On the stability of strange dwarf hybrid stars

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We investigate the stability of stars with a density discontinuity between a high-density core and a very low density mantle. Previous work on “strange dwarfs” suggested that such a discontinuity could stabilize stars that would have been classified as unstable by the conventional criteria based on extrema in the mass-radius relation. We investigate the stability of such stars by numerically solving the Sturm-Liouville equations for the lowest-energy modes of the star. We find that the conventional criteria are correct, and strange dwarfs are not stable.

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I. INTRODUCTION

The density profile of stars whose temperature is low compared to the Fermi energy of their constituent fermions can be understood in terms of a zero-temperature equation of state \( \varepsilon(P) \) which gives the energy density as a function of the pressure. Spherically symmetric configurations can then be obtained by solving the Tolman-Oppenheimer-Volkoff (TOV) equation [1, 2], but this procedure yields both stable and unstable solutions. As we will describe in more detail below, there are two commonly used methods for determining stability of solutions to the TOV equation. One is to explicitly obtain the eigenmodes of radial oscillation by solving the relevant Sturm-Liouville equation. The other, developed by Bardeen, Thorne, and Meltzer (BTM) [3], is based on counting the extrema of the mass-radius relation.

It has generally been accepted that BTM proved the validity of their criterion [3–5]. However, Glendenning and Weber have questioned this [6, 7]. They proposed a class of configurations, “strange dwarfs,” that arose from an equation of state with a phase transition at a relatively low critical pressure, around neutron drip. The phase transition introduces a large density discontinuity between an ultra-dense strange matter phase [8–10] and a much lower density phase of conventional white-dwarf material (a degenerate electron plasma). Glendenning and Weber suggested that this discontinuity invalidated the BTM mass-radius stability criteria, allowing for a family of strange dwarf stars (with a strange matter core and a white-dwarf mantle) that, although occurring in a segment of the mass-radius relation that would be unstable according to the BTM mass-radius criteria, were in fact stable.

In this paper, we investigate Glendenning and Weber’s suggestion by solving the Sturm-Liouville equation for the radial oscillation modes of stars with a regulated discontinuity in their equation of state. By studying the behavior of the modes as the regulating width tends to zero, we confirm that the BTM mass-radius criteria correctly reflect the behavior of the radial eigenmodes, and give an accurate account of the stability of the star. This means that strange dwarfs are not stable.

We work in natural units, where \( \hbar = c = 1 \).

II. TWO METHODS FOR DETERMINING THE STABILITY OF STARS

There are two standard methods for determining whether a solution of the TOV equation represents a stable star: (a) the BTM criteria based on extrema in the mass-radius curves, and (b) analysis of the lowest eigenmodes of radial oscillation [3]. We now describe both methods.

A. The mass-radius stability criteria

For material with zero-temperature equation of state \( \varepsilon(P) \), the spherically symmetric gravitationally bound configurations are described by the TOV equation,

\[
\frac{dP}{dr} = -G \frac{(P(r) + \varepsilon(r))(m(r) + 4\pi r^3 P(r))}{r(r - 2Gm(r))},
\]

\[
\frac{dm}{dr} = 4\pi r^2 \varepsilon(r).
\]

where \( P(r) \) and \( \varepsilon(r) \) are the pressure profile and energy-density profile of the star, and \( m(r) \) is the mass enclosed within radius \( r \). The boundary conditions are \( m(0) = 0 \) and \( P(0) \) is some chosen central pressure \( P_{\text{cent}} \). Integrating the TOV equation from \( r = 0 \) outwards, the pressure drops monotonically until at \( r = R \), the radius of the star, the pressure reaches zero. The mass of the star is \( M = m(R) \).

