On Some Modular Equations in the Spirit of Ramanujan

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ABSTRACT. In this paper, we establish some new $P$-$Q$ type modular equations, by using the modular equations given by Srinivasa Ramanujan.

1. Introduction

In Chapter 16 of his second notebook [9], S. Ramanujan developed, theory of theta-function and his theta-function is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1.$$ 

Note that, if we set $a = q^{2iz}$, $b = q^{-2iz}$, where $z$ is complex and $\text{Im}(\tau) > 0$, then $f(a,b) = \vartheta_3(z,\tau)$, where $\vartheta_3(z,\tau)$ denotes one of the classical theta-functions in its standard notation [16, p. 464]. The three most important special cases of $f(a,b)$ [4, p. 36] are

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (q; q^2)_{\infty}^2 (q^2; q^2)_{\infty} = \frac{(-q; -q)_{\infty}}{(q; -q)_{\infty}},$$

$$\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}},$$

$$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q; q)_{\infty}.$$ 

After Ramanujan, we define

$$\chi(q) := (-q; q^2)_{\infty},$$

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where we employ the customary notation

\[(a; q)_\infty := \prod_{n=0}^{\infty} (1 - aq^n), \quad |q| < 1.\]

We now define a modular equation as given by Ramanujan. The complete elliptic integral of the first kind \(K(k)\) is defined by

\[(1.1) \quad K(k) := \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n}{(n)_n} k^{2n} = \frac{\pi}{2} 2F_1 \left( \frac{1}{2}, 1; 1; k^2 \right),\]

where \(0 < k < 1\). The series representation in (1.1) is found by expanding the integrand in a binomial series and integrating termwise and \(2F_1\) is the ordinary or Gaussian hypergeometric function defined by

\[2F_1(a, b; c; z) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n, \quad |z| < 1,\]

with

\[(a)_k = \frac{\Gamma(a + k)}{\Gamma(a)},\]

where \(a, b\) and \(c\) are complex numbers such that \(c\) is not a nonpositive integer. The number \(k\) is called the modulus of \(K\) and \(k' := \sqrt{1 - k^2}\) is called the complementary modulus. Let \(K, K', L\) and \(L'\) denote the complete elliptic integrals of the first kind associated with moduli \(k, k' l\) and \(l'\) respectively. Suppose that the equality

\[(1.2) \quad n \frac{K'}{K} = \frac{L'}{L}\]

holds for some positive integer \(n\). Then a modular equation of degree \(n\) is a relation between the moduli \(k\) and \(l\) which is implied by (1.2). Ramanujan recorded his modular equations in terms of \(\alpha\) and \(\beta\), where \(\alpha = k^2\) and \(\beta = l^2\). We often say that \(\beta\) has degree \(n\) over \(\alpha\). The multiplier \(m\) is defined by

\[m = \frac{K}{L}.\]

Ramanujan [4, p. 122-124] recorded several formulae for \(\varphi, \psi, f\) and \(\chi\) at different arguments of \(\alpha q\) and \(z := 2F_1\left( \frac{1}{2}, \frac{1}{2}; 1; \alpha \right)\) by using

\[\varphi^2(q) = \frac{2}{\pi} K(k) = 2F_1 \left( \frac{1}{2}, 1; 1; k^2 \right), \quad q = \exp\left(-\pi K'/K\right).\]

Ramanujan’s modular equations involve quotients of function \(f(-q)\) at certain arguments. For example [5, p. 206], let

\[P := \frac{f(-q)}{q^{1/6} f(-q^5)} \quad \text{and} \quad Q := \frac{f(-q^2)}{q^{1/3} f(-q^{10})}.\]
then
\[
PQ + \frac{5}{PQ} = \left(\frac{Q}{P}\right)^3 + \left(\frac{P}{Q}\right)^3.
\]

