Particle Number Fluctuations in the Microcanonical Ensemble

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Particle number fluctuations are studied in the microcanonical ensemble. For the Boltzmann statistics we deduce exact analytical formulae for the microcanonical partition functions in the case of non-interacting massless neutral particles and charged particles with zero net charge. The particle number fluctuations are calculated and we find that in the microcanonical ensemble they are suppressed in comparison to the fluctuations in the canonical and grand canonical ensembles. This remains valid in the thermodynamic limit too, so that the well-known equivalence of all statistical ensembles refers to average quantities, but does not apply to fluctuations. In the thermodynamic limit we are able to calculate the particle number fluctuations in the system of massive bosons and fermions when the exact conservation laws of both the energy and charge are taken into account.

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I. INTRODUCTION

The statistical hadron gas model (see e.g. Ref. [1] and recent review [2]) appears to be rather successful in describing the data of nucleus-nucleus (A+A) collisions for particle multiplicities in a wide range of the collision energies. Usually one considers a thermal system created in A+A collision in the grand canonical ensemble (GCE). There are, however, situations when the canonical ensemble (CE) [3] or even microcanonical ensemble (MCE) [4] with explicit treatment of charge conservations or both charge and energy conservations are required. This happens, for example, when the statistical model is applied to elementary \( pp, p\bar{p}, e^+e^- \) collisions. Different statistical ensembles are not equivalent for small systems created in these collisions.

In A+A collisions one prefers to use the GCE because it is the most convenient one from the technical point of view and due to the fact that both the CE and MCE are equivalent to the GCE in the thermodynamic limit (i.e. when the size of the system tends to infinity). However, the thermodynamic equivalence of ensembles means only that the average values of different physical quantities calculated in different ensembles are equal to each other in the thermodynamic limit. On the other hand, the analysis of fluctuations is also an important tool to study a physical system. Event-by-event analysis of A+A collisions (see e.g. Ref. [5]) can reveal new physical effects not seen in observables averaged over a large sample of events. An essential part of the total fluctuations measured on the event-by-event basis is expected to be the thermal fluctuations. It was demonstrated for the first time in Ref. [6] that particle number fluctuations are different in the CE and GCE even in the thermodynamic limit. In the present paper we extend the results of Ref. [6] and make the analytical calculations of the particle number fluctuations in the MCE.

The statistical hadron gas model is usually limited to the non-relativistic cases and the number of particles are assumed to be fixed in these ensembles. In the relativistic situation one can only fix the conserved charges (in the CE), or both the energy and conserved charges (in the MCE), while the particle numbers still fluctuate both in the CE and MCE. Results of both Ref. [6] and the present paper demonstrate that the particle number fluctuations are different in various ensembles even in the thermodynamic limit.

The paper is organized in the following way. First, we deduce and study the exact analytical expressions in the MCE for massless neutral particles with Boltzmann statistics: the partition function and average number of particles in Sec. II and particle number fluctuations in Sec. III. The extension of these results for the system of charged particles with zero net charge is considered in Sec. IV. In Sec. V we use the method of microscopic correlator proposed in Ref. [7]. This gives us a possibility to study the MCE fluctuations in the thermodynamic limit for a much more general situation. It includes the effects of quantum statistics and non-zero particle mass. We consider the system of neutral particles and then extend the formulation to charged particles too. Both the energy and charge exact conservation laws are imposed. We summarize our consideration in Sec. VI.
II. THE MICROCANONICAL PARTITION FUNCTION AND AVERAGE NUMBER OF PARTICLES

In order to calculate analytically the microcanonical partition function\(^1\) we start in Secs. II-III with the system of non-interacting massless neutral particles and neglect the effects of quantum statistics (the extensions will be treated in Secs. IV-V). The microcanonical partition function for one massless particle with energy \(E\) in the volume \(V\) can be easily calculated:

\[
W_1(E, V) = \frac{gV}{(2\pi)^{3/2}} \int d^3 p \, \delta(E - |\vec{p}|) = \frac{gV}{2\pi^2} \int_0^\infty dp \, p^2 \delta(E - p) = \frac{gV}{2\pi^2} E^2.
\]

(1)

Here \(g\) is the degeneracy factor. In the case of two massless particles:

\[
W_2(E, V) = \frac{1}{2} \frac{gV}{(2\pi)^{3/2}} \int d^3 q \, \frac{gV}{(2\pi)^{3/2}} \int d^3 p \, \delta(E - |\vec{q}| - |\vec{p}|)

= \frac{1}{2} 4\pi \frac{gV}{(2\pi)^3} \int_0^E dq \, q^2 W_1(E - q, V)

= \frac{1}{2} \left( \frac{gV}{2\pi^2} \right)^2 \int_0^E dq \, q^2 (E - q)^2 = \frac{1}{60} \left( \frac{gV}{2\pi^2} \right)^2 E^5.
\]

(2)

The factor \(1/2\) appears because particles are identical. It can be proven that the \(N\)-particle microcanonical partition function \(W_N(E, V)\) has the form (see Appendix A):

\[
W_N(E, V) = \frac{1}{E} \frac{x^N}{(3N - 1)!N!},
\]

(3)

where

\[
x \equiv gVE^3/\pi^2.
\]

(4)

The total partition function in the MCE is:

\[
W(E, V) = \sum_{N=1}^\infty W_N(E, V) = \frac{1}{E} \sum_{N=1}^\infty \frac{x^N}{(3N - 1)!N!} = \frac{x}{E} \sum_{n=0}^\infty \frac{x^n}{(3n + 2)!(n + 1)!}

= \frac{x}{E} \sum_{n=0}^\infty \frac{3x^n}{(3n + 3)!n!} = \frac{x}{2E} \sum_{n=0}^\infty \left( \frac{4}{3} \right)_n \left( \frac{5}{3} \right)_n \left( \frac{2}{3} \right)_n \left( \frac{x}{27} \right)_n

= \frac{x}{2E} \, _0F_3 \left( ; \frac{4}{3}, \frac{5}{3}, \frac{2}{3}; \frac{x}{27} \right),
\]

(5)

where we use (see e.g. Ref. 8 and Appendix B) the Pochhammer symbol \((a)_n\) \(^{12}\) and the generalized hypergeometric function (GHF) \(_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)\) \(^{13}\).

In the GCE the independent variables are the volume \(V\) and temperature \(T\). For neutral massless particles with Boltzmann statistics one easily finds the average number of particles and average energy:

\[
\langle N \rangle_{g.c.e.} = \frac{gVT^3}{\pi^2}, \quad \langle E \rangle_{g.c.e.} = \frac{3gVT^4}{\pi^2}.
\]

(6)

We want to compare the results of the MCE and GCE at equal volumes \(V\) and energies \(\langle E \rangle_{g.c.e.} = E\). From Eq. (6) it follows:

\[
\langle N \rangle_{g.c.e.} = \bar{N} = \left( \frac{x}{27} \right)^{1/4}.
\]

(7)

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\(^1\) We define the microcanonical ensemble as the statistical system with fixed energy. An exact conservation of momentum, angular momentum and parity are neglected here. The MCE with conserving momentum was considered in Ref. 4.
FIG. 1: The ratio of average particle number $\langle N \rangle_{\text{m.c.e.}}$ in the MCE to that $\overline{N}$ in the GCE.

