Nonlinear Modulation of Multi-Dimensional Lattice Waves

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The equations governing weakly nonlinear modulations of \(N\)-dimensional lattices are considered using a quasi-discrete multiple-scale approach. It is found that the evolution of a short wave packet for a lattice system with cubic and quartic interatomic potentials is governed by generalized Davey-Stewartson (GDS) equations, which include mean motion induced by the oscillatory wave packet through cubic interatomic interaction. The GDS equations derived here are more general than those known in the theory of water waves because of the anisotropy inherent in lattices. Generalized Kadomtsev-Petviashvili equations describing the evolution of long wavelength acoustic modes in two and three dimensional lattices are also presented. Then the modulational instability of a \(N\)-dimensional Stokes lattice wave is discussed based on the \(N\)-dimensional GDS equations obtained. Finally, the one- and two-soliton solutions of two-dimensional GDS equations are provided by means of Hirota’s bilinear transformation method.

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I. INTRODUCTION

Since the pioneering work of Fermi, Pasta and Ulam\(^1\) on the nonlinear dynamics in lattices, the understanding of the dynamical localization in ordered, spatially extended discrete systems has experienced considerable progress. In particular, one-dimensional(1D) lattice solitons, which are localized nonlinear excitations due to the balance between nonlinearity and dispersion, are shown to exist\(^2\). Similar to the cases in fluid physics and nonlinear optics, most of the analytical approaches on lattice solitons are based on weakly nonlinear theory. The basic idea of the weakly nonlinear theory is that linearized lattice equations are assumed to provide a satisfactory first approximation for those finite-amplitude disturbances which are, in some sense, sufficiently small. Successive approximations may then be developed by an asymptotic expansion in ascending powers of a characteristic wave amplitude. The weakly nonlinear theory has been shown to be very successful in revealing many important physical processes, e. g. resonant wave-wave interactions, modulational instability, the formation of solitons, etc, in a clear-cut way. A very useful method for the asymptotic expansion is the method of multiple-scales, which in the case of lattices reduces the system to a set of partial differential equations for the envelope (or amplitude) while the original system is a set of differential-difference equations, and usually cannot be solved exactly. There are two basic advantages of the multiple scale expansion: (i) it contains a unique explicit small parameter, and hence is controllable, and (ii) it allows obtaining solutions in an explicit form. It is well known that, for a 1D lattice wave with a large spatial extension, the envelope of the lattice wave is governed by the nonlinear Schrödinger (NLS) equation for a short wavelength packet\(^3\) and the Korteweg-de Vries (KdV) equation for a long wavelength acoustic mode\(^4\).

In recent years, much attention has been paid to coherent structures in multi-dimensional lattices (see e.g. Ref.\(^5\)). In particular we mention a generalization of the KdV equation in a 2D lattice with only a cubic interatomic potential, i.e. the Kadomtsev-Petviashvili (KP) equation, derived for a lattice wave traveling in a given direction\(^6\) and coupled 2D NLS equations describing quadratic solitons due to the second harmonic generation in a 2D lattice of the two-component dipoles\(^7\). However, to the best of our knowledge, up to now 2D and 3D generalization of the NLS equation with a mean motion induced by oscillatory wave packets in lattice systems (i.e. due to long wavelength acoustic mode) has not been developed. Meantime such motion introduces dramatical changes in the lattice dynamics.

In the present paper, using a quasi-discrete multiple-scale approach\(^8,9,10\), we derive generalized Davey-Stewartson (GDS) equations in multidimensional lattices with cubic and quartic interatomic potentials. Because of the anisotropy inherent in lattice systems (i.e. without continuous translation and rotation symmetries), in the
the system we consider is a monoatomic scalar lattice with nearest-neighbor interatomic interactions. The equations of motion describing the system are given by

\[ \frac{d^2}{dt^2} u(n) = \sum_{j=1}^{d} J_{2j}[u(n + a_j) + u(n - a_j) - 2u(n)] \]

\[ + \sum_{j=1}^{d} J_{3j}\{[u(n + a_j) - u(n)]^2 - [u(n - a_j) - u(n)]^2\} \]

\[ + \sum_{j=1}^{d} J_{4j}\{[u(n + a_j) - u(n)]^3 + [u(n - a_j) - u(n)]^3\}. \]  

(1)

Here \( u(n) \) is the displacement from its equilibrium position of the particle having the mass \( M \) and located at the site \( n = \sum_{j=1}^{d} n_j a_j \), \( n_j \) being integers, \( a_j \) being the lattice vectors, and \( d \) being the dimension of the lattice, \( J_{\alpha,j} = K_{\alpha,j}/M (\alpha = 2, 3, 4), K_{2,j}, K_{3,j} \) and \( K_{4,j} \) are harmonic, cubic and quartic nearest-neighbor force constants, respectively. Notice that the anisotropy of the lattice is included in the consideration (i.e. in a generic case \( K_{\alpha,j} \neq K_{\alpha,i} \) for \( i \neq j \)). We include the cubic potential here since most of realistic interatomic potentials (such as the potentials of Born-Mayer-Coulomb, Lennard-Jones, Morse, Toda, etc.) display strong cubic nonlinearity (i.e. \( J_{\alpha,3} \neq 0 \)). In the most direct physical applications (namely, to atomic crystals) the dimension \( d \) can be either 2 or 3, although more formal lattices with \( d \) being bigger than 3 are available. In the present section we deal with the last, more general, case.

In order to investigate weakly nonlinear modulation of a lattice wave packet we use the quasi-discrete multiscale method [12, 13] to derive the envelope equations describing the development of the modulation of the packet along the line of Davey and Stewartson for water waves [14]. Namely, we set

\[ u(n) = \sum_{\nu=1}^{\infty} \epsilon^\nu u_\nu(r, \tau; \phi(n, t)) \]  

(2)

with

\[ r = \epsilon(n - vt), \quad \phi(n, t) = q \cdot n - \omega t, \]  

(3)

where \( \epsilon \) is a formal small parameter representing the relative amplitude of the excitation, \( q \) is the wave vector: \( q = \sum_{j=1}^{d} q_j b_j \), \( b_j \) being the vectors of the reciprocal lattice: \( b_j \cdot a_j = \delta_{ji} \) and \( \omega \) is the frequency of the respective harmonic. The constant vector \( v \) as well as the link between \( \omega \) and \( q \), i.e. the dispersion relation, are to be determined by solvability conditions.

There are some comments to be made here. In a generic case the entries of the expansion (3) depend on the whole hierarchy of ”slow” variables, i.e. one should consider the set of variables \( \{r_\nu, t_\nu\} (\nu = 1, 2, \ldots) \) where \( r_\nu = \epsilon^\nu(n - v_\nu t) \) and \( t_\nu = \epsilon^\nu t_\nu \), which are regarded as independent. In the case one is interested in the effect of quadratic and cubic nonlinearity only the scales up to \( r_2 \) and \( t_2 \) turn out to be relevant. In the present paper we restrict our consideration.
to the solutions independent on $r_2$ and this a reason of introducing only the "lowest order" slow variables $r = r_1$ and $	au = r_2$.

