PARABOLIC SINGULAR INTEGRALS WITH NONHOMOGENEOUS KERNELS

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Abstract. We establish $L^2$ boundedness of all “nice” parabolic singular integrals on “Good Parabolic Graphs”, aka regular Lip$(1, 1/2)$ graphs. The novelty here is that we include non-homogeneous kernels, which are relevant to the theory of parabolic uniform rectifiability. Previously, the third named author had treated the case of homogeneous kernels. The present proof combines the methods of that work (which in turn was based on methods described in Christ’s CBMS lecture notes), with the techniques of Coifman-David-Meyer.

This is a very preliminary draft. Eventually, these results will be part of a more extensive work on parabolic uniform rectifiability and singular integrals.

1. Notation, definitions, and statement of results

Notation: Our ambient space is $(n + 1)$-dimensional space-time:

$$\mathbb{R}^{n+1} = \{ X := (X, t) = (x_0, x, t) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} \} ,$$

and we shall also at times work with $n$-dimensional space-time

$$\mathbb{R}^n = \{ x := (x, t) \in \mathbb{R}^{n-1} \times \mathbb{R} \} .$$

We write $(\xi, \tau) \in \mathbb{R}^{n-1} \times \mathbb{R}$ to denote points on the Fourier transform side of space-time. We also write $\mathbb{R}_x^n$ to denote purely spatial $n$-dimensional Euclidean space. Note in particular that we shall often distinguish one spatial variable in $\mathbb{R}_x^n$, and write, e.g., $X = (x_0, x) \in \mathbb{R} \times \mathbb{R}^{n-1}$, and that we use lower case letters $x, y, z$ to denote spatial points in $\mathbb{R}^{n-1}$, and capital letters $X = (x_0, x), Y = (y_0, y), Z = (z_0, z)$ to denote points in $\mathbb{R}_x^n = \mathbb{R} \times \mathbb{R}^{n-1}$. In accordance with the notation introduced above, we let $||X|| = ||(x, t)|| = ||(x_0, x, t)||$ and $||x|| = ||(x, t)||$ denote the parabolic length of the vectors $X = (X, t) \in \mathbb{R}^{n+1}$ and $x = (x, t) \in \mathbb{R}^n$, respectively.

We let $d = n + 1$ denote the parabolic homogeneous dimension of space-time $\mathbb{R}^n$.

We define a fractional integral operator $I_p$ of parabolic order $1$ on $\mathbb{R}^n$ by means of the Fourier transform:

$$\hat{I}_p f(\xi, \tau) := ||(\xi, \tau)||^{-1} \hat{f}(\xi, \tau) .$$
Definition 1 (Parabolic Calderón-Zygmund kernels). For an integer $N \geq 1$, we shall say that a kernel $K = K(X,t)$ satisfies a $(d$-dimensional) parabolic C-Z($N$) condition, and we write $K \in C-Z(N)$, if it satisfies the following properties:

(i) Smoothness with C-Z estimates: $K \in C^N(\mathbb{R}^{n+1} \setminus \{0\})$, with the estimate

$$\left| \nabla_x^j \partial_t^k K(X,t) \right| \leq C_{j,k} \| (X,t) \|^{-d-j-2k}, \quad \forall 0 \leq j + k \leq N.$$

(ii) Oddness in spatial variables: $K(X,t) = -K(\bar{X},t)$, for each $(\bar{X},t) \in \mathbb{R}^n \times \mathbb{R}$. We shall simply say that $K$ is a C-Z kernel (and we write $K \in C-Z$), if $K \in C-Z(N)$ for some positive integer $N$. We shall also consider analogous kernels $H(x,t)$ defined on $\mathbb{R}^n$, not necessarily odd in the space variables, but still satisfying the $n$-dimensional version of property (i) above, i.e.,

$$(2) \quad \left| \nabla_x^j \partial_t^k H(x,t) \right| \leq C_{j,k} \| (x,t) \|^{-d-j-2k}, \quad \forall 0 \leq j + k \leq N.$$

In this case we shall say that $H$ satisfies the C-Z($N$)(i) condition, or $H \in C-Z(N)(i)$.

