Consistency and renormalization of the $\lambda = \frac{1}{3}$ soft breaking model of the complete Hořava theory

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Abstract. We discuss the $\lambda = \frac{1}{3}$ soft breaking of the complete Hořava theory we have recently introduced. We show the consistency of the theory from the point of view of the algebra of the constraints and of the existence and uniqueness of their solutions. The theory, in distinction to the $\lambda \neq \frac{1}{3}$ theory, has exactly the same physical degrees of freedom as General Relativity. We discuss the structure of the propagator and its behaviour at low and high energies. We comment on the spherically symmetric solutions of the theory.

1. Introduction

The Hořava proposal for a renormalizable theory [1] of gravity give rise to several models of gravity. All of them are in principle renormalizable although a proof of it has to be explicitly given since the models are restricted by second class constraints and one has to show that the quantization procedure may be performed without the introduction of dangerous non-local terms. The main property of Hořava proposal is to consider an anisotropic behaviour of space and time at high energies, the scaling being of the form

$$t \sim [b]^\frac{2}{3},$$
$$x \sim [b]^1.$$

The scaling breaks the relativistic symmetry with the idea of recovering it, at least in an approximated sense without violating the newtonian limit, at low energies.

The main benefit would be to obtain a renormalizable theory of gravity, which is now possible since higher order spatial derivatives terms may be added to the potential without breaking the spatial diffeomorphism invariance of the action. In a relativistic theory together with the higher order spatial derivatives terms one has to include higher order time derivatives terms which introduces ghosts into the formulation and break the unitarity property of the theory. At high energies the propagator is dominated by the higher spatial derivatives terms which improve the ultraviolet properties of the theory. The final goal of obtaining a renormalizable theory may be achieved by considering a suitable scaling $z$ in order to have a dimensionless coupling on the action. For a four dimensional gravity theory the value of $z$ should be $z = 3$.

In the Hořava proposal the space-time breaks into a spatial foliation parametrized by a preferred time, the foliation preserving diffeomorphism of the three dimensional spatial leaves.
are the gauge symmetries of the theory. The natural formulation of the theory is then in terms of the ADM variables $g_{ij}, N, N_i$. The action is given by

$$S = \int dt d^3x \sqrt{g} N \left( G^{ijkl} K_{ij} K_{kl} - \nu \right)$$  \hspace{1cm} (1)$$

where

$$K_{ij} = \frac{1}{2N} \left( \dot{g}_{ij} - 2\nabla_{(i}N_{j)} \right)$$  \hspace{1cm} (2)$$

$$G^{ijkl} = \frac{1}{2} \left( g^{ik}g^{jl} + g^{il}g^{jk} \right) - \lambda g^{ij}g^{kl}.$$  \hspace{1cm} (3)$$

The dimensionless coupling constant of the action has been absorbed by a redefinition of the fields. The coupling $\lambda$ is dimensionless and the potential density $\nu = \nu(g_{ij}, a_i,\ldots)$ is the most general combination of the spatial metric, its curvature tensor, the vector $a_i \equiv \frac{N}{N^2}$ and covariant spatial derivatives of these objects transforming as a scalar under spatial diffeomorphisms. In order to have power-counting renormalizability the potential must include terms up to sixth order in spatial derivatives. The potential may include a cosmological constant but we do not consider it here. The highest order derivatives term in the potential is $\sqrt{g} C_{ij} C^{ij}$, it was introduced by Horava and it is the unique interacting term of that order arising from a detailed balance principle. $C_{ij}$ is the Cotton tensor constructed for the three dimensional metric on the spatial leaves of the foliation. This interacting term has its origin on the three dimensional Chern-Simons theory constructed from the Levi-Civita connection in terms of the three dimensional metric. We will consider as in [1] a detailed balance principle at high energies but we break it at low energies by introducing the most general quadratic terms on spatial derivatives. It is a soft breaking mechanism which preserves the renormalizability properties of the theory. The potential becomes then

$$\nu = -R - \alpha a_i a^i + w C_{ij} C^{ij}$$  \hspace{1cm} (4)$$

where $\alpha$ and $w$ are coupling constants. The second term on the right hand member was introduced in [2].

The Hořava gravities have been developed for projectable theories where $N = N(t)$ and for non-projectable ones when $N = N(x,t)$. They are inequivalent theories, there is no way to transform by using the gauge freedom of the theory, from one theory to the other. We consider the non-projectable case where the full symmetries of the theory may be completely realized. Among the non-projectable theories most of the work has been performed for $\lambda \neq \frac{1}{3}$. The tensor $G^{ijkl}$ in (1) is not invertible for $\lambda = \frac{1}{3}$, hence two different theories with different physical degrees of freedom arises from the two cases. Most of the work has been done for $\lambda \neq \frac{1}{3}$ [3, 4, 5, 6, 7, 8, 9, 10, 11]. In particular the strong coupling problem has been extensively analyzed. In this case the theory has one additional physical degree of freedom compared to General Relativity [3, 4]. The case we consider corresponds to $\lambda = \frac{1}{3}$, the theory contains exactly the physical degrees of freedom as General Relativity, they may be realized in terms of the transverse-traceless components of the three dimensional metric.

