ON NON-CONGRUENT NUMBERS WITH $1$ MODULO $4$ PRIME FACTORS

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Abstract. In this paper, we use the 2-decent method to find a series of odd non-congruent numbers $\equiv 1 \pmod{8}$ whose prime factors are $\equiv 1 \pmod{4}$ such that the congruent elliptic curves have second lowest Selmer groups, which includes Li and Tian’s result [5] as special cases.

1. Introduction

The congruent number problem is about when a positive integer can be the area of a rational right triangle. A positive integer $n$ is a non-congruent number equivalent to that the congruent elliptic curve

$$E := E^{(n)} : y^2 = x^3 - n^2x$$

has Mordell-Weil rank zero. In [3] and [4], Feng obtained several series of non-congruent numbers for $E^{(n)}$ with the lowest Selmer groups. In [5], Li and Tian obtained a series of non-congruent numbers whose prime factors are $\equiv 1 \pmod{8}$ such that $E^{(n)}$ has second lowest Selmer groups. The essential tool of the above results is the 2-descend method of elliptic curves. In this paper, we will use this method to get a series of odd non-congruent numbers whose prime factors are $\equiv 1 \pmod{4}$ such that $E^{(n)}$ has second lowest Selmer groups, which includes Li and Tian’s result as special cases.

Suppose $n$ is a square-free integer such that $n = p_1 \cdots p_k \equiv 1 \pmod{8}$ and primes $p_i \equiv 1 \pmod{4}$, then by quadratic reciprocity law $\left( \frac{p_i}{p_j} \right) = \left( \frac{p_j}{p_i} \right)$. 

Definition 1.1. Suppose $n = p_1 \cdots p_k \equiv 1 \pmod{8}$ and $p_i \equiv 1 \pmod{4}$. The graph $G(n) := (V, A)$ associated to $n$ is a simple undirected graph with vertex set $V := \{\text{prime } p \mid n\}$ and edge set $A := \{pq : \left( \frac{p}{q} \right) = -1\}$.

Recall for a simple undirected graph $G = (V, A)$, a partition $V = V_0 \cup V_1$ is called even if for any $v \in V_i$ ($i = 0, 1$), $\#\{v \to V_{1-i}\}$ is even. $G$ is called an odd graph if the only even partition is the trivial partition $V = \emptyset \cup V$. Then our main result is:

Theorem 1.2. Suppose $n = p_1 \cdots p_k \equiv 1 \pmod{8}$ and $p_i \equiv 1 \pmod{4}$. If the graph $G(n)$ is odd and $\delta(n)$ (as given by (16)) is 1, then for the congruent elliptic curve $E = E^{(n)}$,

$$\text{rank}_2(E(\mathbb{Q})) = 0 \text{ and } \text{III}(E/\mathbb{Q})[2\infty] \cong (\mathbb{Z}/2\mathbb{Z})^2.$$ 

As a consequence, $n$ is a non-congruent number.

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The following Corollary is Li and Tian’s result, cf. [3]:

**Corollary 1.3.** Suppose \( n = p_1 \cdots p_k \) and \( p_i \equiv 1 \pmod{8} \). If the graph \( G(n) \) is odd and the Jacobi symbol \( \left( \frac{1+\sqrt{-1}}{n} \right) = -1 \), then for \( E = E^{(n)} \),

\[
\text{rank}_{\mathbb{Z}}(E(\mathbb{Q})) = 0 \quad \text{and} \quad \text{III}(E/\mathbb{Q})[2^\infty] \cong (\mathbb{Z}/2\mathbb{Z})^2.
\]

As a consequence, \( n \) is a non-congruent number.

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## 2. Review of 2-descent method.

In this section, we recall the 2-descent method of computing the Selmer groups of elliptic curves. This section follows [5] pp 232-233, also cf. [1] §5 and [7] X.4.

For an isogeny \( \varphi : E \to E' \) of elliptic curves defined over a number field \( K \), one has the following fundamental exact sequence

\[
0 \to E'(K)/\varphi E(K) \to S^{(\varphi)}(E/K) \to III(E/K)[\varphi] \to 0.
\]  

(2)

Moreover, if \( \psi : E' \to E \) is another isogeny, for the composition \( \psi \circ \varphi : E \to E \), then the following diagram of exact sequences commutes (cf. [8] p 5):

\[
\begin{array}{ccccccccc}
0 & \to & E'(K)/\varphi E(K) & \to & S^{(\varphi)}(E/K) & \to & III(E/K)[\varphi] & \to & 0 \\
\downarrow \psi & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & E(K)/\psi \varphi E(K) & \to & S^{(\psi \varphi)}(E'/K) & \to & III(E'/K)[\psi \varphi] & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & E(K)/\psi E'(K) & \to & S^{(\psi)}(E'/K) & \to & III(E'/K)[\psi] & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0
\end{array}
\]