Substituting (3) and (3) into Eq. (1) and equating the coefficients of the same powers of $\epsilon$, we obtain the hierarchy of equations as follows

$$Lu_\nu \equiv \omega^2 \frac{\partial^2 u_\nu}{\partial \phi^2} - \sum_j J_{2j} \left( u_\nu^{(j)} + u_\nu^{(-j)} \right) = M_\nu, \quad \nu = 1, 2, \ldots (4)$$

Here $u_\nu^{(\pm j)} = u_\nu (r, \tau; \phi(n, t) \pm q_j) - u_\nu (r, \tau; \phi(n, t))$,

$$M_1 = 0, \quad (5a)$$

$$M_2 = -2 \omega (v \cdot \nabla) \frac{\partial u_1}{\partial \phi} + \sum_j J_{2j} a_j \frac{\partial}{\partial x_j} (u_1^{(j)} - u_1^{(-j)}) + \sum_j J_{3j} \left( (u_1^{(j)})^2 - (u_1^{(-j)})^2 \right), \quad (5b)$$

$$M_3 = -(v \cdot \nabla)^2 u_1 - 2 \omega (v \cdot \nabla) \frac{\partial u_2}{\partial \phi} + 2 \omega \frac{\partial^2 u_1}{\partial \phi \partial \tau} + \sum_j J_{2j} a_j \frac{\partial}{\partial x_j} (u_2^{(j)} - u_2^{(-j)})$$

$$+ \sum_j J_{2j} \left( a_j \frac{\partial}{\partial x_j} (u_1^{(j)} + u_1^{(-j)}) + 2 \sum_j J_{3j} \left( (u_2^{(j)} u_1^{(j)} - u_2^{(-j)} u_1^{(-j)}) \right)$$

$$+ u_1^{(j)} a_j \frac{\partial}{\partial x_j} (u_1^{(j)} + u_1^{(-j)}) - u_1^{(-j)} a_j \frac{\partial}{\partial x_j} (u_1^{(j)} + u_1^{(-j)}) \right) + \sum_j J_{4j} \left( (u_1^{(j)})^3 + (u_1^{(-j)})^3 \right), \quad (5c)$$

$\nabla \equiv \partial / \partial r$, $a_j = |a_j|$, and $x_m$ is the $m$th coordinate of the vector $r$, $r = \sum_m x_m a_m / a_m$.

For further consideration we have to specify the effect we are looking for and this will determine the form of lowest-order $(j = 1)$ solution of Eq. (3). Namely we will be interested in the weakly nonlinear modulation of a lattice wave originated by the interaction between a long wave-length acoustic mode and a high frequency mode. Thus we choose

$$u_1 = A_0 (r, \tau) + \{ A_1 (r, \tau) \exp [i \phi(n, t)] + c.c. \}, \quad (6)$$

where the real function $A_0$ stands for a mean motion induced by the oscillatory wave packet, which has the complex envelope function $A_1$, and c.c. denotes corresponding complex conjugate term. Then

$$u_1^{( \pm j )} = [\exp (\pm iq_j) - 1] A_1 e^{i \phi(n, t)} + c.c.,$$

and in the first order [see Eqs. (3), (5)] we immediately arrive at the dispersion relation of the underline linear lattice

$$\omega^2 \equiv [\omega(q)]^2 = 2 \sum_j J_{2j} (1 - \cos q_j). \quad (7)$$

Next we take into account that

$$v_g \equiv \frac{d\omega}{dq} = \frac{1}{\omega} \sum_j J_{2j} \sin (q_j) a_j / a_j, \quad (8)$$

which is the group velocity of the linear wave. Then, subject to assumption (3) the second order equation of system (3) takes the form

$$Lu_2 = 2i \omega \left\{ (v_g \cdot \nabla) (A_1 e^{i \phi} - \bar{A}_1 e^{-i \phi}) + \chi^{(2)} (A_1^2 e^{2i \phi} - \bar{A}_1^2 e^{-2i \phi}) \right\} \quad (9)$$
where
\[
\chi^{(2)} = \sum_m J_{3m} \frac{\omega}{\omega} (\cos q_m - 1) \sin q_m
\]
(10)
is the effective quadratic nonlinearity.

The solvability condition for the system (9) (in other words the conditions of the absence of secular terms in \(u_2\)) means the orthogonality of the right hand side of Eq. (9) to the kernel of the operator \(L\), i.e. to (6). Hence the r.h.s. of Eq.(6) must do not contain the terms proportional to \(\exp(\pm i\phi)\) and we conclude that \(v = v_g\), i.e. \(v\) introduced in (3) is nothing but the group velocity of the high frequency carrier wave. Next we can look for the solution \(u_2\) (it must be orthogonal to the first order approximation, i.e. to the kernel of the operator \(L\)) in a form of the expansion over the eigenfunctions of the operator \(L\). Having done this one ensures that the only nonzero term of such an expansion is given by
\[
u_2 = i\alpha A_1^2 \exp(2i\phi) + c.c., \quad \alpha = -\frac{2\omega\chi^{(2)}}{4[\omega(q)]^2 - [\omega(2q)]^2}.
\]
(11)

Formula (11) is valid unless the condition \(\omega(2q) = 2\omega(q)\) is satisfied. As it is evident this is the condition of the resonant second harmonic generation [10,14]. It can be satisfied in a lattice with a complex cell, but it is not difficult to ensure that \(\omega(2q) \neq 2\omega(q)\) for all \(q\) in a monoatomic lattice with the nearest neighbor interactions.

Passing to the third order of the multiple scale expansion we introduce the (symmetric) group velocity dispersion tensor (GVDT) by the formula \((v_j = \partial \omega/\partial q_j)\)
\[
\omega_{ij} = \frac{1}{\omega} [J_{2j} \cos(q_j) a_i a_j \delta_{ij} - v_i v_j],
\]
(12)
and the (symmetric) effective GVDT \(\Omega_{ij}\)
\[
\Omega_{ij} = \frac{1}{\omega} [J_{2j} a_i a_j \delta_{ij} - v_i v_j].
\]
(13)

Then the solvability condition for the third order terms gives rise to the closed system of equations for \(A_0\) and \(A_1\):
\[
\sum_{l,m} \Omega_{im} \frac{\partial^2}{\partial x_l \partial x_m} A_0 = -2 \sum_m \delta_m \frac{\partial}{\partial x_m} |A_1|^2,
\]
(14)
\[
i \frac{\partial A_1}{\partial \tau} + \frac{1}{2} \sum_{l,m} \omega_{lm} \frac{\partial^2}{\partial x_l \partial x_m} A_1 = \chi |A_1|^2 A_1 + A_1 \sum_m \delta_m \frac{\partial}{\partial x_m} A_0,
\]
(15)
where
\[
\delta_m = \frac{2\alpha_m}{\omega} J_{3m} (1 - \cos q_m),
\]
(16)
\[
\chi = \frac{2}{\omega} \sum_m [2\alpha J_{3m} (1 - \cos q_m) \sin q_m + 3J_{4m} (1 - \cos q_m)^2].
\]
(17)

