Duality Property for Linear Canonical Transform

Mawardi Bahri\textsuperscript{a,1}, Moh. Ivan Azis\textsuperscript{b}, Amir Kamal Amir\textsuperscript{c}
\textsuperscript{a,b,c} Department of Mathematics, Hasanuddin University, Makassar 90245, Indonesia
E-mail: \textsuperscript{a}mawardibahri@gmail.com, \textsuperscript{b}mohivanazis@yahoo.co.id, \textsuperscript{c}amirkamalamir@yahoo.com

Abstract. In the present we first introduce adjoint operators of the linear canonical transform. It is shown the adjoint of the the linear canonical transform is its inverse. We finally derive duality property of the linear canonical transform.

1. Introduction
As we know, the quaternion Fourier transform \cite{1, 2} is a non-trivial generalization of the Fourier transform in setting of the quaternion algebra. Likewise the linear canonical transformation (LCT) is an extension of the Fourier transformation. In recent years, the LCT has attracted the attention of researchers to investigate both its theory and application. In the view of theory, some useful properties the LCT has been published in a number of papers (see, for example, \cite{3, 4, 5, 6, 7, 8, 9}). In the view of applied, it has be used in applied mathematics, optics, digital information processing, and so on. Nowadays, many general transformations are obtained by replacing the kernel of transforms that will be generalized with the LCT kernel (see \cite{10, 11, 12, 13, 14, 15, 16, 17}).

In the present work we first investigate some properties of the LCT, which have not been established in the literature. We then provide the definition of adjoint operators of the linear canonical transform. Based on the LCT adjoint operators we provide alternative proofs of properties of the LCT. Finally, we derive in detail the duality theorem related to the linear canonical transform.

This article is structured as follows. In Section 2 we derive several properties of the LCT, which has not been established in the literature. Section 3 introduces adjoint operators of the linear canonical transform. Section 4 presents the duality property in the linear canonical transform domain. In Section 5 we conclude the article.

2. Definition of Linear Canonical Transform (LCT)
The linear canonical transformation (LCT) was first proposed by Moshinsky and Quesnee \cite{18} in its attempt to generalize the classical Fourier transform (FT). We start by introducing the LCT definition as follows.

Definition 2.1. Let $C = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2 \times 2}$ be a matrix parameter such that $\det(C) = ad - bc = 1$.\textsuperscript{1}
The LCT of a complex function \( f \in L^1(\mathbb{R}) \) is given by

\[
L_C \{ f \} (u) = F_C(u) \left( \int_{\mathbb{R}} f(z) K_C(z, u) \, dz, \ b \neq 0 \right) = \sqrt{2\pi b} e^{\frac{i}{2} (\frac{a}{b} z^2 - \frac{2}{b} uz + \frac{d}{b} u^2 - \frac{\pi}{2})} K_C(z, u),
\]

(1)

where \( K_C(z, u) \) is given by

\[
K_C(z, u) = \frac{1}{\sqrt{2\pi b}} e^{\frac{i}{2} (\frac{a}{b} z^2 - \frac{2}{b} uz + \frac{d}{b} u^2 - \frac{\pi}{2})}.
\]

(2)

From (2), it is obvious that \( K_{C^{-1}}(u, z) = K_C(z, u) \).

In the next section, we fix \( b \neq 0 \). We see that for \( C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \) the LCT definition (1) will lead to the Fourier transformation (FT) definition. The inverse transform of the LCT is given by

\[
L_{C^{-1}} \{ L_C \{ f \} \} (z) = f(z) = \int_{\mathbb{R}} L_C \{ f \} (u) K_{C^{-1}}(u, z) \, du
\]

\[
= \int_{\mathbb{R}} L_C \{ f \} (u) \frac{1}{\sqrt{2\pi b}} e^{\frac{i}{2} (\frac{a}{b} z^2 - \frac{2}{b} uz + \frac{d}{b} u^2 - \frac{\pi}{2})} du.
\]

(3)

The LCT of a function \( f \in L^1(\mathbb{R}) \) could be computed via associated FT that is

\[
L_C \{ f \} (u) = e^{-i\frac{\pi}{2}} \frac{e^{\frac{i\pi}{2} u^2}}{\sqrt{b}} F \{ e^{\frac{i\pi}{2} z^2} f(z) \} \left( \frac{u}{b} \right),
\]

(4)

where \( F \{ f \} (u) = \hat{f}(u) \) is the Fourier transformation of the complex function \( f \in L^1(\mathbb{R}) \) defined by \([19, 20]\)

\[
F \{ f \} (u) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(z) e^{-iu z} \, dz.
\]

(5)

Let \( h(z) = e^{-\frac{i\pi}{2}} \frac{e^{\frac{i\pi}{2} u^2}}{\sqrt{b}} f(z) \). Then equation (4) can be written in the following form

\[
e^{-\frac{i\pi}{2} u^2} L_C \{ f \} (u) = F \{ h \} \left( \frac{u}{b} \right).
\]

(6)

For further use, we define the inner product for the space of \( L^2(\mathbb{R}) \) as

\[
(g, h) = \int_{\mathbb{R}} g(z) \overline{h(z)} \, dz,
\]

(7)

where \( \overline{g} \) stands for the complex conjugate of \( g \).

