PERMANENCE, EXTINCTION AND PERIODIC SOLUTION OF
A STOCHASTIC SINGLE-SPECIES MODEL WITH LÉVY NOISES

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Abstract. This paper considers a stochastic single-species model with Lévy
noises and time periodic coefficients. By Lyapunov functions and stochastic
estimates, the threshold conditions between the time-average persistence in
probability and extinction for the model are derived where Lévy noises play an
important role in persistence and extinction of populations. It is shown that
the time-average persistence in probability of the model implies the existence
and uniqueness of positive periodic solution and the existence and uniqueness
of periodic measure of the model. An example and its numerical simulations
are given to verify the effectiveness of the theoretical results.

1. Introduction. Populations are often subject to environmental fluctuations [11,
13, 29, 31]. May [29] pointed out that due to environmental noise, the birth rates,
carrying capacities and other parameters which characterize natural biological sys-
tems all, to a greater or lesser degree, exhibit random fluctuations. It is there-
fore important to reveal how stochastic noises affect the dynamics of populations.
The classical stochastic biological model is that with a continuous driving process.
However, the population may suffer sudden environmental shock, e.g., earthquakes,
floods, hurricanes, epidemics and so on. These events are so strong that they
break the continuity of the sample paths. Thus, modeled with white noise cannot
capture these phenomena. Introducing Lévy noises into the model may be a rea-
sonable way to accommodate such phenomena; see [1, 2, 3, 4, 5, 6, 9, 10, 20, 23,
24, 25, 26, 28, 32, 33, 36, 38]. In addition to stochastic disturbances, populations
in ecology are often faced with the seasonal variations from environment and bio-
logical activity, which could be from the changes of weather, food supply, mating
habits, hunting or harvesting seasons. The biological parameters under seasonal
forces could naturally be assumed to be cyclic or periodic [7]. A number of pa-
pers [17, 18, 21, 27, 33, 34, 35, 40] have recently considered the periodic solutions
or almost periodic solutions in stochastic differential systems. In [18], Jiang et al.
considered the stochastic permanence, global stability and periodic solution in a
stochastic logistic equation with white noises and time periodic coefficients. Wang
and Liu [33] studied the existence and uniqueness of square-mean almost periodic

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solutions to stochastic evolution equations perturbed by Lévy noise. Zu et al. [40] considered a stochastic predator-prey model with white noises and time periodic coefficients, and established sufficient conditions for the existence of a positive periodic solution. Zhang et al. [37] studied a class of periodic stochastic differential equations driven by Lévy processes, and obtained sufficient conditions for the existence and uniqueness of periodic solution.

Recently, the time-average persistence in probability of a population was proposed by [15] and is powerful for the analysis of autonomous stochastic models, has been studied for a variety of stochastic models [15, 30]. However there are no results on time-average persistence in probability of stochastic population systems with Lévy noises and time periodic coefficients.

In this paper, we consider a stochastic single-species model with Lévy noises and time periodic coefficients of the form

\[ d\phi(t) = \Lambda(t, \phi(t))dt + \sigma(t)\phi(t)dB(t) + \int_{\mathbb{Z}} \phi(t^-)h(t, z)\tilde{N}(dt, dz), \]

where \( \phi(t) \) denotes the population size at time \( t \). \( \phi(t^-) \) is the left limit of \( \phi(t) \). \( dB(t) \) is the white noise, namely, \( B(t) \) a real-valued standard Brownian motion defined on a complete probability space \((\Omega, \mathcal{F}, P)\) with a filtration \( \{\mathcal{F}_t\}_{t \geq 0} \) satisfying the usual conditions, \( \sigma(t) \) is the intensity of the white noise. \( N \) is a Poisson counting measure with compensator \( \tilde{N} \) and characteristic measure \( \pi \) on a measurable subset \( \mathbb{Z} \) of \( (0, \infty) \) with \( \pi(\mathbb{Z}) < \infty \). \( \tilde{N}(dt, dz) := N(dt, dz) - \pi(dz)dt \), which is a martingale-valued measure. \( h(t, z) \) is the coefficient of the effect of jump noise on the population. \( \Lambda(t, \phi(t)) \) corresponds to the growth of the population. Throughout this paper, we assume that \( N \) and \( B \) are independent, and \( \Lambda : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R} \), \( \sigma : \mathbb{R}_+ \to \mathbb{R}_+ \) and \( h : \mathbb{R}_+ \times \mathbb{Z} \to \mathbb{R} \) are continuous functions, bounded and \( T \)-periodic \( (T > 0) \) in the \( t \)-variable. From the biological point of view, the function \( \Lambda(t, \phi) \) is assumed to be continuously differentiable in \( \phi \) and satisfies the following hypotheses:

**H1:** For all \( t \in [0, T] \) and \( \phi > 0 \), we have \( \Lambda(t, 0) = 0 \), \( \lim_{\phi \to -\infty} \Lambda(t, \phi) = -\infty \),

\[ \lim_{\phi \to \infty} \frac{\Lambda(t, \phi)}{\phi} = -\infty, \quad \lim_{\phi \to 0^+} \frac{\partial \Lambda(t, \phi)}{\partial \phi} = 0 \]

and there exist constants \( d_1 > 0 \) and \( d_2 > 0 \) such that

\[ -d_2 \leq \partial^2 \Lambda(t, \phi) / \partial \phi^2 \leq -d_1. \]

Further, \( h(t, z) \) satisfies the following hypotheses:

**H2:** For any \( z \in \mathbb{Z} \), we have \( h(t, z) > -1 \), and there is a constant \( K > 0 \) such that

\[ \int_{\mathbb{Z}} h^2(t, z) \vee (\ln(1 + h(t, z)))^2 \pi(dz) \leq K \quad \text{for all} \quad t \in [0, T]. \]

Assumption (H1) is quite general; for instance, it is satisfied by a logistic growth of the population. From [2, 3, 20, 23, 25, 26, 28, 36, 38], assumption (H2) is very reasonable. In [2], Bao et al. considered equation (1) when (H2) holds and \( \Lambda(t, \phi(t)) = \phi(t)[r(t) - a(t)\phi(t)] \), and obtained the sufficient conditions for stochastic permanence and extinction of equation (1). If \( h(t, z) \equiv 0 \) and \( \Lambda(t, \phi(t)) = \phi(t)[r(t) - a(t)\phi(t)] \), the stochastic permanence, global stability and periodic solutions of equation (1) were explored in [17, 18, 21].

The purpose of the present paper is to investigate the time-average persistence in probability, periodicity and extinction of equation (1). By applying the method
in [15, 30], the threshold conditions for the time-average persistence in probability and extinction of equation (1) are established, which improves and extends the corresponding results in [2, 17, 18, 21]. Interestingly, we are able to show that equation (1) exhibits a unique positive periodic solution and a unique periodic measure provided it admits the time-average persistence in probability, which improves the earlier studies for the existence of periodic solution where the mathematical models do not include Lévy noise [21, 34, 35].

The organization of this paper is as follows. In the next section, we state our main results. In section 3, we present the proofs for the main theorems. In Section 4, numerical simulations are carried out to illustrate our main results. The last section presents brief discussions on the results of this paper.

2. Main results. Throughout this paper, we denote by \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) a complete probability space with a filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) satisfying the usual conditions (i.e., it is right continuous and increasing while \(\mathcal{F}_0\) contains all \(P\)-null sets). In view of the biological background, we consider the dynamics of equation (1) in \(\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}\). For convenience we let \(g^u = \max_{t \in [0,T]} g(t)\), \(g^l = \min_{t \in [0,T]} g(t)\) and \(g^T = \int_0^T g(s)ds/T\), where \(g(t)\) is a continuous \(T\)-periodic function. Let \(\phi(t)\) denote a solution of equation (1) corresponding to an initial \(\phi_0 = \phi(0) \in \text{Int}\mathbb{R}_+\), where \(\text{Int}\mathbb{R}_+\) is the interior of \(\mathbb{R}_+\).

**Definition 2.1.** (See,[15]) Equation (1) is said to be time-average persistent in probability if for any \(\varepsilon \in (0,1)\), there exists a positive constant \(L := L(\varepsilon) > 1\) such that

\[
\liminf_{t \to \infty} \frac{1}{t} \int_0^t P\{\phi(s) \in [L^{-1}, L]\} ds \geq 1 - \varepsilon,
\]

where \(\phi(t)\) is the solution of equation (1) with the initial condition \(\phi_0 = \phi(0) \in \text{Int}\mathbb{R}_+\).

