Rotation set for maps of degree 1 on sun graphs

Sylvie Ruette

January 6, 2019

Abstract

For a continuous map on a topological graph containing a unique loop $S$, it is possible to define the degree and, for a map of degree 1, rotation numbers. It is known that the set of rotation numbers of points in $S$ is a compact interval and for every rational $r$ in this interval there exists a periodic point of rotation number $r$. The whole rotation set (i.e. the set of all rotation numbers) may not be connected and it is not known in general whether it is closed.

A sun graph is the space consisting in finitely many segments attached by one of their endpoints to a circle. We show that, for a map of degree 1 on a sun graph, the rotation set is closed and has finitely many connected components. Moreover, for all but finitely many rational numbers $r$ in the rotation set, there exists a periodic point of rotation number $r$.

1 Introduction

In [2], a rotation theory is developed for continuous self maps of degree 1 of topological graphs having a unique loop. A rotation theory is usually developed in the universal covering space by using the liftings of the maps under consideration. The universal covering of a graph containing a unique loop is an “infinite tree modulo 1” (see Figure 1). It turns out that the rotation theory on the universal covering of a graph with a unique loop can be easily extended to the setting of infinite graphs that look like the space $\hat{G}$ on Figure 2. These spaces are defined in detail in Section 2.1 and called lifted graphs. Each lifted graph $\hat{T}$ has a subset $\hat{T}$ homeomorphic to the real line $\mathbb{R}$ that corresponds to an “unwinding” of a distinguished loop of the original graph. In the sequel, we identify $\hat{T}$ with $\mathbb{R}$.

![Figure 1: A sun graph on the left, and its universal covering on the right.](image)

Given a lifted graph $T$ and a map $F$ from $T$ to itself of degree one, there is no difficulty to extend the definition of rotation number to this setting in such a way that every periodic point has a rational rotation number as in the circle case. However, the obtained rotation set $\text{Rot}(F)$ may not be connected (see [2, Example 1.12]). Despite of this fact, it is proved in [2] that the set $\text{Rot}_\mathbb{R}(F)$ corresponding to the rotation numbers of all points belonging to $\mathbb{R}$, has properties which are similar to (although weaker than) those of the rotation interval for a circle map of degree one. Indeed, this set is a compact non empty interval, if $p/q \in \text{Rot}_\mathbb{R}(F)$ then there exists
a periodic point of rotation number \( p/q \), and if \( p/q \in \text{Int} (\text{Rot}_R(F)) \) then, for all large enough positive integers \( n \), there exists a periodic point of period \( nq \) of rotation number \( p/q \).

We conjecture that the whole rotation set \( \text{Rot}(F) \) is closed. In this paper, we prove that, when the space \( T \) is the universal covering of a sun graph (consisting in finitely many disjoint segments attached by one of their endpoints to a circle, see Figure 1), then the rotation set is the union of finitely many compact intervals. Moreover, all but finitely many rational points \( r \) in \( \text{Rot}(F) \) are rotation numbers of periodic (mod 1) points. It turns out that the proofs extend to a class of maps on graphs that we call sun-like maps, which are defined in Section 2.3.

This paper is the sequel of [5], which deals with the graph \( \sigma \), i.e., a sun graph with a unique branch. The results obtained for \( \sigma \) in [5] are stronger but the methods cannot be generalised to sun graphs. Here the main tool is the construction of a countable oriented graph, and the symbolic dynamics on this graph reflects enough of the dynamics of the original map to compute rotation numbers and find periodic points. The idea is inspired by the Markov diagram introduced by Hofbauer [4] to study piecewise monotone interval maps, although the goals are very different (the Markov diagram was used to study measures of maximal entropy).

The paper is organised as follows. In Section 2, we give the definitions of the objects we deal with: lifted graphs, maps of degree 1, sun graphs and sun-like maps, rotation numbers and rotation sets; we also recall the main results on the rotation set of \( \mathbb{R} \subset T \) when \( T \) is a lifted graph. In Section 3, we recall the notion of positive covering, which is a key tool to find periodic points. In Section 4, we define a partition \( \mathcal{P} \) of the branches of \( T \) (where \( T \) is the universal covering of a sun graph) according to some dynamical properties and we state that the rotation number of a point can be computed using its itinerary according to the partition \( \mathcal{P} \). Then, in Section 5, we define the covering graph \( \mathcal{G} \) associated to this partition \( \mathcal{P} \) (\( \mathcal{G} \) is a countable oriented graph), which gives a relation between itineraries of points and infinite paths in this graph, and we study the structure of the graph. Finally, in Section 6, we study the rotation set of the covering graph and, in Section 7, we pull back these results on the space \( T \) and we prove the main result about the rotation set of sun-like maps.

2 Definitions and first properties

2.1 Lifted graphs

A topological finite graph is a compact connected set \( G \) containing a finite subset \( V \) such that each connected component of \( G \setminus V \) is homeomorphic to an open interval. The aim of this section is to define in detail the class of lifted graphs where we develop the rotation theory. They are obtained from a topological finite graph by unwinding one of its loops. This gives a new space that contains a subset homeomorphic to the real line and that is “invariant by a
translation” (see Figures 1 and 2). In [2], a larger class of spaces called lifted spaces is defined.

**Definition 2.1** Let \( T \) be a connected topological space. We say that \( T \) is a lifted graph if there exist a homeomorphism \( h: \mathbb{R} \to h(\mathbb{R}) \subset T \), and a homeomorphism \( \tau: T \to T \) such that

i) \( \tau(h(x)) = h(x+1) \) for all \( x \in \mathbb{R} \),

ii) the closure of each connected component of \( T \setminus h(\mathbb{R}) \) is a topological finite graph that intersects \( h(\mathbb{R}) \) at a single point,

iii) the number of connected components \( C \) of \( T \setminus h(\mathbb{R}) \) such that \( C \cap h([0,1]) \neq \emptyset \) is finite.

The class of all lifted graphs will be denoted by \( T^0 \).

To simplify the notation, in the rest of the paper we identify \( h(\mathbb{R}) \) with \( \mathbb{R} \) itself. In this setting, the map \( \tau \) can be interpreted as a translation by 1. So, for all \( x \in T \), we write \( x+1 \) to denote \( \tau(x) \). Since \( \tau \) is a homeomorphism, this notation can be extended by denoting \( \tau^m(x) \) by \( x+m \) for all \( m \in \mathbb{Z} \).

We endow a lifted graph \( T \) with a distance \( d \) invariant by the translation \( \tau \), i.e., \( \forall x, y \in T \), \( d(x+1, y+1) = d(x, y) \).

A loop is a subset homeomorphic to a circle. If \( G \) is a topological finite graph with a unique loop, then its universal covering is an infinite tree (i.e., it has no loop) and belongs to \( T^0 \). Figure 1 illustrates this situation. Because of (ii) in the previous definition, if the topological finite graph \( G \) has several loops, the infinite graph obtained by unwinding a distinguished loop may or may not be a lifted graph. The essential property of the class \( T^0 \) is the existence of a natural retraction from \( T \) to \( \mathbb{R} \).

**Definition 2.2** Let \( T \in T^0 \). The retraction \( r_\mathbb{R}: T \to \mathbb{R} \) is the continuous map defined as follows. When \( x \in \mathbb{R} \), then \( r_\mathbb{R}(x) := x \). When \( x \notin \mathbb{R} \), there exists a connected component \( C \) of \( T \setminus \mathbb{R} \) such that \( x \in C \) and \( C \) intersects \( \mathbb{R} \) at a single point \( z \), and we let \( r_\mathbb{R}(x) := z \).

### 2.2 Maps of degree 1 and rotation numbers

A standard approach to study the periodic points and orbits of a graph map is to work at lifting level with the periodic (mod 1) points. The results on the lifted graph can obviously been pulled back to the original graph (see [2]). Moreover, the rotation numbers have a signification only for maps of degree 1, as in the case of circle maps (see, e.g., [1] for the rotation theory for circle maps). In this paper, we deal only with maps of degree 1 on lifted graphs.

**Definition 2.3** Let \( T \in T^0 \), \( F: T \to T \) a continuous map of degree 1 and \( x \in T \). When the limit exists, the rotation number of \( x \) is

\[
\rho_F(x) := \lim_{n \to \infty} \frac{r_\mathbb{R} \circ F^n(x) - r_\mathbb{R}(x)}{n}.
\]

The next, easy lemma states that two points in the same orbit have the same rotation number, as well as two point equal (mod 1).
Lemma 2.5 Let $T \in \mathbf{T}^0$, $F : T \to T$ a continuous map of degree 1, and $x \in T$ such that $\rho_F(x)$ exists.

i) \( \forall k \in \mathbb{Z}, \rho_F(x + k) = \rho_F(x) \).

ii) \( \forall n \geq 1, \rho_F(F^n(x)) = \rho_F(x) \).

Remark 2.6 If \( F^q(x) = x + p \) with \( q \in \mathbb{N} \) and \( p \in \mathbb{Z} \), then \( \rho_F(x) = p/q \). Therefore all periodic (mod 1) points have rational rotation numbers.

An important object that synthesises the information about rotation numbers is the rotation set.

Definition 2.7 Let $T \in \mathbf{T}^0$ and $F : T \to T$ a continuous map of degree 1. For $E \subset T$, the rotation set of $E$ is:

\[ \text{Rot}_E(F) := \{ \rho_F(x) \mid x \in E \text{ and } \rho_F(x) \text{ exists} \} \]

When $E = T$, we omit the subscript and we write $\text{Rot}(F)$ instead of $\text{Rot}_T(F)$.

We define

\[ T_\mathbb{R} := \bigcup_{n \geq 0} F^n(\mathbb{R}). \]

The next theorem summarises the properties of $T_\mathbb{R}$ and $\text{Rot}_\mathbb{R}(F)$ (see Lemma 5.2 and Theorems 3.1, 5.7, 5.18 in [2]).

