Critical-current density from magnetization loops of finite high-$T_c$ superconductors

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We analyze the effects of demagnetizing fields in the magnetization hysteresis loops of type-II superconductors, by a model that allows the calculation of current and field profiles in finite cylindrically symmetric superconductors in the critical state. We show how the maximum in the magnetization curve gradually shifts from negative applied field values to the central position with decreasing sample thickness. From the analysis of the calculated field profiles, we demonstrate that one can obtain the intrinsic field-dependent critical-current density of the superconductor with great accuracy by measuring the magnetic response of superconductors with large aspect ratio with the applied field parallel to the shortest dimension.

The critical current density $J_c$ is one of the key parameters of high-$T_c$ superconductors. In particular, knowing the dependence of $J_c$ on the internal magnetic field $H_i = B/\mu_0$ in the superconductor is a key concern for the study of the current-carrying states of the superconductors. The $J_c(|H_i|)$ dependence is obtained from two general types of experiments: electrical and magnetic. In electrical transport measurements, the reliability and repeatability of the measured values of the critical current is reduced because of the difficulties in making contacts in the sample and in choosing a voltage criteria for defining $J_c$. Moreover, in general large currents circulate through the sample during the measurements, so that they produce a magnetic field that modifies the applied field in an inhomogeneous way. This makes the estimation of the intrinsic $J_c(|H_i|)$ dependence difficult. The other general method for obtaining $J_c$ is from magnetization measurements, typically isothermal magnetization loops $M(H_a)$, where $H_a$ is the applied field. $J_c$ is obtained from the width $\Delta M$ of the $M(H_a)$ hysteresis loop using

$$J_c(H) = \Delta M(H)/d,$$

where $d$ is a length characteristic of the sample size and geometry; for a cylinder of radius $R$,

$$J_c(H) = 3\Delta M(H)/2R.$$  \hspace{1cm} (1)

This relation follows from the critical-state model [1]. The magnetic method also involves some approximations. First, the method was originally derived for the case of $J_c$ independent of $|H_i|$ [1]. If instead $J_c$ depends on $|H_i|$, as in most actual superconductors, what is obtained from the width of the loop is not the required intrinsic $J_c(|H_i|)$ function, but a different function, $J_c(|H_a|)$. These functions are approximately equal only if some extra conditions concerning internal field homogeneity are met, as discussed in [2] and also below. The second important approximation is that the method is only theoretically justified for infinitely long samples. The validity of the method for extracting $J_c$ from the width of the loop is therefore questionable in the realistic case of finite superconductors.

Calculations of the magnetic response of finite superconductors in the critical state including demagnetizing effects have been recently presented for strips [3] and cylinders [4,5], following previous works on very thin strips [6] and disks [7]. Very recently, Shantsev et al [8] have discussed important features of the effects of demagnetizing fields in the hysteresis loop. In particular, they demonstrated theoretically and experimentally that for very thin strips in perpendicular field the peak that appears in the reverse magnetization curve is located not at negative fields, but at the central position.

In this work, we systematically study the effect of demagnetizing fields in the magnetic response of finite superconducting cylinders in the presence of a uniform applied field, and discuss the implications in $J_c$ extraction. We first introduce a model which allows us to compute current and field profiles and magnetization loops of superconducting cylinders with the same intrinsic parameters but different aspect ratios, in order to study in detail the effects of demagnetizing fields. These will be analyzed in relation to two particular features: the position of the peak in the magnetization curves as a function of the sample aspect ratio, extending the work of Shantsev et al [8], and the relation between the shape of the hysteresis loop and the field dependence of the critical current density for the different aspect ratios.

