Convergent resummed linear $\delta$ expansion in the critical $O(N)$ $(o_t^2)^2_{3d}$ model

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The nonperturbative linear $\delta$ expansion (LDE) method is applied to the critical $O(N)$ $o_t^2$ three-dimensional field theory which has been widely used to study the critical temperature of condensation of dilute weakly interacting homogeneous Bose gases. We study the higher order convergence of the LDE as it is usually applied to this problem. We show how to improve both, the large-$N$ and finite $N = 2$, LDE results with an efficient resummation technique which accelerates convergence. In the large $N$ limit, it reproduces the known exact result within numerical integration accuracy. In the finite $N = 2$ case, our improved results support the recent numerical Monte Carlo estimates for the critical transition temperature of Bose-Einstein condensation.

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Perturbative methods are well established tools to study diverse aspects of physical systems. However, in many important problems one is not allowed to use perturbation theory, or it just breaks down. In field theory, a well-known open problem concerns the description of physical phenomena close to critical points of continuous or second order phase transitions. In this situation the divergence of length and time scales of the fluctuations, associated to infrared (IR) divergences and critical slow-down, respectively, result in the singular behavior of many physical quantities, like correlation lengths, susceptibilities and many other parameters. In these cases one must recur to nonperturbative methods, such as the renormalization group methods, the $\epsilon$-expansion, the $1/N$ approximation and other techniques (for a review, see Ref. [1]).

A timely important problem associated to the breakdown of perturbation theory near a critical point is the evaluation of the critical transition temperature for a weakly interacting homogeneous Bose gas. This apparently simple, but highly nontrivial task has been the source of controversy for many years. Recently, its functional form was found to be $T_c = T_0 \left( 1 + c_1 n^{1/3} + \left[ c_2 \ln(a n^{1/3}) + c_3 a n^{2/3} + O(a^3 n) \right] \right)$, where $T_0$ is the ideal gas condensation temperature, $a$ is the $s$-wave scattering length, $n$ is the density and $c_1, c_2, c_3$ are numerical coefficients [2]. However, a strong debate concerns the values of the numerical coefficients, especially $c_1$, which has been computed by different authors, using different methods. Some analytical predictions included the self-consistent resummation schemes ($c_1 \approx 2.90$) [3], the $1/N$ expansion at leading order ($c_1 \approx 2.33$) [4] and at next to leading order ($c_1 \approx 1.71$) [3] and also the linear $\delta$-expansion (LDE) [3, 4] at second order ($c_1 \approx 3.06$) [3]. The numerical methods include essentially Monte Carlo lattice simulations (MC). The most recent MC results are reported by the authors of Ref. [4] ($c_1 = 1.29 \pm 0.05$) and of Ref. [10] ($c_1 = 1.32 \pm 0.02$). The problem is that these coefficients (except $c_2$) are sensitive to the IR physics at the critical point and so, no perturbative approach can be used to predict them.

At first, one may believe that, due to the complex nature of the problem, it is unlikely that a definitive analytical prediction of the IR coefficients could be made. In general, higher order computations quickly become prohibitively difficult with the traditional nonperturbative analytical methods which, like the $1/N$ expansion, rely on the resummation of an infinite number of particular contributions. At the same time, the LDE great advantage is the fact that all calculations, including renormalization, are performed in a perturbative way. This means that one always deals with a finite set of contributions at each order. This advantage is well illustrated in a recent work by some of the present authors [11, 12], where the LDE was extended to order-$\delta^4$. The results give strong indications that, when properly applied, the LDE leads to very precise analytical predictions for the nonperturbative coefficients $c_1$ and $c_2'$. In fact, the results of Ref. [12] for these coefficients, at second ($c_1 \approx 3.06$, $c_2' \approx 101.4$), third ($c_1 \approx 2.45$, $c_2'' \approx 98.2$) and fourth orders ($c_1 \approx 1.48$, $c_2'' \approx 82.9$) seem to, roughly, converge to the precise MC results of Ref. [10], where $c_2'' = 75.7 \pm 0.4$ was also predicted.

