Uniform $K$-continuity and ISS in $L^\infty$-norm of nonlinear parabolic PDEs

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Abstract

We introduce the notion uniform $K$-continuity (UKC) to describe uniformly continuous dependence of solutions on external disturbances for nonlinear parabolic PDEs. As an application of UKC, input-to-state stability (ISS) estimates in $L^\infty$-norm are established for weak solutions of a class of nonlinear parabolic PDEs with mixed/ Dirichlet/ Robin boundary disturbances. The properties of UKC and ISS estimates are established by the De Giorgi iteration.

Key words: Uniform $K$-continuity; input-to-state stability; boundary disturbances; $L^\infty$-norm; nonlinear PDEs.

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1 Definitions for stabilities

Consider the following parabolic equation:

\begin{align}
L_t u + h(u) &= f(x, t), \quad (x, t) \in (0, 1) \times \mathbb{R}_+,
ku(0, t) - lu_x(0, t) &= p(t), \quad t \in \mathbb{R}_+,
\end{align}

This is a reduced version of our manuscript written in the summer of 2019, which presents the application of the De Giorgi iteration in the analysis of ISS in $L^\infty$-norm for nonlinear parabolic PDEs with Robin or mixed boundary disturbances.

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\[\begin{align*}
m u(1, t) + n u_x(1, t) &= q(t), \quad t \in \mathbb{R}_+, \\
u(x, 0) &= \phi(x), \quad x \in (0, 1),
\end{align*}\]

where \(L_t u := u_t - a(x, t) u_x + b(x, t) u_x + c(x, t) u\) is the linear part of a parabolic equation with continuous functions \(a, b, c\) on \((0, 1) \times \mathbb{R}_+\) and \(a \geq 0\). \(h\) represents the nonlinearity of the equation. \(f, d_0,\) and \(d_1\) are disturbances distributed over the domain \((0, 1)\) and on the boundaries \(x = 0, 1\). \(\phi\) is the initial value. \(k, \ell, n\) are nonnegative continuous functions satisfying \(k + \ell > 0, m + n > 0\).

Suppose that \(f \in F, d \in D, \phi \in U\) for certain Banach spaces \(F, D, U\) endowed with norm \(\| \cdot \|_F, \| \cdot \|_D\) and \(\| \cdot \|_U\), respectively, (1) has a unique global solution \(u(\cdot, t) \in U\). For system (1) with data \((\phi, f, p, q)\), we denote it by \(\Sigma(\phi, f, p, q)\). Without special statement, let \(u_1, u_2\) be solutions of system (1) associated with data \((\phi, f_1, p_1, q_1)\) and \((\phi, f_2, p_2, q_2)\), respectively.

**Definition 1.1** We say that the solution of \(\Sigma(\phi, f, p, q)\) is uniformly continuous (UC) in the norm of \(U\) w.r.t. the boundary disturbances \(p, q\) and the in-domain disturbance \(f\), if for every \(\varepsilon > 0\), there exists \(\delta(\varepsilon) > 0\) such that for any \(t > 0\):

\[\|p_1 - p_2\|_{L^\infty(0, t)} + \|q_1 - q_2\|_{L^\infty(0, t)} + \|f_1 - f_2\|_{L^\infty((0, 1) \times (0, t))} < \delta \Rightarrow \|u_1(\cdot, t) - u_2(\cdot, t)\|_U < \varepsilon.\]

**Definition 1.2** We say that the solution of \(\Sigma(\phi, f, p, q)\) is uniformly \(K\)-continuous (UKC) in the norm of \(U\) w.r.t. the boundary disturbances \(p, q\) and the in-domain disturbance \(f\), if there exist functions \(\gamma_f, \gamma_p, \gamma_q \in K\) such that for any \(t > 0\):

\[\|u_1(\cdot, t) - u_2(\cdot, t)\|_U \leq \gamma_f(\|f_1 - f_2\|_{L^\infty((0, 1) \times (0, t))}) + \gamma_p(\|p_1 - p_2\|_{L^\infty(0, t)}) + \gamma_q(\|q_1 - q_2\|_{L^\infty(0, t)}).\]

**Remark 1.1**

(i) Here we used the terminology “uniformly” in terms of the external disturbances \(f, p, q\), i.e., for every \(\varepsilon > 0\), the positive constant \(\delta\) depends only on \(\varepsilon\), which is independent of \(f, p, q\).

