Algorithm for Optimization of Inverse Problem Modeling in Fuzzy Cognitive Maps

Alina Vladimirovna Petukhova 1, Anna Vladimirovna Kovalenko 2 and Anna Vyacheslavovna Ovsyannikova 3, *

1 COPELABS, Lusófona University, Campo Grande, 376, 1749-024 Lisbon, Portugal
2 Department of Data Analysis and Artificial Intelligence, Kuban State University, Stavropol'skaya St., 149, 350040 Krasnodar, Russia
3 Department of Mathematical, Financial University under the Government of the Russian Federation, Leningradsky Prospekt, 49, 125993 Moscow, Russia
* Correspondence: avovsyannikova@fa.ru

Abstract: Managerial decision-making is a complex process that has several problems. The more heterogeneous the system, the more immeasurable, non-numerical information it contains. To understand the cognitive processes involved, it is important to describe in detail their components, define the dependencies between components, and apply relevant algorithms for scenario modelling. Fuzzy cognitive maps (FCMs) is the popular approach for modeling a system’s behavior over time and defining its main properties. This work develops a new algorithm for scenario analysis in complex systems represented by FCMs to provide support for decision-making. The algorithm allows researchers to analyze system-development scenarios to obtain the required change to the system’s components that leads to the target state. The problem of determining a system’s initial state is most conspicuous when constructing a compound or unbalanced fuzzy maps. Currently, a brute force algorithm is used to calculate the steps needed to approach a target, but that takes exponential time. The paper describes a new algorithm to obtain the initial values of the controlled concepts in fuzzy cognitive maps using the theory of neutrosophic fuzzy equations. This approach reduces the time needed to find the optimal solution to a problem, and it allows inverse problems to be solved in the fuzzy cognitive maps as a part of the scenario-modeling framework.

Keywords: fuzzy cognitive maps; scenario modeling; reverse task; fuzzy relational equations; neutrosophic fuzzy equations

MSC: 93C42

1. Introduction

Cognitive modeling is the universal tool for solving governance problems in semi-structured (weakly formalized) systems. Fuzzy cognitive maps, initially introduced by [1], is one of the most popular cognitive modeling techniques. It combines neural networks and fuzzy logic to predict changes in concepts represented in the causal maps. The structure of fuzzy cognitive maps is based on a list of numerical or abstract concepts determined by experts and the causal relationships among them. These relationships are defined as weights on the edges of the graph and can be obtained from expert interviews or statistical data analysis.

The advantage of fuzzy cognitive maps is its dynamic nature: it allows us to determine the particular state of a system and its development over time. In the past decade, fuzzy cognitive maps has been widely used in various scientific fields such as the social and political sciences, medicine, engineering, governance, information technology, robotics, education, forecasting and the environment [2].

The application of fuzzy cognitive maps is not limited to the representation of causal relationships: they also contain the flows of information and are used to simulate external...
system influences. Additionally, they simulate changes in the system for each time step because each the calculation of each successive state is based on the values of the previous one.

The majority of the authors in the FCM domain describe the problem of direct modelling, in particular ascertaining the new states of target concepts after an external change to the controlled concepts. This approach is widely used to solve problems of complex systems development and time series forecasting [3–10]. Various research groups are focusing on different aspects of the direct modeling problem: Wei et al. describe the learning method in FCMs as a convex optimization problem with constraints and solved it using classic interior-point methods [11]; Concepcion et al. used the theorem to prove that the state space of an FCM shrinks infinitely and converges to a limited state space [12]; and Mazzuto et al. found a way to create FCMs by incorporating the information carried by a large dataset [13]. Another block of research is focused on the selection of subsystems within fuzzy cognitive maps (FCMs) [14] or multi-weighted FCMs [15].

The main focus of this article is on the analysis of the solution to the inverse-modeling problem, which is the process of defining the required change in the concepts of controlled fuzzy cognitive maps so that the system can achieve the target state with a certain level of confidence [16]. In the case of socio-economic FCMs, solving the inverse-modeling task will allow analysts to obtain optimal management decisions during organizational strategic planning [17].

Several articles [18–22] considered the possibility of using neutrosophic fuzzy equations to solve the problem. The main focus of these works was to solve neutrosophic fuzzy equations for fuzzy relational maps [23]. These differ from fuzzy cognitive maps in that they separate causal associations into two disjoint units so that the concepts of the domain and range spaces do not overlap [18]. In addition, this approach assumes that no intermediate relationship exists between elements or domain nodes and elements of range spaces; however, for the many complex systems described in [24–26], this simplification is not applicable.

The article proposed an approach to finding the initial states of the target concepts in the complex FCMs by solving adjusted neutrosophic fuzzy equations. Additionally, minimal correction of the target state and the adjustment of the FCM structure was presented when a solution to the original neutrosophic fuzzy equation did not exist.

This article is structured as follows: the second section provides background information on the theory of fuzzy cognitive maps, gives inverse problem statements, and describes the proposed algorithm, including methods of minimal correction and adjustment to the fuzzy cognitive maps structure. In the third section, the results of applying the proposed method to various complex fuzzy systems described in the literature are presented. Finally, the fourth section contains a discussion of the method and a potential direction for future research.

2. Materials and Methods

Formally defined, a fuzzy cognitive map is a causal network that reflects some area of knowledge and has the following representation (1):

$$G = < E, W >,$$

where $E = e_1, e_2, \ldots, e_n$ (a set of concepts), and $W$ is a matrix of link weights on the set $E$, which defines the relation between them.

The concepts $e_i$ and $e_j$ are considered in relation $w$ ($e_i, e_j$ or $e_i,w,e_j$) such that a change in the state of $e_i$ (cause) leads to a change in the state of $e_j$ (effect). In the terminology of cognitive analysis, in this case, concept $e_i$ influences concept $e_j$. There are three types of weights that exist in fuzzy cognitive maps [18]: $w_{ji} > 0$ represents a positive causal relationship; $w_{ji} < 0$ represents a negative causal relationship; and $w_{ji} = 0$ means there is no relationship between nodes.
The concepts themselves can be presented as both relative (qualitative) indicators (e.g., popularity, social tension) and absolute (quantifiable) values (e.g., price, emissions or distance). A simple fuzzy cognitive map example is presented in Figure 1.

