Contact seaweeds

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Abstract
A $(2k + 1)$–dimensional contact Lie algebra is one which admits a one-form $\varphi$ such that $\varphi \wedge (d\varphi)^k \neq 0$. Such algebras have index one, but this is not generally a sufficient condition. Here we show that index-one type-$A$ seaweed algebras are necessarily contact. Examples, together with a method for their explicit construction, are provided.

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1 Introduction

The index of a Lie algebra $(\mathfrak{g}, [-,-])$ is an important algebraic invariant which was first formally introduced by Dixmier ([8], 1977). It is defined by

$$\text{ind } \mathfrak{g} = \min_{\varphi \in \mathfrak{g}^*} \dim(\ker(B_\varphi)),$$

where $\varphi$ is an element of the linear dual $\mathfrak{g}^*$ and $B_\varphi$ is the associated skew-symmetric Kirillov form defined by

$$B_\varphi(x,y) = \varphi([x,y]), \text{ for all } x, y \in \mathfrak{g}.$$

On a given Lie algebra $\mathfrak{g}$, index-realizing linear forms, i.e., those $\varphi \in \mathfrak{g}^*$ which satisfy $\dim(\ker(B_\varphi)) = \text{ind } \mathfrak{g}$, are called regular and exist in profusion, being dense in both the Zariski and Euclidean topologies of $\mathfrak{g}^*$ (see [7]).

Using the above notation, the index is used to describe certain important classes of algebras.

- If $\dim \mathfrak{g} = 2n$ and if there exists a $\varphi$ such that $(d\varphi)^n \neq 0$, then $\mathfrak{g}$ is said to be Frobenius, and $\varphi$ is called a Frobenius form. Deformation theorists are interested in Frobenius Lie algebras because each such $\mathfrak{g}$ provides a solution to the classical Yang-Baxter equation, which in turn quantizes to a universal deformation formula, i.e., a Drinfel’d twist which deforms any algebra which admits an action of $\mathfrak{g}$ by derivations (see [10]). A Lie algebra is Frobenius precisely when its index is zero.

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• If \( \dim g = 2n + 1 \) and if there exists a \( \varphi \) such that \( \varphi \wedge (d\varphi)^n \neq 0 \), then \( g \) is said to be contact, \( \varphi \) is called a contact form, and \( \varphi \wedge (d\varphi)^n \) is a volume form on the underlying Lie group. The construction and classification of contact manifolds is a central problem in differential topology (see [15]). If a Lie algebra is contact, then its index is equal to one. The converse is not true in general.\(^1\) However, there are a few important families of Lie algebras for which index one identifies if a given Lie algebra is contact. For example, index-one nilpotent Lie algebras and real, compact Lie algebras of index one are necessarily contact (see [11]).

Remark 1. Frobenius and contact Lie algebras are tightly interwoven. Every Frobenius Lie algebra has a codimension-one contact ideal, and every Frobenius Lie algebra is a codimension-one ideal of a contact Lie algebra (see [1]).

Here, we seek to classify contact algebras among a class of matrix algebras called seaweed algebras (or simply, “seaweeds”). These algebras, along with their evocative name, were first introduced by Dergachev and A. Kirillov in ([7], 2000), where they defined such algebras as complex subalgebras of \( \mathfrak{gl}(n) \) preserving certain flags of subspaces of \( C^n \) developed from two compositions of \( n \). The passage to seaweeds of “classical type” is realized by requiring that elements of the seaweed subalgebra of \( \mathfrak{gl}(n) \) satisfy additional algebraic conditions. For example, the type-A case \( (A_{n-1} = \mathfrak{sl}(n, \mathbb{C})) \) is defined by a vanishing trace condition. Ongoing, we will assume that all Lie algebras are over \( \mathbb{C} \).

We can now state the main result of this article, which asserts that index one is sufficient for seaweed subalgebras of \( \mathfrak{sl}(n) = A_{n-1} \) to be contact.

Theorem. If \( g \) is a type-A seaweed, then \( g \) is contact if and only if \( \text{ind} g = 1 \).

The structure of the paper is as follows. In Section 2, we define type-A seaweeds and detail the construction of an associated planar graph, called a meander, which is helpful in computing the seaweed’s index. In Section 3, we discuss a framework for constructing regular one-forms on type-A seaweeds and explicitly compute the kernels of the associated Kirillov forms. In Section 4, we establish that the regular one-forms of Section 3 are, in fact, contact, thus proving the main theorem. In Section 5, the reader will find algebraic technology which can be used to generate a spate of both Frobenius and contact seaweeds of arbitrarily high dimension.

