Seshadri constants and periods of polarized abelian varieties

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0 Introduction

The purpose of this paper is to study the Seshadri constants of abelian varieties. Consider a polarized abelian variety \((A, L)\) of dimension \(g\) over the field of complex numbers. One can associate to \((A, L)\) a real number \(\varepsilon(A, L)\), its Seshadri constant, which in effect measures how much of the positivity of \(L\) can be concentrated at any given point of \(A\). The number \(\varepsilon(A, L)\) can be defined as the rate of growth in \(k\) of the number of jets that one can specify in the linear series \(|O_A(kL)|\). Alternatively, one considers the blow-up \(f : \tilde{X} = \text{Bl}_x(X) \to X\) of \(X\) at a point \(x\) with exceptional divisor \(E \subset \tilde{X}\) over \(x\), and defines

\[
\varepsilon(A, L) = \sup \{ \varepsilon \in \mathbb{R} \mid f^*L - \varepsilon E \text{ is nef} \}.
\]

(Since \(A\) is homogeneous, this is independent of \(x\).) There has been recent interest in finding bounds on the Seshadri constants of abelian varieties and on smooth projective varieties in general (see [6] and [8]). For the case of abelian varieties one has the elementary bounds

\[
1 \leq \varepsilon(A, L) \leq \sqrt{Lg} \, \varepsilon(A, L) = \sup \{ \varepsilon \in \mathbb{R} \mid f^*L - \varepsilon E \text{ is nef} \}.
\]

where by a result of Nakamaye [11] the lower bound is taken on only by abelian varieties which are polarized products of an elliptic curve and an abelian variety of dimension \(g - 1\). Write now, as usual, \(A\) as the quotient \(A = V/\Lambda\) of its universal covering \(V\) and a lattice \(\Lambda \subset V\). Viewing the first Chern class of \(L\) as a positive definite Hermitian form on \(V\), its real part is a positive definite inner product \(b_L\) on \(V\), where we consider \(V\) as a real vector space of dimension \(2g\). We define the minimal period length of \((A, L)\) to be the real number

\[
m(A, L) = \min_{\lambda \in \Lambda, \lambda \neq 0} b_L(\lambda, \lambda) .
\]

So \(m(A, L)\) is the (square of the) length of the shortest non-zero period of \(A\), where the length is taken with respect to the euclidian metric defined by \(b_L\). When \(L\) is a principal polarization, this invariant has been studied by Buser and Sarnak in [2], who use an average argument familiar from the geometry of numbers to get a bound on the maximal value of \(m(A, L)\). Lazarsfeld has recently established in [8] a surprising connection between minimal period lengths and Seshadri constants. Using symplectic blowing up in the spirit of [3] he shows that the Seshadri constant of \((A, L)\) is bounded below in terms of \(m(A, L)\):

\[
\varepsilon(A, L) \geq \frac{\pi}{4} m(A, L) .
\]

(\(L\))

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By generalizing the result of Buser and Sarnak we obtain a lower bound on \( m(A, L) \) in terms of the type of the polarization, which then combined with \( (L) \) leads to:

**Theorem 1.** (a) The maximal value of \( m(A, L) \) as \( (A, L) \) varies over the moduli space \( \mathcal{A}_D \) of polarized abelian varieties of fixed type \((d_1, \ldots, d_g)\) is bounded below by

\[
\max_{(A,L) \in \mathcal{A}_D} m(A, L) \geq \frac{1}{\pi} \sqrt{2Lg} .
\]

(b) For the very general polarized abelian variety \((A, L)\) of fixed type \((d_1, \ldots, d_g)\) one has the inequality

\[
\varepsilon(A, L) \geq \frac{1}{4} \sqrt{2Lg} = \frac{1}{4} \left ( \frac{2g!}{\prod_{i=1}^{g} d_i} \right )^{\frac{1}{g}} .
\]

Note that for \( g \gg 0 \) the upper bound given in the theorem differs from the theoretical upper bound \( \sqrt{Lg} \) only by a factor of approximately 4. Also note that in [8] the inequality \((L)\) is stated for principally polarized abelian varieties. The proof given there, however, extends immediately to abelian varieties with polarizations of arbitrary type.

It is well-known that bounds on Seshadri constants have implications for adjoint linear series. In our situation this applies to the question whether an ample line bundle of some type \((d_1, \ldots, d_g)\) is very ample. One knows by the classical theorem of Lefschetz that \( L \) is very ample whenever \( d_1 \geq 3 \). Further, in case \( d_1 = 2 \) a result of Ohbuchi states that \( L \) is very ample if and only if the linear series \( |\mathcal{O}_A(\frac{1}{2}L)| \) has no fixed divisor. The remaining case of primitive line bundles, i.e. those with \( d_1 = 1 \), seems however to be very hard to deal with. Debarre, Hulek and Spandaw consider in [4] polarizations of type \((1, \ldots, 1, d)\), i.e. pullbacks of principal polarizations under cyclic isogenies, and show that for a generic \((A, L)\) of this type, \( L \) is very ample as soon as \( d > 2g \). Theorem 1 implies a criterion of a similar flavor for polarizations of arbitrary type, although the actual number that one gets in the special case of type \((1, \ldots, 1, d)\) is worse:

**Corollary 2.** Let \((A, L)\) be a generic polarized abelian variety of type \((d_1, \ldots, d_g)\). If

\[
\prod_{i=1}^{g} d_i \geq \frac{(8g)^g}{2g!} \approx \frac{1}{2} (8e)^g ,
\]

then \( L \) is very ample.