The metric of the spherically symmetric, static spacetime inside the star can be written as

\[
ds^2 = e^{2\nu(r)} dt^2 - e^{2\lambda(r)} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\]

where

\[
e^{2\lambda(r)} = \left(1 - \frac{2Gm(r)}{r}\right)^{-1}.
\]
The metric coefficient $\nu(r)$ is given by \cite{11}
\[ \frac{d\nu}{dr} = \frac{G}{r} \left( \frac{m(r) + 4\pi r^3 P(r)}{r - 2Gm(r)} \right), \tag{4} \]
with the boundary condition $\nu(R) = (1/2) \ln(1 - 2GM/R)$ which ensures that it matches to the Schwarzschild solution at $r = R$.

By varying the central pressure $P_{\text{cent}}$ one generates a one-parameter family of stationary configurations, tracing out a curve in the mass-radius plane. A typical curve for stars made of matter that forms a degenerate electron gas at low pressure, and a degenerate neutron gas at high pressure \cite{11, 12}, is shown in Fig. 1. All points on the mass-radius curve are stationary configurations, but not all stationary configurations are stable against radial oscillations. As we will discuss in Sec. \cite{III} a configuration is stable only if all its radial modes are stable. Bardeen, Thorne, and Meltzer (BTM) \cite{3} gave a simple formulation:

**BTM stability criteria:**

1. At each extremum where the $M(R)$ curve rotates counter-clockwise with increasing central pressure, one stable mode becomes unstable. \tag{5}

2. At each extremum where the $M(R)$ curve rotates clockwise with increasing central pressure, one unstable mode becomes stable.

We will now apply these criteria to Fig. 1. The configurations with the lowest central pressure are planet-like, with $M \ll M_\odot$ and $M \propto R^3$. These are stable. As the central pressure rises, the mass rises, giving white dwarf configurations. Then at the Chandrasekhar mass we reach an extremum $(a)$ where the $M(R)$ curve bends counterclockwise, indicating that a stable mode becomes unstable. The $M(R)$ curve then enters an unstable interval, bending counterclockwise again through a second extremum $(b)$, where a second mode becomes unstable. At the third extremum $(c)$ the curve bends clockwise, so one of the two unstable modes becomes stable. At the fourth extremum $(d)$ the curve bends clockwise again, and the remaining unstable mode becomes stable. We are now on a stable branch, the “compact branch”, containing neutron (or hybrid) stars. As the central pressure continues to rise the radius shrinks rapidly and the mass rises, until we reach the fifth extremum $e$ where the curve bends counterclockwise and the star becomes unstable. For typical nuclear matter equations of state, as central pressure is increased further, the curve continues to spiral counterclockwise, so more and more modes become unstable.

As previously noted, Fig. 1 is a typical mass-radius curve for stars made of degenerate electron or neutron matter. One can explore more unusual forms of matter where this typical form of matter has, at some critical pressure, a phase transition to an exotic phase such as quark matter. If this critical pressure is large, then instead of spiraling at high central pressure, the mass-radius curve may feature another stable “twin” or “third family” branch \cite{13, 15}. Alternatively, the phase transition could occur at a low critical pressure: this will be the topic of Sec. \cite{III}

**B. The Sturm-Liouville spectrum**

The BTM stability criteria are a convenient heuristic, but the fundamental criterion for stability is based on computing the spectrum of radial oscillations of the star. The radial oscillations are described \cite{17} by the time-dependent displacement
\[ \delta r_n(r, t) = \frac{e^{\nu(r)}}{r^2} u_n(r) e^{i\omega_n t} \tag{6} \]
where $\nu(r)$ is a solution with eigenvalue $\omega_n^2$ to the Sturm-Liouville eigenvalue problem
\[ \frac{d}{dr} \left( \Pi(r) \frac{du_n}{dr} \right) + \left( Q(r) + \omega_n^2 W(r) \right) u_n(r) = 0, \tag{7} \]
where
\[
\Pi(r) = \frac{e^{\lambda(r) + 3\nu(r)}}{r^2} \Gamma(r) P(r),
\]
\[
Q(r) = -4 \frac{e^{\lambda(r) + 3\nu(r)}}{r^3} \frac{dP}{dr}
- 8\pi \frac{e^{3\lambda(r) + 3\nu(r)}}{r^2} P(r) \left( \varepsilon(r) + P(r) \right)
+ \frac{e^{\lambda(r) + 3\nu(r)}}{r^2} \left( \varepsilon(r) + P(r) \right) \left( \frac{dP}{dr} \right)^2,
\]
\[
W(r) = \frac{e^{3\lambda(r) + \nu(r)}}{r^2} \left( \varepsilon(r) + P(r) \right),
\]
\[
\Gamma(r) = \frac{\varepsilon(r) + P(r)}{P(r)} \frac{dP}{d\varepsilon}.
\]

