Gazeau-Klauder type coherent states for hypergeometric type operators

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Abstract. The hypergeometric type operators are shape invariant, and a factorization into a product of first order differential operators can be explicitly described in the general case. Some additional shape invariant operators depending on several parameters are defined in a natural way by starting from this general factorization. The mathematical properties of the eigenfunctions and eigenvalues of the operators thus obtained depend on the values of the involved parameters. We study the parameter dependence of orthogonality, square integrability and of the monotony of eigenvalue sequence. The obtained results allow us to define certain systems of Gazeau-Klauder coherent states and to describe some of their properties. Our systematic study recovers a number of well-known results in a natural unified way and also leads to new findings.
1. Introduction

Many problems in quantum mechanics and mathematical physics lead to equations of the type

$$\sigma(s)y''(s) + \tau(s)y'(s) + \lambda y(s) = 0$$  \hspace{1cm} (1)

where $\sigma(s)$ and $\tau(s)$ are polynomials of at most second and first degree, respectively, and $\lambda$ is a constant. These equations are usually called equations of hypergeometric type [13], and each of them can be reduced to the self-adjoint form

$$[\sigma(s)\varrho(s)y'(s)]' + \lambda \varrho(s)y(s) = 0$$  \hspace{1cm} (2)

by choosing a function $\varrho$ such that $(\sigma \varrho)' = \tau \varrho$. The equation (1) is usually considered on an interval $(a, b)$, chosen such that $\sigma(s) > 0$ for all $s \in (a, b)$, $\varrho(s) > 0$ for all $s \in (a, b)$ and

$$\lim_{s \to a} \sigma(s)\varrho(s) = \lim_{s \to b} \sigma(s)\varrho(s) = 0.$$  \hspace{1cm} (3)

Since the form of the equation (1) is invariant under a change of variable $s \mapsto cs + d$, it is sufficient to analyze the cases presented in table 1. Some restrictions are imposed on $\alpha$ and $\beta$ in order that the interval $(a, b)$ exists. One can remark that the second derivative $\sigma''$ of $\sigma$ belongs to $\{0, -2, 2\}$.

| $\sigma(s)$ | $\tau(s)$ | $\varrho(s)$ | $(a, b)$ | $\alpha, \beta$ |
|----------|----------|-----------|--------|----------------|
| 1        | $\alpha s + \beta$ | $e^{\alpha s^2/2 + \beta s}$ | $(-\infty, \infty)$ | $\alpha < 0$ |
| $s$      | $\alpha s + \beta$ | $s^{\beta - 1}e^{\alpha s}$ | $(0, \infty)$ | $\alpha < 0$, $\beta > 0$ |
| $1 - s^2$| $\alpha s + \beta$ | $(1 + s)^{-(\alpha - \beta)/2 - 1}(1 - s)^{-(\alpha + \beta)/2 - 1}$ | $(-1, 1)$ | $\alpha < \beta < -\alpha$ |
| $s^2 - 1$| $\alpha s + \beta$ | $(s + 1)^{-(\alpha - \beta)/2 - 1}(s - 1)^{-(\alpha + \beta)/2 - 1}$ | $(1, \infty)$ | $-\beta < \alpha < 0$ |
| $s^2$    | $\alpha s + \beta$ | $s^{\alpha - 2}e^{-\beta/s}$ | $(0, \infty)$ | $\alpha < 0$, $\beta > 0$ |
| $s^2 + 1$| $\alpha s + \beta$ | $(1 + s^2)^{\alpha/2 - 1}e^{\beta \arctan s}$ | $(-\infty, \infty)$ | $\alpha < 0$ |

In the paper [13] our aim was to analyze together in a unified formalism all the cases arising by imposing the conditions [3]. The systems of polynomials $\Phi_l$ obtained in this way can be expressed in terms of the classical orthogonal polynomials, but in some cases we have to consider the classical polynomials for complex values of parameters or outside the interval where they are orthogonal. The literature discussing special function theory and its application to mathematical and theoretical physics is vast, and there is a multitude of different conventions concerning the definition of functions. A unified approach is not possible without a unified definition. The associated special functions we define as

$$\Phi_{l,m}(s) = \kappa^m(s) \frac{d^m}{ds^m} \Phi_l(s) \quad \text{with} \quad \kappa(s) = \sqrt{\sigma(s)}$$  \hspace{1cm} (4)
are eigenfunctions of the hypergeometric type operators
\[ H_m = -\sigma(s)\frac{d^2}{ds^2} - \tau(s)\frac{d}{ds} + \frac{m(m-2)(\sigma'(s))^2}{4} + \frac{m\tau(s)\sigma'(s)}{2\sigma(s)} - \frac{1}{2}m(m-2)\sigma''(s) - m\tau'(s) \]

depending on parameters \(\alpha, \beta\) and \(m\). The operators \(H_m\) satisfy the relations
\[ H_m - \lambda_m = A_m^+ A_m \quad H_m A_m^+ = A_m^+ H_{m+1} \]
\[ H_{m+1} - \lambda_m = A_m^+ A_m \quad A_m H_m = H_{m+1} A_m \]
where \(\lambda_m = -\frac{\sigma''}{2}m(m-1) - \tau' m\) and
\[ A_m = \kappa(s)\frac{d}{ds} - m\kappa'(s) \quad A_m^+ = -\kappa(s)\frac{d}{ds} - \frac{\tau(s)}{\kappa(s)} - (m-1)\kappa'(s). \]

In the particular case when \(\alpha\) and \(\beta\) are such that \(\varphi\) is a power of \(\sigma\) the factorization (6) allows one to prove that the operators \(\tilde{H}_m = H_m - \delta \kappa'(s)\)

depending on an additional parameter \(\delta\) satisfy some relations similar to (6). Certain results concerning the operators \(\tilde{H}_m\) can be found in [7, 11]. The mathematical properties of the operators \(\tilde{H}_m\) depend on the involved parameters, and our main purpose is to investigate this dependence. Our approach is based on the similitude existing between \(H_m\) and \(\tilde{H}_m\). Therefore, we firstly review certain properties of \(H_m\) in a form suitable for our purpose. If the involved parameters satisfy certain restrictions, some Gazeau-Klauder systems of coherent states can be associated in a natural way. We consider the notion of Gazeau-Klauder coherent states in a larger sense. The measure we use is not positive in all the considered cases. In the resolution of identity, certain states bring a positive contribution while the others bring a negative contribution.

It is well-known that the hypergeometric type operators \(H_m\) and \(\tilde{H}_m\) are directly related to some exactly solvable Schrödinger type equations including a large number of potentials (shifted oscillator, three-dimensional oscillator, Pöschl-Teller, generalized Pöschl-Teller, Morse, Scarf hyperbolic, Coulomb, trigonometric Rosen-Morse, Eckart, hyperbolic Rosen-Morse) and their supersymmetric partners. Most of the mathematical formulae occurring in the study of the exactly solvable quantum systems follow from a small number of formulae concerning the hypergeometric type operators. Particularly, almost any factorization used in quantum mechanics for a Schrödinger operator follows through a change of variable [10, 5, 7] from the explicitly described factorization (6).