In what follows, we will investigate several useful properties of the linear canonical transform, which are similar to the properties of the classical Fourier transform.

**Theorem 2.1.** If \( f, g \in L^1(\mathbb{R}) \), then

\[
(f, L_C \{ g \}) = (L_C \{ f \}, \overline{g}).
\]

(8)
Proof. An easy computation yields

\[
(LC\{f\}, \bar{g}) = \int_{\mathbb{R}} LC\{f\}(t) g(t) \, dt = \int_{\mathbb{R}} \int_{\mathbb{R}} f(z) \frac{1}{\sqrt{2\pi b}} e^{\frac{i}{b}(\frac{a}{b}z^2 - \frac{2}{b}zt + \frac{d}{b}t^2 - \frac{\pi}{2})} g(t) \, dt \, dz
\]

\[
= \int_{\mathbb{R}} f(z) \left( \int_{\mathbb{R}} g(t) \frac{1}{\sqrt{2\pi b}} e^{\frac{i}{b}(\frac{a}{b}z^2 - \frac{2}{b}zt + \frac{d}{b}t^2 - \frac{\pi}{2})} \, dt \right) \, dz
\]

\[
= \int_{\mathbb{R}} f(z) LC\{g\}(z) \, dz
\]

\[
= \langle f, LC\{g\} \rangle.
\]

This is the desired result.

Theorem 2.2. If \( f, g \in L^1(\mathbb{R}) \), then one has

\[
\langle f, g \rangle = \langle LC\{f\}, LC\{g\} \rangle.
\]

Proof. By definition, we get

\[
\int_{\mathbb{R}} f(t) \overline{g(t)} \, dt = \int_{\mathbb{R}} f(t) \int_{\mathbb{R}} LC\{g\}(z) \frac{1}{\sqrt{2\pi b}} e^{-\frac{i}{b}(\frac{a}{b}z^2 - \frac{2}{b}zt + \frac{d}{b}t^2 - \frac{\pi}{2})} \, dz \, dt
\]

\[
= \int_{\mathbb{R}} f(t) \frac{1}{\sqrt{2\pi b}} e^{\frac{i}{b}(\frac{a}{b}z^2 - \frac{2}{b}zt + \frac{d}{b}t^2 - \frac{\pi}{2})} \int_{\mathbb{R}} LC\{g\}(z) \, dz \, dt
\]

\[
= \int_{\mathbb{R}} LC\{f\}(z) LC\{g\}(z) \, dz.
\]

The proof is complete.

Theorem 2.3. Assume that \( f, g \in L^1(\mathbb{R}) \) are integrable functions. Then we have the following identity

\[
\int_{\mathbb{R}} f(t) LC\{g\}(t) \, dt = \int_{\mathbb{R}} g(z) LC\{f\}(z) \, dz.
\]

Proof. A simple calculation yields

\[
\int_{\mathbb{R}} f(t) LC\{g\}(t) \, dt = \int_{\mathbb{R}} f(t) \int_{\mathbb{R}} g(z) \frac{1}{\sqrt{2\pi b}} e^{\frac{i}{b}(\frac{a}{b}z^2 - \frac{2}{b}zt + \frac{d}{b}t^2 - \frac{\pi}{2})} \, dz \, dt
\]

\[
= \int_{\mathbb{R}} g(z) \left( \int_{\mathbb{R}} f(t) \frac{1}{\sqrt{2\pi b}} e^{\frac{i}{b}(\frac{a}{b}z^2 - \frac{2}{b}zt + \frac{d}{b}t^2 - \frac{\pi}{2})} \, dt \right) \, dz
\]

\[
= \int_{\mathbb{R}} g(z) LC\{f\}(z) \, dz.
\]

This is the required result.

3. Adjoint Operators of Linear Canonical Transform

In section, we introduce the adjoint operator for the linear canonical transform. It can used to provide an alternative proof of the Plancherel theorem for the linear canonical transform.
Definition 3.1. If $A$ is a bounded linear operator on Hilbert space $H$, the operator $A^* : H \rightarrow H$ defined by

$$ (Ag, h) = (g, A^*h), \quad \forall g, h \in H, $$

is called the adjoint operator of $T$.

The following two results are consequences of the adjoint of the linear canonical transform mentioned above.

