Orbital perturbations due to massive rings

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Abstract

We analytically work out the long-term orbital perturbations induced by a homogeneous circular ring of radius $R_r$ and mass $m_r$ on the motion of a test particle in the cases (I): $r > R_r$ and (II): $r < R_r$. In order to extend the validity of our analysis to the orbital configurations of, e.g., some proposed spacecraft-based mission for fundamental physics like LISA and ASTROD, of possible annuli around the supermassive black hole in Sgr A* coming from tidal disruptions of incoming gas clouds, and to the effect of artificial space debris belts around the Earth, we do not restrict ourselves to the case in which the ring and the orbit of the perturbed particle lie just in the same plane. From the corrections $\Delta \dot{\varpi}_{\text{meas}}$ to the standard secular perihelion precessions, recently determined by a team of astronomers for some planets of the Solar System, we infer upper bounds on $m_r$ for various putative and known annular matter distributions of natural origin (close circumsolar ring with $R_r = 0.02 - 0.13$ au, dust ring with $R_r = 1$ au, minor asteroids, Trans-Neptunian Objects). We find $m_r \leq 1.4 \times 10^{-4} \, m_{\oplus}$ (circumsolar ring with $R_r = 0.02$ au), $m_r \leq 2.6 \times 10^{-6} \, m_{\oplus}$ (circumsolar ring with $R_r = 0.13$ au), $m_r \leq 8.8 \times 10^{-7} \, m_{\oplus}$ (ring with $R_r = 1$ au), $m_r \leq 7.3 \times 10^{-12} \, M_{\odot}$ (asteroidal ring with $R_r = 2.80$ au), $m_r \leq 1.1 \times 10^{-11} \, M_{\odot}$ (asteroidal ring with $R_r = 3.14$ au), $m_r \leq 2.0 \times 10^{-8} \, M_{\odot}$ (TNOs ring with $R_r = 43$ au). In principle, our analysis is valid both for baryonic and non-baryonic Dark Matter distributions.

Keywords: Gravity - Asteroid Belt - Interplanetary Dust - Perturbation Methods

1 Introduction

The giant planets of the Solar System are surrounded by rings [1]. They have widely been studied and have been the object of numerous scientific spacecraft-based missions [2,3,4,5,6,7]. As we will see in more details in Section 2, there is the possibility that circumsolar massive rings also exist at small distances from the Sun [8,9]. Moreover, the dynamical action of the ensemble of the minor asteroids of the asteroid belt between Mars and Jupiter can roughly be modeled as due to a continuous ring [10]. The same holds for the belt [11,12,13] of Trans Neptunian Objects (TNOs) [14,15].

Recent observations suggest that similar structures exist in other stellar systems as well. [16] detected a ring-like structure at about 70 au from the Sun-type star HD 207129 [17] with the IR observations collected by the instruments PACS [18] and SPIRE [19] of the Herschel space observatory [20]. For studies on Kuiper belt-like structures at about 30–50 au or beyond
around other nearby Solar-type main-sequence stars, see, e.g., [21] and [22]. In particular, [21] detected 10 exo-Kuiper belts with masses ranging from to a minimum of \(0.3 \, m_\oplus\) to a maximum of \(48 \, m_\oplus\) by means of far-IR observations from the APEX telescope [24].

On the other hand, rings are common features in different astrophysical objects as well. [25] suggested that supermassive black holes located at the center of galaxies may be shrouded by super-Oort clouds of comets and asteroids. [26] performed numerical solutions to the problem of black holes surrounded by uniformly rotating rings in axially symmetric, stationary space-times. [27] reviewed the properties of gravitating discs around black holes. Moreover, annular structures occur also in several ring galaxies (or R galaxies); they are objects with approximate elliptical rings and no luminous matter visible in their interiors [28, 29]. Furthermore, in some circumstances, as the result of interactions between galaxies, a ring of gas and stars is formed and rotates over the poles of a galaxy, resulting in the polar-ring galaxies [30].

In addition to annular structures of natural origin, in the last decades analogous matter distributions of anthropogenic objects formed around the Earth [31, 32]. Indeed, as first pointed out by [33], collisions among artificial satellites would produce orbiting fragments, each of which would increase the probability of further collisions, leading to the growth of a belt of debris around our planet in the Low Earth Orbit (LEO) region after just decades. [35] showed that the growth of such a debris population is primarily driven by high collision activities around 900 – 1,000 km altitude. Radar-based observational campaigns were implemented to characterize the orbital debris environment [36]. Much more long-lived debris equatorial rings, with lifetimes of the order of millennia, may form in the Geostationary Earth Orbit (GEO) region as well [37, 38, 39]; see, e.g. Figure 1.1 in [34, p. 1]. For a recent overview, see [40].

Self-gravitating toroidal fluid configurations (without a central body) in Newtonian gravity were analytically studied by [41, 42, 43], [44, 45], and [46]. In particular, [44, 45] set up an approximation scheme of uniformly rotating, homogeneous, and axisymmetric rings which turned out to be an extremely good approximation for thin rings; see [48]. For various physical properties of flat and toroidal ring models, see [49] and [50], who exploited the disk schemes by [51], [52] and [53]. For other approaches, see [54] and [55]. [56] analytically considered the problem of a uniformly rotating, self-gravitating ring without a central body in Newtonian gravity. A general relativistic toroid around a black hole was studied by [57]. [58] and [59] considered three-dimensional models for the gravitational field of rings in the context of general relativity. Uniformly rotating, homogeneous and axisymmetric relativistic fluid bodies with a toroidal shape were investigated by [60]. [61] considered uniformly rotating homogeneous rings in post-Newtonian gravity. Uniformly rotating rings in general relativity were investigated also by [62], [63] studied the black hole limit of rotating discs and rings.

As far as the orbital perturbations induced by such mass configurations, [64, p. 58] and [65, p. 195] treated the problem of the attraction of a continuous circular massive ring, and showed that its potential can be approximated by a complete elliptic integral of the first kind [66]. Actually, Gauss had already solved the problem and he used it to state his famous averaging theorem: replacing a perturbing body by an equivalent continuous mass spread over its orbit does not change the secular effects while it removes the periodic terms of the perturbation [67]. See also [68]. [69] analytically treated the equivalent problem of calculating the off-axis electric field of a ring of charge. Numerical computation of the acceleration due to a uniform ring can be found in [70]. The perturbing acceleration on an inner particle due to various external axisymmetric mass distributions (e.g. different kinds of thin rings, uniform and non-uniform

1\(^1\)A lower bound of \(76 \, m_\oplus\) was inferred by [21] for the star HD 95086 [23].

2\(^2\)Satellites moving in it have typically altitudes \(h \lesssim 300 – 1,500 \, \text{km} [34, \text{p. 2}]\).

3\(^3\)Geostationary spacecraft orbit at \(h = 35,800 \, \text{km} [34, \text{p. 4}]\).
disks, tori) was worked out by [71], [72], [73], [74] in the framework of the Pioneer anomaly [71]. [75] analytically worked out the force function of a slightly elliptical Gaussian ring, and treated its generalization to a nearly coplanar system of rings as well. For some approximate calculations of the orbital perturbations on the elliptical motion of a test particle moving inside annular distributions, see, e.g., [76] and [77]. The nature of test particle motion in the presence of a rotating ring of self-gravitating matter around a Kerr black hole was investigated by [78]. The perturbations by a distribution of stars in the Galactic Center were investigated by [79].

In this paper, we analytically work out the long-term, i.e., averaged over one full orbital period, perturbations of a test particle moving about a central body and acted upon by a circular massive ring. More specifically, after having reviewed various baryonic and non-baryonic scenarios involving annular matter distributions in Section 2, we consider the potential induced by the material annulus in the general case of an arbitrary ring-orbit configuration (Section 3). Then, we analytically compute the long-term variations of all the osculating Keplerian orbital elements of a test particle due to a circular ring of homogenous linear density. As a by-product, we preliminarily put model-independent, dynamical constraints on the mass of several annular matter distributions from the orbital motions of some of the planets of the Solar System. In particular, we use the latest determinations of the supplementary secular precessions \( \Delta \dot{\omega}^{\text{meas}} \) of the planetary perihelia [80]. In applying our results to the Solar System, we assume that the perturbed orbital motions occurs in the ring’s plane itself. Table 1 shows that it can be considered a rather reasonable approximation for those planets of the Solar System for which [80] determined \( \Delta \dot{\omega}^{\text{meas}} \) by using the mean Earth’s equator at the epoch J2000.0 as reference \( \{x, y\} \) plane. We consider both the cases in which the planet moves outside the ring (Section 4), and inside it (Section 5). As far as the hypothetical annuli are concerned, a complementary approach which may be followed consists of explicitly modeling them and solving for a dedicated parameter in fitting the models to the observations.

