Hodge–Tate crystals on the logarithmic prismatic sites of semi-stable formal schemes

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Abstract

Let \(O_K\) be a complete discrete valuation ring of mixed characteristic \((0, p)\) with a perfect residue field. In this paper, for a semi-stable \(p\)-adic formal scheme \(X\) over \(O_K\) with rigid generic fibre \(X\) and canonical log structure \(\mathcal{M}_X = \mathcal{O}_X \cap \mathcal{O}_X^\times\), we study Hodge–Tate crystals over the absolute logarithmic prismatic site \((X, \mathcal{M}_X)_\Delta\). As an application, we give an equivalence between the category of rational Hodge–Tate crystals on the absolute logarithmic prismatic site \((X, \mathcal{M}_X)_\Delta\) and the category of enhanced log Higgs bundles over \(X\), which leads to an inverse Simpson functor from the latter to the category of generalised representations on \(X_{\text{pro\acute{e}t}}\).

Contents

1 Introduction
  1.1 Main results .......................................................... 2
  1.2 Notations ............................................................... 3
  1.3 Organizations .......................................................... 6

2 The logarithmic prismatic site

3 The Hodge–Tate crystals on absolute logarithmic prismatic site
  3.1 The \(O_K\) case .......................................................... 14
  3.2 The geometric case .................................................... 16
     3.2.1 The relative case ................................................... 21
     3.2.2 The absolute case .................................................. 23
  3.3 Hodge–Tate crystals as generalised representations ............ 27
  3.4 Local \(p\)-adic Simpson correspondence .......................... 30
     3.4.1 Local Simpson functor for generalised representations .... 30
     3.4.2 Local inverse Simpson functor for enhanced Higgs modules 33
  3.5 A typical example ...................................................... 34

4 Globalizations
  4.1 Definitions and Preliminaries ...................................... 35
  4.2 The inverse Simpson functor for enhanced Higgs bundles ....... 37

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1 Introduction

The theory of prismatic cohomology was introduced by Bhatt–Scholze in \[\text{[BS19]}\], which can be understood as a “universal” \(p\)-adic cohomology theory in the sense that it specialises to most cohomology theories (e.g. \(\text{étale}\) cohomology, crystalline cohomology, \(\text{dé Rham}\) cohomology, \(\text{Hodge–Tate}\) cohomology and so on) in \(p\)-adic geometry. The original paper \[\text{[BS19]}\] was devoted to the relative theory, which has several geometric applications in the (integral) \(p\)-adic Hodge theory. For example, it recovers most works of \[\text{[BMST18]}\] and \[\text{[BMS19]}\] and can be used to study local \(p\)-adic Simpson correspondence (c.f. \[\text{[MT20]}, \text{[Tian21]}\], etc.).

On the other hand, the theory admits a variant which is called the absolute prismatic theory and has more arithmetic applications. For example, for a bounded \(p\)-adic formal scheme \(\mathfrak{X}\), the category of Laurent \(F\)-crystals on the absolute prismatic site \((\mathfrak{X})_\Lambda\) is equivalent to the category of \(\text{étale}\) \(\mathbb{Z}_p\)-local systems on the adic generic fibre \(\mathfrak{X}\) (c.f. \[\text{[Wu21]}, \text{[BS21]}, \text{[MW21a]}\], etc.). In particular, one can recover some results in the classical theory of \((\varphi, \Gamma)\)-modules on Galois representations through this equivalence. If moreover, \(\mathfrak{X}\) is smooth over \(\mathcal{O}_K\), the ring of integers of a finite extension \(K\) of \(\mathbb{Q}_p\), then one can establish an equivalence between the category of prismatic \(F\)-crystals on \((\mathfrak{X})_\Lambda\) and the category of crystalline \(\mathbb{Z}_p\)-local systems on \(\mathfrak{X}_{\text{ét}}\) (c.f. \[\text{[BS21]}, \text{[DL21]}\] for \(\mathfrak{X} = \text{Spf}(\mathcal{O}_K)\) and \[\text{[DLMS22]}, \text{[GR22]}\] for general \(\mathfrak{X}\)). So it seems that the absolute prismatic theory could shed light on studying \(p\)-adic representations.

There is also an application of the absolute theory to \(p\)-adic Simpson correspondence for rigid spaces with good reductions \(\mathfrak{X}\) over \(\mathcal{O}_K\). Indeed, the authors established an equivalence between the category of rational Hodge–Tate crystals on \((\mathfrak{X})_\Lambda\) and the category of enhanced Higgs bundles on \(\mathfrak{X}_{\text{ét}}\) with coefficients in \(\mathcal{O}_X[1/\varpi]\) (c.f. \[\text{[MW22]}\] or Theorem \[3.2\]). On the other hand, the category of rational Hodge–Tate crystals on the sub-site \((\mathfrak{X})_{\text{perf}}\) of perfect prisms is equivalent to the category of generalised representations (i.e. vector bundles with coefficients in \(\hat{\mathcal{O}}_X\)) on \(\mathfrak{X}_{\text{proet}}\). So the restriction gives rise to an inverse Simpson functor from the category of enhanced Higgs bundles to the category of generalised representations, which turns out to be fully faithful. The construction is closely related with Sen theory. For example, when \(\mathfrak{X} = \text{Spf}(\mathcal{O}_K)\), we get classical Sen operators for \(C\)-representations of \(G_K\) (c.f. \[\text{[Gao22]}, \text{[BL22a]}\], etc.).

However, in practice, there is no means that a rigid analytic variety \(\mathfrak{X}\) over \(K\) always admits a good reduction (if this is the case, the Galois representations associated to the \(\text{étale}\) cohomology of \(\mathfrak{X}\) must be crystalline, which is not always true). So it is necessary to generalize stories mentioned above to rigid spaces with non-smooth reductions. A meaningful generalisation is that one may consider those with semi-stable reductions, which can be viewed as smooth objects in logarithmic geometry. Indeed, up to enlarging base field, rigid spaces always admit semi-stable reductions (c.f. \[\text{[Har03]}\]). This suggests us to study the logarithmic analogue of prismatic theory. The fundamental contribution in this direction is due to Koshikawa \[\text{[Kos20]}\]. Indeed, he introduced notions of log prism and logarithmic (relative) prismatic site, and then generalized several results of \[\text{[BS19]}\] to the logarithmic case. Another application of logarithmic prismatic theory is due to Du–Liu \[\text{[DL21]}\]. They showed the category of prismatic \(F\)-crystals on absolute logarithmic prismatic site \((\mathcal{O}_K)^\Lambda_{\text{log}}\) can be identified with the category of semi-stable \(\mathbb{Z}_p\)-representations of \(G_K\), which obviously generalises the story about crystalline representations in the good reduction case. To the best of our knowledge, up to now, one could not see this from the original prismatic theory. So there do exist some new phenomena in the logarithmic setting. This paper is devoted to another direction and more precisely, to generalising main results in \[\text{[MW21b]}\] and \[\text{[MW22]}\]. In other words, we study Hodge–Tate crystals in the logarithmic case, and show these objects can be identified with certain Higgs bundles and construct an inverse Simpson functor for these Higgs bundles (taking values in generalised representations). The specific statements of our results can be founded in the next subsection.

Finally, it is worth pointing out that the usual absolute prismatic theory was recently independently studied by Drinfeld \[\text{[Dr20]}\] and Bhatt–Lurie \[\text{[BL22a]}, \text{[BL22b]}\] in a stacky way. Indeed, one can relate a \(p\)-adic formal scheme \(\mathfrak{X}\) to a certain stack, which is called the prismaticization of \(\mathfrak{X}\), such that studying prismatic theory on
Lemma 1.2

From this, we obtain the following result.

Theorem 4.8

In particular, for any \( A, I \), \( \delta \) is integral, together with a structure map \( \delta \) and a map \( \varphi \) of log ring \((A, I)\).

1.1 Main results

We use notations in Notations 1.2 freely for stating our main results. Throughout this subsection, we fix a complete discretely valued \( p \)-adic field \( K \) and let \( \mathcal{X} \) be a semi-stable \( p \)-adic formal scheme over \( \mathcal{O}_K \) with rigid generic fibre \( X \) and canonical log structure \( \mathcal{M}_X = \mathcal{O}_X \cap \mathcal{O}_X^\times \). We first specify the meaning of absolute logarithmic prismatic site \((\mathcal{X}, \mathcal{M}_X)^\Delta\).

Recall that a (bounded) log prism defined by Koshikawa in [Kos20] is a tuple \((A, I, M, \delta)\) consisting of a bounded prism \((A, I)\) together with a log prism \((A, I, M, \delta)\). For a given prism \((A, I, M, \delta)\), \( \delta \) is a log prism \((A, I, M, \delta)\).

In particular, for any \( m \in M \), \( \varphi(\alpha(m)) = (1 + p\delta)(m) \) and hence \( \varphi \) is a “Frobenius” endomorphism of log ring \((A, M)\).

An object in \((\mathcal{X}, \mathcal{M}_X)^\Delta\) is a log prism \((A, I, M, \delta)\), whose underlying log structure is integral, together with a structure map \( f : \text{Spf}(A/I) \to \mathcal{X} \) which induces an exact closed immersion \((\text{Spf}(A/I), f^*\mathcal{M}_X) \to (\text{Spf}(A/I), \mathcal{M}_X)\) of log \((p, I)\)-adic formal schemes. The morphisms in \((\mathcal{X}, \mathcal{M}_X)^\Delta\) are defined in the obvious way and the topology is generated by \((p\)-completely\) flat covers. See Definition 2.1 for details.

For a given prism \((A, I)\) together with a log structure \( \alpha : M \to A \), it is a strong restriction on \( \alpha \) that there is a \( \log \)-structure \( \delta : M \to A \) making \((A, I, M, \delta)\) a log prism. For example, one can prove that

Lemma 1.1 (Lemma 2.12). If a log prism \((A, I, M, \delta)\) is perfect (i.e. \((A, I)\) is a perfect prism in the usual sense), then for any \( m \in M \), there exists an \( x \in A \) such that \( \alpha(m) = \lceil \kappa \rceil (1 + px) \), where \( \lceil \kappa \rceil \) is the Teichmüller lifting of \( \kappa \).

On the other hand, being an object in \((\mathcal{X}, \mathcal{M}_X)^\Delta\) is also a strong restriction on a log prism \((A, I, M, \delta)\).

Indeed, if this is the case, then the log structure \( M \to A \) is somehow determined by \((\mathcal{X}, \mathcal{M}_X)\) (c.f. Lemma 2.20). This combined with Lemma 1.1 shows that for a perfect prism \((A, I)\) in the usual prismatic site \((\mathcal{X}, \mathcal{M}_X)^\Delta\), there exists a unique way to equip \( A \) with a \( \log \)-structure \( \delta : M \to A \) such that \((A, I, M, \delta) \in (\mathcal{X}, \mathcal{M}_X)^\Delta\) (c.f. Lemma 2.11). In particular, when \( \mathcal{X} = \text{Spf}(R) \) is affine small; that is, \( R \) is étale over \( \mathcal{O}_K(T_0 + \cdots, T_r, T_{r+1} + \cdots, T_d + \cdots)/(T_0 + \cdots - \pi) \), we have the identity of sites

\[
(\mathcal{X}, \mathcal{M}_X)^{\text{perf}}_\Delta = (\mathcal{X})^{\text{perf}}_\Delta.
\]

From this, we obtain the following result.

Lemma 1.2 (Corollary 2.19). For a semi-stable \( p \)-adic formal scheme \( \mathcal{X} \) over \( \mathcal{O}_K \), there is an equivalence of topoi 

\[
\text{Sh}(\mathcal{X})^{\text{perf}}_\Delta \simeq \text{Sh}(\mathcal{X}, \mathcal{M}_X)^{\text{perf}}_\Delta.
\]

As a corollary, we have

Proposition 1.3 (Theorem 4.5). There is a canonical equivalence of categories

\[
\text{Vect}(\mathcal{X}, \mathcal{M}_X)^{\text{perf}}_\Delta, (\mathcal{O}_\Delta^{\text{perf}}_\Delta/p) \simeq \text{Vect}(X_{\text{proét}}, \mathcal{O}_X).
\]
Here, \(\text{Vect}(X_{\text{proet}}, \mathcal{O}_X)\) denotes the category of generalised representations and \(\text{Vect}((\mathcal{X}, \mathcal{M}_X)^{\text{perf}}_{\mathcal{O}_X}[, \mathcal{O}_X[1/\mathcal{p}]]\) denotes the category of rational Hodge–Tate crystals on the perfect site. The notion of Hodge–Tate crystals is similar to that in [MW22] and is specified as follows:

**Definition 1.4 (Definition 4.1).** By a Hodge–Tate crystal on \((\mathcal{X}, \mathcal{M}_X)\), we mean a sheaf \(M\) of \(\mathcal{O}_X\)-modules satisfying the following properties:

1. For any \(\mathfrak{A} = (A, I, M, \delta_{\log}) \in (\mathcal{X}, \mathcal{M}_X)\), \(M(\mathfrak{A})\) is a finite projective \(A/I\)-module.
2. For any morphism \(\mathfrak{A} = (A, I, M, \delta_{\log}) \to \mathfrak{B} = (B, IB, N, \delta_{\log})\) in \((\mathcal{X}, \mathcal{M}_X)\), there is a canonical isomorphism

\[
M(\mathfrak{A}) \otimes_{A/I} B/IB \to M(\mathfrak{B}).
\]

We denote by \(\text{Vect}((\mathcal{X}, \mathcal{M}_X)\), \(\mathcal{O}_X\)\) the category of Hodge–Tate crystals on \((\mathcal{X}, \mathcal{M}_X)\).

Similarly, we define rational Hodge–Tate crystals on \((\mathcal{X}, \mathcal{M}_X)\) (resp. \((\mathcal{X}, \mathcal{M}_X)^{\text{perf}}\), \(\mathcal{O}_X\)) by replacing \(\mathcal{O}_X\) by \(\mathcal{O}_X^\mathcal{p}\) and denote the corresponding category by \(\text{Vect}((\mathcal{X}, \mathcal{M}_X)\), \(\mathcal{O}_X^\mathcal{p}\)) (resp. \(\text{Vect}((\mathcal{X}, \mathcal{M}_X)^{\text{perf}}\), \(\mathcal{O}_X^\mathcal{p}\))).

Thanks to Proposition 1.3, the restriction functor

\[
R : \text{Vect}((\mathcal{X}, \mathcal{M}_X)\), \mathcal{O}_X^\mathcal{p}) \to \text{Vect}((\mathcal{X}, \mathcal{M}_X)^{\text{perf}}\), \mathcal{O}_X^\mathcal{p})
\]

induces a functor from \(\text{Vect}((\mathcal{X}, \mathcal{M}_X)\), \(\mathcal{O}_X^\mathcal{p}\)\) to \(\text{Vect}(X_{\text{proet}}, \mathcal{O}_X)\). It can be shown that this functor is actually fully faithful (c.f. Corollary 4.13).

In order to state our main result, we introduce the notion of “enhanced log Higgs bundles”.

**Definition 1.5 (Definition 4.1).** By an enhanced log Higgs bundle on \(x_{\text{et}}\) with coefficients in \(\mathcal{O}_X\), we mean a triple \((\mathcal{M}, \theta_M, \phi_M)\) satisfying the following properties:

1. \(\mathcal{M}\) is a locally finite free \(\mathcal{O}_X\)-module and

\[
\theta_M : \mathcal{M} \to \mathcal{M} \otimes_{\mathcal{O}_X} \hat{\Omega}_{X, \log}^1 \{-1\}
\]

defines a nilpotent Higgs field on \(\mathcal{M}\), i.e. \(\theta_M\) is a section of \(\text{End}(\mathcal{M}) \otimes_{\mathcal{O}_X} \hat{\Omega}_{X, \log}^1 \{-1\}\) which is nilpotent and satisfies \(\theta_M \wedge \theta_M = 0\). Here “\(-1\)” denotes the Breuil–Kisin twist of \(\zeta\). Denote by \(\text{HIG}(\mathcal{M}, \theta_M)\) the induced Higgs complex.

2. \(\phi_M \in \text{End}(\mathcal{M})\) is “topologically nilpotent” in the following sense:

\[
\lim_{n \to +\infty} \prod_{i=0}^n (\phi_M + i\pi E'(\pi)) = 0
\]

and induces an endomorphism of \(\text{HIG}(\mathcal{M}, \theta_M)\); that is, the following diagram

\[
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\theta_M} & \mathcal{M} \otimes \hat{\Omega}_{X, \log}^1 \{-1\} \xrightarrow{\theta_M} \cdots \xrightarrow{\theta_M} \mathcal{M} \otimes \hat{\Omega}_{X, \log}^d \{-d\} \\
\mathcal{M} & \xrightarrow{\phi_M} & \mathcal{M} \otimes \hat{\Omega}_{X, \log}^1 \{-1\} \xrightarrow{\phi_M + \pi E'(\pi) \text{id}} \cdots \xrightarrow{\phi_M + \pi E'(\pi) \text{id}} \mathcal{M} \otimes \hat{\Omega}_{X, \log}^d \{-d\}
\end{array}
\]

commutes. Denote \(\text{HIG}(\mathcal{M}, \theta_M, \phi_M)\) the total complex of this bicomplex.

Denote by \(\text{HIG}^{\log}(X, \mathcal{O}_X)\) the category of enhanced log Higgs bundles over \(X\). Similarly, we define enhanced log Higgs bundles on \(X_{\text{et}}\) with coefficients in \(\mathcal{O}_X^\mathcal{p}\) and denote the corresponding category by \(\text{HIG}^{\log}(X, \mathcal{O}_X^\mathcal{p})\).

When \(X = \text{Spf}(R)\) is small affine, we also denote \(\text{HIG}^{\log}(X, \mathcal{O}_X)\) (resp. \(\text{HIG}^{\log}(X, \mathcal{O}_X^\mathcal{p})\)) by \(\text{HIG}^{\log}(R)\) (resp. \(\text{HIG}^{\log}(R^\mathcal{p})\)) and call objects in this category enhanced log Higgs modules over \(R\) (resp. \(R^\mathcal{p}\)).
Let \( \text{HIG}^{\text{nil}}_{G_K}(X_C) \) be the category of \( G_K \)-Higgs bundles on \( X_{C, \delta} \) (see Definition \ref{def:HIGnil}). Then there exists a functor
\[
F_1 : \text{HIG}^{\text{log}}_*(X, \mathcal{O}_X[1/p]) \rightarrow \text{HIG}^{\text{nil}}_{G_K}(X_C),
\]
which sends an enhanced log Higgs bundle \((\mathcal{M}, \theta_{\mathcal{M}}, \phi_{\mathcal{M}})\) to the \( G_K \)-Higgs module
\[
(\mathcal{H} = \mathcal{M} \otimes \mathcal{O}_X, \theta_{\mathcal{H}} = \theta_{\mathcal{M}} \otimes \lambda(\zeta_p - 1)\text{id})
\]
with the \( G_K \)-action on \( \mathcal{H} \) induced by the formulae
\[
g \mapsto (1 - c(g)\lambda(1 - \zeta_p)\pi E'(\pi))^{-\delta_{\mathcal{H}}(g)}.
\]
One can check that \( F_1 \) is fully faithful (c.f. Lemma \ref{lem:fullfaith}).

Now, we state our main result as follows:

**Theorem 1.6** (Theorem \ref{thm:mainthm} \cite{MW22} \cite{GT20} \cite{GT18}). Assume \( X \) is a semi-stable \( p \)-adic formal scheme over \( \mathcal{O}_K \) of relative dimension \( d \). Then there is an equivalence from the category \( \text{Vect}((X, \mathcal{M}_X)_{\mathbb{A}}, \mathcal{O}_{\mathbb{A}}[1/p]) \) of rational Hodge–Tate crystals to the category \( \text{HIG}^{\text{log}}_*(X, \mathcal{O}_X[1/p]) \) of enhanced log Higgs bundles with coefficients in \( \mathcal{O}_X[1/p] \), which fits into the following commutative diagram
\[
\begin{array}{ccc}
\text{Vect}((X, \mathcal{M}_X)_{\mathbb{A}}, \mathcal{O}_{\mathbb{A}}[1/p]) & \xrightarrow{R} & \text{Vect}((X, \mathcal{M}_X)_{\mathbb{A}}^{\text{perf}}, \mathcal{O}_{\mathbb{A}}[1/p]) \\
\downarrow{\cong} & & \downarrow{\cong} \\
\text{HIG}^{\text{log}}_*(X, \mathcal{O}_X[1/p]) & \xrightarrow{F_1} & \text{HIG}^{\text{nil}}_{G_K}(X_C).
\end{array}
\]

Here, all arrows are fully faithful and we use "\( \cong \)" to denote equivalences of categories.

If moreover \( X = \text{Spf}(R) \) is small affine, then the above equivalence upgrades to the integral and derived level. More precisely, there is an equivalence (depending on the framing chosen) from the category \( \text{Vect}((X, \mathcal{M}_X)_{\mathbb{A}}, \mathcal{O}_{\mathbb{A}}) \) of Hodge–Tate crystals to the category \( \text{HIG}^{\text{log}}_{\text{perf}}(R) \) of enhanced log Higgs modules over \( R \). In this case, there is a quasi-isomorphism
\[
\text{R} \Gamma((X, \mathcal{M}_X)_{\mathbb{A}}, \mathcal{M}) \simeq \text{HIG}(H, \Theta, \phi)
\]
for any Hodge–Tate crystal \( \mathcal{M} \) with associated enhanced log Higgs module \((H, \Theta, \phi)\).

**Remark 1.7.** In the classical Sen theory, given a \( C \)-representation \( V \) with Sen operator \( \phi_V \), One can always assume the matrix of \( \phi_V \) has coefficients in \( K \) (`Sen\cite{Sen77} Theorem 5`) with a suitable choice of basis of \( V \). However, for a matrix \( A \) with coefficients in \( K \), it does not necessarily come from a \( C \)-representation. Our result shows that if
\[
\lim_{n \to +\infty} \prod_{i=0}^{n} (A + i\pi E'(\pi)) = 0,
\]
then \( A \) can be realized as the Sen operator of some \( C \)-representations. We conjecture a similar result in the relative case (c.f. Conjecture \ref{conj:rel}).

As a corollary, we get the following finiteness result, which is an analogue of \cite{Tian87} Theorem 2.8, 2.9 and \cite{MW22} Corollary 2.21 in the absolute logarithmic case.

**Corollary 1.8.** Keep notations as in Theorem \ref{thm:mainthm} Then for any Hodge–Tate crystal \( \mathcal{M} \in \text{Vect}((X, \mathcal{M}_X)_{\mathbb{A}}, \mathcal{O}_{\mathbb{A}}) \), \( \text{R} \nu_*(\mathcal{M}) \) is a perfect complex of \( \mathcal{O}_X \)-modules with tor-amplitude in \([0, d+1]\), where \( \nu_* : \text{Sh}((X, \mathcal{M}_X)_{\mathbb{A}}) \rightarrow \text{Sh}(X_{\delta}) \) is the natural morphism of topoi. As a consequence, if moreover \( X \) is proper, then \( \text{R} \Gamma((X, \mathcal{M}_X)_{\mathbb{A}}, \mathcal{M}) \) is a perfect complex of \( \mathcal{O}_K \)-modules with tor-amplitude in \([0, 2d + 1]\). Similar results hold for rational Hodge–Tate crystals.
Note that if $\mathfrak{X}$ is smooth, one can regard it as a log formal scheme, whose canonical log structure is actually induced by the composition $(\mathbb{N} \xrightarrow{1 + \pi} \mathcal{O}_K \to \mathcal{O}_X)$. Then one can consider the absolute logarithmic prismatic site $(\mathfrak{X}, \mathcal{M}_X)_{\Delta}$. Clearly, the rule $(A, I, M, \delta_{\log}) \mapsto (A, I)$ defines a functor $(\mathfrak{X}, \mathcal{M}_X)_{\Delta} \to (\mathfrak{X})_{\Delta}$, which gives rise to a natural functor

$$\text{Vect}((\mathfrak{X})_{\Delta}, \mathcal{M}_X(\mathcal{O}_X[1/p])) \to \text{Vect}((\mathfrak{X}, \mathcal{M}_X)_{\Delta}, \mathcal{M}_X(\mathcal{O}_X[1/p])).$$

Comparing Theorem 1.6 with [MW22, Theorem 1.12] (or Theorem 3.2), we get the following result:

**Corollary 1.9** (Corollary 3.31 [MW22]). There is a commutative diagram of categories

$$\begin{array}{ccc}
\text{Vect}((\mathfrak{X})_{\Delta}, \mathcal{M}_X(\mathcal{O}_X[1/p])) & \to & \text{Vect}((\mathfrak{X}, \mathcal{M}_X)_{\Delta}, \mathcal{M}_X(\mathcal{O}_X[1/p])) \\
\cong & & \cong \\
\text{HIG}_{\text{nil}}^*(\mathfrak{X}, \mathcal{O}_X[1/p]) & \to & \text{HIG}^*_{\text{log}}(\mathfrak{X}, \mathcal{O}_X[1/p]),
\end{array}$$

where the bottom functor sends an enhanced Higgs bundle $(H, \theta_H, \phi_H)$ to the enhanced log Higgs bundle $(\mathcal{H}, \theta_{\mathcal{H}}, \pi_{\phi_{\mathcal{H}}})$ (and hence is fully faithful). In particular, when $\mathfrak{X} = \text{Spf}(R)$ is small affine, the diagram also holds on the integral level.

Indeed, the commutative diagram in Corollary 1.9 is also compatible with the equivalence in Proposition 1.13 and Simpson correspondence [c.f. Remark 1.14].

