Adversarial Attacks on Linear Contextual Bandits

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Abstract

Contextual bandit algorithms are applied in a wide range of domains, from advertising to recommender systems, from clinical trials to education. In many of these domains, malicious agents may have incentives to attack the bandit algorithm to induce it to perform a desired behavior. For instance, an unscrupulous ad publisher may try to increase their own revenue at the expense of the advertisers; a seller may want to increase the exposure of their products, or thwart a competitor’s advertising campaign. In this paper, we study several attack scenarios and show that a malicious agent can force a linear contextual bandit algorithm to pull any desired arm \( T - o(T) \) times over a horizon of \( T \) steps, while applying adversarial modifications to either rewards or contexts that only grow logarithmically as \( \mathcal{O}(\log T) \). We also investigate the case when a malicious agent is interested in affecting the behavior of the bandit algorithm in a single context (e.g., a specific user). We first provide sufficient conditions for the feasibility of the attack and we then propose an efficient algorithm to perform the attack. We validate our theoretical results on experiments performed on both synthetic and real-world datasets.

1. Introduction

Recommender systems are at the heart of the business model of many industries like e-commerce or video streaming (Davidson et al., 2010; Gomez-UrIBE & Hunt, 2015). The two most common approaches for this task are based either on matrix factorization (Park et al., 2017) or bandit algorithms (Li et al., 2010), which both rely on a unaltered feedback loop between the recommender system and the user. In recent years, a fair amount of work has been dedicated to understanding how targeted perturbations in the feedback loop can fool a recommender system into recommending low quality items.

Following the line of research on adversarial attacks in deep learning (Goodfellow et al., 2014) and supervised learning (Biggio et al., 2012; Jagielski et al., 2018; Li et al., 2016; Liu et al., 2017), attacks on recommender systems have been focused on filtering-based algorithms (Christakopoulou & Banerjee, 2019; Mehta & Nejdl, 2008) and offline contextual bandits (Ma et al., 2018). The question of adversarial attacks for online bandit algorithms has only started being studied quite recently (Jun et al., 2018; Liu & Shroff, 2019; Immorlica et al., 2018), though solely in the multi-armed stochastic setting. Although the idea of online adversarial bandit algorithms is not new (see EXP3 algorithm in (Auer et al., 2002)), the focus is different from what we are considering here. Indeed, algorithms like EXP3 or EXP4 (Lattimore & Szepesvári, 2018) are designed to find the optimal actions in hindsight to adapt to any stream of rewards without any further assumptions.

The opposition between the adversarial bandit setting and the stochastic setting has sparked interests in studying a middle ground. In (Bubeck & Slivkins, 2012), the learning algorithm has no knowledge of the type of feedback it receives. In (Li et al., 2019; Gupta et al., 2019; Lykouris et al., 2019; Kapoor et al., 2019), the rewards are assumed to be stochastic but can be perturbed by some attacks and the authors focus on constructing algorithms able to find the optimal actions even in the presence of some non-random perturbations. However, those perturbations are bounded and agnostic to the choices of the learning algorithm. There are also some efforts in the broader Deep Reinforcement Learning (DRL) literature, focusing on modifying the observations of different states to fool a DRL system at inference time (Hussenot et al., 2019; Sun et al., 2020).

Contribution. In this work, we first follow the research direction opened by (Jun et al., 2018) where the attacker has the objective of fooling a learning algorithm into taking a specific action as much as possible. Consider a news recommendation problem as described in (Li et al., 2010), a bandit algorithm has to choose between \( K \) articles to recommend to a user, based on some information about them, termed
context. We assume that an attacker sits between the user and the website and can choose the reward (i.e., click or not) for the recommended article. Their goal is to fool the bandit algorithm into recommending a particular target article to most users. We extend the work in (Jun et al., 2018; Liu & Shroff, 2019) to the contextual linear bandit setting showing how to perturb rewards for both stochastic and adversarial algorithms. For the first time, we consider and analyze the setting in which the attacker can only modify the contextual setting associated with the current user (the reward is not altered). The goal of the attacker is still to fool the bandit algorithm into pulling the target arm for most users while minimizing the total norm of their attacks. We show that it is possible to fool the widely known LinUCB algorithm (Abbasi-Yadkori et al., 2011; Lattimore & Szepesvári, 2018) with this new type of attack on the context. Finally, we present a harder setting for the attacker, where the latter can only modify the context associated to a specific user. For example, this situation may occur when a malicious agent has infected some computers with a Remote Access Trojan (RAT). The attacker can thus modify the history of navigation of a specific user and, as a consequence, the information seen by the online recommender system. We show how the attacker can attack the two very common bandit algorithms LinUCB and LinTS (Agrawal & Goyal, 2013) and, in certain cases, have them pull a target arm most of the time when a specific user visits a website.

2. Preliminaries

We consider the standard contextual linear bandit setting with $K \in \mathbb{N}$ arms. At each time $t$, the agent observes a context $x_t \in \mathbb{R}^d$, selects an action $a_t \in [1, K]$ and observes a reward: $r_{t,a_t} = \langle \theta_a, x_t \rangle + \eta_{t,a_t}$, where for each arm $a$, $\theta_a \in \mathbb{R}^d$ is a feature vector and $\eta_{t,a_t}$ is a conditionally independent zero-mean, $\sigma^2$-subgaussian noise. We also make the following assumptions on the contexts and parameter vectors.

**Assumption 1.** There exist $L > 0$ and $D \subset \mathbb{R}^d$, such that for all $t$, $x_t \in D$ and:

$$\forall x \in D, \forall a \in [1, K], \quad ||x||_2 \leq L \text{ and } (\theta_a, x) \in (0, 1]$$

In addition, we assume that there exists $S > 0$ such that $||\theta_a||_2 \leq S$ for all arms $a$.

The agent is interested in minimizing the cumulative regret, $R_T$ after $T$ steps:

$$R_T = \sum_{t=1}^T \langle a^*_t, x_t \rangle - \langle \theta_{a^*_t}, x_t \rangle$$

where $a^*_t := \arg\max_a \langle \theta_a, x_t \rangle$. A bandit learning algorithm $\mathcal{A}$ is said to be no-regret when it satisfies $R_T = o(T)$, i.e., the average expected reward received by $\mathcal{A}$ converges to the optimal one.

Classical bandit algorithms (e.g., LinUCB (Alg. 3) and LinTS (Alg. 4)) compute an estimate of the unknown parameters $\theta_a$ using past observations. Formally, for each arm $a \in [K]$ we define $S'_a$ as the set of times up to $t-1$ (included) where the agent played arm $a$. Then, the estimated parameters are obtained through regularized least-squares regression as $\hat{\theta}_a^t = (X_{t,a}X_{t,a}^\top + \lambda I)^{-1}X_{t,a}Y_{t,a}$, where $\lambda > 0$, $X_{t,a} = (x_i)_{i \in S'_a} \in \mathbb{R}^{d \times |S'_a|}$ and $Y_{t,a} = (r_{t,a_i})_{i \in S'_a} \in \mathbb{R}^{|S'_a|}$. Denote by $V_{t,a} = \lambda I + X_{t,a}X_{t,a}^\top$ the design matrix of the regularized least-square problem and by $\|x\|_V = \sqrt{x^\top V x}$ the weighted norm w.r.t. any positive matrix $V \in \mathbb{R}^{d \times d}$. We define the confidence set

$$C_{t,a} = \left\{ \theta \in \mathbb{R}^d : \|\theta - \hat{\theta}_{t,a}\|_{V_{t,a}} \leq \beta_{t,a} \right\}$$

where $\beta_{t,a} = \sigma \sqrt{d \log \left( \frac{1+L^2(1+|S'_a|)/\delta}{\delta} \right)} + S \sqrt{\lambda}$, which guarantees that $\theta_a \in C_{t,a}$, for all $t > 0$, w.p. $1 - \delta$. This uncertainty is used to balance the exploitation-exploitation trade-off either through optimism (e.g., LinUCB) or through randomization (e.g., LinTS).

3. Online Adversarial Attacks on Rewards

The ultimate goal of a malicious agent is to force the bandit algorithm to perform a desired behavior. An attacker may simply want to induce the bandit algorithm to perform poorly—ruining the users’ experience—or to force the algorithm to suggest a specific arm. The latter case is particularly interesting in advertising where a seller may want to increase the exposure of its product at the expense of the competitors. Note that the users’ experience is also compromised by the latter attack since the suggestions they will receive will not be tailored to their needs. Similarly to (Liu & Shroff, 2019; Jun et al., 2018), we focus on the latter objective, i.e., to fool the bandit algorithm into pulling a target arm $a^t$ for $T - o(T)$ time steps (independently of the user). A way to obtain this behavior is to dynamically modify the reward in order to make the bandit algorithm believe that $a^t$ is optimal. Clearly, the attacker has to pay a price in order to modify the perceived bandit problem and fool the algorithm. If there is no restriction on when and how the attacker can alter the reward, the attacker can easily fool the algorithm. However, this setting is not interesting since the attacker may pay a cost higher than the loss suffered by the attacked algorithm. An attack strategy is thus considered successful when the total cost of the attack is sublinear in $T$.

