Recent advances in opinion propagation dynamics: A 2020 Survey

Hossein Noorazar∗

1Washington State University, Pullman, Washington, United States of America

Abstract

Opinion dynamics have attracted the interest of researchers from different fields. Local interactions among individuals create interesting dynamics for the system as a whole. Such dynamics are important from a variety of perspectives. Group decision making, successful marketing and constructing networks (in which consensus can be reached or prevented) are a few examples of existing or potential applications. The invention of the Internet has made the opinion fusion faster, unilateral, and at a whole different scale. Spread of fake news, propaganda, and election interferences have made it clear there is an essential need to know more about these dynamics.

The emergence of new ideas in the field has accelerated over the last few years. In the first quarter of 2020, at least 50 research papers have emerged, either peer-reviewed and published or on pre-print outlets such as arXiv. In this paper, we summarize these ground-breaking ideas and their fascinating extensions, and introduce newly developed concepts.

Keywords:— Opinion dynamics, social dynamics, social interaction, consensus, polarization

Contents

0 Introduction 1
1 Preliminaries 2
2 Milestones 2
2.1 Continuous opinion space models 2
2.1.1 DeGrootian models 2
2.1.2 Bounded confidence models 2
2.2 Discrete opinion space models 3
2.2.1 Galam model 3
2.2.2 Sznajd model 3
2.2.3 Voter model 4
3 Milestones' extensions 4
3.1 Stubborn agents 4
3.2 Biased agents 4
3.3 Opinion manipulation 4
3.4 Power evolution 5
3.5 Repulsive behavior 5
3.6 Noisy models 6
3.7 Interrelated topics 6
3.8 Expressed vs. private opinions 7
4 Last words 7
5 New questions 8
6 Conclusions 8
7 Acknowledgments 8

0 Introduction

Opinion dynamics studies propagation of opinions in a network through interactions of its agents. Modeling opinion dynamics goes back a few decades. French in 1956 was among the first researchers to devote attention to opinion dynamics; A Formal Theory of Social Power [1]. Almost two decades later, in 1974, DeGroot established one of the most simplest models [2], which has become one of the most well known.

∗hnoorazar@math.wsu.edu
Opinion dynamics take different shapes depending on the nature of the topic under consideration and the purpose of interactions. For example, the topic could be liking or disliking a certain food such as fish; here, binary opinion dynamics comes into play [3-5]. Sometimes the opinion can be represented by continuous variables. For example, the extent to which one supports a cause. Over the years, a few different models for continuous-opinions have been proposed [1,6,7]. Either as an abstract idea or in real life, one can think of a continuous-opinion space in which one must take discrete actions [8,9]. For instance, in an election where each agent’s support for candidates falls on the continuous spectrum, each agent must still cast a discrete vote.

In short, opinion dynamics can be explained as follows. Agents start with an initial opinion. Connected agents interact and update their opinion by a given clear update rule. This process is carried on until a termination criteria is met. An example of termination criteria is reaching a steady state in which agents do not change their opinion anymore.

In this paper, we briefly introduce the main concepts and newly developed ideas of modeling opinion dynamics. In Sec. 2 the major models are presented, then, in Sec. 3 basic results and certain extensions of the well-known models are reviewed. In Sec. 4, we go through some models that have not been studied exhaustively, but are interesting and have contributed novel concepts. Finally, the conclusions are presented in Sec. 6.

1 Preliminaries

The network of agents is denoted by G in which N agents are present. Let A be the adjacency matrix where A_{ij} = 1 if agents i and j are connected and A_{ij} = 0 otherwise. The row-stochastic influence matrix is denoted by W where W_{ij} is the level of influence of agent j on agent i; 0 ≤ W_{ij} ≤ 1.

Let us define the opinion space to be the set of all possible opinions denoted by O. Examples of opinion space are O = {0, 1}, {1, 2, . . . , m}, [0, 1]. The opinion of agent i at time t is denoted by o_{i}^{(t)}. The state of the system at time t is denoted by o^{(t)} = [o_{1}^{(t)}, o_{2}^{(t)}, . . . , o_{N}^{(t)}].

When the system reaches an equilibrium, we say the system has converged. The convergence state may be consensus, polarization or fragmentation. Polarization is the state in which there are only two clusters of agents, and fragmentation is the state there are more than two clusters.

Before moving to the next section, we would like to mention that there is no convergence on terminology in the literature. For example, consider an agent that does not change its opinion over time; some papers refer to such agent as a leader [10], others as a media [11], some as a stubborn agent [12] and a few as an inflexible agent [13,14]. A closed-minded agent is referred to someone who does not change its opinion in [15] and in some papers, a closed-minded agent is an agent whose confidence radius is small compared to other agents [16]. We adhere to the following definitions. An agent that does not change its opinion over time is referred to as a fully-stubborn agent. An agent that weighs its initial opinion, i.e., takes its initial opinion into account, in all interactions over time is referred to as a partially-stubborn agent. If an agent is not stubborn, it is called a non-stubborn agent. In the bounded confidence models, we say an agent is closed-minded if its confidence radius is smaller than that of others.

2 Milestones

In this section we present the main models that have inspired a tremendous amount of research. We start with the models in which the opinion space is continuous and then present the models in which the opinion space is discrete.

2.1 Continuous opinion space models

In this section, we present the DeGroot model and its major extension known as the Friedkin-Johnsen model. Next, we examine the bounded confidence models in which agents interact with those whose opinion are close enough to that of their own.

2.1.1 DeGrootian models

We begin with the simplest model, the DeGroot model. Moreover, since the Friedkin-Johnsen extension of the DeGroot model is well-known and has been studied extensively, we present it here as well.

\[ o^{(t)} = W o^{(t-1)} = W^2 o^{(t-2)} = \cdots = W^t o^{(0)} \] (1)

where W is a row-stochastic weight matrix and W^t is its t-th power. The model is linear and traceable over time. Classical linear algebra tools are sufficient to analyze this model. It has been very well studied and different extensions of it exist. The DeGroot model is an iterative averaging model. If the network is connected then convergence is equivalent to consensus (Fig. 1). Berger [17] showed the DeGroot model will reach consensus if and only if there exists a power t of the weight matrix for which W^t has a strictly positive column. Let W be such a matrix; then, the consensus opinion is given by o^* = \langle \ell, o^{(0)} \rangle where \ell^t_W is the left eigenvector of W associated with 1, constrained to (1_N, \ell^t_A) = 1.

Friedkin-Johnsen model. One of the major extensions of the DeGroot model was introduced by Friedkin and Johnsen [18,19] and is known as the Friedkin-Johnsen (FJ) model. Since it has functioned as a ground-breaking model, we include it here instead of in Sec. 3. In the FJ model, the idea of stubborn-agents is added to the DeGroot model:

\[ o^{(t+1)} = D W o^{(t)} + (I - D) o^{(0)} \] (2)

where D = diag([d_1, d_2, . . . , d_N]) with entries that specify the susceptibility of individual agents to influence, i.e., (1 - d_i) is the level of stubbornness of agent i. For a fully-stubborn agent 1 - d_i = 1, for a partially-stubborn agent 0 < 1 - d_i < 1 and for a non-stubborn agent 1 - d_i = 0. W is a row stochastic influence matrix. The convergence and stability of the FJ model are studied in [20].

These two models are the main two DeGrootian models. We now move on to the bounded confidence models.

2.1.2 Bounded confidence models

In this section, we look at bounded confidence models. A bounded confidence model (BCM) is a model in which agents ignore the ideas that are too far from their own. The well-known pairwise BC model is given by Deffuant et. al. [6] and is called the DW model, while the most well-known synchronous version is given by Hegselmann and Krause [21], and is called the IH model.
In another scenario, let each agent can interact with any other agent at all times, there will be no interaction, and thus, there will be $O$. Intuitively and in actuality, the confidence radius affects the number of clusters at equilibrium. Consider the opinion space $O = [0, 1]$ and confidence radius $r$ to be 1. Then, each agent can interact with any other agent at all times, and the subspace in which agents lie in, will be contractive. In another scenario, let $r < m = \min_{i,j} \{|o_i^{(0)} - o_j^{(0)}|\}$; then, there will be no interaction, and thus, there will be $N$ clusters of size 1. In fact, Fortunato [23] claims $r = 0.5$ is the critical confidence radius above which the agents come to a consensus. Moreover, the number of clusters at equilibrium is approximately $1/2r$ [24]. Lorenz [16] investigates a heterogeneous (in confidence radius) case in which there are two groups of agents. One group is closed-minded, i.e., has a smaller confidence radius, and the other group is open-minded. It is shown in [16] that heterogeneity of confidence radius helps consensus to be reached; the final state will be consensus when the confidence radius of the open-minded group is well below the aforementioned $r = 0.5$ for the homogeneous case. Chen et. al. [25] investigate convergence properties of a heterogeneous (in confidence interval) DW model. An asymmetric DW model is discussed in [26]. Shang [27] has proposed a modified DW model where confidence radius is assigned to edges, as opposed to agents. Equivalently, agent $i$ trusts agent $j$ and $k$ differently; the convergence properties of such a model are studied by Shang [27]. Another work that studies the convergence properties of a modified DW is [28], in which the learning rate is a function of opinion difference.

