Deterministic compressed sensing matrices: Construction via Euler Squares and applications

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Abstract
In Compressed Sensing the matrices that satisfy the Restricted Isometry Property (RIP) play an important role. But to date, very few results for designing such matrices are available. For applications such as multiplier-less data compression, binary sensing matrices are of interest. The present work constructs deterministic and binary sensing matrices using Euler Squares. In particular, given a positive integer $m$ different from $p, p^2$ for a prime $p$, we show that it is possible to construct a binary sensing matrix of size $m \times c(m\mu)^2$, where $\mu$ is the coherence parameter of the matrix and $c \in [1, 2)$. The matrices that we construct have smaller density (that is, percentage of nonzero entries in the matrix is small) with no function evaluation in their construction, which support algorithms with low computational complexity. Through experimental work, we show that our binary sensing matrices can be used for such applications as content based image retrieval. Our simulation results demonstrate that the Euler Square based CS matrices give better performance than their Gaussian counterparts.

Keywords:
Compressed Sensing, Coherence, RIP, Binary sensing matrices, Euler Squares, CBIR.

1. Introduction
Compressed Sensing (CS) aims at recovering high dimensional sparse vectors from considerably fewer linear measurements. The problem of sparse recovery through $l_0$ norm minimization is not tractable. Several researchers E. Candes [3], D. Donoho [12], have made pioneering contributions reposing the problem as a simple Linear Programming Problem

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(LPP). They have then established the conditions that ensure the stated equivalence between the original $l_0$ problem and its reposed version. It is known that Restricted Isometry Property (RIP) is one sufficient condition to ensure the equivalence. Random matrices with Gaussian or Bernoulli entries have been shown to satisfy RIP with high probability [4].

In the recent literature on CS [3, 5, 7], the deterministic construction of CS matrices has gained momentum. A. Amini et. al. [2], R. Devore [11], P. Indyk [15] and S. Li et. al. [18] have constructed deterministic binary sensing matrices using ideas from algebra and graph theory. In the present work, however, we attempt to construct deterministic binary sensing matrices using Euler Squares. The advantages of the proposed methodology are as follows:

- Matrices of general row size (different from a prime and its square) and smaller density are constructed.
- Simplicity in construction is achieved

Here, density is defined as the ratio of number of nonzero entries to the total number of entries of the matrix. Sparse sensing matrices may contribute to fast processing with low computational complexity in compressed sensing [14].

In the recent past, Gaussian matrices (which are shown to admit sparse representations [4]) have been used to project data into lower dimension for the purpose of classification [21]. There are, however, the following advantages of using deterministic binary matrices for the stated purpose:

- Binary matrices being sparse and possessing 0, 1 as elements provide multiplier-less and faster dimensionality reduction operation, which is not possible with Gaussian matrices
- There is a nonzero probability of non-compliance with RIP for Gaussian matrices

The problem (10, 23) of searching for similar images in a large image repository based on content is called Content Based Image Retrieval (CBIR). CBIR has several multimedia applications like text based retrieval (such as google search) and retrieval from medical databases. As images of large size typically involve more complexity, one needs to project them to lower dimensional spaces. It is demonstrated that the proposed binary sensing matrices project data into lower dimensional spaces in such a way that the reduced vectors are useful for the purpose of CBIR. Moreover, the proposed dimensionality reduction technique through binary sensing matrices allows for reconstruction, which is important for tele-medicine (1, 9). The other dimensionality reduction techniques and even the sparsity seeking Dictionary based methods [10] in general do not provide this advantage.

The paper is organized into several sections. In section 2 basic CS theory and the equivalence between $l_0$ norm and $l_1$ norm problems are given. In section 3 the deterministic construction procedure of binary sensing matrices using Euler Squares is presented. Section 4 gives comparison with existing constructions. In section 5 an application to content based image retrieval is demonstrated. Concluding remarks are given in section 6.
2. Sparse recovery from linear measurements

As stated already, CS refers to the problem of reconstruction of an unknown vector \( u \in \mathbb{R}^M \) from linear measurements \( y = (\langle u, \phi_1 \rangle, \ldots, \langle u, \phi_m \rangle) \in \mathbb{R}^m \) with \( \langle u, \phi_j \rangle \) being the inner-product between \( u \) and \( \phi_j \). The basic objective in CS is to design a recovery procedure based on the sparsity assumption on \( u \) when the number of measurements \( m \) is much small compared to \( M \). Sparse representations have merit for various applications [10, 13, 22, 25, 26, 27] in areas such as image/signal processing and numerical computation.

