Formal GNS Construction and WKB Expansion in Deformation Quantization

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1. The concept of deformation quantization has been defined and exemplified in [2]. The existence of star-products on every symplectic manifolds has been established in [3] and in [4]. This gives a reasonable physical picture of the noncommutative algebra of quantum observables with built-in classical limit. However, the discussion of a formal analogue of representations of the deformed algebra in some ‘Hilbert space’ seems to have been restricted to examples in the literature up to now. In [4] the authors have proposed how to construct formal pre-Hilbert spaces for the deformed algebra in the same category by means of a generalized Gel’fand-Naimark-Segal (GNS) construction.

We shall now briefly review this construction and shall apply this in the next section to the Weyl star-product on the cotangent bundle of $\mathbb{R}^n$ (which will give back the usual Schrödinger representation together with the Weyl symmetrization rule; details thereof can be found in [4]). The third section contains new material: We shall describe how to incorporate the usual WKB expansion into the framework of star-products and GNS representations by means of a certain positive linear functional on the deformed algebra having support on a projectable Lagrangean submanifold graph$(dS)$ of $T^*\mathbb{R}^n$. The main trick is to use a suitable form of the star-exponential $e^{*_{\frac{\lambda}{\hbar}S}}$.

Let $\mathbb{C}((\lambda))$ (resp. $\mathbb{R}((\lambda))$) be the field of complex (resp. real) Laurent series in $\lambda$ with finite principal part (i.e. a finite number of negative powers in $\lambda$) and for any complex or real vector space $V$ let $V((\lambda))$ denote the vector space of all formal Laurent series (with finite principal part) in $\lambda$ with coefficients in $V$. Let $(M,\omega)$ be a symplectic manifold equipped with a differential star-product $*$ satisfying $f*g=\bar{g} * \bar{f}$. Let $C^\infty(M)$ be the space of all smooth complex-valued functions on $M$. Then $(C^\infty(M)((\lambda)),*)$ becomes an associative algebra over the Laurent field $\mathbb{C}((\lambda))$. Since

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the field $\mathbb{R}((\lambda))$ is an ordered field by using the real order of the lowest nonvanishing coefficient, a $\mathbb{C}((\lambda))$-linear functional $\omega$ of $C^\infty(M)((\lambda))$ can be called positive iff $\omega(f \ast f)$ is positive in the Laurent field $\mathbb{R}((\lambda))$ for all $f \in C^\infty(M)((\lambda))$. As in the case of $C^*$-algebras it can be shown that the so-called Gel'fand ideal $J_\omega := \{ f \in C^\infty(M)((\lambda)) | \omega(\hat{f} \ast f) = 0 \}$ is a left ideal of the algebra $C^\infty(M)((\lambda))$, that the quotient $\mathcal{H}_\omega := C^\infty(M)((\lambda))/J_\omega$ (with canonical projection denoted by $f \mapsto \Psi_f$) is a left module for $C^\infty(M)((\lambda))$, equipped with a positive definite $\mathbb{C}((\lambda))$-valued inner product defined by $\langle \Psi_f, \Psi_g \rangle := \omega(\hat{f} \ast g)$, and that the GNS representation $\pi_\omega(f)\Psi_g := \Psi_{f \ast g}$ is a star-representation in the sense that $\pi_\omega(f)$ and $\pi_\omega(\hat{f})$ are formal adjoints. In [4] it was shown that on any Kähler manifold equipped with the star-product of Wick type (see e.g. [3]) every delta-functional (evaluation at a point) is a positive functional whose GNS-representation gives rise to a formal Bargmann representation.

