WEAK\(^*\) FIXED POINT PROPERTY
AND THE SPACE OF AFFINE FUNCTIONS

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Abstract. First we prove that if a separable Banach space \(X\) contains an isometric copy of an infinite-dimensional space \(A(S)\) of affine continuous functions on a Choquet simplex \(S\), then its dual \(X^*\) lacks the weak\(^*\) fixed point property for nonexpansive mappings. Then, we show that the dual of a separable Lindenstrauss space \(X\) fails the weak\(^*\) fixed point property for nonexpansive mappings if and only if \(X\) has a quotient isometric to some space \(A(S)\). Moreover, we provide an example showing that “quotient” cannot be replaced by “subspace”. Finally, it is worth to be mentioned that in our characterization the space \(A(S)\) cannot be substituted by any space \(C(K)\) of continuous functions on a compact Hausdorff \(K\).

1. Introduction

Let \(X\) be an infinite-dimensional real Banach space and let us denote by \(B_X\) its closed unit ball and by \(S_X\) its unit sphere. A Banach space \(X\) is called an \(L_1\)-predual (or a Lindenstrauss space) if its dual \(X^*\) is isometric to \(L_1(\mu)\) for some measure \(\mu\). The most widely studied \(L_1\)-preduals are classical Banach spaces \(C(K)\) of continuous functions on a compact Hausdorff space \(K\). In this paper two other subclasses of Lindenstrauss spaces play a crucial role. The first one is the well-known class of spaces \(A(S)\) of continuous affine functions defined on a Choquet simplex \(S\). It is worth to be mentioned that the class of \(A(S)\) spaces is strictly wider than the class of \(C(K)\) spaces. Moreover, an \(L_1\)-predual \(X\) is an \(A(S)\) space if and only if \(B_X\) has an extreme point (\([13]\)). The second class that we are interested in is the collection of all hyperplanes in \(c\), the space of convergent sequences endowed with the standard supremum norm. These hyperplanes were extensively studied in \([1, 2]\). Here we recall some notations and properties about these spaces that will be useful in the sequel. It is well-known that \(c^*\) can be isometrically identified with \(\ell_1\) in the following way: for every \(x^* \in c^*\) there exists a unique element \(f = (f(1), f(2), \ldots) \in \ell_1\) such that

\[
x^*(x) = \sum_{n=0}^{\infty} f(n+1)x(n) = f(x)
\]

with \(x = (x(1), x(2), \ldots) \in c\) and \(x(0) = \lim x(n)\). Let \(f \in S_{c^*}\). Consider the hyperplane in \(c\) defined by

\[
W_f = \{ x \in c : f(x) = 0 \}.
\]

In \([1]\), the following results are proved:

1. \(W_f^*\) is isometric to \(\ell_1\) if and only if there exists \(j_0 \geq 1\) such that \(|f(j_0)| \geq 1/2\).
2. \(W_f\) is isometric to \(c\) if and only if there exists \(j_0 \geq 2\) such that \(|f(j_0)| \geq 1/2\).

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Under the additional assumption $|f(1)| \geq 1/2$ and $|f(j)| < 1/2$ for every $j \geq 2$, Theorem 4.3 in [1] identifies $W_f^*$ and $\ell_1$ by giving the following dual action: for every $x^n \in W_f^*$ there exists a unique element $g \in \ell_1$ such that

$$x^n(x) = \sum_{n=1}^{\infty} g(n)x(n) = g(x),$$

where $x = (x(1), x(2), \ldots) \in W_f$. Moreover, if $\{e^*_n\}$ denotes the standard basis in $\ell_1$, then

$$(\bigvee) \quad e^*_n \frac{\sigma(\ell_1, W_f)}{f(2)} \left( -\frac{f(2)}{f(1)} - \frac{f(3)}{f(1)} - \frac{f(4)}{f(1)} \cdots \right),$$

where $\sigma(X^*, X)$ denotes the weak* topology on $X^*$ induced by $X$.

The aim of this paper is to investigate the relationships between presence of an isometric copy of an $A(S)$ space in a separable space $X$ and the failure of weak* fixed point property for nonexpansive mapping in the dual space $X^*$. We recall that the space $X^*$ is said to have the weak* fixed point property (briefly, $\sigma(X^*, X)$-FPP) if for every nonempty, convex, $\sigma(X^*, X)$-compact subset $C$ of $X^*$, every nonexpansive mapping (i.e., a mapping $T : C \to C$ such that $\|T(x) - T(y)\| \leq \|x - y\|$ for all $x, y \in C$) has a fixed point.

