From Dimension-Free Manifold to Dimension-varying (Control) System

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Abstract—Starting from the projection among Euclidean space of different dimensions (ESDD), the inner product, norm, and distance of two vectors of different dimensions are proposed, which lead to an equivalence. As a quotient space of ESDDs over equivalence, the dimension-free Euclidean space (DFES) and dimension-free Euclidean manifold (DFEM) are obtained, which have bundled vector spaces as the tangent space at each point. Main objects for classical manifolds in differential geometry, such as functions, (co-)vector fields, tensor fields, etc., have been extended to DFEM. Using the natural projection from ESDDs to DFES, a fiber bundle structure is obtained, which has EFDDs as its total space and DFES as its base space. Then the dimension-varying dynamic system (DVDS) and dimension-varying control system (DVCS) are presented, which have DFEM as their state spaces. The realization, which is a lifting of dimension-free dynamic system (DFDS) or dimension-free control system (DFCS) from DFEM into ESDD, and the projection of DVDS or DVCS from ESDD onto DFEM are investigated.

Index Terms—Cross-dimensional projection, Euclidean space of different dimension (ESDD), dimension-varying dynamic (control) system (DVDS and DVCS), dimension-free Euclidean space (manifold) (DFES and DFEM), dimension-free dynamic (control) system (DFDS and DFCS), semi-tensor product (STP) of matrices.

I. INTRODUCTION

DVDSs and DVCSs exist widely in nature and man-made equipments or environments. For instance, in the internet users are joining/withdrawing frequently. In a biological system, cells are producing/dying from time to time. Some man-made mechanical systems are also of varying dimensions. For instance, the docking/undocking of spacecrafts [23], the vehicle clutch systems are connecting/disconnecting while speed changes [8]. The DVDS models are also used for specious population dynamics [22], [13].

Another interesting phenomenon, which stimulates our research interest, is: an object, or a complex system, may be described by models with different dimensions. For instance, in a power system a single generator can be modeled as a 2, 3, or 5, 6, or even 7, dimensional dynamic system [20]. In contemporary physics, the sting theory assumes the dynamics of stings to be the model for universe of time-space. But this model may have dimension 4 (Instain Relativity), 5 (Kalabi-Klein theory), 10 (Type 1 string), 11 (M-theory) or even 26 (Bosonic model) [18]. One observes from this phenomenon that two models with different dimensions might be very close or even equivalent. In other words, dimension-varying model may be proper to describe such dynamics.

So far, a classical way to deal with DVDSs and DVCSs is switching [23]. This approach ignores the dynamics of the system during the dimension-varying process. In practice, the transient period may be long enough so that the dynamics during this process is not ignorable. For instance, automobile clutch takes about 1 second to complete a connection or separation action. Docking/undocking of spacecrafts takes even longer. Not to mention that some processes might be continuously dimension-varying. In the latter cases, switching is almost meaningless.

To our best knowledge, there is no proper theory in existing mathematics to model DVDS and DVCS. No matter ordinal or partial differential equations or difference equations, only fixed dimension dynamic models can be treated. To provide a proper model for formulating DVDS and DVCS, a completely new framework should be created.

The objective of this paper is twofold. The first one is to build the DFES or DFEM, which provides a mathematical framework for DVDSs or DVCSs. The other is to use the geometric structure of DFEM to model, analyze and/or design controls for DVDSs and DVCSs.

This paper is a follow-up of our previous exploring works. In [6], [7] the dimension-free matrix theory has been proposed and investigated. As an application of dimension-free matrix theory, the dimension-varying linear (control) systems have been investigated [8]. The basic concept used there was the equivalence of vectors of different dimensions. This idea is also one of the key techniques in this paper. Two of the major contributions of this paper are (i) proposing DFES

This work is supported partly by the National Natural Science Foundation of China (NSFC) under Grants 62073315, 61074114, and 61273013.

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(DFEM) for the first time, which makes the “state space” of the DVDS/DVCS a mathematically clear geometric object; (ii) the formulation and control problems for general DVDS or DVCS, either linear or nonlinear, are unambiguously defined.

In this paper we first review the inner product, norm, and distance on ESDD $\mathcal{V} := \bigcup_{n=1}^{\infty} \mathcal{V}_n$ (equivalently, $\mathbb{R}^\infty := \bigcup_{n=1}^{\infty} \mathbb{R}^n$), which turns it into a topological space; and the vector space structure is posed on $\mathcal{V}$, which makes it a pseudo-vector space. We refer to [3], [7] for details. Then the projection of a vector to a certain Euclidian space is recalled, which was firstly proposed in [8] and then has been discussed in detail by [10], [11], [24], [25]. Then the cross-dimensional linear control systems are investigated as S-systems (semi-group systems). Though these things have been discussed before, they are re-organized systematically in section 2 of this paper.

The main issue of this paper is to provide a method to construct DFEM, starting from DFES. As a quotient space $(\Omega)$ of $\mathbb{R}^\infty$ over vector equivalence, it becomes a vector space. For each point in $\Omega$ a neighborhood coordinate bundle is proposed. Using neighborhood coordinate bundles, differentiable structure is obtained. Unlike the classical differential manifold, this differential structure poses on each point a bundled Euclidean spaces of different dimensions, called the tangent bundle, as its tangent space. A DFEM is a fiber bundle, which is locally homeomorphic to a coordinate neighborhood bundle of DFES.

Over a DFES (or DFEM), the smooth ($C^n$) function, vector field, co-vector field, distribution, co-distribution, tensor field, etc. are proposed. The integral curves for vector fields, or integral manifolds for distributions, are also properly defined. In one word, a dimension-free differential geometric structure is proposed for DFEMs. Finally, dimension-free tensor fields are also introduced, and using degree 2 covariant tensor fields, dimension-free Riemannian manifold and symplectic manifold are also proposed.

Then a DFDS or DFCS over a DFEM is presented using dimension-free vector bundles. The properties and control design of DFDSs are investigated via differential geometric structure of DFEM.

Lifting the trajectory of DFDS (or DFCS) to MDVS, a set of trajectories over Euclidean spaces of different dimensions are obtained. Using them a cross-dimensional trajectory of DVDS or DVCS can be constructed. Conversely, if a classical dynamic (control) system is defined on a Euclidean space, it can also be projected to DFES via natural projection.

Next, the paper attempts to explore the dynamic and control of DVDSs. The basic idea is as follows: Projection-lift connects the DFDS on DFEM (as the base space) with the DVDS on MDVS (as the total space). That is, the fiber bundle $(\mathbb{R}^\infty, \text{Pr}, \Omega)$ provides the bearing state space for both DFDSs and DVDSs. In addition to general DVDSs, particular attention has been paid to the dynamics of the transient process of classical dimension-varying systems, which have invariant dimensions except the transient period.

The STP of matrices was proposed by the author and his colleagues [3], [5]. It was the fundamental tool in previous works on DVDSs and DVDCSs [6], [7], [8]. It is also a basic tool in the approach of this paper. In this paper the default matrix product is assumed to be STP. We refer to [3], [5] for notations and basic concepts/results.

The rest of this paper is built up as follows. The ESDDs are investigated is Section 2. The vector space structure is proposed, and the equivalence among vectors of different dimensions is obtained. Then the inner product, norm, and distance are also proposed. The topology deduced by distance follows naturally. Section 3 considers the projection of a vector to another vector of different dimension, is also proposed. Section 3 considers the projections among Euclidian Spaces of different dimensions. First, the projection of a vector onto another vector, which may have different dimensions. Then the projection of a linear (control) system on a Euclidean space onto another Euclidean space of different dimension is considered. As a generalization, the projection of a dimension varying linear system onto a fixed dimension Euclidean space is also considered. The DFES is considered in Section 4. First, the quotient space is constructed from ESDDs by using vector equivalence. Then the topology and vector space structure of the quotient space are proposed, which make the quotient space a topological vector space, called the DFES. Using the natural projection from ESDDs to DFES, a fiber bundle structure is obtained. Using the coordinates from ESDDs, smooth functions over DFES are constructed. Section 5 considers the differential structure on DFES, which turns EFES into a DFEM. Then the (co)-vector fields, (co)-distributions, and the integral curves of vector fields over DFEM are proposed and investigated. In Section 6, the tensor-fields over DFEMs are constructed first. Using proper symmetric and skew-symmetric covariant tensor fields, the dimension-free Riemannian manifold and dimension-free Symplectic manifold are constructed respectively. As an application, Section 7 considers dimension-varying dynamic (control) systems. First, the projection of a nonlinear (control) system on an Euclidean space onto another Euclidean space of different dimension is proposed. Then the nonlinear (control) systems over DFESs are considered, which is then used to model dimension-varying nonlinear (control) systems over ESDDs. Finally, the control problems of
dimension-varying linear and nonlinear systems are considered in principle. Section 7 is some concluding remarks. First, the construction of DFESs (DFEMs) is summarized step by step. Then the modeling and control design of dimension-varying systems are also summarized. Finally, a conjecture is presented, which claims that DFESs (DFEMs) might be used as the framework for string theory.

II. EUCLIDEAN SPACES OF DIFFERENT DIMENSIONS

A. From Cross-Dimensional Set to Cross-Dimensional Vector Space

Consider an $n$ dimensional real vector space, denoted by $\mathcal{V}_n$. For simplicity, one can take $\mathcal{V}_n = \mathbb{R}^n$. To construct cross-dimensional state space, the set of mix-dimensional vectors, called ESDD, is defined as

$$\mathcal{V} := \bigcup_{n=1}^{\infty} \mathcal{V}_n.$$  

We similarly consider

$$\mathcal{V} = \mathbb{R}^\infty := \bigcup_{n=1}^{\infty} \mathbb{R}^n.$$ 

First, we define “addition” and “scalar product” over $\mathcal{V}$ to turn it into a vector space.

**Definition 2.1:**

(i) Let $x \in \mathcal{V}_m \subset \mathcal{V}$, $r \in \mathbb{R}$. Then the scalar product is defined as follows:

$$r \times x := rx \in \mathcal{V}_m.$$  

(ii) Let $x \in \mathcal{V}_m$, $y \in \mathcal{V}_n$, and $t = \text{lcm}(m, n)$ be the least common multiple of $m$ and $n$. Then the addition of $x$ and $y$ is defined as follows:

$$x \oplus y := (x \otimes 1_{t/m}) + (y \otimes 1_{t/n}) \in \mathcal{V}_t,$$  

where

$$1_k := [1, 1, \ldots, 1]^T_k.$$ 

is called a one-vector of dimension $k$.

Correspondingly, the subtraction of $y$ from $x$ is defined as follows:

$$x \ominus y := x \oplus (-y).$$  

A straightforward calculation can verify the following result:

**Proposition 2.2:** Set $\mathcal{V}$ with scalar multiplication defined by $[\mathbb{I}]$ and addition-subtraction defined by $[\mathbb{I}]-[\mathbb{I}]$ is a pseudo-vector space.

In fact, in a pseudo-vector space zero is not an element. It is considered as a set of zero elements as

$$0 := \{ [0, 0, \ldots, 0]^T_n \mid n = 1, 2, \ldots \},$$ 

which is called a zero-vector.

Hence, assume $x \in \mathcal{V}_n$, then, $-x \in \mathcal{V}_n$ such that

$$x + (-x) \in 0.$$  

We can also define finite dimensional subspaces as follows:

$$\mathcal{V}[r, n] := \bigcup_{k=n}^{r} \mathcal{V}_k.$$  

It is easy to verify that $\mathcal{V}[r, n]$, is also a pseudo-vector space, called a finite dimensional subspace of $\mathcal{V}$.

**Remark 2.3:** For notational ease, when $x \in \mathcal{V}_n$, we assume $-x \in \mathcal{V}_n$ too. Since $-x$ is not unique, this $-x$ is considered as a representative of the set of $-x$.

B. Equivalent Vectors

**Definition 2.4:**

(i) Let $x, y \in \mathcal{V}$, $x$ and $y$ are said to be equivalent, denoted by $x \leftrightarrow y$, if there exist two one-vectors $1_\alpha$ and $1_\beta$, such that

$$x \otimes 1_\alpha = y \otimes 1_\beta.$$  

(ii) The equivalence class of $x$ is denoted by

$$\bar{x} := \{ y \mid y \leftrightarrow x \}.$$  

**Remark 2.5:** It is necessary to prove that $\leftrightarrow$ is an equivalence relation. That is, (a) $x \leftrightarrow x$, (b) $x \leftrightarrow y \iff y \leftrightarrow x$, (c) $x \leftrightarrow y$ and $y \leftrightarrow z \implies x \leftrightarrow z$. The verification is straightforward.

The following proposition is obvious.

**Proposition 2.6:** Assume $x, y \in \mathcal{V}$. Then $x \leftrightarrow y$, if and only if,

$$x \ominus y \in 0.$$  

A partial order can be defined within an equivalent class.

**Definition 2.7:** Consider an equivalence class $\bar{x}$.

(i) A partial order, denoted by $\leq$, is defined as follows: Let $x, y \in \bar{x}$. If there exists a one-vector $1_s$ such that $x \otimes 1_s = y$, then $x \leq y$.

(ii) $x_1 \in \bar{x}$ is called the smallest element of $\bar{x}$, if $y \in \bar{x}$ and

$$y \leq x_1 \text{ then } y = x_1.$$  

Of course, $\leq$ can also be considered as a partial order on $\mathcal{V}$. But, if two vectors have such a relation, then they must
belong to an equivalence class. Hence, it is more precise to say that the partial order \( \leq \) is defined on \( \bar{x} \).

For the equivalence we have the following properties.

**Theorem 2.8:**

(i) If \( x \leftrightarrow y \), then there exists a \( \gamma \in \mathcal{V} \) such that
\[
x = \gamma \otimes 1_\beta, \quad y = \gamma \otimes 1_\alpha.
\] (8)

(ii) In each equivalence class \( \bar{x} \) there exists unique smallest element \( x_1 \in \bar{x} \).

**Proof:** (ii) is obvious, we prove (i) only. Let \( x \in \mathcal{V}_\alpha \), \( y \in \mathcal{V}_\beta \), \( x \leftrightarrow y \).

- Case 1: Assume \( \alpha \) and \( \beta \) are co-prime, that is, \( \gcd(\alpha, \beta) = 1 \). Since we have
\[
x \otimes 1_\beta = y \otimes 1_\alpha,
\] (9)

and \( \alpha \) and \( \beta \) are co-prime, comparing the right hand side of (9) with its left hand side yields
\[
x_i = y_j, \quad i = 1, \ldots, \alpha; \quad j = 1, \ldots, \beta.
\]

- Case 2: For general case, let \( \gcd(\alpha, \beta) = \ell > 1 \). Then we have \( \alpha = s\ell, \beta = t\ell \), where \( \gcd(s, t) = 1 \). Moreover,
\[
x \otimes 1_\ell = y \otimes 1_s.
\] (10)

Express \( x \) and \( y \) into their component-wise form as
\[
x = [x_1^1, \ldots, x_s^1; x_1^2, \ldots, x_s^2; \ldots; x_1^\ell, \ldots, x_s^\ell]^T,
\]
\[
y = [y_1^1, \ldots, y_s^1; y_1^2, \ldots, y_s^2; \ldots; y_1^\ell, \ldots, y_s^\ell]^T.
\]

Then the Eq. (10) can be expressed as
\[
[x_1^1, \ldots, x_s^1]^T \otimes 1_\ell = [y_1^1, \ldots, y_s^1]^T \otimes 1_s, \quad i = 1, \ldots, \ell.
\]

Since \( \gcd(s, t) = 1 \), according to Case 1, one sees that
\[
x_p^i = y_q^i, \quad p = 1, \ldots, s; \quad q = 1, \ldots, t; \quad i = 1, \ldots, \ell.
\]

Define \( \gamma \in \mathcal{V}_{\ell} \) by
\[
\gamma_i := x_p^i = y_q^i, \quad i = 1, \ldots, \ell.
\]

Then we have
\[
x = \gamma \otimes 1_s; \quad y = \gamma \otimes 1_t.
\]

**Remark 2.9:**

(i) If \( x = y \otimes 1_s \), then \( y \) is called a divisor vector of \( x \), and \( x \) is called a multiplier vector of \( y \). This relation determines the order \( y \preceq x \).

(ii) If Eq. (8) holds, and \( \alpha, \beta \) are co-prime, then the \( \gamma \) in Eq. (8) is called the maximum common divisor vector of \( x \) and \( y \), denoted by
\[
\gamma = \gcd(x, y).
\]

It is easy to prove that if \( z \) is also a common divisor vector of \( x \) and \( y \), then \( z \preceq \gamma \). Moreover, the maximum common divisor vector is unique.

(iii) If Eq. (6) holds and \( \alpha, \beta \) are co-prime, then
\[
\xi := x \otimes 1_\alpha = y \otimes 1_\beta
\] (11)

is called the least common multiple vector of \( x \) and \( y \), denoted by
\[
\xi = \text{lcm}(x, y).
\]

It is also easy to prove that if \( z \) is also a common multiple vector of \( x \) and \( y \), then \( \xi \preceq z \). Moreover, the least common multiple vector is also unique.

(iv) Consider an equivalence class \( \bar{x} \). Let \( x_1 \) be its smallest element. Then the elements in \( \bar{x} \) can be expressed as
\[
x_i = x_1 \otimes 1_i, \quad i = 1, 2, \ldots.
\] (12)

the element \( x_i \) is called the \( i \) th element of \( \bar{x} \). Hence the elements in \( \bar{x} \) can be expressed into a sequence as:
\[
\bar{x} = \{x_1, x_2, x_3, \ldots\}.
\]

According to partial order \( \preceq \), \( \bar{x} \) is a lattice.

**Proposition 2.10:**

(i) Assume \( x \in \mathcal{V} \), then \( (\bar{x}, \preceq) \) is a lattice.

(ii) Assume \( x, y \in \mathcal{V} \), then
\[
(\bar{x}, \preceq) \cong (\bar{y}, \preceq),
\] (13)

where \( \cong \) stands for lattice isomorphism. That is, any two equivalence classes are lattice isomorphic.

**Proof:**

(i) It is straightforward verifiable that for any two elements \( u, v \in \bar{x} \),
\[
\sup(u, v) = \text{lcm}(u, v); \quad \inf(u, v) = \gcd(u, v).
\]

Then the conclusion is obvious.

(ii) Assume \( \bar{x} = \{x_1, x_2, \cdots\} \) and \( \bar{y} = \{y_1, y_2, \cdots\} \). Define \( \pi : \bar{x} \rightarrow \bar{y} \) by
\[
\pi(x_i) = y_i, \quad i = 1, 2, \cdots.
\]

Then one sees easily that \( \pi \) is a lattice isomorphism.

**C. Norm and Distance of ESDD**

**Definition 2.11:** Let \( x \in \mathcal{V}_m \subset \mathcal{V}, \ y \in \mathcal{V}_n \subset \mathcal{V} \), and,
\( t = m \lor n \). Then the inner product of \( x \) and \( y \) is defined by
\[
\langle x, y \rangle_{\mathcal{V}} := \frac{1}{t} \langle x \otimes 1_{t/m}, y \otimes 1_{t/n} \rangle,
\] (14)
where $\langle \cdot , \cdot \rangle$ is the conventional inner product on $\mathbb{R}^t$. That is, if $x, y \in \mathbb{R}^t$, then
\[
\langle x, y \rangle = \sum_{i=1}^{t} x_i y_i.
\]
The inner product defined by (14) is called the weighted inner product, because there is a weight coefficient $1/t$.

**Remark 2.12:** The inner product on a real vector space $V$ should satisfy the following conditions.

(i) **(Distributive Law)**
\[
\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle, \quad x, y, z \in V.
\]

(ii) **(Symmetry)**
\[
\langle x, y \rangle = \langle y, x \rangle, \quad x, y \in V.
\]

(iii) **(Linearity)**
\[
\langle ax, y \rangle = a \langle x, y \rangle, \quad a \in \mathbb{R}, \quad x, y \in V.
\]

(iv) **(Non-negativity)**
\[
\langle x, x \rangle \geq 0, \quad \text{and} \quad \langle x, x \rangle = 0 \Rightarrow x = 0.
\]

Since $V$ is a pseudo-vector space, it is easy to prove that the inner product defined by Eq. (14) satisfies all the above requirements except one, that is, $\langle x, x \rangle = 0 \Rightarrow x = 0$ is replaced by $\langle x, x \rangle = 0 \Rightarrow x \in \overline{0}$. So the inner product defined by Eq. (14) is also called a pseudo-inner product.

Using inner product, the norm of $x \in V$ can be defined.