A vector \( u \in \mathbb{R}^M \) is \( k \)-sparse if it has at most \( k \) nonzero coordinates. Let \( \|v\|_0 \) stand for \( |\{i \mid v_i \neq 0\}| \). The problem of obtaining the sparse vector from its linear measurements may be posed as

\[
P_0 : \min_v \|v\|_0 \text{ subject to } \Phi v = y.
\]

This \( l_0 \)-minimization problem is NP-hard [12] in general. There are greedy algorithms for solving \( P_0 \) problem and orthogonal matching pursuit (OMP) is one of the popular methods [13]. Several researchers (for example, D. Donoho [12] and E. Candes [8]) have established the conditions that ensure the recovery of solution to \( P_0 \) from:

\[
P_1 : \min_v \|v\|_1 \text{ subject to } \Phi v = y.
\]

Here \( \|v\|_1 \) denotes the \( l_1 \)-norm of the vector \( v \in \mathbb{R}^M \). Denote the solution to \( P_1 \) by \( f_{\Phi}(y) \) and solution to \( P_0 \) by \( u^0_{\Phi}(y) \in \mathbb{R}^M \).

2.1. On the equivalence between \( P_0 \) and \( P_1 \) problems

The mutual-coherence \( \mu(\Phi) \) of a given matrix \( \Phi \) is the largest absolute inner-product between different normalized columns of \( \Phi \). Denoting the \( k \)-th column in \( \Phi \) by \( \phi_k \), one defines \( \mu(\Phi) \), the mutual-coherence, as \( \mu(\Phi) = \max_{1 \leq i,j \leq M, i \neq j} |\langle \phi_i, \phi_j \rangle| / \|\phi_i\|_2 \|\phi_j\|_2 \). For a matrix of size \( m \times M \), \( \mu \) satisfies: \( \mu \geq \sqrt{\frac{M-m}{m(M-1)}} \), called the Welch bound. It is known [13] that for \( \mu \)-coherent matrices \( \Phi \), one has \( u^0_{\Phi}(y) = f_{\Phi}(y) = u \), provided \( u \) is \( k \)-sparse with \( k < \frac{1}{2} (1 + \frac{1}{\mu}) \).

The Restricted Isometry Property (RIP) plays an important role [8] in CS as it establishes the equivalence between the \( P_0 \) and \( P_1 \) problems. An \( m \times M \) matrix \( \Phi \) is said to satisfy the Restricted Isometry Property (RIP) of order \( k \) with constant \( \delta_k \) if for all \( k \)-sparse vectors \( x \in \mathbb{R}^M \), we have

\[
1 - \delta_k \|x\|_2^2 \leq \| \Phi x \|_2^2 \leq (1 + \delta_k) \|x\|_2^2.
\]

(1)

It is known [8, 12] that the RIP along with some conditions on \( \delta_k \) imply the equivalence between the \( P_0 \) and \( P_1 \) problems.

One of the important problems in CS theory deals with constructing CS matrices that satisfy RIP for the largest possible range of \( k \). It is known that the widest range possible on sparsity is \( k = O(\log \frac{M}{m}) \) [4, 11, 16]. However, the only known matrices that satisfy the RIP for this range are based on random constructions.
2.2. Existing deterministic constructions

To the best of our knowledge, designing good deterministic constructions of RIP matrices is still a very interesting problem. R. Devore [11] has constructed deterministic binary sensing matrices of size \( p^2 \times p^r+1 \), where \( p \) is a prime power. Later on, S. Li, F. Gao et. al. [18] have generalized the work in [11], constructing binary sensing matrices of size \( |\mathcal{P}|q \times q^{G(\mathcal{G})} \), where \( p \) is any prime power and \( \mathcal{P} \) is the set of all rational points on algebraic curve \( X \) over finite field \( \mathbb{F}_q \). P. Indyk [15] has constructed binary sensing matrices using hash functions and extractor graphs with sizes \( r^{O(\log \log n)^{O(1)}} \times n \), where \( r \ll n \). A. Amini et. al. [2] have constructed binary sensing matrices using OOC codes. In all these constructions, CS matrices are given for specific sets of row sizes. J. Bourgain et. al. [6] have constructed RIP matrices of order \( k \geq m^{1+\epsilon} \), for some \( \epsilon > 0 \) and \( M^{1-\epsilon} \leq m \leq M \) using additive combinatorics. It is remarkable that this construction overcomes the natural barrier \( k = O(m^{1.5}) \) for those based on coherence. J. L. Nelson et. al [24] have given lower and upper bounds on maximal possible column size in terms of fixed row size \( m \) and coherence \( \mu \). They have given a class of deterministic CS matrices of size \( p \times p^r \), where \( p \) is a prime number.

