On Autonilpotent Finite Groups
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Abstract. In the paper autonilpotent groups were characterized as groups $G$ such that $\text{Aut}G$ stabilizes some chain of subgroups of $G$. It was shown that a $p$-group is autonilpotent if and only if its group of automorphisms is also a $p$-group. Analogues of Baer’s theorem about the hypercenter and Frobenius $p$-nilpotency criterion were obtained for autonilpotent groups.

Keywords. Finite groups; nilpotent groups; autonilpotent groups; hypercenter of a group.

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1 Introduction and results

Throughout this paper and all groups are finite and $G$ always denotes a finite group. Recall that $\text{Aut}G$ and $\text{Inn}G$ are the groups of all and inner automorphisms of $G$ respectively.

Let $A$ be a group of automorphisms of a group $G$. Kaloujnine [1] and Hall [2] showed that if $A$ stabilizes some chain of subgroups of $G$ then $A$ is nilpotent. In this paper we will consider the converse question: assume that $A$ stabilizes some chain of subgroups of $G$, what can be said about $G$?

M. R. R. Moghaddam and M. A. Rostamyari [3] introduced the concept of autonilpotent group. Let

$$L_n(G) = \{x \in G \mid [x, \alpha_1, \ldots, \alpha_n] = 1 \quad \forall \alpha_1, \ldots, \alpha_n \in \text{Aut}G\}$$

Then $G$ is called autonilpotent if $G = L_n(G)$ for some natural $n$. Some properties of autonilpotent groups were studied in [4].

Theorem 1.1. A group $G$ is autonilpotent if and only if $\text{Aut}G$ stabilizes some chain of subgroups of $G$.

Recall that $\pi(G)$ is the set of prime divisors of $G$. It is known that a group is nilpotent if and only if it is the direct product of its Sylow subgroups. Here we proved

Theorem 1.2. A group $G$ is autonilpotent if and only if it is the direct product of its Sylow subgroups and the automorphism group of a Sylow $p$-subgroup of $G$ is a $p$-group for all $p \in \pi(G)$.

In [5] all abelian autonilpotent groups were described. In particular, abelian autonilpotent non-unit groups of odd order don’t exist. It was not known about the existence of autonilpotent $p$-groups of odd order.

Corollary 1.1. Let $p$ be a prime. A $p$-group $G$ is autonilpotent if and only if $\text{Aut}G$ is a $p$-group.

An example of a $p$-group $G$ of order $p^5$ ($p > 3$) such that $\text{Aut}G$ is also a $p$-group was constructed in [6]. In the library of small groups of GAP [7] there are 30 groups of order $3^6$ such that their automorphism groups are also 3-groups (for example groups [729, 31], [729, 41] and [729, 46]).

Note that $L_n(G) \subseteq L_{n+1}(G)$ for all natural $n$. Since $G$ is finite, it is clear that $L_n(G) = L_{n+1}(G)$ for some natural $n$. In this case we shall call a subgroup $L_n(G) = L_\infty(G)$ the absolute hypercenter of $G$. Hence $G$ is autonilpotent if and only if $G = L_\infty(G)$. In [8] Baer showed that a $p$-element $g$ of $G$ belongs to the hypercenter $Z_\infty(G)$ of $G$ if and only if it commutes with all $p'$-elements of $G$. It means that $g^x = g$ for any $p'$-element $x$ of $G$. Here we obtain the following analogue of Baer’s result.
Theorem 1.3. Let $g$ be a $p$-element of a group $G$. Then $g \in L_{\infty}(G)$ if and only if $g^a = g$ for every $p'$-element $\alpha$ of Aut$G$.

Corollary 1.2. A group $G$ is autonilpotent if and only if every automorphism $\alpha$ of $G$ fixes all elements of $G$ whose orders are coprime to the order of $\alpha$.

According to Frobenius $p$-nilpotency criterion (see [3, 5E, 5.26]) a group $G$ is nilpotent if and only if $N_G(P)/C_G(P)$ is a $p$-group for every $p$-subgroup $P$ of $G$ and every $p \in \pi(G)$.

