Threshold phenomena for interference with randomly placed sensors

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Abstract

Assume \( n \) sensors are initially placed on the half-infinite interval \([0, \infty)\) according to Poisson process with arrival rate \( n \). Let \( s \geq 0 \) be a given real number. We are allowed to move the sensors on the line, so as that no two sensors are placed at distance less than \( s \). When a sensor is displaced a distance equal to \(|m(i)|\) the cost of movement is proportional to some (fixed) power \( a > 0 \) of the \(|m(i)|\) distance traveled. As cost measure for the displacement of the team of sensors we consider the \( a \)-total movement defined as the sum \( M_a := \sum_{i=1}^{n} |m(i)|^a \), for some constans \( a > 0 \). In this paper we study tradeoffs between interference value \( s \) and the expected minum \( a \)-total movement. For the line, the main results can be summarized as follows.

1. If \( s = \frac{1}{n} \) then we present an algorithm that uses \( \Theta(n^{1-\frac{a}{2}}) \) \( a \)-total expected movement, when \( a \in \mathbb{N} \) and \( O(n^{1-\frac{a}{2}}) \) \( a \)-total expected movement, when \( a > 1 \) (see Algorithm 1, Theorem 5, Theorem 7 and Theorem 20). We also give the lower bound \( \Omega(n^{1-\frac{a}{2}}) \), when \( a \geq 1 \) (see Theorem 8 and Theorem 9).

2. Fix \( \epsilon > 0 \) independent on \( n \). If \( s = \frac{1+\epsilon}{n} \) we prove the upper bound \( O(n) \) on the expected \( a \)-total movement, when \( a > 0 \) (see Theorem 5, Theorem 6 and Theorem 10) and the lower bound \( \Omega(n) \), when \( a \geq 1 \) (see Theorem 11 and Theorem 12).

3. Fix \( \epsilon > 0 \) independent on \( n \). If \( s = \frac{1-\epsilon}{n} \) we prove the upper bound \( O(n^{1-a}) \) on the expected \( a \)-total movement, when \( a > 0 \) (see Algorithm 2 and Theorem 14).

Our investigations explain the threshold phenomena around the interference value \( \frac{1}{n} \) as this affects the expected minimum \( a \)-total movement of the sensors to prevent interference.

Similar results concerning the expected minimum \( d \)-dimensional \( a \)-total movement and interference value \( s \) are obtained when the sensors are displaced in the hyperoctant \([0, \infty)^d\) according to \( d \) identical and independent Poisson processes.

Keywords: Interference, Analysis of algorithms, Random, Sensors

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1. Introduction

Mobile sensors are being deployed for detecting and monitoring events which occur in many instances of everyday life. However, it is often the case that monitoring may not be as effective due to external factors such as harsh environmental conditions, sensor faults, geographic obstacles, etc. In such cases sensor realignments may be required, e.g., sensors must be relocated from their initial positions to new positions so as to attain the desired communication characteristics.

It is very well known that proximity between sensors affects transmission and reception signals and causes degradation of performance (see [7]). The closer their distance the higher the resulting interference and hence performance degradation. Hence to avoid interference a critical interference value, say \( s \geq 0 \), is established and sensors must be a distance of at least \( s \) apart. It is therefore crucial to understand the critical interference value in a sensor network.

The present paper is concerned with random realignments of sensors on the real line. Assume that \( n \) sensors are initially placed on the half-infinite interval \([0, \infty)\) according to Poisson process with arrival rate \( n \). The initial placement of the sensors does not guarantee avoiding interference since the sensors have been placed randomly according to the arrival times of Poisson process. To attain a requirement that no two sensors are placed at distance less than \( s \), for some \( s \geq 0 \), sensors have to move from their initial locations to new positions. Further, fix \( a > 0 \) and consider a set of \( n \) sensors. Suppose that the \( i \)–th sensor’s displacement is equal \( |m(i)| \), for \( i = 1, 2, \ldots, n \). Then the \( a \)-total movement of the whole system of \( n \) sensors is \( \sum_{i=1}^{n} |m(i)|^{a} \). What is the expected minimum \( a \)-total movement which the sensors have to move to satisfy a requirement that no two sensors are placed at distance less than \( s \), for some \( s \geq 0 \)? In this study we derive tradeoffs between the expected minimum \( a \)-total movement and the interference value \( s \) and explain threshold phenomena around the interference value \( \frac{1}{a} \).

1.1. Related work

Interference has been the subject of extensive interest in research community in the last decade. Some papers study interference in relation to network performance degradation [2, 3]. Moscibroda et al. [14] consider the average interference problem while maintaining the desired network properties such as connectivity, multicast trees or point-to-point connections while in [1] the authors proposes connectivity preserving and spanner constructions which are interference optimal. The interference minimization in wireless ad-hoc networks in a plane was studied in [8]. The asymptotic analysis of interference problem on the line using queueing theory was provided in [13].

More importantly, our work is related to the paper [2] where the authors consider the expected minimum total displacement required so that in their final positions every pair of sensors is at distance greater than \( s \) for \( n \) sensors placed uniformly according to Poisson process. It is worth mentioning some asymptotic bounds in [2] are one-sided. Our analysis generalizes the result of the paper [2] from \( a = 1 \) to all exponents \( a > 0 \). We give full asymptotic results (lower and upper bound, exact asymptotics) which explain the threshold phenomena.
1.2. Preliminaries and model

Consider \( n \) sensors initially placed on the half-infinite interval \([0, \infty)\) according to Poisson process with arrival rate \( n \). Assume that, the \( i \)-th event represents the location of the \( i \)-th sensor, for \( i = 1, 2, \ldots, n \). Let \( X_i \) be the arrival time of the \( i \)-th event in this Poisson process, i.e., the position of the \( i \)-th sensor in the interval \([0, \infty)\). We know that the random variable \( X_i \) obeys the Gamma distribution with parameters \( i, n \). Its probability density function is given by

\[
 f_{i,n}(t) = ne^{-nt} \frac{(nt)^{i-1}}{(i-1)!}
\]

Moreover, the probability density function of random variable \( X_j + l - X_j = X_l \) is given by the formula

\[
 f_{l,n}(t) = ne^{-nt} \frac{(nt)^{l-1}}{(l-1)!}
\]

(see [10, 12, 16] for additional details on the Poisson process). Notice that,

\[
 \int_0^\infty f_{l,n}(t) dt = 1 \tag{2}
\]

\[
 \int_0^\infty t f_{l,n}(t) dt = \frac{l}{n} \tag{3}
\]

where \( l, n \) are positive integers (see [5, Chapter 15]).

We define below the concept of interference value.

**Definition 1** (s interference value). Let \( s \geq 0 \) be a given real number. The \( s \) interference value requires that no two sensors are placed at Euclidean distance less than \( s \).

To attain an interference value of at least \( s \) between each pair of sensors is required to move the sensors from their initial locations to new positions. As cost measure for the displacement of the team of sensors we consider a-total movement defined as follows.

**Definition 2** (a-total movement). Let \( a > 0 \) be a constant. Suppose that the \( i \)-th sensor’s displacement is equal \( |m(i)| \). The a-total movement is defined as the sum

\[
 M_a := \sum_{i=1}^n |m(i)|^a
\]

Motivation for this extended cost metric arises from the fact the cost of individual sensor displacement may not be linear in this displacement, but rather be dependent on some power of the distance traversed. Moreover, the parameter \( a \) may well represent various conditions on the barrier: obstacles, lubrication, etc. Therefore, the a-total movement is a more realistic metric than the one previously considered \( a = 1 \).

1.3. Outline and results of the paper

Fix \( \epsilon > 0 \) (independent on \( n \)). Assume that \( n \) mobile sensors are displaced on the interval \([0, \infty)\) according to the arrival times of Poisson process with arrival rate \( n \). We derive tradeoffs between the expected minimum a-total movement and the interference value \( s \). Table 1 summarizes the results proved in Section 3.
Table 1: The expected minimum $a$-total movement of $n$ sensors in the interval $[0, \infty)$ as a function of the interference value $s$.

| Interference value $s$ | Expected $a$-total movement | Algorithm  | Theorem |
|------------------------|-------------------------------|------------|---------|
| $s = \frac{1-\epsilon}{n}$, $\epsilon > 0$ | $O \left( n^{1-a} \right)$ | $IN_1 \left( n, \frac{1-\epsilon}{n} \right)$ | 14 |
| $s = \frac{1}{n}$ | $\Theta \left( n^{\frac{1-a}{d}} \right)$ if $a \geq 1$, $O \left( n^{\frac{1-a}{d}} \right)$ if $a \in (0, 1)$ | $MV_1 \left( n, \frac{1}{n} \right)$ | 5, 7, 8, 9, 20 |
| $s = \frac{1+\epsilon}{n}$, $\epsilon > 0$ | $\Theta (n)$ if $a \geq 1$, $O (n)$ if $a \in (0, 1)$ | $MV_1 \left( n, \frac{1+\epsilon}{n} \right)$ | 5, 6, 10, 11, 12 |

Our investigations explain the presence of a threshold around the interference value $\frac{1}{n}$ as this affects the expected minimum $a$-total movement of the sensors to prevent interference.

Let us consider case $a = 2$. Notice that for interference value $s = \frac{1}{n}$ the expected minimal total movement to power 2 is $\Theta(1)$, below $s < \frac{1}{n}$ it declines sharply to $O \left( \frac{1}{n} \right)$, and above $s > \frac{1}{n}$ it increase to $\Theta(n)$.

Similar sharp decrease and increase hold for all $a > 0$.

Further, we explain threshold phenomena for interference when the sensors are located in the higher dimension. Table 2 summarizes the results obtained in Section 4.