These modular equations are also called Schläfli-type. Since the publication of [5], several authors, including N. D. Baruah [2], [3] M. S. M. Naikia [7], [8] K. R. Vasuki [12], [13] and K. R. Vasuki and B. R. Srivatsa Kumar [14] have found additional modular equations of the type (1.3). Recently C. Adiga, et. al. [1] have established several modular relations for the Rogers-Ramanujan type functions of order eleven which analogous to Ramanuja’s forty identities for Rogers-Ramanujan functions and also they established certain interesting partition-theoritic interpretation of some of the modular relations and H. M. Srivastava and M. P. Chaudhary [11] established a set of four new results which depict the interrelationships between \(q\)-product identities, continued fraction identities and combinatorial partition identities.

On page 366 of his ‘Lost’ notebook [10], Ramanujan has recorded a continued fraction
\[
G(q) := \frac{q^{1/3}}{1 + \frac{q + q^2}{1 + \frac{q^2 + q^4}{1 + \ldots}}}, \quad |q| < 1,
\]
and claimed that there are many results of \(G(q)\) which are analogous to the famous Roger’s-Ramanujan continued fraction. Motivated by Ramanujan’s claim H. H. Chan [6], N. D. Baruah [2], K. R. Vasuki and B. R. Srivatsa Kumar [15] have established new identities providing the relations between \(G(q)\) and seven continued fractions \(G(-q), G(q^2), G(q^3), G(q^5), G(q^7), G(q^{11})\) and \(G(q^{13})\). We conclude this introduction by recalling certain results on \(G(q)\) stated by Ramanujan [4] and H. H. Chan [6].

\[
G(-q) := q^{1/3} \frac{\chi(q)}{\chi(q^3)}
\]

where \(\chi(q)\) is defined as \(\chi(q) = (-q; q^2)_\infty\).

\[
G(q) + G(-q) + 2G^2(-q)G^2(q) = 0
\]

and

\[
G^2(q) + 2G^2(q^2)G(q) - G(q^2) = 0.
\]

For a proof of (1.5) and (1.6), see [6].

Motivated by the above works in this paper, we establish some new \(P-Q\) type modular equations, by employing Ramanujan’s modular equations.
2. Main Results

Theorem 2.1. If
\[ X := q^{1/3} \frac{\chi(q) \chi(q^6)}{\chi(q^3) \chi(q^2)} \quad \text{and} \quad Y := q^{2/3} \frac{\chi(q^2) \chi(q^{12})}{\chi(q^6) \chi(q^4)} \]
then
\[ 2X^2 - 22Y^4 X^3 - 2Y + 4Y^2 X - 18X^2 Y^3 + 17Y^9 X^2 - 10Y^8 X + 17Y^{10} X^3 + Y^{11} X \]
\[ + 34Y^5 X + 328Y^7 X^3 - 160Y^6 X^2 - 30Y^7 X^6 - 30Y^6 X^5 + 12Y^5 X^4 - 371Y^8 X^4 + 328Y^9 X^5 \]
\[ - 10Y^{11} X^4 - 160Y^{10} X^6 + 34Y^{11} X^7 - 22Y^9 X^8 + 12Y^8 X^7 + 4Y^{11} X^{10} - 18Y^{10} X^9 - 2Y^{12} X^{11} \]
\[ + 10Y^2 X^4 + 20Y^4 X^6 + 20Y^6 X^8 + 10X^{10} Y^8 + 2Y^{10} X^{12} = 0. \]

