Abstract

The CP$^{N-1}$ model in three euclidean dimensions is studied in the presence of a Chern-Simons term using the 1/$N$ expansion. The $\beta$-function for the statistics parameter $\theta$ is found to be zero to order 1/$N$ in the unbroken phase by an explicit calculation. It is argued to be zero to all orders. Some remarks on the $\theta$ dependence of the critical exponents are also made.
The 2+1 dimensional CP\(^{N-1}\) model (without Chern-Simons term) has been extensively studied in the last decade since it was proven to be renormalizable in the 1/\(N\) expansion contrary to naive power counting \[1\]. Like many other low dimensional systems, its importance is twofold. First, it provides a theoretical laboratory for the study of phenomena that are expected to occur in 3+1 dimensional gauge theories and secondly, it can be interpreted as an effective theory for the description of condensed matter systems. The addition of a Chern-Simons (CS) term \[2\] is of particular interest in this second aspect. Its addition to the CP\(^{N-1}\) model has recently been considered by S.H. Park \[3\].

Let us first recall the structure of the CP\(^{N-1}\) model without CS term \[4\]. Let \(z\) be an \(N\) component complex field. The condition for \(z\) to represent a point in CP\(^{N-1}\) can be implemented by the introduction of a Lagrange multiplier \(\alpha\) ensuring that \(|z| = \text{const.}\) and a dummy abelian gauge field \(A\) ensuring that \(z \equiv e^{i\phi}z\). This leads to the lagrangian:

\[
\mathcal{L}_{\text{CP}^{N-1}} = |D_A z|^2 + \alpha \left( |z|^2 - \frac{N\Lambda}{g} \right),
\]

(The symbol \(D_A\) represent the covariant derivative of \(z\)). The “length square” of \(z\) has been chosen to be \(N\Lambda/g\) for later convenience; \(\Lambda\) will also play the role of a cut-off and \(g\) is a dimensionless coupling constant.

It is well known that quantum corrections will generate a kinetic term for the “photon” \(A\), making it into a truly propagating degree of freedom. However, field \(A\) can be promoted to a propagating field already at the classical level by simply adding a kinetic term to the lagrangian (1). In three dimensions, beside the usual Maxwell term, we have the option of choosing the CS term:

\[
\mathcal{L}_{\text{CS}} = i \frac{\theta}{4\pi} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda.
\]

Notice that in this model \(\theta\) need not be quantized.

The importance of this term is manifest from expression(2). First of all, it involves only one derivative and it is therefore expected to dominate the usual Maxwell term at low momenta (large distances). Second, the presence of the epsilon tensor brakes parity and time reversal preserving the combination of the two. This is what is expected in a two dimensional system with an external magnetic field along the perpendicular direction.
Finally, the CS term does not depend on the metric. Thus, its addition to the effective field theory is insensitive to the microscopic details that depend on the metric.

We study the system described by $L_{\text{CPN-1}} + L_{\text{CS}}$ in the $1/N$ expansion. As usual [1], the first step consists in integrating out the $z$ degrees of freedom, yielding an effective action for $A$ and $\alpha$. Then, we impose the vanishing of the linear term in the effective action itself, yielding the “gap equation” defining the two phases of the system:

$$\int^\Lambda \frac{d^3p}{(2\pi)^3} \frac{1}{p^2 + \alpha_c} - \frac{\Lambda}{g} + |z_c|^2 = 0 \quad \text{and} \quad z_c \alpha_c = 0. \quad (3)$$

We shall work in the unbroken phase ($g > g_c$), where we assume $\alpha_c = m^2$, $\sigma = \alpha - \alpha_c$, and $<A_\mu> = 0$. In terms of the shifted field $\sigma$, the “photon” field $A$ and the sources $J$ and $K_\mu$ for $z$ and $A_\mu$, the generating functional reads:

$$Z[J,K] = \int D\sigma DA \exp \left\{ - N \left[ \text{Tr} \log(-D^2_A + m^2 + \sigma) - \{\text{linear terms}\} + \int dx L_{\text{CS}} \right] + \int dx dy J(x)^* <x|(-D^2_A + m^2 + \sigma)^{-1}|y>J(y) + \int dx K \cdot A \right\}. \quad (4)$$

Notice that we have chosen the coefficient of the CS term to scale like $N$ in order to enhance its contribution in the $1/N$ expansion.