**Deffuant-Weisbuch model.** The celebrated DW model of Deffuant et. al. [6] is defined by the following rule:

$$
\begin{align*}
o_i^{(t+1)} &= o_i^{(t)} + \mu \cdot (o_i^{(t)} - o_j^{(t)}) \\
o_j^{(t+1)} &= o_j^{(t)} + \mu \cdot (o_i^{(t)} - o_j^{(t)})
\end{align*}
$$

(DW)

where $\mu$ is the so-called learning rate that usually lies in $(0,0.5)$ to avoid crossover. The update takes place only if $|o_i^{(t)} - o_j^{(t)}| \leq r$ where $r$ is called confidence radius. In [6] all agents share the same confidence radius $r$ and the same learning rate $\mu$. Said differently, the system is homogeneous in both $r$ and $\mu$. Obvious variations can be achieved by introducing a heterogeneous confidence radius to the system, adding asymmetry in the confidence radius, or even an agent-specific time-varying confidence radius [22]. Intuitively and in actuality, the confidence radius affects the number of clusters at equilibrium. Reference [21] includes the analyses of convergence and consensus for the HK model. Bhattacharyya et. al. [29] study convergence properties of a multidimensional HK model.

**2.2 Discrete opinion space models**

In this section, we focus on opinion models whose opinion space is discrete. Variations of the Ising model provide examples with binary opinion space. A discretized version of DW [30] is another example. These models have applications in real life. There are at least two studies using binary opinion space to explain Trump’s 2016 victory [31,32].

**2.2.1 Galam model**

In addition to Friedkin, who has left a large footprint in this field since 1986 [33], Galam has spent more than 35 years studying opinion dynamics from a sociophysics perspective [34,35] and his work has inspired other researchers. He has studied a range of different dynamics [36] including “democratic voting in bottom up hierarchical systems, decision making, fragmentation versus coalitions, terrorism and opinion dynamics.” Reference [36] reviews Galam’s work prior to 2008 and further details can be found in his book [37]. Here, we introduce some of the newer works related to the binary opinion space used in the Galam model.

In the Galam model, there are two opinions in the opinion space. The update rule is as follows; (1) agents are randomly distributed in groups of size $r$, (2) each group uses majority rule to update their opinion, then (3) agents are shuffled and the cycle begins again at step (1).

Gärtner and Zehmakan [38] address consensus time and sensitivity of outcome as functions of initial state in the Galam model.

**2.2.2 Sznajd model**

Ising models have a long history in statistical physics. Here, we overview some of the well-known models of this kind in the field of opinion dynamics, namely the Sznajd model [39]. In the Sznajd model, $N$ agents are sitting on a 1-dimensional lattice. Opinion space is given by $O = \{-1, +1\}$. At a given time $t$, two neighbors $i$ and $i + 1$ are selected randomly. If $o_i^{(t)} \times o_{i+1}^{(t)} = 1$ then agents $i - 1$ and $i + 2$ adopt the direction of agents $i$ and $i + 1$, otherwise, the agent $i - 1$ adopts the opinion of agent $i$ and agent $i + 2$ adopts the opinion of its selected neighbor, agent $i + 1$. Steady states of such a model
have all agents in agreement at either +1 or -1 or a stalemate. The time needed to reach equilibrium is discussed in [39] through Monte Carlo simulations. Some results from the original Sznajd model and the Sznajd model on a complete graph are presented in [40].

Phase transition phenomena in the Sznajd model with the presence of anticonformists in complete graphs are examined by [41, 42]. Calvelli et. al. [42] also consider 2D and 3D lattices. To learn more about the Sznajd model please see [43, 44].

2.2.3 Voter model

In a voter model, opinion space is binary. At a given time $t$, a random agent, $i$, is chosen. Then, $i$ chooses a random neighbor and adopts the state of the neighbor.

The voter model on regular lattices has been studied extensively. There are also variations of the voter model on different network topologies. Sood and Redner and [45] investigate the voter model on a heterogeneous graph, and [46] explore the dynamics and convergence time of the voter model on a graph with two cliques. The influence of an external source is investigated in [47]. The impact of “active links” on the convergence of the voter model is investigated in [48]. To learn more about voter models please see [49]. Examples of the other extensions are given below.

3 Milestones’ extensions

We are ready to investigate some of the fascinating extensions of the reviewed models in the previous section.

3.1 Stubborn agents

Stubborn agents in DeGroot model. One major modification of the DeGroot model known as the FJ model adds stubbornness and was presented previously. However, here we introduce other versions that are new and have not studied extensively. Abrahamsson et. al. [12] study the effect of the presence of fully-stubborn agents in the DeGroot model. Wai et. al. [50] propose “an active sensing method to estimate the relative weight (or trust) agents place on their neighbors” and explore the role of stubborn agents in such an environment. Zhou et. al. [51] study the effect of partially-stubborn agents on a modified DeGroot model. In their altered model, an agent not only takes the opinions of its neighbors into account but also takes the opinions of its neighbors’ neighbors into account as well.

Stubborn agents in DW and HK. The effect of Stubborn agents in DW and HK is explored in [52] and [11], respectively. In the latter, stubborn agents are labeled as media. They investigate “how the number of media accounts and the number of followers per media account affect the media impact.” Moreover, one of their novel contributions is “content quality.”

Stubborn agents in Galam model. References [13, 53] study the effect of inflexible agents in the Galam model. Another work [54] is an extension of the Galam model in which they study how the minority wins against the majority in scenarios such as the US election in 2016. The model given in [55] also has the stubbornness ingredient where stubborn agents are called leaders and their power of influence is discussed. Cheon and Morimoto [56] consider a Galam model that includes balancer agents who oppose stubborn agents. Contrarian agents are specific to the Galam model, hence, we include them here. Galam and Cheon [57] investigate the effect of asymmetry in contrarian behavior which is an extension of [13].

Stubborn agents in voter model. References [58] and [59] explore the role of stubborn agents in a voter model and a noisy voter model, respectively. Yıldız et. al. [60] examine the effect of stubborn agents with opposing views on the convergence of the system. Mukhopadhyay et. al. [3] investigates the effect of biased agents in both voter and majority-rule dynamics. They also add stubbornness to the majority-rule case. The bias and stubbornness are implemented by updating probability. They study the relationship between the size of the network and (1) consensus time, and (2) probability of consensus.

3.2 Biased agents

Biased agents are more open to agents’ that hold similar opinions to themselves as opposed to others, i.e., the homophily quality.

Biased agents in DeGroot model. Dandekar et al. [61] incorporate the idea of biased agents in the DeGroot model and turn it into a nonlinear model. Xia et. al. [62] provide some analysis for equilibria in such a model.

Biased agents in DW model. In the DW model a pair of agents is chosen randomly. Sîrbu et. al. [63] modify the DW model to add the bias ingredient. In this model, agent $i$ is chosen randomly and then the interaction partner $j$ is chosen by a probability function that depends on the difference of opinion. The closer the opinion of agent $j$ is to the opinion of $i$, the more the probability of interaction between them. Convergence properties and network size effects are addressed in [63].

Biased agents in HK model. Chen et. al. [64] take the modified HK model of Fu et. al. [65] and extend it to include biased agents. They call the new model the “Social-Similarity-Based HK model.” In this model, for two agents to interact not only do they need to hold close opinions but the criteria of social similarity also must be met, i.e., their other attributes need to be close as well. Social similarity can be measured by considering different attributes such as age, education, and other traits.

3.3 Opinion manipulation

Opinion manipulation is fascinating for different reasons. Maximizing the number of customers in a market or interfering with another country’s election are two examples of opinion manipulation. Below, we review some of the proposed models.

Opinion manipulation in DeGroot model. There are a few works about how to make a network reach a consensus and further still, how to influence the agents toward a predetermined opinion [66–68]. Dong et al. [66] propose a network modification, i.e., adding a minimum number of edges to the network, to reach consensus in the DeGroot model. Zhou et. al. [51] consider the manipulation of public opinion in a modified DeGroot model.

