Integral representations for the Hartman–Watson density

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Abstract

This paper concerns the density of the Hartman–Watson law. Yor (1980) obtained an integral formula that gives a closed-form expression of the Hartman–Watson density. In this paper, based on Yor’s formula, we provide alternative integral representations for the density. As an immediate application, we recover in part a Dufresne’s result (2001) that exhibits remarkably simple representations for densities of the laws of exponential additive functionals of Brownian motion.

1 Introduction

Let $B = \{B_t\}_{t \geq 0}$ be a one-dimensional standard Brownian motion. For every $\mu \in \mathbb{R}$, we denote by $B^{(\mu)} = \{B^{(\mu)}_t := B_t + \mu t\}_{t \geq 0}$ the Brownian motion with constant drift $\mu$ and set

$$A^{(\mu)}_t := \int_0^t e^{2B^{(\mu)}_s} \, ds, \quad t \geq 0;$$

when $\mu = 0$, we simply write $A_t$ for $A^{(\mu)}_t$. This additive functional, together with the geometric Brownian motion $e^{B^{(\mu)}_t}$, $t \geq 0$, plays an important role in a number of areas such as option pricing in mathematical finance, diffusion processes in random environments, probabilistic study of Laplacians on hyperbolic spaces, and so on; see the detailed surveys [7, 8] by Matsumoto–Yor and references therein.

In [10], Yor proved that for every $t > 0$, the joint law of $B_t$ and $A_t$ is given by

$$\mathbb{P}(B_t \in dx, A_t \in dv) = \frac{1}{v} \exp \left\{ -\frac{1}{2v} \left( 1 + e^{2x} \right) \right\} \Theta(e^x/v, t) \, dx \, dv, \quad x \in \mathbb{R}, \; v > 0, \quad (1.1)$$

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or equivalently,
\[ P(e^{B_t} \in du, A_t \in dv) = \frac{1}{uv} \exp \left( -\frac{1 + u^2}{2v} \right) \Theta(u/v, t) \text{d}u \text{d}v, \quad u, v > 0, \]
(1.2)
where for every \( r > 0 \), the function \( \Theta(r, t), t > 0 \), is the (unnormalized) density of the so-called Hartman–Watson distribution \([3]\) which is characterized by the Laplace transform
\[ \int_0^\infty dt \exp \left( -\frac{\chi^2}{2} t \right) \Theta(r, t) = I_{|\lambda|}(r), \quad \lambda \in \mathbb{R}. \]
(1.3)
Here for every index \( \nu \in \mathbb{R} \), the function \( I_{\nu} \) is the modified Bessel function of the first kind of order \( \nu \); see \([4, \text{Section } 5.7]\) for definition. By the Cameron–Martin relation, we see from (1.2) that for every \( \mu \in \mathbb{R} \) and \( t > 0 \), the law of \( A_t^{(\mu)} \) is expressed as
\[ P(A_t^{(\mu)} \in dv) \text{d}v = \frac{1}{v} \exp \left( -\frac{\mu^2}{2v} t \right) \int_0^\infty du \frac{u^{\mu}}{u} \exp \left( -\frac{u^2}{2v} \right) \text{d}u \]
\[ = v^{\mu-1} \exp \left( -\frac{1}{2v} - \frac{\mu^2}{2t} \right) \int_0^\infty dr r^{\mu-1} \exp \left( -\frac{v^2}{2r} \right) \Theta(r, t) \]
(1.4)
for \( v > 0 \), where we changed the variables with \( u = vr, r > 0 \), for the second line.

It is also proven by Yor \([9]\) that the function \( \Theta \) admits the following integral representation: for every \( r > 0 \) and \( t > 0 \),
\[ \Theta(r, t) = \frac{r}{\sqrt{2\pi t}} \int_0^\infty dy \exp \left( -\frac{\pi^2 - y^2}{2t} \right) \exp (-r \cosh y \sinh y \sin \left( \frac{\pi y}{t} \right)), \]
(1.5)
which may be rephrased as
\[ \Theta(r, t) = \frac{r}{2\pi} \exp \left( \frac{\pi^2}{2t} \right) \mathbb{E} \left[ \exp (-r \cosh B_t) \sinh B_t \sin \left( \frac{\pi B_t}{t} \right) \right] \]
(1.6)
due to the fact that the function \( \mathbb{R} \ni y \mapsto \sinh y \sin(\pi y/t) \) is symmetric. Yor obtained the formula (1.5) by inverting the Laplace transform (1.3), reasoning of which is also reproduced in \([7, \text{Appendix A}]\). In \([2, \text{Subsection } A.3]\), we explain (1.5) via the relation
\[ \int_0^\infty dr \frac{r}{r} \exp (-r \cosh x \Theta(r, t)) = \frac{1}{\sqrt{2\pi t}} \exp \left( -\frac{x^2}{2t} \right), \quad t > 0, \quad x \in \mathbb{R}, \]
(1.7)
which is found, e.g., in \([5, \text{Proposition } 4.5 \text{ (i)}]\) and, as was observed in \([7, \text{Proposition } 4.2]\), may be obtained by integrating both sides of (1.4) with respect to \( v \). In this paper, we continue our discussion of \([2, \text{Subsection } A.3]\) and, based on Yor’s formula (1.5) (or (1.6)), aim at providing the following alternative representations of \( \Theta \):
Theorem 1.1. For every $r > 0$ and $t > 0$, it holds that

$$
\Theta(r, t) = \frac{r}{\pi} \exp \left( \frac{\pi^2}{8t} \right) \mathbb{E} \left[ \cosh B_t \cos(r \sinh B_t) \cos \left( \frac{\pi}{2t} B_t \right) \right]
$$

