Fair $k$-Center Clustering for Data Summarization

Matthäus Kleindessner $^1$  Pranjal Awasthi $^1$  Jamie Morgenstern $^2$

Abstract
In data summarization we want to choose $k$ prototypes in order to summarize a data set. We study a setting where the data set comprises several demographic groups and we are restricted to choose $k_i$ prototypes belonging to group $i$. A common approach to the problem without the fairness constraint is to optimize a centroid-based clustering objective such as $k$-center. A natural extension then is to incorporate the fairness constraint into the clustering objective. Existing algorithms for doing so run in time super-quadratic in the size of the data set. This is in contrast to the standard $k$-center objective that can be approximately optimized in linear time. In this paper, we resolve this gap by providing a simple approximation algorithm for the $k$-center problem under the fairness constraint with running time linear in the size of the data set and $k$. If the number of demographic groups is small, the approximation guarantee of our algorithm only incurs a constant-factor overhead. We demonstrate the applicability of our algorithm on both synthetic and real data sets.

1. Introduction
Machine learning algorithms have been rapidly adopted in numerous human-centric domains, from personalized advertising to lending to health care. Fast on the heels of this ubiquity have come a whole host of concerning behaviors from these algorithms: facial recognition has higher accuracy on white, male faces (Buolamwini & Gebru, 2017); DUI arrest help advertisements are shown more regularly to minority profiles (Sweeney, 2013); and criminal recidivism tools are more likely to label African-American low-risk defendants as high-risk (Angwin et al., 2016). There are also several examples of unsavory ML behavior pertaining to unsupervised learning tasks, whether considering the bias evident in word2vec embeddings (Bolukbasi et al., 2016) or the gender imbalance of CEO image search results (Kay et al., 2015). Most of the academic work on fairness in machine learning, however, has investigated how to solve classification tasks subject to various constraints on the behavior of a classifier on different demographic groups (e.g., Hardt et al., 2016; Zafar et al., 2017; Agarwal et al., 2018).

This paper adds to the literature on fair methods for unsupervised learning tasks (see Section 4 for related work). We consider the problem of data summarization (Hesabi et al., 2015) through the lens of algorithmic fairness. The goal of data summarization is to output a small but representative subset of a data set. Think of an image database and a user entering a query that is matched by many images. Rather than presenting the user with all matching images, we only want to show a summary. In such an example, a data summary can be quite unfair on a demographic group. Indeed, Google Images has been found to answer the query “CEO” with a much higher fraction of images of men compared to the real-world fraction of male CEOs (Kay et al., 2015).

One approach to the problem of data summarization is provided by centroid-based clustering, such as $k$-center (formally defined in Section 2) or $k$-medoid (Hastie et al., 2009, Section 14.3.10; sometimes referred to as $k$-median). For a centroid-based clustering objective, an optimal clustering of a data set $S$ can be defined by $k$ points $c_1^*, ..., c_k^* \in S$, called centroids, such that the clusters are formed by assigning every $s \in S$ to its closest centroid. Since the centroids are good representatives of their clusters, the set of centroids can be used as a summary of $S$. This approach of data summarization via clustering is used in numerous domains: in the social sciences (Bartholomew et al., 2008; Cameron & Trivedi, 2010), in psychology (Borgen & Barnett, 1987; Campbell et al., 2008), and in biology (Eisen et al., 1998).

If the data set $S$ comprises several demographic groups $S_1, ..., S_m$, we may consider $c_{i1}^*, ..., c_{ik}^*$ to be a fair summary only if the groups are represented fairly: if in the real world 70% of CEOs are male and we want to output ten images for the query “CEO”, then three of the ten images should show women. Formally, this can be encoded with one parameter $k_{S_i}$ for every group $S_i$. Our goal is then to minimize the clustering objective under the constraint that $k_{S_i}$ many centroids belong to $S_i$. A constraint of this form
can also enforce balanced summaries: even if in the real world there are more male CEOs than female ones, we might want to output an equal number of male and female images to reflect that gender is not definitional to the role of CEO.

Centroid-based clustering under such a constraint has been studied in the theoretical computer science literature (see Sections 2 and 4). However, existing approximation algorithms for this problem run in time \( \omega(|S|^2) \), while the unconstrained \( k \)-center clustering problem can be approximated in time linear in \(|S|\). Since data summarization is particularly useful for massive data sets, such a slowdown may be practically prohibitive. The contribution of this paper is to present a simple approximation algorithm for \( k \)-center clustering under our fairness constraint with running time only linear in \(|S|\) and \( k \). The improved running time comes at the price of a worse guarantee on the approximation factor if the number of demographic groups is large. However, note that in practical situations concerning fairness, the number of groups is often quite small (e.g., when the groups encode gender or race). Furthermore, in our extensive numerical simulations we never observed a large approximation factor, even when the number of groups was large (cf. Section 5), indicating the practical usefulness of our algorithm.

Outline of the paper In Section 2, we formally state the \( k \)-center and the fair \( k \)-center problem. In Section 3, we present our algorithm and provide a sketch of its analysis. The full proofs can be found in Appendix A. We discuss related work in Section 4 and present a number of experiments in Section 5. Further experiments can be found in Appendix B. We conclude with a discussion in Section 6.

2. Definition of \( k \)-Center and Fair \( k \)-Center

Let \( S \) be a finite data set and \( d : S \times S \rightarrow \mathbb{R}_{\geq 0} \) be a metric on \( S \). In particular, we assume \( d \) to satisfy the triangle inequality. The standard \( k \)-center clustering problem is the following minimization problem

\[
\min_{C = \{c_1, \ldots, c_k\} \subseteq S} \max_{s \in S} d(s, C),
\]

where \( k \in \mathbb{N} \) is a given parameter and \( d(s, C) = \min_{c \in C} d(s, c) \). Here, \( c_1, \ldots, c_k \) are called centers. Any set of centers defines a clustering of \( S \) by assigning every \( s \in S \) to its closest center. The \( k \)-center problem is NP-hard and is also NP-hard to approximate to a factor better than 2 (Gonzalez, 1985; Vazirani, 2001, Chapter 5). The famous greedy strategy of Gonzalez (1985) is a 2-approximation algorithm with running time \( O(k|S|) \) if we assume that \( d \) can be evaluated in constant time (this is the case, e.g., if a problem instance is given via the distance matrix \((d(s, s'))_{s,s' \in S}\)). This algorithm randomly selects an element of the data set as first center and then iteratively adds the next center to be the point with maximum distance to the current set of centers.

### Algorithm 1 Approximation algorithm for (3)

1. **Input**: \( d : S \times S \rightarrow \mathbb{R}_{\geq 0}; k \in \mathbb{N}; C_0 \subseteq S \)
2. **Output**: \( C = \{c_1, \ldots, c_k\} \subseteq S \)
3. set \( C = \emptyset \)
4. for \( i = 1 \) to \( i = k \)
5. choose \( c_i \in \text{argmax}_{s \in S} d(s, C \cup C_i^0) \)
6. set \( C = C \cup \{c_i\} \)
7. return \( C \)

We consider a fair variant of \( k \)-center as described in Section 1. Our variant also allows for the user to specify a given subset \( C_0 \subseteq S \) that has to be included in the set of centers (think of the example of the image database and the case that we always want to show five prespecified images as part of the summary). Assuming that \( S = \bigcup_{i=1}^m S_i \), where \( S_1, \ldots, S_m \) are the \( m \) demographic groups, the fair \( k \)-center problem can be stated as the minimization problem

\[
\min_{C = \{c_1, \ldots, c_k\} \subseteq S; C \cap S_i = k, i = 1, \ldots, m} \max_{s \in S} d(s, C \cup C_0),
\]

where \( k_{S_i} \in \mathbb{N} \) with \( \sum_{i=1}^m k_{S_i} = k \) and \( C_0 \subseteq S \) are given. By means of a partition matroid, the fair \( k \)-center problem can be phrased as a matroid center problem, for which Chen et al. (2016) provide a 3-approximation algorithm using matroid intersection (e.g., Cook et al., 1998). Chen et al. (2016) do not discuss the running time of their algorithm, but it requires to sort all distances between elements in \( S \) and hence has running time at least \( \Omega(|S|^2 \log |S|) \). In our experiments in Section 5 we observe a running time in \( \Omega(|S|^{5/2}) \).

**Notation** For \( l \in \mathbb{N} \), we sometimes use \([l] = \{1, \ldots, l\}\).

3. A Linear-time Approximation Algorithm

In this section, we present our approximation algorithm for the minimization problem (2). It is a recursive algorithm with respect to the number of groups \( m \). To increase comprehensibility, we first present the case of two groups and then the general case of an arbitrary number of groups.