The boundary conditions for the eigenvalue problem are
\[
u_n \propto r^3 \quad \text{at } r = 0 \quad (8)
\]
\[
\frac{d\nu_n}{dr} = 0 \quad \text{at } r = R \quad (9)
\]

where \(R\) is the surface of the star.

The solutions to the Sturm-Liouville eigenvalue problem are a discrete set of eigenfunctions \(\nu_n(r)\) with eigenvalues \(\omega_n^2\) which are the squared frequencies of the oscillation modes. The eigenvalues, which are real, form a lower-bounded infinite sequence \(\omega_1^2 < \omega_2^2 < \omega_3^2 < \cdots\). For the \(n\)th mode, if \(\omega_n^2 > 0\), the frequency is real and the mode is stable and oscillatory. However, if \(\omega_n^2 < 0\) then the frequency is purely imaginary and the mode is unstable and exponentially grows or decays.

To determine the overall stability of the star, it is sufficient to look just at the lowest eigenvalue, \(\omega_1^2\). If \(\omega_1^2 > 0\), then all \(\omega_n^2 > 0\) and the star is stable. If \(\omega_1^2 < 0\), then there is (at least) one unstable mode and the star is unstable [22].

\[\text{III. FIRST-ORDER TRANSITIONS AND STRANGE DWARFS}\]

In Refs. [6] [7], Glendenning and collaborators claimed that the BTM mass-radius criteria described in Sec. IIA were not valid for equations of state that contained a sharp first-order transition from a low-density gas of ordinary matter with degenerate electrons to a high-density phase which in their case was strange quark matter. The mass-radius curve for their equation of state was similar to Fig. 1 except that they claimed that the interval of the mass radius curve from \(c\) to \(d\) was stable, and constituted a new family of stars, ‘strange dwarfs’.

To study this claim, we use an equation of state similar to the one proposed in Refs. [6] [7].

\[
v(P) = \begin{cases} 
\varepsilon_{\text{BPS}}(P) & P \leq P_{\text{crit}} \\
kp + 4B & P > P_{\text{crit}}
\end{cases}
\]

This equation of state is plotted in Fig. 2. At low pressure it is the Baym-Pethick-Sutherland (BPS) equation of state for degenerate matter [18]. At a critical pressure \(P_{\text{crit}}\) there is a sharp first-order transition with a very large (by a factor of order \(10^3\)) discontinuity in the energy density to a phase that is modeled using a constant-sound-speed (CSS) equation of state [19] [21]. This could correspond to some exotic phase such as strange quark matter.

Glendenning et al. chose \(P_{\text{crit}} = P_{\text{drip}} = 3742\, \text{MeV}^4\), which is the pressure in the BPS equation of state corresponding to neutron drip density, \(\varepsilon_{\text{drip}} = 4 \times 10^{11}\, \text{g cm}^{-3} = 1.6 \times 10^6\, \text{MeV}^4\). For their physical model this is the highest possible transition pressure, since if it were any larger neutrons would drip out of the crust and be attracted into the strange matter core [22]. Additionally, they chose \(k = 3\) and \(B^{1/4} = 145\, \text{MeV}\).