Proof. From (1.4) and the definition of \( X \) and \( Y \), it can be seen that
\[ B - AX = 0 \quad \text{and} \quad C - BY = 0. \]
where \( A = G(-q) \), \( B = G(-q^2) \) and \( C = G(-q^4) \). On changing \( q \) to \( q^2 \) in (1.5), we have
\[ G(q^2) + G(-q^2) + 2G^2(-q^2)G^2(q^2) = 0 \]
and also change \( q \) to \(-q\) in (1.6), we have
\[ G^2(-q) + 2G^2(q^2)G(-q) - G(q^2) = 0. \]
Eliminating \( G(q^2) \) between (2.2) and (2.3) using Maple,
\[ 2(AB)^4 - 4(AB)^3 + 3(AB)^2 + AB + A^3 + B^3 = 0. \]
Now on using first identity of (2.1) in (2.4), we obtain
\[ 2B^6 - 4B^4 X + 3B^2 X^2 + X^3 + BX + BX^4 = 0. \]
On replacing \( q \) to \( q^2 \) in (2.4) we see that
\[ 2(BC)^4 - 4(BC)^3 + 3(BC)^2 + BC + B^3 + C^3 = 0. \]
Using second identity of (2.1) in the above, it is easy to see that
\[ 2B^6 Y^4 - 4B^4 Y^3 + 3B^2 Y^2 + Y + B + BY^3 = 0. \]
Finally, on eliminating \( B \) between (2.5) and the above, using Maple we obtain

\[ P(X, Y)Q(X, Y) = 0, \]

where

\[
P(X, Y) = X - 16Y^4X^2 - 6XY^3 - 6Y^5X^3 - 2Y^5 + Y^5X^6 + 10Y^3X^4 + 10Y^2X^3 + 5Y^4X^5 - 2Y^6X^2
\]

and

\[
Q(X, Y) = -2Y + 2X^2 - 22Y^4X^3 + 4Y^2X - 18X^2Y^3 + 17Y^9X^2 - 10Y^8X + 17Y^{10}X^3
\]

\[ + Y^{11}X + 328Y^7X^3 + 34Y^5X - 160Y^6X^2 - 30Y^7X^6 - 30Y^6X^5 + 12Y^5X^4 - 18Y^8X^3
\]

\[ - 371Y^8X^4 + 328Y^9X^5 - 10Y^{11}X^4 - 160Y^{10}X^6 + 34Y^{11}X^7 - 22Y^9X^8 + 12Y^8X^7
\]

\[ + 4Y^{11}X^10 - 2Y^{12}X^11 + 2Y^{10}X^{12} + 10Y^2X^4 + 20Y^6X^8 + 20Y^4X^6 + 10X^{10}Y^8.
\]

By examining the behaviour of the first factor near \( q = 0 \), it can be seen that there is a neighbourhood about the origin, where \( P(X, Y) \neq 0 \) and \( Q(X, Y) = 0 \) in this neighbourhood. Hence by the identity theorem, we have \( Q(X, Y) = 0 \). \( \square \)

**Theorem 2.2.** If

\[
X := q^{1/6} \frac{\chi^2(q^3)}{\chi(q) \chi(q^9)} \quad \text{and} \quad Y := q^{1/3} \frac{\chi^2(q^6)}{\chi(q^2) \chi(q^{18})}
\]

then

\[
\left( \frac{X}{Y} \right)^3 + \left( \frac{Y}{X} \right)^3 + \left( \frac{XY}{X} \right)^{5/2} + 11 \left( \frac{XY}{X} \right)^{1/2} + 11 \left( \frac{XY}{X} \right)^{1/2} \left( \frac{XY}{X} \right)^{1/2} \left( \frac{XY}{X} \right)^{1/2}
\]

\[
(2.6)
\]

\[
= (XY)^3 + \frac{1}{(XY)^3} - 11 \left( XY \right)^2 + \frac{1}{(XY)^2} + 44 \left( XY + \frac{1}{XY} \right) + 8 \left( X^3 + \frac{1}{X^3} \right) + 8 \left( Y^3 + \frac{1}{Y^3} \right) - 86.
\]

