A BLOW-UP METHOD TO PRESCRIBED MEAN CURVATURE GRAPHS WITH FIXED BOUNDARIES

HENGYU ZHOU

Abstract. In this paper, we apply a blow-up method of Schoen and Yau in [39] to study a large class of prescribed mean curvature (PMC) Dirichlet problems in \( n(n \geq 2) \)-dimensional Riemannian manifolds. In this process we establish curvature estimates for almost minimizing PMC hypersurfaces, using an approach of Schauder estimates from Simon [41]. We define an \( N_c \)-f domain, where \( f \) is a given function generating from the PMC equation. Combining this condition with a sufficiently mean convex assumption the blow-up method yields corresponding solutions to these PMC Dirichlet problems. Such \( N_c \)-f assumption is almost optimal by an example. An application of our result into the PMC Plateau problem is also presented.

1. Introduction

In this paper we will study the prescribed mean curvature (PMC) graph in Riemannian manifolds under a fixed Dirichlet data, i.e, the Dirichlet problem of some types of PMC equations from a geometrical viewpoint.

The first motivation comes from the connection between PMC equations and the PMC Plateau problem of surfaces in 3-manifolds by Gulliver-Spruck [20] and Giusti [17]. Combining their work together, in 3-manifolds, the existence of PMC graphs is closely related to the existence of PMC disks (with fixed Jordan curve). See Theorem 7.7 and Theorem 7.6.

The second motivation of our study is the PMC graph of functions to solve (1.1) below which appears naturally from minimal graphs in the conformally product manifold (see Remark 6.4). In fact those functions usually satisfy a general form of PMC equations as follows:

\[
\begin{align*}
-\text{div} \left( \frac{Du}{\omega} \right) + G(x, u, -\frac{Du}{\omega}, \frac{1}{\omega}) &= 0 \\
\omega &= \sqrt{1 + |Du|^2}
\end{align*}
\]

on a \( C^2 \) domain \( \Omega \) in \( N \). Here \((x, u)\) denotes the position and \((\frac{-Du}{\omega}, \frac{1}{\omega})\) reflects the normal vector of PMC graphs in product manifolds.

The existence of the solution to (1.1) with Dirichlet data in Riemannian manifold can be explained as to find PMC graphs in product manifolds (with a fixed boundary) as a special case to solve the PMC Plateau problem with an appropriate formulation. However, this explanation is little discussed.
in literature. Within our knowledge, most conditions on the underlying domain for the PMC equations for example (but not limited to), are purely analytic and have little geometric correspondence.

To seek a more geometrical condition to investigate, we turn to the existence of Jang graphs (a special type of PMC graphs) which appear in the proof of the positive mass theorem by Schoen-Yau. They use a blow-up method. A key observation of this method is that the non-existence of apparent horizons in the underlying domain implies the global existence of a Jang graph over this domain under the natural setting in general relativity. This inspires us to propose a new assumption to include non-existence of apparent horizons as a special case.

Suppose $M$ is an $n$-dimensional Riemannian manifold and $f$ is a $C^{1,\alpha}$ function in the tangent bundle $TM$. When $7 \geq n \geq 2$, we say a $C^{2,\alpha}$ domain $\Omega$ in $M$ has the $Nc$-f property or is an $Nc$-f domain if there is no $C^{3,\alpha}$ domain $E$ in the closure of $\Omega$ with mean curvature $f(\cdot, \vec{v})$ on the whole $\partial E$ or $-f(\cdot, -\vec{v})$ on the whole $\partial E$. Here $\vec{v}$ is the outward normal vector of $E$ and the mean curvature of $\partial E$ is the divergence of $\vec{v}$.

A simple version of our main result in Theorem 6.1 is stated as follows.

**Theorem 1.1.** Let $\Omega$ be a $C^2$ compact $Nc$-f domain in an $n$-dimensional Riemannian manifold where $f$ is a $C^{1,\alpha}$ function on the tangent bundle of $\Omega$ and $2 \leq n \leq 7$. Suppose the mean curvature of the boundary of $\partial \Omega$ with respect to the outward normal vector, $H_{\partial \Omega}$, satisfies

$$H_{\partial \Omega} \geq \max\{f(x, \gamma), -f(x, -\gamma)\}$$

where $\gamma$ is the outward normal vector of $\Omega$ at $x$ and the Dirichlet problem takes the form of

$$\begin{cases}
-\text{div} \left( \frac{Du}{\omega} \right)(x) - F(x, -\frac{Du}{\omega}, \frac{1}{\omega}) + \phi(x, u, -\frac{Du}{\omega}, \frac{1}{\omega}) - \frac{1}{\omega} = 0 & \text{on } \Omega \\
u = \psi & \text{on } \partial \Omega
\end{cases}$$

with $\frac{\partial \phi}{\partial n} \geq 0$ and $F(x, X, 0) = f(x, X)$ for $x \in M, X \in T_x M$. Here $F, \phi$ are $C^{1,\alpha}$ with respect to its arguments and $\phi$ is either uniformly bounded or $\frac{\partial \phi}{\partial n} \geq \beta > 0$ for a fixed constant $\beta$.

Then the Dirichlet problem admits a unique solution in $C^{3,\alpha}(\Omega) \cap C(\Omega)$ and for any $\psi \in C(\partial \Omega)$.

We give some remarks on this theorem. The condition $\frac{\partial \phi}{\partial n} \geq 0$ guarantees the uniqueness part. In Theorem 5.7 for fixed any constant function $F$ and $\phi \equiv 0$ we construct a compact manifold in which all conditions of Theorem 1.1 are satisfied except the $Nc$-f property, no solution to the PMC equation in 1.1 exists. This implies that the $Nc$-f property is almost optimal. Applying comparison techniques, bounded domains in Euclidean spaces, Hyperbolic spaces with sufficiently mean convex boundaries are $Nc$-f domains by Theorem 5.3. A special case of PMC graphs from the Dirichlet problem 1.3,
$F \equiv 0, \phi$ is a constant, has been considered by the author in [14] and [50] with a different technique from Giusti [18]. We refer to a continuous method by Casteras-Heinonen-Holopainen-Lira [43] Theorem 1.4 and Spruck [4] Theorem 1.1 in which their assumptions are almost but not completely covered by the Nc-f property (Theorem 5.3). At last, the theorem above also holds when $n > 7$ by adding a singular set requirement in the definition of the Nc-f property in Definition 5.1. See Theorem 6.1 for a complete version.

An application of Theorem 1.1 is that we obtain new manifolds to solve the PMC Plateau problem in 3-manifolds in the setting of Gulliver-Spruck [20]. More study in this topic is working on by Lizhi Chen and the author in another paper [5]. And the Nc-f property is an important condition we will use.

In (1.3) the term $F - \frac{\phi}{\omega}$ can represent many PMC functions, for example, constant mean curvature $c$, $tr(k) - k(Du, Du)$ (the Jang equation) for a symmetric $(0, 2)$-tensor $k$ (see Remark 6.4). For some earlier results on minimal surface equations in Euclidean spaces and hyperbolic spaces we refer to [24, 32, 19]. For the Dirichlet problem of more general PMC functions, we refer to Bergner [3] with a strict condition on the domain $\Omega$. The Dirichlet problem of translating mean curvature equations, $F \equiv 0$ and $\phi$ is a constant, in Euclidean spaces appears in [47, 48, 33] to study the type-II singularity of the mean curvature flow.

The key ingredient to show Theorem 1.1 is the blow-up method in [39] which shows the existence of Jang graphs in the positive mass theorem. Their basic idea to find the solution to (1.3) can be summarized in the following two steps:

1. First consider an auxiliary PMC equation

   \begin{equation}
   (1.4) \quad -\text{div} \left( \frac{Du}{\omega} \right) - F(x, \frac{Du}{\omega}, \frac{1}{\omega}) + \phi(x, u, \frac{Du}{\omega}, \frac{1}{\omega}) + q(t, u, \frac{1}{\omega}) = 0
   \end{equation}

   where $q$ is a smooth function with its arguments satisfying $\frac{\partial q}{\partial u} \geq \alpha(t) > 0$ and $\lim_{t \to 0} q(t, u, \frac{1}{\omega}) = 0$ uniformly when $|u| \leq K$. Under various conditions, there is a classical solution to (1.4), written as $u_t$;

2. Then letting $t$ approach 0 yields the existence of the solution to (1.3) via a series of techniques from PDE and geometric measure theory.

In this paper we choose $q(t, u, \frac{1}{\omega})$ as $\frac{tu}{\omega}$ instead of $tu$ in [39] (4.1). The main difference is that the latter one requires that (1.2) holds strictly to deal with the boundary value of $u$ but in our case (1.2) is sufficient because of Theorem A.3. This will give more room when we consider the blow-up process in domains with piecewise $C^2$ boundary of which mean curvature in its regular part is equal to $f$ only depending its position. This setting happens frequently in the PMC Plateau problem of surfaces in 3-manifolds discussed in our future work.

To realize the first step above, we apply a Perron method from Eichmair
to solve (1.4) (see also Ju-Liu [25]). In the second step above the graph of $u_t$ can be viewed as a graph with bounded mean curvature (called as a $\Lambda$-perimeter minimizer). We apply an curvature estimate in Theorem 3.3 which generalizes a result of Simon [41, Theorem 1] on a curvature estimate of minimal boundary. Its main method is from the Schauder estimate of elliptic equations. For an alternative curvature estimate we refer to the work of Schoen-Simon [38] observed by Eichmair [11, Appendix A]. Our curvature estimate should have independent interest.

In this blow-up process an interior estimate of mean curvature type equations is also established in Theorem A.2. This generalizes various special cases (see [3, 29, 27, 28, 7, 46, 13, 43, 35] etc.) for any $C^2$ function $u$ satisfying (1.1) where $G = G(x, u, X, r)$ is a $C^{1,\alpha}$ function satisfying a monotone condition $\frac{\partial G}{\partial u} \geq 0$.

This paper is organized as follows. In section 2 we collect some preliminary facts on $\Lambda$-perimeter minimizers. In section 3 we establish the curvature estimates in Theorem 3.3. In section 4 we consider an auxiliary Dirichlet problem in Theorem 4.1. In section 5 we discuss the $\text{Nc-f}$ property and provide two examples. In section 6 we proceed the blow-up process and establish Theorem 6.1. In section 7 we apply Theorem 6.1 into the PMC Plateau problem and extend the results of Gulliver-Spruck [20] into Riemannian manifolds. In appendix A we collect some results on PMC equations in Riemannian manifolds.

This project is supported by NSFC no. 11801046 and the Natural Science Foundation of Chongqing, China no. cstc2021jcyj-msxmX0430.

2. Preliminary

A well-known fact is that PMC graphs with bounded mean curvature are $\Lambda$-perimeter minimizers in product manifolds (Lemma 2.11). In this section we collect some facts on perimeters needed for later usage. Our main references are the book of Maggi [34], Giusti [18] and Simon [42]. In those references, most of the statements are described in Euclidean spaces. Their Riemannian versions can be easily obtained by little efforts. In what follows if no confusion we directly state them in Riemannian manifolds without proof.

Fix $n \geq 2$. Let $N \subset \mathbb{R}^{n+k}$ be an $n$-dimensional Riemannian manifold with possible boundary and $TN$ be its tangent bundle. Let $dvol, div, \langle \cdot, \cdot \rangle$ denote the volume form, the divergence, and the inner product of $N$ respectively. Let $B_r(x) \subset N$ be an embedded ball $B_r(x) \subset N$ centering at $x$ with sufficiently small radius $r$. Let $W$ be an open set in $N$. We say a set $G \subset W$ if the closure of $G$ is a compact set in $W$. Let $E$ be a Borel set in $N$ and $\lambda_E$ denotes its characteristic function. Let $H^l$ be the $l$-dimensional Hausdorff measure in $\mathbb{R}^{n+k}$.
Definition 2.1 (P. 122, [34]). Fix an open set $\Omega$ in $\mathbb{R}^n$. We say $E$ is a locally finite perimeter set or a Caccioppoli set if for any compact set $K \subset \Omega$

\begin{equation}
\sup_{spt(X) \subset K} \{ \int_{\Omega} \lambda E div(X) d\text{vol} : \langle X, X \rangle \leq 1 \} < \infty
\end{equation}

is finite. The perimeter of $E$ in $\Omega$ is given by

\begin{equation}
P(E, \Omega) = \sup_{spt(X) \subset \Omega} \{ \int_{\Omega} \lambda E div(X) d\text{vol} : \langle X, X \rangle \leq 1 \}
\end{equation}

If two Caccioppoli sets differ in a Lebesgue measure set their properties are unchanged. Thus we say such two Caccioppoli sets are equivalent. For example, if a Caccioppoli set is not empty, then the volume of such set is positive. In fact we can always choose a unique representation in the equivalent class of a Caccioppoli set with the property in (2.3).

Proposition 2.2 (Proposition 3.1, [18]). Suppose $E$ is a Borel set in $\mathbb{R}^n$, there is a Borel set $\tilde{E}$ (differs with $E$ only a measure zero set) satisfying

\begin{equation}
0 < H^n(\tilde{E} \cap B_r(x)) < H^n(B_r(x))
\end{equation}

for any $x$ in $\partial \tilde{E}$ and sufficiently small $r > 0$.

Remark 2.3. Throughout this paper $\{ x \in \mathbb{R}^n : 0 < H^n(\tilde{E} \cap B_r(x)) < H^n(B_r(x)) \text{ for some } r > 0 \}$ is called the measure-theoretical boundary of $E$, written as $\partial E$.

Definition 2.4. [34, Page 278]. Fix a nonnegative constant $\Lambda$. We say a Caccioppoli set $E$ is a $\Lambda$-perimeter minimizer in an open set $W \subset \mathbb{R}^n$ if for any Caccioppoli set $F$ satisfying $F \Delta E \subset W$ it holds that

\begin{equation}
P(E, W) \leq P(F, W) + \Lambda \text{vol}(E \Delta F)
\end{equation}

Define $\text{reg}(\partial E)$ as the set $\{ x \in \partial E : \partial E$ is a $C^{1,\gamma}$ graph in a neighborhood of $x$ for some $\gamma \in (0, \frac{1}{2})\}$ and $\text{sing}(\partial E) := \partial E \setminus \text{reg}(\partial E)$.

Remark 2.5. If we choose $W$ as the embedded ball $B_r(p)$, a $\Lambda$-perimeter minimizer $E$ is a $(\Lambda, r)$-perimeter minimizer. By [34, Theorem 26.5] the reduced boundary of $\Lambda$-perimeter minimizer is just $\text{reg}(\partial E)$. Moreover in [34] all local properties of $(\Lambda, r)$-perimeter minimizers hold for $\Lambda$-perimeter minimizers.

Theorem 2.6 (Theorem 17.7, [42]). Fix $\nu > 0$. Suppose $K$ is compact in the open set $W \subset \mathbb{R}^n$ and the injective radius of $W$, $\text{inj}(W)$, is positive. Then there is a $d_0 = \min\{\text{dist}(W, \partial N), \text{inj}(W)\}$ and $\kappa$ depending only $\nu, K$ for any $C^2$ hypersurface $\Sigma$ with mean curvature $H_\Sigma$ satisfying $|H_\Sigma| \leq \nu$ it holds that

\begin{equation}e^{\kappa \sigma} \frac{H^{n-1}(\Sigma \cap B_\sigma(p))}{\omega_{n-1} \sigma^{n-1}}\end{equation}

is increasing for all $\sigma \in (0, d_0)$ for any $p$ in $K$. 
Usually, it is hard to detect the regular set in the boundary of a \(\Lambda\)-perimeter minimizer. But there is one exception as follows.

**Lemma 2.7.** Let \(E\) be a Caccioppoli set in a \(C^2\) bounded domain \(\Omega \subset \mathbb{N}\) such that \(\partial E\) is tangent to \(\partial \Omega\) at \(p\). If in a neighborhood of \(p\), \(E\) is a \(\Lambda\)-perimeter minimizer for some positive constants \(\Lambda\). Then \(p\) is contained in \(\text{reg}(\partial E)\).

**Proof.** By the Nash embedding theorem, we assume \(\mathbb{N}\) is an \(n\)-dimensional embedded submanifold in \(\mathbb{R}^{n+k}\). Consider an isometry \(T_\lambda(x) = \frac{x-p}{\lambda}\). Following [44, Lemma 3.3] there is a sequence \(\{\lambda_i\}_{i=1}^\infty\) with \(\lim_{i \to \infty} \lambda_i = 0\) such that \(T_\lambda(E)\) converges locally to a minimal cone \(C\) in \(T_p \mathbb{N} = (\mathbb{R}^n)\). However \(E \subset \Omega\), \(p \in \partial \Omega\) and \(\Omega\) is \(C^2\) near \(p\). Then \(C\) is contained in a half-space of \(T_p \mathbb{N}\). By [18, Lemma 15.5], \(C\) is a half-space. Thus the Hausdorff density of \(\partial E\) at \(p\) is 1. By the Allard regularity theorem ([32, Theorem 24.2]), \(p\) belongs to \(\partial E\). \(\square\)

From Federer’s dimension reduction process and Simons’ minimizing cone, the following regular property holds for \(\Lambda\)-perimeter minimizers.

**Theorem 2.8** (Theorem 28.1, [34]). Let \(N\) be an \(n\)-dimensional Riemannian manifold. Suppose \(E\) is a \(\Lambda\)-perimeter minimizer in an open set \(W \subset \mathbb{N}\). Then

1. if \(2 \leq n \leq 7\), \(\text{sing}(\partial E)\) is empty;
2. if \(n = 8\), \(\text{sing}(\partial E)\) is a discrete set;
3. if \(n \geq 9\), \(H^s(\text{sing}(\partial E)) = 0\) for any \(s > n - 8\).

**Definition 2.9.** We say a sequence of Caccioppoli sets \(\{E_i\}_{i=1}^\infty\) locally converges to a Caccioppoli set \(E\) in an open set \(\Omega \subset \mathbb{N}\) if for any Borel set \(A \subset \subset \Omega\)

\[
\lim_{i \to +\infty} \int_A |\lambda E - \lambda E_i| \text{dvol} = 0
\]

where \(\text{dvol}\) is the volume form of \(\mathbb{N}\).

There are three good properties for the convergence of \(\Lambda\)-perimeter minimizers.

**Lemma 2.10.** Suppose \(\{E_j\}_{j=1}^\infty\) is a sequence of \(\Lambda\)-perimeter minimizers in an open set \(W \subset \mathbb{N}\). Then

1. (Proposition 21.13 and Theorem 21.14 in [34]) there is a \(\Lambda\)-perimeter minimizer \(E\) such that a subsequence of \(\{E_j\}_{j=1}^\infty\), still denoted by \(\{E_j\}_{j=1}^\infty\), converges locally to \(E\) in \(W\);
2. Fix \(B_{r_0}(p)\) as an embedded ball in \(W\). Then there is a dense set \(I\) in \((0, r_0)\) for any \(r\) in \(I\) such that

\[
\lim_{j \to \infty} P(E_j, B_r(p)) = P(E, B_r(p))
\]
(3) (Theorem 1 in [44] and Theorem 26.6 in [34]) Suppose \( v \) is the outward normal vector of \( \partial E \) at some point \( x \) in \( \text{reg}(\partial E) \), \( \{x_j : x_j \in \partial E_j\}_{j=1}^{\infty} \) converges to \( x \), then there is a positive integer \( j_0 \) such that there is a neighborhood of \( x \), \( W \), such that for any \( j \geq j_0 \), \( \partial E_j \) is regular in \( W \), \( \{x_j \in W : x_j \in \partial E_j\}_{j=1}^{\infty} \) converges to \( x \) and \( \{v_j\}_{j=j_0}^{\infty} \) converges to \( v \) in the tangent bundle \( TN \). Here \( v_j \) is the normal vector of \( \partial E_j \) at \( x_j \).