Finally, for a generalised representation $\mathcal{L}$ with associated $G_K$-Higgs bundle $(H, \theta_H)$, one can define an arithmetic Sen operator $\phi_{\mathcal{L}}$ on $H$ such that $(H, \theta_H, \phi_{\mathcal{L}})$ is an arithmetic Higgs bundle in the sense of Definition 4.4. In particular, when $\mathfrak{X} = \text{Spf}(\mathcal{O}_K)$, the arithmetic Sen operator coincides with classical Sen operator.

Now, for a rational Hodge–Tate crystal $\mathcal{M} \in \text{Vect}((\mathfrak{X}, \mathcal{M}_X)_{\Delta}, \mathcal{M}_X(\mathcal{O}_X[1/p]))$ with associated enhanced log Higgs bundle $(H, \theta_H, \phi_H)$ and generalised representation $\mathcal{L}$, we make the following conjecture:

**Conjecture 1.10.** With notations as above, we have $\phi_{\mathcal{L}} = -\frac{\phi_H}{\pi E(\pi)}$.

Similar conjectures have already appeared in [MW21b] and [MW22]. We confirm the above conjecture in the case for $\mathfrak{X} = \text{Spf}(\mathcal{O}_K)$ (c.f. Theorem 3.14).

### 1.2 Notations

In this paper, we fix a complete discrete valuation ring $\mathcal{O}_K$ of mixed characteristic $p$ with perfect residue field $k$ and fractional field $K$. Let $\bar{K}$ be a fixed algebraic closure of $K$, $G_K := \text{Gal}(\bar{K}/K)$ be the absolute Galois group of $K$, $C$ be the $p$-adic completion of $\bar{K}$ and $\pi$ be a fixed uniformizer of $\mathcal{O}_K$ with minimal polynomial $E(u) \in \mathfrak{S} := W(k)[[u]]$ over $W(k)$. We fix compatible systems $\{\zeta_{p^n}\}_{n \geq 0}$ and $\{\pi^{\frac{1}{p^n}}\}_{n \geq 0}$ of primitive $p$-power roots of 1 and $\pi$, respectively. Then we get elements $\epsilon = (1, \zeta_p, \ldots)$ and $\pi^\epsilon = (\pi, \pi^{\frac{1}{p}}, \pi^{\frac{1}{p^2}}, \ldots)$ in $\mathcal{O}_C$. Define $K_{\text{cyc}} := \cup_{n \geq 1} K(\zeta_{p^n})$ and $K_{\infty} := \cup_{n \geq 1} K(\pi^{\frac{1}{p^n}})$. Then $K_{\text{cyc, inf}} := K_{\text{cyc}}K_{\infty} = \cup_{n \geq 1} K_n$ for $K_n = K(\zeta_{p^n}, \pi^{\frac{1}{p^n}})$. Denote by $L$ the $p$-adic completion for $L \in \{K_{\text{cyc}}, K_{\text{cyc, inf}}\}$. Note that both $K_{\text{cyc, inf}}$ and $K_{\text{cyc}}$ are Galois extension of $K$. We denote by $\hat{G}_K = \text{Gal}(K_{\text{cyc, inf}}/K)$ and $\Gamma_K = \text{Gal}(K_{\text{cyc}}/K)$ the corresponding Galois groups and then get an exact sequence\footnote{The authors knew from Hui Gao that he could confirm the conjecture by using a relative version of Kummer Sen theory in [Gao22].}

$$1 \to \mathbb{Z}_p(1) \to \hat{G}_K \to \Gamma_K \to 1.$$

Let $\chi : \Gamma_K \to \mathbb{Z}_p(\times)$ be the cyclotomic character and $e : G_K \to \mathbb{Z}_p$ be the map determined by $e(\sigma \pi^\epsilon) = e(\sigma) \pi^\epsilon$ for any $\sigma \in G_K$. Let $A_{\text{inf}} = W(\mathcal{O}_C)$ be the infinitesimal period ring of Fontaine and $\theta : A_{\text{inf}} \to \mathcal{O}_C$ be the natural map $\theta_{\text{inf}} : A_{\text{inf}}[1/p] \to \mathcal{O}_C[1/p]$. Let $\mathfrak{M} : \Gamma_K \to G_K$ be the map determined by $\mathfrak{M}(\sigma \pi^\epsilon) = \sigma \pi^\epsilon$ for any $\sigma \in G_K$. Let $A_{\text{inf}} = W(\mathcal{O}_C)$ be the infinitesimal period ring of Fontaine and $\theta : A_{\text{inf}} \to \mathcal{O}_C$ be the natural map $\theta_{\text{inf}} : A_{\text{inf}}[1/p] \to \mathcal{O}_C[1/p]$.
surjection. Then \( \ker(\theta) \) is generated by either \( E([\pi^e]) \) or \( \xi = \frac{\varphi(u)}{\varphi(\mu)} \) for \( \mu = [e] - 1 \). Define \( \lambda = \theta(\xi) \), which is a unit in \( \mathcal{O}_C \). We always equip \( \mathcal{O}_K \) with the log structure associated to \( \mathbb{N} \xrightarrow{1+\pi} \mathcal{O}_K \) and equip \( \mathcal{S} \) with the \( \delta \)-structure such that \( \varphi(u) = u^p \).

In this paper, we essentially consider semi-stable \( p \)-adic formal scheme \( x \) over \( \mathcal{O}_K \) with rigid generic fibres \( X \). In other words, étale locally, \( x = \text{Spf}(R) \) is affine such that there exists an étale morphism

\[
\square : \mathcal{O}_K(T_0, \ldots, T_r, T_{r+1}^\pm, \ldots, T_d^\pm) / (T_0 \cdots T_r - \pi) \to R
\]

of \( \mathcal{O}_K \)-algebras for some \( 0 \leq r \leq d \). In this case, we say \( x \) or \( R \) is small affine. We also say \( x \) admits a chart if this is the case. We often consider such an \( x \) as a log formal scheme with the canonical log structure \( \mathcal{M}_X \to \mathcal{O}_X \) for \( \mathcal{M}_X = \mathcal{O}_X \cap \mathcal{O}_X^\times \). When \( x = \text{Spf}(R) \) is small affine as above, \( \mathcal{M}_X \) is induced by the pre-log structure \( \mathbb{N}^{r+1} \xrightarrow{e_i \mapsto T_i^\vee} R \), where \( e_i \) denotes the generator of \( (i+1) \)-th component of \( \mathbb{N}^{r+1} \) for \( 0 \leq i \leq r \).

### 1.3 Organizations

In Section 2, we give a quick review on logarithmic prismatic site and prove some basic properties for logarithmic prismatic theory. In Section 3, we study Hodge–Tate crystals on absolute logarithmic prismatic site for a semi-stable \( p \)-adic formal scheme over \( \mathcal{O}_K \) and show Simpson correspondence in the local case. We first deal with the \( \text{Spf}(\mathcal{O}_K) \) case and then move to the higher dimensional case. In Section 4, we glue local constructions and deduce a global theory.

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### 2 The logarithmic prismatic site

The logarithmic prismatic site was introduced by Koshikawa \([\text{Kos20}]\) as an analogue of the prismatic site defined in \([\text{BS19}]\) in the theory of logarithmic geometry. In this section, we give a quick review of its definition and provide some basic relevant properties.

**Definition 2.1 (\([\text{Kos20}]\) Definition 2.2).** A \( \delta \)-ring is a tuple \((A, \delta, M) \twoheadrightarrow A, \delta(M) : M \to A\) consisting of a \( \delta \)-ring \((A, \delta)\), a prelog structure \( \alpha : M \to A \) and a map \( \delta(M) : M \to A \) such that

1. \( \delta(M)(e_M) = 0 \), where \( e_M \) is the unity of the commutative monoid \( M \);
2. for any \( m \in M \), \( \alpha(m)^p \delta(M)(m) = \delta(\alpha(m)) \);
3. for any \( m_1, m_2 \in M \), \( \delta(M)(m_1 m_2) = \delta(M)(m_1) + \delta(M)(m_2) + p \delta(M)(m_1) \delta(M)(m_2) \).

We often write \((A, M, \delta(M))\) for a \( \delta \)-ring for simplicity.

A morphism \( f : (A, M, \delta(M)) \to (B, N, \delta(N)) \) of \( \delta \)-rings is a morphism of prelog rings compatible with both \( \delta \) and \( \delta(M) \).

Let \((A, M) \twoheadrightarrow A, \delta(M)\) be a \( \delta(M) \)-ring and \( \varphi \) be the induced Frobenius endomorphism on \( A \). Then the rule

\[
m \mapsto 1 + p \delta(M)(m)
\]

defines a morphism of monoids \( M \to 1 + pA \) such that for any \( m \in M \),

\[
\varphi(\alpha(m)) = \alpha(m)^p(1 + p \delta(M)(m)). \tag{2.1}
\]

\(^3\)When we say a \( p \)-adic formal scheme is semi-stable, we always assume it is separated.
If $p$ belongs to the Jacobson radical of $A$, then $1 + pA \subset A^\times$. This implies that for any $n \geq 1$, $\varphi^n(\alpha(m))$ differs from $\alpha(m)p^m$ by a unit. Therefore if $\alpha : M \to A$ is furthermore a log structure (i.e., the restriction of $\alpha$ on $\alpha^{-1}(A^\times)$ induces an isomorphism $\alpha^{-1}(A^\times) \to A^\times$), then the rule $m \mapsto m^p\alpha^{-1}(1 + p\delta_{\log}(m))$ defines an endomorphism $\phi$ of $M$ and $(\varphi : A \to A, \phi : M \to M)$ is an endomorphism of the log structure $(M \to A)$. In this case, $(\varphi, \phi)$ is a lifting of the Frobenius endomorphism of the log structure $(M^a \to A/p)$, where $(M^a \to A/p)$ is the associated log structure on $A/p$ induced by the prelog structure $(M \to A/p)$.

**Definition 2.2 (Kos20 Definition 3.3[)]. A (bounded) prelog prism is a tuple $(A, I, M, \delta_{\log})$ such that $(A, I)$ is a (bounded) prism and $(A, M, \delta_{\log})$ is a $\delta_{\log}$-ring. A bounded prelog prism $(A, I, M, \delta_{\log})$ is called a log prism if $(A, M)$ is $(p, I)$-adically log-affine. Morphisms of (pre-)log prisms are defined as morphisms of underlying $\delta_{\log}$-rings.

Every bounded prelog prism is uniquely associated to a log prism in the following sense:

**Lemma 2.3 (Kos20 Proposition 2.14,Corollary 2.15).** Let $(A, M \xrightarrow{\alpha} A, \delta_{\log})$ be a $\delta_{\log}$-ring and $N$ be the monoid fitting into the following push-out diagram of monoids:

$$
\begin{array}{ccc}
\alpha^{-1}(A^\times) & \xrightarrow{\alpha} & A^\times \\
\downarrow & & \downarrow \\
M & \xrightarrow{\alpha} & N
\end{array}
$$

Then $(A, N)$ admits a unique $\delta_{\log}$-structure compatible with $(A, M, \delta_{\log})$. If moreover $A$ is classically $J$-adically complete for some finite generated ideal $J$ of $A$ containing $p$ (e.g. $(A, I, M, \delta_{\log})$ is a bounded prelog prism and $J = I + pA$), then the above construction is compatible with the étale localization of the log $J$-adic formal scheme $(\text{Spf}(A), M^a)$.

If no ambiguity appears, for a bounded prelog prism $(A, I, M, \delta_{\log})$, we denote its associated log prism by $(A, I, M^a, \delta_{\log})$, where $M^a := \Gamma(\text{Spf}(A), M^a)$.

Now, we follow [Kos20 Definition 4.1] and [DL21 Definition 5.0.6] to define the absolute logarithmic prismatic site for a log $p$-adic formal scheme $(\mathfrak{X}, \mathcal{M}_\mathfrak{X})$.

**Definition 2.4 (Absolute logarithmic prismatic site).** Let $(\mathfrak{X}, \mathcal{M}_\mathfrak{X})$ be a log $p$-adic formal scheme. We denote by $(\mathfrak{X}, \mathcal{M}_\mathfrak{X})_\Delta$ the opposite of the category whose objects are log prisms $\mathfrak{A} = (A, I, M, \delta_{\log})$ with integral log structures together with maps of $p$-adic formal schemes $f_{\mathfrak{A}} : \text{Spf}(A/I) \to \mathfrak{X}$ which induce exact closed immersions of log $(p, I)$-adic formal schemes $(\text{Spf}(A/I), f_{\mathfrak{A}}^*\mathcal{M}_\mathfrak{X}) \to (\text{Spf}(A), M^a)$

A morphism

$$
g : \mathfrak{A} = (A, I, M, \delta_{\log}) \to \mathfrak{B} = (B, J, N, \delta_{\log})
$$

of objects $\mathfrak{A}, \mathfrak{B} \in (\mathfrak{X}, \mathcal{M}_\mathfrak{X})_\Delta$ is a morphism of log prisms making the following diagram

$$
\begin{array}{ccc}
(\text{Spf}(B), N^a) & \xrightarrow{(\text{Spf}(B), N^a)} & (\text{Spf}(A), M^a) \\
\downarrow & & \downarrow \\
(\text{Spf}(A/I), f_{\mathfrak{A}}^*\mathcal{M}_\mathfrak{X}) & \xrightarrow{(\text{Spf}(A/I), f_{\mathfrak{A}}^*\mathcal{M}_\mathfrak{X})} & (\text{Spf}(A/I), f_{\mathfrak{A}}^*\mathcal{M}_\mathfrak{X}) \\
(\mathfrak{X}, \mathcal{M}_\mathfrak{X}) & & (\mathfrak{X}, \mathcal{M}_\mathfrak{X})
\end{array}
$$

of morphisms of log $(p, I)$-adic formal schemes commute. Such a morphism is a cover if and only if $A \to B$ is $(p, I)$-adically faithfully flat and the induced morphism of $(\text{Spf}(B), N^a) \to (\text{Spf}(A), M^a)$ is strict.\footnote{This amounts to that the log structure $f_{\mathfrak{A}}^*\mathcal{M}_\mathfrak{X}$ on $\text{Spf}(A/I)$ coincides with that associated to the composition $M \to A \to A/I$.}

We define the structure sheaf $\mathcal{O}_\Delta$ on $(\mathfrak{X}, \mathcal{M}_\mathfrak{X})_\Delta$ by sending an object $(A, I, M, \delta_{\log})$ to $A$.\footnote{This amounts to that the log structure $N$ on $B$ coincides with that associated to the composition $M \to A \to B$.}
When $\mathfrak{X} = \text{Spf}(R)$ is affine and the log structure $\mathcal{M}_X$ is induced by an integral prelog structure $(M \rightarrow R)$, we also denote the absolute logarithmic prismatic site $(\mathfrak{X}, \mathcal{M}_X)_{\triangle}$ by $(M \rightarrow R)_{\triangle}$.

**Definition 2.5** (The perfect logarithmic prismatic site). A prelog prism $(A, I, M, \delta_{\log})$ is called **perfect** if $(A, I)$ is a perfect prism. For a log $p$-adic formal scheme $(\mathfrak{X}, \mathcal{M}_X)$, we denote by $(\mathfrak{X}, \mathcal{M}_X)_{\text{perf}}$ the full subcategory of $(\mathfrak{X}, \mathcal{M}_X)_{\triangle}$ whose objects are perfect log prisms.

**Remark 2.6** (Perfection). For a prelog prism $(A, I, M, \delta_{\log})$ with induced Frobenius lifting $\varphi$, we denote by $A_{\text{perf}}$ the $(p, I)$-completion of the perfection of $A$ with respect to $\varphi$. Then $(A_{\text{perf}}, I, M \rightarrow A \rightarrow A_{\text{perf}}, \delta_{\log})$ is the initial object in the category of prelog prisms over $(A, I, M, \delta_{\log})$.

Now we give some basic examples of log prisms used in this paper.

**Example 2.7.** (1) Let $M_{\log} \rightarrow \mathcal{E}$ be the log structure on $\mathcal{E}$ associated to the prelog structure $\mathbb{N} \xrightarrow{1-u} \mathcal{E}$ and let $\delta_{\log}(u) = 0$. Then $(\mathcal{E}, (E), M_{\log}, \delta_{\log})$ is a log prism in $(\mathbb{N} \xrightarrow{1-u} \mathcal{O}_K)_{\triangle}$, which will be referred as Breuil–Kisin log prism.

(2) Let $M_{\text{inf}} \rightarrow A_{\text{inf}}$ be the log structure on $A_{\text{inf}}$ associated to the prelog structure $\mathbb{N} \xrightarrow{1-|\pi^j|} A_{\text{inf}}$. Then $(A_{\text{inf}}, (\xi), M_{\text{inf}}, \delta_{\log})$ is a perfect log prism in $(\mathbb{N} \xrightarrow{1-u} \mathcal{O}_K)_{\triangle}$, which will be referred as Fontaine log prism.

(3) The map $i : \mathcal{E} \rightarrow A_{\text{inf}}$ sending $u$ to $[\pi^q]$ induces a morphism

$$(\mathcal{E}, (E), M_{\log}, \delta_{\log}) \rightarrow (A_{\text{inf}}, (\xi), M_{A_{\text{inf}}}, \delta_{\log})$$

of log prisms, which is a cover in $(\mathbb{N} \xrightarrow{1-u} \mathcal{O}_K)_{\triangle}$.

**Example 2.8.** Let $\mathfrak{S}(\mathbb{N}^{r+1} \xrightarrow{T_i \cdot \pi \beta} \mathfrak{O}_K)$ be an étale morphism and $\alpha : \mathbb{N}^{r+1} \rightarrow R$ be the morphism of monoids sending $c_i$ to $T_i$ for any $0 \leq i \leq r$, where $c_i$ denotes the generator of $(i + 1)$-th component of $\mathbb{N}^{r+1}$. Let $\hat{R}_\infty$ be the $p$-complete base change to $\hat{A}_\infty$, which is the $p$-adic completion of the union $\bigcup_{n \geq 0} A_n$ for

$$A_n = \mathcal{O}_C(T_0 \pm \frac{\beta}{\pi}, T_r \pm \frac{\beta}{\pi^2}, T_1 \pm \frac{\beta}{\pi^2}, \ldots, T_d \pm \frac{\beta}{\pi^2})/\langle T_0 \pm \beta, T_r \pm \beta - \pi \pm \beta \rangle,$$

along the étale morphism $\square$.

(1) Let $\mathfrak{S}(R)$ be the unique lifting of $R$ over

$$(\mathfrak{S}(R), (E), M_{\log}(R), \delta_{\log})$$

induced by $\square$ and $\mathbb{N}^{r+1} \rightarrow \mathfrak{S}(R)$ be the morphism of monoids sending $c_i$ to $T_i$ for any $0 \leq i \leq r$ which induces a log structure $M_{\mathfrak{S}(R)}$ on $\mathfrak{S}(R)$. Put $\delta_{\log}(c_i) = \delta(T_i) = 0$ for any $0 \leq i \leq r < j \leq d$. Then $(\mathfrak{S}(R), (E), M_{\mathfrak{S}(R)}, \delta_{\log})$ is a log prism in $(\mathbb{N}^{r+1} \xrightarrow{\alpha} R)_{\triangle}$.

(2) Let $M_{A_{\text{inf}}(\hat{R}_\infty)}$ be the log structure on $A_{\text{inf}}(\hat{R}_\infty)$ associated to the prelog structure $\mathbb{N}^{r+1} \rightarrow A_{\text{inf}}(\hat{R}_\infty)$ sending $c_i$ to $[T_i^q]$, where $T_i^q = (T_i \pm \beta, \ldots)$ in $\hat{R}_\infty$ for any $0 \leq i \leq r$. Then $(A_{\text{inf}}(\hat{R}_\infty), (\xi), M_{A_{\text{inf}}(\hat{R}_\infty)}$, $\delta_{\log})$ is a perfect log prism in $(\mathbb{N}^{r+1} \xrightarrow{\alpha} R)_{\triangle}$.

(3) The map $\mathfrak{S}(R) \rightarrow A_{\text{inf}}(\hat{R}_\infty)$ sending $u$ to $[\pi^q]$ and $T_i$ 's to $[T_i^q]$ 's defines a morphism

$$(\mathfrak{S}(R), (E), M_{\mathfrak{S}(R)}, \delta_{\log}) \rightarrow (A_{\text{inf}}(\hat{R}_\infty), (\xi), M_{A_{\text{inf}}(\hat{R}_\infty)}, \delta_{\log})$$

of log prisms, which is a cover in $(\mathbb{N}^{r+1} \xrightarrow{\alpha} R)_{\triangle}$.

The following structure lemma says that in some cases, log structures of prisms in $(\mathfrak{X}, \mathcal{M}_X)_{\triangle}$ are “uniformly” determined by the log structure $\mathcal{M}_X$ on $\mathfrak{X}$. 
Lemma 2.9. (1) For any log prism \((A, I, M, \delta_{\log}) \in (\mathbb{N}^{1+i} \to \mathcal{O}_K)_{\Delta}\), there exists a lifting \(u\) of \(\pi \in A/I\) such that the log structure \(M\) is associated to the prelog structure \((\mathbb{N}^{1+i} \to A)\).

(2) Let \((\mathbb{N}^{r+1} \to R)\) be as in Example 2.8. Then for any log prism \((A, I, M \to A, \delta_{\log}) \in (\mathbb{N}^{r+1} \to R)_{\Delta}\), there exist liftings \(t_i\)'s of \(T_i\)'s in \(A\) such that \(M\) is associated to the prelog structure \(\mathbb{N}^{r+1} \to A\) which sends \(e_i\) to \(t_i\) for any \(0 \leq i \leq r\).

**Proof.** The item (1) is [DL21] Lemma 5.0.10. The proof of item (2) is similar but for the convenience of readers, we show details here.

By virtues of Definition 2.4, the log structures on \(A/I\) associated to the prelog structures \((\mathbb{N}^{r+1} \to R \to A/I)\) and \((M \to A \to A/I)\) coincide. Therefore, there exists \(m_i\)'s in \(M\) such that \(\alpha(m_i) = \bar{u}_iT_i\) in \((A/I)\) for \(\bar{u}_i\)'s in \((A/I)^\times\). Since \(A\) is \(i\)-adically complete, \(\bar{u}_i\)'s lift to units \(u_i\)'s in \(A^\times\). So replacing \(m_i\)'s by \(m_iu_i\)'s, we may assume \(\alpha(m_i) = T_i\ mod I\).

Let \(M' \to A\) be the log structure associated to the prelog structure \((\mathbb{N}^{r+1} \to A/I\to A)\).

Then we get a morphism \(f : (M' \to A) \to (M \to A)\) of log structures whose induced log structures modulo \(I\) coincide. It remains to show \(f\) is the identity itself. Since \(A\) is \(I\)-complete, this follows from deformation theory of log structures (e.g. [Ols05] Theorem 8.36]) directly.

**Remark 2.10** (Rigidity of log structures). Let \((A, I, M, \delta_{\log}) \in (\mathbb{X}, \mathcal{M}_X)_{\Delta}\) be a log prism. For any \(A/I\)-algebra \(\overline{B}\), which is endowed with the log structure \((M^a \to \overline{B})\) associated to \((M \to A/I \to \overline{B})\), the deformation problem for lifting \((M^a \to \overline{B})\) to a log prism over \((A, I, M, \delta_{\log})\) coincides with that for lifting \(\overline{B}\) to a prism over \((A, I)\).

Indeed, by [Ols05] Lemma 8.22, [Lemma 8.26], there exists a canonical quasi-isomorphism between \((p, I)\)-completions of \(\mathcal{L}_{(M^a \to \overline{B})/\overline{B}}\) and \(\mathcal{L}_{\overline{B}/A/I}\). In particular, for any \((B, IB, N, \delta_{\log}) \in (\mathbb{X}, \mathcal{M}_X)_{\Delta}\) over \((A, I, M, \delta_{\log})\), the log structure \(N \to B\) has to be associated to the prelog structure \((M \to A \to B)\).

**Remark 2.11**. Let \((A, I, M, \delta_{\log}) \to (B, IB, N, \delta_{\log})\) be a cover in \((\mathbb{X}, \mathcal{M}_X)_{\Delta}\). For any log prism \((C, IC, L, \delta_{\log}) \in (\mathbb{X}, \mathcal{M}_X)_{\Delta}\) over \((A, I, M, \delta_{\log})\), let \(D\) denote the \((p, I)\)-complete tensor product \(B \hat{\otimes}_A C\) which is endowed with the log structure \((L^a \to D)\) associated to the composition \(L \to C \to D\). Then \(\delta_{\log} : L \to C \to D\) defines a log prism \((D, ID, L^a, \delta_{\log})\) which is the pushout of the diagram

\[
\begin{align*}
(B, IB, N, \delta_{\log}) \leftarrow (A, I, M, \delta_{\log}) & \to (C, IC, L, \delta_{\log}).
\end{align*}
\]

In particular, let \((B^*, IB^*)\) be the Čech nerve of the cover \((A, I) \to (B, IB)\) of prisms and let \(N^* \to B^*\) be the log structure induced by \(M \to A \to B^*\). Then \((B^*, IB^*, N^*, \delta_{\log})\) is the Čech nerve of the cover \((A, I, M, \delta_{\log}) \to (B, IB, N, \delta_{\log})\) of log prisms in \((\mathbb{X}, \mathcal{M}_X)_{\Delta}\). The argument in the proof of [BS19 Corollary 3.12] shows that the presheaves \((A, I, M, \delta_{\log}) \to A\) and \((A, I, M, \delta_{\log}) \to A/I\) are indeed sheaves. The latter is denoted by \(\underline{\mathcal{O}}_{\Delta}\).

At the end of this section, we want to study structures of (certain) perfect log prisms. We begin with the following lemma:

**Lemma 2.12**. Let \((A, I)\) be a perfect prism and \(R := A/I\). Then for any \(x \in A\) such that \(\varphi(x) = xp(1 + py)\) for some \(y \in A\), \(x\) factors as

\[
x = [a] \prod_{i=1}^{\infty} (1 + p\varphi^{-1}(y))^{p^{i-1}},
\]

where \(a\) is the reduction of \(x\) modulo \(p\).

