W(E_{10}) Symmetry, M-Theory and Painlevé Equations

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Abstract

The Weyl group symmetry $W(E_k)$ is studied from the points of view of the $E$-strings, Painlevé equations and U-duality. We give a simple reformulation of the elliptic Painlevé equation in such a way that the hidden symmetry $W(E_{10})$ is manifestly realized. This reformulation is based on the birational geometry of the del Pezzo surface and closely related to Seiberg-Witten curves describing the $E$-strings. The relation of the $W(E_k)$ symmetry to the duality of M-theory on a torus is discussed on the level of string equations of motion.
1 Introduction

In a recent paper [1], the second order (difference) Painlevé equations have been classified by using the geometry of algebraic surfaces. The classification falls into three types: rational, trigonometric and elliptic. Each case is associated with a special divisor corresponding to one of the Kodaira singular fibers of elliptic fibration (Table 1) [2].

\begin{align*}
\text{ell.} & & I_0 \\
\text{tri.} & & I_1 \rightarrow I_2 \rightarrow I_3 \rightarrow I_4 \rightarrow I_5 \rightarrow I_6 \rightarrow I_7 \rightarrow I_8 \rightarrow I_9 \\
& & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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is that in the $E$-string case the 9 points are chosen in special position so that the surface admits an elliptic fibration.

Furthermore, the $E_n$-series in the above diagram is also well-known in connection with the duality symmetry of $M$-theory compactified on a torus [8]. In this correspondence, the Weyl group part of the $U$-duality group is identified with the Cremona isometry $W(E_k)$ for del Pezzo $B_k$ [9]. (This duality was discussed from the point of view of Little String Theory supersymmetric indices [10].) The aim of this paper is to examine the correspondence by closely looking the way how the Weyl group is realized in each case.

This paper is organized as follows. In section 2, we clarify the special role of the fiber at $u = \infty$ in the Seiberg-Witten geometry [11]. We also give an example of duality map between two Seiberg-Witten curves corresponding to different space-time dimensions. In section 3, we give a reformulation of the elliptic Painlevé equation where the hidden $W(E_{10})$ symmetry is manifestly realized and the relation to the Seiberg-Witten geometry of $E$-strings is discussed. In section 4, we study the Painlevé equations arising from a consistent truncation/reduction of the $M$-theory and compare the Painlevé Bäcklund transformations with the $U$-duality. Finally, section 5 is devoted to the conclusions and discussions.

2 The role of the fiber at $u = \infty$

The equations for the $SU(2)$ $E_8$ flavor Seiberg-Witten curves with two mass parameters have been given by Minahan et. al.[12]

\begin{align*}
\text{rat.} & \quad y^2 = x^3 - 2u(u^2 + m_2^2 x)(u^2 + m_2^2 x), \\
\text{tri.} & \quad y^2 = x^3 + u^2 x^2 - 2u(u^2 + \sin^2 m_1 x)(u^2 + \sin^2 m_2 x), \\
\text{ell.} & \quad y^2 = x^3 + (1 + k^2)u^2 x^2 - 2u(u^2 + \sin^2 m_1 x)(u^2 + \sin^2 m_2 x) + k^2 u^4 x.
\end{align*}

(3)

The discriminants and singular fibers are

\begin{align*}
\text{rat.} & \quad \Delta = u^8(u^2 + \cdots), \quad I_2^* + 2I_1 + (\Pi)_{u=\infty}, \\
\text{tri.} & \quad \Delta = u^8(u^3 + \cdots), \quad I_2^* + 3I_1 + (I_1)_{u=\infty}, \\
\text{ell.} & \quad \Delta = u^8(u^4 + \cdots), \quad I_2^* + 4I_1.
\end{align*}

(4)

The $I_2^*(=D_6)$ singularity at $u = 0$ corresponds to the two mass deformation of $\Pi^*(=E_8)$. Note that the difference among the three cases (rat/tri/ell) appears on the fiber at $u = \infty$. That is, the fiber is a cusp/nodal/smooth curve, respectively.