Table 2: The expected minimum $d$-dimensional $a$-total movement of $n$ sensors in the interval $[0, \infty)^d$ as a function of the interference value $s$.

| Interference value $s$ | Expected $d$-dimensional $a$-total movement | Algorithm  | Theorem |
|------------------------|---------------------------------------------|------------|---------|
| $s = \frac{1-\epsilon}{n^{d/2}}$, $\epsilon > 0$ | $O \left( n^{1-\frac{a}{d}} \right)$ | $IN_d \left( n, \frac{1-\epsilon}{n^{d/2}} \right)$ | 19 |
| $s = \frac{1}{n^{d/2}}$ | $\Theta \left( n^{1-\frac{a}{d}} \right)$ if $a \geq 1$, $O \left( n^{1-\frac{a}{d}} \right)$ if $a \in (0, 1)$ | $MV_d \left( n, \frac{1}{n^{d/2}} \right)$ | 17 |
| $s = \frac{1+\epsilon}{n^{d/2}}$, $\epsilon > 0$ | $\Theta (n)$ if $a \geq 1$, $O (n)$ if $a \in (0, 1)$ | $MV_d \left( n, \frac{1+\epsilon}{n^{d/2}} \right)$ | 18 |

Here is an outline of the paper. In Section 2 we provide several combinatorical facts that will be used in the sequel. In Section 3 we investigate sensors on the line. We show that the expected $a$-total movement of algorithm $MV_1 \left( n, \frac{1}{n} \right)$ is

$$\frac{a!}{2^a \left( \frac{a}{2} + 1 \right)!} n^{1-\frac{a}{2}} + O \left( n^{-\frac{a}{2}} \right),$$

when $a$ is an even natural number. Section 5 explains threshold around the interference.
value $\frac{1}{n}$. In Section 4 we investigate threshold phenomena in the higher dimensions. In Appendix we prove that the expected $\alpha$-total movement of algorithm $MV_1(n, \frac{1}{n})$ is
\[
\frac{a!}{2^\frac{\alpha}{2} \Gamma\left(\frac{\alpha}{2} + 2\right) n^{\frac{\alpha}{2} - 1}} + O\left(\frac{1}{n^{\frac{\alpha}{2} - \frac{1}{2}}}\right),
\]
when $a$ is an odd natural number, where $\Gamma(z)$ is Euler Gamma function.

2. Basic facts and notations

In this section we recall some known facts about special functions and special numbers which will be useful in the analysis in the next sections. We also prove Lemma 3 and Lemma 4.

We will use the following notations for the rising factorial [6]
\[
n^k = \begin{cases} 1 & \text{for } k = 0 \\ n(n+1) \cdots (n+k-1) & \text{for } k \geq 1 \end{cases}
\]
Let $\left[ \begin{array}{c} n \\ k \end{array} \right]$ be the Stirling numbers of the first, which are defined for all integer numbers such that $0 \leq k \leq n$. The Stirling numbers of the first kind arises as coefficients of the rising factorial (see [6, Identity 6.11])
\[
x^m = \sum_{l_2} \left[ \begin{array}{c} m \\ l_2 \end{array} \right] x^{l_2}
\]
Let $\langle\langle \begin{array}{c} n \\ l \end{array} \rangle\rangle$ be the Eulerian numbers of the second kind, which are defined for all integer numbers such that $0 \leq k \leq n$. The following two identities for Eulerian numbers of the second kind are known (see Identities (6.42), and (6.44) in [6]):
\[
\sum_{l} \langle\langle \begin{array}{c} m \\ l \end{array} \rangle\rangle = \frac{(2m)!}{(m)!^2} \frac{1}{2^m}
\]
\[
\left[ \begin{array}{c} m \\ m-p \end{array} \right] = \sum_{l} \langle\langle \begin{array}{c} p \\ l \end{array} \rangle\rangle \left( \begin{array}{c} m+l \\ 2p \end{array} \right)
\]
Let us recall the definition of a finite difference of a function $f$
\[
\Delta f(x) = f(x+1) - f(x).
\]
Then, high-order differences are defined by iteration
\[
\Delta^a f(x) = \Delta \Delta^{a-1} f(x).
\]
It is easy to prove by induction the following formula (see also [6, Identity 5.40])
\[
\Delta^a f(x) = \sum_{j} \left( \begin{array}{c} a \\ j \end{array} \right) (-1)^{a-j} f(x+j)
\]
A crucial observation is the following identity, which will be useful in the asymptotic analysis of Algorithm 1 when $s = \frac{1}{n}$.
Lemma 3. Assume that $a$ is an even positive number. Then

$$
\sum_j \left( \frac{a}{j} \right) (-1)^j \left[ \frac{j}{j-l_1} \right] = \begin{cases} 0 & \text{if } 2l_1 < a \\ \frac{a!}{(\frac{a}{j})!} & \text{if } 2l_1 = a. \end{cases}
$$

Proof. Choosing $f(x) = \lfloor \frac{x}{x-l_1} \rfloor$ in (7) we see that

$$\Delta^a \left[ \frac{x}{x-l_1} \right] \bigg|_{x=0} = \sum_j \left( \frac{a}{j} \right) (-1)^{a-j} \left[ \frac{x+j}{x+j-l_1} \right] = \sum_j \left( \frac{a}{j} \right) (-1)^j \left[ \frac{j}{j-l_1} \right].$$

Applying equations (5), (6) and the following identity

$$\Delta^a \left( \frac{x+l}{2l_1} \right) \bigg|_{x=0} = \begin{cases} 0 & \text{if } 2l_1 < a \\ 1 & \text{if } 2l_1 = a. \end{cases}
$$

we easily derive

$$\Delta^a \left[ \frac{x}{x-l_1} \right] \bigg|_{x=0} = \sum l \left( \frac{a}{l} \right) \Delta^a \left( \frac{x+l}{2l_1} \right) \bigg|_{x=0} = \begin{cases} 0 & \text{if } 2l_1 < a \\ \frac{a!}{(\frac{a}{l})!} & \text{if } 2l_1 = a. \end{cases}
$$

This is enough to prove Lemma 3. \qed

The following lemma provides simple estimations for the general random variable, which are useful in the threshold tight bounds when interference value is greater or equal to $\frac{1}{n}$.

Lemma 4. Assume that $Z$ is positive, absolutely continuous random variable with $\mathbb{E}[Z] < \infty$. Let $f(t)$ be the probability density function of the random variable $Z$ and $D(q) = \int_0^\infty |t-q| f(t) dt$. Then

$$|\mathbb{E}[Z] - q| \leq D(q) \leq \mathbb{E}[Z] + q$$

(8)

Proof. First of all observe that

$$D(q) = \int_0^\infty (t-q) f(t) dt + 2 \int_0^q (q-t) f(t) dt$$

(9)

It is easy to see that

$$\int_0^\infty (t-q) f(t) dt = \mathbb{E}[Z] - q$$

(10)

and

$$0 \leq 2 \int_0^q (q-t) f(t) dt \leq 2q \int_0^q f(t) dt \leq 2q$$

(11)

Combining (9), (10) and (11) one gets

$$\mathbb{E}[Z] - q \leq D(q) \leq \mathbb{E}[Z] + q$$

(12)
Now observe that

\[ D(q) = \int_0^\infty (q - t)f(t)dt + 2\int_q^\infty (t - q)f(t)dt \]  

(13)

It is easy to see that

\[ \int_0^\infty (q - t)f(t)dt = q - E[Z] \]  

(14)

\[ 0 \leq 2\int_q^\infty (t - q)f(t)dt \]  

(15)

Combining (13), (14) and (15) one gets

\[ q - E[Z] \leq D(q) \]  

(16)

Finally putting together (12) and (16) we have

\[ \left| E[Z] - q \right| \leq D(q) \leq E[Z] + q. \]

This is enough to complete the proof of Lemma 4. \qed

We will also use Jensen’s inequality for expectations. If \( f \) is a convex function, then

\[ f\left( E[X] \right) \leq E\left[ f\left( X \right) \right] \]  

(17)

provided the expectations exists (see [16, Proposition 3.1.2]).

We will also use the following notation

\[ |x|^+ = \max\{x, 0\} \]  

(18)

for positive parts of \( x \in \mathbb{R} \).

The Euler Gamma function (see [15])

\[ \Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt \]  

(19)

is defined for \( z > 0 \). Notice that \( \Gamma(z) \) satisfies the following functional equations

\[ \Gamma(z + 1) = \Gamma(z)z \]  

(20)

Moreover, for \( n \) natural number we have

\[ \Gamma(n+1) = n! \]

We will also use the following forms of Stirling’s formula (see [3, page 54])

\[ \sqrt{2\pi e^{-N}e^{N+\frac{1}{2}}} < N! < \sqrt{2\pi e^{-N}e^{N+\frac{1}{2}}} \]  

(21)

Let \( f \) be non-negative integer. Then

\[ \sum_{i=2}^n (i-1)^f = \frac{1}{f+1}n^{f+1} + \sum_{l=0}^f c_l n^l \]  

(22)

where \( c_l \) are some constans independent on \( n \) (see [6, Formula (6.78)]).
3. Sensors on the Line

In this section we analyze interference problem when the sensors are placed on the half-infinite interval \([0, \infty)\).

3.1. Analysis of Algorithm 1

In this subsection we present algorithm \(MV_1(n, s)\) (see Algorithm 1). We show that the expected \(a\)-total movement of algorithm \(MV_1(n, s)\) is

\[
\frac{a!}{2^a (\frac{a}{2} + 1)!} n^{1 - \frac{a}{2}} + O \left( n^{-\frac{a}{2}} \right),
\]

when \(a\) is an even positive number.

Fix \(\epsilon > 0\) independent on \(n\). We prove that the expected \(a\)-total movement of algorithm \(MV_1(n, 1 + \frac{\epsilon}{n})\) is \(\Theta(n)\), when \(a\) is an even positive number or \(a = 1\).

Algorithm 1 \(MV_1(n, s)\) Moving sensors in the \([0, \infty)\); \(s > 0\).

Require: The initial location \(X_1, X_2, \ldots, X_n\) of the \(n\) sensors in the \([0, \infty)\)
Ensure: The final positions of the sensors such that each pair of consecutive sensors
is separated by the distance greater or equal than \(s\)

1: for \(i = 2\) to \(n\) do
2: move the sensor \(X_i\) at the position \(X_1 + (i - 1)s\)
3: end for

We prove the following theorem.