Now suppose \( n \) is a fixed odd positive square-free integer, \( K = \mathbb{Q} \), and \( E/\mathbb{Q}, E'/\mathbb{Q}, \varphi, \psi = \varphi^\vee \) are given by

\[
E = E^{(n)} : y^2 = x^3 - n^2 x, \quad E' = E^{(\hat{n})} : y^2 = x^3 + 4n^2 x,
\]

\[
\varphi : E \to E', \ (x,y) \mapsto \left( \frac{y^2}{x^2}, \frac{y(x^2+n^2)}{x^2} \right),
\]

\[
\psi : E' \to E, \ (x,y) \mapsto \left( \frac{y^2}{4x^2}, \frac{y(x^2-4n^2)}{8x^2} \right).
\]
Similarly, suppose \( \phi \psi = [2], \psi \phi = [2] \). In this case \( \iota_1 \) and \( \iota_2 \) are exact. Let \( \tilde{S}^{(\psi)}(E'/\mathbb{Q}) \) denote the image of \( S^{(\psi \phi)}(E'/\mathbb{Q}) \) in \( S^{(\psi)}(E'/\mathbb{Q}) \). Then

\[
\#\text{III}(E/\mathbb{Q})[\phi] = \frac{\#S^{(\phi)}(E/\mathbb{Q})}{\#E'(\mathbb{Q})/\phi E(\mathbb{Q})}, \quad \#\text{III}(E'/\mathbb{Q})[\psi] = \frac{\#S^{(\psi)}(E'/\mathbb{Q})}{\#E(\mathbb{Q})/\psi E'(\mathbb{Q})},
\]

and

\[
\#\text{III}(E/\mathbb{Q})[2] = \frac{\#S^{(\phi)}(E/\mathbb{Q}) \cdot \#\tilde{S}^{(\psi)}(E'/\mathbb{Q})}{\#E'(\mathbb{Q})/\phi E(\mathbb{Q}) \cdot \#E(\mathbb{Q})/\psi E'(\mathbb{Q})}.
\]

Similarly,

\[
\#\text{III}(E'/\mathbb{Q})[2] = \frac{\#S^{(\psi)}(E'/\mathbb{Q}) \cdot \#\tilde{S}^{(\phi)}(E'/\mathbb{Q})}{\#E'(\mathbb{Q})/\psi E'(\mathbb{Q}) \cdot \#E(\mathbb{Q})/\phi E' \mathbb{Q})}.
\]

The 2-descent method to compute the Selmer groups \( S^{(\phi)}(E/\mathbb{Q}) \) and \( S^{(\psi)}(E'/\mathbb{Q}) \) is as follows (cf. [7] for general elliptic curves). Let

\[
S = \{ \text{prime factors of } 2n \} \cup \{ \infty \},
\]

\[
Q(S, 2) = \{ b \in \mathbb{Q}^+ / \mathbb{Q}^{x^2} : 2 \mid \text{ord}_p(b), \forall p \notin S \}.
\]

Note that \( Q(S, 2) \) is represented by factors of \( 2n \) and we identify these two sets. By the exact sequence

\[
0 \to E'(\mathbb{Q})/\phi E(\mathbb{Q}) \xrightarrow{i} Q(S, 2) \xrightarrow{j} WC(E/\mathbb{Q})[\phi],
\]

where

\[
i : (x, y) \mapsto x, \quad O \mapsto 1, \quad (0, 0) \mapsto 4n^2, \quad j : d \mapsto \{ C_d/\mathbb{Q} \}
\]

and \( C_d/\mathbb{Q} \) is the homogeneous space for \( E/\mathbb{Q} \) defined by the equation

\[
C_d : dw^2 = d^2 + 4n^2 z^4,
\]

the \( \phi \)-Selmer group \( S^{(\phi)}(E/\mathbb{Q}) \) is then

\[
S^{(\phi)}(E/\mathbb{Q}) \cong \{ d \in Q(S, 2) : C_d(\mathbb{Q}) \neq \emptyset, \forall p \in S \}.
\]

Similarly, suppose

\[
C_d' : dw^2 = d^2 - n^2 z^4.
\]

The \( \psi \)-Selmer group \( S^{(\psi)}(E'/\mathbb{Q}) \) is then

\[
S^{(\psi)}(E'/\mathbb{Q}) \cong \{ d \in Q(S, 2) : C_d'(\mathbb{Q}) \neq \emptyset, \forall p \in S \}.
\]

The method to compute \( \tilde{S}^{(\psi)}(E/\mathbb{Q}) \) follows from [1] §5, Lemma 10:

**Lemma 2.1.** Let \( d \in S^{(\psi)}(E/\mathbb{Q}) \). Suppose \((\sigma, \tau, \mu)\) is a nonzero integer solution of \( d \sigma^2 = d^2 \tau^2 + 4n^2 \mu^2 \). Let \( \mathcal{M}_b \) be the curve corresponding to \( b \in \mathbb{Q}^x / \mathbb{Q}^{x^2} \) given by

\[
\mathcal{M}_b : \ dw^2 = d^2 + 4n^2 \phi^4, \quad d \sigma w - d^2 \tau t^2 - 4n^2 \mu z^2 = bu^2.
\]

Then \( d \in \tilde{S}^{(\psi)}(E/\mathbb{Q}) \) if and only if there exists \( b \in Q(S, 2) \) such that \( \mathcal{M}_b \) is locally solvable everywhere.

