POSITIVELY CURVED HOMOGENEOUS METRICS ON SPHERES

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The study of manifolds with positive sectional curvature has a long history and can be considered as the beginning of global Riemannian geometry. Nevertheless there are few general theorems concerning this class of manifolds which do not require further geometric assumptions. Apart from obstructions which already hold for manifolds with non-negative sectional curvature, there are only restrictions on the fundamental group due to the classical theorems of Bonnet-Meyers and Synge.

There are also very few known examples. They all arise as quotients of a compact Lie group, endowed with a left invariant metric, by a subgroup of isometries acting freely. They consist, apart from the rank one symmetric spaces, of certain homogeneous spaces in dimensions 6, 7, 12, 13 and 24 due to Berger [Be], Wallach [Wa], and Allof-Wallach [AW], and of biquotients in dimensions 6, 7 and 13 due to Eschenburg [E1, E2] and Bazaikin [Ba].

The homogeneous spaces which admit homogeneous metrics with positive sectional curvature have been classified ([Wa, BH]). A natural further question is the behavior of the pinching constants, i.e. the quotient of the minimum by the maximum of the sectional curvature. Of particular interest is whether one can determine in each case the homogeneous metric with the best pinching constant. This necessarily involves a classification of all positively curved homogeneous metrics on each space, and the determination of all 2-planes with minimal and maximal sectional curvature. This is a notoriously difficult algebraic problem and has been solved (numerically in some cases) in a remarkable paper by Püttmann [Pü] for all homogeneous spaces not diffeomorphic to a sphere (see also [E1, Gr, He, Hua, Va] for previous particular cases).

In this paper we will consider the only remaining case, that of homogeneous metrics on spheres, and will classify the set of positively curved ones. Apart from the natural challenge that this question poses, we will see that the set of curvature tensors involved is surprisingly complicated, and the methods developed for solving the problem may be of interest in other situations as well.

Homogeneous metrics on spheres were classified in [Zi] and can be described geometrically in terms of the Hopf fibrations:

\[ S^1 \to S^{2n+1} \to \mathbb{C}P^n, \quad S^3 \to S^{4n+3} \to \mathbb{H}P^n, \quad S^7 \to S^{15} \to S^8. \]

By scaling the round sphere metric on the total space by a constant \( t \) in the direction of the fibers, we obtain a one parameter family of metrics \( g_t \). For this family of metrics we will show:

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THEOREM A. The homogeneous metrics \( g_t \) have positive sectional curvature if and only if \( 0 < t < 4/3 \) and their pinching constants are given in (2.2).

This result is well known in the case where the fiber is one dimensional since in that case the curvature operator is diagonal and the maximum and minimum of the sectional curvature is hence obtained at what we call natural two-planes. They consist of vertical, horizontal, and vertizontal two planes, depending on whether the two plane is spanned by two vertical vectors, two horizontal ones, or one vertical and one horizontal. If the fiber dimension is 3 or 7, the curvature operator is not diagonal anymore and if \( 1 < t < 4/3 \) the maximum, and if \( 4/5 < t < 1 \) the minimum of the sectional curvatures is attained at two planes whose projection onto the horizontal and the vertical space are both two dimensional.

The only remaining homogeneous metrics are given (up to scaling) by the family \( g(t_1, t_2, t_3) \) on \( \mathbb{S}^{4n+3} \) where we modify the round sphere metric with an arbitrary left invariant metric on the 3-sphere fiber. Up to isometry, one can assume the metric is diagonal with respect to a fixed basis and \( t_i \) are the lengths squared of the basis vectors.

To describe the positive curvature conditions, let
\[
V_i = \left( t_j^2 + t_k^2 - 3t_i^2 + 2t_it_j + 2t_it_k - 2t_jt_k \right)/t_i \quad \text{and} \quad H_i = 4 - 3t_i,
\]
with \((i, j, k)\) a cyclic permutation of \((1, 2, 3)\). We will show:

THEOREM B. The homogeneous metrics \( g(t_1, t_2, t_3) \) on \( \mathbb{S}^{4n+3} \) have positive sectional curvature if and only if
\[
V_i > 0, \quad H_i > 0 \quad \text{and} \quad 3|t_jt_k - t_j - t_k + t_i| < t_jt_k + \sqrt{H_iV_i}
\]
with \((i, j, k)\) a cyclic permutation of \((1, 2, 3)\).

The cone condition \( V_i > 0 \) ensures that the totally geodesic fibers have positive sectional curvature and \( H_i > 0 \) guarantees that all horizontal curvatures are positive. The third condition is more subtle and again due to two-planes with a two dimensional projection onto the vertical and onto the horizontal space. This extra condition cuts out a very thin slice from the star shaped region \( V_i > 0, \ H_i > 0 \), see Figures 3-6.

In the proof of Theorem A we determine all critical points of the sectional curvature function. For the 3-dimensional family in Theorem B, this direct approach turns out to be intractable. The method used in [P] (which is due to Thorpe [Th]) works by modifying the curvature operator with invariant 4-forms. Although this approach was very successful for the homogeneous metrics studied in [P], it turns out that it gives little information for the metrics \( g(t_1, t_2, t_3) \). Instead, we will determine those metrics where the minimum of the sectional curvature is 0, by explicitly solving for the two planes that achieve this minimum.

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1. Preliminaries

If $G$ is a compact Lie group acting transitively on $S^{n-1} \subset \mathbb{R}^n$ then the action of $G$ is equivalent to a linear action of a subgroup of $\text{SO}(n)$ (see [MS], [Bo]). The set of $G$ invariant metrics was determined in [Zi] and can be described, up to scaling, as follows:

- $G = \text{SO}(n), \text{Spin}(7)$ or $G_2$. The isotropy representation is irreducible and hence all metrics have constant curvature.
- $G = \text{U}(n)$ or $\text{SU}(n)$. The isotropy representation consists of two irreducible summands of dimension 1 and $2n - 2$ and the $G$ invariant metrics are the round sphere metric scaled in the direction of the Hopf fibration with one dimensional fibers.
- $G = \text{Sp}(n)\text{Sp}(1)$ or $G = \text{Spin}(9)$. The isotropy representation consists of two irreducible summands of dimension 3 and $4n - 4$ respectively 7 and 8, and the $G$ invariant metrics are the round sphere metric scaled in the direction of the Hopf fibration with three respectively seven dimensional fibers.
- $G = \text{Sp}(n)$. The isotropy representation consists of a trivial 3-dimensional and a $4n - 4$ dimensional irreducible summand and the $G$ invariant metrics are the round sphere modified on the fibers of the Hopf fibration with three dimensional fibers by an arbitrary left invariant metric. Up to isometries the metric on the fiber can be assumed to be diagonal.
- $G = \text{Sp}(n)\text{U}(1)$. This is the subclass of metrics in the previous case where the metric on the fiber has two diagonal entries of the same length.

