Windows on Quantum Gravity

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These are notes for a graduate course in quantum gravity planned at IFT-UAM/CSIC for the spring term of 2020, but delayed for an indefinite period of time (although a shortened version was given in the Escuela de Física (Institute of Physics) in the University of Costa Rica.). The aim of the course was to highlight the most important conceptual problems in the field and to summarize some of the most imaginative solutions to them that have been proposed in the literature. The course can best be characterized as idiosyncratic rather than encyclopedic.

1. Introduction

The aim of these notes is to assess progress in the field in the last few decades (since the advent of strings, say). I have tried to avoid unnecessary technicalities, to the extent that this is compatible with making precise statements. Previous references written much in the same spirit as this one are [3, 4, 37, 71].

It cannot be denied that progress in this particular field has been largely overemphasized.

The problem of quantum gravity is not by any means a new one. It is not even clear to what extent it is a scientific problem. Certainly, there is not a single experiment that points out where we have to look. In spite of this, the list of people that have attempted to understand something about the relationship between General Relativity (GR) and Quantum Mechanics (QM) is quite large; there are books devoted entirely to it. [16]

Attempts can be roughly classified in two big groups after some coarse graining. The staring point is that we have two theories which have experimental confirmation in their respective regions of validity and up to the often astounding experimental precision, namely General Relativity (GR) and quantum mechanics (QM), valid when $\hbar = 0$ (but $G \neq 0$ and $c = \infty$) and quantum mechanics (QM), valid when $\hbar \neq 0$ but $G = 0$ and $c = \infty$. The region with $G = 0$ but $c \leq \infty$ is also understood in some sense; it corresponds to quantum field theory (QFT).

Then it appears natural to some people to put General Relativity (GR) first, and try in some sense to quantize the space-time geometry.

On the other side of the coin there are those that put Quantum Mechanics (QM) first, and try to change the gravitational theories accordingly. I include in this group theories in which GR is treated as an emergent theory, and the fundamental quantum variables are not necessarily geometrical.

Another contentious point is whether it should be demanded that the ensuing hypothetical theory obeys a correspondence principle of sorts. By this we mean that whatever theory is put forward, one of the essential requisites is that it should be smoothly connected with Quantum Field Theory (QFT) in some smooth limit and with GR in an also smooth way.

As I said, not everybody shares this principle (I do). The aim of these notes has certainly not been to give a complete review of the field. I only touch a few topics which seem to me particularly interesting, and about which I think I am able to make some comments. Also the depth on the treatment is quite uneven. This is sometimes due to my taste, but also sometimes the reason is purely pedagogical. There are some things (like the canonical approach) that in my opinion every serious student should master. Advance apologies to all authors whose works are not duly represented. This is most probably due to ignorance for my part.

The first paragraph is devoted to a short discussion of whatever experimental hard evidence we have (or may be we can have in the future) that gravity has necessarily to be quantized.

Then some of the general arguments on how Einstein’s equations can emerge from a more fundamental theory are reviewed. After that some thought is devoted to what are the questions we would like to have our theory to answer, and we find that even to make a precise statement of those questions is not an easy task.

Then we devote some space to a few comments about the canonical formalism, which is always at the root of any quantum mechanical approach. This is probably the longest, and in my opinion the most important one.

It follows some thoughts on what is the symmetry group of the theory and the related question of what are the observables.

We end we some short comments on what has been the impact of string theory on the topics above.

2. Do We Need Quantum Gravity?

The first thing we have to decide is whether there is any evidence that gravity has to be quantized. (confer references and comments in [18]). If gravity remains a classical field, the second member of Einstein’s equations should be interpreted as some expectation value of a composite operator in QFT.

$$T_{\mu\nu} \rightarrow \langle \psi | T_{\mu\nu} | \psi \rangle$$

(1)
This equation (apparently first proposed by Möller\cite{50}) is not easy to justify from first principles. In spite of this fact, it is frequently used to compute the gravitational back-reaction to some quantum corrections to some particular phenomena under consideration, computed in some initially fixed gravitational background.

It has been argued\cite{7,25} that these modified Einstein’s equations are not invariant under field redefinitions. Besides, renormalization forces to include terms quadratic in curvature that in turn are claimed to imply negative energy\cite{76}.

Perhaps the most precise analysis of Möller’s ansatz is the one of Randjbar-Daemi and Kibble\cite{46} where they deduce it from the variational principle.

\[
S_p \equiv \frac{1}{2\kappa^2} \int d(\text{vol}) R
+ \int dt \left\{ \text{Im} \langle \psi | \psi \rangle - \langle \psi | [H, |\psi\rangle] + a(\langle \psi | |\psi\rangle - 1) \right\}
\tag{2}
\]

The equations of motion (EM) read

\[
i \frac{d}{dt} |\psi\rangle = H |\psi\rangle - a |\psi\rangle
\]

\[
\langle \psi | |\psi\rangle - 1 = 0
\]

\[
G_{\mu\nu} = 2\kappa^2 \langle \psi | \frac{\delta H}{\delta \phi^\mu} |\psi\rangle
\tag{3}
\]

where

\[
G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}
\tag{4}
\]

is Einstein’s tensor.

Here the hamiltonian \( H \) depends on the gravitational field, which depends on the state \( |\psi\rangle \). This then introduces nonlinearities in quantum mechanics. Those se are severely restricted by experiment\cite{22} to be at most of the order of \( 10^{-21} \). Whether this is in contradiction with the semiclassical theory is not clear\cite{18}.

A quick (newtonian) estimate can be done as follows. Assume the equations

\[
i\hbar \frac{d\psi}{dt} = -\frac{\gamma}{2\kappa} \Delta \psi - m\Phi\psi
\]

\[
\Delta \Phi = 4\pi G m |\psi|^2
\tag{5}
\]

Consider a particle of mass \( m \) with a localized initial wavefunction

\[
|\psi(\tau, t = 0)\rangle \equiv \left( \frac{1}{\pi\Delta x^2} \right)^{\frac{i}{4}} e^{-\frac{|\xi|^2}{2\Delta x^2}}
\tag{6}
\]

The time evolution is driven by two effects. The ordinary spread of the wave function in QM on the one hand, and the self-gravitation on the other. Which of those effects dominates depends on the particle mass. The peak probability density for a free particle is known to occur at

\[
\frac{r_p}{\Delta x} \sim \sqrt{1 + \frac{\hbar^2}{m^2 \Delta x^2}}
\tag{7}
\]

This accelerates outward as

\[
\frac{\dot{r}_p}{\Delta x} \sim \frac{\hbar^2}{m^2 \Delta x^2}
\tag{8}
\]

The opposite inward gravitational acceleration is

\[
\frac{a_g}{r_p} \sim \frac{G m}{r_p^2}
\tag{9}
\]

Equality at \( t=0 \) needs

\[
m_c \sim \left( \frac{\hbar^2}{G \Delta x} \right)^{\frac{1}{2}}
\tag{10}
\]

The qualitative behavior is as follows.

- For \( m \ll m_c \) is similar to a free particle; that is, the quantum spread dominates.
- For \( m \gg m_c \), the packet will undergo gravitational collapse.
- For \( m \sim m_c \), the wave packet is unstable.

In [18] it was estimated that the results are almost three orders of magnitude below any possible experimental reach. This presumably means that we will have experimental information in the not-too-distant future.

### 3. Gravity As An Emergent Theory. Einstein’s Equations from Thermodynamics

Recently\cite{69} (although some of the main ideas are older than that\cite{45}) it has been emphasized that Einstein’s equations can also be contemplated as emergent out of some more basic (unknown) degrees of freedom. From this point of view gravity would be like thermodynamics, or fluid mechanics; a macroscopic theory which makes no sense to extrapolate at arbitrary small lengths.

Were these claims to be true, then the whole effort to quantize gravity is misplaced, and we should concentrate in the dynamics of those more basic degrees of freedom. Let us examine how this comes about with a little bit more detail. In string theory, gravity is also emergent, in a sense which in our opinion is not yet completely understood. We will have more to say about this in the sequel.

The starting point is Unruh’s\cite{66} fundamental observation that to every Killing horizon (which by definition is the locus of all points where the Killing becomes null, \( \xi^2 = 0 \)) there is a temperature associated, which is proportional to the acceleration of the said Killing, which is defined through

\[
\xi^\mu \nabla_\mu \xi^\nu = \kappa \xi^\nu
\tag{11}
\]

This expression depends on the normalization of the Killing vector. Here we assume that it is fixed by demanding that \( \xi^2 = 1 \) asymptotically.

This observation allows a thermodynamic interpretation of all horizons, somewhat similar to the one that Bekenstein and Hawking gave to the ordinary Schwarzschild’s horizon.

Let us then translate the thermodynamic formula

\[
\delta Q = T dS
\tag{12}
\]
in the vicinity of a Killing horizon. Consider a point \( P \) at a horizon \( \mathcal{H} \) with a corresponding Killing \( \xi \).

\[
T = \frac{h\kappa}{2\pi} \tag{13}
\]

where the acceleration of the normalized Killing can be computed through the convenient formula.

\[
\kappa^2 \equiv -\frac{1}{2} \nabla^\alpha \xi^\beta \nabla_\alpha \xi_\beta \tag{14}
\]

The flow of energy and momentum is given by the integral of the energy-momentum tensor

\[
\delta Q = \int_{\mathcal{H}} T_{\alpha\beta} \xi^\alpha \, d\sigma^\beta \tag{15}
\]

Call \( k \) the tangent vector to the horizon and \( \lambda \) the affine parameter (negative in the past of the point under consideration). Then

\[
\xi^\alpha = -\kappa \lambda^k \tag{16}
\]

as well as

\[
d\sigma^\alpha = k^\alpha \, d\lambda \, dA \tag{17}
\]

where \( dA \) is the element of area on a cross section of the horizon.

\[
\delta Q = -\kappa \int_{\mathcal{H}} \lambda \, T_{\alpha\beta} \xi^\alpha \, k^\beta \, d\lambda \, dA \tag{18}
\]

Now the essential input is the assumption that the entropy associated to a Killing horizon is not extensive (that is, scaling with the volume); but that it rather scales with the area

\[
dS = \eta \delta A \tag{19}
\]

where \( dA \) is the area variation of a cross section of a pencil of generators of the horizon. It is a fact that

\[
\delta A = \int_{\mathcal{H}} \theta \, d\lambda \, dA \tag{20}
\]

where the expansion of the congruence, \( \theta \), is defined as

\[
\theta \equiv \nabla^\mu k^\nu \tag{21}
\]

The dependence of this expansion is given by Raychaudhuri’s equation

\[
\frac{d\theta}{d\lambda} = -\frac{1}{2} \theta^2 - \sigma^2 - R_{\alpha\beta} k^\alpha k^\beta \tag{22}
\]

which can be trivially integrated, assuming that at the point chosen

\[
\theta = \sigma = 0 \tag{23}
\]

yielding

\[
\theta = -\lambda \, R_{\alpha\beta} k^\alpha k^\beta \tag{24}
\]

which in turn implies that

\[
\delta A = - \int_{\mathcal{H}} \lambda \, R_{\alpha\beta} k^\alpha k^\beta \, d\lambda \, dA \tag{25}
\]

Horizon thermodynamics then is telling us that

\[
T_{\alpha\beta} k^\alpha k^\beta = \frac{h\eta}{2\pi} R_{\alpha\beta} k^\alpha k^\beta \tag{26}
\]

for all null vector \( k \). This means that

\[
T_{\alpha\beta} = \frac{h\eta}{2\pi} R_{\alpha\beta} + \mathcal{F}_{\alpha\beta} \tag{27}
\]

Conservation of the energy momentum tensor then implies Einstein’s equations

\[
R_{\alpha\beta} - \frac{1}{2} (R + 2\lambda) g_{\alpha\beta} = \frac{2\pi}{\eta} T_{\alpha\beta} \tag{28}
\]

Consistency then demands that

\[
\eta = \frac{1}{4\pi G} \tag{29}
\]

In Erik Verlinde’s version this argument is called entropic. The root of it, however, still lies in the hypothesis that the gravitational entropy is proportional to the area. The whole argument does not generalize easily to non-static situations, like for example, to the treatment of cosmology, in which time dependence is essential.

In order to explain the existence of dark matter, Verlinde was forced to introduce a second set of quantum degrees of freedom, which are such that their entropy is extensive (that is proportional to the volume, like in normal laboratory systems).

This in turn allowed him to make some predictions on the static distribution of dark matter (no dynamics yet). It seems that for the time being observations are not supporting this alternative viewpoint. More tests are on the way though.

At any rate, what these works prove is that independently of what the fundamental quantum degrees of freedom could be, black hole (or even horizon) thermodynamics leads to Einstein’s equations in an almost unique way. Black hole physics acquires then an essential role as determining in some sense the whole dynamics of the gravitational field.

### 4. Physics of Gravitons

One of the smoking guns of a fundamental interaction is that there should be an intermediate particle responsible for it; in our case, this would be the graviton. Physics of gravitons in flat space (and even in a curved background) can be studied by gauge theory techniques, as was pioneered long ago by Feynman and DeWitt, and culminating in the famous calculation to one loop by ‘t Hooft and Veltman.\(^\text{[42]}\) This is still a very active topic of research. And this in spite of the fact that graviton cross sections are so small that there is no conceivable way in which they can be measured with present colliders. The reason is that this way of making precise computations, and any inconsistency found here will presumably serve as a hint of what could happen in the non-perturbative regime as well as indications of the region of parameter space in which this non-perturbative physics will be located.
There are indications that in some aspects gravity is similar to a double copy of a gauge theory. Amplitudes diverge much less than expected by naive power-counting arguments, and it is not excluded that the maximum supersymmetric theory with 32 supercharges (N=8 supergravity in 4 dimensions) could be all loop finite.

We can parametrize our ignorance on the fundamental ultraviolet (UV) physics by writing down all local operators of dimension $D$ in the low energy fields $\phi_i(x)$ compatible with the basic symmetries of all matter fields, represented schematically by $\phi_i$

$$L = \sum_{D=0}^{\infty} \frac{\lambda_D}{\Lambda^D} \mathcal{O}^{(D+4)} [\phi_i]$$

Here $\Lambda$ is an ultraviolet (UV) cutoff, which damps the contributions of large euclidean momenta (or small euclidean distances) and $\lambda_D$ is an infinite set of dimensionless couplings.

Standard Wilsonian arguments imply that irrelevant operators, corresponding to $D > 0$, which means that the total dimension of the operator is bigger than four, are less and less important as we are interested in deeper and deeper infrared (IR) (low energy) variables. The opposite occurs with relevant operators, corresponding to $D < 0$, so that the total dimension of the operator is less than four, like the masses, that become more and more important as we approach the IR. The intermediate role is played by the marginal operators, corresponding to precisely $D = 0$, so that the dimension of the operator is exactly four, and whose relevance in the IR is not determined solely by dimensional analysis, but rather by quantum corrections. The range of validity of any finite number of terms in the expansion is roughly $E \leq \Lambda$, where $E$ is a characteristic energy of the process under consideration.