**Setting.** We assume that the attacker has the same knowledge as the bandit algorithm $\mathcal{A}$ about the problem (i.e.,
Algorithm 1 Contextual ACE

for $t = 1, \ldots, T$
\begin{itemize}
  \item Alg. $\mathcal{A}$ chooses arm $a_t$ based on context $x_t$
  \item Environment generates reward: $r_{t,a_t} = \langle \theta_{a_t}, x_t \rangle + \eta_t$ with $\eta_t$ conditionally $\sigma^2$-subgaussian
  \item Attacker observes reward $r_{t,a_t}$ and feeds the perturbed reward $\tilde{r}_{t,a_t}$ (or $\tilde{r}_{t,a_t}^2$) to $\mathcal{A}$
\end{itemize}
end for

knows $\sigma$ and $L$). The attacker is assumed to be able to observe the context $x_t$ and the arm $a_t$ pulled by $\mathcal{A}$, and can modify the reward received by $\mathcal{A}$. When the attacker modifies the reward $r_{t,a_t}$ into $\tilde{r}_{t,a_t}$, the instantaneous cost of the attack is defined as $c_t := |r_{t,a_t} - \tilde{r}_{t,a_t}|$. The goal of the attacker is to fool algorithm $\mathcal{A}$ such that arm $a^\dagger$ is pulled $T - o(T)$ times and $\sum_{t=1}^T c_t = o(T)$.

Attack idea. We leverage the idea presented in (Liu & Shroff, 2019) and (Jun et al., 2018) where the attacker lowers the reward of arms $a \neq a^\dagger$ so that algorithm $\mathcal{A}$ learns that the target arm is optimal for every context. Since $\mathcal{A}$ is assumed to be no-regret, the attacker only needs to modify the rewards $o(T)$ times to achieve this goal.

Lowering the rewards has the effect of shifting the vectors $(\theta_a)_{a \neq a^\dagger}$ to new vectors $(\theta'_a)_{a \neq a^\dagger}$ such that for all arms $a \neq a^\dagger$ and all contexts $x \in D$, $\langle \theta'_a, x \rangle \leq \langle \theta_a, x \rangle$.

Since rewards are assumed to be bounded (see Asm. 1), this objective can be achieved by simply forcing the reward of non-target arms $a \neq a^\dagger$ to the minimum value. Contextual ACE (see Alg. 2) implements a soft version of this idea by leveraging the knowledge of the reward distribution. At each round $t$, Contextual ACE modifies the reward perceived by $\mathcal{A}$ as follows:

$$\tilde{r}_{t,a_t} = \begin{cases} 
\eta'_t & \text{if } a_t \neq a^\dagger \\
\tilde{r}_{t,a_t} & \text{otherwise}
\end{cases}$$

(2)

where $\eta'_t$ is a $\sigma$-subgaussian random variable generated by the attacker independently of all other random variables. By doing this, Contextual ACE transforms the original problem into a stationary bandit problem in which $a^\dagger$ is optimal for all the contexts (having mean $(x, \theta_1)$) and all the other arms have expected reward of 0. Despite this attack may seem expensive, the following proposition shows that the cumulative cost of the attacks is sublinear.

Proposition 1. For any $\delta \in (0, 1/K]$, when using Contextual ACE algorithm (Alg 1) with perturbed rewards $\tilde{r}_t$, with probability at least $1 - K\delta$, algorithm $\mathcal{A}$ pulls arm $a^\dagger$ for $T - o(T)$ time steps and the total cost of attacks is $o(T)$.

The proof of this proposition is provided in App. A.1. While Prop. 1 holds for any no-regret algorithm $\mathcal{A}$, we can provide a more precise bound on the total cost by inspecting the algorithm. For example, we can show (see App. D), that, with probability at least $1 - K\delta$, the number of times LIN-UCB (Abbasi-Yadkori et al., 2011) pulls arms different than $a^\dagger$ is at most:

$$\sum_{j \neq a^\dagger} N_j(T) \leq \frac{64Ks^2\lambda S^2}{\min_{x \in D} \langle \theta_{a^\dagger}, x \rangle^2} \left( d \log \left( \frac{\lambda + TL^2}{\delta^2} \right) \right)^2$$

This directly translates into a bound on the total cost.

Comparison with ACE. In the stochastic setting, the ACE algorithm (Liu & Shroff, 2019) leverages a bound on the expected reward of each arm in order to modify the reward. However, the perturbed reward process seen by algorithm $\mathcal{A}$ is non-stationary and in general there is no guarantee that an algorithm minimizing the regret in a stationary bandit problem keeps the same performance when the bandit problem is not stationary anymore. Nonetheless, transposing the idea of the ACE algorithm to our setting would give an attack of the following form, where at time $t$, Alg. $\mathcal{A}$ pulls arm $a_t$ and receives rewards $\tilde{r}_{t,a_t}^2$:

$$\tilde{r}_{t,a_t}^2 = \begin{cases} 
    r_{t,a_t} + \max(-1, \min(0, C_{t,a_t})) & \text{if } a_t \neq a^\dagger \\
    r_{t,a_t} & \text{otherwise}
\end{cases}$$

with $C_{t,a_t} = (1 - \gamma) \min_{\theta \in C_{t,a_t}} \langle \theta, x_t \rangle - \max_{\theta \in C_{t,a_t}} \langle \theta, x_t \rangle$. Note that $C_{t,a}$ is defined as in Eq. 1 using the non-perturbed rewards, i.e., $Y_{t,a} = (r_{t,a})_{i=1}^S$.

Constrained Attack. When the attacker has a constraint on the instantaneous cost of the attack, using the perturbed reward $\tilde{r}_t^2$ may not be possible as the cost of the attack at time $t$ is not decreasing over time. Using the perturbed reward $\tilde{r}_t^2$ offers a more flexible type of attack with more control on the instantaneous cost thanks to the parameter $\gamma$. However, even this attack does not work when the maximum cost of an attack is too small.

Defense mechanism. The attack based on reward $\tilde{r}_t^1$ is hardly detectable without prior knowledge about the problem. In fact, the reward process associated to $\tilde{r}_t^1$ is stationary and compatible with the assumption about the true reward (e.g., subgaussian). While having very low rewards is reasonable in advertising, in other problems makes the attack easily detectable. On the other hand, the fact that $\tilde{r}_t^2$ is a non-stationary process makes this attack easy to detect. When some data are already available on each arm, the learner can monitor the difference between the average rewards per action computed on new and old data.

Remark 1. It is possible to extend this attack to multiple target arms $a^\dagger \in A^\dagger$. Similarly to (2), we can set $\tilde{r}_{t,a_t}^2 = \eta'_t$ when $a_t \notin A^\dagger$. 
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4. Online Adversarial Attacks on Contexts

In this section, we consider the attacker to be able to alter the context $x_t$ perceived by the algorithm rather than the reward that is to say the attacker is now restricted to change the type of users presented to the learning algorithm $\mathcal{A}$, hence changing their perception of the environment.

Setting. As in Sec. 3, we consider the attacker to have the same knowledge about the problem as $\mathcal{A}$. The main difference with the previous setting is that the attacker acts before the algorithm. It means that the attacker does not know the arm that would have been chosen by $\mathcal{A}$ when presented with the true context $x_t$. Therefore, we need to have knowledge about the way the algorithm $\mathcal{A}$ behaves. We focus on LINUCB and we assume the attacker knows the parameters of the algorithm. This is known as a white-box setting in the adversarial attacks literature (Goodfellow et al., 2014). The goal of the attacker is unchanged: they aim at forcing the algorithm to pull the target arm $a^\dagger$ for $T - o(T)$ time steps while paying a sublinear total cost. We denote by $\tilde{x}_t$ the context after the attack and by $c_t = ||x_t - \tilde{x}_t||_2$ the instantaneous cost. Finally, we assume that the attacker knows a positive lower bound of the expected reward as follows.

Assumption 2. For all $x \in \mathcal{D}$, $0 < \nu \leq \langle x, \theta_{a^\dagger} \rangle$.

Attack Idea: The idea behind the attack in this setting is similar to the idea behind the attack in Sec 3: the attacker builds a bandit problem where arm $a^\dagger$ is optimal for all contexts by lowering the perceived value of all other arms. Here, the attacker cannot modify the reward, but they can scale the contexts in order to decrease the predicted rewards in the original context. At time $t$, the attacker receives the context $x_t$ and computes the attack as follows:

1. Thanks to the white-box setting, the attacker computes the arm $a_t$ that $\mathcal{A}$ would pull if presented with context $x_t$.

2. If $a_t \neq a^\dagger$ then the attacker changes the context to $\tilde{x}_t = \alpha_{a_t} x_t$ with $\alpha_{a_t} = \max_{x \in \mathcal{D}} \langle \theta_{a_t}, x \rangle / \langle \theta_{a^\dagger}, x \rangle$. In practice, for every arm $a$ we set $\alpha_a = \alpha = 2/\nu$.

In other words, the attacker performs a dilatation of the incoming context every time alg. $\mathcal{A}$ does not pull arm $a^\dagger$. The fact that the decision rule used by LINUCB is invariant by dilatation guarantees that the attacker will not inadvertently lower the perceived rewards for arm $a^\dagger$. Because the rewards are assumed to be linear, presenting a large context $\alpha x$ and receiving the reward associated with the normal context $x$ will skew the estimated rewards of LINUCB. The attack protocol is summarized in Alg. 2.