To study the stability of stellar configurations made of matter obeying this equation of state, we regulated the phase transition between nuclear matter and strange quark matter, smoothing the first-order jump into a crossover with width \(\delta P\). This allows us to solve the Sturm-Liouville eigenvalue problem using standard numerical tools. These would fail if one tried to directly tackle the discontinuous equation of state, but by sending
\[ \delta P \to \text{small enough values we can see what the limiting behavior in the discontinuous case will be.} \]

Our equation of state took the form

\[
\begin{align*}
\varepsilon(P) &= \frac{1}{2} \left( 1 - \tanh \left( \frac{P - P_{\text{crit}}}{\delta P} \right) \right) \varepsilon_{\text{BPS}}(P) \\
&\quad + \frac{1}{2} \left( 1 + \tanh \left( \frac{P - P_{\text{crit}}}{\delta P} \right) \right) (kP + 4B)
\end{align*}
\] (11)

We studied the solutions of the TOV equation and their stability, according to both the BTM mass-radius criteria and explicit numerical solution of the Sturm-Liouville eigenvalue problem, in the limit \(\delta P \to 0\) where the crossover became very rapid, approximating a discontinuity. For the low pressure region we fitted the BPS tabulated data to a continuous function.

IV. RESULTS AND CONCLUSIONS

In Fig. 3 we plot the mass radius curve for solutions to the TOV equation for the regulated equation of state with \(\delta P = 100\,\text{MeV}^4\).

This mass-radius curve looks qualitatively like the schematic shown in Fig. 1. The extremum labelled \(b\) occurs where the central pressure in the star reaches the critical pressure \(P_{\text{crit}}\). This means that all stars with central pressure below the value at \(b\) (i.e. on the curve from \(b\), down to \(a\) and further along the white dwarf branch) have only conventional degenerate-electron matter: the center of the star is not yet dense enough to contain any of the high-density phase. For central pressures larger than at \(b\), the star contains a core of the high-density phase.

On the curve from \(b\) through \(c\) and on to \(d\) the core is small relative to the surrounding crust of nuclear matter. Glendenning et al., using a model where the high-density phase is strange quark matter, called these stars “strange dwarfs”. At higher central pressures, on the curve from \(d\) to \(e\), the core becomes large and its gravitational attraction squeezes the crust down to a thin layer: In Glendenning et al.’s model these are strange quark stars with a thin nuclear crust.

If there is a sharp first-order transition in the equation of state then point \(b\) is a cusp in the \(M(R)\) relation. For finite but very small transition width \(\delta P \lesssim 1\,\text{MeV}^4\) the cusp becomes a minimum at which, according to the BTM criteria, the second-lowest mode goes from stable to unstable as central pressure increases. In our calculation we use values of \(\delta P\) in the range 10 to 100\,\text{MeV}^4, in which case the mass radius relation develops a more complicated structure at \(b\) which may have multiple extrema as the curve spirals and then “uncoils” again. This structure occurs in a very small range of masses and radii near \(b\), and is invisible on the scales shown in Fig. 3.

The details of this structure depend on the exact profile of the regulated transition, but, as we will see, (i) the lowest eigenmode remains negative so all these configurations are unstable; (ii) as central pressure increases through \(b\), the net outcome is that the second-lowest mode goes from stable to unstable; (iii) this behavior is not relevant to the stability of strange dwarfs, which lie between \(c\) and \(d\) on the mass-radius curve.

According to the BTM mass-radius criteria (Sec. II A), only the portions of the mass-radius curve denoted by a solid line in Fig. 3 are stable. However, Glendenning et al. claim that the portion between extrema \(c\) and \(d\), corresponding to the strange dwarfs, is also stable. To test this we solved the Sturm-Liouville eigenvalue problem for stars with a range of central pressures.

Our results are displayed in Figs. 4 5 6 7 where we have used an arcsinh scale on the y axes. This has the wide dynamic range of a log scale while also including zero and negative values.