Opinion manipulation in FJ model. To the best of our knowledge, there is no study to influence agents toward consensus in the FJ model. Previously, a missing piece for manipulation of agents in models was preventing a network from reaching a consensus. A very recent study, Gaitonde et al. [69], investigates adversarial manipulation of a network to prevent it from consensus where the dynamics are governed by the FJ model.
Opinion manipulation in DW model. Pineda and Buendia [67] investigate the effect of mass media in both DW and HK models. They consider heterogeneous (in confidence radius) cases for both DW and HK and study conditions under which the effect of mass media is maximized. Another example for affecting the network’s opinion is presented in [70].

Opinion manipulation in HK model. Standard tools in linear algebra enable one to understand the dynamics of the DeGroot model. Such results help to manipulate the opinion of the agents by modifying the topology of the network [66]. However, this is not the only way to manipulate the network’s final state. Brooks and Porter [11] use media to manipulate the outcome of the discussion in a network; “We maximize media impact in a social network by tuning the number of media accounts that promote the content and the number of followers of the accounts.”

Opinion manipulation in voter model. Gupta et al. [71] propose strategies for manipulating the agents’ opinion in a voter model. The influence maximization on a complex network is given in [72], and influence maximization by considering the agents’ power of influence, the Influence Power-based Opinion Framework, is proposed in [73].

More on opinion manipulation. We can mention Refs. [74–77] as other examples of opinion manipulation. Goyal and Manjunath [77] build on [78] and investigate a scenario in which two competing forces try to gain control of the network and maximize the number of their followers. Each “controller” has a budget constraint and Nash control strategies are determined for each controller. Brede [79] investigates a rewiring model in which influencers try to maximize their impact.

3.4 Power evolution

We mentioned two properties of the final state in the DeGroot model. If the network of agents is connected, i.e., the network does not consist of disjoint subgraphs, then the final state is consensus and the consensus value is a weighted average of the initial opinions. These two properties were motivated for the introduction of power evolution in the DeGroot model.

Power evolution in DeGroot model. Jia et al. [80] study the evolution of power in a network. In the work of Jia et al. [80], power of influence of agents evolves over a sequence of topics where the dynamic of each topic discussion is governed by the DeGroot model. Suppose the network discusses a sequence of topics \( s = 0, 1, 2, \cdots \) one after another where the dynamic of each discussion is governed by the DeGroot model. The weight matrix for each topic depends on the outcome of the previous topic:

\[
o^{(t+1)}(s) = W(s) o^{(t)}(s) \quad (3)
\]

In Eq. 3 the weight matrix \( W(s) \) depends on the outcome of the topic \( s-1 \). This model is known as the DeGroot–Friedkin model. The relation between social power and centrality ranking is established in [80]. Moreover, the conditions under which a democratic or autocratic structure are formed is discussed.

Kang et al. [81] use a two-layer network to explore the evolution of social power in the DeGroot–Friedkin model. Convergence properties of such a model are provided. The weight matrix \( W(s) \) is decomposed into a sum of two matrices \( W(s) = D(s) + (I_n - D(s))C \) where \( C \) is called a relative influence matrix. \( C \) is row-stochastic, irreducible with a zero diagonal. In the case of [81], there are two relative influence matrices, one for each layer. It is shown in [80] that for the DeGroot–Friedkin model, a democratic configuration will be reached if and only if the relative matrix \( C \) is doubly-stochastic. In the case of a two-layer network of [81], the democratic configuration will be reached if both of the influence matrices are doubly-stochastic. They both have similar results for emerging autocratic configuration under star topology.

The evolution of individuals’ power is further studied in [82–88]. Ye and Anderson [84] extended the DeGroot–Friedkin model by adding new characteristics to agents; “humbleness” and “unreactiveness.” In this extension, power evolution of agents is “distorted” by these new characteristics. These researchers study the existence and uniqueness of equilibria, and convergence if it exists. Askarzadeh et al. [87] study the power evolution of the DeGroot–Friedkin model using probability and Markov chain theory.

Power evolution in FJ model. Tian et al. [88] study power evolution in the FJ model, including the properties of equilibria and the conditions under which democracy is achieved. Furthermore, it is shown that autocracy cannot be achieved in the presence of stubborn agents.

We mentioned earlier that stubborn agents are also called media or leaders. The reason is that they can influence other agents and have a major impact on the final state of the system. Equivalently, stubbornness translates into social power. This fact is not only observed in the FJ model, but also in the DeGroot model (e.g. [89]).

3.5 Repulsive behavior

The models we observed so far only support two types of behavior–attraction or indifference. Humans are more complicated. If the topic is sensitive then repulsive behavior emerges and causes polarization or fragmentation. Let us look at models that support such a behavior.

Repulsion in the DeGroot model. Chen [90] adds a repulsive behavior to the model of Dandekar et al. [61]. The model proposed in [90] uses a single parameter–entrainment parameter–to capture both bias and backfire effect. Their model also supports polarization that previously did not exist in the original DeGroot model.

Repulsion in DW model. Repulsive behavior in a modified DW model is discussed in [91–93]. The model proposed in [7] is based on minimizing interaction energy between agents. The interaction energy is defined via potential functions. The update rule in [7] is given by:

\[
\begin{align*}
\delta i(t) & = \frac{1}{2} \psi'(o_i(t) - o_j(t)) (o_i(t) - o_j(t)) \\
\delta j(t) & = \frac{1}{2} \psi'(o_i(t) - o_j(t)) (o_i(t) - o_j(t)) 
\end{align*}
\]

(4)

With the proper choice of a potential function, this model collapses to the DW model (see Fig. 2a), or the model of Jager and Amblard [93] (see Fig. 2b). Using the potential function in Fig. 2b, the model will support three types of behavior: attraction, indifference, and repulsion. The potential function given in Fig. 2c produces a modified DW model with repulsive behavior. In these three potential functions the confidence radius is \( r = 0.3 \).

More on repulsive behavior. References [94–103] consider signed graphs and the concept of balance theory [104] in their work of modeling antagonistic or repulsive behavior. In a signed graph each edge is labeled with a positive or negative sign, defining friendship or antagonistic
relationships. Aghbolagh et. al. [103] has implemented three types of behavior—attraction, indifference and repulsion—in a modified HK model. They show their new model can lead to consensus, bipartite consensus, and clustering of opinions.

### 3.6 Noisy models

Noise is injected into models for different purposes. For instance, Mäs [105] uses noise to implement the idea of the tendency for uniqueness in the model of Durkheim [106]. The strength of noise in this model increases as the clusters grow in size. Other forms of noise are implemented to model different traits of humans behavior. Noise can be used to model the death or birth of an agent, and to mimic internal thoughts or interactions with external sources such as media or books. Below we review some of the noisy models.

**Uncertainty in DeGroot model.** A modified DeGroot model, taking into account uncertainty of agents encoded as intervals, is given in [91].

**Noise in DW model.** One can argue humans do not have a sharp threshold like a confidence radius for accepting or rejecting other ideas. Grauwin and Jensen [107] use random noise in the DW model to kill the aforementioned sharp threshold. In [107], two agents interact with a probability that depends on the difference of opinion of the two agents—an interaction noise. Another type of noise introduced in [107] is reminiscent of the death of a person and birth of another; an agent changes its opinion at time $t$ to a random opinion with some probability. Pineda et. al. [108] investigate another type of noise in the DW model: “Individuals are given the opportunity to change their opinion, with a given probability, to a randomly selected opinion inside an interval centered around the present opinion.” References [109–113] also investigate the noise effect in models inspired by the DW model.

**Noise in HK model.** Su et. al. [114] introduce noise to the homogeneous HK model and show how it can help the formation of consensus. A recent study investigates the role of environment and communication noise in the heterogeneous HK model [115]. Phase transition and convergence time are studied in the presence of environmental noise in [115]. Another example of noise in the HK model is discussed in [116]. Nonlinear stability for the HK model in the presence of noise is addressed in [117]. A modification of the HK model uncertainty of agents. In this model, some agents may have an opinion that is actually an interval, not a single number.

**Noise in Galam model.** Hamann [119] explores noise in a modified Galam model in a group of mobile agents with the presence of contrarians.

**Noise in SznaJD model.** Sabatelli and Richmond [120] add noise to a modified Sznajd model where the updates are done in a synchronous fashion. One of their results in that they “predict that consensus can be increased by the addition of an appropriate amount of random noise.”

**Noise in voter model.** One of the earliest noisy voter model is [121], which employs standard statistical physics techniques and examines the critical behavior of the system and its phase transition. References [122,123] examine the role of noise in the voter model on complex networks. The role of “zealots”, i.e., fully-stubborn agents, in a noisy voter model is inspected in [59] where agents form a fully connected graph.