Table 1: Inclination \( I \), longitude of the ascending node \( \Omega \) and argument of pericenter \( \omega \) of the planets of the Solar System for which [80] determined the supplementary perihelion precessions \( \Delta \dot{\omega}^{\text{meas}} \) with respect to the mean Earth’s equator and equinox at J2000.0. \( I \) and \( \Omega \) determine the orientation of the orbital plane in space, while \( \omega \) yields the orientation of the orbit in its plane [81].

|        | Mercury | Venus | Earth | Mars | Jupiter | Saturn |
|--------|---------|-------|-------|------|---------|--------|
| \( I \) (deg) | 28.6    | 24.4  | 23.4  | 24.7 | 23.2    | 22.5   |
| \( \Omega \) (deg) | 10.9    | 7.9   | 0.0   | 3.4  | 3.2     | 5.9    |
| \( \omega \) (deg) | −295.7  | 145.2 | 76.7  | −27.5| 10.4    | 87.1   |

We remark that our analysis is not necessarily limited to the major natural bodies of the Solar System (and of other stellar systems). Indeed, annular massive distributions like, e.g., the minor asteroid ensemble have an impact on several proposed space-based missions for fundamental physics like ASTROD [82, 83, 84], LISA [85, 86, 87, 88], GAIA [89, 90] relying upon spacecraft in Solar orbits of about 1 au. Moreover, man-made space debris belts orbiting the Earth may affect the orbital motion of terrestrial artificial satellites of interest for fundamental physics as well. Section 6 summarizes our findings.

As a final remark, we point out that the present work has no pretensions of exhaustiveness about the whole ring problem. In particular, it purposely does not deal with the domain of
periodic orbits, where the potential in closed form expressions is used. For such a potential, there are orbits rather complicated, which likely could not be represented with only a few terms of the Legendre expansion used here. These orbits have a high variable orbital elements, and an averaging over the true anomaly like that performed in this paper gives no actual information. For studies on some general features of the dynamics around a massive annulus, like orbit stability for various configurations and the equilibrium of the system, see, e.g., 97, 91, 92, 93, 94, 95, and 96.

Moreover, we do not pursue elegant and compact formulations which, if on the one hand may be of interest for the specialized reader, on the other hand may have little practical uses for the broad class of problems outlined here and for the allegedly broader audience interested in them.

2 Baryonic and non-baryonic matter distributions in the Solar System

2.1 Baryonic matter rings

Is the Sun surrounded by a ring of rocky material lying at no more than a few Solar radii (1 $R_\odot = 0.00465$ au)?

Such a hypothesis, along with that of a putative intra-Mercurial planet named Vulcan, was put forth for the first time by 97 in the form of a ring of asteroids, known also as “Vulcanoids” 98, between Mercury and the Sun to explain the anomalous perihelion precession of Mercury of 38 degrees of arc per century (arcsec cty$^{-1}$) discovered by him in 1859. Later, 99, by including more observations in a new analysis of the motion of Mercury, came to an anomalous precession of 43 arc sec cty$^{-1}$. It was explained by 101 in terms of a zodiacal matter distribution whose gravitational attraction was equal to that of two ellipsoids, one inside Mercury’s orbit and the other outside that of Venus. The hypothesis of a matter ring was supported also by 102 in a series of lectures given in 1906-1907 on the limits of the Newton’s law of gravitation. As it is well known, 103 finally explained the anomalous perihelion precession of Mercury within the framework of its general theory of relativity which does not contain any ad-hoc, adjustable parameter. The possible existence of a circumsolar belt of small asteroids at about 0.1 au was revamped by 104 as a consequence of a reanalysis of some photographic plates taken during an eclipse in 1970. For more details, see 105, 106 pointed out that Vulcanoids not smaller than 100 m may exist in a gravitationally stable band inside the orbit of Mercury, at distances of 0.06 – 0.21 au from the Sun, with an inclination to the ecliptic not exceeding 10 deg. See also 107, 108, 109. Their existence would be in agreement with the predictions of the scale relativity theory 110, 111, in their unsuccessful attempt to directly detect Vulcanoids in coronagraph images from the instrument LASCO 112 carried onboard the spacecraft SoHO 113, placed an upper bound on the size of such hypothetical objects which should not exceed 60 km. In a previous SoHO/LASCO-based investigation, 109 concluded that the present-day population of Vulcanoids larger than 1 km should amount to less than about 1,800 – 42,000 objects.

The discovery of rings around some of the outer giant planets of the Solar System in the end of 70s 2, 3, 4 enforced the possibility that a circumsolar massive ring exists as well 8. Various evolutionary, physical, chemical and observational considerations have been provided about such a hypothesis. According to 8, the constituents would be $\gtrsim 10$ km size boulders

\footnote{Its currently accepted value is 42.98 arcsec cty$^{-1}$; see 100 for a discussion.}
made of refractory material like graphite. As far as the radius $R_t$ of a permanent ring structure around the Sun is concerned, the values \[ R_t = \begin{align*} 0.02 \text{ au} &= 4.3 \ R_\odot, \\ 0.13 \text{ au} &= 27.9 \ R_\odot, \end{align*} \]

were proposed. The total mass would be much less than $m_t \lesssim 6 \times 10^{22} \ \text{kg}$ [8]. [114, 115] theoretically predicted the existence of a circumsolar dust ring close to the Sun; for a review on the formation and possible observation of a dust ring, see [116]. Direct searches for a hypothetical circumsolar ring by exploiting its electromagnetic emission in the infrared (IR) portion of the spectrum were conducted yielding contradictory results [117], especially as far as the nature and the dimensions of the putative constituents of the ring are concerned. Earlier observations of the F-corona brightness enhancement near 4 $R_\odot$ by [118] and [119, 120] supported the ring hypothesis. [121] attempted to detect the existence of such a ring structure utilizing the IR observations taken during optimal viewing conditions of the total Solar eclipse of 16 February 1980. The near-IR observations by [122], who exploited the 1983 Solar eclipse, supported the existence of a circumsolar ring of dust lying approximately in the ecliptic plane at about 4 $R_\odot$. [123], in their search for excess IR emission in the Solar equatorial plane during the eclipse of 11 July 1991, concluded that the circumsolar dust ring may be a transient feature, perhaps due to the injection of dust into near-Solar space by a sun-grazing comet. [124] argued against the existence of a circumsolar dust ring. According to [125], the ring’s constituents must be fine particles, instead of 10 km sized boulders at 4 $R_\odot$ from the Sun. [126] concluded that it is not reasonable to suggest the presence of Si nanoparticles in the vicinity of the Sun. The hypothesis of a massive circumsolar ring made of macroscopic chunks of matter was recently reanalyzed by [9] who showed that, while a possible Solar ring structure at 4 $R_\odot$ would not be stable over the age of the Solar System because of the Alfvén drag [127] produced during even moderate Solar magnetic storms, a ring at $\sim 27 \ R_\odot$ would, instead, be stable even during very intense Solar magnetic activity.

[128] found evidence for a dust ring associated with the orbit of Venus in the data from the Helios-2 probe [129]. [130] and [131] pointed out the possible existence of a circumsolar ring of asteroidal dust with $R_t \sim 1 \ \text{au}$; the azimuthal structure of this ring was predicted to be asymmetric, with the region trailing the Earth being substantially more dense than that in the leading direction. [132], who analyzed data collected by the instrument Diffuse Infrared Background Experiment (DIRBE) [133] carried onboard the COsmic Background Explorer (COBE) satellite [134], confirmed the existence of such a ring. [135] further investigated the structure of the Earth’s circumsolar dust ring. For a review about dust bands in the Solar System and in extra-Solar planetary systems, see [136].

