Betti numbers of Stanley–Reisner rings with pure resolutions

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Abstract

Let $\Delta$ be simplicial complex and let $k[\Delta]$ denote the Stanley–Reisner ring corresponding to $\Delta$. Suppose that $k[\Delta]$ has a pure free resolution. Then we describe the Betti numbers and the Hilbert–Samuel multiplicity of $k[\Delta]$ in terms of the $h$–vector of $\Delta$. As an application, we derive a linear equation system for the components of the $h$–vector of the clique complex of an arbitrary chordal graph.

1 Introduction

Let $k$ denote an arbitrary field. Let $R$ be the graded ring $k[x_1, \ldots, x_n]$. The vector space $R_s = k[x_1, \ldots, x_n]_s$ consists of the homogeneous polynomials of total degree $s$, together with 0.

In [9] R. Fröberg characterized the graphs $G$ such that $G$ has a linear free resolution. He proved:

**Theorem 1.1** Let $G$ be a simple graph on $n$ vertices. Then $R/I(G)$ has linear free resolution precisely when $\overline{G}$, the complementary graph of $G$ is chordal.

0 Keywords. Betti number, Hilbert function, Stanley-Reisner ring

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In [6] E. Entander generalized Theorem 1.1 for generalized chordal hypergraphs.

In this article we prove explicit formulas for the Betti numbers of the Stanley–Reisner ring \(k[\Delta]\) such that \(k[\Delta]\) has a pure free resolution in terms of the \(h\)-vector of \(\Delta\).

In Section 2 we collected some basic results about simplicial complexes, free resolutions, Hilbert functions and Hilbert series. We present our main results in Section 3.

2 Preliminaries

2.1 Free resolutions

Recall that for every finitely generated graded module \(M\) over \(R\) we can associate to \(M\) a minimal graded free resolution

\[
0 \rightarrow \bigoplus_{i=1}^{\beta_p} R(-d_{p,i}) \rightarrow \bigoplus_{i=1}^{\beta_{p-1}} R(-d_{p-1,i}) \rightarrow \ldots \rightarrow \bigoplus_{i=1}^{\beta_0} R(-d_{0,i}) \rightarrow M \rightarrow 0,
\]

where \(p \leq n\) and \(R(-j)\) is the free \(R\)-module obtained by shifting the degrees of \(R\) by \(j\).

Here the natural number \(\beta_k\) is the \(k\)’th total Betti number of \(M\) and \(p\) is the projective dimension of \(M\).

The module \(M\) has a pure resolution if there are constants \(d_0 < \ldots < d_p\) such that

\[
d_{0,i} = d_0, \ldots, d_{p,i} = d_p
\]

for all \(i\). If in addition

\[
d_i = d_0 + i,
\]

for all \(1 \leq i \leq p\), then we call the minimal free resolution to be \(d_0\)-linear.

In [20] Theorem 2.7 the following bound for the Betti numbers was proved.

**Theorem 2.1** Let \(M\) be an \(R\)-module having a pure resolution of type \((d_0, \ldots, d_p)\) and Betti numbers \(\beta_0, \ldots, \beta_p\), where \(p\) is the projective dimension of \(M\). Then

\[
\beta_i \geq \binom{p}{i}
\]

for each \(0 \leq i \leq p\).
2.2 Hilbert–Serre Theorem

Let $M = \bigoplus_{i \geq 0} M_i$ be a finitely generated nonnegatively graded module over the polynomial ring $R$. We call the formal power series

$$H_M(z) := \sum_{i=0}^{\infty} h_M(i) z^i$$

the Hilbert–series of the module $M$.

The Theorem of Hilbert–Serre states that there exists a (unique) polynomial $P_M(z) \in \mathbb{Q}[z]$, the so-called Hilbert polynomial of $M$, such that $h_M(i) = P_M(i)$ for each $i >> 0$. Moreover, $P_M$ has degree $\dim M - 1$ and $(\dim M - 1)!$ times the leading coefficient of $P_M$ is the Hilbert–Samuel multiplicity of $M$, denoted here by $e(M)$.