**Theorem 5.** Fix \(\epsilon \geq 0\) independent on \(n\). Let \(a\) be an even natural number. The expected \(a\)-total movement of algorithm \(MV_1(n, \frac{1 + \epsilon}{n})\) is respectively

\[
\begin{cases}
\frac{a!}{2^a (\frac{a}{2} + 1)!} n^{1 - \frac{a}{2}} + O \left( n^{-\frac{a}{2}} \right) & \text{when } \epsilon = 0, \\
\Theta(n) & \text{when } \epsilon > 0.
\end{cases}
\]

**Proof.** Let \(X_i\) be the arrival time of the \(i\)th event in a Poisson process with arrival rate \(n\). We know that the random variable \(X_i - X_1 = X_{i-1}\) obeys the Gamma distribution with density

\[
f_{i-1,n}(t) = \frac{n e^{-nt} (nt)^{i-2}}{(i-2)!}
\]

for \(i = 2, 3, \ldots, n\). (see equation (1) for \(j = 1, l = i - 1\)). Assume that, \(a\) is even natural number. Let \(D_i^{(a)}\) be the expected distance to the power \(a\) between \(X_i - X_1\) and the \(i\)-th sensor position, \(t_{i-1} = (1 + \epsilon)i - 1\), hence given by

\[
D_i^{(a)} = \int_0^\infty |t - t_{i-1}|^a f_{i-1,n}(t)dt = \int_0^\infty (t_{i-1} - t)^a f_{i-1,n}(t)dt.
\]
Observe that
\[ D_i^{(a)} = \sum_j \binom{a}{j} (1 + \epsilon) \frac{j^i - 1}{n} a^{-j} (-1)^j \int_0^\infty t^j f_{i-1,n}(t) dt. \]

Using the integral Identities (2) and (3) we see that
\[ D_i^{(a)} = \frac{1}{n^a} \sum_j \binom{a}{j} (1 + \epsilon)^a a^{-j} (-1)^j \frac{(i + j - 2)!}{(i - 2)!}. \]

Let \( j \in \{0, \ldots, a\} \). Applying Identity (4) we deduce that
\[ (i - 1)^a - j \frac{(i + j - 2)!}{(i - 2)!} = (i - 1)^a - j \frac{(i - 1)!}{(i - 2)!} = \sum_{\ell_1} \left[ \begin{array}{c} j \\ \ell_1 \end{array} \right] (i - 1)^a - \ell_1. \]

Hence
\[ D_i^{(a)} = \frac{1}{n^a} \sum_{\ell_1} \left[ \sum_j \binom{a}{j} (1 + \epsilon)^a a^{-j} (-1)^j \frac{(i - 1)^a - \ell_1}{(i - 1)^a - \ell_1} \right]. \]

Changing the summation we get
\[ D_i^{(a)} = \frac{1}{n^a} \sum_{\ell_1} (i - 1)^a - \ell_1 \left[ \sum_j \binom{a}{j} (1 + \epsilon)^a a^{-j} (-1)^j \frac{(i - 1)^a - \ell_1}{(i - 1)^a - \ell_1} \right]. \]

Now, we will estimate separately when \( \epsilon = 0 \) and when \( \epsilon > 0 \).

Case \( \epsilon = 0 \).

Applying Lemma 3 we get
\[ D_i^{(a)} = \frac{1}{n^a} \frac{a!}{(\frac{a}{2})!} \cdot (i - 1)^a + \frac{1}{n^a} \sum_{\ell_1 > a} C_{1,a - \ell_1} \cdot (i - 1)^a - \ell_1, \]
where \( C_{1,a - \ell_1} \) depends only on \( a \) and \( \ell_1 \). Applying Identity (22) we conclude that the expected sum of displacements to the power \( a \) of algorithm \( MV_1(n, \frac{1}{\epsilon}) \) is
\[ \sum_{i=2}^n D_i^{(a)} = \frac{a!}{2\pi (\frac{a}{2} + 1)!} n^{1 - \frac{a}{2}} + O(n^{-\frac{a}{2}}). \]
This is enough to prove the case when \( \epsilon = 0 \).

Case \( \epsilon > 0 \). Observe that \( \sum_j \binom{a}{j} (-1)^j (1 + \epsilon)^a \frac{[j]}{[j]} = \epsilon^a \). Therefore
\[ D_i^{(a)} = \epsilon^a \cdot (i - 1)^a + \frac{1}{n^a} \sum_{\ell_1 > 0} C_{2,a - \ell_1} \cdot (i - 1)^a - \ell_1, \]
where \( C_{2,a - \ell_1} \) depends only on \( a \) and \( \ell_1 \). Again, applying Identity (22) we conclude that the expected sum of displacements to the power \( a \) of algorithm \( MV_1(n, \frac{1 + \epsilon}{\epsilon}) \) is \( \Theta(n) \). This is enough to prove the case when \( \epsilon > 0 \) and completes the proof of Theorem 5. \( \square \)
We also prove the following tight bound for 1-total movement of Algorithm 1 when interference value is greater than \( \frac{1}{n} \).

**Theorem 6.** Fix \( \varepsilon > 0 \) independent on \( n \). The expected 1-total movement of algorithm \( MV_1(n, \frac{1}{n}) \) is respectively \( \Theta(n) \).

**Proof.** Let \( D_i^{(1)} \) be the expected distance between \( X_i - X_1 \) and the \( i \)-th sensor position, \( t_{i-1} = (1 + \varepsilon) \frac{i-1}{n} \) for \( i = 2, 3, \ldots, n \), hence given by

\[
D_i^{(1)} = \int_{0}^{\infty} |t - t_{i-1}| f_{i-1,n}(t) dt.
\]

Let us recall that \( \mathbb{E}[X_i - X_1] = \frac{i-1}{n} \) (see (3) for \( l = i - 1 \)). Applying Equation (8) in Lemma 4 for \( Z = X_i - X_1, q = t_{i-1} \) and \( \varepsilon > 0 \) we have

\[
\varepsilon \frac{i-1}{n} \leq D_i^{(1)} \leq (2 + \varepsilon) \frac{i-1}{n}.
\]

Since \( \sum_{i=2}^{n} (i - 1) = \frac{(n-1)n}{2} \), we have

\[
\sum_{i=2}^{n} D_i^{(1)} = \Theta(n)
\]  \hspace{1cm} (23)

This completes the proof of Theorem 6. \( \square \)

### 3.2. Expected \( a \)-total movement \( a \) for \( s = \frac{1}{n} \)

In this subsection we look at the expected \( a \)-total movement when the interference value \( s = \frac{1}{n} \). We prove the upper bound \( O(n^{1-\frac{a}{2}}) \) on the expected \( a \)-total movement, when \( a > 0 \) (see Theorem 7). Our Theorem 8 and Theorem 9 give the lower bound \( \Omega(n^{1-\frac{a}{2}}) \) on the expected \( a \)-total movement, when \( a \geq 1 \).

We begin with a theorem which indicates how to apply the results of Theorem 5 to the upper bound on the expected \( a \)-total movement, when \( a > 0 \).

**Theorem 7.** Let \( a > 0 \). The expected \( a \)-total movement of algorithm \( MV_1(n, \frac{1}{n}) \) is respectively \( O(n^{1-\frac{a}{2}}) \).

**Proof.** First we proof the upper bound. Assume that \( a > 0 \). Let \( D_i^{(a)} \) be the expected distance to the power \( a \) between \( X_i - X_1 \) and the \( i \)-th sensor position. Let \( b \) be the even natural number such that \( b - a > 0 \). Then we use discrete Hölder inequality with parameters \( \frac{b}{a} \) and \( \frac{b}{b-a} \) and get

\[
\sum_{i=2}^{n} D_i^{(a)} \leq \left( \sum_{i=2}^{n} \left( D_i^{(a)} \right)^{\frac{b}{a}} \right)^{\frac{a}{b}} \left( \sum_{i=2}^{n} 1 \right)^{-\frac{b-a}{b}} = \left( \sum_{i=2}^{n} \left( D_i^{(a)} \right)^{\frac{b}{a}} \right)^{\frac{a}{b}} (n-1)^{-\frac{b-a}{b}}
\]  \hspace{1cm} (24)
Next we use Jensen’s inequality (see (17)) for \( f(x) = x^\frac{1}{2} \) and \( E[X] = D_i^{(a)} \) and get
\[
\left(D_i^{(a)}\right)^{\frac{1}{b}} \leq D_i^{(b)}.
\] (25)

Combining together (24), (25) and Theorem 5 we deduce that
\[
\sum_{i=2}^{n} D_i^{(a)} \leq \left(\Theta \left(\frac{1}{n^{2-\epsilon}}\right)\right)^{\frac{n}{2}} (n-1)^{\frac{1}{2}} = \Theta \left(n^{1-\frac{1}{2}}\right).
\]

This is enough to prove the upper bound and finishes the proof of Theorem 7.

We are now prove the desired lower bound for expected 1–total movement.

**Theorem 8.** Any sensor’s displacement algorithm which has interference value \( s = \frac{1}{n} \) requires expected 1-total movement of at least \( \Omega \left(\sqrt{n}\right) \).

**Proof.** Before providing the proof of the theorem we make two important observations.

Let \( X_1 < X_2 < \cdots < X_n \) be the initial positions of the sensors. Recall that by the monotonicity lemma no sensor \( X_i \) is ever placed before sensor \( X_j \), for all \( i < j \).

Observe that, there are two classes of algorithms with interference value \( s = \frac{1}{n} \). The first one moves the sensors to the new anchor locations (see Case 1 and Case 3). The second moves the first sensor to the random position. We assume that the final position of the first sensor is the positive, absolutely continuous random variable \( Z \) with \( E[Z] < \infty \) (see Case 2 and Case 4).

We are now ready to prove the theorem. Let \( X_i \) be the arrival times of the \( i \)-th event in Poisson process with arrival rate. Let \( a_i = \frac{1}{n} - \frac{1}{2n}, \) for \( i = 1, 2, \ldots, n \) be the anchor points. The following results was proved in [2, Lemma 1]
\[
\sum_{i=1}^{n} E[|X_i - a_i|] = C_1 \left(\sqrt{n}\right) + o \left(\sqrt{n}\right),
\] (26)
where \( C_1 \) is some constants independent on \( n \). There are four cases to consider.

**Case 1.** The algorithm moves the \( i \)-th sensor to the position \( b_i = \frac{1}{n}(i - 1) + b_1 \), for \( i = 1, 2, \ldots, n \) and \( b_1 > \frac{1}{2} C_1 \).

Combining together Equation (8) in Lemma 4 for \( Z = X_i, q = b_i \), Equation (3) for \( l = i \) and the triangle inequality we have
\[
\sum_{i=1}^{n} E[|X_i - b_i|] \geq \sum_{i=1}^{n} \left|b_i - \frac{i}{n}\right| = \sum_{i=1}^{n} \left|b_1 - \frac{1}{n}\right| \geq \sum_{i=1}^{n} \left|\frac{1}{2} \sqrt{n} - \frac{1}{n}\right| = \Theta \left(\sqrt{n}\right).
\]

This is enough to prove the first case.

**Case 2.** The algorithm moves the sensor \( X_i \) to the position \( Z + b_i \), where \( b_i = \frac{1}{n}(i - 1) \), for \( i = 1, 2, \ldots, n \) and \( E[Z] > \frac{1}{2} C_1 \).

