Integral representations of shallow neural network with Rectified Power Unit activation function

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Abstract

In this effort, we derive a formula for the integral representation of a shallow neural network with the Rectified Power Unit activation function. Mainly, our first result deals with the univariate case of representation capability of RePU shallow networks. The multidimensional result in this paper characterizes the set of function that can be represented with bounded norm and possibly unbounded width.

Keywords: shallow neural network, integral representation, rectified power unit, Radon transform

1 Introduction

A theoretical understanding of which functions can be well approximated by neural networks has been studied extensively in the field of approximation theory with neural networks (e.g., § 6 8 11 12 13 21 31 33 39 41). In particular, integral representation techniques for shallow neural networks have received increasing attention (e.g., § 10 15 17 19 20 23 30 32). Indeed, many authors have focused on the complexity control and generalization capability for shallow neural networks. The main motivation which makes this direction of research active is that "the norm of the weights is more important than the depth of the network" e.g., § 3 37 40 42.

A shallow neural network, with \( k \) neurons can be defined as a real valued function defined on \( \mathbb{R}^d \) of the following form:

\[
\sum_{i=1}^{k} a_i \sigma(w_i, x) - b_i + c
\]

where \( \sigma \) is a non-linear function and \( k \) is called the width of the network. Moreover, \( w_i \in \mathbb{R}^d \) and \( b_i \in \mathbb{R} \) are the inner weights and \( a_i \in \mathbb{R} \) are the outer weights, and \( c \in \mathbb{R} \). The non-linear function which acts componentwise is called activation function. Here we use non-decreasing homogeneous activation functions such as RePU activation function defined as

\[
\sigma(x) = \max(0, x)^p := [x]_+^p
\]

for \( p \in \mathbb{N} \). If \( p = 1 \) then \( \sigma \) is the well known Rectified Linear Unit (ReLU), with has seen considerable recent empirical success. Moreover, if \( p = 2 \), then \( \sigma \) is the Rectified Quadratic Unit (ReQU), which starts to get the attention of researchers thanks to its representation capability. RePU activation functions have many advantages such as homogeneity and regularity. Moreover, the representation capability of a RePU neural network makes it an interesting activation function, for instance, RePU can represent the identity on bounded domains for all \( p \in \mathbb{N} \), the square and hence the product on bounded domains for any integer \( p \geq 2 \). Note that our analysis can be applied to a larger class of activation functions satisfies similar properties to RePU activation function.

In several papers, the study of deep learning shows that the size of the network tends to infinity when the approximation error goes to zero. In our paper, we analyze approximation while controlling the norm of the weights, instead of the size of the networks. Hence we can consider networks with unbounded or infinite size. Thus our focus relies on the understanding of the representation cost which is determined by the minimal norm required to represent a function by shallow neural networks with an architecture of unbounded size.

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In this paper, we develop the characterization of the size of weights required to realize a given function by an unbounded width single hidden-layer with RePU activation function, in the univariate and multivariate cases. For simplicity, we assume that the target function is real valued, but we can easily generalize our results to the multi-dimensional output case.

1.1 Contributions

Mainly, our contribution is motivated by the problem of understanding approximation capability of shallow neural networks. Hence, our approach uses similar techniques as in the papers [35] and [29]. Our contributions can be summarized as follows:

First, we show that for a function \( f \) that can be presented as a shallow neural network \( \sum_{i=1}^{k} a_i [(w_i, x) - b_i]_+^p + c \), such that \( p \geq 1 \), the minimal cost \( \inf \frac{1}{2} \sum_{i=1}^{k} (\|w_i\|_2^2 + |a_i|^p) \) is the same as \( \inf \frac{1}{2} \sum_{i=1}^{k} |a_i|^{1/p} \) (such that \( w_i \in \mathbb{R}^d \) and \( a_i \in \mathbb{R} \)) for any input dimension \( d \in \mathbb{N} \). It is worth to mention that this is a well known result for networks with ReLU activation function.

In Theorem 4.1, we deal with real valued functions defined on \( \mathbb{R} \), that is univariate functions with single-output. We show that the minimal representation cost \( \overline{\mathcal{R}}(f) \) (see (2.7)) of a univariate function \( f \) to be represented as infinite width single-hidden layer neural network with RePU activation function is the same as the following quantity:

\[
\max \left( \frac{1}{p!} \int_{-\infty}^{\infty} \left| f^{(p+1)}(b) \right| \, db, \frac{1}{p!} \left| f^{(p)}(-\infty) + f^{(p)}(\infty) \right|^{1/p} \right).
\]

In the multivariate case, we use tools and techniques from harmonic analysis and measure theory toward the main result. Mainly, we characterize the representational cost for any real valued function defined on \( \mathbb{R}^d \) that can be represented by RePU neural networks such that \( p \) and \( d \) are odd integers. Mainly the cost is given by the following quantity

\[
\|f\|_{\mathcal{R}_d}^p = \left\| \sum_{\ell=0}^{2d} \mathcal{R}^d \{ \Delta^{(d+p)/2} f \} \right\|_{\mathcal{M}_d^1} = \left\| \sum_{\ell=0}^{2d} \partial_b^{d+p} \mathcal{R}_b^d \{ f \} \right\|_{\mathcal{M}_d^1},
\]

where \( \mathcal{R} \) is the Radon transform, we refer to Section 3 and Section 5 for notations and details. Moreover, it is worth to mention that our characterization is valid only for odd dimension and odd power in the RePU activation function. We provide a comprehensive view of the the representational cost \( \|f\|_{\mathcal{R}_d}^p \) of a function \( f \), which is in particular finite if and only if \( f \) well approximated by a bounded norm RePU network, i.e., have finite \( \mathcal{R} \)-norm. It turns out that using RePU activation function leads to the appearance of unregularized monomial units which needed a correction. Instead in the multivariate cases in [29], authors only have dealt with a single unregularized linear unit. Although our results might seem straightforward at first glance, actually proving them is quite delicate.

1.2 Related work

In real life applications, shallow neural networks are less popular than deep neural networks, although theoretically shallow networks are better understood. For instance, integral representation of neural networks have been established to explore the expressive power of shallow neural networks e.g., [2, 26], and to estimate the approximation errors e.g., [22].

In [2], Barron showed that a single hidden-layer neural network with sigmoidal activation function can approximated any continuous function on a bounded domain up to any given precision, where the approximation error related to the number of nodes in the network. Instead, in [7], Candès used single-hidden layer neural network with oscillatory activation function and methods of harmonic analysis to the problem of representing a function in terms of shallow neural networks.

Ito [15] established a uniform approximation on the whole space \( \mathbb{R}^d \), where he used a shallow neural network with step or sigmoid activation function. In the later paper, the inversion formula of the Radon transform plays a crucial role, during the proof of the integral representation. Moreover, Ito used a function as outer weights depends on the inner weights, which also satisfies regularity and growth assumptions.

Kurkova, in [22], showed estimates of network complexity, in representing a multivariate function through shallow neural networks such that the author considered shallow networks as integral transforms with kernels corresponding to network units. Since the classes of functions that can be expressed as integral with kernels are sufficiently large, many authors contributed to characterize and determine...
those classes, for instance, all sufficiently smooth compactly supported functions or functions decreasing sufficiently rapidly at infinity can be expressed as networks with infinitely many Heaviside perceptrons cf. \[33\] \[19\] \[24\].

Our work is also related to \[35\] \[29\]. In \[35\], authors initiated the study of the representational cost of univariate functions in term of weight magnitude for shallow neural networks with ReLU activation function. Ongie et al. have been extended this approach to understand what kind of multivariate functions can be represented by infinite-width shallow neural networks with ReLU activation function, where the key analysis tool used in their main problem is the Radon transform cf. \[29\].

In subsequent papers, shallow networks have been studied from various perspectives for different objectives. For example, in \[16\], generic scheme have been introduced to approximate functions with the so-called neural tangent kernel by sampling from transport mappings between the initial weights and their desired values. Instead, in \[9\], authors have been analyzed the training and generalization behavior of infinitely wide two-layer neural networks with homogeneous activation. From optimization point of view, Pilanci and Ergen showed \[33\] the equivalence between $\ell^2$-regularized empirical risk minimization of shallow ReLU networks and the finite-dimensional convex group-lasso problem.

Lastly, we remark that the theoretical analysis and applications of shallow neural networks is growing in terms of number of published papers and researchers contributing to this field.

### 1.3 Organization of the Paper

The organization of this paper is as follows. In Section 2, we describe the problem setting and define notions of shallow neural networks with RePU activation function. In Section 3 we recall the required preliminary concepts and definitions of Radon measure and Radon transform and its dual. In Section 4, based on the definitions in Section 2, we state our main result for the case of univariate functions. Finally, in Section 5 using definitions in Section 4 and the Radon transform from Section 3 we characterize the norm of weights required to realize a multivariate function as an unbounded single hidden-layer with RePU activation function.

### 2 Infinite width RePU neural networks

Infinite width representations with ReLU activation function have been considered in e.g., \[29\] \[31\] \[35\]. In the current section, we introduce shallow neural networks with RePU activation function in the infinite width setting.

For fixed $p \in \mathbb{N}$, we can write two layers network with an unbounded number of neurons, $d$ dimensional input and one-dimensional output, as a function $g_{\theta,p}: \mathbb{R}^d \rightarrow \mathbb{R}$ given by

$$g_{\theta,p}(x) = \sum_{i=1}^{k} a_i[(w_i, x) - b_i]_+^p + c, \text{ for all } x \in \mathbb{R}^d$$  \hspace{1cm} (2.1)

$k$ represents the width where $k \in \mathbb{N}$, $w_i$ are the rows of $W \in \mathbb{R}^{k \times d}$ and the parameters $\theta = (k, W = (w_1, ..., w_k), b = (b_1, ..., b_k), a = (a_1, ..., a_k), c) \in \Gamma$ such that

$$\Gamma := \{ \theta = (k, W, b, a, c) | k \in \mathbb{N}, W \in \mathbb{R}^{k \times d}, b \in \mathbb{R}^k, a \in \mathbb{R}^k, c \in \mathbb{R} \}.$$  

In the previous definition of the network function $g_{\theta,p}(x)$ we can exclude the bias $c$, since it can be simulated through an additional unit, for instance $b_{k+1} = -1$, $a_{k+1} = c$, and $w_{k+1} = 0$. Instead, the unregularized biases \{$b_i\}_{i=1}^k$ in the hidden layer are crucial to our analysis, thus removing or regularizing them will lead to substantial changes.

For any $\theta \in \Gamma$ we define a cost function $C(\theta)$ by

$$C(\theta) = \frac{1}{2} \left( \|W\|_F^2 + \|a\|_2^{2/p} \right) = \frac{1}{2} \sum_{i=1}^{k} \left( \|w_i\|_2^2 + |a_i|^{2/p} \right),$$  \hspace{1cm} (2.2)

where $\|W\|_F$ is the Frobenius norm of the matrix $W$. Based on the previous definition of the cost, the minimal representation cost necessary to represent a given function $f \in \mathbb{R}^d \rightarrow \mathbb{R}$ in terms of a shallow neural network is given by

$$R(f) := \inf_{\theta \in \Gamma} C(\theta) \text{ such that } f = g_{\theta,p}.$$  \hspace{1cm} (2.3)
Thanks to the $p$-homogeneity of the RePU activation function, we can show (see Appendix A) that minimizing $C(\theta)$ is the same as restricting the inner layer weight vectors $\{w_i\}_{i=1}^k$ to be the unit norm while minimizing the $\ell^1/p$ quasinorm of the outer layer weight $a$, more details can be found in Appendix A.

Let $\Theta$ be the collection of all $\theta \in \Gamma$ such that, for any $i \in \{1, \ldots, k\}$, $w_i$ belongs to the unit sphere $S^{d-1} := \{w \in \mathbb{R}^d : ||w|| = 1\}$, hence we have

$$R(f) = \inf_{\theta \in \Theta} \|w\|_{1/p}^p$$

(2.4)

In the case where $p = 1$, $R(f)$ is finite if $f$ is a continuous piecewise linear function with finitely many pieces that can be represented by a finite width two layers ReLU networks. Similarly, when $p = 2$, $R(f) < \infty$ if $f$ is realizable as a finite width two layers ReQU networks, that is $f$ is a continuous piecewise quadratic function with finitely many pieces. For general $p$, $R(f)$ is finite only if $f$ is a continuous piecewise polynomial of order $p$ with finitely many pieces, which can be represented by a finite width two layers RePU neural networks.

Since we focus on the minimal norm required to represent a function with an infinite-size network, we use a modified representational cost, rather than (2.4), which captures larger space of functions. The following defines the minimal limiting representational cost of all sequences of shallow neural networks converging to $f$ uniformly,

$$\overline{R}(f) := \lim_{\varepsilon \to 0} \left( \inf_{\theta \in \Theta} C(\theta) \text{ s.t. } |(g_{\theta,p} - f)(x)| \leq \varepsilon \text{ for any } ||x|| \leq \frac{1}{\varepsilon} \text{ and } g_{\theta,p}(0) = f(0) \right).$$

(2.5)

Consequently, if $\overline{R}(f)$ is finite, then $f$ can be represented by an infinite width shallow RePU network, where the outer-most weights are described by a density $\mu(w, b)$ for all weights and bias pairs $(w,b) \in S^{d-1} \times \mathbb{R}$. We denote by $M^1(S^{d-1} \times \mathbb{R})$ the space of signed measures $\mu$ defined on $(w, b) \in S^{d-1} \times \mathbb{R}$ with finite total variation norm $||\mu||_{M^1} = \int_{S^{d-1} \times \mathbb{R}} d|\mu|$, more details can be found in Section 3. Let $c \in \mathbb{R}$, and $p \in \mathbb{N}$, we define the infinite width two layers RePU network $H^p_{\mu,c}$ as follows:

$$H^p_{\mu,c}(x) := \int_{S^{d-1} \times \mathbb{R}} \frac{1}{1 + |b|^p} \left( (\langle w, x \rangle - b)_+^p - [-b]^p_+ \right) d\mu(w, b) + c.$$  

(2.6)

In case $c = 0$, we write $H^p_{\mu,0}$ instead of $H^p_{\mu,0}$. Our definition is a correction of the one proposed firstly by [55]. Moreover it is a generalization of the definition given in [29]. Indeed when $p = 1$ then $2H^p_{\mu,c}$ is the same shallow network integral representation given in [29] Equation (8). Moreover, our definition ensures that the integral is well defined, since the integrand $\frac{1}{1 + |b|^p} \left( (\langle w, x \rangle - b)_+^p - [-b]^p_+ \right)$ is continuous and bounded for any $(w, b) \in S^{d-1} \times \mathbb{R}$ and fixed $x \in \mathbb{R}$.