At several points, we will consider the standard (unfair) \( k \)-center problem (1) generalized to the case of initially given centers \( C_0' \subseteq S \), that is

\[
\min_{C = \{c_1, \ldots, c_k\} \subseteq S} \max_{s \in S} d(s, C \cup C_0').
\]

We can adapt the greedy strategy of Gonzalez (1985) while maintaining its 2-approximation guarantee for (3). For the sake of completeness, we provide the algorithm as Algorithm 1 and state the following lemma:

**Lemma 1.** Algorithm 1 is a 2-approximation algorithm for the unfair \( k \)-center problem (3) with running time \( O((k + |C_0'|)|S|) \), assuming \( d \) can be evaluated in constant time.
Algorithm 2 Approximation algorithm for (2) when \( m = 2 \)
1: Input: metric \( d : S \times S \rightarrow \mathbb{R}_{\geq 0}; k_{S_1}, k_{S_2} \in \mathbb{N}_0 \) with \( k_{S_1} + k_{S_2} = k; C_0 \subseteq S; \) group-membership vector \( \in \{1, 2\}[8] \) encoding membership in \( S_1 \) or \( S_2 \\
2: \) Output: \( C^A = \{c^A_1, \ldots, c^A_k\} \subseteq S \\
3: \) run Algorithm 1 on \( S \) with \( k = k_{S_1} + k_{S_2} \) and \( C_0' = C_0; \) \n4: \( \) let \( \bar{c}^A = \{\bar{c}^A_1, \ldots, \bar{c}^A_k\} \) denote its output; \n5: \( \) if \( |\bar{c}^A \cap S_1| = k_{S_1} \) \( \# \) implies \( |\bar{c}^A \cap S_2| = k_{S_2} \)
6: \( \) return \( \bar{c}^A \)
7: \( \) we assume \( |\bar{c}^A \cap S_1| > k_{S_1}; \) otherwise we switch the role of \( S_1 \) and \( S_2 \\
8: \) form clusters \( L_1, \ldots, L_k, L'_1, \ldots, L'_{|C_0|} \) by assigning \n9: \( \) every \( s \in S \) to its closest center in \( \bar{c}^A \cup C_0 \)
10: \( \) while \( |\bar{c}^A \cap S_1| > k_{S_1} \) and there exists \( L_i \) with center \( \bar{c}^A_i \) \( \) in \( S_1 \) and \( \bar{c}^A_i \) \( \) in \( L_i \cap S_2 \\
11: \) \( \) replace center \( \bar{c}^A_i \) with \( y \) by setting \( \bar{c}^A_i = y \\
12: \) if \( |\bar{c}^A \cap S_1| = k_{S_1} \) \( \# \) implies \( |\bar{c}^A \cap S_2| = k_{S_2} \)
13: \( \) return \( \bar{c}^A \\
14: \) let \( S' = \cup_{i \in [k]: \bar{c}^A_i \in S_1} L_i \) \( \# \) we have \( S' \subseteq S_1 \\
15: \) run Algorithm 1 on \( S' \cup C_0' \) with \( k = k_{S_1} \) and \( C_0' = C_0 \cup (\bar{c}^A \cap S_2) \); \( \) let \( \bar{c}^A \) denote its output; \n16: \( \) return \( \bar{c}^A \cup (\bar{c}^A \cap S_2) \) as well as \( (k_{S_2} - |\bar{c}^A \cap S_2|) \) many arbitrary elements from \( S_2 \\

A proof of Lemma 1, similar in structure to a proof in Har-Peled (2011, Section 4.2) for the strategy of Gonzalez (1985) for problem (1), can be found in Appendix A.

3.1. Fair k-Center with Two Groups

Assume that \( S = S_1 \cup S_2 \). Our algorithm first runs Algorithm 1 for the unfair problem (3) with \( k = k_{S_1} + k_{S_2} \) and \( C_0' = C_0 \). If we are lucky and Algorithm 1 picks \( k_{S_1} \) many centers from \( S_1 \) and \( k_{S_2} \) many centers from \( S_2 \), our algorithm terminates. Otherwise, Algorithm 1 picks too many centers from one group, say \( S_1 \), and too few from \( S_2 \). We try to decrease the number of centers in \( S_1 \) by replacing any such a center with an element in its cluster belonging to \( S_2 \). Once we have made all such available swaps, the remaining clusters with centers in \( S_1 \) are entirely contained within \( S_1 \). We then run Algorithm 1 on this subset with \( k = k_{S_1} \) and the centers from \( S_2 \) as well as \( C_0 \) as initial centers, and return both the centers from the recursive call and those from \( S_2 \) from the initial call.

This algorithm is formally stated as Algorithm 2. The following theorem states that it is a 5-approximation algorithm and that our analysis is tight—in general, Algorithm 2 does not achieve a better approximation factor.

Theorem 1. Algorithm 2 is a 5-approximation algorithm for the fair k-center problem (2) with \( m = 2 \), but not a \((5-\varepsilon)\)-approximation algorithm for any \( \varepsilon > 0 \). It can be implemented in time \( O((k + |C_0||S|)) \), assuming \( d \) can be evaluated in constant time.

Proof. Here we only present a sketch of the proof. The full proof can be found in Appendix A. For showing that Algorithm 2 is a 5-approximation algorithm, let \( r_{\text{fair}} \) be the optimal value of (2) and \( r^* \) be the optimal value of (3) for \( C_0' = C_0 \). Clearly, \( r^* \leq r_{\text{fair}} \). Let \( C^A \) be the set of centers returned by Algorithm 2. It is clear that \( C^A \) comprises \( k_{S_1} \) many elements from \( S_1 \) and \( k_{S_2} \) many elements from \( S_2 \). We need to show that \( \min_{s \in C^A \cup C_0} d(s, c) \leq 5r_{\text{fair}} \) for every \( s \in S \). Let \( C^A \) be the output of Algorithm 1 when called in Line 3 of Algorithm 2. Since Algorithm 1 is a 2-approximation algorithm for (3) according to Lemma 1, we have \( \min_{s \in C^A \cup C_0} d(s, c) \leq 2r^* \leq 2r_{\text{fair}}, s \in S \). Assume that \( |\bar{c}^A \cap S_1| > k_{S_1} \). It follows from the triangle inequality that after exchanging centers in the while-loop in Line 9 of Algorithm 2 we have \( \min_{s \in C^A \cup C_0} d(s, c) \leq 4r_{\text{fair}}, s \in S \). Assume that still \( |\bar{c}^A \cap S_1| > k_{S_1} \). We only need to show that \( \min_{s \in C^A \cup C_0} d(s, c) \leq 5r_{\text{fair}}, s \in S' \). Let \( C^A_{\text{fair}} \) be an optimal solution to (2). We split \( S' \) into two subsets \( S' = S'_0 \cup S'_1 \), where \( S'_0 \) comprises all \( s \) for which the closest center in \( C^A_{\text{fair}} \cup C_0 \) is in \( S_2 \cup C_0 \). Using the triangle inequality we can show that \( \min_{s \in C^A \cup C_0} d(s, c) \leq 5r_{\text{fair}}, s \in S'_0 \). We partition \( S'_1 \) into at most \( k_{S_1} \) many clusters corresponding to the closest center in \( C^A_{\text{fair}} \). Each of these clusters has diameter not greater than \( 2r_{\text{fair}} \). If Algorithm 1 in Line 15 of Algorithm 2 chooses one element from each of these clusters, we immediately have \( \min_{s \in C^A \cup C_0} d(s, c) \leq 2r_{\text{fair}}, s \in S'_1 \). Otherwise, Algorithm 1 chooses an element from \( S'_0 \) or two elements from the same cluster of \( S'_1 \). In both cases, it follows from the greedy choice property of Algorithm 1 that \( \min_{s \in C^A \cup C_0} d(s, c) \leq 5r_{\text{fair}}, s \in S'_1 \).

A family of examples shows that Algorithm 2 is not a \((5-\varepsilon)\)-approximation algorithm for any \( \varepsilon > 0 \).
Figure 1. An example illustrating the need for a more sophisticated procedure for exchanging centers in the case of three or more groups compared to the case of only two groups: we would like to exchange a center from $S_1$ for an element from $S_3$, but cannot do that directly. Rather, we have to make a series of exchanges.

while in the case of only two groups this can easily be done, the difficulty with this idea comes from the exchanging process. Formally, we are given $k$ centers $c_1^A, \ldots, c_k^A$ and the corresponding clustering $S \setminus S_{C_0} = \bigcup_{i=1}^k L_i$, where $S_{C_0} = \bigcup_{i=1}^k L_i$ is the union of clusters with a center in $C_0$, and we want to exchange some centers $c_i^A$ for an element in their cluster $L_i$ such that there exists a strict subset of groups $G \subseteq \{S_1, \ldots, S_m\}$ with the following properties:

$$\forall S_j \in \{S_1, \ldots, S_m\} \setminus G : \sum_{i=1}^k 1 \{c_i^A \in S_j\} \leq k_{S_j}.$$  

(4)

While in the case of only two groups this can easily be achieved by exchanging centers from the group that has more than the requested number of centers for elements from the other group, as we do in Algorithm 2, it is not immediately clear how to deal with a situation as shown in Figure 1. There are three groups $S_1, S_2, S_3$ (elements of these groups are shown in blue, green, and red, respectively), and we have $k_{S_2} = k_{S_3} = k_{S_1} = 1$. For the current set of centers (elements at the centers of the circles) there does not exist $G \subseteq \{S_1, S_2, S_3\}$ satisfying (4) and (5). We would like to decrease the number of centers in $S_1$ and increase the number of centers in $S_3$, but the clusters with a center in $S_1$ do not comprise an element from $S_3$. Hence, we cannot directly exchange a center from $S_1$ for an element in $S_3$. Rather, we first have to exchange a center from $S_1$ for an element in $S_2$ (although this increases the number of centers from $S_2$ over $k_{S_2}$) and then a center from $S_2$ for an element in $S_3$. An algorithm that can deal with such a situation is Algorithm 3. It exchanges some centers for an element in their cluster $L_i$ and yields $G \subseteq \{S_1, \ldots, S_m\}$ that provably satisfies (4) and (5), as stated by the following lemma. Its proof can be found in Appendix A.

**Lemma 2.** Algorithm 3 is well-defined, it terminates, and exchanges centers in such a way that the set $G$ that it returns satisfies $G \subseteq \{S_1, \ldots, S_m\}$ and properties (4) and (5).