**Definition 2.2.** (See,[19]) A stochastic process \(\xi(t) = \xi(t, \omega) (-\infty < t < \infty)\) is said to be periodic with period \(T\) if its finite dimensional distributions are periodic with period \(T\), i.e., for any positive integer \(n\) and any moments of time \(t_1, \cdots, t_n\), the joint distribution of random variables \(\xi(t_1 + kT), \ldots, \xi(t_n + kT)\) are independent of \(k\) \((k = \pm 1, \pm 2, \cdots)\).

**Definition 2.3.** Equation (1) is said to be stochastically ultimately bounded if for any \(\varepsilon \in (0,1)\), there is a positive constant \(H := H(\varepsilon)\) such that any solution \(\phi(t)\) of equation (1) with the initial data \(\phi_0 \in \text{Int}\mathbb{R}_+\) satisfies

\[
\limsup_{t \to \infty} P\{\phi(t) > H\} < \varepsilon.
\]

We first provide the well-posedness of equation (1).

**Theorem 2.4.** Suppose that (H1) and (H2) hold, then for any given initial value \(\phi(0) \in \text{Int}\mathbb{R}_+\), equation (1) has a unique global solution \(\phi(t) \in \text{Int}\mathbb{R}_+\) for all \(t \geq 0\) almost surely. Moreover, there is a positive constant \(M\) such that solution \(\phi(t)\) of equation (1) exhibits the property that

\[
\limsup_{t \to \infty} E[\phi^2(t)] \leq M.
\]

The proof is postponed to Appendix A.

By (2) and Chebyshev’s inequality, we can establish the following theorem.
Theorem 2.5. Let (H1) and (H2) hold. Then the positive solutions of equation (1) are stochastically ultimately bounded.

Let $\mathcal{B}(\text{Int} \mathbb{R}_+)$ denote the $\sigma$-algebra of Borel sets of $\text{Int} \mathbb{R}_+$. The transition function of a Markov process $\phi(t)$ is defined by $p(s, \phi(s), t, A) = P\{\phi(t) \in A|\phi(s)\}, t \geq s \geq 0, A \in \mathcal{B}(\text{Int} \mathbb{R}_+)$, a.s. It is called periodic if $p(s, \phi(s), t + s, A)$ is periodic in $s$. We now state the main results of this paper for equation (1).

Theorem 2.6. Assume (H1) and (H2) hold. Let $\phi(t)$ be a solution of equation (1) with initial value $\phi(0) \in \text{Int} \mathbb{R}_+$. Set
\begin{equation}
\lambda(t) = \frac{\partial \Lambda(t, 0)}{\partial x} - \frac{\sigma_1^2(t)}{2} - \int_{\mathbb{Z}} [h(t, z) - \ln(1 + h(t, z))] \pi(dz). \tag{3}
\end{equation}
(1): If $\lambda^T > 0$, then equation (1) is time-average persistent in probability and has a unique positive $T$-periodic solution $\phi^*(t)$, which satisfies
\begin{equation}
\lim_{t \to \infty} \mathbb{E}[|\phi(t) - \phi^*(t)|] = 0. \tag{4}
\end{equation}
In particular, the transition function $p(s, \phi(s), s + t, \cdot)$ of equation (1) converges weakly to a periodic measure $\rho_s(\cdot)$ as $t \to \infty$, where $\rho_s(\cdot)$ is a continuous $T$-periodic function defined on $[0, \infty)$ with the probability measure value. Let $f(t, z) : [0, \infty) \times \text{Int} \mathbb{R}_+ \to \mathbb{R}$ be a $\rho_1$-integrable $T$-periodic function, and there is a positive constant $K$ such that
\[ |f(t, \phi(t)) - f(t, \phi^*(t))| \leq K|\phi(t) - \phi^*(t)|. \]
Then we have
\begin{equation}
P\left\{ \lim_{t \to \infty} \frac{1}{t} \int_0^t f(s, \phi(s)) ds = \frac{1}{T} \int_0^T \int_{\text{Int} \mathbb{R}_+} f(s, \xi) \rho_s(d\xi) ds \right\} = 1. \tag{5}
\end{equation}
(2): If $\lambda^T < 0$, then equation (1) becomes extinct a.s.

Remark 1. Notice that
\[ h(t, z) - \ln(1 + h(t, z)) \geq 0 \]
for $h(t, z) > -1$ and $z \in \mathbb{Z}$. Then by (3) and Theorem 2.6, we conclude that the Lévy noises are unfavorable for the permanence of equation (1).

It is clear that the equation (1) describes the dynamics of population in a random environment with small jump noises [1]. To elucidate effects of large jump noises on the stochastic population models, we consider the following equation:
\begin{equation}
d\psi(t) = \Lambda(t, \psi(t)) dt + \sigma_1(t) \psi(t) dB(t) + \int_{\mathbb{Z}} \psi(t^-) h(t, z) N(dt, dz). \tag{6}
\end{equation}
Since the Poisson random measure $N$ is a nonnegative increasing counting measure, it follows from (6) that $h(t, z) > 0$ ($t \geq 0, z \in \mathbb{Z}$) implies that the jump noises are advantageous for the population (e.g., immigrant, planting and breeding), whereas $-1 < h(t, z) < 0$ means that the jump noises are disadvantageous for the population (e.g., harvesting and emigration).

Recall that $N(dt, dz) = \bar{N}(dt, dz) + \pi(dz) dt$. Then equation (6) takes the form
\begin{align}
d\psi(t) &= \left[ \Lambda(t, \psi(t)) + \int_{\mathbb{Z}} \psi(t) h(t, z) \pi(dz) \right] dt + \sigma_1(t) \psi(t) dB(t) \\
&\quad + \int_{\mathbb{Z}} \psi(t^-) h(t, z) \bar{N}(dt, dz).
\end{align}
Let $\Lambda_0(t, \psi(t)) = \Lambda(t, \psi(t)) + \int_{\mathbb{Z}} \psi(t)h(t, z)\pi(dz)$. Obviously, $\Lambda(t, \psi(t))$ satisfies assumption (H1) implies that $\Lambda_0(t, \psi(t))$ satisfies assumption (H1). Therefore, by Theorem 2.6, we can obtain the following theorem.

**Theorem 2.7.** Assume (H1) and (H2) hold. Let $\psi(t)$ be a solution of equation (6) with initial value $\psi(0) \in \text{Int}\mathbb{R}_+$. Set

$$
\lambda_0(t) = \frac{\partial \Lambda(t, 0)}{\partial x} - \frac{\sigma_1^2(t)}{2} + \int_{\mathbb{Z}} \ln(1 + h(t, z))\pi(dz). \tag{7}
$$

(1): If $\lambda_0^T > 0$, then equation (6) is time-average persistent in probability and has a unique positive $T$-periodic solution $\psi^*(t)$, which satisfies

$$
\lim_{t \to \infty} E|\psi(t) - \psi^*(t)| = 0.
$$

In particular, the transition function $p(s, \psi(s), s + t, \cdot)$ of equation (6) converges weakly to a periodic measure $\tilde{\rho}_s(\cdot)$ as $t \to \infty$, where $\tilde{\rho}_s(\cdot)$ is a continuous $T$-periodic function defined on $[0, \infty)$ with the probability measure value. Let $f(t, z) : [0, \infty) \times \text{Int}\mathbb{R}_+ \to \mathbb{R}$ be a $\tilde{\rho}_t$-integrable $T$-periodic function, and there is a positive constant $K$ such that

$$
|f(t, \psi(t)) - f(t, \psi^*(t))| \leq K|\psi(t) - \psi^*(t)|.
$$

Then we have

$$
P\left\{ \lim_{t \to \infty} \frac{1}{t} \int_0^t f(s, \psi(s))ds = \frac{1}{T} \int_0^T \int_{\text{Int}\mathbb{R}_+} f(s, \xi)\tilde{\rho}_s(d\xi)ds \right\} = 1.
$$

(2): If $\lambda_0^T < 0$, then equation (6) becomes extinct a.s.

**Remark 2.** From Theorem 2.7, we can obtain the following conclusions: (a) If $h(t, z) > 0$ for $t \geq 0$ and $z \in \mathbb{Z}$, the jump noises are favorable for the permanence of equation (6); (b) If $-1 < h(t, z) < 0$ for $t \geq 0$ and $z \in \mathbb{Z}$, the jump noises are unfavorable for the permanence of equation (6).

3. **Proofs of main results.** We start from some auxiliary definitions and results concerning periodic Markov process [19]. Set $U_r = \{ x \in \text{Int}\mathbb{R}_+ : r^{-1} \leq x \leq r, r \geq 1 \}$ and $U_r^c = \{ x \in \text{Int}\mathbb{R}_+ : x \notin U_r \}$. The following lemma is fundamental in establishing the existence of $T$-periodic solution of equation (1).