(1.8)

$$
= \frac{r}{\pi} \exp \left( \frac{\pi^2}{8t} \right) \mathbb{E} \left[ \cosh B_t \sin(r \sinh B_t) \sin \left( \frac{\pi}{2t} B_t \right) \right]
$$

(1.9)

$$
= \frac{r}{2\pi} \exp \left( \frac{\pi^2}{8t} \right) \mathbb{E} \left[ \cosh B_t \cos \left( r \sinh B_t - \frac{\pi}{2t} B_t \right) \right].
$$

(1.10)

More generally, we have for every $r > 0$ and $t > 0$,

$$
\Theta(r, t) = \frac{r}{\pi} \exp \left( \frac{\pi^2}{8t} \right) \mathbb{E} \left[ \cosh B_t \cos(r \sinh B_t - \nu) \cos \left( \frac{\pi}{2t} B_t - \nu \right) \right],
$$

(1.11)

where $\nu \in \mathbb{R}$ is arbitrary. The representations (1.8), (1.9) and (1.10) may be seen as the case $\nu = 0, \pi/2, \pi/4$, respectively.

The third representation (1.10) follows by summing (1.8) and (1.9). If we multiply (1.8) and (1.9) by $\cos^2 \nu$ and $\sin^2 \nu$, respectively, then taking their sum leads to the fourth representation (1.11); for details, see the proof of Theorem 1.1 given in Section 2 whose reasoning will also reveal that $\nu$ may be replaced by any complex number.

Remark 1.1. (1) Taking the difference of (1.8) and (1.9) leads to the following fact of interest:

$$
\mathbb{E} \left[ \cosh B_t \cos \left( r \sinh B_t + \frac{\pi}{2t} B_t \right) \right] = 0 \quad \text{for any } r > 0 \text{ and } t > 0.
$$

(1.12)

The representation (1.10) should be considered jointly with this fact.

(2) Differentiating the representation (1.11) with respect to $\nu$ yields

$$
\mathbb{E} \left[ \cosh B_t \sin \left( r \sinh B_t + \frac{\pi}{2t} B_t - 2\nu \right) \right] = 0, \quad r > 0, \ t > 0,
$$

for any $\nu \in \mathbb{R}$, which extends (1.12).

(3) In view of (1.5), we see from (1.8) that

$$
\lim_{r \to \infty} \mathbb{E} \left[ \cosh B_t \cos(r \sinh B_t) \cos \left( \frac{\pi}{2t} B_t \right) \right] = 0,
$$

which may also be explained by the Riemann-Lebesgue lemma. The same remark is true for (1.9).

Theorem 1.1 has several applications. One of its immediate consequences is that for every fixed $t > 0$, the derivative of $\Theta(r, t)$ of any order at $r = 0+$ vanishes:

Proposition 1.1. Fix $t > 0$. It holds that

$$
\lim_{r \to 0^+} \frac{\partial^n}{\partial r^n} \Theta(r, t) = 0, \quad n = 0, 1, 2, \ldots.
$$

(1.13)

In particular, as $r \to 0+$,

$$
\Theta(r, t) = o(r^\kappa) \quad \text{for any } \kappa > 0.
$$

(1.14)
Therefore from (1.14), we see that the integral in (1.4) does converge even when \( \mu \leq 0 \). The fact (1.13) was deduced in [6, Subsection 2.1] from Yor’s formula (1.5) combined with a remark by Stieltjes in 1894 that for any integer \( n \),

\[
\int_{\mathbb{R}} dx \exp \left( -\frac{x^2}{2t} \right) e^{nx} \sin \left( \frac{\pi x}{t} \right) = 0
\]

(see [6, Equation (6.3)]). Our Theorem 1.1 enables us to obtain (1.13) without relying on Stieltjes’ remark; see Subsection 3.1.

