This is a short and simple introduction into perturbiners, that is, the solutions of field equations which are generating functions for tree amplitudes. The perturbiners have been constructed in Yang-Mills, in SUSY Yang-Mills, in gravity, and in sin(h)-Gordon.
I would like to present here work done in collaboration with Alexei Rosly. This is devoted to
solutions of field equations which describe tree amplitudes with any number of external legs. In works
1,2,3 such solutions were constructed in Yang-Mills (see also a closely related work 4). In works 3,4,5
such solutions were constructed in gravity and in Yang-Mills interacting with gravity. Work 6 is about
the sin(h)-Gordon case and work 7 deals with the case of SUSY Yang-Mills.

Unfortunately, because of lack of the time and because of the interest of the audience, I shall not
be able to explain the constructions itself - you will have to trust me that they are beautiful. I shall
only explain what type of solutions we construct. So to say, I shall only formulate the problem, a
solution for which can be found in the references above.

First of all, I, perhaps, need to explain that multi-leg amplitudes are non-trivial objects even
in the tree approximation. The problem is that, although every given diagram gives quite a simple
contribution, an algebraic function of external momenta, the number of different diagrams grows
enormously with the number of legs, and all these algebraic functions are different, so that the whole
thing becomes untreatable when the number of legs becomes bigger than, say, 10, to say nothing about
arbitrary number of legs.

In principle, one could take this as that the nature is such that the multi-leg amplitudes are
complicated objects and nothing to do about it. However, there are known cases when the final
expression for the amplitudes - if it is available at all - turns out to be much simpler than intermediate
steps. For instance, in 1,2,3,4 it was noticed that in \( \phi^4 \) theory all amplitudes \( 2 \to n \) for \( n \) bigger than
4 vanish at threshold. Notice that every given diagram gives a nontrivial contribution. That is only
sum of them what vanishes.

Other example of the case when the final expression is much simpler than intermediate steps are
the so-called Parke-Taylor, or “maximally helicity violating” amplitudes in Yang-Mills 5,6,7. These are
amplitudes with two positive-helicity gluons in the initial state and any number of the positive-helicity
 gluons in the final state. Kinematics is arbitrary. The explicit expressions for these amplitudes were
conjectured in 8 and were proven in 9. I am not giving here those nice expressions, I just notice that
they contain only the pairwise collinear singularities, those of the type of \( \frac{1}{(p_i \cdot p_j)} \), where \( p_i \) stands for
momentum of \( i \)-th particle, while a separate diagram definitely gives much more singularities.

These remarkable cancellations above motivated us for the work presented here.

Our initial point is that tree amplitudes are related to a solution of the classical field equation of
the model. To make the relation precise we need to consider not the amplitudes themselves, but form-
factors, that is, would be amplitudes with one unamputated off-shell leg and a number of on-shell
ones, which are amputated as they should be according to the LSZ rules. It is convenient to take the
off-shell leg in the coordinate \( (x) \) representation. Of course, the amplitudes can be obtained from the
form-factors applying the LSZ rules to the off-shell leg.

The form-factors can also be written as matrix elements of the field operator between vacuum and
n-particle state,

\[
< p_1, \ldots, p_l | \phi(x) | 0 >_{\text{tree}}
\]

where the subscript indicates that we are interested only in the tree contributions. At tree level, when
analytical structure of the form-factor is very simple (it is just an algebraic function of the momenta),
we can afford not to make a difference between in- and out- states.

Notice that \( x \)-dependence of the form-factors is very simple, it is just a product of the plane
waves corresponding to the momenta of external on-shell legs. This follows from the fact that in
momentum representation, the dependence on the momentum of the off-shell leg is given by the
overall conservation-law \( \delta \)-function:

\[
< p_1, \ldots, p_l | \phi(x) | 0 >_{\text{tree}} = \int dp' e^{ip'x} \delta(p' - \sum_j p_j)(\ldots) = e^{i \sum_j p_jx} (a \ function \ of \ \{p_j\})
\]

Obviously, an individual form-factor does not obey field equations. What obeys field equations is
a generating function for form-factors which we define next. To do this we
1) fix a set of momenta \( \{p_j\} : p_1, \ldots, p_N \);
2) introduce corresponding set of parameters \( \{a_j\} : a_1, \ldots, a_N \);
3) define a function of $x$, $\{p_j\}$ and $\{a_j\}$ such that the individual form-factors appears as coefficients in the Taylor expansion of this function in powers of $a$'s:

$$\Phi(x, \{p\}, \{a\}) = \sum_{l=1}^{L} \sum_{\{J\}} a_{J_1} \cdots a_{J_l} < p_{J_1}, \ldots, p_{J_l} | \phi(x) | 0 >_{\text{tree}}$$

(3)

where the one-particle form-factor is clearly

$$< p | \phi(x) | 0 > = (\ldots) e^{ipx}$$

(4)

($\ldots$) in Eq.(4) stands for a polarization factors in a non-scalar case, as well as for a color matrix in the Yang-Mills case, etc.

