U(1)-invariant special Lagrangian 3-folds. III.
Properties of singular solutions

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1 Introduction

Special Lagrangian submanifolds (SL m-folds) are a distinguished class of real m-dimensional minimal submanifolds in $\mathbb{C}^m$, which are calibrated with respect to the m-form $\text{Re}(dz_1 \wedge \cdots \wedge dz_m)$. They can also be defined in Calabi–Yau manifolds, are important in String Theory, and are expected to play a rôle in the eventual explanation of Mirror Symmetry between Calabi–Yau 3-folds.

This is the third in a suite of three papers [9, 10] studying special Lagrangian 3-folds $N$ in $\mathbb{C}^3$ invariant under the U(1)-action

$$e^{i\theta} : (z_1, z_2, z_3) \mapsto (e^{i\theta}z_1, e^{-i\theta}z_2, z_3) \text{ for } e^{i\theta} \in \text{U}(1).$$

These three papers and [11] are reviewed in [12]. Locally we can write $N$ as

$$N = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : \text{Im}(z_3) = u(\text{Re}(z_3), \text{Im}(z_1z_2)), \text{Re}(z_1z_2) = v(\text{Re}(z_3), \text{Im}(z_1z_2)), |z_1|^2 - |z_2|^2 = 2a\},$$

where $a \in \mathbb{R}$ and $u, v : \mathbb{R}^2 \to \mathbb{R}$ are continuous functions. It was shown in [9] that when $a \neq 0$, $N$ is an SL 3-fold in $\mathbb{C}^3$ if and only if $u, v$ satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -2(v^2 + y^2 + a^2)^{1/2} \frac{\partial u}{\partial y},$$

and then $u, v$ are smooth and $N$ is nonsingular.

Here is what we mean by saying $N$ is locally of the form (2). Every connected U(1)-invariant SL 3-fold $N$ may be written in the form (2) for $u, v$ continuous, multi-valued ‘functions’ (multifunctions). Since (3) is a nonlinear Cauchy-Riemann equation, $u + iv$ is a bit like a holomorphic function of $x + iy$, and so the multifunctions $(u, v)$ behave like holomorphic multifunctions in complex analysis, such as $\sqrt{z}$.

Thus we expect them to have isolated branch points in $\mathbb{R}^2$. Over simply connected open sets $U \subset \mathbb{R}^2$ not containing any branch points, the multifunctions $(u, v)$ decompose into sheets, each of which is a continuous, single-valued function. This is like choosing a branch of $\sqrt{z}$ on a simply-connected open subset
$U \subset \mathbb{C} \setminus \{0\}$. In terms of $N$, we expect there to be a discrete set of ‘branch point’ $U(1)$-orbits, and small $U(1)$-invariant open sets in $N$ away from these can be written in the form (2) for single-valued $(u,v)$.

As $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ there exists $f : \mathbb{R}^2 \to \mathbb{R}$ with $\frac{\partial f}{\partial y} = u$ and $\frac{\partial f}{\partial x} = v$, satisfying

$$\left(\left(\frac{\partial f}{\partial x}\right)^2 + y^2 + a^2\right)^{-1/2} \frac{\partial^2 f}{\partial x^2} + 2 \frac{\partial^2 f}{\partial y^2} = 0. \quad (4)$$

In [9, Th. 7.6] we proved existence and uniqueness for the Dirichlet problem for (4) in strictly convex domains $S$ in $\mathbb{R}^2$ when $a \neq 0$. This gives existence and uniqueness of a large class of nonsingular $U(1)$-invariant SL 3-folds in $\mathbb{C}^3$ satisfying certain boundary conditions.

When $a = 0$, if $v = y = 0$ the factor $-2(v^2 + y^2 + a^2)^{1/2}$ in (3) becomes zero, and then (3) is no longer elliptic. Because of this, when $a = 0$ the appropriate thing to do is to consider weak solutions of (3), which may have singular points $(x,0)$ with $v(x,0) = 0$. At such a point $u,v$ may not be differentiable, and $(0,0,x+iu(x,0))$ is a singular point of the SL 3-fold $N$ in $\mathbb{C}^3$.

In [10, Th. 7.1] we proved existence and uniqueness for a suitable class of weak solutions to the Dirichlet problem for (4) in strictly convex domains $S$ in $\mathbb{R}^2$ when $a = 0$. This gives existence and uniqueness of a large class of singular $U(1)$-invariant SL 3-folds in $\mathbb{C}^3$ satisfying certain boundary conditions.

The goal of this paper is to study these singular SL 3-folds in more detail. It will be shown that under mild conditions the singularities are isolated, and can be roughly classified by a multiplicity $n \geq 1$, and within each multiplicity by one of two types. For singularities of multiplicity $n$, the ‘germ’ of the singularity is described to leading order by real parameters $\gamma_1, \ldots, \gamma_n$.

Examples of singular SL 3-folds will be constructed with every multiplicity $n \geq 1$ and type, and realizing all possible values of $\gamma_1, \ldots, \gamma_n$. They provide an infinite family of local models for singularities of compact SL 3-folds in $\mathbb{C}^3$, including singular fibres, which may be of every multiplicity and type. These will be studied further in [11], in connection with the SYZ Conjecture. For a brief summary of [9, 10, 11] and this paper, see [12].

Section 2 introduces special Lagrangian geometry, and §3 gives some background in analysis and Geometric Measure Theory. Section 4 then reviews the preceding papers [9, 10]. The new material begins in §5 where we classify the possible tangent cones, in the sense of Geometric Measure Theory, at a singular point of an SL 3-fold of the form (2).

Let $(u,v)$ and $(\tilde{u},\tilde{v})$ satisfy (3) in a domain $S$ in $\mathbb{R}^2$, for $a \neq 0$. In §9 we defined the multiplicity of a zero of $(u,v) - (\tilde{u},\tilde{v})$, and related the number of zeroes of $(u,v) - (\tilde{u},\tilde{v})$ in $S^o$, counted with multiplicity, to boundary data on $\partial S$. Section 6 extends these to the singular case $a = 0$, showing that zeroes of $(u,v) - (\tilde{u},\tilde{v})$ are isolated, with multiplicity a positive integer.

As part of the proof, in §6.2 we study singular solutions $(u,v)$ of (3) with $a = 0$ and $v(x,0) \equiv 0$, so that $u,v$ is singular all along the $x$-axis in $S$. We show
that such solutions have the symmetry \( u(x, -y) = u(x, y), \ v(x, -y) = -v(x, y), \) and the corresponding SL 3-folds \( N \) are actually the union of two nonsingular U(1)-invariant SL 3-folds \( N_\pm \) intersecting in a real analytic real curve \( \gamma \), which is the singular set of \( N \) and the fixed point set of the U(1)-action \( \Gamma \).

Section \ref{sect:symmetry} applies the results of \ref{sect:almost} to construct special Lagrangian fibrations on open subsets of \( \mathbb{C}^3 \), which will be the central tool in \ref{sect:SL}. Let \((u, v)\) be a singular solution of \ref{sect:almost} in \( S \) with \( \alpha = 0 \). In \ref{sect:almost} we show that either \( u(x, -y) \equiv u(x, y) \) and \( v(x, -y) \equiv -v(x, y) \), or the singularities of \( u, v \) in \( S^\circ \) are isolated.

For isolated singularities in \( S^\circ \) we define the multiplicity \( n \geq 1 \) and type, discuss the tangent cones of the corresponding SL 3-fold singularities, and give counting formulae for singular points with multiplicity. Section \ref{sect:types} shows that singularities with every multiplicity \( n \geq 1 \) and type exist, and occur in codimension \( n \) in the family of all U(1)-invariant SL 3-folds.

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## 2 Special Lagrangian submanifolds in \( \mathbb{C}^m \)

For introductions to special Lagrangian geometry, see the author \cite{Hitchin} and Harvey and Lawson \cite{HarveyLawson} §III. We begin by defining *calibrations* and *calibrated submanifolds*, following Harvey and Lawson \cite{HarveyLawson}.

**Definition 2.1.** Let \((M, g)\) be a Riemannian manifold. An oriented tangent \( k \)-plane \( V \) on \( M \) is a vector subspace \( V \) of some tangent space \( T_xM \) to \( M \) with \( \dim V = k \), equipped with an orientation. If \( V \) is an oriented tangent \( k \)-plane on \( M \) then combining \( g|_V \) with the orientation on \( V \) gives a natural volume form \( \text{vol}_V \) on \( V \), which is a \( k \)-form on \( V \).

Now let \( \varphi \) be a closed \( k \)-form on \( M \). We call \( \varphi \) a calibration on \( M \) if for every oriented \( k \)-plane \( V \) on \( M \) we have \( \varphi|_V \leq \text{vol}_V \). Here \( \varphi|_V = \alpha \cdot \text{vol}_V \) for some \( \alpha \in \mathbb{R} \), and \( \varphi|_V \leq \text{vol}_V \) if \( \alpha \leq 1 \). Let \( N \) be an oriented submanifold of \( M \) with dimension \( k \). Then each tangent space \( T_xN \) for \( x \in N \) is an oriented tangent \( k \)-plane. We call \( N \) a calibrated submanifold if \( \varphi|_{T_xN} = \text{vol}_{T_xN} \) for all \( x \in N \).

It is easy to show that calibrated submanifolds are automatically minimal submanifolds \cite[Th. II.4.2]{HarveyLawson}. Here is the definition of special Lagrangian submanifolds in \( \mathbb{C}^m \), taken from \cite[§III]{HarveyLawson}.

**Definition 2.2.** Let \( \mathbb{C}^m \) have complex coordinates \((z_1, \ldots, z_m)\), and define a metric \( g \), a real 2-form \( \omega \) and a complex \( m \)-form \( \Omega \) on \( \mathbb{C}^m \) by

\[
\begin{align*}
g & = |dz_1|^2 + \cdots + |dz_m|^2, \quad \omega = \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + \cdots + dz_m \wedge d\bar{z}_m), \\
\text{and} \quad \Omega & = dz_1 \wedge \cdots \wedge dz_m.
\end{align*}
\]
Then $\text{Re} \Omega$ and $\text{Im} \Omega$ are closed real $m$-forms on $\mathbb{C}^m$. Let $L$ be an oriented real submanifold of $\mathbb{C}^m$ of real dimension $m$. We say that $L$ is a special Lagrangian submanifold of $\mathbb{C}^m$, or SL $m$-fold for short, if $L$ is calibrated with respect to $\text{Re} \Omega$, in the sense of Definition 2.1.

Harvey and Lawson [4, Cor. III.1.11] give the following alternative characterization of special Lagrangian submanifolds.

**Proposition 2.3.** Let $L$ be a real $m$-dimensional submanifold of $\mathbb{C}^m$. Then $L$ admits an orientation making it into a special Lagrangian submanifold of $\mathbb{C}^m$ if and only if $\omega|_L \equiv 0$ and $\text{Im} \Omega|_L \equiv 0$.

An $m$-dimensional submanifold $L$ in $\mathbb{C}^m$ is called Lagrangian if $\omega|_L \equiv 0$. Thus special Lagrangian submanifolds are Lagrangian submanifolds satisfying the extra condition that $\text{Im} \Omega|_L \equiv 0$, which is how they get their name.

### 3 Background material from analysis

Here are some definitions we will need to make sense of analytic results from [9, 10]. A good reference for §3.1–§3.2 is Gilbarg and Trudinger [3].

#### 3.1 Banach spaces of functions on subsets of $\mathbb{R}^n$

We shall work in a special class of subsets of $\mathbb{R}^n$ called domains.

**Definition 3.1.** A closed, bounded, contractible subset $S$ in $\mathbb{R}^n$ will be called a domain if it is a disjoint union $S = S^o \cup \partial S$, where the interior $S^o$ of $S$ is a connected open set in $\mathbb{R}^n$ with $S = \overline{S^o}$, and the boundary $\partial S = S \setminus S^o$ is a compact embedded hypersurface in $\mathbb{R}^n$.

A domain $S$ in $\mathbb{R}^2$ is called strictly convex if $S$ is convex and the curvature of $\partial S$ is nonzero at every point. So, for example, $x^2 + y^2 \leq 1$ is strictly convex but $x^4 + y^4 \leq 1$ is not, as its boundary has zero curvature at $(\pm 1, 0)$ and $(0, \pm 1)$.

We will use a number of different spaces of real functions on $S$.

**Definition 3.2.** Let $S$ be a domain in $\mathbb{R}^n$. For $k \geq 0$, define $C^k(S)$ to be the space of continuous functions $f : S \to \mathbb{R}$ with $k$ continuous derivatives, and norm $\|f\|_{C^k} = \sum_{j=0}^k \sup_{S} |\partial^j f|$. Define $C^\infty(S) = \bigcap_{k=0}^{\infty} C^k(S)$. For $k \geq 0$ and $\alpha \in (0, 1]$, define the Hölder space $C^{k, \alpha}(S)$ to be the subset of $f \in C^k(S)$ for which

$$[\partial^k f]_\alpha = \sup_{x \neq y \in S} \frac{|\partial^k f(x) - \partial^k f(y)|}{|x - y|^\alpha}$$

is finite. The Hölder norm on $C^{k, \alpha}(S)$ is $\|f\|_{C^{k, \alpha}} = \|f\|_{C^k} + [\partial^k f]_\alpha$. For $q \geq 1$, define the Lebesgue space $L^p(S)$ to be the set of locally integrable functions $f$ on $S$ for which the norm

$$\|f\|_{L^p} = \left( \int_S |f|^p \, dx \right)^{1/p}$$
is finite. Then \( C^k(S), C^{k,p}(S) \) and \( L^p(S) \) are Banach spaces. Here \( \partial \) is the vector operator \( \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right) \), and lengths \( |\partial^j f| \) are computed using the standard Euclidean metric on \( \mathbb{R}^n \).

### 3.2 Quasilinear elliptic operators and weak solutions

Quasilinear elliptic operators are a class of nonlinear partial differential operators, which are linear and elliptic in their highest-order derivatives.

**Definition 3.3.** Let \( S \) be a domain in \( \mathbb{R}^n \). A second-order quasilinear operator \( Q : C^2(S) \rightarrow C^0(S) \) is an operator of the form

\[
(Qu)(x) = \sum_{i,j=1}^{n} a^{ij}(x,u,\partial u) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) + b(x,u,\partial u),
\]

where \( a^{ij} \) and \( b \) are continuous maps \( S \times \mathbb{R} \times (\mathbb{R}^n)^* \rightarrow \mathbb{R} \), and \( a^{ij} = a^{ji} \) for all \( i,j = 1, \ldots, n \). We call the functions \( a^{ij} \) and \( b \) the coefficients of \( Q \). We call \( Q \) elliptic if the symmetric \( n \times n \) matrix \( (a^{ij}) \) is positive definite at every point of \( S \times \mathbb{R} \times (\mathbb{R}^n)^* \).

A second-order quasilinear operator \( Q : C^2(S) \rightarrow C^0(S) \) is in *divergence form* if it is written as

\[
(Qu)(x) = \sum_{j=1}^{n} \frac{\partial}{\partial x_j}(a^j(x,u,\partial u)) + b(x,u,\partial u)
\]

for functions \( a^j \in C^1(S \times \mathbb{R} \times (\mathbb{R}^n)^*) \) for \( j = 1, \ldots, n \) and \( b \in C^0(S \times \mathbb{R} \times (\mathbb{R}^n)^*) \).

**Definition 3.4.** Let \( Q \) be a second-order quasilinear operator in divergence form on a domain \( S \) in \( \mathbb{R}^n \), given by equation (8). Then we say that \( u \in L^1(S) \) is a weak solution of the equation \( Qu = f \) for \( f \in L^1(S) \) if \( u \) is twice weakly differentiable with weak derivative \( \partial u \), and \( a^j(x,u,\partial u), b(x,u,\partial u) \in L^1(S) \) with

\[
-\sum_{j=1}^{n} \int_S \frac{\partial \psi}{\partial x_j} \cdot a^j(x,u,\partial u) dx + \int_S \psi \cdot b(x,u,\partial u) dx = \int_S \psi \cdot f dx
\]

for all \( \psi \in C^1(S) \) with \( \psi|_{\partial S} = 0 \).

If \( Q \) is a second-order quasilinear operator, we shall interpret the equation \( Qu = f \) in three different senses:

- We say that \( Qu = f \) holds classically, if \( u \in C^2(S) \) and \( f \in C^0(S) \) and \( Qu = f \) in \( C^0(S) \) in the usual way.

- We say that \( Qu = f \) holds with weak derivatives if \( u \) is twice weakly differentiable and \( Qu = f \) holds in \( L^p(S) \) for some \( p \geq 1 \), defining \( Qu \) using the weak derivatives of \( u \).
We say that $Qu = f$ holds weakly if $u$ is a weak solution of $Qu = f$, as in Definition 3.4. Note that this requires only that $u$ be once weakly differentiable, and the second derivatives of $u$ need not exist even weakly.

Clearly the first sense implies the second, which implies the third. If $Q$ is elliptic and $a^j, b, f$ are suitably regular, one can usually show that a weak solution to $Qu = f$ is a classical solution, so that the three senses are equivalent. But we will be dealing with singular equations that are not elliptic at every point, and then the three senses will be distinct.

3.3 Geometric Measure Theory and tangent cones

Much of this paper concerns the singularities of SL 3-folds, which are examples of singular minimal submanifolds. Now there is an elegant theory of singular submanifolds called Geometric Measure Theory, which is especially powerful in dealing with singular minimal submanifolds. An introduction to the subject is provided by Morgan [15] and an in-depth treatment by Federer [2], and Harvey and Lawson [4, §II] relate Geometric Measure Theory to calibrated geometry.

Let $m \leq n$ be nonnegative integers. One defines a class of $m$-dimensional rectifiable currents in $\mathbb{R}^n$, which are measure-theoretic generalizations of compact, oriented $m$-submanifolds $N$ with boundary $\partial N$ in $\mathbb{R}^n$, with integer multiplicities. Here $N$ with multiplicity $k$ is like $k$ copies of $N$ superimposed, and changing the orientation of $N$ changes the sign of the multiplicity. This enables us to add and subtract submanifolds.

If $T$ is an $m$-dimensional rectifiable current, one can define the volume $\text{vol}(T)$ of $T$, by Hausdorff $m$-measure. If $\varphi$ is a compactly-supported $m$-form on $\mathbb{R}^n$ then one can define $\int_T \varphi$. Thus we can consider $T$ as an element $\varphi \mapsto \int_T \varphi$ of the dual space $(\mathcal{D}^m)^*$ of the vector space $\mathcal{D}^m$ of compactly-supported $m$-forms on $\mathbb{R}^n$. This induces a topology on the space of rectifiable currents in $\mathbb{R}^n$. The interior $T^\circ$ of $T$ is $T \setminus \partial T$ (that is, $\text{supp } T \setminus \text{supp } \partial T$).

An $m$-dimensional rectifiable current $T$ is called an integral current if $\partial T$ is an $(m-1)$-dimensional rectifiable current. By [15, 5.5], [2, 4.2.17], integral currents have strong compactness properties.

**Theorem 3.5.** Let $m \leq n$, $C > 0$, and $K$ be a closed ball in $\mathbb{R}^n$. Then the set of $m$-dimensional integral currents $T$ in $K$ with $\text{vol}(T) \leq C$ and $\text{vol}(\partial T) \leq C$ is compact in an appropriate weak topology.

This is significant as it enables us to easily construct minimal (volume-minimizing) integral currents satisfying certain conditions, by choosing a suitable minimizing sequence and extracting a convergent subsequence. The important question then becomes, how close are such minimal integral currents to being manifolds, and how bad are their singularities? One can (partially) answer this using regularity results. Here, for example, is a major result of Almgren [15, 8.3], slightly rewritten.

**Theorem 3.6.** Let $T$ be an $m$-dimensional minimal rectifiable current in $\mathbb{R}^n$. Then $T^\circ$ is a smooth, embedded, oriented, minimal $m$-submanifold with locally...
Next we discuss tangent cones of minimal rectifiable currents, a generalization of tangent spaces of submanifolds, as in [15] 9.7.

**Definition 3.7.** A locally rectifiable current $C$ in $\mathbb{R}^n$ is called a cone if $C = tC$ for all $t > 0$, where $t: \mathbb{R}^n \to \mathbb{R}^n$ acts by dilations in the obvious way. Let $T$ be a rectifiable current in $\mathbb{R}^n$, and let $x \in T^{o}$. We say that $C$ is a tangent cone to $T$ at $x$ if there exists a decreasing sequence $r_1 > r_2 > \cdots$ such that $r_j \to 0$ and $r_j^{-1}(T - x) \to C$ as a locally rectifiable current as $j \to \infty$.

The next result follows from Morgan [15, p. 94-95], Federer [2, 5.4.3] and Harvey and Lawson [4, Th. II.5.15].

**Theorem 3.8.** Let $T$ be a minimal rectifiable current in $\mathbb{R}^n$. Then for all $x \in T^{o}$, there exists a tangent cone $C$ to $T$ at $x$. Moreover $C$ is itself a minimal locally rectifiable current with $\partial C = \emptyset$, and if $T$ is calibrated with respect to a constant calibration $\varphi$ on $\mathbb{R}^n$, then $C$ is also calibrated with respect to $\varphi$.

Note that the theorem does not claim that the tangent cone $C$ is unique, and in fact it is an important open question whether a minimal rectifiable current has a unique tangent cone at each point of $T^{o}$, [15, p. 93]. However, using the idea of density we can constrain the choice of tangent cones.

**Definition 3.9.** Let $T$ be an $m$-dimensional minimal locally rectifiable current in $\mathbb{R}^n$. For each $x \in T^{o}$ define the density $\Theta(T, x)$ of $T$ at $x$ by

$$\Theta(T, x) = \lim_{r \to 0^{+}} \frac{\text{vol}(T \cap B_r(x))}{\omega_m r^m},$$

where $B_r(x)$ is the closed ball of radius $r$ about $x \in \mathbb{R}^n$, and $\text{vol}(\ldots)$ the volume of $m$-dimensional rectifiable currents, and $\omega_m$ the volume of the unit ball in $\mathbb{R}^m$. By [15] 9.4] the limit in (10) exists for all $x \in T^{o}$. By [2, 5.4.5(1)] the density is an upper-semicontinuous function on $T^{o}$.

