COMPOSITIONAL CONSTRUCTION OF INFINITE ABSTRACTIONS FOR
NETWORKS OF STOCHASTIC CONTROL SYSTEMS

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Abstract. This paper is concerned with a compositional approach for constructing infinite abstractions of interconnected discrete-time stochastic control systems. The proposed approach uses the interconnection matrix and joint dissipativity-type properties of subsystems and their abstractions described by a new notion of so-called stochastic storage functions. The interconnected abstraction framework is based on new notions of so-called stochastic simulation functions, constructed compositionally using stochastic storage functions of components. Using stochastic simulation functions one can quantify the distance between original interconnected stochastic control systems and interconnected abstractions in the probabilistic setting. Accordingly, one can leverage the proposed results to perform analysis and synthesis over abstract interconnected systems, and then carry the results over concrete ones. In the first part of the paper, we derive dissipativity-type compositional reasoning for the quantification of the distance in probability between the interconnection of stochastic control subsystems and that of their abstractions. Moreover, we consider specifications expressed as syntactically co-safe linear temporal logic formulae and show how a synthesized policy for the abstract system can be refined to a policy for the original system while providing guarantee on the probability of satisfaction. In the second part of the paper, we focus on a class of discrete-time nonlinear stochastic control systems with independent noises in the abstract and concrete subsystems, and propose a computational scheme to construct abstractions together with their corresponding stochastic storage functions. We demonstrate the effectiveness of the proposed results by constructing an abstraction (totally 3 dimensions) of the interconnection of three discrete-time nonlinear stochastic control subsystems (together 222 dimensions) in a compositional fashion such that the compositionality condition does not require any constraint on the number or gains of the subsystems. We also employ the abstraction as a substitute to synthesize a controller enforcing a syntactically co-safe linear temporal logic specification.

1. Introduction

Large-scale interconnected systems have received significant attentions in the last few years due to their presence in real life systems including power networks and air traffic control. Each complex real-world system can be regarded as an interconnected system composed of several subsystems. Since these large-scale network of systems are inherently difficult to analyze and control, one can develop compositional schemes and employ the abstractions of the given networks as a replacement in the controller design process. In other words, in order to overcome the computational complexity in large-scale interconnected systems, one can abstract the original concrete system by a simpler one with potentially a lower dimension. Those abstractions allow us to design controllers for them, and then refine the controllers to the ones for the concrete complex systems, while provide us with the quantified errors in this controller synthesis detour.

In the past few years, there have been several results on the construction of (in)finite abstractions for stochastic systems. Existing results for continuous-time systems include infinite approximation techniques for jump-diffusion systems [JP09], finite bisimilar abstractions for incrementally stable stochastic switched systems [ZAG15] and randomly switched stochastic systems [ZA14], and finite bisimilar abstractions for incrementally stable stochastic control systems without discrete dynamics [ZMEM14]. Recently, compositional construction of infinite abstractions is discussed in [ZRME17] using small-gain type conditions and of finite bisimilar abstractions in [MSSM17] based on a new notion of disturbance bisimilarity relation.

For discrete-time stochastic models with continuous state spaces, finite abstractions are initially employed in [APLS08] for formal synthesis of this class of systems. The algorithms are improved in terms of scalability in
and implemented in the tool FAUST\cite{SA13, Sou14}. Extension of the techniques to infinite horizon properties is proposed in\cite{TA11} and formal abstraction-based policy synthesis is discussed in\cite{TMKA13}. A new notion of approximate similarity relation is proposed in\cite{HSA17} that takes into account both deviation in stochastic evolution and in outputs of the two systems. Recently, compositional construction of finite abstractions is discussed in\cite{SAM15} and\cite{LSZ18} using dynamic Bayesian networks and finite Markov decision processes, respectively, and of infinite abstractions in\cite{LSMZ17} using small-gain type conditions all for discrete-time stochastic control systems.

In this paper, we provide a compositional approach for the construction of infinite abstractions of interconnected discrete-time stochastic control systems using the interconnection matrix and joint dissipativity-type properties of subsystems and their abstractions. Our abstraction framework is based on a new notion of stochastic simulation functions under which an abstraction, which is itself a discrete-time stochastic control system with potentially a lower dimension, performs as a substitute in the controller design process. The stochastic simulation function is used to quantify the error in probability in this detour controller synthesis scheme. As a consequence, one can leverage our proposed results to synthesize a policy that satisfies a temporal logic property over the abstract interconnected system and then refine this policy back for the concrete interconnected one.

Our proposed approach differs from the one in\cite{LSMZ17} in three directions. First and foremost, rather than using small-gain type reasoning, we employ the dissipativity-type compositional reasoning that may not require any constraint on the number or gains of the subsystems for some interconnection topologies (cf. case study). Second, we provide a scheme for the construction of infinite abstractions for a class of discrete-time nonlinear stochastic control systems whereas the construction scheme in\cite{LSMZ17} only handles linear systems. As our third contribution, we consider a fragment of linear temporal logic (LTL) known as syntactically co-safe linear temporal logic (scLTL)\cite{KV01} whereas the results in\cite{LSMZ17} only deals with bounded invariant. In particular, given such a specification over the concrete system, we construct an epsilon-perturbed specification over the abstract system whose probability of satisfaction gives a lower bound for the probability of satisfaction in the concrete domain.

It should be also noted that we do not put any restriction on the sources of uncertainties in the concrete and abstract systems. Thus our result is more general than\cite{ZRME17}, where the noises in the concrete and abstract systems are assumed to be the same, which means the abstraction has access to the noise of the concrete system. Finally, we show the effectiveness of dissipativity-type compositional reasoning for large-scale systems by first constructing an abstraction (totally 3 dimensions) of the interconnection of three discrete-time nonlinear stochastic control subsystems (together 222 dimensions) in a compositional fashion. Then, we employ the abstraction as a substitute to synthesize a controller enforcing a syntactically co-safe linear temporal logic specification over the concrete network.

2. Discrete-Time Stochastic Control Systems

2.1. Preliminaries. We consider a probability space \((\Omega, \mathcal{F}_\Omega, P_\Omega)\), where \(\Omega\) is the sample space, \(\mathcal{F}_\Omega\) is a sigma-algebra on \(\Omega\) comprising subsets of \(\Omega\) as events, and \(P_\Omega\) is a probability measure that assigns probabilities to events. We assume that random variables introduced in this article are measurable functions of the form \(X : (\Omega, \mathcal{F}_\Omega) \rightarrow (S_X, \mathcal{F}_X)\). Any random variable \(X\) induces a probability measure on its space \((S_X, \mathcal{F}_X)\) as \(\text{Prob}(A) = P_\Omega(X^{-1}(A))\) for any \(A \in \mathcal{F}_X\). We often directly discuss the probability measure on \((S_X, \mathcal{F}_X)\) without explicitly mentioning the underlying probability space and the function \(X\) itself.

A topological space \(S\) is called a Borel space if it is homeomorphic to a Borel subset of a Polish space (i.e., a separable and completely metrizable space). Examples of a Borel space are the Euclidean spaces \(\mathbb{R}^n\), its Borel subsets endowed with a subspace topology, as well as hybrid spaces. Any Borel space \(S\) is assumed to be endowed with a Borel sigma-algebra, which is denoted by \(\mathcal{B}(S)\). We say that a map \(f : S \rightarrow Y\) is measurable whenever it is Borel measurable.
2.2. Notation. The following notation is used throughout the paper. We denote the set of nonnegative integers by \( \mathbb{N} := \{0, 1, 2, \ldots\} \) and the set of positive integers by \( \mathbb{N}_{\geq 1} := \{1, 2, 3, \ldots\} \). The symbols \( \mathbb{R} \), \( \mathbb{R}_{>0} \), and \( \mathbb{R}_{\geq 0} \) denote the set of real, positive and nonnegative real numbers, respectively. Given a vector \( x \in \mathbb{R}^n \), \( \|x\| \) denotes the Euclidean norm of \( x \). Symbols \( I_n \) and \( 1_n \) denote respectively the identity matrix in \( \mathbb{R}^{n \times n} \) and the column vector in \( \mathbb{R}^{n \times 1} \) with all elements equal to one. Given \( N \) vectors \( x_i \in \mathbb{R}^n \), \( n_i \in \mathbb{N}_{\geq 1} \), and \( i \in \{1, \ldots, N\} \), we use \( x = [x_1; \ldots; x_N] \) to denote the corresponding vector of dimension \( \sum_i n_i \). We denote by \( \text{diag}(a_1, \ldots, a_N) \) a diagonal matrix in \( \mathbb{R}^{N \times N} \) with diagonal matrix entries \( a_1, \ldots, a_N \) starting from the upper left corner. Given functions \( f_i : X_i \to Y_i \) for any \( i \in \{1, \ldots, N\} \), their Cartesian product \( \prod_{i=1}^N f_i : \prod_{i=1}^N X_i \to \prod_{i=1}^N Y_i \) is defined as \( (\prod_{i=1}^N f_i)(x_1, \ldots, x_N) = [f_1(x_1); \ldots; f_N(x_N)] \). For any set \( A \) we denote by \( A^\infty \) the Cartesian product of a countable number of copies of \( A \), i.e., \( A^\infty = \prod_{k=0}^\infty A \). A function \( \gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \), is said to be a class \( \mathcal{K} \) function if it is continuous, strictly increasing and \( \gamma(0) = 0 \). A class \( \mathcal{K} \) function \( \gamma \) is said to be a class \( \mathcal{K}_\infty \) if \( \gamma(r) \to \infty \) as \( r \to \infty \).

2.3. Discrete-Time Stochastic Control Systems. We consider stochastic control systems in discrete time (dt-SCS) defined over a general state space and characterized by the tuple

\[
\Sigma = (X, U, W, \varsigma, f, Y_1, Y_2, h_1, h_2),
\]

where \( X \) is a Borel space as the state space of the system. We denote by \( (X, \mathcal{B}(X)) \) the measurable space with \( \mathcal{B}(X) \) being the Borel sigma-algebra on the state space. Sets \( U \) and \( W \) are Borel spaces as the external and internal input spaces of the system. Notation \( \varsigma \) denotes a sequence of independent and identically distributed (i.i.d.) random variables on a set \( V_\varsigma \) as

\[
\varsigma := \{\varsigma(k) : \Omega \to V_\varsigma, \ k \in \mathbb{N}\}.
\]

The map \( f : X \times U \times W \times V_\varsigma \to X \) is a measurable function characterizing the state evolution of the system. Finally, sets \( Y_1 \) and \( Y_2 \) are Borel spaces as the external and internal output spaces of the system, respectively. Maps \( h_1 : X \to Y_1 \) and \( h_2 : X \to Y_2 \) are measurable functions that map a state \( x \in X \) to its external and internal outputs \( y_1 = h_1(x) \) and \( y_2 = h_2(x) \), respectively.