To see the meaning of the fiber at $u = \infty$, let us consider the sections. For all three cases, the Mordell-Weil lattice are $A_1^* \oplus A_1^*$. In fact, we have the following generators...
of the sections,[12]

rat. $x = -\frac{1}{v^2} u^2 \quad y = i \frac{1}{v^3} u^3,$

tri. $x = -\frac{1}{\sin^2 v} u^2 \quad y = i \frac{\cos v}{\sin^2 v} u^3,$

ell. $x = -\frac{1}{\sin^2 v} u^2 \quad y = i \frac{\text{cn} \text{dn} \text{sn} v}{\text{sn}^3 v} u^3,$

where $v = m_1$ or $m_2$. Other sections can be obtained by addition and have the form at $u = \infty$ as

$$x = a_2 u^2 + a_1 u + \cdots, \quad y = b_3 u^2 + b_2 u^2 + \cdots,$$

where the leading term is of the form (5) with $v = k_1 m_1 + k_2 m_2$ ($k_1, k_2 \in \mathbb{Z}$). Let us consider the elliptic case. The fiber at $u = \infty$ is a smooth curve

$$y^2 = x^3 + (1 + k^2) x^2 + k^2 x,$$

which can be parametrized as

$$x = x(v) = -\frac{1}{\text{sn}^2 v}, \quad y = y(v) = i \frac{\text{cn} \text{dn} \text{sn} v}{\text{sn}^3 v}.$$

Note that this is nothing but the leading term of the section. Hence the parameter $v$ represent the point where the section and the fiber at $u = \infty$ intersect. Similarly, for the case of rational or trigonometric, the intersection point is parametrized by the trigonometric or rational functions of parameter $v$. In [7] the coincidence between certain parameter $v$ in sections and mass parameters was observed. The above argument explains the mechanism of this identification.

Finally, we consider the relation between the curves in (3) and $SU(2) N_f = 2$ Seiberg-Witten curve [14]. The $N_f = 2$ Seiberg-Witten curve

$$y^2 = (x^2 - \frac{\Lambda^4}{64})(x-u) + \frac{\Lambda^2}{4} M_1 M_2 x - \frac{\Lambda^4}{64} (M_1^2 + M_2^2),$$

and the elliptic case in (3) are both the generic curves with the $D_6$ singularity. Hence, they should be related with each other. In fact, up to simple change of variables $x, u$, these curves are equivalent. The relations of parameters are

$$\Lambda^2 = 4\left(\frac{1}{\text{sn} m_1} - \frac{1}{\text{sn} m_2}\right),$$

$$\Lambda^2 (M_1 + M_2) = 8 \frac{\text{cn} \text{dn} m_1}{\text{sn}^3 m_1}, \quad \Lambda^2 (M_1 - M_2) = 8 \frac{\text{cn} m_2 \text{dn} m_2}{\text{sn}^3 m_2}.$$ 

This mapping $(\Lambda, M_1, M_2) \leftrightarrow (k, m_1, m_2)$ can be interpreted as a kind of duality which connects different theories (in different dimensions).
3 The elliptic Painlevé equation

On a del Pezzo surface $B_k$ the Weyl group $W(E_k)$ acts as the Cremona isometry [15]. For the case of $k = 9$, the Weyl group $W(E_9)$ is the affine Weyl group of type $W(E_8^{(1)})$ which contains the translation subgroup $\mathbb{Z}^n$ and this is the origin of the elliptic Painlevé equation [1]. This construction can be considered as an example of general strategy to construct discrete Painlevé equations by using affine Weyl groups [16]. We will reformulate the elliptic Painlevé equation in the form where the hidden $W(E_{10})$ symmetry is manifestly realized. Let $M$ be the space of $3 \times 10$ matrix

$$M = \left\{ X = \begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_{10} \\ y_1 & y_2 & y_3 & \cdots & y_{10} \\ z_1 & z_2 & z_3 & \cdots & z_{10} \end{bmatrix} \right\}.$$  (11)

Each column vector $P_i = (x_i : y_i : z_i)$ may be thought of as a projective coordinate of a point $P_i \in \mathbb{P}^2$. In view of this, we make an identification

$$\mathcal{M} = \text{PGL}(3)/\mathbb{C}^{10}.  \quad (12)$$

A representative of this coset can be taken as

$$X = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & u_5 & \cdots & u_{10} \\ 1 & 1 & v_5 & \cdots & v_{10} \end{bmatrix},  \quad (13)$$

where

$$u_i = \frac{\mu_{234}\mu_{13i}}{\mu_{134}\mu_{23i}}, \quad v_i = \frac{\mu_{234}\mu_{12i}}{\mu_{124}\mu_{23i}} \quad (i = 5, \ldots, 10),  \quad (14)$$

and $\mu_{ijk}$ is the minor determinant of $X$ taking $i$, $j$ and $k$th columns. We have an action of the symmetric group $S_{10}$ which act as a permutation of the columns of $X$. In terms of the coordinates $(u_i, v_i)$, $i = 5, \ldots, 10$ the $S_{10}$-action can be written as follows [15]:

The actions of $s_1, s_2, s_3$ are given by

$$s_1(u_i) = \frac{1}{u_i}, \quad s_1(v_i) = \frac{v_i}{u_i},$$

$$s_2(u_i) = v_i, \quad s_2(v_i) = u_i,$$

$$s_3(u_i) = \frac{u_i - v_i}{1 - v_i}, \quad s_3(v_i) = \frac{v_i}{v_i - 1},  \quad (15)$$

The action of $s_4$ is

$$s_4(u_5) = \frac{1}{u_5}, \quad s_4(v_5) = \frac{1}{v_5}, \quad s_4(u_i) = \frac{u_i}{u_5}, \quad s_4(v_i) = \frac{v_i}{v_5}, \quad (i = 6, \ldots, 10)  \quad (16)$$
And \( s_i \) for \( i = 5, \ldots, 9 \) act as

\[

g_i(u_i) = u_{i+1}, \quad g_i(u_{i+1}) = u_i, \quad g_i(u_j) = u_j, \quad (j \neq i, i + 1) \\
g_i(v_i) = v_{i+1}, \quad g_i(v_{i+1}) = v_i, \quad g_i(v_j) = v_j, \quad (j \neq i, i + 1)
\]

Besides the permutations \( s_i \in S_{10} \) \((i = 1, \ldots, 9)\), there exist another important involution \( s_0 \) on the variables \((u_i, v_i)\), namely

\[

s_0(u_i) = \frac{1}{u_i}, \quad s_0(v_i) = \frac{1}{v_i}. \quad (18)
\]

Geometrically, this is a standard Cremona transformation with center \((P_1, P_2, P_3)\). By direct computation, we have

\[

(s_0 s_i)^2 = 1, \quad (i \neq 3) \quad \text{and} \quad (s_0 s_3)^3 = 1. \quad (19)
\]

In summary, the transformations \( s_i, i = 0, 1, \ldots, 9 \) defined by \((15), (16), (17)\) and \((18)\) give a birational representation of the Weyl group \( W(E_{10}) \) on the field of rational functions \( C(u_5, \ldots, u_{10}, v_5, \ldots, v_{10}) \).

\[E_{10}\]

The construction of the elliptic Painlevé equation is very simple. The Weyl group \( W(E_{10}) \) contains \( W(E_8^{(1)}) \) generated by \( s_i \) \((i = 0, \ldots, 8)\). This group \( W(E_8^{(1)}) \) has a translation subgroup \( Z^8 \). The birational action of these translations on \( \mathcal{M} \) is nothing but the Sakai’s elliptic Painlevé equation. The explicit action of these translations on the variables \((u_i, v_i)\) are too complicated and seems to be beyond our computational ability. We give an intermediate formula for one of the translations

\[

T = (pqp)^2, \quad p = s_3 s_4 s_5 s_2 s_3 s_4 s_1 s_2 s_3 s_0, \quad q = s_6 s_7 s_8 s_5 s_6 s_7 s_4 s_5 s_0. \quad (20)
\]

The result is given as follows:

\[

p(u_5, u_6, u_i) = \frac{\mu_{146}}{\mu_{156}} \left( \frac{\mu_{256}}{\mu_{246}}, \frac{\mu_{356}}{\mu_{346}}, \frac{\mu_{56i}}{\mu_{46i}} \right), \quad (i = 7, \ldots, 10)
\]

\[

p(v_5, v_6, v_i) = \frac{\mu_{145}}{\mu_{156}} \left( \frac{\mu_{256}}{\mu_{245}}, \frac{\mu_{356}}{\mu_{345}}, \frac{\mu_{56i}}{\mu_{45i}} \right), \quad (i = 7, \ldots, 10)
\]

\[2\] There exist 240 commuting translations corresponding to \( E_8 \) roots. Any of them can be represented as a composition of 58 simple reflections. Among the 240 translations, only 8 of them are multiplicatively independent.
\[ q(u_5, u_6, u_7, u_8, u_9, u_{10}) = \frac{1}{u_7} (u_8, u_9, 1, u_5, u_6, u_{10}), \]
\[ q(v_5, v_6, v_7, v_8, v_9, v_{10}) = \frac{1}{v_7} (v_8, v_9, 1, v_5, v_6, v_{10}). \]  

