APPLICATION OF MULTIVARIATE SPLINES TO DISCRETE MATHEMATICS

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Abstract. Using methods developed in multivariate splines, we present an explicit formula for discrete truncated powers, which are defined as the number of non-negative integer solutions of linear Diophantine equations. We further use the formula to study some classical problems in discrete mathematics as follows. First, we extend the partition function of integers in number theory. Second, we exploit the relation between the relative volume of convex polytopes and multivariate truncated powers and give a simple proof for the volume formula for the Pitman-Stanley polytope. Third, an explicit formula for the Ehrhart quasi-polynomial is presented.

1. Introduction

Let $M$ be an $s \times n$ integer matrix with columns $m_1, \ldots, m_n \in \mathbb{Z}^s$ such that their convex hull does not contain the origin. For a given $\alpha \in \mathbb{Z}^s$, consider the following system of linear Diophantine equations

$$M\beta = \alpha, \quad \beta \in \mathbb{Z}^n.$$

The number of non-negative integer solutions $\beta$ for this system is denoted by $t(\alpha|M)$ and the resulting function $t(\cdot|M)$ on $\mathbb{Z}^s$ is called a discrete truncated power. Discrete truncated powers, also called vector partition functions, have many applications in various mathematical areas including Algebraic Geometry, Representation Theory, Number Theory, Statistics and Randomized Algorithms. In general, one studies $t(\cdot|M)$ by generating functions and by means of algebraic geometry etc.\[2, 3, 5, 6, 28.\] For example, when $M$ is unimodular, that is when every nonsingular square submatrix of order $s$ has determinant $\pm 1$, two algebraic algorithms for generating the explicit formula for $t(\cdot|M)$ are presented in \[16.\] When $s = 1$, an explicit formula for $t(\cdot|M)$, which counts the integer solutions for the linear Diophantine

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equation, is presented in [2]. Especially, Popoviciu [27] gave a beautiful formula for $t(\cdot | M)$, when $M = (a, b)$ where $a$ and $b$ are relatively prime.

In this paper, an explicit formula for $t(\cdot | M)$ is presented for any integer matrix $M$. In contrast to other ways, our method is based on multivariate spline functions and is inspired by the work of Dahmen, Micchelli and Jia [10, 12, 21, 22], who exploited the relation between $t(\cdot | M)$ and multivariate splines, and demonstrated the piecewise structure of $t(\cdot | M)$. Moreover, we believe that the tool of multivariate splines, which have been developed in this latter theory, shed some light on problems concerning $t(\cdot | M)$.

The main results in this paper are as follows. As the central result of the paper, an explicit formula for $t(\cdot | M)$ in terms of multivariate splines is presented. As applications of our formula, we first generalize the formula for partition functions in [2] and give a simple proof for Popoviciu’s formula. Further, we show that the relative volume of a convex polytope agrees with the value of the multivariate truncated power at a certain point. We also present an efficient method for computing the volume of convex polytopes and re-prove the volume formula for the polytope related to empirical distributions, which is presented in [20] as a central result.

The paper is organized as follows. To help make this paper self-contained, we introduce the multivariate truncated power and the box spline in Section 2. In Section 3, the discrete truncated power $t(\cdot | M)$ is introduced. An explicit formula for $t(\cdot | M)$ is described in Section 4. The proof of the formula is presented in Section 5. In Section 6, we simplify the explicit formula for $t(\cdot | M)$ for some special matrices $M$. The explicit formula for partition functions in [2] follows directly from our formula. Section 7, containing two subsections, uses multivariate truncated powers to investigate the volume of convex polytopes and the Ehrhart quasi-polynomial. Particularly, in Subsection 7.1, an efficient method for computing the volume of convex polytopes is introduced, with re-proving the volume formula for the polytope related to empirical distributions which is the main result in [26]; in Subsection 7.2, an explicit formula for the Ehrhart quasi-polynomial is presented.
It is necessary here to recall some previous notations (see [22]). Throughout the paper, $\mathbb{Z}_+$ and $\mathbb{R}_+$ denote the non-negative integer and non-negative real sets respectively. For given sets $D_1$, $D_2$, let $1_{D_1}(D_2) = 0$ if $D_1 \cap D_2 = \emptyset$, otherwise let $1_{D_1}(D_2) = 1$. The linear space $\mathbb{R}^s$ is equipped with the norm $| \cdot |$ given by $|x| = \sum_{1 \leq j \leq s} |x_j|$, where $x = (x_1, x_2, \ldots, x_s) \in \mathbb{R}^s$. Let $A$ and $B$ be two subsets of $\mathbb{R}^s$. Then $A - B$ is the set of all elements in the form of $a - b$, where $a \in A$ and $b \in B$. The sets $A + B$ and $cA$ are defined analogously, where $c \in \mathbb{R}$. The set $A \setminus B$ is the complement of $B$ in $A$. A subset $\Omega$ of $\mathbb{R}^s$ is called a cone if $\Omega + \Omega \subseteq \Omega$ and $c\Omega \subseteq \Omega$ for all $c > 0$. If a cone is also an open set, then we call it an open cone. Let $Y$ be an $s \times n$ matrix. The linear span of $Y$, denoted by $\text{span}(Y)$, is the set $\{ \sum_{y \in Y} a_y y : a_y \in \mathbb{R} \text{ for all } y \}$. The cone spanned by $Y$, denoted by $\text{cone}(Y)$, is the set $\{ \sum_{y \in Y} a_y y : a_y \geq 0 \text{ for all } y \}$. We denote by $\text{cone}^0(Y)$ the relative interior of $\text{cone}(Y)$. The convex hull of $Y$, denoted by $[Y]$, is the set $\{ \sum_{y \in Y} a_y y : a_y \geq 0 \text{ for all } y \text{ and } \sum_{y \in Y} a_y = 1 \}$. Let $M$ be an $s \times n$ matrix with integer columns $m_1, \ldots, m_n$. We denote by $[[M]]$ the zonotope spanned by $m_1, \ldots, m_n$, i.e., $[[M]] := \{ \sum_{j=1}^n a_j m_j : 0 \leq a_j \leq 1, \forall j \}$. Moreover, set $[[M]] := \{ \sum_{j=1}^n a_j m_j : 0 \leq a_j < 1, \forall j \}$. 

We shall use standard multiindex notations. Specifically, an element $\alpha \in \mathbb{Z}_+^s$ is called an $s$-index, and $|\alpha|$ is called the length of $\alpha$. Define $z^\alpha := z_1^{\alpha_1} \cdots z_s^{\alpha_s}$ for $z = (z_1, \ldots, z_s) \in \mathbb{C}^s$ and $\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{Z}_+^s$. Also, set $\exp(c\alpha) := (\exp(c\alpha_1), \ldots, \exp(c\alpha_s))$, where $c$ is a constant. For $z = (z_1, \ldots, z_s) \in \mathbb{C}^s$ and $w = (w_1, \ldots, w_s) \in (\mathbb{C} \setminus 0)^s$, set $z/w := (z_1/w_1, \ldots, z_s/w_s)$. We denote by $\mathbb{P}_k := \mathbb{P}_k(\mathbb{R}^s)$ the linear space of polynomials in $s$ real variables with total degree $\leq k$, where $k \in \mathbb{Z}$. If $k$ is a negative integer, then we interpret $\mathbb{P}_k$ as the trivial linear space $\{0\}$. Moreover, we also set $\mathbb{P}(\mathbb{R}^s) := \bigcup_{k=0}^{\infty} \mathbb{P}_k(\mathbb{R}^s)$.
2. Multivariate truncated powers and box splines

Let $M$ be an $s \times n$ real matrix with $\text{rank}(M) = s$. Throughout this section we always assume that the convex hull of $M$ does not contain the origin. The multivariate truncated power $T(\cdot|M)$ associated with $M$, introduced firstly by Dahmen [8], is the distribution given by the rule

\[ (2.1) \quad \int_{\mathbb{R}^n} T(x|M) \phi(x) dx = \int_{\mathbb{R}^s_+} \phi(Mu) du, \quad \phi \in \mathcal{D}(\mathbb{R}^s), \]

where $\mathcal{D}(\mathbb{R}^s)$ is the space of test functions on $\mathbb{R}^s$, i.e., the space of all compactly supported and infinitely differentiable functions on $\mathbb{R}^s$. From (2.1), we see that the support of $T(\cdot|M)$ is $\text{cone}(M)$. In fact, $T(\cdot|M)$ is identified with the function (see [15] pp.12)

\[ (2.2) \quad T(x|M) = \frac{\text{vol}_n(M^{-1}x \cap \mathbb{R}^n_+)}{\sqrt{|\det(MM^T)|}}, \]

where $M^{-1}x := \{ y : My = x \}$ and $\text{vol}_n(M^{-1}x \cap \mathbb{R}^n_+)$ denotes the volume of $M^{-1}x \cap \mathbb{R}^n_+$ in $\mathbb{R}^n$. In the following, we review some basic properties of multivariate truncated powers. For more detailed information about this function, the reader is referred to [8] [15].

For $y = (y_1, \ldots, y_s) \in \mathbb{R}^s$ and a function $f$ defined on $\mathbb{R}^s$, we denote by $D_y f$ the directional derivative of $f$ in the direction $y$, i.e., $D_y = \sum_{j=1}^s y_j D_j$, where $D_j$ denotes a partial derivative with respect to the $j$th coordinate. The following differential formula was given in [8]. For $y \in M$, we have $D_y T(\cdot|M) = T(\cdot|M\setminus y)$.