Since the metrics in the case of $G = \text{U}(n)$ have a diagonal curvature operator, it remains to consider two cases:

(a) A one parameter family of metrics where the Hopf fibration with 3 or 7-dimensional fibers is scaled by $t$ in the direction of the fibers of a Hopf fibration.

(b) A three parameter family of metrics where the Hopf fibration with 3-dimensional fibers is modified in the direction of the fibers by a left invariant metric.

We will discuss case (a) in Section 2, and case (b) in Section 3.

We first describe the metrics in the case of $S^{4n-1} = \text{Sp}(n)/\text{Sp}(n-1)$. On the Lie algebra of $\text{Sp}(n)$ we have the biinvariant inner product $g_0(A, B) = -\frac{1}{2}\text{Re}(\text{tr} AB)$. We then consider the orthogonal splitting $\mathfrak{sp}(n) = \mathfrak{sp}(n-1) \oplus \mathfrak{m}$ and identify $\mathfrak{m}$ with the tangent space of $\text{Sp}(n)/\text{Sp}(n-1)$ at the identity coset. A basis of $\mathfrak{m}$, orthonormal in $g_0$, is given by $X_r$, $r = 1 \cdots 3$ and $U_{rs}$, $r = 1 \cdots 4$, $s = 1 \cdots n - 1$. Here $X_r$ are the matrices which have as its only non-zero entries, $\sqrt{2i}$, $\sqrt{2j}$ respectively $\sqrt{2k}$ in row one and column one. $U_{1s}$ has as its only non-zero entries a 1 in row 1 and column $s + 1$ and a $-1$ in row $s + 1$ and column 1. $U_{2s}$ has as its only non-zero entries an $i$ in row 1 and column $s + 1$ and in row $s + 1$ and column 1 as well. Similarly $U_{3s}$ and $U_{4s}$ with $i$ replaced by $j$ or $k$ respectively. In this basis, the metric $g_0$ induced on $S^{4n-1}$ has constant curvature 1. The vectors $X_r$ are tangent to the fibers of the Hopf fibration and $U_{rs}$ is a basis of the horizontal space. The isotropy group $\text{Sp}(n)$ acts trivially on the vertical space and irreducibly on the horizontal space. We can therefore scale the metric such that it is equal to $g_0$ on the horizontal space. On the vertical space the inner product is arbitrary, but as was observed
in $[Zi]$, we can use the gauge group $N(\text{Sp}(n))/\text{Sp}(n) = \text{Sp}(1)$ to make the metric on the vertical space diagonal. In other words, the vectors $X_i$ are orthogonal and have length squared $t_i, i = 1 \cdots 3$. We will denote this metric by $g_{(t_1,t_2,t_3)}$. If $t_1 = t_2$, the metrics are invariant under the bigger group $\text{Sp}(n)U(1)$ and if $t_1 = t_2 = 1$ they are invariant under $U(2n)$. Finally, if $t_1 = t_2 = t_3 := t$, the metric is invariant under $\text{Sp}(n)\text{Sp}(1)$ and will be denoted by $g_t$.

In the case of $S^{15} = \text{Spin}(9)/\text{Spin}(7)$, the basis is more complicated and can be chosen as follows. Let $E_{ij}$ be the standard basis of $\mathfrak{spin}(9)$, and let

$$
\mathfrak{t}_1 = \text{span } \{E_{24} + E_{68}, E_{28} + E_{46}, E_{26} - E_{48}\}
$$

$$
\mathfrak{t}_2 = \text{span } \{E_{23} + E_{67}, E_{27} + E_{36}, E_{34} + E_{78}, E_{38} + E_{47}, E_{37} - E_{48}\}
$$

$$
\mathfrak{t}_3 = \text{span } \{E_{27} - E_{45}, E_{23} + E_{58}, E_{24} - E_{57}, E_{28} + E_{35}, E_{56} - E_{78}, 2E_{25} - E_{38} + E_{47}\}
$$

$$
\mathfrak{t}_4 = \text{span } \{E_{12} + E_{56}, E_{16} + E_{25}, E_{13} + E_{57}, E_{17} + E_{35}, E_{14} + E_{58}, E_{18} + E_{45}, E_{15} - E_{48}\}
$$

$$
\mathfrak{m}_1 = \text{span } \{X_1 = E_{15} + E_{26} + E_{37} + E_{48}, X_2 = E_{17} + E_{28} - E_{35} - E_{46}, X_3 = E_{13} - E_{24} - E_{57} + E_{68}, X_4 = E_{16} - E_{25} - E_{38} + E_{47}, X_5 = E_{18} - E_{27} + E_{36} - E_{45}, X_6 = E_{12} + E_{34} - E_{56} - E_{78}, X_7 = E_{14} + E_{23} - E_{58} - E_{67}\}
$$

$$
\mathfrak{m}_2 = \text{span } \{U_1 = E_{19}, U_2 = E_{29}, U_3 = E_{39}, U_4 = E_{49}, U_5 = E_{59}, U_6 = E_{69}, U_7 = E_{79}, U_8 = E_{89}\}.\]

The basis is chosen such that $\mathfrak{t}_1 \simeq \mathfrak{su}(2), \mathfrak{su}(2) + \mathfrak{t}_2 \simeq \mathfrak{su}(3), \mathfrak{su}(3) + \mathfrak{t}_3 \simeq \mathfrak{g}_2$, and $\mathfrak{g}_2 + \mathfrak{t}_4 \simeq \mathfrak{spin}(7)$. The orthogonal complement to $\mathfrak{spin}(7)$ is $\mathfrak{m}_1 \oplus \mathfrak{m}_2$ and is identified with the tangent space. The isotropy group leaves $\mathfrak{m}_i$ invariant and acts irreducibly on them. Thus we can scale any $\text{Spin}(7)$ invariant inner product such that $U_i$ is an orthonormal basis of $\mathfrak{m}_2$, $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are orthogonal, and the vectors $V_i \in \mathfrak{m}_1$ are orthogonal with length squared equal to $t$. This metric will be denoted by $g_t$.