**Algorithm 3** Algorithm for exchanging centers & finding $G$

1: **Input:** centers $c_1^A, \ldots, c_k^A$ and the corresponding clustering $S \setminus S_{C_0} = \bigcup_{i=1}^k L_i$; $k_{S_1}, \ldots, k_{S_m} \in \mathbb{N}_0$ with $\sum_{i=1}^m k_{S_i} = k$; group-membership vector $\in \{0, 1\}$

2: **Output:** $\tilde{c}_1^A, \ldots, \tilde{c}_k^A$, where some centers $\tilde{c}_i^A$ have been replaced with an element in $L_i$, and $G \subseteq \{S_1, \ldots, S_m\}$ satisfying (4) and (5)

3: set $\tilde{k}_{S_j} = \sum_{i=1}^k 1 \{c_i^A \in S_j\}$ for $S_j \in \{S_1, \ldots, S_m\}$

4: construct a directed unweighted graph $G$ on $V = \{S_1, \ldots, S_m\}$ as follows: we have $S_i \rightarrow S_j$, if and only if there exists $L_i$ with center $\tilde{c}_i^A$ in $S_i$ and $y \in L_i \setminus S_j$

5: compute all shortest paths on $G$

6: while $\tilde{k}_{S_{j'}} \neq k_{S_j}$ for some $S_j$ and there exist $S_r, S_s$ such that $\tilde{k}_{S_r} > k_{S_r}$ and $\tilde{k}_{S_s} < k_{S_s}$ and there exists a shortest path $P = S_{r_0}S_{r_1}\cdots S_{r_w}$ with $S_{r_0} = S_r$, $S_{w_0} = S_s$ that connects $S_r$ to $S_s$ in $G$

7: for $i = 0, \ldots, w - 1$

8: find $L_i$ with center $\tilde{c}_i^A \in S_{r_i}$ and $y \in L_i \cap S_{r_{i+1}}$; replace $\tilde{c}_i^A$ with $y$ by setting $\tilde{c}_i^A = y$

9: update $k_{S_r} = k_{S_r} - 1$ and $\tilde{k}_{S_s} = \tilde{k}_{S_s} + 1$

10: recompute $G$ and all shortest paths on $G$

11: if $\tilde{k}_{S_j} = k_{S_j}$ for all $S_j \in \{S_1, \ldots, S_m\}$

12: return $\tilde{c}_1^A, \ldots, \tilde{c}_k^A$ and $G = \emptyset$

13: else:

14: set $G' = \{S_j \in \{S_1, \ldots, S_m\} : \tilde{k}_{S_j} > k_{S_j}\}$ and $G' = G' \cup \{S_j \in \{S_1, \ldots, S_m\} \setminus G' : there exists S_i \in G' and a path from S_i to S_j in G\}$

15: return $\tilde{c}_1^A, \ldots, \tilde{c}_k^A$ and $G$

Observing that the number of iterations of the while-loop in Line 7 is upper-bounded by $k$ as the proof of Lemma 2 shows, that the number of iterations of the for-loop in Line 8 is upper-bounded by $m$, and that all shortest paths on $G$ can be computed in running time $O(m^3)$ (Cormen et al., 2009, Chapter 25), it is not hard to see that Algorithm 3 can be implemented with running time $O(km|S| + km^2)$.

Using Algorithm 3, it is straightforward to design a recursive approximation algorithm for the fair $k$-center problem (2) as outlined at the beginning of Section 3.2. We state the algorithm as Algorithm 4. Applying, by means of induction, a similar technique as in the proof of Theorem 1 to every (recursive) call of Algorithm 4, we can prove the following:

**Theorem 2.** Algorithm 4 is a $(3 \cdot 2^m - 1)$-approximation algorithm for the fair $k$-center problem (2) with $m$ groups. It can be implemented in time $O((|C_0|m + km^2)|S| + km^4)$, assuming $d$ can be evaluated in constant time.
Algorithm 4 Approximation alg. for (2) for arbitrary m

1: Input: metric \( d : S \times S \to \mathbb{R}_{\geq 0} \); \( k_{S_1}, \ldots, k_{S_m} \in \mathbb{N}_0 \) with \( \sum_{i=1}^{m} k_{S_i} = k \); \( C_0 \subseteq S \); group-membership vector \( \in \{1, \ldots, m\} \)
2: Output: \( C^A = \{\hat{c}^A_1, \ldots, \hat{c}^A_k\} \subseteq S \)
3: run Algorithm 1 on \( S \) with \( k = \sum_{i=1}^{m} k_{S_i} \) and \( C'_0 = C_0 \); let \( C^A = \{\hat{c}^A_1, \ldots, \hat{c}^A_k\} \) denote its output
4: if \( m = 1 \)
5: return \( \hat{C}^A \)
6: form clusters \( L_1, \ldots, L_k, L'_1, \ldots, L'_{|C_0|} \) by assigning every \( s \in S \) to its closest center in \( \hat{C}^A \cup C'_0 \)
7: apply Algorithm 3 to \( \hat{c}^A_1, \ldots, \hat{c}^A_k \) and \( \cup_{i=1}^{k} L_i \) in order to exchange some centers \( \hat{c}^A_i \) and obtain \( G \subseteq \{S_1, \ldots, S_m\} \)
8: if \( G = \emptyset \)
9: return \( \hat{C}^A \)
10: let \( S' = \cup_{i\in[k]} \hat{c}^A_i \) is from a group in \( G \) \( L_i \) and \( C' = \{\hat{c}^A_i \in \hat{C}^A : \hat{c}^A_i \) is from a group not in \( G \} \) recursively call Algorithm 4, where:
11: \( S' \cup C' \cup C'_0 \) plays the role of \( S \) • we assign elements in \( C' \cup C'_0 \) to an arbitrary group in \( G \) and hence there are \( |G| < m \) many groups \( S_j_1, \ldots, S_j_{|G|} \) • the requested numbers of centers are \( k_{S_j_1}, \ldots, k_{S_j_{|G|}} \) • \( C' \cup C_0 \) plays the role of initially given centers \( C_0 \)
12: let \( \hat{C}^R \) denote its output
13: return \( \hat{C}^R \cup C' \) as well as \( \{k_{S_j} - |C' \cap S_j|\} \) many arbitrary elements from \( S_j \) for every group \( S_j \) not in \( G \)

It is not clear to us whether our analysis of Algorithm 4 is tight and the approximation factor achieved by Algorithm 4 can indeed be as large as \( (3 \cdot 2^{m-1} - 1) \) or whether the dependence on \( m \) is actually less severe (compare with Section 5 and Section 6). Although trying hard to find instances for which the approximation factor of Algorithm 4 is large, we never observed a factor greater than 8.

Lemma 3. Algorithm 4 is not a \( (8 - \varepsilon) \)-approximation algorithm for any \( \varepsilon > 0 \) for (2) with \( m \geq 3 \) groups.

The proofs of Theorem 2 and Lemma 3 are in Appendix A.

4. Related Work

Fairness By now, there is a huge body of work on fairness in machine learning. For a recent paper providing an overview of the literature on fair classification see Donini et al. (2018). Our paper adds to the literature on fair methods for unsupervised learning tasks (Chierichetti et al., 2017; Celis et al., 2018a;b;c; Samadi et al., 2018; Schmidt et al., 2018). Note that all these papers assume to know which demographic group a data point belongs to just as we do. We discuss the two works most closely related to our paper.

First, Celis et al. (2018b) also deal with the problem of fair data summarization. They study the same fairness constraint on the summary as we do, that is the summary must contain \( k_{S_i} \) many elements from group \( S_i \). However, while we aim for a representative summary, where every data point should be close to at least one center in the summary, Celis et al. aim for a diverse summary. Their approach requires the data set \( S \) to consist of points in \( \mathbb{R}^n \), and then the diversity of a subset of \( S \) is measured by the volume of the parallelepiped that it spans (Kulesza & Taskar, 2012). Note that the summarization objective of Celis et al. is different from ours, and in different application domains one or the other may be more appropriate. An advantage of our approach is that it only requires access to a metric on the data set, rather than assuming feature representations of the data points.

The second line of work we discuss centers around the paper of Chierichetti et al. (2017). Their paper proposes a notion of fairness for clustering different from ours. Based on the fairness notion of disparate impact (Feldman et al., 2015) for classification (or the \( p\%\)-rule; Zafar et al., 2017), the paper by Chierichetti et al. asks that every group be approximately equally represented in each cluster. In their paper, Chierichetti et al. focus on \( k \)-medoid and \( k \)-center clustering and the case of two groups. Subsequently, Rößner & Schmidt (2018) study such a fair \( k \)-center problem for multiple groups, and Schmidt et al. (2018) build upon the work of Chierichetti et al. to devise algorithms for such a fair \( k \)-means problem. While we certainly consider the fairness notion of Chierichetti et al. (2017), which can be applied to any kind of clustering, to be meaningful in some scenarios, we believe that in certain applications of centroid-based clustering (such as data summarization) our proposed fairness notion provides a more sensible alternative.