(ii) It is obvious that ULG \(\Rightarrow\) UKC \(\Rightarrow\) UC.

**Definition 1.3** We say that \(\Sigma(\phi, 0, 0, 0)\) is stable at zero (0-S) in the norm of \(U\) w.r.t. the initial data \(\phi\), if there exists a function \(\theta \in K_{\infty}\) such that for any \(t > 0\):

\[\|u(\cdot, t)\|_U \leq \theta(\|\phi\|_U).\]

**Definition 1.4** We say that \(\Sigma(\phi, 0, 0, 0)\) is \(K\)-\(L\)-stable at zero (0-KLS) in the norm of \(U\) w.r.t. the initial data \(\phi\), if there exists a function \(\beta \in K\) such that for any \(t > 0\):

\[\|u(\cdot, t)\|_U \leq \beta(\|\phi\|_U, t).\]

Particularly, system \(\Sigma(\phi, 0, 0, 0)\) is said to be exponentially stable at zero (0-ES) in the norm of \(U\) w.r.t. the initial data \(\phi\), if there exists a function constants \(M, \lambda > 0\) such that \(\beta(\|\phi\|_U, t) := M\|\phi\|_U e^{-\lambda t}\) in (3).

**Remark 1.2**

(i) It is obvious that 0-KLS \(\Rightarrow\) 0-S.

(ii) If \(\Sigma(\phi, 0, 0, 0)\) is 0-S (or 0-KLS), then it is 0-UGS (or 0-UGAS) (see [1] for definitions of 0-UGS and 0-UGAS).

**Definition 1.5** System \(\Sigma(\phi, f, p, q)\) is said to be input-to-state stable (ISS) in the norm of \(U\) w.r.t. the boundary disturbances \(p, q\) and the in-domain disturbance \(f\), if there exist functions \(\beta \in K\) and \(\gamma_f, \gamma_p, \gamma_q \in K\) such that the solution of (1) satisfies for any \(T > 0\), it holds:

\[\|u(\cdot, t)\|_U \leq \beta(\|\phi\|_U, t) + \gamma_f(\|f\|_{L^\infty((0, 1) \times (0, t))}) + \gamma_p(\|p\|_{L^\infty(0, t)}) + \gamma_q(\|q\|_{L^\infty(0, t)}).\]

Moreover, system \(\Sigma(\phi, f, p, q)\) is said to be exponential input-to-state stable (EISS) in the norm of \(U\) w.r.t. the boundary disturbances \(p, q\) and the in-domain disturbance \(f\), if there exist constants \(M, \lambda > 0\) such that \(\beta(\|\phi\|_U, t) = M\|\phi\|_U e^{-\lambda t}\) in (4).

**Proposition 1.1** The following statements hold true:
(i) If $\Sigma(\phi, f, p, q) = \text{UKC}$ in the norm of $U$ w.r.t. $f, p, q$, and $\Sigma(u, \phi, 0, 0, 0)$ is $0$-KLS in the norm of $U$ w.r.t. $\phi$, then $\Sigma(\phi, f, p, q)$ is ISS in the norm of $U$ w.r.t. $f, p, q$;

(ii) If $\Sigma(\phi, f, p, q)$ is $\text{UKC}$ in $L^j$-norm w.r.t. $f, p, q$, and $\Sigma(u, \phi, 0, 0, 0)$ is $0$-KLS in $L^j$-norm w.r.t. $\phi$, $1 \leq i \leq j \leq +\infty$, then $\Sigma(\phi, f, p, q)$ is ISS in $L^i$-norm w.r.t. $f, p, q$.