Theorem 2.8 Let $T \in \mathbf{T}^0$ and $F : T \to T$ a continuous map of degree 1. Then $T_\mathbb{R} \in \mathbf{T}^0$, $\text{Rot}_{T_\mathbb{R}}(F) = \text{Rot}_\mathbb{R}(F)$ and the set $\text{Rot}_\mathbb{R}(F)$ is a nonempty compact interval. Moreover, if \( r \in \text{Rot}_\mathbb{R}(F) \cap \mathbb{Q} \), then there exists a periodic (mod 1) point $x \in T_\mathbb{R}$ such that $\rho(x) = r$.

2.3 Sun graphs and sun-like maps

A sun graph is a topological finite graph that looks like the graph $S$ on Figure 1. It is composed of a circle and finitely many disjoint compact intervals, each interval being attached by one of its endpoints to the circle.

Let $T \in \mathbf{T}^0$ and $F : T \to T$ a continuous map of degree 1. We define

\[ X := T \setminus T_\mathbb{R} \cap r_\mathbb{R}^{-1}((0,1)). \]

Then $X$ is composed of finitely many finite graphs and $T = T_\mathbb{R} \cup (X + \mathbb{Z})$. Note that $T_\mathbb{R}$ and $X$ implicitly depend on $F$.

If $T$ is the lifting of a sun graph, then $X$ is either empty, or composed of finitely many disjoint intervals, each intersecting $T_\mathbb{R}$ at one of its endpoint. Maps with the same properties will be called sun-like maps.

Definition 2.9 Let $T \in \mathbf{T}^0$ and $F : T \to T$ a continuous map of degree 1. If $T \setminus T_\mathbb{R} \cap r_\mathbb{R}^{-1}((0,1))$ is composed of finitely many intervals whose closures are disjoint, we say that $F$ is a sun-like map. The intervals

\[ \{ \overline{C} \mid C \text{ is a connected component of } T \setminus T_\mathbb{R} \cap r_\mathbb{R}^{-1}((0,1)) \} \]

are called the branches of $F$ and denoted by $(X^i)_{i \in \Lambda}$, where $\Lambda$ is some finite set of indices. The set of all sun-like maps of degree 1 on $T$ is denoted by $\mathcal{S}_1(T)$. 

Remark 2.10 In a sun graph, two different segments do not meet the circle at the same point because the segments are compact and disjoint. Similarly, two branches of a sun-like map are not allowed to have a common endpoint in $T_R$. This property prevents $F$ from oscillating infinitely many times between two branches. On the contrary, $F$ may oscillate between a branch and $T_R$.

Definition 2.11 Let $F$ be a sun-like map and $(X^i)_{i \in \Lambda}$ its branches. Each branch $X^i$ may be endowed with two opposite orders. We choose the one such that $\min X^i$ is the one-point intersection $X^i \cap T_R$.

Consider a sun-like map $F$. Because of the definition of sun-like maps, all the paths starting in $T_R$ and ending in some branch $X^i$ must pass through the one-point intersection $T_R \cap X^i$. Thus, if $E$ is a connected set in $T$ containing one point of $T_R$ and one point of $X^i$, then $E$ contains $T_R \cap X^i = \{\min X^i\}$. We shall use this property several times.

3 Positive covering

Definition 3.1 Let $F$ be a sun-like map and $(X^i)_{i \in \Lambda}$ its branches. For every $i \in \Lambda$, the retraction map $r_i: T \to X^i$ can be defined in a natural way by $r_i(x) := x$ if $x \in X^i$ and $r_i(x) := \min X^i$ otherwise.

The notion of positive covering for subintervals of $\mathbb{R}$ has been introduced in $[2]$. It can be extended for subintervals of any subset of $T$ on which a retraction can be defined, as in $[5]$. In this paper, we shall use positive covering in the branches of $F$. All properties of positive covering remain valid in this context. In particular, if a compact interval $I$ positively $F$-covers itself, then $F$ has a fixed point in $I$ (Proposition 3.5).

Definition 3.2 Let $T \in \mathcal{T}^0$, $F \in \mathcal{S}_1(T)$ and $I, J$ two non empty compact subintervals with $I \subset X^i$ and $J \subset X^j$, where $X^i, X^j$ are two branches of $F$ ($X^i$ and $X^j$ may be equal). Let $n$ be a positive integer and $p \in \mathbb{Z}$. We say that $I$ positively $F^n$-covers $J+p$ and we write $I \xrightarrow{F^n} J+p$ if there exist $x, y \in I$ with $x \leq y$ (with respect to the order in $X^i$) such that $r_j(F^n(x)-p) \leq \min J$ and $\max J \leq r_j(F^n(y)-p)$ (with respect to the order in $X^j$). In this situation, we also say that $I + q$ positively $F^n$-covers $J + p + q$ for all $q \in \mathbb{Z}$.

Remark 3.3 If $F^n(x) \in T_R$ and $J \subset X^j$ for some $j$, then the inequality $r_j(F^n(x)-p) \leq \min J$ is automatically satisfied. We shall often use this remark to prove that an interval positively covers another.

The next lemma is $[2]$ Lemma 2.2(c)]. It states that positive coverings can be concatenated.

Lemma 3.4 Let $I, J, K$ be non empty compact intervals, each one included in some branch of $F$. Let $n, m$ be positive integers and $p, q \in \mathbb{Z}$. If $I \xrightarrow{F^n} J+p$ and $J \xrightarrow{F^m} K+q$, then $I \xrightarrow{F^{n+m}} K+p+q$.

The next proposition is $[2]$ Proposition 2.3, rewritten in some less general form.

Proposition 3.5 Let $T \in \mathcal{T}^0$ and $F \in \mathcal{S}_1(T)$. Let $I_0, \ldots, I_{k-1}$ be non empty compact intervals, each one included in some branch of $F$, and $p_1, \ldots, p_k \in \mathbb{Z}$. Suppose that we have a chain of positive coverings:

$$ I_0 \xrightarrow{F} I_1 + p_1 \xrightarrow{F} I_2 + p_2 \xrightarrow{F} \cdots \xrightarrow{F} I_{k-1} + p_{k-1} \xrightarrow{F} I_0 + p_k. $$

Then there exists a point $x_0 \in I_0$ such that $F^k(x_0) = x_0 + p_k$ and $F^i(x_0) \in I_i + p_i$ for all $i \in [1, n-1]$.  

5
4 Partition of the branches and itineraries

Since $T_R$ is $F$-invariant, if $F^{n_0}(x) \in T_R$ for some $n_0 \geq 0$, then $F^n(x) \in T_R$ for all $n \geq n_0$ and $\rho_F(x) \in \text{Rot}_{T_R}(F)$. The properties of the rotation set $\text{Rot}_{T_R}(F)$ has been recalled in Theorem 2.8. Consequently, it remains to consider the points whose orbits do not fall in $T_R$. Since $T = T_R \cup (X + Z)$ and $\rho_F(x + 1) = \rho_F(x)$ (Lemma 2.5(i)), we have

$$\text{Rot}(F) = \text{Rot}_{T_R}(F) \cup \text{Rot}_{X}(F),$$

where $X^\infty := \{ x \in X \mid \forall n \geq 0, F^n(x) \in X + Z \}$.  \hspace{1cm} (1)

Our first step consists in dividing each branch of $X$ according to the location of the images in one of the sets $X^i + p, i \in \Lambda, p \in Z$.

**Lemma 4.1** Let $T \in T^2$, $F \in S_1(T)$ and $(X^i)_{i \in \Lambda}$ the branches of $F$. For every branch $X^i$, there exist an integer $N_i \geq 0$, disjoint non empty compact intervals $X^i_1, \ldots, X^i_{N_i} \subset X^i$ and, for every $j \in [1, N_i]$, there exist $\ell(X^i_j) \in \Lambda$ and $p(X^i_j) \in Z$ such that

i) $X^i_1 < X^i_2 < \cdots < X^i_{N_i}$ (with respect to the order in $X^i$),

ii) $F(X^i_j) \subset \left( X^{\ell(X^i_j)} + p(X^i_j) \right) \cup \text{Int}(T_R)$,

iii) $F(\min X^i_j) = \min X^{\ell(X^i_j)} + p(X^i_j) \in T_R$,

iv) $F \left( X \setminus \bigcup_{i \in \Lambda, j \in [1, N_i]} X^i_j \right) \cap (X + Z) = \emptyset$.

**Proof.** We fix $i \in \Lambda$. If $F(X^i) \cap (X + Z) = \emptyset$, we take $N_i = 0$ and there is nothing to do. Otherwise, we can define $a_1 := \min \{ x \in X^i \mid F(x) \in X + Z \}$ and $p_1 \in Z$ such that $F(a_1) \in X + p_1$. Since $F$ is a sum-like map, there is a unique $\ell_1 \in \Lambda$ such that $a_1 \in X^{\ell_1} + p_1$. We define

$$b_1 := \max \{ x \in [a_1, \max X^i] \mid F(x) \in X^{\ell_1} + p_1 \text{ and } F([a_1, x]) \subset (X^{\ell_1} + p_1) \cup \text{Int}(T_R) \},$$

and $X^i_1 := [a_1, b_1]$. Then $X^i_1$ satisfies (ii) with $p(X^i_1) := p_1$ and $\ell(X^i_1) := \ell_1$. Moreover, $F(\min X^i_1) \in T_R$ because $\min X^i \in T_R$, which implies that $F([\min X^i, a_1])$ contains $\min X^{\ell_1} + p_1$ by connectedness. Thus $F(a_1) = \min X^{\ell_1} + p_1$ by minimality of $a_1$, which is (iii) for $X^i_1$.

