GENERALIZED NAVIER-STOKES EQUATIONS WITH INITIAL DATA IN LOCAL $Q$-TYPE SPACES

PENGTAO LI AND ZHICHUN Zhai

Abstract. In this paper, we establish a link between Leray mollified solutions of the three-dimensional generalized Navier-Stokes equations and mild solutions for initial data in the adherence of the test functions for the norm of $Q^β_{α, loc}(\mathbb{R}^3)$. This result applies to the usual incompressible Navier-Stokes equations and deduces a known link.

1. Introduction

This paper studies the relationship between Leray mollified solutions and mild solutions to the generalized Navier-Stokes equations in $\mathbb{R}^3$:

$$
\begin{cases}
\partial_t u + (-\Delta)^β u + (u \cdot \nabla) u - \nabla p = 0, & \text{in } \mathbb{R}_{t,x}^{1+3}, \\
\nabla \cdot u = 0, & \text{in } \mathbb{R}_{t,x}^{1+3}, \\
u|_{t=0} = u_0, & \text{in } \mathbb{R}^3,
\end{cases}
$$

(1.1)

for $β \in (1/2, 1]$, where $(-\Delta)^β$ is the fractional Laplacian with respect to $x$ defined by

$$
\widehat{(-\Delta)^β u}(t, ξ) = |ξ|^{2β} \widehat{u}(t, ξ)
$$

through Fourier transform. Here $u$ and $p$ are non-dimensional quantities corresponding to the velocity of the fluid and its pressure. $u_0$ is the initial data and for the sake of simplicity, the fluid is supposed to fill the whole space $\mathbb{R}^3$. Equations (1.1) have been studied intensively, see Cannone [1], Giga and Miyakawa [6], Kato [7], Koch and Tataru [8], Xiao [10], Lions [12], Wu [14-15], Li and Zhai [13].

When $β = 1$, equations (1.1) become the usual incompressible Navier-Stokes equations. In dimensional 3, the global existence, uniqueness and regularity of the solutions for the usual Navier-Stokes equations are long-standing open problem of fluid dynamics and the regularity problems is of course a millennium prize problem. Generally speaking, there are two specific approaches in the study of the existence of solutions to the three-dimensional incompressible Navier-Stokes equations. The first one is due to Leray [11] and the second is due to Kato [7]. We refer the readers to Cannone [2] and Lemarié-Rieusset [9] for further information.

For general $β$, we can also define the mild and mollified solutions separately as follows.

The generalized Navier-Stokes system is equivalent to the fixed point problem:

$$
u(t, x) = e^{-t(-\Delta)^β} u_0(x) - B(u, u)(t, x),
$$

(1.2)

2000 Mathematics Subject Classification. Primary 35Q30; 76D03; 42B35; 46E30.

Key words and phrases. Generalized Navier-Stokes equations; $Q^β_{α, loc}(\mathbb{R}^3)$; Lorentz spaces.

Project supported in part by Natural Science and Engineering Research Council of Canada.
where the bilinear form $B(u, v)$ is defined by

$$B(u, v) = \int_0^t e^{-(t-s)(-\Delta)\beta} \mathbb{P} \nabla \cdot (u \otimes v)(s, x) ds.$$ 

Here $e^{-(t-s)(-\Delta)\beta}$ denotes the convolution operator generated by the symbol

$$[e^{-(t-s)(-\Delta)\beta}](\xi) = e^{-t|\xi|^{2\beta}}$$

and $\mathbb{P}$ denotes the Leray projector onto the divergence free vector field.

**Definition 1.1.** A mild solution to the generalized Navier-Stokes equations (1.1) is a solution to equations (1.2) obtained via a fixed point procedure.

The mollified solutions are constructed in the same way as mild solutions, but with a slightly different model. Indeed, instead of the term $u \otimes u$ involved in the (GNS) equations, we look for something smoother. Let $\omega \in \mathcal{D}(\mathbb{R}^3)$ with $\omega > 0$ and $\int_{\mathbb{R}^3} \omega(x) dx = 1$. Then for $\varepsilon > 0$, the mollified generalized Navier-Stokes equations are given by:

$$
\begin{align*}
\partial_t u + (-\Delta)^\beta u + ((u * \omega_{\varepsilon}) \cdot \nabla)u - \nabla p &= 0, \\
\nabla \cdot u &= 0, \\

u_{t=x} &= u_0,
\end{align*}
$$

with $\omega_{\varepsilon}(x) = \frac{1}{\varepsilon^3} \omega(\frac{x}{\varepsilon})$.

Similar to the equations (1.1), equations (1.3) can be rewritten as a fixed point problem:

$$u_{\varepsilon} = e^{-(t-s)(-\Delta)\beta} u_0 - B_{\varepsilon}(u_{\varepsilon}, u_{\varepsilon}),$$

where the bilinear operator $B_{\varepsilon}$ is defined by

$$B_{\varepsilon}(u, v) = \int_0^t e^{-(t-s)(-\Delta)\beta} \mathbb{P} \nabla \cdot ((u * \omega_{\varepsilon}) \otimes u)(s) ds.$$

**Definition 1.2.** The mollified solution to equations (1.1) is the sequence $\{u_{\varepsilon}\}_{\varepsilon > 0}$ of the solutions to the system (1.3) for $\varepsilon > 0$.

In [10], for $\beta = 1$ of equations (1.1), that is, the usual incompressible Navier-Stokes equations, Lemarié-Rieusset and Prioux established a link between these two solutions. They proved that if the initial data $u_0 \in \mathcal{D}(\mathbb{R}^3)^{\text{bmo}^{-1}(\mathbb{R}^3)}$, then there exists $T > 0$ such that the mollified solutions $u_{\varepsilon} \in \mathcal{D}((0, T] \times \mathbb{R}^3)^{X_{\beta, 1}^1(\mathbb{R}^3)}$ constructed via the theory of Leray converges, when $\varepsilon$ tends to 0, to the mild solution given by Kato, for $t \in (0, T)$.

In [13], inspired by Xiao’s paper [16], we considered the well-posedness and regularity of equations (1.1) with initial data in some new critical spaces $Q_{\alpha, \infty}^{\beta, 1}(\mathbb{R}^n)$. In that paper, we proved that for initial data $u_0 \in Q_{\alpha, \infty}^{\beta, 1}(\mathbb{R}^n)$ there exists a unique mild solution in the space $X_{\alpha, \infty}^{\beta, 1}$, where the space $Q_{\alpha, \infty}^{\beta, 1}(\mathbb{R}^n)$ occurring above is a class of spaces which own a structure similar to the space $BMO^{-1}(\mathbb{R}^n)$ in [8] and $Q_{\alpha, \infty}^{-1}(\mathbb{R}^n)$ in [16]. It is easy to see that if $\alpha = -\frac{n}{2}$ and $\beta = 1$, $Q_{\alpha, \infty}^{\beta, 1}(\mathbb{R}^n) = BMO^{-1}(\mathbb{R}^n)$, and if $\alpha \in (0, 1)$ and $\beta = 1$, $Q_{\alpha, \infty}^{\beta, 1}(\mathbb{R}^n) = Q_{\alpha, \infty}^{-1}(\mathbb{R}^n)$. Here $Q_{\alpha, \infty}^{-1}(\mathbb{R}^n)$ is the derivative space of $Q_{\alpha}(\mathbb{R}^n)$, see Xiao [16], Dafni and Xiao [3]-[4], Essen, Janson, Peng and Xiao [5]. Therefore our well-posed result generalized the result of Koch and Tataru [8] and that of Xiao [16].
The main goal of this paper is to establish a relation between the mild solutions obtained in [16] and [13] and the mollified solutions for the equations (1.3). In fact, our main results mean that for initial data \( u_0 \in D(\mathbb{R}^3)^{Q_{\alpha,T}^{\beta,-1}(\mathbb{R}^3)} \), with \( \beta \in (1/2, 1] \), there exists \( T > 0 \) such that the sequence \( \{u_\varepsilon\}_{\varepsilon > 0} \in D((0,T) \times \mathbb{R}^3)^{X_{\alpha,T}^1(\mathbb{R}^3)} \) of solutions to (1.3) converges, when \( \varepsilon \to 0 \), to the mild solution obtained in [16] and [13], for \( t \in (0,T) \). For the usual incompressible Navier-Stokes equations, when \( \alpha = 0 \), our main result goes back to Lemarié-Rieusset and Prioux [10, Theorem 1.1].

However, it is worth pointing out that their Theorem does not deduce our results even though \( Q_{\alpha,loc}^{1,-1}(\mathbb{R}^3) \) is a subspace of \( bmo^{-1}(\mathbb{R}^3) \), since \( X_{\alpha,T}^1(\mathbb{R}^3) \) is proper subspace of \( X_{\alpha,T}^1(\mathbb{R}^3) \) when \( 0 < \alpha < 1 \).

In the following, we give some definitions and known results. The first one is the space \( Q_{\alpha,loc}^{\beta,-1}(\mathbb{R}^n) \) defined as follows.

**Definition 1.3.** For \( \alpha > 0 \) and \( \max\{1/2, \alpha\} < \beta \leq 1 \) with \( \alpha+\beta-1 \geq 0 \), \( Q_{\alpha,loc}^{\beta,-1}(\mathbb{R}^n) \) is the space of tempered distributions \( f \) on \( \mathbb{R}^n \) such that, for all \( T \in (0,\infty) \),

\[
\sup_{0 < r^{2\beta} < T} \sup_{x_0 \in \mathbb{R}^n} r^{2\alpha-n+2\beta-2} \int_0^{r^{2\beta}} \int_{|x-x_0| < r} |e^{-t(-\Delta)^\beta} f(y)|^2 \frac{dydt}{t^{\alpha/\beta}} < \infty.
\]

The norm on \( Q_{\alpha,loc}^{\beta,-1}(\mathbb{R}^n) \) is defined by

\[
\|f\|_{Q_{\alpha,loc}^{\beta,-1}(\mathbb{R}^n)} = \left( \sup_{0 < r^{2\beta} < T} \sup_{x_0 \in \mathbb{R}^n} r^{2\alpha-n+2\beta-2} \int_0^{r^{2\beta}} \int_{|x-x_0| < r} |e^{-t(-\Delta)^\beta} f(y)|^2 \frac{dydt}{t^{\alpha/\beta}} \right)^{1/2}.
\]

In [13], it was proved that that the space \( Q_{\alpha,loc}^{\beta,-1}(\mathbb{R}^n) \) consist of the derivatives of functions in \( Q_{\alpha}^{\beta}(\mathbb{R}^n) \) which is composed of all measurable functions with

\[
\sup_I |I(\beta)|^{2(\alpha+\beta-1)-n} \int_I \int |f(x) - f(y)|^2 \frac{dx}{|x-y|^{n+2(\alpha-\beta+1)}} dy < \infty
\]

where the supremum is taken over all cubes \( I \) with the edge length \( l(I) \) and the edges parallel to the coordinate axes in \( \mathbb{R}^n \).

