Finite Element Convergence Analysis of a Schwarz Alternating Method for Nonlinear Elliptic PDEs

Messaoud Boulbrachene

Department of Mathematics, Sultan Qaboos University, P.O. Box 36, PC 123, Al-Khoud, Muscat Sultanate of Oman. Email: boulbrac@squ.edu.om.

ABSTRACT: In this paper, we prove uniform convergence of the standard finite element method for a Schwarz alternating procedure for nonlinear elliptic partial differential equations in the context of linear subdomain problems and nonmatching grids. The method stands on the combination of the convergence of linear Schwarz sequences with standard finite element \( L^\infty \) -error estimate for linear problems.

Keywords: Schwarz Method; Finite elements; Convergence.

1. Introduction

The Schwarz alternating method can be used to solve elliptic boundary value problems on domains that consist of two or more overlapping subdomains. The solution is approximated by an infinite sequence of functions that results from solving a sequence of elliptic boundary value problems in each of the subdomains.

There has been extensive analysis of the Schwarz alternating method for nonlinear elliptic boundary value problems [1-4] and the references therein). Also, the effectiveness of Schwarz methods for these problems (especially those in fluid mechanics) has been demonstrated by many authors.

In this paper, we are concerned with the finite element convergence analysis of overlapping Schwarz alternating methods in the context of nonmatching grids for nonlinear PDEs, where the Schwarz sub problems are linear. This study constitutes, to some extent, an improvement of the one achieved in [5], on a Schwarz method with nonlinear sub problems.

For that, we develop an approach which combines the convergence result of Lui [6], with standard finite element error estimate for linear elliptic equations.

For other works on finite element convergence analysis in the maximum norm of overlapping nonmatching Schwarz method, we refer to [7-12].

The rest of the paper is organized as follows. In section 2, we state the continuous alternating Schwarz sub problems and define their respective finite element counterparts in the context of nonmatching overlapping grids. In section 3, we give \( L^\infty (\Omega) \)- convergence analysis of the method.
2. Preliminaries

We begin by laying down some definitions and classical results related to linear elliptic equations.

2.1 Linear elliptic equations

Let $\Omega$ be a bounded polyhedral domain of $\mathbb{R}^2$ or $\mathbb{R}^3$ with sufficiently smooth boundary $\partial \Omega$. We consider the bilinear form

$$a(u,v) = \int_{\Omega} \nabla u \nabla v \, dx$$

(1)

the linear form

$$(f,v) = \int_{\Omega} f(x)v(x) \, dx$$

(2)

the right hand side:

$$f$$ is a regular function,

(3)

the space

$$V^{(g)} = \{ v \in H^1(\Omega) \text{ such that } v = g \text{ on } \Omega \}$$

(4)

where $g$ is a regular function defined on $\partial \Omega$.

We consider the linear elliptic equation: Find $\xi \in V^{(g)}$ such that

$$a(\xi,v) + c(\xi,v) = (f,v) \quad \forall v \in H^1_0(\Omega)$$

(5)

where

$$c \in \mathbb{R}, \quad c > 0 \text{ such that } c \geq \beta > 0$$

(6)

Let $V_h$ be the space of finite elements consisting of continuous piece-wise linear functions,

$$\phi_s \text{, } s = 1, 2, ..., m(h) \text{ be the basis functions of } V_h,$$

and $m(h)$ denote the number of vertices of the triangulation in $\Omega$. Let also $V^0_h$ be the subspace of $V_h$ defined by

$$V^0_h = \{ v \in V_h \text{ such that } v = 0 \text{ on } \partial \Omega \}$$

(7)

The discrete counterpart of (5) consists of finding $\xi_h \in V^{(g)}_h$ such that

$$a(\xi_h,v) + c(\xi_h,v) = (f,v) \quad \forall v \in V^0_h$$

(8)

where $V^{(g)}_h$ is the space of

$$V^{(g)}_h = \{ v \in V_h \text{ such that } v = \pi_h g \text{ on } \partial \Omega \}$$

(9)

and $\pi_h$ is the Lagrange interpolation operator on $\partial \Omega$.

Theorem 1. [13] Under suitable regularity of the solution of problem (5), there exists a constant $C$ independent of $h$ such that

$$\| \xi - \xi_h \|_{L^2(\Omega)} \leq C h^2 |\ln h|$$

(10)

Lemma 1. [5] Let $w \in H^1(\Omega) \cap C(\overline{\Omega})$ satisfy $a(w,\varphi) + c(w,\varphi) \geq 0$ $\forall$ non-negative $\varphi \in H^1_0(\Omega)$, and $w \geq 0$ on $\partial \Omega$. Then $w \geq 0$ on $\overline{\Omega}$.