2. Polynomials of hypergeometric type

It is well-known [13] that for \(\lambda = \lambda_l\), where \(l \in \mathbb{N}\) and
\[ \lambda_l = -\frac{\sigma''}{2}l(l-1) - \tau' l = \begin{cases} -\alpha l & \text{if } \sigma'' = 0 \\ l(l - \alpha - 1) & \text{if } \sigma'' = -2 \\ l(1 - \alpha - l) & \text{if } \sigma'' = 2 \end{cases} \]
the equation (11) admits a polynomial solution $\Phi_l = \Phi_l^{(\alpha, \beta)}$ of at most $l$ degree

$$\sigma(s)\Phi''_l + \tau(s)\Phi'_l + \lambda_l\Phi_l = 0.$$  \hspace{1cm} (10)

**Theorem 1.** The function $\Phi_l(s)\sqrt{\varrho(s)}$ is square integrable on $(a, b)$ and

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_l$$

for any $l < \Lambda$, where

$$\Lambda = \begin{cases} \infty & \text{if } \sigma'' \in \{0, -2\} \\ \frac{1-\alpha}{2} & \text{if } \sigma'' = 2. \end{cases}$$  \hspace{1cm} (11)

**Proof.** The convergence of the integral $\int_a^b |\Phi_l(s)|^2\varrho(s)ds$ follows from the restrictions imposed to $\alpha$ and $\beta$ (see table 1) and from the relation

$$\left\{ \gamma \mid \lim_{s \to a} \sigma(s)\varrho(s)s^\gamma = 0 \text{ and } \lim_{s \to b} \sigma(s)\varrho(s)s^\gamma = 0 \right\} = \begin{cases} [0, \infty) & \text{if } \sigma'' \in \{0, -2\} \\ [0, -\alpha) & \text{if } \sigma'' = 2. \end{cases}$$  \hspace{1cm} (12)

□

**Theorem 2.** The system of polynomials $\{\Phi_l \mid l < \Lambda\}$ is orthogonal with weight function $\varrho(s)$ in $(a, b)$, and $\Phi_l$ is a polynomial of degree $l$ for any $l < \Lambda$.

**Proof.** Let $l$, $k \in \mathbb{N}$ with $0 \leq l < k < \Lambda$. From the relations

$$[\sigma(s)\varrho(s)\Phi'_l] + \lambda_l\varrho(s)\Phi_l = 0 \quad \text{and} \quad [\sigma(s)\varrho(s)\Phi'_k] + \lambda_k\varrho(s)\Phi_k = 0$$

we get

$$(\lambda_l - \lambda_k)\Phi_l(s)\Phi_k(s)\varrho(s) = \frac{d}{ds}\left\{\sigma(s)\varrho(s)[\Phi_l(s)\Phi'_k(s) - \Phi_k(s)\Phi'_l(s)]\right\}.$$  

Since the Wronskian $W[\Phi_l(s), \Phi_k(s)] = \Phi_l(s)\Phi'_k(s) - \Phi_k(s)\Phi'_l(s)$ is a polynomial of at most $l + k - 1$ degree, from (12) it follows that

$$(\lambda_l - \lambda_k)\int_a^b \Phi_l(s)\Phi_k(s)\varrho(s)ds = \sigma(s)\varrho(s)W[\Phi_l(s), \Phi_k(s)]\big|_a^b = 0.$$  

Each $\Phi_l$ is a polynomial of at most $l$ degree and the polynomials $\{\Phi_l \mid 0 \leq l < \Lambda\}$ are linearly independent. This is possible only if $\Phi_l$ is a polynomial of degree $l$ for any $l < \Lambda$. \hspace{1cm} □

**Theorem 3.** Up to a multiplicative constant

$$\Phi_l^{(\alpha, \beta)}(s) = \begin{cases} H_l \left( \sqrt{\frac{-\alpha}{2}} s - \frac{\beta}{\sqrt{-2\alpha}} \right) & \text{if } \sigma(s) = 1 \\ L_l^{\beta-1}(-\alpha s) & \text{if } \sigma(s) = s \\ P_l^{(-\alpha/\beta+2-1, (-\alpha+\beta)/2-1)}(s) & \text{if } \sigma(s) = 1 - s^2 \\ P_l^{((\alpha-\beta)/2-1, (\alpha+\beta)/2-1)}(-s) & \text{if } \sigma(s) = s^2 - 1 \\ \left(\frac{s}{\beta}\right)^{l/2} L_l^{(\alpha-\beta)/2-1, (-\alpha-\beta)/2-1}(\frac{s}{\beta}) & \text{if } \sigma(s) = s^2 \\ i^l P_l^{((\alpha+\beta)/2-1, (\alpha-\beta)/2-1)}(is) & \text{if } \sigma(s) = s^2 + 1. \end{cases}$$  \hspace{1cm} (13)
where $H_n$, $L_n$ and $P_n^{(p,q)}$ are the Hermite, Laguerre and Jacobi polynomials, respectively.

**Proof.** In the case $\sigma(s) = s^2$ the function $\Phi^{(\alpha,\beta)}_l(s)$ satisfies the equation

$$s^2 y'' + (\alpha s + \beta) y' + [-l(l-1) - \alpha l] y = 0.$$ 

If we denote $t = \beta/s$ then the polynomial $u(t) = t^l y(\beta/t)$ satisfies the equation

$$tu'' + (-\alpha + 2 - 2l - t)u' + lu = 0$$

that is, the equation whose polynomial solution is $L_{l}^{1-\alpha-2l}(s)$ (up to a multiplicative constant). In a similar way one can analyse the other cases. □

3. Special functions of hypergeometric type

Let $l \in \mathbb{N}$, $l < \Lambda$, and let $m \in \{0,1,...,l\}$. If we differentiate (10) $m$ times then we get

$$\sigma(s) \frac{d^{m+2}}{ds^{m+2}} \Phi_l + [\tau(s) + m\sigma'(s)] \frac{d^{m+1}}{ds^{m+1}} \Phi_l + (\lambda_l - \lambda_m) \frac{d^m}{ds^m} \Phi_l = 0. \quad (14)$$

The equation obtained by multiplying this relation by $\sqrt{\sigma(s)}$ can be written as

$$H_m \Phi_{l,m} = \lambda_l \Phi_{l,m} \quad (15)$$

where $H_m$ is the differential operator

$$H_m = -\sigma(s) \frac{d^2}{ds^2} - \tau(s) \frac{d}{ds} + \frac{m(m - 2)}{4} \frac{(\sigma'(s))^2}{\sigma(s)} + \frac{m\tau(s)}{2} \frac{\sigma'(s)}{\sigma(s)} - \frac{1}{2} m(m - 2) \sigma''(s) - m\tau'(s). \quad (16)$$

and the functions

$$\Phi_{l,m}(s) = \kappa^m(s) \frac{d^m}{ds^m} \Phi_l(s) \quad (17)$$

defined by using

$$\kappa(s) = \sqrt{\sigma(s)} \quad (18)$$

are called the associated special functions.

