Bi-Hamiltonian structure of the general heavenly equation

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Abstract. We discover two additional Lax pairs and three nonlocal recursion operators for symmetries of the general heavenly equation introduced by Doubrov and Ferapontov. Converting the equation to a two-component form, we obtain Lagrangian and Hamiltonian structures of the two-component general heavenly system. We discover that in the two-component form we have only a single nonlocal recursion operator. Composing the recursion operator with the first Hamiltonian operator we obtain second Hamiltonian operator. Thus, the general heavenly equation in the two-component form is a bi-Hamiltonian system completely integrable in the sense of Magri.

1. Introduction

In paper [1], Doubrov and Ferapontov introduced the general heavenly equation (GHE)

\[ \alpha u_{12}u_{34} + \beta u_{13}u_{24} + \gamma u_{14}u_{23} = 0, \quad \alpha + \beta + \gamma = 0 \]  

where \( \alpha, \beta, \gamma \) are arbitrary constants satisfying one linear relation given above. Here \( u = u(z_1, z_2, z_3, z_4) \) is a holomorphic function of four complex variables. They also obtained the Lax pair for equation (1.1)

\[ X_1 = u_{34}\partial_1 - u_{13}\partial_4 + \gamma\lambda(u_{34}\partial_1 - u_{14}\partial_3), \]
\[ X_2 = u_{23}\partial_4 - u_{34}\partial_2 + \beta\lambda(u_{34}\partial_2 - u_{24}\partial_3), \]

where \( \partial_i \) means \( \partial/\partial z_i \), while \( u_{ij} = \partial^2 u/\partial z_i \partial z_j \).

There are only few examples of multi-dimensional integrable systems. The so-called heavenly equations make up an important class of such integrable systems since they are obtained by a reduction of the Einstein equations with Euclidean (and neutral) signature for (anti-)self-dual (ASD) gravity which includes the theory of gravitational instantons. All of these equations are of Monge-Ampère type, so that the only nonlinear terms are Hessian \( 2 \times 2 \) determinants. General heavenly equation is an important example of such equations. An explicit description of ASD Ricci-flat vacuum metric governed by GHE, null tetrad and basis 1-forms for this metric were obtained in our paper [2]. Recently, Bogdanov [3] showed important relations between GHE and heavenly equations of Plebański and developed \( \partial \)-dressing scheme for GHE in the
context of the inverse scattering method. This stresses the importance of a further study of the Doubrov-Ferapontov’s general heavenly equation.

In this paper, we obtain recursion operators for symmetries of GHE (1.1) and discover its bi-Hamiltonian structure in a two-component form, where the single second-order PDE (1.1) is presented as an evolutionary system of two first-order PDEs with two unknowns. Therefore, by the theorem of Magri [4] this general heavenly (GH) system is completely integrable.

While completing this paper, we became aware of the preprint [5] by A. Sergyeyev where he discovers a recursion operator for the one-component version of GHE, which coincides with the first one of our recursion operators, as an example of application of his general method for constructing recursion operators for dispersionless integrable systems.

If \( \varphi \) is a symmetry characteristic for (1.1), it satisfies the symmetry condition

\[
\dot{\varphi} = \alpha (u_{34}\varphi_{12} + u_{12}\varphi_{34}) + \beta (u_{24}\varphi_{13} + u_{13}\varphi_{24}) + \gamma (u_{23}\varphi_{14} + u_{14}\varphi_{23}) = 0. \tag{1.3}
\]

A recursion operator maps any symmetry again into a symmetry and, as a consequence, it commutes with the operator \( \dot{A} \) on solutions. For all other heavenly equations in the classification [1] of Doubrov and Ferapontov the symmetry condition has a two-dimensional divergence form which allows us to introduce partner symmetries [6–10], a powerful tool for finding recursion operators and noninvariant solutions [11, 12] which are necessary for the construction of the famous gravitational instanton \( K3 \). It is easy to check that the symmetry condition (1.3) for the general heavenly equation cannot be presented in a two-dimensional divergence form but it can be presented in a three-dimensional divergence form.

Therefore, the method of partner symmetries does not work any more for GHE, so that we have to use here a different approach in order to find a recursion operator which could be regarded as a generalization of the method of partner symmetries. This approach is based on splitting each of the two Lax operators with respect to the spectral parameter \( \lambda \) in two operators and multiplying the first operator by the inverse of the second operator in each pair, obtained by the splitting in \( \lambda \). This method was presented in [5] using somewhat more geometric language. The idea of obtaining recursion operators from Lax pairs was used in 2003 in our paper on partner symmetries of the complex Monge-Ampère equation [7].