2 Seaweeds and Meanders

A type-A seaweed in its “standard (matrix) form”\(^2\) can be described as a subalgebra of \( \mathfrak{sl}(n) \) constructed as follows. First, fix two ordered compositions of \( n \), \( (a) = (a_1, \ldots, a_m) \) and \( (b) = (b_1, \ldots, b_t) \). Let \( D_{(a)} \) be the subalgebra of block-diagonal matrices whose blocks have sizes \( a_1 \times a_1, \ldots, a_m \times a_m \) and similarly for \( D_{(b)} \). A type-A seaweed algebra (or simply, “type-A seaweed”) is the subalgebra of \( \mathfrak{sl}(n) \) spanned by the intersection of \( D_{(a)} \) with the lower-triangular matrices and the intersection of \( D_{(b)} \) with the upper-triangular matrices. We call the the locations of potentially nonzero entries in the seaweed admissible locations. For a type-A seaweed defined by two compositions of \( n \) as above, we write \( p_n^A a_1 \ldots | b_1 \ldots | b_t \). See Example 1.

Remark 2. We tacitly assume a standard (Chevalley) basis for a seaweed \( g = p_n^A a_1 \ldots | b_1 \ldots | b_t \) given by the union of the following two sets of matrix units:

\[ \{ e_{i,i} - e_{i+1,i+1} \mid 1 \leq i \leq n - 1 \} , \text{ and} \]

\(^1\)The Lie algebra \( g = \langle e_1, e_2, e_3 \rangle \) with relations \( [e_1, e_2] = e_2 \) and \( [e_1, e_3] = e_3 \) has index one but is not contact.

\(^2\)Since every seaweed is conjugate to one in standard form, we have presumed this in the definition of a type-A seaweed in order to ease exposition. A basis-free definition reckons seaweed subalgebras of a reductive Lie algebra \( g \) as the intersection of two parabolic algebras whose sum is \( g \) (see [13]). For this reason, seaweed algebras have elsewhere been called biparabolic (see [12]). We do not require the latter definition for our present discussion.
• \( \{ e_{i,j} \mid 1 \leq i \neq j \leq n \text{ and } (i,j) \text{ is an admissible location} \} \).

**Example 1.** We illustrate a type-A seaweed in its standard matrix form – revealing its characteristic wavy seaweed “shape.” The asterisks represent admissible locations and entries in non-admissible locations are zeroes. See Figure 1.

![Figure 1: The seaweed \( p^A_n \)](image)

To each seaweed \( p^A_n \) we associate a planar graph called a *meander*, constructed as follows. First, place \( n \) vertices \( v_1 \) through \( v_n \) in a horizontal line. Next, create two partitions of the vertices by forming *top* and *bottom blocks* of vertices of size \( a_1, a_2, \ldots, a_m \), and \( b_1, b_2, \ldots, b_t \), respectively. Place edges in each top (likewise bottom) block in the same way. Add an edge from the first vertex of the block to the last vertex of the same block. Repeat this edge addition on the second vertex and the second to last vertex within the same block and so on within each block of both partitions. Top edges are drawn concave down and bottom edges are drawn concave up. Let \( M(g) \) denote the meander associated with \( g \). We place a counterclockwise orientation on \( M(g) \) to produce the *directed meander* \( \overrightarrow{M(g)} \). See Example 2.

**Example 2.** We illustrate the meander and directed meander of \( g = p^A_n \). See Figure 2.

![Figure 2: \( M(g) \) and \( \overrightarrow{M(g)} \)](image)

A meander can be visualized inside its associated seaweed \( g \) if one views the diagonal locations \( \{(i,i)\}_{i=1}^n \) of \( g \) as the \( n \) vertices \( \{v_i\}_{i=1}^n \) of the meander and reckons the top edges \( \{(v_i,v_j) \mid i < j\} \) of the meander as the unions of line segments connecting the matrix locations \( (i,i) \rightarrow (j,i) \rightarrow (j,j) \) and the bottom edges \( \{(v_i,v_j) \mid i < j\} \) of the meander as the unions of line segments connecting the matrix locations \( (i,i) \rightarrow (i,j) \rightarrow (j,j) \). See Figure 3.
Remarkably, the index of the seaweed can be computed by counting the number and type of the connected components in the associated meander. In a given meander, we call a connected component a *path* if it is not a cycle. Note that a path may be degenerate, i.e., consist of a single vertex.

**Theorem 1** (Dergachev and A. Kirillov [7], 2000). If \( g \) is a type-A seaweed, then

\[
\text{ind} \ g = 2C + P - 1,
\]

where \( C \) is the number of cycles and \( P \) is the number of paths in \( M(g) \).

