The quantum emission spectra of rapidly-rotating Kerr black holes: discrete or continuous?

Shahar Hod

The Ruppin Academic Center, Emeq Hefer 40250, Israel
and
The Hadassah Institute, Jerusalem 91010, Israel
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Bekenstein and Mukhanov (BM) have suggested that, in a quantum theory of gravity, black holes may have discrete emission spectra. Using the time-energy uncertainty principle they have also shown that, for a (non-rotating) Schwarzschild black hole, the natural broadening $\delta \omega$ of the black-hole emission lines is expected to be small on the scale set by the characteristic frequency spacing $\Delta \omega$ of the spectral lines: $\zeta_{\text{Sch}} \equiv \delta \omega / \Delta \omega \ll 1$. BM have therefore concluded that the expected discrete emission lines of the quantized Schwarzschild black hole are unlikely to overlap. In this paper we calculate the characteristic dimensionless ratio $\zeta(\bar{a}) \equiv \delta \omega / \Delta \omega$ for the predicted BM emission spectra of rapidly-rotating Kerr black holes (here $\bar{a} \equiv J/M^2$ is the dimensionless angular momentum of the black hole). It is shown that $\zeta(\bar{a})$ is an increasing function of the black-hole angular momentum. In particular, we find that the quantum emission lines of Kerr black holes in the regime $\bar{a} \gtrsim 0.9$ are characterized by the dimensionless ratio $\zeta(\bar{a}) \gtrsim 1$ and are therefore effectively blended together. Our results thus suggest that, even if the underlying mass (energy) spectrum of these rapidly-rotating Kerr black holes is fundamentally discrete as suggested by Bekenstein and Mukhanov, the natural broadening phenomenon (associated with the time-energy uncertainty principle) is expected to smear the black-hole radiation spectrum into a continuum.

I. INTRODUCTION

Analyzing the quantum properties of fundamental fields in classical black-hole spacetimes, Hawking [1] has revealed that black holes are actually not completely black. In particular, according to Hawking’s result, semi-classical black holes are characterized by continuous evaporation spectra in which the emitted field quanta have the familiar black-body statistical distribution with a well defined temperature [1, 2]. This intriguing theoretical prediction is certainly one of the most important outcomes of the interplay between quantum field theory and classical general relativity.

It is important to realize, however, that Hawking’s seminal analysis [1] has a semi-classical nature: while the fundamental fields are properly analyzed at the quantum level, the curved black-hole spacetime is treated as a classical background. Although we do not have a self-consistent theory of quantum gravity, it is natural to expect that some modifications to the (semi-classically predicted) Hawking emission spectrum [1] may arise if the black-hole spacetime itself would properly be treated as a dynamical quantum entity [3].

An interesting heuristic quantization of the black-hole area (mass) spectrum was proposed long ago by Bekenstein [3]. In his influential work, Bekenstein has pointed out that the black-hole surface area behaves as a classical adiabatic invariant [3]. Since classical adiabatic invariants are usually related to physical quantities with discrete quantum spectra [4], Bekenstein has suggested that the black-hole surface area has a discrete (quantum) spectrum of the form [3, 5]

$$A_n = 4\gamma \hbar \cdot n \quad ; \quad n = 1, 2, 3, \ldots$$

Here $\gamma$ is a dimensionless constant of order unity. Three possible values of $\gamma$ are often used in the literature: $\gamma = 2\pi$, $\gamma = \ln 2$ [6, 7], and $\gamma = \ln 3$ [8]. It is worth mentioning that, using different quantization schemes, several authors (see [6–27] and references therein) have re-derived the uniformly spaced black-hole area spectrum [1].