Hence there exist integers $m_0, \ldots, m_{d-1}$ such that $h_M(z) = m_0 \cdot \binom{z}{d-1} + m_1 \cdot \binom{z}{d-2} + \ldots + m_{d-1}$, where $\binom{z}{r} = \frac{1}{r!} z(z-1) \ldots (z-r+1)$ and $d := \dim M$. Clearly $m_0 = e(M)$.

We can summarize the Hilbert-Serre theorem as follows:

**Theorem 2.2 (Hilbert–Serre)** Let $M$ be a finitely generated nonnegatively graded $R$–module of dimension $d$, then the following statements hold:

(a) There exists a (unique) polynomial $P(z) \in \mathbb{Z}[z]$ such that the Hilbert–series $H_M(z)$ of $M$ may be written as

$$H_M(z) = \frac{P(z)}{(1 - z)^d}$$

(b) $d$ is the least integer for which $(1 - z)^d H_M(z)$ is a polynomial.

2.3 Simplicial complexes and Stanley–Reisner rings

We say that $\Delta \subseteq 2^{[n]}$ is a simplicial complex on the vertex set $[n] = \{1, 2, \ldots, n\}$, if $\Delta$ is a set of subsets of $[n]$ such that $\Delta$ is a down–set, that is, $G \in \Delta$ and $F \subseteq G$ implies that $F \in \Delta$, and $\{i\} \in \Delta$ for all $i$.

The elements of $\Delta$ are called faces and the dimension of a face is one less than its cardinality. An $r$-face is an abbreviation for an $r$-dimensional face. The dimension of $\Delta$ is the dimension of a maximal face. We use the notation $\dim(\Delta)$ for the dimension of $\Delta$. 
If \( \dim(\Delta) = d - 1 \), then the \((d + 1)\)-tuple \((f_{-1}(\Delta), \ldots, f_{d-1}(\Delta))\) is called the \textit{f-vector} of \(\Delta\), where \(f_i(\Delta)\) denotes the number of \(i\)-dimensional faces of \(\Delta\).

Let \(\Delta\) be an arbitrary simplicial complex on \([n]\). The \textit{Stanley–Reisner ring} \(k[\Delta] := R/I(\Delta)\) of \(\Delta\) is the quotient of the ring \(R\) by the \textit{Stanley–Reisner ideal}

\[
I(\Delta) := \langle x^F : F \notin \Delta \rangle,
\]
generated by the non–faces of \(\Delta\).

The following Theorem was proved in [1] Theorem 5.1.7.

**Theorem 2.3** Let \(\Delta\) be a \((d - 1)\)-dimensional simplicial complex with \(f\)-vector \(f(\Delta) := (f_{-1}, \ldots, f_{d-1})\). Then the Hilbert–series of the Stanley–Reisner ring \(k[\Delta]\) is

\[
H_{k[\Delta]}(z) = \sum_{i=0}^{d-1} \frac{f_i z^i}{(1 - t)^{i+1}}.
\]

**Lemma 2.4** The \(f\)-vector and the \(h\)-vector of a \((d - 1)\)-dimensional simplicial complex \(\Delta\) are related by

\[
\sum_i h_i t^i = \sum_{i=0}^{d} f_{i-1} t^i (1 - t)^{d-i}.
\]

In particular, the \(h\)-vector has length at most \(d\), and

\[
h_j = \sum_{i=0}^{j} (-1)^{j-i} \binom{d-i}{j-i} f_{i-1}
\]
for each \(j = 0,\ldots,d\).
3 Our main result

In the following Theorem we describe the Betti numbers of $k[\Delta]$ in terms of the $h$–vector of $\Delta$.