**Example 3.** Consider the type-A seaweed \( b = p_{26}^{4} \), which is the standard Borel subalgebra of \( \mathfrak{sl}(5) \) (see Figure 4 (left)). Since \( M(b) \) consists of zero cycles and three paths (see Figure 4 (right)), it follows from Theorem 1 that

\[
\text{ind} \ b = 2(0) + 3 - 1 = 2.
\]

Any meander can be contracted, or “wound down,” to the empty meander through a sequence of graph-theoretic moves – detailed in Lemma 1 below – each of which is uniquely determined by the structure of the meander at the time of the move application. Such a sequence is called the *signature* of the meander (see [3]). The signature is essentially a graph theoretic recasting of Panyushev’s reduction algorithm (see [13]).

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3The authors actually established the formula \( 2C + P \) for the index of a seaweed subalgebra of \( \mathfrak{gl}(n) \), but only a minor algebraic argument is required to extend to the type-A case yielding (1). See [4].
Lemma 1 (Coll et al. [3]). Let $g = p_n^{a_1|...|a_m}$ be a type-A seaweed with associated meander $M(g)$ – in this setting, we say the meander $M = M(g)$ is of type $\frac{a_1|...|a_m}{b_1|...|b_t}$. Create a meander $M'$ by one of the following moves.

1. Pure Contraction ($P$): If $a_1 > 2b_1$, then $M \mapsto M'$ of type $\frac{a_1-2b_1|a_2|...|a_m}{b_2|...|b_t}$.
2. Block Elimination ($B$): If $a_1 = 2b_1$, then $M \mapsto M'$ of type $\frac{b_1|a_2|...|a_m}{b_2|...|b_t}$.
3. Rotation Contraction ($R$): If $b_1 < a_1 < 2b_1$, then $M \mapsto M'$ of type $\frac{b_1|a_2|...|a_m}{2b_1-b_2|...|b_t}$.
4. Component Deletion ($C(c)$): If $a_1 = b_1 = c$, then $M \mapsto M'$ of type $\frac{a_2|...|a_m}{b_2|...|b_t}$.
5. Flip ($F$): If $a_1 < b_1$, then $M \mapsto M'$ of type $\frac{b_1|b_2|...|b_t}{a_1|...|a_m}$.

These moves are called winding-down moves. For all moves, except the Component Deletion move, $g$ and $g'$ (the seaweed with meander $M(g') = M'$) have the same index.

If $g$ is as in Lemma 1 and $M(g)$ is a meander for which the collection of component deletions in its signature is $\{C(c_1), ..., C(c_q)\}$, then we say that $M(g)$ and, by an abuse of terminology, $g$ each have homotopy type $\mathcal{H}(c_1, ..., c_q)$. See Example 4.

Example 4. The meander of type $\frac{2|6}{8}$ of Example 2 has signature $FPFRBC(2)$, and so has homotopy type $\mathcal{H}(2)$. The winding-down moves associated with the signature are illustrated in Figure 5.

![Figure 5: Winding down the meander of type $\frac{2|6}{8}$](image)

Since contact seaweeds are the main focus, we need only consider those seaweeds with index one. Theorem 1 tells us how to find them. For emphasis, we record this in the following corollary to Theorem 1.

Theorem 2. A type-A seaweed has index one if and only if its associated meander consists of exactly one cycle or exactly two paths, i.e., its associated meander has homotopy type $\mathcal{H}(2)$ or $\mathcal{H}(1,1)$.

3 Framework for regular forms

In her dissertation, Dougherty ([6], 2019) establishes a complicated inductive framework for the explicit construction of families of regular one-forms on seaweeds of classical type (cf. [2]). When restricted to type-A seaweeds of index one, this framework simplifies considerably. Importantly, some of these regular families consist of one-forms that are also contact. We detail this “type-A framework” in the following subsections.

3.1 Component meanders

Notation: Ongoing, $g$ will be assumed to be a type-A seaweed.
If \( g \) has homotopy type \( \mathcal{H}(c_1, \ldots, c_q) \), define the component meander \( CM(g) \) associated with \( g \) to be the meander with the same signature as \( g \) except that the component deletions are all of size one. The construction of \( CM(g) \) involves an implicit identification of components, i.e., vertices and edges, in \( M(g) \). The vertices of \( CM(g) \) are \( \{v_{I_1}, \ldots, v_{I_t}\} \), where \( I_i \) is the set of indices for the vertices in \( M(g) \) that were collapsed into one vertex in \( CM(g) \). Note that \( |I_i| = c_j \) if \( v_{I_i} \) is a vertex of \( CM(g) \) arising from the collapsing of a component of size \( c_j \). Also note \( CM(g) \) may be oriented as usual to yield \( \overrightarrow{CM(g)} \). See Example 5.