In the present work, binary sensing matrices are constructed using Euler Squares. In particular, given any positive integer \( m \neq p, p^2 \) (for prime \( p \)), the procedure allows to construct binary CS matrix of size \( m \times M \), where \( M = c(m\mu)^2 \), with \( \mu \) being the coherence parameter of the matrix and \( c \in [1, 2) \). As the proposed methodology does not involve any function evaluations and gives matrices of small density, this method involves less complexity as compared to some of the existing ideas stated above. The applicability of construction is demonstrated through an application to CBIR.

3. Euler Squares for constructing CS matrices

3.1. Euler Squares

An Euler Square of order \( n \), degree \( k \) and index \( n, k \) is a square array of \( n^2 \) number of elements, \((a_{ij1}, a_{ij2}, \ldots, a_{ijk})\), where \( a_{ijr} = 0, 1, 2, \ldots, n - 1; r = 1, 2, \ldots, k; i, j = 1, 2, \ldots, n; n > k; a_{ipr} \neq a_{iqr} \) and \( a_{ipjr} \neq a_{qipj} \) for \( p \neq q \) and \((a_{ijr} + 1)(a_{ijr} + 1) \neq (a_{ipqr} + 1)(a_{ipqr} + 1) \) for \( i \neq p \) and \( j \neq q \).

Harris F. MacNeish [19] has constructed Euler Squares for the following cases:
1. Index \( p, p - 1 \), where \( p \) is a prime number.
2. Index \( p', p' - 1 \), for \( p \) prime.
3. Index \( n, k \), where \( n = 2^r p_1^{r_1} p_2^{r_2} \ldots p_t^{r_t} \) for distinct odd primes \( p_1, p_2, \ldots, p_t \). Here, \( k + 1 \) equals the least of the numbers \( 2^r, p_1^{r_1}, p_2^{r_2}, \ldots, p_t^{r_t} \).
For example, the Euler Square of index 3, 2 is as follows:

\[
\begin{array}{cccc}
0 & 0 & 1 & 1 \\
1 & 2 & 0 & 0 \\
2 & 1 & 0 & 2 \\
1 & 0 & 2 & 1 \\
\end{array}
\]

Lemma 3.1. [19] Let \( k' < k \). Then the existence of the Euler Square of index \( n, k \) implies that the Euler Square of index \( n, k' \) exists.
3.2. Deterministic constructions via Euler Squares

Using the concept of Euler Square of index \( n, k \), deterministic binary CS matrices \( \Phi \) that possess small coherence can be constructed. In particular, for \( n \geq 3, k \geq 2 \), a CS matrix of size \( nk \times n^2 \), which may be treated as a block matrix consisting of \( k \) number of \( n \times n^2 \) blocks can be obtained. Each column in \( \Phi \) corresponds to a \( k \)-ad in the Euler Square. \( \Phi \) is defined as follows: For \( 1 \leq i \leq nk, 1 \leq j \leq n^2 \),

\[
\phi_{ij} = \begin{cases} 
1 & \text{if } (a_j)_{\left\lfloor \frac{i-1}{n} \right\rfloor+1} \equiv i-1 \pmod{n} \\
0 & \text{otherwise},
\end{cases}
\]

where \((a_j)\) is the \( j^{th} \) \( k \)-ad, \((a_j)_l\) is \( l^{th} \) element in \( j^{th} \) \( k \)-ad and \( \left\lfloor x \right\rfloor \) is the largest integer not greater than \( x \).

There are exactly \( k \) ones in each column of \( \Phi \) and size of the matrix is \( nk \times n^2 \). For example, the \( 6 \times 9 \) matrix given by the Euler Square of index 3, 2 is as follows:

\[
\begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0
\end{pmatrix}
\]

The following lemma finds the bound on the mutual coherence of \( \Phi \).

