Carleman estimates for the time-fractional advection-diffusion equations and applications

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Abstract
In this article, we prove Carleman estimates for the generalized time-fractional advection-diffusion equations by considering the fractional derivative as perturbation for the first order time-derivative. As a direct application of the Carleman estimates, we show a conditional stability estimate for a lateral Cauchy problem for the time-fractional advection-diffusion equation, and we also investigate the stability of an inverse source problem.

Keywords: time-fractional advection-diffusion equation, Carleman estimate, lateral Cauchy problem, inverse source problem

1. Introduction and main results
Recently, the advection-diffusion equations have been tested by more experiment data to be recognized as better models in a wide range of problems in analyzing mass transport. For example, numerous field experiments for the solute transport in highly heterogeneous media demonstrate that solute concentration profiles indicate anomalous non-Fickian growth rates, skewness and long-tailed profile (see e.g. [4] and [14]), which are poorly characterized by the conventional mass transport equations based on Fick’s law. In order to more accurately interpret these effects, the non-Fickian diffusion model has been proposed as mass transport model, say, time-fractional diffusion equation:

\[ \partial_t u(x,t) + q(x,t) \partial^\alpha_t u(x,t) = \Delta u(x,t), \quad (x,t) \in \mathbb{R}^n \times (0, \infty), \]

\[ (1) \]

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where by $\partial_t^\alpha$ we denote the Caputo derivative with respect to the temporal variable $t > 0$:

$$\partial_t^\alpha g(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{dg(\tau)}{d\tau} d\tau, \quad 0 < \alpha < 1,$$

where $\Gamma(\cdot)$ is the Gamma function. See, e.g. [13] and [21] for the properties of the Caputo derivative, and see, e.g. [8, 23] and the references therein for the time-fractional diffusion equations.

Introducing the time-fractional derivatives of arbitrary orders into the equation of mass transport for a heterogeneous medium achieved great successes, for example, it is shown to be an efficient model for describing some anomalous diffusion processes in the highly heterogeneous media: for example, [9] points out that diffusion equation with time-fractional derivative was well-performed in describing the long-tailed profile of a particle that diffuses in a highly heterogeneous medium, and [20] shows that there holds the non-Fick’s law in the anomalous diffusion. As for the applications of time-fractional advection-diffusion equations, researches are recently extended and we can refer to [23] in which the macrodispersion experiment site mobile tritium mass decline is well modeled by the equation (1) with the time-fractional derivative of order $\alpha = 0.33$. We refer to [24] and [25] as a modified model in analyzing mass transport inside a geothermal reservoir. We refer to [2] and [7] in which the experimental results from solute mass transfer in aquifers and soils seem to be better modeled by the time-fractional advection-diffusion equations than by the classical advection-diffusion equations. We refer to [3] and [5] on the applications of time-fractional advection-diffusion equations in hydrologic modeling and blood flow experiments. As for theoretical analysis and numerical treatment for time-fractional advection-diffusion equations, we can refer to [11, 12, 18, 19] and the references therein.

In this paper, assuming $0 < \alpha_1 < \cdots < \alpha_n < 1$, we consider a generalized time-fractional advection-diffusion equation:

$$\left(Lu\right)(x,t) \equiv \partial_t u + \sum_{j=1}^n q_j(x,t)\partial_t^{\alpha_j} u - \sum_{i,j=1}^n a_{ij}(x,t)\partial_i\partial_j u + \sum_{i=1}^n b_i(x,t)\partial_i u + c(x,t) u = F, \quad (2)$$

where $(x,t) \in \mathbb{R}^n \times (0, \infty)$. The above equation (2) has the spatially and temporally variable coefficients, and such kind of equations simulate the advection diffusion, which is more general than that in [8] and [23], and so can be regarded as more feasible model equation than symmetric fractional diffusion equations in modeling diffusion in heterogeneous media. Here in this article, we study the stability for a lateral Cauchy problem and an inverse source problem for this equation, and the stability is a fundamental theoretical subject. To the best knowledge of the authors, the stability results for both of the lateral Cauchy problem and the inverse source problem of the equation (2) were not yet established. One of the difficulties is that the transform argument or Fourier methods used in the above mentioned references [8] and [23] can not work any more because of the non-symmetry of the system. However, from the shape of the equation (2) we regard the lower fractional order terms as perturbations of the first order time-derivative, so that we can derive the Carleman estimate for the equation (2) based on the parabolic equations, and then consider the stability of the lateral Cauchy problem and the inverse source problem.

To this end, we start from fixing some general settings and notations. Let $T > 0$ be a fixed constant and $\Omega \subset \mathbb{R}^n$ be a bounded domain, $n \geq 1$, with sufficiently smooth boundary $\partial \Omega$, for example, of $C^2$-class. We set $Q := \Omega \times (0, T)$ and $\Sigma := \partial \Omega \times (0, T)$. Assume that $a_{ij} = a_{ji} \in C^1(\overline{Q})$, $1 \leq i,j \leq n$, satisfy that
\[ \rho \sum_{j=1}^{n} \xi_j^2 \leq \sum_{j,k=1}^{n} a_{jk}(x,t)\xi_j\xi_k, \quad (x,t) \in \overline{\Omega}, \xi \in \mathbb{R}^n, \]

where \( \rho > 0 \) is a constant independent of \( x, t, \xi \). We set \( \partial_{\nu_j} u = \sum_{i,j=1}^{n} a_{ij}\nu_i \partial_j u \) where \( (\nu_1, \ldots, \nu_n) \) denotes the unit outwards normal vector to the boundary \( \partial\Omega \). Let \( L^2(\Omega) \) and \( H^k(\Omega) \) denote the usual Lebesgue space and the Sobolev spaces. For \( k = 0, 1, 2 \) and \( l = 0, 1 \), we define

\[ H^{k,l}(Q) = \{ u \in L^2(\Omega); \partial_{\nu_1}^{\gamma_1} \cdots \partial_{\nu_k}^{\gamma_k} u \partial_n^l u \in L^2(\Omega) \} \quad \text{where} \quad \gamma_0 \leq l, \sum_{i=1}^{n} \gamma_i \leq k, \gamma_i \in \mathbb{N} \cup \{0\}. \]

We then apply lemma 4.1 in [27] to find a function \( d \in C^2(\overline{\Gamma}) \) satisfying

\[ d > 0 \text{ in } \Omega, \quad |\nabla d| > 0 \text{ on } \overline{\Omega}, \quad d = 0 \text{ on } \partial\Omega \setminus \Gamma. \]

Let us now first consider the equation (2) with \( \alpha_1 < \frac{1}{2} \) (we call the corresponding diffusion sub-diffusion), we set the weight function \( \varphi_1 \) as follows

\[ \varphi_1(x,t) = e^{\lambda \psi_1(x,t)}, \quad \psi_1(x,t) = d(x) - \beta t^{2-2\alpha_1}, \quad \forall \lambda \geq 0, \quad x \in \overline{\Omega}, \quad t \geq 0, \]

where \( \beta > 0 \) is a positive constant which will be chosen later, and we have the following Carleman type estimate for the equation (2)

**Theorem 1.** We assume \( \alpha_1 < \frac{1}{2} \) and \( q_i, b_j, c \in L^\infty(Q) \) \( (i = 1, \cdots, \ell, j = 1, \cdots, n) \) in the equation (2), and we let \( \Sigma_0 = \Pi \times \{0\} \) and \( D \subseteq Q \) be a bounded domain whose boundary \( \partial D \) is composed of a finite number of smooth surfaces. Then there exists a constant \( \lambda_0 > 0 \) such that for arbitrary \( \lambda \geq \lambda_0 \), we can choose a constant \( s_0(\lambda) > 0 \) satisfying: there exist constants \( C = C(s_0, \lambda_0) > 0 \) and \( C(\lambda) > 0 \) such that

\[ \int_D \left\{ \frac{1}{s^{2\lambda}} |\partial_t u|^2 + s \lambda^2 |\nabla u|^2 + s^2 \lambda^4 |\partial_j^2 u|^2 \right\} e^{2s\varphi_1} \text{d}x \text{d}t \]

\[ \leq C \int_D \left| (\partial_t u^2 + u^2) \right| \text{d}x \text{d}t + C(\lambda) e^{C(\lambda)s} \int_{\partial D \cup \Sigma_0} (|\nabla u|^2 + u^2) \text{d}S + C(\lambda) e^{C(\lambda)s} \int_{\partial D \cup \Sigma_0} |\partial T u|^2 \text{d}S \]

for all \( s \geq s_0 \) and all \( u \in H^{2,1}(D) \), where \( \bar{L} := L - \sum_{j=1}^{\ell} q_j(x,t) \partial_{\nu_j}^n; \)

As for similar works, we can refer also to [15]. From the above Carleman estimate, similarly by the argument used in [27], we further have the conditional stability for the lateral Cauchy problem for the equation (2). However, a bit different from the results in [27], here due to the choice of the weight function in the Carleman estimate in theorem 1, we can only prove the continuous dependency of the solution with respect to initial values, boundary values and source terms in the case of \( \alpha_1 \in (0, \frac{1}{2}) \). More precisely we state

**Theorem 2.** We assume \( \alpha_1 < \frac{1}{2} \) and \( q_i, b_j, c \in L^\infty(Q) \) \( (i = 1, \cdots, \ell, j = 1, \cdots, n) \) in the equation (2). Let \( \Gamma \subseteq \partial\Omega \) be an arbitrary non-empty sub-boundary of \( \partial\Omega \). For any bounded domain \( \Omega_0 \) such that \( \overline{\Omega}_0 \subseteq \Omega \cup \Gamma \), \( \partial\Omega_0 \cap \partial\Omega \subseteq \Gamma \) is a non-empty open subset of \( \partial\Omega \), we can choose a sufficiently small \( \varepsilon = \varepsilon(T, \Omega_0) > 0 \) such that
\[\|u\|_{H^{1,\beta}(\Omega \times (0,T))} \leq C(\|u\|_{H^{1+\theta}(\Omega)}^{1-\theta}D^\theta + D),\]

where \(D := \|u(\cdot,0)\|_{H^1(\Omega)} + \|F\|_{L^2(\Omega)} + \|u(\cdot,0)\|_{L^2(\Omega \times (0,T))} + \|\partial_x u\|_{L^2(\Gamma \times (0,T))}\) and the constants \(C > 0\) and \(\theta \in (0,1)\) may depend on \(T\), the choice of \(\Omega_0\) and the coefficients of the equation (2).

The above arguments used for deriving theorem 2 cannot work anymore for the general fractional order \(\alpha_1 \geq \frac{1}{2}\) (the corresponding diffusion is called super-diffusion), but for some special fractional orders, it is expected that the Carleman estimate still holds true. As a partial affirmative answer, here for example, we focus on deriving the Carleman estimate for the case that the fractional order \(\alpha_1 = \frac{3}{4}\). For this, we consider the following equation

\[(Lu)(x,t) \equiv \partial_t u + q(x)\partial_x^2 u - \sum_{i,j=1}^n a_{ij}(x)\partial_{x_i}\partial_{x_j} u + \sum_{i=1}^n b_i(x)\partial_i u + c(x)u = F \quad (6)\]

in the case of \(\alpha = \frac{3}{4}\), where the coefficients satisfy \(b_i \in L^\infty(\Omega), \; i = 1, \ldots, n, \; c \in L^\infty(\Omega)\) and the source term \(F\) is assumed to be smooth enough. To this end, we first introduce the Riemann–Liouville fractional derivative of order \(\alpha \in [m-1, m]\) with \(m = 1, 2, \ldots\) which is usually defined by

\[D_t^\alpha h(t) := \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t (t-r)^{-\alpha+m-1}h(r)dr\]

and the Riemann–Liouville fractional integral \(D_t^{-\alpha}\) of order \(\alpha \in (0,1)\) which is defined by

\[D_t^{-\alpha} h(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1}h(r)dr.\]

We choose the weight function \(\varphi_2:\)

\[\varphi_2(x,t) = e^{\lambda \varphi_1(x,t)}, \quad \psi_2(x,t) = d(x) - \beta(t-t_0)^2 + c_0, \quad (7)\]

where \(t_0 \in (0, T)\) and \(\beta > 0\) are fixed later, \(c_0 := \max\{\beta t_0^2, \beta(T-t_0)^2\}\). In needs of applications, we establish a Carleman estimate with a cut-off function \(\chi_0 \in C^\infty(\mathbb{R}^{n+1})\) which is defined by

\[\chi_0(x,t) = \begin{cases} 1, & (x,t) \in D_0, \\ 0, & (x,t) \notin D, \end{cases}\]

where \(D\) is an arbitrary subdomain of \(Q_1 := \Omega_1 \times (0,T)\) and \(D_0 \subseteq D\). Now we are ready to state the Carleman estimate in the case of \(\alpha = \frac{3}{4}\).

**Theorem 3.** Let \(F\) satisfy \(D_t^j F \in L^2(Q)\) \((j = -1, 0, 1, 2)\) in the equation (6). Let \(\lambda \geq 1\) be fixed large. Then there exist constants \(s_0 \geq 1\) and \(C > 0\) such that

\[
\int_Q \chi_0^2 \left( \sum_{j=1}^6 (s_\varphi_2)^{1-j} |D_t^{j} u|^2 + \sum_{i,j=1}^n (s_\varphi_2)^{-2} |\partial_i \partial_j u|^2 + |\nabla u|^2 \right) e^{2s_\varphi_2} dx dt \\
\leq C \int_Q \chi_0^2 \left( \sum_{j=1}^2 (s_\varphi_2)^{-1-j} |D_t^{j} F|^2 \right) e^{2s_\varphi_2} dx dt + I_1 + I_2
\]
for all \( s \geq s_0 \) and all \( u \) smooth enough satisfying (6) and \( u(\cdot, 0) = 0 \) in \( \Omega \), where

\[
I_1 = C \int_{Q} \left( |\partial \chi_0|^2 + |\nabla \chi_0|^2 + \sum_{j=1}^{n} |\partial \bar{\partial} \chi_0|^2 \right) \left[ \sum_{j=1}^{2} \left( |\nabla (D_t^j u)|^2 + |D_t^j u|^2 \right) \right] e^{2\nu_2} \, dx \, dr,
\]

\[
I_2 = C e^{\lambda s} \int_{t \times (0, T)} \left( |\nabla (D_t^1 u)|^2 + |D_t^1 u|^2 \right) dS \, dr.
\]

Furthermore, we assume \( D_t^j F \in L^2(Q) \) (\( j = 3, 4, 5, 6 \)). Let \( \lambda \geq 1 \) be fixed large. Then there exist constants \( \tilde{s}_0 \geq 1 \) and \( \tilde{C} > 0 \) such that

\[
\int_{Q} \chi_0^2 \left( \sum_{j=1}^{10} (s \varphi_2)^{6-j} |D_t^j u|^2 \right) e^{2\nu_2} \, dx \, dr \leq \tilde{C} \int_{Q} \chi_0^2 \left( \sum_{j=3}^{6} (s \varphi_2)^{3-j} |D_t^j F|^2 \right) e^{2\nu_2} \, dx \, dr + J_1 + J_2
\]

for all \( s \geq \tilde{s}_0 \) and all \( u \) smooth enough satisfying (6) and \( u(\cdot, 0) = 0 \) in \( \Omega \), where

\[
J_1 = \tilde{C} \int_{Q} \left( |\partial \chi_0|^2 + |\nabla \chi_0|^2 + \sum_{j=1}^{n} |\partial \bar{\partial} \chi_0|^2 \right) \left[ \sum_{j=3}^{6} \left( |\nabla (D_t^j u)|^2 + |D_t^j u|^2 \right) \right] e^{2\nu_2} \, dx \, dr,
\]

\[
J_2 = \tilde{C} e^{\lambda s} \int_{t \times (0, T)} \sum_{j=3}^{6} \left( |\nabla (D_t^j u)|^2 + |D_t^j u|^2 \right) dS \, dr.
\]

We notice that the above Carleman estimate with the regular weight function \( \varphi_2 \) requires the homogeneous initial condition. We refer to [26] and [28] for the Carleman estimate for the diffusion equations with a half order time-fractional derivative where the homogeneous initial value is also needed for deriving the Carleman estimates. Moreover, it should be mentioned here that our method can work for the case where the fractional order \( \alpha \in \{0, \frac{1}{2}\} \) is rational. The proof of this statement will be given in appendix.

As an application of the first part of theorem 3, we can easily derive the stability for the lateral Cauchy problem for the equation (6).

