A new Lax pair for the sixth Painlevé equation associated with \( \hat{so}(8) \)

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Dedicated to Professor Mitsuo Morimoto on his sixtieth birthday

Introduction

In this article, we propose a new representation of Lax type for the sixth Painlevé equation. This representation, formulated in the framework of the loop algebra \( so(8)[z, z^{-1}] \) of type \( D_4^{(1)} \), provides a natural explanation of the affine Weyl group symmetry of \( P_{VI} \). After recalling a standard derivation of \( P_{VI} \), we describe in Section 2 fundamental Bäcklund transformations for \( P_{VI} \). In Section 3, we present our Lax pair for \( P_{VI} \) associated with \( so(8)[z, z^{-1}] \), and explain how the Bäcklund transformations arise from the linear problem. For the general background on Painlevé equations, we refer the reader to [2].

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1 The sixth Painlevé equation

The sixth Painlevé equation is the following nonlinear ordinary differential equation of second order for the unknown function \( y = y(t) \):

\[
y'' = \frac{1}{2} \left( \frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t} \right) (y')^2 - \left( \frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t} \right) y' + \frac{y(y-1)(y-t)}{t^2(t-1)^2} \left( \alpha + \beta \frac{t}{y^2} + \gamma \frac{t-1}{(y-1)^2} + \delta \frac{t(t-1)}{(y-t)^2} \right),
\]

where ‘ stands for the derivation with respect to the independent variable \( t \), and \( \alpha, \beta, \gamma, \delta \) are complex parameters.

A standard way to derive the sixth Painlevé equation is to employ the monodromy preserving deformation of a second order Fuchsian differential equation on \( \mathbb{P}^1 \), with four regular singular points and one apparent singularity. Consider the system of linear differential equations

\[
(\partial^2_x + a_1(x,t)\partial_x + a_2(x,t))u = 0, \quad \partial_t u = (b_1(x,t)\partial_x + b_2(x,t))u,
\]
for the unknown functions \( u = u(x,t) \), where \( \partial_x = \partial/\partial x \) and \( \partial_t = \partial/\partial t \). We assume that the coefficients \( a_j(x,t) \) and \( b_j(x,t) \) are rational functions in \( x \), depending holomorphically on \( t \), and that the first equation is Fuchsian with Riemann scheme

\[
\begin{aligned}
&x = 0 \quad x = 1 \quad x = t \quad x = q \quad x = \infty \\
&\kappa_0 \quad \kappa_1 \quad \kappa_t \quad 2 \quad \kappa_\infty + \rho
\end{aligned}
\]

(3)

with respect to the variable \( x \). In this scheme, \( \kappa_0, \kappa_1, \kappa_t, \kappa_\infty \) and \( \rho \) are generic complex parameters subject to the Fuchs relation

\[
\kappa_0 + \kappa_1 + \kappa_t + \kappa_\infty + 2\rho = 1.
\]

(4)

We also assume that the singularity \( x = q \), which may depend on \( t \), is non-logarithmic. Under these assumptions, the coefficients \( a_1(x,t), a_2(x,t) \) are expressed in the form

\[
a_1(x,t) = \frac{1 - \kappa_0}{x} + \frac{1 - \kappa_1}{x - 1} + \frac{1 - \kappa_t}{x - t} - \frac{1}{x - q},
\]

\[
a_2(x,t) = \frac{1}{x(x - 1)} \left\{ \frac{t(t - 1)H}{x - t} + \frac{q(q - 1)p}{x - q} + \rho(\kappa_\infty + \rho) \right\},
\]

(5)

respectively, where

\[
p = \text{Res}_{x=q}(a_2(x,t)dx), \quad H = -\text{Res}_{x=1}(a_2(x,t)dx).
\]

(6)

Furthermore, the coefficient \( H \) is determined as a polynomial in \( (q,p) \) with coefficients in \( \mathbb{C}(t) \); explicitly, it is given by

\[
H = \frac{1}{t(t - 1)} \left[ p^2q(q - 1)(q - t) - p\{ \kappa_0(q - 1)(q - t) + \kappa_1q(q - t) + (\kappa_t - 1)q(q - 1) \} + \rho(\kappa_\infty + \rho)(q - t) \right].
\]

(7)

The compatibility condition of the linear differential system \((\text{2})\) then turns out to be expressed as the Hamiltonian system

\[
H_{VI} : \quad \partial_t(q) = \frac{\partial H}{\partial p}, \quad \partial_t(p) = -\frac{\partial H}{\partial q},
\]

(8)

with polynomial Hamiltonian \( H \) in \((\text{2})\); namely,

\[
t(t - 1)\partial_t(q) = 2pq(q - 1)(q - t) - \{ \kappa_0(q - 1)(q - t) + \kappa_1q(q - t) + (\kappa_t - 1)q(q - 1) \}
\]

\[
+ \kappa_tq(q - t) + (\kappa_t - 1)q(q - 1) \}
\]

\[
t(t - 1)\partial_t(p) = -p^2(3q^2 - 2(1 + t)q + t) - 2\kappa_0 + \kappa_1 + \kappa_t - 1 \}
\]

\[
- \kappa_0(1 + t) - \kappa_t - \kappa_t + 1 \}
\]

\[- \rho(\kappa_\infty + \rho).
\]

