PLETHYSM AND LATTICE POINT COUNTING

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Abstract. It is shown that the multiplicities in the plethysm of $GL(n)$ are given by the answer to a lattice point counting problem. This problem is attacked with the help of computer algebra. An explicit answer is obtained for $S^\mu(S^k)$, when $\mu$ is any partition of 4 or 5. In the general case, asymptotics of the solution are studied.

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1. INTRODUCTION

The plethysm problem can be stated in different ways. One is to describe the homogeneous polynomials on the spaces $S^kW$ and $\bigwedge^k W$ in terms of representations of the group $GL(W^*)$. This is equivalent to decomposing $S^d(S^kW^*)$ into isotypic components and finding the multiplicity of each isotypic component. The general goal in plethysm is to determine the coefficients of $S^\lambda$ in $S^\mu(S^\nu W)$ as a function of $\lambda, \mu,$ and $\nu$. The term plethysm was coined by Littlewood [Lit36] and this type of problems appears in many branches of mathematics beyond representation theory (consult [LR11] for some recent developments in plethystic calculus). A general explicit solution of plethysm may be intractable as the resulting formulas are simply too complicated. Here we show piecewise quasi-polynomials that describe the plethysm and then focus on two directions. One is explicit descriptions for small $\mu$ which we find with the help of computer algebra. The other direction is asymptotics of our formulas where we can confirm a conjecture of Howe [How87, 3.6(d)] on the lead term. Our explicit formulas for plethysm are not practical to work with by hand (we recommend to not print them on paper), but computers are quick to evaluate them, study their asymptotics, and generally extract different sorts of information from them. Our results are summarized in the following theorem. The proof of the formula is completed after Section 4, while the asymptotics are dealt with in Section 5.

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Theorem 1.1. Let $\lambda$ be a partition of $k \times |\mu|$. The multiplicity of the isotypic component of $S^\mu(S^kW^*)$ corresponding to $\lambda$, as a function of $\lambda$ and $k$, is a piecewise quasi-polynomial:

$$\dim \mu \frac{d!}{d^l} \# P^\lambda_{k,|\mu|} + \left( -1 \right)^{\frac{d}{2} - 1} \sum_{\alpha \vdash d, \alpha \neq (1, \ldots, 1)} \chi_\mu(\alpha) \frac{D_\alpha}{d!} \sum_{\pi \in S_{d-1}} \text{sgn}(\pi) Q_\alpha(k, \lambda_\pi),$$

where $P^\lambda_{k,|\mu|}$ is an explicit polytope, $\chi_\mu$ the character of the symmetric group $S_{|\mu|}$ corresponding to the partition $\mu$, the $Q_\alpha$ are counting functions for the fibers of projections of explicit polyhedral cones, and $\lambda_\pi$ is a linear shift of $\lambda$. Moreover, $\dim \mu \frac{d!}{d^l} \# P^\lambda_{k,|\mu|}$ is the leading term coming from the Littlewood-Richardson rule. When $\mu$ is any partition of 3, 4, or 5, the explicit piecewise quasi-polynomials have been computed and can be downloaded from the project homepage in ISL format.

Our approach can be summarized as follows. As usual, we first compute the character of the representation $S^\mu(S^kW)$. Using known formulas relating Schur polynomials and complete symmetric polynomials we relate the multiplicities of isotypic components of the plethysm to coefficients of monomials of a specific polynomial (Propositions 3.8 and 4.4, Section 4.4). We then reduce the determination of these coefficient to a purely combinatorial problem: lattice point counting in certain rational polytopes related to transportation polytopes (Definition 4.6). For fixed $d$, the final multiplicity is a function of $\lambda_1, \ldots, \lambda_d$ and $k$. These arguments may belong to a finite number of polyhedral chambers. In each chamber, the result is a quasi-polynomial, that is a polynomial with coefficients that depend on the remainders of its arguments modulo a fixed number. Equivalently it is a polynomial in floor functions of linear expressions in the arguments. Software to determine piecewise quasi-polynomials is well-developed due to applications ranging from toric geometry to loop optimization in compiler research. We show how to use BARVINOK [VSB07] and the ISL-library [Ver10] to make the Theorem 1.1 explicit. This yields a concrete decomposition of $S^\mu(S^kW)$ (and $S^\mu(\bigwedge^k W)$) for any partition $\mu$ of 3, 4 or 5. For each fixed $\mu$, the result is a decomposition of $(\lambda, k)$-space into polyhedral chambers where in each chamber the multiplicity is a quasi-polynomial. We have set up a homepage for the results in this paper at

http://www.thomas-kahle.de/plethysm.html

In the appendix (Section 6) we detail our experiences with the software.

Before presenting our methods we now give a short overview of applications of our results as well as different approaches.

Classical results: Our results extend classical theory. For example the description of quadrics on the space $S^k(W)$ is a classical result of Thrall [Thr42], [CGR84, 4.1–4.6].

Example 1.2. One has $GL(W)$-module decompositions

$$S^2(S^kW) = \bigoplus S^3W, \quad \bigwedge^2(S^kW) = \bigoplus S^4W,$$

where the first sum runs over representations corresponding partitions $\lambda$ of $2k$ into two even parts and the second sum runs over representations corresponding to partitions $\delta$ of $2k$ into two odd parts.

The decomposition of cubics $S^3(S^kW^*)$ is also known. Stated in different forms it can be found in [Thr42, Plu72, CGR84, How87, Aga02]. In fact, the latter four have formulas for $S^\mu(S^kW)$ for any partition $\mu$ of 3. Determining $S^4(S^k)$ has been addressed in [Fou54, Dun52, How87].
**Asymptotics:** The explicit formulas for plethysm become complicated quickly, but there is hope for simpler asymptotic formulas. For instance the decomposition of $S^d(S^kW)$ is related to $(S^kW)^{\otimes d}$ by means of the symmetrizing operator $(S^kW)^{\otimes d} \rightarrow S^d(S^kW)$. There the decomposition of the domain of the resulting quasi-polynomial is known from Pieri’s (or more generally the Littlewood-Richardson) rule. In the same vein Howe [How87, 3.6(d)] identified the leading term for $S^3(S^k)$ and $S^4(S^k)$. A different approach by Fulger and Zhou [FZ12] studies the asymptotics of plethysm by considering how many different irreducible representations and which sums of multiplicities can appear. Knowledge of explicit quasi-polynomial formulas allows one to test techniques for studying the asymptotics algorithmically on non-trivial examples. Another insight from Section 5 is that the language of convex discrete geometry may be more useful for proofs than that of piecewise quasi-polynomials.

**Evaluation:** One of the principal uses of our results is evaluation of the plethysm function. While evaluation for individual values can be done in LiE [vLCL92] and other packages, our results are more flexible as they are given as functions on parameter space and can thus be evaluated parametrically.

**Example 1.3.** Let $\mu = (5)$, $\lambda = (31, 3, 2, 2, 2)$, and make the following definitions:

$$p_1 = -\frac{289}{720} s + \frac{1}{20} s^2 + \frac{1}{720} s^3$$

$$p_2 = \frac{5}{8} + \frac{1}{8} s$$

$$p_3 = \frac{1}{3} - \frac{1}{6} s$$

$$p_4 = \frac{7}{12} - \frac{1}{3} s$$

$$A(s) = p_1 + p_2 \left\lfloor \frac{s}{2} \right\rfloor + p_3 \left\lfloor \frac{s}{3} \right\rfloor + \left( p_4 + \frac{1}{2} \left\lfloor \frac{s}{3} \right\rfloor \right) \left\lfloor \frac{1 + s}{3} \right\rfloor + \frac{1}{4} \left( \left\lfloor \frac{1 + s}{3} \right\rfloor^2 + \left\lfloor \frac{s}{4} \right\rfloor - \left\lfloor \frac{3 + s}{4} \right\rfloor \right)$$

With these definitions the coefficient of $S^s\lambda$ in $S^\mu(S^s)$ equals

$$A(s) + \begin{cases} 
1 & \text{if } s \equiv 0 \mod 5 \\
\frac{3}{5} & \text{if } s \equiv 1 \mod 5 \\
\frac{4}{5} & \text{if } s \equiv 2, 3, 4 \mod 5,
\end{cases}$$

which in particular has a polynomial lead term of degree 3 (see Proposition 5.2).

**Remark 1.4.** It can be slightly tricky to automatically extract these kind of formulas from our computational results. In general iscc has somewhat limited capabilities in producing human-readable output. In Example 6.6 we give a complete discussion of how to derive this result using our computational results and iscc.