Note that if $C$ is an $m$-dimensional locally rectifiable cone in $\mathbb{R}^n$, and $S^{n-1}$ is the unit sphere in $\mathbb{R}^n$, then $C \cap S^{n-1}$ is an $(m - 1)$-dimensional rectifiable current in $S^{n-1}$, and $\Theta(C, 0) = \text{vol}(C \cap S^{n-1})/\text{vol}(S^{m-1})$. By [15] 9.9], the density at $x \in T^{o}$ of a minimal rectifiable current $T$ agrees with the density at 0 of any tangent cone $C$ to $T$ at $x$.

**Theorem 3.10.** Let $T$ be a minimal rectifiable current in $\mathbb{R}^n$, and $x \in T^{o}$. Then each tangent cone $C$ to $T$ at $x$ has $\Theta(C, 0) = \Theta(T, x)$.

Therefore all tangent cones $C$ to $T$ at $x$ must have the same density at 0. A multiplicity 1 minimal $m$-submanifold $T$ in $\mathbb{R}^n$ has unique tangent cone $T_x T$ at $x$ and density 1 at every point. Here is a kind of converse to this, [2, 5.4.7].
Theorem 3.11. Let $T$ be an $m$-dimensional minimal rectifiable current in $\mathbb{R}^n$. Then $\Theta(T,x) \geq 1$ for all $x \in T^\circ$. There exists $\Upsilon > 1$ depending only on $m,n$ such that if $x \in T^\circ$ and $\Theta(T,x) < \Upsilon$ then $\Theta(T,x) = 1$, and $T$ is a smooth, embedded, multiplicity 1 minimal $m$-submanifold near $x$.

The way this works is that the only tangent cones $C$ with density 1 at 0 are multiplicity 1 subspaces $\mathbb{R}^m$ in $\mathbb{R}^n$, and if a minimal rectifiable current has this as a tangent cone at $x$, then it is a submanifold near $x$.

4 Review of material from [9] and [10]

We now recapitulate those results from [9] and [10] that we will need later. Readers are referred to [9, 10] for proofs, discussion and motivation. The material of §4.6 is new.

4.1 Finding the equations

The following result [9, Prop. 4.1] is the starting point for everything in [9, 10] and this paper.

Proposition 4.1. Let $S$ be a domain in $\mathbb{R}^2$ or $S = \mathbb{R}^2$, let $u,v : S \to \mathbb{R}$ be continuous, and $a \in \mathbb{R}$. Define

$$N = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1z_2 = v(x,y) + iy, \quad z_3 = x + iu(x,y), \quad |z_1|^2 - |z_2|^2 = 2a, \quad (x,y) \in S\}. \quad (11)$$

Then

(a) If $a = 0$, then $N$ is a (possibly singular) special Lagrangian 3-fold in $\mathbb{C}^3$, with boundary over $\partial S$, if $u,v$ are differentiable and satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -2(v^2 + y^2)^{1/2} \frac{\partial u}{\partial y}, \quad (12)$$

except at points $(x,0)$ in $S$ with $v(x,0) = 0$, where $u,v$ need not be differentiable. The singular points of $N$ are those of the form $(0,0,z_3)$, where $z_3 = x + iu(x,0)$ for $x \in \mathbb{R}$ with $v(x,0) = 0$.

(b) If $a \neq 0$, then $N$ is a nonsingular SL 3-fold in $\mathbb{C}^3$, with boundary over $\partial S$, if and only if $u,v$ are differentiable on all of $S$ and satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -2(v^2 + y^2 + a^2)^{1/2} \frac{\partial u}{\partial y}. \quad (13)$$

By showing that $u,v$ each satisfy second-order elliptic equations and using the maximum principle, we prove [9, Cor. 4.4]:

Corollary 4.2. Let $S$ be a domain in $\mathbb{R}^2$, let $a \neq 0$, and suppose $u,v \in C^1(S)$ satisfy (13). Then the maxima and minima of $u$ and $v$ are achieved on $\partial S$. 8
4.2 Examples

Here are some examples of SL 3-folds \( N \) in the form (11), and the corresponding functions \( u, v \). Let \( a \geq 0 \), and define

\[
N_a = \left\{ (z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 - 2a = |z_2|^2 = |z_3|^2, \quad \text{Im}(z_1z_2z_3) = 0, \quad \text{Re}(z_1z_2z_3) \geq 0 \right\}. \tag{14}
\]

Then \( N_a \) is a nonsingular SL 3-fold diffeomorphic to \( S^1 \times \mathbb{R}^2 \) when \( a > 0 \), and \( N_0 \) is an SL T^2-cone with one singular point at \((0, 0, 0)\). The \( N_a \) are invariant under the U(1)^2-action

\[(e^{i\theta_1}, e^{i\theta_2}) : (z_1, z_2, z_3) \mapsto (e^{i\theta_1}z_1, e^{i\theta_2}z_2, e^{-i\theta_1-i\theta_2}z_3), \tag{15}\]

and are part of a family of explicit U(1)^2-invariant SL 3-folds written down by Harvey and Lawson [4 §III.3.A]. By [9, Th. 5.1], these SL 3-folds can be written in the form (11).

**Theorem 4.3.** Let \( a \geq 0 \). Then there exist unique \( u_a, v_a : \mathbb{R}^2 \to \mathbb{R} \) such that the SL 3-fold \( N_a \) of (14) agrees with \( N \) in (11) with \( u_a, v_a \) in place of \( u, v \) and \( S = \mathbb{R}^2 \). Furthermore:

(a) \( u_a, v_a \) are smooth on \( \mathbb{R}^2 \) and satisfy (13), except at \((0, 0)\) when \( a = 0 \), where they are only continuous.

(b) \( u_a(x, y) < 0 \) when \( y > 0 \) for all \( x \), and \( u_a(x, 0) = 0 \) for all \( x \), and \( u_a(x, y) > 0 \) when \( y < 0 \) for all \( x \).

(c) \( v_a(x, y) > 0 \) when \( x > 0 \) for all \( y \), and \( v_a(0, y) = 0 \) for all \( y \), and \( v_a(x, y) < 0 \) when \( x < 0 \) for all \( y \).

(d) \( u_a(0, y) = -y(|a| + \sqrt{y^2 + a^2})^{-1/2} \) for all \( y \).

(e) \( v_a(x, 0) = x(x^2 + 2|a|)^{1/2} \) for all \( x \).

In [9 Ex. 5.2 & Ex. 5.4] we give two further examples:

**Example 4.4.** Let \( \alpha, \beta, \gamma \in \mathbb{R} \) and define \( u(x, y) = \alpha x + \beta \) and \( v(x, y) = \alpha y + \gamma \). Then \( u, v \) satisfy (12) for any value of \( a \).

**Example 4.5.** Let \( S = \mathbb{R}^2 \), \( u(x, y) = |y| - \frac{1}{2} \cosh 2x \) and \( v(x, y) = -y \sinh 2x \). Then \( u, v \) satisfy (12), except that \( \frac{\partial u}{\partial y} \) is not well-defined on the \( x \)-axis. So equation (11) with \( a = 0 \) defines an explicit special Lagrangian 3-fold \( N \) in \( \mathbb{C}^3 \). It turns out that \( N \) is the union of two nonsingular SL 3-folds intersecting in a real curve, which are constructed in [7 Ex. 7.4] by evolving paraboloids in \( \mathbb{C}^3 \).
4.3 Results motivated by complex analysis

Section 6 of [9] proves analogues for solutions of (13) of results on the zeroes of holomorphic functions, using the ideas of winding number and multiplicity, defined in [9] Def. 6.1 & Def. 6.3.

**Definition 4.6.** Let \( C \) be a compact oriented 1-manifold, and \( \gamma : C \to \mathbb{R}^2 \setminus \{0\} \) a differentiable map. Then the winding number of \( \gamma \) about 0 along \( C \) is

\[
\frac{1}{2\pi} \int_C \gamma^* (d\theta),
\]

where \( d\theta \) is the closed 1-form \((x\,dy - y\,dx)/(x^2 + y^2)\) on \( \mathbb{R}^2 \setminus \{0\} \).

In fact the winding number is simply the topological degree of \( \gamma \). Thus it is actually well-defined for \( \gamma \) only continuous, and is invariant under continuous deformations of \( \gamma \), which will be important in [9].

We gave the \( d\theta \) definition for differentiable \( \gamma \) first only for the sake of explicitness.

**Definition 4.7.** Let \( S \) be a domain in \( \mathbb{R}^2 \), let \( a \neq 0 \), and suppose \((u_1, v_1)\) and \((u_2, v_2)\) are solutions of (13) in \( C^1(S) \). Let \( k \geq 1 \) be an integer and \((b, c) \in S^0\). We say that \((u_1, v_1) - (u_2, v_2)\) has a zero of multiplicity \( k \) at \((b, c)\) if \( \partial^j u_1(b, c) = \partial^j u_2(b, c) \) and \( \partial^j v_1(b, c) = \partial^j v_2(b, c) \) for \( j = 0, \ldots, k - 1 \), but \( \partial^k u_1(b, c) \neq \partial^k u_2(b, c) \) and \( \partial^k v_1(b, c) \neq \partial^k v_2(b, c) \). If \((u_1, v_1) \neq (u_2, v_2)\) then every zero of \((u_1, v_1) - (u_2, v_2)\) has a unique multiplicity.

In [9] Prop. 6.5 we show that zeroes of \((u_1, v_1) - (u_2, v_2)\) resemble zeroes of holomorphic functions to leading order.

**Proposition 4.8.** Let \( S \) be a domain in \( \mathbb{R}^2 \), let \( a \neq 0 \), and let \((u_1, v_1)\) and \((u_2, v_2)\) be solutions of (13) in \( C^1(S) \). Suppose \((u_1, v_1) - (u_2, v_2)\) has a zero of multiplicity \( k \geq 1 \) at \((b, c) \in S^0 \). Then there exists a nonzero complex number \( C \) such that

\[
\lambda u_1(x, y) + iv_1(x, y) = \lambda u_2(x, y) + iv_2(x, y) + C(\lambda(x - b) + i(y - c))^k + O((|x - b|^{k+1} + |y - c|^{k+1})),
\]

(16)

where \( \lambda = \sqrt{2}(v_1(b, c))^2 + c^2 + a^2 \)^{1/4}.

In [9] Th. 6.7 we give a formula for the number of zeroes of \((u_1, v_1) - (u_2, v_2)\).

**Theorem 4.9.** Let \( S \) be a domain in \( \mathbb{R}^2 \) and \((u_1, v_1), (u_2, v_2)\) solutions of (13) in \( C^1(S) \) for some \( a \neq 0 \), with \((u_1, v_1) \neq (u_2, v_2)\) at every point of \( \partial S \). Then \((u_1, v_1) - (u_2, v_2)\) has finitely many zeroes in \( S \). Let there be \( n \) zeroes, with multiplicities \( k_1, \ldots, k_n \). Then the winding number of \((u_1, v_1) - (u_2, v_2)\) about 0 along \( \partial S \) is \( \sum_{i=1}^n k_i \).

4.4 Generating \( u, v \) from a potential \( f \)

In [9] Prop. 7.1 we show that solutions \( u, v \in C^1(S) \) of (13) come from a potential \( f \in C^2(S) \) satisfying a second-order quasilinear elliptic equation.
Proposition 4.10. Let \( S \) be a domain in \( \mathbb{R}^2 \) and \( u, v \in C^1(S) \) satisfy \( (13) \) for \( a \neq 0 \). Then there exists \( f \in C^2(S) \) with \( \frac{\partial f}{\partial y} = u \), \( \frac{\partial f}{\partial x} = v \) and

\[
P(f) = \left( \left( \frac{\partial f}{\partial y} \right)^2 + y^2 + a^2 \right)^{-1/2} \frac{\partial^2 f}{\partial x^2} + 2 \frac{\partial^2 f}{\partial y^2} = 0. \tag{17}
\]

This \( f \) is unique up to addition of a constant, \( f \mapsto f + c \). Conversely, all solutions of \( (17) \) yield solutions of \( (13) \).

Equation \( (17) \) may also be written in divergence form as

\[
P(f) = \frac{\partial}{\partial x} \left[ A(a, y, \frac{\partial f}{\partial x}) \right] + 2 \frac{\partial^2 f}{\partial y^2} = 0, \tag{18}
\]

where \( A(a, y, v) = \int_0^v (w^2 + y^2 + a^2)^{-1/2} \, dw \), so that \( \frac{\partial A}{\partial v} = (v^2 + y^2 + a^2)^{-1/2} \). Note that \( A \) is undefined when \( a = y = 0 \).

In [10, Th. 7.6] we prove existence and uniqueness of solutions to the Dirichlet problem for \( (17) \) in strictly convex domains in \( \mathbb{R}^2 \), as in Definition 3.1.

Theorem 4.11. Let \( S \) be a strictly convex domain in \( \mathbb{R}^2 \), and let \( a \neq 0, k \geq 0 \) and \( \alpha \in (0, 1) \). Then for each \( \phi \in C^{k+2,\alpha}(\partial S) \) there exists a unique solution \( f \) of \( (17) \) in \( C^{k+2,\alpha}(S) \) with \( f|_{\partial S} = \phi \). This \( f \) is real analytic in \( S^\circ \), and satisfies \( \|f\|_{C^1} \leq C\|\phi\|_{C^2} \), for some \( C > 0 \) depending only on \( S \).

After considerable work, this was extended to the case \( a = 0 \) in [10, Th. 7.1].

Theorem 4.12. Let \( S \) be a strictly convex domain in \( \mathbb{R}^2 \) invariant under the involution \( (x, y) \mapsto (x, -y) \), let \( k \geq 0 \) and \( \alpha \in (0, 1) \). Then for each \( \phi \in C^{k+3,\alpha}(\partial S) \) there exists a unique weak solution \( f \) of \( (17) \) in \( C^1(S) \) with \( f|_{\partial S} = \phi \). Furthermore \( f \) is twice weakly differentiable and satisfies \( (17) \) with weak derivatives.

Let \( u = \frac{\partial f}{\partial y} \) and \( v = \frac{\partial f}{\partial x} \). Then \( u, v \in C^0(S) \) are weakly differentiable and satisfy \( (13) \) with weak derivatives, and \( v \) satisfies \( (19) \) weakly with \( a = 0 \). The weak derivatives \( \frac{\partial u}{\partial y}, \frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial v}{\partial x} \) satisfy \( \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \in L^p(S) \) for \( p \in [1, 2] \), and \( \frac{\partial u}{\partial y} \in L^q(S) \) for \( q \in [1, 2] \), and \( \frac{\partial u}{\partial y} \) is bounded on \( S \). Also \( u, v \) are \( C^{k+2,\alpha} \) in \( S \) and real analytic in \( S^\circ \) except at singular points \( (x, 0) \) with \( v(x, 0) = 0 \).

Combined with Propositions 4.9 and 4.10, these two theorems give an existence and uniqueness result for \( U(1) \)-invariant SL 3-folds in \( C^3 \) satisfying certain boundary conditions. In [10, Th. 7.2] we show that in the last two theorems \( f \) depends continuously on \( \phi, a \).

Theorem 4.13. Let \( S \) be a strictly convex domain in \( \mathbb{R}^2 \) invariant under the involution \( (x, y) \mapsto (x, -y) \), let \( k \geq 0 \) and \( \alpha \in (0, 1) \). Then the map \( C^{k+3,\alpha}(\partial S) \times \mathbb{R} \to C^1(S) \) taking \( (\phi, a) \mapsto f \) is continuous, where \( f \) is the unique solution of \( (17) \) (with weak derivatives) with \( f|_{\partial S} = \phi \) constructed in Theorem 4.11 when \( a \neq 0 \), and in Theorem 4.12 when \( a = 0 \). This map is also continuous in stronger topologies on \( f \) than the \( C^1 \) topology.
In [9] Th. 7.11 we prove an analogue of Theorem 4.9 where we count zeroes of \((u_1, v_1) - (u_2, v_2)\) not in terms of a winding number, but in terms of the stationary points of the difference of potentials \(\phi_1 - \phi_2\) on \(\partial S\).

**Theorem 4.14.** Let \(S\) be a strictly convex domain in \(\mathbb{R}^2\), let \(a \neq 0, \alpha \in (0, 1)\), and \(f_1, f_2 \in C^{2,\alpha}(S)\) satisfy (14) with \(f_j|_\partial S = \phi_j\). Set \(u_j = \frac{\partial f_j}{\partial y}\) and \(v_j = \frac{\partial f_j}{\partial x}\), so that \(u_j, v_j \in C^{1,\alpha}(S)\) satisfy (15). Suppose \(\phi_1 - \phi_2\) has exactly \(l\) local maxima and \(l\) local minima on \(\partial S\). Then \((u_1, v_1) - (u_2, v_2)\) has \(n\) zeroes in \(S^o\) with multiplicities \(k_1, \ldots, k_n\), where \(\sum_{i=1}^n k_i \leq l - 1\).

### 4.5 An elliptic equation satisfied by \(v\)

In [9] Prop. 8.1 we show that if \(u, v\) satisfy (18) then \(v\) satisfies a second-order quasilinear elliptic equation, and conversely, any solution \(v\) of this equation extends to a solution \(u, v\) of (18).

**Proposition 4.15.** Let \(S\) be a domain in \(\mathbb{R}^2\) and \(u, v \in C^2(S)\) satisfy (18) for \(a \neq 0\). Then

\[
Q(v) = \frac{\partial}{\partial x} \left[ (v^2 + y^2 + a^2)^{-1/2} \frac{\partial v}{\partial y} \right] + 2 \frac{\partial^2 v}{\partial y^2} = 0. \tag{19}
\]

Conversely, if \(v \in C^2(S)\) satisfies (18) then there exists \(u \in C^2(S)\), unique up to addition of a constant \(u \mapsto u + c\), such that \(u, v\) satisfy (18).

In [9] Prop. 8.7 we show that solutions of (18) satisfying strict inequalities on \(\partial S\) satisfy the same inequality on \(S\).

**Proposition 4.16.** Let \(S\) be a domain in \(\mathbb{R}^2\), let \(a \neq 0\), and suppose \(v, v' \in C^2(S)\) satisfy (18) on \(S\). If \(v < v'\) on \(\partial S\) then \(v < v'\) on \(S\).

In [9] Th. 8.8 we prove existence and uniqueness of solutions to the Dirichlet problem for (19) when \(a \neq 0\) in arbitrary domains in \(\mathbb{R}^2\).

**Theorem 4.17.** Let \(S\) be a domain in \(\mathbb{R}^2\). Then whenever \(a \neq 0\), \(k \geq 0\), \(\alpha \in (0, 1)\) and \(\phi \in C^{k+2,\alpha}(\partial S)\) there exists a unique solution \(v \in C^{k+2,\alpha}(S)\) of (19), with \(v|_{\partial S} = \phi\). Fix a basepoint \((x_0, y_0) \in S\). Then there exists a unique \(u \in C^{k+2,\alpha}(S)\) with \(u(x_0, y_0) = 0\) such that \(u, v\) satisfy (19). Furthermore, \(u, v\) are real analytic in \(S^o\).

This was extended to \(a = 0\) in [10] Th. 6.1, for a restricted class of domains \(S\). Much of the technical work in [10] went into proving \(u, v\) are continuous.

**Theorem 4.18.** Let \(S\) be a strictly convex domain in \(\mathbb{R}^2\) invariant under the involution \((x, y) \mapsto (x, -y)\), let \(k \geq 0\) and \(\alpha \in (0, 1)\). Suppose \(\phi \in C^{k+2,\alpha}(\partial S)\) with \(\phi(x, 0) \neq 0\) for points \((x, 0)\) in \(\partial S\). Then there exists a unique weak solution \(v\) of (19) in \(C^0(S)\) with \(a = 0\) and \(v|_{\partial S} = \phi\).

Fix a basepoint \((x_0, y_0) \in S\). Then there exists a unique \(u \in C^0(S)\) with \(u(x_0, y_0) = 0\) such that \(u, v\) are weakly differentiable in \(S\) and satisfy (12) with...
weak derivatives. The weak derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ satisfy $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \in L^p(S)$ for $p \in [1, \frac{3}{2})$, and $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \in L^q(S)$ for $q \in [1, 2)$, and $\frac{\partial v}{\partial x}$ is bounded on $S$. Also $u, v$ are $C^{k+2, a}$ in $S$ and real analytic in $S^0$ except at singular points $(x, 0)$ with $v(x, 0) = 0$.

Combined with Proposition 4.11 these two theorems give an existence and uniqueness for nonsingular and singular U(1)-invariant SL 3-folds in $\mathbb{C}^3$ satisfying certain boundary conditions. In [10, Th. 6.2] we show that in the last two theorems $u, v$ depend continuously on $\phi, a$.

**Theorem 4.19.** Let $S$ be a strictly convex domain in $\mathbb{R}^2$ invariant under the involution $(x, y) \mapsto (x, -y)$, let $k \geq 0$, $\alpha \in (0, 1)$, and $(x_0, y_0) \in S$. Define $X$ to be the set of $\phi \in C^{k+2, a}(\partial S)$ with $\phi(x, 0) = 0$ for some $(x, 0) \in \partial S$. Then the map $C^{k+2, a}(\partial S) \times \mathbb{R} \setminus X \times \{0\} \to C^0(S)^2$ taking $(\phi, a) \to (u, v)$ is continuous, where $(u, v)$ is the unique solution of (13) (with weak derivatives when $a = 0$) with $v|_{\partial S} = \phi$ and $u(x_0, y_0) = 0$, constructed in Theorem 4.17 when $a \neq 0$, and in Theorem 4.18 when $a = 0$. This map is also continuous in stronger topologies on $(u, v)$ than the $C^0$ topology.

### 4.6 A class of solutions of (12) with singularities

In most of the rest of the paper we shall be studying solutions of (12) with singularities, and the corresponding SL 3-folds $N$. So we need to know just what we mean by a singular solution of (12). We give a definition here, to avoid repeating technicalities about weak derivatives, and so on, many times.

**Definition 4.20.** Let $S$ be a domain in $\mathbb{R}^2$ and $u, v \in C^0(S)$. We say that $u, v$ are a singular solution of (12) if

(i) $u, v$ are weakly differentiable, and their weak derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ in $L^1(S)$ satisfy (12),

(ii) $v$ is a weak solution of (19) with $a = 0$, as in (13),

(iii) Define the singular points of $u, v$ to be those $(x, 0) \in S$ with $v(x, 0) = 0$. Then except at singular points, $u, v$ are $C^2$ in $S$ and real analytic in $S^0$, and satisfy (12) in the classical sense.

