TRANSONIC SHOCKS FOR 3-D AXISYMMETRIC
COMPRESSIBLE INVISCID FLOWS IN CYLINDERS

HYANGDONG PARK AND HYEONGYU RYU

Abstract. We establish the existence of an axisymmetric weak solution to the steady Euler system with a transonic shock, nonzero vorticity, and nonzero swirl in a three-dimensional cylinder. When prescribing the supersonic solution in the upstream region by axisymmetric functions with variable entropy and variable angular momentum density (=swirl), we construct such a solution by using a Helmholtz decomposition of the velocity field and the method of iteration. An iteration scheme is developed using a delicate decomposition of the Rankine-Hugoniot conditions on the transonic shock via Helmholtz decomposition.

1. Introduction

The steady inviscid compressible flow of ideal polytropic gas in \( \mathbb{R}^3 \) is governed by the steady Euler system (cf. [4, 12]):

\[
\begin{aligned}
\text{div}(\rho u) &= 0, \\
\text{div}(\rho u \otimes u + p I_3) &= 0 \quad (I_3 : 3 \times 3 \text{ identity matrix}), \\
\text{div}(\rho u B) &= 0.
\end{aligned}
\]  

(1.1)

In the system above, \( \rho = \rho(x), \ u = (u_1, u_2, u_3)(x), \ p = p(x), \) and \( B = B(x) \) denote the density, velocity, pressure, and the Bernoulli invariant of the flow, respectively, at \( x = (x_1, x_2, x_3) \in \mathbb{R}^3 \). For a constant \( \gamma > 1 \) called the adiabatic exponent, \( B \) is defined by

\[
B := \frac{1}{2} |u|^2 + \frac{\gamma p}{(\gamma - 1)p}.
\]

Let \( \Omega \subset \mathbb{R}^3 \) be an open connected set, and let \( \Gamma \) be a non-self-intersecting \( C^1 \)-surface dividing \( \Omega \) into two disjoint open subsets \( \Omega^\pm \) such that \( \Omega = \Omega^- \cup \Gamma \cup \Omega^+ \).

Definition 1.1. We define \( U = (u, \rho, p) \in [L^\infty_{\text{loc}}(\Omega) \cap C^1_{\text{loc}}(\Omega^\pm) \cap C^0_{\text{loc}}(\Omega^\pm \cup \Gamma)]^5 \) to be a weak solution to (1.1) in \( \Omega \) if the following properties are satisfied: For any
test function \( \xi \in C_0^\infty(\Omega) \) and \( j = 1, 2, 3 \),
\[
\int_{\Omega} \rho u \cdot \nabla \xi \, dx = \int_{\Omega} (\rho u_j u + p e_j) \cdot \nabla \xi \, dx = \int_{\Omega} \rho u B \cdot \nabla \xi \, dx = 0, \tag{1.2}
\]
where \( e_j \) is the unit vector in the \( x_j \)-direction.

By the integration by parts, one can easily check that \( U \) satisfies (1.2) if and only if
- \((w_1)\) \( U \) is a classical solution to (1.1) in \( \Omega^\pm \);
- \((w_2)\) \( U \) satisfies the Rankine-Hugoniot conditions
\[
[\rho u \cdot n]_\Gamma = 0, \quad [\rho (u \cdot n)^2 + p]_\Gamma = 0, \quad [\rho u \cdot n B]_\Gamma = 0, \tag{1.3}
\]
\[
[\rho (u \cdot n)]_\Gamma = 0 \quad \text{for all} \quad k = 1, 2, \tag{1.4}
\]
where \([\cdot]_\Gamma \) is defined by
\[
[F(x)]_\Gamma := F(x)|_{\Omega^+} - F(x)|_{\Omega^-} \quad \text{for} \quad x \in \Gamma,
\]
\( n \) is a unit normal vector field on \( \Gamma \), and \( \tau_k \) \((k = 1, 2)\) are unit tangent vector fields on \( \Gamma \) such that they are linearly independent at each point on \( \Gamma \).

Assume that \( \rho > 0 \) in \( \Omega \). Then, the condition in (1.4) is satisfied if either \( u \cdot n = 0 \) on \( \Gamma \), or \([u \cdot \tau_k]_\Gamma = 0\) for all \( k = 1, 2 \). We are interested in the latter case. For the former case, one can refer to [3, 4] and the references therein.

**Definition 1.2.** We define \( U = (u, \rho, p) \in [L^\infty_{loc}(\Omega) \cap C^1_{loc}(\Omega^\pm) \cap C^0_{loc}(\Omega^\pm \cup \Gamma)]^5 \) to be a weak solution to (1.1) in \( \Omega \) with a transonic shock \( \Gamma \) if we have the following properties:

(i) \( \Gamma \) is a non-self-intersecting \( C^1 \)-surface dividing \( \Omega \) into two open subsets \( \Omega^\pm \) such that \( \Omega = \Omega^+ \cup \Gamma \cup \Omega^- \);

(ii) \( U \) satisfies \((w_1)\), i.e., \( U \) is a classical solution to (1.1) in \( \Omega^\pm \);

(iii) (Positivity of density) \( \rho > 0 \) in \( \Omega \);

(iv) (Rankine-Hugoniot conditions) \( U \) satisfies (1.3) in \((w_2)\) and \([u \cdot \tau_k]_\Gamma = 0\) for all \( k = 1, 2 \);

(v) (Transonic speed) \( |u| > c \) (supersonic speed) in \( \Omega^- \) and \( |u| < c \) (subsonic speed) in \( \Omega^+ \) for the sound speed \( c := \sqrt{\gamma p} \);

(vi) (Admissibility) \( u|_{\Omega^- \cap \Gamma} \cdot n > u|_{\Omega^+ \cap \Gamma} \cdot n > 0 \) for the unit normal vector field \( n \) on \( \Gamma \) pointing toward \( \Omega^+ \).

In this paper, we study the existence of a weak solution to (1.1) with a transonic shock in the sense of Definition 1.2 in a three-dimensional cylinder (Figure 1.1).
Let \((x, r, \theta)\) denote the cylindrical coordinates of \(x = (x_1, x_2, x_3) \in \mathbb{R}^3\), i.e.,

\[(x_1, x_2, x_3) = (x, r \cos \theta, r \sin \theta), \quad r \geq 0, \quad \theta \in \mathbb{T},\]

where \(\mathbb{T}\) is a one-dimensional torus with period \(2\pi\). Then, any function \(f(x)\) and vector-valued function \(\mathbf{F}(x)\) can be represented by

\[f(x) = f(x, r, \theta) \quad \text{and} \quad \mathbf{F}(x) = F_x(x, r, \theta)\mathbf{e}_x + F_r(x, r, \theta)\mathbf{e}_r + F_\theta(x, r, \theta)\mathbf{e}_\theta\]

for orthonormal vectors

\[\mathbf{e}_x = (1, 0, 0), \quad \mathbf{e}_r = (0, \cos \theta, \sin \theta), \quad \mathbf{e}_\theta = (0, -\sin \theta, \cos \theta).\]

**Definition 1.3.** [4, 5]

(i) A function \(f(x)\) is axisymmetric if its value is independent of \(\theta\).
(ii) A vector-valued function \(\mathbf{F}(x)\) is axisymmetric if each of functions \(F_x, F_r,\) and \(F_\theta\) is axisymmetric.

The purpose of this paper is to prove the existence of an axisymmetric weak solution to the steady Euler system \([13]\) with a transonic shock in a three-dimensional cylinder. The existence, uniqueness, and stability of transonic shocks for 3-D steady flows in cylindrical nozzles were studied in \([6, 7, 8, 9, 10, 11]\). In \([6, 7, 8]\), the existence and stability of multidimensional transonic shocks for potential flows were established. In \([9]\), authors proved the uniqueness of solutions with a transonic shock in a class of transonic shock solutions, which are not necessarily small perturbations of the background solution, for potential flow. In \([10, 11]\), the existence and stability of the perturbed compressible flow including a transonic shock were studied for the prescribed pressure at the exit up to a constant. For studies on multidimensional transonic shocks in diverging nozzles, see \([2, 16, 17, 18, 19]\) and the references cited therein.

In this paper, we establish the existence of an axisymmetric weak solution to the steady Euler system with a transonic shock, nonzero vorticity, and nonzero swirl in a three-dimensional cylinder. When prescribing the supersonic solution in the upstream region by axisymmetric functions with variable entropy and variable angular
momentum density (=swirl), we construct such a solution by using a Helmholtz decomposition of the velocity field and the method of iteration. An iteration scheme is developed using a delicate decomposition of the Rankine-Hugoniot conditions on the transonic shock via Helmholtz decomposition. This approach, using Helmholtz decomposition, is a new attempt to investigate multidimensional transonic shock solutions to the Euler system.

The structure of the paper is as follows: In Section 2 the main problem and theorem are stated as Problem 2.1 and Theorem 2.1, respectively. In Section 3 we reformulate the main problem via Helmholtz decomposition, and state its solvability as Theorem 3.1. In Section 4 we prove Theorem 3.1 by using the method of iteration. Finally, in Section 5 we prove Theorem 2.1 by Theorem 3.1.

2. Problems and Main Results

Define a cylinder \( N \) by
\[
N := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : -1 < x_1 < 1, 0 \leq \sqrt{x_2^2 + x_3^2} < 1\}.
\]

As defined in the previous section, let \((x, r, \theta)\) be the cylindrical coordinates of \((x_1, x_2, x_3) \in \mathbb{R}^3\), i.e.,
\[
(x_1, x_2, x_3) = (x, r \cos \theta, r \sin \theta), \quad r \geq 0, \quad \theta \in \mathbb{T},
\]
where \(\mathbb{T}\) denotes a one-dimensional torus with period \(2\pi\). Then, the entrance \(\Gamma_{en}\), exit \(\Gamma_{ex}\), and the wall \(\Gamma_w\) of \(N\) are defined as
\[
\Gamma_{en} := \partial N \cap \{x = -1\}, \quad \Gamma_{ex} := \partial N \cap \{x = 1\}, \quad \Gamma_w := \partial N \cap \{r = 1\}.
\]

We first construct a simple solution. Let \((u^-_0, \rho^-_0, p^-_0)\) be positive constants satisfying
\[
u^-_0 > \sqrt{\frac{\gamma p^-_0}{\rho^-_0}}.
\]
Define a function \(U_0\) by
\[
U_0(x_1, x_2, x_3) := \begin{cases} (u^-_0, \rho^-_0, p^-_0) & \text{for } x_1 < 0, \\ (u^+_0, \rho^+_0, p^+_0) & \text{for } x_1 > 0, \end{cases}
\]
where \(u^+_0 := (u^+_0, 0, 0)\) and \((u^+_0, \rho^+_0, p^+_0)\) are positive constants defined by
\[
u^+_0 := \frac{\rho^+_0}{\rho^-_0} u^-_0 \quad \text{and} \quad p^+_0 := \rho^-_0 |u^-_0|^2 + p^-_0 - \rho^+_0 |u^+_0|^2.
\]
Then one can easily check that $U_0$ is a weak solution to (1.1) in $\mathcal{N}$ with a transonic shock $S_0 := \mathcal{N} \cap \{x = 0\}$ (Figure 2.1). For later use, we set

$$S_0^\pm := p_0^\pm \left(\frac{\rho_0^\pm}{\rho_0^\pm}\right)^{\gamma}, \quad B_0 := \frac{1}{2} |u_0^\pm|^2 + \frac{\gamma p_0^\pm}{(\gamma - 1) \rho_0^\pm},$$

$$\varphi_0^\pm(x) := u_0^\pm x_1 \quad \text{for} \quad x = (x_1, x_2, x_3) \in \mathcal{N}.$$ (2.1)

**Figure 2.1.** Left: Background solution, Right: Problem 2.1

Before we state our problem and main results, we introduce some Hölder norms defined as follows:

(i) (Standard Hölder norms) Let $\Omega \subset \mathbb{R}^n$ be an open bounded connected set. For $\alpha \in (0, 1)$ and $m \in \mathbb{Z}^+$, define the standard Hölder norms by

$$\|u\|_{m, \Omega} := \sum_{0 \leq |\beta| \leq m} \sup_{x \in \Omega} |D^\beta u(x)|, \quad [u]_{m, \alpha, \Omega} := \sum_{|\beta| = m} \sup_{x, y \in \Omega, x \neq y} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x - y|^\alpha},$$

$$\|u\|_{m, \alpha, \Omega} := \|u\|_{m, \Omega} + [u]_{m, \alpha, \Omega}.$$

Here, $D^\beta$ denotes $\partial^{\beta_1}_{x_1} \ldots \partial^{\beta_n}_{x_n}$ for a multi-index $\beta = (\beta_1, \ldots, \beta_n)$ with $\beta_j \in \mathbb{Z}^+$ and $|\beta| = \sum_{j=1}^n \beta_j$.