We call Eqs.(14) and (15) the ND GDS equations.
III. GENERALIZED DAVEY-STEWARTSONS

Let us now focus our attention on a special case of a 2D lattice (i.e., $\mathbf{r} = (x_1, x_2)$). For the sake of simplicity, the lattice will be considered symmetric, $J_{a,j} = J_a$ ($a = 2, 3, 4$ and $j = 1, 2$) and orthogonal: $\mathbf{a}_1 \cdot \mathbf{a}_2 = 0$, with the lattice constant equal to unity, $|a_j| = 1$. In order to diagonalize the effective GVDT $\Omega_{lm}$ in a general case we rotate the original Cartesian system (with the coordinate basis $(1, 0)$ and $(0, 1)$) to a new one with the coordinate basis $\mathbf{e}_1 = (\lambda_1, \lambda_2)$ and $\mathbf{e}_2 = (\lambda_2, -\lambda_1)$, where

$$\lambda_j = \frac{v_j}{v_g} = \frac{\sin q_j}{\sqrt{\sin^2 q_1 + \sin^2 q_2}},$$

and $v_j$ is the $j$-th component of the group velocity defined in (2) (as it is evident $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$). In this way one of the directions of the new basis, namely $\mathbf{e}_1$ coincides with the direction of the group velocity of the carrier wave, i.e., $v_g = v_2 e_1$ the other direction is orthogonal to it. As a result, $x_1$ and $x_2$ in Eqs.(14) and (15) take the form $x_1 = \epsilon(n_1 - \lambda_1 v_g t)$ and $x_2 = \epsilon(n_2 - \lambda_2 v_g t)$, and envelope equations (14) and (15) are reduced to

$$\alpha_{11}\frac{\partial^2 A_0}{\partial \xi^2} + \alpha_{22}\frac{\partial^2 A_0}{\partial \eta^2} = -2 \left( \beta_1 \frac{\partial}{\partial \xi} + \beta_2 \frac{\partial}{\partial \eta} \right) |A_1|^2,$$

$$\frac{\partial A_1}{\partial \tau} + \mathcal{L} A_1 = A_1 \left( \beta_1 \frac{\partial}{\partial \xi} + \beta_2 \frac{\partial}{\partial \eta} \right) A_0 + \chi |A_1|^2 A_1,$$

where

$$\mathcal{L} = \gamma_{11} \frac{\partial^2}{\partial \xi^2} + \gamma_{22} \frac{\partial^2}{\partial \eta^2} + \gamma_{12} \frac{\partial^2}{\partial \xi \partial \eta},$$

$$\xi = \mathbf{r} \cdot \mathbf{e}_1 = \lambda_1 x_1 + \lambda_2 x_2 = \epsilon(\lambda_1 n_1 + \lambda_2 n_2 - v_g t),$$

$$\eta = \mathbf{r} \cdot \mathbf{e}_2 = -\lambda_2 x_1 + \lambda_1 x_2 = \epsilon(-\lambda_2 n_1 + \lambda_1 n_2),$$

$$\alpha_{11} = \frac{1}{\omega}(J_2 - v_g^2), \quad \alpha_{22} = \frac{J_2}{\omega},$$

$$\beta_1 = \frac{2J_3}{\omega}[\lambda_1(1 - \cos q_1) + \lambda_2(1 - \cos q_2)], \quad \beta_2 = \frac{2J_3}{\omega}[\lambda_1(1 - \cos q_2) - \lambda_2(1 - \cos q_1)],$$

$$\gamma_{11} = \frac{1}{\omega}(-v_g^2 + J_2(\lambda_1^2 \cos q_1 + \lambda_2^2 \cos q_2)), \quad \gamma_{22} = \frac{J_2}{2\omega}(\lambda_1^2 \cos q_1 + \lambda_2^2 \cos q_2),$$

$$\gamma_{12} = \frac{J_2}{\omega}\lambda_1 \lambda_2 (\cos q_2 - \cos q_1),$$

$$\chi = \frac{2}{\omega} \left\{ 2J_3 \alpha [\sin q_1(1 - \cos q_1) + \sin q_2(1 - \cos q_2)] + 3J_4 [(1 - \cos q_1)^2 + (1 - \cos q_2)^2] \right\},$$

$$\alpha = \frac{4J_3 [\sin q_1(1 - \cos q_1) + \sin q_2(1 - \cos q_2)]}{4[\omega(q)]^2 - [\omega(2q)]^2}.$$ 

Equations (19) and (20) represent a generalized form of the conventional DS equations. They include the dispersion, diffraction and nonlinearity of the system. One of their important features is that there exists a coupling between the mean field (denoted by $A_0$) and the envelope of the carrier wave (denoted by $A_1$). The mean field $A_0$ generates a strain field in the system. If $J_3 = 0$, a case for a symmetric interatomic potential, we have $A_0 = 0$ thus the mean motion and hence the strain field vanishes. Another important feature for Eqs. (19) and (20) is their property of anisotropy. For different wave vector $\mathbf{q} = (q_1, q_2)$, the coefficients of the equations take different values and some of these coefficients may become vanishing for some particular directions of $\mathbf{q}$.

The conventional DS equations were derived firstly in surface water waves [1] and now are a well-known 2D soliton model in the soliton theory [2]. Note that for water waves, the system is isotropic (i.e. it possesses a continuous rotation symmetry). The envelope equations are the same for all propagating directions of the waves and hence the coefficients appearing in the equations are independent on $q_1$ and $q_2$ and correspondingly $\beta_2$, and $\gamma_{12}$ vanish...
(see also Ref. [13]). However, for the lattice system the modulating equations take a more general form because the lattice is anisotropic (without the continuous rotation symmetry). We mention that although the coefficients \( \alpha_{ij} \) are both positive, signs of the coefficients \( \gamma_{ij} \) may change depending on the choice of the wave-vector in the first Brillouin zone.

We now discuss several particular cases for the 2D GDS equations derived above. In the following circumstances (i.e. in some special points and lines of the Brillouin zone, see Fig. 1) the 2D GDS equations reduce to the conventional DS equations:

1. \( q_1 q_2 = 0 \) (then \( \lambda_1 \lambda_2 = 0 \), and \( \beta_2 = \gamma_{12} = 0 \))
2. \( q_1 = q_2 = q \) (then \( \lambda_1 = \lambda_2 = 2^{-1/2} \) and \( \beta_2 = \gamma_{12} = 0 \))
3. \( q_1 = -q_2 = q \) (then \( \lambda_1 = -\lambda_2 = 2^{-1/2} \) and \( \beta_1 = \gamma_{12} = 0 \))

More precisely, since \( \alpha_{11} > 0 \) and \( \alpha_{22} > 0 \) for any \( q \), at \( \gamma_{11}\gamma_{22} < 0 \) Eqs. (19) and (20) can be classified as DSII equations, while for \( \gamma_{11}\gamma_{22} > 0 \) they form a dynamical system that can be identified neither with DS I nor DSII equations appearing in the theory of water waves (see e.g. [12]).