**Lemma 3.1.** [19] Let $\beta(r)$ be a positive function, which tends to infinity as $r \to \infty$. Assume that the transition function $p(s, \phi(s), t, \cdot)$ is $T$-periodic and there are some $\zeta_0 \in \text{Int}\mathbb{R}_+$ and $s_0 \in [0, T)$ such that

$$
\sup_{\zeta_0 \in U_{\beta(r)}, 0 < t - s_0 < T} p(s_0, \zeta_0, t, U_r^c) \to 0 \quad \text{as} \quad r \to \infty. \tag{8}
$$

Then, equation (1) admits a $T$-periodic solution if and only if

$$
\lim_{r \to \infty} \lim_{\theta \to \infty} \frac{1}{\theta} \int_{s_0}^{\theta + s_0} p(s_0, \zeta_0, u, U_r^c)du = 0. \tag{9}
$$

To prove this theorem, we need the following lemma.
Lemma 3.2. Assume that (H1) and (H2) hold. Let \( \phi(t) \) be the solution of equation (1) with initial value \( \phi(0) \in \text{Int} \mathbb{R}_+ \). If \( \lambda^2 > 0 \), then for any \( \varepsilon \in (0, 1) \) there exist a positive constant \( \kappa_0 := \kappa_0(\varepsilon) \) and a positive integer \( n \) such that

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} P\{\phi(kT) \leq \kappa_0\} \leq \varepsilon. \tag{10}
\]

Proof. Let \( \phi(t) \) be the solution of equation (1) with initial value \( \phi(0) \in \text{Int} \mathbb{R}_+ \). We divide the proof into three steps.

Step 1: Let \( \eta > 0 \). If \( \delta \in (0, \eta] \), we define the stopping time

\[
\tau^n(x) = \inf\{t \geq 0 : \phi(t) \geq \eta\} \quad \text{for any} \quad \phi(0) \in (0, \delta]
\]

For any \( \varepsilon \in (0, 1) \) and positive integer \( n > 0 \), by the exponential martingale inequality with jumps ([2], Lemma 4.3), we have

\[
P\left\{ \sup_{0 \leq t \leq \eta T} \left[ \int_0^t \sigma(s)dB(s) - \frac{1}{2} \int_0^t \sigma^2(s)ds + \int_0^t \int_{\mathbb{R}} \ln(1 + h(s, z))\tilde{N}(ds, dz) \right. \right.
\]

\[
\left. - \left[ \int_0^t \int_{\mathbb{R}} [h(s, z) - \ln(1 + h(s, z))]\pi(dz)ds \right] > \ln \frac{1}{\varepsilon} \right\} \leq \varepsilon. \tag{11}
\]

Therefore, there is an \( \Omega_1 \subset \Omega \) with \( P(\Omega_1) \geq 1 - \varepsilon \) such that for any \( \omega \in \Omega_1 \) and \( t \in [0, \eta T] \)

\[
\int_0^t \sigma(s)dB(s) + \int_0^t \int_{\mathbb{R}} \ln(1 + h(s, z))\tilde{N}(ds, dz)
\]

\[
\leq \frac{1}{2} \int_0^t \sigma^2(s)ds + \int_0^t \int_{\mathbb{R}} [h(s, z) - \ln(1 + h(s, z))]\pi(dz)ds + \ln \frac{1}{\varepsilon}. \tag{12}
\]

From (H1), we have

\[
\Lambda(t, \phi(t)) \leq \frac{\partial \Lambda(t, 0)}{\partial x} \phi(t) - d_1 \phi^2(t). \tag{13}
\]

Applying Itô’s formula with jumps, we have

\[
d \ln \phi(t) = \left[ \frac{\Lambda(t, \phi(t))}{\phi(t)} - 0.5\sigma^2(t) - \int_{\mathbb{R}} (h(t, z) - \ln(1 + h(t, z)))\pi(dz) \right] dt
\]

\[
+ \sigma(s)dB(t) + \int_{\mathbb{R}} \ln(1 + h(t, z))\tilde{N}(dt, dz). \tag{14}
\]

It follows from (13) that

\[
d \ln \phi(t) \leq \left[ \frac{\partial \Lambda(t, 0)}{\partial x} - d_1 \phi(t) - 0.5\sigma^2(t) - \int_{\mathbb{R}} (h(t, z) - \ln(1 + h(t, z)))\pi(dz) \right] dt
\]

\[
+ \sigma(s)dB(t) + \int_{\mathbb{R}} \ln(1 + h(t, z))\tilde{N}(dt, dz)
\]

\[
\leq \left[ \frac{\partial \Lambda(t, 0)}{\partial x} - 0.5\sigma^2(t) - \int_{\mathbb{R}} (h(t, z) - \ln(1 + h(t, z)))\pi(dz) \right] dt
\]

\[
+ \sigma(s)dB(t) + \int_{\mathbb{R}} \ln(1 + h(t, z))\tilde{N}(dt, dz). \]
Therefore, when \( \omega \in \Omega_1 \) we have
\[
\ln \phi(t) < \ln \phi(0) + \int_0^t \left[ \frac{\partial \Lambda(t, 0)}{\partial x} - 0.5\sigma^2(s) - \int_Z (h(s, z) - \ln(1 + h(s, z)))\pi(dz) \right] ds \\
+ \int_0^t \sigma(s)dB(s) + \int_0^t \int_Z \ln(1 + h(s, z))\tilde{N}(ds, dz) \\
\leq \ln \phi(0) + \Lambda^u_0 t + \ln \frac{1}{\varepsilon}
\]
for all \( t \in [0, nT] \), where \( \Lambda^u_0 = \max_{t \in [0, T]} \partial \Lambda(t, 0)/\partial x \). Letting \( \eta > 0 \) and
\( \delta = \eta \varepsilon \exp(-\Lambda^u_0 nT) \),
we can see that if \( \phi(0) \in (0, \delta) \), then
\[
\phi(t) < \eta \text{ for all } 0 \leq t < nT \text{ and } \omega \in \Omega_1,
\]
which implies that
\[
\mathbb{P}\{\tau^{n \eta}_{\phi(0)} \geq nT\} \geq \mathbb{P}(\Omega_1) \geq 1 - \varepsilon \text{ for any } \phi(0) \in (0, \delta].
\]

**Step 2:** Clearly, (H1) implies
\[
\Lambda(t, \phi(t)) \geq \frac{\partial \Lambda(t, 0)}{\partial x} \phi(t) - d_2 \phi^2(t).
\]
This, together with (14), implies that
\[
d\ln \phi(t) \geq \left[ \frac{\partial \Lambda(t, 0)}{\partial x} - d_2 \phi(t) - 0.5\sigma^2(t) \\
- \int_Z (h(t, z) - \ln(1 + h(t, z)))\pi(dz) \right] dt \\
+ \sigma(s)dB(t) + \int_Z \ln(1 + h(t, z))\tilde{N}(dt, dz) \\
= [\lambda(t) - d_2 \phi(t)]dt + \sigma(s)dB(t) + \int_Z \ln(1 + h(t, z))\tilde{N}(dt, dz).
\]
Note that \( \lambda^T > 0 \). We can choose \( \eta > 0 \) sufficiently small such that
\[
\lambda^T/2 - d_2 \eta > 0. \tag{16}
\]
For any positive integer \( n > 0 \), we define \( \Omega_2^{n(0)} = \{\tau^{n \eta}_{\phi(0)} \geq nT\} \) and
\[
\Omega_3 \equiv \left\{ \left| \int_0^{nT} \sigma(s)dB(s) \right| \leq \frac{\lambda^T nT}{8} \right\} \\
\cap \left\{ \left| \int_0^{nT} \int_Z \ln(1 + h(s, z))\tilde{N}(ds, dz) \right| \leq \frac{\lambda^T nT}{8} \right\}.
\]
It follows from Step one and the strong law of large numbers for martingales (see, e.g., [22]) that for any \( \varepsilon \in (0, 1) \) there exist positive integer \( \overline{n} \) and \( \delta := \delta(\varepsilon, \eta, \overline{n}) \) such that \( \mathbb{P}(\Omega_2) \geq 1 - \varepsilon/2 \) and \( \mathbb{P}(\Omega_3) \geq 1 - \varepsilon/2 \) for any \( \phi(0) \in (0, \delta]. \) Then by (15)
and (16), for any \( \phi(0) \in (0, \delta] \) and \( \omega \in \Omega_2^{\phi(0)} \cap \Omega_3^{\pi} \), we have
\[
\ln \phi(\pi T) - \ln \phi(0) \geq \int_0^{\pi T} \left[ \lambda(s) - d_2 \phi(s) \right] ds + \int_0^{\pi T} \sigma(s) dB(s)
\]
\[
+ \int_0^{\pi T} \int_{\mathbb{Z}} \ln(1 + h(s, z)) \tilde{N}(ds, dz)
\]
\[
\geq \int_0^{\pi T} \left[ \lambda(s) - d_2 \eta \right] ds - \left| \int_0^{\pi T} \sigma(s) dB(s) \right|
\]
\[
- \left| \int_0^{\pi T} \int_{\mathbb{Z}} \ln(1 + h(s, z)) \tilde{N}(ds, dz) \right| \geq \frac{\lambda T \pi T}{4},
\]
which means that
\[
P\{ \ln \phi(\pi T) - \ln \phi(0) \geq \lambda T \pi T/4 \} \geq P(\Omega_2^{\phi(0)} \cap \Omega_3^{\pi}) \geq 1 - \varepsilon \text{ for any } \phi(0) \in (0, \delta].
\]