**Theorem 4.** If $0 \leq m \leq l < \Lambda$ then $\Phi_{l,m}(s) \sqrt{\sigma(s)}$ is square integrable on $(a,b)$.

**Proof.** The convergence of the integral $\int_a^b |\Phi_{l,m}(s)|^2 \sigma(s) ds$ follows from the restrictions imposed to $\alpha$ and $\beta$ (see table 1) and from the relation (12). □

By differentiating (10) $m - 1$ times we obtain

$$\sigma(s) \frac{d^{m+1}}{ds^{m+1}} \Phi_l(s) + (m - 1)\sigma'(s) \frac{d^m}{ds^m} \Phi_l(s) + \frac{(m-1)(m-2)}{2} \sigma''(s) \frac{d^{m-1}}{ds^{m-1}} \Phi_l(s)$$

$$+ \tau(s) \frac{d^m}{ds^m} \Phi_l(s) + (m - 1)\tau'(s) \frac{d^{m-1}}{ds^{m-1}} \Phi_l(s) + \lambda_l \frac{d^{m-1}}{ds^{m-1}} \Phi_l(s) = 0.$$ 

If we multiply this relation by $\kappa^{m-1}(s)$ then we get the three term recurrence relation

$$\Phi_{l,m+1}(s) + \left(\frac{\tau(s)}{\kappa(s)} + 2(m-1)\kappa'(s)\right) \Phi_{l,m}(s) + (\lambda_l - \lambda_{m-1}) \Phi_{l,m-1}(s) = 0 \quad (19)$$
Gazeau-Klauder type coherent states for hypergeometric type operators

for \( m \in \{1, 2, \ldots, l-1\} \), and

\[
\left( \frac{\tau(s)}{\kappa(s)} + 2(l-1)\kappa'(s) \right) \Phi_{l,t}(s) + (\lambda_l - \lambda_{l-1})\Phi_{l,t-1}(s) = 0
\]  

(20)

for \( m = l \). For each \( m \in \{0, 1, \ldots, l-1\} \), by differentiating (17), we obtain

\[
\frac{d}{ds} \Phi_{l,m}(s) = m\kappa^{m-1}(s)\kappa'(s) \frac{d^m}{d^m s} \Phi_l + \kappa^m(s) \frac{d^{m+1}}{d^{m+1} s} \Phi_l
\]

that is, the relation

\[
\kappa(s) \frac{d}{ds} \Phi_{l,m}(s) = m\kappa'(s) \Phi_{l,m}(s) + \Phi_{l,m+1}(s)
\]

which can be written as

\[
\left( \kappa(s) \frac{d}{ds} - m\kappa'(s) \right) \Phi_{l,m}(s) = \Phi_{l,m+1}(s).
\]  

(21)

If \( m \in \{1, 2, \ldots, l-1\} \) then by substituting (21) into (19) we get

\[
\left( \kappa(s) \frac{d}{ds} + \frac{\tau(s)}{\kappa(s)} + (m - 2)\kappa'(s) \right) \Phi_{l,m}(s) + (\lambda_l - \lambda_{m-1})\Phi_{l,m-1}(s) = 0
\]

that is,

\[
\left( -\kappa(s) \frac{d}{ds} - \frac{\tau(s)}{\kappa(s)} - (m - 1)\kappa'(s) \right) \Phi_{l,m+1}(s) = (\lambda_l - \lambda_m)\Phi_{l,m}(s).
\]  

(22)

for all \( m \in \{0, 1, \ldots, l-2\} \). From (20) it follows that this relation is also satisfied for \( m = l - 1 \).

The relations (21) and (22) suggest us to consider for \( m + 1 < \Lambda \) the operators

\[
A_m = \kappa(s) \frac{d}{ds} - m\kappa'(s)
\]

\[
A_m^+ = -\kappa(s) \frac{d}{ds} - \frac{\tau(s)}{\kappa(s)} - (m - 1)\kappa'(s).
\]

(23)

satisfying the relations

\[
A_m \Phi_{l,m} = \begin{cases} 
0 & \text{for } l = m \\
\Phi_{l,m+1} & \text{for } m < l < \Lambda
\end{cases}
\]

(24)

\[
A_m^+ \Phi_{l,m+1} = (\lambda_l - \lambda_m)\Phi_{l,m} \quad \text{for } 0 < m < l < \Lambda.
\]

and

\[
\Phi_{l,m}(s) = \begin{cases} 
\kappa'(s) & \text{for } m = l \\
\frac{A_m^+}{\lambda_l - \lambda_m} \frac{A_{m+1}^+}{\lambda_l - \lambda_{m+1}} \cdots \frac{A_{\Lambda-1}^+}{\lambda_l - \lambda_{\Lambda-1}} \kappa'(s) & \text{for } 0 < m < l < \Lambda.
\end{cases}
\]

(25)

**Theorem 5.** The operators \( H_m \) are shape invariant

\[
H_m - \lambda_m = A_m^+ A_m \quad H_{m+1} - \lambda_m = A_m A_m^+
\]

(26)

and satisfy the intertwining relations

\[
H_m A_m^+ = A_m^+ H_{m+1} \quad A_m H_m = H_{m+1} A_m.
\]

(27)

**Proof.** Direct computation. \( \Box \)
Lemma 1. If the functions \( \varphi, \psi : (a, b) \rightarrow \mathbb{R} \) are derivable and if
\[
\lim_{s \to a} \kappa(s) g(s) \varphi(s) \psi(s) = \lim_{s \to b} \kappa(s) g(s) \varphi(s) \psi(s) = 0
\] (28)
then
\[
\langle A_m \varphi, \psi \rangle = \langle \varphi, A^+_m \psi \rangle \quad \text{and} \quad \langle A^+_m \varphi, \psi \rangle = \langle \varphi, A_m \psi \rangle.
\] (29)
If the functions \( \varphi, \psi : (a, b) \rightarrow \mathbb{R} \) are twice derivable and if
\[
\lim_{s \to a} \kappa(s) g(s) (A_m \varphi)(s) \psi(s) = \lim_{s \to b} \kappa(s) g(s) (A_m \varphi)(s) \psi(s) = 0
\] (30)
then
\[
\langle H_m \varphi, \psi \rangle = \langle \varphi, H_m \psi \rangle.
\] (31)

Proof. Integrating by parts we get
\[
\langle A_m \varphi, \psi \rangle = \int_a^b [\kappa(s) \frac{d}{ds} - m \kappa'(s)] \varphi(s) \psi(s) g(s) ds = \kappa(s) g(s) \varphi(s) \psi(s) \bigg|_a^b
\]
\[
+ \int_a^b \varphi(s) \left[ -\kappa(s) \frac{d}{ds} - \frac{\tau(s)}{\kappa(s)} - (m-1) \kappa'(s) \right] \psi(s) g(s) ds = \langle \varphi, A^+_m \psi \rangle
\]
and
\[
\langle H_m \varphi, \psi \rangle = \langle (A^+_m A_m + \lambda_m) \varphi, \psi \rangle = \langle A_m \varphi, A_m \psi \rangle + \langle \varphi, \lambda_m \psi \rangle = \langle \varphi, H_m \psi \rangle.
\]

Theorem 6. For each \( m < \Lambda \), the functions \( \Phi_{l,m} \) with \( m \leq l < \Lambda \) are orthogonal with weight function \( g(s) \) in \( (a, b) \).