The proposition below establishes a Lipschitz continuous dependency of the solution with respect to the data.

Notation 1. Let $(f,g),(\tilde{f},\tilde{g})$ be a pair of data, and $\xi = \sigma(f,g)$ and $\tilde{\xi} = \sigma(\tilde{f},\tilde{g})$ be the corresponding solutions to (5).

Proposition 1. Under the conditions of lemma 1, we have:

$$\| \xi - \tilde{\xi} \|_{L^2(\Omega)} \leq \max \left\{ \frac{1}{\beta} \| f - \tilde{f} \|_{L^2(\Omega)}, \| g - \tilde{g} \|_{L^2(\Omega)} \right\}$$

(11)

Proof. First, set

$$\theta = \max \left\{ \frac{1}{\beta} \| f - \tilde{f} \|_{L^2(\Omega)}, \| g - \tilde{g} \|_{L^2(\Omega)} \right\}$$

(12)
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Then
\[ f \leq f + \| f - \bar{f} \|_{L^\infty(\Omega)} \]
\[ \leq f + \frac{c}{\beta} \| f - \bar{f} \|_{L^\infty(\Omega)} \]
\[ \leq f + c \max \left\{ \frac{1}{\beta} \| f - \bar{f} \|_{L^\infty(\Omega)}, \| g - \tilde{g} \|_{L^\infty(\Omega)} \right\} \]
\[ \leq f + c \theta \]

So
\[ a(\xi, \phi) + a(\xi + \theta, \phi) \leq a(\xi, \phi) + c(\xi, \phi) + c(\xi + \theta, \phi) \quad \forall \phi \geq 0, \quad \phi \in H^1_0(\Omega) \]
\[ \leq a(\xi + \theta, \phi) + c((\xi + \theta), \phi) = (f + c \theta, \phi) \]

On the other hand, we have
\[ \xi + \theta - \xi \geq 0 \text{ on } \partial \Omega \]

So
\[ \left\{ \begin{array}{l}
a(\xi + \theta - \xi, \phi) + c(\xi + \theta - \xi, \phi) \geq 0 \\
\xi + \theta - \xi \geq 0 \text{ on } \partial \Omega
\end{array} \right. \]

Thus, making use of lemma 1, we get
\[ \xi + \theta - \xi \geq 0 \text{ on } \overline{\Omega} \]

Similarly, interchanging the roles of the couples \((f, g)\), \((\bar{f}, \tilde{g})\), we obtain
\[ \xi + \theta - \xi \geq 0 \text{ on } \overline{\Omega} \]

which completes the proof.

**Remark 1.** Lemma 1 holds true in the discrete case.

Indeed, assume that the discrete maximum principle (d.m.p) holds, i.e. the matrix resulting from the finite element discretization is an M-Matrix. Then we have:

**Lemma 2.** Let \( w \in V_h \) satisfy \( a(w, \phi_h) + c(w, \phi) \geq 0 \) \( \forall \phi_h, \phi = 1, 2, ..., m(h) \) and \( w \geq 0 \) on \( \partial \Omega \). Then \( w \geq 0 \) on \( \Omega \).

**Proof.** The proof is a direct consequence of the discrete maximum principle.

Let \((f, g)\) and \((\bar{f}, \tilde{g})\) be a pair of data and \( \zeta_h = \sigma_h(f, g) \) and \( \bar{\zeta}_h = \sigma_h(\bar{f}, \tilde{g}) \) be the corresponding solutions to problem (8).

**Proposition 2.** Let the d.m.p hold. Then, under conditions of lemma 2, we have
\[ \left\| \varphi_h - \varphi_h \right\|_{L^\infty(\Omega)} \leq \max \left\{ \frac{1}{\beta} \| f - \bar{f} \|_{L^\infty(\Omega)}, \| g - \tilde{g} \|_{L^\infty(\Omega)} \right\} \]

**Proof.** The proof is similar to that of the continuous case. Indeed, as the basis functions \( \phi \geq 0 \) of the space \( V_h \) are positive, it suffices to make use of the discrete maximum principle.

Let \((f, g)\) and \((\bar{f}, \tilde{g})\) be a pair of data and \( \zeta_h = \sigma_h(f, g) \) and \( \bar{\zeta}_h = \sigma_h(\bar{f}, \tilde{g}) \) be the corresponding solutions to problem (8).