Note that \([a] \prod_{i=1}^{\infty} (1 + p\varphi^{-1}(y))^{p^{i-1}}\) is well-defined as \(A\) is classically \(p\)-complete.
Lemma 2.14. Let $\mathcal{P}$ be a prelog prism and the category of perfectoid rings. We want to generalise this to the logarithmic case. Let $R$ structure

Now, assume we are given an $x \in R$ and we denote by $\rho := \varphi^{-1}(x)$. This violates to that $l \geq 1$. Then we see that $\rho x$ is not zero in $S/\rho S$, which implies that $\rho^l x$ does not vanish modulo $\rho^l S$. This violates to that $l \geq 1$ together with $x \in S$.

Proof. Put $y_n = \varphi^{-n}(y)$ for any $n \geq 0$. We are reduced to showing that for any $n \geq 1$,

$$x \equiv [a] \prod_{i=1}^{\infty} (1 + py_i)^{p^{i-1}} \mod p^n.$$  

Write $x = [a] + px_1$ for the unique $x_1 \in A$. Then the assumption on $x$ can be restated as follows:

$$[a]^p + p\varphi(x_1) = ([a] + px_1)^p(1 + p\varphi(y)).$$

So we get that $\varphi(x_1) \equiv \varphi([a]y_1) \mod p$ and a fortiori that $x_1 \equiv [a]y_1 \mod p$. Therefore, we can write $x = [a](1 + py_1) + p^2x_2$ for the unique $x_2 \in A$.

Assume $x = [a] \prod_{i=1}^{n} (1 + py_i)^{p^{i-1}} + p^{n+1}x_{n+1}$ for some $n \geq 1$ with $x_{n+1} \in A$. Then the assumption on $x$ says that

$$[a]^p \prod_{i=0}^{n-1} (1 + py_i)^{p^i} + p^{n+1}\varphi(x_{n+1}) = ([a] \prod_{i=1}^{n} (1 + py_i)^{p^{i-1}} + p^{n+1}x_{n+1})^p(1 + py)
\equiv [a]^p \prod_{i=1}^{n} (1 + py_i)^{p^i}(1 + py) \mod p^{n+2}.$$

As a consequence, we have

$$p^{n+1}x_{n+1} \equiv [a] \prod_{i=1}^{n} (1 + py_i)^{p^{i-1}}((1 + py_{n+1})p^n - 1) \mod p^{n+2}.$$  

In particular, we get $x \equiv [a] \prod_{i=1}^{n+1} (1 + py_i)^{p^{i-1}} \mod p^{n+2}$ as desired. 

The following corollary is clear.

Corollary 2.13. Let $(A, I, M, \delta_{log})$ be a perfect log prism and $R = A/I$, then $\alpha(M) \subset [R^p] \cdot (1 + pA)$, where $[R^p] := \{[x]| x \in R^p\}$. In particular, if we equip $A$ with the log structure $(N \to A)$ associated to the prelog structure $R^p \to A$, then there exists a unique morphism of log prisms $(A, I, M, \delta_{log}) \to (A, I, N, \delta_{log}).$

As mentioned in [BS10] Theorem 3.10], there is a canonical equivalence between the category of perfect prisms and the category of perfectoid rings. We want to generalise this to the logarithmic case.

Lemma 2.14. Let $S$ be a $p$-torsion free perfectoid ring. Then for any $x \in S[\frac{1}{p}]$ such that $x^p \in S$, we have $x \in S$.

Proof. This is well-known and we provide a proof here for the convenience of readers.

By [BMS18] Lemma 3.9, there exists a $\varpi \in S^p$ such that $\varpi^p$ is a unit multiple of $p$. Define $\rho := \varphi^{-1}(\varpi^p)$. Then [BMS18] Lemma 3.10 implies that the Frobenius induces an isomorphism

$$S/\rho S \cong S.$$  

Now, assume we are given an $x \in S[\frac{1}{p}]$ such that $x^p \in S$. Then there exists an integer $N$ such that $\rho^N x \in S$ and we denote by $l$ the smallest one satisfying above property. It is enough to show that $l \leq 0$. Otherwise, assume that $l \geq 1$. Then we see that $\rho^l x$ is not zero in $S/\rho S$, which implies that $\rho^l x^p$ does not vanish modulo $\rho^l S$. This violates to that $l \geq 1$ together with $x \in S$.

Lemma 2.15. Let $R$ be a perfectoid ring with tilt $R^p$. Let $a_1, a_2 \in R^p$ and $x \in R$. If $a_1^p = a_2^p(1 + px)$ and $a_1^p \mid p$, then there exists a unit $u \in R^p$ such that $a_1 = a_2u$ and $u^2 = 1 + px$.  

11
Proof. Let $\varpi \in R^p$ such that $\varpi^d$ is a unit multiple of $p$.

We first assume $R$ is $p$-torsion free and hence $R^p$ is $\varpi$-torsion free. In this case, $c := \frac{a_1}{a_2}$ is well-defined in $R^p[\frac{1}{a_2}]$ with $\varpi^d = (1 + px)$. By Lemma 2.14, for any $n \geq 0$, $(\varpi^{-n}(c))^d \in R$. So $c \in R^p$. Combining this with that $\varpi^d \in R^p$, we see that $c \in (R^p)^\times$ and therefore $u = c$ is desired.

In general, by [Bha, Proposition 3.2], if we put $S = R/R[\sqrt{pR}], \overline{R} = R/\sqrt{pR}$ and $\overline{S} = S/\sqrt{pS}$, then there is a fibre square of perfectoid rings:

$$
\begin{array}{c}
R \\
\downarrow \\
R
\end{array}
\begin{array}{c}
S \\
\downarrow \\
\overline{S}
\end{array}
$$

By what we have proved, there exists a $c \in (S^p)^\times$ which is represented by the compatible sequence $\{c_n\}_{n \geq 0}$ such that $a_1 = a_2c$ and $\varpi^d = c_0 = (1 + px)$ in $S^p$. Since $c_0$ coincides with 1 in $\overline{S}$, so are $c_n$’s. In particular, denote by $\tilde{c}_n$ the element $(1, c_n) \in \overline{R} \times \overline{S} \cong R$, then $\tilde{c}_0 = 1 + px$ and $\tilde{c}_{n+1} = \tilde{c}_n$ for all $n$. Put $\tilde{c} \in R^p$ which is represented by $\{\tilde{c}_n\}$. Then $u = \tilde{c}$ is desired. \qed

Remark 2.16. We only proved above Lemma 2.15 for $R$ absolutely integral closed in the early draft. This stronger version is due to some discussions with Henh Duong.

Lemma 2.17. For any perfect log prism $(A, I, M_1, \delta_{log})$ and $(A, I, M_2, \delta_{log})$ in $(N \xrightarrow{1+\tau} O_K)_\Delta$ or $(N^{r+1} \xrightarrow{\alpha} R)_\Delta$ with the same underlying prism, we always have $(A, I, M_1, \delta_{log}) = (A, I, M_2, \delta_{log})$.

Proof. We only deal with the $(N^{r+1} \xrightarrow{\alpha} R)_\Delta$ case and the $(N \xrightarrow{1+\tau} O_K)_\Delta$ case follows from the similar argument.

By Lemma 2.9 for any $0 \leq i \leq r$, there are liftings $t_{1i}, t_{2i} \in A$ of $T_i$ such that the log structures $M_1 \to A$ and $M_2 \to A$ are associated to prelog structures $(N^{r+1} \xrightarrow{\tilde{c}_i t_{1i}, \forall i \leq r} A)$ and $(N^{r+1} \xrightarrow{\tilde{c}_i t_{2i}, \forall i \leq r} A)$ for $0 \leq i \leq r$, respectively. By Corollary 2.13 there are $a_{1i}, a_{2i} \in (A/I)^b$ and $x_{i1}, x_{i2} \in A$ such that $t_{1i} = (a_{1i})(1 + px_{i1})$ and $t_{2i} = (a_{2i})(1 + px_{i2})$ for any $0 \leq i \leq r$. Let $\theta : A \to A/I$ be the natural surjection. Then for all $i$, we have

$$a_{1i}^d(1 + p\theta(x_{i1})) = T_i = a_{2i}^d(1 + p\theta(x_{i2})).$$

Since $A/I$ is classically p-complete, for any $0 \leq i \leq r$, there exists a $y_i \in A/I$ such that $a_{1i}^d = a_{2i}^d(1 + py_i)$. By Lemma 2.15, we deduce that $a_{1i}$ and $a_{2i}$ differ from a unit in $(A/I)^b$. So for any $0 \leq i \leq r$, there exists $u_i \in A^\times$ such that $t_{1i} = t_{2i} u_i$ as desired. \qed

Proposition 2.18. Let $(X, M_X)$ be $(N \xrightarrow{1+\tau} O_K)_\Delta$ or $(N^{r+1} \xrightarrow{\alpha} R)_\Delta$. Then one can identify the sites $(X, M_X)^{perf} = (X)^{perf}$ via the forgetful functor $(A, I, M, \delta_{log}) \to (A, I)$.

Proof. We only deal with the $(N^{r+1} \to R)_\Delta$ case. By Lemma 2.17, it is enough to show that for any perfect prism $(A, I)$, there exists a log structure $M \to A$ making $(A, I, M, \delta_{log})$ an object in $(N^{r+1} \to R)_\Delta$. Let $\varpi \in (A/I)^b$ such that $\varpi^d$ is a unit multiple of $p$ and let $\rho = \varpi^{-1}(\varpi)^d$ as before. By [BMS18, Lemma 3.9], for any $0 \leq i \leq r$, one can find a compatible sequence $\{t_{i,n}\}_{n \geq 0}$ in $A/I$ such that $t_{i,0}$ coincides with $T_i$ and for any $n \geq 0$, $t_{i,n}^{p^{\rho}}$ coincides with $t_{i,n}$ modulo $p\rho A/I$. Let $\tilde{t}_i \in (A/I)^b$ be the element determined by $\{t_{i,n}\}_{n \geq 0}$. Then we obtain that

$$\tilde{t}_i^d = \lim_{n \to +\infty} t_{i,n}^{p^n}.\tag{12}$$

In particular, we have

$$\tilde{t}_i^d \equiv T_i \mod p\rho A/I.$$

Write $\tilde{t}_i = T_i + p\rho x_i$ for some $x_i \in A/I$. For any $0 \leq i \leq r$, put $T_i := \prod_{0 \leq j \leq r, j \neq i} T_j$. Then $T_i T_i^\rho = \pi$. In particular, there exists some $y_i \in R$ such that $p = T_i T_i^\rho y_i$. Now, we obtain that

$$\tilde{t}_i^d = T_i + p\rho x_i = T_i(1 + \rho T_i^\rho x_i)$$

In particular,...
is a unit multiple of $T_i$. It is easy to see that the log structure associated to
\[
\oplus_{i=0}^r \mathbb{N} e_i \xrightarrow{e_i + r_i} A
\]
is desired. We win! \hfill \square

Let $\mathfrak{X}$ be a separated semi-stable formal scheme over $\mathcal{O}_K$ with the rigid analytic generic fibre $X$ equipped with the log structure $\mathcal{M}_X = \mathcal{O}_X \cap \mathcal{O}_X \to \mathcal{O}_X$. Since étale locally on $\mathfrak{X}$, $(\mathfrak{X}, \mathcal{M}_X)$ is induced by the prelog structure $\mathbb{N}^{r+1} \to R$ as described in Example 2.8, then the following result follows from Proposition 2.18 directly.

**Corollary 2.19.** With notations as above, we have $\text{Sh}((\mathfrak{X})_{\Delta}^{\text{perf}}) = \text{Sh}((\mathfrak{X}, \mathcal{M}_X)_{\Delta}^{\text{perf}})$.

**Remark 2.20.** In this subsection, most of results (from Lemma 2.4 to Corollary 2.19) are proved for log formal schemes $(\mathfrak{X}, \mathcal{M}_X)$ whose log structures are locally induced by finite generated free monoids (i.e. locally, $\mathfrak{X} = \text{Spf}(R)$ with $\mathcal{M}_X$ induced by $\mathbb{N}^s \to R$ for some $s \geq 0$). We believe that those relevant results still hold true in a more general setting and for example, hold for those $(\mathfrak{X}, \mathcal{M}_X)$ with $\mathcal{M}_X$ just fine (or even integral). Since results given above are enough for our use, we will not handle the most general case in this paper.

### 3 The Hodge–Tate crystals on absolute logarithmic prismatic site

In this section, we always assume $\mathfrak{X}$ is a semi-stable $p$-adic formal scheme over $\mathcal{O}_K$ with the log structure $\mathcal{M}_X = \mathcal{O}_X \cap \mathcal{O}_X$, where $X$ is the rigid analytic generic fibre of $\mathfrak{X}$. Étale locally, $\mathfrak{X} = \text{Spf}(R)$ and $\mathcal{M}_X$ is induced by the prelog structure $\mathbb{N}^{r+1} \to R$ as described in Example 2.8.

Our purpose is to generalize results in [MW21b] and [MW22] to the logarithmic case. That is, we want to study Hodge–Tate crystals on the absolute logarithmic prismatic site $(\mathfrak{X}, \mathcal{M}_X)_{\Delta}$. Before moving on, we specify the meaning of Hodge–Tate crystals as follows:

**Definition 3.1 (The Hodge–Tate crystals).** By a **Hodge–Tate crystal** on $(\mathfrak{X}, \mathcal{M}_X)_{\Delta}$, we mean a sheaf $\mathcal{M}$ of $\overline{\mathcal{O}}_{\Delta}$-modules satisfying the following properties:

1. For any $\mathfrak{A} = (A, I, M, \delta_{\log}) \in (\mathfrak{X}, \mathcal{M}_X)_{\Delta}$, $\mathcal{M}(\mathfrak{A})$ is a finite projective $A/I$-module.

2. For any morphism $\mathfrak{A} = (A, I, M, \delta_{\log}) \to \mathfrak{B} = (B, IB, N, \delta_{\log})$ in $(\mathfrak{X}, \mathcal{M}_X)_{\Delta}$, there is a canonical isomorphism

$$\mathcal{M}(\mathfrak{A}) \otimes_{A/I} B/IB \to \mathcal{M}(\mathfrak{B}).$$

We denote by $\text{Vect}((\mathfrak{X}, \mathcal{M}_X)_{\Delta}, \overline{\mathcal{O}}_{\Delta})$ the category of Hodge–Tate crystals on $(\mathfrak{X}, \mathcal{M}_X)_{\Delta}$.

Similarly, we define **rational Hodge–Tate crystals** on $(\mathfrak{X}, \mathcal{M}_X)_{\Delta}$ (resp. $(\mathfrak{X}, \mathcal{M}_X)_{\Delta}^{\text{perf}}$) by replacing $\overline{\mathcal{O}}_{\Delta}$ by $\overline{\mathcal{O}}_{\Delta, [1]}$ and denote the corresponding category by $\text{Vect}((\mathfrak{X}, \mathcal{M}_X)_{\Delta}, \overline{\mathcal{O}}_{\Delta, [1]})$ (resp. $\text{Vect}((\mathfrak{X}, \mathcal{M}_X)_{\Delta}^{\text{perf}}, \overline{\mathcal{O}}_{\Delta, [1]})$).

In [MW21b] and [MW22], the authors proved the following theorem:

**Theorem 3.2 (MW22 Theorem 1.6, 1.12).** Assume $\mathfrak{X}$ is a smooth $p$-adic formal scheme over $\mathcal{O}_K$ of relative dimension $d$. Then there is an equivalence from the category $\text{Vect}((\mathfrak{X})_{\Delta}, \overline{\mathcal{O}}_{\Delta, [1]})$ of rational Hodge–Tate crystals to the category $\text{HIG}^{\text{st}}((\mathfrak{X}, \mathcal{O}_X, [1]))$ of triples $(\mathcal{H}, \Theta, \phi)$ consisting of

1. A Higgs bundle $(\mathcal{H}, \Theta)$ with coefficients in $\mathcal{O}_X [1]$ such that $\Theta$ is nilpotent and

2. An endomorphism $\phi$ of $\mathcal{H}$ such that $[\Theta, \phi] = E'(\pi)\Theta$ and $\lim_{n \to +\infty} \prod_{i=0}^n (\phi + iE'((\pi))) = 0$.

Such a triple was referred as an **enhanced Higgs bundle** in loc. cit.
The equivalence fits into the following commutative diagram

\[
\begin{array}{ccc}
\text{Vect}((\mathcal{X}_\Delta, \overline{\mathcal{O}}_{\Delta}[^1_p])) & \xrightarrow{\sim} & \text{Vect}((\mathcal{X}_{\Delta}^{\text{perf}}, \overline{\mathcal{O}}_{\Delta}[^1_p])) \\
\downarrow \cong & & \downarrow \cong \\
\text{HIG}_{\text{nil}}^\Delta(\mathcal{X}, \mathcal{O}_X[1_p]) & \xrightarrow{\sim} & \text{HIG}_{G_K}^\Delta(X_C),
\end{array}
\]

where all arrows are fully faithful functors. Here, we use “\(\cong\)” to denote equivalences of categories.

If moreover \(X = \text{Spf}(R)\) is small affine, then the above equivalence upgrades to the integral and derived level.

More precisely, there is an equivalence (depending on the chosen framing) from the category \(\text{HIG}_{\text{nil}}^\Delta(R)\) of triples \((H, \Theta, \phi)\) consisting of a finite projective \(R\)-module \(H\), a nilpotent Higgs field \(\Theta\) on \(H\) and an endomorphism \(\phi\) of \(H\) satisfying \([\Theta, \phi] = \epsilon^\prime(\pi)\Theta\) and \(\lim_{n \to +\infty} \prod_{i=0}^n (\phi + i\epsilon^\prime(\pi)) = 0\). In this case, there is a quasi-isomorphism

\[
\text{R} \Gamma ((R, M)) \cong \lim_{n \to +\infty} \prod_{i=0}^n (\phi_M + i\epsilon^\prime(\pi)) = 0
\]

for any Hodge–Tate crystal \(M\) with associated triple \((H, \Theta, \phi)\).

In particular, when \(X = \text{Spf}(\mathcal{O}_K)\), the theorem can be formulated as follows:

**Theorem 3.3** ([MW21b Theorem 1.2, 1.3, 1.7, Gao22 Theorem 4.3.3]). The evaluation at Breuil–Kisin prism \((\mathcal{G}, (E))\) induces an equivalence from the category of Hodge–Tate crystals on \((\mathcal{O}_K)_\Delta\) to the category \(\text{HT}((\mathcal{O}_K)_{\Delta})\) of pairs \((M, \phi_M)\) consisting of a finite projective \(\mathcal{O}_K\)-module \(M\) and an \(\mathcal{O}_K\)-linear endomorphism \(\phi_M\) of \(M\) such that

\[
\lim_{n \to +\infty} \prod_{i=0}^n (\phi_M + i\epsilon^\prime(\pi)) = 0.
\]

The above equivalence is still true for rational Hodge–Tate crystals by using finite dimensional \(K\)-vector spaces instead of finite projective \(\mathcal{O}_K\)-modules. Moreover, let \(M\) be a (rational) Hodge–Tate crystal on \((\mathcal{O}_K)_\Delta\) with associated pair \((M, \phi_M)\), then the following assertions are true:

1. There exists a natural quasi-isomorphism

\[
\text{R} \Gamma ((\mathcal{O}_K)_\Delta, M) \cong [M \xrightarrow{\phi_M} M].
\]

2. The restriction to \((\mathcal{O}_K)_\Delta^{\text{perf}}\) induces a fully faithful functor from the category of rational Hodge–Tate crystals over \(\mathcal{O}_K\) to the category of (semi-linear) continuous \(C\)-representations of \(G_K\):

\[
\text{Vect}((\mathcal{O}_K)_\Delta, \overline{\mathcal{O}}_{\Delta}[^1_p]) \to \text{Rep}_{G_K}(C).
\]

If we denote by \(V\) the corresponding \(C\)-representation of \(M\) and \(\Theta_V\) the Sen operator of \(V\), then we have \(V = M \otimes_K C\) with \(\Theta_V = -\frac{\phi_M}{\epsilon^\prime(\pi)}\) and a natural quasi-isomorphism

\[
\text{R} \Gamma ((\mathcal{O}_K)_\Delta, M) \cong \text{R} \Gamma (G_K, V).
\]

The purpose of this section is to generalise these results to the logarithmic case (for semi-stable \(X\) with the canonical log structure \(\mathcal{M}_X\)).

### 3.1 The \(\mathcal{O}_K\) case

We first assume \(X = \text{Spf}(\mathcal{O}_K)\) with the log structure \(\mathcal{M}_X\) corresponding to the prelog structure \((\mathbb{N} \xrightarrow{1 \mapsto \pi} \mathcal{O}_K)\). We start with the following lemma:

---

Such a triple was referred as an enhanced Higgs module in loc. cit.
Lemma 3.4. (1) The Breuil–Kisin log prism $(\mathcal{G}, (E), M_{\mathbb{G}}, \delta_{\log})$ is a cover of the final object of the topos $\text{Sh}((\mathbb{N} \overset{1 \mapsto \pi}{\longrightarrow} \mathcal{O}_K)_{\Delta})$.

(2) The Fontaine log prism $(\mathbb{A}_{\text{inf}}, (\xi), M_{\text{inf}}, \delta_{\log})$ is a cover of the final object of the topos $\text{Sh}((\mathbb{N} \overset{1 \mapsto \pi}{\longrightarrow} \mathcal{O}_K)_{\text{perf}})$.

Proof. (1) This is [DL12, Proposition 5.0.16].

(2) This follows from [MW21b, Lemma 2.2 (2)] and Proposition 2.18.

For any $n \geq 0$, let $A^n = W(k)[[u_0, 1 - \frac{u_1}{u_0}, \ldots, 1 - \frac{u_n}{u_0}]]$ and define

$$\mathcal{G}^n = A^n \langle \frac{1 - u_1/u_0}{E(u_0)}, \ldots, \frac{1 - u_n/u_0}{E(u_0)} \rangle_{\Delta},$$

where the completion is with respect to the $(p, E(u_0))$-adic topology. Then $(\mathcal{G}^n, (E(u_0)))$ is a prism in $(\mathcal{O}_K)_{\Delta}$ and the $\frac{u_i}{u_0}$'s are units in $\mathcal{G}^n$ for all $i$. So the log structures on $\mathcal{G}^n$ associated to $\mathbb{N} \overset{1 \mapsto u_0}{\longrightarrow} \mathcal{G}^n$ are independent of the choice of $i$. We denote this log structure by $(M_{\mathbb{G}^n} \rightarrow \mathcal{G}^n)$. Note that $E(u_i) - E(u_0)$ is divided by $u_i - u_0 = u_0 E(u_0) \frac{u_i}{u_0}^{-1}$. By [BS19, Lemma 2.24], the ideal $E(u_i) \mathcal{G}^n$ is also independent of the choice of $i$.

Lemma 3.5. Keep notations as above.

(1) For any $n \geq 1$, the prism $(\mathcal{G}^n, (E(u_0)))$ is bounded. As a consequence, $(\mathcal{G}^n, (E(u_0)), M_{\mathbb{G}^n}, \delta_{\log}) \in (\mathbb{N} \rightarrow \mathcal{O}_K)_{\Delta}$.

(2) The cosimplicial log prism $(\mathcal{G}^\bullet, (E), M_{\mathbb{G}^\bullet}, \delta_{\log})$ is the Čech nerve associated to the Breuil–Kisin log prism.

Proof. This is essentially [DL12, Lemma 5.0.12], but for the convenience of readers, we repeat the proof here.

Note that $(\mathcal{G}, (E))$ is a bounded prism and $A^n$ is faithfully flat over $\mathcal{G}$. Since $p, E(u_0), \frac{u_i}{u_0} - 1, \ldots, \frac{u_n}{u_0} - 1$ is a regular sequence in $A^n$, by [BS19, Proposition 3.13], $(\mathcal{G}^n, (E))$ is $(p, E)$-faithfully flat over $(\mathcal{G}, (E))$. In particular, it is bounded.

To finish the proof, it remains to show $(\mathcal{G}^n, (E), M_{\mathbb{G}^n}, \delta_{\log})$ is the initial object in the category of log prisms $(A, I, M, \delta_{\log})$ in $(\mathbb{N} \rightarrow \mathcal{O}_K)_{\Delta}$ to which there are $n + 1$ morphisms from $(\mathcal{G}, (E), M_{\mathbb{G}}, \delta_{\log})$. We only deal with the $n = 1$ case and the general case follows from a similar argument.

Put $\mathcal{G}^0 = W(k)[[u_0, \ldots, u_n]][\frac{\frac{u_0}{E(u_0)} - \cdots - \frac{u_n}{E(u_0)}}{u_0}]_{\Delta}$ with log structure induced by the map $\oplus_{i=0}^n N_{c_i} \overset{c_i - u_i, y_i}{\longrightarrow} \mathcal{G}^0$. Then $(\mathcal{G}^0, (E(u_0)))$ is the Čech nerve for Breuil–Kisin prism $(\mathcal{G}, (E))$ in the usual prismatic site $(\mathcal{O}_K)_{\Delta}$. In particular, $(\mathcal{G}^0, (E(u_0)))$ is the self coproduct of $(\mathcal{G}, (E))$ in $(\mathcal{O}_K)_{\Delta}$. Note that there are obvious maps $\delta_{\log} : \mathbb{N}^{n+1} \overset{\oplus}{\longrightarrow} \mathbb{N} c_0 \oplus \cdots \oplus \mathbb{N} c_n$ making $(\mathcal{G}^0, (E(u_0)), \mathbb{N}^{n+1}, \delta_{\log})$ a cosimplicial log prism.

