Rotational Symmetry and Regularization Dependence in the $\Phi_4^4$-Model

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Abstract

We study the one component $\Phi^4$ model for four different lattice actions in the Gaussian limit and for the Ising model in the broken phase. Emphasis is put on the euclidean invariance properties of the boson propagator. A measure of the violation of rotational symmetry serves as a tool to compare the regularization dependence of the triviality bound.

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1 Motivation and introduction

There is ample evidence that for the lattice regularized $\Phi^4$-model the physical quartic coupling decreases logarithmically with increasing cutoff and that the continuum limit describes non-interacting boson fields. Thus the scalar sector of the standard model

$$S_{\text{scalar}} = \frac{1}{2} (\partial_{\mu} \Phi)^2 + \frac{m_0^2}{2} |\Phi|^2 + \frac{\lambda}{4!} (|\Phi|^2)^2,$$

where the Higgs fields live in $O(4)$, has to be considered as an effective low energy model describing the leading cutoff effects and may be studied independently.

The cutoff then represents new, possibly more fundamental physics which would replace the standard-model at higher energies. However, its influence is effective on all scales and manifests itself in the cutoff dependence of all observables. In describing the experiments through a lattice model, the higher in energy we go the stronger we expect such cutoff effects to be. More and more terms would have to be included in the effective action in order to explain experiments with sufficient accuracy. Within the scaling region one expects to find this dependencies to be only weak, but this knowledge is mainly based on rough estimates and there are only very few explicit studies \[1, 2, 3, 4, 5\].

In this article we study the cutoff dependence with emphasis on the euclidean invariance properties of the boson propagator. As a simplified toy model we use the one-component $\Phi^4$ model, where one does not have the additional complication due to Goldstone modes. We expect that most of our qualitative results carry over to $O(N)$.

One particularly interesting observable exhibiting a cutoff dependence as discussed above is the upper bound on the effective coupling $\lambda_R$. This bound emerges at infinite bare coupling (the Ising limit of the model) as a consequence of triviality, and is related to the Higgs mass via the ratio

$$R = \sqrt{\lambda_R / 3} = m_H / f.$$

In first order perturbation-theory and for the $O(4)$ model the $W$ mass may be related to the vacuum condensate $f_{\pi}^{\text{phys}} = \langle \Phi_R \rangle \approx 246$ GeV, and therefore

\[1\]The subscript $R$ denotes renormalized quantities throughout the article.
provides an upper bound for the physical Higgs mass \[ 6 \]. Because this quantity is essentially nonperturbative, we need nonperturbative methods to study its cutoff dependence.

In our further discussion we refer to different lattice regularizations only. In lattice formulations the cut-off dependence enters through the discretization of the action and the geometry of the lattice mesh. In particular we expect regularization dependent lattice artifacts for large values of the dimensionless Higgs mass (low values of the correlation length). Neuberger et al.\[ 7, 8 \] have studied these effects in the \( O(N) \)-model by systematically adding operators of increasing order in \( 1/\Lambda \); they also studied the contribution of similar higher order terms in a \( O(4) \) model on an \( F_4 \)-lattice \[ 9 \].

Different lattice geometries correspond to different lattice regularizations and have been studied mainly in the context of the model with \( O(4) \) symmetry, in connection with the triviality bound on the Higgs mass. First results in \[ 1 \] were followed by a particular careful study of the model on the \( F_4 \) lattice \[ 2, 3 \]. During completion of this article we learned of a detailed study of the Symanzik action \[ 5 \].

Here we present a Monte Carlo study of the one-component \( \Phi^4 \)-model in the broken phase. For the standard hypercubic lattice this model has been extensively studied both, analytically \[ 10 \] and in various Monte Carlo simulations \[ 11 \]. We investigate it on altogether four different lattice geometries, introduced in the context of the \( O(4) \)-model and report on a simultaneous analysis. We compare the rotational invariance (RI) properties of the boson propagator in the broken phase of the corresponding Ising model with those of the related Gaussian model. We use the violation of this invariance as a measure in order to estimate the regularization dependence of the triviality bound.

2 Various lattice regularizations

2.1 Lattice actions

Consider the action for the one-component \( \Phi^4 \)-model defined on a lattice. We study four different kinds of lattice-geometries and corresponding actions, which we call \( C \), \( C' \), \( F_4 \) and \( S \). All four lattices may be embedded into the regular hypercubic one \( \Lambda_L \subset \mathbb{Z}^4 \) of volume \( L^4 \), with different cou-
plings to nearest, next-to-nearest and on-axis distance two neighbour spins. For convenience we present all results on this hypercubic mesh with lattice constant \( a \equiv 1 \). The general action \( S \) may then be written in the form

\[
S = \sum_{x \in \Lambda_L} \left[ -\kappa \sum_{\nu=1}^{N} \eta_{\nu} \hat{\Phi}_{x+e_{\nu}} \hat{\Phi}_{x} + \hat{\Phi}_{x}^{2} + g(\hat{\Phi}_{x}^{2} - 1)^2 \right] .
\]

By \( x + e_{\nu} \) we denote the neighbour site in direction \( e_{\nu} \), which may be a linear combination of the cartesian basis vectors in 4 dimensions \( \{ \hat{e}_{\mu}, \mu = 1, \ldots, 4 \} \).