Roughly speaking, Theorem 1 says that the Seshadri constant of a very general abelian variety \((A, L)\) is quite close to the theoretical upper bound \( \sqrt{Lg} \). In the other direction, one is lead to ask under which geometrical circumstances \( \varepsilon(A, L) \) can become small – apart from the trivial situation when \( A \) contains an elliptic curve of small degree. Lazarsfeld [8] has shown that for the Jacobian \((JC, \Theta)\) of a compact Riemann surface \( C \) of genus \( g \geq 2 \) one has \( \varepsilon(JC, \Theta) \leq \sqrt{g} \). Now the principally polarized abelian varieties which may be considered as being closest to Jacobians are Prym varieties of étale double coverings. Our second result then shows that this intuition is indeed reflected by the fact that Prym varieties have small Seshadri constants:
Theorem 3. Let \((P, \Xi)\) be the Prym variety of an étale double covering \(\tilde{C} \to C\) of a compact Riemann surface \(C\) of genus \(g \geq 3\).

(a) One has
\[
\varepsilon(P, \Xi) \leq \sqrt{2(g-2)} = \sqrt{2(\dim(P) - 1)} .
\]

(b) If \(C\) admits a map \(C \to \mathbb{P}^1\) of degree \(d\), and if \(\tilde{C}\) is not hyperelliptic, then
\[
\varepsilon(P, \Xi) \leq \frac{2(d-1)(g-1)}{d + g - 1} = \frac{2(d-1)\dim(P)}{d + \dim(P)} .
\]

As for the assumptions in (b) note that for hyperelliptic \(\tilde{C}\), \((P, \Xi)\) is a hyperelliptic Jacobian, and then one has \(\varepsilon(P, \Xi) \leq 2 \dim(P)/(1 + \dim(P))\) by [8], which is weaker than the inequality for the non-hyperelliptic case in (b). Already the case \(\dim(P) = 2\) shows however that this cannot be improved in general.

The main result of the paper [2] by Buser and Sarnak states that Jacobians have periods of unusually small length. It is a consequence of Theorem 3 that a similar statement also holds (with larger numbers) for Prym varieties. In fact, combining Theorem 3 with \((L)\) we obtain:

Corollary 4. In the situation of cases (a) and (b) of Theorem 3 one has the following bounds on the minimal period length \(m(P, \Xi)\):

(a) \(m(P, \Xi) \leq \frac{4}{\pi} \sqrt{2(g-2)}\),

(b) \(m(P, \Xi) \leq \frac{8(d-1)(g-1)}{\pi(d + g - 1)} \leq \frac{8(d-1)}{\pi}\).

It would be interesting to know if one can get stronger inequalities by the methods of [2].

Finally, in an appendix (joint with T. Szemberg) we show how one can obtain more refined results on Seshadri constants for the case of abelian surfaces. One knows by work of Steffens [13] that for an abelian surface \((A, L)\) of type \((1, d)\) the Seshadri constant is maximal, i.e. equal to \(\sqrt{2d}\), if \(2d\) is a square and rank \(\text{NS}(A) = 1\). The most surprising result here is that by contrast if \(2d\) is not a square, then \(\varepsilon(A, L)\) is always sub-maximal:

Theorem 5 (with Szemberg). Let \(A\) be an abelian surface and let \(L\) be an ample line bundle of type \((1, d)\), \(d \geq 1\). If \(\sqrt{2d}\) is irrational, then
\[
\varepsilon(L) \leq \frac{2d}{\sqrt{1/k^2_0 + 2d}} ,
\]
where \((\ell_0, k_0)\) is the primitive solution of the diophantine equation \(\ell^2 - 2dk^2 = 1\) (Pell’s equation). In particular \(\varepsilon(L)\) is sub-maximal, i.e. \(\varepsilon(L) < \sqrt{2d}\).

If \(2d + 1\) is a square, then the inequality above is sharp. In fact, in this case the upper bound is taken on whenever \(\text{NS}(A) \cong \mathbb{Z}\).

As a consequence, one obtains:
Corollary 6. The Seshadri constant of an ample line bundle on an abelian surface is rational.

It is not known if Seshadri constants are always rational numbers, not even for the case of abelian varieties or for smooth surfaces. Also, the sub-maximality statement in Theorem 5 suggests the possibility that there is additional structure to these invariants which is not fully understood yet.

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Notation and Conventions. We work throughout over the field \( \mathbb{C} \) of complex numbers.

Numerical equivalence of divisors or line bundles will be denoted by \( \equiv \).

1 Period lengths of abelian varieties

The purpose of this section is to prove Theorem 1 from the introduction. We start with some remarks on polarized abelian varieties. So let \( d_1, \ldots, d_g \) be positive integers such that \( d_i | d_{i+1} \) for \( 1 \leq i < n \), let \( D \) be the diagonal matrix \( D = \text{diag}(d_1, \ldots, d_g) \) and denote as usual by \( \mathcal{A}_D \) the moduli space of polarized abelian varieties of type \( (d_1, \ldots, d_g) \). Recall that it can be realized as a quotient

\[ \mathcal{A}_D = \mathcal{H}_g / \text{Sp}^D_2(\mathbb{Z}) \]

of the Siegel upper half space \( \mathcal{H}_g = \{ Z \in M_g(\mathbb{C}) \mid \, tZ = Z \, \text{and} \, \text{Im} \, Z > 0 \} \) where the symplectic group

\[ \text{Sp}^D_2(\mathbb{Z}) = \left\{ R \in M_{2g}(\mathbb{Z}) \mid R \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}^t R = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} \right\} , \]

acts on \( \mathcal{H}_g \) by

\[ Z \mapsto R \cdot Z = (aZ + bD)(D^{-1}cZ + D^{-1}dD)^{-1} \quad \text{for} \quad R = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Sp}^D_2(\mathbb{Z}) \]