### 3.7 Interrelated topics

It is rarely the case that a given issue exists in an isolated environment. The change of opinion about one topic could cause a change of opinion about another topic. For instance, a change of opinion about health can lead to a change of opinion about exercise and diet. This area has not yet seen much exploration. Below, we present the limited models of this nature.

**Interrelated topics in FJ model.** In 2016, two independent works [7,124] proposed novel ideas for the dynamics of interrelated (coupled or interdependent) topics. The model in [7] is novel and does not fall under the umbrella of classical models; however, the Ref. [124] is a generalization of the FJ model. Friedkin et. al. [124] revisit the idea of interdependent topics in the multidimensional FJ model in [20]. Tian and Wang [125] introduce the idea of sequentially dependent topics in which each topic is discussed in a sequence and the outcome of topic $s$ affects the dynamics of the discussion of topic $s + 1$ where each topic’s dynamic is governed by the FJ model.

**Interrelated topics in DW model.** Fei et. al. [126] propose a model for interdependent topics where interactions are pairwise and follow the bounded confidence concept.

**More on interrelated topics.** Ahn et. al. [127] propose a novel opinion model with interrelated topics. Let $\ell$ and $s$ to be two given coupled topics. In [7], for example, the interaction between agents is based on a given topic. Suppose agents $i$ and $j$ interact about topic $\ell$ and update their
opinion. Consequently, by internal thoughts due to the coupling of \( \ell \) and \( s \), their opinion about \( s \) will be updated as well, despite the fact that they did not discuss topic \( s \). However, in [127] it is possible that topic \( \ell \) of agent \( i \) is coupled with topic \( s \) of agent \( j \). Other novel models have recently been proposed, such as [128], *Bayesian learning* model [129], a model based on *Achlioptas Process* [130] and a model based on *Latané’s social impact theory* [131], to name a few.

### 3.8 Expressed vs. private opinions

The expressed opinion of agents is not always identical to their true internal belief. Social pressure can cause people to express an opinion that aligns with that of others while contradicting their internally held belief. In this section, we present models that consider the co-evolution of expressed and personal opinions.

**Expressed vs. private opinions in FJ.** The dynamics of the co-evolution of expressed and private opinions (EPO) in the FJ model along with its convergence properties are presented in [132].

**Expressed vs. private opinions in voter model.** References [133–135] explore different ideas about expressed and private opinions in the voter model.

**More on EPO.** Some novel models explore the dynamics of private and expressed opinions [136–138]; while they do not fall under the umbrella of well-known models, they are worth mentioning.

We mentioned agents may reveal an opinion that is different from their internal true belief. In [132] the co-evolution of the two (internal and revealed) opinions are studied. We have also talked about manipulating the agents to influence them toward a predetermined target. The model presented by Afshar and Asadpour [140] is somewhere in between. Their model is inspired by the DW model and includes some *informed agents* who pretend their opinion is close to that of other agents. These informed agents influence other agents toward a predetermined target opinion.

Table 1 provides an overview of the material presented in this paper.

### 4 Last words

Before closing the discussion, we would like to cover other interesting models that have not yet been studied extensively and acknowledge the great efforts of other researchers.

Li et. al. [141] propose an interesting model. Unlike BCMs, agents in their model of [141] interact if the difference of opinion is larger than a threshold due to social pressure. In the model given in [142] there is a potential to interact with agents whose opinions outside of the confidence interval.

Zhang and Hong [143] propose two synchronous versions of BCM in which not all neighbors of agent \( i \) participate in the update of the opinion of agent \( i \). Instead, several neighbors are selected randomly. These researchers are interested in the convergence properties of this model, which sits between the pairwise interaction in the DW model and the synchronous HK model. In this model, there is potential to interact with agents beyond one’s confidence radius. More details about the model are given in [144]. Another interesting work [15] studies convergence of a modified HK model in which agents have *inertia*. References [65, 145, 146] contain other examples of the study of the convergence properties of the HK model and its variations. Gang et. al. [147] investigate the final state of the heterogeneous (in confidence radius) HK model.

Rubio et. al. [148] have recently proposed a model for anomaly detection in the Industrial Internet of Things architectures. Other examples of applications of opinion dynamics in engineering are given in [149] and [150], where the later reference studies voting processes inspired by BCM.

Physics has inspired different models of opinion dynamics, of course. There are several works based on the kinetic theory of gases [151–161] where interactions are defined by Boltzmann type equations. Applications of such models in other fields, such as economics, are found in the book by Pareschi and Toscani [162]. Düring et. al. [163] investigate the presence of leaders and Wang et. al. [164] took the effect of noise into account in these dynamics. Furthermore, the mean-field theory has been employed to explore the landscape of dynamics [165–168]. The Ising model is another tool that can be used when the opinion space is binary [32, 169–171], with applications in areas such as group decision making [39, 172].

Lastly, it is worthwhile mentioning the evolution of agents’ *susceptibility to persuasion*, which is examined in [173, 174]. Edge weights are used to implement the frequency of interaction between agents in [175, 176]. For more details about continuous-opinion-space models we refer the reader to the tutorials in Refs. [177, 178].

| Model       | Objection          | References                  |
|-------------|--------------------|------------------------------|
| DeGroot     | Convergence        | [17]                         |
|             | Stubbornness       | [12, 18, 50, 51]             |
|             | Bias               | [61, 62]                     |
|             | Opinion manipulation | [51, 66–68]                 |
|             | Repulsion          | [90]                         |
|             | Power evolution    | [80–88]                      |
| FJ          | Convergence        | [20]                         |
|             | Opinion manipulation | [69]                      |
|             | Power evolution    | [88]                         |
|             | Interrelated topics| [20, 124, 125]              |
|             | EPO                | [132]                        |
| DW          | Convergence        | [16, 23, 25–28]             |
|             | Stubbornness       | [52]                         |
|             | Bias               | [63]                         |
|             | Opinion manipulation | [67]                      |
|             | Repulsion          | [91, 92]                     |
|             | Interrelated topics| [126]                        |
|             | Noise              | [107–113]                    |
| HK          | Convergence        | [21, 29]                     |
|             | Stubbornness       | [41]                         |
|             | Bias               | [64]                         |
|             | Opinion manipulation | [11]                      |
|             | Noise              | [114–117]                    |
| Galam       | Convergence        | [38]                         |
|             | Stubbornness       | [13, 53–56]                 |
|             | Contrary           | [13, 57]                     |
|             | Noise              | [119]                        |
| Voter       | Stubbornness       | [5, 58–60]                   |
|             | Opinion manipulation | [71–73]                     |
|             | EPO                | [133–135]                    |
|             | Noise              | [59, 121–123]                |

*Table 1: Overview of materials presented in the paper.*
5 New questions

While a great deal of progress has been made in the field, there is still great potential for improvement. Humans do not interact with all their neighbors simultaneously, unlike the DeGroot model. Even for a network of computers that can interact quickly and can follow a clear set of rules, there are physical limitations. Balanced graphs are used to model repulsive behavior based on principles such as: “friend of my friend, is my friend” or “enemy of my enemy is my friend,” which are not always true.

While some of the opinion dynamics models are designed to model a certain trait (e.g., homophily) or are tailored to create an interesting dynamics (e.g., preventing consensus by introduction of noise), these models are not universal. Hence, it would be interesting to shrink the gap between simplicity of theoretical models and complexity of humans’ behavior.

Opinion dynamics could be used to detect fake-news resources on social media. Detecting susceptible individuals who might be attracted to terrorist groups via the Internet is another potential domain of work.

It would be helpful to see more applications of opinion dynamics in real world problems. More specifically, it would be fascinating to take advantage of opinion dynamics models to detect and flag computers or processors sending erroneous or corrupted messages in computer buses.

6 Conclusions

In this paper, we reviewed well-known models of opinion dynamics for both continuous and discrete opinion spaces. In the continuous-opinion-space case, we reviewed the DeGroot model, and one of its major extensions, namely the FJ model. Afterwards, we presented the two major bounded confidence models, namely the DW model and the HK model. In the discrete-opinion-space case, we reviewed the Galam model, the Sznajd model and the voter model. Subsequently, for the selected additional models, reviewed some extensions that added extra ingredients (s) to the original model—stubbornness, bias, repulsive behavior, power evolution, interrelated topics, noise, and expressed and private opinions. Finally, we posed new questions for future explorations.

7 Acknowledgments

We would like to acknowledge the insightful inputs of Mohammad Hossein Namaki that immeasurably helped in the development of this manuscript.

