Error propagation in the procedure of pressure reconstruction based on PIV data

Zhongyi Wang¹, Qi Gao¹*, Runjie Wei² and Jinjun Wang¹

¹Key Laboratory of Fluid Mechanics, Ministry of Education, Beijing University of Aeronautics and Astronautics, Beijing, China
²MicroVec., Inc, Beijing, China

*Email: qigao@buaa.edu.cn

Abstract. Pressure reconstruction based on particle image velocimetry (PIV) has become a popular technique in experimental fluid mechanics. Errors in the raw velocity field significantly affect the accuracy of pressure gradient field and further reduce the quality of reconstructed pressure. Thus, error propagation deserves a serious concern. The format and magnitude of errors are investigated using a probability density function (PDF) based method. Theoretical derivation and numerical validation are operated in both Eulerian and Lagrangian descriptions. The influence of spatial and temporal resolutions on error propagation is discussed. A criterion of parameter selection for error suppression in the Lagrangian method is proposed. The results show that error propagation in the Lagrangian method has definite format and magnitude which makes this method more suitable for error control through specific treatment. Time interval during pressure reconstruction should be carefully determined since a large uncertainty appears when the time interval is employed improperly.

1. Introduction

Particle image velocimetry (PIV) based pressure reconstruction has become an important technique in modern experimental fluid mechanics [1]. This technique is realized by solving the Navier-Stokes (N-S) equations. Different methods have been developed for pressure reconstruction which can be categorized into two types: the direct path integration method [2] and the Poisson solver method [3, 4]. For every method, the calculation of pressure gradient is unavoidable. Since the viscous term is generally neglected for at least two-order smaller than the other terms in the N-S equations, the main task for pressure integration is to calculate the material acceleration [1]. Noise propagation from velocity to material acceleration has been investigated by several researchers [5, 6] which mainly focused on the aspect of uncertainty. The format of the errors during noise propagation and how the relevant parameters, such as the temporal and spatial resolutions, affect the error propagation process have not been well investigated.

Therefore, error propagation in the procedure of pressure reconstruction is discussed using a statistical tool of probability density function (PDF) in this work, through which both the magnitude and format in error propagation can simultaneously be estimated. Different error propagation mechanisms are found between the Eulerian and Lagrangian methods. Assessments are conducted using a 2D synthetic solid body rotation flow. Effect of the spatial and temporal resolutions on error propagation is investigated and a criterion for determining the measurement parameters is provided.
2. Methodology

2.1. Basic theory for pressure reconstruction

The standard N-S equation for incompressible flow is [1]
\[ \nabla p = -\rho \frac{Du}{Dt} + \mu \nabla^2 u, \]  
(1)
where the viscous term is neglected in the following discussion for its small magnitude. Both the Eulerian and Lagrangian methods are used to calculate the pressure gradient field (the material acceleration term) in this paper which can be expressed respectively as [7]
\[ \frac{Du}{Dt} = \frac{\partial u}{\partial t} + (u \cdot \nabla) u, \]  
(2)
\[ \frac{Du}{Dt} = \frac{du(x_p(t), t)}{dt}. \]  
(3)

After obtaining pressure gradient, pressure is directly integrated using the following equation
\[ p(s_{end}) = p(s_{start}) + \int_{s_{start}}^{s_{end}} \nabla p \cdot ds. \]  
(4)

In this paper, the Eulerian method uses a central difference scheme to calculate both the local and convective acceleration [8]. The Lagrangian method applies a pseudo-tracing method [2]. Finally, pressure is integrated through a widely used omni-directional integral algorithm [2, 9].

2.2. Error propagation derivation

Two reasonable assumptions about the PIV determined velocity field are proposed:
- The value of error on every grid is independent from its velocity.
- The value of error on every grid is spatially and temporally independent from each other.

It should be reminded that the locally correlated errors due to overlap in PIV correlation would not affect the spatial independency of PIV errors in global statistics analysis. Velocity error \( \varepsilon_u \) with a Gaussian distribution as follows is studied [10]
\[ f(\varepsilon_u) = \frac{1}{\sqrt{2\pi \sigma}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \]  
(5)
Equation (5) is denoted as \( \varepsilon_u \sim N(\mu, \sigma^2) \). Here, \( \mu \) and \( \sigma^2 \) are the mathematical expectation and variance, respectively. \( \mu \) is generally set as 0.

2.2.1. Error propagation to pressure gradient. Error propagation to pressure gradient is derived in both the Eulerian and Lagrangian systems.