For a single-component equation (1.1), we discover three Lax pairs, related by discrete symmetries of both GHE and its symmetry condition, and three recursion operators corresponding to them. However, a two-component form of the equation (1.1) is not invariant under these permutations of indices, so two other recursion operators are related to two other 2-component systems. We do not consider them here because they are obtained from our two-component system just by the permutations of indices.

Another important property of heavenly equations is that they can be presented in a two-component evolutionary form as bi-Hamiltonian systems [13–16]. We show here that GHE also possesses this property. In a two-component form we construct a Lagrangian for this system and discover its symplectic and Hamiltonian structure. Composing the recursion operator in a \( 2 \times 2 \) matrix form with the Hamiltonian operator \( J_0 \) we generate the second Hamiltonian operator \( J_1 = RJ_0 \). We have checked directly that the Jacobi identities for the linear combination \( aJ_0 + bJ_1 \) with arbitrary constant coefficients \( a \) and \( b \) (obviously skew-symmetric) are satisfied which proves the Hamiltonian property of \( J_1 \) and the compatibility of the two Hamiltonian operators \( J_0 \) and \( J_1 \). The proof of the Jacobi identities will be presented in a future publication. We find the corresponding Hamiltonian density \( H_0 \) such that the original general heavenly flow is generated by the action of \( J_1 \) on variational derivatives of the Hamiltonian functional \( H_0 \), so that the GH system turns out to be a bi-Hamiltonian system.

In section 2, we obtain two more Lax pairs for GHE in addition to the Doubrov-Ferapontov Lax pair. We show how to use these three Lax pairs for constructing three nonlocal recursion operators for GHE.
In section 3, we present the general heavenly equation in a two-component evolutionary form and obtain a Lagrangian for this GH system.

In section 4, we discover symplectic and Hamiltonian structure of the GH system.

In section 5, we derive a nonlocal recursion operator for the two component GH system.

In section 6, acting by the recursion operator on the first Hamiltonian operator \( J_0 \) we obtain second nonlocal Hamiltonian operator \( J_1 \). We find the corresponding Hamiltonian density which generates the GH system with the aid of the second Hamiltonian operator. Thus, we obtain a bi-Hamiltonian representation for the general heavenly equation in a two-component form.

2. Recursion operators

We introduce the following first-order differential operators from which the Lax operators (1.2) are constructed

\[
L_{14(3)} = u_{34}D_1 - u_{13}D_4, \quad L_{13(4)} = u_{34}D_1 - u_{14}D_3, \\
L_{24(3)} = u_{34}D_2 - u_{23}D_4, \quad L_{23(4)} = u_{34}D_2 - u_{24}D_3
\]

(2.1)

where \( D_i \) denotes total derivative with respect to \( z_i \). The general definition is \( L_{ij(k)} = u_{jk}D_i - u_{ik}D_j \), so that \( L_{ji(k)} = -L_{ij(k)} \). Lax operators take the form

\[
X_1 = L_{14(3)} + \lambda \gamma L_{13(4)}, \quad X_2 = -L_{24(3)} + \lambda \beta L_{23(4)}.
\]

(2.2)

The symmetry condition (1.3), where we use \( \alpha = - (\beta + \gamma) \), becomes

\[
\hat{A} \phi = \{ \beta (L_{14(2)}D_3 - L_{14(3)}D_2) + \gamma (L_{24(1)}D_3 - L_{24(3)}D_1) \} \phi = 0 \\
\equiv \{ \beta (D_3L_{14(2)} - D_2L_{14(3)}) + \gamma (D_3L_{24(1)} - D_1L_{24(3)}) \} \phi = 0
\]

(2.3)

where \( L_{14(2)} = u_{24}D_1 - u_{12}D_4, \quad L_{24(1)} = u_{14}D_2 - u_{12}D_4 \).

(2.4)

To arrive at two different Lax pairs, we apply two permutations of indices which leave invariant both the equation (1.1) and its symmetry condition (1.3) but which do change the Lax pair and recursion operators. The permutation \( 1 \leftrightarrow 3, \ 2 \leftrightarrow 4 \) yields the second Lax pair

\[
X_1^{(2)} = L_{23(1)} + \lambda \gamma L_{13(2)} , \quad X_2^{(2)} = L_{24(1)} - \lambda \beta L_{14(2)}
\]

where \( L_{13(2)} = u_{23}D_1 - u_{12}D_3, \quad L_{23(1)} = u_{13}D_2 - u_{12}D_3 \).