When inserting the representation (1.5) into (1.4), a double integral emerges in the description of the law of \( A^{(\mu)}_t \). The second application of Theorem 1.1 is that, when \( \mu \) is a nonnegative integer, we easily reduce that apparently complicated double integral to a single integral using Fubini’s theorem, thanks to the well-known formulae (see [4, Equations (4.11.2) and (4.11.3)]) for the Hermite polynomials

\[
H_{2n}(x) = (-1)^n 2^{2n+1} \frac{e^{x^2}}{\sqrt{\pi}} \int_0^\infty ds \ s^{2n} e^{-s^2} \cos(2xs), \quad x \in \mathbb{R},
\]

\[
H_{2n+1}(x) = (-1)^n 2^{2n+2} \frac{e^{x^2}}{\sqrt{\pi}} \int_0^\infty ds \ s^{2n+1} e^{-s^2} \sin(2xs), \quad x \in \mathbb{R},
\]

where \( n \) is any nonnegative integer. Dealing with other values of \( \mu \) as well, we put the above-mentioned reduction in Proposition 1.2 below, which recovers in part Theorem 4.2 of [1] by Dufresne. For every \( \mu \in \mathbb{R} \), we denote by \( H_\mu \) the Hermite function of degree \( \mu \) and recall its integral representation when \( \mu > -1 \):

\[
H_\mu(x) = \frac{2^{\mu+1} e^{x^2}}{\sqrt{\pi}} \int_0^\infty ds \ s^\mu e^{-s^2} \cos \left( 2xs - \frac{\pi \mu}{2} \right), \quad x \in \mathbb{R} \quad (1.15)
\]

(see Section 10.2 and Equation (10.5.5) in [4] for the definition of the Hermite functions and the integral representation (1.15), respectively).

**Proposition 1.2.** Let \( \mu > -1 \). For every \( t > 0 \), the law of \( A^{(\mu)}_t \) admits the density function expressed by

\[
\mathbb{P}(A^{(\mu)}_t \in dv) \frac{dv}{dv} = C_\mu(t) \mathbb{E} \left[ \exp \left( -\frac{\cosh^2 B_t}{2v} \right) H_\mu \left( \frac{\sinh B_t}{\sqrt{2v}} \right) \cosh B_t \cos \left( \frac{\pi}{2} \left( \frac{B_t}{t} - \mu \right) \right) \right] \quad (1.16)
\]

for \( v > 0 \), where \( C_\mu(t) = (1/\sqrt{2^{\mu+1} \pi}) e^{\pi^2/(8t)} - \mu^2 t/2 \). In particular, when \( \mu = 0 \) and 1,

\[
\mathbb{P}(A_t \in dv) \frac{dv}{dv} = C_0(t) \mathbb{E} \left[ \exp \left( -\frac{\cosh^2 B_t}{2v} \right) \cosh B_t \cos \left( \frac{\pi}{2t} B_t \right) \right], \quad (1.17)
\]

\[
\mathbb{P}(A^{(1)}_t \in dv) \frac{dv}{dv} = C_1(t) \mathbb{E} \left[ \exp \left( -\frac{\cosh^2 B_t}{2v} \right) \sinh(2B_t) \sin \left( \frac{\pi}{2t} B_t \right) \right]. \quad (1.18)
\]
The expression (1.17) was obtained by several authors, for which we refer the reader to [1, p. 223] as well as the beginning of [6, Subsection 2.2]; an alternative proof of (1.18) will be found in Example 2.1 below. If we consider the law of \(1/(2A^\mu_t)\), then from (1.16), we partly recover Formula (4.9) in [1, Theorem 4.2] due to Dufresne, who also shows that the formula is valid for \(\mu \leq -1\) as well, by developing a recurrence relation that connects the law of \(1/(2A^\nu_t)\) with that of \(1/(2A^\mu_t)\) when \(\nu < \mu\). We do not pursue it here with generality, however, if we repeat integration by parts as necessary appealing to the fact (1.13), then Theorem 1.1 enables us to reduce the computation of the case \(\mu \leq -1\) to a situation where the formula (1.15) applies or the function \(H_{-1}\) emerges; see Remark 3.1 at the end of this paper.

The rest of the paper is organized as follows: in Section 2, we prove Theorem 1.1. Propositions 1.1 and 1.2 are proven in Section 3.

2 Proof of Theorem 1.1

From now on, we fix \(t > 0\). This section is devoted to the proof of Theorem 1.1. Let two real-valued functions \(F\) and \(G\) on \(\mathbb{R}\) be continuous for simplicity and suppose that they are even functions and satisfy

\[
\mathbb{E}[|F(B_t)|] < \infty \quad \text{and} \quad \mathbb{E}[|G(B_t)| \cosh B_t] < \infty.
\]

(2.1)

Lemma 2.1. If, moreover, \(F\) and \(G\) fulfill the relation

\[
\mathbb{E}\left[\frac{F(B_t)}{\cosh B_t + \cosh x}\right] = \mathbb{E}\left[\frac{G(B_t)}{\cosh(x + B_t)}\right]
\]

for any \(x \in \mathbb{R}\),

(2.2)

then we have

\[
\mathbb{E}[e^{-r \cosh B_t} F(B_t)] = \mathbb{E}[G(B_t) \cosh B_t \cos(r \sinh B_t)] \quad \text{for any } r > 0.
\]

(2.3)

