Abstract

This is an exposition of some of the constructions which have arisen in higher-dimensional category theory. We start with a review of the general theory of operads and multicategories (cf. [Bur], [Her], [Lei1]). Using this we give a new definition of $n$-category (a variation on Batanin’s); we also give an informal definition in pictures. Next we discuss Gray-categories and their place in coherence problems. Finally, we present various constructions relevant to the opetopic definitions of $n$-category.

New material includes our definition of lax $n$-category; a suggestion for a definition of lax cubical $n$-category; a characterization of small Gray-categories as the small substructures of $\mathbf{2-Cat}$; a conjecture on coherence theorems in higher dimensions; a construction of the category of trees and, more generally, of $n$-pasting diagrams; and an analogue of the Baez-Dolan slicing process in the general theory of operads.
Preface to the arXiv version

This paper was written in early 1998. At the time I did not post it on the electronic archive because the files, containing as they do many memory-intensive graphics, were considered too large; and later I did not post it because there were various shortcomings that I wanted to make good before committing it to permanent storage. But technology has progressed, and I have become wise enough to know that I will never get round to revising this paper in its present form: so here it is.

The main aspects I would like to have changed are as follows. The choice of the term ‘lax $n$-category’ was probably misguided, and I would now substitute ‘weak $n$-category’. Chapters I and II are mostly superseded by my thesis (math.CT/0011106), where the ideas are explained more precisely and, I think, more clearly. Chapters III and IV, however, do not yet appear in any form elsewhere. In Chapter III I should have explained more clearly the non-standard usage of $2$-$\text{Cat}$, which denotes 2-categories, homomorphisms, strong transformations and modifications, and in section III.7 I would have liked to add some further points and subtract at least one dubious statement. Finally, the subject has progressed significantly since the time of writing and there are many new references to be added, in particular to the work of Cheng relating the opetopes of Chapter IV to the opetopes of [BD] and the multitopes of [HMP]. The bibliography of my thesis or (better) of my survey paper math.CT/0107188 should provide a decent substitute.

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Introduction

The subject of this essay is higher-dimensional category theory; the theme is to make sense of the diverse structures which have arisen in its pursuit. For instance: various different notions of ‘operad’ have been developed, and we provide a general theory covering many of them. Or: Gray-categories were introduced to address questions of coherence for tricategories, and here we give two characterizations of them (both different from the original definition) in an attempt to see what the pattern is for coherence in higher dimensions.

The description of mathematical structures in primitive terms constitutes much of what is here. Typically, we give informal arguments using pictures (for example, the Batanin operads of Chapter II). Where proofs are necessary—and there are few big theorems or startling results—they tend to be bland, seemingly formal and low in content. We usually omit these.

To date there appear to have been three main approaches to the problem of defining ‘lax $n$-category’. (See also the introduction to [Sim1] for some notes on the evolution of the subject.) Zouhair Tamsamani has made a definition which generalizes the correspondence between a category and its nerve $\Delta^{op} \longrightarrow \text{Set}$ ([Tam]); we say nothing about this, but for some speculation on page 29. Baez and Dolan presented their definition in terms of opetopes, and this inspired a related description by Hermida, Makkai and Power ([BD], [HMP]). Finally, Batanin defined lax $n$- and $\omega$-categories using globular structures ([Bat]). On the second and third approaches we have much more to say.

The contents of this essay are as follows. A preliminary chapter reviews some basic bicategory theory, which we shall call upon later. Chapter I presents a general theory of operads and multicategories. The language alone allows us to give a very concise definition of a Batanin operad; in Chapter II we explain what a Batanin operad is in elementary language, and how lax $n$-categories may then be defined. We also include a parallel (but truncated) development for cubical categories, the higher-dimensional versions of double categories. Chapter III is on Gray-categories, important in coherence for tricategories. Some ideas from Chapter II are employed here. Chapter IV, however, calls only upon Chapter I: using some of the results there, various opetopic structures and methods are illuminated. There is no attempt to describe the opetopic approach as a whole; the theory of Chapter I seems less well suited to this style than to Batanin’s.
Terminology

We distinguish between strict $n$-categories at one end of the scale, and lax $n$-categories at the other. Some authors have used weak instead of lax; ‘lax’ seems more evocative and a better opposite to ‘strict’. Baez has argued convincingly that of the strict and lax versions, the lax are more commonly-occurring and more natural as an idea, so this ought to be the default in the nomenclature. Thus he uses the term ‘$n$-category’ in contrast to ‘strict $n$-category’. However, the terminology for dimension 2 is quite well-established, where bicategories are the lax version and 2-categories the strict. Our policy is always to say ‘lax’ or ‘strict’ except in dimensions 2 and 3, and there use 2-category and 3-category for the strict versions and bicategory and tricategory for the lax.

For us, an operad is a multicategory with just one object; this conflicts with [BD], as explained after Definition I.2.2.

Related Work

Preliminaries on Bicategories. The basic definitions are taken from [Bén] and [Gray], and the outline of the coherence theorem from [St2] and [GPS]. The same material is covered in more detail in [Lei2]; see also [Lack] for another summary. I have not seen the ideas on ‘bias’ elsewhere; however, Tamsamani appears to establish that his definition of $n$-category is ‘equivalent to’ the usual one in the case $n = 2$ ([Tam]), and this must require some consideration of the bias issue.

I: Operads and Multicategories. The material here was first published as [Lei1], of which this chapter is an abbreviated version. At that time the ideas were new to the author; however, the definition of $(S,\ast)$-multicategory was also given by Burroni in 1971 and Hermida in 1997 ([Bur], [Her]). Some of the other ideas in the chapter also appear in one or both of these sources. The condition that a monad be cartesian is close to a condition in [CJ], results from which on familial representability are used here and appear to bear on the construction of the free strict $\omega$-category functor in II.3. It seems that Kelly’s theory of clubs ([Kel1], [Kel2], [Kel3]) also has common ground with this chapter.

II: The Globular Approach. This is a variant of Batanin’s definition of lax $n$-category, which appears in [Bat] and is summarized in [St3]. The operads used in this chapter are the same as Batanin’s, but our notions of contractibility differ. The final section, on cubical structures, is to my knowledge original.

III: Gray-categories. These were defined in [GPS] following [Gray]. We use an equivalent definition, given in [Bat]. Sections III.3 and III.4 are based on these sources, with what appears to be a new emphasis (on which processes are canonical). Sections III.6 and III.7, on Cayley representation, seem to be new.

IV: The Opetopic Approach. Opetopes seem to have been defined first in [BD]; they are also explained in [Baez] and used in the [HMP] approach to lax $n$-categories. My understanding of the latter is based on [Hy]. Various ideas in this chapter also appear in [Her]. Categories of trees are employed in [Bor], [Sny], [KM1], [KM2] and [Soi].
Other. The Tamsamani approach is laid out in [Tam] and explored further in Simpson’s papers [Sim1] and [Sim2]. Another set of questions about lax structures is suggested by the relaxed multilinear categories of Borcherds ([Bor], [Sny]) and the very similar pseudo-monoidal categories of Soibelman ([Soi]); see also I.2.5(d), (e) and IV.3.

Acknowledgements

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I would like to thank Peter Johnstone, Craig Snydal, and especially Martin Hyland, for the help, direction and encouragement they have continued to give me. John Power and Ross Street have imparted valuable advice on tricategories, and I have had useful discussions with Robin Cockett on structured categories.

In previous versions of this paper, the second chapter was billed as an account of Batanin’s definition of lax \(\omega\)-category. This is not the case. I thank Michael Batanin for pointing this out to me, and apologise for the previous misrepresentation of his work. The introduction to Chapter II explains more fully how our two definitions differ.
Preliminaries on Bicategories

Here we review the basic properties of bicategories and state our terminology. The final section, on ‘biased’ vs. ‘unbiased’ bicategories, looks at what happens when composition of arities 2 and 0 is replaced by composition of arbitrary arities.

[Lei2] is a more detailed summary of most of this material.

Basic terminology

We will typically denote 0-cells of a bicategory $\mathcal{B}$ by $A$, $B$, $\ldots$, 1-cells by $f$, $g$, $\ldots$ and 2-cells by $\alpha$, $\beta$, $\ldots$, e.g. $A \xrightarrow{f} B$. The ‘vertical’ composite of 2-cells $\alpha \bullet \beta$ is written $\beta \circ \alpha$ or $\beta \alpha$, and the ‘horizontal’ composite of 2-cells $\alpha \ast \alpha'$ is written $\alpha' \ast \alpha$.

A morphism $\mathcal{B} \longrightarrow \mathcal{B}'$ consists of a map $F$ of the underlying graphs together with ‘coherence’ 2-cells $Fg \circ Ff \longrightarrow F(g \circ f)$ and $1 \longrightarrow F1$ satisfying some axioms. If these 2-cells are all invertible then $F$ is called a homomorphism; if they are identities (so that $Fg \circ Ff = F(g \circ f)$ and $F1 = 1$) then $F$ is called a strict homomorphism.
A transformation $\mathcal{B} \xrightarrow{\sigma} \mathcal{B}'$ between morphisms consists of 1-cells $\sigma_A : FA \rightarrow GA$ and ‘coherence’ 2-cells

```
\[
\begin{array}{c}
FA \\ \sigma_A \\ GA
\end{array}
\xrightarrow{Ff}
\begin{array}{c}
FB \\ \sigma_B \\ GB
\end{array}
\]
```
satisfying axioms. If the $\sigma_f$ are all invertible [identities] then $\sigma$ is a strong [strict] transformation.

A modification $F \xrightarrow{\sigma} G$ between transformations consists of 2-cells $FA \xrightarrow{\sigma_A} GA$ satisfying axioms.

We will chiefly be interested in the versions where the coherence cells are isomorphisms: that is, homomorphisms and strong transformations (and modifications). Plain morphisms and transformations will not be used at all.

**Duality**

Given a bicategory $\mathcal{B}$, we may form a dual bicategory $\mathcal{B}^{op}$ by reversing the 1-cells but not the 2-cells. Thus if $\mathcal{B}$ has a 2-cell $A \xrightarrow{\alpha} B$ then $\mathcal{B}^{op}$ has a 2-cell $A \xleftarrow{f} B$.

**Internal features**

We can make certain definitions in an arbitrary bicategory $\mathcal{B}$ by generalizing from the case $\mathcal{B} = \text{Cat}$. An (internal) equivalence in $\mathcal{B}$ consists of a pair of 1-cells $A \xrightarrow{f} B$ together with invertible 2-cells $1 \rightarrow gf$ and $fg \rightarrow 1$. We also say that $f$ is an equivalence.
A *monad* in a bicategory consists of a 0-cell $A$, a 1-cell $\xymatrix{A \ar[r]^t & A}$, and 2-cells

\[
\begin{array}{c}
\eta \downarrow \\
\mu \\
\hline
\end{array}
\]

such that the diagrams

\[
\begin{array}{c}
t \circ 1 \\
\eta \\
\hline
\end{array} \quad \text{and} \quad \begin{array}{c}
t \circ (t \circ t) \\
\mu \\
\hline
\end{array}
\]

commute.

**Functor bicategories**

Given bicategories $\mathcal{B}$ and $\mathcal{B}'$, there is a bicategory $[\mathcal{B}, \mathcal{B}']$ with 0-cells homomorphisms, 1-cells strong transformations and 2-cells modifications. This is a 2-category if $\mathcal{B}'$ is. In particular, $[\mathcal{B}^{op}, \text{Cat}]$ is a 2-category for any $\mathcal{B}$.

**Biequivalence**

Let $\mathcal{B}$ and $\mathcal{B}'$ be bicategories. A *biequivalence* from $\mathcal{B}$ to $\mathcal{B}'$ consists of a pair of homomorphisms $\mathcal{B} \xrightarrow{F} \mathcal{B}'$ together with an equivalence $1 \xrightarrow{G \circ F} 1$ inside the bicategory $[\mathcal{B}, \mathcal{B}]$ and an equivalence $F \circ G \xrightarrow{1}$ inside $[\mathcal{B}', \mathcal{B}']$. We also say that $F$ is a biequivalence and that $\mathcal{B}$ is biequivalent to $\mathcal{B}'$. Just as for equivalence of plain categories, there is an alternative criterion for biequivalence: namely, that a homomorphism $F : \mathcal{B} \xrightarrow{\eta} \mathcal{B}'$ is a biequivalence if and only if $F$ is locally an equivalence and is surjective-up-to-equivalence on objects. The former condition means that each functor $\mathcal{B}(A, B) \xrightarrow{\eta} \mathcal{B}'(FA, FB)$ is an equivalence; the latter that if $B'$ is any 0-cell of $\mathcal{B}'$ then there is some 0-cell $B$ of $\mathcal{B}$ such that $FB$ is (internally) equivalent to $B'$.
Coherence

We can construct representables for a bicategory $\mathcal{B}$: for each 0-cell $A$, a homomorphism $\mathcal{B}(-, A) : \mathcal{B}^{\text{op}} \to \text{Cat}$, and similarly 1- and 2-cells. Thus we get a Yoneda homomorphism $Y : \mathcal{B} \to [\mathcal{B}^{\text{op}}, \text{Cat}]$. It is straightforward to calculate that $Y$ is locally an equivalence, by showing that locally it is full, faithful and essentially surjective on objects. From this it follows that $Y$ provides a biequivalence from $\mathcal{B}$ to its full image, a 2-category. Hence every bicategory is biequivalent to a 2-category. We call this ‘the coherence theorem for bicategories’; it implies that in a suitable sense, every diagram of coherence 2-cells in a bicategory commutes.

Bias

The traditional definition of a bicategory is ‘biased’ towards binary and nullary compositions, in that only these are given explicit mention. For instance, there is no specified ternary composite of 1-cells, $(h, g, f) \mapsto hgf$, only the derived ones like $h(gf)$ and $((h1)g)(f1)$. It is necessary to be biased in order to achieve a finite axiomatization. However, such finite axiomatizations become progressively more complex in higher dimensions (see [GPS], for instance), so the prevailing approach to defining higher-dimensional categories is an ‘unbiased’ one, treating all arities even-handedly (usually via operads).

We would therefore like to define ‘unbiased bicategory’ and see how this notion compares to the traditional one.

First, let us define an unbiased category. The data consists of a collection $C_0$ of objects, a collection $C(A, B)$ for each $A, B \in C_0$, and then for each sequence $A = (A_0, A_1, \ldots, A_n)$ of objects ($n \geq 0$), a function

$$c_A : C(A_{n-1}, A_n) \times \cdots \times C(A_0, A_1) \to C(A_0, A_n).$$

One then defines the 1-terms to be the class of functions generated from the $c_A$’s by (binary and nullary) products and composition. The axioms are that if

$$C(A_{n-1}, A_n) \times \cdots \times C(A_0, A_1) \xrightarrow{s} C(A_0, A_n)$$

are two 1-terms, then $s = t$. Unbiased functors and natural transformations are defined similarly, to make a 2-category $\text{U-Cat}$. Clearly an unbiased category is uniquely determined by its binary and nullary compositions, so the forgetful 2-functor (strict homomorphism) $\text{U-Cat} \to \text{Cat}$ is an isomorphism.

Next we can define an unbiased bicategory, in the same style as an unbiased category, and unbiased homomorphisms, \ldots. Any unbiased bicategory has an underlying bicategory; conversely, we may non-canonically choose a process by which any bicategory extends to an unbiased bicategory. The crux is that any unbiased bicategory is isomorphic to one coming from a traditional bicategory, in the sense of there being an invertible (non-strict) unbiased homomorphism between the two. Note that this is two levels better than we might have asked
for: it’s an isomorphism, not just an (unbiased) equivalence or biequivalence. With suitable definitions in place, this translates to a statement about the similarity of the tricategories $\text{Bicat}$ and $\text{U-Bicat}$. 
Chapter I

Operads and Multicategories

In this chapter we introduce the language of operads and multicategories to be used in the rest of the essay. The simplest kind of operad—a plain operad—consists of a sequence $C(0), C(1), \ldots$ of sets together with an ‘identity’ element of $C(1)$ and ‘composition’ functions

$$C(n_1) \times \cdots \times C(n_k) \times C(k) \longrightarrow C(n_1 + \cdots + n_k),$$

obeying associativity and identity laws. (In the original definition, [May], the $C(n)$’s were not just sets but spaces with symmetric group action.) The simplest kind of multicategory consists of a collection $C_0$ of objects, and arrows like

$$s_1, \ldots, s_n \xrightarrow{a} s$$

($s_1, \ldots, s_n, s \in C_0$), together with composition functions and identity elements obeying associativity and unit laws. These will be called plain multicategories; a plain operad is therefore a one-object plain multicategory.

The general idea now is that there’s nothing special about sequences of objects: the domain of an arrow might form another shape instead, such as a tree of objects or just a single object (as in a normal category). Indeed, the objects do not even need to form a set. Maybe a graph or a category would do just as well. Together, what these generalizations amount to is the replacement of the free-monoid monad on Set with some other monad on some other category.

This generalization is put into practice as follows. The graph structure of a plain multicategory is a diagram

$$
\begin{array}{ccc}
C_0^* & \xrightarrow{\text{dom}} & C_1 \\
& \searrow & \swarrow \\
& C_0 & \\
\end{array}
$$
in \textbf{Set}, where \((\ )^*\) is the free-monoid monad. Now, just as a (small) category can be described as a diagram

\[
\begin{array}{c}
  \text{D}_1 \\
  \downarrow \quad \downarrow \\
  \text{D}_0 & \text{D}_0
\end{array}
\]

in \textbf{Set} together with identity and composition functions

\[
\text{D}_0 \to \text{D}_1, \quad \text{D}_1 \times \text{D}_0 \to \text{D}_1
\]
satisfying some axioms, so we may describe the multicategory structure on \(C_0 \to C_1 \to C_0\) by manipulation of certain diagrams in \textbf{Set}. In general, we take a category \(S\) and a monad \(\star\) on \(S\) satisfying some simple conditions, and define \((S, \star)\)-multicategory. Thus a category is a \((\textbf{Set}, id)\)-multicategory.

Section I.1 describes the simple conditions on \(S\) and \(\star\) required in order that everything that follows will work. Many examples are given. Section I.2 explains what an \((S, \star)\)-multicategory is and how the examples relate to existing notions of multicategory. In particular, a concise definition of Batanin operads is given. Most of these existing notions carry with them the concept of an \textit{algebra} for an operad/multicategory; I.3 defines algebras in our general setting. Section I.4 is on \((S, \star)\)-structured categories, which are to \((S, \star)\)-multicategories as strict monoidal categories are to plain multicategories; in I.5 we sketch the construction of the free multicategory on a graph. The last two sections are included because of their impact on the opetopic approach to \(n\)-categories: they enable, for instance, a compact construction of the opetopes (IV.1). They will not be used in Chapters II and III.