**Example 1.5.** Actual numerical evaluation is very quick too. For instance the multiplicity of the isotypic component of

$$\lambda = (616036908677580244, 1234567812345678, 12345671234567, 123456123456)$$

in $S^5(S^{123456789123456789})$ equals

$$240963570406235272797673915801061590529381724384546352415930440743659968070016051.$$
Testing conjectures: Although many theoretical formulas for plethysm are known, some basic properties are mysterious. For example, a conjecture of Foulkes states that for $a < b$, $S^a(S^b)$ embeds as a subrepresentation into $S^b(S^a)$. For $a = 2$, this is a classical result. For $a = 3$ it was shown in this century [DS00] and for $a = 4$ in [McK08]. Using explicit quasi-polynomials one can attack this problem for any fixed $a$. One would need to compare two explicit quasi-polynomials, and in particular decide if their difference is a positive quasi-polynomial. At the moment we cannot complete this direction as our methods only work for fixed exponent of the outer Schur functor. Results of Bedratyuk [Bed11] indicate that explicit quasi-polynomials can also be found for fixed exponent of the inner Schur functor, once we fix the group (to be $SL(n)$). Our result can also be considered as a step towards Stanley’s Problem 9 in [Sta00] asking for the combinatorial description of plethysm. For this major breakthrough one would need “positive” formulas.

The zero locus of plethysm coefficients: The question of which isotypic components appear in plethysm is highly nontrivial. Some very special cases follow from the resolution of Weintraub’s conjecture [Wei90, BCI11, MM12]. We hope that our formulas can contribute to finding further regularities among partitions that appear in different plethysms, for instance by studying the zeros of our quasi-polynomials.

Geometric Complexity Theory: The problem of separation of complexity classes is addressed with geometric methods in [MS01, BLMW11]. The crucial point of this program requires to compare closures of orbits of explicit symmetric tensors. The plethysm plays an important role there [MS01, p. 516], [BLMW11, p. 10], [Lan13, p. 10].

Computation of syzygies: The computations of syzygies of homogeneous varieties is related to (inner) plethysm (see [Wey03, p. 63]). Weyman [Wey03, p. 241] applies the explicit computation of plethysm $S^a(S^b)$ to study rank varieties [Wey03, Section 7.1] and topics related to free resolution of the Grassmannian.

Unification: Many specialized methods have been developed to attack plethysm problems and most of them work only in a very restricted set of exponents. For instance, we have already mentioned several methods to compute $S^3(S^k)$ [Thr42, Phi72, CGR84, How87, Aga02]. Quasi-polynomials and convex bodies provide a unifying framework for all of those techniques. Specializing to certain situations is just partially evaluating the quasi-polynomial. In the history of plethysm, people have written a new paper with a new technique for the next higher exponent. Our method, in contrast, stays the same. To get the plethysm $S^6(S^n)$ we just have to wait a few years for computers to catch up.

Representations of $S_n$: The plethysm is also related to representations of the permutation group $S_n$. For two representations of $S_n$ and $S_b$ corresponding to $\lambda$ and $\mu$, respectively, there is a wreath product representation $\lambda \wr \mu$ of the wreath product group $S_\lambda \wr S_\mu$. One has a natural inclusion $S_\lambda \wr S_\mu \subseteq S_{\lambda \mu}$. By the Frobenius characteristic map we can identify representations of symmetric groups with symmetric polynomials and under this identification, the representation of $S_{\lambda \mu}$ induced from $\lambda \wr \mu$ is exactly the plethysm of the representations given by $\lambda$ and $\mu$. The interested reader may consult [Sta00, Vol. 2, Theorem A2.6] and references there. For similar results on the Kronecker coefficients, appearing in the decomposition of the tensor product of representations of symmetric groups, see [BOR09, BOR11].

Classical algebraic geometry: The spaces $S^k W$ and $\bigwedge^k W$ are ambient spaces of the Veronese variety and the Grassmannian. These varieties and related objects, e.g. their secant
and tangential varieties, have been studied classically (see [Zak93] and references therein). The description of the algebra of the Veronese and Grassmannian is well-known. However, as the decomposition of $S^d(S^k W^*)$ is not known, the decomposition of the degree $d$ part of the ideal is a difficult problem—even for quartics! Our results provide such a description. Furthermore, due to problems motivated by determining ranks of tensors, secant varieties are often studied from a computational point of view. It is an open problem to check, if the ideal of the secant (line) variety of any Grassmannian is generated by cubics. A description of all cubics in the ideal was given in [MM14]. It is natural to ask, which quadrics are generated by cubics. To answer this question, the description of all degree four polynomials is helpful. Thus our results provide very practical information. One could argue that we provide the decomposition of very low degree equations. Note however, that on the $k$-th secant variety, no equations of degree less than or equal to $k$ vanish. On the other hand, the equations of degree $k + 1$ sometimes already provide all generators of the ideal (e.g. the Segre-Veronese varieties for $k = 2$ [Rai12]). Thus, knowing their decompositions is an important first step in determining the structure of the whole ideal. The same method can be applied to other ideals defined by objects related to representation theory. One example is the ideal of relations among $k \times k$ minors of a generic matrix studied in [BCV13].

Errors: As the formulas and computations become more and more technically involved the chance of human error rises. To quote from Howe [How87]: Here we will outline what is involved in the computations and list our answers. The details are available from the author on request. The author does hope someone will check the calculations, because he does not have a great deal of faith in his ability to carry through the details in a fault-free manner. He hopes however that the answers are qualitatively correct as stated. We have not checked all historical formulas that overlap with our results, but errors have been identified before (compare [MM14, Appendix] and [CGR84]).

Convention. All representations considered are finite dimensional.

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2. Preliminaries and notation

Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ with $\lambda_1 \geq \cdots \geq \lambda_l > 0$ be an integral partition of $n$, i.e. $\sum_{i=1}^l \lambda_i = n$. We set $|\lambda| = n$. Consider a vector space $W$ of dimension at least $l$. Let $S^\lambda W$ be the irreducible representation of $GL(W)$ corresponding to $\lambda$, obtained by acting with the Young symmetrizer $c_\lambda$ on $W^\otimes n$ [FH91]. We use the convention that the partition $(1, \ldots, 1)$ corresponds to the wedge product representation $\wedge^n W$ and $(\lambda_1)$ to the symmetric power $S^n W$. All irreducible representations of $SL(W)$ can be obtained by considering partitions $\lambda$ with $n$ arbitrary and $l < \dim W$. For $GL(W)$ the theory is similar. There we have to specify an additional integer $r$. The vector space and the group action is the same as for $SL(W)$, but additionally we multiply a given vector by the determinant to the power $r$. 
In general, $S^\lambda$ is an endofunctor on the category of representations, known as the Schur functor. Applying Schur functors to the standard representation $W$ yields all irreducible representations. However, applying $S^\lambda$ to other irreducible representations, in general, we obtain reducible representations. The plethysm is to understand the decomposition of $S^\lambda(S^\mu W)$.

3. Characters

The trace of a $GL(W)$ representation is a symmetric polynomial in the eigenvalues, known as the character of the representation. To determine the character, one considers the action of diagonal matrices.

**Definition 3.1** $(h_k(x^a), \psi_\alpha)$. Consider $d$ variables $x_1, \ldots, x_d$. For $a \in \mathbb{N}$, let $h_k(x^a)$ be the complete symmetric polynomial of degree $k$ in the variables $x_1^a, \ldots, x_d^a$. Let $\alpha$ be a multi-index of length $j$. We define the polynomials

$$
\psi_n = \sum_i x_i^n, \quad \psi_\alpha = \prod_{i=1}^j \psi_{\alpha_i}, \quad \text{and} \quad \psi_\alpha \circ h_k = \prod_{i=1}^j h_k(x^{\alpha_i}).
$$

**Example 3.2.** The character of the representation $S^k(W)$ is the sum of all monomials of degree $k$ in $\dim W$ variables, that is $h_k(x)$. For $\bigwedge^k W$ we obtain the sum of all squarefree monomials of degree $k$, known as the elementary symmetric polynomial.

For any representation $V$ the associated character is denoted by $P_V$. The character of the irreducible representation $S^\lambda W$ is the Schur polynomial $P_\lambda$. Schur polynomials are independent and form a basis of symmetric polynomials. Since the character of a sum of two representations is the sum of their characters, in order to decompose any representation, it is enough to express its character as a sum of Schur polynomials. Precisely: $V = \sum(S^\lambda V)^{\otimes a_\lambda}$ if and only if $P_V = \sum a_\lambda P_\lambda$. Other operations on representations translate too. For instance, the plethysm of two symmetric polynomials $f, g$ is the composition $f \circ g$ (see [Mac98, I.8] for a precise algebraic definition).

**Proposition 3.3** ([Mac98, I.8.3, I.8.4, I.8.6]). For any symmetric polynomial $f$, the mapping $g \rightarrow g \circ f$ is an endomorphism of the ring of symmetric polynomials. For any $n \in \mathbb{N}$, the mapping $g \rightarrow \psi_n \circ g$ is an endomorphism of the ring of symmetric polynomials. Moreover,

$$
\psi_n \circ g = g \circ \psi_n = g(x_1^n, x_2^n, \ldots).
$$

**Remark 3.4.** Proposition 3.3 justifies the notation

$$
\psi_\alpha \circ h_k = \prod_{i=1}^j h_k(x^{\alpha_i}).
$$

From now on assume that $\dim W$ is large enough so that all appearing partitions have at most $\dim W$ parts and fix a partition $\mu$ of an integer $d$. Irreducible representations of the permutation group $S_d$ are indexed by Young diagrams with exactly $d$ boxes. The character corresponding to the Young diagram $\rho \vdash d$ is denoted $\chi_\rho$.

