Fermions in the Presence of the Antisymmetric Fields

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Abstract

In this manuscript we study the Dirac action in the presence of the Ramond-Ramond (R-R) potentials as gauge fields. Therefore, for the R-R field \( A_{\mu_1...\mu_{p+1}} \), we identify the corresponding fermion with an extended \( p \)-dimensional object, which we call it \( F_p \)-brane. Conservation of the tensor currents, associated to these fermionic branes, imposes an external tensor current. This external current enables us to study the R-R fields and their Hodge dual fields as independent degrees of freedom. We observe that an \( F_p \)-brane should live with its dual brane, i.e. \( F(d-p-2) \)-brane. The gauge symmetry and some other properties of a system of an \( F_p \)-brane and its dual object will be discussed.

\textit{PACS}: 11.25.-w

\textit{Keywords}: Dirac action; R-R fields; Branes.
1 Introduction

Supergravity theories in diverse dimensions admit a variety of $p$-branes. In the context of the effective $d = 10$ or $d = 11$ dimensional supergravity theories, a $p$-brane is a $p$-dimensional extended source for a $(p + 2)$-form field strength $F_{p+2}$. This field satisfies the equation

$$\nabla_\mu F^{\mu_1\ldots\mu_{p+1}} = j^{\mu_1\ldots\mu_{p+1}}, \quad (1)$$

where $j^{\mu_1\ldots\mu_{p+1}}$ is a $(p + 1)$-form current. The dual field $F'$ also satisfies the equation

$$\nabla_\mu F'^{\mu_1\ldots\mu_{p'+1}} = j'^{\mu_1\ldots\mu_{p'+1}}, \quad (2)$$

in which $j'^{\mu_1\ldots\mu_{p'+1}}$ is $(p' + 1)$-form current with $p' = d - p - 4$ [1, 2, 3].

From the other side, central extension of the superalgebras induces fermionic charges on the branes [4, 5]. By some assumptions the fermionic charge of D-string is interpreted as a source for the dilaton field [6]. Similarly, modification of the super-AdS algebras imposes fermionic charges for the branes [7]. Note that gauging an scalar field theory by form fields also is possible, e.g. see [8].

We construct a model which incorporates fermionic brane charge, whose existence has been anticipated in the Refs. [4, 5, 6, 7]. Therefore, combining the R-R fields and the fermionic charges as physical bases of our model, we introduce the gauged Dirac action by the R-R fields. This is analog of QED. That is, extended fermions are sources for the R-R fields. Assume there is a fermionic field $\psi_p$ corresponding to a $p$-dimensional brane. A tensor current due to this fermion appears as source of the R-R field $A_{\mu_1\ldots\mu_{p+1}}$. In other words, this R-R field can be emitted by $\psi_p$. We call this $p$-dimensional fermionic object as “F$p$-brane”, where “F” refers to the fermionic properties. For this system the gauge symmetry is obvious.

Generally, an F$p$-brane can be viewed as a new object which is different from a D$p$-brane. In contrast to the D-branes, the Hodge dual of an F$p$-brane is $(d - p - 2)$-dimensional object. We shall observe that an F$p$-brane should live with its dual brane. This system under an external current is formed. The external current generalizes the equations (1) and (2). Therefore, the R-R fields and their Hodge dual fields appear as independent degrees of freedom.

This paper is organized as follows. In section 2, the gauged Dirac action by an R-R field will be studied. In section 3, the Hodge dual of the F$p$-brane, i.e. $F(d - p - 2)$-brane will be introduced. In section 4, a system of an F$p$-brane with $F(d - p - 2)$-brane in the presence of an external current will be analyzed.
2 The gauged action

For the Dirac action in the $d$-dimensional flat spacetime, which is gauged by the R-R field $A_{\mu_1...\mu_{p+1}}$, we introduce the following action

\[
S = \int d^dX \left( \bar{\psi}_p (i \Gamma^\mu D_\mu - m) \psi_p - \frac{1}{2(p+2)!} F_{\mu_1...\mu_{p+2}} F^{\mu_1...\mu_{p+2}} \right),
\]

where $\psi_p$ is a Dirac field, $D_\mu$ is an appropriate operator, $m$ is the mass of the $F_p$-brane and $F_{p+2}$ is field strength of $A_{p+1}$, i.e. $F_{p+2} = dA_{p+1}$. The Dirac matrices are $\{\Gamma^\mu\}$. The spacetime metric also is considered as $\eta_{\mu\nu} = \text{diag}(-1,1,...1)$. Since the pre-factor of the integral will not appear in our calculations, we put it away.