The discrete black-hole area spectrum [1] also implies a discrete mass (energy) spectrum $\{M_n\}$ for quantum black holes. Thus, as pointed out in [3], a quantized black hole is expected to be characterized by a discrete line emission. In particular, Bekenstein and Mukhanov (BM) [4] have advocated the idea that, within the framework of a quantum theory of gravity, the radiation emitted by a quantized Schwarzschild black hole of mass $M_n$ should be at integer multiples of the fundamental frequency [3, 6]

$$\omega_0 \equiv (M_n - M_{n-1})/\hbar = \frac{\gamma T_{\text{BH}}^{\text{Sch}}}{\hbar} ,$$

where

$$T_{\text{BH}}^{\text{Sch}} = \frac{\hbar}{8\pi M}$$

$$\zeta_{\text{Sch}} \equiv \delta \omega / \Delta \omega = \frac{\hbar}{8\pi M} .$$
is the Bekenstein-Hawking temperature \[ T_{\text{BH}} \] of the Schwarzschild black hole.

According to the quantization scheme presented in \[ 3,6 \], the decay of a quantized Schwarzschild black hole of mass \( M_n \) into the \( M_{n-k} \) mass level \[ 28 \] is accompanied by the emission of a field quantum of frequency \( \omega_k = (M_n - M_{n-k})/\hbar = k \cdot \omega_0 \) [see Eq. (1)]. This implies that the discrete line emission \( \{\omega_0, 2\omega_0, 3\omega_0, \ldots\} \) suggested by BM \[ 3,6 \] for a quantized Schwarzschild black hole is characterized by the constant frequency spacing

\[
\Delta \omega^\text{Sch} = \omega_0
\]

between adjacent emission lines.

II. THE NATURAL BROADENING OF THE SCHWARZSCHILD SPECTRAL LINES

The interesting question of the \textit{natural broadening} of the Schwarzschild black-hole emission lines has been addressed by BM \[ 6 \] (see also \[ 29,30 \]). Arguing from the time-energy uncertainty principle, it was suggested in \[ 6 \] to estimate the natural broadening \( \delta \omega \) of the Schwarzschild black-hole spectral lines from the relation \( \delta \omega = 1/\tau \) \[ 4 \], where \( \tau \) is the characteristic lifetime (as measured by asymptotic observers) of the \( n \)th black-hole mass (energy) level \[ 31 \].

As suggested by BM \[ 6 \], the characteristic lifetime, \( \tau \), of the Schwarzschild \( n \)th mass level can be estimated from the relation \( \tau = (dN/dt)^{-1} \), where \( dN/dt \) is the semi-classical emission rate of the black hole \[ 1,32-35 \]. For the emission of (massless) gravitons and photons from a Schwarzschild black hole one finds the emission rate \( (dN/dt)^\text{Sch} \simeq 1.6 \times 10^{-4} M^{-1} \) \[ 32 \], which implies \( \tau^\text{Sch} \simeq 6 \times 10^4 M \) \[ 36 \] for the survival time (lifetime) of the \( n \)th Schwarzschild mass level. Using the time-energy uncertainty relation, \( \delta \omega = 1/\tau \) \[ 4 \], one finds

\[
\delta \omega^\text{Sch} \simeq 1.6 \times 10^{-4} M^{-1}
\]

for the characteristic natural broadening of the Schwarzschild spectral lines.

Taking cognizance of Eqs. \[ 2-5 \], one concludes that, for a Schwarzschild black hole \[ 6 \], the frequency spacing \( \Delta \omega^\text{Sch} \) and the natural width \( \delta \omega^\text{Sch} \) of the spectral lines are well separated in magnitude. In particular, one finds [see Eqs. \( 2-5 \)]

\[
\zeta^\text{Sch} \equiv \frac{\delta \omega^\text{Sch}}{\Delta \omega^\text{Sch}} \simeq 4 \times 10^{-3} \ll 1.
\]

The strong inequality \[ 6 \] implies that the discrete Schwarzschild black-hole emission lines \( \{\omega_0, 2\omega_0, 3\omega_0, \ldots\} \) suggested by BM \[ 3,6 \] are unlikely to overlap.

III. THE SPECTRAL EMISSION LINES OF ROTATING KERR BLACK HOLES AND THEIR NATURAL BROADENING

In this paper we shall generalize the analyzes of \[ 6,29,30 \] to the regime of \textit{rotating} Kerr black holes. In particular, our main goal is to study the dependence of the fundamental dimensionless ratio \( \zeta(\bar{a}) \equiv \delta \omega/\Delta \omega \) on the black-hole rotation parameter \( \bar{a} \equiv J/M^2 \) \[ 37 \].