Proposition 2. Using the attack described in Alg. 2, for any $\delta \in (0, 1/K]$, with probability at least $1 - K\delta$, the number of times LINUCB does not pull arm $a^\dagger$ before time $T$ is at most:

$$\sum_{j\neq a^\dagger} N_j(T) \leq 32K^2 \left( \frac{\lambda}{\alpha^2} + \sigma^2 d \log \left( \frac{\lambda d + TL^2 \alpha^2}{d \delta} \right) \right)^3$$

with $N_j(T)$ the number of times arm $j$ has been pulled during the first $T$ steps. The total cost for the attacker is bounded by:

$$\sum_{t=1}^{T} c_t \leq \frac{64K^2}{\nu} \left( \frac{\lambda}{\alpha^2} + \sigma^2 d \log \left( \frac{\lambda d + TL^2 \alpha^2}{d \delta} \right) \right)^3$$

The proof of Proposition 2 (see App. A.2) assumes that the attacker can attack at any time step, and that they can know in advance which arm will be pulled by Alg. $\mathcal{A}$ in a given context. Thus it is not applicable to random exploration algorithms like LinTS (Agrawal & Goyal, 2013) and $\varepsilon$-greedy. We also observed empirically that randomized algorithms are more robust to attacks (see Sec. 7).

Remark 2. If the attacker wants alg. $\mathcal{A}$ to pull any arm in a set of target arms $A^\dagger$, the same type of attack can still be used with $\nu$ such that $0 < \nu \leq \max_{x \in A^\dagger} \langle x, \theta_a \rangle$ for all $x \in \mathcal{D}$. Then, the context is multiplied by $\alpha = 2/\nu$ when alg. $\mathcal{A}$ is going to pull an arm not in $A^\dagger$.

5. Attacks on a Single Context

Previous sections focused on the man-in-the-middle (MITM) attack either on reward or context. The MITM attack allows the attacker to arbitrarily change the information observed by the recommender system at each round. This attack may be difficulty feasible in practice, since the exchange channels are generally protected by authentication and cryptographic systems. In this section, we consider the scenario where the attacker has control over a single user $u$. As an example, consider the case where the device of the user is infected by a malware (e.g., Trojan horse), giving full control of the system to the malicious agent. The attacker can thus modify the context of the specific user (e.g., by altering the cookies) that is perceived by the recommender.
system. We believe that changes to the context (e.g., cookies) are more subtle and less easily detectable than changes to the reward (e.g., click). Moreover, if the reward is a purchase, it cannot be altered easily by taking control of the user’s device.

Clearly, the impact of the attacker on the overall performance of the recommender system depends on the frequency of the specific user, that is out of the attacker’s control. It may be thus impossible to obtain guarantees on the cumulative regret of algorithm \(\mathcal{A} \). For this reason, we mainly focus on the study of the feasibility of the attack.

Formally, the attacker targets a specific user (i.e., the infected user) associated to a context \(x^t\). Similarly to Sec. 4, the objective of the attacker is to find the minimal change to the context presented to the recommender system \(\mathcal{A} \) such that the target arm \(a^t \) is selected by \(\mathcal{A} \). \(\mathcal{A} \) observes a modified context \(\tilde{x}\) instead of \(x^t\). After selecting an arm \(a_t, \mathcal{A}\) observes the true noisy reward \(r_{t,a_t} = \langle \theta_{a_t}, x^t \rangle + \eta_{a_t}^t\). As before, we study the white-box setting where the attacker has access to all the parameters of \(\mathcal{A} \).

5.1. Optimistic Algorithms

LinUCB chooses the arm to pull by maximizing an upper-confidence bound on the expected reward. For each arm \(a \) and context \(x\), the UCB value is given by

\[
\max_{\theta \in \tilde{C}_{t,a}} \langle x, \theta \rangle = \langle x, \tilde{\theta}^*_a \rangle + \beta_{t,a} \|x\|_{\tilde{V}_{t,a}}^{-1} \tag{3}
\]

where \(x, \theta \in \mathbb{R}^d\) are vectors, \(\beta_{t,a}\) is a parameter of the attacker and \(\tilde{C}_{t,a} := \{ \theta \mid \|\theta - \tilde{\theta}^*_a\|_{\tilde{V}_{t,a}} \leq \beta_{t,a} \}\) is the confidence set constructed by LinUCB. We use the notation \(\tilde{C}, \tilde{V}\) to stress the fact that LinUCB observes only the modified context.

In contrast to Sec. 3 and 4, the attacker may not be able to force the algorithm to pull the desired arm \(a^t\). In other words, Problem 3 may not be feasible. However, we are able to characterize the feasibility of (3).

**Theorem 1.** For any \(\xi > 0\), Problem (3) is feasible at time \(t\) if and only if:

\[
\exists \theta \in \tilde{C}_{t,a^t}, \quad \theta \not\in \text{Conv} \left( \bigcup_{a \neq a^t} \tilde{C}_{t,a} \right) \tag{4}
\]

In other words, the condition given by Theorem 1 says that the attack described here can be done when there exists a vector \(x\) for which the arm \(a^t\) is assumed to be optimal according to LinUCB. The condition mainly stems from the fact that optimizing a linear product on a convex compact set will reach its maximum on the edge of this set. In our case this set is the convex hull described by the confidence ellipsoids of LinUCB.

Although it is possible to use an optimization algorithm for this particular class of non-convex problems—e.g., DC programming (Tuy, 1995)—they are still slow compared to convex algorithms. Therefore, we present a simple convex relaxation of the previous problem that is simple and still enjoys some empirical performance improvement compared to Problem (3). The relaxed problem is the following:

\[
\min_{y \in \mathbb{R}^d} \|y\|_2 \quad \text{s.t.} \quad \max_{\theta \in \tilde{C}_{t,a}} \langle x^t + y, \theta \rangle = \max_{\theta \in \tilde{C}_{t,a}} \langle x^t, \tilde{\theta}^*_a \rangle + \xi \leq \begin{cases} 0, & a = a^t, \\ \|\theta - \tilde{\theta}^*_a\|_{\tilde{V}_{t,a}} \leq \beta_{t,a}, & a \neq a^t. \end{cases} \tag{5}
\]

Since the RHS of the constraint in Problem (3) can be written as \(\max_{\theta \in \tilde{C}_{t,a}} \langle \theta, x^t + y \rangle\) for any \(y\), the relaxation here consists in using \(\langle \theta, x^t + y \rangle\) as a lower-bound to this maximum for any \(\theta \in \tilde{C}_{t,a^t}\).

For the relaxed Problem (5), the same type of reasoning as for Problem (3) gives that Problem (5) is feasible if and only if:

\[
\hat{\theta}_{a^t}(t) \not\in \text{Conv} \left( \bigcup_{a \neq a^t} \tilde{C}_{t,a} \right)
\]
Remark 3. When a set of target arms is available, the feasibility condition is the same except that the attacker now cares about the union of the confidence ellipsoids for each arm in the set of target arms.

When condition (4) is not met, the arm $a^\dagger$ cannot be pulled by LinUCB. Indeed, the proof of Theorem 1 shows that the upper-confidence of the arm $a^\dagger$ is always dominated by another arm for any context. Let us assume that $a^\dagger$ is optimal for some contexts. More formally, there exists a sub-space $V \subset \mathcal{D}$ such that:

$$\forall x \in V, \quad \langle x, \hat{\theta}_{a^\dagger} \rangle > \langle x, \hat{\theta}_a \rangle$$

We also assume that the distribution of the contexts is such that, for all $t$, $\mu := \mathbb{P}(x_t \in V) > 0$. Then, the regret is lower-bounded in expectation by:

$$\mathbb{E}(R_T) = \mathbb{E} \left( \sum_{t=1}^T \mathbb{1}_{\{x_t \in V\}} (\langle x_t, \hat{\theta}_{a^\dagger} - \theta_{a^\dagger} \rangle) \right) \geq \mu m(T) \min_{x \in V, a \neq a^\dagger} \max(\theta_{a^\dagger} - \theta_a, x)$$

where $m(T)$ is the expected number of times $t \leq T$ such that condition (4) is not met. LinUCB guarantees that $\mathbb{E}(R_T) \leq \mathcal{O}(\sqrt{T})$ for every $T$. Hence, $m(T) \leq \mathcal{O}\left( \frac{\min_{x \in V, a \neq a^\dagger} \max(\sigma_{a^\dagger} - \sigma_a, x)}{\mu} \right)$. This means that, in an unattacked problem, condition (4) is met $T = \mathcal{O}(\sqrt{T})$ times. On the other hand, when the algorithm is attacked the regret of LinUCB is not sub-linear as the confidence bound for the target arm is not valid anymore. Hence we cannot provide the same type of guarantees for the attacked problem.

5.2. Random Exploration algorithms

The previous subsection focused on LinUCB, however we can obtain similar guarantees for algorithms with random exploration such as LinTS. In this case, it is not possible to guarantee that a specific arm will be pulled for a given context because of the randomness in the arm selection process. The objective is to guarantee that arm $a^\dagger$ is pulled with probability at least $1 - \delta$.