For given initial state \( x(0) \in X \) and input sequences \( \nu(\cdot) : \mathbb{N} \to U \) and \( w(\cdot) : \mathbb{N} \to W \), evolution of the state of dt-SCS \( \Sigma \) can be written as

\[
\Sigma : \left\{ \begin{array}{l}
x(k + 1) = f(x(k), \nu(k), w(k), \varsigma(k)) \\
y_1(k) = h_1(x(k)) \\
y_2(k) = h_2(x(k)),
\end{array} \right. \quad k \in \mathbb{N}.
\]

Remark 2.1. The above definition can be generalized by allowing the set of valid external inputs to depend on the current state and internal input of the system, i.e., to include \( \{U(x, w) \mid x \in X, w \in W\} \) in the definition of dt-SCS which is a family of non-empty measurable subsets of \( U \) with the property that

\[
K := \{x, u, w : x \in X, w \in W, u \in U(x, w)\}
\]

is measurable in \( X \times U \times W \). For the succinct presentation of the results, we assume in this paper that the set of valid external inputs is the whole external input space: \( U(x, w) = U \) for all \( x \in X \) and \( w \in W \), but the obtained results are generally applicable.

Given the dt-SCS in (2.1), we are interested in Markov policies to control the system.

Definition 2.2. A Markov policy for the dt-SCS \( \Sigma \) in (2.1) is a sequence \( \rho = (\rho_0, \rho_1, \rho_2, \ldots) \) of universally measurable stochastic kernels \( \rho_n \), defined on the input space \( U \) given \( X \times W \) and such that for all \( (x_n, w_n) \in X \times W \), \( \rho_n(U(\{x_n, w_n\})) = 1 \). The class of all such Markov policies is denoted by \( \Pi_M \).

We associate respectively to \( U \) and \( W \) to be collections of sequences \( \{\nu(k) : \Omega \to U, \ k \in \mathbb{N}\} \) and \( \{w(k) : \Omega \to W, \ k \in \mathbb{N}\} \), in which \( \nu(k) \) and \( w(k) \) are independent of \( \varsigma(t) \) for any \( k, t \in \mathbb{N} \) and \( t \geq k \). For any initial state \( a \in X \), \( \nu(\cdot) \in \mathcal{U} \), and \( w(\cdot) \in \mathcal{W} \), the random sequences \( x_{aw} : \Omega \times \mathbb{N} \to X \), \( y_{1aw} : \Omega \times \mathbb{N} \to Y_1 \), and \( y_{2aw} : \Omega \times \mathbb{N} \to Y_2 \),
and $y_{aw}^2 : \Omega \times \mathbb{N} \to Y_2$ that satisfy (2.2) are called respectively the solution process and external and internal output trajectory of $\Sigma$ under external input $\nu$, internal input $w$ and initial state $a$.

In the sequel we assume that the state space $X$ of $\Sigma$ is a subset of $\mathbb{R}^n$. System $\Sigma$ is called finite if $X, U, W$ are finite sets and infinite otherwise. In this paper we are interested in studying interconnected discrete-time stochastic control systems without internal inputs and outputs that result from the interconnection of dt-SCS having both internal and external inputs and outputs. In this case, the interconnected dt-SCS without internal input and output is indicated by the simplified tuple $(X, U, \varsigma, f, Y, h)$ with $f : X \times U \times V_\varsigma \to X$.

3. Stochastic Storage and Simulation Functions

In this section, we first introduce a notion of so-called stochastic storage functions for the discrete-time stochastic control systems with both internal and external inputs which is adapted from the notion of storage functions from dissipativity theory [AMP10]. We then define a notion of stochastic simulation functions for systems with only external input. We use these definitions to quantify closeness of two dt-SCS.

**Definition 3.1.** Consider dt-SCS $\Sigma = (X, U, W, \varsigma, f, Y_1, Y_2, h_1, h_2)$ and $\hat{\Sigma} = (\hat{X}, \hat{U}, \hat{W}, \hat{\varsigma}, \hat{f}, \hat{Y}_1, \hat{Y}_2, \hat{h}_1, \hat{h}_2)$ with the same external output spaces. A function $V : X \times \hat{X} \to \mathbb{R}_{\geq 0}$ is called a stochastic storage function (SSF) from $\hat{\Sigma}$ to $\Sigma$ if there exist $\alpha \in \mathcal{K}_\infty$, $\kappa \in \mathcal{K}$, $\rho_{\text{ext}} \in \mathcal{K}_\infty \cup \{0\}$, some matrices $G, \hat{G}, H$ of appropriate dimensions, and some symmetric matrix $\hat{X}$ of appropriate dimension with conformal block partitions $\hat{X}_{ij}$, $i, j \in \{1, 2\}$, such that for any $x \in X$ and $\hat{x} \in \hat{X}$ one has

$$\alpha(||h(x) - \hat{h}(\hat{x})||) \leq V(x, \hat{x}), \quad (3.1)$$

and $\forall x \in X \ \forall \hat{x} \in \hat{X} \ \forall w \in W \ \exists \nu \in U$ such that $\forall \hat{w} \in \hat{W} \ \forall w \in W$ one obtains

$$E\left[V(x(k+1), \hat{x}(k+1)) \bigg| x(k) = x, \hat{x}(k) = \hat{x}, w(k) = w, \hat{w}(k) = \hat{w}, \nu(k) = \nu, \hat{\nu}(k) = \hat{\nu} \right] - V(x, \hat{x}) \leq -\kappa(V(x, \hat{x})) + \rho_{\text{ext}}(||\hat{\nu}||) + \psi, \quad (3.2)$$

for some $\psi \in \mathbb{R}_{\geq 0}$.

We use notation $\hat{\Sigma} \preceq_{\varphi} \Sigma$ if there exists a storage function $V$ from $\hat{\Sigma}$ to $\Sigma$, in which $\hat{\Sigma}$ is considered as an abstraction of concrete system $\Sigma$.

**Remark 3.2.** The second condition above implies implicitly existence of a function $\nu = \nu_\varsigma(x, \hat{x}, \hat{\nu})$ for satisfaction of (3.2). This function is called the interface function and can be used to refine a synthesized policy $\hat{\nu}$ for $\hat{\Sigma}$ to a policy $\nu$ for $\Sigma$.

For the interconnected dt-SCS, the above notion reduces to the following definition.

**Definition 3.3.** Consider two dt-SCS $\Sigma = (X, U, \varsigma, f, Y, h)$ and $\hat{\Sigma} = (\hat{X}, \hat{U}, \hat{\varsigma}, \hat{f}, Y, \hat{h})$ with the same output spaces. A function $V : X \times \hat{X} \to \mathbb{R}_{\geq 0}$ is called a stochastic simulation function (SSF) from $\hat{\Sigma}$ to $\Sigma$ if

- $\exists \alpha \in \mathcal{K}_\infty$ such that
  $$\forall x \in X, \forall \hat{x} \in \hat{X}, \alpha(||h(x) - \hat{h}(\hat{x})||) \leq V(x, \hat{x}), \quad (3.3)$$

- $\forall x \in X, \hat{x} \in \hat{X}, \hat{\nu} \in \hat{U}, \exists \nu \in U$ such that
  $$E\left[V(x(k+1), \hat{x}(k+1)) \bigg| x(k) = x, \hat{x}(k) = \hat{x}, \nu(k) = \nu, \hat{\nu}(k) = \hat{\nu} \right] - V(x, \hat{x}) \leq -\kappa(V(x, \hat{x})) + \rho_{\text{ext}}(||\hat{\nu}||) + \psi, \quad (3.4)$$

for some $\kappa \in \mathcal{K}$, $\rho_{\text{ext}} \in \mathcal{K}_\infty \cup \{0\}$, and $\psi \in \mathbb{R}_{\geq 0}$.
The next theorem shows the usefulness of SSF in comparing output trajectories of two dt-SCS in a probabilistic sense.

**Theorem 3.4.** Let \( \Sigma = (X, U, \zeta, f, Y, h) \) and \( \hat{\Sigma} = (\hat{X}, \hat{U}, \hat{\zeta}, \hat{f}, \hat{Y}, \hat{h}) \) be two dt-SCS with the same output spaces. Suppose \( V \) is an SSF from \( \hat{\Sigma} \) to \( \Sigma \), and there exists a constant \( 0 < \hat{\kappa} < 1 \) such that function \( \kappa \in \mathcal{K} \) in (3.3) satisfies \( \kappa(r) \geq \hat{\kappa} r \forall r \in \mathbb{R}_{\geq 0} \). For any random variables \( a \) and \( \hat{a} \) as the initial states of the two dt-SCS and any external input trajectory \( \hat{\nu}(...) \in \hat{U} \) preserving Markov property for the closed-loop \( \hat{\Sigma} \), there exists an input trajectory \( \nu(...) \in U \) of \( \Sigma \) through the interface function associated with \( V \) such that the following inequality holds:

\[
P \left\{ \sup_{0 \leq k \leq T_d} \| y_{\nu}(k) - \hat{y}_{\hat{\nu}}(k) \| \geq \varepsilon \mid [a; \hat{a}] \right\} \leq \delta \tag{3.5}
\]

\[
\delta := \left\{ \begin{array}{ll}
1 - (1 - \frac{V(a, \hat{a})}{\alpha(\varepsilon)})T_d & \text{if } \alpha(\varepsilon) \geq \frac{1}{\hat{\kappa}}, \\
\frac{V(a, \hat{a})}{\alpha(\varepsilon)}(1 - \hat{\kappa})T_d + \frac{1}{\alpha(\varepsilon)}(1 - (1 - \hat{\kappa})T_d) & \text{if } \alpha(\varepsilon) < \frac{1}{\hat{\kappa}},
\end{array} \right.
\]

provided that there exists a constant \( \hat{\psi} \geq 0 \) satisfying \( \hat{\psi} \geq \rho_{\text{ext}}(\| \hat{\nu} \|_{\infty}) + \psi \).

The proof of Theorem 3.4 is provided in the Appendix. The results shown in Theorem 3.4 provide closeness of output behaviours of two systems in finite-time horizon. We can extend the result to infinite-time horizon given that \( \hat{\psi} = 0 \) as stated in the following corollary.

