Effect of Primordial Magnetic Field on Seeds for Large Scale Structure

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(Dated: March 31, 2021)

Magnetic field plays a very important role in many astronomical phenomena at various scales of the universe. It is no exception in the early universe. Since the energy density, pressure, and tension of the primordial magnetic field affect gravitational collapses of plasma, the formation of seeds for large scale structures should be influenced by them. Here we numerically investigate the effects of stochastic primordial magnetic field on the seeds of large scale structures in the universe in detail. We found that the amplitude ratio between the density spectra with and without PMF \(|P(k)/P_0(k)|\) at \(k > 0.2\) Mpc\(^{-1}\) lies between 75% and 130% at present for the range of PMF strengths \(0.5\) nG < \(B_\lambda\) < \(1.0\) nG, depending on the spectral index of PMF and the correlation between the matter density and the PMF distributions.

PACS numbers: 98.65.Dx

I. INTRODUCTION

The possible existence of a primordial magnetic field (PMF) is an important consideration in modern cosmology. The origin of PMFs has been intensively studied \[1, 2, 3, 4, 5, 6, 7, 8\]. Although it seems unlikely to survive an epoch of inflation, it is conceivable that large-scale magnetic fields and magnetic inhomogeneities could be generated at the end of that era or in subsequent phase transitions \[9\]. Studies of magnetogenesis are partly motivated by the need to explain the origin of large-scale magnetic fields which are observed in galaxies or in clusters of galaxies \[10\]. If these magnetic fields have the primordial origin, the PMF should have influenced a variety of phenomena in the early universe \[11, 12, 13, 14, 15, 16, 17, 18\]. Several semi-analytic and numerical studies \[13, 14, 15, 16, 17, 18\] pointed out that the effect of the PMF is one of the new physical processes in the early universe.

Large-scale magnetic fields are now considered to be standard components. Even at high redshift, the existence of dynamically significant magnetic fields is suggested from observations of Faraday rotation associated with high redshift Lyman-\(\alpha\) absorption systems \[10\]. However their picture in the early universe is not clear and is still a matter of debate. If dynamically significant large-scale magnetic fields were present in the early universe, they must have affected the formation and evolution of the observed structure, and some signatures of their past should be included in their structure and spectrum. Indeed, high resolution Faraday rotation maps and the study of extragalactic cosmic rays provide direct observational support for this point of view \[21\].

In the evolution of baryonic and dark matter at large scales, accretion shocks form in the infalling flows towards the growing nonlinear structures such as sheets, filaments and clusters (e.g., \[22, 23\]). In fact, some evidence of these accretion shocks has been detected in radio relic sources \[24\]. The properties of the shocks depend on the power spectrum of the initial perturbations of baryonic and dark matter on a given scale as well as the background expansion in a given cosmological model. Therefore, if there existed dynamically significant large-scale magnetic fields in the early universe, the dynamical influence of magnetic fields on the spectrum of perturbations and thus on flow motions should not be ignored. From this point of view, investigations of the physical process in the early universe with PMF provide important suggestions for studies of the formation and evolution of large scale structures (LSS).

In this paper, we assume the existence of a large-scale PMF and analyze the magnetic effects on density inhomogeneities numerically. Effects of PMF on density fields, especially on cosmic microwave background (CMB) anisotropies at the photons’ last scattering surface (PLSS), were studied by considering the scalar-type component of energy momentum tensor in PMF \[25\], or including a new analytical magnetic power spectrum source due to a Lorentz force without previous approximations. Here we include both effects consistently and extend their previous studies.
by expanding the analysis from the epoch of creation of CMB anisotropies toward the present epoch. In particular, we consider the different correlation between the PMF and the matter power spectrum, and investigate their effects on the density fields at large scales in the presence of PMF.

Throughout this paper we fix the cosmological parameters as follows [27]: $h = 0.71$, $\Omega_b = 0.044$, $\Omega_{CDM} = 0.226$, $n_s = 0.93$, and $\tau_c = 0.10$ in flat Universe models (thus $\Omega = 1 - \Omega_b - \Omega_{CDM} = 0.73$), where $h$ denotes Hubble parameter in units of 100 km/s/Mpc, $\Omega_b$ and $\Omega_m$ are the baryon and cold dark matter densities in critical density units, $n_s$ is the spectral index of the primordial scalar fluctuation, and $\tau_c$ is the optical depth of Compton scattering.

II. COSMOLOGICAL PERTURBATIONS WITH STOCHASTIC PRIMORDIAL MAGNETIC FIELD

A. Cosmological MHD

Let us consider the PMF created by some effects during the radiation-dominated epoch. The energy density of the magnetic field is treated as a first order perturbation in a flat Friedmann-Robertson-Walker (FRW) background cosmology. The electromagnetic tensor has the usual form

$$F^{\alpha \beta} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix},$$

(1)

where $E_i$ and $B_i$ are the electric and magnetic fields. The energy momentum tensor for electromagnetism is

$$T^{\alpha \beta}_{[EM]} = \frac{1}{4\pi} \left( F^{\gamma \gamma} F_{\gamma \delta} - \frac{1}{4} g^{\alpha \beta} F_{\gamma \delta} F^{\gamma \delta} \right).$$

(2)

The Maxwell stress tensor, consisting of the space-space components of the energy momentum tensor of the electromagnetic field, is

$$-T^{ik}_{[EM]} = \sigma^{ik} = \frac{1}{a^2} \frac{1}{4\pi} \left( E^i E^k + B^i B^k - \frac{1}{2} \delta^{ik} (E^2 + B^2) \right).$$

(3)

Within the linear approximation, the magnetic field evolves as a stiff source. Therefore we can discard all back reactions from the magneto hydrodynamic (MHD) fluid onto the field itself. The conductivity of the primordial plasma is very large, and the field is "frozen-in" [14]. This is a very good approximation for the epochs in which we are interested. Furthermore, we can neglect the electric field, $E \sim 0$, and also decouple the time evolution of the magnetic field from its spatial dependence, i.e., $B(\tau, \mathbf{x}) = B_0(\mathbf{x})/a^2$ for very large scales. In this way we obtain the following expressions,

$$T^{00}_{[EM]} = \frac{B^2}{8\pi a^6},$$

(4)

$$T^{i0}_{[EM]} = T^{0k}_{[EM]} = 0,$$

(5)

$$-T^{ik}_{[EM]} = \sigma^{ik} = \frac{1}{8\pi a^6} (2B^i B^k - \delta^{ik} B^2).$$

(6)

First and second terms in Eq.(6) are magnetic tension and pressure, respectively.

B. Evolution Equations of Cosmological perturbations

Combining the Einstein equations and linearizing them, we obtain the perturbation evolution equations. In order to consider the effect of the PMF, we should add the electromagnetic energy momentum tensor to that of standard cosmic fluids,

$$T^{\alpha \beta} = T^{\alpha \beta}_{[FL]} + T^{\alpha \beta}_{[EM]}.$$  

(7)

We assume that the baryonic matter is representable as a perfect fluid and neglect the anisotropic pressure perturbations. As an initial condition, we consider adiabatic perturbations and neglect entropy perturbations initially. The
line element in the conformal synchronous gauge for a flat Friedmann-Robertson-Walker (FRW) background is given by
\[ ds^2 = a^2(\tau)[-dt^2 + (\delta_{ij} + h_{ij})dx^i dx^j], \]
where \( x^i \) is the spatial coordinate, \( a(\tau) \) is scale factor, \( \tau \) is the conformal time defined by \( d\tau = dt/a(\tau) \), \( h_{ij} \) are metric perturbations around the background spacetime, and the speed of light is set to unity. We will be working in the Fourier space in this paper. We introduce two fields \( h(\mathbf{k}, \tau) \) and \( \eta(\mathbf{k}, \tau) \) in \( k \)-space and write the scalar mode of \( h_{ij} \) as a Fourier integral
\[ h_{ij}(x, \tau) = \int d^3k e^{i\mathbf{k}\cdot\mathbf{x}} \left[ \delta_{ij} + 6\eta(\mathbf{k}, \tau) \left( \frac{1}{3} \delta_{ij} - \frac{\delta_{ij} T_k^k}{3} \right) \right], \]
where \( k \) is the wave number in the Fourier space and \( \mathbf{k} = \mathbf{k} \) and \( \mathbf{k} \) is a unit vector of wave number.