Throughout the paper, $i, j, k$ in any formula will always denote a cyclic permutation of $(1, 2, 3)$.

2. Pinching constants for $g_t$

We now study the metrics $g_t$ and compute their pinching. We start with the case where the metric is invariant under $G = \text{Sp}(n)\text{Sp}(1)$ and use the basis $X_r, U_{r,s}$ of the tangent space described in Section 1. They are all orthogonal to each other, $|X_r|^2 = t$ and the remaining vectors have unit length. The isotropy group $\text{Sp}(n - 1)\text{Sp}(1)$ acts on the tangent space as follows: $(A, q)(y, v) = (qyq^{-1}, Avq^{-1})$ where $y \in \text{Im } \mathbb{H}$ and $v \in \mathbb{H}^n$. Modulo this isotropy action we can thus assume that the 2-plane is spanned by $X, Y$ with

$$
X = a_1 X_1 + a_2 X_2 + a_3 X_3 + a_4 U_{11} \tag{2.1}
$$

$$
Y = b_1 X_1 + b_2 X_2 + b_3 X_3 + b_4 U_{11} + b_5 U_{21} + b_6 U_{12}.
$$
A computation shows that (see also Section 3):
\[
\langle R(X, Y)X, Y \rangle = t \{ (a_2 b_3 - a_3 b_2)^2 + (a_3 b_1 - a_1 b_3)^2 + (a_1 b_2 - a_2 b_1)^2 \} \\
+ (4 - 3t) a_0^2 b_5^2 + 6t(t - 1) (a_2 b_3 - a_3 b_2) a_4 b_5 \\
+ t^2 \{ (b_1 a_1 - b_2 a_4)^2 + (b_4 a_2 - b_2 a_4)^2 + (b_4 a_3 - b_3 a_4)^2 \} \\
+ t^2 b_5^2 (a_1^2 + a_2^2 + a_3^2) + b_6^2 (t^2 (a_1^2 + a_2^2 + a_3^2) + a_4^2). 
\]

The sectional curvatures of the natural 2-planes are \(1/t\) for vertical 2-planes, \(t\) for vertizantl 2-planes and they lie in \([4 - 3t, 1]\) for horizontal 2-planes. Here we use the convention that in an interval \([a, b]\) we allow \(a \leq b\) or \(b \leq a\). If we want the sectional curvatures to be positive, we need to assume that \(0 < t < 4/3\), and we will see that this is indeed also sufficient. In addition we want to compute the pinching and hence we minimize and maximize all sectional curvatures. Our strategy will be to see if all curvatures lie in the interval given by the values at natural 2-planes. We will see that this is not always the case and will determine the 2-planes that actually achieve the maximum and minimum.

If we set \(A = (a_1, a_2, a_3)\) and \(B = (b_1, b_2, b_3)\), the sectional curvature is given by
\[
\sec(X, Y) \{ (t|A|^2 + a_1^2)(t|B|^2 + b_2^2 + b_3^2) - (t(A \cdot B) + a_4 b_4)^2 + b_6^2 (t|A|^2 + a_4^2) \} \\
= t|A \times B|^2 + (4 - 3t) a_0^2 b_5^2 - 6t(t - 1) ((A \times B) \cdot e_1) a_4 b_5 + \\
+ t^2 \{ b_5^2 |A|^2 + a_2^2 |B|^2 - 2a_4 b_4 (A \cdot B) + b_6^2 |A|^2 \} + b_6^2 (t^2 |A|^2 + a_4^2). 
\]

In the following we will use the observation that a function of the type \(\frac{ax^2 + b}{cx^4 + dx^2}\), where \(a, b, c, d\) are constants, is either decreasing or increasing on \([0, \infty]\). We can apply this for the variables \(b_4\) and \(b_6\) if we subtract a multiple of \(X\) from \(Y\) in order to make \(A \cdot B = 0\), which we will assume from now on. In that case, \(\sec \to t\) when \(b_4 \to \infty\) and \(\sec \to \frac{t^2 |A|^2 + a_4^2}{|A|^2 + a_4^2} \in [1, t]\) when \(b_6 \to \infty\), and we can hence assume that \(b_4 = b_6 = 0\).

Finally, we can make \((A \times B) \cdot e_1\) be equal to any value in the interval \([-|A||B|, |A||B|]\) without changing \(|A|\) or \(|B|\). Since \(a_4\) can take on negative values, we can thus assume that \((A \times B) \cdot e_1 = |A||B|\) in order to determine the maximum and minimum of the sectional curvatures.

If we set \(x = |A|, y = |B|, r = a_4, s = b_5\) we hence have to determine the critical points of
\[
F = \frac{tx^2 y^2 + (4 - 3t) r^2 s^2 + 6t(t - 1) x y r s + t^2 (x^2 s^2 + y^2 r^2)}{(tx^2 + r^2)(ty^2 + s^2)}. 
\]

