Solvable two-particle systems with time-asymmetric interactions in de Sitter space

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Abstract

The two-particle models in de Sitter space-time with time-asymmetric retarded-advanced interactions are constructed. Particular cases of the field-type electromagnetic and scalar interactions are considered. The manifestly covariant descriptions of the models within the Lagrangian and Hamiltonian formalisms with constraints are proposed. It is shown that the models are de Sitter-invariant and integrable. An explicit solution of equations of motion is derived in quadratures by means of projection operator technique.

Keywords: de Sitter space, time-asymmetric models, integrable systems

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1 Introduction

It is known that one has to deal with complex difference-differential equations when considering a relativistic classical dynamics of a system of interacting charges [1, 2]. This is even more the case for scalar [2], gravitational [3] or non-Abelian [4] interactions where the dynamics is governed by integro-differential equations. Such a hereditary dynamics is neither solvable nor appropriate for the Hamiltonian description. In order to avoid these difficulties, Staruszkiewicz [5], Rudd and Hill [6] invented the model describing the following time-asymmetric interaction of two pointlike charged particles: the advanced field of the first particle acts on the second particle, the retarded field of the second particle acts on the first particle, and a radiation reaction is neglected. This model is built of the action-at-a-distance Tetrode-Fokker variational functional [7, 8] via replacing its integrand, the symmetric Green function of d’Alembert equation, by the retarded (or advanced) one. In this way the model was reformulated to the Lagrangian form and then to the Hamiltonian form [9] which was shown integrable [10] due to exact Poincaré-invariance. The Staruszkiewicz-Rudd-Hill model was generalized for a variety of non-electromagnetic time-asymmetric interactions (scalar, gravitational, confining etc.) [11, 13], and corresponding quantum versions [14, 15] revealed their physical adequacy, despite of an artificially broken causality of interactions.
A dynamics of interacting particles in a curved space-time is more complicated than that in the Minkowski space. To author's knowledge, solvable examples of two-particle dynamics are unknown even for the cases of symmetric space-times which occur in cosmology. However, the Staruszkiewicz-Rudd-Hill model can serve as a basis for appropriate generalization. Here a class of two-particle systems with time-asymmetric interactions is considered in de Sitter space-time. It includes, in particular, the models with electromagnetic [16] and scalar interactions built in terms of appropriate Green functions. The representation of de Sitter space-time as a hyperboloid in the 5-dimensional Minkowski space $\text{M}_5$ is exploit. A single particle dynamics is used to introduce elements of this representation. Then for a dynamical system of two particles with the electromagnetic, scalar and more general time-asymmetric interactions the covariant variational principle is constructed, and an appropriate Hamiltonian description with constraints is developed. The dynamics is invariant with respect to de Sitter group O(1,4) and integrable. A solution for this dynamics is obtained in terms of quadratures. This is done by means of projection operators built in terms of conserved canonical generators of O(1,4). The system of free particles as a time-asymmetric model is particularly considered.

2 Manifestly covariant test particle mechanics in de Sitter space

Let us start with the action integral determining the dynamics of a test particle of the mass $m$ in a curved space-time:

$$I = -m \int_{\tau_1}^{\tau_2} d\tau \sqrt{g_{\mu\nu}(x(\tau))\dot{x}^\mu(\tau)\dot{x}^\nu(\tau)}; \quad (2.1)$$

here $\tau$ parameterizes points $x(\tau)$ of a particle world line, i.e., the geodesic, $x^\mu(\tau)$ ($\mu = 0, ..., 3$) are particle coordinates, and $g_{\mu\nu}(x)$ is a metric tensor in a chosen chart of the space-time considered. The action (2.1) is invariant with respect to an arbitrary change of the evolution parameter: $\tau \to \tau' = f(\tau)$, since the parametrization of geodesics has no physical meaning. For the de Sitter space-time [17] geodesics were studied from different viewpoints [17–20] in many coordinate charts introduced for this space-time [20–22].

It is convenient to consider de Sitter space-time as a 4-dimensional hyperboloid $H$:

$$\eta_{MN}y^M y^N := (y^0)^2 - (y^1)^2 - \cdots - (y^4)^2 = -R^2 \quad (2.2)$$

in 5-dimensional Minkowski space $\text{M}_5$ with coordinates $y^M$ ($M = 0, 1, \ldots, 4$) and the metrics $||\eta_{MN}|| = \text{diag}(+, -, \ldots, -)$; [20, 23]. The constant $R$ determines the scalar curvature $\mathcal{R}$ of the de Sitter space, and it is related with the cosmological $\Lambda$-constant: $\mathcal{R} = 12/R^2 = 4\Lambda$; the speed of light is put $c = 1$.

The hyperboloid $H$ is invariant with respect to de Sitter group O(1,4) represented in $\text{M}_5$ by standard linear pseudoorthogonal transformations. Thus we will use standard notations for O(1,4)-invariants $y \cdot z := \eta_{MN}y^M z^N$ and $y^2 := y \cdot y$ built of arbitrary 5-vectors $y, z \in \text{M}_5$. 

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The embedding $\mathbb{H} \hookrightarrow \mathbb{M}_5$ implies, in terms of local coordinates $x^\mu$ in de Sitter space, a set of appropriate functions $y^M(x)$ turning the equation (2.2) into identity [20–22]. Then the pseudo-Euclidian $O(1,4)$-invariant metrics is pulled back naturally from $\mathbb{M}_5$ onto $\mathbb{H}$:

$$\rho(x, x') := (y - y')^2\big|_H.$$  \hspace{1cm} (2.3)

This endows the de Sitter space with a causal structure of the ambient Minkowski space:

- the interval between points $x, x' \in H$ is timelike if $\rho(x, x') > 0$, i.e., if $y' \in H \subset \mathbb{M}_5$ lies inside the light cone with a vertex $y \in H \subset \mathbb{M}_5$ (or the same with $y$ and $y'$ permuted);
- the interval is spacelike if $\rho(x, x') < 0$, i.e., if $y'$ lies outside the light cone;
- the interval is isotropic if $\rho(x, x') = 0$, i.e., if $y'$ lies on the light cone hypersurface.

For infinitely closed 5-vectors $y$ and $y' = y + dy$ the function (2.3) yields the pseudo-Riemannian metrics involved in the action integral (2.1) for the case of de Sitter space:

$$ds^2 := \eta_{MN} dy^M dy^N\big|_H = g_{\mu\nu}(x) dx^\mu dx^\nu.$$  \hspace{1cm} (2.4)

Thus the test particle dynamics in de Sitter space can be reformulated to some variational principle with a constraint, defined in the configuration space $\mathbb{M}_5$ [20, 24, 25]. The simplest version is [20]:

$$I = -\int d\tau \left\{ m\sqrt{\dot{y}^2(\tau)} - \lambda(\tau)(y^2(\tau) + R^2) \right\},$$  \hspace{1cm} (2.5)

where the condition (2.2) is taken into account as a holonomic constraint by means of the Lagrange multiplier $\lambda(\tau)$. The Euler-Lagrange equation for 5-vector $y(\tau)$ representing the particle position $x(\tau) \in \mathbb{H}$ can be written down in the following manifestly covariant form

$$\frac{d}{d\tau} \frac{\dot{y}}{\sqrt{\dot{y}^2}} - \sqrt{\dot{y}^2} \frac{y}{R} = 0$$  \hspace{1cm} (2.6)

which is invariant with respect to both the $O(1,4)$ group and an arbitrary change of the evolution parameter $\tau$. The solution of the geodesic equation (2.5) is

$$y(\tau) = y(0) \cosh \frac{s(\tau)}{R} + R \frac{\dot{y}(0)}{\sqrt{\dot{y}^2(0)}} \sinh \frac{s(\tau)}{R},$$  \hspace{1cm} (2.7)

where the constant 5-vectors $y(0)$ and $\dot{y}(0)$ are subjected to the constraint (2.2) and its differential consequence $y \cdot \dot{y} = 0$, and $s(\tau)$ is the proper time elapsed from $y(0)$ to $y(\tau)$:

$$s(\tau) := \int_0^\tau d\tau \sqrt{\dot{y}^2(\tau)}.$$  \hspace{1cm} (2.8)

The proper time as a function of $\tau$ cannot be determined from the equation (2.5), due to reparametrization invariance, but it can be chosen by hands for a convenience. For example, with the proper time parametrization $s(\tau) := \tau$ we have $\dot{y}^2 = 1$, and the equation (2.6) reproduces the de Sitter geodesic found in Ref. [20].
Due to de Sitter symmetry, there exists 10 integrals of motion collected in the skew-symmetric angular 5-momentum tensor:

\[ J_{MN} = y_M \pi_N - y_N \pi_M = -J_{NM}, \quad (2.8) \]

where

\[ \pi_M = my_M / \sqrt{y^2} \quad (2.9) \]

are components of 5-momentum.