We define $X^i_2, \ldots, X^i_{N_i}$ inductively. Suppose that $X^i_j = [a_j, b_j], p_j = p(X^i_j)$ and $\ell_j = \ell(X^i_j)$ are already defined and that $b_j$ satisfies:

$$b_j = \max \{ x \in [a_j, \max X^i] \mid F(x) \in X^{\ell_j} + p_j \text{ and } F([a_j, x]) \subset (X^{\ell_j} + p_j) \cup \text{Int}(T_R) \}.$$

If $F([b_j, \max X^i]) \subset \text{Int}(T_R)$, we take $N_i := j$ and the construction is over. Otherwise, we define

$$a_{j+1} := \inf \{ x \in (b_j, \max X^i] \mid F(x) \in X + Z \}. \hspace{1cm} (2)$$

We first show that $a_{j+1} > b_j$. By definition, there exists a sequence of points $(x_n)_{n \geq 0}$ in $(b_j, \max X^i)$ such that $\lim_{n \to +\infty} x_n = a_{j+1}$ and $F(x_n) \in X + Z$ for all $n \geq 0$. Since the number of branches is finite, we may assume (by taking a subsequence if necessary) that there exists $\ell_{j+1} \in \Lambda$ such that $F(x_n) \in X^{\ell_{j+1}} + Z$ for all $n \geq 0$. Let $m_n \in Z$ be such that $F(x_n) \in X^{\ell_{j+1}} + m_n$. By continuity, $\lim_{n \to +\infty} r_R \circ F(x_n) = r_R \circ F(a_{j+1})$. Since $r_R \circ F(x_n) = r_R \circ (\min X^{\ell_{j+1}} + m_n)$, this implies that the sequence of integers $(m_n)_{n \geq 0}$ is ultimately constant, and equal to some integer $p_{j+1}$. Then $F(a_{j+1}) = \lim_{n \to +\infty} F(x_n) \in X^{\ell_{j+1}} + p_{j+1}$. By continuity, $F([a_{j+1}, x_n]) \subset$
(X^{\ell_i+1} + p_{j+1}) \cup \text{Int}(T_R) \) for all large enough \( n \). Moreover, \( F((b_j, a_{j+1})) \subset \text{Int}(T_R) \) by definition of \( a_{j+1} \). If \( p_{j+1} = p_j \) and \( \ell_{j+1} = \ell_j \) then, for all large enough \( n \), we would have
\[
F(x_n) \in X^{\ell_j} + p_j \text{ and } F([b_j, x_n]) \subset (X^{\ell_j} + p_j) \cup \text{Int}(T_R),
\]
which would contradict the definition of \( b_j \) because \( x_n > b_j \). Hence \( (p_{j+1}, \ell_{j+1}) \neq (p_j, \ell_j) \). This implies that \( a_{j+1} > b_j \) and, consequently, \( a_{j+1} \) is actually a minimum in \([2]\). Since \( F((b_j, a_{j+1})) \) is non empty and included in \( \text{Int}(T_R) \), necessarily \( F(a_{j+1}) \) is equal to \( \min X^{\ell_{j+1}} + p_{j+1} \) by minimality of \( a_{j+1} \).

Finally, we define
\[
b_{j+1} := \\
\max\{x \in [a_{j+1}, \max X^i] \mid F(x) \in X^{\ell_{j+1}} + p_{j+1} \text{ and } F([a_{j+1}, x]) \subset (X^{\ell_{j+1}} + p_{j+1}) \cup \text{Int}(T_R)\},
\]
and \( X_{j+1}^i := [a_{j+1}, b_{j+1}] \). Then \( X_{j+1}^i > X_j^i \), and (ii) and (iii) are satisfied with \( p(X_{j+1}^i) := p_{j+1} \) and \( \ell(X_{j+1}^i) := \ell_{j+1} \).

Let \( \delta \) be the infimum of \( d(x, y) \) where \( x, y \) belong to two different sets of the form \( X^{\ell} + p, \ell \in \Lambda, p \in \mathbb{Z} \). This is actually a minimum because the sets \( X^{\ell} \) are compact, \( \Lambda \) is finite and the distance \( d \) is invariant by translation by 1. Moreover, \( \delta > 0 \) because the branches are pairwise disjoint.

By uniform continuity of \( F \) on the compact set \( X \), there exists \( \eta > 0 \) such that, if \( x, y \) belong to \( X \) with \( d(x, y) < \eta \), then \( |F(x) - F(y)| < \delta \). This implies that \( |a_{j+1} - b_j| \geq \eta \), otherwise \( F(a_{j+1}) \) and \( F(b_j) \) would be in the same set \( X^{\ell} + p \). This ensures that for a given \( i \in \Lambda \), the number of intervals \( X_j^i \) is finite, and the construction ultimately ends. By construction, (iv) is satisfied. \( \blacksquare \)

**Remark 4.2** The fact that the sets \( X_j^i \) are intervals is very important because it will allow us to use positive coverings. In an ideal situation, we would like to define the sets \( X \)

We call \( \mathcal{P} := \{X_j^i \mid i \in \Lambda, j \in [1, N_i]\} \) the *basic partition* of \( X \) (although the true partition of \( X \) is \( \mathcal{P} \cup \{X \setminus \bigcup X_j^i\} \)). The set \( X \setminus \bigcup X_j^i \), as well as \( T_R \), plays the role of “dustbin”: we do not need to care about points whose orbit falls in these sets because their rotation numbers belong to \( \text{Rot}_R(F) \).

According to Lemma \ref{lem:4.1}(iv), every point \( x \in X \) such that \( F(x) \in X + \mathbb{Z} \) belongs to some \( A \in \mathcal{P} \), and this \( A \) is unique because the elements of \( \mathcal{P} \) are pairwise disjoint. This allows us to code the orbits of the points of \( X^\infty \) with respect to the partition \( \mathcal{P} \).

**Definition 4.3** Let \( X^\infty := \{x \in X \mid \forall n \geq 0, F^n(x) \in X + \mathbb{Z}\} \). If \( x \in X^\infty \) then, for every \( n \geq 0 \), there is a unique \( A_n \in \mathcal{P} \) such that \( F^n(x) \in A_n + \mathbb{Z} \). The sequence \( (A_n)_{n \geq 0} \) is called the *itinerary* of \( x \). Let \( \Sigma \subset \mathcal{P}^{\mathbb{Z}^+} \) be the set of all itineraries of points \( x \in X^\infty \).

The next lemma is straightforward.

**Lemma 4.4** If \( (A_n)_{n \geq 0} \) is the itinerary of \( x \in X^\infty \), then
\[
\forall n \geq 0, \quad F^n(x) \in A_n + p(A_0) + p(A_1) + \cdots + p(A_{n-1}),
\]
and, if \( \rho_F(x) \) exists,
\[
\rho_F(x) = \lim_{n \to +\infty} \frac{p(A_0) + \cdots + p(A_{n-1})}{n}.
\]
5 The covering graph associated to $\mathcal{P}$

Knowing the itinerary of a point $x \in X^\infty$ is enough to compute the rotation number of $x$. Therefore we can focus on the set $\Sigma$ of all itineraries. If 

$$\forall A, B \in \mathcal{P}, \ F(A) \cap (B + p(A)) \implies F(A) \supset B + p(A),$$

then it can be shown that $\Sigma$ is a Markov shift on the finite alphabet $\mathcal{P}$. In this case, the rotation set of $X^\infty$ can be easily computed by the use of the Markov graph of $\Sigma$. In [7] this is done for transitive subshifts of finite type, and it is shown that in this case the rotation set is a compact interval. When the Markov shift is not transitive, or equivalently when its Markov graph is not strongly connected (see Definition 5.17 below), one has to look at the different connected components of the graph, each of which giving an interval.

In general, $\Sigma$ may not be a Markov shift. We are going to build a countable oriented graph, called the covering graph associated to $\mathcal{P}$, that plays the role of the Markov graph: the symbolic itineraries can be read in the graph and the structure of the graph (in particular its connected components) will give the structure of the rotation set.

5.1 Definitions and first properties

The construction of the covering graph is inspired by the Markov diagram of a (non Markov) interval map, first introduced by Hofbauer for piecewise monotone maps [4]. Our definition is closer to the Buzzi’s version of the Markov diagram [3], although the basis of our covering graph is always finite, as in Hofbauer’s graph. In Hofbauer’s and Buzzi’s constructions, the basis consists in monotone intervals, whereas our basis will be the basic partition $\mathcal{P}$ (no monotonicity is involved here).

Definition 5.1 If $A_i \in \mathcal{P}$ for all $i \in [0,n]$, we define 

$$\langle A_0A_1\ldots A_n \rangle := F^n \left( \{ x \in T \mid \forall i \in [0,n], F^i(x) \in A_i + Z \} \right) \cap X.$$

Remark 5.2 If the itinerary of $x$ begins with $A_0A_1\ldots A_n$, then $F^n(x) \in \langle A_0A_1\ldots A_n \rangle + Z$. In this sense, $\langle A_0\ldots A_n \rangle$ is the set of points whose “past itinerary” is $A_0\ldots A_n$.

The next lemma gives an alternative definition of $\langle A_0\ldots A_n \rangle$ and states that this set is actually an interval.

Lemma 5.3 If $A_i \in \mathcal{P}$ for all $i \in [0,n]$, then

i) $\langle A_0\ldots A_n \rangle = F^n(A_0 + Z) \cap F^{n-1}(A_1 + Z) \cap \cdots \cap F(A_{n-1} + Z) \cap A_n$

ii) $\langle A_0A_1\ldots A_n \rangle$ is either empty, or a closed subinterval of $A_n$ containing $\min A_n$.

Proof. By definition,

$$\langle A_0\ldots A_n \rangle = F^n ((A_0 + Z) \cap F^{-1}(A_1 + Z) \cap \cdots \cap F^{-n}(A_n + Z)) \cap X.$$

Thus

$$\langle A_0\ldots A_n \rangle = F^n (A_0 + Z) \cap F^{n-1}(A_1 + Z) \cap \cdots \cap F(A_{n-1} + Z) \cap (A_n + Z) \cap X.$$

Since $(A_n + Z) \cap X = A_n$, this gives the first equality of (i). If we write this equality for $\langle A_0\ldots A_{n-1} \rangle$, we see that $\langle A_0\ldots A_n \rangle = F(\langle A_0\ldots A_{n-1} \rangle + Z) \cap A_n$. Since $\langle A_0\ldots A_{n-1} \rangle \subset A_{n-1}$,
Lemma 5.8 implies that $F(\langle A_0\ldots A_{n-1} \rangle) \subset X + p(A_{n-1})$, and hence $F(\langle A_0\ldots A_{n-1} \rangle + \mathbb{Z}) \cap A_n = F(\langle A_0\ldots A_{n-1} \rangle - p(A_{n-1})) \cap A_n$. This is the second equality of (i).