**Example 5.** Consider \( g = p_6^4 \frac{2^{11}1^2}{6} \). In Figure 6, we illustrate the meander, component meander, and directed component meander of \( g \).

![Figure 6: M(g), CM(g), and \( \overrightarrow{CM(g)} \)](image)

### 3.2 The Core and Peak

The core and peak of \( g \) are sets of sets of admissible locations and their definitions are based on a vector space decomposition of \( g \) developed from the homotopy type of \( g \) as follows. If \( g \) has homotopy type \( \mathcal{H}(c_1, \ldots, c_q) \) then

\[
\begin{align*}
g & = \bigoplus_{i=1}^{q} g|_{c_i},
\end{align*}
\]

where \( g|_{c_i} \) is the subspace of \( g \) corresponding to a particular component of size \( c_i \) in \( M(g) \). See Example 6.

**Example 6.** Consider \( g = p_6^4 \frac{2^{11}1^2}{6} \) of our running Example 5. Note that \( g \) has homotopy type \( \mathcal{H}(2,1) \), which yields the vector space decomposition \( g = g|_2 \oplus g|_1 \). See Figure 7.

![Figure 7: Vector space decomposition of \( g \)](image)

We are now ready to formally define the core and peak of \( g \). First, fix a component \( g|_{c_i} \) of homotopy type \( c_i \) and define the sets

\[
V_{c_i} = \{I \mid v_I \text{ is a vertex on the path of } CM(g) \text{ corresponding to } c_i\},
\]
\[ C_{c_i} = \{ I \times I \mid I \in V_{c_i} \}. \]

\[ P_{c_i} = \{ I \times J \mid I, J \in V_{c_i} \text{ and } (v_I, v_J) \text{ is an edge in } \overrightarrow{CM}(g) \}. \]

The set \( C_{c_i} \) is the core set of \( g|_{c_i} \) — the set of \( c_i \times c_i \) blocks on the diagonal of \( g \) contained in \( g|_{c_i} \) — and the set \( P_{c_i} \) is the peak set of \( g|_{c_i} \). Now, we define the core of \( g \) and the peak of \( g \) as the union of the core sets and peak sets, respectively, of the components in the vector space decomposition (2). In other words,

\[ C_g = \bigcup_{i=1}^{q} C_{c_i} \quad \text{and} \quad P_g = \bigcup_{i=1}^{q} P_{c_i}. \]

Since \( C_g \) and \( P_g \) consist of sets of ordered pairs of indices defining blocks of admissible locations in \( g \), we refer to elements of \( C_g \) and \( P_g \) as core blocks and peak blocks, respectively.

**Example 7.** We illustrate the definitions above by constructing the core and peak sets of the seaweed of Example 5, \( g = p^4_{12}A^2_62^11^11^2 \). Recall \( g \) has homotopy type \( H(2, 1) \). As illustrated in Figure 6 (center) above, the vertices of \( CM(g) \) are written as \( v_{(1,2)}, v_{(3)}, v_{(4)}, \) and \( v_{(5,6)} \). So, we have that

\[ V_2 = \{ \{1, 2\}, \{5, 6\} \} \quad \text{and} \quad V_1 = \{ \{3\}, \{4\} \}, \]

so the core set of \( g \) is

\[ C_g = \left\{ \{(1,1), (1,2), (2,1), (2,2)\}, \{(3,3)\}, \{(4,4)\}, \{(5,5), (5,6), (6,5), (6,6)\} \right\}. \]

Now, to construct the peak set of \( g \), consider \( \overrightarrow{CM}(g) \). As illustrated by Figure 6 (right), we have that

\[ P_2 = \left\{ \{(1,5), (1,6), (2,5), (2,6)\} \right\} \quad \text{and} \quad P_1 = \left\{ \{(3,4)\} \right\}, \]

so the peak set of \( g \) is

\[ P_g = \left\{ \{(1,5), (1,6), (2,5), (2,6)\}, \{(3,4)\} \right\}. \]

The core blocks and peak blocks of \( g \) are bolded and outlined, respectively, in the seaweed in Figure 8.