**Theorem 4.** For any small \( \epsilon > 0 \) and any bounded domain \( \Omega_0 \) such that \( \overline{\Omega_0} \subset \Omega \cup \Gamma \), \( \partial \Omega_0 \cap \partial \Omega \) is a non-empty open subset of \( \partial \Omega \) and \( \partial \Omega_0 \cap \partial \Omega \subsetneq \Gamma \), there exist constants \( C > 0 \) and \( \theta \in (0, 1) \) such that

\[
\| u \|_{H^1(Q \times (0, T - \epsilon))} \leq CD + CM^{1-\theta} D^\theta
\]

for \( u \) satisfying the equation (6) and \( u(x, 0) = 0 \) for \( \forall x \in \Omega \). Here \( M \) and \( D \) are defined as follows

\[
M := \| D_t^1 u \|_{H^1(Q)},
\]

\[
D := \sum_{j=1}^{2} \| D_t^j F \|_{L^2(Q)} + \| D_t^j u \|_{H^1(\Gamma \times (0, T))} + \| \partial_n (D_t^1 u) \|_{L^2(\Gamma \times (0, T))}.
\]

**Remark 1.1.** Here the constants \( C \) and \( \theta \) depend on the choice of \( \Omega_0, \epsilon \) and the coefficients of the equation (6).

Now on the basis of the above Carleman estimate, we consider another application: an inverse source problem for the equation (6) where the source term is in the form of \( F(x, t) = R(x, t) f(x) \), that is:
\[(Lu)(x,t) = \partial_t u + q(x)\partial_x^3 u - \sum_{i,j=1}^n a_{ij}(x)\partial_j\partial_i u + \sum_{i=1}^n b_i(x)\partial_i u + c(x)u = R(x,t)f(x) \quad \text{in } Q. \quad (8)\]

**Problem 1 (Inverse source problem).** Fix an observation time \(t_0 \in (0, T)\). We intend to determine a spatially varying factor \(f\) for given \(R\) by measuring the data on some sub-boundary and the value of the solution \(u\) at \(t = t_0\).

The measurements are of the same type as in the case of a heat equation and see, e.g., [27] for a similar inverse source problem for a heat equation. In our problem, we deal with a parabolic equation with some lower-order time-fractional derivative. The idea is to put the fractional derivative term into source term and give some suitable estimates. We have the following conditional stability result in a level set \(\Omega_\epsilon := \{x \in \Omega; d(x) > \epsilon\}\) for any \(\epsilon > 0\).

**Theorem 5.** Assume that \(R \in L^\infty(Q)\) satisfies
\[
|R(\cdot,t_0)| \geq c_0 \quad \text{on } \overline{\Omega}, \quad D_j^\frac{3}{2} R \in L^\infty(Q), \quad j = -1, 0, ..., 6 \quad (9)
\]
with some constant \(c_0 > 0\). Then for any \(\epsilon > 0\) there exist constants \(C > 0\) and \(\theta \in (0, 1)\) such that
\[
\|f\|_{L^2(\Omega_\epsilon)} \leq CD + CM^{1-\theta}D^\theta
\]
for all \(u\) which is smooth enough and satisfies the equation (8) and
\[
u(\cdot, 0) = 0 \quad \text{in } \Omega. \quad (10)
\]

Here by \(M\) and \(D\) we denote a priori bound and measurements as follows
\[
M := \|f\|_{L^2(\Omega)} + \left\| D_j^\frac{3}{2} u \right\|_{H^{\frac{1}{2}}(Q)}^p,
\]
\[
D := \|u(x,t_0)\|_{H^1(\Omega)} + \sum_{j=3}^6 \left\| D_j^\frac{3}{2} u \right\|_{H^{\frac{1}{2}}(\Gamma \times (0,T))} + \sum_{j=3}^6 \left\| D_j^\frac{3}{2} (\partial_\nu u) \right\|_{L^2(\Gamma \times (0,T))}.
\]
where the constants \(C\) and \(\theta\) depend on \(\epsilon\) and the coefficients in the equation (8).

To the best knowledge of the authors, most of the existing literature focuses on the uniqueness of the inverse problems for the time-fractional diffusion equation, see, e.g., [6, 10, 16, 17, 28] and the references therein. This is a first attempt to attack the stability of the inverse source problem for the time-fractional advection-diffusion equation (8). Moreover, it should be mentioned here that we can still obtain a similar stability result for our inverse source problem via the first estimate in theorem 3 but we need to assume that \(\partial_t u(x, 0) = 0\), which is not requested for the corresponding inverse source problem for classical parabolic equations. However, according to the second part of theorem 3, it is not necessary to assume that both the solution and its time derivatives vanish at the initial time. We only need the homogeneous initial value.

The rest of this paper is composed of three sections. In section 2, by regarding the fractional-order terms as parts of non-homogeneous source and applying the Carleman estimate for the parabolic equations, we will give a proof of theorem 1 in the case where the highest fractional order is strictly less than half, and then as a direct application, the conditional stability of the lateral Cauchy problem for the equation (2) stated in theorem 2 will be established.
In section 3, we will first finish the proof of theorem 3 with a regular weight function which is one common choice when dealing with inverse problems, and on the basis of the Carleman type estimate in theorem 3, we will show that the solution continuously depends on the Cauchy data as well as the source term and prove a stability inequality for the inverse source problem. Concluding remarks will be given in section 4. Finally, we will provide a brief proof of the Carleman estimate for the equation (6) with general rational orders less than \( \frac{3}{4} \) in appendix.

2. Carleman estimate for the sub-diffusion and its applications

In this section, we investigate the equation (2) with fractional order \( \alpha_1 < \frac{1}{2} \). We point out that the equation (2) can be considered to be a parabolic type equation if we regard the lower fractional order terms as new non-homogeneous terms. Therefore, it is expected to employ the Carleman estimate for the parabolic equations to derive the Carleman estimate for our equation, which is the key idea in this section. Owing to this treatment, in section 2.1, we will give the proof of theorem 1, while theorem 2 will be proved as an application in section 2.2.

2.1. Carleman estimate for the sub-diffusion

In this subsection, recalling \( d \in C^2(\Omega), |\nabla d| \neq 0 \) on \( \Omega \), and \( \psi_1 = d(x) - \beta t^{2-2\alpha_1} \) with \( \beta > 0 \), we modify locally the arguments on pp 9–19 of the survey paper [27] to prove the Carleman estimate (5).

**Proof of theorem 1.** It is sufficient for us to discuss the derivation of a Carleman estimate for \( L_0 = \partial_t - \sum_{i,j=1}^n a_{ij}(x,t)\partial_i\partial_j \) with the new weight function \( \varphi_1 = e^{\lambda \psi_1} \). Namely

\[
\int_D \left\{ \frac{1}{s\varphi_1}|\partial_t u|^2 + s\lambda^2 \varphi_1|\nabla u|^2 + s^3 \lambda^4 \varphi^3_1 u^2 \right\} e^{2s\varphi_1} \, dx \, dt \\
\leq C \int_D |L_0 u|^2 e^{2s\varphi_1} \, dx \, dt + C(\lambda)^r \int_{\partial D} (|\nabla u|^2 + |u|^2) dS dt + C(\lambda)^r \int_{\partial D \setminus \Sigma_0} |\partial_t u|^2 dS dt
\]

for all \( s \geq s_0 \) and all \( u \in H^{2,1}(D) \). Henceforth, \( C > 0 \) denotes generic constants which are independent of \( s \) and change line by line.

We note

\[ w(x,t) = e^{s\psi_1(x,t)} u(x,t), \quad \sigma(x,t) = \sum_{i,j=1}^n a_{ij}(x,t) \partial_i \partial_j d, \quad (x,t) \in \partial D \]

and

\[ P_3 \sigma = e^{-s\psi_1} L_0(e^{-s\psi_1} w) = \partial_t w - \sum_{i,j=1}^n a_{ij} \partial_i \partial_j w + 2s\lambda \varphi_1 \sum_{i,j=1}^n a_{ij} \partial_i \partial_j d + s^3 \lambda^4 \varphi^3_1 u^2 \]

\[ - s^2 \lambda^2 \varphi^2_1 \sigma w + s \lambda^2 \varphi_1 \partial \varphi_1 w + s \lambda \varphi_1 w \sum_{i,j=1}^n a_{ij} \partial_i \partial_j - s \lambda \varphi_1 w(\partial_t \psi_1). \]

Now we introduce a new operator \( P_3 \) which is defined by
\[ P_{3w} := Pw + \left( s\lambda^2 \varphi_1 \sigma - s\lambda \varphi_1 \sum_{i,j=1}^{n} a_{ij} \partial_i \partial_j d + s\lambda \varphi_1 (\partial_1 \psi_1) \right) w, \]

and we decompose \( P_1 \) into the parts \( P_1 \) and \( P_2 \),

\[ P_{3w} = P_1 w + P_2 w, \]

where

\[ P_1 w = -\sum_{i,j=1}^{n} a_{ij} \partial_i \partial_j w - s^2 \lambda^2 \varphi_1^2 \sigma w, \]

and

\[ P_2 w = \partial_t w + 2s\lambda \varphi_1 \sum_{i,j=1}^{n} a_{ij} (\partial_i d) \partial_j w + 2s\lambda^2 \varphi_1 \sigma w. \]

From the above notations for \( P, P_1, P_2, P_3 \), it follows that

\[
\left\| e^{s^2 \lambda^2 \varphi_1 \sigma} L_{ij} + \left( s\lambda^2 \varphi_1 \sigma - s\lambda \varphi_1 \sum_{i,j=1}^{n} a_{ij} \partial_i \partial_j d + s\lambda \varphi_1 (\partial_1 \psi_1) \right) w \right\|^2_{L^2(D)} = \left\| P_1 w + P_2 w \right\|^2_{L^2(D)} \geq 2 \int_D (P_1 w)(P_2 w) dx dt + \left\| P_2 w \right\|^2_{L^2(D)}. \tag{11}
\]

We will estimate \( \int_D |P_2 w|^2 + 2(P_1 w)(P_2 w) dx dt \) from below. Firstly, we have

\[
\int_D (P_1 w)(P_2 w) dx dt = -\sum_{i,j=1}^{n} \int_D a_{ij} (\partial_i \partial_j w) \partial_i \partial_j w dx dt - \sum_{i,j=1}^{n} \int_D a_{ij} (\partial_i \partial_j w) 2s\lambda \varphi_1 \sum_{k,l=1}^{n} a_{kl} (\partial_k \partial_l) \partial_i \partial_j w dx dt
\]

\[ - \sum_{i,j=1}^{n} \int_D a_{ij} (\partial_i \partial_j w) 2s^2 \lambda^2 \varphi_1^2 \sigma w dx dt - \int_D s^2 \lambda^2 \varphi_1^2 \sigma w (\partial_1 \psi_1) dx dt
\]

\[ - \int_D 2s^3 \lambda^4 \varphi_1^3 \sigma w \sum_{i,j=1}^{n} a_{ij} (\partial_k \partial_l) \partial_i \partial_j w dx dt - \int_D 2s^3 \lambda^4 \varphi_1^3 \sigma^2 w^2 dx dt = \sum_{k=1}^{6} J_k. \]

Now, applying the integration by parts and the symmetry of \( \{a_{ij}\} : a_{ij} = a_{ji} \), we give the estimates of \( J_k, k = 1, \cdot \cdot \cdot , 6 \) separately.

\[ J_1 = -\sum_{i,j=1}^{n} \int_D a_{ij} (\partial_i \partial_j w) \partial_i \partial_j w dx dt \]

\[ = \sum_{i,j=1}^{n} \int_D (\partial_i a_{ij}) (\partial_j w) \partial_i \partial_j w dx dt + \sum_{i,j=1}^{n} \int_D a_{ij} (\partial_j w) \partial_i \partial_j w dx dt - \sum_{i,j=1}^{n} \int_{\partial D} a_{ij} (\partial_j w) \nu_i \partial_i \Sigma dx dt. \]

Here and henceforth \( \nu := (\nu_1, \cdot \cdot \cdot , \nu_n, \nu_{n+1}) \) denotes the unit normal exterior with respect to the boundary \( \partial D \) of \( D \). In particular, \( \nu_{n+1} \) is the component in the time direction. By noting
\( \nu_i = 0, \forall i = 1, \ldots, n \) on \( \Sigma_0 \), then integration by parts yields
\[
J_1 = \sum_{i,j=1}^{n} \int_D (\partial_i a_{ij})(\partial_j w) \partial_i w \, dx \, dt + \frac{1}{2} \sum_{i,j=1}^{n} a_{ij} (\partial_i w)(\partial_j w) \, dx \, dt
\]

\[
- \sum_{i,j=1}^{n} \int_{\partial D \setminus \Sigma_0} a_{ij} (\partial_i w) \nu_j \partial_i w \, d\Sigma
\]

\[
= \sum_{i,j=1}^{n} \int_D (\partial_i a_{ij})(\partial_j w) \partial_i w \, dx \, dt - \frac{1}{2} \sum_{i,j=1}^{n} \int_D (\partial_i a_{ij})(\partial_j w) \partial_i w \, dx \, dt
\]

\[
+ \frac{1}{2} \sum_{i,j=1}^{n} \int_{\partial D \setminus \Sigma_0} a_{ij} (\partial_i w)(\partial_j w) \nu_j \partial_i w \, d\Sigma - \sum_{i,j=1}^{n} \int_{\partial D \setminus \Sigma_0} a_{ij} (\partial_i w) \nu_j \partial_i w \, d\Sigma.
\]

Thus,
\[
|J_1| \leq C \int_D |\nabla w| |\partial_i w| \, dx \, dt + C \int_D |\nabla w|^2 \, dx \, dt + C \int_{\partial D} |\nabla w|^2 \, d\Sigma + C \int_{\partial D \setminus \Sigma_0} |\nabla w| |\partial_i w| \, d\Sigma.
\]

We apply the Cauchy–Schwarz inequality to the first and fourth terms on the right-hand side by noting \( |\nabla w| |\partial_i w| = s^2 \lambda^2 \varphi^2 |\nabla w| |s^{-2} \lambda^{-2} \varphi^{-2}| |\partial_i w| \), and obtain
\[
|J_1| \leq C \int_D \frac{1}{s} |\nabla w|^2 \, dx \, dt + C \int_D s \lambda \varphi |\nabla w|^2 \, dx \, dt + C \int_{\partial D} |\nabla w|^2 \, d\Sigma + C \int_{\partial D \setminus \Sigma_0} |\nabla w| |\partial_i w| \, d\Sigma.
\]

Next similarly to the argument on pp. 12-13 in [27], we have
\[
J_2 = - \sum_{i,j,k,l=1}^{n} \int_D 2s \lambda \varphi_{i,j,k,l} (\partial_i a_{ij})(\partial_j d)(\partial_k w)(\partial_l w) \, dx \, dt
\]

\[
= 2s \lambda \int_D \sum_{i,j,k,l=1}^{n} \lambda (\partial_i d) \varphi_{i,j,k,l} (\partial_i a_{ij})(\partial_j d)(\partial_k w)(\partial_l w) \, dx \, dt
\]

\[
+ 2s \lambda \int_D \sum_{i,j,k,l=1}^{n} \varphi_{i,j,k,l} (\partial_i a_{ij})(\partial_j d)(\partial_k w)(\partial_l w) \, dx \, dt
\]

\[
+ 2s \lambda \int_D \sum_{i,j,k,l=1}^{n} \varphi_{i,j,k,l} (\partial_i a_{ij})(\partial_j d)(\partial_k w)(\partial_l w) \, dx \, dt
\]

\[
- 2s \lambda \int_{\partial D} \sum_{i,j,k,l=1}^{n} \varphi_{i,j,k,l} (\partial_i a_{ij})(\partial_j d)(\partial_k w)(\partial_l w) \nu_j \, d\Sigma.
\]

We have
\[
(\text{first term}) = 2s \lambda^2 \int_D \left| \sum_{j=1}^{n} a_{ij} (\partial_j d)(\partial_j w) \right|^2 \, dx \, dt \geq 0
\]
and

$$(\text{third term}) = s \lambda \int_D \varphi^1 \sum_{i,j=1}^n \sum_{k,l=1}^n a_{ij}(\partial_i \varphi_k) \partial_j ((\partial_i \varphi_k) (\partial_j \varphi_l)) dx dr$$

$$= s \lambda \int_{\partial D} \varphi^1 \sum_{i,j=1}^n \sum_{k,l=1}^n a_{ij}(\partial_i \varphi_k) (\partial_j \varphi_l) \nu_i d\Sigma$$

$$- s \lambda^2 \int_D \varphi^1 \sum_{i,j=1}^n \sigma a_{ij}(\partial_i \varphi_k) \partial_j \varphi_l dx dr$$

$$- s \lambda \int_D \varphi^1 \sum_{i,j=1}^n \partial_i (a_{ij}(\partial_i \varphi_k)) (\partial_j \varphi_l) \partial_j \varphi_l dx dr,$$

which imply

$$J_2 \geq - \int_D s \lambda^2 \varphi_1 \sigma \sum_{i,j=1}^n a_{ij}(\partial_i \varphi_k) \partial_j \varphi_l dx dr - C \int_D s \lambda \varphi_1 |\nabla \varphi_1|^2 dx dr - C \int_{\partial D} s \lambda \varphi_1 |\nabla \varphi_1|^2 d\Sigma$$

$$J_3 \geq 2 \int_D s \lambda^2 \varphi_1 \sigma \sum_{i,j=1}^n a_{ij}(\partial_i \varphi_k) \partial_j \varphi_l dx dr + 2 \int_D s \lambda^2 \sum_{i,j=1}^n \partial_i (\varphi_1 \sigma a_{ij}) \partial_j \varphi_l dx dr$$

$$- 2 \int_{\partial D} s \lambda \varphi_1 |\nabla \varphi_1|^2 d\Sigma$$

$$\geq 2 \int_D s \lambda^2 \varphi_1 \sigma \sum_{i,j=1}^n a_{ij}(\partial_i \varphi_k) \partial_j \varphi_l dx dr - C \int_D s \lambda^2 |\nabla \varphi_1||w| dx dr - C \int_{\partial D} s \lambda^2 \varphi_1 |\nabla \varphi_1||w| d\Sigma.$$