**Definition 2.13:** The norm of $x \in V$ is defined by
\[
\|x\|_V := \sqrt{\langle x, x \rangle}_V.
\]

**Remark 2.14:** The norm of a vector space $V$ should satisfy the following conditions.

(i) **(Triangle Inequality)**
\[
\|x + y\| \leq \|x\| + \|y\|, \quad x, y \in V.
\]

(ii) **(Linearity)**
\[
\|ax\| = |a| \|x\|, \quad a \in \mathbb{R}, \quad x \in V.
\]

(iii) **(Non-Negativity)**
\[
\|x\| \geq 0, \quad \text{and} \quad \|x\| = 0 \Rightarrow x = 0.
\]

One sees easily that the norm defined by Eq. (19) satisfies all above requirements, except one: that is, $\|x\| = 0 \Rightarrow x = 0$ should be replaced by $\|x\| = 0 \Rightarrow x \in \overline{0}$. It is also called a pseudo-norm.

Finally, we define the distance on $V$.

**Definition 2.15:** Let $x, y \in V$. The distance between $x$ and $y$ is defined by
\[
d(x, y) := \|x - y\|_V. \tag{23}
\]

**Remark 2.16:** The distance on $X$, denoted by $d : X \times X \rightarrow \mathbb{R}$, should satisfy the following conditions.

(i) **(Triangle Inequality)**
\[
d(x, z) \leq d(x, y) + d(y, z), \quad x, y, z \in X. \tag{24}
\]

(ii) **(Symmetry)**
\[
d(x, y) = d(y, x), \quad x, y \in X. \tag{25}
\]

(iii) **(Non-Negativity)**
\[
d(x, y) \geq 0, \quad \text{and} \quad d(x, y) = 0 \Rightarrow x = y. \tag{26}
\]

It is easily verified that the distance defined by (23) satisfies all the above requirements except one, that is, $d(x, y) = 0 \Rightarrow x = y$ should be replaced by $d(x, y) = 0 \Rightarrow x \leftrightarrow y$. Hence, this distance is also called a pseudo-distance.

**Remark 2.17:** The distance defined on a vector space is, in general, required to be invariant under displacement. That is,
\[
d(x + z, y + z) = d(x, y), \quad x, y, z \in X. \tag{27}
\]

**D. Topology on ESDD**

This subsection considers the topology on the ESDD
\[
V = \mathbb{R}^\infty = \bigcup_{n=1}^{\infty} \mathbb{R}^n.
\]

We refer to any standard textbook of topology for the basic concepts of topologies involving in this subsection, for instance, [9], [19]. In the following some topologies are considered.

- **Natural Topology:**

  Naturally, the topology on each $\mathbb{R}^n$ is considered as conventional topology. Precisely speaking, define the open balls in $\mathbb{R}^n$ with center at $c = (c_1, c_2, \cdots, c_n)$, and radius $r > 0$, denoted by
\[
B_r(c) := \left\{ (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \mid \sqrt{\sum_{i=1}^{n} (x_i - c_i)^2} < r \right\}.
\]

Taking
\[
B^n := \left\{ B_r(c) \mid c \in \mathbb{R}^n, r > 0 \right\}
\]
as topological basis, the topology on $\mathbb{R}^n$ generated by topological basis $B^n$ is the conventional topology on $\mathbb{R}^n$. 
Then each $\mathbb{R}^n$, $n = 1, 2 \cdots$, are considered as a set of clopen sets in $\mathcal{V}$. Such a topology is called the natural topology on $\mathcal{V}$, denoted by $\mathbf{N}$.

The following property is obvious.

**Proposition 2.18:**

(i) Assume $\emptyset \neq O_n \subset \mathbb{R}^n$ is an open set, then it is also open in $(\mathcal{V}, \mathbf{N})$.

(ii) $(\mathcal{V}, \mathbf{N})$ is second countable.

(iii) $(\mathcal{V}, \mathbf{N})$ is a Hausdorff space.

- **Distance-Deduced Topology:**

  Define open balls in $\mathcal{V} = \mathbb{R}^\infty$ by

  $$B_r(c) := \{x \in \mathbb{R}^\infty \mid d(x, c) < r\}, \quad c \in \mathbb{R}^\infty.$$ 

  Using $B := \{B_r(c) \mid c \in \mathbb{R}^\infty, r > 0\}$

  as a topological basis, the topology generated by $B$ is called the distance-deduced topology on $\mathbb{R}^\infty$, denoted by $\mathbf{D}$.

**Remark 2.19:**

(i) Assume $\emptyset \neq O_n \subset \mathbb{R}^n$ be an open set, it is not open under distance-deduced topology, i.e., not open in $(\mathcal{V}, \mathbf{D})$. This is because for a point $x \in O_n$ there exists a point $y = x \otimes 1_s \notin O_n$. But $d(x, y) = 0$, which means $x$ is not an internal point of $O_n$. Hence, $O_n$ is not open in $(\mathcal{V}, \mathbf{D})$.

(ii) $(\mathcal{V}, \mathbf{D})$ is not a Hausdorff space. To see this, consider $x$ and $x \otimes 1_s$, $s > 1$, which are two different points. But they are not separable in $(\mathcal{V}, \mathbf{D})$. It is clear that $(\mathcal{V}, \mathbf{D})$ is not even $T_0$.

(iii) It is easy to see that if $O$ is open in $(\mathcal{V}, \mathbf{D})$, then $O$ is also open in $(\mathcal{V}, \mathbf{N})$. Hence,

$$\mathbf{D} \subset \mathbf{N}.$$ 

That is, the distance-deduced topology $\mathbf{D}$ is rougher than the natural topology $\mathbf{N}$, or latter is tinier than the prior.

(iii) **Gluing Topology:**

From a topological space, a quotient space can be obtained by gluing some points together. The topology of such quotient space is called the gluing topology.

Starting from $(\mathcal{V}, \mathbf{N})$, the equivalent points can be glued together to get quotient space. That is, consider each equivalent class $\bar{x}$ as a point. Then the gluing topology is denoted by $\mathbf{Q}$, and the topological space $(\mathcal{V}, \mathbf{Q})$ is the quotient space.

Since

$$d(x, y) = 0 \Leftrightarrow x \leftrightarrow y,$$

it is obvious that the gluing topology is equivalent to distance-deduced topology.

(iv) **Product Topology:**

One way to understand $\mathcal{V} = \mathbb{R}^\infty$ is to consider

$$\mathbb{R}^\infty = \prod_{n=1}^{\infty} \mathbb{R}^n.$$ 

Then the product topology is generated by the topological basis

$$B = \{\prod_{n=1}^{\infty} O_n \mid O_n \subset \mathbb{R}^n \text{ is open, and } O_n = \mathbb{R}^n \text{ except finite } n\}.$$ 

The product topology is denoted by $\mathbf{P}$. It is easy to see that $\mathbf{P} = \mathbf{N}$.

### III. PROJECTIONS ON ESDD

#### A. Projection of A Vector to Another Vector

**Definition 3.1:** Assume $\xi \in \mathcal{V}_n$. A cross-dimensional projection of $\xi$ to $\mathcal{V}_m$, denoted by $\pi^m_n(\xi)$, is defined as follows:

$$\pi^m_n(\xi) := \text{argmin}_{x \in \mathcal{V}_m} \|\xi - x\|_\mathcal{V}. \quad (28)$$

Assume $t = \text{lcm}(n, m) = t$ and denote $\alpha := t/n, \beta := t/m$. Then the distance between $\xi$ and $x \in \mathcal{V}$ is

$$\Delta := \|\xi - x\|_\mathcal{V}^2 = \frac{1}{t} \|\xi \otimes 1_{t/n} - x \otimes 1_\beta\|^2.$$ 

Denote

$$\xi \otimes 1_{t/n} := (\eta_1, \eta_2, \cdots, \eta_t)^T,$$

where

$$\eta_j = \xi_i, \quad (i-1)\alpha + 1 \leq j \leq i\alpha; \ i = 1, \cdots, n.$$ 

Then,

$$\Delta = \frac{1}{t} \sum_{i=1}^m \sum_{j=1}^{\beta} (\eta_{(i-1)\beta+j} - x_i)^2. \quad (29)$$

Setting

$$\frac{\partial \Delta}{\partial x_i} = 0, \quad i = 1, \cdots, m$$

yields

$$x_i = \frac{1}{m} \left( \sum_{j=1}^{\beta} \eta_{(i-1)\beta+j} \right), \quad i = 1, \cdots, m. \quad (30)$$

That is, $\pi^m_n(\xi) = x$. Moreover, it is easy to verify the following orthogonality:

$$\langle \xi \otimes x, x \rangle_\mathcal{V} = 0.$$ 

Figure 1 shows that projection.

The above argument leads to the following conclusion:

**Proposition 3.2:** Let $\xi \in \mathcal{V}_n$. Then the projection of $\xi$ on $\mathcal{V}_m$, say, $x$, can be calculated by Eq. (30). Moreover, $\xi \otimes x$ and $x$ are orthogonal.
Moreover, we try to figure out this matrix. Finally, it is ready to verify that

\[ \xi^T x = \begin{bmatrix} 0.7143, 0.7143, 0.7143, -0.2857, -0.2857, -0.2857, -1.2857, -1.2857, -1.0000, -1.0000, 0, 0, 0, 1, 1, 0.8571, 1.8571, 1.8571, 1.8571, -2.1429, -2.1429, -2.1429 \end{bmatrix}. \]

Example 3.3: Assume \( \xi = [1, 0, -1, 0, 1, 2, -2]^T \in \mathbb{R}^7 \). Consider its projection on \( \mathbb{R}^3 \), denoted by \( \pi^T_3(\xi) := x \). Then \( \eta = \xi \otimes 1_3 \). Denote by \( x = [x_1, x_2, x_3]^T \), then

\[
x_1 = \frac{1}{7} \sum_{j=1}^{7} \eta_j = 0.2857
\]

\[
x_2 = \frac{1}{14} \sum_{j=8}^{21} \eta_j = 0
\]

\[
x_3 = \frac{1}{21} \sum_{j=15}^{21} \eta_j = 0.1429.
\]

Moreover,

\[
\xi^T x = \begin{bmatrix} 0.7143, 0.7143, 0.7143, -0.2857, -0.2857, -0.2857, -1.2857, -1.2857, -1.0000, -1.0000, 0, 0, 0, 1, 1, 0.8571, 1.8571, 1.8571, 1.8571, -2.1429, -2.1429, -2.1429 \end{bmatrix}.
\]

Finally, it is ready to verify that

\[
\left\langle \xi^T x, x \right\rangle_{V} = 0.
\]

Since the projection of a vector to a space of different dimension \( \pi^m_n \) is a linear mapping, it can be expressed by a matrix. Assume there exists a matrix \( \Pi^m_n \), such that the projection of \( \xi \in \mathbb{R}^n \) to \( \mathbb{R}^m \) can be expressed as

\[
\pi^m_n(\xi) = \Pi^m_n \xi, \quad \xi \in V_n.
\]

We try to figure out this matrix.

Let \( \text{lcm}(n, m) = t, \alpha := t/n \), and \( \beta := t/m \). Then

\[
\eta = \xi \otimes 1_\alpha = (I_n \otimes 1_\alpha) \xi
\]

\[
x = \frac{1}{\beta} \left( I_m \otimes 1_\beta \right) \eta = \frac{1}{\beta} \left( I_m \otimes 1_\beta \right) \left( I_n \otimes 1_\alpha \right) \xi.
\]

Hence we have

\[
\Pi^m_n = \frac{1}{\beta} \left( I_m \otimes 1_\beta \right) \left( I_n \otimes 1_\alpha \right).
\]

Using this structure, we have the following result.

**Lemma 3.4:**

(i) Let \( n \geq m \). Then \( \Pi^m_n \) is of full row rank. Hence, \( \Pi^m_n (\Pi^m_n)^T \) is invertible.

(ii) Let \( n \leq m \). Then \( \Pi^m_n \) is of full column rank. Hence, \( (\Pi^m_n)^T \Pi^m_n \) is invertible.

**Proof:**

(i) Assume \( n \geq m \). When \( n = m \), \( \Pi^m_n (\Pi^m_n)^T \) is an identity matrix, the conclusion is trivial. We, therefore, need only to consider the case when \( n > m \). According to the structure of \( \Pi^m_n \) determined by Eq. (32), it is easy to see that each row of \( \Pi^m_n \) contains at least two non-zero elements. Moreover, when \( j > i \) the column of non-zero element in row \( i \) is prior to the column of non-zero element in row \( j \), and only when \( j = i + 1 \) there is an overlapped column. This structure ensures the full row rank of \( \Pi^m_n \). Hence, \( \Pi^m_n (\Pi^m_n)^T \) is invertible.

(ii) According to Eq. (32), one sees easily that

\[
\Pi^m_n = \frac{\beta}{\alpha} (\Pi^m_n)^T.
\]

Hence, the full column rank of \( \Pi^m_n \) comes from the full row rank of \( \Pi^m_n \).

The following proposition shows that the projection from factor dimension space to multiple dimension space does not lose information.

**Proposition 3.5:** Let \( X \in \mathbb{R}^m \). Project it to \( \mathbb{R}^{km} \) and then project the image back to \( \mathbb{R}^m \), the vector \( X \) remains unchanged. That is,

\[
\Pi^m_{km} \Pi^m_k = I_m.
\]

**Proof:**

\[
\Pi^m_{km} \Pi^m_k = \frac{1}{k} \left( I_m \otimes 1_k^T \right) (I_m \otimes 1_k) = I_m.
\]

**B. Least Square Approximation of Linear Systems**

Consider a linear system

\[
\xi(t+1) = A \xi(t), \quad \xi(t) \in \mathbb{R}^n.
\]

Our goal is to find a matrix \( A_x \in \mathcal{M}_{m \times m} \), and construct a linear system on \( \mathbb{R}^m \) as

\[
x(t+1) = A_x x(t), \quad x(t) \in \mathbb{R}^m.
\]

Then take (35) as the project system of (34) on \( \mathbb{R}^m \).

We are mainly concerning about the trajectories. The trajectory of the idea project system should satisfy the same projection relation. That is,

\[
x(t, \pi(\xi_0)) = \pi^m_n (\xi(t, \xi_0)).
\]

Unfortunately, it is, in general, impossible to realize this. So we can only search such a system that makes the error of
Eq. (35) to be smallest. That is, we can only search the least square approximation.

Plugging Eq. (35) into Eq. (33) yields

$$\Pi_m^n \xi(t+1) = A_\pi \Pi_m^n \xi(t).$$  \hspace{1cm} (37)

Using Eq. (34) and noting that $\xi(t)$ is arbitrary, we have

$$\Pi_m^n A = A_\pi \Pi_m^n.$$  \hspace{1cm} (38)

Using Lemma 3.4 the least square approximated system can be obtained.

**Proposition 3.6:** Consider a continuous time linear system

$$\dot{\xi}(t) = A\xi(t), \quad \xi(t) \in \mathbb{R}^n.$$  \hspace{1cm} (39)

Its least square projected system on $\mathbb{R}^m$ is

$$\dot{x}(t) = A_\pi x(t), \quad x(t) \in \mathbb{R}^m,$$  \hspace{1cm} (40)

where,

$$A_\pi = \begin{cases} 
\Pi_m^n A (\Pi_m^n)^T (\Pi_m^n (\Pi_m^n)^T)^{-1} & n \geq m \\
\Pi_m^n A (\Pi_m^n)^T (\Pi_m^n)^T & n < m.
\end{cases}$$  \hspace{1cm} (41)

**Proof:** Assume $n \geq m$, right multiplying both sides of Eq. (38) by $\Pi_m^n (\Pi_m^n)^T$ yields the first equality of Eq. (41).

Assume $n < m$, we search the solution of the following form:

$$A_\pi = \hat{A}(\Pi_m^n)^T.$$  

Then the least square solution $\hat{A}$ can be obtained as

$$\hat{A} = \Pi_m^n A (\Pi_m^n)^T (\Pi_m^n)^T.$$  

Hence, we have

$$A_\pi = \Pi_m^n A (\Pi_m^n)^T (\Pi_m^n)^T,$$

which is the second equality of Eq. (41). \hspace{1cm} \Box

As an application, assume $n$ is very large, that is, system (34) is a large scale one. Then we may project it onto a lower dimensional space $\mathcal{V}_m$, where, $m \ll n$. That is, we have a lower dimensional trajectory to approximate the original one, which might reduce the computational complexity. In the sequel one may see that the projection of lower dimensional system into a higher dimensional vector space is sometimes also necessary.

Similarly, the projection of linear control systems can also be obtained.

**Corollary 3.7:**

(i) Consider a discrete time linear control system

$$\begin{cases} 
\xi(t+1) = A\xi(t) + Bu, \quad \xi(t) \in \mathbb{R}^n \\
y(t) = C\xi(t), \quad y(t) \in \mathbb{R}^p.
\end{cases}$$  \hspace{1cm} (42)

Its least square projected system on $\mathbb{R}^m$ is

$$\begin{cases} 
x(t+1) = A_\pi x(t) + \Pi_m^n Bu, \quad x(t) \in \mathbb{R}^m \\
y(t) = C_\pi x(t),
\end{cases}$$  \hspace{1cm} (43)

where, $A_\pi$ is determined by Eq. (41). Moreover,

$$C_\pi = \begin{cases} 
C(\Pi_m^n)^T (\Pi_m^n (\Pi_m^n)^T)^{-1}, & n \geq p \\
C((\Pi_m^n)^T (\Pi_m^n)^T)^{-1} (\Pi_m^n)^T, & n < p.
\end{cases}$$  \hspace{1cm} (44)

(ii) Consider a continuous time linear control system

$$\begin{cases} 
\dot{\xi}(t) = A\xi(t) + Bu, \quad \xi(t) \in \mathbb{R}^n \\
y(t) = C\xi(t), \quad y(t) \in \mathbb{R}^p.
\end{cases}$$  \hspace{1cm} (45)

Its least square projected system on $\mathbb{R}^m$ is

$$\begin{cases} 
\dot{x}(t) = A_\pi x(t) + \Pi_m^n Bu, \quad x(t) \in \mathbb{R}^m \\
y(t) = C_\pi x(t),
\end{cases}$$  \hspace{1cm} (46)

where, $A_\pi$ is determined by Eq. (41). Moreover, $C_\pi$ is determined by Eq. (44).

C. Approximation of Linear Dimension-varying System

Consider a discrete-time linear dimension-varying system

$$\xi(t+1) = A(t)\xi(t),$$  \hspace{1cm} (47)

where $\xi(t) \in \mathbb{R}^{n(t)}$, $\xi(t+1) \in \mathbb{R}^{n(t+1)}$, $A(t) \in \mathcal{M}_{n(t+1) \times n(t)}$.

Similarly to the constant dimensional system, we search its least square projection on $\mathbb{R}^m$ as

$$x(t+1) = A_\pi(t) x(t),$$  \hspace{1cm} (48)

where,

$$A_\pi(t) = \begin{cases} 
\Pi_m^{n(t)+1} A(\Pi_m^{n(t)})^T (\Pi_m^{n(t)})^T (\Pi_m^{n(t)})^T)^{-1} & n(t) \geq m \\
\Pi_m^{n(t)+1} A (\Pi_m^{n(t)})^T (\Pi_m^{n(t)})^T)^{-1} & n(t) < m.
\end{cases}$$  \hspace{1cm} (49)

The main advantage of this projected system is: it projects a dimension-varying system into a constant dimension system.

Similarly to the constant dimension case, the following result can be obtained.

**Theorem 3.8:**

(i) Consider a discrete-time linear dimension-varying system

$$\xi(t+1) = A(t)\xi(t),$$  \hspace{1cm} (50)

where, $\xi(t) \in \mathbb{R}^{n(t)}$, $A(t) \in \mathcal{M}_{n(t+1) \times n(t)}$. Its least square projected system is

$$x(t+1) = A_\pi(t) x(t),$$  \hspace{1cm} (51)

where, $A_\pi$ is determined by Eq. (49).
Consider a discrete-time linear dimension-varying linear system
\[
\begin{align*}
\xi(t+1) &= A(t)\xi(t) + B(t)u \\
y(t) &= C(t)\xi(t), 
\end{align*}
\] (52)
where \(\xi(t) \in \mathbb{R}^{n(t)}\), \(A(t) \in \mathcal{M}_{n(t+1) \times n(t)}\), \(B(t) \in \mathcal{M}_{n(t+1) \times m}\), \(C(t) \in \mathcal{M}_{p \times n(t)}\). Its least square projected control system is
\[
\begin{align*}
x(t+1) &= A_\pi(t)x(t) + \Pi^n_{m(t+1)}Bu, \quad x(t) \in \mathbb{R}^m \\
y(t) &= C_\pi(t)x(t), \quad y(t) \in \mathbb{R}^p, 
\end{align*}
\] (53)
where \(A_\pi\) is determined by Eq. (49). Moreover,
\[
C_\pi = \begin{cases} 
C(t)(\Pi^n_{m(t)})^T (\Pi^n_{m(t)}(\Pi^n_{m(t)})^T )^{-1}, & n(t) \geq p \\
C(t) (\Pi^n_{m(t)}(\Pi^n_{m(t)})^T )^{-1} (\Pi^n_{m(t)})^T, & n(t) < p.
\end{cases}
\] (54)

Consider a continuous-time linear dimension-varying control system
\[
\begin{align*}
\dot{\xi}(t) &= A(t)\xi(t) + B(t)u \\
y(t) &= C(t)\xi(t), 
\end{align*}
\] (55)
where \(\xi(t) \in \mathbb{R}^{n(t)}\), \(A(t) \in \mathcal{M}_{n(t+1) \times n(t)}\), \(B(t) \in \mathcal{M}_{n(t+1) \times r}\), \(C(t) \in \mathcal{M}_{p \times n(t)}\). Its least square projected control system is
\[
\begin{align*}
\dot{x}(t) &= A_\pi(t)x(t) + \Pi^n_{m(t+1)}Bu, \quad x(t) \in \mathbb{R}^m \\
y(t) &= C_\pi(t)x(t), \quad y(t) \in \mathbb{R}^p, 
\end{align*}
\] (56)
where \(A_\pi\) is determined by Eq. (49), \(C_\pi\) is determined by Eq. (56).

