How to detect the fourth order cumulant of electrical noise

Joachim Ankerhold$^{1,2}$ and Hermann Grabert$^1$

$^1$ Physikalisches Institut, Albert-Ludwigs-Universität, 79104 Freiburg, Germany
$^2$ Service de Physique de l’Etat Condensé, DSM/DRECAM, CEA Saclay, 91191 Gif-sur-Yvette, France

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It is proposed to measure the current noise generated in a mesoscopic conductor by macroscopic quantum tunneling (MQT) in a current biased Josephson junction placed in parallel to the conductor. The theoretical description of this set-up takes into account the complete dynamics of detector and noise source. Explicit results are given for the specific case of current fluctuations in an oxide layer tunnel junction, and it is shown how the device allows to extract the fourth order cumulant of the noise from the MQT data for realistic experimental parameters.

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Within the last decade electrical noise has moved into the focus of research activities on electronic transport in nanostructures [1], since it provides information on microscopic mechanisms of the transport not available from the voltage dependence of the average current. Lately, attention has turned from the noise auto-correlation function to higher order cumulants of the current fluctuations characterizing non-Gaussian statistics [2, 3]. While theoretical attempts to predict these cumulants for a variety of devices are quite numerous [3], experimental observation is hard because of small signals, large bandwidth detection, and strict filtering demands. A first pioneering measurement by Reulet et al. [4] of the third cumulant of the current noise from a tunnel junction has intensified efforts and several new proposals for experimental set-ups have been put forward very recently, some of which are based on Josephson junctions (JJ) as noise detectors. Lindell et al. [5] employed a Coulomb blockaded JJ to demonstrate that the conductance of the junction in the Coulomb gap region is sensitive to the non-Gaussian character of noise applied to the junction. A modification of this set-up was suggested by Heikkilä et al. [6] to get specific information on the third cumulant of the noise. Another recent experiment [7] has observed activated-over-the-barrier-jumps of a JJ biased by a noisy current. The data are consistent with resonant activation produced by the second cumulant of the noise at the plasma frequency of the junction. For a measurement of the full distribution of current fluctuations Tobiska and Nazarov [8] suggested to use an array of overdamped JJs acting as a threshold detector for rare current fluctuations triggering over-the-barrier jumps. Since in the overdamped limit retrapping spoils the built-up of a detectable voltage, it was argued by Pekola et al. [9] that an experimentally more accessible detector would extract the noise characteristics from modifications of the macroscopic quantum tunneling (MQT) rate in an underdamped JJ.

In all experimental set-ups to measure higher order cumulants realized and proposed so far, heating is one of the major experimental obstacles [10]. Thus, experiments have primarily attempted to establish just the unspecified non-Gaussian nature of the noise or to measure the third cumulant (skewness). The latter one is particularly accessible since it can be discriminated from purely Gaussian noise due to its asymmetry, e.g. when inverting the current through the conductor. This is in contrast to the fourth order cumulant (sharpness), which on the one hand due to heating effects may be completely hidden behind the second and the third one, but on the other hand is required to gain an essentially complete characterization of the distribution of current fluctuations. In this Letter we propose and analyze a set-up, with the circuit diagram depicted in Fig. 1 which allows to detect the fourth order cumulant of the current noise generated by a nanoscale conductor. Since this conductor is placed in parallel to a current biased JJ in the zero voltage state, no heating occurs prior to the decay of this state by MQT. However, the MQT rate is modified in a specific way by the even higher order cumulants characterizing the non-Gaussian current fluctuations of the conductor.

The complete statistics of current noise generated by a mesoscopic conductor can be gained from the generating functional

$$G[\phi] = e^{-S_0[\phi]} = \langle \mathcal{T} \exp \left[ \frac{i}{\hbar} \int_C dt I(t) \phi(t) \right] \rangle,$$

where $I(t)$ is the current operator and $\mathcal{T}$ the time ordering operator along the Kadanoff-Baym contour $C$. Time correlation functions of arbitrary order of the current are determined from functional derivatives of $G[\phi]$, in particular, the average current

$$C_1(t) = \langle I(t) \rangle = i e \frac{\partial S_{\mathcal{C}}[\phi]}{\partial \phi(t)} \bigg|_{\phi=0}$$
and the current auto-correlation function
\[ C_2(t, t') = \langle I(t) I(t') \rangle = e^2 \partial^2 S_G[\phi]/\partial \phi(t) \partial \phi(t') \big|_{\phi=0} . \]

Higher order functional derivatives give the cumulants related to non-Gaussian current fluctuations
\[ C_n(t_1, \ldots, t_n) = -(-ie)^n \partial^n S_G[\phi]/\partial \phi(t_1) \cdots \partial \phi(t_n) \big|_{\phi=0} . \]

We remark that the functional \( S_G[\phi] \) carries the full frequency dependence of all current cumulants and not just their time averaged zero frequency values usually studied in the field of full counting statistics \(^2\).