(cf. \cite{[7], Sect. 8.2}). Following the approach of Buser and Sarnak \cite{[2]} we now show:

Theorem 1.1. (a) One has

\[ \max_{(A,L) \in \mathcal{A}_D} m(A,L) \geq \frac{1}{\pi} \sqrt{2L^g} . \]

(b) There is a countable union \( \mathcal{B} \subset \mathcal{A}_D \) of proper closed subvarieties of \( \mathcal{A}_D \) such that for all \( (A,L) \in \mathcal{A}_D - \mathcal{B} \) one has the inequality

\[ \varepsilon(A,L) \geq \frac{1}{4} \sqrt[4]{2L^g} = \frac{1}{4} \left( 2g! \prod_{i=1}^{g} d_i \right)^{\frac{1}{4}} . \]
Proof. Let \( Z \) be an element of \( \mathcal{H}_g \), i.e. \( Z = X + iY \) with real-valued symmetric matrices \( X \) and \( Y \) such that \( Y \) is positive definite. Recall that modulo the action of \( \text{Sp}_D^{2g}(\mathbb{Z}) \) on \( \mathcal{H}_g \) the matrix \( Z \) corresponds to the isomorphism class of the polarized abelian variety \( (A_Z, L_Z) \) whose lattice is

\[
\Lambda_Z = (Z, D) \mathbb{Z}^{2g} \subset \mathbb{C}^g
\]

and whose Hermitian form \( H_Z \) is given by the matrix \( Y^{-1} \) with respect to the standard basis of \( \mathbb{C}^g \). Consider the quadratic form

\[
q_Z : \Lambda_Z \rightarrow \mathbb{R}
\]

\[
\lambda \mapsto H_Z(\lambda, \lambda)
\]

The columns of the matrix \( (Z, D) \) form a symplectic basis for \( \Lambda_Z \), i.e. a basis with respect to which the first Chern class of \( L_Z \), viewed as an alternating form on \( \Lambda_Z \), is given by the matrix

\[
\begin{pmatrix}
0 & D \\
-D & 0
\end{pmatrix}
\]

Now for \( m, n \in \mathbb{Z}^g \) we have

\[
q_Z(Zn + Dm) = (t(Zn + Dm))Y^{-1}(Zn + Dm)
\]

\[
= (t, t) \begin{pmatrix}
D & 0 \\
X & D
\end{pmatrix} \begin{pmatrix}
Y^{-1} & 0 \\
0 & D^{-1}YD^{-1}
\end{pmatrix} \begin{pmatrix}
D & X \\
0 & D
\end{pmatrix} \begin{pmatrix}
m \\
n
\end{pmatrix}
\]

so that \( q_Z \) is given with respect to the symplectic basis by the matrix \( Q_Z = tP_ZP_Z \), where

\[
P_Z = P_{X,Y} = \begin{pmatrix}
\sqrt{Y^{-1}} & 0 \\
0 & \sqrt{YD^{-1}}
\end{pmatrix} \begin{pmatrix}
D & X \\
0 & D
\end{pmatrix} \in M_{2g}(\mathbb{R})
\]

(1.1.1)

Fix now a real number \( R > 0 \) and denote for a polarized abelian variety \( (A, L) \) by \( n_R(X, L) \) the number of non-zero periods in the closed ball \( \overline{B}_R(0) \subset (V, b_L) \), i.e.

\[
n_R(A, L) = \# \{ \lambda \in \Lambda - \{0\} \mid b_L(\lambda, \lambda) \leq R^2 \}
\]

In view of what we found above, one has

\[
n_R(A_Z, L_Z) = \sum_{\ell \in \mathbb{Z}^{2g} \setminus \{0\}} \chi_{R^2}(t\ell Q_Z\ell),
\]

where \( \chi_{R^2} \) is the characteristic function of the interval \([0, R^2]\). We will now consider in particular period matrices of the form \( Z = X + \frac{1}{y^2} \mathbb{I} \) for \( y > 0 \), where \( \mathbb{I} \) denotes the identity matrix. The idea is to study the average of \( n_R(A_Z, L_Z) \) when \( y \) is fixed and \( X \) varies over a suitable compact set. Specifically, let

\[
V \subset \{ X \in M_g(\mathbb{R}) \mid tX = X \}
\]

be the compact subset consisting of the matrices whose entries are bounded by the exponent of \( L_Z \), i.e.

\[
V = \left\{ X \mid 0 \leq X_{ij} \leq d_g \text{ for } 1 \leq i, j \leq g \right\}
\]

(1.1.2)
and consider the average

\[ I(y) = \text{def} \frac{1}{\text{vol}(V)} \int_V \sum_{\ell \in \mathbb{Z}^g, \ell \neq 0} f \left( P_{X_\frac{1}{y}, \ell} \cdot \ell \right) dX \]

\[ = \frac{1}{\text{vol}(V)} \int_V n_R(A_Z, L_Z) dX , \]

where \( f : \mathbb{R}^{2g} \to \mathbb{R} \) is the function \( f(x) = \chi_{\mathbb{R}^2}(x \cdot x) \). It follows from Lemma 1.2 below that

\[ \lim_{y \to \infty} I(y) = \frac{R^{2g} \cdot \sigma_{2g}}{\prod_{i=1}^g d_i} , \]

so that we will have \( \lim_{y \to \infty} I(y) < 2 \) if we choose \( R \) such that

\[ R^2 < \frac{1}{\pi} \sqrt{2g! \prod d_i} . \quad (1.1.3) \]