Centroid-based clustering There are many papers proposing heuristics and approximation algorithms for both \( k \)-center (e.g., Hochbaum & Shmoys, 1986; Mladenović et al., 2003; Ferone et al., 2017) and \( k \)-medoid (e.g., Charikar et al., 2002; Arya et al., 2004; Li & Svensson, 2013) under various assumptions on \( S \) and the distance function \( d \). There are also numerous papers on versions with constraints, such as lower or upper bounds on the size of the clusters (Aggarwal et al., 2010; Cygan et al., 2012; Rößner & Schmidt, 2018). Most important to mention are the works by Hajighayi et al. (2010), Krishnaswamy et al. (2011) and Chen et al. (2016). Hajighayi et al. are the first that consider our fairness constraint (for two groups and without a set \( C_0 \) that has to be included in the set of centers) for \( k \)-medoid.
They present a local search algorithm and prove it to be a constant-factor approximation algorithm. Their work has been generalized by Krishnaswamy et al., who consider \( k \)-medoid under the constraint that the set of centers has to form an independent set in a given matroid. This kind of constraint contains our fairness constraint as a special case (for an arbitrary number of groups and an arbitrary set \( C_0 \)). Krishnaswamy et al. obtain a 16-approximation algorithm for this so-called matroid median problem based on rounding the solution of a linear programming relaxation. Subsequently, Chen et al. study the matroid center problem. Using an algorithm for matroid intersection as black box, they obtain a 3-approximation algorithm. Note that none of Hajiaghayi et al., Krishnaswamy et al. or Chen et al. discuss the running time of their algorithm, except for arguing it to be polynomial time (compare with Section 2).

5. Experiments

In this section, we present a number of experiments. We begin with a motivating example on a small image data set illustrating that a summary produced by Algorithm 1 (i.e., the standard greedy strategy for the unfair \( k \)-center problem) can be quite unfair. We also compare summaries produced by our algorithm to summaries produced by the method of Celis et al. (2018b). We then investigate the approximation factor of our algorithm on several artificial instances for which we know or can compute the optimal value of the fair \( k \)-center problem (2) and compare our algorithm to the one for the matroid center problem by Chen et al. (2016), both in terms of approximation factor and running time. Next, on both synthetic and real data, we compare our algorithm in terms of the cost of its output to two baseline heuristics. Finally, we compare our algorithm to Algorithm 1 more systematically. We study the difference in the costs of the outputs of our algorithm and Algorithm 1, a quantity one may refer to as price of fairness, and measure how unfair the output of Algorithm 1 can be. In the following, all boxplots show results of 200 runs of an experiment.

5.1. Motivating Example and Comparison with Celis et al. (2018b)

Consider the 14 images\(^1\) of medical doctors shown in the first row of Figure 2. Assume we want to generate a summary of size four of these images. One way to do so is to run Algorithm 1. The first column of the table in Figure 2 shows in each row the summary produced in one run of Algorithm 1 (recall that all algorithms considered here are randomized algorithms). These summaries are quite unfair: although there is an equal number of images of female doctors and images of male doctors, all these summaries show three or even four females. To overcome this bias we can apply our algorithm or the method of Celis et al. (2018b), which both allow us to explicitly state the numbers of females and males that we want in the summary. The second and the third column of the table show summaries produced by these algorithms. It is hard to say which of them produces more useful summaries and the results ultimately depend on the feature representations of the images (see the next paragraph). To provide further illustration, we present a similar experiment in Figure 11 in Appendix B. Note that we chose very small numbers of images in these experiments solely for the purpose of easy visual digestion.

For computing feature representations of the images and running the algorithm of Celis et al. we used the code provided by them. The feature vector of an image is a histogram based on the image’s SIFT descriptors; see Celis et al. for details. We used the Euclidean metric between these feature vectors as metric \( d \) for Algorithm 1 and our algorithm.

5.2. Approximation Factor and Comparison with Chen et al. (2016)

We implemented the algorithm by Chen et al. (2016) using the generic algorithm for matroid intersection provided in SageMath\(^2\). To speed up computation, rather than testing all distance values as threshold as suggested by Chen et al., we implemented binary search to look for the optimal value.

In the experiment shown in the left part of Figure 3, we study the approximation factor achieved by our algorithm (Alg. 4) and the algorithm by Chen et al. (M.C.) in various settings of values of \( m, |C_0| \) and \( k_S, i \in [m] \). The data set \( S \) always consists of 25 vertices of a random graph and is small.

\(^1\)All images were found on https://commons.wikimedia.org, https://pexels.com or https://pixnio.com and are in the public domain.

\(^2\)http://sagemath.org/
enough to explicitly compute an optimal solution to the fair $k$-center problem \(^{(2)}\). The random graph is constructed according to an Erdős-Rényi model, where any possible edge between two vertices is contained in the graph with probability $2 \log(|S|)/|S|$. With high probability such a graph is connected (if not, we discard it). We put random weights on the edges, drawn from the uniform distribution on $[100]$, and let the metric $d$ be the shortest-path distance on the graph. We assign every vertex to one of $m$ groups uniformly at random and randomly choose a subset $C_0 \subseteq S$ of initially given centers. As we can see from the boxplots, the approximation factor achieved by our algorithm is never larger than $2.4$. We also see that in each of the seven settings that we consider the median of the achieved approximation factors (indicated by the red lines in the boxes) is smaller for our algorithm than for the algorithm by Chen et al.

In the experiment shown in the right part of Figure 3, we study the running time of the two algorithms as a function of the size of the data set, which is created analogously to the experiment in the left part. We set $m = 5$, $C_0 = \emptyset$ and $k_{S_i} = 4, i \in [5]$. The shown curves are obtained from averaging the running times of 200 runs of the experiment. While our algorithm never runs for more than 0.01 seconds, the algorithm by Chen et al., on average, runs for 240 seconds when $|S| = 250$. Its run time grows at least as $|S|^{5/2}$, which proves it to be inappropriate for massive data sets.

In the experiment of Figure 4, we once more study the approximation factor achieved by our algorithm. We place 100 optimal centers at $(i, j) \in \mathbb{R}^2, i, j \in \{0, \ldots, 9\}$, and sample 10000 points around them such that for every center the farthest point in its cluster is at distance $0.5$ from the center (Euclidean distance). One such a point set can be seen in the left plot of Figure 4. We randomly assign every point and center to one of $m$ groups and set $k_{S_i}$ to the number of centers that have been assigned to group $S_i$. We let $C_0 = \emptyset$. For $m \in \{2, \ldots, 20\}$, the right part of Figure 4 shows boxplots of the approximation factors for our algorithm. Similarly as before, the approximation factor achieved by our algorithm is never larger than 2.6. Most interestingly, the approximation factor increases very moderately with $m$.

5.3. Comparison with Baseline Approaches

We compare our algorithm in terms of the cost of an approximate solution to two baseline heuristics for the fair $k$-center problem \(^{(2)}\). The first one, referred to as Heuristic A, runs Algorithm 1 on each group separately (with $k = k_{S_i}$ and $C'_0 = S_i \cap C_0$ for group $S_i$) and returns the union of the
centers obtained for the groups as output. The second one, Heuristic B, greedily chooses centers similarly to Algorithm 1, but only from those groups for which we have not reached the requested number of centers yet. It is easy to see that the approximation factor achieved by these heuristics can be arbitrarily large on some worst-case instances.

Figure 6 shows boxplots of the costs of the approximate solutions returned by our algorithm and the two heuristics for three data sets: the data set in the left plot consists of 2000 vertices of a random graph constructed similarly as in the experiments of Figure 3. We set \( m = 10, k_{S_i} = 4, i \in [10], \) and \( |C_0| = 10. \) The data set in the middle and in the right plot consists of the first 25000 records of the Adult data set (Dheeru & Karra Taniskidou, 2017). We only use its six numerical features (e.g., age, hours worked per week), normalized to have mean zero and standard deviation one, for representing records and use the \( l_1 \)-distance as metric \( d. \) For the experiment shown in the middle plot, we split the data set into two groups according to the sensitive feature gender (there are 16709 males and 8291 females) and set \( k_{S_1} = k_{S_2} = 200. \) For the experiment shown in the right plot, we split the data set into five groups according to the feature race (#White=21391, #Asian-Pac-Islander=775, #Amer-Indian-Eskimo=241, #Other=214, #Black=2379) and set \( k_{S_j} = 50, i \in [5]. \) In both cases, we let \( C_0 \) be a subset of randomly chosen records of size \( |C_0| = 100. \) In Figure 10 in Appendix B we present results for other choices of \( k_{S_i}. \) The two heuristics perform surprisingly well. Although coming without any worst-case guarantees, overall, the cost of their solutions is comparable to the cost of the output of our algorithm. Still, in four out of seven experiments, our algorithm is superior to Heuristic B, which in turn is superior to Heuristic A in all seven experiments.

5.4. Comparison with Unfair Algorithm 1

We compare the cost of the solution produced by our algorithm to the cost of the (potentially) unfair solution provided by Algorithm 1. Of course, we expect the latter to be lower. We consider the case \( k_{S_i} = k_{S_j}, i, j \in [m], \) and also examine how balanced the numbers of centers from a group \( S_i \) in the output of Algorithm 1 are. Figure 5 shows the results, where the data sets and settings equal the ones in the experiments of Figure 6. Similar experiments with different settings are provided in Figure 12 in Appendix B. Remarkably, the costs of the solution produced by our algorithm and Algorithm 1 have the same order of magnitude in all experiments, showing that the price of fairness is small. On the other hand, the output of Algorithm 1 can be highly unfair.

6. Discussion

We considered \( k \)-center clustering under a fairness constraint that is motivated by the application of centroid-based clustering for data summarization. We presented a simple approximation algorithm with running time only linear in the size of the data set \( S \) and the number of centers \( k. \) We proved our algorithm to be a 5-approximation algorithm when \( S \) consists of two groups. For more than two groups, our analysis yields an upper bound on the approximation factor that increases exponentially with the number of groups. We do not know whether this exponential dependence is necessary or whether our analysis is loose—in our extensive numerical simulations we never observed a large approximation factor. Beside answering this question, in future work it would be interesting to extend our results to other clustering objectives such as \( k \)-medoid or \( k \)-means. It would also be interesting to characterize properties of data sets that guarantee that fast algorithms find an optimal fair clustering.
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Appendix

A. Proofs

Proof of Lemma 1:

It is straightforward to see that Algorithm 1 can be implemented in time $O((k + |C_0|)|S|)$. We only need to show that it is a 2-approximation algorithm for (3).

