Shear tensor and dynamics of relativistic accretion disks around rotating black holes

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Abstract

In this paper we solve the hydrodynamical equations of optically thin, steady state accretion disks around Kerr black holes. Here, fully general relativistic equations are used. We use a new method to calculate the shear tensor in the LNRF (Locally Non-Rotating Frame), BLF (Boyer-Lindquist Frame) and FRF (Fluid Rest Frame). We show that two components of shear tensor in the FRF are nonzero (in previous works only one nonzero component was assumed). We can use these tensors in usual transonic solutions and usual causal viscosity, but we derive solutions analytically by some simplifications. Then we can calculate the four velocity and density in all frames such as the LNRF, BLF and FRF.

KeyWords: black hole accretion disks, relativistic disks, accretion disks, shear tensor, hydrodynamic.

1. Introduction

Accretion disks are important in several astrophysical systems. They can be found around Young Stellar Objects (YSO), around compact stellar objects in our galaxy and around several super-massive black holes in Active Galactic Nuclei (AGN).

In super massive accretion disks the mass of black hole is $(10^5 - 10^9)M_\odot$. To study such disks we use the general relativity with relativistic hydrodynamics in Kerr metric background geometry. In relativistic Navier-Stokes fluid we have stress-energy tensor which is related to viscosity and is the cause of redistribution of energy and momentum in fluid. This tensor is defined by the four velocity and metric. In the previous studies, the relativistic and stationary solutions of standard black hole disks were solved. Lasota (1994) was the first who wrote down slim-disk equations which include relativistic effects. He also assumed that only $r$-$\phi$ component of the stress tensor is nonzero. He used a special form for this component, which was followed by Abramowicz et al (1996 and 1997). Chakraborti (1996) derived the transonic solutions of thick and thin disks for a weak viscosity. He assumed similar form for viscosity as Lasota (1994). Then Mannomto (2000) derived the global two temperatures structure of advection-dominated accretion flows (ADAFs) numerically by using full relativistic hydrodynamical equations including the energy equations for the ions and electrons.

Papaloizou & Szuszkiewicz (1994) introduced a phenomenological and non-relativistic equation for the evolution of viscous stress tensor (causal viscosity) which has been used by many authors. Gammie & Popham (1998) and Popham & Gammie (1998) solved ADAFs with relativistic causal viscosity. They used the Boyer-Lindquist coordinates. Takahashi (2007b) solved the equations of relativistic disks in the Kerr-Schild coordinates by using the relativistic causal viscosity. In the papers of Gammie & Popham and Takahashi they assumed that in the FRF, only the $r$-$\phi$ component of shear viscosity is nonzero, then they used the transformation tensors to derive the components of shear stress tensor in the Boyer-Lindquist or Kerr-Schild frames.

In present study, we concentrate on the stationary axisymmetric accretion flow in the equatorial plane. We use a new method to calculate the shear tensor and azimuthal velocity of fluids in the locally non-rotating frame (LNRF) by using Keplerian angular velocity. We derive two kinds of shear tensors in LNRF and BLF; in the first one the direction of fluid rotation is the same as that of black hole ($\Omega_+^\phi$) and in the second one the direction of fluid rotation is in opposite to that of black hole ($\Omega_-$). We calculate the components of the shear tensor for two kinds of fluids in LNRF, BLF and FRF, these calculations show that in FRF there are two nonzero components ($r$-$t$ and $r$-$\phi$ components). But in previous papers, the only nonzero component in FRF was $r$-$\phi$ component. The $r$-$t$ component results from the relativistic calculations of the shear tensor which changes some components of the four velocity. Then, by using these shear tensor components we calculate the four velocity in LNRF, BLF and density in all frames.

This paper’s agenda is as follows. We introduce the metric and reference frame in §2. In §3 basic equations are given. In §4 the shear tensor is calculated in FRF, LNRF and BLF. We derive four velocity in LNRF and BLF in §5 and the influence of two important parameters on density and four velocity can be seen in this section. Also, the influence of the $r$-$t$ component of shear tensor can be seen in this section. Summery and
conclusion are given in §6.

2. Metric, Reference Frame

2.1. Back Ground Metric

For back ground geometry, we use the Boyer-Lindquist coordinates of the rotating black hole space time. In the Boyer-Lindquist coordinates, the Kerr metric is:

\[ ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt), \]

where \( i, j = r, \theta, \phi \). Nonzero components of the lapse function \( \alpha \), the shift vector \( \beta^i \) and the spatial matrix \( \gamma_{ij} \) are given in the geometric units as:

\[ \alpha = \sqrt{\frac{\Sigma\Delta}{A}}, \quad \beta^\theta = -\omega, \quad \gamma_{rr} = \frac{\Sigma}{\Delta}, \]

\[ \gamma_{\theta\theta} = \Sigma, \quad \gamma_{\phi\phi} = \frac{\text{Asin}^2\theta}{\Sigma}. \quad (2) \]

Here, we use geometric mass \( m = GM/c^2 \), \( \Sigma = r^2 + a^2 \cos^2\theta \), \( \Delta = r^2 - 2Mr + a^2 \) and \( A = \Sigma\Delta + 2Mr^2 - a^2 \). The position of outer and inner horizons, \( r\pm \), are calculated by inserting \( \Delta = 0 \) to get \( r\pm = m \pm (m^2 - a^2)^{1/2} \). The angular velocity of the frame dragging due to the black hole rotation is \( \omega = -\gamma_{t\phi}/\gamma_{\phi\phi} = \frac{2m\omega_a}{A} \), where \( M \) is the black hole mass, \( G \) is the gravitational constant and \( c \) is the speed of light and the angular momentum of the black hole, \( J \), is described as:

\[ a = Jc/GM^2, \quad (3) \]

where \(-1 < a < 1\).