Similarly to the previous subsection, the problem of the attacker can be written as:

$$\min_{y \in \mathbb{R}^d} \|y\| \\ s.t. \quad \mathbb{P}(\forall a \neq a^\dagger, \langle x^\dagger + y, \hat{\theta}_a - \hat{\theta}_{a^\dagger} \rangle \leq -\xi) \geq 1 - \delta$$

(6)

where the $\hat{\theta}_a$ for different arms $a$ are independently drawn from a normal distribution with mean $\theta_a(t)$ and covariance matrix $\nu^2 V_{a^\dagger}^{-1}(t)$ with $\nu = \sigma \sqrt{\ln(T)/\delta}$. Solving this problem is not easy and in general not possible. For a given $x$ and arm $a$, the random variable $\langle x, \hat{\theta}_a \rangle$ is normally distributed with mean $\mu_a(x) := \langle \hat{\theta}_a(t), x \rangle$ and variance $\sigma_a^2(x) := \nu^2 \|x\|^2 V_{a^\dagger}^{-1}(t)$. We can then write $\langle x, \hat{\theta}_a \rangle = \mu_a(x) + \sigma_a(x) Z_a$ with $(Z_a) \sim \mathcal{N}(0, I_K)$. For the sake of clarity, we drop the variable $x$ when writing $\mu_a(x)$ and $\sigma_a(x)$. Thus the constraint in Problem (6) becomes:

$$\mathbb{E}_{Z_{a^\dagger}} \left( \Pi_{a \neq a^\dagger} \Phi \left( \frac{\sigma_{a^\dagger} Z_{a^\dagger} + \mu_{a^\dagger} - \mu_a}{\sigma_a} \right) \right) \geq 1 - \delta$$

where $\Phi$ is the cumulative distribution function of a normally distributed Gaussian random variable. Unfortunately, computing exactly the expectation of the last line is an open problem. Following the idea of (Liu & Shroff, 2019), a possible relaxation of the constraint in Problem (6) is, for every arm $a$:

$$1 - \Phi \left( \frac{\mu_{a^\dagger} - \mu_a - \xi}{\sqrt{\sigma_a^2 + \sigma_{a^\dagger}^2}} \right) \leq \frac{\delta}{K-1}$$

Therefore, the relaxed version of the attack on LinTS is:

$$\min_{y \in \mathbb{R}^d} \|y\| \\ s.t. \quad \forall a \neq a^\dagger, \langle x^\dagger + y, \hat{\theta}_a(t) - \hat{\theta}_{a^\dagger}(t) - \xi \rangle \geq \nu \Phi^{-1} \left( 1 - \delta/(K-1) \right) \|x^\dagger + y\| V_{a^\dagger}^{-1}(t) + V_{a^\dagger}^{-1}(t)$$

(7)

Problem (7) is similar to Problem (5) as the constraint is also a Second Order Cone program but with different parameters (see App. C).

6. Attacks on Adversarial Bandits

In the previous sections, we studied algorithms with sublinear regret $R_T$, i.e., mainly bandit algorithms designed for stochastic stationary environments. Adversarial algorithms like EXP4 do not provably enjoy a sublinear regret $R_T$. In addition, because this type of algorithms are, by design, robust to non-stationary environments, one could expect them to induce a linear cost on the attacker. In this section, we show that this is not the case for most contextual adversarial algorithms. Contextual adversarial algorithms are studied through the reduction to the bandit with expert advice problem. This is a bandit problem with $K$ arms where at every step, $N$ experts suggest a probability distribution over the arms. The goal of the algorithm is to learn which expert gets the best expected reward in hindsight after $T$ steps. The regret in this type of problem is defined as:

$$R_T^{exp} = \mathbb{E} \left( \max_{m \in [1,N]} \sum_{t=1}^T \sum_{j=1}^K E_{m,j}^{(t)} r_{t,j} - r_{t,a^\dagger} \right)$$

where $E_{m,j}^{(t)}$ is the probability of selecting arm $j$ for expert $m$. In the case of contextual adversarial bandits, the experts
first observe the context $x_t$ before recommending an expert $m$. Assuming the current setting with linear rewards, we can show that if an algorithm $\mathcal{A}$, like EXP4, enjoys a sublinear regret $R^\text{EXP}_T$, then, using the Contextual ACE attack with either $\tilde{r}^1$ or $\tilde{r}^2$, the attacker can fool the algorithm into pulling arm $a^1$ a linear number of times under some mild assumptions. However, attacking contexts for this type of algorithm is difficult because, even though the rewards are linear, the experts are not assumed to use a specific model for selecting an action.

**Proposition 3.** Suppose an adversarial algorithm $\mathcal{A}$ satisfies a regret $R^\text{EXP}_T$ of order $o(T)$ for any bandit problem and that there exists an expert $m^*$ such that $T - \sum_{t=1}^{T} E \left( P_{m^*, \alpha_t}^{(t)} \right) = o(T)$. Then attacking alg. $\mathcal{A}$ with Contextual ACE leads to pulling arm $a^1$, $T - o(T)$ of times in expectation with a total cost of $o(T)$ for the attacker.

The proof is similar to the one of Prop. 1 and is presented in App. A.4. The condition on the expert in Prop. 3 means that there exists an expert which believes $a^1$ is optimal most of the time. The adversarial algorithm will then learn that this expert is optimal.

Algorithm EXP4 has a regret $R^\text{EXP}_T$ bounded by $\sqrt{2TK \log(N)}$, thus the total number of pulls of arms different from $a^1$ is bounded by $\sqrt{2TK \log(M)/\gamma}$. This result also implies that for adversarial algorithms like EXP3 (Auer et al., 2002), the same type of attacks could be used to fool $\mathcal{A}$ into pulling arm $a^1$ because the MAB problem can be seen as a reduction of the contextual bandit problem with a unique context and one expert for each arm.

7. Experiments

In this section, we conduct experiments on the attacks on contextual bandit problems with simulated data and two real-world datasets: MovieLens25M (Harper & Konstan, 2015) and Jester (Goldberg et al., 2001). The synthetic dataset and the data preprocessing step are presented in Appendix B.1.

7.1. Attacks on Rewards

We study the impact of the reward attack for 4 contextual algorithms: LINUCB, LINTS, $\varepsilon$-GREEDY and EXP4. As parameters, we use $L=1$ for the maximal norm of the contexts, $\delta = 0.01$, $\nu = \sqrt{d \ln(t/\delta)}/2$, $\varepsilon_t = 1/\sqrt{t}$ at each time step $t$ and $\lambda = 0.1$. For EXP4, we use $N = 10$ experts with $N - 2$ experts returning a random action at each time, one expert choosing action $a^1$ every time and one expert returning the optimal arm for every context. With this set of experts the regret of bandits with expert advice is the same as in the contextual case. To test the performance of each algorithm, we generate 40 random contextual bandit problems and run each algorithm for $T = 10^6$ steps on each. We report the average cost and regret for each of the 40

Figure 2 shows the attacked algorithms using the attacked reward $\tilde{r}^1$ (reported as stationary CACE) and the rewards $\tilde{r}^2$ (reported as CACE). These experiments show that, even though the reward process is non-stationary, usual stochastic algorithms like LINUCB can still adapt to it and pull the optimal arm for this reward process (which is arm $a^1$). The true regret of the attacked algorithms is linear as $a^1$ is not optimal for all contexts. In the synthetic case, for the algorithms attacked with the rewards $\tilde{r}_1$, over 1M iterations and $\gamma = 0.22$, the target arm is drawn more than 99.4% of the time on average for every algorithm and more than 97.8% of the time for the stationary attack $\tilde{r}_1$ (see Table 3 in App. B.2). The dataset-based environments (see Figure 2) exhibit the same behavior: the target arm is pulled more than 94.0% of the time on average for all our attacks on Jester and MovieLens and more than 77.0% of the time in the worst case (for LINTS attacked with the stationary rewards) (see Table 3).

7.2. Attacks on Contexts

We now illustrate the setting of Sec. 4. We test the performance of LINUCB, LINTS and $\varepsilon$-GREEDY with the same parameters as in the previous experiments. Yet since the variance is much smaller in this case, we generate a random problem and run 20 simulations for each algorithm and each attack type. The target arm is chosen to minimize the average expected reward over all contexts and we use the exact lower-bound on the reward for this target arm as $\nu$. In addition, we also test the performance of an attack where the contexts are multiplied by 5 compared to the attack in Sec. 4 but where the attacker is only allowed to attack 20%
Table 1. Percentage of iterations for which the algorithm pulled the target arm $a^\dagger$ for each type of attack, averaged on 20 runs of 1M iterations. In the $CC^*$ version, the contexts are modified using the ContextualConic attack (Sec. 4).

|               | Synthetic | Jester | MovieLens |
|---------------|-----------|--------|-----------|
| LinUCB        | 2.38%     | 1.47%  | 2.24%     |
| CC LinUCB     | 99.99%    | 99.53% | 99.74%    |
| CC20 LinUCB   | 99.96%    | 99.24% | 99.55%    |
| $\varepsilon$-GREEDY | 0.26% | 0.58%  | 0.30%     |
| CC $\varepsilon$-GREEDY | 99.98% | 99.83% | 99.90%    |
| CC20 $\varepsilon$-GREEDY | 99.97% | 99.30% | 99.65%    |
| LinTS         | 3.27%     | 1.27%  | 1.29%     |
| CC LinTS      | 9.08%     | 7.24%  | 9.13%     |
| CC20 LinTS    | 32.22%    | 43.78% | 95.78%    |

Table 2. Percentage of times an algorithm pulled the target arm $a^\dagger$ when context $x^*$ was drawn, averaged on 40 runs of 1M iterations. When applicable, the Relaxed version corresponds to solving a relaxed convex version of the problem while the Full attack corresponds to solving the exact optimization problem.