Theorem 1.4. A group $G$ is autonilpotent if and only if $N_{\text{Aut}_G}(P)/C_{\text{Aut}_G}(P)$ is a $p$-group for every $p$-subgroup $P$ of $G$ and every $p \in \pi(G)$.

2 Preliminaries

For an automorphism $\alpha$ and an element $x$ of a group $B$ we denote by $x^\alpha$ the image of $x$ under $\alpha$. Let $A$ and $B$ be groups and $\varphi$ be a homomorphism form $A$ to Aut$B$. We can define the action of $A$ on $B$ in the following way

$$x^\alpha = x^{\varphi(\alpha)}, \quad x \in B, \quad a \in A.$$ 

In this case $A$ is called a group of operators of $B$. It is known that the following sets are subgroups of $A$ for any subgroup $D$ of $B$.

$$N_A(D) = \{a \in A \mid D^a = D\}, \quad C_A(D) = \{a \in A \mid d^a = d \quad \forall d \in D\} \quad \text{and} \quad \text{Aut}_A(D) = N_A(D)/C_A(D),$$ 

where $\text{Aut}_A(D)$ is a group of automorphisms induced by $A$ on $D$. Note that $\text{Ker} \varphi = C_A(B)$.

Hence the actions of $A$ and $\text{Aut}_A(B) = A/C_A(B)$ on $B$ are the same.

Recall that $[b, a] = b^{-1}a^b$ for $a \in A$ and $b \in B$. Let $y \in B$. We can also consider $y$ as an element of Inn$B$. Assume that $y \in N_{\text{Aut}_B(\text{Aut}_A B)}$. Then we can consider $a^y$ as element of $A$

$$x^{a^y} = x^{y^{-1}a^y} = x^{y^{-1}\varphi(\alpha)y} = x^{\varphi(\alpha)y}, \quad x \in B.$$ 

Now it is clear that

$$[b, a]^y = y^{-1}b^{-1}yy^{-1}b^y = (b^{-1})^y b^y = (b^y)^{-1}(b^y)^{y^{-1}a^y} = [b^y, a^y].$$

3 $R$-nilpotent Groups

Let $R$ be a group of operators for a group $G$ and

$$K_0(G, R) = G \quad \text{and} \quad K_n(G, R) = [K_{n-1}(G, R), R],$$

$$L_0(G, R) = 1 \quad \text{and} \quad L_n(G, R) = \{x \in G \mid [x, \alpha_1, \ldots, \alpha_n] = 1 \quad \forall \alpha_1, \ldots, \alpha_n \in R\}.$$ 

Lemma 3.1. Let $R$ be a group of operators for a group $G$ and $\text{Inn}G \leq N_{\text{Aut}_G}(\text{Aut}_R G)$. Then

1. $L_n(G, R)$ is a normal subgroup of $G$ for all natural $n$;
2. $L_n(G, R) = G$ for some natural $n$ if and only if $K_n(G, R) = 1$.

Proof. Let prove (1). Let $x \in L_n(G, R)$. Note that for every $\alpha \in R$ and $y \in G$ there exists $\beta \in R$ with $\alpha = \beta^y$. From $1 = 1^y = [x, \alpha_1, \ldots, \alpha_n]^y = [x^y, \alpha_1^y, \ldots, \alpha_n^y]$ it follows that $[x^y, \alpha_1, \ldots, \alpha_n] = 1 \quad \forall \alpha_1, \ldots, \alpha_n \in R$. Hence $x^y \in L_n(G, R)$. Thus $L_n(G, R)$ is normal in $G$. 