**Definition 3.5** ($z_\rho$, [Mac98, p.17]). Let $\rho = (\rho_1 \geq \cdots \geq \rho_k)$ be a partition of $d$ and $m_i$ the number of parts equal to $i$. We define

$$
z_\rho = \prod_{i \geq 1} i^{m_i}m_i! = \frac{d!}{D_\rho},
$$
where $D_\rho$ is the number of permutations of cycle type $\rho$.

**Remark 3.6.** With [Mac98, I.7.(7.2) and I.7.(7.5)] the character $P_\mu$ can be expressed in terms of $\psi_n$ as

$$P_\mu = \sum_{\rho \vdash d} z_\rho^{-1} \chi_\mu(\rho) \psi_\rho,$$

where $\chi_\mu(\rho)$ is the value of the character $\chi_\mu$ on (any) permutation of type $\rho$.

**Example 3.7.** As the Young diagram $(d)$ corresponds to the trivial representation of $S_d$ we obtain the formula for the complete symmetric polynomial:

$$h_d = P_{(d)} = \sum_{\rho \vdash d} D_\rho \psi_\rho,$$

where $D_\rho$ is the number of permutations of combinatorial type $\rho$ in the group $S_d$. The Young diagram $(1, \ldots, 1)$ corresponds to the sign representation of $S_d$. Hence, we obtain the formula for the character of the wedge power:

$$P_{(1, \ldots, 1)} = \sum_{\rho \vdash d} \text{sgn}(\rho) D_\rho \psi_\rho.$$

**Proposition 3.8.** The character of the representation $S^\mu(S^kW)$ equals

$$P_{S^\mu(S^kW)} = \sum_\alpha \chi_\mu(\alpha) D_\alpha \psi_\alpha \circ h_k,$$

where the sum is taken over all partitions $\alpha$ of $d := |\mu|$ and $D_\alpha$ is the number of permutations of cycle type $\alpha$ in the group $S_d$.

**Proof.** We have

$$P_{S^\mu(S^kW)} = P_{S^\mu} \circ h_k.$$

By Remark 3.6 this equals

$$\sum_{\rho \vdash d} z_\rho^{-1} \chi_\mu(\rho) \psi_\rho \circ h_k.$$

**Remark 3.9.** A similar formula for arbitrary composition of Schur functors is presented in [Yan98, Theorem 2.2]. We do not apply it directly, as it relies on 'nested inverse Kostka numbers'. As explained in [Yan98, Yan02] the computation of those, although possible in many cases, is a nontrivial task. For this reason, we introduce one more change of basis of symmetric polynomials, relating our results to transportation polytopes. From the algorithmic point of view, although the final result counts the same multiplicities, enumeration of points in dilated polytopes is easier than enumeration of skew Young diagrams with specific properties.

For fixed $d$ all partitions can be listed and the decomposition of $P_{S^\mu(S^kW)}$ into Schur polynomials reduces to the decomposition of each polynomial $\psi_\alpha \circ h_k$. Indeed, the values of $\chi_\mu(\rho)$ can be made explicit by the celebrated Frobenius Formula [FH91, 4.10]. As similar results will be used later, we review the formula in detail.

**Definition 3.10** ($[P]_\alpha, \Delta(x)$). For any polynomial $P$ and partition $\alpha = (\alpha_1, \ldots, \alpha_k)$, define $[P]_\alpha$ as the coefficient of the monomial $x_1^{\alpha_1} \cdots x_k^{\alpha_k}$ in $P$. 


Definition 3.11. For a fixed number of variables \(x_1, \ldots, x_k\), the discriminant is
\[
\Delta(x) = \prod_{i<j}(x_i - x_j).
\]

The value of the character \(\chi_\mu\) on any permutation of cycle type \(\rho\) equals:
\[
(3.1) \quad \chi_\mu(\rho) = \frac{\Delta(x)^{|\rho|}}{\prod_{\mu_i, k-1, \mu_2+k-2, \ldots, \mu_k}}. \quad \text{(Frobenius formula)}
\]

Example 3.12. Consider a permutation \(\pi \in S_4\) of cycle type \((3,1)\), e.g. the permutation that fixes 4 and permutes \(1 \to 2 \to 3 \to 1\). Consider a representation corresponding to the partition \(2 + 2 = 4\). We obtain:
\[
\chi_{(2,2)}(\pi) = [(x_1 - x_2)(x_3^2 + x_2^3)(x_1 + x_2)]_{(3,2)} = [x_1^5 - x_1^3 x_2^2 + x_1^2 x_2^3 - x_2^5]_{(3,2)} = -1.
\]

4. Reductions

To make Proposition 3.8 effective we employ the following simplifications.

1. Reduction of the number of variables.
2. Application of the Littlewood-Richardson rule to the most complicated term.
3. Change of basis of symmetric functions,
4. Reduction to combinatorics of polytopes.

4.1. Reduction of the number of variables. Our aim is to compute the multiplicity of the isotypic component corresponding to \(\lambda\) inside \(S^\mu(S^kW)\). By the Littlewood-Richardson rule, \(\lambda\) can have at most \(|\mu|\) rows, so we can assume \(\dim W = |\mu|\).

Proposition 4.1 ([Car90], [Man98]).
\[
S^\mu(S^{2l}W) = S^\mu(\bigwedge^l W)^\vee, \quad S^\mu(S^{2l+1}W) = S^{\nu^\vee}(\bigwedge^{2l} W)^\vee,
\]
where \((.)^\vee\) stands for the representation arising from \((.)\) by replacing each irreducible component corresponding to a Young diagram \(\nu\) with the component corresponding to the transpose of \(\nu\), denoted \(\nu^\vee\).

Proposition 4.1 says that the multiplicity of an isotypic component corresponding to \(\lambda\) inside \(S^\mu(S^kW)\) is the multiplicity of \(\lambda^\vee\) inside either \(S^{\nu^\vee}(\bigwedge^l W)\) or \(S^\mu(\bigwedge^l W)\). For the wedge power the following well-known reductions hold which we prove for the sake of completeness.

Lemma 4.2 (Reduction Lemma [Car90, 5.8, 5.9], [MM14, Lemma 6.3]). Let \(\mu\) be any Young diagram of weight \(d\), and \(\lambda\) a Young diagram with \(d\) columns and weight \(dk\). Let \(\lambda'\) equal \(\lambda\) with the first row removed. The multiplicity of the component corresponding to \(\lambda\) in \(S^\mu(\bigwedge^k W)\) equals the multiplicity of the component corresponding to \(\lambda'\) in \(S^\mu(\bigwedge^{k-1} W)\).

Proof. Consider the inclusion \(S^\mu(\bigwedge^k W) \subset (\bigwedge^k W)^{\otimes n}\) with a basis given by tensor products of wedge products of basis elements of \(W\). Each vector in the highest weight space corresponding to \(\lambda\) must contain exactly one \(e_1\) in each tensor. We get an isomorphism of highest weight spaces by removing \(e_1\) and decreasing the indices of other basis vectors by one. □

The above facts show that whenever \(\lambda^\vee\) has \(d\) nonzero columns (or equivalently \(\lambda\) has \(d\) nonzero rows) we can express the multiplicity in the plethysm by a multiplicity in a simpler plethysm. It follows that it is enough to determine the multiplicities of isotypic components corresponding to \(\lambda\) with at most \(d - 1\) rows. This is equivalent to the assumption that
dim \( W = d - 1 \) or that the symmetric polynomials are in variables \( x_1, \ldots, x_{d-1} \). From now on we make this assumption, recovering the general case at the end (Remark 4.8).

4.2. Application of Littlewood-Richardson rule. Suppose

\[
\psi_\alpha \circ h_k = \sum_\lambda a_{\alpha, \lambda} S^\lambda,
\]

where \( S^\lambda \) is the Schur polynomial corresponding to \( \lambda \) and the sum is over all partitions \( \lambda \vdash dk \), with at most \( d - 1 \) parts. In the following sections we associate polytopes to the polynomials \( \psi_\alpha \circ h_k \). Although our computer algebraic methods work in general, they are least efficient for the partition \( \alpha = (1, \ldots, 1) \). In this section, we show how to express \( \psi_{(1, \ldots, 1)} \circ h_k \) in terms of Schur polynomials without further computation. While in the end these reductions were not necessary in our computations, we present them as an introduction to the methods in the remaining sections and to better understand the leading term in the plethysm formula.

Fix \( \alpha_0 = (1, \ldots, 1) \vdash d \). By Remark 3.4, \( \psi_{\alpha_0} \circ h_k = (h_k(x))^d \), the \( d \)-th power of the complete symmetric polynomial of degree \( k \). As multiplication of polynomials corresponds to the tensor product of representations, this is the character of the representation \( (S^k W)^\otimes d \). The decomposition of this representation is known due Pieri’s rule (or more generally the Littlewood-Richardson rule). In order to make the formulas explicit, consider the following polytope.