**Proof.** We only show the conclusion (2). Since \( \lambda_{E_j} \) converges to \( \lambda_E \) a.e., the coarea formula implies that

\[
\lim_{i \to \infty} H^{n-1}(\partial B_i(p) \cap (E_j \Delta E)) = 0
\]

for any \( r \) in a dense set \( I \subset (0, r_0) \). The semicontinuity of the perimeter implies that

\[
P(E, B_r(p)) \leq \liminf_{j \to \infty} P(E_j, B_r(p))
\]

for any \( r \in (0, r_0) \). Set \( \tilde{E}_j = E \cap B_r(p) \cup E_j \setminus B_r(p) \), by our definition

\[
\lim_{j \to \infty} \tilde{E}_j \Delta E_j = \emptyset,
\]

Choose any \( r' \in (r, r_0) \). By Definition [24] we have

\[
P(E_j, B_{r'}(p)) \leq P(\tilde{E}_j, B_{r'}(p)) + \Lambda \text{vol}(E_j \Delta \tilde{E}_j)
\]

On the other hand by the trace formula [18] Proposition 2.8 we have

\[
P(\tilde{E}_j, B_{r'}(p)) = P(E, B_r(p)) + P(E_j, B_{r'}(p)) \setminus \partial B_{r'}(p))
\]

\[
+ H^{n-1}((E_j \Delta E) \cap \partial B_r(p))
\]

Combining (2.11) and (2.12), we obtain

\[
P(E_j, B_r(p)) \leq P(E, B_r(p)) + \Lambda \text{vol}(E_j \Delta \tilde{E}_j)
\]

\[
+ H^{n-1}(E_j \Delta E \cap \partial B_r(p))
\]

Recall that we always assume \( n \geq 2 \). For any \( r \) in \( I \) by (2.8) and letting \( j \to \infty \) we obtain

\[
\lim_{j \to +\infty} \sup_{j} P(E_j, B_r(p)) \leq P(E, B_r(p))
\]

Putting (2.9) and (2.14) together we obtain the conclusion. \( \square \)

Next, we give an example of \( \Lambda \)-perimeter minimizers for later use. For more examples see [34] section 21.1. For any function \( f \) let \( \text{gr}(f) \) denote the graph of \( f \). Let \( N \times \mathbb{R} \) denote the product manifold. We use the idea and concepts of Eichmair in [10] Example A.1 to obtain the following result.

**Lemma 2.11.** Suppose \( \Omega \) is a \( C^2 \) bounded domain in \( N \), \( u \) is a \( C^2 \) function on \( \Omega \) such that

\[
|\text{div}(\frac{Du}{\omega})| \leq \mu \text{ on } \Omega,
\]

\[
\text{div}(\frac{Du}{\omega}) = \mu \text{ on } \Omega,
\]

\[
\text{div}(\frac{Du}{\omega}) = \mu \text{ on } \partial \Omega.
\]
where $\omega = \sqrt{1 + |Du|^2}$, $\mu$ is a fixed positive constant. Define its subgraph $U = \{(x, r) : x \in \Omega, r < u(x)\}$.

Then there are two positive constants $\delta := \delta(\Omega)$, $\Lambda := \Lambda(\Omega, \mu)$ such that $U$ is a $\Lambda$-perimeter minimizer in $\Omega_\delta \times \mathbb{R}(\partial gr(u))$. Here $\partial gr(u)$ is the boundary of the graph of $u$ over $\Omega$, $\Omega_\delta := \{x \in N, d(x, \Omega) < \delta\}$ and the metric of $N \times \mathbb{R}$ is the product metric.

**Remark 2.12.** Notice that the boundary of $U$ is composed of two parts: $gr(u)$ in $\Omega \times \mathbb{R}$ and the set $\{(x, t) : x \in \partial \Omega, t < \partial gr(u)\}$, which belongs to the boundary of the $C^2$ domain $\Omega \times \mathbb{R}$. Our derivation does not work in a neighborhood of $\partial gr(u)$. Here we do not require $u$ to be uniformly bounded on $\Omega$.

Comparing to [10] Example A.1, our $\Lambda$-perimeter minimizers can be defined on some neighborhoods of $\partial \Omega \times \mathbb{R} \setminus \partial gr(u)$.

**Proof.** Let $d(x, y)$ be the distance between two points $x, y$ in $N$. Define $d(x)$ as the function $\text{sign}(x)d(x, \partial \Omega)$ in a neighborhood of $\partial \Omega$ satisfying $\text{sign}(x) = 1$ when $x \in \Omega$ and otherwise $\text{sign}(x) = -1$. Define a set $\Gamma_\delta := \{x \in N : -\delta < d(x) \leq \delta\}$ such that for any $y \in \Gamma_\delta$ there is only one $x \in \partial \Omega$ such that $d(x, y) = d(y, \partial \Omega)$. Since $\partial \Omega$ is $C^2$, then so is $d(x)$. Then we extend $d(x)$ into a function in $\Gamma_\delta \times \mathbb{R}$ as $d(q) = d(x)$ where $q = (x, r)$ in $\Gamma_\delta \times \mathbb{R}$. Let $\text{div}$ be the divergence of $N \times \mathbb{R}$. Let $n$ be the dimension of $N$. We define an $n$-form on $\Gamma_\delta \times \mathbb{R}$

\begin{equation}
\sigma_0 = Dd_\omega d\text{vol}
\end{equation}

where $D$ and $\text{vol}$ are the gradient and the volume form of $N \times \mathbb{R}$ respectively. It is easy to see that

\begin{equation}
d\sigma_0 = \text{div}(Dd)d\text{vol}, \quad |\text{div}(Dd)| \leq c(\delta)
\end{equation}

in $\Gamma_\delta \times \mathbb{R}$.

Let $\vec{v}$ be the upward normal vector of $gr(u)$ in $N \times \mathbb{R}$. We extend it into a unit vector field $X$ in $\Omega \times \mathbb{R}$ via vertical translation. Namely

\begin{equation}
X(x, u(x) + t) = \vec{v}(x, u(x))
\end{equation}

Define

\begin{equation}
\sigma_1 = X_\perp(d\text{vol})
\end{equation}

By the definition of the divergence and (2.15), for any point $q = (x, u(x) + r) \in \Omega \times \mathbb{R}$ we have

\begin{equation}
d\sigma_1(q) = -\text{div}(\frac{Du}{\omega})d\text{vol}(q)
\end{equation}

in $\Omega \times \mathbb{R}$.

Suppose $F$ is any Caccioppoli set such that $F \Delta U \subset \subset \Omega_\delta \times \mathbb{R}(\partial gr(u))$. Since $F \cap (\Omega \times \mathbb{R}) \Delta U \subset \subset \Omega_\delta \times \mathbb{R}(\partial gr(u))$, by [15] Theorem 3.9, there is an $(n+1)$-integral current $[[V]]$ in $\Omega_\delta \times \mathbb{R}$ such that

\begin{equation}
\partial[[V]] = \partial[[F \cap (\Omega \times \mathbb{R})]] - \partial[[U]], \quad \text{spt}([[V]]) \subset F \cap (\Omega \times \mathbb{R}) \Delta U
\end{equation}
Here $k$ is an integer and $V$ is a Borel set in $\Omega_3 \times \mathbb{R}$.

Let $W$ be an open bounded set satisfying $F \Delta U \subset W$ disjoint with $gr(\psi)$. Let $D_0^n(W)$ be the collection of the $n$-smooth forms $X_\varepsilon$ with compact support in $W$ for some $\varepsilon > 0$ satisfying

1. $(X_\varepsilon, X_\varepsilon) \leq 1, V \subset \sup(X)$;
2. $X_\varepsilon = \sigma_1$ in the intersection of a neighborhood of $V$ and $\{(x, r) : x \in \Omega, \text{ dist}(x, \partial \Omega) > \varepsilon\}$ for some $\varepsilon > 0$ (if not empty);
3. $X_\varepsilon = \sigma_0$ in the intersection of a neighborhood of $V$ and $\{(x, r) : \text{ dist}(x, \partial \Omega) < \frac{\varepsilon}{2}\}$ (if not empty);

Let $M$ denote the mass of integral currents. By Definition

\[ (2.21) \quad P(F \cap \Omega \times \mathbb{R}, W) = M_W(\partial||U|| + \partial||V||) \geq (\partial||U|| + \partial||V||)(X_\varepsilon) \]

for any $X_\varepsilon \in D_0^n(W)$. Notice that $\partial U$ is $C^2$ in $W$ because $W$ is disjoint with $\partial gr(u)$, it is easy to find a sequence $\{X_\varepsilon\}_{\varepsilon > 0}$ in $D_0^n(W)$ such that

\[ (2.22) \quad \lim_{\varepsilon \to 0} \partial||U|| (X_\varepsilon) = P(U, W) \]

For the same sequence $\{X_\varepsilon\}_{\varepsilon > 0}$, by (2.20) and (2.15) we have

\[ (2.23) \quad \lim_{\varepsilon \to 0} \partial||V|| (X_\varepsilon) = \lim_{\varepsilon \to 0} - \int_V d(X_\varepsilon) d\text{vol} \]

\[ = - \int_V d\sigma_1 \geq -\mu \text{vol}(F \cap (\Omega \times \mathbb{R}) \Delta U) \]

Combining (2.21) with (2.22) and (2.23) we obtain

\[ (2.24) \quad P(F \cap (\Omega \times \mathbb{R}), W) \geq P(U, W) - \mu \text{vol}((F \cap (\Omega \times \mathbb{R})) \Delta U) \]

If $F \Delta U \subset \bar{\Omega} \times \mathbb{R}$, (2.24) directly gives the conclusion in the claim just letting $\Lambda = \mu$. Otherwise $F \cap (\Omega \times \mathbb{R})$ is not equal to $\Omega \times \mathbb{R}$. In (2.21), (2.22) and (2.23), we replace $F \cap (\Omega \times \mathbb{R})$ with $F \cup (\Omega \times \mathbb{R})$, $U$ with $\Omega \times \mathbb{R}$ and $X_\varepsilon$ with $\sigma_0$ respectively. The same derivation gives that

\[ (2.25) \quad P(F \cup (\Omega \times \mathbb{R}), W) \geq P((\Omega \times \mathbb{R}), W) - c(\delta) \text{vol}(F \cup (\Omega \times \mathbb{R}) \Delta (\Omega \times \mathbb{R})) \]

Combining (2.24) with (2.25) we obtain

\[ (2.26) \quad P(U, W) + P(\Omega \times \mathbb{R}, W) \leq P(F \cap (\Omega \times \mathbb{R}), W) + P(F \cup (\Omega \times \mathbb{R}), W) \]

\[ + \max\{c(\delta), \mu\} \text{vol}(F \Delta U) \leq P(F, W) + P(\Omega \times \mathbb{R}, W) + \Lambda \text{vol}(F \Delta U) \]

Here $\Lambda$ is the constant $\max\{c(\delta), \mu\}$ and in the second line we apply \[18\] Lemma 15.1.

By the definition of $\Lambda$-perimeter minimizers the proof is complete. \[\square\]

3. Curvature estimates

In this section, we extend Simon’s curvature estimates \[41\] Theorem 1] on minimal boundaries into the case of the boundary of PMC $\Lambda$-perimeter minimizers.

Let $N$ be an $n(\geq 2)$-dimensional Riemannian manifold with a metric $g$. 
Fix \( \alpha \in (0,1) \) and a set \( N_0 \subset N \). By \([30]\) Page 44 the Hölder space \( C^{1,\alpha}(N_0) \) is the collection of \( C^{1,\alpha} \) functions \( u \) in a neighborhood of \( N_0 \), on which \( C^{1,\alpha} \) norm

\[
||u||_{C^{1,\alpha}(N_0)} = \sup_{x \in N_0} \{|u|(x) + |\nabla u|(x) + \sup_{x,y \in N_0, y \neq x} \frac{|\nabla u(x) - \nabla u(y)|}{d^\alpha(x,y)}\}
\]
is finite. Here the second supremum is taken over all \( x \neq y \) in \( N_0 \) such that \( y \) is in a normal coordinate of \( x \) and \( \nabla u(y) \) is the tensor at \( x \) obtained from \( \nabla u \) at \( y \) by parallel transport along the radial geodesic from \( y \) to \( x \).

Let \( M \) be a \( m(\geq 2) \)-dimensional manifold with a metric \( \sigma \). Let \( TM \) be the tangent bundle with the natural Sasaki metric \( \sigma \) (see [1, 37]) from \( \sigma \). Therefore we have a \( C^{1,\alpha} \) norm for \( C^{1,\alpha} \) functions on \( TM \).

**Definition 3.1.** Let \( W \subset M \) be any open set. Define \( T_u M \) be the unit tangent bundle on \( W \), i.e. \( \{(x,v): x \in W, v \in T_x M, \langle v, v \rangle \leq 1\} \). For a \( C^{1,\alpha} \) function \( f(x,v): TM \rightarrow \mathbb{R} \) is a \( C^{1,\alpha} \) function, we define \( N_W(f) \) as

\[
N_W(f) = ||f||_{C^{1,\alpha}(T_uW)}
\]

**Remark 3.2.** In (3.2) the definition of the Sasaki metric of \( TM \) in general has a complicated form. But this definition is not hard to understand as follows. First suppose \( W \subset \mathbb{R}^m \), then any \( C^{1,\alpha} \) function \( f(x,v) \) on \( T_u W \) can be written as \( f(x^1, \ldots, x^m, v^1, \ldots, v^m) \). Then \( N_W(f) \) is proportional to

\[
\sup_{(x,v) \in T_u W} \{|f| + \sum_{i=1}^m \left| \frac{\partial f}{\partial x_i} \right| + \left| \frac{\partial f}{\partial v^i} \right|}(x,v)
\]

\[
+ \sum_{i=1}^m \left\{ \sup_{(x,v),(y,v) \in T_u W} \frac{|\frac{\partial f}{\partial x_i}(x,v) - \frac{\partial f}{\partial x_i}(y,v)|}{|x-y|^\alpha} \right\}
\]

\[
+ \left\{ \sup_{(x,v),(y,v) \in T_u W} \frac{|\frac{\partial f}{\partial v^i}(x,v) - \frac{\partial f}{\partial v^i}(x,v)|}{|x-v|^\alpha} \right\}
\]

When \( W \) is contained in a normal coordinate, the Sasaki metric on \( TW \) still takes the form

\[
\sigma_s = \sum_{i,j,k,l=1}^m \sigma_{ij} dx^i dx^j + h_{kl} dv^k dv^l + w_{kl} dv^k dv^l
\]

It is not hard to verify that for any \( f \in C^{1,\alpha}(TM) \), \( N_W(f) \) is still proportional to (3.3) up to a constant depending only on (3.4).

The main result of this section is stated as follows.

**Theorem 3.3.** Let \( M \) be a \( m(\geq 2) \)-dimensional Riemannian manifold. Fix any open set \( W \subset M \) and \( K \subset W \) is a compact set. Fix a nonnegative constant \( \Lambda \), a positive constant \( \nu \) and \( \alpha \in (0,1) \). Define \( G \) as the set of

1. when \( 2 \leq m \leq 7 \) all \( C^2 \) \( \Lambda \)-perimeter minimizer \( E \) in \( W \) with the mean curvature of \( \partial E \) equal to a \( C^{1,\alpha} \) function \( f_E(x,\bar{v}) \) satisfying \( N_W(f_E) \leq \nu \);
(2) when \( m \geq 8 \) a sequence of \( C^2 \) \( \Lambda \)-perimeter minimizers \( \{E_i\}_{i=1}^{\infty} \) in \( W \) with the mean curvature of \( \partial E_i \) equal to a \( C^{1,\alpha} \) function \( f_i(x,\vec{v}) \) where \( N_W(f_i) \leq \nu \) and \( \{E_i\}_{i=1}^{\infty} \) converges locally to a Caccioppoli set \( F \) in \( W \) such that \( \text{sing}(\partial F) \) is an empty set.

Then there is a positive constant \( \mu \) such that
\[
(3.5) \quad \sup_{E \in G} \left\{ \max_{\partial E \cap \mathbb{R}^n} |A|^2 \right\} \leq \mu
\]
Here \( A \) is the second fundamental form of \( \partial E \).

Our proof mainly follows from Simon [41 section 1].

Remark 3.4. Letting \( \Lambda = 0 \), Simon [41] established the above result for \( 2 \leq m \leq 7 \) and \( m = 8 \) plus the condition that all sets \( E \) in \( G \) are subgraphs.

In what follows we view \( M \) as an embedded \( m \)-dimensional submanifold with possible boundary \( \partial M \) in the Euclidean space \( \mathbb{R}^{m+k} \) for some \( k \).

3.1. Basic facts. The Allard regularity theorem is one of the fundamental results of integral varifolds. It says if at a point the density of multiplicity one of this point is a \( \in \) integral varifolds is sufficiently close to 1, such varifold in a neighborhood of this point is a \( C^{1,\alpha} \) graph for some \( \alpha \in (0,1) \). The statement of the original version [2, Theorem 8.19] is too complicated for our applications. We record the \( C^2 \) version of [42, Theorem 24.2] without introducing the concept of varifolds. Let \( B_\rho^m(a) \) be the ball \( \{ x : |x - a| < \rho \} \) in the \( s \)-dimensional Euclidean space centered at \( a \) with radius \( \rho \). Let \( \omega_s \) be the volume of \( s \)-dimensional unit ball in \( \mathbb{R}^{s+1} \).

**Theorem 3.5 (Theorem 24.2, [42]).** Fix \( p > n > 0 \) and \( k \). Suppose \( \Sigma \subset \mathbb{R}^{n+k} \) is a \( n \)-dimensional \( C^2 \) embedded submanifold with empty boundary in the embedded Euclidean ball \( B_\rho^{n+k}(q), q \in \Sigma \) satisfying that
\[
(3.6) \quad \frac{\mathcal{H}^n(B_\rho^{n+k}(q) \cap \Sigma)}{\omega_n \rho^n} \leq 1 + \delta, \quad \left( \int_{B_\rho^{n+k}(q)} |\vec{H}|^p d\Sigma \right)^{\frac{1}{p}} \rho^{1-\frac{n}{p}} \leq \delta
\]
where \( d\Sigma \) is the area form of \( \Sigma \) and \( \vec{H} \) is the mean curvature vector of \( \Sigma \) in \( \mathbb{R}^{n+k} \).