More generally,

\[ (2.3) \quad D_Y T(\cdot|M) = T(\cdot|M\setminus Y) \quad \text{for} \quad Y \subset M \]

where $D_Y := \prod_{y \in Y} D_y$.

Let $\mathcal{Y}(M)$ denote the set consisting of those $Y \subset M$ such that $M\setminus Y$ does not span $\mathbb{R}^s$ and let $D(M)$ denote the linear space of those infinitely differentiable complex-valued functions $f$ on $\mathbb{R}^s$ that satisfy the following system of linear partial differential equations:

\[ D_Y f = 0, \quad \forall \quad Y \in \mathcal{Y}(M). \]
Based on the definition of $D(M)$, we can see that $D(M') \subset D(M)$ when $M' \subset M$.

It was also proved in [9, 14] that $D(M) \subseteq \mathbb{P}_{n-s}$.

Let $c(M)$ be the union of all the sets cone$(M \setminus Y)$, as $Y$ runs over $\mathcal{Y}(M)$. A connected component of cone$^o(M) \setminus c(M)$ is called a fundamental $M$-cone according to [12]. Then $T(\cdot|M)$ agrees with some homogeneous polynomial of degree $n - s$ in $D(M)$ on each fundamental $M$-cone. In fact, $T(\cdot|M)$ is generally continuous and positive on cone$^o(M)$.

We now turn to box splines. The box spline $B(\cdot|M)$ associated with $M$ is the distribution given by the rule [13, 14]

$$
\int_{\mathbb{R}^s} B(x|M)\phi(x)dx = \int_{[0,1]^n} \phi(Mu)du, \ \phi \in \mathcal{D}(\mathbb{R}^s).
$$

According to (2.4), the support of $B(\cdot|M)$ is $[[M]]$, and hence, we have (see [15], pp.33)

$$
\{ j \in \mathbb{Z}^s : B(x-j|M) \neq 0 \} = \mathbb{Z}^s \cap (x-[[M]]).
$$

By taking $\phi = \exp(-i\xi \cdot)$ in (2.4), we obtain the Fourier transform of $B(\cdot|M)$ as

$$
\hat{B}(\zeta|M) = \prod_{j=1}^n \frac{1 - \exp(-i\zeta^T m_j)}{i\zeta^T m_j}, \ \zeta \in \mathbb{C}^s.
$$

For more detailed information about box splines, the reader is referred to [15].

3. DISCRETE TRUNCATED POWERS

Let $M$ be an $s \times n$ matrix with integer columns $m_1, \ldots, m_n$ and suppose that $[M]$ does not contain the origin. From the definition of $t(\cdot|M)$ given in Section 1, we have

$$
\sum_{\alpha \in \mathbb{Z}^s} \varphi(\alpha)t(\alpha|M) = \sum_{\beta \in \mathbb{Z}_+^n} \varphi(M\beta), \ \text{for any } \varphi \in \mathcal{D}(\mathbb{R}^s).
$$

A comparison between (3.1) and (2.1) shows that (3.1) is a discrete version of (2.1).

This observation was the motivation to designate $t(\cdot|M)$ a discrete truncated power.

Also the identity

$$
T(x|M) = \sum_{\alpha \in \mathbb{Z}^s} t(\alpha|M)B(x-\alpha|M)
$$
established in [10] shows that these three functions, $T(\cdot | M)$, $B(\cdot | M)$ and $t(\cdot | M)$, are closely related.

We denote by $S$ the linear space of all complex functions on $\mathbb{Z}^s$. Given $y \in \mathbb{Z}^s$, the backward difference operator $\nabla_y$ is defined by the rule $\nabla_y f := f - f(\cdot - y)$, where $f \in S$. More generally, for an integer matrix $Y$, we define

$$\nabla_Y := \prod_{y \in Y} \nabla_y.$$ 

In [12], the following difference formula was given:

$$\nabla_y t(\cdot | M) = t(\cdot | M \setminus y),$$

for any $y \in M$. More generally,

$$\nabla_Y t(\cdot | M) = t(\cdot | M \setminus Y),$$

for any $Y \subseteq M$.

Given $\theta = (\theta_1, \ldots, \theta_s) \in (\mathbb{C} \setminus \{0\})^s$, we set

$$M_\theta := \{y \in M : \theta^y = 1\}.$$ 

Let

$$A(M) := \{\theta \in (\mathbb{C} \setminus \{0\})^s : \text{span}(M_\theta) = \mathbb{R}^s\}.$$ 

As pointed out in [11], $\theta \in A(M)$ if and only if it has the form

(3.3) $\theta = \exp(2\pi i \alpha / \det Y),$ 

for some $Y \in \mathcal{B}(M)$, and the vector $\alpha \in \mathbb{Z}^s$ satisfying $Y^T \alpha = |\det Y| L$, where $L \in \mathbb{Z}^s \cap [Y^T])$, i.e., is a lattice point in the parallelepiped determined by $Y^T$.

Here,

$$\mathcal{B}(M) = \{Y \subseteq M : \#Y = s, \text{span}(Y) = \mathbb{R}^s\}.$$ 

We let

$$\nabla(M) := \{f \in S : \nabla_Y f = 0, \text{ for all } Y \in \mathcal{V}(M)\}.$$ 

In [11], Dahmen and Micchelli showed that a sequence $f \in \nabla(M)$ if and only if it has the form $f(\alpha) = \sum_{\theta \in A(M)} \theta^\alpha p_\theta(\alpha), \alpha \in \mathbb{Z}^s$, where $p_\theta \in D(M_\theta)$ for each $\theta \in A(M)$. For a fundamental $M$-cone $\Omega$, we set

$$v(\Omega|M) := \mathbb{Z}^s \cap (\Omega - [M]).$$ 

Dahmen and Micchelli also proved the following result in [12]:

...
Theorem 3.1. ([12] Thm.3.1) Let $M = \{m_1, \ldots, m_n\}$ be an $s \times n$ integer matrix and suppose that $M$ spans $\mathbb{R}^s$ and the convex hull of $M$ does not contain the origin. Then for any fundamental $M$-cone $\Omega$, there exists a unique element $f_\Omega(\cdot|M) \in \nabla(M)$ such that $f_\Omega(\cdot|M)$ agrees with $t(\cdot|M)$ on $v(\Omega|M)$. Moreover, $f_\Omega(\cdot|M)$ has the following properties: for any $x \in \Omega$ such that $v(x|M) \cap \text{cone}(M) = \{0\}$, $f_\Omega(\cdot|M)$ is uniquely determined by

$$f_\Omega(\alpha|M) = 0, \alpha \in v(x|M),$$

and satisfies the relation

$$(3.4) \quad f_\Omega(\alpha|M) = (-1)^{n-s} f_\Omega(-\alpha - \sum_{j=1}^n m_j|M), \alpha \in \mathbb{Z}^s.$$
4. An explicit formula for discrete truncated powers

The objective of this section is to present an explicit formula for discrete truncated powers \( t(\cdot|\mathcal{M}) \) which is the central result of the paper. Throughout the rest of the paper, we use \( \Omega \) to denote any particular fundamental \( \mathcal{M} \)-cone. According to Theorem 3.1, the discrete truncated power \( t(\cdot|\mathcal{M}) \) agrees, on \( v(\Omega|\mathcal{M}) \), with an element in \( \nabla(\mathcal{M}) \), which is denoted as \( f_{\Omega}(\cdot|\mathcal{M}) \). We use \( \overline{\Omega} \) to denote the closure of \( \Omega \). From Lemma 4.4 in [22], we have \( \overline{\Omega} \cap \mathbb{Z}^s \subset v(\Omega|\mathcal{M}) \). Note that \( t(\alpha|\mathcal{M}) = 0 \), when \( \alpha \in \mathbb{Z}^s \setminus \text{cone}(\mathcal{M}) \). Hence, to present an explicit formula for \( t(\cdot|\mathcal{M}) \), we only need an explicit formula for \( f_{\Omega}(\cdot|\mathcal{M}) \).

In the following theorem, we give the polynomial part of \( f_{\Omega}(\cdot|\mathcal{M}) \), i.e., \( P_{e}f_{\Omega}(\cdot|\mathcal{M}) \).