At this point one can develop the covariant Hamiltonian description on the phase space \( T^* \mathbb{M}_5 \) with variables \( y^M, \pi_N \) \((M, N = 0, \ldots, 4)\) and standard Poisson brackets: \( \{y^M, y^N\} = 0, \{\pi_M, \pi_N\} = 0, \{y^M, \pi_N\} = \delta^M_N \).

The integrals of motion \( J_{MN} \) become canonical generators of \( O(1,4) \) group while the Legendre transform \( (2.9) \) is degenerated due to the reparametrization invariance of the action \( (2.4) \). Thus the canonical Hamiltonian vanishes while the mass-shell constraint arises, \( \pi^2 - m^2 = 0 \), apart to the holonomic constraint \( (2.2) \). Both constrains are primary ones according to Dirac’s terminology of canonical formalism with constraints [26]. They form the primary Dirac’s Hamiltonian: \( H_0 = \lambda(\pi^2 - m^2) + \lambda_1(y^2 + R^2) \), where \( \lambda \) and \( \lambda_1 \) are Lagrange multipliers. The compatibility conditions

\[ \{y^2 + R^2, H'_{D}\} = 4\lambda y \cdot \pi \approx 0, \quad \{\pi^2 - m^2, H'_{D}\} = -4\lambda_1 y \cdot \pi \approx 0, \]

give rise to the secondary constraint \( y \cdot \pi = 0 \) so that Dirac’s Hamiltonian at this stage takes the form: \( H'_D = H_D + \lambda_2 y \cdot \pi \).

Reexamining compatibility conditions gives no new constraints but fixes partially Lagrange multipliers: \( \lambda_1 = 0 \). Putting then \( \lambda_2 = -y \cdot \pi / y^2 \) yields the final Dirac’s Hamiltonian \( H_D = \lambda(\tau)\phi(y, \pi) \) with the unspecified Lagrange multiplier \( \lambda(\tau) \) (due to the reparametrization invariance) and the function \( \phi(y, \pi) \) which determines the modified mass-shell constraint

\[ \phi := \pi^2 - m^2 \equiv \frac{1}{2} J^2 / y^2 - m^2 = 0; \quad (2.10) \]

here \( \pi_{\perp M} := \pi_M - \frac{y \cdot \pi}{y^2} y_M \approx \frac{J_{MN} y_N}{R^2} \) (so that \( y \cdot \pi_{\perp} \equiv 0 \)) and \( J^2 := J_{MN} J^{MN} \).

The symbol “\( \approx \)” denotes a “weak equality”, i.e., by virtue of the holonomic constraint \( (2.2); [26] \).

Let us note that the set of constraints \( (2.2) \) and \( (2.10) \) are the 1st class [26], i.e., they satisfy the identity: \( \{y^2 + R^2, \phi\} \equiv 0 \). Together with the Dirac’s Hamiltonian \( H_D = \lambda \phi \) these constraints endow effectively the system with three degrees of freedom (as it should).

Henceforth the quantity \( y \cdot \pi \) is not involved in the dynamics, and the secondary constraint \( y \cdot \pi = 0 \) can be abandoned.

The Hamiltonian equation for the particle position 5-vector \( y \) reads:

\[ \dot{y} = \lambda \{y, \phi\} = 2\lambda \pi_{\perp} \approx \frac{2\lambda}{R^2} J y. \quad (2.11) \]

Note that the matrix \( J := ||J^M_N|| := ||\eta^{ML} J_{LN}|| \) is conserved, thus the equation \( (2.11) \) is linear.

Its formal solution follows immediately: \( y(\tau) = e^{\frac{\alpha(\tau)}{m^2}} y(0) \), where the unspecified function \( s(\tau) = 2m \int_0^\tau d\tau' \lambda(\tau) \) is the Hamiltonian image for the proper time function \( (2.7) \).

The Cauchy problem becomes solved after the matrix \( J \) is expressed in terms of initial values \( y(0) \) and \( \dot{y}(0) \) by the equalities \( (2.8), (2.9) \) and their consequences \( Jy \approx mR^2 v, \)
\[ Jv = my, \text{ where } v = \dot{y}/\sqrt{\dot{y}^2}. \] Then expanding the exponent in power series reproduces the solution (2.6).

It may seem unreasonable the use of 5-dimensional reparametrization invariant description together with Dirac’s formalism with constraints in order to derive geodesics in de Sitter space. These tools, however, appear effective when considering two-body problems in next sections.

### 3 Action-at-a-distance dynamics of two particles in de Sitter space

In the framework of Wheeler-Feynman electrodynamics [1, 2, 27, 28] a system of charged point-like particles is described by the Tetrode-Fokker action-at-a-distance variational principle [7, 8]. This formalism was generalized for a curved space-time by Hoyle and Narlikar [27] and others [28, 29].

For the system of two charged particles of masses \( m_a \) and charges \( e_a (a = 1, 2) \) the Tetrode-Fokker action integral has a form:

\[ I = I_{\text{free}} + I_{\text{int}}, \quad \text{where} \quad I_{\text{free}} = -\sum_{a=1}^{2} m_a \int ds_a, \quad (3.1) \]

\[ ds_a := \sqrt{g_{\mu\nu}(x_a(\tau_a)) \dot{x}_a^\mu(\tau_a) \dot{x}_a^\nu(\tau_a)} \, d\tau_a, \quad (3.2) \]

\[ I_{\text{int}} = -4\pi e_1 e_2 \int \int dx_1^\mu dx_2^\nu G_{\mu\nu}(x_1, x_2); \quad (3.3) \]

here \( x_a^\mu(\tau_a) (\mu = 0, ..., 3) \) are space-time coordinates of particle world lines parameterized by evolution parameters \( \tau_a (a = 1, 2) \). Free-motion terms \( I_{\text{free}} \) of the action (3.1) have the form (2.1) for each particle. An integrand of the interaction term (3.3) is the symmetric Green function \( G_{\mu\nu}(x, x') \) of the covariant wave equation \( \Box A_\mu + R_{\mu\nu} A_\nu = 0 \) for the electromagnetic potential \( A_\mu \) [30, 31]; here \( \Box \) is the d’Alembertian in a curved space-time considered, and \( R_{\mu\nu} \) is the Ricci tensor. For the curved space-time \( G_{\mu\nu}(x, x') \) is a bi-vector function which construction in general is a complicated problem [30].

For de Sitter space-time the symmetric Green function is known from Ref. [32]. It is presented here in geometric terms which are indifferent to a choice of coordinate chart:

\[ G_{\mu\nu}(x, x') = G^\delta_{\mu\nu}(x, x') + G^\Theta_{\mu\nu}(x, x'); \quad (3.4) \]

here

\[ G^\delta_{\mu\nu}(x, x') := \frac{1}{16\pi} \bar{g}_{\mu\nu}(x, x') \delta(\rho(x, x')) \quad (3.5) \]

\[ G^\Theta_{\mu\nu}(x, x') := -\frac{1}{24\pi R^2} \left\{ \left( 1 + \frac{1}{2Z^2} \right) \bar{g}_{\mu\nu} + \frac{R^2}{Z^3} \left( \partial_\mu Z \right) \left( \partial_\nu Z \right) \right\} \Theta(\rho(x, x')); \quad (3.6) \]

\[ \bar{g}_{\mu\nu}(x, x') := -2R^2 \left\{ \partial_\mu \partial_\nu Z - \frac{1}{Z} \left( \partial_\mu Z \right) \left( \partial_\nu Z \right) \right\}, \quad (3.7) \]

\[ Z(x, x') := 1 + \frac{1}{4} \rho(x, x')/R^2, \quad (3.8) \]

\[ \text{An earlier proposal [31] is unappropriate as it does not meet demands of de Sitter-covariance.} \]
where $\bar{g}_{\mu\nu}(x, x')$ is the parallel propagator [30, 31], and the metric function $\rho(x, x')$ is defined by (2.3). We note that the Green function (3.4) consists of two parts. The local part (3.5) is proportional to the Dirac $\delta$-function and thus supported by the light cone surface $\rho(x, x') = 0$. The non-local part (3.6) is proportional to the Heaviside $\Theta$-function and thus supported by the light cone interior $\rho(x, x') > 0$. This is a common feature of curved space-times [30], contrary to the Minkowski space-time, where Green functions of massless fields have a local part only. But in present case of de Sitter space-time the non-local contribution (3.6) of the Green function (3.4) in the integral (3.3) can be effectively reduced to a local one [16].