The important question regarding the LDE convergence properties has been addressed in the context of the anharmonic oscillator (AO) at zero temperature where rigorous proofs have been produced [12]. Those proofs have been extended to the finite temperature domain by Duncan and Jones [13] who considered the AO partition function. Very recently, the convergence proof has been extended to renormalizable quantum field theories at zero temperature [14]. However, it would be interesting to probe convergence in the vicinity of a phase transition such as for the Bose-Einstein condensation (BEC) model considered here. In most applications one can establish simple relations in between the LDE and other nonperturbative methods already at order-$\delta$ where one-loop diagrams are present. In fact, one can show that in those cases the LDE either exactly reproduces $1/N$ results or produce very close numerical estimates [12, 13]. Here, the BEC problem poses an additional difficulty since the first non-trivial contributions start at the two-loop level in the self-energies. This is a consequence of the Hugenholtz-
Pines theorem which eliminates the one-loop momentum independent contributions. In this case, it is not easy to establish simple analytical relations like those given in Refs. [8, 11] and the problem must be treated differently.

Braaten and Radaucc in Ref. [13] have recently revisited the LDE application to the BEC problem also considering the convergence problem. One of the differences between our present work and also Refs. [8, 11] from their approach regards the choice of physical quantity to be extremized. Moreover, the coefficient $c_L^2$ and convergence acceleration, considered by us, were not investigated in Ref. [13]. We discuss both approaches, in more details, in a longer companion paper [17].

In this Letter, our aim is to mainly investigate the convergence of the LDE, in the critical $O(N)$ \( \langle \phi_i \rangle_{3d}^2 \) model, by considering its behavior at the large-$N$ as well as at the finite $N = 2$ limits. This study will be combined with a simple, but powerful, all order LDE resummation technique [18] so that we can explicitly show convergence.

At the critical point, one can describe a weakly interacting dilute homogeneous Bose gas by an effective action analogous to a $O(2)$ scalar field model in three-dimensions given by

\[
S_{\phi} = \int d^3x \left( \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} r \phi^2 + \frac{u}{4!} \phi^4 \right),
\]

where \( \phi \) is a two-component real scalar field. The parameters \( r \) and \( u \) are related to the original parameters of the nonrelativistic action, the chemical potential \( \mu \), atomic mass \( m \) and temperature \( T \) by \( r = -2 m \mu \) and \( u = 48 \pi a T \) [4, 11]. The leading order coefficient of the critical temperature shift can be expressed as \( \frac{1}{3} \left[ \xi(3/2)^2 \right]^{-4/3} \Delta(\phi^2)/u \), where \( \Delta(\phi^2) = \langle \phi^2 \rangle_u - \langle \phi^2 \rangle_0 \). The subscripts \( u \) and 0 mean that the field fluctuations are to be evaluated in the presence and in the absence of interactions respectively.

The implementation of LDE within this model is reviewed in previous applications [8, 11] (see also Ref. [17]). In practice, one considers the original theory Eq. (1) adding a quadratic (in the fields) term \( 1/2 (1 - \delta) \phi^2 \phi^2 \) where \( \eta \) is an arbitrary mass parameter. At the same time \( u \rightarrow \delta u \) and \( r \rightarrow \delta r \). One ends up with an interpolated theory described by the propagator \( (q^2 + \eta^2)^{-1} \) and vertices \( - \delta u \phi^2 \), \( \delta \eta \phi^2 \) and \( - \delta r \phi^4 \). A physical quantity \( \Phi(k) \) is then evaluated to an order-$\delta^k$ using these new Feynman rules and following the program, including renormalization, of ordinary perturbation theory. This method operates for any \( N \) and is free from IR divergences. Nonperturbative results for the original \( (\eta\text{-independent}) \) theory are obtained by optimizing \( \Phi(k) \) with respect to \( \eta \) at \( \delta = 1 \). One possible optimization procedure, adopted in a part of this work, is the Principle of Minimal Sensitivity (PMS) which requires \( d\Phi(k)/d\eta = 0 \) [24]. For comparison and to check the consistency of the optimization procedure for reproducing physically sensible results, we also consider the alternative optimization scheme known as the Fastest Apparent Convergence criterion (FAC) where one requires that, at a given order \( k \), the \( k \)-th coefficient of the perturbative expansion be zero. As we will see, both methods lead to equivalent results.