We now introduce the space \( X_{\alpha,T}^{\beta}(\mathbb{R}^n) \).

**Definition 1.4.** Let \( \alpha > 0 \) and \( \max\{1/2, \alpha\} < \beta \leq 1 \) with \( \alpha+\beta-1 \geq 0 \).

(i) A tempered distribution \( f \) on \( \mathbb{R}^n \) belongs to \( Q_{\alpha,T}^{\beta,-1}(\mathbb{R}^n) \) provided

\[
\|f\|_{Q_{\alpha,T}^{\beta,-1}(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n, r^{2\beta} \leq T} \left( r^{2\alpha-n+2\beta-2} \int_0^{r^{2\beta}} \int_{|y-x| < r} |K_t^\beta * f(y)|^2 t^{-\frac{\alpha}{\beta}} dydt \right)^{1/2} < \infty;
\]

(ii) A tempered distribution \( f \) on \( \mathbb{R}^n \) belongs to \( VQ_{\alpha,T}^{\beta,-1}(\mathbb{R}^n) \) provided \( \lim_{T \to 0} \|f\|_{Q_{\alpha,T}^{\beta,-1}(\mathbb{R}^n)} = 0 \);  

(iii) A function \( g \) on \( \mathbb{R}^{1+n} \) belongs to the space \( X_{\alpha,T}^{\beta}(\mathbb{R}^n) \) provided

\[
\|g\|_{X_{\alpha,T}^{\beta}(\mathbb{R}^n)} = \sup_{t \in (0,T)} t^{1-\frac{\beta}{\alpha}} \|g(t, \cdot)\|_{L^\infty(\mathbb{R}^n)}
\]

\[
+ \sup_{x \in \mathbb{R}^n, r^{2\beta} \leq T} \left( r^{2\alpha-n+2\beta-2} \int_0^{r^{2\beta}} \int_{|y-x| < r} |g(t, y)|^2 t^{-\alpha/\beta} dydt \right)^{1/2} < \infty.
\]
In Xiao [16] and Li and Zhai [13], they proved the following well-posedness results about the equations (1.1) for $\beta = 1$ and $\frac{1}{2} < \beta < 1$, respectively.

**Theorem 1.5.** Let $n \geq 2$, $\alpha > 0$ and $\max\{\alpha, 1/2\} < \beta \leq 1$ with $\alpha + \beta - 1 > 0$. Then

(i) The generalized Navier-Stokes system (1.1) has a unique small global mild solution in $(X_{\alpha,\infty}^\beta(\mathbb{R}^n))^n$ for all initial data $\mathbf{a}$ with $\nabla \cdot \mathbf{a} = 0$ and $\|\mathbf{a}\|_{(Q_{\alpha,\infty}^{\beta,1}(\mathbb{R}^n))^n}$ being small.

(ii) For any $T \in (0, \infty)$ there is an $\varepsilon > 0$ such that the generalized Navier-Stokes system (1.1) has a unique small mild solution in $(X_{\alpha,\infty}^\beta(\mathbb{R}^n))^n$ on $(0, T) \times \mathbb{R}^n$ when the initial data $\mathbf{a}$ satisfies $\nabla \cdot \mathbf{a} = 0$ and $\|\mathbf{a}\|_{(Q_{\alpha,\infty}^{\beta,1}(\mathbb{R}^n))^n} < \varepsilon$. In particular for all $\mathbf{a} \in (VQ_{\alpha,\infty}^{\beta,1}(\mathbb{R}^n))^n$ with $\nabla \cdot \mathbf{a} = 0$ there exists a unique small local mild solution in $(X_{\alpha,\infty}^\beta(\mathbb{R}^n))^n$ on $(0, T) \times \mathbb{R}^n$.

**Remark 1.6.** The core of the proof of the above theorem is the following inequality: for $u$ and $v \in X_{\alpha,\infty}^\beta(\mathbb{R}^n)$, we have

$$
\|B(u, v)\|_{X_{\alpha,\infty}^\beta(\mathbb{R}^n)} \leq \|u\|_{X_{\alpha,\infty}^\beta(\mathbb{R}^n)} \|v\|_{X_{\alpha,\infty}^\beta(\mathbb{R}^n)}.
$$

The above inequality will play an important role in this paper.

We recall the definition of the Lorentz space $L^{p,\infty}(\mathbb{R}^n)$ for $1 < p < \infty$.

**Definition 1.7.** Let $1 < p < \infty$. A function $f \in L^1_{loc}(\mathbb{R}^n)$ is in the Lorentz space $L^{p,\infty}(\mathbb{R}^n)$ if and only if

$$
\forall \lambda > 0, \quad |\{x \in \mathbb{R}^n, |f(x)| > \lambda\}| \leq \frac{C}{\lambda^p}.
$$

For $p = \infty$, we have $L^{\infty,\infty}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$.

In [10], the authors introduced a new class of Lorentz spaces.

**Definition 1.8.** For $1 < p < \infty$, the weak Lorentz space $\tilde{L}^{p,\infty}((0, T))$ is the adherence of functions in $L^{\infty}((0, T))$ for the norm in the Lorentz space $L^{p,\infty}((0, T))$, that is,

$$
\tilde{L}^{p,\infty}((0, T)) = L^{\infty}((0, T))^{L^{p,\infty}((0, T))}.
$$

Let $\tilde{L}^{p,\infty}((0, T), L^{q,\infty}(\mathbb{R}^3))$ be the space of all measurable functions $f$ on $(0, T) \times \mathbb{R}^3$ such that $\|f(t, \cdot)\|_{L^{q,\infty}(\mathbb{R}^3)} \in \tilde{L}^{p,\infty}((0, T))$. Then the following proposition holds.

**Proposition 1.9.** ([10], Proposition 2.7) Let $T > 0$, $1 < p < \infty$ and $1 < q \leq \infty$. The following properties are equivalent:

1. $f \in \tilde{L}^{p,\infty}((0, T), L^{q,\infty}(\mathbb{R}^3))$;
2. For all $\lambda > 0$, there exists a constant $C(\lambda)$ such that $C(\lambda) \to 0$ as $\lambda \to \infty$ and $\left|\left\{|f(t)|^{q,\infty}(\mathbb{R}^3) > \lambda\}\right| \leq \frac{C(\lambda)}{\lambda^p}$;
3. For all $\varepsilon > 0$, there exists $f_1 \in L^{\infty}((0, T), L^{q,\infty}(\mathbb{R}^3))$ and $f_2 \in \tilde{L}^{p,\infty}((0, T), L^{q,\infty}(\mathbb{R}^3))$ such that $\|f_2\|_{\tilde{L}^{p,\infty}((0, T), L^{q,\infty}(\mathbb{R}^3))} \leq \varepsilon$ and $f = f_1 + f_2$. 
The rest of this paper is organized as follows: In Section 2, we give two technical lemmas: Lemma 2.1 the continuity of the bilinear operator \( B(\cdot, \cdot) \) in Lorentz spaces; Lemma 2.2 a local existence of mild solution to equations (1.1) with initial data in Lorentz spaces. In Section 3, we establish main results of this paper. We only provide a proof for three spatial dimensions, but our proof goes through almost verbatim in higher dimensions.

2. Technical Lemmas

In this section, we prove two preliminary lemmas. The first one can be regarded as an generalization of [10, Lemma 6.1] for the case \( \beta = 1 \).

Lemma 2.1. Let \( T > 0 \) and \( \frac{1}{p} < \beta < 1 \),

\[
B(u, v)(t, x) = \int_0^t e^{-(t-s)(-\Delta)^{\beta}} \nabla (u \otimes v)(s, x) ds.
\]

Let \( \frac{2\beta}{2\beta - 1} \leq p < \infty \) and \( \frac{3}{2\beta - 1} < q \leq \infty \) such that \( \beta - \frac{1}{p} = \frac{\beta}{p} + \frac{3}{2q} \). Then we have

\begin{enumerate}
\item \( B \colon L^p(0, T), L^{q, \infty}(\mathbb{R}^3) \times L^p(0, T), L^{q, \infty}(\mathbb{R}^3) \rightarrow L^p(0, T), L^{q, \infty}(\mathbb{R}^3) \)
\end{enumerate}

with

\[
\|B(u, v)\|_{L^p(0, T), L^{q, \infty}(\mathbb{R}^3)} \leq C \|u\|_{L^p(0, T), L^{q, \infty}(\mathbb{R}^3)} \|v\|_{L^p(0, T), L^{q, \infty}(\mathbb{R}^3)};
\]

\begin{enumerate}
\item \( B \colon L^\infty(0, T), L^{q, \infty}(\mathbb{R}^3) \times L^p(0, T), L^{q, \infty}(\mathbb{R}^3) \rightarrow L^\infty(0, T), L^{q, \infty}(\mathbb{R}^3) \)
\end{enumerate}

with

\[
\|B(u, v)\|_{L^\infty(0, T), L^{q, \infty}(\mathbb{R}^3)} \leq C \|u\|_{L^\infty(0, T), L^{q, \infty}(\mathbb{R}^3)} \|v\|_{L^p(0, T), L^{q, \infty}(\mathbb{R}^3)};
\]

\begin{enumerate}
\item \( B \colon L^\infty(0, T), L^{q, \infty}(\mathbb{R}^3) \times L^\infty(0, T), L^{q, \infty}(\mathbb{R}^3) \rightarrow L^\infty(0, T), L^{q, \infty}(\mathbb{R}^3) \)
\end{enumerate}

with

\[
\|B(u, v)\|_{L^\infty(0, T), L^{q, \infty}(\mathbb{R}^3)} \leq C T^{1/p} \|u\|_{L^\infty(0, T), L^{q, \infty}(\mathbb{R}^3)} \|v\|_{L^\infty(0, T), L^{q, \infty}(\mathbb{R}^3)};
\]

\begin{enumerate}
\item \( B \colon L^\infty(0, T), L^{q, \infty}(\mathbb{R}^3) \times L^p(0, T), L^{q, \infty}(\mathbb{R}^3) \rightarrow L^p(0, T), L^{q, \infty}(\mathbb{R}^3) \)
\end{enumerate}

with

\[
\|B(u, v)\|_{L^p(0, T), L^{q, \infty}(\mathbb{R}^3)} \leq C T^{1/p} \|u\|_{L^\infty(0, T), L^{q, \infty}(\mathbb{R}^3)} \|v\|_{L^\infty(0, T), L^{q, \infty}(\mathbb{R}^3)}.
\]

