the abelian group $\mathbb{Z}^3$ in the Hilbert space $\ell_2((\mathbb{Z}^3)^m)$. Since $\ell_2^s((\mathbb{Z}^3)^m)$ is invariant with respect to the group $\hat{U}_t^m$, $t \in \mathbb{Z}^3$, i.e.,

$$\hat{U}_t^m \ell_2^s((\mathbb{Z}^3)^m) \subset \ell_2^s((\mathbb{Z}^3)^m),$$

we denote by $\hat{U}_t^m$ the restriction of $\hat{U}_t^m$ to the subspace $\ell_2^s((\mathbb{Z}^3)^m)$.

Via the Fourier transform $\mathcal{F}_m$ the unitary representation of $\mathbb{Z}^3$ in $\ell_2^s((\mathbb{Z}^3)^m)$ induces a representation of the group $\mathbb{Z}^3$ in the Hilbert space $L_2^s((\mathbb{T}^3)^m)$ by unitary (multiplication) operators $U_{st}^m = (\mathcal{F}_m)^{-1} \hat{U}_{st}^m \mathcal{F}_m$, $t \in \mathbb{Z}^3$ given by:

$$(U_{st}^m f)(k_1, k_2, ..., k_m) = \exp \left(-i(t, k_1 + k_2 + ... + k_m)\right) f(k_1, k_2, ..., k_m),$$

$$f \in L_2^s((\mathbb{T}^3)^m).$$

Denote by $K = k_1 + k_2 + ... + k_m \in \mathbb{T}^3$ the total quasi-momentum of the $m$ particles and define $\mathbb{F}_K^m$ as follows

$$\mathbb{F}_K^m = \{(k_1, ..., k_{m-1}, K - k_1 - ... - k_{m-1}) \in (\mathbb{T}^3)^m : k_1, k_2, ..., k_{m-1} \in \mathbb{T}^3, K - k_1 - ... k_{m-1} \in \mathbb{T}^3\}.$$

Decomposing the Hilbert space $L_2^s((\mathbb{T}^3)^m)$ into the direct integral

$$L_2^s((\mathbb{T}^3)^m) = \int_{K \in \mathbb{T}^3} \oplus L_2^s(\mathbb{F}_K^m) dK$$

yields the decomposition of the unitary representation $U_{st}^m$, $t \in \mathbb{Z}^3$ into the direct integral

$$U_{st}^m = \int_{K \in \mathbb{T}^3} \oplus U_t(K) dK,$$

where

$$U_t(K) = \exp(-i(t, K))I \quad \text{on} \quad L_2^s(\mathbb{F}_K^m)$$

and $I = L_2^s(\mathbb{F}_K^m)$ denotes the identity operator on the Hilbert space $L_2^s(\mathbb{F}_K^m)$.

The above Hamiltonians $\hat{h}$ and $\hat{H}$ obviously commute with the groups of translations $\hat{U}_{st}^2$, $t \in \mathbb{Z}^3$ and $\hat{U}_{st}^3$, $t \in \mathbb{Z}^d$ respectively, i.e.,

$$\hat{U}_{st}^2 \hat{h} = \hat{h} \hat{U}_{st}^2, \quad t \in \mathbb{Z}^3$$

and

$$\hat{U}_{st}^3 \hat{H} = \hat{H} \hat{U}_{st}^3, \quad t \in \mathbb{Z}^3.$$

Hence, the operators $h$ and $H$ can be decomposed into the direct integrals

$$h = \int_{k \in \mathbb{T}^3} \oplus \hat{h}(k) dk \quad \text{and} \quad H = \int_{K \in \mathbb{T}^3} \oplus \hat{H}(K) dK$$

(3.1)
with respect to the decompositions
\[ L_2^s(T^3) = \int_{k \in T^3} \oplus L_2^s(F^2_k) dk, \quad \text{and} \quad L_2^s((T^3)^2) = \int_{K \in T^3} \oplus L_2^s(F^3_K) dK \]
respectively.

We introduce the mapping
\[ \pi^{(2)} : (T^3)^2 \to T^3, \quad \pi^{(2)}((k_\beta, k_\gamma)) = k_\beta \]
resp.
\[ \pi^{(3)} : (T^3)^3 \to (T^3)^2, \quad \pi^{(3)}((k_\alpha, k_\beta, k_\gamma)) = (k_\alpha, k_\beta). \]

Denote by \( \pi_k^{(2)} \), \( k \in T^3 \) resp. \( \pi_K^{(3)} \), \( K \in T^3 \) the restriction of \( \pi^{(2)} \) resp. \( \pi^{(3)} \) onto \( F^2_k \subset (T^3)^2 \) resp. \( F^3_K \subset (T^3)^3 \), i.e.,
\[ \pi^{(2)}(k, f, \gamma) = \pi_k^{(2)}(f) = (k, f) \]
resp.
\[ \pi^{(3)}(K, f, \gamma) = \pi_K^{(3)}(f) = (K, f). \]

At this point it is useful to remark that \( F^2_k, k \in T^3 \) resp. \( F^3_K, K \in T^3 \) is three resp. six-dimensional manifolds isomorphic to \( T^3 \) resp. \( (T^3)^2 \).

**Lemma 3.1.** The mapping \( \pi_k^{(2)} \), \( k \in T^3 \) resp. \( \pi_K^{(3)} \), \( K \in T^3 \) are bijective from \( F^2_k \subset (T^3)^2 \) resp. \( F^3_K \subset (T^3)^3 \) onto \( (T^3)^2 \) resp. \( T^3 \) with the inverse mapping given by
\[ (\pi_k^{(2)})^{-1}(k_\beta) = (k_\beta, k - k_\beta) \]
resp.
\[ (\pi_K^{(3)})^{-1}(k_\alpha, k_\beta) = (k_\alpha, k_\beta, K - k_\alpha - k_\beta). \]

### 3.1. The Fiber Operators
The fiber operators \( \tilde{h}(k) \), \( k \in T^3 \), from the direct integral decomposition \( (3.1) \) are unitarily equivalent to the operators \( h(k), k \in T^3 \), of the form
\[ (3.2) \quad h(k) = h^0(k) - v. \]

Let \( L^e_2(T^3) \subset L_2(T^3) \) be the subspace of even functions. The operators \( h^0(k) \) and \( v \) are defined on the Hilbert space \( L^e_2(T^3) \) by
\[ (h^0(k)f)(k_\beta) = E_k(k_\beta)f(k_\beta), \quad f \in L^e_2(T^3), \]
where
\[ (3.3) \quad E_k(k_\beta) = \varepsilon\left(\frac{k}{2} - k_\beta\right) + \varepsilon\left(\frac{k}{2} + k_\beta\right) \]
and
\[ (vf)(k_\beta) = (2\pi)^{-\frac{3}{2}} \int_{T^3} v(k_\beta - k_\beta') f(k_\beta') dk_\beta', \quad f \in L^e_2(T^3). \]

The fiber operators \( \tilde{H}(K), K \in T^3 \) from the direct integral decomposition \( (3.1) \) are unitarily equivalent to the operators \( H(K), K \in T^3 \), and are given by
\[ (3.4) \quad H(K) = H_0(K) - V_1 - V_2 - V_3. \]

The operators \( H_0(K) \) and \( V_{\alpha} = V, \alpha = 1, 2, 3, \) are defined on the Hilbert space \( L^e_2((T^3)^2) \equiv L_2(T^3) \otimes L^e_2(T^3) \) and in the coordinates \( (k_\alpha, k_\beta) \in (T^3)^2 \) have form
\[ (3.4) \quad (H_0(K)f)(k_\alpha, k_\beta) = E(K; k_\alpha, k_\beta)f(k_\alpha, k_\beta), \quad f \in L^e_2((T^3)^2), \]
\[ E(K; k_\alpha, k_\beta) = \varepsilon(K - k_\alpha) + \varepsilon\left(\frac{k_\alpha}{2} - k_\beta\right) + \varepsilon\left(\frac{k_\alpha}{2} + k_\beta\right). \]
and
\[ \text{(3.5)} \quad V = I \otimes v, \]
where $\otimes$ is the tensor product and $I = I_{L_2(T^3)}$ is the identity operator in $L_2(T^3)$.

Later on all our calculations will be carried out in the configuration space, i.e., in a six-dimensional manifold $F^3_K \subset (T^3)^3$ isomorphic to $(T^3)^2$. As coordinates in $F^3_K$, we will choose one of the three pairs of vectors $(k_\alpha, k_\beta)$ which run independently through the whole space $T^3$ (if it does not lead to any confusion we will write $(p, q)$ instead of $(k_\alpha, k_\beta)$).