In order to show this let us first introduce the relative position 5-vector $r \equiv y_1 - y_2$, the particle unit 5-velocities $v_a \equiv \dot{y}_a / \sqrt{\dot{y}_a^2}$, and the dimensionless scalars of these 5-vectors $v_1 \cdot v_2$ and $r \cdot v_a / R$ ($a = 1, 2)$ which are homogeneous functions of degree zero of derivatives $\dot{y}_1$ and $\dot{y}_2$. It is convenient for a subsequent interim calculation to present these scalars as follows:

$$
\omega := v_1 \cdot v_2|_{\mathbb{H}} = \frac{1}{2} \frac{d^2 \rho(x_1, x_2)}{ds_1 ds_2}, \quad \nu_a := \frac{r \cdot v_a}{R}|_{\mathbb{H}} = -\frac{(-)^a}{2R} \frac{d \rho(x_1, x_2)}{ds_a}, \quad a = 1, 2, \quad (3.9)
$$

where the function $\rho(x_1, x_2)$ and the interval elements $ds_a$ are defined by eqs. (2.3) and (3.2), respectively. Note that the differentiation over $ds_1$ (or $ds_2$) acts on $x_1(\tau_1)$ (or $x_2(\tau_2)$).

In these terms the integrand of the interaction term (3.3) of the action (3.1) reads:

$$
dx_1^\mu dx_2^\nu G_{\mu\nu}(x_1, x_2) = \frac{ds_1 ds_2}{4\pi} \left\{ \left( \omega - \frac{\nu_1 \nu_2}{2Z} \right) \delta(\rho) - \left( \frac{2Z + 1}{Z^2} \omega - \frac{Z + 1}{Z^3} \nu_1 \nu_2 \right) \Theta(\rho) \right\}.
$$

Then, applying the integration-by-part formula:

$$
\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} ds_1 ds_2 \omega F(\rho) = -\frac{1}{2} \int_{-\infty}^{+\infty} ds_1 ds_2 \frac{d^2 \rho}{ds_1 ds_2} F(\rho)
$$

$$
= -2R^2 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} ds_1 ds_2 \nu_1 \nu_2 \frac{dF(\rho)}{d\rho} \bigg|_{s_1 = -\infty, s_2 = -\infty} - \frac{1}{2} \rho F(\rho) \bigg|_{s_1 = +\infty, s_2 = +\infty},
$$

which holds for any function $F(\rho)$, to the Tetrode-Fokker integral (3.3), one obtains:

$$
I_{\text{int}} = -4\pi e_1 e_2 \int \int d\tau_1 d\tau_2 \dot{x}_1^a \dot{x}_2^b G_{ab}(x_1, x_2) \simeq -e_1 e_2 \int \int ds_1 ds_2 \omega \delta(\rho), \quad (3.10)
$$

where the symbol “$\simeq$” denotes an equality up to boundary terms which do not contribute in variational problem.

It is remarkable that the only local (i.e., light cone surface) contribution of Green function remains in the Tetrode-Fokker integral (3.10): this structure is a necessary starting point for a construction of the model of Staruszkiewicz-Rudd-Hill type in next Section.

Similarly, one can consider a particle system with the scalar interaction. The interaction term of the Fokker-type action (3.1) in this case has a form [27]:

$$
I_{\text{int}} = -4\pi g_1 g_2 \int \int ds_1 ds_2 G(x_1, x_2), \quad (3.11)
$$
where $g_a \ (a = 1, 2)$ are scalar “charges” of particles, and the bi-scalar function $G(x, x')$ is the symmetric Green function of the wave equation $\Box \varphi = 0$ for a scalar field $\varphi$ mediating the interaction and minimally coupled to a gravitation [30]. For the de Sitter space-time the Green function $G(x, x')$ was found by Narlikar [31]:

$$G(x, x') = G^\delta(x, x') + G^\Theta(x, x') := \frac{1}{4\pi} \left\{ \delta(\rho) - \frac{1}{2R^2} \Theta(\rho) \right\}. \quad (3.12)$$

In contrast to the case of electromagnetic interaction, the nonlocal contribution $G^\Theta(x, x')$ of the Green function (3.12) is essential: it cannot be removed from the action (3.11) by means of the integration by parts or another equivalent transformation.

The Penrose-Chernikov-Tagirov equation ($\Box - \frac{R}{6}) \varphi = 0$ corresponds to a conformal coupling of the scalar field to a gravitation [33, 34]. In the case of de Sitter space-time the scalar curvature $R = 12/R^2$ is constant, and the Green function can be found easily by means of distributional methods [30]. It appears purely local:

$$G(x, x') = G^\delta(x, x') := \frac{1}{4\pi} \delta(\rho). \quad (3.13)$$

The electromagnetic (3.10) and scalar (3.11), (3.13) interaction terms of the Fokker-type action admit the obvious de-Sitter-invariant generalization:

$$I_{\text{int}} = -\int \int ds_1 \, ds_2 \, f(\nu_1, \nu_2, \omega) \delta(\rho), \quad (3.14)$$

where $ds_a$ are defined in (3.2), and $f(\nu_1, \nu_2, \omega)$ may be an arbitrary function of its three scalar arguments (3.9), so it is a homogeneous function of degree zero of $\dot{y}_1$ and $\dot{y}_2$. Thus the expression (3.14) possesses both the de Sitter invariance and the double reparametrization invariance. It comprises a variety of interactions which may have a field-theoretical nature or can be introduced phenomenologically.

4 Time-asymmetric models in de Sitter space-time

Staruszkiewicz [5,9], Rudd and Hill [6] replaced in the Tetrode-Fokker action integral the symmetric Green function $G$ of d’Alembert equation by the retarded $G^{(+)}$ or advanced $G^{(-)}$ Green function: $G^{(\pm)}(x_1, x_2) = 2\Theta(\pm(x_1 - x_2^0))G(x_1, x_2)$. This have led them to a two-particle model with the time-asymmetric retarded-advanced interaction. Following this idea, one should insert in the general interaction term (3.11) of the Fokker-type action (3.1) the factor $2\Theta(\eta(x_1^0 - x_2^0)) = 2\Theta(\eta(y_1^0 - y_2^0))$, where $\eta = +1$ or $-1$. Then, similarly to the single-particle case considered in Section 2, it is convenient to present this Fokker-type action via global variables in the ambient Minkowski space $\mathbb{M}_5$. One thus obtains:

$$I_{\text{int}} = -\int \int d\tau_1 \, d\tau_2 \sqrt{\dot{y}_1^2} \sqrt{\dot{y}_2^2} \, f(\nu_1, \nu_2, \omega) \, 2\Theta(\eta r^0) \, \delta(r^2) |_{\mathbb{H}^2}, \quad (4.1)$$

where the integrand in r.-h.s. of (4.1) is constrained on $\mathbb{H}^2 = \mathbb{H} \times \mathbb{H}$, i.e., the particle position 5-vectors $y_a(\tau_a) \ (a = 1, 2)$ are subjected to the hyperboloid conditions for each particle:

$$y_a^2 + R^2 = 0, \quad a = 1, 2. \quad (4.2)$$
An integrand of the double integral $I_{\text{int}}$ in (4.1) is non-zero provided
\[ r^2 := (y_1 - y_2)^2 = 0, \quad \eta^0 := \eta(y_1^0 - y_2^0) > 0. \] (4.3)
This condition can be treated as the equation of past or future light cone, depending on the value $\eta = \pm 1$ and the choice which point $y_1$ or $y_2$ is a vertex of the cone. If the time-symmetric action (3.14) is invariant under a particle permutation, the invariance of the corresponding time-asymmetric action (4.1) is provided by the additional change $\eta \rightarrow -\eta$.

From a physical viewpoint, the choice of the sign factor $\eta = \pm 1$ is indifferent. Both cases correspond to the electromagnetic interaction with a "spoiled" causality. They lead to distinguished two-body problems which differ from one another and from those of Wheeler-Feynman or retarded electrodynamics. It is worth noting that in the case of the flat-space Staruszkiewicz-Rudd-Hill model the particle world lines corresponding to different $\eta = \pm 1$ are distinguishable only in a highly relativistic domain [5, 13].