Using the above observation, one sees again that if any of the 4 variable vanish, the values of \(F\) lie between the curvatures of natural 2-planes. We can thus normalize the vectors such that \(x = y = 1\) and one easily sees that the only non-zero critical points of the remaining function of \(r\) and \(s\) are, besides \((r, s) = (0, 0)\), given by
\[
r = -s = \pm \sqrt{1 - 2t} \quad \text{with} \quad F = 4(1 - t) 
\]
and
\[
r = s = \pm \sqrt{(4t + 1)/t} \quad \text{with} \quad F = \alpha := \frac{16t^2 - 8t + 4}{11t + 1}. 
\]
The value of $F$ for the first critical point lies in between the curvatures of natural 2-planes for $0 < t < 1/2$. But for the second critical point, $\alpha > t$ for $1 < t < 4/3$ and $\alpha < t$ for $4/5 < t < 1$. We thus obtain for the maximum of the sectional curvature

$$\max \sec = \begin{cases} 
1/t & \text{if } 0 < t \leq 1/3, \\
4 - 3t & \text{if } 1/3 \leq t \leq 1, \\
\alpha & \text{if } 1 \leq t \leq 4/3,
\end{cases}$$

and for the minimum

$$\min \sec = \begin{cases} 
t & \text{if } 0 < t \leq 4/5, \\
\alpha & \text{if } 4/5 \leq t \leq 1, \\
4 - 3t & \text{if } 1 \leq t \leq 4/3.
\end{cases}$$

Hence the pinching constants, i.e., $\delta = \min \sec / \max \sec$, are given by

$$\text{pinching } \delta = \begin{cases} 
t^2 & \text{if } 0 < t \leq 1/3, \\
t/(4-3t) & \text{if } 1/3 \leq t \leq 4/5, \\
\alpha/(4-3t) & \text{if } 4/5 \leq t \leq 1, \\
(4-3t)/\alpha & \text{if } 1 \leq t \leq 4/3.
\end{cases}$$

Figure 1 shows a graph of the pinching number on the left. For $4/5 < t < 4/3$ the pinching is slightly less than what one would obtain by using only natural 2-planes. The difference is shown in the picture on the right and is less than 0.008.
Spin(7) acts via the standard representation on \( m_1 \) and via the spin representation on \( m_2 \). Thus, since \( \text{Spin}(7)/G_2 = S^7 \) and \( G_2/\text{SU}(3) = S^6 \), we may assume \( B_1 \in \text{span}(U_1, U_2) \). Since the isotropy group \( \text{SU}(3) \) fixes \( X_1 \), we may hence assume \( A_1 \in \text{span}(X_1, X_2) \), although for symmetry reasons we will choose \( A_1 \in \text{span}(X_1, X_2, X_3) \). Since \( \text{SU}(2) \) fixes \( \text{span}(X_1, X_2, X_3) \), we may choose \( A_2 \in \text{span}(X_1, X_2, X_3, X_4) \).

Let \( X = a_1 X_1 + a_2 X_2 + a_3 X_3 + a_4 U_1 \) and \( Y = b_1 X_1 + b_2 X_2 + b_3 X_3 + x X_4 + b_4 U_1 + b_5 U_5 \). If we set \( A = (a_1, a_2, a_3) \) and \( B = (b_1, b_2, b_3) \) as before, the sectional curvature is given by

\[
\sec(X, Y) \{ (t|A|^2 + a_1^2)(t|B|^2 + b_1^2 + b_2^2) - (t(A \cdot B) + a_4 b_4)^2 + x^2(t|A|^2 + a_3^2) \} = t|A \times B|^2 + (4 - 3t)a_4 b_4 + 6t(t - 1)(\langle A \times B \rangle \cdot e_1) a_4 b_4 +
\]

\[
+ t^2\{ b_1^2|A|^2 + a_3^2|B|^2 - 2a_4 b_4(A \cdot B) + b_4^2|A|^2 \} + x^2(t|A|^2 + t^2a_3^2),
\]

and the argument proceeds as before. In particular, for both cases of \( g_t \), maximum and minimum are assumed at 2-planes tangent to a totally geodesic 7-sphere.

### 3. Positive curvature on \( S^{4n-1} \)

In this section we will study the curvature tensor of the metrics \( g_{(t_1, t_2, t_3)} \) on \( S^{4n-1} \) described in Section 1 and use the basis \( X_r, U_{rs} \) for the tangent space. Modulo the action of the isotropy subgroup \( \text{Sp}(n - 1) \), it is enough to consider 2-planes spanned by vectors of the form

\[(3.1) \quad X = a_1 X_1 + a_2 X_2 + a_3 X_3 + a_4 U_1 \]
\[Y = b_1 X_1 + b_2 X_2 + b_3 X_3 + b_4 U_1 + b_5 U_2 + b_6 U_3 + b_7 U_4 + b_8 U_5 \]

Besides the quantities \( V_i \) and \( H_i \) defined in the introduction, we also set

\[L_i = 6(t_j t_k - t_j - t_k + t_i).\]

Using any one of the curvature formulas for a homogeneous space (see, e.g., [Pil]), one shows that

\[
R(X_i, X_j, X_k, X_l) = V_k , \quad R(U_{1p}, U_{2q}, U_{1p}, U_{2q}) = R(U_{1q}, U_{2p}, U_{1q}, U_{2p}) = H_\gamma \]
\[
R(U_{1p}, U_{2q}, U_{2p}, U_{2q}) = R(U_{1p}, U_{2q}, U_{2p}, U_{2q}) = 1 , \quad R(U_{1p}, U_{2p}, U_{1q}, U_{2q}) = \sigma(\tau \cdot \tau - 2t_2) \]
\[
R(U_{1p}, U_{1q}, U_{2q}, U_{1q}) = R(U_{2p}, U_{2q}, U_{1q}, U_{1q}) = 2(1 - t_2) , \quad R(U_{1p}, U_{2p}, U_{1q}, U_{2q}) = \sigma(2 - t_2) \]
\[
R(U_{1p}, U_{1q}, U_{2q}, U_{1q}) = R(U_{2p}, U_{1q}, U_{1q}, U_{1q}) = R(U_{2p}, U_{1q}, U_{2q}, U_{1q}) = 1 - \tau \]
\[
R(U_{1p}, U_{2q}, U_{2q}, U_{1q}) = (1 - \tau) , \quad R(U_{1p}, U_{2q}, U_{1q}, U_{2q}) = -\sigma(1 - \tau) \]
\[
R(U_{1p}, X_\alpha, U_{1q}, X_\alpha) = R(U_{2p}, X_\alpha, U_{2q}, X_\alpha) = t_2^2 \]
\[
R(U_{1p}, U_{2q}, X_\alpha, X_\beta) = 2R(U_{1p}, X_\alpha, U_{2q}, X_\beta) = 2R(U_{1p}, X_\alpha, X_\alpha, U_{2q}) = -L_\gamma/3 \]
\[
R(U_{1p}, U_{2q}, X_\beta, X_\gamma) = -2R(U_{1p}, X_\gamma, U_{2q}, X_\beta) = -\sigma L_\alpha/3,
\]