Let $(A, I, M, \delta_{\log})$ be a log prism with morphisms $f_0, f_1 : (\mathcal{G}, (E), M_{\mathbb{G}}, \delta_{\log}) \rightarrow (A, I, M, \delta_{\log})$. Then there is a unique morphism of prelog prisms $f : (\mathcal{G}^0, (E(u_0)), \mathbb{N}^2, \delta_{\log}) \rightarrow (A, I, M, \delta_{\log})$. However, $f$ is not a morphism in $(\mathbb{N} \rightarrow \mathcal{O}_K)_{\Delta}$ as $(\mathcal{G}^0, (E(u_0)), \mathbb{N}^2, \delta_{\log})$ is not an object in this category. The reason is that the morphism

$$(\text{Spf}(\mathcal{G}^0/(E(u_0)), (\mathbb{N}^2)^a)) \overset{\mathcal{G}^0}{\longrightarrow} (\text{Spf}(\mathcal{G}^0), (\mathbb{N}^2)^a)$$

is not exact, where the log structure for the former is induced by $\mathbb{N} \rightarrow \mathcal{O}_K \rightarrow \mathcal{G}^0/E(u_0)$.Fortunately, by [Kos20, Proposition 3.7], there exists a log prism $(\mathcal{G}^0, (E(u_0)), N, \delta_{\log}) \in (\mathbb{N} \rightarrow \mathcal{O}_K)_{\Delta}$ over $(\mathcal{G}^0, (E(u_0)), \mathbb{N}^2, \delta_{\log})$ which is initial among log prisms $(C, (E(u_0)), L, \delta_{\log}) \in (\mathcal{O}_K)_{\Delta}$ fitting into the following commutative diagram

$$(\mathcal{G}^0/(E(u_0)), (\mathbb{N} \rightarrow \mathcal{O}_K \rightarrow \mathcal{G}^0/E(u_0))^a) \overset{\mathcal{G}^0}{\longrightarrow} (\mathcal{G}^0/(E(u_0)), (\mathbb{N} \rightarrow \mathcal{O}_K \rightarrow \mathcal{G}^0/E(u_0))^a),$$

The authors knew from Heng Du that the result was first obtained by T. Koshikawa.

9 The log prism $(\mathcal{G}^0/(E(u_0)), N, \delta_{\log})$ is called the exactification of $(\mathcal{G}^0/(E(u_0)), \mathbb{N}^2, \delta_{\log})$ in loc.cit..
where we require the right vertical map induces an exact closed immersion of \((p, E(u_0))-\)adic log formal schemes. In particular, \((\hat{\mathcal{O}}^1_0, (E(u_0)), N, \delta_{\log})\) is the self coproduct of Breuil–Kisin log prism in \((N \to \mathcal{O}_K|_{\Delta})\). It remains to show \((\hat{\mathcal{O}}^1_0, (E(u_0)), N, \delta_{\log}) = (\mathcal{O}^1, (E(u_0)), M_{\mathcal{O}^1}, \delta_{\log})\). By construction, the projection \(p\) appearing in above diagram is induced by \(pr : N^2 \cong N\mathbb{C}_0 + N\mathbb{C}_1 \to N\) with \(pr(e_i) = 1\) for \(i = 1, 2\). Consider the submonoid \(M_{\mathcal{O}^1} \cong \mathbb{Z}\mathbb{C}_0 \oplus \mathbb{Z}\mathbb{C}_1\) generated by \(N^2\) and the kernel \((p\)gp\) : \((N^2)_{\text{gp}} \to (N)_{\text{gp}}\). Then \(M = N\mathbb{C}_0 + N\mathbb{C}_1 + \mathbb{Z}(e_0 - e_1)\). By [Kos20 Construction 2.18] (and [Kos20 Proposition 2.16]), we see that \(\hat{\mathcal{O}}^1_0 = W(k)[u_0, u_1][[\frac{u_0}{u_1}]] \{ 1 - \frac{u_1}{u_0} \}\mathcal{O}_{\mathcal{O}_K^1}\), where the completion is with respect to the \((p, E(u_0))-\)adic topology. Now, [BS19 Proposition 3.13] shows that \(p, E(u_0)\) forms a regular sequence in \(\hat{\mathcal{O}}^1_0\) (as it does in \(W(k)[u_0, u_1][[\frac{u_0}{u_1}]]\)) and hence so is \(u_0, E(u_0)\). Since both \(u_0\) and \(E(u_0)\) divide \(u_0 - u_1\) in \(\hat{\mathcal{O}}^1_0\), we get \(1 - \frac{u_1}{u_0} \in \hat{\mathcal{O}}^1_0\). This implies that \(\hat{\mathcal{O}}^1_0 = \mathcal{O}^1\) as desired.

Proof. This follows from Lemma 3.6 and the proof of [MW21b, Theorem 3.24].

For any \(n \geq 0\), put
\[
A_{\inf}^n := (A_{\inf} \hat{\otimes} W(k) \cdots \hat{\otimes} W(k) A_{\inf})[[1 - \frac{u_1}{u_0}, \ldots, 1 - \frac{u_n}{u_0}], \{ 1 - \frac{u_1}{u_0}/E(u_0), \ldots, 1 - \frac{u_n}{u_0}/E(u_0) \}_{\Delta}^{\text{perf}},
\]
where \(A_{\inf} \hat{\otimes} W(k) \cdots \hat{\otimes} W(k) A_{\inf}\) is the \((p, E)-\)completed tensor product of \((n + 1)-\)copies of \(A_{\inf}\) over \(W(k)\) and \(u_0\) denotes the corresponding \([p^n]\) of the \(i + 1\)-component of this product. Then the log structure associated to \((N \to \mathcal{O}_K|_{\Delta})\) is independent of the choice of \(i\). We denote this log structure by \(M_{\mathcal{O}^1_{\inf}} \to A_{\inf}^n\). The similar argument in the proof of Lemma 2.18 shows that \((A_{\inf}^n, (\xi), M_{\mathcal{O}^1_{\inf}}, \delta_{\log})\) is a log prism and the cosimplicial log prism \((A_{\inf}^n, (\xi), M_{\mathcal{O}^1_{\inf}}, \delta_{\log})\) is the Čech nerve associated to the Fontaine log prism \((A_{\inf}, (\xi), M_{\inf}, \delta_{\log}) \in (N \to \mathcal{O}_K|_{\Delta})\)

Lemma 3.6. The cosimplicial prism \((A_{\inf}^n, (\xi))\) is the Čech nerve associated to \((A_{\inf}, (\xi)) \in (\mathcal{O}_K|_{\Delta})^{\text{perf}}\).

Proof. This is a corollary of Proposition 3.18.

Similar to [MW21b Theorem 3.24], we have the following theorem:

Theorem 3.7. The evaluation at Fontaine log prism \((A_{\inf}, (\xi), M_{\mathcal{O}^1_{\inf}}, \delta_{\log})\) induces an equivalence from the category of rational Hodge–Tate crystals \(\text{Vect}((N \to \mathcal{O}_K|_{\Delta})^{\text{perf}}, \mathcal{O}_{\mathcal{O}_K|_{\Delta}}^{[\frac{1}{p}]}))\) to the category of \(C\)-representations \(\text{Rep}_{\mathcal{O}_K}(C)\) of \(G_K\).

Proof. This follows from Lemma 3.6 and the proof of [MW21b Theorem 3.24].

Now we focus on (rational) Hodge–Tate crystals on \((N \to \mathcal{O}_K|_{\Delta})\). As in [MW21b], the key point is to describe the structure of cosimplicial ring \(G^* / E G^*\).

Lemma 3.8. For any \(1 \leq i \leq n\), denote \(X_i\) the image of \(\frac{1 - u_i}{E(u_0)} \in G^n\) modulo \((E, G)\), then \(G^n / (E, G) \cong \mathcal{O}_K\{X_1, \ldots, X_n\}_{\mathbb{Z}_p}\) is the free \(p\)-d-polynomial ring on the variables \(X_1, \ldots, X_n\). Moreover, for any \(0 \leq i \leq n\), let \(p_i : G^n \to G^{n+1}\) be the structure morphism induced by the order-preserving map

\[
\{0, \ldots, n\} \to \{0, \ldots, i - 1, i + 1, \ldots, n + 1\}
\]

then via above isomorphisms, we have

\[
p_i(X_j) = \begin{cases} (X_{j+1} - X_1)(1 - \pi E^i(\pi)X_1)^{-1}, & i = 0 \\ X_j, & j < i \\ X_{j+1}, & 0 < i \leq j. \end{cases}
\]
Proof. Just modify the proof of [MW21b, Lemma 2.6]. The only difference appears in the formula (3.1) for $p_0$. We need to show

$$p_0(1 - u_j/u_0) = \frac{1 - u_{j+1}/u_1}{E(u_0)} = \frac{u_1/u_0 - u_{j+1}/u_0 u_0 E(u_0)}{E(u_0)}\frac{1}{u_1 E(u_1)}$$

goes to $(X_{j+1} - X_1)(1 - \pi E'(\pi)X_1)^{-1}$ modulo $E$. But this follows from that

$$E(u_0) - E(u_1) \equiv E'(u_1)(u_0 - u_1) \mod (u_0 - u_1)^2$$

$$\equiv u_0 E'(u_0)(1 - u_1/u_0) \mod (E)^2$$

easily. We win! 

In order to compute logarithmic prismatic cohomology for Hodge–Tate crystals, we need the following lemma.

Lemma 3.9. Let $\mathcal{M}$ be a (rational) Hodge–Tate crystal on $\mathbb{N} \to \mathcal{O}_K\Delta$ and $\mathfrak{A} = (A, I, M, \delta_{\log}) \in (\mathbb{N} \to \mathcal{O}_K)\Delta$ be a log prism. Then for any $i \geq 1$, we have

$$H^i(\mathfrak{A}, \mathcal{M}) = 0.$$ 

Proof. For any log prism $\mathfrak{B} = (B, IB, N, \delta_{\log}) \in (\mathbb{N} \to \mathcal{O}_K)\Delta$ which is a cover of $\mathfrak{A}$, by Remark 2.10 the Čech nerve associated to this cover exists and its underlying cosimplicial prism coincides with the Čech nerve associated to the cover $(A, I) \to (B, IB)$ of prisms in $(\mathcal{O}_K)\Delta$. Now the result follows from the same argument used in [Tian21, Lemma 3.11].

Now we are able to prove the logarithmic analogue of [MW21b, Theorem 3.8, Theorem 3.12]:

Theorem 3.10. The evaluation at Breuil–Kisin log prism $(\mathfrak{S}, (E, M_\mathfrak{S}, \delta_{\log}))$ induces an equivalence of the category $\text{Vect}(\mathbb{N} \to \mathcal{O}_K)\Delta, \mathfrak{S}_\Delta)$ of Hodge–Tate crystals on $\mathbb{N} \to \mathcal{O}_K\Delta$ to the category $\text{HT}^{\log}(\mathcal{O}_K)$ of pairs $(M, \phi_M)$ consisting of a finite projective $\mathcal{O}_K$-module $M$ and an $\mathcal{O}_K$-linear endomorphism $\phi_M$ of $M$ satisfying

$$\lim_{n \to +\infty} \prod_{i=0}^n (\phi_M + i\pi E'(\pi)) = 0.$$ 

The logarithmic prismatic cohomology of $\mathcal{M}$ is computed as follows:

$$\text{RF}((\mathbb{N} \to \mathcal{O}_K)\Delta, \mathcal{M}) \cong [M \xrightarrow{\phi_M} M].$$

The similar result holds for rational Hodge–Tate crystals by replacing finite projective $\mathcal{O}_K$-modules by finite dimensional $K$-vector spaces in the above statement.

Proof. By Lemma 3.8 (1), giving a Hodge–Tate crystal $\mathcal{M}$ on $\mathbb{N} \to \mathcal{O}_K\Delta$ amount to giving a finite projective $\mathcal{O}_K$-module $M$ together with a stratification $\varepsilon$, i.e. an $\mathfrak{S}^1/E$-linear isomorphism

$$\varepsilon : M \otimes_{\mathcal{O}_K, p_0} \mathfrak{S}^1/E \to M \otimes_{\mathcal{O}_K, p_1} \mathfrak{S}^1/E$$

satisfying the cocycle condition with respect to the cosimplicial ring $\mathfrak{S}^*/(E)$. Comparing Lemma 3.8 and Lemma 3.9 with [MW21b, Lemma 2.6] and [MW21b, Lemma 3.10], respectively, and using [MW21b, Lemma 3.6] (for $\alpha = \pi E'(\pi)$), we see that the arguments for the proofs of [MW21b, Theorem 3.8] and [MW21b, Theorem 3.12] still work in the logarithmic case. Then the theorem follows.

Remark 3.11. Let $\mathcal{M}$ be a (rational) Hodge–Tate crystal on $\mathbb{N} \to \mathcal{O}_K\Delta$ with associated pair $(M, \phi_M)$. Similar to [MW21b, Remark 3.9], the stratification $\varepsilon$ on $M$ with respect to the cosimplicial ring $\mathfrak{S}^*/(E)$ is given by

$$\varepsilon = (1 - \pi E'(\pi)X_1)^{\frac{\phi_M}{E'(\pi)}} : M\{X_1\}_{\text{pd}}^\wedge \to M\{X_1\}_{\text{pd}}^\wedge$$

via the canonical isomorphisms $M \otimes_{\mathcal{O}_K, p_1} \mathcal{O}_K\{X_1\}_{\text{pd}}^\wedge \cong M(\mathfrak{S}^1, (E), M_{\mathfrak{S}^1}, \delta_{\log}) \cong M \otimes_{\mathcal{O}_K, p_0} \mathcal{O}_K\{X_1\}_{\text{pd}}^\wedge$. 

17
Now we define a restricted site \((\mathbb{N} \to \mathcal{O}_K)_\Delta^\prime\). Its underlying category is the full subcategory of \((\mathbb{N} \to \mathcal{O}_K)_\Delta\) spanned by the prisms admitting maps from the Breuil–Kisin log prism \((\mathfrak{S}, (E), M_{\mathfrak{S}}, \delta_{\log})\) and the coverings are inherited from the site \((\mathbb{N} \to \mathcal{O}_K)_\Delta\).

Using \((\mathbb{N} \to \mathcal{O}_K)_\Delta^\prime\) instead of \((\mathcal{O}_K)_\Delta^\prime\) defined in [MW21b Subsection 3.3], we have the following result for prismatic crystals (which are defined as Hodge–Tate crystals by using \(\mathcal{O}_\Delta^\prime\) instead of \(\mathcal{O}_\Delta\)) on \((\mathbb{N} \to \mathcal{O}_K)_\Delta^\prime:\)

**Corollary 3.12.** For any prismatic crystal \(\mathcal{M}\), the logarithmic prismatic cohomology \(R\Gamma((\mathbb{N} \to \mathcal{O}_K)_\Delta^\prime, \mathcal{M})\) is concentrated in degrees in \([0, 1]\).

**Proof.** Just modify the proof of [MW21b Theorem 3.15].

Now, we want to study the restriction functor

\[
\text{Vect}((\mathbb{N} \to \mathcal{O}_K)_\Delta^\prime, \mathcal{O}_\Delta^\prime[\frac{1}{p}]) \to \text{Vect}((\mathbb{N} \to \mathcal{O}_K)_\Delta^\prime, \mathcal{O}_\Delta[\frac{1}{p}]).
\]

By virtues of Theorem 3.7 and Theorem 3.10, this functor can be viewed as a functor from the category of pairs \((V, \phi_V)\) as mentioned in Theorem 3.10 to the category \(\text{Rep}_{G_K}(C)\) of \(C\)-representations of \(G_K\).

Note that Lemma 3.6 combined with [MW21b Proposition 3.22] says that there is a natural isomorphism of cosimplicial rings

\[
\mathcal{O}_\Delta[\frac{1}{p}](\mathcal{O}_{\mathcal{A}_{\log}}^\circ, (\xi), M_{\mathcal{A}_{\log}}, \delta_{\log}) \cong C(G_K^\circ, C),
\]

where the latter is the cosimplicial ring of continuous functions from \(G_K^\circ\) to \(C\). So for our purpose, we need to specify the function corresponding

\[X_1 \in \mathcal{O}_K \{X_1\}_{\text{pd}}[\frac{1}{p}] \cong \mathcal{O}_\Delta[\frac{1}{p}](\mathfrak{S}^1, (E), M_{\mathfrak{S}^1}, \delta_{\log})\]

via the natural morphism

\[
\mathcal{O}_\Delta[\frac{1}{p}](\mathfrak{S}^1, (E), M_{\mathfrak{S}^1}, \delta_{\log}) \to \mathcal{O}_\Delta[\frac{1}{p}](\mathcal{O}_{\mathcal{A}_{\log}}^\circ, (\xi), M_{\mathcal{A}_{\log}}, \delta_{\log}) \cong C(G_K, C).
\]

**Lemma 3.13.** For any \(g \in G_K\), we have \(X_1(g) = c(g)\lambda(1 - \zeta_p)\), where \(\lambda\) is the image of \(\frac{\xi}{E(\{\pi^r\})}\) in \(\mathcal{O}_C\) and \(c(g) \in \mathbb{Z}_p\) is determined by \(g(\pi^r) = \pi^c c(g)\).

**Proof.** The proof is similar to that of [MW21b Proposition 3.26]. Note that \(X_1\) is the image of \(\frac{1 - \nu_{\mathfrak{S}_{\log}}}{E(\{\nu_{\mathfrak{S}_{\log}}\})}\) modulo \(E\) and that as functions in \(C(G_K, \mathcal{A}_{\log})\), \(u_0(g) = [\pi^0]\) and \(u_1(g) = g([\pi^0]) = [\pi^b c(g)]\). Therefore, we have

\[X_1(g) = \frac{\xi}{E([\pi^r])}(1 - [\pi^r])\frac{[c(g)] - 1}{[\xi] - 1} \mod E.
\]

So the result follows.

**Theorem 3.14.** The restriction functor

\[
\text{Vect}((\mathbb{N} \to \mathcal{O}_K)_\Delta^\prime, \mathcal{O}_\Delta[\frac{1}{p}]) \to \text{Vect}((\mathbb{N} \to \mathcal{O}_K)_\Delta^\prime, \mathcal{O}_\Delta[\frac{1}{p}]) \cong \text{Rep}_{G_K}(C)
\]

is fully faithful. More precisely, for a rational Hodge–Tate crystal \(\mathcal{M}\) with associated pair \((M, \phi_M)\), the resulting \(C\)-representation is given by \(V = M \otimes_K C\), on which \(g \in G_K\) acts via

\[U(g) = (1 - c(g)\pi \lambda(1 - \zeta_p)E'(\pi))^{-\phi_M / \pi E'(\pi)}.
\]

The Sen operator \(\Theta_V\) of \(V\) is \(-\frac{\phi_M}{\pi E'(\pi)}\) and there is natural quasi-isomorphism

\[R\Gamma((\mathbb{N} \to \mathcal{O}_K)_\Delta^\prime, \mathcal{M}) \cong R\Gamma(G_K, V).
\]
Proof. Let $\mathcal{M}$ be a rational Hodge–Tate crystal with associated pair $(M, \phi_M)$ and $C$-representation $V$. Combining Remark 3.11 and Lemma 3.13, we deduce that $V = M \otimes_K C$, on which the action of $g \in G_K$ on $V$ is given by $U(g)$ as desired. The proof of [Gao22 Theorem 4.3.3] shows that $\Theta_V = -\frac{\wedge}{\wedge K(\sigma)}$. Now using the same argument in the proof of [MW21b Proposition 3.31], we get

$$\operatorname{R} \Gamma((\mathcal{N} \to \mathcal{O}_K)_{\Delta}, \mathcal{M}) \cong \operatorname{R} \Gamma(G_K, V).$$

In particular, we have

$$\mathbb{H}^0((\mathcal{N} \to \mathcal{O}_K)_{\Delta}, \mathcal{M}) \cong \mathbb{H}^0(G_K, V).$$

Since the equivalence in Theorem 3.10 preserves tensor products and dualities, the full faithfulness follows directly. The proof is complete.

Finally, we want to compare (rational) Hodge–Tate crystals on usual absolute prismatic site $(\mathcal{O}_K)_{\Delta}$ and logarithmic one $(\mathcal{N} \to \mathcal{O}_K)_{\Delta}$.

Note that the forgetful functor $(A, I, M, \delta_{log}) \mapsto (A, I)$ from $(\mathcal{N} \to \mathcal{O}_K)_{\Delta}$ to $(\mathcal{O}_K)_{\Delta}$ induces a natural functor of the categories of Hodge–Tate crystals on corresponding sites

$$\operatorname{Vect}((\mathcal{O}_K)_{\Delta}, \mathcal{O}_{\Delta}) \to \operatorname{Vect}((\mathcal{N} \to \mathcal{O}_K)_{\Delta}, \mathcal{O}_{\Delta}).$$

By virtues of Theorem 3.10 and Theorem 3.10, we get a commutative diagram of categories

$$\begin{array}{ccc}
\operatorname{Vect}((\mathcal{O}_K)_{\Delta}, \mathcal{O}_{\Delta}) & \xrightarrow{\simeq} & \operatorname{Vect}((\mathcal{N} \to \mathcal{O}_K)_{\Delta}, \mathcal{O}_{\Delta}) \\
\downarrow \simeq & & \downarrow \simeq \\
\operatorname{HT}(\mathcal{O}_K) & \xrightarrow{\simeq} & \operatorname{HT}^{log}(\mathcal{O}_K),
\end{array}$$

where the vertical equivalences are induced by evaluating at Breuil–Kisin prism $(\mathcal{S}, (E))$ and Breuil–Kisin log prism $(\mathcal{S}, (E), M_{\mathcal{S}}, \delta_{log})$, respectively. Let $\mathcal{M}$ be a Hodge–Tate crystal in $\operatorname{Vect}((\mathcal{O}_K)_{\Delta}, \mathcal{O}_{\Delta})$ with associated pairs $(M, \phi_M)$ and $(M, \phi_{log}^M)$ in $\operatorname{HT}(\mathcal{O}_K)$ and $\operatorname{HT}^{log}(\mathcal{O}_K)$, respectively. Comparing Lemma 3.8 with [MW21b Lemma 2.6], we see that $\phi_{log}^M = \pi \phi_M$. So we obtain the following corollary:

**Corollary 3.15.** The functor $\operatorname{Vect}((\mathcal{O}_K)_{\Delta}, \mathcal{O}_{\Delta}) \to \operatorname{Vect}((\mathcal{N} \to \mathcal{O}_K)_{\Delta}, \mathcal{O}_{\Delta})$ constructed above is fully faithful. More precisely, the functor fits into the following commutative diagram:

$$\begin{array}{ccc}
\operatorname{Vect}((\mathcal{O}_K)_{\Delta}, \mathcal{O}_{\Delta}) & \xrightarrow{\simeq} & \operatorname{Vect}((\mathcal{N} \to \mathcal{O}_K)_{\Delta}, \mathcal{O}_{\Delta}) \\
\downarrow \simeq & & \downarrow \simeq \\
\operatorname{HT}(\mathcal{O}_K) & \xrightarrow{\simeq} & \operatorname{HT}^{log}(\mathcal{O}_K),
\end{array}$$

where the bottom functor sends a pair $(M, \phi_M)$ in $\operatorname{HT}(\mathcal{O}_K)$ to $(M, \pi \phi_M)$ in $\operatorname{HT}^{log}(\mathcal{O}_K)$. The result also holds for rational Hodge–Tate crystals.

One can also compare cohomologies in this case. It is easy to deduce from Theorem 3.3 and Theorem 3.10 that for any Hodge–Tate crystal $\mathcal{M} \in \operatorname{Vect}((\mathcal{O}_K)_{\Delta}, \mathcal{O}_{\Delta})$, we get a quasi-isomorphism

$$\operatorname{R} \Gamma((\mathcal{O}_K), \mathcal{M}[\frac{1}{p}]) \cong \operatorname{R} \Gamma((\mathcal{N} \to \mathcal{O}_K)_{\Delta}, \mathcal{M}[\frac{1}{p}]).$$

### 3.2 The geometric case

Now let $\mathfrak{X}$ be a semi-stable $p$-adic formal scheme over $\mathcal{O}_K$ with the log structure $\mathcal{M}_X = \mathcal{O}_X^\wedge \cap \mathcal{O}_X$ as in the beginning of this section. We will investigate both $\operatorname{Vect}((\mathfrak{X}, \mathcal{M}_X)/(\mathcal{S}, E, M_{\mathcal{S}}, \delta_{log}))_{\Delta}, \mathcal{O}_{\Delta})$ and $\operatorname{Vect}((\mathfrak{X}, \mathcal{M}_X)_{\Delta}, \mathcal{O}_{\Delta})$. Since we will not change the log structure, we just write $\operatorname{Vect}((\mathfrak{X}/(\mathcal{S}, (E))_{\Delta}, \mathcal{O}_{\Delta})$ and $\operatorname{Vect}((\mathfrak{X})_{\Delta, \log}, \mathcal{O}_{\Delta})$ respectively for simplicity. Similar notations are also used for sites of perfect log prisms.
3.2.1 The relative case

This part will be a generalization of [11an21] to the semi-stable case. We first assume $X = \text{Spf}(R)$ is small affine with $M_X$ induced by the prelog structure as given in Example 2.8.

**Lemma 3.16.** Keep notations as in Example 2.5.

1. The log prism $(\mathcal{G}(R), (E), M_{\log}(R), \delta_{\log})$ is a cover of the final object in the topos $\text{Sh}((R)_{\log})$.
2. The perfect log prism $(\mathcal{A}_{\log}(\hat{R}_\infty), I, M_{\log}(\hat{R}_\infty), \delta_{\log})$ is a cover of the final object in the topos $\text{Sh}((R)_{\log}^{\text{perf}})$.