Table 1 summarizes our notation, in particular the number \( N \) of interaction neighbours, the weight factors \( \eta_{\nu} \) and a normalization factor \( \alpha \) that has been defined in such a way that the trivial continuum limit of the kinetic term is universal (with a leading \( p^2 \) term in the inverse momentum space propagator).

We denote the ‘lattice field’ by \( \hat{\Phi} \) to distinguish it from the field \( \Phi \) in ‘continuum’ normalization. The naive continuum action is reconstructed in the limit \( a \to 0, L \to \infty \) with the substitutions

\[
\begin{align*}
    m_0^2 & = \frac{1 - 2g}{\alpha \kappa} - \frac{1}{\alpha} \sum_{\nu=1}^{N} \eta_{\nu} \\
    \lambda & = \frac{6g}{(\alpha \kappa)^2} \\
    \Phi_{x} & = \sqrt{2\alpha \kappa} \hat{\Phi}_{x} .
\end{align*}
\]

\( C \) denotes the usual simple hypercubic lattice action with interactions between nearest neighbours along the directions of the cartesian axes. In comparison to \( C \) the \( C' \)-action \( \square \) has additional couplings to the next-to-nearest neighbours in diagonal directions so that there are altogether 32 interaction neighbours. \( \mathcal{F}_4 \) is the nearest neighbour action on an \( F_4 \)-lattice \( \square \). It’s embedding into a hypercubic lattice may be imagined by removing all odd sites of the grid. The symmetries for this case forbid terms like \( \sum_{\mu} p_{\mu}^4 \) in the inverse momentum space propagator and guarantee Lorentz-invariance to a higher order in the momentum cut-off \( (O(\Lambda^{-4})) \) than the first two hypercubic lattice schemes \( (O(\Lambda^{-2})) \). The \( S \) denotes the Symanzik improved action which includes antiferromagnetic couplings to the the next-to-nearest neighbours along the 4 euclidean axes-directions. These additional terms are incorporated in order to remove the undesired \( O(\Lambda^{-2}) \) contributions, like for
$F_4$. The remaining corrections of $O(\Lambda^{-4})$ in the lattice dispersion relation guarantee tree-level improvement [12] (cf. sect. 4).

With

$$L = L_{kin} + L_{pot}, \quad L_{pot} = \dot{\Phi}_x^2 + g \left( \dot{\Phi}_x^2 - 1 \right)^2,$$

the kinetic parts can be written

$$L_{kin}^C = -2\kappa \sum_{\mu=1}^{4} \dot{\Phi}_{x+\dot{e}_\mu} \dot{\Phi}_x,$$

$$L_{kin}^{C'} = -2\kappa \sum_{\mu=1}^{4} \left[ \dot{\Phi}_{x+\dot{e}_\mu} + \sum_{\nu=1}^{\mu-1} \left( \dot{\Phi}_{x+\dot{e}_\mu + \dot{e}_\nu} + \dot{\Phi}_{x+\dot{e}_\mu - \dot{e}_\nu} \right) \right] \dot{\Phi}_x,$$

$$L_{kin}^{F_4} = -2\kappa \sum_{\mu=1}^{4} \sum_{\nu=1}^{\mu-1} \left( \dot{\Phi}_{x+\dot{e}_\mu + \dot{e}_\nu} + \dot{\Phi}_{x+\dot{e}_\mu - \dot{e}_\nu} \right) \dot{\Phi}_x,$$

$$L_{kin}^{S} = -2\kappa \sum_{\mu=1}^{4} \left( \dot{\Phi}_{x+\dot{e}_\mu} - \frac{\dot{\Phi}_{x+2\dot{e}_\mu}}{16} \right) \dot{\Phi}_x.$$

Note that $L_{kin}^{F_4}$ does not couple odd with even sites.

Quantization amounts to summing over all field configurations and taking expectation values with regard to the Gibbs measure $Z^{-1} \exp (-S) d\Phi$ where the partition function $Z$ is the normalization factor such that $\langle 1 \rangle = 1$. For the Gaussian version of the model this can be done explicitly, since the functional integrals factorize in momentum space. For the general theory one has to rely on expansion techniques or Monte Carlo integration.

### 2.2 How to compare two regularization schemes

Comparing results from different actions obtained at finite cut-off is not straightforward, though. Different actions have different scaling violation properties. Choosing couplings such that the observed correlation lengths agree is not necessarily the best definition. Consider e.g. an effective action for block variables (which usually is more complicated). Comparing the results of the original and the block action at a point in their respective coupling constant spaces where the dimensionless correlation lengths agree, the block action will definitely have better continuum properties. It is “closer to the continuum limit”.