4. THE CONCEPT OF A ZERO ENERGY RESONANCE

We introduce the concept of a zero energy resonance (threshold resonance) for the (lattice) two-particle operator $h(0)$ defined by [32].

We remark that under Hypothesis 2.1 the sequence $\{\hat{v}(s)\}_{s \in \mathbb{Z}^2}$ of the Fourier coefficients of the continuous function $v(p)$ is an element of $l_2(\mathbb{Z}^2)$ and then the equality
\[ \text{(4.1)} \quad v(p) = (2\pi)^{-\frac{d}{2}} \sum_{s \in \mathbb{Z}^3} \hat{v}(s)e^{i(p,s)}, \]
should be understood as follows: the $L_2(T^3)$-function $\kappa = (2\pi)^{-\frac{d}{2}} \sum_{s \in \mathbb{Z}^2} \hat{v}(s)e^{i(p,s)}$ has the continuous representative $v(p)$.

Since the function $\mathcal{E}_k(q)$ has a unique non-degenerate minimum at the point $q = \frac{k}{2}$ for any $k \in U_\delta(0)$ and $z \leq \mathcal{E}_{\min}(k)$ the integral
\[ \text{(4.2)} \quad G(p, q; k, z) = \frac{1}{(2\pi)^3} \int_{T^3} \frac{\hat{v}^\frac{1}{2}(p-t)\hat{v}^\frac{1}{2}(t-q)dt}{\mathcal{E}_k(t) - z} \]
is finite, where
\[ \text{(4.3)} \quad \hat{v}^\frac{1}{2}(p) = (2\pi)^{-\frac{d}{2}} \sum_{s \in \mathbb{Z}^3} \hat{v}(s)e^{i(p,s)} \]
is the kernel of the operator $\hat{v}^\frac{1}{2}$.

We define the integral operator $G(k, z)$ in $L_2^p(T^3)$ by
\[ \text{(4.4)} \quad G(k, z)f(p) = \int_{T^3} G(p, q; k, z)f(q)dq. \]

Lemma 4.1. The operator $G(k, z), k \in U_\delta(0), z \leq \mathcal{E}_{\min}(k)$ acts in $L_2^p(T^3)$, is positive, belongs to the trace class $\Sigma_1$, is continuous in $z$ from the left up to $z = \mathcal{E}_{\min}(k)$.

Proof. Lemma 6.1 can be proven in the same way as Theorem 4.5 in [5].

Remark 4.2. Clearly, the operator $h(0)$ has an eigenvalue $z \leq \mathcal{E}_0(0) = 0$, i.e., $\text{Ker}(h(0) - zI) \neq 0$, if and only if the compact operator $G(0, z)$ on $L_2^2(T^3)$ has an eigenvalue 1 and there exists a function $\psi \in \text{Ker}(I - G(0, z))$ such that the function $f$ given by
\[ f(p) = \frac{(\hat{v}^\frac{1}{2}\psi)(p)}{\mathcal{E}_0(p) - z} \text{ a.e. } p \in T^3, \]
belongs to $L_2(T^3)$. In this case $f \in \text{Ker}(h(0) - zI)$.
Moreover, if $z < 0$, then
\[ \dim \text{Ker}(h(0) - zI) = \dim \text{Ker}(I - G(0, z)) \]
and

\[ \text{Ker}(h(0) - zI) = \{ f | f(\cdot) = \frac{(v^2 \psi)(\cdot)}{E_0(\cdot)} - z, \psi \in \text{Ker}(I - G(0, z)) \}. \]

In the case of a threshold eigenvalue \( z = 0 \) equality (4.2) may fail to hold.

Therefore equality (4.2) should be replaced by the inequality

\[ \dim \text{Ker}(h(0)) \leq \dim \text{Ker}(I - G(0, 0)). \]

**Definition 4.3.** The operator \( h(0) \) is said to have a zero energy resonance (at the threshold) if 1 is an eigenvalue (single or multiple) of \( G(0, 0) \) and at least one (up to a normalization) of the associated eigenfunctions \( \psi \) satisfies the condition

\[ (v^2 \psi)(\cdot) \in L^2(T^3), \]

i.e.,

\[ 1 \leq \dim \text{Ker}(I - G(0, z)) \geq \dim \text{Ker}(h(0) - zI) + 1. \]

**Remark 4.4.** The operator \( h(0) \) has a zero energy resonance, if and only if the point 1 is a simple eigenvalue of the operator \( G(0, 0) \) and the corresponding eigenfunction \( \psi \in L^2(T^3) \) satisfies the condition

\[ \int v^2(p) |\psi(p)|^2 \, dp \neq 0. \]

5. **Statement of the Main Results**

We set:

\[ E_{\min}(K) \equiv \min_{k_\alpha, k_\beta \in T^3} E(K, k_\alpha, k_\beta), \quad E_{\max}(K) \equiv \max_{k_\alpha, k_\beta \in T^3} E(K, k_\alpha, k_\beta). \]

Our main results are as follows.

**Theorem 5.1.** Assume Hypotheses 2.1. Let \( h(0) \) have a zero energy resonance. Then for all \( k \in U_0^\theta(0) \) the operator \( h(k) \) has a unique positive eigenvalue \( z(k) \) lying below the essential spectrum. Moreover \( z(k) \) is analytic in \( U_0^\theta(0) \).

**Theorem 5.2.** For the essential spectrum \( \sigma_{\text{ess}}(H(K)) \) of \( H(K) \), \( K \in T^3 \) the following equality holds

\[ \sigma_{\text{ess}}(H(K)) = \bigcup_{p \in T^3} \{ \sigma_d(h(p)) + \varepsilon(K - p) \} \cup [E_{\min}(K), E_{\max}(K)], \]

where \( \sigma_d(h(k)) \) is the discrete spectrum of the operator \( h(k) \), \( k \in T^3 \).

We denote by \( N(K, z) \) the number of eigenvalues of \( H(K) \), \( K \in T^3 \) below \( z \leq \tau(K) \), where

\[ \tau(K) = \inf \sigma_{\text{ess}}(H(K)). \]

**Theorem 5.3.** Assume Hypotheses 2.1 and that the operator \( h(0) \) has a zero energy resonance.

Then the operator \( H(0) \) has infinitely many eigenvalues lying below the bottom of the essential spectrum and the function \( N(0, z) \) obeys the relation

\[ \lim_{z \to -0} \frac{N(0, z)}{\log |z|} = \frac{\lambda_0}{2\pi}, \]

where \( \lambda_0 \) the unique positive solution of the equation

\[ \lambda = \frac{8 \sinh \pi \lambda / 6}{\sqrt{3} \cosh \pi \lambda / 2}. \]
Theorem 5.4. Let the conditions of Theorem 5.3 fulfilled. Then for all $K \in U_0^q(0)$ the number $N(K, 0)$ is finite and the following asymptotics holds

$$\lim_{|K| \to 0} \frac{N(K, 0)}{\log |K|} = \frac{\lambda_0}{\pi}.$$  

6. Spectral properties of the two-particle operator $h(k)$

By Weyl’s theorem the essential spectrum $\sigma_{\text{ess}}(h(k))$ of the operator $h(k), k \in T^3$ defined by (3.2) coincides with the spectrum $\sigma(h_0(k))$ of the non-perturbed operator $h_0(k)$. More specifically,

$$\sigma_{\text{ess}}(h(k)) = [E_{\text{min}}(k), E_{\text{max}}(k)],$$

where

$$E_{\text{min}}(k) \equiv \min_{p \in T^3} E_k(p), \quad E_{\text{max}}(k) \equiv \max_{p \in T^3} E_k(p)$$

and $E_k(p)$ is defined by (5.3).

We denote by $r_0(k, z)$ resp. $r(k, z)$ the resolvent of the operator $h_0(k)$ resp. $h(k)$.

Recall the operators $r_0(k, z)$ and $r(k, z)$ are connected by the following relations

$$r(k, z) = r_0(k, z) + r_0(k, z)v r_0(k, z) = r_0(k, z) + r(k, z)v r_0(k, z).$$

Set

$$w(k, z) = I + v \frac{r_0(k, z)}{v} \frac{r_0(k, z)}{v}.$$

The resolvent relation (6.1) yields

$$w(k, z) = (I - v \frac{r_0(k, z)}{v} \frac{r_0(k, z)}{v})^{-1}.$$

For a bounded self-adjoint operator $A$, we define $n(\lambda, A)$ as

$$n(\lambda, A) = \sup \{ \dim F : \langle Au, u \rangle > \lambda, \, u \in F, \, ||u|| = 1 \}.$$  

The number $n(\lambda, A)$ is equal the infinity if $\lambda$ is in the essential spectrum of $A$ and if $n(\lambda, A)$ is finite, it is equal to the number of the eigenvalues of $A$ bigger than $\lambda$.