**Lemma 3.2.** The coherence of \( \Phi \) is atmost equal to \( \frac{1}{k} \).

**Proof.** Suppose there exist two columns \( \phi_j, \phi_l \) of \( \Phi \) such that the overlap between them is atleast 2. That is, if \( u_j, u_l \) are the support vectors of \( \phi_j, \phi_l \), then \( |u_j \cap u_l| \geq 2 \), which implies that there exist two \( k \)-ads \((a_j) = (a_{ij1}, a_{ij2}, \ldots, a_{ijk}), (a_l) = (a_{pl1}, a_{pl2}, \ldots, a_{plk})\) such that

\[ a_{ijr} = a_{plr} \text{ and } a_{ijs} = a_{pls} \text{ for some } 1 \leq r, s \leq k. \]  

**Case 1:** Suppose \( i = p \) and \( j \neq l \), from (3), \( a_{pjr} = a_{plr} \) and \( a_{ijs} = a_{pls} \) for \( j \neq l \), which is a contradiction from the definition of Euler Square.

**Case 2:** Suppose \( i \neq p \) and \( j = l \). From (3), \( a_{iir} = a_{plr} \) and \( a_{ijs} = a_{pjs} \) for \( i \neq p \), which is again a contradiction.

**Case 3:** Suppose \( i \neq p \) and \( j \neq l \). From (3), \( a_{ijr} = a_{plr} \) and \( a_{ijs} = a_{pls} \), implies that \((a_{ijr}+1)(a_{ijs}+1) = (a_{plr}+1)(a_{pls}+1)\) for \( i \neq p \) and \( j \neq l \) which is absurd. Hence there are no two such \( k \)-ads satisfying (3). Therefore, the overlap between any two columns of \( \Phi \) is atmost 1. Since each column in \( \Phi \) contains fixed number of \( k \) ones, it follows that the coherence of \( \Phi \), \( \mu(\Phi) \), is atmost equal to \( \frac{1}{k} \). \( \square \)
Remark 1: The maximum possible column size of any binary matrix is \( \binom{m}{\ell} \), where \( m \) is the row size, \( k \) is the number of ones in each column and \( r \) is the overlap between any two columns. Euler Square of index \( n, k \) results in a binary sensing matrix of size \( nk \times n^2 \). In this matrix each column contains \( k \) number of ones and coherence is atmost \( \frac{1}{k} \). The maximum possible column size is thus \( \binom{m}{\ell} = O(n^2) = O((m\mu)^2) \). For the Euler Square based matrix \( m = nk, M = n^2 \) and \( \mu = \frac{1}{k} \), and hence \( M = (m\mu)^2 \), which is in the order of maximum possible column size. The density is \( \frac{1}{n} \), which is very small for large \( n \).

The following proposition relates the RIP constant \( \delta \) and \( \mu \).

**Proposition 3.3.** Suppose that \( \phi_1, \ldots, \phi_m \) are the unit norm columns of the matrix \( \Phi \) possessing the coherence \( \mu \). Then \( \Phi \) satisfies the RIP of order \( k' \) with constant \( \delta_{k'} = (k'-1)\mu \).

Using lemma 3.2 and Proposition 3.3 it is shown that the matrix \( \Phi \) so constructed satisfies RIP.

**Theorem 3.4.** The matrix \( \Phi_0 = \frac{1}{\sqrt{k}} \Phi \) satisfies RIP with \( \delta_{k'} = \frac{k'-1}{k-1} \) for any \( k' < k + 1 \).

Remark 2: \( m = nk, M = n^2 \) and \( k' < k + 1 \) gives \( \mu = \frac{m}{n}, n = \sqrt{M} \) which implies that \( k = \frac{m}{\sqrt{M}} \). Consequently \( k'(m, M) < \frac{m}{\sqrt{M}} + 1 \).

The previous procedure can be generalized for any row size \( m \) different from a prime and its square. The following theorem summarizes the main result.

**Theorem 3.5.** Suppose \( m \) is any positive integer different from \( p, p^2 \) for a prime \( p \). Then there exists a CS matrix whose row size is \( m \) and the column size is a function of \( m^2 \).

**Proof.** Case-1: If \( m = p^i, i > 2 \) then \( m \) can be written as \( m = p^{i-1}p \). Since \( i > 2 \), we have \( (p^{i-1} - 1) > p \). From [10], it is known that the Euler Square of index \( p^i, p^{i-1} - 1 \) exists. Since \( (p^{i-1} - 1) > p \), from lemma 3.1 Euler Square of index \( p^{i-1}, p \) exists. If we construct the binary matrix using this Euler Square, then the row size of it becomes \( p^i \).