Other matter distributions, residing far from the Sun, which can be treated with the approach outlined here are the ring of minor asteroids between Mars and Jupiter [10], and the TNOs’ belt [12, 13]. The action of the minor asteroids was modeled as due to a static circular ring at 2.8 au [10, 137, 138, 139, 77] or at 3.14 au [140, 141, 142, 15]. [14, 15] modeled the TNOs belt as a circular massive ring with $R_t = 43 \ \text{au}$.

### 2.2 Non-baryonic Dark Matter distributions

Until now we only considered rings made of ordinary baryonic matter. In principle, our analysis can extend to non-baryonic Dark Matter (DM) as well.
Its existence was postulated long ago to explain the discrepancy between the observed kinematics of some components of astrophysical systems like clusters of galaxies and spiral galaxies, and the predicted one on the basis of the Newtonian dynamics and the matter directly detected from the emitted electromagnetic radiation (visible stars and gas clouds). Postulated the existence of undetected, baryonic matter; today it is believed that the hidden mass is constituted by non-baryonic, weakly interacting particles in order to cope with certain issues pertaining galaxy and galaxy clusters formation, the Cosmic Microwave Background (CMB) and primordial nucleosynthesis. On cosmological scales, DM accounts for about 23% of the mass-energy content of the Universe. A widely popular generic class of new-particle candidates is the Weakly Interacting Massive Particle (WIMP) scenario. In it, DM annihilates with itself and interacts with the rest of the Standard Model (SM) via the weak interaction. The WIMP is typically defined as a stable, electrically neutral, massive particle which arises naturally in supersymmetric SM extensions. A pair of WIMPs can annihilate, producing ordinary particles and gamma rays.

2.2.1 WIMPs

DM particles in the Galactic halo, traversed by the Solar System during its Galactic journey, can end up on orbits gravitationally bound to the Solar System via purely gravitational three-body interactions with the planets. However, when only gravitational interaction is considered, the inverse process of capture from the Galactic halo, leading to ejection of dark-matter particles from the Solar System, must be taken into account as well. As a result, a detailed balance holds, so that there is a rather small limiting density \( \rho_{\text{DM}} \) of dark matter particles in the Solar System, not particularly larger than the mean Galactic value \( \rho_{\text{DM}} \sim 4 \times 10^{-25} \, \text{g cm}^{-3} \). There are, however, different points of view on that issue according to which the situation with the true value for typical life-time of the captured DM in the Solar System would be far from being clear; see . An increase in the population of DM particles bound to the Solar System can occur only if physical mechanisms other than pure gravitational scattering are at work like e.g., weak interactions with atoms in the Sun or the planets. Within such a framework, investigated a scenario in which WIMPs undergo weak scattering with atomic nuclei in layers within the Sun close to its surface in such a way that they lose just enough energy to travel along closed Earth-crossing orbits with high eccentricities. Following such trajectories crossing the Solar surface, most of WIMPs are doomed to experience a second scattering event after just \( 10^3 \sim 10^4 \) revolutions which would significantly reduce their semimajor axes \( a_W \); they will end up in the Sun’s core where they will ultimately annihilate with each other. However, if the layer of Solar matter traversed is quite thin, the lifetime of the grazing WIMPs gets larger. Moreover, during such an increased time the WIMPs experience small orbital perturbations by other planets of the Solar System which, through the mechanism, stabilize them in stable orbital configurations at 1 au which do not intersect the Sun anymore over times comparable to the Solar System’s lifetime. More precisely, obtained that the long-term survival of such WIMPs orbits is greater than 4.5 Gyr if their semimajor axes are smaller than half the semimajor axis of Jupiter, i.e. for \( a_W < a_{\text{Jupiter}}/2 = 2.6 \, \text{au} \). further studied such a scenario: anyway, according to , the overall effect on the bound DM density should be small. The presence of such a stable WIMPs distribution at the Earth’s location would have a great impact for the laboratory-based experimental searches of DM like, e.g., CDMSI, CDMSII, DAMA/NaI and its successor DAMA/LIBRA, XENON10 and ZEPLIN III; for a recent introduction to dedicated DM experiments, see . Thus, it is of great importance to put model-independent, dynamical constraints on
the total mass of such a putative distribution of DM at 1 au.

### 2.2.2 Mirror matter

Other possible stable matter distributions made of another kind of DM, the mirror matter [175, 176, 177, 178, 179, 180, 181, 182, 183, 184, 185, 186], have been hypothesized. It arises if instead of (or in addition to) assuming a symmetry between bosons and fermions, i.e. supersymmetry, one assumes that nature is parity symmetric. For a popular overview of such a scenario, see [187]. Mirror matter was supposed to exist in the Solar System; see [188, 189, 190, 191, 192, 193, 194].

### 2.2.3 CUDOs

Recently, a new class of exotic objects in the Solar System has been proposed: the COmpact Ultra Dense Objects (CUDOs) [195]. They would be gravitationally self-bound objects made of non-baryonic (dark) matter fermions of mass 250 GeV to 100 TeV, or from massless fermions hidden by vacuum structure of similar energy scale [195]. The density at the center of a typical CUDO is more than $10^{12}$ times higher than nuclear density, so that such extremely compact objects are small both in mass (sub-planetary, $M_C < 0.1 M_\oplus$) and radius ($R_C < 10$ cm), making them difficult to observe by conventional astronomical methods [195]. Impact signatures in Solar System bodies have recently been proposed to constrain CUDOs’ local abundances [196]. There may exist also CUDOs of other types, such as supersymmetric Q-balls [197, 198, 199] or CUDOs composed of standard model particles [200, 201]. Q-ball collisions with Earth have been considered by [202].

### 3 Calculation of the potential of the ring for a generic orientation of its plane in space and with respect to the orbit of the test particle

For the sake of generality, in the following we will consider an arbitrary configuration of the ring-particle system. Indeed, in several astronomical and astrophysical scenarios involving, e.g., binary pulsars, exoplanets, etc. the coordinate system adopted does not generally use any characteristic physical and/or orbital plane of the bodies of interest. Moreover, it is reasonable to expect that in several practical situations it cannot be assumed that the perturbing ring structures share the same plane of the perturbed bodies investigated. A striking example of non-coplanarity of several test particles orbiting their common primary is given by the stellar system orbiting the supermassive black hole in Sgr A* at the center of the Milky Way [205]; see [79] for a calculation of the orbital perturbations experienced by a S-type star due to a distribution of bodies. Recently, [206] discovered a dense gas cloud with $m_c = 3 m_\oplus$ which is rapidly approaching the accretion zone of Sgr A* on a highly elliptic orbit ($e_c = 0.94$) with an orbital period of 137 yr: it should reach the perinigricon in 2013. Over the past three years the cloud has begun to disrupt, likely by the tidal forces of the black hole [206]; it might, thus, create an annular matter distribution perturbing the S-type stars orbiting Sgr A*. Other

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It is customarily adopted a system with the reference $z$ axis oriented along the line-of-sight, from the object to the observer, and the reference $\{x, y\}$ plane coinciding with the usual plane of the sky which is tangential to the celestial sphere at the position of the object. In it, the reference $x$ axis is directed towards the North Celestial Pole (see, e.g., [203, p. 287] and [204, p. 446]).
examples are given by the spacecrafts of the LISA [87] and ASTROD [82] constellations, whose orbital planes are planned to have quite different nodes $\Omega$ (see, e.g., [207]). In the case of the space debris belts surrounding the Earth [40], the GEO ring lies in the equatorial plane [37], contrary to the orbits of several satellites potentially of interest for fundamental physics and/or other scopes which supposedly are perturbed by it.

Let us choose a generic inertial frame endowed with a Cartesian coordinate system having the central body of mass $M$ located at its origin $O$. Let $r$ be the position vector of the perturbed test particle, while $R$ denotes the position vector of a mass element $dm_r$ of the annulus. In general, the annulus is inclined by an angle $I\,\hat{r}$ to the reference $\{x, y\}$ plane, and its intersection with the latter one is displaced by an angle $\Omega\,\hat{x}$ with respect to the reference $x$ axis. Thus, $I\,\hat{r}$ and $\Omega\,\hat{x}$ are the same for all the mass elements $dm_r$ of the annulus. Their position along it is picked up by an angle $\varphi\,\hat{r}$, measured in the annulus’s plane from its intersection with the reference $\{x, y\}$ plane. Thus, each $dm_r$ has its own $\varphi\,\hat{r}$. By assuming a homogeneous matter distribution, it will be

$$dm_r = \rho_r \, Rd\varphi\,\hat{r},$$

where $\rho_r$ is the constant linear mass density.