Proof. The proof of this lemma bases on the following inequality:

\[
(2.1) \quad \|B(u, v)(t)\|_{L^{q, \infty}(\mathbb{R}^3)} \leq \int_0^t (t-s)^{\frac{1}{2}(-1-\frac{3}{q})} \|u(s)\|_{L^{q, \infty}(\mathbb{R}^3)} \|v(s)\|_{L^{q, \infty}(\mathbb{R}^3)} ds.
\]

In fact, since

\[
B(u, v)(t, x) = \int_0^t e^{-(t-s)(-\Delta)^{\beta}} \nabla (u \otimes v)(s, x) ds,
\]

we have

\[
\|B(u, v)(t)\|_{L^{q, \infty}(\mathbb{R}^3)} \leq \int_0^t \|e^{-(t-s)(-\Delta)^{\beta}} \nabla (u \otimes v)(s, x)\|_{L^{q, \infty}(\mathbb{R}^3)} ds.
\]
Since $e^{-u(-\Delta)^{\beta}} \mathbb{P}\nabla$ is a convolution operator, the Young inequality tells us for $1 + \frac{1}{q} = \frac{2}{q} + \frac{1}{r}$,

$$\|B(u, v)(t)\|_{L^q(\mathbb{R}^3)} \leq \int_0^t \|e^{-(t-s)(-\Delta)^{\beta}} \mathbb{P}\nabla\|_{L^q(\mathbb{R}^3)}\|u \otimes v(s)\|_{L^{2/q}(\mathbb{R}^3)} ds
$$

$$\leq \int_0^t \|e^{-(t-s)(-\Delta)^{\beta}} \mathbb{P}\nabla\|_{L^q(\mathbb{R}^3)}\|u(s)\|_{L^{q}(\mathbb{R}^3)}\|v(s)\|_{L^{q}(\mathbb{R}^3)} ds.$$

For $n = 3$, the derivation of the generalized Oseen kernel satisfies

$$\left|e^{-(t-s)(-\Delta)^{\beta}} \mathbb{P}\nabla(x, y)\right| \leq \frac{1}{((t-s)^{\frac{1}{p'}} + |x-y|)^{4}}$$

(See [13, Lemma 4.10]). Then we can get

$$\|e^{-(t-s)(-\Delta)^{\beta}} \mathbb{P}\nabla\|_{L^q(\mathbb{R}^3)} \leq \|e^{-(t-s)(-\Delta)^{\beta}} \mathbb{P}\nabla\|_{L^q(\mathbb{R}^3)}$$

$$\leq \left(\int_{\mathbb{R}^3} (t-s)^{\frac{1}{p'}} + |x-y|)^{4r} dy\right)^{1/r}
$$

$$= \left(\int_0^\infty (t-s)^{\frac{1}{p'}} (|x-y|/t^{\frac{1}{p'}})^2 \right)^{1/r}
$$

$$= (t-s)^{\frac{1}{p'} - \frac{1}{p}} \left(\int_0^\infty \frac{\lambda^2}{(1 + \lambda)^{4r}} d\lambda\right)^{1/r}
$$

$$\leq C(t-s)^{\frac{1}{p'}(\beta - 2)}.$$

For $1 = \frac{1}{r} + \frac{1}{q}$, we have

$$\|B(u, v)(t)\|_{L^q(\mathbb{R}^3)} \leq \int_0^t (t-s)^{\frac{1}{p'}(\beta - 2)} \|u(s)\|_{L^q(\mathbb{R}^3)}\|v(s)\|_{L^{q}(\mathbb{R}^3)} ds
$$

$$= \int_0^t (t-s)^{\frac{1}{p'}(-1 + \frac{1}{p})} \|u(s)\|_{L^q(\mathbb{R}^3)}\|v(s)\|_{L^{q}(\mathbb{R}^3)} ds.$$ This completes the proof of (2.1). Now we prove (1) by using (2.1). (1). Since $\beta - \frac{1}{2} = \frac{\beta}{p} + \frac{3}{2q}$, we know $1 + \frac{1}{p} = \frac{1}{p} + \frac{1}{q} + \frac{1}{r_1}$, with $\frac{1}{r_1} = \frac{1}{2p} + \frac{3}{23q}$. By Young’s inequality, we get

$$\|B(u, v)\|_{L^p((0, T), L^q(\mathbb{R}^3))} \leq \|s^{-\frac{1}{r_1}} - \frac{3}{2q} \|_{L^{r_1}(0, T)} \|u\|_{L^p((0, T), L^q(\mathbb{R}^3))} \|v\|_{L^p((0, T), L^q(\mathbb{R}^3))}.$$ Now we compute the norm $\|s^{-\frac{1}{r_1}}\|_{L^{r_1}(0, T)}$, where

$$\|f\|_{L^{r_1}(0, T)} = \sup_{\lambda} \lambda \left\{ t \in (0, T), |f(t)| > \lambda \right\}^{1/p}.$$ If $s \in (0, T)$ and $T < \frac{1}{\lambda_{r_1}}$, then

$$\lambda \left\{ t \in (0, T), s^{-\frac{1}{r_1}} > \lambda \right\}^{1/r_1} \leq \lambda T^{1/r_1} \leq 1.$$

If $T > \frac{1}{\lambda_{r_1}}$, then $T^{-\frac{1}{r_1}} < \lambda$. For $s^{-\frac{1}{r_1}} > \lambda$, we can find a $s_0$ such that $s_0 = \frac{1}{\lambda_{r_1}}$. When $0 < s < s_0$, $s^{-\frac{1}{r_1}} > s_0^{-\frac{1}{r_1}} = \lambda$, then we have

$$\lambda \left\{ s \in (0, T) : s^{-\frac{1}{r_1}} > \lambda \right\}^{1/r_1} \leq \lambda s_0^{1/r_1} = 1.$$
Therefore we get \( s^{-\frac{1}{p}} \in L^{r_1,\infty}((0,T)) \) and \( \|s^{-\frac{1}{r_1}}\|_{L^{r_1,\infty}(0,T)} \leq 1 \).

(2). It follows from (2.1) that
\[
\|B(u,v)(t)\|_{L^q,\infty}(\mathbb{R}^3) \leq \|u\|_{L^\infty((0,T),L^q,\infty)(\mathbb{R}^3)} \int_0^t (t-s)^{-\frac{1}{r_1} - \frac{3}{2q}} \|v(s)\|_{L^\infty(\mathbb{R}^3)} ds.
\]

Then we obtain
\[
\|B(u,v)(t)\|_{L^\infty((0,T),L^q,\infty)(\mathbb{R}^3)} \leq \|u\|_{L^\infty((0,T),L^q,\infty)(\mathbb{R}^3)} \sup_{0<t<T} \int_0^t (t-s)^{-\frac{1}{r_1} - \frac{3}{2q}} \|v(s)\|_{L^\infty(\mathbb{R}^3)} ds.
\]

By Hölder’s inequality with \( 1 = \frac{1}{p} + \frac{1}{2q} + \frac{3}{2q} \), we have
\[
\|B(u,v)(t)\|_{L^\infty((0,T),L^q,\infty)(\mathbb{R}^3)} \leq C\|u\|_{L^\infty((0,T),L^q,\infty)(\mathbb{R}^3)} \|v\|_{L^p,\infty((0,T),L^q,\infty)(\mathbb{R}^3)}.
\]

(3). By (2.1), we get
\[
\|B(u,v)\|_{L^\infty((0,T),L^q,\infty)(\mathbb{R}^3)} \leq \|u\|_{L^\infty((0,T),L^q,\infty)(\mathbb{R}^3)} \|v\|_{L^\infty((0,T),L^q,\infty)(\mathbb{R}^3)} \int_0^t (t-s)^{-\frac{1}{r_1} - \frac{3}{2q}} ds
\]
\[
\leq CT^{1/p}\|u\|_{L^\infty((0,T),L^q,\infty)(\mathbb{R}^3)} \|v\|_{L^\infty((0,T),L^q,\infty)(\mathbb{R}^3)}.
\]

(4). (2.1) and Young’s inequality with \( 1 + \frac{1}{p} = \frac{1}{p} + \frac{1}{2q} + \frac{3}{2q} \) imply that
\[
\|B(u,v)(t)\|_{L^\infty(\mathbb{R}^3)} \leq \int_0^t (t-s)^{-\frac{1}{r_1} - \frac{3}{2q}} \|u(s)\|_{L^\infty(\mathbb{R}^3)} \|v(s)\|_{L^\infty(\mathbb{R}^3)} ds
\]
\[
\leq \|u\|_{L^\infty((0,T),L^\infty(\mathbb{R}^3))} \left( \int_0^t (t-s)^{-\frac{1}{r_1} - \frac{3}{2q}} \|v(s)\|_{L^\infty(\mathbb{R}^3)} ds \right).
\]
Since \( \|f * g\|_{p,\infty} \leq \|f\|_{p,\infty} \|g\|_1 \), we have
\[
\|B(u,v)\|_{L^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3))} \leq \|u\|_{L^\infty((0,T),L^q,\infty)(\mathbb{R}^3)} \|v\|_{L^\infty((0,T),L^q,\infty)(\mathbb{R}^3)} \sup_{0<t<T} \int_0^t s^{-\frac{1}{r_1} - \frac{1}{p}} ds
\]
\[
\leq CT^{1/p}\|u\|_{L^\infty((0,T),L^q,\infty)(\mathbb{R}^3)} \|v\|_{L^\infty((0,T),L^q,\infty)(\mathbb{R}^3)}.
\]

This completes the proof of Lemma 2.1.