Case-2: Let \( m \) be any integer such that \( m \neq p^i \) for \( i > 2 \). From the fundamental theorem of arithmetic \( m \) is factorized as \( m = 2^r p_1^r p_2^r \ldots p_l^r \). Let \( k' = \min\{2^r, p_1^r, p_2^r, \ldots, p_l^r\} \) and \( k = k' - 1 \). With out loss of generality assume that \( k' = 2^r \) (The following arguments hold true even if \( k' = p_s^s \) for some \( s \in \{1, 2, \ldots, l\} \)). From the construction of MacNeish [10], it is known that the Euler Square of index \( m, k \) exists. Let \( m_1 = p_1^r p_2^r \ldots p_l^r \) and without loss of generality assume that \( p_1^r = \min\{p_1^r, p_2^r, \ldots, p_l^r\} \). Since \( 2^r \leq (p_1^r - 1) \), Euler Square of index \( m_1, (p_1^r - 1) \) exists, which in view of lemma 3.1 implies that the Euler Square of index \( m_1, 2^r \) exists. The binary CS matrix constructed through this Euler Square has a row size of \( m_1 2^r \), which is \( m \).

**Remark 3:** In both the cases above \( M = (m\mu)^2 \). Hence this construction reaches the maximum possible column size asymptotically.

The afore-described construction methodology results in a binary CS matrix of size \( nk \times n^2 \) from an Euler Square of index \( n, k \). For \( n \) being of the form \( 2^r p_1^r p_2^r \ldots p_l^r \) (that is, \( n \) is not a power of single prime), it is shown in the next section that the CS matrix with the
and each column of $\Phi^{(1)}$ concatenating $\Phi^{(0)}$ of index $n$ being of size $n$.

Using it we construct a matrix $\Phi^{(1)}$ with coherence atmost $\frac{1}{k'}$. Now using these two matrices $\Phi^{(0)}$ and $\Phi^{(1)}$, a new matrix $\Psi^{(1)}$ is obtained as detailed below:

The matrix $\Phi^{(0)}$ may be viewed as a block matrix possessing $k$ blocks with each block being of size $n \times n^2$. Since $k_1 > k$ and $n_1 k = \frac{n}{k_1} k$, we have $n_1 k < n$. The suitably zero-padded $\Phi^{(1)}$ may be added as an additional column block to $\Phi^{(0)}$ to generate a matrix of size $nk \times (n^2 + n^2_1)$. Since each column of the sub block of $\Phi^{(0)}$ contains only one nonzero entry and each column of $\Phi^{(1)}$ contains $k$ nonzero entries, the coherence parameter of the column-appended matrix is atmost $\frac{1}{k}$. One may obtain $k$ number of zero-padded column blocks by placing $\Phi^{(1)}$ in the corresponding row-block-locations of $\Phi^{(0)}$. Using all these column blocks, we obtain a new binary matrix of size $nk \times (nk_1^2)$, say $\Psi^{(1)}$. The elements of the matrix $\Psi^{(1)}$ may be generated as follows: for $t = 1, 2, \ldots, k; \ (t-1)n_1 + 1 \leq j \leq tn_1$, $\psi_{ij} = 0$, also for $(t-1)n_1 + 1 \leq j \leq tn_1$, $(t-1)n + 1 \leq i \leq (t-1)n + n_1 k$, $\psi_{ij}^{(1)} = \begin{cases} 1 & \text{if } (a_j) \left\lfloor \frac{-(t+1)n_1}{n_1} \right\rfloor + 1 \equiv i - (1 + (t-1)n) \pmod{n_1} \\ 0 & \text{otherwise,} \end{cases}$ 

and for $(t-1)n_1 + 1 \leq j \leq tn_1$, $(t-1)n + n_1 k + 1 \leq i \leq n k$, $\psi_{ij}^{(1)} = 0$, where $(a_j)$ is the $j^{th}$ $k$–ad in the Euler Square of index $n_1$, $k$.