(ii) (Weighted Hölder norms) Let $\Gamma$ be a closed portion of $\partial \Omega$. For $x, y \in \Omega$ and $k \in \mathbb{R}$, set

$$\delta_x := \inf_{z \in \Gamma} |x - z| \quad \text{and} \quad \delta_{x,y} := \min(\delta_x, \delta_y),$$

and define the weighted Hölder norms by

$$\|u\|_{m, \alpha}^{(k, \Gamma)} := \sum_{0 \leq |\beta| \leq m} \sup_{x \in \Omega} \delta_x^{\max(|\beta| + k, 0)} |D^\beta u(x)|,$$

$$[u]_{m, \alpha}^{(k, \Gamma)} := \sup_{x, y \in \Omega, x \neq y} \delta_{x,y}^{\max(m + \alpha + k, 0)} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x - y|^\alpha},$$

$$\|u\|_{m, \alpha}^{(k, \Gamma)} := \|u\|_{m, \alpha}^{(k, \Gamma)} + [u]_{m, \alpha}^{(k, \Gamma)}.$$  

$C_{m, \alpha}^{(k, \Gamma)}(\Omega)$ denotes the completion of the set of all smooth functions whose $\| \cdot \|_{m, \alpha}^{(k, \Gamma)}$ norms are finite.

Our goal is to solve the following problem.
Problem 2.1. Let \((u^-, \rho^-, p^-)\) be an axisymmetric supersonic solution of (1.1) in \(\mathcal{N}\) with \(B = B_0\) for a constant \(B_0 > 0\) defined in (2.1), and suppose that
\[
u^- \cdot e_r = 0 \quad \text{on} \quad \Gamma_w.
\]
Find a weak solution \(U = (u, \rho, p)\) to (1.1) with a transonic shock
\[\mathcal{S}_f : x = f(r) \quad (\text{Figure 2.1})\]
in the sense of Definition 1.2 in \(\mathcal{N}\) such that we have the following properties:
(a) (Positivity of density and velocity along \(x\)-direction)
\[
\rho > 0 \quad \text{and} \quad u \cdot e_x > \frac{u_0^+}{2} > 0 \quad \text{in} \quad \mathcal{N}.
\]
(b) In \(\mathcal{N}_f^- : = \mathcal{N} \cap \{x < f(r)\}\), \(U = (u^-, \rho^-, p^-)\) holds.
(c) In \(\mathcal{N}_f^+ : = \mathcal{N} \cap \{x > f(r)\}\), \(|u| < c\) for the sound speed \(c = \sqrt{\gamma p}\).
(d) \(U\) satisfies the boundary condition
\[
u \cdot e_r = 0 \quad \text{on} \quad \Gamma_{ex} \cup \Gamma_w.
\]
(e) \(U\) satisfies the Rankine-Hugoniot conditions
\[
[u \cdot \tau_f]_{\mathcal{E}_f} = \rho u \cdot n_f]_{\mathcal{E}_f} = [B]_{\mathcal{E}_f} = [\rho(u \cdot n_f)^2 + p]_{\mathcal{E}_f} = 0 \quad \text{on} \quad \mathcal{S}_f,
\]
where \(\tau_f\) and \(n_f\) denote a unit tangent vector field and unit normal vector field on \(\mathcal{S}_f\), respectively.

Problem 2.1 can be rewritten as the following free boundary problem:

Problem 2.2. Under the same assumptions of Problem 2.1, find a radial function
\[
f : B_1(0) := \{y \in \mathbb{R}^2 : |y| < 1\} \to \left(-\frac{1}{4}, \frac{1}{4}\right)
\]
and a weak solution \(U = (u, \rho, p)\) to (1.1) in \(\mathcal{N}_f^+ : = \mathcal{N} \cap \{x > f(r)\}\) such that the following properties hold:
(a) The properties (a) and (c) in Problem 2.1 hold in \(\mathcal{N}_f^+\).
(b) On \(\Gamma_{w,f}^+ (:= \partial \mathcal{N}_f^+ \cap \Gamma_w) \cup \Gamma_{ex}, U\) satisfies the boundary conditions
\[
u \cdot e_r = 0.
\]
(c) On \(\mathcal{S}_f : x = f(r)\), \(U\) satisfies the boundary conditions
\[
\begin{cases}
\nu \cdot \tau_f = \nu^- \cdot \tau_f, \\
\rho\nu \cdot n_f = \rho^- \nu \cdot n_f, \\
\left(\rho(u \cdot n_f)^2 + p\right) = \rho^- (u^- \cdot n_f)^2 + p^-,
\end{cases}
\]
where \(\tau_f\) and \(n_f\) denote a unit tangent vector field and unit normal vector field on \(\mathcal{S}_f\), respectively.
(d) The Bernoulli function $B$ is a constant function,

$$B \equiv B_0 \text{ in } \mathcal{N}_f^-,$$

where $B_0$ is given in (2.1).

The main theorem of this paper is as follows:

**Theorem 2.1.** Let $(u^-, \rho^-, p^-)$ be from Problem 2.2. For any fixed $\alpha \in (\frac{1}{6}, 1)$, there exists a small constant $\sigma_1 > 0$ depending only on $(u_0^\pm, \rho_0^\pm, p_0^\pm, \gamma, \alpha)$ so that if

$$\sigma(u^-, \rho^-, p^-) := \| (u^-, \rho^-, p^-) - (u_0^-, \rho_0^-, p_0^-) \|_{1, \alpha, \mathcal{N}} \leq \sigma_1,$$

then there exists an axisymmetric solution $U = (u, \rho, p)$ of Problem 2.2 with a transonic shock $x = f(r)$ satisfying

$$\| f \|_{2, \alpha, B_1^-(0)} + \| (u, \rho, p) - (u_0^+, \rho_0^+, p_0^+) \|_{1, \alpha, \mathcal{N}_f^+} \leq C\sigma,$$

where the constant $C > 0$ depends only on $(u_0^+, \rho_0^+, p_0^+, \gamma, \alpha)$.

3. Reformulation of Problem 2.2 via Helmholtz decomposition

Suppose that the smooth solution $(u, \rho, p)$ of (1.1) is axisymmetric with $B \equiv B_0$ for a constant $B_0 > 0$ defined in (2.1). Then

$$u = u_x(x, r)e_x + u_r(x, r)e_r + u_\theta(x, r)e_\theta, \quad \rho = \rho(x, r), \quad p = p(x, r),$$

and the system (1.1) can be rewritten as follows:

$$\begin{cases}
\partial_x (\rho u_x) + \partial_r (\rho u_r) + \frac{\rho u_r}{r} = 0, \\
\rho (u_x \partial_x + u_r \partial_r) u_r - \frac{\rho \Lambda^2}{r^3} + \partial_r (S \rho) = 0, \\
\rho (u_x \partial_x + u_r \partial_r) S = 0, \\
\rho (u_x \partial_x + u_r \partial_r) \Lambda = 0.
\end{cases} \tag{3.1}$$

Here, $S := \frac{p}{\rho}$ denotes the entropy and

$$\Lambda(x, r) := ru_\theta(x, r)$$

denotes the angular momentum density.

For a radial function $f : B_1(0) \rightarrow (-\frac{1}{4}, \frac{1}{4})$ to be determined, we express the velocity field $u$ as

$$u = \nabla \varphi + \text{curl} V \text{ in } \mathcal{N}_f^+$$

for axisymmetric functions

$$\varphi(x) = \varphi(x, r), \quad V(x) = h(x, r)e_r + \psi(x, r)e_\theta.$$
Suppose that \((\varphi, V)\) are \(C^2\) in \(N^+_f\). Then a straightforward computation yields

\[
\mathbf{u} = \left( \partial_x \varphi + \frac{1}{r} \partial_r (r \psi) \right) \mathbf{e}_x + \left( \partial_r \varphi - \partial_x \psi \right) \mathbf{e}_r + \left( \frac{\Lambda}{r} \right) \mathbf{e}_\theta
\]

\[=: q(r, \psi, D\psi, D\varphi, \Lambda) \quad \text{for} \quad D = (\partial_x, \partial_r). \tag{3.2}\]

To simplify notations, we set

\[
t(r, \psi, D\psi, \Lambda) := q(r, \psi, D\psi, D\varphi, \Lambda) - \nabla \varphi = \text{curl} \ V. \tag{3.3}\]

Then, as in [4, Section 3], we can rewrite (3.1) as a system for \((\varphi, \psi, S, \Lambda)\):

\[
\begin{cases}
\text{div} \left( H(S, q) q \right) = 0, \\
- \Delta (\psi \mathbf{e}_\theta) = G(S, \Lambda, \partial_r S, \partial_r \Lambda, t, \nabla \varphi) \mathbf{e}_\theta, \\
H(S, q) q \cdot \nabla S = 0, \\
H(S, q) q \cdot \nabla \Lambda = 0,
\end{cases} \tag{3.4}\]

with \(q = q(r, \psi, D\psi, D\varphi, \Lambda)\), and \(t = t(r, \psi, D\psi, \Lambda)\) for \((H, G)\) defined by

\[
H(\eta, q) := \left[ \frac{\gamma - 1}{\gamma \eta} \left( B_0 - \frac{1}{2} |q|^2 \right) \right]^{1/(\gamma - 1)},
\]

\[
G(\eta_1, \eta_2, \eta_3, \eta_4, t, v) := \frac{1}{(t + v) \cdot \mathbf{e}_x} \left( \frac{H^{\gamma - 1}(\eta_1, t + v)}{\gamma - 1} \eta_3 + \frac{\eta_2}{r^2} \eta_4 \right) \tag{3.5}\]

for \(\eta \in \mathbb{R}, q \in \mathbb{R}^3, \eta_1, \eta_2, \eta_3, \eta_4 \in \mathbb{R}, \) and \(t, v \in \mathbb{R}^3\).

For an axisymmetric supersonic solution \((\mathbf{u}^-, \rho^-, p^-)\) given in Problem 2.1, we find \((S^-, \Lambda^-, \varphi^-, \psi^-)\) such that

\[
\mathbf{u}^- = q(r, \psi^-, D\psi^-, D\varphi^-, \Lambda^-), \quad \rho^- = H(S^-, \mathbf{u}^-), \quad p^- = S^- (\rho^-)^\gamma \quad \text{in} \quad N.
\]

For that purpose, we first solve the following linear boundary value problem:

\[
- \Delta W = (\partial_x(\mathbf{u}^- \cdot \mathbf{e}_r) - \partial_r(\mathbf{u}^- \cdot \mathbf{e}_x)) \mathbf{e}_\theta \quad \text{in} \quad N \tag{3.6}\]

with boundary conditions

\[
\partial_x W = 0 \quad \text{on} \quad \Gamma_{en} \cup \Gamma_{ex}, \quad W = 0 \quad \text{on} \quad \Gamma_w \cup \{r = 0\}. \tag{3.7}\]

By [5, Proposition 3.3], the unique axisymmetric solution \(W^- \in C^{2,\alpha}(\overline{N})\) of the boundary value problem (3.6)-(3.7) has the form of

\[
W^- = \psi^-(x, r) \mathbf{e}_\theta
\]
and the axisymmetric function $\psi^-$ is $C^2$ as a function of $(x,r)$ in a two-dimensional rectangle $(-1,1) \times (0,1)$. With such $\psi^-$, we define axisymmetric functions $(\varphi^-, S^-, \Lambda^-)$ by

\[
\varphi^-(x,r) := \int_0^x \left( u^- \cdot e_x - \frac{1}{r} \partial_r (r \psi^-) \right) (y,r) dy, \\
S^-(x,r) := \frac{p^-}{(\rho^-)}\gamma (x,r), \\
\Lambda^-(x,r) := ru^- \cdot e\theta (x,r) \quad \text{in } N.
\]

(3.8)

Then it follows from (3.6) and (3.8) that

\[
\partial_r \varphi^- = u^- \cdot e_r + \partial_x \psi^- \quad \text{in } N.
\]

By direct computations, one can check that there exists a constant $C > 0$ depending only on $(u^-_0, \rho^-_0, p^-_0, \gamma, \alpha)$ such that

\[
\sigma(S^-, \Lambda^-, \varphi^-, \psi^-) \leq C \sigma(u^-, \rho^-, p^-),
\]

where $\sigma(u^-, \rho^-, p^-)$ is defined in (2.3) and

\[
\sigma(S^-, \Lambda^-, \varphi^-, \psi^-) := \|(S^-, \Lambda^-) - (S^-_0, 0)\|_{1,\alpha,N} + \|\varphi^- - \varphi^-_0\|_{2,\alpha,N} + \|\psi^- e\theta\|_{2,\alpha,N}
\]

for $(S^-_0, \varphi^-_0)$ defined by (2.1).