**Step 3:** By (2), there exist a positive constant \( C_0 \) and a positive integer \( n^* > 1/T \) such that any solution \( \phi(t) \) of (1) with the initial data \( \phi(0) \in \text{Int} \mathbb{R}_+ \) satisfies
\[
E \left[ \int_0^{n T} \phi^2(s) ds \right] \leq C_0(\phi^2(0) + n T) \text{ for all } n \geq n^*.
\]
(17)

Therefore, for any \( \phi(0) \in \text{Int} \mathbb{R}_+ \), positive integer \( n \geq n^* \) and \( A \in \mathcal{F} \), it follows from (13), (14), (17) and Hölder’s inequality that
\[
\frac{1}{n T} E[|\ln \phi(n T) - \ln \phi(0)| I_A]
\]
\[
\leq \frac{P(A)}{n T} \int_0^{n T} \lambda(s) ds + \frac{d_1}{n T} E \left[ \int_0^{n T} I_A \phi(s) ds \right] + E \left[ I_A \left| \frac{1}{n T} \int_0^{n T} \sigma(s) dB(s) \right| \right]
\]
\[
+ E \left[ I_A \left| \int_0^{n T} \int_{\mathbb{Z}} \ln(1 + h(s, z)) \tilde{N}(ds, dz) \right| \right]
\]
\[
\leq \lambda T P(A) + d_1 \sqrt{\frac{P(A)}{n T}} \left( E \left[ \int_0^{n T} \phi^2(s) ds \right] \right)^{1/2} + \frac{\sqrt{\lambda T}}{n T} \left( \int_0^{n T} \sigma^2(s) ds \right)^{1/2}
\]
\[
+ \frac{\sqrt{\lambda T}}{n T} \left( \int_0^{n T} \int_{\mathbb{Z}} [\ln(1 + h(s, z))]^2 \pi(dz) ds \right)^{1/2}
\]
\[
\leq [\lambda T + \sigma^u + d_1 \sqrt{C_0(1 + \phi^2(0)) + \sqrt{K}}] \frac{P(A)}{\sqrt{n T}} \leq \bar{K} \phi(0) \sqrt{P(A)}
\]
(18)

where
\[
\bar{K} \phi(0) = \sqrt{\bar{b}_1 + \bar{b}_2 \phi^2(0)}
\]
and
\[
\bar{b}_1 = 2((\lambda T + \sigma^u + \sqrt{K})^2 + d_1^2 C_0), \quad \bar{b}_2 = 2d_1^2 C_0
\]
and \( K \) is defined in (H2). Let \( \eta_0 \in (0, 1/2) \) and \( \overline{N} > 0 \) be chosen later. When \( A = \Omega \) and \( \phi(0) \in (0, \overline{N}) \), by Chebyshev’s inequality we obtain
\[
P \left\{ \frac{\ln \phi(n T) - \ln \phi(0)}{n T} \geq \theta(\overline{N}) := \frac{2 \bar{K}(\overline{N})}{\eta_0} \right\} \leq \eta_0.
\]
(19)

Set
\[
\Omega_0^{n} = \{ n T \lambda T/4 \leq \ln \phi(n T) - \ln \phi(0) \leq \theta(\overline{N}) n T \}.
\]
(20)
It follows from Step two and (19) that there exist a \( \delta_0 > 0 \) and a positive integer \( \bar{n} \geq n^* \) such that for any \( \phi(0) \in (0, \delta_0) \), we have \( P\{\Omega_0^\pi \} > 1 - 2\eta_0 \). Let \( \bar{N} > \delta_0 > 0 \).

Define \( \bar{m} = \ln(N/d_0) + \theta(\bar{N})\pi T \). Since

\[
|\ln \bar{N} - \ln \phi(\pi T)| \leq |\ln \bar{N} - \ln \phi(0)| + |\ln \phi(\pi T) - \ln \phi(0)|,
\]

it follows from (19) that

\[
P\{|\ln \bar{N} - \ln \phi(\pi T)| \geq \bar{m}\} \leq P\{|\ln \phi(\pi T) - \ln \phi(0)| \geq \theta(\bar{N})\pi T\} \leq \eta_0 \tag{21}
\]

for all \( \phi(0) \in (\delta_0, \bar{N}) \). Let \( \bar{\delta}_0 \) and \( \kappa_0 \) satisfy

\[
\ln \bar{N} - \ln \bar{\delta}_0 = \bar{m},
\]

\[
\ln \bar{N} - \ln \kappa_0 = \bar{m} + \theta(\bar{N})\pi T.
\]

Clearly, \( \kappa_0 < \bar{\delta}_0 < \delta_0 \). Define

\[
G(x) = (\ln \bar{N} - \ln x) \lor \bar{m}, \quad \text{for } x > 0.
\]

Let \( \Omega_0^\pi = \Omega/\Omega_0^\pi \). Clearly,

\[
G(x) - G(y) \leq |\ln x - \ln y|, \quad \text{for } x, y > 0. \tag{22}
\]

Therefore, for any \( \phi(0) \in \text{int} \mathbb{R}_+ \), using (18) we have

\[
\frac{1}{\pi T} E[(G(\phi(\pi T)) - G(\phi(0)))I_{\Omega_0^\pi}] \leq \frac{1}{\pi T} E[|\ln \phi(\pi T) - \ln \phi(0)|I_{\Omega_0^\pi}] \leq \bar{K}(\phi(0))\sqrt{P(\Omega_0^\pi)}. \tag{23}
\]

If \( \phi(0) \in (0, \kappa_0) \), we have \( G(\phi(0)) = (\ln \bar{N} - \ln \phi(0)) \geq \bar{m} \). Hence, if \( \omega \in \Omega_0^\pi \) and \( \phi(0) \in (0, \kappa_0) \), we have

\[
\ln \bar{N} - \ln \phi(\pi T) \geq \ln \bar{N} - (\theta(\bar{N})\pi T + \ln \phi(0)) \geq \ln \bar{N} - \ln \kappa_0 - \theta(\bar{N})\pi T = \bar{m}.
\]

It follows that

\[
\frac{1}{\pi T} (G(\phi(\pi T)) - G(\phi(0))) = \frac{1}{\pi T} (\ln \phi(0) - \ln \phi(\pi T)) \leq -\frac{\lambda_T}{4}. \tag{24}
\]

Combining (23) and (24) yields

\[
\frac{1}{\pi T} (E[G(\phi(\pi T)) - G(\phi(0))]) \leq -(1 - 2\eta_0)\frac{\lambda_T}{4} + \sqrt{2\eta_0} \bar{K} \bar{N}. \tag{25}
\]

If \( \phi(0) \in (\kappa_0, \bar{\delta}_0) \), then \( G(\phi(0)) = (\ln \bar{N} - \ln \phi(0)) \geq \bar{m} \). For \( \omega \in \Omega_0^\pi \), we have

\[
\ln \bar{N} - \ln \phi(\pi T) \leq \ln \bar{N} - \ln \phi(0) - \pi T\lambda_T/4 < G(\phi(0)).
\]

It follows that

\[
G(\phi(\pi T)) \leq G(\phi(0)).
\]

This, together with (23), implies that

\[
\frac{1}{\pi T} (E[G(\phi(\pi T)) - G(\phi(0))]) \leq \sqrt{2\eta_0} \bar{K} \bar{N}. \tag{26}
\]