The average number of particles in the MCE equals to:

$$\langle N \rangle_{\text{m.c.e.}} = \frac{1}{W(E, V)} \sum_{N=1}^{\infty} NW_N(E, V) = \frac{1}{W(E, V)} \sum_{N=1}^{\infty} \frac{x^N}{E (3N-1)! (N-1)!}$$

$$= \frac{1}{W(E, V)} \frac{x}{E} \sum_{n=0}^{\infty} \frac{x^n}{(3n+2)! n!} = \frac{1}{W(E, V)} \frac{x}{2E} \sum_{n=0}^{\infty} \frac{(x/27)^n}{n!}$$

$$= \frac{1}{W(E, V)} \frac{x}{2E} _0F_3 \left( ; \frac{4}{3}, \frac{5}{3}, \frac{x}{27} \right) = \frac{aF_3 \left( ; \frac{4}{3}, \frac{5}{3}, \frac{2}{27} \right)}{aF_3 \left( ; \frac{4}{3}, \frac{5}{3}, \frac{2}{27} \right)}.$$  

From Eq. 8, using the asymptotic behavior of the GHF [13], one finds the asymptotic expansion of $\langle N \rangle_{\text{m.c.e.}}$ at $\overline{N} \to \infty$:

$$\langle N \rangle_{\text{m.c.e.}} \simeq \overline{N} \left( 1 + \frac{1}{8 \overline{N}} + \frac{35}{1152 \overline{N}^2} + ... \right).$$

The dependence of the ratio $\langle N \rangle_{\text{m.c.e.}} / \overline{N}$ on $\overline{N}$ is shown in Fig. 1.

As seen from Fig. 1 the ratio $\langle N \rangle_{\text{m.c.e.}} / \overline{N}$ goes to 1 at $\overline{N} \gg 1$. As the smallest number of particles in the MCE equals to 1, one finds that $\langle N \rangle_{\text{m.c.e.}} \simeq 1$ when the volume $V$ and the energy $E$ of the system become small, i.e. if $VE^3 \ll 1$. On the other hand, in the GCE one finds $\langle N \rangle_{\text{g.c.e.}} \propto (VE^3)^{1/4} \ll 1$ in this limit of small systems. Therefore, at $\overline{N} < 1$ the ratio $\langle N \rangle_{\text{m.c.e.}} / \overline{N}$ is larger than 1 and it increases monotonously when $\overline{N} \to 0$. 
III. PARTICLE NUMBER FLUCTUATIONS

To study the particle number fluctuations in the MCE we calculate

$$\langle N^2 \rangle_{\text{m.c.e.}} \equiv \frac{1}{W(E,V)} \sum_{N=1}^{\infty} N^2 W_N(E,V) = \frac{1}{W(E,V)} \frac{1}{E} \sum_{N=1}^{\infty} \frac{N^2 x^N}{(3N-1)!N!} \quad (10)$$

$$= \frac{1}{W(E,V)} \frac{1}{E} \left[ \frac{x}{2} \left(_0 F_3 \left( ; 1, \frac{4}{3}, \frac{5}{3}; \frac{x}{27} \right) \right) + \frac{x^2}{120} \left(_0 F_3 \left( ; \frac{2}{3}, \frac{7}{3}, \frac{8}{3}; \frac{x}{27} \right) \right) \right]$$

$$= \langle N \rangle_{\text{m.c.e.}} + \frac{x}{60} \left(_0 F_3 \left( ; \frac{2}{3}, \frac{5}{3}, 2; \frac{x}{27} \right) \right).$$

The asymptotic expansion of $$\langle N^2 \rangle_{\text{m.c.e.}}$$ (10) can be found using Eq. (B3):

$$\langle N^2 \rangle_{\text{m.c.e.}} \simeq N^2 \left( 1 + \frac{1}{2N} + \frac{11}{144 N^2} + \ldots \right). \quad (11)$$

A measure of the fluctuations, the scaled variance $$\omega$$, is defined as usual,

$$\omega_{\text{m.c.e.}} = \frac{\langle N^2 \rangle_{\text{m.c.e.}} - \langle N \rangle_{\text{m.c.e.}}^2}{\langle N \rangle_{\text{m.c.e.}}}, \quad (12)$$

and it is plotted in Fig. 2.

The asymptotic expansion of $$\omega_{\text{m.c.e.}}$$ (12) has the form:

$$\omega_{\text{m.c.e.}} \simeq \frac{1}{4} \left( 1 - \frac{1}{8N} + \ldots \right). \quad (13)$$

FIG. 2: The scaled variance $$\omega_{\text{m.c.e.}}$$ (12) in the MCE.
FIG. 3: The particle number distributions in the MCE and GCE for $\bar{N} = 0.5$.

FIG. 4: The same as in Fig. 3 but for $\bar{N} = 10$. 
In Figs. 3 and 4 we present the particle number distributions in the MCE and GCE for small ($\overline{N} = 0.5$) and large ($\overline{N} = 10$) systems.

In the MCE the particle number distribution equals

$$P_{m.c.e.}(E, V, N) = \frac{W_N(E, V)}{W(E, V)} = \frac{1}{(3N - 1)!} \frac{x^N}{E W(E, V)}, \quad N = 1, 2, \ldots \quad (14)$$

and it has the Poisson form in the GCE

$$P_{g.c.e.}(\overline{N}, N) = \exp \left( - \overline{N} \right) \frac{\overline{N}^N}{N!}, \quad N = 0, 1, 2, \ldots \quad (15)$$

Note that Poisson distribution $P_{g.c.e.}(\overline{N}, N)$ results in

$$\omega_{g.c.e.} = \frac{\overline{N}^2 - \overline{N}^2}{\overline{N}} = 1. \quad (16)$$

It is seen from Eq. (13) that in the thermodynamic limit $V \to \infty$ the MCE scaled variance equals to $\omega_{m.c.e.} = 1/4$, and it remains quarter as large as the scaled variance $\omega_{g.c.e.}$ in the GCE. We conclude that the particle number distributions are different in the MCE and GCE. The MCE particle number distribution (14) is narrower than that of the GCE (15) in both small ($\overline{N} < 1$) and large ($\overline{N} > 1$) systems. At $\overline{N} < 1$ the probability distributions $P_{m.c.e.}(E, V, N)$ and $P_{g.c.e.}(\overline{N}, N)$ both have their maximums at the smallest values of $N$. Then the crucial difference between two ensembles follows from the fact that the minimal allowed value of $N$ is $N = 0$ in the GCE, but $N = 1$ in the MCE (see Fig. 3).