(iv) For $a \in (0, 1]$ there exist $u_a, v_a \in C^2(S)$ satisfying (13) such that as $a \to 0+$ we have

\[
\begin{align*}
    u_a \to u, & \quad v_a \to v & & \text{in } C^0(S), \quad \frac{\partial u_a}{\partial x} \to \frac{\partial u}{\partial x}, \quad \frac{\partial u_a}{\partial y} \to \frac{\partial v}{\partial y} & & \text{in } L^2(S), \\
    \frac{\partial u_a}{\partial y} \to \frac{\partial v_a}{\partial y} & & \text{in } L^q(S), \quad q \in [1, 2), & & \text{and } \frac{\partial u_a}{\partial x} \to \frac{\partial v_a}{\partial x} & & \text{in } L^r(X), \quad r \in [1, \infty). 
\end{align*}
\]

The reason we choose this definition is that Theorems 4.12 and 4.13 prove existence and uniqueness of singular solutions $u, v$ of (12) with boundary conditions on certain domains $S$. Note that part (iv) holds automatically in these theorems because $u, v$ were constructed as limits of $u_a, v_a$ satisfying (20) in the
Proof. For \(a \in (0,1]\) let \(u_a, v_a\) be as in Definition 4.20(iv), and define \(N_a\) by (11) using \(u_a, v_a\). Then \(N_a\) is a compact, nonsingular SL 3-fold with boundary, and so defines a special Lagrangian integral current. We shall show that \(N_a \to N\) as rectifiable currents as \(a \to 0_+\), so that \(N\) is an SL rectifiable current.

Calculation shows that the volume of \(N_a\) is given by

\[
\text{vol}(N_a) = \int_{N_a} \text{Re} \Omega = \int_S (1 + \frac{\partial u_a}{\partial x} \frac{\partial v_a}{\partial y} - \frac{\partial u_a}{\partial y} \frac{\partial v_a}{\partial x}) \, dx \, dy.
\]

Fix \(q \in [1,2]\) and \(r \in [1,\infty)\) with \(\frac{1}{q} + \frac{1}{r} = 1\). Then by (20), \(\frac{\partial u_a}{\partial x}, \frac{\partial v_a}{\partial y}, \frac{\partial u_a}{\partial y}, \frac{\partial v_a}{\partial x}\) converge in \(L^2, L^2, L^q, L^r(S)\) respectively as \(a \to 0_+\). Hence \(\text{vol}(N_a)\) converges to the well-defined integral \(\int_S (1 + \frac{\partial u_a}{\partial x} \frac{\partial v_a}{\partial y} - \frac{\partial u_a}{\partial y} \frac{\partial v_a}{\partial x}) \, dx \, dy\) as \(a \to 0_+\), by Hölder’s inequality. In particular, \(\text{vol}(N_a)\) is uniformly bounded for small \(a\).

Using (20), one can show that the family of rectifiable currents \(N_a\) is Cauchy in the flat norm on currents as \(a \to 0_+\). But the family of integral flat chains supported in a given compact subset of \(\mathbb{R}^n\) is complete in the flat norm. Thus \(N_a \to N\) in the flat norm as \(a \to 0_+\) for a unique integral flat chain \(N\).

As \(\text{vol}(N_a)\) is uniformly bounded for small \(a\) we have \(\text{vol}(N) < \infty\). So by the Closure Theorem 15.5.4, \(N\) is a rectifiable current. Since \(N_a \to N\) as currents, and \(N_a\) is special Lagrangian for all \(a\), we see that \(N\) is special Lagrangian. And as \(u_a \to u\), \(v_a \to v\) in \(C^0(S)\) the supports of \(\partial N_a\) converge to (21) as \(a \to 0_+\), so \(\partial N\) is supported in (21).

Note that we do not claim that \(N\) is an integral current, nor that \(\partial N\) is rectifiable. This is because our assumptions are not strong enough to ensure that \(\partial N\) has finite area (Hausdorff 2-measure) near points \((0,0,x+iu(x,0))\) for \((x,0) \in \partial S\) with \(v(x,0) = 0\). For \(u, v\) in Theorem 14.12, the estimates of 14 do not imply \(\partial N\) has finite area, which is why we have not supposed it. For \(u, v\) in Theorem 1.18, \(\partial N\) does have finite area, and \(N\) is an integral current.
\section{U(1)-invariant special Lagrangian cones}

Combining Proposition \ref{prop:existence} and Theorems \ref{thm:main} and \ref{thm:existence} we get powerful existence results for singular SL 3-folds of the form \((\ref{eq:form})\) with \(a = 0\). By Proposition \ref{prop:existence} we may regard these as \textit{minimal rectifiable currents}, as in \ref{chap:existence}, and so by Theorem \ref{thm:existence} there exists a \textit{tangent cone} at each singular point, which will be a U(1)-invariant SL cone in \(\mathbb{C}^3\).

In this section we will study the possible tangent cones of singular SL 3-folds of the form \((\ref{eq:form})\) with \(a = 0\), and find there are only a few possibilities, which can be written down very explicitly. We begin by quoting work of the author \cite{6} and Haskins \cite{5} on U(1)-invariant SL cones in \(\mathbb{C}^3\). Our first result comes from \cite[Th. 8.4]{0} with \(a_1 = a_2 = -1\) and \(a_3 = 2\), with some changes in notation.

\textbf{Theorem 5.1.} Let \(N_0\) be a closed special Lagrangian cone in \(\mathbb{C}^3\) invariant under the \(U(1)\)-action \((\ref{eq:action})\), with \(N_0 \setminus \{0\}\) connected. Then there exist \(A \in [-1, 1]\) and functions \(w : \mathbb{R} \to (-\frac{1}{2}, 1)\) and \(\alpha, \beta : \mathbb{R} \to \mathbb{R}\) satisfying

\[
\left(\frac{dw}{dt}\right)^2 = 4((1 - w)^2(1 + 2w) - A^2), \quad \frac{d\alpha}{dt} = \frac{A}{1 - w},
\]

\[
\frac{d\beta}{dt} = -\frac{2A}{1 + 2w} \quad \text{and} \quad (1 - w)(1 + 2w)^{1/2} \cos(2\alpha + \beta) \equiv A,
\]

such that away from points \((z_1, z_2, z_3) \in \mathbb{C}^3\) with \(z_j = 0\) for some \(j\), we may locally write \(N_0\) in the form \(\{\Phi(r, s, t) : r > 0, s, t \in \mathbb{R}\}\), where

\[
\Phi : (r, s, t) \mapsto \left(\Re^{i(\alpha(t) + s)} \sqrt{1 - w(t)}, \Re^{i(\alpha(t) - s)} \sqrt{1 - w(t)}, \Re^{i\beta(t)} \sqrt{1 + 2w(t)}\right).
\]

We can say more about \(N_0\) by dividing into cases, depending on \(A\).

\textbf{Theorem 5.2.} In the situation of Theorem \ref{thm:existence} we have

(a) If \(A = 1\) then \(N_0\) is the \(U(1)^2\)-invariant SL \(T^2\)-cone

\[
\{(\Re^{i\theta_1}, \Re^{i\theta_2}, \Re^{i\theta_3}) : r \geq 0, \quad \theta_1, \theta_2, \theta_3 \in \mathbb{R}, \quad \theta_1 + \theta_2 + \theta_3 = 0\}.
\]

(b) If \(A = -1\) then \(N_0\) is the \(U(1)^2\)-invariant SL \(T^2\)-cone

\[
\{(\Re^{i\theta_1}, \Re^{i\theta_2}, \Re^{i\theta_3}) : r \geq 0, \quad \theta_1, \theta_2, \theta_3 \in \mathbb{R}, \quad \theta_1 + \theta_2 + \theta_3 = \pi\}.
\]

(c) If \(A = 0\) then for some \(\phi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\) either \(N_0 = \Pi^\phi_+\) or \(N_0 = \Pi^\phi_-\) or \(N_0\) is the singular union \(\Pi^\phi_+ \cup \Pi^\phi_-\), where \(\Pi^\phi_{\pm}\) are the SL 3-planes

\[
\Pi^\phi_+ = \{(z, \Re^{i\phi} \bar{z}, \Re^{i\phi}) : z \in \mathbb{C}, \quad r \in \mathbb{R}\}
\]

and \(\Pi^\phi_- = \{(z, -\Re^{i\phi} \bar{z}, \Re^{i\phi}) : z \in \mathbb{C}, \quad r \in \mathbb{R}\}.
\]
Proposition 5.4. Let $N$ under the hypotheses of Theorem 5.1.

1. noting that $\Phi$ above is The SL

Proposition 5.3. $u$ with $T$ action (1) and intersect in the line $T$ The point about the SL 3-planes $\Pi$ constant, and (c) then follows from Theorem 5.1 with a certain amount of work.

§[6, 7.11] shows that $z$ be written

in the next three propositions, we work out which of the SL cones above can be written globally in the form (11). The first follows from Theorem 4.3, as (25)

elliptic functions $\text{sn}(\cdot)$ and $\text{cn}(\cdot)$ are non-constant and periodic with period $T$. The limiting values of $\Phi$ as $A \to 0_+$ and $A \to 1_-$, and $\Phi : (0, 1) \to \mathbb{R}$ is real analytic and strictly monotone decreasing with $\Phi(A) \to -\pi$ as $A \to 0_+$ and $\Phi(A) \to -2\pi/\sqrt{3}$ as $A \to 1_-$, and $\Phi : (-1, 0) \to \mathbb{R}$ is real analytic and strictly monotone decreasing with $\Phi(A) \to \pi$ as $A \to 0_-$ and $\Phi(A) \to 2\pi/\sqrt{3}$ as $A \to -1_+$.

Proof. The division into cases (a)–(d) corresponds to the cases considered in [6, §7.2–§7.5]. Case (a) comes immediately from [6, §7.3], and (b) follows from (a) by replacing $z_3$ by $-z_3$. When $A = 0$, equation (23) shows that $\alpha, \beta$ are constant, and (c) then follows from Theorem 5.1 with a certain amount of work.

The point about the SL 3-planes $\Pi_{\pm}^\phi$ is that they are invariant under the U(1)-action (11) and intersect in the line $\{ (0, 0, re^{i\phi}) : r \in \mathbb{R} \}$. Thus if $N_0 = \Pi_+^\phi \cup \Pi_-^\phi$, then $N_0 \setminus \{ 0 \}$ is connected, which is why these two SL 3-planes can be combined under the hypotheses of Theorem 5.1.

For case (d), when $A \in (0, 1)$ explicit formulae for $w$ in terms of the Jacobi elliptic functions $\text{sn}(t, k)$ are given in [6, Prop. 8.6] and [5, Prop. 4.2], and [6, Prop. 7.11] shows that $w$ and $2\alpha + \beta$ are non-constant and periodic with period $T$. The limiting values of $\Phi$ as $A \to 0_+$ and $A \to 1_-$ follow from [6, Prop. 7.13], noting that $\Phi$ above is $\frac{1}{2} \Psi$ in the notation of [6].

By Haskins [5, p. 20], $\Phi(A)$ is strictly monotone on $(0, 1)$, noting that the U(1)-action (11) corresponds to the case $\alpha = 0$ in [5], and $\Phi$ and $A$ in our notation correspond to $\Theta_2$ and $3\sqrt{3}J$ in his notation. This proves (d) when $A \in (0, 1)$.

The claims for $A \in (-1, 0)$ follow by replacing $z_3$ by $-z_3$.

We are interested in the tangent cones not of arbitrary U(1)-invariant SL 3-folds, but only those which can be written globally in the form (11). Therefore, in the next three propositions, we work out which of the SL cones above can be written globally in the form (11). The first follows from Theorem 1.3. as (24) agrees with the SL 3-fold $N_0$ of (11).

Proposition 5.3. The SL $T^2$-cone of (26) may be written in the form (11) with $u = u_0$ and $v = v_0$, where $u_0, v_0$ are as in Theorem 4.1. Similarly, the SL $T^2$-cone of (27) may be written in the form (11) with $u = -u_0$ and $v = -v_0$.

The second is elementary.

Proposition 5.4. Let $\phi \in (\frac{\pi}{2}, \frac{3\pi}{2})$. Then the SL 3-planes $\Pi_{\pm}^\phi$ of (27) may be written

\begin{align*}
\Pi_{+}^\phi = \{ (z_1, z_2, z_3) \in \mathbb{C}^3 : z_1z_2 = v + iy, \ |z_1|^2 - |z_2|^2 = 0, \\
& \quad u = x \tan \phi, \ v = y \tan \phi, \ x \in \mathbb{R}, \ y \geqslant 0 \} \tag{28}
\end{align*}

\begin{align*}
\Pi_-^\phi = \{ (z_1, z_2, z_3) \in \mathbb{C}^3 : z_1z_2 = v + iy, \ |z_1|^2 - |z_2|^2 = 0, \\
& \quad u = x \tan \phi, \ v = y \tan \phi, \ x \in \mathbb{R}, \ y \leqslant 0 \} \tag{29}
\end{align*}

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Thus the union $\Pi_+^\phi \cup \Pi_-^\phi$ may be written in the form $\mathbf{11}$ with $u = x \tan \phi$, $v = y \tan \phi$ and $a = 0$.

However, the cones $N_0$ in case (d) above cannot be written this way.

**Proposition 5.5.** None of the SL cones $N_0$ of Theorem $\mathbf{11}$ with $0 < |A| < 1$ may be written globally in the form $\mathbf{11}$ with single-valued functions $u, v$.

**Proof.** For the cone $N_0$ of Theorem 5.1 to be closed without boundary, we need the function $\Phi$ of (24) to be periodic in $t$. Since $w$ and $2\omega + \beta$ are periodic with period $T$ by part (d) of Theorem 5.2, the possible periods of $\Phi$ in $t$ are $qT$ for integers $q \geq 1$, and this will happen if $e^{i\beta(t)}$ has period $qT$. Now as $w(t)$ has period $T$, we see from (23) and the definition of $\Phi(A)$ that $\beta(t + T) = \beta(t) + \Phi(A)$ for all $t$, and hence $\beta(t + qT) = \beta(t) + q\Phi(A)$.

Therefore $e^{i\beta(t + qT)} = e^{i\beta(t)}$ if and only if $q\Phi(A) = 2\pi p$ for some $p \in \mathbb{Z}$.

Hence $\Phi$ has period $qT$ if $\Phi(A) = 2\pi p q^{-1}$ for integers $p, q$, where $q \geq 1$ is as small as possible. But $\Phi(A)$ lies in $(\pi, 2\pi/3)$ or $(-2\pi/3, -\pi)$ by part (d) of Theorem 5.2 and thus $\frac{p}{q}$ lies in $(\frac{1}{2}, \frac{1}{\sqrt{3}})$ or $(-\frac{1}{\sqrt{3}}, -\frac{1}{2})$. It easily follows that $q \geq 3$ and $|p| \geq 2$.

Now by (24) we have $x + iu = z_3 = r\sqrt{1 + 2w}e^{i\beta}$ and $v + iy = z_1z_2 = r^2(1-w)e^{(2\omega+\beta)}e^{-i\beta}$. Since $2\omega + \beta$ is periodic, considering the phases of $x + iu$ and $v + iy$ we see that in one period $qT$ of $t$ the phase of $x + iu$ rotates through an angle $2\pi p$, and the phase of $v + iy$ rotates through an angle $-2\pi p$.

It is not difficult to use this to show that for generic $(x, y) \in \mathbb{R}^2$ we expect $|p|$ points $(u, v)$ in $\mathbb{R}^2$ (possibly counting with multiplicity) for which $x + iu$ and $v + iy$ can be written in the form above for some $(z_1, z_2, z_3) \in N_0$. For instance, if $x = 0$, $y > 0$ and $\cos(2\omega(t) + \beta(t)) > 0$ for all $t$, which is reasonable as $2\omega + \beta$ is periodic, then there is one point $(u, v)$ over $(x, y)$ for each $t \in [0, T]$ with $e^{i\beta} = i$. Since $\beta$ increases by $2\pi p$ on $[0, T]$, there will be $|p|$ such points.

Thus, as $|p| \geq 2$ there will be at least 2 points $(u, v)$ over each generic $(x, y)$, so $N_0$ cannot be written in the form $\mathbf{11}$ for single-valued functions $u, v$, but only for multi-valued ‘functions’ $(u, v)$ with at least 2 values at generic points.  

We can now classify the possible tangent cones in our problem.

**Theorem 5.6.** Let $S$ be a domain in $\mathbb{R}^2$ and $u, v \in C^0(S)$ a singular solution of (2), as in (4.6). Define $N$ by $\mathbf{11}$ with $a = 0$. Then by Proposition 4.21 we may regard $N$ as a minimal rectifiable current, as in §3.3.

Let $z \in N^0$ be a singular point, so that $z = (0, 0, z_3)$ by Proposition 4.1(a), and $C$ be a tangent cone to $N$ at $z$. Then the only possibilities for $C$ are

(i) $C$ is given in (24), with multiplicity 1.

(ii) $C$ is given in (25), with multiplicity 1.

(iii) $C$ is $\Pi_+^\phi$ or $\Pi_-^\phi$, or $\Pi_+^\phi \cup \Pi_-^\phi$ for some $\phi \in (-\frac{\pi}{2}, \frac{\pi}{2})$, where $\Pi_+^\phi$ and $\Pi_-^\phi$ are defined in (24) and have multiplicity 1.

(iv) $C$ is the sum of $\Pi_+^{\phi/2}$ with multiplicity $k$ and $\Pi_-^{\phi/2}$ with multiplicity $l$, for nonnegative integers $k, l$ with $k + l \geq 1$.

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Proof. Let $C$ be a tangent cone to $N$ at $z$. Then $C$ is a minimal locally rectifiable cone without boundary, and is special Lagrangian by Theorem 5.8. Since $N$ is invariant under the $U(1)$-action $\mathbb{I}$ which fixes $z$ it easily follows that $C$ is invariant under $\mathbb{I}$. Also, $C$ is an embedded minimal 3-submanifold with positive integer multiplicity outside a closed singular set $S$ of Hausdorff dimension at most 1, by Theorem 5.6.

Since $N \setminus \{z\}$ is locally connected, by considering the limiting process defining $C$ one can show that $C \setminus \{0\}$ is also connected. As $C$ is a $U(1)$-invariant cone, $S$ is also a $U(1)$-invariant cone. Therefore $S$ is a union of $(0, 0, 0)$ and some collection of rays $\{(0, 0, re^{i\theta}) : r > 0\}$, as otherwise $S$ would contain a Hausdorff dimension 2 cone on a $U(1)$-orbit. So $S \subseteq \{(0, 0, z) : z \in \mathbb{C}\}$.

Thus $C$ is a $U(1)$-invariant SL cone in $\mathbb{C}^3$, with multiplicity, outside a singular set $S$. Such cones are locally classified in Theorem 5.1, and so $C \setminus S$ is locally parametrized by $\Phi$ as in (24). Hence, each connected component of $C \setminus S$ fits into the framework of Theorems 5.1 and 5.2 with some value of $A$, functions $u, \alpha$ and $\beta$, and so on, and some positive integer multiplicity.

In cases (a)–(d) of Theorem 5.2 only the cones $N_0$ of case (c) intersect $\{(0, 0, z) : z \in \mathbb{C}\}$ other than at 0. Since $S$ lies in $\{(0, 0, z) : z \in \mathbb{C}\}$, and $C$ is closed and nonsingular except at $S$, we see that if a component of $C \setminus \{0\}$ locally agrees with a cone $N_0$ in cases (a), (b), (d) of Theorem 5.2 then $C$ must contain all of $N_0$, since otherwise the boundary of the included subset of $N_0$ would lie in $S$.

By an elementary calculation we can prove:

**Lemma 5.7.** Let $N$ be an SL 3-fold of the form $\mathbb{I}$ with $u, v : S \to \mathbb{R}$ locally $C^1$ almost everywhere, and let $\chi : \mathbb{R}^2 \to \mathbb{R}$ be smooth and compactly-supported. Then

$$\int_N \chi(\text{Re } z_3, \text{Im } (z_1z_2)) \cdot \text{Re}(dz_1 \wedge dz_2) \wedge \text{Re}(dz_3) = 2\pi \int_S \chi(x, y) \, dx \, dy. \quad (30)$$

The important point here is that the right hand side of (30) is independent of $u, v$. Now $C$ is the limit, as a current, of a sequence $r_j^{-1}(N - z)$ as $j \to \infty$. Each $r_j^{-1}(N - z)$ is of the form $\mathbb{I}$ for single-valued $u, v$, which are locally $C^1$ almost everywhere by Definition 4.20(iii), so that (30) holds for $r_j^{-1}(N - z)$.

We wish to take the limit of (30) for $r_j^{-1}(N - z)$ as $j \to \infty$. Currents and their limits are defined by the integral of smooth, compactly-supported forms, and the 3-form on the l.h.s. of (30) is not compactly-supported. However, if we take $\chi$ to be nonnegative then this 3-form has nonnegative restriction to each $r_j^{-1}(N - z)$. Taking the limit $j \to \infty$ then shows that

$$\int_C \chi(\text{Re } z_3, \text{Im } (z_1z_2)) \cdot \text{Re}(dz_1 \wedge dz_2) \wedge \text{Re}(dz_3) \leq 2\pi \int_{\mathbb{R}^2} \chi(x, y) \, dx \, dy, \quad (31)$$

since the effect of a portion of $r_j^{-1}(N - z)$ going to infinity in the support of $\chi(\text{Re } z_3, \text{Im } (z_1z_2))$ is that a positive contribution to the l.h.s. of (30) for $r_j^{-1}(N - z)$ does not appear in the l.h.s. of (31).
Over a small open neighbourhood of a generic point \((x, y) \in \mathbb{R}^2\), \(C\) is non-singular, and splits into \(k\) components, each of the form \((11)\) for single-valued \(u, v\). But then Lemma 5.7 for \(\chi\) supported near \(x, y\) shows that

\[
\int_C \chi(\Re z_3, \Im(z_1z_2)) \cdot \Re(dz_1 \wedge dz_2) \wedge \Re(dz_3) = 2\pi k \int_{\mathbb{R}^2} \chi(x, y) \, dx \, dy.
\]

Comparing this with \((31)\) shows that the only possibilities are \(k = 0, 1\). Therefore for generic \((x, y) \in \mathbb{R}^2\) there are either 0 or 1 points \((u, v)\), counted with multiplicity, such that there exists \((z_1, z_2, z_3) \in C\) with \(z_1z_2 = v + iy\) and \(z_3 = x + iu\). So by Proposition 5.5 the cones \(N_0\) of part (d) of Theorem 5.2 do not occur, even locally, as tangent cones \(C\) to \(N\).