Quite generally, the Cartesian components of $r$ can straightforwardly be expressed in the usual way [81, p. 51] in terms of the longitude of the ascending node $\Omega$, the inclination $I$ to the reference $\{x, y\}$ plane, the argument of pericenter $\omega$, the true anomaly $f$, which is the time-dependent, fast variable giving the instantaneous position of the test particle, the semimajor axis $a$, and the eccentricity $e$ of the Keplerian ellipse of the test particle.

Likewise, by assuming that the matter annulus has, in general, an elliptic shape, for the components of $R$ we can write

$$X = R (\cos \Omega \cos \varphi - \cos I \sin \Omega \sin \varphi),$$

$$Y = R (\sin \Omega \cos \varphi + \cos I \cos \Omega \sin \varphi),$$

$$Z = R \sin I \sin \varphi.$$  \hspace{1cm} (2)

Note that, in general, $R$ can vary along the massive annulus according to a relation analogous to the standard expression of the Keplerian ellipse. Here we will assume $R = R_r$ throughout the ring.

The potential exerted by a mass element $dm_r$ in a point located at $r$ is

$$dU_r = - \frac{Gdm_r}{|r - R|} = - \frac{Gm_r}{2\pi q} \frac{d\varphi_r}{\sqrt{1 + \alpha^2 - 2\alpha \eta(\varphi_r)}},$$

where

(I) $q = r$, $\alpha = \frac{R_r}{r}$ for $r > R_r$,  \hspace{1cm} (3)

(II) $q = R_r$, $\alpha = \frac{r}{R_r}$ for $r < R_r$,  \hspace{1cm} (4)

and

$$\eta(\varphi) = \hat{r} \cdot \hat{R}$$  \hspace{1cm} (5)

is the cosine of the angle between $r$ and $R$. It explicitly depends on $\varphi$. It can be written as

$$\eta(\varphi) = C \cos \varphi + S \sin \varphi,$$  \hspace{1cm} (6)

with

$$C = \cos \Delta \Omega \cos u - \cos I \sin \Delta \Omega \sin u,$$  \hspace{1cm} (7)

$$S = \cos I \sin \Delta \Omega \cos u + (\cos I \cos I_r \cos \Delta \Omega + \sin I \sin I_r) \sin u,$$
with \( u = \omega + f \). From the point of view of the computation of the potential of the ring, \( I, \Omega, \omega, f, I, \Omega \) entering eq. (7) are to be considered as fixed parameters.

By expanding eq. (3) in powers of \( \alpha < 1 \), it can be obtained

\[
dU_r = (\Psi_0 + \Psi_1 + \Psi_2 + \Psi_3 + \Psi_4 + \cdots) d\varphi_r, \tag{8}
\]

with

\[
\Psi_0 = -\frac{Gm}{2\pi q},
\]

\[
\Psi_1 = -\frac{Gm \eta}{2\pi q} \alpha,
\]

\[
\Psi_2 = \frac{Gm}{4\pi q} (1 - 3\eta^2) \alpha^2,
\]

\[
\Psi_3 = \frac{3Gm \eta}{4\pi q} (1 - \frac{5}{3} \eta^2) \alpha^3,
\]

\[
\Psi_4 = -\frac{3Gm}{16q} \left[ 1 - 10 \eta^2 (1 + \frac{35}{8} \eta^2) \right] \alpha^4.
\]

The contribution of each term of eq. (9) can be treated separately. First, the overall potential has to be computed by integrating over \( \varphi_r \)

\[
U_j = \int_0^{2\pi} \Psi_j(\varphi_r) d\varphi_r, \quad j = 0, 1, 2, \ldots \tag{10}
\]

We have

\[
U_0 = -\frac{Gm}{q},
\]

\[
U_1 = 0,
\]

\[
U_2 = \frac{Gm}{2q} \left[ 1 - \frac{3}{2} \left( C_2^2 + S_2^2 \right) \right] \alpha^2,
\]

\[
U_3 = 0,
\]

\[
U_4 = -\frac{3Gm}{8q} \left\{ 1 + \frac{35}{8} \left( C_4^4 + S_4^4 \right) - 5 \left[ S_2^4 + C_2^2 \left( 1 - \frac{7}{8} S_2^2 \right) \right] \right\} \alpha^4.
\]

Then, the perturbations of the Keplerian orbital elements of the test particle induced by \( U_j(t), \quad j = 0, 1, 2, \ldots \) can straightforwardly be worked out with, e.g., the Lagrange planetary equations [81 pp. 251-252]. To this aim, the averages over one full orbital period \( P_b \) must be worked out as

\[
\langle U_j \rangle = \left( \frac{1}{P_b} \right) \int_0^{P_b} U_j(t) dt, \quad j = 0, 1, 2, \ldots , \tag{12}
\]

where \( U_j, \quad j = 0, 1, 2, \ldots \) are to be evaluated onto the unperturbed Keplerian ellipse of the moving test particle; as such, they are to be thought as functions of time \( t \). Using the eccentric anomaly \( E \) [81 p. 32] as fast variable of integration in eq. (12) turns out to be computationally more convenient when \( q = R_s \) since \( U_j \propto r^j, \quad j = 1, 2, \ldots \)

Instead, when \( q = r \) and \( U_j \propto r^{-(j+1)}, \quad j = 0, 1, 2, \ldots \), the use of the true anomaly \( f \) is computationally more suitable.

Performing the integrations of eq. (12) is, in general, quite cumbersome even for small values of \( j \) in eq. (9).
If the annulus and the test particle’s orbit are coplanar, the calculations are simpler. The potential of the ring can be worked out by using the standard mathematical result, valid for $\alpha < 1$,

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{d\xi}{\sqrt{1 + \alpha^2 - 2\alpha \cos \xi}} = \frac{1}{\pi} \left\{ K \left[ -\frac{4\alpha}{(1-\alpha)^2} \right] + K \left[ \frac{4\alpha}{(1+\alpha)^2} \right] \right\} =$$

$$= 1 + \frac{1}{4} \alpha^2 + \frac{9}{64} \alpha^4 + \frac{25}{256} \alpha^6 + \cdots,$$

(13)

where $K(k)$ is the complete elliptic integral of the first kind [66]. Thus, the potential is

$$U_r = -\frac{Gm_r}{2\pi q} \int_0^{2\pi} \frac{d\varphi_r}{\sqrt{1 + \alpha^2 - 2\alpha \cos \varphi_r}} = -\frac{Gm_r}{q} \left( 1 + \frac{1}{4} \alpha^2 + \frac{9}{64} \alpha^4 + \cdots \right).$$

(14)

[208] yielded an expression in closed form, in terms of complete elliptic integrals. See also [209, 210], and [211] who treated the equivalent problem of the computation of the averaged Hamiltonian in the case of an elliptical orbit perturbed by a circular perturber. It can be shown that eq. (14) yields just the acceleration at $r$ obtained by [139].

4 The inner ring: $r > R_t$

In the case (I) of eq. (4), it is

$$\langle U_0 \rangle = -\frac{Gm_r}{a}.$$

(15)

Thus, provided that $m_r$ is by orders of magnitude less than the mass of the central body, all the Keplerian orbital elements are left unaffected, apart from the mean anomaly $M$. The Lagrange equation for its variation yields

$$\frac{dM}{dt} - n_b = \frac{2Gm_r}{m_b a^3}.$$

(16)

In it, $n_b = \sqrt{GM/a^3}$ is the Keplerian mean motion connected to the orbital period by $n_b = 2\pi/P_b$.

