TOPOLOGICAL CONJUGACY TO GIVEN CONSTANT LENGTH SUBSTITUTION MINIMAL SYSTEMS

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Abstract. We find necessary and sufficient conditions for a symbolic dynamical system to be topologically conjugate to any given constant length substitution minimal system, thus extending the results in [CKL] for the Morse and Toeplitz substitutions.

1. Introduction

In [CKL] three of the authors characterized those symbolic minimal systems that are topologically conjugate to the Morse Minimal System (the closure of the shift-orbit of the Morse-Thue sequence) and those topologically conjugate to the closely related Toeplitz Minimal System. The Morse result is that a symbolic minimal system $(Y, \sigma)$ is topologically conjugate to the Morse system if and only if there exist $N \geq 1$ and $2^N$-blocks $C_0 \neq C_1$ such that every point in $Y$ can be written as a concatenation of $C_0$’s and $C_1$’s in exactly one way and such that the sequences of $C$’s in these concatenations in some sense “mirror” points in the Morse system. However the arguments in [CKL] are valid only for substitutions sharing some of the properties of the Morse or Toeplitz substitutions.

Our main result is Theorem 1, which extends the Morse and Toeplitz results to all infinite substitution minimal systems generated by primitive, constant length substitutions. In Theorem 2 we show that the “mirroring” property in the Morse Dynamical Characterization Theorem in [CKL] holds for the class of primitive, one-to-one substitutions.
\( \theta \) of constant length at least three such that for all \( s \neq t \), \( \theta(s) \) and \( \theta(t) \) disagree at some place other than the first or last. Using ideas from the proof of Theorem 1 we show in Theorem 3 that if \( \theta \) is a primitive substitution of constant length that generates an infinite system \((X_\theta, \sigma)\) and if \( \zeta \) is a primitive, one-to-one substitution of constant length that generates an infinite system \((X_\zeta, \sigma)\) topologically conjugate to \((X_\theta, \sigma)\), then the number of letters in the alphabet of \( \zeta \) is bounded above by the number of 3-blocks that appear in \( X_\theta \). We give an example to show that the bound is attained for the Morse system.

2. Background

In this paper a dynamical system is a pair \((X, T)\), where \( X \) is a compact metric space and \( T : X \to X \) is a homeomorphism. The notion of “sameness” for dynamical systems is topological conjugacy: \((X, T)\) and \((Y, S)\) are topologically conjugate if and only if there exists a homeomorphism \( F : (X, T) \to (Y, S) \) such that \( F \circ T \equiv S \circ F \). In this case \( F \) is called a topological conjugacy and \( F^{-1} : (Y, S) \to (X, T) \) is also a topological conjugacy. A topological semi-conjugacy, also called a factor map, is a continuous, onto map \( F : (X, T) \to (Y, S) \) such that \( F \circ T \equiv S \circ F \).

A dynamical system \((X, T)\) is called minimal if and only if it contains no proper subsystem. Equivalently, if \( X' \subseteq X \) is nonempty, closed, and \( T \)-invariant (i.e. \( T(X') \subseteq X' \)), then \( X' = X \). Equivalently again, every orbit \( \{T^n(x) : -\infty < n < \infty \} \) is dense in \( X \).

A symbolic system is a subsystem of some \((\prod_{-\infty}^{\infty} S, \sigma)\), where \( S \) is a finite set of symbols, also called the alphabet, and \( \sigma : \prod_{-\infty}^{\infty} S \to \prod_{-\infty}^{\infty} S \) is the (left) shift: \( (\sigma(x))_i = x_{i+1} \), \( -\infty < i < \infty \). As is customary, we shall abuse notation and write \((X, \sigma)\) instead of \((X, \sigma|_X)\). A basic fact about symbolic systems is the Curtis-Hedlund-Lyndon Theorem [H]: any topological conjugacy or semi-conjugacy \( F : (X, \sigma) \to (Y, \tau) \) between symbolic systems is given by a local rule \( f : S^m_{-\infty} \to S_Y \), where for every \( x \in X \), \( (F(x))_i = f(x_{i-m}, \ldots, x_{i+a}) \), \( -\infty < i < \infty \). Here \( m \geq 0 \) is called the memory and \( a \geq 0 \) the anticipation of \( F \). In
this case $F$ is called an $(m+1+a)$-block map. Note that by adding superfluous variables an $r$-block map is also an $s$-block map for all $s \geq r$. The powers of $f$ are defined so that for every $n \geq 2$, $f^n$ is a local rule for $F^n$. For example, if $f$ is a 3-block map, then $f^2$ is the 5-block map defined by

$$f^2(x_1, \ldots, x_5) := f(f(x_1, x_2, x_3), f(x_2, x_3, x_4), f(x_3, x_4, x_5)).$$