**Theorem 4.1.** Suppose that \( p_{\mu,\Omega} \) is the homogeneous polynomial part of degree \( n - s - \mu \) of \( P_{e}f_{\Omega}(x|\mathcal{M}) \). Under the conditions of Theorem 3.1 when \( x \in \Omega \), we have

\[
(4.1) p_{\mu,\Omega}(x) = \begin{cases} T(x|\mathcal{M}), & \mu = 0, \\ - \sum_{j=0}^{\mu-1} \left( \sum_{|u|=k-j} D^u p_{j,\Omega}(x)(-i)^{|u|} D^u \hat{B}(0|\mathcal{M})/u! \right), & 1 \leq \mu \leq n - s. \end{cases}
\]

**Proof.** Rewriting of Equation (3.4) yields

\[
T(x|\mathcal{M}) - \sum_{0 < |u| \leq n-s} D^u P_{e}f_{\Omega}(x|\mathcal{M})(-i)^{|u|} D^u \hat{B}(0|\mathcal{M})/u! = P_{e}f_{\Omega}(x|\mathcal{M}), \quad x \in \Omega.
\]

According to (4.2), \( p_{0,\Omega}(x) \) agrees with \( T(x|\mathcal{M}) \) on \( \Omega \). Since both sides of (4.2) are polynomials, we can rewrite (4.2) as

\[
(4.3) p_{0,\Omega}(x) - \sum_{0 < |u| \leq n-s} D^u \sum_{\mu=0}^{n-s} p_{\mu,\Omega}(x)(-i)^{|u|} D^u \hat{B}(0|\mathcal{M})/u! = \sum_{\mu=0}^{n-s} p_{\mu,\Omega}(x), \quad x \in \mathbb{R}^s.
\]

A comparison between both sides of the equation above shows that

\[
p_{\mu,\Omega}(x) = - \sum_{j=0}^{\mu-1} \left( \sum_{|u|=\mu-j} D^u p_{j,\Omega}(x)(-i)^{|u|} D^u \hat{B}(0|\mathcal{M})/u! \right), \quad 1 \leq \mu \leq n - s.
\]

\[\square\]
Based on (3.5), to write down an explicit formula for \( t(\cdot | M) \) on \( \Omega \), we only need a formula for \( P_{\vartheta} f_{\Omega}(\cdot | M) \) for each fixed \( \vartheta \in A(M) \). To describe the formula conveniently, we suppose \( M \setminus M_\vartheta = (m_1, \ldots, m_\kappa) \), where \( \kappa = \#M - \#M_\vartheta \). We select \( r \) as the least integer in the set \( \{ r : (\vartheta^m)^r = 1 \text{ for all } m \in M \setminus M_\vartheta \} \). (According to (3.3), the set is non-empty, since there is at least an integer, \( \prod_{Y \in \mathcal{B}(M)} |\det(Y)| \), in it.) We use \( q_{\mu, r}(x), \mu \in \mathbb{Z}_+ \), to denote homogeneous polynomials satisfying the following condition:

\[
q_{\mu, r}(x) = \sum_{j_1 + \cdots + j_\kappa = \mu} \frac{D_{m_1}^{j_1} \cdots D_{m_\kappa}^{j_\kappa} T(x|M_\vartheta)}{(j_1 + 1)!} \prod_{i=1}^\kappa s_{i+1}(\vartheta^{-m_i}) \prod_{w \in M \setminus M_\vartheta} (1 - \vartheta^{-w}), \quad x \in \Omega.
\]

Throughout the rest of the paper, we set \( s_j(x) := (-1)^j(x + 2^j x^2 + \cdots + (r - 1)!x^{r-1}) \) and \( \tilde{M}_r := \{ \tilde{m}_1, \ldots, \tilde{m}_\kappa \} \), where if \( m_i \in M_\vartheta \), then \( \tilde{m}_i = m_i \), otherwise, \( \tilde{m}_i = r m_i \).

**Theorem 4.2.** Suppose that \( p_{\mu, \Omega}^\vartheta \) is the homogeneous polynomial part of degree \( n - s - \kappa - \mu \) of \( P_{\vartheta} f_{\Omega}(\cdot | M) \). Under the conditions of Theorem 3.1, for each fixed \( \vartheta \in A(M) \), we have

\[
p_{0, \Omega}^\vartheta(\cdot) = q_{0, r}^\vartheta(\cdot),
\]

\[
p_{\mu, \Omega}^\vartheta(\cdot) = q_{\mu, r}^\vartheta(\cdot) - \sum_{j=0}^{\mu-1} \sum_{|u| = \mu - j} D^u p_{j, \Omega}^\vartheta(\cdot)(-i)^{|u|} D^u \tilde{B}(0|\tilde{M}_r)/|u|! \left( 1 - \vartheta^{-w} \right),
\]

where \( r \) is the least positive integer such that \( (\vartheta^m)^r = 1 \) holds for all \( m \in M \setminus M_\vartheta \).

Combining Theorem 4.2 and \( q_{0, r}^\vartheta(x) = T(x|M_\vartheta) \prod_{w \in M \setminus M_\vartheta} \frac{1}{1 - \vartheta^{-w}}, \quad x \in \Omega \), we can easily extend Theorem 3.2 as follows:

**Theorem 4.3.** (31) Under the conditions of Theorem 3.2, the leading part of \( P_{\vartheta} f_{\Omega}(\cdot | M) \) agrees with

\[
T(\cdot | M_\vartheta) \prod_{w \in M \setminus M_\vartheta} \frac{1}{1 - \vartheta^{-w}}
\]
on \( \Omega \).

An alternative method for proving Theorem 4.3 is also given in 31.
Remark 4.4. Since $P_\vartheta f_\Omega(\cdot|M)$ is a polynomial and $\overline{\Omega} \cap \mathbb{Z}^\ast \subset v(\Omega|M)$ (see Lemma 4.4 in [22]), Theorem 3.2 and Theorem 4.3 also hold on $\overline{\Omega}$.

Remark 4.5. In [5], Brion and Vergne give formulas for the lattice point enumerator of a convex rational polytope in terms of certain Todd differential operators. These are interesting connections to topology. The salient difference in approaches lies in our explicit use of splines. Although the formula presented in [5] might eventually be equivalent our, the analysis of the equivalence between our different formulas would be the content of another paper.

Example 4.6. We consider the number of non-negative integer solutions of the linear equations

$$
\begin{pmatrix}
3 & 2 & 1 & 0 \\
0 & 1 & 2 & 2
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{pmatrix}
=\begin{pmatrix}
k_1 \\
k_2
\end{pmatrix}.
$$

Set $M = \begin{pmatrix}3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 2\end{pmatrix}$ and $k = \begin{pmatrix}k_1 \\ k_2\end{pmatrix}$. A simple computation shows that $A(M) = \{(1, 1)\} \cup A_3(M) \cup A_2(M)$, where

$$
A_3(M) = \{(1, -1), (\exp(2\pi i/3), \exp(2\pi i/3)), (\exp(4\pi i/3), \exp(4\pi i/3))\},
$$

$$
A_2(M) = \{(-1, 1), (\exp(2\pi i/3), 1), (\exp(2\pi i/3), -1), (\exp(4\pi i/3), 1),
$$

$$
(\exp(4\pi i/3), -1), (i, -1), (\exp(3\pi i/2), -1), (\exp(2\pi i/3), \exp(5\pi i/3)),
$$

$$
(\exp(4\pi i/3), \exp(\pi i/3))\}.
$$

For $\theta \in A_3(M)$, we have $\#M_\theta = 3$ while $\#M_\theta = 2$ provided that $\theta \in A_2(M)$.

For the matrix $M$, there are 3 fundamental $M$-cones, i.e., $\Omega_1 := \text{cone}^\circ(M_{12})$, $\Omega_2 := \text{cone}^\circ(M_{23})$ and $\Omega_3 := \text{cone}^\circ(M_{34})$, where $M_{12} = \begin{pmatrix}3 & 2 \\ 0 & 1\end{pmatrix}$, $M_{23} = \begin{pmatrix}2 & 1 \\ 1 & 2\end{pmatrix}$ and $M_{34} = \begin{pmatrix}1 & 0 \\ 2 & 2\end{pmatrix}$. 
Using Theorem 4.2, we obtain the explicit formula for \( t(\cdot | M) \) as follows: when \( k \in v(\Omega_1, M) \),

\[
t(k|M) = k_2^2/24 + 5k_2/24 + 11/48 + 5/48 + \left(3 + \sqrt{3}i\right)k_2/36 + (17 + 3\sqrt{3}i)/72 \]
\[
\cdot \exp((2\pi ik_1 + 2\pi ik_2)/3) + (-1)^{k_2}(k_2/24 + 5/48) + \exp((4\pi ik_1 + 4\pi ik_2)/3)
\]
\[
\left((3 - \sqrt{3}i)k_2/36 + (17 - 3\sqrt{3}i)/72\right) + \exp(2\pi ik_1/3)/18 - \sqrt{3}/18i \exp(2\pi ik_1/3)(-1)^{k_2}
\]
\[
+ \exp(4\pi ik_1/3)/18 + \sqrt{3}/18i(-1)^{k_2} \exp(4\pi ik_1/3) + \exp((2\pi ik_1 + 5\pi ik_2)/3)(1/24
\]
\[
+ \sqrt{3}/72i) + \exp((4\pi ik_1 + \pi ik_2)/3)(1/24 - \sqrt{3}/72i);
\]
when \( k \in v(\Omega_2, M) \),

\[
t(k|M) = k_1k_2/18 - (k_1^2 + k_2^2)/72 + 7k_2/72 + k_1/18 + 23/108 + (-1)^{k_2}(k_2/24 + 5/48)
\]
\[
+ \exp((2\pi ik_1 + 2\pi ik_2)/3) \left((3 + \sqrt{3}i)/108 \cdot (2k_1 - k_2) + 19/216 + \sqrt{3}/24i\right)
\]
\[
+ \exp((4\pi ik_1 + 4\pi ik_2)/3) \left((3 - \sqrt{3}i)/108 \cdot (2k_1 - k_2) + 19/216 - \sqrt{3}/24i\right)
\]
\[
+ (-1)^{k_1}/16 + 1/18 \exp(2\pi ik_1/3) - \sqrt{3}/18i \exp(2\pi ik_1/3)(-1)^{k_2} + \exp(4\pi ik_1/3)/18
\]
\[
+ \sqrt{3}/18i(-1)^{k_2} \exp(4\pi ik_1/3) + 1/8 \exp(\pi ik_1/2)(-1)^{k_2} + 1/8(-1)^{k_2} \exp(3\pi ik_1/2)
\]
\[
+ (1/24 + \sqrt{3}/72) \exp((2\pi ik_1 + 5\pi ik_2)/3) + (1/24 - \sqrt{3}/72) \exp((4\pi ik_1 + \pi ik_2)/3);
\]
when \( k \in v(\Omega_3, M) \),

\[
t(k|M) = k_2^2/24 + k_1/4 + 47/144 + (-1)^{k_2}(k_1/12 + 1/4) + (-1)^{k_1}/16 + \exp(2\pi ik_1/3)/18
\]
\[
- (\sqrt{3}/18)i \exp(2\pi ik_1/3)(-1)^{k_2} + (1/18) \exp(4\pi ik_1/3) + (\sqrt{3}/18)i \exp(4\pi ik_1/3)(-1)^{k_2}
\]
\[
+ (1/8) \exp(\pi ik_1/2)(-1)^{k_2} + (1/8) \exp(3\pi ik_1/2)(-1)^{k_2}.
\]

5. Proof of Theorem 4.2

In this section, we shall prove Theorem 4.2. The proof begins with several lemmas, designed to make the proof more readable. The main idea for proving Theorem 4.2 is to generalize Equation (3.6), with the equation playing an important
role in obtaining the polynomial part for \( f_{\Omega}(\cdot|M) \). This is the plan. Our discussion will be broken into three steps.