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Let show that \( L_n(G, R) \) is a subgroup of \( G \). Since \( G \) is finite, it is sufficient to show that if \( x, y \in L_n(G, R) \), then \( xy \in L_n(G, R) \). Let \( \alpha \in R \). It is straightforward to check that \([xy, \alpha] = [x, \alpha][y, \alpha]\). From \( \text{Inn}G \leq N_{\text{Aut}G}(\text{Aut}_RG) \) it follows that \([x, \alpha][y, \alpha] = [x^\gamma, \alpha^\delta][y, \alpha] \), where \( \alpha^\gamma \in R \). It means that \([xy, \alpha_1, \ldots, \alpha_n] = [z, \beta_1, \ldots, \beta_n][y, \alpha_1, \ldots, \alpha_n] \), where \( \beta_1, \ldots, \beta_n \in R \) and \( z \in L_n(G, R) \) as a conjugate of \( x \). Therefore \( xy \in L_n(G, R) \). Thus \( L_n(G, R) \leq G \).

Let prove (2). Assume that \( K_n(G, R) = 1 \). From \([x, \alpha_1, \ldots, \alpha_n] \mid \forall x \in G \) and \( \forall \alpha_1, \ldots, \alpha_n \in R \), it follows that \( L_n(G, R) = G \). Assume now that \( L_n(G, R) = G \). Since \( L_1(G, R) \) is a subgroup, \([L_1(G, R), R] \leq L_{i-1}(G, R) \). From \( L_n(G, R) = G \) it follows that \( K_i(G, R) \leq L_{n-i}(G, R) \). Thus \( K_n(G, R) = 1 \).

We shall call a group \( G \) \( R \)-nilpotent if \( K_n(G, R) = 1 \) for some natural \( n \). Hence if \( R = G \), then \( R \)-nilpotent group \( G \) is nilpotent and if \( R = \text{Aut}G \), then \( R \)-nilpotent group \( G \) is autonilpotent by Lemma 3.1.

**Theorem 3.1.** Let \( R \) be a group of operators for a group \( G \). Then \( G \) is \( R \)-nilpotent if and only if \( R \) stabilizes some chain of subgroups of \( G \).

**Proof.** Assume that \( G \) is \( R \)-nilpotent. Then \( 1 = K_n(G, R) \) for some \( n \). From \([K_i(G, R), R] = K_{i+1}(G, R) \) it follows that \( xK_{i+1}(G, R) = x^\alpha K_{i+1}(G, R) \) for every \( \alpha \in R \) and \( x \in K_i(G, R) \). It means that \( R \) stabilizes
\[
1 = K_n(G, R) \leq K_{n-1}(G, R) \leq \cdots \leq K_0(G, R) = G.
\]
Assume that \( R \) stabilizes the following chain of subgroups
\[
1 = G_n < G_{n-1} < \cdots < G_0 = G
\]
Note tat \( K_0(G, R) = G \leq G_0 \). Assume that \( K_i(G, R) \leq G_i \). Let show that \( K_{i+1}(G, R) \leq G_{i+1} \).

Since \( x^\alpha G_{i+1} = xG_{i+1} \) for every \( \alpha \in R \) and \( x \in G_i \), we see that
\[
K_{i+1}(G, R) = [K_i(G, R), R] \leq [G_i, R] \leq G_{i+1}.
\]
Hence \( K_n(G, R) \leq G_n = 1 \). Therefore \( G \) is \( R \)-nilpotent.

Theorem 1.2 follows from Theorem 3.1 when \( R = \text{Aut}G \). According to [10, A, 13.8(b)] every group \( G \) is \( F(G) \)-nilpotent, where \( F(G) \) is the Fitting subgroup of \( G \).

**Theorem 3.2.** Let \( p \) be a prime and \( R \) be a group of operators for a \( p \)-group \( G \). Then \( G \) is \( R \)-nilpotent if and only if \( \text{Aut}_RG \) is a \( p \)-group.

**Proof.** (a) If a \( p \)-group \( G \) is \( R \)-nilpotent, then \( \text{Aut}_RG \) is a \( p \)-group.

Note that \( R \) stabilizes the chain of subgroups
\[
1 = K_n(G, R) < K_{n-1}(G, R) < \cdots < K_1(G, R) < K_0(G, R) = G.
\]
By [10, Corollary 12.4A(a)], \( \text{Aut}_RG = R/C_R(G) \) is a \( p \)-group.