![Figure 1. Simple fuzzy cognitive map schema.](image)

2.1. Inverse Task for a Controlled Cognitive Process

The state of a complex system is a vector of influences on controlled concepts since only these are accessible to direct influence. The values of the controlled concepts are affected by external flows such as managerial decisions in socio-economic systems. The rest of the concepts, including the targets, are influenced indirectly through the controlled ones. The value of a controlled concept can be increased, decreased, or left unchanged. In the case of the impulse, the values can be changed to varying degrees within the specified scale.

The target state of the system is defined by a vector to which the values of the corresponding target concepts should be as close as possible during dynamic modeling. It is required that the relative change of the target concepts and the consonance (the degree of confidence in the attainability) be set. As the value of the consonance increases, confidence in the sign of influence increases [27].

To solve the inverse problem, let us consider in detail the structure of the fuzzy cognitive maps. From the original matrix, we have

\[
\begin{align*}
W &= \begin{bmatrix} w_{ij} \end{bmatrix}, \quad \text{sub-matrix of conditions;} \\
A &= \begin{bmatrix} a_{ij} \end{bmatrix}_{n \times n}, \quad \text{control} \\
B &= \begin{bmatrix} b_{ij} \end{bmatrix}_{n \times n}, \quad \text{target} \\
C &= \begin{bmatrix} c_{ij} \end{bmatrix}_{m \times n}, \quad \text{and direct influence} \\
D &= \begin{bmatrix} d_{ij} \end{bmatrix}_{m \times n}, \quad \rightarrow s_{k+1} = W \circ s_k
\end{align*}
\]

Equation (2) defines the rules for changing the weight of vertices in the model. The vector \( s_k \) can be split into the corresponding vectors \( \vec{u}_k, \vec{x}_k, \vec{y}_k \), where, by analogy to the traditional theory of linear systems, \( \vec{u}_k \) is the vector of control concepts; \( \vec{x}_k \) is the vector state concepts; and \( \vec{y}_k \) is the vector of target concepts. In the algorithm of selection, elements containing 0 in the rows of the matrix \( W \) belong to the target \( B \), and those containing 0 in the columns belong to the control \( C \). As a result, we have solutions to the direct scenario modeling (3) and (4):

\[
\begin{align*}
\vec{x}_{k+1} &= A \circ \vec{x}_k \lor B \circ \vec{u}_k, \\
\vec{y}_k &= C \circ \vec{x}_k \lor D \circ \vec{u}_k
\end{align*}
\]

or

\[
\begin{align*}
\vec{x}_k &= A^* \circ \vec{u}_0, \\
\vec{y}_k &= C \circ A^* \circ B \circ \vec{u}_0 \lor D \circ \vec{u}_0
\end{align*}
\]

where, \( A^* = \bigvee_{k=0}^{\infty} A_k \) —transitive closure in fuzzy regular matrix algebra. The calculation of the transitive matrix in an FCM defined the indirect connections in the map and displayed them using weights in the state matrix \( X \) for time \( t \), when the system converged to a stable state with an initial change in the constant control vector.
To solve the inverse task in this work, we proposed using the mathematical apparatus of neutrosophic fuzzy equations for which we reviewed its basic theoretical concepts. The idea is built on the theory of neutrosophic sets [28]—of which the FCM is a specific type—which are used to represent knowledge and reason. Neutrosophic sets are generalized fuzzy sets that allow for the inclusion of indeterminacy and uncertainty. This makes them more expressive and flexible than fuzzy sets but also more difficult to work with. For example, by using neutrosophic sets the function of superposition of possible states of electromagnetic waves may be simulated by a neutrosophic function, i.e., a function that has values that are not unique to each argument from the definition domain. Compared to neutrosophic sets, FCMs are more limited in their expressiveness but are easier to work with.

Let \( X, Y, Z \) be discrete, crisp sets of finite cardinality \( m, n \) and \( k \), respectively, and let \( \tilde{A}(X, Y), \tilde{X}(Y, Z), \tilde{B}(X, Z) \) be fuzzy matches. Let there be a composition

\[
\tilde{A} \circ \tilde{X} = \tilde{B},
\]

where \( \circ \) is the operation of fuzzy composition in the theory of fuzzy logic (5) that corresponds to matrix Equation (6).

\[
\mathbf{A} \circ \mathbf{X} = \mathbf{B},
\]

where \( \mathbf{A}_{n \times m}, \mathbf{X}_{m \times k}, \mathbf{B}_{n \times k} \) is the matrix representation of \( \mathbf{A}, \mathbf{X} \) and \( \mathbf{B} \), respectively. Consider the inverse modeling problem for fuzzy correspondences:

1. the definition of \( \mathbf{A} \) with known \( \mathbf{X}, \mathbf{B} \) and \( \circ \)
2. the definition of \( \mathbf{X} \) with known \( \mathbf{A}, \mathbf{B} \) and \( \circ \)

By analogy with the terminology of matrix algebra, the first is called the left and the second the right neutrosophic fuzzy equation. Note that, since the t-norm is commutative for \( \circ = (\max, T) \), problem 2 reduces to 1; that is, in this case, it is sufficient to consider only left-handed equations.

If the neutrosophic fuzzy equation contains a composition of an unknown row vector and a given matrix or a given matrix with an unknown column vector, then the solution is reduced to solving a system of fuzzy polynomial equations. Let us consider the basic concepts using the example of a system of fuzzy polynomial equations with the composition operation max–min and denote it as \( \mathbf{A} \cdot \mathbf{X} = \mathbf{B} \):

\[
\begin{bmatrix}
(a_{11} \land x_{11}) \lor \ldots \lor (a_{1n} \land x_{1n}) = b_1, \\
\ldots \\
(a_{m1} \land x_{m1}) \lor \ldots \lor (a_{mn} \land x_{mn}) = b_m.
\end{bmatrix}
\]

The complete set of solutions of a solvable finite system of left neutrosophic fuzzy equations is determined by the maximum solution and a finite number of minimum solutions. The set of solutions of fuzzy equations is convexly ordered; that is, if \( (x_{ij}) \) and \( (y_{ij}) \) are two different solutions and \( x_{ij} > z_{ij} > y_{ij} \) for any \( i \) and \( j \), then \( (z_{ij}) \) will also be a solution. Therefore, the main task is to find the base and all the branches of this equation. The system of fuzzy polynomial equations is solvable if the system has a maximum solution that can be found with an analytical expression [29] as described in Algorithm 1.
Algorithm 1: Determination of the maximum solution and solvability.

