Constructing Functional Braids for Low-Leakage Topological Quantum Computing

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We discuss how to significantly reduce leakage errors in topological quantum computation by introducing an irrelevant error in phase, using the construction of a CNOT gate in the Fibonacci anyon model as a concrete example. To be specific, we construct a functional braid in a six-anyon Hilbert space that exchanges two neighboring anyons while conserving the encoded quantum information. The leakage error is $\sim 10^{-10}$ for a braid of $\sim 100$ interchanges of anyons. Applying the braid greatly reduces the leakage error in the construction of generic controlled-rotation gates.

I. INTRODUCTION

Topological quantum computing is a novel concept aiming at preventing quantum decoherence in quantum computation on a hardware level [1, 2, 3, 4]. The prototype of a topological quantum computer is a two-dimensional system of topological order with a collection of non-Abelian anyons, which can be created, braided, fused, and measured. Quantum information is stored globally and thus affected only by global operations like braidings of anyons, but not by local perturbations like noises. The leading candidates for topological states of matter are fractional quantum Hall liquids, in which some of the fractionally charged quasiparticles are believed to obey non-Abelian statistics [3]. As a consequence, the architecture and algorithms for topological quantum computers are significantly different from conventional quantum computers such as ion-trap quantum computers. Topological quantum computers are based on an anyon model, shown to be equivalent to a quantum circuit model [2]. Due to its intrinsic connection to the unitary representation of the braid groups, a perfect problem for it is naturally the evaluation of the Jones polynomial [5, 6], which has been shown to be intimately connected to topological quantum field theories (TQFTs) [5]. TQFTs are low-energy effective theories for topological phases of matter, in which non-Abelian anyons might exist. The worldlines of anyons braiding in (2+1)-dimensional space-time (with proper closure) can be regarded as knots or links. To achieve universal topological quantum computing, one may wish to construct a universal set of quantum gates, which are certain braids in (2+1)-dimensional space-time. However, to solve an apparently easier problem such as the realization of a CNOT gate is highly nontrivial for a concrete anyon model [5, 6].

While the discussions here may apply generically to other models that support universal topological quantum computation, we continue by focusing specifically on the simple Fibonacci model (equivalent to the SU(2)$_3$ Chern-Simons-Witten theory for our purpose). Using the simplest model, Bonesteel et al. [8] explicitly showed a CNOT gate can be approximated (to a distance $\sim 10^{-3}$) by the composite braid of an injection weave, an iX weave, and the inverse injection weave (up to a single-qubit rotation). The divide-and-conquer approach avoids the difficulties of the direct search in the space of braids with at least 6 anyons. Later, Hormozi et al. [9] also proposed that a CNOT gate can be alternatively approximated (also to a distance $\sim 10^{-3}$) by the composite braid of an F weave, a $\pi$-phase weave, and the inverse F weave (up to two single-qubit rotations). These three-anyon weaves (a subset of braids in which only one of the three anyons moves) are constructed by brute-force search [8] and the accuracy of the CNOT gate can be systematically improved by the Solovay-Kitaev algorithm [8, 9] at the expense of increasing the braid length significantly. The injection weave and the F weave, which approximate the identity matrix and the $F$ matrix, respectively, are examples of functional braids that might play important roles in future topological quantum computer architecture.

In this letter, we discuss a promising low-leakage construction of generic controlled-rotation gates in the Fibonacci anyon model, which is based on an arbitrary single-qubit phase gate. Regardless of the phase, we can use the underlying braid to construct an exchange braid that swaps two non-Abelian anyons in a six-anyon system without changing the quantum information stored in the system. The freedom in the phase guarantees us to achieve leakage errors smaller than that of a generic single qubit of the same braid length, e.g., a leakage error $\sim 10^{-10}$ (in terms of distance) with no more than 100 interchanges of neighboring anyons. This allows us to construct two-qubit gates with extremely low leakage errors. Other applications of the braid and the generalization of

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this approach will be discussed.