Hence, \(C \setminus \{(0, 0, z) : z \in \mathbb{C}\}\) is a union of connected components, with multiplicity, each of the form \(N_0 \setminus \{(0, 0, z) : z \in \mathbb{C}\}\), where \(N_0\) is one of the SL cones in parts (a)–(c) of Theorem 5.2. If there is more than one component, their closures must intersect in \(S \setminus \{0\}\) to ensure that \(C \setminus \{0\}\) is connected. The only way for this to happen is if \(C = \Pi^0_+ \cup \Pi^0_-\) as in part (c) of Theorem 5.2 with \(S = \Pi^0_+ \cap \Pi^0_-\), so that \(C \setminus S\) has two connected components \(\Pi^0_+ \setminus S\) and \(\Pi^0_- \setminus S\). Thus, \(C\) cannot combine more than one possibility from parts (a)–(c) of Theorem 5.2.

It remains only to pin down the multiplicities of each component of \(C \setminus S\). First note that they are all positive, as SL 3-folds cannot converge in the sense of currents to SL 3-folds with the opposite orientation. In cases (i)–(iii), \(\Theta(C, 0) = k + l\).

Proposition 5.8. In cases (i) and (ii) of Theorem 5.6 we have \(\Theta(C, 0) = \pi/\sqrt{3} \approx 1 \cdot 81\). In case (iii) \(\Theta(C, 0) = 1\) when \(C = \Pi^0_+\) or \(\Pi^0_-\) and 2 when \(C = \Pi^0_+ \cup \Pi^0_-\), and in case (iv) \(\Theta(C, 0) = k + l\).

Proof. For case (i), let \(\Sigma = N_0 \cap S^5\), where \(S^5\) is the unit sphere in \(\mathbb{C}^3\). The metric on \(\Sigma \cong T^2\) is isometric to the quotient of \(\mathbb{R}^2\) with its flat Euclidean metric by the lattice \(\mathbb{Z}^2\) with basis \(2\pi(\frac{1}{\sqrt{3}}, 0), 2\pi(\frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}})\). Taking a \(2 \times 2\) determinant gives \(\text{area}(\Sigma) = 4\pi^2/\sqrt{3}\). The density follows by dividing by \(\text{area}(S^2) = 4\pi\).

Cases (i) and (ii) are isomorphic under \(z_3 \mapsto -z_3\), and so have the same density. Cases (iii) and (iv) are immediate, as \(\Pi^0_\pm\) are \(\mathbb{R}^3\) vector subspaces.

6 Multiplicities of zeroes and counting formulae

We shall now generalize the material of §3.3 to the singular case \(a = 0\). We begin by defining the multiplicity of an isolated zero \((u, v) - (\hat{u}, \hat{v})\), where \(u, v\) and \(\hat{u}, \hat{v}\) are singular solutions of (12).
Definition 6.1. Suppose \( S \) is a domain in \( \mathbb{R}^2 \), and \((u, v), (\hat{u}, \hat{v})\) are singular solutions of (12) in \( S \). We call a point \((b, c) \in S\) a zero of \((u, v) - (\hat{u}, \hat{v})\) in \( S \) if \((u, v) = (\hat{u}, \hat{v})\) at \((b, c)\). A zero \((b, c)\) is called singular if \(c = 0\) and \(v(b, 0) = \hat{v}(b, 0) = 0\), so that \((b, c)\) is a singular point of \( u, v \) and \( \hat{u}, \hat{v} \). Otherwise we say \((b, c)\) is a nonsingular zero. We call a zero \((b, c)\) isolated if for some \( \epsilon > 0 \) there exist no other zeroes \((x, y)\) of \((u, v) - (\hat{u}, \hat{v})\) in \( S \) with \(0 < (x - b)^2 + (y - c)^2 < \epsilon^2\).

Let \((b, c) \in S^a\) be an isolated zero of \((u, v) - (\hat{u}, \hat{v})\). Define the multiplicity of \((b, c)\) to be the winding number \(\gamma_c(b, c)\) of \(u, v\) about \((b, c)\), where \(\epsilon > 0\) is chosen small enough that \(\gamma_c(b, c)\) lies in \( S^a \) and \((b, c)\) is the only zero of \((u, v) - (\hat{u}, \hat{v})\) inside or on \(\gamma_c(b, c)\).

As \((u, v) - (\hat{u}, \hat{v})\) is continuous, though not necessarily differentiable, its winding number about \(0\) along the positively oriented circle \(\gamma_c(b, c)\) is well-defined. Since winding numbers are invariant under continuous deformation of the path, this is independent of \(\epsilon\), so the multiplicity of \((b, c)\) is well-defined.

We shall eventually prove that if \((u, v) \neq (\hat{u}, \hat{v})\) then zeroes \((b, c)\) of \((u, v) - (\hat{u}, \hat{v})\) in \( S^a \) are isolated, with positive multiplicity. It is simple to show that multiplicities are nonnegative, and we do this in §6.2. But if \((b, 0)\) is a singular zero of \((u, v) - (\hat{u}, \hat{v})\), it is more difficult to prove \((b, 0)\) is isolated, or has positive multiplicity.

To do this requires a diversion in §6.2 to study singular solutions \(u, v\) with \(v(x, 0) \equiv 0\). We show such \(u, v\) have the symmetry \(u(x, -y) = u(x, y)\), \(v(x, -y) = -v(x, y)\), and the corresponding SL 3-folds \(N\) are actually the union of two nonsingular U(1)-invariant SL 3-folds \(N_{\pm}\) intersecting in a real analytic real curve \(\gamma\).

In §6.3 we use this to show that if \(v < \hat{v}\) on \(\partial S\), then \(v < \hat{v}\) on \(S\). Section 6.4 then proves that multiplicities of zeroes in Definition 6.1 are positive. This is the essential step in showing zeroes are isolated, which we do in §6.5. Sections 6.5 and 6.7 also prove counting formulae for zeroes in \(S^a\) with multiplicity, in terms of boundary data.

6.1 Nonnegativity of winding numbers and multiplicities

We begin with four propositions. The first implies that intersection multiplicities of singular solutions are nonnegative. Later in Corollary 6.17 we will show that they are in fact positive. Much of the intervening discussion is to exclude the possibility of intersections with multiplicity zero.

Proposition 6.2. Let \(u, v \in C^0(S)\) and \(\hat{u}, \hat{v} \in C^0(S)\) be singular solutions of (12) in a domain \(S\) in \(\mathbb{R}^2\), as in §4.6 such that \((u, v) \neq (\hat{u}, \hat{v})\) at every point of \(\partial S\). Then the winding number of \((u, v) - (\hat{u}, \hat{v})\) about \(0\) along \(\partial S\) is a nonnegative integer.

Proof. By part (iv) of Definition 4.20 we may write \(u, v\) and \(\hat{u}, \hat{v}\) as the limits in \(C^0(S)\) as \(a \to 0_\ast\) of solutions \(u_a, v_a\) and \(\hat{u}_a, \hat{v}_a\) in \(C^2(S)\) of (13) for \(a \in (0, 1]\). As \((u, v) \neq (\hat{u}, \hat{v})\) on \(\partial S\) we see that for small \(a \in (0, 1]\) we have \((u_a, v_a) \neq (\hat{u_a}, \hat{v}_a)\)
shows that the winding number of \((u,v) - (\hat{u}, \hat{v})\) and \((u_a, v_a) - (\hat{u}_a, \hat{v}_a)\) about 0 along \(\partial S\) are equal, as winding numbers are invariant under continuous deformation of the path. But by Theorem 4.4 the winding number of \((u_a, v_a) - (\hat{u}_a, \hat{v}_a)\) about 0 along \(\partial S\) is a nonnegative integer, so the result follows.

**Proposition 6.3.** Let \(u, v \in C^0(S)\) and \(\hat{u}, \hat{v} \in C^0(S)\) be singular solutions of (12) in a domain \(S\) in \(\mathbb{R}^2\), as in Proposition 6.2 such that \((u,v) \neq (\hat{u}, \hat{v})\) at every point of \(\partial S\). Suppose \((b_1, c_1), \ldots, (b_n, c_n)\) are isolated zeroes of \((u,v) - (\hat{u}, \hat{v})\) in \(S^0\) with multiplicities \(k_1, \ldots, k_n\), but not necessarily the only zeroes. Then the winding number of \((u,v) - (\hat{u}, \hat{v})\) about 0 along \(\partial S\) is at least \(\sum_{i=1}^{n} k_i\).

**Proof.** For each \(i = 1, \ldots, n\), choose \(\varepsilon_i > 0\) such that \(\gamma_{c_i}(b_i, c_i)\) lies in \(S^0\) and \((b_i, c_i)\) is the only zero of \((u,v) - (\hat{u}, \hat{v})\) inside or on \(\gamma_{c_i}(b_i, c_i)\), and in addition the circles \(\gamma_{c_i}(b_i, c_i)\) do not intersect. Clearly this is possible if \(\varepsilon_1, \ldots, \varepsilon_n\) are small enough. Examining the proof of Theorem 4.9 in [9] we see that it holds not only for domains \(S\), which are contractible, but also for more general compact 2-submanifolds \(T\) of \(\mathbb{R}^2\) with finitely many boundary components. Thus Proposition 6.2 also holds for such \(T\).

Let \(T = S \setminus \bigcup_{i=1}^{n} B_{\varepsilon_i}(b_i, c_i)\). Then \(T\) is a compact 2-submanifold in \(\mathbb{R}^2\) whose boundary is the disjoint union of \(\partial S\), and the circles \(\gamma_{c_i}(b_i, c_i)\) for \(i = 1, \ldots, n\), with negative orientation. Let the winding number of \((u,v) - (\hat{u}, \hat{v})\) about 0 along \(\partial S\) be \(k\). Then the winding number of \((u,v) - (\hat{u}, \hat{v})\) about 0 along \(\partial T\) is \(k - \sum_{i=1}^{n} k_i\). So \(k \geq \sum_{i=1}^{n} k_i\) by Proposition 6.2.

**Proposition 6.4.** Let \(u, v \in C^0(S)\) and \(\hat{u}, \hat{v} \in C^0(S)\) be singular solutions of (12) in a domain \(S\) in \(\mathbb{R}^2\), such that \((u,v) \neq (\hat{u}, \hat{v})\). Suppose \((b, c) \in S^0\) is a nonsingular zero of \((u,v) - (\hat{u}, \hat{v})\). Then \((b, c)\) is an isolated zero of \((u,v) - (\hat{u}, \hat{v})\), and its multiplicity is a positive integer \(k\), with \(\partial^ju(b,c) = \partial^j\hat{u}(b,c)\) and \(\partial^jv(b,c) = \partial^j\hat{v}(b,c)\) for \(j = 0, \ldots, k-1\), but \(\partial^ku(b,c) \neq \partial^k\hat{u}(b,c)\) and \(\partial^kv(b,c) \neq \partial^k\hat{v}(b,c)\).

**Proof.** As \(u, v\) and \(\hat{u}, \hat{v}\) are nonsingular near \((b,c)\) we can apply the reasoning of [9] [6.1] for the nonsingular case \(a \neq 0\) to \((u,v)\) and \((\hat{u}, \hat{v})\) near \((b,c)\). By [9] Lem. 6.4] we see that if \((u,v) \neq (\hat{u}, \hat{v})\) then \((b,c)\) is an isolated zero of \((u,v) - (\hat{u}, \hat{v})\), and it has a unique multiplicity \(k\), defined as in Definition 6.7 which is a positive integer.

If \(\varepsilon > 0\) is small enough then \((u,v)\) and \((\hat{u}, \hat{v})\) are nonsingular on the closed disc \(\overline{B}_{\varepsilon}(b,c)\), and \((b,c)\) is the only zero of \((u,v) - (\hat{u}, \hat{v})\) there. The proof of Theorem 4.9 in [9] when \(a \neq 0\) is also valid in this case on \(\overline{B}_{\varepsilon}(b,c)\), and shows that the winding number of \((u,v) - (\hat{u}, \hat{v})\) about 0 along \(\gamma_{c}(b,c)\) is \(k\). Thus this multiplicity \(k\) coincides with that in Definition 6.7, and the proof is complete.

When \(\hat{u} = \alpha x + \beta, \hat{v} = \alpha y + \gamma\), intersection multiplicities are positive.

**Proposition 6.5.** Let \((u,v)\) be a singular solution of (12) in a domain \(S\) in \(\mathbb{R}^2\), as in 4.4, and let \(\hat{u}(x,y) = \alpha x + \beta\) and \(\hat{v}(x,y) = \alpha y + \gamma\) for \(\alpha, \beta, \gamma \in \mathbb{R}\), as in Example 4.4. Suppose \((u,v) - (\hat{u}, \hat{v})\) has a zero \((b,c)\) in \(S^0\). Then the winding number of \((u,v) - (\hat{u}, \hat{v})\) about 0 along \(\partial S\) is a positive integer.
Proof. By part (iv) of Definition 6.20 we may write \( u, v \) as the limits in \( C^0(S) \) as \( a \to 0^+ \) of solutions \( u_a, v_a \) in \( C^2(S) \) of (13) for \( a \in (0, 1] \). Define \( \hat{u}_a = \alpha(x-b) + u_a(b,c) \) and \( \hat{v}_a = \alpha(y-c) + v_a(b,c) \) for \( a \in (0, 1] \). Then \( (\hat{u}_a, \hat{v}_a) \) satisfies (13) and \( \hat{u}_a, \hat{v}_a \to \hat{u}, \hat{v} \) in \( C^0(S) \) as \( a \to 0^+ \).

Since \( (u, v) \neq (\hat{u}, \hat{v}) \) on \( \partial S \) we see that for small \( a \in (0, 1] \) we have \( (u_a, v_a) \neq (\hat{u}_a, \hat{v}_a) \) on \( \partial S \), and the winding numbers of \( (u, v) - (\hat{u}, \hat{v}) \) and \( (u_a, v_a) - (\hat{u}_a, \hat{v}_a) \) about 0 along \( \partial S \) are equal. But as \( (u_a, v_a) = (\hat{u}_a, \hat{v}_a) \) at \( (b, c) \in S^o \), by Theorem 6.14 the winding number of \( (u_a, v_a) - (\hat{u}_a, \hat{v}_a) \) about 0 along \( \partial S \) is a positive integer, so the result follows.

\[ \square \]

### 6.2 Solutions \( u, v \) of (12) with \( v(x, 0) \equiv 0 \)

We now study singular solutions \( u, v \in C^0(S) \) of (12) with \( v(x, 0) \equiv 0 \). Consider the following situation.

**Definition 6.6.** Let \( S \) be a domain in \( \mathbb{R}^2 \) which intersects the x-axis in \( [x_1, x_2] \times \{0\} \) for \( x_1 < x_2 \), with \( (x_1, x_2) \times \{0\} \subset S^o \), and \( u, v \in C^0(S) \) a singular solution of (12) in \( S \), with \( v(x, 0) = 0 \) for all \( x \in [x_1, x_2] \). That is, \( (u, v) \) is singular all along the intersection of \( S \) with the x-axis. Define subsets \( N_\pm \subset \mathbb{C}^3 \) by

\[
N_+ = \{ (z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 z_2 = v(x, y) + iy, \quad z_3 = x + iu(x, y), \quad |z_1| = |z_2|, \quad (x, y) \in S, \quad y \geq 0 \},
\]

\[
N_- = \{ (z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 z_2 = v(x, y) + iy, \quad z_3 = x + iu(x, y), \quad |z_1| = |z_2|, \quad (x, y) \in S, \quad y \leq 0 \}.
\]

Then \( N_+ \cap N_- \) is the real curve \( \gamma = \{ (0, 0, x + iu(x, 0)) : x \in [x_1, x_2] \} \) in \( \mathbb{C}^3 \). The end points of \( \gamma \) are \( (0, 0, x_j + iu(x_j, 0)) \) for \( j = 1, 2 \), and the interior \( \gamma^o \) of \( \gamma \) is \( \{ (0, 0, x + iu(x, 0)) : x \in (x_1, x_2) \} \).

We will prove that \( N_\pm \) are U(1)-invariant SL 3-folds with boundaries

\[
\partial N_+ = \{ (z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 z_2 = v(x, y) + iy, \quad z_3 = x + iu(x, y), \quad |z_1| = |z_2|, \quad (x, y) \in \partial S, \quad y \geq 0 \},
\]

\[
\partial N_- = \{ (z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 z_2 = v(x, y) + iy, \quad z_3 = x + iu(x, y), \quad |z_1| = |z_2|, \quad (x, y) \in \partial S, \quad y \leq 0 \},
\]

which are nonsingular on their interiors, including along \( \gamma^o \).

Here is a large class of examples \( (u, v) \) satisfying Definition 6.6. Note also that Example 6.5 has \( v(x, 0) \equiv 0 \). We shall show in Theorem 6.14 that any \( (u, v) \) of this form has the symmetries \( u(x, -y) \equiv u(x, y) \) and \( v(x, -y) \equiv -v(x, y) \) of these examples.

**Example 6.7.** Let \( S \) be a strictly convex domain in \( \mathbb{R}^2 \) invariant under the involution \( (x, y) \mapsto (x, -y) \), let \( k \geq 0 \) and \( \alpha \in (0, 1) \). Suppose \( \phi \in C^{k+3, \alpha}(\partial S) \) with \( \phi(x, y) \equiv -\phi(x, -y) \). Let \( f \in C^1(S) \) be the unique weak solution of (12) with \( a = 0 \) with \( f|_{\partial S} = \phi \), which exists by Theorem 6.12. Then \( f'(x, y) = \)
\(-f(x, -y)\) also satisfies (13) with \(a = 0\), and \(f'|_{\partial S} = \phi\) as \(\phi(x, y) = -\phi(x, -y)\). Hence \(f' = f\) by uniqueness, so that \(f(x, -y) = -f(x, y)\).

Let \(u = \frac{\partial u}{\partial y}\) and \(v = \frac{\partial v}{\partial x}\). Then \(u, v \in C^0(S)\) are singular solutions of (12) in the sense of [4, 6] by Theorem 4.12. Moreover \(f(x, -y) = -f(x, y)\) implies that \(u(x, -y) = u(x, y)\) and \(v(x, -y) = -v(x, y)\). This gives \(v(x, 0) = 0\) for all \((x, 0) \in S\), so \((u, v)\) is singular all along the intersection of \(S\) with the \(x\)-axis.

We shall show the function \(u(x, 0)\) used to define \(\gamma\) is Lipschitz on closed subintervals \([b, c]\) of the \(x\)-axis in \(S^0\).

**Proposition 6.8.** Let \(S, u, v, x_1, x_2\) be as in Definition 6.4, and suppose \(x_1 < b < c < x_2\). Then there exists \(K > 0\) such that if \(b \leq x' < x'' \leq c\) then \(|u(x'', 0) - u(x', 0)| \leq K|x'' - x'|\).

**Proof.** As \(u, v \in C^0(S)\) we have \((u^2 + v^2)^{1/2} \leq A\) on \(S\) for some \(A > 0\). Choose \(K > 0\) large enough that \(K[(x-x')^2 + y^2]^{1/2} \geq 2A\) on \(\partial S\) for all \(x' \in [b, c]\), and \(|u(c, 0) - u(b, 0)| \leq K|c - b|\). This is clearly possible.

Suppose for a contradiction that \(b \leq x' < x'' \leq c\) with \(|u(x'', 0) - u(x', 0)| > K|x'' - x'|\). Define \(\alpha = (u(x'', 0) - u(x', 0))/(x'' - x')\), so that \(|\alpha| > K\), and \(\beta = u(x', 0) - \alpha x'.\) Set \(\hat{u}(x, y) = \alpha(x - x') + \beta\) and \(\hat{v}(x, y) = \beta y\). Then \(u(x', 0) = \alpha x' + \beta\) and \(u(x'', 0) = \alpha x'' + \beta\).

The conditions \(|u(c, 0) - u(b, 0)| \leq K|c - b|\) and \(|\alpha| > K\) imply that \(u(b, 0) = \alpha b + \beta\) and \(u(c, 0) = \alpha c + \beta\) cannot both hold. Thus, \(u(x, 0) = \alpha x + \beta\) holds for a proper closed subset of \([b, c]\) containing \([x', x'']\). If this subset contains \([x', x'']\), then decrease \(x'\) or increase \(x''\) until it no longer lies in this subset, but \(|u(x'', 0) - u(x', 0)| > K|x'' - x'|\) still holds. With these new \(x', x''\) it is not true that \(u(x, 0) = \alpha x + \beta\) for all \(x \in [x', x'']\).

Then \(\hat{u}, \hat{v}\) satisfy (12), and \((u, v) = (\hat{u}, \hat{v})\) at \((x', 0)\) and \((x'', 0)\) by construction. Now on \(\partial S\) we have

\[
(\hat{u}^2 + \hat{v}^2)^{1/2} \geq |\alpha|((x-x')^2 + y^2)^{1/2} - |u(x', 0)|
\]

\[
\geq K[(x-x')^2 + y^2]^{1/2} - A \geq 2A - A = A \geq (u^2 + v^2)^{1/2}.
\]

Hence the winding number of \((u, v) - (\hat{u}, \hat{v})\) about \(\partial S\) is the same as that of \(-f(x, -y)\) about \(\partial S\), which is 1, as \((\hat{u}, \hat{v})\) has a single zero on the \(x\)-axis in \(S^0\).

Write \(E = \{(x, y) \in S : (u, v) = (\hat{u}, \hat{v})\text{ at } (x, y)\}\). Then \(E\) is compact and contained in \(S^0\), with \((x', 0), (x'', 0) \in E\). Proposition 6.3 implies that if \((b, c) \in E\) is nonsingular then it is isolated, and as \((u, v) - (\hat{u}, \hat{v})\) has winding number 1 about \(\partial S\), Proposition 6.3 shows there is at most 1 nonsingular zero \((b, c)\) in \(E\). Hence \(E\) is the union of a closed subset of the \(x\)-axis, and at most one other point.