1. Initialize the vector \( \hat{X} = (\hat{x}_{ij}) \), where \( \hat{x}_{ij} = 1 \) for \( j = 1 : n \).
2. Initialize the boolean vector \( IND \) with \( IND_i = \text{FALSE} \) for \( i = 1 : m \).
3. For each \( j = 1 : n \) and for each \( i = 1 : m \):
   (a) If the \( j \)-th column \( A(j) \) contains \( a_{ij} \), for which \( a_{ij} = 1 \) set \( IND_i = \text{TRUE} \).
   Determine \( \hat{b}_j \) as:
   \[
   \hat{b}_j = \begin{cases} 
   v_{ij}, & \text{if } a_{ij} > b_j, i = 1, \ldots, m, \\
   1, & \text{otherwise}
   \end{cases}
   \]
   (b) Update \( X \)—assign \( x_j = \hat{b}_j \). Check if there is another \( i \) such that \( a_{ij} \) for which \( a_{ij} > b_i, ab_i = b_j \). If so, correct the \( IND_i = \text{TRUE} \) value.
   (c) Check if there is another \( i \) such that \( a_{ij} = \hat{b}_i \). If so, correct the value \( IND_i = \text{TRUE}, x_j = \hat{b}_j \). Move on to the next \( j \).
4. Check that for all \( i = 1 : m \) vector \( IND = \text{TRUE} \):
   (a) If \( IND_i \) is \( \text{FALSE} \) for some \( i \), the system \( A \odot B \) is unsolvable and \( i \) is the number of the unsolvable equation for the system. Go to Step 5.
   (b) If \( IND_i \) is \( \text{TRUE} \) for all \( i = 1 : m \), system \( A \odot B \) is solvable, and its maximum solution is \( X \).
5. End of the algorithm.

If \( H_i \) denotes the number of \( H \)-type coefficients in the \( i \)-th equation \( A \cdot X = B \), then the number of potential minimum solutions of the system of fuzzy polynomial equations (7) does not exceed
\[
P_{N_1} = \prod_{i=1}^{m} |H_i|.
\] (8)

Current work has been focused on the algorithm to obtain the minimum solutions in the optimal time. Currently, various methods are used to find the minimal solutions to the neutrosophic fuzzy equations: algebraic–logical [30–35], characteristic matrix and decomposition [36,37], cover [38,39], bind variables [40,41], partitions and irreducible paths [42] and solutions-based matrix [43]. Algorithms have exponential complexity in relation to consumed memory and time [29]. The algorithm proposed in our work is also exponential but based on the dominance and use of list enumeration operations to find only the lower solutions. Thus, it implements a faster approach and is the most efficient among all available algorithms.

The proposed algorithm is focused on the selection of determinate concepts in the system of fuzzy polynomial equations to minimize the list enumeration operations. This approach is based on the idea of finding all the selected elements \( A_{ij} \) in \( A \) that are solutions of the \( i \)-th (i.e., \( A_{ij} \odot X_j = B_j \)). Next, we assign \( X_j \) the value of the corresponding \( B_j \), when the coefficient \( A_{ij} \) is a part of the solution, or 0 otherwise. Some solutions are neither bottom nor top. To obtain only the minimal solutions and not consider intermediate one in the computation, the method based on the concept of dominance was used [31] in combination with list enumeration operations. The idea of dominant solutions was first considered by Pappis and Sugeno [36]. The algorithm is presented schematically in Figure 2 and in Algorithm 2.
Algorithm 2: Obtaining lower solutions from $\hat{M}$.

1. Form the sets $M$ and $\hat{M}$.
2. Initialize the solution vector $\hat{X}_0(j) = 0$, $j = 1, n$.
3. Initialize a vector of $\text{rows}(i)$, $i = 1, m$, which contains all sequential line numbers—indices $i$ for every $M_i \in M$.
4. Initialize $i$ equal to the first element from $\text{rows}$.
5. Initialize $\text{sols}$ as an empty set of vectors, which will be the set of all minimal solutions for the current problem.
6. Check if $\text{rows} \neq \varnothing$. If yes, add $\hat{X}_{ij}$ into the vector $\text{sols}$ and proceed to step 8.
7. For each $i$ in $\text{rows}$, expand the set $M_i$: for each $j \in M_i$ create a copy $\hat{X}_0 - \hat{X}_{ij}$ and assign its $j$-th element the value $b_i$. Remove $i$ from the $\text{rows}$ vector.
   Return to Step 6 with the new $\text{rows}$ and $\hat{X}_{ij}$ values obtained from this step.
8. End of the algorithm.

Assume that the system $A \circ X = B$ is solvable and $X$ is the largest solution. Using $X$, marks all coefficients that contributed to the solution of the system. For each equation $i = 1, m$, the set $M_i$ is initialized. It would contain a list of all indices $j$ of the coefficients $A_{ij}$ that contributed to the solution of the $i$-th equation $A_{ij} \circ X_j = B_i$.

**Definition 1.** The set $M_i$, has elements that are indices. $j \in \{1, \ldots, n\}$ in the $i$-th equation (7) where $a_{ij} \wedge \hat{x}_j = b_i$, is called the marking set for the $i$-th equation (7) in ascending order. If $j$ is an element of $M_i$, then $a_{ij}$ is an H-type coefficient, but the reverse is not always true: there may be an H-type coefficient $a_{ij}$ but $j$ does not belong to $M_i$. Therefore, $|M_i| \leq |H_i|$, where $|M_i|$ is the power of $M_i$, and $|H_i|$ is the power of $H_i$.