The following lemma is the Birman-Schwinger principle for the two-particle Schrödinger operators on the lattice $Z^3$.

Lemma 6.1. For any $z \leq E_{\text{min}}(k)$ the following equality holds

$$n(-z, -h(k)) = n(1, G(k, z)),$$

where $G(k, z), z \leq E_{\text{min}}(k)$ is defined by (4.4).

Proof. Lemma 6.1 can be proven in the same way as Theorem 4.5 in [5].

Proof of Theorem 5.1 By the assumptions of Theorem 5.1 the equation

$$G(0, 0)\psi = v \frac{r_0(0, 0)}{v} \frac{r_0(0, 0)}{v} \psi = \psi$$

has a nonzero solution $\psi \in L^2_Z(T^3)$. One has

$$\langle \psi, \psi \rangle = \langle r_0(0, 0) v \frac{r_0(0, 0)}{v} \psi, v \frac{r_0(0, 0)}{v} \psi \rangle = \int_{T^3} \frac{|(v \frac{r_0(0, 0)}{v} \psi)(p)|^2}{E_0(p)} dp,$$
Proof. Schwinger principle one concludes that
\begin{equation}
(6.5)
\end{equation}
origin to a neighborhood \( q \) where the operator \( G \) one mapping
5.4 and 5.3)(see also [17]).
\begin{equation}
(6.4)
\end{equation}
\begin{equation}
(6.3)
\end{equation}
\begin{equation}
(6.2)
\end{equation}

By definition of (6.2) this means that \( n(1, v_{1/2} r_{0}(k, E_{\min}(k)) v_{1/2}) > 1 \). By the Birman-Schwinger principle one concludes that \( h(k), k \neq 0 \) has an eigenvalue lying below \( E_{\min}(k) \).

One checks that the equality
\begin{equation}
(h(k)f,f) = \int_{T^3} \varepsilon(k^2 + g) \left| f(q) \right|^2 dq + \int_{T^3} \varepsilon(k^2 - q) \left| f(q) \right|^2 dq
\end{equation}
holds. Since the function \( f \) is even, making a change of variables in the integrals in the r.h.s.of the latter equality, we have
\begin{equation}
(h(k)f,f) = (h(0)g,g) > 0,
\end{equation}
where \( g(q) = f(k^2 - q) \). This means that \( h(k) > 0 \) for any \( k \in T^3 \setminus \{0\} \).

The operator \( \tilde{G}(k,0), k \in U_{\delta}(0) \) is analytic in \( U_{\delta}(0) \) and hence by the theorem \([25]\) the eigenvalue \( z(k) \) is unique and analytic in \( k \in U_{\delta}(0) \).

Denote by \( G_1 \) the operator with the kernel
\begin{equation}
G_1(p,p') = -\frac{1}{4\pi} v_{1/2}(p)v_{1/2}(p').
\end{equation}

The following asymptotics plays a crucial role in the proof of the main result (Theorem 5.4 and 5.5)(see also [17]).

Lemma 6.2. Assume Hypotheses 2.7.
(i) For all \( k \in U_{\delta}(0) \) the following decomposition holds
\begin{equation}
(6.4)
G(k,0) = G(0,0) + \frac{1}{2} \left| k \right| G_1 + \left| k \right|^2 G_2(k),
\end{equation}
where the operator \( G_2(k) \) is continuous in \( k \in U_{\delta}(0) \).
(ii) For all \( z \leq 0 \) the following decomposition holds
\begin{equation}
G(0,z) = G(0,0) + G_1(-z)^{1/2} + (-z)^{1/2 + \theta_1} G_2(z), \quad \theta_1 < \theta,
\end{equation}
where the operator \( G_2(z) \) is continuous in \( z \leq 0 \).

Proof. For any \( k \in U_{\delta}(0) \) the function \( \tilde{E}_k(q) \) has a unique non-degenerate minimum at \( q_k = \frac{k}{\delta} \). Therefore, by virtue of the Morse lemma (see [14]) there exists a regular one-to-one mapping \( q = \varphi(k;t) \) of certain ball \( W_\gamma(0) \) of radius \( \gamma > 0 \) with the center at the origin to a neighborhood \( \tilde{W}(q_k) \) of the point \( q_k \) such that
\begin{equation}
(6.5)
\tilde{E}_k(\varphi(k;t)) = t^2 + E_{\min}(k)
\end{equation}
with $\varphi(k; 0) = 0$ and for the Jacobian $J(\varphi(k; t))$ of the mapping $q = \varphi(k; t)$ the equality

\begin{equation}
J(\varphi(k; 0)) = \frac{1}{\sqrt{\cos \frac{k}{2} \cos \frac{k}{2} \cos \frac{\pi}{2}}}
\end{equation}

holds.

Since the function $E_0(\cdot)$ has a unique non-degenerate minimum at $t = 0$ by dominated convergence the finite limit

\begin{equation}
G(p, q; 0, 0) = \lim_{k \to 0} G(p, q; k, 0)
\end{equation}

exists for all $p, q \in \mathbb{T}^3$.

For all $p, q \in \mathbb{T}^3$ the following inequalities

\begin{equation}
|G(p, q; k, 0) - G(p, q; 0, 0)| \leq C|k|,
\end{equation}

\begin{equation}
\left| \frac{\partial}{\partial |k|} G(p, q; k, 0) - \frac{\partial}{\partial |k|} G(p, q; 0, 0) \right| < C|k|^2, \quad k \in U_\delta(0)
\end{equation}

hold for some positive $C$ independent of $p$ and $q$.

Indeed, the function $G(p, q; \cdot, 0)$ can be represented as

\begin{equation}
G(p, q; k, 0) = G_1(p, q; k, 0) + G_2(p, q; k, 0)
\end{equation}

with

\begin{equation}
G_1(p, q; k, 0) = \frac{1}{(2\pi)^3} \int_{\mathbb{T}^3 \setminus \mathbb{W}(q)} \frac{v^*_q(\rho - t)}{E_k(t)} \, dt, \quad k \in U_\delta(0),
\end{equation}

and

\begin{equation}
G_2(p, q; k, 0) = \frac{1}{(2\pi)^3} \int_{\mathbb{T}^3 \setminus \mathbb{W}(q)} \frac{v^*_q(\rho - t)}{E_k(t)} \, dt, \quad k \in U_\delta(0).
\end{equation}

Since for any $k \in U_\delta(0)$ the function $E_k(\cdot)$ is continuous on the compact set $\mathbb{T}^3 \setminus \mathbb{W}(q)$ and has a unique minimum at $q_0 \in U_\delta(0)$ there exists $M = \text{const} > 0$ such that

\begin{equation}
\inf_{q \in \mathbb{T}^3 \setminus \mathbb{W}(q)} E_k(q) \geq M.
\end{equation}

Then for all $p, q \in \mathbb{T}^3$ we have

\begin{equation}
|G_2(p, q; k, 0) - G_2(p, q; 0, 0)| \leq Ck^2, \quad k \in U_\delta(0)
\end{equation}

for some $C > 0$ independent of $p, q \in \mathbb{T}^3$.

For all $p, q \in \mathbb{T}^3$ we consider

\begin{equation}
G_1(p, q; k, 0) - G_1(p, q; 0, 0) = -\frac{1}{(2\pi)^3} \int_{\mathbb{W}(q)} \frac{(E_0(t) - E_k(t))v^*_q(p - t)v^*_q(t - q)}{E_k(t)} dt.
\end{equation}

In the integral in (6.10) making a change of variable $q = \varphi(k; t)$ and using equality (6.5) we obtain

\begin{equation}
G_1(p, q; k, 0) - G_1(p, q; 0, 0) = -\frac{\epsilon_{\min}(k)}{(2\pi)^3} \int_{\mathbb{W}(0)} \frac{v^*_q(p - \varphi(k; t))v^*_q(\varphi(k; t) - q)J(\varphi(k; t))}{t^2(t^2 + \epsilon_{\min}(k))} dt.
\end{equation}
Going over in the integral in (6.11) to spherical coordinates $t = r\omega$, we reduce it to the form

$$G_1(p, q; k, 0) - G_1(p, q; 0, 0) = -\frac{E_{\min}(k)}{(2\pi)^3} \int_0^\gamma \frac{F(p, q; k, r)}{r^2 + E_{\min}(k)} dr,$$

with

$$F(p, q; k, r) = \int_{\Omega_2} v^\frac{1}{2}(p - \varphi(k, r\omega))v^\frac{1}{2}(\varphi(k, r\omega) - q)J(\varphi(k, r\omega)) d\omega,$$

where $\Omega_2$ is the unit sphere in $\mathbb{R}^3$ and $d\omega$ is the element of the unit sphere in this space. For all $p, q \in \mathbb{T}^3$ we see that

(6.12) $$|F(p, q; k, r) - F(p, q; 0, 0)| \leq C(r^\theta + |k|^2)$$

for some $C > 0$ independent of $p, q \in \mathbb{T}^3$.