By

$$s \lambda \varphi_1 |\nabla \varphi_1||w| = (s \lambda^2 \varphi_1 |w|)(\lambda |\nabla \varphi_1|) \leq \frac{1}{2} s \lambda^2 \varphi_1^2 w^2 + \frac{1}{2} \lambda^2 |\nabla \varphi_1|^2,$$

and

$$s \lambda^2 \varphi_1 |\nabla \varphi_1||w| = (s \lambda^2 \varphi_1 |w|)(\lambda^2 |\nabla \varphi_1|) \leq \frac{1}{2} s \lambda^2 \varphi_1^2 w^2 + \frac{1}{2} \lambda |\nabla \varphi_1|^2,$$

we have

$$J_3 \geq 2 \int_D s \lambda^2 \varphi_1 \sigma \sum_{i,j=1}^n a_{ij}(\partial_i \varphi_k) \partial_j \varphi_l dx dr - C \int_D s \lambda^4 \varphi_1^2 w^2 dx dr - C \int_D \lambda^3 |\nabla \varphi_1|^2 dx dr$$

$$- C \int_{\partial D} \lambda |\nabla \varphi_1|^2 d\Sigma - C \int_{\partial D} s \lambda^3 \varphi_1^2 w^2 d\Sigma.$$

$$|J_4| = \left| - \frac{1}{2} \int_D s \lambda^2 \varphi_1^2 \partial_i (\varphi_l \partial_i \varphi_l) dx dr \right|$$

$$= \left| \int_D s \lambda^2 \varphi_1^2 \partial_i (2a_{ij} - 2) \partial_i \varphi_l \partial_i \varphi_l dx dr + \frac{1}{2} \int_D s \lambda^2 \varphi_1^2 (\partial_i \varphi_l) \partial_i \varphi_l dx dr - \frac{1}{2} \int_{\partial D} s \lambda^2 \varphi_1^2 \partial_i \varphi_l \partial_i \varphi_l dx dr \right|$$

$$\leq C \int_D s \lambda^3 \varphi_1^2 w^2 dx dr + C \int_{\partial D} s \lambda^3 \varphi_1^2 w^2 d\Sigma.$$
\[ \begin{align*}
J_5 &= -\int_{D} s^3 \lambda^3 \varphi_1^3 \sum_{i,j=1}^{n} \sigma a_{ij}(\partial_d d)\partial_j(\varphi^2) d\sigma \ln\! d\sigma \\
&= 3 \int_{D} s^3 \lambda^4 \varphi_1^3 \sigma^2 w^2 d\sigma \ln\! d\sigma + \int_{D} s^3 \lambda^3 \varphi_1^3 \sum_{i,j=1}^{n} \partial_i(\sigma a_{ij} \partial_j w) w^2 d\sigma \\
&- \int_{\partial D} s^3 \lambda^3 \varphi_1^3 \sum_{i,j=1}^{n} \sigma a_{ij}(\partial_d w) w^2 \nu d\Sigma \\
&\geq 3 \int_{D} s^3 \lambda^4 \varphi_1^3 \sigma^2 w^2 d\sigma - C \int_{D} s^3 \lambda^3 \varphi_1^3 w^2 d\sigma - C \int_{\partial D} s^3 \lambda^3 \varphi_1^3 w^2 d\Sigma \\
J_6 &= -\int_{D} 2s^3 \lambda^4 \varphi_1^3 \sigma^2 w^2 d\sigma \ln\! d\sigma.
\end{align*} \]

By the definition of \( P_{\mathcal{S}} \), we have

\[ \begin{align*}
\epsilon \int_{D} \frac{1}{s^3 \varphi_1} |\partial_d w|^2 d\sigma &= \epsilon \int_{D} \frac{1}{s^3 \varphi_1} P_{\mathcal{S}} d\sigma - 2s^3 \lambda^3 \varphi_1 \sum_{i,j=1}^{n} a_{ij}(\partial_d w) \partial_j w - 2s^3 \lambda^3 \varphi_1 \sigma w \\
&\leq C \int_{D} |P_{\mathcal{S}}|^2 d\sigma + C \int_{D} s^3 \lambda^3 \varphi_1 |
abla w|^2 d\sigma + C \epsilon \int_{D} s^3 \lambda^4 \varphi_1 w^2 d\sigma. \quad (12)
\end{align*} \]

By summing up all the above estimates for \( J_k, k = 1, \ldots, 6 \), we can obtain a lower bound for \( \int_{D} (P_{\mathcal{S}})^2 d\sigma \) in (11):

\[ \begin{align*}
\int_{D} s^3 \lambda^4 \varphi_1^3 \sigma^2 w^2 d\sigma + \int_{D} s^3 \lambda^3 \varphi_1^3 \sigma^2 w^2 d\sigma \ln\! d\sigma - C \int_{D} \frac{1}{s^3 \varphi_1} |\partial_d w|^2 d\sigma \\
&\leq \int_{D} (P_{\mathcal{S}})^2 d\sigma + C \int_{D} (s^3 \lambda^3 \varphi_1 + \lambda^3) |
abla w|^2 d\sigma + C \int_{D} (s^3 \lambda^3 \varphi_1 + s^3 \lambda^4 \varphi_1^2) w^2 d\sigma \\
&+ C \int_{\partial D} (s^3 \lambda \varphi_1 |
abla w|^2 + s^3 \lambda^3 \varphi_1^2 w^2) d\Sigma + C \int_{\partial D \setminus \Sigma_0} |\partial_d w|^2 d\Sigma. \quad (13)
\end{align*} \]

We use (12) and (13) to obtain lower bounds of \( \int_{D} |P_{\mathcal{S}}|^2 d\sigma \) and \( \int_{D} (P_{\mathcal{S}})^2 d\sigma \) respectively, and we substitute them into (11) to obtain

\[ \begin{align*}
2 \int_{D} s^3 \lambda^4 \varphi_1^3 \sigma^2 w^2 d\sigma + 2 \int_{D} s^3 \lambda^3 \varphi_1^3 \sigma^2 w^2 d\sigma \ln\! d\sigma + \left( \frac{\epsilon}{C} - \frac{C}{\lambda} \right) \int_{D} \frac{1}{s^3 \varphi_1} |\partial_d w|^2 d\sigma \\
&\leq C \int_{D} |L_{\mathcal{S}}|^2 d\sigma + C \int_{D} (s^3 \lambda \varphi_1 + s^3 \lambda^2 \varphi_1 + \lambda^3) |
abla w|^2 d\sigma + C \int_{D} (s^3 \lambda^3 \varphi_1 + s^3 \lambda^4 \varphi_1^2) w^2 d\sigma \\
&+ C \int_{\partial D} (s^3 \lambda \varphi_1 |
abla w|^2 + s^3 \lambda^3 \varphi_1^2 w^2) d\Sigma + C \int_{\partial D \setminus \Sigma_0} |\partial_d w|^2 d\Sigma.
\end{align*} \]
whose combination with the ellipticity of $a_{ij}$ and $\sigma_0 := \inf_{(x,t) \in Q} \sigma(x,t) > 0$ yields that

$$
\int_D s^3 \lambda^4 \varphi_1^2 \sigma_0^2 w^2 \, dx \, dt + \int_D (\sigma_0 \rho - C\epsilon s) \lambda^2 |\nabla w|^2 \, dx \, dt + \left( \frac{e_1}{C} - \frac{C}{\lambda} \right) \int_D \frac{1}{s^4 \varphi_1} |\partial_\omega w|^2 \, dx \, dt 
\leq C \int_D |L_\omega u|^2 e^{2s\varphi_1} \, dx \, dt + C \int_D (s^3 \lambda^3 \varphi_1 + s^2 \lambda^4 \varphi_1^2) w^2 \, dx \, dt + C \int_D (s\lambda \varphi_1 + \lambda^2 |\nabla w|^2 \, dx \, dt 
+ C \int_{\partial D} (s\lambda \varphi_1 |\nabla w|^2 + s^2 \lambda^3 \varphi_1^2 w^2) d\Sigma + C \int_{\partial D \setminus \Sigma_0} |\partial_\omega w|^2 d\Sigma.
$$

Thus choosing $\epsilon > 0$ small, and choosing $\lambda$ and then $s$ large, we can absorb the terms suitably to obtain

$$
\int_D \left\{ \frac{1}{s^4 \varphi_1} |\partial_\omega w|^2 + s \lambda^2 \varphi_1 |\nabla w|^2 + s^2 \lambda^4 \varphi_1^2 w^2 \right\} \, dx \, dt 
\leq C \int_D |L_\omega u|^2 e^{2s\varphi_1} \, dx \, dt + \int_{\partial D} (s\lambda \varphi_1 |\nabla w|^2 + s^2 \lambda^3 \varphi_1^2 w^2) d\Sigma + \int_{\partial D \setminus \Sigma_0} |\partial_\omega w|^2 d\Sigma.
$$

Noting $w = u e^{\psi_1}$, we have

$$
\int_D \left\{ \frac{1}{s^4 \varphi_1} |\partial_\omega u|^2 + s \lambda^2 \varphi_1 |\nabla u|^2 + s^2 \lambda^4 \varphi_1^2 u^2 \right\} e^{2s\varphi_1} \, dx \, dt 
\leq C \int_D |L_\omega u|^2 e^{2s\varphi_1} \, dx \, dt + C(\lambda) e^{C(\lambda)s} \int_{\partial D} (|\nabla u|^2 + u^2) d\Sigma + C(\lambda) e^{C(\lambda)s} \int_{\partial D \setminus \Sigma_0} |\partial_\omega u|^2 d\Sigma,
$$

which completes the proof of the theorem. \square

2.2. Application to a lateral Cauchy problem for the sub-diffusion

In this section, we will give a proof of theorem 2. To prove this, we follow the usual argument used in analyzing the lateral Cauchy problem for the parabolic equations, that is, we use the Carleman estimate derived in section 2.1. The first problem which we have to overcome is to evaluate the fractional derivative by the first order time-derivative under some suitable norm. Namely, the following lemma holds.

**Lemma 2.1.** Let $T > 0$ and $0 < \alpha \leq \alpha_1 < \frac{1}{2}$ be given constants. Then the following inequality

$$
\int_{C_1} |\partial^\alpha u|^2 e^{2s\varphi_1} \, dx \, dt \leq C \int_{C_1} \frac{1}{s^4 \varphi_1} |\partial_\omega u|^2 e^{2s\varphi_1} \, dx \, dt
$$

holds true for all $u \in H^{2,1}(Q)$, where $\varphi_1 = e^{\lambda\psi_1}$ with $\psi_1(x,t) = d(x) - \beta t^{2-2\alpha}$, and $C_1 := \{(x,t); x \in \Omega, t > 0, \varphi_1(x,t) > c_1\}$, $i = 1, 2$, and $c_i$ are positive constants such that $c_2 < c_1$.

**Proof.** We choose a nonnegative function $\Phi \in L^\infty(\mathbb{R}^{n+1})$ such that $\text{supp}\Phi \subset C_2$ and $\Phi \equiv 1$ in $C_1$. Thus we have

$$
\int_{C_1} |\partial^\alpha u|^2 e^{2s\varphi_1} \, dx \, dt = \int_{C_1} \Phi(x,t) |\partial^\alpha u|^2 e^{2s\varphi_1} \, dx \, dt.
$$
whose combination with the definition of the Caputo derivative implies that
\[
\int_{C_1} \Phi(x,t) [\partial^\alpha_t u]_r^2 e^{2\varphi_1} dx dt = \int_{C_1} \left| \frac{\Phi(x,t)}{\Gamma(1-\alpha)} \int_0^t (t-r)^{-\alpha} \partial_r u(x,r) dr \right|^2 e^{2\varphi_1} dx dt.
\]

Moreover, since for any fixed \( x \in \Omega \), \( \varphi_1(x,t) \) is decreasing with respect to the variable \( t > 0 \), we see that \( \varphi_1(x,t) \leq \varphi_1(x,r) \) for any \( x \in \Omega \) and \( 0 < r \leq t \), so that \( (x,t) \in C_1 \) implies that \( (x,r) \in C_1 \) for \( 0 < r \leq t \), hence that \( \Phi(x,r) = 1 \) if \( (x,t) \in C_1 \) and \( 0 < r < t \). Finally we have
\[
\int_{C_1} \Phi(x,t) [\partial^\alpha_t u]_r^2 e^{2\varphi_1} dx dt = \int_{C_1} \left| \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-r)^{-\alpha} \Phi^2(x,r) \partial_r u(x,r) dr \right|^2 e^{2\varphi_1} dx dt
\leq \int_0^T \left( \int_0^t (t-r)^{-\alpha} \Phi^2(x,r) \partial_r u(x,r) dr \right)^2 e^{2\varphi_1} dx dt.
\]

Now by noting that \( \varphi_1(x,t) \geq c_0 \), where \( c_0 > 0 \) is a constant, we have
\[
\partial_t \psi_1 = \beta(2\alpha_1 - 2)t^{1-2\alpha_1}, \quad \partial_t \varphi_1 = \lambda(\partial_t \psi_1)\varphi_1 = (2\alpha_1 - 2)\beta \lambda \varphi_1 t^{1-2\alpha_1}.
\]
Hence
\[
t^{1-2\alpha_1} e^{2\varphi_1} = -\frac{1}{4\beta \lambda \varphi_1 (1-\alpha_1)} \partial_t (e^{2\varphi_1}). \tag{14}
\]

By the Cauchy–Schwarz inequality and (3), we have
\[
\int_0^T \left( \int_0^t (t-r)^{-\alpha} \Phi^2(x,r) \partial_r u(x,r) dr \right)^2 e^{2\varphi_1} dx dt \leq \frac{T^{2(\alpha_1-\alpha)}}{1-2\alpha} \int_0^T \left( \int_0^t \Phi(x,r) |\partial_r u(x,r)|^2 dr \right)^2 e^{2\varphi_1} dx dt.
\]

Moreover, since \( 0 < \alpha < \alpha_1 \), we see that
\[
\int_0^T \left( \int_0^t (t-r)^{-\alpha} \Phi^2(x,r) \partial_r u(x,r) dr \right)^2 e^{2\varphi_1} dx dt \leq \frac{T^{2(\alpha_1-\alpha)}}{1-2\alpha} \int_0^T \left( \int_0^t \Phi(x,r) |\partial_r u(x,r)|^2 dr \right)^2 e^{2\varphi_1} dx dt.
\]

Now from the formula (14), integration by parts implies
\[
\int_0^T t^{1-2\alpha_1} \left( \int_0^t \Phi(x,r) |\partial_r u(x,r)|^2 dr \right) e^{2\varphi_1} dx dt
= \frac{1}{1-\alpha_1} \left( \int_0^T \Phi(x,t) |\partial_r u(x,r)|^2 dr \right) e^{2\varphi_1} \bigg|_{r=0}^{r=T}
+ \frac{T}{1-\alpha_1} \int_0^T \Phi(x,t) |\partial_r u(x,r)|^2 e^{2\varphi_1} dx dt
\]
\[
= \frac{1}{1-\alpha_1} \left( \int_0^T \Phi(x,t) |\partial_r u(x,r)|^2 e^{2\varphi_1} dx dt \right) + \int_0^T t^{1-2\alpha_1} \left( \int_0^t \Phi(x,r) |\partial_r u(x,r)|^2 dr \right) e^{2\varphi_1} dx dt.
\]
The last term on the right-hand side can be absorbed into the left-hand side by choosing \(s > 0\) large and we have

\[
\int_0^T t^{1-2\alpha_1} \left( \int_0^T \Phi(x,r) |\partial_t u(r)|^2 dr \right) e^{2\beta \varphi_1(x,t)} dt \leq C \int_0^T \frac{\Phi(x,t)}{s \lambda \varphi_1} |\partial_t u(t)|^2 e^{2\beta \varphi_1} dr.
\]

Thus

\[
\int_0^T t^{1-2\alpha_1} \left( \int_0^T \Phi(x,r) |\partial_t u(r)|^2 dr \right) e^{2\beta \varphi_1(x,t)} dx dt \leq C \int_0^T \frac{\Phi(x,t)}{s \lambda \varphi_1} |\partial_t u(t)|^2 e^{2\beta \varphi_1} dx dr.
\]

This, with the fact \(\text{supp} \Phi \subset C_2\) implies that

\[
\int_{C_1} |\partial_t u|^2 e^{2\beta \varphi_1} dx dt \leq C \int_{C_2} \frac{1}{s \lambda \varphi_1} |\partial_t u|^2 e^{2\beta \varphi_1} dx dr,
\]

which completes the proof of the lemma. \(\square\)

Before giving the proof of theorem 2, we introduce some notations.