In the case of gravitation, we assume that general covariance (or diffeomorphism invariance) is the basic symmetry characterizing the interaction. In order to define fermions we need a locally inertial frame, $\bar{e}_\mu \partial_\mu$ as well as a Lorentz (spin) connection, $\omega$ such that the spinorial covariant derivative is given by $D \equiv \partial_\mu + \omega_\mu$. Then, somewhat symbolically

$$L_{\text{eff}} = \sqrt{-g} \left\{ \lambda_0 \Lambda^4 + \lambda_1 \Lambda^2 R + \lambda_2 R^2 + \frac{1}{2} \bar{\psi} \gamma^a \nabla_\mu \psi \gamma^\mu \phi \\
+ \frac{\lambda_1}{\Lambda^2} R^a \nabla_\mu \psi \nabla_\mu \phi + \frac{\lambda_2}{\Lambda^4} R^3 + \lambda_3 \phi^4 \right\} \\
+ \bar{\psi} \bar{e}_a^\mu \gamma^a D_\mu \psi + \frac{\lambda_3}{\Lambda^4} \bar{\psi} \bar{e}_a^\mu \gamma^a R D_\mu \psi + \cdots \right.$$  \hspace{1cm} (30)

We have represented by $R$ any combination of the Riemann tensor and the spacetime metric, leaving implicit the detailed index structure. If we aim to recover General Relativity in the classical IR limit we are forced to match

$$\lambda_1 \Lambda^4 = -\frac{c^3}{16\pi G} \equiv -2M_p^4$$

This in turn, means that if $\lambda_0 \Lambda^4$ is to yield the observed value for the cosmological constant (which is of the order of magnitude of Hubble’s constant, $H_0^2$), which is a very tiny figure when expressed in particle physics units, $H_0 \sim 10^{-33}$ eV) then

$$\lambda_0 \sim 10^{-244}$$

This is one aspect of the cosmological constant problem; it seems most unnatural that the cosmological constant is observationally so small from the effective lagrangian point of view.

This fact can indeed be used as an argument against the whole effective lagrangians philosophy. I do not have anything new to say on this, except the obvious comment that before dismissing the whole idea one has to put on the other side of the balance the enormous successes of the effective lagrangian techniques in describing low energy physics both in QCD and in the electroweak model.

This expansion is fine as long as it is considered a low energy expansion. As Bjerrum-Bohr, Donoghue and Holstein[14,24] have emphasized, even if it is true that each time that a renormalization is made there is a finite arbitrariness, there are physical predictions stemming from the non-local finite parts.

At any rate, when energies are reached that are comparable to Planck’s mass,

$$E \sim M_p$$

Then all couplings in the effective Lagrangian become of order unity, and there is no decoupling limit in which gravitation can be considered by itself in isolation from all other interactions. This then seems the minimum prize one has to pay for being interested in quantum gravity; all couplings in the derivative expansion become important simultaneously.

No significant differences in this respect appear when supergravity is considered.

The root of the problem lies in the assumption of Diff invariance, which affects all spacetime fields. This assumption in turn has its roots on the equivalence principle, which implies the existence of LIF (locally inertial frames) at every point of the spacetime manifold.

On the other hand, it used to be thought that all QFT involving gravity were necessarily divergent in a non-renormalizable way, even when considering gravitons in flat space. For pure Einstein-Hilbert gravity this was first shown by Goroff and Sagnotti[36] who found a divergent piece at two loops that did not vanish on-shell. For example, it was believed that N=8 supergravity (a theory with 32 supercharges) would be divergent starting at 3 loops (10^24 diagrams approximately would have to be computed to check this).

Recent advances in the computation of QFT amplitudes however, (mainly by Zvi Bern and coworkers[13]) have unveiled unsuspected cancellations, that can not be explained by known symmetries.

The key idea of this research was to build on shell amplitudes starting solely from three-particle vertices, making thus irrelevant all the complicated higher point vertices that used to cloud quantum gravity computations. This idea in turn was inspired by string theory, where gravitons appear in the closed string sector, and gauge fields in the open string sector.

Another important insight (stemming also from string theory) was that in many respects, gravity amplitudes behave as a double copy of the much simpler gauge theory amplitudes.
At tree level there is a duality of sorts between color factors on the gauge side and kinematic factors in quantum gravity. Symbolically, if a gauge amplitude is

$$A = ig^{-2} \sum_i c_i n_i D_i$$

(31)

where $c_i$ are color factors, $n_i$ are kinematics factors, and $D_i$ propagator denominators, then the corresponding gravity amplitude is

$$M = i\kappa^{-2} \sum_i n_i^2 D_i$$

(32)

Now there seems to be a consensus that the first divergences will appear in N=8 supergravity (Sugra) not before 7 loops. Some people even believe that the theory could be all loop finite.

Bern’s group[23] has recently proved the finiteness of this very theory up to 5 loops; the critical dimension at which the first divergence appears is $D_c = \frac{25}{4}$.

At any rate, there is ample evidence in gauge theories (QCD in particular) that the non-perturbative sector is quite important (in the case of QCD it dominates the low energy infrared regime).

It is likely that non-perturbative effects will be even more complicated in quantum gravity, to the extent that string theory dualities[41] serve as some indication of the true physics.

4.1. The Background Field Approach in Quantum Field Theory

Bryce DeWitt[20] pioneered the use of covariant methods in QFT. Thanks to those methods we can, for example study quantum fluctuations around an arbitrary gravitational background. The technique he introduced is precisely named the background field method which later on found applications in some gauge theory computations as well.[1] We can give here but a superficial glimpse of what it is about. Consider the vacuum persistence amplitude in the presence of an arbitrary source (sometimes called the partition function by analogy with statistical mechanics).

$$Z[J] \equiv \int D\phi \ e^{\int[\phi + i/\hbar W_\phi]J(\phi)}$$

(33)

Where in this formal analysis we represent all fields (including the gravitational field) by $\phi(x)$, and we add a coupling to an arbitrary external source as a technical device to compute Green functions out of it by taking functional derivatives of $Z[J]$ and then putting the sources equal to zero. This trick was also invented by Schwinger (DeWitt’s former advisor). The partition function generates all Green functions, connected and disconnected. Its logarithm, $W[J]$ sometimes dubbed the free energy (this name also comes from a direct analogy with similar quantities in statistical physics), generates connected functions only.

It is possible to give an intuitive meaning to the path integral in quantum mechanics as a transition amplitude from an initial state to a final state. This is actually the way Feynman derived it.

In quantum field theory (QFT) the integration measure is not mathematically well-defined. For loop calculations, however, it is enough to formally define the gaussian path integral as a functional determinant, that is

$$\int D\phi \ e^{iW[\phi]} = (\det K)^{-\frac{1}{2}}$$

(34)

where the scalar product is defined as

$$\langle \phi | K | \phi' \rangle \equiv \int d(\text{vol}) \ \phi K \phi'$$

(35)

and $K$ is a differential operator, usually

$$K = \nabla^2 + \text{something}$$

(36)

There are implicit indices in the operator to pair the (also implicit) components of the field $\phi$.

The only extra postulate needed is translation invariance of the measure, in the sense that

$$\int D\phi \ e^{i\langle \phi + x | [K, (\phi + x)] \rangle} = \int D\phi \ e^{i\langle \phi | K | \phi \rangle}$$

(37)

This is the crucial property that allows the computation of integrals in the presence of external sources by completing the square.

It is quite useful to introduce a generating function for one-particle irreducible (1-PI) Green functions. This usually called the effective action and is obtained through a Legendre transform, quite analogous to the one performed when passing from the Lagrangian to the hamiltonian ion classical mechanics.

One defines the classical field as a functional of the external current by

$$\phi_c[J] \equiv \frac{1}{i} \frac{\delta W[J]}{\delta J(x)}$$

(38)

This equation allows, by the implicit function theorem, the formal definition of the inverse function, $J = J[\phi_c]$. The Legendre transform then reads

$$\Gamma[\phi_c] \equiv W[J] - i \int d^n x J(x) \phi_c(x)$$

(39)

It is a fact that

$$\frac{\delta \Gamma}{\delta \phi_c(x)} = \int d^n z \ \delta W[J] \frac{\delta J(z)}{\delta \phi_c(x)} - i J(x)$$

$$- i \int d^n z \phi_c(z) \frac{\delta J(z)}{\delta \phi_c(x)} = - i J(x)$$

(40)

Let us introduce now the background field technique first in the language of Yang-Mills theories. The main idea is to split the integration fields into a classical and a quantum piece:

$$W_\mu \equiv \tilde{A}_\mu + A_\mu$$

(41)
The functional integral is performed over the quantum fields only, where for an ordinary gauge theory the action has three pieces. First the gauge invariant piece

\[ L_{\text{gauge}} \equiv -\frac{1}{4} F_{\mu \nu} [W]^2 \]  

with

\[ F_{\mu \nu} [W] \equiv \partial_{\mu} W_{\nu} - \partial_{\nu} W_{\mu} + g f_{\alpha \beta \gamma} W_{\mu}^{\alpha} W_{\nu}^{\beta} W_{\gamma}^{\gamma} \]  

where as usual,

\[ [T_{\alpha}, T_{\beta}] = i f_{\alpha \beta \gamma} T_{\gamma} \]  

The gauge transformations with parameter \( \omega^\alpha \) are

\[ \delta W_\mu^\alpha \equiv \delta ( \Lambda_\mu^\alpha + A_\mu^\alpha ) \equiv -f_{\alpha \beta \gamma} \omega^\beta W_\mu^\gamma + \frac{1}{8} \partial_\mu \omega^\alpha \]

\[ = -f_{\alpha \beta \gamma} \omega^\beta ( \Lambda_\mu^\alpha + A_\mu^\alpha ) + \frac{1}{8} \partial_\mu \omega^\alpha \]  

This can be implemented in two ways. First letting the background field be inert. Those are the quantum gauge

\[ \delta_Q \Lambda_\mu^\alpha \equiv 0 \]

\[ \delta_Q A_\mu^\alpha = -f_{\alpha \beta \gamma} \omega^\beta ( \Lambda_\mu^\alpha + A_\mu^\alpha ) + \frac{1}{8} \partial_\mu \omega^\alpha \]  

Those are the transformations in need of gauge fixing. It is to be remarked that gauge symmetry is realized non-linearly on the quantum fields.

It is also possible to reproduce the total gauge transformations through the classical transformations

\[ \delta_C \Lambda_\mu^\alpha = -f_{\alpha \beta \gamma} \omega^\beta \Lambda_\mu^\alpha + \frac{1}{8} \partial_\mu \omega^\alpha \]

\[ \delta_C A_\mu^\alpha = -f_{\alpha \beta \gamma} \omega^\beta A_\mu^\gamma \]  

under which the quantum field transforms linearly as an adjoint vector field.

Currents transform in such a way that

\[ \delta_C \int J_\nu^\alpha A_\nu^\alpha = 0 \]  

that is

\[ \delta_J^\alpha = -f_{\alpha \beta \gamma} \omega^\beta J_\gamma^\alpha \]  

The beauty of the background field method is that it is possible to gauge fix the quantum symmetry while preserving the classical gauge symmetry. All computations are then invariant under gauge transformations of the classical field, and so are the counterterms. This simplifies the heavy work involved in computing Feynman diagrams in the presence of dynamical gravity.

The simplest background gauge is

\[ \bar{F}[A] \equiv \partial_\mu A_\nu^\alpha + g f_{\alpha \beta \gamma} \Lambda_\mu^\beta A_\nu^\gamma \equiv ( \bar{D}_\mu A )^\alpha \]  

L. Abbott\(^{1)}\) was able to prove a beautiful theorem to the effect that the effective action computed by the background field method is simply related to the ordinary effective action

\[ \Gamma_{BF}[A_{\delta}] = \Gamma_{BF}[A_{\mu}] \mid_{A_{\mu} = A_{\delta} + \lambda} \]  

This means in particular, that

\[ \Gamma_{BF}[A] = \Gamma_{BF}[0, A = A_\Lambda] \]  

At the one loop order all this simplifies enormously. Let us spell in detail the simplest scalar case. Working in euclidean space, introducing sources for the quantum fields only, and in a schematic notation,

\[ e^{-W[\phi]} \equiv \int D\phi e^{-S[\phi]} / d(\text{vol}) J_\phi S[\phi] / d(\text{vol}) J_\phi \phi \]

\[ = e^{-S[\phi] - 1/2 \log \det K[\phi]} / d(\text{vol}) J K^{-1}[\phi] \]

This means that the classical field in the background field formalism is given by

\[ \phi_c = -\frac{1}{2} K^{-1}[\tilde{\phi}] \]

so that the sources read

\[ J = -2 K[\phi] \phi_c \]

and the action can be written as

\[ \Gamma^{BF}[\phi_c, \tilde{\phi}] = W[\bar{J}[\phi_c]] - \int d(\text{vol}) J_\phi S[\phi] + \frac{1}{2} \log \det K[\phi] + \int d(\text{vol}) \phi_c K[\tilde{\phi}] K^{-1}[\tilde{\phi}] K[\phi] \phi_c - 2 \int d(\text{vol}) \phi_c K[\tilde{\phi}] \phi_c - \int d(\text{vol}) \phi_c K[\tilde{\phi}] \phi_c \]

Then by Abbott’s theorem

\[ \Gamma(\phi_c) = \Gamma^{BF}[0, \tilde{\phi} = \phi_c] = W[\tilde{\phi}] \equiv S[\tilde{\phi}] + \frac{1}{2} \log \det K[\tilde{\phi}] \]

The one loop effective action is equal to the background field free energy, and the background field can be equated to the classical field.

The case of gravitations is analogous (but for algebraic complexity). We again split the gravitational field as

\[ g_{\mu \nu} = \bar{g}_{\mu \nu} + \kappa h_{\mu \nu} \]  

This is done so that the mass dimension of the graviton field \( h_{\mu \nu} \) is equal to one. The full diffeomorphism gauge symmetry

\[ \delta g_{\mu \nu} \equiv \delta (\bar{g}_{\mu \nu} + \kappa h_{\mu \nu}) = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \]

\[ = \xi^\mu \partial_\mu \xi_\nu + g_{\mu, \nu} \partial_\mu \xi^\nu + g_{\nu, \mu} \partial_\mu \xi^\nu + g_{\mu, \nu} \partial_\mu \xi^\nu \]
includes both the one-loop background field transformations

\[
\delta g_{\mu\nu} = \epsilon(\xi) g_{\mu\nu}, \\
\delta h_{\mu\nu} = \frac{1}{\kappa} (\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu) \\
\delta \eta_{\mu\nu} = 0
\] (60)

(thelterm\(\xi^i \partial_i h_{\mu\nu}\)is of two-loop order). It also includes the quanti-

tum gauge transformations

\[
\delta g_{\mu\nu} = 0 \\
\delta h_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu
\] (61)

A convenient gauge fixing is the harmonic or de Donder gauge, defined by

\[
\nabla_\mu h^{\mu\nu} = \frac{1}{2} \nabla_\nu h
\] (62)

(where \(h \equiv \nabla^{\alpha} h_{\alpha\beta}\)). The reason for the name is as follows. Con-

sider the coordinates as functions. Then

\[
\Box \psi = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu}) = -\Gamma^{\mu}_{\rho\sigma} g^{\rho\sigma}
\] (63)

When perturbing around flat space

\[
g_{\mu\nu} \equiv \eta_{\mu\nu} + \kappa h_{\mu\nu}
\] (64)

it so happens that

\[
\Gamma^{\mu}_{\rho\sigma} g^{\rho\sigma} = \frac{1}{2} \left( -\partial^\mu h + 2 \partial_\nu h^{\nu\mu} \right)
\] (65)

and the gauge condition is equivalent to the demand that the co-

ordinates are harmonic when considered as functions on the mani-

fold.