If the 9 points \( P_1, \ldots, P_9 \) are in general position, there exist unique elliptic curve \( C \subset \mathbf{P}^2 \) which pass through the 9 points. This curve \( C \) play the role of the fiber at \( u = \infty \) in the previous section and it is invariant under the action of \( W(E_8^{(1)}) \). Using this curve \( C \) as a “ruler”, Sakai introduced another coordinates of the coset \( \mathcal{M} : \theta_1, \ldots, \theta_9, \tau \) and \((x : y : z) \in \mathbf{P}^2\), such that the matrix \( X \) is represented as

\[
X = \begin{bmatrix}
\varphi(\theta_1) & \varphi(\theta_2) & \cdots & \varphi(\theta_9) & x \\
\varphi'(\theta_1) & \varphi'(\theta_2) & \cdots & \varphi'(\theta_9) & y \\
1 & 1 & \cdots & 1 & z
\end{bmatrix}.
\]

(23)

Here \( \varphi(\theta) = \varphi(\theta, \tau) \) is the Weierstrass \( \varphi \) function which parameterize the elliptic curve \( C \). In terms of Sakai’s coordinates, the action of \( W(E_8^{(1)}) \) is given as follows [1]. The \( S_9 \) part is just the permutation of the parameters \( \theta_i \). The only non-trivial one is \( s_0 \) which has been determined explicitly\(^3\) as

\[
s_0(\theta_i) = \theta'_i = \begin{cases} 
\theta_i + \frac{1}{3}(\theta_1 + \theta_2 + \theta_3), & i = 4, \ldots, 9 \\
\theta_i - \frac{2}{3}(\theta_1 + \theta_2 + \theta_3), & i = 1, 2, 3
\end{cases}
\]

(24)

\[
s_0 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x'_1 & x'_2 & x'_3 \\ y'_1 & y'_2 & y'_3 \\ z'_1 & z'_2 & z'_3 \end{bmatrix} \begin{bmatrix} d_{23} l_{31} l_{12} \\ d_{31} l_{12} l_{23} \\ d_{12} l_{23} l_{31} \end{bmatrix},
\]

(25)

\[
l_{jk} = \det \begin{bmatrix} x & x_j & x_k \\ y & y_j & y_k \\ z & z_j & z_k \end{bmatrix}, \quad d_{jk} = \det \begin{bmatrix} x_* & x_j & x_k \\ y_* & y_j & y_k \\ z_* & z_j & z_k \end{bmatrix} \quad \text{det} \begin{bmatrix} x'_* & x'_j & x'_k \\ y'_* & y'_j & y'_k \\ z'_* & z'_j & z'_k \end{bmatrix}.
\]

(26)

Where \((x_*, y_*, z_*)\) and \((x'_*, y'_*, z'_*)\) are any points on the curve \( C \) such that \((x_*, y_*, z_*) = (\varphi(\theta), \varphi'(\theta), 1), (x'_*, y'_*, z'_*) = (\varphi(\theta'), \varphi'(\theta'), 1)\) with \( \theta' = \theta + (\theta_1 + \theta_2 + \theta_3)/3 \).

In these coordinates, the translation \( T \) (20) acts on the parameters \( \theta_i \) as

\[
T(\theta_1, \ldots, \theta_9) = (\theta_1, \ldots, \theta_9) - \frac{1}{3} \theta(2, 2, 2, -1, \ldots, -1),
\]

(27)

\(^3\)This means that these nine \( \theta_i \)'s transform under \( W(E_8) \) as the \( SL(9) \) Cartan subalgebra, and hence correspond to the nine radii in the \( T^9 \) compactification of \( M \)-theory. The extra Weyl reflection \( s_0 \) (called ‘2/5 transformation’ in [33]) is naturally understood via the \( SL(9) \) decomposition of \( E_8 \) [17].
where $\theta = \sum_{i=1}^{9} \theta_i$. When $\theta = 0$ (modulo periods), the first 9 points are in special position such that the curve $C$ passing through the 9 points is given by one parameter family (a ‘pencil’ of cubic)