Proof. By the former condition in (2.1) and Fubini’s theorem, the left-hand side of the relation (2.2) is rewritten as

\[
\int_0^\infty dr \, e^{-r \cosh x} \mathbb{E}[e^{-r \cosh B_t} F(B_t)]
\]

(2.4)

for any \(x \in \mathbb{R}\). On the other hand, by noting that

\[
\frac{1}{\cosh(x + B_t)} = \frac{2 \cosh(x - B_t)}{\cosh(2x) + \cosh(2B_t)} = \frac{\cosh x \cosh B_t - \sinh x \sinh B_t}{\cosh^2 x + \sinh^2 B_t},
\]

the right-hand side of the relation (2.2) is rewritten as

\[
\mathbb{E}\left[\frac{\cosh x}{\cosh^2 x + \sinh^2 B_t} G(B_t) \cosh B_t\right]
\]
thanks to symmetry of $G$. By the simple relation that
\[
\frac{\cosh x}{\cosh^2 x + \sinh^2 B_t} = \int_0^\infty dr \, e^{-r \cosh x} \cos(r \sinh B_t),
\]
Fubini’s theorem entails that the last expectation is further rewritten as
\[
\int_0^\infty dr \, e^{-r \cosh x} \mathbb{E}[G(B_t) \cosh B_t \cos(r \sinh B_t)]
\]
owing to the latter condition in (2.1). Comparing the above expression with (2.4) and appealing to the injectivity of Laplace transform, we arrive at (2.3).

The next lemma provides an alternative expression of the relation (2.2).

**Lemma 2.2.** The relation (2.2) is equivalent to the relation that
\[
\frac{2}{\pi} \mathbb{E} \left[ \frac{F(B_t) \cosh B_t}{\cosh(2B_t) + \cosh(2x)} \right] = \frac{1}{\sqrt{2\pi t}} \exp \left( -\frac{x^2}{2t} \right) G(x) \quad \text{for any } x \in \mathbb{R}. \tag{2.5}
\]

**Proof.** We appeal to the injectivity of Fourier transform. To this end, observe first that
\[
\int_{\mathbb{R}} dx \, \mathbb{E} \left[ \frac{|F(B_t)|}{\cosh B_t + \cosh x} \right] < \infty, \quad \int_{\mathbb{R}} dx \, \mathbb{E} \left[ \frac{|G(B_t)|}{\cosh(x + B_t)} \right] < \infty \tag{2.6}
\]
as well as
\[
\int_{\mathbb{R}} dx \, \mathbb{E} \left[ \frac{|F(B_t)| \cosh B_t}{\cosh(2B_t) + \cosh(2x)} \right] < \infty. \tag{2.7}
\]
The first observation (2.6) is clear by the condition (2.1). By noting that
\[
\sup_{b \in \mathbb{R}} \frac{\cosh b}{\cosh(2b) + \cosh(2x)} = \frac{1}{2} \sup_{b \in \mathbb{R}} \frac{\cosh b}{\cosh^2 b + \sinh^2 x} \leq \frac{1}{4} \min \{2, |\sinh x|^{-1}\}
\]
for any $x \neq 0$, the second observation (2.7) follows from the former condition in (2.1) as well. We also observe that the right-hand side of the relation (2.2) does give an even function in $x$ by symmetry of $G$. Therefore, thanks to the injectivity of Fourier transform, the assertion of the lemma is reduced to the equivalence of the following two relations (2.8) and (2.9):
\[
\int_{\mathbb{R}} dx \, \cos(\xi x) \mathbb{E} \left[ \frac{F(B_t)}{\cosh B_t + \cosh x} \right] = \int_{\mathbb{R}} dx \, \cos(\xi x) \mathbb{E} \left[ \frac{G(B_t)}{\cosh(x + B_t)} \right] \quad \text{for any } \xi \in \mathbb{R}, \tag{2.8}
\]
\[
\frac{2}{\pi} \int_{\mathbb{R}} dx \, \cos(\xi x) \mathbb{E} \left[ \frac{F(B_t) \cosh B_t}{\cosh(2B_t) + \cosh(2x)} \right] = \mathbb{E}[G(B_t) \cos(\xi B_t)] \quad \text{for any } \xi \in \mathbb{R}. \tag{2.9}
\]
Fix $\xi \in \mathbb{R}$ arbitrarily. We may assume $\xi \neq 0$ since both sides of each relation above are clearly continuous at $\xi = 0$. By \eqref{2.6} and Fubini’s theorem, the left-hand and right-hand sides of the relation \eqref{2.8} are equal respectively to

$$
\mathbb{E}\left[F(B_t) \int_{\mathbb{R}} dx \frac{\cos(\xi x)}{\cosh B_t + \cosh x}\right] = \frac{2\pi}{\sinh(\pi \xi)} \mathbb{E}\left[F(B_t) \frac{\sin(\xi B_t)}{\sinh B_t}; B_t \neq 0 \right],
$$

$$
\mathbb{E}\left[G(B_t) \int_{\mathbb{R}} dx \frac{\cos(\xi x)}{\cosh(x + B_t)}\right] = \frac{\pi}{\cosh(\frac{\pi \xi}{2})} \mathbb{E}[G(B_t) \cos(\xi B_t)],
$$

where we used the fact that

$$
\int_{\mathbb{R}} dx \frac{\cos(\xi x)}{\cosh b + \cosh x} = \frac{2\pi \sin(\xi b)}{\sinh(\pi \xi) \sinh b}
$$

for every nonzero real $b$, and that

$$
\int_{\mathbb{R}} dx \frac{\cos(\xi x)}{\cosh x} = \frac{\pi}{\cosh(\frac{\pi \xi}{2})}
$$