This chapter is an abbreviated version of [Lei1].

\section*{I.1 Cartesian Monads}

In this section we introduce the conditions required of a monad \((\ )^*, \eta, \mu)\) on a category \(S\), in order that we may (in I.2) define the notion of an \((S, \star)\)-multicategory. The conditions are that the category and the monad are both cartesian, as defined now.

\begin{definition}
A category is called cartesian if it has all finite limits.
\end{definition}

\begin{definition}
A monad \((\ )^*, \eta, \mu)\) on a category \(S\) is called cartesian if

a. \(\eta\) and \(\mu\) are cartesian natural transformations, i.e. for any \(X \xrightarrow{f} Y\) in
$S$ the naturality squares

\[
\begin{array}{ccc}
X & \xrightarrow{\eta_X} & X^* \\
\downarrow{f} & & \downarrow{f^*} \\
Y & \xrightarrow{\eta_Y} & Y^*
\end{array}
\quad
\begin{array}{ccc}
X & \xrightarrow{\mu_X} & X^* \\
\downarrow{f} & & \downarrow{f^*} \\
Y & \xrightarrow{\mu_Y} & Y^*
\end{array}
\]

are pullbacks, and

b. $(\ )^*$ preserves pullbacks.

Examples I.1.3

a. The identity monad on any category is clearly cartesian.

b. Let $S = \text{Set}$ and let $^*$ be the monoid monad, i.e. the monad arising from the adjunction

\[
\text{Monoid} \xrightarrow{\top} \text{Set}.
\]

Certainly $S$ is cartesian. It is easy to calculate that $^*$, too, is cartesian ([Lei1, 1.4(ii)]).

c. A non-example. Let $S = \text{Set}$ and let $(\ )^*, \eta, \mu$ be the free commutative monoid monad. This is not cartesian: e.g. the naturality square for $\mu$ at $2 \rightarrow 1$ is not a pullback.

d. Let $S = \text{Set}$. Any finitary algebraic theory gives a monad on $S$; which are cartesian? Without answering this question completely, we indicate a certain class of theories which do give cartesian monads. An equation (made up of variables and finitary operators) is said to be strongly regular if the same variables appear in the same order, without repetition, on each side. Thus

\[
(x.y).z = x.(y.z) \quad \text{and} \quad (x \uparrow y) \uparrow z = x \uparrow (y.z),
\]

but not

\[
x + (y + (-(y))) = x, \quad x.y = y.x \quad \text{or} \quad (x.x).y = x.(x.y),
\]

qualify. A theory is called strongly regular if it can be presented by operators and strongly regular equations. In Example (b), the only property of the theory of monoids that we actually needed was its strong regularity: for in general, the monad yielded by any strongly regular theory is cartesian.
This last result, and the notion of strong regularity, are due to Carboni and Johnstone. They show in [CJ, Proposition 3.2 via Theorem 2.6] that a theory is strongly regular iff \( \eta \) and \( \mu \) are cartesian natural transformations and \( (\ )^* \) preserves wide pullbacks. A **wide pullback** is by definition a limit of shape

\[
 \begin{array}{c}
 \vdots \\
 \bullet \\
 \vdots 
\end{array}
\]

where the top row is a set of any size (perhaps infinite). When the set is of size 2 this is an ordinary pullback, so the monad from a strongly regular theory is indeed cartesian. (Examples (e) and (f) can also be found in [CJ].)

e. Let \( S = \text{Set} \), and let \(+\) denote binary coproduct: then the endofunctor \(- + 1\) on \( S \) has a natural monad structure. This monad is cartesian, corresponding to the theory of pointed sets.

f. Let \( S = \text{Set} \), and consider the finitary algebraic theory on \( S \) generated by one \( n \)-ary operation for each \( n \in \mathbb{N} \), and no equations. This theory is strongly regular, so the induced monad \( ((\ )^*, \eta, \mu) \) on \( S \) is cartesian.

If \( X \) is any set then \( X^* \) can be described inductively by:

- if \( x \in X \) then \( x \in X^* \)
- if \( t_1, \ldots, t_n \in X^* \) then \( \langle t_1, \ldots, t_n \rangle \in X^* \).

We can draw any element of \( X^* \) as a tree with leaves labelled by elements of \( X \):

- \( x \in X \) is drawn as \( x \)
- if \( t_1, \ldots, t_n \) are drawn as \( T_1, \ldots, T_n \) then \( \langle t_1, \ldots, t_n \rangle \) is drawn as

\[
 \begin{array}{c}
 T_1 \\
 T_2 \\
 \vdots \\
 T_n \\
 \end{array}
\]

, or if \( n = 0 \), as \( \circ \).

Thus the element \( \langle \langle x_1, x_2, \langle \rangle, x_3, \langle x_4, x_5 \rangle \rangle \rangle \) of \( X^* \) is drawn as

\[
 \begin{array}{c}
 x_1 \\
 x_2 \\
 x_3 \\
 x_4 \\
 x_5 \\
 \end{array}
\]
The unit $X \to X^*$ is $x \mapsto \bullet$, and multiplication $X^{**} \to X^*$ takes an $X^*$-labelled tree (e.g.,

![Diagram of a tree with nodes labeled $t_1$, $x_1$, $x_2$, $x_3$, $x_4$, and $t_2$.]

with

$t_1 = x_1 \cdot x_2$ and $t_2 = \circ x_3 \cdot x_4$

and gives an $X$-labelled tree by substituting at the leaves (here,

![Another diagram showing the tree labeled $x_1$, $x_2$, $x_3$, and $x_4$.]

)g. On the category $\text{Cat}$ of small categories and functors, there is the free strict monoidal category monad. Both $\text{Cat}$ and the monad are cartesian.

h. A globular set is a diagram

\[ \cdots \to X_{n+1} \xrightarrow{s} X_n \xrightarrow{s} \cdots \to X_1 \xrightarrow{s} X_0 \]

in $\text{Set}$ satisfying the ‘globularity equations’ $ss = st$ and $ts = tt : X_{n+1} \to X_{n-1}$.

The underlying graph of a strict $\omega$-category is a globular set: $X_n$ is the set of $n$-cells, and $s$ and $t$ are the source and target functions. One can construct the free strict $\omega$-category monad on the category of globular sets and show that it is cartesian. Moreover, the category of globular sets is cartesian, being a presheaf category (i.e. of the form $[\mathcal{G}^{\text{op}}, \text{Set}]$). This example is explained further in Chapter II.
I.2 Multicategories

We now describe what an \((S, \ast)\)-multicategory is, where \(\ast\) is a cartesian monad on a cartesian category \(S\). As mentioned in the introduction to this chapter, this is a generalization of the (well-known) description of a small category as a monad object in the bicategory of spans.

We will use the phrase ‘\((S, \ast)\) is cartesian’ to mean that \(S\) is a cartesian category and \((\ (\ ), \eta, \mu)\) is a cartesian monad on \(S\).

Construction I.2.1

Let \((S, \ast)\) be cartesian. We construct a bicategory \(B\) from \((S, \ast)\), which in the case \(\ast = id\) is the bicategory of spans in \(S\). Hermida calls \(B\) the ‘Kleisli bicategory of spans’ in [Her]; the formal similarity between the definition of \(B\) and the usual construction of a Kleisli category is evident.

0-cell: Object \(S\) of \(S\).

1-cell \(R \to S\): Diagram

\[
\begin{CD}
A @>R\ast>> S \\
\end{CD}
\]

in \(S\).

2-cell \(A \to A'\): Commutative diagram

\[
\begin{CD}
A @>R\ast>> S \\
A' @AA\ast\quad A \quad A'\AA\ast
\end{CD}
\]

in \(S\).

1-cell composition: To define this we need to choose particular pullbacks in \(S\), and in everything that follows we assume this has been done. Take

\[
\begin{CD}
A @>d>> R\ast \\
\quad @VVcV \quad @VVqV \\
S @>>B<< T
\end{CD}
\]

and

\[
\begin{CD}
B @>p>> T \\
\quad @AAA \quad A \quad A \AAA\ast
\end{CD}
\]
then their composite is given by the diagram

\[
\begin{array}{c}
\text{B} \\
\downarrow \\
\text{A}^* \\
\downarrow \\
\text{B} \end{array}
\begin{array}{c}
\text{d}^* \\
\downarrow \\
\text{c}^* \\
\downarrow \\
\text{q} \\
\downarrow \\
\text{p} \\
\downarrow \\
\text{T} \\
\text{R}^* \\
\text{S}^* \\
\text{R}^{**} \\
\mu_R \\
\text{R}^* \\
\eta_S \\
\text{1} \\
\text{S} \\
\text{S}^* \\
\end{array}
\]

where the right-angle mark in the top square indicates that the square is a pullback.

1-cell identities: The identity on \( S \) is

\[
\begin{array}{c}
\text{S} \\
\downarrow \\
\text{S}^* \\
\downarrow \\
\text{1} \\
\text{S} \\
\end{array}
\]

2-cell identities and compositions: Identities and vertical composition are as in \( S \). Horizontal composition is given in an obvious way.

Because the choice of pullbacks is arbitrary, 1-cell composition does not obey strict associative and identity laws. That it obeys them up to invertible 2-cells is a consequence of the fact that \( ((\ )^*, \eta, \mu) \) is cartesian.

Definition I.2.2 Let \((S, ^*)\) be cartesian. Then an \((S, ^*)\)-multicategory is a monad in the associated bicategory \( B \) of Construction I.2.1.

An \((S, ^*)\)-multicategory therefore consists of a diagram \( C_0^* \overset{d}{\longrightarrow} C_1 \overset{c}{\longrightarrow} C_0 \) in \( S \) and maps \( C_0 \overset{\text{ids}}{\longrightarrow} C_1 \), \( C_1 \overset{\text{comp}}{\longrightarrow} C_1 \) satisfying associative and identity laws. Think of \( C_0 \) as ‘objects’, \( C_1 \) as ‘arrows’, \( d \) as ‘domain’ and \( c \) as ‘codomain’. Such a multicategory will be called an \((S, ^*)\)-multicategory on \( C_0 \), or if \( C_0 = 1 \) an \((S, ^*)\)-operad. (Baez and Dolan, in [BD], use ‘operad’ or ‘typed operad’ for the same kind of purpose as we use ‘multicategory’, and ‘untyped operad’ where we use ‘operad’.)

It is inherent that everything is small: when \( S = \text{Set} \), for instance, the objects and arrows form sets, not classes. For plain multicategories, at least, there seems to be no practical difficulty in using large versions too.

Definition I.2.3 Let \((S, ^*)\) be cartesian.
a. An \((S,*)\)-graph (on an object \(C_0\)) is a diagram \(C_0^* \xleftarrow{C_1} C_0\) in \(S\). A map of \((S,*)\)-graphs

\[
\begin{array}{ccc}
C_0 & \xrightarrow{C_1} & \widetilde{C}_1 \\
\downarrow & & \downarrow & \downarrow \\
C_0^* & \xrightarrow{\widetilde{C}_0} & \widetilde{C}_0
\end{array}
\]

is a pair \((C_0 \xrightarrow{f_0} \widetilde{C}_0, C_1 \xrightarrow{f_1} \widetilde{C}_1)\) of maps in \(S\) such that

\[
\begin{array}{ccc}
C_0 & \xrightarrow{C_1} & \widetilde{C}_1 \\
\downarrow & & \downarrow & \downarrow \\
C_0^* & \xrightarrow{\widetilde{C}_0} & \widetilde{C}_0
\end{array}
\]

commutes.

b. A map of \((S,*)\)-multicategories \(C \xrightarrow{f} \widetilde{C}\) (with graphs as in (a)) is a map \(f\) of their graphs such that the diagrams

\[
\begin{array}{ccc}
C_0 & \xrightarrow{f_0} & \widetilde{C}_0 \\
\downarrow & & \downarrow & \downarrow \\
C_1 & \xrightarrow{f_1} & \widetilde{C}_1 \\
\downarrow & & \downarrow & \downarrow \\
\widetilde{C}_0^* & \xrightarrow{\widetilde{C}_0} & \widetilde{C}_0
\end{array}
\]

\[
\begin{array}{ccc}
C_0 & \xrightarrow{f_0} & \widetilde{C}_0 \\
\downarrow & & \downarrow & \downarrow \\
C_1 & \xrightarrow{f_1} & \widetilde{C}_1 \\
\downarrow & & \downarrow & \downarrow \\
\widetilde{C}_0^* & \xrightarrow{\widetilde{C}_0} & \widetilde{C}_0
\end{array}
\]

commute.

Remarks I.2.4

a. Fix \(S \in S\). Then we may consider the category of \((S,*)\)-graphs on \(S\), whose morphisms \(f = (S \xrightarrow{f_0} S, C_1 \xrightarrow{f_1} \widetilde{C}_1)\) all have \(f_0 = 1\). This is just the slice category \(S\downarrow S\). It is also the full sub-bicategory of \(B\) whose only object is \(S\), and is therefore a monoidal category. The category of \((S,*)\)-multicategories on \(S\) is then the category \(\text{Mon}(S\downarrow S)\) of monoids in \(S\downarrow S\).

b. A choice of pullbacks in \(S\) was made; changing that choice gives an isomorphic category of \((S,*)\)-multicategories.
c. If \((\mathcal{S}', \sharp)\) is also cartesian then a monad functor \((\mathcal{S}, \ast) \rightarrow (\mathcal{S}', \sharp)\) gives a functor \((\mathcal{S}, \ast)\text{-\textit{Multicat}} \rightarrow (\mathcal{S}', \sharp)\text{-\textit{Multicat}}\), and the same is true of monad opfunctors. See [St1] for the terminology and [Lei, 4.4] for more details.

**Examples I.2.5**

a. Let \((\mathcal{S}, \ast) = (\text{Set}, \text{id})\). Then \(\mathcal{B}\) is the bicategory of spans, and a monad in \(\mathcal{B}\) is just a (small) category. Thus categories are \((\text{Set}, \text{id})\)-multicategories. Functors are maps of such. More generally, if \(\mathcal{S}\) is any cartesian category then \((\mathcal{S}, \text{id})\)-multicategories are internal categories in \(\mathcal{S}\).

b. Let \((\mathcal{S}, \ast) = (\text{Set}, \text{free monoid})\). Specifying an \((\mathcal{S}, \ast)\)-graph \(\overset{d}{C_0} \xleftarrow{c} C_1 \xrightarrow{d} C_0\) is equivalent to specifying a set \(C(s_1, \ldots, s_n; s)\) for each \(s_1, \ldots, s_n, s \in C_0\) \((n \geq 0)\); if \(a \in C(s_1, \ldots, s_n; s)\) then we write

\[
s_1, \ldots, s_n \xrightarrow{a} s
\]

or

\[
\begin{array}{c}
  s_1 \\
  s_2 \\
  \vdots \\
  s_n
\end{array} \xrightarrow{a} s
\]

or

\[
\begin{array}{c}
  s_1 \\
  s_2 \\
  \ldots \\
  s_n
\end{array} \xleftarrow{a} s
\]

In the associated bicategory, the identity 1-cell \(\overset{n\text{c}_0}{C_0} \xrightarrow{1} C_0\) on \(C_0\) has

\[
C_0(s_1, \ldots, s_n; s) = \begin{cases} 
1 & \text{if } n = 1 \text{ and } s_1 = s \\
\emptyset & \text{otherwise}.
\end{cases}
\]

The composite 1-cell \(C_1 \circ C_1\) is

\[
\{(a_1, \ldots, a_n, a) \mid da = (ca_1, \ldots, ca_n)\},
\]
i.e. is the set of diagrams

\[
\begin{array}{c}
\vdots \\
| \\
| \\
| \\
| \cdots \\
| \\
| \\
\end{array}
\]

(I.A)

with the evident domain and codomain functions.

We then have a function \( \text{ids} \) assigning to each \( s \in C_0 \) a member of \( C(s; s) \), and a function \( \text{comp} \) composing diagrams like (I.A). These are required to obey associative and identity laws. Thus a \((\text{Set}, \text{free monoid})\)-multicategory is just a plain multicategory and a \((\text{Set}, \text{free monoid})\)-operad is a plain operad.

c. Let \((\mathcal{S}, \ast) = (\text{Set}, \ast + 1)\). It is not hard to see ([Lei1, 2.6(iv)]) that an \((\mathcal{S}, \ast)\)-multicategory is a (small) category \( C \) together with a functor \( C \to \text{Set} \). To put it another way, an \((\mathcal{S}, \ast)\)-multicategory is a discrete opfibration (between small categories); in fact, the category of \((\mathcal{S}, \ast)\)-multicategories is the category of discrete opfibrations.

d. Let \((\mathcal{S}, \ast) = (\text{Set}, \text{tree monad})\), as in I.1.3(f). An \((\mathcal{S}, \ast)\)-multicategory consists of a set \( C_0 \) of objects, and sets like

\[
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
\]

\((s_1, s_2, s \in C_0)\), together with a unit element of each \( C(s) \) and composition functions like
(r_1, r_2, r_3, r_4 \in C_0). These are to satisfy associativity and identity laws.

When C_0 = 1, so that we’re considering (S, * )-operads, the graph structure

is comprised of sets like C

The (S, *)-multicategories are a simpler version of Soibelman’s pseudomonoidal categories ([Soi]); they omit the aspect of maps between trees. A similar relation is borne to Borcherds’ relaxed multilinear categories ([Bor], [Sny]).

e. When S = Cat and * is the free strict monoidal category monad, an (S, * )-operad is what Soibelman calls a strict monoidal 2-operad in [Soi, 2.1]. Such a structure might also be thought of as a plain operad enriched in Cat.

f. Let (S, *) = (Globular sets, free strict 2-category). An (S, * )-operad is exactly what Batanin calls an operad (see Chapter II).

**I.3 Algebras**

Most of the existing notions of operad (e.g. [May], [BD], [Bat]) carry with them the idea of an *algebra* for an operad. We now define a category of algebras for any (S, * )-multicategory.