(9)

This system of nonlinear equations is in fact equivalent to the sixth Painlevé equation \( P_{VI} \) for \( y = q \), with parameters

\[
\alpha = \frac{\kappa_\infty^2}{2}, \quad \beta = -\frac{\kappa_0^2}{2}, \quad \gamma = \frac{\kappa_t^2}{2}, \quad \delta = \frac{1 - \kappa_t^2}{2}.
\]

(10)

We remark that, in place of \((\text{2})\), one can naturally make use of the Schlesinger system of rank two, with regular singular points \( x = 0, 1, \infty \).
2 Discrete symmetry of $H_{VI}$

It is known that the sixth Painlevé equation admits a group of Bäcklund transformations which is isomorphic to the (extended) affine Weyl group of type $D^{(1)}_4$ (see [7], for instance).

In describing the Bäcklund transformations for $H_{VI}$, it is convenient to use the parameters

$$
\begin{align*}
\alpha_0 &= \kappa_t, \\
\alpha_1 &= \kappa_\infty, \\
\alpha_2 &= \rho, \\
\alpha_3 &= \kappa_1, \\
\alpha_4 &= \kappa_0
\end{align*}
$$

(11)

with linear relation

$$
\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1,
$$

so that

$$
\begin{align*}
t(t-1)H &= p^2q(q-1)(q-t) - p\left\{ (\alpha_0 - 1)q(q - 1) \\
&+ \alpha_3q(q-t) + \alpha_4(q-1)(q-t) \right\} + \alpha_2(\alpha_1 + \alpha_2)(q-t).
\end{align*}
$$

(12)

In the following, we identify the parameter space for $H_{VI}$ with the affine space $V = \mathbb{C}^4$ with canonical coordinates $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$, and regard $\alpha_j$ as linear functions of $V$ such that

$$
\begin{align*}
\alpha_0 &= 1 - \varepsilon_1 - \varepsilon_2, \\
\alpha_1 &= \varepsilon_1 - \varepsilon_2, \\
\alpha_2 &= \varepsilon_2 - \varepsilon_3, \\
\alpha_3 &= \varepsilon_3 - \varepsilon_4, \\
\alpha_4 &= \varepsilon_5 + \varepsilon_6.
\end{align*}
$$

(13)

We identify $V$ with the Cartan subalgebra of the simple Lie algebra $\mathfrak{so}(8)$ (of type $D_4$); $\{\varepsilon_1, \ldots, \varepsilon_4\}$ is then a canonical orthonormal basis of $V^*$, and $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4$ are the simple affine roots. Note that the null root $\delta = \alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$ is normalized to be the constant function 1.

By a Bäcklund transformation, we mean a transformation of dependent variables and parameters that leaves the system invariant. Let us show an example of Bäcklund transformation for $H_{VI}$. Define new variables $\tilde{q}, \tilde{p}$ by

$$
\begin{align*}
\tilde{q} &= q, \\
\tilde{p} &= p - \frac{\alpha_0}{q-t}.
\end{align*}
$$

(14)

Then one can verify directly that, if the pair $(q, p)$ satisfies the Hamiltonian system (3), then the pair $(\tilde{q}, \tilde{p})$ again satisfies the same system with parameters $\alpha_0, \alpha_2$ replaced by $-\alpha_0, \alpha_2 + \alpha_0$, respectively; we refer to this Bäcklund transformation as $s_0$. To be more precise, let us consider the field of rational functions

$$
\mathcal{K} = \mathbb{C}(\alpha_1, \alpha_2, \alpha_3, \alpha_4, q, p, t) \quad (\alpha_0 = 1 - \alpha_1 - 2\alpha_2 - \alpha_3 - \alpha_4),
$$

(15)

and the Hamiltonian vector field

$$
\delta = \frac{\partial H}{\partial p} \frac{\partial}{\partial q} - \frac{\partial H}{\partial q} \frac{\partial}{\partial p} + \frac{\partial}{\partial t}
$$

(16)

acting on $\mathcal{K}$ as a derivation. We regard this differential field $(\mathcal{K}, \delta)$ as representing the Hamiltonian system $H_{VI}$. We define the automorphism $s_0 : \mathcal{K} \to \mathcal{K}$ by setting

$$
\begin{align*}
s_0(\alpha_0) &= -\alpha_0, \\
s_0(\alpha_2) &= \alpha_2 + \alpha_0, \\
s_0(\alpha_j) &= \alpha_j \quad (j \neq 0, 2),
\end{align*}
$$

