A general achievable rate region and two certain capacity regions for Slepian–Wolf multiple-access relay channel with non-causal side information at one encoder

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Abstract
Two-user multiple-access relay channel (MARC) with side information (SI) non-causally known at one encoder is studied, where transmitters send a common message with rate $R_0$, and private messages with rates $R_1$ and $R_2$; and derive a general achievable rate region by appropriately using superposition block Markov encoding, binning scheme, Gel’fand–Pinsker (GP) coding and partial decode-forward (PDF) strategy at the relay. Also, it is shown that this capacity inner bound can be tight for two special classes of degraded and reversely degraded multiple-access relay channels with one informed encoder. In order for the theoretical findings to be potentially applicable in communications performance metrics evaluations, the obtained general achievable rate region, while having all of the previous works on the point to point, relay and multiple access channels with and without SI as its special cases, is extended to the corresponding Gaussian version, the dirty paper coding of which has been a challenging and widely studied problem, and the result of which includes a variety of previously studied important Gaussian results. Finally, mathematical results are evaluated numerically.

1 INTRODUCTION

Shannon [1] studied the point to point channel in the presence of causal side information. The non-causal SI at the transmitter was introduced in [2] and the corresponding channel capacity was obtained by Gel’fand–Pinsker (GP) [3] and Costa extended the GP result to the Gaussian version by dirty paper coding (DPC) [4]. In [5], the authors analysed the effect of the correlation between channel input and SI as a general description for DPC. In [7], the authors investigated the multi-layer coding over a dirty-paper channel. The effect of side information (causal or non-causal) known at the source(s) and destination on the capacity region of the multiple-access channel (MAC) was studied in many literatures. A class of time-varying multiple-access channel (TVMAC) with state known to encoders and decoder was studied in [6]. Jafar [8] studied the capacity region of point to point channel and MAC with independent SI causally and non-causally known at the source(s).

In [9–14], the authors derived the achievable rate region for DM-MAC with correlated SI at sources by random binning, also, they extended their work to Gaussian MAC with side information at both encoders (doubly dirty MAC) in the high-SNR and strong interference regime; and proved that positive rates are not achievable by extended DPC to doubly dirty MAC suffering from strong interference, in contrast, they showed that positive rates are achievable by lattice-strategies independent of the interference power. Authors, in [15–18], studied the capacity inner bound, outer bound and the special capacity region of DM-MAC with one informed source in various senarios; also, they investigated the Gaussian case by generalised dirty paper coding (GDPC). Cemal and Steinberg [19] studied the multiple-access channel with complete SI at the receiver and incomplete and rate-limited SI at transmitters. In [20], [21], the authors derived the capacity inner bound for the multiple-access channel with SI partially and non-causally known at the sources, and showed the optimality of lattice coding. The authors in [20–23]...
and [24] have studied the MAC with SI using lattice strategy, and fading MAC, respectively.

The relay channel (RC), first introduced by Van der Meulen [25], was studied for a variety of classes of RC such as degraded, reversely degraded, full feedback and Gaussian degraded relay channels by Cover and El Gamal, also, they defined decode and forward (DF) and estimate or compress and forward (EF or CF) strategies at the relay; finally, they obtained a capacity upper bound and a general capacity lower bound for RC [26]. Then, the RC has been studied widely in the literature [27–33], and also, with SI in [34–36].

The multiple-access relay channel (MARC) where a relay cooperates with some transmitters to communicate with the destination was introduced in [37]. Researchers extensively studied the capacity inner bound for MARC with coding strategies of DF, CF and amplify and forward (AF) in [38], [39] and [40]. A new achievable rate region for Slepian–Wolf MARC was derived in [41], also, the capacity inner bounds for discrete and memoryless MARC with partial decode and forward (PDF) strategy and regular block Markov coding/backward decoding with and without non-causal SI known at one transmitter were obtained in [42] and [43], respectively. The capacity inner bound for DM-MARC with non-causal SI known at the relay and its extension to Gaussian case was studied in [44]. Also, the MARC in the presence of SI at both cooperating encoders and the MARC with relay-source feedback, and the transmission of analog information over MARC in [45–47], respectively, and the MIMO state-dependent channel in which only the helper node has a non-causal knowledge of the state in [48] were studied.

1.1 Our motivation and work

As shown in previous works, the non-causal SI can improve the rate region of the point-to-point channel, MAC and MARC. The MARC with PDF strategy at the relay and without any SI in [41] and [42] and with SI at the relay [44] has been studied.

The MARC with a common message at the transmitters, SI at one encoder and PDF strategy at the relay as a general communication channel, including (Slepian–Wolf MAC, widely studied MAC with SI at one encoder, the RC with PDF strategy at the relay, and point to point channel with SI) has not been studied before.

In this paper, this general channel is analysed from viewpoint of the achievable rate region. Specifically, we consider a two-user multiple-access relay channel with non-causal side information only at one encoder, where, the uninformed and informed encoders transmit a common message with rate \( R_0 \) and a private message with rates \( R_1 \) and \( R_2 \), respectively. To obtain the capacity inner bound for the MARC three different strategies can be used: irregular encoding successive decoding, regular encoding backward decoding and regular encoding sliding-window decoding, none of which has been proved to be optimal. In [43], regular block Markov encoding and backward decoding have been used to study the effect of non-causal side information available at one encoder on the achievable rate region of discrete memoryless MARC, however, in the present paper, the irregular block Markov encoding and successive decoding are used to achieve a capacity inner bound by appropriately applying binning scheme, superposition coding, Markov block encoding and Gel’fand–Pinkev method. As easily seen, in the presence of side information non-causally known at one encoder, none of these achievable rate regions includes the other. Also in this paper, for the first time, we obtain the capacity region for two special classes of degraded and reversely degraded two-user multiple-access relay channels in the presence of non-causally side information known non-causally at one encoder, in addition, our discrete alphabet results have extended to the continuous alphabet Gaussian MARC, using the extended version of dirty paper coding in two layers for PDF strategy, which is useful for analysing the role of side information and the relay in communications performances, such as transmission rates, energy efficiency, coverage region etc. Additionally in this paper, showing the effect of side information on the capacity inner bound, comparisons with the previous works and numerical simulations have been provided.

1.2 Paper organisation

The rest of the paper is organised as follows: In Section 2, the two-user discrete memoryless multiple-access relay channel with non-causal side information only at one encoder is described and we prove a capacity inner bound. In Section 3, capacity regions for two special classes of MARCs with one informed encoder are obtained. In Section 4, the results are extended to the Gaussian version. In Section 5, the numerical results are presented. Finally, we conclude the paper in Section 6.

1.3 Notations

In the whole of this paper, uppercase letters and lowercase letters are used to represent random and their realisations, respectively. The probability mass function (pmf) for random variables \( X \) and \( Y \), where \( x \in \mathcal{X} \) and \( y \in \mathcal{Y} \), are denoted by \( p_X(x) \) and \( p_Y(y) \), respectively. Also, the conditional pmf \( X \) given \( Y \) is represented by \( p_{X|Y}(x|y) \). The \( n \)-sequences random variable with respect to \( P_X(x) \) is shown by \( X^n \triangleq (X_1, X_2, \ldots, X_n) \) and the set of all \( \varepsilon \)-typical \( n \)-sequences \( X^n \) is represented by \( \mathcal{A}_\varepsilon^n \).

2 A GENERAL CAPACITY INNER BOUND FOR DM-MARC WITH SI NON-CAUSALLY KNOWN AT ONE SOURCE

Two-user DM-MARC consists of one relay which aids two sources to communicate with one destination as shown in Figure 1. The relay improves the communication performances between transmitters and receiver such as coverage region and transmission rate. Different coding strategies can be employed at the relay such as DF, PDF, CF or AF.
Markov encoding and random binning in which the capacity inner bound can be obtained by $b$ transmission block, each consisting of $n$ transmissions, where, the relay cooperates in transmitting messages by sending the bin index. Note that the average rate over the $b$ blocks is $R(b-1)/b$, which can be made as close to $R$ as desired. The encoders use superposition coding, enabling them to transmit both the common and individual messages, the informed encoder non-causally knowing the side information uses Gel’fand–Pinsker coding, resulting in canceling the effect of interference in the Gaussian version by dirty paper coding, and the relay and the receiver do the decoding processes in general ways.

We use the binning scheme, Markov block coding, superposition coding to prove the theorem. In Markov block coding the encoders send $b-1$ i.i.d messages $M_{ij} \in \{1:2^nR_j\}$ with $j \in \{1:b-1\}, i \in (1,2)$ in $b$ transmission block. In the binning scheme, the relay helps encoders with sending the bin index $I_{j}$ of the messages $(M_{ij}, M_{j+1})$ in block $j+1$.