The Feynman rules for the $1/N$ expansion can be obtained by expanding the trace in powers of the fields [1], [4]. While doing so, it is convenient to rescale the fields by a factor $\sqrt{N}$ so that the propagators will all be independent on $N$ and the vertices will all scale like inverse powers of $N$. In particular, the $\sigma$-propagator, represented by a dashed line throughout the paper, is given by:

$$\Sigma(p) = -\frac{4\pi|p|}{\arctg(|p|/2m)}, \quad (5)$$

where the peculiar minus sign is a reminder of the fact that $\sigma$ is actually a Lagrange multiplier. The photon propagator, represented by a curly line is given by:

$$D_{\mu\nu}(p) = \left[ \frac{\Gamma(p)}{\Gamma^2(p) + \left( \frac{\theta}{4\pi} \right)^2} \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) + \frac{\theta/4\pi}{\Gamma^2(p) + \left( \frac{\theta}{4\pi} \right)^2} \epsilon_{\mu\nu\rho\sigma} p^\rho \right], \quad (6)$$

with

$$\Gamma(p) = \frac{p^2 + 4m^2}{8\pi|p|} \arctg(|p|/2m) - \frac{m}{4\pi}, \quad (7)$$
When needed, the propagator for the $z$ field itself,
\[
\Delta(p) = \frac{1}{p^2 + m^2}
\] (8)
will be represented by a solid line.

The first question we would like to address is whether or not the statistics parameter $\theta$ is renormalized in the $1/N$ expansion. At first glance it seems like the Coleman-Hill theorem \[5\] is not directly applicable here since its proof relies on perturbative arguments. There are known cases where extra corrections can appear \[6\]. However, we will show that this is not the case in this model, i.e., the statistics parameter is not renormalized.

Let us consider the $1/N$ correction to the photon (inverse) propagator. In terms of the non-local vertices defined by the expansion of (4), this can be represented by the five (amputated) diagrams in fig. (1). Of course, each diagram represents the sum of many local diagrams constructed by substituting $z$-loops to each vertex and attaching to it the propagators in all possible ways.

We write the contribution of each diagram to the inverse propagator as
\[
\Pi_{\mu\nu}^{(i)}(p) = (\delta_{\mu\nu} - \frac{p_{\mu}p_{\nu}}{p^2})\Pi_{S}^{(i)}(p) - \frac{\theta}{4\pi}\Pi_{A}^{(i)}(p)\epsilon_{\mu\nu\rho}p^{\rho} \quad i = 1 \ldots 5.
\] (9)

We are interested in the zero momentum limit of the anti-symmetric part $\Pi_A = \sum \Pi_A^{(i)}$, the renormalization constant for $\theta$ being given by $Z_\theta = 1 + \Pi_A(0)$.

Diagrams 1, 2 and 3 yield manifestly symmetric expressions, i.e., for $i = 1, 2$ and 3, $\Pi_A^{(i)}(p)$ is identically zero for all values of $p$ and we need not consider them any further.