Proof. By using (12) we get
\[
\lim_{s \to a, b} \kappa(s) g(s) (A_m \Phi_{l,m})(s) \Phi_{k,m}(s) = \lim_{s \to a, b} \kappa(s) g(s) \Phi_{l,m+1}(s) \Phi_{k,m}(s) = 0
\]
for \( l, k \in \mathbb{N} \) with \( m \leq l < \Lambda \) and \( m \leq k < \Lambda \). Therefore, we can use lemma 1 and obtain
\[
\langle \lambda_l - \lambda_k \rangle \langle \Phi_{l,m}, \Phi_{k,m} \rangle = \langle H_m \Phi_{l,m}, \Phi_{k,m} \rangle - \langle \Phi_{l,m}, H_m \Phi_{k,m} \rangle = 0.
\]

Theorem 7. If \( 0 \leq m < l < \Lambda \) then \( ||\Phi_{l,m+1}|| = \sqrt{\lambda_l - \lambda_m} ||\Phi_{l,m}|| \).

Proof. Since \( \lim_{s \to a} \kappa(s) g(s) \Phi_{l,m}(s) \Phi_{l,m+1}(s) = \lim_{s \to b} \kappa(s) g(s) \Phi_{l,m}(s) \Phi_{l,m+1}(s) = 0 \) we get
\[
||\Phi_{l,m+1}||^2 = \langle A_m \Phi_{l,m}, \Phi_{l,m+1} \rangle = \langle \Phi_{l,m}, A^+_m \Phi_{l,m+1} \rangle = (\lambda_l - \lambda_m) ||\Phi_{l,m}||^2.
\]

The normalized associated special functions
\[
\phi_{l,m} = \Phi_{l,m} / ||\Phi_{l,m}||
\] (32)
satisfy the relations
\[
A_m \phi_{l,m} = \begin{cases} 
0 & \text{for } l = m \\
\sqrt{\lambda_l - \lambda_m} \phi_{l,m+1} & \text{for } m < l < \Lambda
\end{cases}
\]
\[
A^+_m \phi_{l,m+1} = \sqrt{\lambda_l - \lambda_m} \phi_{l,m} \quad \text{for } 0 \leq m < l < \Lambda
\] (33)
\[
\phi_{l,m} = \frac{A^+_m}{\sqrt{\lambda_l - \lambda_m}} \frac{A^+_m}{\sqrt{\lambda_l - \lambda_{m+1}}} \cdots \frac{A^+_m}{\sqrt{\lambda_l - \lambda_{m-1}}} \phi_l.
\]
4. Shape invariant operators related to $H_m$

Some additional shape invariant operators directly related to $H_m$ can be obtained in the cases when $\alpha$ and $\beta$ are such that there exists $k \in \mathbb{R}$ with $\varrho(s) = \sigma^k(s)$ (see table 2).

| $\sigma(s)$ | $\tau(s)$ | $\varrho(s)$ | $k$ | $(a, b)$ |
|-------------|-----------|-------------|-----|---------|
| $s$         | $\beta$   | $s^{\beta-1}$ | $\beta - 1$ | $(0, \infty)$ |
| $1 - s^2$   | $\alpha s$ | $(1 - s^2)^{-\alpha/2-1}$ | $-\frac{\alpha}{2} - 1$ | $(-1, 1)$ |
| $s^2 - 1$   | $\alpha s$ | $(s^2 - 1)^{\alpha/2-1}$ | $\frac{\alpha}{2} - 1$ | $(1, \infty)$ |
| $s^2$       | $\alpha s$ | $s^{\alpha-2}$ | $\frac{\alpha}{2} - 1$ | $(0, \infty)$ |
| $s^2 + 1$   | $\alpha s$ | $(s^2 + 1)^{\alpha/2-1}$ | $\frac{\alpha}{2} - 1$ | $(-\infty, \infty)$ |

From $(\sigma \varrho)' = \tau \varrho$ we get $\tau(s) = (k + 1)\sigma'(s) = 2(k + 1)\kappa(s)\kappa'(s)$, and

$$A_m = \kappa(s) \frac{d}{ds} - m \kappa'(s) \quad A_m^+ = -\kappa(s) \frac{d}{ds} - (2k + m + 1)\kappa'(s). \quad (34)$$

**Theorem 8.** If $\alpha$ and $\beta$ are such that $\varrho(s) = \sigma^k(s)$ then for any $\delta \in \mathbb{R}$ the operators

$$\tilde{A}_m = A_m + \frac{\delta}{2m + 2k + 1} \quad \tilde{A}_m^+ = A_m^+ + \frac{\delta}{2m + 2k + 1} \quad (35)$$

satisfy the relations

$$\tilde{A}_m^+ \tilde{A}_m = \tilde{H}_m - \tilde{\lambda}_m \quad \tilde{A}_m \tilde{H}_m = \tilde{H}_{m+1} \tilde{A}_m \quad (36)$$

where

$$\tilde{H}_m = H_m - \delta \frac{dk}{ds} \quad \tilde{\lambda}_m = \lambda_m - \frac{\delta^2}{(2m + 2k + 1)^2} \quad (37)$$

for any $m \in \mathbb{R}$ with $2m + 2k + 1 \neq 0$.

**Proof.** Since $A_m^+ A_m = H_m - \lambda_m$ and $A_m A_m^+ = H_{m+1} - \lambda_m$ we obtain

$$(A_m^+ + \varepsilon)(A_m + \varepsilon) = H_m - \lambda_m - \varepsilon(2m + 2k + 1)\kappa'(s) + \varepsilon^2$$

$$(A_m + \varepsilon)(A_m^+ + \varepsilon) = H_{m+1} - \lambda_m - \varepsilon(2m + 2k + 1)\kappa'(s) + \varepsilon^2$$

for any constant $\varepsilon$. If we choose $\varepsilon = \delta/(2m + 2k + 1)$ then we get

$$\tilde{H}_m \tilde{A}_m^+ = (\tilde{A}_m^+ \tilde{A}_m + \tilde{\lambda}_m) \tilde{A}_m^+ = \tilde{A}_m^+ (\tilde{A}_m \tilde{A}_m^+ + \tilde{\lambda}_m) = \tilde{A}_m^+ \tilde{H}_{m+1}$$

$$\tilde{A}_m \tilde{H}_m = \tilde{A}_m (\tilde{A}_m^+ \tilde{A}_m + \tilde{\lambda}_m) = (\tilde{A}_m \tilde{A}_m^+ + \tilde{\lambda}_m) \tilde{A}_m = \tilde{H}_{m+1} \tilde{A}_m. \quad \Box$$