**Corollary 3.5.** Let \( \Sigma \) and \( \hat{\Sigma} \) be two dt-SCS without internal inputs and outputs and with the same output spaces. Suppose \( V \) is an SSF from \( \hat{\Sigma} \) to \( \Sigma \) such that \( \rho_{\text{ext}}(...) \equiv 0 \) and \( \psi = 0 \). For any random variables \( a \) and \( \hat{a} \) as the initial states of the two dt-SCS and any external input trajectory \( \hat{\nu}(...) \in \hat{U} \) preserving Markov property for the closed-loop \( \hat{\Sigma} \), there exists \( \nu(...) \in U \) of \( \Sigma \) through the interface function associated with \( V \) such that the following inequality holds:

\[
P \left\{ \sup_{0 \leq k \leq \infty} \| y_{\nu}(k) - \hat{y}_{\hat{\nu}}(k) \| \geq \varepsilon \mid [a; \hat{a}] \right\} \leq \frac{V(a, \hat{a})}{\alpha(\varepsilon)}.
\]

The proof of Corollary 3.5 is provided in the Appendix.

**Remark 3.6.** Note that \( \psi = 0 \) is possible mainly if concrete and abstract systems are both continuous-space but possibly with different dimensions and share the same multiplicative noise. Depending on the dynamic, function \( \rho_{\text{ext}}(...) \) can be identically zero (cf. Section 4 and case study).

Note that the relation (3.5) lower bounds the probability that the Euclidean distance between any output trajectory of the abstract model and the corresponding one of the concrete model remains close and is different from the probabilistic version discussed for finite state, discrete-time labeled Markov chains in [DLT08], which hinges on the absolute difference between transition probabilities over sets covering the state space. However, one can still use the results in Theorem 3.4 and design controllers for abstractions and refine them to the ones for concrete systems while providing the probability of satisfaction over the concrete domain which is discussed in details in the next section.

### 4. Probability of Satisfaction for Properties Expressed as scLTL

Consider a dt-SCS \( \Sigma = (X, U, \zeta, f, Y, h) \) and a measurable target set \( B \subset Y \). We say that an output trajectory \( \{ y(k) \}_{k \geq 0} \) reaches a target set \( B \) within time interval \([0, T_d]\) \subset \mathbb{N} \), if there exists a \( k \in [0, T_d] \) such that \( y(k) \in B \). This bounded reaching of \( B \) is denoted by \( \Diamond \leq T_d \{ y \in B \} \) or briefly \( \Diamond \leq T_d B \). For \( T_d \to \infty \), we denote the reachability property as \( \Diamond B \), i.e., eventually \( B \). For a dt-SCS \( \Sigma \) with policy \( \rho \), we want to compute the probability that an output trajectory reaches \( B \) within the time horizon \( T_d \in \mathbb{N} \), i.e., \( P(\Diamond \leq T_d B) \). The reachability probability is the probability that the target set \( B \) is eventually reached and is denoted by \( P(\Diamond B) \).
More complex properties can be described using temporal logic. Consider a set of atomic propositions \( AP \) and the alphabet \( \Sigma_a := 2^AP \). Let \( \omega = \omega(0), \omega(1), \omega(2), \ldots \in \Sigma_a^\omega \) be an infinite word, that is, a string composed of letters from \( \Sigma_a \). Of interest are atomic propositions that are relevant to the dt-SCS via a measurable labeling function \( L \) from the output space to the alphabet as \( L : Y \rightarrow \Sigma_a \). Output trajectories \( \{y(k)\}_{k \geq 0} \in Y^\mathbb{N} \) can be readily mapped to the set of infinite words \( \Sigma_a^\omega \), as

\[
\omega = L(\{y(k)\}_{k \geq 0}) := \{ \omega \in \Sigma_a^\omega \mid \omega(k) = L(y(k)) \}.
\]

Consider LTL properties with syntax \([BKL08]\)

\[\omega \]

the satisfaction relation are defined recursively over 

\[L \]

function

\[\exists \]

of models known as Deterministic Finite-state Automata (DFA).

\[\omega \]

Let \( \omega_k = \omega(k), \omega(k+1), \omega(k+2), \ldots \) be a subsequence (postfix) of \( \omega \), then the satisfaction relation between \( \omega \) and a property \( \phi \), expressed in LTL, is denoted by \( \omega \models \phi \) (or equivalently \( \omega_k \models \phi \)). The semantics of the satisfaction relation are defined recursively over \( \omega_k \) and the syntax of the LTL formula \( \phi \). An atomic proposition \( p \in AP \) is satisfied by \( \omega_k \), i.e., \( \omega_k \models p \), iff \( p \in \omega(k) \). Furthermore, \( \omega_k \models \neg \phi \) if \( \omega_k \not\models \phi \) and we say that \( \omega_k \models \phi_1 \land \phi_2 \) if \( \omega_k \models \phi_1 \) and \( \omega_k \models \phi_2 \). The next operator \( \omega_k \models \bigcirc \phi \) holds if the property holds at the next time instance \( \omega_{k+1} \models \phi \). We denote by \( \bigcirc^j \), \( j \in \mathbb{N} \), \( j \) times composition of the next operator. With a slight abuse of the notation, one has \( \bigcirc^0 \phi = \phi \) for any property \( \phi \). The temporal until operator \( \omega_k \models \phi \lor \phi_2 \) holds if \( \exists i \in \mathbb{N} : \omega_{k+i} \models \phi_2 \), and \( \forall j \in \mathbb{N} : 0 \leq j < i, \omega_{k+j} \not\models \phi_1 \). Based on these semantics, operators such as disjunction (\( \lor \)) can also be defined through the negation and conjunction: \( \omega_k \models \phi_1 \lor \phi_2 \iff \omega_k \models \neg (\neg \phi_1 \land \neg \phi_2) \).

We are interested in a fragment of LTL properties known as syntactically co-safe linear temporal logic (scLTL) \([KV01]\). This fragment is defined as follows.

**Definition 4.1.** An scLTL over a set of atomic propositions \( AP \) has syntax

\[\phi := \text{true} \mid p \mid \neg \phi \mid \phi \land \phi \mid \bigcirc \phi \mid \phi \lor \phi \mid \phi \rightarrow \phi \mid \bigcirc \phi \lor \phi \mid \neg \phi \lor \phi \mid \bigcirc \phi \land \phi \mid \phi \land \bigcirc \phi \mid \phi \rightarrow \bigcirc \phi\]

with \( p \in AP \).

Even though scLTL formulas are defined over infinite words (as in LTL formulae), their satisfaction is guaranteed in finite time. Any infinite word \( \omega \in \Sigma_a^\omega \) satisfying an scLTL formula \( \phi \) has a finite word \( \omega_f \in \Sigma_a^n \), \( n \in \mathbb{N} \), as its prefix such that all infinite words with prefix \( \omega_f \) also satisfy the formula \( \phi \). We denote the language of such finite prefixes associated with an scLTL formula \( \phi \) by \( L_f(\phi) \).

In the remainder, we consider scLTL properties since their verification can be performed via a reachability property over a finite state automaton \([KV01] [BYG17]\). For this purpose, in this section we introduce a class of models known as Deterministic Finite-state Automata (DFA).

**Definition 4.2.** A DFA is a tuple \( \mathcal{A} = (Q, q_0, \Sigma_a, F_a, \tau) \), where \( Q \) is a finite set of locations, \( q_0 \in Q \) is the initial location, \( \Sigma_a \) is a finite set (a.k.a. alphabet), \( F_a \subseteq Q \) is a set of accept locations, and \( \tau : Q \times \Sigma_a \rightarrow Q \) is a transition function.

A finite word composed of letters of the alphabet, i.e., \( \omega_f = (\omega_f(0), \ldots, \omega_f(n)) \in \Sigma_a^n \), is accepted by a DFA \( \mathcal{A} \) if there exists a finite run \( q = (q(0), \ldots, q(n+1)) \in Q_f^{n+2} \) such that \( q(0) = q_0, q(i+1) = \tau(q(i), \omega_f(i)) \) for all \( 0 \leq i \leq n \) and \( q(n+1) \in F_a \). The accepted language of \( \mathcal{A} \), denoted \( L(\mathcal{A}) \), is the set of all words accepted by \( \mathcal{A} \). For every scLTL property \( \phi \), cf. Definition 4.1 there exists a DFA \( \mathcal{A}_\phi \) such that

\[L_f(\phi) = L(\mathcal{A}_\phi) = L(\mathcal{A}_\phi) = L(\mathcal{A}_\phi).
\]

As a result, the satisfaction of the property \( \phi \) now becomes equivalent to the reaching of the accept locations in the DFA. We use the DFA \( \mathcal{A} \) to specify properties of dt-SCS \( \Sigma = (X, U, c, f, Y, h) \) as follows. Recall that \( L : Y \rightarrow \Sigma_a \) is a given measurable function. To each output \( y \in Y \) it assigns the letter \( L(y) \in \Sigma_a \). Given a policy \( \rho \), we can define the probability that an output trajectory of \( \Sigma \) satisfies an scLTL property \( \phi \) over time horizon \([0, T_d]\), i.e. \( \mathbb{P}(\omega_f \in L(\mathcal{A}_\phi), |\omega_f| \leq T_d + 1) \), with \( |\omega_f| \) denoting the length of \( \omega_f \).

The following example provides an automaton associated with a reach-avoid specification.
Example 4.3. Consider the DFA depicted in Figure 1 left with $Q = \{q_0, q_1, q_2, q_3\}$, initial location $q_0$, $\Sigma_a = \{a, b, c\}$, and $F_a = \{q_2\}$. This DFA is associated to a reach-avoid specification as described in the sequel. The set of words accepted by the DFA is \{b, ab, a^2b, a^3b, \ldots\}. Any measurable labeling function $L : Y \to \{a, b, c\}$ induces a partition of the output space $Y$ as follows:

$$A, B, C \subseteq Y \text{ and } A \cap B \cap C = \emptyset.$$

Sets $A, B, C$ are measurable subsets of $Y$, and their intersection is empty. We can interpret $A$ as the safe set and $B$ as the target set. Thus output trajectories of a d-t-SCS $\Sigma$ satisfies the specification $(a U b)$ if and only if their associated words are accepted by this DFA.