Similar to Gammie & Popham (1998), we set \( G = M = c = 1 \) for basic scalings. The nonzero components of metric, \( g_{\mu\nu} \), and its inverse, \( g^{\mu\nu} \) are calculated in Appendix 1.

2.2. Reference Frame

In our study, we use three reference frames. The first one is the Boyer-Lindquist frame (BLF) based on the Boyer-Lindquist coordinates describing the metric, in which, our calculations are done. The second one is the locally non-rotating reference frame (LNRF) which is formed by observers with a future-directed unit vector orthogonal to \( t = \text{constant} \). By using the Boyer-Lindquist coordinates, the LNRF observer is moving with the angular velocity of frame dragging (\( \omega \)). The third frame is fluid rest frame (FRF), an orthonormal tetrad basis carried by observers moving along the fluid.

The physical quantities measured in LNRF are described by using the hat such as \( u^\hat{\alpha} \) and \( u_{\hat{\alpha}} \), and in FRF by using parentheses such as \( u^{(\mu)} \) and \( u_{(\mu)} \). The transformation matrices of FRF, LNRF and BLF are given in Appendix 2 (Bardeen 1970; Bardeen, Press & Teukolsky 1972; Frolov & Novikov 1998).

3. Basic Equations

The basic equations for the relativistic hydrodynamics are the baryon-mass conservation \( (\rho u^\mu)_{;\mu} = 0 \) and the energy momentum conservation: \( T_{\nu}^{\mu;\nu} = 0 \), where \( \rho \) is the rest-mass density and \( T^{\mu\nu} \) is the energy-momentum tensor. Basic dynamical equations except the baryon mass conservation are calculated from the energy-momentum tensor, \( T^{\mu\nu} \). We use the energy-momentum tensor written as:

\[ T^{\mu\nu} = \rho u^\mu u^\nu + p g^{\mu\nu} + q^\mu u^\nu + q^\nu u^\mu, \quad (4) \]

where \( p \) is the pressure, \( \eta = (\rho + u + p)/\rho \) is the relativistic enthalpy, \( u \) is the internal energy, \( t^{\mu\nu} \) is the viscous stress-energy tensor and \( q^\mu \) is the heat-flux four vector. The relativistic Navier-Stokes shear stress, \( t^{\mu\nu} \), is written as (Misner, Thorne & Wheeler 1973):

\[ t^{\mu\nu} = -2\lambda \sigma^{\mu\nu} - \zeta \Theta h^{\mu\nu}, \quad (5) \]

where \( \lambda \) is the coefficient of dynamical viscosity, \( \zeta \) is the coefficient of bulk viscosity, \( h^{\mu\nu} = g^{\mu\nu} + u^{\mu}u^{\nu} \) is the projection tensor, \( \Theta = u^{\gamma}_{;\gamma} \) is the expansion of the fluid world line, and \( \sigma^{\mu\nu} \) is the shear tensor of the fluid which is calculated as:

\[ \sigma_{\mu\nu} = \frac{1}{2}(u_{;\mu}^{;\nu} + u_{;\nu}^{;\mu} + a_{\mu}u_{;\nu} + a_{\nu}u_{;\mu}) - \frac{1}{3}\Theta h_{\mu\nu}, \quad (6) \]

where \( a_{\mu} = u_{;\mu}^{;\gamma}u^{;\gamma} \) is the fourth acceleration.

We study a stationary, axisymmetric and equatorially symmetric global accretion flow in the equatorial plane, i.e., we assume \( u_\theta = 0 \). We also assume that the effects of the bulk viscosity and heat-flux four vector is negligible. In the following sections, we derive the basic equations using the vertical averaging procedures around the equatorial plane which were derived in, e.g., Gammie & Popham (1998).