**Proof.** From Entry 12(v) of Chapter 17 [4, p. 124], we have

\[
(2.7)
\]

\[
X = \left( \frac{\alpha \gamma (1 - \alpha) (1 - \gamma)}{\beta^2 (1 - \beta)^2} \right)^{1/24}
\]

where \( \beta \) and \( \gamma \) be of the third and ninth degrees respectively, with respect to \( \alpha \). Let

\[
B := q^{1/3} \frac{\chi^2(q^6)}{\chi(q^2) \chi(q^{18})}.
\]
Then from Entry 12(vii) of Chapter 17 [4, p. 124], we have

\[ B = \left\{ \frac{\alpha^2 \gamma^2 (1 - \beta)^2}{\beta^2 (1 - \alpha)(1 - \gamma)} \right\}^{1/24}. \]

By (2.7) and (2.8), we deduce that

\[ (\alpha \gamma \beta)^{1/8} = XB \quad \text{and} \quad \left\{ \frac{(1 - \alpha)(1 - \gamma)}{(1 - \beta)} \right\}^{1/8} = \frac{X^2}{B}. \]

From Entry 3 (xii) and (xiii) of Chapter 20 [4, p. 352-358], we have

\[ (\beta^2 \alpha \gamma)^{1/4} + \left( \frac{(1 - \beta)^2}{(1 - \alpha)(1 - \gamma)} \right)^{1/4} - \left( \frac{\beta^2 (1 - \beta)^2}{\alpha \gamma (1 - \alpha)(1 - \gamma)} \right)^{1/4} = -\frac{3m}{m'} \]

and

\[ \left( \frac{\alpha \gamma}{\beta^2} \right)^{1/4} + \left( \frac{(1 - \alpha)(1 - \gamma)}{(1 - \beta)^2} \right)^{1/4} - \left( \frac{\alpha \gamma (1 - \alpha)(1 - \gamma)}{\beta^2 (1 - \beta)^2} \right)^{1/4} = \frac{m'}{m}. \]

where \( m = z_1/z_3 \) and \( m' = z_3/z_9 \). Thus (2.9), (2.10) and (2.11) yields

\[ M(2B^4 + X^4 - B^2 X^6) - B^2 = 0 \quad \text{and} \quad X^4 + X^2 B^4 - B^2 + 3M X^6 B^2 = 0. \]

where \( M = m/m' \). Which implies

\[ X^6 B^6 - 6B^4 X^4 - B^2 X^6 + B^6 - X^6 + B^2 X^2 + X^8 B^2 = 0. \]

Let

\[ A := q^{1/6} \frac{\chi^2 (-q^3)}{\chi (-q) \chi (-q^9)} \]

Then, from Entry 12(vi) of Chapter 17 [4, p. 124], we have

\[ A = \left\{ \frac{\alpha^2 (1 - \beta)^4}{\beta^2 (1 - \alpha)^2 (1 - \gamma)^2} \right\}^{1/24}. \]

From (2.7) and (2.13), we obtain

\[ \left\{ \frac{(1 - \alpha)(1 - \gamma)}{(1 - \beta)^2} \right\}^{1/8} = \frac{X}{A} \quad \text{and} \quad \left( \frac{\alpha \gamma}{\beta^2} \right)^{1/8} = AX^2. \]

Using the above in (2.10) and (2.11), we deduce

\[ (X^4 A^4 + X^2 - X^6 A^2)M - A^2 = 0 \quad \text{and} \quad X^2 + X^4 A^4 - A^2 + 3M X^6 A^2 = 0. \]

From the above two identities, we obtain

\[ X^8 A^6 - 6X^4 A^4 - A^6 X^6 + A^6 X^2 - X^2 + A^2 + X^6 A^2 = 0. \]
Changing \( q \) to \( q^2 \) in the above, we have

\[
Y^8B^6 - 6Y^4B^4 - B^8Y^6 + B^6Y^2 - Y^2 + B^2 + Y^6B^2 = 0.
\]