In conclusion, there is a systematic way of studying quantum
fluctuations around an arbitrary background, at least as long as
this background is a solution of the classical Einstein’s equations.
This should be enough to study quantum corrections to gravita-
tional waves, for example.

It is not yet excluded that some supersymmetric versions of this
problem can lead to finite theories, at least when the back-
ground is flat.

Some non-generic subtle points appear in the frequent case
when the background has got horizons and/or singularities.

5. What are the Questions Whose Response We

are Seeking for?

Let us think for a moment what are the questions we would like
to be answered by our theory of quantum gravity.

We are used in quantum mechanics or quantum field theory
(QFT) to be able to answer questions on probability amplitudes
for transitions between states at given times (or given spacetime
points).

Related computations involve correlators of strings of space-
time fields acting on the vacuum. Reduction techniques relate

those correlators to S matrix elements, which are directly related
to cross sections that can be observed in colliders.

In some Hilbert space of sorts (that is, a Fock space) ,
Schödinger’s equation (or equivalently, Heisenberg’s equation of
motion) still holds. Those relate the time derivative of either the
physical state or else the spacetime fields to some second mem-
ber involving the full hamiltonian of the system.

It is quite difficult to generalize any of these concepts to quantum
gravity. This is specially so if we insist (as we do) in the corre-

spondence principle alluded to formerly; that is, demanding a
smooth limit when \(\kappa \to 0^+\)

• The first thing is that there is no a natural definition of time in
GR (and correspondingly of a hamiltonian). We shall review in
a moment why this is so.

• Besides, the starting point for the analysis of causality (and uni-



tarity, more on this later) in QFT is the microcausality condition,

namely

\[(x - y)^2 < 0 \Rightarrow [\phi(x), \phi(y)] = 0\] (66)

that is, fields at spacelike separated points do commute. It is
not known what is the correct generalization of this idea to GR,
or even if there is a natural notion of causality there.

• As a consequence, there is no clear definition of causality. At
the classical level, it can be related to conjectures that aim at
keeping a well-defined Cauchy problem, such as the Hawking’s
chronology conjecture (no closed timelike curves) or Penrose’s cos-
ic censorship (no naked singularities), but even those ideas are
not easy to generalize at the quantum level.

• Something similar happens with the related concept of unitar-

ity. In QFT it is believed to be equivalent to

\[SS^* = 1\] (67)

but in general spacetimes (i.e., not asymptotically flat) there is
no concept of S-matrix, so that it is not clear what is the
physical meaning of amplitudes or even S-matrix.

• Finally, when quantum gravity is applied to the whole universe
(quantum cosmology) besides the conceptual question of what
is the observable meaning of a transition amplitude between
two different 3-space geometries, (we do not have a set of uni-
verses at our disposal) there is unavoidably the general ques-
tion of how to generalize the Copenhagen interpretation of
quantum mechanics.\(^{[39]}\) because there is no possible distinc-
tion between the observer and the system. Then, we are in a
real swampland.

What is the physical meaning of the assertion that the
probability of transition from one 3-space geometry \(\Sigma_i\) to an-
other one \(\Sigma_j\) at some other cosmic time \(T\) is a given number
\(P(\Sigma_i, \Sigma_j, T)\)? How can we verify those hypothetical predictions,
even in principle?

It is quite often mentioned that quantum gravity would
give information on space-time singularities, both inside black
holes (where they are hidden behind a horizon) and at the big
bang, which is classically a naked singularity.

It may well be so. But not necessarily.
It could be the case that quantum gravity yields no informa-
tion beyond the platitude that in quantum mechanics geomet-
rical points are in contradiction with the uncertainty principle.

For example, in quantum electrodynamics (QED) Coulomb’s singular-
ity is hidden by the renormalization procedure, a procedure that tames all diver-
gences, but one that gives no particularly physical intuition on them, other
that the dependence of the fine structure constant with the
renormalization scale, as determined by the renormalization

group.

6. The Canonical Approach

It is widely acknowledged that there is a certain tension be-
tween a (3 + 1) decomposition implicit in any canonical approach, privile-
leging a particular notion of time, and the beautiful geometrical
structure of general relativity, with its invariance under general
coordinate transformations.

To begin with, it is clear that any hypothetical definition of ener-
gy cannot have any tensorial character. In fact, the equivalence
principle guarantees that there is a LIF (locally inertial frame) in
which there cannot be any tensorial quantity with that physical in-
terpretation. The reason is that at any point of spacetime one can
choose a Local Inertial Frame (LIF), in which the gravitational
potential (nothing to do with supersymmetry) has been nicely
reviewed in [19].

Let us now nevertheless explore how far we can go on this road.

6.1. Quasilocal Energy, Pseudotensors and Superpotentials

Before beginning with the canonical formalism proper, let us
make a few general remarks on the concept of energy in gen-
eral relativity (GR). The definition of energy in GR was (and still
is) one of the main problems that kept physicists and mathemat-
icians alike busy for many years, including Einstein, Hilbert,
Noether, Dirac, and so on. And this in spite of the fact that it is clear
that the equivalence principle tells us immediately the there cannot be any tensorial quantity with that physical inter-
pretation. The reason is that at any point of spacetime one can
choose a Local Inertial Frame (LIF), in which the gravitational
field vanishes (locally) and so must do its energy.

It is interesting to remind us of some of those efforts. They
have been nicely reviewed in [19].

Let us start by choosing a particular frame as well as a super-
potential (nothing to do with supersymmetry)

\[ H_{\mu}^{[\alpha]} = H_{\mu}^{[\alpha]} \]  
and split Einstein’s tensor in such a way that

\[ \kappa \sqrt{|g|} N^\mu t_\mu \equiv -N^\mu \sqrt{|g|} G_\mu^{[\alpha]} + \frac{1}{2} \partial_\lambda \left( N^\mu H^{[\alpha\beta]} \right) \]  

Now we particularize to a frame in which the components of the
vector \( N^\mu \) are constant

\[ \partial_\lambda N^\mu = 0 \]  

Then, in this frame, and using Einstein’s equations

\[ G_{\mu\nu} = \kappa T_{\mu\nu} \]  

we get

\[ \partial_\lambda H_{\mu}^{[\alpha]} = 2\kappa \sqrt{|g|} T_{\mu}^{\lambda} \equiv 2\kappa \sqrt{|g|} \left( t_\mu + T_{\mu} \right) \]  

Owing the the skewness of the superpotential, the total energy-
momentum is conserved in the ordinary, non-covariant sense

\[ \partial_\lambda \left( \sqrt{|g|} T_{\mu}^{\lambda} \right) = 0 \]  

This then yields a conserved energy-momentum

\[ N^\mu \mathcal{P}_\mu = \int N^\mu T_{\mu} \, d\Sigma \]  

In particular, the energy in a spacelike volume \( V \) is given by

\[ P_S(V) \equiv \int_V N^\mu T_{\mu} \, d\Sigma = \frac{1}{2\kappa} \int_V \partial_\lambda \left( N^\mu H_{\mu}^{[\alpha\beta]} \right) \, d\Sigma \]

\[ = \int_{\Sigma=\partial V} B^{[\alpha]}(N) \, d\sigma \]

where

\[ B^{[\alpha]}(N) \equiv \frac{1}{2\kappa} N^\mu H_{\mu}^{[\alpha\beta]} \]  

This has led to the authors of [19] to conclude that for any pseudo-
tensor, the associated superpotential is naturally a hamiltonian
boundary term. This energy so defined is quasilocal in the sense
that it does not depend on the values of the frame and the fields
on the whole volume whose energy is being computed; but only
on the corresponding values at the boundary of said volume.

There is no simple way in which this is conserved, though, except
in the familiar Arnowitt-Deser-Misner (ADM) case[9] of asym-
ptotically flat spacetimes.

Let us examine a concrete example that will lead to Møller’s
pseudotensor. Define the connection 1-forms

\[ \omega_{\mu}^{[\alpha]} \equiv \Gamma_{\rho\beta}^{[\alpha]} \, dx^\rho \]  

and the corresponding curvature 2-form

\[ \Omega_{\beta}^{[\alpha]} \equiv d\omega_{\beta}^{[\alpha]} + \omega_{\gamma}^{[\alpha]} \wedge \omega_{\beta}^{[\gamma]} \]  

Then the Einstein-Hilbert lagrangian can be written in the some-
what pedantic form

\[ L_{EH} = \Omega_{\beta}^{[\alpha]} \wedge \eta_{\alpha}^{[\beta]} \equiv \Omega_{\beta}^{\alpha} \wedge \ast \left( dx_{\alpha} \wedge dx^\beta \right) = R \, d(\text{vol}) \]  

The Hamiltonian can be defined as usual

\[ i_N L = i_N \Omega \wedge \eta + \Omega \wedge i_N \eta = \mathcal{L}_N \omega \wedge \eta - \mathcal{H}(N) \]  

where

\[ -\mathcal{H}(N) = -di_N \omega \wedge \eta + i_N \omega \wedge \omega \wedge \eta + \Omega \wedge i_N \eta \]

\[ = -d(i_N \omega \wedge \eta) + i_N \omega \wedge d\eta + i_N (\omega \wedge \omega) \wedge \eta + \Omega \wedge i_N \eta \]
First of all, define
\[ \eta \equiv e^{a_1 \ldots a_n} \epsilon^{1} \land \ldots \land \epsilon^{n} \] (82)

Then, for the metric connection
\[ D\eta \equiv d\eta + \sum e^{a_1 \ldots a_n} \omega^{a_1} \epsilon^{1} \land \ldots \land \epsilon^{n} = 0 \] (83)
owing to the condition
\[ de^a + \alpha^a \land \epsilon^a = 0 \] (84)

Let us now examine
\[ i_a \omega \land d\eta + i_\alpha (\omega \land \omega) \land \eta = i_a \omega \land D\eta = 0 \] (85)

It is useful to distinguish between \( g \)
\[ N^\mu H_\mu \equiv -\Omega^\mu \_g \land N^\mu \eta_\mu \] (86)
which is claimed to project to the usual ADM hamiltonian, and the boundary term
\[ B(N) \equiv i_a \omega \land \eta = N^a \omega^{\mu_1 \ldots \mu_n} \eta^a \equiv N^a \mathcal{M}^{a_1 \ldots a_n} dS_{a_1 \ldots a_n} \] (87)
with
\[ dS_{a_1 \ldots a_n} \equiv \frac{1}{2} \epsilon_{a_1 \ldots a_n} dx^1 \land dx^n \] (88)

To be specific
\[ \mathcal{M}^{a_1 \ldots a_n} = \sqrt{|g|} (g^{a_1 \ldots a_n} \Gamma^a_\mu \_\alpha - g^{a_1 \ldots a_n} \Gamma^a_\mu \_\alpha) \] (89)

This is the superpotential (again, nothing to do with supersymmetry) whose divergence yields Møller’s pseudotensor.

It is not clear what is the conclusion of this canonical analysis. It was clear since the beginning that there is a deep contradiction between the hamiltonian being a geometrical quantity on the one hand, and the principle of general covariance (or diffeomorphism invariance) which forbids privileging certain frames of reference. On the other hand, all known formulations of quantum mechanics are related to the hamiltonian in an essential way. There seems to be no easy way out. On the one hand, to insist that some frames of reference are special in some sense (like Fock’s harmonic coordinate systems) is ugly and goes against the beautiful GR philosophy. On the other hand, as we have already seen, it is very difficult to modify, whatever slightly, quantum mechanics.

The modern theory of hamiltonians for systems with constraints is rooted in the analysis of General Relativity. Anderson and Bergmann\(^{(1)}\) introduced the concepts of primary and secondary constraints, as well as first class and second class constraints. Al this was beautifully explained by Dirac\(^{(1)}\) in his famous Yeshiva lectures. Thereby it was noticed that for a system such that the lagrangian is homogeneous of first order in the velocities, that is
\[ \sum_{i=1}^{N} a^i \frac{\partial L}{\partial \dot{a}^i} = L \] (90)
the naive hamiltonian vanishes, because
\[ H \equiv \sum a^i \dot{a}^i - L = 0 \] (91)
The first thing to notice is that if we redefine the time coordinate as
\[ t \rightarrow \tau(t) \] (92)
then the action is invariant,
\[ \int dt L(q_\cdot, \frac{dq_\cdot}{dt}) = \int d\tau L(q_\cdot, \frac{dq_\cdot}{d\tau}) \] (93)
This means that the time variable can be chosen at will, the action is insensitive to this choice.

The second thing to notice is that necessarily the \( p_\cdot \) are homogeneous functions of degree zero of the velocities. That is, they are functions of the ratios of velocities, of which there are only \( N - 1 \) independent ones. This means that there is at least one primary constraint. The total hamiltonian consists in the sum of all primary first class constraints multiplied with arbitrary coefficients
\[ H_\tau \equiv \sum_\alpha v_\alpha \phi_\alpha \] (94)
The EM would read
\[ \dot{g} \sim \sum_\alpha v_\alpha [g, \phi_\alpha] \] (95)
It is clear that, in spite of the fact that we are in the framework of Newtonian mechanics, there is no absolute time variable here. Any monotonic function of \( t \) is valid as a new time variable. There is no absolute time; this is in agreement with our previous observations.

Now it is a fact that any theory can be put into this form just by defining the time variable as an extra coordinate \( t \equiv \dot{a}^0 \), and introducing a new time variable, \( \tau \).

\[ L_{\text{new}} \left( a^0, a^0, \frac{da^0}{d\tau}, \frac{da^0}{d\tau} \right) \equiv L \left( \frac{df}{d\tau}, \frac{df}{d\tau}, \frac{df}{d\tau}, \frac{df}{d\tau} \right) \] (96)

It is plain that the action is invariant
\[ S \equiv \int L_{\text{new}} \mathrm{d}\tau = \int L \mathrm{d}t \] (97)

6.2. Dirac Universal Brackets

In this section it will prove convenient to reserve the label \( y^a \) for the spacetime coordinates in Minkowski space to tell them apart from the \( 3 + 1 \) coordinates \((t, x')\) to be defined in the sequel.

Let us now consider a foliation of space-time given by the function
\[ t(y^a) = C \] (98)
The level hypersurfaces are spacelike; that is, the normal vector
\[ n_\alpha \equiv N \partial_\alpha t \]
is timelike, and will always be assumed normalized
\[ n^2 = 1 \]
so that
\[ \frac{1}{N^2} \equiv g^{\alpha \beta} \partial_\alpha t \partial_\beta t \]
We shall also assume the existence of a \( \infty \) congruence of curves
\[ y^\alpha \equiv \sigma^\alpha (x^i, \tau) \]
Each curve in the congruence is the presented by
\[ x^i = C^i \]
Tangent vectors on the \( t = \text{constant} \) hypersurface are given by
\[ \xi^\alpha_i \equiv \partial_\alpha \sigma^\alpha (t) \quad i = 1 \ldots 3 \]
The normal vector \( n \) is orthogonal to all tangent vectors
\[ g_{\alpha \beta} \xi^\alpha n^\beta = 0 \quad i = 1, \ldots, 3. \]
The vector tangent to the congruence is not necessarily normal to the hypersurfaces; it can be expanded in general as
\[ N^\alpha \equiv \frac{\partial \sigma^\alpha}{\partial t} \equiv N n^\alpha + N^i \xi^\alpha_i \equiv N n^\alpha + N^\alpha \]
where
\[ N \equiv N^\alpha n_\alpha = \frac{\partial \sigma^\alpha}{\partial t} N \partial_\alpha t \]
is usually denoted as the lapse and \( N \) is the shift in ADM's (Arnowitt-Deser-Misner) notation.