3.1. Mass conservation

The equation for the baryon mass conservation is written as:

\[ (\rho u^\mu)_{;\mu} = 0, \quad (7) \]

where \( u^\mu \) is the four velocity and \( \sqrt{-g} = r^2 \). By averaging the physical quantities around the equatorial plane and assuming constant \( \dot{M} \) (the mass-accretion rate), we can write equation (7) as follows:

\[ (4\pi H_\theta r^2 \rho u^r)_{;r} = 0 \Rightarrow -4\pi H_\theta r^2 \rho u^r = \dot{M}, \quad (8) \]

where \( H_\theta \) is half-thickness of the accretion disk in \( \theta \) direction. If we normalize the rest-mass density, \( \rho \), by setting \( \dot{M} = 1 \), for calculating global structure of accretion flow, we have:

\[ -4\pi H_\theta \rho u^r r^2 = 1, \quad (9) \]

also, by differentiating equation (9) we have:

\[ \frac{d\ln H_\theta}{dr} + 2\frac{1}{r} + \frac{d\ln u^r}{dr} + \frac{d\ln \rho}{dr} = 0. \quad (10) \]

3.2. Killing Vector

Two specific killing vectors of Kerr metric are \( e^\mu_\phi = (1,0,0,0) \) and \( e^\mu_\theta = (0,0,0,1) \), which are used to derive the disk equations. At first, angular momentum conservation can be derived by\( e^\mu_\phi \)

\[ (T_{\mu}^{\nu} e^\mu_\phi)_{;\nu} = 0 \Rightarrow (T_{\nu}^{\nu})_{;\nu} = 0. \quad (11) \]

By vertically averaging equation (11) we have:

\[ \frac{1}{r^2}(r^2 T_{\phi}^{\phi})_{;r} = 0 \Rightarrow \dot{M} \eta l - 4\pi H_\theta \rho^2 T_{\phi}^{\phi} = \dot{M} j, \quad (12) \]
where $\dot{M}$ is the total inward flux of angular momentum. This equation is similar to the angular momentum equation of Gammie & Papaloizou (1998).

Another equation which can be obtained by using $\varepsilon^\mu_{\nu}$ is:

\[(T^\mu_{\nu} \varepsilon^\nu_{\mu})_{,\nu} = 0 \Rightarrow (T^\mu_{\nu})_{,\nu} = 0. \quad (13)\]

Similarly, by vertically averaging and inserting constant $\tilde{E}$, we have:

\[4\pi H \Theta r^2 ((p + \rho + u)u_t u_r + t^r_t) = \dot{E}. \quad (14)\]

This equation expresses the constancy of mass-energy flux $\dot{E}$ in terms of the radius; $\dot{E}$ is the actual rate of change of the black hole mass. If the fluid is cold and slow at large radiiuses, then $\dot{E} \approx \dot{M}$ (Gammie & Papaloizou 1998).

3.3. Energy Equation

The equation of local energy conservation is obtained from $u^\mu T^\mu_{\nu,\nu} = 0$, as follows:

\[-u_r \frac{d}{dr} (\rho + u) - (\rho + u + p)\Theta + (\rho + u + p)u^\nu u^\mu_{,\mu} + t^\nu_{\mu,\mu} = 0, \quad (15)\]

where $\Theta$ is defined as:

\[\Theta = u^\nu_{,\nu}. \quad (16)\]

And from equation (7) we have:

\[(\rho u^\nu_{,\nu}) = 0 \Rightarrow \rho u^\nu_{,\nu} + \rho u^\nu_{,\nu} = \frac{d\rho}{dr} u^r + \rho u^r_{,\nu} = 0. \quad (17)\]

Therefore the energy equation can be written as:

\[u_r \left( \frac{du_r}{dr} \frac{d}{dr} (\rho + u) - \frac{dp}{dr} u^\nu_{,\nu} + t^\nu_{\mu,\mu} u^\mu. \right) = (\rho + u + p)\Theta u^\mu_{,\mu} + t^\nu_{\mu,\mu} u^\mu. \quad (18)\]

In this paper we do not derive the temperature, pressure and internal energy so that, we will not use this equation. If we want to derive these variables we must use a state equation and use the relation of shear tensor components.

4. Shear Tensor

Lasota (1994) used the $r - \phi$ component of stress shear tensor as follows:

\[\tau^r_{\phi} = -\nu \rho \frac{A^{3/2} \Delta^{1/2} \gamma^3_{\phi} d\Omega}{r^2}, \quad (19)\]

\[\gamma^3_{\phi} \equiv (1 - (\nu^3)^2)^{-1/2}, \quad (20)\]

where $\nu^3$ is the azimuthal component of velocity in the LNRF which is introduced by Manmoto (2000):

\[\nu^3 = \frac{A}{r^2 \Delta^{1/2} \tilde{\Omega}}, \quad (21)\]

where $\tilde{\Omega}$ is defined as:

\[\tilde{\Omega} = \Omega - \omega. \quad (22)\]

Abramowicz et al (1996 and 1997), Manmoto (2000) and others also used equation (19) for stress tensor. In 1994, Papaloizou & Szuszkiewicz introduced a non-relativistic causal viscosity which was used in relativistic form by Peitz & Apple (1997a,b), Gammie & Papaloizou (1998) and Papaloizou & Gammie (1998) in Kerr metric and Takahashi (2007b) in Kerr-Schild metric. They assumed that in the FRF all components vanish, except $t_{\phi r} = t_{\phi \theta}$. The $t_{\tau \phi} = S$ can be introduced in relativistic form as:

\[\frac{DS}{D\tau} = -\frac{S - S_0}{\tau_r}, \quad (23)\]

where $D/D\tau = u^\mu_{,\mu}$ and $S_0$ is the equilibrium value of the stress tensor and $\tau_r$ is the relaxation time scale. Therefore, in steady state we have:

\[u_r \frac{dS}{dr} = -\frac{S - S_0}{\tau_r}. \quad (24)\]

