HOMOLOGICALLY AREA MINIMIZING SURFACES WITH NON-SMOOTHABLE SINGULARITIES

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Dedicated to Xunjing Wei

Abstract. Let dimensions \( d \geq 3 \) and codimensions \( c \geq 3 \) be positive integers, and define the exceptional set \( E = \{3, 4\} \). We prove that, for \( d \notin E \) with arbitrary \( c \), or for \( d \in E \) with \( c \leq d \), for any \( d \)-dimensional integral homology class \([\Sigma]\) on a compact (not necessarily orientable) \((d + c)\)-dimensional smooth manifold \( M \), there exist open sets \( \Omega_{[\Sigma]} \) in the space of smooth Riemannian metrics such that all area-minimizing integral currents in \([\Sigma]\) are singular for metrics in \( \Omega_{[\Sigma]} \). This resolves a conjecture by White regarding the generic regularity of area-minimizing surfaces (Problem 5.16 in [3]) and provides a sharp dimension-wise answer. As a corollary, we determine the moduli space of area-minimizing currents in the vicinity of any area-minimizing transverse immersion with dimension \( d \geq 3 \) and codimension \( c \geq 3 \) that satisfies an angle condition of asymptotically sharp order in \( d \). Analogous conclusions apply to mod 2 area-minimizing surfaces.

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1. Introduction

In this paper, area-minimizing surfaces refer to area-minimizing integral currents or mod 2 currents. Roughly speaking they are singular surfaces counted with multiplicity that minimize the area functional for homologous competitors in integral or mod 2 homology classes. These surfaces arise naturally in various geometric contexts, such as special holonomic geometries (e.g., special Lagrangians, associative varieties, etc.,) and in the geometry of positive scalar curvature (area-minimizing hypersurfaces).

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In many cases, area-minimizing surfaces exhibit singularities. However, Almgren’s Big Theorem ([4]) and De Lellis-Spadaro’s new proof ([9][10][11]), control the dimension of the singular set, bounding it by \(d - 2\). In the hypersurface case, the singular set’s dimension can be reduced to \(d - 7\) as per [18]. In general the \(j\)-th strata in the Almgren stratification of the singular set is \(j\)-rectifiable according to [42].

A natural question is whether these singular sets can be generically avoided. We know that 2-dimensional area minimizing integral currents are classical branched minimal immersed surfaces and all the tangent cones are unique by [12][56]. Consequently, 2-dimensional area-minimizing surfaces are smooth in generic metrics by [57]. Similar results hold for area-minimizing hypersurfaces in 8-dimensional manifolds ([45] and [47]).

One might hope for similar results in general dimensions and codimensions, but existing work by Lawlor (Section 6.4 in [30]), Morgan (the first paragraph on page 138 in [39]), and Haskins and Pacini (Theorem 2.2 in [24]) suggests the presence of persistent singularities. Brian White posed the following question (Problem 5.16 in [3]):

"One can also ask whether singularities in a homologically area minimizing cycle in a Riemannian manifold disappear after generic perturbations of the metric."

According to [50], integral homology classes of dimension larger than 6 and codimension larger than 2 may not have smooth representatives, leading to a trivial answer to White’s question in those cases. However, our results show that, for area-minimizers, in general, every homology class can exhibit non-smoothable singularities in both integral cases (Theorem 1.1) and mod 2 cases (Theorem 1.4). Our answer to White’s question is sharp dimension-wise.

**Theorem 1.1.** Let \(d \geq 3, c \geq 3\) be integers, and set the exceptional set \(E = \{3, 4\}\). In case of \(d \not\in E\), \(c\) arbitrary, or \(d \in E, c \leq d\), for every \(d\)-dimensional integral homology class \([\Sigma]\) on a compact (not necessarily orientable) \((d + c)\)-dimensional smooth manifold \(M\), there exist open sets of Riemannian metrics \(\Omega_{[\Sigma]}\), so that all area-minimizing integral currents in \([\Sigma]\) are singular for all metrics in \(\Omega_{[\Sigma]}\). If \([\Sigma]\) admits immersed representatives with multiplicities on their connected components, then the singular set of any area-minimizer has Hausdorff dimension at least \(\max\{d - c, d - 5\}\) for metrics in \(\Omega_{[\Sigma]}\).

Thus, in general dimensions and codimensions, homologically area-minimizing surfaces can remain singular under generic perturbations of metric.

**Remark 1.** The dimension and codimension bounds stem from two types of obstructions to smoothing. When \(3 \leq c \leq d\), transverse self-intersections with not-so-small angles serve as one kind of obstruction. This is demonstrated using Lawlor’s vanishing calibrations (introduced in Section 2.5). For \(c = 2\) or \(d = 2\), even orthogonal transverse intersections can be smoothed.
out, as seen in complex hypersurfaces and holomorphic curves, where Lawlor’s calibration doesn’t exist. For $d \geq 5$ and $c \geq 3$, bordism rings pose obstructions (first observed by Haskins and Pacini in [24]), such as cones with non-boundary links. Our singularity model utilizes the cone over the Veronese embedding of $\mathbb{CP}^2 \subset S^7$ and its product with spheres. As the obstruction arises from bordism, it is insensitive to codimensions. Combining these two cases explains our dimension and codimension requirements.

Remark 2. Transverse intersection singularities deform into nearby transverse intersections in nearby metrics. We do not know the exact nature of singularities that minimizers possess under metric perturbations for bordism ring-induced singularities.

Remark 3. Since analytic metrics are dense, restricting the initial point of perturbation to analytic metrics cannot yield generic regularity.

Remark 4. In contrast to Thom’s results in [50], area-minimizers with non-smoothable singularities exist from dimension 3, while integral homology classes without smooth representatives emerge from dimension 7. Our result encompasses every integral and mod 2 (Theorem 1.4) homology class that meets the dimension and codimension bounds.

Remark 5. Starting from dimension 4, the transversality type singularities used in the theorem can locally be made of special Lagrangians. From dimension 6, the oriented bordism ring type singularities can also locally be made of special Lagrangians. As a result, even when imposing the condition of special Lagrangian calibrations, one should not expect better outcomes.

Remark 6. Most of these singularities are not isolated. To the author’s knowledge, these examples provide the first instances of non-smoothable, non-isolated singularities of area-minimizing currents under metric perturbation. (In fixed metric cases, numerous examples arise by considering the sum of orthogonal subtori in a high-dimensional torus.)

Remark 7. The codimension requirement of $c \geq 3$ may also be sharp. Both transversality and bordism ring obstructions are absent in codimension 2. However, all known examples of codimension 1 and 2 area-minimizing currents are smoothable.

As a by-product, we also determine the moduli space of area-minimizing integral currents near an area-minimizing immersion.

**Theorem 1.2.** Let $T$ be a $d$-dimensional homologically area-minimizing integral current in a $d + c$-dimensional closed compact (not necessarily orientable) Riemannian manifold $M^{d+c}$ with a metric $g$, with $d, c \geq 3$. Suppose $T$ satisfies the following conditions:

- $T$ is a self-transverse immersion of a (not necessarily connected) compact orientable manifold $\Sigma$;
The angles of self-intersections (Definition 2.2) are always larger than $2 \arctan \frac{2}{\sqrt{c^2 - 4}}$.

Near the self-intersection set of $T$, $T$ decomposes into the sum of two embedded submanifolds.

Then there exists an open neighborhood $\Omega_T$ of $g$ in the space of Riemannian metrics and an open set $U \subset M$ containing the support of $T$, with the following properties:

- For any $g' \in \Omega_T$, any homologically area-minimizing current in the homology class $[T]$ with support in $U$ is also a transverse immersion of $\Sigma$.
- The self-intersection set of the minimizers is diffeomorphic to that of $T$.
- If $T$ is the unique minimizer, then we can take $U = M$.

Remark 8. The angle condition is of sharp asymptotic order in view of Lawlor’s necks (31), where the topology of nearby area-minimizing currents can change and self-intersection singularities can disappear when the angle equals $\frac{\pi}{d} = \frac{\pi}{c}$.

Area-minimizing self-transverse immersions, and more generally, area-minimizing $C^\infty$ generic immersions (defined in Section 2.6) are abundant due to the following result by Yongsheng Zhang.

Lemma 1.3. (Theorem 3.20 in 60) Suppose $i: \Sigma \to M$ is a $C^\infty$ immersion of a connected orientable smooth $d$-dimensional manifold $\Sigma$ into $M^{d+c}$ ($d, c \geq 3$), satisfying the following properties:

- $i(\Sigma)$ is homologically non-trivial;
- $i(\Sigma)$ is $C^\infty$ generic;
- Around the singular set of $i(\Sigma)$, it can be decomposed into the sum of embedded submanifolds.

Then there exists a metric $g$ on $M$ so that $i(\Sigma)$ is uniquely homologically area-minimizing, and a smooth form $\psi_\Sigma$ that calibrates $i(\Sigma)$ in a neighborhood of it. Moreover, the intersection angles on self-intersecting sheets are all equal to $\frac{\pi}{d}$.

Remark 9. For $c < d$, such immersions can possess quite complicated singularities, coming from the stratification of points with fixed multiplicity.

We can also consider the case of unoriented area-minimizing surfaces, for which a direct extension of our methods yields the following.

Theorem 1.4. Theorem 1.1, 1.2 and 3.1 hold, if we drop the orientability requirements and replace the integral homology with the mod 2 homology and integral currents with mod 2 currents.

Remark 10. The dimension bound for the mod 2 case in Theorem 1.1 is sharp, as shown in 58. While it is generally expected that singularities
can be perturbed away in codimension 1 (e.g., [33]), it is anticipated that mod 2 area-minimizing currents in codimension 2 may not be smoothable. If one can demonstrate that a mod 2 cone with a non-bounding link in \( \mathbb{R}^5 \) is area-minimizing using Lawlor-type retractions, a direct application of our methods would imply that the mod 2 cases of Theorem 1.1 hold for all \( d \geq 3, c \geq 2 \).

As a by-product, we obtain the following realization result for mod 2 area-minimizing cones.

**Theorem 1.5.** Let \( M \) be a smooth manifold. Suppose we have a mod 2 cycle \( T \), so that

- \( T \) represents a nontrivial mod 2 homology class;
- The regular part of \( T \) is connected and of multiplicity 1;
- The singular set of \( T \) has finitely many connected components \( S_1, \ldots, S_k \);
- There exist smooth open sets \( U_j \) around each \( S_j \), so that \( T \) can be sent to \( C_j \times \Sigma_j \) by a diffeomorphism \( U_j \to B_n^{1j} \times \Sigma_j \), where \( C_j \) is a truncated mod 2 area-minimizing cone with smooth link in the unit ball \( B_n^{1j} \subset \mathbb{R}^n_j \) equipped with Lawlor’s retraction (Section 2.5), and \( \Sigma_j \) is a smooth manifold with boundary.

Then there exists a smooth metric \( g \) on \( M \), so that \( T \) is the unique homologically mod 2 area-minimizing current in its mod 2 homology class.

**Remark 11.** The multiplicity 1 assumption is not a restriction, since we are dealing with mod 2 currents.

A direct corollary follows.

**Corollary 1.** All mod 2 area-minimizing cones that admit Lawlor’s retractions as defined in [30] and the products of the cones with Euclidean space arise as tangent cones of mod 2 homological area-minimizers in smooth manifolds.

**Remark 12.** The integral homology case follows from Main Theorem in [61].

1.1. **Sketch of proof.** For the proof of Theorem 1.1 we only need to consider the case where the homology class is represented by an embedded submanifold, potentially with multiplicity on its connected components. In cases where this cannot happen, the statements are trivial. Given that we do not impose global obstructions on the existence of smooth minimizers, our focus should be on identifying local obstructions. By finding suitably calibrated singularities that cannot be locally perturbed, we can utilize the constructions in [60] and [61] to attach these singularities to the representatives. (Technically, this gluing cannot be applied directly to a multiplicity greater than 1. Instead, we must perform connected sums of the components and use Lemma 1.3 to eliminate the multiplicity.)
We employ two types of local obstructions in this proof. The first involves a transverse intersection with a relatively large intersection angle, as demonstrated by Lawlor ([30]) in the Euclidean case for two orthogonal $\mathbb{R}^d$ intersecting only at 0 within $\mathbb{R}^{2d}$. Our strategy is to generalize Lawlor’s vanishing calibration methods to broader Riemannian metrics and a wider variety of intersections. In essence, the obstruction resulting from extending Lawlor’s method relies on the first-order deviation of comass from the flat case. For smooth minimal submanifolds forming the intersection, this first-order term corresponds exactly to the mean curvature. Thus, meticulous calculations within the Riemannian setting yield the desired results. As a bonus, this also provides Theorem 1.2.

The second local obstruction stems from singularities bounded by nontrivial elements in the oriented bordism ring, as initially identified by Haskins and Pacini ([24]). We construct general singularities by employing cones over $\mathbb{C}P^2$ and its product with spheres. A slicing argument reveals that any nearby minimizer must also be singular.

For the mod 2 cases (Theorem 1.4), we follow a similar approach, replacing calibrations with area non-increasing retractions.

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2. Basic facts and preliminaries

In this section, we will review some fundamental concepts and provide basic definitions.
Given a Euclidean space $\mathbb{R}^{n+m}$ with coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_m)$, we use $x$ to represent the vector of $x$-components in $\mathbb{R}^n$ and $y$ for the $y$-components in a similar manner. The coordinate label $x_1, \ldots, x_n$ will be denoted by a Roman index subscript, i.e., $x_j$. The corresponding vector fields $\partial_{x_j}$ will be represented as $\partial_j$. For the coordinate label among $y_1, \ldots, y_m$, we use a Greek letter, i.e., $y_\alpha$. The vector fields $\partial_{y_\alpha}$ will be denoted by $\partial_\alpha$.

We adopt the convention $r = \sqrt{\sum_j x_j^2}$, $z = \sqrt{\sum_\alpha y_\alpha^2}$, and $t = \frac{z}{r}$, following [30]. The $j$-dimensional unit sphere and unit ball in $\mathbb{R}^{j+1}$ will always be denoted as $S^j$ and $B^j$, respectively.

Consider two transversally intersecting planes, $P_1$ and $P_2$, with non-trivial intersection in Euclidean space. Then, $P_1 \cup P_2$ can be expressed as a product of their intersection $P_1 \cap P_2$ with two planes, $P'_1$ and $P'_2$, intersecting only at the origin in a lower-dimensional Euclidean space.

**Definition 2.1.** The intersection angle $\theta(P_1, P_2)$ is defined to be

$$\theta(P_1, P_2) = \inf_{v \in P'_1, w \in P'_2} \arccos \frac{|\langle v, w \rangle|}{|v||w|}.$$  

Let $X, Y$ be two submanifolds intersecting transversely along a submanifold $L$ inside a Riemannian manifold $M$.

**Definition 2.2.** The intersection angle $\theta(X, Y)$ (at the tangent space level of) $X, Y$ is defined to be

$$\theta(X, Y) = \inf_{p \in L} \theta(T_p X, T_p Y).$$

2.1. Integral currents and calibrations. Since our focus is mainly on the geometric side, the reader may consider integral currents as chains over immersed oriented submanifolds with singularities. The action of an integral current on a differential form is integration on the chain. If the chain is an oriented smooth submanifold $\Sigma$ with mild singularities (possibly with boundary), we typically use the notation $\Sigma$ instead of the conventional literature notation $[\Sigma]$.

For calibrations, the reader should be familiar with the definitions of comass and calibrations (Section II.3 and II.4 in [22]). The primary reference is [22]. The most important concept to keep in mind is the fundamental theorem of calibrations (Theorem 4.2 in [22]), which states that calibrated currents are area-minimizing among homologous competitors. We will apply this theorem numerous times without explicit citation.

2.2. Definition of mod 2 minimizing. We begin by defining the concept of mod 2 minimizing, which is identical to Definition 1.2 in [8].

**Definition 2.3.** An integral current $T$ on a manifold $M$ is called a cycle (or boundary) mod 2 if there exists an integral current $S$ such that $T + 2S$ is a cycle (or boundary, respectively) as an integral current. A mod 2 cycle
This considered area minimizing mod 2 if, for any mod 2 boundary $S$, the following inequality holds:

$$M(T) \leq M(T + S),$$

where $M$ denotes the mass of currents.

**Remark 13.** Corollary 1.5 and 1.6 in [53] show that the above definition is equivalent to the canonical definition of mod 2 flat chains.

### 2.3. Fermi coordinates.

Suppose $\Sigma$ is a smooth submanifold of $M$. Given a coordinate system $(x_1, \ldots, x_n)$ on $\Sigma$, we define the Fermi coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_m)$ by exponentiating sections in the normal bundle of $\Sigma$ (see Chapter 2.1 of [21]). It is essential to understand that $z = \sqrt{\sum_{\alpha} y_\alpha^2}$ represents the distance to $\Sigma$, and $\partial_z = \frac{y_\alpha}{z} \partial_{\alpha}$ integrates to geodesics originating orthogonally from $\Sigma$, always orthogonal to $\partial_j$ (refer to Chapter 2.3 and 2.4 in [21]). We will make use of the following lemma:

**Lemma 2.1.** We have

$$\langle \partial_{x_1} \wedge \cdots \wedge \partial_{x_n}, \partial_{x_1} \wedge \cdots \wedge \partial_{x_n} \rangle(x,y)$$

$$= (1 - 2H^\Sigma y_\alpha + O(|y|^2)) \langle \partial_{x_1} \wedge \cdots \wedge \partial_{x_n}, \partial_{x_1} \wedge \cdots \wedge \partial_{x_n} \rangle(x,0),$$

where $H = H^\Sigma \partial_\alpha$ denotes the mean curvature of $\Sigma$.