**Definition 4.3 (The polytope \( P_{k,d} \)).** Let \((x_1^1, x_1^2, x_2^1, \ldots, x_{d-1}^1, x_1^{d-1}, \ldots, x_{d-1}^{d-1})\) denote coordinates of the vector space \( \mathbb{R}^1 \times \mathbb{R}^2 \times \cdots \times \mathbb{R}^{d-1} \). Denote \( x_1^0 = k \), \( x_{j+1}^j = k - \sum_{i=1}^j x_i^j \) and \( x_i^j = 0 \) for \( i > j + 1 \). Let \( P_{k,d} \) be the polytope defined by the following constraints:

1. \( x_i^j \geq 0 \), for all \( i, j \),
2. \( \sum_{t \leq j} x_i^t \leq \sum_{t \leq j-1} x_i^{t-1} \), for all \( j \) and \( 1 < i \leq j \), and
3. \( k - \sum_{i=1}^j x_i^j \leq \sum_{t \leq j-1} x_j^t \), for all \( j \).

In the following, if \( P \) is a polytope, then \( \# P \) denotes the number of integral points in \( P \). In Definition 4.3, \( x_i^j \) corresponds to the number of boxes added according to Pieri’s rule in the \( j \)-th step in the \( i \)-th row. By Pieri’s rule we obtain the following

**Proposition 4.4.** The coefficient \( a_{\alpha_0, \lambda} \) in the expansion

\[
\psi_{\alpha_0} \circ h_k = \sum_\lambda a_{\alpha_0, \lambda} S^\lambda,
\]

equals the number of integral points in \( P_{k,d} \) intersected with the hyperplanes \( \sum_j x_i^j = \lambda_i \). In particular, it can be computed as the number of points in the fiber of a projection of \( P_{k,d} \). We will denote the intersection by \( P_{k,d}^\lambda \). \( \square \)

**Remark 4.5.** There are other methods to compute the Littlewood-Richardson coefficients, e.g. due to Berenstein and Zelevinsky [BZ92], that could provide other polytopial descriptions. Contrary to plethysm, the question which representations \( S^\nu \) appear (with positive multiplicities) in \( S^\lambda \otimes S^\mu \) is well-understood [Kly98, KT99, KTW04].

4.3. Change of basis. Suppose

\[
\psi_\alpha \circ h_k = \sum_\lambda a_{\alpha, \lambda} S^\lambda,
\]
where $S^\lambda$ is the Schur polynomial corresponding to $\lambda$ and the sum is taken over all partitions $\lambda \vdash dk$, with at most $d - 1$ parts. By the results of [FH91, Appendix A] and [Mac98], the coefficient $a_{\alpha, \lambda}$ is equal to the coefficient of the monomial $x_1^{\lambda_1 + d - 2} \cdots x_{d-1}^{\lambda_{d-1}}$ in the polynomial $(\psi_{\alpha} \circ h_k) \prod_{i<j} (x_i - x_j)$, that is:

$$a_{\alpha, \lambda} = \left[ \Delta(x)(\psi_{\alpha} \circ h_k) \right]_{(\lambda_1 + d - 2, \lambda_2 + d - 3, \ldots, \lambda_{d-1})}.$$

### 4.4. Integral points in polytopes.

When $d$ is fixed, the discriminant $\prod_{i<j} (x_i - x_j)$ is explicit. Our aim is to compute the coefficients of the monomials appearing in $\Delta(x)(\psi_{\alpha} \circ h_k)$.

**Definition 4.6 ((\(\alpha, \lambda\))-matrix).** Fix partitions $\alpha, \lambda$ and suppose that $\alpha$ has $a$ parts. An $a \times (d - 1)$ matrix $M$ with nonnegative integral entries is an $(\alpha, \lambda)$-matrix if

1. each row sums up to $k$, i.e. $\sum_{j=1}^{d-1} M_{i,j} = k$ for each $1 \leq i \leq a$, and
2. the $\alpha$-weighted entries of the $j$-th column sum up to $\lambda_j$, i.e. $\sum_{i=1}^{a} \alpha_i M_{i,j} = \lambda_j$ for each $1 \leq j \leq d - 1$.

**Example 4.7.** Let $d = 3$ and $\alpha = (3)$ and $\lambda = (\lambda_1, \lambda_2, \lambda_3)$. According to Definition 4.6, an $(\alpha, \lambda)$-matrix is a non-negative integral $(1 \times 2)$ matrix $M = (M_{11}, M_{12})$ satisfying $M_{11} + M_{12} = k$ and $3M = (\lambda_1, \lambda_2)$. There is no such matrix unless $\lambda_1 \equiv 0 \mod 3$, and if this is the case, for each $k$ there is exactly one such matrix if and only if $\lambda_2 = 3k - \lambda_1$.

It is a straightforward observation that the coefficient of $x^\lambda$ in $\psi_{\alpha} \circ h_k$ equals the number of different $(\alpha, \lambda)$-matrices, as each matrix encodes the expansion of the product $\prod_{i=1}^{a} h_k(x^{\alpha_i})$. We want to obtain an explicit formula for the number of $(\alpha, \lambda)$-matrices for fixed $\alpha$ as a piecewise quasi-polynomial in $k, \lambda_1, \ldots, \lambda_{d-2}$ ($\lambda_{d-1}$ is determined as $\sum_{i=1}^{d-1} \lambda_i = kd$). Denote this quasi-polynomial by $Q_{\alpha}$ such that

$$\psi_{\alpha} \circ h_k = \sum_{\lambda} Q_{\alpha}(k, \lambda_1, \ldots, \lambda_{d-2}) x^\lambda.$$ 

Hence by the Vandermonde formula,

$$\psi_{\alpha} \circ h_k \prod_{i<j} (x_i - x_j) = \psi_{\alpha} \circ h_k (-1)^{\binom{d-1}{2}} \prod_{i<j} (x_j - x_i)$$

$$= (-1)^{\binom{d-1}{2}} \left( \sum_{\lambda} Q_{\alpha}(k, \lambda_1, \ldots, \lambda_{d-2}) x^\lambda \right) \left( \sum_{\pi \in S_{d-1}} \text{sgn}(\pi) \prod_{i=1}^{d-1} x_{(\pi^{-1}(i))} \right).$$

Consequently the coefficient of $x_1^{\lambda_1 + d - 2} \cdots x_{d-1}^{\lambda_{d-1}}$ in $(\psi_{\alpha} \circ h_k) \prod_{i<j} (x_i - x_j)$ equals:

$$(-1)^{\binom{d-1}{2}} \sum_{\pi \in S_{d-1}} \text{sgn}(\pi) Q_{\alpha}(k, \lambda_1 + d - 1 - \pi(1), \lambda_2 + d - 2 - \pi(2), \ldots, \lambda_{d-2} + 2 - \pi(d - 2)).$$

For each permutation $\pi \in S_{d-1}$ denote $\lambda_\pi = (\lambda_1 + d - 1 - \pi(1), \lambda_2 + d - 2 - \pi(2), \ldots, \lambda_{d-2} + 2 - \pi(d - 2))$. Using this notation we obtain the formula for the multiplicity $a_{\lambda}$ of the isotypic component corresponding to $\lambda$ inside $S^\mu(S^kW)$ for $\mu$ a partition of $d$:

$$(-1)^{\binom{d-1}{2}} \left( \sum_{\alpha \vdash d} \chi_{\mu}(\alpha) \frac{D_\alpha}{d!} \sum_{\pi \in S_{d-1}} \text{sgn}(\pi) Q_{\alpha}(k, \lambda_\pi) \right).$$
The summand for the partition \( \alpha = (1, \ldots, 1) \) can be made explicit:

\[
\dim \frac{\mu}{d!} \#P_{\alpha}^\lambda + (-1)^{\frac{d-1}{2}} \left( \sum_{\alpha \vdash d, \mu \neq (1, \ldots, 1)} \chi_{\mu}(\alpha) \frac{D_{\alpha}}{d!} \sum_{\pi \in S_{d-1}} \text{sgn}(\pi) Q_{\alpha}(k, \lambda_{\pi}) \right),
\]

where \( \dim \mu = \chi_{\mu}(1, \ldots, 1) \) is the value of the character \( \chi_{\mu} \) on the trivial permutation, and thus equal to the dimension of the representation of \( S_{|\mu|} \) corresponding to \( \mu \). We may identify \( S_{d-1} \) with the Weyl group \( \mathcal{W} \). Let \( \rho \) be half of the sum of positive weights. For aesthetic reasons we may rewrite the above formulas as follows

\[
(-1)^{\frac{d-1}{2}} \left( \sum_{\alpha \vdash d} \chi_{\mu}(\alpha) \frac{D_{\alpha}}{d!} \sum_{\pi \in \mathcal{W}} \text{sgn}(\pi) Q_{\alpha}(k, \lambda + \rho - \pi(\rho)) \right).
\]