(17)
Table 1: Bäcklund transformations for $H_{V1}$

|   | $\alpha_0$ | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ | $\alpha_4$ | $q$   | $p$   |
|---|-------------|-------------|-------------|-------------|-------------|-------|-------|
| $s_0$ | $-\alpha_0$ | $\alpha_1$ | $\alpha_2 + \alpha_0$ | $\alpha_3$ | $\alpha_4$ | $q$   | $p - \frac{\alpha_0}{q-t}$ |
| $s_1$ | $\alpha_0$ | $-\alpha_1$ | $\alpha_2 + \alpha_1$ | $\alpha_3$ | $\alpha_4$ | $q$   | $p$   |
| $s_2$ | $\alpha_0 + \alpha_2$ | $\alpha_1 + \alpha_2$ | $-\alpha_2$ | $\alpha_3 + \alpha_2$ | $\alpha_4 + \alpha_2$ | $q + \frac{\alpha_2}{p}$ | $p$   |
| $s_3$ | $\alpha_0$ | $\alpha_1$ | $\alpha_2 + \alpha_3$ | $-\alpha_3$ | $\alpha_4$ | $q$   | $p - \frac{\alpha_3}{q-1}$ |
| $s_4$ | $\alpha_0$ | $\alpha_1$ | $\alpha_2 + \alpha_4$ | $\alpha_3$ | $-\alpha_4$ | $q$   | $p - \frac{\alpha_4}{q}$   |
| $r_1$ | $\alpha_1$ | $\alpha_0$ | $\alpha_2$ | $\alpha_4$ | $\alpha_3$ | $\frac{t(q-1)}{q-t} - \frac{(q-t)((q-1)p + \alpha_2)}{t(t-1)}$ | |
| $r_3$ | $\alpha_3$ | $\alpha_4$ | $\alpha_2$ | $\alpha_0$ | $\alpha_1$ | $\frac{t}{q} - \frac{g(qp + \alpha_2)}{t}$ | |
| $r_4$ | $\alpha_4$ | $\alpha_3$ | $\alpha_2$ | $\alpha_1$ | $\alpha_3$ | $\frac{q-t}{q-1} - \frac{(q-1)((q-1)p + \alpha_2)}{t-1}$ | |

and

$$s_0(q) = q, \quad s_0(p) = p - \frac{\alpha_0}{q-t}, \quad s_0(t) = t. \quad (18)$$

Then one can show that the automorphism $s_0 : \mathcal{K} \to \mathcal{K}$ commutes with the Hamiltonian vector field $\delta$. In this sense, a Bäcklund transformation can be defined alternatively to be an automorphism of the differential field that commutes with the derivation.

Table 1 is the list of fundamental Bäcklund transformations for $H_{V1}$. We consider two subgroups

$W = \langle s_0, s_1, s_2, s_3, s_4 \rangle \subset \tilde{W} = \langle s_0, s_1, s_2, s_3, s_4, r_1, r_3, r_4 \rangle \subset \text{Aut}_4(\mathcal{K}) \quad (19)$

of differential automorphisms of $\mathcal{K}$, generated by the Bäcklund transformations in Table 1. Then it turns out that $W$ and $\tilde{W}$ are isomorphic to the affine Weyl group and the extended affine Weyl group of type $D_4^{(1)}$, respectively. The Bäcklund transformations $s_i \ (i = 0, 1, 2, 3, 4)$ and $r_i \ (i = 1, 3, 4)$ in fact satisfy the fundamental relations

$$s_i^2 = 1 \quad (i = 0, 1, 2, 3, 4),$$

$$s_is_j = s_js_i \quad (i, j = 0, 1, 3, 4),$$

$$s_is_2s_i = s_2s_is_2 \quad (i = 0, 1, 3, 4),$$

$$r_i^2 = 1 \quad (i = 1, 3, 4)$$

$$r_ir_j = r_k \quad (\{i, j, k\} = \{1, 3, 4\})$$

$$r_is_j = s_{\sigma_i(j)}r_i \quad (i = 1, 3, 4; \ j = 0, 1, 2, 3, 4),$$

where $\sigma_i \ (i = 1, 3, 4)$ are the permutations defined by

$$\sigma_1 = (01)(34), \quad \sigma_3 = (03)(14), \quad \sigma_4 = (04)(13). \quad (20)$$
We also remark that each element \( w \in \tilde{W} \) defines a canonical transformation:

\[
w(\{\varphi, \psi\}) = \{w(\varphi), w(\psi)\} \quad (\varphi, \psi \in \mathcal{K}),
\]

where \( \{, \} \) stands for the standard Poisson bracket defined by

\[
\{\varphi, \psi\} = \frac{\partial \varphi}{\partial p} \frac{\partial \psi}{\partial q} - \frac{\partial \varphi}{\partial q} \frac{\partial \psi}{\partial p}.
\]

(23)

**Remark 2.1** The fundamental relations for the generators \( s_0, s_1, s_2, s_3, s_4 \) of the affine Weyl group of type \( D_4^{(1)} \) is described as follows in terms of the Cartan matrix \( A = (a_{ij})_{i,j=0}^4 \):

\[
s_i^2 = 1 \quad (i = 0, 1, 2, 3, 4)
\]

\[
s_is_j = s_js_i \quad \text{if} \quad (a_{ij}, a_{ji}) = (0, 0),
\]

\[
s_is_js_i = s_js_is_j \quad \text{if} \quad (a_{ij}, a_{ji}) = (-1, -1),
\]

(24)

where

\[
A = \begin{bmatrix} 2 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ -1 & -1 & 2 & -1 & -1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{bmatrix}.
\]