We split the private messages $W_i$ for $i = 1, 2$ with rates $R_i$ into two parts $W_i'$ and $W_i''$ with rates $R_i'$ and $R_i''$, respectively.

Now, we prove a theorem to obtain a general inner bound for the capacity region of MARC with PDF strategy at the relay and side information (SI) at one encoder. The theorem, while having all known previous results as its special cases (as explained in the corollaries, below), shows well the effects of the SI and the relay with PDF strategy in comparison with the corresponding special channels; and can be extended to the continuous alphabet Gaussian version in order for the practical aspects of the results to be clarified more.

**Theorem 1.** Consider the two-user MARC with non-causal side information only at one encoder. The transmitters cooperate with each other to send a generic message with rate $R_0$. The uninformed source transmits a personal message with rate $R_1$ and the informed source does the similar transmission with rate $R_2$. The following achievable rate region is established with PDF strategy.

\[
\bigcup (R_0, R_1, R_2) \in \mathbb{R}^3_+ : \\
R_0 \leq \min \{I(U_0; U_1; U_2; Y_R|X_R), I(X_RU_0U_1U_2; Y_D)\} - I(U_2; S|X_RU_0),
\]

\[
R_1 \leq \min \{I(U_1; Y_R; X_RU_0U_2), I(U_1; Y_D; X_RU_0U_2) + I(X_R; Y_D), I(U_1U_2; Y_R|X_RU_0) - I(U_2; S|X_RU_0), \\
I(U_1U_2; Y_D|X_RU_0) + I(X_R; Y_D) - I(U_2; S|X_RU_0)\} + I(X_1; Y_D|X_RU_0U_1U_2V),
\]

\[
R_2 \leq \min \{I(U_2; Y_R|X_RU_0U_1), I(U_2; Y_D|X_RU_0U_1) + I(X_R; Y_D) + I(V; Y_D|X_RU_0U_1U_2X_1) - I(U_2V; S|X_RU_0),
\]

In PDF coding strategy, the relay decodes some parts of the messages, but in DF strategy, the relay decodes the whole message sent by sources. The PDF and DF can be optimal when the relay-sources channel is excellent.

We consider DM-MARC with non-causal SI at one encoder $(X_1 \times X_2 \times X_1^R \times S, p(y_D, y_R|x_1, x_2, x_1^R, s), y_D \times y_R)$ that consists of six finite stes: $X_1, X_2, X_1^R, S, Y_R, Y_D$ and a collection of conditionally probability mass functions $p(y_D, y_R|x_1, x_2, x_1^R, s)$ on $Y_D \times Y_R$.

A $(2^nR_0, 2^nR_1, 2^nR_2, n)$ code consists of:

- Three sets of integers $W_0 \in \{1:2^nR_0\}, W_1 \in \{1:2^nR_1\}$ and $W_2 \in \{1:2^nR_2\}$.
- Two messages $M_1 \in (W_0, W_1)$ and $M_2 \in (W_0, W_2)$.
- Two source encoders $x_1^{R_0}(m_1)$ and $x_2^{R_1}(m_2, s)$ where $m_1 \in M_1$ and $m_2 \in M_2$.
- A set of relay functions $f(\hat{y}_R) = f_y(x_1^{R_0}m_1, x_2^{R_1}m_2) \in \mathbb{N}^y_{y_y}$, $1 \leq y \leq n$.
- And a decoding function $(\hat{W}_0, \hat{W}_1, \hat{W}_2) = (\hat{m}_1, \hat{m}_2) = g(f(\hat{y}_R))$.

The average probability of error is defined as

\[
P_e = \frac{1}{2^n(R_0+R_1+R_2)} \sum_{(m_1, m_2) \in [1:2^nR_0] \times [1:2^nR_1] \times [1:2^nR_2]} P(E),
\]

where $E = ((\hat{m}_1, \hat{m}_2) \neq (m_1, m_2)) \cap (m_1, m_2)_{sent}$.

The region of all $(R_0, R_1, R_2)$ are achievable if $P_e \to 0$, for some $(2^nR_0, 2^nR_1, 2^nR_2, n)$ code.

The best and general strategy for the relay is partial decode and forward (PDF), where, the relay decodes whole or part of the message(s) and cooperates with encoders to communicate the message(s) to the destination. Hence, we use block
Due to the distribution of \( I(X_1;Y_0) \), \( I(X_2;Y_0|U_0,U_1) \) and \( I(X_1;Y_0|U_0,U_1) \), we can extend our results to imperfect SI known non-causally at the informed encoder by transforming \( S \) to \( \tilde{S} \) in Equations (1)–(11).

**Corollaries of Theorem 1:**

Now, we compare our capacity inner bound with previous works and show that our result includes all previous regions.

**Corollary 1.** The achievable region for the Slepian–Wolf multiple-access channel [49] can be obtained by setting \( X_R = S = \emptyset, Y_D = Y_R \), \( U_k = X_k, k = 1, 2, V = X_2 \) and \( U_0 = S \) (common message).

**Corollary 2.** The achievable region of MAC in [50] can be derived by setting \( X_R = U_0 = S = \emptyset, Y_D = Y_R \) and \( V = X_2 \).

**Corollary 3.** The capacity inner bound for MAC with informed sources and independent messages [15], can be achieved by setting \( X_R = \emptyset \), \( U_1 = X_1, V = U_2, Y_R = Y_D \) and \( U_0 = Q \) (common variable).

**Corollary 4.** The achievable rate region for MAC with non-causal side information at informed source in which both encoders transmit a generic message and the individual message is transmitted with informed source [51], can be obtained by setting \( X_R = U_1 = X_1 = \emptyset, U_2 = V = U, U_0 = X_1 \) and \( Y_D = Y_R = Y \).

**Corollary 5.** The inner bound of the two-user MAC with SI only at one transmitter where the uninformred source’s message is known at the informed source [16], can be obtained by setting \( U_0 = X_2, U_2 = U, X_R = U_1 = X_1 = \emptyset, V = U_2, X_2 = X_1 \) and \( Y_D = Y_R \).

**Corollary 6.** It can be shown, the achievable regions for the two-user MAC with SI only at one source where only uninformred node sends an individual message [17], is obtained by setting \( U_0 = U_1, U_1 = X_1, X_R = \emptyset, Y_R = Y_D = Y \) and \( V = U_2 \).

**Corollary 7.** Two lower bounds of DM-RC with non-causal side information known at the source were derived in [35], in which the source transmits two layers of the state description to relay and destination. As can be easily seen the inner bound of [35], Theorem 1, \( Z_R = Z_D = \emptyset, U_R = U_D = \emptyset \) is obtained by setting \( U_0 = U_1 = X_1 = \emptyset, Y_2 = Y_R, Y_D = Y_D, X_2 = V = X_R, U = U_2 \) and \( U_2 = U_1 \).

**Corollary 8.** The inner bound of multiple access relay channel [37], can be obtained by setting \( S = \emptyset \), \( U_0 = \emptyset \), \( U_1 = X_1 \) and \( U_2 = V = X_2 \).

**Corollary 9.** It is obviously seen that our inner bound is reduced to a general achievable rate region in [41] by setting \( S = \emptyset \) and \( V = X_2 \).
3 \ |
CAPACITY REGIONS FOR TWO
SPECIAL CLASSES OF DEGRADED AND
REVERSELY DEGRADED MARCs WITH
ONE INFORMED TRANSMITTER

The general capacity region for multiple-access relay channel or
even relay channel with or without side information is an open
problem. However, in this section, we present two special capacity
regions for multiple-access relay channel in the presence of side
information known non-causally only at one encoder.

We consider a two-user multiple-access relay channel
with SI at the informed transmitter such that all chan-
nels between encoders and relay are better than the chan-
nels between transmitters, or in the other words, \( X_1, X_2, Y -
Y_R X_R - Y_D \), known as degraded MARC. As the channel
between encoder to relay is better than to decoder, the
relay recovers whole the messages, hence, we use the DF
strategy.

Theorem 2. Consider the specially degraded MARC where, the uni-
formed encoder transmits only the common message and the informed encoder
transmits the common and private messages. The capacity region for this
channel is the convex hull of all rates \( (R_1, R_2) \) satisfying:

\[
R_2 \leq \min \{ I(U_2; Y_R | X_1, X_R), I(U_2; Y_D | X_1, X_R) \}
\]

\[
+ I(X_R, Y_D) - I(U_2; Y_D | X_1, X_R),
\]

\[ R_1 + R_2 \leq \min \{ I(X_1; U_2 | Y_R | X_R), I(X_R; U_2 | X_1; Y_D) \}
\]

\[
- I(U_2; Y_D | X_1, X_R).
\]

With constraints:

\[
I(U_2; Y_D | X_1) \leq I(X_R; Y_D),
\]

\[
X_2 = \mathcal{f}(W_0 W_2 Z_D^{-1}),
\]

where \( U_2 = W_0 W_2 Z_D^{-1} \), and the union is taken over:

\[
p(u, x_R, u_2, x_1, x_2, y_2, y_1, y_D) = p(u) p(x_R) p(x_1 | x_R)
\]

\[
p(u_2, x_2 | x_1, x_R, y_1) p(y_1 | y_D, x_2, x_R, y_1).
\]

Proof. See Appendix A.2.