Note that the existence of \( \sigma, \tau, \mu \) follows from Hasse-Minkowski theorem (cf. [6]).
3. Local computation

We need a modification of the Legendre symbol. For \( x \in \mathbb{Q}_p \) or \( x \in \mathbb{Q} \) such that \( \text{ord}_p(x) \) is even, we set
\[
\left( \frac{x}{p} \right) := \left( \frac{xp^{-\text{ord}_p(x)}}{p} \right).
\]
Thus \( (\cdot)^{\frac{1}{2}} \) defines a homomorphism from \( \{ x \in \mathbb{Q}^\times / \mathbb{Q}^\times : \text{ord}_p(x) \text{ is even} \} \) to \( \{ \pm 1 \} \).

### Lemma 3.1. Computation of Selmer groups.

In this subsection, we will find the conditions when \( C_d \) or \( C'_d \) is locally solvable. We will not give details since one only need to consider the valuations and quadratic residue.

#### Lemma 3.1. \( d \in S^{(\circ)}(E/\mathbb{Q}) \) if and only if \( d \) satisfies

1. \( d > 0 \) has no prime factor \( p \equiv 3 \pmod{4} \);
2. \( \left( \frac{n/d}{p} \right) = 1 \) for all odd \( p \mid d \);
3. \( \left( \frac{d}{p} \right) = 1 \) for all odd \( p \mid (2n/d) \);
4. if \( 2 \mid d \), \( n \equiv \pm 1 \pmod{8} \).

**Proof.** In this case \( C_d : dw^2 = d^2t^4 + 4n^2z^4 \). It is obvious that \( C_d(\mathbb{R}) \neq \emptyset \iff d > 0 \).

Assume \( d > 0 \).

(i) If \( 2 \nmid d \mid n \), then \( C_d : w^2 = d(t^4 + 4(n/d)^2z^4) \).

- \( p = 2 \). \( C_d(\mathbb{Q}_2) \neq \emptyset \iff d \equiv 1 \pmod{4} \).
- \( p \mid d \). \( C_d(\mathbb{Q}_p) \neq \emptyset \iff \left( \frac{n/d}{p} \right) = 1 \) and \( p \equiv 1 \pmod{4} \).
- \( p \nmid d \). \( C_d(\mathbb{Q}_p) \neq \emptyset \iff \left( \frac{d}{p} \right) = 1 \).

(ii) If \( 2 \mid d \mid 2n \), then \( C_d : w^2 = d(t^4 + (2n/d)^2z^4) \).

- \( p = 2 \). \( C_d(\mathbb{Q}_2) \neq \emptyset \iff d \equiv 2 \pmod{8} \), \( n \equiv \pm 1 \pmod{8} \).
- \( 2 \nmid p \mid d \). \( C_d(\mathbb{Q}_p) \neq \emptyset \iff \left( \frac{n/d}{p} \right) = 1 \) and \( p \equiv 1 \pmod{4} \).
- \( p \mid d \). \( C_d(\mathbb{Q}_p) \neq \emptyset \iff \left( \frac{d}{p} \right) = 1 \).

Combining (i) and (ii) follows the lemma. \( \square \)

#### Lemma 3.2. \( d \in S^{(\circ)}(E'/\mathbb{Q}) \) if and only if \( d \) satisfies

1. \( d \equiv \pm 1 \pmod{8} \) or \( n/d \equiv \pm 1 \pmod{8} \)
2. \( \left( \frac{n/d}{p} \right) = 1 \) for all \( p \mid d, p \equiv 1 \pmod{4} \);
3. \( \left( \frac{d}{p} \right) = 1 \) for all \( p \mid (n/d), p \equiv 1 \pmod{4} \).

**Proof.** In the case \( C'_d : dw^2 = d^2t^4 - n^2z^4 \).

(i) If \( 2 \mid d \), consider the 2-valuation of each side, we see \( C'_d(\mathbb{Q}_2) = \emptyset \).

(ii) If \( 2 \nmid d \mid n \), then \( C'_d : w^2 = d(t^4 - (n/d)^2z^4) \).

- \( p = 2 \). \( C'_d(\mathbb{Q}_2) \neq \emptyset \iff d \equiv \pm 1 \pmod{8} \) or \( n/d \equiv \pm 1 \pmod{8} \).
- \( p \mid d \). \( C'_d(\mathbb{Q}_p) \neq \emptyset \iff \left( \frac{n/d}{p} \right) = 1 \) or \( \left( \frac{-n/d}{p} \right) = 1 \).
- \( p \nmid d \). \( C'_d(\mathbb{Q}_p) \neq \emptyset \iff \left( \frac{d}{p} \right) = 1 \) or \( \left( \frac{-d}{p} \right) = 1 \).