$$\lambda F(x, y, z) + \mu G(x, y, z) = 0. \quad (28)$$

Then the corresponding del Pezzo $B_9$ admits an elliptic fibration $B_9 \to \mathbb{P}^1 = \{ (\lambda : \mu) \}$ and 9 blown-up $\mathbb{P}^1$’s correspond to 9 sections of the fibration. The parameters $\theta_i$ specify the 9 points $(x_i : y_i : z_i) = (\wp(\theta_i) : \wp'(\theta_i) : 1)$ where the sections intersect with the marked curve $C$ (at $u = \infty$). These data define the Seiberg-Witten curve for $d = 6$ $E_8$-string as explained in [18]. In this special case, by choosing the parameter $(\lambda : \mu)$ suitably $C$ may pass the 10th point $(x : y : z)$ also and we can put $(x : y : z) = (\wp(\theta_{10}) : \wp'(\theta_{10}) : 1)$. Then all the action of $W(E_{10})$ are represented by addition and permutation on the variables $\theta_i$ ($i = 1, \ldots, 10$).

4 Painlevé Bäcklund transformations and $U$-duality

4.1 Relation to $M$-theory duality

The del Pezzo surfaces $B_k$ play crucial role in various context of string compactifications. In a recent paper [9], Iqbal, Neitzke and Vafa observed a duality between $M$-theory on $T^k$ and del Pezzo surfaces $B_k$. In this correspondence, the Weyl group part of the $U$-duality group is identified with the Cremona isometry $W(E_k)$ for del Pezzo $B_k$. As we have seen in the previous section, the Cremona isometry is the origin of the Bäcklund transformation/discrete time evolution of the elliptic Painlevé equation, it is natural to expect some relation between the Painlevé Bäcklund transformations and $U$-duality. In fact, there exist an analogy between these two Weyl group realizations. Namely the permutation part of the duality can be realized as a change of the order of the compactifications [19], correspondingly the Weyl group symmetry of the $E_{10}$ Painlevé equation appears as a change of the blowing down structure [1].

The Painlevé difference equations reduce to the six Painlevé differential equations in the continuum limit. The latter also possess affine Weyl group symmetries generated by the Bäcklund transformations. Thus it will be interesting to explore whether these differential equations have direct connections with the string equations of motion.

In general relativity, it has been known for some time that some static, axisymmetric solutions of Einstein(-Maxwell)’s equation(s) obey Painlevé differential equations [20]-[23]. For example, the Ernst equation, the equation of motion for the scalars in the dimensionally reduced $D = 4$ pure gravity, reduce to the third or the fifth Painlevé equation under certain assumptions. Since $D = 10$, type IIB scalar sigma model is identical to that of the Ernst system, one can exploit the general relativity result to find special IIB scalar solutions that obey Painlevé equations.
Let us consider a consistent truncation of type IIB supergravity

\[ \mathcal{L} = \sqrt{-G^{(10)}} \left( R^{(10)} - \frac{\partial_M \tau \partial^M \tau}{2(\text{Im} \tau)^2} \right), \]  

(29)

where \( \tau = C + ie^{-\Phi} \) with \( C \) and \( \Phi \) being the RR scalar and the dilaton, respectively. We further adopt an ansatz that the ten-dimensional Einstein-frame metric \( G^{(10)}_{MN} \) is of the form

\[ ds^2_{\text{IIB}} = \lambda^2 (dx^2 + d\rho^2) + \rho^2 d\phi^2 + (-dt^2 + \sum_{i=1}^6 dx_i^2), \]  

(30)

and that \( \lambda \) is a real function of \( -\infty < x < \infty \), the coordinate parallel to the symmetric axis, and the radial coordinate \( \rho \geq 0 \). \( \phi \) is the angle coordinate. We also assume that the complex potential \( \tau \) depends only on \( x \) and \( \rho \). In this way, we get a two-dimensional system without enlarging the duality symmetry than \( SL(2, \mathbb{R})/U(1) \).