(25)

The action of \( s_i \) on the simple affine roots \( \alpha_j \) is given by

\[
s_i(\alpha_j) = \alpha_j - \alpha_i a_{ij} \quad (i, j = 0, 1, 2, 3, 4).
\]

(26)

Note also that \( \tilde{W} \) is isomorphic to the semidirect product \( W \rtimes \Omega \) of \( W \) and \( \Omega = \{1, r_1, r_3, r_4\} \) acting on \( W \) through the permutations \( \{1, \sigma_1, \sigma_3, \sigma_4\} \) of indices for the generators \( s_j \); \( \Omega \) is identified with a group of diagram automorphisms of the Dynkin diagram of type \( D_4^{(1)} \). If we set

\[
\varphi_0 = q - t, \quad \varphi_1 = 1, \quad \varphi_2 = -p, \quad \varphi_3 = q - 1, \quad \varphi_4 = q,
\]

(27)

the Bäcklund transformations \( s_i \) are expressed as

\[
s_i(\varphi_j) = \varphi_j + \frac{\alpha_i}{\varphi_i} u_{ij}, \quad u_{ij} = \{\varphi_i, \varphi_j\} \quad (i, j = 0, 1, 2, 3, 4),
\]

(28)

consistently with the birational Weyl group actions discussed previously in [5], [6]. In this particular case, the matrix \( U = (u_{ij})_{i,j=0}^4 \) is given by

\[
U = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.
\]

(29)

Observe that \( u_{12} = u_{21} = 0 \); this degeneracy is caused by the normalization that one of the regular singular points of (2) is placed at \( x = \infty \).
Remark 2.2 The extended affine Weyl group $\widetilde{W}$ of type $D_4^{(1)}$ is expressed as the semidirect product of the weight lattice $P$ of type $D_4$ and the Weyl group $W(D_4) = \langle s_1, s_2, s_3, s_4 \rangle$ acting on $P$:

$$\widetilde{W} \leftarrow P \rtimes W(D_4), \quad P = \bigoplus_{i=1}^{4} \mathbb{Z} \varpi_i,$$

where $\varpi_i$ are the fundamental weights of type $D_4$ defined by

$$\varpi_1 = \varepsilon_1, \quad \varpi_2 = \varepsilon_1 + \varepsilon_2, \quad \varpi_3 = \frac{1}{4}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4), \quad \varpi_4 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4).$$

(31)

Note also that the weight lattice $P$ is the set of all elements

$$\lambda = \frac{1}{2}(n_1 \varepsilon_1 + n_2 \varepsilon_2 + n_3 \varepsilon_3 + n_4 \varepsilon_4) \quad (n_1, n_2, n_3, n_4 \in \mathbb{Z})$$

(32)

such that, either all the $n_j$'s are even, or all the $n_j$'s are odd. The translations $T_{\varpi_i}$ $(i = 1, \ldots, 4)$ corresponding to $\varpi_i$ are expressed as

$$T_{\varpi_1} = r_1 s_1 s_2 s_3 s_4 s_2 s_1, \quad T_{\varpi_2} = s_0 s_2 s_1 s_3 s_4 s_2 s_1 s_3 s_4 s_2, \quad T_{\varpi_3} = r_3 s_3 s_2 s_1 s_4 s_2 s_3, \quad T_{\varpi_4} = r_4 s_4 s_2 s_1 s_3 s_4 s_2,$$

(33)

in terms of the generators $s_j$ and $r_j$. These elements transform the simple affine roots $\alpha_j$ as follows.

|   | $\alpha_0$ | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ | $\alpha_4$ |
|---|---|---|---|---|---|
| $T_{\varpi_1}$ | $\alpha_0 + 1$ | $\alpha_1 - 1$ | $\alpha_2$ | $\alpha_3$ | $\alpha_4$ |
| $T_{\varpi_2}$ | $\alpha_0 + 2$ | $\alpha_1$ | $\alpha_2 - 1$ | $\alpha_3$ | $\alpha_4$ |
| $T_{\varpi_3}$ | $\alpha_0 + 1$ | $\alpha_1$ | $\alpha_2$ | $\alpha_3 - 1$ | $\alpha_4$ |
| $T_{\varpi_4}$ | $\alpha_0 + 1$ | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ | $\alpha_4 - 1$ |

(34)

Regarded as automorphisms of $K = \mathbb{C}(\alpha_1, \alpha_2, \alpha_3, \alpha_4, q, p, t)$, $T_{\varpi_i}$ $(i = 1, 2, 3, 4)$ provide a commuting family of Bäcklund transformations for $H_{V_1}$, which are called Schlesinger transformations (such Bäcklund transformations that act as shift operators on the parameter space).