The proof is analogous to the proof of Case 1. Applying Equation (8) in Lemma 4 for \( Z := X_i - Z, q = b_i \), Equation (3) for \( l = i \) and the triangle inequality we get
\[
\sum_{i=1}^{n} E[|X_i - Z - b_i|] \geq \sum_{i=1}^{n} \left|E[Z] - \frac{1}{n}\right| \geq \sum_{i=1}^{n} \left|\frac{1}{2} \sqrt{n} - \frac{1}{n}\right| = \Theta \left(\sqrt{n}\right).
\]
This is sufficient to prove the second case.

*Case 3.* The algorithm moves the *i*-th sensor to the position \( b_i = \frac{1}{n}(i - 1) + b_1, \) for \( i = 1, 2, \ldots, n \) and \( b_1 \leq \frac{1}{\sqrt{2n}} \).

Let us recall that \( a_i = \frac{1}{n} - \frac{1}{2n} \), for \( i = 1, 2, \ldots, n \). Using the triangle inequality \( |X_i - a_i| \leq |X_i - b_i| + |b_i - a_i| \) we get

\[
\sum_{i=1}^{n} E[|X_i - b_i|] \geq \sum_{i=1}^{n} E[|X_i - a_i|] - \sum_{i=1}^{n} \left| b_i - \frac{1}{2n} \right|.
\]

Putting together Equation (26), assumption \( b_1 \leq \frac{1}{\sqrt{2n}} \) and the triangle inequality \( |b_1 - \frac{1}{2n}| \leq b_1 + \frac{1}{2n} \) we have

\[
\sum_{i=1}^{n} E[|X_i - b_i|] \geq C_1 \left( \sqrt{n} \right) + o(\sqrt{n}) - nb_1 - \frac{1}{2} = \Theta \left( \sqrt{n} \right).
\]

This is sufficient to prove the third case.

*Case 4.* The algorithm moves the sensor \( X_i \) to the position \( Z + b_i \), where \( b_i = \frac{1}{n}(i - 1) \), for \( i = 1, 2, \ldots, n \) and \( E[Z] \leq \frac{1}{\sqrt{2n}} \).

We proceed analogously as in the lower bound treatment from the proof of Case 3. Notice that \( a_i = \frac{1}{n} - \frac{1}{2n} \), for \( i = 1, 2, \ldots, n \). Using the triangle inequality \( |X_i - a_i| \leq |X_i - (Z + b_i)| + |(Z + b_i) - a_i| \) we get

\[
\sum_{i=1}^{n} E[|X_i - (Z + b_i)|] \geq \sum_{i=1}^{n} E[|X_i - a_i|] - \sum_{i=1}^{n} E \left| Z - \frac{1}{2n} \right|.
\]

Combining together Equation (26), assumption \( E[Z] \leq \frac{1}{\sqrt{2n}} \) and the triangle inequality \( E \left[ |Z - \frac{1}{2n}| \right] \leq E[Z] + \frac{1}{2n} \) we have

\[
\sum_{i=1}^{n} E[|X_i - (Z + b_i)|] \in \Omega(n).
\]

This is enough to prove the fourth case and sufficient to complete the proof of Theorem 8. \( \square \)

We now apply Theorem 8 in order to derive the lower bound on the expected \( a \-\text{total movement when } a > 1. \)

**Theorem 9.** Let \( a > 1. \) Then any sensor's displacement algorithm which has interference value \( s = \frac{1}{n} \) requires expected \( a\-\text{total movement of at least } \Omega \left( n^{1-\frac{a}{2}} \right). \)

**Proof.** Assume that \( a > 1. \) Let \( E_i^{(a)} \) be the expected distance to the power \( a \) of \( i \)-th sensor for \( i = 1, 2, \ldots, n. \) Then we use discrete Hölder inequality with parameters \( a \)
and \( a \) and get
\[
\sum_{i=1}^{n} E_i^{(1)} \leq \left( \frac{1}{n} \sum_{i=1}^{n} \left( E_i^{(1)} \right)^a \right)^{\frac{1}{a}} \left( \frac{1}{n} \sum_{i=1}^{n} \right)^{\frac{a-1}{a}}
\]
\[
= \left( \frac{1}{n} \sum_{i=1}^{n} \left( E_i^{(1)} \right)^a \right)^{\frac{1}{a}} \left( \frac{1}{n} \right)^{\frac{a-1}{a}}
\]
(27)

Next we use Jensen’s inequality (see (17)) for \( f(x) = x^a \) and get
\[
\left( E_i^{(1)} \right)^a \leq E_i^{(a)}
\]
(28)

Combining together (27), (28) and Theorem 8 we deduce that
\[
\sum_{i=1}^{n} E_i^{(a)} \geq \left( \frac{1}{n} \sum_{i=1}^{n} \right)^a \frac{1}{n} \frac{a}{a+1} = \left( \Omega \left( \sqrt{n} \right) \right)^a \frac{1}{n} \frac{a}{a+1} = \Omega \left( n^{1-\frac{a}{2}} \right).
\]

This is enough to prove the lower bound and completes the proof of Theorem 9.

3.3. Expected \( a \)-total movement for \( s > \frac{1}{n} \)

In this subsection we study the expected \( a \)-total movement when the interference value \( s \) is greater than \( \frac{1}{n} \). We give the upper bound \( O(n) \) on the expected \( a \)-total movement, when \( a > 0 \) (see Theorem 10) and the lower bound \( \Omega(n) \) on the expected \( a \)-total movement, when \( a \geq 1 \) (see Theorem 11 and Theorem 12).

We begin with a theorem which indicates how to apply the results of Theorem 5 to the upper bound on the expected \( a \)-total movement, when \( a > 0 \).

Theorem 10. Fix \( \varepsilon > 0 \) independent on \( n \). Let \( a > 0 \). The expected \( a \)-total movement of algorithm \( MV_1 \left( n, \frac{\varepsilon n}{a} \right) \) is in \( O(n) \).

Proof. Let \( D_i^{(a)} \) be the expected distance to the power \( a \) between \( X_i - X_1 \) and the \( i \)th sensor position. Let \( b \) be the even natural number such that \( b - a > 0 \). Then we proceed as in the upper bound treatment from the proof of Theorem 7 and get
\[
\sum_{i=2}^{n} D_i^{(a)} \leq \left( \frac{1}{n} \sum_{i=2}^{n} \right)^{\frac{1}{a}} \left( \frac{1}{n} \sum_{i=2}^{n} \right)^{\frac{a-1}{a}} (n - 1)^{\frac{b-a}{a}}
\]
(29)

\[
\left( D_i^{(a)} \right)^{\frac{1}{a}} \leq D_i^{(b)}
\]
(30)

Combining together (29), (30) and Theorem 5 we deduce that
\[
\sum_{i=2}^{n} D_i^{(a)} \leq \left( \Theta(n) \right)^{\frac{1}{a}} (n - 1)^{\frac{b-a}{a}} = \Theta(n).
\]

This is sufficient to complete the proof of Theorem 10.
We can now prove the desired lower bound for expected 1-total movement.

**Theorem 11.** Fix $\epsilon > 0$ independent on $n$. Then any sensor’s displacement algorithm which has interference value $s = \frac{1 + \epsilon}{n}$ requires expected 1-total movement of at least $\Omega(n)$.

**Proof.** The proof of the theorem is analogous to the proof of Theorem 8. Fix $\epsilon > 0$ independent on $n$. Let $X_i$ be the arrival times of the $i$-th event in Poisson process with arrival rate. There are two cases to consider.

**Case 1.** The algorithm moves the $i$-th sensor to the position $b_i$ and $b_i = \frac{1 + \epsilon}{n}(i - 1) + b_1$ for $i = 1, 2, \ldots, n$.

It is sufficient to show that
\[
\sum_{i=1}^{n} E[|X_i - b_i|] \in \Omega(n).
\]

Applying Equation (8) in Lemma 4 for $Z = X_i$, $q = b_i$ and Equation (3) for $l = i$ we have
\[
\sum_{i=1}^{n} E[|X_i - b_i|] \geq \sum_{i=1}^{n} \left| b_i - \frac{i}{n} \right| = \sum_{i=1}^{n} \left| \frac{\epsilon}{n}i + b_1 - \frac{1 + \epsilon}{n} \right| 
\]
\[
\geq \sum_{i \geq \lceil \frac{1}{\epsilon} \rceil} \left( \frac{\epsilon}{n}i + b_1 - \frac{1 + \epsilon}{n} \right) = \Theta(n).
\]

This is enough to prove the first case.

**Case 2.** The algorithm moves the sensor $X_i$ to the position $Z + \frac{1 + \epsilon}{n}(i - 1)$ for $i = 1, 2, \ldots, n$, where $Z$ is the positive, absolutely continuous random variable with $E[Z] < \infty$.

The proof is analogous to the proof of Case 1. Using Equation (8) in Lemma 4 for $Z := X_i - Z$, $q = \frac{1 + \epsilon}{n}(i - 1)$ and Equation (3) for $l = i$ we get
\[
\sum_{i=1}^{n} E\left[\left|X_i - \left(Z + \frac{1 + \epsilon}{n}(i - 1)\right)\right|\right] \geq \sum_{i=1}^{n} \left| \frac{\epsilon}{n}i + E[Z] - \frac{1 + \epsilon}{n} \right|
\]
\[
\geq \sum_{i \geq \lceil \frac{1}{\epsilon} \rceil} \left( \frac{\epsilon}{n}i + E[Z] - \frac{1 + \epsilon}{n} \right) = \Theta(n).
\]

This is enough to prove the second case and completes the proof of Theorem 11. \qed

We now apply Theorem 11 in order to derive the lower bound on the expected $a$-total movement when $a > 1$.

**Theorem 12.** Fix $\epsilon > 0$ independent on $n$. Let $a > 1$. Then any sensor’s displacement algorithm which has interference value $s = \frac{1 + \epsilon}{n}$ requires expected $a$-total movement of at least $\Omega(n)$. 