Thus, the connected components of \(E\) are closed intervals in the \(x\)-axis, plus at most one other point. Proposition 6.3 implies that if \(T \subset S\) is a subdomain with \(\partial T \cap E = \emptyset\) and at least one connected component of \(E\) in \(T^c\), then the winding number of \((u, v) - (\hat{u}, \hat{v})\) about 0 along \(\partial T\) is positive. So by the argument of Proposition 6.3 we see that \(E\) has at most one connected component.
However, \((x',0)\) and \((x'',0)\) must lie in different connected components of \(E\), as it is not true that \(u(x,0) = \alpha x + \beta\) for all \(x \in [x', x'']\), so that \([x', x''] \times \{0\} \not\subseteq E\). So \(E\) has at least two connected components, a contradiction.

Now we can prove the \(N_\pm\) are rectifiable currents, with boundaries \(\{31\}, \{35\}\).

**Proposition 6.9.** In the situation above, \(N_\pm\) are the supports of special Lagrangian rectifiable currents, which we identify with \(N_\pm\), with boundaries \(\partial N_\pm\) supported on \(\{31\}\) and \(\{35\}\). In particular, \(\gamma^o\) lies in the interiors \(N_\pm^o\), not the boundaries.

**Proof.** Observe that \(N_\pm\) are the intersection of \(N\) with the subsets \(\text{Im}(z_1 z_2) \geq 0\), \(\text{Im}(z_1 z_2) \leq 0\) of \(\mathbb{C}^3\), where \(N\) is the support of special Lagrangian rectifiable current with boundary supported on \(\{21\}\), by Proposition 4.21. So by the definition \[15\text{ Th. 4.7}\] of rectifiable \(m\)-current as a countable union of Lipschitz images of bounded measurable subsets of \(\mathbb{R}^m\), we see that \(N_\pm\) are the supports of special Lagrangian rectifiable currents, by making the subsets of \(\mathbb{R}^m\) smaller.

It is also easy to see that

\[
\text{supp}(\partial N_+) \subseteq \text{supp}(\partial N) \cap \{(z_1, z_2, z_3) \in \mathbb{C}^3 : \text{Im}(z_1 z_2) \geq 0\} \cup \\
\text{supp}(N) \cap \{(z_1, z_2, z_3) \in \mathbb{C}^3 : \text{Im}(z_1 z_2) = 0\},
\]

and similarly for \(\partial N_-\). By Definition 6.0 and equations \{11\} and \{21\} we see that the first line of the right hand side of \(\{36\}\) is \(\{31\}\), and the second line is \(\gamma\).

Now by \[15\text{ Th. 4.7}\], if \(T\) is an \(m\)-current in \(\mathbb{R}^n\) and \(\text{supp}(T)\) has zero Hausdorff \(m\)-measure, then \(T = 0\). From Proposition 6.8 we see that \(\gamma^o\) has Hausdorff dimension 1, and zero Hausdorff 2-measure. Hence the portion of the current \(\partial N_+\) supported on \(\gamma^o\) is zero. That is, \(\partial N_+\) is supported on \(\{31\}\), with \(\gamma^o\) in the interior of \(N_+\), and similarly for \(N_-\).

Next we identify the tangent cones to \(N_\pm\) along the singular curve \(\gamma^o\).

**Proposition 6.10.** In the situation above, at each \(z \in \gamma^o\), any tangent cone \(C\) to \(N_\pm\) is \(\Pi^o_\pm\) with multiplicity 1 for some \(\phi \in (-\phi_\pm, \phi_\pm)\), as in part (iii) of Theorem 5.7.

**Proof.** We give the proof for \(N_+\). As \(N_+\) lies in the subset \(y = \text{Im}(z_1 z_2) \geq 0\) in \(\mathbb{C}^3\) it follows that \(\text{Im}(z_1 z_2) \geq 0\) on \(C\). But the cones \(N_0\) in parts (i) and (ii) of Theorem 6.0 and the planes \(\Pi^o_\phi\) in part (iii), all contain points with \(\text{Im}(z_1 z_2) < 0\). The only remaining possibilities for \(C\) are \(\Pi^o_\phi\) with multiplicity 1 as in part (iii) of Theorem 5.6 or part (iv) of Theorem 5.6. So we have to eliminate possibility (iv).

As a current, \(C\) is the limit as \(j \to \infty\) of a sequence \(r_j^{-1}(N_+ - z)\), where \(z = (0, 0, x' + iu(x', 0))\) for some \(x' \in (x_1, x_2)\), and \(r_j \to 0_+\). It is easy to show that \(r_j^{-1}(N_+ - z)\) may be written in the form \{11\} with \(a = 0\) on a neighbourhood of \((0,0)\) in \(\{(x,y) \in \mathbb{R}^2 : y \geq 0\}\), with \(u, v\) replaced by

\[
u_j(x, y) = r_j^{-1}u(r_j(x - x'), r_j^2y) \quad \text{and} \quad v_j(x, y) = r_j^{-2}v(r_j(x - x'), r_j^2y).
\]
Finally, Bryant shows that this formal power series converges near
uniquely by a recursive formula derived from the special Lagrangian condition.

Let \( \delta \) be a nonsingular real analytic curve lying in the subset \( \{(0,0,z_3) : z_3 \in \mathbb{C} \} \) of \( \mathbb{C}^3 \). Then there exist exactly two \( \mathbb{C}^3 \) containing \( \delta \) invariant under the \( \text{U}(1) \)-action, which are exchanged by the involution \( (z_1, z_2, z_3) \mapsto (-z_1, z_2, z_3) \).

Bryant’s proof involves expanding the \( \text{U}(1) \)-invariant \( \mathbb{C}^3 \) containing \( \delta \) as a power series in suitable coordinates. For the first nontrivial term there are two choices locally, representing the two possible \( \text{U}(1) \)-invariant special Lagrangian choices for \( TN \) along \( \delta \). The higher order terms are then defined uniquely by a recursive formula derived from the special Lagrangian condition. Finally, Bryant shows that this formal power series converges near \( \delta \).

We shall use Bryant's result to show that singular solutions \( u, v \) of \( \text{SL}^3 \) in which \( v \) is zero on an open interval in the \( x \)-axis have the symmetries \( u(x, -y) = u(x, y) \) and \( v(x, -y) = -v(x, y) \), as in Example 6.7.
Let $S$ be a domain in $\mathbb{R}^2$ invariant under $(x, y) \mapsto (x, -y)$. Let $u, v \in C^0(S)$ be singular solutions of (12), as in [4.6]. Suppose there exist $b < c$ in $\mathbb{R}$ such that $(x, 0) \in S$ and $v(x, 0) = 0$ for all $x \in [b, c]$. Then $u(x, -y) = u(x, y)$ and $v(x, -y) = -v(x, y)$ for all $(x, y) \in S$.

**Proof.** Let $T \subset S$ be a subdomain invariant under $(x, y) \mapsto (x, -y)$ such that the intersection of $T$ with the $x$-axis is $\{(x, 0) : x \in [b, c]\}$. Define

$$N_+ = \{ (z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 z_2 = v(x, y) + iy, \quad z_3 = x + iu(x, y), \quad |z_1| = |z_2|, \quad (x, y) \in T, \quad y \geq 0 \},$$

and define $N_-$ the same way, but with $y \leq 0$ rather than $y \geq 0$. Then Theorem 6.11 shows that $N_{\pm}$ are compact $U(1)$-invariant $SL_3$-folds with boundary, which contain and are nonsingular along the nonsingular real analytic curve $\delta = \{(0, 0, x + iu(x, 0)) : x \in (b, c)\}$.

By Theorem 6.13 there are locally exactly two $U(1)$-invariant $SL_3$-folds containing $\delta$, which must be $N_{\pm}$, and thus $N_{\pm}$ are exchanged by $(z_1, z_2, z_3) \mapsto (-z_1, z_2, z_3)$ near $\delta$. This implies that $u(x, -y) = u(x, y)$ and $v(x, -y) = -v(x, y)$ in $T$ near the $x$-axis. But $u, v$ are real analytic in $S^0$ except at the $x$-axis, and continuous in $S$, so it easily follows that $u(x, -y) = u(x, y)$ and $v(x, -y) = -v(x, y)$ in $S$.

Later we will use this to prove that $U(1)$-invariant $SL_3$-folds have only isolated singularities, under mild conditions.

### 6.3 Inequalities on $v$ for singular solutions of (12)

Next we generalize Proposition 4.16 to singular solutions of (12).

**Theorem 6.15.** Let $S$ be a strictly convex domain in $\mathbb{R}^2$ invariant under the involution $(x, y) \mapsto (x, -y)$, and let $u, v \in C^0(S)$ and $\hat{u}, \hat{v} \in C^0(S)$ be singular solutions of (12), as in [4.6]. If $v < \hat{v}$ on $\partial S$ then $v < \hat{v}$ on $S$.

**Proof.** First we prove the weaker statement that if $v < \hat{v}$ on $\partial S$ then $v \leq \hat{v}$ on $S$. By part (iv) of Definition 4.20, we may write $v, \hat{v}$ as the limits in $C^0(S)$ as $a \to 0_+$ of solutions $v_a, \hat{v}_a \in C^2(S)$ of (19) for $a \in (0, 1]$. Since $v_a \to v$ and $\hat{v}_a \to \hat{v}$ in $C^0(S)$ as $a \to 0_+$, as $v < \hat{v}$ on $\partial S$ it follows that $v_a < \hat{v}_a$ on $\partial S$ for small $a \in (0, 1]$. Thus $v_a < \hat{v}_a$ on $S$ for small $a \in (0, 1]$ by Proposition 4.16. Taking the limit $a \to 0_+$ then shows that $v \leq \hat{v}$ in $S$.

Choose $\phi' \in C^{k+2,\alpha}(\partial S)$ such that $v < \phi' < \hat{v}$ on $\partial S$ and $\phi'(x, 0) \neq 0$ for points $(x, 0) \in \partial S$. Let $u', v' \in C^0(S)$ be a singular solution of (12) with $v'_{|\partial S} = \phi'$, which exists by Theorem 4.8. Since $v < u' < \hat{v}$ on $\partial S$ we see that $v < v' < \hat{v}$ near $\partial S$, from above. Thus we can choose a slightly smaller domain $T \subset S^0$ such that $v < v' < \hat{v}$ on $S \setminus T^0$, and in particular on $\partial T$.

If $t \in \mathbb{R}$ is small then $(x + t, y) \in S$ whenever $(x, y) \in T$. Define $v'_t \in C^0(T)$ by $v'_t(x, y) = v'(x + t, y)$. Then $v'_t$ is a weak solution of (19) with $a = 0$. Also, since $v'_t$ depends continuously on $t$ we see that there exists $\epsilon > 0$ such that $v'_t$ is
well-defined and \( v < v' < \hat{v} \) on \( \partial T \) for \( t \in (-\epsilon, \epsilon) \). Thus, by the first part of the proof we have \( v \leq v' \leq \hat{v} \) on \( T \) for all \( t \in (-\epsilon, \epsilon) \).

Suppose for a contradiction that \( v(r, s) = \hat{v}(r, s) \) for some \((r, s) \in S\). Then \((r, s) \in T^\circ, \) as \( v < v' < \hat{v} \) on \( S \setminus T^\circ \). As \( v \leq v' \leq \hat{v} \) on \( T \) for all \( t \in (-\epsilon, \epsilon) \) we have \( v'(r, s) = v(r, s) \) for all \( t \in (-\epsilon, \epsilon) \). Hence \( v'(r + t, s) = v(r, s) \) for all \( t \in (-\epsilon, \epsilon) \).

Consider the two cases (a) \( s = 0 \) and \( v(r, 0) = 0 \) and (b) \( s \neq 0 \) or \( v(r, s) \neq 0 \). In case (a) we have shown that \( v' \) is zero on an open interval \((r - \epsilon, r + \epsilon)\) in the \( x\)-axis. So Theorem \( 6.14 \) gives \( v'(x, y) = -v'(x, -y) \) in \( S \). But then \( v'(x, 0) \equiv 0, \) contradicting \( v'(x, 0) \neq 0 \) for \((x, 0) \in \partial S \).

In case (b), as \( v' \) is real analytic where it is nonsingular we see that \( v'(x, s) = v(r, s) \) for all \( x \) such that \((x, s) \in S \). In particular, this implies that \( \phi'(x_1, s) = \phi'(x_2, s) \) for the two points \((x_1, s), (x_2, s) \) in \( \partial S \) of the form \((x, s) \). Choosing \( \phi' \) so that \( \phi'(x_1, s) \neq \phi'(x_2, s) \) we again derive a contradiction. \( \Box \)

### 6.4 Positivity of winding numbers and multiplicities

Using the results of 6.2–6.3 we shall complete the arguments of 6.4 to show that singular solutions intersect with positive multiplicity.

**Proposition 6.16.** Let \( u, v \in C^0(S) \) and \( \hat{u}, \hat{v} \in C^0(S) \) be singular solutions of 6.2 in a domain \( S \in \mathbb{R}^2 \), as in 6.4 such that \((u, v) \neq (\hat{u}, \hat{v})\) at every point of \( \partial S \). Suppose \((u, v) - (\hat{u}, \hat{v})\) has a zero \((b, c) \) in \( S^\circ \). Then the winding number of \((u, v) - (\hat{u}, \hat{v})\) about 0 along \( \partial S \) is a positive integer.

**Proof.** If \((b, c) \) is a nonsingular zero then by Proposition 5.4 the multiplicity of \((b, c) \) is a positive integer \( k \), and by Proposition 6.3 the winding number of \((u, v) - (\hat{u}, \hat{v})\) about 0 along \( \partial S \) is at least \( k \), so it is a positive integer, as we have to prove.

So let \((b, c) \) be a singular zero, giving \( c = 0 \) and \( v(b, 0) = \hat{v}(b, 0) = 0 \). As \((u, v) \neq (\hat{u}, \hat{v}) \) along \( \partial S \), there exists a small \( \delta > 0 \) such that if \( \alpha \in (-\delta, \delta) \) then \((u + \alpha, v) \neq (\hat{u}, \hat{v}) \) along \( \partial S \), and the winding numbers of \((u + \alpha, v) - (\hat{u}, \hat{v}) \) and \((u, v) - (\hat{u}, \hat{v}) \) about 0 along \( \partial S \) are the same.

We shall show that one can choose \( \alpha \in (-\delta, \delta) \) such that \((u + \alpha, v) - (\hat{u}, \hat{v}) \) has a zero \((b', c') \) near \((b, 0) \) in \( S^\circ \) with positive multiplicity. Then by Proposition 6.3 the winding number of \((u + \alpha, v) - (\hat{u}, \hat{v})\) about 0 along \( \partial S \) is a positive integer, and thus the winding number of \((u, v) - (\hat{u}, \hat{v})\) about 0 along \( \partial S \) is a positive integer, as we want.

Let \( \epsilon > 0 \) be small enough that \( B_\epsilon(b, 0) \) lies in \( S^\circ \). Suppose \( v \neq \hat{v} \) at every point of \( \gamma_\epsilon(b, 0) \). Then either \( v < \hat{v} \) on \( \gamma_\epsilon(b, c) \) or \( v > \hat{v} \) on \( \gamma_\epsilon(b, 0) \) by continuity. Now \( \gamma_\epsilon(b, 0) \) is the boundary of a strictly convex domain \( B_\epsilon(b, 0) \) invariant under \((x, y) \mapsto (x, -y) \). Applying Theorem 6.10 shows that \( v < \hat{v} \) or \( v > \hat{v} \) on \( B_\epsilon(b, 0) \).

But this contradicts \( v = \hat{v} = 0 \) at \((b, 0) \).

Thus, for any small \( \epsilon \) there exists a point \((b', c') \) on \( \gamma_\epsilon(b, 0) \) where \( v = \hat{v} \). Set \( \alpha = \hat{u}(b', c') - u(b', c') \). Then \((u + \alpha, v) - (\hat{u}, \hat{v})\) is zero at \((b', c') \). Also, since \( u = \hat{u} \) at \((b, 0) \) and \( u, \hat{u} \) are continuous we see that \( \alpha \to 0 \) as \( \epsilon \to 0_+ \), and so \( \alpha \in (-\delta, \delta) \) if \( \epsilon \) is small enough.

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For each sufficiently small $\epsilon > 0$ we have constructed $\alpha \in (-\delta, \delta)$ and a zero $(b', c')$ of $(u + \alpha, v) - (\hat{u}, \hat{v})$ on the circle $\gamma_{\epsilon}(b, c)$. Consider the two cases

(a) $(b', c')$ is a nonsingular zero for some small $\epsilon > 0$, and
(b) $(b', c')$ is a singular zero for all small $\epsilon > 0$.

In case (a), for this $\epsilon$ Proposition 6.12 shows that the multiplicity of $(b', c')$ is positive, and we are finished.

In case (b) we must have $c' = 0$, so that $b' = b \pm \epsilon$, and so for all small $\epsilon > 0$ we have either $v(b + \epsilon, 0) = \hat{v}(b + \epsilon, 0) = 0$ or $v(b - \epsilon, 0) = \hat{v}(b - \epsilon, 0) = 0$. It is easy to show using continuity of $v$, $\hat{v}$ that $v$, $\hat{v}$ are zero on a positive length closed interval in the $x$-axis, as in [6.2] Theorem 6.14 then shows that $u(x, -y) \equiv u(x, y)$, $v(x, -y) \equiv -v(x, y)$, $\hat{u}(x, -y) \equiv \hat{u}(x, y)$ and $\hat{v}(x, -y) \equiv -\hat{v}(x, y)$ near $(b, 0)$ in $S$.

By Proposition 6.10 the U(1)-invariant SL 3-folds $N, \hat{N}$ corresponding to $u + \alpha$, $v$ and $\hat{u}, \hat{v}$ have a common singular point at $z = (0, 0, b' + iu(b', 0))$, with tangent cones $\Pi^+_u \cup \Pi^+_v$ and $\Pi^+_\hat{u} \cup \Pi^+_{\hat{v}}$. For generic $\epsilon$ it is easy to see that $\phi \neq \hat{\phi}$, and one can then show using tangent cones that the multiplicity of $(b', 0)$ is 1, which is positive. This completes the proof. □

Since the multiplicity of an isolated zero of $(u, v) - (\hat{u}, \hat{v})$ given in Definition 6.1 is the winding number of $(u, v) - (\hat{u}, \hat{v})$ about 0 along the boundary of a domain $\partial_{(b, c)}$ containing a zero $(b, c)$ of $(u, v) - (\hat{u}, \hat{v})$, we deduce:

**Corollary 6.17.** Let $u, v \in C^0(S)$ and $\hat{u}, \hat{v} \in C^0(S)$ be singular solutions of (12) in a domain $S$ in $\mathbb{R}^2$. Then the multiplicity of any isolated zero $(b, c)$ of $(u, v) - (\hat{u}, \hat{v})$ in $S^0$ is a positive integer.

### 6.5 Counting formulae using winding numbers

We can now generalize Theorem 4.13 to the singular case.

**Theorem 6.18.** Let $S$ be a domain in $\mathbb{R}^2$, and let $u, v$ and $\hat{u}, \hat{v}$ be singular solutions of (12) in $C^0(S)$, with $(u, v) \neq (\hat{u}, \hat{v})$ at every point of $\partial S$. Then $(u, v) - (\hat{u}, \hat{v})$ has at most finitely many zeroes in $S^0$, all isolated. Suppose that there are $n$ zeroes with multiplicities $k_1, \ldots, k_n$. Then the winding number of $(u, v) - (\hat{u}, \hat{v})$ about 0 along $\partial S$ is $\sum_{i=1}^{n} k_i$.

**Proof.** Let the winding number of $(u, v) - (\hat{u}, \hat{v})$ about 0 along $\partial S$ be $k$. Then $k \geq 0$ by Proposition 6.2. Suppose there exist $k + 1$ isolated zeroes of $(u, v) - (\hat{u}, \hat{v})$ in $S^0$. The multiplicity of each is at least 1 by Corollary 6.17, so Proposition 6.3 shows that the winding number of $(u, v) - (\hat{u}, \hat{v})$ about 0 along $\partial S$ is at least $k + 1$, a contradiction. Hence there can be at most $k$ isolated zeroes of $(u, v) - (\hat{u}, \hat{v})$ in $S^0$, and in particular there are finitely many.

Next we shall show that there are no nonisolated zeroes of $(u, v) - (\hat{u}, \hat{v})$. Let $Z$ be the set of nonisolated zeroes of $(u, v) - (\hat{u}, \hat{v})$. Then $Z$ is a closed subset of $S^0$, as the zero set of $(u, v) - (\hat{u}, \hat{v})$ is closed and $Z$ is this with finitely many isolated points removed. By Proposition 6.3 any nonsingular zero is isolated,
so $Z$ consists of singular zeroes, which therefore lie on the $x$-axis, and $u = \hat{u}$ and $v = \hat{v} = 0$ on $Z$.

Each connected component of $Z$ is therefore a closed, connected subset of the $x$-axis in $S^0$. By elementary topology, it must be either a point or a closed interval of positive length. Divide into two cases:

(a) a connected component of $Z$ is an interval $[\alpha, \beta] \times \{0\}$ for $\alpha < \beta$, and
(b) all connected components of $Z$ are points.

In case (a), as $v_j = 0$ on $(\alpha, \beta) \times \{0\}$ Theorem 6.14 shows that $u_j(x, -y) = u_j(x, y)$ and $v_j(x, -y) = -v_j(x, y)$ for $j = 1, 2$ near $(\alpha, \beta) \times \{0\}$. As $u = \hat{u}$ and $v = \hat{v} = 0$ on $(\alpha, \beta) \times \{0\}$, Theorem 6.13 implies that $(u, v) \equiv (\hat{u}, \hat{v})$ near $(\alpha, \beta) \times \{0\}$. But the $(u_j, v_j)$ are real analytic where they are nonsingular in $S^o$, and continuous in $S$, so $(u, v) \equiv (\hat{u}, \hat{v})$ in $S$. This contradicts $(u, v) \neq (\hat{u}, \hat{v})$ on $\partial S$, and excludes case (a).