But then there exists a real number \( y > 0 \) and a symmetric matrix \( X \in M_{2g}(\mathbb{R}) \) such that

\[ \sum_{\ell \in \mathbb{Z}^g, \ell \neq 0} f \left( P_{X_\frac{1}{y}, \ell} \cdot \ell \right) = n_R(A_Z, L_Z) < 2 . \]

Since \( n_R(A_Z, L_Z) \) is in any event an even non-negative integer, we must then have \( n_R(A_Z, L_Z) = 0 \). But this just means that for the polarized abelian variety \( (A_Z, L_Z) \) corresponding to \( Z \) one has

\[ m(A_Z, L_Z) > R^2 , \]

and, using (1.1.3) this implies the asserted lower bound on the maximum of \( m(A, L) \).

Assertion (b) follows from (a) and Lazarsfeld’s inequality (L).

\[ \square \]

**Lemma 1.2.** Let \( f : \mathbb{R}^{2g} \to \mathbb{R} \) be an integrable function of compact support. Consider the function \( I_f : \mathbb{R}^+ \to \mathbb{R} \) which is defined by

\[ I_f(y) = \frac{1}{\text{vol}(V)} \int_V \sum_{\ell \in \mathbb{Z}^g, \ell \neq 0} f \left( P_{X_\frac{1}{y}, \ell} \cdot \ell \right) dX \]

where \( P \) and \( V \) are as in (1.1.1) and (1.1.2) respectively. Then

\[ \lim_{y \to \infty} I_f(y) = \frac{1}{\det D} \int_{\mathbb{R}^{2g}} f(x) dx . \]
The proof is completely elementary but somewhat tricky. Here the dependence on the type of the polarization comes in crucially.

**Proof of Lemma 1.2.** The integral \( \text{vol}(V) \cdot I_f(y) \) can be written as

\[
\int_V \sum_{m,n \in \mathbb{Z}} f \begin{pmatrix}
    y(d_1m_1 + \sum_{i=1}^g X_{1i}n_i) \\
    \vdots \\
    y(d_g m_g + \sum_{i=1}^g X_{gi}n_i) \\
    y^{-1}n_1 \\
    \vdots \\
    y^{-1}n_g
\end{pmatrix} dX.
\]

We integrate under the sum and consider first the terms with \( n \neq 0 \). The contribution of such a term, if say \( n_k \neq 0 \), is

\[
c_n = \int_0^{d_g} \cdots \int_0^{d_g} \sum_{m_1} \cdots \sum_{m_2} F_{1,k}(m, n, X)dX',
\]

where we set \( dX' = \prod_{i \leq j, (i,j) \neq (1,k)} dX_{ij} \) and

\[
F_{1,k}(m, n, X) = \sum_{m_1} \int_0^{d_g} \left( yd_1n_k \left( \frac{m_1}{n_k} + \frac{X_{1k}}{d_1} + \lambda_k \right) \right) dX_{1k}
\]

\[
= \sum_{m_1=0}^{n_k-1} \sum_{j=-\infty}^{\infty} \int_0^{d_g} d_1 f \left( yd_1n_k \left( \frac{m_1}{n_k} + j + T_{1k} + \lambda_k \right) \right) dT_{1k}
\]

where \( \lambda_k \) is independent of \( T_{1k} \) and \( m_1 \). We therefore obtain

\[
F_{1,k}(m, n, X) = n_k \frac{d_g}{d_1} \int_{-\infty}^{\infty} d_1 f \left( \frac{yd_1n_k T_{1k}}{d_1} \right) dT_{1k}
\]

\[
= y^{-1} \frac{d_g}{d_1} \int_{-\infty}^{\infty} f \left( \frac{t_1}{d_1} \right) dt_1.
\]

Continuing in the same manner with \( m_2, X_{2k} \) up to \( m_g, X_{2g} \) we find

\[
c_n = \int_0^{d_g} \cdots \int_0^{d_g} y^{-1} \left( \prod_{i=1}^g \frac{d_g}{d_1} \right) \int_{\mathbb{R}^g} f(t_1, \ldots, t_g, y^{-1}n_1, \ldots, y^{-1}n_g) dt \prod_{i \leq j, i \neq k \neq j} dX_{ij}
\]

\[
= \frac{\text{vol}(V)}{\det D} y^{-1} \int_{\mathbb{R}^g} f(t_1, \ldots, t_g, y^{-1}n_1, \ldots, y^{-1}n_g) dt
\]
so that, taking into account that $f$ is of compact support, one gets

$$
\lim_{y \to -\infty} \text{vol}(V) \cdot I_f(y) = \lim_{y \to -\infty} \sum_{n \in \mathbb{Z}^g} c_n + \sum_{m \neq 0} f \left( \begin{array}{c} yDm \\ 0 \end{array} \right) = \frac{\text{vol}(V)}{\det D} \int_{\mathbb{R}^g} f(x) dx
$$

which proves the lemma.

\[\square\]

**Corollary 1.3.** Let $(A, L)$ be a generic polarized abelian variety of type $(d_1, \ldots, d_g)$. If

$$
\prod_{i=1}^{g} d_i \geq \frac{(8g)^g}{2g!} \approx \frac{1}{2} (8e)^g,
$$

then $L$ is very ample.