![Figure 8: The core and peak blocks identified within g](image)

### 3.3 Regular one-forms

The core and peak of an index-one \( g \) facilitate the definition of a family \( \Phi \) of regular one-forms on \( g \) reliant on the homotopy type of \( g \). An element of \( \Phi \) is of the form \( \sum c_{i,j}^* \), where \( c_{i,j}^* \) denotes the dual of the matrix unit \( e_{i,j} \) and the sum is over a restricted set of admissible locations \((i, j)\) of \( g \). The restrictions are determined by the homotopy type of \( g \), which, for an index-one \( g \), must be either \( H(2) \) or \( H(1, 1) \).
3.4 Homotopy type $\mathcal{H}(2)$

Let $g$ have homotopy type $\mathcal{H}(2)$, with attendant peak and core. Here, we construct a regular one-form $\varphi(2) \in \Phi$ using $P_g$ and $C_g$. We find it convenient to mark the admissible locations which define $\varphi(2)$ by black dots—a black dot is placed in location $(i, j)$ if and only if $e_{i,j}^*$ is a nonzero summand of $\varphi(2)$—according to the following schema. Note that the core and peak blocks are all $2 \times 2$. A single black dot is placed in the upper left corner of each of the core blocks. For the peak blocks, black dots are placed along the diagonal. See Example 8.

Example 8. Recall that $g = p_{8,6}^{A,2}$ has homotopy type $\mathcal{H}(2)$. The regular functional $\varphi(2)$ has summands determined by locations of black dots in Figure 9.

![Figure 9: The summands of $\varphi(2)$ identified within $g$](image)

As Figure 9 displays, $\varphi(2)$ is given by

$$\varphi(2) = e_{1,1}^* + e_{3,3}^* + e_{5,5}^* + e_{7,7}^* + e_{1,7}^* + e_{2,8}^* + e_{3,5}^* + e_{4,6}^* + e_{7,3}^* + e_{8,4}^*.$$

3.4.1 The kernel of $\varphi(2)$

In the proof of the main theorem in Section 4, we require a specific description of $\ker(B_{\varphi(2)}) = \text{span}\{k\}$. To identify $k$, we first require the following technical lemmas.

Lemma 2. If $g = p_n^{A_1, \ldots, A_m}$ has homotopy type $\mathcal{H}(2)$, then $a_i$ and $b_j$ are even for all $1 \leq i \leq m$ and $1 \leq j \leq t$.

Proof. Since $g$ has homotopy type $\mathcal{H}(2)$, its meander $M(g)$ consists of exactly one cycle; moreover, each vertex has degree 2. In particular, each vertex is incident with exactly one top edge and exactly one bottom edge. However, if $a_i$ (resp. $b_j$) is odd for some $i$ (resp. $j$), then the middle vertex of block $a_i$ (resp. $b_j$), i.e., $v_{\sum_{k=1}^{i} a_k + \left\lceil \frac{a_i}{2} \right\rceil}$ (resp. $v_{\sum_{k=1}^{j} b_k + \left\lceil \frac{b_j}{2} \right\rceil}$) is not incident to a top (resp. bottom) edge. The result follows.

Lemma 3. Let $g$ have homotopy type $\mathcal{H}(2)$. If $e_{i,j}^*$ occurs as a nontrivial summand in $\varphi(2)$, then $i$ and $j$ have the same parity.

Proof. Let $g = p_n^{A_1, \ldots, A_m}$ have homotopy type $\mathcal{H}(2)$, and without loss of generality, assume there exists $s$ such that $a_s > 2$. Consider the collection of peak blocks determined by $a_s$; in particular, consider the summands of $\varphi(2)$ within each of these peak blocks. Such summands are $e_{i,j}^*$ and $e_{i+1,j+1}^*$, where

$$i = \sum_{r=1}^{s} a_r - 1 + 2t \quad \text{and} \quad j = \sum_{r=1}^{s-1} a_r + 1 + 2t$$

for $0 \leq t \leq \left\lfloor \frac{a_s}{4} \right\rfloor$. Notice that $i$ and $j$ are both odd for all $t$, by Lemma 2. Since $a_s$ was arbitrary and the same argument applies for all $b_s$, the result follows.
With Lemmas 2 and 3 established, the following result identifies the generator of \( \ker(B_{\varphi(2)}) \).