Again, this means that the vector that goes from the point \((t, x) \in \Sigma \) to the point \((t + dt, x) \in \Sigma_{t+dt}\) does not lie necessarily in the direction of the normal to the hypersurface \( t = \text{constant} \).

Actually from consistency of the previous definitions it follows that
\[ N^\alpha \partial_\alpha t = 1 \]
Also our parametrization of the curves of the congruence as \( x^i = C^i \) imply that
\[ \mathcal{L}(N^n) \xi^\alpha_i = 0 \]
Finally, the fact that the coordinates \( x^i \) on each surface are independent means that, considered as spacetime vectors, \( \xi^\alpha_i \equiv \xi^\alpha_i \partial_\alpha \)
\[ \left[ \xi^\alpha_i, \xi^\beta_j \right] = 0 \]
The \( \infty \) dynamical variables \( \sigma^\alpha \) will have some canonically conjugate momenta
\[ \{ \sigma^\alpha (t, x), \pi_\alpha (t, y) \} = \delta^\alpha_\beta \delta^i_1 (x - y) \]
If the generalized coordinates \( \sigma^\alpha \) are to vary at all in the dynamics, the constraints have to involve the conjugate momenta, so that it must be possible to write the constraints as
\[ H_\tau = \int d^3x \ c^\alpha (t, x) (\pi_\alpha + K_\alpha) \]
Then it is a fact that
\[ \dot{\sigma}^\alpha \equiv \{ \sigma^\alpha, H_\tau \} = c^\alpha \]
It is useful to decompose any vector index into normal and tangential components
\[ V_\alpha \equiv V.n \equiv g_{\alpha \beta} V^\beta n_\beta \]
\[ V_i \equiv V.\xi_i \equiv g_{\alpha \beta} V^\alpha \xi_i^\beta \]
There is always the danger of taking \( V_i \) such defined as the space components of the \( n \)-dimensional quantity \( V \), but we shall try not to do so in the sequel. Actually,
\[ V^\alpha = V.n^\alpha + V_i^i \xi^\alpha_i \]
provided we define as usual the \textit{induced metric} on the hypersurface as
\[ h_{ij} \equiv \xi^\mu_i g_{\mu \nu} \xi^\nu_j \]
which is nonsingular if the hypersurfaces are everywhere spacelike. Then we define the inverse matrix
\[ h^{ij} h_{ij} \equiv \delta^i_1 \]
(\text{Please notice that in general, when the shift does not vanish,})
\[ h^{ij} \neq g^{ik} g^{jm} h_{km} \]
The spacetime metric can be reconstructed out of the lapse and shift through
\[ ds^2 = g_{\mu \nu} dx^\mu dy^\nu = g_{\mu \nu} (N^\mu dt + \xi^\mu_i dx^i) (N^\nu dt + \xi^\nu_i dx^i) \]
\[ = N^2 dt^2 + h_{ij} (dx^i + N^i dt) (dx^j + N^j dt) \]
This in turn implies
\[ g_{\mu \nu} = n_\mu n_\nu + \xi_\mu^i \xi_\nu^i \]
Then Dirac showed that for all these systems there is an universal set of Poisson brackets, namely,
\[
\{\pi_n(t, x), \pi_n(t, x')\} = \pi_n(t, x)\delta(x - x') + \pi_n(t, x')\delta(x - x')
\]
\[
\{\pi_n(t, x), \pi_n(t, x')\} = \pi_n(t, x')\delta(x - x')
\]
\[
\{\pi_n(t, x), \pi_n(t, x')\} = -2\pi_n(t, x)\delta(x - x') - \Delta\pi(t, x)\delta(x - x')d
\]
(121)

where we define to save space
\[
\delta(x - x') \equiv \delta^{ij}(x - x')
\]
(122)

and as usual the ordinary thee-dimensional laplacian is
\[
\Delta \equiv \partial_i\partial_j
\]
(123)

Let us work this out in some detail.

0 = \{n_j, n_j, \pi_n, \pi_n\} \xi^\mu_n + n_n \{\xi^\mu_n, \pi_n, \pi_n\}
(124)

We learn that
\[
\{n_j, \pi_n, \pi_n\} \xi^\mu_n = -n_j, \partial_\delta(x - x')
\]
(125)

Again,
\[
0 = \frac{1}{2} \{n_j, n_j, \pi_n, \pi_n\} n_n
\]
(126)

Then
\[
\{n_j, \pi_n, \pi_n\} = \{n_j, \pi_n, \pi_n\} \left(n_n n_i + \xi^\mu_n \xi^\mu_j h^\mu \right)
\]
(127)

Notice that we have defined
\[
\delta_{ij}(x - x') \equiv \delta_{ij}(x - x')
\]
(128)

It follows that
\[
\{n_j, \pi_n, \pi_n\} = \{n_j, \pi_n, n_n\} = -n_n n_j \delta_{ij}
\]
\[
= -\xi^\nu_j \partial_{ij} (n_n n_i \delta(x - x')) + \xi^\nu_j \partial_{ij} (n_n n_n \delta)
\]
\[
= -\xi^\nu_j \partial_{ij} (x - x')
\]
(129)

Before going on, let us show an elementary relationship. It is plain that
\[
\partial_j (n_j, \xi^\mu_n) = 0 = \partial_j n_j, \xi^\mu_n + n_j, \partial_\xi^\mu_n
\]
(130)

as well as
\[
\partial_\xi^\mu_n \partial n_n = 0
\]
(131)

and multiplying by \(\xi^\nu_j\)
\[
\partial_j n^\nu_j = \xi^\nu_j n \xi^\nu_j = \xi^\nu_j \partial n, \equiv n_{\nu_j}
\]
(132)

Let us now define the construct \(n_{\nu_j}\)
\[
\{n_j, n_j, \pi_n, \pi_n\} = n_j, \xi^\nu_j \partial n \xi^\nu_j = \xi^\nu_j \partial n, \equiv n_{\nu_j}
\]
(133)

It follows that
\[
\{n_j, \pi_n, \pi_n\} = \{n_j, \pi_n, \pi_n\} \xi^\mu_j \delta(x - x') = -\delta_i (n_j, \xi^\nu_j \xi^\mu_i \delta(x - x')) + \delta_i (n_j, n_i \delta) \delta(x - x')
\]
(134)

Finally
\[
\{n_j, \pi_n, \pi_n\} = \{n_j, n_n, \pi_n, \pi_n\} = \pi_i n^\nu_i \pi_n, \pi_n, n_n \pi_i, \pi_i, \pi_n \pi_i, n_n \pi_i, \pi_n
\]
\[
= \pi_i n^\nu_i \pi_n, \pi_n + \pi_i n^\nu_i \pi_i, \pi_i, \pi_n \pi_i, n_n \pi_i, \pi_n
\]
\[
+ \pi_i n^\nu_i \pi_i, \pi_i, \pi_n \pi_i, n_n \pi_i, \pi_n
\]
(135)

We have presented Dirac’s results with (perhaps excessive) detail to highlight the generality and beauty of Dirac’s algebra. Notice that no dynamics enters into the proof; all results are purely kinematical as a consequence of having assumed from the very beginning that there is no preferred notion of time. This is an eccentric luxury in flat spacetime, but it will become compulsory in General Relativity.

### 6.3. The Arnowitt-Deser-Misner (ADM) Formalism

Let us apply Dirac’s ideas to the gravitational field. We shall assume that there is a foliation as before. Remember the components of the spacetime metric in terms of the lapse and shift functions.

\[
g_{00} = N^2
\]
\[
g_{0i} = h_i N^\nu
\]
\[
g_{ij} = h_i h_j
\]
(136)

whose inverse reads

\[
g^{00} = N^{-2}
\]
\[
g^{0i} = \frac{N^i}{N^2}
\]
\[
g^{ij} = h^i + \frac{N^i N^j}{N^2}
\]
Let us denote by $D_i$ the covariant derivative with respect to the three-dimensional Levi-Civita connection associated to the induced metric, $h_{ij}$.

It can be easily checked that
\[
D_i A_j = \nabla^\rho A_\rho \varepsilon^\rho_j.
\] (137)

From the definition itself of the induced metric it follows that
\[
\partial_\rho \varepsilon^\rho_j = \varepsilon^\rho_j.
\] (138)

Perform now cyclic permutations in the above
\[
\partial_\rho \varepsilon^\rho_j = \varepsilon^\rho_j \quad (\text{because of } [145]).
\] (148)

First of all let us derive some properties of the extrinsic curvature. It is symmetric, $K_{ij} = K_{ji}$.

On the other hand, the explicit expression for the extrinsic curvature reads
\[
K_{ij} = -\varepsilon^\rho_i \nabla_\rho \varepsilon^\rho_j.
\] (146)

Let us now relate the Riemann tensor on the hypersurface (computed with the induced metric) with the corresponding Riemann tensor of the spacetime manifold. Those are the famous Gauss-Codazzi equations, which we purport now to derive.

They were one of the pillars of Gauss' *theorema egregium*, which asserts that *If a curved surface is developed upon any other surface whatever the measure of curvature in each point remains unchanged.*

We start with
\[
0 = D_j (g_{\alpha\beta} n^\alpha n^\beta) = D_j \sigma^\rho (\{\alpha \rho \beta\} + \{\beta \rho \alpha\}) n^\alpha n^\beta + g_{\alpha\beta} D_j n^\alpha n^\beta = g_{\rho\sigma} n^\rho \nabla_\sigma n^\beta = \varepsilon^\rho_i \nabla_\rho n^\beta.
\] (145)

On the other hand, remembering that
\[
\varepsilon^\rho_i \nabla_\rho n^\beta = \nabla_\rho (\varepsilon^\rho_i n^\beta) = \varepsilon^\rho_i \nabla_\rho n^\beta = K_{ij}
\] (149)

This symmetry implies a very useful formula for the extrinsic curvature, namely
\[
K_{ij} = \nabla_\rho (\varepsilon^\rho_i n^\beta) = \varepsilon^\rho_i \nabla_\rho n^\beta = K_{ij}
\] (150)

By the way, in the physics jargon when $K_{ij} = 0$ it is said that it is a *moment of time symmetry.*

On the other hand, remembering that
\[
\varepsilon^\rho_i \nabla_\rho n^\beta = \varepsilon^\rho_i - n^\rho n_\beta
\] (151)

we deduce
\[
-K_{ij} \varepsilon^\rho_i = - (\varepsilon^\rho_i n^\rho) \nabla_\rho n^\beta = - \nabla_\rho n^\rho n^\beta
\] (152)

(because of [145]).

Let us analyze the definition of extrinsic curvature in even more detail. We follow the explicit computations in the classic reference
\[
D_k \sigma^\rho = \nabla^\rho \sigma^\rho = \nabla^\rho (\{\alpha \rho \beta\} + \{\beta \rho \alpha\}) = \varepsilon^\rho_i \nabla_\rho n^\beta.
\] (143)

so that the quantity we have just defined
\[
K_{ik} = n_\rho \{ \varepsilon^\rho_i \sigma^\rho = - \nabla^\rho \sigma^\rho (\{\alpha \rho \beta\} + \{\beta \rho \alpha\}) = \varepsilon^\rho_i \nabla_\rho n^\beta
\] (144)

This tensor is called the *extrinsic curvature*, and represents the four-dimensional covariant derivative of the normal vector, projected on the surface.
and using again the definition of the extrinsic curvature to eliminate the term with two derivatives,  
\[
\xi^a_m \tilde{R}_{ij}(h) = -\partial_i \left( \frac{\partial^a}{\partial \tau^i} n^\rho \xi^\rho_m - \left( \partial^a_{\rho \tau} n^\rho \xi^\tau_m + K^a_{\rho m} n^\rho \right) \right) \\
+ D_k K_m n^a + \frac{\partial}{\partial \tau^i} \left( \frac{\partial^a}{\partial \tau^i} n^\rho \xi^\rho_m + \left( \frac{\partial^a}{\partial \tau^i} n^\rho \xi^\tau_m + K^a_{\rho m} n^\rho \right) \right) \\
- D_j K_m n^a + \frac{\partial}{\partial \tau^i} \left( \frac{\partial^a}{\partial \tau^i} n^\rho \xi^\rho_m + \left( \frac{\partial^a}{\partial \tau^i} n^\rho \xi^\tau_m + K^a_{\rho m} n^\rho \right) \right) \\
+ K^a_j \left( D_k n^a + \frac{\partial}{\partial \tau^i} \left( \frac{\partial^a}{\partial \tau^i} n^\rho \xi^\rho_m + \left( \frac{\partial^a}{\partial \tau^i} n^\rho \xi^\tau_m + K^a_{\rho m} n^\rho \right) \right) \\
- \frac{\delta^a}{\delta \tau^i} \left( \frac{\partial^a}{\partial \tau^i} n^\rho \xi^\rho_m + \left( \frac{\partial^a}{\partial \tau^i} n^\rho \xi^\tau_m + K^a_{\rho m} n^\rho \right) \right) \right) \\
+ \frac{\delta^a}{\delta \tau^i} \left( \frac{\partial^a}{\partial \tau^i} n^\rho \xi^\rho_m + \left( \frac{\partial^a}{\partial \tau^i} n^\rho \xi^\tau_m + K^a_{\rho m} n^\rho \right) \right) \right)
\]

Using again the definition of the extrinsic curvature, as well as the one of the full Riemann tensor, we get  
\[
\xi^a_m \tilde{R}_{ij}(h) = -n^\rho \xi^a_i \xi^\rho_j \tilde{R}_{ij} \rho \mu [\xi]
\]

This projects into the famous Gauss-Codazzi equations  
\[
R_{ij}(h) + K_i K_j - K_k K_k = n^\rho \xi^a_i \xi^\rho_j \tilde{R}_{ij} \rho \mu [\xi]
\]

This equation is telling us that the Riemann tensor associated to the induced metric is the total tangent projection of the full four-dimensional Riemann tensor plus a couple of terms involving the extrinsic curvature.