A certain part of this discrete symmetry of $H_{V_1}$ can be explained by the monodromy preserving deformation of a second order Fuchsian equation $[3]$. In fact, each $s_i$ $(i = 0, 1, 3, 4)$, except $s_2$, arises from a simple transformation of the unknown function $u = u(x, t)$, and each $r_i$ $(i = 1, 3, 4)$ from a fractional linear transformation of the coordinate $x$. This framework does not seem, however, to explain the particular Bäcklund transformation $s_2$ in Table $[3]$, which is essential in understanding the whole picture of discrete symmetry of $H_{V_1}$. In the next section, we propose a new Lax pair for $H_{V_1}$, from which all the Bäcklund transformations in Table $[3]$ can be understood naturally.
3 Lax pair associated with $\hat{\mathfrak{so}}(8)$

Consider the following system of linear differential equations for the column vector $\vec{\psi} = (\psi_1, \psi_2, \ldots, \psi_8)^t$ of eight unknown functions $\psi_i = \psi_i(z, t)$ ($i = 1, 2, \ldots, 8$):

$$ (z \partial_z + M)\vec{\psi} = 0, \quad \partial_t \vec{\psi} = B\vec{\psi}, $$

(35)

with the compatibility condition

$$ [z \partial_z + M, \partial_t - B] = 0. $$

(36)

We assume that the matrices $M$ and $B$ are in the form

$$ M = \begin{bmatrix}
\varepsilon_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \varepsilon_2 & -p & -1 & -1 & 0 & 0 & 0 \\
0 & 0 & \varepsilon_3 & q - 1 & q & 0 & 0 & 0 \\
0 & 0 & 0 & \varepsilon_4 & 0 & -q & 1 & 0 \\
0 & 0 & 0 & 0 & -\varepsilon_4 & 1 - q & 0 & 0 \\
-\varepsilon_2 & 0 & 0 & 0 & 0 & -\varepsilon_3 & p & 0 \\
(q - t)z & 0 & 0 & 0 & 0 & 0 & -\varepsilon_2 & -1 \\
0 & (q - t)z & z & 0 & 0 & 0 & 0 & -\varepsilon_1 \\
\end{bmatrix}, $$

(37)

and

$$ B = \begin{bmatrix}
u_1 & x_1 & y_1 & 0 & 0 & 0 & 0 & 0 \\
u_2 & x_2 & -y_3 & -y_4 & 0 & 0 & 0 & 0 \\
u_3 & x_3 & x_4 & 0 & 0 & 0 & 0 & 0 \\
u_4 & 0 & 0 & 0 & -x_4 & y_4 & 0 & 0 \\
u_5 & 0 & 0 & 0 & -u_4 & -x_3 & y_3 & 0 \\
u_6 & 0 & 0 & 0 & 0 & -u_3 & -x_2 & -y_1 \\
u_7 & -\varepsilon_2 & 0 & 0 & 0 & 0 & -u_2 & -x_1 \\
u_8 & 0 & -\varepsilon_2 & 0 & 0 & 0 & 0 & -u_1 \\
\end{bmatrix}, $$

(38)

respectively, where $\varepsilon_j$ are complex constants, and the variables $q, p, x_j, y_j$ and $u_j$ are functions in $t$. As before, we set

$$ \alpha_0 = 1 - \varepsilon_1 - \varepsilon_2, \quad \alpha_1 = \varepsilon_1 - \varepsilon_2, \quad \alpha_2 = \varepsilon_2 - \varepsilon_3, $$

$$ \alpha_3 = \varepsilon_3 - \varepsilon_4, \quad \alpha_4 = \varepsilon_3 + \varepsilon_4. $$

(39)

Theorem 3.1 Under the compatibility condition $[M, B]$, the variables $x_j, y_j$ and $u_j$ are determined uniquely as elements of $K = \mathbb{C}(\alpha_1, \alpha_2, \alpha_3, \alpha_4, q, p, t)$. The compatibility condition is then equivalent to the Hamiltonian system $H_{V1}$ of the sixth Painlevé equation.

The variables $x_j, y_j$ and $u_j$ appearing in $B$ are determined explicitly as follows:

$$ x_1 = \frac{q - t}{t(t - 1)}, \quad x_3 = -\frac{q}{t}, \quad x_4 = -\frac{q - 1}{t - 1}, $$

$$ y_1 = \frac{1}{t(t - 1)}, \quad y_3 = -\frac{1}{t}, \quad y_4 = -\frac{1}{t - 1}, $$

$$ x_2 = -\frac{(q - t)p + \alpha_1 + \alpha_2}{t(t - 1)}, $$

(40)
We consider the following realization of the simple Lie algebra \( \mathfrak{so}(8) \) and its loop algebra. With the notation of matrix units \( E_{ij} = (\delta_{ia}\delta_{jb})_{a,b=1} \), we set

\[
J = \sum_{i=1}^{8} E_{i,9-i}.
\]