**Definition 2 ([31]).** Let $A_l$ and $a_k$ be the $l$-th and $k$-th equations, respectively, in (7) and $b_l \geq b_k$. Equation $A_l$ is called dominant with respect to equation $A_k$ if for each $j = 1, n$, and the following holds: if $a_{lj}$ is an H-type coefficient, then $a_{kj}$ is also an H-type coefficient.

If $A_l$ is the dominant equation for $A_k$ in (7), then whenever $A_l$ satisfies the solution to (7), this solution also satisfies $A_k$. Therefore, $A_k$ is a redundant equation.

For example, in

$$
(A^* : B) = \begin{pmatrix}
0 & 0 & 0.9 & 0.9 & 0 & : 0.9 \\
0 & 0.9 & 0 & 0 & 0 & : 0.9 \\
0 & 0 & 0 & 1 & 0 & : 0.7 \\
1 & 0.7 & 0 & 0 & 0.7 & : 0.7 \\
0.5 & 0 & 0.5 & 0 & 0 & : 0.5
\end{pmatrix}
$$

(9)

the third equation dominates the last, and the last equation is redundant. According to (8), $PN_1 = 2 * 2 * 1 * 3 = 24$, and after removing the last redundant equation $PN_2 = 12$.

Instead of working with dominant equations, we used labeling sets because they all have the properties of initial equations for finding solutions. Thus, we reduced the time complexity of building solutions by optimizing the selection of objects for which the search was performed.
Form marking $\mathbf{M}$ and dominant $\mathbf{M}$ sets

Initialize the solution vector $X_0(j) = 0, j = 1, \ldots, n$

Initialize a vector of $\text{rows}(i), i = 1, m$, which contains all sequential line numbers - indices $i$ for every $M_i \in \mathbf{M}$.

Initialize $\mathbf{sol}$ as an empty set of vectors, which will be the set of all minimal solutions for the current problem.

Check if $\text{rows} \neq \emptyset$.

Yes

Add $X_0$ into the vector $\mathbf{sol}$.

No

For each $i$ in $\text{rows}$, expand the set $M_i$: for each $j \in M_i$ create a copy $X_0 - X_j$ and assign its $j$-th element value $b_j$. Remove $i$ from $\text{rows}$ vector.

End of the algorithm.

Figure 2. Algorithm for obtaining a lower solution from $\hat{\mathbf{M}}$.

**Definition 3.** Let $M_l, M_k$ belong to the marking sets for $\mathbf{M}$ (7) and $b_l \geq b_k$. If $M_l$ is a subset of $M_k$, then $M_l$ is called the dominant set over $M_k$, and $M_k$ is called the dominated set by $M_l$.

The set of all dominant labeling sets for (7) is denoted by $\hat{\mathbf{M}}$. Each $M_i$ from $\hat{\mathbf{M}}$ corresponds to the $i$-th row in $\mathbf{A}$ or the $i$-th equation in (7).

The number of lower solutions now does not exceed the estimate of

$$PN_2 = \prod_{i=1}^{m} |M_i|, \text{ where } M_i \in \hat{\mathbf{M}}.$$  \hspace{1cm} (10)

If we compare (8) and (10), clearly, $PN_1 \geq PN_2$.

The new matrix $\check{\mathbf{A}} = (\hat{a}_{ij})$ was obtained from $\mathbf{A}^*$ and $\hat{\mathbf{M}}$. The dominated sets were removed and $\check{\mathbf{A}}$ had the same solutions as $\mathbf{A}$ (11).

For example, in (9), after removing the dominant equations, we had

$$\check{\mathbf{A}} : \mathbf{B} = \begin{pmatrix} 0 & 0 & 0.9 & 0 & 0 : 0.9 \\ 0 & 0 & 0 & 0 & 1 : 0.7 \\ 1 & 0.7 & 0 & 0 & 0.7 : 0.7 \end{pmatrix}$$  \hspace{1cm} (11)
The maximum solution corresponded to the weights of the input concepts in the cognitive map for conservative system modeling; the minimum solution corresponded to democratic or minimum possible effects on the control concepts of the system.

2.2. Method of Minimum Adjustments in the Absence of an Explicit Solution to the Inverse Modeling Problem

When working with real systems presented by FCMs, there are often cases when the initially set target indicators are unattainable and lead to system undecidability. When looking for the maximum solution for neutrosophic fuzzy equations, we determined whether the original system were solvable for defined target concepts.

If \( a_{ij} \land \hat{x}_j = b_i \) (or \( a^*_{ij} \land \hat{x}_j = b_i \)), the coefficients \( a_{ij} \) and \( a^*_{ij} \), respectively, are called the determining coefficients of the neutrosophic fuzzy equation. \( A \circ X = B \) is soluble if there is at least one determining factor for each \( i = 1 : m \); otherwise, it is unsolvable, but it is possible to determine the equation that made the system unsolvable.

In the absence of a solution for the given target indicators, it was suggested that the adjusted values of column \( b \) be found. For that it was necessary to substitute the obtained values into the original equation for the maximum solution to the system of equations. As a result, we obtained the vector of the minimum adjustments to target indicators \( b_{upd} \), which was always \( \leq b_i \).

2.3. Method for Optimizing the Structure of the Fuzzy Cognitive Maps and Allocation of Dedicated Control Concepts in the Primary Structure of the Map

In the initial structure of the map, the control and target vectors were not always highlighted as is required by the structure of the equation solution algorithm for finding the target system state defined by the expert. To highlight these vectors a method was proposed in which additional concepts were injected into the fuzzy cognitive maps that had a single connection with the target or control concepts defined by the expert. Such additional “buffer” concepts kept the values of the parameters and made isolating the concepts of states possible.