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So one has to define what one means by the intrinsic ratio between the scale parameters in two different regularization schemes A and B. In an asymptotically free theory that issue can be resolved by adjusting the behaviour of the $\beta$-function in the weak coupling region \[13, 10, 14\]. In our case in principle we can rely on renormalized perturbation theory. In fact, the two-loop expansion gives

$$m_{\text{phys}}a_s = C_s(\beta_1\lambda_R)^{-\beta_2/\beta_1^2} e^{-1/\beta_1\lambda_R}\{1 + O(\lambda_R)\}$$

where the subscript $s$ denotes the regularization scheme (lattice action or geometry). Asymptotically, at small enough small $\lambda_R$, $a_s/a_{s'} = C_s/C_{s'}$ can be determined by lattice perturbation theory (cf. the results of \[10\] for the hypercubic lattice model).

For a comparison of different effective theories at finite cutoff and not too small $\lambda_R$ we can no longer rely on that approach. It would be appropriate to choose another physical observable $\mathcal{O}$, that may serve as a means to measure the distance from the continuum limit independently. Comparing results due to different actions $S_s$ at the same values of $\mathcal{O}$ then provides an estimate for the regularization dependence. This observable could be for example a $\pi\pi$-scattering cross-section, the $\sigma$-decay width or the violation of euclidean RI of some quantities \[2, 3, 5\]. Different operators $\mathcal{O}$ will usually have different cut-off dependence and their choice needs further motivation: There is no “best” observable. Quantities like the ratio $R$ should be considered versus such a “distance” $\mathcal{O}$ to the continuum limit. Of course we could still use the inverse correlation length $m_{\text{phys}}a_s$, but in order to compare different schemes we have to rescale $a_s$ at each point such that $\mathcal{O}$ agrees.

In this study for four different kinds of lattice actions we compute the envelope of $R$ in the Ising limit ($g = \infty$). In order to relate the respective results we use for $\mathcal{O}$ the RI violations of the inverse correlation length and the inverse 1-particle momentum space propagator.

## 3 Rotational invariance (RI)

The process of regularization by a lattice implies a violation of rotational symmetry for non-vanishing lattice constant. Only in the continuum limit do we hope to restore full euclidean symmetry. Thus the amount of violation of RI provides a measure of how close one is to the continuum.
In order to estimate the violation of RI one can in principle refer to scattering cross sections. However, measuring these in computer simulations is notoriously difficult. For that reason one relied on tree level lattice perturbative calculations in order to estimate the leading violations for e.g. Goldstone-Goldstone scattering in the $O(4)$ model $[13, 10, 2, 3, 5]$ where the Born term is proportional to the massive scalar propagator. In the one component model this would require the study of scattering of massive Higgs particles.

In our approach we want check the violation of RI in the Higgs propagator $[16]$. That quantity is more accessible in the simulation. For simplicity we consider only 2-d slices of our hypercubic lattice. This has the additional advantage that the Monte Carlo data results from operators summed over the other two directions provide better statistics.

The problem then is to quantify the RI of some function $F$ defined on a planar grid. Specifying $F$ to be a real positive function in the plane, with the symmetry property $F(r, \varphi) \equiv F(r, -\varphi) \equiv F(r, \varphi + \frac{k}{2}\pi)$, $k \in \mathbb{Z}$, $\varphi \in [-\pi, \pi)$. Then $F$ has a series-expansion of the form

$$F(r, \varphi) = \sum_{n=0}^{\infty} \alpha_n(r) \cos(4n\varphi) \quad . \quad (11)$$

For fixed $r$ we define $\varphi_0$ to be the directional angle for which $F$ is minimal. For the inverse propagator our models always have $\varphi_0 = 0$; for the inverse correlation length $\varphi_0 = 0$ for $C, C'$ and $\varphi_0 = \pi/4$ for $F_4, S$ (in a d=2 slice). A measure for the deviation from the RI of $F(r, \varphi)$ at the distance $r$ can be defined by

$$\mathcal{N}(r) := \sqrt{\frac{1}{2\pi} \int_{-\pi}^{+\pi} \left[ \frac{F(r, \varphi)}{F(r, \varphi_0)} - 1 \right]^2 d\varphi} = \sqrt{\lambda_0^2(r) + \frac{1}{2} \sum_{n=1}^{\infty} \lambda_n^2(r)} \quad . \quad (12)$$

where

$$\lambda_n(r) = \frac{\alpha_n(r)}{F(r, \varphi_0)} - \delta_{n,0} \quad .$$

The last relation follows from the orthogonality of the cosine functions. The leading contribution to $\mathcal{N}$ may be estimated from the ratio $\frac{F(r, \pi/4)}{F(r, 0)}$ since in leading order $\mathcal{N} \simeq \sqrt{\frac{3}{8}} \left| \frac{F(r, \pi/4)}{F(r, 0)} - 1 \right|$.
We can estimate the violation of RI either in real space or, by Fourier transformation, in momentum space. For infinite lattice size the momentum variables cover finite continuous intervals. In our discussion of the Gaussian model we utilize this property.