By way of example let us consider an Ohmic resistor of resistance \( R \) in thermal equilibrium at inverse temperature \( \beta \). Then, the functional \( S_G[\phi] \equiv S_R[\phi] \) takes the well-known form
\[ S_R[\phi] = \frac{1}{2} \frac{h}{e^2 R} \int_c dt \int_c dt' \alpha(t-t') \phi(t) \phi(t') \]
where
\[ \alpha(t) = \frac{\pi}{2(h\beta)^2 \sinh(\pi t/h\beta)} . \]

The quadratic form reflects the Gaussian nature of the current fluctuations in this case which implies that all cumulants except for \( C_2 \) vanish. On the other hand, for a tunnel junction with many transmission channels, where each channel has a small transmission coefficient \( T_i \) leading to the dimensionless conductance \( g_T = h/(4\pi e^2 R_T) = \pi \sum T_i \), where \( R_T \) is the tunneling resistance, one has \(^11\)
\[ S_T[\phi] = -4g_T \int_c dt \int_c dt' \alpha(t-t') \sin^2 \left( \frac{\phi(t) - \phi(t')}{2} \right) . \]

Here, the periodicity in \( \phi \) reflects the discreteness of the transferred charges associated with non-Gaussian current fluctuations.

To gain information on the noise of the conductor, it may be placed in parallel to a current biased JJ as depicted in the circuit diagram of Fig.\(^1\). For a bias current \( I_b \) below the critical current \( I_c \), the JJ is in its zero voltage state and the bias current flows as a supercurrent entirely through the JJ branch of the circuit. Consequently, no heating occurs in the conductor and the total system can easily be kept at low temperatures, where the decay of the zero voltage state occurs through MQT. The rate of this process depends with exponential sensitivity on the current fluctuations of the conductor so that the JJ acts as a noise detector.

The MQT rate \( \Gamma \) can be calculated in the standard way \(^12\) \(^13\) from the imaginary part of the free energy \( F \), i.e.,
\[ \Gamma = \frac{2}{\hbar} \text{Im} \{ F \} , \]
where \( F = -(1/\beta) \ln(Z) \) is related to the partition function \( Z = \text{Tr} \{ e^{-\beta H} \} \). In the path integral representation one has
\[ Z = \int \mathcal{D}[\theta] e^{-S[\theta]} , \]
which is a sum over all imaginary time paths with period \( h/\beta \) of the phase difference \( \theta \) across the JJ weighted by the dimensionless action \( S[\theta] = S_{JJ}[\theta] + S_G[\theta/2] \). Here
\[ S_{JJ}[\theta] = \frac{1}{h} \int_0^{h/\beta} dt \left[ 1/2 \theta^2 C_J \dot{\theta}(t) + U(\theta) \right] \]
is the action of the bare JJ and \( S_G \) is the generating functional of current fluctuations of the conductor introduced above. In Eq. \(^9\) \( \varphi_T = h/2e \) denotes the reduced flux quantum, \( C_J \) is the capacitance of the JJ, and the tilted washboard potential \( U(\theta) = -E_J[\cos(\theta) - s \theta] \), where \( E_J \) is the Josephson energy and \( s = I_b/I_c \). \( \Gamma \) is calculated by expanding about the unperturbed bounce trajectory, an extremal \( \delta S[\theta] = 0 \) periodic path in the inverted barrier potential. By approximating a well-barrier segment around the well minimum (plasma frequency) at bias current \( s \), one finds an analytic solution in the limit of vanishing temperature, i.e.,
\[ \theta_B(\tau) = -\frac{\delta \theta_0}{\cosh^2(\Omega_T/2)} \theta_0 . \]

Here, \( \Omega = \Omega(s) \) is the frequency for small oscillations around the well bottom (plasma frequency) at bias current \( s \), \( M = \frac{\varphi_0^2}{2} C_J \), and \( \delta \theta_0 \) denotes the exit point determined from \( U(\theta_m) = U(\theta_m + \delta \theta_0) \). The corresponding MQT rate reads
\[ \Gamma_0 = 6\sqrt{6} \Omega \bar{b}/\hbar \pi \exp \left( -\frac{36 \bar{b}^2}{5 M^2} \right) \]
where \( \bar{b} = (2M\Omega^2/27) \delta \theta_0^2 \) is the barrier height.