(1) Eulerian perspective. With a central difference scheme, the errors of local acceleration \( \varepsilon_a^{el} \) along x-direction (only one direction is considered for easy explanation) can be derived as
\[ \varepsilon_a^{el} = \frac{\partial u|_{x+k\Delta x} + \varepsilon_u|_{x+k\Delta x}}{\partial t} - \frac{\partial u|_{x+k\Delta x} + \varepsilon_u|_{x+k\Delta x}}{\partial t} \approx \frac{\varepsilon_u|_{x+k\Delta x} - \varepsilon_u|_{x-k\Delta x}}{2\Delta t}. \]  
(6)
Here, \( \varepsilon_u|_{x+k\Delta x} \) and \( \varepsilon_u|_{x-k\Delta x} \) are random errors of \( u|_{x+k\Delta x} \) and \( u|_{x-k\Delta x} \), respectively. Since the random errors satisfy the PDF of \( N(\mu, \sigma^2) \), the PDF of \( \varepsilon_a^{el} \) could be obtained as
\[ \varepsilon_a^{el} \sim N(\mu\Delta t^{-1}, \sigma^2 \Delta t^{-2}/2). \]  
(7)
Equation (7) indicates that errors propagating from Gaussian velocity errors to local acceleration don't change the format. Definite PDF could be calculated for \( \varepsilon_a^{el} \) (ignoring the truncation errors). This result could also be obtained using the central limit theorem [11].
However, the nonlinear convective acceleration in the Eulerian system produces unclear error distribution. In the same way as the derivation of $\varepsilon^2_a$, errors of the $x$-component of the convective acceleration could be obtained as

$$ \varepsilon^2_a \approx \frac{\partial(u\varepsilon^u_x)}{\partial x} \bigg|_{x,t} \approx \frac{(u\varepsilon^u_x)_{t+\Delta t} - (u\varepsilon^u_x)_{t-\Delta t}}{2\Delta x}. \quad (8) $$

In equation (8), a high-order smaller term has been truncated. It indicates that the distribution of $\varepsilon^2_a$ depends on both $u$ and $\varepsilon^u_x$. As $u$ always changes with different flows, the PDF of $\varepsilon^2_a$ is hard to be determined in advance. Therefore, the PDF of the full material acceleration noise by the Eulerian method is also indefinable which makes it a difficult task to design a targeted error suppression treatment.

(2) Lagrangian perspective. The Lagrangian method calculates acceleration by tracking virtual particles [2, 9]. Random noises propagating from velocity to the Lagrangian acceleration are derived as

$$ \varepsilon^l_a = \frac{D(u_p | + \varepsilon^u_p)}{Dt} - \frac{D(u_p | - \varepsilon^u_p)}{Dt} \approx \varepsilon^u_a \big|_{t+\Delta t} - \varepsilon^u_a \big|_{t-\Delta t}. \quad (9) $$

Equation (9) has the similar form as the local acceleration error which results in

$$ \varepsilon^l_a \sim N(\mu\Delta t^{-1}, \sigma^2\Delta t^{-2}/2). \quad (10) $$

2.2.2. Error propagation to pressure. The pressure error $\varepsilon_p$ is related to the pressure gradient error $\varepsilon_{\nabla p}$ by a linear equation. In the Lagrangian method, $\varepsilon_{\nabla p} = \varepsilon^l_a$. Dividing the integration field into $n$ intervals with a space interval of $\Delta x$ and denoting the averaged value of $\varepsilon_{\nabla p}$ in the $i$th interval as $(\varepsilon_{\nabla p})_i$, we can get

$$ \varepsilon_p \big|_{b_i} = \lim_{n \to \infty} \sum_{i=1}^{n} \varepsilon_{\nabla p} \big|_{b_i} \Delta x. \quad (11) $$

Finally, the PDF of the pressure errors is derived as

$$ \varepsilon_p \sim N(n\mu\Delta x\Delta t^{-1}, n\sigma^2\Delta x^2\Delta t^{-2}/2), \quad (12) $$

where $n$ denotes the integration length. For a measurement domain with a grid size of $n_x \times n_y$, the largest variance for the pressure errors from the expectation should be less than $\sqrt{n_x^2 + n_y^2} \sigma^2\Delta x^2\Delta t^{-2}/2$. The Eulerian method is not included in this pressure error analysis for its unclear error distribution.

In a summary, errors of the pressure gradient and pressure using the Lagrangian approach has a certain distribution which is independent from the flow velocity. However, the Eulerian approach suffers from unclear acceleration errors due to the convective term. This special characteristic of Lagrangian approach makes it a more preferable method in practice since targeted treatment could be designed for noise reduction. Therefore, the Lagrangian method is specifically investigated later.