First, we introduce a set. Let

\[
c(\Omega, H) := \bigcap_{h \in H} \bigcap_{b_h \in [0, 1]} (\Omega + b_h h),
\]

where \( H \) is a finite set of real \( s \)-vectors. For example, if we set \( H := \{1\} \) and \( \Omega := (0, +\infty) \), then \( c(\Omega, H) = (1, +\infty) \). We have

**Lemma 5.1.** Suppose that \( H \) is a finite set of real \( s \)-vectors. Then for the fundamental \( M \)-cone \( \Omega \), the set \( c(\Omega, H) \subset \Omega \) and the volume of \( c(\Omega, H) \) in \( \mathbb{R}^s \) is infinity.

**Proof.** According to the definition of \( c(\Omega, H) \), one has

\[
c(\Omega, H) = \bigcap_{h \in H} \bigcap_{b_h \in [0, 1]} (\Omega + b_h h) \subset \bigcap_{h \in H, b_h = 0} (\Omega + b_h h) = \Omega.
\]

Based on the definition of the fundamental \( M \)-cone, there exist real \( s \)-vectors \( g_1, \ldots, g_\omega, \omega \in \mathbb{N} \), such that \( \text{span}\{g_1, \ldots, g_\omega\} = \mathbb{R}^s \) and \( \Omega = \{a_1 g_1 + \cdots + a_\omega g_\omega : a_i > 0\} \). Hence, for any \( h \in H \), there exist \( \tilde{a}_1(h), \ldots, \tilde{a}_\omega(h) \in \mathbb{R} \) such that

\[
h = \tilde{a}_1(h) g_1 + \cdots + \tilde{a}_\omega(h) g_\omega.
\]

Then

\[
c(\Omega, H) = \bigcap_{h \in H} \bigcap_{b_h \in [0, 1]} \bigcup_{a_i > 0} \{\sum_{i=1}^\omega (a_i + b_h \tilde{a}_i(h)) g_i\}.
\]

Let \( a(h) := \max\{|\tilde{a}_1(h)|, \ldots, |\tilde{a}_\omega(h)|\} \) and let \( a(H) := \max\{a(h) : h \in H\} \). A simple observation is that \( a_i + b_h \tilde{a}_i(h) > 0 \) for any \( b_h \in [0, 1], h \in H \), provided that \( a_i > a(H) \). Hence, we have

\[
S := \{x : x = a_1 h_1 + \cdots + a_\omega h_\omega, a_i \in \mathbb{R}, a_i > a(H)\} \subset c(\Omega, H).
\]

Note that the volume of \( S \) in \( \mathbb{R}^s \) is infinite. So, the volume of \( c(\Omega, H) \) in \( \mathbb{R}^s \) is also infinite. \( \square \)

Second, we shall generalize Equation (3.6), since it is of importance in obtaining the explicit formula for \( P_{\varphi} f_{\Omega}(x|M) \). We recall some notations and definitions. Let \( \varphi \) be an arbitrary but a fixed vector in \( A(M) \). Recall that \( r \) is the least positive integer such that \( (\varphi^m)^r = 1 \) holds for all \( m \in M \setminus M_\varphi \). Let \( \tilde{M}_r = \{\tilde{m}_1, \ldots, \tilde{m}_n\} \),
where if \( m_i \in M_\beta \), then \( \tilde{m}_i = m_i \), otherwise, \( \tilde{m}_i = rm_i \). Before extending Equation \((3.0)\), we first generalize \((3.2)\) as follows:

**Proposition 5.2.**

\[
\sum_{\alpha \in Z^t} \vartheta^{-\alpha} t(\alpha|M) B(x - \alpha|\tilde{M}_r) = \sum_{0 \leq r_1, \ldots, r_n < r} \vartheta^{-\sum_{i=1}^n m_ir_i} T(x - \sum_{i=1}^n r_i)\tilde{M}_r.
\]

**Proof.** Using the fact that \( t(\alpha|M) = \#\{B \in Z^t_+: M\beta = \alpha\} \), we see that

\[
\sum_{\alpha \in Z^t} \vartheta^{-\alpha} t(\alpha|M) B(x - \alpha|\tilde{M}_r) = \sum_{\beta \in Z^t_+} \vartheta^{-\sum_{i=1}^n m_i\beta_i} B(x - \sum_{i=1}^n m_i\beta_i|\tilde{M}_r) = \sum_{\beta_1, \ldots, \beta_n \in Z^t_+} \vartheta^{-\sum_{i=1}^n m_i\beta_i} B(x - \sum_{i=1}^n m_i\beta_i|\tilde{M}_r).
\]

We can write \( \beta_i \) as \( \gamma_i r + r_i \), where \( 0 \leq r_i < r \), \( \gamma_i, r_i \in Z^t_+ \) for \( i \leq \kappa \). Substituting \( \beta_i = \gamma_i r + r_i \), where \( i \leq \kappa \), into \((5.2)\) and noting that \( \vartheta^{rm_i} = 1 \), we obtain that

\[
\sum_{0 \leq r_1, \ldots, r_n < r} \sum_{\gamma_1, \ldots, \gamma_n \in Z^t_+} \sum_{\beta_1, \ldots, \beta_n \in Z^t_+} \vartheta^{-\sum_{i=1}^n m_i\gamma_i} B(x - \sum_{i=1}^n \gamma_i m_i) = \sum_{0 \leq r_1, \ldots, r_n < r} \sum_{\gamma_1, \ldots, \gamma_n \in Z^t_+} \vartheta^{-\sum_{i=1}^n m_i\gamma_i} \sum_{\beta_1, \ldots, \beta_n \in Z^t_+} B(x - \sum_{i=1}^n \beta_i m_i) = \sum_{0 \leq r_1, \ldots, r_n < r} \vartheta^{-\sum_{i=1}^n m_i r_i} T(x - \sum_{i=1}^n m_i r_i|\tilde{M}_r).
\]

The last equation follows from \( \sum_{\beta \in Z^t_+} B(x - \tilde{M}_r|\tilde{M}_r) = T(x|\tilde{M}_r) \), which can be obtained directly from the definitions of \( B(\cdot|M) \) and \( T(\cdot|M) \).

Now we arrive at

\[
\sum_{\alpha \in Z^t} \vartheta^{-\alpha} t(\alpha|M) B(x - \alpha|\tilde{M}_r) = \sum_{0 \leq r_1, \ldots, r_n < r} \vartheta^{-\sum_{i=1}^n m_ir_i} T(x - \sum_{i=1}^n m_i r_i|\tilde{M}_r).
\]

\( \square \)

In fact, if we set \( \vartheta = e \), then \((5.1)\) is reduced to \((3.2)\). So, as said above, \((5.1)\) can be considered as a generalization of \((3.2)\). To simplify the term \( \vartheta^{-\alpha} t(\alpha|M) B(x - \alpha|\tilde{M}_r) \) in \((5.1)\), we need to study \( \theta/\vartheta \) where \( \theta \in A(M) \).

**Proposition 5.3.** For any \( \theta \in A(M) \), we have \( \theta/\vartheta \in A(\tilde{M}_r) \) and \( D(M_\theta) \subset D((\tilde{M}_r)_{\theta/\vartheta}) \).
Proof. The definition of $\tilde{M}_r$ shows that $q^m = \tilde{q}^m = 1$ for any $m \in \tilde{M}_r$. Hence, $(\tilde{M}_r)_\theta = (\tilde{M}_r)_{\theta/\vartheta}$ for any $\theta \in A(M)$. Also, $rM_\theta \subset (\tilde{M}_r)_\theta$ since $\theta^m = 1$ implies $\tilde{q}^m = 1$. So, we have span($(\tilde{M}_r)_\theta) = \text{span}((\tilde{M}_r)_{\theta/\vartheta}) = \text{span}(M_\theta) = \mathbb{R}^s$ which implies $\theta/\vartheta \in A(\tilde{M}_r)$. To prove that $D(M_\theta) \subset D((\tilde{M}_r)_\theta)$, we only need to show that $D(M_\theta) \subset D((\tilde{M}_r)_\theta)$ since $(\tilde{M}_r)_\theta = (\tilde{M}_r)_{\theta/\vartheta}$. Select an $f \in D(M_\theta)$ and consider $D_{\tilde{Y}} f$, where $\tilde{Y} = (\tilde{y}_1, \ldots, \tilde{y}_w) \in \mathcal{V}((\tilde{M}_r)_\theta)$. We claim that $D_{\tilde{Y}} f = 0$. We set $Y = (y_1, \ldots, y_w)$, where $y_j = \tilde{y}_j$ if $\tilde{y}_j \in M_\theta$ otherwise $y_j = \tilde{y}_j/\vartheta$. We can see that $Y \in \mathcal{V}(M_\theta)$ and hence $D_Y f = c_0 D_{\tilde{Y}} f = 0$, where $c_0$ is a non-zero constant. So, we have $f \in D((\tilde{M}_r)_\theta)$, which implies that $D(M_\theta) \subset D((\tilde{M}_r)_\theta)$.