**Theorem 3.** Let \( g \subset sl(n) \) be a type-A seaweed with homotopy type \( \mathcal{H}(2) \). If \( \varphi(2) \) is defined as above, then

\[
\ker(B_{\varphi(2)}) = \text{span}\{k\},
\]

where

\[
k = \sum_{i=1}^{n} (-1)^{i+1} e_{i,i}.
\]

**Proof.** Since \( \varphi(2) \) is regular on \( g \), we need only show that \( k \in \ker(B_{\varphi(2)}) \). As a consequence of Lemma 2, we have that \( n \) is even, so \( \text{tr}(k) = 0 \) and \( k \in g \). Now, to see that \( k \in \ker(B_{\varphi(2)}) \), consider the following:

(a) \( \varphi(2)([k, e_{i,i} - e_{i+1,i+1}]) = 0 \), for all \( 1 \leq i \leq n - 1 \),

(b) \( \varphi(2)([k, e_{i,j}]) = 0 \), for all \( e_{i,j} \in g \) such that \( e_{i,j}^* \) is not a summand of \( \varphi(2) \), and

(c) \( \varphi(2)([k, e_{i,j}]) = \varphi(2)(e_{i,j} - e_{i,j}^*) = 0 \), for all \( e_{i,j} \in g \) such that \( e_{i,j}^* \) is a summand of \( \varphi(2) \).

Equations (a) and (b) follow immediately from \( k \) being a diagonal element of \( g \), and Equation (c) follows from Lemma 3. Therefore, \( k \in \ker(B_{\varphi(2)}) \).

**Example 9.** Returning to the seaweed \( g = p^A_5 \) from Example 8, we have that \( \ker(B_{\varphi(2)}) = \text{span}\{e_{1,1} - e_{2,2} + e_{3,3} - e_{4,4} + e_{5,5} - e_{6,6} + e_{7,7} - e_{8,8}\} \).

### 3.5 Homotopy type \( \mathcal{H}(1,1) \)

**Notation:** Throughout this section, for a given graph \( G \), we denote its vertex set by \( V(G) \) and its edge set by \( E(G) \).

When \( g \) has homotopy type \( \mathcal{H}(1,1) \), the core and peak blocks have dimension one, so \( CM(g) = M(g) \). Moreover, \( M(g) \) must consist of two paths, say \( P_1 \) and \( P_2 \). Let \( P_1 \) be the path which contains \( v_1 \). Now define the one-form \( \varphi(1,1) \in \Phi \) as follows:

\[
\varphi(1,1) = \sum_{(v_i,v_j) \in E(M(g))} e_{i,j}^* + \sum_{v_i \in V(P_1)} e_{i,i}^*.
\]

(3)

Note that all the terms in the first summation of (3) can be read directly from the edges of the directed meander in the obvious way. See Example 10.

**Example 10.** Consider the seaweed \( g = p^A_5 \) with directed meander shown in Figure 10 below. Note that \( g \) has homotopy type \( \mathcal{H}(1,1) \), so the above construction yields the regular one-form

\[
\varphi(1,1) = e_{1,3}^* + e_{4,3}^* + e_{5,2}^* + e_{1,1}^* + e_{3,3}^* + e_{4,4}^*.
\]

Figure 10: The directed meander \( \vec{M}(g) \)
3.5.1 The kernel of $\varphi_{(1,1)}$

By construction, the one-form $\varphi_{(1,1)}$ is regular. As in the $\mathcal{H}(2)$ case, we can also describe the kernel of its associated Kirillov form. See Theorem 4.

**Theorem 4.** If $\mathfrak{g}$ has homotopy type $\mathcal{H}(1,1)$ with $\varphi_{(1,1)}$ defined as in (3), then

$$\ker(B_{\varphi_{(1,1)}}) = \text{span}\{h\},$$

where

$$h = |V(P_2)| \sum_{i \in V(P_1)} e_{i,i} - |V(P_1)| \sum_{j \in V(P_2)} e_{j,j}.$$

**Proof.** Since $\varphi_{(1,1)}$ is regular on $\mathfrak{g}$, we need only show that $h \in \ker(B_{\varphi_{(1,1)}})$. We first establish that $h \in \mathfrak{g}$ by noting that $\text{tr}(h) = |V(P_2)||V(P_1)| - |V(P_1)||V(P_2)| = 0$. Now, to see that $h \in \ker(B_{\varphi_{(1,1)}})$, consider the following:

- $\varphi_{(1,1)}([h, e_{i,i} - e_{i+1,i+1}]) = 0$, for all $1 \leq i \leq |V(P_1)| + |V(P_2)| - 1$,
- $\varphi_{(1,1)}([h, e_{i,j}]) = 0$, for all $e_{i,j} \in \mathfrak{g}$ such that $e_{i,j}$ is not a summand of $\varphi_{(1,1)}$,
- $\varphi_{(1,1)}([h, e_{i,j}]) = \varphi_{(1,1)}([V(P_2)](e_{i,j} - e_{i,j})) = 0$, for all $e_{i,j} \in \mathfrak{g}$ such that $(v_i, v_j) \in E(P_1)$, and
- $\varphi_{(1,1)}([h, e_{i,j}]) = \varphi_{(1,1)}([V(P_1)](e_{i,j} - e_{i,j})) = 0$, for all $e_{i,j} \in \mathfrak{g}$ such that $(v_i, v_j) \in E(P_2)$.