It is also the case that when only one of the components of the four-dimensional Riemann tensor is projected along the normal, and all the others are tangent, then  
\[
D_j K_k - D_k K_j = -n^\rho \xi^a_i \xi^\rho_j \tilde{R}_{ij} \rho \mu [\xi]
\]

Please note that not all components of the full Riemann tensor can be recovered from the knowledge of the Riemann tensor computed on the hypersurface plus the extrinsic curvature. As a matter of fact, our main object of interest, which is the scalar of curvature (which we need for the Einstein-Hilbert (EH) action)  
\[
(4) R = (4) R^{ij} + 2 (4) R_{\mu \nu}^{\rho \sigma} = (3) R + K^2 - K_j K_j + 2 (4) R_{\mu \nu}^{\rho \sigma}
\]

This means that an explicit computation of \( (4) R_{\mu \nu}^{\rho \sigma} \) is needed before the Einstein-Hilbert term could be written in the 1+\(1+\) decomposition. To do that, consider the four-dimensional Ricci’s identity  
\[
\tilde{\nabla}_i \tilde{\nabla}_\rho n_\mu - \tilde{\nabla}_\rho \tilde{\nabla}_i n_\mu = R^{\rho \mu \nu \sigma} n_\nu n_\sigma
\]

Now  
\[
n^\rho \left( \tilde{\nabla}_i \tilde{\nabla}_\rho n^\nu - \tilde{\nabla}_\rho \tilde{\nabla}_i n^\nu \right) = n^\rho \tilde{g}^{\rho \nu} R^{\rho \nu \sigma \tau} n_\sigma n_\tau \equiv R^{\mu \nu}_{\mu \nu}
\]

Besides,  
\[
\tilde{\nabla}_i n^\rho \tilde{\nabla}_\rho n^\nu = \tilde{\nabla}_\rho n_\nu \left( n^\rho n^\nu + \xi^\rho_j \xi^\nu_k \right) \tilde{\nabla}_\rho n_\nu = -K^i \tilde{K}^j
\]

\[
\delta K_{ij} = \frac{1}{4N} \left( \delta^i \delta^j + \delta^j \delta^i \right)
\]

and  
\[
\delta K_{ij} = \frac{1}{2N} \tilde{h}_{ij}
\]

Then  
\[
\tilde{\nabla}_i \tilde{\nabla}_\rho n_\mu = \tilde{\nabla}_\rho \tilde{\nabla}_i n_\mu + R^{\rho \mu \nu \sigma} n_\nu n_\sigma
\]

\[
\delta K_{ij} = \frac{1}{4N} \left( \delta^i \delta^j + \delta^j \delta^i \right)
\]

\[
\delta K_{ij} = \frac{1}{2N} \tilde{h}_{ij}
\]

Summarizing,  
\[
R^{\mu \nu}_{\mu \nu} = n^\rho \tilde{\nabla}_i \tilde{\nabla}_\rho n^\nu - n^\rho \tilde{\nabla}_i \tilde{\nabla}_\rho n^\nu = \tilde{\nabla}_i \left( n^\rho \tilde{\nabla}_\rho n^\nu \right) - \tilde{\nabla}_\rho \left( n^\rho \tilde{\nabla}_i n^\nu \right) + \tilde{\nabla}_\rho \left( n^\rho \tilde{\nabla}_\rho n^\nu \right)
\]

The determinants are related through  
\[
\sqrt{(g)} = \sqrt{(\tilde{g})}
\]

The EH lagrangian can then be written as follows  
\[
L_{EH} = \frac{1}{N} \left( R + K_{ij} K^{ij} - K^2 \right) - \partial_\rho V^\rho \equiv L'_{EH} - \partial_\rho V^\rho
\]

where  
\[
V^\rho \equiv 2 \sqrt{(g)} \left( n^\rho \tilde{\nabla}_\rho n^\nu - n^\nu \tilde{\nabla}_\rho n^\rho \right)
\]

The resulting lagrangian, \( L'_{EH} \), does not contain \( \tilde{N} \) or \( N_i \), and does contain only first time derivatives of \( \tilde{g}_{ij} \). This lagrangian differs from the EH one by a total derivative. This is irrelevant for the EM, but it has importance whenever the spacetime manifold has got a boundary.

At any rate, this is the starting point of the ADM hamiltonian formalism. There are the primary constraints  
\[
\pi^\mu = \frac{\delta L}{\delta \dot{N}^\mu} = 0
\]

In order to compute the spacelike momenta, consider  
\[
\dot{h}_{ij} = \mathcal{L}(N^\sigma) h_{ij} = \xi^\rho_i \xi^\sigma_j \mathcal{L}(N^\sigma) \delta_{\rho \sigma}
\]

= (Remembering that this Lie derivative of the spacelike basis vectors vanishes)  
\[
= \xi^\rho_i \xi^\sigma_j \left( \nabla_\rho n_\sigma + \nabla_\sigma n_\rho \right) - \nabla_\rho \left( n^\rho \tilde{\nabla}_\nu n^\nu \right)
\]

= \xi^\rho_i \xi^\sigma_j \left( \nabla_\rho \left( N n_\sigma + N_\sigma \right) + \nabla_\sigma \left( N n_\rho + N_\rho \right) \right)

= 2N K_{ij} + D_i N_j + D_j N_i
\]

Then  
\[
\pi^j = \sqrt{\tilde{h}} (K^j - K h^j)
\]
Let us compute now the Hamiltonian

$$H \equiv \int d^4x (\pi^\mu, N^\nu + \pi^\nu, h^{ij} - L$$

(169)

where

$$L = N \sqrt{|h|} (R[h] + K_0 K^0 - K^2)$$

(170)

Now, it is clear that

$$\sqrt{h}(K_0 K^0 - K^2) = \frac{1}{\sqrt{h}} (\pi^\nu, \pi^\nu - \frac{1}{2} \pi^2)$$

(171)

We just derived

$$\dot{h}_{ij} = 2NK_j + D_i N_j + D_j N_i$$

(172)

Summarizing,

$$H = \pi^\nu, h^\nu - L = \pi^\nu \left( \frac{2N}{\sqrt{|h|}} \left( \pi^\nu, \pi^\nu - \frac{1}{2} \pi^2 \right) + D_i N_j + D_j N_i \right) - \frac{1}{\sqrt{|h|}} \left( \pi^\nu, \pi^\nu - \frac{1}{2} \pi^2 \right) - \sqrt{|h|} R[h]$$

(173)

where (dropping surface terms)

$$H = \frac{1}{2 \sqrt{|h|}} \left( h_{ij} h^0 + h_j h^0 - h^0 h_{ij} \right) - \sqrt{|h|} R[h]$$

(174)

Now we have the following constraints

$$\pi^\nu \sim 0$$

$$N^\nu - C^\nu \sim 0$$

$$\dot{H} \sim 0$$

$$\dot{H}_i \sim 0$$

(175)

and the corresponding brackets

$$\{\pi^\nu, N^\nu - C^\nu \} \sim g^{0i} \delta(x - x')$$

$$\{\pi^\nu, H \} \sim 0$$

$$\{\pi^\nu, H_i \} \sim 0$$

$$\{N^\nu - C^\nu, H \} \sim 0$$

$$\{N^\nu - C^\nu, H_i \} \sim 0$$

(176)

6.4. Careful Analysis of the Boundary Terms

The purpose of this section is to give a detailed treatment of boundary terms following Brown and York[17] (confer also the careful treatment in [56]).

Consider a tubular domain $D$ of spacetime, whose boundary has three different pieces: The two caps at the initial and final times, $\Sigma_i$ and $\Sigma_f$. Those are spacelike, codimension one hypersurfaces (that is $d = n - 1$). The physical space-time of course has dimension $n = 4$, but the analysis can easily be made for general dimension $n$. Then there is the "boundary at infinity", $r = R \rightarrow \infty$, which is the surface of a cylinder, also of codimension one, but timelike instead of spacelike. We shall call it $B \equiv \partial D$. Now this boundary can be understood as generated by the union of all the codimension two boundaries of the constant time hypersurfaces

$$(B = \partial D) \equiv \cup_i \left( \Sigma_i = \partial \Sigma_i \right)$$

(178)

• An intuitive grasp of the general situation can stem from the trivial example in flat space, to which we are going to refer all the time.

$$D \equiv \{ r \leq R \quad t_1 \leq t \leq t_2 \}$$

(179)

In this way the caps are defined by the solid balls

$$\Sigma_i \equiv \{ r \leq R \quad \text{or} \quad t = \text{constant} \}$$

(180)
The embedding in spacetime is simply
\[ y^0 = t \]
\[ y^i = x^i \]
so that the induced tangent vectors is
\[ \xi^i \equiv \frac{\partial y^i}{\partial x^\alpha} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \] (182)
and the normal vector
\[ n^a \equiv (1, 0, 0, 0) \] (183)

The induced metric reads
\[ h_{ij} = n_\alpha \xi^\alpha \xi^\beta \equiv -\delta_{ij} \] (184)

The normal to \( \Sigma \) in Minkowski space is
\[ n^a = (1, 0, 0, 0) \] (185)

The boundary of such caps are the two-spheres
\[ S_\gamma \equiv \{ r = R \cup \ t = \text{constant} \} \] (186)

We can choose polar coordinates \( \theta \equiv (\theta, \phi) \). The imbedding matrix of the boundary in \( \Sigma \), using these is
\[ \xi^i \equiv \frac{\partial y^i}{\partial \theta^\alpha} = \begin{pmatrix} \cos \theta \cos \phi & -\sin \theta \sin \phi \\ \cos \theta \sin \phi & \sin \theta \cos \phi \\ -\sin \theta & 0 \end{pmatrix} \] (187)

It is equivalent to use
\[ \theta_1 \equiv \frac{y_1}{R} \]
\[ \theta_2 \equiv \frac{y_2}{R} \] (188)

then
\[ y_1 = \theta_2 R \sqrt{1 - \theta_1^2 - \theta_2^2} \] (189)

The embedding matrix is now
\[ \xi^i \equiv \begin{pmatrix} R & 0 & 0 \\ 0 & R & 0 \\ -R \frac{\theta_1}{\sqrt{1 - \theta_1^2 - \theta_2^2}} & -R \frac{\theta_2}{\sqrt{1 - \theta_1^2 - \theta_2^2}} & -R \frac{1}{\sqrt{1 - \theta_1^2 - \theta_2^2}} \end{pmatrix} \] (190)

The induced metric reads
\[ \sigma_{ab} \equiv \xi^i h_{ij} \xi^j = -\xi_\alpha \xi_\beta = -\frac{R^2}{1 - \theta_1^2 - \theta_2^2} \begin{pmatrix} 1 - \theta_1^2 & \theta_1 \theta_2 \\ \theta_1 \theta_2 & 1 - \theta_2^2 \end{pmatrix} \] (191)

The contravariant metric reads
\[ \sigma^{ab} = -\frac{1}{R^2} \begin{pmatrix} 1 - \theta_1^2 & -\theta_1 \theta_2 \\ -\theta_1 \theta_2 & 1 - \theta_2^2 \end{pmatrix} \] (192)

Out of the two embedding matrices we can draw the composition
\[ \xi^i \equiv e^i_a e^j_a = \begin{pmatrix} 0 & 0 & 0 \\ R & 0 & 0 \\ \frac{\theta_1}{\sqrt{1 - \theta_1^2 - \theta_2^2}} & \frac{\theta_2}{\sqrt{1 - \theta_1^2 - \theta_2^2}} & -\frac{1}{\sqrt{1 - \theta_1^2 - \theta_2^2}} \end{pmatrix} \] (193)

The normal to the boundary in \( \Sigma \), is
\[ v^i = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) = \left( \frac{x}{R}, \frac{y}{R}, \frac{z}{R} \right) \] (194)

The extrinsic curvature of \( S_\gamma \rightarrow \Sigma \) reads
\[ k_{ab} \equiv \nabla_i v^i_e x^a_b = \frac{1}{R} \delta_i^e \delta_a^b = -\sigma_{ab} \] (195)

Let us now examine the constructs
\[ v^i v^e e^j_a = \begin{pmatrix} \theta_1 & 0 & 0 \\ 0 & \theta_2 & \theta_1 \theta_2 \sqrt{1 - \theta_1^2 - \theta_2^2} \\ 0 & \theta_1 \theta_2 & \theta_1 \theta_2 \sqrt{1 - \theta_1^2 - \theta_2^2} \end{pmatrix} \] (196)

All this explicitly checks that
\[ n^a n^b - \sigma_{ab} \sigma_{cd} e^a_c e^b_d = \eta^{ab} \] (198)

The timelike boundary is just \( S_\gamma \times \mathbb{R} \)
\[ B \equiv \cup_\gamma S_\gamma \] (199)
Let us now draw from the example to the general case. The surfaces $S^m_{n-2} \equiv \Sigma_{n-1}$ provide a foliation of the timelike boundary $B_{n-1} \leftarrow V_n$ of the domain of spacetime under consideration. The coordinates in $S^m_{n-2}$ will be denoted by $\theta_a \quad a = 1 \ldots n-2$. The imbedding $S_{n-2} \leftarrow \Sigma_{n-1}$ is described by

$$\theta \in S_{n-2} \leftarrow \gamma(\theta^a) \in \Sigma_{n-1} \quad (i = 1 \ldots n-1) \quad (a = 1 \ldots n-2)$$

The imbedding of $S$ in $\Sigma$ defines in a natural way $(n-2)$ tangent space vectors

$$\xi^a \equiv \frac{\partial \gamma^i}{\partial \theta^a}$$

The unit normal to $S_{n-2}$ in $\Sigma_{n-1}$ will be denoted by $\nu^{\phi}$, and out of it we construct a vector

$$\nu^a \equiv \nu^{\phi} \xi^a \in T(S)$$

which is such that it is unitary $\nu \cdot \nu = 1$ and is tangent to $\Sigma_{n-1}$, that is, $\nu \cdot n = 0$. There are also $n-2$ spacelike vectors obtained by combining the two embeddings $S \leftarrow \Sigma$ and $\Sigma \leftarrow M$:

$$\xi^a \equiv \xi_j^a \xi^j$$

The induced metric in $S_{n-2} \equiv \partial S_{n-1}$ is

$$ds^2 \equiv \sum_{a,b=1}^{n-2} \sigma_{\theta^a} d\theta^a d\theta^b = h_{ij} n_i^a \nu^b d\theta^a d\theta^b = g_{\phi \psi} \nu^\phi \nu^\psi d\theta^a d\theta^b$$

The spacetime metric can be recovered from

$$g_{\phi \psi} = -\nu^\phi \nu^\psi + n^a n^b + \sigma_{\phi \psi} \xi^a \xi^b$$

The extrinsic curvature of $S_{n-2} \leftarrow \Sigma_{n-1}$ is defined as usual

$$k_{ab} \equiv \nabla \nu^\phi \xi^a + \nabla \nu^\psi \xi^b$$

It is possible to choose the coordinates $\theta^a$ in such a way that they intersect $S^m_{n-2} \equiv \Sigma_{n-1}$ orthogonally. This means that the vector $\nu^a$ is the tangent vector to the timelike flow

$$N \nu^a = \frac{\partial \nu^a}{\partial t}$$

The set of all $S^m_{n-1}$ for varying $t$ do foliate the timelike boundary of spacetime $B_{n-1} \equiv \partial V_n$. In this boundary $B_{n-1}$ we can also introduce coordinates $z^m \quad m = 1 \ldots n-1$ (one of which is timelike), and the corresponding $(n-1)$ vectors

$$z^m \equiv \partial V_n$$

and we can write the completeness relation

$$g_{\phi \psi} = -\nu \nu^\phi + g_{\phi \psi} z^m z^m$$

It is simplest to choose (as we did in our explicit example)

$$z^m \equiv (t, \theta^a)$$

then

$$ds^2 = \left( \frac{\partial x^a}{\partial t} \right) dt + \left( \frac{\partial x^a}{\partial \theta^b} \right) d\theta^b \equiv N \nu^a dt + \xi^a d\theta^b$$

in such a way that

$$\sum\limits_{a,b} \sigma_{\theta^a} d\theta^a d\theta^b$$

and the determinant obeys

$$|\gamma| = N^2 \sigma$$

Finally, the extrinsic curvature of $B_{n-1} \leftarrow V_n$ is

$$k_{ab} \equiv \nabla \nu^\phi \xi^a + \nabla \nu^\psi \xi^b$$

Let us apply all this mathematics to the Einstein-Hilbert action. We consider a tubular region of the full spacetime bounded by two spacelike hypersurfaces of constant time, $\Sigma_2$ and $\Sigma_1$, and the surface of the asymptotic cylinder, $B$

$$\partial V_n = \Sigma_2 \equiv \Sigma_1 + B$$

This is the generalization to an arbitrary spacetime of the construction made in the example.
This means that the timelike boundary $B$ is foliated by \( \Sigma \),

\[ \kappa = \gamma^i k_i = \gamma^i \nabla_i n^a \epsilon^a_i = \nabla_i (\rho^a - \psi^a \nu^b) \]

This means that

\[ \kappa + \nabla_i \nu_a n^a \nu^b = \nabla_i \nu_a (\rho^a - \psi^a \nu^b + n^a n^b) \]

\[ = \nabla_i \nu_a \epsilon^a_i = \sigma_{ab} k_{ab} = k \]

so that

\[ \int_B = 2 \int_{\Sigma_{-1}^{+1}} kN \sqrt{\sigma} |d^{n-2} \theta | \]

As was already clear from the explicit example, this integral diverges even $R \to \infty$ even in flat space. In order to refer all expressions to this value, so that the action in flat space vanishes, it is often subtracted a term in the action

\[ \Delta D \equiv - \frac{2}{16 \pi G} \int_B k_0 N \]

where $k_0$ represents the extrinsic curvature of $S_{n-2}$ embedded in flat space.