The Fokker-type action integral (3.1), (4.1) is invariant with respect to an arbitrary change of each parameter $\tau_1$ and $\tau_2$. Thus two of ten variables $y_1^M(\tau_1), y_2^M(\tau_2)$ ($M = 0, ..., 4$) to be found remain undetermined within the variational problem. It is profitable to fix partially this functional arbitrariness by hands as follows. Let us choose one of variables, say $y_1^0(\tau_2)$, in such a way that the condition (4.3) turns into identity at $\tau_1 = \tau_2$. This implies that both particle world lines are parameterized by a common evolution parameter, say $\tau_1$, and the simultaneous events $y_1(\tau_1)$ and $y_2(\tau_1)$ lie on the isotropic light cone surface (4.3). Using the equality (see [9])
\[ 2 \Theta\left[\eta(y_1^0(\tau_1) - y_2^0(\tau_2))\right] \delta\left[(y_1(\tau_1) - y_2(\tau_2))^2\right] = \frac{\delta(\tau_1 - \tau_2)}{\left[y_2(\tau_2) \cdot (y_1(\tau_1) - y_2(\tau_2))\right]} \]
in the interaction term (4.1) and integrating over $\tau_2$ reduces the Fokker-type action (3.1) to the single-time form
\[ I = \int d\tau \tilde{L} \] (4.4)
with the lagrangian $\tilde{L} := L|_{TK}$, where
\[ L = -\sum_{a=1}^{2} m_a \sqrt{\dot{y}_a^2} - \sqrt{\dot{y}_1^2} \sqrt{\dot{y}_2^2} f(\nu_1, \nu_2, \omega) \left|\dot{y}_2 \cdot r\right|. \] (4.5)
The Lagrangian $\tilde{L}$ is defined on the tangle bundle $TK$ over the 7-dimensional configuration manifold $K \subset \mathbb{H}^2 \subset M_5^2 \equiv M_5 \times M_5$ described by the conditions (4.2), (4.3). The corresponding variational problem gives rise to second order differential equations of motion and thus the transition to the usual Hamiltonian description is straightforward.

The Lagrangian $\tilde{L}$ (as well as $L$) is the first degree homogeneous function of particle velocities. Thus the action (4.1) possesses a residual invariance with respect to an arbitrary change of the common evolution parameter: $\tau$. This symmetry allows one to fix a remaining timelike variable by hands and, together with the conditions (4.2), (4.3), enables to arrive at the ordinary Lagrangian description in the 6-dimensional configuration space $\mathcal{Q}$. In practice, however, an explicit elimination of redundant variables, say $y_1^0$,
$y_1^4, y_2^0, y_2^4$, breaks a manifest 5-dimensional Lorentz-covariance, and makes a subsequent treatment cumbersome. As usual, a success in solving equations of motion is predetermined by an appropriate parametrization of the configuration space which is not evident in the case of $Q$.

An alternative way is the use of a manifestly covariant Lagrangian description in the 10-dimensional configuration space $\mathbb{M}_5^2$. In this case an unconditional extremum problem of the action (4.4) is modified in favor of an equivalent conditional extremum problem of

$$I = \int d\tau \left\{ L + \lambda_0 r^2 + \sum_{a=1}^{2} \lambda_a (y_a^2 + R^2) \right\}$$

(4.6)

with the Lagrangian function (4.5) defined on $\text{T}\mathbb{M}_5^2$. The Lagrangian multipliers $\lambda_0(\tau), \lambda_a(\tau)$ take the conditions (4.3), (4.2) into account as holonomic constraints; the unilateral constraint $\eta r^0 > 0$ is implied as well.

de Sitter invariance of the Lagrangian (4.5) and constraints (4.2), (4.3) provides the existence of ten Noether integrals of motion, collected in the angular 5-momentum tensor:

$$J_{MN} = \sum_{a=1}^{2} (y_a M \pi_a N - y_a N \pi_a M)$$

(4.7)

where

$$\pi_{aM} = \partial L/\partial \dot{y}_a^M, \quad a = 1, 2.$$  

(4.8)

Besides, the Lagrangian (4.5) satisfies the identity:

$$\sum_{a=1}^{2} \dot{y}_a \cdot \pi_a - L = 0,$$

(4.9)

due to the reparametrization invariance of the action (4.6).

5 Canonical formalism with constraints

The Lagrangian description in the configuration space $\mathbb{M}_5^2$ enables a natural transition to the manifestly covariant Hamiltonian description with constraints [26]. The corresponding 20-dimensional phase space $\text{T}\mathbb{M}_5^2$ with the particle canonical variables $y_a^M, \pi_{bN}$ ($a, b = 1, 2; M, N = 0, \ldots, 4$) is endowed with the standard Poisson brackets: $\{y_a^M, y_b^N\} = 0$, $\{\pi_{aM}, \pi_{bN}\} = 0$, $\{y_a^M, \pi_{bN}\} = \delta_{ab}\delta_N^M$.

Components of the conserved angular 5-momentum tensor (4.7) become, within the Hamiltonian description, the generators $J_{MN}$ of the canonical realization of the de Sitter group, i.e., they satisfy the canonical relations of the Lie algebra of $\text{O}(1,4)$:

$$\{J_{MN}, J_{LK}\} = \eta_{ML}J_{NK} + \eta_{NL}J_{ML} - \eta_{MK}J_{NL} - \eta_{NL}J_{MK}.$$  

(5.1)

Due to the identity (4.9) the Legendre transformation (4.8) is degenerated, the canonical Hamiltonian vanishes, while the additional constraint arises [26], similarly to the mass-shell constraint in the single particle case. The function determining this constraint
constitutes (together with the holonomic constraints (4.2), (4.3)) a primary Dirac’s Hamiltonian.

The subsequent procedure is similar to that of the single particle case in Section 2. The compatibility conditions of the dynamics with primary constraints give rise to secondary constraints which then are combined with the primary constraints in the secondary Dirac’ Hamiltonian etc. In a final compatible form the dynamics is generated by the Dirac’s Hamiltonian

$$H_D = \lambda(\tau) \Phi(y_a, \pi_b)$$

where \(\lambda(\tau)\) is unspecified Lagrange multiplier (due to the reparametrization invariance), and the constraint \(\Phi(y_a, \pi_b) = 0\) is the first class with respect to the holonomic constraints (4.2), (4.3), i.e., the function \(\Phi(y_a, \pi_b)\) satisfies the equalities:

$$\{\Phi, r^2\} = 0, \quad \{\Phi, y_a^2 + R^2\} = 0, \quad a = 1, 2.$$ (5.2)

Besides, this constraint must be de Sitter invariant asinmuch the angular momentum tensor (4.7) must be conserved.

We will refer to \(\Phi(y_a, \pi_b) = 0\) as the dynamical constraint for two reasons. Firstly, the function \(\Phi(y_a, \pi_b)\) generates an evolution via Dirac’s Hamiltonian. Secondly, a specific form of \(\Phi(y_a, \pi_b)\) is determined by the Lagrangian (4.5), in particular, by the form of the interaction function \(f(\nu_1, \nu_2, \omega)\) chosen. However, the equations (5.2) and de Sitter invariance requirements are sufficient to outline a general structure of the dynamical constraint and the corresponding Hamiltonian mechanics.

Let functions of canonical variables \(\varphi(y_a, \pi_b)\) which satisfy the conditions (5.2) be referred to as observables in Dirac’s meaning \[26\]. We will use sometimes the collective canonical variables:

$$Y^M = \frac{1}{2}(y_1^M + y_2^M), \quad r^M = y_1^M - y_2^M, \quad \Pi_M = \pi_{1M} + \pi_{2M}, \quad \pi_M = \frac{1}{2}(\pi_{1M} - \pi_{2M}).$$ (5.3)

The components of position 5-vectors \(Y, r\) are the observables. Solving the equations (5.2) yields other observables, the momentum-type 5-vectors \(\Pi_{\perp}, \pi_{\perp}\) with the components:

$$\Pi_{\perp M} := \frac{Y^L J_{LM}}{Y^2} \approx \Pi_M + \frac{(Y \cdot \Pi)Y_M + (Y \cdot \pi)r_M}{R^2},$$ (5.4)

$$\pi_{\perp M} := \left(\delta^N_M - \frac{r_N \Pi^N_{\perp}}{\Pi_{\perp \cdot r}}\right) \left(\delta^L_N - \frac{Y_N Y^L}{Y^2}\right) \pi_L$$ (5.5)

which are not all independent due to the identities: \(\Pi_{\perp} \cdot Y \equiv 0, \quad \pi_{\perp} \cdot Y \equiv 0, \quad \Pi_{\perp} \cdot \pi_{\perp} \equiv 0\).

A set of functions \(\varphi(Y, r, \Pi_{\perp}, \pi_{\perp})\) constitutes a complete algebra of observables which is closed with respect to Poisson brackets. Indeed, if \(\varphi_1\) and \(\varphi_2\) are observables then \(\{\varphi_1, \varphi_2\}\) is observable due to the Jacobi identity. The particle positions \(y_a\) and the dynamical constraint \(\Phi(Y, r, \Pi_{\perp}, \pi_{\perp})\) are observable, thus the algebra of observables is sufficient to formulate equations of motion generated by the Dirac’s Hamiltonian \(H_D = \lambda \Phi\). Aforementioned secondary constraints are not observable and can be abandoned, similarly to the single particle case.