II. FIBONACCI ANYONS

The Fibonacci anyons emerge from the Yang-Lee model in conformal field theories and are also speculated to exist in the $\nu = 12/5$ fractional quantum Hall liquid \cite{11,12}, as well as in the non-Abelian spin-singlet state \cite{13}. The Fibonacci anyon model contains two types of anyons with topological charges 0 (vacuum) and 1 (Fibonacci anyon) satisfying a nontrivial fusion rule $1 \times 1 = 0 + 1$. We denote $\ket{ab}$ for the state in which anyons $a$ and $b$ fuse into $c$. As in a generic anyon model, the braiding rules for two Fibonacci anyons can be described by the $R$-matrix and the associativity of fusion (satisfying a pentagon relation) can be described by the $F$-matrix \cite{3}.

We use two pairs of Fibonacci anyons with total charge 0 to encode one qubit of information. As illustrated in Fig. 1 (a), the two pairs must have the same total charge 0 or 1, spanning a two-dimensional Hilbert space, or a qubit. We can write down the basis states as

$$\ket{0} = \ket{(11)_0(11)_0} \text{ and } \ket{1} = \ket{(11)_1(11)_1}.$$  

We note that, alternatively, we can use a set of three anyons with total charge 1, such that $\ket{0} = \ket{(11)_01}$ and $\ket{1} = \ket{(11)_11}$. The two encoding schemes are equivalent since $\ket{(11)_a(11)_a} = \ket{(11)_a11}$, which is just $\ket{(11)_a1}$ in the three-anyon encoding scheme, as explained, e.g., in Ref. \cite{3}. The braiding of neighboring anyons in the qubit can be represented by $2 \times 2$ matrices, as shown in Fig. 1 (b), obtained from $F$- and $R$-matrices. There are only two independent matrices, since $\sigma_1 = \sigma_3$, so, in practice, we can leave, e.g., the bottom anyon intact. Therefore, we can use 4 elementary matrices $\sigma_2^a, \sigma_3^a$ (equivalent to the braiding matrices in the three-anyon encoding scheme) to construct any single-qubit gate, since the Fibonacci anyon model supports universal topological quantum computation.

III. SINGLE-QUBIT GATES

A generic single-qubit gate can be represented by a $2 \times 2$ unitary matrix, which can be written in the form

$$e^{i\alpha} \begin{bmatrix} \sqrt{1 - b^2} e^{-i\beta} & be^{i\gamma} \\ -be^{-i\gamma} & \sqrt{1 - b^2} e^{i\beta} \end{bmatrix},$$

(1)

where $b, \alpha, \beta$ and $\gamma$ are all real parameters. Apart from the overall phase factor, one needs 3 parameters $b, \beta$ and $\gamma$ to specify the matrix. To search for any desired gate, we implement the brute-force algorithm \cite{8} with some modifications. In particular, we store the intermediate matrix products, as we generate longer and longer braids, ordered by the parameters $b, \beta$, and $\gamma$ in Eq. (1). When we reach our searching limit (on a desktop PC), we look up in the saved list an optimal braid that can be combined to approach the target gate. In computer algorithms, this is equivalent to a bidirectional search \cite{4}. We find, in general, single-qubit gates [such as the $iX$ gate in Fig. 3(b)] can be approximated by braids of about 100 interchanges of neighboring anyons to a distance $\sim 10^{-6}$, sufficiently small for the application of quantum error-correcting code.

One may notice that when $b = 0$, $\gamma$ drops out automatically. The resulting diagonal matrix is a phase gate in the quantum computing language. Compared with the identity matrix targeted by the injection weave \cite{8}, an additional error in phase is introduced. However, if the phase error is irrelevant (as we will see), the leakage error due to non-zero off-diagonal matrix elements can be greatly reduced. This is because, with only one parameter $b$ to target, some diagonal matrices can be approximated with much higher accuracy within a certain braid length. As an example, we show in Fig. 2(a) a sequence of 99 elementary braids (or a braid of length 99) that approaches a phase gate with off-diagonal matrix elements of order $10^{-10}$ (i.e. more than 3 orders of magnitude improvement over the generic case). Note the braid exchanges the positions of the two center anyons, while rotating the qubit only by a phase. In the following, we show this plain-looking braid can facilitate topological quantum computing with remarkably low leakage error.