Also, we consider a two-user reversely degraded multiple-
access relay channel with only one informed encoder where encoders transmit only individual messages, and the relay uses PDF strategy.

Theorem 3. Consider a multiple-access reversely degraded relay chan-
nel with non-causal SI at one encoder where the relay decodes just part of
the messages and the rest of the messages are decoded only at the desti-
nation. The encoders transmit only private messages with rates \( (R_1 = R_1', R_2 = R_2' + R_2'') \). The capacity of this channel is the convex

hull of rates \( (R_1, R_2') \) satisfying:

\[
R_1 \leq I(U_1; Y_R | U_2, X_R) + I(X_1; Y_D | U_1, U_2, Y_R),
\]

\[
R_2' \leq I(U_2; Y_R | U_1, X_R) + I(V; Y_D | U_1, U_2, Y_R)
\]

\[
- I(U_2; V; Y_D),
\]

\[
R_1 + R_2' \leq I(U_1; U_2, Y_R | X_R) + I(X_1; V | Y_D | U_1, U_2, Y_R)
\]

\[
- I(U_2; V; Y_D).
\]

where the union is taken over:

\[
p(u, x_R, u_2, x_1, x_2, y_2, y_1, y_D) = p(u) p(x_R) p(u_1, x_1 | x_R)
\]

\[
p(u_2, x_2 | x_1, x_R, y_1) p(y_1 | y_D, x_2, x_R, y_1).
\]

And we define, \( U_1 = W_1', U_2 = W_2 Y_R^{-1} \) and \( V = W_2 Y_D^{-1} Y_R^{-1} \).

Proof. See Appendix A.3.

\]

Corollary 10. It is obviously seen that with \( S = \emptyset \), our capacity region
is reduced to the capacity region for multiple-access reversely degraded RC
studied in [41].

4 \ |
THE GAUSSIAN MARC WITH
NON-CAUSAL SIDE INFORMATION AT
ONE ENCORDER (EXTENSION OF THE
THEOREM 1 TO THE GAUSSIAN
VERSION)

The Gaussian rate region is useful for computing practical com-
munications performances. Due to these applications, in this
section, we consider a Gaussian MARC, with SI at one trans-
mmitter and derive the corresponding capacity inner bound. The
outputs of the channel are:

\[
Y_R^n = g_{R_1} X_1^n + g_{R_2} X_2^n + g_{SR} S^n + Z_R^n,
\]

\[
Y_D^n = g_{D_1} X_1^n + g_{D_2} X_2^n + g_{DR} X_R^n + g_{SD} S^n + Z_D^n,
\]

where \( X_1^n \) is the signal transmitted by the uninformed
encoder with average power constraint \( \sum_{i=1}^n X_{1,i} < n P_1 \), \( X_2^n \) is
the informed encoder signal with average power constraints
\( \sum_{i=1}^n X_{2,i} < n P_2 \), \( X_R^n \) is the signal of the relay with average power
constraints \( \sum_{i=1}^n X_{R,i} < n P_R \), \( S^n \) is the side information known
at the informed encoder and it is assumed that \( S \) is zero-mean
Gaussian random variable with variance \( Q_1 \), however, despite
the Costa work, it is assumed to be dependent on \( X_2 \). \( Z_R \) and
\( Z_D \) are independent normal random variables with zero mean
and variance \( N_R \) and \( N_D \), respectively. \( g_{R_1}, g_{R_2}, g_{D_1}, g_{D_2}, g_{DR}, g_{SR}\)
and \( g_{SD} \) are positive constants representing the static gains of
the links.
The distribution of inputs, side information and auxiliary random variables in (8) can be extended to linear continuous version all of which are explained in the proof of Theorem 4.

**Theorem 4.** The achievable rate region for Gaussian MARC with one informed encoder can be found as:

\[
\begin{align*}
R_0 + R_1 + R_2 &\leq \frac{1}{2} \max \log_2 \left\{ \frac{\mathcal{R}_1(1 - P_2^2)}{\mathcal{R}_0 + \mathcal{R}_1 + \mathcal{R}_2} \right\}, \\
R_0 + R_2 &\leq \frac{1}{2} \max \log_2 \left\{ \frac{\mathcal{R}_0(1 - P_2^2)}{\mathcal{R}_0 + \mathcal{R}_2} \right\}, \\
R_0 + R_1 + R_3 &\leq \frac{1}{2} \max \log_2 \left\{ \frac{\mathcal{R}_1(1 - P_2^2)}{\mathcal{R}_0 + \mathcal{R}_1 + \mathcal{R}_3} \right\},
\end{align*}
\]

where

\[
\begin{align*}
f_{R_1} &= \mathcal{R}_1^2 \mathcal{R}_1 P_1 + \mathcal{R}_0^2 (1 - P_2^2) P_2 + \mathcal{R}_0^2 \mathcal{R}_0, \\
f_{R_2} &= \mathcal{R}_2^2 \mathcal{R}_2, \\
f_{R_3} &= \mathcal{R}_3^2 \mathcal{R}_3.
\end{align*}
\]
f_{R_1} = f_{R_2} + g_{R_1} \sqrt{P_1} \beta_1 \rho_{1} + \delta_{R_2} \rho_{2U_0} P_2 + 2 g_{R_1} \rho_{2U_0} \sqrt{P_1} \eta_1 \beta_1 \rho_{1} P_2,

f_{D_1} = g_{D_1} \sqrt{P_1} \beta_1 \rho_{1} + \delta_{D_2} \rho_{2U_0} P_2 + 2 g_{D_2} \rho_{2U_0} \sqrt{P_1} \eta_1 \beta_1 \rho_{1} P_2,

f_{D_2} = g_{D_1} \sqrt{P_1} \beta_1 \rho_{1} + \delta_{D_2} \rho_{2U_0} P_2 + 2 g_{D_2} \rho_{2U_0} \sqrt{P_1} \eta_1 \beta_1 \rho_{1} P_2,

f_{D_3} = f_{D_2} + f_{D_3},

A = P_2 \left(1 \rho_{2U_0}^2 \rho_{2U_0}^2 \right) + \left( \sigma_1 \sqrt{Q_1} + \rho_{2U_0} \sqrt{P_2} \right)^2,

B = P_2 \left(1 \rho_{2U_0}^2 \rho_{2U_0}^2 \rho_{2U_0}^2 \right) + \left( \sigma_1 \eta_2 \sqrt{Q_1} + \rho_{2U_0} \sqrt{P_2} \right)^2,

C = P_2 Q_1 \left( g_{R_1}(\sigma_1 - \sigma_2) - g_{R_2} \right) \left(1 \rho_{2U_0}^2 \rho_{2U_0}^2 \rho_{2U_0}^2 \right),

D = P_2 Q_1 \left( g_{D_1}(\sigma_1 - \sigma_2) - g_{D_2} \right) \left(1 \rho_{2U_0}^2 \rho_{2U_0}^2 \rho_{2U_0}^2 \right),

E = P_2 Q_1 \left( g_{D_2}(\sigma_1 - \sigma_2) - g_{D_2} \right) \left(1 \rho_{2U_0}^2 \rho_{2U_0}^2 \rho_{2U_0}^2 \right).

where: -1 \leq \rho_{2U_0} \rho_{2U_0} \rho_{2U_0} \leq 1 and 0 \leq \sigma_1 = 1 - \alpha, \beta_1 = 1 - \beta_1, \eta_1 = 1 - \eta_1 \leq 1.

Proof. See Appendix A.4

Remark 3. (i) In Theorems 1–3 for discrete alphabet and memory-less MARC with SI at one encoder, the parameters are channel outputs (Y_D, Y_R), channel inputs (X_1, X_2, X_R), side information S, auxiliary random variables (U_1, U_2, U_0, V) characterising the relay strategy and different parts of messages. The rate regions are described by various mutual information between these variables, where, the role of each variable such as S is seen in the rate regions and easily interpreted. (ii) In Theorem 4 for the Gaussian MARC (continuous alphabet version) in addition to the Y_D, Y_R, X_1, X_2, X_R, U_1, U_2, U_0, V, noise variables in the relay and the receiver are added; and due to the dependency of Gaussian entropy, and hence the mutual information, to the intended variables variances, in (23)–(29), we have channel gains, average powers and corresponding correlation coefficients, where, in every term the role of each of these parameters is seen and can be interpreted.