Then there are two constants \( \delta = \delta(n,k,p), \gamma = \gamma(n,k,p) \in (0,1) \) such that \( 3.6 \) implies the existence of the linear isometry \( q_1 \in \mathbb{R}^{n+k} \) and \( u \in C^{1,\frac{1}{\gamma}}(\mathbb{R}^{n+k},\mathbb{R}^{k}) \) with \( u(0) = 0, \Sigma \cap B_{\gamma \rho}^{n+k}(q) = q_1(\text{gr}(u)) \cap B_{\gamma \rho}^{n+k}(q) \) and
\[
(3.7) \quad \rho^{-1} \sup_{B_\rho} |u| + \sup_{B_\rho} |Du| + \rho^{1-\frac{n}{p}} \sup_{x,y \in B_{\gamma \rho}(0)x \neq y} \frac{|Du(x) - D(y)|}{|x - y|^{1-\frac{n}{p}}} \leq c \delta^n
\]
where \( c = c(n,k,p) \).

**Lemma 3.6.** Let \( W \) be an open set in a \( m(\geq 2) \)-dimensional Riemannian manifold \( M \) and \( K \subset W \) be a compact set. Fix constants \( \Lambda, \nu, \alpha \in (0,1), \tau_0 < \text{dist}(W,\partial M) \). Let \( \delta = \delta(m,k,p) \) be the constant given in Theorem 3.5. Let \( G \) be the set of \( C^2 \) \( \Lambda \)-perimeter minimizers in Theorem 3.5.
Then there is a \( \rho_1 = \rho_1(\Lambda, \nu, r_0, \delta, K) \) such that for any \( E \in \mathcal{G} \), any \( q \in K \cap \partial E \) (3.6) holds on the embedded ball \( B_\rho(q) \) for any \( \rho \in (0, \rho_1) \).

We mainly follow the idea of [41, section 1].

**Proof.** Let \( \delta = \delta(m, k, p) \) be the constant given in Theorem 3.5. \( K \subset W \) is compact. Using a finite collection of balls covering \( K \), by Definition 2.4 there is a universal constant \( A \) such that

\[
H^{m-1}(\partial E \cap B_\rho(p)) \leq A
\]

for any \( p \in K \) and any \( E \in \mathcal{G} \). Here \( \dim M = m (\geq 2) \). Let \( n = m - 1 \) and \( k' = k + 1 \). By the definition of \( \mathcal{G} \), for each \( E \in \mathcal{G} \), its mean curvature \( |H_{\partial E}| \leq \nu \) on \( W \). Therefore there is a \( \rho_1 < r_0 \) such that

\[
(\int_{B_\rho^{m+k}(q)} |\vec{H}|^p d(\partial E))^{\frac{1}{p}} \rho^{1-\frac{m-1}{p}} \leq \delta
\]

for any \( \rho \in (0, \rho_1) \), any \( q \in K \) and any \( E \in \mathcal{G} \). This is the second inequality in (3.6).

We argue the first inequality in (3.6) by contradiction. Suppose no \( \rho_0 \in (0, \rho_1] \) as above such that the first inequality in (3.6) holds for some \( E \in \mathcal{G} \) and some \( \rho \in (0, \rho_0) \). Then there is a triple sequence \( \{(p_j, E_j, \rho_j) : p_j \in K, E_j \in \mathcal{G}, \rho_j \in (0, \rho_1)\} \) such that

\[
H^{m-1}(\partial E_j \cap B_{\rho_j}(p_j)) \geq \omega_{m-1}(1 + \delta)\rho_j^{m-1},
\]

\[
\lim_{j \to +\infty} p_j = z, \quad \lim_{j \to +\infty} \rho_j = 0
\]

By Lemma 2.10 \( \{E_j\}_{j=1}^\infty \) converges locally to a \( \Lambda \)-perimeter minimizer \( E \) and \( z \in \partial E \).

From the monotonicity formula in Theorem 2.6 we have

\[
e^{\kappa(r - \rho_j)} \frac{H^{m-1}(\partial E_j \cap B_{\rho_j}(p_j))}{\omega_{m-1}(r - \rho_j)^{m-1}} \geq (1 + \delta)
\]

for all \( r \in [\rho_j, \rho_1] \). Here \( \kappa \) is a constant depending on \( K \) and \( \nu \). By (3.11)

\[
H^{m-1}(\partial E_j \cap B_{\rho_j}(z)) \geq e^{\kappa(\rho_j + t_j - r)}(1 + \delta)
\]

for all \( r \in [\rho_j + t_j, \rho_1 + t_j] \). Notice that \( \{\rho_j\}_{j=1}^\infty \) also converges to 0. By the conclusion (2) of Lemma 2.10 there is a \( r_1 < \rho_1 \) and a dense set \( I \in (0, r_1) \) such that for any \( r \in I \) we have

\[
H^{m-1}(\partial E \cap B_{\rho}(z)) \geq e^{-\kappa r}(1 + \delta) \geq 1 + \frac{\delta}{2}
\]

When \( 2 \leq m \leq 7 \), \( \text{sing}(\partial E) \) is empty in \( W \) by Theorem 2.8. When \( m \geq 8 \), \( \text{sing}(\partial E) \) is empty by the definition of \( \mathcal{G} \) in Theorem 3.3. No matter which
case, \( \partial E \) is regular in \( W \). Thus \( \partial E \) is embedded and \( C^1, \gamma \) in \( W \) for some \( \gamma \in (0, \frac{1}{2}) \). Therefore

\[
\lim_{r \to 0} \frac{H^{m-1}(\partial E \cap B_r(z))}{\omega_{m-1} r^{m-1}} = 1
\]

This is a contradiction to (3.14). Thus we obtain the desirable conclusion. 

\[\square\]

### 3.2. The proof of Theorem 3.3.

**Proof.** Let \( W \) be an open set of \( M \subset \mathbb{R}^{m+k} \) and \( K \subset W \) be compact. Let \( \mathcal{G} \) be the collection of \( C^2 \) \( \Lambda \)-perimeter minimizers \( E \) satisfying the condition (1) or (2) in Theorem 3.3.

By Lemma 3.6 there is a \( \rho = \rho(\Lambda, \nu, r_0, \delta, K) \) such that for any \( E \in \mathcal{G} \), any \( p \in K \cap \partial E \), (3.6) holds on \( B_\rho(p) \). Therefore there is a constant \( \gamma \) in \((0, 1)\) such that (3.7) holds on \( B_\gamma(p) \). Namely for any two points \( p_1, p_2 \) in \( B_\gamma(p) \cap \partial E \),

\[
|\vec{v}(p_1) - \vec{v}(p_2)| \leq C(n, k, \rho, \gamma) \delta \frac{m-1}{4} \text{dist}(p_1, p_2)
\]

where on the left-hand side we use the Euclidean distance in \( \mathbb{R}^{m+k} \), in the right-hand side \( \text{dist} \) is the intrinsic distance of \( M \). Here \( \delta \) is the fixed constant in (3.6).

Because \( K \subset W \) is compact, the following fact is easily obtained.

**Lemma 3.7.** There is a positive radius \( \rho_1 < \min \{ r_0, \gamma \rho \} \) such that for any \( p \in K \) on \( B_{\rho_1}(p) \) there is a local coordinate \( \{ u_1, \cdots, u_m \} \) such that \( g_{ij} = \frac{\partial}{\partial u_i} \cdot \frac{\partial}{\partial u_j} \) satisfying the metric \( g_{ij}(p) = \delta_{ij}, g_{mm} \equiv 1, g_{mi} = 0 \) for all \( i = 1, \cdots, m \) and

\[
C^{-1} I \leq (g_{ij})(p) \leq CI \quad \rho_1^{1+\alpha} \| g_{ij} \|_{C^{1, \alpha}(B_{\rho_1}(p))} \leq C,
\]

Here \( i, j, k = 1, \cdots, m \) for any \( p \) in \( B_{\rho_1}(p) \), \( C \) is a constant only depending on \( K \) and \( I \) is the identity matrix.

We denote \( X^p_m \) by the unit vector field \( \frac{\partial}{\sqrt{\det g}} \) in \( B_{\rho_1}(p) \). By Lemma 3.7 there is a uniformly positive constant \( C_1 \) such that

\[
|X^p_m(p_1) - X^p_m(p_2)| \leq C_1 \text{dist}(p_1, p_2)
\]

for any \( p \in K \) and any \( p_1, p_2 \in B_{\rho_1}(p) \). Here \( |,| \) denotes the Euclidean distance in \( \mathbb{R}^{m+k} \).

For any \( E \in \mathcal{G} \) and any \( p \in \partial E \cap K \), we take the coordinate \( \{ u^1, \cdots, u^m \} \) in \( B_{\rho_1}(p) \) as in Lemma 3.7. With a rotation we can require that \( X^E_m(p) \) is the normal vector of \( \partial E \) at \( p \). Let \( \vec{v}_E \) denote the normal vector of \( \partial E \). Combining (3.16) and (3.18) yields that for any \( p_1 \in B_{\rho_1}(p) \), any \( E \in \mathcal{G} \) and \( p \in \partial E \cap K \)

\[
|\langle \vec{v}_E, X^E_m(p) \rangle - \langle \vec{v}_E, X^E_m(p) \rangle| = |\langle \vec{v}_E, X^E_m(p_1) \rangle - 1| \leq 2C_2 \text{dist}(p_1, p)
\]
Here $C_2 = \max\{C(n, k, \rho, \gamma)\delta^{-\frac{m-1}{2}}, C_1\}$. Since $C_2$ is a fixed constant independent of $E$, we choose $\rho_2 < \rho_1$ sufficiently small such for any $E$ in $\mathcal{G}$, any $p \in K \cap \partial E$, any $q \in B_{\rho_2}(p) \cap \partial E$, we have

\[(3.20) \quad \langle \tilde{v}_E, X^p_m \rangle(q) \geq \frac{1}{2}\]

As a result $B_{\rho_2}(p) \cap \partial E$ is written as $\{y, v(y)\}$ where $y = (u_1, \ldots, u_{m-1})$. Moreover $(0, v(0))$ represents the point $p$. By Lemma 3.7, $X^p_m$ is orthogonal to $\{\frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_{m-1}}\}$. Therefore

\[(3.21) \quad \tilde{v}_E = \frac{-g^{ij}v_i \frac{\partial}{\partial u_i} + X^p_m}{\sqrt{1 + g^{ij}v_i v_j}}, \quad \langle \tilde{v}_E, X^p_m \rangle = \frac{1}{\sqrt{1 + g^{ij}v_i v_j}}\]

where $(g^{ij}) = (g_{ij})^{-1}$ is a $(m-1) \times (m-1)$ matrix. Here $v_i, v_{ij}$ denote the first and second derivatives of $v$ with respect to the coordinate $\{u_1, \ldots, u_{m-1}\}$.

By (3.17) and (3.20), there is a constant $C_3$ such that

\[(3.22) \quad \sum_{i=1}^{m-1} v_i^2 \leq C_3\]

where $(u_1, \ldots, u_{m-1}, v(u_1, \ldots, u_{m-1}))$ belongs to $\partial E \cap B_{\rho_2}(p)$. Thus $\partial E \cap B_{\rho_2}(p)$ contains a graph of $v$ over a domain

\[(3.23) \quad U_{C_4} := \{(u^1, \ldots, u^{m-1}) : u_1^2 + \cdots + u_{m-1}^2 \leq C_4\} \text{ in } \mathbb{R}^{m-1}\]

where $C_4$ only depends on $C_3$. Applying the coordinate in Lemma 3.7 $v$ satisfies the following mean curvature type equation

\[(3.24) \quad \frac{1}{\sqrt{1 + g^{ij}v_i v_j}}(g^{ij} - \frac{v^iv^j}{1 + g^{ij}v_i v_j})v_{ij} = f(y, v, \frac{-g^{ij}v_i \frac{\partial}{\partial u_i} + X^p_m}{\sqrt{1 + g^{ij}v_i v_j}})\]

over the domain $U_{C_4}$. Here $v^i = g^{ik}v_k$. By Lemma 3.7 and (3.22) we have

\[(3.25) \quad C^{-1}I \leq g^{ij}(y, v) \leq CI, \quad p_1^{-1+\alpha}||g_{ij}(y, v)||_{C^{1, \alpha}} \leq C \text{ on } U_{C_4}\]

Recall that the mean curvature of $\partial E$ is $f(p, \vec{v})$ satisfying $N_W(f) \leq \nu$ (see Definition 3.1). By Definition 3.1, Remark 3.2 and Lemma 3.7 on $U_{C_4}$ we have

\[(3.26) \quad ||f||_{C^{1, \alpha}(T^p W)} \leq \nu\]

for $(x, p) \in T^p W$. Notice that the normal vector $\vec{v} = (p_1, \ldots, p_{m-1}, p_m)$, $p_i = -g^{ij}v_i \frac{\partial}{\partial u_i}, i = 1, \ldots, m - 1$ and $p_m = \frac{1}{\sqrt{1 + g^{ij}v_i v_j}}$. By (3.22), (3.24) is uniformly parabolic. Recall that $g_{ij}(p) = \delta_{ij}$. By (3.22), (3.24) is uniformly elliptic over $U_{C_4}$. Because of $g_{ij} = 0$ for $i, j = 1, \ldots, m$ at $p$, with (3.25), (3.26) the classical Schauder estimate implies that

\[(3.27) \quad |A|^2(p) = \sum_{i=1}^{m-1} v_{ii}(0) \leq \mu(C_4, \nu, C)\]
Notice that all constants $C_4, \nu, C$ are independent of any $p \in K$ and any $E \in \mathcal{G}$.

Thus we conclude (3.3). The proof is complete. \qed

4. An auxiliary Dirichlet problem

In this section we use the Perron method to study the following Dirichlet problem

\begin{equation}
\begin{aligned}
\mathcal{L}_H(u) = 0 & \quad \text{on } \Omega \\
u = \psi & \quad \text{on } \partial \Omega
\end{aligned}
\end{equation}

where

\begin{equation}
\mathcal{L}_H(u) := -\text{div} \left( \frac{Du}{\omega} \right) + H_1(x, u, -\frac{Du}{\omega}, \frac{1}{\omega}) + H_2(x, u, -\frac{Du}{\omega}, \frac{1}{\omega}) \frac{1}{\omega}
\end{equation}

$\omega = \sqrt{1 + |Du|^2}$, $\Omega$ is a domain in an $n(\geq 2)$ dimensional Riemannian manifold $N$, $M$ is the product manifold $N \times \mathbb{R}$ and $H_i(x, z, X, r)$ is a $C^{1,\alpha}$ function on $TM$ for $i = 1, 2$ with $(x, z) \in M$ and $X + r\partial_r \in T(x,z)M$ with $X \in T_xN$.

The main result of this section is stated as follows.

**Theorem 4.1.** Fix two constants $\beta > 0$ and $\alpha \in (0, 1)$. Let $\Omega$ be a bounded $C^{2,\alpha}$ domain. Let $H_1(x, z, X, 0) = f(x, z, X)$ for any $x \in N$, $z \in \mathbb{R}$, $X$ in $TN$. Suppose

\begin{equation}
\frac{\partial H_1}{\partial z} + \frac{\partial H_2}{\partial z} \geq \beta > 0
\end{equation}

and

\begin{equation}
H_{\partial \Omega}(x) \geq \max \{ f(x, \psi(x), \gamma(x)), -f(x, \psi(x), -\gamma(x)) \} \quad x \in \partial \Omega
\end{equation}

Here $\gamma$ is the outward normal vector of $\partial \Omega$. Then there is a unique solution to the Dirichlet problem (4.1) in $C^{3,\alpha}(\Omega) \cap C(\bar{\Omega})$ for any $\psi$ in $C(\partial \Omega)$.

**Remark 4.2.** The Dirichlet problem (4.1)-(4.3) can be used to blow up Jang equations. See Sakovich [36, section 4] for more references.

**Remark 4.3.** Bergner [3] first considered the general form of (4.1) in Euclidean spaces with only requiring that $\frac{\partial H}{\partial z}$ is non-negative. His smallness condition upon $H_1$, $H_2$ and $\Omega$ guarantees the a priori $C^0$ estimate of the solution to (4.1) is uniformly bounded. In our setting the strict monotone condition (4.3) gives the corresponding a priori estimate. An improvement of the boundary condition in [3] was later given by Marquardt [35].

The closed version of Theorem 4.1 is stated as follows.

**Theorem 4.4.** If $\Omega$ is a closed $n(\geq 2)$-dimensional manifold without boundary and (4.3) holds. Then there is a unique solution to the problem $\mathcal{L}_H(u) = 0$ on $\Omega$ in $C^{3,\alpha}(\Omega)$. 
We did not consider the variational approach in [18] since the graph of the solution to the Dirichlet problem (4.1) may not be written as the critical point of any area-type functional. The continuous method in [16, section 11.3] did not work in our case because we cannot keep (4.4) unchanged in the continuous process even if the $C^0$ estimate of the solution to (4.1) is available by the maximum principle. Therefore the Perron theory used by Eichmair [10] may be the only choice to show Theorem 4.1.

4.1. The Maximum Principle. We collect two well-known maximum principles about mean curvature equations for later use.

**Theorem 4.5.** Suppose $G(x, u, Du)$ is a $C^1$ function on $N \times \mathbb{R} \times TN$ satisfying $\partial_q G(x, u, Du) \geq 0$. Suppose $u_1, u_2 \in C^2(\Omega) \cap C(\overline{\Omega})$. Define the elliptic operator

$$
Lu := -\text{div}(\frac{Du}{\sqrt{1 + |Du|^2}}) + G(x, u, Du) \quad \text{on} \quad \Omega
$$

If $Lu_1(x) \geq Lu_2(x)$ on $\Omega$ and $u_1(x) \geq u_2(x)$ on $\partial \Omega$, $u_1(x) \geq u_2(x)$ on $\Omega$. Moreover, $u_1(x_0) = u_2(x_0)$ for some point $x_0 \in \Omega$, $u_1(x) \equiv u_2(x)$.

Recall that for a hypersurface $\Sigma$ with its normal vector $\vec{v}$, its mean curvature $H_\Sigma$ is defined by $\text{div}(\vec{v})$.

**Theorem 4.6** (Proposition 4.12). Let $\Omega$ be a domain with $C^2$ boundary. Let $G : TN \to \mathbb{R}$ be a $C^1$ function. Suppose

1. $\Sigma$ is a $C^2$ hypersurface in the complement of $\Omega$;
2. $\Sigma$ is tangent to $\partial \Omega$ at a point $p$ in the interior of $\Sigma$ and $\partial \Omega$;
3. in an embedded ball $B$ containing $p$, for any $q \in B$, $H_\Sigma(q) \leq G(q, \vec{v}_1(q))$, $H_\Sigma(q) \geq G(q, \vec{v}_2(q))$;

where $\vec{v}_1$ and $\vec{v}_2$ denote the normal vector of $\Sigma$ and $\partial \Omega$ respectively satisfying $\vec{v}_1(p) = -\vec{v}_2(p)$ pointing into $\Omega$.