If $k = 0$, there is nothing to show, so assume that $k \geq 1$. Let $C = \{c_1, \ldots, c_k\}$ be the output of Algorithm 1 and $C^* = \{c_1^*, \ldots, c_k^*\}$ be an optimal solution of (3) with objective value $r^*$. Let $s \in S$ be arbitrary. We need to show that $d(s, \hat{c}) \leq 2r^*$ for some $\hat{c} \in C \cup C'_0$. If $s \in C \cup C'_0$, there is nothing to show. So assume $s \notin C \cup C'_0$. If $\hat{c} \in C'_0 \cap \arg\min_{c \in C \cup C'_0} d(s, c) \neq \emptyset$, there exists $\hat{c} \in C'_0$ with $d(s, \hat{c}) \leq r^*$ and we are done. Otherwise, let $c_i \in \arg\min_{c \in C \cup C'_0} d(s, c)$ and hence $d(s, c_i) \leq r^*$.

We distinguish two cases:

- $\exists c_j \in C$ with $c_j^* \in \arg\min_{c \in C \cup C'_0} d(c_j, c)$: We have $d(c_j, c_j^*) \leq r^*$ and hence $d(s, c_j) \leq d(s, c_j^*) + d(c_j^*, c_j) \leq 2r^*$.

- $\nexists c_j \in C$ with $c_j^* \in \arg\min_{c \in C \cup C'_0} d(c_j, c)$: There must be $c' \neq c'' \in C \cup C'_0$, where not both $c'$ and $c''$ can be in $C'_0$, and $\hat{c} \in C \cup C'_0$ such that $\hat{c} \in \arg\min_{c \in C \cup C'_0} d(\hat{c}, c)$ and $\hat{c} \in \arg\min_{c \in C \cup C'_0} d(c', c)$. Since $d(c', \hat{c}) \leq r^*$ and $(c'', \hat{c}) \leq r^*$, it follows that $d(c', c'') \leq d(c', \hat{c}) + d(\hat{c}, c'') \leq 2r^*$.

Without loss of generality assume that in the execution of Algorithm 1, $c''$ has been added to the set of centers after $c'$ has been added. In particular, we have $c'' \in C$ and $c'' = c_l$ for some $l \in \{1, \ldots, k\}$. Due to the greedy choice in Line 5 of the algorithm and since $s$ has not been chosen by the algorithm, we have

$$2r^* \geq d(c', c'') \geq \min_{c \in \{c_1, \ldots, c_{l-1}\} \cup C'_0} d(c'', c) \geq \min_{c \in \{c_1, \ldots, c_{l-1}\} \cup C'_0} d(s, c).$$

\[\square\]

Proof of Theorem 1:

Again it is easy to see that Algorithm 2 can be implemented in time $O((k + |C_0|)|S|)$. We need to prove that it is a 5-approximation algorithm, but not a $(5 - \varepsilon)$-approximation algorithm for any $\varepsilon > 0$.

1. Algorithm 2 is a 5-approximation algorithm:

Let $r^*_\text{fair}$ be the optimal value of the fair problem (2) and $r^*$ be the optimal value of the unfair problem (3). Clearly, $r^* \leq r^*_\text{fair}$. Let $C'_\text{fair} = \{c_{k_{S_1}}^{(1)*}, c_{k_{S_1}}^{(2)*}, \ldots, c_{k_{S_2}}^{(1)*}\}$ with $c_{k_{S_1}}^{(1)*}, \ldots, c_{k_{S_2}}^{(2)*} \in S_2$ be an optimal solution to the fair problem (2) with cost $r^*_\text{fair}$ and $C^A = \{c_1^A, \ldots, c_k^A\}$ be the centers returned by Algorithm 2. It is clear that Algorithm 2 returns $k_{S_1}$ many elements from $S_1$ and $k_{S_2}$ many elements from $S_2$ and hence $C^A = \{c_1^A, \ldots, c_{k_{S_1}}^{(1)*}, c_{k_{S_1}}^{(2)*}, \ldots, c_{k_{S_2}}^{(2)*}\}$ with $c_{k_{S_1}}^{(1)*}, \ldots, c_{k_{S_2}}^{(2)*} \in S_2$. We need to show that $\min_{c \in C^A \cup C'_0} d(s, c) \leq 5r^*_\text{fair}$, $s \in S$. 

\[\]
Let $\tilde{C}^A = \{\tilde{c}_1^A, \ldots, \tilde{c}_A^A\}$ be the output of Algorithm 1 when called in Line 3 of Algorithm 2. Since Algorithm 1 is a 2-approximation algorithm for the unfair problem (3) according to Lemma 1, we have
\[
\min_{c \in C_A \cup C_0} d(s, c) \leq 2r^* \leq 2r_{\text{fair}}^*, \quad s \in S.
\]
(6)

If Algorithm 2 returns $\tilde{C}^A$ in Line 6, that is $C_A = \tilde{C}^A$, we are done. Otherwise assume, as in the algorithm, that $|\tilde{C}^A \cap S_1| > k_{S_1}$. Let $\tilde{c}_i^A \in S_1$ be a center of cluster $L_i$ that we replace with $y \in L_i \cap S_2$ and let $\hat{y}$ be an arbitrary element in $L_i$. Because of (6), we have $d(\tilde{c}_i^A, y) \leq 2r_{\text{fair}}^*$ and $d(\hat{c}_i^A, y) \leq 2r_{\text{fair}}^*$ and hence $d(y, \hat{c}_i^A) + d(\hat{c}_i^A, y) \leq 4r_{\text{fair}}^*$ due to the triangle inequality. Consequently, after the while-loop in Line 9, every $s \in S$ is in distance of $4r_{\text{fair}}^*$ or smaller to the center of its cluster. In particular, we have
\[
\min_{c \in C_A \cup C_0} d(s, c) \leq 4r_{\text{fair}}^*, \quad s \in S,
\]
and if Algorithm 2 returns $\tilde{C}^A$ in Line 13, we are done. Otherwise, we still have $|\tilde{C}^A \cap S_1| > k_{S_1}$ after exchanging centers in the while-loop in Line 9. Let $S' = \bigcup_{i \in [k]} \tilde{c}_i^A, L_i$, that is the union of clusters with a center $\tilde{c}_i^A \in S_1$. Since there is no more center in $S_1$ that we can exchange for an element in $S_2$, we have $S' \subseteq S_1$. Let $S'' = \bigcup_{i \in [k]} \tilde{c}_i^A, L_i$ be the union of clusters with a center $\tilde{c}_i^A \in S_2$ and $S_{C_0} = L'_1 \cup \ldots \cup L'_{|C_0|}$ be the union of clusters with a center in $C_0$. Then we have $S = S' \cup S'' \cup S_{C_0}$. We have $\tilde{C}^A \cap S_2 \subseteq C^A$ and
\[
\min_{c \in C_A \cup C_0} d(s, c) \leq \min_{c \in (\tilde{C}^A \cap S_2) \cup C_0} d(s, c) \leq 4r_{\text{fair}}^*, \quad s \in S'' \cup S_{C_0}.
\]
(7)

Hence we only need to show that $\min_{c \in C_A \cup C_0} d(s, c) \leq 5r_{\text{fair}}^*$ for every $s \in S'$. We split $S'$ into two subsets $S' = S'_a \cup S'_b$, where
\[
S'_a = \left\{ s \in S' : \arg\min_{c \in C_A \cup C_0} d(s, c) \cap (C_0 \cup S_2) \neq \emptyset \right\}
\]
and $S'_b = S' \setminus S'_a$. For every $s \in S'_a$ there is an $c \in (C_0 \cup S_2) \subseteq (S'' \cup S_{C_0})$ with $d(s, c) \leq r_{\text{fair}}^*$ and it follows from (7) and the triangle inequality that
\[
\min_{c \in C_A \cup C_0} d(s, c) \leq \min_{c \in (\tilde{C}^A \cap S_2) \cup C_0} d(s, c) \leq 5r_{\text{fair}}^*, \quad s \in S'_a.
\]
(8)

It remains to show that $\min_{c \in C_A \cup C_0} d(s, c) \leq 5r_{\text{fair}}^*$ for every $s \in S'_b$. For every $s \in S'_b$ there exists $c \in \{c_1^{(1)}, \ldots, c_{k_{S_1}}^{(1)}\}$ with $d(s, c) \leq r_{\text{fair}}^*$. We can write $S'_b = \bigcup_{j=1}^{k_{S_1}} \{ s \in S'_b : d(s, c_j^{(1)}) \leq r_{\text{fair}}^* \}$ (some of the sets in this union might be empty, but that does not matter). Note that for every $j \in \{1, \ldots, k_{S_1}\}$ we have
\[
d(s, s') \leq 2r_{\text{fair}}^*, \quad s, s' \in \left\{ s \in S'_b : d(s, c_j^{(1)}) \leq r_{\text{fair}}^* \right\},
\]
(9)
due to the triangle inequality. It is
\[
S' = S'_a \cup S'_b = S'_a \cup \bigcup_{j=1}^{k_{S_1}} \left\{ s \in S'_b : d(s, c_j^{(1)}) \leq r_{\text{fair}}^* \right\}
\]
and when, in Line 15 of Algorithm 2, we run Algorithm 1 on $S' \cup C_0$ with $k = k_{S_1}$ and initial centers $C_0' = C_0 \cup (\tilde{C}^A \cap S_2)$, one of the following three cases has to happen (we denote the centers returned by Algorithm 1 by $\hat{C}^A = \{c_1^{(1)}, \ldots, c_{k_{S_1}}^{(1)}\}$):