**Proof.** We only prove (ii) with $1 \leq i \leq j < +\infty$. Let $u, v$ be the unique solutions of $\Sigma(\phi, f, p, q)$ and $\Sigma(\phi, 0, 0, 0)$ respectively. By Definition of UKC and 0-KLS, there exist functions $\beta \in KL$ and $\gamma f, \gamma p, \gamma q \in K$ and a function $\beta \in KL$ such that for any $t > 0$:

$$
\|u(\cdot, t)\| \leq \|v\| + \|u - v\| \leq \beta(\|\phi\|, T) + \|u - v\| \leq \beta(\|\phi\|, T) + \gamma f(\|f - 0\|_{L^\infty((0,1) \times (0,t))}) + \gamma p(\|p - 0\|_{L^\infty(0,t)}) + \gamma q(\|q - 0\|_{L^\infty(0,t)})).
$$

**Remark 1.3**

(i) As for linear system with external disturbances, it is easy to see that $\Sigma(\phi, f, p, q)$ is ISS in the norm of $U$ w.r.t. $f, p, q$, if and only if $\Sigma(\phi, f, p, q)$ is UKC in the norm of $U$ w.r.t. $f, p, q$, and $\Sigma(\phi, 0, 0, 0)$ is 0-KLS in the norm of $U$ w.r.t. $\phi$, i.e., ISS $\iff$ UKC $\wedge$ 0-KLS, provided that $\Sigma(\phi, f, p, q)$ is a linear system.

(ii) As for nonlinear system, it is not an easy task to verify its ISS property w.r.t. boundary disturbances. However, noting Proposition 1.1, in order to obtain the ISS property, it suffices to obtain its UKC and 0-KLS properties, which can be often addressed.

Without special statements, we always assume that:

- $f \in L^\Phi((0, 1) \times \mathbb{R}^+), p, q \in L^2_{loc}(\mathbb{R}^+), \phi \in W^1_2((0,1);$
- $a, c, m, n \in \mathbb{R}^+$;
- $h \in C(\mathbb{R})$ is increasing on $\mathbb{R}$ with $h(0) = 0$;
- $|h(u) - h(u)| \leq \mu |u|^{\lambda+2}$ for all $u \in \mathbb{R}$, where $1 \leq \lambda < 3, \mu \in \mathbb{R}^+$.

## 2 UKC and ISS in $L^\infty$-norm for nonlinear parabolic PDEs with in-domain and mixed boundary disturbances

In this section, we study UKC in $L^\infty$-norm and ISS in different norms for the following nonlinear PDEs with mixed boundary disturbances:

$$
\begin{align*}
\frac{d}{dt} u - a u_{xx} + cu + h(u) &= f(x, t), \quad (x, t) \in (0, 1) \times \mathbb{R}^+, \quad \text{(5a)} \\
u(0, t) &= p(t), \quad t \in \mathbb{R}^+, \quad \text{(5b)} \\
u_x(1, t) &= -mu(1, t) + q(t), \quad t \in \mathbb{R}^+, \quad \text{(5c)} \\
u(x, 0) &= \phi(x), \quad x \in (0, 1). \quad \text{(5d)}
\end{align*}
$$

**Definition 2.1** We say $u \in W^1_2((0, 1) \times \mathbb{R}^+)$ is a (weak) solution of (5), if for a.e. $T \in \mathbb{R}^+$ and for any $\varphi \in W^1_2((0, 1) \times (0, T))$ with $\varphi(0, \cdot) = 0$ in $(0, T)$, there holds:

$$
\begin{align*}
\int_0^T \int_0^1 u \varphi dx dt &= \int_0^T \int_0^1 \varphi(1, t) dt + \int_0^T \int_0^1 a u_x \varphi_x dx dt + \int_0^T \int_0^1 (cu + h(u)) \varphi dx dt \\
&+ \int_0^T \int_0^1 f \varphi dx dt,
\end{align*}
$$

and $u(\cdot, 0) = \phi(\cdot)$ a.e. in $(0, 1)$ and $u(0, \cdot) = p(\cdot)$ a.e. in $\mathbb{R}^+$.