Later on, it will be seen that the fixed dimension projected system is a very useful realization of dimension-varying systems.

In the following an example is presented to depict projected system.

**Example 3.9:** Consider a dimension-varying system
\[
\begin{align*}
\xi(t+1) &= A(t)\xi(t) + B(t)u \\
y(t) &= C(t)\xi(t), 
\end{align*}
\] (57)
where
\[
\xi(t) \in \begin{cases} 
\mathbb{R}^5, & t \text{ is even,} \\
\mathbb{R}^4, & t \text{ is odd.}
\end{cases}
\]

A straightforward computation shows that
\[
\Pi^4_3 = (I_3 \otimes 1^T_4)(I_4 \otimes 1_3)/3 = \begin{bmatrix} 1 & 1/3 & 0 & 0 \\
2/3 & 2/3 & 0 \\
0 & 1 & 1/3 & 1
\end{bmatrix}
\]
\[
\Pi^5_3 = (I_3 \otimes 1^T_5)(I_5 \otimes 1_3)/3 = \begin{bmatrix} 1 & 2/3 & 0 & 0 & 0 \\
0 & 1 & 1/3 & 0 & 0 \\
0 & 0 & 2/3 & 1 & 1
\end{bmatrix}
\]
Then the projected system becomes
\[
\begin{align*}
x(t+1) &= A_\pi(t)x(t) + B_\pi(t)u \\
y(t) &= C_\pi(t)x(t), 
\end{align*}
\] (58)
where,
\[
A(t) = \tilde{A}_1; \quad B(t) = \tilde{B}_1; \quad C(t) = \tilde{C}_1 \quad t \text{ is even,}
\]
\[
A(t) = \tilde{A}_2; \quad B(t) = \tilde{B}_2; \quad C(t) = \tilde{C}_2 \quad t \text{ is odd,}
\]
where
\[
\tilde{A}_1 = \Pi^4_3 A_1 (\Pi^5_3 (\Pi^5_3)^T)^{-1}
\]
\[
= \begin{bmatrix} 0.9316 & -0.5556 & 1.6239 \\
1.4325 & -0.3111 & -0.7214 \\
1.0923 & -0.6000 & 0.7077
\end{bmatrix}
\]
\[ \hat{A}_2 = \Pi_2^1 A_2 (\Pi_2^1)^T (\Pi_2^1 (\Pi_2^1)^T)^{-1} \]
\[ = \begin{bmatrix} 0.8333 & 1.3333 & 0.8333 \\ 2.0500 & 1.2500 & -1.0500 \\ 0.9167 & -0.5833 & 0.4167 \end{bmatrix} ; \]
\[ \hat{B}_1 = \Pi_2^1 B_1 = \begin{bmatrix} 2.6667 & 1.3333 & \\ 0.3333 & -0.3333 & \end{bmatrix} ; \]
\[ \hat{B}_2 = \Pi_2^1 B_2 = \begin{bmatrix} 2.3333 & -1.6667 & \\ 1.0000 & -0.6667 & \end{bmatrix} ; \]
\[ \hat{C}_1 = C_1 (\Pi_2^1)^T (\Pi_2^1 (\Pi_2^1)^T)^{-1} \]
\[ = \begin{bmatrix} -0.0359 & 1.7333 & -0.4974 \\ 1.3333 & -2.6667 & 1.3333 \end{bmatrix} ; \]
\[ \hat{C}_2 = C_2 (\Pi_2^1)^T (\Pi_2^1 (\Pi_2^1)^T)^{-1} \]
\[ = \begin{bmatrix} 1.7000 & 2.0000 & -0.7000 \\ 0.0500 & 1.2500 & -2.0500 \end{bmatrix} \].

IV. Constructing DFES From ESDD

A. From Equivalence to Quotient Vector Space

Consider the ESDD \( \mathcal{V} \). Let \( x, y \in \mathcal{V} \). From above arguments it is known that \( x \leftrightarrow y \) if and only if, one of the following two equivalent conditions holds.

(i) There exist \( 1_\alpha \) and \( 1_\beta \), such that
\[ x \otimes 1_\alpha = y \otimes 1_\beta. \]

(ii) \[ d_\mathcal{V}(x, y) = 0. \]

Definition 4.1:

(i) The quotient space of \( \mathcal{V} \) under equivalence relation \( \leftrightarrow \), denoted by \( \Omega \), is called the DFES. That is, DFES is
\[ \Omega = \mathcal{V}/ \leftrightarrow . \] (59)

(ii) Let \( \bar{x}, \bar{y} \in \Omega \). Then the addition of \( \bar{x} \) and \( \bar{y} \) is defined by
\[ \bar{x} \dagger \bar{y} := \bar{x} \dagger y. \] (60)

Correspondingly, the subtraction is defined by
\[ \bar{x} \ddagger \bar{y} := \bar{x} \dagger (-\bar{y}), \] (61)

where, \( -\bar{y} = \bar{y} \).

The following proposition shows Eq. (60) and Eq. (61) are properly defined.

Proposition 4.2: The addition \( \dagger \) over equivalence class is consistent with equivalence \( \leftrightarrow \). That is, if \( x \leftrightarrow x' \) and \( y \leftrightarrow y' \), then \( x \dagger y \leftrightarrow x' \dagger y' \).

Proof: Since \( x \leftrightarrow x' \), according to Theorem 2.8 there exists \( \gamma \), say, \( \gamma \in \mathcal{V}_p \), such that
\[ x = \gamma \otimes 1_\alpha, \quad x' = \gamma \otimes 1_\beta. \]

Similarly, there exists \( \pi \), assume \( \pi \in \mathcal{V}_q \), such that
\[ y = \pi \otimes 1_s, \quad y' = \pi \otimes 1_t. \]

Let \( \xi = \text{lcm}(p, q), \eta = \text{lcm}(p, sq) \), and \( \eta = \xi \ell \). Then
\[ x \dagger y = (\gamma \otimes 1_\alpha) \dagger (\pi \otimes 1_s) = [\gamma \otimes 1_\alpha \otimes 1_{\ell/p}] + [\pi \otimes 1_s \otimes 1_{\ell/q}] = (\gamma \otimes 1_{\xi/p}) + [\pi \otimes 1_{\ell/q}] \otimes 1_\ell = (\gamma \dagger \pi) \otimes 1_\ell. \]

Hence, \( x \dagger y \leftrightarrow \gamma \dagger \pi \). Similarly, we have \( x' \dagger y' \leftrightarrow \gamma \dagger \pi \).

The conclusion follows. \( \blacksquare \)

Corollary 4.3: The addition \( \dagger \) and subtraction \( \ddagger \) over DFES \( \Omega \) defined by Eq. (60) and Eq. (61) respectively are well defined.

Let \( \bar{x} \in \Omega \). Then the scalar product on \( \Omega \) is defined by
\[ a\bar{x} := a\bar{x}, \quad a \in \mathbb{R}. \] (62)

It is also well defined.

The operators defined by Eq. (60), Eq. (61), and Eq. (62) make \( \Omega \) a vector space.

Theorem 4.4: Using the addition defined by Eq. (60) and the scalar product defined by Eq. (62), \( \Omega \) is a vector space.

Proof: It is ready to verify that the requirements of a vector space are satisfied. It is worth noting that unlike \( \mathcal{V} \), the equivalence class of zero \( \bar{0} \) is unique now, hence the inverse of \( \bar{x} = -\bar{x} \), which is also unique.

Consider the subspaces of ESDD and the corresponding subspaces of DFES.

Definition 4.5:

(i) Let \( p \in \mathbb{Z}_+ \) be a positive integer. Define \( p \)-upper truncated ESDD as
\[ \mathcal{V}^{[p]} := \bigcup_{\{s \mid p \mid s\}} \mathcal{V}_s. \] (63)

(ii) Define \( p \)-upper truncated DFES as
\[ \Omega^p := \mathcal{V}^{[p]}/ \leftrightarrow := \{ \bar{x} \mid x_1 \in \mathcal{V}_{p r}, r \geq 1 \}. \] (64)

(iii) Define \( p \)-lower truncated ESDD as
\[ \mathcal{V}^{[p]} := \bigcup_{\{s \mid s \mid p \}} \mathcal{V}_s. \] (65)

(iv) Define \( p \)-lower truncated DFES as
\[ \Omega_p := \mathcal{V}^{[p]}/ \leftrightarrow := \{ \bar{x} \mid x_1 \in \mathcal{V}_s, s \mid p \}. \] (66)
**Proposition 4.6:**

(i) $\Omega^p$, and $\Omega_p$, $p = 1, 2, \cdots$ are subspaces of $\Omega$.

(ii) If $i, j$, then $\Omega^j$ is a subspace of $\Omega^i$, $\Omega_i$ is a subspace of $\Omega_j$.

Next, we consider the lattice structure on Euclidean spaces in $\mathbb{R}^\infty$. We briefly review some basic concepts as follows.

**Definition 4.7:** Let $L$ be a set with a partial ordered $\leq$.

(i) $L$ is said to be a lattice, if for any two elements $a, b$ there exist a common least upper bound $u = \sup(a, b)$ and a common greatest lower bound $v = \inf(a, b)$.

(ii) Assume $(L, \leq)$ is a lattice and $H \subseteq L$. $H$ is said to be a sub-lattice of $L$ if $(H, \leq)$ is also a lattice.

(iii) Assume $(L, \leq)$ is a lattice and $H \subseteq L$ is a sub-lattice. Suppose for any $x \in L$ if there exists an $h \in H$ such that $x \leq h$ then $x \in H$, then $H$ is said to be an ideal of $L$.

(iv) Assume $(L, \leq)$ is a lattice and $H \subseteq L$ is a sub-lattice. Suppose for any $x \in L$ if there exists an $h \in H$ such that $h \leq x$ then $x \in H$, then $H$ is said to be a filter of $L$.

(v) Let $(L, \leq)$ and $(S, \prec)$ be two lattices. If there exists a mapping $\pi : L \rightarrow S$, satisfying

\[
\pi(\sup(a, b)) = \sup(\pi(a), \pi(b)),
\pi(\inf(a, b)) = \inf(\pi(a), \pi(b)),
\]

then $(L, \leq)$ is said to be lattice homomorphic to $(S, \prec)$, and $\pi$ is called a homomorphism. If in addition, $\pi$ is bijective and $\pi^{-1}$ is also a homomorphism, then $(L, \leq)$ and $(S, \prec)$ are said to be lattice isomorphic, and $\pi$ is called an isomorphism.

**Example 4.8:** Consider $\mathbb{R}^\infty$.

(i) A partial order $\prec$ is defined as follows: Let $\mathbb{R}^m, \mathbb{R}^n \in \mathbb{R}^\infty$. If there exists $1_s$ such that

\[
\mathbb{R}^m \otimes 1_s \subseteq \mathbb{R}^n,
\]

then $\mathbb{R}^m \prec \mathbb{R}^n$. Then it is easy to verify that $(\mathbb{R}^\infty, \prec)$ is a lattice. Moreover, for any two Euclidean spaces $\mathbb{R}^p$ and $\mathbb{R}^q$,

\[
\sup(\mathbb{R}^p, \mathbb{R}^q) = \mathbb{R}^{p \vee q}, \quad \inf(\mathbb{R}^p, \mathbb{R}^q) = \mathbb{R}^{p \wedge q}.
\]

(ii) $\mathbb{R}^{[1, p]}$ is an ideal of $\mathbb{R}^\infty$.

(iii) $\mathbb{R}^{[p, \infty]}$ is a filter of $\mathbb{R}^\infty$.

**Example 4.9:** The lattice structure $\mathbb{R}^\infty$ can be transferred to $\Omega$:

(i) Define $\Omega_{(n)} := \mathbb{R}^n / \leftrightarrow$, $n = 1, 2, \cdots$.

Then $\Omega = \bigcup_{n=1}^{\infty} \Omega_{(n)}$. Define $\Omega_{(m)} \prec \Omega_{(n)} \iff \mathbb{R}^m \prec \mathbb{R}^n$.

Then it is obvious that $(\Omega, \prec)$ is a lattice with

\[
\sup(\Omega_{(p)}, \Omega_{(q)}) = \Omega_{(p \vee q)}, \quad \inf(\Omega_{(p)}, \Omega_{(q)}) = \Omega_{(p \wedge q)}.
\]

(ii) $\Omega_p$ is an ideal of $\Omega$.

(iii) $\Omega^p$ is a filter of $\Omega$.

In fact, $\Omega$ has the same lattice structure as its filters.

**Proposition 4.10:** Let $p > 1$. The filter $\Omega^p$ is lattice isomorphic to $\Omega$.

**Proof:**

Define a mapping $\varphi : \Omega^p \rightarrow \Omega$ as

\[
\varphi(\Omega_{(np)}) := \Omega_{(p)}.
\]

Then it is easy to verify that $\varphi$ is a lattice isomorphism.

**B. Topology on DFES**

First, we extend the inner product over ESDD $\mathcal{V}$ to DFES $\Omega$.

**Definition 4.11:** Let $\bar{x}, \bar{y} \in \Omega$. Then their inner product is defined as

\[
\langle \bar{x}, \bar{y} \rangle := \langle \bar{x}, \bar{y} \rangle_{\mathcal{V}}, \quad x \in \bar{x}, \quad y \in \bar{y}.
\]

The following proposition shows Definition 4.11 is well posed.

**Proposition 4.12:** Eq. $\text{(68)}$ is properly defined. That is, it is independent of the choice of representatives $x$ and $y$.

**Proof:**

Assume $x_1 \leftrightarrow x_2$ and $y_1 \leftrightarrow y_2$. According to Theorem 2.8, there exist $x_0 \in \mathcal{V}$ and $y_0 \in \mathcal{V}$, such that

\[
x_1 = x_0 \otimes 1_\alpha; \quad x_2 = x_0 \otimes 1_\beta,
\]

\[
y_1 = y_0 \otimes 1_p; \quad y_2 = y_0 \otimes 1_q.
\]

First, we prove two facts:

- **Fact 1:** Let $s \wedge t = \xi$, and $s = a\xi$, $t = b\xi$, where, $a \wedge b = 1$. If $f, g$ satisfy

\[
sf = tg,
\]

then, $a f \xi = b g \xi$, i.e., $a f = b g$. Since $a \wedge b = 1$, there exists a $c$ such that

\[
f = cb, \quad g = ca.
\]

- **Fact 2:**

\[
\langle x, y \rangle_{\mathcal{V}} = \langle x \otimes 1_s, y \otimes 1_s \rangle_{\mathcal{V}}
\]

Eq. $\text{(69)}$ can be verified by definition directly.

Next, we consider

\[
\langle x_1, y_1 \rangle_{\mathcal{V}} = \langle x_0 \otimes 1_\alpha, y_0 \otimes 1_p \rangle_{\mathcal{V}}
\]

\[
= \left( x_0 \otimes 1_\alpha \otimes \frac{1}{\nu_{\alpha \nu_p}}, y_0 \otimes 1_p \otimes \frac{1}{\nu_{\alpha \nu_p}} \right)_{\mathcal{V}}
\]

\[
= \left( x_0 \otimes 1_{\nu_{\alpha \nu_p}}, y_0 \otimes 1_{\nu_{\alpha \nu_p}} \right)_{\mathcal{V}}.
\]
Hence we have
\[
\frac{s\alpha \vee tp}{s} = t \frac{s\alpha \vee tp}{t}.
\]
Using fact 2, one sees that
\[
\frac{s\alpha \vee tp}{s} = cb; \quad \frac{s\alpha \vee tp}{t} = ca.
\]
Using fact 1 yields
\[
\langle x_1, y_1 \rangle_\mathcal{V} = \langle x_0 \otimes 1, y_0 \otimes 1 \rangle_\mathcal{V} = \langle x_0 \otimes 1, y_0 \otimes 1 \rangle_\mathcal{V}.
\]
Similarly, we have
\[
\langle x_2, y_2 \rangle_\mathcal{V} = \langle x_0 \otimes 1, y_0 \otimes 1 \rangle_\mathcal{V}.
\]
The conclusion follows.

Since \( \Omega \) is a vector space, Eq. (68) defines an inner product on \( \Omega \). This inner product has the following properties.

**Proposition 4.13:** \( \Omega \) with the inner product defined by Eq. (68) is an inner product space. But it is not a Hilbert space.

**Proof:** It is obviously an inner product space. To see that it is not a Hilbert space, we construct a sequence as follows:
\[
\begin{align*}
x_1 &= a \in \mathbb{R} \\
x_{i+1} &= x_i \otimes 1_2 + \frac{1}{i+1} (\delta_{2,i+1}^1 - \delta_{2,i+1}^2), \quad i = 1, 2, \ldots.
\end{align*}
\]
It is obvious that this sequence is a Cauchy sequence. But it does not converge to any point \( x \in \mathcal{V} \). Let \( \bar{x}_i := \bar{x}_i \).

According to Proposition 4.12 it is easy to see that \( \{\bar{x}_i\} \) is also a Cauchy sequence in \( \Omega \), but it cannot converge to any point in \( \Omega \).

Given a point \( x \in \mathcal{V} \), a mapping \( \varphi_x : \mathcal{V} \rightarrow \mathbb{R} \) can be constructed as
\[
\varphi_x : y \mapsto \langle x , y \rangle_\mathcal{V}.
\]
Similarly, a point \( \bar{x} \in \Omega \) can be used to construct a mapping \( \varphi_{\bar{x}} : \Omega \rightarrow \mathbb{R} \) as
\[
\varphi_{\bar{x}} : \bar{y} \mapsto \langle \bar{x} , \bar{y} \rangle_\mathcal{V}.
\]
Conversely, not every linear mapping \( \varphi : \Omega \rightarrow \mathbb{R} \) can be expressed as a mapping deduced by an element as \( \varphi_{\bar{x}} \). This is because \( \Omega \) is an infinite dimensional vector space, while each element \( \bar{x} \in \Omega \) is a finite dimensional element.

Using the inner product defined by Eq. (68), the norm and distance on \( \Omega \) can also be defined.

**Definition 4.14:**
(i) Let \( \bar{x} \in \Omega \). The norm of \( \bar{x} \) is defined as
\[
\|\bar{x}\|_\mathcal{V} := \|x\|_\mathcal{V}. \tag{71}
\]
(ii) Let \( \bar{x}, \bar{y} \in \Omega \). The distance between \( \bar{x} \) and \( \bar{y} \) is defined as
\[
d_\mathcal{V}(\bar{x}, \bar{y}) := d_\mathcal{V}(x, y). \tag{72}
\]

According to Proposition 4.12, Eq. (71) and Eq. (72) are both well defined.

Finally, As a topological space, the topology on \( \Omega \) is deduced by the distance. This topology is equivalent to the quotient topology of \( (\mathcal{V}, \mathcal{N}) \) over equivalence. That is, the glued topology inherited from \( (\mathcal{V}, \mathcal{N}) \).

As a topological space, \( \Omega \) has the following properties.

**Proposition 4.15:** \( \Omega \) is second countable, Hausdorff space.

**Proof:** Since \( \mathcal{V}^n = \mathbb{R}^n \) is second countable, denote by \( \{O_i^n \mid i = 1, 2, \ldots\} \) its countable topological bases. Then \( \bigcup_{n=1}^\infty \bigcup_{i=1}^\infty O_i^n \) is a topological basis of \( \mathcal{V} \), which is also countable. Hence, as its quotient space, \( \Omega = \mathcal{V}/ \equiv \) is also second countable.

Since \( \Omega \) is a metric space, then \( \bar{x} \neq \bar{y} \), if and only if, \( d_\mathcal{V}(\bar{x}, \bar{y}) > 0 \). It is obvious that this space is a Hausdorff space. (In fact, it is easy to see that this space is \( T_4 \).)