In Fig. 8 we show the two lowest eigenvalues, \(\omega_0^2\) and \(\omega_1^2\), as a function of central pressure. We see that these behave in a way that is consistent with the BTM mass-radius criteria: at extremum \(a\) the lowest mode becomes unstable (\(\omega_0^2\) goes below zero). At \(b\) the next-to-lowest mode also becomes unstable, and then at extremum \(c\) it becomes stable again. The lowest mode remains unstable until we reach extremum \(d\), at which point it becomes stable again.

In Fig. 9 we zoom in on the range of central pressures between extremum \(c\) and \(d\), where Glendenning et al. claim that there are stable strange dwarf configurations. In this range the lowest radial eigenmode remains negative,
$\Omega = 6.6 \times 10^{-22} \text{ MeV}$

**Figure 4:** The squared frequencies of the two lowest radial oscillations for the TOV solutions plotted in Fig. 3. For stellar configurations with central pressures between $a$ and $d$ we find that $\omega_0^2 < 0$ so these configurations are unstable. This agrees with the BTM criteria (5).

$\delta p = 100 \text{ MeV}^4$

$\delta p = 50$

$\delta p = 10$

**Figure 6:** The $n = 0, 1$, and 2 eigenfunctions $u_n(r)$ (see Eq. 6) for a TOV solution with central pressure $7 \times 10^6 \text{ MeV}^4$, which lies between points $c$ and $d$ in Fig. 3. At this central pressure, the phase transition is located at $r = 2.4 \text{ km}$. As expected for Sturm-Liouville eigenfunctions, $u_n(r)$ has $n$ nodes. The arcsinh scale exaggerates the sharpness of the zero-crossing of the $n = 2$ eigenfunction.

$\Omega = 6.6 \times 10^{-22} \text{ MeV}$

**Figure 5:** The squared frequencies of the four lowest oscillation modes, as in Fig. 3 but magnified to more clearly show the range of central pressures from $c$ to $d$ where strange dwarfs were hypothesized to occur. Between $c$ and $d$, $\omega_0^2 < 0$ and thus strange dwarfs are unstable.

$\delta p = 100 \text{ MeV}^4$

$\delta p = 50$

$\delta p = 10$

**Figure 7:** Dependence of the lowest squared frequency on the regulator $\delta P$ that gives the phase transition a non-zero width. In the limit $\delta P \to 0$, the lowest eigenvalue $\omega_0^2$ remains negative for all central pressures between $a$ and $d$. 

Phase transition

Central Pressure (MeV$^4$)

Radial distance (km)
indicating that these solutions to the TOV equation are unstable. Comparing with Fig. 2 in Ref. [6], it seems likely that Glendenning et al. mistook the second-lowest eigenmode for the lowest one, giving them the impression that these configurations were stable.

To check that we have found the lowest eigenmode we show in Fig. [6] the eigenfunctions of a configuration with central pressure of $7 \times 10^4$ MeV$^4$, which lies between extrema $c$ and $d$. The $u_0$ eigenmode has no nodes, indicating that it is indeed the lowest mode. In general the $n$-th eigenfunction has $n$ nodes, as expected for solutions to a Sturm-Liouville problem.

The eigenvalue spectra shown in Figs. [4] and [5] were calculated for a regulator width $\delta P = 100$ MeV$^4$. To show that the results carry over to the discontinuous limit, we show in Fig. [2] the dependence of $\omega_0$ on central pressure for several different regulator widths. The spectrum shows some dependence on the regulator width when $P_{\text{crit}}$ is close to $P_{\text{crit}}$ (near $b$ on the $M(R)$ curve), which is expected since this is where a tiny core of the high density phase first appears in the star.

However, the lowest eigenvalue remains negative as the transition becomes sharper ($\delta P \to 0$), and in the strange dwarf region (between $c$ and $d$) there is very little sensitivity to the regulator.

We conclude that the BTM mass-radius criteria for the stability of stars are valid in the presence of an arbitrarily sharp jump in the energy density as a function of pressure. Our results imply that the strange dwarfs proposed in Refs. [6, 7] are not stable.

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