3. Schwarz Alternating Methods for Nonlinear PDEs

3.1 The nonlinear PDE

Consider the nonlinear PDE: Find \( u \in C^2(\Omega) \) such that
\[ \left\{ \begin{array}{l}
-\Delta u + cu = f(u) \text{ in } \Omega \\
u = 0 \text{ on } \partial \Omega
\end{array} \right. \]
or in its weak form: Find $u \in H^1_0(\Omega)$ such that

$$a(u, v) + c(u, v) = (f(u), v) \quad \forall v \in H^1_0(\Omega)$$

where

$$f(.)$$ is a non-decreasing nonlinearity

We assume that $f(.)$ is Lipschitz continuous, that is

$$|f(x) - f(y)| \leq k |x - y| \quad \forall x, y \in \mathbb{R}$$

such that

$$k < \beta$$

where $\beta$ is defined in (6).

**Theorem 2.** [14] Under the above assumptions, Problem (11) has a unique solution.

### 3.2 The Linear Schwarz Subproblems

We decompose $\Omega$ into two overlapping smooth subdomains $\Omega_1$ and $\Omega_2$ such that:

$$\Omega = \Omega_1 \cup \Omega_2$$

We denote by $\partial \Omega_i$ the boundary of $\Omega_i$, $\gamma_i = \partial \Omega_i \cap \Omega_j$, and $\Gamma_j = \partial \Omega \cap \Omega_j$. We assume that the intersection of $\gamma_i$ and $\gamma_j$; $i \neq j$ is empty. Let $u_i^0$ be an initial guess. We define the alternating Schwarz sequences $(u_{i}^{n+1})$ on $\Omega_i$ such that $u_{i}^{n+1} \in C^2(\Omega_i)$ solves

$$
\begin{align*}
-\Delta u_{i}^{n+1} + cu_{i}^{n+1} &= f(u_i^n) \quad \text{in } \Omega_i \\
u_{i}^{n+1} &= 0 \quad \text{on } \Gamma_1 \\
u_{i}^{n+1} &= u_{2}^{n+1} \quad \text{on } \gamma_2
\end{align*}
$$

and the sequence $(u_{2}^{n+1})$ such that $u_{2}^{n+1} \in C^2(\Omega_2)$ solves

$$
\begin{align*}
-\Delta u_{2}^{n+1} + cu_{2}^{n+1} &= f(u_2^n) \quad \text{in } \Omega_2 \\
u_{2}^{n+1} &= 0 \quad \text{on } \Gamma_2 \\
u_{2}^{n+1} &= u_{i}^{n+1} \quad \text{on } \gamma_2
\end{align*}
$$

Note that Schwarz subdomain problems (14) and (15) are linear.

**Theorem 3.** [6] The sequences (14) and (15) converge uniformly in $C^2(\Omega_1)$ and $C^2(\Omega_2)$, respectively, to $u_i = u \cap \Omega_i$, $i = 1, 2$, where $u$ is the solution of (11).

### 3.3 The variational Linear Schwarz Subproblems

The corresponding variational problems read as follows: $u_{1}^{n+1} \in V_1$ solves

$$
\begin{align*}
b_1(u_{1}^{n+1}, v) &= (f(u_i^n), v) \quad \forall v \in V_1 \\
u_{1}^{n+1} |_{\gamma_1} &= u_{2}^{n+1} |_{\gamma_1}
\end{align*}
$$

where

$$b_1(u_{1}^{n+1}, v)$$
and $u^{n+1}_2 \in V_2$ solves
\[
\begin{cases}
    b_2(u^{n+1}_2, v) = (f(u^n_2), v), & \forall v \in V_2 \\
    u^{n+1}_2 |_{\gamma_2} = u^n_2 |_{\gamma_2}
\end{cases}
\]
where
\[
V_i = H^1(\Omega_i); \quad V_i^0 = H^1_0(\Omega_i)
\]
\[
b_i(u, v) = \int_{\Omega_i} (\nabla u \cdot \nabla v + cuv) \, dx
\]
\[
(u, v)_i = \int_{\Omega_i} u(x) v(x) \, dx
\]

### 3.4 The Discretization

For $i = 1, 2$, let $\tau^h$ be a standard regular and quasi-uniform finite element triangulation in $\Omega_i$; $\tau^h_i$, being the mesh size. The two meshes being mutually independent on $\Omega_i \cap \Omega_j$, a triangle belonging to one triangulation does not necessarily belong to the other.

Let us define the discrete analog of spaces $V_i$ and $V_i^0$, respectively, that is
\[
V_h = \left\{ v \in C(\Omega_i) \text{such that } v|_K \in P_i \quad \forall K \in \tau^h \right\}
\]
\[
\forall \quad V_h^0 = \left\{ v \in V_h \text{ such that } v = 0 \text{ on } \partial \Omega_i \right\}
\]
and let $\pi_h$ denote the Lagrange interpolation operator on $\gamma_i$.