Let us first consider, in the large-$N$ limit, the usual analytical way [8, 11] to define a non-trivial LDE, by performing an order by order perturbative evaluation of \( \langle \phi_i \rangle_u(\Phi) \) which, moreover, has the advantage of being straightforwardly generalized for arbitrary \( N \) values. The convergence property issues can be studied by comparing the results with the “exact” \( 1/N \) result \( c_1 = 2.328 \). The details of such a calculation follow from the methods used in Ref. [11] and are given in the accompanying paper [17]. One obtains

\[
\langle \phi_i \rangle_u(20) = -\frac{N \eta}{4 \pi} + \frac{N u}{3} \sum_{i=1}^{19} C_i \left( -\delta u N \right)^i + O(\delta^{21}),
\]

where the coefficients are given by \( C_1 = 7.249 \times 10^{-5} \), \( C_2 = 2.032 \times 10^{-6} \), \( C_3 = 6.400 \times 10^{-8} \), \( C_4 = 2.080 \times 10^{-9} \), \( C_5 = 5.021 \times 10^{-11} \), \( C_6 = 2.760 \times 10^{-12} \), \( C_7 = 3.580 \times 10^{-14} \), \( C_8 = 6.500 \times 10^{-16} \), \( C_9 = 1.090 \times 10^{-17} \), \( C_{10} = 1.040 \times 10^{-19} \), \( C_{11} = 7.030 \times 10^{-22} \), \( C_{12} = 2.810 \times 10^{-24} \), \( C_{13} = 7.300 \times 10^{-26} \), \( C_{14} = 2.800 \times 10^{-28} \), \( C_{15} = 6.000 \times 10^{-31} \), \( C_{16} = 5.780 \times 10^{-33} \), \( C_{17} = 1.100 \times 10^{-35} \), \( C_{18} = 1.130 \times 10^{-37} \) and \( C_{19} = 1.390 \times 10^{-40} \). All coefficients for \( i \geq 2 \) in Eq. (2) were obtained from dimensional integrals over Feynman parameters, that we have performed by using the well-known Monte Carlo multidimensional integration routine VEGAS [24]. We should note that for such high dimension and complicated integrals, the Monte-Carlo statistical integration error cannot be reduced well below the percent level and this will be reflected on our final results.

Here, as in the original LDE applications to the BEC problem [8, 11], our strategy is to evaluate \( \langle \phi_i \rangle_u(\Phi) \) perturbatively to order-$\delta^5$ and then to extremize this quantity following the PMS. By setting \( u = 0 \) in the optimal \( \langle \phi_i \rangle_u(20) \) one immediately obtains the optimal \( \langle \phi_i \rangle_0(20) \) and, at the same time, the optimal \( \Delta(\phi_i) \). By applying the PMS to this quantity, at each order, one obtains solutions that can be grouped into complex families whose first member is real as discussed in the anharmonic oscillator analogous studies (see Bellet, Garcia and Neveu in Refs. [12]) and in Ref. [11]. To order-$\delta^20$, our results are shown in Table I, together with those obtained by the FAC optimization procedure, for the best converging family of solutions (see Ref. [17] for details). In the very last line of Table I, we indicate the corresponding statistical integration accuracy as provided by VEGAS. Note that both, the PMS and FAC, results converge to similar values, attesting that our results are not merely an artifact of the optimization procedure.