So defined generating function can be seen to obey field equations of the model. Indeed, applying the inverse propagator to the off-shell leg of an individual form-factor one obtains a sum of products of two, of three and so on - corresponding to the types of vertices in the theory - form-factors with smaller amount of legs. This relation can be used as a recursion relation. On the other hand, this relation is seen to be equivalent to field equations on the generating function:

$$\left( \frac{\partial^2}{\partial x^2} + m^2 \right) \Phi(x, \{p\}, \{a\}) = \lambda_3 \Phi^2 + \lambda_4 \Phi^3 + \ldots$$

(5)

Indeed, expanding Eq.(5) in powers of $a$’s one obtains the recursion relation among form-factors. (Eq.(5) is written for a scalar theory just for the brevity.)

Having started with the form-factors we have proved that the generating function obeys field equations. We can reverse the logic and use the field equations to find $\Phi(x, \{p\}, \{a\})$. To do this we must specify what solution to pick up, and according to the definition above we should pick up a solution which is a power series in the set of variables

$$\mathcal{E}_j = a_j e^{ip_j x}, \ j = 1, \ldots, N$$

starting with first order terms of the type of

$$\Phi(x, \{p\}, \{a\}) = \sum_{j=1}^{N} (\ldots) \mathcal{E}_j + \text{higher order terms in } \{\mathcal{E}_j\}$$

(7)

($\ldots$) here is the same as in Eq.(4).

Clearly, solution of this type exists and is unique (in gauge theories it is unique after a gauge fixing, or, equivalently, it is unique modulo gauge transformations), provided the operator on r.h.s. of Eq.(5) is invertible on the power series in the variables $\mathcal{E}_j$, which is true when the nonresonantness condition is satisfied, that is, none of linear combinations with integer coefficients of the momenta $p_j$ from the given set gets to mass shell,

$$\left( \sum_j n_j p_j \right)^2 \neq m^2$$

(8)

This is simply a condition that there are no internal lines on-shell in the Feynman diagrams.

It is also convenient to impose the nilpotency condition on the parameters $a_j$,

$$a_j^2 = 0, \ j = 1, \ldots, N$$

(9)

(note that in bosonic case still $a_i a_j = a_j a_i$) which is equivalent to excluding form-factors with identical particles from the generating function. In the massive case this condition is very convenient technically, while in the massless case it is even problematic to proceed without the nilpotency, because in the massless case identical particles necessary break nonresonantness.

I would like to stress at this point that we have introduced a class of solutions of field equations which are as universal as, say, solitonic solutions. We called these solutions perturbiners.
In the classical text-books on quantum field theory, e.g. 15, one can also find a discussion of solutions generating tree amplitudes, but they are defined differently, with use of asymptotic Feynman-type conditions. That definition is not very convenient, and it is not strange that no explicit examples were given in those books.

In the papers cited at the beginning of this talk we constructed perturbiners in various theories. Unfortunately, in Yang-Mills and in gravity the generic perturbiner is unavailable so far. What allows one to proceed is the restriction on polarizations of the external on-shell particles in the form-factors. Restricted in this way perturbiner obeys not the full Yang-Mill (Einstein) equations, but just the self-duality equations instead. It generates just the same-helicity form-factors. Having obtained the specific perturbiner with only same-helicity form-factors included, one can add opposite helicity particles perturbatively. This has been done in 2 in order to describe the Parke-Taylor amplitudes and in 7 to generalize the Parke-Taylor amplitudes to include any number of the same helicity gravitons in addition to the gluons. Our main tool was the so-called zero-curvature representation of the field equations. For the self-duality equations (in Yang-Mills and in gravity) and for the sin(h)-Gordon the zero-curvature representation is one-dimensional and this the case when perturbiner is constructed very efficiently. Remarkably, same-helicity form-factors in Yang-Mills and generic (tree) form-factors in sin(h)-Gordon are described in the same way. Perhaps, this opens a perspective to find Yang-Mills avatars of the many known quantum exact results in sin(h)-Gordon. In 9 a progress has been achieved about generic perturbiner in N = 3 SUSY Yang-Mills (which, of course, contains complete information about all tree form-factors of non-supersymmetric Yang-Mills). However, the zero-curvature representation is two-dimensional in that case, the construction is much more involved, and the complete solution has not been obtained yet.

In conclusion, I would like to stress that perturbiners provide probably the simplest way of describing tree amplitudes, involve nice mathematical constructions and reveal similar structures underlying different theories. I hope they will find their place in future developments.

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