Note that we do not assume homogeneity of $K$ and $H$ in the preceding definition.

Definition 3 (Regular Lip(1,1/2) functions and Good Parabolic Graphs). We say that a function $A : \mathbb{R}^n \to \mathbb{R}$ is a regular Lip(1,1/2) function if it is Lip(1,1/2), and if in addition $\mathbb{D}_n A \in BMO$, where $\mathbb{D}_n := I_n \circ \partial_t$ is the half-order time derivative. Following [H1], we endow the regular Lip(1,1/2) functions with the norm

$$\|A\|_{Lip} := \| \nabla_x A \|_{L^\infty(\mathbb{R}^n)} + \| \mathbb{D}_n A \|_{BMO(\mathbb{R}^n)}.$$  

Of course, BMO is parabolic BMO, defined with respect to parabolic cubes (or balls).

We say that a graph $\Gamma := \{(A(x),x)\}$ is a Good Parabolic Graph, and we write $\Gamma \in GPG$, if $A$ is a regular Lip(1,1/2) function.

“Surface measure” on $\Gamma$ is defined to be $d\sigma(x,t) := \sqrt{1 + |\nabla_x A(x,t)|^2 \, dx \, dt}$.

Given a Good Parabolic Graph $\Gamma$, and a C-Z kernel $K$, we define associated truncated singular integral operators as follows, for each $\varepsilon > 0$:

$$T^x_\varepsilon f(X) := \int_{\Gamma \cap |X-Y| > \varepsilon} K(X-Y) \, f(Y) \, d\sigma(Y), \quad X \in \Gamma,$$

and using graph co-ordinates $\Gamma := \{(A(x),x)\}$, we write the corresponding Euclidean version

$$(4) \quad T^x_\varepsilon f(x) := \int_{|x-y| > \varepsilon} K(A(x) - A(y),x-y) \, f(y) \, dy, \quad x \in \mathbb{R}^n.$$

We have the following.

Theorem 5. Let $K \in C-Z(2N)$, and suppose that $\Gamma := \{(A(x),x)\} \in GPG$. If $N$ is large enough, then for some universal constant $N_1$, we have the uniform $L^2$ bound

$$\sup_{\varepsilon > 0} \| T^x_\varepsilon f \|_{L^2(\Gamma)} \leq C (1 + \| A \|_{Lip})^{N_1} \| f \|_{L^2(\Gamma)},$$

where $C$ depends on $K$ and $n$. 
The case that $K$ is *homogeneous* was previously treated in [H2]; some particular homogeneous kernels arising in the theory of parabolic layer potentials were originally treated by Lewis and Murray [LM].

Combining Theorem 5 with the results of [BHHLN], and using the “big pieces/good-$\lambda$” method of G. David (see [D, Proposition III.3.2]), we obtain as an immediate corollary the following (see [BHHLN, Section 4] for more details):

**Corollary 6.** The conclusion of Theorem 5 continues to hold when the graph $\Gamma$ is replaced by a parabolic uniformly rectifiable set.

In order to prove Theorem 5, we will need the following pair of results.

**Theorem 7.** Let $H$ be a kernel defined on $\mathbb{R}^n \setminus \{0\}$, odd in the space variable, and satisfying (2) with $N = 1$, i.e., $H \in C$-$Z(1)(i)$. Define truncated convolution singular integral operators

$$S_\epsilon f(x) := \int_{|x-y|>\epsilon} H(x-y) f(y) \, dy, \quad x \in \mathbb{R}^n.$$  

Then $S_\epsilon : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ uniformly in $\epsilon$, i.e.,

$$\sup_{\epsilon>0} \|S_\epsilon f\|_{L^2(\mathbb{R}^n)} \leq C \|f\|_{L^2(\mathbb{R}^n)}.$$

The theorem is well known, and we omit the proof. One can either estimate the Fourier transform of the truncated kernel, and use Plancherel’s theorem, or else use the parabolic $T1$ theorem.