If \( \phi(0) \in (\bar{\delta}_0, \delta_0) \), we have \( G(\phi(0)) = \bar{m} \). Obviously, \( G(\phi(\pi T)) = \bar{m} \) in \( \Omega_0^\pi \). This and (23) imply

\[
\frac{1}{\pi T} (E[G(\phi(\pi T)) - G(\phi(0))]) \leq \sqrt{2\eta_0} \bar{K} \bar{N}. \tag{27}
\]
If \( \phi(0) \in (\delta_0, \Omega] \), we have \( G(\phi(0)) = \bar{m} \). Let \( \Pi_1 = \{ |\ln \Omega - \ln \phi(\pi T)| \geq \bar{m} \} \). Then, for \( \omega \in \Omega \setminus \Pi_1 \), we have \( G(\phi(\pi T)) = \bar{m} \). Furthermore, (22) implies
\[
G(\phi(\pi T)) - G(\phi(0)) \leq |\ln \phi(\pi T) - \ln \phi(0)|.
\]
Consequently,
\[
\frac{1}{\pi T} \left( E[|G(\phi(\pi T)) - G(\phi(0))|] \right) = \frac{1}{\pi T} \left( E[|G(\phi(\pi T)) - G(\phi(0))|] \right)_{\Pi_1}
\leq \frac{1}{\pi T} E[|\ln \phi(\pi T) - \ln \phi(0)|]_{\Pi_1}.
\]
From (21), we have \( P(\Pi_1) \leq 2\eta_0 \). Therefore, by (18) and (28), we have
\[
\frac{1}{\pi T} \left( E[|G(\phi(\pi T)) - G(\phi(0))|] \right) \leq \sqrt{2\eta_0 K(\Omega)}.
\]
Now, for any \( \phi(0) \in \text{int}\mathbb{R}_+ \) and non-negative integer \( k \), by the Markov property of \( \phi(t) \) we have
\[
\frac{1}{\pi T} E \left[ G(\phi((k+1)\pi T)) - G(\phi(k\pi T)) \right] = \int_{\text{int}\mathbb{R}_+} P\{\phi(k\pi T) \in \xi \} \left( \frac{1}{\pi T} E[|G(\phi(\pi T)) - G(\xi)|] \right).
\]
This implies that
\[
\frac{1}{\pi T} E \left[ G(\phi((k+1)\pi T)) - G(\phi(k\pi T)) \right] = \frac{1}{\pi T} E[I\{\phi(\pi T) \in (0, \kappa_0)\}] E[\phi(\pi T)) - G(\xi)]
\]
\[
+ \frac{1}{\pi T} E[I\{\phi(\pi T) \in (\kappa_0, \Omega)\}] E[\phi(\pi T)) - G(\xi)]
\]
\[
+ \frac{1}{\pi T} E[I\{\phi(\pi T) > \Omega\}] E[\phi(\pi T)) - G(\xi)].
\]
From (25), (26), (27) and (29), we have
\[
\frac{1}{\pi T} E[I\{\phi(\pi T) \in (0, \kappa_0)\}] E[\phi(\pi T)) - G(\xi)]
\leq \left( - (1 - 2\eta_0) \lambda T / 4 + \sqrt{2\eta_0 K(\Omega)} \right) P\{\phi(k\pi T) \in (0, \kappa_0)\}
\]
and
\[
\frac{1}{\pi T} E[I\{\phi(\pi T) \in (\kappa_0, \Omega)\}] E[\phi(\pi T)) - G(\xi)]
\leq \sqrt{2\eta_0 K(\Omega)} P\{\phi(k\pi T) \in (\kappa_0, \Omega)\}.
\]
For \( \phi(0) = \xi := \phi(k\pi T) \), by (22) we have
\[
\frac{1}{\pi T} E[\phi(\pi T)) - G(\xi)] \leq \frac{1}{\pi T} E[|\ln \phi(\pi T) - \ln \xi|].
\]
Using similar arguments to those in (17) and (18), we get
\[
\frac{1}{\pi T} E[|\ln \phi(\pi T) - \ln \xi|] \leq \sqrt{\tilde{\sigma}_1 + \tilde{\sigma}_2 E[\phi^2(k\pi T)]}.
\]
It follows from (33), (34) and Hölder’s inequality that
\[
\frac{1}{nT} E[I_{\{\xi = \phi(knT) > N\}} E[G(\phi(knT)) - G(\xi)]]
\leq (E[(\frac{1}{nT} E[G(\phi(knT)) - G(\xi)]^2)]^{1/2} \sqrt{P{\{\phi(knT) > N\}}}
\leq \bar{\theta} \sqrt{(1 + E[\phi^2(knT)])} \sqrt{P{\{\phi(knT) > N\}}},
\]
where \(\bar{\theta} = \sqrt{\theta_1 \lor \theta_2}\). Then, by (31), (32) and (35) we have
\[
\frac{1}{nT} E[G(\phi((k + 1)knT)) - G(\phi(knT))]
\leq -\frac{(1 - 2\eta_0)}{4} P{\{\phi(knT) \in (0, \kappa_0]\}
+ \sqrt{2\theta_0 \bar{K}(N)} + \bar{\theta} \sqrt{(1 + E[\phi^2(knT)])} \sqrt{P{\{\phi(knT) > N\}}}.
\]

By Chebyshev’s inequality, we have
\[
P{\{\phi(knT) > N\} \leq \frac{E[\phi^2(knT)]}{N^2} \leq \frac{1 + E[\phi^2(knT)]}{N^2}.
\]

This, together with (36), implies that
\[
\frac{1}{nT} E[G(\phi((k + 1)knT)) - G(\phi(knT))]
\leq -\frac{(1 - 2\eta_0)}{4} P{\{\phi(knT) \in (0, \kappa_0]\}
+ \sqrt{2\theta_0 \bar{K}(N)} + \bar{\theta} \sqrt{(1 + E[\phi^2(knT)])} \sqrt{P{\{\phi(knT) > N\}}}.
\]

It follows from (37) that for any positive integers \(n\),
\[
0 \leq \frac{E[G(\phi(nknT))]}{knT} \leq G(\phi(0)) - \frac{(1 - 2\eta_0)}{4} \frac{1}{n} \sum_{k=0}^{n-1} P{\{\phi(knT) \in (0, \kappa_0]\}
+ \sqrt{2\theta_0 \bar{K}(N)} + \bar{\theta}(1 + \frac{\sum_{k=0}^{n-1} E[\phi^2(knT)]}{N})
\]

By (2), we have
\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} E[\phi^2(knT)] \leq M.
\]

Since \(\lambda^T > 0\), it follows from (38) and (39) that
\[
\liminf_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} P{\{\phi(knT) \in (0, \kappa_0]\} \leq \frac{4}{(1 - 2\eta_0)\lambda^T} \left(\sqrt{2\theta_0 \bar{K}(N)} + \bar{\theta}(1 + M) \frac{\sqrt{\lambda^T}}{N}\right).
\]

For any \(\varepsilon \in (0, 1)\), by choosing \(N := N(\varepsilon)\) sufficiently large and \(\eta_0 := \eta_0(\varepsilon)\) sufficiently small, we can obtain that
\[
\liminf_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} P{\{\phi(knT) \in (0, \kappa_0]\} \leq \varepsilon,
\]
which implies (10). \(\square\)
Now, we are in a positive of proving Theorem 2.6.

Proof of Theorem 2.6-(1). Let \( \phi(t) \) be the solution of equation (1) with initial value \( \phi(0) \in \text{Int} \mathbb{R}_+ \). For \( \varepsilon \in (0, 1) \), it follows from Lemma 3.2 and Theorem 2.5 that there exist a positive constant \( L_0 > 1 \) and a positive integer \( \pi \) such that

\[
\liminf_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} P\{\phi(k\pi T) \in [L_0^{-1}, L_0]\} \geq 1 - \varepsilon/2 \tag{40}
\]

for any \( \phi(0) \in \text{Int} \mathbb{R}_+ \).