Then $\Sigma$ coincides with $\partial \Omega$ in a neighborhood of $p$.

**Remark 4.7.** The condition $\vec{v}_1(p) = -\vec{v}_2(p)$ is essential because two unit spheres in Euclidean spaces can touch each other at a common point $p$ with $\vec{v}_1(p) = -\vec{v}_2(p)$.

4.2. The Perron Theory. We use the following definitions of the Perron method from Eichmair [10, section 3]. In what follows we always assume the operator $L_H$ is defined by (4.2) satisfying (4.3), $B_r(x)$ denotes the embedded ball centering at $x$ with radius $r$.

**Lemma 4.8.** Suppose $\Omega$ is a bounded domain. Fix any $c > 0$. For any $x \in \Omega$ there is a continuous positive function $r(x)$ such that the Dirichlet problem $L_H(u) = 0$ in $B_r(x)$ with $u = \psi$ on $\partial B_r$, $r \in (0, r(x))$ has a unique solution in $C^{2,\alpha}(B_r(x)) \cap C(\overline{B_r(x)})$ for any $\psi \in C(\partial B_r(x))$ satisfying $|\psi| \leq c$. 

Proof. Fix any $x \in \overline{\Omega}$. First we assume that $\psi$ is in $C^{2,\alpha}(\overline{B}_r(x))$. For any $s \in (0,1)$, let $v_s \in C^{3,\alpha}(B_r(x)) \cap C(\overline{B}_r(x))$ be the solution to the Dirichlet problem

$$
\begin{cases}
-\text{div}(\frac{Du}{\omega}) + s\{G(x,u,-\frac{Du}{\omega},1,\frac{1}{\omega})\} = 0 & \text{on } B_r(x), \\
\quad u = s\psi & \text{on } \partial B_r(x)
\end{cases}
$$

where $G(p,Y)$ be a $C^{1,\alpha}$ function on $TM$ given by $G(p,Y) = H_1(x,u,X,t) + H_2(x,u,X,t)\mu$ with $p = (x,u)$ and $Y = X + t\partial_r$ for $X \in TN$. Since $|\psi| \leq c$ and $\frac{\partial H_1}{\partial x} + \frac{\partial H_2}{\partial x} \geq \beta$, by applying the maximum principle

$$
\max_{\overline{B}_r(y)} |v_s| < c_0 := c_0(\mu_0, \beta, c)
$$

Here $\mu_0 := \{x \in \Omega, X \in T\Omega, (X,X) \leq 1, |t| \leq 1, |u| \leq c, |H_1(x,u,X,t)| + |H_2(x,u,X,t)|\}$. Thus on $B_r(x)$ it holds that

$$
v_s(\frac{Dv_s}{\omega}, \frac{1}{\omega}) + H_2(x,u_s,-\frac{Dv_s}{\omega}, \frac{1}{\omega}) \leq c_0 =: c_1(0,c_0,\mu_0),
$$

where $\omega = \sqrt{1 + |Dv_s|^2}$ for any $s \in [0,1]$. Since $\Omega$ is compact, there is a continuous function $r(x)$ depending on $x$ such that for each $r \in (0,r(x))$, the mean curvature of $B_r(x)$ satisfies

$$
H_{\partial B_r(x)} = \frac{n-1}{r} + o(r) > c_1
$$

Therefore there is a positive constant $\mu := \mu(c_0, \beta)$ such that for any $s \in (0,1)$.

$$
sN_W(G) \leq \mu
$$

where $W := \{(x,u) : x \in \overline{\Omega}, |u| \leq c_0\}$. By (4.9), (4.8) and Theorem A.3, there are two positive constants $\kappa, \nu$ (independent of $s \in [0,1]$) and $u_{s, \pm} = s\psi \pm \frac{\log(1 + \kappa d(x))}{\nu}$ such that

$$
\pm L_s(u_{s, \pm}) \geq 0
$$

on the domain $\Gamma_{s, \pm} := \{x' \in B_r(x) : \pm u_{s, \pm}(x') < c_0, d(x') < \frac{1}{\nu}\}$ and $\pm u_{\pm} = c_0$ on $\partial \Gamma_{s, \pm} \setminus \partial B_r(x)$. Applying Theorem A.3 we obtain

$$
v_s(x') - \frac{\log(1 + \kappa d(x'))}{\nu} \leq u_{s, \pm}(x') \leq v_s(x') + \frac{\log(1 + \kappa d(x'))}{\nu}
$$

for any $x'$ in $\Gamma_{s, \pm}$. Here $d(x') = \text{dist}(x', \partial B_r(x))$. Because $v_s = s\psi$ on $\partial B_r(x)$ this implies that

$$
\max_{\partial B_r(x)} |Dv_s| \leq c_3 =: c_3(\kappa, \nu)
$$

for any $s \in [0,1]$. By (4.3), (4.10) and (4.13), the item (2) of Theorem A.2 implies that

$$
\max_{B_r(y)} |Dv_s| \leq c_4 =: c_4(c_3, \mu)
$$
Applying the classical Schauder estimate the $C^{1,\alpha}$ norm of $v_s$ is uniformly bounded. By the continuous method in [10] Theorem 11.4 and Section 11.3] we obtain the existence of the solution $u \in C^{3,\alpha}(B_r(x)) \cap C(\bar{B}_r(x))$ to the Dirichlet problem $\mathcal{L}_H(u) \equiv 0$ on $B_r(x)$ with $u = \psi$ on $\partial B_r(x)$ for any $\psi \in C^{2,\alpha}(\bar{B}_r(x))$ satisfying $|\psi| \leq c$.

Now we consider the case that $\psi \in C(\partial B_r(x))$. Construct two monotone sequences $\{\psi_{i,k}\}_{k=1}^\infty$ for $i = 1, 2$ in $C^{2,\alpha}(\partial B_r(x))$ which converges to $\psi$ increasingly or decreasingly in the sense of $C(\partial B_r(y))$. Let $\{u_{i,k}\}_{k=1}^\infty \subset C^{3,\alpha}(\bar{B}_r(x)) \cap C(\bar{B}_r(x))$ be the solutions to the problem (4.16) for boundary data $\{\psi_{i,k}\}_{k=1}^\infty$. Since $\frac{\partial u}{\partial n} \geq 0$, by Theorem 4.5 $\{u_{i,k}\}_{k=1}^\infty$, $i = 1, 2$ are increasing and decreasing sequences respectively. By Theorem A.3 and Theorem 4.6 $\{u_{1,k}\}_{k=1}^\infty$ locally converges to a function $u(x)$ in $C^{3,\alpha}(\bar{B}_r(x))$ satisfying $\mathcal{L}_H(u) = 0$. Moreover, we have

$$u_{1,k} \leq u \leq u_{2,k} \text{ on } B_r(x) \quad (4.15)$$

Recall that $\{\psi_{i,k}\}_{k=1}^\infty$ for $i = 1, 2$ in $C^{2,\alpha}(\partial B_r(x))$ converges to $\psi$ increasingly or decreasingly in the sense of $C(\partial B_r(y))$. Therefore $u$ belongs to $C(\bar{B}_r(x))$.

This gives the existence in Lemma 4.8. The uniqueness is from (4.3) and the maximum principle in Theorem 4.5. The proof is complete. \qed

**Definition 4.9 (\mathcal{L}_H lift of a Perron subsolution).** [10] Definition 3.2. Let $x$ be a point in $\Omega$ and $r(x)$ be the continuous positive function in Lemma 4.8. Let $\underline{u}$ be a Perron subsolution (supersolution) of $\mathcal{L}_H$ on $\Omega$ if the following property holds.

(P) Fix any $r \in (0, r(x))$ and $B_r(x)$ be the embedded ball centered at $x$ with radius $r$ in $\Omega$. Let $u \in C(\bar{B}_r(x)) \cap C^2(B_r(x))$ be the unique solution of $\mathcal{L}_H(u) = 0$ on $B_r(x)$ with $u = \underline{u}$ on $\partial B_r(x)$. Then $u \geq \underline{u}$ $(u \leq \underline{u})$.

Moreover, the function $\hat{u}$ is defined by

$$\hat{u} = \begin{cases} u(x) & \text{on } B_r(x) \\ \underline{u}(x) & \text{on } \Omega \setminus B_r(x) \end{cases}$$

is called as the $\mathcal{L}_H$ lift of $\underline{u}$ with respect to $B_r(x)$.

**Definition 4.10 ([10], Definition 3.3).** Let $\bar{u}$ be a Perron supersolution of $\mathcal{L}_H$ on $\Omega$. We denote by $\mathcal{S}_{\bar{u}} := \{\underline{u} \in C(\Omega): \underline{u} \text{ is a Perron subsolution of } \mathcal{L}_H \text{ on } \Omega \text{ with } \underline{u} \leq \bar{u}\}$ the class of all Perron subsolution of $\mathcal{L}_H$ lying below $\bar{u}$.

By the maximum principle in Theorem 4.5, we have the following properties on Perron subsolutions.

**Lemma 4.11 (Lemma 3.1, [10]).** Let $\bar{u}$ and $\mathcal{S}_{\bar{u}}$ be given by Definition 4.10. Then the following two basic properties of $\mathcal{S}_{\bar{u}}$ hold

1. If $\underline{u}, \underline{v} \in \mathcal{S}_{\bar{u}}$, then $\max\{\underline{u}, \underline{v}\} \in \mathcal{S}_{\bar{u}}$.
2. If $\underline{u} \in \mathcal{S}_{\bar{u}}$, $x \in \Omega$, $r \in (0, r(x))$ and $\hat{u}$ is the $\mathcal{L}_H$ lift of $\underline{u}$ with respect to $B_r(x)$, then $\hat{u} \in \mathcal{S}_{\bar{u}}$.
Definition 4.12 ([10], Definition 3.4). Let \( \bar{u} \) be a Perron supersolution of \( \mathcal{L}_H \) such that \( S_{\bar{u}} \neq \emptyset \). The function \( u^P(x) := \sup \{ u(x) : u \in S_{\bar{u}} \} \) is called the Perron solution of \( \mathcal{L}_H \) on \( \Omega \) with respect to \( \bar{u} \).

By the interior gradient estimate in Theorem A.2 we obtain the interior regularity of the Perron solution as follows.

Lemma 4.13. Suppose \( \Omega \) is a domain. Let \( \bar{u} \) be a Perron supersolution on \( \Omega \) such that \( S_{\bar{u}} \neq \emptyset \). Let \( u^P(x) \) be the Perron solution of \( \mathcal{L}_H \) on \( \Omega \) with respect to \( \bar{u} \). Then \( u^P \in C^{3,\alpha}(\Omega) \) and \( \mathcal{L}_H u^P = 0 \) on \( \Omega \).

Remark 4.14. In this lemma we do not require that \( \Omega \) is bounded.

Proof. Fix any \( x_0 \in \Omega \) and \( r < \frac{1}{4}r(x_0) \) such that \( B_r(x_0) \) is an embedded ball and for any \( y \) in \( B_r(x_0) \), \( r(y) \geq \frac{1}{4}r(x_0) > 2r \). Here we use the fact that \( r(x) \) is a continuous function in Lemma 4.8. By the definition of the Perron solution, there is a sequence \( \{ v_i(x) \}_{i=1}^{\infty} \) in \( S_{\bar{u}} \) such that \( \lim_{i \to +\infty} v_i(x_0) = u^P(x_0) \).

Now let \( \{ \hat{v}_i(x) \}_{i=1}^{\infty} \) be the \( \mathcal{L}_H \) lift of \( v_i(x) \) with respect to \( B_r(x_0) \). Since \( \hat{v}_i(x) \) is uniformly bounded on \( B_r(x_0) \), Theorem A.2 implies that \( \max_{x \in \partial B_r(x_0)} |D \hat{v}_i| \) is uniformly bounded for any \( r' < r \). By the classical Schauder estimate so is the \( C^{3,\alpha} \) norm of \( \hat{v}_i \) on \( B_r(x_0) \). As a result, possible choose a subsequence, \( \{ \hat{v}_i \}_{i=1}^{\infty} \) converges to \( \hat{u}(x) \) in the locally \( C^{3,\alpha} \) sense. Moreover \( \mathcal{L}_H \hat{u} = 0 \) on \( B_r(x_0) \). By definition \( \hat{u} \leq u^P \) on \( B_r(x_0) \) and \( \hat{u}(x_0) = u^P(x_0) \).

Suppose there is a point \( y_0 \) in \( B_r(x_0) \) such that \( \hat{u}(y_0) < u^P(y_0) \). Now let \( \{ v'_i \}_{i=1}^{\infty} \) in \( S_{\bar{u}} \) such that \( v'_i \geq v_i \) for each \( i \), \( \lim_{i \to +\infty} v'_i(y_0) = u^P(y_0) \). Let \( \hat{v}'_i \) be the \( \mathcal{L}_H \) lift of \( v'_i \) with respect to \( B_r(y_0) \). Since \( r(y_0) > r \), \( x_0 \) is contained in \( B_r(y_0) \). As the above derivation in the previous paragraph, \( \{ \hat{v}'_i \} \) converges to a \( \hat{u}'(x) \) in the local \( C^2 \) sense. Moreover \( \mathcal{L}_H \hat{u}' = 0 \) on \( B_r(y_0) \). Let \( B \) be the set \( B_r(y_0) \cap B_r(x_0) \). Since \( v'_i \geq v_i \) for each \( i \), \( \hat{v}'_i \geq \hat{u}' \) on \( B \). On the other hand, \( \hat{u}(x_0) = u^P(x_0) \geq \hat{u}'(x_0) \) and \( \mathcal{L}_H \hat{u} = \mathcal{L}_H \hat{u}' \equiv 0 \) on \( B \). By Lemma 4.8 \( \hat{u}' \equiv \hat{u} \) on \( B \). This is a contradiction since \( \hat{u}(y_0) < u^P(y_0) = \hat{u}'(y_0) \). As a result \( u^P \equiv \hat{u} \) on \( B_r(x_0) \). The proof is complete. \( \square \)

4.3. Conclusion of Theorem 4.1 and Theorem 4.4

Proof. The proof of Theorem 4.1 Suppose \( \Omega \) is a bounded \( C^{2,\alpha} \) domain. Without loss of generality we can assume \( \psi \in C^{2,\alpha}(\bar{\Omega}) \). The general case \( \psi \in C(\partial \Omega) \) follows from the same proof in the last part of Lemma 4.8 with a monotone approximation process.

Notice that \( \beta > 0 \). Define a finite positive number \( \alpha_1 \) as

\[
\alpha_1 := \frac{1}{\beta} \max \{|H_1(x, \psi(x), X, r)| + |H_2(x, \psi(x), X, r)| : \]
\[
x \in \bar{\Omega}, X \in T\Omega, \langle X, X \rangle \leq 1, |r| \leq 1 \} + \max_{\Omega} |\psi| + 1
\]

(4.16)

Since \( \frac{\partial H_1}{\partial z_2} + \frac{\partial H_2}{\partial z_2} \geq \beta > 0 \), \( \pm \mathcal{L}_H (\pm \alpha_1) \geq 0 \) on \( \Omega \). As a result \( \alpha_1 \) and \( -\alpha_1 \) is the Perron supersolution and the Perron subsolution of \( \mathcal{L}_H u = 0 \) on \( \Omega \).
It is obvious that 

\begin{equation}
(4.18)
\end{equation}

By definition 

\begin{equation}
(4.19)
\end{equation}

\begin{equation}
(4.20)
\end{equation}

\begin{equation}
(4.21)
\end{equation}

respectively.

In Theorem 4.3 let \( c_0 \) be the constant \( \alpha_1 + 1 \). By Theorem 4.3 there are two positive constants \( \kappa, \nu \) and 

\begin{equation}
\pm u \equiv \psi \pm \frac{\log(1+\kappa d(x))}{\nu}
\end{equation}

such that

\begin{equation}
\pm L_H(v_{\pm}) \geq 0
\end{equation}

on the domain \( \Gamma_{\pm} := \{ x \in \Omega : \pm v_{\pm}(x) < \alpha_1 + 1, d(x) < \frac{1}{\nu} \} \) and \( \pm v_{\pm} = \alpha_1 + 1 \) on \( \partial \Gamma_{\pm} \setminus \partial \Omega \). Here \( d(x) \) denotes the distance function from \( \Omega \) which is well-defined because \( \Omega \) is bounded. Define \( \delta \) as the number

\begin{equation}
(4.18)
\end{equation}

It is obvious that \( \delta > 0 \). Now we define two functions

\begin{equation}
(4.19)
\end{equation}

By definition \( u_{\pm} \) and \( v_{\pm} \) are continuous on \( \Omega \) with \( u_{\pm} = \psi \) on \( \partial \Omega \).

Indeed \( u_{\pm} \) and \( v_{\pm} \) are the Perron subsolution and the Perron supersolution of \( L_H(u) \) respectively. Let \( r(x) \) be the positive continuous function in Lemma 4.8. For any \( x \in \Omega \), define

\begin{equation}
(4.20)
\end{equation}

By the definition of \( \delta \) for any fixed \( x \in \Omega \) and fixed \( r < r_0(x) \), there are only two cases happen (a): \( B_r(x) \subset \Gamma_+ \), \( u_+ \leq v_+ \) or (b): \( u_+ \equiv \alpha_1 \) on \( B_r(x) \). Then on \( B_r(x) \) the \( L_H \) lift of \( u_+, \dot{u}_+ \), is well-defined. If \( B_r(x) \subset \Gamma_+ \), \( \dot{u}_+ \equiv v_+ \) on \( B_r(x) \) because \( v_+ \) is a Perron supersolution on \( B_r(x) \). Otherwise \( u_+ \equiv \alpha_1 \) on \( B_r(x) \), then \( \dot{u}_+ \equiv \alpha_1 \equiv u_+ \). No matter which case we have \( \dot{u}_+ \leq \min\{v_+, \alpha_1\} = u_+ \) on \( \Gamma_+ \). As a result, \( u_+ \) is a Perron supersolution on \( \Omega \). As above, \( u_- \) is a Perron subsolution on \( \Omega \).

Since \( u_- \in S_{u_+} \), by Lemma 4.13 the Perron solution \( u^P(x) := \sup\{v(x) : x \in \Omega, v \in S_{u_+}\} \) exists and \( u^P \in C^3(\Omega) \) with \( L_H u^P = 0 \) on \( \Omega \). On the other hand, \( u_- \leq u^P \leq u_+ \) on \( \Omega \). Because both of \( u_\pm \) are continuous on \( \Omega \) and equal to \( \psi \) on \( \partial \Omega \). Thus \( u^P \) belongs to \( C(\Omega) \cap C^3(\Omega) \) and is solution to the Dirichlet problem in (4.1). The uniqueness is obvious because of (4.3) and the maximum principle in Theorem 4.5.

The proof of Theorem 4.4 is complete. \( \square \)

**Proof. The proof of Theorem 4.4** A key fact is that \( \Omega \) is a closed Riemannian manifold without boundary. Thus in what follows there is no derivation involving any boundary. Define a positive number

\begin{equation}
(4.21)
\end{equation}

where \( \beta \) is from (4.3). A direct computation shows that \( \pm L_H(\pm \alpha_2) \geq 0 \). Thus \( \alpha_2, -\alpha_2 \) are the Perron supersolution and the Perron subsolution of
\[ L_H(u) = 0 \] respectively.