- For every $j \in \{1, \ldots, k_{S_1}\}$ there exists $j' \in \{1, \ldots, k_{S_1}\}$ such that $c_j^{(1)} \in \{ s \in S'_b : d(s, c_j^{(1)}) \leq r_{\text{fair}}^* \}$. In this case it immediately follows from (9) that
  \[
  \min_{c \in C_A \cup C_0} d(s, c) \leq \min_{c \in \hat{C}^A} d(s, c) \leq 2r_{\text{fair}}^*, \quad s \in S'_b.
  \]
Appendix to Fair $k$-Center Clustering for Data Summarization

There exist $j' \in \{1, \ldots, k_{S_1}\}$ such that $c_j^{(1)} \in S_{S_0}'$. When Algorithm 1 picks $c_j^{(1)}$, any other element in $S'$ cannot be at a larger minimum distance from a center in $(\hat{C}^A \cap S_2) \cup C_0$ or a previously chosen center in $\hat{C}^A$ than $c_j^{(1)}$. It follows from (8) that

\[
\min_{c \in \hat{C}^A \cup C_0} d(s, c) \leq 5r_{\text{fair}}, \quad s \in S'.
\]

There exist $j \in \{1, \ldots, k_{S_1}\}$ and $j' \neq j'' \in \{1, \ldots, k_{S_1}\}$ such that $c_j^{(1)}$, $c_j^{(1)} \in \{s \in S_{S_0}' : d(s, c_j^{(1)}) \leq r_{\text{fair}}^s\}$. Assume that Algorithm 1 picks $c_{j'}^{(1)}$ before $c_{j''}^{(1)}$. When Algorithm 1 picks $c_{j''}^{(1)}$, any other element in $S'$ cannot be at a larger minimum distance from a center in $(\hat{C}^A \cap S_2) \cup C_0$ or a previously chosen center in $\hat{C}^A$ than $c_{j''}^{(1)}$. Because of $d(c_{j'}^{(1)}, c_{j''}^{(1)}) \leq 2r_{\text{fair}}^s$ according to (9), it follows that

\[
\min_{c \in \hat{C}^A \cup C_0} d(s, c) \leq 2r_{\text{fair}}^s, \quad s \in S'.
\]

In all cases we have

\[
\min_{c \in \hat{C}^A \cup C_0} d(s, c) \leq 5r_{\text{fair}}^s, \quad s \in S_{S_0}',
\]

which completes the proof of the claim that Algorithm 2 is a 5-approximation algorithm.

2. Algorithm 2 is not a $(5 - \varepsilon)$-approximation algorithm for any $\varepsilon > 0$:

Consider the example given by the weighted graph shown in Figure 7, where $0 < \delta < \frac{1}{10}$. We have $S = S_1 \cup S_2$ with $S_1 = \{f_1, f_2, f_3, f_4, f_5\}$ and $S_2 = \{m_1, m_2, m_3, m_4, m_5, m_6\}$. All distances are shortest-path-distances. Let $k_{S_1} = 1$, $k_{S_2} = 3$, and $C_0 = \emptyset$. We assume that Algorithm 1 in Line 3 of Algorithm 2 picks $f_5$ as first center. It then chooses $f_2$ as second center, $f_3$ as third center and $f_1$ as fourth center. Hence, $\hat{C}^A = \{f_5, f_2, f_3, f_1\}$ and $|\hat{C}^A \cap S_1| > k_{S_1}$. The clusters corresponding to $\hat{C}^A$ are $\{f_5\}$, $\{f_2, f_4\}$, $\{f_3, m_3, m_4, m_5, m_6\}$ and $\{f_1, m_1, m_2\}$. Assume we replace $f_3$ with $m_4$ and $f_1$ with $m_2$ in Line 10 of Algorithm 2. Then it is still $|\hat{C}^A \cap S_1| > k_{S_1}$, and in Line 15 of Algorithm 2 we run Algorithm 1 on $\{f_2, f_4, f_5\} \cup \{m_2, m_4\}$ with $k = 1$ and initially given centers $C_0' = \{m_2, m_4\}$. Algorithm 1 returns $\hat{C}^A = \{f_5\}$. Finally, assume that $m_5$ is chosen as arbitrary third center from $S_2$ in Line 16 of Algorithm 2. So the centers returned by Algorithm 2 are $C^A = \{f_5, m_2, m_4, m_5\}$ with a cost of $5 \frac{\delta}{2}$ (incurred for $f_4$). However, the optimal solution $C^*_{\text{fair}} = \{f_5, m_1, m_3, m_6\}$ has cost only $1 + \delta$. Choosing $\delta$ sufficiently small shows that Algorithm 2 is not a $(5 - \varepsilon)$-approximation algorithm for any $\varepsilon > 0$. \hfill \qed
Appendix to Fair $k$-Center Clustering for Data Summarization

Proof of Lemma 2:

We want to show three things:

1. Algorithm 3 is well-defined:
   
   If the condition of the while-loop in Line 7 is true, there exists a shortest path $P = S_{v_0}S_{v_1}\cdots S_{v_w}$ with $S_{v_0} = S_r$, $S_{v_w} = S_s$ that connects $S_r$ to $S_s$ in $G$. Since $P$ is a shortest path, all $S_{v_i}$ are distinct. By the definition of $G$, for every $l = 0, \ldots, w-1$ there exists $L_l$ with center $\tilde{c}^A_l \in S_{v_l}$ and $y \in L_l \cap S_{v_{l+1}}$. Hence, the for-loop in Line 8 is well defined.

2. Algorithm 3 terminates:
   
   Let, at the beginning of the execution of Algorithm 3 in Line 3, $H_1 = \{S_j \in \{S_1, \ldots, S_m\} : \tilde{k}_{S_j} = k_{S_j}\}$, $H_2 = \{S_j \in \{S_1, \ldots, S_m\} : \tilde{k}_{S_j} > k_{S_j}\}$ and $H_3 = \{S_j \in \{S_1, \ldots, S_m\} : \tilde{k}_{S_j} < k_{S_j}\}$. For $S_j \in H_2$, $\tilde{k}_{S_j}$ never decreases during the execution of the algorithm. For $S_j \in H_2$, $\tilde{k}_{S_j}$ never decreases during the execution of the algorithm and decreases at most until it equals $k_{S_j}$. For $S_j \in H_3$, $\tilde{k}_{S_j}$ never decreases during the execution of the algorithm and increases at most until it equals $k_{S_j}$. In every iteration of the while-loop, there is $S_j \in H_3$ for which $\tilde{k}_{S_j}$ increases by one. It follows that the number of iterations of the while-loop is upper-bounded by $k$.

3. Algorithm 3 exchanges centers in such a way that the set $G$ that it returns satisfies $G \subseteq \{S_1, \ldots, S_m\}$ and properties (4) and (5):

   Note that throughout the execution of Algorithm 3 we have $\tilde{k}_{S_j} = \sum_{i=1}^{k} \mathbb{1}\{c_i^A \in S_j\}$ for the current centers $c_1^A, \ldots, c_k^A$. If the condition of the if-statement in Line 13 is true, then $G = \emptyset$ and (4) and (5) are satisfied.

   Assume that the condition of the if-statement in Line 13 is not true. Clearly, the set $G$ returned by Algorithm 3 satisfies (5). Since the condition of the if-statement in Line 13 is not true, there exist $S_j$ with $\tilde{k}_{S_j} > k_{S_j}$ and $S_l$ with $\tilde{k}_{S_l} < k_{S_l}$. We have $S_j \in G$, but since the condition of the while-loop in Line 7 is not true, we cannot have $S_l \in G$. This shows that $G \subseteq \{S_1, \ldots, S_m\}$. We need to show that (4) holds. Let $L_k$ be a cluster with center $\tilde{c}_k^A \in S_f$ for some $S_f \in G$ and assume it contained an element $o \in S_f$ with $S_f \notin G$. But then we had a path from $S_f$ to $S_f'$ in $G$. If $S_f \in G'$, this is an immediate contradiction to $S_f \notin G$. If $S_f \notin G'$, since $S_f \in G$, there exists $S_g \in G'$ such that there is a path from $S_g$ to $S_f$. But then there is also a path from $S_g$ to $S_f'$, which is a contradiction to $S_f' \notin G$.