In the thermodynamic limit the average number of particles goes to infinity and the main contribution to the microcanonical partition function $W(E, V)$ comes from the states with large number of particles $N \gg 1$. In this limit the MCE particle number distribution can be simplified in the following way. Using the Stirling formula for factorials one finds:

$$W_N(E, V) = \frac{1}{E} \frac{x^N}{(3N - 1)!} \frac{1}{N!} \approx \frac{1}{E} \exp[f(N)], \quad (17)$$

where $f(N) \simeq N \log (\overline{N}) - 4N(\log N - 1) + \frac{1}{2} \log 3 - \log(2\pi)$. We expand the function $f(N)$ in Taylor series near the point of its maximum $\overline{N}$. For $\overline{N} \gg 1$ and $|N - \overline{N}| \ll \overline{N}$ we find:

$$f(N) \approx f(\overline{N}) + f'(\overline{N}) \frac{1}{2} (N - \overline{N})^2 = f(\overline{N}) - \frac{2(N - \overline{N}^2)}{\overline{N}}, \quad (18)$$

where $\overline{N} = (x/27)^{1/4}$ is fixed by the condition $f'(\overline{N}) = 0$, and it coincides with the result of Eq. (4). From Eq. (18) it follows that the particle number distribution in the MCE can be approximated as:

$$P_{m.c.e.}(E, V, N) \propto \exp \left[ - \frac{2(N - \overline{N}^2)}{\overline{N}} \right]. \quad (19)$$

For the Gauss distribution $P_G(N) \propto \exp \left[ -(N - \overline{N})^2/2\sigma^2 \right]$ the variance is easily calculated at $\overline{N} \to \infty$ and it equals to $(\overline{N}^2) - (\overline{N})^2 = \sigma^2$. Therefore, from Eq. (19) it follows $(\overline{N}^2)_{m.c.e.} - (\overline{N})^2_{m.c.e.} = \overline{N}/4$, and the MCE scaled variance is $\omega_{m.c.e.} = 1/4$. The Poisson distribution (15) for $\overline{N} \gg 1$ can be approximated by the Gauss distribution, but it equals to $P_{g.c.e.}(\overline{N}, N) \propto \exp \left[ -(N - \overline{N})^2/2\overline{N} \right]$, and this leads to $\omega_{g.c.e.} = 1$. Therefore, at $\overline{N} \gg 1$ both the MCE and GCE particle number distributions can be approximated by the Gauss distributions with the same average value $\overline{N}$, but with different widths: $\sigma^2_{m.c.e.} = \overline{N}/4$ and $\sigma^2_{g.c.e.} = \overline{N}$. A consequence of this is that the MCE scaled variance is a quarter the size of that in the GCE for classical massless neutral particles in the thermodynamic limit.

**IV. THE MCE FOR MASSLESS CHARGED PARTICLES**

The microcanonical partition function discussed in the previous sections can be generalized for the system of charged particles. If the system net charge $Q$ equals zero the number of positively charged $N_+$ and negatively charged $N_-$
particles are equal in each microscopic configuration. The total MCE partition function is \( a \equiv gV/\pi^2 \):
\[
W(E, V, Q = 0) = \sum_{N_+=1}^{\infty} \sum_{N_-=1}^{\infty} \int_0^\infty dE_+ \int_0^\infty dE_- \ W_{N_+}(E_+, V) \ W_{N_-}(E_-, V)
\times \delta(N_+ - N_-) \delta[E - (E_+ + E_-)] = \sum_{N_+=1}^{\infty} \int_0^\infty dE_+ \ W_{N_+}(E_+, V) \ W_{N_+}(E - E_+ V)
= \sum_{N_+=1}^{\infty} \frac{a^{2N_+}}{(3N_+ - 1)!^2 \ N_+!^2} \int_0^E dE_+ \ E_+^{3N_+ - 1} (E - E_+)^{3N_+ - 1} . \tag{20}
\]
The last integral in Eq. (20) can be easily evaluated using the Euler Beta-function:
\[
\int_0^E dE_+ \ E_+^{3N_+ - 1} (E - E_+)^{3N_+ - 1} = E^{6N_+ - 1} \ B(3N_+, 3N_+) = E^{6N_+ - 1} \ \frac{(3N_+ - 1)!^2}{(6N_+ - 1)! \ N_+!^2} . \tag{21}
\]
Finally one finds:
\[
W(E, V, Q = 0) = \frac{x^2}{E} \sum_{n=0}^\infty \frac{x^{2n}}{(6n + 5)! \ (n + 1)^2} = \frac{x^2}{120E} \ \frac{\phi F_7 \left( \frac{7}{6}, \frac{4}{3}, \frac{5}{2}, \frac{5}{3}, \frac{11}{6}, \frac{2}{2}, (\frac{x}{216})^2 \right)}{\phi F_7 \left( \frac{7}{6}, \frac{4}{3}, \frac{5}{2}, \frac{5}{3}, \frac{11}{6}, \frac{2}{2}, (\frac{x}{216})^2 \right)} . \tag{22}
\]
Similarly to Eqs. (23) after some calculations one obtains:
\[
\langle N_{\pm} \rangle_{m.c.e.} = \frac{\phi F_7 \left( \frac{7}{6}, \frac{4}{3}, \frac{5}{2}, \frac{5}{3}, \frac{11}{6}, \frac{2}{2}, (\frac{x}{216})^2 \right)}{\phi F_7 \left( \frac{7}{6}, \frac{4}{3}, \frac{5}{2}, \frac{5}{3}, \frac{11}{6}, \frac{2}{2}, (\frac{x}{216})^2 \right)} , \tag{23}
\]
and
\[
\langle N_{\pm}^2 \rangle_{m.c.e.} = \frac{\phi F_7 \left( \frac{7}{6}, \frac{4}{3}, \frac{5}{2}, \frac{5}{3}, \frac{11}{6}, \frac{2}{2}, (\frac{x}{216})^2 \right)}{\phi F_7 \left( \frac{7}{6}, \frac{4}{3}, \frac{5}{2}, \frac{5}{3}, \frac{11}{6}, \frac{2}{2}, (\frac{x}{216})^2 \right)} . \tag{24}
\]
In the GCE of Boltzmann massless charged particles with \( Q = \langle N_+ \rangle_{g.c.e.} - \langle N_- \rangle_{g.c.e.} = 0 \) one finds:
\[
\langle N_{\pm} \rangle_{g.c.e.} = gVT^3 \ \frac{\pi}{\pi^2} , \quad \langle E \rangle_{g.c.e.} = \frac{6gVT^4}{\pi^2} . \tag{25}
\]
The results of the MCE and GCE are again compared at equal volumes \( V \) and energies \( \langle E \rangle_{g.c.e.} = E \). From Eq. (20) it follows:
\[
\langle N_{\pm} \rangle_{g.c.e.} \equiv \langle \bar{N}_{\pm} \rangle = \langle \bar{N}_{\pm} \rangle = \langle \bar{N}_{2\pm} \rangle = \langle \bar{N}_{3\pm} \rangle . \tag{26}
\]
The asymptotic expansions for \( \langle N_{\pm} \rangle_{m.c.e.} \) and \( \langle N_{\pm}^2 \rangle_{m.c.e.} \) at \( \bar{N}_{\pm} \to \infty \) are found using Eq. (23):
\[
\langle N_{\pm} \rangle_{m.c.e.} \approx \bar{N}_{\pm} \left( 1 + \frac{49}{2304\bar{N}_{\pm}} \ + \frac{49}{9216\bar{N}_{\pm}} + \ldots \right) \tag{27}
\]
and
\[
\langle N_{\pm}^2 \rangle_{m.c.e.} \approx \bar{N}_{\pm}^2 \left( 1 + \frac{1}{8\bar{N}_{\pm}} \ + \frac{49}{1152\bar{N}_{\pm}} + \frac{49}{6144\bar{N}_{\pm}} + \ldots \right) . \tag{28}
\]
The behavior of the ratio \( \langle N_{\pm} \rangle_{m.c.e.}/\bar{N}_{\pm} \) is shown in Fig. 3.
FIG. 5: The ratio of average particle number \( \langle N_\pm \rangle_{m.c.e.} \) in the MCE to that \( N_\pm \) in the GCE for positively (negatively) charged particles at zero net charge \( Q = 0 \).