(2.5)

and operators \( L_{14(2)}, \ L_{24(1)} \) are defined in (2.4). In (2.5) the overall minus in \( X_1^{(2)} \) is skipped.

The permutation \( 1 \leftrightarrow 4, \ 2 \leftrightarrow 3 \) yields the third Lax pair

\[
X_1^{(3)} = L_{41(2)} + \lambda \gamma L_{42(1)} , \quad X_2^{(3)} = -L_{31(2)} + \lambda \beta L_{32(1)}
\]

\[
L_{41(2)} = u_{12}D_4 - u_{24}D_1, \quad L_{42(1)} = u_{12}D_4 - u_{14}D_2, \\
L_{31(2)} = u_{12}D_3 - u_{23}D_1, \quad L_{32(1)} = u_{12}D_3 - u_{13}D_2.
\]

(2.7)

We note that the symmetry condition (2.3) has a three-dimensional divergence form and therefore our definition of partner symmetries, which requires a two-dimensional divergence form of the symmetry condition [6], does not work here, so that we need here a different approach to obtain a recursion operator which is applied to the proof of the following theorem. It is based on our observation that recursion operators for partner symmetries of the heavenly equations, which we studied before, are composed from the operators obtained by splitting the Lax operators with respect to spectral parameter \( \lambda \) and multiplying the first operator by the inverse of the second operator in each pair, that was obtained by splitting in \( \lambda \) [7–9, 14, 15].
Theorem 2.1 The general heavenly equation admits the following three different Lax pairs (2.9a), (2.10a), (2.11a) and three respective recursion operators defined by the relations (2.9b), (2.10b), (2.11b)

\[
\begin{align*}
X_1^{(1)} &= L_{14(3)} + \lambda \gamma L_{13(4)}, & X_2^{(1)} &= -L_{24(3)} + \lambda \beta L_{23(4)} \\
L_{14(3)} \varphi &= \gamma L_{13(4)} \psi, & L_{24(3)} \varphi &= -\beta L_{23(4)} \psi. \\
X_1^{(2)} &= L_{23(1)} + \lambda \gamma L_{14(2)}, & X_2^{(2)} &= L_{24(1)} - \lambda \beta L_{14(2)} \\
L_{23(1)} \varphi &= \gamma L_{14(2)} \psi, & L_{24(1)} \varphi &= -\beta L_{14(2)} \psi. \\
X_1^{(3)} &= L_{41(2)} + \lambda \gamma L_{42(1)}, & X_2^{(3)} &= -L_{31(2)} + \lambda \beta L_{32(1)} \\
L_{41(2)} \varphi &= \gamma L_{42(1)} \psi, & L_{31(2)} \varphi &= -\beta L_{32(1)} \psi.
\end{align*}
\] (2.9a), (2.10a), (2.11a)

Proof

The Lax pair (2.9a) is known from the paper [1] while two other Lax pairs are obtained from (2.9a) by permutations of indices which do not change the equation (1.1).

To prove that the relations (2.9b) indeed represent a recursion operator, we analyze their integrability conditions. We have two sets of integrability conditions for (2.9b)

\[
[L_{24(3)}, L_{14(3)}] \varphi = (\gamma L_{24(3)} L_{13(4)} + \beta L_{14(3)} L_{23(4)}) \psi
\] (2.12)

\[
\beta \gamma [L_{23(4)}, L_{14(3)}] \varphi = (\beta L_{23(4)} L_{14(3)} + \gamma L_{13(4)} L_{24(3)}) \varphi.
\] (2.13)

In condition (2.12), the commutator in the left-hand-side expands as

\[
[L_{24(3)}, L_{14(3)}] \varphi = \frac{1}{u_{34}} \{(u_{34} u_{234} + u_{23} u_{344}) L_{14(3)} + (u_{34} u_{134} - u_{13} u_{344}) L_{24(3)} \} \varphi
\]

which, with the use of equations (2.9b) converts to derivatives of \( \psi \). In condition (2.13), the commutator in the left-hand-side expands as

\[
[L_{23(4)}, L_{13(4)}] \psi = \frac{1}{u_{34}} \{(u_{34} u_{234} - u_{24} u_{344}) L_{13(4)} - (u_{34} u_{134} - u_{14} u_{334}) L_{23(4)} \} \psi
\]