Let $\psi(b) = 1 + |b|^{p-1}$, then we show that $\overline{R}(f)$ in (2.5), is the same as the following:

$$\overline{R}(f) = \min_{\mu \in M^1(S^{d-1} \times \mathbb{R}), c \in \mathbb{R}} ||\mu||_{M^1(1/\psi)}$$

(2.7)

such that $f = H^p_{\mu,c}$,

the norm notation is defined in Section 3 while the details of the proof are in Appendix B.

**Remark 2.1.** There is an equivalence between the loss minimization approach by controlling $C(\theta)$ while fitting some loss function $L(g_{\theta,p})$ to train an infinite width RePU neural network $g_{\theta,p}$ and learning a function $f$ by controlling $\overline{R}(f)$ while fitting the loss $L(f)$. More specifically, for any hyper parameter $\lambda \in \mathbb{R}$, we have the following

$$\min_{\theta \in \Theta} L(g_{\theta,p}) + \lambda C(\theta) \iff \min_{f : \mathbb{R}^d \to \mathbb{R}} L(f) + \lambda \overline{R}(f).$$

(2.8)

The right-hand side of (2.8) gives information about the class of functions we are minimizing with infinite width, moreover it describes what we are minimizing with a finite, but sufficiently large, width.

**3 Radon measure and Radon transform**

In this section we recall some definitions and properties of Radon measure and Radon transform. Both ingredients play an important role in our analysis.

4
3.1 Radon measure

Let $B$ be the $\sigma$-algebra of all Borel sets. A Radon measure is a positive Borel measure $\mu : B \rightarrow [0, +\infty]$ which is finite on compact sets and is inner regular in the sense that for every Borel set $E$ we have

$$\mu(E) = \sup\{\mu(K) : K \subset E, K \in K\}$$

$K$ denoting the family of all compact sets. For a signed measure $\mu$ defined on $\mathbb{S}^{d-1} \times \mathbb{R}$, $\|\mu\|_{M^1} = \int \, d|\mu|$ denotes its total variation norm. Let $\mathcal{M}^1(\mathbb{S}^{d-1} \times \mathbb{R})$ denotes the space of measures on $\mathbb{S}^{d-1} \times \mathbb{R}$ with finite total variation norm. Moreover, for a measure $\mu$ and a positive function $\omega$ defined on $\mathbb{S}^{d-1} \times \mathbb{R}$ such that $\omega$ is integrable with respect to $\mu$, we define a “weighted” version of the total variation norm as follows

$$\|\mu\|_{\omega M^1} := \|\omega \mu\|_{M^1} := \int \omega \, d|\mu|.$$

Since $\mathbb{S}^{d-1} \times \mathbb{R}$ is a locally compact space, $\mathcal{M}^1(\mathbb{S}^{d-1} \times \mathbb{R})$ is the Banach space dual of $C_0(\mathbb{S}^{d-1} \times \mathbb{R})$, the space of continuous functions on $\mathbb{S}^{d-1} \times \mathbb{R}$ vanishing at infinity [25, Chapter 2, Theorem 6.6], and

$$\|\mu\|_{M^1} = \sup \left\{ \int \varphi \, d\mu : \varphi \in C_0(\mathbb{S}^{d-1} \times \mathbb{R}), \|\varphi\|_1 \leq 1 \right\}.$$

The weighted version of the total variational norm is defined in a similar way as (3.1). That is,

$$\|\omega \mu\|_{M^1} = \sup \left\{ \int \varphi \omega \, d\mu : \varphi \in C_0(\mathbb{S}^{d-1} \times \mathbb{R}), \|\varphi\|_1 \leq 1 \right\},$$

if the weight $\omega$ has nice properties then we can relax the assumptions on $\varphi$. In our paper $\omega(w, b) = \omega(b) = \frac{\omega(x - b)^p}{1 + |b|^{p-1}} \in C_0(\mathbb{S}^{d-1} \times \mathbb{R})$, hence it is enough to consider the continuity assumption only on $\varphi$. We write $(\mu, \varphi)$ to denote $\int \varphi \, d\mu$, where $\mu \in \mathcal{M}^1(\mathbb{S}^{d-1} \times \mathbb{R})$ and $\varphi \in C_0(\mathbb{S}^{d-1} \times \mathbb{R})$.

Let $C_0(\mathbb{S}^{d-1} \times \mathbb{R})$ denotes the space of continuous and bounded functions on $\mathbb{S}^{d-1} \times \mathbb{R}$. Hence any measure $\mu \in \mathcal{M}^1(\mathbb{S}^{d-1} \times \mathbb{R})$ can be extended uniquely to a continuous linear functional on $C_0(\mathbb{S}^{d-1} \times \mathbb{R})$. Therefore, for all $x \in \mathbb{R}^d$, the infinite width network

$$\mathbb{R}_{\mu}^p(x) := \int_{\mathbb{S}^{d-1} \times \mathbb{R}} \left( \frac{\omega(x - b)^p}{1 + |b|^{p-1}} \right) \, d\mu(w, b)$$

is well-defined, since $\varphi(w, b) = \frac{\omega(x - b)^p}{1 + |b|^{p-1}}$ belongs to $C_0(\mathbb{S}^{d-1} \times \mathbb{R})$.

In our paper a measure $\mu \in \mathcal{M}^1(\mathbb{S}^{d-1} \times \mathbb{R})$ is called even measure if

$$\int_{\mathbb{S}^{d-1} \times \mathbb{R}} \varphi(w, b) \, d\mu(w, b) = \int_{\mathbb{S}^{d-1} \times \mathbb{R}} \varphi(-w, b) \, d\mu(w, b) \quad \text{for all } \varphi \in C_0(\mathbb{S}^{d-1} \times \mathbb{R})$$

or $\mu$ is odd measure if

$$\int_{\mathbb{S}^{d-1} \times \mathbb{R}} \varphi(w, b) \, d\mu(w, b) = -\int_{\mathbb{S}^{d-1} \times \mathbb{R}} \varphi(-w, b) \, d\mu(w, b) \quad \text{for all } \varphi \in C_0(\mathbb{S}^{d-1} \times \mathbb{R}).$$

Every measure $\mu \in \mathcal{M}^1(\mathbb{S}^{d-1} \times \mathbb{R})$ is uniquely decomposable as $\mu = \mu^+ + \mu^-$ where $\mu^+$ is even and $\mu^-$ is odd, which called the even and odd decomposition of $\mu$. For instance, let $\mu$ be a measure with a density $g(w, b)$ then $\mu^+$ is the measure with density $g^+(w, b) = \frac{1}{2}(g(w, b) + g(-w, -b))$ and $\mu^-$ is the measure with density $g^-(w, b) = \frac{1}{2}(g(w, b) - g(-w, -b))$.

We denote by $\mathcal{M}^1_0(\mathbb{S}^{d-1} \times \mathbb{R})$ the subspace of all even measures in $\mathcal{M}^1(\mathbb{S}^{d-1} \times \mathbb{R})$. The space $\mathcal{M}^1_0(\mathbb{S}^{d-1} \times \mathbb{R})$ is the Banach space dual of $C_{0,e}(\mathbb{S}^{d-1} \times \mathbb{R})$, the subspace of all even functions $\varphi \in C_0(\mathbb{S}^{d-1} \times \mathbb{R})$. Even measures play an important role in our paper, mainly our main results use even measure in the characterization of the representational cost. cf., e.g., Lemma 5.1. The following result can be found in [29].

Lemma 3.1. Let $\mu \in \mathcal{M}^1(\mathbb{S}^{d-1} \times \mathbb{R})$ with $\mu = \mu^+ + \mu^-$ where $\mu^+$ is even and $\mu^-$ is odd. Then $\|\mu^+\|_{M^1} \leq \|\mu\|_{M^1}$ and $\|\mu^-\|_{M^1} \leq \|\mu\|_{M^1}$.

\footnote{We assume $\mu$ is a signed Radon measure; see, e.g., [29] for a formal definition. In this paper a measure means Radon measure to avoid confusion with the Radon transform.}
3.2 The Radon transform and its dual

The Radon transform is a fundamental transform applicable to tomography, i.e., the creation of an image from the scattering data associated with cross-sectional scans of an object. In this section, we recall some aspects and features of the Radon transform mentioned in Helgason’s book, more details about its properties and applications can be found in [14].

In our approach the representational cost \( R_1(f) \) in Section 3 is characterized in terms of the Radon transform. Therefore, we briefly review the Radon transform and its dual. Note that several properties of the Radon transform are analogous to the properties of the Fourier transform.

Let \( f \) be a real-valued function defined on \( \mathbb{R}^d \) and integrable on each hyperplane in \( \mathbb{R}^d \). The Radon transform \( \mathcal{R} \) represents \( f \) in terms of its integrals over all possible hyperplanes in \( \mathbb{R}^d \), in the following way

\[
\mathcal{R}\{f\}(w,b) := \int_{(w,x) = b} f(x) \, dx \quad \text{for all } (w,b) \in \mathbb{S}^{d-1} \times \mathbb{R},
\]

where \( dx \) represents integration with respect to the \( d-1 \) dimensional Lebesgue measure on the hyperplane \( \langle w, x \rangle = b \). In view of the fact that \( \langle w, x \rangle = b \) and \( -\langle w, x \rangle = -b \) determine the same hyperplane, then the Radon transform is an even function, that is, \( \mathcal{R}\{f\}(w, b) = \mathcal{R}\{f\}(-w, -b) \) for all \( (w, b) \in \mathbb{S}^{d-1} \times \mathbb{R} \).

The dual Radon transform \( \mathcal{R}^* \), i.e., the formal adjoint of \( \mathcal{R} \), of a function \( \varphi : \mathbb{S}^{d-1} \times \mathbb{R} \to \mathbb{R} \) is defined as follows:

\[
\mathcal{R}^*\{\varphi\}(x) := \int_{\mathbb{S}^{d-1}} \varphi(w, (w, x)) \, dw \quad \text{for all } x \in \mathbb{R}^d,
\]

where \( dw \) represents integration with respect to the surface measure of the unit sphere \( \mathbb{S}^{d-1} \).

The function \( f \) can be recovered from the Radon transform employing the inverse Radon formula. The inverse Radon transform corresponds to the reconstruction of the function from the projections. Mainly the inverse Radon transform is a composition of the dual Radon transform \( \mathcal{R}^* \) followed by a multiplier operator.

The multiplier operator is given by \( m_{d-1}(D) = (-\Delta)^{(d-1)/2} \), such that for any \( s > 0 \) the operator \( m_{s}(D) = (-\Delta)^{s/2} \) is defined by

\[
m_{s}(D)f(x) = \mathcal{F}^{-1}\{m(\xi) (\mathcal{F} f)(\xi)\}(x)
\]

where \( \mathcal{F} \) and \( \mathcal{F}^{-1} \) are the Fourier transform and the inverse Fourier transform defined respectively by

\[
(\mathcal{F} f)(\xi) = \hat{f}(\xi) \equiv (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-i(x,\xi)} f(x) \, dx
\]

\[
(\mathcal{F}^{-1} f)(x) \equiv (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{i(x,\xi)} f(\xi) \, d\xi
\]

when \( f \in L^1(\mathbb{R}^d) \). When \( d = 2n + 1 \) for \( n \in \mathbb{N} \), \( (-\Delta)^{(d-1)/2} = (-1)^n \Delta^\frac{n}{2} \) which is the Laplacian applied \( n \) times. While if \( d \) is even it is a pseudo-differential operator given by convolution with a singular kernel. In view of the previous operators the explicit inversion formula for the Radon transform is given by

\[
f = \gamma_d(-\Delta)^{(d-1)/2}\mathcal{R}^*\{\mathcal{R}\{f\}\},
\]

where \( \gamma_d \) is a constant depending on the dimension \( d \), the equality in (3.6) holds for \( f \) belonging to many common function spaces e.g., \( f \) is a rapidly decreasing functions on \( \mathbb{R}^d \), more details can be found in e.g., [14].

Although more restriction are imposed on the function space, the inversion formula for the dual Radon transform (3.7) can be given in a similar way to the inversion formula for the Radon transform. Indeed, if \( \varphi \) is an even function belongs to the Schwartz class \( \mathcal{S}(\mathbb{S}^{d-1} \times \mathbb{R}) \), then

\[
\varphi = \gamma_d\mathcal{R}\{(-\Delta)^{(d-1)/2}\mathcal{R}^*\{\varphi\}\},
\]

where \( \gamma_d = \frac{1}{(2\pi)^{d/2}} \). We denote the Fourier transform with respect to \( b \) by \( \mathcal{F}_b \). We recall the Fourier slice theorem for the Radon transform cf. Helgason [14]. The \( \mathcal{F}_b \) of the Radon transform is given by

\[
\mathcal{F}_b\mathcal{R}\{f\}(w, \tau) = \hat{f}(\tau \cdot w)
\]
where \( f \in L^1(\mathbb{R}^d) \), \( \tau \in \mathbb{R} \) and \( w \in S^{d-1} \). From the definition of a multiplier operator in (3.5) and the later equality (3.8) we establish the intertwining property of the Laplacian and the Radon transform: assuming \( f \) and \( \Delta f \) are in \( L^1(\mathbb{R}^d) \), we have
\[
\mathcal{R}\{\Delta f\} = \partial_b^2 \mathcal{R}\{f\}
\] (3.9)
where \( \partial_b \) is the partial derivative with respect to \( b \). More generally for any positive integer \( s \), assuming that \( f \) and \( (-\Delta)^{s/2} f \) are in \( L^1(\mathbb{R}^d) \) we have
\[
\mathcal{R}\{(-\Delta)^{s/2} f\} = (-\partial_b^2)^{s/2} \mathcal{R}\{f\}
\] (3.10)
where fractional powers of \(-\partial_b^2\) can be defined similarly as the fractional powers of the Laplacian.