In the case of pure quadratic potential, \( J_3 = 0 \), we have that the evolution equations for \( A_1 \) and \( A_0 \) are decoupled. Then the GDS equations reduce to a generalized 2D NLS equation (i.e. the NLS equation plus a cross-derivative term \( \partial^2 A_1 / (\partial \xi \partial \eta) \)). Finally, if \( q_2 = 0 \) and \( \partial / \partial \eta = 0 \), the 2D GDS equations (19) and (20) recover the envelope equations derived in Ref. [8].

In the 3D case, Eqs. (19) and (20) are replaced by the 3D GDS equations:

\[
\begin{align*}
\frac{\alpha'_{11}}{2} \frac{\partial^2 A_0}{\partial \xi^2} + \frac{\alpha'_{22}}{2} \frac{\partial^2 A_0}{\partial \eta^2} + \frac{\alpha'_{33}}{2} \frac{\partial^2 A_0}{\partial \zeta^2} &= -2 \left( \beta'_1 \frac{\partial}{\partial \xi} + \beta'_2 \frac{\partial}{\partial \eta} + \beta'_3 \frac{\partial}{\partial \zeta} \right) |A_1|^2, \\
\frac{\partial A_1}{\partial \tau} + \frac{\gamma'_{11}}{2} \frac{\partial^2 A_1}{\partial \xi^2} + \frac{\gamma'_{22}}{2} \frac{\partial^2 A_1}{\partial \eta^2} + \frac{\gamma'_{33}}{2} \frac{\partial^2 A_1}{\partial \zeta^2} &= \left( \gamma'_{12} \frac{\partial^2}{\partial \xi \partial \eta} + \gamma'_{23} \frac{\partial^2}{\partial \eta \partial \zeta} + \gamma'_{31} \frac{\partial^2}{\partial \zeta \partial \xi} \right) A_1 \\
&= A_1 \left( \beta'_1 \frac{\partial}{\partial \xi} + \beta'_2 \frac{\partial}{\partial \eta} + \beta'_3 \frac{\partial}{\partial \zeta} \right) A_0 + \chi |A_1|^2 A_1,
\end{align*}
\]

where \( \alpha'_{jj}, \beta'_j, \gamma'_{ij} \) \( (j = 1, 2, 3) \) and \( \chi' \) are constants dependent on \( q = (q_1, q_2, q_3) \) and the parameters of the system, which are not needed here and not written down explicitly. The definitions of \( \xi, \eta \) and \( \zeta \) are given by

\[
\begin{align*}
\xi &= \epsilon (\lambda_1 n_1 + \lambda_2 n_2 + \lambda_3 n_3 - v_s t), \\
\eta &= \epsilon (-\lambda_2 n_1 + \lambda_1 n_2), \\
\zeta &= \epsilon (-\lambda_3 n_1 + \lambda_1 n_3),
\end{align*}
\]

where \( \omega \) and \( \lambda_j \) are defined by (7) and [13].

### IV. LONG-WAVELENGTH LIMIT

Note that the envelope equations (14) and (15) are invalid for \( q = 0 \) since in this case there is a divergence in their coefficients. From the physical point of view this happens because vanishing \( q \) corresponds to a long wavelength acoustic mode in the lattice. In this case a different asymptotic expansion must be used to obtain divergence-free envelope equations. For simplicity we consider the case of a symmetric 2D square lattice. In this situation the asymptotic expansion (2) must be replaced by

\[
u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots,
\]

with
\[ u_\nu = u_\nu(\xi, \eta, \tau) \quad \nu = 0, 1, 2, \ldots, \] (36)
\[ \xi = \epsilon(\lambda_1 n_1 + \lambda_2 n_2 - c t), \] (37)
\[ \eta = \epsilon^2(-\lambda_2 n_1 + \lambda_1 n_2), \] (38)
\[ \tau = \epsilon^3 t, \] (39)

where \( c = \sqrt{J_2} \) is the speed of sound, \( \lambda_l (l = 1, 2) \) are determined by the solvability conditions required at \( O(\epsilon^2) \)-order. A solvability condition in the fourth-order of the expansion yields the generalized KP equation:

\[
\frac{\partial}{\partial \xi} \left[ \frac{\partial v}{\partial \tau} + \frac{c}{24}(\lambda_1^3 + \lambda_2^3) \frac{\partial^3 v}{\partial \xi^3} + \frac{J_1}{c}(\lambda_1^3 + \lambda_2^3) \frac{\partial v}{\partial \xi} \right] + \frac{3}{2c}(\lambda_1^4 + \lambda_2^4) v^2 \frac{\partial v}{\partial \xi} + c \frac{\partial^2 v}{\partial \eta^2} = 0, \] (40)

where \( v = \partial u_0 / \partial \xi \). In deriving Eq. (40) we have assumed that \( J_3 = \epsilon \tilde{J}_3 \), with \( \tilde{J}_3 \) of order unity. The parameters \( \lambda_l (l = 1, 2) \) are direction-dependent and we find that their values can be obtained by using (38) but taking the limit \( q \to 0 \). Thus the values of the coefficients in Eq. (40) are dependent on the ways of \( q \) approaching zero. For instance

1. \( \lambda_1 = 1, \lambda_2 = 0 \), if \( q_2 = 0, q_1 \to 0 \);
2. \( \lambda_1 = \lambda_2 = 1/\sqrt{2} \), if \( q_1 = q_2 = q \to 0 \).

The reason for appearing different values of the coefficients corresponding to different directions is also due to the anisotropy of the system. It is easy to see that the KP equation obtained in Ref. [1] is our particular case with quartic nonlinearity being absent (i.e. \( J_4 = 0 \)). Eq. (40) admits solitary wave solutions [2].

It is relevant to mention here that the coefficient of the term \( \partial^2 v / \partial \eta^2 \) is positive which means that the line (i.e. \( \eta \)-independent) solitons of Eq. (40) is stable while this equation does not admit any kind of lump (i.e. decaying when \( \xi^2 + \eta^2 \to 0 \)) solution.

In the same way, in 3D case Eq. (40) is generalized to

\[
\frac{\partial}{\partial \xi} \left[ \frac{\partial v}{\partial \tau} + a_1 \frac{\partial^3 v}{\partial \xi^3} + a_2 v \frac{\partial v}{\partial \xi} + a_3 v^2 \frac{\partial v}{\partial \xi} \right] + a_4 \frac{\partial^2 v}{\partial \eta^2} + a_5 \frac{\partial^2 v}{\partial \xi^2} = 0, \] (41)

where \( \xi, \eta \), and \( \zeta \) are the same as (32) - (34). \( a_l (l = 1, 2, 3, 4, 5) \) are real constants dependent on \( \lambda_j (j = 1, 2, 3) \) (given by 18) with \( q \to 0 \).