Remark 4. (i) Consider the Gaussian multiple-access relay channel with partial side information 3 at one informed encoder satisfying (21) and (22). We assume S = 3 + E_3 and E_3 ~ N(0, D), where D = E[(S - \bar{S})^2] and E[|S|] = E[|\bar{S}|] = \bar{Q}. It can be easily shown that the capacity inner bound for MARC with imperfect SI at informed encoder can be obtained by transforming Q \rightarrow Q_1, N_R \rightarrow N_R + g_{RD}(Q - \bar{Q}) and N_D \rightarrow N_D + g_{RD}(Q - \bar{Q}) in (23)–(29) in Theorem 4, and hence showing the effect of imperfectness of side information (ii) In relations (23)–(29) for Gaussian MARC rate region, impacts of correlations between random variables (e.g. SI and input X_2 as \rho_{2U_0}) are seen obviously.

Corollaries of Theorem 4

A variety of previous works for Gaussian channels is included in the Theorem 4 as special cases.

Corollary 11. The capacity inner bound for Gaussian point to point channel with side information non-causally known at source [5, case 1] can be obtained by omitting relay and uninformed encoder and following condition:

\alpha = \beta_1 = \gamma_1 = \theta_2 = P_R = \rho_{2M_2} = \rho_{2U_0} = g_{RD} = 0, \quad g_{R_1} = g_{R_2} = g_{D_1} = g_{D_2} = 0, \quad N_R = N_D = N, \quad \theta_1 = \alpha, \quad \rho_{2U_0} \sqrt{P_2Q} = A_1, \quad Q_1 = Q_1.

And obviously, a special case of [5] and by \rho_{2U_0} = 0, Costa result [4] can be obtained.

Corollary 12. The achievable rate region for Gaussian MAC with the common message in [52] can be obtained by omitting relay and side information at the encoder and the following condition:

\bar{Q}_1 = 0, \quad \alpha = 1, \quad \bar{\beta}_1 = \alpha, \quad \rho_{2U_0} = \bar{\beta}, \quad \rho_{2U_0} = \rho_{2M_2} = 0, \quad \gamma_1 = 0, \quad g_{R_1} = g_{R_2} = g_{D_1} = g_{D_2} = 1, \quad N_R = N_D = N, \quad P_R = 0.

Corollary 13. The achievable rate region for Gaussian MAC [53, 54] can be obtained by omitting relay, common message and side information at encoder:

\bar{Q}_1 = 0, \quad \alpha = 0, \quad \bar{\beta}_1 = 1, \quad \rho_{2U_0} = \rho_{2M_2} = \rho_{2U_0} = 0, \quad \gamma_1 = 0, \quad g_{R_1} = g_{R_2} = g_{D_1} = g_{D_2} = 1, \quad N_R = N_D = N, \quad P_R = 0.

Corollary 14. The achievable rate region for Gaussian MAC with state known to one encoder and independent messages in [15] can be obtained by omitting the relay and the following condition:

\bar{Q}_1 = \alpha = \rho_{2U_0} = \rho_{2M_2} = \theta_2 = 0, \quad \bar{\beta}_1 = 1, \quad \rho_{2U_0} = \rho, \quad \theta_1 = \alpha, \quad g_{R_1} = g_{R_2} = g_{D_1} = g_{D_2} = 1, \quad P_R = 0, \quad N_R = N_D = N, \quad \bar{Q}_1 = \bar{Q}_1.

Corollary 15. It can be shown that the achievable rate region for Gaussian MAC with state known to one encoder (only uninform encoder
transmits private message) in [16] can be obtained by omitting the relay and the following condition:
\[ g_1 = \xi, \quad g_{R1} = g_{R2} = g_{D1} = g_{D2} = 1, \quad N_R = N_D = N, \]
\[ \theta_1 = \alpha(1 - \frac{\rho_2}{Q}) - \frac{\rho_1}{Q} \quad \text{and} \quad Q_1 = Q_2. \]

Corollary 16. The general achievable rate region for Gaussian Skipian–Wolf MARC in [41] can be obtained by omitting side information at the informed encoder and the following condition:
\[ Q_1 = \rho_2 \gamma_1, \quad Q_2 = \rho_2 \gamma_2, \quad \gamma_1 = 0, \quad \rho_2 = 1 - \gamma_2, \quad \gamma_2 = 0 \leq \gamma_2 \leq 1. \]

5 NUMERICAL RESULTS

The obtained discrete alphabet results can be extended, by using linear relations between channel input–output (relations (21) and (22) in Section 4), to continuous alphabet versions, Gaussian versions with constant channel gains (affected by path loss) and wireless band fading version with random channel coefficients (affected by general path loss, shadowing and multi-path fading). In this paper, we have studied the Gaussian version (Theorem 4 in Section 4) assuming the gains to be static and functions of free space path-loss dependent on the distance between transmitter and receiver (Figure 2), modelled by \[ g_{Ri} = d_{Ri}^{-\kappa}, \] (where \( \kappa \) is the free space loss exponent factor and \( d_{Ri} \) is the distance between the uninformed encoder and the relay); and generally by \( (d_{ij}^{-\kappa}) \), where \( \kappa \) is the free-space loss exponent factor and \( d_{ij} \) is the distance between node \( i = 1, 2, R, S \) and node \( j = R, D \).

In this section, we compare our results with the previous papers for MARC with or without SI. As can be seen in the rates of Theorem 4 (as explained in Remarks 3, 4), it is shown that the informed encoder can cancel some part of the interference effect and can achieve better rates. In Figure 3, we are depicted our inner bounds \( R_2, R_1 + R_2 \) and \( R_0 + R_1 + R_2 \), when \( \frac{P_B}{N_D} = \frac{P_B}{N_R} = \frac{Q}{N_D} = \frac{Q}{N_R} = \frac{Q}{N_D} = \frac{Q}{N_R} = \frac{Q}{N_D} = \frac{Q}{N_R} = 5 \) and \( \kappa = 4 \); Also, we compare the achievable rate for the informed encoder in our model with Gaussian MARC without any informed encoder which has been studied in [41]. It can be seen that the informed encoder can cancel some part of the interference effect. In Figure 4, we compare the inner bounds \( R_2, R_1 + R_2 \) and \( R_0 + R_1 + R_2 \) for our channel, when \( \frac{P_B}{N_D} = \frac{P_B}{N_R} = \frac{Q}{N_D} = \frac{Q}{N_R} = \frac{Q}{N_D} = \frac{Q}{N_R} = \frac{Q}{N_D} = \frac{Q}{N_R} = 5 \) and \( \kappa = 4 \); Also, we compare the achievable rate for the informed encoder and uninformed encoder studied in [41] based on interference power \( Q \) in Figure 5, for \( d = 0.5, d = 1 \) and \( d = 1.5 \), where \( d \) is the distance between the relay and the line connecting the two transmitters and \( \frac{P_B}{N_D} = \frac{P_B}{N_R} = \frac{Q}{N_D} = \frac{Q}{N_R} = \frac{Q}{N_D} = \frac{Q}{N_R} = \frac{Q}{N_D} = \frac{Q}{N_R} = 5 \). As can be easily seen our results can be reduced to the results in [41], as we mentioned in corollary 16. Finally, we compare the private achievable rate region of the informed encoder \( R_2 \) for the Gaussian MARC with using the irregular block Markov encoding and successive decoding studied in this paper and the regular block Markov encoding and backward decoding studied in [43], Figure 6, when \( \frac{P_B}{N_D} = \frac{P_B}{N_R} = \frac{Q}{N_D} = \frac{Q}{N_R} = \frac{Q}{N_D} = \frac{Q}{N_R} = \frac{Q}{N_D} = \frac{Q}{N_R} = 5 \) and \( \frac{P_B}{N_D} = \frac{P_B}{N_R} = \frac{Q}{N_D} = \frac{Q}{N_R} = \frac{Q}{N_D} = \frac{Q}{N_R} = \frac{Q}{N_D} = \frac{Q}{N_R} = 5 \). As we mentioned in
previous sections, none of which has been proved to be optimal. Figure 6, illustrated a situation that the coding used in this paper is better than coding studied in [43].