In the case (S, *) = (Set, id), where we are dealing with a normal category C, the category of algebras will be [C, Set]. Now [C, Set] is equivalent to the category of discrete opfibrations over C, where by definition, a discrete opfibration over C is a functor D \longrightarrow C such that for each arrow in C, every lift to
$D$ of the source of the arrow extends uniquely to a lift of the whole arrow. This means precisely that the left-hand half of the diagram

\[
\begin{array}{ccc}
D_1 & \xrightarrow{f_1} & D_0 \\
\downarrow{D_0} & & \downarrow{f_0} \\
C_1 & \xrightarrow{c} & C_0 \\
\downarrow{d} & & \downarrow{d} \\
C_0 & \xrightarrow{c} & C_0 \\
\end{array}
\]

is a pullback square. So by analogy, we will say that a map $D \xrightarrow{f} C$ of $(S, *)$-multicategories is a discrete opfibration if

\[
\begin{array}{ccc}
D^*_0 & \xleftarrow{f^*_0} & D^*_1 \\
\downarrow{f_0} & & \downarrow{f_1} \\
C^*_0 & \xleftarrow{d} & C^*_1 \\
\end{array}
\]

is a pullback square, and define an algebra for $C$ to be a discrete opfibration over $C$. A map $(D \xrightarrow{f} C)$ to $(\tilde{D} \xrightarrow{\tilde{f}} C)$ of algebras for $C$ consists of a map $D \xrightarrow{g} \tilde{D}$ of multicategories such that $\tilde{f}g = f$. We thereby obtain the category of algebras $\text{Alg}(C)$ for any $(S, *)$-multicategory $C$.

There is an alternative definition of algebra, which gives a category of algebras equivalent to the original one. It is longer to state but seems to be easier to use in practice. The initial observation is that if $C$ is a normal category then the forgetful functor $\mathbb{C}, \mathbb{Set} \xrightarrow{\text{forgetful}} \mathbb{Set}$ is monadic, the monad $T$ on $\mathbb{Set}^C_0$ being given by

\[
(TX)s = \coprod_{s' \xrightarrow{a} s \text{ in } C} Xs'
\]

($X \in \mathbb{Set}^C_0$, $s \in C_0$). Transferring $T$ across the equivalence $\mathbb{Set}^C \simeq \mathbb{Set}/C_0$, we get a monad $T'$ on $\mathbb{Set}/C_0$ such that $\mathbb{C}, \mathbb{Set}$ is equivalent to the category of $T'$-algebras. It turns out ([Lei1, 3.2]) that if $(X \xrightarrow{p} C_0)$ is an object of $\mathbb{Set}/C_0$ then $T'(X \xrightarrow{p} C_0)$ is the right-hand diagonal of the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\cap} & C_1 \\
\downarrow{p} & & \downarrow{d} \\
C_0 & & C_0 \\
\end{array}
\]
We are now ready to generalize to an arbitrary cartesian \((\mathcal{S}, \cdot)\).

**Construction I.3.1**

Let \((\mathcal{S}, \cdot)\) be cartesian: then any \((\mathcal{S}, \cdot)\)-multicategory \(C\) gives rise to a monad on \(\mathcal{S}/C_0\). The alternative definition of the category of algebras for \(C\) is as the category of algebras for this monad.

The functor part of the monad on \(\mathcal{S}/C_0\) will be called \((\cdot)^*\); in what follows, we’ll write \((X \xrightarrow{p} C_0)^* = (X, \overset{p}{\longrightarrow} C_0)\). The details omitted below are given in [Lei1, 3.3].

- \((X, \overset{p}{\longrightarrow} C_0)\) is the composite down the right-hand diagonal of

\[
\begin{array}{ccc}
X^* & \xrightarrow{p^*} & C_0^* \\
\downarrow & & \downarrow d \\
C_1 & \xrightarrow{c} & C_0
\end{array}
\]

The functor \((\cdot)^*\) is defined on morphisms in an obvious way.

- The unit at \((X \xrightarrow{p} C_0)\) is given by

\[
\begin{array}{ccc}
X & \xrightarrow{p} & C_0 \\
\downarrow \eta_X & \downarrow \text{unit}_p & \downarrow \text{ids} \\
X^* & \xrightarrow{p^*} & C_0^* \\
\downarrow & & \downarrow d \\
C_1 & \xrightarrow{c} & C_0
\end{array}
\]

The multiplication is defined in a similar, if slightly more complicated, fashion.

It is now straightforward to check that \((\cdot)^*, \text{unit}, \text{mult})\) forms a monad on \(\mathcal{S}/C_0\), and that the category of algebras for \((\cdot)^*, \text{unit}, \text{mult})\) is equivalent to the category of discrete opfibrations over \(C\). We indicate roughly how this isomorphism works. Given an algebra \((X \xrightarrow{p} C_0)^* \xrightarrow{h} (X \xrightarrow{p} C_0)\) for \((\cdot)^*\),
there is a diagram

\[
\begin{array}{c}
X_* \\
\downarrow h \\
X \\
\downarrow p^* \\
X^* \\
\downarrow p \\
C_1 \\
\downarrow d \\
C_0 \\
\end{array}
\]

the left-hand half of which is a pullback square, and a multicategory structure on \(X^* \longrightarrow X \longrightarrow X\) such that this diagram defines a map of multicategories—and therefore a discrete opfibration.

In the case \((\mathcal{S}, \ast) = (\text{Set}, \text{id})\), where \(C_0 \longrightarrow C_1 \longrightarrow C_0\) is a normal category \(C_0 \longrightarrow C_1 \longrightarrow C_0\), the object \((X \longrightarrow C_0)\) of \(\text{Set}/C_0\) gives a map \(C_0 \longrightarrow \text{Set}\) by \(x \mapsto p^{-1}\{x\}\), and the algebra structure \(h\) extends this to a functor \(C \longrightarrow \text{Set}\). The category \(X \longrightarrow X \longrightarrow X\) is then the Grothendieck opfibration of the functor.

Finally, with the \((\text{Set}, \text{id})\) case of plain categories in mind, we would expect a map \(C \longrightarrow C'\) of multicategories to yield a functor \(\text{Alg}(C) \longrightarrow \text{Alg}(C')\). This is indeed the case, as may easily be verified using either of the two definitions.

**Examples I.3.2**

a. When \((\mathcal{S}, \ast) = (\text{Set}, \text{id})\), \(\text{Alg}(C)\) is equivalent to \([C, \text{Set}]\) (or isomorphic, if we use the second definition of \(\text{Alg}\)).

b. When \((\mathcal{S}, \ast) = (\text{Set}, \text{free monoid})\), so that an \((\mathcal{S}, \ast)\)-multicategory is a multicategory of the familiar kind, we already have an idea of what an algebra for \(C\) should be: a ‘multifunctor \(C \longrightarrow \text{Set}\)’. That is, an algebra for \(C\) should consist of:

- for each \(s \in C_0\), a set \(X(s)\)
- for each \(s_1, \ldots, s_n \in C\), a function \(X(s_1) \times \cdots \times X(s_n) \longrightarrow X(s)\), in a way compatible with identities and composition.

In fact, this is the same as either definition of algebra given above. We work with the second one: algebras for \(C\) are algebras for the induced monad \(\ast\) on \(\mathcal{S}/C_0\). If \((X \longrightarrow C_0)\) is an object of \(\mathcal{S}/C_0\) and we put \(X(s) = p^{-1}\{s\}\) then

\[
X_* = \{(x_1, \ldots, x_n, f) \mid df = (px_1, \ldots, px_n)\}
\]

\[
= \{X(s_1) \times \cdots \times X(s_n) \times C(s_1, \ldots, s_n; s) \mid s_1, \ldots, s_n, s \in C_0\},
\]

24
and an algebra structure on \((X \xrightarrow{p} C_0)\) therefore consists of a function

\[X(s_1) \times \cdots \times X(s_n) \longrightarrow X(s)\]

for each member of \(C(s_1, \ldots, s_n; s)\), subject to certain laws.

c. Let \((\mathcal{S}, \ast)\) be the tree monad on \(\text{Set}\); for simplicity, let us just consider operads \(C\) for \((\mathcal{S}, \ast)\)—thus the object-set \(C_0\) is 1. An algebra for \(C\) consists of a set \(X\) together with a function \(X\xrightarrow{\bullet} X\) satisfying some axioms. One can calculate that an element of \(X\bullet\) consists of an \(X\)-labelling of the leaves of a tree \(T\), together with a member of \(C(T)\). An \(X\)-labelling of an \(n\)-leafed tree \(T\) is just a member of \(X^n\), so one can view the algebra structure \(X\xrightarrow{\bullet} X\) on \(X\) as: for each number \(n\), \(n\)-leafed tree \(T\), and element of \(C(T)\), a function \(X^n \longrightarrow X\). These functions are required to be compatible with gluing of trees in an evident way.

d. For \((\mathcal{S}, \ast) = (\text{Globular sets}, \text{free strict } \omega\text{-category})\), Batanin constructs a certain operad \(K\), the ‘universal contractible operad’. He then defines a lax \(\omega\)-category to be an algebra for \(K\). We follow exactly this strategy for defining lax \(\omega\)-category in Chapter II, except that our operad \(K\) is different from his.

e. The graph \(1\xleftarrow{1} \xrightarrow{1} 1\) is terminal amongst all \((\mathcal{S}, \ast)\)-graphs. It carries a unique multicategory structure, since a terminal object in a monoidal category always carries a unique monoid structure. It then becomes the terminal \((\mathcal{S}, \ast)\)-multicategory. The induced monad on \(\mathcal{S}/1\) is just \((\ast, \eta, \mu)\), and so an algebra for the terminal multicategory is just an algebra for \(\ast\). (E.g. an algebra for the terminal plain operad is a monoid.) This observation can aid recognition of when a theory of operads or multicategories fits into our scheme. For instance, if we were to read Batanin’s paper and learn that, in his terminology, an algebra for the terminal operad is a strict \(\omega\)-category ([Bat, p. 51, example 3]), then we might suspect that his operads were \((\mathcal{S}, \ast)\)-operads for the free strict \(\omega\)-category monad \(\ast\) on some suitable category \(\mathcal{S}\)—as indeed they are.

## I.4 Structured Categories

The observation from which this section takes off is that any strict monoidal category has an underlying multicategory. (All monoidal categories and maps between them will be strict in this section. For the time being, ‘multicategory’ means plain multicategory.) Explicitly, if \((\mathcal{C}, \otimes)\) is a monoidal category, then the underlying multicategory \(\mathcal{C}\) has the same object-set as \(\mathcal{C}\) and has homsets defined by

\[\text{Hom}_\mathcal{C}(s_1, \ldots, s_n; s) = \text{Hom}_\mathcal{C}(s_1 \otimes \cdots \otimes s_n, s)\]

for objects \(s_1, \ldots, s_n, s\). Composition and identities in \(\mathcal{C}\) are easily defined.
There is a converse process: given any multicategory $C$, there is a ‘free’
monoidal category $\mathcal{C}$ on it. Informally, an object/arrow of $\mathcal{C}$ is a sequence of
objects/arrows of $C$. Thus the objects of $\mathcal{C}$ are of the form $(s_1, \ldots, s_n)$ ($s_i \in C_0$),
and a typical arrow $(s_1, s_2, s_3, s_4, s_5) \longrightarrow (s'_1, s'_2, s'_3)$ is a sequence $(a_1, a_2, a_3)$
of elements of $A$ with domains and codomains as illustrated:

\[ \begin{array}{c}
  s_1 \\
  \downarrow \scriptstyle a_1 \\
  s' \end{array} \quad \begin{array}{c}
  s_2 \\
  \downarrow \scriptstyle a_2 \\
  s'_2 \\
  \downarrow \scriptstyle a_3 \\
  s'_3 \\
  \downarrow \\
  s_3 \\
  \downarrow \\
  s_4 \\
  \downarrow \\
  s_5 \\
 \end{array} \]

(I.B)

The tensor in $\mathcal{C}$ is just juxtaposition.

For example, the terminal multicategory $\mathbf{1}$ has one object and, for each $n \in \mathbb{N}$,
one arrow of the form

\[ n \begin{array}{c}
  \cdots \\
  \downarrow \\
  \end{array} \]

figure I.B (above) indicates that the ‘free’ monoidal category on the multicat-
egory $\mathbf{1}$ is $\Delta$, the category of finite ordinals (including 0), with addition as
$\otimes$.

The name ‘free’ is justified: that is, there is an adjunction

\[ \text{Monoidal Categories} \quad \xleftarrow{i} \quad \xrightarrow{j} \quad \text{Multicategories} \]

where the two functors are those described above. Moreover, this adjunction
is monadic. (But the forgetful functor does not provide a full embedding of
\textbf{Monoidal Categories} into \textbf{Multicategories}. It is faithful, but not full: there
is a multicategory map $\mathbf{1} \longrightarrow \Delta$ sending the unique object of $\mathbf{1}$ to the object
1 of $\Delta$, and this map does not preserve the monoidal structure.)

Naturally, we would like to generalize from $(\mathcal{S}, *) = (\mathbf{Set}, \text{free monoid})$ to
any cartesian $(\mathcal{S}, *)$. To do this, we need a notion of ‘$(\mathcal{S}, *)$-structured category’,
which in the case $(\mathbf{Set}, \text{free monoid})$ just means monoidal category. A monoidal
category is a category object in \textbf{Monoid}, so it is reasonable to define an
$(\mathcal{S}, *)$-
structured category to be an $(\mathcal{S}(\cdot), \text{id})$-multicategory—that is, a category object
in the category $\mathcal{S}(\cdot)$ of algebras for the monad $(\cdot)^*$ on $\mathcal{S}$.
It is now possible to describe a monadic adjunction

\[
\begin{array}{c}
(\mathcal{S}, \ast)^{-}\text{Struct} \\
\downarrow F \\
(\mathcal{S}, \ast)^{-}\text{Multicat}
\end{array} \\
\begin{array}{c}
\downarrow U \\
\end{array}
\]

generalizing that above. The effect of the functors \(U\) and \(F\) on objects is as outlined now. Given an \((\mathcal{S}, \ast)^{-}\)-structured category \(D\), with algebraic structure \(D_0^* \otimes D_0 \rightarrow D_0\) and \(D_1^* \otimes D_1 \rightarrow D_1\), the graph \(D_0^* \leftarrow C_1 \rightarrow D_0\) of \(UD\) is given by

\[
\begin{array}{c}
C_1 \\
\downarrow \\
D_1 \\
\downarrow \\
D_0 \\
\downarrow \\
D_0 \\
\end{array}
\]

Given an \((\mathcal{S}, \ast)^{-}\)-multicategory \(C\), the category \(FC\) has graph

\[
\begin{array}{c}
C_1^* \\
\downarrow \\
C_{0^*} \\
\downarrow \\
\mu_{C_0} \\
\end{array}
\]

\[
\begin{array}{c}
C_{0^{**}} \\
\downarrow \\
C_1^{**} \\
\downarrow \\
C_1^* \\
\downarrow \\
C_0^* \\
\end{array}
\]

and the algebraic structures \(C_1^{**} \otimes C_1^*, \ C_{0^{**}} \otimes C_0^*\) are components of \(\mu\).

### I.5 The Free Multicategory Monad

Let \((\mathcal{S}, \ast)\) be cartesian. Subject to certain further conditions on \(\mathcal{S}\) and \(\ast\), the following are true:

- the forgetful functor

\[
(\mathcal{S}, \ast)^{-}\text{-Multicat} \longrightarrow (\mathcal{S}, \ast)^{-}\text{-Graph}
\]

has a left adjoint, the ‘free \((\mathcal{S}, \ast)^{-}\)-multicategory functor’

- the adjunction is monadic

- the monad on \((\mathcal{S}, \ast)^{-}\text{-Graph}\) is cartesian (and the category \((\mathcal{S}, \ast)^{-}\text{-Graph}\) is cartesian, as always)
The precise nature of the conditions and the free multicategory construction is not important for our purposes. However, as a gesture towards supporting the assertions above, here is a sketch of a construction of the free functor. Given an \((S, \ast)\)-graph \(G\), one defines a sequence \((G_0 \longleftarrow A^{(n)} \longrightarrow G_0)_{n \in \mathbb{N}}\) of \((S, \ast)\)-graphs by \(A^{(0)} = G_1\) and \(A^{(n+1)} = G_0 + G_1 \circ A^{(n)}\), where + is binary coproduct and \(\circ\) is composition in the bicategory of spans for \((S, \ast)\). The free multicategory \(G_0 \longleftarrow A \longrightarrow G_0\) on \(G\) is given as a colimit of the \(A^{(n)}\)'s. (In the case \((S, \ast) = (\text{Set, id})\) and \(G_0 = 1\) we are forming the free monoid on a set \(G\), and the formula \(A^{(n+1)} = 1 + G_0 A^{(n)}\) expresses the fact that a word on a set is either the empty word or an element of the set followed by a word.) To make this work it is necessary to assume that in \(S\) certain colimits exist and interact with pullbacks in a suitable way, and that the functor \((\_)^\ast\) preserves certain filtered colimits.

As may be apparent from the foregoing sketch of the construction, taking the free multicategory on a graph does not change the objects-object \(S (= G_0)\). It follows that all the statements in the first paragraph of this section hold if we replace the forgetful functor
\[
(S, \ast)\text{-Multicat} \longrightarrow (S, \ast)\text{-Graph}
\]
by the forgetful functor
\[
(S, \ast)\text{-Multicategories on } S \longrightarrow (S, \ast)\text{-Graphs on } S,
\]
for any \(S \in \mathcal{S}\). We therefore have the monad ‘free \((S, \ast)\)-multicategory on \(S\)’ on the cartesian category \(\mathcal{S} = (S, \ast)\text{-Graphs on } S\).

All we will need to know about the conditions on \(S\) and \(\ast\) is:

- the conditions on \(S\) are satisfied if \(S\) is \(\text{Set}\), or any presheaf category
- the conditions on \(\ast\) are satisfied if \(\ast\) is the identity monad, or in fact just as long as the functor \((\_)^\ast\) is finitary
- if \((S, \ast)\) satisfies the conditions then so too does
\[
((S, \ast)\text{-Graphs on } S, \text{free } (S, \ast)\text{-multicategory on } S),
\]
for each \(S \in \mathcal{S}\).

For example, these conditions hold for any cartesian finitary algebraic theory on \(\text{Set}\).
Chapter II

The Globular Approach

Perhaps the most direct approach to defining ‘lax $n$-category’ is the kind suggested by Batanin in [Bat]. In this chapter we present a definition of lax $n$-category which is a variation on Batanin’s. An informal version can be given using no technical terms at all (II.1). The definition also has a very simple structure when described in the language of general operads, as already seen in I.1.3(h), I.2.5(f) and I.3.2(d).