where \( p \neq q \in \{1, (n - 1)\} \), \( r \neq s \in \{1, 2, 3, 4\} \) and \((\alpha, \beta, \gamma)\) a permutation, with sign \( \sigma \), of \((1, 2, 3)\). This easily implies that
\[ \langle R(X, Y)X, Y \rangle = V_1(a_2b_3 - a_3b_2)^2 + V_2(a_3b_1 - a_1b_3)^2 + V_3(a_1b_2 - a_2b_1)^2 \\
+ L_1(a_2b_3 - a_3b_2)a_4b_5 + L_2(a_3b_1 - a_1b_3)a_4b_6 + L_3(a_1b_2 - a_2b_1)a_4b_7 \\
+ H_1a_1^2b_5^2 + H_2a_2^2b_6^2 + H_3a_3^2b_7^2 \\
+ t_1^2(b_4a_1 - b_1a_4)^2 + t_2^2(b_4a_2 - b_2a_4)^2 + t_3^2(b_4a_3 - b_3a_4)^2 \\
+ (b_5^2 + b_6^2 + b_7^2)(t_1^2a_1^2 + t_2^2a_2^2 + t_3^2a_3^2) + b_8^2(t_1^2a_1^2 + t_2^2a_2^2 + t_3^2a_3^2 + a_4^2). \]

We now consider the set of homogeneous metrics

\[ \mathcal{P} = \{g(t_1, t_2, t_3) \mid t_i > 0, V_i > 0, H_i > 0\} \]

where the vertical and the horizontal curvatures are already positive. Notice that (3.2) implies that then all vertizontal curvatures are automatically positive as well. We want to check if these conditions are also sufficient by looking for metrics in \( \mathcal{P} \) where the minimum of the sectional curvature is 0. We thus assume from now that \( g = g(t_1, t_2, t_3) \in \mathcal{P} \) is such that for any choice of the parameters \( a_i, b_i \) we have \( R(X, Y, X, Y) \geq 0 \), with equality for some value of \( a_i \) and \( b_i \). In the following, we will assume that \( X, Y \) span such a zero curvature 2-plane and our strategy will be to try to solve for \( a_i, b_i \) in terms of \( t_i \).

We can clearly assume that \( b_8 = 0 \) since otherwise the two plane spanned by \( \bar{X} = X, \bar{Y} = Y - b_8 U_{21} \) would have negative sectional curvature contradicting our assumption. Thus we can restrict ourselves to metrics on \( S^7 \).

The expression of the curvature in (3.2) suggests that one should consider vectors \( X, Y \) orthonormal with respect to a modified inner product. We will assume in the following that \( X \) and \( Y \) satisfy the conditions

\[
\begin{cases}
  t_1^2a_1^2 + t_2^2a_2^2 + t_3^2a_3^2 + a_4^2 = 1 \\
  t_1^2b_1^2 + t_2^2b_2^2 + t_3^2b_3^2 + b_4^2 + b_5^2 + b_6^2 + b_7^2 = 1 \\
  t_1^2a_1b_1 + t_2^2a_2b_2 + t_3^2a_3b_3 + a_4b_4 = 0.
\end{cases}
\]

For later purposes we observe that, under these conditions, we can assume that \( 0 < a_4 < 1 \). Indeed, if \( a_4 = 0 \), either one of \( b_i, i = 4, \ldots, 7 \) is non-zero or the 2-plane is vertical. In either case, (3.2) implies that all curvatures are positive since \( g \in \mathcal{P} \). Similarly, if \( a_4 = 1 \), the curvatures are positive since \( a_i = 0, i = 1, 2, 3 \) and thus one of \( b_i, i = 1, 2, 3 \) or \( i = 5, 6, 7 \) is non-zero.

Using (3.3), we may rewrite the last two lines in (3.2) as follows:

\[
\begin{align*}
  & t_1^2(b_4a_1 - b_1a_4)^2 + t_2^2(b_4a_2 - b_2a_4)^2 + t_3^2(b_4a_3 - b_3a_4)^2 \\
  & + (b_5^2 + b_6^2 + b_7^2)(t_1^2a_1^2 + t_2^2a_2^2 + t_3^2a_3^2) \\
  & = b_4^2(t_1^2a_1^2 + t_2^2a_2^2 + t_3^2a_3^2) + a_4^2(t_1^2b_1^2 + t_2^2b_2^2 + t_3^2b_3^2) \\
  & \quad - 2a_4b_4(t_1^2a_1b_1 + t_2^2a_2b_2 + t_3^2a_3b_3) + (b_5^2 + b_6^2 + b_7^2)(t_1^2a_1^2 + t_2^2a_2^2 + t_3^2a_3^2) \\
  & = (b_4^2 + b_5^2 + b_6^2 + b_7^2)(1 - a_4^2) + a_4^2(1 - b_4^2 - b_5^2 - b_6^2 - b_7^2) - 2a_4b_4(-a_4b_4) \\
  & = b_4^2 + a_4^2 + (b_5^2 + b_6^2 + b_7^2)(1 - 2a_4^2). 
\end{align*}
\]
We now claim that under the assumptions (3.3), the value of $b_4$ must be 0. Indeed, if we let $\vec{X} = X$ and $\vec{Y} = Y - bV_1$ we obtain
\[
\langle R(X,Y)X,Y \rangle - \langle R(\vec{X},\vec{Y})\vec{X},\vec{Y} \rangle = b_1^2(1 - a_4^2) + 2a_3^3b_4^2 = b_1^2(1 + a_4^2)
\]
and hence $\langle R(\vec{X},\vec{Y})\vec{X},\vec{Y} \rangle < 0$, contradicting our assumption.