In case (b), suppose $Z$ has at least $k + 1$ connected components. As $Z$ is a closed subset of the $x$-axis whose connected components are points, it is easy to see that we can find $k + 1$ disjoint closed discs $D_1, \ldots, D_{k+1}$ in $S^0$ with centres on the $x$-axis, such that $(u, v) - (\hat{u}, \hat{v})$ has no zeroes on $\partial D_j$ and $D_j^o$ contains a connected component of $Z$. By Proposition 6.16 the winding number of $(u, v) - (\hat{u}, \hat{v})$ about 0 along $\partial D_i$ is a positive integer.

Thus the sum of the winding numbers of $(u, v) - (\hat{u}, \hat{v})$ about 0 along $\partial D_i$ for $i = 1, \ldots, k + 1$ is greater than $k$, the winding number about 0 along $\partial S$. Reasoning as in the proof of Proposition 6.13 with $T = S \setminus \bigcup_{i=1}^{k+1} D_i^o$ then gives a contradiction. Hence $Z$ has at most $k$ connected components, all single points, and so all isolated zeroes. Thus $Z = \emptyset$ by definition, and there are finitely many zeroes of $(u, v) - (\hat{u}, \hat{v})$, all isolated, as we have to prove.

So let the zeroes of $(u, v) - (\hat{u}, \hat{v})$ be $(b_1, c_1), \ldots, (b_n, c_n)$, with multiplicities $k_1, \ldots, k_n$. Define $\epsilon_1, \ldots, \epsilon_n$ and $T$ as in the proof of Proposition 6.13. Then the winding number of $(u, v) - (\hat{u}, \hat{v})$ about 0 along $\partial T$ is $k - \sum_{i=1}^n k_i$. But there are no zeroes of $(u, v) - (\hat{u}, \hat{v})$ in $T$, so by the proof of Theorem 6.13 in [9] we see that this winding number is zero, and hence $k = \sum_{i=1}^n k_i$, completing the proof. \hfill \Box

6.6 A criterion for isolated zeroes

Without assuming that $(u, v) \neq (\hat{u}, \hat{v})$ at every point of $\partial S$, we can generalize Theorem 6.13 to show that $(u, v) - (\hat{u}, \hat{v})$ has isolated zeroes in $S^0$.

**Theorem 6.19.** Let $S$ be a domain in $\mathbb{R}^2$, and let $u, v$ and $\hat{u}, \hat{v}$ be singular solutions of (12) in $C^0(S)$, such that $(u, v) \neq (\hat{u}, \hat{v})$. Then there are at most countably many zeroes of $(u, v) - (\hat{u}, \hat{v})$ in $S^0$, all isolated.

**Proof.** We may surround each isolated zero of $(u, v) - (\hat{u}, \hat{v})$ in $S^0$ by a small disc in $S^0$, such that the collection of such discs is disjoint. As $S^0$ can contain only countably many disjoint discs, there are only countably many isolated zeroes.
in $S^\circ$. Since $(u, v) \neq (\hat{u}, \hat{v})$, Proposition 6.14 shows that nonsingular zeroes of $(u, v) - (\hat{u}, \hat{v})$ in $S^\circ$ are isolated.

Let $Z$ be the set of nonisolated zeroes of $(u, v) - (\hat{u}, \hat{v})$ in $S^\circ$. Then $Z$ is closed in $S^\circ$ (though not necessarily in $S$) and is a subset of the $x$-axis in $S^\circ$, by the arguments in the proof of Theorem 6.18. Thus, by elementary topology each connected component of $Z$ is either a point, or an interval of positive length. Case (a) in the proof of Theorem 6.18 shows that if a component is an interval of positive length then $(u, v) \equiv (\hat{u}, \hat{v})$, a contradiction. So, the connected components of $Z$ are all points.

Given any $(b, c) \in Z$, we can find a subdomain $T \subset S^\circ$ such that $(b, c) \in T^\circ$ and $(u, v) \neq (\hat{u}, \hat{v})$ at every point of $\partial T$. We must ensure that $\partial T$ avoids the zeroes of $(u, v) - (\hat{u}, \hat{v})$. There are only countably many isolated zeroes, so a generic $T$ has none on $\partial T$, and we can also arrange for $\partial T$ to avoid $Z$, as $Z$ is closed in the $x$-axis in $S^\circ$ and its connected components are points. Applying Theorem 6.18 on $T$ shows that all zeroes of $(u, v) - (\hat{u}, \hat{v})$ in $T^\circ$ are isolated, a contradiction as $(b, c) \in T^\circ$. Thus $Z = \emptyset$, and there are no nonisolated zeroes.

If there are infinitely many zeroes of $(u, v) - (\hat{u}, \hat{v})$ in $S^\circ$, then they have a limit point in $\partial S$ which is a nonisolated zero. For all $k \geq 1$ one can write down examples of holomorphic functions on domains $S$ in $\mathbb{C}$ which are $C^k$ on $\partial S$, but which have countably many zeroes in $S^\circ$ converging to a limit in $\partial S$. Given the strong analogy between 12 and the Cauchy–Riemann equations, it is likely that there exist examples in Theorem 6.19 in which $(u, v) - (\hat{u}, \hat{v})$ has infinitely many zeroes in $S^\circ$.

From Proposition 4.8 and Theorem 6.19, we see that if $u, v$ and $\hat{u}, \hat{v}$ satisfy 12 or 13 and $(b, c) \in S^\circ$ is a zero of $(u, v) - (\hat{u}, \hat{v})$, then either $(b, c)$ is an isolated zero with a unique multiplicity, or $(u, v) \equiv (\hat{u}, \hat{v})$. In effect, $(u, v) \equiv (\hat{u}, \hat{v})$ means that $(b, c)$ is a zero with ‘multiplicity $\infty$’. So we make the following convention, which will simplify the discussion in §9.

**Definition 6.20.** Let $S$ be a domain in $\mathbb{R}^2$, and let $u, v$ and $\hat{u}, \hat{v}$ be singular solutions of 12 in $C^0(S)$, or solutions of 13 in $C^1(S)$ for some $a \neq 0$. We say that $(u, v) - (\hat{u}, \hat{v})$ has a zero of multiplicity at least $k$ at $(b, c) \in S^\circ$ if either $(u, v) - (\hat{u}, \hat{v})$ has an isolated zero of multiplicity at least $k$ at $(b, c)$, or $(u, v) \equiv (\hat{u}, \hat{v})$.

### 6.7 Counting formulae using potentials

Here is a generalization of Theorem 6.14 to the singular case.

**Theorem 6.21.** Suppose $S$ is a strictly convex domain in $\mathbb{R}^2$ invariant under $(x, y) \mapsto (x, -y)$, and $\phi_1, \phi_2 \in C^{3, \alpha}(\partial S)$ for some $\alpha \in (0, 1)$. Let $u_j, v_j \in C^0(S)$ be the singular solution of 12 constructed in Theorem 4.12 from $\phi_j$.

Suppose $\phi_1 - \phi_2$ has exactly $l$ local maxima and $l$ local minima on $\partial S$. Then $(u_1, v_1) - (u_2, v_2)$ has finitely many zeroes in $S^\circ$, all isolated. Let there be $n$ zeroes in $S^\circ$ with multiplicities $k_1, \ldots, k_n$. Then $\sum_{i=1}^n k_i \leq l - 1$. 

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Proof. For $j = 1,2$ and $a \in (0,1]$, let $f_{j,a} \in C^{3,\alpha}(S)$ be the solution of (17) in $C^{k+2,\alpha}(S)$ with $f_{j,a}|_{\partial S} = \phi_j$ given in Theorem 4.11 and set $u_{j,a} = \frac{\partial}{\partial x}f_{j,a}$ and $v_{j,a} = \frac{\partial}{\partial y}f_{j,a}$. Then Theorem 4.14 shows that for all $a \in (0,1]$ there are no more than $l-1$ zeroes of $(u_{1,a},v_{1,a}) - (u_{2,a},v_{2,a})$ in $S^0$, counted with multiplicity.

Let $T \subset S$ be a subdomain with no zeroes of $(u_{1},v_{1}) - (u_{2},v_{2})$ on $\partial T$. Now $(u_{j,a},v_{j,a}) \to (u_{j},v_{j})$ in $C^0(S)$ as $a \to 0$, by Theorem 4.13. Thus, for small $a \in (0,1]$ there are no zeroes of $(u_{1,a},v_{1,a}) - (u_{2,a},v_{2,a})$ on $\partial T$, and the winding numbers of $(u_{1},v_{1}) - (u_{2},v_{2})$ and $(u_{1,a},v_{1,a}) - (u_{2,a},v_{2,a})$ about $0$ along $\partial T$ are equal.

But from above there are no more than $l-1$ zeroes of $(u_{1,a},v_{1,a}) - (u_{2,a},v_{2,a})$ in $T^0$, counted with multiplicity. Hence by Theorem 4.9 the winding number of $(u_{1,a},v_{1,a}) - (u_{2,a},v_{2,a})$ about $0$ along $\partial T$ is no more than $l-1$. As this is the winding number of $(u_{1},v_{1}) - (u_{2},v_{2})$, Theorem 4.13 shows that there are no more than $l-1$ zeroes of $(u_{1},v_{1}) - (u_{2},v_{2})$ in $T^0$, counted with multiplicity.

Since $\phi_1 - \phi_2$ is not constant we have $(u_{1},v_{1}) \neq (u_{2},v_{2})$, and so by Theorem 4.14 there are at most countably many zeroes of $(u_{1},v_{1}) - (u_{2},v_{2})$ in $S^0$, all isolated. If there were infinitely many we could choose T above to contain at least $l$ zeroes, giving a contradiction. Thus there are only finitely many, we can choose $T$ to contain them all, and the result follows. □

We can also generalize [9] Th. 7.10 to include the number of nonsingular zeroes on $\partial S$ in the inequality. However, there may be a problem with including the singular zeroes of $(u_{1},v_{1}) - (u_{2},v_{2})$ on $\partial S$.

7 Special Lagrangian fibrations

In [16], Strominger, Yau and Zaslow proposed an explanation of Mirror Symmetry between Calabi–Yau 3-folds $X, \tilde{X}$ in terms of the existence of dual special Lagrangian fibrations $f : X \to B, \tilde{f} : \tilde{X} \to B$ over the same base space $B$, a real 3-manifold. This is known as the SYZ Conjecture. These fibrations $f, \tilde{f}$ must necessarily contain singular fibres, which are a source of many of the mathematical difficulties surrounding the SYZ Conjecture, as the singularities of SL 3-folds are not yet well understood.

We will now use our results to construct large families of special Lagrangian fibrations of open subsets of $C^3$ invariant under the $U(1)$-action [11], including singular fibres. These can serve as local models for singularities of SL fibrations of (almost) Calabi–Yau 3-folds. In [11] we will discuss these fibrations at much greater length, and draw some conclusions on the singular behaviour of SL fibrations of (almost) Calabi–Yau 3-folds, and on how to best formulate the SYZ Conjecture.

**Definition 7.1.** Let $S$ be a strictly convex domain in $\mathbb{R}^2$ invariant under $(x,y) \to (x,-y)$, let $U$ be an open set in $\mathbb{R}^3$, and $\alpha \in (0,1)$. Suppose $\Phi : U \to C^{3,\alpha}(\partial S)$ is a continuous map such that if $(a,b,c) \neq (a',b',c')$ in $U$ then $\Phi(a,b,c) - \Phi(a',b',c')$ has exactly one local maximum and one local minimum in $\partial S$.  

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Let \( \alpha = (a, b, c) \in U \). If \( a \neq 0 \), let \( f_\alpha \in C^{3,\alpha}(S) \) be the unique solution of (17) with \( f_\alpha|_{\partial S} = \Phi(\alpha) \), which exists by Theorem 4.11. If \( a = 0 \), let \( f_\alpha \in C^{1}(S) \) be the unique weak solution of (18) with \( f_\alpha|_{\partial S} = \Phi(\alpha) \), which exists by Theorem 14.12. Define \( u_\alpha = \frac{\partial f_\alpha}{\partial y} \) and \( v_\alpha = \frac{\partial f_\alpha}{\partial x} \). Then \((u_\alpha, v_\alpha)\) is a solution of (18) if \( a \neq 0 \), and a singular solution of (12) if \( a = 0 \). Also \( u_\alpha, v_\alpha \) depend continuously on \( \alpha \in U \) in \( C^0(S) \), by Theorem 4.13.

For each \( \alpha = (a, b, c) \) in \( U \), define \( N_\alpha \) in \( C^3 \) by

\[
N_\alpha = \{(z_1, z_2, z_3) \in C^3 : z_1z_2 = v_\alpha(x, y) + iy, \quad z_3 = x + iu_\alpha(x, y), \quad |z_1|^2 - |z_2|^2 = 2a, \quad (x, y) \in S^0\}. \tag{37}
\]

Then \( N_\alpha \) is a noncompact SL 3-fold without boundary in \( C^3 \), which is nonsingular if \( a \neq 0 \), by Proposition 4.11.

We shall show that the \( N_\alpha \) are the fibres of an SL fibration. This is one of the main results of the paper, which will be the central tool in 11.

**Theorem 7.2.** In the situation of Definition 7.1 if \( \alpha \neq \alpha' \) in \( U \) then \( N_\alpha \cap N_{\alpha'} = \emptyset \). There exists an open set \( V \subset C^3 \) and a continuous, surjective map \( F : V \to U \) such that \( F^{-1}(\alpha) = N_\alpha \) for all \( \alpha \in U \). Thus, \( F \) is a special Lagrangian fibration of \( V \subset C^3 \), which may include singular fibres.

**Proof.** Let \( \alpha = (a, b, c) \) and \( \alpha' = (a', b', c') \) be distinct elements of \( U \). As \( |z_1|^2 - |z_2|^2 = 2a \) on \( N_\alpha \) and \( |z_1|^2 - |z_2|^2 = 2a' \) on \( N_{\alpha'} \), clearly \( N_\alpha \cap N_{\alpha'} = \emptyset \) if \( a \neq a' \). Suppose \( a = a' \). Then \( \Phi(a, b, c) - \Phi(a', b', c') \) has exactly one local maximum and one local minimum in \( \partial S \), by the condition in Definition 7.1. Hence by Theorem 4.14 when \( a \neq 0 \) and Theorem 6.21 when \( a = 0 \) we see that \((u_\alpha, v_\alpha) - (u_{\alpha'}, v_{\alpha'})\) has no zeroes in \( S^0 \), and thus \( N_\alpha \cap N_{\alpha'} = \emptyset \).

Let \( V = \bigcup_{\alpha \in U} N_\alpha \), and define \( F : V \to U \) by \( F(z) = \alpha \) if \( z \in N_\alpha \). As \( N_\alpha \cap N_{\alpha'} = \emptyset \) when \( \alpha \neq \alpha' \) this map \( F \) is well-defined, and clearly \( F^{-1}(\alpha) = N_\alpha \). As \( N_\alpha \neq \emptyset \) for all \( \alpha \in U \), we see that \( F \) is surjective. It remains only to show that \( F \) is continuous, and \( V \) is open.

Fix \( \alpha' = (a', b', c') \in U \). As \( U \) is open in \( \mathbb{R}^3 = \mathbb{R} \times \mathbb{R}^2 \) there exist open \( A \subset \mathbb{R} \) and \( B \subset \mathbb{R}^2 \) with \( (a', b', c') \in A \times B \subset U \). Let \( (x, y) \in S^0 \), and for each \( \alpha \in A \) define \( \Psi_{a,x,y} : B \to \mathbb{R}^2 \) by \( \Psi_{a,x,y}(b, c) = (u_\alpha(x, y), v_\alpha(x, y)) \) for \( \alpha = (a, b, c) \). Then \( \Psi_{a,x,y} \) is continuous and depends continuously on \( a, x, y \), as \( u_\alpha, v_\alpha \) are continuous and depend continuously on \( \alpha \) from Definition 7.1. Also \( \Psi_{a,x,y} \) is injective, as \( N_\alpha \cap N_{\alpha'} = \emptyset \) for \( \alpha \neq \alpha' \in U \).

Since \( \Psi_{a,x,y} \) is continuous and injective, it follows by elementary topology in \( \mathbb{R}^2 \) that \( W_{a,x,y} = \Psi_{a,x,y}(B) \) is open in \( \mathbb{R}^2 \), and \( \Psi_{a,x,y}^{-1} : W_{a,x,y} \to B \) is continuous. Furthermore, \( W_{a,x,y} \) and \( \Psi_{a,x,y}^{-1} \) depend continuously on \( a, x, y \).

Now if \( z = (z_1, z_2, z_3) \in V \) and \( F(z) = (a, b, c) \in A \times B \), then \( 2a = |z_1|^2 - |z_2|^2 \), \( x = \text{Re}(z_3) \), \( u = \text{Re}(z_1z_2) \) and \( y = \text{Im}(z_1z_2) \). Therefore

\[
a = \frac{1}{2}(|z_1|^2 - |z_2|^2), \quad (b, c) = \Psi_{a,x,y}^{-1}(\text{Re}(z_3), \text{Im}(z_3), \text{Im}(z_1z_2)).
\]

As \( \Psi_{a,x,y}^{-1} \) is continuous and depends continuously on \( a, x, y \), we see that \( (a, b, c) \) depends continuously on \( z \). Hence \( F : z \mapsto (a, b, c) \) is continuous.
Finally we show $V$ is open. Fix $(z_1', z_2', z_3') \in N_{\alpha'}$, and let $x' = \Re(z_3')$, $u' = \Im(z_3')$, $v' = \Re(z_1'z_2')$ and $y' = \Im(z_1'z_2')$. Then $\Psi_{a', x', y'}(b', c') = (u', v')$, so $(u', v') \in W_{a', x', y'}$. As $W_{a, x, y}$ is open and depends continuously on $a, x, y$, there exist open neighbourhoods $W$ of $(u', v')$ in $\mathbb{R}^2$ and $X$ of $(a', x', y')$ in $A \times S^0$ such that if $(a, x, y) \in X$ then $W \subset W_{a, x, y}$. Define

$$Y = \left\{ (z_1, z_2, z_3) \in \mathbb{C}^3 : (\Im(z_3), \Re(z_1z_2)) \in W, \right.$$

$$\left. \left( \frac{1}{a'}(|z_1|^2 - |z_2|^2), \Re(z_3), \Im(z_1z_2) \right) \in X \right\}.$$ 

Then $Y$ is open in $\mathbb{C}^3$, as $W, X$ are open, and contains $(z_1', z_2', z_3')$.

We claim that $Y \subset V$. Let $z = (z_1, z_2, z_3) \in Y$, and set $a = \frac{1}{2}(|z_1|^2 - |z_2|^2)$, $x = \Re(z_3)$, $u = \Im(z_3)$, $v = \Re(z_1z_2)$ and $y = \Im(z_1z_2)$. Then $(u, v) \in W$ and $(a, x, y) \in X$. Hence $W \subset W_{a, x, y}$, so $(u, v) \in W_{a, x, y}$. Thus $\Psi_{a, x, y}^{-1}(u, v) = (b, c)$ is well-defined. From the definitions we see that $\alpha = (a, b, c)$ lies in $U$ and $(z_1, z_2, z_3) \in N_{\alpha}$, so that $z \in V$. Thus $Y \subset V$, so that any $(z_1', z_2', z_3') \in V$ has an open neighbourhood $Y \subset V$, and $V$ is open.

Note that in the above we have chosen to define the $N_{\alpha}$ over $S^0$, so that they are noncompact SL 3-folds without boundary. The closures $\overline{N}_{\alpha}$ are compact SL 3-folds with boundary, defined over $S$. The main reason for working over $S^0$ rather than $S$ is to avoid difficulties in proving that $\overline{N}_{\alpha} \cap \overline{N}_{\alpha'} = \emptyset$ if $\alpha \neq \alpha'$ in $U$. The problem is that $\overline{N}_{\alpha}, \overline{N}_{\alpha'}$ may intersect in a boundary point, lying over $\partial S$.

If we strengthen the condition on $\Phi$ in Definition 7.1 to say that if $(a, b, c) \neq (a, b', c')$ in $U$ then $\Phi(a, b, c) - \Phi(a, b', c')$ has exactly two stationary points, and the second derivative of $\Phi(a, b, c) - \Phi(a, b', c')$ is nonzero at each, then we can use Theorem 7.10 to show that $\overline{N}_{\alpha} \cap \overline{N}_{\alpha'}$ contains no nonsingular points. But there remains the possibility, when $a = 0$, that $\overline{N}_{\alpha}, \overline{N}_{\alpha'}$ could intersect in a common singular point lying over some $(x, 0) \in \partial S$.

Here is a simple way to produce families $\Phi$ satisfying Definition 7.1.

**Example 7.3.** Let $S$ be a strictly convex domain in $\mathbb{R}^2$ invariant under $(x, y) \rightarrow (x, -y)$, let $a \in (0, 1)$ and $\phi \in C^{3, \alpha}(\partial S)$. Define $U = \mathbb{R}^3$ and $\Phi : \mathbb{R}^3 \rightarrow C^{3, \alpha}(\partial S)$ by $\Phi(a, b, c) = \phi + bx + cy$. If $(a, b, c) \neq (a, b', c')$ then $\Phi(a, b, c) - \Phi(a, b', c') = (b - b')x + (c - c')y \in C^{\infty}(\partial S)$. As $b - b', c - c'$ are not both zero and $S$ is strictly convex, it easily follows that $(b - b')x + (c - c')y$ has exactly one local maximum and one local minimum in $\partial S$. Hence the conditions of Definition 7.1 hold for $S, U$ and $\Phi$, and so Theorem 7.2 defines an open set $V \subset \mathbb{C}^1$ and a special Lagrangian fibration $F : V \rightarrow \mathbb{C}^3$.

Let $a, b, c, c' \in \mathbb{R}$. Then $f(a, b, c) = f(a, b, c') + (c-c')y$, $u(a, b, c) = u(a, b, c') + (c-c')$ and $v(a, b, c) = v(a, b, c')$. It follows from Example 7.1 that $N_{(a, b, c)}$ is the translation of $N_{(a, b, c')}$ by $(0, 0, i(c-c'))$ in $\mathbb{C}^3$. So, changing the parameter $c$ in $U = \mathbb{R}^3$ just translates the fibres $N_{\alpha}$ in $\mathbb{C}^3$.