4 Infinite width bounded norm networks for univariate function

In the next result, we consider a possible model for an infinite width network on one dimension. In particular we characterize \( \overline{\mathcal{R}}(f) \) for univariate function \( f \). Moreover, in this section, we assume that the measure has a density that satisfies certain decay properties. Mainly, we treat an infinite width neural networks \( H^p_{\mu,c} \) with density \( \mu(w,b) \) that decays as \( \frac{1}{1 + |b|^{p-1}} \) when \( |b| \to \infty \), this assumption is needed to ensure convergence of certain integrals in the proof of Theorem 4.1.

**Theorem 4.1.** Let \( p \in \mathbb{N} \) such that \( p \) is odd, \( f \) is a real-valued function defined on \( \mathbb{R} \) and \( \psi(b) = 1 + |b|^{p-1} \) where \( b \in \mathbb{R} \), then we have
\[
\min_{\mu,c\in\mathbb{R}} \|\mu\|_{\mathcal{M}^1(\mathbb{R})}^{1/p} \text{ such that } f = H^p_{\mu,c}
= \max\left( \frac{1}{p!} \int_{-\infty}^{\infty} \left| f^{(p+1)}(b) \right| \, db \right)
\]
where the minimum is taken over all measures \( \mu \in \mathcal{M}^1(\mathbb{S}^{d-1} \times \mathbb{R}) \) with density that decays as \( \frac{1}{1 + |b|^{p-1}} \) when \( |b| \to \infty \).

The derivatives of \( f \) are interpreted in the weak sense, that is we work with distributions, rather than only functions. Hence integrals in the statement and its proof are well defined even if \( f \) is not differentiable. Noting that we use the same symbols to refer to measure and its associated density. Moreover, we denote by \( f^{(p)}(\infty) = \lim_{x\to\infty} f^{(p)}(x) \) and \( f^{(p)}(-\infty) = \lim_{x\to-\infty} f^{(p)}(x) \). Observe that both limits exist if \( \int_{-\infty}^{\infty} |f^{(p+1)}(x)| \, dx \) is finite. Furthermore, in our paper, we deal with more relaxed assumptions on the measure \( \mu \), since we don’t ask \( \mu \) to have a density in the Schwartz space.

**Proof.** In one dimension, \( w \in \{\pm 1\} \), (by transforming \((w,b) \mapsto (w,wb)\), which does not change \( \overline{\mathcal{R}}(f) \), in view of a linear change of variables argument in (2.9) it is more convenient to reparametrize the RePU units as \( |w(x-b)+p| \) instead of \( |wx-b|^p \). Let \( H(z) \) denotes the Heaviside function, whose distributional derivative is the Dirac distribution \( \delta \). Then for any representation \( f = H^p_{\mu,c} \), we have
\[
f(x) = \int_{\mathbb{R}} \left( \mu(1,b) \frac{|x-b|^p - |b|^p}{1 + |b|^{p-1}} + \mu(-1,b) \frac{|x-b|^p - |b|^p}{1 + |b|^{p-1}} \right) \, db + c.
\] (4.1)

For \( \epsilon > 0 \), we let \( x \) be fixed in the ball of center \( 0 \) and radius \( \frac{\epsilon}{4} \), we differentiate \( f \) in (4.1) \( p + 1 \) times with respect to \( x \), then we get for \( k \in \{1, \ldots, p\} \)
\[
f^{(k)}(x) = \frac{p!}{(p-k)!} \int_{\mathbb{R}} \mu(1,b) \frac{|x-b|^{p-k} - |b|^{p-k}}{1 + |b|^{p-1}} + (-1)^k \mu(-1,b) \frac{|x-b|^{p-k} - |b|^{p-k}}{1 + |b|^{p-1}} \, db,
\] (4.2)
hence
\[
f^{(p)}(x) = p! \int_{\mathbb{R}} \mu(1,b) \frac{H(x-b) - \mu(-1,b) H(b-x)}{1 + |b|^{p-1}} \, db
\] (4.3)
\[
f^{(p+1)}(x) = p! \int_{\mathbb{R}} \mu(1,b) \frac{\mu(-1,b)}{1 + |b|^{p-1}} \, db
\]
\[
= p! \mu(1,x) + \mu(-1,x) \frac{1}{1 + |x|^{p-1}} \approx p! \frac{\mu(x)}{\psi(x)}.
\] (4.4)
where we set \( \mu_+(x) := (\mu(1, x) + \mu(-1, x)) \) and \( \psi(x) = 1 + |x|^{p-1} \).

The quantity \( p! \mu_+^{(p)} \), uniquely represents \( f^{(p+1)} \). Moreover, the measure \( \mu \) that represents the function \( f \) is almost unique (up to shifting argument with respect to \( \mu \)). We denote by \( \mu_- (b) = \mu(1, b) - \mu(-1, b) \).

Since \( w \in \{-1, 1\}, \mu_+ \) and \( \mu_- \) define \( \mu \) as following

\[
\mu(w, b) = \frac{1}{2} \left( \mu_+(b) - \frac{1}{p} \sum_{k=1}^{p} (-w)^{2k+1} \cdot \mu_-(b) \right) = \frac{1}{2} (\mu_+(b) + w \cdot \mu_- (b)).
\]

(4.5)

Note that, for odd \( p \), we can write

\[
[x]_+^p = \frac{1}{2} (|x| + x^p), \quad \text{for any } x \in \mathbb{R}.
\]

(4.6)

In view of (4.5) and (4.6), we have the following:

\[
f(x) = \int_{\mathbb{R}} \left( \mu(1, b) \frac{[x - b]_+^p - [-b]_+^p}{1 + |b|^{p-1}} + \mu(-1, b) \frac{[b - x]_+^p - [b]_+^p}{1 + |b|^{p-1}} \right) db + c
\]

\[
= \frac{1}{2} \int_{\mathbb{R}} \frac{1}{\psi(b)} \left( \mu_+(b) \left( [x - b]^p - |b|^p \right) + \mu_-(b) \left( (x - b)^p + b^p \right) \right) db + c
\]

\[
= \frac{1}{2} p! \int_{\mathbb{R}} f^{(p+1)}(b) \left( [x - b]^p - |b|^p \right) db
\]

\[
+ \frac{1}{2} \sum_{k=1}^{p} \left( (-1)^{p-k} \left( \frac{p}{k} \right) \int_{\mathbb{R}} \frac{\mu_-(b)}{\psi(b)} b^{p-k} db \right) x^k + \left( \int_{\mathbb{R}} \frac{\mu_- (b)}{\psi(b)} db + c \right).
\]

(4.7)

From the previous equation (4.7), \( \mu_- \) contributes the coefficients of a clear polynomial of order \( p \) in \( f = \mu_{p,c}^{(p)} \).

Since we can adjust the constant term using the bias \( c \) without changing \( \mathcal{R}(f) \), the only important issue with choosing \( \mu_- \) is to get the correct polynomial term. To get a clear idea about which polynomial correction we need, it is important to consider (4.3) to evaluate \( f^{(p)}(-\infty) \) and \( f^{(p)}(+\infty) \). Due to the fact that \( \mu \in M^1(\mathbb{S}^d \times \mathbb{R}) \), it is clear that \( \int |f^{(p+1)}| \, dx \) is finite. Therefore \( f^{(p)} \) converges at \( \pm \infty \), and

\[
f^{(p)}(-\infty) + f^{(p)}(+\infty) = \int_{\mathbb{R}} (0 - \mu(-1, b)) p! \frac{1}{1 + |b|^{p-1}} db + \int_{\mathbb{R}} (\mu(1, b)) p! \frac{1}{1 + |b|^{p-1}} db - 0 \, db
\]

\[
= p! \int_{\mathbb{R}} \frac{\mu_- (b)}{\psi(b)} db.
\]

(4.8)

Moreover when \( k \in \{1, \ldots, p-1\} \), in view of (4.2), we have

\[
f^{(k)}(0) = \frac{p!}{(p-k)!} \int_{\mathbb{R}} \mu(1, b) \left( \frac{[-b]_+^{p-k}}{1 + |b|^{p-1}} \right) + (-1)^k \mu(-1, b) \left( \frac{[b]_+^{p-k}}{1 + |b|^{p-1}} \right) db
\]

If \( k \) is an even number, using (4.6), then

\[
f^{(k)}(0) = \frac{p!}{(p-k)!} \int_{\mathbb{R}} \mu(1, b) \left( \frac{[-b]_+^{p-k}}{1 + |b|^{p-1}} \right) + \mu(-1, b) \left( \frac{[b]_+^{p-k}}{1 + |b|^{p-1}} \right) db
\]

\[
= \frac{p!}{(p-k)!} \int_{\mathbb{R}} \mu_+(b) \left( \frac{[b]_+^{p-k}}{1 + |b|^{p-1}} \right) + \mu_-(b) \left( \frac{b^{p-k}}{1 + |b|^{p-1}} \right) db
\]

\[
= \frac{p!}{(p-k)!} \int_{\mathbb{R}} \mu_+(b) \left( \frac{b^{p-k}}{1 + |b|^{p-1}} \right) db.
\]

Therefore,

\[
\int_{\mathbb{R}} \frac{\mu_- (b)}{\psi(b)} b^{p-k} db = \frac{1}{p!} \int_{\mathbb{R}} f^{(p+1)}(b) [b]_+^{p-k} db = \frac{(p-k)!}{p!} f^{(k)}(0).
\]

(4.9)

Instead if \( k \) is an odd number, then using the fact that \([x]_+^{2n} = x^{2n} - [-x]_+^{2n}\) for any \( n \in \mathbb{N} \) and any
\( x \in \mathbb{R} \), we get

\[
f^{(k)}(0) = \frac{p!}{(p-k)!} \int_{\mathbb{R}} \mu(1, b) \frac{[-b]^{p-k}}{1 + |b|^{p-k}} - \mu(-1, b) \frac{[b]^{p-k}}{1 + |b|^{p-k}} \, db
\]

\[
= \frac{p!}{(p-k)!} \int_{\mathbb{R}} \mu(1, b) \frac{b^{p-k} - [b]^{p-k}}{1 + |b|^{p-1}} - \mu(-1, b) \frac{b^{p-k} - [-b]^{p-k}}{1 + |b|^{p-1}} \, db
\]

\[
= \frac{p!}{(p-k)!} \int \mu_-(b) b^{p-k} \, db - \frac{p!}{(p-k)!} \int_{\mathbb{R}} \frac{1}{\psi(b)} (\mu(1, b) [b]^{p-k} - \mu(-1, b) [-b]^{p-k}) \, db.
\]

Moreover, using the fact that \( \mu_+ (b) - \mu_- (b) = 2\mu(-1, b) \), we have

\[
\int \frac{1}{\psi(b)} \left( \mu(1, b) [b]^{p-k} - \mu(-1, b) [-b]^{p-k} \right) \, db = \int \frac{1}{\psi(b)} \left( \mu(1, b) [b]^{p-k} - \mu(-1, b) [-b]^{p-k} \right) \, db
\]

\[
= \int \frac{1}{\psi(b)} \left( \mu_+ (b) [b]^{p-k} + \mu(-1, b) [b]^{p-k} - \mu(-1, b) [-b]^{p-k} \right) \, db
\]

\[
= \int \frac{1}{\psi(b)} \left( \mu_+ (b) [b]^{p-k} - \mu(-1, b) \left( [b]^{p-k} + [-b]^{p-k} \right) \right) \, db
\]

\[
= \int \frac{1}{\psi(b)} \left( \mu_+ (b) [b]^{p-k} - \frac{1}{2} (\mu_+ (b) - \mu_- (b)) b^{p-k} \right) \, db.
\]

Then for odd \( k \) we conclude that

\[
f^{(k)}(0) = \frac{p!}{(p-k)!} \left( \int \frac{\mu_-(b)}{\psi(b)} b^{p-k} \, db - \int \frac{1}{\psi(b)} \left( \mu_+ (b) [b]^{p-k} - \frac{1}{2} (\mu_+ (b) - \mu_- (b)) b^{p-k} \right) \, db \right)
\]

\[
= \frac{p!}{(p-k)!} \left( \frac{1}{2} \int \frac{\mu_-(b)}{\psi(b)} b^{p-k} \, db - \int \frac{1}{\psi(b)} \left( \mu_+ (b) [b]^{p-k} - \frac{1}{2} \mu_+ (b) b^{p-k} \right) \, db \right)
\]

\[
= \frac{p!}{(p-k)!} \left( \frac{1}{2} \int \frac{\mu_-(b)}{\psi(b)} b^{p-k} \, db + \int \frac{\mu_+ (b)}{\psi(b)} [-b]^{p-k} \, db \right).
\]

Finally, we get

\[
\int \frac{\mu_-(b)}{\psi(b)} b^{p-k} \, db = 2 \frac{(p-k)!}{p!} f^{(k)}(0) - \frac{2}{p!} \int \frac{f^{(p+1)}(-b)}{\psi(b)} b^{p-k} \, db. \quad (4.10)
\]

In summary, any measure \( \mu_- \) that fulfills (4.8), (4.9), (4.10), in conjunction with (4.5), we have

\[
\|\mu\|_{\mathcal{M}^{(1/\psi)}} = \frac{1}{2} \int \left( \left| \frac{1}{p!} f^{(p+1)}(b) + \frac{\mu_-(b)}{\psi(b)} \right| + \left| \frac{1}{p!} f^{(p+1)}(b) - \frac{\mu_-(b)}{\psi(b)} \right| \right) \, db. \quad (4.11)
\]