First we prove that if a separable Banach space $X$ contains an isometric copy of an infinite-dimensional space $A(S)$, then its dual $X^*$ lacks the $\sigma(X^*, X)$-FPP (see Theorem 2.3). This sufficient condition can be extended by considering a quotient containing an isometric copy of an $A(S)$ space (see Remark 2.5). Our theorem extends, in a separable case, the result by Smyth stating that $C(K)^*$ fails the $\sigma(C(K)^*, C(K))$-FPP (see [15]).

In the last section we discuss the case of separable $L_1$-preduals and we completely characterize the weak* topologies that fail the $\sigma(X^*, X)$-FPP. Indeed, we prove that the dual $X^*$ of a separable Lindenstrauss space $X$ fails the $\sigma(X^*, X)$-FPP if and only if $X$ has a quotient isometric to an infinite-dimensional $A(S)$ space for some Choquet simplex $S$. We also show that the latter condition may be replaced by: $X$ has a quotient containing an isometric copy of an infinite-dimensional $A(S)$ space for some Choquet simplex $S$ (see Theorem 3.6). Finally, one may ask whether in the previous results the space $A(S)$ can be replaced by a space $\mathcal{C}(K)$ or if the quotient can be removed, in a sense that these conditions can be replaced by: $X$ has a subspace isometric to an infinite-dimensional $A(S)$ space. Remark 3.4 and Example 3.1 show that the answers for both questions are negative.

2. **Weak* fixed point property in the dual of separable Banach space**

This section is devoted to study a sufficient condition for the failure of the $\sigma(X^*, X)$-FPP for a generic separable space $X$ in term of the presence of an isometric copy of an $A(S)$ space. In [2], we provided a sufficient condition of similar type but based on the presence of an isometric copy of the so-called bad hyperplane in $c$. Recall that the hyperplane $W_f$ is called *bad* if $f \in \ell_1$ is such that $\|f\| = 1$, $|f(1)| = \frac{1}{2}$ and the set $N^+ = \{ n \in \mathbb{N} : f(n+1) \leq 0 \}$ is infinite. The word *bad* was chosen with respect to the $\sigma(\ell_1, W_f)$-FPP since the dual of every bad $W_f$ lacks the $\sigma(\ell_1, W_f)$-FPP. The statement of our aforementioned result is:

**Theorem 2.1.** (Theorem 3.7 in [2]) Let $X$ be a separable Banach space. If $X$ contains a subspace isometric to a bad hyperplane, then $X^*$ fails the $\sigma(X^*, X)$-FPP.
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The main result of this section aims to provide a sufficient condition for the failure of the \(\sigma(X^*, X)\)-FPP by reformulating Theorem 2.1 where bad hyperplane is replaced by an \(A(S)\) space. In order to prove it, we need to recall a known result that we quote here for the sake of convenience of the reader.

**Lemma 2.2.** ([10], p. 441) Let \(X\) be a closed subspace of \(C(K)\) of all real continuous functions on a compact Hausdorff space \(K\). For each \(q \in K\) let \(x^*_q \in X^*\) be defined by

\[
x^*_q(f) = f(q), \quad f \in X.
\]

Then every extreme point of the closed unit sphere of \(X^*\) is of the form \(\pm x^*_q\) with \(q \in K\).

**Theorem 2.3.** Let \(X\) be a separable Banach space. If \(X\) contains an isometric copy of some infinite-dimensional \(A(S)\) space, then \(X^*\) lacks the \(\sigma(X^*, X)\)-FPP.

**Proof.** We start the proof by considering the case when \(A(S)^*\) is nonseparable and hence \(X^*\) is nonseparable. By Theorem 2.3 in [9] every separable Lindenstrauss space \(X\) with nonseparable dual contains a subspace isometric to the space \(C(\Delta)\), where \(\Delta\) is the Cantor set. Since \(C(\Delta)\) contains an isometric copy of \(c\), by Theorem 2.1 \(X^*\) lacks the \(\sigma(X^*, X)\)-FPP.