The requirement of the dynamical constraint to be de Sitter invariant yields the following its general structure:

$$\Phi(\Pi_{\perp}^2, \pi_{\perp}^2, \Pi \cdot r, \pi \cdot r) = 0,$$ (5.6)
where $\Phi$ may be an arbitrary function of its scalar arguments: $\Pi^2_\perp$, $\pi^2_\perp$ and $\Pi \cdot r \approx \Pi_\perp \cdot r$, $\pi \cdot r \approx \pi_\perp \cdot r$; here the use of weak equality “$\approx$” by virtue of the holonomic constraints (3.1). (3.4) simplifies the dynamical constraint but does not affects the dynamics of observables.

The dynamical constraint (5.6) determines implicitly one of the argument of $\Phi$ as a function in variables of different particles. The reason is that this constraint is concerted from this tangled description.

Fortunately, two physically motivated examples, considered in Section 3, are the cases. For the electromagnetic time-asymmetric interaction one puts in (4.5) $f = \alpha_e \omega$, where $\alpha_e := e_1 e_2$, and arrives at the following dynamical constraint:

$$
\Phi_e := \pi^2_\perp + \frac{1}{4} \Pi^2_\perp + \frac{\frac{(\pi_\perp \cdot r)(\pi_2 \cdot r)}{\eta \Pi \cdot r} - \alpha_e}{\eta \Pi \cdot r} \left( \frac{m^2_1 \pi_2 \cdot r + m^2_2 \pi_1 \cdot r}{\Pi \cdot r} \right) - \alpha_e \frac{\Pi^2_\perp - m^2_1 - m^2_2}{\eta \Pi \cdot r} + \alpha_e^2 \left( \frac{m^2_1}{\eta \pi_1 \cdot r - \alpha_e} + \frac{m^2_2}{\eta \pi_2 \cdot r - \alpha_e} \right) = 0. \tag{5.7}
$$

For the scalar interaction $f = \alpha_s := g_1 g_2$, and the dynamical constraint has the form:

$$
\Phi_s := \pi^2_\perp + \frac{1}{4} \Pi^2_\perp + \frac{(\pi_\perp \cdot r)(\pi_2 \cdot r)}{4 R^2} - \frac{m^2_1 \pi_2 \cdot r + m^2_2 \pi_1 \cdot r + 2 \eta \alpha_s m_1 m_2}{\Pi \cdot r} = 0. \tag{5.8}
$$

Both the constraints (5.7) and (5.8) reduce in the free-particle limit $\alpha \to 0$ to the constraint:

$$
\Phi_{\text{free}} := \pi^2_\perp + \frac{1}{4} \Pi^2_\perp + \frac{(\pi_\perp \cdot r)(\pi_2 \cdot r)}{4 R^2} - \frac{m^2_1 \pi_2 \cdot r + m^2_2 \pi_1 \cdot r}{\Pi \cdot r}. \tag{5.9}
$$

This case of the time-asymmetric system with no interaction (i.e., $f = 0$) deserves a particular consideration. The free-particle dynamical constraint (5.9) is not the additive function in variables of different particles. The reason is that this constraint is concerted with the light cone constraint (4.3) which, in turn, binds in an isotropic interval the positions of even free particles. In Appendix B the dynamics of two free particles is manifested from this tangled description.

The system determined by the set of 1st-class constraints (4.2), (4.3) and (5.6) possesses 6 degrees of freedom. Besides, as it follows from the structure of Lie algebra (5.1) of de Sitter group [35], of ten components of the conserved angular momentum tensor (4.7) one can construct six integrals of motion which are in an involution in terms of Poisson brackets. This is sufficient for the system to be integrable in the Liouville sense. The next natural step would be a transition to the description on a reduced 12-dimensional phase space, and separating degrees of freedom by choosing appropriate canonical variables. It turned out more constructive to analyze the system within the manifestly covariant description on the 20-dimensional phase space $T^* M^5_0$ where de Sitter symmetry is realized in a transparent way.
6 Equations of motion and their integration.

Useful integrals of motion arise from two Casimir functions of the de Sitter algebra (5.1):

\[ J^2 := -\text{tr}(J^2) = J_{MN}J^{MN}, \quad V^2 := V_MV^M, \quad (6.1) \]

where the following 5-pseudo-vector

\[ V_M := \frac{1}{8} \epsilon_{MABCD}J^{AB}J^{CD} \quad (6.2) \]

is introduced by means of the Levi-Chivita symbol \( \epsilon_{MABCD} \). Then using the equalities:

\[ \Pi^2_\perp \approx -\frac{1}{R^2} \left( \frac{1}{2} J^2 + (\pi \cdot r)^2 \right), \quad \pi^2_\perp \approx -\frac{1}{R^2} \left( V^2 - \frac{1}{2} (\pi \cdot r)^2 J^2 - (\pi \cdot r)^4 \right) \quad (6.3) \]

recasts arguments of the dynamical constraint (5.6) into an equivalent set,

\[ \Phi(\Pi \cdot r, \pi \cdot r; J^2, V^2) = 0, \quad (6.4) \]

which is more convenient for a dynamical analysis.

Let us consider the equation of motion for the relative position 5-vector \( r \):

\[ \dot{r} = \lambda \{ r, \Phi(\Pi \cdot r, \pi \cdot r; J^2, V^2) \} = \lambda \left( \frac{\partial \Phi}{\partial \pi \cdot r} - 4 \frac{\partial \Phi}{\partial J^2} J - \frac{\partial \Phi}{\partial V^2} K \right) r; \quad (6.5) \]

here \( J := ||J^M_N|| \) and

\[ K := ||K^M_N|| := ||\epsilon^M_{NABC}V^A J^{BC}|| \quad (6.6) \]

are conserved matrices while the Lagrangian multiplier \( \lambda(\tau) \) as a function of \( \tau \) is unspecified and can be chosen for convenience reason.

If the variable \( \Pi \cdot r = \Psi(\tau) \) was known as a function of \( \tau \) then \( \pi \cdot r = \psi(\tau) \) could be found from the dynamical constraint (5.6) as a solution of the algebraic equation:

\[ \Phi(\Psi(\tau), \psi(\tau); J^2, V^2) = 0 \implies \psi(\tau) := \psi(\Psi(\tau); J^2, V^2) \quad (\text{since } J^2, V^2 \text{ are conserved}). \]

In turn, the Hamiltonian equation for \( \Psi(\tau) \),

\[ \dot{\Psi} = \lambda \{ \Psi, \Phi \} = \lambda \frac{\partial \Phi}{\partial \Psi} \left( \psi(\Psi; J^2, V^2); J^2, V^2 \right) \Psi, \quad (6.7) \]

is self-sufficient, separable in \( \tau \) and \( \Psi \), and reduces obviously to quadratures. Note that the resulting function \( \Psi(\tau) \) depends on a choice of the Lagrange multiplier \( \lambda(\tau) \). Alternatively, one can choose \( \Psi(\tau) \) and then find \( \lambda(\tau) \) from eq. (6.7) without integration. The choice of the function \( \Psi(\tau) \) implies a fixing of the evolution parameter \( \tau \).

At this point the equation (6.3) becomes a closed linear equation with respect to 5-vector \( y \), with known \( \tau \)-dependent matrix coefficients. The substitution

\[ r(\tau) = \frac{\Psi(\tau)}{\Psi(0)} q(\tau) \quad (6.8) \]
simplifies this equation to the form:

\[ \dot{q} = -\lambda \left( \frac{4}{J^2} \frac{\partial \Phi}{\partial J} \frac{\partial \Phi}{\partial V^2} K \right) q. \] (6.9)

A subsequent integration procedure is based on the projection operator techniques described in Appendix C. A structure and the action of projection operators depends on eigenvalues of the matrix \( J \) which, in turn, depend on values of the Casimir functions \( \Sigma \). Here we suppose \( J^2 < 0 \) and \( V^2 < 0 \) so that \( J \) possesses the following eigenvalues:

\[ \pm \Sigma := \pm \sqrt{\Sigma_+^2}, \pm i S := \pm \sqrt{\Sigma_-^2} \] and 0, where \( \Sigma_\pm \) are defined in (C.3). Other cases can be treated similarly; they are omitted here.