Thus we have to minimize the following function, where we have used the further abbreviation $A_{ij} = a_i b_j - a_j b_i$

\[
F(a_i, b_i) = V_1 A_{23}^2 + V_2 A_{31}^2 + V_3 A_{12}^2 + L_1 A_{23} a_4 b_5 + L_2 A_{31} a_4 b_6 + L_3 A_{12} a_4 b_7
\]

subject to the restrictions (3.3). We use Lagrange multipliers to determine the minimizing multipliers. We claim they are given by

\[
\begin{align*}
2V_3 b_2 A_{12} - 2V_2 b_3 A_{31} + L_3 a_4 b_2 b_7 - L_2 a_4 b_3 b_6 + 2t_1^2 \lambda_1 a_1 + t_1^2 \lambda_3 b_1 &= 0 \\
2V_1 b_3 A_{23} - 2V_3 b_1 A_{12} + L_1 a_4 b_3 b_5 - L_3 a_4 b_1 b_7 + 2t_2^2 \lambda_2 a_2 + t_2^2 \lambda_3 b_2 &= 0 \\
2V_2 b_1 A_{31} - 2V_1 b_2 A_{23} + L_2 a_4 b_1 b_6 - L_1 a_4 b_2 b_5 + 2t_3^2 \lambda_3 a_3 + t_3^2 \lambda_2 b_3 &= 0.
\end{align*}
\]

By setting $S_i = 2V_i A_{jk} + L_i a_4 b_{4+i}$, we may rewrite the above equations as

\[
\begin{align*}
S_3 b_2 - S_2 b_3 + 2t_1^2 \lambda_1 a_1 + t_1^2 \lambda_3 b_1 &= 0 \\
S_1 b_3 - S_3 b_1 + 2t_2^2 \lambda_2 a_2 + t_2^2 \lambda_3 b_2 &= 0 \\
S_2 b_1 - S_1 b_2 + 2t_3^2 \lambda_3 a_3 + t_3^2 \lambda_2 b_3 &= 0,
\end{align*}
\]

that is,

\[
-(S_1, S_2, S_3) \times (b_1, b_2, b_3) + 2 \lambda_1 (t_1^2 a_1, t_2^2 a_2, t_3^2 a_3) + \lambda_3 (t_1^2 b_1, t_2^2 b_2, t_3^2 b_3) = (0, 0, 0).
\]

Since $(b_1, b_2, b_3)$ is orthogonal to $(t_1^2 a_1, t_2^2 a_2, t_3^2 a_3)$ this implies that $(b_1, b_2, b_3)$ is orthogonal also to $\lambda_3 (t_1^2 b_1, t_2^2 b_2, t_3^2 b_3)$. If $\lambda_3 \neq 0$ it follows that $b_1 = b_2 = b_3 = 0$. But in this case (3.2) implies that the sectional curvature is positive, contradicting our assumption. Hence $\lambda_3 = 0$ and $(t_1^2 a_1, t_2^2 a_2, t_3^2 a_3)$ is orthogonal to $(S_1, S_2, S_3)$.

In a similar way, using the Lagrange multipliers for $b_1, b_2, b_3$ we also have the orthogonality between $(t_1^2 b_1, t_2^2 b_2, t_3^2 b_3)$ and $(S_1, S_2, S_3)$. Hence there exists a $\lambda$ (we use $-2\lambda$ for convenience) with

\[
(S_1, S_2, S_3) = -2\lambda (t_1^2 a_1, t_2^2 a_2, t_3^2 a_3) \times (t_1^2 b_1, t_2^2 b_2, t_3^2 b_3)
\]

\[
= -2\lambda (A_{23} t_2^2 t_3^2, A_{31} t_3^2 t_1^2, A_{12} t_1^2 t_2^2).
\]

We can thus rewrite the Lagrange multiplier equations for $a_1, a_2, a_3, b_1, b_2, b_3$ as

\[
\begin{align*}
L_1 a_4 b_5 &= -2(\lambda t_2^2 t_3^2 + V_1) A_{23} \\
L_2 a_4 b_6 &= -2(\lambda t_3^2 t_1^2 + V_2) A_{31} \\
L_3 a_4 b_7 &= -2(\lambda t_1^2 t_2^2 + V_3) A_{12}.
\end{align*}
\]

We can now determine the multipliers. We claim they are given by

\[
\begin{align*}
\lambda_1 &= \lambda (1 - b_5^2 - b_6^2 - b_7^2) , \\
\lambda_2 &= \lambda (1 - a_4^3) , \\
\lambda_3 &= 0.
\end{align*}
\]
To see this, say for $\lambda_1$, set

$$v = (t_1^2a_1, t_2^2a_2, t_3^2a_3), \quad w = (t_1^2b_1, t_2^2b_2, t_3^2b_3), \quad z = (b_1, b_2, b_3).$$

We thus have

$$2\lambda_1v = (S_1, S_2, S_3) \times z = -2\lambda(v \times w) \times z.$$

Under the restrictions (3.3), it follows that $v \cdot z = 0$ and $w \cdot z = 1 - b_5^2 - b_6^2 - b_7^2$. Using the identity $(v \times w) \times z = (w \cdot z)v - (v \cdot z)w$ the claim follows.

Consider now the multiplier equations with respect to $b_5, b_6, b_7$:

$$
\begin{align*}
2b_5(1 - 2a_4^2) + 2b_7H_1a_4^2 + L_1A_{23}a_4 + 2\lambda_2b_5 &= 0 \\
2b_6(1 - 2a_4^2) + 2b_6H_2a_4^2 + L_2A_{31}a_4 + 2\lambda_2b_6 &= 0 \\
2b_7(1 - 2a_4^2) + 2b_7H_3a_4^2 + L_3A_{12}a_4 + 2\lambda_2b_7 &= 0
\end{align*}
$$

Using the abbreviation

$$h = \frac{1 - 2a_4^2 + \lambda_2}{a_4}$$

we can rewrite the equations as

$$
\begin{align*}
H_1a_4b_5 &= -b_5h - \frac{L_1}{2}A_{23} \\
H_2a_4b_6 &= -b_6h - \frac{L_2}{2}A_{31} \\
H_3a_4b_7 &= -b_7h - \frac{L_3}{2}A_{12}.
\end{align*}
$$