Therefore, it remains to consider the case when \(A(S)^* = \ell_1\). Let \(\|\|\|\|\) denote the identically equal 1 function defined on \(S\). Let \(\pi\) denote the canonical embedding of \(A(S)\) into \(A(S)^* = \ell_\infty\). Since \(A(S) \subset C(S)\), by Lemma 2.2 there exists a sequence of signs \((\varepsilon(n))_{n \in \mathbb{N}}\), \(\varepsilon(n) = \pm 1\) for all \(n \in \mathbb{N}\) such that \(\pi(1) = (\varepsilon(1), \varepsilon(2), \varepsilon(3), \ldots)\).

Without loss of generality, we may assume that there is a subsequence \((\varepsilon(n_j))_{j \in \mathbb{N}}\) of \((\varepsilon(n))_{n \in \mathbb{N}}\) such that \(\varepsilon(n_j) = 1\) for each \(j \in \mathbb{N}\). Let us consider the set

\[
C = \{f \in \ell_1 : \pi(1)(f) = 1\} \cap B_{\ell_1} = \left\{ f = (f(1), f(2), \ldots) \in B_{\ell_1} : \sum_{i=1}^{\infty} \varepsilon(i)f(i) = 1 \right\}.
\]

It is easy to see that \(C\) is a nonempty, convex and \(\sigma(\ell_1, A(S))\)-compact subset of \(\ell_1\). Now, by choosing a subsequence, we may assume that \(\{e^*_{n_j}\}_{j \in \mathbb{N}}\) is \(\sigma(\ell_1, A(S))\)-convergent to some \(e^*\). Since \(e^* \in C\), we have \(\|e^*\| = 1\) and \(e^*(n_j) \geq 0\) for every \(j \in \mathbb{N}\).

From now on, the proof follows the approach already used in the proof of Theorem 3.7 in [2]. For the convenience of the reader we repeat here the relevant part of that proof. Again, by choosing a subsequence, we may assume that \(w_0 = e^* - w_0 \neq 0\) where \(w_0 = \sum_{j=1}^{\infty} e^*(n_j)e^*_n\). Let \(x^*_{n_j}\) be a norm-preserving extension of \(e^*_n\) to the whole space \(X\). Now we consider the extension of \(w_0\) to the whole space \(X\) defined by \(\tilde{w}_0 = \sum_{j=1}^{\infty} e^*(n_j)x^*_n\) and the elements \(\tilde{w}_0 = x^* - \tilde{w}_0\) and \(\tilde{w} = \frac{\tilde{w}_0}{\|\tilde{w}_0\|}\). Now, by adapting to our framework the approach developed in the last part of the proof of Theorem 8 in [7], we show that the \((X^*, X)\)-compact, convex set

\[
D = \left\{ \mu_1x^* + \mu_2\tilde{w} + \sum_{j=1}^{\infty} \mu_{j+2}x^*_{n_j} : \sum_{k=1}^{\infty} \mu_k = 1, \mu_k \geq 0, k = 1, 2, \ldots \right\}
\]

is equal to

\[
D = \left\{ \lambda_1\tilde{w} + \sum_{j=1}^{\infty} \lambda_{j+1}x^*_{n_j} : \sum_{k=1}^{\infty} \lambda_k = 1, \lambda_k \geq 0, k = 1, 2, \ldots \right\}.
\]

Next, we consider the map \(T : D \to D\) defined by:

\[
T \left( \lambda_1\tilde{w} + \sum_{j=1}^{\infty} \lambda_{j+1}x^*_{n_j} \right) = \sum_{j=1}^{\infty} \lambda_jx^*_{n_j}.
\]
functions on the Choquet simplex

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Proof. Let $A$ which are not

basis ($\sigma$-equivalent:

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Finally, it is easy to see that $T$ has no fixed point in $D$. \qed

Remark 2.4. Theorem 2.3 can be proved in a completely different way. Indeed, from the proof of Theorem 1 in [16], we know that every $A(S)$ space contains an $\ell_1$-predual subspace that is isometric to a hyperplane in $c$ containing the point $(1, 1, 1, \ldots) \in c$. It is easy to observe that such a hyperplane is bad. Therefore, by applying Theorem 2.4, we conclude that $X^*$ lacks the $\sigma(X^*, X)$-FPP. This alternative proof relies on a deep technique developed by Zippin, whereas our approach is direct and self-contained. One may conjecture that each $A(S)$ space contains an isometric copy of the whole space $c$, which is the simplest example of bad hyperplane. However, it occurs that there exists an infinite-dimensional $A(S)$ space that does not contain an isometric copy of $c$ (see [3]).