For arbitrary given domain \(\Omega_0\) such that \(\overline{\Omega_0} \subset \Omega \cup \Gamma\), similar to theorem 5.1 in [27], we will choose a suitable weight function \(\psi_1(x,t) := d(x) - \beta t^{2-2\alpha_1}\). For this, we first choose a bounded domain \(\Omega_1\) with smooth boundary such that

\[
\Omega \subseteq \Omega_1, \quad \Gamma = \overline{\partial \Omega \cap \Omega_1}, \quad \partial \Omega \setminus \Gamma \subset \partial \Omega_1.
\]

We then apply lemma 4.1 in [27] to obtain \(d \in C^2(\overline{\Omega_1})\) satisfying

\[
d(x) > 0, \ x \in \Omega_1, \quad d(x) = 0, \ x \in \partial \Omega_1, \quad |\nabla d(x)| > 0, \ x \in \Omega.
\]

Then we can choose \(\beta > 0\) such that

\[
\beta \left( \frac{T}{2} \right)^{2-2\alpha_1} < ||d||_{C^2(\overline{\Omega_1})} < \beta T^{2-2\alpha_1}.
\]

(15)

Moreover, since \(\overline{\Omega_0} \subset \Omega_1\), we can choose a sufficiently large \(N > 1\) such that

\[
\Omega_0 \subset \Omega_1 \cap \{x \in \Omega_1; \ d(x) > \frac{4}{N} ||d||_{C^2(\overline{\Omega_1})}\}.
\]

(16)

We set \(\mu_k = \exp\left(\lambda(\frac{N}{2} ||d||_{C^2(\overline{\Omega_1})} - 2^{1/4})\right)^{1/2} \), and \(D_k := \{(x,t); \ x \in \Omega_1, \ t > 0, \ \varphi_1(x,t) > \mu_k\}\), \(k = 1, 2, 3, 4\). Then we can verify from (15) and (16) that

\[
\Omega_0 \times \left(0, \frac{T}{2M}\right) \subset D_4 \subset D_3 \subset D_1 \subset \Omega_1 \times (0,T),
\]

(17)

where \(M := N^{-\frac{1}{2M}}\), and

\[
\partial D_1 \subset \Sigma_0 \cup \Sigma_1 \cup \Sigma_2
\]

(18)

are valid. Here \(\Sigma_0 = \{(x,0); \ x \in \Omega\}\), \(\Sigma_1 \subset \Gamma \cup (0,T)\) and \(\Sigma_2 = \{(x,t); \ x \in \Omega, \ t > 0, \ \varphi_1(x,t) = \mu_1\}\).

Now we are ready to give the proof of our main theorem.
Proof of theorem 2. We start from the Cauchy data
\[
\begin{align*}
\begin{cases}
u(x, t) = g_0(x, t) & \text{on } \Gamma \times (0, T], \\
\partial_{\nu} v(x, t) = g_1(x, t) & \text{on } \Gamma \times (0, T]
\end{cases}
\end{align*}
\]
for the equation (2).

Henceforth \(C > 0\) denotes generic constants depending on \(\lambda\), but independent of \(s\) and the choice of \(g_0, g_1, u\). For it, we need a cut-off function because we have no data \(\partial_{\nu} u\) on \(\partial D \setminus \Gamma \times (0, T)\). Let \(\chi \in C^\infty(\mathbb{R}^{n+1})\) satisfy \(0 \leq \chi \leq 1\) and
\[
\chi(x, t) = \begin{cases} 1, & \varphi_1(x, t) > \mu_3, \\
0, & \varphi_1(x, t) < \mu_2. \end{cases}
\]
(19)

Setting \(v := \chi u, \bar{L} := L = -\sum_{j=1}^{\ell} q_j \partial_t^{\alpha_j}\), and then using Leibniz’s formula for the differential of the product we have
\[
\bar{L}v = \chi L u + A_1 u = \chi L u - \chi \sum_{j=1}^{\ell} q_j \partial_t^{\alpha_j} u + A_1 u = \chi F - \chi \sum_{j=1}^{\ell} q_j \partial_t^{\alpha_j} u + A_1 u.
\]
(20)

Here the last term \(A_1 u\) involves only the linear combination of \((\partial_t^\lambda) u, (\partial_t \partial_t^\lambda) u, (\partial_t^\lambda) (\partial_t u)\) and \((\partial_t^\lambda) u, i, j = 1, \ldots, n\).

By (18) and (19), we see that \(v = |\nabla v| = \partial_t v = 0\) on \(\Sigma_2\). Hence using the Carleman estimate in theorem 1 in \(D_1\) to (20), from \(D_3 \subset D_1\) by an argument similar to theorem 3.2 in [27], we find
\[
\int_{D_1} \left\{ \frac{1}{s \varphi_1} |\partial_t v|^2 + s \lambda^2 \varphi_1 |\nabla v|^2 + s^3 \lambda^4 \varphi_1^2 \right\} e^{2s\varphi_1} d\mathbf{x} d\mathbf{t}
\leq C \int_Q F^2 e^{2s\varphi_1} d\mathbf{x} d\mathbf{t} + C \int_{D_1} \sum_{j=1}^{\ell} \chi^2(x, t) |\partial_t^{\alpha_j} u|^2 e^{2s\varphi_1} d\mathbf{x} d\mathbf{t} + C \int_{D_1} |A_1 u|^2 e^{2s\varphi_1} d\mathbf{x} d\mathbf{t}
\]
\[
+ e^{C(\lambda)s} \int_{\Sigma_2 \cup (\Gamma \times (0, T))} (|\nabla v|^2 + v^2) d\Sigma + e^{C(\lambda)s} \int_{\Gamma \times (0, T)} |\partial_t v|^2 d\mathbf{s} d\mathbf{t}
\]
(21)

for all \(s \geq s_0\) and \(\lambda \geq \lambda_0\).

By (19), \(A_1 u\) does not vanish only if \(\mu_2 \leq \varphi_1(x, t) \leq \mu_3\) and so
\[
\int_{D_1} |A_1 u|^2 e^{2s\varphi_1} d\mathbf{x} d\mathbf{t} \leq C e^{2s\mu_3} ||u||^2_{H^{\mu_3}(Q)}.
\]

Moreover, from (17) and lemma 2.1, letting \(C_1 = \{(x, t); x \in \bar{\Omega}, t > 0, \varphi_1(x, t) > \frac{\mu_3 + \mu_4}{2}\}\) and \(C_2 = D_3\) in lemma 2.1, for \(\lambda\) being large fixed, we conclude that the integration \(\int_Q \sum_{j=1}^{\ell} \chi^2(x, t) |\partial_t^{\alpha_j} u|^2 e^{2s\varphi_1} d\mathbf{x} d\mathbf{t}\) can be absorbed by the left-hand side of (21), which implies
\[
\int_{\Omega} \left( \frac{1}{s\varphi_1} |\partial_t u|^2 + s\varphi_1 |\nabla u|^2 + s^3 \varphi_1 u^2 \right) e^{2s\varphi_1} \, dx \, dt \\
\leq Ce^{C_1} \|F\|_{L^2(\Omega)} + Ce^{2\mu_1} \|u\|_{H^{s}(\Omega)}^2 + C \sum_{j=1}^{T} \int_{D_j} |\partial_t^{\alpha} u|^2 e^{2s\varphi_1} \, dx \, dt \\
+ e^{C_4} \int_{\Sigma_0 \cup \{0 \times (0,T)\}} (|\nabla v|^2 + v^2) \, d\Sigma + e^{C_4} \int_{\Gamma \times (0,T)} |\partial_\nu v|^2 \, dS \, dt.
\]

By (16), we can directly verify that \(\varphi_1(x,t) \leq \frac{e^{2s\varphi_1}}{s}\) in \(D_j \backslash C_i\), and if \((x,t) \in \Omega_0 \times (0,\varepsilon)\), then \(\varphi_1(x,t) > \mu_4\). Then combining with (17) and (18), by Hölder's inequality, we have

\[
e^{2\mu_4} \int_{\Omega} \left( \frac{1}{s} |\partial_t u|^2 + s|\nabla u|^2 + s^3 u^2 \right) \, dx + e^{C_4} \int_{\Omega} (|\nabla v(x,0)|^2 + v^2(x,0)) \, dx + e^{C_4} \int_{\Gamma \times (0,T)} (|\partial_\nu v|^2 + |\nabla v|^2 + v^2) \, dS \, dt
\]

for \(s \geq s_0\). Then dividing both sides by \(e^{2\mu_4}\), since

\[
se^{-2\mu_4 \frac{e^{2s\varphi_1}}{s}} \leq Ce^{-\frac{(\mu_4-\mu_3)}{s}}, \quad se^{-2(\mu_4-\mu_3)} \leq Ce^{-s(\mu_4-\mu_3)} \leq Ce^{-\frac{(\mu_4-\mu_3)}{s}}
\]

by replacing \(C\) by \(Ce^{C_1}\), we have

\[
\|u\|_{H^{s}(\Omega_0 \times (0,\frac{T}{m}))}^2 \leq Ce^{-\frac{(\mu_4-\mu_3)}{s}} \|u\|_{H^{s}(\Omega)}^2 + Ce^{C_4} D^2
\]

for all \(s \geq 0\) and \(u \in H^2(\Omega)\), where the constant \(C > 0\) depends on \(T, \Omega_0\) and the coefficients of the equation (2).

Firstly, if \(D = 0\), letting \(s \to \infty\), we conclude that \(u = 0\) in \(\Omega_0 \times (0,\frac{T}{m})\), so that the conclusion of theorem 2 holds true. Next let \(D \neq 0\). First let \(D \geq \|u\|_{H^{s}(\Omega)}\). Then (22) implies

\[
\|u\|_{H^{s}(\Omega_0 \times (0,\frac{T}{m}))} \leq Ce^{C_4} D, \quad s \geq 0.
\]

Second let \(D \leq \|u\|_{H^{s}(\Omega)}\). Then we choose \(s > 0\) minimizing the right-hand side of (22), that is,

\[
e^{-\frac{(\mu_4-\mu_3)}{s}} \|u\|_{H^{s}(\Omega)}^2 = e^{C_4} D^2.
\]

By \(D \neq 0\), we can choose

\[
s = \frac{4}{2C + \mu_4 - \mu_3} \log \frac{\|u\|_{H^{s}(\Omega)}}{D} > 0.
\]

Then (22) gives

\[
\|u\|_{H^{s}(\Omega_0 \times (0,\frac{T}{m}))} \leq 2C\|u\|_{H^{s}(\Omega)}^{1-\theta} D^\theta.
\]
where \( \theta := \frac{\mu_4 - \mu_3}{2 + \mu_3 - \mu_4} \), and the constant \( C \) depends on \( \Omega_0, T \) and the coefficients of the equation (2). We complete the proof of the theorem by setting \( \varepsilon = \frac{T}{2M} \).

3. Carleman estimate for a super-diffusion and its applications

In this section, we pay attention to the Carleman estimate for the equation (6) in the case of \( \alpha = \frac{3}{4} \), and its applications to the lateral Cauchy problem and the inverse source problem. Without loss of generality, we set \( a_{ij} = \delta_{ij} \) here, say,

\[
\partial_t u + q(x)\partial_t^{\frac{3}{4}} u - \Delta u + B(x) \cdot \nabla u + c(x) u = F \quad \text{in } Q.
\]

In the following two subsections, we will give the proofs of theorems 3–5 respectively.

3.1. Carleman estimate for a super-diffusion

In this subsection, we will give a proof of theorem 3. Similarly to the case of \( \alpha < \frac{1}{2} \), we construct the Carleman estimate for the following parabolic type equation

\[
\partial_t u - \Delta u + B \cdot \nabla u + c(x) u = F - q(x)\partial_t^{\frac{3}{4}} u,
\]

whereas here we further multiply both sides of the above equation by several Riemann–Liouville fractional derivatives in order to deal with the new source term \( \partial_t^{\frac{3}{4}} u \). We have the following details of the proof. 

**Proof of theorem 3.** Because of the zero initial condition, we can rewrite the equation (23) as follows

\[
\partial_t u - \Delta u + B \cdot \nabla u + c u = F - q(x)D_t^{\frac{3}{4}} u \quad \text{in } Q.
\]

Recalling the definition of Riemann–Liouville (R–L) fractional integral operator \( D_t^{-p} \) (e.g. see Podlubny [21]):

\[
D_t^{-p} u := \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} u(s) ds,
\]

and we have the semigroup property for the R–L fractional integral operators:

\[
D_t^{-p_1} D_t^{-p_2} u = D_t^{-p_1 - p_2} u, \quad \forall p_1, p_2 > 0.
\]

We first apply the fractional integral operator \( D_t^{-\frac{3}{4}} \) on both sides of equation (24) to derive

\[
D_t^{-\frac{3}{4}} (\partial_t u) - D_t^{-\frac{3}{4}} (\Delta u) + D_t^{-\frac{3}{4}} (B \cdot \nabla u) + D_t^{-\frac{3}{4}} (c u) = D_t^{-\frac{3}{4}} F - D_t^{-\frac{3}{4}} (qD_t^{\frac{3}{4}} u).
\]

Moreover, the homogeneous initial value implies that the differential operators and the R–L integral operator are commutable, which along with the formula \( D_t^{-\frac{3}{4}} (D_t^{\frac{3}{4}} u) = D_t^{-\frac{1}{4} + \frac{1}{4}} u = D_t^{\frac{1}{4}} u \) implies that

\[
\partial_t y - \Delta y + B \cdot \nabla y + cy = D_t^{-\frac{3}{4}} F - qD_t^{\frac{3}{4}} u
\]

(25)
where \( y := D_{\frac{1}{\lambda}}^\frac{1}{z} u \). Similarly, we apply the R–L fractional differential operators \( D^\frac{1}{\lambda} \) and \( D_{\frac{1}{\lambda}}^\frac{1}{z} \) to the equation (24) separately to obtain the following equations:

\[
\partial_t v - \Delta v + B \cdot \nabla v + cv = D^\frac{1}{\lambda} F - q\partial_t u
\]

and

\[
\partial_t w - \Delta w + B \cdot \nabla w + cw = D_{\frac{1}{\lambda}}^\frac{1}{z} F - qD^\frac{1}{\lambda} w
\]

where \( v := D^\frac{1}{\lambda} u \) and \( w := D_{\frac{1}{\lambda}}^\frac{1}{z} u \).

For simplicity, we denote

\[
\tilde{L} u := \partial_t u - \Delta u + B \cdot \nabla u + cu
\]

and

\[
\tilde{u} := \chi_0 u, \quad \tilde{v} := \chi_0 v, \quad \tilde{w} := \chi_0 w.
\]

Then by noting that

\[
\tilde{L}(\chi_0 u) - \chi_0 \tilde{L} u = (\partial_t \chi_0) u - 2\nabla \chi_0 \cdot \nabla u - (\Delta \chi_0) u + (B \cdot \nabla \chi_0) u,
\]

the equations (24)–(27) can be rewritten by:

\[
\tilde{L}(\tilde{u}) = \chi_0 F - \chi_0 q(\partial_t y) - 2\nabla \chi_0 \cdot \nabla u + (B \cdot \nabla \chi_0 - \Delta \chi_0 + \partial_t \chi_0) u,
\]

\[
\tilde{L}(\tilde{v}) = \chi_0 D^\frac{1}{\lambda} F - \chi_0 q(\partial_t w) - 2\nabla \chi_0 \cdot \nabla v + (B \cdot \nabla \chi_0 - \Delta \chi_0 + \partial_t \chi_0) v
\]

and

\[
\tilde{L}(\tilde{w}) = \chi_0 D_{\frac{1}{\lambda}}^\frac{1}{z} F - \chi_0 q(\partial_t v) - 2\nabla \chi_0 \cdot \nabla w + (B \cdot \nabla \chi_0 - \Delta \chi_0 + \partial_t \chi_0) w.
\]

Now we use the Carleman estimate for parabolic type stated in the following lemma.