Combining (i) and (ii) follows the lemma. \( \square \)
3.2. Computation of the images of Selmer groups. Suppose $0 < 2d \in S^{(\varphi)}(E/Q)$, $d$ is odd with no $\equiv 3 \pmod{4}$ prime factor, we want to find a necessary condition for $2d \in S^{(\varphi)}(E/Q)$. Write $2d = r^2 + \mu^2$ and select the triple $(\sigma, \tau, \mu)$ in Lemma 2.1 to be $(2n, \sigma \tau/d, \mu)$. Then the defining equations of $M_{4ndb}$ in (1) can be written as

$$w^2 = 2d(t^4 + (n/d)^2)z^4, \quad w - \tau t^2 - (n/d)\mu z^2 = bu^2.$$  

By abuse of notations, we denote the above curve by $M_b$. We use the notation $O(p^m)$ to denote a number with $p$-adic valuation $\geq m$.

The case $p | d$. For $i_p \equiv \tau / \mu \pmod{p\mathbb{Z}_p}$, $i_p \in \mathbb{Z}_p$ and $i_p^2 = -1$, then $p \mid (\tau - i_p\mu), \quad p \nmid (\tau + i_p\mu)$.

It’s easy to see $v(t) = v(z)$, we may assume that $z = 1$, $t^2 \equiv \pm \frac{i_\beta n}{d} \pmod{p}$, then $M_b$ is given by

$$M_b : w^2 = 2d(t^4 + (n/d)^2), \quad w - \tau t^2 - (n/d)\mu = bu^2.$$

(i) If $v(bu^2) = m \geq 3$, then by $w^2 = (\tau t^2 + \frac{\tau \mu}{d} + O(p^m))^2 = 2d(t^4 + \frac{\tau^2}{d^2})$,

$$\left(\frac{\mu t^2 - \frac{n\tau}{d}}{d^2}\right)^2 = O(p^m).$$

Let $t^2 = \frac{\tau}{d^2} + \beta$, where $v(\beta) = \alpha \geq \frac{m}{2}$, then

$$w^2 = 2d\left(\frac{n}{d}\right)^2 + \left(\frac{n\tau}{d\mu}\right)^2 + 2\frac{n\tau}{d\mu} \beta + \beta^2,$$

$$= \frac{4n^2}{\mu^2}(1 + \frac{\tau \mu}{n\beta} + \frac{d\mu^2}{2n^2\beta^2}).$$

Take the square root on both sides, then

$$w = \pm \frac{2n}{\mu}\left(1 + \frac{1}{2}\frac{\tau \mu}{n\beta} + \frac{d\mu^2}{2n^2\beta^2} - \frac{1}{8}\left(\frac{\tau \mu}{n\beta}\right)^2 + O(p^{3\alpha - 3})\right)$$

$$= \pm \left(\frac{2n}{\mu} + \tau \beta + n\mu\left(\frac{\beta}{2n}\right)^2 + O(p^{3\alpha - 2})\right),$$

but on the other hand,

$$w = \tau t^2 + \frac{n\mu}{d} + bu^2 = \frac{2n}{\mu} + \tau \beta + bu^2.$$

The sign must be positive and

$$bu^2 = n\mu\left(\frac{\beta}{2n}\right)^2 + O(p^{3\alpha - 2}),$$

thus $p \mid b$, $\left(\frac{\beta}{p}\right) = \left(\frac{n\mu}{p}\right), \quad \left(\frac{\beta}{p}\right) = \left(\frac{\mu}{p}\right) = \left(\frac{2n}{p}\right)$.

(ii) If $v(bu^2) = m \leq 2$ and $t^2 \equiv \frac{i_\beta n}{d} \pmod{p}$, let $t^2 = \frac{i_\beta n}{d} + p\alpha i_p$, then

$$w^2 = 2d \cdot p\alpha i_p \cdot \left(\frac{2i_\beta n}{d} + p\alpha i_p\right) = -4p^2 \cdot \frac{n\alpha}{p}(1 + \frac{p\alpha}{2n}$$.
and
\[ w_1 = \frac{w}{p} = \pm 2ip\sqrt{\frac{na}{p} \left( 1 + \frac{pd\alpha}{4n} + O(p^2) \right)}, \]
\[ bu^2 = w - \tau t^2 - \frac{n\mu}{d} \]
\[ = \pm 2pi_p\sqrt{\frac{na}{p} \left( 1 + \frac{pd\alpha}{4n} \right)} - \frac{i_p\tau n}{d} - \frac{n\mu}{d} - \tau \alpha i_p + O(p^3) \]
\[ = -\frac{p^2i_p\tau}{n} \left( \frac{na}{p} \pm \frac{n}{p\tau} \right)^2 - \frac{i_p}{2d\tau} (\tau - i_p\mu)^2 \pm 2p^2i_p\sqrt{\frac{na}{p} \alpha \, d\alpha}{4n} + O(p^3). \]

If \( v(bu^2) = 2 \), then \( \sqrt{\frac{na}{p}} \equiv \pm \frac{n}{p\tau} \pmod{p} \), and
\[ bu^2 = -\frac{n\mu i_p}{2d\tau} (\tau - i_p\mu)^2 \pm 2p^2i_p\sqrt{\frac{na}{p} \alpha \, d\alpha}{4n} + O(p^3) \]
\[ = -\frac{n\mu i_p (\tau - i_p\mu)^3 (3\tau + i_p\mu)}{8d\tau^3} + O(p^3) \]
\[ = -\frac{n\mu i_p (\tau - i_p\mu)^3}{2d\tau^2} + O(p^3) = O(p^3), \]
which is impossible! Thus \( v(bu^2) = 1 \) and \( p \mid b \),
\[ \left( \frac{b/p}{p} \right) = \left( -\frac{pi_p\tau/n}{p} \right) = \left( 2p\tau/n \right), \quad \text{or} \quad \left( \frac{n/b}{p} \right) = \left( \frac{2\tau}{p} \right). \]