Let us decompose the 5-vector \( q \) (and then other position 5-vectors) by means of the projection operators (C.12)-(C.13) defined in Appendix C:

\[ q = \left( \Omega^{(\Sigma)} + \Omega^{(S)} + \Omega^{(0)} \right) q := q^{(\Sigma)} + q^{(S)} + q^{(0)}. \] (6.10)

The projectors (C.12)-(C.13) commute with the matrix \( J \). Using this fact and the properties (C.16) of the matrix (6.6) permits one to split the equation (6.9) into the set:

\[ \dot{q}^{(i)}(\tau) = f^{(i)}(\tau) J q^{(i)}(\tau), \quad \tau = \Sigma, S, 0, \] (6.11)

where \( f^{(0)}(\tau) \equiv 0, \)

\[ f^{(\Sigma)}(\tau) := -\lambda \left( \frac{4}{J^2} \frac{\partial \Phi}{\partial J^2} + 2S^2 \frac{\partial \Phi}{\partial V^2} \right), \quad f^{(S)}(\tau) := -\lambda \left( \frac{4}{J^2} \frac{\partial \Phi}{\partial J^2} - 2\Sigma^2 \frac{\partial \Phi}{\partial V^2} \right). \] (6.12)

Then formal solutions to the equations (6.11)-(6.12) are:

\[ q^{(i)}(\tau) = \exp\{ F^{(i)}(\tau) J \} r^{(i)}(0), \quad \text{where} \quad F^{(i)}(\tau) := \int_0^\tau d\tau f^{(i)}(\tau), \quad \tau = \Sigma, S, 0. \] (6.13)

Matrix exponents in these solutions can be unraveled by means of eqs. (C.14):

\[ \begin{align*}
q^{(\Sigma)}(\tau) &= \left( \cosh \left( \Sigma F^{(\Sigma)}(\tau) \right) + \frac{J}{\Sigma} \sinh \left( \Sigma F^{(\Sigma)}(\tau) \right) \right) r^{(\Sigma)}(0), \\
q^{(S)}(\tau) &= \left( \cos \left( SF^{(S)}(\tau) \right) + \frac{J}{S} \sin \left( SF^{(S)}(\tau) \right) \right) r^{(S)}(0), \\
q^{(0)}(\tau) &= r^{(0)}(0).
\end{align*} \] (6.14)

A convolution of 5-vector \( r \) with the angular momentum tensor (4.7) expressed in terms of the collective variables (5.3) yields the equality for the 5-vector \( Y \):

\[ Y \approx J - \psi \Psi \frac{\partial}{\partial \Psi} r. \] (6.17)

Then eqs. (6.8), (6.10), (6.14)-(6.17) lead to the expressions for particle positions

\[ y^{(i)}_a(\tau) = \frac{1}{\Psi(0)} \left\{ J - \psi(\tau) - \frac{1}{2}(-)^a \Psi(\tau) \right\} q^{(i)}(\tau), \quad a = 1, 2, \quad \tau = \Sigma, S, 0. \] (6.18)
where all the quantities in r.-h.s. are known functions of τ at this point.

In order to have a complete solution for Cauchy problem, it is sufficient to express the angular momentum matrix \( J \) and its invariants \( \Sigma, S \) in terms of initial values \( y_a(0), y_a(0) \) by eqs. (4.7), (4.8) and (6.1), (6.2), (C.3). If the initial point belongs to \( T \mathcal{K} \), i.e., the initial values \( y_a(0), y_a(0) \) are subjected to the conserved holonomic constraints (1.2), (1.3) and their differential consequences (see also (A.7), (A.8)), then the particle world lines (6.18) lie in \( \mathcal{K} \) by construction. The momentum-type variables (5.4), (5.5) are subsidiary and not important within the classical consideration.

7 Conclusion

Green functions of massless fields in the Minkowski space-time are located on the light cone surface. This field-theoretical outcome was basic for a construction of the original Staruszkiewicz-Rudd-Hill model and its non-electromagnetic generalizations.

In the curved space-time the Green function of electromagnetic and other massless fields possesses a non-local tail spread in the light cone interior [30]. It is shown here that in particular case of de Sitter space-time the nonlocal contribution of the electromagnetic Green function in the Tetrode-Fokker action integral can be converted to a dynamically equivalent local contribution. The nonlocal contribution of the scalar Green function is unavoidable, if the theory of minimal coupling is implied. Instead, the Green function of the scalar field conformally coupled to de Sitter metrics is shown to be purely local. These two examples of field-theoretical nature are included in a wide class of time-asymmetric models built from general demands of de Sitter symmetry and self-consistency of the Hamiltonian dynamics.

Every time-asymmetric model possesses 6 degrees of freedom and 6 integrals of motion in involution which are independent functions of canonical generators \( J_{MN} \) of \( O(1,4) \) group [35]. Thus these dynamical systems are integrable in the Liouville sense. In practice, the integrability presupposes a choice of appropriate canonical variables in terms of which degrees of freedom separates. In the case of curved de Sitter space-time this task encounters technical difficulties when constructing the description in a 12-dimensional phase space.

Thus in the present paper the time-asymmetric models are treated as constrained systems in 20-dimensional phase space \( T^\ast \mathcal{M}_5 \); de Sitter invariance of all the constraints admits a formulation of equations of motion in a manifestly covariant 5-dimensional form. Moreover, there exists some analogy between a dynamics of the relativistic particle in a constant electromagnetic field [36,37] and the present problem. As the Maxwell tensor in the first case, the conserved angular 5-momentum tensor in the second case is skew-symmetric, treated as constant and covariantly “mounted” into equations of motion. Thus the projection operator technique, used in the first case [36,37], is adapted here to the present 5-dimensional case. In such a way the equations of motion are split and solved in quadratures.

It was noted in Introduction that the Staruszkiewicz-Rudd-Hill model in a flat space-time endow corresponding two-particle systems with physically meaningful features. What distinguishes the model from the retarded or Wheeler-Feynman electrodynamics is the time-asymmetric retarded-advanced causal structure of interaction, a price for a
solvability of the model. Even so, the classical model represents properly relativistic effects in a system of two charged particles within the moderately relativistic domain where the radiation reaction is minor. The quantum versions of this model and some other time-asymmetric models yield relativistic spectra which accord well with results of the quantum field theory [14] and actual meson spectroscopy [15].

A study of de-Sitter-relativistic effects in systems of single gravitating bodies and test particles [20, 24, 25] deepen understanding of the expanding Universe. The next step in this direction would be a prospective elaboration of de Sitter invariant two-particle models with electromagnetic and other interactions. A quantization of time-asymmetric models in de Sitter space is addressed to future works.

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Appendix

A. The relation between the Lagrangian function and the dynamical constraint of time-asymmetric models

Chosen the sign $\eta = 1$ or $\eta = -1$ in the model (see Section 4), let us present the Lagrangian (4.5) in the equivalent form:

$$L = \vartheta F(\nu_1, \nu_2, \omega),$$

$$\vartheta := \eta \dot{y}_1 \cdot r = \eta \dot{y}_2 \cdot r = \eta (\dot{y}_1 + \dot{y}_2) \cdot r / 2 > 0,$$

$$F := - \sum_{a=1}^{2} \frac{\eta m_a}{R \nu_a} - \frac{f(\nu_1, \nu_2, \omega)}{R^2 \nu_1 \nu_2},$$

where $F$ is a function of the scalar arguments (3.9) and thus is a homogeneous function of degree zero of particle velocities $\dot{y}_a$. The scalar factor $\vartheta$ is homogeneous of degree one and positive on timelike world lines. It is presented in (A.2) diversely by accounting a differential consequence $\dot{y}_1 \cdot r = \dot{y}_2 \cdot r$ of the light cone constraint (4.3). In this regards an apparent particle asymmetry of the interaction term of the Lagrangian (4.5) is seeming.

In terms of the functions (A.2), (A.3) and the collective variables (5.3) the Legendre transform (4.8) acquires the manifestly covariant 5-vector form:

$$\Pi = \frac{\partial L}{\partial \dot{y}} = A(\nu_1, \nu_2, \omega) \dot{Y} \vartheta + B(\nu_1, \nu_2, \omega) \dot{r} \vartheta + D(\nu_1, \nu_2, \omega) \eta r,$$

$$\pi = \frac{\partial L}{\partial \dot{y}} = B(\nu_1, \nu_2, \omega) \dot{Y} \vartheta + C(\nu_1, \nu_2, \omega) \dot{r} \vartheta,$$

where

$$A := A_1 + A_2, \quad B := \frac{A_1 - A_2}{2}, \quad C := \frac{A}{4} + \frac{\partial f}{\partial \omega}, \quad D := \frac{1}{R^2} \left[ \frac{f}{\nu_1 \nu_2} - \frac{1}{\nu_2} \frac{\partial f}{\partial \nu_1} - \frac{1}{\nu_1} \frac{\partial f}{\partial \nu_2} \right],$$

$$A_a := -\eta R m_a \nu_a - \frac{\nu_a}{\nu_{\bar{a}}} \frac{\nu_{\bar{a}}}{\nu_{\bar{a}}} \frac{\partial f}{\partial \nu_a} + \left[ \frac{\nu_a}{\nu_{\bar{a}}} \omega - 1 \right] \frac{\partial f}{\partial \omega}; \quad \nu_a \cdot \bar{\nu}_a = 1.$$
The right-hand side of eqs. (A.4), (A.5) is evidently zero-degree homogeneous in \( \dot{Y}, \dot{y} \), thus the Legendre transform is degenerated.