C. Fiber Bundle Structure over ESDD and DFES

First, we recall the definition of a fibre bundle.

**Definition 4.16:** [14] Let \( T \) and \( B \) be two topological spaces, \( Pr : T \rightarrow B \) is a continuous surjective mapping. Then
\[
T \xrightarrow{Pr} B
\]
is called a fiber bundle, where \( T \) is the total space, \( B \) is the base space. For each \( b \in B \), \( Pr^{-1}(b) \) is called the fiber at \( b \).

The following result comes from the definition immediately.

**Proposition 4.17:** Let \( T = (\mathcal{V}, \mathcal{N}) \) be the total space, \( B = (\Omega, \mathcal{D}) \) be the base space, and \( Pr : T \rightarrow B \) be the natural projection, i.e., \( \bar{x} \mapsto \bar{x} \). Then
\[
(\mathcal{V}, \mathcal{N}) \xrightarrow{Pr} (\Omega, \mathcal{D})
\]
is a fiber bundle, which is called the dimension-free Euclidean bundle (DFEB).

The DFEB is said to be a discrete bundle, because the bundle at each point \( \bar{x} \) is a discrete countable (topological) subspace of ESDD \( \mathcal{V} \).

**Definition 4.18:**
(i) Two fiber bundles \( (T_i, Pr_i, B_i), \ i = 1, 2 \) are said to be homomorphic, if there exist two continuous mappings \( \pi : T_1 \rightarrow T_2 \) and \( \varphi : B_1 \rightarrow B_2 \), such that the Fig. 2 is commutative. In addition, if both \( \pi \) and \( \varphi \) are bijective, and \( \pi^{-1} : T_2 \rightarrow T_1 \) and \( \varphi^{-1} : B_2 \rightarrow B_1 \) are also making the Fig. 2 commutative, \( (T_i, Pr_i, B_i), \ i = 1, 2 \) are said to be isomorphic.

(ii) Two fiber bundles on \( B \), denoted by \( (T_i, Pr_i, B), \ i = 1, 2 \), are said to be homomorphic, if there exists a continuous mapping \( \pi : T_1 \rightarrow T_2 \), such that the Fig. 3 is commutative. In addition, if \( \pi \) is bijective, and
is obvious that "local coordinate neighborhood" for each point \(\bar{x}\) is called a leaf of the bundle of coordinate neighborhood bundle of \(\bar{x}\).

An example is given in the following to depict the bundle of coordinate neighborhood.

**Example 4.22:** Assume \(x = (\alpha, \alpha, \beta, \beta)^T \in \mathbb{R}^4\), then \(\bar{x} = \{x_1, x_2, \cdots\}\), where, \(x_i = (\alpha, \beta)^T \in \mathbb{R}^2\). Hence \(\dim(\bar{x}) = 2\). Consider \(O_{\bar{x}} = B_r(\bar{x}) \subseteq \Omega\), which is an open ball neighborhood of \(\bar{x}\). Then the set of coordinate charts, deduced by \(O_{\bar{x}}\), is

\[
V_O = \{B_{r_1}(x_1), B_{r_2}(x_2), \cdots\},
\]

where, \(r_i = 1/\sqrt{2r}, x_i = (\alpha, \beta)^T \otimes 1_i, i = 1, 2, \cdots\). The bundle of coordinate neighborhood of \(\bar{x}\) is \(V_O \xrightarrow{P_r} O_{\bar{x}}\).

Fig. 5 demonstrates the bundle of coordinate neighborhood of \(\bar{x}\).

D. Coordinate Neighborhood and Continuous Function on DFES

To establish a differential structure on DFES, we need a "local coordinate neighborhood" for each point \(\bar{x} \in \Omega\). Since \(\Omega\) is a dimension-free space, the coordinate neighborhoods are not classical ones in standard differential manifold. In fact, they are sub-fiber bundles of DFEB.

**Definition 4.20:** Let \(\bar{x} \in \Omega\). The dimension of \(\bar{x}\), denoted by \(\dim(\bar{x})\), is the dimension of the smallest element in \(\bar{x}\). That is,

\[
\dim(\bar{x}) = \dim(x_1) = \min_{x \in \bar{x}} \dim(x) \quad (73)
\]

**Definition 4.21:** Let \(\bar{x} \in \Omega\), and \(\dim(\bar{x}) = p\). Assume \(O_{\bar{x}}\) is an open neighborhood of \(\bar{x} \in \Omega\). That is, \(\bar{x} \in O_{\bar{x}}\), and \(O_{\bar{x}} \subseteq \Omega\) is open. Then

\[
V_{O_{\bar{x}}} := P_r^{-1}(O_{\bar{x}}) \cap V^{\rho p}, \quad r = 1, 2, \cdots \quad (76)
\]

is called a leaf of the bundle of coordinate neighborhood bundle of \(\bar{x}\).

Note that the set of coordinate charts \(V_O\) does not include all the inverse image of \(O\), i.e.,

\[
V_O \subseteq \bigcap_{\bar{x} \notin V} P_r^{-1}(O).
\]

But it can provide coordinates for all points within \(O\). The following proposition shows this.

**Proposition 4.23:** Assume \(\bar{y} \in O\), then,

\[
Pr^{-1}(\bar{y}) \cap V_O \neq \emptyset \quad (77)
\]

**Proof:** Assume \(\bar{y} \in O\), \(\dim(\bar{x}) = p\), \(\dim(\bar{y}) = q\), \(r = p \lor q\), then

\[
y_{r/q} \in Pr^{-1}(O) \cap V^r \subseteq V_O.
\]
Now we are ready to define continuous function on $\Omega$.

**Definition 4.24:** Let $f : \Omega \to \mathbb{R}$ be a real function on $\Omega$.

(i) Define

$$f(x) := f(\bar{x}), \quad x \in \mathcal{V}.$$  \hfill (78)

Then $f : \mathcal{V} \to \mathbb{R}$ is a real function on $\mathcal{V}$.

(ii) If for each point $\bar{x} \in \Omega$ there exists a neighborhood $O_{\bar{x}}$ of $\bar{x}$ such that on each leaf $\mathcal{V}_{\bar{x}} \subset \mathbb{R}^r$ $f \in C(\mathcal{V}_{\bar{x}})$, then $f$ is called a continuous function on $\Omega$.

(iii) If on each leaf of the bundle of coordinate neighborhood $f \in C^r(\mathcal{V}_{\bar{x}})$, then $f$ is called a $C^r$ function on $\Omega$, where $r = 1, 2, \cdots, \infty, \omega$, $r = \omega$ means $f$ is an analytic function.

**Remark 4.25:** In definition 4.21 the set of coordinate neighborhood is used. In fact, up to now only global coordinates are used. So the definition can also be global coordinates. That is, consider $\mathbb{R}^rp$ as each leaf. Using local coordinate neighborhood is for the definition to be applicable to DFEM, which will be discussed in the sequel.

Constructing a differentiable function on $\Omega$ is very difficult. Our technique to construct such a function is to transfer a smooth function on $\mathcal{V}$ to $\Omega$. Note that $\mathcal{V}^n = \mathbb{R}^n$ is a clopen subset of $\mathcal{V}$. $f : \mathcal{V} \to \mathbb{R}$ is continuous, if and only if, $f_n := f|\mathcal{V}^n$, $n = 1, 2, \cdots$, is continuous. Hence, it is reasonable to transfer an $f \in C^r(\mathbb{R}^n)$ to $\Omega$.

**Definition 4.26:** Let $f \in C^r(\mathbb{R}^n)$. Define $\tilde{f} : \Omega \to \mathbb{R}$ as follows: Let $\bar{x} \in \Omega$ and dim($\bar{x}$) = $m$. Then

$$\tilde{f}(\bar{x}) := f(\Pi^m_n(x_1)), \quad \bar{x} \in \Omega,$$  \hfill (79)

where $x_1 \in \bar{x}$ is the smallest element in $\bar{x}$.

**Proposition 4.27:** Assume $f \in C^r(\mathbb{R}^n)$, then the function $\tilde{f}$ defined by Eq. (79) is $C^r$, that is, $\tilde{f} \in C^r(\Omega)$.

**Proof:** Given $\bar{x} \in \Omega$, where $\text{dim}(\bar{x}) = m$. Consider a leaf of the bundle of coordinate neighborhood $\mathcal{V}_{\bar{x}}$ of $\bar{x}$. Assume $y \in \mathcal{V}_{\bar{x}}$, consider the following two cases:

- Case 1: $y \in \mathbb{R}^m$ is the smallest element of $\bar{y}$. By definition,

$$\tilde{f}(y) = f(\Pi^m_n y).$$  \hfill (80)

- Case 2: $y_1 \in \bar{y}$ is the smallest element of $\bar{y}$ and dim($y_1$) = $\xi$. Then there exists $s$ such that $y = y_1 \otimes 1_s$. Since $y \in \mathbb{R}^m$, then $\xi s = mr$. By definition,

$$\tilde{f}(y) = f(\bar{y}) = f(\Pi^s_n y_1).$$  \hfill (81)

Denote

$$z_0 := \Pi^s_n y_1 \in \mathbb{R}^m.$$

Then $z_0$ is the point on $\mathbb{R}^m$, which is closest to $y_1$. Since $y \leftrightarrow y_1$. According to Proposition 4.12 we know

$$d_{\mathcal{V}}(z, y) = d_{\mathcal{V}}(z, y_1), \quad z \in \mathbb{R}^m.$$  \hfill (82)

Hence, $z_0$ is also the point on $\mathbb{R}^m$ which is closest to $y$. That is,

$$\Pi^m_n y = z_0 = \Pi^s_n y_1.$$  \hfill (83)

Hence, Eq. (81) becomes Eq. (80). It is obvious that $\tilde{f}$ is a $C^r$ function on $\mathcal{V}_{\bar{x}}$.

The following is a simple example.

**Example 4.28:** Given

$$f(x_1, x_2, x_3) = x_1 + x_2^2 - x_3 \in C^\omega(\mathbb{R}^3).$$  \hfill (84)

(i) Assume $\bar{y} \in \Omega$, where $y_1 = (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5)^T \in \mathbb{R}^5$. It is easy to calculate that

$$\Pi^5_3 = \frac{1}{5} \begin{pmatrix} I_3 \otimes 1^T_2 \end{pmatrix}, \quad \Pi^5_3 = \Pi^5_3,$$

where $\mathcal{V}^3 \subset \mathcal{V}^5$.

Hence we have

$$\tilde{f}(y) = f(\Pi^5_3 y_1) = \frac{1}{5}(3\xi_1 + 2\xi_2) + \frac{1}{5}(\xi_2 + 3\xi_3 + \xi_4)^2 - \frac{1}{5}(3\xi_1 + 3\xi_5).$$

(ii) Assume $\bar{y} \in \Omega$, where $y_1 = (\xi_1, \xi_2) \in \mathbb{R}^2$.

- Consider $\mathcal{V}^2$:

$$\Pi^2_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0.5 & 0.5 \\ 0 & 1 \end{pmatrix}$$

Fig. 5. Bundle of coordinate neighborhood
Then,
\[ \hat{f}|_{\nu^0} = \xi_1 + \frac{1}{4}(\xi_1 + \xi_2)^2 - \xi_2. \]

- Consider \( \nu^0 \): Since
\[
\Pi^4 = \frac{1}{4} \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}
\]
We have
\[
\hat{f}|_{\nu^0} = \frac{1}{4}(3\xi_1 + \xi_2) + \frac{1}{16}(\xi_2 + \xi_3)^2 - \frac{1}{4}(\xi_3 + 3\xi_4).
\]

V. DIFFERENTIAL STRUCTURE ON DFES/DFEM

A. From DFES to DFEM

Definition 5.1: Given a fiber bundle \( T \xrightarrow{\pi} B \).

(i) Let \( \emptyset \neq O \subset B \) be an open set of \( B \). Then
\[
\pi^{-1}(O) \xrightarrow{\pi} O
\]
is called the open sub-bundle (over \( O \)).

(ii) Let \( b \in B \). \( O_b \subset B \) is an open neighborhood of \( b \).

Then the open sub-bundle (over \( O_b \)) is called an open neighborhood bundle of \( b \).

Definition 5.2: Assume

(i) \( M = \bigcup_{n=1}^{\infty} M_n \), where \( M_n, \ n = 1, 2, \ldots \) are \( n \) dimensional \( C^r \) manifolds;

(ii) there is an equivalence relation \( \sim \) over \( M \), and \( B = M/\sim \) is the quotient space;

(iii) under the natural projection \( \pi, M \xrightarrow{\pi} B \) becomes a fiber bundle.

\( M \xrightarrow{\pi} B \) is called a DFEB with \( B \) as a DFEM, if the following conditions are satisfied.

(i) There is an open cover \( \{O_\lambda | \lambda \in \Lambda \} \) of \( B \), that is,
\[
\bigcup_{\lambda \in \Lambda} O_\lambda = B.
\]

(ii) For each \( O_\lambda \), there exists an open sub-bundle \( (\Pr^{-1}(U_\lambda) \xrightarrow{\Pr} U_\lambda) \) of \( (\mathbb{R}^\infty \xrightarrow{\Pr} \Omega) \), which is bundle isomorphic to \( (\Pr^{-1}(O_\lambda) \xrightarrow{\pi} O_\lambda) \).

Let \( x \in O_\lambda \). Then \( (\Pr^{-1}(U_\lambda) \xrightarrow{\Pr} U_\lambda) \) is called the bundle of coordinate neighborhood of \( x \).

(iii) Assume \( W_1 = O_{\lambda_1}, W_2 = O_{\lambda_2}, W_1 \cap W_2 \neq \emptyset, x \in W_1 \cap W_2 \). \( \Phi_i : O_{\lambda_i} \to U_{\lambda_i} = \Pi^{-1}(O_i), \phi_i(O_i) \to B \)
\( i = 1, 2 \), are the mappings for corresponding bundle isomorphisms, then

(a) the Fig. 6 is commutative. That is,
\[
\pi(x) = \phi_i \circ \Pr \circ \Psi_i(x), \quad i = 1, 2;
\]

(b) \( \Psi_1 \circ \Psi_2^{-1} : \Psi_2(W_1 \cap W_2)) \to \Psi_1(W_1 \cap W_2), \)
\( \Psi_2 \circ \Psi_1^{-1} : \Psi_1(W_1 \cap W_2) \to \Psi_2(W_1 \cap W_2), \)

are \( C^r \).

The following example provides a DFEM.

Example 5.3: Consider
\[
S_\infty := \bigcup_{n=1}^{\infty} S_n,
\]
where
\[
S_n = \{x \in \mathbb{R}^{n+1} | x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\}, \quad n = 1, 2, \ldots
\]
are the \( n \) th dimensional sphere in \( \mathbb{R}^{n+1} \) space. Denote by \( P_n = (0, \cdots, 0, -1) \) and \( Q_n = (0, \cdots, 0, 1) \) the north and south poles of \( n \) dimensional sphere respectively.

(i) Set \( M_n := S_n \setminus P_n \), and define a mapping \( \Psi_n : M_n \to \mathbb{R}^n \) by
\[
\xi_i = \frac{x_i}{1 + x_{n+1}}, \quad i = 1, 2, \ldots, n.
\]

Define
\[
M = \bigcup_{n=1}^{\infty} M_n,
\]
and using the inherent topology from \( \mathbb{R}^{n+1} \) for \( M_n \), and assume \( M_n \) are clopen in \( M \). Then the mapping \( \Psi : M \to \mathbb{R}^\infty \) is a topological isomorphism.
To see this, we have only to show $\Psi$ is bijective and $\Psi^{-1}$ is also continuous. It is clear from definition (88) that

$$(\xi_1^2 + \cdots + \xi_n^2)(1 + x_{n+1})^2 = \sum_{i=1}^n x_i^2.$$  

Then we have

$$\|\xi\|^2(1 + x_{n+1})^2 + x_{n+1}^2 = 1.$$  

Solving Equation (89) and noting that $x_{n+1} \neq -1$ yield

$$x_{n+1} = \frac{1 - \|\xi\|^2}{1 + \|\xi\|^2},$$  

and

$$x_i = (1 + x_{n+1})\xi_i, \quad i \in [1, n].$$  

Eq.s (90)-(91) show that $\Psi^{-1}$ is also continuous.

Next, for $a, b \in M$ we define

$$a \sim_M b \iff \Psi(a) \leftrightarrow \Psi(b).$$  

Then we can define a mapping $\psi : M/ \sim_M \to \Omega$ by

$$\Psi(a) = x \mapsto \psi(a) := \bar{x}.$$  

Because of (92), Eq. (93) is properly defined.

Finally, we define $\pi : M \to M/ \sim_M$ as $\pi = \psi^{-1} \circ \text{Pr} \circ \Psi$.

Then it is ready to verify that $(M, \pi, M/ \sim_M)$ is a DFEM and $M/ \sim_S$ is a DFEM.

(ii) Set $N_n := S_n \setminus Q_n$, and define a mapping $\Phi_n : N_n \to \mathbb{R}^n$ by

$$\eta_i = \frac{x_i}{1 - x_{n+1}}, \quad i = 1, 2, \ldots, n.$$  

Similarly to case (i), for $x, y \in N$ we define

$$x \sim_N y \iff \Phi(x) \leftrightarrow \Phi(y),$$  

and $\pi : N \to N/ \sim_N$ as $\pi = \phi^{-1} \circ \text{Pr} \circ \Phi$, where $\phi$ can be constructed similarly as for $\psi$. Then $(N, \pi, N/ \sim_N)$ is a DFEM and $N/ \sim_S$ is a DFEM.

(iii) It is natural to consider $S_\infty = \bigcup_{n=1}^{\infty} S_n$. We may consider $\{M, N\}$ as an open cover of $S$. Then we use $\sim_M$ and $\sim_N$ as equivalence relations on $M$ and $N$ respectively. Unfortunately, the equivalences relations $\sim_M$ and $\sim_N$ are not consistent. That is, they defines different equivalent classes on $M \cap N$. It seems that it is impossible to build a DFEM structure over whole $S_\infty$.

**B. Vector Fields on DFES**

First, we define the tangent space of $\Omega$.

**Definition 5.4:** Let $\bar{x} \in \Omega$ and $\dim(\bar{x}) = m$. Then the tangent space of $\bar{x}$, called the tangent bundle at $\bar{x}$ and denoted by $T_{\bar{x}}(\Omega)$, is defined by

$$T_{\bar{x}}(\Omega) := \mathcal{Y}^{[m\cdot]}.$$  

Fig. 7. Tangent Bundle on Dimension-Free Manifold

**Remark 5.5:** When $\Omega$ is replaced by a DFEM, Definition 5.4 can only be considered as for given fixed set of coordinate charts.

Recall the definition of bundle of coordinate neighborhood of DFES, (refer to Figure 5), it is easily seen that for each $\bar{x} \in \Omega$ the bundle of coordinate neighborhood coincides with its tangent bundle. The only difference is: each leaf of the bundle of coordinate neighborhood is a sublattice of part of the tangent bundle.

When a DFEM $M$ is considered, Let $\bar{x} \in M$ and $\dim(\bar{x}) = m$, then the tangent bundle $T_{\bar{x}}(M)$ is depicted at Figure 7 where

$$T_{\bar{x}} = \mathbb{R}^{[m\cdot]}, \quad i = 1, 2, \cdots .$$

That is,

$$T_{\bar{x}}(M) = \mathbb{R}^{[m\cdot]}.$$  

If we consider the tangent space over whole $\Omega$, that is,

$$T(\Omega) := \bigcup_{\bar{x} \in \Omega} T_{\bar{x}},$$

Then it is obvious that

$$T(\Omega) = \mathbb{R}^{\infty}.$$  

Next, we define vector fields on $\Omega$. The following definition is also available for DFEMs.

**Definition 5.6:** $\bar{X}$ is called a $C^r$ vector field on $\Omega$, denoted by $X \in V^r(\Omega)$, if it satisfies the following condition:

(i) At each point $\bar{x} \in \Omega$, there exists $p = p_{\bar{x}} = \mu_{\bar{x}} \dim(\bar{x})$, called the dimension of the vector field $\bar{X}$ at $\bar{x}$ and denoted by $\dim(\bar{X}_{\bar{x}})$, such that $\bar{X}$ assigns to the bundle of coordinate neighborhood at $\bar{x}$ a $p$ sub-lattice,
\[ \mathcal{V}_O^{[p]} = \{O^p, O^{2p}, \ldots\}, \text{ then at each leaf of this sub-lattice assigns a vector } X^j \in T_{x^\mu}(O^{jp}), \ j = 1, 2, \ldots \]

(ii) \( \{X^j \mid j = 1, 2, \ldots\} \) satisfy consistence condition, that is,

\[ X^j = X^1 \otimes 1_j, \quad j = 1, 2, \ldots \quad (97) \]

(iii) At each leaf \( O^{jp} \subset \mathbb{R}^{\mu \cdot \text{dim}(x)} \),

\[ \bar{X}|_{O^{jp}} \in V^r(O^{jp}). \quad (98) \]

**Definition 5.7:** A vector field \( \bar{X} \in V^r(\Omega) \) is said to be dimension bounded, if

\[ \max_{\bar{x} \in \Omega} \text{dim}(\bar{X}_{\bar{x}}) < \infty. \quad (99) \]

In the following a method is presented to construct a \( C^r \) vector field on \( \Omega \). The method is similar to the construction of continuous functions. It is first built on \( V^m = \mathbb{R}^m \), and then extended to \( T(\Omega) = \mathbb{R}^\infty \).