Diagram 4 gives
\[
\Pi_{\mu\nu}^{(4)}(q) = \frac{2}{N} \int \frac{d^3l}{(2\pi)^3}v_{\mu\sigma}(q,l)v_{\nu\tau}(q,l)D_{\sigma\tau}(l)\Sigma(l - q),
\] (10)
where $v$ represents the non-local vertex:
\[
v_{\mu\sigma}(q,l) = \int \frac{d^3p}{(2\pi)^3} \left[ -\delta_{\mu\sigma} \Delta(p+l)\Delta(p+q) + (2p_\mu + q_\mu)(2p_\sigma + l_\sigma)\Delta(p)\Delta(p+l)\Delta(p+q) \right].
\] (11)

It is a simple matter to check that $v$ satisfies the relations $q_\mu v_{\mu\sigma}(q,l) = v_{\mu\sigma}(q,l)l_\sigma = 0$ and $v_{\mu\sigma}(q,l) = v_{\sigma\mu}(l,q)$ and that it can be written as:
\[
v_{\mu\sigma}(q,l) = (\delta_{\mu\sigma}(l \cdot q) - l_\mu q_\sigma)\phi(q,l) + (l^2 q^2 \delta_{\mu\sigma} - q_\mu q_\sigma l^2 - l_\mu l_\sigma q^2 + q_\mu l_\sigma l \cdot q)\psi(q,l),
\] (12)
where the two scalar functions in (12) have Feynman parametrization:

\[
\phi(q,l) = -\frac{1}{4\pi} \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \frac{\alpha\beta}{(m^2 + \alpha q^2 + \beta l^2 - (\alpha q + \beta l)^2)^{3/2}},
\]

(13)

and

\[
l^2 \psi(q,l) = \frac{1}{8\pi} \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \frac{\alpha - 2\alpha^2}{(m^2 + \alpha q^2 + \beta l^2 - (\alpha q + \beta l)^2)^{3/2}}.
\]

(14)

The anti-symmetric part of \(\Pi^{(4)}_{\mu\nu}\) can be extracted in the usual way, by contracting with the epsilon tensor. In our notations:

\[
\Pi^{(4)}_A(q) = -\frac{1}{N} \int d^3 l \frac{\Sigma(l - q)}{(2\pi)^3 \Gamma^2(l) + (\frac{\mu}{4\pi})^2 l^2} \left[ 2l \cdot q l^2 \phi^2 + 2l^2 (l^2 q^2 + (l \cdot q)^2) \phi \psi + 2(l \cdot q) q^2 l^4 \psi^2 \right].
\]

(15)

The functions \(\phi\) and \(\psi\) are analytic at \(q = 0\) and their asymptotic behaviour for large \(l\) is such that the integral in (15) converges. Hence, \(\Pi^{(4)}_A(0) = 0\) and there is no renormalization for \(\theta\) coming from the fourth diagram in fig. (1).

Diagram number (5) can be treated similarly. It can be written in compact form as:

\[
\Pi^{(5)}_{\mu\nu} = \frac{1}{N} \int \frac{d^3 k}{(2\pi)^3} D_{\rho\lambda}(k) v_{\mu\nu\rho\lambda}(k, q),
\]

(16)

where \(v_{\mu\nu\rho\lambda}(k, q)\) is the highly non-local expression representing the four photon vertex. Since there are no \(\sigma\)-lines in this diagram, this is the same expression that we would obtain by summing all the two loop diagrams of scalar QED
[7]. In other words, the only substantial difference between the expression of \(\Pi^{(5)}_{\mu\nu}\) and the analogous quantity of scalar QED is in the form of the internal photon propagator \(D_{\rho\lambda}(k)\). But the expression for the vertex alone is enough to guarantee a vanishing contribution to the renormalization of \(\theta\), i.e., without having to perform the integral over \(k\) in (16) it is easy to check that

\[
\Pi^{(5)}_A(0) = \left[ -\frac{2\pi}{3\theta} \epsilon_{\mu\nu\tau} \frac{\partial}{\partial q_\tau} \Pi^{(5)}_{\mu\nu}(q) \right]_{q \to 0} = 0.
\]

(17)

This was the last possible source of divergence to order \(1/N\), so we have explicitly checked that there is no renormalization of \(\theta\) to this order.