FIG. 6: The scaled variance \( \omega^\pm_{m.c.e.} \) for positively (negatively) charged particles in the MCE at zero net charge \( Q = 0 \).
FIG. 7: The particle number distributions of positively (negatively) charged particles in the MCE (31) and in the GCE (the Poisson distribution) at zero net charge \( Q = 0 \) and \( N_\pm = 0.5 \).

The behavior of the scaled variance,

\[
\omega_{m,c.e.}^\pm = \frac{\langle N_\pm^2 \rangle_{m,c.e.} - \langle N_\pm \rangle_{m,c.e.}^2}{\langle N_\pm \rangle_{m,c.e.}},
\]

is shown in Fig. 6. The asymptotic expansion of \( \omega_{m,c.e.}^\pm \) at \( N_\pm \to \infty \) has the form:

\[
\omega_{m,c.e.}^\pm \approx \frac{1}{8} \left( 1 - \frac{49}{1152 N_\pm^2} + \ldots \right),
\]

while \( \omega_{c.e.}^\pm = 1 \), the same as for neutral particles, and \( \omega_{c.e.} = 1/2 \) [6]. From Eq. (30) and Fig. 6 one sees that \( \omega_{m,c.e.}^\pm = 1/8 \) in the thermodynamic limit, and this is by a factor of 1/8 smaller than the scaled variance in the GCE and by a factor of 1/4 than in the CE. Therefore, for the system of massless particles with Boltzmann statistics the exact energy conservation leads to the MCE suppression of the particle number scaled variance in the thermodynamic limit by a factor of 1/4, and the exact charge conservation makes an additional suppression by a factor of 1/2.

From Eq. (22) we find the positive (negative) particle number distribution in the MCE at net charge \( Q = 0 \) equal to zero:

\[
P_{m,c.e.}(E, V, N_\pm, Q = 0) = \frac{1}{E W(E, V, Q = 0)} \frac{x^{2N_\pm}}{(6N_\pm - 1)! N_\pm!^2}.
\]

In Figs. 7 and 8 we present the particle number distributions \( P_{m,c.e.}(E, V, N_\pm, Q = 0) \) (31) for \( N_\pm = 0.5 \) and \( N_\pm = 10 \), respectively, and compare them with the GCE Poisson distributions which remain the same as for the case of neutral particles. Similar to Eqs. (17, 19) we find the Gauss approximation for the MCE particle number distribution in the thermodynamic limit:

\[
P_{m,c.e.}(E, V, N_\pm, Q = 0) \propto \exp \left[ - \frac{4(N_\pm - \overline{N}_\pm)^2}{\overline{N}_\pm} \right],
\]

where \( \overline{N}_\pm \) is the average number of particles.

\( P \) and \( Q \) on the axes of the diagram represent the probability and the net charge, respectively. The points on the graph correspond to the calculated particle number distributions in the MCE and GCE, while the solid line represents the Poisson distribution.
from which it evidently follows $\langle N_{\pm}^2 \rangle_{m.c.e.} - \langle N_{\pm} \rangle_{m.c.e.}^2 = \overline{N}_{\pm}/8$ and $\omega_{m.c.e.}^\pm = 1/8$.

V. PARTICLE NUMBER FLUCTUATIONS FOR BOSONS AND FERMIONS

To study the effects of quantum statistics in the MCE we use the technique proposed in Ref. [7]. This method allows us to calculate the particle number fluctuations in the systems with the exact conservation laws imposed in the thermodynamic limit $V \to \infty$. We reproduce the results of the previous sections for massless particles with Boltzmann statistics and study the MCE particle number fluctuations in a general case of the system of massive charged particles with Bose and/or Fermi statistics taken into account.

Let us start with a system of neutral bosons or fermions, then we extend our formulation to a system of charged particles. The system of neutral non-interacting identical Bose or Fermi particles can be characterized by the occupation numbers $n_p$ of single quantum states labeled by momenta $p$. The occupation numbers run over $n_p = 0, 1, 2, \ldots$ for the bosons. The GCE average values and fluctuations of $n_p$ equal to [9]:

\begin{align}
\langle n_p \rangle_{g.c.e.} &= \frac{1}{\exp(\epsilon_p/T) - \gamma}, \\
\langle (\Delta n_p)^2 \rangle_{g.c.e.} &= \langle n_p^2 \rangle_{g.c.e.} - \langle n_p \rangle_{g.c.e.}^2 = \langle n_p \rangle_{g.c.e.} (1 + \gamma \langle n_p \rangle_{g.c.e.}) \equiv v_p^2,
\end{align}

where $\gamma = +1$ and $\gamma = -1$ for Bose and Fermi statistics, respectively, $\epsilon_p = \sqrt{p^2 + m^2}$ and $m$ is the particle mass. Note that $\gamma = 0$ in Eqs. (33) corresponds to the Boltzmann approximation which is valid if $\langle n_p \rangle_{g.c.e.} \ll 1$ for all $p$.

\[2\text{ There are examples of neutral bosons, like photon, } \pi^0, \rho^0, \text{ etc., which are identical to their antiparticles. For fermions such a consideration has only illustrative purposes. One always needs to introduce some kind of charge to distinguish fermions from their antiparticles.}\]
Expressions (33, 34) are microscopic in the sense that they describe the average values and fluctuations of single modes with momentum $p$. However, the fluctuations of macroscopic quantities of the system can be determined through the fluctuations of these single modes. To be more precise, we will demonstrate that the particle number fluctuations can be written in terms of the microscopic correlator $\langle \Delta n_p \Delta n_k \rangle_{g.c.e.}$, where $\Delta n_p \equiv n_p - \langle n_p \rangle_{g.c.e.}$. This correlator can be presented as:

$$\langle \Delta n_p \Delta n_k \rangle_{g.c.e.} = v_p^2 \delta_{pk},$$

due to the fact that the GCE fluctuations of the occupation numbers for different quantum states $p \neq k$ are statistically independent. The variance $\langle (\Delta N)^2 \rangle_{g.c.e.} \equiv \langle N^2 \rangle_{g.c.e.} - \langle N \rangle_{g.c.e.}^2$ of the total number of particles, $N \equiv \sum_p n_p$, equals to:

$$\langle (\Delta N)^2 \rangle_{g.c.e.} = \sum_{p,k} \langle n_p n_k \rangle_{g.c.e.} - \langle n_p \rangle_{g.c.e.} \langle n_k \rangle_{g.c.e.} = \sum_{p,k} \langle \Delta n_p \Delta n_k \rangle_{g.c.e.} = \sum_p v_p^2 .$$