The mapping $m \mapsto \tilde{\lambda}_m$ is an increasing function on the set $\{ m \mid \frac{d}{dm} \tilde{\lambda}_m > 0 \}$, where

$$\frac{d}{dm} \tilde{\lambda}_m = \begin{cases} 
4\delta^2(2m+2\beta-1)^{-3} & \text{if } \sigma(s) = s \\
(2m-\alpha-1) [1+4\delta^2(2m-\alpha-1)^{-4}] & \text{if } \sigma(s) = 1-s^2 \\
(2m+\alpha-1) [-1+4\delta^2(2m+\alpha-1)^{-4}] & \text{if } \sigma'' = 2
\end{cases} \quad (38)$$
and, up to a multiplicative constant, the solution $\tilde{\Phi}_{m,m}$ of the equation $\tilde{A}_m \tilde{\Phi}_{m,m} = 0$ is

$$
\tilde{\Phi}_{m,m}(s) = \begin{cases} 
(\sqrt{s})^m e^{-\frac{2\sqrt{s}}{2m+2\beta-1}} & \text{if } \sigma(s) = s \\
(\sqrt{1-s^2})^m e^{-\frac{2\sqrt{s}}{2m+2\alpha-1}} & \text{if } \sigma(s) = 1-s^2 \\
(\sqrt{s^2-1})^m (s+\sqrt{s^2-1})^{-\frac{2}{2m+2\alpha-1}} & \text{if } \sigma(s) = s^2-1 \\
(\sqrt{s^2+1})^m (s+\sqrt{s^2+1})^{-\frac{2}{2m+2\alpha-1}} & \text{if } \sigma(s) = s^2+1.
\end{cases}
$$

The set $\mathcal{M} = \{ m \mid \frac{d}{dm} \tilde{\lambda}_m > 0 \text{ and } \int_a^b \tilde{\Phi}_{m,m}^2(s)ds < \infty \}$ of all the values of $m$ for which $\frac{d}{dm} \tilde{\lambda}_m > 0$ and $\tilde{\Phi}_{m,m} \sqrt{\vartheta}$ is square integrable on $(a, b)$ is presented in table 3.

**Table 3.** The set $\mathcal{M}$ of all the values of $m$ for which $\frac{d}{dm} \tilde{\lambda}_m > 0$ and $\tilde{\Phi}_{m,m} \sqrt{\vartheta}$ is square integrable

| $\sigma(s)$ | $\tau(s)$ | $\mathcal{M}$ |
|-------------|-----------|---------------|
| $s$         | $\beta$  | $\emptyset$   |
| $s^2$       | $\alpha$ | $\emptyset$   |
| $s^2 - 1$   | $\alpha$ | $(-\beta + \frac{1}{2}, \infty)$ for $\delta > 0$ |
| $s^2 + 1$   | $\alpha$ | $(-\infty, \frac{1}{2}) - \sqrt{\frac{1}{2}}$ for any $\delta \in \mathbb{R}$ |
| $1-s^2$     | $\alpha$ | $(\frac{1+\alpha}{2}, \infty)$ for any $\delta \in \mathbb{R}$ |

**Lemma 2.** If $l \in \mathcal{M}$ and $n \in \mathbb{N}$ are such that $\{l-n, l-n+1, \ldots, l\} \subseteq \mathcal{M}$ then for each $m \in \{l-n, l-n+1, \ldots, l-1\}$ the function

$$
\tilde{\Phi}_{l,m} = \frac{\tilde{A}_l^+}{\lambda_l - \lambda_m} \frac{\tilde{A}_{l+1}^+}{\lambda_l - \lambda_{m+1}} \ldots \frac{\tilde{A}_{l-1}^+}{\lambda_l - \lambda_{l-1}} \tilde{\Phi}_{l,l}
$$

has the form

$$
\tilde{\Phi}_{l,m} = \begin{cases} 
\sum_{j=0}^{l-m} c_j (\sqrt{s})^{l-j} e^{-\frac{2\sqrt{s}}{2l+2\beta-1}} & \text{if } \sigma(s) = s \\
\sum_{j=0}^{l-m} c_j s^j (\sqrt{1-s^2})^{l-j} e^{-\frac{2\sqrt{s}}{2l+2\alpha-1}} & \text{if } \sigma(s) = 1-s^2 \\
\sum_{j=0}^{l-m} c_j s^j (\sqrt{s^2-1})^{l-j} (s+\sqrt{s^2-1})^{-\frac{2}{2m+2\alpha-1}} & \text{if } \sigma(s) = s^2-1 \\
\sum_{j=0}^{l-m} c_j s^j (\sqrt{s^2+1})^{l-j} (s+\sqrt{s^2+1})^{-\frac{2}{2m+2\alpha-1}} & \text{if } \sigma(s) = s^2+1
\end{cases}
$$
Proof. The statement follows from the relations
\[
\frac{d}{ds} \sqrt{s} = \frac{1}{2 \sqrt{s}} \quad \frac{d}{ds} \sqrt{1 - s^2} = -\frac{s}{\sqrt{1 - s^2}} \quad \frac{d}{ds} s^{\frac{\alpha - \beta - 1}{\alpha - \beta}} = \frac{s^\beta \Gamma (\frac{1}{2} - \frac{\alpha - \beta - 1}{2})}{\Gamma (\frac{1}{2})} \frac{d}{ds} s^{\frac{\alpha - \beta - 1}{\alpha - \beta}}
\]
\[
\frac{d}{ds} \sqrt{s^2 + 1} = \frac{s}{\sqrt{s^2 + 1}} \quad \frac{d}{ds} \sqrt{1 - s^2} \frac{d}{ds} (s + \sqrt{s^2 + 1})^{\frac{\alpha - \beta - 1}{\alpha - \beta}} = -\frac{\delta}{2l + 1} (s + \sqrt{s^2 + 1})^{\frac{\alpha - \beta - 1}{\alpha - \beta}}
\]
\[\square\]

Theorem 9. If \(l \in \mathcal{M}\) and \(n \in \mathbb{N}\) are such that \(\{l - n, l - n + 1, \ldots, l\} \subset \mathcal{M}\) then for each \(m \in \{l - n, l - n + 1, \ldots, l\}\) the function \(\tilde{H}_m \tilde{\Phi}_{l,m} \sqrt{\rho}\) is square integrable and
\[
\tilde{H}_m \tilde{\Phi}_{l,m} = \tilde{\lambda}_l \tilde{\Phi}_{l,m}
\]
\[
\tilde{A}_m \tilde{\Phi}_{l,m} = \begin{cases} 0 & \text{for } m = l \\ \tilde{\Phi}_{l,m+1} & \text{for } m < l \end{cases}
\]
\[
\tilde{H}_m \tilde{\Phi}_{l,m+1} = (\tilde{\lambda}_l - \tilde{\lambda}_m) \tilde{\Phi}_{l,m}
\]

Proof. The square integrability follows from (41). The definition (40) of \(\tilde{\Phi}_{l,m}\) can be re-written as
\[
\tilde{\Phi}_{l,m} = \frac{\tilde{A}_m}{\lambda_l - \lambda_m} \tilde{\Phi}_{l,m+1}
\]
and \(\tilde{H}_l \tilde{\Phi}_{l,m} = (\tilde{A}_m \tilde{H}_l + \tilde{\lambda}_l) \tilde{\Phi}_{l,m} = \tilde{\lambda}_l \tilde{\Phi}_{l,m}\). The relation \(\tilde{H}_m \tilde{\Phi}_{l,m} = \tilde{\lambda}_l \tilde{\Phi}_{l,m}\) follows by recurrence
\[
\tilde{H}_{m+1} \tilde{\Phi}_{l,m+1} = \tilde{\lambda}_l \tilde{\Phi}_{l,m+1} \quad \implies \quad \tilde{H}_m \tilde{\Phi}_{l,m} = \frac{\tilde{H}_m \tilde{A}_m}{\lambda_l - \lambda_m} \tilde{\Phi}_{l,m+1} = \frac{\tilde{A}_m \tilde{H}_{m+1}}{\lambda_l - \lambda_m} \tilde{\Phi}_{l,m+1} = \tilde{\lambda}_l \tilde{\Phi}_{l,m}.
\]
From the relation (44) we get
\[
\tilde{A}_m \tilde{\Phi}_{l,m} = \frac{\tilde{A}_m \tilde{A}_m}{\lambda_l - \lambda_m} \tilde{\Phi}_{l,m+1} = \frac{\tilde{H}_{m+1} - \tilde{\lambda}_m}{\lambda_l - \lambda_m} \tilde{\Phi}_{l,m+1} = \tilde{\Phi}_{l,m+1}. \quad \square
\]