Indeed, applying equality (4.3) and taking into account that the function $v^\frac{1}{2}(\cdot)$ is even on $\mathbb{T}^3$ we get

(6.13) $$|v^\frac{1}{2}(p - t)v^\frac{1}{2}(t - q) - v^\frac{1}{2}(p)v^\frac{1}{2}(q)| \leq \frac{1}{(2\pi)^3} \sum_{s \in \mathbb{Z}^3} |\hat{v}(s)||e^{-2i(t, s)} - 1|.$$

For any $0 < \theta \leq 1$ the inequality $|e^{-2i(t, s)} - 1| \leq C|\theta|^\theta |s|^\theta$, $p \in \mathbb{T}^3$, $s \in \mathbb{Z}^3$ holds. The Hypothesis (2.1) and inequality (6.13) yield the inequality

$$|v^\frac{1}{2}(p - t)v^\frac{1}{2}(t - q) - v^\frac{1}{2}(p)v^\frac{1}{2}(q)| \leq C|t|^\theta, \quad \frac{1}{2} < \theta \leq 1,$$

for some $C > 0$, independent on $p, q \in \mathbb{T}^3$.

Since the function $\varphi(k, \cdot)$ (see (1.4)) is regular and from (6.6) we have

$$|J(\varphi(k, r)) - J(\varphi(k, 0))| \leq C|r|^2 \quad \text{and} \quad |J(\varphi(k, 0)) - J(\varphi(0, 0))| \leq C|k|^2.$$

Thus

$$|F(p, q; k, r) - F(p, q; 0, 0)| \leq |F(p, q; k, r) - F(p, q; k, 0)| + |F(p, q; k, 0) - F(p, q; 0, 0)| \leq$$

$$\int_{\Omega_2} |v^\frac{1}{2}(p - \varphi(k, r\omega))v^\frac{1}{2}(\varphi(k, r\omega) - q) - v^\frac{1}{2}(p)v^\frac{1}{2}(q)||J(\varphi(k, 0))| d\omega +$$

$$\int_{\Omega_2} |v^\frac{1}{2}(p - \varphi(k, r\omega))v^\frac{1}{2}(\varphi(k, r\omega) - q)||J(\varphi(k, r\omega)) - J(\varphi(k, 0))| d\omega +$$

$$\int_{\Omega_2} |v^\frac{1}{2}(p)||v^\frac{1}{2}(q)||J(\varphi(k, 0)) - J(\varphi(0, 0))| d\omega \leq C(r^\theta + |k|^2).$$

For any $p, q \in \mathbb{T}^3$ the function $G_1(p, q; k, 0) - G_1(p, q; 0, 0)$ can be written in the form

(6.14) $$G_1(p, q; k, 0) - G_1(p, q; 0, 0) = -\frac{F(p, q; 0, 0)}{(2\pi)^3} \int_0^\gamma \frac{E_{\min}(k) dr}{r^2 + E_{\min}(k)} -$$

$$\frac{1}{(2\pi)^3} \int_0^\gamma \frac{E_{\min}(k)(F(p, q; k, r) - F(p, q; 0, 0))}{r^2 + E_{\min}(k)} dr.$$

By inequality (6.12) for all $p, q \in \mathbb{T}^3$ we have

(6.15) $$\int_0^\gamma \frac{F(p, q; k, r) - F(p, q; 0, 0)}{r^2 + E_{\min}(k)} dr \leq C \int_0^\gamma \frac{r^\theta + |k|^2}{r^2 + E_{\min}(k)} dr.$$

The asymptotics

$$E_{\min}(k) = \frac{1}{4} k^2 + O(|k|^4) \quad \text{as} \quad k \to 0.$$
Theorem 6.3. Assume Hypotheses 2.7

a) If zero is a regular point of \( h(0) \), then the operator \( w(k, 0) \) resp. \( w(0, z) \) is bounded for all \( k \in U_\delta(0) \) resp. \( z \leq 0 \).

b) If \( h(0) \) has a zero energy resonance, then the following equalities hold

\[
\frac{1}{k} \left( \int_0^\gamma \frac{\mathcal{E}_{\min}(k) dr}{r^2 + \mathcal{E}_{\min}(k)} \right) - \frac{1}{4} \int_0^\gamma \frac{k^2 dr}{r^2 + \frac{1}{4} k^2} \to 0 \text{ as } k \to 0.
\]

Computing the integrals

\[
\int_0^\gamma \frac{|k|}{r^2 + \frac{1}{4} k^2} dr \quad \text{and} \quad \int_0^\gamma \frac{r^\theta + \frac{1}{4}|k|^2}{r^2 + \frac{1}{4} k^2} dr
\]
we obtain

\[
\int_0^\gamma \frac{|k|}{r^2 + \frac{1}{4} k^2} dr \to \pi \text{ as } |k| \to 0 \quad \text{and} \quad \int_0^\gamma \frac{|k|(r^\theta + |k|^2)}{r^2 + \frac{1}{4} k^2} dr \to 0 \text{ as } |k| \to 0.
\]

Using equality (6.14) and inequality (6.15) we have

\[
\frac{\partial}{\partial |k|} G_1(p, q; 0, 0) = \lim_{|k| \to 0^+} \frac{G_1(p, q; k, 0) - G_1(p, q; 0, 0)}{|k|} = -\frac{1}{8\pi} F(p, q; 0, 0).
\]

with the continuous function \( F(p, q; 0, 0) \). Therefore the inequality

\[
|G_1(p, q; k, 0) - G_1(p, q; 0, 0)| < C|k|, \quad k \in U_\delta(0)
\]
holds for some positive \( C \) independent of \( p, q \in \mathbb{T}^3 \).

Then from (6.9) and (6.16) it follows that the right-hand derivative of \( G_1(p, q; \cdot, 0) \) at \( |k| = 0 \) and for all \( p, q \in \mathbb{T}^3 \) we have

\[
\frac{\partial}{\partial |k|} G(p, q; 0, 0) = -\frac{1}{8\pi} v^\perp(p) v^\perp(q).
\]

Comparing (6.9) and (6.16) we get (6.7). In the same way one can prove the inequality (6.8). Then we obtain (6.4). The part (ii) of Theorem 6.2 can be proven in the same way as part (i)(see also [17]).

\[\square\]

**Theorem 6.3.** Assume Hypotheses 2.7

a) If zero is a regular point of \( h(0) \), then the operator \( w(k, 0) \) resp. \( w(0, z) \) is bounded for all \( k \in U_\delta(0) \) resp. \( z \leq 0 \).

b) If \( h(0) \) has a zero energy resonance, then the following equalities hold

\[
w(k, 0) = \frac{8\pi(\cdot, \psi)}{\varphi^2(0)||\psi||^2|k|} + w_1(k), \quad k \in U_\delta^0(0),
\]

and

\[
w^\frac{1}{2}(k, 0) = \frac{2\sqrt{2\pi(\cdot, \psi)}}{\varphi(0)||\psi||\sqrt{|k|}} + \tilde{w}_1(k), \quad k \in U_\delta^0(0),
\]

resp.

\[
w(0, z) = \frac{4\pi(\cdot, \psi)}{\varphi^2(0)||\psi||^2\sqrt{-z}} + w_2(z), \quad z < 0,
\]

and

\[
w^\frac{1}{2}(0, z) = \frac{2\sqrt{4\pi(\cdot, \psi)}}{\varphi(0)||\psi||\sqrt{-z}} + \tilde{w}_2(z), \quad z < 0,
\]

where \( \varphi(0) = (v^\perp, \psi) \), the operators \( w_1(k) \) and \( \tilde{w}_1(k) \) are continuous in \( k \in U_\delta(0) \) resp. \( w_2(z) \) and \( \tilde{w}_2(z) \) are continuous in \( z \leq 0 \).
Proof. We shall prove the part of Theorem 6.3 concerning the operator \( w(k, 0) \). The part of Theorem 6.3 concerning \( w(0, z) \) can be proven in the same way.

(a) Let zero be a regular point. Since the operator \( G(k, 0) \) is analytic in \( \mathcal{U}_\delta(0) \), compact and the number 1 is not an eigenvalue for \( G(0, 0) \), the operator \( (I - G(k, 0))^{-1} \) exists for sufficiently small \( |k|, k \in \mathcal{U}_\delta(0) \).