Our model, which simulates the process of penetration of supercurrents inside a superconductor, is based in the fact that any modification of the applied field results in a change in the superconductor current distribution in order to minimize the change in the magnetic energy. The current distribution in the initial magnetization curve (after the sample has been zero-field cooled and a magnetic field is applied) can therefore be obtained from the magnetic energy minimization. We will obtain the reverse current distributions (once the field has reached a maximum value and is decreased) by the conventional procedure of superposing a current distribution with opposite sign to the "frozen" field profiles as in [10]. In all cases, we assume that $B = \mu_0 H$, which holds for fields $H_{c1} << H << H_{c2}$, as is usually done in critical-state modeling.
Consider a cylindrical type-II superconductor of radius $R$ and length $L$ located in a uniform applied field, $H_a$, directed along its axis. We use common cylindrical coordinates $(\rho, \theta, z)$, $z$ being the direction of the axis of the superconductor. Owing to the symmetry of the system (which reduces the problem to a two-dimensional one), all supercurrents flow with angular direction. We divide the superconductor in a regular grid of $n \times m$ coaxial rings in which linear currents can flow. The magnetic flux that threads one of these linear circuits at the position $(\rho, z)$ due to the external applied field is \( \Phi(\rho, z) = \mu_0 H_a \pi \rho^2 \), while the flux that threads the same circuit due to all currents circulating in the superconductor is \( \Phi(\rho, z) = \sum_{\rho', z'} M(\rho, z, \rho', z') I(\rho', z') \), where \( I(\rho', z') \) indicates the current in the circuit \((\rho', z')\), and \( M(\rho, z, \rho', z') \) the mutual inductance between the circuits \((\rho', z')\) and \((\rho, z)\). The self-inductances \( M(\rho, z, \rho, z) \) are calculated from the mutual inductance between two close linear circuits \([9]\). The model details are described in Refs. \([1,13]\). In this paper we have used typical values of $n \times m \simeq 120 \times 20$ for thin samples and $n \times m \simeq 60 \times 60$ for samples with larger $L/R$ ratios.

Let us assume we have a given current distribution corresponding to an applied field $H_a$ (if we are calculating the first point after the initial state, then the initial current distribution is zero everywhere). Setting a current $I$ at a circuit $(\rho, z)$ requires an energy \( E = \Phi I(\rho, z) \) while it contributes to reduce the energy (current has opposite sign to \( I \)) due to the external applied field $H_a$ by a quantity \( I \Phi(\rho, z) \). We find in this way the circuit that yields the largest decrease of energy and set a current $I$ there. The new currents are set accomplishing the chosen material law $J = J_c(H/|H|)$, $|H|$ being the modulus of the total field $\mathbf{H}$, and $I = J_c(H/|H|)(RL/nm)$. When no new currents minimize further the energy, we calculate the magnetic field inside the superconductor and change the value (not the distribution) of the already induced currents to accomplish the material law. At this point, we calculate the magnetic moment resulting from all the circulating currents and all the other relevant magnitudes. After this, the applied field can be increased again, and the process is restarted from the existing current distribution. The reverse stage (corresponding to decreasing $H_a$ from $H_{\text{max}}$ to $-H_{\text{max}}$) is calculated by superposing the frozen current penetration set at $H_{\text{max}}$ to the one induced in the reverse stage (calculated in a similar manner as described for the initial curve). This procedure is typical of the critical state model \([10]\).

Our model allows the implementation of an arbitrary $J_c(H/|H|)$ dependence. The following discussions and conclusions are valid for any dependence as long as $J_c$ is a decreasing function on $|H|$, which is physically reasonable. For illustrating our results, we choose an exponential dependence $J_c = J_{c0} \exp(-|H|/H_0)$, where $J_{c0}$ and $H_0$ are positive constants. The exponential dependence has been successfully applied to high-$T_c$ superconductors \([14,15]\). A useful parameter for the analysis is $p = J_{c0} R/H_0$, which takes on large values for a strong dependence of $J_c$ on $|H|$ and tends to 0 for $J_c$ independent of $H$ \([4]\).

Realistic values of $p$ for high-$T_c$ superconductors range from 1 to 10 \([1,13]\).