**Theorem 8.** Let $H$ be a kernel defined on $\mathbb{R}^n \setminus \{0\}$, even in the space variable, and satisfying (2) with $N = 1$, i.e., $H \in C$-$Z(1)(i)$. Let $A$ be a regular Lip$(1,1/2)$ function. Define the truncated first commutator

$$C_\epsilon f(x) := \int_{|x-y|>\epsilon} \frac{A(x) - A(y)}{|x-y|} H(x-y) f(y) \, dy, \quad x \in \mathbb{R}^n.$$  

Then $C_\epsilon : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ uniformly in $\epsilon$, i.e.,

$$\sup_{\epsilon>0} \|C_\epsilon f\|_{L^2(\mathbb{R}^n)} \leq C \|A\|_{\text{comm}} \|f\|_{L^2(\mathbb{R}^n)}.$$  

In the case that $H$ is homogeneous, the result is proved in [H1]. We shall prove the general case here (which is not much harder). We defer the proof to the end of this note, and take Theorem 8 for granted for the moment.

2. Proofs of the Theorems

We take Theorem 8 for granted until the end of this section.

**Proof of Theorem 5.** Note that it is equivalent to prove

$$\sup_{\epsilon>0} \|T_\epsilon f\|_{L^2(\mathbb{R}^n)} \leq C(1 + \|A\|_{\text{comm}})^N \|f\|_{L^2(\mathbb{R}^n)},$$  

(9)
for some universal constant $N_0$, where $T_x$ is defined as in (4). To this end, we first use the technique of [CDM] to reduce matters to the case of “singular integrals of Calderón type”. The method is nowadays well-known, but for the reader’s convenience, we provide the details. Observe that

$$K(x_0, x) = K\left(\frac{x_0}{\|x\|}, \frac{x}{\|x\|}\right),$$

so therefore

$$K(x_0, x) = \int_{\mathbb{R}} e^{2\pi i \frac{\langle x_0 \rangle}{\|x\|}} \hat{K}\left(\|x\|, \frac{x}{\|x\|}\right) d\zeta = \int_{\mathbb{R}} e^{2\pi i \frac{\langle x_0 \rangle}{\|x\|}} H_{\zeta}(x) d\zeta,$$
Taking even and odd parts, and using (10), we see that

\[(11) \quad K(x_0, x) = \int_{\mathbb{R}} \sin \left(\frac{2\pi x_0}{|x|} \zeta \right) H^\text{even}_{\zeta} \, d\zeta + \int_{\mathbb{R}} \cos \left(\frac{2\pi x_0}{|x|} \zeta \right) H^\text{odd}_{\zeta} \, d\zeta .\]

To conclude the proof of Theorem 5, it is therefore enough to prove the following.

**Theorem 12.** Let $H$ be a singular kernel satisfying (2), with $N = 1$. Define truncated singular integrals of “Calderón-type” by

\[(13) \quad T_\varepsilon f(x) := \int_{|x-y|>\varepsilon} E \left(\frac{A(x) - A(y)}{|x-y|} \right) H(x-y) f(y) \, dy, \quad x \in \mathbb{R}^n ,\]

where either $H(x,t)$ is odd in $x$, and $E = \cos$, or $H(x,t)$ is even in $x$, and $E = \sin$

Then there is a universal constant $N_0$ such that

\[
\sup_{\varepsilon>0} \|T_\varepsilon f\|_{L^2(\mathbb{R}^n)} \leq C(1 + \|A\|_{\text{comm}})^{N_0} \|f\|_{L^2(\mathbb{R}^n)} ,
\]

where $C$ depends on $n$ and the constants in (2).