We use a new method to derive the shear tensor approximately. We calculate $\nu^3$ from equation (21) in LNRF assuming $u^r = 0$ (the azimuthal velocity is much greater than accretion velocity or radial velocity, except very close to the inner edge, and for fluids with a small viscosity). In order to simplify, we assume $\Omega = \Omega_k$, where Keplerian angular velocity $\Omega_k$, is defined as:

\[\Omega_k = \pm \frac{M}{r^{3/2} \pm a M^{1/2}}, \quad (25)\]

therefore, for azimuthal velocity we have:

\[v^\phi = \frac{A}{r^2 \Delta^{1/2} (1 - \frac{a}{r} \pm \frac{2a}{a})}. \quad (26)\]

First we calculate the $u^i = \frac{\dot{r}}{r^2}$ = $\sqrt{1 - (\nu^3)^2}$. We can calculate four velocity in LNRF (Appendix 3). The four velocity for $\Omega^-$ (direction of fluid rotation is opposite to the black hole rotation) is:

\[u^-_{\mu} = \left( \frac{-r\sqrt{(r^2 - a)} - \frac{A}{r^2 \Delta^{1/2} \tilde{\Omega}}}{\sqrt{r^4 (r^2 - a)^2 \Delta - A^2 - 4a^2 r^2 (r^2 - a)^2 - 4arA(r^2 - a)}}, 0, 0, \frac{-r^3 + ra^2 + 2ar^2}{\sqrt{r^4 (r^2 - a)^2 \Delta - A^2 - 4a^2 r^2 (r^2 - a)^2 - 4arA(r^2 - a)}} \right). \quad (27)\]

Calculations show that in LNRF $\Gamma^\phi_{\mu\nu} = 0$ and $\Theta = 0$. The shear tensor can be calculated by inserting the four velocity of LNRF in equation (46). The nonzero components of $u^-_{\mu\nu}$, $a^-_{\mu\nu} = u^-_{\mu\nu} u^\nu_{,\nu}$) and $\sigma^-_{\mu\nu}$ for $\Omega^-$ in LNRF are:

\[a_{\mu} = u^-_{\tau\tau} u^\tau_{,\tau} + u^-_{r\tau} u^r_{,\tau} + u^-_{\rho\tau} u^\rho_{,\tau} + u^-_{\chi\tau} u^\chi_{,\tau} = 0, \quad (28)\]
where

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from equation (5) by zero bulk viscosity and \( \lambda = \rho \nu \), we have:

\[ \sigma_{\phi \phi}^+ = \frac{r^2}{2} \frac{d \Omega}{dr}, \]

therefore, \( \Omega^+ \) is

\[ \Omega^+ = \int \frac{2}{r^2} \sigma_{\phi \phi}^+ dr. \]  

For calculation the \( \Omega^\pm \) we use equations 30 and 33 with a simplification in \( \frac{1}{r^2} \) and \( \frac{1}{\Omega^\pm} \) as:

\[ \frac{1}{B^+} = \left( r^7 + 2r^5 a - r^4 a^2 - 3r^6 - 6r^3 a^2 + r^5 a^2 \right) \left( r^7 + 2r^5 a - r^4 a^2 + 3r^6 - 6r^3 a^2 - 3r^2 a^3 - 5r^4 a^4 \right) \]

5. Deriving four velocity

Assuming special values for \( \lambda \) (for example in Takahashi (2007b), \( \lambda = 1.5, 1.7, 2.1, 2.2, 2.3, 2.4 \) and 2.5), we can solve equations (10), (12), (14) and (18) to derive four velocity, density, pressure and etc. numerically. In numerical solutions we must insert values of physical variables on boundary conditions specially on horizon. Some of these boundary conditions are nonphysical, therefore we want to derive analytical solutions without any boundary conditions. We use some non-relativistic relations to calculate four velocity and density in LNRF and BLF analytically. At first, the \( t_{\phi \phi} \) relation of stress tensor in Takahashi (2007a) is used in a relativistic disk, this relation is:

\[ t_{\phi \phi} = -\nu \frac{r^2}{2} \Omega. \]  

Similar to Abramowicz et al.(1996) we assume \( \eta = 1 \), therefore from equation (5) by zero bulk viscosity and \( \lambda = \rho \nu \), we have:

\[ \sigma_{\phi \phi}^+ = \frac{r^2}{2} \frac{d \Omega}{dr}, \]

therefore, \( \Omega^\pm \) is

\[ \Omega^\pm = \int \frac{2}{r^2} \sigma_{\phi \phi}^+ dr. \]  

Influences of this approximation in shear tensor components are seen in figure 1 for \( \Omega^+ \) and \( \alpha = 9 \). Solid curves have no simplifying and dotted curves are with this simplifying.