Now on eliminating \( B \), between (2.12) and (2.14), using Maple we obtain

\[
C(X, Y)D(X, Y) = 0.
\]

where

\[
C(X, Y) = X^4Y + X^3 + XY + 6Y^2X^2 + Y^3X^3 + Y^3 + XY^4
\]

and

\[
D(X, Y) = X^8Y^5 - Y^7X^7 - 8Y^4X^7 + 7X^7Y + 11X^6Y^6 + 11Y^3X^6 + X^5Y^8 - 44Y^5X^5
+ 11X^5Y^2 - 8X^4Y^7 + 86Y^4X^4 - 8X^4Y + 11Y^6X^3 - 44Y^3X^3 + X^3 + 11Y^5X^2
+ 11Y^2X^2 + XY^7 - 8XY^4 - XY + Y^3.
\]

By examining the behaviour of \( C(X, Y) \) near \( q = 0 \), it can be seen that there is a neighbourhood about the origin, where this factor is not zero. Then the second factor \( D(X, Y) = 0 \) in this neighbourhood. Hence by the identity theorem, we have

\[
D(X, Y) = 0.
\]

On dividing the above throughout by \( (XY)^4 \), we obtain the result.

**Theorem 2.3.** If

\[
X := q^{1/3} \frac{\chi(q^3)\chi(q^3)}{\chi(q)\chi(q^{15})} \quad \text{and} \quad Y := q^{2/3} \frac{\chi(q^6)\chi(q^{10})}{\chi(q^2)\chi(q^{30})}
\]

then

\[
\left( \frac{X}{Y} \right)^3 + \left( \frac{Y}{X} \right)^3 + \left( (XY)^{5/2} + \frac{1}{(XY)^{5/2}} + (XY)^{1/2} + \frac{1}{(XY)^{1/2}} \right) \left( \left( \frac{X}{Y} \right)^{3/2} + \left( \frac{Y}{X} \right)^{3/2} \right)
\]

\[
= (XY)^3 + \frac{1}{(XY)^3} - 5 \left( (XY)^2 + \frac{1}{(XY)^2} \right) + 10 \left( XY + \frac{1}{XY} \right) + 4 \left( X^3 + \frac{1}{X^3} + Y^3 + \frac{1}{Y^3} \right) - 20.
\]

**Proof.** Let

\[
B := q^{2/3} \frac{\chi(-q^6)\chi(-q^{10})}{\chi(-q^2)\chi(-q^{30})}
\]

By Entry 12(v) and (vii) of Chapter 17 [4, p. 124], we have

\[
(2.16)
\]

\[
X = \left\{ \frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)} \right\}^{1/24} \quad \text{and} \quad B = \left\{ \frac{\alpha^2\delta^2(1-\beta)(1-\gamma)}{\beta^2\gamma^2(1-\alpha)(1-\delta)} \right\}^{1/24},
\]
where \( \alpha, \beta, \gamma \) and \( \delta \) are of the first, third, fifth and fifteenth degrees respectively. From (2.16), we deduce that

\[
(\alpha \delta)_{\frac{1}{8}} = XB, \quad \left( \frac{(1 - \alpha)(1 - \delta)}{(1 - \beta)(1 - \gamma)} \right)_{\frac{1}{8}} = \frac{X^2}{B}.
\]

From Entry 11(viii) and (ix) of Chapter 20 [4, p. 383-397], we have

\[
(\alpha \delta)_{\frac{1}{8}} + \left( \frac{(1 - \alpha)(1 - \delta)}{(1 - \beta)(1 - \gamma)} \right)_{\frac{1}{8}} = \sqrt{\frac{m}{m'}}
\]

and

\[
(\beta \gamma)_{\frac{1}{8}} + \left( \frac{(1 - \beta)(1 - \gamma)}{(1 - \alpha)(1 - \delta)} \right)_{\frac{1}{8}} = -\sqrt{\frac{m}{m'}}.
\]