V. MODULATIONAL INSTABILITY OF A PLANE LATTICE WAVE WITH A MEAN MOTION

In recent years, the use of nonlinear envelope (or amplitude) equations for studying the stability of patterns and waves in systems in and outside of equilibrium is widely employed [12, 13, 14]. The modulational stability of a plane water wave (e.g. a uniform Stokes wave) was analyzed by Davey and Stewartson based on the DS equations they derived [14]. In the same way the ND GDS equations (14) and (15) obtained here can be used to study the modulational stability of a uniform Stokes lattice wave in \( N \) dimensions. A Stokes lattice wave here means a linear plane lattice wave with the wave vector \( q \).

Note that the uniform vibrating solution of Eqs. (14) and (15) reads

\[
A_0 = 0, \quad A_1 = U_0 \exp(-i \Omega \tau), \] (42)

which, when incorporating the carrier wave (see (13)), corresponds a plane lattice wave with the wave vector \( q \) and the frequency \( \omega(q) + \Omega \) excited in the system, where \( U_0 \) is a constant and \( \Omega = \chi U_0^2 \). Assume that a perturbation is added into the uniform vibrating solution (12), i.e.

\[
A_0(x_1, x_2, ..., \tau) = \hat{k}_+ \exp(i \sum_m Q_m x_m) + \hat{k}_- \exp(-i \sum_m Q_m x_m), \] (43)
\[
A_1(x_1, x_2, ..., \tau) = U_0 \exp(-i \Omega \tau) \left[ 1 + \hat{\varepsilon}_+ \exp(i \sum_m Q_m x_m) + \hat{\varepsilon}_- \exp(-i \sum_m Q_m x_m) \right], \] (44)

7
with \( \hat{\kappa}_\pm = \kappa_\pm(0) \exp(\sigma_R \pm i \sigma_I) \) and \( \hat{\varepsilon}_\pm = \varepsilon_\pm(0) \exp(\sigma_R \pm i \sigma_I) \), where \( \mathbf{Q} = (Q_1, Q_2, \ldots, Q_N) \) and \( \sigma_I = \sigma_I(\mathbf{Q}) \) are respectively the wave vector and frequency of the perturbation, \( \sigma_R = \sigma_R(\mathbf{Q}) \) denotes the growth rate of the perturbation, \( \kappa_\pm(0) \) and \( \varepsilon_\pm(0) \) are small constants with the condition \( \kappa_+ = \kappa_0(0) \) because of the reality of \( A_0 \). Substituting (43) and (44) into the Eqs. (14) and (15) we obtain a set of linear equations on \( \kappa_\pm(0) \) and \( \varepsilon_\pm(0) \):

\[
- (\alpha_{11} Q_1^2 + \alpha_{22} Q_2^2) \kappa_+(0) + 2 i U_0^2 (\beta_1 Q_1 + \beta_2 Q_2) \varepsilon_+(0) = 0,
\]

(45)

\[
(\Omega + i \sigma - \gamma_{11} Q_1^2 - \gamma_{22} Q_2^2 - \gamma_{12} Q_1 Q_2 - 2 \chi U_0^2 \varepsilon_+(0) - \chi U_0^2 \varepsilon_+(0) - i (\beta_1 Q_1 + \beta_2 Q_2) \kappa_+(0) = 0,
\]

(46)

\[
(\Omega + i \sigma^* - \gamma_{11} Q_1^2 - \gamma_{22} Q_2^2 - \gamma_{12} Q_1 Q_2 - 2 \chi U_0^2 \varepsilon_+(0) - \chi U_0^2 \varepsilon_+(0) + i (\beta_1 Q_1 + \beta_2 Q_2) \kappa_+(0) = 0,
\]

(47)

\[
(\Omega + i \sigma^* - \gamma_{11} Q_1^2 - \gamma_{22} Q_2^2 - \gamma_{12} Q_1 Q_2 - 2 \chi U_0^2 \varepsilon_+(0) - \chi U_0^2 \varepsilon_+(0) + i (\beta_1 Q_1 + \beta_2 Q_2) \kappa_+(0) = 0,
\]

(48)

where \( \sigma = \sigma_R + i \sigma_I \). A solvability condition of Eqs. (45-48) results in

\[
(\sigma_R + i \sigma_I)^2 = \left( \sum_{l,m} \omega_{lm} Q_l Q_m \right) U_0^2 \left[ -\chi + 2 \left( \sum_{m} \delta_m Q_m \right)^2 \right] - \frac{1}{4} \sum_{l,m} \omega_{lm} Q_l Q_m \right] \cdot
\]

(49)

Note that the right side of Eq. (49) is real. Thus when

\[
\left( \sum_{l,m} \omega_{lm} Q_l Q_m \right) \left[ U_0^2 \left[ -\chi + 2 \left( \sum_{m} \delta_m Q_m \right)^2 \right] - \frac{1}{4} \sum_{l,m} \omega_{lm} Q_l Q_m \right] > 0,
\]

(50)

one has \( \sigma_I = 0 \). As a result if the condition (50) is satisfied we have the growth rate

\[
\sigma_R = \pm \left\{ \sum_{l,m} \omega_{lm} Q_l Q_m \right\} \left[ U_0^2 \left[ -\chi + 2 \left( \sum_{m} \delta_m Q_m \right)^2 \right] - \frac{1}{4} \sum_{l,m} \omega_{lm} Q_l Q_m \right] \right\}^{1/2}
\]

(51)

Thus one always has a positive \( \sigma_R \) branch if the condition (50) is satisfied. In this case the perturbation grows exponentially and hence the uniform vibrating solution (12) is modulational unstable.

For the 2D GDS equations (19) and (20), the condition of the modulational instability (50) reads

\[
\left\{ U_0^2 \left[ -\chi + \frac{2 (\beta_1 Q_1 + \beta_2 Q_2)^2}{\alpha_{11} Q_1^2 + \alpha_{22} Q_2^2} \right] - \frac{1}{2} (\gamma_{11} Q_1^2 + \gamma_{22} Q_2^2 + \gamma_{12} Q_1 Q_2) \right\} > 0.
\]

(52)

Thus due to the anisotropy of the lattice (i.e. \( \beta_2 \gamma_{12} \neq 0 \)) the criterion (52) gives much richer behavior for the stability of the Stokes wave than that in isotropic systems (e.g. water waves). In particular, for a given Stokes lattice wave there exist two (or may be four depending on the Stokes lattice wave) wave vectors \( \mathbf{Q} \) for which the instability evolves with the biggest increment. This phenomenon reminds the so-called strengthening of inhomogeneities, known in the theory of beam propagation in Kerr medium (13). There is however an essential difference originated by the anisotropy: the biggest exponent is characterized by the amplitude of the value of the wave vector and also by the lattice direction. The position of the points providing the largest increment depends on the choice of the wave vector of the Stokes lattice wave.