As far as the predictions of Ref. [16] are concerned we note that our results, optimized via PMS or FAC, are also very stable. Like those authors, at about 18th order we achieve \( \sim 7\% \) accuracy. Let us now present a way
of improving the above results with an efficient LDE resummation technique. Performing the usual LDE interpolation with \( \eta^* = \eta(1 - \delta)1/2 \) and \( u \to \delta u \), expanded to order \( p \), defines a partial sum \( \Phi^{(p)}(\eta, u, \delta) \equiv \sum_{n=0}^{\infty} s_n \delta^n \), which for \( \delta \to 1 \) is given formally, from the simple pole residues, as

\[
\Phi^{(p)}(\eta, u, \delta \to 1) = \frac{1}{2\pi i} \oint d\delta \frac{\delta^{-p-1}}{1 - \delta} \Phi(\eta, u, \delta),
\]

where the anticlockwise contour encircles the origin. Now, one performs a change of variables \( \frac{v}{u} \) for the relevant \( \delta \to 1 \) limit: \( \delta \equiv 1 - v/p \), together with a similarly order-dependent rescaling of the arbitrary mass parameter, \( \eta \to \eta \times p^{1/2} \), where the power 1/2 is dictated by the scalar mass interaction term. For \( p \to \infty \) this resummation can be summarized as the replacement \( \eta^* \to \eta^* \times u^{1/2} \), followed by the contour integration

\[
\langle \phi^2 \rangle^{(4)}_{\eta \to \infty} = \frac{1}{2\pi i} \oint \frac{dv}{v} \exp(v) \langle \phi^2(\eta^* \to \eta^* u^{1/2}) \rangle,
\]

where the “weight” \( \exp(v)/v \) originates from \( d\delta (1 - \delta)^{-1} \to dv/v; \lim_{p \to \infty} (1 - v/p)^{p-1} = \exp(v) \), and the original contour was deformed to encircle the branch cut \( \text{Re}[v] < 0 \). Noting that the LDE produces a power series in \( u^{(k+1)}/\eta^2 \) and by using

\[
\int dv \exp(v) v^a = 2\pi i \Gamma(-a),
\]

one sees that the main effect of this resummation is to divide the original expansion coefficients at order \( k \) by terms \( \Gamma(1 + k/2) \sim (k/2)! \) for large \( k \). This damping of the perturbative coefficients at large order, as implied by this specific resummation, is rather generic and was exploited recently in the completely different context of asymptotically free models \(^{12}\) where it was shown to accelerate convergence of the LDE. When applied to the anharmonic oscillator, it is in fact (asymptotically) equivalent to the more direct LDE resummation with an order-dependent rescaling of the arbitrary mass, as employed in some of the Refs. \(^{2}\) to establish rigorous convergence of the LDE for the oscillator energy levels, which is itself an extension of the order-dependent mapping (ODM) resummation technique \(^{3}\). The contour integral resummation is however very convenient since, algebraically, it is simpler than the direct LDE summation, in particular to recover the original theory for \( \eta^* \to 0 \). The results obtained through this contour integral resummation technique (CIRT) applied to Eq. (2) are also shown in Table I. Fast convergence happens already at order-\( \delta^{10} \), within \( \sim 1\% \) of the exact large-\( N \) result for \( c_1 \). Note the complete stability of results from this order onwards. Within the intrinsic numerical integration error involved in the computation of the coefficients in Eq. (2) we can conclude on the actual convergence of the CIRT towards the exact \( 1/N \) result. Note how the CIRT-PMS predictions quickly become more accurate, than the ordinary PMS (FAC) results, when more approximants are considered.

We finally turn to the finite \( N = 2 \) case, for which the quantity \( \langle \phi^2 \rangle_{u}^{(k)} \) has been evaluated, up to order-\( \delta^4 \), in Ref. \(^{11}\). Its perturbative expansion is

\[
\langle \phi^2 \rangle_{u}^{(4)} = -\frac{\eta^*}{2\pi} + \delta u \sum_{i=1}^{3} K_{i} \left( \frac{\delta u}{\eta^*} \right)^{i} + \mathcal{O}(\delta^5),
\]

where the coefficients are given by \( K_1 = 3.222 \times 10^{-6}, K_2 = 1.524 \times 10^{-6} \) and \( K_3 = 1.042 \times 10^{-7} \). The contour integral technique applied at large-\( N \) also improves the finite \( N = 2 \) result of Ref. \(^{11}\) showing that the complex family of solutions has real parts that converge to \( c_1 = 1.15 \pm 0.01 \), whereas the most recent Monte Carlo predictions are \(^{6}\) \( c_1 = 1.29 \pm 0.05 \) and \(^{10}\) \( c_1 = 1.32 \pm 0.02 \). Apart from \( \langle \phi^2 \rangle_{u}^{(k)} \), the quantity \( \delta r_c^{(k)} = -\delta r_c^{(k)}(0) \) also enters the evaluation of the order-\( \delta^2 \) coefficient \( c_0^2 \) which appears in the \( T_c \) expansion \(^{2}\). For \( N = 2 \) its order-\( \delta^4 \) perturbative evaluation is