References

[1] Jr. French, J. R. P. A formal theory of social power. Psychological Review, 63(3):181–194, 1956.
[2] Morris DeGroot. Reaching a consensus. Journal of the American Statistical Association, 69(345):118–121, 1974.
[3] Soham Biswas and Parongama Sen. Model of binary opinion dynamics: Coarsening and effect of disorder. Physical Review E-Statistical, Nonlinear, and Soft Matter Physics, 80(2):4–7, 2009.
[4] Fei Ding, Yun Liu, Bo Shen, and Xia-Meng Si. An evolutionary game theory model of binary opinion formation. Physica A: Statistical Mechanics and its Applications, 389(8):1745–1752, 2010.
[5] Arpan Mukhopadhyay, Ravi R. Mazumdar, and Rahul Roy. Opinion dynamics under voter and majority rule models with biased and stubborn agents, 2020.
[6] Guillaume Deffuant, David Neau, Frederic Amblard, and Gérard Weisbuch. Mixing beliefs among interacting agents. Advances in Complex Systems, 03(01n04):87–98, 2000.
[7] Hossein Noorazar, Matthew J. Sottile, and Kevin R. Vieix. An energy-based interaction model for population opinion dynamics with topic coupling. International Journal of Modern Physics C, 29(11):1850115, 2018.
[8] André C. R. Martins. Continuous opinions and discrete actions in opinion dynamics problems. International Journal of Modern Physics C, 19(4):617–624, 2007.
[9] André C. R. Martins. Discrete opinion dynamics with M choices. The European Physical Journal B, 93(1):1, 2020.
[10] Yuhao Yi and Stacy Patterson. Disagreement and polarization in two-party social networks, 2019.
[11] Heather Z. Brooks and Mason A. Porter. A model for the influence of media on the ideology of content in online social networks, 2019.
[12] Olle Abrahamsson, Danyo Danev, and Erik G. Larsson. Opinion dynamics with random actions and a stubborn agent, 2019.
[13] F. Jacobs and S. Galam. Two-opinions-dynamics generated by inflexibles and non-contrarian and contrarian floaters. Advances in Complex Systems, 22(04):1950008, 2019.
[14] Serge Galam and Frans Jacobs. The role of inflexible minorities in the breaking of democratic opinion dynamics. Physica A: Statistical Mechanics and its Applications, 381:366 – 376, 2007.
[15] Bernard Chazelle and Chu Wang. Inertial hegselmann-krause systems. IEEE Transactions on Automatic Control, 62(8):3905–3913, Aug, 2017.
[16] Jan Lorenz. Heterogeneous bounds of confidence: Meet, discuss and find consensus! Complexity, 15(4):43–52, Apr, 2010.
[17] Roger L. Berger. A necessary and sufficient condition for reaching a consensus using degroot’s method. Journal of the American Statistical Association, 76(374):415–418, 1981.
[18] Noah Friedkin and Eugene Johnsen. Social influence and opinions. Journal of Mathematical Sociology - J MATH SOCIOL, 15:193–206, Jan. 1990.
[19] Noah Friedkin and Eugene Johnsen. Social Influence Networks and Opinion Change Models of opinion formation. Advances in Group Processes, 16:1–29, Jan. 1999.
[20] S. E. Parsegov, A. V. Proskurnikov, R. Tempo, and N. E. Friedkin. Novel multidimensional models of opinion dynamics in social networks. IEEE Transactions on Automatic Control, 62(5):2270–2285, May 2017.
[21] Rainer Hegselmann and Ulrich Krause. Opinion dynamics and bounded confidence: Models, analysis and simulation. *Journal of Artificial Societies and Social Simulation*, 5(3), 2002.

[22] Pawel Sobkowicz. Extremism without extremists: Diffuuent model with emotions. *Frontiers in Physics*, 3:17, 2015.

[23] Santo Fortunato. Universality of the threshold for complete consensus for the opinion dynamics of deffuant et al. *International Journal of Modern Physics C*, 15(09):1301–1307, 2004.

[24] Claudio Castellano, Santo Fortunato, and Vittorio Loreto. Statistical physics of social dynamics. *Reviews of modern physics*, 81(2):591, 2009.

[25] Ge Chen, Wei Su, Wenjun Mei, and Francesco Bullo. Convergence Properties of the Heterogeneous Diffuuent-Weibsch Model. arXiv:1901.02092, 2019.

[26] Jiangbo Zhang and Ge Chen. Convergence rate of the asymmetric diffuuent-weibsch dynamics. *Journal of Systems Science and Complexity*, 28(4):773–787, 2015.

[27] Yilun Shang. An agent based model for opinion dynamics with random confidence threshold. *Communications in Nonlinear Science and Numerical Simulation*, 19(10):3766–3777, 2014.

[28] Changwei Huang, Qionglin Dai, Wenchen Han, Yuee Feng, Hongyan Cheng, and Haihong Li. Effects of heterogeneous convergence rate on consensus in opinion dynamics. *Physica A: Statistical Mechanics and its Applications*, 499:428–435, Jun. 2018.

[29] Arnab Bhattacharyya, Mark Braverman, Bernard Chazelle, and Huy L. Nguyen. On the convergence of the hegelmann-krause system. In *Proceedings of the 4th Conference on Innovations in Theoretical Computer Science, ITCS ’13*, page 61–66, New York, NY, USA, 2013. Association for Computing Machinery.

[30] Dietrich Stauffer, A. O. Sousa, and C. Schulze. Discretized opinion dynamics of Diffuuent on scale-free networks. *Journal of Artificial Societies and Social Simulation*, 7(3):21, 2003.

[31] Serge Galam. The Trump phenomenon: An explanation from sociophysics. *International Journal of Modern Physics B*, 31(10), Apr. 2017.

[32] Soumyajoyti Biswas and Parongama Sen. Critical noise can make the minority candidate win: The U.S. presidential election cases. *Physical Review E*, 96(3), Sep. 2017.

[33] Noah E. Friedkin. A formal theory of social power. *The Journal of Mathematical Sociology*, 12(2):103–126, 1986.

[34] Serge Galam, Yuval Gefen (Feigenblat), and Jonathan Shapir. Sociophysics: A new approach of sociological collective behaviour. i. mean-behaviour description of a strike. *The Journal of Mathematical Sociology*, 9(1):1–13, 1982.

[35] Serge Galam. Majority rule, hierarchical structures, and democratic totalitarianism: A statistical approach. *Journal of Mathematical Psychology*, 30(4):426 – 434, 1986.

[36] Serge Galam. Sociophysics: a review of galam models. *International Journal of Modern Physics C*, 19(03):409–440, 2008.

[37] Serge Galam. *Sociophysics, A Physicist’s Modeling of Psycho-political Phenomena*. Springer-Verlag New York, 2012.

[38] Bernd Gärtner and Ahad N. Zehmakan. Threshold behavior of democratic opinion dynamics. *Journal of Statistical Physics*, 2020.

[39] Katarzyna Sznajd-Weron and Józef Sznajd. Opinion evolution in closed community. *International Journal of Modern Physics C*, 11(06):1157–1165, 2000.

[40] Frantisek Slanina and Hynek Lavicka. Analytical results for the sznajd model of opinion formation. *The European Physical Journal B-Condensed Matter and Complex Systems*, 35(2):279–288, 2003.

[41] Roni Muslim, Rinto Anugrah, Sholihin Sholihun, and Muhammad Farchani Rosyid. Phase transition of the sznajd model with anticonformity for two different agent configurations. *International Journal of Modern Physics C*, 0(0):2050052, 2020.

[42] Matheus Calvelli, Nuno Crokidakis, and Thadeu J.P. Penna. Phase transitions and universality in the sznajd model with anticonformity. *Physica A: Statistical Mechanics and its Applications*, 513:518 – 523, 2019.

[43] Katarzyna Sznajd-Weron. Sznajd model and its applications. *arXiv preprint physics/0503239*, 2005.

[44] Dietrich Stauffer. Sociophysics: the sznajd model and its applications. *Computer Physics Communications*, 146(1):93 – 98, 2002. Proceedings of the STAT-PHYS Satellite Conference: Challenges in Computational Statistical Physics in teh 21st Century.

[45] Vishal Sood and Sidney Redner. Voter model on heterogeneous graphs. *Physical review letters*, 94(17):178701, 2005.

[46] Michael T Gastner and Kota Ishida. Voter model on networks partitioned into two cliques of arbitrary sizes. *Journal of Physics A: Mathematical and Theoretical*, 52(50):505701, Nov. 2019.

[47] Jimit R. Majmudar, Stephen M. Krone, Bert O. Baumgaertner, and Rebeca C. Tyson. Voter models and external influence. *The Journal of Mathematical Sociology*, 44(1):1–11, 01 2020.

[48] Inés Caridi, Sergio Manterola, Viktoriya Semeshenko, and Pablo Balenzuela. Topological study of the convergence in the voter model. *Applied Network Science*, 4(1):1–13, 2019.