### 2.1 ULC in $L^\infty$-norm w.r.t. in-domain and mixed boundary disturbances

**Proposition 2.1** System (5) is ULC in $L^\infty$-norm w.r.t. boundary disturbances $p, q$ and in-domain disturbance $f$.  

Proof Let \( u_i (i = 1, 2) \) be the solution of the following equations:

\[
\begin{align*}
\dot{u}_t - au_{xx} + cu + h(u) &= f_i (x, t) \text{ in } (0, 1) \times \mathbb{R}_+, \\
\dot{u}(0, t) &= p_i (t), \quad t \in \mathbb{R}_+, \\
\dot{u}_x (1, t) &= -mu (1, t) + q_i (t), \quad t \in \mathbb{R}_+, \\
\dot{u}(x, 0) &= \phi (x), \quad x \in (0, 1),
\end{align*}
\]

where \( f_i \in C([0, 1] \times \mathbb{R}_{\geq 0}), p_i, q_i \in C(\mathbb{R}_{\geq 0}), i = 1, 2 \).

Consider \( w = u_1 - u_2 \), which satisfies:

\[
\begin{align*}
\dot{w}_t - aw_{xx} + cw + h(u_1) - h(u_2) &= f (x, t) \text{ in } (0, 1) \times \mathbb{R}_+, \\
\dot{w}(0, t) &= p(t), \quad t \in \mathbb{R}_+, \\
\dot{w}_x (1, t) &= -mw (1, t) + q(t), \quad t \in \mathbb{R}_+, \\
\dot{w}(x, 0) &= 0, \quad x \in (0, 1),
\end{align*}
\]

where \( f := f_1 - f_2, p := p_1 - p_2, q := q_1 - q_2 \).

We proceed by De Giorgi iteration. Specifically, for \( t > 0 \), let \( k_0 = \max \left\{ \max_{s \in [0, t]} p (s), \frac{1}{m} \max_{s \in [0, t]} q (s), 0 \right\} \). For \( k \geq k_0 \), let \( \eta (x, s) = (w (x, s) - k)_+ \chi_{[t_1, t_2]} (s) \), where \( \chi_{[t_1, t_2]} (s) \) is the character function on \([t_1, t_2]\) and \( 0 \leq t_1 < t_2 \leq t \). Let \( A_k (s) = \{ x \in (0, 1) ; w (x, s) > k \} \) and \( \varphi_k = \sup_{s \in (0, t)} |A_k (s)| \), where \( |B| \) denotes the 1-dimensional Lebesgue measure of a set \( B \subset (0, 1) \).

Multiplying (6) by \( \eta \), and noting that \((w (0, s) - k)_+ = 0\) for \( k \geq k_0 \) and \( s \in [0, t] \), we get

\[
\int_0^t \int_0^1 (w - k)_+(w - k)_+ \chi_{[t_1, t_2]} (s) dx ds - a \int_0^t (-mw (1, s) + q(s))(w (1, s) - k)_+ \chi_{[t_1, t_2]} (s) ds
\]

\[
+ \frac{a}{2} \int_0^t \int_0^1 |(w - k)_+ x|^2 \chi_{[t_1, t_2]} (s) dx ds + c \int_0^t \int_0^1 w (w - k)_+ \chi_{[t_1, t_2]} (s) dx ds
\]

\[
+ \int_0^t \int_0^1 (h (u_1) - h (u_2)) (w - k)_+ \chi_{[t_1, t_2]} (s) dx ds = \int_0^t \int_0^1 f (w - k)_+ \chi_{[t_1, t_2]} (s) dx ds.
\]