In a straightforward application one could study the real space propagator for given fixed coupling $\kappa$. However, it is clear that small and large distances (for periodic b.c.) trivially show strongest RI violation. In order to compare results at different values of the coupling (and the correlation length) we have to specify a typical value of $|x|$ or $|p|$ where we determine $N$ for that coupling. We choose $|p| = m(\kappa)$, a momentum value equal to the inverse correlation length in the direction of the cartesian axes.

For all lattice actions considered we have calculated $N$ for

i) the inverse momentum-space propagator $\tilde{G}^{-1}$, denoted by $N_G$, and

ii) the inverse directional correlation-length $\xi^{-1}$, denoted by $N_\xi$.

In the continuum limit $m \to 0$ we have $N \to 0$.

We determined $N$ for the Gaussian model both analytically with Taylor-expansion and by use of conventional numerical methods. For the Monte Carlo results of our simulation of the Ising model we relied on suitable interpolation of the numerically determined momentum space propagator, as discussed later.

i) Inverse momentum-space propagator

We restrict the momentum-space to two dimensions by considering a particle which moves only in 2 of the 4 space-time dimensions with 4-momentum $p = (p_t, p_x, p_y = 0, p_z = 0)$, with the notation $p_t = m \cos \varphi$, $p_x = m \sin \varphi$, i.e. $|p| = m$.

If the propagator is known explicitly, as it is the case for the Gaussian model, the simplest way to calculate $N_G(m)$ is by the series-expansion of $\tilde{G}^{-1}$ in the functions $\cos(4n\varphi)$ ($s$ denotes the lattice-action). The corresponding coefficients $\alpha_n$ and $\lambda_n$ may then be expressed through Bessel functions and decrease rapidly. For $n \geq 1$ and $p \leq 1$ the ratio $\lambda_{n+1}/\lambda_n \approx 10^{-3n}$. Therefore it is sufficient to consider only the first 2 coefficients [16]. For propagator values given numerically one has to rely on an adequate interpolation or fitting procedure.
ii) Inverse directional correlation length

The inverse directional correlation length \( \xi^{-1}_s(m_s | \hat{r}) \) \[17\] for the lattice-action \( s \) in direction \( \hat{r} = (\hat{r}_1, \hat{r}_2, \hat{r}_3, \hat{r}_4) \), \( |\hat{r}| = 1 \), is defined implicitly by the equation

\[
\tilde{G}_s^{-1}(\kappa_s | p = i\xi^{-1}_s\hat{r}) \equiv 0 . \tag{13}
\]

In that expression \( \xi^{-1}_s \) describes the asymptotic, exponential decay of the configuration-space propagator in direction \( \hat{r} \). In particular, along the main axes, e.g. \( \xi^{-1}(m|(1,0,0,0)) = m(\kappa) \). Again we study the behaviour in a 2-d plane.

Even in the Gaussian model the coefficients \( \alpha_n \) and \( \lambda_n \) cannot be represented exactly, but one may express them through a power-series. The qualitative behaviour for \( m < 1 \) does not differ from the one of \( \tilde{G}_s^{-1} \). For numerically given propagators one may determine the directional correlation length from e.g. a fit along the specified direction (to the suitably interpolated propagator). If the deviation from RI is correctly described by the equivalent Gaussian model, one may rely to a fit to the corresponding propagator.

4 The Gaussian model

Removing the quartic term in (3) we are left with the Gaussian model describing a free scalar particle. The action is a positive-definite quadratic form,

\[
S = \sum_{x,y \in \Lambda_l} \tilde{\Phi}_x Q_{x,y} \tilde{\Phi}_y , \quad Q_{x,y} = \delta_{x,y} - \kappa \sum_{\nu=1}^N \eta_\nu \delta_{x,y-e_\nu} , \tag{14}
\]

and it is possible to solve explicitly for quantities like the propagator. For the momentum-space propagator one finds

\[
\tilde{G}(\kappa | p)^{-1} = 2\left[1 - \kappa N \sum_{\nu=1}^N \eta_\nu \cos(p \cdot e_\nu)\right] , \tag{15}
\]

where the components of \( p \) assume values \( \frac{2n\pi}{L} \), \( n = 0, \ldots, L - 1 \) on an \( L^4 \) lattice with periodic boundary conditions. The configuration-space propagator \( \langle \Phi_y \Phi_x \rangle \) cannot be represented in similar closed analytical form. When
it is summed up over 3-space, however, it is possible to give an explicit expression. For periodic boundary conditions $\mathbf{x} + L \hat{e}_\mu \equiv \mathbf{x}$ and if there are only ferromagnetic interactions (like in the cases $C$, $C'$ and $F_4$), the well-known result (cf. [18]) is

$$
\sum_{\mathbf{x}} \langle \Phi_0 \Phi_\mathbf{x} \rangle = \frac{2\alpha \kappa}{L} \sum_{n_0=0}^{L-1} \mathcal{G} \left( \kappa \mid p_t = \frac{2n_0 \pi}{L}, \bar{0} \right) e^{ip_t t} = \frac{e^{-mt} + e^{-m(L-t)}}{2(1 - e^{-Lm}) \sinh m}.
$$

(16)

(Note that this is the propagator for fields in the continuum notation.) The relation between the lattice mass $m$ and the hopping parameter $\kappa$ is defined implicitly by the pole of the momentum-space propagator in the complex energy-plane

$$
\mathcal{G}^{-1} \left( \kappa \mid im, \bar{0} \right) = 0
$$

with the solution

$$
\sinh \frac{m}{2} = \frac{1}{2} \sqrt{\frac{1 - N \kappa}{\alpha \kappa}}.
$$

(17)

This implies the critical values for $\kappa_c = \frac{1}{N}$ for these three actions (with corresponding values of $N$).