Our definition of lax $n$-category is very close to Batanin’s, although not (as asserted previously) the same. The operads he uses are the same as the $(\mathcal{S},^\ast)$-operads here (for a certain choice of $(\mathcal{S},^\ast)$), and part of the purpose of this chapter is to explain in elementary language and pictures what these $(\mathcal{S},^\ast)$-operads are, so that the knowledgeable reader may understand that the two kinds of operad coincide. However, the other main concept used in the definition, contractibility, is defined differently in our two accounts.

The final section of the chapter is on another theme. It is a brief explanation of how we might go about defining (strict and lax) $n$-tuple categories, which are to $n$-categories as double categories are to 2-categories. In other words, it is a cubical rather than a globular approach, and the development is exactly analogous. Conceivably there is some connection here with Tamsamani’s $n$-categories.

A category is a finite-limit-preserving functor $\Delta^{\text{op}} \to \text{Set}$. In [Tam], a lax $n$-category is a functor $(\Delta^n)^{\text{op}} \to \text{Set}$ satisfying three axioms. The first is a condition called ‘truncatability’. The second is a generalization of the ‘finite-limit-preserving’ condition. The third is a degeneracy condition, analogous to the fact that a 2-category is a double category with a certain degeneracy in its graph. The present author’s ignorance prevents further discussion.

Section II.1 is an informal definition of lax $n$-category; II.2 describes the
formal structure of the definition. Section II.3 concerns globular diagrams like

![Diagram](image)

and how they may be manipulated: these are the cells of the free strict \(\omega\)-category \(1^*\) on the terminal globular set. One way to define \(X^*\) for general \(X\) is to define \(1^*\) and then each fibre in the map \(X^* \xrightarrow{1^*} 1^*\), and we discuss this here. In II.4, operads and their algebras are explained in elementary terms. We then reach the definition of lax \(n\)- and \(\omega\)-categories in II.5, via the concept of a contraction on an operad (which is not part of the general theory of operads). Finally, section II.6 is on cubical structures, as explained above.

The chief reference for this chapter is [Bat], for the material on operads and a different view of contractibility. Street has also produced an account of Batanin’s work, [St3].

II.1 Informal Outline

As in the traditional conception, the graph structure of a lax \(n\)-category consists of 0-cells \(A\), 1-cells \(f\), 2-cells \(\alpha\), . . . . There are then various ways of composing these cells; just how many ways and how they interact depends on whether we are dealing with strict or lax \(n\)-categories, or something in between. In a strict \(n\)-category, there will be precisely one way of composing a diagram like

![Diagram](image)

(II.A)

to obtain a 2-cell; that is, any two different ways of doing it (e.g. compose \(\alpha'\) with \(\alpha\), and \(\gamma\) with \(g\), then the two of these together) give exactly the same resulting 2-cell. In a lax \(n\)-category there will be many ways, but the resulting 2-cells will all be equivalent in a suitably weak sense.
Our method of describing what ways of composing are available in a lax \( n \)-category depends on one simple principle, the ‘contraction principle’. Take, for example, the diagram (II.A) above. Suppose we have already constructed two ways of composing a generic diagram

\[
\begin{tikzpicture}
  \node (A) at (0,0) {A};
  \node (B) at (1.5,0) {B};
  \node (C) at (3,0) {C};
  \node (D) at (4.5,0) {D};
  \draw[-stealth] (A) -- (B) node[pos=0.5, above] {\(f\)};
  \draw[-stealth] (B) -- (C) node[pos=0.5, above] {\(g\)};
  \draw[-stealth] (C) -- (D) node[pos=0.5, above] {\(h\)};
  \draw[-stealth] (D) -- (A) node[pos=0.5, above] {\(f'\)};
\end{tikzpicture}
\]

of 1-cells, namely \((rq)p\) and \(r(qp)\). Then we deduce that there is a way of composing diagram (II.A) to get a 2-cell of the form \(A\) \begin{tikzpicture}
  \node (A) at (0,0) {A};
  \node (B) at (1.5,0) {B};
  \node (C) at (3,0) {C};
  \node (D) at (4.5,0) {D};
  \draw[-stealth] (A) -- (B) node[pos=0.5, above] {\(f\)};
  \draw[-stealth] (B) -- (C) node[pos=0.5, above] {\(g\)};
  \draw[-stealth] (C) -- (D) node[pos=0.5, above] {\(h\)};
  \draw[-stealth] (D) -- (A) node[pos=0.5, above] {\(f'\)};
\end{tikzpicture} (II.B)

As another example of the principle, this time in one higher dimension, take a diagram

\[
\begin{tikzpicture}
  \node (A) at (0,0) {A};
  \node (B) at (1.5,0) {B};
  \node (C) at (3,0) {C};
  \node (D) at (4.5,0) {D};
  \draw[-stealth] (A) -- (B) node[pos=0.5, above] {\(f\)};
  \draw[-stealth] (B) -- (C) node[pos=0.5, above] {\(g\)};
  \draw[-stealth] (C) -- (D) node[pos=0.5, above] {\(h\)};
  \draw[-stealth] (D) -- (A) node[pos=0.5, above] {\(f'\)};
\end{tikzpicture}
\]

Suppose we have constructed two ways of composing a diagram of the shape of (II.A) to a 2-cell, each of which invokes the same way of composing the 1-cells along the top and bottom. Say, for instance, that the first way of composing (II.A) results in a 2-cell \(A\) \begin{tikzpicture}
  \node (A) at (0,0) {A};
  \node (B) at (1.5,0) {B};
  \node (C) at (3,0) {C};
  \node (D) at (4.5,0) {D};
  \draw[-stealth] (A) -- (B) node[pos=0.5, above] {\(f\)};
  \draw[-stealth] (B) -- (C) node[pos=0.5, above] {\(g\)};
  \draw[-stealth] (C) -- (D) node[pos=0.5, above] {\(h\)};
  \draw[-stealth] (D) -- (A) node[pos=0.5, above] {\(f'\)};
\end{tikzpicture} and the second way in a 2-cell \(A\) \begin{tikzpicture}
  \node (A) at (0,0) {A};
  \node (B) at (1.5,0) {B};
  \node (C) at (3,0) {C};
  \node (D) at (4.5,0) {D};
  \draw[-stealth] (A) -- (B) node[pos=0.5, above] {\(f\)};
  \draw[-stealth] (B) -- (C) node[pos=0.5, above] {\(g\)};
  \draw[-stealth] (C) -- (D) node[pos=0.5, above] {\(h\)};
  \draw[-stealth] (D) -- (A) node[pos=0.5, above] {\(f'\)};
\end{tikzpicture}. Then the contraction principle says that there is a way of
composing (II.B) to get a 3-cell of the form

\[
\begin{array}{c}
\delta' \\
A & \quad \delta \\
\end{array}
\]

\[
\begin{array}{c}
h' \cdot (g f'') f \cdot hg \\
D & \quad A
\end{array}
\]

In general, we may state the contraction principle as follows. Suppose we are given an \((n + 1)\)-dimensional diagram and two ways of composing the \(n\)-dimensional diagram at its source/target, such that these two ways match on the \((n - 1)\)-dimensional source and target. Then there's a way of composing the \((n + 1)\)-dimensional diagram, inducing the first way on its source and the second way on its target. (In our first example, we implicitly used the fact that the two ways of composing

\[
\begin{array}{c}
p \\
A & \quad q & \quad r
\end{array}
\]

\((rq)p\) and \(r(qp)\), do the same thing to the bounding 0-cells: nothing at all.)

Let us see how the contraction principle generates the ways of composing in a lax \(\omega\)-category. First of all, there’s a way of composing a (‘generic’) 0-cell \(A\) to get a 0-cell: do nothing. Now, given a diagram

\[
\begin{array}{c}
f_1 \\
A_0 & \quad f_2 & \quad \ldots & \quad f_n \\
A_1 & \quad \ldots & \quad A_n
\end{array}
\]

of 1-cells, note that there are two (identical) ways of composing the 0-cell diagram \(\bullet\), both of which are the ‘do nothing’ way just referred to. So by the contraction principle, there arises a way of composing

\[
\begin{array}{c}
f_1 \\
A_0 & \quad f_2 & \quad \ldots & \quad f_n \\
A_1 & \quad \ldots & \quad A_n
\end{array}
\]

to a 1-cell \(A_0 \rightarrow A_n\); we call this 1-cell \(\bullet \rightarrow \left(\cdot f_n \ldots f_2 f_1\right) A_n\). Putting together these ways of composing 1-cells, we can take a diagram like

\[
\begin{array}{c}
f_1 \\
A_0 & \quad f_2 & \quad f_3 & \quad f_4 & \quad f_5
\end{array}
\]

and do a composition like

\[
\begin{array}{c}
\rightarrow f_1 & \quad f_2 & \quad f_3 & \quad f_4 & \quad f_5
\end{array}
\]

\[
\rightarrow (f_5 f_4) 1 (f_3 f_2 f_1)
\]

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where to get ‘1’ we’ve taken \( n = 0 \) in the above. Next, we may take a 2-dimensional diagram like (II.A) and find a way of composing it by using its 1-dimensional boundary, as on page 31. For another example, take the operation ‘do nothing’ on \( \bullet \) to get a way of composing \( \bullet \), and similarly the operation

\[
\begin{array}{c}
\frown f \quad \frown g \\
\frown g \frown f
\end{array}
\]

on

\[
\begin{array}{c}
\frown f \quad \frown g
\end{array}
\]

to get a way of composing

\[
\begin{array}{c}
\frown f \quad \frown g \quad \frown f
\end{array}
\]

putting these together gives two different ways of composing

\[
\begin{array}{c}
\frown f \quad \frown g \quad \frown f
\end{array}
\]

familiar from the interchange law. There is also the all-at-once way of composing this shape, given by applying the contraction principle to

\[
\begin{array}{c}
\frown f \quad \frown g \quad \frown g f
\end{array}
\]

At this point the reader might be wondering what has happened to the coherence conditions that exist in, say, bicategories. The answer lies in how we realise lax \( n \)-categories as lax \( \omega \)-categories: namely, a lax \( n \)-category is a lax \( \omega \)-category all of whose \( m \)-cells are identities when \( m > n \). Take, for instance, lax 2-categories and the two traditional ways of composing a diagram
to give a 2-cell, i.e.
\[
\begin{array}{c}
\begin{array}{c}
A \\
gf
\end{array}
\end{array}
\]
and
\[
\begin{array}{c}
\begin{array}{c}
A \\
gf
\end{array}
\end{array}
\]
where \(\gamma_1 = (\beta' \circ \beta) \circ (\alpha' \circ \alpha)\) and \(\gamma_2 = (\beta' \circ \alpha) \circ (\beta \circ \alpha)\). As these two ways restrict to the same way on the 1-dimensional source and target, there is a 3-cell of the form

\[
\begin{array}{c}
\begin{array}{c}
A \\
gf
\end{array}
\end{array}
\]

for apply the contraction principle to the degenerate 3-cell diagram

\[
\begin{array}{c}
\begin{array}{c}
A \\
gf
\end{array}
\end{array}
\]

But in a (lax) 2-category the only 3-cells are identities, so \(\gamma_1 = \gamma_2\), as expected. In this manner we can see that the different ways of composing a diagram of 2-cells in a lax 2-category depend only on what compositions they induce on the diagrams of 1-cells at its source and target, just as in a traditional bicategory; and, indeed, that bicategories and our lax 2-categories are the same but for inessential detail (the ‘bias’ issue mentioned on page 9).

\section{II.2 Formal Outline}

As indicated previously, the language of general operads allows us to state a definition of lax \(\omega\)-category quite easily. A certain category \(\mathcal{S}\) and monad \(*\) are defined, and a certain \((\mathcal{S},*)\)-operad \(K\); lax \(\omega\)-categories are then just \(K\)-algebras. The point where our definition departs from Batanin’s is the choice of the operad \(K\). For us, \(K\) will be the structure formed by the ‘ways of composing’ discussed in the previous section.

The category \(\mathcal{S}\) is that of globular sets, as defined in I.1.3(h); thus the underlying graph of a lax \(\omega\)-category will be a globular set. The monad \(*\) is the free strict \(\omega\)-category monad. In outline, a \textit{strict \(\omega\)-category} consists of a globular set \(X\) and for each \(n \geq k\) a binary composition function

\[
X(n) \times_{X(k)} X(n) \longrightarrow X(n)
\]

and an identity function

\[
X(k) \longrightarrow X(n),
\]
satisfying the appropriate source-target relations, and such that the compositions and identities obey strictly all the possible associative, identity and interchange laws. The forgetful functor from strict \( \omega \)-categories to globular sets has a left adjoint, described, essentially, in [Bat] (and see also II.3 below). It is straightforward to calculate that the adjunction is monadic and that the monad \( (\ )^* \) on the (cartesian) category of globular sets is cartesian. We may therefore speak of \( (S, \ast)^* \)-multicategories and operads.

Our major task now is to define the operad \( K \). \textit{En route}, I will attempt to explain what is going on in elementary terms.

**II.3 Trees**

It is instructive to contemplate the globular set \( 1^* \), where

\[
1 = (\cdots \overset{1}{\longrightarrow} \cdots \overset{1}{\longrightarrow}).
\]

The free strict \( \omega \)-category functor takes a globular set \( X \) and creates formally all possible composites in it, to make \( X^* \). Thus a typical element of \( 1^*(2) \) looks like

\[
\text{(II.C)}
\]

where each \( k \)-cell drawn represents the unique member of \( 1(k) \). Note that because of identities (which we think of throughout as nullary composites), this diagram might be thought of as representing an element of \( 1^*(n) \) for any \( n \geq 2 \). Let us call a globular diagram representing an element of \( 1^*(n) \) an \( n \)-glob. (The collections of \( m \)-globs and of \( n \)-globs are considered disjoint, when \( m \neq n \).) This 2-glob (II.C) has a source and a target, both of which are the 1-glob

\[
\text{(II.C)}
\]

(Since all cells in \( 1 \) have the same source and target—are ‘endomorphisms’—it is inevitable that the same should be true in \( 1^* \).)

Having described \( 1^* \) as a globular set, we next turn to its strict \( \omega \)-category structure: in other words, how globs may be composed.
Typical binary compositions are illustrated by

\[ \begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{binary-composition1.png}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{binary-composition2.png}
\end{array}
\end{array} \]

and

\[ \begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{binary-composition3.png}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{binary-composition4.png}
\end{array}
\end{array} \]

These compositions are possible because the sources/targets match appropriately: e.g. in the first calculation, where we are gluing along 1-cells (indicated by $\otimes_1$), the 1-dimensional parts of the two arguments are the same. A typical nullary composition—identity—is

\[ \begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{nullary-composition1.png}
\end{array}
\end{array} \rightarrow \begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{nullary-composition2.png}
\end{array}
\end{array} \]

Now in keeping with the operadic philosophy, we do not wish to restrict our attention merely to binary and nullary compositions, but rather to treat all shapes of composition even-handedly. We may think of the first binary composition above as indexed by $\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{indexed-binary-composition1.png}
\end{array}
\end{array}$, because we were composing one 2-cell with another by joining along their bounding 1-cells. The composition can be represented as

\[ \begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{indexed-binary-composition2.png}
\end{array}
\end{array} \rightarrow \begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{indexed-binary-composition3.png}
\end{array}
\end{array} \]

(II.D)
In general, the ways of composing globs are indexed by globs themselves. For instance,

\[
\begin{array}{ccccccc}
/ & \sim & / & / \sim / & ? & / \sim / & / \sim / & / \sim / & ? & / & ? & / & / \sim / & / \sim / & / \sim / & ? & / \sim / & ? & \sim O \sim O \sim O \sim O \sim (II.E)
\end{array}
\]

represents the composition

\[
\begin{array}{cc}
\left( \begin{array}{c}
\odot_1 \\
\odot_0
\end{array} \right)
\end{array}
\]

We have now described the strict $\omega$-category $1^*$. The next observation is that given a globular set $X$ and a glob $\tau$, there arises a set $X^\tau$. Formally, this is the fibre over $\tau$ in the map $X^* \to 1^*$. Informally, it’s a labelling of $\tau$ in $X$. For example, if $\tau$ is the glob of diagram (II.C) then $X^\tau$ has elements $(A, B, C, D, f, f', f'', g, h, h', \alpha, \alpha', \alpha'', \beta)$ where $A, \ldots, D \in X(0)$, $f, \ldots, h' \in X(1)$, $\alpha, \ldots, \beta \in X(2)$, and $s(\alpha) = f$, $t(\alpha) = f'$, etc., as in the picture

There is an alternative way to represent elements of $1^*(n)$, much exploited by Batanin: as trees. (These trees differ from those of Example I.1.3(f) and

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Chapter IV, in both minor details and intent.) For instance, we translate the glob into the tree. The thinking here is that the glob is 3 1-cells long, so we start the tree as; then the first column is 3 2-cells high, the second 0, and the third 1, so the tree becomes; finally, there are no 3-cells so the tree stops there. Formally, we may define an $n$-stage tree $(n \in \mathbb{N})$ to be a diagram

$$
\tau(n) \rightarrow \tau(n-1) \rightarrow \cdots \rightarrow \tau(1) \rightarrow \tau(0) = 1
$$

in the category $\Delta$ of all finite ordinals. The tree $\in 1^*(2)$ corresponds to a certain diagram $4 \rightarrow 3 \rightarrow 1$ in $\Delta$, for example; note that if $\tau$ is an $n$-stage tree with $\tau(n) = 0$ then the height of the picture of $\tau$ will be less than $n$. The source/target $\partial \tau$ of an $n$-stage tree $\tau$ is the $(n-1)$-stage tree obtained by removing all the nodes at height $n$, or formally, truncating 

$$
\tau(n) \rightarrow \tau(n-1) \rightarrow \cdots \rightarrow \tau(1) \rightarrow \tau(0)
$$

to

$$
\tau(n-1) \rightarrow \cdots \rightarrow \tau(1) \rightarrow \tau(0).
$$

It can be proved that the elements of $1^*(n)$ correspond one-to-one with the $n$-stage trees, and that $\partial$ provides the source and target (see [Bat] and the following paragraphs). Henceforth we assume this, write $\text{Tr}$ for $1^*$, and prefer the word ‘tree’ to ‘glob’.

