VIRTUAL POINCARÉ POLYNOMIAL OF THE SPACE OF STABLE PAIRS SUPPORTED ON QUINTIC CURVES

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Abstract. Let $M^\alpha(d, \chi)$ be the moduli space of $\alpha$-stable pairs $(s, F)$ on the projective plane $\mathbb{P}^2$ with Hilbert polynomial $\chi(F(m)) = dm + \chi$. For sufficiently large $\alpha$ (denoted by $\alpha^\infty$), it is well known that the moduli space is isomorphic to the relative Hilbert scheme of points over the universal degree $d$ plane curves. For the general $(d, \chi)$, the relative Hilbert scheme does not have a bundle structure over the Hilbert scheme of points. In this paper, as the first non-trivial such a case, we study the wall crossing of the $\alpha$-stable pairs space when $(d, \chi) = (5, 2)$. As a direct corollary, by combining with Bridgeland wall crossing of the moduli space of stable sheaves, we compute the virtual Poincaré polynomial of $M^\infty(5, 2)$.

1. Introduction

1.1. Introduction and results. By definition, a pair $(s, F)$ consists of a sheaf $F$ on $\mathbb{P}^2$ and one-dimensional subspace $s \subset H^0(F)$. Let us fix $\alpha \in \mathbb{Q}[m]$ with a positive leading coefficient. A pair $(s, F)$ is called $\alpha$-semistable if $F$ is pure and for any subsheaves $F' \subset F$, the inequality

$$\frac{\chi(F'(m)) + \delta \cdot \alpha}{r(F')} \leq \frac{\chi(F(m)) + \alpha}{r(F)}$$

holds for $m \gg 0$. Here $r(F)$ is the leading coefficient of the Hilbert polynomial $\chi(F(m))$ and $\delta = 1$ if the section $s$ factors through $F'$ and $\delta = 0$ otherwise. When the strict inequality holds, $(s, F)$ is called $\alpha$-stable.

With the help of the general result of the geometric invariant theory ([18]), Le Potier proved that there exist projective schemes $M^\alpha(d, \chi)$ parameterizing $S$-equivalence classes of $\alpha$-semistable pairs with Hilbert polynomial $P(m) = dm + \chi$. Also, M. He [13] study the wall crossings (or flips) of the moduli spaces $M^\alpha(d, \chi)$ as $\alpha$ varies. In two external case,

- If $\deg(\alpha) \geq 2$, then $M^{\alpha = \infty}(d, \chi)$ is isomorphic to the relative Hilbert scheme of $n := \chi - \frac{d(3-d)}{2}$ points on the universal degree $d$ curve ([13] §4.4), [21 Proposition B.8]).
- If $\alpha$ is sufficiently small (denoted by $\alpha = 0^+$), the moduli space has a natural forgetting morphism

$$\xi : M^{0^+}(d, \chi) \to M(d, \chi)$$
which associates to the $0^+$-stable pair $(s, F)$ the sheaf $F$. The later moduli space $M(d, \chi)$ parameterizes semistable sheaves with Hilbert polynomial $dm + \chi$.

When $\alpha = \infty$, $\alpha$-stable pairs are precisely stable pairs in the sense of Pandharipande-Thomas. Let us denote by $B(d, n)$ the relative Hilbert scheme of $n$ points on the universal degree $d$ curve. In [6], when $d \leq 5$ and $\chi = 1$, the moduli spaces are birational among each other and thus we obtain the cohomology group of the space $M(d, \chi)$ by studying the wall crossing of the moduli spaces $M^\alpha(d, \chi)$. In this case, the work is well going since the relative Hilbert scheme $B(d, n)$ is a projective bundle and all of the wall crossings are simple, that is, the number of the JH-filterations of $\alpha$-stable pairs is two. But, for the large $(d, \chi)$, the wall crossings among the $\alpha$-stable pairs space become very complicate because the wall crossing may not not be simple. Also it is hard to understand the geometry of the relative Hilbert scheme $B(d, \chi)$ (cf. [6]). Hence we need more careful study to get some geometric information of the $M(d, \chi)$ from the $\alpha$-stable pairs spaces or its wall crossings. In this paper, we study the wall crossings when $(d, \chi) = (5, 2)$. This is the first case such that the relative Hilbert scheme is not a projective bundle and the wall crossings may not be simple. That is, we will show that

**Theorem 1.1.**

1. There are five times wall-crossing between $M^\infty(5, 2)$ and $M^+(5, 2)$ with geometric meaningful centers.
2. The forgetting map $M^+(5, 2) \to M(5, 2)$ is a projective bundle map on the $M_2^+(5, 2)$ with fiber $\mathbb{P}^1$. In the complement of $M_2^+(5, 2)$, it is a $\mathbb{P}^2$-bundle map.