**Proof.** The inner product evaluates to $\det(\partial_{x_i}, \partial_{x_j})$. We then find

$$\partial_{y_\alpha} \langle \partial_{x_1}, \partial_{x_j} \rangle = \langle \nabla \partial_{y_\alpha} \partial_{x_1}, \partial_{x_j} \rangle + \langle \partial_{x_1}, \nabla \partial_{y_\alpha} \partial_{x_j} \rangle$$

$$= \langle \nabla \partial_{x_1} \partial_{y_\alpha}, \partial_{x_j} \rangle + \langle \partial_{x_1}, \nabla \partial_{x_j} \partial_{y_\alpha} \rangle$$

$$= -2A^\alpha_{ij},$$

with $A$ the second fundamental form of $\Sigma$, i.e., $A(\partial_i, \partial_j) = A^\alpha_{ij} \partial_\alpha$. Thus, by Taylor’s theorem with remainder, we have

$$\det(\partial_{x_i}, \partial_{x_j})(x,y) = \det(g_{ij}(x,0) - 2A^\alpha_{ij} y_\alpha + O(|y|^2))$$

$$= \det g_{ij}(x,0) \det(\delta_i^j - 2g^{il} A^\alpha_{lj} y_\alpha + O(|y|^2))$$

$$= \det(\partial_{x_i}, \partial_{x_j})(x,0)(1 - 2g^{il} A^\alpha_{lj} y_\alpha + O(|y|^2))$$

$$= (1 - 2H^\Sigma y_\alpha + O(|y|^2)) \det(\partial_{x_i}, \partial_{x_j})(x,0).$$

\[\Box\]

### 2.4. Non-continuous calibration forms.

We need the following fact about non-smooth differential forms as calibrations.

**Lemma 2.2.** Let $\phi$ be an $l$-dimensional $L^\infty$-differential form defined on a smooth open set $U$ within a closed compact manifold $M$. Assume that $\phi$ has a comass no larger than 1 and that, locally, we have $\phi = d\omega$ for some Lipschitz form $\omega$. Let $S$ be a current supported in $\bar{U}$ with a smooth boundary on $\partial U$ and calibrated by $\phi$. If there exists a local choice of $\omega$ such that the
singular set of \(\omega\) has zero \(l\)-dimensional Hausdorff measure in the support of \(S\), then \(S\) is area-minimizing among homologous competitors in \(U\).

Remark 14. Here the singular set of \(\omega\) means the complement of the set where \(\omega\) is \(C^1\).

Proof. This result is a Riemannian adaptation of Theorem A8 in [30]. Utilizing Nash’s embedding theorem, we can isometrically embed \(M\) into a high-dimensional Euclidean space. Let \(\pi\) denote the smooth projection of the tubular neighborhood of radius \(r_0 > 0\) around \(M\), with \(r_0\) sufficiently small. Note that \(\pi\) is induced by the normal exponential map, so \(d\pi\) is the identity on the tangent bundle of \(M\). Consequently, \(\pi^*(\phi)\) will have comass \(1 + O(r)\) in \(\pi^{-1}(U)\), where \(r < r_0\) denotes the distance to \(M\). By the naturalness of the exterior derivative, we obtain

\[
d(\pi^*(\omega)) = \pi^*(d\omega) = \pi^*(\phi).
\]

We then proceed with the same mollifier argument as in the proof of Lemma A8 in [30], using \(\epsilon < \frac{\pi}{2}\) and the pullback of the forms by \(\pi\). The only difference is that \(\pi^*\phi\) is not exact. To remedy this, observe that locally it still holds that  

\[
d(\pi^*\omega) = (\pi^*\phi)\epsilon,
\]

so we have \(d(\pi^*\phi)\epsilon = 0\), with subscript \(\epsilon\) denoting mollifications with parameter \(\epsilon\). Thus, we use the homologous condition, i.e., \(T - S = \partial Q\), to deduce that \((T - S)((\pi^*\phi)\epsilon) = Q(d(\pi^*\phi)\epsilon) = 0\). 

2.5. Lawlor’s vanishing calibrations and retractions. We need to refine some facts from [30]. Lawlor’s vanishing calibration is roughly speaking a discontinuous form that has Lipschitz anti-derivative locally and is zero outside an angular wedge around the surfaces, i.e., for \(\frac{\pi}{r} = t\) larger than some angle, and smooth on the inside of the wedge. It can serve as a calibration due to Lemma 2.2. The following are roughly from Section 2.2 and 2.3 in [30].

Definition 2.4. For parameters \((n, a) \in \mathbb{N} \times \mathbb{R}\), satisfying constraints

\[
n \in \mathbb{Z}, n \geq 3, a \in \mathbb{R}, \frac{4n}{n + 2} < a < n(n - 2),
\]

define \(g_{(n,a)} : [0, \infty) \to \mathbb{R}\) as follows. For \(t \in [0, \tan \theta(n, a)]\), with

\[
\theta(n, a) = \arctan \sqrt{\frac{a}{n(n - 2)}},
\]

define

\[
g_{(n,a)}(t) = 1 - c(n, a)t^2,
\]

with \(c(n, a) = \frac{n(n-2)}{a}\). On \((\tan \theta(n, a), \infty)\) define

\[
g_{(n,a)}(t) \equiv 0.
\]

Lemma 2.3. The function \(g_{(n,a)}\) has the following properties.

- \(g_{(n,a)}\) is smooth and monotonically decreasing on \([0, \tan \theta(n, a)]\) and \(\theta(n, a) < \frac{\pi}{4}\).
Moreover, we can also ensure that \( g \) satisfies the following differential inequality for \( t \leq \tan \theta(n, a) \),
\[
0 < \kappa(n, a) \leq \left( g(n, a) - \frac{t}{n} g(n, a)' \right)^2 + \left( \frac{g(n, a)'}{n} \right)^2 \leq 1 - \delta(n, a)t^2,
\]
with
\[
\delta(n, a) = \frac{(n - 2)^2(a(n + 2) - 4n)}{a^2n} > 0, \quad \kappa(n, a) = \frac{4(a - 1)}{a^2} > 0.
\]

Proof. Since \( c(n, a) = \frac{n(n - 2)}{a} \) with \( 0 < a < n(n - 2) \) and \( \tan \theta(n, a) = \sqrt{\frac{a}{n(n - 2)}} \), we clearly have \( g(n, a)' \leq 0 \), and \( \tan \theta(n, a) < 1 \). From now on, we restrict to \( 0 \leq t \leq \tan \theta(n, a) \).

Direct calculation gives
\[
\left( g(n, a) - \frac{t}{n} g(n, a)' \right)^2 + \left( \frac{g(n, a)'}{n} \right)^2 = 1 - 2\left( \frac{n - 2}{a} \right) (a - 2)t^2 + \frac{(n - 2)^4}{a^2} t^4.
\]

Regarding the above as a quadratic polynomial in \( t^2 \). The discriminant can be calculated to be
\[
\Delta = -\frac{16(a - 1)(n - 2)^4}{a^4} < 0,
\]
since \( a > \frac{4n}{n^2 + 2} > 1 \). Note that when \( t = 0 \), \( (g(n, a) - \frac{t}{n} g(n, a)')^2 + \left( \frac{g(n, a)'}{n} \right)^2 = 1 \). Since \( \Delta < 0 \), \( (2.3) \) will always be positive. The minimum of \( (g(n, a) - \frac{t}{n} g(n, a)')^2 + \left( \frac{g(n, a)'}{n} \right)^2 \) for \( t^2 \in \mathbb{R} \) is reached at its symmetric axis \( t^2 = \frac{a - 2}{(n - 2)^2} \), and the minimum can be calculated to be
\[
\kappa(n, a) = \frac{4(a - 1)}{a^2}.
\]

The coefficient of the \( t^4 \) term in \( (2.3) \) is positive. Using \( t^4 \leq (\tan \theta(n, a))^2 t^2 = \frac{a}{n(n - 2)} t^2 \), we have
\[
\left( g(n, a) - \frac{t}{n} g(n, a)' \right)^2 + \left( \frac{g(n, a)'}{n} \right)^2 \leq 1 - \frac{(n - 2)^2(a(n + 2) - 4n)}{a^2n} t^2.
\]

For \( \delta \) to exist, clearly \( a > \frac{4n}{n^2 + 2} \) suffices. \( \square \)

Remark 15. The taking \( a \) arbitrarily close to \( \frac{4n}{n^2 + 2} \), we see that as long as \( \theta(n, a) > \arctan \frac{2}{\sqrt{n^2 + 4}} \), we can find such \( g(n, a) \).

With \( g(n, a) \) in hand, we can construct a Lipschitz area-nonincreasing retraction.
Lemma 2.4. There exists a 1-homogeneous area-nonincreasing Lipschitz retraction $\Pi$ from $\mathbb{R}^{n+m}$ to the $x_1 \cdots x_n$-plane, defined by,

$$\Pi(x, y) = (g_{(n,a)}(t))^{1/n}(x, 0).$$

$\Pi$ maps $\{\arctan t > \theta(n,a)\}$ smoothly to 0, and $\{\arctan \leq \theta(n,a)\}$ smoothly onto the $x_1 \cdots x_n$-plane.

Proof. The corresponding integral curve in the notation of Section 1.1 in [30] is

$$r_{\theta} = (r_\theta \cos \theta) = g_{(n,a)}(\tan \theta).$$

Direct calculation as in Section 2.4 in [30] using inequality (2.2) shows that $\Pi$ is area-nonincreasing. \qed

Now we can construct Lawlor’s vanishing calibration. Let $dx = dx_1 \wedge \cdots \wedge dx_n$, and consider

$$\psi = \frac{1}{n} (x_1 dx_2 \wedge \cdots \wedge dx_n - x_2 dx_1 dx_3 \wedge \cdots \wedge dx_n + \cdots).$$

Direct calculation gives

$$d\psi = dx_1 \wedge \cdots \wedge dx_n, \quad \psi = \frac{r}{n} \partial_r (dx_1 \wedge \cdots \wedge dx_n),$$

with $\partial_r = \frac{x_i}{r} \partial_i$, $dr = \frac{z}{r} dx_i$, and $i$ the interior product by inserting a vector. In other words, $\frac{r}{n} \psi$ is dual to the tangent plane to standard spheres centered at the origin in $\mathbb{R}^n$.

Lemma 2.5. Define Lawlor’s vanishing calibration as

$$\phi = d(g_{(n,a)} \frac{\psi}{n}) = \left((g_{(n,a)} - \frac{t}{n}g_{(n,a)}')dr + \left(\frac{g_{(n,a)}}{n}\right)'dz\right) \wedge \left(\frac{r}{n} \psi\right).$$

Then $\phi$ is a simple form, dual to the n-plane $\phi^*$ spanned by $((g_{(n,a)} - \frac{t}{n}g_{(n,a)}')\partial_r + (\frac{g_{(n,a)}}{n})'dz) \wedge \left(\frac{r}{n} \psi\right)^*$ (the tangent planes to the standard spheres centered at the origin). The dual n-plane $\phi^*$ to $\phi$ is the orthogonal complement of the tangent space of level sets of the retraction $\Pi$.

Proof. The form is simple by construction. By Lemma 2.2 and inequality (2.2), $\phi$ can serve as a calibration form and calibrates the $x_1 \cdots x_n$-plane. And by 2.4.1 Proposition in [30], the dual plane to $\phi$ is the orthogonal complement to the level sets of the retraction $\Pi$. \qed

Lemma 2.6. If $f$ is a smooth function on $x_1 \cdots x_n$-plane, then $(f \circ \Pi)\phi$ is closed. Moreover, if $|f| \leq 1$ then $(f \circ \Pi)\phi$ is a calibration form that satisfies the assumptions in Lemma 2.2.
Proof. By discussion in 6.2.5 Definition in [30], \( d((f \circ \Pi)\phi) = 0 \), whenever it is defined. To show the claim about satisfying assumptions in Lemma 2.2, we want to show that \( (f \circ \Pi)\phi \) is the exterior derivative of a Lipschitz form. Using the same reasoning as in Corollary A9 in [30], it suffices to find a Lipschitz form \( \eta \) so that
\[
- d((f \circ \Pi)\phi) \wedge (g(n,a)\psi) = d\eta,
\]
and take the antiderivative of \( (f \circ \Pi)d(g(n,a)\psi) \) to be \( (f \circ \Pi)g(n,a)\psi + \eta \). As shown in the proof of Corollary A9 in [30], this reduces to showing that \( d((f \circ \Pi)\phi) \wedge g(n,a)\psi \) is Lipschitz. Since both factors are smooth on the two closed sets \( 0 \leq t \leq \tan \theta(n,a) \) and \( t \geq \tan \theta(n,a) \), we only need to deal with \( t = \tan \theta(n,a) \). This follows from direct calculations using Leibnitz rule's calculation, and the fact that \( g(n,a)\psi((x + x_0, y + y_0)) = O(\sqrt{x^2 + y^2}) \) for \( |y_0|/|x_0| = \tan \theta(n,a) \). □

2.6. Transversality, generic immersions and proof of Lemma 1.3

First of all, by transverse (self-transverse), we mean that the intersection set has only double points, and at any point of the intersection set, the tangent planes to the two pieces sum to give the tangent space of the ambient manifold.

By \( C^\infty \) generic immersions, we mean immersions with normal crossings (Definition 3.1 in [19]). They are also called strongly self-transverse in [57] and completely regular in [26]. For a rigorous introduction please see the above references. Generic immersions form an open and dense subset of the space of immersions, as detailed in Definition 1.1 and Proposition 3.2 of [26].

Heuristically, generic immersions imply that the immersed sheets are locally pairwise transverse along the intersections, any sheet is transverse to the intersection of transverse pairs of sheets, and so on. Furthermore, the points of multiplicity at most \( j + 1 \) are contained within the points of multiplicity \( j \), with codimension \( c \). There can be at most \( \lfloor \frac{d+c}{c} \rfloor \) sheets meeting at a single point, as stated in the first paragraph on Page vii of [26]. Additionally, by Lemma 1.9 in [26], a coordinate system exists around any point in the image of the immersion, such that the immersion becomes linear subspaces in general positions.

2.7. Zhang’s gluing constructions for calibrations. Consider a sequence of boundaryless currents \( T^l \geq 1 \) of dimension \( d \geq 1 \) and codimension \( c \geq 1 \) on a smooth compact manifold \( M \), converging to an integral current \( T^\infty \) in the flat norm. Suppose they satisfy the following properties. (Note that from now on, the superscript \( l \) can be \( \infty \).)

1) Each \( T^l \),
   a) represents the same nontrivial integral homology class;
   b) has density 1 \( d \)-dimensional Hausdorff measure a.e.;
   c) i) is either irreducible in the sense that there are no two different boundaryless multiplicity 1 currents that sum to it;
ii) or decomposes into a finite sum of irreducible currents
\[ T^l = \sum_{a=1}^{b} T^l_1, \]
with the collection of homology classes \( \{[T^l_1]\} \) linearly independent over \( \mathbb{Z} \).

2) There exists four fixed (not necessarily connected) smooth compact open sets \( \tilde{O} \supset O \supset O' \supset O'' \) and sequences of shrinking smooth open sets \( V^l_1 \supset V^l_2 \supset \cdots \), so that
a) each \( O \)-symbol set is compactly contained in the previous one;
b) the singular set of \( T^l \) is contained in \( O'' \), each \( T^l \) is smooth near \( \partial O'' \) and \( T^l \) restricted to \( (O'')^C \) is the same for all \( l \).
c) \( \lim \text{dist}(\text{spt} T^l, (O'')^C, V^l_j \cap (O'')^C) = 0 \) and each \( V^l_j \) contains the support of \( T^l \).
d) each \( V^l_j \cup O'' \) is the same set for all \( l \).
e) i) in case of 1.a.i, if \( V^l_j \) converges to \( \text{spt} T \) in Hausdorff distance, then we have \( H_d(V^l_j, Z) = Z[T^l] \). Otherwise in case 1.a.i, we have \( H_d(V^l_j \cup O', Z) = Z[T^l] \).
ii) in case of 1.a.ii, if \( V^l_j \) converges to \( \text{spt} T \) in Hausdorff distance, then we have \( H_d(V^l_j, Z) = \oplus_j Z[T^l_j] \). Otherwise in case 1.a.ii, we have \( H_d(V^l_j \cup O', Z) = \oplus_j Z[T^l_j] \).
f) \( \partial O \) intersect \( T^l \) transversely,
g) each \( \partial V^l_j \) intersects \( \partial O \) and \( \partial O' \) transversely, and \( \partial O \cap V^l_j \) deformation retracts onto \( \partial O \cap \text{spt} T^l \).

3) there exists a sequence of smooth metric \( \{g^l_O\} \) on \( \tilde{O} \) and a sequence of bounded measurable \( d \)-dimensional form \( \{\phi^l\} \) on \( O \) that satisfies the assumptions in Lemma 2.2, so that
a) \( \phi^l \) calibrates \( T^l \), \( \phi^l \) is smooth \( d \)-dim a.e. on the support of \( T^l \), and \( \phi^l \) is smooth in \( (O \setminus O') \cap V^l_j \) for some \( j \geq k_0 \), with the same \( k_0 \) for all \( l \).
b) \( g^l_O \) converges smoothly to a smooth limit \( g^\infty_O \), and \( \phi^l \) converges in \( L^\infty \) to \( \phi^\infty \).

**Remark 16.** For the condition of \( H_d(U_j, Z) = Z[T] \), in most cases, \( T \) can be triangulated into a simplicial complex, so one can apply Theorem 1 in [28] to construct such \( U_j \) (Lemma 2.8). For irreducibility, generically speaking, the connectedness of the regular part suffices (Lemma 2.10).

**Remark 17.** Roughly speaking, the sequence \( V^l_j \) is to provide tube-like neighborhoods for \( T^l \). The \( O \) sets are created for the convenience of extensions and gluings.

We will use the results from [60] and [61] to prove the following lemma.

**Lemma 2.7.** With the above assumptions, we have the following constructions.
• Up to truncating the first few terms of all the sequences above, there exists a sequence of smooth Riemannian metrics \( \{ g^l \} \) on \( M \), so that \( T^l \) is homologically area-minimizing in \( g^l \), and the metric is unchanged near the singular set of \( T^l \).

• The sequence \( \{ g^l \} \) converges smoothly to a smooth limit \( g^\infty \), in which \( T^\infty \) is homologically area-minimizing.

Moreover, we can achieve more in the following cases.

• Suppose no current contained in \( O \) can be in the same homology class as \( T^l \). In case of \( 2[T^l] \neq 0 \), \( T^l \) is the unique homological area-minimizer. In case of \( 2[T^l] = 0 \), \( T^l \) and \( -T^l \) are all the area-minimizers in the class \( [T^l] \).

• If \( T^l \) belongs to a nontrivial real homology class on \( M \) and each \( \phi^l \) is smooth, then we can find a smooth form \( \phi^l \) that calibrates \( T^l \). Moreover, \( \phi^l \) converges to \( \phi^\infty \) smoothly.