which again with the use of (2.9b) converts to derivatives of \( \varphi \). Thus, the condition (2.12) becomes the equation for \( \psi \) only, which can be straightforwardly checked to coincide with \( \dot{A} \psi = 0 \), while the condition (2.13) is the equation \( \dot{A} \varphi = 0 \) for \( \varphi \) only. Therefore, the integrability conditions of equations (2.9b) are symmetry conditions for \( \psi \) and \( \varphi \), which means that both \( \psi \) and \( \varphi \) are symmetry characteristics for GHE (1.1). Hence, the relations (2.9b) are recursion relations between symmetries \( \psi \) and \( \varphi \) and, consequently, the relations (2.9b) determine a recursion operator. Since two other relations (2.10b) and (2.11b) are obtained from the recursion relations (2.9b) by permutations of indices, which leave invariant the equation (1.1) and its symmetry condition, these relations also determine recursion operators.

We note that the recursion relations (2.9b) look like an auto-Bäcklund transformation between the symmetry conditions \( \dot{A} \varphi = 0 \) and \( \dot{A} \psi = 0 \). Later, we have found that the same observation was made earlier in [5] in a more general context. We have checked that our recursion relations (2.9b) coincide with the ones obtained somewhat earlier by A. Sergyeyev in the version 1 of [5]. The approach we used here can be regarded as a generalization of the concept of partner symmetries.
3. Two-component evolutionary form of the general heavenly equation

In order to discover Hamiltonian structure of GHE we need to convert it to a two-component evolutionary form. For this purpose, we transform \( z_1, z_2 \) into the “time” and “space” variables \( t \) and \( x \) as \( t = z_1 + z_2, \quad x = z_1 - z_2, \quad y = z_3, \quad z = z_4 \). GHE (1.1) becomes

\[
\alpha(u_{tt} - u_{xx})u_{yz} + \beta(u_{ty} + u_{xy})(u_{tz} - u_{xz}) + \gamma(u_{ty} - u_{xy})(u_{tz} + u_{xz}) = 0. \tag{3.1}
\]

To convert (3.1) into an evolutionary system, we define the second component as \( v = u_t \) with the following final result for (3.1)

\[
\begin{cases}
  u_t = v \\
  v_t = \frac{1}{u_{yz}} [u_{xx}u_{yz} - u_{xy}u_{xz} + v_yv_z + b(v_yu_{xx} - v_zu_{xy})]
\end{cases} \tag{3.2}
\]

where \( b = \frac{\beta - \gamma}{\alpha} \) is the single parameter constructed from the coefficients of GHE (1.1). It is easy to check that (3.2) are Euler-Lagrange equations with the Lagrangian density

\[
L = \left( u_{tt} - \frac{v^2}{2} \right) u_{yz} - \frac{1}{2} u_yu_zu_{xx} + \frac{b}{3} u_t(u_zu_{xy} - u_yu_{xz}). \tag{3.3}
\]

4. First Hamiltonian structure

The momenta are defined as

\[
\Pi_u = \frac{\partial L}{\partial u_t} = vu_{yz} + b(u_zu_{xy} - u_yu_{xz}), \quad \Pi_v = \frac{\partial L}{\partial v_t} = 0. \tag{4.1}
\]

Hence, the Lagrangian (3.3) is degenerate since the momenta cannot be inverted for the velocities. Therefore, following Dirac [17] we impose (4.1) as constraints

\[
\Phi_u = \Pi_u - vu_{yz} - b(u_zu_{xy} - u_yu_{xz}), \quad \Phi_v = \Pi_v \tag{4.2}
\]

and calculate the Poisson bracket of the constraints

\[
K_{ij} = [\Phi_i(x, y, z), \Phi_j(x', y', z')], \quad \text{where } i, j = 1, 2, u_1 = u, \ u_2 = v, \ \Phi_1 = \Phi_u, \ \Phi_2 = \Phi_v \quad \text{using}
\]

\[
[\Pi_i(x, y, z), \ u_k(x', y', z')] = \delta_i^k \delta(x - x')\delta(y - y')\delta(z - z').
\]

The result in a matrix form reads

\[
K = \begin{pmatrix}
    b(u_zD_y - u_yD_z) + v_zD_y + v_yD_z + v_{yz}, & -u_{yz} \\
    u_{yz} & 0
  \end{pmatrix} \tag{4.3}
\]

which is obviously a skew symmetric matrix. The symplectic structure is defined in terms of (4.3) by the differential two-form \( \omega = \frac{1}{2} du_t \wedge K_{ij} du_j \) with the final result

\[
\omega = \frac{b}{2}(u_{xx}du_y \wedge du_y - u_{xy}du_y \wedge du_z) + \frac{1}{2}(v_xdu_y \wedge du_y + v_ydu_y \wedge du_z) - u_{yz}du_y \wedge dv. \tag{4.4}
\]