In order to minimize \( \|\mu\|_{\mathcal{M}^{(1/\psi)}} \) we have to solve the following

\[
\min_{\mu} \frac{1}{2} \int \left( \left| \frac{1}{p!} f^{(p+1)}(b) + \frac{\mu_-(b)}{\psi(b)} \right| + \left| \frac{1}{p!} f^{(p+1)}(b) - \frac{\mu_-(b)}{\psi(b)} \right| \right) \, db
\]

such that

\[
\frac{1}{p!} \sum_{k=1}^{p-1} \int_{\mathbb{R}} f^{(p+1)}(b) [b]^{p-2k} \, db - \frac{(p-2k)!}{p!^2} f^{(2k)}(0) = \int \frac{\mu_-(b)}{\psi(b)} \, db.
\]

\[
\sum_{k=0}^{p-3} \frac{2}{p} \left( f^{(p+1)}(-b) \right) (0) = \sum_{k=0}^{p-3} \frac{2}{p} \int_{\mathbb{R}} f^{(p+1)}(-b) [b]^{p-2k} \, db.
\]

\[
\sum_{k=0}^{p-3} \frac{2}{p} \int_{\mathbb{R}} f^{(p+1)}(-b) [b]^{p-2k} \, db.
\]
We let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ be the Lagrange multipliers and $\mathcal{L}$ is the Lagrangian, given by

$$
\mathcal{L}(\mu_-, f', \ldots, f^{(p+1)}, \lambda_1, \lambda_2, \lambda_3) = \frac{1}{2} \int_{\mathbb{R}} \left( \frac{1}{p!} f^{(p+1)}(b) + \frac{\mu_-(b)}{\psi(b)} \right) db + \lambda_1 \left( \int_{\mathbb{R}} \frac{\mu_-(b)}{\psi(b)} db - \frac{f^{(p)}(-\infty)}{p!} \right) + \lambda_2 \left( \sum_{k=1}^{\infty} \int_{\mathbb{R}} \frac{\mu_-(b)}{\psi(b)} db^{p-2k} - \frac{1}{p!} \int_{\mathbb{R}} f^{(p+1)}(b) db^{p-2k} + \frac{1}{p!} \int_{\mathbb{R}} f^{(2k)}(0) \right) + \lambda_3 \left( \sum_{k=0}^{\infty} \int_{\mathbb{R}} \frac{\mu_-(b)}{\psi(b)} db^{p-(2k+1)} - \frac{2(p-(2k+1)!)}{p!} f^{(2k+1)}(0) + \frac{2}{p!} \int_{\mathbb{R}} f^{(p+1)} [-b]_{p}^{p-(2k+1)} db \right).
$$

In order to determine the functional derivative of $\mathcal{L}$ with respect to $\mu_-$, we recall that a functional derivative of $\mathcal{L}$ is a functional of functional that we denote by

$$
\frac{\partial \mathcal{L}(\mu_-, f', \ldots, f^{(p+1)}, \lambda_1, \lambda_2, \lambda_3)[h]}{\partial \mu_-} = \frac{d}{d\gamma} \mathcal{L}(\mu_- + \gamma h, f', \ldots, f^{(p+1)}, \lambda_1, \lambda_2, \lambda_3) \bigg|_{\gamma=0}.
$$

Hence in our case, the functional derivative of $\mathcal{L}$ with respect to $\mu_-$ is the functional that maps the functions $\mu_-, f', \ldots, f^{(p+1)}, \lambda_1, \lambda_2, \lambda_3$ and $h$ to the number $\frac{\partial \mathcal{L}}{\partial \mu_-} (\mu_-, f', \ldots, f^{(p+1)}, \lambda_1, \lambda_2, \lambda_3)[h]$. Moreover, we choose the preceding function $h$ to be the Dirac delta $\delta_b(t)$, hence we get

$$
\frac{\partial \mathcal{L}}{\partial \mu_-} (\mu_-, f', \ldots, f^{(p+1)}, \lambda_1, \lambda_2, \lambda_3) = \frac{1}{2} \int_{\mathbb{R}} \left( \sign \left( \frac{1}{p!} f^{(p+1)}(t) + \frac{\mu_-(t)}{\psi(t)} \right) + \sign \left( \frac{1}{p!} f^{(p+1)}(t) - \frac{\mu_-(t)}{\psi(t)} \right) \right) \delta_b(t) dt + \lambda_1 \left( \int_{\mathbb{R}} \frac{\delta_b(t)}{\psi(t)} dt \right) + \lambda_2 \left( \sum_{k=1}^{\infty} \int_{\mathbb{R}} \frac{\delta_b(t)}{\psi(t)} db^{p-2k} \right) + \lambda_3 \left( \sum_{k=0}^{\infty} \int_{\mathbb{R}} \frac{\delta_b(t)}{\psi(t)} db^{p-(2k+1)} \right).
$$

Furthermore, we set the functional derivative of the Lagrangian $\mathcal{L}$ with respect to $\mu_-$ to zero, that is

$$
0 = \frac{\partial \mathcal{L}}{\partial \mu_-} (\mu_-, f', \ldots, f^{(p+1)}, \lambda_1, \lambda_2, \lambda_3) = \frac{1}{2} \left( \sign \left( \frac{1}{p!} f^{(p+1)} + \frac{\mu_-}{\psi} \right) - \sign \left( \frac{1}{p!} f^{(p+1)} - \frac{\mu_-}{\psi} \right) \right) \frac{\psi}{\psi^2} + \lambda_1 \frac{\psi}{\psi^2} \sum_{k=1}^{\infty} \int_{\mathbb{R}} b^{p-2k} + \lambda_2 \frac{\psi}{\psi^2} \sum_{k=0}^{\infty} \int_{\mathbb{R}} b^{p-(2k+1)}. \quad (4.13)
$$

Let $\Lambda = \frac{\psi}{\psi^2} \sum_{k=1}^{\infty} \int_{\mathbb{R}} b^{p-2k} + \frac{\psi}{\psi^2} \sum_{k=0}^{\infty} \int_{\mathbb{R}} b^{p-(2k+1)}$. Therefore the possible values of $\Lambda$ depend on $\lambda_1, \lambda_2, \lambda_3$ and $b$, which can be treated as follows:

$\Lambda = 0$ : Then sign $\left( \frac{1}{p!} f^{(p+1)} + \frac{\mu_-}{\psi} \right) = \sign \left( \frac{1}{p!} f^{(p+1)} - \frac{\mu_-}{\psi} \right)$, which implies that $\left| \frac{\mu_-}{\psi} \right| \leq \frac{1}{p!} |f^{(p+1)}|$ point-wise and from (4.11) we have

$$
\|\mu\|_{\mathcal{M}(1/\psi)} = \frac{1}{p!} \int_{\mathbb{R}} |f^{(p+1)}(b)| db.
$$

Furthermore, in order to satisfy (4.12), we have

$$
\frac{1}{p!} \left| f^{(p)}(-\infty) + f^{(p)}(\infty) \right| = \left| \int_{\mathbb{R}} \frac{\mu_-}{\psi(b)} db \right| \leq \int_{\mathbb{R}} \left| \frac{\mu_-}{\psi(b)} \right| db \leq \frac{1}{p!} \int_{\mathbb{R}} |f^{(p+1)}(b)| db.
$$

and

$$
\frac{1}{p!} \sum_{k=1}^{\infty} \int_{\mathbb{R}} |f^{(p+1)}(b)| b^{p-2k} db - \frac{(p-2k)!}{p!} f^{(2k)}(0) \leq \sum_{k=1}^{\infty} \int_{\mathbb{R}} \left| \frac{\mu_-}{\psi(b)} \right| b^{p-2k} db \leq \frac{1}{p!} \sum_{k=1}^{\infty} \int_{\mathbb{R}} |f^{(p+1)}(b)| b^{p-2k} db.
$$
Moreover,
\[
\sum_{k=0}^{\infty} \frac{(p - (2k + 1))!}{p^k} | f^{(2k+1)}(0) - \frac{2}{p^k} \int_{\mathbb{R}} f^{(p+1)}[-b]^{p-(2k+1)} \, db | \leq \sum_{k=0}^{\infty} \int_{\mathbb{R}} | \frac{\mu_p(-b)}{\psi(b)} | b^{p-(2k+1)} \, db \\
\leq \frac{1}{p^k} \sum_{k=0}^{\infty} \int_{\mathbb{R}} | f^{(p+1)}(b) | b^{p-(2k+1)} \, db.
\]

\(\Lambda < 0\): In this case \(\frac{1}{p^k} f^{(p+1)} + \frac{\mu_p}{\psi} \geq 0\) and \(\frac{1}{p^k} f^{(p+1)} - \frac{\mu_p}{\psi} \leq 0\) pointwise, then \(\frac{\mu_p}{\psi} \geq \frac{1}{p^k} | f^{(p+1)} |\) and from (4.11) and the constraint in (4.12): we get \(\|\mu\|_{M^1(1/\psi)} = \int_{\mathbb{R}} \frac{\mu_p(-b)}{\psi(b)} \, db = \frac{1}{p^k} \left( f^{(p)}(-\infty) + f^{(p)}(\infty) \right)\),

this happens if \(f^{(p)}(-\infty) + f^{(p)}(\infty) \geq \int_{\mathbb{R}} f^{(p+1)}(b) \, db\) and

\[
\sum_{k=0}^{\infty} \frac{(p - (2k + 1))!}{p^k} f^{(2k+1)}(0) - \frac{2}{p^k} \int_{\mathbb{R}} f^{(p+1)}[-b]^{p-(2k+1)} \, db = \sum_{k=0}^{\infty} \int_{\mathbb{R}} \frac{\mu_p(-b)}{\psi(b)} b^{p-(2k+1)} \, db \\
\geq \frac{1}{p^k} \sum_{k=0}^{\infty} \int_{\mathbb{R}} | f^{(p+1)}(b) | b^{p-(2k+1)} \, db,
\]

where we used the fact that \(p - (2k + 1)\) is an even number.

\(\Lambda > 0\): In this case \(\frac{1}{p^k} f^{(p+1)} + \frac{\mu_p}{\psi} \leq 0\) and \(\frac{1}{p^k} f^{(p+1)} - \frac{\mu_p}{\psi} \geq 0\) pointwise, then \(\frac{\mu_p}{\psi} \leq \frac{1}{p^k} | f^{(p+1)} |\) and from (4.11) and the constraint in (4.12): we get

\[
\|\mu\|_{M^1(1/\psi)} = \int_{\mathbb{R}} \left( -\frac{\mu_p}{\psi} \right) \, db = -\frac{1}{p^k} \left( f^{(p)}(-\infty) + f^{(p)}(\infty) \right)
\]

which is true when \(f^{(p)}(-\infty) + f^{(p)}(\infty) \leq -\int_{\mathbb{R}} | f^{(p+1)}(b) | \, db\).

In view of the previous cases, we conclude that

\[
\|\mu\|_{M^1(1/\psi)} = \frac{1}{p^k} \max(\int_{\mathbb{R}} | f^{(p+1)} | \, db, | f^{(p)}(-\infty) + f^{(p)}(+\infty) |).
\]

\[\square\]

5 Infinite width bounded norm networks for functions in higher dimension

The main aim of our paper is to determine \(\overline{R}(f)\) for any function \(f : \mathbb{R}^d \to \mathbb{R}\), and characterize when it is finite in order to know what are the functions that can be approximated arbitrarily well with bounded norm but unbounded width RePU neural networks.

For odd \(p\) every two layers RePU networks decompose into the sum of a network with the \(p\) power of absolute units plus a polynomial part of order \(p\). Indeed, if \(p \in \mathbb{N}\) is odd, then \([x]_+^p = \frac{1}{2} (|x|^p + x^p)\). Given the proof of Theorem 4.1, i.e., in the univariate case, the weights on the \(p\) power of the absolute value units determine the representational cost, with a correction term needed if the polynomial part has a large weight.

In our approach we propose to consider adding unregularized monomial units \(\sum_{k=1}^{p} (v_k, x^k)\) to “absorb” any representational cost due to the polynomial part. In other words, for any \(\theta \in \Theta\) and \(v \in \mathbb{R}^{d \times p}\) we define the class of unbounded width two layers RePU neural networks \(g_{\theta, v, p}\) with monomial units by

\[
g_{\theta, v, p}(x) = g_{\theta, p}(x) + \sum_{k=1}^{p} (v_k, x^k),
\]

where \(g_{\theta, p}\) is as defined in (2.1). Moreover we associate \(g_{\theta, v, p}\) with the same weight norm \(C(\theta)\) as defined in (2.2). Namely, we exclude the norm of the weight \(v = (v_1, \ldots, v_p)\) on the additional units from the
cost. Then we define the representational cost \( \overline{R}_1(f) \) for this class of neural networks in the following way

\[
\overline{R}_1(f) := \lim_{\varepsilon \to 0} \left( \inf_{\theta \in \Theta} C(\theta) : |g_{\theta, w, p}(x) - f(x)| \leq \varepsilon, \forall \|x\| \leq 1/\varepsilon, \text{ and } g_{\theta, w, p}(0) = f(0) \right).
\]  

(5.1)

Likewise, for all \( \mu \in M^1(\mathbb{S}^{d-1} \times \mathbb{R}) \), \( v \in \mathbb{R}^{d \times p} \), \( c \in \mathbb{R} \), we define an infinite width network with monomial units by

\[
\mathcal{H}^p_{\mu, v, c}(x) := \mathcal{H}^p_{\mu, c}(x) + \sum_{k=1}^{p} \langle r_k, x^k \rangle.
\]

Let \( \psi(x) = 1 + |x|^{p-1} \) where \( p \) is an odd number, we prove in Appendix B that \( \overline{R}_1(f) \) can be expressed as follows:

\[
\overline{R}_1(f) = \min_{\mu \in M^1(\mathbb{S}^{d-1} \times \mathbb{R}), v \in \mathbb{R}^{d \times p}, c \in \mathbb{R}} \|\mu\|^{1/p}_{M^1(1/\psi)} \text{ such that } f = \mathcal{H}^p_{\mu, v, c}.
\]  

(5.2)

Consequently, we show that the minimizer of (5.2) is unique and can be determined in the next lemma, where its proof is given in Appendix B.