The boundary terms in the Hamiltonian read

\[ H_{\text{boundary}} = -2 \frac{1}{16 \pi G} \int_{S_{n-2}} \left( N (k - k_0) - N_1 (K^{ij} - K^i \nu^j) \psi_j \right) \]

\[ \times \sqrt{|\sigma|} \ d^{n-2} \theta \]

(227)

(228)

(229)

This boundary term yields the value of the energy for the gravitational field. It depends on the foliation chosen as well as on the lapse and shift which are arbitrary. When the space is asymptotically flat, representing flat asymptotic coordinates as \((T, X^i)\), it is possible to choose \( \Sigma \), so that goes into $T = \text{constant}$. It is clear that

\[ N^a \to N \left( \frac{\partial \alpha^a}{\partial T} \right) + N^i \left( \frac{\partial \alpha^a}{\partial X^i} \right) \]

(230)

It is then natural to define the ADM mass associated to a given solution by choosing a FIDO at rest at infinity, that is, $N = 1$, $N^i = 0$, so that

\[ N^a \to \left( \frac{\partial \alpha^a}{\partial T} \right) \]

(231)

and the flow generates a time translation at infinity. Then

\[ M \equiv \lim_{R \to \infty} \frac{1}{8 \pi G} \int_{S_{n-2}} (k - k_0) \sqrt{\sigma} |d^{n-2} \theta | \]

(232)

- As the simplest of all possible exercises, let us compute the ADM mass for the four-dimensional Schwarzschild’s spacetime.

\[ ds^2 = \left( 1 - \frac{r_S}{r} \right) dt^2 - \frac{dr^2}{1 - \frac{r_S}{r}} - r^2 d\Omega_2^2 \]

(233)

Let us choose $\Sigma_0$ to be really the surfaces of constant Schwarzschild time. Then the unit normal is given by

\[ n^a = \frac{1}{\sqrt{1 - \frac{r_S}{r}}} (1, 0, 0, 0) \]

(234)

The induced metric in $\Sigma_0$ is

\[ h_{ij} dx^i dx^j = \frac{1}{1 - \frac{r_S}{r}} dr^2 + r^2 d\Omega_2^2 \]

(235)
The ADM mass does not capture the mass loss due to radiation. In order to do that, it is necessary to choose the boundary at null infinity, instead of at spatial infinity. The corresponding mass is called the Bondi mass\(^\text{[15]}\).

The matter represented by the energy-momentum tensor as above is referred to as null dust. The solution of Einstein’s equations is called the Vaidya metric and reads

\[
\begin{align*}
\sigma_{ab} \, d\theta^a \, d\theta^b &= R^2 \, d\Omega_2^2, \\
\sigma_{ab} &= R^2 \, d\Omega^2.
\end{align*}
\]

The contravariant metric in the sector \((u, r)\) reads

\[
g^{\alpha\beta} = \begin{pmatrix} 0 & \frac{1}{r} \frac{2GM(u)}{r} \\ \frac{1}{r} \frac{2GM(u)}{r} & 1 - \frac{2GM(u)}{r} \end{pmatrix}
\]

Let us consider again the surface \(\Sigma\), where \(u + r = \text{constant}\). Its covariant normal reads

\[
n_u \sim (1, 1)
\]

so that the normal vector

\[
n \sim (g^{\mu\nu} + g^{\nu\mu}, g^{\mu\nu} + g^{\nu\mu}) = \left( 1, \frac{2GM(u)}{r} \right)
\]

Normalizing

\[
n = \frac{1}{\sqrt{1 + \frac{2GM(u)}{r}} \frac{\partial u}{\partial r}}
\]

The induced metric in \(\Sigma\) is obtained by substituting \(du = -dr\), so that

\[
ds^2 = -\left( 1 + \frac{2GM(u)}{r} \right) dr^2 - r^2 \, d\Omega^2
\]

The boundary \(\partial \Sigma\) is just the sphere \(r = R\). The normal is

\[
n_u \equiv \frac{1}{\sqrt{1 + \frac{2GM(u)}{r}}} \frac{\partial}{\partial r}
\]

The extrinsic curvature reads

\[
k \equiv \nabla_a n^a = \frac{2}{R \sqrt{1 + \frac{2GM(u)}{r}}} \sim \frac{2}{R} \left( 1 - \frac{GM(u)}{R} + \ldots \right)
\]

The induced metric on the boundary is just

\[
ds^2 = -R^2 \, d\Omega^2
\]

The extrinsic curvature of a surface of the same intrinsic geometry, only that embedded in flat space is

\[
k_u = \frac{1}{r} \frac{\partial}{\partial r} (r^2) = \frac{2}{R}
\]
so that
\[ k - k_0 = -\frac{2GM(u)}{R^2} \quad (259) \]

If we integrate now on spatial infinity \( R \to \infty \), this means that we keep \( t \equiv u + r \) constant, so that \( u \sim -R \to -\infty \). This means that
\[ M_{ADM} = M(u = -R) \quad (260) \]

If we integrate now on null infinity \( \nu \to \infty \), while \( u \) is kept fixed, then
\[ M_B = M(u) \quad (261) \]

the mass function.

6.5. Canonical Quantum Gravity

The first attempt to quantize the gravitational field\(^{20}\) stems from the preceding canonical approach, just by converting the Poisson (or Dirac) brackets into commutators as in
\[ \{\pi_g(x), h^{ij}(x')\} = -i\hbar \delta_{ij}\delta(x, x') \quad (262) \]

There are lots of mathematical ambiguities of operator ordering, and also defining in a precise way products of distributions, and so on, and also physical problems, to which now turn. But still, the approach offers some glimpses of what a true quantum geometry theory might be.

From the physical point of view, it has been realized since long that this whole approach suffers from the frozen time problem, i.e., the generic Hamiltonian reads

\[ H \equiv \int d^3x \left( NH + N' H' \right) \]

so that acting on physical states of the Hilbert space with the corresponding operator
\[ \hat{H}\psi = 0 \quad (263) \]
in such a way that Schrödinger’s equation
\[ i\hbar \frac{\partial}{\partial t} \psi = \hat{H}\psi \quad (264) \]

seemingly forbids any time dependence. There is no known way out of this dilemma. Some concept of time can be recovered however in the semiclassical approximation,\(^{111}\) although it is not clear how to connect with the minkowskian time.

There are many unsolved problems in this approach, which has been kept at a formal level. The first one is an obvious operator ordering ambiguity owing to the nonlinearity of the classical expression for the hamiltonian. In the same vein, it is not clear whether it is possible to make the constraints hermitian. There is no clear candidate for a positive semi-definite scalar product. Besides, it is not clear that one recovers the full diffeomorphism invariance from the Dirac–Schwinger algebra. Actually, it is not even known whether this is necessary; that is, what is the full symmetry of the quantum theory.

We can proceed further, still formally\(^1\), using the Schrödinger representation defined in such a way that
\[ (\hat{h}_j\psi)[h] \equiv h_j(x)\psi[h] \quad (265) \]

and
\[ (\hat{\pi}_j\psi)[h] \equiv -i\hbar \frac{\delta\psi}{\delta h_j[x]}[h] \quad (266) \]

If we assume that diffeomorphisms act on wave functionals as:
\[ \psi[\hat{f}\psi] = \psi[h] \quad (267) \]

then the whole setup for the quantum dynamics of the gravitational field lies in Wheeler’s superspace (nothing to do with supersymmetry) which is the set of three-dimensional metrics modulo three-dimensional diffeomorphisms: \( \text{Riem}(\Sigma)/\text{Diff}(\Sigma) \).

The Hamiltonian constraint then implies the famous Wheeler–DeWitt equation:
\[ -\hbar^2 2\kappa^2 C_{ijkl}\delta^{ij}_{kl}\frac{\delta^3 \psi}{\delta h_i \delta h_j}[h] = \frac{\hbar}{2\kappa^2} R^{(3)}[h] \psi[h] = 0 \quad (268) \]

where the DeWitt tensor is:
\[ C_{ijkl} \equiv \frac{1}{\sqrt{\hbar}} \left( h_{ij} h_{kl} - \frac{1}{2} h_{ik} h_{lj} \right) \quad (269) \]

Needless to say, this equation, suggestive as it is, is plagued with ambiguities. The manifold of positive definite metrics has been studied by DeWitt. He showed that it has signature \((-1, +1)\), where the timelike coordinate is given by the breathing mode of the metric:
\[ \zeta = \sqrt{\frac{32}{3}} h^{1/4} \quad (270) \]

\(^1\) It is bound to be formal as long as the problem of the infinities is not fully addressed. We know from the analysis of this representation for gauge theories in the lattice that those are the most difficult problems to solve.
and in terms of other five coordinates $\zeta^a$ orthogonal to the time-like coordinate, the full metric reads

$$ds^2 = -d\zeta^2 + \frac{3}{32} \zeta^2 g_{ab} d\zeta^a d\zeta^b$$  \hfill (271)

with

$$g_{ab} = tr h^{-1} \partial_a h h^{-1} \partial^b$$  \hfill (272)

The five dimensional submanifold with metric $g_{ab}$ is the coset space

$$SL(3, \mathbb{R})/SO(3)$$  \hfill (273)

It has been much speculated whether the timelike character of the dilatations lies at the root of the concept of time. The Wheeler-deWitt equation can be written in a form quite similar to the Klein-Gordon equation:

$$\left( -\frac{\partial^2}{\partial \zeta^2} + \frac{3}{32} \zeta^2 g^{ab} \partial_a \zeta \partial_b \zeta + \frac{3}{32} \zeta^2 R(3) \right) \psi = 0$$  \hfill (274)

The analogy goes further in the sense that also here there is a naturally defined scalar product which is not positive definite:

$$\langle \psi, \chi \rangle \equiv \int_{\Sigma} \psi^* \delta \Sigma \ G_{ij} \frac{\delta x}{\partial h_{ij}} - \chi^* \delta \Sigma \ G_{ij} \frac{\delta x}{\partial h_{ij}}$$  \hfill (275)

There has been a lot of activity in canonical quantum gravity following the discovery by Ashtekar\cite{10} of a new set of variables cf. for example, Rovelli’s book in [16].

It is my opinion that despite some formal interest in many cases, this approach fails to comply with the correspondence principle, in the sense that it is not connected smoothly with either classical general relativity or else perturbative quantum corrections. This does not mean that some concepts and techniques developed in this approach could not be useful in the quantum regime very far from the classical limit; we don’t know yet.

At any rate, we shy away from treating this approach further in these lectures and refer to the literature to the interested reader. For a thorough review of this viewpoint cf. Thiemann’s book.\cite{65}

### 6.6. The Hartle-Hawking State

Let us first review Vilenkin’s idea of creation of universes out of nothing.\cite{70} Consider a charged particle of mass $m$ and charge $q$ in a constant electric field $E$ moving in the $(t, x)$ plane. Its trajectory is given by the hyperbola

$$(t - t_0)^2 - (x - x_0)^2 = -R^2$$  \hfill (276)

where $R \equiv | \frac{m}{qE} |$. When

$$x - x_0 = \pm R$$  \hfill (277)

there is a turning point at which $t = t_0$; the solution does not exist for

$$x - x_0 < R$$  \hfill (278)

The euclidean trajectory (a compact instanton or bounce) is obtained by the change

$$t \rightarrow i \tau$$  \hfill (279)

and is just a circle

$$(r - r_0)^2 + (x - x_0)^2 = R^2$$  \hfill (280)

This instanton yields the amplitude for pair production in the presence of an electric field. Their action provides the dominant term in the quantitative formula for this amplitude.

Proceeding by analogy, Alex Vilenkin tried to apply this idea in the creation of universes from nothing. Assume the familiar Friedmann metric for space-time

$$ds^2 = dt^2 - a^2(t) \delta_{ij} dx^i dx^j$$  \hfill (281)

The lorentzian solution for the scale factor of de Sitter space of radius $L = \frac{1}{H}$ is given by

$$a(t) = \frac{1}{H} \cosh Ht$$  \hfill (282)

where $H^2$ is proportional to the vacuum energy density. The corresponding euclidean solution (the de Sitter instanton of Gibbons and Hawking\cite{12}) is

$$a(\tau) = \frac{1}{H} \cos H\tau$$  \hfill (283)

Vilenkin interpreted this instanton by analogy with the Schwinger process by indicating an amplitude for the creation of the universe from nothing whatsoever. The euclidean manifold is glued to the lorentzian one at a moment of time symmetry where the extrinsic curvature vanishes.

We have already pointed out the difficulties of physical interpretation of the wave function $\psi[h_{ij}]$ in quantum gravity. Nevertheless Hartle and Hawking proposed a concrete way to compute this wavefunction of the universe. The no-boundary state (now best known as the Hartle-Hawking state) is characterized by the wave functional (that yields some amplitude for a three-manifold $\Sigma$ en-
such that they are sometimes referred to as transverse diffeomorphisms. In a Taylor expansion in a local patch
\[ \xi^{\mu} \equiv \frac{\partial}{\partial x} + \frac{\partial^{\mu}}{\partial x^\mu} x^\alpha + \frac{\partial^{\mu}}{\partial x^\mu} x^\beta + \cdots \] (287)
(\text{where} \ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \ \text{are arbitrary constants}) this means that all coefficients are totally traceless.
\[ \phi^{\mu} = \phi^{\mu} = \cdots = 0 \] (288)
That is
\[ \phi^{\mu} \rightarrow \phi^{\mu} \sim \frac{1}{n+1} \left( \phi^{\mu} \right) \] (289)
and so on. Einstein himself was fond of the gauge condition \(|g| = 1; \) and was the first to propose some incomplete version of the unimodular theory. Those area preserving diffeomorphisms leave invariant a given measure, such as the Lebesgue measure, \( d^n x \), although they share problems with their bigger cousin \( \text{Diff}(M) \).