We need the following local existence of solution to equations (1.1) with initial data in Lorentz spaces.

**Lemma 2.2.** Let \( \frac{1}{2} < \beta \leq 1 \), \( \frac{3}{2\beta - 1} < q \leq \infty \), \( \frac{2\beta}{2\beta - 1} < p \leq \infty \) and \( u_0 \in L^{q,\infty}(\mathbb{R}^3) \).

For \( T > 0 \) such that \( 4T^{1/p}\|u_0\|_{L^\infty(\mathbb{R}^3)} < 1 \), there exists a mild solution \( u \in L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3)) \) to equations (1.1), which is unique in the ball centered at \( 0 \), of radius \( 2\|u_0\|_{L^{q,\infty}(\mathbb{R}^3)} \).

**Proof.** We construct \( \{e_n\} \) as follows:

\[
e_{n+1} = e_0 - B(e_n, e_n),
\]
\[
e_0 = e^{-t(-\Delta)^\beta} u_0.
\]
We claim that \( \|e_n\|_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))} \leq 2\|u_0\|_{L^{q,\infty}(\mathbb{R}^3)} \). For \( n = 0 \), by Young’s inequality, we have
\[
\|e_0\|_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))} = \|e^{-t(-\Delta)^{3/4}}u_0\|_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))} \\
= \sup_{t \in (0,T)} \|e^{-t(-\Delta)^{3/4}}u_0\|_{L^{q,\infty}(\mathbb{R}^3)} \\
\leq \|e^{-t(-\Delta)^{3/4}}\|_{L^1(\mathbb{R}^3)} \|u_0\|_{L^{q,\infty}(\mathbb{R}^3)} \\
\leq \left( \int_{\mathbb{R}^3} \frac{1}{t^{3/4}} \left( 1 + \frac{|x|^2}{t^{3/2}} \right) dy \right) \|u_0\|_{L^{q,\infty}(\mathbb{R}^3)} \\
\leq 2\|u_0\|_{L^{q,\infty}(\mathbb{R}^3)}.
\]
Assume that the estimate is true for some \( n \in \mathbb{N} \). For \( n + 1 \), we get
\[
\|e_{n+1}\|_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))} \leq \|e_0\|_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))} + \|B(e_n, e_n)\|_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))} \\
\leq \|e_n\|_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))} + T^{1/p}\|e_n\|_{L^{q,\infty}(\mathbb{R}^3)}^2 + 4T^{1/p}\|u_0\|_{L^{q,\infty}(\mathbb{R}^3)}^2 \\
\leq 2\|u_0\|_{L^{q,\infty}(\mathbb{R}^3)}.
\]
This tells us
\[
\|e_{n+1} - e_n\|_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))} = \|B(e_n, e_n) - B(e_n-1, e_n-1)\|_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))} \\
\leq T^{1/p}\|e_n - e_n-1\|_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))} (\|e_n\|_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))} + \|e_{n-1}\|_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))}) \\
\leq 4T^{1/p}\|u_0\|_{L^{q,\infty}(\mathbb{R}^3)}(\|e_n - e_{n-1}\|_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))}).
\]
Since \( 4T^{1/p}\|u_0\|_{L^{q,\infty}(\mathbb{R}^3)} < 1 \), the Picard contraction principle guarantees this lemma.

3. Main Results

In this section, we state and prove our main results. First, we need the following proposition which generalizes the case \( \beta = 1 \) established by Lemarié-Rieusset and Prioux [10].

**Proposition 3.1.** Let \( T > 0 \), \( \frac{1}{2} < \beta < 1 \) and \( u, v \) be two mild solutions to the equations \((1, 7)\) belonging to the space \( \mathcal{L}^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3)) \) with \( \frac{3}{2\beta - 1} < p < \infty \) and \( \frac{3}{2\beta - 1} < q < \infty \) such that \( \beta - \frac{1}{2} = \frac{\beta}{p} + \frac{3}{2q} \). Assume that there exists \( \theta \in (0,T) \) such that \( u(\theta) = v(\theta) \). Then \( u \) and \( v \) are equal for \( t \in (\theta, T] \).

**Proof.** Let \( t_0 > 0 \) and \( \lambda > 0 \). We can split \( u \) and \( v \) into:
\[
u = u_\lambda + u'_\lambda \quad \text{and} \quad v = v_\lambda + v'_\lambda
\]
where \( u_\lambda = u\chi_{\{t\|u(t)\|_{L^{q,\infty}(\mathbb{R}^3)} > \lambda\}} \) and \( v_\lambda = v\chi_{\{t\|v(t)\|_{q,\infty}(\mathbb{R}^3)} > \lambda\} \). By construction and the definition of the Lorentz spaces (see Proposition 1.3) we have
\[
\|u'_\lambda\|_{L^\infty((\theta,\theta+t_0),L^{q,\infty}(\mathbb{R}^3))} \leq \lambda, \quad \|u_\lambda\|_{L^{p,\infty}((\theta,\theta+t_0),L^{q,\infty}(\mathbb{R}^3))} \leq C(\lambda)
\]
and the same estimates hold true for $v'_\lambda$ and $v_\lambda$. Then, we compute by Lemma 2.11,
\[
\begin{align*}
&\|u - v\|_{L^p,\infty((\theta, \theta + t_0), L^q,\infty(\mathbb{R}^3))} \\
\leq &\ |B(u, u) - B(u, v)|_{L^p,\infty((\theta, \theta + t_0), L^q,\infty(\mathbb{R}^3))} \\
\leq &\ C_0 \|B(u - v, u)\|_{L^p,\infty((\theta, \theta + t_0), L^q,\infty(\mathbb{R}^3))} + C_0 \|B(v, u - v)\|_{L^p,\infty((\theta, \theta + t_0), L^q,\infty(\mathbb{R}^3))} \\
\leq &\ C \|u - v\|_{L^p,\infty((\theta, \theta + t_0), L^q,\infty(\mathbb{R}^3))} \left( \|u\|_{L^p,\infty((\theta, \theta + t_0), L^q,\infty(\mathbb{R}^3))} + \|v\|_{L^p,\infty((\theta, \theta + t_0), L^q,\infty(\mathbb{R}^3))} \right).
\end{align*}
\]

Since $u = u_\lambda + u'_\lambda$ with $\|u_\lambda\|_{L^p,\infty((\theta, \theta + t_0), L^q,\infty(\mathbb{R}^3))} \leq C(\lambda)$ and $\|u'_\lambda\|_{L^p,\infty((\theta, \theta + t_0), L^q,\infty(\mathbb{R}^3))} \leq \lambda$, we get $\|u\|_{L^p,\infty((\theta, \theta + t_0), L^q,\infty(\mathbb{R}^3))} \leq C(\lambda) + \lambda t_0^{1/p}$. The same estimate holds for $v$. Hence we can obtain
\[
\|u - v\|_{L^p,\infty((\theta, \theta + t_0), L^q,\infty(\mathbb{R}^3))} \leq C_0 \left(2C(\lambda) + 2\lambda t_0^{1/p}\right) \|u - v\|_{L^p,\infty((\theta, \theta + t_0), L^q,\infty(\mathbb{R}^3))}.
\]

We choose $\lambda > 0$ large enough to guarantee $2C_0 C(\lambda) < 1/4$ and choose $t_0 > 0$ small enough such that $C_0 t_0^{1/p} < 1/4$. Thus there exists $\delta < 1$ satisfies
\[
\|u - v\|_{L^p,\infty((\theta, \theta + t_0), L^q,\infty(\mathbb{R}^3))} \leq \delta \|u - v\|_{L^p,\infty((\theta, \theta + t_0), L^q,\infty(\mathbb{R}^3))}.
\]

So $u = v$ for $t \in (\theta, \theta + t_0)$. For $T$, there exists $n$ such that $T < \theta + nt_0$. Thus $u = v$ for $t \in (\theta, T)$. \qed

**Lemma 3.2.** ([10] Proposition 2.9) Let $T > 0$ and $1 \leq p, q \leq \infty$. If $u$ satisfies
\[
(3.1) \begin{cases}
\sup_{t \in (0, T)} t^{1/p} \|u(t)\|_{L^q,\infty(\mathbb{R}^3)} < \infty, \\
t^{1/p} \|u(t)\|_{L^q,\infty(\mathbb{R}^3)} \rightarrow 0, \text{ as } t \rightarrow 0,
\end{cases}
\]
then the function $u$ belongs to the space $\bar{L}^p,\infty((0, T), L^q,\infty(\mathbb{R}^3))$.

To establish the equivalence between the mild and mollified solution to the (GNS) equations, we need the following lemma.

**Lemma 3.3.** Let $v \in D((0, T] \times B(0, R))$ and $R > 0$ such that supp $v \subset (0, T] \times B(0, R)$ where supp denotes the supports of the function $v$ and $B(0, R)$ the closed ball of radius $R$ centered at $0$. Then, for $t \in (0, T)$ and $y \in \mathbb{R}^3$ such that $|y| \geq \lambda R$ for $\lambda > 1$, we have for some constant $C > 0$,
\[
|B(u, v)(t, y)| \leq \frac{C}{(\lambda R)^2} \|u\|^2_{L^2((0, T] \times \mathbb{R}^3)}.
\]
Proof. Since \(|y| \geq \lambda R > \lambda |z|, \ |y - z| \geq (1 - \frac{1}{\lambda}) |y|\). Then, we can get

\[
|B(u, u)(t, y)| = \left| \int_0^t e^{-(t-s)(-\Delta)^\beta} \nabla (u \otimes u) ds \right|
\leq C \int_0^t \int_{\mathbb{R}^3} \frac{1}{((t-s)^{\frac{1}{2\beta}} + |z-y|)^4} |v(s, z)|^2 ds dz
\leq C \int_0^t \int_{\mathbb{B}(0, R)} |z-y|^4 |v(s, z)|^2 ds dz
\leq C \int_0^t \int_{\mathbb{B}(0, R)} |v(s, z)|^2 ds dz
\leq C \frac{1}{|y|} \int_0^t \int_{\mathbb{B}(0, R)} |v(s, z)|^2 ds dz
= C \frac{1}{|y|} \|v\|_{L^2((0, T) \times \mathbb{R}^3)}^2 \lesssim \frac{1}{|\lambda R|^4} \|v\|_{L^2((0, T) \times \mathbb{R}^3)}^2.
\]

\[
\square
\]

Theorem 3.4. Let \(\alpha > 0\), max \(\{\frac{1}{2}, \alpha\} < \beta \leq 1\) with \(\alpha + \beta - 1 \geq 0\) and let \(u_0 \in D(\mathbb{R}^3))\) such that \(v \cdot u_0 = 0\) and \(T > 0\) small enough to ensure \(\|e^{-t(-\Delta)^\beta} u_0\|_{X^\beta_{\alpha, T}(\mathbb{R}^3)} < \frac{1}{\lambda C}\). Then there exists a mild solution \(u \in D(\mathbb{R}^3)\) to equations (1.1).