Following the theory of the effect of an electromagnetic environment on MQT \(^12\), the partition function can now be calculated for arbitrary coupling between detector and conductor based on a numerical scheme developed in \(^13\). Analytical progress is made when the noise generating element has a dimensionless conductance \( g \ll E_J/\hbar \Omega \) so that the influence of the noise on the MQT rate can be calculated by expanding about the unperturbed bounce which gives
\[ \Gamma = \Gamma_0 e^{-S_G[\theta_B/2]} . \]

The correction \( S_G[\theta_B/2] \) is usually dominated by the second cumulant \( C_2 \) (width) and the fourth cumulant \( C_4 \) (sharpness). Note that this approximation still contains the full dynamics of detector and noise source since any approximation relying on a time scale separation, as e.g. the adiabatic limit considered in \(^9\), is usually not applicable.

Now, in case of a tunnel junction \(^14\) as noise element one finds for \( S_G[\theta_B/2] \equiv S_T[\theta_B/2] \) from \(^11\) and \(^2\)
\[ S_T[\theta_B/2] = \frac{GT}{4\pi} \int_0^\infty d\omega \omega |\tilde{\rho}(\omega)|^2 \]
\[ + \frac{4\pi}{87} \frac{h^2}{e^2 R_T} \int_c dt \int_c dt' \alpha(t-t') \sin^2 \left( \frac{\phi(t) - \phi(t')}{2} \right) . \]
with
\[ \tilde{\rho}(\omega) = \int_{-\infty}^{\infty} d\tau \, e^{i\theta_B(\tau)/2} \, e^{i\omega \tau}. \]

By expanding the first exponential and performing the Fourier transform for each power of \( \theta_B(\tau) \) separately, the relevant part \( \rho(\omega) = \tilde{\rho}(\omega) - 2\pi \delta(\omega) \) reads
\[ \rho(\omega) = \frac{\pi}{4} \frac{\omega}{\sinh(\pi \omega/\Omega)} \sum_{k=1}^{\infty} \left( \frac{2i\delta \theta_0}{k! (2k-1)!} \right)^k \prod_{l=1}^{k-1} \left( \frac{\omega^2}{\Omega^2} + l^2 \right). \]

This way, Eq. (5) can be cast into
\[ S_T[\theta_B/2] = \frac{g_T}{4\pi^3} \sum_{k,k'} I_{k,k'} \, \delta \theta_0^{k+k'} \] (7)

with the coefficients
\[ I_{k,k'} = \frac{(-1)^{(3k+k')/2} \, 2^{k+k'}}{k! \, k'! (2k-1)! (2k'-1)!} \, A_{kk'}. \]

Here,
\[ A_{kk'} = \int_0^{\infty} dy \, \frac{y^3 e^y}{(e^y - 1)^2} \left[ \prod_{l=1}^{k-1} \left( \frac{y^2}{4\pi^2} + l^2 \right) \right] \times \left[ \prod_{l=1}^{k'-1} \left( \frac{y^2}{4\pi^2} + l^2 \right) \right] \]

with \( A_{kk'} = A_{k'k} \) so that \( I_{k,k'} \neq 0 \) only for \( k + k' \) even. This means that all odd cumulants of the fluctuating current vanish according to a vanishing net current \( \langle I(t) \rangle = 0 \) through the conductor. Specifically, one finds
\[
\begin{align*}
A_{11} &= 6\zeta(3) \\
A_{22} &= 6\zeta(3) + \frac{51\zeta(5)}{2\pi^2} + \frac{71\zeta(7)}{16\pi^4} \\
A_{31} &= 24\zeta(3) + \frac{69\zeta(5)}{4\pi^2} + \frac{71\zeta(7)}{16\pi^4}.
\end{align*}
\]

The terms in the sum (7) related to a contribution of order \( \delta \theta_0^{k+k'} \) determine the impact of the \( (k+k') \)th-moment of the current fluctuations of the tunnel junction onto the MQT process. Since in Eq. (5) the term of order \( \delta \theta_0^k \) contains contributions centered around \( \omega = 0, \Omega, \ldots, k\Omega \), the influence of the \( (k+k') \)th-moment results from mode mixing between fluctuations with frequencies \( l \Omega \) and \( l' \Omega \) where \( l \leq k, k' \leq k' \).

In lowest order, \( k + k' = 2 \), one gains from Eq. (7) the Gaussian noise contribution providing a correction to the bare MQT rate
\[ \Gamma_T^{(2)} = \Gamma_0 \exp \left[ -\frac{6\zeta(3) \, g_T}{\pi^3} \, \delta \theta_0(s)^2 \right]. \] (8)

Apparently, this reflects the well-known fact that Gaussian noise leads to a reduction of the tunneling rate [12].