IV. FROM ONE TO TWO QUBITS

Let us create an auxiliary pair of Fibonacci anyons with total charge 0 next to the two pairs that encode a qubit. The whole Hilbert space of the 6 Fibonacci anyons with total charge 0 is 5-dimensional. The additional 3 basis states have a total charge 1 for the 4 encoding anyons, thus are noncomputational basis states. In the enlarged
space, we note the computational basis states for the 6 anyons can be rewritten as \( |((11)_a(11)_b)(11)_b⟩ = |((11)_a1)(1(11)_b)⟩ \). Since a set of 3 nontrivial anyons with total charge 1 is equivalent to a set of 4 with total charge 0 in terms of computing basis and corresponding braiding matrices, the six-anyon system can thus be regarded as two qubits (with total charge 0), except that the second (right) qubit is in the definite state of \(|((11)⟩_a1)⟩ \). Suppose we now implement the braid in Fig. 2(a) to the second qubit, as shown in Fig. 2(b). We obtain just a phase factor since the braid is represented by a diagonal matrix in the single-qubit representation. As a result, the positions of the fourth and fifth anyons are exchanged, but the quantum information stored in the qubit remains intact.

In the full 5-dimensional space, the braiding of the second set of anyons leads to a \( 2 \times 2 \) (computational) and a \( 3 \times 3 \) (noncomputational) matrices. They are decoupled (block-diagonal) to the order of \( 10^{-10} \), the same order as the off-diagonal matrix elements of the single-qubit phase gate in Fig. 2(a). The \( 2 \times 2 \) submatrix is exactly proportional to the identity matrix. The phase we introduced as an error drops out and is indeed irrelevant. The braid in the 6-anyon system shown in Fig. 2(b) is thus dubbed the exchange braid, since its sole function is to exchange one anyon in the encoded qubit with another in the auxiliary pair without distorting the quantum information encoded in the qubit (though with a leakage error \( \sim 10^{-10} \)). The exchange braid has many applications. For example, it can facilitate topological quantum computing with only a few mobile anyons. Compared to the work published earlier \[13\] based on the injection weave (with a distance \( \sim 10^{-3} \) and a length of 48) \[8\], with the exchange braid approach, we can realize the function of injection (with a phase to be canceled by its inverse) with much lower leakage errors. Since we introduce pairs of anyons with total topological charge 0, it is also possible to insert the pairs at any location.

FIG. 2: (Color online) (a) A braid with 99 anyon interchanges that approximates a phase gate, where \( |δ_1| = |δ_2| = \sqrt{1 - r^2} \approx 6.2 \times 10^{-16} \). (b) The exchange braid in the six-anyon braid space, in which \( σ_4 \) and \( σ_5 \) are the corresponding 5 \( × \) 5 braiding matrices. Note that the sequence of interchanges for the top 3 anyons is exactly the same as in (a). The phase factor \( e^{iθ} \) inherited from (a) is not important and can be canceled out by the inverse exchange braid. The upper 2 \( × \) 2 block is exactly proportional to the identity matrix, and 1 \( - r \approx 2 \times 10^{-19} \). The lower 3 \( × \) 3 block \( A \) (in the noncomputational space) is irrelevant, if the quantum state remains in the computational space. The leakage to the noncomputational space spanned by \( |((11)_a(11)_b)(11)_b⟩, |((11)_a(11)_b)(11)_b⟩ \) and \( |((11)_a(11)_b)(11)_b⟩ \) is determined by \( |ε_1| \leq 6.2 \times 10^{-16} \) for \( i = 1-6 \).

V. CONTROLLED GATES

We can imagine the two Fibonacci anyons of the auxiliary pair are themselves two pairs of Fibonacci anyons, with total charge 1 each. The four anyons thus form a second qubit in the state \(|((11)_a(11)_b⟩ \). Let us treat the two subpairs as single anyons and perform the exchange braid to the two-qubit system. The result is one composite anyon exchanges its position with the neighboring anyon in the encoded qubit. Suppose we then perform two consecutive \( σ_3 \) braidings, followed by the inverse of the exchange braid in the end. The composite braid performs a \( σ_2^2 \) to the encoded qubit. On the other hand, we can also encode the second qubit in the state \( |((11)_a(11)_b)⟩ \). Since the two composite anyons in the auxiliary pair each have trivial total charge, the composite braid thus makes no change to the encoded qubit. Combining the two scenarios, we conclude the composite braid (of length 200) for the 8-anyon system, as shown in Fig. 2(a), approximates the controlled-\( σ_3^2 \) operation to a distance \( \sim 10^{-9} \).

