Stretch-Twist-Fold and slow filamentary dynamos in liquid sodium Madison Dynamo Experiment

by

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Abstract

Recently Ricca and Maggione [MHD (2008)] have presented a very simple and interesting model of stretch-twist-fold dynamo in diffusive media based on numerical simulations of Riemannian flux tubes. In this paper we present a yet simpler way of analytically obtaining fast and slow dynamo, generated by the curvature energy of magnetic filaments in diffusive media. A geometrical model for the galactic or accretion disk dynamo in shear flows is presented. In the fast dynamo case it is shown that the absence of stretching leads to the absence of fast dynamos and when torsion of filaments vanishes the dynamo action cannot be supported as well. This is the Cowling-Zeldovich theorem for planar flows. Isotropy of the magnetic fields hypothesis is used to compute the fast nature of dynamo. A similar result using non-holonomic Frenet frame has been recently obtained for filamentary dynamos [Garcia de Andrade, AN (2008)]. The stretch-twist-fold (STF) filamented models discussed here may serve to formulate future experiments in the Madison Dynamo Experiment (MDE) facility [Nornberg et al, Phys Plasmas 13 (2008)]. Though their running experiments deal with magnetic fields that are orthogonal to the flow, other topologies are also used here, with an eye in future experiments. Unstretched filaments in MDE shows Vishik anti-fast dynamo action applies [Garcia de Andrade[Phys Plasmas 15 (2008)].
I Introduction

Earlier Chui and Moffatt [1] have investigated the role of Riemannian metric and its back reaction on the helicity of magnetic twisted flux tubes. More recently the effects of the Riemann metric on the stretching and dynamo action have been also investigated [2, 3] in plasma dynamo flows [4]. The role of stretching is known to be of utmost importance for dynamo action [5]. Presence of diffusion is actually fundamental in the Vainshtein-Zeldovich stretch-twist-fold (STF) dynamo mechanism [6] for folding and merging processes in flux tubes, result that has been shown by Ricca and Maggione [7]. STF has been recently demonstrated experimentally by Carey Foster and the Madison Dynamo Experiment (MDE) group [8]. Thus this fast dynamo mechanism is one of the most well tested methods of obtaining fast dynamos. Contrarily to what appears, the existence and discovered of fast dynamos are not so easy due to the fact that simplicity criteria are in general related with the symmetry processes that in general, due to Cowling’s and Zeldovich’s anti-dynamo theorems, joint with diffusion destroys dynamo action enhanced by advection. Presence of Frenet holonomic frame turns problem geometry still more simple, than the other techniques, however, not so simple as the Riemannian flux tubes. By the way this seems to be the main reason, that the Riemannian plasma kinematic dynamo solutions of the self-induction equation [2, 3], have ended with slow rather than fast dynamos. In this paper, a thoroughly investigation of the kinematic dynamo problem in the holonomic (depending only of the coordinate along the magnetic filaments), shows that a fast dynamo can be obtained where the magnetic Reynolds number is bounded from below by the inverse of the constant curvature, while slow dynamos can be obtained by another bound based on the total kinematic curvature energy. The growth rate is computed in the fast dynamo case but mainly our conclusions including also the slow dynamo case, are based on the magnetic energy of the dynamos, obtained from the self-induction equation. The slow dynamo is obtained by the limit of the vanishing of diffusion limit and the growth rate \( \gamma > 0 \). Actually the fact of the matter is that, the filament investigation can also produce a better understanding of the magnetic axis behaviour of twisted magnetic flux tubes, where the role played by the twist is here, played by torsion, of course having in mind that, twist may have a torsion contribution. For example, a straight flux tube cannot have a torsioned magnetic
geometrical axis. This paper is organised as follows: Section 2 addresses a brief review of Frenet frame equations, useful for the reader to be able to follow the rest of the paper. Section 3 presents the investigation of the absence of stretching by only constraining the advection-stretching term to vanish. Section 4 discusses the fast dynamo and slow cases in terms of the magnetic energy and its relation with curvature kinetic energy integral. Section 5 presents the orthogonal MDE case between magnetic field and flows in diffusive media. Conclusions are presented in section 6.
II Dynamo filamented flows in holonomic Frenet frame

This section presents a brief review of the Serret-Frenet holonomic frame [9] equations that are specially useful in the investigation of fast and slow filamented dynamos in magnetohydrodynamics (MHD) endowed with magnetic diffusion. Here, the Frenet frame is attached along the magnetic flux tube axis, which possesses Frenet torsion and curvature [?]. These mathematical objects completely topologically determine the filaments. Besides, one needs some dynamical relations from vector analysis and differential geometry of curves, such as the Frenet frame \((\mathbf{t}, \mathbf{n}, \mathbf{b})\) equations

\[
\mathbf{t}' = \kappa \mathbf{n} \quad (\text{II.1})
\]
\[
\mathbf{n}' = -\kappa \mathbf{t} + \tau \mathbf{b} \quad (\text{II.2})
\]
\[
\mathbf{b}' = -\tau \mathbf{n} \quad (\text{II.3})
\]