A. The quantized emission spectrum of the Kerr black hole

The Bekenstein-Hawking temperature \( T_{\text{BH}} \) and the angular velocity \( \Omega_H \) of a rotating Kerr black hole are respectively given by the relations \[ 1,3,5 \]

\[
T_{\text{BH}} = \frac{\hbar (r_+ - r_-)}{4\pi (r_+^2 + a^2)} \quad \text{and} \quad \Omega_H = \frac{a}{2Mr_+},
\]

where

\[
r_\pm = M \pm (M^2 - a^2)^{1/2}
\]

are the black-hole horizon radii.
Taking cognizance of the discrete black-hole area spectrum (1) suggested by Bekenstein [3], and using the first law of black-hole mechanics [38]

\[ \Delta M = \frac{1}{4\hbar} T_{\text{BH}} \Delta A + \Omega_{\text{H}} \Delta J , \]  

one finds that a quantized rotating Kerr black hole is expected to be characterized by the discrete emission frequencies [39]

\[ \omega_{k,m} = \frac{\gamma T_{\text{BH}}}{\hbar} \cdot k + m \Omega_{\text{H}} ; \quad k \in \mathbb{Z} . \]  

Here \( m \) is the azimuthal harmonic index of the emitted field quanta. Note that the discrete black-hole radiation spectrum (10) is characterized by the frequency spacing [40]

\[ \Delta \omega = \omega_{k+1,m} - \omega_{k,m} = \frac{\gamma T_{\text{BH}}}{\hbar} . \]  

B. The natural broadening of the Kerr spectral lines

Following the Bekenstein-Mukhanov analysis of the Schwarzschild black-hole emission spectrum presented in [6], we shall now use the time-energy uncertainty principle [4] in order to estimate the natural broadening \( \delta \omega = 1/\tau \) of the Kerr black-hole emission lines. In particular, following [6] we shall assume that the characteristic lifetime \( \tau \) of the Kerr \( n \)th mass level (that is, the average time gap between quantum leaps) can be estimated as the reciprocal of the semi-classical [1, 32–35] black-hole emission rate [see Eq. (15) below] [41].

The semi-classical emission rate of a rotating Kerr black hole (that is, the number of quanta emitted from the black hole per unit of time) is given by the Hawking relation [1, 32, 42]

\[ \dot{N} \equiv \frac{dN}{dt} = \frac{1}{2\pi} \sum_{s,l,m} \int_0^\infty d\omega \frac{\Gamma}{e^{\hbar(\omega-m\Omega_{\text{H}})/T_{\text{BH}}} - 1} . \]  

Here \( s, m \) and \( l \geq \max(s, |m|) \) are respectively the spin parameter and the harmonic indices (azimuthal and spheroidal) of the emitted field quanta. The energy-dependent grey-body factors (absorption probabilities) \( \Gamma = \Gamma_{slm}(\omega; \bar{a}) \) [32] in (12) quantify the interaction of the emitted field quanta with the effective curvature potential that surrounds the emitting black hole.

The dependence of the semi-classical emission rate \( \dot{N}(\bar{a}) \) [see Eq. (12)] on the black-hole rotation parameter \( \bar{a} \) can be computed along the lines of the numerical procedure described in [32]. In particular, one finds that, in the regime of rapidly-rotating Kerr black holes, the semi-classical radiation spectrum of a massless spin-\( s \) field is greatly dominated by the [32, 43]

\[ l = m = s \]  

angular mode. In addition, the characteristic thermal (exponential) factor that appears in the expression (12) for the black-hole emission rate implies that, in the regime of rapidly-rotating near-extremal, \( T_{\text{BH}} \to 0 \) black holes, the emission of high energy quanta with \( \omega > m \Omega_{\text{H}} \) is exponentially suppressed. Thus, the semi-classical emission spectra of rapidly-rotating Kerr black holes are dominated by field quanta in the energy interval [44]

\[ 0 \leq \omega \lesssim m \Omega_{\text{H}} + O(T_{\text{BH}}/\hbar) . \]  