The Lagrange multiplier equation for the variable $a_4$ is given by

$$
2H_1a_4b_5^2 + 2H_2a_4b_6^2 + 2H_3a_4b_7^2 + L_1A_{23}b_5 + L_2A_{31}b_6 + L_3A_{12}b_7 \\
+ 2a_4 - 4a_4(b_5^2 + b_6^2 + b_7^2) + 2\lambda_1a_4 = 0.
$$

Using (3.7) it follows that

$$
-2(b_5^2 + b_6^2 + b_7^2)h + 2a_4 - 4a_4(b_5^2 + b_6^2 + b_7^2) + 2\lambda_1a_4 = 0.
$$

If in turn we replace the value of $\lambda_1, h$ and $\lambda_2$ in this expression, and multiplying by $a_4$, the multiplier equation for $a_4$ becomes

$$
(a_4^2 - b_5^2 - b_6^2 - b_7^2)(\lambda + 1) = 0.
$$
We have as a last condition that the sectional curvature of the 2-plane is 0. Using (3.4)-(3.7), we get

\[ 0 = V_1 A_{23}^2 + V_2 A_{31}^2 + V_3 A_{12}^2 + \frac{L_1}{2} a_4 b_5 A_{23} + \frac{L_2}{2} a_4 b_6 A_{31} + \frac{L_3}{2} a_4 b_7 A_{12} \]

\[- h a_4 (b_5^2 + b_6^2 + b_7^2) + a_4^2 + (b_5^2 + b_6^2 + b_7^2)(1 - 2a_4^2) \]

\[ = V_1 A_{23}^2 + V_2 A_{31}^2 + V_3 A_{12}^2 - (\lambda t_2^2 t_3^2 + V_1) A_{23}^2 - (\lambda t_3^2 t_1^2 + V_2) A_{31}^2 - (\lambda t_1^2 t_2^2 + V_3) A_{12}^2 \]

\[ + a_4^2 + (b_5^2 + b_6^2 + b_7^2)(1 - 2a_4^2 - h a_4) \]

\[ = -\lambda (A_{23}^2 t_2^2 t_3^2 + A_{31}^2 t_3^2 t_1^2 + A_{12}^2 t_1^2 t_2^2) + a_4^2 - (b_5^2 + b_6^2 + b_7^2) \lambda_2 \]

\[ = -\lambda (1 - b_5^2 - b_6^2 - b_7^2)(1 - a_4^2) + a_4^2 - \lambda (b_5^2 + b_6^2 + b_7^2)(1 - a_4^2) \]

\[ = a_4^2 - \lambda (1 - a_4^2). \]

In the above computation we have used the consequence of (3.3) that the vector with components \( t_4 a_i, i = 1, 2, 3 \) and the one with components \( t_4 b_i, i = 1, 2, 3 \) are orthogonal to each other and hence their cross product has length squared \((1 - b_5^2 - b_6^2 - b_7^2)(1 - a_4^2)\). It thus follows that

\[ \lambda = \frac{a_4^2}{1 - a_4^2} \]

which in turn implies

\[ h = \frac{1 - a_4^2}{a_4}. \]

Furthermore, comparing with (3.8), we obtain

\[ a_4^2 = b_5^2 + b_6^2 + b_7^2. \]

Altogether, we see that we can find a minimizing 2-plane with zero sectional curvature if and only if we can solve the following system:

\[
\begin{align*}
L_i a_4 b_{4+i} &= -2 \left( \frac{t_2^2 t_3^2}{1-a_4^2} + V_i \right) A_{jk} \\
H_i a_4 b_{4+i} &= -\frac{1-a_4^2}{a_4} a_4 b_{4+i} - \frac{L_i}{2} A_{jk} \\
&= a_4^2 = b_5^2 + b_6^2 + b_7^2
\end{align*}
\]

under the restrictions (3.3). Since we assumed that \( a_4 > 0 \), there must be one \( b_i, i > 4 \) which does not vanish. If \( b_{4+i} \neq 0 \), we can solve the first equation for \( \frac{A_{jk}}{a_4 b_{4+i}} \), substitute into the second, and obtain a quadratic equation in \( Z = \frac{a_4^2}{1-a_4^2} \):

\[
\begin{align*}
t_2^2 t_3^2 H_i Z^2 + \left( t_3^2 t_1^2 + H_i V_i - \frac{L_i^2}{4} \right) Z + V_i = 0.
\end{align*}
\]

In this equation, the leading coefficient and the last term are positive by assumption. Since we also assumed that \( a_4 < 1 \), we need a positive solution, which implies in particular that \( E_i := t_2^2 t_3^2 + H_i V_i - \frac{L_i^2}{4} < 0 \). We now claim that there can be at most one \( i \) with
$E_i < 0$. For this purpose, we will show that $E_1 > 0$ if $0 < t_1 \leq t_2 \leq 4/3$. It then follows, by symmetry, that the same is true for $t_1 \leq t_3$ and hence $E_1$ can be negative only in the region where $t_1 > \max\{t_2, t_3\}$. Similarly for $E_2$ and $E_3$ in the regions $t_2 > \max\{t_1, t_3\}$ and $t_3 > \max\{t_1, t_2\}$ respectively, which will finish our claim.

The fact that $E_1 > 0$ for $0 < t_1 \leq t_2 \leq 4/3$ can be proved by somewhat tedious estimates. We indicate the reason via a picture. We factor $E_1$ as follows:

$$t_1 E_1 = A t_2^2 + B t_3 + C = (-12t_1 + 4 - 8t_1 t_2 + 18t_1 t_2) t_3^2 + (-12t_1 t_2 - 8t_2 + 12t_1^2 + 18t_1 t_2 + 8t_1 - 18t_1 t_2) t_3 - 12t_1^2 + 4t_2^2 + 12t_2 t_3 - 12t_1 t_2^2 + 8t_1 t_2.$$

One can now draw a graph of the coefficients $A$ and $C$ to check that in the region $0 < t_1 \leq t_2 \leq 4/3$ the function $A$ is positive and $C$ is non-negative. Hence $t_1 E_1$ is an upward pointing parabola in the $t_3$ variable which thus has as its minimum $C - \frac{B^2}{4A}$. Figure 2 shows where this minimum is positive in the white area in the right hand side picture. The left hand side picture shows in the black area where the coefficient $B$ is negative. Altogether, this implies that $E_1$ is positive.