We show (ii) by induction on $n$. If $n = 0$, then $\langle A_0 \rangle = A_n$ and there is nothing to prove.

Suppose that $\langle A_0 \ldots A_n \rangle \neq \emptyset$ and that $\langle A_0 \ldots A_{n-1} \rangle$ is a closed subinterval of $A_{n-1}$ containing $\min A_{n-1}$ (note that $\langle A_0 \ldots A_{n-1} \rangle$ is not empty if $\langle A_0 \ldots A_n \rangle \neq \emptyset$). By (i),

$$\langle A_0 \ldots A_n \rangle = (F(\langle A_0 \ldots A_{n-1} \rangle) - p(A_{n-1})) \cap A_n.$$

By continuity, $F(\langle A_0 \ldots A_{n-1} \rangle)$ is compact and connected, and thus $(F(\langle A_0 \ldots A_{n-1} \rangle) - p(A_{n-1})) \cap A_n$ is a closed subinterval of $A_n$, which is non empty by assumption. Moreover, $\langle A_0 \ldots A_{n-1} \rangle$ contains $\min A_{n-1}$ by the induction hypothesis, and $F(\min A_{n-1}) \in T_{\mathbb{R}}$ by Lemma 4.1(iii). This implies that the interval $F(\langle A_0 \ldots A_{n-1} \rangle) - p(A_{n-1})$ contains a point of $T_{\mathbb{R}}$ and a point of $A_n$, and thus it contains $\min A_n$ by connectedness. Therefore (ii) holds for $\langle A_0 \ldots A_n \rangle$. This ends the induction.

We define an equivalence relation between the finite sequences of elements of $\mathcal{P}$.

**Definition 5.4** Let $A_0, \ldots, A_n, B_0, \ldots, B_m \in \mathcal{P}$. We set $A_0 \ldots A_n \sim B_0 \ldots B_m$ if there is $k \in [0, \min(n, m)]$ such that

$$\begin{cases} A_{n-i} = B_{m-i} & \text{for all } i \in [0, k] \\ \langle A_0 \ldots A_{n-k} \rangle = \langle B_0 \ldots B_{m-k} \rangle \end{cases}$$

(3)

**Remark 5.5** It follows from Lemma 5.3(i) that

if $A_0 \ldots A_n \sim B_0 \ldots B_m$, then $\langle A_0 \ldots A_n \rangle = \langle B_0 \ldots B_m \rangle$.

This means that, although the two sets come from different “past itineraries”, their futures are indistinguishable. If $\alpha = A_0 \ldots A_n / \sim$ is an equivalence class, then $\langle \alpha \rangle$ denotes $\langle A_0 \ldots A_n \rangle$, which is well defined according to what precedes.

The next result follows straightforwardly from Lemma 5.3 and the fact that the elements of $\mathcal{P}$ are disjoint.

**Lemma 5.6** If $\alpha = A_0 \ldots A_n / \sim$ and $\langle \alpha \rangle \neq \emptyset$, then $A_n$ is the unique element $A \in \mathcal{P}$ such that $\langle \alpha \rangle \subset A$.

Now we have all the notations to define the covering graph.

**Definition 5.7** We define the oriented graph $\mathcal{G}$ as follows:

- the set of vertices is the set of equivalence classes $\alpha = A_0 \ldots A_n / \sim$, where $n \geq 0$, $A_0, \ldots, A_n \in \mathcal{P}$ and $\langle A_0 \ldots A_n \rangle \neq \emptyset$,
- if $\alpha, \beta$ are two vertices, there is an arrow $\alpha \rightarrow \beta$ iff there exist $A_0, \ldots, A_n, A_{n+1} \in \mathcal{P}$ such that $\alpha = A_0 \ldots A_n / \sim$ and $\beta = A_0 \ldots A_n A_{n+1} / \sim$.

$\mathcal{G}$ is called the covering graph associated to $\mathcal{P}$.

The next lemma justifies the name “covering graph”.

**Lemma 5.8** If $\alpha \rightarrow \beta$ in $\mathcal{G}$ and if $A \in \mathcal{P}$ is such that $\langle \alpha \rangle \subset A$, then $\langle \alpha \rangle \xrightarrow{p} (\beta) + p(A)$.
Proof. Let $A_0, \ldots, A_n, A_{n+1} \in \mathcal{P}$ be such that $\alpha = A_0 \ldots A_n / \sim$ and $\beta = A_0 \ldots A_n A_{n+1} / \sim$. Necessarily, $A_n = A$ (Lemma 5.6). According to Lemma 5.3(ii), $\langle \alpha \rangle$ is a closed subinterval of $A$ and $\min(\alpha) = \min A$. Thus Lemma 4.1(iii) implies that

$$F(\min(\alpha)) \in T_{\mathbb{R}}. \quad (4)$$

Moreover, $\langle \beta \rangle = F(\langle \alpha \rangle - p(A)) \cap A_{n+1}$ by Lemma 5.3(i). Thus

$$\exists x_0 \in \langle \alpha \rangle \text{ such that } \max(\beta) = F(x_0) - p(A). \quad (5)$$

Since $\langle \beta \rangle$ is a closed subinterval of $A_{n+1}$, Equations (4) and (5) imply that $\langle \alpha \rangle \xrightarrow{F} \langle \beta \rangle + p(A)$. □

**Definition 5.9** The **significant part** of $A_0 \ldots A_n$ is $A_{i} \ldots A_{n}$, where $i \in [0, n]$ is the greatest integer such that $A_0 \ldots A_n \sim A_i \ldots A_n$. If $\alpha$ is the equivalence class of $A_0 \ldots A_n$, the significant part of $\alpha$ is defined as the significant part of $A_0 \ldots A_n$. This does not depend on the representative of $\alpha$.

If $A_0 \ldots A_n$ is the significant part of a vertex $\alpha$, the **height** of $\alpha$ is $H(\alpha) = n$. The **basis** of $\mathcal{G}$ is the set of vertices of height 0, that is, $\{A/ \sim | A \in \mathcal{P}\}$. We identify it with $\mathcal{P}$.

The next result, quite natural, will simplify the handling of arrows.

**Lemma 5.10** Let $\alpha = A_0 \ldots A_n / \sim$ be a vertex of $\mathcal{G}$ and let $\alpha \rightarrow \beta$ be an arrow in $\mathcal{G}$. Then there exists $A_{n+1} \in \mathcal{P}$ such that $\beta = A_0 \ldots A_n A_{n+1} / \sim$.

Proof. By definition, there exist $B_0, \ldots, B_m, B_{m+1} \in \mathcal{P}$ such that $\alpha = B_0 \ldots B_m / \sim$ and $\beta = B_0 \ldots B_m B_{m+1} / \sim$. Let $A_{n-k} \ldots A_0$ be the significant part of $\alpha$ (with $K = H(\alpha) \in [0, n]$). According to the definitions, we have $m \geq k$ and

$$A_{n-i} = B_{m-i} \text{ for all } i \in [0, k],$$

$$\langle B_0 \ldots B_{m-k} \rangle = B_{m-k} = A_{n-k} = \langle A_0 \ldots A_{n-k} \rangle.$$  

This implies that $B_0 \ldots B_m B_{m+1} \sim A_0 \ldots A_n B_{m+1}$. This proves the lemma with $A_{n+1} := B_{m+1}$. □

We shall need some notions about paths in oriented graphs.

**Definition 5.11** A **(finite) path** in $\mathcal{G}$ is a sequence of vertices $\alpha_0 \ldots \alpha_n$ such that $\alpha_i \rightarrow \alpha_{i+1}$ is an arrow in $\mathcal{G}$ for all $i \in [0, n-1]$. A **loop** is a path $\alpha_0 \ldots \alpha_n$ such that $\alpha_n = \alpha_0$. An **infinite path** in $\mathcal{G}$ is an infinite sequence of vertices $\bar{\alpha} = (\alpha_n)_{n \geq 0}$ such that $\alpha_i \rightarrow \alpha_{i+1}$ for all $i \geq 0$.

If $K$ is a subgraph of $\mathcal{G}$, let $\Gamma(K)$ be the set of all infinite paths in $K$.

In the following, the infinite paths in $\mathcal{G}$ will be denoted with a bar (e.g. $\bar{\alpha}$) to distinguish them from vertices (e.g. $\alpha_n, \beta$).

**Remark 5.12** Endowed with the shift map $\sigma: (\alpha_n)_{n \geq 0} \rightarrow (\alpha_n)_{n \geq 1}$, $\Gamma(\mathcal{G})$ is a topological Markov chain on a countable graph (see e.g. [3]). We shall not explicitly use this structure of dynamical system, although it will underlie the definition of rotation numbers of elements of $\Gamma(\mathcal{G})$ and the relation between $\Gamma(\mathcal{G})$ and $X^\infty$.  

10
5.2 Relation between itineraries of points of $X^\infty$ and infinite paths in $G$

The next result states that there is a correspondence between itineraries of points of $X^\infty$ and infinite paths in $G$. This is a key property of the covering graph: it will allow us to pull back on $X^\infty$ the results obtained for $G$.

**Proposition 5.13** If $(A_n)_{n \geq 0}$ is the itinerary of some point $x \in X^\infty$ and $\alpha_n = A_0 \ldots A_n/ \sim$, then $(\alpha_n)_{n \geq 0}$ is an infinite path in $G$. Reciprocally, if $(\alpha_n)_{n \geq 0}$ is an infinite path in $G$ with $H(\alpha_0) = k$, then there exists a point $x \in X^\infty$ of itinerary $(A_n)_{n \geq 0}$ such that, for all $n \geq 0$, $\alpha_n = A_0 \ldots A_n + k/ \sim$.