Fock proposed a long time ago [30] that harmonic coordinates should be privileged, and that in some sense they were the only with a physical meaning. They are defined by
\[ \Box x^{(i)} = 0 \] (290)
\[ \text{(where the coordinates are considered as a set of functions on the manifold} \]
\[ x^{(i)} : M \rightarrow \mathbb{R}^n \] At the linear level this is equivalent to the de Donder gauge condition.
\[ \partial_j h^{\mu^i} = \frac{1}{2} \partial^j h \] (292)
This condition is left invariant by linearized diffeomorphisms such that the generic vector is harmonic
\[ \Box \xi^{(i)} = 0 \] (293)
that is; all contractions of the flat metric with the covariant indices should vanish
\[ \eta^{\gamma \delta} \phi^{\mu} = \eta^{\gamma \delta} \phi^{\mu} = \cdots = 0 \] (294)
That is
\[ \phi^{\mu} \rightarrow \frac{1}{n+1} \left( \phi^{\mu} \right) \] (295)
\[ \text{(where} \ \frac{V^\mu}{\eta^\mu} \equiv \phi^{\mu} \text{and so on).} \]
It also seems clear that when we are integrating upon a restricted class of spacetimes with some specific type of boundary, or asymptotic behavior, then the gauge group is restricted to the subgroup consisting on those diffeomorphisms that act trivially.
on the boundary (or leave invariant the boundary conditions). The subgroup that act not-trivially is related to the set of conserved charges, if any. In the asymptotically flat case this is precisely the Poincaré group, $SO(1, 3)$ that gives rise to the ADM mass and also the BMS group acting on null infinity.

In the asymptotically anti-de Sitter case, this is related to the conformal group $SO(2, 3)$.

It is nevertheless not clear what is the physical meaning of keeping constant the boundary of spacetime (or keeping constant some set of boundary conditions) in a functional integral of some sort. This is related to the issue of whether the functional integral over geometries allows for topology change.

Incidentally, it is very difficult to define what could be observables in a diffeomorphism invariant theory, other than global ones defined as integrals of scalar densities composite operators $O(\phi_\alpha(x))$ (where $\phi_\alpha$, $\alpha = 1 \ldots N$ parametrizes all physical fields) with the pseudo-riemannian measure

$$\mathcal{O} \equiv \int \sqrt{|g|} \, d^4x \, O(\phi_\alpha(x))$$

Some people claim that there are no local observables whatsoever, but only pseudolocal ones\cite{33}; the fact is that we do not know. Again, the exception to this stems from keeping the boundary conditions fixed; in this case it is possible to define an $S$-matrix in the asymptotically flat case, and a conformal quantum field theory (CFT) in the asymptotically anti-de Sitter case. Unfortunately, the most interesting case from the cosmological point of view, which is when the space-time is asymptotically de Sitter is not well understood.

In the mathematical front, it is well known that the equivalence problem in four-dimensional geometries is undecidable.\cite{45} This theorem, which was first proved by Kolmogorov, states that given two four-dimensional manifolds, there is no systematic procedure to determine whether those two manifolds are diffeomorphic or not. In three dimensions Thurston’s geometrization conjecture has recently been put on a firmer basis by Hamilton and Perelman, but it is still not clear whether it can be somehow implemented in a functional integral without some drastic restrictions. Those caveats should be kept in mind when reading the sequel.

Gauge theories can be formulated in the background field approach, as introduced by B. DeWitt and others (cf. \cite{20}). In this approach, the quantum field theory depends on a background field, but not on any one in particular, and the theory enjoys background gauge invariance.

Is it enough to have the functional integral of quantum gravity formulated in such a way?

It can be argued that the only vacuum expectation value consistent with diffeomorphism invariance is

$$\langle 0 | g_{\alpha \beta}(x) | 0 \rangle = 0 \quad (296)$$

in which case the answer to the above question ought to be in the negative, because this is a singular background and curvature invariants do not make sense. It all boils down to whether the ground state of the theory is diffeomorphism invariant or not. There is an example, namely three-dimensional gravity in which invariant quantization can be performed.\cite{75} In this case at least, the ensuing theory is almost topological, although the issue is not completely clear owing to subleties related to the Bañados-Henneaux-Teitelboim (BHT) black hole.

In all attempts of a canonical quantization of the gravitational field, one always ends up with an (constraint) equation corresponding physically to the fact that the total hamiltonian of a parametrization invariant theory should vanish. When expressed in the Schrödinger picture, this equation is often dubbed the Wheeler-de Witt equation. This equation is plagued by operator ordering and all other sorts of ambiguities. It is curious to notice that in ordinary quantum field theory there also exists a Schrödinger representation, which came recently to be controlled well enough as to be able to perform lattice computations (\cite{17}).

Gauge theories can be expressed in terms of gauge invariant operators, such as Wilson loops. They obey a complicated set of equations, the loop equations, which close in the large $N$ limit as has been shown by Makeenko and Migdal (\cite{17}). These equations can be properly regularized, e.g. in the lattice. Their explicit solution is one of the outstanding challenges in theoretical physics. Although many conjectures have been advanced in this direction, no definitive result is available.

8. Do Strings Answer Any of Our Questions?

It should be clear by now that we probably still do not know what is exactly the problem to which string theories\cite{18,58} are the answer. This fact has been repeatedly emphasized by the late Joe Polchinski.\cite{33} We shall concentrate in just one aspect that we believe to be important and which turns out to be quite contentious. Ever since Maldacena’s conjecture (more on this in a moment) some people put forward the idea the gravity is emergent in a holographic way from a conformal field theory (CFT) defined in the boundary of the bulk spacetime (one dimension less). This is a fascinating topic, which drives much of the research in the field. We shall give here just the general idea, and then comment on some aspects of it. Perhaps just one comment on the way quantum gravity appears in string theory.

The starting point of the whole topic is a one-dimensional object living in some D-dimensional flat space. Consistency of the quantization is believed to be possible only when conformal symmetry is maintained in the process. This implies that $D = 26$. Absence of tachyons is only possible (through a projection first invented by Gliozzi, Scherk and Olive (GSO))\cite{33} when supersymmetry is implemented and this implies $D = 10$.

There are two places in which gravitation enters into the game. There are two types of strings: closed and open. Open strings also include closed strings as a subsector; but the opposite is not true: there are consistent theories of closed strings only. When the spectrum of states is analyzed, one finds gauge fields in the open string sector, and gravitons in the closed string sector. One can indeed study on-shell perturbative string amplitudes, and take the limit of infinite string tension (where strings degenerate into points) and in this limit one gets quantum field theory (QFT) amplitudes (usually with much supersymmetry).

There is another way to understand gravitation in the framework of string theory. We can try to understand the quantum behavior of strings in some curved space-time. Demanding conformal invariance (the vanishing of the corresponding beta functionals)\cite{31} imply that the background has to obey some field
equations, which correspond to some (supersymmetric) generalization of Einstein’s equations.

Finally, there is a still more indirect way. As we shall see in a moment, there are indications of some dualities strong/weak coupling in string theories. They suggest that there is some as yet unknown theory (dubbed M-theory) which explains all these symmetries. Not much is known about this theory, unfortunately. One of the main problems is that it is always strongly coupled; there is no weak coupling regime. Suggestive as all this might be, we lack ideas on how to do convincing computations.

Even less is known of how to make contact with the low-energy, non-supersymmetric world. Here the main problem is the huge arbitrariness that necessarily enters when breaking supersymmetry. This implies a huge loss of predictivity.

### 8.1. The Maldacena Conjecture and Gravity/CFT Duality

Maldacena\[48\] proposed as a conjecture that IIB string theories in a background $AdS_5 \times S_5$, with common radius $L \sim l_s(N)$\(^{1/4}\) (where $l_s$ is the characteristic length of string theory defined by the string tension through $\alpha’ \equiv \frac{l_s}{2\pi}$, and $g_s$ is the string coupling constant, related to the value of the dilaton field) and $N$ units of RR flux that is, $\int F_5 = N$ (which implies that $F_5 \sim \frac{N}{2\pi}$) is equivalent to a four dimensional ordinary gauge theory in flat four-dimensional Minkowski space, namely $N = 4$ super Yang-Mills with gauge group $SU(N)$ and coupling constant $g = g_s^{1/2}$.

Although there is much supersymmetry in the problem and the kinematics largely determine correlators, (in particular, the symmetry group $SO(2,4) \times SO(6)$ is realized as an isometry group on the gravity side and as an $R$-symmetry group as well as conformal invariance on the gauge theory side) this is not fully $\sigma^2$ and the conjecture has passed many tests in the semiclassical approximation to string theory, which corresponds to large $\frac{\sigma^2}{\sigma_0}$, dual to large $N$ on the CFT side.

The action of the RR field, given schematically by $\int F_5 \wedge \star F_5$, scales as $N^2$, whereas the ten-dimensional Einstein-Hilbert $\int R$, depends overall on the geometric scale as the eighth power of the common radius, $L^8$. The ‘t Hooft coupling is $\lambda = g_5^2 N \sim \frac{\sigma^4}{\sigma_0}$ and the tenth dimensional Newton’s constant is

$$\kappa_{10}^2 \sim G_{10} \sim \frac{\sigma^2}{\sigma_0} \sim \frac{L^8}{N^2}. \quad (297)$$

If we consider the effective five dimensional theory after compactifying on a five sphere of radius $r$, the RR term yields a negative contribution $\sim \frac{N}{(2\pi)^2 r^5}$, whereas the positive curvature of the five sphere $S^5$ gives a positive contribution, $\sim \frac{1}{4} r^5$. The competition between these two terms in the effective potential is responsible for the minimum with negative cosmological constant.

### 8.2. String Dualities and Branes

The so-called T-duality is the simplest of all dualities and the only one which can be shown to be true, at least in some contexts\[37\]. At the same time it is a very stringy characteristic, and depends in an essential way on strings being extended objects. In a sense, the web of dualities rests on this foundation, so that it is important to understand clearly the basic physics involved. Let us consider strings living on an external space with one compact dimension, which we shall call $Y$, with topology $S^1$ and radius $R$. The corresponding field in the imbedding of the closed string (where we identify in the word sheet of the string the spatial coordinate $\sigma \sim \sigma + 2\pi$), which we shall still call $Y$, (i.e. we are dividing the target-space dimensions as $(X^\mu, Y)$, where $Y$ parameterizes the circle), has then the possibility of winding around it:

$$Y(\sigma + 2\pi, \tau) = Y(\sigma, \tau) + 2\pi R m.$$  

A closed string can close in general up to an isometry of the external spacetime.

The zero mode expansion of this coordinate (that is, forgetting about oscillators) would then be

$$Y = Y_c + 2P_c \tau + mR \sigma.$$  

Canonical quantization leads to $[Y_c, P_c] = i$, and single-valuedness of the plane wave $e^{iP_c \tau}$ enforces as usual $P_c \in \mathbb{Z}/R$, so that $P_c = \frac{n}{R}$.

\begin{align}
\text{Area of the horizon for a Schwarzschild black hole is given by:} \\
A = \frac{8\pi G^2}{c^4} M^2 \quad \text{ (299)}
\end{align}

---

2 The only correlators that are completely determined through symmetry are the two and three-point functions.
The zero mode expansion can then be organized into left and right movers in the following way

\[ Y_+ (r + \sigma) = Y_+ / 2 + \left( \frac{n}{R} + mR \right) (r + \sigma), \]

\[ Y_- (r - \sigma) = Y_- / 2 + \left( \frac{n}{R} - mR \right) (r - \sigma). \]  

(303)

The mass shell conditions reduce to

\[ m_L^2 = \frac{1}{2} \left( \frac{n}{R} + mR \right)^2 + N_L - 1, \]

\[ m_R^2 = \frac{1}{2} \left( \frac{n}{R} - mR \right)^2 + N_R - 1. \]  

(304)

Level matching, \( m_L = m_R \), implies that there is a relationship between momentum and winding numbers on the one hand, and the oscillator excess on the other

\[ N_R - N_L = nm. \]  

(305)

At this point it is already evident that the mass formula is invariant under

\[ R \rightarrow R' \equiv \frac{2l^2}{R}, \]  

(306)

provided that at the same time one exchanges momentum and winding numbers. This is the simplest instance of T-Duality.

On the other hand, it is an old observation (which apparently originated in Schrödinger) that Maxwell’s equations are almost symmetrical with respect to interchange between electric and magnetic degrees of freedom (electromagnetic duality). This idea was explored by Dirac and eventually lead to the discovery of the consistency conditions between electric and magnetic charges that have to be fulfilled if there are magnetic monopoles in nature. The fact that nonsingular magnetic monopoles appear as classical solutions in some gauge theories led further support to this duality viewpoint. In order to be able to make a consistent conjecture, first put forward by Montonen and Olive, super-symmetry is needed, as first remarked by Osborn.

Now in strings there are the so-called Ramond-Ramond (RR) fields, which are p-forms of different degrees. In the same way that one forms (i.e., the Maxwell field) couples to charged particles that is, from the spacetime point of view, to objects of dimension 0 with one-dimensional trajectories, a p-form

\[ A_{\mu_1 \ldots \mu_p} \]  

(307)

would couple to a (p – 1)-dimensional object, whose world history is described by a p-dimensional hypersurface

\[ x^\mu = x^\mu (\xi_1, ..., \xi_p) \]  

(308)

These objects are traditionally denoted by the name p-branes (it all originated in a dubious joke). That is, ordinary particles are 0-branes, a string is a 1-brane, a membrane is a 2-brane, and so on.

Dualities relate branes of different dimensions in different theories; this means that if one is to take this symmetry seriously, it is not clear at all that strings are the more fundamental objects: in the so called M-theory (to be introduced in a moment) branes appear as fundamental as strings.

If we are willing to make the hypothesis that supersymmetry is not going to be broken whilst increasing the coupling constant, \( g_s \), some astonishing conclusions can be drawn. Assuming this, massless quanta can become massive as \( g_s \) grows only if their number, charges and spins are such that they can combine into massive multiplets (which are all larger than the irreducible massless ones). The only remaining issue, then, is whether any other massless quanta can appear at strong coupling.

Now, in the IIA string theory there are states associated to the Ramond-Ramond (RR) one form, \( A_1 \), namely the D-0-branes, whose tension goes as \( m - \frac{1}{k} \). This clearly gives new massless states in the strong coupling limit.

There are reasons to think that this new massless states are the first level of a Kaluza-Klein tower associated to compactification on a circle of an 11-dimensional theory. Actually, assuming an 11-dimensional spacetime with an isometry \( k = \frac{e}{29} \), an Ansatz which exactly reproduces the dilaton factors of the IIA string is

\[ ds_{11}^2 = e^{\tau/2} (dx - A_1^{(3)} dx^3)^2 + e^{-\tau/2} g_{\mu\nu} dx^\mu dx^\nu. \]  

(309)

Equating the two expressions for the D0 mass,

\[ \frac{1}{g_s} = \frac{1}{R_{11}}, \]  

(310)

leads to \( R_{11} = e^{\phi} = R_A^{2/3} \) (where \( g_A \) is the gauge coupling constant).

This means that a new dimension appears at strong coupling, and this dimension is related to the dilaton. The only reason why we do not see it at low energies is precisely because of the smallness of the string coupling, related directly to the dilaton field. The other side of this is that this eleven dimensional theory, dubbed M-theory does not have any weak coupling limit; it is always strongly coupled. Consequently, not much is known on this theory, except for the fact that its field theory, low curvature limit is \( \mathcal{N} = 1 \) supergravity in \( d = 11 \) dimensions.