(An equivalent definition of tree uses the fact that an $(n+1)$-stage tree is just a sequence of $n$-stage trees. Thus we may define $\text{Tr}(0) = 1$, $\text{Tr}(n+1)$ to be the free monoid on $\text{Tr}(n)$, and $\text{Tr}(n+1) \xrightarrow{\partial} \text{Tr}(n)$ to be the free monoid functor applied to $\text{Tr}(n) \xrightarrow{\partial} \text{Tr}(n-1)$.)

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As a demonstration of the tree notation, the binary compositions of page 36 look like

\[
\begin{align*}
\bigotimes_1 &\bigotimes & &\bigotimes_0 \\
&= & &_
\end{align*}
\]

and the nullary one looks like

\[
\begin{align*}
\in \mathcal{Tr}(1) &\in \mathcal{Tr}(2).
\end{align*}
\]

The indexing shape of the first binary composition is \(\in \mathcal{Tr}(2)\), and the whole composition is represented as

\[
\begin{align*}
\bigotimes_1 &\bigotimes & &\bigotimes_0 \\
&= & &_
\end{align*}
\]

The more general, tree/glob-indexed composition on page 37 translates to
representing the composition

\[
\left( \begin{array}{c}
\text{\vdots} \\
\tau_1 \\
\vdots \\
\tau_0 \\
\vdots \\
\tau_0
\end{array} \right) \otimes_0 \left( \begin{array}{c}
\text{\vdots} \\
\vdots \\
\vdots \\
\tau_0 \\
\vdots \\
\tau_0
\end{array} \right) = \left( \begin{array}{c}
\text{\vdots} \\
\tau_0 \\
\vdots \\
\tau_0 \\
\vdots \\
\vdots \\
\tau_0 \\
\vdots \\
\tau_0 \\
\vdots \\
\vdots \\
\tau_0
\end{array} \right).
\]

The foregoing considerations get us most of the way to a rigorous definition of the free strict $\omega$-category monad $(\ )^*$, as follows.

- A tree is defined as a certain kind of diagram in $\Delta$ (as above).
- For each tree $\tau$ is defined a globular set $\hat{\tau}$. This seems to be most easily done with the the second, free-monoid, definition of tree, but we omit the details. (Or see [Bat], where $\hat{\tau}$ is called $T^\tau$.) Pictorially, we take a tree such as

\[
\begin{array}{c}
\text{\vdots} \\
\vdots \\
\tau \\
\vdots \\
\vdots \\
\vdots \\
\tau \\
\vdots \\
\vdots \\
\vdots
\end{array}
\]

and draw a corresponding globular diagram

\[
\begin{array}{c}
A \\
\downarrow f \\
B \\
\downarrow g \\
C \\
\downarrow h \\
D
\end{array}
\]

where $A, f, \beta, \ldots$ are formal symbols. This in turn yields a globular set

\[
\hat{\tau} = (\cdots \emptyset \Rightarrow \{\beta, \beta', \beta'', \gamma, \gamma'\} \Rightarrow \{f, g, g', g'', h, h', h''\} \Rightarrow \{A, B, C, D\}).
\]

- Let $X$ be a globular set: then $X^\tau$ is defined as the set of maps $\hat{\tau} \to X$ of globular sets.
- Finally, define $X^*(n) = \coprod_{\tau \in \text{Tr}(n)} X^\tau$.

With the rest of the details in place, it can then be shown that $(\ )^*$ deserves the name ‘free strict $\omega$-category monad’, and in particular that $\text{Tr}$ is the free strict $\omega$-category on $1$.

**II.4 Operads and Algebras**

It has already been stated that if $S$ is the category of globular sets and $^*$ the free strict $\omega$-category monad, then $(S,^*)$-operads and their algebras are what Batanin calls operads and algebras.\(^1\) We now explain these in elementary terms.

---

\(^1\)In fact, [Bat] is presented in the wider context of monoidal globular categories, but this need not concern us here. The Batanin operads referred to here are sometimes given the fuller name of ‘$\omega$-operads in $\text{Span}$’ in [Bat].
A collection is a graph

\[
\begin{array}{c}
\text{Tr} = 1^* \\
\end{array}
\]

in \( S \). In other words, a collection consists of a set \( C(\tau) \) for each tree \( \tau \), together with a pair of functions \( C(\tau) \xrightarrow{s} C(\partial \tau) \), satisfying the usual globularity relations \( ss = st \) and \( ts = tt \). An operad structure on the collection \( C \) consists of identities and compositions with suitable properties. The identities consist of an element of \( C(v_n) \) for each \( n \), where \( v_n \in \text{Tr}(n) \) is the tree

\[
\vdots \\
n \text{ edges} \\
\]

(corresponding to the glob which is a single \( n \)-cell). For composition in \( C \), consider a diagram

\[
\tau_1 = \text{Diagram} \\
\tau_2 = \text{Diagram} \\
\tau_3 = \text{Diagram} \\
\tau = \text{Diagram} \\
f_1, f_2, f_3 \text{ match suitably on their sources and targets (e.g. } t(f_1) = s(f_2)) \text{. Then composition should produce from this data a diagram}
\]

\[
\tau_o(\tau_1, \tau_2, \tau_3) = \text{Diagram} \\
\]

i.e. an element of \( C(\tau_o(\tau_1, \tau_2, \tau_3)) \).

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According to the general theory, composition is a map $C \circ C \to C$ over $\mathbf{Tr}$, where $C \circ C$ is given by the diagram

$$
\begin{array}{ccc}
\mu \downarrow & & \downarrow \mu_1 \\
\mathbf{Tr} & \to & \mathbf{Tr} \\
\mu_1 \downarrow & & \downarrow 1^{**} \\
C & \xleftarrow{d} & C^* \\
\downarrow d & & \downarrow !^* \\
\mathbf{Tr} & \xleftarrow{!} & \mathbf{Tr} \\
\end{array}
$$

To see what a typical element of $C \circ C$ is, consider $\tau$ above, an element of $\mathbf{Tr}(2)$. In the map $C^* \to \mathbf{Tr}$, a typical element of the fibre over $\tau$ is the left-hand half of diagram (II.F). In the map $C \to \mathbf{Tr}$, a typical element of the fibre over $\tau$ is the right-hand half of (II.F). The map $\mu_1 \circ d \circ \text{pr} : C \circ C \to \mathbf{Tr}$ sends (II.F) to the tree $\tau \circ (\tau_1, \tau_2, \tau_3)$. So, as hoped for above, composition takes data such as (II.F) and produces an element of $C(\tau \circ (\tau_1, \tau_2, \tau_3))$. This composition is moreover required to be associative and to obey unit laws with respect to the identities. For example, if we have a diagram

$$
\begin{array}{ccc}
& f_{13} & \\
& f_{12} & 1 \\
& f_{11} & \\
\downarrow f_2 & & \downarrow f_1 \\
& f_1 & \\
\downarrow f_{12} & & \downarrow f_{13} \\
& f_{11} & \\
\end{array}
$$

of the kind (II.F), then

$$
f \circ (f_1 \circ (f_{11}, f_{12}, f_{13}), f_2) = (f \circ (f_1, f_2)) \circ (f_{11}, f_{12}, f_{13}, 1).
$$

We have now seen that an operad consists of a set $C(\tau)$ for each $\tau$, with source and target functions, and compositions between the $C(\tau)$'s according to the compositions for trees. Think of an element of $C(\tau)$ as a way of composing a diagram shaped like the glob $\tau$; then it makes sense that an algebra for $C$ should be a globular set $X$ with for each $f \in C(\tau)$ a function $X^\tau \xrightarrow{f} X(n)$, such that composition of these functions $f$ commutes with composition in the
operad. So for instance, if

\[ f \in C \]

and

is a picture in \( X \), then \( \bar{f} \) composes this picture to give a 2-cell of \( X \). This is indeed what the general theory says an algebra is. For an algebra structure on \( X \) is a suitable map \( \xrightarrow{h} X \), where \( X \) is the pullback

\[
\begin{array}{c}
\xymatrix{
X^* \ar[r]^h \ar[dr] & X \ar[d] \ar[dl] \\
& C \\
\text{Tr} \ar[u] & \end{array}
\]

this means that \( X_*(n) = \prod_{\tau \in \text{Tr}(n)} (X^* \times C(\tau)) \) and that \( h \) is a sequence of functions \( \left( \prod_{\tau \in \text{Tr}(n)} (X^* \times C(\tau)) \rightarrow X(n) \right)_{n \in \mathbb{N}} \) obeying the usual laws.

We have now discussed what operads and their algebras look like, and it is time to come to the main point of the chapter.

II.5 Contractions and Lax \( n \)-Categories

This section formalizes the ‘contraction principle’ of II.1, allowing us to define the operad \( K \) whose algebras are the lax \( \omega \)-categories.

If \( C \) is a collection, \( \sigma \) a tree, and \( f, f' \in C(\sigma) \), let us say \( (f, f') \) is a matching pair for \( \sigma \) if \( s(f) = s(f') \in C(\partial \sigma) \) and \( t(f) = t(f') \in C(\partial \sigma) \). Here it is intended that if \( \sigma \) is the 0-stage tree then any pair \( f, f' \in C(\sigma) \) is matching.
Definition II.5.1 Let \((C \xrightarrow{\psi} \text{Tr})\) be a collection. A contraction \(\psi\) on \(C\) is a function assigning to each \(\tau \in \text{Tr}(n)\) \((n \geq 1)\) and matching pair \((f, f')\) for \(\partial \tau\), an element \(\psi(f, f') \in C(\tau)\) such that \(s(\psi(f, f')) = f\) and \(t(\psi(f, f')) = f'\).

We use the category of operads-with-contraction, in which an object is an operad with a specified contraction (on the underlying collection), and the maps are the operad maps preserving the specified contractions.

It should be apparent that the operad for lax \(\omega\)-categories, described informally in II.1, carries a contraction. Indeed, the presence of the contraction was all that we used to generate the operad: thus we define \(K\) to be the initial operad-with-contraction. It is of course necessary to verify that such an initial object exists. Here it is hoped that the reader will be content with the heuristic argument laid out in II.1, where starting from the empty collection it was suggested how the operad and contraction structures could be added in freely.

Let us say a globular set \(X\) is \(n\)-dimensional if for all \(m \geq n\),

\[
s = t : X(m + 1) \longrightarrow X(m)
\]

and this map is an isomorphism.

Definition II.5.2 A lax \(\omega\)-category is an algebra for \(K\). A lax \(n\)-category is a lax \(\omega\)-category whose underlying globular set is \(n\)-dimensional.

We remark that despite being able to make this definition, there is no immediately obvious way to define lax functors, transformations, \ldots. In particular, the category of \(K\)-algebras consists of lax \(\omega\)-categories and strict \(\omega\)-functors.

Our final observation is that the terminal operad \((\text{Tr} \xrightarrow{1} \text{Tr})\) has a unique contraction on it, and is then the terminal operad-with-contraction. As always, the algebras for the terminal operad are the algebras for the monad \(\ast\) on \(S\) (see I.3.2(e)). So the algebras for the terminal operad-with-contraction are the strict \(\omega\)-categories, and the algebras for the initial operad-with-contraction are the lax \(\omega\)-categories. One might therefore hope that in some sense the category of operads-with-contraction indexes the different possible strengths for theories of \(\omega\)-categories.

II.6 The Cubical Approach

2-categories, strict and lax, are not the only 2-dimensional generalization of categories: there is also the notion of a double category. This section presents the beginnings of an approach to \(n\)-tuple categories. It runs in very close parallel with our approach to \(n\)-categories, hence its inclusion in this chapter.

A double category may be defined as a category object in \(\text{Cat}\). More descriptively, the graph structure consists of collections of

- 0-cells \(A\)
- horizontal 1-cells \(f\)
vertical 1-cells $p$

2-cells $\alpha$

and various source and target functions, as illustrated by the picture

\[
\begin{array}{cccc}
A_1 & f_1 & A_2 \\
p_1 & & p_2 \\
A_3 & f_2 & A_4
\end{array}
\]

The category structure consists of identities and composition functions for 2-cells and both kinds of 1-cell, obeying strict associativity, identity and interchange laws; see [KS] for more details. It is worth keeping in mind that a 2-category is a double category in which all the vertical 1-cells are identities, and similar degeneracy properties should hold in higher dimensions.

More generally, let us define $n$-cubical set for any $n \in \mathbb{N}$; the intention is that a 2-cubical set will be the underlying graph of a double category. So, let $\text{Cube}_n$ be the category with

**objects:** subsets $D$ of \{0, 1, $\ldots$, $n - 1$\}

**maps** $D \longrightarrow D'$: the inclusion $D \subseteq D'$, together with a function $D \setminus D \longrightarrow \{0, 1\}$

(in which context 0 should be read as ‘source’ and 1 as ‘target’)

**composition:** place functions side-by-side.

Then we define an $n$-cubical set to be a functor $\text{Cube}_n^{op} \longrightarrow \text{Set}$. For instance, we may think of a 2-cubical set $X$ as:

- $X\emptyset = \{0$-cells$\}$
- $X\{0\} = \{\text{horizontal 1-cells}\}$
- $X\{1\} = \{\text{vertical 1-cells}\}$
- $X\{0, 1\} = \{2$-cells$\}$

and, for instance, the map $\{1\} \longrightarrow \{0, 1\}$ given by

\[
\{0, 1\} \setminus \{1\} = \{0\} \overset{0}{\longrightarrow} \{0, 1\}
\]

sends $\alpha \in X\{0, 1\}$ to $p_1 \in X\{1\}$, in the diagram above.

The category $\mathcal{S}$ of $n$-cubical sets is a presheaf category, therefore cartesian. A strict $n$-tuple category is an $n$-cubical set together with various compositions and identities, as for double categories, all obeying strict laws. The free strict $n$-tuple category $X^*$ on an $n$-cubical set $X$ consists of all formal composites of
cells in $X$. In particular, cells of $1^*$ look like cuboids. For example, a typical element of $1^*\{0,1,2\}$ may be depicted as

(Where the device on the right is just a ‘compass’ to show which element of $\{0,1,2\}$ points in which direction). Equally, this cuboid could be used to illustrate an element of $1^*(D)$ for any $D \supseteq \{0,1,2\}$. Composition in $1^*$ is indicated by

![Diagram of composition](image)

(II.G)

The indexing shape of this composition being

![Indexing Shape](image)

(II.H)

(cf. the discussion of indexing on page 36). Given an $n$-cubical set $X$ and a cuboid $\tau \in 1^*(D)$, there arises a set $X^\tau$. Formally, this is the fibre over $\tau$ in the map $X^* \to 1^*$; informally, $X^\tau$ consists of ways of labelling the cuboid $\tau$ in $X$.

In parallel to the representation of a glob as a diagram $\tau(n) \to \cdots \to \tau(0)$ in $\Delta$, a cuboid can be represented as its sequence of edge-lengths. So, define a $D$-cuboid $(D \subseteq \{0,1,\ldots,n-1\})$ to be a map $D \to N$. To each cuboid $\tau$ is associated an $n$-cubical set $\hat{\tau}$: e.g. if $n = 2$ and $\tau$ is the cuboid (II.H) above then $\hat{\tau}$ is a certain 2-cubical set with 6 2-cells, 8 horizontal 1-cells, 9 vertical 1-cells and 12 0-cells. Then define $X^\tau$ to be the set of maps $\hat{\tau} \to X$, for any $n$-cubical set $X$, and define $X^*(D) = \bigsqcup_{(D \text{-cuboids } \tau)} X^\tau$. This goes most of the way towards a rigorous definition of the functor $(\ )^*$, the monad structure of which can then be described; the resulting monad $^*$ on $\mathcal{S}$ is cartesian. (The manipulations involved are rather easier here than in the globular setting, because we have cuboids instead of trees.)
($S,^*$)-operads can be understood in much the same way as Batanin’s operads. A Batanin operad associates to each tree a set, and has composition functions corresponding to the composition of trees; a cubical operad associates to each cuboid a set, and has composition functions corresponding to the composition of cuboids. If $C$ is an operad and $\tau$ a cuboid then an element of $C(\tau)$ can be thought of as a way of composing a diagram of shape $\tau$; thus an algebra for $C$ consists of a cubical set $X$ together with suitably compatible functors $X^\tau \xrightarrow{f} X(D) \ (f \in C(\tau), \ \tau \in \mathbf{1}^*(D))$. (Note that the restriction to finite sets $\{0, 1, \ldots, n - 1\}$ has been quite unnecessary, and was only done for the sake of simplicity.)

The final step of the analogy would be to develop an appropriate notion of contraction in the cubical setting, in order to define lax $n$-tuple categories. This has not, as far as I know, been carried out.
Chapter III

Gray-Categories

Gray-categories are significant because of their place in coherence for tricategories, itself perhaps the most challenging feature of the higher-dimensional landscape. Whereas any bicategory is biequivalent to a 2-category, it is not true that any tricategory is triequivalent to a 3-category. Instead, as revealed in [GPS], any tricategory is triequivalent to a Gray-category.

In this chapter we take the view that Gray-categories are structures in their own right. It is only by making a non-canonical choice that we can embed the class of Gray-categories in the class of tricategories, and so speak of Gray-categories as a particular kind of tricategory, as in the coherence theorem just stated. An important case of this is that 2-categories (and homomorphisms, ...) naturally form a Gray-category, not a tricategory; this boils down to the fact that there is no canonical way to define the horizontal composite of strong transformations. Similarly, bicategories do not naturally form a tricategory, only a ‘near-Gray-category’. This raises questions over the conventional wisdom that lax $n$-categories should form a lax $(n + 1)$-category.

The necessity of this non-canonical choice seems intimately related to the coherence issue. Braided monoidal categories (BMC’s; see [JS]) are a case in point. Any BMC ‘is’ (after some non-canonical choices) a tricategory with one 0-cell and one 1-cell. Equivalences of BMC’s then correspond to triequivalences of their tricategories. Moreover, the property of a braiding being a symmetry is preserved under BMC-equivalence, and the braiding is a symmetry if the tricategory is a 3-category. It follows that any BMC whose braiding is not a symmetry provides an example of a tricategory not triequivalent to a 3-category. (It doesn’t quite follow immediately as the 3-category needn’t be a BMC; see [GPS, chapter 8].) Now braidings on a given monoidal category come in pairs $(\gamma_{AB}, \gamma_{BA}^{-1} : A \otimes B \to B \otimes A$, so to speak); to choose one particular braiding over its partner is non-canonical. The exception is when the braiding is equal to its partner, which happens exactly when the braiding is a symmetry. It is hard not to wonder if this non-canonical choice is related to the non-canonical choice involved (above) for Gray-categories, particularly given the nature of the latter choice. However, these connections remain mysterious, so we leave the matter
there.