6 Conclusion

In this paper, we have derived a general achievable rate region for discrete alphabet and memoryless two-user multiple-access relay channel, where only one of the encoders knows non-causally the side information, the relay utilises the PDF strategy and decodes just part of the message and both encoders transmit common and private messages. The obtained general capacity inner bound was shown to include all the previous results, emphasising the effects of SI and relay PDF strategy. There is no known capacity region for MARC with or without side information; here, we have shown that our capacity inner bound can be tight for two special classes of multiple-access relay channels with non-causal side information known at only one source. Finally, we have extended our discrete alphabet theorem to the Gaussian case (with constant channel gains), for clarifying the practical aspects of the theorem, and proved that previously studied various important Gaussian results are all special cases of our results; certainly, the derived discrete alphabet results can be extended to continuous alphabet block fading wireless versions, the same as for Gaussian case, however, with random channel coefficients. The obtained theoretical results have been compared numerically with two of the previous papers.

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APPENDICES A

In details for Appendix A.2 and Appendix A.3:

a: follows since Fano’s inequality.
b: follows since Csiszár-Körner’s sum identity.
c: follows since $X_{R_i} = W_0^i W_1^{j+i} Y_{D_i}^{j+i-1} - W_0^n W_2^n S_i$.
d: follows since $X_{R_i} = f(W_0^n W_2^n Y_{D_i}^{j+i-1})$.
e: follows since $Y_{R_i} = W_0^n W_2^n S_i Y_{D_i}^{j+i-1} - X_{R_i}, X_1, X_2, S_i - Y_{R_i} Y_{D_i}$.
f: follows since $X_{R_i} = f(Y_{R_i}^{j+i-1})$.
g: follows since $H(S_i) = H(S_i|X_{R_i}^{j+i-1})$.
h: follows since $X_{I_i} = f(W_0^n W_1^n Y_{D_i}^{j+i-1})$.
i: follows since the channel is degraded.
j: follows since the channel is reversed degraded.
k: follows since $W_1^n Y_{D_i}^{j+i-1} = X_{I_i} - Y_{R_i} Y_{D_i}$.

A.1 Proof of theorem 1

Codebook generation: Fix the pmf $p(s, x_{R_i}, u_0, u_1, u_2, x_{1}, v)$ in (8) and function $x_{2} = (s, x_{R}, u_0, u_1, u_2, x_{1}, v)$:

$$
p(s, x_{R_i}, u_0, u_1, u_2, x_{1}, 0, i) = p(s)p(x_{R_i}|u_0)p(u_1|x_{R_i})p(u_2|x_{R_i}, x_{1}, 0).
$$

1) Generate $2^{nR}$ independent identically distributed n-sequence $x_{R_i}^n$, each according to $p(x_{R_i}^n) = \prod_{i=1}^{m} p(x_{R_i})$ and index them as $x_{R_i}^n(m)$, $m \in [1 : 2^{nR}]$.

2) For each $x_{R_i}^n(m)$, generate $2^{nR_0}$ conditionally independent $n$-sequence $u_0^n$, each drawn according to $p(u_0^n|x_{R_i}^n(m))$, index them as $u_0^n(j, m)$, $j \in [1 : 2^{nR_0}]$.

3) For each $x_{R_i}^n(m)$, $u_0^n(j, m)$, generate $2^{nR_1}$ conditionally independent $n$-sequence $u_1^n$, each drawn according to $p(u_1^n|x_{R_i}^n(m), u_0^n(j, m)) = \prod_{i=1}^{m} p(u_1^n|x_{R_i}(m), u_0^n(j, m))$, index them as $u_1^n(l_1, l_2, j, m)$, $l_1 \in [1 : 2^{nR_1}], l_2 \in [1 : 2^{nR_2}]$.

4) For each $x_{R_i}^n(m), u_0^n(j, m)$, generate $2^{n(R_2 + I(\sum_{i=1}^{m}x_{R_i}(m), u_0^n(j, m)))}$ conditionally and independent $n$-sequence $u_2^n$ , each drawn according to $p(u_2^n|x_{R_i}^n(m), u_0^n(j, m)) = \prod_{i=1}^{m} p(u_2^n|x_{R_i}(m), u_0^n(j, m))$, index them as $u_2^n(l_2, l_3, j, m)$, $l_2 \in [1 : 2^{nR_2}], l_3 \in [1 : 2^{nR_2}]$. How to cite this article: Etminan J, Mohanna F, Hodtani GA. A general achievable rate region and two certain capacity regions for Slepian–Wolf multiple-access relay channel with non-causal side information at one encoder. IET Commun. 2021;15:497–512. https://doi.org/10.1049/cmu2.12082
5) For each \( \{x^*_i(m), u^*_i(j, k, l), u^*_i(l_1, j, m)\} \), generate \( 2^{nR_i} \) conditionally independent n-sequence \( x^*_i \), each drawn according to \( p(x^*_i|m, u^*_i(j, k, l), u^*_i(l_1, j, m)) = \prod_{j=1}^{m} p(x_j|X_R, X_L, m, u^*_i(j, k, l), u^*_i(l_1, j, m)), \) index them as \( x^*_i \).

6) For each \( \{x^*_i(m), u^*_i(j, k, l), u^*_i(l_2, j, m)\} \), generate \( 2^{nR_i} \) conditionally and independently n-sequence \( v^* \), each drawn according to \( p(v^*|X_R, X_L, m, u^*_i(j, k, l), u^*_i(l_2, j, m)) = \prod_{j=1}^{m} p(v_j|X_R, X_L, m, u^*_i(j, k, l), u^*_i(l_2, j, m)), \) index them as \( v^* \).

7) Partition the set of messages, \( w_i, k = 1, 2 \), into \( 2^{nR_i} \) equal size parts randomly and name the partitions of \( w_i \) as \( w_i^\beta(m), m \in [1 : 2^{nR_i}] \).

8) Partition the set of messages, \( (w_i^1, w_i^2) \), into \( 2^{nR_i} \) equal size parts randomly and name the partitions of \( (w_i^1, w_i^2) \) as \( w_i^\beta(m), m \in [1 : 2^{nR_i}] \).

9) Partition the set of messages, \( (w^1, w^2) \), into \( 2^{nR_i} \) equal size parts randomly and name the partitions of \( (w^1, w^2) \) as \( w^\beta(m), m \in [1 : 2^{nR_i}] \).

**Encoding in source terminals:** Let \( w_i \in [1 : 2^{nR_i}] \), \( w_i^1 \in [1 : 2^{nR_i}] \), \( w_i^2 \in [1 : 2^{nR_i}] \), \( w_i^1 \in [1 : 2^{nR_i}] \), \( w_i^2 \in [1 : 2^{nR_i}] \) be the new messages to be sent in \( B + 1 \) blocks. Split the messages to \( B \) equally sized blocks. Transmitter 1 sends \( x^1_{1,b}(w^1, w^1, w^1, m_{i-1}) \) and transmitter 2 with knowing \( S^\gamma \), finds \( x^2_{2,b}(w^1, w^1, w^1, m_{i-1}) \) such that:

\[
\{ (x^a_{R,b}(m_{i-1}), u^a_{0,b}(w_{0,1}, m_{i-1}), u^a_{2,b}(l_{2,b}, l_{1,b}, m_{i-1}), v^a) \in A^\gamma \},
\]

if more than one sequence exists, it picks the sequence with the minimum number and if no such sequence exists, it picks, \( l_{2,b} = 1 \). After that finds \( v^b_{2,b}(q_{2,b}, w^1_{2,b}, l_{2,b}, l_{1,b}, m_{i-1}) \) such that:

\[
\{ (x^a_{R,b}(m_{i-1}), u^a_{0,b}(w_{0,1}, m_{i-1}), u^a_{2,b}(l_{2,b}, l_{1,b}, m_{i-1}), v^a) \in A^\gamma \},
\]

if more than one sequence exists, it picks the sequence with the minimum number and if no such sequence exists, it picks, \( l_{2,b} = 1 \). Transmitter 2 sends \( x^3_{2,b}(x^a_{R,b}(m_{i-1}), u^a_{0,b}(w_{0,1}, m_{i-1}), u^a_{2,b}(l_{2,b}, l_{1,b}, m_{i-1}), v^a) \).