2. Consider the cotangent bundle $\pi : T^*\mathbb{R}^n \to \mathbb{R}^n$ of $\mathbb{R}^n$ with its canonical symplectic structure, the standard Weyl product $\ast$, and the standard volume form $\omega := dq^1 \cdots dq^n$ on $\mathbb{R}^n$. We define

$$C^\infty(T^*\mathbb{R}^n)_{\mathbb{R}^n} := \{ f \in C^\infty(T^*\mathbb{R}^n) | \text{supp}(f) \cap i(\mathbb{R}^n) \text{ is compact.} \}$$

where $i : \mathbb{R}^n \to T^*\mathbb{R}^n$ is the canonical embedding of $\mathbb{R}^n$ as the zero section. Then we define the $\mathbb{C}((\lambda))$-linear functional $\omega_0 : C^\infty(T^*\mathbb{R}^n) \to \mathbb{C}((\lambda))$ by

$$\omega_0(f) := \int_{\mathbb{R}^n} dq^i f^i,$$

and let $\mathcal{S} : C^\infty(T^*\mathbb{R}^n)((\lambda)) \to C^\infty(T^*\mathbb{R}^n)((\lambda))$ be defined as $\mathcal{S} := \exp\left(\frac{i}{2} \Delta \right)$ where $\Delta := \frac{\partial^2}{\partial q^i \partial q^k}$. Then the following Lemma is found by integration by parts:

**Lemma 1** For any two $f, g \in C^\infty(T^*\mathbb{R}^n)_{\mathbb{R}^n}((\lambda))$ we have

$$\omega_0(f \ast g) = \int_{\mathbb{R}^n} dq^i (i^* \mathcal{S} f)(i^* \mathcal{S} g)$$

whence $\omega_0$ is a positive linear functional. The Gel'fand ideal of $\omega_0$ is given by

$$J_0 := \{ f \in C^\infty(T^*\mathbb{R}^n)_{\mathbb{R}^n}((\lambda)) | i^* \mathcal{S} f = 0 \}.$$

In [4] we have shown that $J_0$ is the left ideal generated by the $n$ momentum functions $p_1, \ldots, p_n$.

Moreover we can determine the GNS representation induced by $\omega_0$ explicitly using the map $\mathcal{S}$ and the fact that $\mathcal{S} \circ \pi^* = \pi^*$:

**Theorem 2** Let $f \in C^\infty(T^*\mathbb{R}^n)((\lambda))$, $g \in C^\infty(T^*\mathbb{R}^n)_{\mathbb{R}^n}((\lambda))$, and $\varphi, \psi \in C^\infty(\mathbb{R}^n)((\lambda))$.

1. There is a canonical isomorphism $\Phi : C^\infty(T^*\mathbb{R}^n)_{\mathbb{R}^n}((\lambda))/J_0 \to \mathcal{H}_0 := C^\infty(\mathbb{R}^n)((\lambda))$ given by $\Phi(\Psi_g) := i^* \mathcal{S} g$ whose inverse is simply given by $\Phi^{-1}(\varphi) := \Psi_{\pi^* \varphi}$. The hermitean product on $\mathcal{H}_0$ induced by the pull-back of the GNS inner product with respect to $\Phi$ is the standard $L^2$-inner product $\langle \varphi, \psi \rangle = \int_{\mathbb{R}^n} dq^i \hat{\varphi} \hat{\psi}$.

2. $J_0$ is even a left ideal of $C^\infty(T^*\mathbb{R}^n)((\lambda))$ whence the GNS representation is defined for all its elements $f$ and given by the following formula on elements of $\mathcal{H}_0$:

$$\pi_0(f) \varphi = i^* \mathcal{S} (f \ast (\pi^* \varphi)) = \sum_{r=0}^{\infty} \frac{1}{r!} \left( \frac{\lambda}{i} \right)^r \frac{\partial^r(\mathcal{S} f)}{\partial p_{i_1} \cdots p_{i_r}} \frac{\partial^r \varphi}{\partial q^{i_1} \cdots q^{i_r}}.$$
We had also shown in \[\text{[4]}\] that the representation \(\Phi\) exactly corresponds to the Weyl symmetrization rule of polynomials in \(q^1, \ldots, q^n, p_1, \ldots, p_n\).