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Proof. Assume that \( a > 1 \). Let \( E_i^{(a)} \) be the expected distance to the power \( a \) of \( i \)-th sensor for \( i = 1, 2, \ldots, n \). As in the proof of Theorem 9 we get two inequalities:

\[
\sum_{i=1}^{n} E_i^{(1)} \leq \left( \sum_{i=1}^{n} E_i^{(1)} \right)^{\frac{1}{a}} n^{\frac{a-1}{a}} \tag{31}
\]

\[
\left( E_i^{(1)} \right)^{\frac{1}{a}} \leq E_i^{(a)} \tag{32}
\]

Combining together (31), (32) and Theorem 11 we deduce that

\[
\sum_{i=1}^{n} E_i^{(a)} \geq \left( \sum_{i=1}^{n} E_i^{(1)} \right)^{a} n^{-a+1} = \Omega(n^{1-a}).
\]

This is enough to prove the lower bound and completes the proof of Theorem 12.

3.4. Expected \( a \)-total movement for \( s < \frac{1}{n} \)

In this subsection we present algorithm \( IN_{1}(n, s) \) (see Algorithm 2) for interference problem. Fix \( \epsilon > 0 \) independent on \( n \). Let \( a > 0 \). We show that expected \( a \)-total movement of algorithm \( IN_{1}(n, \frac{1-\epsilon}{n}) \) is in \( O\left(n^{1-a}\right) \).

We begin with the following lemma which will be helpful in the proof of Theorem 14.

**Algorithm 2** \( IN_{1}(n, s) \) Moving sensors in the \([0, \infty)\); \( s > 0 \).

**Require:** The initial location \( X_1 \leq X_2 \leq \cdots \leq X_n \) of the \( n \) sensors in the \([0, \infty)\).

**Ensure:** The final positions of the sensors such that \( \forall_{i=2,3,\ldots,n} X_i - X_{i-1} \geq s \) (so as that the distance between consecutive sensors is greater or equal to \( s \))

1: for \( i = 2 \) to \( n \) do
2: if \( X_i - X_{i-1} < s \) then
3: move right-to-left sensor \( X_i \) at the new position \( s + X_{i-1} \).
4: else
5: do nothing
6: end if
7: end for

**Lemma 13.** Fix \( \epsilon > 0 \) independent on \( n \). Let \( a > 0 \) and let \( s = \frac{1-\epsilon}{n} \). Assume that random variable \( X_1 \) obeys the Gamma distribution with parameters \( l, n \). Then

\[
\sum_{i=1}^{n} \frac{1}{n} \mathbb{E} \left[ (|sl - X_i|^+)^a \right] = O(n^{1-a}).
\]

**Proof.** First of all observe that

\[
\mathbb{E} \left[ (|sl - X_i|^+)^a \right] = \int_{0}^{sl} (sl - t)^a f_{1,n}(t) dt \leq (sl)^a \int_{0}^{sl} f_{1,n}(t) dt \tag{33}
\]
where \( f_{l,n}(t) = ne^{-nt/(n)!} \). The substitution \( tn = x \) yields
\[
\int_0^{sn} f_{l,n}(t) dt = \int_0^{sn} e^{-x} \frac{x^{l-1}}{(l-1)!} dx. \tag{34}
\]

Observe that, the function \( g(x) = e^{-x}x^{l-1} \) is monotonically increasing over the interval \([0, l-1]\) and monotonically decreasing over the interval \([l-1, \infty)\). Therefore, we easily derive the following inequality
\[
\int_0^{sn} e^{-x} \frac{x^{l-1}}{(l-1)!} dx \leq \max_{x \in [0,sn]} e^{-x} \frac{x^{l-1}}{(l-1)!} = \begin{cases} \left( \frac{sn}{e^{ns}} \right)^l \frac{l^l}{(l-1)!} & \text{if } sn \leq l-1, \\ \frac{sn}{e^{ns}} \frac{l^{l-1}}{(l-1)!} & \text{if } sn > l-1 \end{cases} \tag{35}
\]

Applying Stirling’s formula \((21)\) for \( N = l \) we get
\[
\frac{l!}{(l-1)!} \leq e^{l/2}. \tag{36}
\]

Using \((36)\) with assumption \( s = \frac{1-\varepsilon}{n} \) in Inequality \((35)\) we imply
\[
\int_0^{sn} e^{-x} \frac{x^{l-1}}{(l-1)!} dx \leq \begin{cases} \left( \frac{sn}{e^{ns}} \right)^l \frac{l^l}{(l-1)!} & \text{if } l \geq \frac{1}{\varepsilon}, \\ \frac{sn}{e^{ns}} \frac{l^{l-1}}{(l-1)!} & \text{if } l < \frac{1}{\varepsilon} \end{cases} \tag{37}
\]

Putting together \((33)\), \((34)\) and \((37)\) we have
\[
\sum_{l=1}^{n} \frac{n}{l} E \left[ (|sl - X_l|)^a \right] \leq n^{1-a} \left( (ns)^a \sum_{l \geq \frac{1}{\varepsilon}} \left( \frac{ns}{e^{ns}} \right)^l l^{a-\frac{1}{2}} + (ns)^{a+1} \sum_{l < \frac{1}{\varepsilon}} \frac{l^a(l-1)^{l-1}}{e^{(l-1)!}} \right) \tag{38}
\]

Combining assumption \( sn < 1 \) with the elementary inequality \( xe < e^x \) when \( x < 1 \) we deduce that \( \frac{ns}{e^{ns}} < 1 \). Hence
\[
(ns)^a \sum_{l \geq \frac{1}{\varepsilon}} \left( \frac{ns}{e^{ns}} \right)^l l^{a-\frac{1}{2}} + (ns)^{a+1} \sum_{l < \frac{1}{\varepsilon}} \frac{l^a(l-1)^{l-1}}{e^{(l-1)!}} = O(1) \tag{39}
\]

Finally, putting together \((38)\) and \((39)\) we get
\[
\sum_{l=1}^{n} \frac{n}{l} E \left[ (|sl - X_l|)^a \right] = O \left( n^{1-a} \right).
\]

This is enough to prove Lemma 13. □

We can now prove the desired result.
Theorem 14. Fix $\epsilon > 0$ independent on $n$. Let $a > 0$. The expected $a$-total movement of algorithm $IN_1(n, \frac{1+\epsilon}{n})$ is in $O\left(n^{1-a}\right)$.

Proof. Let $s = \frac{1-\epsilon}{n}$. Observe that Algorithm 2 is the sequence of the two phases $P$ and $Q$. During phase $P$ Algorithm 2 moves the sensors $X_{i+1}, X_{i+2}, \ldots X_{i+k}$ at the new position. Then in phase $Q$ Algorithm 2 leaves the sensors $X_{i+k+1}, X_{i+k+2}, \ldots X_{i+k+p}$ at the same position. Notice that the sensors $X_{i+1}, X_{i+2}, \ldots X_{i+k}$ have to move cumulatively. Let $T_{P,Q}$ be the cost of movement to the power $a$ in the phase $P$ and $Q$ of Algorithm 2. Observe that

$$T_{P,Q} = \sum_{l=1}^{k} (|sl + X_i - X_{i+l}|^a).$$

Using Identity $X_{i+l} - X_i = X_l$ (see (1)) we get

$$T_{P,Q} = \sum_{l=1}^{k} (|sl - X_l|^a). \quad (40)$$

Let $T(a, s)$ be the expected $a$-total movement of Algorithm 2. Applying (40) we have the following upper bound

$$T(a, s) \leq \max_{0 \leq k_1 + k_2 + \ldots + k_m \leq n} \sum_{j=1}^{m} \sum_{l=1}^{k_j} E\left([|sl - X_l|^a]\right). \quad (41)$$

Observe that the expected cost $E\left([|sl - X_l|^a]\right)$ can appear in the double sum (41) at most $\frac{n}{m}$ times. Therefore

$$T(a, s) \leq \sum_{l=1}^{n} \frac{n}{l} E\left([|sl - X_l|^a]\right).$$

Finally, using Lemma 13 for $s = \frac{1-\epsilon}{n}$ we conclude $T(a, s) = O\left(n^{1-a}\right)$. This is enough to prove Theorem 14. $\square$

4. Sensors in the Higher Dimension.

In this section we analyze interference problem when the sensors are placed on the hyperoctant $[0, \infty)^d$.

4.1. Preliminaries and notation

Let $d$ be a natural number greater than 1. We define below the concept $d$-dimensional $a$-total movement which refers to a movement of sensors only according to the axes.

Definition 15 ($d$-dimensional $a$-total movement). Let $a > 0$ be a constant. Consider a sensor $S_i$ located in the position $(y_1(i), y_2(i), \ldots, y_d(i))$, where $y_1(i), y_2(i), \ldots, y_d(i) \in \mathbb{R}_+$. We move the sensor $S_i$ to the position $(y_1(i) + m_1(i), y_2(i) + m_2(i), \ldots, y_d(i) + m_d(i))$. Then the $d$-dimensional $a$-total movement is defined as the double sum $M_{d,a} := \sum_{i=1}^{n} \sum_{j=1}^{d} |m_j(i)|^a$. 

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The following corollary shows the asymptotic equivalence of $d$-dimensional $a$-total movement and Euclidean expected total movement to power $a$, when $d \in \mathbb{N} \setminus \{1\}$ and $a \in \mathbb{N}_+$.

**Corollary 16.** Fix $a \in \mathbb{N}_+$. Assume that $d \in \mathbb{N} \setminus \{1\}$. Let $T_{\text{axes}}(n)$ be the expected $d$-dimensional $a$-total movement of $n$ sensors according to the axes and let $T_{\text{Euclidean}}(n)$ be the Euclidean expected total movement to power $a$ of $n$ sensors. Then $T_{\text{axes}}(n) = \Theta(T_{\text{Euclidean}}(n))$.

**Proof.** Observe that if the sensor $S_i$ moves from position $(y_1(i), y_2(i), \ldots, y_d(i))$ to the position $(y_1(i) + m_1(i), y_2(i) + m_2(i), \ldots, y_d(i) + m_d(i))$ then the movement to the power $a$ only according to the axes is $\sum_{j=1}^d |m_j(i)|^a$ and the Euclidean distance to the position $a$ is $\left(\sqrt{\sum_{j=1}^d m_j(i)^2}\right)^a$.

