Comments on scalar-tensor representation of non-locally corrected gravity.

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Abstract

The scalar-tensor representation of nonlocally corrected gravity is considered. Some special solutions of the vacuum background equations were obtained that indicate to the nonequivalence of the initial theory and its scalar-tensor representation.

1 Introduction

Recent observations have shown that the universe is presently accelerating. Most attempts to explain this acceleration involve the introduction of dark energy, as a source of the Einstein field equations. Standard $\Lambda CDM$ model, in which dark energy is considered as a small positive cosmological constant, provides a very good fit to the supernovae data [1] as well as CMB measurements [2] and observations of large scale structure [3], but the small and fine-tuned value of the cosmological constant cannot be explained within current particle physics [4]. As a result a many other cosmological models have been proposed to giving a dynamical origin to dark energy. Pure phenomenologically one can consider dark energy as a perfect fluid with a sufficiently negative pressure [5]. There is also a large class of scalar field models in the literature including quintessence [6] and phantom [7] fields.

Scalar-tensor theories of gravity called an attention in connection with attempts to give an geometrical explanation to a dark energy phenomenon [8], [9], [10], [11]. They are conformally coupled with intensively studied $f(R)$ generalized gravity theories in metric and Palatini formalism. Although several $f(R)$ modified gravity models have been proposed which realize the correct cosmological evolution and satisfy solar system tests, for the current moment there are some considerable problems in construction of available models of dark energy on the basis of theories of this type [12] (see also ref. [13]).
Several papers have appeared recently containing scalar-tensor representation of nonlocal theories \cite{14}, \cite{15}, \cite{16}, \cite{17}. The simple example of the nonlocal action \cite{18} is

\[ S = \int d^4x \sqrt{-g} \left\{ \frac{R}{2\kappa^2} (1 + f^{-1}(R)) + L_m \right\}. \quad (1) \]

where \( R \) is the Ricci scalar, \( \Box \) is the d’Alembertian, and \( L_m \) is the Lagrangian of matter. More general action involving \( m \) different powers \( \Box^{-1} \) acting on \( R \) also can be considered \cite{15}.

The standard approach is based on introducing the Lagrange multiplier \( \xi \) and rewriting the initial theory in the local form

\[ S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} \left( R(1 + f(\phi)) + \xi(\Box \phi - R) \right) + L_m \right\}. \quad (2) \]

Validity of such approach raises the doubts. On the one hand, the constraint equation

\[ \Box \phi - R = 0 \quad (3) \]

has the formal solution \( \phi = \Box^{-1} R \). Substituting the above solution into \( (2) \) one reobtains \( (1) \).

On the other hand, unlike classical mechanics, constraint \( (3) \) contains the second order time derivatives. Besides, this constraint equation does not allow to determine \( \varphi(R) \) uniquely, since it remain valid after the transformation \( \varphi \rightarrow \varphi + \varphi_D \), where function \( \varphi_D(x) \) is an arbitrary solution of the d’Alembert’s equation. For the proof of equivalence of theories it is necessary to check up equivalence of the dynamical equations that has not been done earlier. The purpose of this work is a more detailed analysis of the relationship between models \( (1) \) and \( (2) \).

2 The basic equations

In this section we shortly describe some equivalent form of action \( (2) \) and the basic equations. Integrating by parts and neglecting the boundary terms, the action \( (2) \) can be rewritten as

\[ S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} \left( R(1 + f(\phi) - \xi) - \xi_{,\alpha} \phi^{,\alpha} \right) + L_m \right\}. \quad (4) \]

It is convenient to introduce a new scalar field \( \Psi = f(\phi) - \xi \), which is one nonminimally coupled to gravity in physical (Jordan) frame. The action then becomes \cite{10}

\[ S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left\{ (1 + \Psi) R - f'(\phi) \phi_{,\alpha} \phi^{,\alpha} + \Psi_{,\alpha} \phi^{,\alpha} + 2\kappa^2 L_m \right\}. \quad (5) \]