**Lemma 3.1.** Let \( F \in L^2(Q) \). Then there exist constants \( \lambda \geq 1, \tilde{s} \geq 1 \) and \( C > 0 \) such that

\[
\int_Q \left\{ s^{\tilde{s} - 1} \varphi_2^{\tilde{s} - 1} \left( (\partial_t u)^2 + \sum_{i,j=1}^n (\partial_i \partial_j u)^2 \right) + s^{\tilde{s} + 1} \lambda^2 \varphi_2^{\tilde{s} + 1} |\nabla u|^2 + s^{\tilde{s} + 3} \lambda^4 \varphi_2^{\tilde{s} + 3} |u|^2 \right\} e^{2\lambda s} \, dx \, dt \leq C \int_Q s^{\tilde{s} + 3} |F|^2 e^{2\lambda s} \, dx \, dt + C(\lambda) e^{C(\lambda) s} \int_Q (|\nabla u|^2 + |u|^2) \, dx \, dt
\]

for all \( \tilde{s} \geq \tilde{s}, \lambda \geq \lambda, \tau \in \mathbb{Z} \) and all \( u \) smooth enough satisfying the equation:

\[
\tilde{L} u = F
\]

with conditions \( u(\cdot, 0) = u(\cdot, T) = 0 \) on \( \tilde{\Omega} \).
The proof of this lemma is similar to that of theorem 3.2 in [27]. Here we assume \( u(t, 0) = u(\cdot, T) = 0 \), because we consider the Carleman estimate in the whole cylindrical domain \( Q = \Omega \times (0, T) \), so that the integration by parts in \( t \) for the proof produces terms of \( u(t, 0) \) and \( u(\cdot, T) \). For eliminating those terms we need the assumption. We here omit the details of the proof.

Applying the above lemma to the equations (28)–(31) respectively, we have the following Carleman estimates:

\[
\int_Q \left\{ \frac{1}{s^2 \phi_2^2} \left( |\partial_t \tilde{v}|^2 + \sum_{i,j=1}^n |\partial_i \partial_j \tilde{v}|^2 \right) + \lambda^2 |\nabla \tilde{v}|^2 + s \lambda \phi_2 |\tilde{v}|^2 \right\} e^{2s \tilde{v}} \, ds dt
\]

\[
\leq C \int_Q \left\{ \frac{1}{s^2 \phi_2^2} \left( |\chi_0 \partial_t \tilde{u}|^2 + |\chi_0 (\partial \tilde{u})|^2 + |\tilde{L}(\chi_0 \tilde{u}) - \chi_0 \tilde{L} \tilde{u}|^2 \right) e^{2s \tilde{v}} \, ds dt + C(\lambda) e^{\lambda s} \int_\Sigma \left( |\nabla_x \tilde{u}|^2 + |\tilde{u}|^2 \right) dSdt
\]

for all \( \lambda \geq \lambda_1 \) and \( s \geq s_1 \).

\[
\int_Q \left\{ \frac{1}{s^2 \phi_2^2} \left( |\partial_t \tilde{w}|^2 + \sum_{i,j=1}^n |\partial_i \partial_j \tilde{w}|^2 \right) + \lambda^2 |\nabla \tilde{w}|^2 + s \lambda \phi_2 |\tilde{w}|^2 \right\} e^{2s \tilde{v}} \, ds dt
\]

\[
\leq C \int_Q \left\{ \frac{1}{s^2 \phi_2^2} \left( |\chi_0 \partial_t \tilde{u}|^2 + |\chi_0 (\partial \tilde{u})|^2 + |\tilde{L}(\chi_0 \tilde{u}) - \chi_0 \tilde{L} \tilde{u}|^2 \right) e^{2s \tilde{v}} \, ds dt + C(\lambda) e^{\lambda s} \int_\Sigma \left( |\nabla_x \tilde{u}|^2 + |\tilde{u}|^2 \right) dSdt
\]

for all \( \lambda \geq \lambda_2 \) and \( s \geq s_2 \).

\[
\int_Q \left\{ \frac{1}{s^2 \phi_2^2} \left( |\partial_t \tilde{y}|^2 + \sum_{i,j=1}^n |\partial_i \partial_j \tilde{y}|^2 \right) + \lambda^2 |\nabla \tilde{y}|^2 + s \lambda \phi_2 |\tilde{y}|^2 \right\} e^{2s \tilde{v}} \, ds dt
\]

\[
\leq C \int_Q \left\{ \frac{1}{s^2 \phi_2^2} \left( |\chi_0 \partial_t \tilde{u}|^2 + |\chi_0 (\partial \tilde{u})|^2 + |\tilde{L}(\chi_0 \tilde{u}) - \chi_0 \tilde{L} \tilde{u}|^2 \right) e^{2s \tilde{v}} \, ds dt + C(\lambda) e^{\lambda s} \int_\Sigma \left( |\nabla_x \tilde{u}|^2 + |\tilde{u}|^2 \right) dSdt
\]

for all \( \lambda \geq \lambda_3 \) and \( s \geq s_3 \).

\[
\int_Q \left\{ \frac{1}{s^2 \phi_2^2} \left( |\partial_t \tilde{z}|^2 + \sum_{i,j=1}^n |\partial_i \partial_j \tilde{z}|^2 \right) + s \lambda \phi_2 |\nabla \tilde{z}|^2 + s \lambda \phi_2 |\tilde{z}|^2 \right\} e^{2s \tilde{v}} \, ds dt
\]

\[
\leq C \int_Q \left\{ \frac{1}{s^2 \phi_2^2} \left( |\chi_0 \partial_t \tilde{u}|^2 + |\chi_0 (\partial \tilde{u})|^2 + |\tilde{L}(\chi_0 \tilde{u}) - \chi_0 \tilde{L} \tilde{u}|^2 \right) e^{2s \tilde{v}} \, ds dt + C(\lambda) e^{\lambda s} \int_\Sigma \left( |\nabla_x \tilde{u}|^2 + |\tilde{u}|^2 \right) dSdt
\]

for all \( \lambda \geq \lambda_4 \) and \( s \geq s_4 \).

Moreover, we observe that a direct calculation implies that
\[
| \partial_t \tilde{u} |^2 = |(\partial_t \chi_0) u + \chi_0 (\partial_t u) |^2 \geq \frac{1}{2} |\chi_0 (\partial_t u) |^2 - |(\partial_t \chi_0) u |^2.
\]
\[
| \partial_t \tilde{v} |^2 = |\chi_0 (\partial_v \chi_0) |^2 - 3 |(\partial_t \chi_0) (\partial_t u) |^2 - 3 |(\partial_t \chi_0) |^2 \geq \frac{1}{2} |\chi_0 (\partial_t u) |^2 - |\chi_0 | |\partial_t \chi_0 | |u |^2.
\]
\[
| \nabla \tilde{u} |^2 = |\chi_0 \nabla u + u \nabla \chi_0 |^2 \geq \frac{1}{2} |\chi_0 \nabla u |^2 - |\nabla \chi_0 | |u |^2.
\]

The same calculations yield
\[
| \partial_t \tilde{v} |^2 \geq \frac{1}{2} |\chi_0 (\partial_v \chi_0) |^2 - |(\partial_t \chi_0) |^2,
\]
\[
| \nabla \tilde{v} |^2 \geq \frac{1}{2} |(\partial_t \chi_0) |^2 - |\chi_0 | |\partial_t \chi_0 | |w |^2.
\]

Thus, we can write the four Carleman estimates as
\[
\int_Q \left\{ s^{-3} \varphi_2^3 |(\chi_0 \partial_t u) |^2 + |\sum_{i,j=1}^n (\partial_t \partial_t \partial_j \partial_j u) |^2 + \lambda^2 |\chi_0 \nabla u |^2 + s \lambda^2 |\varphi_2 | |\chi_0 u |^2 \right\} e^{2\varphi_2} \, dx \, dr 
\]
\[
\leq C \int_Q s^{-1} \varphi_2^3 (|\chi_0 |^2 + |\chi_0 |^2) e^{2\varphi_2} \, dx \, dr + C (\lambda) s^{-1} e^{C(\lambda) s} \int_Q ((|\nabla \chi_0 (\chi_0 u) |^2 + |\chi_0 u |^2) e^{2\varphi_2} \, dx \, dr + C \int_Q s^{-1} \varphi_2^3 (|\tilde{u}(\chi_0 u) |^2 - |\chi_0 \tilde{u} |^2 + |\partial_t \chi_0 |^2 |v |^2 + s \lambda^2 |\varphi_2 | |\chi_0 u |^2 |w |^2) e^{2\varphi_2} \, dx \, dr
\]
\[
(32)
\]
for all \( \lambda \geq \lambda_1 \) and \( s \geq s_1 \).
\[
\int_Q \left\{ s^{-3} \varphi_2^3 |\chi_0 (\partial_t u) |^2 + s^{-1} \lambda^2 \varphi_2^2 |\chi_0 \nabla u |^2 + s \lambda^2 |\varphi_2 | |\chi_0 u |^2 \right\} e^{2\varphi_2} \, dx \, dr 
\]
\[
\leq C \int_Q s^{-1} \varphi_2^3 (|\chi_0 |^2 + |\chi_0 |^2) e^{2\varphi_2} \, dx \, dr + C (\lambda) s^{-1} e^{C(\lambda) s} \int_Q ((|\nabla \chi_0 (\chi_0 u) |^2 + |\chi_0 u |^2) e^{2\varphi_2} \, dx \, dr + C \int_Q s^{-2} \varphi_2^2 (|\tilde{u}(\chi_0 u) |^2 - |\chi_0 \tilde{w} |^2 + |\partial_t \chi_0 |^2 |v |^2 + s \lambda^2 |\varphi_2 | |\chi_0 u |^2 |w |^2) e^{2\varphi_2} \, dx \, dr
\]
\[
(33)
\]
for all \( \lambda \geq \lambda_2 \) and \( s \geq s_2 \).
\[
\int_Q \left\{ s^{-3} \varphi_2^3 |\chi_0 (\partial_t w) |^2 + s^{-2} \lambda^2 \varphi_2^2 |\chi_0 \nabla w |^2 + \lambda^4 |\chi_0 w |^2 \right\} e^{2\varphi_2} \, dx \, dr 
\]
\[
\leq C \int_Q s^{-1} \varphi_2^3 (|\chi_0 |^2 + |\chi_0 |^2) e^{2\varphi_2} \, dx \, dr + C (\lambda) s^{-3} e^{C(\lambda) s} \int_Q ((|\nabla \chi_0 (\chi_0 w) |^2 + |\chi_0 w |^2) e^{2\varphi_2} \, dx \, dr + C \int_Q s^{-3} \varphi_2^3 (|\tilde{w}(\chi_0 w) |^2 - |\chi_0 \tilde{w} |^2 + |\partial_t \chi_0 |^2 |v |^2 + s \lambda^2 |\varphi_2 | |\chi_0 w |^2 |w |^2) e^{2\varphi_2} \, dx \, dr
\]
\[
(34)
\]
for all \( \lambda \geq \lambda_4 \) and \( s \geq s_4 \). And
\[
\int_0^s \left\{ s^{-1} \varphi_2^{-1} |\chi_0(\partial_t y)|^2 + s \lambda^2 \varphi_2 |\chi_0 \nabla y|^2 + s^3 \lambda^4 \varphi_2^3 |\chi_0 y|^2 \right\} e^{2s\varphi_2} \, dx \, dr \\
\leq C \int_0^s \left\{ (|\chi_0 D_{s-1}^{-1} F|^2 + |\chi_0 w|^2) e^{2s\varphi_2} \, dx \, dr \right. \\
+ \left. C(\lambda) e^{C(\lambda)s} \int_\Sigma (|\nabla_x (\chi_0 y)|^2 + |\chi_0 y|^2) \, dS \, dr \right\} e^{2s\varphi_2} \, dx \, dr
\]
(35)

for all \( \lambda \geq \lambda_4 \) and \( s \geq s_4 \). Combining the Carleman inequalities (32)–(35), we calculate (32) \( \times \frac{1}{\lambda} + (33) \times \frac{1}{\lambda^2} + (34) \times \frac{1}{\lambda^3} + (35) \), and we note that
\[
\int_0^s \chi_0^2 \left( s^{-4} \lambda^{-3} \varphi_2^{-4} |\partial_t w|^2 + s^{-3} \lambda^{-2} \varphi_2^{-3} |\partial_t \partial_y |^2 + s^{-2} \lambda^{-1} \varphi_2^{-2} |\partial_t u|^2 + s^{-1} \varphi_2^{-1} |\partial_y |^2 \right) + \lambda |w|^2 + s \lambda^2 \varphi_2 |v|^2 \\
+ s^2 \lambda^3 \varphi_2^3 |u|^2 + s^3 \lambda^4 \varphi_2^4 |\nabla u|^2 + s^2 \lambda^2 \varphi_2^2 |\partial_t \partial_y |^2 \left( \sum_{i=1}^{n} |\partial_y \partial_x |^2 \right) \right) e^{2s\varphi_2} \, dx \, dr \\
\leq C \int_0^s \chi_0^2 \left( \sum_{j=1}^{2} s^{-j} \lambda^{-j-1} \varphi_2^{-j-1} |D_j^1 F|^2 \right) e^{2s\varphi_2} \, dx \, dr + \text{LOW + BDY} \\
+ C \int_0^s \chi_0^2 \left( s^{-3} \lambda^{-3} \varphi_2^{-3} |\partial_t v|^2 + s^{-2} \lambda^{-2} \varphi_2^{-2} |\partial_t u|^2 + s^{-1} \varphi_2^{-1} |\partial_y |^2 \right) + \lambda |w|^2 + s \lambda^2 \varphi_2 |v|^2 \right) e^{2s\varphi_2} \, dx \, dr
\]
for all \( \lambda \geq \lambda_4 \) := \max_{1 \leq i \leq 4} \lambda_i \) and \( s \geq s_5 \) := \max_{1 \leq i \leq 4} s_i \). Here LOW and BDY are the lower order terms and boundary terms determined as
\[
\text{LOW} = C \int_0^s \left( |\partial_t \chi_0|^2 + |\nabla \chi_0|^2 + \sum_{i=1}^{n} |\partial_y \chi_0 |^2 \right) \left( \sum_{j=1}^{2} \left( |\nabla(D_j^1 u)|^2 + |D_j^1 u|^2 \right) \right) e^{2s\varphi_2} \, dx \, dr, \\
(36)
\]
\[
\text{BDY} = C e^{C^3} \int_{\Gamma \times (0, T)} \sum_{j=1}^{2} \left( |\nabla_x (D_j^1 u)|^2 + |D_j^1 u|^2 \right) \, dS \, dr. \\
(37)
\]
From the choice of cut-off function \( \chi_0 \), we note that \( \chi_0 \) and its derivatives vanish on the parts \( \Omega \times \{0, T\} \) and \( (\partial \Omega \setminus \Gamma) \times (0, T) \). Thus the boundary terms BDY survive only on the part \( \Gamma \times (0, T) \).