The result follows. \(\square\)

**Example 11.** Returning to the seaweed $\mathfrak{g} = p_A^{1,1,1}$ from Example 10, we have that

$$\ker(B_{\varphi_{(1,1)}}) = \text{span}\{2e_{1,1} - 3e_{2,2} + 2e_{3,3} + 2e_{4,4} - 3e_{5,5}\}.$$

4 Main results

In this section, we establish the main result of this article. We first need some notation and a few general results about contact Lie algebras.

**Notation:** In this section, we will use $\mathfrak{f}$ to denote an arbitrary Lie algebra and we will continue with our established convention of letting $\mathfrak{g}$ represent a type-A seaweed.

Let $\mathfrak{f}$ be a $(2k + 1)$-dimensional contact Lie algebra with contact form $\varphi$. Fix an ordered basis $\mathcal{B} = \{E_1, \ldots, E_{2k+1}\}$ of $\mathfrak{f}$ and define $C(\mathfrak{f}, \mathcal{B}) = ([E_i, E_j])_{1 \leq i, j \leq 2k+1}$ to be the *commutator matrix* associated to $\mathfrak{f}$ and indexed by basis $\mathcal{B}$. The contact form $\varphi$ can be applied to each element of $C(\mathfrak{f}, \mathcal{B})$ to yield the matrix

$$[B_\varphi] = \varphi(C(\mathfrak{f}, \mathcal{B})).$$

Denote by $[\varphi] = (x_1 \ldots x_{2k+1})^t$ the coordinate vector in $\mathbb{C}^{2k+1}$ such that $\varphi = \sum_{i=1}^{2k+1} x_i E_i^*$ where $\{E_1^*, \ldots, E_{2k+1}^*\}$ is the “dual basis” associated to $\mathcal{B}$ and consider the square $(2k + 2)$-dimensional skew-symmetric matrix

$$\begin{bmatrix} 0 & [\varphi]^t \\ -[\varphi] & \varphi(C(\mathfrak{f})) \end{bmatrix}.$$

A straightforward computation yields the following technical lemma.

**Lemma 4** (Salgado [14], 2019). Let $\mathfrak{f}$ be a Lie algebra with dim $\mathfrak{g} = 2k + 1$ and $\varphi \in \mathfrak{f}^*$. Using the notation developed above,

$$\varphi \wedge (d\varphi)^k = \det \left( \left[ B_\varphi \right] \right) E_1^* \wedge \cdots \wedge E_{2k+1}^*.$$

Therefore, $\varphi$ is a contact form on $\mathfrak{f}$ if and only if

$$\det \left( \left[ B_\varphi \right] \right) \neq 0.$$
From Lemma 4 above, we obtain the following useful characterization of a contact Lie algebra.

**Lemma 5.** An index-one Lie algebra $\mathfrak{f}$ is contact if and only if there exists a regular one-form $\varphi \in \mathfrak{f}^*$ for which there is an element $x \in \mathfrak{f}$ with $\ker(B_{\varphi}) = \text{span}\{x\}$ and $\varphi(x) \neq 0$.

**Proof.** To establish the forward implication, we need only recall that every contact form $\varphi$ has a unique Reeb vector, $x_\varphi \in \mathfrak{f}$ defined by the equations

$$B_{\varphi}(x_\varphi, -) = 0$$

and

$$\varphi(x_\varphi) = 1.$$  

For the reverse implication, let $\varphi \in \mathfrak{f}^*$ be a regular one-form, i.e., $\dim \ker(B_{\varphi}) = \text{ind } \mathfrak{f} = 1$. Let $x \in \mathfrak{f}$ denote a generator of $\ker(B_{\varphi})$, then extend $x$ to a basis of $\mathfrak{f}$, and consider $\det [\hat{B}_{\varphi}]$ with respect to this basis. Computing, we have that

$$\det [\hat{B}_{\varphi}] = \varphi(x)^2 \det [B'_{\varphi}],$$

where $B'_{\varphi}$ is the submatrix of $[B_{\varphi}]$ with the row and column indexed by $\varphi(x)$ removed. Since $\varphi$ is a regular one-form, $B'_{\varphi}$ has full rank, so $\det [B'_{\varphi}] \neq 0$. The result follows from Lemma 4. \qed

We are now in a position to prove the main theorem of this article.

