Study on stress concentration of elastic plates under special constraints

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Abstract. The dual form differential governing equations for elastic thin plates widely used in the engineering practice are established in the state space formalism, and the complete solution space is obtained, including zero eigensolutions and nonzero eigensolutions. Moreover, the problems of the non-homogeneous lateral boundary conditions and the non-homogeneous governing equations are studied systematically. In numerical examples, we discussed stress distributions for the bending problems of thin plates, and analyzed the stress concentrations due to displacement boundary constraints using the analytical method.

1. Introduction

In modern engineering design, the application of elastic plate is more and more extensive [1, 2]. Due to the need of engineering development, the traditional structures between slab and beam, slab and column can not fully meet the structural requirements. In some specific building structures, the elastic plate will produce stress concentration because of the special constraints or because of the gravity and other loads [3]. This kind of stress concentration will have a great influence on the fatigue life of elastic plates, and even cause serious engineering accidents.

The influence of transverse shear force on deformation is neglected in the classical elastic plate study. If the thickness of the elastic plate is large enough to form a medium thick plate or the aperture of the hole in the plate is the same order of magnitude as the thickness of the plate, there will be a considerable error in the research of the elastic plate. Considering the study of the edge of the elastic plate under special constraints, the thickness of the elastic plate and the openings, the famous Reissner type plate shell theory has been widely used [4]. The theory can be applied to thin and thick plates and shells by considering the transverse shear of plates. It provides a scientific theoretical guidance for the study of stress concentration of elastic plates. Based on the theory of Reissner type plates and shells, the elastic plates with special constraints can be explored comprehensively.

In recent years, there has been a great breakthrough in solving the problem of elasticity. Zhong took the lead in introducing Hamiltonian system and symplectic mathematics into elasticity to solve the problems of poor accuracy, incomplete analytical solution and small range of analytical solution brought by traditional methods [5]. The problems in Lagrange system have been successfully transferred to symplectic system, which makes the research on the related problems of elastic plates rise a lot. Among them, the research on plate bending has realized the exploration of plate bending, and the successful research results on plate bending have been obtained. With the help of symplectic system, this paper studies the stress concentration problem of elastic plates under special constraints, and further obtains...
the stress concentration effect of elastic plates under different conditions in relevant projects, so as to make a reasonable analysis.

2. Solution method

For plane elastic problems, the energy density function is

\[ L = \frac{1}{2} (\lambda + 2G) \left[ \ddot{u}^2 + \left( \frac{\partial v}{\partial y} \right)^2 \right] + \lambda \dot{u} \frac{\partial v}{\partial y} + \frac{1}{2} G \left( \frac{\partial u}{\partial y} + \ddot{v} \right)^2 \]  

in which \( \lambda \) and \( G \) are elastic constants, \( u \) and \( v \) are displacement components. According to the properties of Hamiltonian operator matrix, the general solution of basic governing equation can be expressed as

\[ \psi = \psi_j(y) e^{\mu_j x} \]  

By substituting Eq. (2) into Eq. (1), we get

\[ \mathbf{H}\psi_j(y) = \mu_j \psi_j(y) \]  

where \( \mu_j \) is the eigenvalue and \( \psi_j \) is the corresponding eigensolution vector. Then the problem is transformed into solving eigenvalues and eigensolutions. Since the operator matrix \( \mathbf{H} \) is Hamiltonian type, the functional can be defined as:

\[ \langle \psi_j, \mathbf{J}, \psi_j \rangle = \int_{-1}^{1} \psi_j^T \mathbf{J} \psi_j dy \]  

Here \( \psi_j \) and \( \psi_j \) are any two eigensolution vectors, \( \mathbf{J} \) is the unit rotation matrix. It is not difficult to prove that both eigenvalues and eigenvectors come in pairs, that is, if \( \mu \) is the eigenvalue, then \( -\mu \) is also the eigenvalue. For the convenience of discussion, the eigenvalues can be divided into two categories, they are

\[ \mu_a : \text{Re}(\mu_a) \leq 0, \text{Im}(\mu_a) < 0 \]  

and

\[ \mu_b : \mu_b = -\mu_a \]  

In this way, the eigensolutions corresponding to the two kinds of eigenvalues have the following symplectic orthogonal relations:

\[ (\psi_i^{(\alpha)}, \mathbf{J}, \psi_j^{(\beta)}) = - (\psi_i^{(\beta)}, \mathbf{J}, \psi_j^{(\alpha)}) = \delta_{ij} \]  

\[ (\psi_i^{(\alpha)}, \mathbf{J}, \psi_j^{(\alpha)}) = (\psi_i^{(\beta)}, \mathbf{J}, \psi_j^{(\beta)}) = 0 \]  

Therefore, the eigensolution space composed of all eigensolution vectors is complete, that is to say, any state vector \( \psi \) can be represented by linear combination, that is

\[ \psi_i = \sum [a_n \psi_i^{(\alpha)}(y) e^{\mu_n x} + b_n \psi_i^{(\beta)}(y) e^{-\mu_n x}] \]  

where \( a_n \) and \( b_n \) are coefficients to be determined. According to the symplectic orthogonal normalization relation, the coefficients can be expressed as

\[ a_n = \langle \psi_i, \mathbf{J}, \psi_j^{(\beta)} \rangle, \quad b_n = - \langle \psi_i, \mathbf{J}, \psi_j^{(\alpha)} \rangle \]
3. Numerical example
Let’s consider the cantilever beam in the rectangular coordinate system shown in Fig. 1. The free end is under the action of couple $M$. Dimensions of cantilever beams are taken as: $h = 50\text{mm}$, $l/h = 8$, $M = 4\text{KN}$.

![Cantilever beam diagram](image1.png)

**Figure 1.** Cantilever beam diagram

Fig. 2 is the shear stress distribution curve. In this figure, it can be found that the stress near the upper and lower edges of the fixed end is much greater than that far away from the fixed end, which shows obvious stress concentration.

![Shear stress distribution](image2.png)

**Figure 2.** Shear stress distribution

4. Conclusion
The stress concentration of materials under special boundary conditions is a mechanical problem of great concern in engineering. By using the Hamiltonian system method, the solution of the boundary condition problem can be expressed as a linear combination of the eigensolutions, and the combination coefficients can be accurately described according to the symplectic orthogonal normalization relationship. The method used in this paper can effectively deal with quasi-static mechanical problems and provide a new idea for solid mechanics.

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