The linearized perturbation equations are obtained from the Einstein equations up to first order \[25, 31, 32\]:
\[ k^2\eta - \frac{1}{2} H \dot{h} = 4\pi G a^2 \delta \tau_0^0, \]  \[ k^2\dot{\eta} = 4\pi G a^2 k^3 \delta T^0_j, \]  \[ \ddot{h} + 2H \dot{h} - 2k^2\eta = 8\pi G a^2 \dot{T}_i^i, \]  \[ \ddot{\eta} + 6\ddot{\eta} + 2H(\dot{h} + 6\dot{\eta}) - 2k^2\eta = 24\pi G a^2 \left( \delta_{ij} k_i \cdot k_j - \frac{1}{3} \delta_{ij} \right) \left( \delta T^i_j - \frac{\delta_{ij} T^k_k}{3} \right), \]
where
\[ \delta T^0_0 = \delta \rho = \delta \rho_{[\text{FL}]} + \delta \rho_{[\text{EM}]}, \]  \[ ik^3 \delta T^0_j = (\rho + p)v, \]  \[ \dot{T}^i_j = \delta T^i_j_{[\text{FL}]} + \delta T^i_j_{[\text{EM}]} , \]
\( Z_{[\text{EM,SI}]}(k) \) is a scalar part of the magnetic shear stress, and \((\rho + p)\sigma\) is a fluid shear stress. We assume that the PMF \( B_0 \) is statistically homogeneous, isotropic and random. For such a magnetic field, the power spectrum can be taken as a power-law \( S(k) = \langle B(k)B^*(k) \rangle \propto k^n \) \[14\] where \( n \) is the power-law spectral index of the PMF. The index \( n \) can be either negative or positive depending on the physical processes of the creation. From ref. \[14\], a two-point correlation function for PMF is defined by
\[ \langle B^i(k)B^i*(k') \rangle = \frac{2\pi n + 8}{2k^{n+3}_\lambda} \frac{B^2}{(\frac{n}{n+2})} k^n \rho_{ij}(k) \delta(k - k'), \quad k < k_C, \]
where
\[ P^{ij}(k) = \delta^{ij} - \frac{k^i k^j}{k^2}, \]
\( B_\lambda \) is the magnetic comoving mean-field amplitude obtained by smoothing over a Gaussian sphere of comoving radius \( \lambda \), and \( k_\lambda = 2\pi/\lambda \) (\( \lambda = 1 \) Mpc in this paper). The cutoff wave number \( k_C \) in the magnetic power spectrum is defined by \[28\],
\[ k^{-5-n}_C(\tau) = \begin{cases} \frac{B^2}{4\pi(n+3)} \int_{\tau}^{\tau_{\text{dec}}} d\tau l_{\lambda}^3, & \tau < \tau_{\text{dec}} \\ k^{-5-n}_C(\tau_{\text{dec}}), & \tau > \tau_{\text{dec}}, \end{cases} \]
where \( l_\lambda \) is the mean free path of photons, and \( \tau_{\text{dec}} \) is the time of the decoupling of photons from baryons (see appendix B). Evaluating the two-point correlation function of the electromagnetic stress-energy tensor in the Fourier space, we obtain the power spectrum of PMF energy density, Lorenz force and shear stress as the following (see appendix)
\[ |E_{[\text{EM,SI}]}(\mathbf{k}, \tau)|^2 \delta(k - k') = \frac{1}{(2\pi)^3} \langle T(\mathbf{k}, \tau)_{[\text{EM,SI}]} T^*(\mathbf{k'}, \tau)_{[\text{EM,SI}]} \rangle, \]
and
\[ |Z(k)_{[EM:S]}|^2 \delta(k-k') = \frac{1}{(2\pi)^3} \left\langle \left( \frac{2}{3} T(k, \tau)_{[EM:S1]} - T(k, \tau)_{[EM:S2]} \right) \left( \frac{2}{3} T^*(k', \tau)_{[EM:S1]} - T^*(k', \tau)_{[EM:S2]} \right) \right\rangle, \]  \hspace{1cm} (22)

respectively. Explicit expressions for the ensemble averages to evaluate the above spectra, in the case of power law stochastic magnetic field, are given as (also see appendix)

\[
\left\langle T(k, \tau)_{[EM:S1]} T^*(k, \tau)_{[EM:S1]} \right\rangle = \frac{1}{4(2\pi)^{n+8}} \left\langle \frac{B^2}{2k^{n+3}} \frac{(n+2)}{2} \right\rangle^2 \int \frac{k'k'^n+2}{(k'k+n+2)(n+4)} \left\{ (k+k')^{n+2} - |k-k'|^{n+2} \right\} \]

\[
- \frac{1}{k'^2n(n+4)} \left\{ |k-k'|^{n+2} + |k+k'|^{n+2} \right\} + \frac{k}{k'^2n(n+4)} \left\{ (k+k')^{n+2} - |k-k'|^{n+2} \right\}, \]  \hspace{1cm} (23)

\[
\left\langle T_{[EM:S1]}(k)T_{[EM:S2]}^*(k) \right\rangle + \left\langle T_{[EM:S2]}(k)T_{[EM:S1]}^*(k) \right\rangle = \frac{1}{(2\pi)^{n+8}} \left\langle \frac{B^2}{2k^{n+3}} \frac{(n+2)}{2} \right\rangle^2 \int \frac{k'k'^n+3}{(k'k')^n(n+2)} \left\{ (k+k')^{n+3} - |k-k'|^{n+3} \right\}

- \frac{3}{k'^2k'^n(n+2)(n+4)} \left\{ |k-k'|^{n+4} + (k+k')^{n+4} \right\} - \frac{1}{k'^3k'^n(n+2)(n+4)} \left\{ (k+k')^{n+4} - |k-k'|^{n+4} \right\}

+ \frac{3}{k'^4k'^n(n+2)(n+4)(n+6)} \left\{ (k+k')^{n+6} - |k-k'|^{n+6} \right\}, \]  \hspace{1cm} (24)

and

\[
\left\langle T_{[EM:S2]}(k)T_{[EM:S2]}^*(k) \right\rangle = \frac{1}{(2\pi)^{n+8}} \left\langle \frac{B^2}{2k^{n+3}} \frac{(n+2)}{2} \right\rangle^2 \int \frac{k'k'^n+4}{(k'k')^n(n+2)(n+4)} \left\{ (k+k')^{n+4} - |k-k'|^{n+4} \right\}

- \frac{3}{(k'k')^n(n+6)} \left\{ |k-k'|^{n+6} + (k+k')^{n+6} \right\} + \frac{3}{(k'k')^n(n+2)(n+8)} \left\{ (k+k')^{n+8} - |k-k'|^{n+8} \right\}. \]  \hspace{1cm} (25)

The evolutions of fluid variables can be obtained by imposing the conservation of energy-momentum, which is a consequence of the Einstein equations

\[ T^\mu_{\nu; \mu} = \delta_{\nu} T^\mu_{\nu} + \Gamma^\nu_{\alpha \beta} T^\alpha_{\beta} + \Gamma^\alpha_{\alpha \beta} T^\nu_{\beta} = 0. \]  \hspace{1cm} (26)