(b) Let \( P \) be a \( p \)-group of operators of a \( p \)-group \( G \). Then \( G \) has a \( P \)-admissible maximal subgroup.

Note that maximal subgroups of \( G \) are in one-to-one correspondence with maximal subgroups of \( G/\Phi(G) \). It is well known that \( G/\Phi(G) \) is isomorphic to the direct product of \( n \) copies of \( \mathbb{Z}_p \) for some natural \( n \). So the number of maximal subgroups of \( G/\Phi(G) \) is equal to the number of maximal subspaces of a vector space of dimension \( n \) over \( \mathbb{F}_p \). It is known (for example see [11]) that this number \( k = (p^n - 1)/(p - 1) \). So the number of maximal subgroups of \( G \) is coprime to \( p \). Note that if \( M \) is a maximal subgroup of \( G \), then
\[
M^z = \{m^z \mid m \in M\}
\]
is also maximal subgroup of $G$. Hence a $p$-group $P$ acts on the set of maximal subgroups of $G$. From $(k, p) = 1$ it follows that $P$ has a fixed point on it. It means that there exists a maximal subgroup $M$ of $G$ with $M^p = M$ for all $x \in P$.

(c) Let $P$ be a group of operators of a $p$-group $G$. If $\text{Aut}_P G$ is a $p$-group, then $G$ has $P$-composition series with simple factors.

Note the actions of $P$ and $\text{Aut}_P G$ on $G$ are the same. From (b) it follows that every $P$-admissible subgroup $M$ of $G$ has maximal $P$-admissible subgroup $N$. Hence $G$ has $P$-composition series with simple factors.

(d) Let $P$ be a group of operators of a $p$-group $G$. If $\text{Aut}_R G$ is a $p$-group, then $G$ is $R$-nilpotent.

By (c) $G$ has $R$-composition series with simple factors. Let

$$1 = G_n < G_{n-1} < \cdots < G_1 < G_0 = G$$

be this series. Note that $G = K_0(G, R) \leq G$. Assume that we show that $K_i(G, R) \leq G_i$ for some $i$. Let show that $K_{i+1}(G, R) \leq G_{i+1}$ (note that the order of $\text{Aut} G_i / G_{i+1} \simeq \text{Aut} Z_p / Z_{p-1}$ is coprime to $p$). Hence $R$ acts trivially on $G_i / G_{i+1}$, i.e.

$$(gG_{i+1})^{-1}(gG_{i+1})^\alpha = [g, \alpha]G_{i+1} = G_{i+1}$$

for all $g \in G_i$ and $\alpha \in R$. So $[g, \alpha] \in G_{i+1}$ for all $g \in G_i$ and $\alpha \in R$. It means that

$$K_{i+1}(G, R) = [K_i(G, R), R] \leq [G_i, R] \leq G_{i+1}.$$ 

Thus $K_n(G, R) \leq G_n = 1$. Hence $G$ is $R$-nilpotent.

It is clear that $L_n(G, R) \subseteq L_{n+1}(G, R)$. Since $G$ is finite, we see that there is a natural $n$ such that $L_n(G) = L_{n+1}(G)$ for all $i > 1$. In this case let $L_n(G, R) = L_\infty(G, R)$. Note that if $R = \text{Inn} G$, then $L_\infty(G, R) = Z_\infty(G)$ the hypercenter of $G$, and if $R = \text{Aut} G$, then $L_\infty(G, R) = L_\infty(G)$ the absolute hypercenter of $G$.

**Theorem 3.3.** Let $R$ be a group of operators for a group $G$ with $\text{Inn} G \leq \text{Aut}_R G$ and $g$ be a $p$-element of $G$. Then $g \in L_\infty(G, R)$ if and only if $g^\alpha = g$ for every $p'$-element $\alpha$ of $R$.