(7)

Noting that for \( w (1, s) \geq k \), it follows \(-w (1, s) \leq -k_0 \leq -\frac{1}{m} \sup_{s \in [0, t]} q (s)\), which implies

\[
\int_0^t (-mw (1, s) + q(s))(w (1, s) - k)_+ \chi_{[t_1, t_2]} (s) ds \leq \int_0^t \left( -\max_{s \in [0, t]} q (s) \right)(w (1, s) - k)_+ \chi_{[t_1, t_2]} (s) ds \leq 0.
\]

For \( w (x, s) \geq k \geq k_0 \geq 0 \), it follows \( u_1 = u_2 + w \geq u_2 \), which and the increasing property of \( h \) give

\[
\int_0^t \int_0^1 (h (u_1) - h (u_2)) (w - k)_+ \chi_{[t_1, t_2]} (s) dx ds \geq 0.
\]

It is obvious that

\[
\int_0^t \int_0^1 w (w - k)_+ \chi_{[t_1, t_2]} (s) dx ds \geq 0.
\]

Then by (7), we obtain

\[
\int_0^t \int_0^1 (w - k)_+(w - k)_+ \chi_{[t_1, t_2]} (s) dx ds + a \int_0^t \int_0^1 |(w - k)_+ x|^2 \chi_{[t_1, t_2]} (s) dx ds
\]

\[
= \int_0^t \int_0^1 f (w - k)_+ \chi_{[t_1, t_2]} (s) dx ds.
\]

4
\[
\int_0^t \int_0^1 f(w - k) + \chi_{[t_1, t_2]}(s)dxds.
\] (8)

Let \(I_k(s) = \int_0^1 ((w(x, s) - k)_+, 2)dx\). Suppose that \(I_k(t_0) = \max_{s \in [0, t]} I_k(s)\) with some \(t_0 \in [0, t]\). Due to \(I_k(0) = 0\) and \(I_k(s) \geq 0\), we can assume that \(t_0 > 0\) without loss of generality.

For \(\epsilon > 0\) small enough, choosing \(t_1 = t_0 - \epsilon\) and \(t_2 = t_0\), it follows that

\[
\frac{1}{2\epsilon} \int_{t_0 - \epsilon}^{t_0} \int_0^1 ((w - k)_+^2)dxds + \frac{a}{\epsilon} \int_{t_0 - \epsilon}^{t_0} \int_0^1 ((w - k)_+ x)^2 dxds \leq \frac{1}{\epsilon} \int_{t_0 - \epsilon}^{t_0} \int_0^1 |f(w - k)_+|dxds.
\]

Note that

\[
\frac{1}{2\epsilon} \int_{t_0 - \epsilon}^{t_0} \int_0^1 ((w - k)_+^2)dxds = \frac{1}{2\epsilon}(I_k(t_0) - I_k(t_0 - \epsilon)) \geq 0.
\]

Then we have

\[
a \int_0^1 ((w(x, t_0) - k)_+ x)^2 dx \leq \int_0^1 (f(x, t_0))((w(x, t_0) - k)_+ x) dx.
\]

Then proceeding exactly as the proof of [2, Lemma 6], we obtain

\[
w(x, s) \leq \max \left\{ \max_{s \in [0, t]} p(s), \frac{1}{m} \max_{s \in [0, t]} q(s), 0 \right\} + \frac{4\sqrt{2}}{a} \max_{(x, s) \in [0, 1] \times [0, t]} |f(x, s)|.
\] (9)