A substitution of constant length $L \geq 2$ is a mapping $\theta : S \rightarrow S^L$ from a finite set of symbols $S$ to the set of $L$-blocks, i.e. blocks of length $L$, with entries from $S$. The classic example is the Morse substitution: $0 \mapsto 01, 1 \mapsto 10$. A substitution $\theta$ can be extended to a mapping of finite blocks by concatenation. For example, $\theta(st) := \theta(s)\theta(t)$. The powers of $\theta$ are defined in the obvious way. For example, if $\theta(s) = tuv$, then $\theta^2(s) := \theta(t)\theta(u)\theta(v)$.

A symbolic minimal system $(X, \sigma)$ is called a substitution minimal system if and only if it can be generated by a substitution $\theta$, i.e. for every $s \in S_X$, the alphabet of $X$, and for every $n \geq 1$, $\theta^n(s)$ appears in $X$. The substitution $\theta$ defined on alphabet $S$ is called primitive if and only if there exists $n \geq 1$ such that for every $s, t \in S$, $s$ appears in $\theta^n(t)$.

Lemmas 1-3 below hold for all substitutions, not just substitutions of constant length. Lemma 1 shows that when studying substitution minimal systems, there is no loss in generality in assuming that the generating substitution is primitive.

**Lemma 1.** [DMK] [Q, Prop. 5.5] Every primitive substitution generates a unique substitution minimal system. Conversely, every substitution minimal system can be generated by a primitive substitution.

An important property of substitution minimal systems generated by primitive substitutions is recognizability, also called unique decipherability, defined in the following lemma.

**Lemma 2.** [M Théorèmes 1 et 2] Let $\theta$ be a primitive substitution that generates an infinite symbolic minimal system $(X_\theta, \sigma)$. Then every point in $X_\theta$ can be written as a concatenation of the blocks $\theta(s)$ in exactly one way.
Lemma 2 is not true for primitive substitutions that generate finite systems. For example, \(0 \mapsto 010, 1 \mapsto 101\).

The unique substitution minimal system generated by \(\theta\) is denoted \((X_\theta, \sigma)\).

A basic fact about substitution minimal systems is

**Lemma 3.** [Q, Prop. 5.4] Let \((X_\theta, \sigma)\) be the substitution minimal system generated by \(\theta\). Then for every \(n \geq 1\), \(X_\theta = X_{\theta^n}\).

Lemma 4 shows that we can assume, whenever it is useful, that a substitution \(\theta\) is one-to-one, i.e. if \(s \neq t\), then \(\theta(s) \neq \theta(t)\).

**Lemma 4.** [BDM, Prop. 2.3] For any primitive, constant length substitution \(\theta\) such that \((X_\theta, \sigma)\) is infinite, there is a primitive, one-to-one substitution \(\zeta\) of the same constant length such that \((X_\theta, \sigma)\) and \((X_\zeta, \sigma)\) are topologically conjugate.

For references on substitutions, see [F] for arbitrary substitutions and [G] for constant length substitutions.

## 3. Conjugacy to a given substitution minimal system

In this section we find in Theorem 1 necessary and sufficient conditions for a symbolic minimal system to be topologically conjugate to a given constant length substitution minimal system. For a subclass of substitutions, including the Morse substitution, we find in Theorem 2 a result with a simpler statement and a simpler proof than for Theorem 1.

**Theorem 1.** Let \(\theta\) be a primitive, one-to-one substitution of constant length \(L \geq 2\) such that \(X_\theta\) is infinite, and let \((Y, \sigma)\) be a symbolic minimal system. Then \((Y, \sigma)\) is topologically conjugate to \((X_\theta, \sigma)\) if and only if

1. there exist \(N \geq 1\) and a collection \(\mathcal{B}\) of \(L^N\)-blocks such that every point in \(Y\) can be written as a concatenation of blocks in \(\mathcal{B}\) in exactly one way,
2. there exists a 2-block semiconjugacy \(G : (X_\theta, \sigma) \to (Y_0, \sigma^{L^N})\), where \(Y_0\) is the set of points in \(Y\) such that the blocks in \(\mathcal{B}\) start at multiples of \(L^N\), and
(3) if $stu$ and $s't'u'$ are 3-blocks appearing in $X_\theta$ and $t \neq t'$, then $g(stu) \neq g(s't'u')$, where $g$ is a local rule of $G$.