Theorem 3.1. The adjoint of the linear canonical transform is inverse of the linear canonical transform, i.e.

$$ (LC\{f\}, g) = (f, L^{-1}C[g]). $$

Proof. For any $f, g \in L^2(\mathbb{R})$ applying (7) results in

$$ (LC\{f\}, g) = \int_{\mathbb{R}} LC\{f\}(u) g(u) du $$

$$ = \int_{\mathbb{R}} \int_{\mathbb{R}} f(z) \frac{1}{\sqrt{2\pi b}} e^{i(\frac{a}{2}z^2 - \frac{b}{2}zu + \frac{d}{2}u^2 - \frac{\pi}{2})} g(u) du dz $$

$$ = \int_{\mathbb{R}} f(z) \left( \int_{\mathbb{R}} g(u) \frac{1}{\sqrt{2\pi b}} e^{i(\frac{a}{2}z^2 - \frac{b}{2}zu + \frac{d}{2}u^2 - \frac{\pi}{2})} du \right) dz $$

$$ = \int_{\mathbb{R}} f(z) L^{-1}_C[g] dz $$

$$ = (f, L^{-1}_C[g]), $$

which completes the proof.

Theorem 3.2 (Plancherel formula). Given $g, h \in L^2(\mathbb{R})$, then we have

$$ (LC\{g\}, LC\{h\}) = (g, h) $$

and

$$ (L^{-1}_C[LC\{g\}], L^{-1}_C[LC\{h\}]) = (g, h). $$

Proof. With the aid of (15) we get

$$ (LC\{f\}, LC\{g\}) = (f, L^{-1}_C[LC\{g\}]) $$

$$ = (f, g). $$

Equation (18) can be obtained in a similar way.

4. Duality of Linear Canonical Transform

One of properties of the Fourier transformation is duality. In the following we shall show that the duality property is still valid in the LCT domain with some changes.

Theorem 4.1. Suppose that $f \in L^2(\mathbb{R})$. Then we have

$$ LC\{F_C(u)\} = -f(u). $$

where $C^* = \begin{bmatrix} d & -b \\ c & a \end{bmatrix}$. 
Proof. It directly follows from (1) that

\[ L_{C^*}\{F_C(u)\} = \int_{\mathbb{R}} L_C\{f\}(u) \frac{1}{\sqrt{2\pi b}} e^{\frac{i}{2b}(\frac{a^2}{b}z^2 + \frac{2}{b}zu - \frac{2}{b}u^2 - \frac{2}{b}z^2 + \frac{2}{b}u^2 - \frac{2}{b})} du. \] (21)

Interchanging the role of \( u \) by \( z \) in (21), we obtain

\[ L_{C^*}\{F_C(u)\} = \int_{\mathbb{R}} L_C\{f\}(z) \frac{1}{\sqrt{2\pi b}} e^{\frac{i}{2b}(\frac{a^2}{b}z^2 + \frac{2}{b}zu - \frac{2}{b}u^2 - \frac{2}{b}z^2 + \frac{2}{b}u^2 - \frac{2}{b})} dz \]

\[ = \int_{\mathbb{R}} L_C\{f\}(z) \frac{1}{\sqrt{2\pi b}} e^{-\frac{i}{2b}(\frac{a^2}{b}z^2 - \frac{2}{b}zu + \frac{4}{b}u^2 + \frac{2}{b})} dz. \] (22)

Because \( \frac{1}{i} = e^{-\frac{2\pi}{b}} \), (22) can be expressed as

\[ L_{C^*}\{F_C(u)\} = \int_{\mathbb{R}} L_C\{f\}(z) \frac{e^{-\frac{2\pi}{b}}}{\sqrt{2\pi b}} e^{-\frac{i}{2b}(\frac{a^2}{b}z^2 - \frac{2}{b}zu + \frac{4}{b}u^2 + \frac{2}{b})} dz \]

\[ = \int_{\mathbb{R}} L_C\{f\}(z) \frac{e^{-\frac{2\pi}{b}}}{\sqrt{2\pi b}} e^{-\frac{i}{2b}(\frac{a^2}{b}z^2 - \frac{2}{b}zu + \frac{4}{b}u^2 + \frac{2}{b})} dz \]

\[ = - \int_{\mathbb{R}} L_C\{f\}(z) \frac{1}{\sqrt{2\pi b}} e^{-\frac{i}{2b}(\frac{a^2}{b}z^2 - \frac{2}{b}zu + \frac{4}{b}u^2 + \frac{2}{b})} dz \]

\[ = - f(u). \] (23)

This is the desired result. \( \square \)

5. Conclusion

We have proposed the new properties of the LCT. Applying its properties we have established duality property for the LCT, which is an extension of duality property for the classical Fourier transformation with modifications.

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