Theorem 10. If \(l \in \mathcal{M}\) and \(n \in \mathbb{N}\) are such that \(\{l - n - 1, l - n, \ldots, l\} \subset \mathcal{M}\) then the functions \(\tilde{\Phi}_{m,m}, \tilde{\Phi}_{m+1,m}, \ldots, \tilde{\Phi}_{l,m}\), where \(m = l - n\), are orthogonal.

Proof. By using (41) and lemma 1 we get
\[
(\tilde{\lambda}_m - \tilde{\lambda}_m') (\tilde{\Phi}_{n',m}, \tilde{\Phi}_{n'',m}) = (\tilde{H}_m \tilde{\Phi}_{n',m}, \tilde{\Phi}_{n'',m}) - (\tilde{\Phi}_{n',m}, H_m \tilde{\Phi}_{n'',m}) = 0.
\]
for any \(n', n'' \in \{m, m + 1, \ldots, l\}\). \(\square\)

Theorem 11. If \(l \in \mathcal{M}\) and \(n \in \mathbb{N}\) are such that
\[
\{l - n, l - n + 1, \ldots, l\} \subset \mathcal{M}
\]
then for each \(m \in \{l - n, l - n + 1, \ldots, l - 1\}\) we have
\[
(\tilde{A}_m \tilde{\Phi}_{l,m}, \tilde{\Phi}_{l,m+1}) = (\tilde{\Phi}_{l,m}, \tilde{A}_m \tilde{\Phi}_{l,m+1})
\]
and
\[ ||\tilde{\Phi}_{l,m+1}|| = \sqrt{\lambda_l - \lambda_m} \cdot ||\tilde{\Phi}_{l,m}||. \] (46)

**Proof.** We have \( \lim_{s \to a} \kappa^{2k+1}(s) \Phi_{l,m}(s) \Phi_{l,m+1}(s) = \lim_{s \to b} \kappa^{2k+1}(s) \Phi_{l,m}(s) \Phi_{l,m+1}(s) = 0 \).

In view of lemma 1 we get
\[ ||\tilde{\Phi}_{l,m+1}||^2 = \langle \tilde{A}_m \tilde{\Phi}_{l,m}, \tilde{\Phi}_{l,m+1} \rangle = \langle \tilde{\Phi}_{l,m}, \tilde{A}_m^+ \tilde{\Phi}_{l,m+1} \rangle = (\tilde{\lambda}_l - \tilde{\lambda}_m) ||\tilde{\Phi}_{l,m}||^2. \]

\[ \square \]

The normalized associated special functions
\[ \tilde{\phi}_{l,m} = \tilde{\Phi}_{l,m} / ||\tilde{\Phi}_{l,m}|| \] (47)

satisfy the relations
\[ \tilde{A}_m \tilde{\phi}_{l,m} = \begin{cases} 0 & \text{for } l = m \\ \sqrt{\lambda_l - \lambda_m} \tilde{\phi}_{l,m+1} & \text{for } m < l \end{cases} \]
\[ \tilde{A}_m^+ \tilde{\phi}_{l,m+1} = \sqrt{\lambda_l - \lambda_m} \tilde{\phi}_{l,m} \text{ for } 0 \leq m < l \] (48)

5. Gazeau-Klauder coherent states

Let \( m < \Lambda \) be a fixed natural number, and let
\[ \Lambda_m = \Lambda - m = \begin{cases} \infty & \text{if } \sigma'' \in \{0, -2\} \\ \frac{1-\alpha}{2} - m & \text{if } \sigma'' = 2. \end{cases} \] (49)

The functions
\[ |n\rangle = \phi_{m+n,m} \text{ with } 0 \leq n < \Lambda_m \] (50)

form an orthonormal system, and satisfy the relation
\[ (H_m - \lambda_m)|n\rangle = e_n |n\rangle \] (51)

where
\[ e_n = \lambda_{m+n} - \lambda_m = \begin{cases} -\alpha n & \text{if } \sigma'' = 0 \\ n(n + 2m - \alpha - 1) & \text{if } \sigma'' = -2 \\ n(1 - \alpha - 2m - n) & \text{if } \sigma'' = 2. \end{cases} \] (52)

One can remark that
\[ 0 = e_0 < e_1 < e_2 < \ldots < e_n \text{ for any } n < \Lambda_m. \]

By following the method presented in [8] we define the Gazeau-Klauder coherent states
\[ |J, \gamma\rangle = N(J)^{-1} \sum_{n<\Lambda_m} \frac{J^{n/2}}{\sqrt{\rho_n}} e^{-i\alpha_n \gamma} |n\rangle \]
described by the real two-parameter \((J, \gamma) \in [0, \infty) \times \mathbb{R}\), where \(N(J)\) is a normalizing constant and

\[
\rho_n = \begin{cases} 
1 & \text{if } n = 0 \\
e_1 e_2 \ldots e_n & \text{if } n > 0 
\end{cases}
\]

satisfies the relation \(\sigma'' = -2\)

\[
\sum_{n<\Lambda_m} J_n \rho_n = \sqrt{\sum_{n<\Lambda_m} J_n}. \tag{54}
\]

Theorem 12. There exists a real function \(k(J)\) such that

\[
\int_0^\infty k(J) \, dJ \int_{-\infty}^\infty |J, \gamma \rangle \langle J, \gamma| \, d\nu(\gamma) = \sum_{n<\Lambda_m} |n\rangle \langle n| = \mathbb{I} \tag{55}
\]

with \(d\nu(\gamma)\) defined by

\[
\int_{-\infty}^{\infty} \ldots d\nu(\gamma) \equiv \lim_{R \to \infty} \frac{1}{2R} \int_{-R}^{R} \ldots d\gamma. \tag{56}
\]

**Proof.** Looking for a function of the form \(k(J) = N(J)^2 \rho(J)\) we get

\[
\int_0^\infty k(J) \, dJ \int_{-\infty}^\infty |J, \gamma \rangle \langle J, \gamma| \, d\nu(\gamma) = \sum_{n<\Lambda_m} \left( \frac{1}{\rho_n} \int_0^\infty J^n \rho(J) \, dJ \right) |n\rangle \langle n|. 
\]