For the case $S$, with the additional antiferromagnetic interaction term, (17) assumes the form

$$
8 - \kappa(45 + 16 \cosh m - \cosh 2m) = 0,
$$

(19)

which has, for $\frac{1}{39} < \kappa < \kappa_c = \frac{2}{15}$, two real solutions $m_1$ and $m_2$ obeying the relation $\cosh m_1 + \cosh m_2 = 8$. Only $m_1$ approaches zero at the critical point. The other pole corresponds to a ‘ghost’ with a negative residue, approaching $m_2 \to \cosh^{-1}(7)$ at the critical point. In the continuum-limit the corresponding physical mass goes to infinity and this state decouples.

For this action (17) assumes the form

$$
\sum_{\mathbf{x}} \langle \Phi_0 \Phi_\mathbf{x} \rangle =
\frac{3}{\cosh m_2 - \cosh m_1} \left[ \frac{e^{-m_1 t} + e^{-m_1(L-t)}}{(1 - e^{-Lm_1}) \sinh m_1} - \frac{e^{-m_2 t} + e^{-m_2(L-t)}}{(1 - e^{-Lm_2}) \sinh m_2} \right].
$$

(20)

The closer to the phase transition we are, the more the physical mode dominates the propagator. Practically the ghost becomes noticeable only at small
distances $t$. E.g. for $m_1 = 1$ at $t = 0$ the ghost term contributes about 20% to the propagator, at $t = 1$ it is 4%, but at $t = 8$ its contribution has already decreased by several orders of magnitude and may be neglected.

We now turn to the RI properties of the propagator. In the Gaussian model we have the advantage that we can work on both, finite and infinite volumes. We consider a two-dimensional submanifold, either in real space or in momentum space.

Fig. 1a gives the inverse momentum space propagator $\tilde{G}^{-1}(p, \varphi)$ for the Gaussian models and a given value of $m = 0.407$, and in Fig. 1b we plot the corresponding $\varphi$-dependence for $p = \frac{\pi}{2}$ as an example.

Following the above discussion and requiring $|p| = m$, in Fig. 2 we show lines of constant values of $\tilde{G}^{-1}(p_x = m \sin \varphi, p_t = m \cos \varphi)$, as derived for infinite lattice size. For small $m$ (small lattice spacing) rotational symmetry is restored.

All this information may be summarized by plotting $N_G$ and $N_\xi$, shown in Fig. 3 for $p \leq 1$. In each case the RI is violated only up to a very small amount, especially for $F_4$, where $N_G, N_\xi < 0.0004$ for that region of values. Even for the standard hypercubic action $C$ where the largest violations appear, $N$ does not exceed the value 0.012.

Generally in two or more dimensions $\xi_s^{-1}$ is only calculable approximately, e.g. as a Taylor-series in the lattice-mass $m_s$,

$$\xi_s^{-1}(m_s | \hat{r}) = m_s \left[ 1 + a_s(\hat{r})m_s^2 + O(m_s^4) \right]. \quad (21)$$

Both RI-functions $N_G$ and $N_\xi$ may be expanded

$$N_F(m_s) = A_s(m_s^2)^{\gamma_s} \left[ 1 + B_{l,F}m_s^2 + O(m_s^4) \right]. \quad (22)$$

The leading coefficient $A_s$ is independent of $F \in \{ \tilde{G}_s^{-1}, \xi_s^{-1} \}$, but it is not allowed to conclude that $A_s$ is a universal constant [2]. The same holds for $\gamma_s$; it is a measure for the degree of euclidean symmetry restoration in the limit $m_s \to 0$ and has for $C, C'$ the value 1 and for $F_4, S$ the value 2. There are no $O(a^2)$ corrections for $F_4$ and the tree-level improved Symanzik action $S$, which shows the higher degree of RI for these actions, one of the reasons for their construction.

Postulating

$$N_B(m_B) \overset{!}{=} N_A(m_A) \quad (23)$$
we could rescale the corresponding definitions of the lattice constants and get asymptotic ratios

\[
\frac{a_{C'}}{a_C} \rightarrow \sqrt{7} \quad , \quad \frac{a_{F_4}}{a_S} \rightarrow \sqrt{6} ,
\]

for the Gaussian models.

## 5 Results for the Ising limit

The upper bound on the renormalized coupling and on the Higgs boson mass is realized for bare coupling \( g \to \infty \), the Ising limit. Thus it has become customary to study \( \Phi^4 \)-models in that simple limit restricting the range of values of the field variable to \( \pm 1 \).