**Proof.** (1) For a log prism $(A, I, M, \delta_{\log})$ in $(R)_{\log}$, we need to show there exists a cover $(C, IC, N, \delta_{\log})$ of $(A, I, M, \delta_{\log})$ which admits a morphism from $(\mathcal{G}(R), (E), M_{\log}(R), \delta_{\log})$.

As $R$ is étale over $O_K(T_0, \ldots, T_r, T_{r+1}^\pm, \ldots, T_d^\pm)/(T_0 \cdots T_r - \pi)$, we may assume

$$R = O_K(T_0, \ldots, T_r, T_{r+1}^\pm, \ldots, T_d^\pm)/(T_0 \cdots T_r - \pi).$$

By Lemma 2.3, there are liftings $t_i$'s of $T_i$'s in $A$ such that the log structure $M$ is associated to the prelog structure over $\mathbb{N}$

$\begin{array}{c}
\varepsilon_{i=0}^{r-2} \quad 0 \leq i \leq r-1 \quad A \end{array}$

Put $B = (A[u, T_0, \ldots, T_r, T_{r+1}^\pm, \ldots, T_d^\pm]/(T_0 \cdots T_r - u))[(\frac{u}{T_0})^\pm, \ldots, (\frac{u}{T_d})^\pm]$. We claim $B$ is a $\mathbb{C}$-completely faithfully flat over $A$ and that $(\frac{u}{T_0} - 1, \ldots, \frac{u}{T_d} - 1)$ forms a $(p, I)$-completely regular sequence relative to $A$. For simplicity, we assume $r = d = 1$. The general case follows from the same argument. Note that we have $B = A[u, T_0, T_1, Z_0^\pm, Z_1^\pm]/(T_0 T_1 - u, Z_0 T_0 - t_0, Z_1 T_1 - t_1) = A[Z_0^\pm, Z_1^\pm]$. Now the claim easily follows.

Write $J = (I, \frac{T_0}{u} - 1, \ldots, \frac{T_d}{u} - 1) \subset B$. Then by [BS19] Proposition 3.1.3, if we put $C = B(\frac{1}{T_0})^\gamma$, then $(C, IC)$ is a flat cover of $(A, I)$. By virtue of Lemma 2.3 if we equip $C$ with the log structure $(N \to C)$ induced by $M \to A \to C$, then $(C, IC, N, \delta_{\log})$ is a flat cover of $(A, I, M, \delta_{\log})$. Note that $E(u) \equiv E(t_0 \cdots t_r) \equiv 0 \mod I$ in $C$. We have $E(u) \in IC$ and hence $E(u)C = IC$. In particular, $C$ is also $(p, E(u))$-complete and $\frac{T_i}{u}$'s are invertible in $C$. So $T_i$'s can be viewed as elements in $N$. This gives us a morphism from $(\mathcal{G}(R), (E), M_{\log}(R), \delta_{\log})$ to $(C, IC, N, \delta_{\log})$. We are done.

(2) By Proposition 2.18 it is enough to show $(A_{\log}(\hat{R}_\infty), (\xi))$ is a cover of the final object in the topos $\text{Sh}((R)_{\log}^{\text{perf}})$. For any perfect prism $(A, I) \in (R)_{\log}^{\text{perf}}$, the $p$-completed tensor product $A/\hat{R}_\infty$ is a $\mathbb{C}$-completely faithfully flat quasi-syntomic algebra over $A/I$. By [BS19] Proposition 7.11 (2), there is a cover of perfect prism $(A, I) \to (B, IB)$ with $B/IB$ an $\hat{R}_\infty$-algebra. By deformation theory, there is a morphism of prisms $(A_{\log}(\hat{R}_\infty), (\xi)) \to (B, I)$. Now the result follows. □

As a corollary, we also have similar results in the relative prismatic site.

**Corollary 3.17.** (1) The log prism $(\mathcal{G}(R), (E), M_{\log}(R), \delta_{\log})$ is a cover of the final object in the topos $\text{Sh}((R)/(\mathcal{G}, E))_{\log})$.

(2) The perfect log prism $(A_{\log}(\hat{R}_\infty), I, M_{\log}(\hat{R}_\infty), \delta_{\log})$ is a cover of the final object in the topos $\text{Sh}((R)/(\mathcal{G}, E))_{\log}^{\text{perf}}$.

Now we study the cosimplicial log prisms $(\mathcal{G}(R)^n_{\log}, (E), M_{\log}(R)_{\log})$ associated with the log prism $(\mathcal{G}(R), (E), M_{\log}(R), \delta_{\log})$ in $(\text{Sh}((R)/(\mathcal{G}, E)))_{\log}$. Explicitly, write $B_{n, \log} = (\mathcal{G}(R)^{\otimes n}[[1 - \frac{T_1}{T_1}, \ldots, (1 - \frac{T_n}{T_n})]]_{0 \leq i \leq r})$, where $(\mathcal{G}(R)^{\otimes n}^\gamma$ denotes the tensor product of $n + 1$-copies of $\mathcal{G}(R)$ over $\mathcal{G}$. Then we have

$$\mathcal{G}(R)^n_{\log} = B_{n, \log} \frac{1}{E(u)} \left( \frac{1}{E(u)} \right)^{r+1}$$

where $\frac{1}{E(u)}$ with $1 \leq i \leq n$ means adding all $\frac{1 - T_i}{E(u)}$ for $1 \leq j \leq d$. Here, we write $T_{j,i}$ for the corresponding element in the $i$-th component in $B_{n, \log}$. Note that the element $\frac{1 - T_{m,i}}{E(u)}$ automatically belongs to $\mathcal{G}(R)^n_{\log}$ by the relation $u = \prod_{0 \leq j \leq r} T_{j,i}$. Now, one can check that for any $1 \leq m \leq n$ and any $0 \leq i \leq r$, $\frac{T_{m,i}}{E(u)}$ is a unit in $\mathcal{G}(R)^n_{\log}$. So the log structure induced by $\mathbb{N}^{r+1} \to \mathcal{G}(R)^n_{\log}$ sending $e_i$ to $T_{i,m}$ for any $0 \leq i \leq r$ is independent of the choice of $m$ and will be denoted by $M^n_{\log}$.
Lemma 3.18. The cosimplicial log prism \((\mathcal{S}(R)_{rel}^\ast, (E(u)), M_{rel}^\ast, \delta_{log})\) is the Čech nerve of \((\mathcal{S}(R), (E(u)), M_{\mathcal{S}(R)}, \delta_{log})\) in \((R/(\mathcal{S}, E))_{\Delta_{log}}\).

Proof. The proof is similar to that of Lemma 3.3. We only show \((\mathcal{S}(R)_{rel}^1, (E(u)), M_{rel}^1, \delta_{log})\) is the self coproduct of \((\mathcal{S}(R), (E(u)), M_{\mathcal{S}(R)}, \delta_{log})\) and the general case follows from a similar argument. Since \(B_{rel}^1\) is faithfully flat over \(\mathcal{S}(R)\), we can check that

\[
p, E(u), 1 - \frac{T_{1,1}}{t_{1,0}}, \ldots, 1 - \frac{T_{n,1}}{t_{n,0}}, T_{r+1,i,0} - T_{r+1,0,0}, \ldots, T_{d,1} - T_{d,0}
\]

forms a regular sequence in \(B_{rel}^1\). So \([\text{BS19}]\) Proposition 3.13] implies that \((\mathcal{S}(R)_{rel}^1, (E))\) is a faithfully flat cover of \((\mathcal{S}(R), (E))\) and hence a bounded prism such that \(p, E\) forms a regular sequence in \(\mathcal{S}(R)_{rel}^1\).

Consider the monoid \(N^{r+1} \oplus_{R} N^{r+1} \cong (\oplus_{i=0}^{r} \mathbb{N}e_i, 0) \oplus_{N} (\oplus_{i=0}^{r} \mathbb{N}e_{i,1})\), which is the push-out of the diagram

\[
\delta_{i,0}^\ast \oplus_{R} \mathbb{N}e_{i,1} \leftarrow \mathbb{N}^r \oplus_{R} \mathbb{N}e_{i,1} \rightarrow \delta_{i,0}^\ast \oplus_{R} \mathbb{N}e_{i,1}.
\]

There is an obvious way to make \((\mathcal{S}(R)_{rel}^1, \mathbb{N}^{r+1} \oplus_{R} \mathbb{N}^{r+1}, \delta_{log})\) a prelog prism over \((\mathcal{S}, (E(u)), \mathbb{N}, \delta_{log})\). There exists a “multiplication” map of prelog rings

\[
(\mathcal{S}(R)_{rel}^1, \mathbb{N}^{r+1} \oplus_{R} \mathbb{N}^{r+1}) \rightarrow (R, \mathbb{N}^{r+1})
\]

such that the induced surjection on monoids \(pr : \mathbb{N}^{r+1} \oplus_{R} \mathbb{N}^{r+1} \rightarrow \mathbb{N}^{r+1}\) is given by \(pr(e_i, 1) = pr(e_{i,2}) = e_i\). Let \(N\) be the submonoid of \(Z^{r+1} \oplus_{R} Z^{r+1}\) generated by \(\mathbb{N}^{r+1} \oplus_{R} \mathbb{N}^{r+1}\) and \((\mathbb{N}^r)^{-1}(0)\). Then \(N = (\oplus_{i=0}^{r} \mathbb{N}e_{i,0} \oplus_{N} \oplus_{i=0}^{r} \mathbb{N}e_{i,1}) + \sum_{i=1}^{r} e_{i,0} - e_{i,1}\).

By [Koz20] Construction 2.18], the exactification of \((\mathcal{S}(R)_{rel}^1, (E(u)), N^{r+1} \oplus_{R} N^{r+1}, \delta_{log})\) is exactly \((\mathcal{S}(R)_{rel}^1, (\mathcal{S}(R)_{rel}^1, \mathbb{N}^{r+1} \oplus_{R} \mathbb{N}^{r+1}, \delta_{log})\), where the completion with respect to the \((p, E(u))-adically\) and the prelog structure \(\mathbb{N}^{r+1} \oplus_{R} \mathbb{N}^{r+1} \rightarrow \mathbb{N}^{r+1} \oplus_{R} \mathbb{N}^{r+1}\) is given by the map sending \(e_i\)’s to \(T_{i,0}\)’s.

By construction, for any log prism \((A, I, M, \delta_{log}) \in (R/(\mathcal{S}, (E)))_{\Delta_{log}}\), there exists a unique morphism of log rings

\[
f : (\mathcal{S}(R)_{rel}^1, (E(u)), (\mathbb{N}^{r+1})^a, \delta_{log}) \rightarrow (A, I, M, \delta_{log}).
\]

Since \(1 - \frac{T_{1,1}}{t_{1,0}}, \ldots, 1 - \frac{T_{n,1}}{t_{n,0}}, T_{r+1,i,0} - T_{r+1,0,0}, \ldots, T_{d,1} - T_{d,0}\) is a \((p, E(u))-adically\) regular sequence relative to \(\mathcal{S}(R)\), by \([\text{BS19}]\) Proposition 1, \(\mathcal{S}(R)_{rel}^1 = (\mathcal{S}(R)_{rel}^1, \delta_{i,0})_{E(u)}\) exists and is \((p, E(u))-completely\) flat over \(\mathcal{S}(R)\). In particular, \(p, E(u)\) is a regular sequence in \(\mathcal{S}(R)_{rel}^1\) and hence so is \(u, E(u)\). By construction, \((\mathcal{S}(R)_{rel}^1, (E(u)), (\mathbb{N}^{r+1})^a, \delta_{log})\) belongs to \((R/(\mathcal{S}, (E)))_{\Delta_{log}}\) and \(f\) factors over the natural map

\[
(\mathcal{S}(R)_{rel}^1, (E(u)), (\mathbb{N}^{r+1})^a, \delta_{log}) \rightarrow (\mathcal{S}(R)_{rel}^1, (E(u)), (\mathbb{N}^{r+1})^a, \delta_{log})
\]

uniquely. In other words, \((\mathcal{S}(R)_{rel}^1, (E(u)), (\mathbb{N}^{r+1})^a, \delta_{log})\) is the self coproduct of \((\mathcal{S}(R), (E(u)), M_{\mathcal{S}(R)}, \delta_{log})\) in \((R/(\mathcal{S}, (E)))_{\Delta_{log}}\).

It remains to show that \((\mathcal{S}(R)_{rel}^1, (E(u)), (\mathbb{N}^{r+1})^a, \delta_{log}) \cong (\mathcal{S}(R)_{rel}^1, (E(u)), M_{rel}^1, \delta_{log})\). The universal property for the former provides a unique morphism from \((\mathcal{S}(R)_{rel}^1, (E(u)), (\mathbb{N}^{r+1})^a, \delta_{log})\) to \((\mathcal{S}(R)_{rel}^1, (E(u)), M_{rel}^1, \delta_{log})\). We need to construct its inverse. By definition of \(\mathcal{S}(R)_{rel}^1\), it is enough to show that \(1 - \frac{T_{i,1}}{T_{i,0}}\) is topologically nilpotent and that \(\frac{1 - T_{i,1}/T_{i,0}}{E(u)}\) exists in \(\mathcal{S}(R)_{rel}^1\) for any \(i\). By \((E(u))-adicity\) of \(\mathcal{S}(R)_{rel}^1\), we only need to show that \(\frac{1 - T_{i,1}/T_{i,0}}{E(u)}\) exists for any \(i\). Note that both \(\frac{T_{i+1} - T_{i,0}}{E(u)}\) and \(\frac{T_{i+1} - T_{i,0}}{T_{i,0}}\) exists in \(\mathcal{S}(R)_{rel}^1\). Since \(T_{i,0}\) and \(E(u)\) meet transversely in \(\mathcal{S}(R)_{rel}^1\) for \(1 \leq i \leq r\) (as \(u\) and \(E(u)\) do), so \(\frac{T_{i,1} - T_{i,0}}{E(u)}\) exists for any \(1 \leq i \leq d\). The existence is trivial because \(T_{i,0}\) is invertible in this case.

Lemma 3.19. For any \(1 \leq i \leq n\) and \(1 \leq j \leq d\), let \(Y_{j,i}\) denote the images of \(\frac{1 - T_{i,1}/T_{i,0}}{E(u)}\) in \(\mathcal{S}(R)_{rel}^1/E(u_0)\) respectively. Then we have

\[
\mathcal{S}(R)_{rel}/E(u) \cong R[Y_{1,1}, \ldots, Y_{n,d}]_{pd}
\]
where the right hand side is the p-adic completion of the free pd-algebra over $R$ with variables $\{Y_1, \cdots, Y_n\}$. Moreover, let $p_i : \mathcal{S}(R)_{rel}/E \to \mathcal{S}(R)_{rel}/E$ be the structure morphism induced by the order-preserving injection

$$\{0, \ldots, n\} \to \{0, \ldots, i - 1, i + 1, \ldots, n + 1\},$$

then we have

$$p_i(Y_j) = \begin{cases} 
\sum_{j=1}^{i-1} Y_j, & i = 0 \\
\sum_{j=1}^{i-1} Y_j, & 1 \leq j < i \\
\sum_{j=1}^{i+1} Y_j, & 0 < i \leq j
\end{cases} \quad (3.2)$$

**Proof.** This can essentially reduce to the proofs of [MW21b, Lemma 2.6 and 2.7] (note that all $T_{j,i}$’s are of rank 1). \hfill \Box

**Definition 3.20.** A log Higgs module over $R$ is a finite projective module $M$ over $R$ together with an $R$-linear morphism

$$\theta : M \to M \otimes_R \tilde{\Omega}^1_{R/\mathcal{O}_{X_{\log}} \{-1\}}$$

such that $\theta \wedge \theta = 0$. If we write $\theta = \sum_{i=1}^{d} \theta_i \otimes \text{dlog}T_i \{-1\}$ with $\theta_i : M \to M$, then this condition is equivalent to saying $\theta \theta_j = \theta_i \theta_i$.

For any $n$-tuple $m = (m_1, \cdots, m_n) \in \mathbb{N}^n$, we put

$$\theta^m_i = \prod_{i=1}^{n} \theta^m_i \in \text{End}_R(M).$$

We say $\theta$ is topologically nilpotent if $\theta^m_i$ tends to 0 as $|m| := \sum_{i=1}^{n} m_i$ tends to infinity.

Let $\text{HIG}^{\text{log}}(R)$ denote the category of topologically nilpotent log Higgs modules over $R$.

**Theorem 3.21.** There is an equivalence between the categories

$$\text{Vect}((R)/\mathcal{S}(E))_{\Delta, \log} / \overline{\Omega}_\Delta \simeq \text{HIG}^{\text{log}}(R).$$

**Proof.** Given a Hodge–Tate crystal $M$, we have the corresponding stratification

$$\epsilon : M \otimes_{R, p_{0}} R\{\Delta\} \to M \otimes_{R, p_{0}} R\{\Delta\}.$$ 

Note that $p_0$ and $p_1$ are both the natural inclusion $R \hookrightarrow R_{\{\Delta\}}$. Without loss of generality, we may assume $M$ is finite free over $R$ with a basis $\{e_1, \cdots, e_d\}$. Then we can write $\epsilon(e) = \sum_{I \in \mathbb{N}^d} A_I \sum_{|I|}^{i}$ with $A_I \in M_I(R)$. Then one can show (by putting all $X$ to be 0 in the calculations after Lemma 2.3) that the cocycle condition is equivalent to the following conditions:

1. $A_I = \text{id}$ when $I = (0, \cdots, 0)$;
2. if we write $\theta_{e,i} = A_I$ with $i$-th component in $I$ being 1 and other components being 0, then $\theta_{e,i} \theta_{e,j} = 0$;
3. $A_I = \prod_{i=1}^{d} \theta_{e,i}^{m_i}$ with $I = (m_1, \cdots, m_d)$ satisfying $\lim_{|I| \to +\infty} A_I = 0$.

So we can define a topologically nilpotent log Higgs module as $\theta = \sum_{i=1}^{d} \theta_{e,i} \otimes \text{dlog}T_i \{-1\}$. Conversely, given a Higgs module $(M, \theta) \in \text{HIG}^{\text{log}}(R)$ with $\theta = \sum_{i=1}^{d} \theta_{i} \otimes \text{dlog}T_i \{-1\}$, we can get a stratification by setting $A_I = \prod_{i=1}^{d} \theta_{e,i}^{m_i}$ with $I = (m_1, \cdots, m_d)$.

Now we compare the cohomology of a Hodge–Tate crystal and its associated Higgs complex.

**Theorem 3.22.** Let $M \in \text{Vect}((R)/\mathcal{S}(E))_{\Delta, \log} / \overline{\Omega}_\Delta$ be a Hodge–Tate crystal with associated topological quasi-nilpotent log Higgs module $(M, \theta_M)$. There is a quasi-isomorphism

$$\text{R} \Gamma_{\Delta}(M) \simeq \text{HIG}(M, \theta_M).$$
Proof. The proof is the same as the proof of [Tian21, Theorem 4.12] (because the results in [BJ] still hold true in the semi-stable case by considering log-crystalline cohomology and log-de Rham cohomology). □

In particular, one can get the following corollary directly.

**Theorem 3.23.** Let \( X \) be a semi-stable p-adic formal scheme over \( \mathcal{O}_K \) of relative dimension \( d \), with the log structure \( \mathcal{M}_X = \mathcal{O}_X^* \cap \mathcal{O}_X \) where \( X \) is the generic fiber. Let \( \nu : \text{Sh}((\mathcal{X}/((\mathcal{E}, (E))))_{\text{log}}) \to \text{Sh}(\mathcal{X}_{\text{et}}) \) be the natural morphism of topoi.\(^{10}\)

Then for any Hodge–Tate crystal \( \mathcal{M} \) in \( \text{Vect}((\mathcal{X}/((\mathcal{E}, (E))))_{\text{log}}, \mathcal{O}_\Delta) \), \( \mathcal{R}_\nu \mathcal{M} \) is a perfect complex of \( \mathcal{O}_X \)-modules with tor-amplitude \([0, d]\). If moreover \( X \) is proper, then \( \mathcal{R}_\Delta \mathcal{M} \) is a perfect complex of \( \mathcal{O}_K \)-modules with tor-amplitude \([0, 2d]\).

**Remark 3.24.** Similar results as stated in Theorem 3.23 about the cohomological finiteness for Hodge–Tate crystals also hold for prismatic crystals (i.e., crystals on \( (\mathcal{X}/((\mathcal{E}, (E))))_{\text{log}} \) with coefficients in \( \mathcal{O}_\Delta \)). More precisely, if \( X \) is proper semistable of relative dimension \( d \), then for any prismatic crystal \( \mathcal{M} \) on \( (\mathcal{X}/((\mathcal{E}, (E))))_{\text{log}}, \mathcal{R}_\Delta \mathcal{M} \) is a perfect complex with tor-amplitude \([0, 2d]\). This can be deduced from Theorem 3.23 together with derived Nakayama’s lemma directly.

### 3.2.2 The absolute case

Now we study the cosimplicial log prisms \((\mathcal{E}(\mathcal{R}), (E), M_{\mathcal{E}(\mathcal{R})}, \delta_{\log})\) corresponding to the Čech nerve of the log prism \((\mathcal{E}(\mathcal{R}), (E), M_{\mathcal{E}(\mathcal{R})}, \delta_{\log})\) in \((\mathcal{R}, \text{log})\).

Explicitly, write \( B^n = \mathcal{E}(\mathcal{R}) \otimes \mathbb{Z} \mathbb{N} \left[ (1 - \frac{u_{0}}{u_{i}}), \ldots, (1 - \frac{u_{0}}{u_{n}}), (1 - \frac{T_{0,i} + \cdots T_{d,i}}{E(u_{0})}, \ldots, (1 - \frac{T_{0,i} + \cdots T_{d,i}}{E(u_{0})}) \right]_{0 \leq i \leq r} \), where \( \mathcal{E}(\mathcal{R}) \otimes \mathbb{N} \) means the tensor product of \( n + 1\)-copies of \( \mathcal{E}(\mathcal{R}) \) over \( \mathcal{W}(k) \). Then we have

\[
\mathcal{E}(\mathcal{R})^n = B^n \{ \frac{1}{E(u_{0})}, \ldots, \frac{1}{E(u_{0})}, 1 - \frac{1}{E(u_{0})}, 1 - \frac{1}{E(u_{0})}, \ldots, 1 - \frac{1}{E(u_{0})} \} \]

where \( \frac{1}{E(u_{0})} \) with \( 1 \leq i \leq n \) means adding all \( \frac{1}{E(u_{0})} \) for \( 1 \leq j \leq d \). Here, we write \( T_{j,i} \) for the corresponding element in the \( i\)-th component in \( B^n \). Note that the element \( \frac{1}{E(u_{0})} \) automatically belongs to \( \mathcal{E}(\mathcal{R})^n \) by the relation \( u_{i} = \prod_{0 \leq j \leq r} T_{j,i} \). So the log structures induced by the map \( \mathbb{N}^{n+1} \to \mathcal{E}(\mathcal{R})^n \) sending \( e_{j}'s \) to \( T_{1,m}'s \) are independent of the choice of \( 0 \leq m \leq n \). We denote this log structure by \( M_{\mathcal{E}(\mathcal{R})}^n \). A similar argument for the proof of Lemma 3.18 shows that \( (\mathcal{E}(\mathcal{R}), (E(u_{0})), M_{\mathcal{E}(\mathcal{R})}, \delta_{\log}) \) is the Čech nerve of \((\mathcal{E}(\mathcal{R}), (E(u_{0})), M_{\mathcal{E}(\mathcal{R})}, \delta_{\log})\) in \( (\mathcal{R}, \text{log}) \). Then the following Lemma can be deduced from the same argument as in the proof of Lemma 3.18 and therefore we omit its proof here.

**Lemma 3.25.** For any \( 1 \leq i \leq n \) and \( 1 \leq j \leq d \), let \( X_{i,i}, Y_{j,i} \) denote the images of \( \frac{1}{E(u_{0})} \) and \( \frac{1}{E(u_{0})} \) in \( \mathcal{E}(\mathcal{R})^n/E(u_{0}) \) respectively. Then we have

\[
\mathcal{E}(\mathcal{R})^n/E(u_{0}) \cong R\{X_{1}, \ldots, X_{n}, Y_{1}, \ldots, Y_{n}\}^{\mathbb{Z} \mathbb{N}} \]

where the right hand side is the p-adic completion of the free \( \mathbb{Z} \mathbb{N} \)-algebra over \( R \) with variables \( \{X_{1}, \ldots, X_{n}, Y_{1}, \ldots, Y_{n}\} \).

Moreover, let \( p_{i} : \mathcal{E}(\mathcal{R})^n/E \to \mathcal{E}(\mathcal{R})^n/E \) be the structure morphism induced by the order-preserving injection

\[
\{0, \ldots, n\} \to \{0, \ldots, i - 1, i + 1, \ldots, n + 1\},
\]

\(^{10}\)See [Kos20, Remark 4.4].
then we have
\[
p_i(X_j) = \begin{cases} 
(X_{j+1} - X_1)(1 - \pi E'(\pi)X_1)^{-1}, & i = 0 \\
X_j, & 1 \leq j < i \\
X_{j+1}, & 0 < i \leq j;
\end{cases} 
\]
\[
p_i(Y_j) = \begin{cases} 
(Y_{j+1} - Y_1)(1 - \pi E'(\pi)X_1)^{-1}, & i = 0 \\
Y_j, & 1 \leq j < i \\
Y_{j+1}, & 0 < i \leq j.
\end{cases} 
\]

(3.3)

Next we want to give an equivalent description of the Hodge–Tate crystals on the absolute site \( (R)_{\Delta, \log} \).