Now by noting that
\[
||D_\alpha^\beta h||_{L^2(0, T)} \leq C ||D_\alpha^\beta h||_{L^2(0, T)}, \quad 0 < \alpha < \beta < 2, \\
(38)
\]
provided that \( h(0) = 0 \), which actually can be derived by using the definition of \( D_\alpha^\beta \) and Young’s inequality, we see that
\[
\text{BDY} \leq C e^{C^3} \int_{\Gamma \times (0, T)} \left( |\nabla_x (D_\alpha^\beta u)|^2 + |D_\alpha^\beta u|^2 \right) \, dS \, dr.
\]
We will write the right hand side of the above inequality simply by BDY when no confusion can arise.
By fixing $\lambda$ large enough, we can absorb the last term on the right-hand side into the left-hand side which leads to

$$
\int_Q \lambda^2 \left( \sum_{j=1}^6 s^{2-j} \nu_2^{2-j} |D_t^j u|^2 + |\nabla u|^2 + s^{-2} \nu_2^{-2} \sum_{i=1}^n |\partial_i \partial_j u|^2 \right) e^{2s^2 \nu_2^2} \mathrm{d}x \mathrm{d}t \leq C \int_Q \lambda^2 \left( s^{-1-j} \nu_2^{1-j} \sum_{j=1}^2 |D_t^j F|^2 \right) e^{2s^2 \nu_2^2} \mathrm{d}x \mathrm{d}t + \text{LOW} + \text{BDY}.
$$

Next we apply a similar argument to obtain estimates of the time derivatives of higher orders. Let us apply R–L fractional differential operators $D_t^k$, $k = 3, 4, 5, 6$ to equation (24) respectively and obtain

$$
D_t^k (\partial_t u) - D_t^k (\Delta u) + D_t^k (B \cdot \nabla u) + D_t^k (cu) = D_t^k F - D_t^k (qD_t^3 u), \quad k = 3, 4, 5, 6.
$$

By definition of R–L fractional derivatives, we have the following equations:

$$
D_t^\gamma (D_t^\alpha u) = D_t^{\gamma+\alpha} u, \quad \gamma > 0, \quad 0 < \alpha < 1,
$$

$$
D_t^\gamma (\partial_t u) = \partial_t (D_t^\gamma u) = D_t^{\gamma+1} u, \quad \gamma > 0
$$

under the additional condition

$$
u(0) = 0.
$$

The above relations can be verified assuming enough regularity of $u$ and we refer to Podlubny [21] for the detailed calculations. Therefore we rewrite the equation (24) to obtain

$$
\partial_t u_k - \Delta u_k + B \cdot \nabla u_k + cu_k = D_t^k F - qD_t^{12} u_k
$$

with $u_k := D_t^k u$, $k = 3, 4, 5, 6$. Again by denoting $\tilde{u}_k := \chi_0 u_k$, we have

$$
\tilde{L} (\tilde{u}_k) = \chi_0 D_t^k F - \chi_0 qD_t^{12} u - 2\nabla \chi_0 \cdot \nabla u_k + (B \cdot \nabla \chi_0 - \Delta \chi_0 + \partial_t \chi_0) u_k, \quad k = 3, 4, 5, 6.
$$

We employ lemma 3.1 to the above equations with $\tau$ taking $\tau_k = 3 - k$, $k = 3, 4, 5, 6$ respectively. Thus we have the following Carleman estimates:

$$
\int_Q \left( s^{2-k} \nu_2^{-2} |\partial_t \tilde{u}_k|^2 + s^{2-k} \nu_2^{-2} |\tilde{u}_k|^2 \right) e^{2s^2 \nu_2^2} \mathrm{d}x \mathrm{d}t \leq C \int_Q s^{3-k} \nu_2^{1-k} \left( |\chi_0 D_t^k F|^2 + |\chi_0 qD_t^{12} u_k|^2 \right) e^{2s^2 \nu_2^2} \mathrm{d}x \mathrm{d}t
$$

$$
+ C \int_Q s^{3-k} \nu_2^{1-k} |\tilde{L} (\chi_0 u_k) - \chi_0 \tilde{L} (u_k)|^2 e^{2s^2 \nu_2^2} \mathrm{d}x \mathrm{d}t + C(\lambda) e^{C(\lambda)s^{3-k}} \int_{\Sigma} |\nabla_x \tilde{u}_k|^2 + |\tilde{u}_k|^2 \mathrm{d}S d\tau
$$

for all $\lambda \geq \lambda$ and $s \geq \hat{s}$, $k = 3, 4, 5, 6$. Direct calculation yields

$$
|\partial_t \tilde{u}_k|^2 = |(\partial_t \chi_0) u_k + \chi_0 (\partial_t u_k)|^2 \geq \frac{1}{2} |(\partial_t \chi_0) u_k|^2 - |(\partial_t \chi_0) u_k|^2.
$$
Then the above estimates imply
\[
\int \lambda^\frac{3}{2} \sum_{k=3}^{10} (s \lambda \varphi_2)^{6-k} |D_7^k u|^2 e^{2s\varphi_2^2} dx dr 
\leq C \int \lambda_0^2 \sum_{k=3}^{6} (s \lambda \varphi_2)^{3-k} \left( |D_7^k u|^2 + |D_7^k u|^2 \right) e^{2s\varphi_2^2} dx dr 
+ C \int \lambda_0^2 \sum_{k=3}^{6} (s \lambda \varphi_2)^{3-k} \left( \tilde{L}(\lambda_0 u_k) - \chi_0 \tilde{L}(u_k) \right)^2 + |(\partial_1 \lambda_0 u_k)|^2 e^{2s\varphi_2^2} dx dr 
+ C(\lambda) e^{C(\lambda)s} \int \sum_{k=3}^{6} \left( \|
abla_{x,t} \tilde{u}_k \|^2 + |\tilde{u}_k|^2 \right) dS dr
\]
for all \( \lambda \geq \hat{\lambda} \) and \( s \geq \hat{s} \), \( k = 3, 4, 5, 6 \). Summing up the above estimates with weight \( \lambda^{3-k} \) leads to
\[
\lambda \int \lambda_0^2 \sum_{k=3}^{10} (s \lambda \varphi_2)^{6-k} |D_7^k u|^2 e^{2s\varphi_2^2} dx dr 
\leq C \int \lambda_0^2 \sum_{k=3}^{6} (s \lambda \varphi_2)^{3-k} \left( |D_7^k u|^2 + |D_7^k u|^2 \right) e^{2s\varphi_2^2} dx dr 
+ C \int \lambda_0^2 \sum_{k=3}^{6} (s \lambda \varphi_2)^{3-k} \left( \tilde{L}(\lambda_0 u_k) - \chi_0 \tilde{L}(u_k) \right)^2 + |(\partial_1 \lambda_0 u_k)|^2 e^{2s\varphi_2^2} dx dr 
+ C(\lambda) e^{C(\lambda)s} \int \sum_{k=3}^{6} \left( \|
abla_{x,t} \tilde{u}_k \|^2 + |\tilde{u}_k|^2 \right) dS dr
\]
for all \( \lambda \geq \hat{\lambda} \) and \( s \geq \hat{s} \). By fixing \( \lambda \) large enough, we can absorb the term
\[
C \int \lambda_0^2 \sum_{k=3}^{6} (s \lambda \varphi_2)^{3-k} |D_7^k u|^2 e^{2s\varphi_2^2} dx dr = C \int \lambda_0^2 \sum_{k=6}^{9} (s \lambda \varphi_2)^{6-k} |D_7^k u|^2 e^{2s\varphi_2^2} dx dr
\]
on the RHS into the LHS. This yields
\[
\int \lambda_0^2 \sum_{k=3}^{10} (s \varphi_2)^{6-k} |D_7^k u|^2 e^{2s\varphi_2^2} dx dr 
\leq C \int \lambda_0^2 \sum_{k=3}^{6} (s \lambda \varphi_2)^{3-k} |D_7^k F|^2 e^{2s\varphi_2^2} dx dr + \text{LOW1} + \text{BDY1}
\]
for all \( s \geq \hat{s} \) with
\[
\text{LOW1} = C \int \left( |\partial_1 \chi_0|^2 + \|
abla \chi_0\|^2 + \sum_{i,j=1}^{n} |\partial_i \partial_j \chi_0|^2 \right) \left[ \sum_{k=3}^{6} \left( |\nabla (D_7^k u)|^2 + |D_7^k u|^2 \right) \right] e^{2s\varphi_2^2} dx dr 
\]
\[
\text{BDY1} = C e^{C(\lambda)} \int_{\Gamma \times (0,T)} \sum_{k=3}^{6} \left( |\nabla_{x,t} (D_7^k u)|^2 + |D_7^k u|^2 \right) dS dr.
\]
This completes the proof of theorem 3.

**Remark 3.1.** According to our arguments, we point out that the diffusion equation with fractional order \( \alpha = \frac{3}{2} \) is the critical case which our methods can deal with. In fact, our methods can work for any positive rational fractional order which is less than \( \frac{3}{2} \). The idea of proof
is given in the appendix. Moreover, our methods can also work in the case when the equation (8) has more than one fractional derivatives of rational orders less than $\frac{1}{2}$.

3.2. Application to a lateral Cauchy problem for the super-diffusion

In this subsection, we employ the Carleman estimate in theorem 3 to investigate the conditional stability for the lateral Cauchy problem. For this, we recall the partial differential equation:

$$
\partial_t u + q(x)\partial_x^2 u - \Delta u + B \cdot \nabla u + cu = F \quad \text{in } Q
$$

with the zero initial condition:

$$
u(x, 0) = 0, \quad x \in \Omega.
$$

We mainly follow the steps in [27] theorem 5.1. Instead of a classical Carleman estimate for parabolic equation, we should apply our Carleman estimate established in section 3.1.

**Proof of theorem 4.** By the choice of $\Omega_0$, we have $\overline{\Omega_0} \subset \Omega_1$ where $\Omega_1$ is defined in section 1. Then we can find a sufficiently large $N > 1$ such that

$$
\left\{ x \in \Omega_1; d(x) > \frac{4}{N} \|d\|_{C(\overline{\Omega})} \right\} \cap \overline{\Omega} \supset \Omega_0.
$$

Moreover we choose $\beta > 0$ such that

$$
\beta e^2 \leq \|d\|_{C(\overline{\Omega})} \leq 2\beta e^2.
$$

For any $t_0 \in [\sqrt{2}e, T - \sqrt{2}e)$, we set $\mu_k := \exp\{\lambda(\frac{1}{N} \|d\|_{C(\overline{\Omega})} - \frac{\alpha^2}{N} + c_0)\}$, $E_k := \{(x, t) \in Q_1; \varphi_2(x, t) > \mu_k\}$ and $D_k := E_k \cap Q_1, k = 1, 2, 3, 4$. Recall that $c_0$ is the same constant as that in (7). Then we can easily verify the following two facts:

(i) $\Omega_0 \times (t_0 - \frac{\sqrt{2}e}{N}, t_0 + \frac{\sqrt{2}e}{N}) \subset D_4 \subset D_3 \subset D_2 \subset D_1 \subset \Omega \times (t_0 - \sqrt{2}e, t_0 + \sqrt{2}e)$,

(ii) $\partial D_1 \subset \Sigma_1 \cup \Sigma_2, \quad \Sigma_1 \subset \Gamma \times (0, T), \quad \Sigma_2 = \{(x, t) \in Q; \varphi_2(x, t) = \mu_1\}.

Next, we specify the cut-off function $\chi_0$ with $D_0 = E_3$ and $D = E_2$. In detail, let $\chi_0 \in C^\infty(\mathbb{R}^{n+1})$ satisfy $0 \leq \chi_0 \leq 1$ and

$$
\chi_0(x, t) = \begin{cases} 1, \quad \varphi_2(x, t) > \mu_3, \\
0, \quad \varphi_2(x, t) \leq \mu_2. 
\end{cases}
$$

Thus from theorem 3, it follows that

$$
\int_{D_2} \chi_0^2 \left( \sum_{j=1}^{6} s^{-j} |D_j^2 u|^2 + |\nabla u|^2 + s^{-2} \sum_{i,j=1}^n |\partial_i \partial_j u|^2 \right) e^{2\varphi_2} \, dx \, dt 
\leq C \int_{D_2} \chi_0^2 \left( \sum_{j=1}^{2} |D_j^2 F|^2 \right) e^{2\varphi_2} \, dx \, dt + \text{LOW} + \text{BDY}
$$

(39)

for all $s \geq s_0$ where LOW and BDY are the same notations as (36) and (37). In the above inequality, we have used the fact that $\varphi_2$ is upper and lower bounded by constants which are
independent of \(s\). Since the second term LOW on the RHS includes some derivatives of \(\chi_0\), it vanishes in \(E_3\) and outside of \(E_2\), which implies that \(\varphi_2(x, t) \leq \mu_3\) in \(E_2 \setminus E_3 \supset D_2 \setminus D_3\). Together with the relation \(\Omega_0 \times (t_0 - \frac{\epsilon}{\sqrt{N}}, t_0 + \frac{\epsilon}{\sqrt{N}}) \subset D_4 \subset D_3 \subset D_2 \subset D_1\) and \(\chi_0 = 1\) in \(D_4 \subset E_4 \subset E_3\), we obtain

LHS of (39) \(\geq \int_{D_4} \chi_0 \left( \sum_{j=1}^{n} s^{2-j}[D_1^j u]^2 + |\nabla u|^2 + s^{-2} \sum_{j=1}^{n} |\partial_j \partial_\alpha u|^2 \right) e^{2\mu_2} dx dt \)

\(= \int_{D_4} \left( \sum_{j=1}^{n} s^{2-j}[D_1^j u]^2 + |\nabla u|^2 + s^{-2} \sum_{j=1}^{n} |\partial_j \partial_\alpha u|^2 \right) e^{2\mu_2} dx dt \)

\(\geq e^{2\mu_2} \int_{0}^{t_1} \int_{\Omega_0} \left( \sum_{j=1}^{n} s^{2-j}[D_1^j u]^2 + |\nabla u|^2 + s^{-2} \sum_{j=1}^{n} |\partial_j \partial_\alpha u|^2 \right) dx dt\),

and

RHS of (39) \(\leq C \int_{D_4} \chi_0 \left( \sum_{j=1}^{n} [D_1^j F]^2 \right) e^{2\mu_2} dx dt + C \int_{D_4} \left( \sum_{j=1}^{n} |D_1^j u|^{2} + |D_1^j \nabla u|^2 \right) e^{2\mu_2} dx dt + C e^{C_\mu_2} \int_{(0,T)} \left( \sum_{j=1}^{n} |D_1^j u|^{2} + |D_1^j \nabla u|^2 \right) dx dt\)

\(\leq C e^{C_\mu_2} \left( \sum_{j=1}^{n} \|D_1^j u\|_{L^2(\Omega \times (0,T))} + \sum_{j=1}^{n} \|D_1^j \nabla u\|_{L^2(\Omega \times (0,T))} \right)\)

\(\leq C e^{C_\mu_2} M^2 + C e^{C_\mu_2} D^2\)

where \(M\) and \(D\) are defined in theorem 4.

Therefore (39) yields

\[\|u\|_{H^1(\Omega \times (t_0 - \frac{\epsilon}{\sqrt{N}}, t_0 + \frac{\epsilon}{\sqrt{N}}))} \leq C e^{-2(\mu_4 - \mu_3)} M^2 + C e^{C_\mu_2} D^2\]

for all \(s \geq s_0\). Moreover, we choose \(s \rightarrow s_1\) satisfying \(s^2 e^{-s(\mu_4 - \mu_3)} \leq C\) and \(s^2 \leq e^{C_\mu_2}\) in order that \(s^2 e^{-2(\mu_4 - \mu_3)} \leq C e^{-s(\mu_4 - \mu_3)}\). We choose again large \(C > 0\), and we have

\[\|u\|_{H^1(\Omega \times (t_0 - \frac{\epsilon}{\sqrt{N}}, t_0 + \frac{\epsilon}{\sqrt{N}}))} \leq C^{-C_\mu_4} M^2 + C e^{C_\mu_2} D^2\]

for all \(s \geq s_2 := \max\{s_0, s_1\}\), \(C_0 := \mu_4 - \mu_3 > 0\). After the change of the variable \(s \rightarrow s_2\) \(\rightarrow s\), we obtain

\[\|u\|_{H^1(\Omega \times (t_0 - \frac{\epsilon}{\sqrt{N}}, t_0 + \frac{\epsilon}{\sqrt{N}}))} \leq C^{-C_\mu_4} M^2 + C e^{C_\mu_2} D^2\]

for all \(s \geq 0\). The generic constant \(C\) depends on \(s_2, \epsilon\) and the choice of \(\Omega_0\), etc, but is still independent of \(s\).

Let \(m \in \mathbb{N}\) satisfy

\[\sqrt{2} \epsilon + \frac{me}{\sqrt{N}} \leq T - \sqrt{2} \epsilon < \sqrt{2} \epsilon + \frac{(m+1) \epsilon}{\sqrt{N}} < T.\]
We here notice that the constant $C$ is also independent of $t_0$ provided that $t_0 \in [\sqrt{2}\epsilon, T - \sqrt{2}\epsilon]$.

Thus, by taking $t_0 = \sqrt{2}\epsilon + \frac{k\epsilon}{\sqrt{N}}$, $k = 0, 1, 2, \ldots, m$ in (40), summing up over $k$ and replacing $\sqrt{2}\epsilon$ by $\epsilon$, we obtain

$$\|u\|_{H^1(\Omega_0 \times (\epsilon, T - \epsilon))} \leq Ce^{-C_0 M^2} + Ce^{C_0 D^2}$$

for all $s \geq 0$.

Finally, we show the estimate of Hölder type.

Firstly, let $D = 0$. Then letting $s \to \infty$ in (41), we see that $u = 0$ in $\Omega_0 \times (\epsilon, T - \epsilon)$. Thus the conclusion holds true.

