Correlation Exponent and Anomalously Localized States at the Critical Point of the Anderson Transition

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We study the box-measure correlation function of quantum states at the Anderson transition point with taking care of anomalously localized states (ALS). By eliminating ALS from the ensemble of critical wavefunctions, we confirm, for the first time, the scaling relation \( z(q) = d + 2\tau(q) - \tau(2q) \) for a wide range of \( q \), where \( q \) is the order of box-measure moments and \( z(q) \) and \( \tau(q) \) are the correlation and the mass exponents, respectively. The influence of ALS to the calculation of \( z(q) \) is also discussed.

KEYWORDS: Anderson transition, critical states, multifractal, anomalously localized states, two-dimensional symplectic system, forced oscillator method

1. Introduction

It is widely accepted that critical wavefunctions at the Anderson metal-insulator transition are multifractal,\(^1,2\) since the multifractal nature at criticality forms a basis of the single-parameter scaling argument of the Anderson transition,\(^3\) properties of critical wavefunctions have been extensively studied.\(^4\) Multifractal properties of wavefunctions are usually characterized by the mass exponent \( \tau(q) \) or the multifractal spectrum \( f(\alpha) \), where \( \tau(q) \) describes the system-size dependence of the local moments \( Z_q \) of the wavefunction distribution and \( f(\alpha) \) is the fractal dimension of the spatial distribution of boxes with the Lipschitz-Hölder exponent \( \alpha \). It is well known that \( \tau(q) \) is related to \( f(\alpha) \) through the Legendre transform. This relation provides a key criterion in judging whether numerically calculated exponents are appropriate. However, difficulty in calculating \( f(\alpha) \) from its definition prevents us from utilizing actually this relation for checking the accuracy of the exponent. For this purpose, we should, thus, provide other relations between exponents characterizing multifractality of a critical wavefunction. A possible candidate is the scaling relation \( z(q) = d + 2\tau(q) - \tau(2q) \), where \( z(q) \) is the correlation exponent describing the box-measure correlation function. The box-measure correlation function is a generalization of the local moment \( Z_q \) and can be easily obtained in numerical calculations. Although Pracz et al. have tried to confirm this scaling relation for the quantum Hall transition,\(^5\) \( z(q) \) coincides with \( 2 + 2\tau(q) - \tau(2q) \) only in a narrow range of \( q \) around \( q = 0 \) in their result. Since the profile of \( z(q) \) near \( q = 0 \) is not sensitive to details of the multifractal distribution, it is important to confirm the scaling relation for large \( q \) values.

In order to clarify the reason why the calculated correlation exponent \( z(q) \) deviates from \( 2 + 2\tau(q) - \tau(2q) \) at large \( q \)'s in their work, we notice the fact that some states are spatially "localized" due to statistical fluctuations of disorder even not in an insulating phase. These quantum states are called anomalously localized states (ALS).\(^6-14\) Major interest in ALS is focused on them in a metallic phase because of the lack of theoretical tools for critical states. There should, however, exist ALS also at the critical point, because the statistical fluctuation of disorder at criticality easily yields a pre-localized state compared to ALS in the metallic regime. In fact, several previous works directly and indirectly support the existence of ALS at the critical point.\(^15-17\) Recently, we have shown that the ratio of the number of ALS at criticality to the number of the total states increases with system size \( L \) and saturates at a certain value for \( L \rightarrow \infty \), while typical states are kept to be multifractal.\(^18\) Since the scaling theory and the renormalization group theory are based on the multifractal nature of critical wavefunctions (i.e., divergence of the correlation length at the transition point), conclusions concerning properties just at the critical point obtained from these theories are valid only for typical states. Therefore, the scaling relation on the correlation exponent \( z(q) \) should be considered for typical critical states. This would be a primary reason why the scaling relation has not been provided for large \( q \)'s in the previous numerical work.