3. Results and discussion

Assessment is conducted using a synthetic solid body rotation flow [2] since quantitative evaluation of errors is impossible in experiment with unknown noise value. Two main tests are shown: one is the qualitative investigation about error sensitivity of different methods; the other one is a quantitative validation about the PDF of errors as derived in section 2, as well as a criterion for measurement parameter selection. Synthetic velocity fields with a size of $101 \times 101$ grids are produced. The time and space intervals are set as $dt = 0.01$ s and $h = 1$ in both directions. Rotation angular velocity is $\omega = 4$ rad/s. Gaussian errors as equation (5) are added into accurate velocity fields and $3\sigma$ value is used to represent the maximum of velocity errors. As the flow is axisymmetric, only the $x$-component of a
vector is discussed in the following description. Parameters of the Gaussian distribution are chosen as 
\[ \mu = 0 \quad \text{and} \quad 3\sigma_u = 0.02\sqrt{u_0} \cdot u_0^2 \] 
represents the average of the accurate velocity component \( u_0^2 \).

3.1. Error propagation in both the Eulerian and Lagrangian system

Contours of the theoretical and calculated total material accelerations (\( \|Du/Dt\| \)) are given in figure 1. As can be seen, the Eulerian material acceleration suffers from more fluctuation than the Lagrangian acceleration which indicates that the Eulerian method is more prone to errors [5, 7]. Correlation coefficients between each of the calculated acceleration and the theoretical acceleration are 0.86 and 0.95, respectively. Although the Lagrangian method has better accuracy than the Eulerian method, obvious fluctuations caused by errors are still visible which indicates that an error suppression treatment is necessary.

![Figure 1](image1.png)

(a) Theoretical acceleration  
(b) Eulerian acceleration  
(c) Lagrangian acceleration

**Figure 1.** Comparison between total material acceleration. \( \|Du/Dt\| \) is non-dimensionalized by \( \omega^2 R_{\text{max}}^2 \).

The calculated and theoretical pressure values have much smaller difference than that of the total acceleration which is undetectable directly from the pressure contours. To show the difference clearly, the contours of the pressure errors are shown in figure 2. The result is similar to what figure 1 shows that the Eulerian pressure has larger errors.

![Figure 2](image2.png)

(a) Eulerian pressure error  
(b) Lagrangian pressure error

**Figure 2.** Comparison between pressure errors. Pressure errors are non-dimensionalized by \( 0.5\rho \omega^2 R_{\text{max}}^2 \).

3.2. Validation and control of the error propagation

Considering the better accuracy and certain error distribution by Lagrangian method, the error propagation to the Lagrangian acceleration has the highest priority for investigation. Figure 3 gives the PDF of the Lagrangian acceleration errors with different temporal and spatial resolutions. In figure 3(a), the space interval is fixed to 1. As can be seen, the Lagrangian acceleration errors have a Gaussian distribution. The maximum acceleration error is inversely proportional to \( dr \) [12], which is consistent with equation (10). In figure 3(b), \( dr \) is specially set as 0.001 s so that the maximum displacement of virtual particles in each direction is about 0.2. As a result, PDF of the errors at \( h \leq 0.2 \) coincide with each other, which suggests that these two spatial resolutions (\( h = 0.1 \) and 0.2) are high enough for an accurate particle tracking when \( dr = 0.001 \) s. On the contrary, space interval of \( h \) larger than 0.2 would cause increasing errors as \( h \) increases.
PDFs of the acceleration errors are fitted using the normfit tool of MATLAB and result is shown in Table 1. A noise enlarged multiple $k$ is defined as $k = \frac{\sigma_a}{\sigma_u/dt}$ to describe how the error propagates from velocity to acceleration. According to equation (10), $k \approx 0.71$. Statistically, a value of $k \approx 0.87$ is obtained for properly selected time and space intervals. The larger value of $k$ is caused by tracking errors, truncation errors and Gaussian fitting errors. Noting that the maximum displacement of virtual particles in each direction is about 0.2 when $dt = 0.001$ s, 1 when $dt = 0.005$ s, and 2 when $dt = 0.01$ s. It can be summarized from the table that:

1. For fixed space interval of 1, $k$ increases as $dt$ decreases when $dt < 0.005$ s and increases as $dt$ increases when $dt > 0.005$ s. Therefore, if the spatial resolution is considered to be high enough for an accurate particle tracking, two conclusions could be obtained as

- When $dt < h/u_{\text{max}}$, uncertainty increases dramatically as $dt$ decreases, which results in dramatically increased Lagrangian acceleration errors.
- When $dt > h/u_{\text{max}}$, tracking errors and truncation errors increase slowly as $dt$ increases, which results in slowly increased Lagrangian acceleration errors.