In Fig. 1 we show the calculated $M(H_a)$ curves for different values of $L/R$, for the cases $p = 0$ (Bean’s model), 3, and 10. $M$ and $H_a$ are normalized to $H_p = H_0 \ln(1 + p)$, which corresponds to the penetration field for an infinite cylinder. We find that the calculated $M(H_a)$ loops for sufficiently large samples coincide between our numerical accuracy with the known analytical results for infinite cylinders \([13]\). Let us now discuss the differences observed in the loops for each value of $p$, which are only due to the sample geometry. In all cases, the demagnetizing field enhances the initial slope of both the initial and reverse curves. We have analyzed this effect in detail, observing good agreement with experimental data measured for niobium cylinders of different lengths in \([13]\); we obtain the same values for the initial slope of the $M(H_a)$ curves as those calculated by Brandt \([7]\) and Chen et al. \([10]\) for a wide range of $L/R$ values, with less than 1% deviation. The shape of the central part of the loop is not changed in the Bean case ($p = 0$), since even in a finite sample the width of the loop is proportional to the constant value of $J_c$. However, important variations are observed for non-constant $J_c$. When $p \neq 0$, a peak appears in the reverse magnetization curve. In Fig. 2 we show the calculated dependence of the peak position $H_{\text{peak}}$ on $L/R$ for $p=3$ and 10.

The general trend of the peak position is that it tends towards $H_a = 0$ with decreasing sample $L/R$ ratio. A similar tendency was found in \([1]\) for a three parameter $J_c(H)$ function and using an iterative method valid only for the fully penetrated stage. Our calculated results give the correct known limit when the sample is very large (solid line in Fig. 2 indicates the limit for infinite cylinders \([13]\)). Our results are also compatible with \([13]\), where it is said that the peak tends towards $H_a = 0$ for thin samples without reaching exactly the central position (this is in contrast with the strip case, for which the peak position was shown to be zero for very thin strips \([12,19]\).

The peak position and the other observed features of the hysteresis loop can be understood as follows. We start by discussing the situation in the simpler case of an infinitely long superconductor with a given $J_c(H)$ function. After reaching the maximum applied field $H_{\text{max}}$, when we reverse the magnetic field sweeping direction in the hysteresis loop, supercurrents are induced at the surface of the cylinder with a direction opposite to the shielding currents of the initial magnetization curve. These reverse supercurrents gradually enter the sample, confining the original currents to the interior. When $H_a$ decreases, the internal field in the region penetrated by reverse currents also decreases, these currents become larger, and therefore the magnetization increases. Passing through $H_a = 0$, the internal field becomes negative at the surface regions of the superconductor. Further de-
creasing $H_a$ makes the $H_i$ profile negative, so that $|H_i|$ again increases. Since $J_c$ decreases with the absolute value of $H_i$, the currents become low and $M$ decreases. Then, a peak appears in the magnetization at some negative value of $H_a$, for which an averaged value of $|H_i|$ is minimum. This process is depicted in Fig. 3a, where we show the calculated field profiles in the cylinder midplane corresponding to the case $p = 10$ for a long cylinder with $L/R = 10$, which schematically represents the infinite case.

There is a key difference for this case and the behavior observed for a realistic finite sample. Whereas in an infinite sample the internal field at a given point in the superconductor has only contributions from $H_a$ and the field created by the currents exterior to this point [14], there is contribution from all the circulating currents at all points in a finite superconductor. This self-field contribution produces an effect in the central region of the loop $(|H_i|)$ small similar to shifting the applied field upwards to higher values for $H_a > 0$ and downwards to lower values for $H_a < 0$. This explains the increase in the initial slopes of the loops shown in Fig. 1, as well as the appearance of a peak in $M$ for applied field values $H_a$ much closer to the central position. We show in Fig. 3b the calculated field profiles for the same case, $p = 10$, as in Fig. 3a, but for a thin disk with $L/R = 0.1$. The large effect of demagnetizing fields is made manifest by the almost constant field profiles characteristic of high applied fields already achieved for low values of $|H_i|$.