Remark 18. If \( T \) is homologically trivial, then we can make \( T \) area-minimizing in a neighborhood and we do not need the irreducibility and homological assumptions.

Remark 19. Compared to [60] and [61], the main difference is the requirement of neighborhoods with homology spanned by \( T \). This is due to the fact that \( T \) might represents a torsion class in our cases, so we need more than a calibration locally to compare area.

Remark 20. The strongest form as state above involving sequences of constructions is never used in this manuscript. However, it is useful in many other settings, so we include it here for completeness.

Proof. The idea of the proof is a combination of the methods in [60], [61], and [49]. It involves three steps. The first one is to make \( T^l \) minimizing in a neighborhood. The next step is to rescale the metric to make \( T^l \) globally minimizing. The last two correspond to the two added assumptions.

2.7.1. Set up. In order to avoid cumbersome notations, we will drop the sup/subscripts \( l \). Before the proof, let us do some setup.

First, equip \( M \) with an arbitrary smooth metric \( g_0 \). Then we glue \( g_0 \) to \( g_O \) in \( \tilde{O} \setminus O \) by a fixed 1 to 0 transition function \( \tau \) that is 1 in \( \tilde{O}^\infty \) and 0 in \( O \) to define \( g_{\tilde{O}\setminus O} = \tau g_0 + (1 - \tau) g_O \). (By point 2.a, such \( \tau \) exist.) Define our base metric

\[
\bar{g}_0 = \begin{cases} 
g_0, & \text{in } (\tilde{O})^\infty, 
g_{\tilde{O}\setminus O}, & \text{in } \tilde{O} \setminus O, 
g_O, & \text{in } O. 
\end{cases}
\]

By construction, \( \bar{g}_0 \) is smooth.

Due to 2.b and 2.c, we can let \( j_0 \) be large enough so that \( \text{dist}(V_j \cap (O'')^\infty, \text{spt} T \cap (O'')^\infty) \) is smaller than the focal radius of \( T \cap (O'')^\infty \) for \( j \geq j_0 \). By 2.c, 2.d,
and 3.b, such $j_0$ can be chosen to be the same for all $l$ up to truncating the first few terms in all sequences. Thus, we can always talk about pullbacks of volume forms of $T$ under normal bundle projection of $T$ in $V_j \cap (O' \setminus \tilde{C})$ for $j \geq j_0$. Now fix one $j \geq \max\{j_0, k_0\}$ ($k_0$ defined in 3.a).

2.7.2. Step 1: making $T$ minimizing in a neighborhood. First, let us construct a metric and a calibration in a neighborhood of the current $T$ so that $T$ becomes area-minimizing in this neighborhood.

We already have a calibration at hand in $O$. In $V_j \cap (O' \setminus \tilde{C})$, the pullback of volume form $\omega_T$ of $T$ under normal bundle projection $\pi_T$ in $V_j$ is closed and one can rescale the metric to make it a calibration (Remark 3.5 in [60]). Denote the form by $\pi_T^*(\omega_T)$.

\begin{equation}
(2.11)
\end{equation}

(Note that such a rescale depends smoothly on on $g_0$ by assumption 2.b and 2.d.) Now we need to glue the metric and calibration forms together smoothly (mostly Section 4 in [60]). The proof of Section 4 in [60] covered the case where $O$ is a ball of radius $\epsilon$, and the gluing happens at radius $\epsilon/3$ to $\epsilon$. The proof carries over to our case with two conditions:

- a smooth transition region $\Omega$ between the two calibrations (corresponding to $B_\epsilon(0) \setminus B_{\epsilon/3}(0)$ in Zhang’s proof) with $H^d(\Omega, \mathbb{R}) = 0$ to get primitives of calibration forms
- and a smooth function $\rho$ defined in $\Omega$ with no critical point and level sets foliating $\Omega$ (corresponding to the radius function in Zhang’s proof) whose level sets foliate $\Omega$ to construct smooth transition functions.

We will show how to construction $\Omega$ and $\rho$.

Consider a fixed smooth vector field $v$ that is nonzero inward pointing on $\partial O$, tangent to $\partial V_j$ on $\partial V_j$ and 0 outside of a neighborhood of $\partial O$. The existence of $v$ can be deduced by transversality and a partition of unity argument. Now extend $v$ smoothly. By 2.b and 2.d, such $v$ can be chosen to be the same for all $l$.

Now if we flow $M$ according to $v$ for a short time $t \in [0, \epsilon]$, so that $\partial O$ is flowed into $O \setminus \tilde{O}$. Then $V_j$ is unchanged under the flow and $O$ is flowed into a slightly smaller open set $\tilde{O}'$.

Now set $\Omega = (O \setminus \tilde{O}') \cap V_j$. By construction, $\Omega$ is a smooth manifold with corner and $\Omega$ admits a deformation retract onto $\partial O \cap V_j$. By 2.f, $\partial O \cap V_j$ deformation retracts onto $\partial O \cap \text{spt} T$, which by 2.g and 2.b is a smooth $d-1$-dimensional submanifold. By homotopy invariance of homology, we deduce that $H^d(U, \mathbb{R}) = 0$. By de Rham’s theorem, every $d$-dimensional closed form in $\Omega$ is exact. Thus by Theorem 3.1.1 in [44], there exist forms $\omega'_T$ and $\phi'$
on \( \Omega \) that depends smoothly on \( \pi_T^* (\omega_T) \) and \( \phi \) respectively, so that
\[
d\omega'_T = \pi_T^* (\omega_T), \ d\phi' = \phi.
\]

Implicit function arguments show that \( \partial O \times [0, \epsilon] \) admits a diffeomorphism onto \( M \) by the flow of \( v \). Thus, if we set \( \rho \) to be the time function, then \( \rho \) can substitute the radius function \( r \) used in Zhang’s proof. Note that \( \rho \) is the same for all \( l \).

Run through the argument in Section 4 of [60]. In the process, we glue the two forms together using the transition of antiderivatives in (2.12).

Then we restrict to a smaller \( V_j \) with larger \( j \) uniformly using 3.c (in Zhang’s proof, a smaller \( h \)-disk normal bundle) to achieve condition (c) on page 664 of [60]. Then decompose the metric to horizontal and vertical directions of \( T \) and change accordingly using his proof. In the end, we deduce that there exists a smooth metric
\[
g \label{eq:2.13}
\]
and a smooth calibration form
\[
\overline{\phi} \label{eq:2.14}
\]
defined in \( V_j \cup \tilde{O}' \) (for a possibly larger \( j \) than we strated), so that \( \overline{\phi}|_{O'} = \phi \) and \( \overline{g}|_{O'} = g_{O} \) and and \( \overline{\phi}|_{V_j \cap (O')^c} \) restrict to pullbacks of volume forms of \( T \) under normal bundle projection in the metric \( \overline{g}_{O} \), i.e., a simple form. It is straightforward to verify that in this process, every auxiliary function and vector space splitting used depends geometrically and smoothly on \( T, \phi, g \) restricted to \( (O \setminus O') \cap V_j \), which themselves are smooth. The auxiliary constants depend continuously on \( T, \phi, g \). Thus, after direct calculation, we can conclude the gluing process preserves the claimed convergences with respect to \( l \).

Remark 21. The auxiliary constants in Zhang’s proof depend on the comass of the forms, which depend continuously on the geometrical objects involved. This is the main reason that we cannot turn the sequential convergence into real parameter families of smooth convergence.

2.7.3. Step 2: making \( T \) a homological minimizer. By 2.d, \( V_j \cup \tilde{O}' = V_j \cup O'' \cup \tilde{O}' \) is the same for all \( l \). First, we need to set up an inner and an outer copy of \( V_j \cup O' \) in order to give room for our arguments. Just like in the previous subsection, take a vector field \( v_{\overline{\phi}} \) that is nonzero inward pointing on \( \partial \tilde{O}' \) and tangential to \( V_j \), then flow for a short time gives a smaller open set \( \tilde{O}' \subset \tilde{O}' \) while keeping \( V_j \) unchanged. Now do this with \( V_j \) and \( \tilde{O}' \) reversed, we dedeuce the existence of a diffeomorphism sending \( V_j \cup \tilde{O}' \) to a smaller open set
\[
\tilde{V}_j \cup O',
\]
with each component of \( \tilde{V}_j \cup \tilde{O}' \) compactly contained in that of \( V_j \cup \tilde{O}' \). Repeating the argument with outward pointing vectors, we deduce the
existence of a diffeomorphism sending $V_j \cup \tilde{O}'$ to a larger open set
$$\hat{V}_j \cup \hat{O},$$
each component of the latter compactly containing that of the previous one.
Now we can glue $\check{g}$ (defined in (2.13)) and $\bar{g}$ (defined in (2.10)) together in $V_j \cup \hat{O}' \setminus (V_j \cup \hat{O})$. Again this gluing can be made to depend smoothly on the objects involved and preserves the convergence of sequences.
We are ready to run the arguments of Section 4 in [61]. First, let us deal with the harder case of 1.a.i. The basic idea of Section 4 of [61] is to make the metric small near $T$ to force any minimizer to stay in a neighborhood. For the case of $H_d(V_j \cup O, \mathbb{Z}) = \mathbb{Z}[T]$, in his notation in the proof of Theorem 4.1 in [61], we set $W'' = \hat{V}_j \cup \hat{O}'$, $W' = V_j \cup \hat{O}'$, $W = \hat{V}_j \cup \hat{O}'$. By homotopy invariance of homology, all three $W$ variant sets have $H_d(\cdot, \mathbb{Z}) = \mathbb{Z}[T]$. Then in his notation after changing the metric to $\check{g}$, the rescaled form $\gamma \hat{\phi}$ remains a calibration form in $U$ and calibrates $T$ ($\gamma$ as defined in last line of Page 180 in [61]). By his argument, any area-minimizer $S$ in metric $\check{g}$ homologous to $T$ must stay in $W$. However, $H_d(W, \mathbb{Z}) = \mathbb{Z}[T]$. Since $S$ is not a boundary, we deduce that $S$ is homologous to $\alpha T$ for some integer $\alpha \neq 0$ in $W$. (Warning: restricting to open subsets can change homological relations between cycles in general, so we don’t know that $S$ and $T$ are homologous in $W$.) Thus, we have $|\alpha| M(T) = |S(\check{\phi})| \leq M(S)$. This implies $M(T) \leq M(S)$, with equality if and only if $\alpha = \pm 1$ and $S$ calibrated by $\check{\phi}$. Thus, we have constructed a smooth metric so that $T$ is homologically area-minimizing. For the case of $H_d(V_j, \mathbb{Z}) = \mathbb{Z}[T]$, just do the above argument with $W'' = \hat{V}_j$, $W' = V_j$, $W = \hat{V}_j$. It is straightforward to verify that all auxiliary functions involved depend smoothly on $T, \phi, g$ and all auxiliary constants depend continuously so, thus preserving the convergence.
In the case of 1.a.ii, the above warning does not hold by linear independence of $\{T_j\}$ over $\mathbb{Z}$. In other words, if we run the argument above, then for all $S \in [T]$ and spt$S \subset W$, we must have $S$ homologous to $T$ in $W$. Thus, we must have $M(T) = T(\check{\phi}) = S(\check{\phi}) \leq M(S)$ directly.

2.7.4. Step 3.1: uniquely minimizing and global calibration. Now we need to prove that $[T]$ is the unique homological area-minimizer when no current in $\tilde{O}$ is homologous to $T$. In the proof above, any area-minimizer $T'$ homologous to $T$ must stay in $W$. Then consider a smooth bump $\beta$ function that is 0 outside of $\exp^1 T \cap (\hat{V}_j \cap O^L)$, 1 on $\exp^1 T \cap (\hat{V}_j \cap O^L)$ and with value always in $[0, 1]$. Let $\nu$ denote the distance to $T$ on the regular part and consider the conformal metric $(1 + \beta z^2)\check{g}$. The area of everything is at least that in $\check{g}$. Moreover, $T$ has an unchanged area. However, with the positive $z^2$ bump, we deduce that any current passing through $W \cap O^L$ has increased area unless it has the same support as $T$. By the added assumption on $O$, any other minimizer $T'$ must pass through $W \cap O^L$, otherwise they are contained in $O$, a contradiction. Thus $T'$ has the same support as $T$. In case of 1.a.i,
by irreducibility, we must have $T' = -T$ or $T$. In case of 1.a.ii, by linear independence of the summands $T_j$, we must have $T' = T$. Again, all auxiliary functions depend geometrically on $\phi, T, g$, thus preserving convergence.

2.7.5. Step 3.2: global calibration. If $T$ lies in a nontrivial real homology class, then argue as Section 3.3 and 3.4 in [60] to transit the calibration we produced near $T$ to an arbitrary closed form $\psi$ with $[T](\psi) = 1$ away from $T$. Such $\psi$ exists by the reasoning in the first paragraph of Section 3.2 in [60]. Then we use the universal coefficient theorem instead of retraction to deduce that $H^d(V_j \cup O, \mathbb{R}) \cong H^d(T, \mathbb{R})$ or $H^d(V_j, \mathbb{R}) \cong H^d(T, \mathbb{R})$. Use Theorem 3.1.1 in [44] to get antiderivatives of exact forms that depend smoothly on the form. This gives (3.4) in [60]. The only thing that needs modification is a smooth distance function for the transition. In the case of $H^d(V_j, \mathbb{R}) = \mathbb{R}[T]$, just use the distance function to $\partial V_j$ and reverse the transition. In the case of $H^d(V_j \cup O, \mathbb{R}) = \mathbb{Z}[T]$, note that one can smooth the corners of $V_j \cup O$ to get smooth open subsets that converge to $V_j \cup O$ in Hausdorff distance. Use the distance function to these smooth approximations instead and reverse the transition. Then just chase the argument and replace $\|\Phi\|_g$ in his notation with 1. Again, we can preserve the convergence. □

It is natural to ask when a current satisfies the conditions listed before the lemma. The following comes in handy.

**Lemma 2.8.** Let $Z$ be a closed subset of a compact closed smooth manifold $M$. Suppose $Z$ admits a Whitney stratification. Then there exist a sequences of smooth neighborhoods $V_1 \supset V_2 \supset V_3 \cdots$ so that each $V_j$ is compactly contained in the previous ones and each $V_j$ deformation retracts onto $S$.

**Proof.** Suppose the Whitney stratification of $Z$ is $Z_1 \supset Z_2 \supset \cdots$. (Here we use the convention that $Z_j \setminus Z_{j+1}$ is the smooth open submanifolds that form the stratification.) Then it is straightforward to check that $M \supset Z_1 \supset Z_2 \supset \cdots$ is a Whitney stratification of $M$.

By Proposition 5 on page 199 of [20], we deduce that $M$ admits a triangulation so that $Z$ is a subcomplex of $M$. (The proposition is stated for a stratified object, which is precisely the axiomatization of desirable properties of Whitney stratifications as stated in Section 8 of [41].)

By 3.1.1, 3.1.2 and 6.3.1 of [48], we know that the $n$-th barycentric subdivision for $n \geq 2$ gives a neighborhood $V^n$ of $Z$ that deformation retracts onto $Z$. By construction we have $V^{n+1} \subset V^n$ with compact containment as sets and $V_j$ converges to $Z$ in Hausdorff distance.

Now by the claim (1a)' in the proof of Theorem 1 in [28], for every such neighborhood of the $Z$, there exist smooth regular neighborhoods $V_j$ that can be made arbitrarily close to $V_j$ in Hausdorff distance and $V_j$ deformation retracts onto $Z$. We are done. □

As a corollary, we deduce the following.
Lemma 2.9. Suppose we have a compact closed Hausdorff \( d \)-dimensional set \( S \) that is smooth \( d \)-dimensional a.e. on a manifold \( M \). Suppose there exists a neighborhood \( U \) containing all the singular points of \( S \) with finitely many connected components \( U_1, \ldots, U_l \). For each connected component \( U_j \), we have the following dichotomy:

- either \( S \cap U_j \) can be sent to \( C \times \Sigma \subset B^n_1 \times \Sigma \) by a diffeomorphism. Here \( \Sigma \) is a smooth manifold and \( C \) an \( m \)-dimensional cone over a smooth \( m-1 \)-dimensional submanifold of \( S^{n-1} \) with \( 1 < m < n \), truncated inside the unit ball \( B^n_1 \).
- or \( S \cap U_j \) decomposes into finitely many embedded submanifolds that sum to represent a \( C^\infty \) generic immersion.

Then there exists sequences of smooth neighborhoods \( V_1 \supset V_2 \cdots \supset S \) each compactly contained in the previous one, so that \( V_j \) deformation retracts onto \( S \).