**Algorithm 5.8:**

- Step 1: Assume there exists a smallest dimension \( m > 0 \), such that \( \bar{X} \) is defined over whole \( \mathbb{R}^m \). That is,

\[ \bar{X}|_{\mathbb{R}^m} := X \in V^r(\mathbb{R}^m). \quad (100) \]

From the constructing point of view: A vector field \( X \in V^r(\mathbb{R}^m) \) is firstly given, such that the value of \( \bar{X} \) at leaf \( \mathbb{R}^m \) is uniquely determined by Eq. (100).

- Step 2: Extend \( X \) to \( T_\bar{y} \). Assume \( \text{dim}(\bar{y}) = s \), denote \( m \vee s = t, t/m = \alpha, t/s = \beta \). Then \( \text{dim}(T_\bar{y}) = t \). Let \( y \in \bar{y} \cap R^{[t, \cdot]} \), and \( \text{dim}(y) = kt, k = 1, 2, \ldots \). Define

\[ \bar{X}(y) := \Pi_{\\kappa=m}^m X(\Pi_{m}^{\kappa}y), \quad k = 1, 2, \ldots \quad (101) \]

**Theorem 5.9:**

(i) The \( \bar{X} \) generated by Algorithm 5.8 is a \( C^r \) vector field, that is, \( \bar{X} \in V^r(\Omega) \).

(ii) If \( \bar{X} \in V^r(\Omega) \) is dimension bounded, then \( \bar{X} \) can be generated by Algorithm 5.8.

**Proof:**

(i) By definition, for any \( \bar{y} \in \Omega \) and assume \( \text{dim}(\bar{y}) = s \), then on a sub-lattice \( \mathbb{R}^{[t, \cdot]} \) of the bundle of coordinate neighborhood of \( \bar{y} \) (Since only the DFES is considered now, each leaf of the bundle of coordinate neighborhood can be whole Euclidean space.) a vector \( \bar{X}_{\bar{y}} \) is assigned. In the following we prove that the set of such vectors are consistent. Assume \( \text{dim}(y) = kt = k\beta m \), when \( k = 1 \), \( y = y_\beta \), then

\[ \bar{X}(y_\beta) = \Pi_{m}^{\kappa=m} X(\Pi_{m}^{\beta} y_\beta) \]

\[ = (I_{m}^{\beta} \otimes 1_t^2)(I_{m} \otimes 1_\beta)X(\Pi_{m}^{\beta} y_\beta) \]

\[ = (I_{m} \otimes 1_\beta)X(\Pi_{m}^{\beta} y_\beta) \]

\[ = X(\Pi_{m}^{\beta} y_\beta) \otimes I_\beta. \]

Similar calculation shows that

\[ \bar{X}(y_{\beta}) = X(\Pi_{m}^{\beta} y_\beta) \otimes I_k. \]

Since \( y_{\beta} \leftrightarrow y_\beta \), then \( \Pi_{m}^{\beta} y_\beta = \Pi_{m}^{\beta} y_\beta \). Hence,

\[ \bar{X}(y_{\beta}) = \bar{X}(y_\beta) \otimes 1_k. \]

The consistence is proved.

Finally, we prove Eq. (98). That is, to show that on leaf \( \mathbb{R}^{jp} \), \( \bar{X} \) is a \( C^r \) vector field. Since on a leaf all the points are of the same dimension, then the definition Eq. (101) ensures \( \bar{X}|_{\mathbb{R}^{jp}} \) is a \( C^r \) vector field.

(ii) Assume \( \bar{X} \) is dimension bounded, set

\[ m := \text{lcm}\{\text{dim}(\bar{X}_{\bar{x}}) \mid \bar{x} \in \Omega\}. \]

Then it is clear that \( X := \bar{X}|_{\mathbb{R}^m} \in C^r(\mathbb{R}^m) \). Moreover, since \( \bar{X} \) satisfies Definition 5.6, then starting from this \( X \), the vector field constructed by Eq. (101) will coincide with \( \bar{X} \).

Hereafter, we consider only dimension bounded vector fields. This is because note only they are easily constructible, but also they are practically useful in modeling dynamic systems.

We consider an example.

**Example 5.10:** Let \( X = (x_1 + x_2, x_2^2) \in C^r(\mathbb{R}^2) \). Assume \( \bar{X} \in C^r(\Omega) \) is generated by \( X \).

(i) Consider \( \bar{y} \in \Omega \), \( \text{dim}(\bar{y}) = 3 \), Denote \( y_1 = (\xi_1, \xi_2, \xi_3)^T \in \mathbb{R}^3 \). Since \( 2 \vee 3 = 6 \), \( \bar{X} \) at \( \bar{y} \cap \mathbb{R}^6 \) is well defined.

Now consider \( y_2 \).

\[ \bar{X}(y_2) = \Pi_{3}^{2} X(\Pi_{3}^{2}(y_2)) \]

\[ = \Pi_{2}^{3} X(\Pi_{2}^{3}(y_2)) \]

\[ = (I_{2} \otimes I_{13})(I_{2} \otimes I_{13})(y_1 \otimes I_{2}). \]

\[ = \left[ \begin{array}{c}
\frac{2}{3}(\xi_1 + \xi_2 + \xi_3) \\
\frac{2}{3}(\xi_1 + \xi_2 + \xi_3) \\
\frac{1}{3}(\xi_2 + \xi_3) \\
\frac{1}{3}(\xi_2 + \xi_3) \\
\frac{1}{3}(\xi_2 + \xi_3) \\
\frac{1}{3}(\xi_2 + \xi_3)
\end{array} \right]. \]

Consider \( y_4 \), similar calculation shows that

\[ \bar{X}(y_4) = \Pi_{4}^{3} X(\Pi_{4}^{3}(y_4)) \]

In fact, we have

\[ \bar{X}(y_{2k}) = \bar{X}(y_2) \otimes 1_k, \quad k = 1, 2, \ldots. \]

(ii) Consider \( \bar{X}|_{\mathbb{R}^0} \):
Assume $z = (z_1, z_2, z_3, z_4, z_5, z_6)^T \in \mathbb{R}^6$. Then

$$X^6 := \bar{X}_z = \Pi^6_5X(\Pi^6_5z) = \begin{bmatrix} \frac{1}{\beta}(z_1 + 2z_2 + z_3 + z_4 + z_5 + z_6) \\
\frac{1}{\beta}(z_1 + z_2 + 3z_3 + z_4 + z_5 + z_6) \\
\frac{1}{\beta}(z_1 + 2z_2 + 3z_3 + z_4 + z_5 + z_6) \\
\frac{1}{\beta}(z_1 + 2z_2 + z_3 + 3z_4 + z_5 + z_6) \\
\frac{1}{\beta}(z_4 + z_5 + z_6)^2 \\
\frac{1}{\beta}(z_4 + z_5 + z_6)^2 \\
\frac{1}{\beta}(z_4 + z_5 + z_6)^2 \\
\frac{1}{\beta}(z_4 + z_5 + z_6)^2 \end{bmatrix}.$$

(102)

$X^6 \in V^w(\mathbb{R}^6)$ is a classical vector field.

Next, we consider the integral of vector fields on $\Omega$.

**Definition 5.11:** Assume $\bar{X} \in C^r(\Omega)$, $X \in C^r(\mathbb{R}^n)$ is its generator, if

$$X = \bar{X}|_{\mathbb{R}^n}.$$  \hspace{1cm} (103)

The generator of smallest dimension is called the minimum generator.

The following result is an immediate consequence of the definition and Theorem 5.9.

**Proposition 5.12:** Assume $\bar{X} \in V^r(\Omega)$.

(i) If $X \in V^r(\mathbb{R}^n)$ is its generator, then $X \otimes 1_s \in V^r(\mathbb{R}^n)$ is also its generator.

(ii) If $X \in V^r(\mathbb{R}^n)$ is its generator, $Y \in V^r(\mathbb{R}^m)$, $m < n$ is also its generator, then $m|n$, and $X = Y \otimes 1_{n/m}$.

(iii) Assume $\bar{X} \in V^r(\Omega)$ is dimension bounded, then it has a generator, and hence has minimum generator.

**Definition 5.13:** Let $\bar{X} \in C^r(\Omega)$. $\bar{x}(t, \bar{x}_0)$ is called the integral curve of $\bar{X}$ with initial value $\bar{x}_0$, denoted by $\bar{x}(t, \bar{x}_0) = \Phi^\bar{X}_t(\bar{x}_0)$, if for each initial value $\bar{x}_0 \in \bar{x}_0 \cap \mathbb{R}^n$, and each generator of $\bar{X}$, denoted by $X = \bar{X}|_{\mathbb{R}^n}$, the following condition holds:

$$\Phi^\bar{X}_t(\bar{x}_0)|_{\mathbb{R}^n} = \Phi^X_t(x_0), \quad t \geq 0. \hspace{1cm} (104)$$

Next, we consider the existence and the properties of integral curve. First, assume $X = \bar{X}|_{\mathbb{R}^n}$ is the smallest generator of $\bar{X}$. Then, all the generators of $\bar{X}$ are $X_k = \bar{X}|_{\mathbb{R}^n}$, $k = 1, 2, \cdots$. Now assume $\bar{x}_0 \in \Omega$, $\dim(\bar{x}_0) = j$, and $j \lor n = s$, then

$$\bar{x} \cap \mathbb{R}^\ell \neq \emptyset,$n

if and only if, $\ell = ks, k = 1, 2, \cdots$. Denote $x_0 = \bar{x}_0 \cap \mathbb{R}^s$, then

$$\Phi^\bar{X}_t(x_0) = \Phi^X_t(x_0)|_{\mathbb{R}^s}.$$  \hspace{1cm}

Moreover,

$$\Phi^\bar{X}_t(\bar{x}_0)|_{\mathbb{R}^ks} = \Phi^{X_k s}_t(x_0^{ks}) = \Phi^{X_j}_t(x_0^s) \otimes 1_k.$$

Hence, the integral curve of $\bar{X}$ with initial value $\bar{x}_0$ is a set of integral curves defined on the sublattice bundle $\mathbb{R}^{[s]} = \{ \mathbb{R}^{ks} | k = 1, 2, \cdots \}$, and they are all equivalent. That is, for any $0 \leq k$, $k' < \infty$

$$\Phi^{X_{k s}}_t(x_0^{ks}) \leftrightarrow \Phi^{X_{k' s}}_t(x_0^{k' s}), \quad \forall t \geq 0.$$

**Example 5.14:** Recall Example 5.10: Let $\bar{X} \in \Omega$ be generated by $X = (x_1 + x_2, x_2^3)^T \in C^o(\mathbb{R}^2)$, and assume the initial value is $\bar{x}_0 \in \Omega$, $\dim(\bar{x}_0) = 3$, i.e., $x_1 = (\xi_1, \xi_2, \xi_3)^T$. Find the integral curve of $\bar{X}$ initiated at $\bar{x}_0$.

Since $2 \lor 3 = 6$, the integral curve is a set of equivalent curves defined on $\mathbb{R}^{6k}$, $k = 1, 2, \cdots$. We can first calculate the one defined on $\mathbb{R}^6$, $X'|_{\mathbb{R}^6} := X^6$, it is calculated by Eq. (102). Note that $x_0^T = \bar{x}_0 \cap \mathbb{R}^6$, then $x_2^0 = (\xi_1, \xi_2, \xi_3, \xi_3)^T$. Hence the integral curve is

$$\Phi^{X_6}_t(x_0^6).$$

It follows that

$$\Phi^{\bar{X}}_t(\bar{x}_0) = \left\{ \Phi^{X_k s}_t(x_0^6) \otimes 1_k | k = 1, 2, \cdots \right\}. \hspace{1cm} (105)$$

**C. Co-Vector Fields on DFES**

First, we define the cotangent space on DFES $\Omega$.

**Definition 5.15:** Let $\bar{x} \in \Omega$ and $\dim(\bar{x}) = m$. Then the cotangent space at $\bar{x}$, called the cotangent bundle at $\bar{x}$ and denoted by $T^*_\bar{x}(\Omega)$, is defined by

$$T^*_\bar{x}(\Omega) := V^{*[m]}(\bar{x}). \hspace{1cm} (106)$$

**Remark 5.16:** When $\Omega$ is replaced by DFEM, Definition 5.15 remains available.

Similarly to tangent bundle, for a given $\bar{x} \in \Omega$, each leaf of its cotangent bundle is an Euclidean space. Moreover, the cotangent bundle at each point is a sub-lattice of $\mathbb{R}^{\infty}$. If $\Omega$ is replaced by a DFEM, then the cotangent bundle is similar to tangent bundle. Hence, the Figure may also be considered as a description of cotangent bundle of DFEM.

Next, we define co-vector field on $\Omega$. The following definition is also applicable to DFEM.

**Definition 5.17:** $\bar{\omega}$ is called a $C^r$ co-vector field on $\Omega$, denoted by $\bar{\omega} \in V^{*[r]}(\Omega)$, if it satisfies the following conditions:

(i) At each point $\bar{x} \in \Omega$, there exists a $p = p_{\bar{x}} = \mu_{\bar{x}} \dim(\bar{x})$, called the dimension of co-vector field $\bar{\omega}$ at $\bar{x}$, denoted by $\dim(\bar{\omega}_{\bar{x}})$, such that $\bar{\omega}$ assigns a $p$-upper sub-lattice of the bundle of coordinate neighborhood at $\bar{x}$ as $V^{*[p]}\{0p, 02p, \cdots\}$, and then a set of co-vectors $\omega^j \in T^*_\bar{x}(\mathbb{R}^{jp})$, $j = 1, 2, \cdots$.

(ii) $\{\omega^j | j = 1, 2, \cdots\}$ satisfy consistent condition, that is,

$$\omega^j = \omega^j \otimes \frac{1}{j} T_j, \quad j = 1, 2, \cdots. \hspace{1cm} (107)$$
(iii) On each leaf $O^{ip} \subset \mathbb{R}^{|\mu \dim(x)|}$, 
\[ \bar{\omega}|_{O^{ip}} \in V^{*r}(O^{ip}). \]  

**Definition 5.18:** $\bar{\omega} \in V^{*r}(\Omega)$ is said to be dimension bounded, if 
\[ \max_{x \in \Omega} \dim(\bar{\omega}) < \infty. \]  

Similarly to the vector field, the co-vector field can be constructed as follows: First, define it on a leaf $\mathcal{V}^m = \mathbb{R}^m$, then extend it to $T^*(\Omega) = \mathbb{R}^\infty$.

**Algorithm 5.19:**
- Step 1: Assume there exists a smallest $m$, such that $\bar{\omega}$ is defined on $\mathbb{R}^m$. That is 
\[ \bar{\omega}|_{\mathbb{R}^m} := \omega \in V^{*r}(\mathbb{R}^m). \]  
  From constructing point of view, assume $\omega \in V^{*r}(\mathbb{R}^m)$, then $\bar{\omega}$ is defined on $\mathbb{R}^m$ as Eq. (110).
- Step 2: Extend $\omega$ to $T^*_y$. Assume $\dim(\bar{y}) = s$, $m \vee s = t$, $t/m = \alpha$, $t/s = \beta$, and then $\dim(T^*_y) = t$. Let $y \in \bar{y} \cap \mathbb{R}^{[n]}$, and $\dim(y) = kt$, $k = 1, 2, \ldots$. Define 
\[ \bar{\omega}(y) := \omega(\Pi^{kt}_m y) \Pi^{kt}_m, \quad k = 1, 2, \ldots. \]  

Similarly to vector fields, hereafter we consider only dimension bounded co-vector fields.

Similarly to vector fields, co-vector fields are defined on a sub-lattice of the bundle of coordinate neighborhood of a point $\bar{x} \in \Omega$. Assume $\omega = \bar{\omega}|_{\mathbb{R}^n}$ is the smallest generator of $\bar{\omega}$. Then, all the generators of $\bar{\omega}$ are $\omega_k = \bar{\omega}|_{\mathbb{R}^{km}}$, $k = 1, 2, \ldots$. Now assume $\bar{x}_0 \in \Omega$ and $\dim(\bar{x}_0) = j$. Denote $j \vee n = s$, then 
\[ \bar{x} \cap \mathbb{R}^{f} \neq \emptyset, \]  
if and only if, $\ell = ks$, $k = 1, 2, \ldots$.

In fact, a co-vector field can also be considered as a function of vector field. Hence, the consistence of co-vector fields and vector fields is important. The following proposition shows this consistence.

**Proposition 5.21:** Let $\bar{X} \in V^r(\Omega)$, $\bar{\omega} \in V^{*r}(\Omega)$, and $\dim(\bar{X}) = \dim(\bar{\omega})$. Then at any point $\bar{x} \in \Omega$ and the sub-lattice of the bundle of coordinate neighborhood of $\bar{x}$ where both $\bar{X}$ and $\bar{\omega}$ are well defined, the action of $\bar{\omega}$ on $\bar{X}$, denoted by $\bar{\omega}(\bar{X})$, is uniquely defined. That is, on $x_k = \bar{x} \cap \mathbb{R}^{kp}$, $k = 1, 2, \ldots$, 
\[ \bar{\omega}(\bar{X})|_{x_k} = \text{const.}, \quad k = 1, 2, \ldots. \]  

**Proof:** Denote $\dim x = s$, $\dim(\bar{X}) = \dim(\bar{\omega}) = m$. According to the previous argument, it is clear that the sub-lattice, where both $\bar{X}$ and $\bar{\omega}$ are defined, is $\{x_p, x_{2p}, \ldots\}$, where, $p = s \vee m$. To prove Eq. (112), it is enough to show that 
\[ \bar{\omega}(\bar{X})|_{x_k} = \bar{\omega}(\bar{X})|_x, \quad k = 1, 2, \ldots. \]  

Assume $p = rm$, then 
\[ \omega_1 = \omega(\Pi^{km}_m(x_p)) \Pi^{km}_m \]  
\[ \omega_k = \omega(\Pi^{km}_m(x_{kp})) \Pi^{km}_m \]  
\[ X_1 = \Pi^{rm}_m \Pi^{km}_m \Pi^{km}_m X(\Pi^{km}_m(x_p)) \]  
\[ X_k = \Pi^{rm}_m \Pi^{km}_m \Pi^{km}_m X(\Pi^{km}_m(x_{kp})) \]  
Using Eq. (114), we have 
\[ \omega_1(X_1) = \omega(\Pi^{km}_m(x_p)) \Pi^{km}_m \Pi^{km}_m X(\Pi^{km}_m(x_p)) \]  
\[ \omega_k(X_k) = \omega(\Pi^{km}_m(x_{kp})) \Pi^{km}_m \Pi^{km}_m X(\Pi^{km}_m(x_{kp})) \]  
Since $x_p \leftrightarrow x_{kp}$, then $\Pi^{km}_m(x_p) = \Pi^{km}_m(x_{kp})$. Hence, to prove Eq. (115) it is enough to show 
\[ \Pi^{rm}_m \Pi^{km}_m = \Pi^{km}_m \Pi^{rm}_m = \Pi^{km}_m. \]  

A straightforward computation shows 
\[ \Pi^{rm}_m \Pi^{km}_m = \Pi^{km}_m \Pi^{rm}_m = 1. \]  

Co-vector field is also called one-form. Assume $\bar{f} \in C^r(\Omega)$, then on each leaf $R^m f^m := \bar{f}|_{\mathbb{R}^m}$ has its differential 
\[ df^m = \left( \frac{\partial f^m}{\partial x_1}, \frac{\partial f^m}{\partial x_2}, \ldots, \frac{\partial f^m}{\partial x_m} \right). \]  

Then one sees easily that 
**Proposition 5.22:** Eq. (116) generates a co-vector field.