All together we have reduced the problem of finding the coefficients of the plethysm to computing the piecewise quasi-polynomials \( Q_{\alpha} \) that count the number of \((\alpha, \lambda)\)-matrices. Let \( \alpha \) be a partition with \( a \) parts. The integral \((a \times (d - 1))\)-matrices form an \( a(d - 1)\)-dimensional lattice and the linear equations in Definition 4.6 define hyperplanes in this lattice. When \( L \) denotes the resulting affine sublattice, the \((\alpha, \lambda)\)-matrices are simply the non-negative integer points in \( L \). Alternatively, let \( P_{\alpha, \lambda} \) (not to be confused with \( P_{k,d} \)) be the (rational) polytope \((L \otimes_{\mathbb{Z}} \mathbb{Q}) \cap \mathbb{Q}_{\geq 0}^{a(d-1)}\). It is a polytope since each coordinate is non-negative and bounded from above by \( \max \lambda_i \). The number of \((\alpha, \lambda)\)-matrices equals \( \#P_{\alpha, \lambda} \), the number of integral points in \( P_{\alpha, \lambda} \). It is also worth noting that for any partition \( \alpha \), the polytope \( P_{\alpha, \lambda} \) can be obtained from the \( P_{(1, \ldots, 1), \lambda} \) by a series of hyperplane cuts given by equalities of coordinates. The polytopes \( P_{(1, \ldots, 1), \lambda} \) are transportation polytopes, well studied objects in combinatorics and optimization [KW68, Bol72, BR93, DLK13, Liu13].

The following remark follows by combining Proposition 4.1 and Lemma 4.2.

**Remark 4.8.** Let us fix \( \mu \) be a partition of \( d \) and \( \lambda = (\lambda_1, \ldots, \lambda_d) \) with \( \sum \lambda_i = dk \). The multiplicity of \( \lambda \) in \( S^\mu(S^k) \) equals

1. the multiplicity of \((\lambda_1 - \lambda_d, \ldots, \lambda_{d-1} - \lambda_d, 0)\) in \( S^\mu(S^{k-\lambda_d}) \) if \( \lambda_d \) is even,
2. the multiplicity of \((\lambda_1 - \lambda_d, \ldots, \lambda_{d-1} - \lambda_d, 0)\) in \( S^{\mu^\vee}(S^{k-\lambda_d}) \) if \( \lambda_d \) is odd, where \( \mu^\vee \) is the transpose of \( \mu \).

Additionally, the value \( \lambda_1 - \lambda_d \) is determined by the equation \( d(k - \lambda_d) = \sum_{i=1}^{d-1} \lambda_i - (d-1)\lambda_d \).

Consequently, our implementation uses arguments

\[
(b_1, \ldots, b_{d-2}, s) = (\lambda_{d-1} - \lambda_d, \ldots, \lambda_2 - \lambda_d, k - \lambda_d).
\]

**Remark 4.9** (Stable multiplicites). Fix an integer \( d \) and let \( \lambda \) be a Young tableau. For every sufficiently large \( k \), we can construct another tableau \( \lambda'(k) \) by adding a new first row to \( \lambda \) such that \(|\lambda'(k)| = dk \). As a function of \( k \), the multiplicity of the isotypic component \( \lambda'(k) \) in \( S^d(S^k) \) becomes eventually constant as \( k \) grows [Wei90, CT92, Man98]. This fact follows easily in our setting. Indeed, note that the desired multiplicity is a function of counts of \((\alpha, \lambda')\)-matrices. Now when \( k \) is very large each possible filling of the columns 2 to \( a \) of an \((\alpha, \lambda')\)-matrix (restricted by the conditions coming from \( \lambda \)) can be uniquely completed.

**Remark 4.10** (Formal power series). A different well-studied approach to plethysm is through formal power series. As before we work with polynomials in variables \( x_1, \ldots, x_{d-1} \). We already justified that the multiplicity of the isotypic component corresponding to \( \lambda \) inside
$S^\mu(S^k)$ equals:
\[
\left[ \sum_{\alpha \vdash d} \frac{D_{\alpha}}{d!} \chi_\mu(\alpha) \Delta(x) \psi_\alpha \circ h_k \right]_{(\lambda_1+d-2, \ldots, \lambda_d-1)}.
\]
As a formal power series, $\sum_k h_k = \prod_{i=1}^{d-1} \frac{1}{1-x_i}$ and hence
\[
\sum_k \psi_\alpha \circ h_k = \prod_j \prod_i \frac{1}{1-x_i^{\alpha_j}}.
\]
Consider the power series:
\[
PS_\mu = \sum_{\alpha \vdash d} \frac{D_{\alpha}}{d!} \chi_\mu(\alpha) \frac{\Delta(x)}{x_1^{d-2} x_2^{d-3} \cdots x_{d-2}} \prod_j \prod_i \frac{1}{1-x_i^{\alpha_j}}.
\]
It follows that the multiplicity of the isotypic component corresponding to $\lambda$ inside $S^\mu(S^k)$ is the coefficient of $x^\lambda$ in $PS_\mu$. This formula is theoretically explicit for any fixed $\mu$ but it can not be evaluated easily. Of course, for given $\lambda$ one can differentiate $PS_\mu$, but it is unclear how to carry this out as a function of $\lambda$. In contrast, our methods give the coefficients parametrized.

**Remark 4.11.** Another possible approach to lattice point counting problems is through Brion-Vergne formula [BV97, p. 802 Theorem (ii)] for vector partition functions. It provides an expression for the number of lattice points in polytopes depending on shifts of facets. Our approach here is much more elementary.

5. **Asymptotic behavior**

Our main formula (4.1) also provides insight into the asymptotical properties of plethysm. The main aim is to identify the leading terms of the piecewise quasi-polynomials that we obtain. As already conjectured by Howe [How87] it is natural to expect that the leading terms come from the polytope of highest dimension, i.e. from the coefficient in the tensor product. This is not obvious since the contribution of a polytope in the quasi-polynomial is not of degree equal to the dimension of the polytope. The reason are the signed summations in the formula which decrease the degree. Below we show how to control this type cancellation, which allows us to obtain the asymptotics. Our strategy is as follows:

1. Introduce a new variable $s$,
2. Multiply each variable in the quasi-polynomial by $s$ and ask for the leading term with respect to the degree of $s$ in order to identify the leading term.
3. Show that the contribution from polytopes of smaller dimension is strictly smaller than the contribution from the Littlewood-Richardson rule.

More precisely, we compute the multiplicity of $s\lambda$ inside $S^\mu(S^k)$ for $s \in \mathbb{N}$. In this case all polytopes appearing in the computation of (4.1) are $s$-th dilatations of $P_{\alpha,\lambda}$ and $P_{k,|\mu|}$ obtained for $s = 1$. The Hilbert-Ehrhart quasi-polynomials of these polytopes are particularly important for us. We can compute the leading term of $\#P_{sk,|\mu|}^\lambda$, which is $\text{Vol} P_{k,|\mu|}^\lambda s^{\text{dim} P_{k,|\mu|}}$. One expects this term to be the leading term of the entire formula, as the dimension of $P_{\alpha,\lambda}$ is largest when $\alpha = (1, \ldots, 1)$, the Littlewood-Richardson contribution. Indeed, assume that $\alpha$ has $a$ parts and $\lambda$ had $l$ parts. As we are only interested in partitions $s\lambda$ we can assume
that we work with exactly $l$ variables. We have $\dim P_{a,\lambda} = (a-1)(l-1)$. In contrast, assuming $\lambda_1 > k$, the intersection of $P_{k,d}$ with the hyperplanes $\sum_j x_j^j = \lambda_i$ has dimension
\[ \dim P_{k,d}^\lambda = 1 + \cdots + (l-1) + (l-1)(d-l) = (l-1)(d-l/2). \]

With $\alpha = (1, \ldots, 1)$ and $d = |\mu| = |\alpha|$, the difference $\dim P_{a,\lambda} - \dim P_{k,d}^\lambda$ is $(l-1)(l/2-1) = \frac{(l-1)(l-2)}{2}$. The following lemma suggests the general approach.

**Lemma 5.1.** Let $Q$ be any degree $r$ polynomial in variables $x_1, \ldots, x_l$. Then, for any constants $c_1, \ldots, c_l$, the polynomial
\[ \sum_{\sigma \in S_l} \text{sgn}(\sigma)Q(x_1 + c_1 - \sigma(1), \ldots, x_l + c_l - \sigma(l)) \]
is of degree at most $r - \frac{l(l-1)}{2}$.