3. We shall now use these results to formulate the WKB expansion for \(T^*\mathbb{R}^n\) in the framework of deformation quantization. Let \(H \in C^\infty(T^*\mathbb{R}^n)\) be a classical real-valued Hamiltonian and assume that there is a smooth real-valued function \(S : \mathbb{R}^n \to \mathbb{R}\) satisfying the following (in general nonlinear) first order partial differential equation, the so-called Hamilton-Jacobi equation, for some chosen real number \(E\):

\[
H(dS(q)) = E \quad \text{for all } q \in \mathbb{R}^n.
\]

Let \(L := \text{graph}(dS) \subset T^*\mathbb{R}^n\) be the graph or \(dS\) which is known to be a projectable Lagrangian submanifold of \(T^*\mathbb{R}^n\) (see e.g. \[\text{[4]},\text{p.28-31}\]). Equation \(\text{(1)}\) implies \((H - E)|_L = 0\), and the local flow of the Hamiltonian vector field of \(H\) preserves \(L\).

We are now going to construct a suitable GNS functional having support in \(L\). First, consider the ‘Heisenberg equation’ with respect to the Hamiltonian \(\pi^*S\):

\[
\frac{df}{dt}(t) = \frac{i}{\lambda}\left((\pi^*S) \ast f(t) - f(t) \ast (\pi^*S)\right)
\]

which can be shown to have a unique solution \(f(t) = A_t(f_0)\) for each initial function \(f_0 \in C^\infty(T^*\mathbb{R}^n)((\lambda))\) where \(t \mapsto A_t\) is a one-parameter group of \(\mathbb{C}((\lambda))\)-linear automorphisms of the \(C^\infty(T^*\mathbb{R}^n)((\lambda))\) commuting with complex conjugation. It allows for the factorization

\[
A_t = \Phi^*_t \circ T_t
\]

where \(\Phi_t\) is the Hamiltonian flow of \(\pi^*S\) which is positive since \(\omega_1\) is a left ideal in the algebra \(f \in T^*\mathbb{R}^n\) whose support has compact intersection with \(L\), and let \(\omega_1 : C^\infty(T^*\mathbb{R}^n)_L((\lambda)) \to \mathbb{C}((\lambda))\) be the following \(\mathbb{C}((\lambda))\)-linear functional (where \(i_L\) denotes the canonical injection \(L \subset T^*\mathbb{R}^n\)):

\[
\omega_1(f) := \int_\mathbb{R}^n(\Phi^*_t d^nq) \ i_L^*(T_{-1}f) = \omega_0 \circ A_{-1}(f)
\]

which obviously has support in \(L\) and is easily seen to be positive since \(A_{-1}\) is a real automorphism. Moreover, the Gel’fand ideal of \(\omega_1\) is simply given by \(J_1 = A^*_1 J_0\), and the GNS representation \(\pi_1\) induced by \(\omega_1\) can be constructed as follows: since \(C^\infty(T^*\mathbb{R}^n)_L((\lambda)) = A_1(C^\infty(T^*\mathbb{R}^n)_{\mathbb{R}^n}((\lambda)))\) there is a well-defined unitary map \(U : \mathcal{H}_0 \to \mathcal{H}_1\) between the two formal GNS pre-Hilbert spaces given by \(U(\Psi^{(0)}_{A_1}) := \Psi^{(1)}_{A_1}\) whence the vector space \(\mathcal{H}_1\) is isomorphic to \(C^\infty_0(L)((\lambda))\). Observing that the Gel’fand ideal of \(\omega_1\) is again a left ideal in the algebra \(C^\infty(T^*\mathbb{R}^n)((\lambda))\) we get the following formula for the GNS representation of \(f \in C^\infty(T^*\mathbb{R}^n)((\lambda))\):

\[
\pi_1(f) = U \pi_0(A_{-1} f) U^{-1}.
\]