Define \( V_1(\phi(t)) = \phi(t) - \ln \phi(t) \). By suitable modification to the proof of Theorem 2.4, we can show that there exists a positive constant \( C_1 \) such that

\[
E[V_1(\phi(t))] \leq V_1(\phi(0)) + C_1 t \tag{41}
\]

for all \( t \geq 0 \). It follows that for any \( L > 1 \), \( \phi(0) \in [L_0^{-1}, L_0] \) and \( t \in [0, \pi T] \),

\[
P(\phi(t) \notin [L^{-1}, L]) \leq \frac{\sup_{\phi(0) \in [L_0^{-1}, L_0]} V_1(\phi(0)) + C_1 \pi T}{\inf_{\phi(t) \notin [L^{-1}, L]} V_1(\phi(t))}
\]

\[
\leq \frac{(L_0 - \ln L_0) + C_1 \pi T}{1/L + \ln L}.
\]

This means that there is an \( L := L(\varepsilon, L_0, \pi T) > 1, L \to \infty \) as \( \varepsilon \to 0 \) such that

\[
P(L^{-1} \leq \phi(t) \leq L) \geq 1 - \varepsilon/2 \tag{42}
\]

for all \( \phi(0) \in [L_0^{-1}, L_0] \) and \( t \in [0, \pi T] \). It follows from the Markov property and (42) that

\[
P(L^{-1} \leq \phi(k\pi T + t) \leq L) \geq (1 - \varepsilon/2)P(\phi(k\pi T) \in [L_0^{-1}, L_0])
\]

for all \( t \in [0, \pi T] \). This, together with (40), implies that for any \( \phi_0 \in \text{Int} \mathbb{R}_+ \),

\[
\liminf_{n \to \infty} \frac{1}{nT} \int_0^{nT} P(L^{-1} \leq \phi(s) \leq L)ds \geq (1 - \varepsilon/2)(1 - \varepsilon/2) \geq 1 - \varepsilon.
\]

As a result,

\[
\liminf_{t \to \infty} \frac{1}{t} \int_0^t P(L^{-1} \leq \phi(s) \leq L)ds \geq 1 - \varepsilon, \tag{43}
\]

which implies that equation (1) is time-average persistent in probability.

Since \( L \to \infty \) as \( \varepsilon \to 0 \), there exists a strictly decreasing function \( r(\varepsilon) > 1 \), \( r(\varepsilon) \to \infty \) as \( \varepsilon \to 0 \), such that

\[
\liminf_{t \to \infty} \frac{1}{t} \int_0^t P(r^{-1}(\varepsilon) \leq \phi(s) \leq r(\varepsilon))ds \geq 1 - \varepsilon.
\]

As a result,

\[
\lim_{r(\varepsilon) \to \infty} \liminf_{t \to \infty} \frac{1}{t} \int_0^t P(\phi(s) \notin [r^{-1}(\varepsilon), r(\varepsilon)])ds
\]

\[
= \lim_{\varepsilon \to 0} \liminf_{t \to \infty} \frac{1}{t} \int_0^t P(\phi(s) \notin [r^{-1}(\varepsilon), r(\varepsilon)])ds = 0.
\]

Hence, the condition (9) of Lemma 3.1 holds.
Let \( \beta(r) = \ln(\ln r) \) and consider large \( r \) satisfying \( \beta(r) > 1 \). Then by (41), for \( t \in (0, T) \) and \( \phi(0) \in \beta^{-1}(r), \beta(r)] \) we have

\[
P(\phi(t) \notin [r^{-1}, r] \mid \phi(0)) \leq \frac{\sup_{\phi(0) \in \beta^{-1}(r), \beta(r]} V_1(\phi(0)) + C_1 t}{\inf_{\phi(t) \notin [r^{-1}, r]} V_1(\phi(t))} \leq \frac{(\beta(r) - \ln \beta(r)) + C_1 T}{1/r + \ln r}.
\]

It follows that condition (8) of Lemma 3.1 holds. Furthermore, the transition function of equation (1) is \( T \)-periodic due to the periodic coefficients of equation (1) (see Lemma 5 in [37]). Consequently, we conclude from Lemma 3.1 that equation (1) has a \( T \)-periodic Markov solution \( \phi^*(t) \).

The proof of (4) is omitted because it is similar to it for Lemma 10 in [25].

We now turn to prove (5). By Definition 2.2, the periodicity of \( \phi^*(t) \) and the continuity of \( E(\phi^*(s), \cdot) \), equation (1) has a \( T \)-periodic measure \( \rho_s(\cdot) \), where \( \rho_s(\cdot) \) is defined by \( \rho_s(A) = P(\phi^*(s) \in A) \) for all \( A \in B(\Int\RR_+) \). Clearly, (4) implies \( T \)-periodic solution \( \phi^*(t) \) of equation (1) is globally asymptotically stable. Then by similar procedure as in the proof of Theorem 2.8 in [16], the transition function \( p(s, \phi(s), s + t, \cdot) \) of equation (1) converges weakly to the periodic measure \( \rho_s(\cdot) \) as \( t \to \infty \), which means that equation (1) has a unique \( T \)-periodic measure \( \rho_s(\cdot) \). Let \( f(t, z) : [0, \infty) \times \Int\RR_+ \to \RR \) be a \( \rho_\tau \)-integrable \( T \)-periodic function, and there is a positive constant \( K \) such that

\[
|f(t, \phi(t)) - f(t, \phi^*(t))| \leq K|\phi(t) - \phi^*(t)|.
\]

Then by (4), we have

\[
\lim_{t \to \infty} E[|f(t, \phi(t)) - f(t, \phi^*(t))|] = 0,
\]

which implies that

\[
\lim_{t \to \infty} E\left[ \frac{1}{T} \int_0^T f(s, \phi(s))ds - \frac{1}{T} \int_0^T f(s, \phi^*(s))ds \right] = 0.
\]

This, together with Fatou’s lemma and Chebyshev’s inequality, implies that

\[
P\left\{ \lim_{t \to \infty} \frac{1}{T} \int_0^T f(s, \phi(s))ds - \frac{1}{T} \int_0^T f(s, \phi^*(s))ds = 0 \right\} = 1.
\]

Let \( Y^*(t) = f(t, \phi^*(t)) \) and \( Y(t) = f(t, \phi(t)) \). Since \( f(t, z) \) is a \( T \)-periodic function in \( t \) and \( \phi^*(t) \) is a \( T \)-periodic stochastic process, it follows that \( Y^*(t) \) is also \( T \)-periodic stochastic process (see [19], p.44). Using the same arguments as in the proof of equation (1) has a unique \( T \)-periodic measure \( \rho_s(\cdot) \), we can obtain that (44) implies stochastic process \( Y(t) \) has a unique \( T \)-periodic measure \( \pi_t(\cdot) \), where \( \pi_t(\cdot) \) is defined by \( \pi_t(A) = P(Y^*(s) \in A) \) for all \( A \in B(\Int\RR_+) \), \( s \geq 0 \). This, together with the Theorem 3.2 in [12], that the time average of the periodic measure \( \pi_t(\cdot) \) defined by \( \pi(\cdot) = T^{-1} \int_0^T \pi_s(\cdot)ds \) is a unique invariant measure of \( Z(t) \). Then by Theorems 3.2.6 and 3.3.1 in [8], we have \( \pi(\cdot) \) is ergodic. Define \( D = \{ z : z = f(t, \xi), t \in [0, T], \xi \in \Int\RR_+ \} \). It follows that

\[
P(\lim_{t \to \infty} \frac{1}{T} \int_0^T Z^*(s)ds = \int_D z\pi(dz)) = 1,
\]
This, together with the strong law of large numbers for martingales, implies that

\[
\mathbb{P}\left\{ \lim_{t \to \infty} \frac{1}{t} \int_0^t Z^*(s)ds = \frac{1}{T} \int_0^T \int_D z \pi_s(dz)ds \right\} = 1.
\]

Then, by a simple measurable transformation method in [14], we have

\[
\mathbb{P}\left\{ \lim_{t \to \infty} \frac{1}{t} \int_0^t f(s, \phi^*(s))ds = \frac{1}{T} \int_0^T \int_{\mathbb{R}^+} f(s, \xi) \rho_s(d\xi)ds \right\} = 1.
\]

This, together with (45), implies (5).

**Proof of Theorem 2.6-(2).** By Itô’s formula with jumps and (13), we obtain

\[
t^{-1}(\ln \phi(t) - \ln \phi(0)) = \frac{1}{t} \int_0^t \left[ \Lambda(s, \phi(s)) - 0.5 \sigma^2(s) - \int_{\mathbb{Z}} (h(s, z) - \ln(1 + h(s, z))) \pi(dz) \right] ds
\]

\[
+ \frac{1}{t} \int_0^t \sigma_1(s) dB_1(s) + \frac{1}{t} \int_0^t \int_{\mathbb{Z}} \ln(1 + h_1(s, z)) \tilde{N}(ds, dz)
\]

\[
\leq \frac{1}{t} \int_0^t \left[ \frac{\partial \Lambda(s, 0)}{\partial x} - 0.5 \sigma^2(s) - \int_{\mathbb{Z}} (h(s, z) - \ln(1 + h(s, z))) \pi(dz) \right] ds
\]

\[
+ \frac{1}{t} \int_0^t \sigma_1(s) dB_1(s) + \frac{1}{t} \int_0^t \int_{\mathbb{Z}} \ln(1 + h_1(s, z)) \tilde{N}(ds, dz)
\]

\[
= \frac{1}{t} \int_0^t \lambda(s) ds + \frac{1}{t} \int_0^t \sigma_1(s) dB_1(s) + \frac{1}{t} \int_0^t \int_{\mathbb{Z}} \ln(1 + h_1(s, z)) \tilde{N}(ds, dz).
\]

This, together with the strong law of large numbers for martingales, implies that

\[
\limsup_{t \to \infty} \frac{\ln \phi(t)}{t} \leq \limsup_{t \to \infty} \frac{1}{t} \int_0^t \lambda(s) ds = \lambda^T < 0 \quad \text{a.s.}
\]

Therefore, we have \( \phi(t) \to 0 \) as \( t \to \infty \) a.s.