**Proof:** Taking $df^m$ as the smallest generator of a co-vector field. Consider the differential of $\bar{f}$ on $\mathbb{R}^{km}$. Assume $y \in \mathbb{R}^{km}$, consider $f(\Pi^{km}_m y)$. A simple computation shows that 
\[ df(\Pi^{km}_m y) = \left( \frac{\partial f^m|_{\Pi^{km}_m y}}{\partial x_1}, \frac{\partial f^m|_{\Pi^{km}_m y}}{\partial x_2}, \ldots, \frac{\partial f^m|_{\Pi^{km}_m y}}{\partial x_m} \right) \Pi^{km}_m \]  
\[ = df(\Pi^{km}_m y) \Pi^{km}_m y. \]  

This fact shows that the differential of $\bar{f}$ on leaf $\mathbb{R}^{km}$ is exactly the co-vector field deduced by $df^m$. 

If a co-vector field is deduced by a function, it is called an exact form.
D. Distribution and Co-distribution on DFES

1) Distribution on DFES:

Definition 5.23: A distribution \( \tilde{D} \) on \( \Omega \) is a rule, which assigns at each point \( \tilde{x} \in \Omega \) a sub-lattice of its bundle of coordinate neighborhood \( O_{\tilde{x}} \), denoted by \( O_{\tilde{x}} = O_{\tilde{x}} \cap \mathbb{R}^{jr_s} \), \( r \in \mathbb{Z}_+ \), \( s = \text{dim}(\tilde{x}) \), \( j = 1, 2, \cdots \), and on the tangent space of \( x_{jr} \in O_{jr}, T_{x_{jr}}(\mathbb{R}^{jr_s}) \), a subspace \( D_{x_{jr}}(x_{jr}) \subset T_{x_{jr}}(\mathbb{R}^{jr_s}) \). Moreover, this set of subspaces satisfies the consistence condition, i.e.,
\[
D_j(x_{jr}) = D_1(x_r) \otimes 1_j, \quad j = 1, 2, \cdots . \tag{118}
\]
Similarly to vector fields and co-vector fields, a distribution can be construct as follows: First, a distribution can be defined on a leaf \( V^m = \mathbb{R}^m \), then it is extended to \( T(\Omega) = \mathbb{R}^\infty \).

Algorithm 5.24:

- Step 1: Assume \( m \) is the smallest one, such that \( \tilde{D} \) is defined on leaf \( \mathbb{R}^m \). That is,
\[
\tilde{D}_{|\mathbb{R}^m} := D(x) \subset T^r(\mathbb{R}^m). \tag{119}
\]
- Step 2: Extend \( D(x) \) to \( T_{\tilde{y}}(\Omega) \). Let \( \dim(\tilde{y}) = s \), and \( m \vee s = t, t/m = \alpha, t/s = \beta \). Then \( \dim(T_{\tilde{y}}) = t \).
Assume \( y \in \tilde{y} \cap \mathbb{R}^{[k]} \), and \( \dim(y) = kt, k = 1, 2, \cdots \).
Define
\[
D(y) := \Pi^k_m D(\Pi^k_m y), \quad k = 1, 2, \cdots . \tag{120}
\]
Similarly to vector fields and co-vector fields, the following result can be obtained.

Theorem 5.25: The \( \tilde{D} \) constructed by Algorithm 5.24 is a distribution on \( \Omega \). That is, \( \tilde{D}(\tilde{x}) \subset T_x(\Omega), \forall \tilde{x} \in \Omega \).

The most commonly used distributions are expanded by a set of vector fields.

Definition 5.26: Assume \( \tilde{X}_i \in V^r(\Omega) \) and \( \dim(\tilde{X}_i) = m_i, i \in [1, n], \) and \( m = \text{lcm}\{m_i | i \in [1, n]\} \). Moreover, let \( \tilde{X}_i|_{\mathbb{R}^m} = X_i \), and \( D_m(x) \subset T^r(\mathbb{R}^m) \) be the distributions generated by the expansions of \( X_i \), \( i \in [1, n] \). Then the distribution \( \tilde{D} \subset T(\Omega) \) constructed by \( D_m(x)s, s = 1, 2, \cdots \) is called the distribution spanned by extended by \( \tilde{X}_i \), \( i \in [1, m] \).

Definition 5.27: Assume \( \tilde{X}_i \in V^\infty(\Omega) \), \( \dim(\tilde{X}_i) = m_i, i = 1, 2, \) and \( m = m_1 \vee m_2 \). Then the Lie bracket of \( \tilde{X}_1 \) and \( \tilde{X}_2 \) is defined by
\[
[\tilde{X}_1, \tilde{X}_2] := \tilde{X} \in V^\infty(\Omega) \tag{121}
\]
where \( \tilde{X} \) is the vector field determined by generator \( X \), and
\[
X = [\tilde{X}_1|_{\mathbb{R}^m}, \tilde{X}_2|_{\mathbb{R}^m}]. \tag{122}
\]

Example 5.28: Assume \( \tilde{X}, \tilde{Y} \in V^\infty(\Omega) \), \( \tilde{X} \) and \( \tilde{Y} \) are generated by \( X_0 \in V^\infty(\mathbb{R}^2) \) and \( Y_0 \in V^\infty(\mathbb{R}^3) \), where
\[
X_0(x) = [x_1 + x_2, x_3]^T, \quad Y_0(y) = [y_1^2, 0, y_2 + y_3]^T. \tag{122}
\]
Then \( m = \text{lcm}(2, 3) = 6 \). On leaf \( \mathbb{R}^6 \), we have
\[
X(z) := \Pi^2_x X_0(\Pi^2_z z) = [\alpha, \alpha, \alpha, \beta, \beta]^T, \quad \alpha = \frac{1}{3}(z_1 + z_2 + z_3 + z_4 + z_5 + z_6), \quad \beta = \frac{1}{9}(z_4 + z_5 + z_6), \tag{123}
\]
\[
Y(z) := \Pi^3_y Y_0(\Pi^3_z z) = [\gamma, \gamma, 0, 0, \mu, \mu]^T, \quad \gamma = \frac{1}{4}(z_1 + z_2)^2, \quad \mu = \frac{1}{2}(z_3 + z_4 + z_5 + z_6). \tag{124}
\]

2) Co-distribution on DFES:

Definition 5.30: A co-distribution \( \tilde{D}^* \) on \( \Omega \) is a rule, which assigns at each point \( \tilde{x} \in \Omega \) a sub-lattice \( O_{\tilde{x}} = O_{\tilde{x}} \cap \mathbb{R}^{jr_s} \), \( r \in \mathbb{Z}_+ \), \( s = \text{dim}(\tilde{x}) \), \( j = 1, 2, \cdots \), of its bundle of coordinate neighborhood \( O_{\tilde{x}} \), and a sub-space \( D^*_j(x_{jr}) \subset T^*_x_{jr}(\mathbb{R}^{jr_s}) \) at \( x_{jr} \in O_{jr} \). Moreover, this set of sub-spaces of \( T^*_x_{jr}(\mathbb{R}^{jr_s}) \) satisfy the consistence condition, i.e.,
\[
D^*_j(x_{jr}) = D^*_1(x_r) \otimes 1^T_j, \quad j = 1, 2, \cdots . \tag{123}
\]
Similarly to distribution, a \( C^r \) co-distribution on \( \Omega \) can be constructed as follows:

Algorithm 5.31:

- Step 1: Assume \( m \) is the smallest one, such that \( \tilde{D}^* \) is defined on \( \mathbb{R}^m \). That is,
\[
\tilde{D}^*_{|\mathbb{R}^m} := D^*(x) \subset T^{*r}(\mathbb{R}^m). \tag{124}
\]
A. Tensor Fields on Quotient Space

Let

\[ \phi: V(\mathbb{R}^m) \times \cdots \times V(\mathbb{R}^m) \times V^*(\mathbb{R}^m) \times \cdots \times V^*(\mathbb{R}^m) \to \mathbb{R} \]

be an \((r, s)\)th order tensor field on \(\mathbb{R}^m\), where \(r\) is the covariant order and \(s\) is the controvariant order. The set of \((r, s)\)th order tensor fields is denoted by \(T^r_s(\mathbb{R}^m)\). Let \(\{e_1, e_2, \ldots, e_m\}\) be a basis of \(V(\mathbb{R}^m)\), and \(\{d_1, d_2, \ldots, d_m\}\) be a basis of \(V^*(\mathbb{R}^m)\). Then

\[
\gamma_{i_1, j_1, j_2, \ldots, j_s} := \phi(e_{i_1}, e_{i_2}, \ldots, e_{i_r}, d_{j_1}, d_{j_2}, \ldots, d_{j_s}),
\]

\[1 \leq i_1, \ldots, i_r \leq m, \quad 1 \leq j_1, \ldots, j_s \leq m, \quad (r, s) \]

are called the structure parameters of \(\phi\). Using structure parameters, the structure matrix is construct as follows:

\[
\Gamma_{\phi} := \begin{bmatrix}
\gamma_{1\ldots1} & \gamma_{1\ldots1} & \gamma_{1\ldots1} & \cdots & \gamma_{1\ldots1} \\
\gamma_{1\ldots1} & \gamma_{1\ldots1} & \gamma_{1\ldots1} & \cdots & \gamma_{1\ldots1} \\
\gamma_{1\ldots1} & \gamma_{1\ldots1} & \gamma_{1\ldots1} & \cdots & \gamma_{1\ldots1} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\gamma_{1\ldots1} & \gamma_{1\ldots1} & \gamma_{1\ldots1} & \cdots & \gamma_{1\ldots1}
\end{bmatrix}
\]

Using this structure matrix, we have the evaluation formula for \(\phi\) as

\[
\phi(X_1, \ldots, X_r, \omega_1, \ldots, \omega_s) = \omega_s \cdots \omega_1 \Gamma_{\phi} X_1 \cdots X_r.
\]

In the following we construct a tensor field on \(\Omega\), denoted by \(\Xi \in T^2_1(\Omega)\). Assume \(\Xi \in T^2_1(\mathbb{R}^m)\) is the smallest generator of \(\Xi \in T^2_1(\mathbb{R}^m)\), and denote

\[
\Xi_{|\mathbb{R}^m} := \Xi_k.
\]

Then it is enough to construct the structure matrix \(\Xi_k, k = 1, 2, \ldots\). It is clear that \(\Xi_k\) should satisfy the following requirement: for any \(X_1, \ldots, X_r \in V(\mathbb{R}^m)\) and \(\omega_1, \ldots, \omega_s \in V^*(\mathbb{R}^m)\), their tensor value of \(\Xi\) should be the same as the value of \(\Xi_k\) with its arguments as the projected vectors and co-vectors to \(\mathbb{R}^{km}\). That is,

\[
\Xi(x)(X_1(x), \ldots, X_r(x), \omega_1(x), \ldots, \omega_s(x)) = \Xi_k(y)(\pi_{km}^k(X_1(x)), \cdots, \pi_{km}^k(X_r(x)), \pi_{km}^k(\omega_1(x)), \cdots, \pi_{km}^k(\omega_s(x))),
\]

\[
\pi_{km}^k(x(y)) = \Pi_{mk}^k(y).
\]

where \(y = \Pi_{mk}^m(x), x(y) = \Pi_{km}^m(y)\).

Let \(\Gamma(x)\) be the structure matrix of \(\Xi\) and \(\Gamma_k(y)\) be the structure matrix of \(\Xi_k\). Then Eq. \((129)\) can be expressed as

\[
\omega_s(x) \cdots \omega_1(x) \Gamma(x) X_1(x) \cdots X_r(x) = \omega_s(\Pi_{km}^m(y)) \Pi_{km}^m \cdots \omega_1(\Pi_{km}^m(y)) \Pi_{km}^m \Gamma_k(y)
\]

\[
\Pi_{km}^m X_1(\Pi_{km}^m(y)) \cdots \Pi_{km}^m X_r(\Pi_{km}^m(y)) = \omega_s(x) \cdots \omega_1(x) \Gamma(x) (I_{(s-1)m} \otimes \Pi_{km}^m) \cdots (I_{(r-1)m} \otimes \Pi_{km}^m) X_1(x) \cdots X_r(x).
\]

Hence we have,

\[
\Gamma(x) = \left( (I_{(s-1)m} \otimes \Pi_{km}^m) \cdots \right) (I_{(r-1)m} \otimes \Pi_{km}^m)
\]

\[
\Xi_k(\Pi_{km}^m(y)) = \left( (I_{(s-1)m} \otimes \Pi_{km}^m) \cdots \right) (I_{(r-1)m} \otimes \Pi_{km}^m).
\]

It follows immediately that

\[
\Gamma_k(y) = \Pi_{km}^m(I_{(s-1)m} \otimes \Pi_{km}^m) \cdots (I_{(r-1)m} \otimes \Pi_{km}^m) \Pi_{km}^m(I_{(r-1)m} \otimes \Pi_{km}^m).
\]

A straightforward verification shows that the \(\Gamma_k\) defined by Eq. \((131)\) satisfies Eq. \((130)\).

Next, set \(\bar{x} \in \Omega\), \(\dim(\bar{x}) = s, s \vee m = p\), and let \(p = \mu s = \lambda m\). Then, \(\Xi_k\) is defined at \(\bar{x} \cap \mathbb{R}^p\), \(k = 1, 2, \ldots\). Denote \(x_k = \bar{x} \cap \mathbb{R}^p\), then

\[
\tilde{\Gamma}(x_k) := \Pi_{km}^m \Pi_{km}^{k\lambda m} \cdots \Pi_{km}^{k\lambda m} \Pi_{km}^m(I_{(s-1)m} \otimes \Pi_{km}^{k\lambda m}) \cdots (I_{(r-1)m} \otimes \Pi_{km}^{k\lambda m}) \Pi_{km}^m(I_{(r-1)m} \otimes \Pi_{km}^{k\lambda m}).
\]

\[
\Xi_k(\bar{x}) = \Pi_{km}^m \Pi_{km}^{k\lambda m} \cdots \Pi_{km}^{k\lambda m} \Pi_{km}^m(I_{(s-1)m} \otimes \Pi_{km}^{k\lambda m}) \cdots (I_{(r-1)m} \otimes \Pi_{km}^{k\lambda m}) \Pi_{km}^m(I_{(r-1)m} \otimes \Pi_{km}^{k\lambda m}).
\]

Example 6.1: Assume \(\Xi \in T^2_1(\Omega)\), and its smallest generator is \(\Xi \in T^2_1(\mathbb{R}^2)\), which has its structure matrix as

\[
\Gamma(x) = \begin{bmatrix}
0 & \sin(x_1 + x_2) & 0 & \cos(x_1 + x_2) \\
-\cos(x_1 + x_2) & 0 & 0 & (x_1 + x_2) \quad 0
\end{bmatrix}
\]

\[
(133)
\]

(i) Find the structure matrix of \(\Xi|_{\mathbb{R}^4}\).

Using formula \((131)\), we have

\[
\Xi|_{\mathbb{R}^4} = \Pi_{km}^m \Pi_{km}^{k\lambda m} \Pi_{km}^m \Pi_{km}^{k\lambda m} \cdots \Pi_{km}^{k\lambda m} \Pi_{km}^m(I_{(s-1)m} \otimes \Pi_{km}^{k\lambda m}) \cdots (I_{(r-1)m} \otimes \Pi_{km}^{k\lambda m}) \Pi_{km}^m(I_{(r-1)m} \otimes \Pi_{km}^{k\lambda m}).
\]

\[
= \frac{1}{4} \begin{bmatrix}
0 & 0 & 0 & 0 \\
C & 0 & -C & 0 \\
0 & 0 & 0 & 0 \quad 0
\end{bmatrix}
\]
where
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Consider dynamic system (143). Its projection result.

\[
R = \left( \frac{\partial x}{\partial \xi} \right)^T J_\xi \left( \frac{\partial \omega}{\partial x} \right)
\]

where

\[
\left\| \frac{\partial x}{\partial \xi} \right\|^2 = \left( \frac{\partial x}{\partial \xi_1} \right)^2 + \left( \frac{\partial x}{\partial \xi_2} \right)^2 + \left( \frac{\partial x}{\partial \xi_3} \right)^2
\]

and

\[
\begin{align*}
\frac{\partial x_1}{\partial \xi_1} &= \frac{1 - 4 \xi_1^2 - (\xi_1^2 + \xi_3^2)^2}{(1 + \xi_1^2 + \xi_3^2)^2} \\
\frac{\partial x_2}{\partial \xi_1} &= \frac{-4 \xi_1 \xi_3}{(1 + \xi_1^2 + \xi_3^2)^2} \\
\frac{\partial x_3}{\partial \xi_1} &= \frac{-4 \xi_1 \xi_3}{(1 + \xi_1^2 + \xi_3^2)^2} \\
\frac{\partial x_1}{\partial \xi_2} &= \frac{-4 \xi_3}{(1 + \xi_1^2 + \xi_3^2)^2} \\
\frac{\partial x_2}{\partial \xi_2} &= \frac{-4 \xi_3}{(1 + \xi_1^2 + \xi_3^2)^2} \\
\frac{\partial x_3}{\partial \xi_2} &= \frac{-4 \xi_3}{(1 + \xi_1^2 + \xi_3^2)^2}
\end{align*}
\]

Finally, (Ω, ω) is a dimension-free Riemannian Manifold.

VII. DIMENSION-VARYING DYNAMIC (CONTROL) SYSTEMS

A. Dynamic (Control) Systems over DFEMs

1) Projection of Dynamic (Control) Systems: Consider a dynamic system over \( \mathbb{R}^p \), described as

\[
\Sigma : \dot{x} = F(x), \quad x \in \mathbb{R}^p.
\]

Definition 7.1: Consider dynamic system (143). Its projection onto \( \mathbb{R}^q \) is a dynamic system over \( \mathbb{R}^q \), described as

\[
\pi_\Sigma^\mathbb{R}(\Sigma): \dot{z} = \tilde{F}(z), \quad z \in \mathbb{R}^q,
\]

where

\[
\tilde{F}(z) = \Pi_q^\mathbb{R} (\Pi_q^\mathbb{R}) (z).
\]

Consider a control system

\[
\Sigma^C : \dot{x} = F(x, u), \quad x \in \mathbb{R}^p, \quad u \in \mathbb{R}^r.
\]

Definition 7.2: Consider control system (146). The \( u = u_1, \cdots, u_r \) can be considered as parameters. Then its projection to \( \mathbb{R}^q \) can still be considered as a projection of vector field as

\[
\pi_\Sigma^\mathbb{R}(\Sigma^C): \dot{z} = \tilde{F}(z, u), \quad z \in \mathbb{R}^q, \quad u \in \mathbb{R}^r,
\]

where

\[
\tilde{F}(z, u) = \Pi_q^\mathbb{R} F (\Pi_q^\mathbb{R}) (z, u).
\]

Remark 7.3: The projection from \( \mathbb{R}^p \) to \( \mathbb{R}^q \) can be extended to a projection from \( p \) dimensional manifold to \( q \) dimensional manifold. Then the above descriptions can be considered as the expression over local coordinate charts.

The following is an example.

Example 7.4: Consider the following control system \( \Sigma \):

\[
\begin{cases}
\dot{x}_1 = u_1 \sin(x_1 + x_2), \\
\dot{x}_2 = u_2 \cos(x_1 + x_2).
\end{cases}
\]

(i) Project (149) onto \( \mathbb{R}^3 \). It is ready to calculate that

\[
\begin{align*}
\Pi_2^3 &= \frac{1}{3} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}, \\
\Pi_3^3 &= \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}
\end{align*}
\]

Then the projected system \( \pi_2^\mathbb{R}(\Sigma) \) is calculated as

\[
\begin{align*}
\dot{z}_1 &= u_1 \sin\left(\frac{2}{3} (z_1 + z_2 + z_3)\right), \\
\dot{z}_2 &= \frac{1}{2} \left(u_1 \sin\left(\frac{2}{3} (z_1 + z_2 + z_3)\right) + u_2 \cos\left(\frac{2}{3} (z_1 + z_2 + z_3)\right)\right), \\
\dot{z}_3 &= u_2 \cos\left(\frac{2}{3} (z_1 + z_2 + z_3)\right).
\end{align*}
\]

(ii) Project (149) onto \( \mathbb{R}^4 \). We have

\[
\begin{align*}
\Pi_2^4 &= \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \\
\Pi_3^4 &= \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}
\end{align*}
\]

Then the projected system \( \pi_2^\mathbb{R}(\Sigma) \) is easily obtained as

\[
\begin{align*}
\dot{z}_1 &= u_1 \sin\left(\frac{2}{4} (z_1 + z_2 + z_3 + z_4)\right), \\
\dot{z}_2 &= u_1 \sin\left(\frac{2}{4} (z_1 + z_2 + z_3 + z_4)\right), \\
\dot{z}_3 &= u_2 \cos\left(\frac{2}{4} (z_1 + z_2 + z_3 + z_4)\right), \\
\dot{z}_4 &= u_2 \cos\left(\frac{2}{4} (z_1 + z_2 + z_3 + z_4)\right).
\end{align*}
\]

(iii) Project (150) (i.e., \( \pi_2^\mathbb{R}(\Sigma) \)) back to \( \mathbb{R}^2 \), we have

\[
\begin{align*}
\dot{x}_1 &= \frac{1}{6} \left(5 u_1 \sin(x_1 + x_2) + \cos(x_1 + x_2)\right), \\
\dot{x}_2 &= \frac{1}{6} (u_1 \sin(x_1 + x_2) + 5 \cos(x_1 + x_2)).
\end{align*}
\]

System (152) differs from the original system, which means the transfer loses information.