This leads the following equations in k-space:

\[
\dot{\delta} = -(1+w) \left( v + \frac{\dot{h}}{2} \right) - 3H \left( \frac{\dot{\rho}}{\rho} - w \right) \delta - \frac{3}{8\pi \rho} \left\{ \dot{E}_{[EM:S]}(k, \tau) + 6HE_{[EM:S]}(k, \tau) \right\}, \]  \hspace{1cm} (27)

\[
\dot{v} = -H(1-3w)v - \frac{\dot{w}}{1+w} v + \frac{\dot{\rho}}{\rho} k^2 \delta - k^2 \sigma + k^2 \frac{\Pi_{[EM:S]}(k, \tau)}{4\pi \rho}, \]  \hspace{1cm} (28)

where \( w \equiv p/\rho \). In the continuity and Euler equations for the scalar mode, we can just add the energy density and pressure of the PMF to the energy density and pressure of cosmic fluids, respectively. Since baryon fluid behaves as a nonrelativistic fluid in the epoch of interest, we may neglect \( w \) and \( \delta p_b/\rho_b \), except the acoustic term \( c_s^2 k^2 \delta \). Also, the shear stress of baryons is far smaller, and we can neglect it \([32]\). Since we concentrate on scalar type perturbations in this paper, we do not consider the magneto-rotational instability from the shear stress of the PMF and baryon fluid \([33]\).
From equations (27) and (28), and by considering Compton interaction between baryons and photons, we obtain the same form of the evolution equations of photons and baryons as in previous works [26, 31, 32],

\begin{align}
\dot{\delta}_{\text{CDM}} &= -\frac{1}{2} \dot{h}, \\
\dot{\delta}_\gamma &= -\frac{4}{3} v_\gamma - \frac{2}{3} \dot{h}, \\
\dot{v}_\gamma &= k^2 \left( \frac{1}{4} \delta_\gamma - \sigma_\gamma \right) + a n_e \sigma_T (v_b - v_\gamma), \\
\dot{\delta}_b &= -v_b - \frac{1}{2} \dot{h}, \\
v_b &= -\frac{\dot{\gamma}}{\delta} v_b + \frac{\gamma}{\delta} k^2 \delta_b + \frac{4\dot{\rho}_b}{3\rho_b} a n_e \sigma_T (v_\gamma - v_b) + k^2 \frac{\Pi_{\text{EM:S}}(k, \tau)}{4\pi \rho_b},
\end{align}

where \( n_e \) is the free electron density, \( \sigma_T \) is the Thomson scattering cross section, and \( \sigma_\gamma \) of the second term on the right hand side of equation (31) is the shear stress of the photon with the PMF. Since \( n \lesssim 0 \) is favored by constraints from the gravitational wave background [35] and the effect of PMF is not influenced by the time evolution of the cut off scale \( kC \) for this range of \( n \), we approximately set \( E \propto a^{-6} \) in the following analysis.

### III. CORRELATION IN POWER SPECTRA

In this section, we define a power spectrum function of matter density with the PMF. In the linear approximation, solutions of eqs. (29) + (33) are divided into those with and without PMF by Green’s function method as,

\[ \delta(k) = \delta_{\text{FL}}(k) + \delta_{\text{PMF}}(k). \] (34)

Possible origins of PMF have been studied by many authors, however, we have not reached critical conclusions on the origin of PMF. Thus we cannot know how the PMF correlates with the primordial density fluctuations. However, almost all of previous works investigated the effects of PMF on density perturbations with the assumption that there is no correlation between them [20]. In order to study the PMF effect in a more general manner, we have to consider a correlation between the PMF and the primordial density fluctuations. Therefore we introduce "s" to parameterize the correlation between the PMF and the primordial density fluctuations. The power spectra of baryon \( (P_b(k)) \) and CDM \( (P_{\text{CDM}}(k)) \) density with the PMF in the linear approximation are,

\begin{align}
P_b(k) &= \left\langle \delta_{[b, \text{FL}]}(k) \delta_{[b, \text{FL}]}^*(k) \right\rangle + \left\langle \delta_{[b, \text{PMF}]}(k) \delta_{[b, \text{PMF}]}^*(k) \right\rangle \\
&\quad + 2 \left\langle \delta_{[b, \text{FL}]}(k) \delta_{[b, \text{PMF}]}^*(k) \right\rangle, \\
P_{\text{CDM}}(k) &= \left\langle \delta_{[\text{CDM}, \text{FL}]}(k) \delta_{[\text{CDM}, \text{FL}]}^*(k) \right\rangle + \left\langle \delta_{[\text{CDM}, \text{PMF}]}(k) \delta_{[\text{CDM}, \text{PMF}]}^*(k) \right\rangle \\
&\quad + 2 \left\langle \delta_{[\text{CDM}, \text{FL}]}(k) \delta_{[\text{CDM}, \text{PMF}]}^*(k) \right\rangle,
\end{align}

where, we define the cross correlations as,

\begin{align}
\left\langle \delta_{[b, \text{FL}]}(k) \delta_{[b, \text{PMF}]}^*(k) \right\rangle &= s \sqrt{ \left\langle \delta_{[b, \text{FL}]}(k) \delta_{[b, \text{FL}]}^*(k) \right\rangle \left\langle \delta_{[b, \text{PMF}]}(k) \delta_{[b, \text{PMF}]}^*(k) \right\rangle }, \\
\left\langle \delta_{[\text{CDM}, \text{FL}]}(k) \delta_{[\text{CDM}, \text{PMF}]}^*(k) \right\rangle &= s \sqrt{ \left\langle \delta_{[\text{CDM}, \text{FL}]}(k) \delta_{[\text{CDM}, \text{FL}]}^*(k) \right\rangle \left\langle \delta_{[\text{CDM}, \text{PMF}]}(k) \delta_{[\text{CDM}, \text{PMF}]}^*(k) \right\rangle },
\end{align}

where \( \delta_\alpha, \alpha \in ([b : \text{FL}], [\text{CDM} : \text{FL}]) \) are baryon and CDM density fluctuations without the PMF respectively, and \( \delta_{\beta, \beta} \in ([b : \text{PMF}], [\text{CDM} : \text{PMF}]) \) are baryon and CDM density fluctuations with the PMF, respectively. When \( 0 < s \leq 1, s = 0, \) and \( -1 \leq s < 0 \) on eqs. (37) and (38), they stand for the positive, no, and negative correlations, respectively.

The square root of power spectrum functions of Lorenz force \( \Pi_{\text{EM:S}}(k, \tau) \) in Eq. (33) does not have information about negative or positive, in other words, there is no information which of the magnetic pressure or tension is dominant, and whether the directions of forces from them are same or different. However, there must be such information, so we should take it into account. The Lorenz force term in Eq. (33) can be divided into two terms, the magnetic pressure
and the tension, of which amplitudes are decided by eqs. (23) and (25), respectively. Comparing those equations, we can decide which of them is dominant in the Lorenz force term. To answer the first question we show in which of the magnetic pressure or the tension dominates in the Lorentz force term. The former dominates when \( n < -1.5 \), the latter does when \( n > -1.5 \).