Proof. (only if) Let $F : (X_\theta, \sigma) \rightarrow (Y, \sigma)$ be a topological conjugacy. By composing with a power of the shift, we may assume that $F$ has no memory. Let $f$ be a local rule of $F$. Choose $N$ so that $L^N$ is greater than the anticipation of $F$ and

\[(\ast)\text{ for all 3-blocks } stu \text{ and } s't'u' \text{ appearing in } X_\theta \text{ with } t \neq t', f(\theta^N_n(s)\theta^N_n(t)\theta^N_n(u)) \neq f(\theta^N_n(s')\theta^N_n(t')\theta^N_n(u')).\]

To see that $N$ can be chosen so that $(\ast)$ holds, suppose not. Then equality holds for some $stu$ and $s't'u'$ with $t \neq t'$ and infinitely many $n$. For every $n$ and every $s,t,u$, $\theta^n(s)\theta^n(t)\theta^n(u)$ can be extended to a doubly infinite sequence, that we also call $\theta^n(s)\theta^n(t)\theta^n(u)$, with the zeroth coordinate coming in a place of disagreement between $\theta^n(t)$ and $\theta^n(t')$. Using the fact that $\theta$ and hence every $\theta^n$ is one-to-one, there is a subsequence $(\tilde{n})$ of $(n)$ such that both $(\theta^{\tilde{n}}(s)\theta^{\tilde{n}}(t)\theta^{\tilde{n}}(u))$ and $(\theta^{\tilde{n}}(s')\theta^{\tilde{n}}(t')\theta^{\tilde{n}}(u'))$ converge, say to $x \neq x'$. Then $F(x) = F(x')$, contradicting $F$ being one-to-one.

By adding superfluous variables if necessary, we may assume that the anticipation of $F$ is exactly $L^N$. Define

$$B := \{f(\theta^N_n(st)) : st \text{ is a 2-block appearing in } X_\theta\}.$$ 

We show that condition (1) holds. Since $\theta$ and hence $\theta^N$ are primitive, every point in $X_{\theta^N} = X_\theta$ can be written as a concatenation of blocks $\theta^N(s)$ in exactly one way [M]. Then, since $F$ is a topological conjugacy, every point in $Y$ can be written as a concatenation of blocks in $B$ in exactly one way.

To show that condition (2) holds, let $G : (X_\theta, \sigma) \rightarrow (Y_0, \sigma^{L^N})$, where $Y_0$ is the set of points in $Y$ such that the blocks in $B$ start at multiples of $L^N$, be the 2-block semiconjugacy with local rule $g := f \circ \theta^N$.

Condition (3) follows from $(\ast)$.

(if) It follows from (3) that the topological semiconjugacy $G$ is one-to-one and hence a topological conjugacy.
Let $X_{\theta,0}$ be the set of points in $X_{\theta}$ such that the blocks $\theta^N(s)$ start at multiples of $L^N$. Then $(X_{\theta}, \sigma)$ is topologically conjugate to $(X_{\theta,0}, \sigma^{L^N})$ via the map $s \mapsto \theta^N(s)$.

Standard arguments (see, e.g., the proof of the Morse Dynamical Characterization Theorem in [CKL]) show that $(X_{\theta}, \sigma)$ is topologically conjugate to $(Y, \sigma)$. \hfill \square

Now we consider a special case, substitutions $\theta$ of constant length at least three (see Remark below) for which for all $s \neq t$, $\theta(s)$ and $\theta(t)$ disagree in some entry other than the first or last. This class contains the square of the Morse substitution.

For such substitutions, condition $(\ast)$ in the proof of Theorem 1 can be replaced by the stronger condition $(\ast\ast)$ below and we have the following improvement of Theorem 1 and generalization of the Morse Dynamical Characterization Theorem [CKL].