We now generalize Equation (5.3). Let $H_\theta := \{(r-1)m_i : 1 \leq i \leq \kappa\}$ and

$$Q_{\theta,r}(x) := \sum_{0 \leq r_1, \ldots, r_\kappa < r} \theta^{-\sum_{i=1}^\kappa m_i r_i} T(x - \sum_{i=1}^\kappa m_i r_i |\tilde{M}_r).$$

Then we have the following result.

**Lemma 5.4.** When $x \in c(\Omega, H_\theta)$,

$$Q_{\theta,r}(x) = P_\theta f_{\Omega}(x|M) + \sum_{|u| = 1}^{\kappa-s} D^{u} P_\theta f_{\Omega}(x|M)(-i)^{|u|} D^{u} B(\cdot |\tilde{M}_r)(0)/u!.$$  (5.3)

**Proof.** By Proposition 5.2 we have

$$Q_{\theta,r}(x) = \sum_{\alpha \in \mathbb{Z}^s} \theta^{-\alpha t(\alpha|M)} B(x - \alpha |\tilde{M}_r).$$

So, to this end, we only need to prove that

$$\sum_{\alpha \in \mathbb{Z}^s} \theta^{-\alpha t(\alpha|M)} B(x - \alpha |\tilde{M}_r) = P_\theta f_{\Omega}(x|M) + \sum_{|u| = 1}^{\kappa-s} D^{u} P_\theta f_{\Omega}(x|M)(-i)^{|u|} D^{u} B(\cdot |\tilde{M}_r)(0)/u!.$$  (5.4)

The definition of $c(\Omega, H_\theta)$ shows that $v(x|\tilde{M}_r) \subset v(\Omega|M)$ provided that $x \in c(\Omega, H_\theta)$. Noting that $\{ j \in \mathbb{Z}^s : B(x - j |\tilde{M}_r) \neq 0 \} = v(x|\tilde{M}_r)$ (see [15], pp.33), we have

$$\sum_{\alpha \in v(\Omega|M)} \theta^{-\alpha t(\alpha|M)} B(x - \alpha |\tilde{M}_r) = \sum_{\alpha \in \mathbb{Z}^s} \theta^{-\alpha t(\alpha|M)} B(x - \alpha |\tilde{M}_r),$$  (5.4)
where \( x \in c(\Omega, H_\theta) \). Hence,

\[
(5.5) \quad \sum_{\alpha \in v(\Omega)} \theta^{-\alpha} t(\alpha|\tau) B(x - \alpha|\tau) = \sum_{\alpha \in Z^*} \theta^{-\alpha} t(\alpha|\tau) B(x - \alpha|\tau)
\]

\[
= \sum_{\alpha \in Z^*} \sum_{\theta \in A(\tau)} (\theta/\tau)^\alpha P_\theta f_\tau(\alpha|\tau) B(x - \alpha|\tau) + \sum_{\alpha \in Z^*} P_\theta f_\tau(\alpha|\tau) B(x - \alpha|\tau)
\]

\[
= \sum_{\alpha \in Z^*} P_\theta f_\tau(\alpha|\tau) B(x - \alpha|\tau) \quad \text{where } x \in c(\Omega, H_\theta).
\]

The last equation follows from \( \sum_{\alpha \in Z^*} \rho(\alpha) B(x - \alpha|\tau) = 0 \) for any \( \rho(\alpha) \in E(\tau) \) (see Proposition 5.2 in [12]) and \( \sum_{\theta \in A(\tau)} (\theta/\tau)^\alpha P_\theta f_\tau(\alpha|\tau) \in E(\tau) \), where \( E(\tau) \) is the space of functions \( \rho(\alpha) \) of the form

\[
\rho(\alpha) = \sum_{\theta \in A(\tau)} (\theta/\tau)^\alpha, \quad \rho(\alpha) \in D(\tau).
\]

We now consider the sum

\[
(5.6) \quad \sum_{\alpha \in Z^*} P_\theta f_\tau(\alpha|\tau) B(x - \alpha|\tau), \quad x \in c(\Omega, H_\theta).
\]

Let \( \Psi(y) = P_\theta f_\tau(y|\tau) B(x - y|\tau) \). Then

\[
\hat{\Psi}(\xi) = \int_{R^*} \Psi(y) \exp(-iy\xi) dy
\]

\[
= \int_{R^*} P_\theta f_\tau(y|\tau) B(x - y|\tau) \exp(-iy\xi) dy
\]

\[
= \int_{R^*} P_\theta f_\tau(x - t|\tau) B(t|\tau) \exp(-i(x - t)\xi) dt
\]

\[
= \exp(-ix\xi) \int_{R^*} P_\theta f_\tau(x - t|\tau) B(t|\tau) \exp(it\xi) dt
\]

\[
= \exp(-ix\xi) P_\theta f_\tau(-iD + x|\tau) \hat{B}(-\xi|\tau).
\]

Taking into account that (see [9])

\[
q(D)\hat{B}(2\pi \alpha|\tau) = 0, \quad \text{where } \alpha \in Z^* \setminus 0 \quad \text{and} \quad q \in D(\tau),
\]

Poisson’s summation formula converts the sum (5.6) into

\[
(5.7) \quad P_\theta f_\tau(-iD + x|\tau) \hat{B}(-\xi|\tau)(0),
\]
since \( P_\vartheta f_\Omega (\cdot + x|M) \in D(M_\vartheta) \) for each fixed \( x \) and \( D(M_\vartheta) \subset D(\widehat{M}_r) \). Expanding \( P_\vartheta f_\Omega \) in a Taylor series for each fixed \( x \) and noting that \( \widehat{B}(0|\widehat{M}_r) = 1 \), one has

\[
P_\vartheta f_\Omega (-iD + x|M) \widehat{B}(|\widehat{M}_r|)(0) = P_\vartheta f_\Omega (x|M) + \sum_{0 < |u| \leq n - \kappa - s} D^u P_\vartheta f_\Omega (x|M)(-i)^{|u|} D^u \widehat{B}(|\widehat{M}_r|)(0)/u!,
\]

where \( x \in c(\Omega, H_\vartheta) \). So, combining (5.1), (5.4), (5.5), (5.6), (5.7) and (5.8), we have

\[
Q_{\vartheta, r}(x) = P_\vartheta f_\Omega (x|M) + \sum_{|u| = 1}^{n - \kappa - s} D^u P_\vartheta f_\Omega (x|M)(-i)^{|u|} D^u \widehat{B}(|\widehat{M}_r|)(0)/u!.
\]

Finally, we present an explicit formula for \( Q_{\vartheta, r}(x) \).

**Lemma 5.5.** When \( x \in c(\Omega, H_\vartheta) \),

\[
Q_{\vartheta, r}(x) = \sum_{0 \leq j_1 + \cdots + j_s \leq n - s - \kappa} \frac{1}{j_1^{m_1} \cdots j_s^{m_s}} T(x|M_\vartheta) \prod_{i=1}^{\kappa} \sum_{j_i=1}^{s_i+1} \frac{(-r_i D_{m_i})^{j_i}}{j_i!} T(x|\widehat{M}_r).
\]

**Proof.** When \( x \in c(\Omega, H_\vartheta) \), we can see that \( Q_{\vartheta, r}(x) \) is a polynomial of degree less than \( n - s + 1 \). Using the Taylor expansion, we obtain

\[
T(x - \sum_{i=1}^{\kappa} r_i m_i |\widehat{M}_r) = T(x|\widehat{M}_r) + \sum_{j=1}^{n-s} \frac{1}{j!} (-\sum_{i=1}^{\kappa} r_i D_{m_i})^j T(x|\widehat{M}_r),
\]

where \( x \in c(\Omega, H_\vartheta) \). Noting that \( \sum_{r_i=0}^{r-1} \vartheta^{-r_i m_i} = 0 \), we have

\[
Q_{\vartheta, r}(x)
= \sum_{0 \leq r_1, \ldots, r_s < r} \vartheta^{-\sum_{i=1}^{\kappa} r_i m_i} T(x - \sum_{i=1}^{\kappa} m_i r_i |\widehat{M}_r)
= \sum_{0 \leq r_1, \ldots, r_s < r} \vartheta^{-\sum_{i=1}^{\kappa} r_i m_i} \left( T(x|\widehat{M}_r) + \sum_{j=1}^{n-s} \frac{1}{j!} \left( -\sum_{i=1}^{\kappa} r_i D_{m_i} \right)^j T(x|\widehat{M}_r) \right)
= \sum_{0 \leq r_1, \ldots, r_s < r} \vartheta^{-\sum_{i=1}^{\kappa} r_i m_i} \sum_{0 \leq j_1 + \cdots + j_s \leq n-s} \left( \prod_{i=1}^{\kappa} \frac{(-r_i D_{m_i})^{j_i}}{j_i!} \right) T(x|\widehat{M}_r)
= \sum_{0 \leq j_1 + \cdots + j_s \leq n-s} \sum_{0 \leq r_1, \ldots, r_s < r} \vartheta^{-\sum_{i=1}^{\kappa} r_i m_i} \left( \prod_{i=1}^{\kappa} \frac{(-r_i D_{m_i})^{j_i}}{j_i!} \right) T(x|\widehat{M}_r).
\]
Moreover, if there exists an index $j_h = 0$, where $1 \leq h \leq \kappa$, then

\begin{equation}
\sum_{0 \leq r_1, \ldots, r_s < r} \vartheta^s \sum_{i=1}^{\kappa} r_i m_i \Pi_{i=1}^{\kappa} (-r_i D_{m_i})^{j_i} \equiv 0,
\end{equation}

since $\sum_{r_h=0}^{r-1} \vartheta^{-r_h m_h} (-r_h D_{m_h})^{j_h} = \sum_{r_h=0}^{r-1} \vartheta^{-r_h m_h} \equiv 0$.