We shall see that the gauge symmetry of the action (3) admits the following form for the operator $D_\mu$,

\[
D_\mu = \partial_\mu + iq_p \Gamma^{\mu_1...\mu_p} A_{\mu_1...\mu_p},
\]

where $\Gamma^{\mu_1...\mu_p} = [\Gamma_\mu_1\Gamma_\mu_2...\Gamma_\mu_p]$ is totally antisymmetric. The constant factor $q_p$ is the R-R charge corresponding to the $F_p$-brane. Thus, the action (3) describes the fermionic field $\psi_p$ and the R-R field $A_{\mu_1...\mu_{p+1}}$, which interact with each other. The interaction term of $\psi_p$ with $A_{\mu_1...\mu_{p+1}}$ reveals that $\psi_p$ can emit (absorb) this R-R field. In addition, $A_{\mu_1...\mu_{p+1}}$ can also split to $\psi_p$ and $\bar{\psi}_p$. Since this R-R field has a $p$-dimensional brane source, we can say that $F_p$-brane is an extended fermionic object.

Note that for simplicity we consider flat spacetime. It is straightforward to write the action $S$ in the curved background.

2.1 Field equations

Vanishing the variation of the action (3) gives the following equations of motion

\[
(i \Gamma^\mu D_\mu - m) \psi_p = 0,
\]

\[
\partial_\nu F^{\nu\mu_1...\mu_p} - q_p j_\nu^{\mu_1...\mu_p} = 0,
\]

where the $(p+1)$-form current $j_\nu^{\mu_1...\mu_p}$ has the definition

\[
j_\nu^{\mu_1...\mu_p} = \bar{\psi}_p \Gamma^{\mu_1...\mu_p} \psi_p.
\]

This is an extended current corresponding to the $F_p$-brane. The equation (6) emphasizes that this current reveals a fermionic source for the R-R field $A_{\mu_1...\mu_p}$.
Derivative of the equation (6) gives the conservation law
\[ \partial_\mu j^\mu_1...\mu_p = 0. \] (8)

Note that effect of any \( \partial_\mu \) with \( \mu \in \{\mu, \mu_1, ..., \mu_p\} \) leads to the equation (8). In fact, this equation puts some conditions on the fermionic field \( \psi_p \). Since the indices are chosen from the set \( \{0, 1, ..., d-1\} \), the equation (8) (for \( p \geq 1 \)) puts \( \frac{d!}{p!(d-p)!} \) conditions on \( \psi_p \). By modifying the action we shall remove these conditions.

### 2.2 Gauge symmetry

Now we discuss the gauge symmetry of the action (3). For this, consider the gauge transformations
\[ A_{p+1} \rightarrow A_{p+1} + d\Lambda, \]
\[ \psi_p \rightarrow e^{i\alpha \cdot \Lambda} \psi_p, \] (9)

where \( \Lambda(X) \) and \( \alpha(X) \) are local \( p \)-forms. The dot product \( \alpha \cdot \Lambda \) means \( \alpha \cdot \Lambda \equiv \alpha^{\mu_1...\mu_p} \Lambda_{\mu_1...\mu_p} \).

If \( D_\mu \) really is covariant derivative, \( D_\mu \psi_p \) should have gauge transformation like the field \( \psi_p \). Therefore, having the transformation
\[ D_\mu \psi_p \rightarrow e^{i\alpha \cdot \Lambda} D_\mu \psi_p, \] (10)

leads to the following equation
\[ \partial_\mu (\alpha \cdot \Lambda) = -(p+1)q_\mu \Gamma^{\mu_1...\mu_p} \partial_\mu \Lambda_{\mu_1...\mu_p}. \] (11)

According to this equation, the action \( S \) is invariant under the transformations (9).

The equation (11) gives \( d \) relations between the components of \( \alpha \), while \( \Lambda \) remains arbitrary. Thus, up to these relations, the functions \( \{\alpha^{\mu_1...\mu_p}(X)\} \) also are arbitrary. As an example, for F0-brane, the solution of (11) is
\[ \alpha(X) = -q_0 + \frac{C}{\Lambda(X)}, \] (12)

where \( C \) is an arbitrary constant.

### 3 Action of the dual fields

Now we introduce the Hodge duals of the fields \( \psi_p \) and \( A_{p+1} \). Let denote them by \( \psi'_p \) and \( A'_{p'+1} \). The R-R field \( A'_{\mu_1...\mu_{p'+1}} \) is given by
\[ A'_{\mu_1...\mu_{p'+1}} = \frac{1}{(p+1)!} \epsilon_{\mu_1...\mu_{p'+1}}^{\nu_1...\nu_{p'+1}} A_{\nu_1...\nu_{p'+1}}. \] (13)
In this section we do not use the equation (13). That is, for the next purposes we study the behavior of $A'_{\mu_1...\mu_{p'}+1}$ as an independent degree of freedom. This implies that the action should have the term $|dA'_{\mu_1...\mu_{p'}+1}|^2$. In addition, $A'_{\mu_1...\mu_{p'}+1}$ appears in the covariant derivative. Thus, the action of the dual variables also has the structure of $S$, i.e.,

$$S' = \int d^dX \left( \bar{\psi}'_{p'}(i\Gamma^\mu D'_\mu - m')\psi'_{p'} - \frac{1}{2(p' + 2)!} F'_{\mu_1...\mu_{p'+2}} F'^{\mu_1...\mu_{p'+2}} \right).$$