![Figure 2](image)

Thus we have shown that only one of the equations (3.10) can have a positive solution. Assume that this is the equation for $i = 1$, in other words $b_5 \neq 0$. We then necessarily have $b_6 = b_7 = 0$ and hence $b_5^2 = a_4^2$. Furthermore, (3.9) with $i = 2, 3$ can only hold if $A_{13} = a_1 b_3 - a_3 b_1 = 0$ and $A_{12} = a_1 b_2 - a_2 b_1 = 0$ and hence $a_1 = b_1 = 0$. Thus the normalization (3.3) takes on the form

$$t_2^2 a_2^2 + t_3^2 a_3^2 = t_2^2 b_2^2 + t_3^2 b_3^2 = 1 - a_4^2, \quad t_2^2 a_2 b_2 + t_3^2 a_3 b_3 = 0,$$

which implies that $(t_2 b_2, t_3 b_3) = \pm (t_3 a_3, -t_2 a_2)$ and hence $A_{23} = a_2 b_3 - a_3 b_2 = \mp \frac{1-a_2^2}{t_2 t_3}$. 


Instead of solving the system (3.9) directly, we can now substitute the values of $a_i$ and $b_i$ into the sectional curvature formula (3.4) and obtain:

\[
F = \frac{t_2^2}{(1-a_i^2)^2} = t_2^2 H_1 Z^2 + 2t_2 t_3 (\pm \frac{L_1}{2} + t_2 t_3) Z + V_1.
\]

Since we require that $F \geq 0$, a positive solution $Z$ to $F = 0$ corresponds to a double root which exists if and only if

\[
|L_1| = 2(t_2 t_3 + \sqrt{H_1 V_1}).
\]

Altogether we have shown that a metric in $\mathcal{P}$ with non-negative sectional curvature admits a 0 curvature plane if and only if $|L_i| = 2(t_j t_k + \sqrt{H_i V_i})$ for one $i$. Finally, observe that (3.11) also implies that a metric with $|L_i| > 2(t_j t_k + \sqrt{H_i V_i})$ has 2-planes with negative sectional curvature. Hence a metric in $\mathcal{P}$ with $\sec > 0$ is characterized by $|L_i| < 2(t_j t_k + \sqrt{H_i V_i})$, which proves Theorem B.

Remark. If we have a metric with non-negative, but not positive curvature, the proof also classifies the set of 2-planes with 0 curvature. One easily sees that the 0-curvature 2-planes are all vertical if $V_i = 0$ and for $H_i = 0$ they are all horizontal. For the remaining metrics in the boundary of $\mathcal{P}$ the 0-curvature 2-planes are described as follows. As we saw, only one of the quadratic equations (3.10) has a positive solution. Assuming that this is the case for $i = 1$, the 0 curvature planes consist of two disjoint circles where each 2-plane is spanned by:

\[
X = \frac{\cos(\theta)}{t_2} X_2 + \frac{\sin(\theta)}{t_3} X_3 + \sqrt{\frac{Z}{1+Z}} U_1, \quad Y = -\frac{\sin(\theta)}{t_2} X_2 + \frac{\cos(\theta)}{t_3} X_3 \pm \sqrt{\frac{Z}{1+Z}} U_2
\]

for some $\theta$, and where $Z$ is a positive solution of (3.10).

We illustrate the set of positively curved metrics in Theorem B with some pictures. Figure 3 shows the cone of metrics where horizontal and vertical curvatures are positive and Figure 4 a cross section in the plane $t_1 + t_2 + t_3 = 1$. The cone touches the coordinate planes in the diagonals $t_i = t_j , t_k = 0$. Thus in this case $t_k$ can go to 0, corresponding to a Cheeger deformation in the direction of the extra circle of isometries, i.e. contracting in the orbits of the circle action. Such Cheeger deformations preserve positive curvature.

Figure 5 shows the surfaces $V_i = 0$ and in dark the extra small slice cut out by the additional inequality in Theorem B. Figure 6 shows the difference in the $t_3$ coordinates of the two surfaces as a function of $t_1, t_2$. The difference is at most 0.0085. The curve that separates the two surfaces is given by $4t_1 t_2 = 4t_1 + 4t_2 - 3$. A typical metric with some negative sectional curvatures in between these two surfaces is given by $(t_1, t_2, t_3) = (0.25, 0.25, 0.33)$. The 2-plane spanned by $(a_1, a_2, a_3, a_4) = (4, 0, 0, \frac{7}{10})$ and $(b_1, b_2, b_3, b_4, b_5, b_6, b_7) = (0, 4, 0, 0, 0, 0, \frac{7}{10})$ then has $\sec(X, Y) = -0.69$.

Figure 7 shows a cross section of the set of positively curved metrics by the plane $t_1 = t_2$ which corresponds to $\text{Sp}(n) \ U(1)$ invariant metrics. The dotted curve goes from $(0, 0)$ to $(\frac{1}{2}, \frac{2}{3})$ and is given by the equation $t_3 = \frac{t_1 (4t_1^2 - 12t_1^2 - 4 + 9t_1)}{3(2t_1 - 2t_1^2 - 1)}$. It is slightly less than
$t_3 = \frac{4}{3} t_1$ which corresponds to $V_3 = 0$. Figure 8 again shows the difference between these two curves. A dot in the pictures represents the biinvariant metric.

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Figure 3. Positive cone $\mathcal{P} = \{g_{t_1,t_2,t_3} \mid V_i > 0, H_i > 0\}$.

Figure 4. Cross section of $\mathcal{P}$ in 2-plane $t_1 + t_2 + t_3 = 1$. 
Figure 5. One face of modified $\mathcal{P}$.

Figure 6. Difference of the two surfaces.
Figure 7. Slice with $t_1 = t_2$.

Figure 8. Difference in height.