Employing (2.17) in (2.18) and (2.19), we obtain

\[
M(XB^2 + X^2 - X^3B) - B = 0 \quad \text{and} \quad X^2 + B^2X - B + MBX^3 = 0,
\]

where \( M = \sqrt{m/m'} \). Which implies

\[
4X^2B^2 + X^3 - X^4B + XB^4 - X^3B^3 - B^3 - BX = 0.
\]

Let

\[
A := q^{1/3} \frac{\chi(-q^3)\chi(-q^5)}{\chi(-q)\chi(-q^{15})}.
\]

Then, by employing Entry 12(vi) of Chapter 17 [4, p. 124] and (2.16) we deduce that

\[
\left( \frac{(1 - \alpha)(1 - \delta)}{(1 - \beta)(1 - \gamma)} \right)_{\frac{1}{8}} = \frac{X}{A} \quad \text{and} \quad (\alpha \delta)_{\frac{1}{8}} = AX^2.
\]

Using these in (2.18) and (2.19), upon simplifying the resulting identities, and then replacing \( q \) by \( q^2 \), we obtain

\[
4B^2Y^2 + Y - BY^3 + B^4Y^3 - B^3Y^4 - B^3Y - B = 0.
\]

Eliminating \( B \) from (2.20) and (2.21), using Maple we obtain

\[
C(X,Y)D(X,Y) = 0.
\]

where

\[
C(X,Y) = X^4Y + X^3 + XY + 4Y^2X^2 + Y^3X^3 + Y^3 + XY^4
\]

and

\[
D(X,Y) = X^8Y^5 - X^7Y^7 - 4X^7Y^4 + X^7Y + 5X^6Y^6 + X^6Y^3 + X^5Y^2 + X^3Y^8
\]
\[-10Y^5X^5 - 4X^4Y^7 + 20X^4Y^4 - 4X^4Y + X^3Y^6 - 10Y^3X^3 + X^3 + X^2Y^5 + 5Y^2X^2 + XY^7 - 4XY^4 - XY + Y^3.\]

It is same as discussed in Theorem 2.2, that \(C(X, Y) \neq 0\) near \(q = 0\) whereas \(D(X, Y) = 0\) in some neighbourhood \(q = 0\). Hence by identity theorem, we have\(D(X, Y) = 0.\)

Finally, on dividing the above throughout by \((XY)^4\), we obtain the result.

**Theorem 2.4.** If

\[X := \frac{q^{2/3} \chi(q)\chi(q^7)}{\chi(q^3)\chi(q^{21})} \quad \text{and} \quad Y := \frac{q^{4/3} \chi(q^2)\chi(q^{14})}{\chi(q^6)\chi(q^{42})},\]

then

\[p_{12} + 14p_{11} + 229p_{10} + 1328p_9 + 1635p_8 - 15550p_7 - 8529p_6 - 177752p_5 - 37641p_4 + 764070p_3 + 2368728p_2 + 4125694p_1 - 2(2q_{23} + 24q_{21} + 158q_{19} + 586q_{17} + 663q_{15} + 13509q_{13} + 43169q_{11} + 36801q_9 - 14490q_7 - 612613q_6 - 1259739q_5 - 1742545q_4)\]

\[= 2(2q_{21} - 6q_{19} + 51q_{17} - 111q_{15} - 2275q_{11} - 8880q_9 - 22108q_7 - 43267q_5 + 65339q_3 - 79989q_1)r_9 - 2(q_{15} + 2q_{13} + 4q_{11} + 20p_9 + 78q_7 + 88q_5 + 38q_3 + 155q_1)r_{15} + (6p_{11} + 60p_{10} + 162p_9 - 560p_8 - 5129p_7 - 11254p_6 + 10488p_5 + 126726p_4 + 406080p_3 + 828738p_2 + 1238441p_1 + 1410116)s_3 + (p_{10} - 10p_9 + 11p_8 + 60p_7 + 218p_6 + 896p_5 + 2022p_4 + 3816p_3 + 7277p_2 + 111558p_1 + 13838)s_6 + (p_5 - 2p_4 - 3p_3 + 8p_2 + 2p_1 - 12)s_9 + 4907562 = 0.\]