**Proof.** Without loss of generality we can assume that $Q$ is a monomial $x_1^{\beta_1} \cdots x_l^{\beta_l}$ with $\sum_i \beta_i = r$. We can expand the product
\[ (x_1 + c_1 - \sigma(1))^{\beta_1} (x_2 + c_2 - \sigma(2))^{\beta_2} \cdots (x_l + c_l - \sigma(l))^{\beta_l} \]
as a sum by choosing in each bracket $(x_i + c_i - \sigma(i))$ either the term $x_i + c_i$ or the $\sigma$-term $-\sigma(i)$. If for two different indices $i_0, i_1 \in \{1, \ldots, l\}$ we choose the $\sigma$-term the same number of times then, in the summation over permutations we can pair the terms of permutations differing by the transposition $(i_0, i_1)$. The sum of each pair is zero and thus the only contribution comes from summands in which the $\sigma$-term is chosen a different number of times inside each product $(x_i + c_i - \sigma(i))^{\beta_i}$. Consequently the highest degree terms result from choosing the $\sigma$-term $0 + 1 + 2 + \cdots + l - 1 = \frac{l(l-1)}{2}$ times. \hfill $\Box$

In the main formula (4.1) the functions counting the integral points in the polytopes $P_{a,\lambda}$ appear as sums exactly as in Lemma 5.1. One is tempted to conjecture, as in [How87, 3.6(d)], that the leading term of the multiplicity of the isotypic component corresponding to $s\lambda$ comes from $#P_{k,d}^\lambda$, as above. It is obvious that this term appears, due to the Littlewood-Richardson rule, and, although the assumptions of Lemma 5.1 are not satisfied for $#P_{k,d}^\lambda$, the degree bound holds. The main difficulty in bounding the contributions from the other terms is that the counting function is a piecewise quasi-polynomial: the shifts from the permutation $\sigma$ may change both the chamber and the coefficients of the polynomial. Hence the proof of Lemma 5.1 does not apply, although the dimensions of the polytopes $P_{a,\lambda}$ are strictly smaller, and one expects their contribution to be smaller.

We now provide the estimates for the function counting $\sum_{\pi \in S_{d-1}} \text{sgn}(\pi)Q_{\alpha}(k, \lambda_\pi)$.

**Proposition 5.2.** Suppose $\alpha$ has $a < d$ parts and $\lambda$ has $l$ parts. The leading coefficient of
\[ \sum_{\pi \in S_l} \text{sgn}(\pi)Q_{\alpha}(sk, (s\lambda)_\pi) \]
has degree strictly smaller than $(l-1)(d-l/2)$ with respect to the variable $s$.

**Proof.** Suppose that $\alpha$ has $w$ parts greater than 1 and $h$ parts equal to 1. In particular $2w + h \leq d$. Each $(\alpha, (s\lambda)_\pi)$-matrix $M$ is uniquely determined by two matrices $(M_1, M_2)$, where $M_1$ is the $(w \times l)$-submatrix of $M$, corresponding to rows with coefficients not equal to one and $M_2$ the complimentary $(h \times l)$-submatrix. Let $\alpha'$ be the partition of $d - h$
obtained from $\alpha$ by forgetting the singletons. Introducing parameters $i_j$ for $1 \leq j \leq l - 1$ corresponding to column sums of $M_1$ we obtain:

$$Q_\alpha(sk, (s\lambda)_\pi) = \sum_{i_1=1}^{s\lambda_1+l} \cdots \sum_{i_{l-1}=1}^{s\lambda_{l-1}+2} Q_{\alpha'}(sk, (i_1 \ldots i_{l-1})) Q_{(1, \ldots, 1)}(sk, (s\lambda)_\pi - (i_1 \ldots i_{l-1})).$$

Note that if $i_j > s\lambda_j + (l + 1 - j) - \pi(j)$ then $Q_{(1, \ldots, 1)}(sk, (s\lambda)_\pi - (i_1 \ldots i_{l-1})) = 0$, so we could restrict the summation indices, however we prefer not to. We obtain:

$$\sum_{\pi \in S_l} sgn(\pi)Q_\alpha(sk, (s\lambda)_\pi) = \sum_{i_1=1}^{s\lambda_1+l} \cdots \sum_{i_{l-1}=1}^{s\lambda_{l-1}+2} Q_{\alpha'}(sk, (i_1 \ldots i_{l-1})) \left( \sum_{\pi \in S_l} sgn(\pi)Q_{(1, \ldots, 1)}(sk, (s\lambda)_\pi - (i_1 \ldots i_{l-1})) \right).$$

Let $i_l$ be defined by $\sum_{j=1}^l i_j = sk(d - h)$. The term $\sum_{\pi \in S_l} sgn(\pi)Q_{(1, \ldots, 1)}(sk, (s\lambda)_\pi - (i_1 \ldots i_{l-1}))$ is positive, since, by the arguments in Section 4, it equals the multiplicity of the isotypic component corresponding to the (possibly unordered) partition $s\lambda - (i_1 \ldots i_l)$ inside $(S^sk)^{\otimes h}$. This allows us to bound the degree with which $s$ may appear separately. The degree of $s$ in the term $Q_{\alpha'}(sk, (i_1 \ldots i_{l-1}))$ can be naively bounded by $(w-1)(l-1)$, as each entry of the $j$-th column of $M_1$ is bounded by $s\lambda_j$ plus a constant, and the row and column sums of $M_1$ are fixed. It thus remains to bound the degree of $s$ in

$$\sum_{i_1=1}^{s\lambda_1+l} \cdots \sum_{i_{l-1}=1}^{s\lambda_{l-1}+2} sgn(\pi)Q_{(1, \ldots, 1)}(sk, (s\lambda)_\pi - (i_1 \ldots i_{l-1})).$$

We distinguish two cases depending on the whether $h$ or $l$ are larger.

**Case 1** ($h > l$): By Pieri’s rule the multiplicities of isotypic components corresponding to partitions with at most $l$ parts in $(S^sk)^{\otimes h}$ are determined by the following parameters:

- 1 parameter for the number of boxes added to the first row in the first step (the remaining boxes going into the second row),
- 2 parameters for the number of boxes added to the first and second row in step 2,
- $i \leq l - 1$ parameters for the number of boxes added to rows 1 to $i$ in step $i$,
- $(h - l)(l - 1)$ parameters for the numbers of boxes added to rows from 1 to $l - 1$ in steps $l$ to $h$.

This bounds the exponent of $s$ by $1 + \cdots + l - 1 + (h - l)(l - 1) = (l - 1)(h - l/2)$. All together we obtain the bound $(l - 1)(w + h - 1 - l/2)$ which is strictly smaller than $(l - 1)(d - l/2)$.

**Case 2** ($h \leq l$): The degree of $s$ inside

$$\sum_{i_1=1}^{s\lambda_1+l} \cdots \sum_{i_{l-1}=1}^{s\lambda_{l-1}+2} \sum_{\pi \in S_l} sgn(\pi)Q_{(1, \ldots, 1)}(sk, (s\lambda)_\pi - (i_1 \ldots i_{l-1}))$$

is bounded by the degree of $s$ in the total sum of all multiplicities in the decomposition of $(S^sk)^{\otimes h}$. We could now proceed as above, but [FZ12, Theorem 1.2(ii)] directly gives that this degree equals $\binom{l}{2}$ and hence the total degree in which $s$ can appear is at most

$$(w - 1)(l - 1) + \binom{h}{2}.$$
After estimating \((h_2) \leq (l - 1)h/2\) this is seen as strictly smaller than \((l - 1)(d - l/2)\). □

**Theorem 5.3.** Fix a partition \(\mu\) of \(d\). The multiplicity of the isotypic component corresponding to \(\lambda\) inside \(S^\mu(S^k(V))\) is a piecewise quasi-polynomial in \(k\) and \(\lambda\). Its leading term is the multiplicity of \(\lambda\) inside \(S^k(V)\otimes d\) times \(\frac{\dim \mu}{d}\).

### 6. Appendix

**6.1. Vector partition functions.** Consider a polyhedral cone in standard representation \(C = \{Ax \geq b\} \subset \mathbb{Q}^N\), and a linear map \(\pi : C \rightarrow \mathbb{Q}^n\). The image of \(\pi\) is a polyhedral cone denoted \(D\). In this situation, the preimage of an integral point in \(D\) is a polyhedron and we are interested in the (possibly infinite) number of integral points it contains. The counting function is

\[ \phi : D \cap \mathbb{N}^n \rightarrow \mathbb{N}_0 \cup \{\infty\} \]

\[ \phi(d) = \#\{c \in C \cap \mathbb{N}^N : \pi(c) = d\} \]

The preimage of any rational point \(d \in D\) under \(\pi\) is a polyhedron and there are only finitely many combinatorial types of polyhedra appearing among all preimages (see [VSB+04] for an overview on the history of this result with a focus on implementation). The type depends on which supporting hyperplanes of \(C\) intersect a given preimage \(\pi^{-1}(d)\) non-trivially. This yields a decomposition of \(D\) known as the chamber decomposition of \(D\). As in each chamber the combinatorial type of each fiber is the same, the counting function is a quasi-polynomial since in general the fiber is a rational polytope. All-together, \(\phi\) is a piecewise quasi-polynomial.

In full generality Sturmfels has shown that the lattice point enumerator of a parametric polyhedron \(\{x : Ax \leq b(t)\}\) is a piecewise quasi-polynomial in the parameters \(t\), whenever \(b(t) \in \mathbb{Z}[t]\) is a linear polynomial [Stu95]. He calls \(\phi\) the vector partition function as it counts the number of ways to write a vector \(d\) in terms of generators of \(C\).