(see, e.g., \cite{2} Subsection A.3 and references cited there; these formulae are also able to be verified by standard residue calculus). Consequently, the relation \eqref{2.8} for $\xi \neq 0$ is rephrased as

$$
\frac{1}{\sinh(\frac{\pi \xi}{2})} \mathbb{E}\left[F(B_t) \frac{\sin(\xi B_t)}{\sinh B_t}; B_t \neq 0 \right] = \mathbb{E}[G(B_t) \cos(\xi B_t)].
$$

On the other hand, by \eqref{2.7} and Fubini’s theorem, the left-hand side of the relation \eqref{2.9} is rewritten as

$$
\frac{2}{\pi} \mathbb{E}\left[F(B_t) \cosh B_t \int_{\mathbb{R}} dx \frac{\cos(\xi x)}{\cosh(2B_t) + \cosh(2x)}\right].
$$

Since the integral with respect to $x$ inside the expectation is calculated, by changing the variables with $x = y/2, y \in \mathbb{R}$, as

$$
\frac{1}{2} \int_{\mathbb{R}} dy \frac{\cos(\frac{\pi}{2} y)}{\cosh(2B_t) + \cosh y} = \frac{\pi \sin(\xi B_t)}{\sinh(\frac{\pi \xi}{2}) \sinh(2B_t)}
$$

for $B_t \neq 0$ by \eqref{2.10}, the expression \eqref{2.12} above agrees with the left-hand side of \eqref{2.11}. Hence the relation \eqref{2.9} for $\xi \neq 0$ is also restated as \eqref{2.11} and the proof completes. \hfill \square

In what follows, we denote by $\mathbb{C}$ the complex plane and write $i = \sqrt{-1}$. A pair of functions $F$ and $G$ fulfilling the relation \eqref{2.5} may be obtained by the residue theorem applied to a meromorphic function $f$ of the form

$$
f(z) = \frac{J(z)}{\cosh(2z) + \cosh(2x)} \exp\left(-\frac{z^2}{2t}\right), \quad z \in \mathbb{C},
$$
where \( x \in \mathbb{R} \) and \( J(z) \), \( z \in \mathbb{C} \), is an odd entire function which will be taken to be either \( \sinh(2z) \) or \( \sinh z \) below. When \( x \neq 0 \), the poles \( w \) of \( f \) each of whose imaginary part \( \text{Im} \, w \) lies between \(-\pi\) and \( \pi \), are four points \( \pm x \pm (\pi/2)i \). By taking a rectangular contour circling these poles and having its two sides on the two lines \( \text{Im} \, z = \pm \pi \), residue calculus yields, at least heuristically,

\[
\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{d\xi}{\cosh(2\xi) + \cosh(2x)} \exp \left( -\frac{\xi^2}{2t} + \frac{\pi^2}{8t} \right) \times \left\{ J(\xi - \pi i) \exp \left( \frac{\pi\xi}{t} \right) - J(\xi + \pi i) \exp \left( -\frac{\pi\xi}{t} \right) \right\} = -\frac{1}{\sinh(2x)} \exp \left( -\frac{x^2}{2t} + \frac{\pi^2}{8t} \right) \left\{ J(x + \pi i/2) \exp \left( -\frac{\pi x}{2t} \right) + J(x - \pi i/2) \exp \left( \frac{\pi x}{2t} \right) \right\}
\]

for \( x \neq 0 \). When \( J(z) = \sinh(2z) \) and \( \sinh z \), the above computation is justified, yielding the following lemma:

**Lemma 2.3.** It hold that for any \( x \in \mathbb{R} \),

\[
\frac{1}{\pi} \mathbb{E} \left[ \frac{\sinh B_t \cosh B_t \sin \left( \frac{\pi B_t}{t} \right)}{\cosh(2B_t) + \cosh(2x)} \right] = \frac{1}{\sqrt{2\pi t}} \exp \left( -\frac{x^2}{2t} - \frac{3\pi^2}{8t} \right) \cos \left( \frac{\pi x}{2t} \right), \tag{2.13}
\]

\[
\frac{1}{\pi} \mathbb{E} \left[ \frac{\sinh B_t \sin \left( \frac{\pi B_t}{t} \right)}{\cosh(2B_t) + \cosh(2x)} \right] = \frac{1}{\sqrt{2\pi t}} \exp \left( -\frac{x^2}{2t} - \frac{3\pi^2}{8t} \right) S(x), \tag{2.14}
\]

where in the latter identity, the function \( S(x) \equiv S(x, t) \), \( x \in \mathbb{R} \), is given by

\[
S(x) = \begin{cases} 
\sin \left( \frac{\pi x}{2t} \right) / \sinh x & \text{for } x \neq 0, \\
\pi / (2t) & \text{for } x = 0.
\end{cases}
\]

We remark that these two identities (2.13) and (2.14) are found in Lemmas 3.1 and 3.2 of the paper [6] by Matsumoto–Yor, in which those two lemmas are used to show that, in the case \( \mu = 0 \) and 1, the expression (1.14) with Yor’s formula (1.5) inserted in coincides with (1.17) and (1.18), respectively.