**Theorem 2.** Let $\theta$ be a primitive, one-to-one substitution of constant length $L \geq 3$ such that $X_{\theta}$ is infinite, and let $(Y, \sigma)$ be a symbolic minimal system. Suppose also that any two substitution blocks $\theta(s)$ and $\theta(t)$ with $s \neq t$ disagree somewhere other than in the first or last entry.

Then $(Y, \sigma)$ is topologically conjugate to $(X_{\theta}, \sigma)$ if and only if there exist $N \geq 1$, $a \geq 0$, a collection $B$ of $(L^N - a)$-blocks that are in one-to-one correspondence with the symbols in $X_{\theta}$, and a collection $B'$ of $a$-blocks such that

1. every point in $Y$ can be written as a concatenation of alternating blocks in $B$ and blocks in $B'$ in exactly one way,
2. the blocks in $B'$ are determined by their nearest neighbors in $B$,
3. every second block in a concatenation (1), thought of as an infinite bilateral sequence with letters from $B$, “mirrors” a point in $X_{\theta}$ via the one-to-one correspondence above.

**Remark.** We require $L \geq 3$ because condition (3) is vacuous for $L = 2$. However, this requirement is harmless, for in this case look at $\theta^2$ rather than $\theta$.

The proof of the only if direction proceeds as in the proof of Theorem 1, with $F : (X, \sigma) \to (Y, \sigma)$ being a topological conjugacy with no
memory and anticipation $a$ and condition (***) below taking the place of condition (*).

(***) there exists $N$ such that $L^N$ is greater than the anticipation of $F$ and such that for all symbols $t \neq t'$ appearing in $X_\theta$, $f(\theta^N(t)) \neq f(\theta^N(t'))$.

Let $\mathcal{B} := \{f(\theta^N(s))\}$ and let $\mathcal{B}'$ be the $a$-blocks appearing in points of $Y$ between consecutive blocks in $\mathcal{B}$.

To see that condition (2) holds, note that every block in $\mathcal{B}'$ appears in some $f(\theta^N(st))$.

The proof of the if direction is much the same as it is in the proof of Theorem 1.

4. Number of Symbols

In this section we find a relation between the number of symbols in constant length substitutions that generate topologically conjugate systems.

**Theorem 3.** Let $\theta$ be a primitive, constant length substitution such that $X_\theta$ is infinite. Then the number of symbols in a primitive, one-to-one constant length substitution that generates a substitution minimal system topologically conjugate to $(X_\theta, \sigma)$ is at most the number of 3-blocks appearing in $X_\theta$.

**Proof.** By [CDL] we may assume that the lengths of $\theta$ and $\zeta$ are the same. Let $F : (X_\theta, \sigma) \rightarrow (X_\zeta, \sigma)$ be a topological conjugacy with no memory and local rule $f$. Then, with notation as in the proof of Theorem 1, for every symbol $s'$ appearing in $X_\zeta$, $\zeta^N(s')$ is a subblock of some $B_1B_2$, where $B_1, B_2 \in \mathcal{B}$. But $B_1B_2 = f(\theta^N(stu))$ for some 3-block $stu$ appearing in $X_\theta$. Then we have

$$\#s' = \#\zeta^N(s') \leq \#f(\theta^N(stu)) \leq \#stu,$$

the equality because $\zeta$ is one-to-one, and the first inequality because $\theta$ and $\zeta$ are uniquely decipherable. \qed
The following six-symbol example shows that the bound is attained for the Morse substitution. It is one of the “3-block presentations” of the Morse substitution and so generates a system topologically conjugate to the Morse system (see [CDK]).

\[
\begin{align*}
\zeta(001) &= (101)(011) \\
\zeta(010) &= (110)(100) \\
\zeta(011) &= (110)(101) \\
\zeta(100) &= (001)(010) \\
\zeta(101) &= (001)(011) \\
\zeta(110) &= (010)(100)
\end{align*}
\]

With \( \theta \) the Morse substitution \( 0 \mapsto 01, 1 \mapsto 10 \), \( \zeta(stu) := (s_2t_1t_2)(t_1t_2u_1) \), where \( \theta(s) = s_1s_2 \), etc.

It follows from [CKL, Toeplitz Corollary] that the bound (five) given by Theorem 3 for the Toeplitz substitution \( 0 \mapsto 01, 1 \mapsto 00 \) cannot be attained.

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