It is easy to check that the two-form (4.4) is closed, \( d\omega = 0 \), up to a total divergence and hence it determines a symplectic structure. The inverse to the symplectic operator is the Hamiltonian operator \( J_0 \) because the closed condition for \( \omega \) is equivalent to satisfaction of Jacobi identities for \( J_0 \) [18]. Thus, we obtain the first Hamiltonian operator in the form

\[
J_0 = K^{-1} = \frac{1}{\sqrt{\det K_{ij}}} \begin{pmatrix}
    0 & u_{yz} \\
    -u_{yz} & K_{11}
  \end{pmatrix} \frac{1}{\sqrt{\det K_{ij}}} \tag{4.5}
\]
where \( \det K_{ij} = u_{yz}^2 \). In a final form, the first Hamiltonian operator reads

\[
J_0 = \begin{pmatrix}
0 & \frac{1}{u_{yz}} \\
-\frac{1}{u_{yz}} & J_0^{22}
\end{pmatrix}
\]

where

\[
J_0^{22} = \frac{1}{u_{yz}} \left[ b(u_{xy}D_y - u_{xy}D_z) + v_zD_y + v_yD_z + v_{yz} \right] \frac{1}{u_{yz}}, \quad b = \frac{\beta - \gamma}{\alpha}.
\]

We note that \( J_0 \) is skew symmetric. The Hamiltonian density for the system (3.2) corresponding to the Lagrangian (3.3) is determined by

\[
H_1 = \Pi_u u_t + \Pi_v v_t - L = \frac{1}{2} (v^2 u_{yz} + u_y u_x u_{xx}) \quad \iff \quad H_1 = \frac{1}{2} (v^2 + u_x^2) u_{yz}
\]

the two expressions being equivalent because their difference is a total divergence. The system (3.2) in the first Hamiltonian form becomes

\[
\begin{pmatrix}
u_t \\ v_t
\end{pmatrix} = J_0 \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{u_{yz}} \\
-\frac{1}{u_{yz}} & J_0^{22}
\end{pmatrix} \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix}
\]

with \( J_0^{22} \) defined in (4.7). Here \( \delta_u \) and \( \delta_v \) are Euler-Lagrange operators with respect to \( u \) and \( v \), respectively, closely related to the variational derivatives of the functional

\[
\mathcal{H}_1 = \int_{-\infty}^{+\infty} H_1 dx dy dz
\]

with respect to \( u \) and \( v \) [18]. Here we change the notation \( E_\alpha \) of [18] to \( \delta_u^\alpha \).

5. Recursion operator for the two-component form of the general heavenly equation

Recursion operators in a two-component form are required for constructing new Hamiltonian operators for the two-component system (3.2).

Recursion relations (2.9b) become

\[
\begin{align*}
u_{yz} (\varphi_t + \varphi_z) - (v_y + u_{xy}) \varphi_z &= \gamma \{ u_{yz} (\psi_t + \psi_x) - (v_z + u_{xz}) \psi_y \} \\
u_{yz} (\varphi_t - \varphi_x) - (v_y - u_{xy}) \varphi_z &= -\beta \{ u_{yz} (\psi_t - \psi_x) - (v_z - u_{xz}) \psi_y \}.
\end{align*}
\]

(5.1)

Now we change the notation: \( \psi \) is now reserved for the second component of the symmetry characteristic, so that Lie equations read

\[
\begin{pmatrix} u \\ v
\end{pmatrix} = \begin{pmatrix} \varphi \\ \psi
\end{pmatrix}
\]

(5.2)

where \( v = u_t \) implies \( \psi = \varphi_t \) and

\[
\begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi}
\end{pmatrix} = R \begin{pmatrix} \varphi \\ \psi
\end{pmatrix}
\]