The remainder is the same as the last part in the proof of Theorem 4.1.

We arrive the conclusion. \qed

5. The \textit{Nc-f} property

In this section we first discuss the relationship between the \textit{Nc-f} property and the Ricci assumptions of a domain. Then we construct two examples. The first one says that a \textit{Nc-f} domain can contain a minimal hypersurface when \( f \equiv 0 \). The second one says that if the \textit{Nc-f} assumption is almost optimal. That is if it is removed, there are a compact manifold such that there is no solution to corresponding PMC equations.

**Definition 5.1** \textit{(the \textit{Nc-f} property)}. Fix \( n \geq 2 \), \( \alpha \in (0,1) \) and \( N \) is a \( n \)-dimensional Riemannian manifold. Suppose \( f \) is a \( C^{1,\alpha} \) function in the tangent bundle \( TN \) and \( \Omega \) is a \( C^{2,\alpha} \) domain with embedded boundary. We say that \( \Omega \) has the \textit{Nc-f} property if there is no Caccioppoli set \( E \) in the closure of \( \Omega \) satisfying

1. for any \( p \in \partial E \), \( E \) is a \( \Lambda \)-perimeter minimizer (see Definition 2.4) in a neighborhood of \( p \) for some positive constant \( \Lambda \);
2. the regular part \( \text{reg}(\partial E) \) (see Definition 2.4) is \( C^{2,\alpha} \), embedded and orientable with its mean curvature equal to \( f(x,\bar{v}(x)) \) for any \( x \) in \( \text{reg}(\partial E) \) (\(-f(x,-\bar{v}(x))\) for any \( x \) in \( \text{reg}(\partial E) \));

Here \( \bar{v} \) is the outward normal vector of \( x \) in \( \text{reg}(\partial E) \), the mean curvature is defined as \( \text{div}(\bar{v}) \).

**Remark 5.2.** In this definition we do not require that \( \Omega \) is bounded. By Theorem 2.8 the dimension of the Hausdorff dimension of \( \partial E \setminus \text{reg}(\partial E) \) is at most \( n-8 \).

In the case \( 7 \geq n \geq 2 \), \( \Omega \) is a \textit{Nc-f} domain if and only if in its closure there is no \( C^2 \) domain \( E \) with mean curvature \( f(x,\bar{v}(x)) \) for all \( x \in \partial E \) or \(-f(x,-\bar{v}(x))\) for all \( x \in \partial E \). Here \( \bar{v} \) is the outward normal vector of \( E \).

5.1. \textbf{Examples of \textit{Nc-f} domains.}

**Theorem 5.3.** Suppose \( \Omega \) is a \( C^{2,\alpha} \) \( n \)-dimensional \( (n \geq 2) \) bounded domain with connected embedded boundary \( \partial \Omega \) in \( N \). Let \( \gamma(x) \) be the outward normal vector of \( \partial \Omega \) at \( x \). Then \( \Omega \) has the \textit{Nc-f} property if

1. \( f \) is a \( C^{1,\alpha} \) function on \( TN \) with \( \max_{x \in \Omega, \bar{v} \in T_x N, |\bar{v}| \leq 1} |f(x,\bar{v})| \leq \mu \) for some constant \( \mu \) such that neither the mean curvature of \( \partial \Omega \) is equal to \( f(.,\gamma(\cdot)) \) on the whole \( \partial \Omega \) or \((-f(.,-\gamma(\cdot))\) on the whole \( \partial \Omega \) .
2. one of the following holds:
   a. \( H_{\partial \Omega} \geq \mu, \min_{x \in \Omega, e \in T_x N, |e| \leq 1} \text{Ric}(e,e) > -\frac{\mu^2}{n-1} \); 
   b. \( H_{\partial \Omega} > \mu, \min_{x \in \Omega, e \in T_x N, |e| \leq 1} \text{Ric}(e,e) > -\frac{\mu^2}{n-1} \);
Proof. We argue the conclusion by contradiction. Suppose a non-empty set \( E \) is a Caccioppoli set in \( \Omega \) such that (a) for any \( p \in \partial E \), \( E \) is a \( \Lambda_{p} \)-perimeter minimizer in a neighborhood of \( p \) for some positive constant \( \Lambda_{p} \); (b) \( \text{reg}(\partial E) \) is \( C^{2,\alpha} \) embedded such that \( H_{\text{reg}(\partial E)}(x) \equiv f(x,\vec{v}) \) for any \( x \) in \( \text{reg}(\partial E) \) (or \( H_{\text{reg}(\partial E)}(x) \equiv -f(x,-\vec{v}) \) for any \( x \) in \( \text{reg}(\partial E) \)). Here \( \vec{v} \) is the normal outward vector of \( \text{reg}(\partial E) \).

Now define \( l = \inf_{x \in E} \text{dist}(\partial \Omega, x) \) where \( \text{dist} \) is the distance function induced by the metric of \( N \). Let \( l_{m} \) be the number \( \sup_{x \in \Omega} \text{dist}(x, \partial \Omega) \). We have \( l_{m} \geq l \) because \( E \) is not empty and \( \text{vol}(E) > 0 \) and \( \text{vol}(\Omega \setminus E) > 0 \). First we assume \( l > 0 \).

We use an idea from Kasue \[26\] Page, 120. Because \( l \in (0, l_{m}) \), there is a \( \varepsilon > 0 \) and a geodesic \( \beta(s) \) for the arc-length parameter \( s \in (-\varepsilon, l + \varepsilon] \) connecting \( \beta(0) = y_{0} \in \partial \Omega \) and \( \beta(l) \in \partial E \) such that \( \text{dist}(\partial \Omega, \beta(t)) = t \) for \( t \in (0, l + \varepsilon) \). Let \( \xi(y) \) be the inner normal vector of the point \( y \) on \( \partial \Omega \). Define the map

\[
(5.1) \quad \exp : \partial \Omega \times (-\varepsilon, l + \varepsilon) \to N, \quad \text{given by} \quad \exp(y, t) = \exp_{y}(t \xi(y))
\]

Here \( \exp_{y} \) denotes the exponential map at \( y \). By the definition of \( \beta(t) \), there is no conjugate point on \( \{ \beta(t) : t \in [0, l + \varepsilon) \} \). Therefore there is a connected open set \( V \subset \partial \Omega \) containing \( y_{0} \) and \( \varepsilon \in (0, l) \) such that \( \exp \) is a diffeomorphism from \( V \times (\varepsilon, l + \varepsilon) \) into its image.

Let \( V_{t} \) denote the image of \( V \times \{ t \} \) under the map \( \exp \) for each \( t \in [0, l] \). Since \( \partial \Omega \) is \( C^{2,\alpha} \), \( V_{t} \) is \( C^{2,\alpha} \) (see \[31\]). Moreover for all \( t \in (0, l) \), \( V_{t} \) is disjoint with \( \partial E \) because \( \text{dist}(V_{t}, \partial \Omega) \leq t < l \). Let \( H_{V_{t}} \) denote the mean curvature of \( V_{t} \) at \( \exp_{y}(t \xi(y)) \) with respect the outward unit normal vector \(-d(\exp)(\partial_{t}) \). We can view \( \{ V_{t} \}_{t \geq 0} \) as a flow with the velocity \( \partial_{t} \) as a function on \( V \times (0, l + \varepsilon) \). Choose \( \vec{v} = -d(\exp)(\partial_{t}) \) and \( f = -1, \partial_{t} V_{t} = f \vec{v} \) and by \[23\] Theorem 3.2 (v) we have

\[
(5.2) \quad \frac{\partial H_{V_{t}}}{\partial t} = |A|^{2} + \text{Ric}(-d(\exp)(\partial_{t}), -d(\exp)(\partial_{t})) \\

\geq \frac{H_{V_{t}}^{2}}{n-1} - \frac{\mu^{2}}{n-1} \quad \text{by the condition (2)}
\]

where \( A \) is the second fundamental form of \( V_{t} \) in \( N \). From the condition (2) and \( V_{0} \subset \partial \Omega \), we always have \( H_{\partial \Omega} \geq \mu \). By the maximum principle of ordinary differential equation we obtain that

\[
(5.3) \quad H_{V_{t}} \geq \mu
\]

in a neighborhood of \( \beta(l) \). Since in neighborhood of \( \beta(l) \) \( E \) is a \( \Lambda \)-perimeter minimizer, by Lemma \[27\] \( \beta(l) \) belongs to \( \text{reg}(\partial E) \). Because \( \text{reg}(\partial E) \) is \( C^{2,\gamma} \) with \( H_{\text{reg}(\partial E)} = \pm f(x, \pm \vec{v}) \) near \( \beta(l) \) for some \( \gamma \in (0, \frac{1}{2}) \). By Theorem \[4.10\] and \[5.3\] \( V_{t} \) coincides with \( \text{reg}(\partial E) \) near \( \beta(l) \) and

\[
(5.4) \quad H_{V_{t}} \equiv \mu, \quad \text{Ric}_{\exp(t \xi(y))}(-d(\exp)(\partial_{t}), -d(\exp)(\partial_{t})) \equiv \frac{\mu^{2}}{n-1}
\]
for any $t \in [0, l]$ and $y \in V$. This is a contradiction to condition (2).

Second we assume $l = 0$. Then $\partial E$ is tangent to $\partial \Omega$ at some point $p \in \partial \Omega$. By Lemma 2.7 in a neighborhood of $p \partial E$ is contained in $\text{reg}(\partial E)$. By the condition (2) and Theorem 4.6 $\partial \Omega$ is contained in $\text{reg}(\partial E)$ and its mean curvature is equal to $f(x, \bar{v})$ near $p$ or $-f(x, -\bar{v})$ near $p$ where $\bar{v}$ is the outward normal vector of $\partial \Omega$. Since $\partial \Omega$ is connected, this means curvature conclusion holds on the whole $\partial \Omega$. However, this is a contradiction to condition (1). No matter which case we will obtain the contradiction.

Therefore $\Omega$ has the Nc-f property. The proof is complete. □

The following example shows that the Nc-f domain can have nontrivial topology even when $f \equiv 0$.

Example 5.4. Let $N$ be any $n$($\geq 2$)-dimensional closed Riemannian manifold with a smooth metric $\sigma$. Fix $a > 0$. Define a warped product manifold (5.5) 

$$\{N \times (-a, a), \phi^2(r)(\sigma + dr^2)\}$$

where $\phi(r) : (-a, a) \times \mathbb{R}$ is a positive smooth function satisfying (5.6) 

$$\phi'(0) = 0, \pm \phi'(\pm t) > 0, t \in (0, a)$$

By [50] Lemma 3.1 the mean curvature of the slice $N_t := N \times \{t\}$ with respect to the direction $\partial / \partial t$ is

(5.7) $$H_{N_t} = \frac{n\phi'(t)}{\phi^2(t)} \quad \text{for any} \quad t \in (-a, a)$$

Therefore by (5.6) $H_{\partial(N \times (-s, s))} > 0$ with respect to the outward normal vector. Notice that $N_0$ is a minimal hypersurface in $N \times (-s, s)$. Indeed we claim that

Lemma 5.5. Let $f \equiv 0$. For any $s \in (0, a)$, $N \times (-s, s)$ has the Nc-f property.

Proof. Suppose $E \subset N \times (-s, s)$ is a Caccioppoli set satisfying (a) for any $p \in \partial E$, $E$ is a $\Lambda$-perimeter minimizer in a neighborhood of $p$ for some positive constant $\Lambda$, (b) $\text{reg}(\partial E)$ is embedded, $C^2$ and minimal. Since $E$ is not empty, then $\text{vol}(E) > 0$ and $\text{vol}(N \times (-s, s) \setminus E) > 0$. Then there are two constants $a$ and $b$ in $[-s, s]$ such that $E \subset N \times [a, b]$, both $\partial E \cap N_a$ and $\partial E \cap N_b$ are not empty. By Lemma 2.7 both $\partial E \cap N_a$ and $\partial E \cap N_b$ are contained in $\text{reg}(\partial E)$. By (5.6) and (5.7), at least one of the mean curvature of $N_a$ and $N_b$ is strictly positive with respect to the outward normal vector from $N \times (a, b)$. By Theorem 4.6 this positive mean curvature slice is contained in $\text{reg}(\partial E)$ with vanishing mean curvature. This is a contradiction.

Therefore $N \times (-s, s)$ has the Nc-f property when $f \equiv 0$. □

5.2. Our condition is almost optimal. Now we construct an example to illustrate that the Nc-f property is almost optimal in Theorem 1.1 and Theorem 6.1. If keep other conditions in Theorem 1.1 unchanged and remove the Nc-f property, there are examples that no solution to the PMC equation
Recall that \( n (n \geq 2) \)-dimensional Euclidean space \( \mathbb{R}^n \) with Euclidean metric \( g_E \) takes the following warped product metric.

\[
(R^n, g_E) = \{ S^{n-1} \times [0, \infty), r^2 \sigma_{n-1} + dr^2 \}
\]

where \( \sigma_{n-1} \) is the standard canonical metric on \( S^{n-1} \). Fix any positive \( \beta > 0 \). Define a positive smooth function \( h(r) \) by

\[
h(r) = \begin{cases} 
  r & \text{for } r \in [0, \frac{\beta}{r}) \\
  e^{kr}, & \text{for } r \in [k, \infty)
\end{cases}
\]

where \( k \) is a positive constant strictly greater than \( \max \{ \beta, \frac{n}{\beta} \} \). Now define the manifold \( M \) by the set \( S^{n-1} \times [0, k) \) equipped with the metric \( h^2(r) \sigma_{n-1} + dr^2 \). Let \( M_r \) be the domain \( S^{n-1} \times [0, r) \). By [50, Lemma 3.1], the mean curvature of \( \partial M_r \) with respect to the metric \( h^2(r) \sigma_{n-1} + dr^2 \) and the outward normal vector is

\[
H_{\partial M_r} = (n-1) \frac{h'(r)}{h(r)}
\]

Therefore it is easily verified that the following three properties hold for \( M \).

**Lemma 5.6.** Let \( f = \beta \) and \( n \geq 2 \). It holds that

1. \( H_{\partial M} = (n-1)k > \beta \);
2. \( H_{\partial M_{\frac{\beta}{\rho}}} = \beta \) and \( M \) does not have the \( Nc-f \) property;
3. \( M_{\frac{\beta}{\rho}} \) is a Euclidean ball in \( \mathbb{R}^n \) with radius \( \frac{n}{\beta} \).

As a result \( M \) satisfies the conditions of Theorem 6.1 except the \( Nc-f \) assumption.

**Theorem 5.7.** Let \( M, \beta \) be defined as above and \( n \geq 2 \). There is no \( C^2 \) solution \( u \) to solve the PMC equation

\[- \text{div}(Du) + \beta = 0 \]

in the interior of \( M \). Here \( \omega = \sqrt{1 + |Du|^2} \).

**Proof.** We argue it by contradiction. Suppose there is a \( C^2 \) function \( u \) satisfying \( -\text{div}(\frac{Du}{\omega}) + \beta = 0 \) on \( M \). Let \( \omega_{n-1} \) denote the volume of \( S^{n-1} \). For any set \( E, |E| \) denotes the volume of \( E \). Applying the divergence

\[
\beta |M_{\frac{\beta}{2}}| = \int_{M_{\frac{\beta}{2}}} \text{div}(\frac{Du}{\omega}) = \int_{\partial M_{\frac{\beta}{2}}} \langle \frac{Du}{\omega}, \bar{v} \rangle d\sigma_{n-1} < |\partial M_{\frac{\beta}{2}}| = \omega_{n-1}(\frac{n}{\beta})^{n-1}
\]

Here we use the fact that \( |\frac{Du}{\omega}| < 1 \) on the interior of \( M \). On the other hand, \( M_{\frac{\beta}{2}} \) is a Euclidean ball, \( \beta |M_{\frac{\beta}{2}}| = \omega_{n-1}(\frac{n}{\beta})^{n-1} \). This is a contradiction. The proof is complete. 

\[\square\]
The main difficulty to show Theorem 6.1 is that there is no $C^0$ a priori estimate for the solution to the Dirichlet problem (1.3). The main idea is to use the Nc-f property to prevent the “real” blow-up and obtain the existence of the finite solution as in Schoen-Yau [39, Corollary 1]. See the work of Eichmair [10, 11] for further applications and references.

The main result of this paper is stated as follows.

**Theorem 6.1.** Fix $\alpha \in (0,1)$. Suppose $N$ is an $n(\geq 2)$-dimensional Riemannian manifold. Let $F(x,X,0) = f(x,X)$ for any $(x,X)$ in $TN$ where $x \in N, X \in T_xN$. Suppose $\Omega$ is a bounded $n$-dimensional $C^2$ Nc-f domain in $N$ and the mean curvature of the boundary $\partial \Omega$ satisfies

\[
H_{\partial \Omega}(x) \geq \max\{\pm f(x, \pm \gamma(x))\}
\]

where $\gamma(x)$ is the outward normal vector of $\partial \Omega$ at $x$.

Then the Dirichlet problem

\[
\begin{cases}
-\text{div}\left(\frac{Du}{\omega}\right)(x) - F(x, -\frac{Du}{\omega}, \frac{1}{\omega}) + \phi(x,u, -\frac{Du}{\omega}, \frac{1}{\omega}) \frac{1}{\omega} = 0 & \text{on } \Omega \\
u = \psi & \text{on } \partial \Omega
\end{cases}
\]

admits a unique solution in $C^{3,\alpha}(\Omega) \cap C(\bar{\Omega})$ for any $\psi(x) \in C(\partial \Omega)$ where $\omega = \sqrt{1 + |Du|^2}$. Here $\phi(x,z,Y,r) : T(N \times \mathbb{R}) \to \mathbb{R}$ is a smooth function with $\frac{\partial \phi}{\partial u} \geq 0$ which is either uniformly bounded or $\frac{\partial \phi}{\partial u} \geq \beta$ for some positive constant $\beta > 0$.

We record the results of Spruck [43] on constant mean curvature equations and Casteras-Heinonen-Holopainen [4] on weighted minimal graphs as follows.

**Theorem 6.2.** Suppose $\Omega$ is a bounded $C^{2,\alpha}$ domain in an $n$-dimensional manifold and $\mu$ is a positive constant satisfying

\[
\min_{x \in \bar{\Omega},e \in T_xN,\langle e,e\rangle \leq 1} \text{Ric}(e,e) \geq -\frac{\mu^2}{n-1}, \quad H_{\partial \Omega} \geq \mu
\]

then the Dirichlet problem (6.2) has a solution in $C^{2,\alpha}(\Omega) \cap C(\bar{\Omega})$

1. when $F \equiv \mu, \phi \equiv 0$ by Spruck [43, Theorem 1.4] ;
2. when $F = \langle \frac{Du}{\omega}, Dm(x) \rangle, \phi = h'(z)$ with $\sup_{x \in \bar{\Omega},z \in \mathbb{R}}\{|F| + |\phi|\} \leq \mu$ by Casteras-Heinonen-Holopainen [4, Theorem 1.2].