We need also to prove the lower boundedness of \(w(x, t)\). Indeed, setting \(\overline{w} = -w = u_2 - u_1\), we get

\[
\overline{w}_t - a\overline{w}_{xx} + c\overline{w} + h(u_2) - h(u_1) = -f(x, t) \text{ in } (0, 1) \times \mathbb{R}_+,
\]

\[
\overline{w}(0, t) = -p(t), \quad t \in \mathbb{R}_+,
\]

\[
\overline{w}_x(1, t) = -m\overline{w}(1, t) - q(t), \quad t \in \mathbb{R}_+,
\]

\[
\overline{w}(x, 0) = 0, \quad x \in (0, 1).
\]

Let \(k_0 = \max \left\{ \max_{s \in [0, t]} (-p(s)), \frac{1}{m} \max_{s \in [0, t]} (-q(s)), 0 \right\}\). For \(k \geq k_0\), let \(\eta(x, s) = (w(x, s) - k)_+ \chi_{[t_1, t_2]}(s)\). Proceeding as above and noting that

\[
\int_0^1 ((w(1, s) - q(s))(\overline{w}(1, s) - k)_+ \chi_{[t_1, t_2]}(s))ds \leq \int_0^1 \left( -\max_{s \in [0, t]} q(s) \right)((w(1, s) - k)_+ \chi_{[t_1, t_2]}(s))ds \leq 0,
\]

\[
\int_0^1 \int_0^1 \left( h(u_2) - h(u_1) \right)(\overline{w} - k)_+ \chi_{[t_1, t_2]}(s)dxds \geq 0,
\]

\[
\int_0^1 \int_0^1 (\overline{w} - k)_+ \chi_{[t_1, t_2]}(s)dxds \geq 0,
\]

we obtain

\[
-w(x, s) = \overline{w}(x, s) \leq \max \left\{ \max_{s \in [0, t]} (-p(s)), \frac{1}{m} \max_{s \in [0, t]} (-q(s)), 0 \right\} + \frac{4\sqrt{2}}{a} \max_{(x, s) \in [0, 1] \times [0, t]} |f(x, s)|.
\] (10)
Finally, by (9) and (10), we have
\[
\max_{(x,s)\in[0,1] \times [0,t]} |w(x,s)| \leq \max \left\{ \max_{s \in [0,t]} |p(s)|, \frac{1}{m} \max_{s \in [0,t]} |q(s)| \right\} + \frac{4\sqrt{2}}{a} \max_{(x,s)\in[0,1] \times [0,t]} |f(x,s)|, \tag{11}
\]
which implies that system (5) is ULC in $L^\infty$-norm w.r.t. $f, p, q$.

### 2.2 ISS in $L^\infty$-norm for nonlinear parabolic PDEs with mixed boundary disturbances

**Theorem 2.2** System (5) is EISSL in $L^\infty$-norm w.r.t. boundary disturbances $p, q$ and in-domain disturbance $f$, having the following estimate:
\[
\max_{(x,s)\in[0,1] \times [0,t]} |u(x,s)| \leq \max_{x \in [0,1]} |\phi(x)|e^{-ct} + \max_{s \in [0,t]} |p(s)| + \frac{1}{m} \max_{s \in [0,t]} |q(s)| + \frac{4\sqrt{2}}{a} \max_{(x,s)\in[0,1] \times [0,t]} |f(x,s)|, \forall t > 0. \tag{12}
\]

**Proof** Thanks to Proposition 1.1, it suffices to prove that system (5) is $0$-ES in $L^\infty$-norm w.r.t. $\phi$, provided $f = p = q = 0$. Indeed, let $v$ be the solution of the following equation:
\[
\begin{align*}
v_t - av_{xx} + cv + h(v) &= 0 \text{ in } (0, 1) \times \mathbb{R}_+, \tag{13a} \\
v(0, t) &= 0, \quad t \in \mathbb{R}_+, \tag{13b} \\
v_x(1, t) &= -mv(1, t), \quad t \in \mathbb{R}_+, \tag{13c} \\
v(x, 0) &= \phi(x), \quad x \in (0, 1). \tag{13d}
\end{align*}
\]