Proof. We construct \(\{v_n\}_{n \in \mathbb{N}}\) by

\[
\begin{cases}
    v_n = v_0 - B(v_{n-1}, v_{n-1}), & \text{for } n \geq 1, \\
    v_0 = e^{-t(-\Delta)^\beta} u_0.
\end{cases}
\]

For \(n = 0\). By assumption, if \(u_0 \in D(\mathbb{R}^3))\) such that \(\|u_0 - u_n\|_{Q_{\alpha, T}^\beta(\mathbb{R}^3)} \rightarrow 0\) as \(m \rightarrow \infty\). From the definition of \(Q_{\alpha, T}^{\beta, -1}(\mathbb{R}^3)\), if \(f \in Q_{\alpha, T}^{\beta, -1}(\mathbb{R}^3)\),

\[
\sup_{0 < t^{\frac{1}{\alpha}} < T} \sup_{x_0 \in \mathbb{R}^3} t^{2\alpha - 3 + \frac{\beta}{\alpha} - 2} \int_0^{t^{\frac{1}{\alpha}}} \int_{|x-x_0| < t} |e^{-s(-\Delta)^\beta} f(x)|^2 ds dx < \infty.
\]

Hence, as \(m \rightarrow \infty\),

\[
\sup_{0 < t^{\frac{1}{\alpha}} < T} \sup_{x_0 \in \mathbb{R}^3} t^{2\alpha - 3 + \frac{\beta}{\alpha} - 2} \int_0^{t^{\frac{1}{\alpha}}} \int_{|x-x_0| < t} |e^{-s(-\Delta)^\beta} (u_0 - u_m)(x)|^2 ds dx \rightarrow 0.
\]

From the embedding: \(Q_{\alpha, T}^{\beta, -1}(\mathbb{R}^3) \hookrightarrow B_{1, \infty}^{2\beta}(\mathbb{R}^3)\) (see [2] Theorem 4.6), we obtain

\[
t^{\frac{\beta}{\alpha}} \|e^{-t(-\Delta)^\beta} f\|_{L^\infty(\mathbb{R}^3)} \leq C \|f\|_{Q_{\alpha, T}^{\beta, -1}(\mathbb{R}^3)}.
\]

By the definition of \(X_{\alpha, T}^{\beta}(\mathbb{R}^3)\), we get \(\|e^{-t(-\Delta)^\beta} f\|_{X_{\alpha, T}^{\beta}(\mathbb{R}^3)} \leq C \|f\|_{Q_{\alpha, T}^{\beta, -1}(\mathbb{R}^3)}\). Then we have

\[
\|e^{-t(-\Delta)^\beta} (u_0 - u_m)\|_{X_{\alpha, T}^{\beta}(\mathbb{R}^3)} \leq C \|u_0 - u_m\|_{Q_{\alpha, T}^{\beta, -1}(\mathbb{R}^3)}
\]
and \(\|e^{-(\Delta)^{\beta}}(u_0-u_0^m)\|_{X_{\alpha,T}^{\beta-1}(\mathbb{R}^3)} \to 0\) as \(m \to 0\). So \(e^{-(\Delta)^{\beta}}u_0 = \lim_{m \to \infty} e^{-(\Delta)^{\beta}}u_0^m\) in \(X_{\alpha,T}^{\beta}(\mathbb{R}^3)\). It follows from \(e^{-(\Delta)^{\beta}}u_0^m \in \mathcal{D}((0, T) \times \mathbb{R}^3)\) that

\[ v_0 = e^{-(\Delta)^{\beta}}u_0 \in \mathcal{D}((0, T) \times \mathbb{R}^3) X_{\alpha,T}^{\beta}(\mathbb{R}^3). \]

Let us assume \(v_{n-1} \in \mathcal{D}((0, T) \times \mathbb{R}^3) X_{\alpha,T}^{\beta}(\mathbb{R}^3)\). For \(v_n\), since \(v_n = e^{-(\Delta)^{\beta}}u_0 - B(v_{n-1}, v_{n-1})\),

\[ u_0 \in \mathcal{D}(\mathbb{R}^3) \Rightarrow v_0 = e^{-(\Delta)^{\beta}}u_0 \in \mathcal{D}((0, T) \times \mathbb{R}^3) X_{\alpha,T}^{\beta}(\mathbb{R}^3). \]

We only need to prove \(B(v_{n-1}, v_{n-1}) \in \mathcal{D}((0, T) \times \mathbb{R}^3)\) such that

\[ \|v_{n-1} - v_{n-1}^m\|_{X_{\alpha,T}^{\beta}(\mathbb{R}^3)} \to 0, \quad \text{as} \quad m \to \infty. \]

Since \(v_{n-1}^m\) is compact supported in time and space, we have \(B(v_{n-1}^m, v_{n-1}^m) \in C^\infty((0, T] \times \mathbb{R}^3)\) and is of compact support in time. Let \(\{\varphi_m\}_{m \in \mathbb{N}}\) be a sequence of functions in \(\mathcal{D}(\mathbb{R}^3)\) such that for each \(m \in \mathbb{N}\), \(\varphi_m \in \mathcal{D}(\mathbb{R}^3)\) and \(\varphi_m(x) = 1\) if \(x \in B(0, \lambda_m R_m)\) where \(R_m > 0\) is such that \(\sup \varphi_m^m \subset (0, T) \times B(0, R_m)\) and \(\lambda_m > \|v_{n-1}^m\|_{L^2((0,T) \times \mathbb{R}^3)}^{1/2}\). We denote \(B^m(v_{n-1}, v_{n-1}) = \varphi_m \times B(v_{n-1}^m, v_{n-1}^m)\) and get

\[ \|B(v_{n-1}, v_{n-1}) - B^m(v_{n-1}^m, v_{n-1}^m)\|_{X_{\alpha,T}^{\beta}(\mathbb{R}^3)} \]

\[ \leq \|B(v_{n-1}, v_{n-1}) - B(v_{n-1}^m, v_{n-1}^m)\|_{X_{\alpha,T}^{\beta}(\mathbb{R}^3)} + \|((1 - \varphi_m)B(v_{n-1}^m, v_{n-1}^m))\|_{X_{\alpha,T}^{\beta}(\mathbb{R}^3)} \]

\[ \leq C\|v_{n-1} - v_{n-1}^m\|_{X_{\alpha,T}^{\beta}(\mathbb{R}^3)} \left[ \|v_{n-1}\|_{X_{\alpha,T}^{\beta}(\mathbb{R}^3)} + \|v_{n-1}^m\|_{X_{\alpha,T}^{\beta}(\mathbb{R}^3)} \right] \]

\[ + \|((1 - \varphi_m)B(v_{n-1}^m, v_{n-1}^m))\|_{X_{\alpha,T}^{\beta}(\mathbb{R}^3)}. \]

Since \(\varphi_m\) is supported on \(B(0, \lambda_m R_m + 1)\) and \(\varphi_m = 1\) on \(B(0, \lambda_m R_m)\), \((1 - \varphi_m(y))\) is supported on \(B^c(0, \lambda_m R_m)\) and \(\varphi_m = 1\) on \(B(0, \lambda_m R_m)\). Then, we obtain

\[ \|(1 - \varphi_m)B(v_{n-1}^m, v_{n-1}^m)\|_{X_{\alpha,T}^{\beta}(\mathbb{R}^3)} \]

\[ \leq \sup_{t \in (0,T)} \frac{1}{\lambda_m R_m^4} \frac{1}{\lambda_m R_m^2} \left[ \|v_{n-1}^m\|_{L^2((0,T) \times \mathbb{R}^3)} \right] \]

\[ + \sup_{t \in (0,T)} \sup_{x_0 \in \mathbb{R}^3} \left( \int_0^t \int_{|y - x_0| < 1} \right) \left(1 - \varphi_m\right)B(v_{n-1}^m, v_{n-1}^m)(s, y) ds dy \]

\[ \leq C \frac{1}{\lambda_m R_m^4} \frac{1}{\lambda_m R_m^2} \left[ \|v_{n-1}^m\|_{L^2((0,T) \times \mathbb{R}^3)} \right] \]

\[ + \sup_{t \in (0,T)} \sup_{x_0 \in \mathbb{R}^3} \left( \int_0^t \int_{|y - x_0| < 1} \right) \left(1 - \varphi_m\right)B(v_{n-1}^m, v_{n-1}^m)(s, y) ds dy \]

\[ \leq C T \frac{1}{\lambda_m R_m^4} \frac{1}{\lambda_m R_m^2} \left[ \|v_{n-1}^m\|_{L^2((0,T) \times \mathbb{R}^3)} \right] \]

\[ + \sup_{t \in (0,T)} \sup_{x_0 \in \mathbb{R}^3} \left( \int_0^t \int_{|y - x_0| < 1} \right) \left(1 - \varphi_m\right)B(v_{n-1}^m, v_{n-1}^m)(s, y) ds dy \]

\[ \leq C T \frac{1}{\lambda_m R_m^4} \frac{1}{\lambda_m R_m^2} \left[ \|v_{n-1}^m\|_{L^2((0,T) \times \mathbb{R}^3)} \right] \]