We are in a position to prove Theorem 1.1.

**Proof of Theorem 1.1.** First we prove (1.8). The identity (2.13) tells us that we may take

\[
F(x) = \frac{1}{2} \sinh x \sin \left( \frac{\pi x}{t} \right) \quad \text{and} \quad G(x) = \exp \left( -\frac{3\pi^2}{8t} \right) \cos \left( \frac{\pi x}{2t} \right)
\]

in the relation (2.5). It is clear that these functions fulfill the integrability condition (2.1). Therefore by Lemmas 2.1 and 2.2 we have for every \( r > 0 \),

\[
\frac{1}{2} \mathbb{E} \left[ e^{-r \cosh B_t} \sinh B_t \sin \left( \frac{\pi B_t}{t} \right) \right] = \exp \left( -\frac{3\pi^2}{8t} \right) \mathbb{E} \left[ \cosh B_t \cos(r \sinh B_t) \cos \left( \frac{\pi B_t}{2t} \right) \right].
\]
Now the representation (1.8) follows from this and Yor’s formula (1.6).

We proceed to the proof of (1.9). The second identity (2.14) in Lemma 2.3 shows that we may take in (2.5)

\[ F(x) = \frac{1}{2} \sinh x \sin \left( \frac{\pi x}{2t} \right) \quad \text{and} \quad G(x) = \exp \left( -\frac{3\pi^2}{8t} \right) S(x), \]

which pair also fulfills (2.1). Therefore by Lemmas 2.1 and 2.2, we have for every \( r > 0, \)

\[ \frac{1}{2} \mathbb{E} \left[ e^{-r \cosh B_t} \sinh B_t \sin \left( \frac{\pi}{t} B_t \right) \right] = \exp \left( -\frac{3\pi^2}{8t} \right) \mathbb{E} \left[ \cosh B_t \cos(r \sinh B_t)S(B_t) \right]. \]

Differentiating both sides with respect to \( r \) and appealing to the formula (1.6) again, we arrive at (1.9).

The representation (1.10) is a consequence of summation of (1.8) and (1.9). To prove (1.11), fix \( \nu \in \mathbb{R} \). Using the addition theorem, we develop

\[
\cos(r \sinh B_t - \nu) \cos \left( \frac{\pi}{2t} B_t - \nu \right) \\
= \cos(r \sinh B_t) \cos \left( \frac{\pi}{2t} B_t \right) \cos^2 \nu + \sin(r \sinh B_t) \sin \left( \frac{\pi}{2t} B_t \right) \sin^2 \nu \\
+ R(B_t) \sin \nu \cos \nu
\]

with \( R(x), x \in \mathbb{R}, \) an odd function such that \( \mathbb{E} \left[ |R(B_t)| \cosh B_t \right] < \infty. \) Hence the right-hand side of the claimed identity (1.11) is equal to

\[
\frac{r}{\pi} \exp \left( \frac{\pi^2}{8t} \right) \left\{ \mathbb{E} \left[ \cosh B_t \cos(r \sinh B_t) \cos \left( \frac{\pi}{2t} B_t \right) \right] \cos^2 \nu \\
+ \mathbb{E} \left[ \cosh B_t \sin(r \sinh B_t) \sin \left( \frac{\pi}{2t} B_t \right) \right] \sin^2 \nu \right\}
\]

by (1.8) and (1.9), which shows (1.11) and completes the proof of the theorem.

We close this section with another instance of a pair of functions \( F \) and \( G \) fulfilling (2.2).

**Example 2.1.** In [2] Subsection A.3, we have observed the relation

\[
\mathbb{E} \left[ \sinh B_t \sin \left( \frac{\pi}{2t} B_t \right) \cosh B_t + \cosh x \right] = \exp \left( -\frac{\pi^2}{8t} \right) \mathbb{E} \left[ \frac{1}{\cosh(x + B_t)} \right] \quad \text{for every} \ x \in \mathbb{R}, \quad (2.15)
\]

and hence the pair of functions

\[
F(x) = \sinh x \sin \left( \frac{\pi x}{2t} \right), \quad G(x) = \exp \left( -\frac{\pi^2}{8t} \right), \quad x \in \mathbb{R}, \quad (2.16)
\]
enjoys the relation (2.2). It is obvious that these two functions satisfy the condition (2.1). Therefore Lemma 2.1 entails that for any $r > 0$,

$$
\mathbb{E}\left[e^{-r \cosh B_t} \sinh B_t \sin \left(\frac{\pi}{2t} B_t\right)\right] = \exp\left(-\frac{\pi^2}{8t}\right) \mathbb{E}\left[\cosh B_t \cos(r \sinh B_t)\right].
$$