**Lemma 5.1.** Let \( p \in \mathbb{N} \) be an odd number, then \( \overline{R}_1(f) = \|\mu^+\|^{1/p}_{M^1(1/\psi)} \) where \( \mu^+ \in M^1_+ \) is the unique even measure such that \( f = \mathcal{H}^p_{\mu^+, v, c} \) for some \( v \in \mathbb{R}^{d \times p}, c \in \mathbb{R} \).

### 5.1 Representational cost for infinite width RePU neural networks

We now observe that the Laplacian of an infinite width network can be related to the dual Radon transform. First, we consider a measure that has a density, then we define the representational cost.

In particular, we consider the following identity:

\[
\Delta \mathcal{H}^p f(x) = \int_{\mathbb{S}^{d-1} \times \mathbb{R}} \mathcal{H}^p_{\mu, v, c}(x) \text{ such that } f = \mathcal{H}^p_{\mu, v, c}.
\]

(5.3)

Taking a differentiation of (5.3), with respect to \( x \), \( (p+1) \) times inside the integral, by applying the Laplacian \( (p+1)/2 \) times, we get

\[
\Delta \mathcal{H}^p f(x) = \int_{\mathbb{S}^{d-1} \times \mathbb{R}} \mathcal{H}^p_{\mu, v, c}(x) \text{ such that } f = \mathcal{H}^p_{\mu, v, c}.
\]

(5.4)

where \( \delta(\cdot) \) denotes a Dirac delta. In view of the dual Radon transform (5.3) and the previous identity (5.1), \( \Delta \mathcal{H}^p f \) can be considered as a dual Radon transform of the weighted density \( \mathcal{H}^p_{\mu, v, c}(x) \). That is, we have the following identity

\[
\Delta \mathcal{H}^p f = p!\mathcal{R}^*\left( \frac{\mu}{\psi} \right), \text{ where } \psi(b) = 1 + |b|^{p-1} \text{ and } b \in \mathbb{R}.
\]  

(5.5)

Using the characterization of \( \overline{R}_1(f) \) given in Lemma 5.1 and the inversion dual Radon transform given in \( \text{[5.7]} \) to \( \text{[5.5]} \), we can determine \( \overline{R}_1(f) \) for a given function \( f \) as next result shows.

**Lemma 5.2.** Let \( p \in \mathbb{N} \) be an odd number, \( f = \mathcal{H}^p_{\mu, v, c} \) where \( \mu \) is an even measure given by a density in \( \mathcal{S} \) \((\mathbb{S}^{d-1} \times \mathbb{R}) \) and \( v = (v_1, \ldots, v_p) \in \mathbb{R}^{d \times p}, c \in \mathbb{R} \) and let \( \psi(x) = 1 + |x|^{p-1} \). Then

\[
\frac{\mu}{\psi} = \frac{\gamma_d}{p!} \mathcal{R}((-\Delta)^{(d+1)/2}) f, \text{ and } \overline{R}_1(f) = \frac{\gamma_d}{p!} \mathcal{R}((-\Delta)^{(d+1)/2}) f \|^{1/p}_{M^1}.
\]

where \( \gamma_d = \frac{1}{2^{(d+1)/2}} \).

Using the fact that \( \mathcal{R}((-\Delta)^{(d+1)/2}) = \partial_0^{d+1} \mathcal{R}(f) \), and that \( \overline{R}_1(f) \) can be computed in terms of the \( M^1 \) norm of \( \partial_0^{d+1} \mathcal{R}(f) \) we discuss the question of its the convergence in the sequel.

\(^2\text{Roughly speaking, a measure } \mu \text{ is even if } \mu(w, b) = \mu(-w, -b) \text{ for all } (w, b) \in \mathbb{S}^{d-1} \times \mathbb{R} \.)
Proposition 5.3. Let $d, p \in \mathbb{N}$ be odd numbers, if $f \in L^1(\mathbb{R}^d)$ and $\Delta^{(d+p)/2}f \in L^1(\mathbb{R}^d)$, then

$$\mathcal{R}_1(f) = \left\| \gamma_d \mathcal{R} \{ \Delta^{(d+p)/2}f \} \right\|^{1/p}_{M^1} = \left\| \gamma_d \partial_0^{d+p} \mathcal{R} \{ f \} \right\|^{1/p}_{M^1} < \infty.$$  

(5.6)

Proof. Recalling that the Radon transform is a bounded linear operator from $L^1(\mathbb{R}^d)$ to $L^1(\mathbb{S}^{d-1} \times \mathbb{R})$ cf. [5]. Then, the fact that $\Delta^{(d+p)/2}f \in L^1(\mathbb{R}^d)$ ensures that $\mathcal{R} \{ \Delta^{(d+p)/2}f \} \in L^1(\mathbb{S}^{d-1} \times \mathbb{R})$.

Therefore, by Definition 5.6 of $\|f\|_\mathcal{R}$ we have

$$\|f\|_\mathcal{R} = \sup \left\{ \left( -\frac{7d}{p} \langle f, (\Delta)^{(d+p)/2} \mathcal{R}^* \{ \phi \} \rangle \right)^{1/p} : \phi \in \mathcal{S}(\mathbb{S}^{d-1} \times \mathbb{R}), \|\phi\|_\infty \leq 1 \right\}$$

$$= \sup \left\{ \left( -\frac{7d}{p} \langle (\Delta)^{(d+p)/2} f, \phi \rangle \right)^{1/p} : \phi \in C_0, \mathcal{S}(\mathbb{S}^{d-1} \times \mathbb{R}), \|\phi\|_\infty \leq 1 \right\}$$

where we used the fact that the Schwartz class $\mathcal{S}(\mathbb{S}^{d-1} \times \mathbb{R})$ is dense in $C_0, \mathcal{S}(\mathbb{S}^{d-1} \times \mathbb{R})$. If we assume that $\mu \in M(\mathbb{S}^{d-1} \times \mathbb{R})$ given by a density then in view of Lemma 5.2 we have $\|\mu\|_{M(1/\phi)} = 2^d\|\mathcal{R} \{ (\Delta)^{(d+p)/2} f \}\|_{M^1}$. Considering the dual definition of the total variation norm (5.1) it follows that

$$\|f\|_\mathcal{R} = \sup \left\{ \left( \frac{\mu}{\phi} \right)^{1/p} : \phi \in C_0, \mathcal{S}(\mathbb{S}^{d-1} \times \mathbb{R}), \|\phi\|_\infty \leq 1 \right\}$$

$$= \|\mu\|_{M^*(1/\phi)} = \left( \|\gamma_d \mathcal{R} \{ (\Delta)^{(d+p)/2} f \}\|_{M^1} \right)^{1/p}.$$  

In view of (5.10), if $f \in L^1(\mathbb{R}^d)$, then, for $s = d + p$, $\mathcal{R} \{ (\Delta)^{(d+p)/2} f \} = (\partial_t)^{d+p} \mathcal{R} \{ f \}$ which gives $\|f\|_\mathcal{R} = \left( \|2^d (\partial_t)^{d+p} \mathcal{R} \{ f \}\|_{M^1} \right)^{1/p}$. \hfill \Box

The definition of $\mathcal{R}_1(f)$ suffers from functions $f$ that are non-integrable along hyperplanes or non-smooth, therefore we need to extend the equalities in (5.6) to a more general case. Mainly, thanks to a duality argument we define a functional “$\mathcal{R}$-norm” that extends (5.6) to the case where $f$ is possibly non-smooth or not integrable along hyperplanes, which is the case if $f$ is a finite width ReLU neural network, that is $p = 1$.

Definition 5.4. Let $\text{Lip}^\kappa(\mathbb{R}^d)$ denotes the space of $\kappa$-order Lipschitz functions on $\mathbb{R}^d$ for $\kappa \in \mathbb{N}_0$ in the sense that there exists $\alpha \in \mathbb{N}_0^d$ such that $f \in C^\alpha(\mathbb{R}^d)$ and $\partial^\alpha f$ is Lipschitz function where $\kappa$ is the smallest integer satisfies $|\alpha| = \kappa$.

Note that, in view of the previous definition, the ReLU is a 0-order Lipschitz function. For any $f \in \text{Lip}^\kappa(\mathbb{R}^d)$, we let $\|f\|_{\mathcal{L}} := \sup_{|\alpha| = \kappa} \sup_{x \neq y} \frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|}{\|x - y\|}$, be the smallest possible Lipschitz constant for $\partial^\alpha f$ such that $|\alpha| = \kappa$.

Remark 5.5. Infinite width networks considered in our paper are $\kappa$-order Lipschitz functions.

Next we define a functional on the space of all $\kappa$-order Lipschitz continuous functions, where $\kappa < p$. Mainly the aim of the following definition is to re-express the $L^1$-norm in (5.6) as a supremum of the inner product over a space of dual functions $\phi : \mathbb{S}^{d-1} \times \mathbb{R} \to \mathbb{R}$, then restrict $\phi$ to a space where $\Delta^{(d+p)/2} \mathcal{R}^* \{ \phi \}$ is always well-defined. Where the restriction comes from the fact that

$$\|\mathcal{R} \{ \Delta^{(d+p)/2} f \}\|_{M^1} = \sup_{\|\phi\|_\infty \leq 1} \langle \mathcal{R} \{ \Delta^{(d+p)/2} f \}, \phi \rangle = \sup_{\|\phi\|_\infty \leq 1} \langle f, \Delta^{(d+p)/2} \mathcal{R}^* \{ \phi \} \rangle,$$

where we use the fact that $\mathcal{R}^*$ is the adjoint of $\mathcal{R}$ and the Laplacian $\Delta$ is self-adjoint.

Definition 5.6. Let $p \in \mathbb{N}$ be an odd number, $\kappa \in \mathbb{N}_0$ such that $\kappa < p$, then for any $\kappa$-order Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ we define its $\mathcal{R}$-norm [\text{\textbf{1}}] by

$$\|f\|_\mathcal{R} := \sup \left\{ \left( -\frac{7d}{p} \langle f, (\Delta)^{(d+p)/2} \mathcal{R}^* \{ \phi \} \rangle \right)^{1/p} : \phi \in \mathcal{R}(\mathbb{S}^{d-1} \times \mathbb{R}), \phi \text{ even }, \|\phi\|_\infty \leq 1 \right\}.$$  

(5.7)

where $\gamma_d = \frac{1}{\sqrt{(2\pi)^d}}$ and $\langle f, \phi \rangle := \int_{\mathbb{R}^d} f(x)\phi(x)dx$. If $f$ is not a $\kappa$-order Lipschitz function we set $\|f\|_\mathcal{R} = +\infty$.

\footnote{\text{Strictly speaking, the functional $\|\cdot\|_\mathcal{R}$ is not a norm, but it is a semi-norm on the space of functions for which it is finite; more details can be found in the Appendix.}}
In Appendix B.1 we show that the $\mathcal{R}$-norm is well-defined, though not always finite, for any $\kappa$-order Lipschitz functions where $\kappa < p$. The main result, in higher dimension, in this paper is the following theorem, where its proof is in Appendix B.1.

**Theorem 5.7.** Let $d, p \in \mathbb{N}$ be an odd numbers, $\kappa \in \mathbb{N}_0$ such that $\kappa < p$, then $(p!)^{1/p} \mathcal{R}_{\kappa}(f) = \|f\|_{\mathcal{R}}$ for any real valued function $f$ defined on $\mathbb{R}^d$. In particular, $\mathcal{R}_{\kappa}(f)$ is finite if and only if $f$ is $\kappa$-order Lipschitz and $\|f\|_{\mathcal{R}}$ is finite.

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Appendix

A Equivalence of overall control to control on the output layer

For the sake of completeness, we prove that regularizing the $\ell_2^p$ quasinorm of the weights in the output layer and the $\ell_2$ norm of weights of the hidden layer, (i.e., $C(\theta)$) is equivalent to restricting the $\ell_2$ norm of incoming weights for each unit in the hidden layer, and regularizing the $\ell_p^R$ quasinorm of weights in the output layer. The idea behind the proof was first used in [27 Theorem 1] for networks without an unregularized bias. Moreover, [28 Theorem 10] extended the equivalence for convex neural networks. In, [35 Lemma A.1] the authors showed that the equivalence is valid for their setting. A similar argument works in our case with the RePU activation function as [35, Lemma A.1] the authors showed that the equivalence is valid for their setting. A similar argument works in our case with the RePU activation function as [35, Lemma A.1] the authors showed that the equivalence is valid for their setting. A similar argument works in our case with the RePU activation function as [35, Lemma A.1] the authors showed that the equivalence is valid for their setting. A similar argument works in our case with the RePU activation function as [35, Lemma A.1] the authors showed that the equivalence is valid for their setting.

Recall the definition of two layers RePU networks $g_{\theta,p}$ for $\theta \in \Gamma$:

$$g_{\theta,p}(x) = \sum_{i=1}^{k} a_i [w_i x - b_i]_+^p + c$$

**Lemma A.1.** Let $\Gamma = \{ \theta = (k, W, b, a, c) \mid k \in \mathbb{N}, W \in \mathbb{R}^{k \times d}, b \in \mathbb{R}^k, a \in \mathbb{R}^k, c \in \mathbb{R} \}$, then we have

$$\inf_{\theta \in \Gamma} \frac{1}{2} \sum_{i=1}^{k} (a_i)^{\frac{2p}{p}} + \|w_i\|_2^2 = \inf_{\theta \in \Gamma} \|a\|_{\ell_p^R}$$

such that $g_{\theta,p} = f$ such that $g_{\theta,p} = f$, for all $i : \|w_i\|_2 = 1$

**Proof.** Let $\theta \in \Gamma$, we consider a rescale by a factor $r$ (which we precise later) of $\theta$, that is we set $\tilde{\theta} = (k, \tilde{W}, \tilde{b}, \tilde{a}, c)$ such that $\tilde{a}_i = r_i w_i$, $\tilde{a}_i = \frac{a_i}{(r_i)^{\frac{1}{p}}}$, $\tilde{b}_i = r_i b_i$. Now, check that, for all $i$:

$$\tilde{a}_i [\tilde{w}_i x - \tilde{b}_i]_+^p = \frac{a_i}{(r_i)^{\frac{1}{p}}} [r_i (w_i x - b_i)]_+^p = a_i [w_i x - b_i]_+^p .$$

The previous equality clearly shows that $g_{\theta,p} = g_{\tilde{\theta},p}$. Moreover, we have that, from the inequality between arithmetic and geometric means:

$$\frac{1}{2} \sum_{i=1}^{k} (a_i)^{\frac{2p}{p}} + \|w_i\|_2^2 \geq \sum_{i=1}^{k} |a_i|^{\frac{2p}{p}} \cdot \|w_i\|_2 ,$$

obviously a rescaling given by $r_i = \sqrt{|a_i|^{\frac{2p}{p}} / \|w_i\|_2}$ minimizes the left-hand side and achieves equality. Since the right-hand side is invariant to rescaling, we can arbitrarily set $\|w_i\|_2 = 1$ for all $i$, then $\sum_{i=1}^{k} |a_i|^{\frac{2p}{p}} = \|a\|_{\ell_p^R}$.