*Proof.* Suppose that $(A_n)_{n \geq 0}$ is the itinerary of $x \in X^\infty$. Then $(A_0 \ldots A_n)$ contains $F^n(x) - (p(A_0) + \cdots + p(A_{n-1}))$. Thus $(A_0 \ldots A_n) \neq \emptyset$ and $\alpha_n = A_0 \ldots A_n/ \sim$ is a vertex of $G$. It follows from the definition that $\alpha_n \to \alpha_{n+1}$ is an arrow in $G$, that is, $(\alpha_n)_{n \geq 0}$ is an infinite path in $G$.

Reciprocally, suppose that $(\alpha_n)_{n \geq 0}$ is an infinite path in $G$. Let $A_0 \ldots A_k$ be the significant part of $\alpha_0$, with $k = H(\alpha_0)$. According to Lemma 5.10 we can find inductively $A_{n+k} \in \mathcal{P}$ such that $\alpha_n = A_0 \ldots A_{n+k}/ \sim$ for all $n \geq 0$. For every $n \geq 0$, let

$$E_n := \{ x \in X \mid \forall i \in [0, n], F^i(x) \in A_i + \mathbb{Z} \} = \{ x \in X \mid \forall i \in [0, n], F^i(x) \in A_i + p(A_0) + \cdots + p(A_{i-1}) \}$$

Then $E_n$ is a compact set and $E_{n+1} \subset E_n$. Moreover, $(F^n(E_n + \mathbb{Z})) \cap X = (A_0 \ldots A_n) \neq \emptyset$ (Lemma 5.3(i)). Hence $E_n \neq \emptyset$. Therefore, the set $\bigcap_{n \geq 0} E_n$ is non empty and every point $x$ in this set satisfies: $x \in X$ and $\forall n \geq 0, F^n(x) \in A_i + \mathbb{Z}$, that is, $x \in X^\infty$ and its itinerary is $(A_n)_{n \geq 0}$.

5.3 Structure of the covering graph

The oriented graph $G$ is usually infinite. However its infinite part is “small” and we shall exploit the particular structure of the covering graph. Proposition 5.13 gives the main properties of the structure of $G$. It implies that “most” infinite paths come back infinitely many times to the basis, which is rigorously stated in Proposition 5.19.

**Lemma 5.14** Let $\alpha, \beta$ be vertices of $G$ such that there is an arrow $\alpha \to \beta$. Then there exist $A_0, \ldots, A_n, A_{n+1} \in \mathcal{P}$ and $k \in [1, N(\alpha)]$ such that $\alpha = A_0 \ldots A_n/ \sim$, $\beta = A_0 \ldots A_n A_{n+1}/ \sim$ and $A_{n+1} = X_k^\ell(\alpha)$. Moreover, for all $j \in [1, k-1]$, $(\alpha) \xrightarrow{F} X_j^\ell(\alpha) + p(A_n)$ and $\alpha \to X_j^\ell(\alpha)$ is an arrow in $G$.

*Proof.* We write $\alpha = A_0 \ldots A_n/ \sim$, $p := p(A_n)$ and $\ell := \ell(A_n)$. Lemma 5.10 states that there is $A_{n+1} \in \mathcal{P}$ such that $\beta = A_0 \ldots A_n A_{n+1}/ \sim$. The set $(A_0 \ldots A_n A_{n+1})$ is non empty and satisfies

$$(\alpha) \xrightarrow{F} (A_0 \ldots A_n - p) \cap A_{n+1} \quad (Lemma \, 5.3(i))$$

$$(A_0 \ldots A_n A_{n+1}) = F((A_0 \ldots A_n - p) \cap A_{n+1}) \subset F(A_n - p) \cap A_{n+1} \subset X_\ell \cap A_{n+1}$$

Thus $A_{n+1}$ is necessarily of the form $X_k^\ell$ for some $k \in [1, N_\ell]$. According to Lemma 5.8 $(\alpha) \xrightarrow{F} (A_0 \ldots A_n X_k^\ell + p$. Thus there exists $x \in (\alpha)$ such that $F(x) = \min (A_0 \ldots A_n X_k^\ell + p = \min X_k^\ell + p$ (the second equality comes from Lemma 5.3(ii)). Moreover, $\min(\alpha) = \min A_n$. 

11
(by Lemma 5.3(ii) again), and thus $F(\min(\alpha)) \in T_R$ by Lemma 4.1(iii). This implies that

$$\langle \alpha \rangle \xrightarrow{F} \text{min } X^j_f, \text{min } X^k_f + p.$$

Thus, for all $j \in [1, k-1]$, we have

$$\langle \alpha \rangle \xrightarrow{\text{min } X^j_f} X^j_f + p,$$

and hence $F(\langle \alpha \rangle + Z) \cap X^j_f = X^j_f$. According to Lemma 5.3(i), $F(\langle \alpha \rangle + Z) \cap X^j_f = \langle A_0 \ldots A_n X^j_f \rangle$. Consequently, $A_0 \ldots A_n X^j_f \sim X^j_f$ and $\alpha \xrightarrow{\text{min } X^j_f}$ is an arrow in $G$. □

**Proposition 5.15** Let $\alpha, \beta$ be vertices of $G$.

i) All but at most one arrows starting from $\alpha$ end at a vertex in the basis.

ii) If $\alpha \xrightarrow{} \beta$ with $H(\beta) > 0$, then $H(\beta) = H(\alpha) + 1$.

**Proof.** We write $\alpha = A_0 \ldots A_n/ \sim$ and $p = p(A_n)$.

i) If $\alpha \xrightarrow{} \beta$, Lemma 5.14 states that there exists $k \in [1, N_\ell]$ such that $\beta = A_0 \ldots A_n X^j_f/ \sim$ and, for all $j \in [1, k-1]$, $\alpha \xrightarrow{} X^j_f$ is an arrow in $G$. This implies that there is at most one vertex of the form $A_0 \ldots A_n X^j_f/ \sim$ of height different from 0. This proves (i).

ii) Suppose that $\alpha \xrightarrow{} \beta$ with $H(\beta) > 0$. Let $A_{n+1} \in P$ be such that $\beta = A_0 \ldots A_n A_{n+1}/ \sim$ (Lemma 5.10). The significant part of $\beta$ is $A_i \ldots A_n A_{n+1}$ with $i := n + 1 - H(\beta) \leq n$ because $H(\beta) \geq 1$. Then by definition $\langle A_0 \ldots A_i \rangle = A_i$, and the significant part of $\alpha$ is $A_i \ldots A_n$. Hence $H(\alpha) = n - i = H(\beta) - 1$. This proves (ii). □

**Remark 5.16** The structure of the graph $G$ can be deduced from Proposition 5.15: if we start from a vertex $A$ in the basis and go up into the heights, there is a unique, finite or infinite, path $(\alpha_n)$ starting at $A$ and such that $H(\alpha_n) = i$. Two such paths starting at two different vertices in the basis are disjoint because if $\alpha$ is a vertex of height $n > 0$ and significant part $A_0 \ldots A_n$ then $\alpha$ belongs only to the path starting at $A_0 \in P$ (this paths begins with $A_0, A_0 A_1/ \sim, \ldots, A_0 \ldots A_n/ \sim = \alpha$). The only other arrows in $G$ end in the basis. This is illustrated in Figure 3.

![Figure 3: The structure of the covering graph. At the top: the finite basis.](image-url)
Definition 5.17 An oriented graph is strongly connected if for every pair of vertices \((u, v)\), there exists an oriented path of positive length from \(u\) to \(v\).

The connected components of an oriented graph are the maximal strongly connected subgraphs.

Remark 5.18 Two connected components are either disjoint or equal.

Some vertices (called inessential vertices) may belong to no connected component, see e.g. the vertex \(\alpha\) in the middle of Figure 13.

Proposition 5.19  
\(\star\) Every connected component of \(G\) meets the basis and the number of connected components is finite and bounded by \(\#P\).

ii) Let \(\mathcal{I} := \{(\alpha_n)_{n \geq 0} \in \Gamma(G) \mid H(\alpha_0) = 0 \quad \text{and} \quad \forall n \geq 1, H(\alpha_n) > 0\}\).

If \((\alpha_n)_{n \geq 0} \in \mathcal{I}\), then \(H(\alpha_n) = n\) for all \(n \geq 0\). Moreover, \(\mathcal{I}\) is a finite set with \(\#\mathcal{I} \leq \#P\).

iii) If \((\alpha_n)_{n \geq 0}\) is an infinite path in \(G\) then, either there exists a connected component \(C\) of \(G\) such that all vertices \(\alpha_n\) belong to \(C\) for all great enough \(n\), or there exist \((\beta_n)_{n \geq 0} \in \mathcal{I}\) and \(N, M \geq 0\) such that, for all \(n \geq 0\), \(\alpha_{N+n} = \beta_{M+n}\).

Proof. i) If the vertex \(v\) belongs to some connected component, there exists a loop starting at \(v\). According to the structure of \(G\) (see Remark 5.16), every loop goes through the basis, which is finite. Thus every connected component meets the basis and the number of connected components of \(G\) is at most \(\#P < +\infty\).

ii) Let \((\alpha_n)_{n \geq 0} \in \mathcal{I}\). By Proposition 5.15(ii), \(H(\alpha_{n+1}) = H(\alpha_n) + 1\) for all \(n \geq 0\), and thus \(H(\alpha_n) = n\) for all \(n \geq 0\). According to Proposition 5.15(i), each vertex \(\alpha_{n+1}\) is uniquely determined by the properties that \(\alpha_n \rightarrow \alpha_{n+1}\) and \(H(\alpha_{n+1}) \neq 0\). Since \(\alpha_0 \in \mathcal{P}\), the number of such infinite paths is less than or equal to \(\#P < +\infty\).

iii) Let \((\alpha_n)_{n \geq 0}\) be an infinite path in \(G\). There are two cases.