Here $M_2^+(5, 2)$ is the locus of $0^+$-stable pairs $(s, F)$ with $h^0(F) = 2$.

On the other hand, the moduli space $M(5, 2)$ has another wall-crossings, that is, the Bridgeland wall crossing. This was done by a general setting by many authors (For example, [24, 3]). To get the cohomology group of the space $M(5, 2)$ from the Bridgeland wall crossing as did in [7], it is essential to know the final birational (or wall-crossing model) of the moduli space $M(5, 2)$. By studying the nef cone of one of the birational model of $M(5, 2)$, we obtain

**Proposition 1.2.** The final birational model of $M(5, 2)$ is isomorphic to the Grassmannian variety $\text{Gr}(2, 15)$.

The wall crossings of two different types are summarized into the following diagram.

As a direct corollary,
Corollary 1.3. The virtual Poincaré polynomial of the space $M^\infty(5,2)$ is given by
\[ 1 + 3p + 9p^2 + 22p^3 + 50p^4 + 99p^5 + 173p^6 + 256p^7 + 330p^8 + 379p^9 + 407p^{10} + 420p^{11} + 426p^{12} + 428p^{13} + 429p^{14} + 428p^{15} + 423p^{16} + 410p^{17} + 382p^{18} + 333p^{19} + 259p^{20} + 176p^{21} + 101p^{22} + 51p^{23} + 22p^{24} + 9p^{25} + 3p^{26} + p^{27}. \]

Remark 1.4. Specially, the virtual Euler number of $M^\infty(5,2)$ is $e(M^\infty(5,2)) = 6030$. But the virtual Euler number of the PT-space of local $\mathbb{P}^2$ (that is, the total space of the canonical line bundle $K_{\mathbb{P}^2}$) is 6060 which is done by using the torus localization technique ([1]). The difference 30 comes from the Euler number of the sheaves supported on $\mathbb{P}^1$ which is the outside of the zero section of local $\mathbb{P}^2$. This is informed to the author by J. Choi. The author would like to thank J. Choi for the comment.

1.2. Stream of the paper. In §2, we study the wall crossing of the moduli spaces of $\alpha$-stable pairs on $\mathbb{P}^2$ by using the classification of semistable sheaves in [10, 17]. Also, we analyze the forgetting map $\xi$, by considering the Brill-Noether locus in $M(5,2)$. In §3, we find the last birational model of $M(5,2)$ by studying the effective cone of the moduli space $M(5,2)$. As a corollary, we obtain the Poincaré polynomial of the space $M(5,2)$ which reproves the result of [25]. In §4, we compute the Poincaré polynomial of the relative Hilbert scheme $B(5,7)$ by using the result of the previous sections.

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2. Wall crossings of the spaces $M^\infty(5,2)$

In this section, we firstly study the wall crossing between $M^\infty(5,2)$ and $M^+(5,2)$. Secondly, we analyze the forgetting map $M^+(5,2) \to M(5,2)$ by studying at the Brill-Noether locus. For the convinces of the readers, we leave the following useful lemmas which will be used several times in the paper.

Lemma 2.1. ([13] Corollary 1.6) Let $\Lambda = (s,F)$ and $\Lambda' = (s',F')$ be pairs on a smooth projective variety $X$. There exists a long exact sequence
\[ 0 \to \text{Hom}(\Lambda, \Lambda') \to \text{Hom}(F, F') \to \text{Hom}(s, H^2(F'))/s' \to \text{Ext}^1(\Lambda, \Lambda') \to \text{Ext}^1(F, F') \to \text{Hom}(s, H^1(F')) \to \text{Ext}^2(\Lambda, \Lambda') \to \text{Ext}^2(F, F') \to \text{Hom}(s, H^2(F')) \to \cdots. \]

On the other hand, let $X$ be a quasi-projective variety. Let us define
\[ P(X) = \sum_i (-1)^i \dim H^i(X) q^i \]
by the virtual Poincaré polynomial of $X$. Let $e(X) := \sum_i (-1)^i \dim H^i(X)$ be the virtual Euler number of the variety $X$. The virtual Poincaré polynomial has the following motivic properties.