Here $m,h$ are $C^2$ functions on $\Omega$.

For an earlier similar result we refer to Dajczer-Hinojosa-de Lira [8].

**Remark 6.3.** If the inequality in (6.3) holds strictly, by Theorem 5.3 $\Omega$ has the Nc-f property. Therefore Theorem 6.1 recovers partially Theorem 6.2.
Remark 6.4. The equation (6.2) appears in the following setting. Let $M_f$ be the conformally product manifold

$$(N \times (a, b), e^{2f}(\sigma + dr^2))$$

where $f$ is a $C^\infty$ function on $N \times (a, b)$. Let $u$ be a $C^3$ function over a domain in $N$. By \([50]\), Lemma 3.1, the graph of $u$ is minimal in $M_f$ if and only if $u$ satisfies

$$(6.5) \quad -\text{div} \left( \frac{Du}{\omega} \right) + n\langle Df, -\frac{Du}{\omega} \rangle + n\frac{\partial f}{\partial r} = 0 \quad \text{on } \Omega$$

Here $D$ denotes the gradient of $f$ with respect to $x$ for fixed $u$.

Now we are ready to prove Theorem 6.1.

**Proof.** Suppose the solution $u$ to (6.2) exists. By the condition upon $\phi$ and the maximum principle, there is a positive constant $\beta_1$ only depending on $\beta$ and $\psi$ such that $|\phi| \leq \beta_1$ for any $(x, u(x))$. Fix any $\psi \in C(\partial \Omega)$. By the condition upon $\phi$ and Theorem 4.1, for any $t \in (0, 1)$, there is a unique solution $u_t$ in $C^{2,\alpha}(\Omega) \cap C(\bar{\Omega})$ solving

$$(6.6) \quad -\text{div} \left( \frac{Du_t}{\omega} \right) - F(x, \frac{Du_t}{\omega}, t) + (\phi(x, u_t, \frac{Du_t}{\omega}, t) + tu_t \frac{1}{\omega}) = 0 \quad \text{on } \Omega$$

with $u = \psi$ for any $\psi \in C(\partial \Omega)$. Define

$$(6.7) \quad F_t(p, Y) := F(x, X, r) - (\phi(x, z, X, r) + tz)r$$

for $t \in (0, 1)$ where $p = (x, z)$, $Y = X + r\partial_r$. It is obvious that $F_t$ is a $C^{1,\alpha}$ function on $T(N \times \mathbb{R})$. Define the subgraph of $u_t$ as

$$(6.8) \quad U_t := \{(x, r) : x \in \Omega, r < u_t(x)\}$$

**Lemma 6.5.** It holds that

1. for each $t \in (0, 1)$ the mean curvature of $\partial U_t$ in $\Omega \times \mathbb{R}$ is

$$(6.9) \quad H_{\partial U_t}(p) = F_t(p, \bar{v}),$$

where $p = (x, u_t(x)) \in \Omega \times \mathbb{R}$ and $\bar{v} = \frac{-Du_t + \theta}{\sqrt{1 + |Du_t|^2}}$ is the upward normal vector at $p$.

2. Define $\beta_2 = \max \{x \in \Omega, |x| \leq 1\} \{|\psi(x)|, |F(x, x, r) + \phi(x, \psi(x), X, r)|\}$. For any $t \in (0, 1)$

$$(6.10) \quad |tu_t| \leq \beta_2, \quad |\text{div} \left( \frac{Du_t}{\omega} \right)| \leq 2\beta_2 \quad \text{on } \Omega$$

where $\omega = \sqrt{1 + |Du_t|^2}$.

**Proof.** The item (1) follows from (6.6) and (6.7). The item (2) follows from applying the maximum principle in (6.6) and the assumption on $\phi$. \(\square\)
**General Case:** First we assume that Theorem 6.1 holds in the case of \( \phi \equiv 0 \). Then there are two solutions \( v_\pm \) in \( C^{2,\alpha}(\Omega) \cap C(\bar{\Omega}) \) to the Dirichlet problems given by

\[
-\text{div}(\frac{Du}{\omega}) - F(x, -\frac{Du}{\omega}, \frac{1}{\omega} \pm \beta_1 + \beta_2) = 0 \quad \text{on } \Omega
\]

with \( u = \psi \) on \( \partial \Omega \). Here we use the fact that \( G_{\pm}(x,X,r) = F(x,X,r) \pm (\beta_1 + \beta_2) r \), \( G(x,X,0) = f(x,X) \)

and for such \( G_{\pm} \) the conditions in Theorem 6.1 still hold for the same \( f \). By Theorem 4.5 we obtain

\[
 v_- \leq u_t \leq v_+ \quad \text{on } \Omega
\]

for any \( t \in (0,1) \). Thus

\[
 \sup_{t \in (0,1)} |u_t| \leq \kappa
\]

for a positive constant \( \kappa \) independent of \( t \).

Let \( W \) be the set \( \{(x,z) \in M : x \in \bar{\Omega}, |z| \leq \kappa\} \). By (6.7) it is obvious that

\[
 N_W(F_t) \leq c
\]

where \( c \) is a positive constant independent of \( t \in (0,1) \). By the conclusion (1) in Theorem A.2 the uniformly \( C^0 \) estimate implies that \( |Du_t| \) is locally uniformly bounded in \( \Omega \). By the classical Schauder estimate the \( C^{3,\alpha} \) norm of \( \{u_t\}_{t \in (0,1)} \) is locally uniformly bounded in \( \Omega \). Letting \( t \to 0 \), a sequence of \( \{u_t\}_{t>0} \) will converge to a \( C^{3,\alpha}(\Omega) \) function \( u \) locally as \( t \to 0 \). This means that \( u \in C^{3,\alpha}(\Omega) \) solves

\[
-\text{div}(\frac{Du}{\omega}) - F(x, -\frac{Du}{\omega}, \frac{1}{\omega}) + \phi(x,u, -\frac{Du}{\omega}, \frac{1}{\omega}) = 0 \quad \text{on } \Omega
\]

On the other hand from (6.13) we have

\[
 v_- \leq u \leq v_+ \quad \text{on } \Omega
\]

Therefore \( u \) belongs to \( C(\bar{\Omega}) \). Then \( u \) is the solution to the Dirichlet problem (6.2) for fixed boundary data \( \psi \in C(\partial \Omega) \). The uniqueness follows from the condition that \( \frac{\partial \phi}{\partial \nu} \geq 0 \). Thus we obtain Theorem 6.1 for general \( \phi \) just assuming Theorem 6.1 holds for \( \phi \equiv 0 \).

**The special case** \( \phi \equiv 0 \). From now on we assume \( \phi \equiv 0 \). It suffices to establish (6.14) for \( \phi \equiv 0 \) under the \( \text{No-f} \) property of \( \Omega \).

Define the set \( \Omega_a \) by

\[
 \Omega_a = \{ x \in N : d(x,\Omega), d(x, \Omega) < a \}
\]

for some positive constant \( a \). Recall that \( gr(\psi) := \{(x,\psi(x)) : x \in \partial \Omega\} \). Since \( \partial gr(u_t) = gr(\psi) \) in \( \partial \Omega \times \mathbb{R} \), by Lemma 2.11 and (6.10) one sees that

**Lemma 6.6.** There are two positive constants \( \delta := \delta(\Omega) \) and \( \Lambda := \Lambda(\Omega, \mu) \) such that \( U_t \) is a \( \Lambda \)-perimeter minimizer in \( \Omega_{\delta \times \mathbb{R}} \setminus gr(\psi) \) for all \( t \in (0,1) \).
Notice that $\delta$ and $\Lambda$ are independent of $t \in (0, 1)$. By Lemma 2.10, there is a sequence $\{ t_l \}_{l=1}^{\infty}$ converging to 0 such that $\{ U_{t_l} \}$ converges locally to $U_0$. Moreover $U_0$ is a $\Lambda$-perimeter minimizer in $\Omega_{\delta} \times \mathbb{R} \setminus gr(\psi)$. By Lemma 15.1 in [18] $U_0$ is the subgraph of a generalized function $u_0$ over $\Omega$. Here $u_0$ may take infinity value on $\Omega$. We define two sets

$$\Omega_{\pm} = \{ x \in \Omega : u_0(x) = \pm \infty \}$$

**Lemma 6.7.** Let $\Lambda, \delta$ be the two constants in Lemma 6.6. It holds that

1. $\Omega_{\pm}$ are $\Lambda$-perimeter minimizers (see Definition 2.4) in $\Omega_{\delta}$;
2. $\text{reg}(\partial \Omega_{\pm})$ are embedded, $C^2$ and orientable with

$$H_{\text{reg}(\partial \Omega_{\pm})}(x) = \pm f(x, \pm \gamma(x))$$

Here $x \in \text{reg}(\partial \Omega_{\pm})$ and $\gamma(x)$ is outward normal vector of $x \in \partial \Omega_{\pm}$ with respect to $\Omega_{\pm}$.

**Proof.** Define $T_s : N \times \mathbb{R} \rightarrow N \times \mathbb{R}$ as $T_s(x, z) = (x, z-s)$ for any $(x, z) \in N \times \mathbb{R}$ and $s \in \mathbb{R}$. Since $T_s$ is an isometry of $N \times \mathbb{R}$, by Lemma 6.6 $T_s(U_0)$ are $\Lambda$-perimeter minimizers in $\Omega_{\delta} \times \mathbb{R} \setminus gr(\psi - s)$. By (6.19) and Lemma 2.10 we have

$$\lim_{s \rightarrow +\infty} T_s(U_+) = \Omega_+ \times \mathbb{R},$$

Since $\psi \in C(\partial \Omega)$ and $\Omega$ is bounded, $gr(\psi - s)$ converges to $-\infty$ uniformly as $s \rightarrow +\infty$. By Lemma 2.11 and Lemma 2.10 $\Omega_+ \times \mathbb{R}$ is a $\Lambda$-perimeter minimizer in $\Omega_{\delta} \times \mathbb{R}$.

Without confusion, we use $P$ to denote the perimeter in $N$ and $N \times \mathbb{R}$ in the following derivation. Suppose $\Omega_{\pm}$ is not a $\Lambda$-perimeter minimizer in $\Omega_{\delta}$. Then there is a Caccioppoli set $E$ in $\Omega_{\delta}$ satisfying $E \Delta \Omega_{\pm} \subset \subset \Omega_{\delta}$ and

$$P(E, \Omega_{\delta}) \leq P(\Omega_{\pm}, \Omega_{\delta}) - \Lambda \text{vol}(E \Delta \Omega_{\pm}) - \varepsilon$$

for some $\varepsilon > 0$. Now we define a Caccioppoli set $E_T$ in $N \times \mathbb{R}$ as

$$\begin{align*}
\lambda_{E_T}(p) &= \lambda_{E \times [-T, T]}(p), \quad p \in N \times [-T, T] \\
\lambda_{E_T}(p) &= \lambda_{\Omega_{\pm} \times \mathbb{R}}(p), \quad p \in N \times (-\infty, -T) \cup N \times (T, +\infty)
\end{align*}$$

Here $\lambda_{E_T}$ is the characteristic function of $E_T$. It is easy to see that $E_T \Delta (\Omega_+ \times \mathbb{R}) \subset \subset \Omega_{\delta} \times (-2T, 2T)$. Let $W_{2T}$ be the open set $\Omega_{\delta} \times (-2T, 2T)$. By (6.22) one sees that

$$P(E_T, W_{2T}) + \Lambda \text{vol}(E_T \Delta (\Omega_+ \times \mathbb{R})) - P(\Omega_+ \times \mathbb{R}, W_{2T})$$

$$\leq P(E_T, \Omega_{\delta} \times (-T, T)) - P(\Omega_+ \times \mathbb{R}, \Omega_{\delta} \times (-T, T)) + 2\text{vol}(\Omega_{\delta}) + 2T\Lambda \text{vol}(\Omega_+ \Delta E)$$

$$\leq 2T(P(E, \Omega_{\delta}) - P(\Omega_+, \Omega_{\delta})) + 2\text{vol}(\Omega_{\delta}) + 2T\Lambda \text{vol}(\Omega_+ \Delta E)$$

$$\leq -2\varepsilon T + 2\text{vol}(\Omega_{\delta}) \leq 0$$

for any $T > \frac{\text{vol}(\Omega_{\delta})}{2\varepsilon}$. This is a contradiction that $\Omega_+ \times \mathbb{R}$ is a $\Lambda$-perimeter minimizer in $\Omega_{\delta} \times \mathbb{R}$. Thus our assumption is false and $\Omega_+$ is also a $\Lambda$-perimeter minimizer in $\Omega_{\delta}$. We obtain the conclusion (1).
By the conclusion (1) \( \text{reg}(\partial \Omega_+) \times \mathbb{R} = \text{reg}(\partial (\Omega_+ \times \mathbb{R})) \). Thus the conclusion (2) is equivalent that
\[
(6.24) \quad H_{\text{reg}(\partial \Omega_+) \times \mathbb{R}}(x, r) = F(x, \gamma(x), 0)
\]
Fix any point \((x, r)\) in \( \text{reg}(\partial \Omega_+) \times \mathbb{R} \). Since \( U_t \) locally converges to \( U_0 \), there is a \( \{ s_j \}_{j=1}^{\infty} \) converging to \( +\infty \) and \( t_j \) converging to 0 such that
\[
(6.25) \quad \lim_{j \to \infty} T_{s_j}(U_{t_j}) = \Omega_+ \times \mathbb{R}
\]
in \( W \) where \( W \) is a bounded neighborhood of \((x, r)\) in \( \Omega_+ \times \mathbb{R} \). Moreover, there is a positive number \( j_0 \) such that for any \( j \geq j_0 \) \( W \) lies above the subgraph of \( \psi - s_j \) in \( \partial \Omega_+ \times \mathbb{R} \). Thus for any \( j \geq j_0 \) \( \partial T_{s_j}(U_{t_j}) \cap W \) is contained in the graph of \( u_{t_j} - s_j \) over \( \Omega \). By (6.6) the mean curvature of \( \partial T_{s_j}(U_{t_j}) \) is
\[
(6.26) \quad H_{\partial T_{s_j}(U_{t_j})} = F(x, -\frac{Du_{t_j}}{\omega}, \frac{1}{\omega}) - t_j \frac{(u_{t_j} - s_j)}{\omega}
\]
where \((x, u_{t_j} - s_j) \in W \) and \( \omega = \sqrt{1 + |Du_{t_j}|^2} \). Now write \( F_j(x, X, r, z) = F(x, X, r) - t_j(z - s_j)r \). It is easy to see that
\[
(6.27) \quad N_{W}(F_j) \leq \nu
\]
independent of \( j \geq j_0 \). For any compact set \( K \) in \( W \setminus \text{sing}(\partial \Omega_+ \times \mathbb{R}) \), By Theorem 3.3 there is a positive constant \( \kappa \) such that
\[
(6.28) \quad \max_{\partial T_{s_j}(U_{t_j}) \cap K} |A|_{\partial T_{s_j}(U_{t_j})}^2 \leq \kappa
\]
for any \( j \geq j_0 \). By Lemma 2.10 the upward normal vector of \( \{ \partial T_{s_j}(U_{t_j}) \} \) should converge to the outward normal vector of \( \Omega_+ \times \mathbb{R} \) written as \( \gamma(x) \) for \( x \in \text{reg}(\partial \Omega_+) \). Moreover for any point \((x, u_{t_j} - s_j) \) in \( W \), \( \lim_{j \to \infty} t_j(u_{t_j} - s_j) = 0 \). By (6.28) and (6.26), we conclude (6.24).

As for the case \( \Omega_- \), we consider \( \tilde{u}_t = -u_t \). Let \( \tilde{U}_t \) be the subgraph of \( \tilde{u}_t \) over \( \Omega \) with the mean curvature of \( \partial \tilde{U}_t \) in \( \Omega \times \mathbb{R} \) given by
\[
(6.29) \quad H_{\partial \tilde{U}_t}(x, r) = \tilde{F}(x, -\frac{D\tilde{u}_t}{\omega}, \frac{1}{\omega}) - t \frac{\tilde{u}_t}{\omega}
\]
By (6.6), \( \tilde{F}(x, z, r) = -F(x, -z, r) \). As the case of \( \Omega_+ \) \( \{ \tilde{U}_t \}_{t>0} \) converges locally to \( \tilde{U}_0 \) which is a subgraph of \( \tilde{u}_0 = -u_0 \) over \( \Omega \). In this setting
\[
(6.30) \quad \tilde{\Omega}_+ = \{ \tilde{u}_0 = +\infty \} = \Omega_-
\]
Arguing exactly as the argument in the case of \( \Omega_+ \), \( \Omega_- \) is a \( \Lambda \)-perimeter minimizer in \( \Omega_\delta \) and the mean curvature of \( \text{reg}(\partial \Omega_-) \) is
\[
(6.31) \quad H_{\text{reg}(\partial \Omega_-)}(x) = \tilde{F}(x, \gamma(x), 0) = -F(x, -\gamma(x), 0) = -f(x, -\gamma(x))
\]
We conclude in the case of \( \Omega_- \). We conclude Lemma 6.7 \( \square \)

**Theorem 6.8.** By the \( Nc-f \) property of \( \Omega \) the number \( \sup_{\Omega, t \in (0,1)} |u_t| \) is finite.
Proof. By the Ne-f property of $\Omega$ and the definition of the perimeter in 2.1 vol$(\Omega_{+}) = 0$. Here vol is the volume of $N$.

Suppose sup$_{\Omega,t\in(0,1)}\{|u_t|\}$ is $+\infty$. Then there is a sequence $\{t_i\}_{i=1}^{\infty}$ converging to 0 and $\{x_i\}_{i=1}^{\infty} \in \Omega$ converging to a point $z \in \Omega$ such that (a) $u_{t_i}(x_i) > i$ or (b) $u_{t_i}(x_i) < -i$. The proof of both cases is similar. Thus we only give the proof of case (a). Let $U_i$ be the subgraph of $u_{t_i} - i$ over $\Omega$. By Lemma 6.6 $U_i$ is a $\Lambda$-perimeter minimizer in $\Omega_{\text{dr}} \times \mathbb{R} \setminus \text{gr}(\psi - i)$. Define $p_i = (x_i,0)$. Then $p_i \in U_i$. Fix $R > 0$. Because $\psi \in C(\partial \Omega)$ is uniformly bounded, there is $i_0 > 0$ such that for all $i \geq i_0$ $\Omega_{\text{dr}} \times (-R,R)$ is disjoint with the subgraph of $\psi - i$ in $\partial \Omega \times \mathbb{R}$. Thus for any $i \geq i_0$ $U_i$ are $\Lambda$-perimeter minimizers in $\Omega_{\text{dr}} \times (-R,R)$. Notice that $p_i \in U_i$. From [34] Theorem 21.11 we have

$$\text{Vol}(U_i \cap B_r(p_i)) \geq C(n) r^{n+1}$$

for sufficiently small $r < R$ independent of $i \geq i_0$. Here $n = \text{dim} N \geq 2$, Vol is the volume of the product manifold $N \times \mathbb{R}$. As $i \to +\infty$, $U_i$ should converge to the set $\Omega_{+} \times \mathbb{R}$. As a result, one sees that

$$\text{Vol}(\Omega_{+} \times \mathbb{R}) \geq C(n) R^{n+1}$$

This is a contradiction to $\text{vol}(\Omega_{+}) = 0$. With a similar derivation, case (b) will give a contradiction to that $\text{vol}(\Omega_{-}) = 0$.