In the rest of this article, we focus on the computation of probability of $\omega \in L(A_\phi)$ over bounded intervals. In other words, we fix a time horizon $T_d$ and compute $P(\omega(0)\omega(1)\ldots\omega(T_d) \in L(A_\phi))$. Suppose $\Sigma$ and $\hat{\Sigma}$ are two d-t-SCS for which the results of Theorem 3.4 hold. Consider a labeling function $L$ defined on their output space and an scLTL specification $\phi$ with DFA $A_\phi$. In the following we show how to construct a DFA $A_{\hat{\phi}}$ of another specification $\hat{\phi}$ and a new labeling function $L_\epsilon$ such that the satisfaction probability of $\hat{\phi}$ by output trajectories of $\hat{\Sigma}$ and labeling function $L_\epsilon$ gives a lower bound on the satisfaction probability of $\phi$ by output trajectories of $\Sigma$ and labeling function $L$.

Consider the DFA $A_\phi = (Q_\phi, q_0, \Sigma_a, F_a, t)$ and the labeling function $L : Y \to \Sigma_a$. The new DFA $A_{\hat{\phi}} = (\hat{Q}_\phi, q_0, \hat{\Sigma}_a, \hat{F}_a, \hat{t})$ will be constructed by adding one absorbing location $q_{abs}$ and one letter $\phi_o$ as $\hat{Q}_\phi := Q_\phi \cup \{q_{abs}\}$ and $\hat{\Sigma}_a := \Sigma_a \cup \{\phi_o\}$. The initial and accept locations are the same with $A_\phi$. The transition relation is defined, $\forall q \in Q, \forall \sigma \in \Sigma_a$, as

$$\hat{t}(q, \sigma) := \begin{cases} t(q, \sigma) & \text{if } q \in Q_\phi, \sigma \in \Sigma_a, \\ q_{abs} & \text{if } \sigma = \phi_o, q \in \hat{Q}_\phi, \\ q_{abs} & \text{if } q = q_{abs}, \sigma \in \hat{\Sigma}_a. \end{cases}$$

In other words, we add an absorbing state $q_{abs}$ and all the states will jump to this absorbing state with label $\phi_o$. As an example the modified DFA of the reach-avoid specification in Figure 1 left is plotted in Figure 1 right.

Figure 1. DFA $A_\phi$ (left) and modified DFA $A_{\hat{\phi}}$ (right) of the reach-avoid specification $(a U b)$.

The new labeling function $L_\epsilon : Y \to \hat{\Sigma}_a$ is constructed using the $\epsilon$-perturbation of subsets of $Y$. Define for any Borel measurable set $A \subseteq Y$, its $\epsilon$-perturbed version $A'_\epsilon$ as

$$A'_\epsilon := \{y \in A \mid \|\tilde{y} - y\| \geq \epsilon \text{ for all } \tilde{y} \in Y \setminus A\}.$$
Remark that set \( \mathcal{A}' \) is just an \( \epsilon \)-deflated version of \( \mathcal{A} \) and without loss of generality we assume it is nonempty. Note that Borel measurability of \( \mathcal{A}' \) is guaranteed for normed spaces. Then \( L'(y) = L(y) \) for any \( y \in \cup_{a \in \Sigma_1}[L^{-1}(a)]' \), otherwise \( L'(y) = \phi \). Such a construction enables us to have the next lemma that relates satisfaction of specifications by output trajectories of two dt-SCS.

**Lemma 4.4.** Suppose output trajectories of two dt-SCS \( \Sigma \) and \( \tilde{\Sigma} \) satisfy the inequality
\[
\sup_{0 \leq k \leq T_d} \|y(k) - \tilde{y}(k)\| < \epsilon,
\]
for some time bound \( T_d \) and \( \epsilon > 0 \). Then \( y(\cdot) \models \phi \) with labeling function \( L \) if \( \tilde{y}(\cdot) \models \phi \) over time interval \([0, T_d]\) with labeling function \( L' \) and modified specification \( \hat{\phi} \).

The proof of Lemma 4.4 is provided in the Appendix. Next theorem presents the core result of this section.

**Theorem 4.5.** Suppose \( \Sigma \) and \( \tilde{\Sigma} \) are two dt-SCS for which inequality (3.5) holds with the pair \((\epsilon, \delta)\) and any time bound \( T_d \). Suppose a specification \( \phi \) and a labeling function \( L \) is defined for \( \Sigma \). The following inequality holds for the labeling function \( L' \) on \( \tilde{\Sigma} \) and modified specification \( \hat{\phi} \):
\[
P(\tilde{y}(\cdot) \models \hat{\phi}) - \delta \leq P(y(\cdot) \models \phi),
\]
where the satisfaction is over time interval \([0, T_d]\).

The proof of Theorem 4.5 is provided in the Appendix. In order to get an upper bound for \( P(y(\cdot) \models \phi) \), we need to define for any Borel measurable set \( A \subset Y \), its \((\epsilon)\)-perturbed version \( A^- \) as
\[
A^- := \{ y \in Y \mid \exists \hat{y} \in A \text{ with } \|y - \hat{y}\| < \epsilon \}.
\]
A new labeling map \( L^- : Y \rightarrow 2^{\Sigma_i} \) is constructed using the \((\epsilon)\)-perturbation of subsets of \( Y \) as
\[
L^-(y) := \bigcup \{ a \in \Sigma_i \mid y \in [L^{-1}(a)]^- \}.
\]

**Theorem 4.6.** Suppose \( \Sigma \) and \( \tilde{\Sigma} \) are two dt-SCS for which inequality (3.5) holds with the pair \((\epsilon, \delta)\) and any time bound \( T_d \). Suppose a specification \( \phi \) and a labeling function \( L \) is defined for \( \Sigma \). The following inequality holds for the labeling function \( L^- \) defined in (4.2) on \( \tilde{\Sigma} \):
\[
P(y(\cdot) \models \phi) \leq P(\tilde{y}(\cdot) \models \phi) + \delta,
\]
where the satisfaction is over time interval \([0, T_d]\) and the probability in the right-hand side is computed for having \( \tilde{y}(\cdot) \models \phi \) for any choice of non-determinism introduced by the labeling map \( L^- \).

The proof is similar to that of Theorem 4.5 and is omitted here due to lack of space.

5. Compositional Abstractions for Interconnected Systems

In this section, we analyze networks of control systems and show how to construct their abstractions together with the corresponding simulation functions by using stochastic storage functions for the subsystems.

First, we first provide a formal definition of interconnection between discrete-time stochastic control subsystems.

**Definition 5.1.** Consider \( N \in \mathbb{N}_{\geq 1} \) stochastic control subsystems \( \Sigma_i = (X_i, U_i, W_i, \varsigma_i, f_i, Y_{i1}, Y_{i2}, h_{i1}, h_{i2}) \), \( i \in \{1, \ldots, N\} \), and a static matrix \( M \) of an appropriate dimension defining the coupling of these subsystems. The interconnection of \( \Sigma_i \) for any \( i \in \{1, \ldots, N\} \), is the interconnected stochastic control system \( \Sigma = (X, U, \varsigma, f, Y, h) \), denoted by \( I(\Sigma_1, \ldots, \Sigma_N) \), such that \( X := \prod_{i=1}^N X_i \), \( U := \prod_{i=1}^N U_i \), function \( f := \prod_{i=1}^N f_i \), \( Y := \prod_{i=1}^N Y_{i1} \), and \( h = \prod_{i=1}^N h_{i1} \), with the internal variables constrained by:
\[
[w_1; \ldots; w_N] = M[h_{21}(x_1); \ldots; h_{2N}(x_N)].
\]
5.1. Compositional Abstractions of Interconnected Systems. This subsection contains one of the main contributions of the paper. Assume that we are given \( N \) stochastic control subsystems \( \Sigma_i = (X_i, U_i, W_i, \varsigma_i, f_i, Y_{1i}, Y_{2i}, h_{1i}, h_{2i}) \) together with their corresponding abstractions \( \tilde{\Sigma}_i = (\tilde{X}_i, \tilde{U}_i, \tilde{W}_i, \tilde{\varsigma}_i, \tilde{f}_i, \tilde{Y}_{1i}, \tilde{Y}_{2i}, \tilde{h}_{1i}, \tilde{h}_{2i}) \) with SStF \( V_i \) from \( \tilde{\Sigma}_i \) to \( \Sigma_i \). We use \( \alpha_i, \kappa_i, \rho_{\text{ext}}, H_i, G_i, \hat{G}_i, \bar{X}_i, \bar{X}_i^{11}, \bar{X}_i^{12}, \bar{X}_i^{21}, \text{and} \bar{X}_i^{22} \) to denote the corresponding functions, matrices, and their corresponding conformal block partitions appearing in Definition 3.1.

In the next theorem, as one of the main results of the paper, we quantify the error between the interconnection of stochastic control subsystems and that of their abstractions in a compositional way.

**Theorem 5.2.** Consider interconnected stochastic control system \( \Sigma = \mathcal{I}(\Sigma_1, \ldots, \Sigma_N) \) induced by \( N \in \mathbb{N}_{\geq 1} \) stochastic control subsystems \( \Sigma_i \) and the coupling matrix \( M \). Suppose stochastic control subsystems \( \tilde{\Sigma}_i \) are abstractions of \( \Sigma_i \) with the corresponding SStF \( V_i \). If there exist \( \mu_i > 0, i \in \{1, \ldots, N\} \), and matrix \( \hat{M} \) of appropriate dimension such that the matrix (in)equality

\[
\begin{bmatrix}
GM & I_q \\
I_q & \hat{M}
\end{bmatrix} \geq 0,
\]

(5.1)

are satisfied, where \( \hat{q} = \sum_{i=1}^{N} q_{2i} \) and \( q_{2i} \) are dimensions of internal outputs of subsystem \( \Sigma_i \), and

\[
G := \text{diag}(G_1, \ldots, G_N), \quad \hat{G} := \text{diag}(\hat{G}_1, \ldots, \hat{G}_N), \quad H := \text{diag}(H_1, \ldots, H_N),
\]

(5.3)

then

\[
\tilde{V}(x, \dot{x}) := \sum_{i=1}^{N} \mu_i V_i(x, \dot{x}_i)
\]

(5.5)

is a stochastic simulation function from the interconnected control system \( \hat{\Sigma} = \mathcal{I}(\hat{\Sigma}_1, \ldots, \hat{\Sigma}_N) \), with the coupling matrix \( \hat{M} \), to \( \Sigma \).

The proof of Theorem 5.2 is provided in the Appendix.

**Remark 5.3.** Note that linear matrix inequality (LMI) (5.1) with \( G = I \) is exactly similar to the LMI appearing in \[\text{AMP16}\] for compositional stability condition based on dissipativity theory. As discussed in \[\text{AMP16}\], the LMI holds independently of the number of subsystems in many physical applications with specific interconnection structures including communication networks, flexible joint robots, power generators, and so on. We refer the interested readers to \[\text{AMP16}\] for more details on the satisfaction of this type of LMI.