Let $w = ve^{ct}$. By direct computations, we have
\[
\begin{align*}
w_t - aw_{xx} + h(we^{-ct})e^{ct} &= 0, \tag{14a} \\
w(0, t) &= 0, \tag{14b} \\
w_x(1, t) &= -mw(1, t), \tag{14c} \\
w(x, 0) &= \phi(x). \tag{14d}
\end{align*}
\]

Let $k_0$ be a nonnegative constant which will be chosen later. For $k \geq k_0$, let $\eta(x,s) = (w(x,s) - k)\chi_{[t_1, t_2]}(s)$ be defined as in the proof of Proposition 2.1. Note that for fixed $t > 0$, $\tilde{h}(w) := h(we^{-ct})e^{ct}$ is increasing in $w$ and $\tilde{h}(0) = 0$. Choosing $k_0 = \max \left\{ \max_{x \in [0,1]} \phi(x), 0 \right\}$, it follows that
\[
\int_0^t \int_0^1 \tilde{h}(w)(w - k)\chi_{[t_1, t_2]}(s)dxds \geq 0, \forall t > 0.
\]

Applying De Giorgi iteration and proceeding as in the proof of the upper boundedness in Proposition 2.1, we obtain
\[
\max_{(x,s)\in[0,1] \times [0,t]} w(x,s) \leq \max_{x \in [0,1]} \phi(x).
\]

Choosing $k_0 = \max \left\{ \max_{x \in [0,1]} (-\phi(x)), 0 \right\}$, it follows that
\[
-\int_0^t \int_0^1 \tilde{h}(-w)(w - k)\chi_{[t_1, t_2]}(s)dxds \geq 0.
\]

Proceeding as in the proof of the lower boundedness in Proposition 2.1, we obtain
\[
\max_{(x,s)\in[0,1] \times [0,t]} (-w(x,s)) \leq \max_{x \in [0,1]} (-\phi(x)).
\]
Then we have
\[
\max_{(x,s) \in [0,1] \times [0,T]} |w(x,s)| \leq \max_{x \in [0,1]} |\phi(x)|,
\]
which implies that
\[
\max_{x \in [0,1]} |v(x,t)| \leq e^{-ct} \max_{x \in [0,1]} |\phi(x)|. \tag{15}
\]

Finally, by Proposition 2.1, (15) and Proposition 1.1, we conclude that system (6) is EISS in \(L^\infty\)-norm w.r.t. boundary disturbances \(p, q\) and in-domain disturbance \(f\), having the estimate (12) due to (11) and (15).

3 UKC and ISS in \(L^\infty\)-norm for nonlinear parabolic PDEs with in-domain and Dirichlet/Robin boundary disturbances

In this section, we present several results on UKC and ISS in \(L^\infty\)-norm for certain nonlinear PDEs with in-domain and Dirichlet/Robin boundary disturbances. Note that these results can be obtained in the same way as in Section 2 based on De Giorgi iteration, thus the details of proofs are omitted.

3.1 ULC and ISS in \(L^\infty\)-norm w.r.t. in-domain and Dirichlet boundary disturbances

Consider the following parabolic equation with Dirichlet boundary disturbances:
\[
\begin{align*}
u_t - au_{xx} + cu + h(u) &= f(x,t) \quad \text{in} \quad (0,1) \times \mathbb{R}_+, \\
u(0,t) &= p(t) \quad t \in \mathbb{R}_+, \\
u(1,t) &= q(t) \quad t \in \mathbb{R}_+, \\
u(x,0) &= \phi(x) , \quad x \in (0,1).
\end{align*}
\tag{16a}
\]