One can also show that $v(a, b, c)(x, y) \rightarrow \pm \infty$ as $b \rightarrow \pm \infty$, for fixed $a, c \in \mathbb{R}$ and $(x, y) \in S^0$. Combining these facts about changing $b, c$ and taking $A = \mathbb{R}$.
and $B = \mathbb{R}^2$ we find that $\Psi_{a,x,y} : \mathbb{R}^2 \to \mathbb{R}^2$ is surjective for all $a \in \mathbb{R}$ and $(x,y) \in S^o$, so that $W_{a,x,y} = \mathbb{R}^2$. From this we easily prove that

$$V = \left\{ (z_1, z_2, z_3) \in \mathbb{C}^3 : (\Re(z_3), \Im(z_1 z_2)) \in S^o \right\}.$$ 

This example, and other families of maps $\Phi$ one can readily construct, generate many special Lagrangian fibrations of open subsets of $\mathbb{C}^3$.

8 A rough classification of singular points

We can now use the work of 4.3 to study singular points of $u, v$.

**Definition 8.1.** Let $S$ be a domain in $\mathbb{R}^2$, and $u, v \in C^0(S)$ a singular solution of (12), as in 4.6. Suppose for simplicity that $S$ is invariant under $(x, y) \mapsto (x, -y)$. Define $u', v' \in C^0(S)$ by $u'(x, y) = u(x, -y)$ and $v'(x, y) = -v(x, -y)$. Then $u', v'$ is also a singular solution of (12).

A singular point, or singularity, of $(u, v)$ is a point $(b, 0) \in S$ with $v(b, 0) = 0$. Observe that a singularity of $(u, v)$ is automatically a zero of $(u, v) - (u', v')$. Conversely, a zero of $(u, v) - (u', v')$ on the $x$-axis is a singularity. A singularity of $(u, v)$ is called isolated if it is an isolated zero of $(u, v) - (u', v')$. Define the multiplicity of an isolated singularity $(b, 0)$ of $(u, v)$ in $S^o$ to be the multiplicity of $(u, v) - (u', v')$ at $(b, 0)$, in the sense of Definition 6.1. By Corollary 6.17 this multiplicity is a positive integer.

If $S$ is not invariant under $(x, y) \mapsto (x, -y)$ then we can still define what it means for a singular point $(b, 0)$ to be isolated, and the multiplicity of an isolated singular point, by restricting to $T = \{(x, y) \in S : (x, -y) \in S\}$ before defining $u', v'$.

From Theorem 6.14 we see that a singular solution $(u, v)$ of (12) in $S$ either has the symmetries $u(x, -y) \equiv u(x, y)$ and $v(x, -y) \equiv -v(x, y)$, as in 6.2, or else its singular points in $S^o$ are all isolated, and so have a well-defined, positive multiplicity.

**Theorem 8.2.** Let $S$ be a domain in $\mathbb{R}^2$, and $u, v \in C^0(S)$ a singular solution of (12), as in 4.6. If $u(x, -y) \equiv u(x, y)$ and $v(x, -y) \equiv -v(x, y)$ near the $x$-axis in $S$ then $(u, v)$ is singular along the intersection of the $x$-axis with $S$, and the singularities are nonisolated. Otherwise there are at most countably many singular points of $(u, v)$ in $S^o$, all isolated.

We may divide isolated singularities $(b, 0)$ into four types, depending on the behaviour of $v(x, 0)$ near $(b, 0)$.

**Definition 8.3.** Let $S$ be a domain in $\mathbb{R}^2$, and $u, v \in C^0(S)$ a singular solution of (12), as in 4.6. Suppose $(b, 0)$ is an isolated singular point of $(u, v)$ in $S^o$. Then there exists $\epsilon > 0$ such that $B_\epsilon(b, 0) \subset S^o$ and $(b, 0)$ is the only singularity of $(u, v)$ in $B_\epsilon(b, 0)$. Thus, for $0 < |x-b| \leq \epsilon$ we have $(x, 0) \in S^o$.
and $v(x,0) \neq 0$. So by continuity $v$ is either positive or negative on each of $[b - \epsilon, b] \times \{0\}$ and $(b, b + \epsilon) \times \{0\}$.

(i) If $v(x) < 0$ for $x \in [b - \epsilon, b)$ and $v(x) > 0$ for $x \in (b, b + \epsilon]$ we say the singularity $(b, 0)$ is of increasing type.

(ii) If $v(x) > 0$ for $x \in [b - \epsilon, b)$ and $v(x) < 0$ for $x \in (b, b + \epsilon]$ we say the singularity $(b, 0)$ is of decreasing type.

(iii) If $v(x) < 0$ for $x \in [b - \epsilon, b)$ and $v(x) < 0$ for $x \in (b, b + \epsilon]$ we say the singularity $(b, 0)$ is of maximum type.

(iv) If $v(x) > 0$ for $x \in [b - \epsilon, b)$ and $v(x) > 0$ for $x \in (b, b + \epsilon]$ we say the singularity $(b, 0)$ is of minimum type.

The type determines whether the multiplicity of $(b, 0)$ is even or odd.

**Proposition 8.4.** Let $u, v \in C^0(S)$ be a singular solution of (12) on a domain $S$ in $\mathbb{R}^2$, and $(b, 0)$ be an isolated singularity of $(u, v)$ in $S^0$ with multiplicity $k$. If $(b, 0)$ is of increasing or decreasing type then $k$ is odd, and if $(b, 0)$ is of maximum or minimum type then $k$ is even.

**Proof.** Let $\epsilon$ be as in Definition 8.3 and define $u', v'$ on $\overline{B}_{\epsilon}(b, 0)$ by $u'(x, y) = u(x, -y)$ and $v'(x, y) = -v(x, -y)$. Then by definition, $k$ is the winding number of $(u, v) - (u', v')$ about $0$ along $\gamma_{\epsilon}(b, 0)$. Divide the circle $\gamma_{\epsilon}(b, 0)$ into upper and lower semicircles, with $y \geq 0$ and $y \leq 0$.

By the involution $(x, y) \mapsto (x, -y)$ we see that $(u, v) - (u', v')$ rotates through the same angle on the upper and lower semicircles. Now $u(b \pm \epsilon, 0) = u'(b \pm \epsilon, 0)$ and $v(b \pm \epsilon, 0) = -v'(b \pm \epsilon, 0)$, so $(u, v) - (u', v')$ lies on the positive $y$-axis at $(b \pm \epsilon, 0)$ if $v(b \pm \epsilon, 0) > 0$, and on the negative $y$-axis if $v(b \pm \epsilon, 0) < 0$.

Using this we see that in cases (i) and (ii) of Definition 8.3 $(u, v) - (u', v')$ rotates through an angle $(2n + 1)\pi$ on the upper semicircle, for $n \in \mathbb{Z}$. Thus $(u, v) - (u', v')$ rotates through $(2n + 1)\pi$ about $\gamma_{\epsilon}(b, 0)$, and $k = 2n + 1$ is odd. Similarly, in cases (iii) and (iv), $(u, v) - (u', v')$ rotates through an angle $2n\pi$ on the upper semicircle and $4n\pi$ about $\gamma_{\epsilon}(b, 0)$, and $k = 2n$ is even.

Now an isolated singular point of $u, v$ in $S^0$ yields an isolated singular point $z$ in the interior of the corresponding $U(1)$-invariant SL 3-fold $N$. The possible tangent cones $C$ to $N$ at $z$ were classified in Theorem 5.6. For cases (i) and (ii) we can use Proposition 6.3 and Theorem 7.7 to identify the multiplicity and type of $(b, 0)$.

**Proposition 8.5.** Let $u, v \in C^0(S)$ be a singular solution of (12) on a domain $S$ in $\mathbb{R}^2$, and $N$ the corresponding $U(1)$-invariant SL 3-fold in $\mathbb{C}^3$. Let $(b, 0)$ be an isolated singularity of $(u, v)$ in $S^0$, and $z = (0, 0, b + iu(b, 0))$ the corresponding singular point of $N$. Suppose $C$ is a tangent cone to $N$ at $z$. If $C$ is as in case (i) of Theorem 5.6 then $(b, 0)$ has multiplicity 1 and is of increasing type. If $C$ is as in case (ii) of Theorem 7.7 then $(b, 0)$ has multiplicity 1 and is of decreasing type.
The author can also show that if \((b, 0)\) has multiplicity \(n \geq 2\) then the unique tangent cone \(C\) to \(N\) at \(z\) is \(\Pi^+ \cup \Pi^-\) for some \(\phi \in (-\frac{\pi}{2}, \frac{\pi}{2})\), where \(\Pi^+\) and \(\Pi^-\) are defined in (24) and have multiplicity 1.

Next we give counting formulae for singularities. Theorem 6.18 yields:

**Theorem 8.6.** Let \(S\) be a domain in \(\mathbb{R}^2\) invariant under \((x, y) \mapsto (x, -y)\), and \(u, v \in C^0(S)\) a singular solution of (14). Define \(u', v' \in C^0(S)\) by \(u'(x, y) = u(x, -y)\) and \(v'(x, y) = -v(x, -y)\). Suppose \((u, v) \neq (u', v')\) at every point of \(\partial S\). Then \((u, v)\) has finitely many singularities in \(S\), all isolated. Let there be \(n\) singularities with multiplicities \(k_1, \ldots, k_n\). Then the winding number of \((u, v) - (u', v')\) about 0 along \(\partial S\) is at least \(\sum_{i=1}^n k_i\).

Here in the last line we say at least rather than exactly \(\sum_{i=1}^n k_i\), as \((u, v) - (u', v')\) may have other zeroes not on the \(x\)-axis, and so not singular points, which contribute to the winding number. Similarly, Theorem 6.21 gives:

**Theorem 8.7.** Suppose \(S\) is a strictly convex domain in \(\mathbb{R}^2\) invariant under \((x, y) \mapsto (x, -y)\), and \(\phi \in C^3, \alpha(\partial S)\) for some \(\alpha \in (0, 1)\). Let \(u, v \in C^0(S)\) be the singular solution of (12) constructed in Theorem 4.12 from \(\phi\).

Define \(\phi' \in C^3, \alpha(\partial S)\) by \(\phi'(x, y) = -\phi(x, -y)\). Suppose \(\phi - \phi'\) has exactly \(l\) local maxima and \(l\) local minima on \(\partial S\). Then \((u, v)\) has finitely many singularities in \(S^0\), all isolated. Let there be \(n\) singularities in \(S^0\) with multiplicities \(k_1, \ldots, k_n\). Then \(\sum_{i=1}^n k_i \leq l - 1\).

9 **Singularities exist with all multiplicities**

We now prove that there exist singularities with every multiplicity \(n \geq 1\), and every possible type, and that singularities of multiplicity \(n\) occur in codimension \(n\) in the family of all \(U(1)\)-invariant SL 3-folds, in a certain sense. For simplicity we work not on a general domain \(S\), but on the unit disc \(D\) in \(\mathbb{R}^2\).

**Definition 9.1.** Let \(D\) be the unit disc \(\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}\) in \(\mathbb{R}^2\), with boundary \(S^1\), the unit circle. Define a coordinate \(\theta : \mathbb{R}/2\pi\mathbb{Z} \to S^1\) by \(\theta \mapsto (\cos \theta, \sin \theta)\). Then \(\cos(j\theta), \sin(j\theta) \in C^\infty(S^1)\) for \(j \geq 1\).

We shall use the functions \(\cos(j\theta), \sin(j\theta)\) for \(j = 1, \ldots, n\) as a family of perturbations to the boundary data \(\phi\) in Theorem 4.12 and show that for any suitable \(\phi\) there exists a family of perturbations of \(\phi\) such that the corresponding singular solution \(u, v\) has a singularity of multiplicity at least \(n\) at \((0, 0)\).

9.1 **Counting stationary points of Fourier sums**

We begin with two propositions on the stationary points of linear combinations of \(\cos(j\theta), \sin(j\theta)\). We say that \(\phi\) has a stationary point of multiplicity \(k \geq 1\) at \(\theta_0\) if \(d^j \phi/d\theta^j(\theta_0) = 0\) for \(j = 1, \ldots, k\), but \(d^{k+1} \phi/d\theta^{k+1}(\theta_0) \neq 0\).

**Proposition 9.2.** Let \(\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n \in \mathbb{R}\) be not all zero for \(n \geq 1\), and define \(\phi \in C^\infty(S^1)\) by \(\phi = \sum_{j=1}^n (\alpha_j \cos(j\theta) + \beta_j \sin(j\theta))\). Then \(\phi\) has at most \(2n\) stationary points in \(S^1\), counted with multiplicity.
Theorem 9.4. Let $a \neq 0$, $n \geq 1$, $\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n \in \mathbb{R}$ and $\psi, \hat{\phi} \in C^\infty(S^1)$. Define $\phi = \psi + \sum_{j=1}^{n} (\alpha_j \cos(j \theta) + \beta_j \sin(j \theta))$ in $C^\infty(S^1)$. Let $f, \tilde{f} \in C^\infty(D)$
be the unique solutions of (17) on $D$ with $f|_{S^1} = \phi$, $\hat{f}|_{S^1} = \hat{\phi}$, which exist by
Theorem 4.7. Let $u = \frac{\partial \hat{f}}{\partial y}$, $v = \frac{\partial \hat{f}}{\partial x}$, $\hat{u} = \frac{\partial f}{\partial y}$, and $\hat{v} = \frac{\partial f}{\partial x}$.

Then for each $a \neq 0$, $n \geq 1$ and $\psi, \phi \in C^\infty(S^1)$ there exist unique $\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n$ with $\sum_{j=1}^n (\alpha_j^2 + \beta_j^2) \leq K_n \|\psi - \hat{\phi}\|_{C^2_{2n}}^2$, where $K_n$ is as in Proposition 4.8 such that $(u,v)- (\hat{u}, \hat{v})$ has a zero of multiplicity at least $n$ at $(0,0)$. Furthermore these $\alpha_j, \beta_j$ depend continuously on $a, \psi$.

**Proof.** We shall prove the theorem by induction on $n$. The first and inductive steps will be shown together. Let $k \geq 0$ and suppose the theorem holds for all $n \leq k$. We will prove it when $n = k + 1$. Fix $a \neq 0$ and $\psi, \phi \in C^\infty(S^1)$.

Let $\gamma = \alpha_{k+1} + i\beta_{k+1} \in \mathbb{C}$. When $k = 0$, let $f_\gamma \in C^\infty(D)$ be the unique solution of (17) with $f_\gamma|_{S^1} = \psi + \alpha_1 \cos \theta + \beta_1 \sin \theta$. When $k \geq 1$, apply the inductive hypothesis for $n = k$ with $\psi$ replaced by $\psi + \alpha_{k+1} \cos(k+1)\theta + \beta_{k+1} \sin(k+1)\theta$. This gives unique $\alpha_1, \beta_1, \ldots, \alpha_k, \beta_k \in \mathbb{R}$ and a solution $f_\gamma \in C^\infty(D)$ of (17) with

$$f_\gamma|_{S^1} = \psi + \sum_{j=1}^{k+1} (\alpha_j \cos(j\theta) + \beta_j \sin(j\theta)).$$

(38)

Define $u_\gamma, v_\gamma$ in the obvious way.

When $k \geq 1$ we know by induction that $(u_\gamma, v_\gamma) - (\hat{u}, \hat{v})$ has a zero of multiplicity at least $k$ at $(0,0)$. Therefore by Proposition 4.8 we may write

$$\lambda u_\gamma(x,y) + iv_\gamma(x,y) = \lambda \hat{u}(x,y) + iv(x,y) + C(\lambda x + iy)^k + O(|x|^{k+1} + |y|^{k+1})$$

(39)

near $(0,0)$, where $\lambda > 0$, and $C \in \mathbb{C}$ is zero if and only if $(u_\gamma, v_\gamma) - (\hat{u}, \hat{v})$ has a zero of multiplicity more than $k$. Define $F(\gamma) = C$, giving a map $F : \mathbb{C} \to \mathbb{C}$. When $k = 0$ this formula is still valid, giving $F(\gamma) = (\lambda(u_\gamma - \hat{u}) + i(v_\gamma - \hat{v}))|_{(0,0)}$. Here are some properties of the maps $F$.

**Proposition 9.5.** This map $F : \mathbb{C} \to \mathbb{C}$ is continuous and injective, and if $|\gamma|^2 > K_{k+1}\|\psi - \hat{\phi}\|_{C^2_{2k+1}}^2$ then $F(\gamma) \neq 0$, where $K_{k+1}$ is as in Proposition 4.8.

**Proof.** By the inductive hypothesis $\alpha_1, \beta_1, \ldots, \alpha_k, \beta_k$ depend continuously on $a, \psi, \phi$ and $\gamma$. Hence $f_\gamma, u_\gamma, v_\gamma$ depend continuously in $C^\infty(D)$ on $a, \psi, \gamma$ by [9, Th. 7.7]. It follows that $F$ is continuous as a function of $\gamma$, as we have to prove, and also varies continuously with $a, \psi, \phi$.

Next we show $F$ is injective. Suppose for a contradiction that $\gamma \neq \gamma' \in \mathbb{C}$ and $F(\gamma) = F(\gamma')$. Then (39) holds for $u_\gamma, v_\gamma$ and $u_{\gamma'}, v_{\gamma'}$ with the same value of $C$. This implies that $v_\gamma(0,0) = v_{\gamma'}(0,0)$, so the values of $\lambda = \sqrt{2}(v(0,0)^2 + a^2)^{1/4}$ for $\gamma$ and $\gamma'$ are the same as well. Thus, subtracting (39) for $\gamma, \gamma'$ gives

$$\lambda u_\gamma(x,y) + iv_\gamma(x,y) = \lambda u_{\gamma'}(x,y) + iv_{\gamma'}(x,y) + O(|x|^{k+1} + |y|^{k+1}).$$

Hence $(u_\gamma, v_\gamma) - (u_{\gamma'}, v_{\gamma'})$ has a zero of multiplicity at least $k + 1$ at $(0,0)$, by Definition 4.7.
Now by (38) the potentials $f_\gamma, f_\gamma'$ satisfy
\[(f_\gamma - f_\gamma')|_{S^1} = \sum_{j=1}^{k+1} \left( (\alpha_j - \alpha'_j) \cos(j\theta) + (\beta_j - \beta'_j) \sin(j\theta) \right).\]
Thus $(f_\gamma - f_\gamma')|_{S^1}$ has at most $2k+2$ stationary points by Proposition 9.2 and so at most $k+1$ local maxima and $k+1$ local minima. Therefore by Theorem 4.14 the number of zeroes of $(u_\gamma, v_\gamma) - (u_{\gamma'}, v_{\gamma'})$ in $D^2$, counted with multiplicity, is at most $k$. But this contradicts $(0, 0)$ being a zero of multiplicity at least $k+1$, so $F$ is injective.

Finally, if $|\gamma|^2 > K_{k+1}||\psi - \phi||^2_{C_{2k+2}}$ then $(f_\gamma - \hat{f})|_{S^1}$ has at most $2k+2$ stationary points by Proposition 9.3 so there are at most $k$ zeroes of $(u_\gamma, v_\gamma) - (\hat{u}, \hat{v})$ in $D^2$ with multiplicity by Theorem 4.14. But $C = F(\gamma) = 0$ if and only if $(0, 0)$ has multiplicity more than $k$, giving $F(\gamma) \neq 0$, as we have to prove. \(\square\)

We shall show that $F$ has a zero in $C$. Consider the case $\psi = \hat{\phi} = 0$. Then when $\gamma = 0$ we have $v_0 = v_0 = \hat{u} = \hat{v} = 0$, so that $F(0) = 0$ by (39). As $F$ is injective and continuous by Proposition 9.6 it follows that the winding number of $F$ about 0 along the circle $\gamma_r(0)$ in $C$ with radius $r > 0$ and centre 0 is $\pm 1$.

We may deform any $\psi, \hat{\phi} \in C^\infty(S^1)$ continuously to $0, 0$ through $t\psi, t\hat{\phi}$ for $t \in [0, 1]$. This gives a 1-parameter family of functions $F_t : C \to C$, with $F_0(0) = 0$, and we seek a zero $\gamma$ of $F_1 = F$. Now $F_t$ depends continuously on $t$, as $F$ depends continuously on $\psi, \hat{\phi}$, and $F_t(\gamma) \neq 0$ if $|\gamma|^2 > K_{k+1}||\psi - \hat{\phi||^2_{C_{2k+2}}}$ by Proposition 9.7.

Thus, if $r^2 > K_{k+1}||\psi - \hat{\phi||^2_{C_{2k+2}}}$ then $F_t$ is nonzero on $\gamma_r(0)$ for all $t \in [0, 1]$, and so the winding number $F_1$ about 0 along $\gamma_r(0)$ is the same as the winding number of $F_0$, which is $\pm 1$. Therefore $F = F_1$ must have a zero inside $\gamma_r(0)$, as otherwise the winding number would be 0.

We have shown that the function $F$ has a zero $\gamma = \alpha_{k+1} + i\beta_{k+1} \in C$. This $\gamma$ is unique by injectivity in Proposition 9.8. The construction above then yields $\alpha_1, \beta_1, \ldots, \alpha_{k+1}, \beta_{k+1}$ such that $(u, v) - (\hat{u}, \hat{v}) = (u_\gamma, v_\gamma) - (\hat{u}, \hat{v})$ has a zero of multiplicity at least $k + 1$, as we have to prove. The $\alpha_j, \beta_j$ are unique for $j = k + 1$ by uniqueness of $\gamma$, and for $1 \leq j \leq k$ by the inductive hypothesis.

Now $\gamma = \alpha_{k+1} + i\beta_{k+1}$ depends continuously on $a, \psi, \hat{\phi}$ as $F$ does, and $\alpha_1, \beta_1, \ldots, \alpha_{k+1}, \beta_{k+1}$ depend continuously on $a, \psi, \hat{\phi}$ and $\gamma$ by the inductive hypothesis. Thus $\alpha_1, \beta_1, \ldots, \alpha_{k+1}, \beta_{k+1}$ depend continuously on $a, \psi, \hat{\phi}$.