At order \( \delta \theta_0^3 \) the sum (7) gives three contributions, namely, \( k = 1, k' = 3 \) and \( k = 3, k' = 1 \) with \( A_{13} = A_{31} \) as well as \( k = 2, k' = 2 \) with \( A_{22} \). This leads to
\[ \Gamma_T^{(4)} = \Gamma_T^{(2)} \exp \left[ \frac{4 \, g_T}{\pi^3} \, (2A_{31} - A_{22}) \, \delta \theta_0(s)^4 \right] \] (9)

so that the fourth order cumulant of the current noise contains both, fluctuations that suppress tunneling (related to \( A_{22} \)) and fluctuations that increase MQT (related to \( A_{31} \)). Since \( 2A_{31} - A_{22} > 0 \), the total impact of the fourth moment leads to an enhancement of the MQT rate.

To obtain explicit results, one derives for the harmonic+cubic potential \( V(\theta) \) an expression for the amplitude \( \delta \theta_0(s) \) of the bounce, namely,
\[ \delta \theta_0(s) = \frac{2\sqrt{1-s^2}}{s}. \]

Accordingly, the barrier height scales with the dimensionless current \( s \) as \( V_b(s) = (2E_f/3)(1-s^2)^{3/2}/s^2 \), and the plasma frequency reads \( \Omega(s) = \sqrt{2E_f/E_C/h} \) \((1-s^2)^{1/4} \) with charging energy \( E_C = 2e^2/C \). Experimentally, in the standard procedure [13] to measure the MQT rate, a bias current pulse of height \( I_b \) and duration \( t \) adiabatically turned on and a voltage pulse is detected when the JJ switches to its finite voltage state. This procedure is performed a few thousand times to build up switching histograms that determine the switching probabilities
\[ P(s) = 1 - e^{-\Gamma(s) \, t}. \]

In Fig. 2 these so-called s-curves are shown for various values of the duration of the current pulse \( t \). The parameters chosen are accessible in realistic experiments. Apparently, for short pulses when the switching occurs for values of the bias current close to the critical current of the JJ, the effect of the non-Gaussian noise fluctuations is completely suppressed due to a decreasing amplitude \( \delta \theta_0(s) \) of the bounce. However, for longer pulses (or equivalently, for shorter pulses in the tails towards lower \( s \) values) the fourth order cumulant leads to a substantial influence, mostly dominated by a shift to smaller
In the standard analysis of MQT data, the rate is purely Gaussian noise and the same JJ are not available directly in the s-curves since reference curves for non-Gaussian fluctuations.

The sensitivity of the JJ in the MQT range even to weak height around the tunnel junction is larger than 10% for values of the pulse presence of purely Gaussian noise and in presence of a contrast in the switching probabilities between MQT in non-Gaussian fluctuations of the tunnel junction. The due to an effective decrease of the barrier height by the Gaussian noise contribution. As seen from Eq. (9) this is

$$s = \{ -\ln[\Gamma(s)/a_q] \}^{2/3}$$

with the prefactor $a_q = 6\sqrt{3}\gamma V_b/\hbar\pi$ gives rise to an essentially straight line, thus reflecting the $s$-dependence of $V_b$. This holds true also for weak purely Gaussian noise, but any non-Gaussian contributions to the MQT rate can be expected to lead to deviations from this scaling behavior. For this purpose, s-curves for shorter pulses, i.e. larger $s$-values, where non-Gaussian effects are absent, can be used to determine the slope $b$ of $B(s) - B(s^*) = b(s - s^*)$ (reference point $s^*$ close to 1). For longer pulses corresponding to smaller values of the switching current, this straight line $b(s - s^*)$ is then compared with the actual $B(s)$ as depicted in Fig. 4. Purely Gaussian noise shows basically no deviations from the linear behavior over a wide range of $s$-values, while closer inspection of the analytical rate expression reveals that small deviations lead to a slight increase of the actual $B(s)$ towards lower $s$. In contrast, non-Gaussian noise displays substantial deviations already for much larger values of $s$ and leads to a decrease of the actual $B(s)$ compared to the reference line extracted for $s$ close to 1. Thus, plotting $B(s)$ allows to directly discriminate the impact of higher than second order cumulants in the noise fluctuations of the conductor. Further, by fitting $B(s)$ with Eq. (3), the coefficient $2A_{31} - A_{22}$ related to $C_4$ in Eq. (3) can be extracted.

The formulation developed above in Eq. (3) is completely general and applies to any nanoscale conductor in parallel to a JJ. A systematic expansion of the action $S_C[\theta B/2]$ in powers of $(\theta B/2)^n$ determines the dynamical impact of the $n$th order cumulant $C_n$ onto the MQT process.

To summarize we have proposed a nanoelectrical circuit where a JJ placed in parallel to an arbitrary conductor acts as detector for non-Gaussian current noise. Since no net current flows through the noise source, heating effects are suppressed and one obtains access to the even order cumulants of the distribution function which are notoriously difficult to detect. For experimentally realistic parameters we have explicitly shown how the fourth order cumulant of a tunnel junction can be extracted.

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