**Proof.** From $\text{Inn} G \leq \text{Aut}_R G$ it follows that $L_\infty(G, R)$ is nilpotent. Let $g$ be a $p$-element of $L_\infty(G, R)$. Note that if $g \in L_1(G, R)$, then $[g, \alpha] = 1$ or $g = g^\alpha$ for all $p'$-elements $\alpha$ of $R$. Assume that if $g \in L_k(G, R)$, then $g^m = g^\alpha$ for all $p'$-elements $\alpha$ of $R$. Let show that if $g \in L_{k+1}(G, R)$, then $g = g^\alpha$ for all $p'$-elements $\alpha$ of $R$.

Since $L_\infty(G, R)$ is nilpotent, $[g, \alpha]$ is a $p'$-element for all $\alpha \in R$. Assume now that $\alpha \in R$ is a $p'$-element. From $g \in L_{k+1}(G, R)$ it follows that $[g, \alpha] \in L_k(G, R)$. By induction $[g, \alpha]^m = [g, \alpha]$. Let $m$ be the order of $\alpha$. From $g^\alpha = g[g, \alpha]$ it follows that $g = g^{\alpha^m} = g[g, \alpha]^m$ or $[g, \alpha]^m = 1$. Since $[g, \alpha]$ is a $p'$-element and $(p, m) = 1$, we see that $[g, \alpha] = 1$ or $g = g^\alpha$. Thus if $g \in L_{k+1}(G, R)$, then $g = g^\alpha$ for all $p'$-elements $\alpha$ of $R$.

From $L_\infty(G, R) = L_n(G, R)$ for some natural $n$ it follows that if $g$ is a $p$-element of $L_\infty(G, R)$, then $g^{\alpha^m} = g$ for every $p'$-element $\alpha$ of $R$.

Let $G_p$ be the set of all elements of $G$ such that $g^\alpha = g$ for every $p'$-element $\alpha$ of $R$ and every $g \in G_p$. Note that if $x, y \in G_p$, then $xy \in G_p$. Hence $G_p$ is a subgroup of $G$. Let $g \in G_p$, $\alpha, \beta \in R$ and $\beta$ is a $p'$-element. Then $\beta^{\alpha^{-1}}$ is a $p'$-element too. Hence

$$(g^\alpha)^\beta = g^\alpha \beta^{-1} \alpha = g^{\alpha \beta^{-1}} = g^\alpha.$$ 

It means that $g^\alpha \in G_p$. Thus $G_p$ is a $R$-admissible subgroup of $G$. Let $x$ be a $p'$-element of $G_p$. From $\text{Inn} G \leq \text{Aut}_R G$ it follows that

$$\alpha_x : g \mapsto g^x.$$
is a $p'$-element of $R$. Hence it acts trivially on $G_p$. Let $P$ be a Sylow $p$-subgroup of $G_p$. Then $P \leq G_p$. It means that all $p$-elements of $G_p$ form a subgroup. So $\text{Pchar} G_p$. Hence $P$ is $R$-admissible. Now $\text{Aut}_R P$ is a $p$-group. It means that $P$ is $R$-nilpotent by Theorem 3.2. So $K_n(P, R) = 1$ for some natural $n$. Thus $[g, \alpha_1, \ldots, \alpha_n] = 1$ for any $g \in P$ and $\alpha_1, \ldots, \alpha_n \in R$. Therefore $P \leq L_{\infty}(G, R)$.

**Corollary 3.1** (Baer [8]). Let $p$ be a prime and $G$ be a group. Then a $p$-element $g$ of $G$ belongs to $Z_{\infty}(G)$ if and only if it permutes with all $p'$-elements of $G$.

**Theorem 3.4.** Let $R$ be a group of operators for a group $G$ with $\text{Inn} G \leq \text{Aut}_R G$. Then $G$ is $R$-nilpotent if and only if $\text{Aut}_R P$ is a $p$-group for all $p$-subgroups $P$ of $G$ and $p \in \pi(G)$.