**Error analysis at the relay:** Assume without loss of the generality \( w_{0,1} = w^1_{1,b} = w^1_{2,b} = m_{i-1} + 1 \) and let \( l_{2,b} = 1 \). Let the sequence satisfying (A.2). Hence the probability of the relay error is:

\[
P(E_R) = P\left( \bigcup_{j=1}^{s} E_{R,j} \right) \leq P(E_{R,1}) + P(E_{R,2})
\]

\[
+ \sum_{j=3}^{s} P(E_{R,3}|E_{R,1} \land E_{R,2})
\]

where

\[
E_{R,1} = \left\{ (X^a_{R,b}(1), U^a_{0,b}(1,1), U^a_{2,b}((l_{2,b}, 1, 1)), S^\gamma) \in A^\gamma \right\}
\]

for all \( l_{2,b} \in [1 : 2^{nR_i(1, l_{1,b})}] \).

\[
E_{R,2} = \{ \hat{m}_{i-1} \neq 1 \}
\]

\[
E_{R,3} = \left\{ (X^a_{R,b}(1), U^a_{0,b}(1,1), U^a_{1,b}(1,1,1)) \right\}
\]
\[ U^a_{R,b}(I_{2,b}, 1, 1, 1), Y^a_{R,b} \notin A^e_{\xi} \}, \quad (A.10) \]

\[ E_{R4} = \{(X^a_{R,b}(1), U^a_{0,b}(w_{0,b}, 1), U^a_{2,b}(l_{2,b}, w_{2,b}, w_{0,b}, 1)), \] \[ U^a_{1,b}(w_{1,b}, w_{0,b}, 1), Y^a_{R,b} \in A^e_{\xi} \text{ for some } w_{0,b} \neq 1 \}, \quad (A.11) \]

\[ E_{R5} = \{(X^a_{R,b}(1), U^a_{0,b}(1, 1), U^a_{1,b}(l_{2,b}, 1, 1), \] \[ U^a_{2,b}(I_{2,b}, 1, 1, 1), Y^a_{R,b} \in A^e_{\xi} \text{ for some } w_{1,b} \neq 1 \}, \quad (A.12) \]

\[ E_{R6} = \{(X^a_{R,b}(1), U^a_{0,b}(1, 1), U^a_{1,b}(l_{2,b}, 1, 1, 1), \] \[ U^a_{2,b}(l_{2,b}, w_{2,b}, 1, 1), Y^a_{R,b} \in A^e_{\xi} \text{ for some } w_{2,b} \neq 1 \}, \quad (A.13) \]

\[ E_{R7} = \{(X^a_{R,b}(1), U^a_{0,b}(1, 1), U^a_{1,b}(l_{2,b}, 1, 1, 1), \] \[ U^a_{2,b}(l_{2,b}, w_{2,b}, 1, 1), Y^a_{R,b} \in A^e_{\xi} \text{ for some } w_{2,b} \neq 1 \}, \quad (A.14) \]

\[ E_{R8} = \{(X^a_{R,b}(1), U^a_{0,b}(1, 1), U^a_{1,b}(l_{2,b}, 1, 1, 1), \] \[ U^a_{2,b}(l_{2,b}, w_{2,b}, 1, 1), Y^a_{R,b} \in A^e_{\xi} \text{ for some } w_{2,b} \neq 1 \}, \quad (A.15) \]

\[ U^a_{1,b}(w_{1,b}, 1, 1), Y^a_{R,b} \in A^e_{\xi} \text{ for some } w_{1,b} \neq 1, l_{2,b} \neq I_{2,b} \}, \quad (A.16) \]

**Error analysis at the receiver:** Assume without loss of generality that \( w_{0,b} = w_{1,b} = w_{2,b} = w_{n,b} = w_{m,b} = m_0 = m_{l-1}, m_b \in \beta(1) \) and let \( I_{2,b} \) and \( l_{2,b} \) denotes the sequence satisfying (A.1) and (A.2), respectively. Hence the probability of error at the destination is bounded by:

\[
P(E_D) = P\left(\bigcup_{j=1}^{10} E_{Dj}\right) \leq \sum_{j=1}^{6} P(E_{Dj}) \]

\[
+ \sum_{j=7}^{17} P\left(\bigcap_{j=1}^{6} E_{Dj}\right),
\]

where

\[ E_{D1} = \{(X^a_{R,b}(1), U^a_{0,b}(1, 1), U^a_{2,b}(l_{2,b}, 1, 1, 1), s^e) \notin A^e_{\xi} \text{ for all } l_{2,b} \in [1 : 2^d(I_{2,b})]\}\]

\[ E_{D2} = \{(X^a_{R,b}(1), U^a_{0,b}(1, 1), U^a_{2,b}(l_{2,b}, 1, 1, 1), \] \[ V^a(l_{2,b}, 1, 1, 1), s^e) \notin A^e_{\xi} \text{ for all } l_{2,b} \in [1 : 2^d(V \setminus I_{2,b})]\}\]

\[ E_{D3} = \{(X^a_{R,b}(1), Y^a_{D,b}) \notin A^e_{\xi} \text{ for some } w_{1,b} \neq 1\}, \quad (A.18) \]

\[ E_{D4} = \{(X^a_{R,b}(1), Y^a_{D,b+1}) \notin A^e_{\xi} \text{ for some } m_{b-1} \neq 1\}, \quad (A.19) \]

\[ E_{D5} = \{\tilde{n}_{b-1} \neq 1\} \in \{(X^a_{R,b-1}(m_{b-1}), Y^a_{D,b+1}) \notin A^e_{\xi} \text{ for some } m_{b-1} \neq 1\}, \quad (A.20) \]

\[ E_{D6} = \{\tilde{n}_{b} \neq 1\} \in \{(X^a_{R,b}(m_{b}), Y^a_{D,b+1}) \notin A^e_{\xi} \text{ for some } m_{b} \neq 1\}, \quad (A.21) \]

\[ E_{D7} = \{(X^a_{R,b}(1), U^a_{0,b}(1, 1), U^a_{1,b}(l_{2,b}, 1, 1, 1), \] \[ U^a_{2,b}(l_{2,b}, w_{2,b}, 1, 1), Y^a_{D,b} \in A^e_{\xi} \text{ for some } w_{2,b} \neq 1\}, \quad (A.22) \]

\[ E_{D8} = \{(X^a_{R,b}(1), U^a_{0,b}(w_{0,b}, 1), U^a_{2,b}(l_{2,b}, w_{2,b}, w_{0,b}, 1), \] \[ U^a_{1,b}(w_{1,b}, w_{0,b}, 1), Y^a_{D,b} \in A^e_{\xi} \text{ for some } w_{0,b} \neq 1\}, \quad (A.23) \]

\[ E_{D9} = \{(X^a_{R,b}(1), U^a_{0,b}(1, 1), U^a_{2,b}(l_{2,b}, w_{2,b}, 1, 1), \] \[ U^a_{1,b}(w_{1,b}, w_{0,b}, 1), Y^a_{D,b} \in A^e_{\xi} \text{ for some } w_{1,b} \neq 1\}, \quad (A.24) \]

\[ E_{D10} = \{(X^a_{R,b}(1), U^a_{0,b}(1, 1), U^a_{1,b}(w_{1,b}, 1, 1), \] \[ U^a_{2,b}(l_{2,b}, w_{2,b}, 1, 1), Y^a_{D,b} \in A^e_{\xi} \text{ for some } w_{1,b} \neq 1\}, \quad (A.25) \]

\[ E_{D11} = \{(X^a_{R,b}(1), U^a_{0,b}(1, 1), U^a_{2,b}(l_{2,b}, l_{2,b}, w_{2,b}, 1, 1), \] \[ U^a_{1,b}(1, 1, 1), Y^a_{D,b} \in A^e_{\xi} \text{ for some } w_{2,b} \neq 1\}, \quad (A.26) \]

\[ E_{D12} = \{(X^a_{R,b}(1), U^a_{0,b}(1, 1), U^a_{1,b}(w_{1,b}, 1, 1), \] \[ U^a_{2,b}(l_{2,b}, w_{2,b}, 1, 1), Y^a_{D,b} \in A^e_{\xi} \text{ for some } w_{2,b} \neq 1, w_{2,b} \neq 1\} \in \beta(1)\}, \quad (A.27) \]

\[ E_{D13} = \{(X^a_{R,b}(1), U^a_{0,b}(1, 1), U^a_{1,b}(1, 1, 1), X^a_{1,b}(1, 1, 1, 1), \] \[ U^a_{2,b}(l_{2,b}, 1, 1, 1), V^a(l_{2,b}, 1, 1, 1), Y^a_{D,b} \in A^e_{\xi} \text{ for some } w_{2,b} \neq 1\}, \quad (A.28) \]

\[ E_{D14} = \{(X^a_{R,b}(1), U^a_{0,b}(1, 1), U^a_{1,b}(1, 1, 1), \] \[ U^a_{2,b}(l_{2,b}, 1, 1, 1), X^a_{1,b}(w_{1,b}, 1, 1, 1), V^a(l_{2,b}, 1, 1, 1), Y^a_{D,b} \in A^e_{\xi} \text{ for some } w_{1,b} \neq 1\}, \quad (A.29) \]

\[ E_{D15} = \{(X^a_{R,b}(1), U^a_{0,b}(1, 1), U^a_{2,b}(l_{2,b}, 1, 1, 1), \] \[ U^a_{1,b}(1, 1, 1), X^a_{1,b}(w_{1,b}, 1, 1, 1), V^a(l_{2,b}, 1, 1, 1), Y^a_{D,b} \in A^e_{\xi} \text{ for some } w_{1,b} \neq 1\}, \quad (A.30) \]
Finally, we obtain the inequalities (1)-(11) with constraints $0 \leq R'_1 \leq R_1$, $0 \leq R'_2 \leq R_2$, substituting $R'_1 = R_1 - R'_2$ and $R'_2 = R_2 - R'_2$ and using the Fourier–Motzkin procedure. Hence, the proof is completed.