In fact, this truncation is equivalent to the dimensional reduction of \( D = 4 \) pure gravity to \( D = 2 \) with a four-dimensional metric

\[ ds^2_{4D} = e^\Phi \left( \lambda^2 (dx^2 + d\rho^2) + \rho^2 d\phi^2 \right) - e^{-\Phi} (dt + A_\phi d\phi)^2 \]  

(31)

with

\[ \partial_\xi A_\phi = -i\rho e^{2\Phi} \partial_\xi C, \quad \xi \equiv x + i\rho. \]  

(32)

The equation of motion for \( \tau \) is given by the Ernst equation

\[ e^{-\Phi} \delta^{\mu\nu} \partial_\mu (\rho \partial_\nu \tau) = -2\rho \delta^{\mu\nu} \partial_\mu \tau \partial_\nu \tau, \]  

(33)

where \( x^\mu = (x, \rho) \). If \( \tau \) is known, the conformal factor \( \lambda \) is consistently determined by integrating the first-order ‘Virasoro constraint’. (See [25] for related technology.)

The metric ansatz (30) is close to that for the D7-brane solutions [26], but the crucial difference is the appearance of \( \rho^2 \) in \( G^{(10)}_{\phi\phi} \). Owing to this explicit coordinate dependence (‘the Weyl canonical coordinate’), \( \tau \) cannot be holomorphic, and the solution does not preserve supersymmetry.

### 4.2 Painlevé III and S-duality

To reduce (33) to a Painlevé equation, we first switch from the \( SL(2, \mathbb{R}) \) variable \( \tau \) to the \( SU(1, 1) \) variable \( F \), defined by [21]

\[ \tau = i \frac{1 + F}{1 - F}. \]  

(34)

In terms of \( F \), the Ernst equation becomes

\[ (1 - F T) \delta^{\mu\nu} \partial_\mu (\rho \partial_\nu F) = -2\rho \delta^{\mu\nu} \partial_\mu F \partial_\nu F. \]  

(35)

One may also trade \( \rho \) for the time \( t \) to discuss colliding string wave solutions. See e.g. [24] for recent discussions and further references.
We further assume the coordinate dependence of $F(x, \rho)$ as

$$F(x, \rho) = f(\rho) e^{i\omega x}, \quad (36)$$

where $f(\rho)$ is a real function, and $\omega$ is a real constant. The equation (35) reduces to

$$(f^2 - 1)(f'' + \frac{f'}{\rho} - \omega^2 f) = 2f(f'^2 - \omega^2 f^2). \quad (37)$$

Here the prime denotes differentiation with respect to $\rho$. Then by the replacement

$$y \equiv \frac{1 + f}{1 - f} \quad (38)$$

we obtain

$$y'' = \frac{y'^2}{y} - \frac{y'}{\rho} + \frac{\omega^2}{4} \left( y^3 - \frac{1}{y} \right). \quad (39)$$

This is Painlevé III

$$y'' = \frac{y'^2}{y} - \frac{y'}{\rho} + \frac{\alpha y^2 + \beta}{\rho} + \gamma y^3 + \frac{\delta}{y} \quad (40)$$

with special parameters

$$\alpha = \beta = 0, \quad \gamma = -\delta = \frac{\omega^2}{4}. \quad (41)$$

Painlevé III (with generic parameters) is known to have a symmetry of Bäcklund transformations isomorphic to the Weyl group of type $(A_1 \oplus A_1)^{(1)}$ generated by three independent Weyl reflections [1]. One of them is

$$y \mapsto \frac{1}{y}, \quad (\alpha, \beta, \gamma, \delta) \mapsto (-\beta, -\alpha, -\delta, -\gamma) \quad (42)$$

which leaves the condition (41) unchanged. Since this implies $\tau \mapsto -1/\tau$, we see that this Bäcklund transformation of Painlevé III precisely corresponds to $S$-duality of IIB theory. On the other hand, the second Bäcklund transformation is simply

$$y \mapsto -y, \quad \rho \mapsto -\rho, \quad (\alpha, \beta, \gamma, \delta) \mapsto (\alpha, \beta, \gamma, \delta) \quad (43)$$

It just flips the sign of $\tau$, and hence is a physically irrelevant transformation. Finally, Painlevé III has yet another independent Bäcklund transformation. It shifts the parameters $\alpha, \beta$ to nonzero values, and therefore the differential equation does not keep its form of what has been reduced from the Ernst equation. Thus it does not correspond to duality, either.