The next non-vanishing averaged potential is $\langle U_2 \rangle$, which yields the following non-zero long-
It turns out that all the long-term rates of change are non-zero, explicitly depending on $I$ which the inclinations are the same and the planes are shifted apart (inclinations but equal displacements with respect to the reference adopted. In general, they do not vanish, and explicitly depend on $I$)

$$\dot{\Omega} = \sin I_r (\cos I \cos I_r + \cos \Delta \Omega \sin I \sin I_r) \sin \Delta \Omega,$$

$$\dot{\omega} = -2 \cos 2I \cos \Delta \Omega \cos I \sin 2I_r + \cos I \left[ \cos 2I_r (3 + \cos 2\Delta \Omega) + 2 \sin^2 \Delta \Omega \right],$$

$$\dot{M} = \frac{3Gm_r}{16n_b (1 - e^2)^3 a^3} \left( \frac{R_e}{a} \right)^2 b_0,$$

$$p_0 = \frac{3M_r}{64n_b (1 - e^2)^3 a^3} \left( \frac{R_e}{a} \right)^2 b_0,$$

$$g_0 = \frac{1}{1 + 3 \cos 2I} \left[ \sin \left( \frac{3I}{2} \right) + 5 \sin \left( \frac{I}{2} \right) \right] \sin 2I_r,$$

$$\omega = \Omega + \omega$$ is the longitude of pericenter. As expected, all the perturbations vanish in the limit $R_e \to 0$. If the ring and the planetary orbit share the same inclination to the reference $\{x, y\}$ plane, but not the same orientation, i.e. for $I = I_r, \Omega \neq \Omega_r$, in general, non-vanishing perturbations depending on $I$ and $\Omega - \Omega_r$ occur. In the case $\Omega = \Omega_r, I \neq I_r$, the secular precession of the inclination $I$ vanishes, while the rates of $\Omega, \omega, M$ do not vanish, being functions of $I, I_r$. Finally, if both the ring and the orbit lie in the same plane, i.e. for $I = I_r, \Omega = \Omega_r$ (cfr. Table 1), then the inclination $I$ and the node $\Omega$ remain unaffected, while the other non-zero precessions are

$$\left\langle \frac{d\omega}{dt} \right\rangle = \frac{3Gm_r}{4a^3 (1 - e^2)^2 n_b} \left( \frac{R_e}{a} \right)^2,$$

$$\left\langle \frac{dM}{dt} \right\rangle - n_b = \frac{3Gm_r}{4a^3 (1 - e^2)^3/2 n_b} \left( \frac{R_e}{a} \right)^2.$$

It can be noticed that eq. [21] and eq. [22] are independent of the specific values of $I$ and $\Omega$.

In the case of $\langle U_4 \rangle$, the resulting long-term rates of variation of the osculating Keplerian orbital elements for a generic ring-orbit configuration cannot be displayed in full since they are quite cumbersome; they are exact in the sense that no approximations in $e, I, I_r, \Omega, \Omega_r$ were adopted. In general, they do not vanish, and explicitly depend on $I$ and $\Delta \Omega$. For different inclinations but equal displacements with respect to the reference $x$ direction ($I \neq I_r, \Omega = \Omega_r$), it turns out that all the long-term rates of change are non-zero, explicitly depending on $I$ and $I_r$. In particular, the shifts in $e$ and $I$ are of order $O(e)$ and $O(e^2)$, respectively. The case in which the inclinations are the same and the planes are shifted apart ($I = I_r, \Omega \neq \Omega_r$) is more complicated.
Much simpler expressions are obtained for $I = I_r$, $\Omega = \Omega_r$.

\[
\left\langle \frac{da}{dt} \right\rangle = 0, \tag{23}
\]

\[
\left\langle \frac{de}{dt} \right\rangle = 0, \tag{24}
\]

\[
\left\langle \frac{dI}{dt} \right\rangle = 0, \tag{25}
\]

\[
\left\langle \frac{d\Omega}{dt} \right\rangle = 0, \tag{26}
\]

\[
\left\langle \frac{d\varpi}{dt} \right\rangle = \frac{45Gm_r (1 + \frac{3}{2}e^2)}{32a^3 (1 - e^2)^4 n_b} \left( \frac{R_r}{a} \right)^4, \tag{27}
\]

\[
\left\langle \frac{dM}{dt} \right\rangle - n_b = \frac{135Gm_r e^2}{128a^3 (1 - e^2)^{7/2} n_b} \left( \frac{R_r}{a} \right)^4. \tag{28}
\]

Also in this case, the non-vanishing rates neither depend on $I$ nor on $\Omega$.

### 4.1 The circumsolar ring with $R_r = 0.13$ au

For the sake of simplicity, we will assume coplanarity between the ring and the perturbed planetary orbit; eq. \(21\) and eq. \(27\) yield

\[
\left\langle \frac{d\varpi}{dt} \right\rangle = \frac{3Gm_r}{4a^3 (1 - e^2)^2 n_b} \left( \frac{R_r}{a} \right)^2 \left[ 1 + \frac{15 (1 + \frac{3}{4}e^2)}{8 (1 - e^2)^2} \left( \frac{R_r}{a} \right)^2 \right]. \tag{29}
\]

In the case of Mercury \((a_{\text{☿}} = 0.387\ \text{au})\), and for \(R_r = 0.13\ \text{au}\) \([9]\), eq. \(29\) yields

\[
\left\langle \frac{d\varpi}{dt} \right\rangle = m_r \times 3.08 \times 10^{-20} \text{ mas cty}^{-1} \text{ kg}^{-1}, \tag{30}
\]

where mas cty\(^{-1}\) is a shorthand for milliarcseconds per century. An upper bound on \(m_r\) can be obtained from the latest determinations \(\Delta \varpi^{(\text{meas})}_{\text{☿}}\) of the maximum allowed value for any unmodeled effect impacting the Hermean secular perihelion precession recently inferred by \([80]\) from observations. \([80]\), who did not model the Solar ring, obtained

\[
\Delta \varpi^{(\text{meas})}_{\text{☿}} = 0.4 \pm 0.6 \text{ mas cty}^{-1}. \tag{31}
\]

Actually, there is also a general relativistic effect, caused by the angular momentum of the Sun, which was not modeled by \([80]\): it is the gravitomagnetic precession of the perihelion \([212]\), which is predicted to be as large as

\[
\dot{\varpi}^{(\text{LT})}_{\text{☿}} = -2 \text{ mas cty}^{-1} \tag{32}
\]
by using the value of the Solar angular momentum by helioseismology \[213\]. In the case of Mercury, also the impact of the mismodeling in the Sun’s even zonal harmonic $J_2$, modeled by \[80\], must be taken into account. By assuming a 10\% uncertainty in it \[214\], it is

$$\omega^{(\delta J_2)} = 2.5 \text{ mas cty}^{-1}. \quad (33)$$

Thus, both eq. (32) and eq. (33) must be subtracted from eq. (31) in inferring an upper bound on $m_r$. As a result, from

$$0.5 \text{ mas cty}^{-1} + m_r \times 3.08 \times 10^{-20} \text{ mas cty}^{-1} \text{ kg}^{-1} \leq 1 \text{ mas cty}^{-1} \quad (34)$$

we have

$$m_r \leq 1.6 \times 10^{19} \text{ kg} = 4.3 \times 10^{-6} m_\oplus. \quad (35)$$

It is 3708 times smaller than the upper bound by \[9\].

One may legitimately wonder if the use of eq. (29) is accurate enough, given the relatively small distance of the Hermean orbit from the putative perturbing ring considered. Actually, the answer is positive since eq. (35), inserted in eq. (27), yields just 0.57 mas cty\(^{-1}\). Thus, we conclude that the truncation level chosen in the expansion of the annulus’ potential yielding eq. (29) is fully adequate to deal with Mercury and a hypothetical perturbing massive ring with $R_r = 0.13$ au, in view of the present-day level of accuracy in constraining the supplementary Hermean perihelion precession.

### 4.2 The circumsolar ring with $R_r = 0.02$ au

The same approach for Mercury and $R_r = 0.02$ au yields

$$m_r \leq 8.3 \times 10^{20} \text{ kg} = 1.4 \times 10^{-4} m_\oplus; \quad (36)$$

it is tighter by about one order of magnitude than the upper limit proposed by \[9\]. Note that eq. (36) and eq. (27) yield just 0.01 mas cty\(^{-1}\).