\[ \leq C T \frac{1}{(mR_m)^4} \to 0 \quad \text{as} \quad m \to \infty. \]
Theorem 3.5. Let $\alpha > 0$, max $\left\{ \frac{1}{2}, \alpha \right\} < \beta \leq 1$ with $\alpha + \beta - 1 \geq 0$ and let $u_0 \in \overline{D(\mathbb{R}^3)}^{X^\beta_{\alpha,T}(\mathbb{R}^3)}$ such that $\nabla \cdot u_0 = 0$ and $T > 0$ is small enough to ensure $\| e^{-t(-\Delta)^\beta} u_0 \|_{X^\beta_{\alpha,T}(\mathbb{R}^3)} < \frac{1}{4C}$. Then for $\varepsilon > 0$, there exists a solution $u_\varepsilon \in \overline{D((0,T] \times \mathbb{R}^3)}^{X^\beta_{\alpha,T}(\mathbb{R}^3)}$ to the mollified generalized Navier-Stokes equations (1.3).
Proof. We only need to prove \( \|f * \omega_{\varepsilon}\|_{X^\beta_{\alpha,T}(\mathbb{R}^3)} \leq \|f\|_{X^\beta_{\alpha,T}(\mathbb{R}^3)} \). In fact, we have \( \|\omega_{\varepsilon} * f\|_{L^\infty(\mathbb{R}^3)} \leq \|\omega_{\varepsilon}\|_{L^1(\mathbb{R}^3)} \|f\|_{L^\infty(\mathbb{R}^3)} \) and

\[
\left( \int_0^{2\alpha - 3 + 2\beta - 2} \frac{dtdx}{t^{\alpha/\beta}} \right)^{1/2} \leq \left( \int_0^{2\alpha - 3 + 2\beta - 2} \frac{dtdx}{t^{\alpha/\beta}} \right)^{1/2},
\]

which converges strongly as \( \varepsilon \) tends to 0,

\[
\leq \int_{\mathbb{R}^3} |\omega_{\varepsilon}(y)| \left( \int_0^{2\alpha - 3 + 2\beta - 2} \frac{dtdx}{t^{\alpha/\beta}} \right)^{1/2} dx dy.
\]

Hence we have

\[
\|\omega_{\varepsilon} * f\|_{X^\beta_{\alpha,T}(\mathbb{R}^3)} \leq \|f\|_{X^\beta_{\alpha,T}(\mathbb{R}^3)},
\]

Similar to the proof of Theorem 3.4, we can complete the proof.

\[\square\]

Theorem 3.6. For \( \alpha > 0 \), max \( \left\{ \frac{1}{2}, \beta \right\} < \beta \leq 1 \) with \( \alpha + \beta - 1 \geq 0 \), let \( u_0 \in D(\mathbb{R}^3) \cap X^{\beta}_{\alpha,T}(\mathbb{R}^3) \) and \( T > 0 \) be given in Theorem 3.4. Then the sequence of solutions \( \{u_{\varepsilon}\}_{\varepsilon > 0} \) to the mollified equations \( (1.3) \) obtained by Theorem 3.3 converges strongly, as \( \varepsilon \) tends to 0, to the mild solution \( u \) to equations \( (1.7) \) obtained by Picard contraction principle, of Theorem 3.4.

Proof. For the bilinear form \( B(u, v) \), we have

\[
u - u_{\varepsilon} = B(u, u) - B_x(u_{\varepsilon}, u_{\varepsilon}) = B(u, u) - B(u_{\varepsilon} * \omega_{\varepsilon}, u_{\varepsilon}) = B(u, u - u_{\varepsilon}) + B(u - (u * \omega_{\varepsilon}), u_{\varepsilon}) + B((u - u_{\varepsilon}) * \omega_{\varepsilon}, u_{\varepsilon})
\]

and

\[
\|u - u_{\varepsilon}\|_{X^\beta_{\alpha,T}(\mathbb{R}^3)} \leq C \|u\|_{X^\beta_{\alpha,T}(\mathbb{R}^3)} \|u - u_{\varepsilon}\|_{X^\beta_{\alpha,T}(\mathbb{R}^3)} + C \|u - (u * \omega_{\varepsilon})\|_{X^\beta_{\alpha,T}(\mathbb{R}^3)} \|u_{\varepsilon}\|_{X^\beta_{\alpha,T}(\mathbb{R}^3)} + C \|(u - u_{\varepsilon}) * \omega_{\varepsilon}\|_{X^\beta_{\alpha,T}(\mathbb{R}^3)} \|u_{\varepsilon}\|_{X^\beta_{\alpha,T}(\mathbb{R}^3)}
\]

where \( A_3 \leq C \|\omega_{\varepsilon}\|_{L^1(\mathbb{R}^3)} \|u - u_{\varepsilon}\|_{X^\beta_{\alpha,T}(\mathbb{R}^3)} \|u_{\varepsilon}\|_{X^\beta_{\alpha,T}(\mathbb{R}^3)} \). Hence we have

\[
\|u - u_{\varepsilon}\|_{X^\beta_{\alpha,T}(\mathbb{R}^3)} \leq 2C \|u\|_{X^\beta_{\alpha,T}(\mathbb{R}^3)} \|u_{\varepsilon}\|_{X^\beta_{\alpha,T}(\mathbb{R}^3)} + 2C \|u - (u * \omega_{\varepsilon})\|_{X^\beta_{\alpha,T}(\mathbb{R}^3)} \|u_{\varepsilon}\|_{X^\beta_{\alpha,T}(\mathbb{R}^3)} + 4C |e^{-t(-\Delta)^{\beta/2}}| u_0|X^\beta_{\alpha,T}(\mathbb{R}^3)| \|u - u_{\varepsilon}\|_{X^\beta_{\alpha,T}(\mathbb{R}^3)} + 2C |e^{-t(-\Delta)^{\beta/2}}| u_0|X^\beta_{\alpha,T}(\mathbb{R}^3)| \|u - (u * \omega_{\varepsilon})\|_{X^\beta_{\alpha,T}(\mathbb{R}^3)}.
\]
This tells us
\[ \| u - u_\varepsilon \|_{X^\alpha_{0, T}(\mathbb{R}^3)} \leq \frac{2C \| e^{-t(-\Delta)^\beta} u_0 \|_{X^\alpha_{0, T}(\mathbb{R}^3)}}{1 - 4C \| e^{-t(-\Delta)^\beta} u_0 \|_{X^\alpha_{0, T}(\mathbb{R}^3)}} \| u - (u * \omega_\varepsilon) \|_{X^\alpha_{0, T}(\mathbb{R}^3)}. \]

Since \( \omega_\varepsilon \in \mathcal{D}(\mathbb{R}^3), \omega_\varepsilon * u \in \mathcal{D}(\mathbb{R}^3 \times (0, T)) \). Thus, for \( u \in \mathcal{D}(\mathbb{R}^3 \times (0, T)) \),
\[ \| u - (u * \omega_\varepsilon) \|_{X^\alpha_{0, T}(\mathbb{R}^3)} \to 0 \quad \text{as} \quad \varepsilon \to 0. \]

Now we recall a class of weak Besov spaces which can be found in [10].

**Definition 3.7.** Let \( \alpha > 0, 1 < q < \infty \). We denote by \( \widetilde{B}^{-\alpha, \infty}_q(\mathbb{R}^3) \) the adherence of functions in \( L^q(\mathbb{R}^3) \) for the norm of \( B^{-\alpha, \infty}_q(\mathbb{R}^3) \) and by \( \widetilde{B}^{-\alpha, \infty}_q(\mathbb{R}^3) \) for functions in \( L^{\infty}(\mathbb{R}^3) \) for the norm of \( B^{-\alpha, \infty}_q(\mathbb{R}^3) = B^{3 - \alpha}_q(\mathbb{R}^3) \), that is,
\[ \widetilde{B}^{-\alpha, \infty}_q(\mathbb{R}^3) = \left\{ u \in L^q(\mathbb{R}^3) \mid \sup_{t > 0} \left\| t^{\alpha/2} e^{-t(-\Delta)^\beta} u(t) \right\|_{L^q(\mathbb{R}^3)} < \infty \right\}, \]
and by \( \widetilde{B}^{-\alpha, \infty}_q(\mathbb{R}^3) \) for functions in \( L^{\infty}(\mathbb{R}^3) \) such that
\[ \left\| (u - u_n)(t) \right\|_{B^{-\alpha, \infty}_q(\mathbb{R}^3)} \to 0, \quad \text{as} \quad t \to 0. \]

**Lemma 3.8.** Let \( \frac{1}{2} < \beta < 1 \) and let \( \alpha > 0 \) and \( 1 < q < \infty \). If \( u \in \widetilde{B}^{-\alpha, \infty}_q(\mathbb{R}^3) \), then
\[
\begin{align*}
\sup_{0 < t < 1} t^{\alpha/2} \| e^{-t(-\Delta)^\beta} u(t) \|_{L^q(\mathbb{R}^3)} < \infty \\
\sup_{t > 0} t^{\alpha/2} \| e^{-t(-\Delta)^\beta} u(t) \|_{L^q(\mathbb{R}^3)} \to \infty, \quad \text{as} \quad t \to 0.
\end{align*}
\]

**Proof.** Since \( u \in \widetilde{B}^{-\alpha, \infty}_q(\mathbb{R}^3) \), we have \( u \in B^{-\alpha, \infty}_q(\mathbb{R}^3) \). Then
\[
\sup_{t > 0} t^{\alpha/2} \| e^{-t(-\Delta)^\beta} u(t) \|_{L^q(\mathbb{R}^3)} < \infty
\]
and there exists a sequence \( \{u_n\}_{n \in \mathbb{N}} \) of functions in \( L^{\infty}(\mathbb{R}^3) \) such that
\[ \| (u_n - u)(t) \|_{B^{-\alpha, \infty}_q(\mathbb{R}^3)} \to 0, \quad \text{as} \quad t \to 0. \]

So there exists \( N > 0 \) such that for \( n > N \),
\[ \sup_{t > 0} t^{\alpha/2} \| e^{-t(-\Delta)^\beta} (u_n - u)(t) \|_{L^q(\mathbb{R}^3)} < \frac{\varepsilon}{2}. \]

Then for all \( t > 0 \) we have, by Young’s inequality,
\[ t^{\alpha/2} \| e^{-t(-\Delta)^\beta} u(t) \|_{L^q(\mathbb{R}^3)} \leq \frac{\varepsilon}{2} + t^{\alpha/2} \| e^{-t(-\Delta)^\beta} u_{N+1}(t) \|_{L^q(\mathbb{R}^3)} < \frac{\varepsilon}{2} + Ct^{\alpha/2} \| u_{N+1}(t) \|_{L^q(\mathbb{R}^3)}. \]

Let \( t_0 = \varepsilon^{2/\beta} (2C \| u_{N+1}(t) \|_{L^q(\mathbb{R}^3)})^{-(2\beta/\alpha)} \), we see that for \( t < t_0 \),
\[ t^{\alpha/2} \| e^{-t(-\Delta)^\beta} u(t) \|_{L^q(\mathbb{R}^3)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \]

The following result gives us a condition for initial data under which the solution to equations (1.1) for \( \beta \in (1/2, 1) \) belongs to the weak Lorentz spaces. Similar results hold for \( \beta = 1 \), see Lemarié-Rieusset and Prioux [10].