[49] Sidney Redner. Reality-inspired voter models: A mini-review. *Comptes Rendus Physique*, 20(4):275 – 292, 2019.

[50] H. Wai, A. Scaglione, and A. Leshem. Active sensing of social networks. *IEEE Transactions on Signal and Information Processing over Networks*, 2(3):406–419, 2016.

[51] Qinyue Zhou, Zhibin Wu, Abdulrahman H. Altalhi, and Francisco Herrera. A two-step communication opinion dynamics model with self-persistence and influence index for social networks based on the degroot model. *Information Sciences*, 519:363 – 381, 2020.

[52] Shiru Huang, Baoxin Xiu, and Yanghe Feng. Modeling and simulation research on propagation of public opinion. In *2016 IEEE Advanced Information Management, Communicates, Electronic and Automation Control Conference (IMCEC)*, pages 380–384. IEEE, 2016.
[53] Taksu Cheon and Serge Galam. Dynamical galam model. *Physics Letters A*, 382(23):1509 – 1515, 2018.

[54] Serge Galam and Taksu Cheon. Tipping point dynamics: a universal formula, 2019.

[55] Shen Qian, Yijun Liu, and Serge Galam. Activeness as a key to counter democratic balance. *Physica A: Statistical Mechanics and its Applications*, 432:187 – 196, 2015.

[56] Taksu Cheon and Jun Morimoto. Balancer effects in opinion dynamics. *Physics Letters A*, 380(3):429 – 434, 2016.

[57] Taksu Galam, Serge; Cheon. Asymmetric contrarians in opinion dynamics. *Entropy*, 22(1):25, 2020.

[58] M Mobilia, A Petersen, and S Redner. On the role of zealotry in the voter model. *Journal of Statistical Mechanics: Theory and Experiment*, 2007(08):P08029–P08029, aug 2007.

[59] Nagi Khalil, Maxi San Miguel, and Raul Toral. Zealots in the mean-field noisy voter model. *Phys. Rev. E*, 97:012310, Jan 2018.

[60] Ercan Yildiz, Asuman Ozdaglar, Daron Acemoglu, Weiguo Xia, Mengbin Ye, Ji Liu, Ming Cao, and Xi Chen, Xiao Zhang, Yong Xie, and Wei Li. Polarization. *Complex Networks XI*, pages 306–313, Cham, 2020.

[61] Pranav Dandekar, Ashish Goel, and David T. Lee. Biased assimilation, homophily, and the dynamics of polarization. *Proceedings of the National Academy of Sciences*, 110(15):5791–5796, 2013.

[62] Weiguo Xia, Mengbin Ye, Ji Liu, Ming Cao, and Ximeng Sun. Analysis of a nonlinear opinion dynamics model with biased assimilation, 2019.

[63] Alina Sîrbu, Dino Pedreschi, Fosca Giannotti, and János Kertész. Algorithmic bias amplifies opinion fragmentation and polarization: A bounded confidence model. *PLoS ONE*, 14(3), Mar. 2019.

[64] Xi Chen, Xiao Zhang, Yong Xie, and Wei Li. Opinion Dynamics of Social-Similarity-Based Hegselmann–Krause Model. *Complexity*, 2017:1820257, 2017.

[65] Guiyuan Fu, Weidong Zhang, and Zhijun Li. Opinion dynamics of modified hegselmann–krause model in a group-based population with heterogeneous bounded confidence. *Physica A: Statistical Mechanics and its Applications*, 419:558 – 565, 2015.

[66] Yucheng Dong, Zhaogang Ding, Luis Martínez, and Francisco Herrera. Managing consensus based on leadership in opinion dynamics. *Information Sciences*, 397-398:187 – 205, 2017.

[67] M. Pineda and G. M. Buendia. Mass media and heterogeneous bounds of confidence in continuous opinion dynamics. *Physica A: Statistical Mechanics and its Applications*, 420:73–84, Feb. 2015.

[68] Dario Bauso and Mark Cannon. Consensus in opinion dynamics as a repeated game. *Automatica*, 90:204–211, Apr. 2018.

[69] Jason Gaitonde, Jon Kleinberg, and Eva Tardos. Adversarial perturbations of opinion dynamics in networks, 2020.

[70] T Carletti, D Fanelli, S Grolli, and A Guarino. How to make an efficient propaganda. *Europhysics Letters (EPL)*, 74(2):222–228, Apr. 2006.

[71] Anmol Gupta, Sharayu Moharir, and Neeraja Saaszrabudhe. Influencing opinion dynamics in networks with limited interaction, 2020.

[72] Guillermo Romero Moreno, Edoardo Manin, Long Tran-Thanh, and Markus Brede. Zealotry and influence maximization in the voter model: When to target partial zealots? In Hugo Barbosa, Jesus Gomez-Gardenes, Bruno Gonçalves, Giuseppe Mangioni, Ronaldo Menezes, and Marcos Oliveira, editors, *Complex Networks XI*, pages 107–118, Cham, 2020. Springer International Publishing.

[73] Q. He, X. Wang, B. Yi, F. Mao, Y. Cai, and M. Huang. Opinion maximization through unknown influence power in social networks under weighted voter model. *IEEE Systems Journal*, pages 1–12, 2019.

[74] Rainer Hegselmann, Stefan König, Sascha Kurz, Christoph Niemann, and Jörg Rambau. Optimal opinion control: The campaign problem. *Jasss*, 18(3), 2015.

[75] IC Moráreescu, VS Varma, L Bușoniu, and S Lasaulce. Space-time budget allocation policy design for viral marketing. *Nonlinear Analysis: Hybrid Systems*, 37:100899, 2020.

[76] Florian Dietrich, Samuel Martin, and Marc Jungers. Control via leadership of opinion dynamics with state and time-dependent interactions. *IEEE Transactions on Automatic Control*, 63(4):1200–1207, Apr. 2018.

[77] M. Goyal and D. Manjunath. Opinion control competition in a social network. In 2020 *International Conference on Communication Systems NETworkS (COM-SNETS)*, pages 306–313, 2020.

[78] B. Aditya Prakash, Alex Beutel, Roni Rosenfeld, and Christos Faloutsos. Winner takes all: Competing viruses or ideas on fair-play networks. In *Proceedings of the 21st International Conference on World Wide Web*, WWW ’12, page 1037–1046, New York, NY, USA, 2012. Association for Computing Machinery.

[79] Markus Brede. How does active participation effect consensus: Adaptive network model of opinion dynamics and influence maximizing rewiring. Jun. 2019.

[80] P. Jia, A. MirTabatabaei, N. Friedkin, and F. Bullo. Opinion dynamics and the evolution of social power in influence networks. *SIAM Review*, 57(3):367–397, 2015.

[81] R. Kang, C. Li, and X. Li. Social power convergence on duplex influence networks with self-appraisals. In 2019 *IEEE 58th Conference on Decision and Control (CDC)*, pages 5611–5612, 2019.

[82] Noah E. Friedkin, Peng Jia, and Francesco Bullo. A theory of the evolution of social power: Natural trajectories of interpersonal influence systems along issue sequences. *Sociological Science*, 2016.

[83] P. Jia, N. Friedkin, and F. Bullo. Opinion dynamics and social power evolution over reducible influence networks. *SIAM Journal on Control and Optimization*, 55(2):1280–1301, 2017.

[84] Mengbin Ye and Brian David Outram Anderson. Modelling of individual behaviour in the degroot–friedkin self-appraisal dynamics on social networks. In 2019 *18th European Control Conference (ECC)*, pages 2011–2017, Jun. 2019.
[85] Mengbin Ye, Ji Liu, Brian D.O. Anderson, Changbin Yu, and Tamer Başar. Evolution of Social Power in Social Networks with Dynamic Topology. *IEEE Transactions on Automatic Control*, 63(11):3793–3808, Nov. 2018.

[86] Zahra Askarzadeh, Rui Fu, Abhishek Halder, Yongxin Chen, and Trypton T. Georgiou. Opinion dynamics over influence networks. In *2019 American Control Conference (ACC)*, pages 1873–1878, July 2019.

[87] Zahra Askarzadeh, Rui Fu, Abhishek Halder, Yongxin Chen, and Trypton T. Georgiou. Stability Theory of Stochastic Models in Opinion Dynamics. *IEEE Transactions on Automatic Control*, pages 1–1, Apr. 2019.

[88] Ye Tian, Peng Jia, Anahita Mirtabatabaei, Long Wang, Noah E. Friedkin, and Francesco Bullo. Social power evolution in influence networks with stubborn individuals. *arXiv:1901.08727*, 2019.

[89] Serge Galam. Stubbornness as an unfortunate key to win a public debate: an illustration from sociophysics. *Mind & Society*, 15(1):117–130, 2016.