**Remark 5.4.** One can relax condition (5.2) and employ the linear least square approach instead of solving the equality exactly. In this case, an additional error resulting from the least square approach is added to \( \psi \) in (5.2) which is left for the future investigations.

6. Discrete-Time Nonlinear Stochastic Control Systems

In this section, we focus on a specific class of discrete-time nonlinear stochastic control systems \( \Sigma_{\text{nl}} \) and quadratic stochastic storage functions \( V \) and provide an approach on the construction of their abstractions.

In the next subsection, we first formally define the class of discrete-time nonlinear stochastic control systems.
6.1. A Class of Discrete-Time Nonlinear Stochastic Control Systems. The class of discrete-time nonlinear stochastic control systems, considered here, is given by
\[
\Sigma_{nl}: \begin{cases} 
    x(k+1) = Ax(k) + E\varphi(Fx(k)) + B\nu(k) + Dw(k) + R\zeta(k), \\
    y_1(k) = C_1 x(k), \\
    y_2(k) = C_2 x(k),
\end{cases}
\]
where the additive noise \( \zeta(k) \) is a sequence of independent random vectors with multivariate standard normal distributions, and \( \varphi : \mathbb{R} \to \mathbb{R} \) satisfies
\[
a \leq \frac{\varphi(v) - \varphi(w)}{v - w} \leq b, \ \forall v, w \in \mathbb{R}, v \neq w,
\]
for some \( a \in \mathbb{R} \) and \( b \in \mathbb{R}_{\geq 0} \cup \{\infty\} \), \( a \leq b \).

We use the tuple
\[
\Sigma_{nl} = (A, B, C_1, C_2, D, E, F, R, \varphi),
\]
to refer to the class of discrete-time nonlinear stochastic control systems of the form \( (6.1) \).

Remark 6.1. If \( \varphi \) in \( (6.1) \) is linear including the zero function (i.e. \( \varphi \equiv 0 \)) or \( E \) is a zero matrix, one can remove or push the term \( E\varphi(Fx) \) to \( Ax \) and, hence, the tuple representing the class of discrete-time nonlinear stochastic control systems reduces to the linear one \( \Sigma_{nl} = (A, B, C_1, C_2, D, R) \). Therefore, every time we use the tuple \( \Sigma_{nl} = (A, B, C_1, C_2, D, E, F, R, \varphi) \), it implicitly implies that \( \varphi \) is nonlinear and \( E \) is nonzero.

Remark 6.2. Similar to what is shown in [AK01], without loss of generality, we can assume \( a = 0 \) in \( (6.2) \) for the class of nonlinear control systems in \( (6.1) \). If \( a \neq 0 \), one can define a new function \( \hat{\varphi}(r) := \varphi(r) - ar \) which satisfies \( (6.2) \) with \( \hat{a} = 0 \) and \( \hat{b} = b - a \), and rewrite \( (6.1) \) as
\[
\Sigma_{nl}: \begin{cases} 
    x(k+1) = \hat{A}x(k) + E\hat{\varphi}(Fx(k)) + B\nu(k) + Dw(k) + R\zeta(k), \\
    y_1(k) = C_1 x(k), \\
    y_2(k) = C_2 x(k),
\end{cases}
\]
where \( \hat{A} = A + aEF \).

Remark 6.3. For the sake of simple presentation, we restrict ourselves to systems with a single nonlinearity as in \( (6.1) \). However, it would be straightforward to obtain analogous results for systems with multiple nonlinearities as
\[
\Sigma_{nl}: \begin{cases} 
    x(k+1) = Ax(k) + \sum_{i=1}^{\tilde{M}} E_i \varphi_i(F_i x(k)) + B\nu(k) + Dw(k) + R\zeta(k), \\
    y_1(k) = C_1 x(k), \\
    y_2(k) = C_2 x(k),
\end{cases}
\]
where \( \varphi_i : \mathbb{R} \to \mathbb{R} \) satisfies \( (6.2) \) for some \( a_i \in \mathbb{R} \) and \( b_i \in \mathbb{R}_{\geq 0} \cup \{\infty\} \), for any \( i \in \{1, \ldots, \tilde{M}\} \).

In the next subsection, we provide conditions under which a candidate \( V \) is an SStF facilitating the construction of an abstraction \( \Sigma_{nl} \).

6.2. Quadratic Stochastic Storage Functions. Here, we employ the following quadratic SStF
\[
V(x, \hat{x}) = (x - P\hat{x})^T \tilde{M}(x - P\hat{x}),
\]
where \( P \) and \( \tilde{M} \succ 0 \) are some matrices of appropriate dimensions. In order to show that \( V \) in \( (6.3) \) is an SStF from \( \Sigma_{nl} \) to \( \Sigma_{nl} \), we require the following key assumption on \( \Sigma_{nl} \).

Assumption 1. Let \( \Sigma_{nl} = (A, B, C_1, C_2, D, E, F, R, \varphi) \). Assume that for some constant \( 0 < \tilde{r} < 1 \) and \( \tilde{k} > 0 \) there exist matrices \( \tilde{M} \succ 0 \), \( K \), \( L_1 \), \( Z \), \( G \), \( \tilde{X}^{11} \), \( \tilde{X}^{12} \), \( \tilde{X}^{21} \), and \( \tilde{X}^{22} \) of appropriate dimensions such that the matrix equality
\[
D = ZG,
\]
and inequality \( (6.5) \) hold.
can be chosen arbitrarily. As an example, one can choose

Remark 6.6.

minimizes

sufficient to satisfy Assumption 6.4 in where matrices

Note that the results in Theorem 6.4 do not impose any condition on matrix

is not the case when one assumes the noise of the concrete subsystem and its abstraction are the same as in

(6.3)

\[
\begin{bmatrix}
(A + BK)^T \hat{M} (A + BK) & (A + BK)^T \hat{M} Z & (A + BK)^T \hat{M} (BL_1 + E) & (A + BK)^T \hat{M} (BR - PB) \\
* & Z^T M Z & Z^T \hat{M} (BL_1 + E) & Z^T \hat{M} (BR - PB) \\
* & * & (BL_1 + E)^T \hat{M} (BL_1 + E) & (BL_1 + E)^T \hat{M} (BR - PB) \\
* & * & * & (BR - PB)^T \hat{M} (BR - PB)
\end{bmatrix}
\]

(6.5)

Now, we provide one of the main results of this section showing under which conditions \( V \) in (6.3) is an SSfT from \( \hat{\Sigma}_{nl} \) to \( \Sigma_{nl} \).

**Theorem 6.4.** Let \( \Sigma_{nl} = (A, B, C_1, C_2, D, E, F, R, \varphi) \) and \( \hat{\Sigma}_{nl} = (\hat{A}, \hat{B}, \hat{C}_1, \hat{C}_2, \hat{D}, \hat{E}, \hat{F}, \hat{R}, \varphi) \) be two stochastic control subsystems with the same external output space dimension. Suppose Assumption [7] holds and there exist matrices \( P, Q, H, L_2, \) and \( \hat{G} \) such that

\[
\begin{align*}
AP &= P\hat{A} - BQ, \\
C_1 P &= \hat{C}_1, \\
\hat{X}^{11} &= \hat{X}^{11}, \\
\hat{X}^{21} &= \hat{X}^{21}, \\
FP &= \hat{F}, \\
E &= P\hat{E} - B(L_1 - L_2), \\
PD &= Z\hat{G},
\end{align*}
\]

(6.6a), (6.6b), (6.6c), (6.6d), (6.6e), (6.6f), (6.6g)

hold. Then, function \( V \) defined in (6.3) is an SSfT from \( \hat{\Sigma}_{nl} \) to \( \Sigma_{nl} \).

The proof of Theorem [6.4] is provided in the Appendix. Note that the functions \( \alpha \in \mathcal{K}_\infty \), \( \kappa \in \mathcal{K} \), \( \rho_{ext} \in \mathcal{K}_\infty \cup \{0\} \), and the matrix \( \hat{X} \) in Definition 3.1 associated with the SSfT in (6.3) are

\[
\alpha(s) = \frac{\lambda_{max}(\hat{M})}{\lambda_{max}(C_1^T X^1)} s^2, \\
\kappa(s) := (1 - \tilde{\kappa})s, \\
\rho_{ext}(s) := \tilde{\kappa} \sqrt{\hat{M}} (BR - PB) \|s\| s^2, \quad \forall s \in \mathbb{R},
\]

Moreover, positive constant \( \psi \) in (3.2) is

\[
\psi = \text{Tr}(R^T \hat{M} R + \hat{R}^T P^T \hat{M} P \hat{R}).
\]

**Remark 6.5.** Note that for any linear system \( \Sigma_{nl} = (A, B, C_1, C_2, D, R) \), stabilizability of the pair \( (A, B) \) in matrices \( E, F \), and \( L_1 \) are identical.

**Remark 6.6.** One can readily verify from the result of Theorem 6.4 that choosing \( \hat{R} \) equal to zero results in smaller constant \( \psi \) and, hence, more closeness between subsystems and their abstractions. Remark that this is not the case when one assumes the noise of the concrete subsystem and its abstraction are the same as in [ZemT17] [Zam13].

**Remark 6.7.** Note that the results in Theorem 6.4 do not impose any condition on matrix \( \hat{B} \) and, hence, it can be chosen arbitrarily. As an example, one can choose \( \hat{B} = I_n \) which makes the abstract system \( \hat{\Sigma}_{nl} \) fully actuated and, hence, the synthesis problem over it much easier.

**Remark 6.8.** Since Theorem 6.4 does not impose any condition on matrix \( \hat{R} \), one can choose \( \hat{R} \) such that it minimizes function \( \rho_{ext} \) for \( V \) as suggested in [GP09]. The following expression for \( \hat{R} \)

\[
\hat{R} = (B^T MB)^{-1} B^T M \hat{P} \hat{B},
\]

minimizes \( \rho_{ext} \).