Proof of Theorem 2:

For showing that Algorithm 4 is a $(3 \cdot 2^{m-1} - 1)$-approximation algorithm let $r^*_{fשית}$ be the optimal value of problem (2) and $C^*_f$ be an optimal solution with cost $r^*_{fshit}$. Let $C^A$ be the centers returned by Algorithm 4. A simple proof by induction over $m$ shows that $C^A$ actually comprises $k_{S_i}$ many elements from every group $S_i$. We need to show that

$$\min_{c \in C^A \cup C_0} d(s, c) \leq (3 \cdot 2^{m-1} - 1)r^*_{fshit}, \quad s \in S. \tag{10}$$

Let $T$ be the total number of calls of Algorithm 4, that is we have one initial call and $T - 1$ recursive calls. Since with each recursive call the number of groups is decreased by at least one, we have $T \leq m$. For $1 \leq j \leq T$, let $S^{(j)}$ be the data set in the $j$-th call of Algorithm 4. We additionally set $S^{(T+1)} = \emptyset$. We have $S^{(1)} = S$ and $S^{(j)} \supseteq S^{(j+1)}$, $1 \leq j \leq T$. For $1 \leq j < T$, let $G^{(j)}$ be the set of groups in $G$ returned by Algorithm 3 in Line 8 in the $j$-th call of Algorithm 4. If in the $T$-th call of Algorithm 4 the algorithm terminates from Line 10 (note that in this case we must have $T < m$), we also let $G^{(T)} = \emptyset$ be the set of groups in $G$ returned by Algorithm 3 in the $T$-th call. Otherwise we leave $G^{(T)}$ undefined. Setting $G^{(0)} = \{S_1, \ldots, S_m\}$, we have $G^{(j)} \supseteq G^{(j+1)}$ for all $j$ such that $G^{(j+1)}$ is defined. For $1 \leq j < T$, let $C_j$ be the set of centers returned by Algorithm 3 in Line 8 in the $j$-th call of Algorithm 4 that belong to a group not in $G^{(j)}$ (in Algorithm 4, the set of these centers is denoted by $C''$). We analogously define $C_T$ if in the $T$-th call of Algorithm 4 the algorithm terminates from Line 10. Note that the centers in $C_j$ are comprised in the final output $C^A$ of Algorithm 4, that is
We first prove by induction that for all \( j \geq 1 \) such that \( G^{(j)} \) is defined, that is \( 1 \leq j < T \) or \( 1 \leq j \leq T \), we have

\[
\min_{c \in C_0 \cup \bigcup_{i=1}^j C_i} d(s, c) \leq (2^{j+1} + 2^j - 2)r^*_\text{fair}, \quad s \in \left( S^{(j)} \setminus S^{(j+1)} \right) \cup \left( C_0 \cup \bigcup_{i=1}^j C_i \right).
\]

(11)

**Base case** \( j = 1 \): In the first call of Algorithm 4, Algorithm 1, when called in Line 3 of Algorithm 4, returns an approximate solution to the unfair problem (3). Let \( r^* \leq r^*_\text{fair} \) be the optimal cost of (3). Since Algorithm 1 is a 2-approximation algorithm for (3) according to Lemma 1, after Line 3 of Algorithm 4 we have

\[
\min_{c \in C_0} d(s, c) \leq 2r^* \leq 2r^*_\text{fair}, \quad s \in S.
\]

Let \( \tilde{c}_1^A \in \tilde{C}^A \) be a center and \( s_1, s_2 \in L_i \) be two points in its cluster. It follows from the triangle inequality that \( d(s_1, s_2) \leq d(s_1, \tilde{c}_1^A) + d(\tilde{c}_1^A, s_2) \leq 4r^*_\text{fair} \). Hence, after running Algorithm 3 in Line 8 of Algorithm 4 and exchanging some of the centers in \( \tilde{C}^A \), we have \( d(s, c(s)) \leq 4r^*_\text{fair} \) for every \( s \in S \), where \( c(s) \) denotes the center of its cluster. In particular,

\[
\min_{c \in C_0 \cup C_1} d(s, c) \leq (2^{j+1} + 2^j - 2)r^*_\text{fair} = 4r^*_\text{fair}
\]

for all \( s \in S \) for which its center \( c(s) \) is in \( C_0 \) or in a group not in \( G^{(1)} \), that is for \( s \in \left( S^{(1)} \setminus S^{(2)} \right) \cup \left( C_0 \cup C_1 \right) \).

**Inductive step** \( j \mapsto j + 1 \): Recall property (4) of a set \( G \) returned by Algorithm 3. Consequently, \( S^{(j+1)} \) only comprises items in a group in \( G^{(j)} \) and, additionally, the given centers \( C_0 \cup \bigcup_{i=1}^j C_i \).

We split \( S^{(j+1)} \) into two subsets \( S^{(j+1)}_a = S^{(j+1)} \cup S^{(j+1)}_b \), where

\[
S^{(j+1)}_a = \left\{ s \in S^{(j+1)} : \arg\min_{c \in C_0 \cup C_0} d(s, c) \cap \left( C_0 \cup \bigcup_{W \in \{S_1, \ldots, S_m\} \setminus G^{(j)}} W \right) \neq \emptyset \right\}
\]

and \( S^{(j+1)}_b = S^{(j+1)} \setminus S^{(j+1)}_a \). For every \( s \in S^{(j+1)}_a \) there exists \( c \in C_0 \cup \bigcup_{W \in \{S_1, \ldots, S_m\} \setminus G^{(j)}} W \) with \( d(s, c) \leq r^*_\text{fair} \). It follows from the inductive hypothesis that there exists \( c' \in C_0 \cup \bigcup_{i=1}^j C_i \) with \( d(c, c') \leq (2^{j+1} + 2^j - 2)r^*_\text{fair} \) and consequently

\[
d(s, c') \leq d(s, c) + d(c, c') \leq r^*_\text{fair} + (2^{j+1} + 2^j - 2)r^*_\text{fair} = (2^{j+1} + 2^j - 1)r^*_\text{fair}.
\]

Hence,

\[
\min_{c \in C_0 \cup \bigcup_{i=1}^j C_i} d(s, c) \leq (2^{j+1} + 2^j - 1)r^*_\text{fair}, \quad s \in S^{(j+1)}_a.
\]

(12)

For every \( s \in S^{(j+1)}_b \) there exists \( c \in C^{*}_\text{fair} \cap \bigcup_{W \in G^{(j)}} W \) with \( d(s, c) \leq r^*_\text{fair} \). Let \( C^{*}_\text{fair} \cap \bigcup_{W \in G^{(j)}} W = \{ \tilde{c}_1^*, \ldots, \tilde{c}_{\tilde{k}}^* \} \) with \( \tilde{k} = \sum_{W \in G^{(j)}} k_W \), where \( k_W \) is the number of requested centers from group \( W \). We can write

\[
S^{(j+1)}_b = \bigcup_{l=1}^{\tilde{k}} \left\{ s \in S^{(j+1)}_b : d(s, \tilde{c}^{(j)}_l) \leq r^*_\text{fair} \right\},
\]
where some of the sets in this union might be empty, but that does not matter. Note that for every \( l = 1, \ldots, \tilde{k} \) we have
\[
d(s, s') \leq 2r_{\text{fair}}^*, \quad s, s' \in \left\{ s \in S_b^{(j+1)} : d(s, \tilde{c}_l^*) \leq r_{\text{fair}}^* \right\}
\]
due to the triangle inequality. It is
\[
S^{(j+1)} = S_a^{(j+1)} \cup S_b^{(j+1)} = S_a^{(j+1)} \cup \bigcup_{l=1}^{\tilde{k}} \left\{ s \in S_b^{(j+1)} : d(s, \tilde{c}_l^*) \leq r_{\text{fair}}^* \right\}
\]
and when, in Line 3 of Algorithm 4, we run Algorithm 1 on \( S^{(j+1)} \) with \( k = \tilde{k} \) and initial centers \( C_0 \cup \bigcup_{l=1}^{j} C_l \), one of the following three cases has to happen (we denote the centers returned by Algorithm 1 in this \((j+1)\)-th call of Algorithm 4 by \( \tilde{F} = \{ \tilde{f}_1^A, \ldots, \tilde{f}_{\tilde{k}}^A \} \) and assume that for \( 1 \leq l < l' \leq \tilde{k} \) Algorithm 1 has chosen \( \tilde{f}_l^A \) before \( \tilde{f}_{l'}^A \):

- For every \( l \in \{1, \ldots, \tilde{k}\} \) there exists \( l' \in \{1, \ldots, \tilde{k}\} \) such that \( \tilde{f}_{l'}^A \in S_b^{(j+1)} \). In this case it immediately follows that
  \[
  \min_{c \in \tilde{F}^A} d(s, c) \leq 2r_{\text{fair}}^*, \quad s \in S_b^{(j+1)},
  \]
  and using (12) we obtain
  \[
  \min_{c \in C_0 \cup \bigcup_{l=1}^{j} C_l \cup \tilde{F}^A} d(s, c) \leq (2^{j+1} + 2^j - 1)r_{\text{fair}}^*, \quad s \in S^{(j+1)}.
  \]

- There exists \( l' \in \{1, \ldots, \tilde{k}\} \) such that \( \tilde{f}_{l'}^A \in S_a^{(j+1)} \). When Algorithm 1 picks \( \tilde{f}_{l'}^A \), any other element in \( S^{(j+1)} \) cannot be at a larger minimum distance from a center in \( C_0 \cup \bigcup_{l=1}^{j} C_l \) or an already chosen center in \( \{ \tilde{f}_1^A, \ldots, \tilde{f}_{l' - 1}^A \} \) than \( \tilde{f}_{l'}^A \). It follows from (12) that
  \[
  \min_{c \in C_0 \cup \bigcup_{l=1}^{j} C_l \cup \tilde{F}^A} d(s, c) \leq (2^{j+1} + 2^j - 1)r_{\text{fair}}^*, \quad s \in S^{(j+1)}.
  \]