**Theorem 3.9.** Let \( \frac{1}{2} < \beta < 1 \) and let \( \frac{2\beta}{2\beta - 1} \leq p \leq \infty \) and \( \frac{3}{2\beta - 1} < q \leq \infty \) such that
\[ \frac{\beta}{p} + \frac{3}{q} = \beta - \frac{1}{2} \]
and \( u_0 \in \widetilde{B}^{-\alpha, \infty}_q(\mathbb{R}^3) \) such that \( \nabla \cdot u_0 \). Then there exists \( T > 0 \) and a mild solution \( u \) to equations (1.1) in the space \( \tilde{L}^{p, \infty}((0, T), L^q(\mathbb{R}^3)) \).
Proof. We construct the sequence \( \{v_n\}_{n \in \mathbb{N}} \) as follows:

\[
\begin{align*}
  v_n &= v_0 - B(v_{n-1}, v_{n-1}) \quad \text{for } n \geq 1, \\
  v_0 &= e^{-t(-\Delta)^{\beta}} u_0.
\end{align*}
\]

We prove that for every \( n \in \mathbb{N} \), the function \( v_n \) belongs to the space \( \tilde{L}^{p,\infty}((0,T), L^{q,\infty}(\mathbb{R}^3)) \). Then we will use an induction argument on \( n \).

For \( n = 0 \), by assumption, \( u_0 \in \tilde{B}^{-a,\infty}(\mathbb{R}^3) \). By Lemma 3.8

\[
(3.5)
\begin{align*}
  \sup_{t \in (0,T)} t^{1/p} \|v_0(t)\|_{L^{q,\infty}(\mathbb{R}^3)} &< \sup_{t > 0} t^{1/p} \|e^{-t(-\Delta)^{\beta}} u_0(t)\|_{L^{q,\infty}(\mathbb{R}^3)} < \infty, \\
  t^{1/p} \|v_0(t)\|_{L^{q,\infty}(\mathbb{R}^3)} &= t^{1/p} \|e^{-t(-\Delta)^{\beta}} u_0(t)\|_{L^{q,\infty}(\mathbb{R}^3)} \to 0, \quad (\text{as } t \to 0).
\end{align*}
\]

By Lemma 3.2, we have \( v_0 \in \tilde{L}^{p,\infty}((0,T), L^{q,\infty}(\mathbb{R}^3)) \).

Next we assume that \( v_{n-1} \in \tilde{L}^{p,\infty}((0,T), L^{q,\infty}(\mathbb{R}^3)) \). Let \( \varepsilon > 0 \), as \( v_{n-1} \in \tilde{L}^{p,\infty}((0,T), L^{q,\infty}(\mathbb{R}^3)) \), there exist two functions \( v_1^{n-1} \in L^{\infty}((0,T), L^{q,\infty}(\mathbb{R}^3)) \) and \( v_2^{n-1} \in L^{p,\infty}((0,T), L^{q,\infty}(\mathbb{R}^3)) \) such that \( \|v_1^{n-1}\|_{L^{p,\infty}((0,T), L^{q,\infty}(\mathbb{R}^3))} \leq \varepsilon \) and \( v_{n-1} = v_1^{n-1} + v_2^{n-1} \). We have

\[
B(v_{n-1}, v_{n-1}) = B(v_1^{n-1}, v_1^{n-1}) + B(v_2^{n-1}, v_1^{n-1}) + B(v_1^{n-1}, v_2^{n-1}) + B(v_2^{n-1}, v_2^{n-1})
= M_1 + M_2
\]

with

\[
M_1 = B(v_1^{n-1}, v_1^{n-1}) + B(v_2^{n-1}, v_1^{n-1}) + B(v_1^{n-1}, v_2^{n-1}) \quad \text{and} \quad M_2 = B(v_2^{n-1}, v_2^{n-1}).
\]

By Lemma 2.1, we get

\[
\|M_1\|_{L^{\infty}((0,T), L^{q,\infty}(\mathbb{R}^3))} \leq \|B(v_1^{n-1}, v_1^{n-1})\|_{L^{\infty}((0,T), L^{q,\infty}(\mathbb{R}^3))} + \|B(v_2^{n-1}, v_1^{n-1})\|_{L^{\infty}((0,T), L^{q,\infty}(\mathbb{R}^3))}
+ \|B(v_1^{n-1}, v_2^{n-1})\|_{L^{\infty}((0,T), L^{q,\infty}(\mathbb{R}^3))}
\leq C \|v_1^{n-1}\|_{L^{\infty}((0,T), L^{q,\infty}(\mathbb{R}^3))} + 2 \|v_1^{n-1}\|_{L^{\infty}((0,T), L^{q,\infty}(\mathbb{R}^3))} \|v_2^{n-1}\|_{L^{\infty}((0,T), L^{q,\infty}(\mathbb{R}^3))}
\leq C.
\]

and

\[
\|M_2\|_{L^{p,\infty}((0,T), L^{q,\infty}(\mathbb{R}^3))} \lesssim \|v_2^{n-1}\|_{L^{p,\infty}((0,T), L^{q,\infty}(\mathbb{R}^3))} \lesssim \varepsilon^2.
\]

Thus, according to Proposition 1.9, we have \( B(v_{n-1}, v_{n-1}) \in \tilde{L}^{p,\infty}((0,T), L^{q,\infty}(\mathbb{R}^3)) \).

We will prove that for every \( n \in \mathbb{N} \),

\[
\|v_n\|_{L^{p,\infty}((0,T), L^{q,\infty}(\mathbb{R}^3))} \leq 2\|e^{-t(-\Delta)^{\beta}} u_0\|_{L^{p,\infty}((0,T), L^{q,\infty}(\mathbb{R}^3))}.
\]

Since \( v_0 = e^{-t(-\Delta)^{\beta}} u_0 \), it is obvious that

\[
\|v_0\|_{L^{p,\infty}((0,T), L^{q,\infty}(\mathbb{R}^3))} \leq 2\|e^{-t(-\Delta)^{\beta}} u_0\|_{L^{p,\infty}((0,T), L^{q,\infty}(\mathbb{R}^3))}.
\]

Assume that this is true for a \( n \in \mathbb{N} \). Then, we have

\[
\|v_{n+1}\|_{L^{p,\infty}((0,T), L^{q,\infty}(\mathbb{R}^3))} \leq 2\|e^{-t(-\Delta)^{\beta}} u_0\|_{L^{p,\infty}((0,T), L^{q,\infty}(\mathbb{R}^3))}.
\]
Taking $4C \| e^{-t(\Delta)} u_0 \|_{L^p,\infty((0,T),L^q,\infty(\mathbb{R}^3))} < 1$, we get

$$\| v_{n+1} \|_{L^p,\infty((0,T),L^q,\infty(\mathbb{R}^3))} \leq 2 \| e^{-t(\Delta)} u_0 \|_{L^p,\infty((0,T),L^q,\infty(\mathbb{R}^3))},$$

that is, $v_{n+1} \in L^p,\infty((0,T),L^q,\infty(\mathbb{R}^3))$ in the ball centered at 0, of radius $2 \| e^{-t(\Delta)} u_0 \|_{L^p,\infty((0,T),L^q,\infty(\mathbb{R}^3))}$. Then,

$$\| v_n - v_{n-1} \|_{L^p,\infty((0,T),L^q,\infty(\mathbb{R}^3))} \leq C \| v_{n-1} - v_{n-2} \|_{L^p,\infty((0,T),L^q,\infty(\mathbb{R}^3))} \| v_{n-2} \|_{L^p,\infty((0,T),L^q,\infty(\mathbb{R}^3))} \| v_{n-1} \|_{L^p,\infty((0,T),L^q,\infty(\mathbb{R}^3))}.$$

Thus, the Picard contraction principle completes the proof. □

Now, we want to give the reverse result of Theorem 3.9. To do this, we need the following lemma.

**Lemma 3.10.** Let $\frac{2}{3} < \beta < 1$ and let $\frac{3}{2(3-1)p} < q < \infty$ and $\frac{2\beta}{(3-1)p} < p < \infty$ such that $\frac{2\beta}{(3-1)p} + \frac{3}{2(3-1)q} = 1$ and $u \in \tilde{L}^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3))$ be a mild solution to equations (1.1). Then, for $0 < \varepsilon < 1$, there exists $0 < t_0 < T$ such that $\forall t \in (0,t_0]$, $\| u(t) \|_{L^q,\infty(\mathbb{R}^3)} \leq \frac{\varepsilon}{2C_0 t^{1/p}}$.