This has important consequences for the method used to extract $J_c(|H_i|)$ from the width of the magnetization loop $M(H_a)$. As discussed by Chen and Goldfarb [3] there are two requirements for using the method: (1) The magnetization on ascending and descending branches of the hysteresis loop at a given field $H_a$ must correspond to fully penetrated states; and (2) the maximum deviation of $J_c(|H_i|)$ in the sample from the value of $J_c(|H_i|)$ for $H_i = H_a$ must be small. Condition (1) is easily fulfilled as long as one takes care to measure a reverse magnetization curve starting from a sufficiently large maximum applied field value. The key condition is then the second, which is understandable taking into account the fact that the formula for $J_c$ extraction is based on the Bean's model for constant $J_c$ so large inhomogeneities in the local currents will yield wrong results. Comparing our results for field profiles calculated for the same material but with different dimensions, see Figs. 3a and 3b, one can clearly see that the field profiles in the case of thin samples are much more spatially uniform than in the case of long superconductors. (Actually, the value of the internal field for thin samples is basically equal to the external $H_a$ value except within a narrow region around $|H_a| = 0$.) There can be several reasons for that. Currents are low, except when $|H_a|$ is very small, owing to the contribution of the demagnetizing fields to $H_i$ in the $J_c(|H_i|)$ function, producing small contribution to the total field. Moreover, in a thin disk a much smaller total current is flowing as compared with a bulk cylinder. Therefore, thin samples in transverse geometry are an optimum case for obtaining $J_c$ from the width of the magnetization loop. This conclusion is confirmed in Fig. 4, where we compare the $J_c(H_a)$ function extracted from the width of the loop with the intrinsic $J_c(|H_i|)$, for the cases $p = 3$ and $10$, and for a long sample ($L/R = 10$) and a thin one ($L/R = 0.1$) for each case. The agreement between both $J_c$ functions is clearly better for the thin samples than for the long ones for all magnetic field values, and particularly for the low-field region which involves in the long sample a large measurement error. Our model also shows that for higher values of $p$ the agreement between $J_c$ functions is better, which implies that measuring thin superconductors is specially important when the expected dependence of $J_c$ on $|H_i|$ is strong.

We conclude from this analysis of demagnetizing effects in realistic finite superconductors that the extraction of the critical current density from magnetization measurements is best done in thin film geometry in perpendicular field, or, if thin films are not available, with the field applied along the sample shortest dimension. In this way, the superconductor critical current density and its dependence on the internal field can be precisely obtained by measuring the hysteresis loops of superconductors using the non-destructive, reproducible, and widespread magnetization measurements.

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FIG. 1. Magnetization loops for the exponential dependence of the critical current, for (a) \( p = 0 \) (Bean model), (b) \( p = 3 \), and (c) \( p = 10 \), and for different values of the length-to-radius, \( L/R = \infty \) (solid), 1 (dashed), and 0.1 (dotted).

FIG. 2. Calculated applied field at which a peak in the reversal magnetization of superconducting cylinders occurs, \( H_{\text{peak}} \), as a function of \( L/R \). Filled circles correspond to \( p = 3 \) and open circles to \( p = 10 \). Horizontal lines represent the known values for infinite cylinders.

FIG. 3. Internal magnetic field \( H_i \) profiles in the midplane of the superconductor cylinder for the case \( p = 10 \) and \( L/R = (a) \) 10 and (b) 0.1, corresponding to the reverse curve after a maximum applied field of \( H_{\text{max}} = 1.5H_p \). The values of the applied field \( H_a \) range from \( 1.4H_p \) to \(-1.4H_p \) in steps of \( 0.2H_p \) (from top to bottom).

FIG. 4. Critical current density calculated from the width of the magnetization loop \( \Delta M \) using Eq. (2) for \( L/R = 0.1 \) (dashed line), and for \( L/R = 10 \) (dotted line), and from the analytical expression of the exponential dependence (solid line). The case (a) corresponds to \( p = 10 \) and (b) to \( p = 3 \). \( H_a \) is the external applied field at which \( \Delta M \) is evaluated, whereas \( H_i \) is the internal field upon which \( J_c \) depends.
\[ \frac{M_z}{H_p} \]

- **(a)**: \( p = 0 \)
- **(b)**: \( p = 3 \)
- **(c)**: \( p = 10 \)

\[ \frac{H_a}{H_p} \]
(a) \( p = 10 \)

(b) \( p = 3 \)