### A.2 Proof of theorem 2

\[
n(R_2) = H(W_2) = H(W_2; W_0)
\]

\[
\begin{align*}
&\leq I(W_2; Y_0; Y_D) - I(W_0; W_2; S^a) + n\varepsilon_a \\
&\leq \sum_{i=1}^{n} \left[ I(W_2; Y_{i+1}; S; Y_D) - I(W_0; W_2; S)|S|_{y_i+1} \right] \\
&= \sum_{i=1}^{n} \left[ I(W_2; Y_{i+1}; S; Y_D) - I(W_0; W_2; S)|S|_{y_i+1} \right] \\
&+ I(W_2; Y_{i+1}; S; Y_D) - I(W_0; W_2; S)|S|_{y_i+1} \\
&\leq \sum_{i=1}^{n} \left[ I(W_2; Y_{i+1}; S)|S|^y_{y_i+1} - I(W_0; W_2; S)|S|_{y_i+1} \right] \\
&- I(W_0; W_2; S)|S|^y_{y_i+1} - I(W_0; W_2; S)|S|_{y_i+1} \\
&\leq \sum_{i=1}^{n} \left[ I(W_2; Y_{i+1}; S)|S|^y_{y_i+1} - I(W_0; W_2; S)|S|_{y_i+1} \right] \\
&+ I(W_2; Y_{i+1}; S; Y_D) - I(W_0; W_2; S)|S|_{y_i+1} \\
&+ I(Y_{y_i}; Y_{i+1}; W_2; S)|S|^y_{y_i+1} - I(W_0; W_2; S)|S|_{y_i+1} \\
&= \sum_{i=1}^{n} \left[ I(W_2; Y_{i+1}; S)|S|^y_{y_i+1} - I(W_0; W_2; S)|S|_{y_i+1} \right] \\
&= \sum_{i=1}^{n} \left[ I(W_2; Y_{i+1}; S)|S|^y_{y_i+1} - I(W_0; W_2; S)|S|_{y_i+1} \right] \\
&= \sum_{i=1}^{n} \left[ I(W_2; Y_{i+1}; S)|S|^y_{y_i+1} - I(W_0; W_2; S)|S|_{y_i+1} \right] \\
&\leq \sum_{i=1}^{n} \left[ I(W_2; Y_{i+1}; S)|S|^y_{y_i+1} - I(W_0; W_2; S)|S|_{y_i+1} \right] \\
&+ I(W_2; Y_{i+1}; S; Y_D) - I(W_0; W_2; S)|S|_{y_i+1} \\
&+ I(Y_{y_i}; Y_{i+1}; W_2; S)|S|^y_{y_i+1} - I(W_0; W_2; S)|S|_{y_i+1} \\
&- I(W_0; W_2; S)|S|^y_{y_i+1} - I(W_0; W_2; S)|S|_{y_i+1} \\
&= \sum_{i=1}^{n} \left[ I(W_2; Y_{i+1}; S)|S|^y_{y_i+1} - I(W_0; W_2; S)|S|_{y_i+1} \right] \\
&\leq \sum_{i=1}^{n} \left[ I(W_2; Y_{i+1}; S)|S|^y_{y_i+1} - I(W_0; W_2; S)|S|_{y_i+1} \right] \\
&\leq \sum_{i=1}^{n} \left[ I(W_2; Y_{i+1}; S)|S|^y_{y_i+1} - I(W_0; W_2; S)|S|_{y_i+1} \right]
\end{align*}
\]
\[
\begin{align*}
&= \sum_{i=1}^{n} \left[ I(W_2, Y_D; Y_D | W_1, Y_D | W_1 Y_D^{-1}, T_e) - I(s^n_i, Y_D | W_2 Y_D^{-1}, T_e) \right] \\
&- I(W_0, Y_D; \mathcal{S}_l, \mathcal{S}_l') \\
&\leq \sum_{i=1}^{n} \left[ I(W_2, s^n_i, Y_D^{-1}, X_{R_1}; Y_D | X_{l}) \\
&- I(W_2, s^n_i, X_{R_1}, Y_D^{-1}; \mathcal{S}_l) \right] \\
&= \sum_{i=1}^{n} \left[ I(X_{R_1}; Y_D | X_{l}) + I(W_2, s^n_i, Y_D^{-1}, X_{R_1}) + I(U_2; Y_D | X_{l}) \right] \\
&- I(U_2; \mathcal{S}_l | X_{R_1}, X_{l}). \quad (A.49)
\end{align*}
\]

For brevity, only some bounds are proved. The other bounds can be done similarly.

**A.3 Proof of theorem 3**

\[
\begin{align*}
\rho_2 &= H(W_2 W_2' W_2') = H(W_2' W_2') + H(W_2' W_2 W_2' W_2') \\
&\leq I(W_2', Y_0 | W_2') + I(W_2' Y_0 W_2 W_2' W_2') + \rho_2 \\
&\leq I(W_2' Y_0 W_2 W_2' W_2') + I(W_2' Y_0 | W_2' W_2 W_2' W_2') + \rho_2 \\
&\cong \sum_{i=1}^{n} \left[ I(W_2', Y_0 | W_2' Y_D^{-1}) \\
+ I(W_2' Y_0 Y_D | W_2' W_2' W_2' W_2' Y_D^{-1} Y_D^{-1}) \right] \\
&= \sum_{i=1}^{n} \left[ I(W_2', Y_0 | W_2' Y_D^{-1}) \\
+ I(W_2', Y_0 Y_D | W_2' W_2' W_2' W_2' Y_D^{-1} Y_D^{-1}) \\
- I(Y_0 | W_2', Y_0 Y_D | W_2' W_2' W_2' W_2' Y_D^{-1} Y_D^{-1}) \right] \\
&\leq \sum_{i=1}^{n} \left[ I(W_2', Y_0 | W_2' Y_D^{-1}) - I(W_2', Y_0 | Y_D | W_2' W_2' W_2' W_2' Y_D^{-1} Y_D^{-1}) \right] \\
&+ I(W_2' S_{n+1} Y_D^{-1}; Y_0 Y_D | W_2' W_2' W_2' W_2' Y_D^{-1} Y_D^{-1}) \\
&- I(Y_0 Y_D | W_2' W_2' W_2' W_2' Y_D^{-1} Y_D^{-1}) \\
&\cong \sum_{i=1}^{n} \left[ I(W_2', Y_0 | W_2' Y_D^{-1}, X_{R_1}) \\
+ I(Y_0 Y_D | W_2' W_2' W_2' W_2' Y_D^{-1} Y_D^{-1}) \\
- I(Y_0 Y_D | W_2' W_2' W_2' W_2' Y_D^{-1} Y_D^{-1}, X_{R_1}) \right] \\
&\Rightarrow \sum_{i=1}^{n} \left[ I(W_2', Y_0 | W_2' Y_D^{-1}, X_{R_1}) \\
+ I(Y_0 Y_D | W_2' W_2' W_2' W_2' Y_D^{-1} Y_D^{-1}) \\
- I(Y_0 Y_D | W_2' W_2' W_2' W_2' Y_D^{-1} Y_D^{-1}, X_{R_1}) \right] \\
&\sim \sum_{i=1}^{n} \left[ I(W_2', Y_0 | W_2' Y_D^{-1}, X_{R_1}) \right. \\
&\left. + I(Y_0 Y_D | W_2' W_2' W_2' W_2' Y_D^{-1} Y_D^{-1}) \\
- I(Y_0 Y_D | W_2' W_2' W_2' W_2' Y_D^{-1} Y_D^{-1}, X_{R_1}) \right] \\
&\Rightarrow \sum_{i=1}^{n} \left[ I(W_2', Y_0 | W_2' Y_D^{-1}, X_{R_1}) \right. \\
&\left. + I(Y_0 Y_D | W_2' W_2' W_2' W_2' Y_D^{-1} Y_D^{-1}) \\
- I(Y_0 Y_D | W_2' W_2' W_2' W_2' Y_D^{-1} Y_D^{-1}, X_{R_1}) \right] \\
&\sim \sum_{i=1}^{n} \left[ I(W_2', Y_0 | W_2' Y_D^{-1}, X_{R_1}) \right. \\
&\left. + I(Y_0 Y_D | W_2' W_2' W_2' W_2' Y_D^{-1} Y_D^{-1}) \\
- I(Y_0 Y_D | W_2' W_2' W_2' W_2' Y_D^{-1} Y_D^{-1}, X_{R_1}) \right].
\end{align*}
\]
\[ \sum_{i=1}^{n} \left[ I(U_2; Y_R | U_1, X_R) + I(V_i; Y_D | U_1, U_2, X_1, X_R) \right], \tag{A.51} \]