We consider the time-independent Schrödinger equation \(\pi_1(H)\phi^{(1)} = E\phi^{(1)}\) for \(\phi^{(1)}\) in the formal distribution space \(\mathcal{H}_1' := (C^\infty_0(L))'((\lambda))\). This is well-defined since \(\pi_1(H)\) is a series of differential operators. Via the map \(U\), this problem is related to the time-independent Schrödinger equation

\[
\pi_0(A_{-1} H)\phi = E\phi
\]

where \(\phi = \sum_{r=0}^{\infty} \lambda^r \phi_r\) and all the \(\phi_r\) are distributions in \(C^\infty_0(\mathbb{R}^n)\). The above equation leads to a system of partial differential equations for the \(\phi_r\) given in the following
Theorem 3 (Formal WKB Expansion) With the above notations we find that the system of equations (9) is equivalent to the following system of partial differential equations for $\phi$:

$$
(i^* \Phi_{-1}^{(1)} H) \phi_r - \frac{i}{2} \frac{\Delta((\Phi_{-1}^{*} H)(\pi^* \phi_r)) + \frac{i}{2} \pi^* \{\Phi_{-1}^{*} H, \pi^* \phi_r\}}{\lambda} = \sum_{a+b+c+d=r+1, \ a, b, c, d \geq 0, \ r > d} \frac{(-i/2)^a (i/2)^b}{a! b!} \Delta^a (M_b(\Phi_{-1}^{*} T^{(c)} H, \pi^* \phi_d))
$$

(10)

where $M_b$ are the bidifferential operators of the standard Weyl star product in $T^* \mathbb{R}^n$, i.e. $M_b(f, g) := (\partial_q \partial_{p_k} - \partial_{p_k} \partial_q) f (q, p) g(q', p')$, $|q = q', p = p'|$.

In the particular case of a Hamiltonian of the physical form $H(q, p) = \sum_{i=1}^n (p_i)^2 + V(q)$ for a smooth real-valued function $V$ on $\mathbb{R}^n$ the system of equations (11) reduces to the well-known WKB transport equations (see e.g. [1, 13-14]) (which is mainly due to the fact that $A_t H = \Phi_t^* H$ for $H$ at most quadratic in the momenta) where $\phi_r := 0$ for negative $r$:

$$
\frac{\partial^2 S}{\partial q^k \partial q^r} \phi_r + 2 \frac{\partial S}{\partial q^k} \frac{\partial \phi_r}{\partial q^k} = i \frac{\partial^2 \phi_{r-1}}{\partial q^k \partial q^k}.
$$

(11)

Remarks:

1. Since we assumed that the energy surface $H^{-1}(E)$ contains a graph of a one-form, namely $dS$, the physical situation in the above Theorem is in so far simplified as there are no classically forbidden regions or turning points. It is an interesting problem to perform the same GNS analysis for general cotangent bundles and more general Lagrangean submanifolds incorporating Maslov corrections.

2. The usual WKB phase function $e^{i S/\hbar}$ is recovered in the following way: assume first that $H$ is polynomial in the momenta. Then e.g. $\exp(\frac{i}{\hbar} (\pi^* S)) * H$ is a well-defined formal Laurent series in $1/\lambda$ converging for all $\lambda = \hbar \in \mathbb{R} \setminus \{0\}$. Hence $A_t H = \exp(\frac{i}{\hbar} (\pi^* S)) * H * \exp(- \frac{i}{\hbar} (\pi^* S)) = (\pi^* \exp(\frac{i}{\hbar} S)) * H * (\pi^* \exp(- \frac{i}{\hbar} S))$ converges for all $\lambda = \hbar \in \mathbb{R} \setminus \{0\}$. Assuming that the power series $\phi$ converges for some $\hbar \in \mathbb{R} \setminus \{0\}$ to a distribution, we readily see that eqn (11) is equivalent to $\pi_0 (H) \exp(\frac{i}{\hbar} S) \phi = \exp(\frac{i}{\hbar} S) \phi$ since $\pi_0$ is a representation.

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