(iii) Project (151) (i.e., \( \pi_2^\mathbb{R}(\Sigma) \)) back to \( \mathbb{R}^2 \), we have \( \Sigma \), which means the transfer is lossless.

Motivated by the above example, we can prove the following result.
Proposition 7.5: Let \( f(x) \in V^\infty(\mathbb{R}^p) \) and \( q = kp \). Then
\[
\pi_p^q \circ \pi_p^p(f(x)) = f(x). \tag{153}
\]

Proof: First, a straightforward computation can prove the following equality:
\[
\Pi_p^{kp} \Pi_p^{kp} = I_p. \tag{154}
\]

Using it, we have that
\[
f(x) \xrightarrow{\pi_p^p} \Pi_p^{kp} f (\Pi_p^{kp} z) \xrightarrow{\pi_p^{kp}} \Pi_p^{kp} \Pi_p^{kp} f (\Pi_p^{kp} \Pi_p^{kp} x) = f(x).
\]

Remark 7.6:
(i) Proposition 7.5 shows that when a vector field is projected onto its multiple-dimension Euclidean space there is no information losing. This is essential for constructing a control system on dimension-free manifolds.
(ii) In previous sections, according to the definition of a vector field on \( \mathbb{R}^\infty \), for a vector field on \( \mathbb{R}^p \) only its integral curves over \( \mathbb{R}^{kp} \) are considered. That means only the projection of the vector field to \( \mathbb{R}^{kp} \) are considered. In current definition, the projection to ant \( \mathbb{R}^s \) is allowed. In fact, only when \( s = kp \), the extension is lossless. When \( s \neq kp \), the projected system can only be considered as an approximated system of the original one. Its integral curve can not be considered as the integral curve of the original system, but only an approximation.

2) Nonlinear Control Systems: To avoid counting the degrees of differentiability, the functions, vector fields, etc. are assumed to be of \( C^\infty \).

Definition 7.7: Assume \( \mathbf{F}(u) \in V^\infty(\Omega) \), \( \bar{h}_s \), \( s \in [1,p] \) \( \in C^\infty(\Omega) \), where \( u = (u_1, u_2, \cdots, u_m) \) are controls, which can be considered as parameters in \( \mathbf{F} \).

(i) A nonlinear control system over \( \Omega \), denoted by \( \Sigma \), is described by
\[
\begin{align*}
\dot{x} &= \mathbf{F}(\bar{x}, u) \\
\bar{y}_s &= \bar{h}_s(x), \quad s \in [1,p],
\end{align*}
\tag{155}
\]
where, \( u = (u_1, \cdots, u_m) \in \mathbb{R}^m \) is the set of controls, \( \bar{y}_s \), \( s \in [1,m] \) are outputs.

(ii) Let \( \bar{f}, \bar{g}_j \), \( j \in [1,m] \) \( \in V^\infty(\Omega) \),
\[
\mathbf{F}(u) = \bar{f} + \sum_{j=1}^{m} \bar{g}_j u_j. \tag{156}
\]
Then \eqref{155} is called an affine nonlinear control system over \( \Omega \).

(iii) Assume
\[
q := \text{lcm} (\text{dim}(\mathbf{F}(u)), \text{dim}(\bar{h}_s), s \in [1,p]).
\]
Then \( \Sigma |_{\mathbb{R}^q} := \Sigma \) is called the minimum generator of \( \Sigma \), denoted by
\[
\begin{align*}
\dot{x} &= F(x,u), \quad x \in \mathbb{R}^q \\
y_s &= h_s(x), \quad s \in [1,p].
\end{align*}
\tag{157}
\]

(iv) \( \Sigma \) is said to be completely controllable (observable), if \( \Sigma \) is completely controllable (observable).

Remark 7.8:
(i) If the state space of minimum generator is \( \mathbb{R}^q \), then, \( \Sigma \) is well posed on \( \mathbb{R}^{kp}, k = 1,2,\cdots \). They will be called the realizations of \( \bar{F}(u) \). Unfortunately, the control properties, such as controllability, observability, etc., of the realizations with different dimensions are not the same. Hence the controllability and observability of \( \Sigma \) are defined by corresponding properties of its minimum generator.

(ii) Hereafter all the control properties of \( \bar{\Sigma} \) are referred to the corresponding properties of its minimum generator.

Since the controllability and observability of a general nonlinear system are challenging problems (we refer to \cite{15} for detailed discussion), and we are now interested mainly in the dimension-related problem, to avoid complexity, we consider only a weak version of controllability and observability.

Denote
\[
\bar{F} = \{ \bar{F}(x,u) | u = \text{const.} \}. \tag{158}
\]

Let
\[
F := \bar{F}|_{\mathbb{R}^q} = \{ \bar{F}|_{\mathbb{R}^q}(x,u) | u = \text{const.} \}. \tag{159}
\]

Definition 7.9: Consider system \( \bar{\Sigma} \), defined by \eqref{155}. Assume \( \bar{\Sigma}|_{\mathbb{R}^q} \) is its minimum generator.

(i) System \eqref{155} is said to be weakly controllable at \( \bar{x} \), if at \( x_q := \bar{x} \cap \mathbb{R}^q \),
\[
\text{dim}(\langle F \rangle_{LA}(x_q)) = q. \tag{161}
\]

System \eqref{155} is said to be weakly controllable, if it is weak controllable at \( \forall x \in \mathbb{R}^q \).

(ii) System \eqref{155} is said to be weakly observable, at \( \bar{x} \), if at \( x_q := \bar{x} \cap \mathbb{R}^q \), \( h_j^{\bar{x}} = \bar{h}_j(x_q), j \in [1,p], \)
\[
\text{dim} (\langle d(LA h_j^{\bar{x}}) (x_q) \rangle) = q. \tag{162}
\]

System \eqref{155} is weakly observable, if it is weakly observable at \( \forall x \in \mathbb{R}^q \).

Example 7.10: Consider a control system \( \Sigma \) on \( \Omega \), its dynamics is
\[
\begin{align*}
\dot{x} &= \bar{f}(\bar{x}) + \bar{g}_1 u_1 + \bar{g}_2 u_2, \\
\bar{y} &= \bar{h}(\bar{x}).
\end{align*}
\tag{163}
\]
where $\dim(f) = 2$, $\dim(g_1) = \dim(g_2) = 4$, $\dim(h) = 2$. $\bar{f}$, $g_1$, $g_2$, $h$ are generated by $f^0$, $g_1^0$, $g_2^0$, $h^0$, which are

$$
f^0 = (\sin(x_1 + x_2), \cos(x_1 + x_2))^T, \\
g_1^0 = (1, 0, 0, 0)^T, \\
g_2^0 = (0, x_1, x_1^2, x_1^3)^T, \\
h^0 = x_1 - x_2.
$$

It is easy to calculate that

$$f(z) = \Pi_z^2 f^0 (\Pi_z^2 z) = \begin{bmatrix} 
\sin\left(\frac{21z_1 + 2z_2 + z_3 + z_4}{2}\right) \\
\sin\left(\frac{21z_1 + 2z_2 + z_3 + z_4}{2}\right) \\
\cos\left(\frac{21z_1 + 2z_2 + z_3 + z_4}{2}\right) \\
\cos\left(\frac{21z_1 + 2z_2 + z_3 + z_4}{2}\right) 
\end{bmatrix};$$

$$h(z) = h^0 (\Pi_z^2 z) = \frac{z_1 + z_2 - z_3 - z_4}{2}.$$

Denote its controllability Lie algebra by

$$\mathcal{L} = \langle f(z), g_1(z), g_2(z) \rangle_{LA}.$$

Since

$$g_1 = (1, 0, 0, 0)^T \in \mathcal{L},$$

$$[g_1, g_2] = (0, 1, 2x_1, 3x_1^2)^T \in \mathcal{L},$$

$$[g_1, [g_1, g_2]] = (0, 0, 2, 6x_1)^T \in \mathcal{L},$$

$$\dim(\mathcal{L}) = 4.$$

System (163) is weakly controllable.

Similarly, it is ready to verify that system (163) is weakly observable.

B. Linear Systems over Dimension-Free Manifolds

1) STP and Lattice Structure over Matrices: Recall STP of matrices [4], [5].

**Definition 7.11:** Assume $A \in \mathcal{M}_{m \times n}$, $B \in \mathcal{M}_{p \times q}$, $t = n \lor p$.

(i) Type 1 Left Matrix-Matrix STP (briefly denoted by MML-1) is defined by

$$A \times B := (A \otimes I_{t/n}) (B \otimes I_{t/p}).$$

(ii) Type 1 Right Matrix-Matrix STP (briefly denoted by MMR-1) is defined by

$$A \times B := (I_{t/n} \otimes A) (I_{t/p} \otimes B).$$

(iii) Type 2 Left Matrix-Matrix STP (briefly denoted by MML-2) is defined by

$$A \circ \ell B := (A \otimes J_{t/n}) (B \otimes J_{t/p}),$$

where,

$$J_k := \frac{1}{k} \mathbf{1}_{k \times k}, \quad k = 1, 2, \cdots .$$

(iv) Type 2 Right Matrix-Matrix STP (briefly denoted by MMR-2) is defined by

$$A \circ \ell B := (J_{t/n} \otimes A) (J_{t/p} \otimes B).$$

Denote the set of all matrices by

$$\mathcal{M} := \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \mathcal{M}_{m \times n}.$$

Essentially, the MM-STPs are products of two sets of matrices. These products lead to the following equivalences of matrices.

**Definition 7.12:** Let $A, B \in \mathcal{M}$.

(i) The matrices $A$ and $B$ are said to be type 1 left equivalent, denoted by $A \sim \ell B$, if there exist $I_\alpha$ and $I_\beta$, such that

$$A \otimes I_\alpha = B \otimes I_\beta.$$

The equivalence class of $A$ is denoted by $\langle A \rangle_\ell$.

(ii) The matrices $A$ and $B$ are said to be type 1 right equivalent, denoted by $A \sim_\ell B$, if there exist $I_\alpha$ and $I_\beta$, such that

$$I_\alpha \otimes A = B \otimes I_\beta.$$

The equivalence class of $A$ is denoted by $\langle A \rangle_\ell$.

(iii) The matrices $A$ and $B$ are said to be type 2 left equivalent, denoted by $A \approx \ell B$, if there exist $J_\alpha$ and $J_\beta$, such that

$$A \otimes J_\alpha = B \otimes J_\beta.$$

The equivalence class of $A$ is denoted by $\langle\langle A \rangle \rangle_\ell$.

(iv) The matrices $A$ and $B$ are said to be type 2 right equivalent, denoted by $A \approx_\ell B$, if there exist $J_\alpha$ and $J_\beta$, such that

$$J_\alpha \otimes A = J_\beta \otimes B.$$

The equivalence class of $A$ is denoted by $\langle\langle A \rangle \rangle_\ell$.

For convenience, in the following we consider only left equivalence only. The results are easily extendable to right equivalence. Hence we assume

$$A \sim B := A \sim_\ell B, \quad A \approx B := A \approx_\ell B,$$

$$\langle A \rangle := \langle A \rangle_\ell, \quad \langle\langle A \rangle \rangle := \langle\langle A \rangle \rangle_\ell.$$

Set

$$\mathcal{M}_\mu = \{ A \in \mathcal{M}_{m \times n} \mid m/n = \mu \}, \quad \mu \in \mathbb{Q}_+.$$

Then,

$$\mathcal{M} = \bigcup_{\mu \in \mathbb{Q}_+} \mathcal{M}_\mu.$$

This is a partition, that is,

$$\mathcal{M}_{\mu_1} \cap \mathcal{M}_{\mu_2} = \emptyset, \quad \mu_1 \neq \mu_2.$$
Consider the equivalence class. Denote

\[ \Sigma = \mathcal{M} / \sim, \quad \Xi = \mathcal{M} / \approx, \]
\[ \Sigma_\mu = \mathcal{M}_\mu / \sim, \quad \Xi_\mu = \mathcal{M}_\mu / \approx. \]

It is obvious that

\[ \Sigma = \bigcup_{\mu \in \mathbb{Q}_+} \Sigma_\mu, \quad \left( \text{or} \quad \Xi = \bigcup_{\mu \in \mathbb{Q}_+} \Xi_\mu \right) \]

is also a partition.

Next, we consider the lattice structure of \( \Sigma_\mu \). The lattice structure of \( \Xi_\mu \) is exactly the same.

Define the order over \( \Sigma_\mu \) as follows:

\[ (A) \prec (B) \iff (A) \supset (B). \] (172)

As a partial order, Eq. (172) also poses a partial order on \( \Sigma \).

The following proposition is an immediate consequence of the definition.

**Proposition 7.13:**

(i) If \( \langle A \rangle, \langle B \rangle \) satisfies the order determined by (172), then there exists a \( \mu \in \mathbb{Q}_+ \), such that \( \langle A \rangle, \langle B \rangle \in \Sigma_\mu \).

(ii) Assume the minimum elements of \( \langle A \rangle \) and \( \langle B \rangle \) are \( A_1 \in \mathcal{M}_{m \times n} \) and \( B_1 \in \mathcal{M}_{p \times q} \) respectively, where \( m/n = p/q = \mu \). Then \( \langle A \rangle \prec \langle B \rangle \), if and only if, there exists \( I_k \) such that \( A_1 \otimes I_k = B_1 \).

2) **Linear Vector Fields:** Let \( \bar{X} \in V^\infty(\Omega) \) be a linear vector field and \( \dim(\bar{X}) = m \). Then there exists \( A \in \mathcal{M}_{m \times m} \) such that \( X := \bar{X}|_{\mathbb{R}^m} = Ax \). Consider \( \bar{X}|_{\mathbb{R}^m} \): Let \( y \in \mathbb{R}^m \). Then

\[ X_k := \bar{X}(y) = \Pi_{km}^m (X (\Pi_{km}^m (y))) = \Pi_{km}^m A \Pi_{km}^m y \]
\[ := A_k y, \] (173)

where,

\[ A_k = \Pi_{km}^m A \Pi_{km}^m = \frac{1}{k} \left( I_m \otimes 1_k \right) A \left( I_m \otimes 1_k^T \right). \] (174)

Then we consider the integral curve of \( \bar{X} \).

Assume \( \bar{X} \in V^\infty(\Omega) \) be a linear vector field and \( \dim(\bar{X}) = m \). \( X := \bar{X}|_{\mathbb{R}^m} = Ax \). Consider \( \bar{x}^0 \in \Omega \).

Case 1: Assume \( \dim(\bar{x}^0) = m \). Then the integral curve of \( \bar{X} \) with initial value \( \bar{x}^0 \) is defined only on a filter of the tangent bundle

\[ T_{\bar{x}^0} \cap \mathbb{R}^{km}, \quad k = 1, 2, \ldots. \]

On \( T_{\bar{x}^0} \cap \mathbb{R}^{km}, \) at \( x_0^0 = x_1^0 \otimes 1_k \), the vector field is determined by Eq. (174). Then the integral curve becomes

\[ x_k(t, x_0^0) = e^{X_k t} x_0 = \left( I_{km} + t(I_m \otimes 1_k)A(I_m \otimes 1_k^T) \right) + \frac{t^2}{2!} (I_m \otimes 1_k)A^2(I_m \otimes 1_k^T) + \cdots \] \[ (x_0^0 \otimes 1_k) \]
\[ = \frac{1}{k} (I_m \otimes 1_k) e^{A(t)(I_m \otimes 1_k)}(x_0^0 \otimes 1_k) \]
\[ = (I_m \otimes 1_k) e^{A t} x_0 \]
\[ = e^{A t} x_0 \otimes 1_k. \] (175)

Case 2: Assume \( \dim(\bar{x}^0) = s, m \vee s = p = km = rs \).

Then the integral curve of \( \bar{X} \) with initial value \( \bar{x}^0 \) is defined on a filter of its tangent bundle

\[ T_{\bar{x}^0} \cap \mathbb{R}^{jp}, \quad j = 1, 2, \ldots. \]

On leaf \( T_{\bar{x}^0} \cap \mathbb{R}^{jp} \), the initial value \( z_0^j = z_1^0 \otimes 1_r \), where \( z_0^j \) is its smallest element. The vector field is \( A_k \), where \( A_k \) is determined by Eq. (174). Hence, the integral curve is

\[ z_r(t, z_0^j) = e^{X_{k1} t} z_0^j \]
\[ = \frac{1}{k} (I_m \otimes 1_k) e^{A(t)(I_m \otimes 1_k)}(z_0^j \otimes 1_r) \]. (176)

On leaf \( T_{\bar{x}^0} \cap \mathbb{R}^{jp} \), the integral curve with initial value \( z_0^j \) is

\[ z_{jr}(t, z_0^j) = e^{X_{kr} t} z_0^j \]
\[ = \frac{1}{k} (I_m \otimes 1_k) e^{A(t)(I_m \otimes 1_k)}(z_1^0 \otimes 1_{jr}). \] (177)

Summarizing the above argument, we have the following result.

**Proposition 7.14:** Let \( \bar{X} \in V^\infty(\Omega) \) be a linear vector field, and \( \dim(\bar{X}) = m \). \( X := \bar{X}|_{\mathbb{R}^m} = Ax \). Assume \( \bar{x}^0 \in \Omega \), \( \dim(\bar{x}^0) = s \).

(i) If \( s = m \), then the integral curve of \( \bar{X}|_{\mathbb{R}^m} \) is

\[ \Phi^X_t(x_1^0) = e^{X_1^0 t} x_1^0. \] (178)

Hence, the integral curve of \( \bar{X}|_{\mathbb{R}^m} \) becomes

\[ \Phi^X_t(x_1^0) = [e^{X_1^0 t} x_1^0] \otimes 1_r. \] (179)

Finally the integral curve of \( \bar{X} \) with initial value \( \bar{x}^0 \) is \( \Phi^X_t(x_1^0) \subset \Omega \).

(ii) If \( s = km \), then the integral curve of \( \bar{X}|_{\mathbb{R}^m} \) is

\[ \Phi^X_t(x_1^0) = e^{X_1^0 t} x_1^0, \] (180)

where, \( X_k \) is determined by Eq. (173). Hence the integral curve of \( \bar{X} \) with initial value \( \bar{x}^0 \) is \( \Phi^X_t(x_1^0) \subset \Omega \).
(iii) If \( m \lor s = p = km = rs \), then the integral curve of \( \overline{X} \) is

\[
\Phi^X_t(x_0) = e^{X_t} (x_0 \otimes I_s).
\]

Hence, the integral curve of \( \overline{X} \) with initial value \( \overline{x}^0 \) is \( \Phi^X_t(x_0^0) \subset \Omega \).

3) **Linear Control Systems:** First, we consider the relationship among equivalent matrices, equivalent vectors, and linear vector fields.

**Proposition 7.15:** Let \( \overline{X} \in V^\infty(\Omega) \) be a linear vector field and \( \dim(\overline{X}) = m \). \( X := \overline{X}|_{\mathbb{R}^m} = Ax \). Assume \( \overline{x}^0 \in \Omega \), \( \dim(\overline{x}^0) = s \), \( m \lor s = p = km = rs \). Then \( \overline{X} \) is defined only on the filter of its tangent bundle

\[
x^0_j = T \overline{x}^0 \cap \mathbb{R}^p, \quad j = 1, 2, \ldots .
\]

Moreover, on leaf \( x^0_\mu \) it is

\[
\overline{X}(x^0_\mu) = A_k x^0_\mu,
\]

where \( A_k \) is determined by Eq. (173). On \( x^0_j \) it is

\[
\overline{X}(x^0_j) = A_{jk} x^0_j, \quad j = 1, 2, \ldots ,
\]

where the two sets of consistent matrices are

\[
A_{jk} = A_k \otimes I_j \approx A_k,
\]

and

\[
A_{jk} = A_k \otimes I_j \approx A_k,
\]

respectively. The available variables are

\[
x^0_j = x^0_\mu \otimes 1_j \leftrightarrow x^0_\mu.
\]

**Proof:** In fact, what do we need is that the tangent vectors on the bundle leaves are consistent. That is,

\[
\overline{X}(x^0_\mu) = \overline{X}(x^0_j) \otimes 1_j, \quad j = 1, 2, \ldots .
\]

It is obvious that Eq. (184) + Eq. (186) or Eq. (185) + Eq. (186) can ensure the Eq. (187) to be true.