Proof. By Lemma 2.8 it suffices to show that \( S \) admits a Whitney stratification. Set the first stratum \( Z_d = S \). For case one (diffeomorphic to \( C \times \Sigma \)) in each \( U_j \), denote \( Z_{d-1}^{\dim \Sigma} \) to the set \( \{0\} \times \Sigma \). For case two (generic immersions), in each \( U_j \), let \( Z_{d-k-1} \) to be the points of multiplicity at least \( k \) intersection set. (Here the lower index always denotes the dimension of the strata.) Then set \( Z_l = \sum_j \sum_{m \leq l} Z_m^{\dim \Sigma} \). We claim that \( Z_d \supset Z_{d-1} \supset \cdots \) is a Whitney stratification of \( S \). (If \( Z_{d-1} \) is empty then just omit it in the sequence.) By the first paragraph of page vii of [26], the points of multiplicity at least \( k \) is closed and the points of multiplicity precisely \( k \) is an embedded open submanifold of dimension \( d - (k - 1)c \). Thus, by construction we always have that each \( Z_l \) is closed and for two consecutive \( Z_l \supset Z_{l'} \) in the filtration, \( Z_l \setminus Z_{l'} \) is an open submanifold of dimension \( l \). Since \( k \) fold points of generic immersion can only have limit of \( k' \) points of intersection for \( k' \geq k \), we deduce that the stratification \( Z_d \supset \cdots \) satisfies the frontier condition of Definition 4.1.3 in [52]. To show this is a Whitney stratification, by Definition 4.2.3 in [52], we have to show that the stratification satisfies Whitney’s condition (b). For case one \( (C \times \Sigma) \), a sequence of smooth points \( \{x_n\} \) converges to the limit of a sequence \( \{y_n\} \) in \( \{0\} \times \Sigma \). Take a coordinate system on \( \Sigma \) centered at \( \lim_n y_n \). Then \( C \times \Sigma \) becomes \( C \times \mathbb{R}^{\dim \Sigma} \subset \mathbb{R}^{n + \dim \Sigma} \). Note that \( C \times \mathbb{R}^{\dim \Sigma} \) is a cone that is translation invariant along \( \mathbb{R}^{\dim \Sigma} \). We deduce that line segments \( [x_n, y_n] \) lie in \( C \times \mathbb{R}^{\dim \Sigma} \). By definition of the tangent plane, we deduce that the line segment \( [x_n, y_n] \) and thus \( \frac{y_n - x_n}{y_n - x_n} \) lie in the tangent space of \( S \) to \( x_n \). Thus, Whitney’s condition (b) is trivially satisfied. In the immersion case, by Lemma 1.9 in [26], at any point \( y \) of the \( k \)-fold intersection set, there exists a coordinate system centered at \( y \), so \( S \) becomes \( d \)-dimensional vector subspaces of \( \mathbb{R}^{d+c} \) in general position. Let \( \{x_n\} \) be a sequence of points in a stratum, with \( \lim_n x_n = y \) and \( \{y_n\} \) be a sequence of points in the same stratum as \( y \) with \( \lim_n y_n = y \). Note that the stratum containing \( x_n \) by construction consists of points of multiplicity...
$k'$ with $k' < k$ and is an open subset of a vector subspace of dimension $d - (k' - 1)c$, whose limit set contains $\{y_n\}$. Thus, the line segment $[x_n, y_n]$ is tangential to the tangent space of $x_n$ to its stratum. Again, by taking the limit, we see that Whitney’s condition (b) is trivially satisfied. □

The following criterion for irreducibility is also useful.

**Lemma 2.10.** Let $T$ be a $d$-dimensional integral cycle $T$ of multiplicity 1 on the regular part (or a mod 2 cycle) on a compact manifold $M$. Suppose the singular set of $T$ has $d$-dimensional Hausdorff measure 0 and the regular part of $T$ is connected. Then $T$ is irreducible.

*Proof.* By our assumptions, there exists a sequence of finite covers of balls around the singular set with radius tending to 0 and volume tending to 0. Without loss of generality, we can substitute the covers with a sequence of smooth open sets $\{V_j\}$ so that the singular set of $T$ is contained in $V_j$ and its Hausdorff distance to $V_j$ converges to 0. Moreover we can assume that $T$ restricted to $V_j^C$ is a smooth manifold with boundary. (Just mollify the collection of spheres and use a generic perturbation to get transverse intersections with $T$.) For $j$ large enough, note that $T$ restricted to $V_j^C$ is connected. Suppose $T'$ is another integral or mod 2 cycle with support contained in the support of $T$ with multiplicity 1 a.e. Then $T'$ restricted to $V_j^C$ for $j$ large enough must be multiples of $T$ restricted to $V_j^C$ by connectedness and the Constancy Theorem on page 357 of [17]. Since $T'$ is of multiplicity one, the multiple must be 1 or $-1$. Since $V_j$ converges to the singular set of $T$ in Hausdorff distance, we deduce that $T' = \pm T$. If $T$ is not irreducible, then we have two multiplicity 1 cycles $T', T''$ so that $T' = \pm T, T'' = \pm T$, and $T' + T'' = T$. However, for $T \neq 0$, this cannot happen. □

2.8. **Proof of Lemma 1.3.** With these basic setups, we are ready to prove Lemma 1.3. For a sum of any finite number of embedded submanifolds of codimension at least 3, that overall represents a generic immersion, Theorem 3.20 in [60] constructs a smooth calibration form and a smooth metric in a neighborhood of these embeddings. However, by our assumptions in Lemma 1.3, near the singular set of the immersion, the image of the immersion decomposes into a sum of embedded submanifolds. Thus, by Lemma 2.7, Lemma 2.10 and Lemma 2.9, we can find a smooth metric and a smooth calibration form near $T$, so that $T$ is area-minimizing.

2.9. **Unique continuation.** We also need the following versions of unique continuation theorems.

**Lemma 2.11.** (Strong unique continuation) If two properly embedded minimal submanifolds in some Riemannian manifold touch at one point of infinite order, then in a neighborhood they coincide.
Proof. This is folklore, but the author could not find a reference for general codimension in arbitrary ambient metrics. We will only sketch proof. Say the two surfaces are $F,G$. By setting up Fermi coordinates adapted to $G$, (Chapter II in [21]) we can express $F$ as a smooth graph. Direct calculation shows

$$\Delta G F^\alpha = O(|F| + (r + |D^2 F|)|DF|)$$

for coordinate components of $F$ normal to $G$. Then use Theorem 1.8 in [29]. \qed

**Lemma 2.12.** (Uniquely minimizing of multiplicity 1 interior regions of minimizers) Let $T$ be an area-minimizing integral or mod 2 current with a possibly nonempty boundary in a compact (not necessarily boundaryless) manifold $M$, satisfying the following properties,

- the support of $T$ lie in the interior of $M$;
- for a smooth open set $U \subset (\text{spt } \partial T)^\complement$, we have $T$ restricted to $U$ is smooth with multiplicity 1 near $\partial U$.

Then $TLU$ is the unique area-minimizer in $M$ among competitors $\partial T' = \partial TL U$ and $T' + TLU^\complement$ homologous to $T$.

**Proof.** First, let us do the integral case. Let $\Gamma$ denote the boundary of $T$ restricted to $U$. Now suppose $\Sigma$ is another integral current with $\partial \Sigma = \Gamma$ and $\Sigma + TLU^\complement$ homologous to $T$. By Theorem 2.1 in [5], we deduce that $\Sigma$ has regular boundary points on an open dense subset $D$ of $\Gamma$. In other words, for any point $b \in D$, there exists a small geodesic ball $B_r(p)$, so that $\Sigma$ equals $\sum_{1 \leq j \leq c} \Sigma_j^+ + \sum_{0 \leq i \leq c-1} \Sigma_i^-$, where $\Sigma_j^+$ are smooth multiplicity 1 submanifolds with boundary $\Gamma \cup D$ and $\Sigma_i^-$ are smooth multiplicity 1 submanifolds with boundary $-\Gamma \cup D$. By Theorem 0.3 in [11], $\Sigma$ is smooth outside of a set of Hausdorff codimension 1 of $\Gamma$ on $\Gamma$. Since $\Sigma + TLU^\complement$ is area-minimizing among competitors homologous to $T$, we deduce that $\Sigma$ must have the same area as $TLU$. Thus we can cut $TLU$ away and paste in $\Sigma$ to get an area-minimizing current. Using the non-existence of $m$-dimensional area-minimizing cones with $m-1$-dimensional translation invariance, we deduce that the tangent cone of the pasted together current at regular points of $T$ is $c$ times the original tangent plane for some integer $c$. Thus, at any regular point of $T$, for any $i$, $\Sigma_i^+$ touches $T$ of $C^1$ order. By Allard regularity ([1]), the patched-together surface must be smooth through $\Sigma_i^+$. This gives infinite order touching and by Lemma 2.11, the patched-together surface coincides with $T$ on the regular part. This implies $c = 1$ and non-existence of $\Sigma_i^-$. Repeat this at each regular point and each component of $\Gamma$. We deduce that $\Sigma$ coincides with $T$ restricted to $U$. For mod 2 cases, the reasoning is the same. Just use the boundary regularity of Theorem 5 in [59]. \qed

2.10. **Allard type regularity.** We will frequently use Allard-type regularity arguments under the following assumptions.
(a) $T$ is an area-minimizing current on (a possibly non-complete non-compact) Riemannian manifold $M$ with metric $g$;
(b) for $\partial T \neq 0$, we assume that $\partial T$ is a smooth (possibly not connected) compact submanifold with multiplicity 1;
(c) $p$ is a point in the interior of $T$, so that $T$ is smooth and multiplicity 1 near $p$;
(d) for $\partial T \neq 0$, $b$ is a point in support of $\partial T$, so that $T$ is a multiplicity 1 smooth manifold with boundary $\partial T$ in a neighborhood of $b$.

Use $B_r(p)$ and $B_r(b)$ to denote the geodesic ball around $p$, $b$ in $g$. We endow the space of Riemannian metrics with the weak $C^\infty$ topology, i.e., the usual one induced by $C^k$ norms for all $k \geq 0$.

**Lemma 2.13.** Let $T$ satisfy our assumptions (a) to (d) above. For any constant $\delta > 0$, there exists a constant $\epsilon > 0$, an open subset $U \subset M$ containing the support of $T$, a radius $r > 0$, and an open set $\Omega_T$ in the space of Riemannian metrics with the following properties.

- For any $g' \in \Omega_T$, and any current $T'$ supported in $U$ and homologous to $T$ in $U$, if $T'$ is area-minimizing among homologous competitors in $M$, then $T'$ is a smooth submanifold of multiplicity 1 in $B_r(p)$, and graphical as exponentiated sections with $C^1$ norm smaller than $\delta$, in the normal bundle of smooth part of $T$.
- If $\partial T'$ is a smooth submanifold of multiplicity 1 and is a graph of an exponentiated section in the normal bundle of $\partial T$ with $C^1$ norm smaller than $\epsilon$, then $T'$ is a smooth manifold with boundary in $B_r(b)$.
- If $T$ is the unique homological area-minimizer on $M$ in its class, then we can take $U = M$.

**Proof.** Embed $M$ isometrically into Euclidean space. Direct calculation shows that any such $T$ in any nearby metrics will have bounded generalized mean curvature with a uniform bound. Then using the monotonicity formula as in [1] and [2], we get upper semi-continuity of densities at the boundary and in the interior. Now argue by contradiction and use Allard regularity in [1] and [2].

3. **Proof of Theorem 1.2**

In this section, we will prove Theorem 1.2. First, we will do a reduction to the local case.

3.1. **Basic setup.** Let $n \geq 3, k \geq 0$ be integers. We need the following basic setup.

(A) $(M^{2n+k}, g)$ be a (possibly non-complete non-orientable) smooth Riemannian manifold, with a (possibly non-orientable) $k$-dim compact closed submanifold $L$ in the interior.
(B) we have two orientable minimal submanifolds $X$ and $Y$ of dimension $n + k$ intersecting transversally along $L$, with boundaries outside of $B_r(L)$.

(C) the intersection of $B_r(L)$ with $X$ and $Y$ are orientable submanifolds in $X$ and $Y$ for a.e. $r$ small.

Remark 22. Here $B_r(L)$ denotes a tubular neighborhood of a small radius $r$ in $M$. The point (C) follows from the first two by Sard’s theorem and the fact that the boundaries of smooth open sets in orientable manifolds are orientable.

At any point $p \in L$, consider the tangent space $T_pM$. Then $T_pX$ and $T_pY$ splits off as products over $T_pL$ into two $n$-dimension planes $X', Y'$ intersecting transversally in $\mathbb{R}^{2n}$. We define the intersecting angle $\theta(p)$ to be $\theta(T_pX, T_pY)$ by Definition 2.1.

**Theorem 3.1.** Suppose $\inf_{p \in L} \theta(p) > 2 \arctan \frac{2}{\sqrt{n^2-4}}$. With the condition (A) to (C), we have the following.

- By restricting to a smaller radius in the tubular neighborhood, the current $X + Y$ is area-minimizing, calibrated by the sum of Lawlor type vanishing calibrations, smooth near $X + Y$ away from $X \cap Y$.
- There exists an open set $\Omega$ in the space of Riemannian metrics containing $g$, and a slightly smaller neighborhood $U$ around $L$, whose boundary $\partial U$ intersects $X + Y$ smoothly and transversally.
- Any integral area-minimizing current $T'$ in $U$ with metric $g' \in \Omega$, and $\partial T'$ smooth and graphical as an exponentiated section in the normal bundle of $\partial((X + Y)L \cup U)$ with $C^1$ norm smaller than some positive constant $c$, must also decompose into $T' = X' + Y'$.
- $X', Y'$ are smooth minimal immersions diffeomorphic to $X \cup U, Y \cup U$, respectively, and $X'$ intersects $Y'$ transversally at a submanifold $L'$ diffeomorphic to $L$.

For the proof of Theorem 3.1 in the next few subsections, we will construct Lawlor’s vanishing calibrations with vanishing angles of less than $\arctan \frac{2}{\sqrt{n^2-4}}$ for both $X$ and $Y$. Then summing the two together gives the calibration. The decomposition for nearby currents comes by summing nearby minimal submanifolds together using similar constructions of vanishing calibrations.

3.2 Proof of Theorem 1.2 assuming Theorem 3.1. First, let us show that we indeed have a reduction. Let $U$ be the open set in Theorem 3.1 around $L$. By Lemma 2.13 and covering the regular part of $T$ in $U^C$ using finitely many small enough balls, for any small $\delta > 0$, there exists an open neighborhood $q_0$ in the space of Riemannian metrics and an open set $O$ containing the support of $T$, so that any homologically area-minimizing current $T'$ contained in $O$, and homologous to $T$ is regular outside of $U$. Moreover,
$T'$ is graphical with $C^1$ norm smaller $\delta$, over $T$ near $\partial(T\cap U)$. Now apply Theorem 3.1 to $T'\cap U$. We are done. In case $T$ is a unique minimizer, just argue by contradiction and use the fact that convergence of area-minimizing current implies convergence of support in Hausdorff distance.

Now we go on to prove Theorem 3.1.

3.3. Fermi coordinates. Let $X,Y,L = X \cap Y$ be as above in the introduction to this section. Start with $X$. Let $\exp_X$ be the exponential map from the normal bundle of $X$ to $M$. For some small $z_0 > 0$, this will be a diffeomorphism for sections no longer than $r$. Let $\pi_X$ denote the corresponding projection induced by the projection in the normal bundle to the zero section.

Similarly, let $\exp_L^X$ be the exponential map of the normal bundle of $L$ inside $x$ under the intrinsic metric of $X$. Again, we can ensure that it is a diffeomorphism up to radius $r_0 > 0$ and use $\pi_L$ to denote the associated projection.

Let $U = \exp_X^X(\exp_L^X(L, \{|\nu_L| \leq r_0\}), \{|\nu_X| \leq z_0\})$, with $\nu_L$ the vectors in the normal bundle of $L$ inside $X$ and similarly for $X$.

We need a cover of $U$ by Fermi coordinates neighborhoods. First, cover $L$ by coordinate neighborhoods $(l_1, \cdots, l_k)$ so that $dl_1 \wedge \cdots \wedge dl_k$ equals the volume form for $L$ orientable or the volume density up to signs for $L$ unorientable (by Moser [11]). By compactness of $L$, we can cover it by finitely many such coordinate neighborhoods $V_1, \cdots, V_b$. For each $V_j$, construct a Fermi coordinate $V_j^X$ over $V_j$ with orthonormal frames in the normal bundle of $L$ in $X$, denoted by $(l_1, \cdots, l_k, x_1, \cdots, x_n)$. Moreover, we require that $dx_1 \wedge \cdots \wedge dx_n \wedge dl_1 \wedge \cdots \wedge dl_k$ are positive multiples of $\omega$. Finally, construct a Fermi coordinate adapted to $X$ with orthonormal frames in the normal bundle of $X$ by $(l_1, \cdots, l_k, x_1, \cdots, x_n, y_1, \cdots, y_n)$. Sometimes we will write it more compactly as $(x,y,l)$. By shrinking the radius and taking finer coverings $V_j$, we can ensure the existence of these orthonormal frames and the Fermi coordinates. Use $U_j$ to denote these coordinate neighborhoods. They form a covering of $U$.

3.4. The projection. Note that for each point $p$ in $U$, there exists a unique geodesic in the intrinsic metric on $X$, starting from $\pi_L \circ \pi_X(p)$ to $\pi_X(p)$, stemming from $L$ orthogonally, and then another unique geodesic starting from $\pi_X(p)$ to $p$, stemming from $X$ orthogonally. We use $r$ to denote the length of the first geodesic and $z$ to denote the length of the second geodesic. Then by the generalized Gauss lemma (Lemma 2.11 in [21]) the vector field $\partial_t$ dual to the 1-form $dz$ has unit length and its integral curves are geodesics normal to $X$. For the same reasoning, the vector field $\partial_t$ dual to the form $dr$ on $X$ has unit length and integrates to the geodesics in $X$ normal to $L$. Note that $r \circ \pi_X = r$. This gives $\pi_X^*dr = dr$. Let $t = \frac{z}{r}$. Again, we let $t = \frac{z}{r}$.

We will define a retraction $\Pi$ from $U$ to $X$ as follows.
Definition 3.1. Consider the unique geodesic from $\pi_L \circ \pi_X(p)$ to $\pi_X(p)$. There exists a unique point $g_{(n,a)}(t)^{1/n} \pi_X(p)$ defined to be the point $g_{(n,a)}(t)^{1/n} \pi_X(p)$ away from $\pi_L \circ \pi_X(p)$ on this geodesic. Define

$$\Pi(p) = g_{(n,a)}(t)^{1/n} \pi_X(p).$$  \hspace{1cm} (3.1)

Lemma 3.2. In any $U_j$ coordinate and using the notation of Section 2.5, we have

$$\pi_X(x,y,l) = (x,0,l), \pi_L \circ \pi_X(x,y,l) = (l,0,0),$$

$$dr = \frac{x_j}{r} dx_j, dz = \frac{y_\alpha}{z} dy_\alpha, \partial_r = \frac{x_j}{r} \partial_j, dz = \frac{y_\alpha}{z} \partial_\alpha,$$

$$\Pi = \Pi.$$  

Proof. Follows from the discussion in Section 2.3 and $\pi^*_L dr = dr$. \qed

Let $\omega$ denote the volume form on $X$. Note that on $X$, $dr$ and $\partial_r$ are both of length 1, so we have $dr \wedge i_{\partial_r} \omega = \omega$, where $i$ is the interior product.