**4. Applications and numerical results.** In this section, we shall use the Milne-Steins method mentioned in [39] to illustrate our main results in section 2 and elucidate effects of Lévy noises on the dynamics of population. Let us consider a periodic logistic equation with Lévy noise

\[
dX(t) = X(t)[(c(t) - b(t), X(t))]dt + \sigma(t) dB(t) + \int_{\mathbb{Z}} h(t, z) X(t^-) N(dt, dz), \quad (46)
\]

Fix the parameters by

\[
b(t) = 0.5 + 0.2 \sin t, \quad \sigma(t) = \sqrt{0.01 + 0.01 \sin t},
\]

\[
h(t, z) \equiv h \quad (h > -1), \quad Z = (0, \infty), \quad \pi(Z) = 1.
\]

**Case 1.** \( c(t) = r(t) - \int_{\mathbb{Z}} h(t, z) \pi(dz) \). Then equation (46) is a special case of equation (1) with \( \Lambda(t, x) = x(r(t) - b(t), x) \). Let

\[
\lambda(t) = r(t) - 0.5 \sigma^2(t) - \int_{\mathbb{Z}} [h(t, z) - \ln(1 + h(t, z))] \pi(dz).
\]

Under the hypothesis (H2), the following conclusions are established by Bao et al. in [2]:

(A): If \( \lambda^T < 0 \), then the solutions of (46) become extinct a.s.
(B): If there exists a constant $c_1 > 0$ such that, for any $t \geq 0$,

$$
\tilde{\lambda}(t) := r(t) - \sigma^2(t) - \int_Z \frac{h^2(t, z)}{1 + h(t, z)} \pi(dz) \geq c_1,
$$

then the solutions of (46) are stochastically permanent. Notice that

$$
\lambda(t) = \tilde{\lambda}(t) + 0.5\sigma^2(t) + \int_Z \left[ \frac{1}{1 + h(t, z)} - 1 - \ln \frac{1}{1 + h(t, z)} \right] \pi(dz) > \tilde{\lambda}(t). \quad (48)
$$

Since Theorem 2.6 means that $\lambda^T = 0$ is the threshold value for the permanence of the population, we see from (48) that the result (B) of [2] is improved. For demonstration purpose, we fix $r(t) = 0.85 + 0.75\sin t$. Then we have

$$
\lambda(t) = 0.845 + 0.745\sin t - h + \ln(1 + h).
$$

If $h = 0.4$, then $\tilde{\lambda}(t) \geq \min_{s \in [0, 2\pi]} \tilde{\lambda}(s) \approx -0.014$ for all $t \geq 0$, which means that the condition of (B) is not satisfied. Nevertheless, since $\lambda^T > 0.78 > 0$, Theorem 2.6-(1) implies that model (46) is time-average persistent in probability and has a positive $2\pi$-periodic solution. The numerical simulations of stochastic persistence and periodicity are shown in Figure. 1-(a). Furthermore, if $h = 2$, then $\lambda^T < -1.155 + \ln 3 < -0.056 < 0$. It follows from Theorem 2.6-(2) that model (46) is extinct with probability one, which is shown in Figure. 1-(b).

In this case, these simulations suggest that the jump noises with large positive coefficients can force the population for equation (46) to be extinct with probability one (see Figure. 1).

**Case 2.** $c(t) = 0.85 + 0.75\sin t$. Then equation (46) is a special case of equation (6) with $\Lambda(t, x) = x(c(t) - b(t)x)$. It follows from (7) that

$$
\lambda_0(t) = 0.845 + 0.745\sin t + \ln(1 + h).
$$
If $h = 0.4$, then $\lambda^T > 1.18 > 0$. This, together with Theorem 2.7-(1) implies that equation (46) is time-average persistent in probability and exhibits a positive $2\pi$-periodic solution, which is shown in Figure. 2-(a). Furthermore, if $h = -0.6$, then $\lambda^T < -0.071 < 0$. It follows from Theorem 2.7-(2) that equation (46) is extinct with probability one, which is shown in Figure. 2-(b).

In this case, these simulations suggest that $h(t, z) > 0$ $(t \geq 0, z \in \mathbb{Z})$ implies that the jump noises are favorable for the permanence of equation (46) (see Figure. 2-(a)), whereas $-1 < h(t, z) < 0$ means that the jump noises are unfavorable for the permanence of equation (46) (see Figure. 2-(b)).

![Figure. 2. Dynamical behaviors of model (46) with $c(t) = 0.85 + 0.75 \sin t$: (a) $h = 0.4$; (b) $h = -0.6$. Other parameters are given by (47).](image)

5. Conclusion. In this paper we have studied the periodic single-species model with Lévy noises. We show that the model admits a unique global positive solution, and investigate stochastic ultimate boundedness, the time-average persistence in probability, periodicity and extinction.

The literature on ecological models in random environments is quite large. But most of the works are concerned with the persistence in mean and a few studies focus upon the time-average persistence in probability and periodicity of stochastic systems with Lévy noises. In this paper we investigate the time-average persistence in probability and periodicity of equation (1). By applying the method in [15, 30], the threshold conditions for time-average persistence in probability of the equation are established, which improves the earlier studies where the persistence of population is obtained (see [2, 17, 18]). We show that there exist a unique positive periodic solution and a unique periodic measure in equation (1) provided that it is time-average persistent in probability, which improves the earlier studies for the existence of periodic solution where the mathematical models do not include Lévy noise [21, 34, 35].

Some interesting results from numerical simulations deserve more emphasis. In the case one of section 4, we have shown that the jump noises with large positive coefficients can force the population for equation (1) to be extinct with probability
globally, it is sufficient to show that the general theorems of existence and uniqueness cannot be applied to this system. However, since they are locally Lipschitz continuous, for any given initial condition \( \phi_0 = \phi(0) \in \text{Int}\mathbb{R}_+ \), there is a unique local positive solution \( \phi(t) \) for \( t \in [0, \tau_e) \), where \( \tau_e \) is the explosion time. In order to show that this solution exists globally, it is sufficient to show \( \tau_e = \infty \) a.s. Let \( k_0 > 1 \) be sufficient large so that \( \phi_0 \in \{1/k_0, k_0 \} \). For each integer \( k > k_0 \), we define the stopping time by

\[
\tau_k = \inf \left\{ t \in [0, \tau_e) : \phi(t) \not\in \left(\frac{1}{k}, k\right) \right\}.
\]

Set \( \inf \emptyset = \infty \), where \( \emptyset \) is the empty set. Clearly, \( \tau_k \) is increasing as \( k \to \infty \). Let \( \tau_\infty = \lim_{k \to \infty} \tau_k \). Then \( \tau_\infty \leq \tau_e \) a.s. If we can show \( \tau_\infty = \infty \) a.s., then \( \tau_e = \infty \) a.s. and \( \phi(t) \in \text{Int}\mathbb{R}_+ \) a.s. for all \( t \geq 0 \). Let \( T_0 > 0 \) be arbitrary. Define a functional \( V \) by

\[
V(\phi(t)) = \phi(t) - \ln \phi(t).
\]  

(49)

For any \( 0 \leq t \leq \tau_k \wedge T_0 \), applying Itô’s formula with jumps, we obtain

\[
dV(\phi(t)) = LVdt + (\phi(t) - 1)\sigma(t)dB(t) + \int_{\mathbb{R}}[\phi(t^-)h(t, z) - \ln(1 + h(t, z))]N(dt, dz),
\]

where \( LV \) is given by

\[
LV(\phi(t)) = (\phi(t) - 1)\frac{\Lambda(t, \phi(t))}{\phi(t)} + 0.5\sigma^2(t) + \int_{\mathbb{R}}[h(t, z) - \ln(1 + h(t, z))]\pi(dz).
\]  

(50)

For each \( t \in [0, T] \), we define

\[\eta(x) = (x - 1)\frac{\Lambda(t, x)}{x}, \quad x > 0.\]

Suppose that (H1) holds. Then we have

\[
\lim_{x \to \infty} \eta(x) = -\infty, \quad \lim_{x \to 0^+} \eta(x) = -\frac{\partial\Lambda(t, 0)}{\partial x}.
\]

It follows that for each \( t \in [0, T] \), there exists a positive constant \( K(t) \) such that \( \eta(x) \leq K(t) \) for all \( x > 0 \). Since \( \Lambda(t, x) \) is a \( T \)-periodic function with time \( t \), it follows that

\[
(x - 1)\frac{\Lambda(t, x)}{x} \leq K^u := \max_{t \in [0, T]} K(t) \text{ for all } t \geq 0 \text{ and } x > 0.
\]

This, together with (50), one can select \( K \geq K^u \) large enough such that

\[
LV(\phi(t)) \leq K \text{ for } t \in [0, \tau_k \wedge T_0].
\]

Therefore, we have

\[
dV(\phi(t)) \leq Kdt + (\phi(t) - 1)\sigma(t)dB(t) + \int_{\mathbb{R}}[\phi(t^-)h(t, z) - \ln(1 + h(t, z))]N(dt, dz)
\]

one. Interestingly, we have found in the case two of section 4 that the jump noises with positive coefficients facilitate the permanence of the specie for equation (6).