The holonomic dynamics considered here considers a steady evolution of the tubes, which implies that the Frenet frame \((\mathbf{t}, \mathbf{n}, \mathbf{b})\), legs are vanish with respect to time derivative. As one shall see in the next section this considerably simplifies our task of finding solutions of self-induction equations, avoiding some of the problems introduces by curvature and torsion of the magnetic field topology and the flux tube axis. Frenet frame with the help of equations

\[
\partial_t \mathbf{t} = [\kappa' \mathbf{b} - \kappa \tau \mathbf{n}] \quad (\text{II.4})
\]
\[
\partial_t \mathbf{n} = \kappa \tau \mathbf{t} \quad (\text{II.5})
\]
\[
\partial_t \mathbf{b} = -\kappa' \mathbf{t} \quad (\text{II.6})
\]

Together with the flow derivative

\[
\dot{\mathbf{t}} = \partial_t \mathbf{t} + (\mathbf{v}, \nabla) \mathbf{t} \quad (\text{II.7})
\]

From these equations one is able in the next sections to write down the expressions for the solenoidal magnetic and filamented flows which allows us to split the self-induced magnetic equation generic flow

\[
\nabla \cdot \mathbf{v} = 0 \quad (\text{II.8})
\]
\[
\nabla \cdot \mathbf{B} = 0 \quad (\text{II.9})
\]
\[ \mathbf{B} = B_s(r) \mathbf{t} + B_\theta(r, s) e_\theta \]  

(II.10)

In the section 3 one shall solve the diffusion equation in the steady case in the holonomic Frenet frame as

\[ \partial_t \mathbf{B} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \Delta \mathbf{B} \]  

(II.11)

Therefore in the next section one shall address the stretching term \((\mathbf{B} \cdot \nabla) \mathbf{v}\), which is present in equation (II.11) due to the vector analysis identity

\[ \nabla \times (\mathbf{v} \times \mathbf{B}) = - (\mathbf{v} \cdot \nabla) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{v} \]  

(II.12)

The first term is advection and the second is the stretching term.
III Unstretching magnetic filaments

In this section we shall consider the stretching part of the self-induction equation

\[ \frac{dB}{dt} = (B \cdot \nabla)v + \eta \nabla^2 B \]  

(III.13)

By assuming, in the first example, that the filamentary stationary flow is given by

\[ v(s) = v_s t + v_n n + v_b b \]  

(III.14)

while the magnetic field is purely toroidal as \( B \)

\[ B(t, s) = B_s(t) t \]  

(III.15)

The only time dependence of \( B_s \) comes from the solenoidal property of the magnetic field above. Thus the stretching of the filamented flow is given by

\[ (B \cdot \nabla)v = B_s \partial_s v = B_s [(v'_s - \kappa v_n)t + (v'_n + \kappa v_s + \tau v_b)n(v'_b - \tau v_n)b] \]  

(III.16)

The first expression on the RHS of this last expression vanished due to the incompressibility condition

\[ \nabla \cdot v = v'_s - \kappa v_n = 0 \]  

(III.17)

By assuming that the flow is planar, to test Cowling-Zeldovich theorem, \((\tau = 0)\) the stretching condition yields the two simple ODEs

\[ v'_s - \kappa v_n = 0 \]  

(III.18)

\[ v'_n + \kappa v_s = 0 \]  

(III.19)

Multiplying the equation (III.13) by \( v_s \) and the equation (III.14) by \( v_n \) and subtracting the resulting equation

\[ T = v^2_s + v^2_n = c_0 \]  

(III.20)

where \( c_0 \) is an integration constant. This shows that the kinetic energy is constant, which shows that the absence of stretching cannot yield a growing kinetic flow energy capable to induce dynamo action. To simplify the above computations one assumes that the filament is helical, where the constant torsion \( \tau_0 \) equals the constant curvature \( \kappa_0 \). Thus Cowling theorem
of non-dynamo action for planar flows is valid. The next example is much simpler. This case uses the following fields

\[ B = B_s t + B_n n \]  
(III.21)

\[ v = v_s t \]  
(III.22)

Due to the solenoidal (incompressibility) of the flow, \( v_s' \) vanishes and therefore the stretching term becomes

\[ (\mathbf{B} \cdot \nabla) v = \kappa_0 v_s B_n \]  
(III.23)

Now imposing the vanishing of stretching yields either that curvature vanishes or normal component vanishes or yet that the toroidal flow vanishes, each relation of these yields a weakness of dynamo action.