In the case \(\sigma'' = 0\) we can choose \(\rho(J) = -\frac{1}{a} e^{-\frac{1}{a} J}\) since the change of variable \(J = -\alpha t\) leads to

\[
\frac{1}{\rho_n} \int_0^\infty J^n \rho(J) \, dJ = \frac{1}{n!} \int_0^\infty t^n e^{-t} \, dt = 1 \quad \text{for any } n \in \mathbb{N}.
\]

The modified Bessel function

\[
K_{\nu}(z) = \frac{\pi I_{-\nu}(z) - I_{\nu}(z)}{2 \sin(\nu \pi)} \quad \text{where } I_{\nu}(z) = \sum_{n=0}^{\infty} \frac{(\frac{1}{2} z)^{\nu+2n}}{n! \Gamma(\nu + n + 1)}. \tag{57}
\]

satisfies the relation \([3]\)

\[
\int_0^\infty 2 x^{\eta+\xi} K_{\eta-\xi}(2 \sqrt{x}) x^{n-1} \, dx = \Gamma(2 \eta + n) \Gamma(2 \xi + n)
\]

which for \(x = J, \eta = \frac{1}{2}, \xi = m - \frac{\alpha}{2}\) becomes

\[
2 \int_0^\infty J^n J^{m-\frac{\alpha}{2}} K_{\frac{\alpha+1}{2} - m}(2 \sqrt{J}) \, dJ = n! \Gamma(n + 2m - \alpha). \tag{58}
\]

The last formula shows that in the case \(\sigma'' = -2\) we can choose \([2] [1]\)

\[
k(J) = \frac{2}{\Gamma(2m - \alpha)} J^{m-\frac{\alpha}{2}} K_{\frac{\alpha+1}{2} - m}(2 \sqrt{J}).
\]

The Bessel function \(J_{\nu}\) satisfies the relation \([3]\)

\[
\int_0^\infty x^\mu J_{\nu}(ax) \, dx = 2^\mu a^{-\mu-1} \frac{\Gamma \left( \frac{1 + \nu + \mu}{2} \right)}{\Gamma \left( \frac{1 + \mu}{2} \right)} \tag{59}
\]
for \(-\Re \nu - 1 < \Re \mu < \frac{1}{2}\). If we use the substitutions \(a = 2\), \(x = \sqrt{J}\), \(\nu = 1 - \alpha - 2m\) and \(\mu = \alpha + 2m + 2n\) then we get the relation

\[
\int_{0}^{\infty} J^n J^{m - \frac{1}{2}} J_{1 - \alpha - 2m}(2\sqrt{J}) \, dJ = \frac{n!}{\Gamma(1 - \alpha - 2m - n)}
\]

which shows that in the case \(\sigma'' = 2\) we can choose \[14\]

\[
\rho(J) = \Gamma(1 - \alpha - 2m)J^{m - \frac{1}{2}} J_{1 - \alpha - 2m}(2\sqrt{J}). \quad \Box
\]

Let \(m \in \mathcal{M}\) and

\[
\tilde{\Lambda}_m = \begin{cases} \infty & \text{if } \sigma(s) = s \text{ or } \sigma(s) = 1 - s^2 \\ \sup \mathcal{M} - m & \text{if } \sigma(s) = s^2 - 1 \text{ or } \sigma(s) = s^2 + 1 \end{cases}
\]

The functions

\[
|\tilde{n}\rangle = \tilde{\phi}_{m+n,m} \quad \text{with } 0 \leq n < \tilde{\Lambda}_m
\]

form an orthonormal system, and satisfy the relation

\[
(H_m - \tilde{\lambda}_m)|\tilde{n}\rangle = \tilde{e}_n|\tilde{n}\rangle
\]

with \(\tilde{e}_n\) defined by

\[
\tilde{e}_n = \tilde{\lambda}_{m+n} - \tilde{\lambda}_m = \begin{cases} \frac{\delta^2 n(n+\beta_m)}{\beta_m (n+\beta_m/2)^2} & \text{if } \sigma(s) = s \\ \frac{n(2\alpha_m)(n-\alpha_m-\delta/2\alpha_m)(n-\alpha_m+\delta/2\alpha_m)}{(n-\alpha_m)^2} & \text{if } \sigma(s) = 1 - s^2 \\ \frac{n(2\alpha_m')(n-\alpha_m'-\delta/2\alpha_m')(n-\alpha_m'+\delta/2\alpha_m')}{(n-\alpha_m')^2} & \text{if } \sigma(s) = s^2 \pm 1. \end{cases}
\]

where \(\beta_m = 2m + 2\beta - 1\), \(\alpha_m = \frac{1+\alpha}{2} - m\) and \(\alpha_m' = \frac{1-\alpha}{2} - m\).

We define the Gazeau-Klauder coherent states

\[
|J, \gamma\rangle = N(J)^{-1} \sum_{n<\tilde{\lambda}_m} J^{n/2} e^{-i\delta n\gamma} |\tilde{n}\rangle
\]

where \(N(J)\) is a normalizing constant and

\[
\tilde{\rho}_n = \frac{\delta^2 n(n+\beta_m)}{(n+\beta_m/2)^2} \frac{\Gamma^2(\beta_m/2+1)}{\Gamma(\beta_m+1)} \frac{n!\Gamma(n+\beta_m+1)}{\Gamma^2(n+\beta_m/2+1)}
\]

in the case \(\sigma(s) = s\),

\[
\tilde{\rho}_n = \frac{\Gamma^2(1-\alpha_m)}{\Gamma(1-2\alpha_m)\Gamma(1-\alpha_m-\delta/2\alpha_m)\Gamma(1-\alpha_m+\delta/2\alpha_m)} \frac{n!\Gamma(n+\alpha_m+\delta/2\alpha_m+1)}{\Gamma^2(n+\alpha_m+1)}
\]

in the case \(\sigma(s) = 1 - s^2\), and

\[
\tilde{\rho}_n = \frac{(-1)^n \Gamma^2(1-\alpha_m')}{\Gamma(1-2\alpha_m')\Gamma(1-\alpha_m'-\delta/2\alpha_m')\Gamma(1-\alpha_m'+\delta/2\alpha_m')} \frac{n!\Gamma(n+\alpha_m'+\delta/2\alpha_m'+1)}{\Gamma^2(n+\alpha_m'+1)}
\]

in the cases \(\sigma(s) = s^2 \pm 1\). One can remark that

\[
N(J)^2 = {}_2F_1 \left( 1 + \frac{\beta_m}{2}, \frac{\beta_m}{2}; 1 + \beta_m; \frac{\beta_m^2}{\delta^2} J \right)
\]

in the case \(\sigma(s) = s\),

\[
N(J)^2 = {}_2F_3 \left( 1 - \alpha_m, 1 - \alpha_m, 1 - 2\alpha_m, 1 - \alpha_m - \frac{\delta i}{2\alpha_m}, 1 - \alpha_m + \frac{\delta i}{2\alpha_m}; J \right)
\]
in the case $\sigma(s) = 1 - s^2$, and $N(J)^2$ is a finite sum in the cases $\sigma(s) = s^2 \pm 1$. If $\sigma(s) = s$ then the coherent states (64) can be defined only for $|J| < \frac{e^2}{\eta}$.