3.5. The vanishing calibration form. First, we need the following lemma

Lemma 3.3. There is a smooth function $\lambda$ on $X$ so that $\lambda = 1 + O(r)$ and satisfies

$$\partial_r \log \lambda = \frac{n}{r} \left(1 - \text{div} \left(\frac{r}{n} \partial_r\right)\right).$$  \hspace{1cm} (3.2)

Proof. In each patch $V^X_m$, we have $\frac{r}{n} \partial_r = \frac{1}{n} (x_j \partial_j)$. The volume form can be written as

$$\omega = \sqrt{\det} dl_1 \wedge \cdots \wedge dl_k \wedge dx_1 \wedge \cdots \wedge dx_n,$$

where

$$\sqrt{\det} = \sqrt{(\partial_{t_1} \wedge \cdots \wedge \partial_{t_k} \wedge \partial_{x_1} \wedge \cdots \wedge \partial_{x_n}, \partial_{t_1} \wedge \cdots \wedge \partial_{t_k} \wedge \partial_{x_1} \wedge \cdots \wedge \partial_{x_n})}.$$

We will verify that $\lambda = (\sqrt{\det})^{-1}$ is the $\lambda$ we want and the definition is independent of coordinate choices.

We have

$$\text{div} \left(\frac{r}{n} \partial_r\right) = \frac{1}{n \sqrt{\det}} \partial_i (\sqrt{\det}) x_i = 1 + \frac{1}{n} \frac{\sqrt{\det}}{\sqrt{\det}} x_j,$$

This gives

$$\partial_r \log \lambda = -\partial_r \log \sqrt{\det}.$$

and thus $\partial_r \log (\lambda \sqrt{\det}) = 0$. Note that by our construction, $\sqrt{\det} = 1$ on $L$ in all patches $V^X$. Thus, if we set $\lambda = 1$ on $L$, then integrating along the $\partial_r$ directions, we must have $\lambda = (\sqrt{\det})^{-1}$ in each patch $V^X$. Taylor’s theorem gives $\sqrt{\det} = 1 + O(r)$. This shows the smoothness and existence of $\lambda$.

It is not obvious that $\sqrt{\det}$ is independent of the choice of Fermi coordinates. We can see this as follows. $dl_1 \wedge \cdots \wedge dl_k$ equal to $\pi_L^\ast \omega_L$ by Lemma 3.2.
where $\omega_L$ is up to signs the volume density on $L$. This is invariant up to signs. Thus the choice of $\partial_{l_1}, \cdots, \partial_{l_k}$ does not matter. For any two different choices of local orthonormal frames $\{e_j\}$ and $\{e'_j\}$ in the normal bundle of $L$ inside $X$, we can assume that with reference to $\{e_j\}$, the frames $\{e'_j\}$ are smooth fiberwise $SO(n)$ transformations of $\{e_j\}$ (taking orientation into account). In Fermi coordinates generated by those two frames, we have $(x'_1, \cdots, x'_{n}) = O(x_1, \cdots, x_n)$ with $O$ a smooth varying function in $SO(n)$ depending only on $l$-coordinate directions. Taking the exterior derivative gives $dx' = O dx + (dO)x$. However, $O$ only depends on the $l_j$ directions. Thus $(dO)x \wedge \pi^*(\omega_L) = 0$. This implies that $dx_1 \wedge \cdots \wedge dx_n \wedge \pi^*(\omega_L) = (\det O) dx'_1 \wedge \cdots \wedge dx'_n \wedge \pi^*(\omega_L) = dx'_1 \wedge \cdots \wedge dx'_n \wedge \pi^*(\omega_L)$. Thus, the choice of orthonormal frames does not matter either. This shows that $dl_1 \wedge \cdots \wedge dl_k \wedge dx_1 \wedge \cdots \wedge dx_n$ is independent of the choice of the Fermi coordinates, so is its ratio over $\omega$.

Lemma 3.4. Define

\[
\psi = \pi_X^* \left( \frac{r}{n} \lambda i_{\partial_r} \omega \right),
\]

where $i$ is the interior product of inserting a vector. Then $\psi$ is a smooth form with

\[
dr \wedge \left( \frac{n}{r} \psi \right) = d\psi.
\]

Proof. By Lemma 3.2 we deduce that $r \partial_r$ is smooth, so $\psi$ is also smooth. By Cartan’s magic formula, the naturalness of exterior derivative, and equation (3.2), we have

\[
d\psi = \pi_X^* \left( \frac{r}{n} \lambda i_{\partial_r} \omega \right)
= \frac{1}{n} \pi_X^* (L_{\lambda r \partial_r} \omega - \lambda i_{r \partial_r} d\omega)
= \frac{1}{n} \pi_X^* (\text{div}_X (\lambda r \partial_r) \omega)
= \frac{1}{n} \pi_X^* ((r \partial_r \lambda + \lambda \text{div}_X (r \partial_r)) \omega)
= \pi_X^* \left( \lambda \left( \frac{r}{n} \partial_r \log \lambda + \text{div}_X \left( \frac{r}{n} \partial_r \right) \right) \omega \right)
= \pi_X^* (\lambda \omega)
= \pi_X^* \left( dr \wedge \frac{n}{r} \lambda i_{\partial_r} \omega \right)
= dr \wedge \frac{n}{r} \psi.
\]
By our assumption, we have \( \inf_{p \in L} \theta(p) = 2 \arctan \frac{2}{\sqrt{n^2 - 4}} + \epsilon \) for some \( \epsilon > 0 \). Now choose \( a \), so that
\[
\arctan \sqrt{\frac{a}{n(n - 2)}} = \arctan \frac{2}{\sqrt{n^2 - 4}} + \epsilon.
\]
By monotonicity of \( \tan \) such \( a \) exists and \( n(n - 2) > a > \frac{4n}{n + 2} \). Moreover, by construction, we have
\[
2 \arctan \sqrt{\frac{a}{n(n - 2)}} < \inf_{p \in L} \theta(p) = 2 \arctan \frac{2}{\sqrt{n^2 - 4}} + \epsilon. \tag{3.4}
\]
With the choice of \( a \), we can construct the cut-off functions \( g_{(n, a)} \) using Lemma 2.3.

**Lemma 3.5.** The form \( \phi \), defined by
\[
\Psi = \frac{1}{\lambda \circ \Pi} d(g_{(n, a)}(t)\psi),
\]
with \( t = \frac{t}{r} \) can serve as a vanishing calibration form, and \( \Psi \) is locally exterior derivatives of Lipschitz forms, i.e., satisfying the assumptions in Lemma 2.2. Moreover, we have
\[
\Psi = \frac{1}{\lambda \circ \Pi} \left( \left( g_{(n, a)}(t) - g_{(n, a)}'(t) \frac{t}{n} \right) dt + g_{(n, a)}'(t) \frac{dz}{n} \right) \wedge \left( \frac{n}{r} \psi \right) \tag{3.5}
\]

**Proof.** The calculation for (3.5) is as follows
\[
\Psi = \frac{1}{\lambda \circ \Pi} \left( g_{(n, a)}(t) dt + g_{(n, a)}'(t) \frac{dz}{r} \right) \wedge \psi
\]
\[
= \frac{1}{\lambda \circ \Pi} \left( g_{(n, a)}(t) dt + g_{(n, a)}'(t) \frac{dz}{r} \right) \wedge \psi
\]
\[
= \frac{1}{\lambda \circ \Pi} \left( g_{(n, a)}(t) dt \wedge \left( \frac{n}{r} \psi \right) + g_{(n, a)}'(t) \frac{dz}{n} - g_{(n, a)}'(t) \frac{t}{n} dt \right) \wedge \left( \frac{n}{r} \psi \right)
\]
\[
= \frac{1}{\lambda \circ \Pi} \left( g_{(n, a)}(t) dt \wedge \left( \frac{n}{r} \psi \right) + g_{(n, a)}'(t) \frac{dz}{n} \right) \wedge \left( \frac{n}{r} \psi \right).
\]
In each Fermi coordinate patch, we have \( \omega = \sqrt{\det dx_1 \wedge \cdots \wedge dx_n \wedge dl_1 \wedge \cdots \wedge dl_k} \). This simplifies \( \Psi \) to
\[
\Psi = \frac{1}{\lambda \circ \Pi} \left( g_{(n, a)}(t) - g_{(n, a)}'(t) \frac{t}{n} \right) dt + g_{(n, a)}'(t) \frac{dz}{n} \wedge \left( \frac{n}{r} \psi \wedge dl_1 \wedge \cdots \wedge dl_k \right), \tag{3.6}
\]
where \( \bar{\psi} \) is defined before Lemma 2.5. This implies \( d\Psi = 0 \), since the \( l_b \) direction derivatives \( \frac{1}{\lambda \circ \Pi} \) are killed by \( dl_1 \wedge \cdots \wedge dl_k \), and by Lemma 2.6 the \( x_j \) direction derivatives are killed by \( \left( g_{(n, a)}(t) - g_{(n, a)}'(t) \frac{t}{n} \right) dt + \).
\( g_{(n,a)}'(t) \frac{dz}{n} \wedge \frac{\sqrt{g}}{r} \). Now we can argue as in the proof of Lemma 2.6 to find Lipschitz primitives locally, in order to use Lemma 2.2. We are done. \( \square \)

By construction \( \Psi \) gives value 1 on the tangent spaces to \( X \), and the singular set of \( \Psi \) is \( L \), which is of codimension \( n \) in \( X \). Thus, if we can show that the comass of \( \Psi \) is no larger than 1, then we know that \( X \) is area-minimizing by Lemma 2.2. It suffices to do this in each coordinate patch.

### 3.6. Comass estimates of \( \Psi \): the scaling factor \( f \)

Now fix a Fermi coordinate neighborhood \( U_j \) defined in Section 3.3. We will write the coordinate as \( (x,y,l) \). Equation (3.6) gives

\[
\Psi = \frac{1}{\lambda \circ \Pi} \left( (g_{(n,a)}(t) - g_{(n,a)}'(t) \frac{t}{n}) \frac{dr}{n} + g_{(n,a)}'(t) \frac{dz}{n} \right) \wedge \left( \frac{n}{r} \sqrt{\gamma} \wedge dl_1 \wedge \cdots \wedge dl_k \right).
\]

Define

\[
\begin{align*}
    c &= g_{(n,a)} - \frac{t}{n} g_{(n,a)}, \\
    s &= \frac{g_{(n,a)}}{n}.
\end{align*}
\]

(3.7) \hspace{1cm} (3.8)

Let \( e_2, \ldots, e_n \) denote orthonormal tangent vectors to the standard spheres in the \( x_1 \cdots x_n \)-planes with \( y_1 = \cdots = y_n = 0 \), and let \( e_2^*, \ldots, e_n^* \) denote their dual in the flat metric. Note that pointwise we can choose \( e_j \) to be linear combinations of \( \{ \partial_{x_j} \} \), thus we can extend \( e_j \) constantly in \( y \)-directions to \( U_j \). This gives \( e_2^* \wedge \cdots \wedge e_n^* = (\frac{2}{r} \sqrt{\gamma}) \).

Similarly, use \( v_2, \ldots, v_n \) to denote the orthonormal tangent vectors to the standard spheres in the \( y_1 \cdots y_n \)-planes with \( x_1 = \cdots = x_n = 0 \). Again pointwise we can choose \( v_j \) to be linear combinations of \( \{ \partial_{y_j} \} \), thus we can extend \( v_a \) constantly in \( x \)-directions to \( U_j \).

The form \( \Psi \) can be written as

\[
\Psi = f(c dr + s dz) \wedge e_2^* \wedge \cdots \wedge e_n^* \wedge dl_1 \wedge \cdots \wedge dl_k.
\]

Now, let us compare \( f(x,y,l) \) with \( f(x,0,l) \). By definition of the retraction (Lemma 2.4), we have

\[
f(x,y,l) = f((g(t))^{1/n} x, 0, l).
\]

However,

\[
f(x,0,l) = \sqrt{\det(x,0,l)}
\]

\[
= \sqrt{\langle \partial_{x_1} \wedge \cdots \wedge \partial_{x_n} \wedge \partial_{l_1} \wedge \cdots \wedge \partial_{l_k}, \partial_{x_1} \wedge \cdots \wedge \partial_{x_n} \wedge \partial_{l_1} \wedge \cdots \wedge \partial_{l_k} \rangle}
\]

equals the coefficient of volume form on \( X \) in this coordinate since \( \partial_{x_j} \) is orthonormal on \( X \). Thus, by Taylor’s theorem and shrinking using finer covers, we can assume that

\[
|f(x,y,l) - f(x,0,l)| \leq O((1 - (g(t))^{1/n})|x|).
\]
with the constant in $O$ depending on the geometry of $L$, $X$ and $M$. On the other hand

$$0 \leq 1 - (g(t))^{1/n} = \frac{1 - g(t)}{\sum_{0 \leq j \leq n} g(t)^{j/n}} \leq c_n t^2.$$ 

Also, $1$ is close to $f(x, 0, l)$, so by enlarging the constant a bit, we have

$$f(x, y, l) = f(x, 0, l)(1 + O(rt^2)).$$ (3.9)

### 3.7. Comass estimates of $\Psi$: the comass on the orthogonal complement of $\ker \Psi$.

Now define the kernel of $\Psi$ to be

$$\ker \Psi = \{w| i_w \Psi = 0\}.$$

By Lemma 7.5 in Section II of [22], to evaluate the comass of $\Psi$, it suffices to evaluate the comass of $\Psi$ restricted to the orthogonal complements of $\ker \Psi$.

Note that by definition $\ker \Psi$ is generated by $v_2, \ldots, v_n, s\partial_r - c\partial_z$. Use $v_1$ to denote the last one. Set $v_j = \frac{v_j}{\langle v_j, v_j \rangle}$ and $h_{ij} = \langle v_i, v_j \rangle$. (Here $\langle \cdot, \cdot \rangle$ is the Riemannian metric.) Using $\partial_r \perp v_j, \partial_r$ everywhere, $\kappa_n \leq c^2 + s^2 \leq 1$, and $\langle \partial_z, \partial_z \rangle = 1$ (Lemma 2.11 in [21]), we can calculate directly by Taylor’s theorem to give,

$$h_{ij} = \delta_{ij} + O(rt),$$

$$h_{ij} = \delta_{ij} + O(rt),$$

$$v_i(x, y, l) = (1 + O(rt))v_i \sqrt{\langle v_i, v_i \rangle(x, 0, l)},$$

with constants depending only on the geometry of $X$ and $\kappa_n$. Then by dimension counting, the orthogonal complement to $\ker \Psi$ is spanned by

$$c\partial_r + s\partial_z - h_{ij} \langle c\partial_r + s\partial_z, v_i \rangle v_j,$$

(3.10)

$$e_2 - h_{ij} \langle e_2, v_i \rangle v_j, \ldots, e_n - h_{ij} \langle e_n, v_i \rangle v_j,$$

(3.11)

$$\partial_1 - h_{ij} \langle \partial_1, v_i \rangle v_j, \ldots, \partial_k - h_{ij} \langle \partial_k, v_i \rangle v_j,$$

(3.12)

Use $w^{\perp \Psi}$ to denote the component of $w$ in the orthogonal complement of $\ker \Psi$. Since the vectors above are all orthogonal to $\ker \Psi$ on $X$, by Taylor’s
theorem, we have
\[ (c\partial_r + s\partial_z)^+\Psi \]
\[ = c\partial_r + s\partial_z - h^{ij}(1 + O(rt)) \frac{1}{\sqrt{c^2 + s^2}} (cs(\langle \partial_r, \partial_r \rangle - \langle \partial_z, \partial_z \rangle) + (c^2 + s^2)\langle \partial_r, \partial_z \rangle) v_j \]
\[ - \sum_{a \geq 2} a^{ij}(c\partial_r + s\partial_z, v_a)(x, 0, l) + O(|y|) v_j \]
\[ = c\partial_r + s\partial_z - h^{ij}(1 + O(rt)) \frac{1}{\sqrt{c^2 + s^2}} (cs(\langle \partial_r, \partial_r \rangle(x, 0, l) + O(|y|) - 1) \]
\[ + (c^2 + s^2)(\langle \partial_r, \partial_z \rangle(x, 0, l) + O(|y|)) v_j - \sum_{a \geq 2} a^{ij}O(rt) v_j \]
\[ = c\partial_r + s\partial_z + \sum_j O(rt) v_j, \]
and similarly \( e_a^{+\Psi} = e_a + \sum_j O(rt) v_j, \quad \partial_{ta}^{+\Psi} = \partial_{ta} + \sum_j O(rt) v_j. \)

To simplify calculations, we will take the simple \( n + k \)-vector representative \( P \) of \( \ker \Psi^+ \), so that
\[ (3.13) \quad \frac{1}{f} \Psi(P) = 1, \]
i.e.,
\[ P = \frac{1}{(\sqrt{c^2 + s^2})^2} (c\partial_r + s\partial_z - h^{ij}(c\partial_r + s\partial_z, v_i) v_j) \]
\[ \wedge (e_2 - h^{ij}(e_2, v_i) v_j) \wedge \cdots \wedge (e_n - h^{ij}(e_n, v_i) v_j) \]
\[ \wedge (\partial_{t1} - h^{ij}(\partial_{t1}, v_i) v_j) \wedge (\partial_{tk} - h^{ij}(\partial_{tk}, v_i) v_j). \]

By the generalized Gauss lemma (Lemma 2.11 in [21]), we have \( \partial_z \) perpendicular to all vectors tangent to \( X \) and their constant translations in Fermi coordinate. This implies the same statement when we take \( \perp \Psi \) of every vector. Thus, expanding out the determinant gives
\[ \langle P, P \rangle = \frac{1}{c^2 + s^2} (c^2 - \frac{d_1}{c^2 + s^2} d_1 + \frac{s^2}{c^2 + s^2} d_2). \]

Here
\[ d_1 = \langle \partial_r^{+\Psi} \wedge e_2^{+\Psi} \wedge \cdots \wedge e_n^{+\Psi} \wedge \partial_{t1}^{+\Psi} \wedge \cdots \wedge \partial_{tk}^{+\Psi} \rangle(x, y, l, \).
\]