As discussed above, following [6] we shall assume that the lifetime \( \tau \) of the meta-stable black-hole state (that is, the average time gap between the emissions of successive black-hole quanta) is given by the reciprocal of the black-hole semi-classical emission rate [32]. Namely,

\[ \tau = \frac{1}{\dot{N}} , \]  

where \( \dot{N} \) is given by Eq. (12). Using the time-energy uncertainty principle [4], one finds (see also [6])

\[ \delta \omega = \frac{1}{\tau} = \dot{N} . \]
for the natural broadening of the black-hole emission lines.

In Table I we display the dimensionless ratio \( \zeta(\tilde{a}) \equiv \frac{\delta \omega}{\Delta \omega} \) [45] for the emission of massless gravitons \((s = 2)\) and photons \((s = 1)\) [13, 46] by rapidly-rotating Kerr black holes [Here the natural broadening \( \delta \omega \) of the black-hole spectral lines is given by Eqs. (12) and (16), and the characteristic frequency spacing \( \Delta \omega \) between adjacent emission lines [40] is given by (11)]. One finds that \( \zeta(\tilde{a}) \) is an increasing function of the dimensionless black-hole rotation parameter \( \tilde{a} \). In particular, we find that rapidly-rotating Kerr black holes in the regime \( \tilde{a} \gtrsim 0.9 \) are characterized by the relation \[ \delta \omega \gtrsim \Delta \omega. \] (17)
The inequality (17) implies that the emission lines of rapidly-rotating Kerr black holes are effectively blended together.

| \( \tilde{a} \equiv J/M^2 \) | 0.80 | 0.90 | 0.96 | 0.99 | 0.999 |
| \( \delta \omega/\Delta \omega \) | 0.12 | 0.39 | 1.34 | 5.39 | 27.98 |

TABLE I: The characteristic dimensionless ratio \( \zeta(\tilde{a}) \equiv \frac{\delta \omega}{\Delta \omega} \) of rapidly-rotating Kerr black holes. Here \( \delta \omega \) is the natural broadening of the black-hole emission lines [see Eqs. (12) and (16)] and \( \Delta \omega \) is the characteristic frequency spacing between adjacent emission lines [see Eq. (11)]. One finds that \( \zeta(\tilde{a}) \) is an increasing function of the black-hole angular momentum. In particular, we find that the dimensionless ratio \( \delta \omega/\Delta \omega \) becomes of order unity at \( \tilde{a} \simeq 0.9 \) [47, 48].

IV. SUMMARY

Starting with the seminal work of Bekenstein [3], many authors (see [6–27] and references therein) have predicted the existence of a uniformly spaced area spectrum [see Eq. (1)] for quantized black holes. This intriguing prediction suggests that, in a quantum theory of gravity [49], black holes may have discrete emission spectra.

Using the time-energy uncertainty principle [4], Bekenstein and Mukhanov [6] have shown that, for (non-rotating) Schwarzschild black holes, the natural broadening \( \delta \omega^{\text{Sch}} \) of the spectral lines is small on the scale set by the characteristic frequency spacing \( \Delta \omega^{\text{Sch}} \) of the lines: \( \delta \omega^{\text{Sch}}/\Delta \omega^{\text{Sch}} \ll 1 \). It was therefore concluded in [6] that the discrete emission lines which are expected to characterize a quantized Schwarzschild black hole are unlikely to overlap. In other words, the Schwarzschild line spectrum is expected to be sharp.

Motivated by this important conclusion, in this paper we have analyzed the natural broadening of the emission lines which, according to [3], are expected to characterize quantized rotating Kerr black holes. In particular, we have studied the dependence of the dimensionless ratio \( \zeta(\tilde{a}) \equiv \frac{\delta \omega}{\Delta \omega} \) on the black-hole rotation parameter \( \tilde{a} \equiv J/M^2 \). It was shown that \( \zeta(\tilde{a}) \) is an increasing function of the black-hole angular momentum. In particular, one finds that the quantum emission lines of rapidly-rotating Kerr black holes in the regime \( \tilde{a} \gtrsim 0.9 \) are characterized by the dimensionless ratio \( \zeta(\tilde{a}) \gtrsim 1 \) [47, 48]. These emission lines are therefore effectively blended together.