Now we derive boundary conditions for $(f,S,\Lambda,\varphi,\psi)$ to satisfy the conditions (b)-(c) in Problem 2.2.

(i) Boundary conditions on $\Gamma_{w,f}^+ \cup \Gamma_{ex}$:

We prescribe boundary conditions for $(\varphi,\psi)$ on $\Gamma_{w,f}^+ \cup \Gamma_{ex}$ as

\[
\begin{aligned}
&\partial_r \varphi = 0 \quad \text{and} \quad \psi = 0 \quad \text{on } \Gamma_{w,f}^+, \\
&\varphi = \varphi^+_0 \quad \text{and} \quad \partial_r \psi = 0 \quad \text{on } \Gamma_{ex}
\end{aligned}
\]

(3.9)

for $\varphi^+_0$ given by (2.1) so that the boundary condition (b) in Problem 2.2 holds on $\Gamma_{w,f}^+ \cup \Gamma_{ex}$.

(ii) The Rankine-Hugoniot conditions on $\mathcal{S}_f$:

In terms of $(f,S,\Lambda,\varphi,\psi)$, the Rankine-Hugoniot conditions in (2.2) become

\[
\begin{aligned}
q \cdot \tau_f &= u^- \cdot \tau_f, \\
q \cdot e\theta &= u^- \cdot e\theta, \\
H(S,q) \cdot (q \cdot n_f) &= \rho^- (u^- \cdot n_f), \\
H(S,q) (q \cdot n_f)^2 + SH^+(S,q) &= \rho^- (u^- \cdot n_f)^2 + p^- 
\end{aligned}
\]

(3.10)-(3.12)

for

\[
\tau_f = \frac{f'(r)e_x + e_r}{\sqrt{1 + |f'(r)|^2}}, \quad n_f = \frac{e_x - f'(r)e_r}{\sqrt{1 + |f'(r)|^2}}.
\]

We derive the conditions of $(f,S,\Lambda,\varphi,\psi)$ to satisfy (3.10)-(3.12).
On $S_f$, if $(f, \Lambda, \varphi, \psi)$ satisfy

$$\varphi = \varphi^-, \quad \Lambda = \Lambda^-,$$

$$- \nabla (\varphi e_\theta) \cdot n_f = \left( \frac{\left( -\frac{1}{r} \psi + \frac{1}{r} \psi^- + \partial_r \psi^- \right) f'(r) - \partial_x \psi^-}{\sqrt{1 + |f'(r)|^2}} \right) e_\theta,$$  \hspace{1cm} (3.13)

then we have

$$\nabla \varphi \cdot \tau_f = \nabla \varphi^- \cdot \tau_f, \quad t \cdot \tau_f = \frac{1}{r} \partial_r (r \psi^-) f'(r) - \partial_x \psi^-, \quad q \cdot e_\theta = \frac{\Lambda^-}{r},$$

from which we get the conditions in (3.10). Note that the first condition in (3.13) is equivalent to

$$f(r) = \frac{(\varphi - \varphi_0^\circ) - (\varphi^- - \varphi_0^-)}{u_0^- - u_0^+(f(r), r)} \text{ for } 0 \leq r \leq 1.$$  

This will be used to find the location of the transonic shock $S_f$ (See Lemma 4.1).

On $S_f$, if $(f, S, \Lambda, \varphi, \psi)$ satisfy

$$\nabla \varphi \cdot n_f = \frac{K_s(f')}{u^- \cdot n_f} - t \cdot n_f,$$

$$S = \left( \rho^-(u^- \cdot n_f)^2 + p^- - \rho^- K_s(f') \right) \left( \frac{\rho^-(u^- \cdot n_f)^2}{K_s(f')} \right)^{-\gamma},$$  \hspace{1cm} (3.14)

for

$$K_s(f') := \frac{2(\gamma - 1)}{\gamma + 1} \left( \frac{1}{2} |u^- \cdot n_f|^2 + \frac{\gamma p^-}{(\gamma - 1) \rho^-} \right),$$  \hspace{1cm} (3.15)

then one can directly check that

$$q \cdot n_f = \frac{K_s(f')}{u^- \cdot n_f},$$  \hspace{1cm} (3.16)

$$\rho^- K_s(f') + S \left( \frac{\rho^-(u^- \cdot n_f)^2}{K_s(f')} \right)^\gamma = \rho^- (u^- \cdot n_f)^2 + p^-.$$  \hspace{1cm} (3.17)

By (3.14)-(3.16) and the definition of $H$ in (3.5), we have

$$H(S, q) = \frac{\rho^-(u^- \cdot n_f)^2}{K_s(f')} \text{ on } S_f.$$  \hspace{1cm} (3.18)

Then it follows from (3.16) and (3.18) that the condition in (3.11) holds. Finally, the condition in (3.12) holds by (3.11) and (3.16)-(3.18).

We gather all the boundary conditions for $(f, S, \Lambda, \varphi, \psi)$ on $S_f$ as follows:

$$\begin{cases}
S = S_{sh}(f'), \quad \Lambda = \Lambda^-, \quad \varphi = \varphi^-,

- \nabla \varphi \cdot n_f = - \frac{K_s(f')}{u^- \cdot n_f} + t \cdot n_f,

- \nabla (\psi e_\theta) \cdot n_f = A(r, \psi, f') e_\theta \quad \text{on } S_f,
\end{cases}$$  \hspace{1cm} (3.19)
with
\[ S_{sh}(f') := \left( \rho^{-1}(u^- \cdot n_f)^2 + p^- - \rho^{-1}K_s(f') \right) \left( \frac{\rho^{-1}(u^- \cdot n_f)^2}{K_s(f')} \right)^{-\gamma}, \]
\[ A(r, \psi, f') := \frac{-\frac{1}{2} \psi + \frac{1}{2} \psi^2 + \partial_x \psi^-}{\sqrt{1 + |f'(r)|^2}}. \] (3.20)

**Theorem 3.1.** Let \((u^-, \rho^-, p^-)\) be from Problem 2.1. For simplicity of notations, let \(\sigma\) denote \(\sigma(u^-, \rho^-, p^-)\) defined in (2.3). Then, for any \(\alpha \in (\frac{1}{6}, 1)\), there exists a small constant \(\sigma_2 > 0\) depending only on \((u^+_0, \rho^+_0, p^+_0, \gamma, \alpha)\) so that if \(\sigma \leq \sigma_2\), then the free boundary problem (3.4) with boundary conditions (3.9) and (3.19) has an axisymmetric solution \((f, S, \Lambda, \phi, \psi)\) that satisfies
\[ \|f\|_{2, \alpha, B_1(0)} + \|(S, \Lambda) - (S^+_0, \phi)\| \leq C\sigma, \] (3.21)
where \((S^+_0, \phi^+_0)\) are given in (2.1) and the constant \(C > 0\) depends only on \((u^+_0, \rho^+_0, p^+_0, \gamma, \alpha)\).

Hereafter, a constant \(C\) is said to be chosen depending only on the data if it is chosen depending only on \((u^+_0, \rho^+_0, p^+_0, \gamma, \alpha)\). Unless otherwise specified, each estimate constant \(C\) is regarded to be depending only on the data for the rest of the paper.

### 4. Proof of Theorem 3.1

This section is devoted to proving Theorem 3.1 by using the method of iteration. In Section 4.1, we first define iteration sets and prove one lemma which concerns the location of the transonic shock. In Sections 4.2, 4.3, we prove two lemmas that will be used to prove Theorem 3.1. Finally, in Section 4.4, we prove Theorem 3.1.

#### 4.1. Iteration sets.
For a fixed \(\alpha \in (\frac{1}{6}, 1)\), we define iteration sets as follows:

(i) Iteration set for \(\phi - \phi^+_0\) (=: \(\phi\)): For \(M_1 > 0\) to be determined later, we define
\[ \mathcal{I}(M_1) := \left\{ \phi = \phi(x, r) \in C^{2, \alpha}_{(-1, -\alpha, \Gamma_w)}(\mathcal{N}_{-1/2}) : \|\phi\|_{2, \alpha, \mathcal{N}_{-1/2}} \leq M_1 \sigma \right\}, \] (4.1)
where \(\mathcal{N}_{-1/2} := \mathcal{N} \cap \{-\frac{1}{2} < x < 1\}\).

(ii) Iteration set for \((S, \Lambda)\): For \(M_2 > 0\) to be determined later, we define
\[ \mathcal{I}(M_2) := \mathcal{I}_1(M_2) \times \mathcal{I}_2(M_2) \] (4.2)
Proof. For the definitions of $\phi$ and $\psi$ in Lemma 4.1, we have $\|S - S_0\|_{1, \alpha, \mathcal{N}_{1/2}}^+ \leq M_2 \sigma$.

Moreover, $f$ satisfies the estimate

$$\|f\|_{2, \alpha, \Omega^+_{1/2}} \leq M_3 \sigma,$$

where $\Omega^+_{1/2}$ is a two-dimensional rectangular domain defined by

$$\Omega^+_{1/2} := \left\{ (x, r) \in \mathbb{R}^2 : -\frac{1}{2} < x < 1, 0 < r < 1 \right\}.$$

(iii) Iteration set for $\psi$: For $M_3 > 0$ to be determined later, we define

$$I(M_3) := \left\{ \psi(x, r) \in C^{2, \alpha}_{(-1, \{r=1\})}({\Omega^+_{1/2}}) : \psi(x, 1) = 0, \partial_r^k \psi(x, 0) = 0 \text{ for } k = 0, 2, \forall x \in \left[-\frac{1}{2}, 1\right] \right\}.$$

Lemma 4.1. Suppose that

$$\sigma \leq \min \left\{ \frac{u_0^- - u_0^+}{3M_1}, \frac{u_0^- - u_0^+}{4(M_1 + 1)} \right\} =: \sigma_3. \tag{4.4}$$

Then, for each $\phi \in I(M_1)$, there exists a radial function $f : B_1(0) \to (-\frac{1}{4}, \frac{1}{4})$ such that

$$f(r) = \frac{\phi - (\varphi^- - \varphi^-)}{u_0^- - u_0^+}(f(r), r).$$

Moreover, $f$ satisfies the estimate

$$\|f\|_{2, \alpha, \partial B_1(0)} \leq C(1 + M_1) \sigma$$

for a constant $C > 0$ depending only on the data.

Proof. For $\sigma \leq \sigma_3$, one can directly check that

$$\partial_r(f + \varphi_0^+ - \varphi^-) = \partial_r \phi + \partial_r(\varphi_0^+ - \varphi^-) + \partial_r(\varphi^- - \varphi^-) < 0.$$
Then, it follows from (4.4) and (4.6) that
\[
\|f\|_{L^2(\Omega)} \leq \frac{1}{u_0 - u_0^*} (M_1 \sigma + \sigma) \leq \frac{1}{4}.
\]
The proof is completed. \hfill \Box

4.2. **Free boundary problem for \( \varphi \).** In this subsection, we solve the following free boundary problem.

**Problem 4.1.** For each \((S_*, \Lambda_*, \psi_*) \in \mathcal{I}(M_2) \times \mathcal{I}(M_3)\), set
\[
\mathbf{q}_* := (q(r, \psi_*, D\psi_*, D\varphi, \Lambda_*), \; \mathbf{t}_* := (t(r, \psi_*, D\psi_*, \Lambda_*)
\]
for \((q, t)\) given by (3.2) and (3.3), respectively. Find \((f, \varphi)\) satisfying
\[
\begin{align*}
\text{div}(H(S_*, \mathbf{q}_*)\mathbf{q}_*) &= 0 \quad \text{in} \quad \mathcal{N}_f^+, \\
\varphi &= \varphi^-, \quad \nabla \varphi \cdot \mathbf{n}_f = -\frac{K_*(f')}{u^* \cdot \mathbf{n}_f} + \mathbf{t}_* \cdot \mathbf{n}_f \quad \text{on} \quad \mathcal{S}_f, \\
\partial_r \varphi &= 0 \quad \text{on} \quad \Gamma_{w,f}, \\
\varphi &= \varphi^+_0 \quad \text{on} \quad \Gamma_{ex},
\end{align*}
\]
where \( H \) and \( K_*(f') \) are given in (3.5) and (3.15), respectively.