Let us take Theorem 12 for granted momentarily. Setting $x_0 = A(x) - A(y)$ in (11), we see that the truncated singular integral $T_\varepsilon$ defined in (4) can be represented as a sum of two terms of the form

\[
\int_{\mathbb{R}} T_\varepsilon^\zeta f \, d\zeta ,
\]

where $T_\varepsilon^\zeta$ is defined as in (13), with $A$ replaced by $\zeta A$, and either with $E = \cos$ and $H = H^\text{odd}_{\zeta}$, or with $E = \sin$ and $H = H^\text{even}_{\zeta}$. Using our previous observation that $(1 + |\zeta|)^N H^\zeta$ satisfies the bounds in (2), uniformly in $\zeta$, and invoking Theorem 12, we find that

\[
\sup_{\varepsilon>0} \|T_\varepsilon f\|_{L^2(\mathbb{R}^n)} \leq \int_{\mathbb{R}} (1 + |\zeta|)^{-N} (1 + |\zeta|)^N \|A\|_{\text{comm}}) \|f\|_{L^2(\mathbb{R}^n)} \, d\zeta \leq (1 + \|A\|_{\text{comm}})^{N_0} \|f\|_{L^2(\mathbb{R}^n)} ,
\]

provided that $N > N_0 + 1$. Thus (9) holds, and the conclusion of Theorem 5 follows.

To complete the proof of Theorem 5, it therefore remains to prove Theorem 12.

**Proof of Theorem 12.** The theorem was already proved in [H2], in the case that $H$ is homogeneous. The present proof is a modified version of the argument in [H2], which in turn follows that of [Ch]. For specificity, we treat the case that $E = \cos$, and $H(x,t)$ is odd in $x$, for each fixed $t$; the proof in the other case is essentially the same. To simplify notation, we set

\[
M := \|A\|_{\text{comm}} .
\]

It is enough to verify the localized $T1$ estimate

\[(14) \quad \int_B |T_\varepsilon f| \leq C(1 + M)^{N_0}(B)
\]

(and the same for the transpose of $T_\varepsilon$, but the latter is of the same form as $T_\varepsilon$), uniformly in $\varepsilon > 0$ and in every parabolic ball

\[
B = B_r(x_B) := \{x \in \mathbb{R}^n : \|x - x_B\| < r\} ,
\]
where \( \eta_B \in C_0^{\infty}(5B) \), with \( \eta_B \equiv 1 \) on \( 4B \), and \( 0 \leq \eta_B \leq 1 \).

By scale-invariance\(^1\), we may suppose that the ball \( B \) in (14) has radius \( r = 1 \).

Following [Ch], we choose \( \varphi \in C_0^{\infty}(1/4, 1) \) such that

\[
(15) \quad \int_0^\infty \varphi(\rho) \frac{d\rho}{\rho} = 1,
\]

We may then replace \( T_\epsilon \eta_B(x) \) by

\[
(16) \quad \int_\mathbb{R}^n \int_\mathbb{R}^n \varphi \left( \frac{|x - y|}{\delta} \right) \cos \left( \frac{A(x) - A(y)}{|x - y|} \right) H(x - y) \, dy \frac{d\delta}{\delta},
\]

since for \( x \in B \), the error is bounded by a uniform constant depending only on \( n \) and \( H \). For \( \delta > 0 \), we define a nice parabolic approximate identity \( P_\delta f := \Phi_\delta * f \), where \( \Phi \in C_0^{\infty}(B(1)), 0 \leq \Phi, \int \Phi = 1 \), and

\[
\Phi_\delta(x, t) := \delta^{-d} \Phi \left( \frac{x}{\delta}, \frac{t}{\delta^2} \right).
\]

We then write

\[
\cos \left( \frac{A(x) - A(y)}{|x - y|} \right) = \cos \left( \frac{(x - y) \cdot P_\delta \nabla_x A(x)}{|x - y|} \right) + \frac{A(x) - A(y) - (x - y) \cdot P_\delta \nabla_x A(x)}{|x - y|} \sin \left( \frac{(x - y) \cdot P_\delta \nabla_x A(x)}{|x - y|} \right)
\]

\[
+ \mathcal{O} \left( \frac{|A(x) - A(y) - (x - y) \cdot P_\delta \nabla_x A(x)|^2}{|x - y|^2} \right) =: I(x, y) + II(x, y) + III(x, y)
\]