Setting \( l = \Omega r^2 \) (Takahashi 2007a), the angular momentum can be calculated. From equations (36) and (12), \( \rho \) can be derived as:

\[ \rho = \frac{j - l}{8\pi H_\rho r^2 \nu \sigma_{\phi \phi}^+}, \]

where \( \nu \) is (Takahashi 2007a):
We can calculate four velocity in BLF using the transformation

\[
\nu = \alpha a_s^2 / \Omega_k,
\]

which is the usual \( \alpha \) prescription for viscosity (Shakura & Sunyaev 1973). In some papers \( H_\theta \) is calculated by studying the vertical structure such as Riffert & Herold (1995), Abramowicz et al. (1997) and Takahashi (2007a,b). In these papers \( H_\theta \) was calculated by vertical averaging, then introduced by other variable such as \( l, u_s, p, \rho \) and etc. If we use each of those relations of \( H_\theta \), we must solve all equations numerically. But we want to derive an analytical solution therefore similar by other variable such as \( \alpha \) prescription which influences on redistribution of energy and momentum. The influence of these parameters are shown in figure 5 with \( a_s = .1 \)

The four velocity and density can be derived for various total inward flux of angular momentum (for example \( j = 1, 2 \) and 3) which are shown in figure 6.

5.1. Influence of \( r-t \) component on the four velocity

In pervious papers such as Abramowicz et al. (1996 and 1997), Gammie & Popham (1998), Manmoto (2000) and Takahashi (2007a,b), only the \( r-\phi \) component of shear tensor was assumed to be nonzero in FRF, but in section 4 we showed that the \( r-t \) component is also nonzero. The \( r-t \) component of shear tensor in LNRF results from covariant derivative of \( u_t \). In general relativity the gravitating field causes time dilation. Due to this time dilation, the coordinate time(\( t \)) and proper time(\( \tau \)) are not the same(\( dt > d\tau \)), therefor \( u_t \) in LNRF can be derived as:

\[
u = \alpha a_s^2 / \Omega_k,
\]

In equation (42), \( 0 < \alpha < 1 \) is viscosity coefficient in \( \alpha \) prescription which influences on redistribution of energy and momentum. The influence of these parameters are shown in figure 5 with \( a_s = .1 \)

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u = \alpha a_s^2 / \Omega_k,
\]
Fig. 2. Surface density, left: $\Omega^-$, right: $\Omega^+$ ($\alpha = .01$, $j = 3$, solid: $a = .9$, dotted: $a = .4$, dash-dotted: $a = 0$, long-dashed: $a = -.4$ and dash-spaced: $a = -.9$)

Fig. 3. Four velocity in LNRF and BLF. Left: $\Omega^-$, right: $\Omega^+$ ($\alpha = .01$, $j = 3$, $a = -.9$ and $a_s = .1$, solid: in BLF and dotted: in LNRF)
Fig. 4. Four velocity and $|\Omega|$ in the BLF. Left: $\Omega^-$, right: $\Omega^+(\alpha = 0.01, j = 3$, solid: $\alpha = 0.9$, dotted: $\alpha = 4$, dash-dotted: $\alpha = 0$, long-dashed: $\alpha = -4$ and dash-spaced: $\alpha = -0.9$)
Fig. 5. Influence of $\alpha$ on density and radial four velocity for $\Omega^+(\alpha = .9, j = 3$ and $a_s = .1$. Solid: $\alpha = .01$, dotted: $\alpha = .1$ and dash-dotted: $\alpha = .5$)

Fig. 6. Influence of $j$ in density and the four velocity of $\Omega^+$ in BLF($\alpha = .01$, $\alpha = .9$ and $a_s = .1$. Solid: $j = 1$, dotted: $j = 2$ and dash-spaced: $j = 3$)
Fig. 7. Influence of the $r-t$ component of shear tensor on $|u_t|$ (in LNRF) and $|u_t|$, $|u^r|$ (in BLF), $|\Omega|$ and $|v_r|$ for $\Omega^\pm$. solid with the $r-t$ component and dotted without it ($\alpha = .01$, $a = .9$, $a_s = .1$ and $j = 3$).
Obviously, $r - t$ component of shear tensor have more influence on $u_t$ and $u^t$. Therefore, for deriving $u_t$ and $u^t$ we can not vanish the $r - t$ component.

6. SUMMARY AND CONCLUSION

In causal viscosity method, only $r - \phi$ component of shear tensor is assumed to be nonzero in FRF. We do not use this method for viscosity because our calculations show that there are two nonzero components of shear tensor in FRF. In our method, we use azimuthal velocity of LNRF and Keplerian angular velocity, then we calculate all components of shear tensor in LNRF for two kinds of fluids (rotation in the direction of black hole $\Omega^+$) and rotation in the opposite direction of black hole ($\Omega^-$). Using transformation tensor we can calculate all components of shear tensor in BLF and FRF.