We have assumed above that the quantum $p$-levels are non-degenerate. In fact each this level should be further specified by the the projection of a particle spin $j$. Thus, each $p$-level splits into $g = 2j + 1$ sub-levels. It will be assumed that the $p$-summation includes all sub-levels too. This does not change the above formulation because of statistical independence of these quantum sub-levels. The degeneracy factor $g$ enters explicitly when one substitutes, in the thermodynamic limit, the summation over discrete levels by the integration:

$$\sum_p \ldots \approx \frac{gV}{2\pi} \int_0^\infty p^2 dp \ldots .$$

The scaled variance $\omega_{g.c.e.}$ in the thermodynamic limit $V \to \infty$ reads:

$$\omega_{g.c.e.} \equiv \frac{\langle (\Delta N)^2 \rangle_{g.c.e.}}{\langle N \rangle_{g.c.e.}} = \frac{\sum_{p,k} \langle \Delta n_p \Delta n_k \rangle_{g.c.e.}}{\sum_p \langle n_p \rangle_{g.c.e.}} = \frac{\sum_p v_p^2}{\sum_p \langle n_p \rangle_{g.c.e.}} \approx \frac{\int_0^\infty p^2 dp \ v_p^2}{\int_0^\infty p^2 dp \ \langle n_p \rangle_{g.c.e.}} .$$

The formula for the microscopic correlator will be modified if we impose the exact conservation laws in our equilibrated system. We introduce the equilibrium probability distribution $W(n_p)$ of the different sets $\{n_p\}$ of the occupation numbers. In the GCE each $n_p$ fluctuates independently according approximately to the Gauss distribution law for $\Delta n_p$ with mean square deviation $v_p^2$:

$$W(n_p) \propto \prod_p \exp \left[ -\frac{(\Delta n_p)^2}{2v_p^2} \right].$$

To justify Eq. (38) one can consider (see Ref. 3) the sum of $n_p$ in small momentum volume $(\Delta p)^3$ with the center at $p$. At fixed $(\Delta p)^3$ and $V \to \infty$ the average number of particles inside $(\Delta p)^3$ becomes large. Each particle configuration inside $(\Delta p)^3$ consists of $(\Delta p)^3 \cdot gV/(2\pi)^3 >> 1$ statistically independent terms, each with average value $\langle n_p \rangle_{g.c.e.}$ (33) and variance $v_p^2$ (34). From the central limit theorem it then follows that the probability distribution for the fluctuations inside $(\Delta p)^3$ should be Gaussian. In fact, we always convolve $n_p$ with some smooth function of $p$, so instead of writing the Gaussian distribution for the sum of $n_p$ in $(\Delta p)^3$ we can use it directly for $n_p$.

Now we want to impose the exact conservation laws. The conserved quantity $A$ (the energy and/or conserved charge) can be written in the form $A \equiv \sum_p a(p) n_p$. An exact conservation law means the restriction on the sets $\{n_p\}$ of the occupation numbers: only those sets which satisfy the condition $\Delta A = \sum_p a(p) \Delta n_p = 0$ can be realized. Let us consider an exact energy conservation. Then $A \to E$ (i.e. $a(p) \to \epsilon_p$) and the distribution (38) will be modified because of the exact energy conservation as:

$$W(n_p) \propto \prod_p \exp \left[ -\frac{(\Delta n_p)^2}{2v_p^2} \right] \ \delta \left( \sum_p \epsilon_p \Delta n_p \right) \propto \int_{-\infty}^{\infty} d\lambda \ \prod_p \exp \left[ -\frac{(\Delta n_p)^2}{2v_p^2} + i\lambda \ \epsilon_p \Delta n_p \right],$$

where $\delta(\epsilon_p \Delta n_p)$ is the Dirac delta-function. It is convenient to generalize distribution (40) using further the integration along imaginary axis in $\lambda$-space. After completing squares one gets:

$$W(n_p, \lambda) \propto \prod_p \exp \left[ -\frac{(\Delta n_p - \lambda v_p^2 \epsilon_p)^2}{2v_p^2} + \frac{\lambda^2}{2} \epsilon_p^2 \right],$$

where $\lambda \in \mathbb{R}$.
and the average values (i.e. the MCE averages) are now calculated as:

\[
\langle ... \rangle_{\text{m.c.e.}} = \frac{\int_{-\infty}^{\infty} d \lambda \int_{-\infty}^{\infty} \prod_p d n_p ... W(n_p, \lambda)}{\int_{-\infty}^{\infty} d \lambda \int_{-\infty}^{\infty} \prod_p d n_p W(n_p, \lambda)} .
\]

(42)

Using Eq. 14 one easily deduces

\[
\langle (\Delta n_p - v_p^2 \lambda_p)(\Delta n_k - v_k^2 \lambda_k) \rangle_{\text{m.c.e.}} = \delta_{pk} v_p^2 , \quad \langle \lambda^2 \rangle_{\text{m.c.e.}} = - \left( \sum_p v_p^2 \right)^{-1} , \quad \langle (\Delta n_p - v_p^2 \lambda_p)\lambda \rangle_{\text{m.c.e.}} = 0 .
\]

Therefore, one finds the MCE average for the microscopic correlator

\[
\langle \Delta n_p \Delta n_k \rangle_{\text{m.c.e.}} = \delta_{pk} - v_p^2 \lambda_p v_k^2 \epsilon_k \langle \lambda^2 \rangle + \langle \Delta n_p \lambda \rangle v_k^2 \epsilon_k + \langle \Delta n_k \lambda \rangle v_p^2 \epsilon_p
\]

(43)

\[
= \delta_{pk} + v_p^2 \lambda_p v_k^2 \epsilon_k \langle \lambda^2 \rangle = \delta_{pk} v_p^2 - \frac{v_p^2 \lambda_p v_k^2 \epsilon_k}{\sum_p v_p^2} .
\]

By means of Eq. 15 one obtains:

\[
\omega_{\text{m.c.e.}} = \frac{\langle (\Delta N^2) \rangle_{\text{m.c.e.}}}{\langle N \rangle_{\text{m.c.e.}}} = \frac{\sum_p k \langle (\Delta n_p) \rangle_{\text{m.c.e.}}}{\sum_p \langle n_p \rangle_{\text{m.c.e.}}} \simeq \frac{\sum_p v_p^2}{\sum_p \langle n_p \rangle_{\text{g.c.e.}}} - \frac{\left( \sum_p v_p^2 \epsilon_p \right)^2}{\sum_p \langle n_p \rangle_{\text{g.c.e.}} \sum_p v_p^2} .
\]

(44)

Comparing Eq. 16 and Eq. 34 one notices two changes of the microscopic correlator due to the exact energy conservation. First, the MCE fluctuations of each mode is reduced, i.e. the value of \( \langle (\Delta n_p)^2 \rangle_{\text{m.c.e.}} \) calculated from Eq. 16 at \( p = k \) is smaller than that of \( \langle (\Delta n_p)^2 \rangle_{\text{g.c.e.}} = v_p^2 \), given by Eq. 34 in the GCE. Second, in the MCE the anticorrelations appear between different modes \( p \neq k \) (they are absent in the GCE). Both these changes result in the MCE suppression of the scaled variances 14 in a comparison with those in the GCE 35. In fact, the first term in the r.h.s. of Eq. 16 equals to the GCE scaled variance \( \omega_{\text{g.c.e.}} \) 35 and the second negative term corresponds to the MCE suppression effects. Note also that according to Eq. 16 the MCE fluctuations in the thermodynamic limit \( V \to \infty \) can be presented in terms of the GCE quantities. The exact energy conservation should also lead to the differences between \( \langle n_p \rangle_{\text{m.c.e.}} \) and \( \langle n_p \rangle_{\text{g.c.e.}} \). However, in the thermodynamic limit it follows: \( \sum_p \langle n_p \rangle_{\text{m.c.e.}} \simeq \sum_p \langle n_p \rangle_{\text{g.c.e.}} \). It can be also proven by the straightforward calculations that differences between \( \langle n_p \rangle_{\text{m.c.e.}} \) and \( \langle n_p \rangle_{\text{g.c.e.}} \) lead to the corrections of the order of \( \frac{V}{N} \) to both \( \langle N^2 \rangle_{\text{m.c.e.}} \) and \( \langle N \rangle_{\text{m.c.e.}} \), but these corrections are equal to each other and they are cancelled out in the calculation of \( \langle (\Delta N^2) \rangle_{\text{m.c.e.}} \).