### 4.3 The ring with $R_r = 1$ au

The perihelion precession of Mars ($a_\sigma = 1.524$ au) can be used to effectively constrain the mass of a ring with $R_r = 1$ au. \[80\] obtained

$$\Delta \omega^{(\text{meas})} = -0.04 \pm 0.15 \text{ mas cty}^{-1} \quad (37)$$

for the Arean supplementary perihelion precession. In the case of Mars, eq. (29) and eq. (37) yield

$$m_r 2.1 \times 10^{-20} \text{ mas cty}^{-1} \text{ kg}^{-1} \leq 0.11 \text{ mas cty}^{-1}, \quad (38)$$

which implies

$$m_r \leq 5.3 \times 10^{18} \text{ kg} = 8.8 \times 10^{-7} m_\oplus. \quad (39)$$

Since the Arean Lense-Thirring precession amounts to just $\omega^{(\text{LT})} = -0.03 \text{ mas cty}^{-1}$, i.e. 5 times smaller than the present-day uncertainty \[80\] in determining the Arean supplementary perihelion, we did not include it in eq. (38). The same holds for the mismodeling in the Sun’s quadrupole moment, since $\omega^{(\delta J_2)} = 0.01 \text{ mas cty}^{-1}$.

Also in this case, it turns out that the level of truncation chosen is adequate, given the current level of accuracy in constraining the supplementary Arean perihelion precession. Indeed, eq. (39), inserted in eq. (27), yields 0.05 mas cty\(^{-1}\), which is 3 times smaller than 0.15 mas cty\(^{-1}\).
5 The outer ring: \( r < R_t \)

In the case (II) of eq. (4), the calculations are, in general, more involved.

It turns out that, although \( \langle U_0 \rangle \) does not vanish, it does not contribute to the long-term variations of the osculating Keplerian orbital elements since it is just made of \( m_r \) and \( R_t \). Indeed, it is \( U_0 = -\frac{G m_e}{R_t} \). Moreover, \( \langle U_1 \rangle = 0 \). As far as \( \langle U_2 \rangle \) is concerned, it contributes to the orbital changes with

\[
\begin{align*}
\left\langle \frac{d e}{d t} \right\rangle &= \frac{15 e G m_e \sqrt{1 - e^2}}{32 n_r R_t^2} E_1, \\
E_1 &= \cos 2 \omega \left( -4 \sin I \sin 2 I \sin \Delta \Omega + 4 \cos I \sin^2 I \sin 2 \Delta \Omega \right) + \\
&\quad + \left( 1 + 3 \cos 2 I \right) \sin I^2 + (3 + \cos 2 I) \sin 2 \Delta \Omega \sin^2 I - 2 \cos \Delta \Omega \sin 2 I \sin 2 \omega, \\
\left\langle \frac{d I}{d t} \right\rangle &= \frac{3 G m_e}{8 n_r \sqrt{1 - e^2 R_t^2}} \left( 2 I_0 + 12 e^2 \right), \\
I_0 &= \sin I \left( \cos I \sin \Delta \Omega \sin I + \sin I \sin I \right) \sin \Delta \Omega, \\
\left\langle \frac{d \Omega}{d t} \right\rangle &= -\frac{3 G m_e \cos I}{8 n_r \sqrt{1 - e^2 R_t^2}} \left( 00 + 02 e^2 \right), \\
00 &= \cos I_k \left( \cos I \cos \Delta \Omega \sin I + \sin I \sin I \right) \cos \Delta \Omega, \\
02 &= \cos I_k \left( \cos I \sin I \sin \Delta \Omega \sin I \sin I \right) \left( -3 + 5 \cos 2 \omega \right) \left( \cos I \sin I - \cos I \cos \Delta \Omega \sin I \sin I \right) - 5 \sin I \sin \Delta \Omega \sin 2 \omega, \\
\left\langle \frac{d e}{d t} \right\rangle &= \frac{3 G m_e}{128 \sqrt{1 - e^2 R_t^2}} \left( P_0 + P_2 e^2 \right), \\
P_0 &= 2 \left( 1 + 3 \cos 2 I \right) \left( 3 + 20 \cos 2 \omega \sin^2 \left( \frac{I}{2} \right) \right) + 6 \cos 2 \Delta \Omega \left( 2 - 2 \cos I + 5 \cos 2 \omega \right) \sin I \left( 2 I + 5 \cos 2 I \cos 2 \Delta \Omega \cos 2 \omega \sin^2 I \right) + \\
&\quad + 40 \sin I \sin 2 I \sin \Delta \Omega \sin 2 \omega + \cos I \left[ 4 \left( 1 + 3 \cos 2 I \right) \left( -1 + 5 \cos 2 \omega \sin^2 \left( \frac{I}{2} \right) \right) + 40 \cos 2 \Delta \Omega \sin^2 \left( \frac{I}{2} \right) \sin^2 I - 40 \sin^2 I \sin 2 \Delta \Omega \sin 2 \omega \right] + \\
&\quad + 4 \cos \Delta \Omega \left[ 3 + 5 \cos 2 I + \cos I \left( -6 - 20 \cos^2 \left( \frac{I}{2} \right) \cos 2 \omega \right) \sin 2 I \tan \left( \frac{I}{2} \right) \right], \\
P_2 &= -4 \left( 1 + 3 \cos 2 I \right) \left( -1 + 3 \cos I + 10 \cos 2 \omega \sin^2 \left( \frac{I}{2} \right) \right) \left( -16 \cos 2 \Delta \Omega \left( 5 \cos^2 \left( \frac{I}{2} \right) \cos 2 \omega + 3 \sin^2 \left( \frac{I}{2} \right) \right) \sin^2 I \right) + \\
&\quad + 8 \cos \Delta \Omega \left( -3 + 5 \cos 2 \omega \right) \sec \left( \frac{I}{2} \right) \sin \left( \frac{I}{2} \right) \sin \Delta \Omega \sin 2 I \sin 2 \omega + \\
&\quad + 80 \cos^2 \left( \frac{I}{2} \right) \sin^2 I \sin 2 \Delta \Omega \sin 2 \omega, \\
\left\langle \frac{d \Delta \Omega}{d t} \right\rangle &= -\frac{G m_e}{256 n_r R_t^2} \left( M_0 + M_2 e^2 \right), \\
M_0 &= 4 \left( 1 + 3 \cos 2 I \right) \left( 7 + 21 \cos 2 I + 30 \cos 2 \omega \sin^2 I \right) + 12 \cos 2 \Delta \Omega \left[ 10 \left( 3 + \cos 2 I \right) \cos 2 \omega + 28 \sin^2 I \right] \sin^2 I - \\
&\quad - 48 \cos \Delta \Omega \left( -7 + 5 \cos 2 \omega \right) \sin 2 I \sin 2 I + 480 \sin I \sin 2 I \sin \Delta \Omega \sin 2 \omega - 480 \cos I \sin^2 I \sin 2 \Delta \Omega \sin 2 \omega, \\
M_2 &= 6 \left( 2 \left( 1 + 3 \cos 2 I \right) \left( 1 + 3 \cos 2 I + 10 \cos 2 \omega \sin^2 I \right) \right) + 2 \cos 2 \Delta \Omega \left[ 10 \left( 3 + \cos 2 I \right) \cos 2 \omega + 12 \sin^2 I \right] \sin^2 I - \\
&\quad - 8 \cos \Delta \Omega \left( -3 + 5 \cos 2 \omega \right) \sin 2 I \sin 2 I + 80 \sin I \sin 2 I \sin \Delta \Omega \sin 2 \omega - 80 \cos I \sin^2 I \sin 2 \Delta \Omega \sin 2 \omega \right).
the semimajor axis is left unaffected. As expected, all the perturbations of eq. (40)-eq. (44) vanish in the limit \( R_r \rightarrow \infty \). If the ring and the planetary orbit share the same inclination to the reference \( \{x, y\} \) plane, but not the same orientation, i.e. for \( I = I_r, \Omega \neq \Omega_r \), in general, all the long-term changes of eq. (40)-eq. (44) do not vanish, depending on \( I, \Delta \Omega, \omega \); it is the case of, e.g., the three spacecrafts of LISA [207]. If \( \Omega = \Omega_r, I \neq I_r \), then eq. (40)-eq. (44) are not zero, being functions of \( I, I_r, \Delta \Omega, \omega \). Finally, if both the ring and the orbit lie in the same plane, i.e. for \( I = I_r, \Omega = \Omega_r \) (cfr. Table 1), then \( e, I, \Omega \) remain unaffected, while the long-term variations of \( \varpi \) and \( \mathcal{M} \) do not vanish, being equal to

\[
\left\langle \frac{d\varpi}{dt} \right\rangle = \frac{3\sqrt{1 - e^2} G m_r}{4n_b R_t^3},
\]

(45)

\[
\left\langle \frac{d\mathcal{M}}{dt} \right\rangle - n_b = -\frac{7 (1 + \frac{3}{7} e^2) G m_r}{4n_b R_t^3}
\]

(46)

Note that eq. (45) and eq. (46) do not depend on the specific values of \( I \) and \( \Omega \).