Similarly, using $k_2 = \max\{2^r, p_{r_1}^2, p_{r_2}^2, \ldots, p_{r_{t-1}}^2\} = p_{r_{t-1}}^2$, $n_2 = \frac{n}{k_2}$, a CS matrix of size $n_2 k \times n_2^2$ (say $\Phi^{(2)}$) is obtained. Now consider $k^2$ row permutations and generate a new matrix, say $\Psi^{(2)}$, of size $nk \times k^2 n_2^2$. Continuing this way till the stage of Euler Square of index $n_1$, $k$, say $\Phi^{(l)}$, of size $nk \times k^l n_2^l$ is obtained at the $l^{th}$ stage. Finally, concatenating $\Phi^{(0)}, \Phi^{(1)}, \ldots, \Phi^{(l)}$, results in a matrix with coherence atmost $\frac{1}{k}$ and column size $(n^2 + n_1^2 k + n_2^2 k^2 + n_3^2 k^3 + \ldots + n_l^2 k^l) \geq n^2 \left\lceil \frac{1 - (k^l - 1)}{1 - \frac{k^l}{k}} \right\rceil$. This methodology is explained diagrammatically in Figure 1 for $l = 2$, that is, $n = 2^r p_{r_1}^2 p_{r_2}^2$.

The column extended matrix has RIP compliance as summarized by the following theorem.

**Theorem 3.6.** The matrix $[\Phi^{(0)} \ \Psi^{(1)} \ldots \ \Psi^{(l)}]$ satisfies the RIP with $\delta_{k'} = \frac{k' - 1}{k}$ for any $k' < k + 1$.  

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Remark 4: Combining the results in Sections 3.2 and 3.3, it can be concluded that the constructed matrix attains column size $m \times c(m\mu)^2$ for some $c \in [1, 2)$. In particular, when $m$ is power of a single prime (that is, $m = p^l, l > 2$), $c = 1$, and in other cases $c$ can be made to lie in $(1, 2)$, as justified in Section 3.3.

4. Comparison with existing constructions

As discussed already in the Section 2.2, existing methods provide binary CS matrices for specific set of integers. The present work, however, provides a procedure for constructing binary matrices for a large class of row sizes. From Remark 1, the maximum possible column size is between $n^2$ and $2n^2$. Since $k \geq 2$, it follows that $(nk)^2 \geq 4n^2$ and hence $M < m^2$. In the constructions provided by [2, 6, 28, 29], $M < m^2$ holds, albeit for specific set of values of $m$. While in the construction of [11], $M > m^2$ holds. It is known [20] that the Welch bound is not sharp in this case. Consequently the gain obtained in terms of increased column size does affect the coherence, which in turn restricts the sparsity of the solution to be recovered. In addition to providing general row size, the present construction is simple in the sense that it does not involve function evaluations like in [11] and gives matrices with small density, which support algorithms with low computational complexity.

With a view towards comparing the numerical performance against the standard Gaussian random matrix, binary matrices of size $55 \times 121$ and $230 \times 529$ are generated using the Euler Squares of indices $(11, 5)$ and $(23, 10)$. The OMP algorithm is used to solve the $l_0$ minimization problem for a signal $x$. Let $\tilde{x}$ denote the recovered solution. From the reconstruction, the Signal-to-noise ratio (SNR) of $x$ is computed using

$$SNR(x) = 10 \log_{10} \left( \frac{\|x\|_2}{\|x - \tilde{x}\|_2} \right) dB.$$
Figure 2: Comparison of the reconstruction performances of Euler Square (dashed line) based matrix and the Gaussian matrix (continuous line) when the matrices are of size (a) \(55 \times 121\) (top plot) and (b) \(230 \times 529\) (bottom plot). These plots indicate that the Euler Square based matrix shows superior performance for some sparsity levels, while for other levels both matrices result in the same performance. The \(x\) and \(y\) axes in both plots refer respectively to the sparsity level and the success rate (in % terms).

For recovery at each sparsity level, 1000 input signals \(x\) are considered. The recovery is considered to be good if \(SNR(x) \geq 100 dB\). Simulation results on both matrices indicate (shown in Figure 3) that the Euler Square matrix gives better performance than the Gaussian matrix for certain higher sparsity levels. While for other sparsity levels, both matrices give the same recovery performance. It is observed that the CS matrices constructed from the Euler Squares of indices \(p, s\) for a prime \(p\) and \(s = \lfloor \frac{p}{2} \rfloor \) to \(p - 1\), give relatively superior performance than the other choices of \(s\) against their Gaussian counterparts.