Recall that $s_j(x) = (-1)^j (x + 2j + 2 \cdots + (r - 1)^j x^{r-1})$. Based on (5.9), (5.10) and (2.3), we have

\begin{align*}
Q_{\vartheta, r}(x) &= \sum_{0 \leq r_1, \ldots, r_s < r} \vartheta^s \sum_{i=1}^{\kappa} r_i m_i \Pi_{i=1}^{\kappa} (-r_i D_{m_i})^{j_i} T(x|\tilde{M}_r) \\
&= \sum_{\kappa \leq j_1 \leq \kappa} \sum_{0 \leq r_1, \ldots, r_s < r} \left( \prod_{i=1}^{\kappa} (-r_i D_{m_i})^{j_i} \right) T(x|\tilde{M}_r) \\
&= \frac{1}{\vartheta} \sum_{0 \leq r_1 < r} \cdots \sum_{0 \leq r_s < r} \left( \prod_{i=1}^{\kappa} (-r_i)^{j_i} D_{m_i}^{j_i} \right) T(x|\tilde{M}_r) \\
&= \sum_{0 \leq j_1, \ldots, j_s \leq n-s-\kappa} \frac{1}{\vartheta} \prod_{i=1}^{\kappa} (-r_i)^{j_i+1} D_{m_i}^{j_i} T(x|\tilde{M}_r) \\
&= \sum_{0 \leq j_1, \ldots, j_s \leq n-s-\kappa} \frac{1}{\vartheta} D_{m_1}^{j_1} \cdots D_{m_s}^{j_s} T(x|\tilde{M}_r) \prod_{i=1}^{\kappa} s_{j_i+1}(\vartheta^{-m_i})/(j_i + 1)!,
\end{align*}

provided that $x \in c(\Omega, H_\vartheta)$.

We now have all the ingredients for the proof of our main theorem.

\begin{proof}[Proof of Theorem 4.2] Since $P_\vartheta f_\Omega$ is a polynomial, it can be determined by its values on $c(\Omega, H_\vartheta)$ according to Lemma 5.1. We recall (4.2) that, for $\mu \in \mathbb{Z}_+$, the homogeneous polynomials $q^\vartheta_{\mu,r}(x)$ of degree $n-s-\kappa - \mu$ satisfy

\begin{equation}
q^\vartheta_{\mu,r}(x) = \sum_{j_1+\cdots+j_s = \mu} \frac{1}{\vartheta} D_{m_1}^{j_1} \cdots D_{m_s}^{j_s} T(x|\tilde{M}_r) \prod_{i=1}^{\kappa} s_{j_i+1}(\vartheta^{-m_i})/(j_i + 1)!, \quad x \in \Omega.
\end{equation}

Based on Lemma 5.3, we have

\begin{equation}
Q_{\vartheta, r}(x) = \sum_{0 \leq \mu \leq n-s-\kappa} q^\vartheta_{\mu,r}(x), \quad x \in c(\Omega, H_\vartheta).
\end{equation}

Note that $P_\vartheta f_\Omega(x|M)$ can be written in the form $\sum_{\mu=0}^{n-s-\kappa} P^\vartheta_{\mu,r}(x)$. By Lemma 5.4, one has

\begin{equation}
\sum_{\mu=0}^{n-s-\kappa} q^\vartheta_{\mu,r}(x) = P_\vartheta f_\Omega(x|M) + \sum_{|u|=1}^{n-s} D^u P_\vartheta f_\Omega(x|M)(-i)^{|u|} D^u \tilde{B}(0|\tilde{M}_r)/u!.
\end{equation}

\end{proof}
Comparing the homogeneous polynomials on both sides of (5.11), we arrive at
\[ p_{\vartheta_0, \Omega}(x) = q_{\vartheta_0, r}(x), \]
\[ p_{\mu, \Omega}(x) = q_{\mu, r}(x) - \sum_{j=0}^{\mu-1} \left( \sum_{|u| = \mu - j} \sum_{|\nu| = \mu - j} D_u p_{\vartheta_0, \mu}(x)(-i)^{|u|} D^\nu \tilde{B}(0) / u! \right), \quad \mu \geq 1. \]

\[ \square \]

6. Discrete truncated powers associated with special matrices

In this section, we shall present more detailed information concerning \( t(\cdot |M) \) for some particular matrices \( M \). We first introduce some definitions. Let
\[ \mathcal{S}_k(M) := \{ Y \subseteq M : \#Y = s + k, \text{ span}(Y) = \mathbb{R}^s \}. \]
In this notation, \( \mathcal{B}(M) = \mathcal{S}_0(M) \). If gcd\(|\det(X)| : X \in \mathcal{B}(Y)\) = 1 for any \( Y \in \mathcal{S}_k(M) \), then \( M \) is called a \( k \)-prime matrix. In particular, when \( M \) is a 1-prime matrix, it is also called a pairwise relatively prime matrix. When \( s = 1 \), the \( k \)-prime matrix has the property that any \( k \) integers in \( \{m_1, \ldots, m_n\} \) have no common factor.

An explicit formula for \( t(\cdot |M) \) is presented in [2] when \( s = 1 \) and \( M \) is a 1-prime matrix.

**Theorem 6.1.** (2) Suppose that \( M = (a_1, \ldots, a_n) \in \mathbb{Z}^n \) with \( a_1, \ldots, a_n \) pairwise relatively prime. Then
\[ t(\alpha |M) = R_{-\alpha}(a_1, \ldots, a_n) + (-1)^n \sum_{j=1}^{n} a_j \sum_{\sigma_j = 1 \neq k} \prod_{k \neq j} \frac{\theta_{-\alpha}}{\theta_{\sigma k} - 1}, \]
where \( R_{-\alpha}(a_1, \ldots, a_n) = -\text{Res}(F_{-\alpha}(z), z = 1) \), and \( F_{-\alpha}(z) = \frac{z^{-\alpha - 1}}{(1 - z a_1) \cdots (1 - z a_n) (1 - z)}. \)

Based on Theorem 4.2 we can extend the theorem above to higher dimensions.

**Theorem 6.2.** Under the conditions of Theorem 6.1,
\[ f_{\Omega}(\alpha |M) = P_{e} f_{\Omega}(\alpha |M) + \sum_{\theta \in A(M) \cup c} \theta^\theta \frac{1}{|\det(M_\theta)|} \prod_{w \in M \setminus M_\theta} \frac{1}{1 - \theta^{-w}} 1_{\text{cone}(M_\theta)}(\Omega), \]
provided that \( M \) is a 1-prime matrix, where \( P_{e} f_{\Omega}(\alpha |M) \) is presented in Theorem 4.1.
Proof. Select a \( \theta \in A(M) \setminus e \). To this end, we need to prove that \( \#M_\theta = s \). By (3.3), \( \theta \) has the form
\[
\theta = \exp(2\pi i \alpha / |\det Y|),
\]
where the definitions of \( Y = (y_1, \ldots, y_s) \) and \( \alpha \) are identical to those of (3.3). Note that \( \theta^y = 1 \) provided that \( y \in Y \), i.e., \( Y \subset M_\theta \). We assert that \( M_\theta = Y \).

Hence, there is a \( \beta := (\beta_1, \ldots, \beta_s) \in \mathbb{Z}^s \) such that \( \theta \) is in the form of
\[
\theta = \exp(2\pi i \beta / |\det Y'|).
\]
Without loss of generality, we suppose \( \beta_1 \neq 0 \). Since \( 0 < \alpha_1 \leq |\det Y'| - 1 \) and \( 0 < \beta_1 \leq |\det(Y')| - 1 \), we have \( \gcd(|\det(Y)|, \det(Y')) > 1 \) due to \( \alpha_1 / |\det Y'| = \beta_1 / |\det Y'| \), which contradicts \( \gcd(|\det(Y')|, \det(Y)) = 1 \). That is, \( M_\theta = Y \).

We next show one example, as the application of Theorem 6.2.