The long-term variations of the Keplerian orbital elements due to \( \langle U_4 \rangle \) are quite involved for the general case \( I \neq I_r, \Omega \neq \Omega_r \): they cannot be explicitly displayed. Also in this case, no approximations in \( e, I, I_r \) were adopted. In the simpler coplanar case, i.e. for \( I = I_r, \Omega = \Omega_r \), they are

\[
\left\langle \frac{da}{dt} \right\rangle = 0,
\]

(47)

\[
\left\langle \frac{de}{dt} \right\rangle = 0,
\]

(48)

\[
\left\langle \frac{dI}{dt} \right\rangle = 0,
\]

(49)

\[
\left\langle \frac{d\Omega}{dt} \right\rangle = 0,
\]

(50)

\[
\left\langle \frac{d\varpi}{dt} \right\rangle = \frac{45G m_r \sqrt{1 - e^2} (1 + \frac{3}{4} e^2)}{32n_b R_t^3} \left( \frac{a}{R_t} \right)^2,
\]

(51)

\[
\left\langle \frac{d\mathcal{M}}{dt} \right\rangle - n_b = -\frac{81G m_r \left[ 1 + \frac{25}{12} e^2 (1 + \frac{e^2}{3}) \right]}{32n_b R_t^3} \left( \frac{a}{R_t} \right)^2.
\]

(52)

Also in this case, the precessions of eq. (51) and eq. (52) are independent of \( I \) and \( \Omega \).

For other analytical calculation, see, e.g., [77].
5.1 The ring with $R_r = 1$ au

As far as the perihelion precession in the coplanar case is concerned, eq. (45) and eq. (51) yield

$$\left\langle \frac{d\varpi}{dt} \right\rangle = \frac{3Gm_r\sqrt{1-e^2}}{4n_bR_r^3} \left[ 1 + \frac{15}{8} \left( 1 + \frac{3}{4}e^2 \right) \left( \frac{a}{R_r} \right)^2 \right]. \quad (53)$$

By proceeding with eq. (31) as in the case of the inner ring, a tight constrain can be obtained by using the perihelion of Venus ($a_\oplus = 0.723$ au), for which [80] obtained

$$\Delta \dot{\varpi}_\oplus^{(\text{meas})} = 0.2 \pm 1.5 \text{ mas cty}^{-1}. \quad (54)$$

By neglecting the Cytherean unmodeled Lense-Thirring and mismodeled $J_2$ precessions $\dot{\varpi}_\oplus^{(\text{LT})} = -0.3 \text{ mas cty}^{-1}$, $\dot{\varpi}_\oplus^{(4J_2)} = 0.2 \text{ mas cty}^{-1}$, eq. (53) and eq. (54) yield

$$m_r \leq 2.1 \times 10^{19} \text{ kg} = 3.4 \times 10^{-6} m_\oplus. \quad (55)$$

It must be noticed that, for $R_r = 1$ au, the Cytherean bound of eq. (55) is weaker than the Arean constraint of eq. (39). Anyway, the use of eq. (53) to deal with Venus and the hypothetical ring at $R_r = 1$ au is adequate, given the current level of accuracy in constraining the supplementary Cytherean perihelion precession. Indeed, eq. (51), computed with eq. (55), yields just $0.6$ mas cty$^{-1}$.

5.2 The ring of minor asteroids between Mars and Jupiter

The overall mass of the ring with which the dynamical action of the minor asteroids orbiting between Mars and Jupiter is usually modeled can effectively be constrained by using the perihelion of Mars and the result of eq. (37) for its supplementary precession [80]. The impact of the individual largest perturbers of the asteroid belt on the motion of the inner planets, especially Mars and the Earth, was studied by, e.g., [215, 216, 217, 218, 219, 220]. As far as their impact on some proposed spacecraft-based missions for fundamental physics are concerned, see, e.g., [221, 222, 223].

By adopting [10, 137, 138, 139, 77] $R_r = 2.80$ au, eq. (53), computed for Mars, yields

$$m_r \leq 7.5 \times 10^{-21} \text{ mas cty}^{-1} \text{ kg}^{-1} \leq 0.11 \text{ mas cty}^{-1}, \quad (56)$$

from which it can be inferred

$$m_r \leq 1.46 \times 10^{19} \text{ kg} = 7.3 \times 10^{-12} M_\odot. \quad (57)$$

Also in this case, we did not consider the Arean Lense-Thirring effect in eq. (56) because of its smallness.

Our result of eq. (57) is smaller than the values obtained by [10] ($m_r = (5 \pm 1) \times 10^{-10} M_\odot$), [137] ($m_r = (6 \pm 1) \times 10^{-11} M_\odot$), [138] ($m_r \lesssim 2.4 \times 10^{-10} M_\odot$), [139] ($m_r = (3.4 \pm 1.5) \times 10^{-11} M_\odot$), and [142] ($m_r = (6 \pm 2) \times 10^{-11} M_\odot$). In the case of [140, 141, 142, 15] $R_r = 3.14$ au, we obtain from Mars

$$m_r \leq 2.33 \times 10^{19} \text{ kg} = 1.1 \times 10^{-11} M_\odot. \quad (58)$$

[142] and [141] obtained $m_r = (1.0 \pm 0.3) \times 10^{-10} M_\odot$, which is one order of magnitude larger than eq. (58). [140] and [15] obtained $m_r = (3.35 \pm 0.35) \times 10^{-10} M_\odot$ and $m_r = (8.7 \pm 3.5) \times 10^{-11} M_\odot$, respectively.
The use of eq. (8) and eq. (9) for \( r < R \), yielding eq. (45) and eq. (51) in the coplanar case, might be questioned because of the closeness of the Arean orbit to the asteroid ring. Actually, it is not the case, as it can be a-posteriori inferred from eq. (57) and eq. (58). Indeed, it turns out that eq. (51), computed with both eq. (57) (\( R_r = 2.80 \) au) and eq. (58) (\( R_r = 3.14 \) au), yields just 0.03 mas cty\(^{-1}\), which is 5 times smaller than 0.15 mas cty\(^{-1}\). Note that the same conclusion holds also if we adopt the upper limits of the figures for \( m_r \) obtained by other authors. Indeed, for \( m_r = 8 \times 10^{-11} M_\odot \) [142] and \( R_r = 2.80 \) au, eq. (51) yields 0.11 mas cty\(^{-1}\). Moreover, \( m_r = 1.22 \times 10^{-10} M_\odot \) and \( R_r = 3.14 \) au [15], inserted in eq. (51), give 0.07 mas cty\(^{-1}\). Thus, we conclude that our expansion to \( \alpha^4 \), giving eq. (53), is fully adequate to treat the case of Mars perturbed by the ring of minor asteroids within the present-day level of accuracy in constraining the supplementary precession of the Arean perihelion.