where

\[(2.22) \quad p_n = (XY)^n + \frac{1}{(XY)^n}, \quad q_n = (XY)^{n/2} + \frac{1}{(XY)^{n/2}}, \quad r_n = \left(\frac{X}{Y}\right)^{n/2} + \left(\frac{Y}{X}\right)^{n/2}, \quad s_n = \left(\frac{X}{Y}\right)^n + \left(\frac{Y}{X}\right)^n.\]

**Proof.** Let

\[B := \frac{q^{4/3} \chi(-q^2)\chi(-q^{14})}{\chi(-q^6)\chi(-q^{42})}.\]

Then from Entry 12 (v) and (vii) of Chapter 17 [4, p. 124], we have

\[(2.23) \quad X = \left\{\frac{\beta \delta (1 - \beta)(1 - \delta)}{\alpha \gamma (1 - \alpha)(1 - \gamma)}\right\}^{1/24} \quad \text{and} \quad B = \left\{\frac{\beta^2 \delta^2 (1 - \alpha)(1 - \gamma)}{\alpha^2 \gamma^2 (1 - \beta)(1 - \delta)}\right\}^{1/24}.

where \(\alpha, \beta, \gamma\) and \(\delta\) are of the degrees 1, 3, 7 and 21 respectively. From (2.23), we deduce that

\[(2.24) \quad XB = \left(\frac{\beta \delta}{\alpha \gamma}\right)^{1/8}, \quad \frac{X^2}{B} = \left(\frac{(1 - \beta)(1 - \delta)}{(1 - \alpha)(1 - \gamma)}\right)^{1/8}.
\]
From Entry 13 of Chapter 20 [4, p. 400-403], we have

$$
\left(\frac{\beta\delta}{\alpha\gamma}\right)^{1/4} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/4} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/4}
$$

(2.25)

$$
-2\left(\frac{\beta\delta}{\alpha\gamma}(1-\delta)\right)^{1/8} \left\{ 1 + \left(\frac{\beta\delta}{\alpha\gamma}\right)^{1/8} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/8} \right\} = mm'
$$

and

$$
\left(\frac{\alpha\gamma}{\beta\delta}\right)^{1/4} + \left(\frac{1-\alpha}{1-\delta}\right)^{1/4} + \left(\frac{1-\alpha}{1-\delta}\right)^{1/4}
$$

(2.26)

$$
-2\left(\frac{\alpha\gamma}{\beta\delta}(1-\delta)\right)^{1/8} \left\{ 1 + \left(\frac{\alpha\gamma}{\beta\delta}\right)^{1/8} + \left(\frac{1-\alpha}{1-\delta}\right)^{1/8} \right\} = \frac{9}{mm'}.
$$

Employing (2.24) in (2.25) and (2.26), we obtain

$$
X^2B^4 + X^4 + B^2X^6 - 2BX^3(B + XB^2 + X^2) - B^2M = 0
$$

and

$$
(X^2B^4 + X^4 + B^2 - 2BX(BX^2 + X + B^2))M - 9B^2X^6 = 0.
$$

where $M = mm'$, which implies

$$
X^6 + B^6 + 6B^4X^4 + X^8B^2 - 2X^7B + X^2B^8 + X^6B^6 - 2X^4B^7
$$

(2.27)

$$
+ B^2X^2 - 2B^4X - 2B^4X^7 - 2BX^4 - 2B^2X = 0.
$$

Let

$$
A := q^{2/3} \frac{\chi(-q)\chi(-q^7)}{\chi(-q^3)\chi(-q^{21})}
$$

From Entry 12 (vi) of Chapter 17 [4, p. 124] and (2.23), we deduce that

$$
\frac{X}{A} = \left(\frac{1-\beta}{1-\alpha}\right)^{1/8} \left(\frac{1-\beta}{1-\alpha}\right)^{1/8}
$$