As for the second question, we found that the roles of magnetic field pressure and tension in the Lorentz force term are different from each other for the random primordial magnetic field defined in Eq.(17). In other words, the scalar force from magnetic field tension acts on density field in the opposite direction in a statistical sense from what magnetic field pressure does. The reason is that, the variance of the Lorenz force \( \Pi(k) \) is always smaller than the simple sum of the variances of magnetic field pressure and tension, because the cross correlation between the two, \( T[EM:S1]^*T[EM:S2] \), always gives the positive values for all \( k \) and \( n \) considered in this paper. Since the force field from the magnetic field pressure is directly related to the magnetic field energy density distributions, that from the tension can also be related to the magnetic field energy distributions. Thus we can decompose the factor as

\[
s = s_{[LF]} \times s_{[DF]},
\]

where

\[
s_{[LF]} = \begin{cases} 
-1, & n < -1.5 \quad \text{(I)}, \\
1, & n > -1.5 \quad \text{(II)}, 
\end{cases}
\]

and

\[
0 < s_{[DF]} \leq 1 \quad \text{(i)}, \\
0 = s_{[DF]} \quad \text{(ii)}, \\
-1 \leq s_{[DF]} < 0 \quad \text{(iii)}. 
\]

Here \( s_{[LF]} \) represents (I) pressure dominant case, (II) tension dominant case. On the other hand, \( s_{[DF]} \) represents (i) positive correlation between the matter and PMF distributions, (ii) no correlation, and (iii) negative correlation. Thus if \( s < 0 \), it means that the the matter and PMF distributions are positively correlated (\( s_{[DF]} > 0 \)) and the PMF pressure dominates in the Lorenz term (\( n < -1.5 \)), or the matter and PMF distributions negatively correlate (\( s_{[DF]} < 0 \)) and the PMF tension dominates in the Lorenz term (\( n > -1.5 \)). In these cases the PMF effect acts against the gravitational collapse and makes the evolution of density perturbations slower like the gas pressure. On the other hand, if \( s > 0 \), the matter and PMF distributions positively correlate (\( s_{[DF]} > 0 \)) and the PMF tension dominates in the Lorenz term (\( n > -1.5 \)), or the the matter and PMF distributions negatively correlate (\( s_{[DF]} < 0 \)) and the PMF pressure dominates in the Lorenz term (\( n < -1.5 \)). In these cases the Lorenz force of PMF accelerates the gravitational collapse. The above discussion is transparent especially in the perturbation evolution after decoupling, because \( \delta \) does not oscillate there.

**IV. NUMERICAL RESULTS**

In this section, we show our numerical results. Since the recent upper limit on PMF amplitude by CMB is \( B_\lambda \sim 7.7 \) nG at \( n_B = -1.5001 \)\(^{[18]} \). In order to be consistent with this result, we use the PMF parameter sets in the following calculations as, \( (B_\lambda, n_B) = (0.5 \text{ nG}, -1.0), (0.5 \text{ nG}, -2.001), (1 \text{ nG}, -1.0), \) and \( (1 \text{ nG}, -2.001) \)\(^{[41]} \).

**A. Before decoupling**

The effect of PMF on CDM is much smaller than that on baryon before decoupling because the PMF cannot affect CDM directly. Moreover, the density of baryons oscillates with that of photons and their gravitational effects on CDM are very small. So we only consider the PMF effect on baryons here.

Since the PMF can both effectively increase or decrease the gas pressure, the PMF affects the frequency of acoustic oscillations before the baryon fluid decouples from the photon fluid. This is important when considering the CMB anisotropies. Since the photon tightly couples with the baryon before the PLSS, it is natural that the frequency of oscillation of the photon density is indirectly increased or decreased by the presence of PMF. In this way the PMF affects the CMB photons which we can observe at present (see \(^{[17] [18]} \)).
B. After decoupling

After baryons decoupling from photons, the baryons density evolution starts to affect CDM density evolution through gravitational interaction \[36\]. So the PMF effect on the CDM increases with time through baryons (left panels in Figs. 2 and 3). Once the CDM density fluctuations are generated by the PMF indirectly, they grow due to the gravitational instability so that the growthrate is the same as that of the primordial CDM density fluctuations. The baryons density fluctuations are generated by PMF directly, and therefore, the gravitational potential does not dominate the evolution of baryon density fluctuation until \(4\pi G\rho \delta \sim k^2 \Phi_{\text{PMF}}(k,\tau)\). The baryons density fluctuations with the PMF increase with the wavenumber \(k\) (right panels in Figs. 4 and 5). The reason is that the effect of the PMF is relatively larger at smaller scales (larger \(k\) values) in Eq.(33). From Figs. 4 and 5 we found that the amplitude ratio between the density spectra with and without PMF \(|P(k)/P_0(k)|\) at \(k > 0.2\ Mpc^{-1}\) lies between 75% and 130% at present for the range of PMF parameters \(n = -2.001\) and \(-1.5 < nG < B_\lambda < 1.0\ nG,\) and \(-1 < s < 1\).

V. DISCUSSIONS OF PMF MODEL

Taking into account the stochastic PMF power spectrum sets (eqs.(20)-(22)), we can investigate the most accurate effect of PMF on the evolution of photons, baryons and CDM density fluctuation. From figs.2-5, it is turned out that there is a strong degeneracy of the correlation factor \(s_{\text{DF}}\) and the PMF amplitude \(B_\lambda\). The correlation factor \(s_{\text{DF}}\) depends on the PMF origin. So, in order to find a clue for solving such degeneracy problem, we shall discuss a relation of the correlation factor \(s_{\text{DF}}\) and the PMF origin.

A. Negative correlation of the density perturbations

When \(s_{\text{DF}} < 0\), the tension of PMF delays the evolution of the matter density fluctuation, and the pressure of PMF accelerates it. In this case, therefore, when the PMF pressure for \(n < -1.5\) dominates, the evolution of the matter density fluctuation is accelerated by the PMF. When \(n > -1.5\), on the other hand, the evolution is delayed by the PMF. In order to substantiate such condition, the PMF must have been generated in the lower density regions. In order to substantiate such condition, the PMF must have been generated in the lower density regions. Some previous works mention that the PMF is directly proportional to the matter density on cosmological scale\[39, 40\]. In these cases, since it is natural that there are more amplitude of PMF on the higher density regions in the frozen-in, this condision would be difficult to be realized with the causally generated PMF.

B. No correlation

In the case of the no correlation between the PMF and the matter density fluctuations, the power spectrum function of the matter density fluctuation with the PMF is increased by the PMF, independently dominances of PMF pressure or tension. Here we must notice the difference between the power spectrum function \(P(k)\) and the matter density fluctuation \(\delta\). While the total density fluctuation \(\delta\) can be smaller or larger if the effect of PMF is dominated by its pressure or by its tension, respectively, the power spectrum function \(P(k)\) is always increased when PMF does not correlate with primordial density fluctuations. If the PMF is generated by density fluctuations proposed by \(\delta\), peaks of PMF are at peaks of the pressure gradient of cosmic fluids. Consequently, the PMF is created along the border between high and low density regions, i.e., \(\delta \sim 0\). In this case there would be no (or very weak) correlation between the PMF and density perturbation statistically.

C. Positive correlation

When \(s_{\text{DF}} > 0\), the pressure of PMF delays the evolution of the matter density fluctuation, and the tension of PMF accelerates it. So, in this case, when the PMF pressure dominates for \(n < -1.5\), the evolution of the matter density fluctuation is delayed by the PMF. In order to substantiate such condition, the PMF may be generated on the higher energy density regions. As an example, let us consider the PMF which was generated by a vector potential generated from the dilaton during inflation\[3\]. If the coupling between fields of dilaton and inflaton is negligibly small, there would be no correlation between the PMF and density perturbations. This is because the PMF was generated by the vector field coupled with dilaton, while the inflaton was responsible for density perturbations.
However, if we consider the case that a curvature coupling (like $R_{\mu\nu}F^{\mu\nu}$) generates the PMF in the same time, the positive correlation between PMF and density fields would be expected, since the electromagnetic fields are coupled with $R$ or Hubble parameter, which is determined by the density fields. Because, as mentioned above Subsection IV.A, the positive correlation between the magnetic field and the matter density at present is reasonable without any surprising PMF generations after the inflation, the positive correlation is the most natural consequence of such (inflationary) PMF generation models.