We consider the following realization of the simple Lie algebra \( \mathfrak{so}(8) \):

\[
\mathfrak{so}(8) = \{ X \in \text{Mat}(8; \mathbb{C}) \mid JX + X^t J = 0 \},
\]

where \( X^t \) denotes the transposition of \( X \). Let us define the Chevalley generators \( E_j, H_j, F_j \) \( (j = 0, 1, 2, 3, 4) \) for the loop algebra \( \mathfrak{so}(8)[z, z^{-1}] \) by

\[
\begin{align*}
E_0 &= z(E_{82} - E_{11}) , & E_1 &= E_{12} - E_{78} , & E_2 &= E_{23} - E_{67} , \\
E_3 &= E_{34} - E_{56} , & E_4 &= E_{35} - E_{46} , \\
F_0 &= z^{-1}(E_{28} - E_{17}) , & F_1 &= E_{21} - E_{87} , & F_2 &= E_{32} - E_{76} , \\
F_3 &= E_{43} - E_{65} , & F_4 &= E_{53} - E_{64} ,
\end{align*}
\]

and \( H_j = [E_j, F_j] \) \((j = 0, 1, 2, 3, 4)\). For a vector \( a = (a_1, a_2, a_3, a_4) \) given, we also use the notation

\[
H(a) = \sum_{i=1}^{4} a_i (E_{ii} - E_{9-i,9-i})
\]

for the corresponding element in the Cartan subalgebra of \( \mathfrak{so}(8) \), so that

\[
\begin{align*}
H_0 &= H(-1, -1, 0, 0) , & H_1 &= H(1, -1, 0, 0) , & H_2 &= H(0, 1, -1, 0) \\
H_3 &= H(0, 0, 1, -1) , & H_4 &= H(0, 0, 1, 1) .
\end{align*}
\]

Notice that the two matrices \( M, B \) belong to a Borel subalgebra of the loop algebra \( \mathfrak{so}(8)[z, z^{-1}] \); in fact, they are expressed in the form

\[
M = H(\varepsilon) + (q - t)E_0 + E_1 - pE_2 + (q - 1)E_3 + qE_4
+ [E_0, E_2] + [E_3, E_2] + [E_4, E_2],
\]

\[
B = H(u) + E_0 + x_1E_1 + x_2E_2 + x_3E_3 + x_4E_4
+ y_1[E_1, E_2] + y_3[E_3, E_2] + y_4[E_4, E_2],
\]

where \( \varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \) and \( u = (u_1, u_2, u_3, u_4) \).
Remark 3.2 The affine Lie algebra $\hat{\mathfrak{so}}(8)$ is realized as a central extension of the loop algebra $\mathfrak{so}(8)[z,z^{-1}]$, together with the derivation $d = z\partial_z$ (see [4], for the detail):

$$\hat{\mathfrak{so}}(8) = \mathfrak{so}(8) \otimes \mathbb{C}[z,z^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

(48)

where $c$ denotes the canonical central element. In this context, the simple affine roots $\alpha_j$ ($j = 0, 1, 2, 3, 4$) are defined as linear functionals of the Cartan subalgebra

$$\mathfrak{h} = \bigoplus_{i=0}^{4} \mathbb{C}h_i \oplus \mathbb{C}d, \quad h_0 = H_0 \otimes 1 + c, \quad h_i = H_i \otimes 1 \quad (i = 1, 2, 3, 4)$$

(49)

such that

$$\langle h_i, \alpha_j \rangle = a_{i,j} \quad (i = 0, 1, 2, 3, 4), \quad \langle d, \alpha_j \rangle = \delta_{0,j}$$

(50)

for $j = 0, 1, 2, 3, 4$; also they are extended to linear functionals on the whole affine Lie algebra $\hat{\mathfrak{so}}(8)$ through the triangular decomposition. Note that our Lax pair mentioned above is formulated in fact in the framework of $\hat{\mathfrak{so}}(8)/\mathbb{C}c = \mathfrak{so}(8)[z,z^{-1}] \otimes \mathbb{C}d$:

$$\mathcal{M} = d + M = z\partial_z + M \in \hat{\mathfrak{so}}(8)/\mathbb{C}c.$$  

(51)

Since $\langle c, \alpha_j \rangle = 0$, we can regard $\alpha_j$ ($j = 0, 1, 2, 3, 4$) and the null root $\delta = \alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$ as linear functionals on $\mathfrak{so}(8)/\mathbb{C}c$. Then we have

$$\alpha_0(\mathcal{M}) = 1 - \varepsilon_1 - \varepsilon_2, \quad \alpha_1(\mathcal{M}) = \varepsilon_1 - \varepsilon_2, \quad \alpha_2(\mathcal{M}) = \varepsilon_2 - \varepsilon_3,$$

$$\alpha_3(\mathcal{M}) = \varepsilon_3 - \varepsilon_4, \quad \alpha_4(\mathcal{M}) = \varepsilon_3 + \varepsilon_4, \quad \delta(\mathcal{M}) = 1.\quad (52)$$

In this sense, our notation for the parameters \([33]\) is consistent with that of simple roots for $\hat{\mathfrak{so}}(8)$.

In our framework, the Bäcklund transformations for $H_{VI}$ are obtained as the gauge transformations

$$s_k \psi = G_k \psi \quad (k = 0, 1, 2, 3, 4), \quad r_k \psi = \Gamma_k \psi \quad (k = 1, 3, 4)$$

(53)

of the linear problem \([35]\), defined by certain matrices $G_k, \Gamma_k$ in the loop group of

$$SO(8) = \{ X \in SL(8; \mathbb{C}) \mid X^4 JX = J \}.$$

(54)

The matrices $G_k$ and $\Gamma_k$ will be specified below.