Remark 2.5. Let $X$ be a separable Banach space. Suppose that some infinite-dimensional $A(S)$ space is a subspace of a quotient $X/Y$ of $X$. Then Theorem 2.3 shows that $Y^\perp$ fails the $\sigma(Y^\perp, X/Y)$-FPP. It follows easily that also $X^*$ lacks the $\sigma(X^*, X)$-FPP.

3. Weak* fixed point property in the dual of separable Lindenstrauss space

In this section we show that the sufficient condition stated in Remark 2.3 is equivalent to the failure of the $\sigma(X^*, X)$-FPP whenever we consider a separable Lindenstrauss space $X$. Moreover, our result is linked to the characterization obtained in [2]. Indeed, we applied the above mentioned notion of bad hyperplane to state the following equivalence.

**Theorem 3.1** (Theorem 4.1 in [2]). Let $X$ be a predual of $\ell_1$. Then the following are equivalent:

1. $\ell_1$ lacks the $\sigma(\ell_1, X)$-FPP for nonexpansive mappings;
2. there is a quotient of $X$ isometric to a bad $W_f$;
3. there is a quotient of $X$ that contains a subspace isometric to a bad $W_g$.

In the following result we replace bad $W_f$ by the space $A(S)$ of affine continuous functions on the Choquet simplex $S$. One can easily observe that there are bad $W_f$ which are not $A(S)$ spaces since their unit balls have no extreme points.

**Theorem 3.2.** Let $X$ be a predual of $\ell_1$. The following statements are equivalent:

1. $\ell_1$ lacks the $\sigma(\ell_1, X)$-FPP for nonexpansive mappings;
2. there is a quotient of $X$ isometric to some $A(S)$ space;
3. there is a quotient of $X$ containing an isometric copy of some $A(S)$ space.

**Proof.** We start by proving that (1) implies (2). From the implication (1) $\Rightarrow$ (4) of Theorem 4.1 in [2], we obtain that there is a subsequence $(e^*_n)_{k \in \mathbb{N}}$ of the standard basis $(e^*_n)_{n \in \mathbb{N}}$ in $\ell_1$ which is $\sigma(\ell_1, X)$-convergent to a norm-one element $e^* \in \ell_1$ with $e^*(n_k) \geq 0$ for all $k \in \mathbb{N}$. From the proof of the implication (4) $\Rightarrow$ (5) of Theorem

Since $x = \lambda_1 \tilde{w} + \sum_{j=1}^{\infty} \lambda_j + 1 x^*_n \in D$ has a unique representation, the map $T$ is well defined. Moreover it is a nonexpansive map. Indeed, for every $\alpha = (\alpha_1, \alpha_2, \ldots)$ such that $\sum_{j=1}^{\infty} |\alpha_j| < \infty$, it holds

$$\left\| \alpha_1 \tilde{w} + \sum_{j=1}^{\infty} \alpha_{j+1} x^*_n \right\| \geq \left\| \alpha_1 \frac{u_0}{\|u_0\|} + \sum_{j=1}^{\infty} \alpha_{j+1} e^*_n \right\| = \sum_{j=1}^{\infty} |\alpha_j|$$

Finally, it is easy to see that $T$ has no fixed point in $D$. \qed

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4.1 in [2], we know that $X$ has a quotient isometric to an $\ell_1$-predual hyperplane $W_f$ containing the point $(1, 1, 1, \ldots) \in c$. By applying Corollary 2 in [2] and ($\forall$) one can prove that the positive face $S$ of the unit sphere of $\ell_1 = W_f^*$ is $\sigma(\ell_1, W_f)$-compact and $W_f$ is isometric to $A(S)$.

The implication (2) $\Rightarrow$ (3) is trivial.

Finally, by applying Remark 2.5 we conclude that (3) $\Rightarrow$ (1).

The following example shows that the quotient in conditions (2) and (3) in Theorem 3.2 cannot be removed in a sense that these conditions can be replaced by: $X$ has a subspace isometric to an infinite-dimensional $A(S)$ space.