**The discrete Maximum principle** (see [15,16]).

We assume that the respective matrices resulting from the discretization of problems (16) and (17) are M-matrices.

### 3.5 The Finite Element Linear Schwarz Sub problems

Let $u^0_h \in V_0$ be the discrete analog of $u^0$; that is, $u^0_h = r_h(u^0)$ where $r_h$ denotes the finite element interpolation operator in $\Omega$. We define the sequence $\left(u^{n+1}_h\right)$ such that $u^{n+1}_h \in V_h$ solves
\[
\begin{cases}
    b_h(u^{n+1}_h, v) = (f(u^n_h), v), & \forall v \in V_h \\
    u^{n+1}_h |_{\gamma_1} = \pi_h(u^n_h)
\end{cases}
\]
and the sequence $\left(u^{n+1}_h\right)$ such that $u^{n+1}_h \in V_h$ solves
\[
\begin{cases}
    b_2(u^{n+1}_h, v) = (f(u^n_h), v), & \forall v \in V_h \\
    u^{n+1}_h |_{\gamma_2} = \pi_h(u^{n+1}_h)
\end{cases}
\]

### 4. $L^\infty$ - Convergence Analysis

This section is devoted to the proof of the main result of the present paper. To that end, we begin by introducing two discrete auxiliary Schwarz sequences and prove a fundamental lemma.
4.1 Auxiliary Discrete Schwarz Sub problems

We construct a sequence \( (\omega_h^{n+1}) \) such that \( \omega_h^{n+1} \in V_h \) solves
\[
\begin{align*}
\begin{cases}
b_1(\omega_h^{n+1}, v) = (f(u^n) , v) \quad \forall v \in V_h^0 \\
u_h^{n+1} |_{\gamma_1} = \pi_h(u^n |_{\gamma_1})
\end{cases}
\end{align*}
\] (23)

and the sequence \( (\omega_h^{n+1}) \) such that \( \omega_h^{n+1} \in V_h \) solves
\[
\begin{align*}
\begin{cases}
b_2(\omega_h^{n+1}, v) = (f(u^n) , v) \quad \forall v \in V_h^0 \\
u_h^{n+1} |_{\gamma_2} = \pi_h(u^{n+1} |_{\gamma_2})
\end{cases}
\end{align*}
\] (24)

Then, it is clear that \( \omega_h^{n+1} \) and \( \omega_h^{n+1} \) are the finite element approximation of \( u_1^{n+1} \) and \( u_2^{n+1} \) defined in (16) and (17), respectively.

**Notation 2.** From now on, we shall adopt the following notations:
\[
\begin{align*}
\| \|_1 & = L^\infty(\gamma_1), \| \|_2 = L^\infty(\gamma_2) \\
\| \|_H & = L^\infty(\Omega), \| \|_{H_1} = L^\infty(\Omega_1), \| \|_{H_2} = L^\infty(\Omega_2) \\
\pi_h & = \pi_{h_2} = \pi_h
\end{align*}
\] (25-27)

4.2 The Main Result

The following lemma will play a key role in proving the main result of this paper.

**Lemma 3.**
\[
\| u_1^n - u_{1h}^n \|_2 \leq \sum_{i=0}^{n} \| u_i^1 - \omega_{1h}^i \|_2 + \sum_{i=0}^{n} \| u_i^2 - \omega_{2h}^i \|_2
\]
\[
\| u_2^n - u_{2h}^n \|_2 \leq \sum_{i=0}^{n} \| u_i^2 - \omega_{1h}^i \|_2 + \sum_{i=0}^{n} \| u_i^2 - \omega_{2h}^i \|_2
\]

**Proof.** The proof will be carried out by induction. For \( n=1 \), we have in \( \Omega_1 \).
\[
\| u_1^1 - u_{1h}^1 \| \leq \| u_1^1 - \omega_{1h}^1 \| + \| \omega_{1h}^1 - u_{1h}^1 \|
\]
and, making use of Proposition 2, we obtain
\[
\| u_1^1 - u_{1h}^1 \| \leq \| u_1^1 - \omega_{1h}^1 \| + \max \left\{ \frac{1}{\beta} \| f(u_1^0) - f(u_{1h}^0) \|, \| \pi_1 u_1^0 - \pi_{1h} u_{1h}^0 \| \right\}
\]
\[
\leq \| u_1^1 - \omega_{1h}^1 \| + \max \left\{ \frac{k}{\beta} \| u_1^0 - u_{1h}^0 \|, \| u_2^0 - u_{2h}^0 \| \right\}
\]