In our Monte Carlo simulation we worked on lattices of size \( 16^4 \) and periodic boundary conditions. Each of the 4 actions was considered at 4–6 values of the coupling parameter \( \kappa > \kappa_{cr} \) in the broken phase, close to the critical point \( \kappa_{cr} \) such that \( m \in (0.2, 0.9) \). For the updating we used the cluster algorithm \[19\] (actions \( C, C' \) and \( F_4 \)) or the Metropolis method (action \( S \), where we had problems with the cluster updating due to presence of additional antiferromagnetic interaction terms). In the determination of the propagators we utilized reduced variance operators \[20\]. For each data point we considered 40 000 (cluster algorithm) or 100 000 (Metropolis) update-measurement pairs, in addition to a suitable number of equilibrating updates.

In this way we obtained the boson propagator summed over \( y, z \) directions

\[
G(\kappa|t, x) := \langle \Phi_{0,0,0,0} \sum_{y,z} \Phi_{t,x,y,z} \rangle
\]

and its Fourier transform. We also determined the susceptibility

\[
\chi = \sum_{t,x} G(\kappa|t, x) = \frac{Z_R}{m_R^2} .
\]

In order to extract the required quantities \( m_s = a_s m_{phys} \) and \( R \), we used \[16\] and \[21\] for the Gaussian propagator

\[
G_s(t) := \sum_x G_s(t, x) \approx a(e^{-mt} + e^{-m(L-t)}) + b ,
\]
where $b$ gives the square of the vacuum expectation value, $\langle \Phi \rangle^2$. For our analysis we therefore use this value for $| \langle \Phi \rangle |$. The direct estimate of $\langle \Phi \rangle$ is not possible due to tunneling between the two ground states.

For the tree level improved action $S$ we also add another term to (27),

$$a'(e^{-m'(m)t} + e^{-m'(m)(L-t)})$$

(28)

where $m'(m)$ is assumed to be given by the relation $\cosh m + \cosh m' = 8$ which is valid for the corresponding Gaussian model.

The maximal relative statistical errors for the fit parameters amount to $\Delta m \approx \pm 2 - 3\%$, $\Delta a \approx \pm 1 - 2\%$, $\Delta b \approx \pm 0.1\%$ for results due to cluster updating and measuring. For the conventional updating and measurement (case $S$) the errors are typically larger by a factor 4.

In Fig. 4 we plot $\kappa_I$ vs. $\kappa_G$ such that the masses determined for the Ising- and the Gaussian model at corresponding values of the reduced coupling $\tau = |\kappa - \kappa_c| / \kappa_c$ agree. On an infinite lattice one expects $\tau_G \propto |\ln \tau_I|^{-\frac{1}{2}}$ and the extrapolation to the critical values $\kappa_{G,c}$ thus should provide an estimate of $\kappa_{I,c}$. On finite size however, the mass does not vanish at the phase transition. Still, within the accuracy of our results the points obey the scaling law quite accurately and we confirm that the leading critical exponents of $m(\tau)$ agree in the two cases; the logarithmic factor cannot be resolved.

Comparing the RI properties of our Ising model results for $G(t, x)$ with those of the Gaussian version for same mass we find surprising agreement (cf. [16]). In Fig.s 1a and 1b we compare the inverse Fourier transformed propagator $\tilde{G}^{-1}(p, \phi)$ with the results due to the Gaussian model, to point out the good agreement. Indeed, within the error bounds we could not find any systematic deviations from the Gaussian behaviour. This conforms with the expectation that the boson propagator in the broken phase of the Ising model is in leading order that of a free particle like in the Gaussian model – up to logarithmic corrections. But for the rotational symmetry properties of the propagator and the considered quantities $N'$ those seem to be small enough to permit neglecting them. This justifies to use the Gaussian values as our measure for violation of RI at given mass, at least for the mass values considered here. This is what we do in the subsequent discussion.

In Table 2 we summarize our results for mass, $f = \langle \Phi_R \rangle = \langle \Phi \rangle / \sqrt{Z}$, wave function renormalization constant determined from (24), the ratio $R$ of (2) and the value of $N'_G$, one of our measures of violation of rotational symmetry.
of the boson propagator. The large errors for the results $S$ are due to the different updating method used there. In Figs. 5 and 6 we plot $m_R$ and $f$ vs. the coupling and compare with the fit to the expected asymptotic behaviour \[21\] \[10\] for $\tau \to 0$

\[m_R \sim C_4 t^{\frac{1}{2}} |\ln t|^{\frac{1}{2}} / \bar{\alpha},\]

\[f \sim C_f t^{\frac{1}{2}} |\ln t|^{\frac{1}{2}}.\] \tag{29}

In Tab. 3 we give the results of our joint fit to the data for $m_R$ and $f$. The ratios $C_4/C_f$ are compatible with the asymptotic value $4\pi \sqrt{2}/3 \ [10]$. Fig. 7 gives $R$ vs. $1/m_R$, where one could in principle read off the triviality bound. Comparing the results for actions $C$ and $S$ we find a difference similar to that observed in the recent study of $O(4)$ models in \[1\]. However, as discussed, it may be more appropriate to use another quantity than $1/m_R$, in order to compare the action (and regularization) dependence of that bound. We return to that point later.