**Lemma 4.2.** There exists a small constant \( \sigma_4 > 0 \) depending only on the data and \((M_2, M_3)\) so that if
\[
\sigma \leq \sigma_4,
\]
then, for each \((S_*, \Lambda_*, \psi_*) \in \mathcal{I}(M_2) \times \mathcal{I}(M_3)\), Problem 4.1 has a unique axisymmetric solution \((f, \varphi)\) that satisfies
\[
\|f\|_{L^2(\Omega)} + \|\varphi - \varphi^+_0\|_{L^2(\Omega)} \leq C(1 + M_2 + M_3)\sigma,
\]
where the constant \( C > 0 \) depends only on the data.

**Proof.** The proof of Lemma 4.2 is divided into five steps.

1. Fix \( \phi_* \in \mathcal{I}(M_1) \). By Lemma 4.1, if \( \sigma \leq \sigma_3 \), then there exists a radial function \( f : B_1(0) \rightarrow (-\frac{1}{4}, \frac{1}{4}) \) such that
\[
f(r) = \phi_* - \frac{(\varphi^- \varphi^+_0)}{u_0 - u_0^*}(f(r), r).
\]

2. (Linearized boundary value problem for \( \varphi \)) For a function \( H \) defined in (3.5), we define \( \tilde{H} \) and \( \mathbf{A} = (A_1, A_2, A_3) \) as follows:
\[
\tilde{H}(\xi, \mathbf{s}, \mathbf{v}) := H(\xi, \mathbf{s} + \mathbf{v}), \quad A_j(\xi, \mathbf{s}, \mathbf{v}) := \tilde{H}(\xi, \mathbf{s}, \mathbf{v})s_j \quad (j = 1, 2, 3),
\]
for \( \xi \in \mathbb{R}, \; s = (s_1, s_2, s_3) \in \mathbb{R}^3, \) and \( \mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3 \). Let us set
\[
\mathbf{V}_0 := (S_0^+, D\varphi^+_0, 0) \quad \text{and} \quad a_{ij} := \partial s_j A_i (\mathbf{V}_0) \quad \text{for} \quad i, j = 1, 2, 3.
\]
Then the constant matrix $[a_{ij}]_{i,j=1}^{3}$ is diagonal and there exists a positive constant $\nu \in (0, 1/10]$ satisfying
\[
0 < \nu < a_{ii} < \frac{1}{\nu} \quad \text{for all} \quad i = 1, 2, 3.
\]

We set $\phi := \varphi - \varphi_0^+$ and rewrite the equation $\text{div} \ (H(S, q)q) = 0$ as
\[
\mathcal{L}(\phi) = \text{div} \mathbf{F}(S - S_0^+, D\phi, t),
\]
where $\mathcal{L}$ and $\mathbf{F} = (F_1, F_2, F_3)$ are defined by
\[
\mathcal{L}(\phi) := \sum_{i=1}^{3} \partial_i (a_{ii} \partial_i \phi),
\]
\[
F_i(Q) := -\tilde{H}(\mathbf{V}_0 + Q)v_i - \int_{0}^{1} D\xi \cdot A_i(\mathbf{V}_0 + tQ) dt \cdot (\xi, \mathbf{v})
- \mathbf{s} \cdot \int_{0}^{1} D_s A_i(\mathbf{V}_0 + tQ) dt - D_s A_i(\mathbf{V}_0),
\]
with $Q = (\xi, \mathbf{s}, \mathbf{v}) \in \mathbb{R} \times (\mathbb{R}^3)^2$. In the above, $\partial_{x_i}$ is abbreviated as $\partial_i$. By the boundary conditions for $\varphi$ in (4.7) and definition of $\varphi_0^+$ given in (2.1), the boundary conditions for $\phi$ on $\partial N_f^+$ become
\[
\begin{cases}
(a_{ii} \partial_i \phi) \cdot (-\mathbf{n}_f) = \mathcal{B}(t_*, f', \nabla \phi) & \text{on } \mathcal{S}_f, \\
(a_{ii} \partial_i \phi) \cdot \mathbf{e}_r = 0 & \text{on } \Gamma_{w,f}^+, \\
\phi = 0 & \text{on } \Gamma_{ex},
\end{cases}
\]
with
\[
\mathcal{B}(t, f', \nabla \phi) := a_{11} \left( -\frac{K_s(f')}{\mathbf{u} \cdot \mathbf{n}_f} + \mathbf{t} \cdot \mathbf{n}_f + \nabla \varphi_0^+ \cdot \mathbf{n}_f \right) + \sum_{j=2}^{3} (a_{11} - a_{jj}) \partial_j \phi (\mathbf{n}_f \cdot \mathbf{e}_j).
\]

We consider the linear boundary value problem
\[
\begin{cases}
\mathcal{L}(\phi) = \text{div} \mathbf{F}_* & \text{in } N_f^+, \\
(a_{ii} \partial_i \phi) \cdot (-\mathbf{n}_f) = \mathcal{B}_* & \text{on } \mathcal{S}_f, \\
(a_{ii} \partial_i \phi) \cdot \mathbf{e}_r = 0 & \text{on } \Gamma_{w,f}^+, \\
\phi = 0 & \text{on } \Gamma_{ex},
\end{cases}
\]
with
\[
\mathbf{F}_* := \mathbf{F}(S_* - S_0^+, D\phi_*, t_*) \quad \text{and} \quad \mathcal{B}_* := \mathcal{B}(t_*, f', \nabla \phi_*).
\]
The standard elliptic theory (cf. Evans [13], Gilbarg-Trudinger [14]) yields that (4.11) has a unique weak solution $\phi \in H^1(N_f^+)$ satisfying
\[
\mathcal{L}[\phi, \zeta] = \langle (\mathbf{F}_*, \mathcal{B}_*), \zeta \rangle \quad \text{for all } \zeta \in \left\{ \zeta \in H^1(N_f^+) : \zeta = 0 \text{ on } \Gamma_{ex} \right\}
\]
where

\[ \mathcal{L}[\phi, \zeta] := \int_{N_f^+} \sum_{i=1}^{3} a_{ii}(\partial_i \phi)(\partial_i \zeta) dx, \]

\[ \langle (F_*, B_*), \zeta \rangle := \int_{N_f^+} F_* \cdot \nabla \zeta dx - \int_{\partial N_f^+ \setminus \Gamma_{\text{ex}}} (F_* \cdot n_{\text{out}}) \zeta dS + \int_{\partial \Sigma_f} B_* \zeta dS. \]

Furthermore, \( \phi \) satisfies

\[ \| \phi \|_{H^1(\Sigma_f^+)} \leq C \left( \| F_* \|_{0, \alpha, \Sigma_f^+} + \| B_* \|_{0, \alpha, \Sigma_f} \right), \]

3. (Estimate of \( \phi \)) For each \( x_0 \in \Sigma_f^+ \) and \( \eta \in \mathbb{R} \) with \( 0 < \eta < \frac{1}{10} \), let us set

\[ B_\eta(x_0) := \{ x \in \mathbb{R}^3 : |x_0 - x| < \eta \}, \quad D_\eta(x_0) := B_\eta(x_0) \cap \Sigma_f^+, \]

\[ (\nabla \phi)_{x_0, \eta} := \frac{1}{|D_\eta(x_0)|} \int_{D_\eta(x_0)} \nabla \phi dx. \]

Since there exists a constant \( \lambda_0 \in (0, 1/10) \) such that

\[ \lambda_0 \leq \frac{|D_\eta(x_0)|}{|B_\eta(x_0)|} \leq \frac{1}{\lambda_0}, \]

we will follow the proofs of [15] Theorem 3.13 and [1] Lemma 3.5] to get

\[ \int_{D_\eta(x)} |\nabla \phi - (\nabla \phi)_{x_0, \eta}|^2 dx \leq C \left( \| F_* \|_{1, \alpha, \Sigma_f^+}^{(-\alpha + \frac{1}{2}, f)} + \| B_* \|_{1, \alpha, \Sigma_f}^{(-\alpha, \partial \Sigma_f)} \right)^2 \eta^{3+2\alpha} \tag{4.12} \]

for any \( x \in \Sigma_f^+ \). Once (4.12) is proved, we obtain from [15] Theorem 3.1] that

\[ \| \phi \|_{1, \alpha, \Sigma_f^+} \leq C \left( \| F_* \|_{1, \alpha, \Sigma_f^+}^{(-\alpha + \frac{1}{2}, f)} + \| B_* \|_{1, \alpha, \Sigma_f}^{(-\alpha, \partial \Sigma_f)} \right). \tag{4.13} \]

By the scaling argument and Schauder estimate with (4.13), we get

\[ \| \phi \|_{2, \alpha, \Sigma_f^+} \leq C \left( \| F_* \|_{1, \alpha, \Sigma_f^+}^{(-\alpha + \frac{1}{2}, f)} + \| B_* \|_{1, \alpha, \Sigma_f}^{(-\alpha, \partial \Sigma_f)} \right). \tag{4.14} \]

We prove (4.12) only for the case \( x \in \Gamma_{w,f} \cap \Sigma_f \). Fix \( x_0 \in \Gamma_{w,f} \cap \Sigma_f \) and \( \chi \in \mathbb{R} \) with \( 0 \leq \chi < \frac{1}{10} \). Let \( \phi_h \) be a weak solution of the following problem

\[
\begin{cases}
\mathcal{L}(\phi_h) = 0 & \text{in } D_{\chi}(x_0), \\
(a_i \partial_i \phi_h) \cdot (-n_f) = B_*(x_0) & \text{on } \partial D_{\chi}(x_0) \cap \Sigma_f, \\
(a_i \partial_i \phi_h) \cdot e_r = 0 & \text{on } \partial D_{\chi}(x_0) \cap \Gamma_{w,f}, \\
\phi_h = \phi & \text{on } \partial D_{\chi}(x_0) \cap N_f^+. 
\end{cases}
\]

Then \( \phi_{nh} := \phi - \phi_h \) satisfies

\[
\int_{D_{\chi}(x_0)} a_{ij} \partial_i \phi_{nh} \partial_j \zeta dx = \int_{D_{\chi}(x_0)} (\text{div} F_*) \zeta dx + \int_{\partial D_{\chi}(x_0) \cap \Sigma_f} (B_* - B_*(x_0)) \zeta dS
\]
for any $\xi \in \{\zeta \in \mathcal{H}^1(D_{x_0}) : \zeta = 0 \text{ on } \partial D_{x_0} \cap \mathcal{N}^+_f \}$. By taking the test function $\xi = \phi_{nh}$ and using the Hölder inequality, Sobolev inequality, and the Poincaré inequality, we get

$$\int_{D_{x_0}} |\nabla \phi_{nh}|^2 \, dx$$

$$\leq C \left( \int_{D_{x_0}} |\text{div} F_*|^{6/5} \, dx \right)^{5/3} + C \left( \chi \int_{\partial D_{x_0} \cap \mathcal{E}_f} |\mathfrak{B}_* - \mathfrak{B}_*(x_0)|^2 \, dS \right)$$

$$=: (i) + (ii).$$

(4.15)

Since $\text{div} F_* \in C_{0, \alpha, \Gamma_{w,f}}^{0, \alpha, \Gamma_{w,f}^+}(\mathcal{N}^+_f)$, the definition of the weighted Hölder norms yields that

$$|\text{div} F_*(x)| \leq \delta_x^{-1+\alpha} ||\text{div} F_*||_{0, \alpha, \Gamma_{w,f}^+}^{1-\alpha} \text{ for } x \in \mathcal{N}^+_f, \quad \delta_x = \inf_{z \in \Gamma_{w,f}^+} |x - z|,$$

from which we obtain that

$$(i) \leq C \chi^{3+2\alpha} \left( ||\text{div} F_*||_{0, \alpha, \mathcal{N}^+_f}^{1-\alpha} \right)^2 \text{ for } \alpha \in \left( \frac{1}{6}, 1 \right).$$