The Cayley representation functor for categories has an obvious analogue in bicategories and tricategories. We examine its properties, and thus obtain a representation theorem for Gray-categories: any small Gray-category ‘is’ a substructure of \( \mathbf{2-Cat} \). This means, of course, that any manipulation of diagrams in a general Gray-category might as well be done in \( \mathbf{2-Cat} \) (in the same sense as is true for categories and \( \mathbf{Set} \)); however, once one knows the definition of a Gray-category, it is probably less distracting to work in the general context.

A more interesting aspect of our representation theorem is its relation to the coherence theorem. This leads us to make a conjecture on coherence theorems for higher-dimensional categories.

Section III.1 is on sesquicategories, which provide our route into defining a Gray-category (III.2). We exhibit \( \mathbf{2-Cat} \) as a Gray-category in III.3, and any Gray-category as a tricategory (non-canonically) in III.4. In III.5 we pause to gather some definitions, before moving on to the Cayley characterization of Gray-categories (III.6) and a conjecture on coherence in higher dimensions (III.7).

### III.1 Sesquicategories

In short, a sesquicategory consists of a category \( \mathcal{C} \) and a factorization

\[
\begin{array}{ccc}
\mathcal{C}^{\text{op}} \times \mathcal{C} & \longrightarrow & \text{Cat} \\
\downarrow \text{Hom} & & \downarrow \text{ob} \\
\text{ob} & \longrightarrow & \text{Set}
\end{array}
\]

Thus for each pair of objects \( A, B \in \mathcal{C}_0 \) there is a category \( \mathcal{C}(A, B) \) whose objects are the morphisms from \( A \) to \( B \). We may think of the objects of \( \mathcal{C} \) as 0-cells, the morphisms in \( \mathcal{C} \) as 1-cells, and the morphisms in the categories \( \mathcal{C}(A, B) \) as 2-cells. For instance, any 2-category has an underlying sesquicategory.

The definition of a sesquicategory can be recast as follows. The graph structure of a sesquicategory \( \mathcal{C} \) is the same as that of a bicategory: that is, a globular set

\[
\begin{array}{ccc}
\mathcal{C}_2 & \longrightarrow & \mathcal{C}_1 \\
\longrightarrow & \longrightarrow & \longrightarrow \\
\mathcal{C}_1 & \longrightarrow & \mathcal{C}_0
\end{array}
\]

truncated at dimension 2. For 1-cells there are binary compositions and identities,

\[
\begin{array}{ccc}
\bullet & \longrightarrow & \bullet \\
\bullet & \longrightarrow & \bullet
\end{array}
\]

and

\[
\begin{array}{ccc}
\bullet & \longrightarrow & \bullet
\end{array}
\]
and for 2-cells there are vertical binary compositions and identities,

\[
\begin{array}{c}
  \bullet \\
  \downarrow \\
\end{array}
\quad \mapsto
\begin{array}{c}
  \\
  \downarrow \\
\end{array}
\]

and

\[
\begin{array}{c}
  \bullet \\
\end{array}
\quad \mapsto
\begin{array}{c}
  \downarrow \\
\end{array}
\]

Finally, there are compositions as illustrated by

\[
\begin{array}{c}
  f \\
  \downarrow \alpha \\
\end{array}
\quad \mapsto
\begin{array}{c}
  \alpha f \\
  \downarrow \\
\end{array}
\]

and

\[
\begin{array}{c}
  \alpha \\
  \downarrow f \\
\end{array}
\quad \mapsto
\begin{array}{c}
  f \alpha \\
  \downarrow \\
\end{array}
\]

(This last kind of composition arises because if \( B \xrightarrow{\alpha} B' \) in \( \mathbb{C} \) then the factorization \( \mathbb{C}^{op} \times \mathbb{C} \xrightarrow{\text{Cat}} \mathbb{C} \), \( \text{Cat} \) gives a functor \( \mathbb{C}(A, B) \xrightarrow{\text{Cat}} \mathbb{C}(A, B') \).) The compositions and identities are subject to various axioms. 1-cell composition and vertical 2-cell composition are to obey associative and identity laws—in other words, the 0- and 1-cells must form a category, as must the 1- and 2-cells. The remaining axioms are indicated in Figure III.a. There, the last two diagrams are binary and left-handed in character; we should really have drawn six more to cover the nullary and/or right-handed versions too.

The crucial difference between a sesquicategory and a 2-category is that in a sesquicategory there is no specified horizontal composition of 2-cells. We can derive one, according to the picture

\[
\text{(III.A)}
\]

(cf. page 36, diagrams (II.D) and (II.E)), but equally we could have used the symmetrically opposite version, and these two horizontal compositions are not
Figure III.a: Some of the axioms for a sesquicategory. The first diagram, for instance, says that \((g\alpha)f = g(\alpha f)\).
the same. So if

\[ \begin{array}{c}
\alpha \\
\beta \\
\end{array} \]

\[ \begin{array}{c}
f \\
g \\
\end{array} \]

is a diagram in a sesquicategory then there are 2-cells

\[ \begin{array}{c}
gf \\
\beta f \downarrow \alpha \downarrow g' \\
gf' \downarrow \\
\end{array} \] and \[ \begin{array}{c}
gf \\
\alpha \downarrow \beta f' \\
gf' \downarrow \\
\end{array} \]

but in general they will be different. We return to this point repeatedly.

Our elementary description of sesquicategories allows us to realise them as algebras for a certain Batanin operad. In this chapter we will be concerned with finite-dimensional structures, so use \( n \)-globular sets

\[ X(n) \rightarrow X(n-1) \rightarrow \cdots \rightarrow X(0) \]

and \( n \)-operads (that is, \((S, \ast)\)-operads where \( \ast \) is the free strict \( n \)-category monad) rather than the \( \omega \)-versions favoured in Chapter II. The 2-operad \( \text{Ssq} \), whose algebras are sesquicategories, is generated by a single element of \( \text{Ssq}(\tau) \) for the tree \( \tau \) taking any one of the values

\[ \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\end{array} , \quad \bullet \quad \text{(1-stage trees)}, \]

\[ \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\end{array} , \quad \text{(1-stage trees)}, \]

\[ \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\end{array} , \quad \text{(2-stage trees)}, \]

subject to the relations described above. That it is possible to specify an operad by generators and relations is established in [Bat]; given that (by I.5) the category of operads is monadic over the category of collections, itself a presheaf category, this does not seem surprising.

**III.2 The Definition of a Gray-Category**

The original definition ([GPS]) of a Gray-category was as a category enriched in \( \text{Gray} \), where \( \text{Gray} \) is the category of 2-categories with a certain symmetric
monoidal structure. Here we ignore this definition, favouring the equivalent one of [Bat, p. 59], and therefore write ‘Gray-category’ rather than ‘Gray-category’.

Gray-categories are defined to be algebras for the 3-operad \( \text{Gy} \). This in turn is defined to be the 3-operad got by taking the 2-operad \( \text{Ssq} \) and contracting from dimension 2 to dimension 3. In other words, for \( \tau \in \text{Tr}(n) \),

\[
\text{Gy}(\tau) = \begin{cases} 
\text{Ssq}(\tau), & n \in \{0, 1, 2\}, \\
\text{Ssq}(\partial\tau) \times \text{Ssq}(\partial\tau), & n = 3.
\end{cases}
\]

(Note that if \( \tau \) is a 3-stage tree then \( \text{Ssq}(\partial^2\tau) \) has just one element; thus any two elements of \( \text{Ssq}(\partial\tau) \) automatically match at their source and target.) The operad structure on \( \text{Gy} \) is uniquely determined by saying that the 2-operad it restricts to is \( \text{Ssq} \).

The underlying graph of a Gray-category is therefore a 3-globular set. In dimensions 0–2 there are the same compositions as in \( \text{Ssq} \), obeying the same laws. A way of composing a 3-cell diagram such as

![Diagram](image)

to a single 3-cell is given exactly as a way of composing up its 2-dimensional source to a single 2-cell and a way of composing its 2-dimensional target to a single 2-cell. In particular, this holds when the 3-cell diagram is a degenerate one such as

![Diagram](image)

so if

![Diagram](image)

is a diagram in a Gray-category then there is a designated 3-cell
where $\gamma_1 = \beta f' \circ g \alpha$ and $\gamma_2 = g' \alpha \circ \beta f$ (the two horizontal composites mentioned in III.1). Similarly, there is a designated 3-cell $\gamma_2 \rightarrow \gamma_1$; since elements of

\[
\text{Gy}
\]

are determined by their source and target, the two 3-cells are mutually inverse. Thus $\gamma_1$ and $\gamma_2$ differ by a (canonical) invertible 3-cell.

Having given this near-elementary description of what a Gray-category is, we will feel free to speak of Gray-categories even when the underlying graphs are large.

### III.3 2-Cat as a Gray-Category

Consider all small 2-categories and all homomorphisms, strong transformations and modifications between them: what kind of structure do they form? Our answer is 'a Gray-category'. As we shall see in the next section, they also form a tricategory, but only in a non-canonical way.

Recall that homomorphisms of bicategories can be composed, and that this composition obeys strict associativity and identity laws. (For the proof of this see [Bén]; for bicategory terminology see the preliminary chapter.) Next, strong transformations can be composed vertically ($\bullet \bullet \bullet$); in general the associativity and identity laws for this composition hold only up to invertible modification, but if the codomain bicategory is a 2-category then they hold strictly. There is no canonical horizontal composition of strong transformations, even when we are only dealing with 2-categories, as we shall discuss later. However, a strong transformation can be composed canonically with a homomorphism on either side—

---

—and the laws indicated in Figure III.a (page 51) then hold. It follows that 2-categories, homomorphisms and strong transformations form a sesquicategory.

We next add in modifications to make 2-Cat into a Gray-category. Modifi-
cations can be composed in all the plausible ways:

and these compositions behave as well as is allowed by the compositions of the transformations that bound them. For instance, there are four obvious ways to compose a diagram

into a single modification, as there are two obvious ways to compose the source transformations

and similarly the target. The point is that the data for a modification consists just of 2-cells, and a diagram of 2-cells in a bicategory either commutes or it doesn’t—there’s no possibility of it ‘commuting up to isomorphism’. To supply a proof that 2-Cat forms a Gray-category would require either an ad hoc inductive argument or a finite axiomatization of Gray-categories,\(^1\) which we could then check 2-Cat against.

We have argued that 2-Cat forms a Gray-category, but in fact it’s almost true that Bicat (=bicategories, homomorphisms, strong transformations and modifications) forms one too. The only obstacle is that a Gray-category \(C\) is required to be locally a 2-category: that is, for each pair of 0-cells, \(C(A,B)\) must be a 2-category. This holds in 2-Cat because the vertical composition of transformations obeys strict associativity and identity laws there. But suppose we take the axioms for a sesquicategory and drop the associativity and identity axioms for vertical composition of 2-cells: then we obtain a more general notion of Gray-category, which we call a near-Gray-category. Thus Bicat forms a near-Gray-category. The difference between near-Gray-categories and Gray-categories may be regarded as inessential because of the coherence theorem for bicategories. By analogy, Gordon, Power and Street prove that any tricategory \(T\) is triequivalent to one which is locally a 2-category, by replacing each bicategory \(T(A,B)\) with a biequivalent 2-category. The next section shows how (near-)Gray-categories can, in fact, be realised as tricategories.

\(^1\)Such an axiomatization appears to have been carried out—see [Cra, section 2]—but I have not investigated this.
III.4 Gray-Categories as Tricategories

A tricategory is defined in [GPS] in the same style as a bicategory is defined in [Bén, 1.1]. That is, a tricategory $\mathcal{T}$ consists of a collection $\mathcal{T}_0$ of objects, a bicategory $\mathcal{T}(A,B)$ for each pair $(A,B)$ of objects, composition and identity homomorphisms $\mathcal{T}(B,C) \times \mathcal{T}(A,B) \to \mathcal{T}(A,C)$ and $1 \to \mathcal{T}(A,A)$, and then various coherence cells satisfying some axioms. In particular, there is a specified horizontal composition of 2-cells. This feature is lacking in Gray-categories, and we saw on page 53 that there are two symmetrically opposite ways of deriving such a composition, so in order to turn a Gray-category into a tricategory we need to make a non-canonical choice. Let us make this choice once and for all: define the horizontal composite $\beta \ast \alpha$ of

$$
\begin{array}{c}
\begin{tikzpicture}
\begin{scope}[xshift=-0.5cm]
\node (a) at (0,0) {$f$};
\node (b) at (0,2) {$g$};
\node (c) at (0,4) {$\beta$};
\node (d) at (0,6) {$\alpha$};
\node (e) at (-1,0) {$f'$};
\node (f) at (-1,2) {$g'$};
\end{scope}
\end{tikzpicture}
\end{array}
$$

as $\beta f' \circ g \alpha$, as in diagram (III.A) on page 50. It is routine to verify that every Gray-category then becomes a tricategory.

Contractions provide a way of discussing this point. To do this we must first note that this chapter’s approach to finite-dimensional structures (page 52) varies from that in Chapter II: there, a lax $n$-category was a lax $\omega$-category whose graph was trivial beyond dimension $n$, whereas here we never even contemplate dimensions beyond $n$. If $C$ is an $n$-collection, define a contraction on $C$ to be a function $\psi$ just as before (page 44) for $\tau$ in dimensions 1 to $n$, together with the condition that if $\sigma$ is an $n$-stage tree and $(f,f')$ a matching pair for $\sigma$ then $f = f'$. We assume (cf. [Bat]) that lax $n$-categories are the algebras for $K_n$, the initial $n$-operad-with-contraction. For now all we actually need to know is that $K_3$-algebras are ‘the same as’ small tricategories, which the informal argument of II.1 makes seem plausible. The phrase ‘the same as’ must be qualified by the fact that in a $K_3$-algebra there are specified compositions of arbitrary shape and arity, whereas in a tricategory there are just a few, all binary or nullary. But this is just the biased-unbiased distinction of page 9 and ought to be harmless.

Now, any contraction on the 3-operad $\text{Gy}$ gives an operad map $K_3 \to \text{Gy}$, since $K_3$ is initial. This in turn yields a functor $\text{Alg}(\text{Gy}) \to \text{Alg}(K_3)$, which on objects sends each Gray-category to a tricategory. But a contraction on $\text{Gy}$ entails a choice of an element of $\text{Gy}(\tau_n)$ for each $n$, where

$$
\tau_n = \begin{array}{c}
\begin{tikzpicture}
\begin{scope}[xshift=-0.5cm]
\node (a) at (0,0) {\cdot \cdot \cdot}
\node (b) at (0,3) {\cdot \cdot \cdot}
\node (c) at (0,6) {\cdot \cdot \cdot}
\node (d) at (0,9) {\cdot \cdot \cdot}
\node (e) at (-1,0) {\cdot \cdot \cdot}
\node (f) at (-1,2) {\cdot \cdot \cdot}
\node (g) at (-1,4) {\cdot \cdot \cdot}
\node (h) at (-1,6) {\cdot \cdot \cdot}
\end{scope}
\end{tikzpicture}
\end{array}
$$

and we know there is no canonical way to do that. What we did at the end of the first paragraph of this section was to choose a particular element of
\( \text{Gy}(\tau_2) \), and that was enough to turn Gray-categories into tricategories because we were using traditional, ‘biased’ tricategories. There are now infinitely many choices, but from our chosen element of \( \text{Gy}(\tau_2) \) we extract one of the two most obvious choices and settle on that. For instance, the chosen element of \( \text{Gy}(\tau_4) \) is obtained by vertical composition according to the diagram

![Diagram](image)

We have shown that any Gray-category is a tricategory, albeit non-canonically, so \( \textbf{2-Cat} \) is a tricategory. In fact, everything above holds just as well for near-Gray-categories, so \( \textbf{Bicat} \) is a tricategory too. We will now see exactly why \( \textbf{2-Cat} \) is an example of a Gray-category which is not canonically a tricategory, as asserted all along. All this says is that there is no canonical way to define the horizontal composite of strong transformations. So, take a diagram

![Diagram](image)

of 2-categories, homomorphisms and strong transformations. The components of a putative \( \sigma' \ast \sigma \) are 1-cells \( (\sigma' \ast \sigma)_B : F'FB \rightarrow G'GB \) in \( B'' \), and the different routes round the square

![Diagram](image)

provide the two possibilities. If \( \sigma' \) were strict then this square would commute and we would be home, but as \( \sigma' \) is only strong there’s just an isomorphism
between the two possibilities. Making our standard choice, we obtain a tricategory $\textbf{2-Cat}$, and similarly $\textbf{Bicat}$. The other possible tricategory $\textbf{Bicat}$ is triequivalent to this one ([GPS, p. 7]).

We finally record that when $\mathcal{T}$ is a tricategory, we will use the phrase ‘$\mathcal{T}$ is a [near]-Gray-category’ to mean that $\mathcal{T}$ is equal to the tricategory arising from some [near]-Gray-category. Amongst all tricategories, the class of [near]-Gray-categories is closed under strict isomorphism (=invertible strict homomorphism) but not plain isomorphism (=invertible homomorphism), or certain dualities.

### III.5 Homomorphisms and Triequivalence

Before moving on to Cayley representation, we need some more definitions.

Objects in a tricategory have the chance of being equal, isomorphic, equivalent or biequivalent (or, of course, none of these); 1-cells can be equal, isomorphic or equivalent; 2-cells can be equal or isomorphic; 3-cells can only be equal or not. A homomorphism of tricategories is a map of their underlying graphs such that all kinds of composition are preserved up to the weakest type of equivalence, and with specified cells providing those equivalences—e.g. if $A \xrightarrow{f} B \xrightarrow{g} C$ is a diagram of 1-cells in the domain category, there’s a specified equivalence between $Fg \circ Ff$ and $F(g \circ f)$. This is usually expressed ([GPS]) by saying that a homomorphism $F : \mathcal{T} \rightarrow \mathcal{T}'$ consists of a function $F_0 : \mathcal{T}_{0} \rightarrow \mathcal{T}'_0$ on the object-collections, a homomorphism $F_{AB} : \mathcal{T}(A,B) \rightarrow \mathcal{T}'(F_0A,F_0B)$ of bicategories for each $A$ and $B$, and various extra data and axioms.