In this paper, we investigate the scaling relation for the correlation exponent \( z(q) \) by taking into account the influence of ALS at the critical point. Critical wavefunctions are prepared in two-dimensional symplectic systems which are described by the SU(2) model.\(^19\) In order to obtain \( z(q) \) for typical critical states, we compose an ensemble of critical wavefunctions from which ALS are eliminated. The exponent \( z(q) \) calculated for such an ensemble satisfies the scaling relation for large \( q \) values. We quantify how well the ensemble from which ALS are eliminated describes statistical properties of typical critical wavefunctions by a parameter \( \Gamma^* \). It is found that the range of \( q \) for which \( z(q) \) satisfies the scaling relation depends on \( \Gamma^* \). This paper is organized as follows. In §2, we give a quantitative definition of ALS at the critical point based on the idea that ALS at criticality do not show multifractality. In §3, the scaling relation for the correlation exponent is reminded in the scaling argument for the box-measure correlation function. We briefly explain, in §4, the SU(2) model and a numerical method to obtain eigenstates of this model. Results of our numerical study are given in §5. The insensitive na-
ture of \( z(q) \) near \( q = 0 \) to ALS is also discussed in this section. Section 6 is devoted to conclusions.

2. Definition of Anomalously Localized States

In order to study the influence of ALS quantitatively, we employ a definition of ALS proposed in ref. 18. This definition is based on the idea that ALS are not multifractal as a consequence of their localized nature. At first, we introduce the box-measure correlation function \( G_q(l, L, r) \) defined by\(^{20,21} \)

\[
G_q(l, L, r) = \frac{1}{N_b N_b} \sum_{b} \sum_{b'} \mu^q_{b(l)} \mu^{q}_{b'(l)}, \tag{1}
\]

where \( \mu_{b(l)} = \sum_{i \in b(l)} |\psi_i|^2 \) and \( \mu_{b'(l)} = \sum_{i \in b'(l)} |\psi_i|^2 \) are the box measures for wavefunction amplitudes \( \psi_i \), in a box \( b(l) \) of size \( l \) and in a box \( b'(l) \) of size \( l \) fixed distance \( r - l \) away from the box \( b(l) \), respectively. \( N_b \) (or \( N_{b'} \)) is the number of boxes \( b(l) \) [or \( b'(l) \)], and the summation \( \sum_b \) (or \( \sum_{b'} \)) is taken over all boxes \( b(l) \) [or \( b'(l) \)] in the system of size \( L \). If a wavefunction is multifractal, \( G_q(l, L, r) \) should behave as\(^{21} \)

\[
G_q(l, L, r) \propto l^{x(q) - y(q)} r^{-z(q)}, \tag{2}
\]

where \( x(q) \), \( y(q) \), and \( z(q) \) are exponents describing multifractal correlations of the amplitude distribution. This relation is sensitive to ALS as demonstrated in ref. 18 and then suitable for defining ALS. To find the \( l \) and \( r \) dependences of \( G_q(l, L, r) \), we concentrate on the following functions,

\[
Q_q(l) = G_q(l, L, r = l) \propto l^{x(q) - z(q)}, \tag{3}
\]

and

\[
R_q(r) = G_q(l = 1, L, r) \propto r^{-z(q)}. \tag{4}
\]

In order to quantify non-multifractality of a specific wavefunction, it is convenient to introduce variances \( \text{Var} (\log Q_2) \) and \( \text{Var} (\log R_2) \) from the linear functions of \( \log l \) and \( \log r \), \( \log Q_2(l) = [x(2) - z(2)] \log l + c_Q \) and \( \log R_2(r) = -z(2) \log r + c_R \), respectively, calculated by the least-square fit. From these variances, a quantity \( \Gamma \) is defined by

\[
\Gamma(L, \lambda) = \lambda \text{Var} (\log Q_2) + \text{Var} (\log R_2), \tag{5}
\]

where \( \lambda \) is a factor to compensate the difference between average values of \( \text{Var} (\log Q_2) \) and \( \text{Var} (\log R_2) \). Using \( \Gamma \) given by eq. (5), the quantitative and expeditious definition of ALS at criticality is presented by

\[
\Gamma > \Gamma^*, \tag{6}
\]

where \( \Gamma^* \) is a criterial value of \( \Gamma \) to distinguish ALS from multifractal states. We can compose a refined ensemble by eliminating ALS from a set of critical wavefunctions. The quality of the refined ensemble is controlled by \( \Gamma^* \).