The outcome of this type of instability may result in the formation of solitons (2) or the appearence of homoclinic structures (see Sec. 3.3 of Ref. (12)).

VI. SOLITON SOLUTIONS

We now consider the soliton solutions of the nonlinear evolution equations derived above. Taking 2D GDS equations (19) and (20) as an example, to obtain the soliton solutions we employ Hirota’s bilinear transformation method,
an ingenious technique of finding exact multi-soliton solitons for nonlinear evolution equations \[17,18\]. Introducing the dependent variable transformation

\[ A_0 = -4 \left( \beta_1 \frac{\partial}{\partial \xi} + \beta_2 \frac{\partial}{\partial \eta} \right) \log F, \quad A_1 = G/F \] (53)

with \( F \) (real) and \( G \) (complex) being the functions of \( \tau, \xi \) and \( \eta \), Eqs.(61) and (62) give rise to the “dispersion relations”

\[ (\alpha_{11} D_{11}^2 + \alpha_{22} D_{12}^2) F \cdot F = |G|^2, \] (54)

\[ (i D_\tau + \gamma_{11} D_\xi + \gamma_{22} D_\eta) G \cdot F = 0, \] (55)

\[ [(\gamma_{11} - 2\beta_1^2) D_\xi^2 + (\gamma_{22} - 2\beta_2^2) D_\eta^2 + + (\gamma_{12} - 4\beta_1 \beta_2) D_\xi D_\eta] F \cdot F + \chi |G|^2 = 0, \] (56)

where \( D_\tau, D_\xi \) and \( D_\eta \) are Hirota’s bilinear operators defined by \[17,18\]

\[ D_\xi^n D_\eta^m D_\tau^p G \cdot F = \left( \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \xi'} \right)^m \left( \frac{\partial}{\partial \eta} - \frac{\partial}{\partial \eta'} \right)^n \left( \frac{\partial}{\partial \tau} - \frac{\partial}{\partial \tau'} \right)^p G(\xi, \eta, \tau) F(\xi', \eta', \tau') \big|_{\xi' = \xi, \eta' = \eta, \tau' = \tau}. \] (57)

In order to get a one-soliton solution we assume

\[ F = 1 + L \exp(\Phi + \Phi^*) , \quad G = \exp(\Phi) \] (58)

with

\[ \Phi = (p_R + i p_I) \xi + (q_R + i q_I) \eta + (s_R + i s_I) \tau + \Phi_{0R} + i \Phi_{0I}, \] (59)

where \( L, p_R, p_I, q_R, q_I, s_R, s_I, \Phi_{0R} \) and \( \Phi_{0I} \) are real, yet to be determined constants. Substituting (58) into (54)-(56) we obtain the set of algebraic equations

\[ 8 L (\alpha_{11} p_R^2 + \alpha_{22} q_R^2) - 1 = 0, \] (60)

\[ \gamma_{11} (p_{11}^2 - p_{12}^2) + \gamma_{12} (p_{R1} q_R - p_{12} q_R) + \gamma_{22} (q_{12}^2 - q_R^2) - s_I = 0, \] (61)

\[ 2 (\gamma_{11} p_{R1} p_{12} + \gamma_{22} q_{R1} q_R + \gamma_{12} p_{R1} q_{R1} + p_{12} q_R) + s_R = 0, \] (62)

\[ \chi + 8 L [ (\gamma_{11} - 2\beta_1^2) p_{12}^2 + (\gamma_{22} - 2\beta_2^2) q_R^2 + (\gamma_{12} - 4\beta_1 \beta_2) p_{R1} q_R ] = 0. \] (63)

From Eq.(60) we get

\[ L = \frac{1}{8 (\alpha_{11} p_{11}^2 + \alpha_{22} q_R^2)}. \] (64)

Eqs.(61) and (62) give rise to the “dispersion relations”

\[ s_I = \gamma_{11} (p_{11}^2 - p_{12}^2) + \gamma_{12} (p_{R1} q_R - p_{12} q_R) + \gamma_{22} (q_{12}^2 - q_R^2), \] (65)

\[ s_R = -2 (\gamma_{11} p_{R1} p_{12} + \gamma_{22} q_{R1} q_R - \gamma_{12} p_{R1} q_R + p_{12} q_R), \] (66)

with \( p_R, p_I, q_R \) and \( q_I \) being arbitrary constants. Eqs.(63) gives a condition for the one-soliton solution. From (53) and the results given above we have

\[ A_0 = -4 (\beta_1 p_R + \beta_2 q_R) [1 + \tanh(\theta - \delta_0)], \] (67)

\[ A_1 = [2 (\alpha_{11} p_{11}^2 + \alpha_{22} q_R^2)]^{1/2} \text{sech}(\theta - \delta_0) \exp(i \varphi), \] (68)

with \( \theta = p_R \xi + q_R \eta + s_R \tau + \Phi_{0R}, \) \( \varphi = p_I \xi + q_I \eta + s_I \tau + \Phi_{0I} \) and \( \delta_0 = (1/2) \log[8 (\alpha_{11} p_{11}^2 + \alpha_{22} q_R^2)] (\Phi_{0R} \) and \( \Phi_{0I} \) are arbitrary constants). Thus the single-soliton solution obtained is a line soliton, which consists of two parts, a vibrating wave packet \( (A_1, \) an envelope soliton) and a mean displacement field \( (A_0, \) a kink).
The two-soliton solutions of the Eqs. (19) and (20) can be obtained by choosing
\[ F = 1 + L_1 \exp(\Phi_1 + \Phi_2^s) + L_2 \exp(\Phi_2 + \Phi_2^s) \]
\[ + (L_3 + i L_4) \exp(\Phi_1 + \Phi_2^s) + (L_3 - i L_4) \exp(\Phi_1^s + \Phi_2) + L_5 \exp(\Phi_1 + \Phi_2 + \Phi_2^s), \]
\[ G = \exp(\Phi_1) + \exp(\Phi_2) + (M_1 + i M_2) \exp(\Phi_1 + \Phi_2 + \Phi_2^s) + (M_3 + i M_4) \exp(\Phi_1 + \Phi_2 + \Phi_2^s), \]
with \( \Phi_j = (p_{jR} + ip_{jI})\xi + (q_{jR} + iq_{jI})\eta + (s_{jR} + is_{jI})\tau + \Phi_{jL}^R + i\Phi_{jL}^\pm (j = 1, 2) \), where \( p_{jR}, p_{jI}, q_{jR}, q_{jI}, s_{jR}, s_{jI}, \Phi_{jL}^R \) and \( \Phi_{jL}^\pm \) are real constants. When (68) and (71) are substituted into the bilinear equations (54)-(56) we obtain a set of nonlinear algebraic equations for the real coefficients \( L_j (j = 1, 2, ..., 5) \) and \( M_j (j = 1, 2, 3, 4) \) appearing in (68) and (71). Solving these equations one can get the expressions of \( L_j \) and \( M_j \) as well as the "dispersion relations", \( s_{jL,R} = s_{jL,R} (p_{jL,R}, q_{jL,R}, q_{jL}, q_{jR}) (j = 1, 2) \), which have been given in the Appendix A. To guarantee (68) and (71) are two-soliton solution, the following conditions must be imposed
\[ \gamma_{12} = 4\beta_1 \beta_2, \]
\[ \frac{\alpha_{11}}{\gamma_{11} - 2\beta_1^2} = \frac{\alpha_{22}}{\gamma_{22} - 2\beta_2^2} = \frac{1}{\chi} \]
\[ (71) \]
\[ (72) \]
In addition, for \( p_{2R} \) and \( q_{2R} \), there is a constraint
\[ \alpha_{22} (\alpha_{11} p_{2R}^2 + \alpha_{22} q_{2R}^2) \chi = (\alpha_{11} \beta_1^2 + \alpha_{22} \beta_2^2) p_{2R}^2 + 2 \alpha_{22} \beta_2^2 q_{2R}^2. \]
\[ (73) \]
It is easy to show that the integrable conditions of the standard DS equations (i.e. the ones amenable to be solved by the inverse scattering transform) derived in water wave problem are the particular case of the conditions (71) and (72) (see Appendix B). This fact implies that the GDS equations (19) and (20) may be integrable under the conditions (71) and (72).