VI. SUMMARY

We numerically investigated the effect of the PMF on the energy density fields by considering the stochastic one that depends on scales, and we quantitatively discuss the effect of the PMF on the seeds of LSS in the early universe. We considered more general effects of the PMF than those considered in the previous works. We considered not only the magnetic field tension but also the increases of pressure and energy density perturbations from the field. Furthermore, by considering the correlation between the PMF and the matter density fluctuation, and taking the mathematically exact stochastic PMF power spectrum sets, we obtained reasonable and accurate evolutions of baryon, CDM, photon, and therefore the large scale structure. We show that the PMF can play very different roles on the evolution of density perturbations according to the correlation. After decoupling, CDM is also influenced indirectly by the PMF through gravitational interaction. We have estimated the effects, and we found that the amplitude ratio between the density spectra with and without PMF ($|P(k)/P_0(k)|$ at $k > 0.2$ Mpc$^{-1}$) lies between 75% and 130% at present for the range of PMF parameters $n = -2.001$ and $-1.0$, $0.5$ nG $< B_{\lambda} < 1.0$ nG, and $-1 < s < 1$.

Interestingly, it is reported that the magnetic field at large scales ($\lambda =$1Mpc) around $B_{\lambda} \sim nG$ [14, 15, 17, 18, 35] provides a new interpretation for the excess of CMB anisotropies at smaller angular scales. If the PMF with such strength was present, it is very likely that it has affected the formations at large scale structure as shown in the present paper. Yoshida, Sugiyama and Hernquist [37] suggested that in order to avoid false coupling of the baryon and CDM for small scales, using independent transfer functions for the baryon and CDM is preferable. The PMF would be another source of this difference in the transfer function for baryon and CDM. Since the density perturbations in the early universe have evolved to the LSS at present age, the evolution of the LSS with the PMF becomes more different than that without the PMF. We have shown that the baryon and CDM energy density perturbations follow very different evolutions in the presence of the PMF; with the PMF taken into consideration, the evolution of large scale structure should become more complicated.

Acknowledgments

We thank K. Omukai for a critical comment on the early version of our paper, and K. Bamba for discussion on the PMF generation during inflation. We are also grateful to S. Kawagoe, M. Kusakabe, S. Inoue, G. J. Mathews and T. Kajino for their valuable discussions. D.G.Y. and K. I. acknowledge the support by Grant-in-Aid for JSPS Fellows.
FIG. 1: Ratio of stochastic PMF pressure and tension sources, $\frac{\langle T^{[EM:S1]}(k)T^{[EM:S1]}_*(k) \rangle}{\langle T^{[EM:S2]}(k)T^{[EM:S2]}_*(k) \rangle}$, as a function of $n_B$. For illustration, a cut off scale $k_C$ is fixed to $k_C = 10 \text{ Mpc}^{-1}$. The pressure dominates for $n_B < -1.5$, and the tension dominates for $n_B > -1.5$.

APPENDIX A: POWER SPECTRUM OF PMF

In this section we derive power spectral of PMF, $Z$, $\Pi$ and $E$, which we used in sec[II]. The electro-magnetic energy momentum tensor can be decomposed into three parts, i.e., scalar, vector, and tensor parts. The scalar part of electro-magnetic energy momentum tensor $T_{ij[EM:S]}$ is defined as,

$$ T_{ij[EM:S]} = S \mathcal{P}_{ijlm} T^{lm}_{[EM]}, $$

(A1)

where $^S\mathcal{P}_{ijlm}$ is a scalar project tensor:

$$ ^S\mathcal{P}_{ijlm} = \hat{k}_j (\hat{k}_m P_{il} - \hat{k}_l P_{jm}) + \frac{1}{2} P_{ij} P_{lm} + \hat{k}_i \hat{k}_l \hat{k}_m $$

(A2)

and

$$ P^{ij}(k') = \delta^{ij} - \frac{k'^i k'^j}{k'^2}. $$

(A3)

Therefore it is easily shown that

$$ \hat{k}^i \hat{k}^j ^S\mathcal{P}_{ijlm} = \hat{k}_l \hat{k}_m. $$

(A4)
FIG. 2: Differences of transfer functions of CDM and baryon with \((P(k))\) and without \((\Pi_0(k))\) the PMF, normalized by \(\Pi_0(k)\), with various strength of magnetic fields as functions of scale factor \(a\) at each wave number \(k\) as indicated. The curves in right and left panels correspond to the differences in CDM and baryon, respectively. The red thick, light-blue thin, orange normal, blue dotted bold, black dotted thin, and green dotted middle curves in each both side panels show the differences for parameters \((B_\lambda, s) = (1.0nG, 1), (1.0nG, 0), (1.0nG, -1), (0.5nG, 1), (0.5nG, 0), \) and \((0.5nG, -1)\), respectively. In all figures, the power spectral index of the PMF is fixed to \(n = -2.001\). The curves with \(s = 0\) in all panels are slightly above unity, although it is difficult to read off from the figure.

1. Power spectrum of Lorenz Force: \(\Pi(k)\)

Using Eqs. (A2)-(A4), we obtain a two-point correlation function of scalar part

\[
\langle T_{EM,0}(k)T_{EM,0}'(k') \rangle = \hat{k}_i \hat{k}_j \hat{k}_l \hat{k}_m \langle T^i_{EM}(k)T^l_{EM}'(k') \rangle = (2\pi)^3 \Pi_{EM,0}(k)|^2 \delta(k - k'). 
\]

(A5)

where we define a power spectrum of Lorenz force \(\Pi_{EM,0}(k)\). The electromagnetic stress-energy tensor in \(k\) space is given by

\[
T^j_{ij}(k)[EM] = \frac{1}{4\pi a^3} \int \frac{d^3k'}{(2\pi)^3} \left\{ \frac{1}{2} \delta^i_j B^i(k')B_j(k-k') - B^i(k')B_j(k-k') \right\}. 
\]

(A6)
For convenience, we decompose $T_j^i(k)_{[EM]}$ into two parts as follows,

$$T_j^i(k)_{[EM]} = T_j^i(k)_{[EM:1]} - T_j^i(k)_{[EM:2]}$$  \hspace{1cm} (A7)

$$T_j^i(k)_{[EM:1]} = \frac{1}{4\pi a^4} \int \frac{d^3k'}{(2\pi)^3} \frac{1}{2} \delta_j^i B_l^i(k') B_l(k-k')$$ \hspace{1cm} (A8)

$$T_j^i(k)_{[EM:2]} = \frac{1}{4\pi a^4} \int \frac{d^3k'}{(2\pi)^3} B^i(k') B_j(k-k').$$ \hspace{1cm} (A9)

Correspondingly, we define $T_{[EM:S1]} = \hat{k}_i \hat{k}_j T_j^i_{[EM:1]}$ and $T_{[EM:S2]} = \hat{k}_i \hat{k}_j T_j^i_{[EM:2]}$. Using Eq. (A7-9), we can rewrite the two point correlation function of Lorenz force of scalar part as,

$$\langle T(k)_{[EM:S]} T^*(k')_{[EM:S]} \rangle = \langle (T(k)_{[EM:S1]} - T(k)_{[EM:S2]}) (T^*(k')_{[EM:S1]} - T^*(k')_{[EM:S2]}) \rangle$$

$$= \langle T(k)_{[EM:S1]} T^*(k')_{[EM:S1]} \rangle - \langle T(k)_{[EM:S1]} T^*(k')_{[EM:S2]} \rangle - \langle T(k)_{[EM:S2]} T^*(k')_{[EM:S1]} \rangle + \langle T(k)_{[EM:S2]} T^*(k')_{[EM:S2]} \rangle.$$ \hspace{1cm} (A10)
FIG. 4: Differences of transfer functions of CDM and baryon with \( (P(k)) \) and without \( (P_0(k)) \) the PMF, normalized by \( P_0(k) \) as a function of wave number \( k \) at each redshift as indicated. The curves in right and left panels correspond to the differences in CDM and baryon, respectively. The red thick, light-blue thin, orange normal, blue dotted bold, black dotted thin, and gree dotted middle curves in each both side panels show \((B \lambda, s) = (1.0nG,1), (1.0nG,0), (1.0nG,-1), (0.5nG,1), (0.5nG,0), \) and \((0.5nG,-1), \) respectively. In all figures, the power spectral index of the PMF is fixed to \( n = -2.001. \) The curves \( s = 0 \) in all panels look like unit, actually, those is more than unit.