(5.3)

where \( \varphi_t = \psi \) and \( \tilde{\varphi}_t = \tilde{\psi} \). Recursion relations (5.1) take the form

\[
\begin{align*}
u_{yz} (\psi + \varphi_x) - (v_y + u_{xy}) \varphi_z &= \gamma \{ u_{yz} (\tilde{\psi} + \tilde{\varphi}_x) - (v_z + u_{xz}) \tilde{\psi}_y \} \\
(v_y - u_{xy}) \varphi_z - u_{yz} (\psi - \varphi_x) &= \beta \{ u_{yz} (\tilde{\psi} - \tilde{\varphi}_x) - (v_z - u_{xz}) \tilde{\psi}_y \}.
\end{align*}
\]
The recursion operator is explicitly defined by the relation

\[
\begin{pmatrix}
    u_{yz}D_x - (v_y + u_{xy})D_z & u_{yz} \\
    (v_y - u_{xy})D_x + u_{yz}D_z & -u_{yz}
\end{pmatrix}
\begin{pmatrix}
    \varphi \\
    \psi
\end{pmatrix}
= \begin{pmatrix}
    \gamma \{u_{yz}D_x - (v_z + u_{xz})D_y\} & \gamma u_{yz} \\
    \beta \{-u_{yz}D_x + (u_{xz} - v_z)D_y\} & \beta u_{yz}
\end{pmatrix}
\begin{pmatrix}
    \tilde{\varphi} \\
    \tilde{\psi}
\end{pmatrix}
\]  

(5.4)

To obtain the recursion operator \( R \) explicitly, we need to invert the matrix differential operator on the right-hand side of the relation (5.4). Note that the inverse \( \begin{pmatrix} e & f \\ g & h \end{pmatrix} \) for the matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) with non-commuting entries \( a, b, c, d \), defined by the equation

\[
\begin{pmatrix} e & f \\ g & h \end{pmatrix}
\begin{pmatrix} a & b \\ c & d \end{pmatrix}
= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]

is determined by the formulas

\[
e = (a - bd^{-1}c)^{-1}, \quad f = (c - db^{-1}a)^{-1}, \quad g = (b - ac^{-1}d)^{-1}, \quad h = (d - ca^{-1}b)^{-1}.
\]

(5.5)

The result for \( R \) is convenient to express in terms of the two differential operators

\[
w = u_{yz}D_x - u_{xz}D_y, \quad \zeta = u_{yz}D_x - u_{xy}D_z.
\]

(5.6)

Using the corresponding entries of the matrix on the right-hand side of (5.4) for \( a, b, c, d \) in the equations (5.5), we obtain \( e, f, g, h \)

\[
e = \frac{1}{2\gamma}w^{-1}, \quad f = -\frac{1}{2\beta}w^{-1}, \quad g = \frac{1}{2\gamma u_{yz}}(1 + v_zD_yw^{-1}) \quad \text{and} \quad h = \frac{1}{2\beta u_{yz}}(1 - v_zD_yw^{-1}).
\]

The recursion operator is explicitly defined by the relation

\[
R = \frac{1}{\alpha} \begin{pmatrix} \beta & -\gamma \\ u_{yz}(w + v_zD_y) & u_{yz}(w - v_zD_y) \end{pmatrix} w^{-1} \begin{pmatrix} \zeta - v_yD_z & u_{yz} \\ \zeta + v_yD_z & -u_{yz} \end{pmatrix}
\]

which we have multiplied by the overall constant factor \((2\beta\gamma)/\alpha\). After performing multiplication in (5.7) we obtain the matrix elements of \( R \) explicitly

\[
R_{11} = w^{-1}(b\zeta + v_yD_z), \quad R_{12} = -w^{-1}u_{yz}
\]

\[
R_{21} = \frac{1}{u_{yz}}(v_zD_yw^{-1}v_yD_z - \zeta) + \frac{b}{u_{yz}}(v_zD_yw^{-1}\zeta - v_yD_z)
\]

\[
R_{22} = b - \frac{v_z}{u_{yz}}D_yw^{-1}u_{yz}, \quad b = (\beta - \gamma)/\alpha.
\]

(5.7)

We do not consider the second and third recursion operators \( R^2 \) and \( R^3 \) defined in (2.10b) and (2.11b) in the two-component form because they refer to two different two-component systems.

The definition of the recursion operator in the two-component form (5.7) contains arbitrariness related to the definition of the inverse operator \( w^{-1} \) which involves addition of an arbitrary element of the kernel of \( w \) defined by \( w(f) = (u_{yz}D_x - u_{xz}D_y)f = 0 \). The general solution for \( w(f) = 0 \) reads \( \ker w = \{ f(z, u_z) \} \) with an arbitrary smooth \( f \). Thus, \( w^{-1} \) is defined up to the addition of an arbitrary function \( f(z, u_z) \) which plays the role of an arbitrary integration constant. This arbitrariness is eliminated by the condition that not only relation \( ww^{-1} = I \) is satisfied, but also \( w^{-1}w = I \) has to be satisfied.