In the next section, we demonstrate the effectiveness of the proposed results for an interconnected system consisting of three nonlinear stochastic control subsystems in a compositional fashion.
Consider a discrete-time nonlinear stochastic control system $\Sigma_{nl}$ satisfying

$$\begin{align*}
\Sigma_{nl}: \begin{cases}
x(k+1) = \hat{G}x(k) + \varphi(x(k)) + \nu(k) + R_1(k), \\
y(k) = Cx(k),
\end{cases}
\end{align*}$$

for some matrix $\hat{G} = (I_n - \tau L) \in \mathbb{R}^{n \times n}$ where $\tau L$ is the Laplacian matrix of an undirected graph with $0 < \tau < 1/\Delta$, where $\Delta$ is the maximum degree of the graph [46]. Moreover, $R = \text{diag}(0.0071_{n_1}, \ldots, 0.0071_{n_N})$, $\varsigma(k) = [\varsigma_1(k) : \ldots : \varsigma_N(k)]$, $\varphi(x) = [\varphi_1(x_1(k)) : \ldots : \varphi_N(x_N(k))]$ where $n = \sum_{i=1}^{N} n_i$, $\varphi_i(x) = \sin(x)$, and $F_i^T = [1 \ 0 \ \ldots \ 0]^T \in \mathbb{R}^{n_i} \ \forall i \in \{1, \ldots, N\}$, and $C$ has the block diagonal structure as $C = \text{diag}(C_{11}, \ldots, C_{1N})$. We partition $x$ as $x = [x_1; \ldots ; x_N]$ and $\nu$ as $\nu = [\nu_1; \ldots ; \nu_N]$, where $x_i, \nu_i \in \mathbb{R}^{n_i}$. Now, by introducing $\Sigma_{nli} = (I_{n_{1i}}, I_{n_{2i}}, C_{1i}, I_{n_i}, \widehat{C}_{1i}, 1, 1, 0.1, 1, \varphi_i)$ satisfying

$$\begin{align*}
\Sigma_{nli}: \begin{cases}
x_{i}(k+1) = x_{i}(k) + \nu_i(k) + 0.0071_{n_i} \varsigma_i(k), \\
y_{i1}(k) = C_{1i}x_i(k), \\
y_{i2}(k) = x_i(k),
\end{cases}
\end{align*}$$

one can readily verify that $\Sigma_{nl} = \mathcal{T}(\Sigma_{nli}, \ldots, \Sigma_{nliN})$ where the coupling matrix $M$ is given by $M = -\tau L$. Our goal is to aggregate each $x_i$ into a scalar-valued $\hat{x}_i$, governed by $\hat{\Sigma}_{nli} = (0.5, 1, \hat{C}_{1i}, 1, 1, 0.1, 1, \varphi_i)$ which satisfies:

$$\begin{align*}
\hat{\Sigma}_{nli}: \begin{cases}
\hat{x}_{i}(k+1) = 0.5\hat{x}_i(k) + 0.1\varphi_i(\hat{x}_i(k)) + \hat{\nu}_i(k) + \hat{w}_i(k), \\
\hat{y}_{i1}(k) = \hat{C}_{1i}\hat{x}_i(k), \\
\hat{y}_{i2}(k) = \hat{x}_i(k),
\end{cases}
\end{align*}$$

where $\hat{C}_{1i} = C_{1i}1_{n_i}$. Note that here $\hat{R}_i, \forall i \in \{1, \ldots, N\}$, are considered zero in order to reduce constants $\psi_i$ for each $V_i$. One can readily verify that, for any $i \in \{1, \ldots, N\}$, conditions (6.4) and (6.5) are satisfied with $\hat{M}_i = I_{n_i}, \hat{\varsigma}_i = 0.95, \hat{\varsigma}_i = 1, \hat{K}_i = (\lambda_i - 1)I_{n_i}, \lambda_i = 0.5, \hat{Z}_i = G_i = I_{n_i}, \hat{L}_{1i} = -1_{n_i}, \hat{R}_i = 1_{n_i}, \hat{X}^{11} = I_{n_i}, \hat{X}^{22} = 0_{n_i}$, and $\hat{X}^{12} = \hat{X}^{21} = \lambda_i 1_{n_i}$. Moreover, for any $i \in \{1, \ldots, N\}$, $\hat{P}_i = 1_{n_i}$ satisfies conditions (6.6) with $\hat{Q}_i = -0.51_{n_i}, \hat{L}_{2i} = -0.11_{n_i}$, and $\hat{H}_i = \hat{G}_i = I_{n_i}$. Hence, function $V_i(x_i, \hat{x}_i) = (x_i - 1_{n_i}\hat{x}_i)^T(x_i - 1_{n_i}\hat{x}_i)$ is an SSstF from $\Sigma_{nli}$ to $\Sigma_{nl}$ satisfying condition (3.1) with $\alpha_i(s) = \frac{1}{2\max(\hat{C}_{1i}, \hat{C}_{2i})s^2}$ and condition (3.2) with $\kappa_i(s) := 0.05s, \rho_{ext}(s) = 0, \forall s \in \mathbb{R}_{\geq 0}, G_i = I_{n_i}, H_i = 1_{n_i}$, and

$$\begin{align*}
\hat{X}_i &= \begin{bmatrix} I_{n_i} & \lambda_i 1_{n_i} \\ \lambda_i 1_{n_i} & 0_{n_i} \end{bmatrix}, 
\end{align*}$$

(7.1)

where the input $\nu_i$ is given via the interface function in (9.4) as

$$\nu_i := (\lambda_i - 1)(x_i - 1_{n_i}\hat{x}_i) - 0.51_{n_i}\hat{x}_i + 1_{n_i}\hat{\nu}_i - 1_{n_i}\varphi_i(F_i x_i) + 0.11_{n_i}\varphi_i(F_i 1_{n_i}\hat{x}_i).$$

Now, we look at $\hat{\Sigma}_{nl} = \mathcal{T}(\hat{\Sigma}_{nli}, \ldots, \hat{\Sigma}_{nliN})$ with a coupling matrix $\hat{M}$ satisfying condition (5.2) as follows:

$$-\tau L \text{diag}(1_{n_1}, \ldots, 1_{n_N}) = \text{diag}(1_{n_1}, \ldots, 1_{n_N})\hat{M}. $$

(7.2)

Note that the existence of $\hat{M}$ satisfying (7.2) for graph Laplacian $\tau L$ means that the $N$ subgraphs form an equitable partition of the full graph [46]. Although this restricts the choice of a partition in general, for the complete graph any partition is equitable.

Choosing $\mu_1 = \cdots = \mu_N = 1$ and using $\hat{X}_i$ in (7.1), matrix $\hat{X}_{cmp}$ in (5.4) reduces to

$$\hat{X}_{cmp} = \begin{bmatrix} I_{n_i} & \lambda_i 1_{n_i} \\ \lambda_i 1_{n_i} & 0_{n_i} \end{bmatrix},$$

where $\lambda = \lambda_1 = \cdots = \lambda_N = 0.5$, and condition (5.1) reduces to

$$\begin{align*}
[\begin{bmatrix} -\tau L^T & 0 \\ 0 & -\tau L \end{bmatrix}] \hat{X}_{cmp} = \tau^2 L^TL - \lambda\tau L - \lambda\tau L^T = \tau L(\tau L - 2\lambda I_n) \leq 0,
\end{align*}$$

(7.3)
The specification with closed loop output trajectories of $\Sigma_{nl}$ (black one) and $\hat{\Sigma}_{nl}$ (red one). The sets $S, O_i, i \in \{1, 2, 3\}$, and $\bar{T}_i, i \in \{1, 2\}$ are given by: $S = [-14, 14]^3$, $O_1 = [-10, -6] \times [6, 10] \times [10, 10]$, $O_2 = [-5, 5]^3$, and $O_3 = [6, 10] \times [-10, -6] \times [10, 10]$, $\bar{T}_1 = [-10, -6] \times [-10, -6] \times [-10, -6]$ and $\bar{T}_2 = [6, 10] \times [6, 10] \times [6, 10]$.

For the sake of simulation, we assume $L$ is the Laplacian matrix of a complete graph and $\tau = 0.001$. We fix $N = 3, n = 222, n_i = 74$, and $C_{ij} = [1 \ 0 \ 0 \ldots \ 0], i \in \{1, 2, 3\}$. By using inequality (3.5) and starting the interconnected systems $\Sigma_{nl}$ and $\hat{\Sigma}_{nl}$ from initial states $-131_{222}$ and $-131_3$, respectively, we guarantee that the distance between outputs of $\Sigma_{nl}$ and $\hat{\Sigma}_{nl}$ will not exceed $\varepsilon = 1$ during the time horizon $T_d = 10$ with probability at least 90%, i.e.

$$\mathbb{P} \left( \|y_{a\nu}(k) - \hat{y}_{a\nu}(k)\| \leq 1, \forall k \in [0, 10] \right) \geq 0.9.$$  

Let us now synthesize a controller for $\Sigma_{nl}$ via the abstraction $\hat{\Sigma}_{nl}$ to enforce the specification, defined by the following scLTL formula (cf. Definition 4.1):

$$\omega = \bigwedge_{j=0}^{T_d} \left( S \land \left( \bigwedge_{i=1}^{3} (-O_i) \right) \right) \land \bar{T}_1 \land \bar{T}_2, \quad (7.3)$$

which requires that any output trajectory $y$ of the closed loop system evolves inside the set $S$, avoids sets $O_i, i \in \{1, 2, 3\}$, indicated with blue boxes in Figure 2, over bounded time interval $[0, T_d]$, and visits each $\bar{T}_i, i \in \{1, 2\}$, indicated with red boxed in Figure 2. We want to satisfy $\omega$ over bounded time interval $[0, 10]$ and take $T_d = 10$. We use SCOTS [RZ16] to synthesize a controller for $\hat{\Sigma}_{nl}$ to enforce (7.3). In the synthesis process we restricted the abstract inputs $\hat{\nu}_1, \hat{\nu}_2, \hat{\nu}_3$ to $[-4, 4]$. We also set the initial states of $\Sigma_{nl}$ to $x_i = P_i \hat{x}_i$, so that $V_i(x_i, \hat{x}_i) = 0$. A realization of closed-loop output trajectories of $\Sigma_{nl}$ and $\hat{\Sigma}_{nl}$ is illustrated in Figure 2. Also, several realizations of the norm of error between outputs of $\Sigma_{nl}$ and $\hat{\Sigma}_{nl}$ are illustrated in Figure 4. Note that it would not have been possible to synthesize a controller using SCOTS for the original 222-dimensional system $\Sigma_{nl}$, without the 3-dimensional intermediate approximation $\hat{\Sigma}_{nl}$. 

![Figure 2](image-url)
Figure 3. A few realizations of the norm of the error between the outputs of $\Sigma_{nl}$ and of $\hat{\Sigma}_{nl}$, e.g. $\|y - \hat{y}\|$, for $T_d = 10$.