(4.16)

Since $\mathfrak{B}_* \in C_{(-\alpha, \partial \mathcal{E}_f)}^{1, \alpha}(\mathcal{G}_f)$, the definition of the weighted Hölder norms yields that

$$(ii) \leq C \chi^{3+2\alpha} \left( ||\mathfrak{B}_*||_{1, \alpha, \mathcal{E}_f}^{(-\alpha, \partial \mathcal{E}_f)} \right)^2.$$

(4.17)

By (4.15)–(4.17), we get

$$\int_{D_{x_0}} |\nabla \phi_{nh}|^2 \, dx \leq C \chi^{3+2\alpha} \left( ||\text{div} F_*||_{0, \alpha, \mathcal{N}^+_f}^{1-\alpha} + ||\mathfrak{B}_*||_{1, \alpha, \mathcal{E}_f}^{(-\alpha, \partial \mathcal{E}_f)} \right)^2.$$

Then [13 Corollary 3.11] implies for $0 < \eta < \chi$

$$\int_{D_{x_0}} |\nabla \phi - (\nabla \phi)_{x_0, \eta}|^2 \, dx \leq C \left( \frac{\eta}{\chi} \right)^5 \int_{D_{x_0}} |\nabla \phi - (\nabla \phi)_{x_0, \eta}|^2 \, dx$$

$$+ C \chi^{3+2\alpha} \left( ||\text{div} F_*||_{0, \alpha, \mathcal{N}^+_f}^{1-\alpha} + ||\mathfrak{B}_*||_{1, \alpha, \mathcal{E}_f}^{(-\alpha, \partial \mathcal{E}_f)} \right)^2.$$

According to [13 Lemma 3.4], we can replace $\chi^{3+2\alpha}$ by $\eta^{3+2\alpha}$. Hence the proof of (4.12) is completed and we finally get the weighted $C^{2,\alpha}$-estimate (4.14) of $\phi$.

By direct computations, one can check that there exists a constant $\epsilon_1 \in (0, \frac{1}{4})$ depending only on the data so that if

$$M_1 \sigma + M_2 \sigma + M_3 \sigma \leq \epsilon_1,$$
then we have
\[ \|F_*\|_{1,0,N^+_{f,j}} (\alpha, \Gamma, \omega) \leq C (M_1 \sigma)^2 + M_2 \sigma + M_3 \sigma), \]  
\[ \|B_*\|_{1,0,\mathcal{E}_j} (\alpha, \Gamma, \omega) \leq C (\sigma + (M_1 \sigma)^2 + M_2 \sigma + M_3 \sigma). \]  
\[ (4.18) \]

It follows from (4.14) and (4.18) that
\[ \|\phi\|_{2,0,N^+_{f,j}} (\alpha, \Gamma, \omega) \leq C (\sigma + (M_1 \sigma)^2 + M_2 \sigma + M_3 \sigma). \]  
\[ (4.19) \]

For any \( \theta \in [0, 2\pi) \), define a function \( \phi^\theta \) by
\[ \phi^\theta(x_1, x_2, x_3) := \phi(x_1, x_2 \cos \theta - x_3 \sin \theta, x_2 \sin \theta + x_3 \cos \theta). \]

Since \( a_{22} = a_{33} \) by definitions of \( a_{ij} \) in (4.9), \( \phi^\theta \) is a solution to (4.11). The uniqueness of a solution to (4.11) implies that \( \phi = \phi^\theta \). Hence \( \phi \) is axisymmetric.

4. (Extension of \( \phi \) into \( N^+_{-1/2} \)) To set an iteration mapping, we need to extend \( \phi \) into \( C^0_{(-1, -\Gamma, \omega)}(N^+_{-1/2}) \). Define an extension function \( E_\theta(\phi) \) of \( \phi \) by
\[ E_\theta(\phi) := E_\theta(\phi) \circ T_\theta, \]  
\[ (4.20) \]

where \( T_\theta : N^+_{2j-1} \rightarrow N \) \( is an invertible transformation defined by \)
\[ T_\theta(x_1, x_2, x_3) = \left( \frac{1}{1 - f(r)}(x_1 - 1) + 1, x_2, x_3 \right) \]

and \( E_\theta(\phi) : N \rightarrow \mathbb{R} \) \( is a function defined by \)
\[ E_\theta(\phi)(y_1, y_2, y_3) := \begin{cases} \phi \circ T_\theta^{-1}(y_1, y_2, y_3) & \text{for } 0 \leq y_1 < 1, \\ \sum_{i=1}^{3} c_i \left( \phi \circ T_\theta^{-1} \right) \left( -\frac{y_1}{i}, y_2, y_3 \right) & \text{for } -1 < y_1 < 0, \end{cases} \]

where \( c_1 = 6, c_2 = -32, \) and \( c_3 = 27, \) which are constants determined by the system of equations
\[ \sum_{i=1}^{3} c_i \left( -\frac{1}{i} \right)^m = 1, \quad m = 0, 1, 2. \]

Since \( |f| < \frac{1}{2} \), the transformation \( T_\theta \) is well-defined and \( N^+_{-1/2} \subset N^+_{2j-1} \). By a direct computation, we have
\[ \|E_\theta(\phi)\|_{2,0,N^+_{-1/2}} (\alpha, \Gamma, \omega) \leq C \|\phi\|_{2,0,N^+_{f,j}} (\alpha, \Gamma, \omega). \]  
\[ (4.21) \]

Then, it follows from (4.19) and (4.21) that
\[ \|E_\theta(\phi)\|_{2,0,N^+_{-1/2}} \leq C_* (\sigma + (M_1 \sigma)^2 + M_2 \sigma + M_3 \sigma) \]

for a constant \( C_* > 0 \) depending only on the data.
5. For fixed \((S_*, \Lambda_*, \psi_*) \in \mathcal{I}(M_2) \times \mathcal{I}(M_3)\), define an iteration mapping \(\mathcal{J}^{(S_*, \Lambda_*, \psi_*)}: \mathcal{I}(M_1) \rightarrow C^{2,\alpha/2}_{(-1-\alpha, \Gamma_\omega)}(\mathcal{N}^+_{-1/2})\) by

\[
\mathcal{J}^{(S_*, \Lambda_*, \psi_*)}(\phi_*) = \mathcal{E}_f(\phi),
\]

where \(\phi\) is the solution to (4.11) associated with \(\phi_*\). We choose constants \(M_1\) and \(\sigma_*^+\) as

\[
M_1 := 4\epsilon_0 + 4\epsilon_0^* M_2 + 4\epsilon_0^* M_3 \quad \text{and} \quad \sigma_*^+ := \min\left\{\sigma_3, \frac{1}{4\epsilon_0^* M_1}, \frac{\epsilon_1}{M_1 + M_2 + M_3}\right\}
\]

with \(\sigma_3\) defined in (4.4) so that the mapping \(\mathcal{J}^{(S_*, \Lambda_*, \psi_*)}\) maps \(\mathcal{I}(M_1)\) into itself whenever \(\sigma \leq \sigma_*^+\).

The iteration set \(\mathcal{I}(M_1)\) given in (4.1) is a convex and compact subset of \(C^{2,\alpha/2}_{(-1-\alpha, \Gamma_\omega)}(\mathcal{N}^+_{-1/2})\). By using the uniqueness of a solution of (4.11), one can show that the iteration mapping \(\mathcal{J}^{(S_*, \Lambda_*, \psi_*)}\) is continuous in \(C^{2,\alpha/2}_{(-1-\alpha, \Gamma_\omega)}(\mathcal{N}^+_{-1/2})\). Then the Schauder fixed point theorem implies that \(\mathcal{J}^{(S_*, \Lambda_*, \psi_*)}\) has a fixed point \(\phi_0 \in \mathcal{I}(M_1)\). For such \(\phi_0\), there exists a radial function \(f: B_1(0) \rightarrow (-\frac{1}{4}, \frac{1}{4})\) satisfying

\[
f(r) = \frac{\phi_0 - (\varphi^0 - \varphi_0^*)}{u_0 - u_0^*}(f(r), r)
\]

by Lemma 4.1. Then \((f, \phi_0, \varphi^0 + \varphi_0^*)\) solves Problem 4.1 and satisfies the estimate (4.3) for \(\sigma \leq \sigma_*^+\).

Let \((f^{(1)}, \varphi^{(1)})\) and \((f^{(2)}, \varphi^{(2)})\) be two solutions of Problem 4.1 and suppose that each solution satisfies the estimate (4.3). Set

\[
\begin{align*}
\bar{\phi} := (\varphi^{(1)} - \varphi^0) - \left((\varphi^{(2)} - \varphi^0) \circ \mathcal{T}\right), \quad \bar{f} := f^{(1)} - f^{(2)}, \\
\bar{\psi} := \psi_* - \left(\psi_* \circ \mathcal{T}\right), \quad \bar{S} := S_* - (S_* \circ \mathcal{T}), \quad \bar{\Lambda} := \Lambda_* - (\Lambda_* \circ \mathcal{T})
\end{align*}
\]

for an invertible transformation \(\mathcal{T}: \mathcal{N}^\perp_{f^{(1)}} \rightarrow \mathcal{N}^\perp_{f^{(2)}}\) defined by

\[
\mathcal{T}(x_1, x_2, x_3) := \left(\frac{1 - f^{(2)}(r)}{1 - f^{(1)}(r)}(x_1 - 1) + 1, x_2, x_3\right).
\]

Then, a direct computation with (4.23) yields that

\[
\begin{align*}
\|\bar{f}\|_{1, \alpha, B_1(0)} & \leq C\|\bar{\phi}\|_{1, \alpha, B_1(0)} \quad \text{for} \quad \sigma \leq \frac{|u_0^+ - u_0^-|}{3}, \\
\|\bar{\psi}\|_{1, \alpha, \Omega_1} + \|(\bar{S}, \bar{\Lambda})\|_{0, \alpha, \mathcal{N}_1} & \leq C(M_2 + M_3)\sigma \|\bar{f}\|_{1, \alpha, B_1(0)},
\end{align*}
\]

where we set \((\mathcal{N}_1, \Omega_1, \Gamma_1, D_1)\) as

\[
\begin{align*}
\mathcal{N}_1 := \mathcal{N}^\perp_{f^{(1)}}, \quad \Gamma_1 := \partial \mathcal{N}_1 \cap \{r = 1\}, \\
\Omega_1 := \{(x, r) \in \mathbb{R}^2 : f^{(1)}(r) < x < 1, 0 < r < 1\}, \quad D_1 := \partial \Omega_1 \cap \{r = 1\}.
\end{align*}
\]
To prove the uniqueness of a solution, we estimate \( \bar{\phi} \) in \( \mathcal{N}_{f(1)}^{+} \). For that purpose, we first get a boundary value problem for \( \bar{\phi} \) in \( \mathcal{N}_{f(1)}^{+} \) by subtracting the boundary value problem (1.11) for \( (\varphi^{(2)} - \varphi_{0}^{+}) \circ \Sigma \) in \( \mathcal{N}_{f(1)}^{+} \) from the boundary value problem (4.11) for \( (\varphi^{(1)} - \varphi_{0}^{+}) \) in \( \mathcal{N}_{f(1)}^{+} \). By using (4.25) and similar methods used to obtain the estimate (4.14), we get
\[
\|\bar{\phi}\|_{1,\alpha_{N_i}^{+}} \leq C_{**} (1 + M_{2} + M_{3}) \|\bar{\phi}\|_{1,\alpha_{N_i}^{+}}
\] (4.26)
for a constant \( C_{**} > 0 \) depending only on the data. Finally, we choose \( \sigma_{4} \) to be
\[
\sigma_{4} := \min \left\{ \sigma_{4}^{*}, \frac{|u_{0}^{+} - u_{0}^{-}|}{3}, \frac{1}{2C_{**}(1 + M_{2} + M_{3})} \right\}
\] (4.27)
for \( \sigma_{4}^{*} \) defined in (4.22) so that (4.25)-(4.26) imply that \( (f^{(1)}, \varphi^{(1)}) = (f^{(2)}, \varphi^{(2)}) \) for \( \sigma \leq \sigma_{4} \). This finishes the proof of Lemma 4.2.