We note that different equalities in these conditions, however, reflect different physical properties. In particular, (71) and the first equality in (72) result in an equation for the wavevector only (i.e. having the form \( f(q_1, q_2) = 0 \) where \( f(q_1, q_2) \) does not depend on the lattice parameters, i.e. on \( J_2 \) for which the existence of solitons is possible. Then first two equalities in (71) and (72) allows one to find the particular values of the nonlinear coefficients. In other words, the above conditions specify the set of points in the first Brillouin zone, and necessary values of the nonlinear forces. What is important for the next consideration, that such points in the Brillouin zone do exist. Indeed, as an example we mention that the above conditions are satisfied for all points \( q = (q_1, 0) \) and \( q = (0, q_2) \).

Eqs. (68) and (71) describe two obliquely interacting solitons in the \((\xi, \eta)\) space. The interaction results in a phase shift (i.e. position shift) for each soliton.

It is possible to get \( N \)-soliton solutions of the 2D GDS equations (19) and (20) using their bilinear representation, Eqs. (54)-(56) under the integrable conditions (71) and (72). We note that due to the anisotropy inherent in the lattice system (i.e. \( \beta_2 \gamma_{12} \neq 0 \) ), the existence of the two-soliton solution needs the condition \( \gamma_{12} = 4 \beta_1 \beta_2 \) (Eq. 71), which is absent for isotropic systems (e.g. water waves).

VII. CONCLUSION

Using a quasi-discrete multiple-scale method we have derived the envelope equations of weakly nonlinear modulations of \( N \)-dimensional lattice waves. The equations are obtained for the case of interaction of a highly frequency modes with a long-wave-length acoustic one (also called mean field) and can be classified as generalized Davey-Stewartson equations. In the case at hand, due to the anisotropy of the lattice system, the GDS equations in two dimensions are reduced either to the DS equations or to a form which does not appears in the theory of water waves [11]. The mean field coupled to the oscillatory short wave packet results from the cubic interatomic potential in the lattice. Additionally, generalized Kadomtsev-Petviashvili equations describing the evolution of a long wavelength acoustic mode in the lattice are also presented. We have also studied the modulation instability of Stokes waves and provided some exact soliton solutions for the two-dimensional GDS equations based on Hirota’s bilinear transformation method.

The results reported here recover the known ones in one-dimensional systems, which give rise to standard lattice solitons. On the other hand the method can also be used to study the weakly nonlinear modulations of the wave packets in vector lattices or in lattices with a complex cell. The derivation procedure involves more cumbersome calculation but the envelope equations obtained take still a form similar to (14) and (15) for high frequency wave packets and (10) and (11) for long wave acoustic modes.
VIII. ACKNOWLEDGEMENTS

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Appendix A

The expressions of $L_j$ and $M_j$ for two-soliton solutions appearing in (69) and (70) are given by

\[
L_1 = \frac{1}{8(\alpha_{11}p_{1R}^2 + \alpha_{22}q_{1R}^2)},
\]

\[
L_2 = \frac{1}{8(\alpha_{11}p_{2R}^2 + \alpha_{22}q_{2R}^2)},
\]

\[
L_3 = -\frac{1}{2} \left( \alpha_{11} \Gamma_{-}^{\perp} + \alpha_{22} \Sigma_{-}^{\perp} \right) + \frac{4}{3} \left( \alpha_{11} \Delta_{-}^{\perp} + \alpha_{22} \Delta_{-}^{\perp} \right)^2,
\]

\[
L_4 = -\frac{1}{2} \left( \alpha_{11} \Gamma_{-}^{\perp} + \alpha_{22} \Sigma_{-}^{\perp} \right) + \frac{4}{3} \left( \alpha_{11} \Delta_{-}^{\perp} + \alpha_{22} \Delta_{-}^{\perp} \right)^2,
\]

\[
L_5 = \frac{1}{64} A_5n,
\]

\[
M_1 = \frac{1}{8} M_{1n},
\]

\[
M_2 = -\frac{1}{2} M_{2n},
\]

\[
M_3 = \frac{1}{8} M_{3n},
\]

\[
M_4 = -\frac{1}{2} M_{4n},
\]

\[
L_{5n} = \alpha_{11}^2 (\Gamma_{+}^{\perp})^2 + \alpha_{22} (\Sigma_{+}^{\perp})^2 + 2\alpha_{11}\alpha_{22} (\Gamma_{-}^{\perp} \Sigma_{-}^{\perp} + 4\Delta_{-}^{\perp} \Delta_{-}^{\perp}),
\]

\[
L_{5d} = \frac{1}{64} A_5d,
\]

\[
M_{1d} = M_{2d} = (\alpha_{11} p_{1R}^2 + \alpha_{22} q_{1R}^2) [\alpha_{11}^2 (\Gamma_{+}^{\perp})^2 + \alpha_{22} (\Sigma_{+}^{\perp})^2 + 2\alpha_{11}\alpha_{22} (\Gamma_{-}^{\perp} \Sigma_{-}^{\perp} + 4\Delta_{-}^{\perp} \Delta_{-}^{\perp})],
\]

\[
M_{2n} = \alpha_{11} p_{1R} (p_{1R} - p_{2R}) [(p_{1R} - p_{2R})^2 - 2(3p_{1R}^2 - p_{2R}^2)] + \alpha_{22} q_{1R} (q_{1R} - q_{2R}) [(q_{1R} - q_{2R})^2 - 2(3q_{1R}^2 - q_{2R}^2)]
\]

\[
+ 2\alpha_{11}\alpha_{22} [(p_{1R} - p_{2R})^2 - 2(3p_{1R}^2 - p_{2R}^2)] [(q_{1R} - q_{2R})^2 - 2(3q_{1R}^2 - q_{2R}^2)]
\]