We are interested in relations between scalars on \( T^*M^2_3 \) and \( TK \), where \( TK \subset TM^2_3 \) is described by the holonomic constraints (1.2), (1.3) and their differential consequences expressed for a convenience in terms the collective variables (5.3):

\[
\begin{align*}
Y^2 &= -R^2; \\
Y \cdot r &= 0; \\
r^2 &= 0, \quad \eta r^0 > 0; \\
\dot{Y} \cdot Y &= 0; \\
\dot{Y} \cdot r &= -\dot{r} \cdot Y; \\
\dot{r} \cdot r &= 0.
\end{align*}
\]  

(A.7)

(A.8)

Multiplying eqs. (A.4), (A.5) by \( r \) and \( Y \) and accounting (A.7), (A.8) yields the relations:

\[
\begin{align*}
\Pi \cdot r &= \eta A(\nu_1, \nu_2, \omega); \\
\pi \cdot r &= \eta B(\nu_1, \nu_2, \omega); \\
\Pi \cdot Y &= -\eta B(\nu_1, \nu_2, \omega); \\
\pi \cdot Y &= -\eta C(\nu_1, \nu_2, \omega).
\end{align*}
\]  

(A.9)

(A.10)

Among these scalars on \( T^*M^2_3 \) two of them, \( \Pi \cdot r \) and \( \pi \cdot r \), are observables, and they are arguments of the dynamical constraint (5.6). Squaring eq. (A.4), one can express the scalar \( \Pi^2 \) in terms of \( \nu_1, \nu_2, \omega \). Scalars \( \Pi \cdot Y, \pi \cdot Y \) and \( \Pi^2 \) are not the observables, but they are related with the third argument \( \Pi^2_\perp \) of (5.6) via the following equality derived by squaring eq. (5.4): \( \Pi^2_\perp = \Pi^2 + [(\Pi \cdot Y)^2 + 2(\Pi \cdot r)(\pi \cdot Y)]/R^2 \). Using this and previous equations yields:

\[
\Pi^2_\perp = \frac{1}{R^2} \left[ A_1^2 + \frac{A_2^2}{\nu_1^2} + 2\omega A_1 A_2 \nu_1 \nu_2 + B^2 - 2AC \right] + 2AD.
\]  

(A.11)

In general, three equations (A.9) and (A.11) can be inverted yielding \( \nu_1, \nu_2 \) and \( \omega \) as functions of \( \Pi \cdot r, \pi \cdot r \) and \( \Pi^2_\perp \). We will use for these functions the notations \( \bar{\nu}_1, \bar{\nu}_2, \bar{\omega} \), and \( \bar{A} := A(\bar{\nu}_1, \bar{\nu}_2, \bar{\omega}) = \eta \Pi \cdot r, \ldots, \bar{D} := D(\bar{\nu}_1, \bar{\nu}_2, \bar{\omega}) \) etc. At this point the set of equations (A.4), (A.5) can be formally inverted yielding velocities in terms of canonical variables:

\[
\begin{align*}
\dot{Y}/y &= \frac{\bar{C}}{\Delta} (\Pi - \bar{D} \eta r) - \frac{\bar{B}}{\Delta} \pi, \\
\dot{r}/y &= \frac{\bar{A}}{\Delta} \pi - \frac{\bar{B}}{\Delta} (\Pi - \bar{D} \eta r),
\end{align*}
\]  

(A.12)

where \( \Delta := AC - B^2 \). Then the l.h.s. of eq. (1.9) can be regarded as the Hamiltonian, proportional to the dynamical constraint: \( H_D \propto \Phi = 0 \). Inserting the expressions (A.12) for the particle velocities \( \dot{y}_a = \dot{Y} - \frac{1}{2}(-)^a \dot{r} \) (\( a = 1, 2 \)) into l.h.s. of eq. (1.9) yields the dynamical constraint of the form (5.6):

\[
\pi^2 + \frac{\bar{C}^2}{R^2} + \frac{\bar{C}}{\eta \Pi \cdot r} \left[ \Pi^2_\perp - \frac{(\pi \cdot r)^2}{R^2} \right] - \frac{\Delta(\bar{F} + \bar{D})}{\eta \Pi \cdot r} = 0.
\]  

(A.13)

It determines the scalar observable \( \pi^2_\perp \) via three other arguments \( \Pi \cdot r, \pi \cdot r \) and \( \Pi^2_\perp \) of the dynamical constraint (5.6). In the free-particle case \( f = 0 \) we arrive at eq. (5.9).

One can obtain other relations between canonical variables, such as \( \Pi \cdot Y + \pi \cdot r = 0 \), following from (A.9), (A.10). These relations represent secondary constraints, mentioned in Section 5, which involve unobservable quantities and thus have not a physical meaning.
B. The free-particle system

The free-particle dynamical constraint (5.9) can be presented diversely:

\[ \Phi_{\text{free}} \approx \pi_{a \perp} \cdot r \Phi_{\text{free}} = 0, \]  

(B.1)

where \( \phi_a := \pi_{a \perp}^2 - m_a^2 \) and \( \pi_{a \perp} := \pi_a - \frac{y_a \cdot \pi}{y_a^2} y_a \) \((a = 1, 2)\). This form is more convenient here. It does not imply, however, that both the expressions \( \phi_1 \) and \( \phi_2 \) vanish (as one could opine by Section 2), so that \( \phi_- := \frac{1}{2}(\phi_1 - \phi_2) \neq 0 \).

The Hamilton equations for the position 5-vectors read:

\[ \dot{y}_a = \lambda \{ y_a, \Phi_{\text{free}} \} \approx 2\lambda \frac{\pi_a \cdot r}{\Pi \cdot r} \left( \pi_{a \perp} + (-)^a \frac{\phi_-}{\Pi \cdot r} \right), \quad a = 1, 2, \quad \bar{a} = 3 - a, \]  

(B.2)

and yield the expressions for the unit 5-velocities of particles:

\[ v_a \equiv \frac{\dot{y}_a}{\sqrt{\dot{y}_a^2}} \approx \frac{1}{m_a} \left( \pi_{a \perp} + (-)^a \frac{\phi_-}{\Pi \cdot r} \right) \]  

(B.3)

which are free of the unspecified Lagrangian multiplier \( \lambda \); here the symbol \( \approx \) denotes a weak equality by virtue of all the constraints (4.2), (4.3) and (5.9).

Differentiating the equalities (B.3) and using the Hamiltonian equations (B.2) and corresponding equations for \( \pi_a \) yields the expressions for derivatives \( \ddot{v}_a \):

\[ \ddot{v}_a \approx 2\lambda \frac{\pi_a \cdot r}{\Pi \cdot r} \sqrt{\dot{y}_a^2} R^2 y_a. \]  

(B.4)

From (B.2) and (B.4) the 2nd-order equations of motion follow:

\[ \frac{d}{d\tau} \frac{\dot{y}_a}{\sqrt{\dot{y}_a^2}} - \sqrt{\dot{y}_a^2} y_a = 0, \quad a = 1, 2. \]  

(B.5)

They are split in variables of different particles, and coincide for each particle with the test body equation (2.5). The solutions \( y_a(\tau) \) have the form (2.6), (2.7) for each particle \( a = 1, 2 \).

C. The angular 5-momentum tensor and projection operators

Components of the angular 5-momentum tensor form the skew-symmetric odd-dimensional matrix \( \|J_{MN}\| \), thus one of its eigenvalue is zero. The same is concerned with the matrix \( J := \|J^M_N\| := \|\eta^{ML}J_{LN}\| \). In order to find other eigenvalues of \( J \) one can use the Hamilton-Cayley theorem and construct for \( J \) the characteristic equation. It obviously includes odd degrees of \( J \) up to five with de Sitter invariant coefficients. Then one arrives by direct calculations at the desirable identity:

\[ J^5 + \frac{1}{2} J^2 J^3 + V^2 J \equiv 0, \]  

(C.1)

where \( J^2 \) and \( V^2 \) are two Casimir functions of de Sitter algebra, defined by eqs. (6.1), (6.2). The l.-h.s. of (C.1) can be formally factorized:

\[ (J^2 - \Sigma_a^2)(J^2 - \Sigma_{\bar{a}}^2)J = 0, \]  

(C.2)
where
\[ \Sigma_\pm := -\frac{1}{4}J^2 \pm \sqrt{\mathcal{D}}, \quad \mathcal{D} := J^4/16 - V^2. \] (C.3)

Thus the matrix $J$ possesses 5 eigenvalues $\pm \Sigma_+, \pm \Sigma_-, 0$.