The Einstein frame action can be obtained by conformal transformation such that
\[ \dot{g}_{\mu\nu} \equiv (1 + \Psi) g_{\mu\nu} \equiv e^\lambda g_{\mu\nu}. \]  

Then, one can obtain [16]:

\[ S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-\tilde{g}} \left\{ \tilde{R} - \frac{3}{2} \lambda \tilde{\alpha} \tilde{\alpha} - e^{-\lambda} f'(\phi) \phi \tilde{\alpha} \right\}, \]

Applying the field redefinition

\[ \lambda = \frac{1}{3} \phi - \sqrt{\frac{2}{3}} \varphi, \]

yields:

\[ S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left\{ \tilde{R} - \varphi \tilde{\alpha} \varphi \tilde{\alpha} - \left( e^{\sqrt{\frac{2}{3}} \varphi} \frac{1}{2} f'(\phi) \right) \phi \tilde{\alpha} \right\}, \]

Let’s notice that the above definition [8] is chosen slightly different from used in [19] to provide more evident transition to the limiting case \( f(\phi) = f_0 = \text{const} \). It should be emphasized that in the limiting case \( f(\phi) = f_0 \) this model reduce to general relativity with one canonnic, one phantom scalar field and the nonminimally coupled to gravity matter. The correct limiting transition will occur only if the additional condition \( e^{\sqrt{\frac{2}{3}} \varphi} = 1 + f_0 \) is added. In this case contributions of scalar fields compensate each other, and the matter coupled to gravity minimally.

By the variation over \( \phi \) and \( \Psi \), one can obtain the modified Einstein equations

\[ \frac{1}{2} R(1 + \Psi) g_{\mu\nu} - R_{\mu\nu}(1 + \Psi) - \frac{1}{2} f' \phi,\alpha \phi^\alpha g_{\mu\nu} + \frac{1}{2} \psi,\alpha \phi^\alpha g_{\mu\nu} + f' \phi,\mu \phi,\nu \\
- \frac{1}{2} \left( \psi,\mu \phi,\nu + \psi,\nu \phi,\mu \right) - g_{\mu\nu} \psi,\alpha + \psi,\mu \psi,\nu + \kappa^2 T_{\mu\nu} = 0, \]

where \( T_{\mu\nu} \) is the energy-momentum tensor of the fluid matter.

Consider now the spatially flat Friedmann-Robertson-Walker Universe, described by the action [5] with the line element

\[ ds^2 = -dt^2 + a^2(t) dx_i dx^i. \]
The Friedmann equations can be written as [15]:

\[ 3H^2(1 + \Psi) = \kappa^2 \rho + \frac{1}{2} \left( f'(\phi)\dot{\phi}^2 - \dot{\Psi}\dot{\phi} \right) - 3H\dot{\Psi}, \]  

(14)

\[ - (2\dot{H} + 3H^2)(1 + \Psi) = \kappa^2 p + \frac{1}{2} \left( f'(\phi)\dot{\phi}^2 - \dot{\Psi}\dot{\phi} \right) + \dot{\Psi} + 2H\dot{\Psi}, \]  

(15)

where \( H = \frac{\dot{a}}{a} \) is the Hubble rate, and energy density \( \rho \) and pressure \( p \) of the background perfect fluid are satisfy the continuity equation

\[ \dot{\rho} + 3H(1 + w)\rho = 0. \]  

(16)

The background field equations are

\[ \ddot{\phi} + 3H\dot{\phi} = -6(\dot{\Psi} + 2H^2), \]  

\[ \ddot{\Psi} + 3H\dot{\Psi} = 0, \]  

(17)

(18)