All supermultiplets of massive one-particle states of the IIB string supersymmetry algebra contains states of at least spin 4. This means that under the previous set of hypothesis, the set of massless states at weak coupling must be exactly the same as the corresponding set at strong coupling. This means that there must be a symmetry mapping weak coupling into strong coupling.

There is a well-known candidate for this symmetry: Let us call, as usual, \( l \) the RR scalar and \( \phi \) the dilaton (NSNS). We can pack them together into complex scalar

\[ S = l + ie^{-\phi/2}. \]  

(311)

The IIB supergravity action in d=10 is invariant under the \( SL(2, \mathbb{R}) \) transformations

\[ S \rightarrow aS + b \]  

(312)

\[ cS + d. \]

\[ ^4 \text{In particular: The fact that there is the possibility of a central extension in the IIA algebra, related to the Kaluza-Klein compactification of the d=11 Supergravity algebra.} \]
if at the same time the two two-forms, $B$ and $A^{(2)}$, the RR field transform as
\[
\begin{pmatrix}
  B \\
  A^{(2)}
\end{pmatrix}
\rightarrow
\begin{pmatrix}
  d & -c \\
  -b & a
\end{pmatrix}
\begin{pmatrix}
  B \\
  A^{(2)}
\end{pmatrix},
\]
(313)

Both the, Einstein frame, metric $g_{\mu \nu}$ and the four-form $A^{(4)}$ are inert under this $SL(2, \mathbb{R})$ transformation.

A discrete subgroup $SL(2, \mathbb{Z})$ of the full classical $SL(2, \mathbb{R})$ is believed to be an exact symmetry of the full string theory. The exact imbedding of the discrete subgroup in the full $SL(2, \mathbb{R})$ depends on the vacuum expectation value of the RR scalar.

The particular transformation
\[
g = \begin{pmatrix}
  0 & 1 \\
  -1 & 0
\end{pmatrix},
\]
(314)

maps $\phi$ into $-\phi$ (when $l = 0$), and $B$ into $A^{(2)}$. This means that the string coupling
\[
g_s \rightarrow \frac{1}{g_s},
\]
(315)

This is a strong/weak coupling type of duality, similar to the electromagnetic duality in that sense. The standard name for it is an $S$-duality type of transformation, mapping the ordinary string with NS charge, to another string with RR charge (which then must be a $(D - 1)$-brane, and is correspondingly called a $D$-string), and, from there, is connected to all other D-branes by T-duality.

Using the fact that upon compactification on $S^1$, IIA at $R_A$ is equivalent to IIB at $R_B \equiv 1/R_A$, and the fact that the effective action carries a factor of $e^{-2\phi}$ we get
\[
R_A g^{2/3}_B = R_B g^{2/3}_A,
\]
(316)

which combined with our previous result, $g_A = R^{1/2}_A$ implies that $g_B = R^{1/2}_B$. Now the Kaluza-Klein Ansatz implies that from the eleven dimensional viewpoint the compactification radius is measured as
\[
R^{10}_B \equiv \frac{R^A}{e^{-2\phi}},
\]
yielding
\[
g_B = \frac{R_A}{R^{10}_B}.
\]
(318)

From the effective actions written above it is easy to check that there is a (S-duality type) field transformation mapping the SO(32) Type I open string into the SO(32) Heterotic one namely
\[
g_{\mu \nu} \rightarrow e^{-\phi} g^{Het}_{\mu \nu}, \]
\[
\phi \rightarrow -\phi, \]
\[
B' \rightarrow B.
\]
(319)

This means that physically there is a strong/weak coupling duality, because coupling constants of the compactified theories

![Figure 5. All string theories are related by T or S-dualities.](image)

Figure 6. Eternal AdS black hole.

would be related by
\[
B_{het} = 1/g_I, \]
\[
R_{het} = R_I/g^{1/2}_I.
\]
(320)

We have only scratched the surface here, but we can refer to the several excellent books treating this subject in more depth such as for example.\[53]\n
8.3. It from bit. Spacetime = Entanglement, ER = EPR, and All That

Ever since Wheeler\[73\] wrote the sentence "...every physical quantity, every it derives its ultimate significance from bits, binary yes-or no indications, a conclusion which we epitomize in the phrase it from bit ", many people have attempted to give a precise meaning to this idea.

For example, van Raamsdonk\[68\] has proposed a radical reinterpretation in the context of gauge/gravity duality of Israel’s\[44\] thermofield black hole state
\[
|\psi_\beta\rangle \equiv \sum_n e^{-\beta E_n/2} |E_n\rangle_L \otimes |E_n\rangle_R
\]
(321)

as representing the eternal AdS-Schwarzschild black hole. As is well-known the thermofield state\[64\] is a clever form of reinterpreting statistical mechanics by postulating another copy of the physical Hilbert space, and consider the product Hilbert space
\[
H_{extended} \equiv H_{phys} \otimes H_{copy}
\]
(322)
In this extended Hilbert space we consider the thermofield state

$$|\psi_\beta\rangle \equiv \sum_n e^{-\beta E_n^\text{phys}} |E_n^\text{phys}\rangle \otimes |E_n^\text{copy}\rangle \quad (323)$$

It so happens that when computing the expectation value of a physical observable (which acts solely on $H^\text{phys}$) on the extended Hilbert space, we get

$$\langle \psi_\beta | O^\text{phys} | \psi_\beta \rangle = \sum_n e^{-\beta E_n^\text{phys}} \langle E_n^\text{phys} | O^\text{phys} | E_n^\text{phys}\rangle \equiv \text{tr} (\rho_\beta O^\text{phys}) \quad (324)$$

owing to the assumed orthogonality of the set of states

$$\langle E_n^\text{phys} | E_m^\text{phys}\rangle = \delta_{nm} \quad (325)$$

and $\rho_\beta \equiv e^{-\beta H^\text{phys}}$ is the thermal density matrix.

The rationale of the reinterpretation is as follows. Assume that the thermofield state is in some sense a true state with a gravity dual. The gravity dual should be a spacetime with two asymptotically AdS regions, each of them corresponding to a copy of the CFT. In one of this asymptotic regions the dual should be just an AdS black hole, which is just a thermal state of the CFT. This is consistent with the fact that, as we have just seen,

$$\text{tr}_L |\psi_\beta\rangle \langle \psi_\beta| = \sum_n e^{-\beta E_n^\text{phys}} \langle E_n^\text{phys} | E_n^\text{phys}\rangle \equiv \rho_\beta \quad (326)$$

The presence of horizons in the gravity dual that prevent classical communication between both asymptotic regions are consistent with the absence of interactions between both copies of the CFT.

This means that the state $|\psi_\beta\rangle$ which could be naively though as being dual to a superposition of disconnected spacetimes, is reinterpreted as a classically connected spacetime. In this example, connectivity stems from entanglement of the two components present in the thermofield state.

Another argument starts by considering a CFT on a sphere, which we again divide in two different hemispheres, which we keep denoting by $L$ and $R$. Assume that the physical Hilbert space is of the form

$$H = H_L \otimes H_R \quad (327)$$

(more on this later).

Assume also that a measure of entanglement between both hemispheres is given by the entanglement (von Neumann) entropy associated to the physical state $|\Psi\rangle$

$$S(L) \equiv -\text{tr}_L \log \rho_L \quad (328)$$

where

$$\rho_L \equiv \text{tr}_R |\Psi\rangle \langle \Psi| \quad (329)$$

This quantity should be regularized. In fact, in the context of AdS/CFT duality Ryu and Takayanagi\cite{39} have proposed that the entropy associated to a finite area $A$ in the CFT side (defined on the boundary $\partial M$ of the bulk space-time $M$) is given by area of the minimal surface $S$ in the bulk $M$ whose boundary on $\partial M$ is precisely $A$: $\partial S = A$, the area under consideration in the CFT.

Assume that this entropy $S(L)$ decreases. Then according to Ryu and Takayanagi’s formula, the area of the minimal surface that separates both components $L$ and $R$ should decrease as well. This means that as the entanglement goes to zero, both regions $L$ and $R$ are pinching off, and in the limit we get two disconnected pieces.

This viewpoint has been forcefully supported by Maldacena and Susskind\cite{40} that proposed that the gravity dual of quantum mechanical entanglement (symbolically, EPR, for Einstein, Podolsky and Rosen\cite{26}) should involve wormholes (symbolically, ER, for Einstein-Rosen bridges\cite{26}). This is the famous slogan $\text{ER} = \text{EPR}$. In this way they are able to question the necessity of firewalls\cite{2} in some cases by considering the entanglement between the black hole and the radiation after the Page time.

General attempts of this type to define spacetime starting from entanglement have been criticized\cite{41} on the basis that in order to define entanglement one needs to define a notion of subsystems and entanglement entropy depends on such definition of subsystems.

Some people still maintains that spacetime structure appears more fundamental than entanglement.

### 8.4. Diffeomorphism Invariant Observables

Dirac himself\cite{22} worried about the general problem of how to define gauge invariant observables in the context of (abelian) gauge theories and introduced dressed operators such as, for a charged scalar field,

$$\Phi_\beta(x) \equiv \phi(t, \vec{x}) e^{iC(x)} \quad (330)$$

He assumed that

$$C(x) \equiv \int c_i(x, x^\prime) A^i(x^\prime) dx^\prime \quad (331)$$

and demanded that it commuted with the generator of gauge transformations

$$G \equiv \int d^3 x f(x) (\partial_x E^i + q\bar{\sigma}\phi_i) \quad (332)$$
Using
\[ [A, G] = -\partial f \]
\[ [\phi, G] = -iq\phi \]

The condition we need to fulfill is
\[ [\Phi_0, G] = -i\Phi_0 \left( df - \int \partial_i c_i(x, x') f(x') d^4 x' \right) \]

That is
\[ \partial_i c_i(x, x') = c(x, x') \delta(x - x') \]

There are many solutions of this. One of the simplest is
\[ C(x) = iq \int \frac{(x - x')^i}{4\pi|x - x'|^4} \Phi_i(t, \bar x') \]

This operator creates a Coulomb field at time t
\[ \left[ E(x), \Phi_i(x') \right]_{\gamma,\tau} = -q \frac{(x - x')^i}{4\pi|x - x'|^4} \Phi_i(x) \]

This dressing can be generalized in various ways. One of the simplest is to introduce a Dirac string (Dirac called it a Faraday line of force). That is,
\[ C = -q \int_{-\infty}^{\infty} A(x') dx' \]

where the path of integration extends from infinity to the location of the charge at the point x. In this case the electric field is concentrated in the Dirac string.

In [23] Donnelly and Giddings have generalized this construction to the gravitational field, albeit to first order in the coupling constant, \( \kappa \). For a scalar field the analogous dressed field is
\[ \Phi(x) \equiv \phi(x' + V^i(x)) = \phi(x) + V^i \partial_i \phi + O(V^2) \]

where
\[ \delta V^i = -\kappa \xi^i(x) \]

We can find vectors such that
\[ V^i(x) = \kappa \int d^4 x' f^{i,\lambda}(x, x') h_{\lambda}(x') \]

In order for this definition to be consistent with linearized diffeomorphism (LDiff)
\[ \delta h_{\lambda} = \partial_\alpha \xi_\lambda + \partial_\mu \xi^{\mu} + O(\kappa) \]
\[ \delta f^{i,\lambda} = \frac{1}{2} \delta^i (x - x') \eta^{\lambda} \]

Preceding along these lines, Giddings finds an obstruction to commutativity of gauge invariant observables associated to different regions of spacetime. Then locality fails for such dressed operators. Calculations are done using standard Dirac commutators stemming from the canonical theory.

8.5. Subsystems

From the Haag algebraic QFT theory, subalgebras of observables may be associated to regions of spacetime, and the subalgebra structure mirrors the topology of the spacetime manifold.

A natural idea is then in the quantum first approach, to try to derive the spacetime structure from the net structure of the subalgebras of the von Neumann algebra of operators.

We have just seen the difficulties for doing so in the presence of gravity, at least if the low energy limit os to be reached in a smooth way.

Difficulties in the definition of subsystems stem from a couple of reasons: one is technical namely the so called type III\( _1 \) property of the adequate subalgebras (which do not contain projectors at all), and the second is the existence of long range (gauge) fields.

In QED it is possible to define localized charged observables, such as
\[ D(x, x') \equiv \phi(x) \phi^{\dagger}(x') \]

which creates a nontrivial field only in the string from the point \( x \) to the point \( x' \). Then it commutes with all observables space-like separated from this region. Interactions modify this property however. This string decays into a dipole field that extends to infinity.

In the presence of gravity, moreover, all localized excitations carry energy in such a way that gravity cannot be screened, so there is not even the analog of the QED charged operator.

In the absence of any hint from experiment it is not then clear how far can we go in this quantum first viewpoint.

9. Conclusions

It should be hopefully clear from the preceding discussions that there are no definite conclusions as yet. Almost all avenues are still open.

Even the most straightforward idea of quantizing the metric tensor is not completely excluded, and some of the most relevant work in the field is done by exploring just ordinary quantum field theory (QFT) amplitudes. It is curious that Veltman’s old idea that diagrams are more fundamental than the lagrangians themselves is now revamped in the work of Arkani-Hamed and coworkers.

It is nevertheless clear that in QFT there is a rich non-perturbative sector, of which essentially nothing is known, in spite of the attempts of a great number of brilliant physicists in the second half of last century.

In the particular case of supersymmetric gauge theories there are some set of dualities (many of them inspired or even implied by string theory), but we are shy of understanding the detailed mechanism at work in the strong coupling sector. It should be said, however, that the only clear image of confinement we have (due to Seiberg and Witten) is precisely in theories with several (more than eight) supersymmetric charges.

It is even conceivable that QFT contains string theory in some sense. The quantum field theoretical framework seems nowadays much richer than was formerly believed to be the case. In the non-supersymmetric case, it is true that most of our information on the non-perturbative sector in quantum chromodynamics (QCD)
comes from the plethora of low energy data in hadronic physics. One wonders what would be our image of the infrared (IR) limit of QCD if we were not aware of this experimental information.

Quantum general relativity (quantizing the metric) is in some sense a gauge theory, but it is also a quite special one. Probably more complicated. It seems that it will be very difficult to understand precisely how a black hole is made out of self-interacting gravitons (or whatever elementary quanta are appropriate) as a bound state or confined state of sorts, before understanding the presumably simpler problem of how a glueball is made out of gluons in a Yang Mills theory such as QCD.

Nevertheless, even in the absence of any experimental information, we cannot avoid to keep thinking on the relationship between two of the most successful theories in physics, namely General Relativity and Quantum Mechanics, in whatever common ground they might share.

We are well aware that the chances of success are very slim, but, like in the famous poem Ihaka by Kavafis, the journey is fascinating, and we have discovered many interesting vistas in our way. This is particularly true for string theory. Even if the main purpose of the theory (a unified theory of all interactions, including gravity) is not fulfilled, it cannot be denied that around this field many topics have flourished. In pure mathematics, of course, but not only. For example most of the modern techniques for doing advanced computations of amplitudes in gauge theories, associated to the names of Zvi Bern, Lance Dixon, David Kosower, and many others stem from string theory. For example, and against superficial appearances, to concentrate in on shell amplitudes and cubic vertices has proved enormously fruitful.

But in spite of all that, Ihaka is still far, far away.

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