Since \( H(x, t) \) is odd in \( x \), term \( I(x, y) \) contributes zero to the integral in (16), as does the part of term \( II(x, y) \) involving \( (x - y) \cdot P_\delta \nabla_x A(x) \) in the numerator of the first factor; thus, the contribution of term \( III(x, y) \) equals

\[
(17) \quad \int_\mathbb{R}^n \int_\mathbb{R}^n \varphi \left( \frac{|x - y|}{\delta} \right) \frac{A(x) - A(y)}{|x - y|} \sin \left( \frac{(x - y) \cdot P_\delta \nabla_x A(x)}{|x - y|} \right) H(x - y) \, dy \frac{d\delta}{\delta}
\]

\[
= \sum_{j,k} a_{k,j} (P_\delta \nabla_x A(x)) \int_\mathbb{R}^n \int_\mathbb{R}^n \varphi \left( \frac{|x - y|}{\delta} \right) \frac{A(x) - A(y)}{|x - y|} \tilde{H}_{k,j}(x - y) \, dy \frac{d\delta}{\delta},
\]

where \( \tilde{H}_{k,j} = Y_{k,j} H \), and with \( \tilde{b} = P_\delta \nabla_x A(x) \),

\[
\sum_{j,k} a_{k,j}(\tilde{b}) Y_{k,j} \left( \frac{y}{|y|} \right) = \sin \left( \frac{y \cdot \tilde{b}}{|y|} \right) =: f(\tilde{b}, y)
\]

is the orthonormal spherical harmonic expansion (with \( k \) being the degree of \( Y_{k,j} \)) of the function \( f(\tilde{b}, y) := \sin(|y|^{-1} y \cdot \tilde{b}) \), which is homogeneous of degree zero (with respect to parabolic dilations), and thus is determined by its values on the

\(^1\)Even though our individual kernels are not homogeneous, they do form a scale-invariant class, with scale-invariant control of the various relevant constants.
therefore, since $H$ has the same property, it follows that $\tilde{H}_{k,j}(x,t) = Y_{k,j}(x,t) H(x,t)$ is even in $x$. Furthermore, for every non-negative integer $m$,

$$\sup_{|\theta| \leq M} \sup_{y \in S^{n-1}} |\nabla^m_x f(\hat{\theta}, y)| \leq C_m M^m$$

so by a well known property of the spherical harmonics (see, e.g., [S, pp. 70-71]) we have

$$\left\| a_{k,j} \right\|_{L^\infty(|\theta| \leq M)} \leq C_m M^m k^{-m}$$

with $m$ arbitrarily large. Let us recall also that by standard facts for spherical harmonics (see, e.g., [CZ]), the dimension $h_k$ of the space of spherical harmonics of degree $k$, satisfies

$$h_k \leq C_n k^{n-2},$$

and also, the normalized spherical harmonics $Y_{k,j}$ satisfy

$$\sup_{y \in S^{n-1}} |Y_{k,j}(y)| + \sup_{y \in S^{n-1}} k^{-1} |\nabla_y Y_{k,j}(y)| \leq C_n k^{(n-2)/2}.$$  

Let us further note that for $x \in B$, the integral in (17) is unchanged if we insert the factor $\eta_B(y)$, since the latter equals 1 in $4B$. Combining these observations, we find that (17) is bounded in absolute value by

$$F(x) := C_M M^m \sum_{k \geq 1} k^{-m+1+(n-2)/2} \sum_{j=1}^{h_k} \left| C_{k,j}^e \eta_B(x) \right|,$$

for $x \in B$, where $C_{k,j}^e$ is a smoothly truncated version of a parabolic Calderón-type commutator, defined as in Theorem 8, with respect to the normalized kernel $k^{-1-(n-2)/2} \tilde{H}_{k,j}$, which as noted above is even in $x$. By this normalization, (20), and Theorem 8, we have

$$\sup_{j,k \geq 0} \sup_{x \in \mathbb{R}^n} \|C_{k,j}^e\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \leq C M.$$