We can now say what a triequivalence is: a homomorphism $F : \mathcal{T} \rightarrow \mathcal{T}'$ which is locally a biequivalence and is surjective-up-to-biequivalence on objects. The first condition means each $F_{AB}$ is a biequivalence of bicategories, and the second that for each $A' \in \mathcal{T}'_0$ there exists $A \in \mathcal{T}_{0}$ such that $FA$ is biequivalent to $A'$ (in $\mathcal{T}'$). Unsurprisingly, this definition of a triequivalence amounts to the same thing as a ‘pseudo-inverse’ definition, as in the 1- and 2-dimensional cases (see page 8 and [GPS, 3.5]). In particular, triequivalence is a symmetric relation.

### III.6 The Cayley Characterization of Gray-Categories

We know $\textbf{2-Cat}$ is a Gray-category, so any sub-tricategory of $\textbf{2-Cat}$ is also a Gray-category. Here we establish a converse, that every small Gray-category is strictly isomorphic to a sub-tricategory of $\textbf{2-Cat}$. So the small Gray-categories are characterized as the small sub-tricategories of $\textbf{2-Cat}$, up to strict isomorphism.

To achieve this converse we use the Cayley homomorphism $\text{Cay} : \mathcal{T} \rightarrow \textbf{Bicat}$, sending an object $A$ of a small tricategory $\mathcal{T}$ to $\coprod_{B \in \mathcal{T}_{0}} \mathcal{T}(B,A)$. We show that if $\mathcal{T}$ is a Gray-category then

a. the image of $\text{Cay}$ lies in $\textbf{2-Cat}$
b. **Cay** is injective (in each of the 4 dimensions) on the underlying graphs

c. **Cay** is a strict homomorphism.

From (b) and (c) it follows that the image of **Cay** is closed under taking identities, composites and coherence cells in **Bicat**. Thus the image of **Cay** is a sub-tricategory of **Bicat**—and in fact of **2-Cat**, by (a). Hence **Cay** provides a strict isomorphism from \( T \) to a sub-tricategory of **2-Cat**.

We first describe the Cayley homomorphism. Given a 3-cell

\[
\begin{array}{ccc}
A & \xrightarrow{\alpha} & A' \\
\downarrow{f} & & \downarrow{g} \\
\end{array}
\]

in a small tricategory \( T \), there arises a diagram

\[
\begin{array}{ccc}
\prod_{B \in T_0} T(B, A) \xrightarrow{\alpha_x} \prod_{B \in T_0} T(B, A') \\
\alpha_x \\
\end{array}
\]

in **Bicat**, which is (by definition) the image of the original 3-cell under **Cay**. Just as for the Yoneda embedding of a bicategory, the coherence data for the homomorphism is provided by the coherence data for \( T \). For instance, if \( A \xrightarrow{f} A' \xrightarrow{f'} A'' \) in \( T \) then the equivalence between the homomorphisms \( f' \circ f \) and \( (f'f)_* \) in **Bicat** is provided by the associativity equivalence in \( T \).

Now suppose \( T \) is a Gray-category, and let us establish claims (a), (b) and (c).

a. **The image of Cay lies in 2-Cat.** All this says is that \( T \) is locally a 2-category, i.e. the vertical composition of 2-cells obeys strict associativity and identity laws. This is precisely the difference between a Gray-category and a near-Gray-category. (Since we are treating \( T \) as a tricategory, rather than a Gray-category as a structure in its own right, we should not really just say ‘composition obeys the associativity law’ but that ‘the associativity isomorphism for composition is the identity’. The latter statement is part of the definition of the tricategory arising from a Gray-category, although we didn’t say it explicitly.)

b. **Cay is injective on the underlying graphs.**

**0-cells** By convention, \( T(B, A) \) and \( T(B', A') \) are disjoint unless \( B = B' \) and \( A = A' \). Thus if \( A \neq A' \) then \( 1_A \notin \text{Cay}(A) \setminus \text{Cay}(A') \).
1-cells Take $A \xrightarrow{f} B$ in $T$. If $Cay(f) = Cay(g)$ then $f1_A = g1_A$; 1-cells in a Gray-category obey the unit law $h1 = h$, so $f = g$.

2-cells Take $A \xrightarrow{\alpha} B$ in $T$. If $\alpha_* = \tilde{\alpha}_*$ then in particular

$$\alpha_*(1_A) = \tilde{\alpha}_*(1_A) : f1_A \longrightarrow g1_A,$$

that is, $\alpha1_A = \tilde{\alpha}1_A : f \longrightarrow g$. But $\alpha1_A = \alpha$ and $\tilde{\alpha}1_A = \tilde{\alpha}$ by the nullary version of the second law in Figure III.a (page 51).

3-cells Take $A \xrightarrow{x} B$ in $T$. If $x_* = y_*$ then in particular

$$x1_A = y1_A.$$ But $x = x1_A$ and $y = y1_A$ by the Gray-category axioms, with the principle in mind that axioms for parallel cells in the top dimension either hold up to equality or not at all.

c. *Cay is a strict homomorphism.* This is the step that takes most work, although none of it is difficult. Strictly speaking, our task is to show that all the cells which make up the coherence data of the homomorphism are identities, where we are thinking of the homomorphism as consisting of a graph map plus coherence data. The force of this is that *Cay* strictly preserves all kinds of composition and identities. We give two instances of this:

**Binary composition of 1-cells** Take $A \xrightarrow{f} A' \xrightarrow{f'} A''$, giving

$$f'_* \circ f_* = \prod_{B \in T_0} T(B, A) \longrightarrow \prod_{B \in T_0} T(B, A'').$$

On a 0-cell $B \xrightarrow{p} A$ of $\prod T(B, A)$, we have $(f'_* \circ f_*)(p) = f'(fp)$ and $(f'f)_*(p) = (f'f)p$, and these are equal because 1-cell composition is associative. On a 1-cell $B \xrightarrow{\gamma} A$ of $\prod T(B, A)$, we have

$$(f'_* \circ f_*)(\gamma) = f'(f\gamma)$$ and $$(f'f)_*(\gamma) = (f'f)\gamma,$$ which are equal by the right-handed version of the second diagram in Figure III.a (page 51).

A 2-cell $x$ of $\prod T(B, A)$ is a 3-cell of $T$, and the 3-cells $f'(fx)$, $(f'f)x$ are equal by the Gray-category axioms ("top dimension principle").

We have thus shown that the graph maps underlying the homomorphisms $f'_* \circ f_*$, $(f'f)_*$ are equal. Having also established that they have the same coherence data (certain 3-cells), we would be able to conclude that $f'_* \circ f_* = (f'f)_*$. 

60
**Horizontal composition of 2-cells** This part is slightly different because a Gray-category does not naturally carry a horizontal composition. Take

![Horizontal Composition Diagram](image)

in \(\mathcal{T}\), yielding

\[
\coprod \mathcal{T}(B, A) \circ \coprod \mathcal{T}(B, A') \coprod \mathcal{T}(B, A'')
\]

in \(\text{Bicat}\). We have to show that

\[
\alpha' \ast \alpha_\ast : f'_\ast \circ f_\ast \longrightarrow g'_\ast \circ g_\ast
\]

and

\[
(\alpha' \ast \alpha)_\ast : (f'f)_\ast \longrightarrow (g'g)_\ast,
\]

strong transformations of bicategories, are equal. (We already know that their domains and codomains are equal.) Since \(\text{Bicat}\) is a Gray-category, \(\alpha' \ast \alpha_\ast\) is by definition equal to

\[
f'_\ast \circ f_\ast \overset{\alpha'_\ast}{\rightarrow} f'_\ast \circ g_\ast \overset{\alpha'_\ast}{\rightarrow} g'_\ast \circ g_\ast.
\]

Since \(\mathcal{T}\) is a Gray-category, \(\alpha' \ast \alpha\) is defined as

\[
f'f \overset{\alpha'}{\longrightarrow} f'g \overset{\alpha'}{\longrightarrow} g'g,
\]

so \((\alpha' \ast \alpha)_\ast = ((\alpha'g) \circ (f'\alpha))_\ast\). The substance of the assertion is that the components of \(\alpha' \ast \alpha_\ast\) and \((\alpha' \ast \alpha)_\ast\) at a 0-cell \(B \xrightarrow{p} A\) of \(\coprod \mathcal{T}(B, A)\) are equal. This says that the 2-cells \((\alpha'\circ (gp))\) and \(((\alpha'g)p)\circ (f'\alpha)p\) of \(\mathcal{T}\) are equal, which follows from the Gray-category axioms.

We have now sketched out a proof that up to strict isomorphism, the small Gray-categories are the small sub-tricategories of \(\textbf{2-Cat}\). If \(\mathcal{T}\) is in fact a (strict) 3-category then the image of the Cayley map lies in the 3-category \(\textbf{2-Cat}_{\text{strict}}\) of 2-categories, strict homomorphisms, strict transformations, and modifications: thus up to strict isomorphism, the small 3-categories are the small sub-tricategories of \(\textbf{2-Cat}_{\text{strict}}\).
III.7 A Conjecture on Coherence in Higher Dimensions

Several points arise from the proof in the previous section. Firstly, we can go through the same arguments in lower dimensions. For small categories \( C \), the Cayley functor \( C \rightarrow \text{Set} \) is always injective (=injective in both dimensions) and so an isomorphism to its image. For small bicategories \( B \), the homomorphism \( \text{Cay} : B \rightarrow \text{Cat} \) is injective if \( f \circ 1 = f \) for 1-cells \( f \), and a strict homomorphism if \( 1 \circ f = f \) and \( (h \circ g) \circ f = h \circ (g \circ f) \). Thus \( \text{Cay} \) is an injective strict homomorphism if \( B \) is a 2-category, and up to strict isomorphism, the small 2-categories are the small sub-bicategories of \( \text{Cat} \).

Secondly, because we assume \( T \) is a Gray-category, the image of \( \text{Cay} : T \rightarrow \text{Bicat} \) is already a tricategory—it doesn’t need closing under composition or enlarging in any other way. Thus \( \text{Cay} \) provides a triequivalence of \( T \) with its genuine image. For an arbitrary tricategory \( T \), the image of \( \text{Cay} \) might not be a tricategory. One possibility is that if we take a suitably enlarged ‘image’ (e.g. the full image) then \( \text{Cay} \) is always a triequivalence to its ‘image’; as remarked in [GPS, p. 4], ‘this may be true, but a proof is not so easy’. (In fact, they were not speaking of \( \text{Cay} \) but the Yoneda homomorphism \( Y : T \rightarrow [\text{Top}, \text{Bicat}] \). Many of the arguments above apply just as well to \( Y \) as to \( \text{Cay} \), and indeed \( \text{Cay} = \coprod Y \).

Thirdly, let us assume that for all small tricategories \( T \), the Cayley map \( \text{does} \) provide a triequivalence to some sub-tricategory of \( \text{Bicat} \). Then we have a proof of the coherence theorem for tricategories in the small case—that is, that any small tricategory is triequivalent to a Gray-category—since the coherence theorem for bicategories implies that the inclusion \( \text{2-Cat} \rightarrow \text{Bicat} \) is a triequivalence. (Again, we may be able to work with \( Y \) instead of \( \text{Cay} \) and get a proof for tricategories which are just \( \text{locally} \) small.)

Finally, we conjecture a coherence theorem for higher-dimensional categories. Let \( R \) be a small tetracategory; assume both that \( \text{Tricat} \) forms a tetracategory (perhaps after making some non-canonical choices), and that there is a Cayley homomorphism \( \text{Cay} : R \rightarrow \text{Tricat} \) which provides a tetaequivalence to some sub-tetracategory of \( \text{Tricat} \). By the coherence theorem for tricategories, the inclusion \( \text{Gray-Cat} \rightarrow \text{Tricat} \) is a tetaequivalence. Thus \( R \) is tetaequivalent to a sub-tetracategory \( R \) of \( \text{Gray-Cat} \). The image of \( R \) in the tetaequivalence \( R \rightarrow R \) is small, so the sub-tetracategory \( R' \) of \( R \) it generates is small, and we then have a tetaequivalence from \( R \) to the small sub-tetracategory \( R' \) of \( \text{Gray-Cat} \).

**Definition III.7.1** A small Gray\(^{(0)}\)-category is a set. A small Gray\(^{(n+1)}\)-category is a small lax \((n+1)\)-category strictly isomorphic to a sub-lax-\((n+1)\)-category of Gray\(^{(n)}\)-Cat.

Here, Gray\(^{(n)}\)-Cat denotes a full sub-lax-\((n+1)\)-category of Lax-\(n\)-Cat, which we are assuming has been defined in a meaningful way. For the first few values of \( n \):
A small Gray\(^{(1)}\)-category is a small category isomorphic to a subcategory of \(\textbf{Set}\), i.e. a small category.

A small Gray\(^{(2)}\)-category is a small bicategory strictly isomorphic to a sub-bicategory of \(\textbf{Cat}\), i.e. a small 2-category.

A small Gray\(^{(3)}\)-category is a small tricategory strictly isomorphic to a sub-tricategory of \(\textbf{2-Cat}\), i.e. a small Gray-category.

The argument on tetracategories above seeks to show that any small tetracategory is tetraequivalent to a small Gray\(^{(4)}\)-category. In dimension 2, it is the usual proof of the coherence theorem but using Cayley rather than Yoneda; in dimension 3, it is our conjectured proof of coherence for tricategories (‘thirdly’ above). The same argument, with the same assumptions as for tetracategories, may be repeated in dimensions 5, 6, . . . . We therefore guess:

**Conjecture III.7.2** Any small lax \(n\)-category is laxly \(n\)-equivalent to a small Gray\(^{(n)}\)-category.

With luck, we may be able to discard our ‘small’s. The issue is presumably a distraction; this is especially the case if we are just using a coherence theorem to ease our manipulation of diagrams, as when we ignore the distinction between \((A \otimes B) \otimes C\) and \(A \otimes (B \otimes C)\) in a lax monoidal category.
Chapter IV

The Opetopic Approach

Opetopes are the backbone of the Baez-Dolan school of higher-dimensional category theory ([Baez], [BD], [Hy], [HMP]). In a nutshell, there is one 0-opetope and an \((n+1)\)-opetope is a pasting of \(n\)-opetopes. Of course, it is the meaning of ‘a pasting’ that matters, and we explain this now.

The unique 0-opetope (with ‘ope’ pronounced as in ‘operation’) is drawn as •. The only pasting-together of •’s (‘0-pasting diagram’) is • itself, so there’s just one 1-opetope, which we like to draw as •

\[
\text{•} \quad \text{and} \quad \text{•}
\]

\((k = 3 \text{ and } k = 0)\). The set of 2-opetopes is therefore isomorphic to \(\mathbb{N}\). However, in our pictures we distinguish between 1-pasting diagrams (as drawn above) and 2-opetopes, even though there’s a natural one-to-one correspondence. For instance, the two 2-opetopes corresponding to the two 1-pasting diagrams above are drawn as

\[
\text{and}
\]

A typical 2-pasting diagram looks like

\[
\text{Note that the arrows go in compatible directions. One can think of the top edges of a 2-opetope as being inputs and the bottom edge as an output; in forming a pasting diagram the rule is that input edges paste to output edges (compare diagram (I.A) on page 20). A 3-opetope is formally just a 2-pasting diagram.}
\]
The best picture we can do on a sheet of paper is

where the 2-opetope on the right is the boundary of the pasting diagram on the left: we imagine a 3-dimensional figure with a flat bottom face and four curved top faces. And so the process continues.

Formally, the sets \( S_n \) of \( n \)-opetopes admit a definition in terms of free multicategories; this is section IV.1. In doing this we obtain a sequence \((\text{Set}/S_n, T_n)\) of cartesian monads. By looking at \((\text{Set}/S_n, T_n)\)-structured categories, we see (IV.2) that for any fixed \( n \) there is a category of \( n \)-pasting diagrams. An arrow in this category can be thought of as the process of ‘composing together’ some of the adjacent \( n \)-opetopes in the domain, to obtain the codomain: e.g. the codomain may be the domain with one of its internal edges erased. When \( n = 2 \) this gives a category of trees (IV.3).

The final section, IV.4, is on a different tack. Baez and Dolan place great emphasis on the process of slicing, in which any operad \( C \) gives rise to its slice operad \( C^+ \). Although their operads are not obviously part of our general scheme, we describe an analogous process for \((S, \ast)\)-multicategories.

We concentrate on describing the structures the opetopic approach gives rise to, rather than its overall shape or its place in \( n \)-category theory. A discussion of the latter can be found in [Baez]; the ideas we do set out below have much in common with [Her].

### IV.1 Opetopes

In this section we construct for each \( n \in \mathbb{N} \) the set \( S_n \) of \( n \)-opetopes, and a cartesian monad \( T_n \) on \( \text{Set}/S_n \).

Start with \( S_0 = 1 \) and \( T_0 = \text{id} \), the identity monad on \( \text{Set} \cong \text{Set}/S_0 \); note that \((\text{Set}/S_0, T_0)\) satisfies the conditions of I.5 for the formation of free \((\text{Set}/S_0, T_0)\)-multicategories. Now suppose, inductively, that \( S_n \) and a cartesian monad \( T_n \) on \( \text{Set}/S_n \) are constructed, satisfying the conditions of I.5. The terminal object of \( \text{Set}/S_n \) is \( (S_n \xrightarrow{1} S_n) \), so if we put

\[
\begin{array}{c}
S_{n+1} \\
\downarrow \quad = T_n \left( \begin{array}{c} S_n \\ S_n \end{array} \right) \\
S_n
\end{array}
\]

then the category of \((\text{Set}/S_n, T_n)\)-graphs on 1 is

\[
\text{Set}/S_n \xrightarrow{S_{n+1}} \text{Set}/S_n \cong \text{Set}/S_{n+1}.
\]
Then define $T_{n+1}$ to be the free $(\text{Set}/S_n, T_n)$-operad monad on $\text{Set}/S_{n+1}$. By I.5, $(\text{Set}/S_{n+1}, T_{n+1})$ is cartesian and satisfies the conditions.