4.3. Free boundary problem for \( (\varphi, S, \Lambda) \).

Problem 4.2. For each \( \psi_{*} \in \mathcal{I}(M_{3}) \), set
\[
\mathbf{q}^{*} := \mathbf{q}(r, \psi_{*}, D\psi_{*}, D\varphi, \Lambda), \quad \mathbf{t}^{*} := \mathbf{t}(r, \psi_{*}, D\psi_{*}, \Lambda)
\]
for \( (\mathbf{q}, \mathbf{t}) \) given in (3.2) and (3.3), respectively. Find \((f, \varphi, S, \Lambda)\) satisfying
\[
\begin{align*}
H(S, \mathbf{q}^{*})\mathbf{q}^{*} \cdot \nabla(S, \Lambda) &= 0 \quad \text{in} \quad \mathcal{N}_{f}^{+}, \\
(S, \Lambda) &= (S_{sh}(f'), \Lambda^{-}) \quad \text{on} \quad \mathcal{E}_{f},
\end{align*}
\]
and
\[
\begin{align*}
\text{div} \left( H(S, \mathbf{q}^{*})\mathbf{q}^{*} \right) &= 0 \quad \text{in} \quad \mathcal{N}_{f}^{+}, \\
\varphi &= \varphi^{*}, \quad -\nabla \varphi \cdot \mathbf{n}_{f} = -\frac{K_{s}(f')}{\mathbf{u} \cdot \mathbf{n}_{f}} + \mathbf{t}^{*} \cdot \mathbf{n}_{f} \quad \text{on} \quad \mathcal{E}_{f}, \\
\partial_{r} \varphi &= 0 \quad \text{on} \quad \Gamma_{w,f}^{+}, \\
\varphi &= \varphi_{0}^{+} \quad \text{on} \quad \Gamma_{ex},
\end{align*}
\] (4.28)
where \( H, S_{sh}, \) and \( K_{s}(f') \) are given in (3.5), (3.20), and (3.13), respectively.

Lemma 4.3. There exists a small constant \( \sigma_{5} > 0 \) depending only on the data and \( M_{3} \) so that if
\[
\sigma \leq \sigma_{5},
\]
then, for each \( \psi_{*} \in \mathcal{I}(M_{3}) \), Problem 4.2 has a unique axisymmetric solution \((f, \varphi, S, \Lambda)\) that satisfies
\[
\begin{align*}
\|f\|_{1,\alpha(B_{1}(0))} + \|\varphi - \varphi_{0}^{+}\|_{1,\alpha_{N_{f}^{+}}^{+}} &\leq C(1 + M_{3}) \sigma, \\
\|(S, \Lambda) - (S_{0}^{+}, 0)\|_{1,\alpha_{N_{f}^{+}}^{+}} &\leq C(1 + M_{3}) \sigma,
\end{align*}
\] (4.29)
where the constant \( C > 0 \) depends only on the data.
Proof: The proof of Lemma 4.3 is divided into three steps.

1. Fix \((S_*, \Lambda_*) \in \mathcal{I}(M_2)\). By Lemma 4.2 if \(\sigma \leq \sigma_4\), then there exists a unique axisymmetric solution \((f, \varphi)\) satisfying (4.28) associated with \((S, \Lambda) = (S_*, \Lambda_*)\). Moreover, the solution \((f, \varphi)\) satisfies

\[
\|f\|_{2, \infty, B_1(0)} + \|\varphi - \varphi_0^+\|_{2, \infty, \mathcal{N}_f^+} \leq C(1 + M_2 + M_3)\sigma. \tag{4.30}
\]

2. (Initial value problem for \((S, \Lambda)\)) We find a solution \((S, \Lambda)\) of the following initial value problem:

\[
\begin{cases}
H(S_*, q_*)q_\cdot \nabla(S, \Lambda) = 0 & \text{in } \mathcal{N}_f^+,
(S, \Lambda) = (S_{sh}(f'), \Lambda^-) & \text{on } \partial \mathcal{N}_f^+,
\end{cases} \tag{4.31}
\]

with \(q_* := q(r, \psi_*, D\psi_*, D\varphi_*, \Lambda_*)\). To solve this problem, we apply the proof of [5, Proposition 3.5]. For that purpose, we first rewrite (4.31) as a problem defined in \(\mathcal{N}_0^+ := \mathcal{N} \cap \{x > 0\}\) by using the change of variables with a flattening map \(\Phi_f: \mathcal{N}_f^+ \longrightarrow \mathcal{N}_0^+\) defined by

\[
\Phi_f(x_1, x_2, x_3) = \left(\frac{x_1 - 1}{1 - f(r)} + 1, x_2, x_3\right).
\]

Since \(|f| \leq \frac{1}{4}\), \(\Phi_f\) is invertible and

\[
\Phi_f^{-1}(y_1, y_2, y_3) = ((y_1 - 1)(1 - f(t)) + 1, y_2, y_3),
\]

where \((y, t, \theta)\) denote the cylindrical coordinates of \((y_1, y_2, y_3) \in \mathcal{N}_0^+\), i.e.,

\[
(y_1, y_2, y_3) = (y, t \cos \theta, t \sin \theta), \quad t \geq 0, \quad \theta \in \mathbb{T}.
\]

For \(y \in \mathcal{N}_0^+\) and \(y_0 \in \partial \mathcal{N}_0^+ \cap \{y = 0\}\), set

\[
(S^*, \Lambda^*)(y) := (S, \Lambda) \circ \Phi_f^{-1}(y), \quad (S_{sh}, \Lambda_{sh})(y_0) := (S_{sh}(f'), \Lambda^-) \circ \Phi_f^{-1}(y_0),
\]

\[
M_x(y) := (H(S_*, q_*)q_* \cdot e_x) \circ \Phi_f^{-1}(y), \quad M_r(y) := (H(S_*, q_*)q_* \cdot e_r) \circ \Phi_f^{-1}(y),
\]

(4.32)

to rewrite the initial value problem (4.31) as follows:

\[
\begin{cases}
(N_y \partial_y + N_t \partial_t)(S^*, \Lambda^*) = 0 & \text{in } \mathcal{N}_0^+,
(S^*, \Lambda^*) = (S_{sh}, \Lambda_{sh}) & \text{on } \partial \mathcal{N}_0^+ \cap \{y = 0\},
\end{cases} \tag{4.33}
\]

with

\[
N_y := M_x + (y - 1)f'(t)M_r \quad \text{and} \quad N_t := (1 - f(t))M_r.
\]

In the below (i)-(iii), we check that the sufficient conditions to apply [5, Proposition 3.5] are hold:

(i) From the equation \(\text{div}(H(S_*, q_*)q_*) = 0\) in \(\mathcal{N}_f^+\), one can directly check that

\[
\partial_y(tN_y) + \partial_t(tN_t) = 0 \quad \text{in } \mathcal{N}_0^+. \tag{4.34}
\]
(ii) Since \((S_\ast, \Lambda_\ast, \psi_\ast) \in \mathcal{I}(M_2) \times \mathcal{I}(M_3)\) and the estimate \([4.30]\) holds, there exists a constant \(\epsilon_2 > 0\) depending only on the data so that if
\[
(M_2 + M_3)\sigma \leq \epsilon_2,
\]
then we have
\[
\|N_y - \rho_0^+ u_0^+\|_{0, \mathcal{N}_0^+} \leq C_\sigma (1 + M_2 + M_3)\sigma
\]
for a constant \(C_\sigma > 0\) depending only on the data. Thus
\[
0 < \frac{\rho_0^+ u_0^+}{2} \leq N_y \leq \frac{3\rho_0^+ u_0^+}{2} \quad \text{in } \overline{\mathcal{N}_0^+}
\]
for
\[
\sigma \leq \min \left\{ \frac{\epsilon_2}{M_2 + M_3}, \frac{\rho_0^+ u_0^+}{2C_\sigma (1 + M_2 + M_3)} \right\} =: \tilde{\sigma}_5.
\]
(iii) The conditions
\[
\partial_r \varphi = 0 \quad \text{and} \quad \psi_\ast = 0 \quad \text{on } \Gamma^+_{w, f} \cup (\mathcal{N}_f^+ \cup \{r = 0\})
\]
imply that
\[
N_t \equiv 0 \quad \text{on } \left( \partial \mathcal{N}_0^+ \cap \{t = 1\} \right) \cup (\mathcal{N}_0^+ \cap \{t = 0\}).
\]
Now we regard \([4.33]\) as a problem defined in a two-dimensional rectangular domain \(\Omega^+: = \{(y, t) \in \mathbb{R}^2 : 0 < y < 1, 0 < t < 1\}\), and apply \([5, \text{Proposition 3.5}]\). Then the initial value problem \([4.33]\) has the unique solution \((S^\ast, \Lambda^\ast)\) defined by
\[
(S^\ast, \Lambda^\ast)(y, t) := (\widetilde{S}_{sh}, \widetilde{\Lambda}_{sh})(\mathcal{R}_{sh}(y, t)) \quad \text{for } (y, t) \in \overline{\Omega^+}.
\]
Here, \(\mathcal{R}_{sh} : \overline{\Omega^+} \rightarrow [0, 1]\) is a function defined by
\[
\mathcal{R}_{sh}(y, t) := G^{-1} \circ w(y, t)
\]
for \(w : \overline{\Omega^+} \rightarrow \mathbb{R}^+\) given by
\[
w(y, t) := \int_0^t z N_y(y, z) dz
\]
and an invertible function \(G : [0, 1] \rightarrow [w(0, 0), w(0, 1)]\) given by
\[
G(r) := w(0, r).
\]
By \([4.35]\) and the definition of \(\mathcal{R}_{sh}\), there exists a constant \(\mu \in (0, 1)\) such that
\[
\mu \leq \frac{\mathcal{R}_{sh}(y, 1) - \mathcal{R}_{sh}(y, t)}{1 - t} = \frac{1 - \mathcal{R}_{sh}(y, t)}{1 - t} \leq \frac{1}{\mu}.
\]
By \([4.39]\) and \([5, \text{Proposition 3.5}]\), the solution \((S^\ast, \Lambda^\ast)\) satisfies
\[
\|(S^\ast, \Lambda^\ast) - (S_0^+, 0)\|_{1, \alpha, \Omega^+} \leq C\|(\widetilde{S}_{sh}, \widetilde{\Lambda}_{sh}) - (S_0^+, 0)\|_{1, \alpha, \partial \Omega^+ \cap \{y = 0\}}.
\]
Then $(S, \Lambda) := (S^*, \Lambda^*) \circ \mathcal{T}_f$ is the unique solution of the initial value problem \eqref{4.31} and the solution satisfies

$$
\|(S, \Lambda) - (S_0^+, 0)\|_{\|\cdot\|_{1,\alpha, \mathcal{N}_f^+}} \leq C\|(S_{ab}(f'), \Lambda^\cdot) - (S_0^+, 0)\|_{\|\cdot\|_{1,\alpha, \mathcal{E}_f}}. \tag{4.40}
$$

3. For a fixed $\psi_\ast \in \mathcal{I}(M_3)$, define an iteration mapping $\mathcal{J}^{\psi_\ast} : \mathcal{I}(M_2) \rightarrow [\mathcal{C}(\alpha, \mathcal{N}_f^+)^+\mathcal{N}_f^-_{-1/2}]^2$ by

$$
\mathcal{J}^{\psi_\ast}(S_\ast, \Lambda_\ast) = (\mathcal{E}_f(S), \mathcal{E}_f(\Lambda))
$$

where $\mathcal{E}_f$ is given by \eqref{4.20} and $(S, \Lambda)$ is the solution to \eqref{4.31} associated with $(S_\ast, \Lambda_\ast)$.

A direct computation with using \eqref{4.40} yields that

$$
\|(\mathcal{E}_f(S), \mathcal{E}_f(\Lambda)) - (S_0^+, 0)\|_{\|\cdot\|_{1,\alpha, \mathcal{N}_f^+}} \leq C_{\mathcal{H}} (\sigma + (M_1 \sigma)^2)
$$

for $M_1 > 0$ given by \eqref{4.22} and $C_{\mathcal{H}} > 0$ depending only on the data.