3. Experimental Results

The algorithm was applied to the FCM of the transport problem Figure 3.
The control concepts in this system were “fuel economy” and “population size”, and the target concepts were “cost of travel”, “air pollution”, “number of accidents” and “probability of being late”.

\[
A = \begin{pmatrix}
0.00 & 0.42 & 0.693 & 0.007 \\
0.42 & 0.00 & 0.007 & 0.693 \\
0.00 & 0.42 & 0.693 & 0.007 \\
0.42 & 0.00 & 0.007 & 0.693 \\
0.00 & 0.00 & 0.300 & 0.005 \\
0.00 & 0.00 & 0.005 & 0.300 \\
0.00 & 0.00 & 0.200 & 0.015 \\
0.00 & 0.00 & 0.015 & 0.200 \\
\end{pmatrix}
\]  

(12)

The target system state was defined by the vector (13):

\[
B = (0.10 \ 0.082 \ 0.10 \ 0.82 \ 0.10 \ 0.82 \ 0.10 \ 0.082)
\]

(13)

With this formulation of the target vector, the solution does not exist since the equations in (12) are unsolvable. Let us find a new target vector (14) according to 1.

\[
B = (0.10 \ 0.082 \ 0.10 \ 0.82 \ 0.043 \ 0.035 \ 0.029 \ 0.024)
\]

(14)

This system is solvable and is represented by the minimum and maximum solution in Tables 1 and 2:

**Table 1.** Minimum solution to the inverse problem in the transportation system.

| Concept         | Weight | Consonance |
|-----------------|--------|------------|
| Fuel economy \(X_2\) | 0.00   | 0.00       |
| Population size \(X_3\) | 0.144  | 1          |

**Table 2.** Maximum solution of the inverse problem in the transportation system.

| Concept         | Weight | Consonance |
|-----------------|--------|------------|
| Fuel economy \(X_2\) | -0.238 | 1          |
| Population size \(X_3\) | 0.144  | 1          |

The maximum solution corresponded to the conservative scenario, which imposed maximum conditions on changing the control concepts to achieve the system target state. The minimum solution showed the mildest conditions for the initial effect on the system. There were also many solutions between the maximum and minimum; in the presented example, they were in the 0.238 – 0 range for the concept “fuel saving”.

Consider, for example, the fuzzy cognitive map of the ecological state of a city Figure 4. The concepts of this map are represented in Table 3:
As we can see, for the fuzzy cognitive map on Figure 4, the control and target concepts were not distinguished in the initial FCM. We solved the problem of how to improve the “sanitary condition” (X5) by influencing the concept of “landfill” (X4). To do this, we introduced additional nodes into the initial fuzzy cognitive map in Table 4:

**Table 4.** Additional concepts injected to the fuzzy cognitive map determining the ecological state of the city.

| Concept                          | Map Designation |
|---------------------------------|-----------------|
| Landfills (inserted)            | X8              |
| Sanitary condition (inserted)   | X9              |

Let’s solve the problem regarding “landfill” by setting a target change of −0.2 with a consonance of 0.8. In this case, the maximum and minimum solutions of the neutrosophic fuzzy equation coincide with Table 5:

**Table 5.** Maximum solution in the fuzzy cognitive map determining the ecological state of the city.

| Concept                          | Weight | Consonance |
|---------------------------------|--------|------------|
| Landfills (inserted)            | −0.47  | 1          |

For the reproducibility of the research results, the R library was developed to allow us to solve the inverse fuzzy cognitive map problem. Within the framework of the library, the following features were implemented:
• system settings by choosing different operations of fuzzy intersection and union (t-norm, s-norm, composition, transitive closure),
• solution of the direct problem of fuzzy cognitive map modeling,
• solution of the inverse problem of fuzzy cognitive map modeling,
• method of minimal adjustments in the target vector of neutrosophic fuzzy equation,
• method for optimizing the structure of the fuzzy cognitive map,
• visualization of the results using modern graphic methods,

For a visual representation of the algorithm, we solved the inverse problem for the FCMs presented in the [44]. In the article, the authors introduced an FCM presented in Figure 5 to determine the size of a brain tumor, classify it and plan the patient’s treatment. The concepts of the FCM are presented in Table 6.

![Figure 5. The fuzzy cognitive map to determine the grade of a brain tumor.](image)

**Table 6.** Concepts in the fuzzy cognitive map determining the grade of brain tumor.

| Concept               | Map Designation |
|-----------------------|-----------------|
| Cellularity           | $C_1$           |
| Mitoses               | $C_2$           |
| Apoptosis             | $C_3$           |
| Multinucleated cells  | $C_4$           |
| Giant cells           | $C_5$           |
| Vascular proliferation| $C_6$           |
| Necrosis              | $C_7$           |
| Pleomorphism          | $C_8$           |
| Tumor grade           | $C_9$           |

We considered the problem-solving goal to be a reduction in tumor grade. A fuzzy cognitive map was represented by control concepts $C_1$ and $C_6$. To solve the problem regarding the $C_9$ concept, we optimized the map structure and introduced the $C_{10}$ concept, which had a single connection with $C_9$. Let us set the target change of the $C_{10}$ concept of the duplicate $C_9$ to $-0.2$ with a consonance of 0.3. After the transformation, we obtained the vector of positive–negative changes in concept $C_{10}$, Table 7:

![Table 7. Vector of positive–negative changes in concept $C_{10}$.](image)

| Concept | Weight | Consonance |
|---------|--------|------------|
| $C_{10}$| 0.108  | 0.200      |
For this problem definition, the neutrosophic fuzzy equations could not be solved, so we applied minimal adjustments and got a new target vector as shown in Table 8.

Table 8. Vector of positive–negative changes in concept $C_{10}$ with minimal adjustments.

| Concept | Weight | Consonance |
|---------|--------|------------|
| $C_{10}$ | 0.108  | 0.158      |

The maximum and minimum solutions for the updated system are presented in Table 9:

Table 9. Maximum and minimum solutions for the fuzzy cognitive map determining the grade of brain tumor.

| $C_1$ | Consonance | $C_6$ | Consonance |
|-------|------------|-------|------------|
| Max solution | −0.396 | 1 | 0.000 | 0 |
| Min solution | 0.000 | 0 | −0.396 | 1 |
| Max solution | −0.396 | 1 | −0.396 | 1 |

To reduce the tumor grade it was necessary to reduce the “cellularity” or “vascular proliferation” concept at −0.369.