and

$$
AX^2 = \left(\frac{\beta\delta}{\alpha\gamma}\right)^{1/8}.
$$

Employing these in (2.25) and (2.26) up on simplifying, the resulting identities and then replacing $q$ by $q^2$, we obtain

$$
B^2Y^6 + B^2 - 2BY + B^8Y^6 + B^6Y^8 - 2B^2Y^7 + B^6Y^2 - 2B^4Y
$$

(2.28)

$$
-2B^4Y^7 - 2B^7Y^4 - 2BY^4 + Y^2 + 6B^4Y^4 = 0.
$$
On eliminating $B$ between (2.27) and (2.28), using Maple we obtain

$$C(X,Y)D(X,Y)E(X,Y) = 0.$$ 

where

$$C(X,Y) = X^6 Y^6 - 2X^4 Y^7 + X^2 Y^8 - 2XY^7 - 2X^4 Y - 2Y^6 + Y^2 X^8$$

$$+X^2 Y^2 + 6Y^4 X^4 - 2Y^4 X^7 + X^6.$$ 

$$D(X,Y) = Y^6 + 256X^6 Y^6 + 38X^4 Y^7 + 2X^2 Y^8 - 2X^4 Y + 2Y^2 X^8 + X^2 Y^2 + 29Y^4 X^4$$

$$-2Y^4 X + 38Y^4 X^7 - 16X^{14} Y^5 + 14X^4 Y^5 - 10X^5 Y^{11} - 10X^{11} Y^5$$

$$+X^{14} Y^2 + 72X^7 Y^7 - 20Y^{13} X^4 - 35X^4 Y^{10} + 66Y^7 X^6 - 16X^3 Y^{14} - 35X^{10} Y^4 + X^{10} Y^{10}$$

$$-2X^6 Y^{15} - 35X^6 Y^{12} - 4X^7 Y^{13} + 66X^7 Y^{10} + 2X^8 Y^{14} + 14X^8 Y^{11} + 66X^8 Y^8 + 38X^9 Y^{12}$$

$$+72X^9 Y^9 + 256X^{10} Y^{10} + 12X^{11} Y^{11} + 14X^{11} Y^8 - 2X^{12} Y^{15} + 29X^{12} Y^{12} + 38X^{12} Y^9$$

$$-35X^{12} Y^6 - 14X^{10} Y^{13} - 2X^{15} Y^{10} - 2X^{13} Y^{13} - 14X^{13} Y^{10} - 4X^{13} Y^7 + X^{14} Y^{14} + 2X^{14} Y^8$$

$$-2X^{15} Y^{12} + X^{16} Y^{10} + 14X^5 Y^{11} - 4X^3 Y^9 + Y^{14} X^2 - 2XY^{10} - 2YX^{10} - 16X^2 Y^{11} + X^6$$

$$-2X^3 Y^3 - 20X^{13} Y^4 + 66X^9 Y^6 - 16X^2 Y^{11} - 14X^6 Y^3 - 20Y^3 X^{12} - 4Y^3 X^9 + 12X^7 Y^5 - 14X^3 Y^6.$$ 

and $E(X,Y)$ is as in (2.22).

As discussed in Theorem 2.2, by examining the behaviour of $C(X,Y)$ and $D(X,Y)$ near $q = 0$, it can be seen that there is a neighbourhood about the origin, where these factors are not zero. Then the third factor $E(X,Y) = 0$ in this neighbourhood. Hence by identity theorem, we have $E(X,Y) = 0$. Finally, on dividing $E(X,Y)$ throughout by $(PQ)^{16}$ and then simplifying we have the result.

\[\Box\]

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