We then have to distinguish between two cases

1: \( \max \left\{ \frac{k}{\beta} \| u_1^0 - u_{1h}^0 \|, \| u_2^0 - u_{2h}^0 \| \right\} = \frac{k}{\beta} \| u_1^0 - u_{1h}^0 \|
and

$$\begin{align*}
2: \max \left\{ \frac{k}{\beta} \left\| u_i^0 - u_{1h}^0 \right\|, \left\| u_i^0 - u_{2h}^0 \right\| \right\} = \left\| u_i^0 - u_{2h}^0 \right\|
\end{align*}$$

Case 1. implies that

$$\left\| u_1^1 - u_{1h}^1 \right\| \leq \left\| u_1^1 - \omega_{1h}^1 \right\| + \frac{k}{\beta} \left\| u_1^0 - u_{1h}^0 \right\|$$

and hence

$$\left\| u_1^1 - u_{1h}^1 \right\| \leq \left\| u_1^1 - \omega_{1h}^1 \right\| + \left\| u_1^0 - u_{1h}^0 \right\| + \left\| u_2^0 - u_{2h}^0 \right\|$$

Case 2. implies that

$$\left\| u_1^1 - u_{1h}^1 \right\| \leq \left\| u_1^1 - \omega_{1h}^1 \right\| + \left\| u_2^0 - u_{2h}^0 \right\|$$

and, in both cases, we have

$$\left\| u_1^1 - u_{1h}^1 \right\| \leq \left\| u_1^1 - \omega_{1h}^1 \right\| + \left\| u_1^0 - u_{1h}^0 \right\| + \left\| u_2^0 - u_{2h}^0 \right\| \quad (28)$$

Similarly, we have in $\Omega_2$

$$\left\| u_2^1 - u_{2h}^1 \right\| \leq \left\| u_2^1 - \omega_{2h}^1 \right\| + \left\| \omega_{2h}^1 - u_{2h}^1 \right\|$$

$$\leq \left\| u_2^1 - \omega_{2h}^1 \right\| + \max \left\{ \frac{k}{\beta} \left\| u_2^0 - u_{2h}^0 \right\|, \left\| u_1^0 - u_{1h}^1 \right\| \right\}$$

$$\leq \left\| u_2^1 - \omega_{2h}^1 \right\| + \max \left\{ \frac{k}{\beta} \left\| u_2^0 - u_{2h}^0 \right\|, \left\| u_1^0 - u_{1h}^1 \right\| \right\}$$

Here also we need to consider the following two cases:

1. $\max \left\{ \frac{k}{\beta} \left\| u_2^0 - u_{2h}^0 \right\|, \left\| u_1^0 - u_{1h}^1 \right\| \right\} = \frac{k}{\beta} \left\| u_2^0 - u_{2h}^0 \right\|$

2. $\max \left\{ \frac{k}{\beta} \left\| u_2^0 - u_{2h}^0 \right\|, \left\| u_1^0 - u_{1h}^1 \right\| \right\} = \left\| u_1^0 - u_{1h}^1 \right\|$.

Case 3. implies that

$$\left\| u_2^1 - u_{2h}^1 \right\| \leq \left\| u_2^1 - \omega_{2h}^1 \right\| + \frac{k}{\beta} \left\| u_2^0 - u_{2h}^0 \right\|$$

so

$$\left\| u_2^1 - u_{2h}^1 \right\| \leq \left\| u_2^1 - \omega_{2h}^1 \right\| + \left\| u_2^0 - u_{2h}^0 \right\|$$
Case 4. implies that
\[
\left\| u_i^2 - u_{2h}^1 \right\|_2 \leq \left\| u_i^2 - \omega_{2h}^1 \right\|_2 + \left\| u_i^1 - u_{ih}^1 \right\|
\]
\[
\leq \left\| u_i^2 - \omega_{2h}^1 \right\|_2 + \left\| u_i^1 - \omega_{ih}^1 \right\| + \left\| u_i^0 - u_{ih}^0 \right\| + \left\| u_i^2 - u_{2h}^0 \right\|_2
\]
Thus, in both cases, we have
\[
\left\| u_i^2 - u_{ih}^1 \right\|_2 \leq \left\| u_i^2 - \omega_{2h}^1 \right\|_2 + \left\| u_i^1 - \omega_{ih}^1 \right\| + \left\| u_i^0 - u_{ih}^0 \right\| + \left\| u_i^2 - u_{2h}^0 \right\|_2
\] (29)