[90] Xi Chen, Panayiotis Tsparas, Jefrey Lijffijt, and Tjij De Bie. Opinion Dynamics with Backfire Effect and Biased Assimilation. Mar. 2019.

[91] Evgenii Kurmyshev, Héctor A. Juárez, and Ricardo A. González-Silva. Dynamics of bounded confidence opinion in heterogeneous social networks: Concord against partial antagonism. *Physica A: Statistical Mechanics and its Applications*, 390(16):2945–2955, 2011.

[92] Sylvie Huet, Guillaume Deffuant, and Wander Jager. A rejection mechanism in 2d bounded confidence provides more conformity. *Advances in Complex Systems*, 11(04):529–549, 2008.

[93] Wander Jager and Frédéric Amblard. Uniformity, bipolarization and pluriformity captured as generic stylized behavior with an agent-based simulation model of attitude change. *Computational & Mathematical Organization Theory*, 10(4):295–303, 2005.

[94] C. Altafini. Dynamics of opinion forming in structurally balanced social networks. In *Proceedings of the IEEE Conference on Decision and Control*, 2012.

[95] Claudio Altafini. Consensus problems on networks with antagonistic interactions. *IEEE Transactions on Automatic Control*, 2013.

[96] Claudio Altafini and Francesca Ceragioli. Signed bounded confidence models for opinion dynamics. *Automatica*, 93:114–125, Jul. 2018.

[97] Simon Schweighofer, David Garcia, and Frank Schweitzer. An agent-based model of multidimensional opinion dynamics and opinion alignment, 2020.

[98] Anton V. Proskurnikov, Alexey S. Matveev, and Ming Cao. Opinion Dynamics in Social Networks with Hostile Camps: Consensus vs. Polarization. *IEEE Transactions on Automatic Control*, 61(6):1524–1536, 2016.

[99] Deepak Bhat and S. Redner. Opinion formation under antagonistic influences, 2019.

[100] Guang He, Jing Liu, Huimin Hu, and Jian-An Fang. Discrete-time signed bounded confidence model for opinion dynamics. *Neurocomputing*, 2019.

[101] Hongwei Zhang and Jie Chen. Bipartite consensus of linear multi-agent systems over signed digraphs: An output feedback control approach. In *IFAC Proceedings Volumes (IFAC-PapersOnline)*, volume 19, pages 4681–4686. IFAC Secretariat, 2014.

[102] Deyuan Meng, Ziyang Meng, and Yiguang Hong. Disagreement of Hierarchical Opinion Dynamics with Changing Antagonisms. *SIAM Journal on Control and Optimization*, 57(1):718–742, Jan. 2019.

[103] Hassan Dehghani Aghbolagh, Mohsen Zamani, Stefania Paolini, and Zhijing Chen. Balance seeking opinion dynamics model based on social judgment theory. *Physica D: Nonlinear Phenomena*, 403:132336, 2020.

[104] Dorwin Cartwright and Frank Harary. Structural balance: a generalization of heider’s theory. *Psychological review*, 63(5):277–293, Sep. 1956.

[105] Michael Mäs, Andreas Flache, and Dirk Helbing. Individualization as driving force of clustering phenomena in humans. *PLoS Computational Biology*, 2010.

[106] The division of labour in society. *New York: The Free Press*, 1893.

[107] Sébastien Glaunin and Pablo Jensen. Opinion group formation and dynamics: Structures that last from nonlasting entities. *Physical Review E - Statistical, Nonlinear, and Soft Matter Physics*, 85(6), Jun. 2012.

[108] M. Pineda, R. Toral, and E. Hernández-Garcia. Diffusing opinions in bounded confidence processes. *European Physical Journal D*, 62(1):109–117, Mar. 2011.

[109] Adrián Carro, Raúl Toral, and Maxi San Miguel. The Role of Noise and Initial Conditions in the Asymptotic Solution of a Bounded Confidence, Continuous-Opinion Model. *Journal of Statistical Physics*, 151(1-2):131–149, 2013.

[110] Walter Quattrociocchi, Guido Caldarelli, and Antonio Scala. Opinion dynamics on interacting networks: media competition and social influence. *Scientific Reports*, 4:4938, May 2014.

[111] Francois Baccelli, Avhishek Chatterjee, and Sriram Vishwanath. Pairwise Stochastic Bounded Confidence Opinion Dynamics: Heavy Tails and Stability. *IEEE Transactions on Automatic Control*, 62(11):5678–5693, Nov. 2017.

[112] Jiangbo Zhang and Yiyi Zhao. The robust consensus of a noisy deffuant-weisbuch model. *Mathematical Problems in Engineering*, 2018.

[113] Miguel Pineda, Raúl Toral, and Emilio Hernández-Garcia. Noisy continuous-opinion dynamics. *Journal of Statistical Mechanics: Theory and Experiment*, 2009(08):P08001, 2009.

[114] Wei Su, Ge Chen, and Yiguang Hong. Noise leads to quasi-consensus of hegselmann–krause opinion dynamics. *Automatica*, 2017.

[115] G. Chen, W. Su, S. Ding, and Y. Hong. Heterogeneous hegselmann-krause dynamics with environment and communication noise. *IEEE Transactions on Automatic Control*, pages 1–1, 2019.

[116] Miguel Pineda, Raúl Toral, and Emilio Hernández-Garca. The noisy Hegselmann-Krause model for opinion dynamics. *European Physical Journal B*, 86(12), Dec. 2013.
[117] Bernard Chazelle, Quansen Jiu, Qianxiao Li, and Chu Wang. Well-posedness of the limiting equation of a noisy consensus model in opinion dynamics. *Journal of Differential Equations*, 263(1):365 – 397, 2017.

[118] Haiming Liang, Yucheng Dong, and Cong Cong Li. Dynamics of uncertain opinion formation: An agent-based simulation. *JASSS*, 2016.

[119] Heiko Hamann. Opinion dynamics with mobile agents: Contrarian effects by spatial correlations. *Frontiers in Robotics and AI*, 5:63, 2018.

[120] Lorenzo Sabatelli and Peter Richmond. Non-monotonic spontaneous magnetization in a sznajd-like consensus model. *Physica A: Statistical Mechanics and its Applications*, 334(1):274 – 280, 2004.

[121] Boris L. Granovsky and Neal Madras. The noisy voter model. *Stochastic Processes and their Applications*, 55(1):23 – 43, 1995.

[122] Adrián Carro, Raúl Toral, and Maxi San Miguel. The noisy voter model on complex networks. *Scientific reports*, 6(1):1–14, 2016.

[123] A. F. Peralta, A. Carro, M. San Miguel, and R. Toral. Analytical and numerical study of the non-linear noisy voter model on complex networks. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 28(7):075516, 2018.

[124] Noah E. Friedkin, Anton V. Proskurnikov, Roberto Tempo, and Sergey E. Parsegov. Network science on belief system dynamics under logic constraints. *Science*, 354(6310):321–326, 2016.

[125] Ye Tian and Long Wang. Opinion dynamics in social networks with stubborn agents: An issue-based perspective. *Automatica*, 96:213 – 223, 2018.

[126] Fei Xiong, Yun Liu, Liang Wang, and Ximeng Wang. Analysis and application of opinion model with multiple topic interactions. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 27(8):083113, 2017.

[127] H. Ahn, Q. Van Tran, M. H. Trinh, M. Ye, J. Liu, and K. L. Moore. Opinion dynamics with cross-coupling topics: Modeling and analysis. *IEEE Transactions on Computational Social Systems*, pages 1–16, 2020.

[128] W. S. Rossi and P. Frasca. Opinion dynamics with topological gossiping: Asynchronous updates under limited attention. *IEEE Control Systems Letters*, 4(3):566–571, 2020.

[129] Jinhua Geng, Xinjiang Wei, Kehua Yuan, and Aili Fang. Opinion dynamics with bayesian learning. *complexity*, pages 1–5, Feb. 2020.

[130] Wenting Wang and Fuzhong Chen. The opinion dynamics on the evolving complex network by achlioptas process. *IEEE Access*, 7:172928–172937, 2019.

[131] Agnieszka Kowalska-Styczeń and Krzysztof Malarz. Opinion formation and spread: Does randomness of behaviour and information flow matter?, 2020.

[132] Mengbin Ye, Yuzhen Qin, Alain Govaert, Brian David Outram Anderson, and Ming Cao. An Influence Network Model to Study Discrepancies in Expressed and Private Opinions. *Automatica*, 107(7):371–381, Sep. 2019.

[133] Michael T Gastner, Beáta Oborny, and Máté Gulyás. Consensus time in a voter model with concealed and publicly expressed opinions. *Journal of Statistical Mechanics: Theory and Experiment*, 2018(6):063401, Jun 2018.