**Example 6.1.** Fix \(d \in D\) and consider points \(kd, k \in \mathbb{N}\) on the ray generated by \(d\). In this case \(\phi(kd)\) equals \(P_{\pi^{-1}(d)}(k)\), the Ehrhart quasi-polynomial of \(\pi^{-1}(d)\). If \(\pi^{-1}(d)\) happens to be an integral polytope, then so are the polytopes \(\pi^{-1}(kd)\) and in this case \(P_{\pi^{-1}(kd)}\) is an honest polynomial [Ehr77].

**Example 6.2.** Consider the two-dimensional cone \(C\) over the matrix \(\begin{pmatrix} 5 & 3 \\ 1 & 2 \end{pmatrix}\), depicted in Figure 1. Let \(\pi\) be the projection to the first coordinate such that the image cone \(D\) is just the \(x\)-axis. The vector partition function counting the number of integer points in a vertical slice of \(C\) is the piecewise quasi-polynomial

\[ \phi(x) = \begin{cases} 0 & x < 0, \\
(x + 1) - \left\lfloor \frac{x + 2}{3} \right\rfloor - \left\lfloor \frac{x + 4}{5} \right\rfloor & x \geq 0. \end{cases} \]

Note that in general a quasi-polynomial can be written as a polynomial expression in the variables and floor functions of linear functions in the variables.

Vector partition functions can be computed symbolically. The first step is usually to compute the chamber decomposition for which several algorithms exist. Once the problem is reduced to determining a quasi-polynomial for each chamber, interpolation may be the first idea that comes to mind. This is known as Claus’s method and was indeed the first method suggested for the determinations of vector partition functions. This method has problems...
since it can be difficult to find sufficiently many lattice points for interpolation. A more efficient approach is to use Barvinok’s method which, in the setting of vector partition functions, was first suggested by Verdooldaege et al. [VSB+07]. Their software BARVINOK (together with the isl library) is the most advanced tool available today. The software is very well developed because of its applications in computer science, for instance to loop optimization in compiler development. It is one of the mathematical software tools with which run time is short compared to the time that humans need to learn something from the result. The introduction of [VBBC05] contains many references.

6.2. Computation of the plethysm coefficient quasi-polynomials. As a proof of concept we evaluated equation (4.1) using BARVINOK. We describe the necessary steps for $d = 5$ here. The input files for $d = 3, 4$ are also available on our project homepage.

In (4.1) the innermost evaluation is the quasi-polynomial function $Q_\alpha$ which depends on $\alpha$ (a partition of $d$) and a permutation $\pi$. $Q_\alpha$ is a function in $\lambda$, but through the series of reductions in Section 4, the final formula has different arguments, called $b_1, \ldots, b_{d-1}, s$ (see Remark 4.8). By convention, the $b_i$ are ordered increasingly with $b_1$ the smallest.

Example 6.3. Suppose we want to determine the multiplicity of the isotypic component for $\lambda = (3, 2, 1)$ in $S^3(S^2)$. This multiplicity is equal to the multiplicity of the isotypic component of $(2, 1, 0)$ in $\bigwedge^3(S^1)$ (Remark 4.8). To evaluate it using our programs, we plug $(b, s) = (1, 1)$ into the quasi-polynomial stored in 111.qpoly. We find that the multiplicity is equal to 0. See the last item in Section 6.4 for how this 0 is presented, though.

In the following we describe the steps necessary to repeat our computations and determine the quasi-polynomials in (4.1). The procedure consists of roughly four steps:

1. Determine $Q_\alpha$ enumerators with barvinokEnumerate.
2. Sum over the partition $\alpha$ using precomputed coefficients $\chi_\mu(\alpha)D_{\alpha}$.
3. Sum over permutations $\pi$.
4. Division by $\pm d!$ and postprocessing.
6.2.1. **Enumeration.** The directory `barvinok_enumeration` contains input files for the program `barvinok_enumerate`, corresponding to determination of the $Q_\alpha$ for different values of $\alpha$ and shifted $\lambda$. These files need to be processed individually with the command

```
barvinok_enumerate --to-isl < "${input}" > "${input}".result
```

where `${input}` is a filename of one of the `.barv` files. We provide the bash script `do.sh` which runs on four parallel processors for this job. In our experience, this step, for $d = 5$, should complete in less than 12 hours even on laptop computers. At this point one could argue that we are doing a lot of computation that is not strictly necessary, since the $Q_\alpha(k, \lambda_\pi)$ for different $\pi$ are the same quasi-polynomials, evaluated at slightly shifted arguments. In principle one would like to compute one quasi-polynomial and evaluate it on shifted arguments. This point is legit, but it seemed more convenient using recomputation with modified constraints. Due to the parallelization, computing all $Q_\alpha(k, \lambda_\pi)$ individually by modifying the input was quick. The result of this computation are stored in `.result` files which we need for the next, computationally more demanding, step: summation.

6.2.2. **Summation.** At this point the `.result` files should contain all lattice point enumerators $Q_\alpha(k, \lambda_\pi)$ and our next task is to sum them with appropriate coefficients. After some experimentation it turned out to be advantageous to first sum over the partition $\alpha$ and only after that do the summation over the permutations $\pi$. For this we use iscc, an interactive (and programmable) isl calculator which comes with the isl distribution. Place the result files in the summation folder which already contains the appropriate summation scripts. Their names are `sum11111.iscc` for $\mu = (11111)$ and so on. These scripts run for a while. Here are some approximate run times that we measured on an Intel Core i7-4770 (3.4GHz):

| script      | runtime in hours |
|-------------|------------------|
| sum11111.iscc | 118              |
| sum2111.iscc  | 38               |
| sum221.iscc   | 7                |
| sum311.iscc   | 2                |
| sum32.iscc    | 7                |
| sum41.iscc    | 38               |
| sum5.iscc     | 118              |

In principle this computation could be parallelized too by structuring summation hierarchically in the form of a tree. At the moment iscc has no native support for parallelism so the only way to parallelize this computation would be on the OS level. This in turn means that intermediate results have to be written to disk and read again. Reading large quasi-polynomials is very slow (see Section 6.4), and consequently we ran each summation on one thread, but different summations at the same time.

The script `sumX.iscc` stores its result in `X.result`. This file then contains the quasi-polynomial we are looking for, multiplied with a signed factorial. In the case of $d = 5$, the factor is $5! = 120$.

6.2.3. **Postprocessing.** In the final step we divide the quasi-polynomial by the appropriate factorial and sign and use a text editor to convert the results from parametric sets of constant functions into functions (see Section 6.4).
6.3. Experiences and limitations. The results of our computation are piecewise quasi-polynomials and their representations are far from unique. The most basic phenomenon bothering us is that divisibility conditions may be obfuscated by existential quantifiers. For one example, if $s$ is a variable, then $s \equiv 0 \mod 5$ may appear as $\exists e_0 = \lfloor (-1 + s)/5 \rfloor, \exists e_1 = \lfloor s/5 \rfloor$ such that $5e_1 = s$ and $5e_0 \leq -2 + s$ and $5e_0 \geq -5 + s$. To see the equivalence, note that the $e_0$ condition is that $s$ leaves any remainder except 1 modulo 5, and thus redundant. At the moment there seems to be no automatic way to remove such redundant conditions while they do appear frequently (this one is taken from Example 6.6).

Another challenge to be addressed in the future is the number of chambers that appears after doing arithmetic with quasi-polynomials. In principle there can be chambers $C_1, C_2$ with corresponding quasi-polynomials $p_1, p_2$ such that $C_2$ is a face of $C_1$, and $p_1$ when restricted to $C_2$ equals $p_2$. At the moment BARVINOK has no means to detect this case during the computation, or rectify it a posteriori. We can not precisely estimate how much this effect hits us. We did run ISC’s coalesce function on each of our results which uses simple tests to detect empty chambers. This has reduced the output of the summation part to approximately a fifth of its original size.

6.4. Quirks. Using BARVINOK and ISC, the following things occurred to us:

- It can take very long to read quasi-polynomials from disk. Our largest result files are 11111.qpoly and 5.qpoly which on a Core i7 needed 5 hours 51 minutes to be parsed. In contrast they need only a second to be written to disk! We asked on the ISL development mailing list and it was confirmed that the parser is not very efficient.
- Reading a quasi-polynomial and then writing it out again need not yield the same representation. The parser that is used to read piecewise quasi-polynomials from files applies certain transformations that are not applied when computing the quasi-polynomials from scratch.
- Mathematically speaking our results are simply functions $\mathbb{N}^d \rightarrow \mathbb{N}$, but in the computer things are not that simple. The program barvinok_enumarate which we use as the first step in our computation does not return functions on $\mathbb{N}^d$—it returns sets of constant functions, parametrized over $\mathbb{N}^d$. It is technically impossible to “evaluate” these parametric sets of constants, because only functions can be evaluated in ISL. To fix this we simply used text editing on the output files to convert expressions like

\[
[b1, s] \rightarrow \{ [] \rightarrow (1/2 * b1 + 1/2 * b1^2) : ... \}
\]

into

\[
\{ [b1, s] \rightarrow (1/2 * b1 + 1/2 * b1^2) : ... \}
\]

- If the result of a quasi-polynomial evaluation is a nonzero integer $n$, then the result is formatted as $\{n\}$. If, however, the result is zero, the empty set is returned: $\{ \}$.