B Characterization of representational cost

Here we establish the optimization equivalents of the representational costs $\overline{R}(f)$ and $\overline{R}_1(f)$ given in [27] and [52].

As an intermediate step, we first give equivalent expressions for $\overline{R}(f)$ and $\overline{R}_1(f)$ in terms of sequences finite width two layers ReLU networks converging pointwise to $f$.

Next, we state our definition for a discrete measure, our definition is slightly different than the one used in [29]. Indeed we use an additional outer-weight generated by a continuous function on $\mathbb{R}$.

**Definition B.1.** Let $\mathcal{D}_\psi(S^{d-1} \times \mathbb{R})$ denotes the space of all measures given by a finite linear combination of Diracs, i.e., all $\mu \in \mathcal{M}(S^{d-1} \times \mathbb{R})$ of the form $\mu = \sum_{i=1}^{k} a_i \psi(b_i) \delta_{(w_i, b_i)}$ for some $a_i \in \mathbb{R}$, $(w_i, b_i) \in S^{d-1} \times \mathbb{R}$, $\psi \in C(\mathbb{R})$, $i = 1, \ldots, k$, where $\delta_{(w,b)}$ denotes a Dirac delta at location $(w, b) \in S^{d-1} \times \mathbb{R}$. We call any $\mu \in \mathcal{D}_\psi(S^{d-1} \times \mathbb{R})$ a discrete measure.
One important property of discrete measures is the one-to-one correspondence between them and finite width two layers RePU networks. Namely, for any $\theta \in \Theta$ a finite width RePU network $g_{\theta,p}(x) = \sum_{i=1}^{k} a_i w_i x - b_i + c$, setting $\mu = \sum_{i=1}^{k} a_i (1 + |b_i|^{p-1}) \delta_{(w_i,b_i)}$ we have $f = \mu^{p,c} = g_{\theta,p}$ with $c' = g_{\theta,p}(0)$. We denote the correspondence by $\theta \in \Theta \leftrightarrow \mu \in \mathcal{D}_{1+|b|^{p-1}}(S^{d-1} \times \mathbb{R})$. Consequently, in this situation

$$\|\mu\|_{\mathcal{M}^1(1/\psi)}^{1/p} \leq C(\theta) = \sum_{i=1}^{k} |a_i|^{1/p} \leq \|\mu\|_{\mathcal{M}^1(1/\psi)},$$

where $\psi(b) = 1 - |b|^{p-1}$ for any $b \in \mathbb{R}$.

In the following, we recall one of the notions of convergence provided with $\mathcal{M}^1(S^{d-1} \times \mathbb{R})$ and the definition of tight measure.

**Definition B.2** (Narrowly convergence in $\mathcal{M}^1(S^{d-1} \times \mathbb{R})$). A sequence of measures $\{\mu_n\}$, with $\mu_n \in \mathcal{M}^1(S^{d-1} \times \mathbb{R})$ is said to converge narrowly to a measure $\mu \in \mathcal{M}^1(S^{d-1} \times \mathbb{R})$ if $f \varphi d\mu_n \to f \varphi d\mu$ for all $\varphi \in C_b(S^{d-1} \times \mathbb{R})$, where $C_b(S^{d-1} \times \mathbb{R})$ denote the set of all continuous and bounded functions on $S^{d-1} \times \mathbb{R}$.

**Definition B.3.** A sequence of measures $\{\mu_n\}$ is called tight if for all $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset S^{d-1} \times \mathbb{R}$ such that $|\mu_n(K_\varepsilon^c) \leq \varepsilon$ for all $n$ sufficiently large.

Next result can be found in [25, Theorem 6.8] and [3] Theorem 8.6.2.

**Lemma B.4.** Every narrowly convergent sequence is tight. Moreover, any sequence of measures $\{\mu_n\}$ that is tight and uniformly bounded in total variation norm has a narrowly convergent subsequence.

Next statement contributes towards the main result of Appendix B concerning equal expressions for the representational costs $\mathcal{R}(\cdot)$ and $\mathcal{R}_1(\cdot)$, cf. Lemma [B.7]

**Lemma B.5.** Let $p \in \mathbb{N}$, $\psi : \mathbb{R} \to \mathbb{R}$ such that $\psi(b) = 1 - |b|^{p-1}$ and $f : \mathbb{R}^d \to \mathbb{R}$ let $f_0$ denotes the function $f_0(x) = f(x) - f(0)$. For $\mathcal{R}(f)$ as defined in (2.5) and $\mathcal{R}_1(f)$ as defined in (5.1), we have

$$\mathcal{R}(f) = \inf \left\{ \limsup_{n \to \infty} \|\mu_n\|_{\mathcal{M}^1(1/\psi)}^{1/p} : \mu_n \in \mathcal{D}_\psi(S^{d-1} \times \mathbb{R}), \ H^p_{\mu_n} \to f_0 \text{ pointwise}, \ \{\mu_n\} \text{ tight} \right\}. \quad (B.1)$$

and

$$\mathcal{R}_1(f) = \inf \left\{ \limsup_{n \to \infty} \|\mu_n\|_{\mathcal{M}^1(1/\psi)}^{1/p} : \mu_n \in \mathcal{D}_\psi(S^{d-1} \times \mathbb{R}), \ \nu_n = (v_{n,1}, \ldots, v_{n,p}) \text{ such that } v_{n,i} \in \mathbb{R}^d, \ H^p_{\mu_n,\nu_n,0} \to f_0 \text{ pointwise}, \ \{\mu_n\} \text{ tight} \right\}. \quad (B.2)$$

**Proof.** First we prove (B.1) for $\mathcal{R}(f)$. Similar arguments lead to (B.2) for $\mathcal{R}_1(f)$, therefore details are left for the reader. We define $R_c(f)$ such that $\mathcal{R}(f) = \lim_{c \to 0} R_c(f)$, thus

$$R_c(f) := \inf_{\theta \in \Theta} C(\theta) \text{ such that } \|g_{\theta,p} - f\|_{L^\infty} = \varepsilon \text{ and } g_{\theta,p}(0) = f(0). \quad (B.3)$$

Also, let $L(f)$ denotes the right-hand side of (B.1).

Step 1: We start by assuming that $\mathcal{R}(f)$ is finite and we set $\varepsilon_n = 1/n$. Then by the definition of $\mathcal{R}(f)$ in (2.5), for all $n$ there exists $\theta_n \in \Theta$ such that $C(\theta_n) \leq R_{c_n}(f) + \varepsilon_n$, where $\|g_{\theta_n,p} - f\|_{L^\infty} \leq \varepsilon_n$ and $g_{\theta_n,p}(0) = f(0)$. Moreover, in view of the correspondence between the parameters of a network and measures, we have $\theta_n \in \Theta \leftrightarrow \mu_n \in \mathcal{M}^1(S^{d-1} \times \mathbb{R})$ with $g_{\theta_n,p} = \mu_n^{p,c}$ where the outer-bias $c = g_{\theta_n,p}(0) = f(0)$ and $\|\mu_n\|_{\mathcal{M}^1(1/\psi)}^{1/p} \leq C(\theta_n)$.

As a result of the previous correspondence, $H^p_{\mu_n}(x) \to g_{\theta_n,p}(x) - f(0)$ and we have $|H^p_{\mu_n}(x) - f(x)| = |g_{\theta_n,p}(x) - f(x)| \leq \varepsilon_n$ for any $x \in \mathbb{R}^d$. Therefore, $H^p_{\mu_n} \to f_0$ pointwise, and since $\mu_n \in \mathcal{D}_\psi(S^{d-1} \times \mathbb{R})$ we have

$$\limsup_{n \to \infty} \|\mu_n\|_{\mathcal{M}^1(1/\psi)}^{1/p} \leq \limsup_{n \to \infty} (R_{c_n}(f) + \varepsilon_n) = \mathcal{R}(f). \quad (B.4)$$

In view of the previous arguments, we conclude that $L(f) \leq \mathcal{R}(f)$. Last step towards our result, is to show that there exists a sequence of measures $\{\mu_n\}$ which is tight. Therefore, let $Q^m_{\gamma,n,p}(x) = \partial^m \int_{S^{d-1} \times \mathbb{R}} \frac{(w_{\gamma,n,p})^b_\mu}{\psi(b)} d\mu_n(w,b)$ for any $m \in \mathbb{N}_0, p \in \mathbb{N}$ and $\gamma \in \mathbb{N}_0^d$ such that $|\gamma| = m$. It is clear that
$Q_{n,p}^m$ is well defined for all $p$ and $m$, since $\mu_n \in \mathcal{D}_\psi(S^{d-1} \times \mathbb{R})$. In the case where $m = p - 1$ then $Q_{n,p}^{p-1}$ is Lipschitz with $\|Q_{n,p}^{p-1}\|_L \leq p\|\mu_n\|_{\mathcal{M}_1(1/\psi)} \leq B < \infty$, consequently $\{Q_{n,p}^{p-1}\}$ is uniformly Lipschitz. Then by Arzela-Ascoli Theorem, $\{Q_{n,p}^{p-1}\}$ has a subsequence $Q_{n,k}^{p-1}$ that converges uniformly on compact subsets, hence $\{\mu_n\}$ is tight.

Step 2: Let $L(f)$ finite and we fix any $\varepsilon > 0$. Then by definition of $L(f)$ there exists a sequence

$$\mu_n \in \mathcal{D}_\psi(S^{d-1} \times \mathbb{R}) \subset \mathcal{M}_1(S^{d-1} \times \mathbb{R}) \leftrightarrow \theta_n \in \Theta$$

such that $\lim_{n \to \infty} \|\mu_n\|_{\mathcal{M}_1(1/\psi)}$ exists with $\lim_{n \to \infty} \|\mu_n\|_{\mathcal{M}_1(1/\psi)} < L(f) + \varepsilon$, then there exists an $N_1$ such that for all $n \geq N_1$ we have $\|\mu_n\|_{\mathcal{M}_1(1/\psi)} \leq L(f) + \varepsilon$. Moreover, $f_n := \mathbb{H}_{\mu_n,p} = g_{\theta_n,p}$ converges to $f$ pointwise where $c = f(0)$ and satisfies $f_n(0) = f(0)$ for all $n$.

Since $\varepsilon$ is fixed then $\{x \in \mathbb{R}^d : \|x\| \leq 1/\varepsilon\}$ is a compact set. We can choose $\mu_n \in \mathcal{D}_\psi(S^{d-1} \times \mathbb{R})$ such that $f_n$ is a monotonic sequence. Hence Dini’s theorem implies that $f_n$ converges to $f$ uniformly on compact subsets. That is there exists an $N_2$ such that $|f_n(x) - f(x)| \leq \varepsilon$ for all $\|x\| \leq 1/\varepsilon$ and for all $x \geq N_2$, $f_n$ satisfies the constraints in the definition of $R_c(\cdot)$. Therefore, for all $n \geq \max\{N_1, N_2\}$ we have

$$0 \leq R_c(f)^p \leq C(\theta_n)^p \leq \|\mu_n\|_{\mathcal{M}_1(1/\psi)} \leq (L(f) + \varepsilon)^p.$$  \hspace{1cm} (B.5)

Hence $R_c(f) \leq L(f) + \varepsilon$, if $\varepsilon \to 0$, we get $\overline{R}(f) \leq L(f)$. Then, in view of Step 1 and Step 2, $\overline{R}(f)$ is finite if and only if $L(f)$ is finite and $\overline{R}(f) = L(f)$.

Based on [25, Theorem 6.9] and that $(w, b) \to \frac{[\langle w, x \rangle - b]^p - |b|^p}{1 + |b|^p}$ is continuous and bounded, we can show the following lemma, the proof is left for the reader.

**Lemma B.6.** Let $p \in \mathbb{N}$, $f = \mathbb{H}_{\mu,\nu,c}^p$ for any $\mu \in \mathcal{M}_1(S^{d-1} \times \mathbb{R})$, $\nu \in \mathbb{R}^{d \times p}$ i.e., $\nu = (v_1, \ldots, v_p)$ such that $v_i \in \mathbb{R}^d$, $i = 1, \ldots, p$, $c \in \mathbb{R}$ and let $\psi(b) = 1 + |b|^{p-1}$ where $b \in \mathbb{R}$. Then there exists a sequence of discrete measures $\mu_n \in \mathcal{D}_\psi(S^{d-1} \times \mathbb{R})$ with $\|\mu_n\|_{\mathcal{M}_1(1/\psi)} \leq \|\mu\|_{\mathcal{M}_1(1/\psi)}$ such that $f_n = \mathbb{H}_{\mu_n,w,c}^p$ converges to $f$ pointwise.

Mainly the previous Lemma [B.6] states that for function $f$ that can be presented as infinite width neural network, the pointwise limit of certain sequence of finite width neural network $f_n$ where $f_n$ defined through sequence of measures uniformly bounded in total variation norm.