The GCE 35 and MCE 14 scaled variances for different statistics are shown as functions of \( m/T \) in Fig. 9. In the Boltzmann approximation \( \gamma = 0 \) from Eq. 38 one finds:

\[
\omega_{\text{g.c.e.}}^{\text{Boltz}} = 1 ,
\]

(45)

which coincides with Eq. 16 used in Section III for \( m = 0 \). The Eq. 35 remains valid for all values of \( m/T \). This is due to the fact that it follows from the Poisson particle number distribution \( P_{\text{g.c.e.}}^{\text{Boltz}}(N, N) \) in the GCE, which is given by Eq. 15 at all values of \( m/T \) (only the average value of particle number \( \langle N \rangle \) decreases with increasing of \( m/T \)). From Eq. 35 it follows that the effects of quantum statistics lead to the Bose enhancement, \( \omega_{\text{g.c.e.}}^{\text{Bose}} > 1 \) at \( \gamma = 1 \), and the Fermi suppression, \( \omega_{\text{g.c.e.}}^{\text{Fermi}} < 1 \) at \( \gamma = -1 \), of the particle number fluctuations. The strongest quantum statistic effects correspond to the \( m/T \to 0 \) limit:

\[
\omega_{\text{g.c.e.}}^{\text{Bose}}(m = 0) = \frac{\pi^2}{6 \zeta(3)} \approx 1.368 ,
\]

(46)

\[
\omega_{\text{g.c.e.}}^{\text{Fermi}}(m = 0) = \frac{\pi^2}{9 \zeta(3)} \approx 0.912 .
\]

(47)

For the particle number fluctuations, as seen from Fig. 9 the Bose enhancement \( \omega_{\text{g.c.e.}}^{\text{Bose}}/\omega_{\text{g.c.e.}}^{\text{Boltz}} > 1 \) and the Fermi suppression \( \omega_{\text{g.c.e.}}^{\text{Fermi}}/\omega_{\text{g.c.e.}}^{\text{Boltz}} < 1 \) factors decrease monotonously with increasing of \( m/T \), in both the GCE and the
FIG. 9: Three upper lines present the GCE scaled variances $\omega_{\text{g.c.e.}}$ \((38)\) at different values of $m/T$, whereas three lower lines correspond to the MCE scaled variances $\omega_{\text{m.c.e.}}$ \((44)\). The dashed lines correspond to the Bose statistics, the dashed-dotted lines to the Fermi statistics and the solid lines to the Boltzmann approximation.

MCE. These effects of quantum statistics in both ensembles become negligible at $m/T \gg 1$, as in this limit one finds $\langle n_p \rangle_{\text{g.c.e.}} \simeq \exp(-\epsilon_p/T) \ll 1$, so that $v_p^2 \simeq \langle n_p \rangle_{\text{g.c.e.}}$ for both the Bose and Fermi statistics.

For the Boltzmann approximation all momentum integrals in Eq. \((44)\) for $\omega_{\text{m.c.e.}}$ can be calculated analytically:

$$
\int_0^{\infty} p^2 dp \exp\left(-\frac{\epsilon_p}{T}\right) = T m^2 K_2\left(\frac{m}{T}\right), \quad \int_0^{\infty} p^2 dp \epsilon_p \exp\left(-\frac{\epsilon_p}{T}\right) = \frac{m^4}{8} \left[ K_4\left(\frac{m}{T}\right) - K_0\left(\frac{m}{T}\right) \right],
$$

$$
\int_0^{\infty} p^2 dp \epsilon_p^2 \exp\left(-\frac{\epsilon_p}{T}\right) = \frac{m^5}{16} \left[ K_5\left(\frac{m}{T}\right) + K_3\left(\frac{m}{T}\right) - 2 K_1\left(\frac{m}{T}\right) \right].
$$

Making use of the asymptotic behavior of the modified Hankel function $K_n(x)$ at $x \to 0$ \((K_0 (x) \simeq -\ln x \text{ and } K_n(x) \simeq \frac{1}{2} \Gamma (n) \left(\frac{x}{2}\right)^{-n} \text{ for } n \geq 1)\) one gets in the $m/T \to 0$ limit:

$$
\omega_{\text{m.c.e.}}^{\text{Boltz}}(m = 0) = \frac{1}{4}, \quad (48)
$$

i.e. for classical massless particles the MCE scale variance is quarter as large as the corresponding scaled variance in the GCE. This result coincides, of course, with that of Eq. \((13)\) obtained in Section III from the MCE partition function of massless particles with Boltzmann statistics. For the case of Bose and Fermi statistics we obtain:

$$
\omega_{\text{m.c.e.}}^{\text{Bose}}(m = 0) = \frac{\pi^2}{6 \xi(3)} - \frac{135 \xi(3)}{2 \pi^4} \simeq 0.535, \quad (49)
$$

$$
\omega_{\text{m.c.e.}}^{\text{Fermi}}(m = 0) = \frac{\pi^2}{9 \xi(3)} - \frac{405 \xi(3)}{7 \pi^4} \simeq 0.198. \quad (50)
$$
The Bose enhancement and Fermi suppression of the fluctuations exist in both the GCE and the MCE. However, the effects of quantum statistics for the particle number fluctuations are stronger in the MCE than those in the GCE. As it follows from Eqs. 15-18 the Bose enhancement of the scaled variance in the MCE at \( m/T \to 0 \) is approximately equal to 2.142, and the Fermi suppression 0.793. These numbers can be compared with 1.368 and 0.912, respectively, found from Eqs. 15-17 in the GCE.

As seen from Fig. 8 the MCE suppression of the particle number fluctuations for massive particles is stronger than that for massless ones, and all MCE scaled variances \( \omega_{\text{m.c.e.}} \) decrease monotonously with increasing of \( m/T \). From the asymptotic expansion at \( x \gg 1 \) 6

\[
K_n(x) \approx \sqrt{\frac{\pi}{2x}} \exp(-x) \sum_{k \geq 0} \frac{1}{(2x)^k} \left( 4n^2 - 1 \right) \ldots \left( 4n^2 - (2k - 1)^2 \right) \frac{1}{k! 2^{2k}}
\]

one finds the behavior of the scaled variance \( \omega_{\text{m.c.e.}}^{\text{Boltz}} \approx \frac{3}{4} (m/T)^{-2} \ll 1 \) at \( m/T \gg 1 \) for the Boltzmann approximation (to obtain this result one needs to keep terms up to \( k = 3 \) in Eq. (51)). The MCE scaled variances for the Bose and Fermi statistics have the same behavior, as the effects of quantum statistics are negligible at \( m/T \gg 1 \), so that \( \omega_{\text{m.c.e.}}^{\text{Bose}} \approx \omega_{\text{m.c.e.}}^{\text{Fermi}} \approx \omega_{\text{m.c.e.}}^{\text{Boltz}} \).