Let us analyze the definition of the inverse operator \( w^{-1} \) in more detail. It determines a solution \( f = w^{-1}g \) to the first-order ODE: \( wf = u_{zy}f_x - u_{xx}f_y = g \) for any given \( g \). Introduce
parameter $s$ along the curves tangent to the vector field $w$, $x = x(s), y = y(s)$ on the plane $z = \text{const}$

\[ w f = \frac{df(x(s), y(s), z)}{ds} = \frac{dx}{ds} f_x + \frac{dy}{ds} f_y = u_{yz} f_x - u_{zx} f_y = g \]

which implies \[ \frac{dx}{ds} = u_{yz}, \quad \frac{dy}{ds} = -u_{zx} \] and so

\[ ds = \frac{dx}{u_{yz}} = -\frac{dy}{u_{zx}} \] (5.8)

The second of these equations is the characteristic ODE with the integrals

\[ z = \text{const}, \quad u_z(x, y, z) = \text{const} \] (5.9)

while the first one determines the parameter $s$ along the characteristics

\[ s = \int \frac{dx}{u_{zy}} = -\int \frac{dy}{u_{zx}} = \frac{1}{2} \int \left( \frac{dx}{u_{zy}} - \frac{dy}{u_{zx}} \right) \] (5.10)

where the integrals are taken under conditions (5.9) and the equations (5.8) have been used.

Under the constraints (5.9), we have $w f = \frac{df}{ds} = D_s f = g$, so that \[ f = w^{-1} g = D_s^{-1} g = \int ds g. \]

We define $w^{-1}$ as the integral operator $\int ds$ with $ds$ defined by (5.8)

\[ w^{-1} = \left\{ \int_a^x \frac{d\xi}{u_{zy}(\xi, y(\xi, c, z), z)} \right\}_{c=u_z(x,y,z)} \quad \text{or} \quad w^{-1} = -\left\{ \int_b^y \frac{d\eta}{u_{zx}(x(\eta, c, z), \eta, z)} \right\}_{c=u_z(x,y,z)} \] (5.11)

where all the integrals are taken at the conditions (5.9) so that $y = y(x, c, z)$ and $x = x(y, c, z)$ are determined by the equation $u_z(x, y, z) = c = \text{const}$ and $z$ is a parameter of the integrations. Here $a$ and $b$ are arbitrarily fixed points such that the functions $f(x, y, z)$ are subject to the boundary condition $f|_{x=a} = 0$ or $f|_{y=b} = 0$, respectively. Then for these classes of functions one can check that $w^{-1} w f(x, y, z) = f$ and also that $w w^{-1} f(x, y, z) = f$, so that $w^{-1} w = I$ and $w w^{-1} = I$.

Indeed, using characteristic equations (5.8) and definition (5.6) of $w$ for the first integral in (5.11) we have

\[ w^{-1} w f(x, y, z) = \left\{ \int_a^x \frac{d\xi}{u_{zy}(\xi, y(\xi, c, z), z)} \partial_{\xi} f(\xi, y, z) - \int_a^x \frac{d\xi}{u_{zy}(\xi, y(\xi, c, z), z)} \partial_y f(\xi, y, z) \right\}_{c=u_z} = \left\{ \int_a^x d\xi D_\xi f(\xi, y(\xi, c, z), z) \right\}_{c=u_z} = [f(x, y(x, c, z), z)|_{c=u_z}]^x_a = f(x, y, z) - f(a, y, z) = f(x, y, z) \]

for all $f$ satisfying $f|_{x=a} = 0$, so that $w^{-1} w = I$. Next, define

\[ F(x, u_z(x, y, z), z) = \left\{ \int_a^x d\xi \frac{f(\xi, y(\xi, c, z), z)}{u_{zy}(\xi, y(\xi, c, z), z)} \right\}_{c=u_z(x,y,z)} \]
where we should use the equations of the characteristics (5.8) for the integral. Then

\[
ww^{-1} f(x, y, z) = (u_{xy}D_x - u_{xx}D_y) \left\{ \int_a^x \frac{d\xi}{u_{zy}(\xi, \zeta, x, y, z)} \right\} = (u_{xy}D_x - u_{xx}D_y)F(x, u_x(x, y, z), z) = u_{zy}F_x^1 + u_{zy}F_u u_{xx} - u_{xx}F_u u_{zy}
\]

so that \(ww^{-1} = I\). The check of the second definition of \(w^{-1}\) in (5.11) could be made in a similar way. With a more symmetric definition

\[
w^{-1} = \frac{1}{2} \left\{ \int_{-\alpha}^x \frac{d\xi}{u_{zy}(\xi, \zeta, x, y, z)} + \int_x^a \frac{d\xi}{u_{zy}(\xi, \zeta, x, y, z)} \right\} = u_{zy}F_x^1 + u_{zy}F_u u_{xx} - u_{xx}F_u u_{zy}
\]

so that \(w^{-1}w = I\) implies the boundary condition \(f(-\alpha, y, z) = -f(\alpha, y, z)\) for an arbitrarily fixed \(\alpha\).