|               | Synthetic | Jester | MovieLens |
|---------------|-----------|--------|-----------|
| LinUCB        | 0.07%     | 0.01%  | 0.39%     |
| LinUCB Relaxed| 13.76%    | 97.81% | 4.09%     |
| LinUCB Full   | 88.30%    | 99.98% | 99.99%    |
| $\varepsilon$-GREEDY | 0.01% | 0.00%  | 0.03%     |
| $\varepsilon$-GREEDY Full | 99.98% | 99.95% | 99.97%    |
| LinTS         | 0.02%     | 0.01%  | 0.05%     |
| LinTS Relaxed | 18.21%    | 80.48% | 5.56%     |

7.3. Attacks on a Single Context

We now move to the setting described in Sec. 5 and test the same algorithms as in Sec. 7.2. We run 40 simulations for each algorithm and each attack type. The target context $x^\dagger$ is chosen randomly and the target arm as the arm minimizing the expected reward for $x^\dagger$. The attacker is only able to modify the incoming context for the target context (which corresponds to the context of one user) and the incoming contexts are sampled uniformly from the set of all possible contexts (of size 100). Table 2 shows the percentage of success for each attack. We observe that the non-relaxed attacks on $\varepsilon$-GREEDY and LinUCB work well across all datasets. However, the relaxed attack for LinUCB and LinTS are not as successful, on the synthetic dataset and MovieLens25M. The Jester dataset seems to be particularly suited to this type of attacks because the true feature vectors are well separated from the convex hull formed by the feature vectors of the other arms. Only 5% of Jester’s feature vectors are contained in the convex hull of the others while this number amounts to 8% for MovieLens and 20% on average for the synthetic dataset.

As expected, the cost of the attacks is linear on all the datasets (see Figure 6 in App. B.4). The cost is also lower for the non-relaxed than for the relaxed version of the attack on LinUCB. Unsurprisingly, the cost of the attacks on LinTS is the highest due to the need to guarantee that the arm $a^\dagger$ will be chosen with high probability (95% in our experiments).

8. Conclusion

We presented several settings for online attacks on contextual bandits. We showed that an attacker can force any contextual bandit algorithm to almost always pull an arbitrary target arm $a^\dagger$ with only sublinear modifications of the rewards. When the attacker can only modify the contexts, we prove that LinUCB can still be attacked and made to
almost always pull $a_t$ by adding sublinear perturbations to the contexts. When the attacker can only attack a single context, we derive a feasibility condition for the attacks and we introduce a method to compute some attacks of small instantaneous cost for LINUCB, $\varepsilon$-GREEDY and LINTS. To the best of our knowledge, this paper is the first to describe effective attacks on the contexts of contextual bandit algorithms. Our numerical experiments, conducted on both synthetic and real-world data, validate our results and show that the attacks on all contexts are actually effective on several algorithms and with more permissible settings.

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A. Proofs

In this appendix, we present the proofs of different theoretical results presented in the paper.

A.1. Proof of Proposition 1

**Proposition.** For any $\delta \in (0, 1/K]$, when using Contextual ACE algorithm (Alg. 1) with perturbed rewards $\hat{r}$, with probability at least $1 - K\delta$, algorithm $\mathcal{A}$ pulls arm $a^\dagger$, $T - o(T)$ times and the total cost of attacks is $o(T)$.

**Proof.** Let us consider the contextual bandit problem $A_1$, with $K$ arms with contexts $x \in \mathcal{D}$ such that the optimal arm has mean reward $(\theta_{a^\dagger}, x)$ and all $K - 1$ other arms have mean 0. Then the regret of algorithm $\mathcal{A}$ for this bandit problem is upper-bounded with probability at least $1 - \delta$ by a function $f_\mathcal{A}(T)$ such that $f_\mathcal{A}(T) = o(T)$. In addition, the reward process fed to Alg. $\mathcal{A}$ by the attacker is a stationary reward process with $\sigma^2$-subgaussian noise. Therefore, the number of times algorithm $\mathcal{A}$ pulls an arm different from $a^\dagger$ is upper-bounded by $f_\mathcal{A}(T) / \min_{x \in \mathcal{D}} \langle x, \theta_{a^\dagger} \rangle$.

In addition, the total cost of the attack is upper-bounded by $\max_{a \in [1, K]} \max_{x \in \mathcal{D}} \langle x, \theta_a \rangle (T - N_a^\dagger(T))$ where $N_a^\dagger(T)$ is the number of times arm $a^\dagger$ has been pulled up to time $T$. Thanks to the previous argument, $T - N_a^\dagger(T) \leq f_\mathcal{A}(T) / \min_{x \in \mathcal{D}} \langle x, \theta_{a^\dagger} \rangle$.

A.2. Proof of Proposition 2

**Proposition.** Using the attack described in Alg. 2, for any $\delta \in (0, 1/K]$, with probability at least $1 - K\delta$, the number of times LINUCB does not pull arm $a^\dagger$ is at most:

$$\sum_{j \neq a^\dagger} N_j(T) \leq 32K^2 \left( \frac{\lambda}{\alpha^2} + \sigma^2 d \log \left( \frac{\lambda d + TL^2 \alpha^2}{d\delta} \right) \right)^3$$

with $N_j(T)$ the number of times arm $j$ has been pulled after $T$ steps, $||\theta_a|| \leq S$ for all arms $a$, $\lambda$ the regularization parameter of LINUCB and for all $x \in \mathcal{D}$, $||x||_2 \leq L$. The total cost for the attacker is bounded by:

$$\sum_{t=1}^{T} c_t \leq \frac{64K^2}{\nu} \left( \frac{\lambda}{\alpha^2} + \sigma^2 d \log \left( \frac{\lambda d + TL^2 \alpha^2}{d\delta} \right) \right)^3$$

**Proof.** Let $a_t$ be the arm pulled by LINUCB at time $t$. For each arm $a$, let $\hat{\theta}_a(t)$ be the result of the linear regression with the attacked context and $\hat{\theta}_a(t, \lambda/\alpha^2)$ the one with the unattacked context and a regularization of $1/\alpha^2$. At any time step $t$, we can write, for all $a \neq a^\dagger$:

$$\hat{\theta}_a(t) = \left( \lambda I_d + \sum_{l=0, a_l=a}^t \alpha^2 x_l x_l^T \right)^{-1} \sum_{k=0}^t r_k x_k = \frac{1}{\alpha} \left( \frac{\lambda}{\alpha^2} I_d + \sum_{k=0}^t x_k x_k^T \right)^{-1} \sum_{k=0}^t r_k x_k = \frac{\hat{\theta}_a(t, \lambda/\alpha^2)}{\alpha}$$

We also note that, since the contexts are not modified for arm $a^\dagger$: $\hat{\theta}_{a^\dagger}(t) = \hat{\theta}_{a^\dagger}(t, \lambda)$. In addition, for any context $x$ and arm $a \neq a^\dagger$, the exploration term used by LINUCB becomes:

$$||x||_{V_{a, t}^{-1}} = \frac{1}{\alpha} ||x||_{\hat{V}_{a, t}^{-1}}$$

where $\hat{V}_{a, t} = \lambda I_d + \sum_{l=0, a_l=a}^t \alpha^2 x_l x_l^T$ and $\hat{V}_{a, t}^{-1} = \lambda/\alpha^2 I_d + \sum_{k=0}^t x_k x_k^T$. For a time $t$, if presented with context $x_t$ LINUCB pulls arm $a_t \neq a^\dagger$, we have:

$$\alpha \left( \langle \hat{\theta}_{a^\dagger}(t), x_t \rangle + \beta_{a^\dagger}(t) ||x_t||_{\hat{V}_{a^\dagger, t}^{-1}} \right) \leq \langle \hat{\theta}_{a^\dagger}(t, \lambda/\alpha^2), x_t \rangle + \beta_{a^\dagger}(t) ||x_t||_{\hat{V}_{a^\dagger, t}^{-1}}$$

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As \( \alpha = \frac{2}{v} \geq \frac{2}{\theta_{a^*} x_t} \), we deduce that on the event that the confidence sets (Theorem 2 in (Abbasi-Yadkori et al., 2011)) hold for arm \( a^* \):

\[
2 \leq \langle \hat{\theta}_{a^*}(t, \lambda / \alpha^2), x_t \rangle + \beta_{a^*}(t) ||x_t||_{\psi_{a^*,t}^{-1}} \leq \langle \theta_{a^*}, x_t \rangle + 2\beta_{a^*}(t) ||x_t||_{\psi_{a^*,t}^{-1}}
\]

Thus, \( 1 \leq 2 - \langle \theta_{a^*}, x_t \rangle \leq 2\beta_{a^*}(t) ||x_t||_{\psi_{a^*,t}^{-1}} \). Therefore,

\[
\sum_{t=1}^{T} \mathbb{I}_{\{a_t \neq a^*\}} \leq \sum_{t=1}^{T} \min(2\beta_{a^*}(t)||x_t||_{\psi_{a^*,t}^{-1}}, 1) \mathbb{I}_{\{a_t \neq a^*\}} \leq \sum_{j \neq a^*} \sum_{t=1}^{T} \beta_j(T) \sum_{t=1}^{T} \mathbb{I}_{\{a_t = j\}} \min(1, ||x_t||_{\psi_{j,t}^{-1}}^2)
\]