In fact, the bound on $\prod_{i=1}^{g} d_i$ guarantees by Theorem 1.1 that for the very general polarized abelian variety $(A, L)$ of the given type one has $\varepsilon(A, L) \geq 2g$. This implies by a standard application of Kawamata-Viehweg vanishing (cf. [3, Sect. 4] and [3, Proposition 6.8]) that $L$ is very ample. Note that since very ampleness is an open condition on $A_D$, the corollary holds for generic $(A, L)$, even if we have the lower bound on $\varepsilon(A, L)$ only for very general $(A, L)$.

## 2 Seshadri constants of Prym varieties

Let $f : \tilde{C} \rightarrow C$ be an étale double cover of a compact Riemann surface $C$ of genus $g$. Identify as usual the Jacobians $JC$ and $J\tilde{C}$ with their respective dual abelian varieties and consider the pullback map $f^* : JC \rightarrow J\tilde{C}$. The Prym variety $P$ of the given double cover is the complementary abelian subvariety of the image of $f^*$ in $J\tilde{C}$ (see [7, Chap. 12] and [10]). The canonical principal polarization $O_{J\tilde{C}}(\tilde{\Theta})$ on $J\tilde{C}$ restricts to twice a principal polarization $O_P(\Xi)$ on the $(g-1)$-dimensional abelian variety $P$. We prove in this section the following bounds on the Seshadri constant of $(P, \Xi)$:

**Theorem 2.1.** Assume $g \geq 3$. Then:

(a) One has

$$
\varepsilon(P, \Xi) \leq \sqrt{2(g-2)}.
$$

(b) If $C$ admits a map $C \rightarrow \mathbb{P}^1$ of degree $d$, and if $\tilde{C}$ is not hyperelliptic, then

$$
\varepsilon(P, \Xi) \leq \frac{2(d-1)(g-1)}{d + g - 1}.
$$

Corollary 4 in the introduction follows from the theorem and Lazarsfeld’s result $(L)$.

**Proof of Theorem 2.1.** (a) Note first that we may assume $\dim(P) \geq 3$, since for $\dim(P) = 2$ the inequality is clear. Further, we may assume that $\tilde{C}$ is not hyperelliptic: in fact,
otherwise \((P, \Xi)\) is a Jacobian (see [4, Corollary 12.5.7]) and then one has \(\varepsilon(P, \Xi) \leq (\dim(P))^{\frac{1}{2}} \leq (2 \dim(P) - 2)^{\frac{1}{2}}\).

We will as usual identify \(J\tilde{C}\) and \(P\) with their respective dual abelian varieties via the isomorphisms defined by the principal polarizations \(\mathcal{O}_{J\tilde{C}}(\Theta)\) and \(\mathcal{O}_P(\Xi)\). The dual map of the inclusion \(\iota: P \hookrightarrow J\tilde{C}\) gives then a surjective morphism \(\tilde{\iota}: J\tilde{C} \twoheadrightarrow P\). Note that the composition \(\iota \circ \tilde{\iota}\) is just the norm endomorphism \(N_P\) of \(P\) (cf. [7, Sect. 12.2]). We will study the image \(S \subset P\) of the composed map

\[
\psi: \tilde{C} \times \tilde{C} \xrightarrow{\tilde{s}} J\tilde{C} \xrightarrow{\tilde{\iota}} P,
\]

where \(\tilde{s}\) is the subtraction map \((x, y) \mapsto \mathcal{O}_{\tilde{C}}(x - y)\). We verify first that

\[
\dim(S) = 2.
\]

In fact, we have

\[
S = \psi(\tilde{C} \times \tilde{C}) = N_P(\tilde{C}) - N_P(\tilde{C}),
\]

so if \(S\) were a point, then \(\tilde{C}\) would be contained in the kernel of \(\tilde{\iota}\) which is certainly impossible. And if \(S\) were a curve, then one would have \(N_P(\tilde{C}) - N_P(\tilde{C}) = N_P(\tilde{C})\), so that \(N_P(\tilde{C})\) would be an elliptic curve; but this would imply \(N_P(\tilde{\Theta}) = N_P(\tilde{C} + \ldots + \tilde{C}) = N_P(\tilde{C})\), contradicting the fact that the fibres of \(N_P\) are of dimension \(g(\tilde{C}) - \dim(P) \leq g(\tilde{C}) - 3\).

We claim next that

\[
\deg(\psi)(S) = \Xi^2 \cdot S = \frac{8}{\deg(\psi)}(g - 1)(g - 2).
\] (2.1.1)

For the proof of (2.1.1) recall first that \(2\tilde{\Theta} \equiv Nm^*\Theta + \tilde{\tau}^*\Xi\), where \(\mathcal{O}_{J\tilde{C}}(\Theta)\) is the canonical principal polarization on \(J\tilde{C}\) and \(Nm: J\tilde{C} \longrightarrow J\tilde{C}\) is the norm map associated with \(f\) (see [7, Proposition 12.3.4]). So we have

\[
\psi^*\Xi \equiv 2\tilde{s}^*\tilde{\Theta} - \tilde{s}^*Nm^*\Theta.
\]

Let now \(F_1, F_2 \subset C \times C\) be fibres of the two projections and \(\Delta \subset C \times C\) the diagonal. For the pullback \(s^*\Theta\) of \(\Theta\) under the subtraction map \(s: C \times C \longrightarrow J\tilde{C}\) one has \(s^*\Theta \equiv (g - 1)(F_1 + F_2) + \Delta\) (cf. [12, Theorem 4.2]). So, using the commutative diagram

\[
\begin{array}{ccc}
\tilde{C} \times \tilde{C} & \xrightarrow{f \times f} & C \times C \\
\downarrow \tilde{s} & & \downarrow s \\
J\tilde{C} & \xrightarrow{Nm} & J\tilde{C}
\end{array}
\]

we find

\[
\tilde{s}^*Nm^*\Theta \equiv (f \times f)^*s^*\Theta \equiv 2(g - 1)(\tilde{F}_1 + \tilde{F}_2) + \tilde{\Delta} + \Gamma_r,
\]
where \( \tilde{F}_1, \tilde{F}_2 \subset \tilde{C} \times \tilde{C} \) are fibres of the projections, \( \tilde{\Delta} \) is the diagonal, and \( \Gamma_\tau \) is the graph of the covering involution \( \tau : \tilde{C} \to \tilde{C} \). We conclude that

\[
\psi^*\Xi \equiv (2g - 2)(\tilde{F}_1 + \tilde{F}_2) + \tilde{\Delta} - \Gamma_\tau
\]