(iii) If \( v(bu^2) = m \leq 2 \) and \( t^2 \equiv -i_p(n/d) \pmod{p} \), then
\[ bu^2 = w - \tau t^2 - (n/d)\mu = (\tau i_p - \mu)n/d + O(p) \]
\[ = 2i_p\tau n/d + O(p) = (1 + i_p)^2 \cdot \frac{n}{d} \cdot \tau + O(p), \]
thus \( p \nmid b \) and \( \left( \frac{b}{p} \right) = \left( \frac{n/d}{p} \right). \)
Note that \( 2\tau \equiv \tau + \mu i_p \pmod{p} \) and \( \left( \frac{2n/d}{p} \right) = 1 \), hence we have

Lemma 3.3. The curve \( M_b \) defined by \( \mathbf{11} \) is locally solvable at \( p \mid d \) if and only if
\[ \text{either} \quad p \mid b, \quad \left( \frac{n/b}{p} \right) = \left( \frac{\tau + \mu i_p}{p} \right); \quad \text{or} \quad p \nmid b, \quad \left( \frac{b}{p} \right) = \left( \frac{\tau + \mu i_p}{p} \right). \]

The case \( p \mid \frac{n}{d} \). In this case \( t \) is a \( p \)-adic unit if and only if \( w \) is so.

(i) If \( v(w) = v(t) = 0 \), then \( w = \pm \sqrt{2d} t^2 \pmod{p} \) and \( (\pm \sqrt{2d} - \tau) t^2 \equiv bu^2 \pmod{p} \). Since \( (\sqrt{2d} - \tau)(\sqrt{2d} + \tau) = 2d - \tau^2 = \mu^2 \) and \( \sqrt{2d} \pm \tau \) are co-prime, \( \text{ord}_p(\sqrt{2d} - \tau) \) is even and \( \left( \frac{\sqrt{2d} - \tau}{p} \right) \) is well defined. Then \( M_b \) is locally solvable if and only if
\[ p \nmid b, \quad \left( \frac{2d}{p} \right) = 1 \quad \text{and} \quad \left( \frac{b}{p} \right) = \left( \frac{\sqrt{2d} - \tau}{p} \right). \]

(ii) If \( v(z) = 0 \) and \( w = pw_1, t = pt_1 \), then \( w_1^2 = 2d(p^2t_1^2 + (\frac{a}{p})^2 z^2) \), \( w_1 \equiv \pm \sqrt{2d} \frac{a}{pd} z^2 \pmod{p} \) and \( bu^2/p \equiv (\pm \sqrt{2d} - \mu) \frac{a}{pd} z^2 \pmod{p} \). Thus \( M_b \) is locally solves.
solvable if and only if

\[ p \mid b, \left( \frac{2d}{p} \right) = 1 \text{ and } \left( \frac{n/(db)}{p} \right) = \left( \frac{\sqrt{2d} - \mu}{p} \right). \]

Note that

\[ 2(\sqrt{2d} - \tau)(\sqrt{2d} - \mu) = (\tau + \mu - \sqrt{2d})^2 \Rightarrow \left( \frac{\sqrt{2d} - \mu}{p} \right) = \left( \frac{2(\sqrt{2d} - \tau)}{p} \right). \]

From now on, suppose \( n = p_1 \cdots p_k \equiv 1 \pmod{8} \) and \( p_i \equiv 1 \pmod{4} \). Pick \( i_p \in \mathbb{Z}/p \) such that \( i_p^2 = -1 \), then

\[ \sqrt{2d} - \tau = -\left( \frac{\tau + \mu i_p}{p} \right) \cdot \frac{1}{2} \left( 1 - \frac{\sqrt{2d}}{\tau + \mu i_p} \right)^2. \]

Note that \( \left( \frac{2d}{p} \right) = 1 \), we have

**Lemma 3.4.** \( M_b \) defined by (11) is locally solvable at \( p \mid \frac{n}{d} \) if and only if

\[ p \mid b, \left( \frac{2d}{p} \right) = 1 \text{ and } \left( \frac{n/b}{p} \right) = \left( \frac{\tau + \mu i_p}{p} \right) \left( \frac{2}{p} \right), \]

or \( p \nmid b, \left( \frac{2d}{p} \right) = 1 \) and \( \left( \frac{b}{p} \right) = \left( \frac{\tau + \mu i_p}{p} \right) \left( \frac{2}{p} \right). \)

By Lemmas 2.1, 3.1, 3.3 and 3.4, and we have

**Proposition 3.5.** Suppose \( n = p_1 \cdots p_k \equiv 1 \pmod{8} \) and \( p_i \equiv 1 \pmod{4} \), then \( 2d \in S^{(\varphi)}(E/\mathbb{Q}) \) if and only if \( d > 0 \) and \( \left( \frac{2n/d}{p} \right) = 1 \) for \( p \mid d \), \( \left( \frac{2d}{p} \right) = 1 \) for \( p \nmid \frac{n}{d} \).