By Lemma [3.10] we see that giving a Hodge–Tate crystal \( M \) of rank \( l \) in \( \text{Vect}((R)_{\Delta, \log}, \Omega^\bullet_{\Delta}) \) is equivalent to giving a finite projective \( R \)-module \( M \) of rank \( l \) with a stratification
\[
M \otimes_{R,p_1} R[X, Y]^\vartriangleleft_{pd} \Rightarrow M \otimes_{R,p_0} R[X, Y]^\vartriangleleft_{pd}
\]
satisfying the cocycle condition with respect to the cosimplicial log prisms \( (\mathfrak{S}(R)^*, (E), M_{\mathfrak{S}(R)^*}, \delta_{\log}) \).

By choosing an affine covering of \( R \), we may assume \( M \) is a finite free \( R \)-module. Let \( e_1, \ldots, e_l \) be an \( R \)-basis of \( M \), then
\[
\varepsilon(e) = \varepsilon \sum_{i \geq 0, I \in \mathbb{N}^d} A_{i,I} X^{[i]} Y^{[I]} \tag{3.4}
\]
for some \( A_{i,I} \in M_i(R) \) satisfying \( \lim_{i+|I| \to +\infty} A_{i,I} = 0 \), where \( X^{[i]} \) denotes the \( i \)-th pd–power \( R^{\otimes i} \) of \( X \) and for any \( I = (i_1, \ldots, i_d) \in \mathbb{N}^d \), \( X^{[I]} \) denotes \( Y^{[i_1]} \cdots Y^{[i_d]} \). Then we have
\[
p^*_2(\varepsilon) \circ p_0^*(\varepsilon)(e) = p^*_2(\varepsilon)(e) \sum_{i \geq 0, I \in \mathbb{N}^d} A_{i,I} (1 - \pi E'(\pi)X_1)^{-i-|I|} (X_2 - X_1)^{|I|} (Y_2 - Y_1)^{|I|}
\]
\[
= \varepsilon \sum_{i \geq 0, I, J \in \mathbb{N}^d} A_{j,J} A_{i,I} X^{[i]} (1 - \pi E'(\pi)X_1)^{-i-|I|} (X_2 - X_1)^{|I|} Y^{[J]} (Y_2 - Y_1)^{|J|}
\]
\[
= \varepsilon \sum_{i \geq 0, I, J \in \mathbb{N}^d} P_{i,J}(X_1, Y_1) X^{[i]} Y^{[J]},
\]
where for any \( i \geq 0 \) and \( I \in \mathbb{N}^d \),
\[
P_{i,I}(X_1, Y_1) = \sum_{j \geq 0, J, L \in \mathbb{N}^d} A_{j,J} A_{i+I+L}(-1)^{|J|+|L|} (1 - \pi E'(\pi)X_1)^{-i-|I|-|J|-|L|} X^{[j]} X^{[i]} Y^{[J]} Y^{[L]}.
\]

On the other hand, we have
\[
p^*_1(\varepsilon)(e) = \varepsilon \sum_{i \geq 0, I \in \mathbb{N}^d} A_{i,I} X^{[i]} Y^{[I]}.
\]

So the stratification \( \varepsilon \) satisfies the cocycle condition if and only if \( A_{0, \emptyset} = I \) and for any \( i \geq 0 \) and \( I \in \mathbb{N}^d \),
\[
\sum_{j \geq 0, J, L \in \mathbb{N}^d} A_{j,J} A_{i+I+L}(-1)^{|J|+|L|} (1 - \pi E'(\pi)X_1)^{-i-|I|-|J|-|L|} X^{[j]} X^{[i]} Y^{[J]} Y^{[L]} = A_{i,I} \tag{3.5}
\]

Let \( Y_1 = 0 \) in (3.3), we get
\[
A_{i,I} = \sum_{j \geq 0} A_{j, \emptyset} A_{i+I+L}(-1)^j (1 - \pi E'(\pi)X_1)^{-i-|I|} X^{[i]} X^{[j]} X^{[i]} Y^{[J]} Y^{[L]}
\]
\[
= \sum_{j \geq 0} A_{j, \emptyset} A_{i+I+L}(-1)^j \sum_{k \geq 0} \binom{j}{k} (-\pi E'(\pi)X)^k X^{[i]} X^{[j]} X^{[i]} Y^{[J]} Y^{[L]}.
\]

(3.6)

Comparing the coefficients of \( X_1 \), we get
\[
A_{1, \emptyset} A_{i,I} - A_{0, \emptyset} A_{i+1,I} + (i + |I|) \pi E'(\pi) A_{0, \emptyset} A_{i,I} = 0
\]

24
which shows that for any \( n \geq 0 \),
\[
A_{n+1,I} = (A_{1,0} + |I|\pi E'(\pi) + n\pi E'(\pi))A_{n,I} = \prod_{i=0}^{n}(A_{1,0} + |I|\pi E'(\pi) + i\pi E'(\pi))A_{0,I}.
\] (3.7)

Let \( X_1 = 0 \) in (3.6), we get
\[
A_{i,I} = \sum_{J,L \in \mathbb{N}^d} A_{0,J}A_{i,I+L}(-1)^{|J|}Y_1^{[J]}Y_1^{[L]}.
\] (3.8)

Comparing the coefficient of \( Y_{k,1} \), we get
\[
A_{0,E_k}A_{i,I} - A_{0,0}A_{i,I+E_k} = 0
\]
which shows that for any \( n \geq 0 \),
\[
A_{n,I} = A_{0,E_k}A_{n,I-E_k} = A_{0,E_k}A_{n,I-E_k} = A_{0,E_k}A_{n,I-E_k},
\] (3.9)
where \( E_k \) denotes the generator of the \( k \)-th component of \( \mathbb{N}^d \).

Combining (3.7) with (3.9), we deduce that for any \( 1 \leq i, j \leq d \),
\[
[A_{0,E_i}, A_{0,E_j}] = A_{0,E_i}A_{0,E_j} - A_{0,E_j}A_{0,E_i} = 0,
\] (3.10)
and that for any \( n \geq 0 \) and \( I = (i_1, \ldots, i_d) \in \mathbb{N}^d \),
\[
A_{n,I} = \prod_{i=0}^{n}(A_{1,0} + |I|\pi E'(\pi) + i\pi E'(\pi))A_{0,E_1}^{i_1}\cdots A_{0,E_d}^{i_d}
\] (3.11)
\[
= A_{0,E_1}^{i_1}\cdots A_{0,E_d}^{i_d} \prod_{i=0}^{n-1}(i\pi E'(\pi) + A_{1,0}).
\]

**Proposition 3.26.** Let \( M \) be a finite free \( R \)-module with a basis \( e_1, \ldots, e_l \) and
\[
\varepsilon : M \otimes_{R,p_1} R\{X,Y\} \rightarrow M \otimes_{R,p_0} R\{X,Y\}
\]
be a stratification on \( M \) with respect to \( \mathfrak{S}(R)^\ast \) such that
\[
\varepsilon(e) = \prod_{n \geq 0, I \in \mathbb{N}} A_{n,I}X^{[n]}Y^{[I]}
\]
for \( A_{n,I} \)'s in \( M_I(R) \). Then the following are equivalent:

1. \((M, \varepsilon)\) is induced by a Hodge–Tate crystal \( M \in \text{Vect}((R)_{\Delta, \log}, \mathcal{O}_{\Delta}) \).
2. There are matrices \( A, \Theta_1, \ldots, \Theta_d \in M_I(R) = \text{End}_R(M) \) satisfying the following conditions:
   - (a) For any \( 1 \leq i, j \leq d \), \([\Theta_i, \Theta_j] = 0\);
   - (b) For any \( 1 \leq i, j \leq d \), \([\Theta_i, A] = \pi E'(\pi)\Theta_i\);
   - (c) \( \lim_{n \to +\infty} \prod_{i=0}^{n-1}(i\pi E'(\pi) + A) = 0 \).

In this case, \( \Theta_i \)'s are nilpotent and for any \( n \geq 0 \), \( I = (i_1, \ldots, i_d) \in \mathbb{N}^d \),
\[
A_{n,I} = \Theta_1^{i_1}\cdots \Theta_d^{i_d} \prod_{i=0}^{n-1}(i\pi E'(\pi) + A).
\]

**Proof.** The proof is the same as that of [MW22 Proposition 2.6]. In particular, we put \( \alpha = \pi E'(\pi) \) in [MW22] Lemma 2.7. \( \square \)
Definition 3.27. By an enhanced log Higgs module over $R$, we mean a triple $(M, \theta_M, \phi)$ where

1. $M$ is a finite projective module over $R$;
2. $\theta_M$ is an $R$-linear homomorphism

$$\theta_M : M \to M \otimes_R \widehat{\Omega}_{/\log}^{1}(-1)$$

such that $\theta_M \wedge \theta_M = 0$. Let $\text{HIG}(M, \theta_M)$ denote the complex

$$M \xrightarrow{\theta_M} M \otimes_R \widehat{\Omega}_{/\log}^{1}(-1) \xrightarrow{\theta_M} M \otimes_R \widehat{\Omega}_{/\log}^{2}(-2) \xrightarrow{\theta_M} \cdots$$

(3.12)

is commutative.

Denote by $\text{HIG}(M, \theta_M, \phi)$ the total complex of $(3.12)$ and by $\text{HIG}^{\log}_{*}(R)$ the category of enhanced log Higgs modules over $R$.

Now, the following theorem follows from Proposition 3.2.24 directly.

Theorem 3.28. The evaluation at $(S(R), (E), M_{S(R)}, \delta_{\log})$ induces an equivalence from the category $\text{Vect}((R)_{\Delta, \log}, \overline{\Omega}_{\Delta})$ of Hodge–Tate crystals to the category $\text{HIG}^{\log}_{*}(R)$ of enhanced log Higgs modules. A similar result holds for rational Hodge–Tate crystals after replacing $\text{HIG}^{\log}_{*}(R)$ by $\text{HIG}^{\log}_{*}(R_{\mathbb{Q}})$.

We can also compare the prismatic cohomology of a Hodge–Tate crystal and the Higgs complex associated with the corresponding Higgs modules.

Theorem 3.29. Let $\mathcal{M} \in \text{Vect}((R)_{\Delta, \log}, \overline{\Omega}_{\Delta})$ be a Hodge–Tate crystal with associated enhanced log Higgs module $(M, \theta_M, \phi_M)$. Then there is a quasi-isomorphism

$$\text{RI}^\alpha_{\Delta}(\mathcal{M}) \simeq \text{HIG}(M, \theta_M, \phi_M).$$

A similar result holds for rational Hodge–Tate crystals.

Proof. The proof is the same as that of [MW21b, Theorem 2.12], except that we need to put $\alpha = \pi E' (\pi)$ and replace modules of continuous differentials by modules of continuous log differentials.

A direct corollary is the following theorem.

Theorem 3.30. Let $\mathcal{X}$ be a semi-stable $p$-adic formal scheme over $\mathcal{O}_K$ of relative dimension $d$, with the log structure $\mathcal{M}_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}}^+ \cap \mathcal{O}_{\mathcal{X}}$ where $\mathcal{X}$ is the generic fiber. Let $\nu : \text{Sh}((\mathcal{X}, \mathcal{M}_{\mathcal{X}})_{\Delta}) \to \text{Sh}(\mathcal{X}_{\text{et}})$ be the natural morphism of topoi. Then for any Hodge–Tate crystal $\mathcal{M} \in \text{Vect}((\mathcal{X}, \mathcal{M}_{\mathcal{X}})_{\Delta}, \overline{\Omega}_{\Delta})$, $R\nu_{*}\mathcal{M}$ is a perfect complex of $\mathcal{O}_{\mathcal{X}}$-modules with tor-amplitude $[0, d + 1]$. If moreover $\mathcal{X}$ is proper, then $R\Gamma_{\Delta}(\mathcal{M})$ is a perfect complex of $\mathcal{O}_{\mathcal{K}}$-modules with tor-amplitude $[0, 2d + 1]$. Similar results also hold for rational Hodge–Tate crystals after replacing $\mathcal{O}_{\mathcal{X}}$-modules and $\mathcal{O}_{\mathcal{K}}$-modules by $\mathcal{O}_{\mathcal{X}}[1/p]$-modules and $K$-vector spaces, respectively.

26
If \( X = \text{Spf}(R) \) is small affine and smooth over \( O_K \) (i.e., we take \( r = 0 \) in Example 2.8), then the log structure \( M_X \) is induced by the composition \( (\mathbb{N} \xrightarrow{\gamma} O_K \xrightarrow{r} R) \). In this case, we still have a natural functor

\[
\text{Vect}((R)_\Delta, \overline{\mathcal{O}}_\Delta) \to \text{Vect}((N \to R)_\Delta, \overline{\mathcal{O}}_\Delta),
\]

which is induced by restriction. By Theorem 3.23 and Theorem 3.28, this gives rise to a functor

\[
\text{HIG}^\text{nil}_*(R) \to \text{HIG}^\text{log}_*(R).
\]

By comparing Lemma 3.25 with [MW22, Lemma 2.2], we see that the functor sends an enhanced Higgs module \((M, \theta_M, \phi_M)\) to an enhanced log Higgs module \((M, \theta_M, \pi \phi_M)\). So we get the following result:

**Corollary 3.31.** The restriction functor \(\text{Vect}((R)_\Delta, \overline{\mathcal{O}}_\Delta) \to \text{Vect}((N \to R)_\Delta, \overline{\mathcal{O}}_\Delta)\) is fully faithful and fits into the following commutative diagram:

\[
\begin{array}{ccc}
\text{Vect}((R)_\Delta, \overline{\mathcal{O}}_\Delta) & \xrightarrow{\cong} & \text{HIG}^\text{nil}_*(R) \\
\downarrow & & \downarrow \text{HIG}^\text{log}_*(R) \\
\text{Vect}((N \to R)_\Delta, \overline{\mathcal{O}}_\Delta) & &
\end{array}
\]

where the bottom functor sends \((M, \theta_M, \phi_M)\) to \((M, \theta_M, \pi \phi_M)\). Similar results hold for rational Hodge–Tate crystals and in this case, for any \(M \in \text{Vect}((R)_\Delta, \overline{\mathcal{O}}_\Delta[\frac{1}{r}])\), we get a quasi-isomorphism

\[
\text{RIG}((R)_\Delta, M) \cong \text{RIG}((N \to R)_\Delta, M).
\]

**Proof.** It remains to prove the desired quasi-isomorphism in the rational case. By virtues of Theorem 3.3 and Theorem 3.20, we only need to check

\[
\text{HIG}(M, \theta_M, \phi_M) \cong \text{HIG}(M, \theta_M, \pi \phi_M),
\]

where \((M, \theta_M, \phi_M)\) is the enhanced Higgs module over \(R[\frac{1}{r}]\) corresponding to the given rational Hodge–Tate crystal \(M \in \text{Vect}((R)_\Delta, \overline{\mathcal{O}}_\Delta[\frac{1}{r}])\). Now the result is clear as \(\pi\) is invertible. \(\square\)

### 3.3 Hodge–Tate crystals as generalised representations

We still assume \( X = \text{Spf}(R) \) is small affine with \( M_X \) induced by the prelog structure given in Example 2.8. For simplicity, we write \( R^\mathcal{O} := O_K(T_0, \ldots, T_r, T_r^{\pm 1}, \ldots, T_d^{\pm 1})/(T_0 \cdots T_r - \pi) \) and there is a fixed framing, i.e., an étale map \( \square : R^\mathcal{O} \to R \).

Let \( R^\mathcal{O}_m \) denote the base change \( R^\mathcal{O} \otimes_{O_K} O_C = O_C(T_0, \ldots, T_r, T_r^{\pm 1}, \ldots, T_d^{\pm 1})/(T_0 \cdots T_r - \pi) \). We fix a compatible family of primitive \( p \)-power roots \( \{\pi^{\frac{1}{p^m}}\}_{m \geq 0} \) of \( \pi \) as in Notations. For each \( m \geq 0 \), we then consider the \( R^\mathcal{O}_C \)-algebra

\[
R^\mathcal{O}_{C,m} := O_C(T_0^{\frac{1}{p^m}}, \ldots, T_r^{\frac{1}{p^m}}, T_r^{\pm \frac{1}{p^m}}, \ldots, T_d^{\pm \frac{1}{p^m}})/(T_0^{\frac{1}{p^m}} \cdots T_r^{\frac{1}{p^m}} - \pi^{\frac{1}{p^m}})
\]

and let \( \tilde{R}^\mathcal{O}_C := (\lim_{\rightarrow} R^\mathcal{O}_{C,m})^{\wedge} \), which is a perfectoid ring over \( O_C \).

Let \( R_C := R^\mathcal{O} \otimes_{R^\mathcal{O}} \tilde{R}^\mathcal{O}_C \) and \( \tilde{R}_C := R_C \otimes_{R^\mathcal{O}} \tilde{R}^\mathcal{O}_C \). We consider the adic generic fibers \( X_C = \text{Spa}(R_C[\frac{1}{p}], R_C) \) of \( X_C = \text{Spf}(R_C) \) and put \( X_\infty = \text{Spa}(\tilde{R}_C[\frac{1}{p}], \tilde{R}_C) \).

Then \( X_\infty \) is a Galois cover of both \( X_C \) and \( X \) with the Galois groups \( \Gamma_{\text{geo}} \) and \( \Gamma \), respectively. For any \( 1 \leq i, j \leq d \) and \( n \geq 0 \), let \( \gamma_i \in \Gamma_{\text{geo}} \) such that \( \gamma_i(T_j^{\frac{1}{p^m}}) = \gamma_i(T_j^{\frac{1}{p^m}}) = \zeta_{p^m}^{-\epsilon} T_j^{\frac{1}{p^m}} \) and \( \gamma_i(T_j^{\frac{1}{p^m}}) = \zeta_{p^m}^{-\epsilon} T_j^{\frac{1}{p^m}} \) for \( 1 \leq i \leq r \) and \( \epsilon = 0 \) for \( r + 1 \leq i \leq d \), then

\[
\Gamma_{\text{geo}} \cong \mathbb{Z}_p\gamma_1 \oplus \cdots \oplus \mathbb{Z}_p\gamma_d.
\]  

\[1\] It is more natural to write \( \Gamma_{\text{geo}} \) as the subgroup of \( \prod_{i=r+1}^d \mathbb{Z}_p \gamma_i \to \mathbb{Z}_p \) such that \( \gamma = \prod_{i=0}^r \gamma_i \in \Gamma_{\text{geo}} \) if and only if \( n_0 + \cdots + n_r = 0 \), where \( \gamma_i(T_j^{\frac{1}{p^m}}) = \zeta_{p^m}^{-\epsilon} T_j^{\frac{1}{p^m}} \) for any \( n \geq 0 \) and \( 0 \leq i, j \leq d \). Clearly, if we put \( \gamma_i = \gamma_i^{-1} \gamma_i \) for \( 1 \leq i \leq r \) and \( \gamma_i = \gamma_i \) for \( r + 1 \leq i \leq d \), then \( \Gamma_{\text{geo}} \cong \mathbb{Z}_p\gamma_1 \oplus \cdots \oplus \mathbb{Z}_p\gamma_d \).
The group \( \Gamma_{\text{geo}} \) is a normal subgroup of \( \Gamma \) fitting into the following exact sequence

\[
0 \to \Gamma_{\text{geo}} \to \Gamma \to G_K \to 1,
\]  

where \( G_K = \text{Gal}(\bar{K}/K) \) is the absolute Galois group of \( K \). Moreover, we have \( \Gamma \cong \Gamma_{\text{geo}} \times G_K \) such that for any \( g \in G_K \) and \( 1 \leq i \leq d \),

\[
g^n_i g^{-1} = \gamma_i^{(g)},
\]

where \( \chi : G_K \to \mathbb{Z}_p^\times \) denotes the \( p \)-adic cyclotomic character.

In this subsection, we want to relate Hodge–Tate crystals on the absolute logarithmic prismatic site of \( (X, \mathcal{M}_X) \) to the generalised representations on \( X_{\text{proet}} \), i.e. vector bundles with coefficients in \( \mathcal{O}_X \). In the local case, the latter is equivalent to the category \( \text{Rep}_T(\hat{R}_\infty[\frac{1}{p}]) \), i.e. the category of finite projective \( \hat{R}_\infty[\frac{1}{p}] \)-modules with semi-linear \( \Gamma \)-actions.

Consider the log prism \( (A_{\text{inf}}(\hat{R}_\infty), (\xi), M_{\log}, \delta_{\log}) \) as in Example 2.8. For simplicity, we write \( A_\infty := A_{\text{inf}}(\hat{R}_\infty) \). Let \( (A_\infty^\bullet, (\xi), M_{\log}, \delta_{\log}) \) denote the cosimplicial perfect log prisms associated with \( (A_{\text{inf}}(\hat{R}_\infty), (\xi), M_{\log}, \delta_{\log}) \) in \( (\mathbb{N}^{r+1} \to R)_\Delta^\perflat \). Ignoring the log structures, we let \( (A_\infty^\bullet, (\xi), M_{\log}, \delta_{\log}) \) denote the cosimplicial perfect prisms associated with \( (A_{\text{inf}}(\hat{R}_\infty), (\xi)) \) in \( (R)_\Delta^\perflat \). Then we have the following lemma, which is a direct consequence of Lemma 2.10.

**Lemma 3.32.** For any \( n \geq 0 \), we have \( A_\infty^{(n)} \cong A_\infty^{[n]} \).

By virtues of the above lemma, we can get the following theorem.

**Theorem 3.33.** Evaluating on the perfect log prism \( (A_{\text{inf}}(\hat{R}_\infty), (\xi), M_{\log}, \delta_{\log}) \) induces an equivalence between the category \( \text{Vect}((R)_\Delta^\perflat, \mathcal{O}_\Delta[\frac{1}{p}]) \) and the category \( \text{Rep}_T(\hat{R}_\infty[\frac{1}{p}]) \).

**Proof.** This follows directly from the above lemma and [MW22, Theorem 2.23].

Now we can compose the following functors

\[
\text{Vect}((R)_\Delta^\log, \mathcal{O}_\Delta) \to \text{Vect}((R)_\Delta^\perflat, \mathcal{O}_\Delta) \to \text{Rep}_T(\hat{R}_\infty[\frac{1}{p}]).
\]

For any Hodge–Tate crystal \( M \in \text{Vec}((R)_\Delta^\log, \mathcal{O}_\Delta) \), we denote the associated representation of \( \Gamma \) by \( V(M) \). Then we want to describe \( V(M) \) in an explicit way.

Let \( A_{\text{inf}}(R) \) be the lifting of \( R_C \) over \( A_{\text{inf}} \) determined by the framing \( \square \). Then \( (A_{\text{inf}}(R), (\xi), M_{\log}, \delta_{\log}) \) with the log structure induced by the log structure on \( \mathcal{G}(R) \) is a log prism in \( (R)_\Delta^\log \) by setting \( \delta(T_i) = 0 \) for any \( 1 \leq i \leq d \). A similar argument used in the proof of [MW21a, Lemma 2.11] shows that \( (A_{\infty}, (\xi), M_{\log}, \delta_{\log}) \) is exactly the perfection of \( (A_{\text{inf}}(R), (\xi), M_{\log}, \delta_{\log}) \). So there is a morphism of cosimplicial rings

\[
\mathcal{G}(R)^\bullet \to A_\infty^\bullet.
\]

As a consequence, we get a natural morphism

\[
\mathcal{G}(R)^\bullet/(E) \xrightarrow{\cong} R[X_1, \ldots, X_\bullet, Y_1, \ldots, Y_\bullet, \gamma_1, \ldots, \gamma_d]_{\text{pd}} \to C(\Gamma^\bullet, \hat{R}_\infty[\frac{1}{p}])
\]  

of cosimplicial rings.

**Lemma 3.34.** Regard \( X_1, Y_1, \ldots, Y_{d,1} \) as functions from \( \Gamma \) to \( \hat{R}_\infty[\frac{1}{p}] \). For any \( g \in G_K \) and \( 1 \leq i \leq d \) in \( \mathbb{Z}_p \), if we put \( \sigma = \gamma_1^{n_1} \cdots \gamma_d^{n_d} g \), then

\[
X_1(\sigma) = c(g)\lambda(1 - \zeta_p),
\]

\[
Y_{i,1}(\sigma) = n_i \lambda(1 - \zeta_p), \quad \forall 1 \leq i \leq d,
\]

where \( \lambda \) is the image of \( \frac{\xi}{E([\pi^j])} \) in \( C \) and \( c(g) \in \mathbb{Z}_p \) is determined by \( g(\pi^j) = c^{(g)}(\pi^j) \).
Proposition 3.37. imply the following result.

Proof. By the formulae of stratification given in Proposition 3.26, the cocycle is given by

\[
X_i(\sigma) \equiv \frac{1 - \varphi([c])}{E([\pi^2])} \mod (E)
\]

\[
\equiv \frac{(1 - [c]^{\varphi(c)})}{E([\pi^2])} \frac{\xi}{1 - [c]} \mod (E)
\]

\[
= c(g)\lambda(1 - \zeta_p).
\]

Similarly, we conclude that for any \(1 \leq i \leq d\),

\[
Y_i,1(\sigma) \equiv \frac{[T_i^p - \sigma([T_i^p])]}{E([\pi^2])|T_i^p|} \mod (E)
\]

\[
\equiv \frac{(1 - [c]^{n_i})}{E([\pi^2])} \frac{\xi}{1 - [c]} (1 - [c]^{\frac{1}{p}}) \mod (E)
\]

\[
= n_i\lambda(1 - \zeta_p).
\]

These complete the proof.

Theorem 3.35. Let \(M \in \text{Vect}((R)_{\Delta,log} \bar{\Theta}_{\Delta})\) be a Hodge–Tate crystal with the associated enhanced Higgs module \((M, \theta_M, \phi_M)\). Let \(\Theta_i\)'s be the matrices of \(\theta_M\) as defined in the proof of Theorem \ref{thm:MW22}. Then the cocycle in \(H^1(\Gamma, \text{GL}(M_{\frac{1}{p}}))\) corresponding to \(V(M)\) is given by

\[
U(\sigma) = \exp(\lambda(1 - \zeta_p) \sum_{i=1}^{d} n_i \Theta_i(1 - c(g)\lambda(1 - \zeta_p)\pi E'(\pi)^{-\frac{\varphi_M}{\pi E'(\pi)}}),
\]

for any \(\sigma = \gamma_1 \cdots \gamma_d g \in \Gamma\), where \(\lambda\) and \(c(g)\) are defined in Lemma \ref{lem:3.34}.