Next, we consider the linear control system on \( \Omega \). Recall a classical linear system \[17\]

\[
\begin{aligned}
\dot{x} &= Ax + \sum_{i=1}^{m} b_i u_i, \\
y &= Cx, & x \in \mathbb{R}^n, & y \in \mathbb{R}^p.
\end{aligned}
\]

One sees that a classical linear control system consists of three ingredients: linear vector field \( Ax \), a set of constant vector fields \( B = \{b_1, \ldots , b_m\} \), and linear function \( Cy \). To extend a classical linear control system to \( \Omega \), it is enough to create these three kind of objects to \( \Omega \). The key of this extension is to make them consistent at each \( x \in \Omega \).

(i) **Linear Vector Field:** Assume the smallest generator of linear vector field \( \overline{X} \) is \( X = Ax \in V^\infty(\mathbb{R}^m) \), \( \dim(\overline{x}^0) = s \), \( m \lor s = p = \mu m = rs \). Then according to the argument in previous subsection, we know

\[
\overline{X} \cap T \overline{x}^0 = \{ \overline{X}(x_j) \mid j = 1, 2, \ldots \}.
\]

Moreover,

\[
\overline{X}(x_j) = A_{j\mu} x_j, \quad j = 1, 2, \ldots ,
\]

where, \( A_{j\mu} \) is determined by Eq. (174) with \( k = j\mu \).

(ii) **Constant Vector Field:** Assume the smallest generator of the constant vector field \( \overline{X} \) is \( X = v \in V^\infty(\mathbb{R}^m) \), \( \dim(\overline{x}^0) = s \), \( m \lor s = p = \mu m = rs \). Then \( \overline{X} \) holds, and

\[
\overline{X}(x_j) = \Pi_{j\mu m}^m X (\Pi_{j\mu m}^m x_j) = \Pi_{j\mu m}^m b
\]

\[
= b \otimes 1_j.
\]

(iii) **Linear Function:** Let \( \tilde{h} \in C^\infty(\Omega) \). \( \dim(\overline{x}^0) = m \), \( \tilde{h} \) is expressed at \( x^0_1 \) as \( hx = c_m x \), where \( c^T \in \mathbb{R}^m \). Let \( \bar{z} \in \Omega \), \( \dim(\bar{z}) = s \), \( m \lor s = p = rs = \mu m \). Then \( \tilde{h} \) is expressed at \( z_1 \) as

\[
\tilde{h}(z_1) = \tilde{h}(\Pi_{s1}^m z_1) = 1 \mu c_m (I_m \otimes 1^T_p) (I_s \otimes 1_r) z_1.
\]

Hence \( \tilde{h} \) can be expressed on leaf \( \mathbb{R}^s \) as

\[
\tilde{h}|_{\mathbb{R}^s} = c_s \bar{z},
\]

where

\[
c_s = 1 \mu c_m (I_m \otimes 1^T_p) (I_s \otimes 1_r).
\]

Particularly, when \( s = km \), we have

\[
c_{km} = 1 \mu c_m (I_m \otimes 1^T_k).
\]

**Definition 7.16:** Assume \( f(x) \) is a linear vector field, \( \tilde{B} = [\tilde{b}_1, \ldots , \tilde{b}_m] \) is a set of constant vector fields, \( \tilde{C} = [c_1, \ldots , c_p]^T \) is a set of linear functions, then

\[
\begin{aligned}
\dot{x} &= \tilde{f}(x) + \tilde{B} u, \\
\tilde{y} &= \tilde{C} x,
\end{aligned}
\]

is a linear control system over \( \Omega \).

**Example 7.17:** Consider a linear control system \( \Sigma \) over \( \Omega \), which has its dynamic equation as Eq. (156), where \( \tilde{f} \) has its smallest generator \( f(x) = 2[x_1 + x_2, x_2]^T \in V^\infty(\mathbb{R}^2) \), \( m = 2 \), the smallest generator of \( \tilde{g}_1 = g_1 = [1, 0, 0, 1]^T \in V^\infty(\mathbb{R}^4) \), the smallest generator of \( \tilde{g}_2 = g_2 = [0, 1, 0, 0]^T \in V^\infty(\mathbb{R}^4) \). \( p = 1 \), \( \tilde{h}|_{\mathbb{R}^2} = x_2 - x_1 \).
Then, \( q = 4 \).

\[
\vec{f}|_{\mathbb{R}^4} = \Pi_4^2 f (\Pi_4^2 [z_1, z_2, z_3, z_4]^T) = \begin{bmatrix} z_1 + z_2 + z_3 + z_4 \\ z_1 + z_2 + z_3 + z_4 \\ z_3 + z_4 \\ z_3 + z_4 \end{bmatrix} := A z,
\]

where,

\[
A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.
\]

\[
\vec{h}|_{\mathbb{R}^4} = h(\Pi_4^2 z) = h(z_1 + z_2, z_3 + z_4) = z_1 + z_2 - z_3 - z_4 := C z,
\]

where,

\[
C = [1, 1, -1, -1].
\]

Then the smallest generator of system \( \Sigma \), denoted by \( \Sigma := \Sigma|_{\mathbb{R}^4} \), is

\[
\begin{cases}
\dot{z} = A z + Bu, \\
y = C z.
\end{cases}
\]

Then it is easy to calculate that the controllability matrix of \( \Sigma \) is

\[
C = \begin{bmatrix} 1 & 0 & 2 & 1 & 6 & 2 & 16 & 4 \\ 0 & 1 & 2 & 1 & 6 & 2 & 16 & 4 \\ 0 & 0 & 1 & 0 & 2 & 0 & 4 & 0 \\ 1 & 0 & 1 & 0 & 2 & 0 & 4 & 0 \end{bmatrix}.
\]

Since \( \text{rank}(C) = 4 \), \( \Sigma \) is completely controllable. By definition, \( \Sigma \) is completely controllable.

The observability matrix of \( \Sigma \) is

\[
\mathcal{O} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 2 & 2 & 0 & 0 \\ 4 & 4 & 4 & 4 \\ 8 & 8 & 16 & 16 \end{bmatrix}
\]

Since \( \text{rank}(\mathcal{O}) = 2 < 4 \), \( \Sigma \) is not completely observable, and so is \( \Sigma \).

C. Dimension-Varying Dynamic (Control) Systems

1) Continuous time Dimension-varying Dynamic Systems:

Consider a continuous time dynamic system

\[
\dot{x} = F(x), \quad x \in \mathcal{X},
\]

where \( F \) is considered as a vector field on \( \mathcal{X} \). Then the solution (integral curve) is expressed as

\[
x(t, x_0) = \Phi_t^F(x_0).
\]

It is well known that if \( (197) \) is a dynamic system, then \( x(t, x_0) \) must be continuous with respect to \( t \). Hence a continuous time dimension-varying dynamic system can not be defined on ESDD \( \mathbb{R}^\infty \). It can only be defined on DFES \( \Omega \) (or in general, DFEM).

**Definition 7.18:** Consider a dynamic system

\[
\dot{x} = F(x), \quad x \in \mathbb{R}^\infty,
\]

is called a realization (or a lifting) of \( (199) \), if for each \( \bar{x} \) there exists \( x \in \bar{x} \), such that the corresponding vector field \( F(x) \in \bar{F}(\bar{x}) \). Meanwhile, system \( (199) \) is called the project system of \( (200) \).

The following result is an immediate consequence of the definition.

**Proposition 7.19:** \( \bar{x}(t) = \bar{x}(t, x_0) \) is the solution of \( (199) \), if and only if, \( x(t) = x(t, x_0) \) is the solution of \( (200) \), where \( x(t) \in \bar{x}(t), t \in [0, \infty) \).

**Definition 7.20:** System \( (200) \) is called a dimension-varying system, if there are at least two points \( x_1, x_2 \) such that \( F(x_1) \in V(\mathbb{R}^{d_1}), F(x_2) \in V(\mathbb{R}^{d_2}) \) and \( d_1 \neq d_2 \).

**Remark 7.21:** The Definitions \( 7.18 \) and \( 7.20 \) can easily be extended to corresponding control systems in a natural way. The Proposition \( 7.19 \) has also its corresponding version for control systems.

2) Constructing Dimension-varying Control Systems:

This section consider how to construct dimension-varying control systems. We start from a dimension-varying control system

\[
\dot{x} = F(x, u), \quad x \in \mathbb{R}^\infty,
\]

aims at designing a dynamic control system on \( \Omega \) as

\[
\dot{x} = \bar{F}(\bar{x}, u), \quad \bar{x} \in \Omega,
\]

such that \( (201) \) is its realization.

We consider several cases.

- Switching Dimension-varying Control System:

Assume the original control system is

\[
\dot{x} = F(x, u), \quad x \in \mathbb{R}^m, \quad u \in \mathbb{R}^p.
\]

The target system is

\[
\dot{z} = G(z, v), \quad z \in \mathbb{R}^n, \quad v \in \mathbb{R}^q.
\]

Our purpose is to switch system \( (203) \) to system \( (204) \) at time \( t = T \). To get a continuous trajectory over \( \Omega \), the following condition is necessary:

\[
\bar{x}(T) = \bar{z}(T) \in \Omega.
\]
Proposition 7.22: Assume (205) is satisfied, and assume system (203) is controllable. Then the dynamic switching from system (203) to system (204) at time \( t = T \) is realizable.

Proof: We construct the following system over \( \Omega \):

\[
\dot{\xi} = \begin{cases} F(\xi, u), & t < T, \\ G(\xi, v), & t > T. \end{cases}
\]

(206)

Since system (204) is controllable, there exists \( u(t), t < T \), such that (206) is controllable to

\[
\xi(T) = \bar{x}(T) = \bar{z}(T),
\]

where \( \dim(\bar{x}) = p \land q \).

Then the minimum realization of (206) becomes the required dimension-varying system.

Example 7.23: Assume \( \Sigma_1 \) is

\[
\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad x \in \mathbb{R}^2,
\]

(207)

\( x(0) = (0, 0)^T \), and \( \Sigma_2 \)

\[
\dot{z} = Az + Bv, \quad z \in \mathbb{R}^3.
\]

(208)

Design a control such that \( \Sigma_1 \) is switched to \( \Sigma_2 \) at \( T = 1 \).

Since \( \Sigma_1 \) is completely controllable, and \( 2 \land 3 = 1 \), we have to design a control which can drive the system from \( x(0) \) to \( x(T) \) with \( \dim(\bar{x}(T)) = 1 \). We may choose \( x(T) = (1, 1)^T \). Then it is easy to calculate that the controllability Gramian matrix is

\[
W_C(t) = \int_0^t e^{-A^*}\begin{bmatrix} B^T & e^{-A^*} \end{bmatrix} \begin{bmatrix} 2t^3 \\ -3t^2 \end{bmatrix} dt.
\]

Then the control is

\[
u(t) = -B^T e^{-A^*} t W_C^{-1}(T) \left(x(0) - e^{-A^T} x(T)\right)
\]

(209)

\[= -\phi t.\]

Using this control, the system can be switched from \( \Sigma_1 \) to \( \Sigma_2 \) at \( T = 1 \).

Continuous Dimension-varying Control System:

In this case we require the designed system (202) has continuous \( F(\bar{x}, u) \). For instance, in a docking/undocking process, we want the dimension-transient process to be as smooth as possible.

Assume there are two original control systems as \( \Sigma_1 \):

\[
\dot{x} = F(x, u), \quad x \in \mathbb{R}^m, \ u \in \mathbb{R}^p;
\]

(209)

and \( \Sigma_2 \):

\[
\dot{z} = G(z, v), \quad z \in \mathbb{R}^n, \ u \in \mathbb{R}^q.
\]

(210)

They will be docked into the target system \( \Omega \) as

\[
\dot{\xi} = H(\xi, w), \quad \xi \in \mathbb{R}^s, \ w \in \mathbb{R}^l.
\]

(211)

It is required that the docking happens during a transient period \([T_0, T_1]\), and the process is smooth.

Definition 7.24: System (209) and system (210) are said to be docked into (211) smoothly during the transient period \([T_0, T_1]\). If

(i) there exists a smooth monotonically non-decreasing function \( \lambda(t), t \in [T_0, T_1] \), such that

\[
\lambda(t) = \begin{cases} 0, & t = T_0, \\ 1, & t = T_1; \end{cases}
\]

and

(ii) there exists a control deformation function

\[w = \varphi(u, v),\]

and using it a control system over \( \Omega \), called a transient system, is constructed as

\[
\begin{pmatrix} \dot{x}(T_1) \\ \dot{z}(T_2) \end{pmatrix} = \xi(T_1),
\]

(212)

with \( \dim(\xi(T_1)) = p \land q \land s \), which is the greatest common divisor of \( p, q, s \).

Remark 7.25:

(i) From definition 7.24 one sees easily that before \( T_0 \) we have the minimum realization of (212) as two systems (209) and (210), and after \( T_1 \) we have the minimum realization (211).

(ii) The minimum realization of (212) is on \( \mathbb{R}^\mu \), where \( \mu = p \lor q \lor s \) is the least common multiple of \( p, q, s \). A numerical algorithm needs to be developed to solve the control problem for docking.

(iii) The function \( \lambda(t) \), called a linear homotopic function, and the deformation function \( \varphi(u, v) \) depend on special docking systems, which may be obtained by engineering experiments. For instance, an example for vehicle clutch system was presented in [3].

(iv) The simplest \( \lambda(t) \) is

\[
\lambda(t) = \frac{t - T_0}{T_1 - T_0}, \quad T_0 \leq t \leq T_1 - T_0.
\]

(v) The \( \lambda(t) \) may be replaced be a more general homotopic function.

Undocking Dimension-varying Control System:

Assume there is an original system \( \Sigma \) as

\[
\dot{\xi} = H(\xi, w), \quad \xi \in \mathbb{R}^s, \ w \in \mathbb{R}^l.
\]

(213)
It will be undocked into two destination systems as $\Sigma_1$: 
\[
\dot{x} = F(x, u), \quad x \in \mathbb{R}^m, \ u \in \mathbb{R}^p; \tag{214}
\]
and $\Sigma_2$: 
\[
\dot{z} = G(z, v), \quad z \in \mathbb{R}^n, \ u \in \mathbb{R}^q. \tag{215}
\]

It is required that the undocking happens during a transient period $[T_0, T_1]$, and the process is smooth.

**Definition 7.26:** System (213) is said to be undocked into (214) and (215) smoothly during the transient period $[T_0, T_1]$. If

(i) there exists a smooth monotonically linear homotopic function $\lambda(t)$, $t \in [T_0, T_1]$; and

(ii) there exist two deformation functions

\[
u = \psi_1(w),
\]
\[
v = \psi_2(w),
\]

such that the following control system (216) over $\Omega$ can be controlled to

\[
\begin{bmatrix}
\dot{x}(T_1) \\
\dot{z}(T_1)
\end{bmatrix} = \tilde{\xi}(T_1),
\]

with $\dim(\tilde{\xi}(T_1)) = p \land q \land s$.

\[
\begin{bmatrix}
\dot{x} \\
\dot{z}
\end{bmatrix} = (1 - \lambda(t))H(\xi, w) + \lambda(t) \begin{bmatrix}
\bar{F}(\bar{x}, \psi_1(w)) \\
\bar{G}(\bar{z}, \psi_2(w))
\end{bmatrix}, \quad t \in [T_0, T_1]. \tag{216}
\]

Similarly to docking case, the arguments in Remark 7.25 are also true for undocking case.

**VIII. CONCLUDING REMARKS**

The main purpose of this paper is to construct a new geometric space called the DFES (or DFEM), which provides a framework (i.e., the state space) for DVDS.

The DFES is constructed as follows:

- **Step 1:** Define inner product for two vectors of different dimensions. It turns the ESDD $\mathcal{V} = \mathbb{R}^\infty = \bigcup_{n=1}^\infty \mathbb{R}^n$ into a distance space.

- **Step 2:** Two vectors $x, y \in \mathbb{R}^\infty$ are said to be equivalent, denoted by $x \leftrightarrow y$, if their distance is zero. Then the quotient space $\Omega = \mathbb{R}^\infty/\leftrightarrow$, called the DFES, is obtained. In fact, $d\mathcal{V}(x, y) = 0$, if and only if, there exist $1_\alpha$ and $1_\beta$ such that $x \otimes 1_\alpha = y \otimes 1_\beta$. Here we may understand that two vectors are equivalent if they contain same information. In other words, vector form is a way for a set of data to express itself. It may be expressed as vectors of different dimensions, but from information point of view, they are equivalent.

- **Step 3:** By posing proper a vector structure $\Omega$ becomes a topological vector space. Let $Pr : \mathbb{R}^\infty \rightarrow \Omega$ be the natural projection. Then $(\mathbb{R}^\infty, Pr, \Omega)$ becomes a fiber bundle.

- **Step 4:** Using the fiber bundle structure of $(\mathbb{R}^\infty, Pr, \Omega)$, each $\bar{x} \in \Omega$ have a coordinate neighborhood, which is a set of coordinate charts of various dimensions. It is called a bundle of coordinate charts. Hence $\Omega$ is called a DFES.

- **Step 5:** Using the bundles of coordinate charts, a differentiable structure can be posed on $\Omega$. Then the continuous functions, (co)-vector fields, (co)-distributions, and tensor fields can be built for $\Omega$. Eventually, the dimension-free Riemannian manifold and dimension-free Symplectic manifold can be properly constructed.

Note that the gluing topology on DFES $\Omega = \mathbb{R}^\infty/\leftrightarrow$ makes it a path-wise connected topological space. Therefore, intuitively, the trajectories of dynamic systems over $\Omega$ can continuously move “cross” Euclidian spaces of different dimensions. This is the main idea for using DFEM to design DVDS and DVCS. Lifting the trajectory of a dynamic system over $\Omega$ to a leaf (a Euclidian space of fixed dimension) is called a realization. As a dynamic system over $\Omega$ is lifted onto leaves of different dimensions, a dimension-varying realization is obtained. Conversely, we can also project the trajectory of a dynamic system on a Euclidian space of fixed dimension onto $\Omega$.

The design of dimension-varying dynamic (control) systems can be described as follows:

- **Step 1:** Project a dimension-varying dynamic system, which has broken vector fields over Euclidian spaces of different dimensions, onto $\Omega$ to form a dynamic system over $\Omega$, which consists of several (finite number) of vector fields.

- **Step 2:** Lifting the dynamic system on $\Omega$ to a Euclidian space of proper dimension, where all the vector fields involved by the dynamic system on $\Omega$ can be properly lifted into this Euclidian space.

- **Step 3:** All the analysis and control design can be done in conventional way for this lifting system on its Euclidian space.

- **Step 4:** Project the resulting manipulated system back to $\Omega$ and then lifting it into several original Euclidean spaces, where the original dimension-varying system lies on.

Finally, we would like to present a conjecture: The DFEM might provide a framework (i.e., the state space) for string theory in physics. The idea is sketched as follows:

Consider a subspace of DFEM as

$$\Omega_3 := \{\bar{x} \in \Omega \mid \dim(\bar{x}) \leq 3\}.$$
We choose 3 because it is the dimension of real physical world. Now $(\mathbb{R}^\infty, Pr, \Omega_3)$ is a sub-bundle of the fiber-bundle $(\mathbb{R}^\infty, Pr, \Omega)$. If we consider all possible realization of dynamic systems over $\Omega_3$, then the minimum total subspace which allow all possible realizations is $\mathbb{R}^{[6]}$. Hence, if we want a space which is of minimum dimension and also is able to contain all moves (or dynamic systems), then it is
\[
B := (\mathbb{R}^{[6]} \rightarrow \Omega_3).
\]
Since this manifold is of dimension 9, plus a dimension for $t$, a manifold of dimension 10 is reasonable for describing state-motion-time. This might be the string space.

Some further arguments are the following:

(i) It is well known in classical differential geometry that a dimension $n$ manifold $M$ has $n$ dimensional tangent space at each point. Hence, if taking both $M$ and $T(M)$ into consideration, an $n$ dimensional manifold with its tangent bundle is a $2n$ dimensional manifold. This is a well known fact in differential geometry. So consider the bundle $B$ as a 9 dimensional manifold is reasonable.

(ii) It seems that there is no static particle in the world.
That is, particles are always joined with their moves. Moves can be described by vector fields. So to describe a particle, a position plus a vector field on its tangent space may be reasonable to describe it, as the particle is small and its movement is very fast. Using string to describe a particle might essentially is a description for both the position and the trajectory of a particle.

(iii) Now taking the position and moving trajectory into consideration. We may consider the extra 6 dimensions are used to describe open string (or open movement of a particle). In addition, we need $SU(3)$ to describe the gauge group and $SU(1)$ for rotation. Then we have $B + SU(3) + SU(1) + time$, which is of dimension 26. This manifold might be proper for Bosonic super-string model.

In one word, DFEM could provide a framework for systems with arbitrary dimensions. Dimension-Free Reimannian manifold may overcome the crisis of classical Reimannian geometry[26].

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