4.1. Phase transition

The phase transition diagrams depict the number of measurements \(m\) needed as a function of sparsity \(k\) for faithfully recovering sparse vectors via \(l_1\)-norm minimization. In Figure 3, the region above the curve is the one in which successful reconstruction is not possible, while the region below consists of points at which successful recovery is possible. Again, recovery is considered to be successful if SNR is greater than 100dB. The smallest \(m\) is found at which a successful recovery at least 90 times is achieved by considering the average recovery over 1000 iterations. Figure 3 provides phase transition for Euler and Gaussian matrices of size \(m \times 121\) where \(m = 22, 33, 44, 55, 66, 77, 88, 99, 110\). It may be inferred from Figure 3 that the Euler Square based recovery is very competitive.
Figure 3: Comparison of the reconstruction performances of Euler Square (dashed line) based matrix and the Gaussian matrix (continuous line) through phase transition. This plot indicates that the Euler Square based matrices provide wider recovery region than their Gaussian counterparts. The x and y axes in the plot respectively represent the percentages of number of measurements \((100m/M)\) and sparsity \((100k/M)\). This plot is generated for \(M = 121\), \(m = 22, 33, 44, 55, 66, 77, 88, 99\) and \(110\).

4.2. Reconstruction of images

The efficacy of Euler Square based matrix is demonstrated using image reconstruction from lower dimensional patches, where the patches are generated via the sensing matrices (as explained in more detail in the next section and Figure 6).

The reconstructions shown in Figure 5 correspond to different down-sampling factors, viz 2.6 and 1.6. The associated reconstruction errors in term of SNR are shown in Table 1. Here by down-sampling factor, we mean the ratio of original patch size to reduced patch size which is same as \(M/m\) (where, \(m \times M\) is the size of the matrix used for projecting data to the lower dimensional space). From Figure 5 and Table 1, it may be concluded that the Euler Square based matrices provide better reconstruction performance.

| Down-sampling factor \(\frac{M}{m}\) | Euler recovery SNR error | Gaussian recovery SNR error (Average error) |
|-------------------------------------|--------------------------|--------------------------------------------|
| 4                                   | 13.36                    | 13.44                                      |
| 2.6                                 | 16.44                    | 15.03                                      |
| 2                                   | 19.63                    | 18.14                                      |
| 1.6                                 | 20.61                    | 19.74                                      |

Table 1: A comparative error analysis of reconstruction by Euler based and Gaussian matrices for different down-sampling factors \(\frac{M}{m}\) 4, 2.6, 2, 1.6. The average error over 1000 iterations is reported for Gaussian matrices.

5. Content based image retrieval (CBIR) using Euler Square matrices

In this section, the usefulness of binary matrices so constructed for the CBIR of medical databases is demonstrated. As more and more hospitals use picture archiving and communication systems (PACS), the medical imagery world wide is increasingly acquired, transferred
Figure 4: Original image

Figure 5: For the original image in Figure 4, the images in (b) and (d) are those reconstructed via the Euler Square based matrices with down-sampling factors 2.6 and 1.6 respectively. The images in (c) and (e) are those obtained via the corresponding Gaussian matrices. This figure states that Euler Square based CS matrices provide competitive reconstruction performance. The corresponding reconstruction errors are reported in Table 1.
and stored digitally \[17\]. The increasing dependence on modern medical diagnostic techniques like radiology, histopathology and computerized tomography has led to an explosion in the number of medical images stored in hospitals. Digital image retrieval technique is crucial in the emerging field of medical image databases for clinical decision making process. It can retrieve images of similar nature (like same modality and disease) and characteristics.

A typical CBIR system involves 2 steps, namely: 1) feature extraction or dimensionality reduction and 2) retrieval of relevant images through similarity metric. In what follows, relevant features are extracted by projecting image content into lower dimensional space. As most of the naturally occurring images admit sparse representation \[13\] in bases such as wavelets, the CS based dimensionality reduction technique allows for reconstruction, which is important for tele-medicine (\[1, 9\] and the references therein). The other dimensionality reduction techniques and even the sparsity seeking Dictionary based method \[10\] in general do not provide this advantage.

The database members \( \{I_l\}_{l=1}^N \) are divided into smaller patches \( \{I_{l,p}\}_{l=1}^N \) of equal size. Here, \( M' \) stands for the number of patches being carved out of each of database members. The vectorized versions of \( I_{l,p} \) are then decomposed into the wavelet domain to generate sparse vectors, say \( I'_{l,p} \) for each patch. It is to be emphasized here that one may use any transformation that sparsely represents each patch. A down-sampled copy of \( I'_{l,p} \) is generated via the binary sensing matrix \( T \) as \( I''_{l,p} = TI'_{l,p} \). If \( I'_{l,p} \) is sparse enough, \( I''_{l,p} \) (and consequently \( I_{l,p} \)) can be recovered from the reduced vector \( I''_{l,p} \) using \( l_1 \) norm minimization technique. Finally, the feature vector of \( l^{th} \) database member is obtained as \( \{I''_{l,p}\}_{l=1}^N \), which is very small in size compared to \( \{I_{l,p}\}_{l=1}^N \). Given the query image \( Q \), its feature vector \( \{Q''_{p}\}_{p=1}^{M'} \) is generated similarly. For retrieving the images of database that are similar in content with the query image, cross-correlation is employed as a similarity metric (other criteria are also possible). The proposed CBIR method is shown in Figure 6.