But using Lemma 11 from (Abbasi-Yadkori et al., 2011) and the bound on the \( \beta_j(T) \) for all arms \( j \), we have with Jensen inequality:

\[
\sum_{t=1}^{T} \mathbb{I}_{\{a_t \neq a^*\}} \leq 4 \sqrt{K} \sum_{t=1}^{T} \mathbb{I}_{\{a_t \neq a^*\}} d \log \left( 1 + \frac{\alpha^2TL^2}{\lambda d} \right) \left( \sqrt{\frac{\lambda}{\alpha^2}} S + \sigma \sqrt{2 \log(1/\delta)} + d \log(1 + \frac{\alpha^2TL^2}{\lambda d}) \right)
\]

\( \square \)

A.3. Proof of Theorem 1

**Theorem.** For any \( \xi > 0 \), Problem (3) is feasible if and only if:

\[
\exists \theta \in \mathcal{C}_{t,a^*}, \quad \theta \notin \text{Conv} \left( \bigcup_{a \neq a^*} \mathcal{C}_{t,a} \right)
\]

where for every arm \( a \), \( \mathcal{C}_{t,a} := \{ \theta \mid ||\theta - \hat{\theta}_{a}(t)||_{\psi_{a,t}} \leq \beta_a(t) \} \) with \( \hat{\theta}_{a}(t) \) the least squares estimate for arm \( a \) built by LINUCB and

\[
\tilde{V}_{a,t} = \lambda I_d + \sum_{l=1, x_l \neq x^*}^{t} \mathbb{I}_{\{a_l = a\}} x_l x_l^T + \sum_{l=1, x_l = x^*}^{t} \mathbb{I}_{\{a_l = a\}} \tilde{x}_l \tilde{x}_l^T
\]

the design matrix of LINUCB at time \( t \) for all arms \( a \) (where \( \tilde{x}_l \) is the modified context)

**Proof.** The proof of Theorem 1 is decomposed in two parts.

First, let us assume that Equation (9) is satisfied. Then, let \( \theta \in \mathcal{C}_{t,a^*} \setminus \text{Conv} \left( \bigcup_{a \neq a^*} \mathcal{C}_{t,a} \right) \), then by the theorem of separation of convex sets applied to \( \mathcal{C}_{t,a^*} \) and \( \{ \theta \} \). There exists a vector \( v \) and \( c_1 < c_2 \) such that for all \( y \in \text{Conv} \left( \bigcup_{a \neq a^*} \mathcal{C}_{t,a} \right) \):

\[
\langle y, v \rangle \leq c_1 < c_2 \leq \langle \theta, v \rangle.
\]

Hence, for \( \xi > 0 \) we have that for \( \bar{v} = \frac{\xi}{c_2 - c_1} v \) that:

\[
\langle y, \bar{v} \rangle + \xi \leq \langle \theta, \bar{v} \rangle
\]

Secondly, let us assume that an attack is feasible. Then there exists a vector \( y \) such that:

\[
\max_{\theta \in \mathcal{C}_{t,a^*}} \langle y, \theta \rangle > c_1 := \max_{a \neq a^*} \max_{\theta \in \mathcal{C}_{t,a}} \langle y, \theta \rangle
\]

Let us reason by contradiction. We assume that \( \mathcal{C}_{t,a^*} \subset \text{Conv} \left( \bigcup_{a \neq a^*} \mathcal{C}_{t,a} \right) \) and consider \( \theta \in \mathcal{C}_{t,a^*} \). There exists \( n \in \mathbb{N}^*, \lambda_1, \ldots, \lambda_n \geq 0 \) and \( \theta_1, \ldots, \theta_n \in \bigcup_{a \neq a^*} \mathcal{C}_{t,a} \) such that \( \theta = \sum_{i=1}^{n} \lambda_i \theta_i \) and \( \sum_{i=1}^{n} \lambda_i = 1 \). Thus

\[
\langle y, \theta \rangle = \sum_{i} \lambda_i \langle y, \theta_i \rangle \leq c_1 \sum_{i} \lambda_i = c_1
\]

The problem is feasible, so \( c_1 < \max_{\theta \in \mathcal{C}_{t,a}} \langle y, \theta \rangle \). This contradicts Eq. 10.

\( \square \)
A.4. Proof of Proposition 3

**Proposition.** For an adversarial algorithm $\mathfrak{A}$, satisfying a regret $R_T^{\text{exp}}$ of order $o(T)$ for any bandit problem and if there exists an expert $m^*$ such that $T - \sum_{t=1}^{T} E \left[ E_{m^*,a^*}^{(t)} \right] = o(T)$. Then attacking Alg. $\mathfrak{A}$ with Contextual ACE lead to pull arm $a^1$, $T - o(T)$ times in expectation with a total cost of order $o(T)$ for the attacker.

**Proof.** Similarly to the proof of Proposition 1, let’s define the bandit with expert advice problem, $\mathcal{A}_i$, such that at each time $t$ the reward vector is $(\tilde{r}_{t,a})_a$ (with $i \in \{1, 2\}$). The regret of this algorithm is: $R_T^{\text{exp}} = E \left( \max_{m \in \{1, N\}} \sum_{t=1}^{T} E_{m}^{(t)} \tilde{r}_{t,a} - \tilde{r}_{t,a_i} \right)$ where $a_t$ are the actions taken by algorithm $\mathcal{A}_i$ to minimize $R_T^{i,\text{exp}}$. Then we have:

$$R_T^{i,\text{exp}} \geq E \left( \sum_{t=1}^{T} \sum_{j=1}^{K} (E_{m^*,j}^{(t)} - \mathbb{1}_{\{j \neq a^1\}}) \tilde{r}_{t,j} + \sum_{t=1}^{T} \tilde{r}_{t,a^1} - \tilde{r}_{t,a_i} \right)$$

Therefore,

$$E \left( \sum_{t=1}^{T} \tilde{r}_{t,a^1} - \tilde{r}_{t,a_i} \right) \leq R_T^{i,\text{exp}} + E \left( \sum_{t=1}^{T} \sum_{j=1}^{K} (\mathbb{1}_{\{j \neq a^1\}} - E_{m^*,j}^{(t)}) \tilde{r}_{t,j} \right) \leq R_T^{i,\text{exp}} + E \left( \sum_{t=1}^{T} (1 - E_{m^*,a^1}^{(t)}) \tilde{r}_{t,j} \right) \leq R_T^{i,\text{exp}} + E \left( \sum_{t=1}^{T} (1 - E_{m^*,a^1}^{(t)}) \right)$$

For strategy $i = 1$, we have:

$$E \left( \sum_{t=1}^{T} \tilde{r}_{t,a^1} - \tilde{r}_{t,a_i} \right) = E \left( \sum_{t=1}^{T} (r_{t,a^1} \mathbb{1}_{a^1 \neq a^*}) \right) \geq (T - E(N_{a^1}(T))) \min_{x \in D} \langle x, \theta_{a^1} \rangle$$

Then, as $R_T^{1,\text{exp}} \in o(T)$ and $E \left( \sum_{t=1}^{T} (1 - E_{m^*,a^1}^{(t)}) \right) \in o(T)$, we deduce that $E(N_{a^1}(T)) = T - o(T)$. For this strategy the cost is therefore bounded by:

$$E \left( \sum_{t=1}^{T} c_t \right) \leq E \left( \sum_{t=1}^{T} \mathbb{1}_{a^1 \neq a^*} + |\eta_{a_i,t}| + |\eta_{a^1,t}'| \right) \leq (1 + 2\sigma)E \left( N_{a^1}(T) \right)$$

For strategy $i = 2$, and $\delta > 0$, let us denote by $E_\delta$ the event that all confidence intervals hold with probability $1 - \delta$. But on the event $E_\delta$, for a time $t$ where $a_t \neq a^1$ and such that $-1 \leq C_{t,a_i} \leq 0$:

$$\tilde{r}_{t,a_i}^2 = r_{t,a_i} + C_{t,a_i} = (1 - \gamma) \min_{\theta \in C_{t,a^1}} \langle \theta, x_t \rangle + \eta_{a_i,t} + \langle \theta_{a_i}, x_t \rangle - \max_{\theta \in C_{t,a^1}} \langle \theta, x_t \rangle \leq (1 - \gamma) \langle \theta_{a^1}, x_t \rangle + \eta_{a_i,t}$$

when $C_{t,a_i} > 0$ (still on the event $E_\delta$):

$$\tilde{r}_{t,a_i}^2 = r_{t,a_i} \leq (1 - \gamma) \langle \theta_{a_i}, x_t \rangle + \eta_{a_i,t}$$

because $C_{t,a_i} > 0$ means that $(1 - \gamma) \langle \theta_{a_i}, x_t \rangle \geq (1 - \gamma) \min_{\theta \in C_{t,a^1}} \langle \theta, x_t \rangle \geq \max_{\theta \in C_{t,a^1}} \langle \theta, x_t \rangle \geq \langle \theta, x_t \rangle$. But finally, when $C_{t,a_i} \leq -1$, $\tilde{r}_{t,a_i}^2 = r_{t,a_i} - 1 \leq \eta_{a_i,t} \leq (1 - \gamma) \langle \theta_{a^1}, x_t \rangle + \eta_{a_i,t}$. But on the complementary event $E_\delta^c$, $\tilde{r}_{t,a_i}^2 \leq r_{t,a_i}$. Thus, given that the expected reward is assumed to be bounded in $(0, 1]$ (Assumption 1):

$$E \left( \sum_{t=1}^{T} \tilde{r}_{t,a_i}^2 - \tilde{r}_{t,a_i}^2 \right) = E \left( \sum_{t=1}^{T} (r_{t,a_i} - \tilde{r}_{t,a_i}^2) \mathbb{1}_{a_i \neq a^1} \right) \geq E \left( \sum_{t=1}^{T} \gamma \min_{x \in D} \langle x, \theta_{a^1} \rangle \mathbb{1}_{a_i \neq a^1} \mathbb{1}_{E_\delta} \right) - T\delta$$
Finally, putting everything together we have:

$$\mathbb{E}\left( \sum_{t=1}^{T} \min_{x \in D}(x, \theta_{a_t}) \mathbb{I}_{\{a_t \neq a^1\}} \right) \leq \tilde{R}_{\ast}^{\text{exp}} + \mathbb{E}\left( \sum_{t=1}^{T} (1 - E_{m^*, a^1}^{(t)}) \right) + \delta T \left( \min_{x \in D}(x, \theta_{a^1}) + 1 \right)$$