For \( n = 2 \)
\[
\left\| u_i^2 - u_{ih}^2 \right\| \leq \left\| u_i^2 - \omega_{ih}^2 \right\| + \left\| \omega_{ih}^2 - u_{ih}^2 \right\|
\]
\[
\leq \left\| u_i^2 - \omega_{ih}^2 \right\| + \max \left\{ \frac{1}{\beta} \left\| f(u_i^2) - f(u_i^1) \right\|, \left\| \sigma_{ih} u_i^2 - \sigma_{ih} u_i^1 \right\| \right\}
\]
\[
\leq \left\| u_i^2 - \omega_{ih}^2 \right\| + \max \left\{ \frac{k}{\beta} \left\| u_i^1 - u_{ih}^1 \right\|, \left\| u_i^2 - u_{2h}^1 \right\| \right\}
\]
Case 1.
\[
\max \left\{ \frac{1}{\beta} \left\| u_i^1 - u_{ih}^1 \right\|, \left\| u_i^1 - u_{ih}^1 \right\| \right\} = \frac{k}{\beta} \left\| u_i^1 - u_{ih}^1 \right\|
\]
\[
\left\| u_i^2 - u_{ih}^2 \right\| \leq \left\| u_i^2 - \omega_{ih}^2 \right\| + \frac{k}{\beta} \left\| u_i^1 - u_{ih}^1 \right\|
\]
\[
\leq \left\| u_i^2 - \omega_{ih}^2 \right\| + \left\| u_i^1 - u_{ih}^1 \right\|
\]
\[
\leq \left\| u_i^2 - \omega_{ih}^2 \right\| + \left\| u_i^1 - \omega_{ih}^1 \right\| + \left\| u_i^0 - u_{ih}^0 \right\| + \left\| u_i^2 - u_{2h}^0 \right\|_2
\]
Case 2.
\[
\max \left\{ \frac{1}{\beta} \left\| u_i^1 - u_{ih}^1 \right\|, \left\| u_i^2 - u_{2h}^1 \right\| \right\} = \left\| u_i^2 - u_{2h}^1 \right\|_2
\]
\[
\left\| u_i^2 - u_{ih}^2 \right\| \leq \left\| u_i^2 - \omega_{ih}^2 \right\| + \left\| u_i^2 - u_{2h}^2 \right\|_2 + \left\| u_i^1 - \omega_{ih}^1 \right\| + \left\| u_i^0 - u_{ih}^0 \right\| + \left\| u_i^2 - u_{2h}^0 \right\|_2
\]
So in both cases
\[
\left\| u_i^2 - u_{ih}^2 \right\| \leq \left\| u_i^2 - \omega_{ih}^2 \right\| + \left\| u_i^2 - u_{2h}^2 \right\|_2 + \left\| u_i^1 - \omega_{ih}^1 \right\| + \left\| u_i^0 - u_{ih}^0 \right\| + \left\| u_i^2 - u_{2h}^0 \right\|_2
\] (30)
or
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\[ |u^2_i - u_{ih}^2| \leq \sum_{i=0}^{2} |u^i_1 - \omega_{ih}^i| + \sum_{i=0}^{1} |u^i_2 - \omega_{2h}^i| \tag{31} \]

On the other hand

\[ |u^2_1 - u_{2h}^2| \leq |u^2_2 - \omega_{2h}^2| + |\omega_{2h}^2 - u_{2h}^2| \]

\[ \leq |u^2_2 - \omega_{2h}^2| + \max \left\{ \frac{k}{\beta} |u^1_1 - u_{2h}^1|, |u^1_2 - u_{2h}^2| \right\} \]

Case 1.

\[ |u^1_2 - u_{2h}^1| \leq |u^2_2 - \omega_{2h}^2| + \frac{k}{\beta} |u^1_2 - u_{2h}^1| \]

\[ \leq |u^2_2 - \omega_{2h}^2| + |u^1_2 - u_{2h}^1| \]

\[ \leq |u^2_2 - \omega_{2h}^2| + |u^1_2 - \omega_{2h}^1| + |u^1_2 - \omega_{2h}^1| + |u^1_0 - u_{1h}^0| + |u^0_2 - u_{2h}^0| \]

Case 2.

\[ |u^2_2 - u_{2h}^2| \leq |u^2_2 - \omega_{2h}^2| + |u^1_2 - u_{2h}^1| \]

\[ \leq |u^2_2 - \omega_{2h}^2| + |u^1_2 - \omega_{2h}^1| + |u^1_2 - \omega_{2h}^1| + |u^1_0 - u_{1h}^0| + |u^0_2 - u_{2h}^0| \]