In this case \( 2d \in \tilde{S}^{(\varphi)}(E/\mathbb{Q}) \) only if there exists \( b \in \mathbb{Q}(S, 2) \) satisfying:

1. If \( p \mid d, i_p \equiv \tau/\mu \pmod{p\mathbb{Z}/p} \), \( i_p^2 = -1 \),

\[ p \mid b, \left( \frac{n/b}{p} \right) = \left( \frac{\tau + \mu i_p}{p} \right), \text{ or } p \nmid b, \left( \frac{b}{p} \right) = \left( \frac{\tau + \mu i_p}{p} \right) \left( \frac{2}{p} \right). \]

2. If \( p \mid \frac{n}{d}, i_p^2 = -1 \),

\[ p \mid b, \left( \frac{n/b}{p} \right) = \left( \frac{2(\tau + \mu i_p)}{p} \right), \text{ or } p \nmid b, \left( \frac{b}{p} \right) = \left( \frac{2(\tau + \mu i_p)}{p} \right). \]

4. **Proof of the main result**

4.1. **Some facts about graph theory.** We now recall some notations and results in graph theory, cf. [3, 4].

**Definition 4.1.** Let \( G = (V, A) \) be a simple undirected graph. Suppose \( |V| = k \).

The **adjacency matrix** \( M(G) = (a_{ij}) \) of \( G \) is the \( k \times k \) matrix defined as

\[ a_{ij} := \begin{cases} 0, & \text{if } \overline{v_i v_j} \notin A; \\ 1, & \text{if } \overline{v_i v_j} \in A. \end{cases} \]

(12)

The **Laplace matrix** \( L(G) \) of \( G \) is defined as

\[ L(G) = \text{diag}\{d_1, \ldots, d_k\} - M(G) \]

(13)

where \( d_i \) is the degree of \( v_i \).
Theorem 4.2. Let $G$ be a simple undirected graph and $L(G)$ its Laplace matrix.
(1) The number of even partitions of $V$ is $2^{k-1-r}$, where $r = \text{rank}_{\mathbb{F}_2} L(G)$.
(2) The graph $G$ is odd if and only if $r = k - 1$.
(3) If $G$ is odd, then the equations

$$L(G) \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} = \begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix}$$

has solutions if and only if $t_1 + \cdots + t_k = 0$.

Proof. The proof of the first two parts follows from [3]. We have a bijection

$$\mathbb{F}_2^k / \{(0, \ldots, 0), (1, \ldots, 1)\} \sim \text{\{partitions of $V$\}}$$

$$(c_1, \ldots, c_k) \mapsto (V_0, V_1)$$

where $V_i = \{v_j : c_j = i \ (1 \leq j \leq k)\}$, $i \in \{0, 1\}$.

Regard $L(G) = \text{diag}(d_1, \ldots, d_k) - (a_{ij})$ as a matrix over $\mathbb{F}_2$. If

$$L(G) \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_k \end{pmatrix} \in \mathbb{F}_2^k,$$

then if $v_i \in V_t, t \in \{0, 1\}$,

$$b_i = d_ic_i + \sum_{j=1}^k a_{ij}c_j = \sum_{j=1}^k a_{ij}(c_i + c_j) = \sum_{j=1}^k a_{ij}(t + c_j) = \sum_{c_j = 1-t} a_{ij} = \# \{v_i \rightarrow V_{1-t} \} \in \mathbb{F}_2.$$

(1) The number of even partitions is

$$\frac{1}{2} \# \left\{(c_1, \ldots, c_k) \in \mathbb{F}_2^n : L(G) \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \right\} = 2^{k-1-r}.$$ 

(2) follows from (1) easily.

(3) Since $L$ is of rank $k-1$, the image space of $L$ is of dimensional $k-1$, but it lies in the hyperplane $x_1 + \cdots + x_k = 0$, thus they coincide and the result follows. \(\square\)

4.2. Graph $G(n)$ and Selmer groups of $E$ and $E'$. From now on, we suppose

$$n = p_1 \cdots p_k \equiv 1 \pmod{8}$$

and $p_1 \equiv 1 \pmod{4}$.

Recall for an integer $a$ prime to $n$, the Jacobi symbol $\left(\frac{a}{n}\right) = \prod_{p|n} (\frac{a}{p})$, which is extended to a multiplicative homomorphism from $\{a \in \mathbb{Q}^\times / \mathbb{Q}^\times 2 : \text{ord}_p(a) \text{ even for } p | n\}$ to $\{\pm 1\}$. Set

$$\left[\frac{a}{n}\right] := \frac{1}{2} \left(1 - \left(\frac{a}{n}\right)\right).$$ 

(14)

The symbol $[\frac{n}{a}]$ is an additive homomorphism from $\{a \in \mathbb{Q}^\times / \mathbb{Q}^\times 2 : \text{ord}_p(a) \text{ even for } p | n\}$ to $\mathbb{F}_2$.