- There exist \( l \in \{1, \ldots, \tilde{k}\} \) and \( l', l'' \in \{1, \ldots, \tilde{k}\} \) with \( l' < l'' \) such that \( \tilde{f}_{l'}^A, \tilde{f}_{l''}^A \in S_b^{(j+1)} \) : \( d(s, \tilde{c}_l^*) \leq r_{\text{fair}}^* \). When Algorithm 1 picks \( \tilde{f}_{l''}^A \), any other element in \( S^{(j+1)} \) cannot be at a larger minimum distance from a center in \( C_0 \cup \bigcup_{l=1}^{j} C_l \) or an already chosen center in \( \{ \tilde{f}_1^A, \ldots, \tilde{f}_{l' - 1}^A \} \) than \( \tilde{f}_{l''}^A \). Because of \( d(\tilde{f}_{l''}^A, \tilde{f}_{l'}^A) \leq 2r_{\text{fair}}^* \) according to (13), it follows that
  \[
  \min_{c \in C_0 \cup \bigcup_{l=1}^{j} C_l \cup \tilde{F}^A} d(s, c) \leq 2r_{\text{fair}}^* \leq (2^{j+1} + 2^j - 1)r_{\text{fair}}^*, \quad s \in S^{(j+1)}.
  \]
In any case, we have
\[
\min_{c \in C_0 \cup \bigcup_{l=1}^{j} C_l \cup \tilde{F}^A} d(s, c) \leq (2^{j+1} + 2^j - 1)r_{\text{fair}}^*, \quad s \in S^{(j+1)}, \quad (14)
\]
Similarly to the base case, it follows from the triangle inequality that after running Algorithm 3 in Line 8 of Algorithm 4 and exchanging some of the centers in \( \tilde{F}^A \), we have
\[
d(s, c(s)) \leq 2(2^{j+1} + 2^j - 1)r_{\text{fair}}^* = (2^{j+2} + 2^{j+1} - 2)r_{\text{fair}}^*
\]
for every \( s \in S^{(j+1)} \), where \( c(s) \) denotes the center of its cluster. In particular, we have
\[
\min_{c \in C_0 \cup \bigcup_{l=1}^{j} C_l} d(s, c) \leq (2^{j+2} + 2^{j+1} - 2)r_{\text{fair}}^*, \quad s \in \left( S^{(j+1)} \setminus S^{(j+2)} \right) \cup \left( C_0 \cup \bigcup_{l=1}^{j+1} C_l \right),
\]
and this completes the proof of (11).
If in the $T$-th call of Algorithm 4 the algorithm terminates from Line 10, it follows from (11) that
\[
\min_{c \in C_0 \cup \bigcup_{l=1}^{T-1} C_l} d(s, c) \leq (2^{T+1} + 2^T - 2)r_{\text{fair}}^*, \quad s \in S.
\] (15)
In this case, since $T < m$, we have
\[
2^{T+1} + 2^T - 2 \leq 2^m + 2^{m-1} - 2 < 2^m + 2^{m-1} - 1,
\]
and (15) implies (10). If in the $T$-th call of Algorithm 4 the algorithm does not terminate from Line 10, it must terminate from Line 5. It follows from (11) that
\[
\min_{c \in C_0 \cup \bigcup_{l=1}^{T} C_l} d(s, c) \leq (2^T + 2^{T-1} - 2)r_{\text{fair}}^*, \quad s \in \left(S \setminus S^{(T)}\right) \cup \left(C_0 \cup \bigcup_{l=1}^{T-1} C_l\right).
\] (16)
In the same way as we have shown (14) in the inductive step in the proof of (11), we can show that
\[
\min_{c \in C_0 \cup \bigcup_{l=1}^{T} C_l \cup \tilde{H}^A} d(s, c) \leq (2^T + 2^{T-1} - 1)r_{\text{fair}}^* \leq (2^m + 2^{m-1} - 1)r_{\text{fair}}^*, \quad s \in S^{(T)},
\] (17)
where $\tilde{H}^A$ is the set of centers returned by Algorithm 1 in the $T$-th call of Algorithm 4. Since $\bigcup_{l=1}^{T-1} C_l \cup \tilde{H}^A$ is contained in the output $C^A$ of Algorithm 4, (17) together with (16) implies (10).

Since running Algorithm 4 involves at most $m$ (recursive) calls of the algorithm and the running time of each of these calls is dominated by the running times of Algorithm 1 and Algorithm 3, it follows that the running time of Algorithm 4 is $O((|C_0| m + km^2)|S| + km^4)$.

Proof of Lemma 3:
Consider the example given by the weighted graph shown in Figure 8, where $0 < \delta < \frac{1}{10}$. We have $S = S_1 \cup S_2 \cup S_3$ with $S_1 = \{m_1, m_2, m_3, m_4, m_5, m_6\}$, $S_2 = \{f_1, f_2, f_3, f_4\}$ and $S_3 = \{z_1, z_2\}$. All distances are shortest-path-distances. Let $k_{S_1} = 4, k_{S_2} = 1, k_{S_3} = 1$ and $C_0 = \emptyset$. We assume that Algorithm 1 in Line 3 of Algorithm 4 picks $f_1$ as first center. It then chooses $f_4$ as second center, $z_1$ as third center, $f_3$ as fourth center, $f_2$ as fifth center and $z_2$ as sixth center. Hence, $\tilde{C}^A = \{f_1, f_4, z_1, f_3, f_2, z_2\}$ and the corresponding clusters are \{f_1, m_1, m_2, m_5\}, \{f_4, m_3, m_4, m_6\}, \{z_1\}, \{f_3\}, \{f_2\} and \{z_2\}. When running Algorithm 3 in Line 8 of Algorithm 4, it replaces $f_1$ with one of $m_1, m_2$ or $m_5$ and it replaces $f_4$ with one of $m_3, m_4$ or $m_6$. Assume that it replaces $f_1$ with $m_2$ and $f_4$ with $m_4$. Algorithm 3 then returns $G = \{S_2, S_3\}$.

Figure 8. An example showing that Algorithm 4 is not a $(8 - \varepsilon)$-approximation algorithm for any $\varepsilon > 0$. 

\[ 
\begin{array}{c}
\text{m1} - 1 + \frac{3\delta}{2} \text{f2} - 4 + \frac{\delta}{2} \text{f3} - 4 \text{f4} - 1 + \delta \text{m3} \\
\text{f1} - 1 \text{\quad} 2 - \delta \quad 12 \quad 8 \quad 2 - \delta \quad 1 \quad \text{f3} - 4 \\
\text{m2} - \delta \quad \text{m0} \quad \text{m0} \quad \text{m0} \end{array} \]
and when recursively calling Algorithm 4 in Line 12, we have $S' = \{f_2, f_3, z_1, z_2\}$ and $C' = \{m_2, m_4\}$. In the recursive call, the given centers are $C'$ and Algorithm 1 chooses $f_3$ and $f_2$. The corresponding clusters are $\{f_2, z_1, z_2\}$, $\{f_2\}$, $\{m_2\}$ and $\{m_4\}$. When running Algorithm 3 with clusters $\{f_3, z_1, z_2\}$ and $\{f_2\}$, it replaces $f_3$ with either $z_1$ or $z_2$ and returns $G = 0$, that is afterwards we are done. Assume Algorithm 3 replaces $f_3$ with $z_2$. Then the centers returned by Algorithm 4 are $z_2, f_2, m_2, m_4$ and two arbitrary elements from $S_1$, which we assume to be $m_5$ and $m_6$. These centers have a cost of 8 (incurred for $z_1$). However, an optimal solution such as $C'_{\text{fair}} = \{m_1, m_2, m_3, m_4, f_3, z_1\}$ has cost only $1 + \frac{3\delta}{2}$. Choosing $\delta$ sufficiently small shows that Algorithm 4 is not a $(8 - \varepsilon)$-approximation algorithm for any $\varepsilon > 0$. □

B. Further Experiments

In Figure 9 we show the costs of the approximate solutions produced by our algorithm (Alg. 4) and the algorithm by Chen et al. (2016) (M.C.) in the run time experiment shown in the right part of Figure 3. Apparently, both methods perform similarly. In Figure 10, Figure 11 and Figure 12 we provide similar experiments as shown in Figure 6, Figure 2 and Figure 5, respectively.

![Figure 9. Cost of the output of our algorithm (Alg. 4) in comparison to the algorithm by Chen et al. (M.C.) in the run time experiment shown in the right part of Figure 3.](image)

![Figure 10. Similar experiments on the Adult data set as shown in Figure 6, but with different values of $k_{S_i}$. 1st plot: $m = 2$, $k_{S_1} = 300$, $k_{S_2} = 100$ ($S_1$ corresponds to male and $S_2$ to female). 2nd plot: $m = 2$, $k_{S_1} = k_{S_2} = 25$. 3rd plot: $m = 5$, $k_{S_1} = 214$, $k_{S_2} = 8$, $k_{S_3} = 2$, $k_{S_4} = 2$, $k_{S_5} = 24$ ($S_1 \sim$ White, $S_2 \sim$ Asian-Pac-Islander, $S_3 \sim$ Amer-Indian-Eskimo, $S_4 \sim$ Other, $S_5 \sim$ Black). 4th plot: $m = 5$, $k_{S_1} = k_{S_2} = k_{S_3} = k_{S_4} = k_{S_5} = 10$.](image)
Figure 11. Similar experiments as shown in Figure 2. A data set consisting of 16 images of faces (8 female, 8 male) and six summaries computed by the unfair Algorithm 1, our algorithm and the algorithm of Celis et al. (2018b). The images are taken from the FEI face database publicly available on https://fei.edu.br/~cet/facedatabase.html.
Figure 12. Similar experiments on the Adult data set as shown in Figure 5, but with different values of $k_s$. **Top left:** $m = 2, k_{S_1} = 300$, $k_{S_2} = 100$ ($S_1$ corresponds to male and $S_2$ to female). **Top right:** $m = 2, k_{S_1} = k_{S_2} = 25$. **Bottom left:** $m = 5, k_{S_1} = 214$, $k_{S_2} = 8, k_{S_3} = 2, k_{S_4} = 24$ ($S_1 \sim $ White, $S_2 \sim $ Asian-Pac-Islander, $S_3 \sim $ Amer-Indian-Eskimo, $S_4 \sim $ Other, $S_5 \sim $ Black). **Bottom right:** $m = 5, k_{S_1} = k_{S_2} = k_{S_3} = k_{S_4} = k_{S_5} = 10$. 