**Theorem 3.1** Let $\Delta$ be a $(d-1)$–dimensional simplicial complex. Suppose that the Stanley–Reisner ring $k[\Delta]$ has a pure free resolution

$$
\mathcal{F}_\Delta : 0 \rightarrow R(-d_p)^{\beta_p} \rightarrow \ldots \rightarrow
$$

$$
\rightarrow R(-d_1)^{\beta_1} \rightarrow R(-d_0)^{\beta_0} \rightarrow R \rightarrow k[\Delta] \rightarrow 0.
$$

Here $p$ is the projective dimension of the Stanley–Reisner ring $k[\Delta]$.

If $h(\Delta) := (h_0(\Delta), \ldots, h_d(\Delta))$ is the $h$–vector of the complex $\Delta$, then

$$
\beta_i = \sum_{\ell=0}^{d_i} (-1)^{\ell+i+1} \binom{n-d}{\ell} h_{d_i-\ell}
$$

for each $0 \leq i \leq p$.

**Remark.** Clearly $h_i = 0$ for each $i > d$.

**Remark.** J. Herzog and M. Kühl proved similar formulas for the Betti number in [16] Theorem 1. Here we did not assume that the Stanley–Reisner ring $k[\Delta]$ with pure resolution is Cohen–Macaulay.

**Proof.** Let $M := k[\Delta]$ denote the Stanley–Reisner ring of $\Delta$. Then we infer from Theorem 2.3 that

$$
H_M(z) = \frac{\sum_{i=0}^{d} h_i z^i}{(1 - z)^d}.
$$

(5)

Since the Hilbert–series is additive on short exact sequences, and since

$$
H_R(z) = \frac{1}{(1 - z)^n},
$$

we conclude...
and consequently
\[ H_{R(-s)}(z) = \frac{z^s}{(1 - z)^n}, \]
the pure resolution
\[ \mathcal{F}_\Delta : 0 \rightarrow R(-d_p)\beta_p \rightarrow \ldots \rightarrow \]
\[ R(-d_1)\beta_1 \rightarrow R(-d_0)\beta_0 \rightarrow R \rightarrow M \rightarrow 0. \] (6)
\[ \quad \rightarrow R(0) \rightarrow R(0) \rightarrow R(0) \rightarrow R(0) \rightarrow 0. \] (7)
yields to
\[ H_M(z) = \frac{1}{(1 - z)^n} + \sum_{i=0}^{p} (-1)^{i+1} \beta_i \frac{z^{d_i}}{(1 - z)^n}, \] (8)
where \( p = \text{pdim}(M). \)

Write \( d := \text{dim}M, \) and let \( m := \text{codim}(M) = n - d. \) It follows from the Auslander–Buchbaum formula that \( m \leq p. \)

Comparing the two expressions (8) and (5) for \( H_M, \) we find
\[ (1 - z)^m \left( \sum_{i=0}^{d} h_i z^i \right) = \sum_{i=0}^{p} (-1)^{i+1} \beta_i z^{d_i} + 1 \] (9)

Using the binomial Theorem we get that
\[ \left( \sum_{j=0}^{n-d} (-1)^j \binom{n-d}{j} z^j \right) \left( \sum_{i=0}^{d} h_i z^i \right) = \sum_{i=0}^{p} (-1)^{i+1} \beta_i z^{d_i}. \] (10)

Comparing the coefficients on the two sides of (10), we get the result.