For further estimate, set $\mathcal{V}$ as

$$
\mathcal{V}(x, r) := \frac{\mathcal{E}_f(\Lambda)(x, r)}{r} \text{ for } (x, r) \in \Omega^-_{1/2}.
$$

Then one can directly check from \eqref{4.31}, \eqref{4.37} and \eqref{4.38} that

$$
\lim_{r \rightarrow 0^+} \partial_r \mathcal{V} = \lim_{r \rightarrow 0^+} \partial_r \left( \frac{\Lambda^* \circ \mathcal{T}}{r} \right) = 0 \text{ in } \Omega^-_f := \Omega^-_{1/2} \cap \{ x \geq f(r) \}.
$$

By the definition of $\mathcal{E}_f$ given by \eqref{4.20}, $\mathcal{V}$ satisfies

$$
\|\mathcal{V}\|_{\|\cdot\|_{1,\alpha, \Omega^+_{1/2}}} \leq C_{\mathcal{H}_\mathcal{H}} \sigma
$$

for a constant $C_{\mathcal{H}_\mathcal{H}} > 0$ depending only on the data.

Choose $M_2$ and $\sigma_\ast^\ast$ as

$$
M_2 := 2(C_{\mathcal{H}} + C_{\mathcal{H}_\mathcal{H}}) \quad \text{and} \quad \sigma_\ast^\ast := \min \left\{ \sigma_4^\ast, \frac{1}{1 + M_2 + M_3}, \frac{M_2}{2C_{\mathcal{H}} M_2^2} \right\}. \tag{4.41}
$$

for $(\sigma_4^\ast, \sigma_5^\ast)$ given in \eqref{4.27} and \eqref{4.36}. Then, under such choices of $(M_2, \sigma_\ast^\ast)$, the mapping $\mathcal{J}^{\psi_\ast}$ maps $\mathcal{I}(M_2)$ into itself whenever $\sigma \leq \sigma_\ast^\ast$.

The iteration set $\mathcal{I}(M_2)$ defined in \eqref{4.2} is a compact and convex subset of $[\mathcal{C}(\alpha/4, \mathcal{N}_f^+) \mathcal{N}_f^-_{-1/2}]^2$. By Lemma \eqref{4.12} and the uniqueness of a solution for \eqref{4.31}, one can prove that $\mathcal{J}^{\psi_\ast}$ is continuous in $[\mathcal{C}(\alpha/4, \mathcal{N}_f^+) \mathcal{N}_f^-_{-1/2}]^2$. Then the Schauder fixed point theorem yields that there exists a fixed point of $\mathcal{J}^{\psi_\ast}$, say $(S_\bar{\psi}, \Lambda_\bar{\psi})$.

By Lemma \eqref{4.2} there exists a solution $(f, \varphi)$ of the free boundary problem \eqref{4.1} associated with $(S_\ast, \Lambda_\ast) = (S_\bar{\psi}, \Lambda_\bar{\psi})$ and the solution satisfies the estimate \eqref{4.7}. Then $(f, \varphi, S_\bar{\psi}|_{\mathcal{N}_f^+}, \Lambda_\bar{\psi}|_{\mathcal{N}_f^+})$ solves Problem \eqref{4.2} and satisfies the estimate \eqref{4.20} for $\sigma \leq \sigma_\ast^\ast$. 

Let \( U^{(k)} := (f^{(k)}, \varphi^{(k)}, S^{(k)}, \Lambda^{(k)}) \) \( k = 1, 2 \) be two solutions to Problem 1.2 and suppose that each solution satisfies the estimate (4.29). Set

\[
\begin{align*}
\tilde{\psi} &:= \psi - (\psi \circ \overline{T}), \\
\tilde{f} &:= f^{(1)} - f^{(2)}, \\
\tilde{\phi} &:= (\varphi^{(1)} - \varphi^{(2)} - \left( (\varphi^{(2)} - \varphi^{(1)}) \circ \overline{T} \right), \\
\tilde{S} &:= S^{(1)} - (S^{(2)} \circ \overline{T}), \\
\tilde{\Lambda} &:= \Lambda^{(1)} - \left( \Lambda^{(2)} \circ \overline{T} \right)
\end{align*}
\]

for a transformation \( \overline{T} : \mathcal{N}^{+}_{f^{(1)}} \to \mathcal{N}^{+}_{f^{(2)}} \) defined in (4.24). For simplicity of notations, let \( (N_k, \Omega_k, \Gamma_k, D_k) \) \( k = 1, 2 \) denote

\[
N_k := N^+_{f^{(k)}}, \quad \Gamma_k := \partial N_k \cap \{ r = 1 \}, \\
\Omega_k := \{(x, r) \in \mathbb{R}^2 : f^{(k)}(r) < x < 1, 0 < r < 1 \}, \quad D_k := \partial \Omega_k \cap \{ r = 1 \}.
\]

By a method similar to the proof of Lemma 4.2 with

\[
\| \tilde{f} \|_{1, \alpha, B_1^0(0)}^{(a, \partial B_1^0(0))} \leq C \| \tilde{\phi} \|_{1, \alpha, N_1}^{(1, \alpha, \Gamma_1)} \quad \text{and} \quad \| \tilde{\psi} \|_{1, \alpha, \Omega_1}^{(1, \alpha, \Omega_1)} \leq CM_3 \sigma \| \tilde{f} \|_{1, \alpha, B_1^0(0)},
\]

one can show that

\[
\| \tilde{\phi} \|_{1, \alpha, N_1}^{(1, \alpha, \Gamma_1)} \leq C^* (1 + M_3) \left( \sigma \| \tilde{\phi} \|_{1, \alpha, N_1}^{(1, \alpha, \Gamma_1)} + \| \langle \tilde{S}, \tilde{\Lambda} \rangle \|_{0, \alpha, N_1}^{(1, \alpha, \Gamma_1)} \right)
\]

for a constant \( C^* > 0 \) depending only on the data. If it holds that

\[
\sigma \leq \frac{1}{2C^*(1 + M_3)},
\]

then

\[
\| \tilde{\phi} \|_{1, \alpha, N_1}^{(1, \alpha, \Gamma_1)} \leq C^* (1 + M_3) \| \langle \tilde{S}, \tilde{\Lambda} \rangle \|_{0, \alpha, N_1}^{(1, \alpha, \Gamma_1)}.
\]

For \( k = 1, 2 \), let \( \tilde{S}_{sh}^{(k)} \) and \( \tilde{R}_{sh}^{(k)} \) \( k = 1, 2 \) be defined in (4.32) and (4.37) associated with \( U^{(k)} \). Then a direct computation with using (4.29) and (4.42)-(4.43) yields that

\[
\| S_{sh}^{(1)} (R_{sh}^{(1)}) - \tilde{S}_{sh}^{(1)} (R_{sh}^{(2)}) \|_{0, \alpha, N_0}^{(1, \alpha, \{ t = 1 \})} \leq C (1 + M_3) \sigma \| \tilde{R}_{sh}^{(1)} \|_{0, \alpha, \Omega_1^+}^{(1, \alpha, \{ r = 1 \})} \leq C (1 + M_3) \sigma \| \langle \tilde{S}, \tilde{\Lambda} \rangle \|_{0, \alpha, N_1}^{(1, \alpha, \Gamma_1)},
\]

and

\[
\| \langle \tilde{S}, \tilde{\Lambda} \rangle \|_{0, \alpha, N_1}^{(1, \alpha, \Gamma_1)} \leq C^{**} (1 + M_3) \sigma \| \langle \tilde{S}, \tilde{\Lambda} \rangle \|_{0, \alpha, N_1}^{(1, \alpha, \Gamma_1)}
\]

for a constant \( C^{**} > 0 \) depending only on the data. Finally, we choose \( \sigma_5 \) as

\[
\sigma_5 := \min \left\{ \sigma_5^*, \frac{1}{2C^*(1 + M_3)}, \frac{1}{2C^{**}(1 + M_3)} \right\}
\]

for \( \sigma_5^* \) defined in (4.41) so that (4.42)-(4.43) and (4.44) imply that \( (f^{(1)}, \varphi^{(1)}, S^{(1)}, \Lambda^{(1)}) = (f^{(2)}, \varphi^{(2)}, S^{(2)}, \Lambda^{(2)}) \) for \( \sigma \leq \sigma_5 \). This completes the proof of Lemma 4.3. \( \square \)
4.4. **Proof of Theorem 3.1**  The proof of Theorem 3.1 is divided into three steps.

1. Fix \( \psi \in \mathcal{I}(M_3) \). According to Lemma 4.3, there exists a unique solution \((f, \varphi, S, \Lambda)\) of Problem 4.2 for \( \sigma \leq \sigma_5 \), and the solution satisfies the estimate (4.29). For such \((f, \varphi, S, \Lambda)\), we consider the following boundary value problem for \( W \):

\[
\begin{cases}
-\Delta W = G_\varphi e_\varphi & \text{in } \mathcal{N}_f^+,

-\nabla W \cdot n_f = A_\varphi e_\varphi & \text{on } \mathcal{S}_f,

W = 0 & \text{on } \Gamma_{w,f}^+,

\partial_x W = 0 & \text{on } \Gamma_{ex},
\end{cases}
\]

with

\[ t_0 := t(f, \psi, D\psi, \Lambda), \quad G_\varphi := G(S, \Lambda, \partial_r S, \partial_r \Lambda, t_0, \nabla \varphi), \quad A_\varphi := A(r, \psi, f') \]

for \((t, G, A)\) given by (3.3), (3.4), and (3.20), respectively.

Let \( \mathcal{H} := \{ \zeta \in H^1(\mathcal{N}_f^+) : \zeta = 0 \text{ on } \Gamma_{w,f}^+ \} \). For \( k = 1, 2, 3 \), if each \( W_k \in \mathcal{H} \) satisfies

\[ \mathcal{L}[W_k, \zeta] = \langle (G_\varphi e_\varphi \cdot e_k), (A_\varphi e_\varphi \cdot e_k), \zeta \rangle \]

for all \( \zeta \in \mathcal{H} \), where

\[ \mathcal{L}[W_k, \zeta] := \int_{\mathcal{N}_f^+} \sum_{i=1}^{3} (\partial_i W_k)(\partial_i \zeta) \, dx, \]

\[ \langle (G_\varphi e_\varphi \cdot e_k), (A_\varphi e_\varphi \cdot e_k), \zeta \rangle := \int_{\mathcal{N}_f^+} (G_\varphi e_\varphi \cdot e_k) \zeta \, dx + \int_{\mathcal{S}_f} (A_\varphi e_\varphi \cdot e_k) \zeta \, ds, \]

then we call \( W = (W_1, W_2, W_3) \) a weak solution of the problem (4.46).

A direct computation implies that

\[ \mathcal{L}[W_k, \zeta] \leq C \|W_k\|_{H^1(\mathcal{N}_f^+)} \|\zeta\|_{H^1(\mathcal{N}_f^+)} \]

and the Poincaré inequality yields that there exists \( \nu_0 > 0 \) depending only on the data such that

\[ \mathcal{L}[\zeta, \zeta] \geq \nu_0 \|\zeta\|_{H^1(\mathcal{N}_f^+)}^2 \]

for all \( \zeta \in \mathcal{H} \).