We have
\[ \langle \partial_r^{+\Psi}, e_j^{+\Psi} \rangle = \langle \partial_r + \sum_a O(rt) v_a, e_j + \sum_b O(rt) v_b \rangle \]
\[ = \langle \partial_r, e_j \rangle + O(rt) (\langle \partial_r, v_a \rangle + \langle e_j, v_a \rangle) + O(r^2 t^2) \]
\[ = \langle \partial_r, e_j \rangle + O(r^2 t^2). \]
And similarly for the inner product of any two terms in the wedge. Thus, we have
\[
d_1 = \langle \partial_r \wedge e_2 \wedge \cdots \wedge e_n \wedge \partial_{t_1} \wedge \cdots \wedge \partial_{t_k}, \partial_r \wedge e_2 \wedge \cdots \wedge e_n \wedge \partial_{t_1} \wedge \cdots \wedge \partial_{t_k} \rangle (1 + O(r^2t^2))
\]
\[
= \langle \partial_{x_1} \wedge \cdots \wedge \partial_{x_n} \wedge \partial_{t_1} \wedge \cdots \wedge \partial_{t_k}, \partial_{x_1} \wedge \cdots \wedge \partial_{x_n} \wedge \partial_{t_1} \wedge \cdots \wedge \partial_{t_k} \rangle (1 + O(r^2t^2))
\]
By Lemma 2.1, the mean curvature of \(X\), \(H_X = 0\) and \(|y| = z = tr\) gives
\[
d_1 = (f(x,0,l))^2(1 + O(t^2r^2)),
\]
with the constant in \(O\) depending on the geometry of \(L, X\) and \(M\). On the other hand,
\[
d_2 = \langle e_2^\perp \wedge \cdots \wedge e_n^\perp \wedge \partial_{t_1}^\perp \wedge \cdots \wedge \partial_{t_k}^\perp, e_2^\perp \wedge \cdots \wedge e_n^\perp \wedge \partial_{t_1}^\perp \wedge \cdots \wedge \partial_{t_k}^\perp \rangle (x, y, l).
\]
A similar calculation as above gives
\[
d_2 = \langle e_2 \wedge \cdots \wedge e_n \wedge \partial_{t_1} \wedge \cdots \wedge \partial_{t_k}, e_2 \wedge \cdots \wedge e_n \wedge \partial_{t_1} \wedge \cdots \wedge \partial_{t_k} \rangle (x, y, l)(1 + O(r^2t^2)).
\]
Denote the plane spanned by \(e_2, \cdots, e_n, \partial_{t_1}, \cdots, \partial_{t_k}\) by \(P'\), and let \((P')^\perp\) denote the projection into the orthogonal complement of \(P'\). Then
\[
d_1 = \langle P, P \rangle (1 + O(r^2t^2)) = \langle (P')^\perp \partial_r, (P')^\perp \partial_r \rangle \langle P', P' \rangle (1 + O(r^2t^2)).
\]
Thus to estimate \(d_2\), it suffices to estimate
\[
\langle (P')^\perp \partial_r, (P')^\perp \partial_r \rangle.
\]
For this, note that \(\partial_r \perp P'\) on \(F_j\), so by Taylor’s theorem, we have \((P')\partial_r = \partial_r + O(rt)\). This gives
\[
\langle (P')^\perp \partial_r, (P')^\perp \partial_r \rangle (x, y, l) = \langle \partial_r, \partial_r \rangle (x, y, l) + O(tr) = 1 + O(tr),
\]
with the constant in \(O\) depending on the geometry of \(L, X\) and \(M\). This gives the estimate
\[
d_2 = d_1(1 + O(tr)).
\]
Combining these gives
\[
\langle P, P \rangle = \frac{1}{c^2 + s^2} d_1(1 + \frac{s^2}{c^2 + s^2} O(tr)).
\]
Note that in all the above big \(O\) terms, they all have a factor of \(r\). Thus, shrinking \(r\), if necessary we have
\[
\frac{1}{\sqrt{\langle P, P \rangle}} = \frac{1}{f(x,0,l)} \sqrt{c^2 + s^2} \frac{1}{1 + \frac{s^2}{c^2 + s^2} O(tr)}.
\]
However, by Lemma 2.3, we have
\[
c^2 + s^2 \leq 1 - \delta_n t^2, \quad \frac{s^2}{c^2 + s^2} \leq \frac{4c_n^2 l^2}{n^2 \kappa_n}.
\]
Inserting these and \((3.9)\) and \((3.13)\), we get
\[
\frac{1}{\sqrt{\langle P, P \rangle}} \leq \frac{1}{f(x,0,l)} \frac{1}{(1 + O(t^2r^2))(1 - \delta_n t^2)(1 + O(t^3r))}.
\]
Thus, the comass of $\Psi$ satisfies
\begin{equation}
\text{comass } \Psi = \Psi\left(\frac{P}{\sqrt{\langle P, P \rangle}}\right) \leq (1 + O(t^2r))(1 + O(t^2r^2))(1 - \delta_n t^2)(1 + O(t^3r)),
\end{equation}
where all the bounds in big $O$ depend only on $n, k$ and the geometry of $X, L$ and $M$.

Note that $\Psi \equiv 0$ for $t > \tan \theta(n, a)$, so we only have to deal with $t \leq \tan \theta(n, a) < 1$ using the above inequality. With $r \to 0$, and no zeroth and first order term in $t$ among the factors, the term $(1 - \delta_n t^2)$ dominates, so we have $\text{comass } \Psi \leq 1$ for $r$ small enough.

### 3.8. Vanishing calibration for $Y$

Now if we do the same reasoning for $Y$, then we can get a vanishing calibration $\Phi$ that calibrates $Y$ for $r$ small enough. By (3.14), we see that the supports of $\Phi$ and $\Psi$ only intersect at $L$. Thus, it can be used to calibrate $X + Y$, which gives its area-minimality.

### 3.9. Nearby currents and finishing the proof of Theorem 3.1

To sum it up, we have proved the following result,

**Corollary.** $X + Y$ is calibrated by the sum of two Lawlor’s vanishing calibration forms $\Phi + \Psi$ in a tubular neighborhood $B_r(L)$. The radius $r$ and comass estimates depend only on the geometry of $L, X + Y$, and $M$.

This already gives the first two bullet points in Theorem 3.1. Now we will the rest of that theorem, i.e., to account for nearby currents denoted as $T'$.

To achieve this, we first restrict our focus to a smaller open set $B_{r_0}(L)$, such that $r_0 < r$ and the restriction of $(X + Y)$ on $B_{r_0}(L)$ still maintains a smooth boundary.

Given the boundary of any nearby current $T' \cup B_{r_0}(L)$ is, by assumption, $\epsilon$ close to the boundary of $(X + Y) \cup B_{r_0}(L)$ in $C^1$, it can be decomposed into two connected, smooth, orientable submanifolds, $A$ and $B$. We then consider the area minimizing current $X'$, bounded by $A$, and $Y'$, bounded by $B$.

Applying Allard’s regularity lemma (Lemma 2.13), we can infer that $X'$ and $Y'$ are diffeomorphic to and graphical over $X$ and $Y$ within $B_{r_0}(L)$. Following the Transversality Theorem 2.1 presented in [27], it can be deduced that $X'$ and $Y'$ intersect along a submanifold $L'$ that is in close proximity to $L$, provided $r$ is small.

By arguing in coordinates and applying the implicit function theorem for any point on $L$, we can demonstrate that $L'$ is locally graphical over $L$. This effectively shows that $L'$ is diffeomorphic to $L$.

Since the constants in the preceding Corollary are contingent on the geometry of $L, X + Y$, and $M$, we can apply the same process to deduce that $X'$ and $Y'$ will also minimize, provided that $r$ is small, and be calibrated by vanishing calibrations with angles close to (3.4). As a consequence, the
support of the two vanishing calibrations will only intersect at $L'$ for $\epsilon$ small. Their sum calibrates $X' + Y'$.

Finally, we apply Lemma 2.12 to infer that $T' \setminus B_{r_0}(L)$, being the unique minimizer bounded by its boundary, must equate to $X' + Y'$. Hence, our proof is complete.

4. The model singularities

In this section, we will construct the model cases of non-smoothable singularities. They will be glued to homology class representatives to prove Theorem 1.1.

4.1. The link of product cones. Let $C_n$ be a cone with a smooth (not necessarily orientable or connected) link in $\mathbb{R}^{n+k}$.

**Definition 4.1.** Define the $l$-th product link over $C_n$ to be

$$\sigma^l(C) = (C \times \mathbb{R}^{l+1}) \cap S^{n+k+l}.$$  \hspace{2cm} (4.1)

Here $S^{n+k+l}$ is the unit sphere in $\mathbb{R}^{n+k+l+1}$ and $C \times \mathbb{R}^{l+1}$ is embedded naturally in $\mathbb{R}^{n+k} \times \mathbb{R}^{l+1} \cong \mathbb{R}^{n+k+l+1}$.

The product link $\sigma^l(C)$ will serve as our model singularities for suitable cones.

**Lemma 4.1.** $\sigma^l(C)$ has the following properties.

- The singular set of $\sigma^l(C)$ as an integral current (or mod 2 current) is totally geodesic $l$-dimensional sphere $\{0\} \times S^l \subset S^{n+k+l+1}$.
- Small enough tubular neighborhoods $U_r$ around the singular set are diffeomorphic to $B_1^{n+k} \times S^l$, with $B_1^{n+k}$ the unit ball in $\mathbb{R}^{n+k}$.
- The diffeomorphisms $\Gamma$ can be chosen so that $\sigma^l(C)$ is sent to $(C \cap B_1^{n+k}) \times S^l$.

**Remark 23.** Here we regard $S^0$ as two points.

**Proof.** Note that $C \times \mathbb{R}^{l+1}$ is also a cone. Thus, we deduce that $\sigma^l(C)$ is smooth outside of the intersection of $S^{n+k+l}$ with the singular set of $C \times \mathbb{R}^{l+1}$, i.e., $\{0\} \times \mathbb{R}^{l+1}$. The intersection is precisely a totally geodesic $l$-dimensional sphere in $S^{n+k+l}$.

Denote the first $n+k$ factors by $x = (x_1, \cdots, x_{n+k})$ and the last $l+1$ factors by $y = (y_1, \cdots, y_{l+1})$. Consider the following map

$$\Gamma(x, y) = \left(\frac{x}{|y|}, \frac{y}{|y|}\right).$$

$\Gamma$ is smooth away from $|y| = 0$. 

Direct calculation shows that
\[ D\Gamma(x,y)(v,w) = \frac{1}{|y|} \left( v - \frac{y}{|y|} w \right) \frac{x}{|x|} , w - \frac{y}{|y|} w \right). \]
Thus, \( D\Gamma \) is not bijective if and only if there exists nontrivial \( v, w \) with \( v = \langle \frac{w}{|y|}, \frac{w}{|y|} \rangle, w = \langle \frac{w}{|y|}, \frac{w}{|y|} \rangle \). Solving these two gives
\[ v = \frac{x}{|y|}, w = \frac{y}{|y|}. \]
If we restrict \( D\Gamma \) to the tangent bundle of \( S^n \), then this gives
\[ 0 = \langle v, x \rangle + \langle w, y \rangle = |x|^2 + |y|, \]
which is impossible as long as \( |y| \neq 0 \). Thus, as long as \( |y| \neq 0 \), \( D\Gamma \) has rank \( n + k + l \) on \( S^n \).
We claim that \( \Gamma \) maps \( S^n \setminus \{ |y| = 0 \} \) bijectively to \( \mathbb{R}^{n+k} \times S^l \).
For injectivity, \( \frac{x}{|y|} = \frac{x'}{|y'|}, \frac{y}{|y|} = \frac{y'}{|y'|} \) implies that \( y' = \frac{|y|}{|y'|} y, x' = \frac{|y|}{|y'|} x \). However, \( x^2 + y^2 = (x')^2 + (y')^2 = 1 \) gives \( |y'| = |y| \), which implies \( (x, y) = (x', y') \).
For surjectivity, consider
\[ \Gamma'(x, y) = \left( \frac{x}{\sqrt{1 + |x|^2}}, \frac{y}{\sqrt{1 + |x|^2}} \right). \]
\( \Gamma' \) is smooth everywhere. Direct calculation shows for \( |y| = 1 \) we have
\[ |\Gamma'(x, y)| = 1, \Gamma \circ \Gamma'(x, y) = (x, y). \]
This implies that \( \Gamma \) is surjective.
By Theorem 4.14 in \cite{32}, we deduce that \( \Gamma \) is a diffeomorphism when restricted to \( S^n \setminus \{ |y| = 0 \} \). Moreover, the inverse diffeomorphism is defined by \( \Gamma' \).
Reducing by the symmetry group \( O(n + k) \times O(l + 1) \), we deduce that the tubular neighborhoods around \( \{0\} \times S^l \) are implicitly defined by
\[ U_r = \{(x, y)||x|^2 + |y|^2 = 1, |x| < \sin r \}. \]
Direct calculation shows that \( U_{\frac{r}{4}} \) is mapped by \( \Gamma' \) bijectively to \( B_1^{n+k} \times S^l \).
Let \( v \) be a point in \( C \) with \( |v| \leq 1 \). For any point \( |y| = 1 \), we have
\[ \Gamma'(v, y) = \left( \frac{v}{\sqrt{1 + |v|^2}}, \frac{y}{\sqrt{1 + |v|^2}} \right) \in C \times \mathbb{R}^{l+1}, \]
by the homogeneity of \( C \). Similarly, for \( |y| \leq 1 \), we have
\[ \Gamma(\sqrt{1 - |y|^2} v, y) = \left( \frac{1 - |y|^2}{|y|} v, \frac{y}{|y|} \right) \in C \times S^l. \]
Thus, we deduce that \( \Gamma \) sends \( \sigma'(C) \cap U_{\frac{r}{4}} \) to \( C \times S^l \). \( \square \)
4.2. Non-smoothable singularities. Let $C$ be an integral or mod 2 area-minimizing cone of dimension $n$ in $\mathbb{R}^{n+k}$. In this section, we always assume the following:

(i) the link of $C$ is non-trivial in the oriented bordism ring if $C$ is an integral current;
(ii) or the link of $C$ is non-trivial in the unoriented bordism ring if $C$ is a mod 2 current;
(iii) or $n \geq 6$ is even and $C = \{0\} \times \mathbb{R}^\frac{n}{2} + \mathbb{R}^\frac{n}{2} \times \{0\}$ as an integral or mod 2 current, where we decompose $\mathbb{R}^n = \mathbb{R}^\frac{n}{2} \times \mathbb{R}^\frac{n}{2}$.

Remark 24. By Theorem 2 in [49] and the explanation of $g = 3$ case on page 370 therein, there exists an embedding of $\mathbb{C}P^2$ into $S^7$, so that the cone $C(\mathbb{C}P^2)$ over it satisfies Lawlor’s curvature criterion (1.3.5 Theorem in [30]), thus area-minimizing both as integral and mod 2 current in $\mathbb{R}^8$. Note that $\mathbb{C}P^2$ cannot bound any orientable or unorientable manifold by Euler characteristic arguments or Stiefel-Whitney class calculations as in [38]. Thus, the cone $C(\mathbb{C}P^2)$ satisfies both the first and the second assumptions.

Remark 25. By [63], the special Lagrangian cones over Wu-manifold links $SU(n)/SO(n)$ for $n \geq 3$ satisfy the first two conditions. (The cohomology ring of $SU(n)/SO(n)$ has been calculated in [35]. The Stiefel-Whitney number is nonzero as a result, thus giving nontrivial bordism class by [50].)

Lemma 4.2. Suppose we have an area-minimizing integral or mod 2 current $T$ on a smooth Riemannian manifold $(M, g)$ satisfying the following properties,

- $T$ is the unique minimizer in its homology class;
- there exists a smooth diffeomorphism $\Phi$ sending a smooth open set $U$ in $M$ to $B^{n+k} \times \Sigma$, with $\Sigma$ a smooth compact closed manifold of dimension $l$;
- $\Phi_\#(T|_U) = (C \cap B^{n+k}) \times \Sigma$ with $C$ satisfying any of the assumptions (i) to (iii).

Then there exists an open set $\Omega[T]$ in the space of Riemannian metrics so that any homological area-minimizers homologous to $T$ with metrics in $\Omega[T]$ are singular. Moreover, the singular set has Hausdorff dimension at least $l$.

Proof. We stay in $B^{n+k} \times \Sigma$ using $\Phi$. In the last case where $C$ is a sum of orthogonal planes, this is a direct consequence of Theorem 4.1 for the integral current case and Theorem 1.4 for mod 2 current case. For $C$ integral with a non-bounding link, by Lemma 2.13 there exists an open set $\Omega[T]$ in the space of Riemannian metrics on $M$ containing $g$, so that any homological area-minimizer $T'$ homologous to $T$ is smooth and graphical over $T$ outside of $(\frac{1}{10}B^{n+k}) \times \Sigma$. Consider the function $f : B^{n+k} \times \Sigma \to \Sigma$ defined by projection into the $\Sigma$ factor. The function $f$ slices $T$ into $C \cap B^{n+k}$. Now suppose $T'$ is smooth near some slice $f^{-1}(v)$ for $v \in \Sigma$. Then by Sard’s theorem (Theorem 6.10 in [32]), for $l$-dimensional a.e. $v'$ near $v$, we deduce
that $f^{-1}(v')$ will slice $T'$ into a smooth submanifold that is graphical over $C \backslash B^{n+k}$ outside of a neighborhood of the conical point. However, this implies that we have a smooth manifold bounded by the link of $C$, which is a contradiction. Thus, the singular set of $T'$ must intersect each slice of $f$. By 3.2.30 in [17], we deduce that the singular set of $T'$ must have Hausdorff dimension at least $l$. For $C \mod 2$ with a non-bounding link, the reasoning is the same. \hfill \Box

5. Proof of Theorem 1.5

In this subsection, we will give a gluing construction for mod 2 currents which is a retraction version of the results in [61]. First, we need a fundamental lemma that turns the constructions in [61] into retractions.

**Lemma 5.1.** Suppose we have a $C^1$ retraction $\Psi$ from a neighborhood of a smooth oriented $d$-dimensional submanifold $\Sigma$ to $\Sigma$. Let $\omega$ denote the volume form of $\Sigma$. Then $\Psi$ is $d$-dimensional area non-increasing at a point $p$ if and only if $\Psi^* \omega$ has comass no larger than 1 at $p$.

**Remark 26.** The lemma holds if $\Psi$ is Lipschitz, $\Sigma$ is a sufficiently smooth current, $\Sigma$ carries a suitable volume form $\omega$ and $\Psi^* \omega$ can serve as a calibration in the sense of Lemma 2.2. In fact, Lawlor’s vanishing calibration is precisely the pullback of the volume form under Lawlor’s retraction.