By definition, the adjacency matrix $M(G(n))$ has entries $a_{ij} = [\frac{p_i}{p_j}]$. For $0 < d | n$, we denote by $\{d, \frac{n}{d}\}$ the partition $\{p : p | d\} \cup \{p : p | \frac{n}{d}\}$ of $G(n)$.

The following proposition is a translation of results in Lemma 4.1 and Lemma 3.2

Proposition 4.3. Given a factor $d$ of $n$.
(1) For the Selmer group $S^{(d)}(E/\mathbb{Q})$, 

(1-a) \( d \in S^{(\nu)}(E/\mathbb{Q}) \) if and only if \( d > 0 \) and \( \{d, n/d\} \) is an even partition of \( G(n) \);

(1-b) Suppose

\[
c_i = \begin{cases} 
1, & \text{if } p_i \mid d, \\
0, & \text{if } p_i \nmid \frac{n}{d}; 
\end{cases}
t_i = \left\lfloor \frac{2}{p_i} \right\rfloor.
\]

Then \( 2d \in S^{(\nu)}(E/\mathbb{Q}) \) if and only if \( d > 0 \) and

\[
L(G) \left( \begin{array}{c} c_1 \\ \vdots \\ c_k \end{array} \right) = \left( \begin{array}{c} t_1 \\ \vdots \\ t_k \end{array} \right).
\]

(2) For the Selmer group \( S^{(\nu)}(E'/\mathbb{Q}) \),

(2-a) \( d \in S^{(\nu)}(E'/\mathbb{Q}) \) if and only if \( d \equiv \pm 1 \pmod{8} \) and \( \{d, n/d\} \) is an even partition of \( G(n) \);

(2-b) \( 2d \notin S^{(\nu)}(E'/\mathbb{Q}) \).

Proof. One only has to show (1-b), the rest is easy. For any \( i \) let \([i]\) be the set of \( j \) such that \( p_i \) and \( p_j \) are both prime divisors of \( d \) or \( n/d \). Then

\[
d_i c_i + \sum_{j \neq i} a_{ij} c_j = \sum_{j \neq i} a_{ij} (c_i + c_j) = \sum_{j \notin [i]} a_{ij} = \left\lfloor \frac{d}{p_i} \right\rfloor \text{ or } \left\lfloor \frac{n/d}{p_i} \right\rfloor.
\]

Then (1-b) follows from Lemma 3.1.

Applying Theorem 4.2(3) to Proposition 4.3, then we have

**Corollary 4.4.** If \( G(n) \) is odd, there exists a unique factor \( 0 < d < \sqrt{2n} \) of \( n \) such that

\[
S^{(\nu)}(E/\mathbb{Q}) = \{1, 2d, 2n/d, n\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},
\]

and

\[
S^{(\nu)}(E'/\mathbb{Q}) = \{\pm 1, \pm n\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.
\]

For the \( d \) given in Corollary 4.4 write \( 2d = \tau^2 + \mu^2 \). If \( 2d \in S^{(\nu)}(E/\mathbb{Q}) \), we suppose \( b \) satisfies the condition that \( M_b \) defined by (11) is locally solvable everywhere. Suppose \( c' = (c'_1, \ldots, c'_k)^T \) and \( t' = (t'_1, \ldots, t'_k)^T \) are given by

\[
c'_j = \begin{cases} 
1, & \text{if } p_j \mid b, \\
0, & \text{if } p_j \nmid b;
\end{cases}
t'_j = \begin{cases} 
\left\lfloor \frac{\tau + \mu i p_j}{p_j} \right\rfloor, & \text{if } p_j \mid d, \\
\left\lfloor \frac{2(\tau + \mu i p_j)}{p_j} \right\rfloor, & \text{if } p_j \nmid \frac{n}{d}.
\end{cases}
\]

By Proposition 3.5 \( Lc' = t' \), i.e., \( Lv = t' \) has a solution \( v = c' \), which means that the summation of \( t'_j \) must be zero in \( \mathbb{F}_2 \) by Theorem 4.2(3).

**Definition 4.5.** Suppose \( n \) is given such that \( G(n) \) is an odd graph. For the unique factor \( d \) given in Corollary 4.4 write \( 2d = \tau^2 + \mu^2 \) and \( \frac{2d}{d} = \tau^2 + \mu^2 \). Let \( i \in \mathbb{Z}/n\mathbb{Z} \) be defined by

\[
i \equiv \frac{\tau}{\mu} \pmod{d}, \quad i \equiv \frac{\tau'}{\mu'} \pmod{\frac{n}{d}}.
\]

We define

\[
\delta(n) := \left\lfloor \frac{\tau + \mu i}{n} \right\rfloor + \left\lfloor \frac{2}{d} \right\rfloor \in \mathbb{F}_2.
\]

Then the following is a consequence of Proposition 3.3.
Corollary 4.6. If $G(n)$ is odd and $\delta(n) = 1$, then
$$\tilde{S}(\psi)(E/Q) = \{1\}.$$ 

Proof. Let $\lambda^*$ be the $\mathbb{F}_2$-rank of $\tilde{S}(\psi)(E/Q)$, $\lambda$ be the $\mathbb{F}_2$-rank of $S(\psi)(E/Q)$, then $\lambda = 2$. The existence of the Cassels’ skew-symmetric bilinear form on III implies that the difference $\lambda - \lambda^*$ is even.