6.5. Evaluation. Evaluation of explicit plethysm coefficients can be done in LiE [vLCL92] and other packages like SAGE:

**Example 6.4.** To evaluate plethysm in SAGE [S+14], first one sets up the ring of symmetric functions in the Schur basis with

```
sage: s = SymmetricFunctions(QQ).schur()
```

After this the (Schur function) plethysm can be computed by plugging in as follows:

```
sage : s([2,1,1])(s[3,1])
```
For both parametric partial and complete evaluation of our stored results, the most practical tool is iscc.

**Example 6.5.** In iscc, to evaluate a quasi-polynomial $P$ (created, for instance with
\[ P := \text{read "111.qpoly"}; \]
at arguments $(3, 2)$ use the following input to iscc:
\[ P (\{[3,2]\}); \]
The () brackets are used to trigger evaluation on an isl domain introduced with {} which in turn consists of only one isolated point $[3,2]$.

**Example 6.6.** In this example we explain how to arrive at the result in Example 1.3 using the provided result files. Note that by means of the reductions in Remark 4.8, this formula could in principle also be derived from the Cayley-Sylvester formula in $SL_2(\mathbb{C})$ representation theory, but this formula is not as explicit as ours. It involves counting tableaux under side constraints.

Let again $\mu = (5)$, and $\lambda = (31, 3, 2, 2, 2)$. The following code loads the quasi-polynomial for $\mu$ from the file $5.qpoly$ and evaluates it along the line $s \cdot \lambda$ for $s \in \mathbb{Z}$. Note that the read command will take very long (up to several hours) since the parser for quasi-polynomials is not very optimized.

\[ P := \text{read "5.qpoly"}; \]
\[ \{s\} \rightarrow [0, 0, s, 6*s] \} . P; \]
The result looks like this:

\[
\begin{align*}
\text{Chamber 1} & \\
\text{exists (e_0 = floor((-1 + s)/5): 5e0 = -1 + s and s >= 1)} & \\
\text{which means } s \geq 1 \text{ and } s \equiv 1 \mod 5. \\
\end{align*}
\]

To parse this, first observe that a new chamber starts whenever we see $[s] \rightarrow$. The first step towards understanding this output is to isolate the four chambers and to reformulate their constraints. The constraints are the items after the colon in each chamber.
exists \((e_0 = \lfloor(-1 + s)/5\rfloor, e_1 = \lfloor(s)/5\rfloor)\):  
\[5e_1 = s \text{ and } s \geq 5 \text{ and } 5e_0 \leq -2 + s \text{ and } 5e_0 \geq -5 + s\]

which, except from \(s \geq 5\), translates into the requirement that \(s\) should leave remainder zero modulo 5, and additionally \(s - 5 \leq \lfloor \frac{s-1}{5} \rfloor \leq s - 2\). The second condition is that \(s\) leaves any remainder except 1 modulo 5, and thus redundant. At the moment our computational tools are unable to carry out this simplification automatically.

**Chamber 3**

exists \((e_0 = \lfloor(-1 + s)/5\rfloor, e_1 = \lfloor(s)/5\rfloor)\): \(s \geq 1\) and \(5e_0 \leq -2 + s\) and \(5e_0 \geq -5 + s\) and \(5e_1 \leq -1 + s\) and \(5e_1 \geq -4 + s\).

The conditions are \(s \geq 1\), \(s - 5 \leq 5 \lfloor \frac{s-1}{5} \rfloor \leq s - 2\) and \(s - 4 \leq \lfloor \frac{s}{5} \rfloor \leq s - 1\). They are both satisfied if and only if \(s\) leaves remainder 2, 3, or 4 modulo 5.

**Chamber 4**

This chamber is singleton: \(s = 0\) and thus the case distinction is complete.

The output of our program has each quasi-polynomial written in an expression involving floor functions. To simplify the presentation, let us introduce the following shorthands which appear in the output

\[p_1 = \frac{3}{5} - \frac{289}{720}s + \frac{1}{20}s^2 + \frac{1}{720}s^3,\]
\[p_2 = \frac{5}{8} + \frac{1}{8}s,\]
\[p_3 = \frac{1}{3} - \frac{1}{6}s,\]
\[p_4 = \frac{7}{12} - \frac{1}{3}s,\]
\[q_1 = 1 - \frac{289}{720}s + \frac{1}{20}s^2 + \frac{1}{720}s^3,\]
\[r_1 = \frac{4}{5} - \frac{289}{720}s + \frac{1}{20}s^2 + \frac{1}{720}s^3.\]

Using these shorthands and only trivial manipulations of the output we arrive at the following three quasi-polynomials in the three non-trivial chambers:

\[s \equiv 1 \mod 5\]
\[
p_1 + p_2 \left\lfloor \frac{s}{2} \right\rfloor + p_3 \left\lfloor \frac{s}{3} \right\rfloor + \left( p_4 + \frac{1}{2} \left\lfloor \frac{s}{3} \right\rfloor \right) \left[ \left\lfloor \frac{1+s}{3} \right\rfloor \right] + \frac{1}{4} \left( \left[ \frac{1+s}{3} \right] \right)^2 + \left[ \frac{s}{4} \right] - \left[ \frac{3+s}{4} \right] \]

\[s \equiv 0 \mod 5\]
\[
q_1 + p_2 \left\lfloor \frac{s}{2} \right\rfloor + p_3 \left\lfloor \frac{s}{3} \right\rfloor + \left( p_4 + \frac{1}{2} \left\lfloor \frac{s}{3} \right\rfloor \right) \left[ \left\lfloor \frac{1+s}{3} \right\rfloor \right] + \frac{1}{4} \left( \left[ \frac{1+s}{3} \right] \right)^2 + \left[ \frac{s}{4} \right] - \left[ \frac{3+s}{4} \right] \]

\[s \equiv 2, 3, 4 \mod 5\]
\[
r_1 + p_2 \left\lfloor \frac{s}{2} \right\rfloor + p_3 \left\lfloor \frac{s}{3} \right\rfloor + \left( p_4 + \frac{1}{2} \left\lfloor \frac{s}{3} \right\rfloor \right) \left[ \left\lfloor \frac{1+s}{3} \right\rfloor \right] + \frac{1}{4} \left( \left[ \frac{1+s}{3} \right] \right)^2 + \left[ \frac{s}{4} \right] - \left[ \frac{3+s}{4} \right] \]

There is an obvious pattern here, but unfortunately the ISL engine has problems with factoring out, or simplifying these expressions automatically. For instance in the third chamber
it actually returns the expression

\[
\begin{align*}
\left( r_1 - p_2 \left\lfloor \frac{s}{2} \right\rfloor - p_3 \left\lfloor \frac{s}{3} \right\rfloor \right) & - \left( p_4 + \frac{1}{2} \left\lfloor \frac{s}{3} \right\rfloor \right) \left( \frac{1}{4} \left\lfloor \frac{1+s}{3} \right\rfloor - \frac{1}{4} \left\lfloor \frac{3+s}{4} \right\rfloor \right) + \left( s - 3 \right) \left\lfloor s - 3 \right\rfloor + \left( \frac{s}{3} - \frac{3+s}{4} \right).
\end{align*}
\]

Not only can we simplify the presentation, in fact the above expression looks like the lead term would be of quasi-polynomial nature while in reality it is not since

\[
\left( \left\lfloor \frac{s}{5} \right\rfloor - \left\lfloor \frac{3+s}{5} \right\rfloor \right) = -1 \quad \text{if } s \equiv 2, 3, 4 \mod 5.
\]

Applying all simplifications and the shortcuts

\[
p = -\frac{289}{720} s + \frac{1}{20} s^2 + \frac{1}{720} s^3,
\]

\[
p_2 = \frac{5}{8} + \frac{1}{8}, \quad p_3 = \frac{1}{3} - \frac{1}{6} s, \quad p_4 = \frac{7}{12} - \frac{1}{3} s,
\]

\[
A(s) = p + p_2 \left\lfloor \frac{s}{2} \right\rfloor + p_3 \left\lfloor \frac{s}{3} \right\rfloor + \left( p_4 + \frac{1}{2} \left\lfloor \frac{s}{3} \right\rfloor \right) \left( \frac{1+s}{3} \right) + \frac{1}{4} \left( \frac{1+s}{3} \right)^2 + \frac{s}{4},
\]

the final result is

\[
Q(s) = A(s) + \begin{cases} 1 & \text{if } s \equiv 0 \mod 5 \\ \frac{2}{5} & \text{if } s \equiv 1 \mod 5 \\ \frac{4}{5} & \text{if } s \equiv 2, 3, 4 \mod 5. \end{cases}
\]

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