Using the inequality (for $a \in \mathbb{N}_+$)

$$\left(\sqrt{\sum_{j=1}^d m_j^2(i)}\right)^a \leq \left(\sum_{j=1}^d |m_j(i)|\right)^a \leq d^{a-1} \sum_{j=1}^d |m_j(i)|^a$$

we get

$$\sum_{i=1}^n \left(\sqrt{\sum_{j=1}^d m_j^2(i)}\right)^a \leq d^{a-1} \sum_{i=1}^n \sum_{j=1}^d |m_j(i)|^a \quad (42)$$

Applying the inequality (for $a \in \mathbb{N}_+$)

$$d^{-\frac{a}{2}} \sum_{j=1}^d |m_j(i)|^a \leq \left(\frac{\sum_{j=1}^d |m_j(i)|}{\sqrt{d}}\right)^a \leq \left(\sqrt{\sum_{j=1}^d m_j^2(i)}\right)^a$$

we have

$$d^{-\frac{a}{2}} \sum_{i=1}^n \sum_{j=1}^d |m_j(i)|^a \geq \sum_{i=1}^n \left(\sqrt{\sum_{j=1}^d m_j^2(i)}\right)^a \quad (43)$$

Putting together (42) and (43) we conclude that $T_{\text{axes}}(n) = \Theta(T_{\text{Euclidean}}(n))$. This is enough to prove Corollary 16. \[\square\]

### 4.2. Deriving a threshold

We consider $n$ sensors are placed in the hyperoctant $[0, \infty)^d$ according to $d$ identical and independent Poisson processes $X_i^{(1)}, X_i^{(2)}, \ldots, X_i^{(d)}$, for $i = 1, 2, \ldots, n^{1/d}$ each with arrival rate $n^{1/d}$. The position of a sensor in the $\mathbb{R}_+$ is determined by the $d$ coordinates $(X_{i_1}^{(1)}, X_{i_2}^{(2)}, \ldots, X_{i_d}^{(d)})$, where $1 \leq i_1, i_2, \ldots, i_d \leq n^{1/d}$.

To avoid interference sensors have to move in the hyperoctant $[0, \infty)^d$ so that the Euclidean distance of each pair of sensors is greater than interference value $s$.

We now embark to extend the results from Section 3 to the high dimensions. We can prove the following sequences of Theorem.

The interference value $s = \frac{1}{n^{1/d}}$ as this affects the expected $a$-total movement we clarify in the next theorem.
Theorem 17. Let $a > 0$ be a constant. Fix $d \in \mathbb{N} \setminus \{1\}$. Assume that $n$ sensors are placed in the $[0, \infty)^d$ according to $d$ independent identical Poisson processes, each with arrival rate $n^{1/d}$. If the interference value $s = \frac{1}{n^{1/d}}$ then the expected $d$-dimensional $a$-total movement is in

$$\begin{cases} \Theta \left( n^{1\frac{a}{d}} \right) & \text{if } a \geq 1, \\ O \left( n^{1\frac{a}{d}} \right) & \text{if } a \in (0, 1). \end{cases}$$

Proof. First of all, we discuss the proof of the upper bound. By Theorem 7 applied to $n := n^{1/d}$ and for $n^{(d-1)/d}$ columns and $n^{(d-1)/d}$ rows we have that the expected $a$-total movement of algorithm $MV_d(n, \frac{1}{n^{1/d}})$ is

$$2n^{(d-1)/d}O \left( \left( n^{1/d} \right)^{1\frac{a}{d}} \right) = O \left( n^{1\frac{a}{d}} \right), \text{ when } a > 0.$$  

Next we prove the lower bound. Since the sensors move only according to the axes to attain the interference value $s = \frac{1}{n}$ in the $[0, \infty)^d$ the sensors have to attain the interference value $s = \frac{1}{n}$ on each column and each row. By Theorem 8 and Theorem 9 applied to $n := n^{1/d}$ and for $n^{(d-1)/d}$ columns and $n^{(d-1)/d}$ rows we have the following lower bound

$$2n^{(d-1)/d}O \left( \left( n^{1/d} \right)^{1\frac{a}{d}} \right) = \Omega \left( n^{1\frac{a}{d}} \right), \text{ when } a \geq 1.$$  

This is sufficient to complete the proof of Theorem 17. \qed

We now analyze the interference value $s > \frac{1}{n^{1/d}}$. 

---

**Algorithm 3** $MV_d(n, s)$ Moving sensors in the $[0, \infty)^d$, $d \geq 2$, $s > 0$.

**Require:** The initial location $(X^{(1)}_{i_1}, X^{(2)}_{i_2}, \ldots, X^{(d)}_{i_d})$ of the $n$ sensors in the $[0, \infty)^d$, $1 \leq i_1, i_2, \ldots, i_d \leq n^{1/d}$

**Ensure:** The final positions of the sensors such that each pair of the sensors is separated by the distance greater or equal than $s$

1: For each column and row in the $[0, \infty)^d$ apply algorithm $IN_d(n, 1/s)$

**Algorithm 4** $IN_d(n, s)$ Moving sensors in the $[0, \infty)^d$, $d \geq 2$, $s > 0$.

**Require:** The initial location $(X^{(1)}_{i_1}, X^{(2)}_{i_2}, \ldots, X^{(d)}_{i_d})$ of the $n$ sensors in the $[0, \infty)^d$, $1 \leq i_1, i_2, \ldots, i_d \leq n^{1/d}$

**Ensure:** The final positions of the sensors such that each pair of the sensors is separated by the distance greater or equal than $s$

1: For each column and row in the $[0, \infty)^d$ apply algorithm $IN_1(n^{1/d}, s)$. 

---

We now analyze the interference value $s > \frac{1}{n^{1/d}}$. 

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Theorem 18. Let \( a > 0 \) be a constant. Fix \( d \in \mathbb{N} \setminus \{1\} \) and \( \epsilon > 0 \) independent on \( n \). Assume that \( n \) sensors are placed in the \([0, \infty)^d\) according to \( d \) independent identical Poisson processes, each with arrival rate \( n^{1/d} \). If the interference value \( s = \frac{1 + \epsilon}{n^{1/d}} \) then the expected \( d \)-dimensional \( a \)-total movement is in

\[
\begin{cases} 
\Theta(n) & \text{if } a \geq 1, \\
O(n) & \text{if } a \in (0, 1).
\end{cases}
\]

Proof. Fix \( \epsilon > 0 \) independent on \( n \). The proof is analogous to the proof of Theorem 17. By Theorem 10 applied to \( n := n^{1/d} \) and for \( n^{(d-1)/d} \) columns and \( n^{(d-1)/d} \) rows and have the expected \( a \)-total movement of algorithm \( MV_d \left( n, \frac{1 + \epsilon}{n^{1/d}} \right) \) is

\[ 2n^{(d-1)/d}O \left( n^{1/d} \right) = O(n), \text{ when } a > 0. \]

This completes the prove of upper bound.

By Theorem 11 and Theorem 12 applied to \( n := n^{1/d} \) and for \( n^{(d-1)/d} \) columns and \( n^{(d-1)/d} \) rows we have the following lower bound

\[ 2n^{(d-1)/d} \Omega \left( n^{1/d} \right) = \Omega(n), \text{ when } a \geq 1. \]

This is enough to prove Theorem 18.

Theorem 19. Let \( a > 0 \) be a constant. Fix \( d \in \mathbb{N} \setminus \{1\} \) and \( \epsilon > 0 \) independent on \( n \). Assume that \( n \) sensors are placed in the \([0, \infty)^d\) according to \( d \) independent identical Poisson processes, each with arrival rate \( n^{1/d} \). If the interference value \( s = \frac{1 - \epsilon}{n^{1/d}} \) then the expected \( d \)-dimensional \( a \)-total movement is in \( O \left( n^{1 - \frac{a}{d}} \right) \).

Proof. Fix \( \epsilon < 0 \) independent on \( n \). By Theorem 13 applied to \( n := n^{1/d} \) and for \( n^{(d-1)/d} \) columns and \( n^{(d-1)/d} \) rows we have the following upper bound

\[ 2n^{(d-1)/d}O \left( n^{1/d} \right)^{1-a} = O \left( n^{1 - \frac{a}{d}} \right), \text{ when } a > 0 \]

which proves the theorem.

5. Conclusion

In this paper we studied tradeoffs between interference value \( s \) and the expected minimum \( a \)-total movement of \( n \) sensors. We obtained bounds on the movement depending on the interference which explained the threshold phenomena.
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6. Appendix

In this Appendix we give an exact asymptotic on the expected $a$-total movement of algorithm $MV_1(n, \frac{1}{n})$, when $a$ is an odd natural number. We prove the following theorem.

**Theorem 20.** Let $a$ be an odd natural number. The expected $a$-total movement of algorithm $MV_1(n, \frac{1}{n})$ is respectively

$$\frac{a!}{2^a \Gamma \left(\frac{a}{2} + 2\right)} \frac{1}{n^{\frac{a}{2}-1}} + O\left(\frac{1}{n^{\frac{a}{2} - \frac{1}{2}}}\right).$$

Theorem 20 together with Theorem 5 give us full asymptotic results on the expected $a$-total movement of algorithm $MV_1(n, \frac{1}{n})$, when $a$ is an odd natural number.

It is worthwhile to mention that, even though there is the simple asymptotic formula on the expected $a$-total movement of algorithm $MV_1(n, \frac{1}{n})$ when $a$ is an odd natural number, the analysis of asymptotic is not combinatorically trivial.

6.1. Preliminaries

We recall some known facts about special functions and special numbers which will be used in the proof of Theorem 20.

We will use the following notation for the falling factorial [6]

$$n^k = \begin{cases} 1 & \text{for } k = 0 \\ n(n-1)\ldots(n-(k-1)) & \text{for } k \geq 1 \end{cases}.$$

The following equation for Stirling numbers of the first kind are well known (see [6, Identity 6.13])

$$x^m = \sum_l \binom{m}{l} (-1)^{m-l} x^l$$

Using integration by parts we can easily derive

$$\int_0^c t^j f_{l,n}(t) dt = \frac{j + l - 1}{n} \int_0^c t^{j-1} f_{l,n}(t) dt - e^{-nc} (\frac{nc}{l-1})^{l-1}$$

where $l, n, j$ are positive integers, $c$ is positive real number and $f_{l,n}(t) = ne^{-nt}(\frac{nt}{l-1})^{l-1}$.

The following useful identity involving alternating sum and Gamma function can be checked by any mathematical software that performs symbolic calculation or can be calculated using the Rice method (see [11] for details).

Assume that $d$ be non-negative integer. Then

$$\sum_{l=0}^d \binom{d}{l} (-1)^l \left(\frac{1}{2}\right)^l \frac{1}{d+1+l} = \frac{d!\sqrt{\pi}}{2^{1+d}\Gamma \left( \frac{d}{2} \right)}$$

We will also use the following forms of Stirling’s formula (see [6, Formula 9.40])

$$m! = \sqrt{2\pi} m^{m+\frac{1}{2}} e^{-m} \left(1 + O\left(\frac{1}{m}\right)\right)$$
6.2. Deriving the exact asymptotic

This section is devoted to the proof of Theorem 20. We begin with the following
sequences of lemmas which will be helpful in the proof of Theorem 20.