(14)

The form $F'_{p'+2}$ is field strength of $A'_{p'+1}$,

$$F'_{p'+2} = dA'_{p'+1}. \quad (15)$$

The covariant derivative $D'_\mu$ is given by (4) with $A'_{\mu_1...\mu_{p'}}$ instead of $A_{\mu_1...\mu_{p}}$ and also $p'$ instead of $p$, where $p' = d - p - 2$ is the dimension of an F$p'$-brane. This brane is source of the R-R field $A'_{p'+1}$. It has the R-R charge $q_{p'}$ and the mass $m'$. The corresponding fermionic field is $\psi'_{p'}$. This field is defined as the dual of $\psi_{p}$. In fact, through the equation of motion $\psi'_{p'}$ depends on the Hodge dual of $A_{p+1}$.

Note that in the action $S'$ we applied the field strength $F'_{p'+2} = \ast dA_{p+1}$. This is different from the D-brane case that $F'_{p'+2} = \ast dA_{p+1}$. Therefore, for a D$p$-brane the Hodge dual is $D(d-p-4)$-brane, while in our model the dual of an F$p$-brane is F$(d-p-2)$-brane. In other words, an F$p$-brane is different from a D$p$-brane.

The action $S'$ under the gauge transformations

$$A'_{p'+1} \rightarrow A'_{p'+1} + d\Lambda', \quad \psi'_{p'} \rightarrow e^{i\alpha'.\Lambda'} \psi'_{p'}, \quad (16)$$

is invariant. This invariance implies the following relation between the components of $\alpha'^{\mu_1...\mu_{p'}}$,

$$\partial_\mu (\alpha'.\Lambda') = -(p' + 1)q_{p'}\Gamma^{\mu_1...\mu_{p'}} \partial_\mu \Lambda'_{\mu_1...\mu_{p'}}, \quad (17)$$

According to the equations of motion, extracted from the action $S'$, the fermionic field $\psi'_{p'}$ should satisfy the condition

$$\partial_\mu j'^{\mu_1...\mu_{p'}} = 0. \quad (18)$$

In deed, this is conservation law for the current $j'^{\mu_1...\mu_{p'}}$. This current originates from the dual fermions. By combining the actions $S$ and $S'$ the conditions (8) and (18) will be removed.
4 A system of Fp-brane and F(d − p − 2)-brane

Now we proceed to study a system of an Fp-brane with an Fp'-brane where \( p' = d − p − 2 \). That is, the fields \( A_{\mu_1...\mu_{p+1}} \) and \( A'_{\mu_1...\mu_{p'+1}} \) (and also \( \psi_p \) and \( \psi'_p \)) appear as independent degrees of freedom. This can be done by combining the actions \( S \) and \( S' \). In addition, we consider the equation (13) as a constraint. Adding all these together, we obtain the action

\[
I = \int d^dX \left[ \bar{\psi}_p (i\Gamma^\mu D_\mu - m) \psi_p - \frac{1}{2(p + 2)!} F_{\mu_1...\mu_{p+2}} F^{\mu_1...\mu_{p+2}} + \bar{\psi}'_p (i\Gamma^\mu D'_\mu - m') \psi'_p - \frac{1}{2(p' + 2)!} F'_{\mu_1...\mu'_{p'+2}} F'^{\mu_1...\mu'_{p'+2}} \right. \\
\left. + (\mu' + 1)! J^{\mu_1...\mu_{p+1}} (A_{\mu_1...\mu_{p+1}} - \frac{\eta}{(\mu' + 1)!} \epsilon_{\mu_1...\mu_{p+1}} \nu_1...\nu_{p'+1} A'_{\nu_1...\nu_{p'+1}}) \right]. 
\]

(19)

The tensor field \( J^{\mu_1...\mu_{p+1}} \) with the conventional factor \((\mu' + 1)!\) is Lagrang multiplier. The factor \( \eta \) is the effect of double Hodge duality, i.e. on a \((p + 1)\)-form it is \( \eta = ** = (-1)^{p(d-p)+d} \).