**Proof.** By definition of retraction, the image of $d\Psi$ lies in the tangent spaces to $\Sigma$. Thus, for any $d$-dimensional unit simple vector $P$, we have

$$\Psi^* \omega(P) = \omega(\Psi_* P) = \pm \sqrt{\langle \Psi_* P, \Psi_* P \rangle}.$$

Thus, the comass of $\Psi^* \omega$ is precisely the maximum of $\sqrt{\langle \Psi_* P, \Psi_* P \rangle} / \sqrt{\langle P, P \rangle}$. \hfill \Box

Now we are ready to give a proof of Theorem 1.5. We will first prove that we can make $T$ area-minimizing in a neighborhood of its support by using area non-increasing retractions, then use the constructions in Section 4 of [61] to find global smooth metrics.

5.1. The normal bundle retraction for the smooth part. To facilitate clearer presentations of the proof of Theorem 1.5 we will omit the subscript $j$ from $\Sigma_j, n_j$, etc. We initiate by constructing a base metric $g_0$. We assign each $\Sigma$ with an arbitrary metric and equip $\mathbb{R}^n$ with the standard flat metric. $\mathbb{R}^n \times \Sigma$ is endowed with the product metric. Using a partition of unity, we provide $M$ with a smooth metric $g$ such that each connected component of a smaller open set $U_j$ is exactly $B^d_j \times \Sigma$ and $T$ is $C \times \Sigma$.

Let $\pi_T$ denote the projection in the normal bundle of the smooth part of $T$. Note that as long as we stay away from the singular set, we can find tubular neighborhoods of uniform radius around $T$, so that $\pi_T$ exists.
Around any point \( p \) on the regular part of \( T \), locally there exists a volume form \( \omega \) of \( T \).

**Remark 27.** **Warning:** the regular part of \( T \) can be unorientable, so volume forms only exist locally up to a sign. From now on, by volume forms, we always mean such local volume forms up to a sign.

Denote

\[ \kappa = \text{comass } \pi_T^* \omega \]

in a tubular neighborhood of \( T \). Note that even if \( T \) is not orientable, \( \kappa \) is still smooth. No matter the orientation of \( T \), the comass of \( \pi_T^* \omega \) equals \( |\pi_T^*(\ker \pi_T^\perp)| \) i.e., reached on the orthogonal complement to the kernel of \( \pi_T \) properly oriented, which varies smoothly.

Then \( \pi_T \) becomes area non-increasing in the metric

\[
(5.1) \quad g_1 = \kappa^2 g_0.
\]

As there always exists a local volume form \( \omega \) on \( T \), by Lemma 3.5 in \cite{60}, \( \pi_T^* \omega \) is a calibration form in \( g_1 \). Following this, Lemma 5.1 completes this step. Consequently, we obtain a smooth area-non-increasing retraction \( \pi_T \) in the smooth metric \( g_1 \) on the regular part of \( T \).

### 5.2. Cases of \( \Sigma \) a point, link of \( C \) orientable.

We will first address the case where each \( \Sigma \) is a point and \( C \) is orientable of dimension \( m \). This implies that the diffeomorphism maps \( T \) around \( S_j \) to a cone \( C \subset B^n_1 \) with an orientable link. The proof of the general case will be presented in the subsequent subsection.

Let us fix some notations. Use \( m \) to denote the dimension of \( C \). For any point \( p \in B^n_1 \), decompose \( p \) as

\[ p = x + \nu, \]

with \( x \) the nearest projection into the cone, and \( \nu = p - x \) the normal vector obtained after subtracting the projection. Denote \( r = |x| \), and \( t = |\nu| \).

By our assumption in \( U_j \) we have Lawlor’s area non-increasing retraction of the cone \( C \). We just need to glue it to the normal bundle retraction \( \pi_T \). Let us first review the constructions in Section 5 of \cite{60}. For singularities being precisely cones, Zhang first uses a modified version of Lawlor’s calibration with

\[ \phi_1 = d\tilde{\psi}_1 = d(f_1 \omega^* \psi), \]

with \( f_1 \) a modified version of \( g_{(m,a)}(t) \), \( d\omega_C \) the volume form of the cone and

\[ \psi = \frac{r}{m} \partial_t d\omega_C \]

the volume form of the link times \( \frac{r}{m} \). By construction, we have

\[
(5.2) \quad d\psi = d\omega_C,
\]
on $C$, where $dvol_C$ is the volume form of the cone (the first formula on page 186 of [61] or just by direct calculation using Cartan’s magic formula $\text{d}V + i\text{d}V = \mathcal{L}_V$).

Zhang then glues the form into the pullback of the volume form under the normal bundle exponential map. Consider

$$\Phi = [\tau(r)(f_1\varpi^\ast\psi) + (1 - \tau(r))\varpi^\ast\psi],$$

with $\tau(r)$ a smooth transition function from 1 to 0, with $r$ the length of the projection to the cone.

Finally, he modifies the metric into a smooth metric $g_2$ so that $\Phi$ becomes a calibration. An important point to note is that the metric patches smoothly to $g_1$ when $\tau$ reaches zero (the fourth line from the bottom of page 186 of [61] with $\sigma = 1$). Thus, define a smooth metric $g$ around $T$ by setting

$$g = \begin{cases} g_2 & \text{on } U_j \\ g_1 & \text{on } \cup_j U_j^C. \end{cases}$$

Now define a retraction in $U_j$ to the cone $C$ as follows:

$$\Pi_0 : x + \nu \mapsto (f_2(t))^{\frac{\nu}{\|\nu\|}} x,$$

with

$$f_2(t) = \tau(r)f_1(t) + (1 - r)$$

We have the following property.

Lemma 5.2. $\Pi_0$ is area non-increasing in the metric $g$.

Proof. In view of Lemma 5.1 and the construction of $g$, it suffices to prove that

$$\Pi_0^*dvol = \Phi.$$

Thus, by naturality of exterior derivative and (5.2), this is equivalent to proving that

$$\Pi_0^*(\frac{\nu}{m}i_\partial, dvol_C) = f_2\varpi^\ast(\frac{\nu}{m}i_\partial, dvol_C).$$

First, let us decompose the tangent space at any point $p \in B^m_1$ into subspaces $TC$ the tangent space to $C$ at $x$, and subspaces $T^+C$ the normal space to $C$ at $x$. Then use $e_2, \ldots, e_m$ to denote the orthonormal basis of $TC/x$ and $v_2, \ldots, v_{n-m}$ to denote the orthonormal basis of $T^+C/v$. Since $C$ is a cone we can make $e_2, \ldots, e_d, v_2, \ldots, v_c$ the same along any ray.

Then we have

$$dvol_C = e_2^\ast \wedge \cdots \wedge e_m^\ast.$$

Use $e_1$ to denote $\frac{\nu}{\|\nu\|}$. Then we have

$$A_C(e_1, e_j) = 0,$$

with $A_C$ the second fundamental form of $C$, by 1-homogeneity of $C$. 

By 2.3.2 Lemma of [30], we have
\[ d\varpi(Y) = [I - \langle A_C(e_i, e_j), \nu \rangle]^{-1}(TC(Y)), \]
for any vector \( Y \), where the factor is regarded as the inverse of a matrix and we express the projection of \( Y \) into \( TC \), i.e., \( TC(Y) \), as a column vector in basis \( e_1, \cdots, e_n \). However, the matrix \( [\langle A_C(e_i, e_j), \nu \rangle] \) is all zero in the first row and first column. Thus, the matrix factor \( [I - \langle A_C(e_i, e_j), \nu \rangle]^{-1} \) has is zero in the first row and the first column except for the (1, 1) element being 1. This implies that \( d\varpi \) sends \( e_1 \) to \( e_1 \) and \( TC/\partial r \) to \( TC/\partial_r \).

On the other hand, \( f_2 \) has only derivatives in the \( e_1 \) direction and \( \nu/|\nu| \) direction. Thus, from
\[ d\Pi_0(Y) = d\left( f_2^{1/2} \varpi \right)(Y) = d(f_2^{1/2})(Y)x + f_2^{1/2}d\varpi(Y), \]
we deduce that
\[ d\Pi_0(\frac{r}{m}e_2^* \wedge \cdots \wedge e_m^*) = f_2^{1/2}d\varpi(e_j), \]
for \( j \neq 2 \). Thus for any vectors \( w_2, \cdots, w_m \subset \text{span}\{e_2, \cdots, e_m\} \), we have
\[ \Pi_0^*(\frac{r}{m}e_2^* \wedge \cdots \wedge e_m^*)(v_2, \cdots, v_m) = f_2^{1/2}\frac{|x|}{m}e_2^* \wedge \cdots \wedge e_m^*(d\Pi_0(v_2), \cdots, d\Pi_0(v_m)) \]
\[ = f_2\frac{|x|}{m}e_2^* \wedge \cdots \wedge e_m^*(d\varpi(v_2), \cdots, d\varpi(v_m)) \]
\[ = f_2\varpi^*(\frac{r}{m}i_{\partial_r}dvol_C). \]

We are done.

Thus, \( \Pi_0 \) is area non-increasing on \( U_j \). However, by construction, \( \Pi_0 \) patches smoothly to \( \pi_T \) on near \( \partial U_j \).
Define
\[ \overline{\Pi} = \begin{cases} 
\Pi_0 & \text{on } U_j \\
\pi_T & \text{on } \cup_j U_j^C \end{cases}, \]
then \( \overline{\Pi} \) is a Lipschitz area non-increasing retraction onto \( T \) in the metric \( g \) defined both in a neighborhood around \( T \).
5.3. The general case. First, if $C$ has a smooth link, but is unorientable, then we no longer have well-defined volume forms of $C$ globally. However, locally $dvol_C$ is always well-defined in small enough conical regions up to a sign. We can still carry out the argument in the previous section, and deduce the existence of a smooth metric $g$ around $T$ and an area non-increasing retraction $\Pi$ onto $T$.

Now we are left to deal with the cases where $T$ becomes $C \times \Sigma$ for nontrivial $\Sigma$ in $U_j$, i.e., $\dim \Sigma \geq 1$. We can take the projection $\Pi_0$ in the previous subsection in $B^n_1$, to define

$$\Pi'_0 = (\Pi_0 \circ \pi_B, \pi_\Sigma),$$

where $\pi_B$ is the projection onto the $B^n_1$ factor and $\pi_\Sigma$ is the projection onto the $\Sigma$ factor. By construction $\Pi'_0$ is a retraction of $U_j$ onto $C \times \Sigma$. If $dvol_\Sigma$ denotes the local volume forms of $\Sigma$, then

$$\pi_B^* \Phi \land \pi_\Sigma^* dvol_\Sigma,$$

have comass at most 1 in the metric $g \times g_\Sigma$, by Proposition 7.10 in [22]. Here $g_\Sigma$ is the metric on $\Sigma$ factor and $dvol_\Sigma$ is the volume form locally of $\Sigma$. However, by the product structure, we have

$$\Pi'_0(dvol_T) = \Pi'_0(dvol_C \land dvol_\Sigma) = \pi_B^* \Phi \land \pi_\Sigma^* dvol_\Sigma.$$

Again, we regard every volume form only as local forms, instead of defined globally. By Lemma 5.1 this implies that $\Pi'_0$ is an area non-increasing retraction.

Since the normal vectors to $T$ lie completely in $n$-dimensional subspaces tangent to the factor $B^n_1$, this implies that geodesics starting from $T$ remain in the $B^n_1$ factor and are just geodesics in $B^n_1$ by Proposition 38 of Chapter 7 in [33]. Thus, by construction, $\Pi'_0$ transits smoothly to the projection $\pi_T$ associated with the normal bundle of $T$ near $\partial U_j$.

Now equip the smooth part of $T$ with the normal bundle projection in Section 5.1 and change the metric accordingly in $\cap_j U_j^c$. While the projection $\pi_T$ transit smoothly to $\Pi'_0$, the metric does not. Since outside of $U_j$, the metric has a conformal factor $\kappa \frac{m-s}{m}$ in the notation of Section 5.1 (if the dimension of $\Sigma$), while near the boundary inside $\partial U_j$, the metric is unchanged in the $\Sigma$ factor while having a conformal factor $\kappa \frac{m}{m}$ in the $B^n_1$ factor.

To resolve this, let $\lambda$ denote the distance to the $\Sigma$ factor in $U_j$ and $\lambda_m$ denote its maximum in $U_j$. By our construction, $\Pi'_0$ becomes $\pi_T$ after some $\lambda'_m < \lambda_m$. Let $\gamma$ be a non-negative smooth transition function that is 0 on $[0, \lambda'_m]$ and 1 on $[\lambda'_m + \lambda_m, \lambda_m]$. Then consider the new metric $g'$ in $U_j$,

$$g' = \kappa \frac{m}{m} \gamma(\lambda + (1-\gamma(\lambda)) \frac{2}{m-s} g_2 + \kappa \left(1 - \frac{m}{2} (\frac{2}{m} \gamma(\lambda) + (1-\gamma(\lambda)) \frac{2}{m-s}) \right) g_\Sigma.$$ 

For $\gamma = 1$, $g' = (\frac{2}{m} g) \times g_\Sigma$. For $\gamma = 0$, $g' = \kappa \frac{m-s}{m} (g \times g_\Sigma)$. For any $m+s$-dimensional plane $P_1 \times P_2$, so that $P_1$ is tangent to $B^n_1$, $P_2$ is tangent to $\Sigma$, ...
we have
\[ \frac{|P_1 \times P_2|_{g'}}{|P_1 \times P_2|_{\kappa^{2/\ell} g \times g_{\Sigma}}} = \left( \frac{\kappa \frac{2}{\ell} \gamma(\lambda) + (1 - \gamma(\lambda)) \frac{2}{m+2}}{\kappa} \right) \left( \frac{1 - \frac{m}{2} \frac{2}{\ell} \gamma(\lambda) + (1 - \gamma(\lambda)) \frac{2}{m+2}}{\kappa} \right) = 1. \]
Thus, for product planes, the area has not changed. However, \( \pi_T^* (dvol_T) = \pi_B^* dvol_C \wedge \pi_T^* dvol_{\Sigma} \) decomposes as a product. (Again, all forms are understood to exist only locally.) Using Proposition 7.10 in [22], we deduce that its comass is still at most 1 in \( g' \). Thus, the associated projection is area non-increasing in the transition region by Lemma 5.1. In the non-transition region everything is not changed, thus remains area non-increasing. Now define
\[
g = \begin{cases} 
g' & \text{on } U_j \\
g_1 & \text{on } \cup_j U_j^C.
\end{cases}
\]
Then \( g \) is smooth by construction. If we define
\[
\Pi = \begin{cases} 
\Pi'_0 & \text{on } U_j \\
\pi_T & \text{on } \cup_j U_j^C.
\end{cases}
\]
Then \( \Pi \) is an area non-increasing Lipschitz retraction onto \( T \) in \( g \).

5.4. Conclusions. To sum it up, we have proven that for \( T \) satisfying the assumptions in Theorem 1.5, there exists a neighborhood \( U \) of \( T \), a smooth metric \( g \) in \( U \) and an area non-increasing Lipschitz projection \( \Pi \). Then invoke Lemma 2.9 to deduce the existence of an open set \( V \subset U \) containing the support of \( T \), so that \( V \) deformation retracts onto \( T \). Flow \( V \) with respect to the inward normal for a short time to induce a smooth open set \( V' \subset V \). \( V' \) also deformation retracts onto \( T \) by composing the retraction of \( V \) with the flow sending \( V \) to \( V' \). Now apply Section 4 of [61] up to a constant conformal factor, to deduce the existence of a smooth metric \( \overline{g} \) that coincides with \( g \) on the smaller open set \( V' \subset V \), so that any area-minimizing mod 2 current \( \overline{T}' \) in the homology class \([T]\) lies in \( V' \). Since \( V' \) deformation retracts onto \( T \), we deduce that \( T' \) must have the same homology class as \( T \) in the region \( V' \). Consider \( \Pi_* T' \). It must be a cycle contained in \( T \) and share the same homology class as \([T]\). However, since \( \Pi \) is area non-increasing, we deduce that the area of \( T' \) is at least that of \( T \).

Note that small tubular neighborhoods around the singular set retract to the \( \Sigma \), so they must have vanishing homology in the dimension of the current. Thus, no current homologous to \( T \) can be supported near the singular set. Now apply the reasoning in Section 2.7.4 in the proof of Lemma 2.7 to make \( T \) the unique minimizer by adding positive bumps to the area away from \( T \). We are done.
In this section, we will prove Theorem 1.1. First of all, we need a reduction to the model cases.

**Lemma 6.1.** Theorem 1.1 holds if we can prove it for cases where the homology class $[\Sigma]$ has a representative that satisfies the assumptions in Lemma 1.3.

**Proof.** Note that a smooth non-orientable submanifold cannot support an integral current with no boundary. Otherwise, suppose a non-orientable submanifold $N$ supports an integral current $T$ with no boundary. By Theorem 1 (6) in [36], there exists a smooth hypersurface $\Sigma$ so that $N \setminus \Sigma$ is orientable. By Constancy Theorem on page 357 of [17], we deduce that $T$ restricted to $N \setminus \Sigma$ equals integer multiples of it. However, $\partial (N \setminus \Sigma) = 2\Sigma$ by construction. This is a contradiction.

Now suppose $[\Sigma]$ has a representative $\Sigma'$ of the homology class so that $\Sigma' = a_1\Sigma'_1 + a_2\Sigma'_2 + \cdots + a_l\Sigma'_l$, where each $\Sigma'_j$ are connected orientable submanifolds counted with multiplicity $a_j > 0$, and they are pairwise disjoint. If we regard $\Sigma$ as an immersion of a surface with many connected components, then with a small perturbation, we can destroy the multiplicities almost everywhere and get a surface $\Sigma''$ homologous to $\Sigma'$ with $\Sigma'' = \Sigma_1 + \cdots + \Sigma_l$, and all the summands are in general position. Since the codimension is larger than 1, by transversality, we can do successive connected sums of $\Sigma_i$ to $\Sigma$, getting a generic immersion of a connected oriented surface $\Sigma$, homologous to $\Sigma'$. (Just connect multiplicity 1 points using a curve, which will avoid intersecting except for endpoints by transversality. Then widen the curve to get a neck that does the job.) By construction, the structure of the current near the self-intersection set satisfies the assumptions in Lemma 1.3. □

**Remark 28.** By [50], some $[\Sigma]$ cannot be represented by continuous maps from smooth manifolds.

Thus, we can assume that $[\Sigma]$ has a multiplicity 1 a.e. representative that is smoothly calibrated in its neighborhood in some smooth metric.