From these data one may obtain the constant $C'$ (cf. \[10\]) relating $m_R$ to the renormalized coupling,

\[C' = \lim_{\lambda_R \to 0} \left[ m_R (\beta_1 \lambda_R)^{\beta_2 / \beta_1} e^{1/\beta_1 \lambda_R} \right]. \tag{30}\]

Depending on the closeness to the continuum limit, this fit to the asymptotic behaviour may be not very reliable. Also, $C'$ is determined just from our data for lattice size $16^4$ and should be extrapolated to infinite size in a finite size scaling study. Within these limitations we obtain the numbers given in Tab. 3. For action $C$ Lüscher and Weisz\[10\] determined the asymptotic value of $C = 5.3(1.2)$; this is definitely larger than ours. However, fitting their results in the region of mass values considered here, we get a value of $\simeq 3.0(3)$. We are obviously not asymptotic enough for $C$.

As discussed it may not be wise to compare the values of $R$ for different actions with each other at some lattice mass $m_R$. Instead we follow \[2\] \[5\] and plot in Fig. 8 the renormalized coupling vs. $N_G$, our measure for the amount of violation of RI.

First we notice, that the difference between the results for $C$ and $C'$ and between those for $F_4$ and $S$ appears to become smaller, as compared to the presentation in Fig. 7; it is typically less than 10%. However, the difference between $C$, $C'$ on one hand and $F_4$, $S$ on the other appears to be sizeable, about 30-40%.
In the results for the $O(4)$ model for $F_4$ and $S$ the authors estimated the regularization dependence by comparing the results at given value of another measure of RI, derived from a perturbation expansion of the Goldstone-Goldstone scattering cross-section. There the difference between $F_4, S$ compared to $C$ was smaller than ours, of the order of 10-20% for 1-10% violation of RI. Although this variation could be due to the study of different models (one component vs. four component fields) this may also indicate a strong dependence on the choice of measure for RI. Maybe a systematic study of higher order terms, like the one initiated recently helps to clarify this issue.

Let us summarize our results for the Ising model:

- The similarities in the propagator between the Gaussian version and the Ising version within the same type of action (regularization) are bigger than between Gauss ↔ Gauss or between Ising ↔ Ising for different lattice actions. Therefore the RI behaviour for the discussed range of couplings seems to be dominated mainly by leading kinematical effects, which are already given in the corresponding Gaussian model.

- The difference between the two measures of RI, $N_G$ and $N_\xi$ are less than 1%. This result is due to the close relationship between the correlation length and the momentum space propagator.

- The regularization dependence of $R$ comparing $C$ with $C'$ and $F_4$ with $S$ is less than 10% in the considered domain of couplings.

- The regularization dependence of $R$ comparing $C$ or $C'$ with $F_4$ or $S$ is about 30-40 % in the considered domain of couplings.

The last mentioned result allows two interpretations: The first is, that the regularization dependence of the envelope indeed may be about 40%. A similar conclusion was put forward in [22, 7]. On the other hand the studies for the $O(4)$ model do not support this strong variation. So the second interpretation may be that there is substantial sensitivity on the definition of a measure of RI.

Leaving aside the issue of the regularization dependence of the triviality bound, we found that different lattice actions lead to propagators with remarkable rotational symmetry, quite contrary to the naive expectation. The
leading violations are well described already by the corresponding Gaussian models.

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Tables

Table 1: The parameters for the four actions considered, defined by embedding into the hypercubic lattice.

| l  | N   | η_ν | α     | \{e_ν, ν = 1, ..., N\} |
|----|-----|-----|-------|------------------------|
| C  | 8   | 1   | 1     | ±e_μ                   |
| C' | 32  | 1   | 7     | ±e_μ, ±(e_μ ± e_λ), λ < μ |
| F_4| 24  | 1   | 6     | ±(e_μ ± e_λ), λ < μ    |
| S  | 16  | 1   | 3/4   | ±e_μ, ±2e_μ            |

Table 2: The parameters for our results in the broken phase of the Ising limit.