We now argue that there is no renormalization for \(\theta\) to all orders in \(1/N\). This is an extension of the well-known Coleman-Hill theorem [5] to the \(1/N\) expansion. Let us briefly
recall how the theorem is proven in ordinary perturbation theory. We restrict ourselves to
the case of scalar QED with CS term [7].

The most general diagram that appears in the correction for the photon propagator
$\Pi_{\mu\nu}(q)$ is made of two external photon lines and an arbitrary number of bosonic loops
and internal photon lines. Each bosonic loop defines a non-local vertex for the photons.
Consider a particular vertex $\Gamma_{\mu_1\mu_2\cdots\mu_m}(k_1, k_2, \cdots, k_m)$ with $m$
incoming photon lines of momentum $k_1, \cdots, k_{m-1}$ and $k_m = \sum_{i=1}^{m-1} k_i$. By gauge invariance and analyticity of $\Gamma$, it
is easy to see that

$$\Gamma_{\mu_1\mu_2\cdots\mu_m}(k_1, k_2, \cdots, k_m) = O(|k_1||k_2|\cdots|k_{m-1}|).$$

We have to consider two possible cases. In the first case, the two external photon legs
(carrying momentum $q$) are attached to two different bosonic loops. In this case, by
applying (18) twice, we obtain $\Pi_{\mu\nu}(q) = O(q^2)$. From this fact and (17) $\Pi_A(0) = 0$
immediately follows.

In the second case, both external legs are attached to the same loop and (18) also
implies $\Pi_{\mu\nu}(q) = O(q^2)$ unless $m = 2$. But the case $m = 2$ corresponds to the one loop
diagrams, for which it can be explicitly checked that the anti-symmetric part $\Pi_A$ vanishes
identically. This ends our sketchy review of the Coleman-Hill theorem.

In our model the vertices are even better behaved. For example, vertices with only
two incoming photon lines are forbidden in the sense of the large $N$ limit [1]. The only
two types of vertex involving photons are either those with two photon lines and $r \geq 1$
$\sigma$-lines:

$$\Gamma_{\mu_1\mu_2}(k_1, k_2, p_1, \cdots p_r) \quad p_r = k_1 + k_2 + \sum_{i=1}^{r-1} p_i,$$

or those with $m \geq 3$ photons and $s \geq 0$ $\sigma$-lines

$$\Gamma_{\mu_1\mu_2\cdots\mu_m}(k_1, k_2, \cdots, k_m, p_1, \cdots p_r),$$

(the last momentum in (20) is also fixed by momentum conservation). But for both types
of vertex (19) and (20) it is now true that

$$\Gamma = O(|k_1||k_2|)$$

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and therefore, any diagram involving only two external photon legs of momentum $q$ is $O(q^2)$, yielding again $\Pi_A(0) = 0$.

This proves the validity of the Coleman-Hill theorem in the $1/N$ expansion. If there is any non-perturbative effect that invalidates the theorem, it must be non-perturbative in $1/N$ as well as in the coupling constant. The result can also be generalized to the fermionic case. Recall that in the presence of fermions there are non-zero radiative corrections coming from one-loop graphs in ordinary perturbation theory. In the large $N$ limit, the same correction will be present in the photon propagator at leading order in $1/N$. There will be no corrections coming from lower orders as long as we stay in the analyticity region.

Now that we have established the non-renormalization of the statistics parameter $\theta$ in the $1/N$ expansion it makes sense to calculate the dependence of the critical exponents on $\theta$ itself. This was first done in [3]. There, dimensional regularization was used throughout the calculation and the critical exponent $\nu$ was obtained by studying the $zz\sigma$ three-point function. This may cause some concern because of the presence of an epsilon tensor that is a truly three-dimensional object and because of the fact that $\sigma$ is not a propagating field, so it is a little distressing to consider external $\sigma$-lines. We recover the same results by following a slightly different approach that does not use either assumptions.