Hence, because $\tilde{R}_{\ast}^{\text{exp}} = o(T)$ and $\mathbb{E}\left( \sum_{t=1}^{T} (1 - E_{m^*, a^1}^{(t)}) \right) = o(T)$ we have that for $\delta \leq 1/T$, the expected number of pulls of arm $a^1$ is of order $o(T)$. In addition, the cost for the attacker is bounded by:

$$\mathbb{E}\left( \sum_{t=1}^{T} c_t \right) = \mathbb{E}\left( \sum_{t=1}^{T} \mathbb{I}_{\{a_t \neq a^1\}} \max(-1, \min(C_t, a_t, 0)) \right) \leq \mathbb{E}\left( \sum_{t=1}^{T} \mathbb{I}_{\{a_t \neq a^1\}} \right)$$

□

B. Experiments

B.1. Datasets and preprocessing

We present here the data and how we preprocess them for the numerical experiments of Section 7.

We consider two types of experiments, one on synthetic data with a contextual MAB problems with $K = 10$ arms such that for every arm $a$, $\theta_a$ is drawn from a folded normal distribution in dimension $d = 30$. We also use a finite number of contexts (10), each of them is drawn from a folded normal distribution projected on the unit circle multiplied by a uniform radius variable (i.i.d. across all contexts). Finally, we scale the expected rewards in $(0, 1]$ and the noise is drawn from a centered Gaussian distribution $\mathcal{N}(0, 0.01)$.

The second type of experiments is conducted in the real-world datasets Jester (Goldberg et al., 2001) and MovieLens25M (Harper & Konstan, 2015). Jester consists of joke ratings on a continuous scale from $0$ to $10$. Jester consists of joke ratings on a continuous scale from $0$ to $10$ for 100 jokes from a total of 73421 users. We use the features extracted via a low-rank matrix factorization ($d = 35$) to represent the actions (i.e., the jokes). We consider a complete subset of 40 jokes and 1918 users. Each user rates all the 40 jokes. At each time, a user is randomly selected from the 1918 users and mean rewards are normalized in $[0, 1]$. The reward noise is $\mathcal{N}(0, 0.01)$. The second dataset we use is MovieLens25M. It contains 25000095 ratings created by 162541 users on 62423 movies. We perform a low-rank matrix factorization to compute users features and movies features. We keep only movies with at least 1000 ratings, which leave us with 162539 users and 3794 movies. At each time step, we present a random user, and the reward is the scalar product between the user feature and the recommend movie feature. All rewards are scaled to lie in $[0, 1]$ and a Gaussian noise $\mathcal{N}(0, 0.01)$ is added to the rewards.

B.2. Attacks on Rewards

In this appendix, we present empirical evolution of the total cost and the number of draws of the target arm as a function of the attack parameter $\gamma$ for the Contextual ACE attack with perturbed rewards $\tilde{r}^{\gamma}$ on generated data.

Fig. 4 (left) shows that the total cost of attacks seems to be quite invariant w.r.t. $\gamma$ except when $\gamma \to 0$ because the difference between the target arm and the other becomes negligible. This is also depicted by the total number of draws (Fig. 4, Right) as the number of draws plummet when $\gamma \to 0$.

B.3. Attacks on all Contexts

Fig. 5 shows the total cost for all the attacks (that is to say including CC LINTS and CC20 LINTS compared to Fig 3). This figure shows that even though the total cost of attacks is linear for the synthetic and Jester dataset, it seems that for MovieLens the attacker achieves their goal with a logarithmic total. Therefore, despite the fact that the estimate of $\theta_{a^1}$ can be polluted by attacked samples, it seems that LINTS can still pick up $a^1$ as being optimal for this particular instance.

B.4. Attack on a single context

The attacks are computed by solving the optimization problems 3 and 5 (Sec. 5). We choose the libraries according to their efficiency for each problem we need to solve. For Problem (5) and Problem (7) we use CVXPY (Agrawal et al., 2018) and...
Problem (7) and Problem (8) are both SOC programs. We can see the similarities between both problems as follows. Let us define for every arm $a \neq a^\dagger$, the ellipsoid:

$$C_{t,a} := \left\{ y \in \mathbb{R}^d \mid \| y - \hat{\theta}_a(t) \|_{A_a^{-1}(t)} \leq \nu \Phi^{-1} \left( 1 - \frac{\delta}{K - 1} \right) \right\}$$

with $A_a(t) = \tilde{V}_a^{-1}(t) + \tilde{V}_{a^\dagger}(t)$ with $\tilde{V}_a(t)$ and $\tilde{V}_{a^\dagger}(t)$ the design matrix built by LinTS and $\hat{\theta}_a(t)$ the least squares estimate of $\theta_a$ at time $t$. Therefore for an arm $a$, the constraint in Problem (8) can be written for any $y \in \mathbb{R}^d$ as:

$$\langle x^* + y, \hat{\theta}_a(t) \rangle - \xi \geq \max_{z \in C_{t,a}} \langle z, x^* + y \rangle$$
Adversarial Attacks on Linear Contextual Bandits

Figure 5. Total cost of the attacks for the attack of Sec. 4 on our synthetic dataset, Jester and MovieLens

![Graphs](image)

Figure 6. Total cost of the attacks for the attacks one one context on our synthetic dataset, Jester and MovieLens. As expected, the total cost is linear.

Indeed for any \( x \in \mathbb{R}^d \),

\[
\max_{y \in C'_{t,a}} \langle y, x \rangle = \langle x, \hat{\theta}_a(t) \rangle + \nu \Phi^{-1} \left( 1 - \frac{\delta}{K-1} \right) \times \max_{\|u\|_2 \leq 1} \langle u, x \rangle \\
= \langle x, \hat{\theta}_a(t) \rangle + \nu \Phi^{-1} \left( 1 - \frac{\delta}{K-1} \right) \max_{\|z\|_2 \leq 1} \langle z, A^{1/2}_a(t) x \rangle \\
= \langle x, \hat{\theta}_a(t) \rangle + \nu \Phi^{-1} \left( 1 - \frac{\delta}{K-1} \right) \|A^{1/2}_a(t) x \|_2
\]

Thus, the constraint is feasible if and only if:

\[
\hat{\theta}_{\alpha^1}(t) \not\in \text{Conv} \left( \bigcup_{\alpha \neq \alpha^1} C'_{t,\alpha} \right)
\]

D. Contextual Bandit Algorithms

In this appendix, we present the different bandit algorithms studied in this paper. All algorithms we consider except EXP4 uses disjoint models for building estimate of the arm feature vectors \( (\theta_a)_{a \in [1,K]} \). Each algorithm (except EXP4) builds least squares estimates of the arm features.

E. Semi-Online Attacks

(Liu & Shroff, 2019) studies what they call the offline setting for adversarial attacks on stochastic bandits. They consider a setting where a bandit algorithm is successively updated with mini-batches of fixed size \( B \). The attacker can tamper with some of the incoming mini-batches. More precisely, they can modify the context, the reward and even the arm that was...
Adversarial Attacks on Linear Contextual Bandits

Algorithm 3 Contextual LinUCB

**Input:** regularization $\lambda$, number of arms $K$, number of rounds $T$, bound on context norms: $L$, bound on norms $\theta_a$: $D$

Initialize for every arm $a$, $V_a^{-1} = \frac{1}{\lambda} I_d$, $\hat{a}_a(t) = 0$ and $b_a(t) = 0$

for $t = 1, \ldots, T$ do

Observe context $x_t$

Compute $\beta_a(t) = \sqrt{d \log \left( \frac{1+N_a(t)L^2/\lambda}{\delta} \right)}$ with $N_a(t)$ the number of pulls of arm $a$

Pull arm $a_t = \arg\max_a (\hat{a}_a(t), x_t) + \beta_a(t)||x_t||_{V_{a_t}^{-1}(t)}$