| Action | κ     | m_R     | f ≡ ⟨Φ_R⟩ | Z_R   | R    | N_G |
|--------|-------|---------|-----------|-------|------|-----|
| C      | 0.1507| 0.244(4)| 0.0832(1) | 0.929(2) | 2.93(4) | 7.6(2)E-4 |
|        | 0.1513| 0.328(3)| 0.1043(6) | 0.924(4) | 2.14(4) | 1.37(2)E-3 |
|        | 0.1520| 0.407(3)| 0.1227(5) | 0.918(6) | 2.31(3) | 2.09(3)E-3 |
|        | 0.1525| 0.449(1)| 0.1339(2) | 0.909(3) | 2.35(1) | 2.55(1)E-3 |
|        | 0.1550| 0.637(2)| 0.1769(1) | 0.886(2) | 2.60(1) | 5.07(3)E-3 |
|        | 0.1560| 0.698(5)| 0.1914(3) | 0.873(3) | 2.65(3) | 6.06(8)E-3 |
| C'     | 0.0336| 0.231(1)| 0.0896(1) | 0.974(2) | 2.58(1) | 9.80(9)E-5 |
|        | 0.0338| 0.319(2)| 0.1198(2) | 0.965(2) | 2.66(2) | 1.88(9)E-4 |
|        | 0.0340| 0.393(6)| 0.1415(3) | 0.964(5) | 2.78(5) | 2.88(9)E-4 |
|        | 0.0346| 0.560(10)| 0.1903(3) | 0.955(3) | 2.95(4) | 6.0(2)E-4 |
|        | 0.0354| 0.738(4)| 0.2372(4) | 0.944(3) | 3.11(2) | 1.07(1)E-3 |
| F_4    | 0.0460| 0.255(3)| 0.0875(1) | 0.959(2) | 2.91(3) | 1.79(8)E-6 |
|        | 0.0470| 0.474(1)| 0.1460(1) | 0.936(1) | 3.27(1) | 2.12(2)E-5 |
|        | 0.0482| 0.670(2)| 0.1873(2) | 0.930(2) | 3.58(1) | 8.34(9)E-5 |
|        | 0.0502| 0.917(4)| 0.2368(6) | 0.913(4) | 3.87(2) | 2.85(5)E-4 |
| S      | 0.1653| 0.23(2) | 0.078(2) | 0.88(4) | 3.0(4) | 0.7(2)E-5 |
|        | 0.1660| 0.33(1) | 0.100(2) | 0.89(2) | 3.3(2) | 3.0(4)E-5 |
|        | 0.1675| 0.47(2) | 0.134(2) | 0.85(3) | 3.5(2) | 1.2(2)E-4 |
|        | 0.1683| 0.55(3) | 0.147(4) | 0.85(4) | 3.7(3) | 2.3(5)E-4 |
|        | 0.1694| 0.61(2) | 0.165(2) | 0.82(2) | 3.7(1) | 3.4(4)E-4 |
Table 3: Results for $\kappa_{c}^{(16)}$, $C_4$ and $C_f$ from a joint fit to $m_R(\tau)$ and $f(\tau)$ according to (29); we also give $C'$ according to a fit to (30).

| Action | $\kappa_{c}^{(16)}$ | $C_4$  | $C_f$    | $C'$  |
|--------|---------------------|--------|----------|-------|
| $C$    | 0.14984(2)          | 4.24(1)| 0.636(1) | 2.9(1) |
| $C'$   | 0.03340(1)          | 3.76(1)| 0.674(1) | 6.4(2) |
| $F_4$  | 0.04559(1)          | 3.33(1)| 0.550(1) | 3.2(1) |
| $S$    | 0.16447(8)          | 4.38(8)| 0.624(5) | 2.4(1) |
Figures

Fig. 1: (a) The inverse momentum space propagator $\tilde{G}^{-1}(p, \varphi)$ for the Gaussian model (action type $C$) for $m = 0.407$; the points give the numeric results from a simulation of the corresponding Ising model at coupling 0.1520 such that $m$ agrees (cf. Table 2). (b) The values of $\tilde{G}^{-1}(p = \pi/2, \varphi)$ compared to the corresponding values of the Ising model simulation (resulting from an interpolation of points along the cartesian directions on the $16^4$ lattice grid).

Fig. 2: Lines of constant values of $\xi^{-1}(m, \varphi)$, as derived in the Gaussian model for infinite lattice size, for the four actions considered. For small $m$ (small lattice spacing) rotational symmetry is restored.

Fig. 3: Our measures for the violation of RI $N_G$ (full lines) and $N_\xi$ (dashed lines for $m \leq 1$).

Fig. 4: We plot $\kappa_I$ vs. $\kappa_G$ such that the masses determined at corresponding values of $\tau$ agree (action $C$; triangles: $L = 8$, squares: $L = 16$, circles: results for infinite size due to [10]).

Fig. 5: $f$ vs. $\kappa$ for the four actions considered; the curves give the fits to the expected critical behaviour [23]. All data is for $L = 16$; in the figure for $C$ the triangles denote the results for infinite lattice size due to [10].

Fig. 6: $m_R$ vs. $\kappa$ for the four actions considered; the curves give the fits to the expected critical behaviour [23]. All data is for $L = 16$; in the figure for $C$ the triangles denote the results for infinite lattice size due to [10].

Fig. 7: $R$ vs. $1/m_R$ for the four actions considered (bars: $C$, squares: $C'$, triangles: $F_4$, asterisks: $S$); the curves give the fits to the asymptotic behaviour [30]. All data is for $L = 16$.

Fig. 8: The ratio $R$ vs. $N_G$; the curves denote fits to the asymptotic behaviour $R(m_R)$, where $m_R$ is related to the corresponding value of $N_G$ as discussed. The left-hand part gives the results on an extended scale. Symbols are like in Fig. 7.