**Lemma B.7.** Let $\psi(b) = 1 + |b|^{p-1}$ where $b \in \mathbb{R}$ and $p \in \mathbb{N}$, then we have the following

$$\overline{R}(f) = \min_{\mu \in \mathcal{M}_1(S^{d-1} \times \mathbb{R}), c \in \mathbb{R}} \|\mu\|_{\mathcal{M}_1(1/\psi)} \text{ such that } f = \mathbb{H}_{\mu,c}^p,$$  \hspace{1cm} (B.6)

and

$$\overline{R}_1(f) = \min_{\mu \in \mathcal{M}_1(S^{d-1} \times \mathbb{R}), c \in \mathbb{R}} \|\mu\|_{\mathcal{M}_1(1/\psi)} \text{ such that } f = \mathbb{H}_{\mu,w,c}^p.$$  \hspace{1cm} (B.7)

**Proof.** We show the first equivalence [B.6] for $\overline{R}(f)$, the second equivalence [B.7] for $\overline{R}_1(f)$ can be proved with similar arguments and therefore it is left to the reader. In similar way to the proof of Lemma B.5 we argue in two steps.

Step 1: Let $L(f)$ be the right-hand side of [B.6] and we assume that $\overline{R}(f)$ is finite. Then, by the equivalence of $\overline{R}(f)$ given in [B.1], it exists a tight sequence of measures $\mu_n \in \mathcal{D}_\psi(S^{d-1} \times \mathbb{R})$, where $\psi(b) = 1 + |b|^{p-1}$ and $b \in \mathbb{R}$, such that $\mu_n$ is uniformly bounded in total variation norm and we have $\mathbb{H}_{\mu_n}^p \to f_0$ pointwise. Since in $\mathcal{M}_1(S^{d-1} \times \mathbb{R})$ weak convergence of measures is equivalent to narrow convergence cf. [25, Theorem 6.7], then by Prohorov’s Theorem, $\{\mu_n\}$ has a subsequence $\{\mu_{n_k}\}$ converging narrowly to a measure $\mu$, cf. [4, Section 8.6 Chapter 8]. Therefore $f_0 = \mathbb{H}_\mu^p$. Since the sequence of measures $\{\mu_{n_k}\}$ converges narrowly, we have

$$\|\mu\|_{\mathcal{M}_1(1/\psi)} \leq \limsup_{k \to \infty} \|\mu_{n_k}\|_{\mathcal{M}_1(1/\psi)} \leq \limsup_{n \to \infty} \|\mu_n\|_{\mathcal{M}_1(1/\psi)}.$$ 

Therefore $L(f) \leq \limsup_{n \to \infty} \|\mu_n\|_{\mathcal{M}_1(1/\psi)}$, which implies that $L(f) \leq \overline{R}(f)$ after taking the infimum over all such sequences $\{\mu_n\}$.
Step 2: In this step we assume that $L(f)$ is finite, and let $\mu \in \mathcal{M}^1(S^{d-1} \times \mathbb{R})$ such that $f_0 = H^p_\mu$. In view of Lemma 5.6 there exists a sequence of measures $\mu_n \in D_\psi(S^{d-1} \times \mathbb{R})$ with $\|\mu_n\|_{M^1(1/\psi)}^{1/p} \leq \|\mu\|_{M^1(1/\psi)}^{1/p}$, such that $H^p_\mu \rightarrow f_0$ pointwise. Therefore,

$$\mathcal{R}(f) \leq \limsup_{n \to \infty} \|\mu_n\|_{M^1(1/\psi)}^{1/p} \leq \|\mu\|_{M^1(1/\psi)}^{1/p}.$$  

Seeing that $\mu$ is arbitrarily chosen in $\mathcal{M}^1(S^{d-1} \times \mathbb{R})$, with $f_0 = H^p_\mu$, then $\mathcal{R}(f) \leq L(f)$.

Step 1 and Step 2 clearly show that $\mathcal{R}(f) = L(f)$, which conclude the claim of our lemma.

Next result shows that the optimization problem describing $\mathcal{R}_1(f)$ in (B.7) reduces to (B.8). That is, if $f$ can be represented as an infinite width network, then $\mathcal{R}_1(f)$ is equal to the minimal total variation norm of all even measures defining $f$.

**Lemma B.8.** Let $p \in \mathbb{N}$ be odd and $\psi(b) = 1 + |b|^{p-1}$ for any $b \in \mathbb{R}$, then

$$\mathcal{R}_1(f) = \min \|\mu^+\|_{M^1(1/\psi)}^{1/p} \text{ such that } f = H^p_{\mu^+,v,c},$$  

(B.8)

where the minimum is taken over all even measure $\mu^+ \in \mathcal{M}_e^1(S^{d-1} \times \mathbb{R})$, $v \in \mathbb{R}^{d \times p}$ and $c \in \mathbb{R}$.

The proof of Lemma B.8 can be conclude in similar way as in [29 Lemma 7]. For the seek of completeness we give the proof.

**Proof.** Let $f = H^p_{\mu^+,v,c}$ for some $\mu \in \mathcal{M}^1(S^{d-1} \times \mathbb{R})$, $v \in \mathbb{R}^{d \times p}, c \in \mathbb{R}$. If $\mu$ has even and odd decomposition $\mu = \mu^+ + \mu^-$ then $f = H^p_{\mu^+,0,0} + H^p_{\mu^-,v,c} = H^p_{\mu^+,v',c}$ for some $v' \in \mathbb{R}^d$. Also, by Lemma 5.1 we have $\|\mu^+\|_{M^1} \leq \|\mu^+ + \mu^-\|_{M^1} = \|\mu\|_{M^1}$ for any $\mu^-$ odd.

**B.1 Proof of Theorem 5.7.**

Let $\mathcal{E}(S^{d-1} \times \mathbb{R})$ denote the space of even Schwartz functions on $S^{d-1} \times \mathbb{R}$, i.e., $\phi \in \mathcal{E}(S^{d-1} \times \mathbb{R})$ if $\phi \in \mathcal{E}(S^{d-1} \times \mathbb{R})$ with $\phi(w,b) = \phi(-w,-b)$ for all $(w,b) \in S^{d-1} \times \mathbb{R}$. In the sequel we need the following property of the Radon transform and its dual when acting on Schwartz function, which can be founded in e.g., [38].

**Lemma B.9.** [38 Theorem 7.7] Let $\phi \in \mathcal{E}(S^{d-1} \times \mathbb{R})$ and define $\varphi = \gamma_d(-\Delta)^{(d-1)/2}R^*\phi$. Then $\varphi \in C^\infty(\mathbb{R}^d)$ with $\varphi(x) = O(\|x\|^{-d})$ and $\Delta \varphi(x) = O(\|x\|^{-d-2})$ as $\|x\| \to \infty$. Moreover, $R^* \varphi = \phi$.

Using the above result we show the $R^*$-norm given in Definition 5.4 is well-defined:

**Proposition B.10.** Let $p \in \mathbb{N}$ be an odd number, $\kappa \in \mathbb{N}_0$ such that $\kappa < p$, for $f \in Lip^\kappa(\mathbb{R}^d)$, if the map $L_f(\phi) := -\frac{\partial}{\partial \mu}(f, -\Delta)^{(d+p)/2}R^* \phi$ is finite for all $\phi \in \mathcal{E}(S^{d-1} \times \mathbb{R})$, then $\|f\|_{R^*} = \sup(L_f(\phi))^{1/p}$ is a well-defined functional taking values in $[0, +\infty]$, where the supremum is taken over all $\phi \in \mathcal{E}(S^{d-1} \times \mathbb{R})$ such that $\|\phi\|_{\infty} \leq 1$.

**Proof.** Using the definition of $\kappa$-order Lipschitz function $f$, then there exists a multi-index $\alpha$ such that $|D^\alpha f(x)| = O(\|x\|^{|\alpha|})$ where $|\alpha| = \kappa$. Therefore, we have $|f(x)| = O(\|x\|^{|\alpha|})$, moreover for any $\phi \in \mathcal{E}(S^{d-1} \times \mathbb{R})$, by Lemma B.9 we get $|(-\Delta)^{(d+p)/2}R^* \phi| = O(\|x\|^{-d})$. We conclude that $|f(x)(-\Delta)^{(d+p)/2}R^* \phi(x)| = O(\|x\|^{-d-1})$ is integrable, then $\langle f, (-\Delta)^{(d+p)/2}R^* \phi \rangle$ is finite. If $\langle f, (-\Delta)^{(d+p)/2}R^* \phi \rangle \neq 0$, we can choose the sign of $\phi$ so that $-\frac{\partial}{\partial \mu}(f, -\Delta)^{(d+p)/2}R^* \phi$ is positive, which implies that $\|f\|_{R^*} \geq 0$.

The following lemma analyses the case where using an even measure belongs to $\mathcal{M}_e(S^{d-1} \times \mathbb{R})$ instead of a measure with density in the Schwartz class can show that equality (5.3) holds true also in the distributional case.

**Lemma B.11.** Let $p \in \mathbb{N}$ be an odd number, $f = H^p_{\mu^+,v,c}$ for any $\mu \in \mathcal{M}_e(S^{d-1} \times \mathbb{R})$, $v \in \mathbb{R}^{d \times p}, c \in \mathbb{R}$. Then we have $\langle f, \Delta^{d-1} \varphi \rangle = \langle \mu, p!R^*(\varphi) \rangle$ for all $\varphi \in C^\infty(\mathbb{R}^d)$ such that $\varphi(x) = O(\|x\|^{-d})$ and $\Delta^{d-1} \varphi(x) = O(\|x\|^{-d-1})$ as $\|x\| \to \infty$ and $\psi(b) = 1 + |b|^{p-1}$ where $b \in \mathbb{R}$.
Proof. We start by setting, for odd $p \in \mathbb{N}$, $f = H_{\mu,v,c}$ with $\mu \in \mathcal{M}_c(S^{d-1} \times \mathbb{R})$, $v \in \mathbb{R}^{d \times p}$, $c \in \mathbb{R}$. The number $\frac{p+1}{2}$ is even, and since polynomials up to order $p$ vanish under the application of $\frac{p+1}{2}$ Laplacian, then $(f, \Delta^{\frac{p+1}{2}} \varphi) = (H_{\mu}^p, \Delta^{\frac{p+1}{2}} \varphi)$, for all $\varphi \in C(\mathbb{R}^d)$ with $\varphi(x) = O(||x||^{-d})$ and $\Delta^{\frac{p+1}{2}} \varphi(x) = O(||x||^{-d-p-1})$ when $||x|| \to \infty$. Hence the problem is reduced to the case where $f = H_{\mu}^p$ and

$$
\int_{\mathbb{R}^d} f(x) \Delta^{\frac{p+1}{2}} \varphi(x) \, dx = \int_{\mathbb{R}^d} \int_{S^{d-1} \times \mathbb{R}} 1 b p \varphi(x) \, dx.
$$

Last equality holds thanks to the fact that for any $t \in \mathbb{R}$ and any odd $p \ l_p^p = \frac{1}{2} (t^p + t^p)$ and that $H_{\mu}^p = H_{\mu, +}^p + H_{\mu, -}^p$, where $H_{\mu, -}^p$ is a polynomial in $x$ of order $p$ that can be neglected in this situation since it vanishes under the application of $\frac{p+1}{2}$ Laplacian. Moreover, we have

$$
\int_{\mathbb{R}^d} f(x) \Delta^{\frac{p+1}{2}} \varphi(x) \, dx = \int_{S^{d-1} \times \mathbb{R}} \left( \int_{\mathbb{R}^d} \frac{1}{2} \frac{|w x - b|^p - |b|^p}{1 + |b|^p} d\mu(w, b) \Delta^{\frac{p+1}{2}} \varphi(x) \, dx \right) \, d\mu(w, b)
$$

where we applied Fubini’s theorem to exchange the order of integration, which is justified by

$$
\frac{1}{2} \int_{S^{d-1} \times \mathbb{R}} \frac{|w x - b|^p - |b|^p}{1 + |b|^p} d\mu(w, b) \leq ||\mu||_{M^1} \sum_{k=1}^{p} \binom{p}{k} ||x||^k
$$

and by the fact that $\Delta^{\frac{p+1}{2}} \varphi(x) = O(||x||^{-d-p-1})$, as it is mentioned in the statement. Then, we get

$$
\frac{1}{2} \int_{S^{d-1} \times \mathbb{R}} \frac{|w x - b|^p - |b|^p}{1 + |b|^p} d\mu(w, b) |\Delta^{\frac{p+1}{2}} \varphi(x)| = O(||x||^{-d-1}),
$$

which leads to

$$
\int_{\mathbb{R}^d} \frac{1}{2} \int_{S^{d-1} \times \mathbb{R}} \frac{|w x - b|^p - |b|^p}{1 + |b|^p} d\mu(w, b) |\Delta^{\frac{p+1}{2}} \varphi(x)| \, dx < \infty.
$$

Let $r_{w,b}(x) := \frac{|w x - b|^p - |b|^p}{1 + |b|^p}$ for any $(w, b) \in S^{d-1} \times \mathbb{R}$ and $x \in \mathbb{R}^d$. In the distributions sense we can show that $\Delta^{\frac{p+1}{2}} r_{w,b}(x) = p! \frac{\delta(w x - b)}{\psi(b)} = p! \frac{\delta(w x - b)}{\psi(b)}$. Namely, for any test function $\varphi \in \mathcal{S}(\mathbb{R}^d)$ we have the following identity

$$
\int_{\mathbb{R}^d} r_{w,b}(x) \Delta^{\frac{p+1}{2}} \varphi(x) \, dx = \frac{p!}{\psi(b)} \int_{w x = b} \varphi(x) \, ds(x) = \frac{p!}{\psi(b)} \mathcal{R} \{ \varphi \}(w, b).
$$

The Radon transform $\mathcal{R} \{ \varphi \}$ is well-defined for smooth functions that decay as $O(||x||^{-d})$. Hence, by continuity $\Delta^{\frac{p+1}{2}} r_{w,b}(x)$ extends uniquely to a distribution acting on $C^\infty$ functions decay like $O(||x||^{-d})$. In view of the previous calculus, we get

$$
\int_{\mathbb{R}^d} f(x) \Delta^{\frac{p+1}{2}} \varphi(x) \, dx = \int_{S^{d-1} \times \mathbb{R}} \left( \int_{\mathbb{R}^d} r_{w,b}(x) \Delta^{\frac{p+1}{2}} \varphi(x) \, dx \right) \, d\mu(w, b)
$$

which shows the claimed statement in the lemma.