Solid curves of figure 1 show unphysical treatments in shear tensor components close to the inner edge. This unphysical treatment in $\sigma_{r\phi}$ may be concerned to assuming $u^r < u^{\phi}$, therefore if we find a suitable $v^r$ this treatment may be resolved. We may have a suitable $v^r$ if we put $u^r = \dot{\gamma} = \sqrt{1 - (v^r)^2} > 0 \Rightarrow 1 - (v^\phi)^2 > (v^r)^2$. According to equation (21), the most important origin of unphysical treatments in $\sigma_{r\phi}$ is the $1/\Delta$ term ($1/\Delta$ has a singularity in the inner edge). Therefore by using Taylor expansion the unphysical treatment of $\sigma_{ij}$ and $\sigma_{r\phi}$ were solved as it can be seen in dotted curves of figure 1.

We solved the hydrodynamical equations in LNRF with assuming $\eta = 1$, we derived the density and four velocity in BLF and FRF. The density which is calculated analytically in LNRF is the same in all frames such as BLF. But comparison of four velocity in LNRF and BLF (figure 3) shows that $u_r$ and $u_t$ have a small difference in two frames. In equation (44), $\sqrt{\Sigma/\Delta} \approx \sqrt{r^2/(r^2 - 2r + a^2)}$ is near 1, specially in larger radiuses, therefore $u_r$ is similar in two frames. Following equation (43), for $u_\phi$ we have two parts, the first part is $(\sqrt{\Delta/\Sigma}/A \approx (r - 2)/r)u_t$ and the second part is $( -2ar/\sqrt{\Delta/\Sigma} \approx -2a/\sqrt{r})u_\phi$. Because value of $u_\phi$ is greater than $u_t$ (dotted curves in figure 3), the first term has more influence on $u_\phi$ than the second term (the second term is important just in inner radiuses).

If we ignore the $r - \phi$ component of shear tensor (solid curves in figure 7), the figures are similar to the figures of the previous works such as Gammie & Popham (1998) and Takahashi (2007b). But, with $r - t$ component of shear tensor (solid curves in figure 7) a few number of four velocities change in all frames, specially $u^t$, $u^\phi$, $u^\theta$ and $v^r$ will change more than the others. The influences of the $r - t$ component on all physical variables can be derived if we solve all the equations of the disk with a state equation numerically.

When we have time dilation $u^t$ has to be greater than 1($dt > d\tau \Rightarrow u^t = \rho > 1$). According to figure 4, the effect of time dilation is much greater in the inner edge than the other places. If we ignore the $r - t$ component of shear tensor or put $|u_t| = 1$(dotted curve in the first panel of first column of figure 7), it is equivalent to ignoring time dilation. Equations (42) and (44) show that the $r - t$ component of shear tensor have no influence on $u^r$ and $u^\phi$. Near the black hole $u^t$ increases, which causes a decreases in $v^r$ ($v^r = \frac{\dot{\gamma}}{\Delta}$).

Energy can be calculated from $T^{tt} = \rho u^t u^t + p u^t u^t$, where the first term of energy is related to $u_t$. Figure 2 shows that $\rho$ is greater in inner edge than the other places and also is the same with the $r - t$ component of shear tensor and without it. The $u_t$ which uses the $r - t$ component of shear tensor is greater than the $u_t$ without it. Therefore, the first term of energy with $r - t$ component of shear tensor is greater than the this term of energy without $r - t$ component.

Far from the black hole the general relativistic influences are vanished, therefore in outer edge, the influences of $r - t$ component of shear tensor are too low (as it is seen in outer radiuses of figure 7).

If we want to calculate the temperature, pressure, internal energy, cooling and heating or radiation, we must use a suitable state equation with equations (9), (12), (14) and (18)(energy equation), then, to solve the equations numerically suitable boundary conditions are needed.

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Appendix 1. Metric Components

Nonzero components of Kerr metric are given as:

\[ g_{tt} = -\alpha^2 + \beta_\phi \beta_\phi = -(1 - \frac{2mr}{\Sigma}), \quad g_{rr} = \gamma_{rr} = \frac{\Sigma}{\Delta}, \]

\[ g_{t\phi} = \beta_\phi = -\frac{2mar \sin^2 \theta}{\Sigma}, \quad g_{\theta\theta} = \gamma_{\theta\theta} = \Sigma, \]

\[ g_{\phi\phi} = \gamma_{\phi\phi} = \frac{A \sin^2 \theta}{\Sigma}, \]

and inverse components of metric \( g^{\mu\nu} \) are:

\[ g^{tt} = -\frac{1}{\alpha^2} = -\frac{A}{\Sigma \Delta}, \quad g^{t\phi} = \frac{\beta_\phi}{\alpha^2} = -\frac{2mar}{\Sigma \Delta}, \]

\[ g^{rr} = \gamma^{rr} = \frac{\Delta}{\Sigma \Delta}, \quad g^{\theta\theta} = \gamma^{\theta\theta} = \frac{1}{\Sigma}, \]

\[ g^{\phi\phi} = \gamma^{\phi\phi} - (\frac{\beta_\phi}{\alpha^2})^2 = \frac{1}{\Delta \sin^2 \theta} (1 - \frac{2mr}{\Sigma}). \]