Thus \( \mathcal{L} \) is a bounded bilinear functional on \( \mathcal{H} \times \mathcal{H} \) and coercive. Since \( G_\varphi e_\varphi \in C^{0,\alpha}_{(1-\alpha, \Gamma_{w,f}^+)}(\mathcal{N}_f^+) \), the definition of the weighted Hölder norms yields that

\[ |G_\varphi e_\varphi(x)| \leq \delta_x^{-1+\alpha} \|G_\varphi e_\varphi\|_{0,\alpha, \mathcal{N}_f^+} \quad \text{for } x \in \mathcal{N}_f^+, \quad \delta_x = \inf_{z \in \Gamma_{w,f}^+} |x - z|, \]

from which we obtain that

\[ \int_{\mathcal{N}_f^+} |G_\varphi e_\varphi|^{6/5} \, dx \leq C \left( \|G_\varphi e_\varphi\|_{0,\alpha, \mathcal{N}_f^+}^{(1-\alpha, \Gamma_{w,f}^+)} \right)^{6/5} \]

for \( \alpha \in \left( \frac{1}{6}, 1 \right) \). (4.47)
By the Hölder inequality, Sobolev inequality, trace inequality, Poincaré inequality, and (4.47), we have
\[ \|(G_\theta e_\theta \cdot e_k), (A_\theta e_\theta \cdot e_k), \zeta\| \leq C \left( \|G_\theta e_\theta\|_{0,\alpha,\mathcal{N}_f^+}^{(1-\alpha,1)} + \|A_\theta e_\theta\|_{0,\alpha,\mathcal{E}_f} \right) \|\zeta\|_{H^1(\mathcal{N}_f^+)} \]
for all \( \zeta \in \mathcal{H} \). Then the Lax-Milgram theorem yields that the boundary value problem (4.46) has a unique weak solution \( W \in H^1(\mathcal{N}_f^+) \) satisfying
\[ \|W\|_{H^1(\mathcal{N}_f^+)} \leq C \left( \|G_\theta e_\theta\|_{0,\alpha,\mathcal{N}_f^+}^{(1-\alpha,1)} + \|A_\theta e_\theta\|_{1,\alpha,\mathcal{E}_f} \right). \]
By a method similar to the proof of Lemma 4.2, one can obtain
\[ \|W\|_{1,\alpha,\mathcal{N}_f^+} \leq C \left( \|G_\theta e_\theta\|_{0,\alpha,\mathcal{N}_f^+}^{(1-\alpha,1)} + \|A_\theta e_\theta\|_{1,\alpha,\mathcal{E}_f} \right). \]
Then, the Schauder estimate with scaling implies that
\[ \|W\|_{2,\alpha,\mathcal{N}_f^+} \leq C \left( \|G_\theta e_\theta\|_{0,\alpha,\mathcal{N}_f^+}^{(1-\alpha,1)} + \|A_\theta e_\theta\|_{1,\alpha,\mathcal{E}_f} \right). \]

2. By adjusting the proof of [5, Proposition 3.3], one can show that \( W \) has the form of
\[ W = \psi e_\theta \]
and \( \psi \) solves the boundary value problem
\[
\begin{cases}
- \left( \frac{\partial_{xx}}{r} + \frac{1}{r} \partial_r (r \partial_r) - \frac{1}{r^2} \right) \psi = G_\theta & \text{in } \mathcal{N}_f^+, \\
- \nabla \psi \cdot n_f = A_\theta & \text{on } \mathcal{E}_f, \\
\psi = 0 & \text{on } \Gamma_w^+ \cup (\mathcal{N}_f^+ \cap \{ r = 0 \}), \\
\partial_r \psi = 0 & \text{on } \Gamma_{ex}.
\end{cases}
\]
Furthermore, \( \psi \) satisfies
\[ \partial_r \psi \equiv 0 \quad \text{on } \mathcal{N}_f^+ \cap \{ r = 0 \} \] (4.48)
and
\[
\begin{align*}
\|\psi\|_{1,\alpha,\Omega_f^+} & \leq C \left( \|G_\theta\|_{0,\alpha,\mathcal{N}_f^+}^{(1-\alpha,1)} + \|A_\theta\|_{1,\alpha,\mathcal{E}_f} \right), \\
\|\psi\|_{2,\alpha,\Omega_f^+} & \leq C \left( \|G_\theta\|_{0,\alpha,\mathcal{N}_f^+}^{(1-\alpha,1)} + \|A_\theta\|_{1,\alpha,\mathcal{E}_f} \right).
\end{align*}
\] (4.49)
where \( \Omega_f^+ \) is a two-dimensional space defined by
\[ \Omega_f^+ := \{(x, r) \in \mathbb{R}^2 : f(r) < x < 1, 0 < r < 1 \}. \]
A direct computation yields that there exists a constant \( \epsilon_3 > 0 \) depending only on
the data so that if
\[ \mathcal{M}_3 \sigma \leq \epsilon_3, \]
then
\[ \|G\|_{(1-\alpha, r^+, \Omega^+)}^{(1-\alpha, r^+, \Omega^+)} \leq C M_2 \sigma \leq C \sigma, \] (4.50)
\[ \|A\|_{(r^-, \partial\Omega)} \leq C (\sigma + (1 + M_3)\sigma^2) \leq C (\sigma + (1 + M_3)^2 \sigma^2). \]

It follows from (4.49)-(4.50) that
\[ \|\psi\|_{(2, \alpha, \Omega^+)} \leq C (\sigma + (1 + M_3)^2 \sigma^2). \] (4.51)

3. Define an iteration map \( J: I(M_3) \rightarrow C^{2, \alpha}_{(r=1, \{r=1\}, \Omega^+)} \) by
\[ J(\psi) = \mathcal{E}_f(\psi), \]
where \( \mathcal{E}_f \) is defined in (4.20) and \( \psi_0 \) is the solution to (4.46) associated with \( \psi_* \).

By (4.48) and the condition \( f'(0) = 0 \), we have
\[ \partial_r \mathcal{E}_f(\psi) \equiv 0 \quad \text{on} \quad \{r = 0\}. \]

By the definition of \( \mathcal{E}_f \) and (4.51), we also have
\[ \|\mathcal{E}_f(\psi)\|_{(1-\alpha, \Omega^+)} \leq C_b (\sigma + (1 + M_3)^2 \sigma^2) \]
for a constant \( C_b > 0 \) depending only on the data. Now, we choose constants \( M_3 \) and \( \sigma_2 \) as
\[ M_3 := 4C_b \quad \text{and} \quad \sigma_2 := \min \left\{ \sigma_5, \frac{\epsilon_3}{M_3}, \frac{M_3}{4C_b}, \frac{1}{8C_b}, \frac{1}{4C_b M_3} \right\} \]
for \( \sigma_5 \) given in (4.45). Then, under such choices of \( (M_3, \sigma_2) \), the mapping \( J \) maps \( I(M_3) \) into itself whenever \( \sigma \leq \sigma_2 \).

The iteration set \( I(M_3) \) defined in (4.3) is a compact and convex subset of \( C^{2, \alpha/2}_{(r=1, \Omega^+)} \). By Lemma 4.3 and the uniqueness of a solution for the boundary value problem (4.40), one can prove that \( J \) is continuous in \( C^{2, \alpha/2}_{(r=1, \Omega^+)} \). Then the Schauder fixed point theorem implies that \( J \) has a fixed point \( \psi_* \in I(M_3) \). According to Lemma 4.3, there exists a solution \((f, \varphi, S, \Lambda)\) of Problem 4.2 associated with \( \psi_* = \psi_* \). Then \((f, \varphi, S, \Lambda, \psi|_{\partial\Omega})\) is a solution of the free boundary problem (3.4) with (3.9) and (3.19), and the solution satisfies the estimate (3.21) by (4.29) and (4.51). The proof of Theorem 3.1 is completed.

5. Proof of Theorem 2.1

According to Theorem 3.1, for \( \sigma \leq \sigma_2 \), the free boundary problem (3.4) with (3.9) and (3.19) has a solution \((f, S, \Lambda, \varphi, \psi)\) that satisfies the estimate (3.21). For
such a solution \((f, S, \Lambda, \varphi, \psi)\), we define \((\mathbf{u}, \rho, p)\) by

\[
\mathbf{u} := \left( \partial_x \varphi + \frac{1}{r} \partial_r (r\psi) \right) \mathbf{e}_x + (\partial_r \varphi - \partial_x \psi) \mathbf{e}_r + \frac{\Lambda}{r} \mathbf{e}_\theta,
\]

\[
\rho := H(S, \mathbf{u}), \quad p := S\rho^\gamma \quad \text{in} \quad \mathcal{N}_f^+,
\]

where \(H\) is given by (3.5). Then, due to the estimate (3.21), \((f, \mathbf{u}, \rho, p)\) satisfy the estimate (2.4). So one can find a small constant \(\sigma_1 \in (0, \min\{\sigma_2, \frac{\hat{c}_0^+}{\hat{c}}\})\) depending only on the data so that if \(\sigma \leq \sigma_1\), then \((f, \mathbf{u}, \rho, p)\) satisfy

\[
\rho \geq \frac{\rho_0^+}{2} > 0, \quad \mathbf{u} \cdot \mathbf{e}_x \geq \frac{u_0^+}{2} > 0, \quad c^2 - |\mathbf{u}|^2 \geq \frac{1}{2} ((c_0^+)^2 - (u_0^+)^2) > 0 \quad \text{in} \quad \mathcal{N}_f^+,
\]

\[
\mathbf{u}^- \cdot \mathbf{n}_f > \mathbf{u} \cdot \mathbf{n}_f > 0 \quad \text{on} \quad \partial \mathcal{N}_f^+.
\]

for \(c_0^+ := \sqrt{\frac{2\rho_0^+}{\rho_0^+}}\) and the unit normal vector field \(\mathbf{n}_f\) on \(\partial \mathcal{N}_f^+\) pointing toward the interior of \(\mathcal{N}_f^+\). The proof of Theorem 2.1 is completed. \(\square\)

Acknowledgements: The research of Hyeongyu Ryu was supported in part by Samsung Science and Technology Foundation under Project Number SSTF-BA1502-02.

References

[1] M. Bae, B. Duan, and C. Xie, Subsonic flow for the multidimensional euler–poisson system, Archive for Rational Mechanics and Analysis, 220 (2016), pp. 155–191.

[2] M. Bae and M. Feldman, Transonic shocks in multidimensional divergent nozzles, Archive for rational mechanics and analysis, 201 (2011), pp. 777–840.

[3] M. Bae and H. Park, Contact discontinuities for 2-dimensional inviscid compressible flows in infinitely long nozzles, SIAM Journal on Mathematical Analysis, 51 (2019), pp. 1730–1760.

[4] M. Bae and H. Park, Contact discontinuities for 3-d axisymmetric inviscid compressible flows in infinitely long cylinders, Journal of Differential Equations, 267 (2019), pp. 2824–2873.

[5] M. Bae and S. Weng, 3-d axisymmetric subsonic flows with nonzero swirl for the compressible euler–poisson system, in Annales de l’Institut Henri Poincaré C, Analyse non linéaire, vol. 35, Elsevier, 2018, pp. 161–186.

[6] G.-Q. Chen and M. Feldman, Multidimensional transonic shocks and free boundary problems for nonlinear equations of mixed type, Journal of the American Mathematical Society, 16 (2003), pp. 461–494.

[7] G.-Q. Chen and M. Feldman, Steady transonic shocks and free boundary problems in infinite cylinders for the euler equations, Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences, 57 (2004), pp. 310–356.

[8] G.-Q. Chen and M. Feldman, Existence and stability of multidimensional transonic flows through an infinite nozzle of arbitrary cross-sections, Archive for Rational Mechanics and Analysis, 184 (2007), pp. 185–242.

[9] G.-Q. Chen and H. Yuan, Uniqueness of transonic shock solutions in a duct for steady potential flow, Journal of Differential Equations, 247 (2009), pp. 564–573.
[10] S. Chen, *Transonic shocks in 3-d compressible flow passing a duct with a general section for euler systems*, Transactions of the American Mathematical Society, 360 (2008), pp. 5265–5289.

[11] S. Chen and H. Yuan, *Transonic shocks in compressible flow passing a duct for three-dimensional euler systems*, Archive for Rational Mechanics and Analysis, 187 (2008), pp. 523–556.

[12] R. Courant and K. O. Friedrichs, *Supersonic flow and shock waves*, vol. 21, Springer Science & Business Media, 1999.

[13] L. C. Evans, *Partial differential equations*, ams, Graduate Studies in Mathematics, 19 (2002).

[14] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Springer, 2015.

[15] Q. Han and F. Lin, *Elliptic partial differential equations*, vol. 1, American Mathematical Soc., 2011.

[16] J. Li, Z. Xin, and H. Yin, *On transonic shocks in a conic divergent nozzle with axi-symmetric exit pressures*, Journal of Differential Equations, 248 (2010), pp. 423–469.

[17] L. Liu, G. Xu, and H. Yuan, *Stability of spherically symmetric subsonic flows and transonic shocks under multidimensional perturbations*, Advances in Mathematics, 291 (2016), pp. 696–757.

[18] L. Liu and H. Yuan, *Global uniqueness of transonic shocks in divergent nozzles for steady potential flows*, SIAM Journal on Mathematical Analysis, 41 (2009), pp. 1816–1824.

[19] Z. Xin and H. Yin, *The transonic shock in a nozzle, 2-d and 3-d complete euler systems*, Journal of Differential Equations, 245 (2008), pp. 1014–1085.

Center for Mathematical Analysis and Computation (CMAC), Yonsei University, 50 Yonsei-Ro, Seodaemun-Gu, Seoul 03722, Republic of Korea

E-mail address: hyangdong.park@yonsei.ac.kr

Department of Mathematics, POSTECH, 77 Cheongam-Ro, Nam-Gu, Pohang, Gyeongbuk 37673, Republic of Korea

E-mail address: hgray@postech.ac.kr