First we prove the following lemma.

Lemma 21. Assume that \(a\) is an odd positive number. Then

\[
\sum_j \left( \begin{array}{c} a \\ j \end{array} \right) (-1)^j \left[ \begin{array}{c} j \\ j - l_1 \end{array} \right] = 0, \quad \text{when} \quad 2l_1 < a + 1.
\]

Proof. The proof is analogous to the proof of Lemma 3. Notice that

\[
\Delta^a \left[ \begin{array}{c} x \\ x - l_1 \end{array} \right] \bigg|_{x=0} = \sum_j \left( \begin{array}{c} a \\ j \end{array} \right) (-1)^{a-j} \left[ \begin{array}{c} x + j \\ x + j - l_1 \end{array} \right] \bigg|_{x=0} = -\sum_j \left( \begin{array}{c} a \\ j \end{array} \right) (-1)^j \left[ \begin{array}{c} j \\ j - l_1 \end{array} \right].
\]

Applying Equations (5), (6) and the following identity

\[
\Delta^a \left[ \begin{array}{c} x \\ x + 2l_1 \end{array} \right] = 0, \quad \text{when} \quad 2l_1 < a + 1
\]

we easily derive

\[
\Delta^a \left[ \begin{array}{c} x \\ x - l_1 \end{array} \right] \bigg|_{x=0} = \sum l \left( \begin{array}{c} a \\ l \end{array} \right) \Delta^a \left( \frac{x + l}{2l_1} \right) \bigg|_{x=0} = 0, \quad \text{when} \quad 2l_1 < a + 1.
\]

This is enough to prove Lemma 21.

As a consequence of Lemma 21 and the probabilistic inequality (49) is the following Lemma.

Lemma 22. Assume that \(a\) is an odd natural number. Let \(t_{i-1} = \frac{i-1}{n}\) and let

\[
A^{(a)}_i = \sum_{j=0}^n \frac{2}{n^a} \left( \begin{array}{c} a \\ j \end{array} \right) (-1)^j (i-1)^{a-j} (i-1)^j.
\]

Then

\[
\sum_{i=2}^n A^{(a)}_i \int^t_{t_{i-1}} f_{i-1,n}(t) dt = O \left( \frac{1}{n^{\frac{a+2}{2}(i-1)}} \right).
\]

Proof. Using (41) and changing the summation we deduce that

\[
A^{(a)}_i = \frac{2}{n^a} \sum_{l_{i-1}} \sum_{j=0}^n \left( \begin{array}{c} a \\ j \end{array} \right) (-1)^j \left[ \begin{array}{c} j \\ j - l_1 \end{array} \right].
\]

Applying Lemma 21 we get

\[
A^{(a)}_i = \frac{2}{n^a} \sum_{2l_1 \geq a+1} C_{2,a-l_1} (i-1)^{a-l_1}
\]

where \(C_{2,a-l_1}\) depends only on \(a\) and \(l_1\). Since \(f_{i-1,n}(t)\) is the probability density we have

\[
\int^t_{t_{i-1}} f_{i-1,n}(t) dt \leq 1
\]

(49)
Then we show that several coefficient of the polynomial $C$ where polynomial of variable $i$. This completes the proof of Lemma 22.

Putting together (48), (49) as well Identity (22) we get
\[
\sum_{i=2}^{n} A_i^{(a)} \int_{0}^{t_{i-1}} f_{i-1,n}(t) dt = O \left( \frac{1}{n^{\alpha-1}} \right).
\]

This completes the proof of Lemma 22.

The proof of the next lemma is technically complicated. Before starting the proof, we briefly explain the overall strategy of the analysis. Firstly, we write $B_i$ as the polynomial of variable $i - 1$ (see (50)). Using the property (7) of high-order difference we show that several coefficient of the polynomial $B_i$ are zero. Finally, we extract the leading coefficient $C_{3, \frac{a+1}{2+1}}$.

**Lemma 23.** Assume that $a$ is an odd natural number. Let
\[
B_i^{(a)} = \sum_{j=0}^{a} \binom{a}{j} (-1)^{j+1} (i-1)^{a-j} \sum_{k=1}^{j} (i+j-2)^{j-k}(i-1)^k.
\]
Then
\[
B_i^{(a)} = \frac{a! \sqrt{\pi}}{2^{a+1} \Gamma(\frac{a}{2} + 1)} (i-1)^{\frac{a+1}{2}} + \sum_{l_1, l_2} C_{3, l_1} (i-1)^{l_1}
\]
where $C_{3, l_1}$ depends only on $a$.

**Proof.** First of all, observe that $B_i^{(a)}$ is the polynomial of variable $i - 1$. Therefore
\[
B_i^{(a)} = \sum_{l_1 < \frac{a+1}{2+1}} C_{3, l_1} (i-1)^{a-l_1} + C_{3, \frac{a+1}{2+1}} (i-1)^{\frac{a+1}{2+1}} + \sum_{l_1 > \frac{a+1}{2+1}} C_{3, l_1} (i-1)^{a-l_1} \quad (50)
\]
Applying Identity (44) we deduce that
\[
(i+j-2)^{j-k} = \sum_{l_2} \left[ \frac{j-k}{j-k-l_2} \right] (-1)^{l_2} (i+j-2)^{j-k-l_2} = \sum_{l_2} \sum_{l_3} \left[ \frac{j-k}{j-k-l_2-l_3} \right] (-1)^{l_2} (i-1)^{j-k-l_2-l_3} (j-1)^{l_3} \left( \frac{j-k-l_2}{j-k-l_2-l_3} \right).
\]
Hence, the coefficient of the term $(i-1)^{a-l_2-l_3}$ in the polynomial $B_i^{(a)}$ equals
\[
d_{l_2, l_3} = \sum_{j=0}^{a} \binom{a}{j} (-1)^{j+1} \sum_{k=1}^{j} \left[ \frac{j-k}{j-k-l_2-l_3} \right] (-1)^{l_2} (j-1)^{l_3} \left( \frac{j-k-l_2}{j-k-l_2-l_3} \right).
\]
Since $a$ is odd natural number we have $(-1)^{j+1} = (-1)^{a-j}$. Therefore, choosing
\[
f_{l_2, l_3}(x) = \sum_{k=1}^{x} \left[ \frac{x-k}{x-k-l_2} \right] (-1)^{l_2} (x-1)^{l_3} \left( \frac{x-k-l_2}{x-k-l_2-l_3} \right)
\]
Observe that \( \binom{x-k}{l_3-l_2} \) is the polynomial of variable \( x-k \) of degree \( l_3 \) and has the coefficient \( \frac{1}{(l_3)!} \) of the term \((x-k)^{l_3} \). Applying Identity (6) for \( m = x-k, p = l_2 \) and Identity (5) for \( m = l_2 \) we observe that \( \binom{x-k}{l_2} \) is the polynomial of variable \( x-k \) of degree \( 2l_2 \) and has the coefficient \( \frac{1}{(l_2)!^2} \) of the term \((x-k)^{2l_2} \).

Therefore, \( f_{l_2,l_3}(x) \) is the polynomial of variable \( x \) of degree \( 2l_2 + 2l_3 + 1 \) and has coefficient \( (-1)^{l_2} \frac{1}{l_2!} \frac{1}{l_3!} \frac{1}{2^{l_2 + l_3 + 1}} \) of the term \( x^{2l_2 + 2l_3 + 1} \) (see (22)).

Using this and the following equation \( \Delta^a \left( x^l \right) = \begin{cases} 0 & \text{if } l < a \\ a! & \text{if } l = a \end{cases} \) we have

\[
\begin{align*}
d_{l_2,l_3} &= \Delta^a f_{l_2,l_3}(x) \bigg|_{x=0} = 0, \text{ when } 2l_2 + 2l_3 + 1 < a.
\end{align*}
\]

Therefore

\[
C_{3, l_1} = 0, \text{ when } l_1 < \frac{a+1}{2} \quad (51)
\]

\[
C_{3, \frac{a+1}{2}} = a! \sum_{2l_2 + 2l_3 + 1 = a} (-1)^{l_2} \frac{1}{l_2!} \frac{1}{l_3!} \frac{1}{2^{l_2 + l_3 + 1}}
\]

Applying Identity (46) for \( d = \frac{a+1}{2} \) we have

\[
C_{3, \frac{a+1}{2}} = a! \sum_{l_2} \left( \frac{a+1}{l_2!} \right) (-1)^{l_2} \frac{1}{2^{l_2 + \frac{a+1}{2} + l_2}} = \frac{a! \sqrt{\pi}}{2^{\frac{a+1}{2}} \Gamma \left( \frac{a+1}{2} \right)} \quad (52)
\]

Together (51), (52), and (50) complete the proof of Lemma 23

We are now ready to prove the following precise asymptotic result.

**Lemma 24.** Assume that \( a \) is an odd natural number. Let

\[
E_i^{(a)} = \frac{1}{n^a} \frac{(i-1)^{\frac{a+1}{2}} (i-1)^{i-2}}{(i-2)!} e^{-(i-1)}.
\]

Then

\[
\sum_{i=2}^{n} E_i^{(a)} = \frac{1}{\sqrt{2\pi}} \left( \frac{\sqrt{\pi}}{n^a - \frac{a}{2}} + O \left( \frac{1}{n^{a - \frac{a}{2}}} \right) \right).
\]

**Proof.** Let \( E_i = \frac{1}{n^a} (i-1)^{\frac{a+1}{2}} (i-1)^{i-2} e^{-(i-1)} \). We divide the sum into three parts

\[
\sum_{i=2}^{n} E_i = \sum_{i=2}^{\lfloor \sqrt{n} \rfloor} E_i + \sum_{\lfloor \sqrt{n} \rfloor + 1}^{n - \lfloor \sqrt{n} \rfloor} E_i + \sum_{n - \lfloor \sqrt{n} \rfloor + 1}^{n} E_i \quad (53)
\]
We approximate the three parts separately. For the first and third term, we use Stirling’s formula \((21)\) for \(m = i - 1\) and the inequality \(e^{\frac{1}{2}\left(c-1\right)^2} > 1\) to deduce that

\[
E_i \leq \frac{1}{\sqrt{2\pi}} \frac{(i - 1)^{\frac{3}{2}}}{n^a}.
\]

Therefore

\[
\sum_{i=2}^{\sqrt{n}} E_i + \sum_{n - \sqrt{n} + 1}^{n} E_i = O\left(\frac{1}{n^{\frac{a}{2} - \frac{1}{2}}}\right).
\]

Hence the first and third term contribute \(O\left(\frac{1}{n^{\frac{a}{2} - \frac{1}{2}}}\right)\) and the asymptotic depends on the second term.