Appendix 2. Transformation Between BLF, LNRF and FRF

Components of \( e_\mu^\phi \) connecting between BLF(Boyer-Lindquist Frame) and LNRF(Locally non-rotating Frame) are calculated as:

\[
\left(\begin{array}{cccc}
\epsilon_t^1 & \epsilon_t^r & \epsilon_t^\theta & \epsilon_t^\phi \\
\epsilon_r^t & \epsilon_r^r & \epsilon_r^\theta & \epsilon_r^\phi \\
\epsilon_\theta^t & \epsilon_\theta^r & \epsilon_\theta^\theta & \epsilon_\theta^\phi \\
\epsilon_\phi^t & \epsilon_\phi^r & \epsilon_\phi^\theta & \epsilon_\phi^\phi \\
\end{array}\right) =
\left(\begin{array}{cccc}
\frac{(\Sigma \Delta)^{1/2}}{A} & 0 & 0 & -\frac{2Mar \sin \theta}{(\Sigma \Delta)^{1/2}} \\
0 & \frac{(\Sigma \Delta)^{1/2}}{A} & 0 & 0 \\
0 & 0 & \Sigma^{1/2} & 0 \\
0 & 0 & 0 & \left(\frac{\Sigma}{A}\right)^{1/2} \sin \theta \\
\end{array}\right),
\] (A3)

\[
\left(\begin{array}{cccc}
\epsilon_t^1 & \epsilon_t^r & \epsilon_t^\theta & \epsilon_t^\phi \\
\epsilon_r^t & \epsilon_r^r & \epsilon_r^\theta & \epsilon_r^\phi \\
\epsilon_\theta^t & \epsilon_\theta^r & \epsilon_\theta^\theta & \epsilon_\theta^\phi \\
\epsilon_\phi^t & \epsilon_\phi^r & \epsilon_\phi^\theta & \epsilon_\phi^\phi \\
\end{array}\right) =
\left(\begin{array}{cccc}
\frac{(\Sigma \Delta)^{1/2}}{A} & 0 & 0 & \frac{2Mar}{(\Sigma \Delta)^{1/2}} \\
0 & \frac{(\Sigma \Delta)^{1/2}}{A} & 0 & 0 \\
0 & 0 & \Sigma^{1/2} & 0 \\
0 & 0 & 0 & \left(\frac{\Sigma}{A}\right)^{1/2} \frac{1}{\sin \theta} \\
\end{array}\right),
\] (A4)

The transformation between LNRF and FRF(Fluid Rest Frame) are as follows:

\[
\left(\begin{array}{cccc}
\epsilon_t^i & \epsilon_r^i & \epsilon_\theta^i & \epsilon_\phi^i \\
\epsilon_r^i & \epsilon_r^r & \epsilon_r^\theta & \epsilon_r^\phi \\
\epsilon_\theta^i & \epsilon_\theta^r & \epsilon_\theta^\theta & \epsilon_\theta^\phi \\
\epsilon_\phi^i & \epsilon_\phi^r & \epsilon_\phi^\theta & \epsilon_\phi^\phi \\
\end{array}\right) =
\left(\begin{array}{cccc}
\frac{\gamma_{\phi t}}{\gamma_{tt}} & \frac{\gamma_{\phi r}}{\gamma_{rr}} & \frac{\gamma_{\phi \theta}}{\gamma_{\theta\theta}} & \frac{\gamma_{\phi \phi}}{\gamma_{\phi\phi}} \\
0 & 0 & 1 & \frac{-\gamma_{\phi \phi} \gamma_{\phi t}}{\gamma_{tt}} \\
\frac{-\gamma_{\phi \phi} \gamma_{\phi r}}{\gamma_{rr}} & \frac{-\gamma_{\phi \phi} \gamma_{\phi \theta}}{\gamma_{\theta\theta}} & \frac{-\gamma_{\phi \phi} \gamma_{\phi \phi}}{\gamma_{\phi\phi}} & 1 \\
0 & 0 & \frac{-\gamma_{\phi \phi} \gamma_{\phi t}}{\gamma_{tt}} & \frac{-\gamma_{\phi \phi} \gamma_{\phi r}}{\gamma_{rr}} \\
\end{array}\right),
\] (A5)

Where we use \( u^i = -u_i = \alpha u^t \), then the Lorentz factor \( \gamma \) is calculated as:

\[
\gamma = (1 - \frac{u^2}{c^2})^{-\frac{1}{2}} = \alpha u^t, \quad (\dot{v}^2 = v_i v^i = v_r^2 + v_\theta^2 + v_\phi^2). \]

In LNRF, three velocity components are calculated as:

\[
v^r = \frac{u^i}{u^t}, \quad (i = r, \theta, \phi).
\] (A8)

After some calculation we have (\( \Omega = \frac{\mathbf{\Omega}}{m} \)):

\[
v^\phi = \frac{A}{\sqrt{\Delta}} \frac{u^r}{u^t}, v^\theta = 0, \quad \dot{v}^\phi = \sqrt{\gamma \dot{\phi} \phi} (\beta_\phi^2 + \Omega). \]

Appendix 3. Deriving Four Velocity in LNRF

From equation (26) we have:

\[
v^\phi \pm = \frac{A}{r\sqrt{\Delta}} (\frac{\pm 1}{r^{3/2} \pm a} - \frac{2ar}{A}).
\] (A10)