Proof. By the formulae of stratification given in Proposition \ref{prop:3.26}, the cocycle is given by

\[
U(\sigma) = \sum_{i_1, \ldots, i_d, n \geq 0} \Theta^{i_1}_{1} \cdots \Theta^{i_d}_{d} \prod_{i=0}^{n-1} (i\pi E'(\pi) + A) Y_1(\sigma)^{[i_1]} \cdots Y_d(\sigma)^{[i_d]} X(\sigma)^{[n]}.
\]

Now the result follows from Lemma \ref{lem:3.34} directly.

As a consequence of the above theorem, we see that the functor \(M \mapsto V(M)\) factors through

\[
\text{Vect}((R)_{\Delta,log, \bar{\Theta}_{\Delta}}) \to \text{Rep}_T(R_C) \to \text{Rep}_T(R_C_{\frac{1}{p}}) \to \text{Rep}_T(R_{\infty, \frac{1}{p}}).\]

From now on, the story is essentially the same as that in [MW22], only with minor changes.

Proposition 3.36. The functor \(\text{Vect}((R)_{\Delta,log, \bar{\Theta}_{\Delta}}) \to \text{Rep}_T(R_C)\) is fully faithful.

Proof. The proof is the same as that of [MW22 Proposition 2.28].

Since \(\text{Spa}(R_{\frac{1}{p}}, R)\) is smooth over \(K\), the similar arguments for the proof of [MW22 Proposition 2.29] imply the following result.

Proposition 3.37. There is an equivalence from the category \(\text{Rep}_T(R_C_{\frac{1}{p}})\) to the category \(\text{Rep}_T(R_{\infty, \frac{1}{p}})\) induced by the base change

\[
V \mapsto V_{\infty} := V \otimes_{R_C_{\frac{1}{p}}} R_{\infty, \frac{1}{p}} \text{,}
\]

which induces a quasi-isomorphism

\[
R\Gamma(\Gamma, V) \cong R\Gamma(\Gamma, V_{\infty}).
\]
As in the good reduction case, we will give another equivalent description of this proposition. For any \( n \geq 0 \), let \( K_n = K(\zeta_p^n, \pi^{1/p^n}) \). Put
\[
R_n^\square = O_K((T_0 \frac{1}{p^n}, \ldots, T_r \frac{1}{p^n}, T_{r+1} \frac{1}{p^n}, \ldots, T_d \frac{1}{p^n})/(T_0 \frac{1}{p^n} \ldots T_r \frac{1}{p^n} - \pi^{1/p^n})).
\]
Then \( R_n^\square[p^n] = K_n(T_1 \frac{1}{p^n}, \ldots, T_d \frac{1}{p^n}) \).

Let \( X_{K_n} = \text{Spa}(R_{K_n}[\frac{1}{p}], R_{K_n}) \) and \( X_n = \text{Spa}(R_n[\frac{1}{p}], R_n) \) denote the base changes of \( X \) to \( \text{Spa}(K_n, O_{K_n}) \) and \( \text{Spa}(R_n[\frac{1}{p}], R_n) \) respectively.

Let \( K_{cyc, \infty} = \bigcup_n K_n \) and \( K_{cyc, \infty} \) be its \( p \)-adic completion, \( R_{cyc, \infty}^\square := \bigcup_n R_n^\square \) and \( \hat{R}_{cyc, \infty}^\square \) be its \( p \)-adic completion.

Consider \( X_{K_{cyc, \infty}} = \text{Spa}(\hat{R}_{K_{cyc, \infty}}[\frac{1}{p}], \hat{R}_{K_{cyc, \infty}}) \) and \( X_{cyc, \infty} = \text{Spa}(\hat{R}_{cyc, \infty}[\frac{1}{p}], \hat{R}_{cyc, \infty}) \) which are the base changes of \( X \) to \( \text{Spa}(\hat{K}_{cyc, \infty}, O_{K_{cyc, \infty}}) \) and \( \text{Spa}(\hat{R}_{cyc, \infty}[\frac{1}{p}], \hat{R}_{cyc, \infty}) \) respectively. Then we see that \( X_{cyc, \infty} \) is a Galois cover of \( X \) with Galois group \( \Gamma_{cyc, \infty} \) and is also a Galois cover of \( X_{K_{cyc, \infty}} \) whose Galois group can be identified with \( \Gamma_{geo} \). We still have a short exact sequence
\[
0 \to \Gamma_{geo} \to \Gamma_{cyc, \infty} \to \hat{G}_K \to 1,
\]
where \( \hat{G}_K = \text{Gal}(K_{cyc, \infty}/K) \). Let \( \text{Rep}_{\Gamma_{cyc, \infty}}(\hat{R}_{K_{cyc, \infty}}[\frac{1}{p}]) \) and \( \text{Rep}_{\Gamma_{cyc, \infty}}(\hat{R}_{cyc, \infty}[\frac{1}{p}]) \) be the category of \( (\text{semi-linear}) \) representations of \( \Gamma_{cyc, \infty} \) over \( 
\hat{R}_{K_{cyc, \infty}}[\frac{1}{p}] \) and \( 
\hat{R}_{cyc, \infty}[\frac{1}{p}] \), respectively. Then by Faltings' almost purity theorem, the following functors
\[
\text{Rep}_{\Gamma_{cyc, \infty}}(\hat{R}_{K_{cyc, \infty}}[\frac{1}{p}]) \to \text{Rep}_{\Gamma}((R_{\hat{R}_{cyc, \infty}}[\frac{1}{p}])
\]
and
\[
\text{Rep}_{\Gamma_{cyc, \infty}}(\hat{R}_{cyc, \infty}[\frac{1}{p}]) \to \text{Rep}_{\Gamma}((R_{\hat{R}_{cyc, \infty}}[\frac{1}{p}])
\]
are both equivalences induced by the corresponding base changes. So Proposition 3.37 can be reformulated as follows.

**Proposition 3.38.** There is an equivalence from the category \( \text{Rep}_{\Gamma_{cyc, \infty}}(R_{K_{cyc, \infty}}[\frac{1}{p}]) \) to the category \( \text{Rep}_{\Gamma_{cyc, \infty}}(\hat{R}_{cyc, \infty}[\frac{1}{p}]) \)
induced by the base change
\[
V \mapsto V_{cyc, \infty} := V \otimes_{R_{K_{cyc, \infty}}} \hat{R}_{cyc, \infty}.
\]

Moreover, the natural morphism \( V \to V_{cyc, \infty} \) induces a quasi-isomorphism
\[
R\Gamma(\Gamma_{cyc, \infty}, V) \simeq R\Gamma(\Gamma_{cyc, \infty}, V_{cyc, \infty}).
\]

**Proof.** The proof is similar to that of [MW21b Proposition 2.31]. \( \square \)

Note that Proposition 3.36 also holds after inverting \( p \), so combining this with Proposition 3.37, we get

**Theorem 3.39.** The natural functor
\[
\text{Vect}((R_{\Delta, \log}^\square[\Delta[1/p]] \to \text{Rep}_{\Gamma}((R_{\hat{R}_{cyc, \infty}}[\frac{1}{p}])
\]
is fully faithful.

### 3.4 Local \( p \)-adic Simpson correspondence

#### 3.4.1 Local Simpson functor for generalised representations

In this subsection, we shall assign to every representation \( V_{\infty} \in \text{Rep}_{\Gamma}(\hat{R}_{\infty}[\frac{1}{p}]) \) a \( G_K \)-Higgs module over \( R_{\Delta[1/p]} \) and an arithmetic Higgs module over \( R_{K_{cyc, \infty}}[\frac{1}{p}] \) in the sense of [MW22 Definition 3.4, 3.6].
As in the good reduction case, the main ingredient for constructing the desired local Simpson functor is a period ring $S_{\text{cyc,}\infty}$ over $\hat{R}_{\text{cyc,}\infty}[\frac{1}{p}]$ together with a universal $\hat{R}_{\text{cyc,}\infty}[\frac{1}{p}]$-linear $\Gamma_{\text{cyc,}\infty}$-equivariant Higgs field

$$\Theta : S_{\text{cyc,}\infty} \to S_{\text{cyc,}\infty} \otimes_R \hat{\Omega}_R^1(-1).$$

This ring can be viewed as sections on $X_{\text{cyc,}\infty}$ of a certain period sheaf $\mathcal{O}_C$ on the pro-étale site $X_{\text{pro ét}}$ (c.f. the $\text{gr}^0\mathcal{O}_{\text{BDR}}$ in \textbf{Sch13} Corollary 6.15) or \textbf{LZ17} Remark 2.1). The $S_{\text{cyc,}\infty}$ and $\Theta$ can be described in a more explicit way; namely, there exists an isomorphism of $\hat{R}_{\text{cyc,}\infty}[\frac{1}{p}]$-algebras

$$S_{\text{cyc,}\infty} \cong \hat{R}_{\text{cyc,}\infty}[\frac{1}{p}][Y_1, \ldots, Y_d]$$

(3.17)

where $\hat{R}_{\text{cyc,}\infty}[\frac{1}{p}][Y_1, \ldots, Y_d]$ is the free algebra over $\hat{R}_{\text{cyc,}\infty}[\frac{1}{p}]$ on the set $\{Y_1, \ldots, Y_d\}$. Via the isomorphism (3.17), the Higgs field $\Theta$ can be written as

$$\Theta = \sum_{i=1}^d \frac{\partial}{\partial Y_i} \otimes \frac{\log(T_i)}{t},$$

(3.18)

where $t$ denotes the $p$-adic analogue of $2\pi i$ (and one identifies Tate twist $\mathbb{Z}_p(n)$ with $\mathbb{Z}_p t^n$ for any $n \in \mathbb{Z}$), and for any $\gamma = \gamma_1^{n_1} \cdots \gamma_d^{n_d} \in \Gamma_{\text{geo}}$, $g \in G_K$ and any $1 \leq j \leq d$,

$$g(Y_i) = \chi(g)^{-1} Y_j,$$

$$\sigma(Y_j) = Y_j + n_j.$$

(3.19)

\textbf{Remark 3.40.} \textit{The local description of the period sheaf $\mathcal{O}_C$ is studied in \textbf{Sch13} for affinoid spaces admitting a standard étale map to the torus. Our case is slightly different. Locally, we have an étale map $\text{Spa}(R[\frac{1}{p}], R) \to \text{Spa}(R[\frac{1}{p}], R^2)$. But one can check that the proofs in \textbf{Sch13} still work in our case and so one can get descriptions of $\mathcal{O}_C$ on $X_{\text{cyc,}\infty}$ as above.}

Then we have the following theorem.

\textbf{Theorem 3.41 (MW22 Theorem 2.39).} \textit{Keep notations as before. Then for any $V_{\text{cyc,}\infty} \in \text{Rep}_{\text{cyc,}\infty}(\hat{R}_{\text{cyc,}\infty}[\frac{1}{p}])$, the restriction $\theta_H(V_{\text{cyc,}\infty})$ of

$$\Theta_{V_{\text{cyc,}\infty}} := \text{id}_{V_{\text{cyc,}\infty}} \otimes \Theta : V_{\text{cyc,}\infty} \otimes_{\hat{R}_{\text{cyc,}\infty}} S_{\text{cyc,}\infty} \to V_{\text{cyc,}\infty} \otimes_{\hat{R}_{\text{cyc,}\infty}} S_{\text{cyc,}\infty} \otimes_R \hat{\Omega}_R^1(-1)$$

is a $\Gamma_{\text{cyc,}\infty}$-equivariant isomorphism and identifies $\Theta_H(V_{\text{cyc,}\infty})$ with $\Theta_{V_{\text{cyc,}\infty}}$.

(1) Define $\Theta_H(V_{\text{cyc,}\infty}) = \theta_H(V_{\text{cyc,}\infty}) \otimes \text{id}_{S_{\text{cyc,}\infty}} + \text{id}_{H(V_{\text{cyc,}\infty})} \otimes \Theta$. Then the natural map

$$H(V_{\text{cyc,}\infty}) \otimes_{\hat{R}_{\text{cyc,}\infty}} S_{\text{cyc,}\infty} \to V_{\text{cyc,}\infty} \otimes_{\hat{R}_{\text{cyc,}\infty}} S_{\text{cyc,}\infty}$$

is a $\Gamma_{\text{cyc,}\infty}$-equivariant isomorphism and defines $\Theta_H(V_{\text{cyc,}\infty})$ with $\Theta_{V_{\text{cyc,}\infty}}$.

(2) Let $HIG(H(V_{\text{cyc,}\infty}), \theta_H(V_{\text{cyc,}\infty}))$ be the associated Higgs complex of $(H(V_{\text{cyc,}\infty}), \theta_H(V_{\text{cyc,}\infty}))$. Then there is a $\hat{G}_K$-equivariant quasi-isomorphism

$$RT(\Gamma_{\text{geo}}, V_{\text{cyc,}\infty}) \simeq HIG(H(V_{\text{cyc,}\infty}), \theta_H(V_{\text{cyc,}\infty})).$$

As a consequence, we get a quasi-isomorphism

$$RT(\Gamma_{\text{cyc,}\infty}, V_{\text{cyc,}\infty}) \simeq RT(\hat{G}_K, HIG(H(V_{\text{cyc,}\infty}), \theta_H(V_{\text{cyc,}\infty}))).$$

(3) Let $\text{HIG}_{\hat{G}_K}(R_{\text{cyc,}\infty}[\frac{1}{p}])$ be the category of nilpotent Higgs modules $(H, \theta_H)$ over $R_{\text{cyc,}\infty}[\frac{1}{p}]$, which are endowed with continuous $\hat{G}_K$-actions such that $\theta_H$'s are $\hat{G}_K$-equivariant. Then the functor

$$H : \text{Rep}_{\text{cyc,}\infty}(\hat{R}_{\text{cyc,}\infty}[\frac{1}{p}]) \to \text{HIG}_{\hat{G}_K}(R_{\text{cyc,}\infty}[\frac{1}{p}])$$

(3.21)
defines an equivalence of categories and preserves tensor products and dualities. More precisely, for a Higgs module \((H, \theta_H)\) over \(R_{\Gamma, \text{cyc}, \infty, p}\) with a \(\hat{G}_K\)-action, define \(\Theta_H = \theta_H \otimes \text{id} + \text{id} \otimes \Theta\). Then the quasi-inverse \(V_{\text{cyc}, \infty}\) of \(H\) is given by

\[V_{\text{cyc}, \infty}(H, \theta_H) = (H \otimes_{R_{\Gamma, \text{cyc}, \infty}} S_{\text{cyc}, \infty})^{\Theta_H = 0}.
\]

**Remark 3.42.** A similar local result in the log smooth case was also obtained in [Tsu18, Theorem 15.1, 15.2].

**Remark 3.43.** The functor \(H\) depends on the choice of toric chart for the moment. However, by using the period sheaf \(\mathcal{O}_X\) on \(X_{\text{pro\acute{e}t}}\), we can consider the functor

\[H : \text{Vect}(\text{Spa}(R_{1/p}, R_{\text{pro\acute{e}t}}), \tilde{\mathcal{O}}_X)) \to \text{HIG}_{G_K}^{\text{nil}}(R_{C_{1/p}}),
\]

which is independent of the choice of charts. Therefore, the local constructions glue. This leads to a global version of Theorem 3.41, which appears in Theorem 4.10 and [MW22, Theorem 3.13].

Similar to \(\text{HIG}_{G_K}^{\text{nil}}(R_{\Gamma, \text{cyc}, \infty, p})\), we define the category \(\text{HIG}_{\Gamma, K}^{\text{nil}}(R_{\Gamma, \text{cyc}, \infty, p})\) consisting of pairs \((H, \theta_H)\), where \(H\) is a representation of \(\Gamma_K\) over \(R_{\Gamma, \text{cyc}, \infty, p}\) (resp. \(R_{\Gamma, \text{cyc}, \infty, p}\)), i.e. a finite projective module over \(R_{\Gamma, \text{cyc}, \infty, p}\) (resp. \(R_{\Gamma, \text{cyc}, \infty, p}\)), endowed with a continuous semi-linear \(\Gamma_K\)-action, and \(\theta_H\) is a Higgs field on \(H\) which is \(\Gamma_K\)-equivariant. Here, \(R_{\Gamma, \text{cyc}} := R \otimes_{\mathcal{O}_X} \mathcal{O}_{\text{Kcyc}}\) and \(R_{\text{Kcyc}}\) is the p-adic completion of \(R_{\text{Kcyc}}\).

For our convenience, we denote \(\text{Rep}_{\Gamma, K}(R_{\Gamma, \text{cyc}, \infty, p})\) (resp. \(\text{Rep}_{\Gamma, K}(R_{\Gamma, \text{cyc}, \infty, p})\)) the category of representations of \(\Gamma_K\) over \(R_{\Gamma, \text{cyc}, \infty, p}\) (resp. \(R_{\Gamma, \text{cyc}, \infty, p}\)).

**Corollary 3.44.** The functor \(H : \text{Rep}_{\Gamma, \text{cyc}, \infty}(\tilde{R}_{\text{Kcyc}, \infty, p}) \to \text{HIG}_{G_K}^{\text{nil}}(R_{\text{Kcyc}, \infty, p})\) upgrades to an equivalence, which is still denoted by \(H\), from \(\text{Rep}_{\Gamma, \text{cyc}, \infty}(\tilde{R}_{\text{Kcyc}, \infty, p})\) to \(\text{HIG}_{G_K}^{\text{nil}}(R_{\text{Kcyc}, \infty, p})\).

**Proof.** By Faltings’ almost purity theorem, we have a natural equivalence

\[\text{Rep}_{\Gamma, K}(R_{\Gamma, \text{cyc}, \infty, p}) \simeq \text{Rep}_{\Gamma, K}(R_{\Gamma, \text{cyc}, \infty, p}),
\]

which induces an equivalence between \(\text{HIG}_{G_K}^{\text{nil}}(R_{\text{Kcyc}, \infty, p})\) and \(\text{HIG}_{\Gamma, K}^{\text{nil}}(R_{\text{Kcyc}, \infty, p})\). Now the result follows from Proposition 3.45 (1).

The other ingredient to construct the Simpson functor is the following decomposition result.

**Proposition 3.45.** Keep notations as above.

1. The base change induces an equivalence from \(\text{Rep}_{\Gamma, K}(R_{\text{Kcyc}, \infty, p})\) to \(\text{Rep}_{\Gamma, K}(R_{\text{Kcyc}, \infty, p})\) such that for any \(V \in \text{Rep}_{\Gamma, K}(R_{\text{Kcyc}, \infty, p})\), there is a canonical quasi-isomorphism

\[R\Gamma(\Gamma_K, V) \simeq R\Gamma(\Gamma_K, V \otimes_{\text{Kcyc}} R_{\text{Kcyc}}).
\]

2. For any \(V \in \text{Rep}_{\Gamma, K}(R_{\text{Kcyc}, \infty, p})\), there exists a unique element \(\phi_V \in \text{End}_{R_{\text{Kcyc}, \infty, p}}(V)\) such that for any \(v \in V\), there exists some \(n \gg 0\) satisfying that for any \(g \in \Gamma_{K(\zeta^n)} := \text{Gal}(K_{\text{cyc}}/K(\zeta^n))\),

\[g(v) = \exp(\log \chi(g) \phi_V)(v).
\]

Such a \(\phi_V\) is called the arithmetic Sen operator of \(V\).

**Proof.** Note that that \(\{R_{K(\zeta^n)}(\tilde{\lambda})\}_{n \gg 0}, \{\Gamma_{K(\zeta^n)}\}_{n \gg 0}\) forms a stably decomposing pair in the sense of [DLLZ18, Definition 4.4]. So the result follows from the argument in the proof of [MW22, Proposition 2.4].

Now we can state and prove the main result in this subsection.
**Theorem 3.46.** There exists a faithful functor

\[ H : \text{Rep}_\Gamma(\hat{\mathbb{R}}_\infty[\frac{1}{p}]) \to \text{HIG}^{\text{arith}}(R_{K_{cyc}}[\frac{1}{p}]), \]

which preserves tensor products and dualities. Here, \( \text{HIG}^{\text{arith}}(R_{K_{cyc}}[\frac{1}{p}]) \) is the category of arithmetic Higgs modules over \( R_{K_{cyc}}[\frac{1}{p}] \) in the sense of [MW22, Definition 2.37].

**Proof.** The result follows from a similar argument in the proof of [MW22, Theorem 2.45]. The only difference is that we have to use Corollary 3.44 instead of [MW22, Corollary 2.43] in loc.cit..

**Remark 3.47.** Replacing \( K_{cyc} \) be \( C \), we obtain from Theorem 3.46 a faithful functor

\[ H : \text{Rep}_\Gamma(\hat{\mathbb{R}}_\infty[\frac{1}{p}]) \to \text{HIG}^{\text{arith}}(R_C[\frac{1}{p}]), \]

which factors through an equivalence

\[ \text{Rep}_\Gamma(\hat{\mathbb{R}}_\infty[\frac{1}{p}]) \to \text{HIG}^{\text{nil}}_{G_K}(R_C[\frac{1}{p}]) \]

where the categories \( \text{HIG}^{\text{nil}}_{G_K}(R_C[\frac{1}{p}]) \) and \( \text{HIG}^{\text{arith}}(R_C[\frac{1}{p}]) \) are defined in the obvious way. We also use this “C-version” of above theorem from now on.

We can make the following conjecture as in the good reduction case.

**Conjecture 3.48.** Let \((H, \theta_H, \phi_H)\) be an arithmetic Higgs module over \( R_C[\frac{1}{p}] \) coming from a representation \( V \in \text{Rep}_\Gamma(\hat{\mathbb{R}}_\infty[\frac{1}{p}]) \). Then there is a quasi-isomorphism

\[ R\Gamma(\Gamma, V) \otimes_{R[\frac{1}{p}]} R_C[\frac{1}{p}] \simeq \text{HIG}(H, \theta_H, \phi_H). \]

### 3.4.2 Local inverse Simpson functor for enhanced Higgs modules

Now we go ahead to construct an inverse Simpson functor defined on the category of enhanced log Higgs modules. The first result is the following theorem.

**Theorem 3.49.** There is a fully faithful functor

\[ V : \text{HIG}^{\text{log}}(R) \to \text{Rep}_\Gamma(R_C) \]

from the category of enhance Higgs modules over \( R \) to the category of representations of \( \Gamma \) over \( R_C \).

**Proof.** Consider the composition of functors

\[ \text{HIG}^{\text{log}}(R) \xrightarrow{\sim} \text{Vect}((R)_\Delta, \overline{\mathcal{O}}_\Delta) \to \text{Vect}((R)^{\text{perf}}_\Delta, \overline{\mathcal{O}}_\Delta[\frac{1}{p}]) \xrightarrow{\sim} \text{Rep}_\Gamma(\hat{\mathbb{R}}_\infty[\frac{1}{p}]). \]

Then by the explicit description of \( \Gamma \)-action given in Theorem 3.35 the above composition factors over \( \text{Rep}_\Gamma(R_C) \to \text{Rep}_\Gamma(\hat{\mathbb{R}}_\infty[\frac{1}{p}]) \) and hence induces a functor

\[ V : \text{HIG}^{\text{log}}(R) \to \text{Rep}_\Gamma(R_C). \]

Then the result is a consequence of Theorem 3.28 combined with Proposition 3.36.

We also have the rational version of the above theorem.

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33
Theorem 3.50. There is a fully faithful functor

\[ V : \text{HIG}^{\log}(R[p]) \to \text{Rep}_R(\hat{R}_\infty[p]) \]

from the category of enhance log Higgs modules over \( R[p] \) to the category of representations of \( \Gamma \) over \( \hat{R}_\infty[p] \). Moreover, let \( V(M) := V(M, \theta_M, \phi_M) \) be the representation associated with the enhance log Higgs module \((M, \theta_M, \phi_M)\). Then we have a quasi-isomorphism

\[ R\Gamma(\Gamma_{\text{geo}}, V(M)) \cong \text{HIG}(M \otimes_R R_C, \theta_M) \]

which is compatible with \( G_K \)-actions.

Proof. The functor \( V \) is given by the composition

\[ \text{HIG}^{\text{nil}}(R[p]) \xrightarrow{\cong} \text{Vect}((R_\Delta, \overline{\mathcal{O}}_R[p]) \to \text{Vect}((R^{\text{perf}}_\Delta, \overline{\mathcal{O}}_R[p]) \cong \text{Rep}_R(\hat{R}_\infty[p]). \]

The proof then is the same as that of [MW22, Theorem 2.51].

\[ \square \]

Remark 3.51. Theorem [3.41] and Theorem [3.53] all depend on the choice of the framing \( \square \) on \( R \). In the rational case, we know that \( \text{Rep}_R(\hat{R}_\infty[p]) \) is equivalent to the category \( \text{Vect}(X, \hat{\mathcal{O}}_X) \) of generalised representations on \( X \). We will see later that this point of view will provide us with the possibility of getting a global functor.

In Theorem 3.50 the full faithfulness of the functor \( V \) amounts to the isomorphism

\[ H^0(\text{HIG}(M, \theta_M, \phi_M)) \cong H^0(\Gamma, V(M)). \]

More generally, we can also ask whether the following conjecture is true.