**Theorem 3.3** The Bäcklund transformations $s_k$ and $r_k$ for $H_{VI}$ are recovered from the compatibility conditions

$$s_k(M) = G_k MG_k^{-1} - z\partial_z(G_k)G_k^{-1}, \quad s_k(B) = G_k BG_k^{-1} + \partial_z(G_k)G_k^{-1}, \quad (55)$$

and

$$r_k(M) = \Gamma_k MG_k^{-1} - z\partial_z(\Gamma_k)\Gamma_k^{-1}, \quad r_k(B) = \Gamma_k BG_k^{-1} + \partial_z(\Gamma_k)\Gamma_k^{-1}. \quad (56)$$
In (53), the matrices $G_k$ are determined as

$$
G_0 = 1 + \frac{\alpha_0}{q-t} F_0, \quad G_1 = 1 + \alpha_1 F_1, \quad G_2 = 1 - \frac{\alpha_2}{p} F_2, \\
G_3 = 1 + \frac{\alpha_3}{q-1} F_3, \quad G_4 = 1 + \frac{\alpha_4}{q} F_4.
$$

(57)

The matrices $\Gamma_1, \Gamma_3$ and $\Gamma_4$ are given explicitly by

$$
\Gamma_1 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & \frac{q-t}{\sqrt{t(t-1)}} & \frac{1}{q-t} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{\sqrt{t(t-1)}}{q-t} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{q-t}{\sqrt{t(t-1)}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{t(t-1)}}{q-t} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{t(t-1)}}{q-t} & 0 \\
z\sqrt{t(t-1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix},
$$

(58)

$$
\Gamma_3 = \begin{bmatrix}
0 & 0 & 0 & \frac{1}{\sqrt{-t}\sqrt{z}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{q}{\sqrt{-t}\sqrt{z}} & \frac{1}{\sqrt{-t}\sqrt{z}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{-t}\sqrt{z}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{-t}\sqrt{z}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{-t}\sqrt{z}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{-t}\sqrt{z}} & 0 & 0 \\
\end{bmatrix},
$$

(59)

$$
\Gamma_4 = \begin{bmatrix}
0 & 0 & 0 & 0 & \frac{1}{\sqrt{1-t}\sqrt{z}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{1-t}\sqrt{z}} & \frac{1}{\sqrt{1-t}\sqrt{z}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{1-t}\sqrt{z}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{1-t}\sqrt{z}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{1-t}\sqrt{z}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{1-t}\sqrt{z}} & 0 \\
\end{bmatrix}.
$$

(60)
We remark that the matrices $\Gamma_k$ ($k = 1, 3, 4$) are expressed as
\[
\begin{align*}
\Gamma_1 &= D(a_1) \exp \left( \frac{1}{q^2} F_2 \right) z^{-\infty_1} C_1, \\
\Gamma_3 &= D(a_2) \exp \left( \frac{1}{q} F_2 \right) z^{-\infty_3} C_3, \\
\Gamma_4 &= D(a_3) \exp \left( \frac{1}{q} F_2 \right) z^{-\infty_4} C_4,
\end{align*}
\]
where we have used the notation
\[
D(a) = \text{diag}(a_1, a_2, a_3, a_4, a_4^{-1}, a_3^{-1}, a_2^{-1}, a_1^{-1})
\]
for a vector $a = (a_1, a_2, a_3, a_4)$, $a_j \neq 0$. In (61), $z^{\infty_k}$ denote the following diagonal matrices associated with the fundamental weights of $\mathfrak{so}(8)$:
\[
\begin{align*}
z^{\infty_1} &= z^{H(1,0,0,0)} = D(z, 1, 1, 1), \\
z^{\infty_2} &= z^{H(1,1,0,0)} = D(z, z, 1, 1), \\
z^{\infty_3} &= z^{H(1,1,1,0)} = (z^{\frac{1}{2}}, z^{\frac{1}{2}}, z^{\frac{1}{2}}, z^{-\frac{1}{2}}), \\
z^{\infty_4} &= z^{H(1,1,0,1)} = (z^{\frac{1}{2}}, z^{\frac{1}{2}}, z^{\frac{1}{2}}, z^{\frac{1}{2}}).
\end{align*}
\]
\[\text{The matrices } C_k \text{ are essentially permutation matrices; with the notation of permutation matrices } S_\sigma = (\delta_{\sigma(i),j})_{i,j=1}^8 \text{ for } \sigma \in S_8, \]
\[\begin{align*}
C_1 &= S_{(18)(45)}, \\
C_3 &= D(1, -1, 1, -1) S_{(14)(26)(37)(58)}, \\
C_4 &= D(1, -1, -1, -1) S_{(15)(26)(37)(48)}.
\end{align*}\]
The matrices $D(a_1), D(a_3), D(a_4)$ in (62) are defined by
\[
\begin{align*}
a_1 &= \begin{pmatrix} 1 & -\sqrt{t(t-1)} & \sqrt{t(t-1)} & 1 \\
1 & q & -q & 1 \\
1 & q & -q & 1 \\
1 & q & -q & 1 \\
\end{pmatrix}, \\
a_3 &= \begin{pmatrix} q & 1 & -q^{-1} & 1 \\
1 & q^{-1} & -q & 1 \\
1 & q & -q & 1 \\
1 & q & -q & 1 \\
\end{pmatrix}, \\
a_4 &= \begin{pmatrix} q & 1 & -q^{-1} & 1 \\
1 & q^2 & -q & 1 \\
1 & q & -q & 1 \\
1 & q & -q & 1 \\
\end{pmatrix}.
\end{align*}
\]
We also remark that, for each $k = 1, 3, 4$, the adjoint action of the matrix $z^{-\infty_1} C_k$ induces the automorphism of the loop algebra $\mathfrak{so}(8)[z, z^{-1}]$ corresponding to the diagram automorphism $\sigma_1 = (01)(34), \sigma_3 = (03)(14)$ or $\sigma_4 = (04)(13)$, respectively. The remaining part of $\Gamma_k$ concerns the normalization of the matrices $M$ and $B$. Note that the system of differential equations (35) has a regular singularity at $z = 0$ with exponents $\pm \varepsilon_j$ ($j = 1, 2, 3, 4$), and an irregular singularity at $z = \infty$. Assuming that $\varepsilon_j$ are generic, let us take a fundamental system of solutions $\Psi = \Psi(z, t)$ of (35) with normalization such that
\[
\Psi(z, t) = \sum_{n=0}^{\infty} \Psi_n(t) z^{-H(z) + n}
\]
around $z = 0$, and that $\Psi_0(t)$ is upper triangular; such a $\Psi$ is determined up to
the multiplication of constant diagonal matrices. Then, for each $k = 0, 1, 2, 3, 4$,
the Bäcklund transformation $s_k$ is interpreted as the transformation