3. Scaling Relation for the Correlation Exponent

In this section, we give a brief explanation of the scaling relations between the exponents \( x(q) \), \( y(q) \), \( z(q) \), and \( \tau(q) \). The mass exponent \( \tau(q) \) which is commonly used in the multifractal analysis is defined by

\[
Z_q(l) \equiv \sum_{b} \mu^q_{b(l)} \propto l^{\tau(q)}. \tag{7}
\]

The local moment \( Z_q(l) \) is related to the correlation function \( G_q(l, L, r) \). Comparing definitions of \( G_q \) and \( Z_q \) [eqs. (1) and (7)], we have

\[
G_q(l, L, r = l) = \frac{Z_q(l)}{N_b}. \tag{8}
\]

Therefore, the relation \( N_b = (L/l)^d \) and eqs. (2) and (7) lead

\[
x(q) - z(q) = y(q) = d + \tau(2q). \tag{9}
\]

For the case of \( r = L \), eq. (1) gives

\[
G_q(l, L, r = L) = \frac{1}{N_b^2} \left( \sum_{b} \mu^q_{b(l)} \right)^2. \tag{10}
\]

Since \( G_q(l, L, r = L) \propto l^{x(q)} L^{-y(q) - z(q)} \) from eq. (2), the exponents are related as

\[
x(q) = y(q) + z(q) = 2d + 2\tau(q). \tag{11}
\]

From eqs. (9) and (11), we obtain

\[
x(q) = 2d + 2\tau(q), \tag{12}
\]

\[
y(q) = d + \tau(2q), \tag{13}
\]

and for the correlation exponent

\[
z(q) = d + 2\tau(q) - \tau(2q). \tag{14}
\]

The last relation with \( q = 1 \) is equivalent to the well-known scaling relation \( \eta = d - D_2 \), where \( \eta \) is the exponent describing the two-particle correlation function \( S(r) = \rho(|\psi(0)|^2|\psi(r)|^2) \), \( \rho \) is the density of states, and \( D_2 \) is the correlation dimension of critical wavefunctions. This equivalency can be understood from the relations \( z(1) = \eta \) because of \( G_1(l = 1, L, r) \sim S(r), \tau(1) = 0, \) and \( \tau(2) = D_2 \).

The above relations (12)-(14) hold for any \( q \) in principle if the wavefunction is multifractal. However, the previous work confirming the scaling relation eq. (14) for the quantum Hall transition\(^5 \) shows that the correlation exponent \( z(q) \) coincides with \( 2 + 2\tau(q) - \tau(2q) \) for a narrow range of \( q \) (\( |q| < 1.5 \)). As mentioned in §1, we consider that the disagreement at large \( |q| \)'s is a consequence of ALS. It should be noted that the agreement for small \( |q| \)'s is rather trivial. Even if the wavefunction is not multifractal, the exponents satisfy the scaling relation eqs. (12)-(14) for \( q = 0 \), because \( \tau(0) = -d \) and \( x(0) = y(0) = z(0) = 0 \) for any \( \mu_{b(l)} \). Increasing \( |q| \) slightly from zero, the correlation exponent \( z(q) \), for example, is well approximated by the parabolic form

\[
z(q) = 2(\alpha_0 - d)q^2, \tag{15}
\]

where \( \alpha_0 \) is the Lipschitz-Hölder exponent giving the maximum of \( f(\alpha) \). If \( \alpha_0 \) is insensitive to details of the distribution of measures, \( z(q) \) approximately satisfies eq. (14) near \( q = 0 \), because \( \tau(q) \) is given by \( (1 - q)|q(\alpha_0 - d) - d| \).
in the parabolic approximation. We should, thus, confirm the scaling relations for large \(|q|\)'s. Since the exponent \(z(q)\) characterizes directly the spatial correlation of wavefunction amplitudes, we focus only on the relation (14) hereafter.

4. System and Numerical Method

Considering the advantage of system sizes, we focus our attention on the Anderson transition in two-dimensional electron systems with strong spin-orbit interactions, in which systems have no spin-rotational symmetry but have the time-reversal one. Hamiltonians describing these systems belong to the symplectic ensemble. Among several models belonging to this universality class, we adopt the SU(2) model because of its small scaling corrections. The Hamiltonian of the SU(2) model is given by

\[ H = \sum_i \varepsilon_i c_i^\dagger c_i - V \sum_{i,j} R_{ij} c_i^\dagger c_j, \]

(16)

where \(c_i^\dagger (c_i)\) is the creation (annihilation) operator acting on a quaternion state vector, \(R_{ij}\) is the quaternion-real hopping matrix element between the sites \(i\) and \(j\), and \(\varepsilon_i\) denotes the on-site random potential distributed uniformly in the interval \([-W/2, W/2]\). Bold symbols represent quaternion-real quantities. The matrix element \(R_{ij}\) is given by

\[ R_{ij} = \cos \alpha_{ij} \cos \beta_{ij} \tau^0 + \sin \gamma_{ij} \sin \beta_{ij} \tau^1 - \cos \gamma_{ij} \sin \beta_{ij} \tau^2 + \sin \alpha_{ij} \cos \beta_{ij} \tau^3, \]