Also, from (A5), \( \Pi_{\text{EM:S}}(k) \) in Eq. (28) and (33) can be written by,

\[
\Pi_{\text{EM:S}}(k) \delta(k-k') = \sqrt{\frac{1}{(2\pi)^3}} \langle T(k)T^*(k') \rangle_{\text{EM:S}}.
\]  
(A11)
FIG. 5: Same as Fig. 4 but for a different power spectral index of the PMF $n_B = -1.0$.

2. Power spectrum of Shear Stress: $Z(k)$

A power spectrum of Loreze force for the scalar part is decomposed into trace-trace and traceless (a shear pressure of PMF) parts. A traceless part of electromagnetic stress-energy tensor can be written by,

$$\Sigma_{ij}^{[\text{EM}]} = T_{ij}^{[\text{EM}]} - \frac{1}{3} \delta_{ij} T_{kk}^{[\text{EM}]}$$

$$= \frac{1}{4\pi a^4} \int \frac{d^3k'}{(2\pi)^3} \left\{ \frac{1}{3} \delta_{ij} B^i(k')B_j(k - k') - B^i(k')B_j(k - k') \right\} ,$$

(A12)
Thus, a scalar part of traceless component is given by,

\[ \Sigma_{[EM:S]}(k) = \left( \hat{k}_i \hat{k}_j - \frac{1}{3} \delta_i^j \right) \Sigma_{j[EM]} \]

\[ = \frac{1}{4\pi a^4} \int \frac{d^3k'}{(2\pi)^3} \hat{k}_i \hat{k}_j \left\{ \frac{1}{3} \delta^{ij} B^i(k') B_i(k-k') - B^i(k') B_j(k-k') \right\} \]

\[ = \left\{ \frac{2}{3} T(k)_{[EM:S1]} - T(k)_{[EM:S2]} \right\} \]  \hspace{1cm} (A13)

Similarly as the above subsection, we obtain the two point correlation function of PMF energy for scalar part as the following,

\[ \langle \Sigma(k)_{[EM:S]} \Sigma(k')^*_{[EM:S]} \rangle \]

\[ = \frac{4}{9} \langle T(k)_{[EM:S1]} T^*(k')_{[EM:S1]} \rangle - \frac{2}{3} \langle T(k)_{[EM:S1]} T^*(k')_{[EM:S2]} \rangle \]

\[ - \frac{2}{3} \langle T(k)_{[EM:S2]} T^*(k)_{[EM:S1]} \rangle + \langle T(k)_{[EM:S2]} T^*(k)_{[EM:S2]} \rangle \]

\[ = (2\pi)^3 |Z(k)_{[EM:S]}|^2 \delta(k-k'), \] \hspace{1cm} (A14)

where \( Z(k)_{[EM:S]} \) is a power spectrum of PMF shear stress. We rewrite Eq. (A14) as following,

\[ Z^2(k)_{[EM:S]} \delta(k-k') = \frac{1}{(2\pi)^3} \langle \Sigma(k)_{[EM:S]} \Sigma(k')^*_{[EM:S]} \rangle. \] \hspace{1cm} (A15)

3. Power spectrum of PMF energy: \( E(k) \)

The PMF energy is defined by Eq. (4). The PMF energy in \( k \) space is given by

\[ T_0^0(k)_{[EM]} = \frac{1}{8\pi a^4} \int \frac{d^3k'}{(2\pi)^3} B^i(k') B_i(k-k') = T(k)_{[EM:S1]} \] \hspace{1cm} (A16)

Similarly as the above subsection, we obtain the two point correlation function of PMF energy for scalar part as the following,

\[ \langle T(k)_{[EM:S1]} T(k')^*_{[EM:S1]} \rangle = (2\pi)^3 |E(k)_{[EM:S]}|^2 \delta(k-k'), \] \hspace{1cm} (A17)

where \( E(k)_{[EM:S]} \) is a power spectrum of PMF energy. We rewrite Eq. (A17) as following,

\[ E^2(k)_{[EM:S]} \delta(k-k') = \frac{1}{(2\pi)^3} \langle T(k)_{[EM:S1]} T(k')^*_{[EM:S1]} \rangle. \] \hspace{1cm} (A18)

In order to obtain \( \Pi(k)_{[EM:S]}, \ Z(k)_{[EM:S]}, \) and \( E(k)_{[EM:S]}, \) we calculate \( \langle T_{[EM:S1]}(k) T^*_{[EM:S1]}(k) \rangle, \) \( \langle T_{[EM:S1]}(k) T^*_{[EM:S2]}(k) \rangle + \langle T_{[EM:S2]}(k) T^*_{[EM:S1]}(k) \rangle \), and \( \langle T_{[EM:S2]}(k) T^*_{[EM:S2]}(k) \rangle \) in Eqs. (A10), (A14) and (A17) the following subsections.

4. \( \langle T_{[EM:S1]}(k) T^*_{[EM:S1]}(k) \rangle \)

Using Eq. (A1), (A2) and (A8), the scalar part of \( T(k)_{[EM:1]} \) becomes

\[ T(k)_{[EM:S1]} = \hat{k}_i \hat{k}_j T_{[EM:1]} \]

\[ = \hat{k}_i \hat{k}_j \frac{1}{8\pi a^4} \int \frac{d^3k'}{(2\pi)^3} \delta^{ij} B^i(k') B_i(k-k') \]

\[ = \frac{1}{2^2(2\pi)^4a^2} \int d^3k' B^i(k') B_i(k-k'). \] \hspace{1cm} (A19)
Thus the two point correlation function of Eq. (A19) given by

\[
\langle T(k_{EM:S1}) \rangle_{[EM:S1]} = \frac{1}{2^{2\pi}a^8} \int d^3k' d^3p' \langle B^i(k')B_i(k-k')B^k*p(p-p') \rangle
\]

We assume the random magnetic field is Gaussian and apply the Wick's theorem

\[
\langle B^i(k')B_i(k-k')B^k*p(p-p') \rangle = \langle B^i(k')B_i(k-k') \rangle \langle B^k*p(p-p') \rangle + \langle B^i(k')B^k*p(p-p') \rangle \langle B_i(k-k') \rangle
\]

and

\[
\langle B^i(k')B^k*p(p-p') \rangle \langle B_i(k-k') \rangle = \langle B^i(k') \rangle \langle B^k*p(p-p') \rangle \langle B_i(k-k') \rangle + \langle B^i(k')B_i(k-k') \rangle \langle B^k*p(p-p') \rangle
\]

A two-point correlation function for PMF is defined by Ref. [14]

\[
\langle B^i(k)B^k*p(k') \rangle = \frac{(2\pi)^{n+8}}{2k_\lambda^{n+3} \Gamma \left(\frac{n+3}{2}\right)} P^{ij}(k)\delta(k-k')
\]

Using the reality condition

\[
B^i(k) = B^i*(-k)
\]

and substituting (A22) into (A21), first, second, and third terms become

\[
\langle B^i(k')B_i(k-k') \rangle \langle B^k*p(p-p') \rangle = \left\{ \frac{(2\pi)^{n+8}}{2k_\lambda^{n+3} \Gamma \left(\frac{n+3}{2}\right)} \right\}^2 k'^n(-p')^n 4\delta(k)\delta(-p)
\]

\[
\langle B^i(k')B^k*p(p-p') \rangle \langle B_i(k-k') \rangle = \left\{ \frac{(2\pi)^{n+8}}{2k_\lambda^{n+3} \Gamma \left(\frac{n+3}{2}\right)} \right\}^2 k'^n \langle k-k' \rangle^n P^{ij}(k')P_{ij}(k-k')
\]

and

\[
\langle B^i(k')B_i^* (p-p') \rangle \langle B^k*(p-p') \rangle = \left\{ \frac{(2\pi)^{n+8}}{2k_\lambda^{n+3} \Gamma \left(\frac{n+3}{2}\right)} \right\}^2 k'^n \langle k-k' \rangle^n P^{ij}_i(k)P^j_i(k-k')\delta((k-k')-(p-p'))
\]

respectively. Because \(k \neq 0, x = 2\pi/k \neq \infty\), \(\delta(k) = \delta(-p) = 0\), the first term on the r.h.s.in Eq. (A21) is 0. From Eq. (A3)

\[
P^{ij}(k')P_{ij}(k-k') = 1 + \frac{\langle k' \cdot (k-k') \rangle^2}{k'^2(k-k')^2}
\]