- Suppose that there exists \(N\) such that \(\forall n \geq N, \alpha_n \notin \mathcal{P}\). Then, by Proposition 5.15(ii), \(\forall n \geq N, H(\alpha_{n+1}) = H(\alpha_n) + 1\), and thus \(\forall n \geq 0, H(\alpha_{N+n}) = H(\alpha_N) + n\). Let \(A_0 \ldots A_M\) be the significant part of \(\alpha_N\), with \(M = H(\alpha_N)\). By Lemma 5.10, there exists a sequence \((A_n)_{n \geq M+1}\) of elements of \(\mathcal{P}\) such that, for all \(n \geq 0\), \(\alpha_{N+n} = A_0 \ldots A_{M+n}/\sim\). We define \(\beta_n := A_0 \ldots A_n/\sim\) for all \(n \geq 0\). Since \(H(\beta_M) = H(\alpha_N) = M\), we have \(H(\beta_n) = n\) for all \(n \in [0, M]\), and \(H(\beta_{M+n}) = H(\alpha_{N+n}) = M + n\) for all \(n \geq 0\). Hence \((\beta_n)_{n \geq 0} \in \mathcal{I}\) and \(\alpha_{N+n} = \beta_{M+n}\) for all \(n \geq 0\).

- Otherwise, there exist infinitely many \(n\) such that \(\alpha_n \in \mathcal{P}\). Since \(\mathcal{P}\) is finite, there exists \(A \in \mathcal{P}\) such that \(\alpha_n = A\) for infinitely many \(n\). Consequently, \(A\) belongs to some connected component \(C\) and \(\alpha_n\) belongs to \(C\) for all great enough \(n\).

6 The rotation set of the covering graph

6.1 Rotation numbers of infinite paths

Definition 6.1 Let \(\alpha \rightarrow \beta\) be an arrow in \(G\) and let \(A\) be the unique element of \(\mathcal{P}\) such that \(\langle \alpha \rangle \subset A \in \mathcal{P}\). The weight of the arrow \(\alpha \rightarrow \beta\) is defined as \(W(\alpha \beta) := p(A)\).

This naturally leads to the following definition of rotation numbers for infinite paths (see [7] for a similar notion in the case of subshifts of finite type).
Definition 6.2 If $\alpha_0 \ldots \alpha_n$ is a finite path in $\mathcal{G}$, its length is $L(\alpha_0 \ldots \alpha_n) := n$ and its weight is $W(\alpha_0 \ldots \alpha_n) := \sum_{i=0}^{n-1} W(\alpha_i\alpha_{i+1})$.

If $\alpha = (\alpha_n)_{n \geq 0} \in \Gamma(\mathcal{G})$, then its rotation number is

$$\rho(\alpha) := \lim_{n \to +\infty} \frac{W(\alpha_0 \ldots \alpha_n)}{n}$$
when this limit exists.

If $\Gamma' \subset \Gamma(\mathcal{G})$, let $\text{Rot}_{\Gamma'} := \{ \rho(\alpha) \mid \alpha \in \Gamma' \text{ and } \rho(\alpha) \text{ exists} \}$.

Definition 6.3 If $\gamma = \alpha_0 \ldots \alpha_n$ and $\gamma' = \beta_0 \ldots \beta_m$ are two paths in $\mathcal{G}$ with $\beta_0 = \alpha_n$, let $\gamma \cdot \gamma'$ denote the concatenation of the two paths, that is,

$$\gamma \cdot \gamma' := \alpha_0 \ldots \alpha_n \beta_1 \ldots \beta_m.$$

If $\gamma$ is a loop then $\gamma^n := \gamma \cdot \gamma \ldots \gamma$ is the $n$-time concatenation of $\gamma$. We define similarly the concatenation of infinitely many finite paths, and the concatenation of a finite path with an infinite path; in these two cases, the resulting paths are infinite.

If $\gamma$ is a loop in $\mathcal{G}$, let $\tilde{\gamma} := \gamma^\infty$ be the corresponding periodic infinite path.

The next lemma is straightforward.

Lemma 6.4 i) If $\gamma, \gamma'$ are two finite paths that can be concatenated, then $L(\gamma \cdot \gamma') = L(\gamma) + L(\gamma')$ and $W(\gamma \cdot \gamma') = W(\gamma) + W(\gamma')$.

ii) If $\gamma$ is a loop, then $\rho(\tilde{\gamma}) = \frac{W(\gamma)}{L(\gamma)} \in \mathbb{Q}$.

6.2 The rotation set of a connected component

We have seen in Proposition 5.19 that an infinite path either ultimately belongs to some connected component of $\mathcal{G}$, or ultimately coincide with some infinite path belonging to some finite set $\mathcal{I}$. In this subsection, we focus on the first case and we study the rotation set of a given connected component of $\mathcal{G}$.

We start with a lemma that uses the concatenation of loops to get rotation numbers.

Lemma 6.5 i) Let $\gamma_1, \gamma_2$ be two loops in $\mathcal{G}$ starting at the same vertex $\alpha$. If $r \in [\rho(\tilde{\gamma}_1), \rho(\tilde{\gamma}_2)] \cap \mathbb{Q}$, there exists a loop $\gamma$ starting at $\alpha$ such that $\rho(\gamma) = r$.

ii) Let $\alpha$ be a vertex of $\mathcal{G}$ and, for every $n \geq 0$, let $\gamma_n$ be a loop in $\mathcal{G}$ starting at $\alpha$. If $\lim_{n \to +\infty} \rho(\tilde{\gamma}_n) = s \in \mathbb{R}$, there then exists $\tilde{\alpha} = (\alpha_n)_{n \geq 0} \in \Gamma(\mathcal{G})$ such that $\alpha_0 = \alpha$, $\alpha_n = \alpha$ for infinitely many $n$ and $\rho(\tilde{\alpha}) = s$.

Proof. i) We write $r = p/q$ with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. Let $L$ be a multiple of $L(\gamma_1), L(\gamma_2)$ and $q$, and let $k_1, k_2, k$ be such that $L = k_1 L(\gamma_1) = k_2 L(\gamma_2) = kq$. We set $\gamma'_i := (\gamma_i)^{k_i}$ and $W_i := W(\gamma'_i)$ for $i \in \{1, 2\}$, and $p' := kp$. In this way, $L(\gamma'_i) = L$ and $\rho(\tilde{\gamma}_i) = \rho(\tilde{\gamma}'_i) = W_i/L$ for $i \in \{1, 2\}$, and $r = p'/L$. Since $\rho(\tilde{\gamma}_1) \leq r \leq \rho(\tilde{\gamma}_2)$, we have $W_1 \leq p' \leq W_2$. We define $\gamma := (\gamma'_1)^{W_2-p'} \cdot (\gamma'_2)^{p'-W_1}$. This is a loop starting at $\alpha$, and

$$\rho(\gamma) = \frac{W(\gamma)}{L(\gamma)} = \frac{W_1(W_2-p') + W_2(p'-W_1)}{L(W_2-p') + L(p'-W_1)} = \frac{(W_2 - W_1)p'}{(W_2 - W_1)L} = r.$$

This proves (i).

ii) Let $(i_n)_{n \geq 0}$ be a sequence of positive integers and $\alpha := \gamma_0^{i_0} \cdot \gamma_1^{i_1} \ldots \gamma_n^{i_n} \ldots$. This is an infinite path starting at $\alpha$ and passing at $\alpha$ infinitely many times. It can be shown that, if the sequence $(i_n)_{n \geq 0}$ increases sufficiently fast, then $\rho(\tilde{\alpha}) = \lim_{n \to +\infty} \rho(\tilde{\gamma}_n) = s$ (see e.g. the proof of [2], Theorem 3.1) for a similar proof exploiting the growth of $(i_n)_{n \geq 0}$.)
Proposition 6.6 Let $C$ be a connected component of $\mathcal{G}$. Then

i) $\text{Rot}_{\Gamma(C)}$ is a non empty compact interval.

ii) $\forall r \in \text{Int}(\text{Rot}_{\Gamma(C)}) \cap \mathbb{Q}$, there exists a loop $\gamma$ in $C$ such that $\rho(\gamma) = r$.

Proof. The set $C \cap \mathcal{P}$ is non empty by Proposition 5.19(i), and thus we can fix $A \in C \cap \mathcal{P}$. For every $B, B' \in C \cap \mathcal{P}$, we choose a path $u(B, B')$ from $B$ to $B'$. Since $\mathcal{P}$ is finite, we can bound $L(u(B, B'))$ and $W(u(B, B'))$ by some quantities $L_0$ and $W_0$ respectively, independently of $B, B'$. Let $\mathcal{L}_A$ be the set of all loops starting at $A$. If $\gamma \in \mathcal{L}_A$, then the periodic path $\tilde{\gamma}$ belongs to $\Gamma(C)$.

Let $\tilde{\alpha} = (\alpha_n)_{n \geq 0} \in \Gamma(C)$ such that $\rho(\tilde{\alpha})$ exists. We are going to show that

$$\forall \varepsilon > 0, \exists \gamma \in \mathcal{L}_A \text{ such that } |\rho(\tilde{\alpha}) - \rho(\gamma)| < \varepsilon. \quad (6)$$

If $u$ is a path from $A$ to $\alpha_0$ then $u \cdot (\alpha_n)_{n \geq 0} \in \Gamma(C)$ and it has the same rotation number as $\tilde{\alpha}$. Thus we can assume that $\alpha_0 = A$.

We first show that there exist a sequence of integers $(n_i)_{i \geq 0}$ increasing to infinity, and finite paths $v_i$ from $\alpha_{n_i}$ to $A$ such that $\forall i \geq 0, W(v_i) \leq W$ and $L(v_i) \leq L$, where

$$W := W_0 + \max\{p(B) \mid B \in \mathcal{P}\} \text{ and } L := L_0 + 1.$$ 

Since $\tilde{\alpha}$ is in $\Gamma(C)$, for all integers $N \geq 0$, there exists a path in $C$ from $\alpha_N$ to $A$. Because of the structure of $\mathcal{G}$ (see Proposition 5.15), this implies that there exists $n \geq N$ and $B \in C \cap \mathcal{P}$ such that $\alpha_n \to B$. Thus we can find a sequence $(n_i)_{i \geq 0}$ increasing to infinity and vertices $(B_i)_{i \geq 0}$ in $C \cap \mathcal{P}$ such that $\alpha_{n_i} \to B_i$ for all $i \geq 0$. We set $v_i := \alpha_{n_i}B_i \cdot u(B_i, A)$. Then $L(v_i) \leq L$ and $W(v_i) \leq W$.