2. For fixed time interval of 0.001 s, $k$ increases as space interval increases when $h > 0.2$ and does not change when $h \leq 0.2$. For fixed time interval of 0.01 s, $k$ does not change when $h \leq 2$. Therefore, if the time interval is considered to be high enough for an accurate pseudo-tracking, another two conclusions could be obtained as

- When $h > u_{\text{max}}dt$, tracking errors and truncation errors increase as $h$ increases, which results in increased Lagrangian acceleration errors.
- When $h < u_{\text{max}}dt$, Lagrangian acceleration errors are independent from space interval. The reason that decreased $h$ would not cause increased uncertainty like $dt$ lies in the different roles of $h$ and $dt$ during particle tracking. As equation (9) shows, $dt$ is directly used as a denominator for Lagrangian acceleration determination, while $h$ is mainly related to the interpolation during particle tracking. Therefore, $h$ would not affect the Lagrangian acceleration accuracy as long as the spatial resolution is high enough for an accurate interpolation.

| $\sigma_u$ | 0.1 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|------------|-----|-----|-----|-----|-----|-----|
| $\sigma_a$ | 700.13 | 703.66 | 778.21 | 883.30 | 954.67 | 961.67 |
| $\sigma_{a/dt}$ | 777.53 | 777.53 | 777.53 | 777.53 | 777.53 | 777.53 |
| $k$ | 0.90 | 0.90 | 1.00 | 1.14 | 1.23 | 1.24 |

**Table 1.** Gaussian fitting parameter and its relationship with the maximum of velocity errors. (a) Influence of $dt$ on $k$ ($h = 1$); (b) Influence of $h$ on $k$ under two time intervals.
A proper selection of the temporal and spatial intervals helps suppress the error enlargement [5, 7]. A simple criterion is proposed as $dt \geq h/u_{ref}$, where $u_{ref}$ is the characteristic velocity of a structure in flow. It suggests that $dt$ is better to be larger than smaller around the optimal $dt$ since uncertainty increases dramatically when $dt$ is very small.

4. Conclusions
The paper proposed a probability density function (PDF) based method for error propagation analysis through which the magnitude and format of the errors could be simultaneously estimated. Error propagation in pressure reconstruction is theoretically and statistically investigated in both the Eulerian and Lagrangian perspectives. Influence of the spatial and temporal resolutions on the error propagation is discussed and assessed using a 2D synthetic solid body rotation flow. A criterion is developed for the parameter determination. The main conclusions are summarized as follows. (1) Acceleration errors determined by the Lagrangian method have a certain PDF which is independent from the flow velocity, while the Eulerian material acceleration errors depend on the velocity fields. (2) The Lagrangian acceleration errors is related to the velocity errors by a noise enlarged multiple which is about 0.71 in an ideal condition and is about 0.87 in statistical result. (3) The spatial and temporal resolutions have an important effect on error propagation. A criterion is developed as $dt \geq h/u_{ref}$ that helps suppressing the error enlargement. Small $dt$ should be carefully used in case of high uncertainty.

Acknowledgements
This work is supported by the National Natural Science Foundation of China (11472030, 11327202, 11490552).

References
[1] van Oudheusden B W 2013 PIV-based pressure measurement Meas Sci Technol 24(3)
[2] Liu X and Katz J 2006 Instantaneous pressure and material acceleration measurements using a four-exposure PIV system Exp Fluids 41(2) 227-40
[3] Gresho P M and Sani R L 1987 On pressure boundary conditions for the incompressible Navier-Stokes equations Int J Numer Meth Fl 7(10) 1111-45
[4] Gurka R, Liberezon A, Hefetz D, Rubinstein D and Shavit U 1999 Computation of pressure distribution using PIV velocity data Workshop on particle image velocimetry
[5] Violato D, Moore P and Scarano F 2011 Lagrangian and Eulerian pressure field evaluation of rod-airfoil flow from time-resolved tomographic PIV Exp Fluids 50(4) 1057-70
[6] de Kat R and Ganapathisubramani B 2013 Pressure from particle image velocimetry for convective flows: a Taylor’s hypothesis approach Meas Sci Technol 24(2) 024002
[7] de Kat R and van Oudheusden B 2012 Instantaneous planar pressure determination from PIV in turbulent flow Exp Fluids 52(5) 1089-106
[8] Christensen K and Adrian R 2002 Measurement of instantaneous Eulerian acceleration fields by particle image accelerometry: method and accuracy Exp Fluids 33(6) 759-69
[9] Liu X and Katz J 2013 Vortex-corner interactions in a cavity shear layer elucidated by time-resolved measurements of the pressure field J Fluid Mech 728 417-57
[10] Foucaut J-M, Carlier J and Stanislas M 2004 PIV optimization for the study of turbulent flow using spectral analysis Meas Sci Technol 15(6) 1046
[11] JCGM 2008 Evaluation of measurement data - guide to the expression of uncertainty in measurement available online: http://wwwbipmorg/en/publications/guides/gumhtml
[12] Measurement science and technology joint committee for guides in metrology
[13] Jensen A and Pedersen G K 2004 Optimization of acceleration measurements using PIV Meas Sci Technol 15(11) 2275