(2.1.2)

and, using the fact that \( \tau \) is fixed-point free, this implies with a calculation

\[
\deg \Xi(S) = \Xi^2 \cdot S = \frac{1}{\deg(\psi)} \Xi^2 \cdot \psi_*(\tilde{C} \times \tilde{C}) = \frac{1}{\deg(\psi)} (\psi^*\Xi)^2 = \frac{8}{\deg(\psi)} (g - 1)(g - 2),
\]

as claimed.

The diagonal \( \tilde{\Delta} \) is the scheme-theoretic inverse image of 0 under \( \psi \). In fact, one has \( N_P = 1 - \tilde{\tau} \), where \( \tilde{\tau} \) is the involution on \( J\tilde{C} \) induced by \( \tau \), thus for \( x, y \in \tilde{C} \)

\[
i \circ \psi(x, y) = i \circ \tilde{i} \circ \tilde{s}(x, y) = N_P \mathcal{O}_{\tilde{C}}(x - y) = \mathcal{O}_{\tilde{C}}(x - y - \tau(x) + \tau(y)),
\]

so that, since \( \tilde{C} \) is not hyperelliptic, \( \psi(x, y) = 0 \) implies \( x = y \). So we have \( \psi^{-1}(0) = \tilde{\Delta} \) set-theoretically and, using the fact that the Abel-Prym map

\[
\tilde{C} \hookrightarrow J\tilde{C} \xrightarrow{\tilde{i}} P
\]

is an embedding, one checks that this also holds scheme-theoretically. Let now \( \tilde{P} \to P \) be the blow-up of \( P \) at 0 with exceptional divisor \( E \), and let \( \tilde{S} = \text{Bl}_0(S) \) be the proper transform of \( S \). One has a commutative diagram

\[
\begin{array}{ccc}
\tilde{\Delta} & \to & \mathbb{P}C_0(S) \\
\downarrow & & \downarrow \\
\tilde{C} \times \tilde{C} & \xrightarrow{\psi} & \tilde{S} \\
\downarrow & & \downarrow \\
\tilde{C} \times \tilde{C} & \xrightarrow{\psi} & S \\
\downarrow & & \downarrow \\
\tilde{C} \times \tilde{C} & \xrightarrow{\psi} & P
\end{array}
\]

where \( \mathbb{P}C_0(S) \) is the projective tangent cone of \( S \) at 0. So, using \( \tilde{\psi}^*E = \tilde{\Delta} \), we obtain

\[
\text{mult}_0(S) = \int_E \mathcal{O}_E(1) \cdot [\mathbb{P}C_0(S)]
\]

\[
= \int_{\tilde{P}} \mathcal{O}_{\tilde{P}}(-E) \cdot [\mathbb{P}C_0(S)] = -\int_{\tilde{P}} \mathcal{O}_{\tilde{P}}(E)^2 \cdot [\tilde{S}]
\]

\[
= -\frac{1}{\deg(\psi)} \int_{\tilde{C} \times \tilde{C}} \tilde{\psi}^* \mathcal{O}_{\tilde{P}}(E)^2 = -\frac{1}{\deg(\psi)} \tilde{\Delta}^2
\]

\[
= \frac{1}{\deg(\psi)} (2g(\tilde{C}) - 2) = \frac{4}{\deg(\psi)} (g - 1).
\]

(2.1.3)
Now recall that by [5, (6.7)] any singular subvariety of $P$ leads to an upper bound on the Seshadri constant of $(P, \Xi)$. Applying this to $S$ we find upon using (2.1.1) and (2.1.3)

$$
\varepsilon(P, \Xi) \leq \sqrt{\frac{\text{deg}_{\Xi}(S)}{\text{mult}_0(S)}} = \sqrt{2(g-2)} .
$$

(b) Suppose now that there exists a map $\phi : C \rightarrow \mathbb{P}^1$ of degree $d$. This implies that there is an effective divisor $D \in |d(F_1 + F_2) - \Delta|$, namely the closure of $\{(x, y) \mid \phi(x) = \phi(y), x \neq y\}$. It pulls back to an effective divisor

$$(f \times f)^*D \in |2d(\tilde{F}_1 + \tilde{F}_2) - \tilde{\Delta} - \Gamma_\tau| .$$

Since by assumption $\tilde{C}$ is not hyperelliptic, we have again $\tilde{\Delta} = \psi^{-1}(0)$ scheme-theoretically, and therefore the $\mathbb{R}$-divisor $\psi^\ast \Xi - \varepsilon(P, \Xi) \cdot \tilde{\Delta}$ is nef, so that

$$
\left(\psi^\ast \Xi - \varepsilon(P, \Xi) \cdot \tilde{\Delta}\right) \cdot (f \times f)^*D \geq 0 .
$$

Upon using (2.1.2) one obtains the asserted inequality for $\varepsilon(P, \Xi)$.