Conjecture 3.52. There is a quasi-isomorphism

\[ R\Gamma(\Gamma, V(M, \theta_M, \phi_M)) \cong \text{HIG}(M, \theta_M, \phi_M). \]

3.5 A typical example

As in the case of good reduction, we end this section by constructing two faithful functors from the category \( \text{HIG}^{\log}(R[p]) \) to the category of \( \text{HIG}^{\text{arith}}(R_C[p]). \)

Proposition 3.53. (1) The functor \( \text{HIG}^{\log}(R[p]) \to \text{HIG}^{\text{arith}}(R_C[p]) \) sending \((M, \theta_M, \phi_M)\) to \((M \otimes_R R_C[p], (\zeta_p - 1)\theta_M, -\frac{1}{\pi E'(\pi)} \phi_M)\) is faithful and induces quasi-isomorphisms

\[ \text{HIG}(M, \theta_M) \otimes_R R_C[p] \cong \text{HIG}(M \otimes_R R_C[p], (\zeta_p - 1)\theta_M) \]

and

\[ \text{HIG}(M, \theta_M, \phi_M) \otimes_R R_C[p] \cong \text{HIG}(M \otimes_R R_C[p], (\zeta_p - 1)\theta_M, -\frac{1}{\pi E'(\pi)} \phi_M). \]

(2) The functor \( \text{HIG}^{\log}(R[p]) \to \text{HIG}^{\text{nil}}(R_C[p]) \) sending \((M, \theta_M, \phi_M)\) to \((M \otimes_R R_C[p], (\zeta_p - 1)\theta_M)\) together with the \( G_K \) action on \( H(M) \) by

\[ g \mapsto (1 - c(g)\lambda(1 - \zeta_p)\pi E'(\pi))^{-\frac{\phi_M}{\pi E'(\pi)}} \]

is fully faithful. Its composition with the faithful functor \( \text{HIG}^{\text{nil}}(R_C[p]) \to \text{HIG}^{\text{arith}}(R_C[p]) \) then gives the second faithful functor desired.
We guess these two faithful functors are actually the same. More precisely, we make the following conjecture.

**Conjecture 3.54.** For a Hodge–Tate crystal $\mathcal{M}$ with associated enhanced log Higgs module $(M, \theta_M, \phi_M)$ and representation $V_\infty(\mathcal{M})$ of $\Gamma$ over $\tilde{R}_\infty^{[1/p]}$, the arithmetic Sen operator of $V_\infty(M)$ is $-\frac{\lambda_M}{\pi E'(\pi)}$.

The above conjecture together with Conjecture 3.48 implies the following quasi-isomorphisms

\[ R\Gamma_\Delta(\mathcal{M}) \otimes_R R_C[p] \simeq \text{HIG}(M, \theta_M, \phi_M) \otimes_R R_C[p] \]

\[ \simeq \text{HIG}(M \otimes_R R_C[p], (\zeta_p - 1)\lambda_M, -\frac{1}{\pi E'(\pi)} \phi_M) \]

\[ \simeq R\Gamma(\Gamma, V_\infty(\mathcal{M})) \otimes_R R_C, \]

where the first two quasi-isomorphisms follow from Theorem 3.29 and Proposition 3.53 (1). In particular, after taking $G_K$-equivariants, we get

\[ R\Gamma_\Delta(\mathcal{M}) \simeq \text{HIG}(M, \theta_M, \phi_M) \simeq R\Gamma(\Gamma, V_\infty(\mathcal{M})) \]

which is exactly what we guess in Conjecture 3.52. We state the result as a lemma.

**Lemma 3.55.** Conjecture 3.54 is a consequence of Conjecture 3.48 and Conjecture 3.54.

### 4 Globalizations

#### 4.1 Definitions and Preliminaries

In this section, we globalise local constructions in the previous section. From now on, we always assume $\mathcal{X}$ is a semi-stable $p$-adic formal scheme over $\mathcal{O}_K$ of relative dimension $d$ with rigid analytic generic fibre $X$. For any $p$-adic complete subfield $L$ of $C$, we denote $\mathcal{X}_L$ and $X_L$ the base-changes of $\mathcal{X}$ and $X$ along natural morphisms $\text{Spf}(\mathcal{O}_L) \to \text{Spf}(\mathcal{O}_K)$ and $\text{Spa}(L, \mathcal{O}_L) \to \text{Spa}(K, \mathcal{O}_K)$, respectively.

**Definition 4.1.** By an enhanced log Higgs bundle on $\mathcal{X}_{\text{et}}$ with coefficients in $\mathcal{O}_X$, we mean a triple $(\mathcal{M}, \theta_M, \phi_M)$ satisfying the following properties:

1. $\mathcal{M}$ is a finite locally free $\mathcal{O}_X$-module and $\theta_M : \mathcal{M} \to \mathcal{M} \otimes_{\mathcal{O}_X} \widehat{\Omega}_X^1 \{-1\}$ defines a nilpotent Higgs field on $\mathcal{M}$, i.e. $\theta_M$ is a section of $\text{End}(\mathcal{M}) \otimes_{\mathcal{O}_X} \widehat{\Omega}_X^1 \{-1\}$ which is nilpotent and satisfies $\theta_M \wedge \theta_M = 0$. Here "$\{-1\}$" denotes the Breuil–Kisin twist of $\bullet$. Denote by $\text{HIG}(\mathcal{M}, \theta_M)$ the induced Higgs complex.

2. $\phi_M \in \text{End}(\mathcal{M})$ is “topologically nilpotent” in the following sense:

\[ \lim_{n \to +\infty} \prod_{i=0}^n (\phi_M + i\pi E'(\pi)) = 0 \]

and induces an endomorphism of $\text{HIG}(\mathcal{M}, \theta_M)$; that is, the following diagram

\[ \begin{array}{cccccc}
\mathcal{M} & \overset{\theta_M}{\longrightarrow} & \mathcal{M} \otimes \widehat{\Omega}_X^1 \{-1\} & \cdots & \overset{\theta_M}{\longrightarrow} & \mathcal{M} \otimes \widehat{\Omega}_X^d \{-d\} \\
\phi_M & \downarrow & (\phi_M + \pi E'(\pi))\text{id} & \downarrow & (\phi_M + d\pi E'(\pi))\text{id} \\
\mathcal{M} & \overset{\theta_M}{\longrightarrow} & \mathcal{M} \otimes \widehat{\Omega}_X^1 \{-1\} & \cdots & \overset{\theta_M}{\longrightarrow} & \mathcal{M} \otimes \widehat{\Omega}_X^d \{-d\}
\end{array} \]

commutes. Denote $\text{HIG}(\mathcal{M}, \theta_M, \phi_M)$ the total complex of this bicomplex.

Denote by $\text{HIG}^\text{log}_{\text{et}}(\mathcal{X}, \mathcal{O}_X)$ the category of enhanced log Higgs bundles over $\mathcal{X}$. Similarly, we define enhanced log Higgs bundles on $\mathcal{X}_{\text{et}}$ with coefficients in $\mathcal{O}_X^{[1/p]}$ and denote the corresponding category by $\text{HIG}^\text{log}_{\text{et}}(\mathcal{X}, \mathcal{O}_X^{[1/p]})$.  

35
**Definition 4.2** ([MW22 Definition 3.4]). By an $G_K$-Higgs bundle on $X_{C,\alpha}$, we mean a pair $(\mathcal{H}, \theta_{\mathcal{H}})$ such that

1. $\mathcal{H}$ is a locally finite free $\mathcal{O}_{X_C}$-module and
   \[
   \theta_{\mathcal{H}} : \mathcal{H} \to \mathcal{H} \otimes_{\mathcal{O}_X} \Omega^1_X(-1)
   \]
defines a nilpotent Higgs field on $\mathcal{H}$, where $\bullet(-1)$ denotes the Tate twist of $\bullet$. Denote by $\text{HIG}(\mathcal{H}, \theta_{\mathcal{H}})$ the induced Higgs complex.

2. $\mathcal{H}$ is equipped with a semi-linear continuous $G_K$-action such that $\theta_{\mathcal{H}}$ is $G_K$-equivariant.

Denote by $\text{HIG}^{nil}_{G_K}(X_C)$ the category of $G_K$-Higgs bundles on $X_{C,\alpha}$. Similarly, one can define $\Gamma_K$-Higgs bundles on $X_{\text{cyc},\text{ét}}$ and denote the corresponding category by $\text{HIG}^{nil}_{\Gamma_K}(X_{\text{cyc}})$.

**Remark 4.3.** By Faltings’ almost purity theorem, there is a natural equivalence of categories

\[
\text{HIG}^{nil}_{\Gamma_K}(X_{\text{cyc}}) \to \text{HIG}^{nil}_{G_K}(X_C)
\]
induced by base-change. We shall use this equivalence freely and identify these two categories in the rest of this paper.

Recall the following definition appearing essentially in [Pet20 Section 3].

**Definition 4.4** ([MW22 Definition 3.6]). Let $X_{\text{cyc}}$ be the ringed space whose underlying space coincides with $X$’s and the structure sheaf is $\mathcal{O}_{X_{\text{cyc}}} := \mathcal{O}_X \otimes_K K_{\text{cyc}}$ (note that this notation does not contradict our convention at the beginning of this section as $K_{\text{cyc}}$ is not $p$-complete).

By an arithmetic Higgs bundle on $X_{\text{cyc},\alpha}$, we mean a triple $(\mathcal{H}, \theta_{\mathcal{H}}, \phi_{\mathcal{H}})$ such that

1. $\mathcal{H}$ is a finite locally free $\mathcal{O}_{X_{\text{cyc}}}$-module and
   \[
   \theta_{\mathcal{H}} : \mathcal{H} \to \mathcal{H} \otimes_{\mathcal{O}_X} \Omega^1_X(-1)
   \]
defines a nilpotent Higgs field on $\mathcal{H}$. Denote by $\text{HIG}(\mathcal{H}, \theta_{\mathcal{H}})$ the induced Higgs complex.

2. $\phi_{\mathcal{H}} \in \text{End}(\mathcal{H})$ induces an endomorphism of $\text{HIG}(\mathcal{H}, \theta_{\mathcal{H}})$; that is, it makes the following diagram

\[
\begin{array}{cccc}
\mathcal{H} & \xrightarrow{\theta_{\mathcal{H}}} & \mathcal{H} \otimes_{\mathcal{O}_X} \Omega^1_X(-1) & \xrightarrow{\theta_{\mathcal{H}}} & \cdots & \xrightarrow{\theta_{\mathcal{H}}} & \mathcal{H} \otimes_{\mathcal{O}_X} \Omega^d_X(-d) \\
\phi_{\mathcal{H}} & & \phi_{\mathcal{H}} & & \cdots & & \phi_{\mathcal{H}}
\end{array}
\]
commute. Denote $\text{HIG}(\mathcal{H}, \theta_{\mathcal{H}}, \phi_{\mathcal{H}})$ the total complex of this bicomplex.

Denote $\text{HIG}_{\text{arith}}(X_{\text{cyc}})$ the category of arithmetic Higgs bundles on $X_{\text{cyc}}$.

Now we have the following (**NOT necessarily commutative**) diagram which is the global version of functors in Subsection 3.5.

\[
\begin{array}{ccc}
\text{HIG}_{\text{logs}}(X, \mathcal{O}_X[\frac{1}{p}]) & \xrightarrow{F_1} & \text{HIG}^{nil}_{G_K}(X_C) \\
& \xrightarrow{F_3} & \text{HIG}_{\text{arith}}(X_{\text{cyc}}) \\
& \xrightarrow{F_2} &
\end{array}
\]

We specify the meaning of functors $F_i$’s as follows:

The functor $F_3$ sends a Hodge–Tate crystal $(\mathcal{M}, \theta_{\mathcal{M}}, \phi_{\mathcal{M}})$ in $\text{HIG}_{\text{logs}}(X, \mathcal{O}_X[\frac{1}{p}])$ to the arithmetic Higgs bundle

\[
(\mathcal{H} = \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_{\text{cyc}}}, \theta_{\mathcal{H}} = \theta_{\mathcal{M}} \otimes \text{id}, \phi_{\mathcal{H}} = -\frac{\phi_{\mathcal{M}}}{\pi E(\pi)} \otimes \text{id}).
\]
The definition of $F_1$ is suggested by the local case (c.f. Proposition 3.53 (2)): It sends an enhanced log Higgs bundle $(\mathcal{M}, \theta_M, \phi_M)$ in $\text{HIG}^{\log}_\ast(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(1))$ to the $G_K$-Higgs bundle

$$(\mathcal{H} = \mathcal{M} \otimes \mathcal{O}_X \mathcal{O}_{X_G}, \theta_H = \theta_M \otimes \lambda(\zeta_p - 1)\text{id})$$

with the $G_K$-action on $\mathcal{H}$ being induced by the formulae

$$g \mapsto (1 - c(g))\lambda(1 - \zeta_p)\pi E'(\pi)^{-\delta_M/\pi E'(\pi)}.$$  

$$\tag{4.4}$$

**Lemma 4.5.** The $F_1$ given above is a well-defined fully faithful functor.

**Proof.** The proof is the same as that of [MW22 Lemma 3.8], which relies on Proposition 3.53. □

The functor $F_2$ is already defined in [MW22] (in the paragraph below [MW22, Remark 3.10]), which again appears essentially in [Pet20 Proposition 3.2].

**Theorem 4.6.** The functors $F_2$ and $F_3$ are faithful and $F_1$ is fully faithful.

**Proof.** This reduces to Theorem 3.39 and Proposition 3.53. □

Now, there are two faithful functors

$$F_2 \circ F_1, F_3 : \text{HIG}^{\log}_\ast(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(1)) \to \text{HIG}^{\text{arith}}(X_{K\text{cyc}}).$$

We have the following conjecture which is true in the case of $\text{Spf}(\mathcal{O}_K)$:

**Conjecture 4.7.** Keep notations as above. Then $F_3 \simeq F_2 \circ F_1$.

### 4.2 The inverse Simpson functor for enhanced Higgs bundles

In this subsection, we construct an inverse Simpson functor from the category $\text{HIG}^{\log}_\ast(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(1))$ to $\text{Vect}(X, \hat{\mathcal{O}}_X)$ by using prismatic methods. The first main result is the following theorem, which establishes a bridge between Hodge–Tate crystals and generalised representations.

**Theorem 4.8.** There is a canonical equivalence between the category $\text{Vect}((X)_{\Delta, \log}^{\text{perf}}, \overline{\mathcal{O}}_{\Delta}(1))$ of Hodge–Tate crystals on $(X)_{\Delta, \log}^{\text{perf}}$ and the category $\text{Vect}(X, \hat{\mathcal{O}}_X)$ of generalised representations on $X_{\text{proet}}$.

**Proof.** This follows from Lemma 3.33 and [MW22 Theorem 3.17]. □

**Remark 4.9.** By Remark 2.10 and [BS21 Proposition 2.7], the presheaf $\overline{\mathcal{O}}_{\Delta}(1)_{\text{perf}}$ on $(X, \mathcal{M}_X)_{\Delta}$ sending $(A, I, M, \delta_{\log})$ to $A(1)_{\text{log}}^{\text{perf}}$ is indeed a sheaf. Then one may define the category $\text{ Vect}^F((X, \mathcal{M}_X)_{\Delta}, \mathcal{O}_{\Delta}(1))$ of Laurent $F$-crystals $(M, \varphi_M)$ on $(X, \mathcal{M}_X)_{\Delta}$ by requiring that $M$ is a sheaf of $\mathcal{O}_{\Delta}(1)$ such that the following conditions hold:

1. For any $A = (A, I, M, \delta_{\log}) \in (X, \mathcal{M}_X)_{\Delta}$, $M(A)$ is a finite projective $A(1)_{\text{log}}^{\text{perf}}$-module together with an $A(1)_{\text{log}}^{\text{perf}}$-linear isomorphism

$$\varphi_M(A) : M(A) \otimes A(1)_{\text{log}}^{\text{perf}} \to M(A).$$

2. For any morphism $A = (A, I, M, \delta_{\log}) \to B = (B, I_B, N, \delta_{\log})$ in $(X, \mathcal{M}_X)_{\Delta}$, there is a canonical isomorphism

$$M(A) \otimes A(1)_{\text{log}}^{\text{perf}} \to M(B),$$

which is compatible with $\varphi_M$.
Now, Lemma 3.33 implies that there is a canonical equivalence of categories

$$\text{Vect}^F((\mathcal{X})_{\Delta}^\text{perf}, O_{\Delta} [\frac{1}{T_{\Delta}^p}]) \rightarrow \text{Vect}^F((\mathcal{X}, \mathcal{M}_X)_{\Delta}^\text{perf}, O_{\Delta} [\frac{1}{T_{\Delta}^p}]),$$

where both categories can be defined in the obvious way. Then [BS21 Corollary 3.7] provides natural equivalences

$$\text{Vect}^F((\mathcal{X})_{\Delta}, O_{\Delta} [\frac{1}{T_{\Delta}^p}]) \rightarrow \text{Vect}^F((\mathcal{X})_{\Delta}^\text{perf}, O_{\Delta} [\frac{1}{T_{\Delta}^p}]) \rightarrow \text{LS}_{Z_p}(X),$$

where $\text{LS}_{Z_p}(X)$ denotes the category of étale $Z_p$-local systems on $X$, the generic fibre of $\mathcal{X}$. In particular, we get an equivalence

$$\text{Vect}^F((\mathcal{X}, \mathcal{M}_X)_{\Delta}^\text{perf}, O_{\Delta} [\frac{1}{T_{\Delta}^p}]) \rightarrow \text{LS}_{Z_p}(X).$$

Without checking details, we still believe that arguments for the proof of [MW21a Theorem 3.1] give an equivalence.\(^{12}\)

$$\text{Vect}^F((\mathcal{X}, \mathcal{M}_X)_{\Delta}, O_{\Delta} [\frac{1}{T_{\Delta}^p}]) \rightarrow \text{Vect}^F((\mathcal{X}, \mathcal{M}_X)_{\Delta}^\text{perf}, O_{\Delta} [\frac{1}{T_{\Delta}^p}]).$$

In particular, this suggests a commutative diagram of equivalent categories

\[
\begin{array}{ccc}
\text{Vect}^F((\mathcal{X})_{\Delta}, O_{\Delta} [\frac{1}{T_{\Delta}^p}]) & \rightarrow & \text{Vect}^F((\mathcal{X})_{\Delta}^\text{perf}, O_{\Delta} [\frac{1}{T_{\Delta}^p}]) \\
| & & |
\text{LS}_{Z_p}(X) & \rightarrow & \text{LS}_{Z_p}(X) \\
| & & |
\text{Vect}^F((\mathcal{X}, \mathcal{M}_X)_{\Delta}, O_{\Delta} [\frac{1}{T_{\Delta}^p}]) & \rightarrow & \text{Vect}^F((\mathcal{X}, \mathcal{M}_X)_{\Delta}^\text{perf}, O_{\Delta} [\frac{1}{T_{\Delta}^p}]).
\end{array}
\]

We also believe that the approach to showing [MW21a Theorem 4.11] also works in the logarithmic case; that is, for a Laurent $F$-crystal $\mathcal{M}$ on $(\mathcal{X}, \mathcal{M}_X)_{\Delta}$ with associated $Z_p$-local system $\mathcal{L}$ on $X_{\text{et}}$, there is a quasi-isomorphism

$$\Gamma((\mathcal{X}, \mathcal{M}_X)_{\Delta}, \mathcal{M})_{\omega=1} \simeq \Gamma(X_{\text{et}}, \mathcal{L}).$$

We will not deal with these problems in this paper.

Now we can get the following important theorem as in the case of good reduction, whose proof is the same as that of [MW22 Theorem 3.19] after we have established all the needed results in the previous sections.

**Theorem 4.10.** Let $\mathcal{X}$ be a semistable $p$-adic formal scheme over $O_K$. Then there is a canonical equivalence of the categories

$$M : \text{Vect}((\mathcal{X})_{\Delta, \log}, \overline{\mathcal{O}}_{\Delta} [\frac{1}{p}]) \rightarrow \text{HIG}_x^\text{log}(\mathcal{X}, \overline{\mathcal{O}}_{\Delta} [\frac{1}{p}]),$$

which makes the following diagram commute

\[
\begin{array}{ccc}
\text{Vect}((\mathcal{X})_{\Delta, \log}, \overline{\mathcal{O}}_{\Delta} [\frac{1}{p}]) & \xrightarrow{R} & \text{Vect}((\mathcal{X})_{\Delta, \log}^\text{perf}, \overline{\mathcal{O}}_{\Delta} [\frac{1}{p}]) \\
\xrightarrow{M} & & \xrightarrow{\phi} \\
\text{HIG}_x^\text{log}(\mathcal{X}, \overline{\mathcal{O}}_{\Delta} [\frac{1}{p}]) & \xrightarrow{F_1} & \text{HIG}_x^\text{null}(X_C).
\end{array}
\]

\(^{12}\)The authors know from Heng Du that he has established this equivalence.
Remark 4.12. At first glance, it seems that we can get the inverse Simpson functor without passing through the prismatic world. Namely, we can compose the functors $F_1$ and $H^{-1}$. But we want to emphasize that the local version of the functor $F_1$ can not be detected without diving into the categories of Hodge–Tate crystals, which reflect the shape of the $G_K$-actions.

As a corollary of Theorem 4.11, the full faithfulness of $R$ is straightforward, as all other arrows in diagram 4.13 are fully faithful.

Corollary 4.13. The restriction $\text{Vect}((\mathcal{X})_{\Delta, \log}, \overline{\mathcal{O}}_{\Delta}[\frac{1}{p}]) \rightarrow \text{Vect}((\mathcal{X})_{\Delta, \log}^{\text{perf}}, \overline{\mathcal{O}}_{\Delta}[\frac{1}{p}])$ is fully faithful.

Since full faithfulness is a local property, by Corollary 4.31 when $\mathcal{X}$ is smooth (so the log structure $\mathcal{M}_X$ is induced by the composition $\mathbb{N} \rightarrow \mathcal{O}_K \rightarrow \mathcal{O}_X$), we have

Corollary 4.14. Assume $\mathcal{X}$ is smooth. There is a fully faithful functor

$$\text{Vect}((\mathcal{X})_{\Delta}, \overline{\mathcal{O}}_{\Delta}[\frac{1}{p}]) \rightarrow \text{Vect}((\mathcal{X}, \mathcal{M}_X)_{\Delta}, \overline{\mathcal{O}}_{\Delta}[\frac{1}{p}])$$

which is induced by restriction and fits into the following commutative diagram:

$$\begin{array}{ccc}
\text{HIG}^{\text{nil}}(\mathcal{X}, \mathcal{O}_X[\frac{1}{p}]) & \longrightarrow & \text{HIG}^{\text{log}}(\mathcal{X}, \mathcal{O}_X[\frac{1}{p}]), \\
\downarrow & & \downarrow \\
\text{Vect}((\mathcal{X})_{\Delta}, \overline{\mathcal{O}}_{\Delta}[\frac{1}{p}]) & \longrightarrow & \text{Vect}((\mathcal{X}, \mathcal{M}_X)_{\Delta}, \overline{\mathcal{O}}_{\Delta}[\frac{1}{p}])
\end{array}$$

where the bottom functor sends an enhanced Higgs bundle $(\mathcal{H}, \theta_\mathcal{H}, \phi_\mathcal{H})$ to the enhanced log Higgs bundle $(\mathcal{H}, \theta_\mathcal{H}, \pi_\phi_\mathcal{H})$.

Remark 4.15. By checking definitions of $F_1$ in both good and semi-stable reduction cases, we can easily see that we indeed have the following commutative diagram:

$$\begin{array}{ccc}
\text{HIG}^{\text{nil}}(\mathcal{X}, \mathcal{O}_X[\frac{1}{p}]) & \longrightarrow & \text{HIG}^{\text{log}}(\mathcal{X}, \mathcal{O}_X[\frac{1}{p}]), \\
\downarrow & & \downarrow \\
\text{Vect}((\mathcal{X})_{\Delta}, \overline{\mathcal{O}}_{\Delta}[\frac{1}{p}]) & \longrightarrow & \text{Vect}((\mathcal{X}, \mathcal{M}_X)_{\Delta}, \overline{\mathcal{O}}_{\Delta}[\frac{1}{p}])
\end{array}$$

Each arrow above is actually a full faithful functor and is an equivalence if it is labelled by “$\simeq$”.

Now, there are two ways to assign to a Hodge–Tate crystal an arithmetic Higgs bundles:

$$F_3 \circ M, F_2 \circ H \circ R = F_2 \circ F_1 \circ M : \text{Vect}((\mathcal{X})_{\Delta, \log}, \overline{\mathcal{O}}_{\Delta}[\frac{1}{p}]) \rightarrow \text{HIG}^{\text{arith}}(\mathcal{X}_C).$$

Then the Conjecture 4.17 can be restated as follows:

Conjecture 4.16. For any rational Hodge–Tate crystal $\mathbb{M} \in \text{Vect}((\mathcal{X})_{\Delta, \log}, \overline{\mathcal{O}}_{\Delta}[\frac{1}{p}])$, denote by $(\mathcal{M}, \theta_\mathcal{M}, \phi_\mathcal{M})$ and $V(\mathbb{M})$ the corresponding enhanced log Higgs bundle and generalised representation. Then the arithmetic Sen operator of $V(\mathbb{M})$ is $-\frac{\phi_\mathcal{M}}{\pi_{\text{Et}}(\mathcal{X}_C)}$.

Remark 4.17. The conjecture is a global version of Conjecture 3.54 and can be checked étale locally on $\mathcal{X}$. The authors knew from Hui Gao that he could confirm Conjecture 4.16 by using a relative version of Kummer Sen theory appearing in [Gao22]. So we will not try to attack the conjecture in this paper.

Finally, we make the following conjecture on the global prismatic cohomology:

Conjecture 4.18. Keep notations as above. Then there exist quasi-isomorphisms

$$R\Gamma(\mathcal{M}) \simeq R\Gamma(\mathcal{X}_\text{et}, \text{HIG}(\mathcal{M}, \theta_\mathcal{M}, \phi_\mathcal{M})) \simeq R\Gamma(X_\text{proét}, V(\mathbb{M})).$$
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