Example 6.3. Consider the following partition function
\[
p_{\{a,b\}}(n) = \#\{ (x, y) \in \mathbb{Z}_+^2 : ax + by = n, a, b \in \mathbb{Z}_+ \},
\]
where \( a \) and \( b \) are relatively prime. We shall prove that \( p_{\{a,b\}}(n) = \frac{n}{ab} - \left\{ \frac{b^{-1}n}{a} \right\} - \left\{ \frac{a^{-1}n}{b} \right\} + 1 \), which is “the beautiful formula due to Popoviciu” [3], where \( \{ x \} \) is the
fractional part of $\frac{1}{a}$, $b^{-1}$ and $a^{-1}$ denote two integers satisfying $b^{-1}b \equiv 1 \pmod{a}$, and $a^{-1}a \equiv 1 \pmod{b}$.

When $n \geq 0$, $p_{(a,b)}(n) = t(n|(a,b))$. Note that $T(x|(a,b)) = \frac{x+d}{ab}$ and $D^1\tilde{B}(0|(a,b)) = -\frac{1}{2}(a+b)$. By Theorem 6.2 when $n \geq 0$, $p_{(a,b)}(n)$ equals to

$$\frac{n}{ab} + \frac{a+b}{2ab} + \frac{1}{a} \sum_{k=1}^{b-1} \frac{\exp(\frac{2\pi i nk}{a})}{1 - \exp(-\frac{2\pi i nk}{a})} + \frac{1}{b} \sum_{k=1}^{a-1} \frac{\exp(\frac{2\pi i nk}{b})}{1 - \exp(-\frac{2\pi i nk}{b})}. \tag{6.1}$$

Based on the discrete Fourier analysis, one has (see [3], pp.144)

$$-\left\{\frac{t}{a}\right\} = \frac{1}{2a} + \frac{1}{a} \sum_{k=1}^{a-1} \frac{\exp(\frac{2\pi i nk}{a})}{1 - \exp(-\frac{2\pi i nk}{a})}, \tag{6.2}$$

where $t, a \in \mathbb{Z}$. According to (6.2), (6.1) can be reduced to $\frac{n}{ab} - \left\{\frac{b-1}{a}\right\} - \left\{\frac{a-1}{b}\right\} + 1$.

### 7. Exact Volume of Polytopes and the Ehrhart Polynomial

#### 7.1. Exact Volume of Polytopes

A convex polytope $P$ is the convex hull of a finite set of points in $\mathbb{R}^n$. In this section, we shall omit the qualifier “convex” since we confine our discussion to such polytopes. An integer polytope is a polytope whose vertices have integer coordinates. Similarly, a rational polytope is a polytope whose vertices have rational coordinates. The exact computation of the volume of $P$ is an important and difficult problem which has close ties to various mathematical areas [11, 31, 47, 29]. If $P$ is a $d$-dimensional polytope in $\mathbb{R}^n$, then let $\text{vol}_n(P)$ denote the $d$-dimensional volume of $P$ in $\mathbb{R}^n$, i.e., the $d$-dimensional measure of $P$ in $\mathbb{R}^n$.

Let $\mathbb{R}_P$ denote the affine space that is spanned by the vertex vectors of $P$. The lattice points in $\mathbb{R}_P$ form an Abelian group of rank $d$, i.e., $\mathbb{R}_P \cap \mathbb{Z}^d$ is isomorphic to $\mathbb{Z}^d$. Hence, there exists an invertible affine linear transformation $T : \mathbb{R}_P \to \mathbb{R}^d$ satisfying $T(\mathbb{R}_P \cap \mathbb{Z}^n) = \mathbb{Z}^d$. The relative volume of $P$, denoted as $\text{vol}(P)$, is just the $d$-dimensional volume of the image $T(P) \subset \mathbb{R}^d$. For more detailed information about the relative volume, the reader is referred to [29].

We next introduce a method for computing the relative volume of polytopes, which depends on the counting function for the integer points in a polytope. We consider the function of an integer-valued variable $g$ that describes the number of
lattice points lying inside the dilated polytope $gP$:

$$L_P(g) := \# \left( gP \cap \mathbb{Z}^n \right).$$

In [18], Ehrhart inaugurated the study of the general properties of $L_P(g)$. He proved that for an $n$-dimensional polytope $P$, $L_P(g)$ is a polynomial in the positive integer variable $g$ and that in fact

$$(7.1) \quad L_P(g) = \text{vol}(P)g^n + \frac{1}{2}\text{vol}(\partial P)g^{n-1} + \cdots + \chi(P).$$

Here, $\chi(P)$ is the Euler characteristic of $P$ and $\text{vol}(\partial P)$ is the surface area of $P$ normalized with respect to the sublattice on each face of $P$. Moreover, the leading coefficient of $L_P(g)$ is the relative volume of $P$. Hence, if we obtain the leading coefficient of $L_P(g)$, we can know the relative volume of $P$. In [11] and [7], the leading coefficient of $L_P(g)$ was computed by the interpolation and the residue theorem respectively. We can even present a formula for the leading coefficient of $L_P(g)$ using discrete truncated powers. Our result is

**Theorem 7.1.** Suppose that $P_{M,b} := \{x : Mx = b, x \in \mathbb{R}^n_+\}$ is an $(n-s)$-dimensional rational polytope. Here, $M$ is an $s \times n$ integer matrix and $b$ is an $s$-vector. Then $\text{vol}(P_{M,b}) = C_0 \cdot T(b|M)$, where $C_0 = \gcd\{|\det(Y)| : Y \in \mathcal{B}(M)\}$.

**Proof.** To prove the theorem, we only need to prove that

$$\text{vol}(P_{M,b}) = \# \{ [[M^T]] \cap \mathbb{Z}^n \} \cdot T(b|M)$$

since $C_0 = \# \{ [[M^T]] \cap \mathbb{Z}^n \}$ (see [30]). Since the dimension of $P_{M,b}$ is $n-s$, we have $b \in \text{cone}^\circ(M)$. By (2.2), $T(b|M)/\text{vol}(P_{M,b})$ is a constant which is independent of $b \in \text{cone}^\circ(M)$. For the integer matrix $M$, there exists an integer vector $b_0 \in \text{cone}^\circ(M)$, such that $P_{M,b_0}$ is an integer polytope. We now consider $L_{P_{M,b_0}}(g)$. Through the definition of $L_{P_{M,b_0}}(g)$ and $t(\cdot|M)$, we have

$$L_{P_{M,b_0}}(g) = t(gb_0|M),$$
with \( gb_0 \in \text{cone}^\circ(M) \) when \( g \in \mathbb{Z}_+ \). Based on Theorem 3.2 and Theorem 4.3, the leading term of \( L_{P_M,b_0}(g) \) is

\[
T(gb_0|M) \left( \sum_{\{\theta \in A(M): M_\theta = M\}} \theta^{gb_0} \right) = g^{n-s}T(b_0|M) \left( \sum_{\{\theta \in A(M): M_\theta = M\}} \theta^{gb_0} \right).
\]

Since \( L_{P_M,b_0}(g) \) is a polynomial, we obtain that \( \sum_{\{\theta \in A(M): M_\theta = M\}} \theta^{gb_0} \) is a constant, i.e., \( \theta^{gb_0} = 1 \). Hence, we have

\[
\sum_{\{\theta \in A(M): M_\theta = M\}} \theta^{gb_0} = \#\{\theta \in A(M): M_\theta = M\} = \#\{[[M^T]] \cap \mathbb{Z}^n\}.
\]

So,

\[
\#\{[[M^T]] \cap \mathbb{Z}^n\} \cdot T(b_0|M) = \text{vol}(P_{M,b_0}).
\]

It follows that

\[
T(b|M)/\text{vol}(P_{M,b}) = T(b_0|M)/\text{vol}(P_{M,b_0}) = 1/\#\{[[M^T]] \cap \mathbb{Z}^n\},
\]

when \( b \in \text{cone}^\circ(M) \). Hence \( \text{vol}(P_{M,b}) = \#\{[[M^T]] \cap \mathbb{Z}^n\} \cdot T(b|M) \).

\[\square\]

**Theorem 7.2.** Suppose that \( P_A^b = \{x : Ax \leq b, x \in \mathbb{R}_+^n\} \) is a \( n \)-dimensional polytope, where \( A \) is an \( s \times n \) integer matrix and \( b \) is an \( s \)-vector. Then \( \text{vol}_n(P_A^b) = T(b|M), \) where \( M = (A, E_{s \times s}) \).

**Proof.** Since \( P_A^b \) is an \( n \)-dimensional polytope in \( \mathbb{R}^n \), the relative volume of \( P_A^b \) is equal to the volume of \( P_A^b \). Let \( P_{A,b} = \{(x, y) : Ax + E_{s \times s}y = b, x \in \mathbb{R}_+^n, y \in \mathbb{R}_+^s\} \).

Since \( E_{s \times s} \subseteq M \), we have \( \gcd(|\det(Y)| : Y \in \mathcal{B}(M)) = 1 \). If each component of \( b \) is rational, then \( P_{A,b} \) is a rational polytope. Note that \( \text{vol}_n(P_A^b) = \text{vol}(P_A^b) = \text{vol}(P_{A,b}) \). By Theorem 7.1, \( \text{vol}_n(P_A^b) = \text{vol}(P_{A,b}) = T(b|M) \) when \( P_{A,b} \) is a rational polytope. Since both \( T(b|M) \) and \( \text{vol}_n(P_{A,b}) \) are continuous at \( b \) and the real number can be approximated by rational numbers, one has \( \text{vol}_n(P_A^b) = T(b|M) \) for any \( b \in \text{cone}^\circ(M) \).

\[\square\]

How can \( T(\cdot|M) \) be computed? In [24], an efficient method for calculating the multivariate truncated power is presented.
Theorem 7.3. (24) Let $M$ be an $s \times n$ matrix with columns $m_1, \ldots, m_n \in \mathbb{Z}^s \setminus \{0\}$ such that the origin is not contained in $\text{conv}(M)$. For any $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, and $x = \sum_{j=1}^{n} \lambda_j m_j$,

\begin{equation}
T(x|M) = \frac{1}{n - s} \sum_{j=1}^{n} \lambda_j T(x|M \setminus m_j).
\end{equation}

Hence, one can compute $T(x|M)$ according to the recurrence (7.2). Combining Theorem 7.2 and Theorem 7.3 we can present an iterative method for computing $\text{vol}(\Pi(x))$. In fact, the computational complexity is $O(s^n)$.