8. DISCUSSION

In this paper, we provided a compositional approach for infinite abstractions of interconnected discrete-time stochastic control systems, with independent noises in the abstract and concrete domains. To do so, we leveraged the interconnection matrix and joint dissipativity-type properties of subsystems and their abstractions. We introduced new notions of stochastic storage and simulation functions in order to quantify the distance in probability between original stochastic control subsystems and their abstractions and their interconnections, respectively. Using those notions, one can employ the proposed results here to synthesize policies enforcing certain temporal logic properties over abstract systems and then refine them to the ones for the concrete systems while quantifying the satisfaction errors in this detour process. We also addressed a fragment of LTL known as syntactically co-safe LTL, and showed how to quantify the probability of satisfaction for such specifications. Furthermore, we provided a computational scheme for a class of discrete-time nonlinear stochastic control systems to construct their abstractions together with their corresponding stochastic storage functions. Finally, we demonstrated the effectiveness of the results by constructing an abstraction (totally 3 dimensions) of the interconnection of three discrete-time nonlinear stochastic control subsystems (together 222 dimensions) in a compositional fashion. We also employed the abstraction as a substitute to synthesize a controller enforcing a syntactically co-safe LTL specification.

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The equality holds due to $\alpha$ (Theorem 3.4).

\textbf{Proof.} (Theorem 3.4) Since $V$ is an SSF from $\hat{\Sigma}$ to $\Sigma$, we have

$$
\mathbb{P} \left\{ \sup_{0 \leq k \leq T_d} \| y_{\alpha\hat{\nu}}(k) - \hat{y}_{\alpha\hat{\nu}}(k) \| \geq \varepsilon \mid [\alpha; \hat{\alpha}] \right\} = \mathbb{P} \left\{ \sup_{0 \leq k \leq T_d} \alpha \left( \| y_{\alpha\hat{\nu}}(k) - \hat{y}_{\alpha\hat{\nu}}(k) \| \right) \geq \alpha(\varepsilon) \mid [\alpha; \hat{\alpha}] \right\}
$$

$$
\leq \mathbb{P} \left\{ \sup_{0 \leq k \leq T_d} V \left( x_{\alpha\hat{\nu}}(k), \hat{x}_{\alpha\hat{\nu}}(k) \right) \geq \alpha(\varepsilon) \mid [\alpha; \hat{\alpha}] \right\}.
$$

(9.1)

The equality holds due to $\alpha$ being a $K_\infty$ function. The inequality is also true due to condition (3.3) on the SSF $V$. The results follows by applying Theorem 3 in [Kus67] pp. 81 to (9.1) and utilizing inequality (3.4). \qed
Proof. (Corollary 4.4) Since $V$ is an SSF from $\hat{\Sigma}$ to $\Sigma$ with $\rho_{\text{ext}}(\cdot) \equiv 0$ and $\psi = 0$, for any $x(k) \in X$ and $\hat{x}(k) \in \hat{X}$ and any $\nu(k) \in U$, there exists $\nu(k) \in U$ such that

$$E[V(x(k+1), \hat{x}(k+1) | x(k), \hat{x}(k), \nu(k), \hat{\nu}(k)] - V(x(k), \hat{x}(k)) \leq -\kappa(V(x(k), \hat{x}(k))),$$

implying that $V(x_{av}(k), \hat{x}_{av}(k))$ is a nonnegative supermartingale [Oks13] for any initial condition $a$ and $\hat{a}$ and inputs $\nu, \hat{\nu}$. Following the same reasoning as in the proof of Theorem 3.4, we have

$$P\left\{ \sup_{0 \leq k < \infty} \|y_{av}(k) - \hat{y}_{av}(k)\| \geq \varepsilon | [a; \hat{a}] \right\} \leq P\left\{ \sup_{0 \leq k < \infty} \alpha \left( \|y_{av}(k) - \hat{y}_{av}(k)\| \right) \geq \alpha(\varepsilon) | [a; \hat{a}] \right\} \leq \frac{V(a, \hat{a})}{\alpha(\varepsilon)},$$

where the last inequality is due to the nonnegative supermartingale property [Kus67].

Proof. (Lemma 4.4) Suppose $\hat{y}(\cdot) \vDash \hat{\phi}$ over time interval $[0, T_d]$. According to the construction of DFA $A_{\hat{q}}$, $q_{abs}$ is an absorbing state and not an accepting state, thus $L^r(\hat{y}(k)) \neq \phi_o$, $\forall k \in [0, T_d]$. Then $L^r(\hat{y}(k)) \in \Sigma_a$, $\forall k \in [0, T_d]$. Assume $L^r(\hat{y}(k)) = a$ then $\hat{y}(k) \in [L^{-1}(a)]^r$. Since we know that

$$\sup_{0 \leq k \leq T_d} \|y(k) - \hat{y}(k)\| < \epsilon,$$

then according to the definition of $\epsilon$-perturbed sets $y(k) \in L^{-1}(a)$ which gives $L(y(k)) = a$. Thus $L(y(\cdot)) = L^r(\hat{y}(\cdot))$ and having $\hat{y}(\cdot) \vDash \hat{\phi}$ guarantees $y(\cdot) \vDash \phi$ due to the particular construction of $\hat{\phi}$.

Proof. (Theorem 4.5) According to Lemma 4.4, $y(\cdot) \nvdash \phi$ results in $\hat{y}(\cdot) \nvdash \hat{\phi}$ over time interval $[0, T_d]$ or

$$\sup_{0 \leq k \leq T_d} \|y(k) - \hat{y}(k)\| \geq \epsilon.$$

Then

$$P(y(\cdot) \nvdash \phi) \leq P(\hat{y}(\cdot) \nvdash \hat{\phi}) + P\left( \sup_{0 \leq k \leq T_d} \|y(k) - \hat{y}(k)\| \geq \epsilon \right) \Rightarrow 1 - P(y(\cdot) \vDash \phi) \leq 1 - P(\hat{y}(\cdot) \vDash \hat{\phi}) + \delta \Rightarrow P(\hat{y}(\cdot) \vDash \hat{\phi}) - \delta \leq P(y(\cdot) \vDash \phi),$$

which completes the proof.

Proof. (Theorem 5.2) We first show that inequality 5.3 holds for some $K_\infty$ function $\alpha$. For any $x = [x_1; \ldots; x_N] \in X$ and $\hat{x} = [\hat{x}_1; \ldots; \hat{x}_N] \in \hat{X}$, one gets:

$$\|h(x) - \hat{h}(\hat{x})\| = \|[h_{11}(x_1); \ldots; h_{1N}(x_N)] - [\hat{h}_{11}(\hat{x}_1); \ldots; \hat{h}_{1N}(\hat{x}_N)]\| \leq \sum_{i=1}^{N} \|h_{1i}(x_i) - \hat{h}_{1i}(\hat{x}_i)\|$$

$$\leq \sum_{i=1}^{N} \alpha_i^{-1}(V_i(x_i, \hat{x}_i)) \leq \bar{\alpha}(V(x, \hat{x})),$$

with function $\bar{\alpha} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ defined for all $r \in \mathbb{R}_{\geq 0}$ as

$$\bar{\alpha}(r) := \max \left\{ \sum_{i=1}^{N} \alpha_i^{-1}(s_i) \mid s_i \geq 0, \sum_{i=1}^{N} \mu_i s_i = r \right\}.$$

It is not hard to verify that function $\bar{\alpha}(\cdot)$ defined above is a $K_\infty$ function. By taking the $K_\infty$ function $\alpha(r) := \bar{\alpha}^{-1}(r)$, $\forall r \in \mathbb{R}_{\geq 0}$, one obtains

$$\alpha(\|h(x) - \hat{h}(\hat{x})\|) \leq V(x, \hat{x}),$$
satisfying inequality (3.3).

Now we prove that function $V$ in (6.5) satisfies inequality (3.4). Consider any $x = [x_1; \ldots; x_N] \in X$, $\hat{x} = [\hat{x}_1; \ldots; \hat{x}_N] \in \hat{X}$, and $\tilde{v} = [\tilde{v}_1; \ldots; \tilde{v}_N] \in U$. For any $i \in \{1, \ldots, N\}$, there exists $\nu_i \in U_i$, consequently, a vector $\nu = [\nu_1; \ldots; \nu_N] \in U$, satisfying (3.2) for each pair of subsystems $\Sigma_i$ and $\tilde{\Sigma}_i$ with the internal inputs given by $[w_1; \ldots; w_N] = M[h_{21}(x_1); \ldots; h_{2N}(x_N)]$ and $[\tilde{w}_1; \ldots; \tilde{w}_N] = \tilde{M}[\tilde{h}_{21}(\hat{x}_1); \ldots; \tilde{h}_{2N}(\hat{x}_N)]$. Then we have the chain of inequalities in (9.2) using conditions (5.1) and (5.2) and by defining $\kappa(\cdot), \rho_{\text{ext}}(\cdot), \psi$ as

$$\kappa(r) := \min \left\{ \sum_{i=1}^{N} \mu_i \kappa_i(s_i) \mid s_i \geq 0, \sum_{i=1}^{N} \mu_i s_i = r \right\}$$

$$\rho_{\text{ext}}(r) := \max \left\{ \sum_{i=1}^{N} \mu_i \rho_{\text{ext}}(s_i) \mid s_i \geq 0, \|s_1; \ldots; s_N\| = r \right\}$$

$$\psi := \sum_{i=1}^{N} \mu_i \psi_i.$$ 

Note that $\kappa$ and $\rho_{\text{ext}}$ in (3.2) belong to $K$ and $K_\infty \cup \{0\}$, respectively, because of their definition provided above. Hence, we conclude that $V$ is an SSF from $\tilde{\Sigma}$ to $\Sigma$. \hfill \Box

**Proof. (Theorem 6.4)** Here we first show that $\forall x, \forall \hat{x}, \forall \tilde{v}, \exists \nu, \exists \hat{w}$, and $\forall w$, $V$ satisfies $\frac{\lambda_{\min}(\tilde{M})}{\lambda_{\max}(C_1^T C_1)} \|C_1 x - \hat{C}_1 \hat{x}\|^2 \leq V(x, \hat{x})$ and then

$$E\left[ V(x(k+1), \hat{x}(k+1) \mid x(k) = x, \hat{x}(k) = \hat{x}, w(k) = w, \hat{w}(k) = \hat{w}, \nu(k) = \nu \right] - V(x, \hat{x}) \leq -(1 - \nu)(V(x, \hat{x}))$$

$$+ k \|\sqrt{M(BR - P\tilde{B})}\|^2 \|\nu\|^2 + \left[ Gw - \hat{G}\tilde{w} \right] \left[ \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \right] \left[ \begin{bmatrix} Gw - \hat{G}\tilde{w} \\ \hat{h}_2(x) - \hat{h}_2(\hat{x}) \end{bmatrix} \right] + \text{Tr}\left( \hat{R}^T \tilde{M} R + \hat{R}^T P^T \tilde{M} \hat{P} \right).$$