Corollary 3.2 Let \( \Delta \) be a \((d-1)\)-dimensional simplicial complex. Then
\[ e(k[\Delta]) = f_{d-1}. \]

Proof. It follows from [1] Proposition 4.1.9 and (2) that
\[ e(k[\Delta]) = \left( \sum_{i=0}^{d} h_i z^i \right) \bigg|_{z=1} = \sum_{i=0}^{d} h_i = f_{d-1}. \]
Corollary 3.3 Let $\Delta$ be a $(d-1)$-dimensional simplicial complex. Suppose that the Stanley–Reisner ring $k[\Delta]$ has an $t$-linear free resolution

$$
\mathcal{F}_{\Delta} : 0 \rightarrow R(-t - p)^{\beta_p} \rightarrow \ldots \rightarrow R(-t - 1)^{\beta_1} \rightarrow R(-t)^{\beta_0} \rightarrow R \rightarrow k[\Delta] \rightarrow 0. \tag{11}
$$

Here $p$ is the projective dimension of the Stanley–Reisner ring $k[\Delta]$.

If $h(\Delta) := (h_0(\Delta), \ldots, h_d(\Delta))$ is the $h$-vector of the complex $\Delta$, then

$$
\beta_i = \sum_{\ell=0}^{t+i} (-1)^{\ell+i+1} h_{t+i-\ell} \binom{n-d}{\ell}
$$

for each $0 \leq i \leq p$.

Corollary 3.4 Let $\Delta$ be a $(d-1)$-dimensional simplicial complex. Suppose that the Stanley–Reisner ring $k[\Delta]$ has an $t$-linear free resolution

$$
\mathcal{F}_{\Delta} : 0 \rightarrow R(-t - p)^{\beta_p} \rightarrow \ldots \rightarrow R(-t - 1)^{\beta_1} \rightarrow R(-t)^{\beta_0} \rightarrow R \rightarrow k[\Delta] \rightarrow 0. \tag{13}
$$

Here $p$ is the projective dimension of the Stanley–Reisner ring $k[\Delta]$.

If $h(\Delta) := (h_0(\Delta), \ldots, h_d(\Delta))$ is the $h$-vector of the complex $\Delta$, then

$$
\sum_{\ell=0}^{j} (-1)^{\ell} h_{j-\ell} \binom{n-d}{\ell} = 0.
$$

for each $j > p+t$.

Proof. Let

$$
P(z) := 1 + \sum_{i=0}^{p} (-1)^{i+1} \beta_i z^{t+i}
$$

Clearly deg$(P) \leq p + t$. Comparing the coefficients of both side of (10), we get the result. \qed
Corollary 3.5 Let \( G \) be an arbitrary chordal graph. Let \( \Delta := \Delta(G) \) be the clique complex of \( G \) and \( d := \dim(\Delta) + 1 \). Let \( h(\Delta) := (h_0(\Delta), \ldots, h_d(\Delta)) \) denote the \( h \)-vector of the complex \( \Delta \). Let \( p \) be the projective dimension of the Stanley–Reisner ring \( k[\Delta] \). Then
\[
\sum_{\ell=0}^{j} (-1)^{\ell} h_{j-\ell} \binom{n-d}{\ell} = 0
\]
for each \( j > p + 2 \).

**Proof.** This follows easily from Theorem 1.1 and Corollary 3.4. \( \square \)

Corollary 3.6 Let \( \Delta \) be a \( (d-1) \)-dimensional simplicial complex. Suppose that the Stanley–Reisner ring \( k[\Delta] \) has a pure free resolution
\[
\mathcal{F}_\Delta : 0 \rightarrow R(-d_p)^{\beta_p} \rightarrow \ldots \rightarrow
\]
\[
\rightarrow R(-d_1)^{\beta_1} \rightarrow R(-d_0)^{\beta_0} \rightarrow R \rightarrow k[\Delta] \rightarrow 0.
\]
(15)

Here \( p \) is the projective dimension of the Stanley–Reisner ring \( k[\Delta] \). Then
\[
\sum_{\ell=0}^{d_i} (-1)^{\ell+i+1} \binom{n-d}{\ell} h_{d_i-\ell} \geq \binom{p}{i}
\]
(17)
for each \( 0 \leq i \leq p \).

**Proof.** This follows easily from Theorem 2.1 and Theorem 3.1. \( \square \)

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