In the following, we shall present a simple proof for the volume formula for a polytope using this method, which is the central result in (26).

Let

$$\Pi_n(x) := \{ y \in \mathbb{R}^n : y_i \geq 0, y_1 + \cdots + y_i \leq x_1 + \cdots + x_i, \text{ for all } 1 \leq i \leq n \}$$

for arbitrary $x := (x_1, \ldots, x_n)$ with $x_i > 0$ for all $i$. Let $V_n(x) := \text{vol}(\Pi_n(x))$. The function $V_n(x)$ is a homogeneous polynomial of degree $n$ in the variables $x_1, \ldots, x_n$.

This polynomial arises in a variety of mathematical fields, such as the calculation of probabilities derived from empirical distribution functions, the study of parking functions, and plane partitions. In (26), an explicit formula for $V_n(x)$ is presented using a probabilistic method. Based on (7.2), we shall show a simple proof for the explicit formula for $V_n(x)$.

Theorem 7.4. (26) For each $n = 1, 2, \ldots$,

$$V_n(x) = \sum_{k \in K_n} \frac{x^k}{k!} = \frac{1}{n!} \sum_{k \in K_n} \binom{n}{k} x^k,$$

where $K_n := \{ k \in \mathbb{Z}^+_n : \sum_{i=1}^{j} k_i \geq j \text{ for all } 1 \leq j \leq n - 1 \text{ and } \sum_{i=1}^{n} k_i = n \}$.

Proof. Let $b = (b_1, \ldots, b_n)^T$, where $b_i = \sum_{h=1}^{i} x_h$, let $I_i$ denote the $i$th column of the $n \times n$ identity matrix, and let $I_i = \sum_{h=1}^{n} I_h$. Let $M_0 = (\tilde{I}_1, \ldots, \tilde{I}_n)$ and
$M = (M_0, E_{n \times n})$. Using these notations, we have

$$\Pi_n(x) = \{ y \in \mathbb{R}^n_+ : M_0 y \leq b \}.$$ 

By Theorem 7.2, $V_n(x) = T(b|M)$. To this end, we present an explicit formula for $T(b|M)$. Note that $b = x_1 I_1 + (x_1 + x_2) I_2 + \cdots + (\sum_{h=1}^n x_h) I_n$. By Theorem 7.3,

$$T(b|M) = \frac{1}{n} (x_1 T(b|M \setminus I_1) + \cdots + (\sum_{h=1}^n x_h) T(b|M \setminus I_n)).$$

Note that

$$b = x_1 I_1 + (x_1 + x_2) I_2 + \cdots + (\sum_{h=1}^n x_h) I_n$$

$$= x_1 I_1 + \cdots + (\sum_{h=1}^{i-1} x_h) I_{i-1} + (\sum_{h=1}^i x_h) I_i + x_{i+1} I_{i+1} + \cdots + (\sum_{h=i+1}^n x_h) I_n$$

$$= x_1 I_1 + \cdots + (\sum_{h=1}^{i-1} x_h) I_{i-1} + (\sum_{h=1}^i x_h) I_i + x_{i+1} I_{i+1} + \cdots + (\sum_{h=i+1}^n x_h) I_n.$$ 

For any fixed integer $i$, where $1 \leq i \leq n$, we have $T(b|M \setminus I_i \cup \tilde{I}_i) = 0$ since $b \notin \text{cone}(M \setminus I_i \cup \tilde{I}_i)$. Hence,

$$T(b|M \setminus I_i) = \frac{1}{n-1} (x_1 T(b|M \setminus I_i \cup I_1) + \cdots + (\sum_{h=1}^{i-1} x_h) T(b|M \setminus I_i \cup I_{i-1}) + x_{i+1} (b|M \setminus I_i \cup I_{i+1}) + (\sum_{h=i+1}^n x_h) T(b|M \setminus I_i \cup I_n)).$$

Moreover, for any $i > j$, we have

$$b = x_1 I_1 + \cdots + (\sum_{h=1}^{j-1} x_h) I_{j-1} + (\sum_{h=1}^j x_h) I_j + x_{j+1} I_{j+1} + \cdots + (\sum_{h=j+1}^n x_h) I_n$$

$$= (\sum_{h=j+1}^{i-1} x_h) I_{i-1} + (\sum_{h=j+1}^i x_h) I_i + x_{i+1} I_{i+1} + \cdots + (\sum_{h=i+1}^n x_h) I_n.$$ 

Substitute $b$ into $T(b|M \setminus (I_i \cup I_j))$ and then expand $T(b|M \setminus (I_i \cup I_j))$ by Theorem 7.3. Continuing the process, we have

$$V_n(x) = T(b|M) = \frac{1}{n!} \sum_{i \in \sigma_n} x(i),$$

where $\sigma_n$ is the set of permutations of the set $\{1, 2, \ldots, n\}$. $x(i) = x(i_1, \ldots, i_n) = x(i_1) \cdots x(i_n)$ and $x(i_k) = x_{j_k+1} + \cdots + x_{i_k}$. Here, $j_k = \max J_k$ (when $J_k = \emptyset, j_k = 0$), and $J_k = \bigcup_{j \leq k-1, j \leq i_k} \{i_j\}$ for a fixed permutation $i = (i_1, \ldots, i_n)$. To this end,
we prove
\[(7.3) \quad \sum_{i \in \sigma_n} x(i) = \sum_{k \in K_n} \binom{n}{k} x^k\]
by induction. When \(n = 1\), it is easy to prove that the conclusion holds. Suppose that the conclusion holds for \(n = m\), i.e.,
\[(7.4) \quad \sum_{i \in \sigma_m} x(i) = \sum_{k \in K_m} \binom{m}{k} x^k.\]
We now consider the case where \(n = m + 1\). Note that \(x_{m+1}\) appears in \(x(i_k)\) if and only if \(i_k = m + 1\). Hence, by (7.4), we have
\[\sum_{i \in \sigma_{m+1}} x(i) = \sum_{k \in K_{m+1}} a_k x^k,\]
where, if \(k_{m+1} = 0\),
\[a_k = \binom{m}{k_1 - 1, \ldots, k_m} + \ldots + \binom{m}{k_1, \ldots, k_m - 1} = \binom{m+1}{k},\]
if \(k_{m+1} = 1\), then \(a_k = (m + 1) \binom{m}{k_1, \ldots, k_m} = \binom{m+1}{k_1, \ldots, k_m, k_{m+1}} = \binom{m+1}{k}.\) Noting \(k_{m+1} \leq 1\), we have
\[\sum_{i \in \sigma_{m+1}} x(i) = \sum_{k \in K_{m+1}} \binom{m+1}{k} x^k.\]
Hence, (7.3) holds when \(n = m + 1\). \(\square\)

7.2. An explicit formula for Ehrhart polynomials. The explicit formula for \(L_P(g)\) is interesting. For any rational polytope, Ehrhart proved that \(L_P(g)\) is a quasipolynomial in \(g\). Here a quasipolynomial is an expression of the form \(c_n(g)t^n + \cdots + c_0(g)\), where the \(c_i(g)\) are periodic functions in \(g\). In (7.1), three coefficients of \(L_P(g)\) are presented. The other coefficients of \(L_P(g)\) have remained a mystery, even for a general lattice 3-simplex, until rather recently with the work of Pommersheim [25] in \(\mathbb{R}^3\), Kantor and Khovanski [23] in \(\mathbb{R}^4\), Cappell and Shaneson [6] in \(\mathbb{R}^n\), and Diaz and Robins [17] in \(\mathbb{R}^n\). In the following theorem, the explicit formula for \(L_P(g)\) is presented in terms of multivariate truncated powers, where \(P\) is a rational polytope.
Theorem 7.5. Let \((n - s)\)-dimensional polytopes \(P = \{x \in \mathbb{R}_+^n : Mx = b\}\), where \(M\) is an \(s \times n\) integer matrix and \(b\) is an integer \(s\)-vector. We have

\[
L_P(g) = \sum_{j=0}^{n-s} p_{j,\Omega}(b) g^{n-s-j} + \sum_{\theta \in A(M) \setminus e} \sum_{j=0}^{n-s-(\#M - \#M_\theta)} (\theta^b)^{p_{j,\Omega}^\theta(b)} g^{n-s-(\#M - \#M_\theta) - j},
\]

where \(\Omega\) is a fundamental \(M\)-cone, \(p_{j,\Omega}(x)\) and \(p_{j,\Omega}^\theta(x)\) are presented in Theorem 4.1 and Theorem 4.2 respectively.

According to the definition of \(L_P(g)\) and \(t(\cdot | M)\), we have \(L_P(g) = t(g b | M)\). By Theorem 4.1, Theorem 4.2 and the properties of multivariate truncate powers, the theorem follows.

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