\[ n(R_1 + R_2) = H(W''_1 W''_2 Y_R^{\gamma_1}) = H(W''_1 W''_2) \]

\[ + H(W''_1 W''_2 | W''_1 W''_2) \]

\[ + I(W''_1 W''_2; Y_R^{\gamma_1}| W''_1 W''_2) - I(W''_1 W''_2; W''_1 W''_2; S^0) + \varepsilon_n \]

\[ = \sum_{i=1}^{n} \left[ I(U_i W''_1 W''_2; Y_R^{\gamma_1}| W''_1 W''_2; S^0) \right] \]

\[ \leq \sum_{i=1}^{n} \left[ I(U_i W''_1 W''_2 Y_R^{\gamma_1}; Y_R^0 | X_R) - I(W''_1 W''_2 W''_1 W''_2; S^0) \right] \]

\[ + I(W''_1 W''_2 Y_R^{\gamma_1}; Y_R^0 | X_R) \]

\[ + I(W''_1 W''_2; Y_R^{\gamma_1}| W''_1 W''_2 Y_R^{\gamma_1}) - I(W''_1 W''_2; W''_1 W''_2; S^0) \]

\[ \leq \sum_{i=1}^{n} \left[ I(U_i W''_1 W''_2 Y_R^{\gamma_1}; Y_R^0 | X_R) \right] \]

\[ - I(U_2; V_i; Y_D | U_1, U_2, X_2) \]

\[ - I(U_2; V_i; S | U_0, X_R) \]. \tag{A.52} \]

### A.4 Proof of theorem 4

The results of theorem 1 can be extended to Theorem 4. The distribution in Theorem 1 can be extended to the linear continuous alphabet channel, that is, the Gaussian version, enabling us to generate inputs and auxiliary random variables as follows:

\[ U_0 = W + \sqrt{\frac{P_0}{P_R}} X_R, \quad U_1 = N_1 + \sqrt{\frac{\beta_1}{P_0}} U_0, \]

\[ X_1 = M_1 + \sqrt{\frac{\beta_1}{P_0}} U_1, \quad X_2 = X_2 + \frac{\sigma_{2x_2}}{P_0} U_0, \]

\[ U_2 V = X_2 - \frac{\sigma_{2x_2}}{\lambda P_2} M_2 - \beta_2 S = X_2 - \frac{\sigma_{2x_2}}{\lambda P_2} M_2 + (\beta_1 - \beta_2) S. \]

Where, in order for the rates to be maximised, the inputs should be Gaussian random variables:

\[ X_R \sim \mathcal{N}(0, P_R), X_1 \sim \mathcal{N}(0, P_1), X_2 \sim \mathcal{N}(0, P_2), \]

\[ U_0 \sim \mathcal{N}(0, P_0), U_1 \sim \mathcal{N}(0, \alpha R_0), \]

\[ N_1 \sim \mathcal{N}(0, \beta_1 P_1), M_1 \sim \mathcal{N}(0, \gamma_1 R_1), M_2 \sim \mathcal{N}(0, \lambda P_2), \]

\[ X_2 \sim \mathcal{N}(0, (1 - \rho^2_{2x_2}) P_2) \]

and \( X_R, M_1, N_1, M_2 \) and \( W \) are mutually independent random variables and \( 0 \leq \alpha = 1 - \beta_1, \beta_1 = 1 - \gamma_1, \rho_1 = 1 - \gamma_1 \leq 1 \).

The informed encoder exploits a more general Gaussian DPC (GDPC). Hence, the aware source input and the known side information are correlated with parameter \( -1 \leq \rho_{2x_2} \leq 1 \).

By splitting the private message in two parts, we generate \( U_2 V \) as the total private message which should be decoded completely just by destination and \( U_2 \) as part of the message is decodable by both relay and destination.

Also, we define:

\[ \sigma_{2x_2} = E[X_2 S] = E[X_2] = \rho_{2x_2} \sqrt{P_2 Q}, \]

\[ \sigma_{2x_0} = E[X_2 U_0] = \rho_{2x_0} \sqrt{P_1 P_0}, \]

\[ \sigma_{2x_2} = E[X_2 M_2] = E[X_2] = \rho_{2x_2} \sqrt{P_2 Q}. \]

Hence:

\[ b(Y_R | X_R) = b(g_{R_1} X_1 + g_{R_2} X_2 + g_{R_0} V + Z_R | X_R) \]

\[ = b \left( g_{R_1} \left( M_1 + \sqrt{\frac{\gamma_1 P_1}{P_{R_1}}} N_1 + \sqrt{\frac{\gamma_1 \beta_1 P_1}{P_0}} W \right) + g_{R_2} \left( X_2 + \frac{\sigma_{2x_0}}{P_0} W \right) + g_{R_0} S + Z_R \right) = \frac{1}{2} \log_2 (2\pi e f_{R_1}). \]
Also, it can be seen that:

\[ b(Y_R | X_R U_0) = \frac{1}{2} \log_2 \left( 2\pi e f_{R_0} \right), \]

\[ b(Y_R | X_R U_0 U_1) = \frac{1}{2} \log_2 \left( 2\pi e f_{R_0} \right), \]

\[ b(Y_D | X_R) = \frac{1}{2} \log_2 \left( 2\pi e f_{D_2} \right), \]

\[ b(Y_D | X_R) = \frac{1}{2} \log_2 \left( 2\pi e f_{D_2} \right), \]

\[ b(Y_D | X_R U_0) = \frac{1}{2} \log_2 \left( 2\pi e f_{D_2} \right), \]

\[ b(Y_D | X_R U_0 U_1) = \frac{1}{2} \log_2 \left( 2\pi e f_{D_2} \right), \]

Also:

\[ b(Y_D | X_R U_0 U_1 U_2 X_1) = b(g_{D_2} \bar{X}_2 + g_{D_2} S + Z_D | U_2) \]

\[ = \frac{1}{2} \log_2 \left( 2\pi e \right) \left[ E(\bar{Y}_D - E(\bar{Y}_D | U_2))^2 \right] \]

\[ = \frac{1}{2} \log_2 \left( 2\pi e \right) \left( \frac{g^2_{R_2} \rho_{2M_2} P_2 + N_D}{B} \right), \]

where, we define \( \bar{Y}_D = g_{R_2} \bar{X}_2 + g_{D_2} S + Z_D \) and \( \bar{Y}_R = g_{R_2} \bar{X}_2 + g_{R_2} S + Z_R \). Hence, it can be shown that:

\[ b(Y_D | X_R U_0 U_2) \]

\[ = \frac{1}{2} \log_2 \left( 2\pi e \right) \left( \frac{g^2_{R_2} \rho_{2M_2} P_2 + N_D}{B} \right), \]

\[ b(Y_D | X_R U_0 U_2) \]

\[ = \frac{1}{2} \log_2 \left( 2\pi e \right) \left( \frac{g^2_{R_2} \rho_{2M_2} P_2 + N_D}{B} \right), \]

\[ b(Y_D | X_R U_0 U_2 V) \]

\[ = \frac{1}{2} \log_2 \left( 2\pi e \right) \left( \frac{g^2_{R_2} \rho_{2M_2} P_2 + N_D}{B} \right), \]

\[ b(U_2 | X_R U_0) \]

\[ = \frac{1}{2} \log_2 \left( 2\pi e B \right), \]

\[ b(U_2 | X_R U_0 S) \]

\[ = \frac{1}{2} \log_2 \left( 2\pi e \left( 1 - \rho^2_{2U_0} - \rho^2_{2M_2} - \rho^2_{2S} \right) \right), \]

\[ b(U_2 V | X_R U_0) \]

\[ = \frac{1}{2} \log_2 \left( 2\pi e \right), \]

\[ b(U_2 V | X_R U_0 S) \]

\[ = \frac{1}{2} \log_2 \left( 2\pi e \left( 1 - \rho^2_{2U_0} - \rho^2_{2M_2} - \rho^2_{2S} \right) \right). \]

And the proof is completed.