We conclude this subsection with a remark on how the Geroch group [27] is related to the Painlevé Bäcklund transformations. An affine Lie group symmetry of a two-dimensional reduced nonlinear sigma model is a general phenomenon [28], and in the
present system (29)(30) the symmetry is $A_1^{(1)}$, the Geroch group. So the natural question is: how does its Weyl group piece act on the Painlevé equation? The answer is as follows: Among two independent Weyl reflections of the Geroch group, one is manifestly realized in the sigma model (29) as S-duality; this is also a symmetry of the Painlevé equation, as we have seen above. The other is obtained by conjugating with the Kramer-Neugebauer (KN) involution \[^{(5)}\] [30, 31]; this Weyl reflection is not the symmetry of the Painlevé equation because the KN involution does not preserve the metric ansatz (36).

### 4.3 Comments on Painlevé V

The Ernst equation is also known to reduce to the fifth Painlevé equation by using a different ansatz [21]. We again start from the equations (35)(36), but this time we allow $f(\rho)$ to take complex values. $\omega$ is a real constant, as before. In this case, the equation (37) is replaced by

$$\left(f\overline{f} - 1\right)(f'' + \frac{f'}{\rho} - \omega^2 f) = 2\overline{f}(f'^2 - \omega^2 f^2). \quad (44)$$

Multiplying $\overline{f}$ and subtracting the complex conjugate, we find an integral

$$\frac{\rho(\overline{f}f' - f\overline{f}')}{(1 - f\overline{f})^2} = ia, \quad (45)$$

where $a$ is a real integration constant. Writing

$$f(\rho) = r(\rho)e^{iu(\rho)} \quad (46)$$

in terms of two real functions $r(\rho)$ and $u(\rho)$, we may express $u'$ as

$$u' = \frac{a(1 - r^2)^2}{2\rho r^2}. \quad (47)$$

Plugging them into (44), we obtain a second order differential equation of a single variable $r(\rho)$. After a short calculation we find

$$Y'' = \left(\frac{1}{2Y} + \frac{1}{Y-1}\right)Y'^2 - \frac{Y'}{\rho} - \frac{a^2(Y - 1)^2}{2\rho^2} \left(Y - \frac{1}{Y}\right) + \frac{2\omega^2 Y(Y + 1)}{Y - 1} \quad (48)$$

for $Y \equiv r^2$. This is the fifth Painlevé equation with parameters

$$\alpha = -\frac{a^2}{2}, \quad \beta = \frac{a^2}{2}, \quad \gamma = 0, \quad \delta = 2\omega^2 \quad (49)$$

in the standard notation.

The $S$-duality transformation $\tau \mapsto -1/\tau$ acts on $r$ as $r \mapsto -r$, which leaves $Y$ invariant. Therefore, it does not correspond to any of the Bäcklund transformations of Painlevé V, but is trivially realized.

\[5\]The image of (31) under the KN involution is nothing but the four-dimensional piece of the dual $M$-theory metric [29].
5 Conclusions and Discussions

In this paper, we studied the Weyl group symmetries from the point of view of Seiberg-Witten theory, the elliptic Painlevé equation and duality symmetry of $M$/string theory. The results are summarized as follows:

- We have clarified the special role of the fiber at $u = \infty$ of the Seiberg-Witten curves. The mass parameters, on which the Weyl group such as $W(E_8^{(1)})$ acts as the flavor symmetry, are identified with the points where the sections intersect the fiber.

- We have given a simple formulation of the elliptic Painlevé equation in which the hidden $W(E_{10})$ symmetry is manifestly realized. The Seiberg-Witten geometry appears as a special case of this, where the solutions reduce to the elliptic functions.

- We have studied some Painlevé differential equations arising from dimensionally reduced equations of motion of strings. In some special case, the Bäcklund transformation of the Painlevé equation can be identified with a duality symmetry of $M$/string theory.

A property of the singularity confinement is proposed as a discrete analog of the Painlevé property [32]. The singularity confinement demands that a singularity depending on the initial data disappears after finite iteration of the mapping and the memory of initial data is recovered. Of course, the $E_{10}$ Painlevé equation has this property. On the other hand, in [33, 34] it is argued that in $M$-theory, the apparent cosmological singularities can be resolved by the duality transformations. This phenomenon may be considered as the Painlevé property. It should be also noted that the hyperbolicity of $W(E_{10})$ is crucial for the chaotic behavior of the cosmological singularity in $M$-theory [35]. In view of this and the symmetry structure, it is natural to guess that the elliptic $E_{10}$ Painlevé equation, which is chaotic but integrable in some sense, may play some role in certain effective dynamics of $M$-theory on $T^{10}$.

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