If $\sum_{j=1}^{k+1} (\alpha_j^2 + \beta_j^2) > K_{k+1}||\psi - \hat{\phi||^2_{C_{2k+2}}}$ then $\phi - \hat{\phi}$ has at most $2k+2$ stationary points on $S^1$ by Proposition 9.9 so there are at most $k$ zeroes of $(u, v) - (\hat{u}, \hat{v})$ in $D^2$ with multiplicity by Theorem 4.14. But $(0, 0)$ is a zero with multiplicity at least $k+1$, a contradiction. Hence $\sum_{j=1}^{k+1} (\alpha_j^2 + \beta_j^2) \leq K_{k+1}||\psi - \hat{\phi||^2_{C_{2k+2}}}$. This completes the induction, and the proof of Theorem 9.3 \(\square\)

The following lemma is easily proved using Proposition 4.8 by subtracting (16) for $u_1, v_1, u_2, v_2$ from (16) for $u_1, v_1, u_3, v_3$.  

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Lemma 9.6. Let $S$ be a domain in $\mathbb{R}^2$, let $u_j, v_j \in C^1(S)$ be solutions of \eqref{13} for $j = 1, 2, 3$ and some $a \neq 0$, and let $(b, c) \in S^0$. Suppose $(u_1, v_1) - (u_2, v_2)$ has a zero of multiplicity at least $k$ and $(u_1, v_1) - (u_3, v_3)$ a zero of multiplicity at least $l$ at $(b, c)$. Then $(u_2, v_2) - (u_3, v_3)$ has a zero of multiplicity at least $\min(k, l)$ at $(b, c)$.

Combining the last two results, we prove:

Theorem 9.7. Let $a \neq 0$, $n \geq 1$, $\gamma_1, \ldots, \gamma_n \in \mathbb{R}$ and $\psi \in C^\infty(S^1)$. Then there exists $K > 0$ depending only on $n, \gamma_1, \ldots, \gamma_n$ and $\psi$ and unique $\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n$ with $\sum_{j=1}^n (\alpha_j^2 + \beta_j^2) \leq K$ such that the following holds. Furthermore, for fixed $a, n, \psi$ the map $(\gamma_1, \ldots, \gamma_n) \mapsto (\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n)$ is injective. Define

$$\phi = \psi + \sum_{j=1}^n (\alpha_j \cos(j\theta) + \beta_j \sin(j\theta)) \quad \text{and} \quad \hat{\phi} = \sum_{j=1}^n \gamma_j \sin(j\theta) \quad \text{in } C^\infty(S^1).$$

Let $f, \hat{f} \in C^\infty(D)$ be the unique solutions of \eqref{14} on $D$ with $f|_{S^1} = \phi, \hat{f}|_{S^1} = \hat{\phi}$, which exist by Theorem 4.11. Define $u = \frac{\partial f}{\partial y}, v = \frac{\partial f}{\partial x}, \hat{u} = \frac{\partial \hat{f}}{\partial y}, \hat{v} = \frac{\partial \hat{f}}{\partial x}$, $u'(x, y) = u(x, -y)$ and $v'(x, y) = -v(x, -y)$, so that $(u, v), (\hat{u}, \hat{v})$ and $(u', v')$ satisfy \eqref{13} in $C^\infty(D)$. Then $(u, v) - (\hat{u}, \hat{v})$ and $(u, v) - (u', v')$ both have a zero of multiplicity at least $n$ at $(0, 0)$.

Proof. Most of the theorem is immediate from Theorem 9.4. We need only prove that $(u, v) - (u', v')$ has a zero of multiplicity at least $n$ at $(0, 0)$, and that $\gamma_j \mapsto \alpha_j, \beta_j$ is injective. As $\phi(x, -y) \equiv -\phi(x, y)$ we have $\hat{f}(x, -y) \equiv -\hat{f}(x, y)$, $\hat{u}(x, -y) = \hat{u}(x, y)$ and $\hat{v}(x, -y) = -\hat{v}(x, y)$. Therefore, as $(u, v) - (\hat{u}, \hat{v})$ has a zero of multiplicity at least $n$ at $(0, 0)$, we see by applying the symmetry $(x, y) \mapsto (x, -y)$ that $(u', v') - (\hat{u}, \hat{v})$ has a zero of multiplicity at least $n$ at $(0, 0)$. Lemma 9.6 then shows that $(u, v) - (u', v')$ has a zero of multiplicity at least $n$ at $(0, 0)$.

Now suppose that for fixed $a, n, \psi$, two distinct $n$-tuples $\gamma_1, \ldots, \gamma_n$ and $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_n$ yield the same $\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n$. Let $(\hat{u}, \hat{v})$ and $(\tilde{u}, \tilde{v})$ be the solutions of \eqref{13} corresponding to the $\gamma_j$ and $\tilde{\gamma}_j$, with potentials $\hat{f}, \tilde{f}$, and $(u, v)$ the solution corresponding to the $\alpha_j, \beta_j$.

From above, $(u, v) - (\hat{u}, \hat{v})$ and $(u, v) - (\tilde{u}, \tilde{v})$ both have a zero of multiplicity at least $n$ at $(0, 0)$. So $(\hat{u}, \hat{v}) - (\tilde{u}, \tilde{v})$ has a zero of multiplicity at least $n$ at $(0, 0)$ by Lemma 9.6. But the corresponding potentials $\hat{f}, \tilde{f}$ satisfy

$$\left(\hat{f} - \tilde{f}\right)|_{S^1} = \sum_{j=1}^n (\gamma_j - \tilde{\gamma}_j) \sin(j\theta).$$

So $\left(\hat{f} - \tilde{f}\right)|_{S^1}$ has at most $2n$ stationary points in $S^1$ by Proposition 9.2 as $\gamma_j \neq \tilde{\gamma}_j$. Hence $(\hat{u}, \hat{v}) - (\tilde{u}, \tilde{v})$ has at most $n - 1$ zeroes in $D^0$ with multiplicity, a contradiction. So the map $(\gamma_1, \ldots, \gamma_n) \mapsto (\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n)$ is injective. \hfill $\Box$

9.3 Constructing singularities with multiplicity $n$

By taking the limit $a \to 0$ in Theorem 9.7 we prove:
Theorem 9.8. Let \( n \geq 1, \gamma_1, \ldots, \gamma_n \in \mathbb{R} \) and \( \psi \in C^\infty(S^1) \). Then there exist unique \( \alpha_1, \beta_1, \ldots, \alpha_n, \beta_n \in \mathbb{R} \) such that the following holds. Furthermore, for fixed \( a, \psi \) the map \((\gamma_1, \ldots, \gamma_n) \mapsto (\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n)\) is injective. Define

\[
\phi = \psi + \sum_{j=1}^{n} (\alpha_j \cos(j\theta) + \beta_j \sin(j\theta)) \quad \text{and} \quad \hat{\phi} = \sum_{j=1}^{n} \gamma_j \sin(j\theta) \quad \text{in} \quad C^\infty(S^1).
\]

Let \( f, \hat{f} \in C^1(D) \) be the unique weak solutions of (13) on \( D \) with \( f|_{S^1} = \phi \), \( \hat{f}|_{S^1} = \hat{\phi} \) and \( a = 0 \), which exist by Theorem 4.12. Define \( u = \frac{\partial f}{\partial y}, v = \frac{\partial f}{\partial x} \), \( \hat{u} = \frac{\partial \hat{f}}{\partial y} \) and \( \hat{v} = \frac{\partial \hat{f}}{\partial x} \), so that \((u, v)\) and \((\hat{u}, \hat{v})\) are singular solutions of (12) in \( C^0(D) \). Then \((u, v) - (\hat{u}, \hat{v})\) has a zero of multiplicity at least \( n \) at \((0, 0)\), and either \((u, v)\) has an isolated singularity of multiplicity at least \( n \) at \((0, 0)\), or \(u(x, -y) \equiv u(x, y)\) and \(v(x, -y) \equiv -v(x, y)\).

**Proof.** Fix \( n, \gamma_j \) and \( \psi \) as above. For each \( a \neq 0 \), Theorem 9.7 gives unique \( \alpha_1, \beta_1, \ldots, \alpha_n, \beta_n \) with \( \sum_{j=1}^{n} (\alpha_j^2 + \beta_j^2) \leq K \) satisfying certain conditions. As the set of \( \alpha_j, \beta_j \) with \( \sum_{j=1}^{n} (\alpha_j^2 + \beta_j^2) \leq K \) is compact, we can choose a sequence \((a^i)_{i=1}^{\infty} \in \{0, 1\}\) such that \( a^i \rightarrow 0 \) as \( i \rightarrow \infty \), such that the corresponding \( \alpha_j^i, \beta_j^i \)

converge to limits \( \alpha_j, \beta_j \) injective in \( \mathbb{R}^{2n} \).

Let \( f^i, u^i, v^i, \hat{f}^i, \hat{u}^i, \hat{v}^i \in C^\infty(D) \) be the corresponding solutions \( f, u, v, \hat{f}, \hat{u}, \hat{v} \) in Theorem 9.7 for \( i = 1, 2, \ldots \), so that \( f^i, u^i, v^i, \hat{f}^i, \hat{u}^i, \hat{v}^i \) satisfy (17) and \( u^i, v^i, a^i \) and \( \hat{u}^i, \hat{v}^i, \hat{a}^i \) satisfy (13) in \( D \). Define \( \phi = \psi + \sum_{j=1}^{n} (\alpha_j \cos(j\theta) + \beta_j \sin(j\theta)) \) in \( C^\infty(S^1) \), and let \( f, \hat{f} \in C^1(D) \) be the unique weak solutions of (13) on \( D \) with \( f|_{S^1} = \phi \), \( \hat{f}|_{S^1} = \hat{\phi} \), which exist by Theorem 4.12. Let \( u, v \) and \( \hat{u}, \hat{v} \) be the corresponding singular solutions of (12).

Then \( \hat{f}|_{S^1} = \hat{f}|_{S^1} = \hat{\phi} \) for all \( i \geq 1 \). Hence by Theorem 4.12, we see that \( f^i \rightarrow f \) in \( C^1(D) \) as \( i \rightarrow \infty \), so that \( u^i, v^i \rightarrow \hat{u}, \hat{v} \) in \( C^0(D) \) as \( i \rightarrow \infty \). Similarly, as \( \alpha_j^i, \beta_j^i \rightarrow \alpha_j, \beta_j \) as \( i \rightarrow \infty \), we see that \( f^i|_{S^1} \rightarrow f|_{S^1} = \phi \) in \( C^\infty(S^1) \) as \( i \rightarrow \infty \). Hence by Theorem 4.12 we see that \( f^i \rightarrow f \) in \( C^1(D) \) and \( u^i, v^i \rightarrow u, v \) in \( C^0(D) \) as \( i \rightarrow \infty \).

Thus \((u^i, v^i) - (\hat{u}^i, \hat{v}^i)\) has a zero of multiplicity at least \( n \) at \((0, 0)\), and \( u^i, v^i \rightarrow u, v \) in \( C^0(S) \) as \( i \rightarrow \infty \). So \((u, v) - (\hat{u}, \hat{v}) \) has a zero at \((0, 0)\). By Theorem 9.21 either \((u, v) \equiv (\hat{u}, \hat{v}) \) or \((0, 0)\) is an isolated zero. If it is isolated, then by \( C^0 \) convergence we find that \((u^i, v^i) - (\hat{u}^i, \hat{v}^i) \) and \((u, v) - (\hat{u}, \hat{v}) \) have the same winding number about \( 0 \) along \( \gamma_n(0, 0) \) for small \( \epsilon > 0 \) and large \( i \). Therefore \((u, v) - (\hat{u}, \hat{v}) \) has multiplicity at least \( n \) at \((0, 0)\).

In the same way, if \( u^i(x, y) = u(x, -y) \) and \( v^i(x, y) = -v(x, -y) \) we find that \((u, v) - (u^i, v^i) \) has a zero of multiplicity at least \( n \) at \((0, 0)\), and so by definition either \((u, v) \equiv (u^i, v^i) \) or \((u, v) \) has an isolated singularity of multiplicity at least \( n \) at \((0, 0)\). It remains only to show that \( \alpha_1, \beta_1, \ldots, \alpha_n, \beta_n \) are unique, and the map \( \gamma_j \mapsto \alpha_j, \beta_j \) injective.

Suppose \( \alpha_j, \beta_j, \phi, f, u, v \) and \( \tilde{\alpha}_j, \tilde{\beta}_j, \tilde{\phi}, \tilde{f}, \tilde{u}, \tilde{v} \) are two distinct solutions. Then

\[
\phi - \tilde{\phi} = \sum_{j=1}^{n} ((\alpha_j - \tilde{\alpha}_j) \cos(j\theta) + (\beta_j - \tilde{\beta}_j) \sin(j\theta)),
\]
so as \( \alpha_j - \tilde{\alpha}_j, \beta_j - \tilde{\beta}_j \) are not all zero \( \phi - \tilde{\phi} \) has at most \( 2n \) stationary points by Proposition 9.2. Theorem 6.21 then shows that there are at most \( n - 1 \) zeroes of \((u, v) - (\tilde{u}, \tilde{v}) \in D^o\), counted with multiplicity.

But as \((u, v) - (\tilde{u}, \tilde{v}) \) and \((\tilde{u}, \tilde{v}) - (\tilde{u}, \tilde{v}) \) both have a zero of multiplicity at least \( n \) at \((0, 0)\), by a version of Lemma 9.6 for \( a = 0 \) we see that \((u, v) - (\tilde{u}, \tilde{v}) \) has a zero of multiplicity at least \( n \) at \((0, 0)\), a contradiction. Thus the \( \alpha_j, \beta_j \) are unique. A similar proof, extending that in Theorem 9.4, shows that the map \((\gamma_1, \ldots, \gamma_n) \mapsto (\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n) \) is injective.

For particular \( \psi \) we can pin down the multiplicity exactly.

**Theorem 9.9.** Take \( \psi = \alpha_{n+1} \cos(n+1)\theta + \beta_{n+1} \sin(n+1)\theta \in C^\infty(S^1) \), for \( \alpha_{n+1} \neq 0 \) and \( \beta_{n+1} \in \mathbb{R} \). Then for all \( \gamma_1, \ldots, \gamma_n \in \mathbb{R} \) the singular solution \((u, v)\) of (12) constructed in Theorem 9.8 from \( \psi \) and \( \gamma_1, \ldots, \gamma_n \) has an isolated singularity at \((0, 0)\) of multiplicity \( n \), and no other singularities in \( D^o \).

**Proof.** Let \( u, \alpha_j, \beta_j, \phi, f, u, v \) be as in Theorem 9.8 and set \( \phi'(x, y) = -\phi(x, -y) \), \( f'(x, y) = -f(x, -y) \), \( u'(x, y) = u(x, -y) \) and \( v'(x, y) = -v(x, -y) \). Then \( \phi - \phi' = 2\sum_{j=1}^{n+1} \alpha_j \cos(j\theta) \). As \( \alpha_{n+1} \neq 0 \), Proposition 9.2 shows that \( \phi - \phi' \) has at most \( 2n + 2 \) stationary points in \( S^1 \), and Theorem 6.21 shows there are at most \( n \) zeroes of \((u, v) - (u', v') \) in \( D^o \), counted with multiplicity. But \((0, 0)\) is a zero of multiplicity at least \( n \). So it has multiplicity exactly \( n \), and there are no other zeroes in \( D^o \). The result follows.

Thus there exist singular solutions \((u, v)\) of (12) with isolated singularities of all multiplicities \( n \geq 1 \) at \((0, 0)\). Now by Proposition 8.4 a singularity of multiplicity \( n \) has one of two types. Clearly, if \((u, v)\) has one type, then \((-u, -v)\) is also singular of multiplicity \( n \) at \((0, 0)\), but with the other type. So we prove:

**Corollary 9.10.** There exist examples of singular solutions \( u, v \) of (12) with isolated singularities of every possible multiplicity \( n \geq 1 \), and with both possible types allowed by Proposition 8.4.

Combining this with Proposition 4.4 gives examples of singular SL 3-folds in \( \mathbb{C}^3 \) with an infinite number of different geometrical/topological types. This is one of the main results of the paper. Also, by combining Example 7.3 with Corollary 9.10 we can construct SL fibrations including singular fibres with every possible multiplicity and type.

If \( \psi(x, -y) \equiv \psi(x, y) \) in Theorem 9.8 we can set \( \beta_j, \gamma_j, \tilde{\phi}, \tilde{f}, \tilde{u}, \tilde{v} \) to zero, and get a simpler theorem constructing solutions \( u, v \) with the symmetries \( u(x, -y) \equiv -u(x, y) \) and \( v(x, -y) \equiv v(x, y) \).

**Theorem 9.11.** Let \( \psi \in C^\infty(S^1) \) with \( \psi(x, -y) \equiv \psi(x, y) \) and \( n \geq 1 \). Then there exist unique \( \alpha_1, \ldots, \alpha_n \in \mathbb{R} \) such that the following holds. Define \( \phi = \psi + \sum_{j=1}^{n} \alpha_j \cos(j\theta) \) in \( C^\infty(S^1) \). Let \( f \in C^1(D) \) be the unique weak solution of (12) on \( D \) with \( f|_{S^1} = \phi \) and \( a = 0 \), which exists by Theorem 1.12. Define \( u = \frac{\partial f}{\partial y} \) and \( v = \frac{\partial f}{\partial x} \), so that \((u, v)\) is a singular solution of (12) in \( C^0(D) \) with \( u(x, -y) \equiv -u(x, y) \) and \( v(x, -y) \equiv v(x, y) \). Then either \((u, v)\) has an isolated singularity of multiplicity at least \( n \) at \((0, 0)\), or \((u, v) \equiv (0, 0)\).
9.4 Discussion

In [13] we have put together a detailed picture of the singularities of singular solutions \( u, v \) of (12). Here are some remarks on this, beginning with how to interpret Theorem 9.8.

First note that the singular solution \((\hat{u}, \hat{v})\) in Theorem 9.8 has the symmetries \( \hat{u}(x,-y) = \hat{u}(x,y) \) and \( \hat{v}(x,-y) = -\hat{v}(x,y) \), and so was studied in [8]. In particular, \( \hat{u}(x,0) \) is a real analytic function of \( x \in (-1,1) \). One can show that \( \gamma_1, \ldots, \gamma_n \) are determined uniquely by \( \frac{\partial^m}{\partial x^m} \hat{u}(0,0) \) for \( m = 0, \ldots, n-1 \), and vice versa. It seems likely that \( \frac{\partial^m}{\partial x^m} u(0,0) \) exists for \( m = 0, \ldots, n-1 \), and equals \( \frac{\partial^m}{\partial x^m} \hat{u}(0,0) \).

Thus, singularities with multiplicity \( n \) at \((0,0)\) are locally described to a first approximation by \( n \) real parameters, the \( \gamma_j \) in Theorem 9.8 and perhaps equivalently by \( \frac{\partial^m}{\partial x^m} u(0,0) \) for \( m = 0, \ldots, n-1 \).

Theorem 9.8 implies that singularities with multiplicity \( n \) at \((0,0)\) and prescribed values of \( \gamma_j \) occur in codimension \( 2n \) in the family of all singular solutions \( u, v \) of (12). Hence, singularities with multiplicity \( n \) at \((0,0)\) but without prescribed \( \gamma_j \) should occur in codimension \( n \) in the family of all singular solutions \( u, v \) of (12).

More generally, if we consider all solutions of (12) and (13), and allow singularities anywhere, then we add one codimension for \( a \in \mathbb{R} \), and subtract one because singularities can occur at \((x,0)\) for any \( x \in \mathbb{R} \). This gives:

**Principle.** Singular points with multiplicity \( n \geq 1 \) should occur in real codimension \( n \) in the family of all SL 3-folds invariant under the U(1)-action (11).

We can relate this to the special Lagrangian fibrations constructed in [7]. If \( \Phi \) is chosen generically in Definition 7.1, or if \( \phi \) is chosen generically in Example 7.3, then we expect \( N_\alpha \) to have singular points of multiplicity \( n \) for \( \alpha \) in a (possibly empty) codimension \( n \) subset of \( U \subseteq \mathbb{R}^3 \). As \( \dim U = 3 \), it follows that \( N_\alpha \) has multiplicity \( n \) singularities for \( n = 1, 2, 3 \) and \( \alpha \) in a \((3-n)\)-dimensional subset of \( U \), and no multiplicity \( n \) singularities for \( n > 3 \).

It is also interesting to consider how the singularities of U(1)-invariant SL 3-folds studied in [8]–[13] relate to the broader current state of knowledge on singularities of SL m-folds in (almost) Calabi–Yau m-folds, which is surveyed in [14]. The class of singular SL m-folds which are best understood theoretically are SL m-folds with isolated conical singularities. They are the subject of a series of papers by the author. For a survey and further references, see [13].

Let \( N \) be an SL m-fold in \( M \) with isolated conical singularities, and \( x \) a singular point of \( N \). The definition [13] Def. 3.7] implies that \( N \) is an SL rectifiable current with \( x \in N^\circ \), and the (unique) tangent cone \( C \) to \( N \) at \( x \) is an SL cone in \( T_x M \) with multiplicity 1 and an isolated singularity at 0.

From [8] if \( N \) is a U(1)-invariant SL 3-fold in \( \mathbb{C}^3 \) and \( x \) an isolated singular point of multiplicity \( n \geq 2 \), in the sense of Definition 8.1 then the (unique) tangent cone \( C \) to \( N \) at \( x \) is \( \Pi_x^+ \cup \Pi_x^- \) for \( \Pi_x^\circ \) defined in [20], with multiplicity 1. That is, \( C \) is the union of two copies of \( \mathbb{R}^3 \) intersecting in \( \mathbb{R} \). So \( C \) does not
have an isolated singularity at 0, but is singular along $\mathbb{R}$.

Therefore the singularities of multiplicity $n \geq 2$ above are examples of isolated singularities of SL 3-folds which are not isolated conical singularities in the sense of [13]. We have been able to understand them well in this series of papers by assuming U(1)-invariance. However, there is no known general theory of special Lagrangian singularities of this kind, similar to the isolated conical case, without assuming U(1)-invariance.

Constructing such a theory appears to the author to be very difficult, because of the lack of good local models for the singularities. (The singularities constructed above are not yet suitable as local models, as we do not understand their asymptotic behaviour near the singular points.) One interesting feature of the results above is that they provide a rich class of fairly generic special Lagrangian singularities, which are not adequately covered by any known general analytic theory. This shows we still have a long way to go in understanding special Lagrangian singularities, if this is a feasible goal.

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