**Example 3.1.** Let $f = (1/2, -1/4, 1/8, -1/16, \ldots) \in \ell_1$. Since $W_f$ is a bad hyperplane, $\ell_1$ fails the $\sigma(\ell_1, W_f)$-FPP. Moreover, $W_f$ does not have a quotient containing an isometric copy of $c$ (see Example 2.4 in [2]).

We claim that the hyperplane $W_f$ does not contain any infinite-dimensional $A(S)$ space. By contradiction, suppose that $A(S) \subset W_f$. Let $\{e_n^*\}$ be the standard basis in $\ell_1 = A(S)^*$. Since $A(S) \subset C(S)$, by Lemma 2.4 there exists a sequence of signs $(\varepsilon(n))_{n \in \mathbb{N}}$, $\varepsilon(n) = \pm 1$ for all $n \in \mathbb{N}$, such that $e_n^*(1) = \varepsilon(n)$, where $1$ denotes the constant function equal to 1 on $S$. Let $\widetilde{e}_n$ denote the norm-preserving extension of $e_n^*$ to the whole $W_f$. Then

$$2 = \|e_n^* + e_m^*\| \leq \|\widetilde{e}_n^* + \widetilde{e}_m^*\| \leq 2.$$  

These relations mean that $\{\varepsilon_n\}$ is represented in $\ell_1 = W_f^*$ by a sequence of disjoint blocks of norm 1. Moreover, for every $n \in \mathbb{N}$, it holds $\widetilde{e}_n^*(1) = e_n^*(1) = \varepsilon(n)$. This shows that $1$ is represented in $W_f$ by $x = (x(1), x(2), \ldots) \in W_f^*$ such that for every $n \in \mathbb{N}$ we have

$$x(i) = \text{sgn} e_n^*(i)$$  

if $i \in \text{supp} e_n^* := \{i \in \mathbb{N} : \varepsilon_n^*(i) \neq 0\}$. Since $x \in B_c$, we have $\lim_{n \to \infty} x(n) = 1$ or $\lim_{n \to \infty} x(n) = -1$. However, there is no such $x \in W_f$.

Theorem 3.2 can be easily extended from the case of $\ell_1$-preduals to the whole class of separable $L_1$-preduals.

**Theorem 3.3.** Let $X$ be a separable Lindenstrauss space. The following statements are equivalent:

1. $X^*$ lacks the $\sigma(X^*, X)$-FPP for nonexpansive mappings;
2. there is a quotient of $X$ isometric to some $A(S)$ space;
3. there is a quotient of $X$ containing an isometric copy of some $A(S)$ space.

**Proof.** By taking into the account Theorem 3.2 it is enough to consider the case when $X^*$ is nonseparable. Theorem 2.3 in [2] states that a separable Lindenstrauss space $X$ with nonseparable dual contains a subspace isometric to the space $C(\Delta)$, where $\Delta$ is the Cantor set. Since $C(\Delta)$ contains an isometric copy of $c$, by Proposition 3.1 in [2] there is a 1-complemented copy of $c$. Therefore, $X$ has a quotient isometric to $c$. This shows that (2) and (3) hold true. Finally, by applying Corollary 3.4 in [2] we conclude the proof.

**Remark 3.4.** In Theorem 3.2 the space $A(S)$ cannot be replaced by any space $C(K)$ of continuous functions on the compact Hausdorff set $K$. Indeed, if $K$ is finite, then $C(K) = \ell_k^{(n)}$ for some $n \in \mathbb{N}$. By [3,13], we know that every separable Lindenstrauss space contains an isometric copy of $\ell_k^{(n)}$ for every $n$. Since $\ell_k^{(n)}$ is always 1-complemented, $X$ has a quotient isometric to $\ell_k^{(n)}$. Moreover, if $K$ is an infinite countable set, then, by Mazurkiewicz-Sierpiński Theorem (10), we know...
that \( C(K) \) contains an isometric copy of \( c \). However, by Example \([31]\) we know that there is an \( \ell_1 \)-predual such that \( \ell_1 \) fails the weak*\-FPP, whereas it does not have a quotient containing an isometric copy of \( c \). It remains to consider the case where \( K \) is uncountable. However, under this assumption \( C(K)^* \) is nonseparable and therefore \( X \) cannot be an \( \ell_1 \)-predual.

We conclude our paper by pointing out that some other equivalent conditions for the weak*\-FPP are known in the literature. We refer the interested reader to \([2, 4, 5, 11, 12]\).

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