Now we define $\gamma_i \in \mathcal{L}_A$ by concatenating $\alpha_0 \ldots \alpha_{n_i}$ with $v_i$. Fix $\varepsilon > 0$ and let $i$ be great enough such that

$$\left| \frac{\rho(\tilde{\alpha}) - W(\alpha_0 \ldots \alpha_{n_i})}{n_i} \varepsilon \right| < \frac{\varepsilon}{3}, \quad \left| \frac{W}{n_i} \varepsilon \right| < \frac{\varepsilon}{3}, \text{ and } \left( \frac{\rho(\tilde{\alpha})}{\varepsilon/3} + \frac{L}{n_i} \right) \left( \frac{\varepsilon/3}{n_i} \right) < \varepsilon.$$ 

Since $\rho(\gamma_i) = \frac{W(\alpha_0 \ldots \alpha_{n_i}) + W(v_i)}{n_i + L(v_i)}$, we have

$$|\rho(\tilde{\alpha}) - \rho(\gamma_i)| \leq \left| \frac{\rho(\tilde{\alpha})}{n_i} - \frac{W(\alpha_0 \ldots \alpha_{n_i})}{n_i} + \frac{W(\alpha_0 \ldots \alpha_{n_i})}{n_i + L(v_i)} + \frac{W(\alpha_0 \ldots \alpha_{n_i})}{n_i} \right|$$

$$< \frac{\varepsilon}{3} + \frac{W(v_i)}{n_i + L(v_i)} - \frac{W(\alpha_0 \ldots \alpha_{n_i})}{n_i(n_i + L(v_i))} \left| \frac{L}{n_i} \right|$$

$$< \frac{\varepsilon}{3} + \frac{W}{n_i} + \frac{W(\alpha_0 \ldots \alpha_{n_i})}{n_i} \left| \frac{L}{n_i} \right|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \left( \frac{\rho(\tilde{\alpha})}{\varepsilon/3} + \frac{L}{n_i} \right) \left( \frac{\varepsilon/3}{n_i} \right)$$

$$< \varepsilon.$$ 

This proves Equation (6). In other words, $\{\rho(\gamma) \mid \gamma \in \mathcal{L}_A\}$ is dense in $\text{Rot}_{\Gamma(C)}$.

The set $\text{Rot}_{\Gamma(C)}$ is non empty because there exists a loop $\gamma$ starting at $A \in C \cap \mathcal{P} \neq \emptyset$, and $\rho(\gamma)$ exists. We set $a := \inf \text{Rot}_{\Gamma(C)}$ and $b := \sup \text{Rot}_{\Gamma(C)}$. We suppose $a < b$, otherwise there is nothing to prove. Let $r \in (a, b) \cap \mathbb{Q}$ and let $\varepsilon > 0$ be such that $a + 2\varepsilon < r < b - 2\varepsilon$. Let $\tilde{\alpha}, \tilde{\beta} \in \Gamma(C)$ be such that $|\rho(\tilde{\alpha}) - a| < \varepsilon$ and $|\rho(\tilde{\beta}) - b| < \varepsilon$. By Equation (6), there exist $\gamma_1, \gamma_2 \in \mathcal{L}_A$ such that $|\rho(\gamma_1) - \rho(\tilde{\alpha})| < \varepsilon$ and $|\rho(\gamma_2) - \rho(\tilde{\beta})| < \varepsilon$, and hence

$$\rho(\gamma_1) < r < \rho(\gamma_2).$$
Then by Lemma 6.5(i) there exists \( \gamma \in \mathcal{L}_A \) such that \( \rho(\gamma) = r \). Now let \( s \in [a, b] \) and let \( (r_n)_{n \geq 0} \) be a sequence in \((a, b) \cap \mathbb{Q}\) such that \( \lim_{n \to +\infty} r_n = s \). What precedes implies that, for all \( n \geq 0 \), there exists \( \gamma_n \in \mathcal{L}_A \) such that \( \rho(\gamma_n) = r_n \). Then, according to Lemma 6.5(ii), there exists an infinite path \( \bar{\alpha} \) such that \( \rho(\bar{\alpha}) = s \) and \( \bar{\alpha} \) starts at \( A \) and passes infinitely many times at \( A \), which implies that \( \bar{\alpha} \in \Gamma(C) \). This ends the proof of the proposition. \( \square \)

### 6.3 The rotation numbers of infinite paths not in connected components

**Proposition 6.7**

Let

\[
\mathcal{J} := \{ \bar{\alpha} = (\alpha_n)_{n \geq 0} \in \Gamma(\mathcal{G}) \mid \exists N(\bar{\alpha}) \geq 0, \forall n \geq N(\bar{\alpha}), \exists B \in \mathcal{P}, \alpha_n \to B \}.
\]

Let \( \bar{\alpha} = (\alpha_n)_{n \geq 0} \in \mathcal{J} \) and let \((A_n)_{n \geq 0}\) be the sequence of elements of \( \mathcal{P} \) such \( \alpha_n = A_0 \ldots A_n / \sim \) for all \( n \geq 0 \). Then \( \rho(\bar{\alpha}) \) exists and is a rational number. More precisely, there exist \( q \geq 1, B_0, \ldots, B_{q-1} \in \mathcal{P} \) and \( M \geq N(\bar{\alpha}) \) such that

\[
\forall n \geq 0, \forall r \in [0, q-1], A_{M+qn+r} = B_r
\]

and \( \rho(\bar{\alpha}) = \frac{p}{q} \), where \( p := p(B_0) + p(B_1) + \cdots + p(B_{q-1}) \). Moreover, there exists \( x \in X^\infty \) such that \( F^q(x) = x + p \), that is, \( x \) is periodic (mod 1) and \( \rho_F(x) = \frac{p}{q} = \rho(\bar{\alpha}) \).

**Proof.** Let \( n \geq N(\bar{\alpha}) \). By Lemma 5.14, there exists \( i \in [1, \ell(A_n)] \) such that \( A_{n+1} = X_1^{\ell(A_n)} \) and, if \( i \geq 2 \), then \( \alpha \to X_1^{\ell(A_n)} \). By definition of \( \mathcal{J} \) and choice of \( n \), there is no arrow from \( \alpha \) to some element of \( \mathcal{P} \). Therefore \( i = 1 \). This implies that, for all \( n \geq N(\bar{\alpha}) \), \( A_{n+1} \) is uniquely determined by \( A_n \). Since \( \mathcal{P} \) is finite, there exist \( M \geq N(\bar{\alpha}) \) and \( q \geq 1 \) such that \( A_{M+q} = A_M \).

If we set \( B_0 \ldots B_{q-1} := A_M \ldots A_{M+q-1} \), we get

\[
\forall n \geq 0, \forall r \in [0, q-1], A_{M+qn+r} = B_r
\]

Then, if we set \( p := p(B_0) + p(B_1) + \cdots + p(B_{q-1}) \), it is clear that \( \rho(\bar{\alpha}) = \frac{p}{q} \). This proves the first part of the proposition.

For all \( n \geq 1 \), we set

\[
\beta_n := 
\begin{array}{c}
B_0 \ldots B_{q-1} \ldots B_0 \ldots B_{q-1} \\
\small{\text{repeated } n \text{ times}}
\end{array} / \sim.
\]

Then \( \beta_n = A_M \ldots A_{M+qn-1} / \sim \). Since \( \langle A_M \ldots A_{M+m} \rangle \xrightarrow{\mathcal{F}} \langle A_M \ldots A_{M+m+1} \rangle + p(A_{M+m}) \) for all \( m \geq 0 \), Lemma 3.4 implies that \( \langle \beta_n \rangle \xrightarrow{F^q} \langle \beta_{n+1} \rangle + p \) for all \( n \geq 1 \). We set \( a_0 := \min B_{q-1} \). Then \( a_0 = \min(\beta_n) \) for all \( n \geq 1 \) by Lemma 5.3(ii) and \( F(a_0) \in T_\mathbb{R} \) by Lemma 4.1(iii). Let \( \ell \in \Lambda \) be such that \( B_{q-1} \subseteq X^\ell \). By Proposition 5.13, there exists \( y' \in X^\infty \) of itinerary \( (A_m)_{m \geq 0} \). Thus the itinerary of \( y := F^{M+q-1}(y') \) is \( (A_{M+q-1+m})_{m \geq 0} \). Let \( G := F^q - p \). It is clear that \( G(T_\mathbb{R} \setminus T_\mathbb{Q}) \subseteq T_\mathbb{R} \). For all \( n \geq 0 \), \( G^n(y) \in B_{q-1} \subseteq X^\ell \), and in particular \( G^n(y) \geq a_0 \) for the order in \( X^\ell \). We define inductively a sequence of points \((a_i)_{i \geq 1}\) such that \( a_i \in [a_{i-1}, y] \) and \( G^i(a_i) = a_0 \) for all \( i \geq 1 \).