Therefore second and third terms in Eq. (A21) become

\[
\langle B^i(k')B^k*p(p-p') \rangle \langle B_i(k-k') \rangle = \left\{ \frac{(2\pi)^{n+8}}{2k_\lambda^{n+3} \Gamma \left(\frac{n+3}{2}\right)} \right\}^2 k'^n \langle k-k' \rangle^n \left\{ 1 + \frac{\langle k' \cdot (k-k') \rangle^2}{k'^2(k-k')^2} \right\} \delta((k-k')-(p-p'))
\]
and
\[
\langle B'(k') B^*(p - p') \rangle \left\langle B_i(k - k') B^{i*}(p') \right\rangle
= \left( \frac{2\pi}{2k_n^{n+3}} \frac{B_\lambda^2}{\Gamma \left( \frac{n+3}{2} \right)} \right)^2 \langle k' | k - k' \rangle \left\{ 1 + \frac{\langle k' \cdot (k - k') \rangle^2}{k'^2 |k - k'|^2} \right\} \delta(k' - (p - p')) \delta((k - k') - p'),
\]
(A29)
respectively. Using Eq. (A28) and (A29), Eq. (A30) becomes
\[
\left\langle T(k, \tau)_{[EM:S1]} T^*(p, \tau)_{[EM:S1]} \right\rangle
= \frac{1}{2^4 (2\pi)^6 a^8} \int d^3 k^' d^3 p^' \left( \frac{2\pi}{2k_n^{n+3}} \frac{B_\lambda^2}{\Gamma \left( \frac{n+3}{2} \right)} \right)^2 \langle k' | k - k' \rangle \left\{ 1 + \frac{\langle k' \cdot (k - k') \rangle^2}{k'^2 |k - k'|^2} \right\} \delta(k - p).
\]
(A30)
Integrating Eq. (A30) by \( p' \), we obtain following equation
\[
\left\langle T(k, \tau)_{[EM:S1]} T^*(p, \tau)_{[EM:S1]} \right\rangle
= \frac{2}{2^4 (2\pi)^6 a^8} \int d^3 k^' \left( \frac{2\pi}{2k_n^{n+3}} \frac{B_\lambda^2}{\Gamma \left( \frac{n+3}{2} \right)} \right)^2 \langle k' | k - k' \rangle \left\{ 1 + \frac{\langle k' \cdot (k - k') \rangle^2}{k'^2 |k - k'|^2} \right\} \delta(k - p).
\]
(A31)
We define that
\[
\mathcal{C} = \cos c = \frac{k' \cdot k}{k'^2 k},
\]
(A32)
Substituting Eq. (A32) into Eq. (A31), The two point function of Lorenz Force becomes
\[
\left\langle T(k, \tau)_{[EM:S1]} T^*(p, \tau)_{[EM:S1]} \right\rangle
= \frac{1}{2^4 (2\pi)^6 a^8} \left( \frac{2\pi}{2k_n^{n+3}} \frac{B_\lambda^2}{\Gamma \left( \frac{n+3}{2} \right)} \right)^2 \times \int d^3 k'^n |k - k'|^{n-2} \left\{ (1 + C^2)k^2 - 4kk' \mathcal{C} + 2k'^2 \right\} \delta(k - p).
\]
(A33)
Choosing \( \hat{k} \) to the polar axis as
\[
d^3 k' = k'^2 d\hat{k} \sin c \; dc \; d\phi,
\]
(A34)
and integrating Eq. (A33) by \( \phi \), the two point function of Lorenz Force is given by
\[
\left\langle T(k, \tau)_{[EM:S1]} T^*(p, \tau)_{[EM:S1]} \right\rangle
= \frac{1}{2^4 (2\pi)^6 a^8} \left( \frac{2\pi}{2k_n^{n+3}} \frac{B_\lambda^2}{\Gamma \left( \frac{n+3}{2} \right)} \right)^2 \times \int d\hat{k} k'^{n+2} \int_{-1}^{1} d\mathcal{C} |k - k'|^{n-2} \left\{ (1 + C^2)k^2 - 4kk' \mathcal{C} + 2k'^2 \right\} \delta(k - p)
\]
(A35)
Almost all of the previous works have treated terms which include \( \mathcal{C} \) in the middle parenthesis as unity. In this paper, however, we calculate Eq. (A35) further by integrating by parts. In addition they have calculated integration of \( k' \) using the Taylor expansion by \( k'/k(k' \ll k) \) or \( k/k'(k \ll k') \). If we would want to estimate Eq. (A35) for only \( k' \ll k \) or \( k \ll k' \), such approximation would be useful. However there is the value \( k' \sim k \) in the integration range, so we must calculate Eq. (A35) without such Taylor expansion. Integrating Eq. (A33) by \( \mathcal{C} \), after long but straightforward
calculation, we can obtain following expression,

\[
\langle T(k, \tau)_{\text{EM:S1}} | T(p, \tau)_{\text{EM:S1}} \rangle = \frac{1}{4(2\pi)^2 a^8} \left\{ \frac{(2\pi)^{n+8} B_3^2}{2k^{n+3} \Gamma \left( \frac{n+3}{2} \right)} \right\}^2 \times \int dk' k'^{n+2} \left[ \frac{n^2 + 4n + 1}{kk'n(n+2)(n+4)} \right] \delta(k - k') \]

\[
\times \left\{ (k + k')^{n+2} - |k - k'|^{n+2} \right\} \delta(k - k')
\]

\[\text{(A36)}\]

5. \(\langle T_{\text{EM:S1}} T^*_{\text{EM:S2}} \rangle + \langle T_{\text{EM:S2}} T^*_{\text{EM:S1}} \rangle\)

Using Eqs. (A1), (A2) and (A9), the scalar part of \(T(k)_{\text{EM:2}}\) becomes

\[
T(k, \tau)_{\text{EM:S2}} = \tilde{k} J_4(k, \tau)_{\text{EM:2}} = \frac{1}{4\pi a^4} \int \frac{d^4k'}{(2\pi)^3} \tilde{k} \tilde{k}' B'(k') B_1(k - k')
\]