(17)

for the nearest neighbor sites \(i\) and \(j\), and \(R_{ij} = 0\) for otherwise. Here, \(\tau^\mu (\mu = 0, 1, 2, 3)\) is the primitive element of quaternions. Random quantities \(\alpha_{ij}\) and \(\gamma_{ij}\) are distributed uniformly in the range of \([0, 2\pi]\), and \(\beta_{ij}\) is distributed according to the probability density \(P(\beta)d\beta = \sin(2\beta)d\beta\) for \(0 \leq \beta \leq \pi/2\). Randomly distributed hopping matrix elements shorten the spin relaxation length which is a dominant irrelevant length scale, and scaling corrections can be negligible in the SU(2) model. It is known that the localization length exponent \(\nu\) of this model is 2.73 \(\pm\) 0.02 and the critical disorder \(W_c\) is 5.952\(V\) at \(E = 1.0V\). \(^19\)

Critical wavefunctions of the SU(2) model have been calculated by using the forced oscillator method (FOM) \(^24\) extended to the eigenvalue problem of quaternion-real matrices. Of course, the Hamiltonian eq. (16) can be represented by complex numbers, and we can use the usual FOM for complex Hermitian matrices to solve the eigenvalue problem. The modified FOM for quaternion-real matrices, however, enables us to calculate eigenvalues and eigenvectors within about a half of CPU time. \(^25\) It should be remarked that the obtained eigenvector is a quaternion-real vector. This vector represents two physical states simultaneously, which correspond to the Kramers doublet. Since the amplitude distribution of these degenerate states are the same, we analyze one of the calculated Kramers doublet.

5. Results

In order to study the correlation exponent \(z(q)\) for critical wavefunctions of the SU(2) model, we calculate \(10^4\) critical wavefunctions of this model at \(E = 1.0V\) and \(W = 5.952V\) by the FOM. Each eigenstate is obtained for a single disorder realization. Periodic boundary conditions are imposed in the \(x\) and \(y\) directions in systems of size \(L = 120\).

Figure 1(a) shows the \(q\) dependence of the correlation exponent \(z(q)\) and the right-hand side of eq. (14), i.e., \(d + 2\tau(q) - \tau(2q)\), calculated for the whole set of the prepared critical wavefunctions (the original ensemble). The exponents \(z(q)\) and \(\tau(q)\) are calculated from the geometric means of \(R_q(r)\) and \(Z_q(l)\) by the least-square fit, respectively. It is found that \(z(q)\) deviates from \(z_r(q)\) at large \(|q|\)'s. The degree of agreement is similar to the result of the previous work for the quantum Hall transition. \(^5\) We see that \(z(q)\) for \(q \geq 5\) exceeds the value of 2 which is the theoretical bound of \(z(q)\).

It should be noted that the scaling relation eq. (14) is valid for typical critical wavefunctions. As shown in our previous work, \(^18\) the distribution function of \(\Gamma\) defined by eq. (5) has its peak at \(\Gamma = 0\), which implies that typical critical wavefunctions are multifractal. Since the geometric mean usually represents the typical value (mode value), it seems that the geometric mean \(\langle R_q(r)\rangle\) used in our calculations gives the correct exponent \(z(q)\) of multifractal critical wavefunctions without influence from ALS. However, the distribution function of \(\Gamma\) de-

![Fig. 1. Calculated correlation exponent \(z(q)\) (open circles) and the right-hand side of eq. (14) \([z_r(q) = d + 2\tau(q) - \tau(2q)]\) (filled circles). Figures labelled by (a) and (b) correspond to the results for the original ensemble and the refined ensemble, respectively.](image-url)
fined for the correlation function \( G_q \) becomes broad with increasing \( |q| \), and then the geometric mean could not represent the typical value for an ensemble composed of an insufficient number of elements due to enhanced statistical fluctuations. In fact, we have confirmed that the correlation exponent calculated from the typical profile of \( R_q(r) \) largely deviates from the exponent computed from \( \langle R_q(r) \rangle \) for large \( |q| \). Although the correlation exponent appeared in eq. (14) should be computed from the typical profile of \( R_q(r) [R_q^{\text{typ}}(r)] \), it is actually quite difficult in numerical calculations to calculate precise value of \( z(q) \) from \( R_q^{\text{typ}}(r) \).