By the representation (6.4) we have

\[
||w(k, 0)|| < C < \infty.
\]

(b). Let the \( h(0) \) have a zero energy resonance. Denote by \( P_0 \) the one dimensional projector onto the subspace \( \mathcal{H}_0 \) associated with \( \psi \) and by \( P_1 \) the projector onto its orthogonal complement \( \mathcal{H}_1 \), so that \( P_0 \oplus P_1 = I \). Let us write the operator \( A = A(k, 0) = I - v^\frac{1}{2} r_0(k, 0)v^\frac{1}{2} \) in the matrix form:

\[
A = \begin{pmatrix}
A_{00} & A_{01} \\
A_{10} & A_{11}
\end{pmatrix},
\]

where \( A_{ij} = P_i AP_j : \mathcal{H}_j \rightarrow \mathcal{H}_i, \ i, j = 0, 1, \)

Since \( G(0, 0)\psi = \psi, P_0(I - G(0, 0))P_1 = 0 \), by Lemma 6.2 we have that

\[
A_{00} = |k|(-P_0G_1P_0 - |k|P_0G_2(k) P_0),
\]

\[
A_{01} = |k|(P_0[-G_1 - |k|G_2(k)] P_1),
\]

\[
A_{10} = A^*_{01},
\]

(6.19)

\[
A_{11} = P_1(I - G(0, 0))P_1 - |k|P_1[G_1 + |k|G_2(k)] P_1.
\]

It is more convenient to consider the operator

\[
B = PAP, \quad P := \begin{pmatrix}
P_0 & 0 \\
\sqrt{|k|} & P_1
\end{pmatrix},
\]

instead of \( A \).

Then from (6.19) we obtain that

\[
B_{00} = -P_0G_1P_0 - |k|P_0G_2(k) P_0,
\]

\[
B_{01} = -\sqrt{|k|}P_0[G_1 + |k|G_2(k)] P_1,
\]

\[
B_{10} = B_{01}^*,
\]

\[
B_{11} = P_1(I - G(0, 0))P_1 - |k|P_1[G_1 + |k|G_2(k)] P_1.
\]

Therefore \( B = B^{(0)} + \tilde{B} \), where

\[
B^{(0)} = \begin{pmatrix}
-P_0G_1P_0 & 0 \\
0 & P_1(I - G(0, 0)) P_1
\end{pmatrix}
\]

and \( \tilde{B} = O(|k|^{\frac{1}{2}}) \) as \( k \rightarrow 0 \).

By the definition of \( P_1 \) the operator \( F = (P_1(I - G(0, 0))P_1)^{-1} \) exists in \( \mathcal{H}_1 \). By definition \( P_0 = ||\psi||^{-2}(, \psi)\psi \), and we obtain from \( (\hat{\psi}, \psi) = \varphi(0) \) in (6.3) that

\[
P_0G_1P_0 = -\frac{1}{8\pi||\psi||^2}P_0(\psi, v^\frac{1}{2} P_0)^2 = -\frac{\varphi^2(0)}{8\pi||\psi||^2} P_0.
\]

Thus

\[
(-P_0G_1P_0)^{-1} = \frac{8\pi||\psi||^2}{\varphi^2(0)} P_0 = \frac{8\pi}{\varphi^2(0)}(, \psi)\psi.
\]
Now since $B = (I + \tilde{B}(B(0)^{-1})B(0)$ and $\tilde{B} = O(|k|^{\frac{1}{2}})$ as $k \to 0$, we have
$$B^{-1} = (B(0)^{-1} + O(|k|^{\frac{1}{2}}) = \left( \frac{8\pi(\psi, \psi)}{\varphi^2(0) ||\psi||^4 |k|} \right) + O(|k|^{\frac{1}{2}}).$$

Taking into account that $w(k, 0) = (A(k, 0))^{-1} = PB^{-1}P$, we complete the proof of (6.17).

Let us prove (6.18). Since $r(k, 0) \geq 0$ for $h(0) \geq 0$ we have $w(k, 0) \geq I \geq 0$. Further, note that
$$\left( \frac{8\pi(\psi, \psi)}{\varphi^2(0) ||\psi||^4 |k|} \right)^{\frac{1}{2}} = \frac{2\sqrt{2\pi}(\psi, \psi)}{\varphi(0) ||\psi|| \sqrt{|k|}},$$
and recall the well known inequality for arbitrary positive operators $A, B$ (see [7]): $||B^{\frac{1}{2}} - A^{\frac{1}{2}}|| \leq ||B - A||^{\frac{1}{2}}$. In combination with (6.17) this yields
$$\left( |\psi(0)||\psi|| \sqrt{|k|} \right) \leq C.$$  

\[
\text{\square}
\]

7. Spectrum of the Operator $H(K)$

Since the particles are identical we have only one channel operator $H_{ch}(K), K \in T^3$
acting in the Hilbert space $L^2_T((T^3)^2) \cong L^2(T^3) \otimes L^2(T^3)$ as

$$H_{ch}(K) = H_0(K) - V,$$

where $H_0(K)$ resp. $V$ is defined by (3.4) resp. (3.5).

The decomposition of the space $L^2_T((T^3)^2)$ into the direct integral

$$L^2_T((T^3)^2) = \int_{k \in T^3} \oplus L^2_T(T^3)dk$$

yields for the operator $H_{ch}(K)$ the decomposition into the direct integral

$$H_{ch}(K) = \int_{k \in T^3} \oplus H_{ch}(K, k)dk.$$

The fiber operator $H_{ch}(K, k)$ acts in the Hilbert space $L^2_T(T^3)$ and has the form

$$H_{ch}(K, k) = h(k) + \varepsilon(K - k)I,$$

where $I$ is identity operator and $h(k)$ is the two-particle operator defined by (3.2). The representation of the operator $H_{ch}(K, k)$ implies the equality

$$\sigma(H_{ch}(K, k)) = \sigma_d(h(k)) \cup [E_{\min}(k), E_{\max}(k)] + \varepsilon(K - k),$$

where $\sigma_d(h(k))$ is the discrete spectrum of the operator $h(k)$.

**Lemma 7.1.** The following equality holds

$$\sigma(H_{ch}(K)) = \cup_{k \in T^3} \{\sigma_d(h(k) + \varepsilon(K - k)) \cup [E_{\min}(K), E_{\max}(K)].$$

**Proof.** The theorem (see, e.g., [25]) on the spectrum of decomposable operators and above obtained structure for the spectrum of $H_{ch}(K, k)$ complete the proof.  

\[
\text{\square}
\]
Lemma 7.2. The following equality holds
\[ \sigma(H_{\text{ch}}(K)) = \sigma_{\text{ess}}(H(K)). \]

Proof. Lemma 7.2 can be proven in the same way as Theorem 3.2 in [5]. \qed

8. Birman-Schwinger Principle for the Operator \( H(K) \).

Recall that (see (3.5)) the operator \( V \) acting in \( L^2_2((\mathbb{T}^3)^2) \) has form
\[ Vf(k_\alpha, k_\beta) = (I \otimes v)f(k_\alpha, k_\beta) \]
and hence the operator \( V^{1/2} \) is of the form
\[ V^{1/2}f(k_\alpha, k_\beta) = (I \otimes v^{1/2})f(k_\alpha, k_\beta) \]

Let \( W(K, z) \) and \( W^{1/2}(K, z) \), \( K \in \mathbb{T}^3, z \leq \tau(K) \), be the operators in \( L^2_2((\mathbb{T}^3)^2) \)
defined as
\[ W(K, z)f(k_\alpha, k_\beta) = \left( I \otimes w(k_\alpha, z - \varepsilon(K - k_\alpha)) \right)f(k_\alpha, k_\beta), \]
\[ W^{1/2}(K, z)f(k_\alpha, k_\beta) = \left( I \otimes w^{1/2}(k_\alpha, z - \varepsilon(K - k_\alpha)) \right)f(k_\alpha, k_\beta). \]

The operator \( W(K, z), K \in \mathbb{T}^3, z \leq \tau(K) \), as related to the resolvent \( R_{\text{ch}}(K, z) \) as
\[ W(K, z) = I + V^{1/2}R_{\text{ch}}(K, z)V^{1/2}, \]
where \( R_{\text{ch}}(K, z) \) is the resolvent of \( H_{\text{ch}}(K), K \in \mathbb{T}^3 \).
One checks that
\[ W(K, z) = (I - V^{1/2}R_0(K, z)V^{1/2})^{-1} \]
where \( R_0(K, z) \) the resolvent of the operator \( H_0(K) \).