□

The following lemma shows $\|f\|_{\mathcal{M}}$ is finite if and only if $f$ is an infinite width net, in which case $\|f\|_{\mathcal{M}}$ is given by the total variation norm of the unique even measure defining $f$.

**Lemma B.12.** Let $p \in \mathbb{N}$ be an odd number, $\kappa \in \mathbb{N}_0$ such that $\kappa < p$, for $f \in Lip^p(\mathbb{R}^d)$. Then $\|f\|_{\mathcal{M}}$ is finite if and only if there exists a unique even measure $\mu \in \mathcal{M}_c(S^{d-1} \times \mathbb{R})$, $v \in \mathbb{R}^{d \times p}$ and $c \in \mathbb{R}$ with $f = H_{\mu,v,c}$, such that $\|f\|_{\mathcal{M}} = \|\mu\|_{\mathcal{M}^+(\mathbb{R}^d)}^{p/2} \psi^p/\psi$.

**Proof.** We start by the direct sense, that is, we assume that $\|f\|_{\mathcal{M}}$ is finite. In view of Definition 5.6 there exists $\kappa < p$ such that the function $f$ is a $\kappa$-order Lipschitz function. Moreover the linear functional

$$
L_f(\phi) = \frac{(-1)^{(p+1)/2} \gamma_d(f, (-\Delta)^{(d+p)/2} \mathcal{R} \{ \phi \})}{\phi \in \mathcal{S}(\mathbb{R}^d)}
$$

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is continuous on $\mathscr{S}(\mathbb{S}^{d-1} \times \mathbb{R})$ with norm $\|f\|_{p,\mathscr{S}}$. Using an extension argument, and the fact that $\mathscr{S}(\mathbb{S}^{d-1} \times \mathbb{R})$ is a dense subspace of $C_{0,e}(\mathbb{S}^{d-1} \times \mathbb{R})$, there exists a unique extension $\tilde{L}_f$ to all of $C_{0,e}(\mathbb{S}^{d-1} \times \mathbb{R})$ under the same norm. Riesz representation theorem declares that $\tilde{L}_f$ can be seen as integration against a measure, that is, there is a unique even measure $\nu \in \mathcal{M}_e(\mathbb{S}^{d-1} \times \mathbb{R})$ such that the extension $\tilde{L}_f(\phi) = \int \phi \, d\nu$ for all $\phi \in C_{0,e}(\mathbb{S}^{d-1} \times \mathbb{R})$ and $\|f\|_{p,\mathscr{S}} = \|\nu\|_{p,\mathcal{M}_e}^{1/p}$.

Without loss of generality, we can write $\Phi_x = p! \int \phi_x \frac{d\mu}{\psi}$ for all $\Phi \in C_{0,e}(\mathbb{S}^{d-1} \times \mathbb{R})$, $\psi(b) = 1 + |b|^{p-1}$, and $\Phi_x \in \mathcal{L}(\mathscr{S}(\mathbb{R}^d))$ under the same norm. Riesz representation theorem declares that $\tilde{L}_f(\phi) = \int \phi \, d\nu$ for all $\phi \in C_{0,e}(\mathbb{S}^{d-1} \times \mathbb{R})$ and $\|f\|_{p,\mathscr{S}} = \|\nu\|_{p,\mathcal{M}_e}^{1/p}$.

In this step it remains to show that $f$ can be expressed as $\mathcal{H}_{\mu,\nu,c}$, where $\mu \in \mathcal{M}_e(\mathbb{S}^{d-1} \times \mathbb{R})$, $\nu \in \mathbb{R}^{d \times p}$ and $c \in \mathbb{R}$. We argue in the sense of tempered distribution. Namely, we show that $\langle f, \phi \rangle = \langle \mathcal{H}_{\mu,\nu,c}, \phi \rangle$ for any $\phi \in \mathcal{S}(\mathbb{R}^d)$. In view of the discussion in the proof of Lemma [B.11] for any test function $\phi \in \mathcal{S}(\mathbb{R}^d)$, it holds that $\langle \Delta^{(p+1)/2} \mathcal{H}_\mu, \phi \rangle = p! \langle \mu, \frac{\mathcal{D}_x}{\psi} \rangle$. Since for any $\phi \in \mathcal{S}(\mathbb{S}^{d-1} \times \mathbb{R})$ the Radon transform $\mathcal{R}_\phi$ belongs to $\mathcal{S}(\mathbb{S}^{d-1} \times \mathbb{R})$ cf. [13] Theorem 2.4. Hence $\tilde{L}_f(\mathcal{R}_\phi)$ can be written as $p! \langle \mu, \frac{\mathcal{D}_x}{\psi} \rangle$ which affirms that $\langle \Delta^{(p+1)/2} \mathcal{H}_\mu, \phi \rangle = L_f(\mathcal{R}_\phi)$. Seeing that $L_f(\phi)$ can be written as

\[-(1)^{(p+1)/2} \gamma_d(f, -\Delta)^{(d+p)/2} \mathcal{R}^* \{ \mathcal{R} \{ \phi \} \} \]

which equals to

\[-\gamma_d(f, \Delta^{(p+1)/2} - (d+1)/2 \mathcal{R}^* \{ \mathcal{R} \{ \phi \} \}. \]

Moreover, using the inversion formula for Radon transform, i.e., $-\gamma_d(-\Delta)^{(d-1)/2} \mathcal{R}^* \{ \mathcal{R} \{ \phi \} \} = \varphi$ for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$, it follows that $L_f(\phi) = \langle f, \Delta^{(p+1)/2} \varphi \rangle = \langle \Delta^{(p+1)/2} f, \phi \rangle$. Hence $\langle \Delta^{(p+1)/2} \mathcal{H}_\mu, \phi \rangle = \langle \Delta^{(p+1)/2} f, \phi \rangle$ for any $\phi \in \mathcal{S}(\mathbb{R}^d)$.

The previous equality yields that $\Delta^{(p+1)/2} f = \Delta^{(p+1)/2} \mathcal{H}_\mu$ holds true in the sense of tempered distributions. Therefore $\Delta^{(p+1)/2} f = 0$, hence $f = \mathcal{H}_\mu$ is a polynomial satisfies the following $\Delta^{(p+1)/2} \mathcal{R}(x) = 0$ for any $x \in \mathbb{R}^d$. Observing that $f$ and $\mathcal{H}_\mu$ are $\kappa$-order Lipschitz, means that $\mathcal{R}$ is a polynomial in $x$ at most of order $p$. Therefore, there exists $v = (v_1, \ldots, v_p) \in \mathbb{R}^{d \times p}$ and $c \in \mathbb{R}$ such that $\mathcal{R}(x) = \sum_{k=1}^p c_k x^k + c$. Consequently, the previous argument concludes that $f = \mathcal{H}_\mu$.

We now prove the inverse implication: we assume that for some $\mu \in \mathcal{M}_e(\mathbb{S}^{d-1} \times \mathbb{R})$, $v \in \mathbb{R}^{d \times p}$, $c \in \mathbb{R}$, $f$ can be expressed as $\mathcal{H}_{\mu,\nu,c}$. Given an even test function $\phi \in \mathcal{S}(\mathbb{S}^{d-1} \times \mathbb{R})$ and let

\[ \varphi = (-1)^{(p+1)/2} \gamma_d(-\Delta)^{(d-1)/2} \mathcal{R}^* \{ \phi \}. \]

Then the function $\varphi$ satisfies the claims in Lemma [B.9] hence $\varphi \in C^\infty(\mathbb{R}^d)$ with $\varphi(x) = O(|x|^{-d})$, $\Delta \varphi(x) = O(|x|^{-d-2})$ as $||x|| \to \infty$ and $\phi = \mathcal{R}_\varphi$. Moreover, by Lemma [B.11] the linear functional $L_f(\phi)$ can be characterized as follows

\[ L_f(\phi) = \langle f, \Delta^{(p+1)/2} \varphi \rangle = \langle \mu, \mathcal{R}_\varphi \rangle = \langle \mu, \mathcal{R}_\varphi \rangle. \]

Using Lemma [B.11] the characterization of the linear functional $L_f(\phi)$ given in (B.12) and the fact that $\mathcal{S}(\mathbb{R}^d)$ is a dense subspace of $C_0(\mathbb{R}^d)$ we get

\[ \|f\|_{p,\mathscr{S}} = \sup \left\{ \left( \|\mu\|_{p,\mathcal{M}_e}^{1/p} \right)^{1/p} : \phi \in \mathcal{S}(\mathbb{S}^{d-1} \times \mathbb{R}), \|\phi\|_{\infty} \leq 1 \right\} \]

\[ = \sup \left\{ \left( \|\mu\|_{p,\mathcal{M}_e}^{1/p} \right)^{1/p} : \phi \in C_{0,e}(\mathbb{S}^{d-1} \times \mathbb{R}), \|\phi\|_{\infty} \leq 1 \right\} \]

then by the dual characterization of the total variation norm we conclude that $\|f\|_{p,\mathscr{S}} = \|\mu\|_{p,\mathcal{M}_e}^{1/p}$.

Uniqueness can be achieved by assuming the existence of $\mu, \nu \in \mathcal{M}_e(\mathbb{S}^{d-1} \times \mathbb{R})$, $v, v' \in \mathbb{R}^{d \times p}$, $c, c' \in \mathbb{R}$. This holds true since any $\phi \in C_{0,e}(\mathbb{S}^{d-1} \times \mathbb{R})$ can be written as $\frac{1}{\psi} \Phi(w, b)$ where $\Phi(w, b) = \frac{1}{\psi} \phi(w, b) \in C_{0,e}(\mathbb{S}^{d-1} \times \mathbb{R})$. 

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4This holds true since any $\phi \in C_{0,e}(\mathbb{S}^{d-1} \times \mathbb{R})$ can be written as $\frac{1}{\psi} \Phi(w, b)$ where $\Phi(w, b) = \frac{1}{\psi} \phi(w, b) \in C_{0,e}(\mathbb{S}^{d-1} \times \mathbb{R})$. 

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such that $\mathcal{H}_{\mu,v,c}^p = \mathcal{H}_{\mu,v',c'}^p$. Since the difference $\mathcal{H}_{\mu,v,c}^p - \mathcal{H}_{\nu,v',c'}^p = \mathcal{H}_{\mu-v,v'-c',c'-c}^p$ hence by the previous arguments

$$\|H_{\mu-v,v'-c',c'-c}^p\| = \|\mu - \nu\|_{\mathcal{M}^1(p/\psi)}^{1/p} = 0.$$ 

Then $\mu = \nu$ which implies that $v' = v$ and $c = c'$ and so the uniqueness. 

Lemma 5.1 is an immediate consequence of the characterization in Lemma B.8 and the uniqueness result given in Lemma B.12. In the case where $p = 1$ a similar result was proved by Ongie et al. cf. [29] for the sake of completeness we give the proof.

**Proof of Lemma 5.1.** We treat the case where $R_1(f)$ is finite. By Lemma B.8, we can characterize the $\mathcal{R}$-norm for a given function $f$ expressed by a neural network $H_{\mu,v,c}^p$ where $\mu \in \mathcal{M}_1^e(S^{d-1} \times \mathbb{R})$ is an even measure, $v \in \mathbb{R}^{d \times p}$, $c \in \mathbb{R}$, as follows: $R_1(f)$ is the minimum of $\|\mu^+\|_{\mathcal{M}^1(1/\psi)}^{1/p}$ over all even measures $\mu^+ \in \mathcal{M}_1^e(S^{d-1} \times \mathbb{R})$, $v' \in \mathbb{R}^{d \times p}$ and $c' \in \mathbb{R}$ such that $f = H_{\mu^+,v',c'}^p$. The uniqueness can be justified in a similar way to the proof of Lemma B.12, hence there is a unique even measure $\mu^+ \in \mathcal{M}_1^e(S^{d-1} \times \mathbb{R})$, $v \in \mathbb{R}^{d \times p}$, and $c \in \mathbb{R}$ such that $f = H_{\mu^+,v,c}^p$. To sum up, $R_1(f) = \|\mu^+\|_{\mathcal{M}^1(1/\psi)}^{1/p}$ which conclude the lemma.

**Proof of Theorem 5.7.** Assume that $R_1(f)$ is finite. Thanks to Lemma 5.1 there exists a unique even measure $\mu \in \mathcal{M}_c(S^{d-1} \times \mathbb{R})$ such that $R_1(f) = \|\mu\|_{\mathcal{M}^1(1/\psi)}^{1/p}$ and $f = H_{\mu,v,c}^p$ for some $v \in \mathbb{R}^{d \times p}$, $c \in \mathbb{R}$. In view of Lemma B.12 we have $\|f\|_{\mathcal{R}} = \|\mu\|_{\mathcal{M}^1(1/\psi)}^{1/p}$. Consequently, $(p/\psi)^{1/p} R_1(f) = \|f\|_{\mathcal{R}}$. In reverse, let $\|f\|_{\mathcal{R}}$ be finite, then by Lemma B.12 $f = H_{\mu,v,c}^p$ for a unique even measure $\mu \in \mathcal{M}_c(S^{d-1} \times \mathbb{R})$, $v \in \mathbb{R}^{d \times p}$ and $c \in \mathbb{R}$, with $\|f\|_{\mathcal{R}} = \|\mu\|_{\mathcal{M}^1(1/\psi)}^{1/p}$. Moreover using Lemma 5.1 $(p/\psi)^{1/p} R_1(f) = \|\mu\|_{\mathcal{M}^1(1/\psi)}^{1/p} = \|f\|_{\mathcal{R}}$. 

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