We also applied the algorithm to solve the inverse problem for the fuzzy cognitive map to determine the failure of a banking project. The target concept changes in the FCM are presented in Table 10.

Table 10. Targeted concept changes in the fuzzy cognitive map determining the failure of the banking project.

| Concept | Weight | Consonance |
|---------|--------|------------|
| Process failure $P_1$ | −0.3 | 0.3 |
| Expectation failure $P_2$ | −0.3 | 0.3 |
| Interaction failure $P_3$ | −0.1 | 0.3 |
| Correspondence failure $P_4$ | −0.2 | 0.3 |

The values of the FCM control concepts for determining the failure of the project are presented in Table 11.

Table 11. Maximum solution for control concepts of the fuzzy cognitive map determining the failure of the project.

| Concept | Weight | Consonance |
|---------|--------|------------|
| Max solution | Environmental pressures $F_{13}$ | −0.2 | 1 |

For this fuzzy cognitive map, the targeting strategy was to increase funding for a bank branch. Let’s set target changes to the concept “Funding” ($C_4$) in Table 12:

Table 12. Targeted concept changes in the fuzzy cognitive map determining the work of a banking system.

| Concept | Weight | Consonance |
|---------|--------|------------|
| Funding $C_4$ | 0.5 | 0.6 |
As a result of solving the system of neutrosophic fuzzy equations, we obtained values for the control concepts of the banking system fuzzy cognitive map that were responsible for the implementation of the strategy in Table 13:

**Table 13.** Maximum and minimum solution for the fuzzy cognitive map determining the work of the banking system.

| Concept                  | Weight | Consonance |
|--------------------------|--------|------------|
| Min solution             |        |            |
| Rules and regulations C2 | 0.25   | 1          |
| New IT solutions C14     | 0      | 0          |
| Min solution             |        |            |
| Rules and regulations C2 | 0      | 0          |
| New IT solutions C14     | 0.25   | 1          |
| Min solution             |        |            |
| Rules and regulations C2 | 0.25   | 1          |
| New IT solutions C14     | 0.25   | 1          |

Thus, the amplification concepts “Rules and regulations”, and “New IT solutions” equally affected the ability to achieve the desired state of the system. The solutions were tested by substituting the changes in the control concepts in the FCM and solving the direct problem. The results of the simulation scenario of increasing the “Rules and regulations” concept corresponded to the minimum solution for the fuzzy cognitive map’s determination of the work of the banking system and are presented in Table 14.

**Table 14.** Simulation results of the scenario of increasing the “Rules and regulations” concept in the fuzzy cognitive map determining the work of the banking system.

| Concept                  | $t_1$ | $t_2$ | $t_3$ | $t_4$ | $t_5$ | $t_6$ |
|--------------------------|-------|-------|-------|-------|-------|-------|
| Client                   | $C_1$ | 0     | 0.25  | 0.25  | 0.75  | 1.141 |
| Rules and regulations C2 |       | 0.25  | 0     | 0     | 0     | 0     |
| New IT solutions C14     |       | 0     | 0.125 | 0.5   | 0.813 | 2.469 |
| Funding                  | $C_4$ | 0     | 0.25  | 0.625 | 1.094 | 2.625 |
| Cost reduction           | $C_5$ | 0     | 0     | 0     | $-0.031$ | 0 |
| Profit/loss              | $C_6$ | 0     | 0.063 | 0     | 0     | 0.813 |
| Investments              | $C_7$ | 0     | 0.25  | 0.313 | 0.594 | 1.344 |
| Staff                    | $C_8$ | 0     | 0.25  | 0.063 | $-0.094$ | 0.125 |
| New services             | $C_9$ | 0     | 0.125 | 0.375 | 1     | 1.859 |
| Quality                  | $C_{10}$ | 0     | 0     | 0.188 | 0.063 | 0.141 |
| Client development       | $C_{11}$ | 0     | 0.25  | 0.5   | 1.219 | 2.844 |
| Service development      | $C_{12}$ | 0     | 0.25  | 0.313 | 0.813 | 1.656 |
| Productivity             | $C_{13}$ | 0     | 0     | 0.125 | 0.813 | 1.672 |

The solution already allowed us to achieve the target state of the banking system at the second step; however, after further analyzing the system development in this scenario, we saw that the system became unbalanced and a sharp increase in the values of the target concept “Funding” ($C_4$) was clear. Such development may indicate the need to train and balance the original fuzzy cognitive map or possibly expand the map by introducing additional concepts.
4. Discussion

FCMs are a type of specific type of neutrosophic sets that uses fuzzy logic to represent the relationships between different concepts. They have been used in a variety of fields, including decision-making, artificial intelligence, and knowledge representation.

An FCM can be used to optimize the behavior of a system by simulating the propagation of information through a network of nodes. Forward propagation can be done manually while the backward requires a specific algorithm. The proposed algorithm allowed us to obtain initial values of the control vector in a more efficient way compared to existing methods. This study included a description of the algorithm for the problem described in reference [17]. After determining the model’s behavior by choosing the most optimal of the possible solutions to the neutrosophic fuzzy equations, we had the opportunity to regulate system development for the chosen strategy. This approach allowed us to estimate the resources needed to achieve target system state and select the most optimal initial impact on the system.

The work improves our knowledge of the mechanisms underlying complex systems, thereby increasing understanding of specific problems in areas such as economics, socio-politics, medical systems and chemical networks.

As a potential for future work, the proposed algorithm can be used for inverse problem modeling in other fuzzy cognitive maps to test its applicability. However, our work did not resolve the issue of finding the time when the system reached a given target state. Another important point to consider is time delays during the concept’s interference. Taking into account the passage of time in the systems described by the FCMs, the influence between cognitive concepts laged, increased or decreased. For this, it was necessary to introduce time dependence—$A(t)$, $B(t)$, $C(t)$, $D(t)$—into the interaction matrices, and introduce the necessary adjustments into the proposed algorithm.

As another addition to current research, we also plan to evaluate the methods of constructing FCMs and introducing the possibility of learning the structure of a map by analyzing nonlinguistic models and extracting the degree of interconnection and level of mutual influence of concepts in an linguistic context. It could reduce the time of FCM construction—for example, organizations could determine the initial weights of concept interactions without peer review—and help identify missing concepts in the initial map structure if certain of them were not linguistically related to the rest of the system.