6. Second Hamiltonian structure

The second Hamiltonian operator \(J_1\) is obtained by composing the recursion operator (5.7) with the first Hamiltonian operator \(J_0 = RJ_0\) or explicitly

\[
\begin{pmatrix}
J_{11}^{(1)} & J_{12}^{(1)} \\
J_{11}^{(2)} & J_{12}^{(2)}
\end{pmatrix} =
\begin{pmatrix}
R_{11} & R_{12} \\
R_{21} & R_{22}
\end{pmatrix}
\begin{pmatrix}
0 & \frac{1}{u_{yz}} \\
-\frac{1}{u_{yz}} & J_0^{(2)}
\end{pmatrix}
\]

where we have used the formula (4.6) and \(J_0^{(2)}\) is defined in (4.7). Utilizing also the expressions (5.7), we obtain the matrix elements of \(J_1\) in the form

\[
J_{11}^{(1)} = w^{-1}, \quad J_{12}^{(1)} = \frac{b}{u_{yz}} - w^{-1}D_y\frac{v_z}{u_{yz}}, \quad J_{12}^{(2)} = \frac{b}{u_{yz}} + \frac{v_z}{u_{yz}}D_yw^{-1}
\]

\[
J_{12}^{(2)} = -\frac{1}{u_{yz}}(\zeta - w)\frac{1}{u_{yz}} + \frac{b}{u_{yz}}(D_yv_z + v_Dy)\frac{1}{u_{yz}}
\]

\[
-\frac{v_z}{u_{yz}}D_yw^{-1}D_y\frac{v_z}{u_{yz}}, \quad \text{where} \quad b = \frac{(\beta - \gamma)}{a}
\]

which is explicitly skew-symmetric. We have also proved the Hamiltonian property of the operator \(J_1\) and the compatibility of the two Hamiltonian structures \(J_0\) and \(J_1\) by checking directly the Jacobi identities for the operator \(J_1 + aJ_0\) with an arbitrary constant \(a\). This proof will be presented in a future publication.

The Hamiltonian density corresponding to the general heavenly system (3.2) with respect to the second Hamiltonian operator \(J_1\) is

\[
H_0 = \frac{1}{2(b^2 - 1)} \left[ 2u_x v + b(v^2 + u_x^2) \right] u_{yz}.
\]

Here we eliminate the case of the degenerate GHE with \(\beta \cdot \gamma = 0\), so that \(b^2 - 1 \neq 0\). Thus, the general heavenly flow takes the bi-Hamiltonian form

\[
\begin{pmatrix}
\delta_u H_1 \\
\delta_v H_1
\end{pmatrix} = J_0 \begin{pmatrix}
\delta_u H_0 \\
\delta_v H_0
\end{pmatrix} = J_1 \begin{pmatrix}
\delta_u H_0 \\
\delta_v H_0
\end{pmatrix}
\]

and hence presents a multidimensional completely integrable system in the sense of Magri [4].
Conclusion
We have constructed two additional Lax pairs and three nonlocal recursion operators for the general heavenly equation (GHE) obtained by the splitting of the Lax pairs with respect to the spectral parameter. Converting GHE to a two-component evolutionary form, we have discovered Lagrangian, symplectic and Hamiltonian structures of this GH system. We have converted the first recursion operator to a matrix $2 \times 2$ form appropriate for our two-component evolutionary system, while the other two operators refer to different two-component systems obtained from the first one by permutations, so that we end up with a single nonlocal recursion operator for the first GH system. Composing the recursion operator $R$ with the first Hamiltonian operator $J_0$ we have obtained the second Hamiltonian operator $J_1 = RJ_0$ and found the corresponding Hamiltonian density generating the two-component flow of GHE. We have checked the Jacobi identities for $J_1$ and compatibility of the two Hamiltonian structures $J_0$ and $J_1$. The proof will be presented in a future publication. Thus, we have shown that the GHE two-component flow is a bi-Hamiltonian system completely integrable in the sense of Magri.

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