$$\Psi \mapsto \tilde{\Psi} = G_k \Psi S_k$$

(67)
of the fundamental system of solutions, where $S_k = \exp(-E_k) \exp(F_k) \exp(-E_k)$
denote a lift of $s_k$ to the loop group of $SO(8)$. Similarly, for each $k = 1, 3, 4$,
the Bäcklund transformation $r_k$ is interpreted as the transformation

$$\Psi \mapsto \tilde{\Psi} = \Gamma_k \Psi R_k,$$

(68)

where $R_k = z^{-\omega_k} C_k$.

**Remark 3.4** The system of differential equations (65) can be equivalently
rewritten into a chain of systems of rank 2. We first extend the indexing set for
$\psi_i$ and $\varepsilon_i$ to $\mathbb{Z}$ by imposing the periodicity condition

$$\psi_{i+8} = z \psi_i, \quad \varepsilon_{i+8} = \varepsilon_i - 1, \quad \varepsilon_{9-i} = -\varepsilon_i$$

for $i \in \mathbb{Z}$. (69)

Then (65) is equivalent to a system for the 2-vectors $\vec{\psi}_i = (\psi_{2i}, \psi_{2i+1})^t$ $i \in \mathbb{Z}$
in the following form:

$$(z \partial_z + M_i) \vec{\psi}_i + N_i \vec{\psi}_{i+1} = 0, \quad (-\partial_t + A_i) \vec{\psi}_i + B_i \vec{\psi}_{i+1} = 0,$$

(70)

where $M_i, N_i, A_i, B_i$ are $2 \times 2$ matrices whose entries depend only on $t$. This
system is formally transformed into

$$(\lambda + M_i) \vec{\varphi}_i + N_i \vec{\varphi}_{i+1} = 0, \quad (-\partial_t + A_i) \vec{\varphi}_i + B_i \vec{\varphi}_{i+1} = 0,$$

(71)

where $\vec{\varphi}_i = \vec{\varphi}_i(\lambda, t)$, by the change of coordinates $z = e^w$ and the Laplace
transformation $\partial_w \leftrightarrow \lambda$, $w \leftrightarrow -\partial_\lambda$. Noting that $N_i$ are invertible, one can rewrite (71) as

$$\partial_t \vec{\varphi}_i = U_i \vec{\varphi}_i, \quad \vec{\varphi}_{i+1} = W_i \vec{\varphi}_i$$

for $i \in \mathbb{Z}$, (72)

where

$$U_i = A_i - B_i N_i^{-1} (\lambda + M_i), \quad W_i = -N_i^{-1} (\lambda + M_i)$$

(73)

This type of $2 \times 2$ nonlinear chains is investigated in [1] in relation to Painlevé
equations. It is not clear yet, however, how our system (72) can be related to
the one employed there for obtaining $P_{VI}$.

In this paper, we have presented a new Lax pair for the sixth Painlevé
equation in the framework of the loop algebra $\mathfrak{so}(8)[z, z^{-1}]$ of type $D_4^{(1)}$. We
also explained how the affine Weyl group symmetry of $P_{VI}$ can be obtained from
the linear problem. We expect that the Lax pair discussed in this paper could
be applied as well to other problems concerning Painlevé equations. Also, it
would be an important problem to understand properly the relationship of our
representation with various approaches to the sixth Painlevé equation as in [2].
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