(2.17)

Observe from [2, Proposition 3.3] that

$$
\mathbb{E}\left[\exp \left(-\lambda \cosh B_t \cos(r \sinh B_t)\right)\right] = \mathbb{E}\left[e^{B_t} \exp\left(-\frac{r^2}{2} A_t\right)\right]
$$

(2.18)

for every $\lambda \geq 0$ and $r \geq 0$, which entails, by differentiating both sides at $\lambda = 0$, that

$$
\mathbb{E}\left[\cosh B_t \cos(r \sinh B_t)\right] = \mathbb{E}\left[e^{B_t} \exp\left(-\frac{r^2}{2} A_t^{(1)}\right)\right]
$$

(2.19)

thanks to the Cameron–Martin relation. Inserting the rewriting

$$
e^{-r \cosh B_t} = \int_{0}^{\infty} \frac{dv}{\sqrt{2\pi v^3}} \exp\left(-\frac{r^2}{2} v\right) \exp\left(-\frac{\cosh^2 B_t}{2v}\right) \cosh B_t
$$

into the left-hand side of (2.17) and using Fubini’s theorem, we obtain from (2.17) and (2.19) the relation that for any $r > 0$,

$$
\int_{0}^{\infty} \frac{dv}{\sqrt{2\pi v^3}} \exp\left(-\frac{r^2}{2} v\right) \mathbb{E}\left[\exp\left(-\frac{\cosh^2 B_t}{2v}\right) \cosh B_t \sinh B_t \sin \left(\frac{\pi}{2t} B_t\right)\right] = \exp\left(-\frac{\pi^2}{8t} + \frac{r^2}{2} A_t^{(1)}\right),
$$

which indicates the formula (1.18) by the injectivity of Laplace transform. Since, in [2], the above two relations (2.15) and (2.18) have been obtained independently of the formulae (1.4) and (1.5), the argument developed above provides an independent, alternative proof of (1.18). Finally, we also remark that in view of Lemma 2.2, the pair (2.16) of $F$ and $G$ may also be obtained from the relation

$$
\frac{1}{2\pi} \exp\left(\frac{\pi^2}{2t}\right) \mathbb{E}\left[\sinh B_t \sin \left(\frac{\pi B_t}{t}\right)\right] = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right), \quad t > 0, \quad x \in \mathbb{R},
$$

by replacing $t$ and $x$ herein by $4t$ and $2x$, respectively, and using the scaling property of Brownian motion. The above relation is a rewriting of (1.7) in terms of (1.6).

3 Proof of propositions

In the sequel, we set the function $g(r) \equiv g(r, t)$, $r > 0$, by

$$
g(r) := \mathbb{E}\left[\cosh B_t \cos(r \sinh B_t) \cos\left(\frac{\pi}{2t} B_t\right)\right].
$$
As seen in the previous section, it holds that for any $r > 0$,

$$g(r) = \mathbb{E}\left[ \cosh B_t \sinh (r \sinh B_t) \sin \left( \frac{\pi}{2t} B_t \right) \right]$$

and

$$\Theta(r, t) = \frac{r}{\pi} \exp \left( \frac{\pi^2}{8t} \right) g(r). \quad (3.1)$$

### 3.1 Proof of Proposition 1.1

In this subsection, we prove Proposition 1.1.

**Proof of Proposition 1.1.** By the relation (3.1), it suffices to show that

$$\lim_{r \to 0^+} g^{(n)}(r) = 0, \quad n = 0, 1, 2, \ldots. \quad (3.2)$$

By observing the fact that

$$\mathbb{E}[\cosh B_t \sinh B_t^n] < \infty$$

for any nonnegative integer $n$, successive differentiation of the above two representations of $g$ yields

$$g^{(2n)}(r) = (-1)^n \mathbb{E}\left[ \cosh B_t \sinh^{2n} B_t \sin (r \sinh B_t) \sin \left( \frac{\pi}{2t} B_t \right) \right],$$

$$g^{(2n+1)}(r) = (-1)^{n+1} \mathbb{E}\left[ \cosh B_t \sinh^{2n+1} B_t \sin (r \sinh B_t) \cos \left( \frac{\pi}{2t} B_t \right) \right]$$

for every nonnegative integer $n$, from which (3.2) follows readily. 

### 3.2 Proof of Proposition 1.2

In this subsection, we prove Proposition 1.2.