\[
\delta r_c^{(4)} = \frac{\delta u \eta^*}{6\pi} + \delta^2 u^2 A_2 \left( \ln \left( \frac{M}{\eta^*} \right) - 0.59775 \right) - \delta^3 u^3 A_3 + \delta^4 u^4 A_4 + \mathcal{O}(\delta^5),
\]

TABLE I: PMS, FAC and CIRT-PMS results for \( c_1 \) at large-\( N \), at different orders \( k \), obtained with the best converging families (real part) toward the exact result \( c_1 = 2.328 \).
where the coefficients are $A_2 = 1.407 \times 10^{-3}$, $A_3 = 8.509 \times 10^{-5}$ and $A_4 = 3.523 \times 10^{-6}$. Treating Eq. \ref{eq:4} with the CIRT one obtains the result \( \text{Re} \left[r^{(4)}_{\text{crit}}(M = u/3)\right] \approx 0.0010034u^2 \) which, together with the CIRT improved \( \langle \phi^2 \rangle_{\text{crit}} \) result, leads to $c_2'' = 84.9 \pm 0.8$ \cite{17}, while the Monte Carlo estimate is $c_2'' = 75.7 \pm 0.4$ \cite{2}. Note that the scale $M = u/3$ was originally chosen in those Monte Carlo applications.

In conclusion, the LDE has been applied successfully to many different problems in field theory where standard perturbation theory does not apply. But despite of its successes, its applicability to higher orders and the study of its convergence properties in field theory, beyond the anharmonic oscillator problem \cite{12}, have proven to be a challenge. Here, we have studied the convergence of the LDE as applied to the critical point of Bose-Einstein condensation. We have shown that the perturbative LDE for the large-$N$ case converges towards the exact result, once resummed to all orders. We have also used the LDE to explicitly evaluate, in the large-$N$ limit, the coefficient $c_1$ to order-$\delta^2$ using two different optimization procedures (PMS and FAC), leading to equivalent converging results, but in both cases $\sim 7\%$ accuracy is only achieved at order-$\delta^{18}$. Then, we have shown how the powerful contour integral resummation technique (CIRT) accelerates convergence already at order-$\delta^{10}$ within $1\%$ accuracy or less, which appears in fact only limited by the intrinsic accuracy of the Monte-Carlo integration used to evaluate the coefficients of the relevant series. This same technique was extended to the relevant finite $N = 2$ case where the recent order-$\delta^2$ results for $c_1$ and $c_2''$ \cite{11} were improved. It is worth remarking that, the $c_1$ values 3.06, 2.45 and 1.48 obtained at orders-$\delta^2$, $\delta^3$ and $\delta^4$ in Ref. \cite{11} are close to 2.90, 2.33 and 1.71 obtained by resumming “setting sun” contributions \cite{3}, leading order \cite{4} and next to leading order $1/N$ \cite{4} contributions, respectively. The similarity between the LDE values at a given order and the values produced by each one of those analytical nonperturbative approximations should come as no surprise if one considers the type of graphs resummed by the LDE, at each order. At order-$\delta^2$, only “setting sun” contributions are considered while more typical $1/N$ leading order contributions appear at order-$\delta^3$. At order-$\delta^4$, graphs which would belong to the $1/N$ at next to leading order also contribute. This simple consideration shows that the hierarchy of LDE numerical values is not a mere coincidence but a consequence of the type of graphs considered at each order. The numerical differences are due to the fact that the LDE actually mixes up those contributions, at each order in $\delta$, irrespective of their $1/N$ order. Here, apart from supporting the recent Monte Carlo predictions, our improved results reinforce the potential of the LDE as a powerful tool to treat nonperturbative problems in field theory.

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