Thus sup$_{\Omega,t\in(0,1)}\{|u_t|\}$ is a finite number. We arrive at the conclusion. □

Since $u_t$ is uniformly bounded on $\Omega$ for $t \in (0,1)$, as the proof of the general case of Theorem 6.1 in page 27 letting $t \to 0$ the sequence $\{u_t\}_{t\in(0,1)}$ converges to a $C^{3,\alpha}$ bounded function $u_0$ on $\Omega$ satisfying

$$-\text{div}\left(\frac{Du}{\omega}\right) - F(x,\frac{-Du}{\omega},\frac{1}{\omega}) = 0 \quad \text{on} \quad \Omega, \quad \omega = \sqrt{1 + |Du|^2}$$

To complete the proof of Theorem 6.1 it remains to show that the limit of $u_0(x)$ as $x$ approaches any boundary point $z \in \partial \Omega$ is $\psi(z)$. Suppose there is a point $x_0 \in \partial \Omega$ the previous statement does not hold. Recall that by Lemma 6.6 $u_0$ is a $\Lambda$-perimeter minimizer in $\Omega_{\text{dr}} \times \mathbb{R} \setminus \text{gr}(\psi)$. There is a point $(x_0,r_0) \in \partial U_0$ with $r_0 \neq \psi(x_0)$. Since $\psi$ is continuous on $\partial \Omega_0$ and $\Omega_{\text{dr}} \times \mathbb{R}$ is a $C^2$ domain, by Lemma 2.7 $\partial U_0$ is regular in $W$ which is a sufficiently small neighborhood of $(x_0,r_0)$. As in 6.27 and 6.28, in $W \partial U_0$ is $C^{3,\alpha}$ with its mean curvature equal to $F(x,X,\langle \vec{v}, \partial_r \rangle)$. Here $\vec{v}$ is the outward normal vector of $\partial U_0$ in $(x,r) \in W$ with $\vec{v} = X + \langle \vec{v}, \partial_r \rangle \partial_r$ for $X \in T_x \Omega$. At $(x_0,r_0)$ there are two cases $\vec{v} = \pm \gamma(x_0)$. However by the condition (i) in Theorem 6.1 we have

$$H_{\partial \Omega \times \mathbb{R}}(x,z) \geq \pm F(x,\pm \gamma(x),0)$$

No matter which case, applying the maximum principle in Theorem 4.6 we obtain that $\partial U_0$ coincides with $\partial \Omega \times \mathbb{R}$ in $W$. Since we can repeat the above process to obtain that $\partial U_0$ coincides with $\Omega \times \mathbb{R}$ far from $\text{gr}(\psi)$. Namely, the boundary of the graph of $u_0$ is unbounded. This is a contradiction because
\{u_t\}_{t \in (0,1)} \text{ is uniformly bounded.}

Therefore \( u_0 \in C(\Omega) \) and \( \lim_{x \in \Omega, x \to z} u_0(x) = \psi(z) \) for any \( z \in \partial \Omega \). With \( u_0 \) belongs to \( C(\Omega) \cap C^{3,\alpha}(\Omega) \) solving the Dirichlet problem \((6.2)\) satisfying \( u_0 = \psi \) on \( \partial \Omega \). The uniqueness is from the maximum principle. We conclude Theorem \([6.1]\) in the case that \( \phi \equiv 0 \).

The proof of Theorem \([6.1]\) is complete. \(\square\)

7. The PMC Plateau problem of disks

In this section we discuss the application of Theorem \([6.1]\) into the PMC Plateau problem in 3-manifolds. We mainly answer the following question with the Nc-f property.

**Question 7.1.** Let \( \Gamma \) be a null-homotopic Jordan curve in a \( C^2 \) compact 3-manifold \( M \) with connected boundary \( \partial M \) and \( f(x) \) be a \( C^{1,\alpha} \) function on \( M \). Can we find an immersed disk with boundary \( \Gamma \) in 3-manifold?

7.1. **Existence.** The main result of this subsection is stated as follows.

**Theorem 7.2.** Let \( \Gamma, f, M \) be given as in Question \([7.1]\). Suppose \( M \) is an Nc-f domain with the mean curvature of its boundary satisfying

\[
H_{\partial M}(x) \geq |f(x)| \quad \forall x \in \partial M
\]

Then there is an immersed disk solution to the PMC Plateau problem with boundary \( \Gamma \) in the sense of Definition \([7.5]\) below.

**Remark 7.3.** Gulliver \([21]\) studied the PMC Plateau problem of disks in a star-shaped compact 3-manifold satisfying a Ricci and mean curvature conditions. The author thought that the result of Gulliver-Spruck \([20]\) includes that \([20]\). The result of Duzaar-Steffen \([9]\) on PMC Plateau problems is different with \([21], [20]\) and Theorem \([7.2]\) in the following point: to make the PMC functional in \((7.3)\) non-negative, \([9]\) first fixes the domain and put appropriate condition on PMC functions; in our setting we reverse the sequence.

Let \( f \) be the \( C^{1,\alpha} \) function given as above. First we recall the definition of PMC functionals on 3-manifold \( M \) from Gulliver-Spruck \([20]\). First assume there is a \( C^1 \) vector \( Q \) satisfying

\[
\text{div}(Q) = f \quad \langle Q, Q \rangle < 1 \quad \text{on the interior of} \quad M
\]

Let \( B \) denote the unit open disk \( \{(u,v) : u^2 + v^2 < 1\} \). Fix a null-homotopic Jordan curve \( \Gamma \) in \( \partial M^3 \). Define \( B_\Gamma \) be the set of functions \( X : B \to M^3 \) in \( C(B) \cap W^{1,2}(B) \) such that \( X \) maps \( \partial B \) monotonely into \( \Gamma \). Consider the following functional

\[
\mathcal{F}(X) = \frac{1}{2} \int_B \langle X_u, X_u \rangle + \langle X_v, X_v \rangle du dv + \int_B \langle Q, X_u \wedge X_v \rangle du dv
\]

where \( X \in B_\Gamma \), \( X_u = X_s(\frac{\partial}{\partial u}) \) and \( X_v = X_s(\frac{\partial}{\partial v}) \) and \( Q \) is a \( C^1 \) vector field on \( M \) satisfying \( \text{div}(Q) = f(x) \).
Remark 7.4. Because we define the mean curvature of a smooth hypersurface by $\text{div}(\vec{v})$, our choice of $Q$ differs that of [20] by a constant.

Consider the minimizing problem

\begin{equation}
\min_{X \in B} \mathcal{F}(X)
\end{equation}

Definition 7.5. Suppose there is a map $X \in B$ realizes the minimum in (7.4). Such $X$ is called the disk solution to the PMC Plateau problem in (7.4).

For more geometric properties of $X(B)$ we refer to the work of Gulliver in [22]. On the other hand, the existence of $Q$ is the key ingredient of the PMC Plateau problem. Actually the main result of Gulliver-Spruck [20] can be summarized as the following existence.

Theorem 7.6. ([20], Proposition 4.1) Let $\Gamma, M, f$ be given in Question 7.1. Suppose there is a $C^1$ vector field $Q$ on $M$ satisfying (7.2) and $\langle Q, Q \rangle < 1 - \beta$ for some positive constant $\beta$. Then there is a disk solution to the PMC Plateau problem with boundary $\Gamma$ in the sense of Definition 7.5.

First we show that the equivalence between PMC graphs and a $C^1$ vector field $Q$ satisfying (7.2). Then it holds that

Theorem 7.7. Fix $M, f(x)$ as in Question 7.1. The existence of a $C^1$ vector $Q$ on $M$ satisfying (7.2) is equivalent to the existence of the graph of a $C^3, \alpha$ function $u$ on $M$ satisfying the PMC equation $-\text{div}(\frac{Du}{\omega}) + f(x) = 0$ on $M$ where $\omega = \sqrt{1 + |Du|^2}$.

Proof. First we show the right hand-side implies the left hand-side. If the function $u$ satisfying the PMC equation above, define $Q = \frac{Du}{\omega}$. Since $u$ is $C^3, \alpha$, then $\langle Q, Q \rangle < 1$ and $\text{div}(Q) = f(x)$.

Suppose there is a $C^1$ vector field $Q$ satisfying $\langle Q, Q \rangle < 1$ on $M$ and $\text{div}(Q) = f(x)$. By the classical divergence theorem, for any Caccioppoli set $A \neq \emptyset, M$, it holds that

\begin{equation}
| \int_A f(x) \text{dvol} | < P(A, M')
\end{equation}

where $M'$ is an open Riemannian manifold strictly containing $M$ and $P$ is the perimeter of $A$ in $M'$, $\text{dvol}$ is the volume form of $M$. Notice that all conclusions in [17] hold on Riemannian manifolds without changing any words. From [17, Theorem 1.1], there is a $C^2$ solution $u$ to the PMC equation $-\text{div}(\frac{Du}{\omega}) + f = 0$ on the interior of $M$. The $C^3, \alpha$ property of $u$ follows from the classical regularity theory when $f$ is $C^{1, \alpha}$. The proof is complete. \hfill \Box

Now we are ready to show Theorem 7.2

Proof. By Theorem 7.6 it is suffices to show that there is a vector field $Q$ satisfying (7.2) and $\langle Q, Q \rangle < 1 - \beta$ on $M$. Since $M$ is Nc-f and the mean curvature of its boundary satisfies (7.1), given any constant $e$, by
theorem [6.1] there is a $C^{3,\alpha}$ function $u \in C^{3,\alpha}(int(M)) \cap C(M)$ solving 
$-\text{div}(\frac{Du}{\omega}) + f = 0$ in the interior of $M$ and $u \equiv \epsilon$ on $\partial M$. Define $Q = \frac{Du}{\omega}$ 
where $\omega = \sqrt{1 + |Du|^2}$.

If $\sup_{x \in \text{int}(M)} \langle Q, Q \rangle = 1$. There is a sequence of $(x_k)_{k=1}^{\infty}$ in $\text{int}(M)$ converges to $y \in \partial M$ and $\lim_{k \to \infty} |Du_k| = +\infty$. Since $M$ is $C^2$, then $gr(u)$ is a $C^2$ surface with $C^2$ boundary. As a result, by the continuity of normal vector fields, $gr(u)$ is tangent to $\partial M$ at the point $(y, u(y))$. Without loss generality, we assume the normal vector of $gr(u)$ points outward of $M \times \mathbb{R}$ at $(y, u(y))$.

By (7.1), it holds that $gr(\gamma)$ is equal to $\pm f(x)$ and $H_{\partial M \times \mathbb{R}} \geq H_{gr(u)}$. From the Hopf Lemma, $\partial M \times \mathbb{R}$ coincides with $\Sigma \times \mathbb{R}$ in a neighborhood of $(y, u(y))$.

By the connectedness of $\partial M$, $H_{\partial M} = f$ on the whole $\partial M$ or $H_{\partial M} = -f$ on the whole $\partial M$. This is a contradiction that $M$ is an $N_c$-f domain. Therefore $\sup_{x \in \text{int}(M)} \langle Q, Q \rangle < 1$.

The proof is complete.

\textbf{Appendix A. Some results on mean curvature equations}

In this section we derive the interior and boundary gradient estimates of the solution to a certain class of mean curvature equations.

Throughout this section we suppose $\Omega$ is a $C^2$ bounded domain in a Riemannian manifold $N$ with dimension $n \geq 2$.

Let $M$ be the product manifold $N \times \mathbb{R}$. Suppose $u$ is a function in $C^3(\Omega)$ satisfying $\omega = \sqrt{1 + |Du|^2}$ and

\begin{equation}
-\text{div}(\frac{Du}{\omega}) + f(x, u, -\frac{Du}{\omega}, \frac{1}{\omega}) = 0
\end{equation}

where $f(x, z, X, r)$ is a $C^{1,\alpha}$ function on the tangent bundle of $M$ (written as $TM$), $p = (x, z) \in M$ for $x \in N$, $z \in \mathbb{R}$, $v = X + r\partial_r \in TM$ for $X \in TN$ and $r \in \mathbb{R}$. Here $\partial_r$ is the unit tangent vector filed of $\mathbb{R}$.

\textbf{Theorem A.1.} Fix two positive numbers $c_0$ and $\mu$. Suppose (a) $u \in C^3(\Omega)$ satisfies (A.1) and $\sup_\Omega |u| \leq c_0$; (b) $\Sigma$ is the graph of $u$ over $\Omega$ and

\begin{equation}
N_W(f) \leq \mu
\end{equation}

where $W = \{(x, z) : x \in \tilde{\Omega}, |z| \leq c_0\} \subset M$ and $N_W(f)$ is from Definition 3.4. Set $\Theta = \frac{1}{\omega}$. For any $\beta \in (0, 1)$, there is a nonnegative constant $c_1$ only depending on $\mu$, $\min_{x \in \Omega} \text{Ric}_x$ and $\beta$ such that

\begin{equation}
\Delta \Theta + \beta |A|^2 \Theta + \frac{\partial f}{\partial z}(1 - \Theta^2) - c_1 \Theta - \frac{\partial f}{\partial r} \langle \nabla \Theta, \partial_r \rangle \leq 0,
\end{equation}

where $A$ is the second fundamental form of $\Sigma$, $\Delta$ and $\nabla$ is the Laplacian and the covariant derivative of $\Sigma$.

\textbf{Proof.} By [50, Lemma 2.2] (its conclusion is true for any dimension greater than two) we have

\begin{equation}
\Delta \Theta + (|A|^2 + \bar{\text{Ric}}(\bar{v}, \bar{v})) \Theta - \langle \nabla H, \partial_r \rangle = 0
\end{equation}

where $\bar{v}$ is the mean curvature vector of $\Sigma$ in $N \times \mathbb{R}$.
where $\tilde{v}$ is the upward normal vector of $\Sigma$, $H$ is the mean curvature of $\Sigma$ with respect to $\tilde{v}$ and $Ric$ is the Ricci curvature of $M$. Notice that $\Theta = \langle \tilde{v}, \partial_r \rangle$.

Let $\partial_r^T$ be the tangent component of $\partial_r$ in $T\Sigma$. By definition we have

\begin{equation}
\label{eq:5}
\partial_r^T = \partial_r - \Theta \tilde{v}
\end{equation}

By (A.1), $H = -f$ with respect to $\tilde{v}$. As a result, we have

\begin{equation}
\label{eq:6}
\Delta \Theta + (|A|^2 + \tilde{Ric}(\tilde{v}, \tilde{v}) - \tilde{v}(f)) \Theta = 0
\end{equation}

Let $\{\partial_1, \cdots, \partial_n\}$ be a local frame in $T\Omega$, $\sigma_{ij} = \langle \partial_i, \partial_j \rangle$, $(\sigma^{ij}) = (\sigma_{ij})^{-1}$. Let $u_i$ and $u_{ij}$ be the corresponding first and second covariant derivatives of $u$. Set $u_i = \sigma^{ik} u_k$. The gradient of $u$ is represented by $Du = u^k \partial_k$. As a result, $v^k = \frac{u^k}{\omega}$, $\tilde{v} = \frac{1}{\omega} \partial_r - v^k \partial_k$. Then $\{X_i = \partial_i + u_i \partial_r, i = 1, \cdots, n\}$ is a frame on $T\Sigma$. The metric of $\Sigma$ is $g_{ij} = \langle (X_i, X_j) \rangle = (\sigma_{ij} + u^i u^j)$ with its inverse matrix $g^{-ij} = (\sigma^{ij} - u^i u^j)$.

The following two formulas are useful (see [49], section 2).

\begin{equation}
\label{eq:7}
v^k_i = \frac{1}{\omega} g^{kl} u_i \quad |A|^2 = v^k_i v^i_k
\end{equation}

Let $X = p^1 \partial_1 + \cdots + p^n \partial_n$. Let $f(x, z, X, r)$ be the function $f(x, z, p^1, \cdots, p^n, r)$. We compute

\[-\tilde{v}(f) \Theta = \Theta^2 (Df, Du) = \Theta v^k \partial_k f\]

\[= \Theta v^k (\partial_k f + \frac{\partial f}{\partial p_j} v^j_k + \frac{\partial f}{\partial r} \Theta_k)\]

\[= (v^k \partial_k f + \frac{\partial f}{\partial p_i} v^i_k v^k) \Theta + \Theta^2 |Du|^2 \frac{\partial f}{\partial z} - \frac{\partial f}{\partial r} (\nabla \Theta, \partial_r)\]

In what follows we apply (A.2) (also see Remark 3.2). For any $\varepsilon > 0$, we have

\[\frac{\partial f}{\partial p_i} v^i_k v^k \geq -\varepsilon |A|^2 - \frac{\mu}{4 \varepsilon}, \tilde{Ric}(\tilde{v}, \tilde{v}) \geq \tilde{Ric}(\frac{Du}{\omega}, \frac{Du}{\omega}) \geq \min_{x \in \Omega} Ric_x \]

Notice that $\Theta^2 |Du|^2 = (1 - \Theta^2)$. As a result, we have

\[(\tilde{Ric}(\tilde{v}, \tilde{v}) - \tilde{v}(f)) \Theta \geq \left( \min_{x \in \Omega} Ric_x - \mu(1 + \frac{1}{4 \varepsilon}) - \varepsilon |A|^2 \right) \Theta - \frac{\partial f}{\partial r} (\nabla \Theta, \partial_r) + \frac{\partial f}{\partial z} (1 - \Theta^2)\]

Putting this into (A.6) and letting $\varepsilon = 1 - \beta$ and $c_1 = \max_{x \in \Omega} |Ric_x| + \mu(1 + \frac{1}{4 \varepsilon})$, we obtain the conclusion. \hfill \Box

Let $B_r(x)$ denote the embedded ball in $N$ centering at $x$ with radius $r$. We roughly extend the result of Wang’s estimate [46] Theorem 1.1] into Riemannian manifolds as follows.