5.3 The belt of the Trans-Neptunian Objects

Concerning the mass of the TNO belt, modeled as a circular ring with [14 15] \( R_r = 43 \) au, it can be constrained with Saturn \( (a_\text{♄} = 9.582 \) au) by using [80]

\[
\Delta \dot{\varpi}_\text{♄}^{\text{meas}} = 0.15 \pm 0.65 \text{ mas cty}^{-1}
\]  

(59)

and eq. (53). Also in this case, there is no need to take into account the Kronian Lense-Thirring and \( \delta J_2 \) precessions since they are far smaller than 0.65 mas cty\(^{-1}\) (\( \dot{\varpi}_\text{♄}^{\text{(LT)}} = -1 \times 10^{-4} \) mas cty\(^{-1}\), \( \dot{\varpi}_\text{♄}^{(\delta J_2)} = 2 \times 10^{-5} \) mas cty\(^{-1}\)). Thus, from

\[
m_r \leq 1.98 \times 10^{-23} \text{ mas cty}^{-1} \text{ kg}^{-1} \leq 0.8 \text{ mas cty}^{-1},
\]

(60)

it turns out

\[
m_r \leq 4.03 \times 10^{22} \text{ kg} = 2.0 \times 10^{-8} M_\odot = 6.6 \times 10^{-3} m_\oplus = 42.7 m_{\text{Ceres}},
\]

(61)

where we adopted [224]

\[
m_{\text{Ceres}} = 9.43 \times 10^{20} \text{ kg}
\]

(62)

for the dwarf planet Ceres.

For a comparison, according to [14], the minimum mass of the TNO belt would be as large as 110 masses of Ceres, while the maximum mass of the ring is expected to be 100 times the minimum mass, i.e. \( 1.1 \times 10^4 m_{\text{Ceres}} \). In other tests by [14], the maximum mass surpasses the minimum mass by 25, 50, and 75 times. Thus, the Kronian bound of eq. (61) is much more stringent than that by [14, 15] inferred for the mass of the TNO ring the smaller value \( m_r = (4.98 \pm 0.14) \times 10^{-8} M_\odot = (105 \pm 3) m_{\text{Ceres}} \). The bound of eq. (61) is also much tighter than those obtained some years ago by [76] from earlier corrections \( \Delta \dot{\varpi}^{\text{meas}} \) to the standard precessions of the perihelia of the inner planets [225].

We remark that, also in this case, the level of truncation considered is fully adequate, given the current level of accuracy in constraining the Kronian perihelion precession. Indeed, even by using the largest admissible value \( m_r = 5.12 \times 10^{-8} M_\odot \) by [15], it turns out that eq. (51) yields just 0.008 mas cty\(^{-1}\).
6 Summary and conclusions

We considered a test particle orbiting a central body of mass $M$ lying at the center of a circular massive ring of radius $R_r$. We analytically worked out the long-term variations of the osculating Keplerian orbital elements $a, e, I, \Omega, \varpi, M$ of the particle induced by the first terms in the expansion of the gravitational potential $U_r$ of the annulus. We considered both the case in which the particle moves outside ($r > R_r$) and inside ($r < R_r$) it. We did not restrict to any specific mutual orientation of the ring and the particle’s orbital plane which, in general, neither coincide ($I \neq I_r, \Omega \neq \Omega_r$) nor lie in the reference $\{x, y\}$ plane of the coordinate system adopted ($I \neq 0, I_r \neq 0$). In general, the resulting perturbations affect all the particle’s orbital elements, apart from the semimajor axis $a$. They depend in a quite intricate way on the inclinations $I, I_r$ and the nodes $\Omega, \Omega_r$ of both the particle and the ring, apart from the case in which they are coplanar ($I = I_r, \Omega = \Omega_r$).

By assuming coplanarity, we applied our results to some hypothesized and existing annular distributions like circumsolar matter rings (Vulcanoids), a ring at 1 au, and the rings with which the dynamical actions of the minor asteroids in the asteroid belt and of the Trans-Neptunian Objects are usually modeled. By comparing our analytical results to the latest observational determinations of the supplementary precessions $\Delta \dot{\varpi}^{(\text{meas})}$ of the longitudes of the perihelia $\varpi$ of some planets of the Solar System, we tentatively inferred upper bounds on the masses $m_r$ of such massive annular distributions. As a complementary approach which could certainly be followed, a standard covariance-based analysis would consist of processing the same observational data records by explicitly modeling the annular mass distributions whose mass one wants to determine/constrain, and treat it as a solve-for parameter to be estimated in the fit of the models to the observations. In any case, it should not be considered as the only legitimate method which must mandatorily be followed to obtain meaningful results. In general, the issue of potentially occurring mutual cancelation of various competing effects can meaningfully be inspected analytically as well, provided that one is able to work out proper expressions adequate to the accuracy level of the data. In this respect, it is not true that the effects caused by the rings are the same as for, say, the Sun’s oblateness: indeed, their analytical expressions are quite different. Moreover, in most cases, several competing effects were actually modeled, so that they can impact the supplementary perihelion precessions only with their mismodeling, which can well be taken into account in evaluating their “residual” effect. We stress that the constraints obtained by us cannot be considered as unrealistically tight; on the contrary, they might be regarded as relatively “generous” since, in inferring them, we assumed just that the entire range of variation of the supplementary perihelion precessions is due to the dynamical action of the rings. We also tested the validity of such an assumption by checking in each case if competing mismodeled/unmodeled known effects like the Sun’s $J_2$ and the Lense-Thirring effect were actually below the uncertainties released for the supplementary perihelion precessions; whenever it was not the case, like for Mercury, we took them into account as well. Claiming a-priori that there might still be room, in principle, for the action of any other unknown force conspiring in partly canceling allegedly larger ring effects makes little sense. After all, the effect of any sort of “Russell teacups” may well creep into the the solved-for rings’ masses estimated in a covariance analysis, if they really existed in Nature and they were not modeled at all. Moreover, we successfully checked that the level of approximation used for our formulas is adequate, given the present-day level of accuracy in constraining the supplementary precessions of the planetary perihelia. On the other hand, apart from the fact that our constraints are compatible with other, previously obtained bounds, it is certainly not strange that they are smaller then them, given that we adopted the latest ephemerides.
Our results are summarized in Table 2.

Table 2: Upper bounds on the mass \( m_r \) of different circular matter rings of natural origin, whose radii \( R_r \) are listed in au, inferred from the corrections \( \Delta \dot{\omega}^{(\text{meas})} \) to the standard perihelion precessions of some planets of the Solar System by [80]. We used eq. (29) and eq. (53) by assuming coplanarity between the rings and the planetary orbits. The upper bound for the TNOs (\( R_r = 43 \) au) is equivalent to \( 6.6 \times 10^{-3} \ m_{\oplus} = 42.7 \ m_{\text{Ceres}} \).

| \( R_r \) (au) | \( m_r^{(\text{max})} \) (kg) | \( m_r^{(\text{max})} (m_{\oplus}, M_{\odot}) \) | Planet adopted |
|-----------------|-----------------|-----------------|-----------------|
| 0.02            | \( 8.3 \times 10^{20} \) | \( 1.4 \times 10^{-4} \ m_{\oplus} \) | Mercury         |
| 0.13            | \( 1.6 \times 10^{19} \) | \( 2.6 \times 10^{-6} \ m_{\oplus} \) | Mercury         |
| 1               | \( 5.3 \times 10^{18} \) | \( 8.8 \times 10^{-7} \ m_{\oplus} \) | Mars            |
| 2.80 (minor asteroids) | \( 1.5 \times 10^{19} \) | \( 7.3 \times 10^{-12} \ \ M_{\odot} \) | Mars            |
| 3.14 (minor asteroids) | \( 2.3 \times 10^{19} \) | \( 1.1 \times 10^{-11} \ \ M_{\odot} \) | Mars            |
| 43 (TNOs)       | \( 4.0 \times 10^{22} \) | \( 2.0 \times 10^{-8} \ \ M_{\odot} \) | Saturn          |

In principle, our results are not limited just to baryonic matter distributions, being valid also for non-baryonic rings made of Dark Matter. Concerning putative circumsolar massive rings, which may, in principle, be made of non-baryonic Dark Matter as well, their mass is no larger than about \( 1 \times 10^{-4} - 3 \times 10^{-6} \ m_{\oplus} \), while the mass of a ring at 1 au is \( m_r \lesssim 9 \times 10^{-7} \ m_{\oplus} \). Moreover, due their generality, our findings can be applied to different astronomical and astrophysical scenarios like Earth satellites, exoplanetary systems, stellar orbits around supermassive black holes, etc.

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