For \( z < \tau(K), K \in \mathbb{T}^3 \), the operator \( W(K, z) \) is positive.
Let \( T(K, z), K \in \mathbb{T}^3, z \leq \tau(K) \), be the operator in \( L^2_2((\mathbb{T}^3)^2) \) defined by
\[ T(K, z) = 2W^{1/2}(K, z)V^{1/2}R_0(K, z)V^{1/2}W^{1/2}(K, z), \]
By the definition of \( N(K, z), K \in \mathbb{T}^3, z \leq \tau(K) \), we have
\[ N(K, z) = n(-z, -H(K)), -z > -\tau(K). \]
The following lemma is a realization of well known Birman-Schwinger principle for the three-particle Schrödinger operators on the lattice \( \mathbb{Z}^3 \) (see [26, 28]).

Lemma 8.1. The operator \( T(K, z), K \in \mathbb{T}^3, z < \tau(K), \) is compact and continuous in \( z < \tau(K) \) and the following equality holds
\[ N(K, z) = n(1, T(K, z)). \]

Proof. We first verify the equality
\[ N(K, z) = n(1, 3R_0^2(K, z)VR_0^2(K, z)). \]
Assume that \( u \in \mathcal{H}_{-H(K)}(-z) \), i.e., \( ((H_0(K) - z)u, u) < 3(Vu, u) \). Then
\[ (y, y) < 3(R_0^2(K, z)VR_0^2(K, z)y, y), \quad y = (H_0(K) - z)^{1/2}u. \]
Thus $N(K, z) \leq n(1, 3R_0^{1/2}(K, z)VR_0^{1/2}(K, z))$. Reversing the argument we get the opposite inequality, which proves (8.3). Any nonzero eigenvalue of $R_0^{1/2}(K, z)VR_0^{1/2}$ is an eigenvalue for $V^{1/2}R_0^{1/2}(K, z)$ with the same algebraic and geometric multiplicities.

Therefore we get

$$n(1, 3R_0^{1/2}(K, z)VR_0^{1/2}(K, z)) = n(1, 3V^{1/2}R_0(K, z)V^{1/2}).$$

Let us check that

$$n(1, 3R_0^{1/2}(K, z)VR_0^{1/2}(K, z)) = n(1, T(K, z)).$$

We shall show that for any $u \in \mathcal{H}(3R_0^{1/2}(K, z)VR_0^{1/2}(K, z))$ there exists $y \in \mathcal{H}(T(K, z))$ such that $(y, y) < (T(K, z)y, y)$. Let $u \in \mathcal{H}(3R_0^{1/2}(K, z)VR_0^{1/2}(1))$ i.e.,

$$(u, u) < 3(V^{1/2}R_0(K, z)V^{1/2}u, u)$$

and hence

$$(I - V^{1/2}R_0(K, z)V^{1/2}(u, u)) < 2(V^{1/2}R_0(K, z)V^{1/2}u, u).$$

Setting $y = (I - V^{1/2}R_0(K, z)V^{1/2})u$ we have

$$(y, y) < (2W^{1/2}(K, z)V^{1/2}R_0(K, z)V^{1/2}W^{1/2}(K, z)y, y),$$

i.e., $(y, y) \leq (T(K, z)y, y)$. Thus $n(1, 3R_0^{1/2}(K, z)VR_0^{1/2}(K, z)) \leq n(1, T(K, z))$.

In the same way one checks that $n(1, T(K, z)) \leq n(1, 3R_0^{1/2}(K, z)VR_0^{1/2}(K, z))$.

Finally we note that for any $z < \tau(K)$ the operator $T(K, z)$ is compact and continuous in $z$.

$$\square$$

**Remark 8.2.** On the left hand side of (8.4) the operator $V^{1/2}R_0(K, z)V^{1/2}$ is a partial integral operator, since the operator $V^{1/2}f(k_\alpha, k_\beta) = V^{1/2}_\alpha f(k_\alpha, k_\beta) = (I \otimes v^{1/2})f(k_\alpha, k_\beta)$

is written in the coordinate $(k_\alpha, k_\beta)$ i.e., it is an integral operator with respect to $k_\beta$.

The right hand side of (8.4) can be written as sum of $V^{1/2}_1R_0(K, z)V^{1/2}_2$ and $V^{1/2}_1R_0(K, z)V^{1/2}_3$, where the operator $V = V_1$ is written in the coordinate $(k_\alpha, k_\beta)$, i.e., it is integral operator with respect to $k_\beta$. But the operators $V = V_2$ and $V = V_3$ in the coordinates $(k_\beta, k_\alpha)$ constitute it is an integral operators with respect to $k_\alpha$ and hence the operator $2V^{1/2}R_0(K, z)V^{1/2}$ on the right hand side of (8.4) is an integral operator.

9. The Number of Eigenvalues of the Operator $H(K)$

In this section we shall prove Theorems 5.3 and 5.4. First we prove that the number $N(K, 0)$ is finite.

**Theorem 9.1.** The following equality holds

$$\lim_{|z| \to 0} \frac{n(1, T(0, z))}{|log|z||} = \frac{\lambda_0}{\pi} \quad \text{resp.} \quad \lim_{|z| \to \infty} \frac{n(1, T(z, z))}{|log|z||} = \frac{\lambda_0}{2\pi}$$

where $\lambda_0$ is defined in Theorem 5.3.

Theorem 9.1 will be deduced by a perturbation argument based on Lemma 4.7, which has been proven in [26]. For completeness, we here reproduce the lemma.
Lemma 9.2. Let \( A(z) = A_0(z) + A_1(z) \), where \( A_0(z) \) (resp. \( A_1(z) \)) is compact and continuous in \( z < 0 \) (resp. \( z \leq 0 \)). Assume that for some function \( f(\cdot) \), \( f(z) \to 0 \), \( z \to 0^- \) one has
\[
\lim_{z \to 0^-} f(z) n(\lambda, A_0(z)) = l(\lambda),
\]
and \( l(\lambda) \) is continuous in \( \lambda > 0 \). Then the same limit exists for \( A(z) \) and
\[
\lim_{z \to 0^-} f(z) n(\lambda, A(z)) = l(\lambda).
\]
Remark 9.3. According to Lemma 9.2 any perturbation of the operator \( A_0(z) \) defined in Lemma 9.2, which is compact and continuous up to asymptotics \( z = 0 \) (resp. \( z \to 0^- \)), \( \lim_{z \to 0^-} f(z) n(\lambda, A(z)) = l(\lambda) \).

Let the operator \( h(0) \) have a zero energy resonance, i.e., the number 1 is an eigenvalue for \( G(0, 0) \) and \( \psi \) is an associated eigenfunction.

Let \( \Psi : L_2(\mathbb{T}^3) \to L_2^2((\mathbb{T}^3)^2) \) be the operator defined by
\[
(\Psi f)(k_1, k_2) = \frac{1}{||\psi||} \psi(k_2) f(k_1)
\]
and \( \Psi^* : L_2^2((\mathbb{T}^3)^2) \to L_2(\mathbb{T}^3) \) be adjoint operator, i.e.,
\[
(\Psi^* f)(k_1) = \frac{1}{||\psi||} \int_{\mathbb{T}^3} \psi(k_2') f(k_1, k_2') dk_2'.
\]

Let \( \Delta^{-1/4} \) be the operator of multiplication by the function
\[
(\varepsilon_{\min}(p) + \varepsilon(K - p) - z)^{-1/4}.
\]

Theorem 6.3 and equality (8.1) yield
\[
W^\frac{1}{2}(K, 0) = \frac{2\sqrt{\pi}||\psi||}{\varphi(0)} \Psi \Delta^{-1/4} \Psi^* + \tilde{W}_1(K),
\]
resp.
\[
W^\frac{1}{2}(0, z) = \frac{2\sqrt{\pi}||\psi||}{\varphi(0)} \Psi \Delta^{-1/4} \Psi^* + \tilde{W}_2(z),
\]
where \( \tilde{W}_1(K) \) resp. \( \tilde{W}_2(z) \) is a continuous operator in \( K \in U_\delta(0) \) resp. in \( z \leq 0 \).

We define the operator \( T^{(1)}(K, z) \) acting in \( L_2^2((\mathbb{T}^3)^2) \) by
\[
T^{(1)}(K, z) = \frac{8\pi||\psi||^2}{\varphi^2(0)} \Psi \Delta^{-1/4} \Psi^* V^\frac{1}{2} R_0(K, z) V^\frac{1}{2} \Psi \Delta^{-1/4} \Psi^*.
\]

Lemma 9.4. For all \( K \in U_\delta(0) \) resp. \( z \leq 0 \) the operator \( T(K, 0) - T^{(1)}(K, 0) \) resp. \( T(0, z) - T^{(1)}(0, z) \) is a compact operator.