It is interesting to notice that the particular theory with a potential density $-R$ describes for $\lambda = 1$ General Relativity while for $\lambda \neq 1$ General Relativity on a particular gauge $\pi = 0$ (in the space-time region where this is admissible) [7]

2. The $\lambda = \frac{1}{3}$ soft breaking theory

The hamiltonian of the theory is [12]

$$\int d^3x \left( \frac{N}{\sqrt{g}} \pi^{ij} \pi_{ij} + \sqrt{g} N \nu + N_i \mathcal{H}^i + \sigma \phi + \mu \pi \right) + E_{ADM}$$  \hspace{1cm} (5)$$
where
\[ H^i = -2 \nabla_j \pi^{ij} + \phi \partial^i N. \] (6)

\( \phi \) is the conjugate momentum associated to \( N \) and \( \sigma, \mu \) are Lagrange multipliers.

\( E_{ADM} \) denotes the ADM energy
\[ E_{ADM} = \oint d\Sigma_i \left( \partial_j g_{ij} - \partial_i g_{jj} \right). \] (7)

It has to be included in the Hamiltonian, following [13], in order to obtain the equations of motion compatible with the boundary conditions from the most general variations of the Hamiltonian.

The preservation of the primary constraints give rise to two new constraints, the Hamiltonian constraint
\[ H \equiv \frac{1}{\sqrt{g}} \pi^{ij} \pi_{ij} - \sqrt{g} R + \alpha \sqrt{g} \left( 2 \nabla_i a^i + a_i a^i \right) + w \sqrt{g} C_{ij} C^{ij} = 0 \] (8)
and
\[ C \equiv \frac{3N}{2 \sqrt{g}} \pi^{ij} \pi_{ij} + \frac{1}{2} \sqrt{g} N R - \sqrt{g} N \left( 2 \nabla_i a^i + \left( 2 - \frac{\alpha}{2} \right) a_i a^i \right) + \frac{3w}{2} \sqrt{g} N C_{ij} C^{ij} = 0. \] (9)

The conservation of these new constraints provides two elliptic partial differential equations for the Lagrange multipliers \( \sigma \) and \( \mu \). The Dirac algorithm ends up at this stage.

From (8) and (9) we obtain
\[ \left( 1 - \frac{\alpha}{2} \right) \nabla^2 N = N \left( \frac{1}{g} \pi^{ij} \pi_{ij} + w C_{ij} C^{ij} \right). \] (10)

The constraint \( H^i = 0 \) is a first class constraint while \( \phi = 0, \pi = 0, H = 0 \) and \( C = 0 \) are second class ones. The primary constraint
\[ \pi = 0 \] (11)
is a particular property of the Hořava theory formulated for \( \lambda = \frac{1}{3} \).

One way to analyze these constraints is to solve \( N \) from (10) in terms of the metric and its conjugate momentum and then go back to the remaining constraints and solve them in terms of the trace of the transverse components of the metric. The longitudinal components of the metric and its conjugate momentum are eliminated, as in General Relativity, from the first class constraints and a gauge fixing condition. It is then crucial to show existence and uniqueness of the solution for \( N \) from (10).

We finally remark that the value \( \lambda = \frac{1}{3} \) is protected against quantum corrections by the second class constraints \( \pi = 0 \) which may be imposed for every time.

3. The existence and the uniqueness of the solution for \( N \) and the physical degrees of freedom

Given a Riemannian metric \( g_{ij} \) its conjugate momentum \( \pi^{ij} \), the equation (10) is a strongly elliptic equation for \( N \). Let us consider the Dirichlet problem for this partial differential equation. We consider an open bounded region \( \Omega \) with smooth boundary \( \partial \Omega \). The Dirichlet form
\[ D(u, v) \equiv \left( 1 - \frac{\alpha}{2} \right) \nabla u \nabla v + u \left( \frac{1}{g} \pi^{ij} \pi_{ij} + w C_{ij} C^{ij} \right) v \]
defined for all \( u \) and \( v \) on the Sobolev space \( H_1(\Omega) \) is selfadjoint, bounded and coercive for \( \alpha < 2 \).
Without loss of generality we may consider \( w > 0 \), this condition arises from the Chern-Simon action from which the interacting term \( \sqrt{g}C_{ij}C^{ij} \) is obtained via a detailed balance principle. The change of sign of \( w \) corresponds to a change of orientability of the three dimensional manifold on which the Chern-Simon action is constructed. We can always impose then \( w > 0 \).

The potential of the Dirichlet form is thus positive. Consequently the operator

\[
-\left(1 - \frac{\alpha}{2}\right)\nabla^2 + \frac{1}{g}\pi^{ij}\pi_{ij} + wC_{ij}C^{ij}
\]

has trivial kernel on \( \Omega \).