5. Conclusions

In this work, we presented an algorithm to solve the inverse fuzzy cognitive map problem using the apparatus of neutrosophic fuzzy equations. Mathematical software and algorithms for intelligent decision support based on fuzzy cognitive maps were developed and presented. We believe that this work can simplify a decision-making process for complex semi-structured systems and with this knowledge gain a better understanding of the possible scenarios for system development. A qualitative analysis of the scenario modeling of a complex system allowed us to determine the minimum necessary changes to the control concepts and to select the optimal scenario for a specific situation.

The proposed algorithm has a number of advantages over other approaches to solving the inverse problem in FCMs. First, it found lower solutions faster and with less memory consumption. Second, since the set of solutions is small, it was relatively easy to construct and interpret. Finally, the algorithm can be used to generate a range of different scenarios between a minimum and maximum solution, which can be useful for exploring the potential consequences of different actions or events.

The main weakness of the proposed method is that the results were strongly related to the choice of the fuzzy operator used in the FCM. The wrong choice can lead to the absence of a solution or to inaccurate predictions. Another weakness is that the suggested algorithm did not take into account non-linear dependencies between concepts and other more complex relations including time dependant influences.
Future development will likely focus on expanding different methods of choosing a fuzzy operator for the FCM model. Another area of focus might be to add time delays to the concept influences to create a function of the concept’s influences instead of the currently used matrix of weights. This may allow further calibration of the algorithm and has the potential to improve scenario planning with FCMs.

**Author Contributions:** Conceptualization, A.V.P. and A.V.K.; methodology, A.V.P.; software, A.V.P.; validation, A.V.K. and A.V.O.; formal analysis, A.V.O.; resources, A.V.O.; data curation, A.V.P.; writing—original draft preparation, A.V.P.; writing—review and editing, A.V.P.; supervision, A.V.K. All authors have read and agreed to the published version of the manuscript.

**Funding:** The research did not had any funding.

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Data Availability Statement:** Not applicable.

**Conflicts of Interest:** The authors declare no conflict of interest.

**References**

1. Kosko, B. *Fuzzy Thinking: The New Science of Fuzzy Logic*; Flamingo London: London, UK, 1993.
2. Felix, G.; Nápoles, G.; Falcon, R.; Froelich, W.; Vanhoof, K.; Bello, R. A review on methods and software for fuzzy cognitive maps. *Artif. Intell. Rev.* **2019**, *52*, 1707–1737.
3. Papakostas, G.A.; Boutalis, Y.S.; Koulouriotis, D.E.; Mertzios, B.G. Fuzzy cognitive maps for pattern recognition applications. *Int. J. Pattern Recognit. Artif. Intell.* **2008**, *22*, 1461–1486.
4. Groumpos, P.P.; Stylios, C.D. Modelling supervisory control systems using fuzzy cognitive maps. *Chaos Solitons Fractals* **2000**, *11*, 329–336.
5. Lu, W.; Yang, J.; Liu, X. Numerical prediction of time series based on FCMs with information granules. *Int. J. Comput. Commun. Control* **2014**, *9*, 313–324.
6. Homenda, W.; Jastrzębska, A.; Pedrycz, W. Joining concept’s based fuzzy cognitive map model with moving window technique for time series modeling. In Proceeding of the IFIP International Conference on Computer Information Systems and Industrial Management, Warsaw, Poland, 24–26 September 2015; Springer: New York, NY, USA, 2015; pp. 397–408.
7. Nápoles, G.; Grau, I.; Falcon, R.; Bello, R.; Vanhoof, K. A granular intrusion detection system using rough cognitive networks. In *Recent Advances in Computational Intelligence in Defense and Security*; Springer: New York, NY, USA, 2016; pp. 169–191.
8. Gataullin, T.M.; Gataullin, S.T.; Ivanova, K.V. Modeling an electronic auction. In Proceedings of the Institute of Scientific Communications Conference, Volgograd, Russia, 19–20 March 2020; Springer: New York, NY, USA, 2020; pp. 1108–1117.
9. Petukhova, A.; Fachada, N. Retail System Scenario Modeling Using Fuzzy Cognitive Maps. *Information* **2022**, *13*, 251. https://doi.org/10.3390/info13050251.
10. Petukhova, A.; Kovalenko, A.; Teunaev, D. Review of dynamic properties and learning algorithms of fuzzy cognitive maps. *Polythematic Netw. Electron. Sci. J. Kuban State Agrar. Univ.* **2021**, *167*. https://doi.org/10.21515/1990-4665-167-004.
11. Lu, W.; Feng, G.; Liu, X.; Pedrycz, W.; Zhang, L.; Yang, J. Fast and Effective Learning for Fuzzy Cognitive Maps: A Method Based on Solving Constrained Convex Optimization Problems. *IEEE Trans. Fuzzy Syst.* **2020**, *28*, 2958–2971. https://doi.org/10.1109/TFUZZ.2019.2946119.
12. Concepción, L.; Nápoles, G.; Bello, R.; Vanhoof, K. On the State Space of Fuzzy Cognitive Maps using Shrinking Functions. In Proceedings of the 32th Benelux Conference on Artificial Intelligence and the 29th Belgian Dutch Conference on Machine Learning, BNAIC/BENELEARN 2020, Leiden, The Netherlands, 19–20 November 2020.
13. Mazzuto, G.; Stylios, C. Empower Fuzzy Cognitive Maps Decision Making abilities with Swarm Intelligence Algorithms. In Proceedings of the 2019 IEEE International Conference on Systems, Man and Cybernetics (SMC), Bari, Italy, 6–9 October 2019; pp. 2602–2607. https://doi.org/10.1109/SMC.2019.8914456.
14. Kochkarov, R. Research of NP-Complete Problems in the Class of Prefractal Graphs. *Mathematics* **2021**, *9*, 2764. https://doi.org/10.3390/math9212764.
15. Kochkarov, R. Multicriteria Optimization Problem on Prefractal Graph. *Mathematics* **2022**, *10*, 930. https://doi.org/10.3390/math10060930.
16. Sikov, V. *Strategic Decision-Making in a Fuzzy Environment*; INPRO-RES: Moscow, Russia, 1995; Volume 228.
17. Vasilev, V.; Savina, I.; Sharipova, I. Construction of fuzzy cognitive maps for the analysis and management of information risks of the university. *Vestn. UGATU* **2008**, *10*, 199–209.
18. Kandasamy, W.V.; Smarandache, F. *Fuzzy Cognitive Maps and Neutrosophic Cognitive Maps*; ProQuest Information and Learning: Ann Arbor, MI, USA, 2003.
19. Kandasamy, W.V.; Smarandache, F. *Basic Neutrosophic Algebraic Structures and Their Application to Fuzzy and Neutrosophic Models*; Infinite Study; ProQuest Information & Learning: Ann Arbor, MI, USA, 2004; Volume 4.
20. Ye, J.; Cui, W. Neutrosophic Compound Orthogonal Neural Network and Its Applications in Neutrosophic Function Approximation. *Symmetry* 2019, 11, 147. https://doi.org/10.3390/sym11020147.