For the second term \(\left(\left\lfloor\sqrt{n}\right\rfloor + 1 \leq i \leq n - \left\lfloor\sqrt{n}\right\rfloor\right)\) we use Stirling’s formula \((47)\) for \(m = i - 1\) to deduce that

\[
E_i = \frac{1}{\sqrt{2\pi}} \frac{(i - 1)^{\frac{3}{2}}}{n^a} = \frac{1}{\sqrt{2\pi}} \frac{(i - 1)^{\frac{3}{2}}}{n^a} \left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right).
\]

Notice that

\[
\sum_{i=2}^{\sqrt{n}} \frac{1}{\sqrt{2\pi}} \frac{(i - 1)^{\frac{3}{2}}}{n^a} + \sum_{n - \sqrt{n} + 1}^{n} \frac{1}{\sqrt{2\pi}} \frac{(i - 1)^{\frac{3}{2}}}{n^a} = O\left(\frac{1}{n^{\frac{a}{2} + \frac{1}{2}}}\right).
\]

Therefore, we can add the terms back in, so we have

\[
\sum_{i=2}^{n - \sqrt{n}} E_i = \sum_{i=1}^{n} E_i + O\left(\frac{1}{n^{\frac{a}{2} - \frac{1}{2}}}\right)
\]

\[
= \frac{1}{\sqrt{2\pi}} \left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right) \sum_{i=1}^{n} \frac{(i - 1)^{\frac{3}{2}}}{n^a} + O\left(\frac{1}{n^{\frac{a}{2} - \frac{1}{2}}}\right) \quad (54)
\]

The remaining sum we approximate with integral. Hence

\[
\sum_{i=2}^{n} \frac{(i - 1)^{\frac{3}{2}}}{n^a} = \int_{2}^{n} \frac{(x - 1)^{\frac{3}{2}}}{n^a} dx + \Delta \quad (55)
\]

with \(|\Delta| \leq \sum_{i=1}^{n-1} \max_{x \leq i < i+1} |f(x) - f(i)|\) (see [4, Page 179].) Observe that, the function \(f(x) = \frac{(x-1)^{\frac{3}{2}}}{n^a}\) is monotonically increasing over the interval \([2, n]\) Hence the error term \(|\Delta|\) telescopes on the interval \([2, n]\). Therefore \(|\Delta| = O\left(\frac{1}{n^{\frac{a}{2}}}\right)\). We derive easily

\[
\int_{2}^{n} \frac{(x - 1)^{\frac{3}{2}}}{n^a} dx = \frac{1}{\frac{3}{2} + 1} \frac{1}{n^{\frac{a}{2} - 1}} + O\left(\frac{1}{n^{\frac{a}{2} - 2}}\right) \quad (56)
\]
Combining together (54), (55) and (56) we deduce that the second term contributes

\[
\sum_{n=\lfloor \sqrt{n} \rfloor +1}^{n-1} E_i = \frac{1}{\sqrt{2\pi \left( \frac{a}{2} + 1 \right)}} \frac{1}{n^{a/2}} + O \left( \frac{1}{n^{a/2}} \right).
\]

This completes the proof of Lemma 24.

**Proof.** (Theorem 20) Assume that \(a\) is an odd natural number. Let \(X_i\) be the arrival time of the \(i\)th event in a Poisson process with arrival rate \(n\). We know that the random variable \(X_i - X_1 = X_{i-1}\) obeys the Gamma distribution with density

\[
f_{i-1,n}(t) = ne^{-nt} (nt)^{i-2} (i-2)!
\]

for \(i = 2, 3, \ldots, n\). (see equation (1) for \(j = 1, l = i - 1\)). Let \(D_{i}^{(a)}\) be the expected distance to the power \(\alpha\) between \(X_i - X_1\) and the \(i\)-th sensor position, \(t_{i-1} = \frac{i-1}{n}\), hence given by

\[
D_{i}^{(a)} = \int_0^\infty |t - t_{i-1}|^\alpha f_{i-1,n}(t) dt.
\]

First of all, observe that

\[
D_{i}^{(a)} = D_{i,2}^{(a)} - D_{i,1}^{(a)}
\]

where

\[
D_{i,1}^{(a)} = \int_0^\infty (t_{i-1} - t)^\alpha f_{i-1,n}(t) dt
\]

\[
D_{i,2}^{(a)} = 2 \int_0^{t_{i-1}} (t_{i-1} - t)^\alpha f_{i-1,n}(t) dt.
\]

The proof of Theorem 20 proceeds along the following steps. Applying Lemma 21 we show that the sum \(\sum_{i=2}^{n} D_{i,1}^{(a)}\) is negligibly and contributes \(O \left( \frac{1}{n^{a/2-1}} \right)\) (see Equation (58)). Then we write \(D_{i,2}^{(a)}\) as the sum of \(D_{i,3}^{(a)}\) and \(D_{i,4}^{(a)}\) (see Equation (59)). Using Lemma 22 we get that the sum \(\sum_{i=2}^{n} D_{i,3}^{(a)}\) is also negligibly and contributes \(O \left( \frac{1}{n^{a/2-1}} \right)\) (see Equation (60)). Thus the asymptotic depends on the expression given by the summand \(\sum_{i=2}^{n} D_{i,4}^{(a)}\). Combining together Lemma 23 and Lemma 24 we deduce the asymptotic result (see Equation (61)).

Firstly we estimate \(D_{i,1}^{(a)}\). We show that

\[
\sum_{i=2}^{n} D_{i,1}^{(a)} = O \left( \frac{1}{n^{a/2-1}} \right)
\]

Now we define

\[
D_{i,1}^{(a)}(j) = \binom{a}{j} t_{i-1}^{a-j} (-1)^j \int_0^\infty t^j f_{i-1,n}(t) dt
\]
for \( j \in \{0, 1, \ldots, a\} \) and \( i \in \{2, \ldots, n\} \). Observe that

\[
D_{i,1}^{(a)} = \sum_{j=0}^{a} D_{i,1}^{(a)}(j).
\]

Using (2), (3) and (4) we have

\[
D_{i,1}^{(a)}(j) = \frac{1}{n^a} \sum_{l_1} (i-1)^{a-l_1} \sum_{j} \binom{a}{j} (-1)^j (i-1)^{a-l_1} \left[ \frac{j}{j-l_1} \right].
\]

Changing the summation we get

\[
D_{i,1}^{(a)} = \frac{1}{n^a} \sum_{l_1} (i-1)^{a-l_1} \sum_{j} \binom{a}{j} (-1)^j \left[ \frac{j}{j-l_1} \right] \sum_{l_1} (i-1)^{a-l_1}.
\]

Applying Lemma 21 we get

\[
D_{i,1}^{(a)} = \frac{1}{n^a} \sum_{l_1} (i-1)^{a-l_1} \sum_{l_1} (i-1)^{a-l_1} \cdot C_{2,a-l_1} \left[ \frac{j}{j-l_1} \right].
\]

where \( C_{2,a-l_1} \) depends only on \( a \) and \( l_1 \). Using Identity (22) we conclude

\[
\sum_{j=0}^{a} \sum_{i=2}^{n} D_{i,1}^{(a)}(j) = O \left( \frac{1}{n^{a+1}} \right).
\]

This finishes the proof of equation (58).

Now we estimate \( D_{i,2}^{(a)} \). Let

\[
D_{i,2}^{(a)}(j) = 2 \binom{a}{j} j^{a-j} (-1)^j \int_{0}^{t_{i-1}} t^{j-1} f_{i-1,n}(t) dt
\]

for \( j \in \{0, 1, \ldots, a\} \) and \( i \in \{2, \ldots, n\} \).

On the other hand, from Equation (45) we get

\[
\int_{0}^{t_{i-1}} t^{j} f_{i-1,n}(t) dt = \frac{(i-1)^2}{n^j} \int_{0}^{t_{i-1}} f_{i-1,n}(t) dt - \sum_{k=1}^{j} \frac{(i+j-2)j-k}{n^{j-k}} e^{-nt_{i-1}} k_{i-1}^{j-k} \frac{(nt_{i-1})^{j-2}}{(i-2)!}.
\]

Therefore

\[
D_{i,2}^{(a)}(j) = D_{i,3}^{(a)}(j) + D_{i,4}^{(a)}(j),
\]

where

\[
D_{i,3}^{(a)}(j) = \frac{2}{n^a} \binom{a}{j} (-1)^j (i-1)^{a-j} (i-1)^j \int_{0}^{t_{i-1}} f_{i-1,n}(t) dt
\]

This finishes the proof of equation (58).
\[
D_{i,4}^{(a)}(j) = \frac{2}{n^a} \binom{a}{j} (-1)^{j+1} (i-1)^{a-j} \sum_{k=1}^{j} (i+j-2)^{j-k}(i-1)^{k} (i-1)^{i-2} e^{-(i-1)}.
\]

Let \(D_{i,3}^{(a)} = \sum_{j=0}^{a} D_{i,3}^{(a)}(j)\) and \(D_{i,4}^{(a)} = \sum_{j=0}^{a} D_{i,4}^{(a)}(j)\). Hence

\[
D_{i,2}^{(a)} = D_{i,3}^{(a)} + D_{i,4}^{(a)} \tag{59}
\]

Using Lemma 22 we get

\[
\sum_{i=2}^{n} D_{i,3}^{(a)} = O \left( \frac{1}{n^{\frac{a+1}{2}-1}} \right) \tag{60}
\]

From Lemma 23, Lemma 24 and Identity (20) for \(z = \frac{a}{2} + 1\) we deduce that

\[
\sum_{i=2}^{n} D_{i,4}^{(a)} = \frac{a!}{2^{a+1} \left( \frac{a}{2} + 2 \right)} \frac{1}{n^{\frac{a}{2}-1}} + O \left( \frac{1}{n^{\frac{a}{2}+\frac{1}{2}}} \right) \tag{61}
\]

Finally, combining together Equations (58-61) finishes the proof of Theorem 20. \(\Box\)