Our results thus suggest that, even if the underlying mass (energy) spectrum of these rapidly-rotating Kerr black holes is fundamentally discrete as suggested by Bekenstein and Mukhanov, the natural broadening phenomenon (associated with the time-energy uncertainty principle [4]) is expected to smear the black-hole emission spectrum into a continuum.

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As emphasized earlier [39], only positive frequency (energy) modes (as measured by asymptotic observers) can be emitted by the black hole (that is, can reach infinity). Thus, the allowed values of the discrete parameters \( \{k, m\} \) are restricted by the requirement

\[
\omega_{k,m} = \gamma T_{BH}/h \cdot k + m \Omega_{BH} \geq 0
\]

We are interested in rapidly-rotating Kerr black holes with \( a = O(M) \). These black holes are characterized by the relation [see Eq. (7)]

\[
T_{BH}/h \ll \Omega_{BH}
\]

Note that this strong inequality implies [see Eq. (10)]

\[
\omega_{k,m+1} - \omega_{k,m} \ll \omega_{k,m+1} - \omega_{k,m}
\]

In particular, one finds that, for rapidly-rotating black holes, the semi-classical emission spectra are dominated by the gravitational (\( s = 2 \)) field quanta with \( l = m = s = 2 \). Note that the exponential factor that appears in the denominator of (12) implies that, for a given value of \( \omega \), the emission of modes with \( m \Omega_{BH} < 0 \) is exponentially suppressed as compared to the emission of modes with \( m \Omega_{BH} > 0 \). In particular, for rapidly-rotating black holes with \( \Omega_{BH}/T_{BH} \gg 1 \), the suppression in the emission of modes with negative values of \( m \Omega_{BH} \) is described by the exponential factor

\[
\sim e^{-h|m\Omega_{BH}|/T_{BH}} \ll 1
\]

Here we take \( \gamma = 1 \) in the expression (11) for \( \Delta \omega \). This order of magnitude is consistent with the most commonly used values of the dimensionless parameter \( \gamma \), see [3, 6].

We assume that the emitting black hole is large enough (that is, cold enough [see Eq. (7)]) such that the emission of massive particles of masses \( M \mu \gg h \) is exponentially suppressed [32]. [Note that in order to reach spatial infinity (that
is, in order to be in an unbound state), the energy of emitted massive quanta should be characterized by the inequality \( \hbar \omega \geq \mu \). Taking cognizance of Eqs. (12) and (14), one finds that the emission of such quanta (that is, massive quanta in the regime \( \hbar \omega \geq \mu \gg \hbar / M \)) is exponentially suppressed]. It is worth emphasizing that extending the family of the emitted particles (for example, taking into account the emission of massive neutrinos from the black hole) would decrease the lifetime \( \tau \) of the black-hole meta-stable state, and thus would increase the natural broadening \( \delta \omega \) of the black-hole emission lines [see Eq. (15)]. This effect would merely support our final conclusion [see Eq. (17) below] that the emission lines of rapidly-rotating Kerr black holes are effectively blended together.

[47] Note that the exact value of \( \tilde{a} \) at which \( \delta \omega / \Delta \omega = 1 \) depends on the chosen value \[3, 6–8\] of the dimensionless parameter \( \gamma \).

[48] In fact, from (11) one finds the asymptotic relation \( \zeta(\tilde{a} \to 1) \to \infty \). Note that this asymptotic behavior is independent of the value of the dimensionless parameter \( \gamma \).

[49] It is worth emphasizing that by “a quantum theory of gravity” we mean a theory in which the black hole itself is properly quantized. This should be contrasted with the semi-classical analysis of Hawking [1], in which the matter fields are properly analyzed at the quantum level, but the black-hole spacetime itself is treated as a classical background.