We use gauge invariant projection with an explicit cut-off $\Lambda$. The critical exponents $\eta$ and $\nu$ are both calculated from the inverse $\sigma$ propagator. The relevant diagrams to order $1/N$ are shown in fig. (2). We have explicitly included all the tadpole diagrams that arise to this order. Also, we have chosen to draw the contribution in terms of local diagrams, (i.e., with the $\sigma$-loops explicitly indicated) because their number is not as large as it was for the photon propagator. One can always recover the previous picture by shrinking each $\sigma$-loop into a non-local vertex. In this case, the last two diagrams of fig. (2) would be combined into a single one.

Applying the ordinary Feynman rules and evaluating the diagrams asymptotically, we obtain the following expression for the divergent part of the $\sigma$ propagator: ($\bar{\theta} = 4\theta/\pi$)

$$\Delta^{-1}(p) = \frac{1}{N} \left[ -\frac{20}{\pi^2} \left(1 - \frac{16}{15} \bar{\theta}^2 \right) \log(\Lambda/m)p^2 + \frac{76}{\pi^2} \left(1 - \frac{64}{76} \frac{\bar{\theta}^2(\bar{\theta}^2 + 5)}{(1 + \bar{\theta}^2)^2} \right) \log(\Lambda/m)m^2 \right].$$

(22)

Actually, a naive evaluation of the diagrams in fig. (2) will give rise to a linear divergence
in the mass term. But this divergence simply redefines the (non universal) critical coupling constant \( g_c \) and does not affect any of the universal quantities.

From (22) one can immediately read off the critical exponents:

\[
\eta = - \frac{20}{\pi^2 N} \left( 1 - \frac{16}{15} \frac{\tilde{\theta}^2}{1 + \tilde{\theta}^2} \right),
\]

\[
\nu = 1 - \frac{48}{\pi^2 N} \left( 1 - \frac{8}{9} \frac{\tilde{\theta}^2 (\tilde{\theta}^2 + 4)}{(1 + \tilde{\theta}^2)^2} \right).
\]

Setting \( \theta = 0 \) we recover the usual \( \text{CP}^{N-1} \) result \( \eta_{\text{CP}^{N-1}} = -20/\pi^2 N \) and \( \nu_{\text{CP}^{N-1}} = 1 - 48/\pi^2 N \), whereas, taking the limit \( \theta \to \infty \), we obtain the critical exponents for the \( S^{2N-1} \) model \( \eta_{\text{S}^{2N-1}} = 4/3\pi^2 N \) and \( \nu_{\text{S}^{2N-1}} = 1 - 16/3\pi^2 N \). This last result is simply explained by recalling that the CS term acts as a mass term for the photon and by letting \( \theta \) go to infinity we simply remove the photon from the spectrum, leaving only the constraint \( |z| = \text{const.} \), which is just the \( S^{2N-1} \) model.

This fact, together with the non renormalization of \( \theta \) raises some interesting issues. We have showed that to all orders in \( 1/N \) there is a line of fixed points, described by \( g = g_c \) and \( \theta \), that interpolates between the ordinary \( \text{CP}^{N-1} \) model without CS term and the \( S^{2N-1} \) model. It would be interesting to investigate whether this line is “broken” by some phenomenon that is non-perturbative in \( 1/N \). This can be answered by studying the object equivalent to the equation of motion, which, in our case is the gap equation describing the two phases.

Another point worth studying is what happens at the critical point, where the photon propagator is no longer analytic. According to the general argument presented in [7], the \( \beta \)-function for \( \theta \) will still be zero but we expect a finite renormalization to occur. This point, however, has to be studied by numerical analysis since already in the simpler setting of \( \text{QED}_3 \) it has proven impossible to evaluate the Feynman integrals.

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Figure Captions

Fig. (1): Corrections to the photon propagator to order $1/N$.

Fig. (2): Corrections to the $z$ propagator to order $1/N$. 