So in both cases

\[ |u^2_2 - u_{2h}^2| \leq |u^2_2 - \omega_{2h}^2| + |u^1_2 - \omega_{2h}^1| + |u^1_2 - \omega_{2h}^1| + |u^1_0 - u_{1h}^0| + |u^0_2 - u_{2h}^0| \]

or

\[ |u^2_2 - u_{2h}^2| \leq \sum_{i=0}^{2} |u^i_1 - \omega_{ih}^i| + \sum_{i=0}^{2} |u^i_2 - \omega_{2h}^i| \tag{32} \]

Now assume that

\[ |u^1_i - u_{ih}^i| \leq \sum_{i=0}^{n} |u^i_1 - \omega_{ih}^i| + \sum_{i=0}^{n-1} |u^i_2 - \omega_{2h}^i| \]

and

\[ |u^2_i - u_{2h}^i| \leq \sum_{i=0}^{n} |u^i_1 - \omega_{ih}^i| + \sum_{i=0}^{n} |u^i_2 - \omega_{2h}^i| \]

and let us prove that

\[ |u^{n+1}_1 - u_{ih}^{n+1}| \leq \sum_{i=0}^{n+1} |u^i_1 - \omega_{ih}^i| + \sum_{i=0}^{n} |u^i_2 - \omega_{2h}^i| \]

and

\[ |u^{n+1}_2 - u_{2h}^{n+1}| \leq \sum_{i=0}^{n+1} |u^i_1 - \omega_{ih}^i| + \sum_{i=0}^{n} |u^i_2 - \omega_{2h}^i| \]
Indeed, we have

\[
\|u_1^{n+1} - u_{1h}^{n+1}\| \leq \|u_1^{n+1} - \omega_h^{n+1}\| + \|\omega_h^{n+1} - u_{1h}^{n+1}\|
\]

\[
\leq \|u_1^{n+1} - \omega_h^{n+1}\| + \max \left\{ \frac{k}{\beta} \|u_1^n - u_{1h}^n\|, \|\pi_h u_2^n - \pi_h u_{2h}^n\| \right\}
\]

\[
\leq \|u_1^{n+1} - \omega_h^{n+1}\| + \max \left\{ \frac{k}{\beta} \|u_1^n - u_{1h}^n\|, \|u_2^n - u_{2h}^n\| \right\}
\]

and, as above, we need to distinguish between two cases:

1: \[
\max \left\{ \frac{k}{\beta} \|u_1^n - u_{1h}^n\|, \|u_2^n - u_{2h}^n\| \right\} = \frac{k}{\beta} \|u_1^n - u_{1h}^n\|
\]

and

2: \[
\max \left\{ \frac{k}{\beta} \|u_1^n - u_{1h}^n\|, \|u_2^n - u_{2h}^n\| \right\} = \|u_2^n - u_{2h}^n\|
\]

**Case 1.** implies that

\[
\|u_1^{n+1} - u_{1h}^{n+1}\| \leq \|u_1^{n+1} - \omega_h^{n+1}\| + \frac{k}{\beta} \|u_1^n - u_{1h}^n\|
\]

\[
\leq \|u_1^{n+1} - \omega_h^{n+1}\| + \|u_1^n - u_{1h}^n\|
\]

\[
\leq \|u_1^{n+1} - \omega_h^{n+1}\| + \sum_{i=0}^n \|u_i^j - \omega_h^i\| + \sum_{i=0}^n \|u_2^j - \omega_2h^j\|
\]

while **Case 2.** implies that

\[
\|u_1^{n+1} - u_{1h}^{n+1}\| \leq \|u_1^{n+1} - \omega_h^{n+1}\| + \|u_2^n - u_{2h}^n\|
\]

\[
\leq \|u_1^{n+1} - \omega_h^{n+1}\| + \sum_{i=0}^n \|u_i^j - \omega_h^i\| + \sum_{i=0}^n \|u_2^j - \omega_2h^j\|
\]

So in both cases

\[
\|u_1^{n+1} - u_{1h}^{n+1}\| \leq \|u_1^{n+1} - \omega_h^{n+1}\| + \sum_{i=0}^n \|u_i^j - \omega_h^i\| + \sum_{i=0}^n \|u_2^j - \omega_2h^j\|
\]

Likewise

\[
\|u_2^{n+1} - u_{2h}^{n+1}\| \leq \|u_2^{n+1} - \omega_2h^{n+1}\| + \|u_2^{n+1} - u_{2h}^{n+1}\|
\]

\[
\leq \|u_2^{n+1} - \omega_2h^{n+1}\| + \max \left\{ \frac{k}{\beta} \|u_2^n - u_{2h}^n\|, \|\pi_h u_1^n - \pi_h u_{2h}^n\| \right\}
\]

\[
\leq \|u_2^{n+1} - \omega_2h^{n+1}\| + \max \left\{ \frac{k}{\beta} \|u_2^n - u_{2h}^n\|, \|u_1^n - u_{1h}^n\| \right\}
\]