**Proof.** It follows from $u \in \tilde{L}^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3))$ that for all $\lambda > 0$, there exists a constant $C(\lambda)$, depending on $\lambda$, such that $C(\lambda) \rightarrow 0$ (as $\lambda \rightarrow \infty$) and

$$\left| \left\{ \| u(t) \|_{L^q,\infty(\mathbb{R}^3)} > \lambda \right\} \right| < \frac{C(\lambda)}{\lambda^p}.$$

Let $0 < \varepsilon < 1$ and $0 < t_0 < 1$. Denote $\lambda_0 = \frac{\varepsilon}{4C_0 t_0^{1/p}}$. When $t_0 \rightarrow 0$ and $C(\lambda_0) \rightarrow 0$, we choose $t_0$ small enough such that $C(\lambda_0) < \frac{\varepsilon}{2C_0 t^{1/p}}$. Let $t \leq t_0$ such that $\lambda_t = \frac{\varepsilon}{4C_0 t^{1/p}} \geq \lambda_0$, then $C(\lambda_t) \leq C(\lambda_0)$. We can get

$$\left| \left\{ t \in (0,T), \| u(t) \|_{L^q,\infty(\mathbb{R}^3)} > \lambda_t \right\} \right| < \frac{C(\lambda_t)}{\lambda_t^p} < \frac{t}{2 \times 4^p}.$$

We claim that there exists $\theta$ such that

$$t - \frac{t}{4^p} \leq \theta \leq t \quad \text{and} \quad \| u(\theta) \|_{L^q,\infty(\mathbb{R}^3)} \leq \frac{\varepsilon}{4C_0 t^{1/p}} = \lambda_t.$$

Otherwise

$$\left| \left\{ t \in (0,T), \| u(t) \|_{L^q,\infty(\mathbb{R}^3)} > \lambda_t \right\} \right| \geq \left| t - \frac{t}{4^p}, t \right| = \frac{t}{4^p}.$$

This is a contraction to (3.6). Let $T^* = \left( 4C_0 \| u(\theta) \|_{L^q,\infty(\mathbb{R}^3)} \right)^{-p}$. Taking $0 < \varepsilon < 1$, we have

$$\| u(\theta) \|_{L^q,\infty(\mathbb{R}^3)} \leq \frac{1}{4C_0 t^{1/p}} \Rightarrow t \leq \left( 4C_0 \| u(\theta) \|_{L^q,\infty(\mathbb{R}^3)} \right)^{-p}.$$

Applying Lemma 2.2 in the interval $[\theta, \theta + T^*]$, there exists a solution $\tilde{u} \in L^\infty(\{\theta, \theta + T^*\}, L^{q,\infty}(\mathbb{R}^3))$ to the equations (1.1). Note that (3.7) implies that

$$\left( \theta, t \right) \subset (\theta, \theta + t) \subset (\theta, \theta + T^*).$$
By Proposition 3.1, we know $u = \tilde{u}$ on $(\theta, t]$. So for $t \leq t_0$, there exists $0 < \theta < t$ such that $u \in L^\infty((\theta, t], L^{q, \infty}(\mathbb{R}^3))$ and

$$\forall s \in (\theta, t], \|u(s)\|_{L^{q, \infty}(\mathbb{R}^3)} \leq 2\|u(\theta)\|_{L^{q, \infty}(\mathbb{R}^3)} \leq \frac{\varepsilon}{2C_0 t_0^{1/p}}.$$  

This completes the proof of this lemma.  

\[\square\]

**Theorem 3.11.** For $\frac{1}{2} < \beta < 1$ and let $\frac{3}{2(\beta - 1)} < q < \infty$ and $\frac{2\beta}{2(\beta - 1)} < p < \infty$ such that $0 < \frac{q}{p} < 1$, if $u \in \hat{L}^{p, \infty}((0, T), L^{q, \infty}(\mathbb{R}^3))$ be a mild solution to equations (1.1). Then

\[
\begin{cases}
\sup_{t \in (0, T)} t^{1/p} \|u(t)\|_{L^{q, \infty}(\mathbb{R}^3)} < \infty, \\
t^{1/p} \|u(t)\|_{L^{q, \infty}(\mathbb{R}^3)} \to 0 \quad (\text{as } t \to 0).
\end{cases}
\]

**Proof.** By Lemma 3.10 for every $\varepsilon > 0$, there exists $t_0$ such that, for all $t \in (0, t_0)$,

$$t^{1/p} \|u(t)\|_{L^{q, \infty}(\mathbb{R}^3)} \leq \frac{\varepsilon}{2C_0},$$

that is, $\lim_{t \to 0} t^{1/p} \|u(t)\|_{L^{q, \infty}(\mathbb{R}^3)} = 0$.

Now we prove the first assertion of (3.8). Checking the proof of Lemma 3.10 and taking $\varepsilon = \frac{\varepsilon}{2}$, we can see that there exist $t_0$ such that for every $t \leq t_0$ and $0 < \theta < t$ such that $u \in L^\infty((\theta, t], L^{q, \infty}(\mathbb{R}^3))$ and

$$\forall s \in (\theta, t], \|u(s)\|_{L^{q, \infty}(\mathbb{R}^3)} \leq 2\|u(\theta)\|_{L^{q, \infty}(\mathbb{R}^3)} \leq \frac{1}{4C_0 t_0^{1/p}}.$$  

On the other hand, Lemma 3.10 and $\lim_{t \to 0} t^{1/p} \|u(t)\|_{L^{q, \infty}(\mathbb{R}^3)} = 0$ tell us that there exists $t_1$ such that for $s \in (0, t_1)$, $t^{1/p} \|u(t)\|_{L^{q, \infty}(\mathbb{R}^3)} \leq C$. If $t_0 > t_1$, take $t_2 < t_1 < t_0$ (otherwise take $t_2 = t_0$). By (3.9), there exists $\theta_2$ such that for every $s \in (\theta_2, t_2)$, $\|u(s)\|_{L^{q, \infty}(\mathbb{R}^3)} \leq \frac{1}{4C_0 t_2^{1/p}}$. Because $t^{1/p} \|u(t)\|_{L^{q, \infty}(\mathbb{R}^3)}$ is bounded on $(0, \theta_2) \subset (0, t_1)$, now we restrict $t \in (\theta_2, T]$. Define a new function $\tilde{u}(s) = u(t_2 - \theta_2 + s)$. Then we only need to prove the assertion for $\tilde{u}(s)$ on $s \in (\theta_2, T + t_2 - \theta_2]$.

Since $u$ is a solution to equations (1.1),

$$u \in \hat{L}^{p, \infty}((t_2, T), L^{q, \infty}(\mathbb{R}^3)) \implies \tilde{u} \in \hat{L}^{p, \infty}((\theta_2, T - t_0 + \theta_2), L^{q, \infty}(\mathbb{R}^3))$$

implies that $\tilde{u}$ is also a solution to the equations (1.1). By Lemma 3.10 for $\varepsilon = \frac{\varepsilon}{2}$ again, we can get that for $\forall t \in (\theta_2, t_2)$, $\|\tilde{u}(t)\|_{L^{q, \infty}(\mathbb{R}^3)} \leq \frac{1}{4C_0 t_2^{1/p}}$. That is, $\forall t \in (t_2, 2t_2 - \theta_2)$, $\|u(t)\|_{L^{q, \infty}(\mathbb{R}^3)} \leq \frac{1}{4C_0 t_2^{1/p}}$. We conclude that

$$\forall t \in (\theta_2, 2t_2 - \theta_2), \|u(t)\|_{L^{q, \infty}(\mathbb{R}^3)} \leq \frac{1}{4C_0 t_2^{1/p}}.$$  

Since $T$ is finite, we can find a constant $n \in \mathbb{N}$ such that $nt_2 < T < (n + 1)t_2$. Hence repeating this argument finite many times, we get

$$\forall t \in (\theta_2, T], \|u(t)\|_{L^{q, \infty}(\mathbb{R}^3)} \leq \frac{1}{4C_0 t_2^{1/p}} < \frac{1}{4C_0 t_2^{1/p}} T^{1/p}.$$  

This completes the proof of this theorem.  

\[\square\]

**Acknowledgements.** We would like to thank our supervisor Professor Jie Xiao for suggesting the problem and kind encouragement.
References

[1] M. Cannone, A generalization of a theorem by Kato on Navier-Stokes equations, Rev. Mat. Iberoam., 13 (1997), 673-697.

[2] M. Cannone, Harmonic analysis tools for solving the incompressible Navier-Stokes equations, In: Handbook of Mathematical Fluid Dynamics Vol 3 (eds. S. Friedlander, D. Serre), Elsevier, 2004, pp. 161-244.

[3] G. Dafni and J. Xiao, Some new tent spaces and duality theorem for fractional Carleson measures and $Q_{\alpha}(\mathbb{R}^n)$, J. Funct. Anal., 208 (2004), 377-422.

[4] G. Dafni and J. Xiao, The dyadic structure and atomic decomposition of $Q$ spaces in several variables, Tohoku Math. J., 57 (2005), 119-145.

[5] M. Essen, S. Janson, L. Peng and J. Xiao, $Q$ space of several real variables, Indiana Univ. Math. J., 49 (2000), 575-615.

[6] Y. Giga, T. Miyakawa, Navier-Stokes flow in $\mathbb{R}^3$ with measures as initial vorticity and Morry spaces, Comm. Partial Differential Equations, 14 (1989), 577-618.

[7] T. Kato, Strong $L^p$-solutions of the Navier-Stokes in $\mathbb{R}^n$ with applications to weak solutions, Math. Zeit., 187 (1984), 471-480.

[8] H. Koch and D. Tataru, Well-posedness for the Navier-Stokes equations, Adv. Math., 157 (2001), 22-35.

[9] P. G. Lemarié-Rieusset, Recent Development in the Navier-Stokes Problem, in: Research Notes in Mathematics, 431, Chapman-Hall/CRC, 2002.

[10] P. G. Lemarié-Rieusset and N. Prioux, The Navier-Stokes equations with data in $bmo^{-1}$, Nonlinear Analysis 70 (2009), 280-297.

[11] L. Leray, Sur le mouvement d’un liquide visqueux emplissant l’espace, Acta Math., 63 (1934), 193-248.

[12] J. L. Lions, Quelques méthodes de résolution des problèmes aux limites non linéaires, (French) Paris: Dunod/Gauthier-Villars, 1969.

[13] P. Li and Z. Zhai, Well-posedness and Regularity of Generalized Navier-Stokes Equations in Some Critical $Q$-spaces, Submitted.

[14] J. Wu, The generalized incompressible Navier-Stokes equations in Besov spaces, Dyn. Partial Differ. Eq., 1 (2004), 381-400.

[15] J. Wu, Lower Bounds for an integral involving fractional Laplacians and the generalized Navier-Stokes equations in Besov spaces, Commun. Math. Phys., 263 (2005), 803-831.

[16] J. Xiao, Homothetic variant of fractional Sobolev space with application to Navier-Stokes system, Dynamic of PDE., 2 (2007), 227-245.

School of Mathematics, Peking University, Beijing, 100871, China
Current address: Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John’s, NL A1C 5S7, Canada
E-mail address: liptao@163.com

Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John’s, NL A1C 5S7, Canada
E-mail address: a64zz2mun.ca