For the system of charged particles and antiparticles in the GCE, similar to Eqs. (28-31), one has:

\[
\langle n_p^{\pm} \rangle_{\text{g.c.e.}} = \frac{1}{\exp[(\epsilon_p + \mu)/T] - \gamma},
\]

\[
((\Delta n_p^{\pm})^2)_{\text{g.c.e.}} \equiv \left( \langle n_p^{\pm} \rangle_{\text{g.c.e.}} - \langle n_p^{\pm} \rangle_{\text{g.c.e.}} \right)^2 = \langle n_p^{\pm} \rangle_{\text{g.c.e.}} \left( 1 + \gamma \langle n_p^{\pm} \rangle_{\text{g.c.e.}} \right) \equiv v_p^{\pm 2},
\]

where \( \mu \) is the chemical potential connected with the conserved charge \( Q \). The microscopic correlator 30 can be then generalized as:

\[
\langle \Delta n_p^\alpha \Delta n_k^\beta \rangle_{\text{g.c.e.}} = v_p^{\alpha 2} \delta_{\alpha \beta},
\]

where \( \alpha, \beta \) are + and(or) -. The average values of the energy \( E = \sum_{p,\alpha} \epsilon_p n_p^\alpha \) and charge\(^3 \) \( Q = \sum_{p,\alpha} q^\alpha n_p^\alpha \) are regulated in the GCE by the temperature \( T \) and the chemical potential \( \mu \), respectively. Similar to Eq. (30) the MCE distribution for the occupation numbers \( n_p^\alpha \) can be presented as:

\[
W(n_p) \propto \prod_{p,\alpha} \exp \left[ - \frac{(\Delta n_p^\alpha)^2}{2v_p^{\alpha 2}} \right] \delta \left( \sum_{p,\alpha} \epsilon_p n_p^\alpha \right) \delta \left( \sum_{p,\alpha} q^\alpha n_p^\alpha \right) \int_{-\infty}^{\infty} d\lambda_E \int_{-\infty}^{\infty} d\lambda_Q \prod_{p,\alpha} \exp \left[ - \frac{(\Delta n_p^\alpha)^2}{2v_p^{\alpha 2}} + i \lambda_E \epsilon_p \Delta n_p^\alpha + i \lambda_Q q^\alpha \Delta n_p^\alpha \right],
\]

where \( \delta \left( \sum_{p,\alpha} \epsilon_p n_p^\alpha \right) \) and \( \delta \left( \sum_{p,\alpha} q^\alpha n_p^\alpha \right) \) correspond to the exact energy and charge conservations, respectively, in the MCE. After the straightforward calculations, similar to Eqs. (41), one gets an expression for the microscopic correlator in the MCE for the charged particles with the exact charge conservation law imposed:

\[
\langle \Delta n_p^\alpha \Delta n_k^\beta \rangle_{\text{m.c.e.}} = v_p^{\alpha 2} \delta_{p,k} \delta_{\alpha \beta} - \frac{v_p^{\alpha 2} v_k^{\beta 2}}{|A|} \left[ q^\alpha q^\beta \sum_{p,\alpha} v_p^{\alpha 2} \epsilon_p^2 + \epsilon_p \epsilon_k \sum_{p,\alpha} v_p^{\alpha 2} q^2 - (q^\alpha \epsilon_k + q^\beta \epsilon_p) \sum_{p,\alpha} v_p^{\alpha 2} \epsilon_p q^\alpha \right],
\]

where

\[
|A| \equiv \left( \sum_{p,\alpha} v_p^{\alpha 2} \epsilon_p^2 \right) \cdot \left( \sum_{p,\alpha} v_p^{\alpha 2} q^2 \right) - \left( \sum_{p,\alpha} v_p^{\alpha 2} \epsilon_p q^\alpha \right)^2.
\]

\(^3\) In what follows we assume \( q^+ = 1 \) and \( q^- = -1 \), so that \( Q = N_+ - N_- \). However, other values with \( q^+ = -q^- \) can be also used.
Therefore, one finds:

\[
\omega_{\text{m.c.e.}}^{\alpha} = \frac{\langle (\Delta N_{\text{m.c.e.}}^2) \rangle_{\text{m.c.e.}}}{\langle N_{\text{m.c.e.}} \rangle_{\text{m.c.e.}}} = \frac{\sum_{p,k} \langle \Delta n_{\text{m.c.e.}}^{\alpha} \Delta n_{\text{m.c.e.}}^{\alpha} \rangle_{\text{m.c.e.}}}{\sum_{p} \langle n_{\text{m.c.e.}}^{\alpha} \rangle_{\text{m.c.e.}}}.
\]

(58)

The first term in the r.h.s. of Eq. (58) corresponds to the GCE scaled variance \(\omega_{\text{g.c.e.}}^{\alpha}\) for the positive (\(\alpha = 1\)) or negative (\(\alpha = -1\)) particles. Note that in the Boltzmann approximation \(\omega_{\text{g.c.e.}}^{\alpha, \text{Boltz}}\) coincides with the scaled variance for the neutral particles \(\omega_{\text{g.c.e.}}^{\text{Boltz}}\) and, therefore, equals 1. The second and third terms in the r.h.s. of Eq. (58) correspond to the MCE suppression of the fluctuations due to the exact conservations of energy (compare to Eq. (44)) and charge, respectively.

In the case of zero net charge \(Q = 0\) (this means \(\mu = 0\) and, therefore, \(\langle n_{\text{m.c.e.}}^{+} \rangle_{\text{g.c.e.}} = \langle n_{\text{m.c.e.}}^{-} \rangle_{\text{g.c.e.}}\) one finds for charged particles:

\[
\omega_{\text{m.c.e.}}^{\pm}(Q = 0) = \frac{\sum_{p} v_{p}^{2}}{\sum_{p} \langle n_{\text{g.c.e.}}^{\pm} \rangle_{\text{g.c.e.}}} - \frac{1}{2} \frac{\sum_{p} v_{p}^{2}}{\sum_{p} \langle n_{\text{g.c.e.}}^{\pm} \rangle_{\text{g.c.e.}}} - \frac{1}{2} \frac{\sum_{p} v_{p}^{2}}{\sum_{p} \langle n_{\text{g.c.e.}}^{\pm} \rangle_{\text{g.c.e.}}} = \frac{1}{2} \omega_{\text{m.c.e.}}\,.
\]

(59)

where \(\omega_{\text{m.c.e.}}\) in Eq. (59) corresponds to the MCE scaled variance found for the neutral particles with the parameters \(m\) and \(q\) equal to those of the charged particles. Thus, we came to the conclusion that in the MCE at \(Q = 0\) an exact charge conservation leads to the scaled variances of (negative) positive particles which are by a factor of 1/2 smaller than the corresponding MCE scaled variances for neutral particles. This result is an agreement with the CE suppression of the particle number fluctuations found in Ref. [1]: the scaled variance in the CE at \(Q = 0\) in the thermodynamic limit equals to \(\omega_{\text{c.c.e.}}^{\pm} = \frac{1}{2} \omega_{\text{g.c.e.}}^{\pm}\) (note that at \(\mu = 0\) the GCE scaled variances for charged and neutral particles are equal to each other). In the case of Boltzmann statistics one finds for massless particles \(\omega_{\text{m.c.e.}}^{\pm, \text{Boltz}}(m = 0) = 1/8\), which coincides, of course, with the result of Eq. (59) obtained in Sec. IV.