By the above analysis, $\delta(n) = \sum_j t_j^i \neq 0$, thus $2d \notin \tilde{S}(\psi)(E/Q)$, we have $\lambda^* < \lambda$, $\lambda^* = 0$. \hfill \square

Remark. If we replace $d$ by $\frac{n}{d}$ in the definition, $\delta(n)$ is invariant. Indeed, $[\frac{2}{d}] = [\frac{2}{n/d}]$. For the other term,
$$\left[\frac{\tau + \mu i}{n}ight] = \left[\frac{\tau + \mu i}{d}\right] + \left[\frac{\tau + \mu i}{n/d}\right]$$
where $i \equiv \tau / \mu \pmod d$, $i' \equiv \tau' / \mu' \pmod {n/d}$. Let $u = (\tau \tau' + \mu \mu')/2$, $v = (\tau \mu' - \mu \tau')/2$, then
$$u + vi = (\tau + \mu i)(\tau' + \mu' i)/2 \equiv \tau (\tau' + \mu' \cdot \frac{\tau}{\mu})$$
$$\equiv \tau (\tau' + \mu' \cdot \mu^2/\mu \cdot v/2 \pmod d).$$
Similarly, $u + vi' \equiv (\tau' + \mu' \cdot \mu^2/\mu \cdot v/2 \pmod {(n/d)}$. If we interchange $d$ and $n/d$, $\delta(n)$ will differ
$$\left[\frac{\tau + \mu i}{d}\right] + \left[\frac{\tau + \mu i}{n/d}\right] + \left[\tau' + \mu' i\right]$$
$$\equiv \left[\frac{2(u + vi)}{d}\right] + \left[\frac{2(u + vi)}{n/d}\right] = \left[\frac{u}{d}\right] + \left[\frac{v}{n/d}\right]$$
$$= \left[\frac{v}{n}\right] = \left[\frac{n}{v}\right] = 0 \in \mathbb{F}_2.$$ 
Thus $\delta(n)$ does not change, which implies that $\delta(n)$ does not depend on the choice of $d, \tau, \mu$ and only depend on $n$.

4.3. Proof of the main result.

Proof of Theorem 1.2. We shall use the fundamental exact sequence (2) and the commutative diagram in (2) frequently.

Since $E(Q)_{\text{tor}} \cap \psi E'(Q) = \{O\}$ and $#E(Q)_{\text{tor}} = 4$, $#E(Q)/\psi E'(Q) \geq 4$. Since $G(n)$ is odd, $#S(\psi)(E'/Q) = 4$ and $#E(Q)/\psi E'(Q) = 4$, by (2), III$(E'/Q)[\psi] = 0$. Apparently $\tilde{S}(\psi)(E'/Q) \supset E(Q)/\psi E'(Q)$ and thus $#\tilde{S}(\psi)(E'/Q) = 4$.

By Corollary 4.6 $\tilde{S}(\psi)(E/Q) = \{1\}$, then $#E'(Q)/\psi E(Q) = 1$. The facts $#E(Q)/\psi E'(Q) = 4$ and $E(Q)_{\text{tor}} \cong \mathbb{Z}/2\mathbb{Z}^2$ imply that $#E(Q)/2E(Q) = 4$ and $\text{rank}_{\mathbb{Z}} E(Q) - \text{rank}_{\mathbb{Z}} E'(Q) = 0$.

From III$(E'/Q)[\psi] = E'(Q)/\psi E(Q) = 0$, the diagram tells us that
$$\text{III}(E'/Q)[2] \cong \text{III}(E'/Q)[\psi] \cong \tilde{S}(\psi)(E/Q) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},$$
and (41) tells us that
$$\text{III}(E'/Q)[2] \cong \text{III}(E'/Q)[\psi] \cong 0.$$ 
Hence III$(E'/Q)[2^{\infty}] = 0$ and III$(E'/Q)[2^k \psi] = 0$. By the exact sequence
$$0 \to \text{III}(E/Q)[\psi] \to \text{III}(E/Q)[2^k] \to \text{III}(E'/Q)[2^{k-1} \psi],$$
we have for every $k \in \mathbb{N}_+$,

$$\text{III}(E/\mathbb{Q})[2^k] \cong \text{III}(E/\mathbb{Q})[\varphi] \cong (\mathbb{Z}/2\mathbb{Z})^2,$$

and thus $\text{III}(E/\mathbb{Q})[2^\infty] \cong (\mathbb{Z}/2\mathbb{Z})^2$.

\[ \square \]

\textbf{Proof of Corollary 1.3} In this case, $d = 1$ and $\tau = \mu = 1$, $\delta(n) = \left[ \frac{1 + \sqrt{-d}}{n} \right]$, thus the result follows. \[ \square \]

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