**Theorem 5.** If $\mathfrak{g}$ is a type-A seaweed, then $\mathfrak{g}$ is contact if and only if $\text{ind } \mathfrak{g} = 1$.

**Proof.** Let $\mathfrak{g}$ be an index-one seaweed subalgebra of $\mathfrak{sl}(n)$. Recall from Theorem 2 that this means $\mathfrak{g}$ has one of the following homotopy types: $\mathcal{H}(2)$ or $\mathcal{H}(1, 1)$. We proceed by treating each homotopy type as its own case, using the constructions and notation of Section 3 and then applying Lemma 5.

**Case 1:** $\mathfrak{g}$ has homotopy type $\mathcal{H}(2)$. Using the notation of Section 3.4, we claim that $\varphi(2)$ is a contact form on $\mathfrak{g}$. Recall from Theorem 3 that

$$\ker(B_{\varphi(2)}) = \text{span}\{k\} = \text{span}\left\{\sum_{i=1}^{n}(-1)^{i+1}e_{i,i}\right\},$$

and notice that $\varphi(2)(k) = \frac{n}{2} \neq 0$. An application of Lemma 5 establishes the claim.

**Case 2:** $\mathfrak{g}$ has homotopy type $\mathcal{H}(1, 1)$. Using the notation of Section 3.5, we claim that $\varphi(1, 1)$ is a contact form on $\mathfrak{g}$. Recall from Theorem 4 that

$$\ker(B_{\varphi(1, 1)}) = \text{span}\{h\} = \text{span}\left\{\sum_{i \in V(P_2)} |V(P_2)| e_{i,i} - |V(P_1)| \sum_{j \in V(P_2)} e_{j,j}\right\},$$

and notice that $\varphi(1, 1)(h) = |V(P_2)||V(P_1)| \neq 0$. An application of Lemma 5 establishes the claim. \qed

The following corollary identifies the only $n$ for which $\mathfrak{sl}(n, \mathbb{C})$ is contact, providing another proof for a well-known result (cf. [11]).

**Corollary 1.** The Lie algebra $\mathfrak{sl}(n, \mathbb{C})$ is contact if and only if $n = 2$.

**Proof.** Note that $\mathfrak{sl}(n, \mathbb{C})$ is a type-A seaweed; in particular, it is the seaweed $\mathfrak{p}^A_{n, \pi}$. Further, its meander consists of $\left\lfloor \frac{n}{2} \right\rfloor$ cycles and, if $n$ is odd, a degenerate path. By Theorem 1, we have that $\text{ind } \mathfrak{sl}(n, \mathbb{C}) = 1$ if and only if $n = 2$, and the result follows from an application of Theorem 5. \qed
5 Examples

Theorem 1 provides an elegant combinatorial formula for the index – and is critical to the proof heuristics of our main Theorem 5 – but to use it to quickly identify, or construct, contact seaweeds is nettlesome, having to first construct the associated meander and then count the number and type of connected components. Fortunately, for seaweeds consisting of a small number of parts, one can determine the index with some dispatch. In particular, the following theorem provides an explicit index formula presented in terms of a linear common divisor of two arguments, each of which is a linear combination of the terms in the seaweed’s defining composition (see [5]). We can use these formulas to manufacture an unlimited supply of contact seaweeds, of arbitrarily large dimension.

**Theorem 6** (Coll et al. [3], 2015). The seaweed $p_n^{a|b|c}$ has index $\gcd(a + b, b + c) - 1$.

So, to generate contact seaweeds of the form $p_n^{a|b|c}$, we need only ensure $\gcd(a + b, b + c) = 2$. This is easy to do; consider, for example, $p_5^{1|1|3}$, $p_7^{1|3|3}$, $p_{12}^{4|2|0}$, etc. Moreover, if $b = 0$, an immediate corollary to Theorem 6 gives an index formula in the maximal parabolic case.

**Theorem 7** (Elashvili [9], 1990). The seaweed $p_n^{a|c}$ has index $\gcd(a, c) - 1$.

Using Theorem 7, the reader will have no difficulty constructing examples of maximally parabolic type-A contact seaweeds. Of course, Theorems 6 and 7 can also be used to generate Frobenius seaweeds since, by Theorem 1, their associated meanders must consist of a single path. Here are a couple of examples: $p_5^{4|2|3}$ and $p_8^{4|2|5}$. See Figure 11, where the meanders of these two Frobenius seaweeds are displayed.

![Figure 11: Meanders](image)

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