Projection operators onto 1-dimensional subspaces corresponding to eigenvalues $j$ of $J$ can be introduced by a standard technique; see for example [37]:
\[ \mathcal{P}(j) = \prod_{j' \neq j} \frac{J - j'}{J - j'}, \quad j = \pm \Sigma_+, \pm \Sigma_-, 0; \] (C.4)

here $j'$ in the product runs over all eigenvalues except $j$.

In general, the Casimir functions $J^2$ and $V^2$ and thus the discriminant $\mathcal{D}$ can acquire arbitrary real (positive or negative) values, so that the eigenvalues $j$ can be real or complex. Here, however, we limit this arbitrariness by natural physical restrictions.

For the single-particle case $J^2 \approx -m^2R^2$ while $V = 0$. In the case of two free particles one obtains from (4.7), (4.8) and (4.5) (with $f = \nu$):

$$
\chi := \frac{J^2}{4m_1m_2R^2} = \mu + \omega - \nu_1\nu_2, \quad \text{(C.5)}
$$

$$
\chi := \frac{V^2}{m_1^2m_2^2R^4} = \nu_1^2 + \nu_2^2 - 2\nu_1\nu_2\omega + \nu_1^2\nu_2^2, \quad \text{(C.6)}
$$

$$
\delta := \frac{\mathcal{D}}{m_1^2m_2^2R^4} = \chi^2 - \chi = (\mu + \omega)^2 - \nu_1^2 - \nu_2^2 - 2\mu\nu_1\nu_2, \quad \text{(C.7)}
$$

where $\omega$ and $\nu_a$ ($a = 1, 2$) are defined by eq. (3.2), and $\mu := \frac{1}{2}[\frac{m_1}{m_2} + \frac{m_2}{m_1}] \geq 1$.

Let us evaluate $J^2$ and $\mathcal{D}$ (or $\chi$ and $\delta$) on the time-like world lines, for which $v_a^2 = 1$, $v_a^0 \geq 1$ ($a = 1, 2$). Since the Casimir functions are integrals of motion and $O(1,4)$-invariants, it is sufficient to evaluate r.h.-s. of (C.5)-(C.7) at the initial moment $\tau = 0$ in arbitrary reference frame.

We will use for 5-vectors the 3-vector notations: $y = \{y^0, y^1, y^2, y^3, y^4\} := \{y^0, \mathbf{y}, y^4\}$.

Let us start with the case $\eta = +1$, i.e., $y^0_1 > y^0_2$.

The action of the group $O(1,4)$ on the hyperboloid $\mathbb{H}$ is transitive [20]. Thus there exists a reference frame where the starting 5-position $y_1$ of the 1st particle and its 5-velocity $v_1$ are as follows:

$$
y_1 = \{0, \mathbf{0}, R\}, \quad v_1 = \{1, \mathbf{0}, 0\}. \quad \text{(C.8)}
$$

Thus $\omega = v_2^0 \geq 1$. Besides, it follows from (C.8) and the constrains (4.2), (4.3): $y_2^0 = R$, $y_2^0 = -|\mathbf{y}_2|$ with arbitrarily chosen 3-vector $\mathbf{y}_2$, i.e.,

$$
y_2 = \{-|\mathbf{y}_2|, \mathbf{y}_2, R\}. \quad \text{(C.9)}
$$

Now, using the differential consequence $y_2 \cdot v_2 = 0$ of the constrains (4.2) yields $v_2^2 < 0$.

Thus $\nu_1 = -y_2 \cdot v_2 = |\mathbf{y}_2|/R > 0$, $\nu_2 = y_1 \cdot v_2 = -v_2^4 > 0$, and $\omega^2 - v_2^2 \geq 1$, $\omega - \nu_2 > 0$.

Finally we impose the additional condition $y_1^0 + y_1^4 > 0$. It selects a half of the hyperboloid $\mathbb{H}$ which corresponds to the flat exponentially expanding Friedmann universe [20]. It is obviously from (C.8) $y_1^0 + y_1^4 > 0$. If the second particle belongs to the same
Using all these inequalities yields the estimates:

\[
\begin{align*}
\kappa & = \mu + \omega - \nu_1 \nu_2 > \mu + \omega - \nu_2 > \mu, \\
\delta & = (\mu + \omega)^2 - \nu_1^2 - \nu_2^2 - 2\mu \nu_1 \nu_2 > (\mu + \omega)^2 - \omega^2 - 2\mu \omega = \mu^2,
\end{align*}
\]

so that

\[
J^2 < -2(m_1^2 + m_2^2)R^2 < 0, \quad \mathcal{D} > (m_1^2 + m_2^2)^2R^4/4 > 0 \Rightarrow \Sigma_+^2 > (m_1^2 + m_2^2)R^2 > 0 \quad (C.10)
\]

while \(\Sigma_2^2\) can be negative or positive.

For \(\eta = -1\) the same estimates can be obtained by the particle permutation \(1 \leftrightarrow 2\).

If an interaction of particles is present but not too strong to close up the gaps \(\propto m_1^2 + m_2^2\) in \(C.10\), the inequalities \(J^2 < 0, \mathcal{D} > 0\) may hold, and we have again \(\Sigma_+^2 > 0\) and \(\Sigma_-^2 \leq 0\).

Here we consider the case \(\Sigma_2^2 := \Sigma^2 > 0, \Sigma_2^2 := -S^2 < 0\) in detail. The matrix \(J\) possesses 5 eigenvalues: \(\pm \Sigma, \pm iS\) (where \(\Sigma > S > 0\) and 0).

Projection operators \((C.4)\) onto 1-dimensional subspaces corresponding to these eigenvalues have the form:

\[
\begin{align*}
p^{(\pm \Sigma)} & := \frac{(J \pm \Sigma)(J^2 + S^2)J}{2\Sigma^2(\Sigma^2 + S^2)}, \\
p^{(\pm iS)} & := \frac{(J \pm iS)(J^2 - \Sigma^2)J}{2S^2(\Sigma^2 + S^2)}, \\
p^{(0)} & := -\frac{(J^2 + S^2)(J^2 - \Sigma^2)}{\Sigma^2S^2}.
\end{align*}
\]

Instead of projectors \((C.11)\), it is convenient to use analogs of Fradkin operators \((36, 37)\):

\[
\begin{align*}
\mathcal{O}^{(\Sigma)} & := p^{(+\Sigma)} + p^{(-\Sigma)} = \frac{(J^2 + S^2)J}{\Sigma^2(\Sigma^2 + S^2)}, \\
\mathcal{O}^{(S)} & := p^{(+iS)} + p^{(-iS)} = \frac{(J^2 - \Sigma^2)J}{S^2(\Sigma^2 + S^2)}
\end{align*}
\]

which project onto the corresponding 2-dimensional subspaces. We note the important properties of these operators:

\[
J^2\mathcal{O}^{(\Sigma)} = \Sigma^2\mathcal{O}^{(\Sigma)}, \quad J^2\mathcal{O}^{(S)} = -S^2\mathcal{O}^{(S)}, \quad Jp^{(0)} = 0. \quad (C.14)
\]

In order to derive important properties of the matrix \(K\) defined by eq. \((6.6)\) it should be simplified. Accounting \((6.2)\) in \((6.6)\) and unraveling the convolution of Levi-Civita symbols \(\epsilon_{\cdots\epsilon\cdots}\) in terms of products of Kronecker symbols \(\delta_1 \cdots \delta_5\) yields the formula:

\[
K = 2J^3 + J^2J. \quad (C.15)
\]

The action of the projectors \((C.12), (C.13)\) onto \((C.15)\) results in the relations:

\[
\begin{align*}
\mathcal{O}^{(\Sigma)}K & = 2S^2\mathcal{O}^{(\Sigma)}J, \\
\mathcal{O}^{(S)}K & = -2\Sigma^2\mathcal{O}^{(S)}J, \\
p^{(0)}K & = 0.
\end{align*}
\]

The properties \((C.14)\) and \((C.16)\) are used in Section 7 for the integration of the system.

The case \(\Sigma_+^2 > 0, \Sigma_-^2 > 0\) can be considered similarly.
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