VI. SUMMARY

We have studied the particle number fluctuations in the MCE. First, in Sec. II we have obtained the partition function for the system of non-interacting massless neutral particles with Boltzmann statistics. This allows us to calculate the particle number fluctuations for this system in Sec. III. These MCE fluctuations are suppressed in comparison to those in the GCE. In the thermodynamic limit the MCE scaled variance of the multiplicity distribution equals 1/4 and this is a quarter the size of that in the GCE. As a second step we consider in Sec. IV the system of the charged massless particles with zero net charge and calculate its partition function. This leads to the scaled variance equal to 1/8 in the thermodynamic limit. Thus, an exact charge conservation makes an additional suppression of the scaled variance by a factor of 1/2. In Sec. V we use an approach proposed in Ref. [4]. This method does not work accurately for finite (small) systems, but it does allow us to calculate the correct values of the scaled variances in the thermodynamic limit \(V \to \infty\) for a much broader class of statistical systems – massive bosons and fermions – when both the energy and charge exact conservations are taken into account. The effects of quantum statistics and non-zero particle mass lead to the significant changes of the MCE particle number fluctuations, and they are studied in details in Sec. V. We have also reproduced the limiting \(V \to \infty\) behavior of the MCE scaled variances for massless particles with Boltzmann statistics obtained in Secs. III–IV from the exact analytical expressions for the partition functions.

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APPENDIX A

The microcanonical partition function $W_N(E, V)$ can be recursively calculated from $W_{N-1}(E, V)$:

$$W_N(E, V) = \frac{1}{N!} \frac{gV}{(2\pi)^3} \int d^3 p^{(N)} \frac{gV}{(2\pi)^3} \int d^3 p^{(N-1)} \ldots \frac{gV}{(2\pi)^3} \int d^3 p^{(1)} \delta(E - \sum_{k=1}^{N} |\vec{p}(k)|)$$  \hspace{1cm} (A1)

$$= \frac{1}{N} 4\pi \frac{gV}{(2\pi)^3} \int_0^E dp^{(N)} \left( \frac{p^{(N)}}{2} \right)^2 \frac{1}{(N-1)!} \frac{gV}{(2\pi)^3} \int d^3 p^{(N-1)} \ldots \frac{gV}{(2\pi)^3} \int d^3 p^{(1)} \delta(E - p^{(N)} - \sum_{k=1}^{N-1} |\vec{p}(k)|)$$

$$= \frac{1}{N} \frac{gV}{2\pi^2} \int_0^E dp^2 W_{N-1}(E - p, V).$$

Now we shall prove by induction method that for arbitrary $N \geq 1$ the microcanonical partition function has the form of Eq. (3). It can be checked directly that Eq. (3) at $N = 1$ and $N = 2$ coincides with (1) and (2), respectively. Now assume that it is correct for $N - 1$,

$$W_{N-1}(E, V) = \frac{2^{N-1}}{(3N-4)!(N-1)!} \left( \frac{gV}{2\pi^2} \right)^{N-1} E^{3N-4},$$  \hspace{1cm} (A2)

and substitute (2) into Eq. (A1):

$$W_N(E, V) = \frac{1}{N} \frac{gV}{2\pi^2} \int_0^E dp^2 \frac{2^{N-1}}{(3N-4)!(N-1)!} \left( \frac{gV}{2\pi^2} \right)^{N-1} (E - p)^{3N-4}$$  \hspace{1cm} (A3)

$$= \frac{1}{N} \left( \frac{gV}{2\pi^2} \right)^N \frac{2^{N-1}}{(3N-4)!(N-1)!} \int_0^E dp^2 (E - p)^{3N-4}. $$

The last integral can be easily evaluated after substitution $p = E - q$:

$$\int_0^E dp^2 (E - p)^{3N-4} = \int_0^E dq (E - q)^2 q^{3N-4} = \frac{2E^{3N-1}}{(3N-1)(3N-2)(3N-3)}. $$  \hspace{1cm} (A4)

Eqs. (A3 A4) result in Eq. (3).

APPENDIX B

The generalized hypergeometric function (GHF), also known as the Barnes extended hypergeometric function, is defined by the following series,

$$pF_q(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k \ldots (a_p)_k}{(b_1)_k(b_2)_k \ldots (b_q)_k} \frac{z^k}{k!}, $$  \hspace{1cm} (B1)

where $(a)_k$ is the Pochhammer symbol:

$$(a)_k = \frac{\Gamma(a + k)}{\Gamma(a)} = a(a + 1) \ldots (a + k - 1). $$  \hspace{1cm} (B2)

The asymptotic behavior of GHF at $z \to \infty$ is given at $p=0$ by

$$\ {}^0F_q(b_1, b_2, \ldots, b_q; z) \simeq \frac{\prod_{j=1}^{q} \Gamma(b_j)}{\sqrt{(q + 1)(2\pi)^q}} \left( \frac{1}{\sqrt{z}} \right)^{b_1} \left( \frac{1}{\sqrt{z}} \right)^{b_2} \ldots \left( \frac{1}{\sqrt{z}} \right)^{b_q} + \sum_{n=0}^{\infty} \frac{c_1}{\sqrt{z}^n} + \frac{c_2}{(\sqrt{z})^{2n}} + \frac{c_3}{(\sqrt{z})^{3n}} + \ldots, $$  \hspace{1cm} (B3)
where

\[ c_1 = \frac{1}{2} D_2 + \frac{q(q + 2)}{24(q + 1)}, \quad (B4) \]

\[ c_2 = \frac{1}{8} D_2^2 + \frac{1}{6} D_3 - \frac{q(q + 2) + 12}{48(q + 1)} D_2 + \frac{q(q + 2)[q(q + 2) + 24]}{1152(q + 1)^2}, \quad (B5) \]

\[ c_3 = -\frac{1}{48} D_2^3 - \frac{1}{12} D_2 D_3 - \frac{1}{12} D_4 + \frac{q(q + 2) + 48}{48(q + 1)} \left( \frac{1}{4} D_2^2 + \frac{1}{3} D_3 \right) - \frac{q^2(q + 2)^2 + 768}{2304(q + 1)^2} D_2 + \frac{q(q + 2) \left[ 5q^2(q + 2)^2 - 288q(q + 2) + 10944 \right]}{414720(q + 1)^3}, \quad (B6) \]

\[ D_2 = B_2 - \frac{1}{(q + 1)} B_1^2, \quad (B7) \]

\[ D_3 = B_3 - \frac{3}{(q + 1)} B_1 B_2 + \frac{2}{(q + 1)^2} B_1^2, \quad (B8) \]

\[ D_4 = B_4 - \frac{4}{(q + 1)} B_1 B_3 + \frac{6}{(q + 1)^2} B_2^2 B_1 - \frac{3}{(q + 1)^3} B_3^2, \quad (B9) \]

and

\[ B_m = 1 + \sum_{j=1}^{q} (b_j)^m. \quad (B10) \]