Secondly, let $D \neq 0$. We divide it into two cases. In the case of $D \geq M$, we see that (41) implies $\|u\|_{H^1(\Omega_0 \times (\epsilon, T - \epsilon))} \leq Ce^{C_0 D}$ for all $s \geq 0$. This has already proved the theorem. Now let $D \leq M$. We choose $s > 0$ minimizing the right-hand side of (41), that is,

$$e^{-C_0 M^2} = e^{C_0 D^2},$$

which yields

$$s = \frac{2}{C + C_0} \log \frac{M}{D}$$

in view of that $D \neq 0$. Then (41) gives

$$\|u\|_{H^1(\Omega_0 \times (\epsilon, T - \epsilon))} \leq 2CM^{\frac{C_0}{C + C_0}}D^{\frac{2C_0}{C + C_0}}.$$  

The proof of theorem 4 is completed. \hfill \Box

3.3. Application to an inverse source problem for the super-diffusion

In this subsection, we consider another application of theorem 3, say, the stability for the inverse source problem 1. We point out here that the uniqueness result for this kind of inverse problem can be proved by a similar argument in [10].

Before giving the proof of theorem 5, we first recall the notation of the bounded domain $\Omega_1$ defined in (3), and a function $d \in C^2(\Omega_1)$ can be chosen so that (4) holds. Next for any $\epsilon > 0$, we recall the level set of $\epsilon$ related to the function $d$ as follows

$$\Omega_\epsilon := \{x \in \Omega; d(x) > \epsilon\}.$$  

Note that $\Omega_\epsilon \neq \emptyset$ for small $\epsilon$ and $\overline{\Omega_\epsilon} \cap \partial \Omega \subset \Gamma$. Now we are ready to give a proof of the stability result.

**Proof of theorem 5.** Recall that our weight function is chosen as

$$\varphi_2(x, t) = e^{\lambda \psi_2(x, t)}, \quad \psi_2(x, t) = d(x) - \beta(t - t_0)^2 + c_0.$$  

Here, $c_0$ is a constant such that $\psi_2$ is always nonnegative and $s, \lambda$ are two large parameters while the parameter $\beta > 0$ will be fixed later. Then on both sides of our governing equation (8), we integrate over the domain $\Omega_{3\epsilon}$ at time $t = t_0$ and obtain

$$\|u\|_{H^1(\Omega_0 \times (\epsilon, T - \epsilon))} \leq Ce^{-C_0 M^2} + Ce^{C_0 D^2}$$
\[
\int_{\Omega_c} |R(x, t_0) f(x)|^2 e^{2\varphi_2(x, t_0)} \, dx \leq \int_{\Omega_c} |\partial_t u(x, t_0)|^2 e^{2\varphi_2(x, t_0)} \, dx + C \int_{\Omega_c} |D^2 u(x, t_0)|^2 e^{2\varphi_2(x, t_0)} \, dx \\
\quad + \int_{\Omega_c} |-\Delta u(x, t_0) + B(x) \cdot \nabla u(x, t_0) + c(x) u(x, t_0)|^2 e^{2\varphi_2(x, t_0)} \, dx.
\]

(42)

Easily, we see that the third term on the RHS is bounded by \( Ce^C ||u(\cdot, t_0)||^2_{L^p(\Omega_c)} \) and the LHS can be estimated with some \( C_0 > 0 \):

\[
\int_{\Omega_c} |R(x, t_0) f(x)|^2 e^{2\varphi_2(x, t_0)} \, dx \geq C_0 \int_{\Omega_c} |f(x)|^2 e^{2\varphi_2(x, t_0)} \, dx
\]

(43)

under the first assumption in (9). The key point is how to estimate the first and second terms on the RHS of (42).

By our notation \( \Omega_c = \{ x \in \Omega; d(x) > \epsilon \} \) for any \( \epsilon > 0 \), we further set

\[
Q_\epsilon := \{ (x, t) \in Q; \psi_2(x, t) > \epsilon + c_0 \}, \quad \epsilon > 0.
\]

Then we have the following relations:

(i) \( Q_\epsilon \subset \Omega_c \times (0, T) \),

(ii) \( Q_\epsilon \supset \Omega_c \times \{ t_0 \} \).

In fact, if \( (x, t) \in Q_\epsilon \), we have \( d(x) - \beta (t - t_0)^2 > \epsilon \), i.e. \( d(x) > \beta (t - t_0)^2 + \epsilon > \epsilon \). This means \( x \in \Omega_c \). (i) is verified. On the other hand, if \( x \in \Omega_c \) and \( t = t_0 \) then \( \psi_2(x, t) = d(x) - \beta (t - t_0)^2 + c_0 = d(x) + c_0 > \epsilon + c_0 \). That is, \( (x, t) \in Q_\epsilon \). (ii) is verified.

Furthermore, we choose \( \beta = \frac{\|d\|_{C(\Gamma)}}{\delta^2} \) where \( \delta := \min \{ t_0, T - t_0 \} \) so that

(iii) \( Q_\epsilon \cap (\Omega \times (0, T)) = \emptyset \)

is valid. Indeed, for \( \forall (x, t) \in \Omega_c \times (0, T) \), \( \psi_2(x, t) = d(x) - \beta (t - t_0)^2 + c_0 \leq \|d\|_{C(\Gamma)} - \beta \delta^2 + c_0 = c_0 \). This leads to \( (x, t) \notin Q_\epsilon \).

Relations (i)–(iii) guarantee that \( Q_\epsilon \) is a sub-domain of \( Q \) and \( \partial Q_\epsilon \cap \partial Q \subset \Gamma \times (0, T) \). Moreover, we assert that

(iv) \( \Omega_{2\epsilon} \times (t_0 - \delta_\epsilon, t_0) \subset Q_{2\epsilon} \), \( \delta_\epsilon := \sqrt{\frac{c_0}{\|d\|_{C(\Gamma)}}} \).

Actually, for any \( (x, t) \in \Omega_{2\epsilon} \times (t_0 - \delta_\epsilon, t_0) \), we have

\[
\psi_2(x, t) = d(x) - \beta (t - t_0)^2 + c_0 > 3\epsilon - \beta \delta^2_\epsilon + c_0 = 2\epsilon + c_0.
\]

That is, \( (x, t) \in Q_{2\epsilon} \).

Now we construct a function \( \eta \in C^2[0, T] \) such that \( 0 \leq \eta \leq 1 \) and

\[
\eta = \begin{cases} 
1 & \text{in } [t_0 - \frac{1}{2} \delta_\epsilon, t_0 + \frac{1}{2} \delta_\epsilon], \\
0 & \text{in } [0, t_0 - \delta_\epsilon] \cup [t_0 + \delta_\epsilon, T]\end{cases}
\]

for any small \( \epsilon < ||d||_{C(\Gamma)} \). Then by the use of the condition (10), noting that \( \eta(t_0 - \delta_\epsilon) = 0 \), \( \eta(t_0) = 1 \), we have
\[
\int_{\Omega_t} |\delta u(x, t_0)|^2 e^{2\psi(x, t_0)} \, dx = \int_{\Omega_t} |\eta(t_0)\delta u(x, t_0)|^2 e^{2\psi(x, t_0)} \, dx
\]
\[
= \int_{\Omega_t} \frac{d}{dt} \int_{\Omega_t} |\eta \partial u|^2 e^{2\psi} \, dx \, dr
\]
\[
= \int_{\Omega_t} \int_{\Omega_t} 2\eta \partial u(\eta \partial_x^2 u + \partial_t \eta \partial_t u) e^{2\psi} \, dx \, dr
\]
\[
- \int_{\Omega_t} \int_{\Omega_t} 4(t - t_0)\beta \epsilon \lambda \phi_2 |\eta \partial u|^2 e^{2\psi} \, dx \, dr
\]
\[
\leq C \int_{\Omega_t} \int_{\Omega_t} (|\delta u||\partial_x^2 u + |\partial_t u|^2 + s|\partial_t u|^2) e^{2\psi} \, dx \, dr
\]
\[
\leq C \int_{\Omega_t} (s^{-2}|\partial_x^2 u|^2 + s^2|\partial_t u|^2) e^{2\psi} \, dx \, dr
\]  
(44)

and
\[
\int_{\Omega_t} |D_t^2 u(x, t_0)|^2 e^{2\psi(x, t_0)} \, dx = \int_{\Omega_t} |\eta(t_0)D_t^2 u(x, t_0)|^2 e^{2\psi(x, t_0)} \, dx
\]
\[
= \int_{\Omega_t} \frac{d}{dt} \int_{\Omega_t} |\eta D_t^2 u|^2 e^{2\psi} \, dx \, dr
\]
\[
= \int_{\Omega_t} \int_{\Omega_t} 2\eta D_t^2 u(\eta \partial_x^2 (D_t^2 u) + \partial_t \eta D_t^2 u) e^{2\psi} \, dx \, dr
\]
\[
- \int_{\Omega_t} \int_{\Omega_t} 4\beta \epsilon \lambda \phi_2 (t - t_0)|\eta D_t^2 u|^2 e^{2\psi} \, dx \, dr
\]
\[
\leq C \int_{\Omega_t} \int_{\Omega_t} (|D_t^2 u||\partial_x^2 u + |D_t^2 u|^2 + s|D_t^2 u|^2) e^{2\psi} \, dx \, dr
\]
\[
\leq C \int_{\Omega_t} (s^{-2}|D_t^2 u|^2 + s^2|D_t^2 u|^2) e^{2\psi} \, dx \, dr.
\]  
(45)

For the last inequalities above, we actually used the Cauchy–Schwarz inequality and the relation (iv).

Now we employ the Carleman estimate established in section 3.1 to evaluate the RHS of (44) and (45). By theorem 3, we obtain
\[
\int_0^\beta s^{-2}|\partial_x^2 u|^2 + s^{-1}|D_t^2 u|^2 + s^2|\partial u|^2 + s^3|D_t^2 u|^2 e^{2\psi} \, dx \, dr
\]
\[
\leq C \int_0^\beta \left( \sum_{j=3}^6 |D_t^2 R|^2 \right) |f|^2 e^{2\psi} \, dx \, dr + \text{LOW1} + \text{BDY1}
\]
for all \( s \geq s_1 \geq 1 \), where the terms LOW1 and BDY1 are defined as
\[
\text{LOW1} = C \int_0^\beta (|\partial_x \chi_0|^2 + |\nabla \chi_0|^2 + \sum_{j=1}^n |\partial_j \partial_x \chi_0|^2) \left[ \sum_{j=3}^6 (|\nabla (D_t^2 u)|^2 + |D_t^2 u|^2) \right] e^{2\psi} \, dx \, dr,
\]
\[
\text{BDY1} = C e^C \int_{T \times (0,T)} \sum_{k=3}^6 (|\nabla_x (D_t^2 u)|^2 + |D_t^2 u|^2) \, dS \, dr.
\]  
28
We choose the cut-off function \( \chi_0 \in C^\infty(\mathbb{R}^{n+1}) \) such that \( 0 \leq \chi_0 \leq 1 \) and
\[
\chi_0 = \begin{cases} 
1, & \psi_2(x,t) > 2\epsilon + c_0, \\
0, & \psi_2(x,t) \leq \epsilon + c_0.
\end{cases}
\]
Therefore \( \chi_0 = 1 \) in \( Q_{2s} \) while its derivatives vanish in \( Q_{2s} \), which enables us to rewrite the above Carleman inequality as follows:
\[
\int_{Q_{2s}} (s^{-2} |\partial_x^2 u|^2 + s^{-1} |D_t^2 u|^2 + s^2 |\partial_t u|^2 + s^3 |D_{tt}^2 u|^2) e^{2\varphi_2} \, dx \, dt \leq \frac{C}{s_1} \int_{Q_{s_1}} \left( \sum_{j=3}^{6} |D_j^2 R|^2 \right) |f|^2 e^{2\varphi_2} \, dx \, dt + \text{LOW2+BDY2}
\]
for all \( s \geq s_1 \geq 1 \), where
\[
\text{LOW2} = Ce^{2\epsilon^2(x,t_0)} \int_{Q_s \backslash Q_{2s}} (|\nabla (D_t^2 u)|^2 + |D_t^2 u|^2) \, dx \, dt,
\]
\[
\text{BDY2} = Ce^{C_1} \int_{Q \times (0,T)} \sum_{k=3}^{6} \left( |\nabla_{x\omega} (D_k^2 u)|^2 + |D_k^2 u|^2 \right) \, ds \, dt.
\]
LOW2 can be derived from the choice of \( \chi_0 \) and the estimate (38). Now substituting (46) into (44) and (45) implies that
\[
\int_{Q_s} |\partial_x u(x,t_0)|^2 e^{2\varphi_2(x,t_0)} \, dx + \int_{Q_s} |D_t^2 u(x,t_0)|^2 e^{2\varphi_2(x,t_0)} \, dx 
\leq \frac{C}{s_1} \int_{Q_{s_1}} \left( \sum_{j=3}^{6} |D_j^2 R|^2 \right) |f|^2 e^{2\varphi_2} \, dx \, dt + \text{LOW2+BDY2}.
\]
Combined with (42) and (43), we obtain
\[
\int_{Q_s} |f(x)|^2 e^{2\varphi_2(x,t_0)} \, dx \leq Ce^{C_2} \| u(\cdot,t_0) \|_{H^1(\Omega_s)} + C \int_{Q_s} |f|^2 e^{2\varphi_2} \, dx \, dt + \text{LOW2+BDY2}.
\]
Moreover, we divide the second term on the RHS into two parts:
\[
C \int_{Q_s} |f|^2 e^{2\varphi_2} \, dx \, dt = C \int_{Q_s} |f|^2 e^{2\varphi_2} \, dx \, dt + C \int_{Q_s \backslash Q_{2s}} |f|^2 e^{2\varphi_2} \, dx \, dt 
\leq \frac{C}{s_1} \int_{Q_{s_1}} \left( \sum_{j=3}^{6} |D_j^2 R|^2 \right) |f|^2 e^{2\varphi_2} \, dx \, dt + Ce^{2\epsilon^2(x,t_0)} \int_{Q_{s_1} \backslash Q_{2s}} |f|^2 \, dx \, dt,
\]
which leads to
\[
\int_{Q_s} |f(x)|^2 e^{2\varphi_2(x,t_0)} \, dx \leq C \int_{Q_{s_1}} |f|^2 e^{2\varphi_2} \, dx \, dt + Ce^{2\epsilon^2(x,t_0)} \int_{Q_{s_1} \backslash Q_{2s}} |f|^2 \, dx \, dt + Ce^{C_2} D^2 + Ce^{2\epsilon^2(x,t_0)} \int_{Q_s \backslash Q_{2s}} |f|^2 e^{2\varphi_2} \, dx \, dt + Ce^{C_2} D^2
\]
\[
\leq C \int_{Q_s} |f|^2 e^{2\varphi_2} \, dx \, dt + Ce^{2\epsilon^2(x,t_0)} M^2 + Ce^{C_2} D^2
\]
for all \(s \geq s_1 \geq 1\). Here \(M\) and \(\mathcal{D}\) denote a priori bound and measurements defined in theorem 5. Since \(\varphi_2(x,t)\) attains its maximum at \(t = t_0\), we can absorb the first term on the RHS into the LHS by taking \(s\) large enough (e.g. \(s \geq s_2\)). That is
\[
\int_{\Omega_\epsilon} |f(x)|^2 e^{2s\varphi_2(x,t_0)} \, dx \leq C e^{2e^{\lambda(s_2+\alpha)}} M^2 + C e^{\epsilon\mathcal{D}^2}
\]
for all \(s \geq \tilde{s} = \max\{s_1, s_2\}\). On the other hand,
\[
\int_{\Omega_\epsilon} |f(x)|^2 e^{2\varphi_2(x,t_0)} \, dx \geq \int_{\Omega_\epsilon} |f(x)|^2 e^{2\varphi_2(x,t_0)} \, dx \geq e^{2e^{\lambda(s_2+\alpha)}} \|f\|^2_{L^2(\Omega_\epsilon)}.
\]
Therefore, we obtain
\[
\|f\|^2_{L^2(\Omega_\epsilon)} \leq C e^{-\epsilon\tilde{s}a} M^2 + C e^{\epsilon\mathcal{D}^2} \tag{47}
\]
for all \(s \geq \tilde{s}\). Here \(\epsilon_0 := 2(e^{\lambda(\epsilon_2+\alpha)} - e^{\lambda(\epsilon_0+\alpha)}) = 2 e^{\lambda(\epsilon_2+\alpha)} (e^{\lambda\epsilon} - 1) > 0\). By substituting \(s\) by \(s + \tilde{s}\), inequality (47) holds for all \(s \geq 0\) with a larger generic constant \(Ce\) which is again denoted by \(C\).

Finally, we repeat the argument in the proof of theorem 4 to show the estimate of Hölder type:
\[
\|f\|_{L^1(\Omega_\epsilon)} \leq C (M^{1-\theta} \mathcal{D}^\theta + \mathcal{D}).
\]
The proof of theorem 5 is completed. \(\square\)

4. Conclusions and remarks

In this paper, we discussed the Carleman estimates for the time-fractional advection-diffusion equation (2) and the applications.