\[
- 4(p_{1R} - p_{2R}) (q_{1R} - q_{2R}) (p_{1R} q_{1R} - p_{2R} q_{2R}) - 4 q_{1R} q_{2R} p_{1R} p_{2R},
\]

\[
M_{3n} = \alpha_{11}^2 [(p_{1R}^2 - p_{2R}^2)^2 + (p_{1R} - p_{2R})^2 [(p_{1R} - p_{2R})^2 - 2(3p_{1R}^2 - p_{2R}^2)]
\]

\[
+ 2\alpha_{11}\alpha_{22} [(p_{1R} - p_{2R})^2 - 2(3p_{1R}^2 - p_{2R}^2)] [(q_{1R} - q_{2R})^2 - 2(3q_{1R}^2 - q_{2R}^2)]
\]

\[
- 4(p_{1R} - p_{2R}) (q_{1R} - q_{2R}) (p_{1R} q_{1R} - p_{2R} q_{2R}) - 4 q_{1R} q_{2R} p_{1R} p_{2R},
\]

\[
M_{4n} = -\alpha_{11} (p_{1R} - p_{2R})^2 [(p_{1R} - p_{2R})^2 - 2(3p_{1R}^2 - p_{2R}^2)] - \alpha_{22} (q_{1R} - q_{2R})^2 (q_{1R} - q_{2R}) [(q_{1R} - q_{2R})^2 - 2(3q_{1R}^2 - q_{2R}^2)]
\]

\[
- \alpha_{11}\alpha_{22} [(q_{1R} - q_{2R})^2 - 2(3q_{1R}^2 - q_{2R}^2)] (p_{1R} q_{1R} - p_{2R} q_{2R}) - 4 q_{1R} q_{2R} (p_{1R} q_{1R} - p_{2R} q_{2R})
\]

where

\[
\Gamma_{\pm,\pm} = (p_{1R} + \sigma_{1R} p_{2R})^2 \pm (p_{1R} + \sigma_{2R} p_{2R})^2,
\]

\[
\Sigma_{\pm,\pm} = (q_{1R} + \sigma_{1R} q_{2R})^2 \pm (q_{1R} + \sigma_{2R} q_{2R})^2,
\]

\[
\Delta_{\pm,\pm} = (p_{1R} + \sigma_{1R} p_{2R}) (p_{1R} + \sigma_{2R} p_{2R}),
\]

\[
\Delta_{\pm,\pm} = (q_{1R} + \sigma_{1R} q_{2R}) (q_{1R} + \sigma_{2R} q_{2R}),
\]

with $\sigma_j = \pm 1$ ($j = 1, 2$).

The “dispersion relations” are given by

\[
s_{1R} = -4\beta_1 \beta_2 (p_{1R} q_{1R} + p_{1R} q_{1R}) + 2\beta_2^2 \left( \frac{\alpha_{11} p_{1R} p_{1R}}{\alpha_{22}} - q_{1R} q_{1R} \right) + 2\beta_1^2 \left( \frac{\alpha_{22} q_{1R} q_{1R}}{\alpha_{11}} - p_{1R} p_{1R} \right),
\]

\[
s_{1I} = \frac{1}{\alpha_{11} \alpha_{22}} \left( -a_{11}^2 \beta_2 (p_{1R} - p_{1I}) - a_{22}^2 \beta_2 (q_{1R} - q_{1I}) \right)
\]

12
\[ s_{2R} = -4\beta_1\beta_2(p_{2R}q_{2I} + p_{2I}q_{2R}) + 2\beta_2^2 \left( \frac{\alpha_{11}p_{2I}p_{2R}}{\alpha_{22}} - q_{2I}q_{2R} \right) + 2\beta_1^2 \left( \frac{\alpha_{22}q_{2I}q_{2R}}{\alpha_{11}} - p_{2I}p_{2R} \right), \]

\[ s_{2I} = \frac{1}{\alpha_{11}\alpha_{22}} \left\{ -\alpha_{11}^2\beta_2^2(p_{2R}^2 - p_{2I}^2) - \alpha_{22}^2\beta_1^2(q_{2R}^2 - q_{2I}^2) + \alpha_{11}\alpha_{22}[\beta_1^2(p_{2R}^2 - p_{2I}^2) + 4\beta_1\beta_2(p_{2R}q_{2R} - p_{2I}q_{2I}) + \beta_2^2(q_{2R}^2 - q_{2I}^2)] \right\}, \]

where \( p_{jR}, p_{jI}, q_{jR} \) and \( q_{jI} \) (\( j = 1, 2 \)) are arbitrary constants.
Appendix B

One type of the standard DS equations which can be solved by the inverse scattering transform is (see p. 240 in Ref.[12] for the case of \( r = -q^* \))

\[
\frac{\partial^2 \phi}{\partial x^2} - \sigma^2 \frac{\partial^2 \phi}{\partial y^2} = -2 \frac{\partial^2}{\partial x^2} (|q|^2),
\]

\[
i \frac{\partial q}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 q}{\partial x^2} + \frac{1}{2} \frac{\partial^2 q}{\partial y^2} = q \phi + |q|^2 q
\]

with \( \sigma^2 = \pm 1 \). Taking the transformation \( x \rightarrow \xi, y \rightarrow \eta, t \rightarrow \frac{2}{\sigma^2} \tau, q \rightarrow \frac{1}{\sigma^2} A_1, \) and \( \phi \rightarrow -\frac{\sigma^2}{2} \frac{\partial A_0}{\partial \xi} \), above equations become

\[
\sigma^2 \frac{\partial^2 A_0}{\partial \xi^2} - \frac{\partial^2 A_0}{\partial \eta^2} = 2 \frac{\partial}{\partial \xi} (|A_1|^2),
\]

\[
i \frac{\partial A_1}{\partial \tau} + \frac{\partial^2 A_1}{\partial \xi^2} + \sigma^2 \frac{\partial^2 A_1}{\partial \eta^2} = \sigma^{-2} |A_1|^2 A_1 - A_1 \frac{\partial A_0}{\partial \xi}.
\]

Comparing with Eqs. (19) and (20), for the last two equations we have

\[
\alpha_{11} = \sigma^2, \quad \alpha_{22} = -1, \quad \beta_1 = -1, \quad \beta_2 = 0,
\]

\[
\gamma_{11} = 1, \quad \gamma_{12} = 0, \quad \gamma_{22} = \sigma^{-2},
\]

\[
\chi = \sigma^{-2},
\]

which satisfy the integrable conditions (71) and (72).

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FIG. 1. The first Brillouin zone for the 2D quadratic lattice. The filled in and empty polygons correspond to the operator \( L \) (it is defined by (21)) of the elliptic and hyperbolic types, respectively. Along the intervals shown by the bold lines (i.e. in the directions \([100]\), \([010]\), \([110]\), and \([1\bar{1}0]\)) indicates the system (19), (20) is reduced to the conventional DSII equation.