### IV Fast and slow filamented dynamo flow

In this section one shall consider the two examples considered above in full detail and solve the self-induction equation in the presence of stretching and diffusion to check for the existence of fast or slow kinematical dynamos here the action of back reaction of Lorentz flow is neglected. Let us now consider first the slow dynamo case which is present in the first example of the last section. Since the stretching terms were computed in the previous section, here one shall only compute the diffusive and growth rate of the magnetic fields as

\[ \frac{1}{2} d_t (\int B_s^2 ds) = -\eta \int B_s^2 \kappa^2 ds \]  
(IV.26)

Since due to the solenoidal character of a monopole free magnetic field, the magnetic field \( B_s \) toroidal component just depends on the time coordinate, which reduces the last equation to

\[ \frac{1}{2} d_t (\int B_s^2 ds) = -\eta B_s^2 \int \kappa^2 ds \]  
(IV.27)
This expression shows clearly that the magnetic energy is proportional to the curvature kinematic energy

\[ K := \int \kappa^2 ds \]  

(IV.28)

Note however that when the diffusion constant, vanishes, the toroidal magnetic energy is constant and therefore the dynamo is toroidally slow. The other expressions for the energy can be obtained for the other directions and we are able to say that the dynamo is actually slow. Combining these three equations one obtains the dynamo relation

\[ \frac{1}{2} \frac{d}{dt} (\int B_s^2 ds) = v_s \kappa_0 \int B_s^2 ds - \frac{\eta}{2} B^2 \]  

(IV.29)

In the limit of \( \eta \to 0 \) from this expression, is easy to see that the magnetic energy may grow under appropriated constraints. Thus the fast dynamo action is possible under the STF method exactly in the MDE. In the next section a more detailed explicify example shall be done without the assumption of isotropy \( B_s = B_n \) used here to simplify computations.

V STF-MDE filament experiment

In this section, we have considered the MDE to test STF, providing a filamented model analogous to the previous ones, with only the difference that the magnetic and velocity fields are orthogonal \( (B, v) = 0 \), which is used in the initial stages of the MDE. The choice given is

\[ B = B_s t + B_n n \]  

(V.30)

and for the flow velocity

\[ v = v_b b \]  

(V.31)

Note that the advective term above does not vanish, since the cross product \( v \times B \) does not vanish. This yields the possibility of the fast dynamo action as obtained in the Madison experiment. The stretching term is given by

\[ (B \cdot \nabla) v = B_s [v'_b b - \kappa_0 v_b n] \]  

(V.32)

In the unstretched case, where this expression vanishes, and since the curvature \( \kappa_0 \) is assumed constant and non-vanishing the only solution is the vanishing of \( v_b \) and since, this is the only
flow component, the magnetic filaments flow is static which does not support any dynamo 
action. This is Vishik's anti-fast dynamo theorem. Since curvature is the Frenet curvature of the 
filaments, and not curvature of liquid metal device, the curvature $\kappa(s)$ may be non-constant. 
This yields the following equation

$$ (\mathbf{B} \cdot \nabla) \mathbf{v} = B_s [v'_b \mathbf{b} - \tau v_b \mathbf{n}] $$  \hspace{1cm} (V.33)

The vanishing of this expression implies that the $v_b$ is constant and that the torsion $\tau$ vanishes, 
which forces the filament flow to be planar and by Cowling's theorem forbides dynamo action. 
Now let us consider the whole self-induction equation. With the help of the solenoidal property 
of the magnetic field, one obtains

$$ \nabla \cdot \mathbf{B} = \partial_s B_s - \kappa_0 B_n = 0 $$  \hspace{1cm} (V.34)

after a long computation one obtains the following magnetic energy relation for the $B_n$ com-
ponent

$$ \frac{1}{2} \frac{d}{dt} \int B_n^2 ds = -2\eta\kappa_0^3 \int B_n^2 \frac{(\kappa_0 - v_b)}{v'_b} ds + \eta \int [B'_n B_n - (2B_n' B_n + B_n^2)\kappa_0^2] ds $$  \hspace{1cm} (V.35)

This expression shows that if the toroidal component did vanish and only normal component of 
the magnetic field survives the dynamo would be slow and not fast. Actually the by considering 
the remaining component of the self-induction equation along the binormal direction $\mathbf{b}$

$$ \frac{B_s}{B_n} = -2\eta \frac{\kappa_0^2}{v'_b} $$  \hspace{1cm} (V.36)

This equation together with the solenoidal character of $\mathbf{B}$, given by expression (V.34) yields 
the following expression for the toroidal magnetic energy growth rate

$$ \frac{1}{2} \frac{d}{dt} \int B_s^2 ds = \int B_s^2 (2v'_b + v''_b) ds $$  \hspace{1cm} (V.37)

This shows that the toroidal magnetic part of the of the growth rate of the energy does not 
depend on diffusion and thus the possibility of the fast dynamo through the STF method is 
guaranteed, such as in MDE experiment.
VI Conclusions

In the case of STF filamentary MDE model, the initial magnetic field in the laminar non-turbulent phase is orthogonal to the flow. Note that expressions for the magnetic filaments may be chosen with appropriate topology and boundary conditions in order to guarantee, dynamo action in the presence of magnetic resistivity MHD equations. In this paper we propose alternative topology between the magnetic lines and the flow in order to guarantee fast kinematic dynamo action through the STF method. Slow dynamo is obtained in some situations with curvature energy integral bounds while, as shown in the MDE experiment, the initial orthogonal relation between magnetic and flow fields, seems to grant the existence of fast dynamos in the STF mechanism. The other examples treated here, may be useful for designing another experiments that could possibly be running in Madison dynamo experimental facility in near future.

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