According to (6.6a), we have $\|C_1 x - \hat{C}_1 \hat{x}\|^2 = (x - \hat{x})^T C_1^T C_1 (x - \hat{x})$. Since $\lambda_{\min}(C_1^T C_1) \|x - \hat{x}\|^2 \leq (x - \hat{x})^T C_1^T C_1 (x - \hat{x}) \leq \lambda_{\max}(C_1^T C_1) \|x - \hat{x}\|^2$ and similarly $\lambda_{\min}(\tilde{M}) \|x - \hat{x}\|^2 \leq \lambda_{\max}(\tilde{M}) \|x - \hat{x}\|^2$, it can be readily verified that $\frac{\lambda_{\min}(\tilde{M})}{\lambda_{\max}(C_1^T C_1)} \|C_1 x - \hat{C}_1 \hat{x}\|^2 \leq V(x, \hat{x})$ holds $\forall x, \forall \hat{x}$, implying that inequality (3.1) holds with $\alpha(s) = \frac{\lambda_{\min}(\tilde{M})}{\lambda_{\max}(C_1^T C_1)} s^2$ for any $s \in \mathbb{R}_{\geq 0}$. We proceed with showing that the inequality (3.2) holds, as well. Given any $x, \hat{x}$, and $\hat{v}$, we choose $\nu$ via the following interface function:

$$\nu = \nu_0(x, \hat{x}, \hat{v}) := K(x - \hat{x}) + Q\hat{x} + \tilde{R} \hat{v} + L_1 \varphi(Fx) - L_2 \varphi(FP\hat{x}),$$

(9.4)

for some matrix $\tilde{R}$ of appropriate dimension. By employing the equations (6.1), (6.6a), (6.6c), (6.6b) and also the definition of the interface function in (6.1), we simplify

$$Ax + E\varphi(Fx) + Bv_0(x, \hat{x}, \hat{v}) + Dw - P(\hat{A} \hat{x} + \hat{E}\varphi(\hat{F} \hat{x}) + \hat{B} \hat{v} + \hat{D} \hat{w}) + (R \hat{z} - P \hat{R} \hat{z})$$

to

$$(A + BK)(x - P \hat{x}) + Z(Gw - \hat{G}\tilde{w}) + (B\tilde{R} - P\tilde{B})\nu + (BL_1 + E)(\varphi(Fx) - \varphi(FP\hat{x})) + (R \hat{z} - P \hat{R} \hat{z}).$$

(9.5)

From the slope restriction (3.2), one obtains

$$\varphi(Fx) - \varphi(FP\hat{x}) = \delta(Fx - FP\hat{x}) = \delta F(x - \hat{x}),$$

(9.6)

where $\delta$ is a constant and depending on $x$ and $\hat{x}$ takes values in the interval $[0, \delta]$. Using (9.6), the expression in (9.5) reduces to

$$((A + BK) + \delta(BL_1 + E)F)(x - P \hat{x}) + Z(Gw - \hat{G}\tilde{w}) + (B\tilde{R} - P\tilde{B})\nu + (R \hat{z} - P \hat{R} \hat{z}).$$

Using Cauchy- Schwarz inequality, (6.5), (6.6e), and (6.6c), one can obtain the chain of inequalities in (9.3) in order to acquire an upper bound. Hence, the proposed $V$ in (6.3) is an SSF from $\Sigma_{nl}$ to $\Sigma_{nl}$, which completes the proof. \hfill \Box
\[
E \left[ \sum_{i=1}^{N} \mu_i V_i(x_i(k+1), \dot{x}_i(k+1)) \mid [x(k) = x, \dot{x}(k) = \dot{x}, \dot{\nu}(k) = \dot{\nu}] \right] - \sum_{i=1}^{N} \mu_i V_i(x_i, \dot{x}_i)
\]

\[
= \sum_{i=1}^{N} \mu_i \left[ V_i(x_i(k+1), \dot{x}_i(k+1)) \mid [x_i(k) = x_i, \dot{x}_i(k) = \dot{x}_i, \dot{\nu}_i(k) = \dot{\nu}_i] \right] - \sum_{i=1}^{N} \mu_i V_i(x_i, \dot{x}_i)
\]

\[
\leq \sum_{i=1}^{N} \mu_i \left( -\kappa_i (V_i(x_i, \dot{x}_i)) + \rho_{eext}(\|\dot{\nu}_i\|) + \psi_i + \left[ \begin{array}{c}
G_i w_i - \hat{G}_i \dot{w}_i \\
\vdots
\end{array} \right]
\begin{bmatrix}
X_1^1 & \cdots & X_1^2 \\
\vdots & \ddots & \vdots \\
X_N^1 & \cdots & X_N^2
\end{bmatrix}
\begin{bmatrix}
\dot{X}_1^1 & \cdots & \dot{X}_1^2 \\
\vdots & \ddots & \vdots \\
\dot{X}_N^1 & \cdots & \dot{X}_N^2
\end{bmatrix}
\begin{bmatrix}
G_i w_i - \hat{G}_i \dot{w}_i \\
\vdots
\end{bmatrix}
\right)
\]

\[
= \sum_{i=1}^{N} -\mu_i \kappa_i (V_i(x_i, \dot{x}_i)) + \sum_{i=1}^{N} \mu_i \rho_{eext}(\|\dot{\nu}_i\|) + \sum_{i=1}^{N} \mu_i \psi_i
\]

\[
+ \sum_{i=1}^{N} \mu_i \rho_{eext}(\|\dot{\nu}_i\|) + \sum_{i=1}^{N} \mu_i \psi_i + \left[ \begin{array}{c}
h_21(x_1) - H_1 \dot{h}_21(\dot{x}_1) \\
\vdots \\
h_2N(x_N) - H_N \dot{h}_2N(\dot{x}_N)
\end{array} \right]
\begin{bmatrix}
X_{cmp}^T \\
\vdots \\
X_{cmp}^N
\end{bmatrix}
\begin{bmatrix}
GM & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & GM & 0
\end{bmatrix}
\begin{bmatrix}
h_21(x_1) - H_1 \dot{h}_21(\dot{x}_1) \\
\vdots \\
h_2N(x_N) - H_N \dot{h}_2N(\dot{x}_N)
\end{bmatrix}
\]

\[
\leq \sum_{i=1}^{N} -\mu_i \kappa_i (V_i(x_i, \dot{x}_i)) + \sum_{i=1}^{N} \mu_i \rho_{eext}(\|\dot{\nu}_i\|) + \sum_{i=1}^{N} \mu_i \psi_i - \kappa (V(x, \dot{x})) + \rho_{eext}(\|\dot{\nu}\|) + \psi
\]
\[ \mathbf{E}\left[ V(x(k+1), \dot{x}(k+1)| x(k) = x, \dot{x}(k) = \dot{x}, w(k) = w, \dot{w}(k) = \dot{w}, \nu(k) = \nu \right] - V(x, \dot{x}) \]
\[ = (x - P\dot{x})^T \left[ (A + BK) + \delta(BL_1 + E)F \right] M ((A + BK) + \delta(BL_1 + E)F) (x - P\dot{x}) + 2 \left( (x - P\dot{x})^T ((A + BK) + \delta(BL_1 + E)F) \right) M (B\tilde{R} - P\tilde{B}) \nu \]
\[ + 2 \left( (Gw - \tilde{G}\dot{w})^T Z^T \right) M (B\tilde{R} - P\tilde{B}) \nu + (Gw - \tilde{G}\dot{w})^T Z^T M Z (Gw - \tilde{G}\dot{w}) \]
\[ + \text{Tr}(R^T \tilde{M} R + \tilde{R}^T P^T \tilde{M} P\tilde{R}) - V(x, \dot{x}) \leq \begin{bmatrix} x - P\dot{x} \\ Gw - \tilde{G}\dot{w} \\ \delta F(x - P\dot{x}) \end{bmatrix}^T \]
\[ \begin{bmatrix} (A + BK)^T \tilde{M} (A + BK) & (A + BK)^T \tilde{M} Z & (A + BK)^T \tilde{M} (BL_1 + E) \\ * & Z^T \tilde{M} Z & Z^T \tilde{M} (BL_1 + E) \\ * & * & (BL_1 + E)^T \tilde{M} (BL_1 + E) \end{bmatrix} \text{Tr} \left( R^T \tilde{M} R + \tilde{R}^T P^T \tilde{M} P\tilde{R} \right) \]
\[ - \text{V}(x, \dot{x}) = -(1 - \tilde{\kappa})(V(x, \dot{x})) - 2\delta(1 - \frac{\delta}{b})(x - P\dot{x})^T F(x - P\dot{x}) + \tilde{\kappa} \| M(B\tilde{R} - P\tilde{B}) \nu \|^2 + \begin{bmatrix} Gw - \tilde{G}\dot{w} \\ C_2 x - H\tilde{C}_2 \dot{x} \end{bmatrix}^T \]
\[ \begin{bmatrix} X^{11} & X^{12} \\ X^{21} & X^{22} \end{bmatrix} \left[ \begin{bmatrix} Gw - \tilde{G}\dot{w} \\ C_2 x - H\tilde{C}_2 \dot{x} \end{bmatrix} \right] + \text{Tr} \left( R^T \tilde{M} R + \tilde{R}^T P^T \tilde{M} P\tilde{R} \right) \leq -(1 - \tilde{\kappa})(V(x, \dot{x})) + \tilde{\kappa} \| M(B\tilde{R} - P\tilde{B}) \| \| \nu \|^2 \]
\[ + \begin{bmatrix} Gw - \tilde{G}\dot{w} \\ C_2 x - H\tilde{C}_2 \dot{x} \end{bmatrix}^T \begin{bmatrix} X^{11} & X^{12} \\ X^{21} & X^{22} \end{bmatrix} \begin{bmatrix} Gw - \tilde{G}\dot{w} \\ C_2 x - H\tilde{C}_2 \dot{x} \end{bmatrix} + \text{Tr} \left( R^T \tilde{M} R + \tilde{R}^T P^T \tilde{M} P\tilde{R} \right) \]
\[ (9.3) \]