Appendix A. The proof of Theorem 2.4.

Proof. The drift coefficients do not satisfy the linear growth condition, which means that the general theorems of existence and uniqueness cannot be applied to this system. However, since they are locally Lipschitz continuous, for any given initial condition \( \phi_0 = \phi(0) \in \text{Int}\mathbb{R}_+ \), there is a unique local positive solution \( \phi(t) \) for \( t \in [0, \tau_e) \), where \( \tau_e \) is the explosion time. In order to show that this solution exists globally, it is sufficient to show \( \tau_e = \infty \) a.s. Let \( k_0 > 1 \) be sufficient large so that \( \phi_0 \in \{1/k_0, k_0 \} \). For each integer \( k > k_0 \), we define the stopping time by

\[
\tau_k = \inf \left\{ t \in [0, \tau_e) : \phi(t) \not\in \left(\frac{1}{k}, k\right) \right\}.
\]

Set \( \inf \emptyset = \infty \), where \( \emptyset \) is the empty set. Clearly, \( \tau_k \) is increasing as \( k \to \infty \). Let \( \tau_\infty = \lim_{k \to \infty} \tau_k \). Then \( \tau_\infty \leq \tau_e \) a.s. If we can show \( \tau_\infty = \infty \) a.s., then \( \tau_e = \infty \) a.s. and \( \phi(t) \in \text{Int}\mathbb{R}_+ \) a.s. for all \( t \geq 0 \). Let \( T_0 > 0 \) be arbitrary. Define a functional \( V \) by

\[
V(\phi(t)) = \phi(t) - \ln \phi(t).
\]  

(49)

For any \( 0 \leq t \leq \tau_k \wedge T_0 \), applying Itô’s formula with jumps, we obtain

\[
dV(\phi(t)) = LVdt + (\phi(t) - 1)\sigma(t)dB(t) + \int_{\mathbb{R}}[\phi(t^-)h(t, z) - \ln(1 + h(t, z))]N(dt, dz),
\]

where \( LV \) is given by

\[
LV(\phi(t)) = (\phi(t) - 1)\frac{\Lambda(t, \phi(t))}{\phi(t)} + 0.5\sigma^2(t) + \int_{\mathbb{R}}[h(t, z) - \ln(1 + h(t, z))]\pi(dz).
\]  

(50)

For each \( t \in [0, T] \), we define

\[\eta(x) = (x - 1)\frac{\Lambda(t, x)}{x}, \quad x > 0.\]

Suppose that (H1) holds. Then we have

\[
\lim_{x \to \infty} \eta(x) = -\infty, \quad \lim_{x \to 0^+} \eta(x) = -\frac{\partial\Lambda(t, 0)}{\partial x}.
\]

It follows that for each \( t \in [0, T] \), there exists a positive constant \( K(t) \) such that \( \eta(x) \leq K(t) \) for all \( x > 0 \). Since \( \Lambda(t, x) \) is a \( T \)-periodic function with time \( t \), it follows that

\[
(x - 1)\frac{\Lambda(t, x)}{x} \leq K^u := \max_{t \in [0, T]} K(t) \text{ for all } t \geq 0 \text{ and } x > 0.
\]

This, together with (50), one can select \( K \geq K^u \) large enough such that

\[
LV(\phi(t)) \leq K \text{ for } t \in [0, \tau_k \wedge T_0].
\]

Therefore, we have

\[
dV(\phi(t)) \leq Kdt + (\phi(t) - 1)\sigma(t)dB(t) + \int_{\mathbb{R}}[\phi(t^-)h(t, z) - \ln(1 + h(t, z))]N(dt, dz)
\]
for \( t \in [0, \tau_k \wedge T_0] \). Taking expectations of the above inequality leads to
\[
EV(\phi(\tau_k \wedge T_0)) \leq V(\phi_0) + KT_0.
\] (51)

Define for each positive integer \( k > 1 \),
\[
\varpi(k) = \inf \left\{ V(\phi) : \phi \notin \left( \frac{1}{k}, k \right) \right\}.
\]

It is easy to see that
\[
\lim_{k \to \infty} \varpi(k) = \infty.
\] (52)

By virtue of (51), we have
\[
\varpi(k)P\{\tau_k \leq T_0\} \leq E[V(\phi(\tau_k))I_{\{\tau_k \leq T_0\}}] \leq EV(\phi(\tau_k \wedge T_0)) \leq V(\phi_0) + KT_0.
\]

Recalling (52) and letting \( k \to \infty \) yields \( P\{\tau_\infty \leq T_0\} = 0 \). Since \( T_0 \) is arbitrary, it follows that \( \tau_\infty = \infty \) almost surely.

Next, we give the proof to (2). By Itô’s formula with jumps, we obtain
\[
d(e^t \phi^2(t)) = e^t \phi^2(t)dt + e^t d\phi^2(t)
\]
\[
= e^t \phi^2(t) \left[ 1 + \frac{2\Lambda(t, \phi(t))}{\phi(t)} + \sigma^2(t) + \int_Z h^2(t, z)\pi(dz) \right] dt
\]
\[
+ 2\sigma(t)\phi^2(t)dB(t) + \int_Z \phi^2(t^-)(1 + h(t, z))^2 - 1] \tilde{N}(dt, dz).
\] (53)
\[
\leq e^t \phi^2(t) \left[ c + \frac{2\Lambda(t, \phi(t))}{\phi(t)} \right] dt + 2\sigma(t)e^t \phi^2(t)dB(t)
\]
\[
+ \int_Z e^t \phi^2(t^-)(1 + h(t, z))^2 - 1] \tilde{N}(dt, dz),
\]
where
\[
c = \max_{t \in [0, T]} \left\{ \sigma^2(t) + \int_Z h^2(t, z)\pi(dz) \right\}.
\]

For each \( t \in [0, T] \), we define
\[
\rho(x) = x^2 \left( c + \frac{2\Lambda(t, x)}{x} \right), \quad x > 0.
\]

Suppose that (H1) holds. Then we have
\[
\lim_{x \to \infty} \rho(x) = -\infty, \quad \lim_{x \to 0^+} \rho(x) = 0.
\]

This implies that for each \( t \in [0, T] \), there exists a positive constant \( \tilde{K}(t) \) such that \( \rho(x) \leq \tilde{K}(t) \) for all \( x > 0 \). Note that \( \Lambda(t, x) \) is a \( T \)-periodic function with time \( t \), it follows that
\[
x^2 \left( c + \frac{2\Lambda(t, x)}{x} \right) \leq \tilde{K}^u := \max_{t \in [0, T]} \tilde{K}(t) \text{ for all } t \geq 0 \text{ and } x > 0.
\]

This, together with (53), one can select \( M \geq \tilde{K}^u \) large enough such that
\[
d(e^t \phi^2(t)) \leq Me^t dt + e^t \left[ 2\sigma(t)\phi^2(t)dB(t)
\]
\[
+ \int_Z \phi^2(t^-)(1 + h(t, z))^2 - 1] \tilde{N}(dt, dz) \right].
\] (54)
Integrating (54) from 0 to $t$ and taking the expectations for both sides lead to

$$E[e^t \phi^2(t)] \leq \phi^2_0 + \int_0^t e^s M ds = \phi^2_0 + M(e^t - 1).$$

Therefore, we have

$$\limsup_{t \to \infty} E[\phi^2(t)] \leq M.$$

This completes the proof. \qed

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