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Here also we need to discuss two cases:

Case 1: \[
\max \left\{ \frac{k}{\beta} \left\| u_2^n - u_{2h}^n \right\|_2, \left\| u_4^{n+1} - u_{4h}^{n+1} \right\|_2 \right\} = \frac{k}{\beta} \left\| u_2^n - u_{2h}^n \right\|_2
\]
implies
\[
\left\| u_2^{n+1} - u_{2h}^{n+1} \right\|_2 \leq \left\| u_2^n - \omega_{2h}^{n+1} \right\|_2 + \left\| u_2^n - u_{2h}^n \right\|_2
\]
\[
\leq \left\| u_2^{n+1} - \omega_{2h}^{n+1} \right\|_2 + \sum_{i=0}^{n} \left\| u_i^i - \omega_{ih}^i \right\|_2 + \sum_{i=0}^{n} \left\| u_i^i - \omega_{2h}^i \right\|_2
\]

Case 2: \[
\max \left\{ \frac{k}{\beta} \left\| u_2^n - u_{2h}^n \right\|_2, \left\| u_4^{n+1} - u_{4h}^{n+1} \right\|_2 \right\} = \left\| u_1^{n+1} - u_{1h}^{n+1} \right\|_2
\]
implies
\[
\left\| u_2^{n+1} - u_{2h}^{n+1} \right\|_2 \leq \left\| u_2^n - \omega_{2h}^{n+1} \right\|_2 + \left\| u_1^{n+1} - u_{1h}^{n+1} \right\|_2
\]
\[
\leq \left\| u_2^{n+1} - \omega_{2h}^{n+1} \right\|_2 + \sum_{i=0}^{n} \left\| u_i^i - \omega_{ih}^i \right\|_2 + \sum_{i=0}^{n} \left\| u_i^i - \omega_{2h}^i \right\|_2
\]

Then, in both cases, we have
\[
\left\| u_2^{n+1} - u_{2h}^{n+1} \right\|_2 \leq \left\| u_2^n - \omega_{2h}^{n+1} \right\|_2 + \sum_{i=0}^{n} \left\| u_i^i - \omega_{ih}^i \right\|_2 + \sum_{i=0}^{n} \left\| u_i^i - \omega_{2h}^i \right\|_2
\]
which completes the proof.

Theorem 4. There exists \( h_n > 0 \) with \( \lim_{n \to \infty} h_n = 0 \), such that
\[
\lim_{n \to \infty} \left\| u_1^i - u_{1h}^i \right\|_{L^\infty(\Omega)} = 0, \quad i = 1, 2
\]

Proof. Let us give the proof for \( i=1 \), the case \( i=2 \) being similar.

Indeed, as
\[
\left\| u_1^n - u_{1h}^n \right\| \leq \sum_{i=0}^{n} \left\| u_i^i - \omega_{ih}^i \right\|_2 + \sum_{i=0}^{n} \left\| u_i^i - \omega_{2h}^i \right\|_2
\]
and
\[
\left\| u_1^n - \omega_{1h}^n \right\| \leq Ch^2 |\ln h|
\]
\[
\left\| u_2^n - \omega_{2h}^n \right\|_2 \leq Ch^2 |\ln h|
\]

Then,
\[
\left\| u_1^n - u_{1h}^n \right\| \leq (2n + 1)Ch^2 |\ln h|
\] (33)
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Also
\[ \|u_i^n - u_{ih}^n\| \leq \|u_{i}^n - u_{ih}^n\| + \|u_{i}^n - u_i^n\| \]

Let \( \varepsilon > 0 \). Theorem 3 implies that there exists \( n_0 \in N \) such that, \( \forall \ n \geq n_0 \)
\[ \|u_i^n - u_i^n\| \leq \frac{\varepsilon}{2} \]

Taking account of (33), the Theorem follows by choosing \( h_n > 0 \) such that
\[ h_n^2 |\ln h_n| \leq \frac{\varepsilon}{(2n + 1)C}, \ \forall n \geq n_0 \]

Conclusion

We have proved convergence of the standard finite element approximation for alternating Schwarz procedure in the context of nonmatching grids. Other type of discretizations may also be considered like mixing finite elements and finite differences. Also, the knowledge of a rate of convergence of the Schwarz procedure will enable derivation of error estimate, in each subdomain, between the discrete Schwarz sequence and the exact solution of the nonlinear PDE.

Conflict of interest

The author declares no conflict of interest.

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