Consequently, by (19), we have

$$\int_B |F(x)|dx \leq \left( \int_B |F(x)|^2dx \right)^{1/2} \leq M^{m+1},$$

provided that we fix $m > 1 + 3(n-2)/2$. For the contribution of term $II(x,y)$, we therefore obtain the desired bound in (14), with $N_0 = m + 1$.

Finally, we consider term $III(x,y)$. Define

$$\gamma_A(x, \delta) := \left( \int_{|A(x) - A(y) - (x-y) \cdot P_B \nabla A(x)|^2} dy \right)^{1/2}.$$ 

After plugging $III(x,y)$ into the integral in (16), and then averaging over the (unit) ball $B$, we obtain the bound

$$|B|^{-1} \int_B \int_0^1 \gamma_A^2(x, \delta) d\delta dx \leq CM^2,$$
by the parabolic Dorronsoro-type result proved in [H2, p.249, estimate (35)]. □

With Theorem 12 in hand, this concludes the proof of Theorem 5. □

Proof of Theorem 8. As above, it is enough to verify the localized $T_1$ estimate (14), but now with $C_e$ in place of $T$. Once again by dilation invariance (within the class of kernels under consideration), we may suppose that $B$ has radius $r = 1$. By the localization lemma in [H2, Appendix, Lemma 2], we may assume that $A$ is supported in $10B$, and that $\nabla_s A$ and $\mathbb{D}_n A$ belong to $L^2(\mathbb{R}^n)$, with

$$
\left( |B|^{-1} \int_{\mathbb{R}^n} \left( |\nabla_s A(x)|^2 + |\mathbb{D}_n A(x)|^2 \right) \, dx \right)^{1/2} \leq \|A\|_{\text{comm}} = M.
$$

For notational convenience, with $x = (\omega, t)$, we shall denote the parabolic dilations as follows:

$$
\rho^{(1,2)} x := (\rho x, \rho^2 t), \quad \rho > 0.
$$

We also denote points on the unit sphere as $\omega = (\omega', \omega_n)$, where $\omega_n$ is the component of $\omega$ in the time direction. Using parabolic polar coordinates (see, e.g., [FR1]), and integrating by parts, we may write

$$
C_e \eta_H(x) = \int_{S^{n-1}} \int_0^\infty A(x) - A(x - \rho^{(1,2)} \omega) H(\rho^{(1,2)} \omega) \rho^{d-1} d\rho \left( 1 + \omega_n^2 \right) d\sigma(\omega)
$$

$$
= \int_{S^{n-1}} \int_0^\infty \omega' \cdot \nabla_s A(x - \rho^{(1,2)} \omega) \tilde{H}(\rho^{(1,2)} \omega) \rho^{d-1} d\rho \left( 1 + \omega_n^2 \right) d\sigma(\omega)
$$

$$
+ \int_{S^{n-1}} \int_0^\infty 2\rho \omega_n \partial_1 A(x - \rho^{(1,2)} \omega) \tilde{H}(\rho^{(1,2)} \omega) \rho^{d-1} d\rho \left( 1 + \omega_n^2 \right) d\sigma(\omega)
$$

$$
+ \text{boundary terms} =: I + H + \text{boundary terms},
$$

where the boundary term at $\rho = \varepsilon$ is uniformly bounded by $CM$, and where

$$
\tilde{H}(\rho^{(1,2)} \omega) := \rho^{1-d} \int_0^\infty r^d H(\rho^{(1,2)} \omega) \, dr = \int_1^\infty r^d H((\tau \rho)^{(1,2)} \omega) \, \frac{dr}{\tau^d},
$$

i.e., with $\rho^{(1,2)} \omega = x - y$,

$$
\tilde{H}(x - y) = \int_1^\infty r^d H((\tau)^{(1,2)}(x - y)) \, \frac{dr}{\tau^d}
$$