**Theorem A.2.** Take the assumptions in Theorem A.1. Suppose $\frac{\partial f}{\partial r} \geq 0$.

1. Let $B_{\rho}(x_0) \subset \Omega$. Then there is a positive constant $\mu_1$ depending on $\mu$, $\varepsilon_0$ and the Ricci curvature on $B_{\rho}(x_0)$ such that

\begin{equation}
\label{eq:7}
\max_{x \in B_{\frac{1}{2} \rho}(x_0)} |Du| \leq \mu_1
\end{equation}


Furthermore suppose \( \Omega \) is \( C^2 \) bounded and \( u \in C^1(\overline{\Omega}) \). Then there is a positive constant \( \mu_2 \) depending on \( \mu, c_0 \) and the Ricci curvature on \( \Omega \) such that
\[
\max_{\overline{\Omega}} |Du| \leq \mu_2 (1 + \max_{\partial \Omega} |Du|)
\]

**Proof.** Recall that \( \omega = \sqrt{1 + |Du|^2} = 1/\Theta \). We just give the sketch of the proof. Since \( \partial f / \partial z \geq 0 \), from (A.3) one sees that
\[
\Delta \Theta - c_1 \Theta - \frac{\partial f}{\partial r} \langle \nabla \Theta, \partial_r \rangle \leq 0,
\]
In the conclusion (1) set \( q(x) = 1 + \frac{u}{2c_0} - \frac{3}{4c_0} \text{dist}^2(x_0, x) \), let \( \eta = e^{Kq(x)} \) where \( K \) is a sufficiently large constant only depending on \( \rho, c_0 \) and \( \min_{x \in \Omega} \text{Ric}_x \). Checking the maximum point of \( \eta \omega \) will yield the estimate.

As the conclusion (2), let \( q(x) = u(x) \) and \( \eta = e^{Ku} \) where \( K \) is determined later. If \( \eta \omega \) achieves its maximum in \( \Omega \) to obtain that
\[
e^{Ku} \omega \leq \max \{ C(K), e^{Kc_0} \sqrt{1 + \max_{\partial \Omega} |Du|^2} \}
\]
Here \( K \) is a sufficiently large constant only depending on \( \mu \) in (A.2). This gives the conclusion (2). \( \square \)

Next, we record two boundary barrier functions used to construct the Perron subsolution and supsolution in section 4.3. Our derivation is similar as that of Marquardt [35].

**Theorem A.3.** Let \( L(u) \) be the operator given by
\[
L(u) := -\text{div} \left( \frac{Du}{\omega} \right) - H_1(x, u, -\frac{Du}{\omega}, \frac{1}{\omega}) + H_2(x, u, -\frac{Du}{\omega}, \frac{1}{\omega}) \frac{1}{\omega}
\]
where \( H_i(x, z, X, r) \) for \( i = 1, 2 \) are \( C^{1,\alpha} \) functions in the tangent bundle of \( N \times \mathbb{R} \). Let \( d(x) = \inf_{y \in \partial \Omega} d(x, y) \). Fix \( \varphi \in C^2(\overline{\Omega}) \) and any positive constant \( c_0 > \max_{\Omega} |\varphi| \).

(1) Supppose
\[
H_{\partial \Omega}(x) \geq H_1(x, \varphi, \gamma(x), 0)
\]
where \( \gamma(x) \) is the outward normal vector of \( \partial \Omega \). Then there are two positive constants \( \kappa > 0, \nu \) depending on \( c_0, H_1, H_2 \) and the \( C^2 \) norm of \( \varphi \) and \( d \) such that for \( u_+(x) = \varphi + \frac{\log(1 + \kappa d(x))}{\nu} \) it holds that
\[
L(u_+) \geq 0
\]
on the domain \( \Gamma_+ := \{ x \in \Omega : u_+(x) < c_0 \} \) with \( \pm u_\pm = c_0 \) on \( \partial \Gamma_+ \setminus \partial \Omega \);

(2) Suppose
\[
H_{\partial \Omega}(x) \geq -H_1(x, \varphi, -\gamma(x), 0)
\]
then there are two positive constants $\kappa > 0, \nu$ depending on $c_0$, $H_1, H_2$ and the $C^2$ norm of $\varphi$ and $d$ such that for $u_+(x) = \varphi + \frac{\log(1+\kappa d(x))}{\nu}$ it holds that

\begin{equation}
L(u_-) \geq 0
\end{equation}

on the domain $\Gamma_- := \{ x \in \Omega : u_-(x) < c_0 \}$ with $-u_- = c_0$ on $\partial \Omega \setminus \partial \Omega$.

**Remark A.4.** To construct the subsolution and supersolution of $L(u) = 0$ on $\Omega$, we do not require that (A.11)-(A.13) strictly holds on $\partial \Omega$.

**Proof.** Let $\{e_1, \cdots, e_n\}$ be any local orthonormal frame of $TN$. As a result any vector field $p$ in $TN$ can be rewritten as $p = p_ie_i$. Thus $|p|^2 = p_i^2$. Then we follow the notation in [16 (14.3)]. For any $u \in C^2(\Omega)$ we write

\begin{equation}
L'(u) := -(1 + |Du|^2) \frac{\nu}{2} L(u) = a^{ij} u_{ij} + b
\end{equation}

where the arguments of all functions are $x, u, p = Du$, $\omega = \sqrt{1 + |p|^2}$, $\Lambda = 1 + |p|^2$ and

\begin{align}
\alpha^{ij}(x, u, p) &= \Lambda a^{ij}_\omega + a_0^{ij}, \quad b(x, u, p) = |p| \Lambda b_\omega + b_0 \\
\alpha^{ij}_\omega &= \delta_{ij} - \frac{p_i p_j}{|p|^2}, \quad a_0^{ij} = \frac{p_i p_j}{|p|^2}, \quad b_\omega = -H_1(x, u, -\frac{p}{\omega}, \frac{1}{\omega}), \\
b_0(x, u, p) &= -\Lambda(\frac{1}{\omega + |p|} H_1(x, u, \frac{p}{\omega}, \frac{1}{\omega}) + H_2(x, u, \frac{p}{\omega}, \frac{1}{\omega}))
\end{align}

Now define $\phi(r) = \frac{\log(1+\kappa r)}{\nu}$. It holds that

\begin{equation}
\phi'(r) = \frac{\kappa}{\nu (1 + \kappa r)}, \quad \phi''(r) = -\nu
\end{equation}

Let $d(x)$ be the function $\inf_{y \in \partial \Omega} d(x, y)$ for $x$ in $\Omega$. We define $u_{\pm}(x) = \varphi(x) \pm \phi(d(x))$. Define $\Gamma := \{ x \in \Omega : d(x) < d_0 \}$ where $d_0$ is a sufficiently small positive constant determined later such that for any $x$ in $\Gamma_{\pm}$ there is a unique point $y \in \partial \Omega$ such that $d(x, y) = d(x, \partial \Omega)$. Let $\gamma(y)$ be the outward normal vector of $y$ which is equal to $-Du(y)$. From now on let $u = u_{\pm}$ and $p$ be $Du_{\pm}$.

In what follows we use $q(s)$ denote different continuous functions satisfying $q(0) = 0$, $|\frac{dq(s)}{s}| \leq C$ for all $r \in [-d_0, d_0]$ and some positive constant $C$. Let $\mu$ be the constant

\begin{equation}
\max_{x \in \Omega, |z| \leq c_0, (X, X) \leq 1, |r| \leq 1} \{|H_1(x, z, X, r)| + |H_2(x, z, X, r)|\}
\end{equation}
Let $\Delta$ be the Laplacian on $N$. Then on $\Gamma$ it holds

$$\tag{A.19} |b_0(x, u, Du)| \leq \mu \Lambda, \quad \Delta d(x) = -H_{\partial \Omega}(y) + q(d(x))$$

$$H_1(x, u, -\frac{p}{\omega}, \frac{1}{\omega}) = H_1(y, \varphi(y), \pm \gamma(y), 0) + q(d(x) + |\varphi'|^{-2}(d(x)))$$

$$p = D\varphi \pm \varphi'(d(x)) Dd = \varphi'(d(x))(1 + q(\frac{1}{\varphi'(d(x))}))$$

$$a^{ij}(x, u, p)(\varphi')^2 d_i d_j = \Lambda(1 + q(\frac{1}{\varphi'(d(x))}))$$

As a result

$$\tag{A.20} \pm L'(u)(x) = \Lambda \varphi' \Delta d(x) \pm a^{ij} \varphi_{ij} + \varphi'(1 + q(\frac{1}{\varphi'})) \Lambda \{ \mp H_1(y, \varphi(y), \pm \gamma(y), 0)$$

$$+ q(d + |\varphi'|^{-2}) \} + b_0 - \nu \Lambda(1 + q(\frac{1}{\varphi'}))$$

$$\leq \Lambda \varphi'(-H_{\partial \Omega}(y) \mp H_1(y, \varphi(y), \pm \gamma(y), 0))$$

$$+ \Lambda(C \varphi' + q(\frac{1}{\varphi'}) - \nu)$$

Since $\nu \geq 1$, we have $\varphi' = \frac{1 + \kappa d}{\nu + \kappa d} \leq 1$. Now there are two positive constants $C_2 \geq 1, C_3$ depending $\mu, d(x)$ and $\varphi$ such that if

$$\tag{A.21} \varphi' \geq C_2, \nu \geq \max\{1, C_3\}$$

then for any $x$ in $\Gamma$ we have

$$\tag{A.22} \pm L'(u)(x) \leq \Lambda \varphi'(-H_{\partial \Omega}(y) \mp H_1(y, \varphi(y), \pm \gamma(y), 0)) \leq 0$$

In the last inequality, we use the condition (A.11) and (A.13). Thus $\pm Lu_\pm \geq 0$ on $\Gamma$.

Now to achieve the goal in (A.22), first fix $\nu \geq \max\{1, C_3\}$. For $x$ in $\Gamma$, we require that

$$\tag{A.23} \varphi'(d(x)) = \frac{\kappa}{\nu(1 + \kappa d(x))} \geq \frac{\kappa}{\nu(1 + \kappa d_0)} \geq C_2;$$

$$\phi(d_0) = \frac{\log(1 + \kappa d_0)}{\nu} \geq c_0 + \max_{\Omega} |\varphi| := C_4$$

The above two inequalities are satisfied by choosing $d_0$ small such that $d_0 C_2 \nu \leq \frac{1}{2}$ and choosing $\kappa$ such that

$$\tag{A.24} \kappa = \max\{\frac{C_2 \nu}{1 - C_2 \nu d_0}, \frac{1}{d_0} (e^{C_4 \nu} - 1)\}$$

With those fixed $\kappa, \nu$ one sees that $\pm L(u_\pm) \geq 0$ on $\Gamma = \{x : d(x) < d_0\}$. Now define $\Gamma_\pm = \{x \in \Gamma : \pm u_\pm(x) < c_0\}$. We obtain the desirable conclusion. \qed
References

[1] R. Albuquerque. Notes on the Sasaki metric. *Expo. Math.*, 37(2):207–224, 2019.

[2] William K. Allard. On the first variation of a varifold. *Ann. of Math. (2)*, 95:417–491, 1972.

[3] Matthias Bergner. The Dirichlet problem for graphs of prescribed anisotropic mean curvature in $\mathbb{R}^{n+1}$. *Analysis (Munich)*, 28(2):149–166, 2008.

[4] Jean-Baptiste Casteras, Esko Heinonen, and Ilkka Holopainen. Dirichlet problem for $f$-minimal graphs. *J. Anal. Math.*, 138(2):917–950, 2019.

[5] Lizhi Chen and Hengyu Zhou. Prescribed mean curvature surfaces in 3-manifolds by a barrier assumption, i: Dehn’s lemma.

[6] Chiara Corsato, Colette De Coster, and Pierpaolo Omari. The Dirichlet problem for a prescribed anisotropic mean curvature equation: existence, uniqueness and regularity of solutions. *J. Differential Equations*, 260(5):4572–4618, 2016.

[7] M. Dajczer, J. H. de Lira, and J. Ripoll. An interior gradient estimate for the mean curvature equation of Killing graphs and applications. *J. Anal. Math.*, 129:91–103, 2016.

[8] Marcos Dajczer, Pedro A. Hinojosa, and Jorge Herbert de Lira. Killing graphs with prescribed mean curvature. *Calc. Var. Partial Differential Equations*, 33(2):231–248, 2008.

[9] Frank Duzaar and Klaus Steffen. Parametric surfaces of least $H$-energy in a Riemannian manifold. *Math. Ann.*, 314(2):197–244, 1999.

[10] Michael Eichmair. The Plateau problem for marginally outer trapped surfaces. *J. Differential Geom.*, 83(3):551–583, 2009.

[11] Michael Eichmair. Existence, regularity, and properties of generalized apparent horizons. *Comm. Math. Phys.*, 294(3):745–760, 2010.

[12] Michael Eichmair, Lan-Hsuan Huang, Dan A. Lee, and Richard Schoen. The space-time positive mass theorem in dimensions less than eight. *J. Eur. Math. Soc. (JEMS)*, 18(1):83–121, 2016.

[13] Michael Eichmair and Jan Metzger. Jenkins-Serrin-type results for the Jang equation. *J. Differential Geom.*, 102(2):207–242, 2016.

[14] Qiang Gao and Hengyu Zhou. The area minimizing problem in conformal cones, II. *Sci. China Math.*, 63(12):2523–2552, 2020.

[15] Qiang Gao and Hengyu Zhou. The area minimizing problem in conformal cones, I. *J. Funct. Anal.*, 280(3):Paper No. 108827, 39, 2021.

[16] Enrico Giusti. On the equation of surfaces of prescribed mean curvature. *Invent. Math.*, 46(2):111–137, 1978.

[17] Enrico Giusti. *Minimal surfaces and functions of bounded variation*, volume 80 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1984.

[18] Elias M. Guio and Ricardo Sa Earp. Errata: “Existence and non-existence for a mean curvature equation in hyperbolic space” [Commun. Pure Appl. Anal. 4 (2005), no. 3, 549–568; mr2167187]. *Commun. Pure Appl. Anal.*, 7(2):465, 2008.

[19] Robert Gulliver and Joel Spruck. Existence theorems for parametric surfaces of prescribed mean curvature. *Indiana Univ. Math. J.*, 22:445–472, 1972/73.

[20] Robert Gulliver, II. The Plateau problem for surfaces of prescribed mean curvature in a Riemannian manifold. *J. Differential Geometry*, 8:317–330, 1973.

[21] Robert D. Gulliver, II. Regularity of minimizing surfaces of prescribed mean curvature. *Ann. of Math. (2)*, 97:275–305, 1973.
[23] Gerhard Huisken and Alexander Polden. Geometric evolution equations for hypersurfaces. In Calculus of variations and geometric evolution problems (Cetraro, 1996), volume 1713 of Lecture Notes in Math., pages 45–84. Springer, Berlin, 1999.
[24] Howard Jenkins and James Serrin. The Dirichlet problem for the minimal surface equation in higher dimensions. J. Reine Angew. Math., 229:170–187, 1968.
[25] Hongjie Ju and Yannan Liu. Dirichlet problem for anisotropic prescribed mean curvature equation on unbounded domains. J. Math. Anal. Appl., 439(2):709–724, 2016.
[26] Atsushi Kasue. Ricci curvature, geodesics and some geometric properties of Riemannian manifolds with boundary. J. Math. Soc. Japan, 35(1):117–131, 1983.
[27] N. Korevaar. An easy proof of the interior gradient bound for solutions to the prescribed mean curvature equation. In Nonlinear functional analysis and its applications, Part 2 (Berkeley, Calif., 1983), volume 45 of Proc. Sympos. Pure Math., pages 81–89. Amer. Math. Soc., Providence, RI, 1986.
[28] Nicholas J. Korevaar. A priori interior gradient bounds for solutions to elliptic Weingarten equations. Ann. Inst. H. Poincaré Anal. Non Linéaire, 4(5):405–421, 1987.
[29] Nicholas J. Korevaar and Leon Simon. Continuity estimates for solutions to the prescribed-curvature Dirichlet problem. Math. Z., 197(4):457–464, 1988.
[30] John M. Lee and Thomas H. Parker. The Yamabe problem. Bull. Amer. Math. Soc. (N.S.), 17(1):37–91, 1987.
[31] Yanyan Li and Louis Nirenberg. Regularity of the distance function to the boundary. Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. (5), 29:257–264, 2005.
[32] Fang-Hua Lin. On the Dirichlet problem for minimal graphs in hyperbolic space. Invent. Math., 96(3):593–612, 1989.
[33] Rafael López. The Dirichlet problem on a strip for the $\alpha$-translating soliton equation. C. R. Math. Acad. Sci. Paris, 356(11-12):1179–1187, 2018.
[34] Francesco Maggi. Sets of finite perimeter and geometric variational problems, volume 135 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2012. An introduction to geometric measure theory.
[35] Thomas Marquardt. Remark on the anisotropic prescribed mean curvature equation on arbitrary domains. Math. Z., 264(3):507–511, 2010.
[36] Anna Sakovich. The Jang equation and the positive mass theorem in the asymptotically hyperbolic setting. Comm. Math. Phys., 386(2):903–973, 2021.
[37] Shigeo Sasaki. On the differential geometry of tangent bundles of Riemannian manifolds. Tohoku Math. J. (2), 10:338–354, 1958.
[38] Richard Schoen and Leon Simon. Regularity of stable minimal hypersurfaces. Comm. Pure Appl. Math., 34(6):741–797, 1981.
[39] Richard Schoen and Shing Tung Yau. Proof of the positive mass theorem. II. Comm. Math. Phys., 79(2):231–260, 1981.
[40] J. Serrin. The problem of Dirichlet for quasilinear elliptic differential equations with many independent variables. Philos. Trans. Roy. Soc. London Ser. A, 264:413–496, 1969.
[41] Leon Simon. Remarks on curvature estimates for minimal hypersurfaces. Duke Math. J., 43(3):545–553, 1976.
[42] Leon Simon. Lectures on geometric measure theory, volume 3 of Proceedings of the Centre for Mathematical Analysis, Australian National University. Australian National University, Centre for Mathematical Analysis, Canberra, 1983.
[43] Joel Spruck. Interior gradient estimates and existence theorems for constant mean curvature graphs in $M^n \times \mathbb{R}$. Pure Appl. Math. Q., 3(3, Special Issue: In honor of Leon Simon, Part 2):785–800, 2007.
[44] Italo Tamanini. Boundaries of Caccioppoli sets with Hölder-continuous normal vector. J. Reine Angew. Math., 334:27–39, 1982.
[45] Yuki Tsukamoto. The Dirichlet problem for a prescribed mean curvature equation. Hiroshima Math. J., 50(3):325–337, 2020.
[46] Xu-Jia Wang. Interior gradient estimates for mean curvature equations. *Math. Z.*, 228(1):73–81, 1998.

[47] Xu-Jia Wang. Convex solutions to the mean curvature flow. *Ann. of Math. (2)*, 173(3):1185–1239, 2011.

[48] Brian White. Subsequent singularities in mean-convex mean curvature flow. *Calc. Var. Partial Differential Equations*, 54(2):1457–1468, 2015.

[49] Hengyu Zhou. Nonparametric mean curvature type flows of graphs with contact angle conditions. *Int. Math. Res. Not. IMRN*, (19):6026–6069, 2018.

[50] Hengyu Zhou. The boundary behavior of domains with complete translating, minimal and CMC graphs in $\mathbb{N}^2 \times \mathbb{R}$. *Sci. China Math.*, 62(3):585–596, 2019.

(H. Z) College of Mathematics and Statistics, Chongqing University, Huxi Campus, Chongqing, 401331, P. R. China

Chongqing Key Laboratory of Analytic Mathematics and Applications, Chongqing University, Huxi Campus, Chongqing, 401331, P. R. China

Email address: zhouhyu@cqu.edu.cn