The equations of motion, extracted from the action (19), are as in the following

\[
(i\Gamma^\mu D_\mu - m) \psi_p = 0,
\]

\[
(i\Gamma^\mu D'_\mu - m') \psi'_p = 0,
\]

\[
A_{\mu_1...\mu_{p+1}} - \frac{\eta}{(\mu' + 1)!} \epsilon_{\mu_1...\mu_{p+1}} \nu_1...\nu_{p'+1} A'_{\nu_1...\nu_{p'+1}} = 0,
\]

\[
\partial_\nu F^{\nu\mu_1...\mu_p} - q_{\nu} j_{\mu_1...\mu_p} = (\mu' + 1)! J^{\mu_1...\mu_p},
\]

\[
\partial_\nu F'^{\nu\mu_1...\mu_{p'}} - q_{\nu} j'_{\mu_1...\mu_{p'}} = (p + 1)! J'^{\mu_1...\mu_{p'}},
\]

(20)

where the \((\mu' + 1)\)-form \( J'_{\mu'+1} \) is Hodge dual of the \((p + 1)\)-form \( J_{p+1} \),

\[
J^{\mu_1...\mu_{p}} = \frac{1}{(p + 1)!} \epsilon^{\mu_1...\mu_{p}} \nu_1...\nu_{p+1} J^{\nu_1...\nu_{p+1}}.
\]

(21)

In the fourth and fifth equations of (20) the tensor fields \( J^{\mu_1...\mu_p} \) and \( J'^{\mu_1...\mu_{p'}} \) act as the external currents. We can also see this from the action (19). Therefore, the third line of (19) can be written as

\[
\mathcal{L}_c = -((\mu' + 1)! J.A + (p + 1)! J'.A').
\]

(22)

This Lagrangian density reveals that \( J^{\mu_1...\mu_p} \) and \( J'^{\mu_1...\mu_{p'}} \) are external sources corresponding to the fields \( A_{\mu_1...\mu_p} \) and \( A'_{\mu_1...\mu_{p'}} \), respectively. In other words, the constraint-terms in the action find the feature of the source-terms.

The fourth and fifth equations of (20) lead to the following conservation laws

\[
\partial_\mu (q_\nu j^{\mu_1...\mu_p}) + (\mu' + 1)! J^{\mu_1...\mu_p} = 0,
\]

\[
\partial_\mu (q'_\nu j'^{\mu_1...\mu_{p'}}) + (p + 1)! J'^{\mu_1...\mu_{p'}} = 0.
\]

(23)
The first equation of (23) implies that the combination of the external current and the current due to the fermionic field $\psi_p$ is conserved. By the second equation, this also holds for the dual case. These equations also removed the conditions (8) and (18) from the fields $\psi_p$ and $\psi'_p$.

In fact, the conservation laws (23) impose the external current $J^{\mu_1 \cdots \mu_p}$. This current interacts with the R-R fields, emitted by the F$p$-brane and the F$(d-p-2)$-brane. Therefore, the action (19) implies that in the presence of the external current a system of the F$p$-brane with the F$(d-p-2)$-brane is formed.

4.1 Symmetries of the action $I$

Under the gauge transformations (9) and (16) (with the equations (11) and (17) for the elements of $\alpha$ and $\alpha'$) the action $I$ is invariant if the forms $\Lambda$ and $\Lambda'$ have the relation

$$d\Lambda' = *d\Lambda.$$  \hspace{1cm} (24)

Note that the external current $J_{p+1}$, and hence $J'_{p'+1}$, should have trivial transformations.

Another symmetry is as follows. Under the exchanges of the variables with their dual variables, \textit{i.e.},

$$A_{p+1} \leftrightarrow A'_{p'+1},$$
$$\psi_p \leftrightarrow \psi'_{p'},$$
$$J_{p+1} \leftrightarrow J'_{p'+1},$$ \hspace{1cm} (25)

and also the exchanges $m \leftrightarrow m'$ and $p \leftrightarrow p'$, the action (19) is symmetric. In fact, the first exchange of (25) gives $q_p \leftrightarrow q'_{p'}$. The invariance of the source terms of (19), by the equation (22), is more obvious.

5 Conclusions

Gauging the Dirac action by the R-R fields produces the fermionic branes. That is, coupling of a fermionic field with an R-R field leads to an effective tensor current as a source for the R-R field. Thus, this fermion describes an extended object, \textit{i.e.} F$p$-brane. The gauged action has the gauge symmetry, as expected.

For an F$p$-brane there is a dual brane, \textit{i.e.} F$(d-p-2)$-brane. The associated gauge fields of these branes are Hodge dual of each other. They have different fermionic fields.
The dual theory has its own gauge symmetry. In general, the F-branes are different from
the D-branes.

Conservation of the tensor currents puts some conditions on the above fermionic fields.
In other words, only under these conditions a single F_p-brane can exist. Removing these
conditions imposes a system of an F_p-brane with an F(d − p − 2)-brane in the presence of
an external current. Therefore, the source of an R-R field is combination of the internal and
the external currents. In addition to the gauge symmetry, this combined system under the
exchange of the variables with their dual variables is invariant.

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