Note that $\tilde{H}$ satisfies (2) with $N = 1$, and $\tilde{H}$ is even in the space variable (since these properties holds for $H$, by assumption). Returning to rectangular coordinates, we see that

$$
I = \int_{|x-y|<\varepsilon} \frac{x-y}{|x-y|} \cdot \nabla_s A(y) \tilde{H}(x - y) \, dy =: S_e(\nabla A)(x),
$$

where $S_e$ is the truncated convolution singular integral operator with the vector-valued C-Z kernel $H_0(x) := x|x|^{-1} H(x)$, which is odd in $x$. Thus,

$$
\int_B |I| \, dx \leq \left( \int_B |S_e(\nabla A)(x)|^2 \, dx \right) \leq CM,
$$
by Theorem 7 and (21).

Finally, writing term \( II \) in rectangular coordinates yields

\[
II = 2 \int_{\|x-y\| > \varepsilon} \frac{t-s}{\|x-y\|} \partial_s A(y) \widetilde{H}(x-y) dy =: \tilde{J}_\varepsilon * \partial_s A(x),
\]

where

\[
\tilde{J}_\varepsilon(x) := \frac{t-s}{\|x-y\|} \widetilde{H}(x-y) \mathbf{1}_{\|x\| > \varepsilon}.
\]

A routine computation shows that

\[
|\tilde{J}_\varepsilon(\xi, \tau)| \lesssim \|\xi, \tau\|^{-1},
\]

uniformly in \( \varepsilon \), and hence that

\[
\|\tilde{J}_\varepsilon * \partial_s A\|_{L^2(\mathbb{R}^n)} \lesssim \|\mathcal{D}_n A\|_{L^2(\mathbb{R}^n)} \lesssim M,
\]

by (21) and the fact that \( B \) has radius 1. Consequently,

\[
\int_B |II| \, dx \lesssim M.
\]

\( \square \)

References

[BHHLN] S. Bortz, J. Hoffman, S. Hofmann, J.-L. Luna, and K. Nyström, Coronizations and big pieces in metric spaces, preprint arXiv:2008.11544

[CZ] A. P. Calderón and A. Zygmund, On singular integrals with variable kernels, Applicable Analysis 7 (1978), pp. 221-238.

[Ch] M. Christ, Lectures on Singular Integral Operators, CBMS regional conference series, #77, Amer. Math. Soc., Providence, 1999.

[CDM] R. Coifman, G. David, and Y. Meyer, La solution des conjectures de Calderón, Adv. in Math. 48 (1983), 144-148.

[D] G. David, Wavelets and singular integrals on curves and surfaces. Lecture Notes in Mathematics, 1465. Springer-Verlag, Berlin, 1991. x+107 pp.

[FR1] E.B. Fabes and N.M. Riviere, Symbolic Calculus of Kernels with Mixed Homogeneity, in “Singular Integrals”, A.P. Calderón, Ed., Proc. Symp. Pure Math., 10, Amer. Math. Soc., Providence, 1967, pp. 106-127.

[H1] S. Hofmann, A characterization of commutators of parabolic singular integrals, Proceedings of conference on Harmonic Analysis and PDE, held at Miraflores de la Sierra, Spain, 1992, J. Garcia-Cuerva, E. Hernandez, F. Soria, editors, CRC press, Boca Raton (1995), pp. 195-210.

[H2] S. Hofmann, Parabolic singular integrals of Calderon-type, rough operators, and caloric layer potentials, Duke Math. J., Vol. 90 (1997), pp 209-259.

[LM] J. L. Lewis and M. A. M. Murray, The method of layer potentials for the heat equation in time-varying domains, Mem. Amer. Math. Soc. 545 (1995) 1 - 157.

[S] E.M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton, 1970.