In this paper, we show that the elimination of ALS from the original ensemble of critical wavefunctions is efficient technique to obtain accurate value of \( z(q) \) even for large \( |q| \). At first, we compose a refined ensemble by eliminating ALS from the original ensemble. The refined ensemble is characterized by the parameters \( \lambda = 3 \) and \( \Gamma^* = 0.06 \), and contains 6962 critical wavefunctions. The correlation exponent calculated from \( \langle R_q(r) \rangle \) for this refined ensemble is presented in Fig. 1(b). The exponent \( z(q) \) is in agreement with \( z_\tau(q) \) for a wider range of \( q \), and does not exceed the theoretical bound 2. This demonstrates the efficiency of the elimination scheme of ALS to calculate precise exponents defined at the critical point. We find, however, that the exponent \( z(q) \) deviates from \( z_\tau(q) \) for \( q \lesssim -2 \) even for the refined ensemble. What is even worse, \( z(q) \) becomes negative for \( q \ll -2 \). To see the reason of this unreasonable result, let us consider a simple case of \( R_q(r) \), namely, \( R_q(r) \) for quantities \( \mu_i \) distributed uniformly in the range of \([0,1]\) without any spatial correlations. From the definition of \( R_q(r) \) [or \( G_q(l, L, r) \)], we have

\[
R_q(r) = \frac{1}{NN_r} \sum_{i} \sum_{j \in r(i)} \mu_i^q \mu_j^q,
\]

where \( N \) is the total site number, \( \sum_{j \in r(i)} \) represents the summation over sites \( j \) away from the site \( i \) by a distance \( r \), and \( N_r \) is the number of such sites. The uncorrelated distribution of \( \mu_i \) makes it possible to replace \( \mu_i^q \) by its average value \( \langle \mu^q \rangle \), and we obtain

\[
R_q(r) = \frac{\langle \mu^q \rangle^2}{NN_r} \sum_{i} \sum_{j \in r(i)} = \langle \mu^q \rangle^2.
\]

From this relation, it seems that the exponent \( z(q) \) becomes zero for any \( q \), because \( \langle \mu^q \rangle \) does not depend on \( r \). This is, however, not true for negative \( q \). The distribution function of \( \mu^q \) is given by \( \theta(t)/[q^t(q-1)/q] \) for positive \( q \) and \([1-\theta(t)]/[q^t(q-1)/q] \) for negative \( q \), where \( t = \mu^q \) and \( \theta(t) \) is the step function defined as \( \theta(t) = 1 \) for \( 0 \leq t \leq 1 \) and \( \theta(t) = 0 \) for \( t > 1 \). For \( q > 0 \), the distribution is truncated at \( t = 1 \) and exponent \( (q-1)/q \) in the distribution function is less than unity, which gives a finite average value of \( \mu^q \). On the contrary, the distribution function for negative \( q \) extends over the range of \([1, \infty) \). The exponent \( (q-1)/q \) becomes less than 2 for \( q < -1 \), which leads an infinite average value of \( \mu^q \). Thus we cannot define the exponent \( z(q) \) from eq. (19) for \( q < -1 \). The divergence of \( \langle \mu^q \rangle \) is a consequence that the distribution function of \( \mu \) is finite at \( \mu = 0 \). Since a distribution of multifractal measures is not uniform (but log-normal approximately) and spatial correlations of them must be taken into account, the above argument cannot be applied straightforwardly to refined ensembles of critical wavefunctions. Nevertheless, the log-normal distribution function taking relatively large values near \( \mu = 0 \) induces a numerical instability for \( q < 0 \). This instability leads the unreasonable result for \( q \ll -2 \).