**Definition 3.1** We say \(u \in W^{1,1}_2((0,1) \times \mathbb{R}_+)\) is a (weak) solution of (16), if for a.e. \(T \in \mathbb{R}_+\) and for any \(\varphi \in W^{1,0}_2((0,1) \times (0,T))\) with \(\varphi(0,\cdot) = \varphi(1,\cdot) = 0\) in \((0,T)\), there holds:
\[
\int_0^T \int_0^1 u_t \varphi dxdt + \int_0^T \int_0^1 au_x \varphi dxdt + \int_0^T \int_0^1 (cu + h(u)) \varphi dxdt = \int_0^T \int_0^1 f \varphi dxdt,
\]
and \(u(\cdot,0) = \phi(\cdot)\) a.e. in \((0,1)\), \(u(0,\cdot) = p(\cdot)\) and \(u(1,\cdot) = q(\cdot)\) a.e. in \(\mathbb{R}_+\).

**Theorem 3.1** The following statements hold true:

(i) System (16) is ULC in \(L^\infty\)-norm w.r.t. the boundary disturbances \(p, q\) and the in-domain disturbance \(f\).

(ii) System (16) is EISS in \(L^\infty\)-norm w.r.t. boundary disturbances \(p, q\) and in-domain disturbance \(f\), having the following estimate:
\[
\max_{(x,s) \in [0,1] \times [0,t]} |u(x,s)| \leq \max_{x \in [0,1]} |\phi(x)|e^{-ct} + \max_{s \in [0,t]} |p(s)| + \max_{s \in [0,t]} |q(s)| + \frac{4\sqrt{2}}{a} \max_{(x,s) \in [0,1] \times [0,t]} |f(x,s)|, \quad \forall t > 0.
\]

3.2 ULC and ISS in \(L^\infty\)-norm w.r.t. in-domain and Robin boundary disturbances

Consider the following parabolic equation with Robin boundary disturbances:
\[
\begin{align*}
u_t - au_{xx} + cu + h(u) &= f(x,t) \quad \text{in} \quad (0,1) \times \mathbb{R}_+, \\
u_x(0,t) &= nu(0,t) + p(t), \quad t \in \mathbb{R}_+, \\
u_x(1,t) &= -mu(1,t) + q(t), \quad t \in \mathbb{R}_+.
\end{align*}
\tag{17a}
\]
\[ u(x, 0) = \phi(x), \quad x \in (0, 1). \]  

(17d)

**Definition 3.2** We say \( u \in \mathbb{W}^{1,1}((0,1) \times \mathbb{R}^+) \) is a (weak) solution of (17), if for a.e. \( T \in \mathbb{R}^+ \) and for any \( \varphi \in \mathbb{W}^{1,0}_2((0,1) \times (0,T)) \), there holds:

\[
\begin{align*}
\int_0^T \int_0^1 u_t \varphi dx dt - \int_0^T (-mu(1,t) + q(t))\varphi(1,t) dt &+ \int_0^T (nu(0,t) + p(t))\varphi(0,t) dt + \int_0^T \int_0^1 au_x \varphi_x dx dt \\
+ \int_0^T \int_0^1 (cu + h(u))\varphi dx dt &= \int_0^T \int_0^1 f \varphi dx dt,
\end{align*}
\]

and \( u(\cdot, 0) = \phi(\cdot) \) a.e. in \((0,1)\).

**Theorem 3.2** The following statements hold true:

(i) System (17) is ULC in \( L^\infty \)-norm w.r.t. boundary disturbances \( p, q \) and in-domain disturbance \( f \).

(iii) System (17) is EISS in \( L^\infty \)-norm w.r.t. the boundary disturbances \( p, q \) and the in-domain disturbance \( f \), having the following estimate:

\[
\max_{(x,s) \in [0,1] \times [0,t]} |u(x,s)| \leq \max_{x \in [0,1]} |\phi(x)| e^{-ct} + \frac{1}{n} \max_{s \in [0,t]} |p(s)| + \frac{1}{m} \max_{s \in [0,t]} |q(s)| + \frac{4\sqrt{2}}{a} \max_{(x,s) \in [0,1] \times [0,t]} |f(x,s)|, \quad \forall t > 0.
\]

**References**

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