We now impose a boundary value on \( \partial\Omega \) for the solution of (10). We consider a function \( f \) defined on \( \Omega \) and satisfying the boundary value condition. The weak solution of (10) with \( N \) satisfying the given boundary value is obtained from the solution on \( H_1(\Omega) \) to

\[
D(u, v) = \langle G, v \rangle
\]  

for all \( v \in H_1(\Omega) \), where \( G = (1 - \frac{\alpha}{2})\nabla^2 f - \left(\frac{1}{g}\pi^{ij}\pi_{ij} + wC_{ij}C^{ij}\right) f \) and \( N = u + f \). We now applied general results for Dirichlet forms with the above properties to show that there exist a unique \( u \in H_1(\Omega) \), with zero boundary value on \( \partial\Omega \), solution of (12). Hence \( N = u + f \) is the unique solution of (10) satisfying the specified boundary value on \( \partial\Omega \).

Using regularity theorems for elliptic operators we then conclude that the solution for \( N \) exist, is unique and it is \( C^\infty \) if the metric and its conjugate momentum are \( C^\infty \) on the three dimensional spatial leaf of the foliation.

We then have a consistent formulation for the gravity theory obtained for a soft breaking of the Hořava original proposal. In fact, the Poisson bracket of the constraints closed on the submanifold of constraints for suitable values of the Lagrange multipliers and the solution for \( N \) exist and it is unique.

The number of physical degrees of freedom is obtained directly by noting that after solving \( N \) we are left with the second class constraints \( \pi = 0 \) and the hamiltonian constraint together with the first class constraint \( \mathcal{H}_i = 0 \). This counting is analogous to the case of General Relativity, where the hamiltonian constraints is of first class. In fact, a gauge fixing for this constraint, which may even be \( \pi = 0 \) (on regions of space-time where it is admissible), leave us with exactly the same number of physical degrees of freedom on both theories. Moreover if we consider a perturbative analysis around a Minkowski background, which is an exact solution of the field equations of the action (1), we obtain that the hamiltonian at quadratic level and at low energies is exactly the same as the hamiltonian of linearized General Relativity.

4. The propagator of the theory

The hamiltonian may also be rewritten as a sum of constraints plus boundary terms

\[
H = \int d^3x \left( \mathcal{N}H + N_i\mathcal{H}^i + \sigma\phi + \mu\pi \right) + E_{ADM} - 2\alpha\Phi_N
\]  

where

\[
\Phi_N \equiv \int d\Sigma_i \partial_i N.
\]  

The role of the boundary terms ensure the existence of the Gateaux derivative of the Hamiltonian under variations of \( N \) that behave as \( \delta N = \mathcal{O}(r^{-1}) \) at infinity. The energy of the theory is given by the value of the Hamiltonian on the constrained submanifold and it is thus equal to \( [E_{ADM} - 2\alpha\Phi_N] \).
We now consider the quadratic contribution to the Hamiltonian, subject to the linearized
constraints. We take
\[ g_{ij} = \delta_{ij} + h_{ij}, \quad \pi^{ij} = p_{ij}, \quad N = 1 + n \tag{15} \]
and obtain from the linearized constraints
\[ h^T = n = 0. \tag{16} \]
\( C^{ij} \) only depends on the transverse-traceless part of \( h_{ij} \) and it appears quadratically in the
constraints. The final expression of the Hamiltonian at the quadratic level, from which we directly
obtain the propagator of the theory, is
\[
H = \int d^3x \left( p_{ij}^T p_{ij}^T + \frac{1}{4} \partial_i h_{jk}^{TT} \partial_j h_{ij}^{TT} + w \partial_i h_{jk}^{TT} \Delta^2 \partial_j h_{ij}^{TT} \right). \tag{17}
\]
We then notice that at high energies the anisotropic third term dominates over the relativistic
second one, the propagator acquires additional powers of the spatial momentum \( (|k|^2)^3 \),
compared to \( |k|^2 \) in the relativistic propagator. This additional powers of momentum improve
the UV properties of the theory. At low energies the theory flows to the \( z = 1 \) theory which
exactly coincides with General Relativity at the quadratic level. The relativistic scaling for the
propagator is then restored at low energies. There is no dependence on the \( \alpha \) coupling in the
expression of the propagator.

5. Conclusions
The soft breaking model of Hořava gravity we are considering is a consistent theory which
propagates the same number of physical degrees of freedom as General Relativity. It is in
principle a renormalizable gravity theory. In order to achieve this final goal one should include
also the \( z = 2 \) interacting term in the theory. Although these terms do not contribute to the low
energy nor to the high energy behaviour of the theory they have to be included in the action
and it must be shown their consistency at any energy level.

We expect due to the structure of the second class constraints that no problem will arise from
their insertion in the quantization process.

From the point of view of the classical solutions of the theory, we have recently obtained
the complete space of spherically symmetric solutions [14] see also [15]. They depend only on
the \( \alpha \) coupling. The “external” solutions for small values of \( \alpha \) are “small” perturbations of
the external Schwarzschild solution corresponding to \( \alpha = 0 \). Depending on the value of \( \alpha \) the
solutions presents wormholes or naked singularities.

Acknowledgments
A. R. and A. S. are partially supported by Projects Fondecyt 1121103 and Mecesup ANT398
(Chile).

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