Thus cross correlations between Eqs. (A19) and (A37) are given by

\[
\langle T_{\text{EM:S1}}(k) T^*_{\text{EM:S2}}(p) \rangle = \frac{1}{2^{3}(2\pi)^{8} a^{8}} \int d^{3}k' \int d^{3}p' \tilde{k} m' B'(k') B_{q}(k - k') B^{*}_{q}(p - p'),
\]

\[\text{(A38)}\]

and

\[
\langle T_{\text{EM:S2}}(k) T^*_{\text{EM:S1}}(p) \rangle = \frac{1}{2^{3}(2\pi)^{8} a^{8}} \int d^{3}k' \int d^{3}p' \tilde{k} m' B'(k') B_{q}(k - k') B^{*}_{q}(p - p'),
\]

\[\text{(A39)}\]

respectively. Using Eq. (A21), Eq. (A38) and (A39) become

\[
\langle T_{\text{EM:S1}}(k) T^*_{\text{EM:S2}}(p) \rangle = \frac{1}{2^{3}(2\pi)^{8} a^{8}} \left\{ \frac{(2\pi)^{n+8} B_3^2}{2k^{n+3} \Gamma \left( \frac{n+3}{2} \right)} \right\}^2 \int d^{3}k' \int d^{3}p' k'^{n} |k - k'|^{n} \hat{p} \hat{p}^{n} \times \left\{ P^{il}(k') P_{qm}(k' - k') \delta(k' - p') \delta((k' - (p - p'))) + P_{q}^{il}(k' - k') P_{qm}^{il}(k') \delta((k' - (p - p'))) \right\}
\]

\[\text{(A40)}\]

and

\[
\langle T_{\text{EM:S2}}(k, \tau) T^*_{\text{EM:S1}}(p, \tau) \rangle = \frac{1}{2^{3}(2\pi)^{8} a^{8}} \left\{ \frac{(2\pi)^{n+8} B_3^2}{2k^{n+3} \Gamma \left( \frac{n+3}{2} \right)} \right\}^2 \int d^{3}k' \int d^{3}p' k'^{n} |k - k'|^{n} \hat{k} \hat{k}' \hat{P}^{in}(k') P_{q}(k - k') \times \left\{ \delta(k' - p') \delta((k' - k') - (p - p')) + \delta((k' - (p - p'))) \delta((k' - k') - p') \right\},
\]

\[\text{(A41)}\]

respectively. Integrating Eq. (A40) and Eq. (A41) by \(p'\), we obtain following equations,

\[
\langle T_{\text{EM:S1}}(k) T^*_{\text{EM:S2}}(p) \rangle = \frac{1}{2^{3}(2\pi)^{8} a^{8}} \delta(k - p) \left\{ \frac{(2\pi)^{n+8} B_3^2}{2k^{n+3} \Gamma \left( \frac{n+3}{2} \right)} \right\}^2 \int d^{3}k' k'^{n} |k - k'|^{n} \hat{p} \hat{p}^{m} \left\{ P^{il}(k') P_{qm}(k' - k') + P_{q}^{il}(k' - k') P_{qm}^{il}(k') \right\}
\]

\[\text{(A42)}\]
Using Eqs. (A21) and (A22), we therefore integrate by $p$.

Substituting Eq. (A44) into Eqs. (A42) and (A43), and combining Eqs. (A42) and (A43), we obtain the following equation:

$$\hat{k}_l P^{\mu\nu}(k') \hat{k}_m P^\nu_q(k - k') = \frac{k'(1 - C^2)(k' - kC)}{|k - k'|^2}. \quad (A44)$$

Similarly, substituting Eq. (A49) for Eq. (A48), we obtain the following equation:

$$\hat{k}_l P^{\mu\nu}(k') \hat{k}_m P^\nu_q(k - k') \equiv \frac{k'(1 - C^2)(k' - kC)}{|k - k'|^2}. \quad (A45)$$

Similarly as above the subsection, the term containing the unit vector $\hat{k}$ in Eqs. (A42) and (A43) becomes

$$\hat{k}_l P^{\mu\nu}(k') \hat{k}_m P^\nu_q(k - k') = \frac{k'(1 - C^2)(k' - kC)}{|k - k'|^2}. \quad (A46)$$

Using Eq. (A37), a two point correlation function of $T_{[EM:S2]}(k)$ becomes

$$\langle T_{[EM:S2]}(k) T_{[EM:S2]}^*(p) \rangle = \frac{1}{2^3(2\pi)^8 a^8} \int d^3k' d^3p' \hat{k}_l \hat{p} \hat{p}^m \langle B^i(k') B_j(k' - k') B^*_{l'}(p') B^*_m(p - p') \rangle \quad (A47)$$

Using Eqs. (A21) and (A22), furthermore integrating by $p'$, we get

$$\langle T_{[EM:S2]}(k, \tau) T_{[EM:S2]}^*(p, \tau) \rangle$$

$$= \delta(k - p) \frac{1}{2^3(2\pi)^8 a^8} \left\{ \frac{(2\pi)^{n+8}}{2k^{n+3}_\lambda} \frac{B^2_\lambda}{\Gamma \left( \frac{n+3}{2} \right)} \right\}^2 \int d^3k' d^3p' \hat{k}_l \hat{p} \hat{p}^m \{ P^{\mu\nu}(k') P^{\mu\nu}(k - k') + P^{\nu\mu}_m(k') P^\nu_j(k - k') \} \quad (A48)$$

Similarly as above the subsection, the term containing the unit vector $\hat{k}$ on Eq. (A48) is expressed as

$$\hat{k}_l \hat{k}_m \hat{p} \hat{p}^m P^{\mu\nu}(k') P^\nu_j(k - k') \equiv \frac{k^2}{|k - k'|^2} (1 - C^2)^2. \quad (A49)$$

Similarly, substituting Eq. (A49) for Eq. (A48), we obtain the following equation:

$$\langle T_{[EM:S2]}(k, \tau) T_{[EM:S2]}^*(k, \tau) \rangle$$

$$= \frac{1}{2^3(2\pi)^8 a^8} \left\{ \frac{(2\pi)^{n+8}}{2k^{n+3}_\lambda} \frac{B^2_\lambda}{\Gamma \left( \frac{n+3}{2} \right)} \right\}^2 \int dk' \int d\chi k'^n d^3k' \hat{k}_l \hat{k}_m \hat{p} \hat{p}^m \{ P^{\mu\nu}(k') P^{\mu\nu}(k - k') + P^{\nu\mu}_m(k') P^\nu_j(k - k') \} \quad (A50)$$
Similarly, integrating Eq. (A50) by $C$, after long calculating, we can obtain the following expression,

$$\langle T_{EM-S2}(k)T_{EM-S2}^\ast(k) \rangle = \frac{1}{(2\pi)^3 a^8} \left( \frac{2\pi}{2k\sigma^3} \frac{B_k^2}{\Gamma \left( \frac{n+3}{2} \right)} \right)^2 \int dk' k'^{n+4} \frac{4}{(kk')^3 n(n+2)(n+4)} \left\{ (k+k')^{n+4} - |k-k'|^{n+4} \right\} $$

$$- \frac{3}{(kk')(n+6)} \left\{ |k-k'|^{n+6} + (k+k')^{n+6} \right\} + \frac{3}{(kk')^2 (n+6)(n+8)} \left\{ (k+k')^{n+8} - |k-k'|^{n+8} \right\}$$

(A51)

**APPENDIX B: CUT OFF SCALE OF PMF**

The photon mean free path $l_\gamma$ is

$$l_\gamma = \frac{1}{\sigma_T n_e(\tau)} \propto a^3, \quad (B1)$$

where $n_e$ is the electron number density. From Refs. [28, 29], the cut off scale of PMF is defined by

$$k^{-2}_C(\tau) = \begin{cases} \frac{B^2}{4\pi(\rho+p)} \int_0^\tau d\tau' \frac{l_\gamma}{a}, \tau < \tau_{dec}, \\ \frac{B^2}{4\pi(\rho+p)} \int_0^{\tau_{dec}} d\tau' \frac{l_\gamma}{a}, \tau > \tau_{dec} \end{cases}$$

(B2)

where $\tau_{dec}$ is the time of decoupling of photons from baryons. Following Ref. [14], for a stochastic magnetic field with a power-law power spectrum, the relation between the effective homogeneous field $B$ and $B_\lambda$ is written by

$$B = B_\lambda \left( \frac{k_C}{k_\lambda} \right)^{n+3}.$$  

(B3)

Substituting Eq. (B2) into Eq. (B3), we obtain

$$k^{-5-n}_C(\tau) = \begin{cases} \frac{B^2}{4\pi(\rho+p)} \int_0^\tau d\tau' \frac{l_\gamma}{a}, \tau < \tau_{dec}, \\ \frac{B^2}{4\pi(\rho+p)} \int_0^{\tau_{dec}} d\tau' \frac{l_\gamma}{a}, \tau > \tau_{dec} \end{cases}$$

(B4)

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