Proof. From (9.1), (9.3) and (8.2) we have
\[
T(K, 0) - T^{(1)}(K, 0) = \frac{4\sqrt{\pi}||\psi||}{\varphi(0)} \Psi \Delta^{-\frac{1}{2}} \Psi^* V^\frac{1}{2} R_0(K, 0) V^\frac{1}{2} \tilde{W}_1(K) +
\]
\[
+ \frac{4\sqrt{\pi}||\psi||}{\varphi(0)} \tilde{W}_1(K) V^\frac{1}{2} R_0(K, 0) V^\frac{1}{2} \Psi \Delta^{-\frac{1}{2}} \Psi^* + 2\tilde{W}_1(K) V^\frac{1}{2} R_0(K, 0) V^\frac{1}{2} \tilde{W}_1(K).
\]
The kernels of the operators $\Psi \Delta - \Psi^* V^z R_0(K, 0) V^z \Psi$, $V^z R_0(K, 0) V^z \Psi \Delta - \Psi^*$ and $V^z R_0(K, 0) V^z$ are bounded by

$$\frac{C_1}{|p|(p^2 + q^2)} \quad \frac{C_2}{(p^2 + q^2)|q|} \quad \text{and} \quad \frac{C_3}{(p^2 + q^2)},$$

respectively.

By passing on to a spherical coordinates system one checks that the functions $|p|^{-1}(p^2 + q^2)^{-1}$, $(p^2 + q^2)^{-1}|q|^{-1}$ and $(p^2 + q^2)^{-1}$ are square integrable. By Lemma 6.3 the operator $\tilde{W}_1(K)$ is uniformly bounded in $K \in U_\delta(0)$. Therefore the right-hand side of (9.4) is a compact operator for any $K \in U_\delta(0)$.

In the same way one checks that $T(0, z) - T^{(1)}(0, z), z \leq 0$, belongs to the Hilbert-Schmidt class.

The proof of Lemma 8.1 is completed. \hfill \Box

For all $K \in U_\delta(0)$ and $z \leq 0$ we define the operator $T^{(1)}(K, z): L_2(\mathbb{T}^3) \to L_2(\mathbb{T}^3)$, by

$$T^{(1)}(K, z) = 8\pi \frac{||\psi||^2}{\varphi^2(0)} \Delta^{-1/4} \Psi^* V^z R_0(K, z) V^z \Psi \Delta^{-1/4}. \tag{9.5}$$

The kernel of $T^{(1)}(K, z)$ in the coordinates $(p, q)$ has form:

$$\frac{1}{\varphi^2(0)\pi^2} \frac{(E(K, \frac{2K}{3} + p, \frac{p}{2} + q) - z)^{-1} \varphi(\frac{p}{2} + q) \varphi(p + \frac{q}{2})}{(E_{\min}(\frac{2K}{3} + p) + \varepsilon(\frac{K}{3} - p) + z)^{1/4}(E_{\min}(\frac{2K}{3} + q) + \varepsilon(\frac{K}{3} - q - z))^{1/4}} \tag{9.6}$$

Lemma 9.5. The discrete spectrum of $T^{(1)}(K, 0)$ and $T^{(1)}(K, 0)$ resp. $T^{(1)}(0, z)$ and $T^{(1)}(0, z)$ coincides.

Proof. According to (9.5) and (9.3) we get

$$T^{(1)}(K, 0) = \Psi T^{(1)}(K, 0) \Psi^*. \tag{9.7}$$

Since any nonzero eigenvalue of $\Psi^* T^{(1)}(K, 0) \Psi$ resp. $\Psi^* T^{(1)}(0, z) \Psi$ is an eigenvalue of $\Psi \Psi^* T^{(1)}(K, 0)$ as well, with the same algebraic and geometric multiplicities, and $\Psi \Psi^* = I$, where $I$ is the identity operator on $L_2(\mathbb{T}^3)$, we have $n(1, T^{(1)}(K, 0)) = n(1, T^{(1)}(K, 0))$.

On the other hand $\Psi^* \Psi$ is the identity operator in $L_2(\mathbb{T}^3)$ and hence using equality (9.6) we have $\sigma_d(T^{(1)}(K, 0)) = \sigma_d(T^{(1)}(K, 0))$.

In the same way one proves that $\sigma_d(T^{(1)}(0, z)) = \sigma_d(T^{(1)}(0, z))$. The Lemma 9.5 is proven. \hfill \Box

Lemma 9.6. The function $E(K, \frac{2K}{3} + p, \frac{p}{2} + q)$ resp. $E_{\min}(\frac{2K}{3} + p) + \varepsilon(\frac{K}{3} - p)$ has the following asymptotics

$$E(K, \frac{2K}{3} + p, \frac{p}{2} + q) = \frac{K^2}{6} + p^2 + (p, q) + q^2 + O(|p|^4) + O(|q|^4) + O(|K|^4) \to 0 \tag{9.7}$$

as $K, p, q \to 0$.

resp.

$$E_{\min}(\frac{2K}{3} + p) + \varepsilon(\frac{K}{3} - p) = \frac{K^2}{6} + \frac{3p^2}{4} + O(|K|^4) + O(|p|^4), \tag{9.8}$$

as $K, p \to 0$.
Proof. The asymptotics
\begin{equation}
\varepsilon(p) = \frac{1}{2} p^2 + O(|p|^4) \quad \text{as} \quad p \to 0
\end{equation}
of the function \( \varepsilon(p) \) yields (9.7). The definition of \( \varepsilon_{\text{min}}(k) \) and the representation (3.3) gives the asymptotics
\begin{equation}
\varepsilon_{\text{min}}(k) = \frac{1}{4} k^2 + O(|k|^4) \quad \text{as} \quad k \to 0,
\end{equation}
which yields (9.8).

Denote by \( \chi_\delta(\cdot) \) the characteristic function of \( U_\delta(0) = \{ p \in \mathbb{R}^3 : |p| < \delta \} \).

For all \( K \in U_\delta(0) \) and \( z \leq 0 \) we define the operator \( T^{(1)}(\delta; K_2^0 + |z|) \), on \( L_2(\mathbb{T}^3) \) with the kernel
\[
\frac{1}{\pi^2} \frac{\chi(p)\chi(q)}{(\frac{2\pi^2}{3} + K_6^0 + |z|)^{1/4}(p^2 + (p, q) + q^2 + \frac{K_2^0}{6} + |z|)(\frac{3q^2}{4} + \frac{K_2^0}{6} + |z|)^{1/4}}
\]
Lemma 9.7. The operator \( T^{(1)}(K, 0) - T^{(1)}(\delta; \frac{K_2^0}{6}) \) resp. \( T^{(1)}(0, z) - T^{(1)}(\delta; |z|) \) belongs to the Hilbert-Schmidt class and is continuous in \( K \in U_\delta(0) \) resp. \( z \leq 0 \).

Proof. Applying the asymptotics (9.7) and (9.8) one can estimate the kernel of the operator \( T(K, 0) - T(\delta; \frac{K_2^0}{6}) \) by
\[
C[(p^2 + q^2)^{-1} + |p|^{-\frac{3}{2}}(p^2 + q^2)^{-1} + (|q|^{-\frac{3}{2}}(p^2 + q^2)^{-1} + \frac{|p|^6 + |q|^6}{|q|^2|p|^2(p^2 + q^2)} + 1]
\]
and hence the operator \( T^{(1)}(K, 0) - T^{(1)}(\delta; \frac{K_2^0}{6}) \) belongs to the Hilbert-Schmidt class for all \( K \in U_\delta(0) \). In combination with the continuity of the kernel of the operator in \( K \in U_\delta(0) \) this gives the continuity of \( T^{(1)}(K, 0) - T^{(1)}(\delta; \frac{K_2^0}{6}) \) in \( K \in U_\delta(0) \).

In the same way one checks that \( T^{(1)}(0, z) - T^{(1)}(\delta; |z|) \) belongs to the Hilbert-Schmidt class and is continuous in \( z \leq 0 \).