In the same fashion, we can construct any controlled-rotation gate by replacing the \( σ_2^2 \) braid by one that approximates the appropriate single-qubit gate. For example, we find a braid of length 108 that approximates the iX gate to a distance \( \sim 1.5 \times 10^{-6} \) as shown in Fig. 2(b). Sandwiching the iX braid between the exchange braid and its inverse, we obtain a braid (of length 306) that approximates a CNOT gate (up to a single-qubit rotation) to a distance \( \sim 1.5 \times 10^{-6} \) with a leakage error \( \sim 10^{-9} \). One can also achieve a more accurate CNOT gate by only improving the iX gate (to distance \( \sim 10^{-9} \)) using, e.g., the Solavay-Kitaev algorithm \[10\].

VI. DISCUSSION

To summarize, we introduce a scheme to construct a functional braid that can be interpreted either as a phase
FIG. 3: (Color online) (a) A braid that approximates a controlled-$\sigma^z$ gate to a distance $\approx 1.2 \times 10^{-9}$. The control (top) qubit can be treated as two composite anyons and each composite anyon can move as a single anyon. In the six-anyon braid space, we apply the exchange braid [see Fig. 2(b)], $\sigma^x$ (two counterclockwise interchanges of the third and the fourth anyons from the bottom), and the inverse exchange braid. If the control qubit is in the state $|0\rangle$, the encoded qubit remains unchanged; otherwise, a $\sigma^z$ rotation is performed. (b) A braid with 108 interchanges that approximates an iX gate to a distance $\approx 1.5 \times 10^{-6}$. Replacing the $\sigma^x$ sandwiched in (a) by the iX braid, we obtain a braid that approximates a CNOT gate (up to a single-qubit rotation) to a distance $\approx 10^{-6}$.

gate (with off-diagonal matrix elements $\sim 10^{-10}$) or as an exchange braid, which swaps two neighboring anyons without affecting the encoded quantum information (except for a leakage error $\sim 10^{-10}$). The intriguing equivalence between the single- and two-qubit constructions holds the key to the remarkable reduction in leakage errors. We illustrate this in Fig. 4 where we plot the distance we can approach in searching for the identity matrix, as well as the distance for phase gates (with arbitrary phases). We point out the curve for the identity matrix (dashed line) extends the earlier approach (up to $L = 48$ as in Fig. 7 of Ref. 9) with the algorithm improvement, which reduces the leakage errors by 3 orders of magnitude by doubling the braid length (achieved with similar computing power). Remarkably, the phase gate based approach further reduces the leakage errors by another 3 orders of magnitude without extra cost on the braid length. Gain in the approach happens at all braid lengths and is based mathematically on the increase of the target space from 0 to 1 dimension, which leads to a faster exponential decay in distance as illustrated in Fig. 4 (quantitatively, a factor of $\sim 1.7$ reduction in decay length).

This study reveals an interesting aspect of the topological quantum computation: errors (in leakage) can be reduced by introducing an additional error (in phase). Based on the guideline, one can engineer more functional braids for Fibonacci anyons $^{17}$, e.g., one that approximates a generalized NOT gate (an off-diagonal matrix) or a generalized Hadamard gate to high accuracy. These functional braids can readily find applications in low-leakage topological quantum algorithms. While we have used the simple Fibonacci anyon model as an example to illustrate our ideas to achieve low-leakage topological quantum computing, such functional braids can be constructed in similar fashions in a generic anyon model that supports universal topological quantum computing.

![Graph showing distance versus braid length](image)

FIG. 4: (Color online) Distance versus braid length in approaching the identity matrix (dots) or any phase gate, i.e., unitary diagonal matrix (squares). The decays can be fit by $1.6e^{-L/7.3}$ (dashed line) and $0.76e^{-L/4.3}$ (solid line), respectively. The significant reduction in distance due to the freedom in phase is key to the low-leakage realization of topological quantum computation based on the exchange braid.

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