**Proof of Proposition 1.2.** We insert the representation (1.11) into (1.4) putting $\nu = \pi \mu / 2$. Then for $\mu > -1$, Fubini’s theorem entails that (1.4) is rewritten as

$$\frac{1}{\pi} \exp \left( \frac{\pi^2}{8t} - \frac{\mu^2}{2} t \right) \nu^{\mu-1} \exp \left( - \frac{1}{2 \nu} \right) \mathbb{E}\left[ h(B_t) \cosh B_t \cos \left( \frac{\pi}{2} \left( \frac{B_t}{l} - \mu \right) \right) \right], \quad (3.3)$$

where we set the function $h(x), x \in \mathbb{R}$, by

$$h(x) = \int_0^\infty dr \, r^{\nu} \exp \left( - \frac{\nu}{2} r^2 \right) \cos \left( r \sinh x - \frac{\pi \mu}{2} \right),$$
which is equal, by changing the variables with \( r = \sqrt{(2/v)s} \), to

\[
\left( \frac{2}{v} \right)^{(\mu+1)/2} \int_0^\infty ds \ s^\mu e^{-s^2} \cos \left( 2\frac{\sinh x}{\sqrt{2v}} s - \frac{\pi \mu}{2} \right)
= \sqrt{\frac{\pi}{(2v)^{\mu+1}}} \exp \left( -\frac{\sinh^2 x}{2v} \right) H_\mu \left( \sinh \frac{x}{\sqrt{2v}} \right)
\]

by the formula (1.15) for \( H_\mu \) with \( \mu > -1 \). Inserting the last expression of \( h \) into (3.3) and rearranging terms lead to (1.16). The representations (1.17) and (1.18) follow by noting that

\[
H_0(x) = 1, \quad H_1(x) = 2x, \quad x \in \mathbb{R}
\]

(see, e.g., [4, p. 60]). The proof is complete.

We end this paper with a remark on the case \( \mu \leq -1 \).

**Remark 3.1.** We take \( \mu = -3/2 \) and \( -2 \) as an illustration. Noting the relation (3.1), we use the function \( g \) to rewrite the integral in (1.4) with respect to \( r \) as

\[
\frac{1}{\pi} \exp \left( \frac{\pi^2}{8t} \right) I(\mu) \quad \text{with} \quad I(\mu) := \int_0^\infty dr \ r^\mu \exp \left( -\frac{v}{2} r^2 \right) g(r).
\]

The proof of Proposition 1.2 shows that \( I(\mu) \) may be expressed in terms of \( H_\mu \) when \( \mu > -1 \). If we take \( \mu = -3/2 \), then integration by parts yields

\[
I(-3/2) = -2v I(1/2) + 2 \int_0^\infty \frac{dr}{r^{1/2}} \exp \left( -\frac{v}{2} r^2 \right) g'(r)
\]

(3.4)

owing to (1.13). Recalling (1.11), we have

\[
g'(r) = -E \left[ \cosh B_t \sinh B_t \sin(r \sinh B_t - \nu) \cos \left( \frac{\pi}{2t} B_t - \nu \right) \right]
= -E \left[ \cosh B_t \sinh B_t \cos \left( r \sinh B_t - \nu - \frac{\pi}{2} \right) \cos \left( \frac{\pi}{2t} B_t - \nu \right) \right]
\]

for every \( r > 0 \) and \( \nu \in \mathbb{R} \). Therefore choosing \( \nu = -(3/4)\pi \) and appealing to Fubini’s theorem and (1.15), we may express the second term on the right-hand side of (3.4) in terms of \( H_{-1/2} \). In the case \( \mu = -2 \), we have in the same way as above,

\[
I(-2) = -v I(0) + \int_0^\infty \frac{dr}{r} \exp \left( -\frac{v}{2} r^2 \right) g'(r).
\]

(3.5)

Noting that

\[
g'(r) = -E \left[ \cosh B_t \sinh B_t \sin(r \sinh B_t) \cos \left( \frac{\pi}{2t} B_t \right) \right], \quad r > 0,
\]
and that $|\sin(r \sinh B_t)/r| \leq |\sinh B_t|$ for any $r > 0$, Fubini’s theorem entails that the second term on the right-hand side of (3.5) is written as

$$
-\mathbb{E}\left[ \cosh B_t \sinh B_t \cos\left(\frac{\pi}{2t} B_t\right) \int_0^\infty \frac{dr}{r} \exp\left(-\frac{r^2}{2}\right) \sin(r \sinh B_t) \right].
$$

By the formulae

$$
\int_0^\infty \frac{ds}{s} e^{-s^2} \sin(2xs) = \sqrt{\pi} \int_0^x dy e^{-y^2}, \quad x \geq 0, \quad (3.6)
$$

$$
H_{-1}(x) = e^{x^2} \int_x^\infty dy e^{-y^2}, \quad x \in \mathbb{R}, \quad (3.7)
$$

the integrand in the last expectation may be expressed in terms of $H_{-1}$. As for the formula (3.7), see [4, Equation (10.5.3)]. The formula (3.6) may easily be verified by writing

$$
\frac{\sin(2xs)}{s} = 2 \int_0^x dy \cos(2sy), \quad s > 0,
$$

and using Fubini’s theorem.

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