Observe reward $r_t$ and update parameters $\hat{a}_a(t)$ and $V_a^{-1}(t)$ such that:

$$V_{a_t}(t+1) = V_{a_t}(t) + x_t x_t^T, \quad b_{a_t}(t+1) = b_{a_t}(t) + r_t x_t, \quad \theta_{a_t}(t+1) = V_{a_t}^{-1}(t+1) b_{a_t}(t+1)$$

end for

Algorithm 4 Linear Thompson Sampling with Gaussian prior

**Input:** regularization $\lambda$, number of arms $K$, number of rounds $T$, variance $\nu$

Initialize for every arm $a$, $V_a^{-1} = \lambda I_d$ and $\hat{a}_a(t) = 0$, $b_a(t) = 0$

for $t = 1, \ldots, T$ do

Observe context $x_t$

Draw $\hat{a}_a = \mathcal{N}(\hat{a}_a(t), \nu^2 V_{a_a}^{-1}(t))$

Pull arm $a_t = \arg\max_{a \in [1, K]} \langle \hat{a}_a, x_t \rangle$

Observe reward $r_t$ and update parameters $\hat{a}_a(t)$ and $V_a^{-1}(t)$

$$V_{a_t}(t+1) = V_{a_t}(t) + x_t x_t^T, \quad b_{a_t}(t+1) = b_{a_t}(t) + r_t x_t, \quad \theta_{a_t}(t+1) = V_{a_t}^{-1}(t+1) b_{a_t}(t+1)$$

end for

pulled for any entry of the attacked mini-batches. The main difference between this type of attacks and the online attacks we considered in the main paper is that we do not assume that we can attack from the start of the learning process: the bandit algorithm may have already converged by the time we attack.

We can still study the cumulative cost for the attacker to change the mini-batch in order to fool a bandit algorithm to pull a target arm $a^\dagger$. Contrarily to (Liu & Shroff, 2019), we call this setting semi-online. We first study the impact of an attacker on LinUCB where we show that, by modifying only $(K-1)d$ entries from the batch $B$, the attacker can force LinUCB to pull arm $a^\dagger$, $M'B - o(M'B)$ times with $M'$ the number of remaining batches updates. The cost of our attack is $\sqrt{MB}$ with $M$ the total number of batches.

**Cost of an attack:** If presented with a mini-batch $B$, with elements $(x_t, a_t, r_t)$ composed of the context $x_t$ presented at time $t$, the action taken $a_t$ and the reward received $r_t$, the attacker modifies element $i$, namely $(x_i^t, a_i^t, r_i^t)$ into $(\tilde{x}_i^t, \tilde{a}_i^t, \tilde{r}_i^t)$. The cost of doing so is $c_i^t = ||x_i^t - \tilde{x}_i^t||_2 + ||\tilde{r}_i^t - r_i^t|| + \mathbf{1}_{\{a_i^t \neq \tilde{a}_i^t\}}$ and the total cost for mini-batch $B$ is defined as $c_B = \sum_{i \in B} c_i^t$.

Finally, we consider the cumulative cost of the attack over $M$ different mini-batches $B_1, \ldots, B_M$, $\sum_{t=1}^{M} c_{B_t}$. The interaction between the environment, the attacker and the learning algorithm is summarized in Alg. 7.

The attack presented here is based on the Ahlber-Nilson-Varah bound (Varah, 1975), which gives a control on the sup norm of a matrix with dominant diagonal elements. More precisely, when presented with a mini-batch $B$, the attacker needs to modify the contexts and the rewards. We assume that the attacker knows the number of mini-batch updates $M$ and has access to a lower-bound on the reward of the target arm, $\nu$ as in Assumption 2.

The attacker changes $(K-1) \times d$ rows of the first mini-batch to rewards of 0 with a context $\delta_a e_i$ for each arm $a \neq a^\dagger$ with $(e_i)$ the canonical basis of $\mathbb{R}^d$. Moreover, $\delta_a$ is chosen such that:

$$\delta_a > \max \left( \sqrt{\frac{2MBL^2d}{\nu} + dMB}, \sqrt{\frac{4L^2d}{\nu^2} + dMB} \right)$$

(11)
Algorithm 5 $\varepsilon$-GREEDY

**Input:** regularization $\lambda$, number of arms $K$, number of rounds $T$, exploration parameter $\varepsilon_t$

Initialize, for all arms $a$, $V_a^{-1}(t) = \lambda I_d$ and $\theta_a(t) = 0$, $\varepsilon_t = 1$, $b_a(t) = 0$

for $t = 1, \ldots, T$

Observe context $x_t$

With probability $\varepsilon_t$, pull $a_t \sim \mathcal{U}([1, K])$, or pull $a_t = \arg\max(\theta_a, x_t)$

Observe reward $r_t$ and update parameters $\tilde{\theta}_a(t)$ and $V_a^{-1}(t)$

$$V_a(t + 1) = V_a(t) + x_t x_t^T, \quad b_a(t + 1) = b_a(t) + r_t x_t, \quad \theta_a(t + 1) = V_a^{-1}(t + 1)b_a(t + 1)$$

end for

Algorithm 6 EXP4

**Input:** number of arms $K$, experts: $(E_m)_{m \in [1,N]}$, parameter $\eta$

Set $Q_1 = (1/N)_{j \in [1,N]}$

for $t = 1, \ldots, T$

Observe context $x_t$ and probability recommendation $(E_m^{(t)})_{m \in [1,N]}$

Pull arm $a_t \sim P_t$ where $P_{t,j} = \sum_{k=1}^{N} Q_{t,k} E_{j,k}^{(t)}$

Observe reward $r_t$ and define for all arms $i \neq a_t = 1 - 1_{\{a_t = i\}}(1 - r_t)/P_{t,i}$

Define $\bar{X}_{t,k} = \sum_a E_{k,a}^{(t)}$ and $\hat{r}_{t,a}$

Update $Q_{t+1,j} = \exp(\eta \hat{r}_{t,j})/\sum_{j=1}^{N} \exp(\eta \hat{r}_{t,j})$ for all experts $i$

end for

with $\beta_{\text{max}} = \max_{t \in [0]} \beta_a(t)$ and $M$ the number of mini-batch updates.

**Proposition 4.** After the first attack, with probability $1 - \delta$, LinUCB always pulls arm $a^\dagger$.

**Proof.** After having poisoned the first mini-batch $B$, the latter can be partitioned into two subsets, $B_c$ (with non-perturbed rows) and $B_{nc}$ (with the poisoned rows). The design matrix of arm $a \neq a^\dagger$ for every time $t$ after the poisoning is:

$$V_{i,a} = \lambda I_d + \sum_{t=1,a_{t,a}}^{t} x_t x_t^T + \delta_a^2 \sum_{i=1}^{d} e_i e_i^T$$

(12)

For every time $t$, non diagonal elements of $V_{i,a} = (v_{i,j})_{i,j}$ are bounded by:

$$\forall i, r_i := \sum_{j \neq i} v_{i,j} \leq \sum_{j \neq i} \sum_{t=1,a_{t,a}}^{t} \|x_t x_t^T\|_\infty \leq dN_a(kB)$$

(13)

Whereas for all diagonal elements, $v_{i,i} \geq \delta_a^2 > r_i$. Thus $V_{i,a}$ is strictly diagonal dominant and by the AhlbergNilsonVarah bound (Varah, 1975):

$$\|V_{i,a}^{-1}\|_\infty \leq \frac{1}{\min_i (\|v_{i,i}\| - r_i)} \leq \frac{1}{\delta_a^2 - dMB}$$

(14)

Then, for every arm $a \neq a^\dagger$ and any context $x \in \mathcal{D}$ and any time $t$ after the attack:

$$\langle \tilde{\theta}_a(t, x) + \beta_a(t) \|x\|_{V_a^{-1}} \rangle$$

$$\leq \sum_{t=1,a_{t,a}}^{t} r_i (V_{i,a}^{-1} x_t)^T x + \beta_a(t) \|x\|_1\sqrt{\|V_{i,a}^{-1}\|_\infty}$$

$$\leq \|V_{i,a}^{-1}\|_\infty dN_a(\alpha) \sup_{y \in \mathcal{D}} \|y\|_2 + \beta_{\text{max}} \sqrt{d} \sup_{y \in \mathcal{D}} \|y\|_2 \sqrt{\|V_{i,a}^{-1}\|_\infty} < \nu$$
**Algorithm 7** Semi-Online Attack Setting.

**Input:** Bandit alg. $\mathcal{A}$, size of a mini-batch: $B$

Set $t = 0$

**while** True

$\mathcal{A}$ observe context $x_t$

$\mathcal{A}$ pulls arm $a_t$ and observes reward $r_t$

Interaction $(x_t, a_t, r_t)$ is saved in mini-batch $\mathcal{B}$

**if** $|\mathcal{B}| = B$ **then**

Attacker modifies mini-batch $\mathcal{B}$ into $\tilde{\mathcal{B}}$

Update alg. $\mathcal{A}$ with poisoned mini-batch $\tilde{\mathcal{B}}$

**end if**

**end while**

We have shown that for any arm $a \neq a^\dagger$ and any time step $t$ after the attack, the upper confidence bound computed by LinUCB is upper-bounded by $\nu$ the arm $a^\dagger$. Then, with probability $1 - \delta$, the confidence set for arm $a^\dagger$ holds and, for all $x \in \mathcal{D}$, arm $a^\dagger$ is chosen by LinUCB. The total cost of this attack is $d \sum_{a \neq a^\dagger} \delta_a L = O(\sqrt{MB})$