[134] Arkadiusz Jędrzejewski, Grzegorz Marcjasz, Paul R. Nail, and Katarzyna Sznajd-Weron. Think then act or act then think? *PLOS ONE*, 13(11):1–19, 11 2018.

[135] Naoki Masuda, N. Gibert, and S. Redner. Heterogeneous voter models. *Phys. Rev. E*, 82:010103, Jul. 2010.

[136] Sheng-Wen Wang, Chung-Yuan Huang, and Chuen-Tsai Sun. Modeling self-perception agents in an opinion dynamics propagation society. *SIMULATION*, 90(3):238–248, 2014.

[137] Chung-Yuan Huang and Tzai-Hung Wen. A novel private attitude and public opinion dynamics model for simulating pluralistic ignorance and minority influence. *Journal of Artificial Societies and Social Simulation*, 17(3):8, 2014.

[138] Francisco J. León-Medina, Jordi Tena-Sánchez, and Francisco J. Miguel. Fakers becoming believers: how opinion dynamics are shaped by preference falsification, impression management and coherence heuristics. *Quality & Quantity*, Jul. 2019.

[139] Yilun Shang. Consensus and clustering of expressed and private opinions in dynamical networks against attacks. *IEEE Systems Journal*, pages 1–7, 2019.

[140] Mohammad Afshar and Masoud Asadpour. Opinion formation by informed agents. *JASSS*, 2010.

[141] Dandan Li, Dun Han, Jing Ma, Mei Sun, Lixin Tian, Timothy Khouw, and H. Eugene Stanley. Opinion dynamics in activity-driven networks. *EPL*, 2017.

[142] Qipeng Liu and Xiaofan Wang. Opinion dynamics with similarity-based random neighbors. *Scientific Reports*, 3(1):2968, 2013.

[143] Jiangbo Zhang and Yiguang Hong. Opinion evolution analysis for short-range and long-range defruant–weisbuch models. *Physica A: Statistical Mechanics and its Applications*, 392(21):5289 – 5297, 2013.

[144] Jiangbo Zhang. Opinion limits study for the multi-selection bounded confidence model. *Plos One*, 14(1):e0210745, 2019.

[145] Hendrik Schawe and Laura Hernández. When open mindedness hinders consensus, 2020.

[146] Young-Pil Choi, Alessandro Paoloucci, and Cristina Fignotti. Consensus of the Hegselmann-Krause opinion formation model with time delay. Sep. 2019.

[147] Gang KOU, YiYi Zhao, Yi Peng, and Yong Shi. Multi-Attitude and private opinions in dynamical networks against attacks. *IEEE Transactions on Industrial Informatics*, 17(9):72928–172937, 2019.

[148] J. E. Rubio, R. Roman, and J. Lopez. Integration of a threat traceability solution in the industrial internet of things. *IEEE Transactions on Industrial Informatics*, 2020.

[149] Michael Kuhn, Christoph Kirse, and Heiko Briesen. Population balance modeling and opinion dynamics—a mutually beneficial liaison? *Processes*, 6(9):164, 2018.

[150] Sergei Yu. Pilyugin and M C Campi. Opinion formation in voting processes under bounded confidence. *Networks & Heterogeneous Media*, 14(3):617–632, 2019.
[151] Dirk Helbing. Boltzmann-like and boltzmann-fokker-plankck equations as a foundation of behavioral models. *Physica A: Statistical Mechanics and its Applications*, 196(4):546–573, 1993.

[152] Giuseppe Toscani et al. Kinetic models of opinion formation. *Communications in mathematical sciences*, 4(3):481–496, 2006.

[153] L. Boudin, Roberto Monaco, and F.Salvarani. A kinetic approach to the study of opinion formation. *ESAIM: Mathematical Modelling and Numerical Analysis*, 43(3):507–522, 2009.

[154] Soumyajyoti Biswas, Arnab Chatterjee, and Parongama Sen. Disorder induced phase transition in kinetic models of opinion dynamics. *Physica A: Statistical Mechanics and its Applications*, 391(11):3257–3265, Jun. 2012.

[155] Lorenzo Pareschi, Pierluigi Vellucci, and Mattia Zanella. Kinetic models of collective decision-making in the presence of equality bias. *Physica A: Statistical Mechanics and its Applications*, 503:1256–1262, 2018.

[156] A. L. Oestereich, M. A. Pires, S. M. Duarte Queirós, and N. Crokidakis. Hysteresis and disorder-induced order in continuous kinetic-like opinion dynamics in complex networks, 2020.

[157] H. Lachowicz, M.; Leszczyński. Modeling asymmetric interactions in economy. *Mathematics*, 8(523), 2020.

[158] Martina Fraia and Andrea Tosin. The Boltzmann legacy revisited: kinetic models of social interactions, 2020.

[159] Miroslaw Lachowicz, Henryk Leszczyński, and Elżbieta Puźniakowska-Gałuch. Diffusive and anti-diffusive behavior for kinetic models of opinion dynamics. *Symmetry*, 11(8):1024, 2019.

[160] F Welington SLima and JA Plascak. Kinetic models of discrete opinion dynamics on directed barabási–albert networks. *Entropy*, 21(10):942, 2019.

[161] Lorenzo Pareschi and Giuseppe Toscani. *Interacting multagent systems: kinetic equations and Monte Carlo methods*. OUP Oxford, 2013.

[162] B Düring, P Markowich, J F Pietschmann, and M T Wolfram. Boltzmann and fokker-planck equations modeling opinion formation in the presence of strong leaders. *Proceedings of the Royal Society aMathematical Physical and Engineering Sciences*, 465(2112):3687–3708, 2009.

[163] Peng Wang, Jia Song, Jie Huo, Rui Hao, and Xu-Ming Wang. Towards understanding what contributes to forming an opinion. *International Journal of Modern Physics C*, 28(11):28, 2017.

[164] Soumyajyoti Biswas, Anjan Kumar Chandra, Arnab Chatterjee, and Bikas K Chakrabarti. Phase transitions and non-equilibrium relaxation in kinetic models of opinion formation. *Journal of Physics: Conference Series*, 297(1):012004, 2011.

[165] Krishanu Roy Chowdhury, Asim Ghosh, Soumyajyoti Biswas, and Bikas K Chakrabarti. Kinetic exchange opinion model: Solution in the single parameter map limit. In *Econophysics of Agent-Based Models*, pages 131–143, Cham, 2014. Springer International Publishing.

[166] Lorenzo Pareschi, Michael Herty, and Giuseppe Visconti. Mean field models for large data-clustering problems. *arXiv preprint arXiv:1907.03585*, 2019.

[167] Bing-Chang Wang and Yong Liang. Robust mean field social control problems with applications in analysis of opinion dynamics, 2020.

[168] A Chmiel, T Gradowski, and A Krawiecki. q-neighbor ising model on random networks. *International Journal of Modern Physics C*, 29(06):1850041, 2018.

[169] Lucas Böttcher, Jan Nagler, and Hans J Herrmann. Critical behaviors in contagion dynamics. *Physical review letters*, 118(8):088301, 2017.

[170] Soham Biswas and Parongama Sen. A new model of binary opinion dynamics: coarsening and effect of disorder. *arXiv preprint arXiv:0904.1498*, 2009.

[171] Serge Galam. Rational group decision making: A random field ising model at t= 0. *Physica A: Statistical Mechanics and its Applications*, 238(1-4):66–80, 1997.

[172] T-H. Hubert Chan, Zhibin Liang, and Mauro Sozio. Revisiting opinion dynamics with varying susceptibility to Persuasion. Jan. 2018.

[173] Rediet Abebe, Jon Kleinberg, David Parkes, and Charalampos E. Tsourakakis. Opinion Dynamics with Varying Susceptibility to Persuasion. Jan. 2018.

[174] Stacy Patterson and Bassam Bamieh. Interaction-driven opinion dynamics in online social networks. In *Proceedings of the First Workshop on Social Media Analytics*, SOMA ’10, pages 98–105, New York, NY, USA, 2010. ACM.

[175] Hossein Noorazar, Matthew Tottile, and Kevin Vixie. Loss of community identity in opinion dynamics models as a function of inter-group interaction strength. *CoRR*, abs/1708.03317, 2017.

[176] Josué Tonelli-Cueto, Kevin R. Vixie, Arghavan Talebanpour, and Yunfeng Hu. From classical to modern opinion dynamics, 2019.

[177] Anton V. Proskurnikov and Roberto Tempo. A tutorial on modeling and analysis of dynamic social networks. part II. *Annual Reviews in Control*, 45:166–190, 2018.