- Since \( G(a_0) \in T_\mathbb{R} \) and \( G'(y) \geq a_0 \), we have \( a_0 \in G([a_0, y]) \) by continuity. Thus there exists \( a_1 \in [a_0, y] \) such that \( G(a_1) = a_0 \).
- Assume that \( a_0, \ldots, a_i \) are already defined. Since \( G^{i+1}(a_i) = G(a_0) \in T_\mathbb{R} \) and \( G^{i+1}(y) \geq a_0 \), the point \( a_0 \) belongs to \( G^{i+1}([a_i, y]) \) by continuity. Thus there exists \( a_{i+1} \in [a_i, y] \) such that \( G^{i+1}(a_{i+1}) = a_0 \). This concludes the construction of \( a_{i+1} \).
The sequence \((a_i)_{i \geq 0}\) is non-decreasing and contained in the compact interval \(X^\ell\). Therefore \(x := \lim_{n \to +\infty} a_i\) exists and belongs to \(X^\ell\). Since \(G(a_{i+1}) = a_i\), we get that \(G(x) = x\). In other words, \(F^n(x) = x + p\). Thus \(F^{nm}(x) \in X\) for all \(n \geq 0\), which implies that \(x \in X^\infty\). Clearly, \(x\) is periodic (mod 1) and \(\rho_F(x) = \frac{p}{q}\).

\[\□\]

### 6.4 The rotation set of \(\Gamma(\mathcal{G})\)

Proposition \ref{prop:rotation_set_of_graph} implies that the rotation set of \(\Gamma(\mathcal{G})\) can be decomposed as the finite union of the rotation sets of the connected components plus a finite set (see Equation \[7\] in the proof below). This decomposition, together with the study of the rotation set of a connected component in the previous subsection, leads to the following theorem.

**Theorem 6.8** The set \(\text{Rot}_{\Gamma(\mathcal{G})}\) is compact and has finitely many connected components.

There exists a finite set \(D\) such that, for every rational number \(p/q\) in \(\text{Rot}_{\Gamma(\mathcal{G})} \setminus D\), there exists a loop \(\gamma\) in \(\mathcal{G}\) such that \(\rho(\gamma) = p/q\).

**Proof.** Let \(C\) be the set of connected components of \(\mathcal{G}\). Let \(\mathcal{I}\) and \(\mathcal{J}\) be the sets defined in Proposition \ref{prop:connected_components} and \ref{prop:intersection}, respectively. Let \(\bar{\alpha} = (\alpha_n)_{n \geq 0} \in \Gamma(\mathcal{G})\) such that \(\rho(\bar{\alpha})\) exists. By Proposition \ref{prop:connected_components}(iii):

- Either there exist a connected component \(C \in \mathcal{C}\) and an integer \(N\) such that, \(\forall n \geq N, \alpha_n \in C\). In this case, \(\rho(\bar{\alpha}) \in \text{Rot}_{\Gamma(C)}\).
- Or there exist \(\bar{\beta} = (\beta_n)_{n \geq 0} \in \mathcal{I} \cap \mathcal{J}\) and integers \(N, M\) such that, \(\forall n \geq 0, \alpha_{N+n} = \beta_{M+n}\). Thus \(\rho(\bar{\beta})\) exists and is a rational number (Proposition \ref{prop:intersection}) and is equal to \(\rho(\bar{\alpha})\), and so \(\rho(\bar{\alpha}) \in \text{Rot}_{\mathcal{I} \cap \mathcal{J}}\).

Consequently,

\[
\text{Rot}_{\Gamma(\mathcal{G})} = \bigcup_{C \in \mathcal{C}} \text{Rot}_{\Gamma(C)} \cup \text{Rot}_{\mathcal{I} \cap \mathcal{J}}.
\]

The sets \(\mathcal{C}\) and \(\mathcal{I}\) are finite by Proposition \ref{prop:connected_components}(i)-(ii), and thus \(\text{Rot}_{\mathcal{I} \cap \mathcal{J}}\) is finite. Moreover, for every \(C \in \mathcal{C}\), \(\text{Rot}_{\Gamma(C)}\) is a compact interval by Proposition \ref{prop:intersection}(i). Therefore \(\text{Rot}_{\Gamma(\mathcal{G})}\) is a finite union of compact intervals (some intervals may be reduced to a single point). This proves the first point of the theorem.

According to Proposition \ref{prop:intersection}(ii), for every rational number \(p/q \in \bigcup_{C \in \mathcal{C}} \text{Int} \ (\text{Rot}_{\Gamma(C)})\), there exists a loop \(\gamma\) such that \(\rho(\gamma) = p/q\). Since the set \(D := \bigcup_{C \in \mathcal{C}} \partial \text{Rot}_{\Gamma(C)} \cup \text{Rot}_{\mathcal{I} \cap \mathcal{J}}\) is finite, this implies the second point of the theorem. \[\□\]

**Remark 6.9**

- The intervals \((\text{Rot}_{\Gamma(C)})_{C \in \mathcal{C}}\) may not be disjoint.
- The number of connected components of \(\text{Rot}_{\Gamma(\mathcal{G})}\) is at most \(2\#\mathcal{P}\) and at most \(#\mathcal{P}\) are not reduced to a single point. This is because both \(#\mathcal{C}\) and \(#\mathcal{I}\) are bounded by \(#\mathcal{P}\).

### 7 The rotation set of a sun-like map

Now that we have studied the rotation set of the covering graph, we are ready to prove that the rotation set of a sun-like map is composed of finitely many compact intervals, and that all but finitely many rational numbers in the rotation set are rotation numbers of some periodic (mod 1) points.

According to Equation \[11\], the rotation set of \(F\) has the following decomposition:

\[
\text{Rot}(F) = \text{Rot}_{\Gamma}(F) \cup \text{Rot}_{X^\infty}(F).
\]

First, we pull back the results from \(\text{Rot}_{\Gamma(\mathcal{G})}\) to \(\text{Rot}_{X^\infty}(F)\).
Proposition 7.1 Let $F \in S_1(T)$ and $G$ its covering graph. Then $\text{Rot}_{X^\infty}(F) = \text{Rot}_{\Gamma(G)}$.

If $\gamma$ is a loop in $G$, then $\rho(\bar{\gamma}) \in \mathbb{Q}$ and there exists a periodic (mod 1) point $x \in X^\infty$ such that $\rho_F(x) = \rho(\bar{\gamma})$. If $\bar{\alpha} \in J$, then $\rho(\bar{\alpha}) \in \mathbb{Q}$ and there exists a periodic (mod 1) point $x \in X^\infty$ such that $\rho_F(x) = \rho(\bar{\alpha})$.

Proof. If we put together Lemma 4.4, Proposition 5.13 and the definition of the rotation numbers in $\Gamma(G)$, then it is clear that

$$\exists x \in X^\infty \text{ such that } \rho_F(x) = r \iff \exists \bar{\alpha} \in \Gamma(G) \text{ such that } \rho(\bar{\alpha}) = r.$$ 

Hence $\text{Rot}_{X^\infty}(F) = \text{Rot}_{\Gamma(G)}$.

Suppose that $\gamma = \alpha_0 \ldots \alpha_n$ is a loop in $G$. According to Lemma 5.8 we have the following chain of positive coverings:

$$(\alpha_0) \xrightarrow{F} (\alpha_1) + W(\alpha_0) (\alpha_1) \xrightarrow{F} (\alpha_2) + W(\alpha_0 \alpha_1) (\alpha_2) \ldots \xrightarrow{F} (\alpha_n) + W(\alpha_0 \ldots \alpha_n).$$

Since $\alpha_0 = \alpha_n$, Proposition 5.5 implies that there exists $x \in (\alpha_0)$ such that $F^m(x) = x + W(\alpha_0 \ldots \alpha_n)$ and $F^i(x) \in (\alpha_i) + W(\alpha_0 \ldots \alpha_i)$ for all $i \in [1, n-1]$. This implies that $x$ is a periodic (mod 1) point, $x \in X^\infty$ because $(\alpha_i) \subset X$ for all $i \in [0, n]$, and

$$\rho_F(x) = \frac{W(\alpha_0 \ldots \alpha_n)}{n} = \frac{W(\gamma)}{L(\gamma)} = \rho(\bar{\gamma}).$$

If $\bar{\alpha} \in J$, the conclusion is given by Proposition 6.7.

The main result of this paper is now a mere consequence of Proposition 7.1, Proposition 6.7, Theorem 6.8 and Theorem 2.8.

Theorem 7.2 Let $F \in S_1(T)$. Then $\text{Rot}(F)$ is a nonempty compact set and has finitely many connected components. There exists a finite set $E$ such that, for every rational number $p/q$ in $\text{Rot}(F) \setminus E$, there exists a periodic (mod 1) point $x \in T$ such that $\rho_F(x) = p/q$. More precisely, $\#E \leq 2\#\mathcal{P}$ and $E \subset \bigcup_{C \in \mathcal{C}} \partial \text{Rot}(\Gamma(C))$, where $\mathcal{C}$ is the set of connected components of $G$.

We conjecture that the exceptional set $E$ is empty. This conjecture is true for the graph $\sigma^3$. In the general case of a degree-one map on a graph with a unique loop, it is not known if the rotation set is closed, if the number of connected components is finite or if (almost) all rational rotation numbers are rotation numbers of periodic points.

References

[1] Ll. Alsedà, J. Llibre, and M. Misiurewicz. Combinatorial dynamics and entropy in dimension one, volume 5 of Advanced Series in Nonlinear Dynamics. World Scientific Publishing Co. Inc., River Edge, NJ, second edition, 2000.

[2] Ll. Alsedà and S. Ruette. Rotation sets for graph maps of degree 1. Ann. Inst. Fourier (Grenoble), 58(4):1233–1294, 2008.

[3] J. Buzzi. Intrinsic ergodicity of smooth interval maps. Israel J. Math., 100:125–161, 1997.

[4] F. Hofbauer. On intrinsic ergodicity of piecewise monotonic transformations with positive entropy. Israel J. Math., I, 34(3):213–237, 1979. II. 38(1–2):107–115, 1981.

[5] S. Ruette. Rotation set for maps of degree 1 on the graph sigma. Israel Journal of Mathematics, 184:275–299, 2011.
[6] D. Vere-Jones. Geometric ergodicity in denumerable Markov chains. *Quart. J. Math. Oxford Ser. (2)*, 13:7–28, 1962.

[7] K. Ziemian. Rotation sets for subshifts of finite type. *Fund. Math.*, 146(2):189–201, 1995.

Laboratoire de Mathématiques d’Orsay, CNRS UMR 8628, Bâtiment 307, Université Paris-Sud 11, 91405 Orsay cedex, France. <Sylvie.Ruette@math.u-psud.fr>