**Appendix: Seshadri constants of abelian surfaces**

Thomas Bauer and Tomasz Szemberg

Our purpose here is to show how one can get more refined results on Seshadri constants for the case of abelian surfaces. In particular it follows that, somewhat surprisingly, Seshadri constants on abelian surfaces are always rational.

Consider an abelian surface $A$ and an ample line bundle $L$ on $A$. Since $\varepsilon(kL) = k\varepsilon(L)$ for any integer $k > 0$, we may assume that $L$ is primitive, i.e. of type $(1,d)$ for some integer $d \geq 1$. Recall the elementary bounds

$$
1 \leq \varepsilon(L) \leq \sqrt{2d} .
$$

One knows moreover by [11, Theorem 1.2] that $\varepsilon(L) \geq \frac{1}{3}$, unless $A$ is a product of elliptic curves. Further, if $\sqrt{2d}$ is rational and rank $\text{NS}(A) = 1$, then by [13] the Seshadri constant $\varepsilon(L)$ is maximal, i.e. $\varepsilon(L) = \sqrt{2d}$, which shows that the upper bound in (*) cannot be improved in general. On the other hand, if $\sqrt{2d}$ is irrational, then our result shows that one does have a better upper bound:

**Theorem A.1.** Let $A$ be an abelian surface and let $L$ be an ample line bundle of type $(1,d)$, $d \geq 1$.

(a) If $\sqrt{2d}$ is irrational, then

$$
\varepsilon(L) \leq \frac{2d}{\sqrt{1/k_0^2 + 2d} ,}
$$

where $(\ell_0, k_0)$ is the primitive solution of the diophantine equation $\ell^2 - 2dk^2 = 1$ (Pell’s equation). In particular $\varepsilon(L)$ is sub-maximal, i.e. $\varepsilon(L) < \sqrt{2d}$. 

(b) One has the lower bound
\[ \varepsilon(L) \geq \min \left\{ \varepsilon_0, \frac{\sqrt{7}}{2}d \right\}, \]
where \( \varepsilon_0 \) is the minimal degree (with respect to \( L \)) of the elliptic curves in \( X \).

(c) If \( 2d + 1 \) is a square, then the inequality in (a) is sharp. In fact, in this case the upper bound is taken on whenever \( \text{NS}(A) \cong \mathbb{Z} \).

At first sight the bound in (a) might appear non-constructive because it involves the primitive solution of Pell’s equation. This solution, however, can be effectively computed via continued fractions (see [\ref{3}] for the numerical values for polarizations of small degree). As for (b) note that it is inevitable that small values of \( \varepsilon(L) \) occur for non-simple abelian surfaces regardless how large the type of the polarization may be, since for any given integer \( e \geq 1 \) there are abelian surfaces \( (A, L) \) of arbitrarily high degree \( L^2 \) containing an elliptic curve of degree \( e \). We do not expect that the particular bound in (b) is optimal; on the other hand it is tempting, in view of (c), to wonder whether the bound in (a) might be sharp in general.

**Proof of Theorem A.1.** (a) Since \( \varepsilon(L) \) is invariant under algebraic equivalence, we may assume \( L \) to be symmetric. Recall that for any \( n \geq 1 \) the space of sections of \( \mathcal{O}_A(nL) \) admits a decomposition
\[ H^0(A, \mathcal{O}_A(nL)) = H^0(A, \mathcal{O}_A(nL))^+ \oplus H^0(A, \mathcal{O}_A(nL))^– \]
into the spaces of even and odd sections whose dimensions are given by the formula
\[ h^0(A, \mathcal{O}_A(nL))^\pm = 2 + \frac{n^2d^2}{2} - \frac{n^\pm(\mathcal{O}_A(nL))}{4}, \]
where \( n^\pm(\mathcal{O}_A(nL)) \) is the number of odd respectively even halfperiods of the line bundle \( \mathcal{O}_A(nL) \) (cf. \[\ref{4}\] Corollary 4.6.6] and \[\ref{1}\] Theorem 3.1]). So for even multiples \( n = 2k \) of \( L \) we have in particular \( h^0(A, \mathcal{O}_A(2kL))^+ = 2 + 2dk^2 \). On the other hand, since an even section vanishes in halfperiods to even orders, it is at most
\[ 1 + 3 + \ldots + (m - 1) = \left( \frac{m}{2} \right)^2 \]
conditions on an even section to vanish at a fixed halfperiod \( x \) to an even order \( m \). This implies that
\[ H^0(A, \mathcal{O}_A(2kL) \otimes \mathcal{T}_x^m)^+ \neq 0 \]
provided that \( m \leq 2\sqrt{2dk^2} + 1 \). Thus there exists an even divisor \( D \in |2kL|^+ \) with multiplicity
\[ \text{mult}_x(D) \geq \left\lfloor 2\sqrt{2dk^2} + 1 \right\rfloor \]
at $x$. The crucial point is now to avoid the round-down in this expression. To this end consider the diophantine equation

$$\ell^2 - 2dk^2 = 1 ,$$

a special case of Pell’s equation. Since $\sqrt{2d}$ is by assumption irrational, Pell’s equation has a primitive solution $(\ell_0, k_0)$. We conclude that

$$\varepsilon(L) \leq \frac{L \cdot D}{\text{mult}_x(D)} = \frac{4dk_0}{2\sqrt{2dk_0^2} + 1} = \frac{2d}{\sqrt{1/k_0^2 + 2d}} < \sqrt{2d} ,$$

as claimed.