**Proof.** Assume that a group $G$ is $R$-nilpotent. Then $G = L_{\infty}(G, R)$ by Lemma 3.1. Let $P$ be a $p$-subgroup of $G$. Then $g = g^\alpha$ for all $p'$-elements $\alpha$ of $R$ and all $g \in P$ by Theorem 3.3. It means that $\text{Aut}_R P = N_R(P)/C_R(P)$ is a $p$-group.

Assume that $\text{Aut}_R P$ is a $p$-group for every $p$-subgroup $P$ of $G$ and every $p \in \pi(G)$. Suppose that $G$ is non-nilpotent. So there is a Schmidt subgroup $S$ of $G$. Then $S$ has a normal $q$-subgroup $Q$ for some prime $q$ and there is a $q'$-element $x$ of $S$ with $x \notin C_G(Q)$. Since

$$\alpha_x : g \to g^x$$

is a non-identity inner automorphism of $G$ of $q'$-order and $\text{Inn} G \leq \text{Aut}_R G$, $\text{Aut}_R Q$ is not a $q$-group for a $q$-subgroup $Q$, a contradiction.

Hence $G$ is nilpotent. Let $P$ be a Sylow $p$-subgroup of $G$. Then $P$ contains all $p$-elements of $G$ and $\text{Pchar} G$. Since $\text{Aut}_R P = R/C_R(P)$ is a $p$-group, $g = g^\alpha$ for all $p'$-elements $\alpha$ of $R$ and all $g \in P$. From Theorem 3.3 it follows that $P \leq L_{\infty}(G, R)$. Hence $G \leq L_{\infty}(G, R)$. Therefore $G = L_{\infty}(G, R)$. Thus $G$ is $R$-nilpotent by Lemma 3.1.

**Theorem 3.5.** Let $R$ be a group of operators for a group $G$ with $\text{Inn} G \leq \text{Aut}_R G$. Then $G$ is $R$-nilpotent if and only if it is the direct product of its Sylow subgroups and $\text{Aut}_R P$ is a $p$-group for every Sylow $p$-subgroup $P$ of $G$ and all $p \in \pi(G)$.

**Proof.** Assume that $G$ is $R$-nilpotent. From $\text{Inn} G \leq \text{Aut}_R G$ it follows that $G$ is nilpotent. Hence it is the direct product of its Sylow subgroups. Note that $\text{Aut}_R P$ is a $p$-group for every Sylow $p$-subgroup $P$ of $G$ and all $p \in \pi(G)$ by Theorem 3.3.

The proof of the converse statement is the same as in the end of proof of Theorem 3.4.

**Proof of Theorem 1.2.** The automorphism group of a direct product of groups was described in [12]. In particular, if $G = P \times H$, where $P$ is a Sylow subgroup of $G$, then $\text{Aut} G = \text{Aut} P \times \text{Aut} H$ and $\text{Aut}_{\text{Aut} G} P = \text{Aut} P$. Now Theorem 1.2 directly follows from Theorem 3.5.

**Final Remarks**

In [13, 14] it was shown that if $G$ has has $A$-composition series with prime indexes then $A$ is supersoluble. Shemetkov [14] and Schmid [15] studied $\mathfrak{S}$-stable groups of automorphisms for a (solubly) saturated formation $\mathfrak{S}$.

Let $\mathfrak{S}$ be a class of groups, $R$ be a group of automorphisms of a group $G$ and $H/K$ be a $R$-composition factor of $G$. We shall call $H/K$ $R$-$\mathfrak{S}$-central if

$$H/K \triangleleft \text{Aut}_R H/K \in \mathfrak{S}.$$ 

Hence if $R = \text{Inn} G$, then $R$-$\mathfrak{S}$-central factor is just $\mathfrak{S}$-central.
Definition 1. We shall call a group $G$ auto-$\mathfrak{F}$-group if every $\text{Aut}G$-composition factor of $G$ is $\text{Aut}G$-$\mathfrak{F}$-central.

Question 1. Describe the class of all autosupersoluble groups.

Question 2. Describe the class of all auto-$\mathfrak{F}$-groups, where $\mathfrak{F}$ is a hereditary saturated formation.

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