3 The analysis of the background equations at \( f'(\varphi) \equiv 0 \)

In what follows we consider the background equations in the simple special case \( f'(\varphi) \equiv 0 \). The equations of motion for the homogeneous background scalar fields and scale factor \( a(t) \) are simplify to

\[ 3H^2(1 + \Psi) = \kappa^2 \rho - \frac{1}{2} \dot{\Psi}\dot{\phi} - 3H\dot{\Psi}, \]  

(19)

\[ - (2\dot{H} + 3H^2)(1 + \Psi) = \kappa^2 p - \frac{1}{2} \dot{\Psi}\dot{\phi} + \dot{\Psi} + 2H\dot{\Psi}, \]  

(20)

\[ \ddot{\phi} + 3H\dot{\phi} = -6(\dot{\Psi} + 2H^2), \]  

\[ \ddot{\Psi} + 3H\dot{\Psi} = 0, \]  

(21)

(22)

where \( p = p(\rho) \).

In absence of matter we obtain a set of the equations describing dynamics of three variables \( \phi, \Psi \) and \( H \).

\[ 3H^2(1 + \Psi) = -\frac{1}{2} \dot{\Psi}\dot{\phi} - 3H\dot{\Psi}, \]  

(23)

\[ - 2\dot{H}(1 + \Psi) = -\dot{\Psi}\dot{\phi} - 4H\dot{\Psi}, \]  

(24)

\[ \ddot{\phi} + 3H\dot{\phi} = -6(\dot{\Psi} + 2H^2), \]  

\[ \ddot{\Psi} + 3H\dot{\Psi} = 0. \]  

(25)

(26)

This system has the trivial solution

\[ \phi(t) = \phi_0, \quad \Psi(t) = \Psi_0, \quad H(t) = 0, \]  

(27)

that corresponds to the Minkowski metric. However other solutions are also available.

Assuming \( \dot{H} \equiv 0 \), the system of equations (23) - (26) is reduced to
\[ 3H^2(1 + \Psi) = -\frac{1}{2} \dot{\Psi} \dot{\phi} - 3H \ddot{\Psi}, \quad (28) \]

\[ 0 = \dot{\Psi} \dot{\phi} + 4H \ddot{\Psi}, \quad (29) \]

\[ \ddot{\phi} + 3H \dot{\phi} = -12H^2, \quad (30) \]

\[ \ddot{\Psi} + 3H \dot{\Psi} = 0, \quad (31) \]

with the de Sitter solution

\[ \Psi = Ae^{-3H_0 t} - 1, \quad \phi = -4H_0 t + \phi_0, \quad H = H_0, \quad (32) \]

where \( A, \phi_0, H_0 \) are arbitrary constants.

Analogously, considering the case \( \dot{\phi} \equiv 0 \), one can obtain

\[ \Psi = \frac{A}{(C + 2t)^{1/2}} - 1, \quad \phi = \phi_0, \quad H = \frac{1}{C + 2t}. \quad (33) \]

It is remarkable, that the initial theory (1) in the case under consideration reduce to

\[ S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left\{ \frac{R}{2\kappa^2}(1 + f_0) + L_m \right\}. \quad (34) \]

This is the action of general relativity with a renormalized gravitational constant. The only homogeneous and isotropic solution of the corresponding vacuum Einstein field equations is the Minkowski metric. Hence, background solutions (32) and (33) are artifacts of the scalar-tensor representation.

The existence of these solutions show that generalized Einstein gravity theories (1) and (2) are not equivalent even at \( f'(\phi) \equiv 0 \). In the scalar-tensor representation there are new solutions for the metrics which are unavailable in the initial theory.

4 Conclusion

We consider the general equations of motion and the background equations for scalar-tensor theory (2). We have obtained particular solutions of the background equations, which show the nonequivalence of theories (1) and (2) even in the limiting case \( f'(\phi) \equiv 0 \). An available problem is the existence of redundant solutions in the scalar-tensor representation of nonlocally corrected gravity. It remains an open question whether constraints can also be satisfied to disregard unphysical solutions.

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