(b) Let $C \subset A$ be an irreducible curve of arithmetic genus $p_a(C) > 1$, let $x \in C$ and $m = \text{mult}_x(C)$. We have to show that

$$\frac{L \cdot C}{m} \geq \frac{\sqrt{7}}{2} \sqrt{d} \quad \text{for all } x \in A .$$

First observe that for the geometric genus one has

$$p_g(C) \geq 2 .$$

In fact, suppose to the contrary that $p_g(C) \leq 1$ and consider the normalization $N \rightarrow C$. Abelian varieties do not contain any rational curves, so $p_g(C) = 1$ and the composed map

$$N \rightarrow C \hookrightarrow A$$

is − after possibly translating $C$ − a homomorphism of abelian varieties, and hence an embedding, which is absurd.

The adjunction formula and the inequality $p_a(C) = p_g(C) + (m_2)$ then yield

$$m \leq \sqrt{C^2 - \frac{7}{4} + \frac{1}{2}} ,$$

Combining this bound on the multiplicities of irreducible curves with Hodge index gives

$$\varepsilon(L) \geq \inf_{L \cdot C} \left\{ \frac{L \cdot C}{\sqrt{(L \cdot C)^2 - \frac{7}{4} + \frac{1}{2}}} \right\}$$

where the infimum is taken over the degrees $L \cdot C$ of the irreducible non-elliptic curves $C \subset A$. But the real-valued function

$$f(t) = \frac{t}{\sqrt{t^2/2d - \frac{7}{4} + \frac{1}{2}}}$$

takes on its minimum at $t_0 = 2\sqrt{7d}$ with minimal value $\frac{\sqrt{7}}{2} \sqrt{d}$ at $t_0$. This implies the assertion.
(c) By assumption we have $2d + 1 = \ell^2$ for some integer $\ell \geq 1$. Then $d$ is an even number, and after possibly replacing $L$ by another symmetric translate one has

$$h^0(A, L)^+ = \frac{d}{2} + 1.$$ 

Since the line bundle $L$ is primitive, it has both even and odd halfperiods. Thus we may choose an odd halfperiod $x$, so that the number of conditions on a section in $H^0(A, L)^+$ to vanish at $x$ to order $2p + 1$ is $2 + 4 + \ldots + 2p = p(p + 1)$. Therefore there exists a divisor $D \in |L|^+$ with

$$\text{mult}_x(D) \geq 2 \left\lfloor \frac{1}{2} \sqrt{2d + 1} - \frac{1}{2} \right\rfloor + 1 = \ell,$$

Suppose now that there is an irreducible curve $C \subset A$ with $L \cdot C / \text{mult}_x(C) < 2d/\ell$. Then the assumption $\text{NS}(A) \cong \mathbb{Z}$ implies that $D$ is irreducible, so that $C$ and $D$ intersect properly, and hence

$$L \cdot C = D \cdot C \geq \text{mult}_x(D) \cdot \text{mult}_x(C) > (L \cdot D)(L \cdot C)^{\ell^2 / (2d)^2} = (L \cdot C)^{2d + 1 / 2d},$$

a contradiction.

In all known examples Seshadri constants have turned out to be rational. While it is unclear if this is true in general, Theorem A.1 implies:

**Corollary A.2.** The Seshadri constant of an ample line bundle on an abelian surface is rational.

In fact, as shown in [13], this follows from the submaximality statement in part (a) of Theorem A.1 by applying the Nakai-Moishezon criterion for $\mathbb{R}$-divisors [3]. It would be interesting to have a more conceptual explanation for the sub-maximality statement in Theorem A.1 and also for the rationality of Seshadri constants on abelian surfaces.

**Remark A.3.** In order to convey some feeling for the numbers involved here, we list for $1 \leq d \leq 20$ the (truncated) numerical values of the upper bound $\varepsilon_{\text{upper}}(d)$ given in part (a) of Theorem A.1 along with the lower bound $\varepsilon_{\text{lower}}(d)$ from part (b) and the theoretical upper bound $\sqrt{2d}$. Note that the lower bound holds for simple $A$ only. Numbers in boldface indicate the cases where one knows the exact value of $\varepsilon(A)$ when rank $\text{NS}(A) = 1$.

| $d$ | $\varepsilon_{\text{lower}}(d)$ | $\varepsilon_{\text{upper}}(d)$ | $\sqrt{2d}$ |
|-----|-------------------------------|-------------------------------|-------------|
| 1   | 1.3228                        | **1.3333**                    | 1.4142      |
| 2   | 1.8708                        | **2.6666**                    | 2.8284      |
| 4   | 2.6457                        | **3.6178**                    | 3.1622      |
| 5   | 2.9580                        | **3.7333**                    | 3.4641      |
| 7   | 3.5000                        | **3.7333**                    | 3.7416      |
| 8   | 3.7416                        | **4.4444**                    | 4.4721      |
| 9   | 3.9686                        | **4.4444**                    | 4.4721      |
| 10  | 4.1833                        | **4.4444**                    | 4.4721      |
| 11  | 4.3874                        | 4.6903                        | 4.6904      |
| 12  | 4.5825                        | **4.8000**                    | 4.8989      |
| 13  | 4.7696                        | 5.0990                        | 5.0990      |
| 14  | 4.9497                        | 5.2913                        | 5.2915      |
| 15  | 5.1234                        | 5.4772                        | 5.4772      |
| 16  | 5.2915                        | 5.6568                        | 5.6568      |
| 17  | 5.4543                        | 5.8309                        | 5.8309      |
| 18  | 5.6124                        | 6.1644                        | 6.1644      |
| 19  | 5.7662                        | 6.3157                        | 6.3245      |
| 20  | 5.9160                        | 6.3157                        | 6.3245      |
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