**Theorem 13.** If $\sigma(s) = 1 - s^2$ or $\sigma(s) = s^2 \pm 1$ then there exists a real function $k(J)$ such that

$$
\int_0^\infty k(J) \, dJ \int_{-\infty}^{\infty} |J, \gamma\rangle \langle J, \gamma| \, d\nu(\gamma) = \sum_{n<\Lambda_m} |\tilde{n}\rangle \langle \tilde{n}| = \mathbb{I}
$$

(65)

with $d\nu(\gamma)$ defined by

$$
\int_{-\infty}^{\infty} \ldots \, d\nu(\gamma) \equiv \lim_{R \to \infty} \frac{1}{2R} \int_{-R}^{R} \ldots \, d\gamma.
$$

**Proof.** Looking for a function of the form $k(J) = N(J)^2 \rho(J)$ we get

$$
\int_0^\infty k(J) \, dJ \int_{-\infty}^{\infty} |J, \gamma\rangle \langle J, \gamma| \, d\nu(\gamma) = \sum_{n<\Lambda_m} \left( \frac{1}{\tilde{\rho}_n} \int_0^{\infty} J^n \rho(J) \, dJ \right) |\tilde{n}\rangle \langle \tilde{n}|.
$$

In the case $\eta = 1$, $m = q$, $n = 0$, the relation involving the Meijer’s $G$-function [12]

$$
\int_0^\infty z^{s-1} \, G_{p,m}^{m,n} \left( \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \bigg| \eta z \right) \, dz = \frac{\eta^{-s} \prod_{j=1}^{m} \Gamma(b_j + s) \prod_{j=1}^{n} \Gamma(1 - a_j - s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j - s) \prod_{j=n+1}^{p} \Gamma(a_j + s)}
$$

(67)

becomes

$$
\int_0^\infty z^{s-1} \, G_{p,0}^{q,0} \left( \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \bigg| z \right) \, dz = \frac{\Gamma(b_1 + s) \ldots \Gamma(b_q + s)}{\Gamma(a_1 + s) \ldots \Gamma(a_p + s)}.
$$

(68)

In order to get $\int_0^{\infty} J^n \rho(J) \, dJ = \tilde{\rho}_n$ we choose [4]

$$
\rho(J) = \frac{\Gamma^2(1-\alpha_m)}{\Gamma(1-2\alpha_m) \Gamma(1-\alpha_m-i\delta/2\alpha_m) \Gamma(1-\alpha_m+i\delta/2\alpha_m)} \, G_{2,4}^{4,0} \left( \begin{array}{c} -\alpha_m, -\alpha_m \\ 0, -2\alpha_m, -\alpha_m - i\delta/2\alpha_m, -\alpha_m + i\delta/2\alpha_m \end{array} \bigg| J \right)
$$

in the case $\sigma(s) = 1 - s^2$, and

$$
\rho(J) = \frac{\Gamma^2(1-\alpha'_m)}{\Gamma(1-2\alpha'_m) \Gamma(1-\alpha'_m-i\delta/2\alpha'_m) \Gamma(1-\alpha'_m+i\delta/2\alpha'_m)} \, G_{2,4}^{4,0} \left( \begin{array}{c} -\alpha'_m, -\alpha'_m \\ 0, -2\alpha'_m, -\alpha'_m - \delta/2\alpha'_m, -\alpha'_m + \delta/2\alpha'_m \end{array} \bigg| J \right)
$$

in the cases $\sigma(s) = s^2 \pm 1$. □

6. Concluding remarks

If we use in equation $H_m \Phi_{l,m} = \lambda_l \Phi_{l,m}$ a change of variable $(a', b') \rightarrow (a, b) : x \mapsto s(x)$ such that $ds/dx = \kappa(s(x))$ or $ds/dx = -\kappa(s(x))$ and define the new functions

$$
\Psi_{l,m}(x) = \sqrt{\kappa(s(x))} \, g(s(x)) \, \Phi_{l,m}(s(x))
$$

(69)

then we get an equation of Schrödinger type

$$
-\frac{d^2}{dx^2} \Psi_{l,m}(x) + V_m(x) \Psi_{l,m}(x) = \lambda_l \Psi_{l,m}(x).
$$

(70)
Since
\[ \int_{a'}^{b'} \Psi_{t,m}(x) \Psi_{k,m}(x) dx = \int_{a'}^{b'} \Phi_{t,m}(s(x)) \Phi_{k,m}(s(x)) \frac{d}{dx} s(x) dx = \int_{a}^{b} \Phi_{t,m}(s) \Phi_{k,m}(s) \varrho(s) ds \]
the functions \( \Psi_{t,m}(x) \) are square integrable (resp. orthogonal) if and only if the corresponding functions \( \Phi_{t,m}(s) \) are square integrable (resp. orthogonal).

If \( ds/dx = \kappa(s(x)) \) then the operators corresponding to \( A_m \) and \( A_m^+ \) are
\[ A_m = [\kappa(s) \varrho(s)]^{1/2} A_m [\kappa(s) \varrho(s)]^{-1/2} \big|_{s=s(x)} = \frac{d}{dx} + W_m(x) \]
\[ A_m^+ = [\kappa(s) \varrho(s)]^{1/2} A_m^+[\kappa(s) \varrho(s)]^{-1/2} \big|_{s=s(x)} = -\frac{d}{dx} + W_m(x) \]
(71)
where the superpotential \( W_m(x) \) is given by the formula
\[ W_m(x) = -\frac{\tau(s(x))}{2\kappa(s(x))} - \frac{2m-1}{2\kappa(s(x))} \frac{d}{dx} \kappa(s(x)). \]
(72)

We have
\[ A_m \Psi_{t,m}(x) = \Psi_{t,m+1}(x) \quad A_m^+ \Psi_{t,m+1}(x) = (\lambda_l - \lambda_m) \Psi_{t,m}(x) \]
\[ -\frac{d^2}{dx^2} + V_m(x) - \lambda_m = A_m A_m^+ \quad -\frac{d^2}{dx^2} + V_{m+1}(x) - \lambda_m = A_m A_m^+ \]
(73)
(74)
and
\[ \Psi_{t,m}(x) = \frac{A_m^+ A_{m+1} \cdots A_{l-2}^+ A_{l-1}^+}{\lambda_l - \lambda_m \lambda_l - \lambda_{m+1} \cdots \lambda_l - \lambda_{l-2} \lambda_l - \lambda_{l-1}} \Psi_{t,l}(x) \]
(75)
for each \( l \in \{0, 1, \ldots, \nu\} \) and each \( m \in \{0, 1, \ldots, l-1\} \).

If we choose the change of variable \( s = s(x) \) such that \( ds/dx = -\kappa(s(x)) \), then
\[ A_m = -\frac{d}{dx} + W_m(x) \quad A_m^+ = \frac{d}{dx} + W_m(x) \]
(76)
and
\[ W_m(x) = -\frac{\tau(s(x))}{2\kappa(s(x))} + \frac{2m-1}{2\kappa(s(x))} \frac{d}{dx} \kappa(s(x)). \]
(77)
Some very similar results can be obtained in the case of operators \( \bar{H}_m \). The Gazeau-Klauder systems of coherent states defined in the previous section correspond through the considered change of variables to some systems of coherent states useful in quantum mechanics.

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