First we derive four velocity for \( \Omega_k = (\Omega^2 + 1/(r^2 + a)) \) as

\[
(v^\phi)^2 = \frac{A^2 + 4a^2 r^2 (r^2 + a)^2 - 4arA (r^2 + a)}{r^4 (r^2 + a)^2 \Delta}.
\] (A11)

In LNRF \( \dot{\gamma} = u^t = \sqrt{1 - \frac{1}{c^2}} \) and in our study we put \( u_\theta = 0 \). If we suppose \( u_\phi > u_i \) (as it was in pervious papers such as Popham & Gammie 1998 and Takahashi 2007a,b), then \( \dot{v}^2 \approx (v^\phi)^2 \) and we have:

\[
\dot{\gamma} = u^t = -u_i = r \sqrt{\Delta} (r^2 + a) \frac{1}{2} (r^4 (r^2 + a)^2 \Delta - A^2 - 4a^2 r^2 (r^2 + a)^2 + 4arA (r^2 + a))^{-\frac{1}{2}},
\] (A12)

also we know that \( v^\phi = u^\phi / u^t \) therefore,
\[
\begin{align*}
\dot{u}^\phi = u^\theta v^\phi &= (r^3 + ra^2 - 2ar^2) \times \\
(r^4(r^2 + a)^2 - A^2 - 4a^2r^2(r^2 + a)^2 + 4arA(r^2 + a))^{-\frac{1}{2}}.
\end{align*}
\] (A13)

For \(\Omega_k^-\) (\(\Omega^+\)) we have:
\[
\begin{align*}
\dot{v} = -u_i &= \frac{1}{\sqrt{1 - (v^\phi)^2}} = r\sqrt{\Delta}(r^2 + a) \times \\
(r^4(r^2 + a)^2 - A^2 - 4a^2r^2(r^2 + a)^2 - 4arA(r^2 + a))^{-\frac{1}{2}},
\end{align*}
\] (A14)

\[
\begin{align*}
\sigma_{\gamma} = u^i v^\phi &= (r^3 + ra^2 + 2ar^2) \times \\
(r^4(r^2 + a)^2 - A^2 - 4a^2r^2(r^2 + a)^2 - 4arA(r^2 + a))^{-\frac{1}{2}}.
\end{align*}
\] (A15)

Appendix 4. Nonzero components of shear tensor in BLF and FRF

Nonzero components of shear tensor in BLF can be calculated by \(\sigma_{\alpha\beta} = e^\alpha_{\alpha} e^\beta_{\beta} \sigma_{\mu\nu}\) where \(e^\alpha\) and \(e^\beta\) are given in Appendix B. Therefore, these components for \(\Omega^\pm\) are:
\[
\begin{align*}
\sigma_{\tau\tau}^+ &= \sigma_{\hat{r}\hat{r}}^+ = \frac{1}{4B + \Delta^\pm} (6r^2 a^6 + 8r^3 a^3 + 36r^5 a^2 + 4r^7 a^6 \\
&- 24r^5 a^4 - 4r^3 a^5 - 10r^7 a^3 - 8r^9 a^2 + 2r^{10} a + 6r^2 a^5 \\
&- 18r^5 a^2 - 4r^2 a^4 + 36r^2 a^3) \\
&+ \sqrt{1 + \frac{2}{B + \Delta^\pm}} (3a^5 r^2 + 2a^5 r^3 + 3a^4 r^5 - 9r^{15} a \\
&+ 18r^5 a^2 - 12r^2 a^4 + 18r^5 a^3 - 2r^{12} a^3 - 3r^3 a^4 - 20r^4 a^2 \\
&- 4r^4 a^4 + 4r^4 a^2 + r^9),
\end{align*}
\] (A15)

Non-zero components of shear tensor in FRF can be calculated by \(\sigma_{(\alpha)(\beta)} = e^\rho_{(\alpha)} e^\phi_{(\beta)} \sigma_{\mu\nu}\) where \(e^\rho_{(\alpha)}\) and \(e^\phi_{(\beta)}\) are transformation matrices of Appendix B.
\[
\sigma_{(\alpha)(\beta)} = \left(\begin{array}{cccc}
\hat{\gamma} & 0 & 0 & \dot{v}^\phi \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\dot{u}^\phi & 0 & 0 & 1 + \frac{u^\phi_0}{1 + \gamma}
\end{array} \right) \times
\left(\begin{array}{cccc}
0 & \sigma_{\tau\tau} & 0 & 0 \\
\sigma_{\hat{r}\hat{r}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array} \right).
\] (A16)

Also, we have:
\[
\begin{align*}
\sigma_{(\gamma)(\phi)} &= \sigma_{(\phi)(\gamma)} = \sigma_{\tau\tau} \sigma_{\hat{r}\hat{r}} + \sigma_{\phi\phi} \frac{u^\phi_0}{1 + \gamma}.
\end{align*}
\] (A17)

Therefore, in FRF, two components of shear tensor are non-zero.