We see from Fig. 1 that the exponent \( z_\tau(q) \) is not sensitive to ALS. This is consistent with the fact that the local moment \( Z_q(l) \) is not enough to distinguish ALS from multifractal wavefunctions. The multifractal spectra calculated for the original and the refined ensembles also indicate this fact as shown in Fig. 2, where \( f(\alpha) \) has been calculated by the \( q \)-microscope. Especially, these two spectra almost collapse in the vicinity of \( \alpha_0 \) giving the maximum of \( f(\alpha) \), which corresponds to \( q = 0 \). Since the profile of \( f(\alpha) \) close to \( \alpha = \alpha_0 \) can be approximated by the parabolic form

\[
f(\alpha) = 2 - \frac{(\alpha_0 - \alpha)^2}{4(\alpha_0 - 2)},
\]

this implies that \( \alpha_0 \) is hardly affected by ALS. (Dashed line in Fig. 2 shows this parabolic approximation with \( \alpha_0 = 2.18 \).) Using this value of \( \alpha_0 \), we can draw the parabolic approximation of \( z(q) \) given by eq. (15) as shown in Fig. 1. The insensitivity of \( \alpha_0 \) to ALS leads sensitive \( z(q) \) to ALS near \( q = 0 \). In fact, \( z(q) \) shown in Fig. 1 coincides with \( z_\tau(q) \) and the parabolic approximation in the vicinity of \( q = 0 \) even for the original ensemble.

Apparently, the quality of the refined ensemble depends on the choice of the value of \( \Gamma^* \) in eq. (6). Figure 3 exhibits this quantitatively. The longitudinal axis \( q^* \) of this figure represents the value of \( q \) \((> 0)\) above which \( |z(q) - z_\tau(q)| \) is larger than 0.04. The data points indicated by arrows correspond to the original ensemble and the refined ensemble used in Figs. 1 and 2. This result indicates that the exponent \( z(q) \) coincides with \( z_\tau(q) \) in the wider range of \( q \) when the degree of refining the en-

\[
\text{Fig. 2. Multifractal spectra } f(\alpha) \text{ for the original ensemble (circles) and the refined ensemble (crosses). Crossed error bars indicate representative values of errors in } f(\alpha) \text{ and } \alpha. \text{ The dashed line indicates the parabolic approximation eq. (20) with } \alpha_0 = 2.18.
\]
ensemble becomes higher. Namely, the high quality refined ensemble ensures the scaling relation eq. (16) with large \( q \). It seems that \( q^* \) becomes infinite when \( \Gamma^* \) goes to zero. The quantity \( q^* \), however, does not diverge actually. Too small \( \Gamma^* \) makes the quality of the refined ensemble rather worse, because the number of samples in the refined ensemble becomes insufficient to provide a good statistics.

6. Conclusions

We have investigated the box-measure correlation function \( G_q(l, L, r) \) of critical wavefunctions of the SU(2) model which belongs to the two-dimensional symplectic class and confirmed the scaling relation for the correlation exponent \( z(q) \) in a wide range of \( q \). It is found that ALS prevent us from calculating accurate values of the correlation exponent with large \( q \)'s. To obtain \( z(q) \), we eliminated ALS from the original ensemble of critical wavefunctions and composed a refined ensemble for which the geometric mean of the box-measure correlation functions represents properly the typical profile of \( G_q(l, L, r) \). It is also found that the range of \( q \) within which the calculated \( z(q) \) satisfies the scaling relation becomes wider when the quality of the refined ensemble becomes better (namely, decreasing \( \Gamma^* \)). We have elucidated that a direct calculation of the exponent \( z(q) \) encounters a numerical instability for \( q \ll -2 \), which originates from the fact that the distribution function of multifractal measure \( \mu \) has relatively large values near \( \mu = 0 \). Some ingenious techniques in numerical calculations would be required to obtain precise values of \( z(q) \) at negative \( q \)'s. In the present paper, the dependence of the correlation function \( G_q(l, L, r) \) on the box-box distance \( r \) has been examined only for \( l = 1 \). If we can deal with huge systems, \( G_q(l, L, r) \) with \( l > 1 \) would provide a more precise value of \( z(q) \) for \( q \ll -2 \), because the numerical instability at negative \( q \)'s might be moderated due to an average effect within each box. In our demonstrations by using the SU(2) model, the value of \( q \) to eliminate ALS has been fixed at 2, while \( G_q \) has been analyzed for many \( q \) values. Essentially, the choice of the value of the moment \( q \) for the elimination does not influence the quality of the obtained refined ensembles. It is, however, better to use a positive small value of \( q \) for ensuring a numerical stability.

The composition of the refined ensemble is efficient also for evaluating other properties at the critical point than the scaling relation treated in this paper. We emphasize that the scaling relation eq. (14) can be used for judging how well the refined ensemble represents typical critical wavefunctions. It is also interesting to study statistical properties of the ensemble of ALS, namely, the complementary set of the refined ensemble. Further investigations of ALS provide us deep insight into the nature of criticality.

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