The performance of the image retrieval task is measured in terms of recall \( R = N_c/N_m \) and precision \( P = N_c / (N_c+N_f) \) where \( N_m \) is the total number of actual (or similar) images, \( N_c \) is the number of images detected correctly, and \( N_f \) is the number of false alarms. A good performance requires both recall and precision to be high, that is, close to unity. Recall is the portion of total relevant images retrieved whereas precision indicates the capability to retrieve relevant images.

5.1. Experimental results

For evaluating proposed CBIR, a database of 821 images present in the form of 12 classes (Figure 7) has been chosen. These classes of images containing skull, breast, chest and hand etc are taken from the popular IRMA database\[2\]. Another 240 are considered as query images.

From the database, \( \{I''_{l,p}\}_{l=1}^N \) and \( \{I''_{l,p}\}_{p=1}^{M'} \) have been generated. \( I''_{l,p}, I''_{p} \) are obtained using binary and Gaussian matrices respectively for the purpose of comparison of their performances in CBIR.

\[2\text{www.irma-project.org}\]
The query image is compared with all database members in compressed domain using cross-correlation as similarity measure for retrieving top 10 similar images. The precision and recall of binary and Gaussian based feature vector with 10 query images per class are shown in Table 2. In the simulation work, Haar wavelets have been used for sparsely representing each patch.

| Classes | 10 query images per class |
|---------|---------------------------|
|         | Euler matrix | Gaussian matrix |
|         | Precision    | Recall   | Precision    | Recall   |
| C1      | 81           | 47.8     | 90           | 58.1     |
| C2      | 100          | 97       | 100          | 85.5     |
| C3      | 98           | 89       | 97           | 81       |
| C4      | 89           | 58.7     | 92           | 75.2     |
| C5      | 76           | 48.9     | 74           | 45       |
| C6      | 95           | 69       | 98           | 71       |
| C7      | 77           | 39       | 66           | 36       |
| C8      | 91           | 61       | 92           | 45       |
| C9      | 98           | 82       | 95           | 79       |
| C10     | 100          | 69       | 100          | 61       |
| C11     | 100          | 85       | 100          | 82       |
| C12     | 100          | 46       | 94           | 37       |
| Average | 92           | 67.4     | 91.5         | 63.4     |

Table 2: Performances (in % terms) of the proposed CBIR method based on the Gaussian and binary matrices.

The classification performances of different classes by binary and Gaussian matrices are shown in Figures 8 and 9 in terms of a confusion matrix. The confusion matrix gives the accuracy of the classification results. The diagonal elements of the confusion matrix indicate if each class is classified correctly. The corresponding nonzero column entries indicate the presence of mis-classification by the CBIR scheme. From Table 2, it may be concluded that the binary based retrieval gives the performance comparable to that of its Gaussian counterpart. The retrieval performances by both matrices may be improved by fine-tuning the associated parameters (like patch size) and by taking different sparsifying basis. Since the main focus of the present work is to construct CS matrices of general size, we do not go into the details of improvement of performance of CBIR and the reconstruction of database member from the reduced dimensional vectors any further.

6. Concluding Remarks

In this paper, Euler Squares of index \((n, k)\) are used to construct binary sensing matrices of size \(nk \times n^2\). Using these and with the help of prime factorization, matrices of general row size with asymptotically optimal column size and good coherence have been constructed.
Further, using block wise extension, the column size of the constructed matrices has been enhanced. Simulation results on two test cases show that the generated matrices capture the support of the unknown vector and give better performance than Gaussian matrices. It has also been demonstrated that the proposed binary sensing matrices exploit the pattern of inherent sparsity present in natural images and project data into lower dimensional spaces in such a way that the reduced vectors are useful for the purpose of CBIR.

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Figure 6: Block diagram of the proposed CBIR method.
Figure 7: Some of the images from each of 12 classes of database.
Figure 8: Confusion matrix of binary matrix based CBMIR.

Figure 9: Confusion matrix of Gaussian matrix based CBMIR.