21. Ashraf, S.; Abdullah, S.; Smarandache, F.; Amin, N.u. Logarithmic Hybrid Aggregation Operators Based on Single Valued Neutrosophic Sets and Their Applications in Decision Support Systems. *Symmetry* 2019, 11, 364. https://doi.org/10.3390/sym11030364.

22. Al-Quran, A.; Hassan, N.; Marei, E. A Novel Approach to Neutrosophic Soft Rough Set under Uncertainty. *Symmetry* 2019, 11, 384. https://doi.org/10.3390/sym11030384.

23. Kandasamy, W.; Sultana, Y. Knowledge processing with fuzzy relational maps. *Ultra Sci. Phys. Sci.* 2000, 12, 242–246.

24. Papageorgiou, E.I.; Markinos, A.T.; Gemtos, T.A. Fuzzy cognitive map based approach for predicting yield in cotton crop production as a basis for decision support system in precision agriculture application. *Appl. Soft Comput.* 2011, 11, 3643–3657.

25. Papageorgiou, E.I.; Kannappan, A. Fuzzy cognitive map ensemble learning paradigm to solve classification problems: Application to autism identification. *Appl. Soft Comput.* 2012, 12, 3798–3809.

26. Alghzawi, A.Z.; Nápoles, G.; Sammour, G.; Vanhoof, K. Forecasting social security revenues in Jordan using fuzzy cognitive maps. In Proceedings of the International Conference on Intelligent Decision Technologies, Algarve, Portugal, 21–23 June 2017; pp. 246–254.

27. Cox, E.D. *Fuzzy Logic for Business and Industry*; Charles River Media, Inc.: Boston, MA, USA, 1995.

28. Smarandache, F. Neutrosophic Set—A Generalization of the Intuitionistic Fuzzy Set. *Int. J. Pure Appl. Math.* 2004, 24. https://doi.org/10.1109/GRC.2006.1635754.

29. Peeva, K. Fuzzy linear systems. *Fuzzy Sets Syst.* 1992, 49, 339–355.

30. Peeva, K.; Kyosev, Y. *Fuzzy Relational Calculus: Theory, Applications and Software (with CD-ROM)*; World Scientific: Singapore, 2004; Volume 22.

31. Peeva, K. Universal algorithm for solving fuzzy relational equations. *Ital. J. Pure Appl. Math.* 2006, 19, 169.

32. De Baets, B. Analytical solution methods for fuzzy relational equations. In *Fundamentals of Fuzzy Sets*; Springer: New York, NY, USA, 2000; pp. 291–340.

33. Di Nola, A.; Lettieri, A.; Perfilieva, I.; Novák, V. Algebraic analysis of fuzzy systems. *Fuzzy Sets Syst.* 2007, 158, 1–22.

34. Nola, A.D.; Sanchez, E.; Pedrycz, W.; Sessa, S. *Fuzzy Relation Equations and Their Applications to Knowledge Engineering*; Springer Science & Business Media: New York, NY, USA, 1989.

35. Li, P.; Fang, S.C. A survey on fuzzy relational equations, part I: Classification and solvability. *Fuzzy Optim. Decis. Mak.* 2009, 8, 179–229.

36. Pappis, C.; Sugeno, M. Fuzzy relational equations and the inverse problem. *Fuzzy Sets Syst.* 1985, 15, 79–90.

37. Miyakoshi, M.; Shimbo, M. Lower solutions of systems of fuzzy equations. *Fuzzy Sets Syst.* 1986, 19, 37–46.

38. Yeh, C.T. On the minimal solutions of max–min fuzzy relational equations. *Fuzzy Sets Syst.* 2008, 159, 23–39.

39. Wu, Y.K.; Guu, S.M. An Efficient Procedure for Solving a Fuzzy Relational Equation with Max–Archimedean t-Norm Composition. *IEEE Trans. Fuzzy Syst.* 2008, 16, 73–84.

40. Lin, J.L. On the relation between fuzzy max–Archimedean t-norm relational equations and the covering problem. *Fuzzy Sets Syst.* 2009, 160, 2328–2344.

41. Lin, J.J.; Wu, Y.K.; Guu, S.M. On fuzzy relational equations and the covering problem. *Inf. Sci.* 2011, 181, 2951–2963.

42. Higashi, M.; Klir, G.J. Resolution of finite fuzzy relation equations. *Fuzzy Sets Syst.* 1984, 13, 65–82.

43. Chen, L.; Wang, P.P. Fuzzy relation equations (I): The general and specialized solving algorithms. *Soft Comput.* 2002, 6, 428–435.

44. Papageorgiou, E.; Spyridonos, P.; Glotsos, D.T.; Stylios, C.; Ravazoula, P.; Nikiforidis, G.; Groumpos, P. Brain tumor characterization using the soft computing technique of fuzzy cognitive maps. *Appl. Soft Comput.* 2008, 8, 820–828. https://doi.org/10.1016/j.asoc.2007.06.006.