### 6.1. The case of $c \leq d$.

The idea is that we can insert a transverse intersection into the multiplicity one surface $\Sigma$. Choose $C$ to be the sum of two transversal orthogonal $c$-dimensional planes in $\mathbb{R}^{2c}$. Then consider $\sigma^{d-c}(C)$ in Lemma 4.1. It is embedded in $S^{d+c}$. Then we do a simultaneous connected sum of $\Sigma$ with $\sigma^{d-c}(C)$ on the smooth part and $M$ with $S^{d+c}$. Since we are connected summing with standard sphere, $M$ is diffeomorphic to $M \# S^{d+c}$ and by homotopy invariance, we have $[\Sigma \# \sigma^{d-c}(C)] = [\Sigma]$ as homology classes. To make the regular part connected, we do a further connected sum of the two summands of $\sigma^{d-c}(C)$.

Note that $\phi_C = dx_1 \wedge \cdots \wedge dx_c + dy_1 \wedge \cdots \wedge dy_c$ calibrates $C$. Thus, $\phi_C \wedge \pi^*_{S^{d-c}} dvol_{S^{d-c}}$ calibrates $C \times S^{d-c} \subset \mathbb{R}^{2c} \times S^{d-c}$ with product metric by
Proposition 7.10 in [22]. Here \( \pi_{S^{d-c}} \) is the projection into the spherical factor and \( d\text{vol}_{S^{d-c}} \) the volume form of \( S^{d-c} \) extended constantly over the product. By Lemma 4.1, a tubular neighborhood around the singular set of \( \sigma^l(C) \) is calibrated by a smooth form in a smooth metric.

Now apply Lemma 2.7. We deduce that in a smooth metric \( \Sigma^l \sigma^l(C) \) is the unique minimizer. Then Lemma 4.2 finishes the proof.

6.2. The case of \( c \geq d \geq 5 \). Here we choose \( C \) to the cone over \( \mathbb{CP}^2 \) mentioned in Remark 24. Instead of using the inclusion \( C \subset \mathbb{R}^8 \), we use \( C \subset \mathbb{R}^{8+m} \) with \( m \geq 0 \). Then \( \sigma^l(C) \subset S^{8+m+l} \). Now connect sum \( \Sigma \) with \( \sigma^l(C) \subset S^{8+m+l} \). \( C \) is still calibrated by Lawlor’s vanishing calibration \( \phi_C \) in \( \mathbb{R}^{8+m} \), so we have an \( L^\infty \) calibration \( \phi_C \wedge \pi^*_g d\text{vol}_{S^l} \) satisfying the assumptions of Lemma 2.2 on \( U_{\pi^4} \times S^l \) in the notation of the proof of Lemma 4.1. Moreover, the calibration is smooth near the boundary of the region \( U_{\pi^4} \times S^l \). The rest is the same as the previous subsection.

Now set \( d = 5 + l \) and \( c = 3 + m \), which are precisely the dimensions and codimensions of \( \sigma^l(C) \). With \( l, m \geq 0 \), we cover all \( c \geq 3, d \geq 5 \).

6.3. Dimension lower bound of the singular set. This is a straightforward consequence of Lemma 4.2 and dimension counting. For \( c \geq d \geq 5 \), \( \max\{d-c, d-5\} = d-5 \), note that the singularity in the last subsection has precisely dimension \( d-5 \). For \( 3 \leq c \leq d, d \geq 5 \), we do the construction of the previous two subsections simultaneously. This gives the \( \max\{d-c, d-5\} \) dimension lower bound on the singular set. For \( d \leq 4 \), we are left with only intersection type singularities, but \( \max\{d-c, d-5\} = d-c \) gives the dimension of the singular set correctly. We have covered all cases. We are done.

7. Proof of Theorem 1.4

Roughly speaking the proof of Theorem 1.4 is the same as the proof for integral homology cases. We just need to replace calibrations with retractions.

7.1. Proof of mod 2 version of Theorem 1.2 and Theorem 3.1 The proof of Theorem 1.2 relies on a cut-and-paste argument. For nearby minimizers, we cut away the surface near the singular set and substitute the sum of two minimal embeddings by Allard regularity. Then we use two vanishing calibrations with support intersecting only at the intersection to calibrate the sum. This implies the sum is just the original minimizer by unique continuation. For the mod 2 case, the only argument that needs modification is the vanishing calibration. In the non-orientable case, the calibration form does not exist. However, the projection defined in the formula (3.1) still exists.

Lemma 7.1. \( \Pi \) is area non-increasing on \( n + k \)-dimensional planes.
Proof. Note that the projection sends the points with $t \geq \theta(n,a)$ to the intersection set, so this holds trivially in that region. For $t \leq \theta(n,a)$, the tangent space to the level sets of $\Pi$ are annihilated by $\Pi$. Moreover, dimension counting gives that the orthogonal complement $P$ to the tangent of level sets has precisely dimension $n + k$. Thus, it suffices to show that $\langle \Pi_*P, \Pi_*P \rangle / \langle P, P \rangle \leq 1$. The same argument as in the orientable case shows that $P$ is spanned by the two lines of vectors following (3.10). The same calculation gives the length of $P$ the same in (3.14). Direct calculation gives
\[ d\Pi(\partial_l) = \partial_l, d\Pi(v_a) = 0, d\Pi(e_b) = (g(n,a))^{\frac{1}{n}} e_b, \]
\[ d\Pi(\partial_r) = g(n,a)^{\frac{1-n}{n}} c \partial_r, d\Pi(\partial_z) = g(n,a)^{\frac{1-n}{n}} s \partial_r, \]
in the notation of (3.10). This yields
\[ \Pi_*P = \partial_r \wedge e_2 \wedge \cdots \wedge e_n \wedge dl_1 \wedge \cdots \wedge dl_k. \]
Thus, we have
\[ \langle \Pi_*P, \Pi_*P \rangle = \frac{1}{\lambda \circ \pi} \]
in the notation of Lemma 3.2. This gives the same upper bound of $\sqrt{\langle \Pi_*P, \Pi_*P \rangle / \langle P, P \rangle}$ as (3.15), which is smaller than 1 provided with small $r$. □

Again this retraction depends only on the geometry of the objects involved. Thus, we can define a retraction to the sum of the two pasted submanifolds using the juxtaposition of the two retractions, which remain a valid retraction by the angle conditions. By area non-increasing property, we deduce that the pasted pieces are also minimizing. Now just argue by unique continuation. We are done.

7.2. Proof of mod 2 version of Theorem 1.1. In the mod 2 case, if a homology class admits smooth representatives in the current sense, then it must be represented by sums of embedded submanifolds with multiplicity one. Now connect summing them together gives a smoothly embedded connected representative $\Sigma$. This gives us the homology representative to glue to.

The rest of the reasoning is the same with Section 7 by using Theorem 1.5 to glue these singularities to the homology class representatives and use necks to connect all the components.

8. Proof of Corollary 1.1

Just apply the construction in the previous section to the chosen cone $C$ replacing the cone over $\mathbb{CP}^2$. 
References

[1] William K. Allard, *First Variation of a Varifold*, Annals of Mathematics Second Series, Vol. 95, No. 3 (May, 1972), pp. 417-491.

[2] William K. Allard, *First Variation of a Varifold: Boundary Behavior*, Annals of Mathematics Second Series, Vol. 101, No. 3 (May, 1975), pp. 418-446.

[3] Some open problems in geometric measure theory and its applications suggested by participants of the 1984 AMS summer institute. Edited by J. E. Brothers. Proc. Sympos. Pure Math., 44, Geometric measure theory and the calculus of variations (Arcata, Calif., 1984), 441–464, Amer. Math. Soc., Providence, RI, 1986.

[4] Frederick J. Almgren, Jr. *Almgren's big regularity paper. Q-valued functions minimizing Dirichlet's integral and the regularity of area-minimizing rectifiable currents up to codimension 2.* With a preface by Jean E. Taylor and Vladimir Scheffer. World Scientific Monograph Series in Mathematics, 1. World Scientific Publishing Co., Inc.

[5] Sheldon Xu-Dong Chang, *Two-dimensional area minimizing integral currents are classical minimal surfaces*, J. Amer. Math. Soc. 1 (1988), no. 4, 699–778.

[6] Benny N. Cheng *Area-minimizing cone-type surfaces and coflat calibrations*, Indiana Univ. Math. J. 37 (1988), no. 3, 505–535.

[7] J. Dadok, R. Harvey and F. Morgan, *Calibrations on ,* Transactions of the American Mathematical Society, May 1988, Vol. 307, No. 1 (May, 1988), pp. 1-40

[8] C. De Lellis; G. De Philippis; J. Hirsch; A. Massaccesi, *On the boundary behavior of mass-minimizing integral currents*, available at https://www.math.ias.edu/delellis/node/148.

[9] C. De Lellis; E. Spadaro, *Regularity of area-minimizing currents I: $L^p$ gradient estimates*, Geom. Funct. Anal. 24 (2014), no. 6.

[10] C. De Lellis; E. Spadaro. *Regularity of area-minimizing currents II: center manifold*, Ann. of Math. (2) 183 (2016), no. 2, 499–575.

[11] C. De Lellis; E. Spadaro, *Regularity of area-minimizing currents III: blow-up*, Ann. of Math. (2) 183 (2016), no. 2, 577–617. Geom. Funct. Anal. 24 (2014), no. 6, 1831–1884.

[12] C. De Lellis; E. Spadaro; L. Spolaor, *Uniqueness of tangent cones for 2-dimensional almost minimizing currents*, Comm. Pure Appl. Math. 70, 1402-1421

[13] C. De Lellis; E. Spadaro; L. Spolaor, *Regularity theory for 2-dimensional almost minimal currents I: Lipschitz approximation*, Trans. Amer. Math. Soc. 370 (2018), no. 3, 1783–1801

[14] C. De Lellis; E. Spadaro; L. Spolaor, *Regularity theory for 2-dimensional almost minimal currents II: branched center manifold*, Ann. PDE 3 (2017), no. 2, Art. 18, 85 pp.

[15] C. De Lellis; E. Spadaro; L. Spolaor, *Regularity theory for 2-dimensional almost minimal currents III: blowup*, to appear in Jour. Diff. Geom.

[16] Lawrence C. Evans, *Partial differential equations*, Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 1998.

[17] Herbert Federer, *Geometric Measure Theory* Springer, New York, 1969.

[18] Herbert Federer, *The singular sets of area minimizing rectifiable currents with codimension one and of area minimizing flat chains modulo two with arbitrary codimension*, Bull. Amer. Math. Soc. 76 (1970), 767–771.

[19] M. Golubitsky; V. Guillemin, *Stable mappings and their singularities*, Graduate Texts in Mathematics, Vol. 14. Springer-Verlag, New York-Heidelberg, 1973. x+209 pp.

[20] Mark R. Goresky, *Triangulation of stratified objects* Proc. Amer. Math. Soc. 72 (1978), no. 1, 193–200.

[21] Alfred Gray *Tubes.* Second edition. With a preface by Vicente Miquel. Progress in Mathematics, 221. Birkhäuser Verlag, Basel, 2004.

[22] Reese Harvey; H. Blaine Lawson, Jr. *Calibrated geometries*. Acta Math. 148 (1982),
[23] Reese Harvey; H. Blaine Lawson, Jr. Calibrated foliations (foliations and mass-minimizing currents). Amer. J. Math. 104 (1982), no. 3, 607–633.
[24] Mark Haskins; Tommaso Pacini, Obstructions to special Lagrangian desingularizations and the Lagrangian prescribed boundary problem, Geom. Topol. 10 (2006), 1453–1521.
[25] Allen Hatcher, Algebraic topology, Cambridge University Press, Cambridge, 2002.
[26] Ralph J. Herbert, Multiple points of immersed manifolds, Mem. Amer. Math. Soc. 34 (1981), no. 250, xiv+60 pp.
[27] Morris W. Hirsch, Differential topology, Corrected reprint of the 1976 original. Graduate Texts in Mathematics, 33. Springer-Verlag, New York, 1994.
[28] Morris W. Hirsch, Smooth regular neighborhoods, Ann. of Math. (2) 76 (1962), 524–530.
[29] Jerry L. Kazdan, Unique Continuation in Geometry, Communications on pure and applied mathematics, 1988, Vol.41(5), p.667-681.
[30] Gary Lawlor, A sufficient criterion for a cone to be area-minimizing, Mem. Amer. Math. Soc. 91 (1991), no. 446.
[31] Gary Lawlor, The angle criterion, Invent. Math. 95 (1989), 437–446.
[32] John M. Lee, Introduction to smooth manifolds, Second edition. Graduate Texts in Mathematics, 218. Springer, New York, 2013. xvi+708 pp.
[33] Yangyang Li, Zhihan Wang, Minimal hypersurfaces for generic metrics in dimension 8, preprint available at arxiv.org/abs/2205.01047
[34] Zhenhua Liu, On a conjecture of Almgren: area-minimizing surfaces with fractal singularities, preprint available at arxiv.org/abs/2110.13137
[35] Mamoru Mimura, Hirosi Toda, Topology of Lie groups. I, II. Translated from the 1978 Japanese edition by the authors. Translations of Mathematical Monographs, 91. American Mathematical Society, Providence, RI, 1991.
[36] William H. Meeks III, Representing codimension-one homology classes on closed nonorientable manifolds by submanifolds, Illinois J. Math. 23 (1979), no. 2.
[37] John Mather, Notes on topological stability, (English summary) Bull. Amer. Math. Soc. (N.S.) 49 (2012), no. 4, 475–506.
[38] John W. Milnor;James D. Stasheff, Characteristic classes, Annals of Mathematics Studies, No. 76. Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1974.
[39] Frank Morgan, On the singular structure of three-dimensional, area-minimizing surfaces, Trans. Amer. Math. Soc. 276 (1983), no. 1, 137–143.
[40] Charles B. Morrey, Jr. Multiple integrals in the calculus of variations. Classics in Mathematics. Springer-Verlag, Berlin, 2008.
[41] Jürgen Moser, On the volume elements on a manifold, Trans. Amer. Math. Soc. 120 (1965), 286–294.
[42] Aaron Naber, Daniele Valtorta, The singular structure and regularity of stationary varifolds, J. Eur. Math. Soc., Volume 22, Issue 10, 2020
[43] Barrett O’Neill, Semi-Riemannian geometry, Pure and Applied Mathematics, 103. Academic Press, Inc.
[44] Günter Schwarz, Hodge decomposition—a method for solving boundary value problems. Lecture Notes in Mathematics, 1607. Springer-Verlag, Berlin, 1995.
[45] Leon Simon, Lectures on Geometric Measure Theory, Proceedings for the Centre for Mathematical Analysis, Australian National University, Canberra, 1983.
[46] Leon Simon, Asymptotics for a class of nonlinear evolution equations, with applications to geometric problems, Ann. of Math. (2) 118 (1983), no. 3, 525–571.
[47] Nathan Smale, Generic regularity of homologically area minimizing hypersurfaces in eight-dimensional manifolds., Comm. Anal. Geom. 1 (1993), no. 2, 217–228.
[48] John R. Stallings, *Lectures on polyhedral topology*, Notes by G. Ananda Swarup. Tata Institute of Fundamental Research Lectures on Mathematics, No. 43 Tata Institute of Fundamental Research, Bombay 1967 iv+260 pp.

[49] Zizhou Tang, Yongsheng Zhang, *Minimizing cones associated with isoparametric foliations*, J. Differential Geom. 115(2): 367-393.

[50] R. Thom, *Quelques propriétés globales des variétés différentiables*, Comment. Math. Helv. 28 (1954), 17–86.

[51] David J. A. Trotman, *Geometric versions of Whitney regularity for smooth stratifications*, Ann. Sci. École Norm. Sup. (4) 12 (1979), no. 4.

[52] David Trotman, *Stratification theory. Handbook of geometry and topology of singularities. I*, 243–273, Springer

[53] Robert Young, *Quantitative nonorientability of embedded cycles*, Duke Math. J. 167(1): 41-108

[54] C. T. C. Wall, *Determination of the cobordism ring*, Ann. of Math. (2) 72 (1960), 292–311.

[55] C. T. C. Wall, *Differential topology*, Cambridge Studies in Advanced Mathematics, 156. Cambridge University Press, Cambridge, 2016.

[56] Brian White, *Tangent cones to two-dimensional area-minimizing integral currents are unique*, Duke Math. J. 50 (1983), no. 1, 143–160.

[57] Brian White, *Generic Transversality of Minimal Submanifolds and Generic Regularity of Two-Dimensional Area-Minimizing Integral Currents*, preprint available at [https://arxiv.org/abs/1901.05148](https://arxiv.org/abs/1901.05148).

[58] Brian White, *Generic regularity of unoriented two-dimensional area minimizing surfaces*, Ann. of Math. (2) 121 (1985), no. 3, 595–603.

[59] Brian White, *Stratification of minimal surfaces, mean curvature flows, and harmonic maps*, J. Reine Angew. Math. 488 (1997).

[60] Yongsheng Zhang, *On extending calibration pairs*, Adv. Math. 308 (2017), 645–670.

[61] Yongsheng Zhang, *On realization of tangent cones of homologically area-minimizing compact singular submanifolds*, J. Differential Geom. 109 (2018), no. 1, 177–188.

[62] *Complex Projective Space*, nLab, available at [https://ncatlab.org/nlab/show/complex+projective+space](https://ncatlab.org/nlab/show/complex+projective+space).

[63] Mathoverflow question about Wu manifolds, available at [https://mathoverflow.net/questions/422416/are-the-symmetric-spaces-operatornamesun-operatornameson-a](https://mathoverflow.net/questions/422416/are-the-symmetric-spaces-operatornamesun-operatornameson-a).

[64] Poincaré lemma, nLab, available at [https://ncatlab.org/nlab/show/Poincar%C3%A9+lemma](https://ncatlab.org/nlab/show/Poincar%C3%A9+lemma).