Let us look at the first few values of $n$. We have $S_0 = 1$ and $T_0 = \text{id}$; the member of $S_0$ may be depicted as the 0-dimensional diagram $\bullet$. Then $(S_1 \rightarrow S_0) = T_0(S_0 \rightarrow S_0)$, so $S_1 = 1$, and the member of $S_1$ may be depicted as $\bullet$. The monad $T_1$ is ‘free $(\text{Set}/S_0, T_0)$-operad’, that is, ‘free monoid’. So $T_1$ sends a set $X$ to the set of all diagrams

$$\begin{array}{cccc}
     \bullet & x_1 & \cdots & x_n
\end{array}$$

with $n \in \mathbb{N}$ and $x_i \in X$. Next, $(S_2 \rightarrow S_1) = T_1(S_1 \rightarrow S_1)$, so $S_2 = \mathbb{N}$, the free monoid on 1. We will draw members of $S_2$ as 2-opetopes, e.g. $\begin{array}{ccc}
     \begin{array}{ccc}
     & & a \\
     & \downarrow & \\
     \bullet & & \bullet
     \end{array}
\end{array}$ for $4 \in \mathbb{N}$ or $\begin{array}{ccc}
     \begin{array}{ccc}
     & & a \\
     \downarrow & & \\
     \bullet & & \bullet
     \end{array}
\end{array}$ for $0 \in \mathbb{N}$. (Sometimes we omit the arrows.) A $(\text{Set}/S_1, T_1)$-operad is a plain operad, so $T_2$ is the monad ‘free plain operad’ on $\text{Set}/S_2 = \text{Set}/\mathbb{N}$. To see what $T_2$ does, let $A = (A(n))_{n \in \mathbb{N}}$ be an object of $\text{Set}/S_2$. Then $A$ consists of a set for each 2-opetope, which can be thought of as a set of labels: thus $a \in A(n)$ is drawn as

$$
\begin{array}{ccc}
     \begin{array}{ccc}
     & & a \\
     & \downarrow & \\
     \bullet & & \bullet
     \end{array}
\end{array}
$$

$T_2$ sends $A$ to the family of pictures obtained by sticking together members of the $A(n)$’s, so if $T_2(A \rightarrow \mathbb{N}) = (A' \rightarrow \mathbb{N})$ then a typical member of $A'(4)$ is

$$
\begin{array}{ccc}
     \begin{array}{ccc}
     \begin{array}{ccc}
     \begin{array}{ccc}
     \begin{array}{ccc}
     \begin{array}{ccc}
     \begin{array}{ccc}
     a_1 & a_2 & a_3 \\
     a_4
     \end{array}
     \end{array}
     \end{array}
     \end{array}
     \end{array}
     \end{array}
     \end{array}
\end{array}
$$

$(a_1 \in A(3), a_2 \in A(3), a_3 \in A(0), a_4 \in A(1))$.

The last paragraph has partially revealed what $(\text{Set}/S_n, T_n)$-operads look like; how about $(\text{Set}/S_n, T_n)$-multicategories? A $(\text{Set}/S_0, T_0)$-multicategory is a category. A $(\text{Set}/S_1, T_1)$-multicategory is a plain multicategory $C$, in which objects look like $t$ (in $C_0$) and arrows like

$$
\begin{array}{ccc}
     \begin{array}{ccc}
     \begin{array}{ccc}
     \begin{array}{ccc}
     t_1 & t_2 & t_3 \\
     \downarrow a \\
     t
     \end{array}
     \end{array}
     \end{array}
\end{array}
$$
In a \((\Set / S_2, T_2)\)-multicategory, a typical object looks like \(\begin{array}{c}
\end{array}\) (and this lies over the element 3 of \(\mathbb{N} = S_2\)), and a typical arrow looks like

\[
\begin{array}{c}
\begin{array}{c}
\text{Source}
\end{array}
\end{array} \xrightarrow{a} \begin{array}{c}
\begin{array}{c}
\text{Target}
\end{array}
\end{array}
\]

(IV.A)

(Here, the opetope on the right-hand side has been drawn to resemble the boundary of the left-hand side, but of course it is just the opetope \(1 \in \mathbb{N} = S_2\).) Composition in a \((\Set / S_2, T_2)\)-multicategory looks like

\[
\begin{array}{c}
\begin{array}{c}
\text{Source}
\end{array}
\end{array} \xrightarrow{a} \begin{array}{c}
\begin{array}{c}
\text{Target}
\end{array}
\end{array}
\]

and the identities consist of an arrow

\[
\begin{array}{c}
\begin{array}{c}
\text{Identity}
\end{array}
\end{array} \xrightarrow{1} \begin{array}{c}
\begin{array}{c}
\text{Identity}
\end{array}
\end{array}
\]

for each \(t\).

Continuing into higher dimensions, in a \((\Set / S_n, T_n)\)-multicategory the objects are labelled \(n\)-opetopes, the domain of an arrow is a labelled \(n\)-pasting diagram, and the codomain is an object whose underlying \(n\)-opetope is the boundary of the domain. This is illustrated in diagram (IV.A); one may also think of an arrow as an \((n+1)\)-opetope with its \(n\) and \((n+1)\)-dimensional parts.
labelled. The next two sections examine \((\text{Set}/\mathcal{S}_n, T_n)\)-structured categories and show how, amongst other things, consideration of them leads naturally to a category of trees.

### IV.2 Pasting Diagrams

A strict monoidal category can be thought of as a plain multicategory in which each sequence \(t_1, \ldots, t_k\) of objects has a representing object \(t_1 \otimes \cdots \otimes t_k\): that is, there’s a natural correspondence

\[
\text{Hom}(t_1, \ldots, t_k; t) \cong \text{Hom}(t_1 \otimes \cdots \otimes t_k; t).
\]

In pictures, a strict monoidal category consists of a category with objects \(t\) and arrows \(t' \xrightarrow{a} t\) together with a monoidal structure

\[
\begin{array}{ccc}
  t_1 & t_2 & \cdots & t_k \\
  \downarrow & & & \downarrow \\
  t_1 \otimes \cdots \otimes t_k
\end{array}
\]

(and similarly for arrows), all obeying the familiar rules. Recall (I.4) that a strict monoidal category is a \((\text{Set}/\mathcal{S}_1, T_1)\)-structured category. One dimension up, a \((\text{Set}/\mathcal{S}_2, T_2)\)-structured category consists of a category with objects \(t\), arrows \(a\) like

\[
\begin{array}{ccc}
  t' & \xrightarrow{a} & t \\
  \downarrow & & \downarrow \\
  t_1 & t_2 & \cdots & t_k
\end{array}
\]

and an algebraic structure, ‘tensor’, making assignments like

\[
\begin{array}{ccc}
  t_1 & t_2 & t_3 \\
  \downarrow & \downarrow & \downarrow \\
  t_1 \otimes t_2 \otimes t_3
\end{array}
\]

and similarly for arrows. (Note the special case of nullary tensoring, \(\cdot \xrightarrow{\text{id}} I\).)

It is hoped that the pattern for \(n = 3, 4, \ldots\) is clear, if difficult to draw. Formally, a \((\text{Set}/\mathcal{S}_n, T_n)\)-structured category is a category object in

\[
(\text{Set}/\mathcal{S}_n)^{T_n} = (\text{Set}/\mathcal{S}_{n-1}, T_{n-1})\text{-Operad}.
\]

For \(n = 1\), a \((\text{Set}/\mathcal{S}_1, T_1)\)-structured category is a category object in \textbf{Monoid} = \((\text{Set}, \text{id})\text{-Operad} \): that is, a strict monoidal category. A \((\text{Set}/\mathcal{S}_2, T_2)\)-structured
category is a category object in the category of plain operads. How does this square with the description in the last paragraph? What we said there, effectively, that a \((\text{Set}/S_2, T_2)\)-structured category consists of a sequence \((C(n))_{n \in \mathbb{N}}\) of categories, together with functors

\[
C(n_1) \times \cdots \times C(n_k) \times C(k) \longrightarrow C(n_1 + \cdots + n_k)
\]

obeying associativity and identity laws. (So it is just a ‘\text{Cat-enriched operad}’, or ‘strict monoidal 2-operad’, as in I.2.5(e).) The object-sets \(C(n)_0\) of each category \(C(n)\) thus form a plain operad \(C_0\), as do the arrow-sets to form \(C_1\). The domain, codomain, composition and identity functions in the \(C(n)\)’s provide \(C_0 \longrightarrow C_1 \longrightarrow C_0\) with the structure of a category object in the category of plain operads.

To form the free \((\text{Set}/S_n, T_n)\)-structured category on a given \((\text{Set}/S_n, T_n)\)-multicategory, we paste together objects and arrows of the multicategory. For instance, if \(n = 1\), \(t_1\) are objects in the given multicategory, and

\[
\begin{array}{ccc}
\downarrow a_1 & & \\
t_1 & \longrightarrow & t_3 \\
\downarrow & & \\
t_6 & \longrightarrow & t_8 \\
\end{array}
\]

are arrows, then

\[
\begin{array}{ccc}
\downarrow a_1 & a_2 & a_3 \\
t_1 & \longrightarrow & t_5 \\
\downarrow & \longrightarrow & \\
t_6 & \longrightarrow & t_8 \\
\end{array}
\]

is an arrow in the free structured category. (This is diagram (I.B), page 26, drawn in a different way.) In particular, the free \((\text{Set}/S_1, T_1)\)-structured category on \(1\) is the simplicial category \(\Delta\). In dimension 2, the terminal \((\text{Set}/S_2, T_2)\)-multicategory \(1\) has objects like # and arrows like

\[
\begin{array}{ccc}
\downarrow & & \\
\text{#} & \longrightarrow & \text{#} \\
\downarrow & & \\
\end{array}
\]

and

\[
\begin{array}{ccc}
\downarrow & & \\
\text{#} & \longrightarrow & \text{#} \\
\downarrow & & \\
\end{array}
\]

(see the description of \((\text{Set}/S_2, T_2)\)-multicategories on page 67). Thus an object of the free \((\text{Set}/S_2, T_2)\)-structured category on \(1\) is a 2-pasting diagram, like

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Roughly speaking, an arrow is a removal of some internal edges. It’s only in the nullary case that this becomes inaccurate; it is more precise to say that an arrow is the replacement of some sub-2-pasting diagrams with their bounding 2-opetopes. So here,

are replaced with their bounding 2-opetopes

We now make the general definition:

**Definition IV.2.1** The structured category of $n$-pasting diagrams, $\text{PD}_n$, is the free $(\text{Set}/S_n,T_n)$-structured category on the terminal $(\text{Set}/S_n,T_n)$-multicategory.

If $S_1$ and $S_2$ are categories with pullbacks and $S_1 \rightarrow S_2$ a functor preserving pullbacks, then there’s an induced functor from $S_1\text{-Cat}$ to $S_2\text{-Cat}$ (where $S\text{-Cat}$ is the category of internal categories in $S$). Applying this to the forgetful functors

\[ (\text{Set}/S_n)^{T_n} \rightarrow \text{Set}/S_n \rightarrow \text{Set} \]

gives a functor

\[ (\text{Set}/S_n,T_n)\text{-Struc} \rightarrow \text{Cat}. \]

The category of $n$-pasting diagrams, $\text{pd}_n$, is defined to be the image of $\text{PD}_n$ under this functor. The definition of free structured category in I.4 tells us that the objects-object of $\text{PD}_n$ is $S_n^+1$, so the object-set of $\text{pd}_n$ is $S_{n+1}$, the set of $(n+1)$-opetopes or, as we prefer to think of them at the moment, $n$-pasting diagrams. An arrow in $\text{pd}_n$ consists of the removal of some $(n-1)$-dimensional interior faces, with a warning about the nullary case as given for $n = 2$.

**IV.3 The Category of Trees**

We are familiar with the categories $\text{pd}_0 = 1$ and $\text{pd}_1 = \Delta$. Here we take a closer look at $\text{pd}_2$. 
As suggested by Figure IV.a, there is a one-to-one correspondence between 2-pasting diagrams and trees. (The trivial or nullary 2-pasting diagram, \( \bullet \), corresponds to the tree \( \bullet \). The present trees are not the Batanin trees of Chapter II.) We may define a tree to be a 2-pasting diagram: that is, a member of \( S_3 \). In turn, \( S_3 \) is defined by

\[
S_3 = T_2 \left( \begin{array}{c} S_2 \\ S_2 \end{array} \right),
\]

which just means that (\( S_3 \) \( \rightarrow \) \( \mathbb{N} \)) is the graph of the free plain operad on (\( \mathbb{N} \) \( \rightarrow \) \( \mathbb{N} \)). But in this case, the free multicategory construction sketched in I.5 reduces to the inductive description of the set \( 1^* \) of unlabelled trees, given in Example I.1.3(f): a tree is either \( \bullet \) or a sequence of trees. We can thus prove that the two sets of trees are isomorphic.

The objects of \( \text{pd}_2 \) are trees, so we define the category of trees to be \( \text{pd}_2 \). Loosely, a morphism of trees consists of the contraction of some internal edges—see Figure IV.b—where an edge is called internal if it does not end in a leaf. Once again, saying this ignores the nullary case (Figure IV.c). Replacing a 2-pasting diagram by its bounding 2-opetope corresponds to replacing an \( n \)-leafed tree by the \( n \)-leafed tree \( \bullet \ldots \bullet \) of height 1, so strictly speaking, a map of trees consists of the replacement of some subtrees by their corresponding height-1 trees. In particular, a subtree which is just a node may be replaced by an edge, \( \bullet \).

Categories of trees have been used by Borcherds to define relaxed multilinear categories, by Soibelman for his very similar pseudo-monoidal categories, and by Kontsevich and Manin for purposes unknown to me ([Bor], [Soi], [KM1], [KM2]). In at least the first two cases it seems that maps of trees are only meant to contract internal edges, not add new edges at nodes. I do not know if this is intentional or the oversight of an apparently trivial case. One dimension down, it corresponds to omitting the injections (face maps) in \( \Delta \), and only taking the surjections (degeneracy maps).

We have been examining the underlying category \( \text{pd}_2 \) of the structured category \( \text{PD}_2 \); let us note finally what \( \text{PD}_2 \) is. Interpreted as a \textbf{Cat}-enriched
Figure IV.b: Dotted edges are those to be removed/contracted

Figure IV.c: Marked regions/nodes are those to have an edge added
operad (see page 69), $PD_2$ consists of the categories $TR(n)$ of $n$-leafed trees, for each $n \in \mathbb{N}$, together with the gluing functors

$$TR(n_1) \times \cdots \times TR(n_k) \times TR(k) \longrightarrow TR(n_1 + \cdots + n_k)$$

and the ‘identity’ object $\bullet$ of $TR(1)$.

**IV.4 Slicing**

At the heart of Baez and Dolan’s explanation of $n$-categories is the process of slicing. Given a multicategory $C$, they construct a multicategory $C^+$ whose algebras are multicategories on $C_0$ over $C$. That is, an algebra for $C^+$ consists of a multicategory $D$ with the same object-set as $C$, together with a map from $D$ to $C$ leaving the object-set fixed. (In their language, a multicategory on $S$ is an ‘$S$-operad’ and $C^+$ is called the slice operad of $C$.) Although the operads they use do not appear to fit neatly into the general theory presented here (because of the symmetric group action), we can nevertheless present an analogous construction.

The construction proceeds in two steps, just as in [BD]. Firstly, we show how to slice a multicategory by an algebra: given a multicategory $D$ and an algebra $E$ for $D$, we find a multicategory $D_E$ such that $Alg(D_E) \cong Alg(D)/E$. Secondly, for any object $S$ we construct the $S$-multicategory $\mathbb{C}$, the crucial property of which is that its algebras are the multicategories on $S$. Given a multicategory $C$, take $D$ to be the $C_0$-multicategory: then $C$ is an algebra for $D$, and $Alg(D_C) \cong Alg(D)/C$ is the category of multicategories on $C_0$ over $C$. Thus we define the slice multicategory $C^+$ to be $D_C$.

For the first step, let $(S, *)$ be cartesian, $D$ an $(S, *)$-multicategory, and $(E \longrightarrow D)$ an algebra for $D$: that is, $E$ is an $(S, *)$-multicategory and $f$ a discrete opfibration. If $H \longrightarrow E$ is a map of multicategories, then $f \circ g$ is a discrete opfibration just when $g$ is, and consequently an algebra for $E$ is an algebra for $D$ over $(E \longrightarrow D)$. Thus in the notation above, $D \longrightarrow (E \longrightarrow D) = E$.

(If we take $(S, *) = (\text{Set}, \text{id})$ then we recover the fact that for a category $D$, a functor $D \longrightarrow \text{Set}$, and its Grothendieck opfibration $E \longrightarrow D$, $[E, \text{Set}] \cong [D, \text{Set}]/k$.)

The construction of the $S$-multicategory multicategory is just as simple. Suppose $(S, *)$ satisfies the free multicategory conditions of I.5 (e.g. if we take $(\text{Set}/S_n, T_n)$ or any finitary cartesian algebraic theory on $\text{Set}$). Let $S \in \mathcal{S}$, and let

$$(S', \mathcal{V}) = ((S, *)\text{-Graphs on } S, \text{free } (S, *)\text{-multicategory on } S).$$

The algebras for the terminal $(S', \mathcal{V})$-multicategory are just the algebras for the monad $\mathcal{V}$ on $S'$ (by I.3.2(e)), i.e. the $(S, *)$-multicategories on $S$. 

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The reader will have observed that in taking the slice multicategory we have shifted up a level: that is, if \( C \) is an \((\mathcal{S}, \ast)\)-multicategory then \( C^+ \) is an \((\mathcal{S}', \sharp)\)-multicategory. In our approach this seems perfectly natural. Applied to the sequence of 'opetopic' monads it means that for any monoid \( M \) there’s a plain multicategory \( M^+ \) whose algebras are monoids over \( M \); for any plain operad \( C \) there’s a \((\text{Set}/\mathcal{S}_2, T_2)\)-multicategory \( C^+ \) whose algebras are plain operads over \( C \); . . . . In contrast, the Baez-Dolan attack manages to keep \( C^+ \) at the same level as \( C \). The price they pay is the introduction of symmetric group actions: thus all their operads are symmetric. Roughly speaking, what happens is that the domain of an arrow like

![Diagram](image)

in a \((\text{Set}/\mathcal{S}_2, T_2)\)-operad is turned into a sequence, so that the arrow becomes, say,

\[ t', \hat{t}, \tilde{t}, \check{t} \overset{a}{\longrightarrow} t. \]

But there is no way to order the constituent 2-opetopes of 2-pasting diagrams which is stable under composition (of the kind illustrated in diagram (IV.B), page 67), so the objects need to be permutable. In this explanation the method of Baez and Dolan is made to look clumsy, but we would expect it to have its own, internal, explanation, which made natural the use of symmetric operads.
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