First, in the case of sub-diffusion, that is, the largest fractional order is strictly less than half, the Carleman estimate for the equation (2) was established by regarding the fractional order terms as perturbation of the first order time-derivative and the use of the Carleman estimate for the parabolic equations. As an application, the conditional stability for a lateral Cauchy problem was obtained, that is, the solution of the equation (2) is dependent continuously on not only the partial Cauchy data and the source term but also the initial value. Due to our choice of the weight function \(\psi_1(x, t) = d(x) - \beta t^{-2(\alpha_1)}\), we do not know whether the estimate is valid without the initial value, and this remains open. On the other hand, the choice of the new weight function \(\psi_1\) is not suitable for the study of the inverse problems. As is well known, for dealing with the inverse problems, the Carleman type estimate derived by \(d(x) - \beta (t - t_0)^2 + c_0 (t_0 \in (0, T))\) should be better according to the series of theories in [27]. The inverse problems for the equation (2) in the case of \(\alpha_1 < \frac{1}{4}\) remain not well solved.

However, the weight function used in [27] can deal with the inverse source problem for the equation (6) in the case of the fractional order \(\alpha = \frac{1}{4}\). For this, we first constructed a Carleman estimate with a cut-off function for the equation (6) by using the regular weight function \(d(x) - \beta (t - t_0)^2 + c_0 (t_0 \in (0, T))\). Then by an argument similar to that in [27], the conditional stability was proved for the inverse source problem (1) as well as the lateral Cauchy problem. Here we point out that our methods in dealing with the case of \(\alpha = \frac{1}{4}\) also work for
the case of rational fractional orders which are smaller than $\frac{3}{4}$. The detailed arguments were given in appendix.

The order $\alpha = \frac{3}{4}$ is the largest one which one can deal with based on our arguments. The case of general orders remains open.

Finally, it should be mentioned that the stability inequality in theorem 4 immediately gives the unique continuation result of (6), i.e. the solution of the equation (6) must vanish in the whole domain $\Omega$ if its initial value is identically 0 in $\Omega$ and the partial boundary data $u|_{\Gamma} \times (0, T)$ and $\partial_{t} u|_{\Gamma} \times (0, T)$ are zero. This principle is called a weak type unique continuation for the equation (6) since the homogeneous initial value is not essential for the unique continuation (UC) (e.g. UC for the elliptic equations or UC for the parabolic equations). However, we cannot repeat this argument to derive the weak unique continuation for the case $\alpha < \frac{1}{2}$ because the constant $\varepsilon > 0$ depends on the choice of $\Omega_0$ in the estimate in theorem 2. We refer to [10] and [22] for another kind of weak type unique continuation where the initial value does not vanish but the homogeneous Dirichlet or Neumann boundary condition is required. On the other hand, the arguments used to derive theorem 3 cannot work for the equation with $t$-dependent coefficients. The general case remains open.

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Appendix. Carleman estimates for the equation (6) with rational orders

In this part, we give the idea about how to establish Carleman estimate for any rational number $\alpha \in (0, \frac{3}{4}]$.

Here is our governing equation:

$$
\begin{align*}
\partial_t u + q(x) \partial_t^\alpha u - \Delta u + B(x) \cdot \nabla u + c(x) u &= F - q D_{t}^{m/k} \frac{\partial u}{\partial t}, \\
\end{align*}
$$

(A.1)

with initial value $u(\cdot, 0) = 0$ in $\Omega$. The sufficiently smooth coefficients $B, c, q$ are supposed to depend on only spatial variable $x$. We divide the problem into two cases:

(i) $\alpha = \frac{m}{2k+1}, m, k \in \mathbb{Z}, k \geq 1, m = 1, 2, \ldots, 2k$,

(ii) $\alpha = \frac{m}{2k}, m, k \in \mathbb{Z}, k \geq 1, m = 1, 2, \ldots, 2k - 1$.

For the case (i), by noting the zero initial condition, similar to the argument used in the above section, the equation (A.1) can be rephrased as

$$
Lu := \partial_t u - \Delta u + B \cdot \nabla u + cu = F - q D_{t}^{m/k} \frac{\partial u}{\partial t}.
$$

(A.2)
We denote $u_j := D^{j}_{-k} u$, $j \in \mathbb{Z}$, and apply the operators $D^{j}_{-k}$ to the equation (A.2) for $j = -k, ..., k$ to derive

$$L(u_j) = D^{j}_{-k} F - q D^{j}_{-k} u, \quad j = -k, ..., k. \quad (A.3)$$

Now employing the Carleman estimate of parabolic type (i.e. lemma 3.1) to each equation in (A.3) and then modifying the orders of $\lambda$ to $\tau_j$, we obtain

$$\int_Q \{ s^{1-\lambda} \varphi^{1-\lambda} (|\partial_\nu u_j|^2 + \sum_{i=1}^n |\partial_i \partial_\nu u_j|^2) + s^{\gamma+2} \varphi^{\gamma+1} |\nabla u_j|^2 + s^{\gamma+3} \varphi^{\gamma+4} |u_j|^2 \} e^{2\rho} \, dx dt \leq C \int_Q \sum_{j=-k}^k \left\{ (s \lambda \varphi) \int_{Q_j} (\sum_{i=1}^n |\partial_i \partial_\nu u_j|^2 + |u_j|^2) e^{2\rho} \, dx dt \right\} \leq C \int_Q \sum_{j=-k}^k \left\{ (s \lambda \varphi) \int_{Q_j} (\sum_{i=1}^n |\partial_i \partial_\nu u_j|^2 + |u_j|^2) e^{2\rho} \, dx dt \right\} \leq C \int_Q \sum_{j=-k}^k \left\{ (s \lambda \varphi) \int_{Q_j} (\sum_{i=1}^n |\partial_i \partial_\nu u_j|^2 + |u_j|^2) e^{2\rho} \, dx dt \right\}$$

for all $s, \lambda \geq 1$ large enough. Here the weight function $\varphi$ is chosen as (7). Next we choose $\tau_j = -\frac{4}{2k+\gamma}(j + k) < 0$, $j = -k, ..., k$ and take the summation of the above estimates to obtain

$$\lambda \int_Q \{ \sum_{j=-k}^k (s \lambda \varphi) \frac{4}{2k+\gamma} |\partial_\nu u_j|^2 + (s \lambda \varphi) \frac{4}{2k+\gamma} |u_j|^2 \} e^{2\rho} \, dx dt \leq C \int_Q \sum_{j=-k}^k (s \lambda \varphi) \frac{4}{2k+\gamma} |u_j|^2 e^{2\rho} \, dx dt + \text{BDY} 1$$

with $u_{j+2k+1} = \partial_\nu D^{j}_{-k} u = \partial_\nu u_j$ and the term BDY 1 is defined by

$$\text{BDY} 1 := C(\lambda) e^{C(\lambda) \gamma} \int_{\Sigma} \sum_{j=-k}^k (|\nabla u_j|^2 + |u_j|^2) e^{2\tau_j} \, dx dt,$$

Meanwhile, a direct calculation yields that

$$\sum_{j=-k}^k (s \lambda \varphi) \frac{4}{2k+\gamma} |u_{j+2k+1}|^2 = \sum_{j=-k+1}^{3k+1} (s \lambda \varphi) \frac{4}{2k+\gamma} |u_j|^2,$$

$$\sum_{j=-k}^k (s \lambda \varphi) \frac{4}{2k+\gamma} |u_{j+m}|^2 = \sum_{j=-k+m}^{k+m} (s \lambda \varphi) \frac{4}{2k+\gamma} |u_j|^2,$$

which imply that

$$\lambda \int_Q \{ \sum_{j=-k}^k (s \lambda \varphi) \frac{4}{2k+\gamma} |\partial_\nu u_j|^2 + (s \lambda \varphi) \frac{4}{2k+\gamma} |\nabla u_j|^2 + \sum_{j=-k}^{3k+1} (s \lambda \varphi) \frac{4}{2k+\gamma} |u_j|^2 \} e^{2\rho} \, dx dt \leq C \int_Q \sum_{j=-k}^k (s \lambda \varphi) \frac{4}{2k+\gamma} |u_j|^2 e^{2\rho} \, dx dt + \text{BDY} 1$$

for all $s, \lambda \geq 1$ large enough. Since $\alpha = \frac{m}{2k+1} \in \{0, \frac{3}{4} \}$ implies that

$$-\frac{4j - 2k - 3}{2k+1} \geq -\frac{4j + 4k - 4m}{2k+1} \quad \text{for} \quad j = -k + m, ..., k + m,$$

32
we can then fix $\lambda$ large so that the second terms on the RHS of (A.4) can be absorbed into the LHS of it. Therefore we have the following Carleman inequality
\[
\int_Q \left\{ \sum_{i=1}^n (x\varphi) - \frac{2\pi i}{\lambda} |\partial_i \partial u|^2 + (x\varphi) - \frac{2\pi i}{\lambda} |\nabla u|^2 + \sum_{j=-k}^{3k+1} (x\varphi) - \frac{2\pi i}{\lambda} |D_{x\varphi} u|^2 \right\} e^{2\varphi} \, dx \, dr 
\]
\[\leq C \int_Q \sum_{j=-k}^k (x\varphi) - \frac{2\pi i}{\lambda} |D_{x\varphi} F|^2 e^{2\varphi} \, dx \, dr + BDY1\]
for all $s$ large enough and all integer $1 \leq m \leq \frac{3}{4}(2k + 1)$.

For the case (ii), we repeat the strategy of case (i) with some necessary changes. Rewrite the equation (A.1) by using the zero initial condition
\[Lu := \partial_t u - \Delta u + B \cdot \nabla u + cu = F - qD_{x\varphi} u.\] (A.5)
Denote $u_j := D_{x\varphi} u, \ j \in \mathbb{Z}$. We apply operators $D_{x\varphi}$ to equation (A.5) for $j = -k + 1, ..., k$ and obtain
\[L(u_j) = D_{x\varphi} F - qD_{x\varphi} u, \ j = -k + 1, ..., k.\] (A.6)
We apply lemma 3.1 to (A.6) for each $j$ and then multiply $\lambda$ on both sides, which leads to
\[\int_Q \left\{ \sum_{j=1}^n (s\varphi) - \frac{2\pi i}{\lambda} |\partial_i \partial u|^2 + (s\varphi) - \frac{2\pi i}{\lambda} |\nabla u|^2 \right\} e^{2\varphi} \, dx \, dr \]
\[\leq C \int_Q (s\varphi) |D_{x\varphi} F|^2 e^{2\varphi} \, dx \, dr + C \int_Q (s\varphi) |u_{j+m}|^2 e^{2\varphi} \, dx \, dr + C(\lambda)s e^{C(\lambda)s} \int_Q (|\nabla u_j|^2 + |u_j|^2) \, dx \, dr\]
for all $s, \lambda \geq 1$ large enough. Next we choose $\tau_j = -\frac{3}{2}(j - k - 1) \leq 0, \ j = -k + 1, ..., k$ and take the summation of the above estimates to obtain
\[\lambda \int_Q \left\{ \sum_{j=1}^n (s\varphi) - \frac{2\pi i}{\lambda} |\partial_i \partial u|^2 + (s\varphi) - \frac{2\pi i}{\lambda} |\nabla u|^2 \right\} e^{2\varphi} \, dx \, dr \]
\[+ \lambda \int_Q \left\{ \sum_{j=-k+1}^k \left( (s\varphi) - \frac{3\pi i}{\lambda} |u_{j+2k}|^2 + (s\varphi) - \frac{3\pi i}{\lambda} |u_j|^2 \right) \right\} e^{2\varphi} \, dx \, dr \]
\[\leq C \int_Q \sum_{j=-k+1}^k (s\varphi) - \frac{3\pi i}{\lambda} |D_{x\varphi} F|^2 e^{2\varphi} \, dx \, dr + C \int_Q \sum_{j=-k+1}^k (s\varphi) - \frac{3\pi i}{\lambda} |u_{j+m}|^2 e^{2\varphi} \, dx \, dr + BDY2\]
where
\[BDY2 := C(\lambda)e^{C(\lambda)s} \int_Q \sum_{j=-k+1}^k (|\nabla u_j|^2 + |u_j|^2) \, dx \, dr.\]
Meanwhile, by a direct calculation, we have
\[\sum_{j=-k+1}^k (s\varphi) - \frac{3\pi i}{\lambda} |u_{j+2k}|^2 = \sum_{j=-k+1}^k (s\varphi) - \frac{3\pi i}{\lambda} |u_j|^2,\]
\[\sum_{j=-k+1}^k (s\varphi) - \frac{3\pi i}{\lambda} |u_{j+m}|^2 = \sum_{j=-k+m}^{k+m} (s\varphi) - \frac{2\pi i}{\lambda} |u_j|^2,\]
which imply
\[
\lambda \int_Q \left\{ \sum_{j,l=1}^n (s\lambda \varphi)^{-\frac{2m}{k+1}} |\partial_t \partial_l u|^2 + (s\lambda \varphi)^{-\frac{2m}{k+1}} |\nabla u|^2 + \sum_{j=-k+1}^{3k} (s\lambda \varphi)^{-\frac{2m-1}{2}} |u_j|^2 \right\} e^{2\lambda \varphi} \, dx \, dt \\
\leq C \int_Q \sum_{j=-k+1}^{k} (s\lambda \varphi)^{-\frac{2m-1}{2}} |D_j^2 F|^2 e^{2\lambda \varphi} \, dx \, dt + C \int_Q \sum_{j=-k+m+1}^{k} (s\lambda \varphi)^{-\frac{2m-1}{2}} |u_j|^2 e^{2\lambda \varphi} \, dx \, dt + BDY\tag{A.7}
\]
for all \( s, \lambda \geq 1 \) large enough. Since \( \alpha = \frac{m}{k} \in (0, \frac{3}{4}] \) implies that
\[
-\frac{2j-k-2}{k} \geq -\frac{2j+2k-2m-2}{k} \quad \text{for } j = -k + m + 1, \ldots, k + m,
\]
we can then fix \( \lambda \) large so that the second terms on the RHS of (A.7) can be absorbed into its LHS. Therefore we have the following Carleman inequality
\[
\int_Q \left\{ \sum_{j,l=1}^n (s\varphi)^{-\frac{2m}{k+1}} |\partial_t \partial_l u|^2 + (s\varphi)^{-\frac{2m}{k+1}} |\nabla u|^2 + \sum_{j=-k+1}^{3k} (s\varphi)^{-\frac{2m-1}{2}} |u_j|^2 \right\} e^{2\varphi} \, dx \, dt \\
\leq C \int_Q \sum_{j=-k+1}^{k} (s\varphi)^{-\frac{2m-1}{2}} |D_j^2 F|^2 e^{2\varphi} \, dx \, dt + BDY\tag{A.8}
\]
for all \( s \) large enough and all integer \( 1 \leq m \leq \frac{3}{2} k \).

In the end, we combine the above two cases and give the Carleman estimate.

**Theorem A.1.** For any rational number \( \alpha = \frac{m}{k} \in (0, \frac{3}{4}] \), suppose \( D_j^2 F \in L^2(\Omega) \) for \( j = j_1, \ldots, j_l \), where \( j_l := \left\lceil -\frac{k+1}{2} \right\rceil + 1 \), \( l = 1, \ldots, k \). Then there exist constants \( s_0 \geq 1 \) and \( C > 0 \) such that
\[
\int_Q \left\{ \sum_{j,l=1}^n (s\varphi)^{\frac{2m}{2k+1}} |\partial_t \partial_l u|^2 + (s\varphi)^{\frac{2m}{2k+1}} |\nabla u|^2 + \sum_{j=-k+1}^{3k} (s\varphi)^{-\frac{2m}{2k+1}} |u_j|^2 \right\} e^{2\varphi} \, dx \, dt \\
\leq C \int_Q \sum_{j=-k+1}^{k} (s\varphi)^{-\frac{2m}{2k+1}} |D_j^2 F|^2 e^{2\varphi} \, dx \, dt + BDY
\]
for all \( s \geq s_0 \) and all \( u \) smooth enough satisfying (A.1) and conditions \( u(\cdot, 0) = u(\cdot, T) = 0 \) on \( \Omega \). Here
\[
BDY := Ce^{Cs} \int_{\Sigma} \left( \sum_{j=j_l}^{j_h} (|\nabla_{x,t} D_j^2 u|^2 + |D_j^2 u|^2) \right) dS dt.
\]

**Remark A.1.** In view of the estimate (38), we can also rewrite \( BDY \) with
\[
BDY \leq C_1 e^{Cs} \int_{\Sigma} \left( |\nabla_{x,t} D_j^2 u|^2 + |D_j^2 u|^2 \right) d\Sigma.
\]
However, constant \( C_1 \) now depends on \( k \), which is the main difficulty in dealing with the equation (6) with the time-fractional derivative of irrational order by using the density of rational numbers in \( \mathbb{R} \). The Carleman estimate for the equation (6) with the general-order time-fractional derivatives remains open.
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