Let
\[ S(r) : L_2((0, r), \sigma_0) \to L_2((0, r), \sigma_0), \quad \sigma_0 = L_2(S^2), \]
\[ r = 1/2 \log \frac{K_2^0}{6} \quad \text{resp.} \quad r = 1/2 \log |z|, \]
\( S^2 \) – being the unit sphere in \( \mathbb{R}^3 \), be the integral operator with the kernel
\[
S(t; y) = \frac{2}{\sqrt{3\pi^2} \cos h y} \frac{1}{t},
\]
\[ y = x - x', \quad x, x' \in (0, r), \quad t = < \xi, \eta >, \quad \xi, \eta \in S^2, \]
and let
\[ \hat{S}(\lambda) : \sigma_0 \to \sigma_0, \quad \lambda \in (-\infty, +\infty) \]
be the integral operator with the following kernel
\begin{equation}
\hat{S}(t; \lambda) = \int_{-\infty}^{+\infty} \exp\{ -i\lambda r \} S(t; r) dr = \frac{1}{\sqrt{3\pi}} \frac{\sinh[\lambda(\arccos \frac{r}{t})]}{(1 - \frac{1}{4}t^2)\frac{r}{2} \sinh(\pi\lambda)}
\end{equation}
For $\mu > 0$, define

$$U(\mu) = (4\pi)^{-1} \int_{-\infty}^{+\infty} n(\mu, S(y)) dy.$$  

Lemma 9.8. The function $U(\mu)$ is continuous in $\mu > 0$, the following limit

$$\lim_{r \to \infty} \frac{1}{2} r^{-1} n(\mu, S(r)) = U(\mu)$$

exists.

Remark 9.9. This lemma can be proven in the same way as the corresponding results of [26]. In particular, the continuity of $U(\mu)$ in $\mu > 0$ is a result of Lemma 3.2, Theorem 4.5 states the existence of the limit

$$\lim_{r \to \infty} \frac{1}{2} r^{-1} n(\mu, S(r)) = U(\mu).$$

Lemma 9.10. The equalities

$$\lim_{|K| \to 0} \frac{n(1, T(1)(\delta, \frac{K^2}{6}))}{|\log|K||} = \frac{\lambda_0}{\pi}$$

and

$$\lim_{|z| \to 0} \frac{n(1, T(1)(\delta, |z|))}{|\log|z||} = \frac{\lambda_0}{2\pi}$$

hold, where $\lambda_0$ is unique positive solution of the equation (5.3).

Proof. The space of functions having support in $U_\delta(0)$ is an invariant subspace for the operator $T(1)(\delta, \frac{K^2}{6})$.

Let $T(1)(\delta, \frac{K^2}{6})$, be the restriction of the operator $T(1)(\delta, \frac{K^2}{6})$ on the invariant subspace $L_2(U_\delta(0))$.

The operator $T(1)(\delta, \frac{K^2}{6})$ is unitarily equivalent with the operator $T(2)(\delta, \frac{K^2}{6})$ acting in $L_2(B_r)$ by

$$T(2)(\delta, \frac{K^2}{6})w(p) = \int_{B_r} f(q) dq$$

where $B_r = \{ p \in T^3 : |p| < r, \quad r = (\frac{|K^2|}{6})^{-\frac{1}{4}} \}$.

The equivalence is performed by the unitary dilation

$$U_r : L_2(U_\delta(0)) \to L_2(B_r), (U_r f)(p) = \left( \frac{r}{\delta} \right)^{-3/2} f\left( \frac{\delta}{r} p \right).$$

Denote by $\chi(\cdot)$ the characteristic function of $U_1(0)$. Further, we may replace $(\frac{3p^2}{4} + 1)^{-1/4}, (\frac{3q^2}{4} + 1)^{-1/4}$ and $q^2 + (p, q) + p^2 + 1$

by $(\frac{3p^2}{4})^{-1/4}(1 - \chi(p)), \quad (\frac{3q^2}{4})^{-1/4}(1 - \chi(q))$ and $q^2 + (p, q) + p^2,$

respectively, since the error will be a Hilbert-Schmidt operator continuous up to $K = 0$.

Then we get the operator $T(2)(r)$ in $L_2(U_\delta(0) \setminus U_1(0))$ with the kernel

$$\frac{2}{\sqrt{3\pi^2 q^2 + (p, q) + p^2}}.$$
By the dilation
\[ M : L_2(U_r(0) \setminus U_1(0)) \rightarrow L_2((0, \mathbb{R}) \times \sigma_0), \quad r = 1/2|\log |K|^2/6|, \]
where \((M f)(x, w) = e^{3x/2} f(e^{x}w), x \in (0, \mathbb{R}), w \in \mathbb{S}^2\), one sees that the operator \(T^{(2)}(r)\) is unitary equivalent to the integral operator \(S(r)\). The difference of the operators \(S(r)\) and \(T^{(1)}(\delta, K^2/\alpha)\) is compact (up to unitarily equivalence). Hence Lemma 9.8 yields
\[ \lim_{|K| \rightarrow 0} \frac{n(1, T^{(1)}(\delta, K^2/\alpha))}{|\log |K||} = U(1). \]

It is convenient to calculate the coefficient \(U(1)\) by means of a decomposition of the operator \(\hat{S}(y)\) into the orthogonal sum over its invariant subspaces.

Denote by \(L_1 \subset L_2(S^2)\) the subspace of the harmonics of degree \(l = 0, 1, \ldots\). It is clear that \(L_2(S^2) = \bigoplus_{l=0}^{\infty} L_l\), \(\dim L_l = 2l + 1\). Let \(P_l : L_2(S^2) \rightarrow L_l\) be the orthogonal projector onto \(L_l\). The kernel of \(P_l\) is expressed via the Legendre polynomial \(P_l(\cdot)\):
\[ P_l(\xi, \eta) = \frac{2l + 1}{4\pi} P_l(<\xi, \eta>). \]

The kernel of \(\hat{S}(y)\) depends on the scalar product \(<\xi, \eta>\) only, so that the subspaces \(L_l\) are invariant for \(\hat{S}(y)\) and
\[ \hat{S}(y) = \bigoplus_{l=0}^{\infty} (\hat{S}^{(l)}(y) \otimes P_l), \]
where \(\hat{S}^{(l)}(y)\) is the multiplication operator by the number
\[ (9.13) \quad \hat{S}^{(l)}(y) = 2\pi \int_{-1}^{1} P_l(t) \hat{S}(t; y) dt \]
in \(L_l\) the subspace of the harmonics of degree \(l\), and \(P_l(t)\) is a Legendre polynomial. Therefore
\[ n(\mu, \hat{S}(y)) = \sum_{l=0}^{\infty} (2l + 1)n(\mu, \hat{S}^{(l)}(y)), \quad \mu > 0. \]

By (9.11) we first calculate \(\hat{S}^{(0)}(y)\):
\[ \hat{S}^{(0)}(y) = \frac{2}{\sqrt{3}} \int_{-1}^{1} \frac{\sinh[y(\arccos(t/2))] \sinh(\pi y) \sqrt{1 - t^2}}{\sqrt{\sinh(\pi y)}} dt = 8 \cdot \frac{3}{4} \cdot \frac{\sinh \frac{\pi}{2} y}{y \cosh \frac{\pi}{2} y}. \]

It follows from (9.12) and (9.13) that
\[ U(1) = \frac{1}{4\pi} \int_{-\infty}^{+\infty} n(1, \hat{S}^{(0)}(y)) dy = \frac{1}{4\pi} \int_{\hat{S}^{(0)}(\lambda) > 1} d\lambda = \frac{\lambda_0}{2\pi}. \]

In the same way proves the second statement of Lemma 9.10. Lemma 9.10 is proven. □

**Proof of Theorem 9.1** Lemmas 9.2, 9.4, 9.5, 9.7 and 9.10 yield the proof of Theorem 9.1.
Proof of Theorem 5.3 and 5.4. Let the conditions of Theorems 5.3 and 5.4 be fulfilled. Then the proof of Theorems 5.3 and 5.4 concerning the asymptotics of the number $N(0, z)$ resp. $N(K, 0)$ of eigenvalues follows from Lemma 8.1 and Theorem 9.1.

Now we shall prove the finiteness of $N(K, 0)$, $K \in U^0_\delta(0)$.

For any $K \in U^0_\delta(0)$ the kernel $T^{(1)}(\delta, K^2; p, q)$ of $T^{(1)}(\delta, K^2)$ estimated by

$$|T^{(1)}(\delta, K^2; p, q)| \leq \frac{C}{|K|^3}, \quad K \in U^0_\delta(0),$$

i.e., $T^{(1)}(\delta, K^2)$ belongs to the Hilbert-Schmidt class.

One concludes from Lemmas 9.4, 9.5 and 9.7 that the operator $T(K, 0)$ splits in to the sum of two compact (up to unitarily equivalence) operators

$$T(K, 0) = T^{(1)}(K, 0) + \tilde{T}(K, 0), \quad K \in U^0_\delta(0).$$

Applying Lemma 8.1 and Weyl's inequality

$$n(\lambda_1 + \lambda_2, A_1 + A_2) \leq n(\lambda_1, A_1) + n(\lambda_2, A_2)$$

we have

$$N(K, 0) = n\left(\frac{1}{2}, T^{(1)}(K, 0)\right) + n\left(\frac{1}{2}, \tilde{T}(K, 0)\right), \quad K \in U^0_\delta(0).$$

□

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