Proposition 2.2. (1) $P(X) = P(X-Z) + P(Z)$ for a closed subvariety of $Z \subset X$.
(2) Let $\pi: X \to Y$ be a Zariski locally trivial fibration with the fiber $F$. Then $P(X) = P(Y) \cdot P(F)$. 

Lemma 2.1 again, one can see that fiberation over using the resolution of Lemma 2.1, we obtain that

\[ \dim \text{Ext}^1((1, F_{4m-2}), (0, F_{m+4})) = \dim \text{Ext}^0(F, F') - \dim \text{Ext}^1(F, F'). \]

All of the wall crossings except at \( \alpha = 3 \) is simple. The wall occurs by following the configuration of points in quintic curves. In this subsection, we will describe the wall crossing for the computation of the virtual Poincaré polynomial of the space \( M^\infty(5, 2) \).

Let us denote the \( C^\alpha_8 \) (resp. \( C^-\alpha_8 \)) by the wall crossing locus of the moduli space \( M^{\alpha - \epsilon}(5, 2) \) (resp. \( M^{\alpha + \epsilon}(5, 2) \)) for sufficient small \( \epsilon > 0 \). During the following lemmas, we use that \( M(1, \chi) \cong M(1, 1) \) by \( F \to F(-\chi + 1) \) and \( M(1, 1) \cong \mathbb{P}^2 \). Let us start with the study of the wall crossing at \( \alpha = 18 - \epsilon \). We remark that the wall crossing locus \( C^-_{18} \) is not a projective bundle over its base space.

**Lemma 2.3.** The wall crossing locus \( C^+_{18} \) at \( \alpha = 18 \) is a \( \mathbb{P}^7 \)-bundle over the product space \( \mathbb{P}^2 \times \mathbb{P}^{14} \). The locus \( C^-_{18} \) is a \( \mathbb{P}^3 \)-bundle over \( \mathbb{P}^2 \times \mathbb{P}^{14} - D \) where \( D = \mathbb{P}^2 \times \mathbb{P}^9 \) and a \( \mathbb{P}^4 \)-bundle over \( D \).

**Proof.** By the analysis of the wall at \( \alpha = 18 \), the \( 18 + \epsilon \)-stable pairs \((1, F)\) in \( C^+_{18} \) fits into a non-split exact sequence

\[ 0 \to (0, F_{m+4}) \to (1, F) \to (1, F_{4m-2}) \to 0, \]

where \( F_{d_{m+\chi}} \) denotes the semistable sheaf with Hilbert polynomial \( d_m + \chi \). Also, one can easily check that all pairs fitting in (2.1) is \( \alpha + \epsilon \)-stable. Thus the wall \( C^+_{18} \) is a \( \mathbb{P}(\text{Ext}^1((1, F_{4m-2}), (0, F_m))) \)-bundle over \( M^\infty(4, -2) \times M^\infty(1, 4) \cong \mathbb{P}^{14} \times \mathbb{P}^2 \). Let \( \chi(F) = h^0(F) - h^1(F) \) and \( \chi(F, F') = \dim \text{Ext}^0(F, F') - \dim \text{Ext}^1(F, F') \). Since \( \text{Ext}^0((1, F_{4m-2}), (0, F_{m+4})) = 0 \) and \( \text{Ext}^1(F_{m+4}) = 0 \), from the exact sequence in Lemma 2.1 we obtain that

\[ \dim \text{Ext}^1((1, F_{4m-2}), (0, F_{m+4})) = h^0(F_{m+4}) - \chi(F_{4m-2}, F_{m+4}). \]

Note that \( F_{4m-2} \cong \mathcal{O}_C \) and \( F_m \cong \mathcal{O}_L(3) \) for some quartic curve \( C \) and a line \( L \). By using the resolution of \( F_{4m-2} \), we obtain that \( \dim \text{Ext}^1((1, F_{4m-2}), (0, F_{m+4})) = 8 \).

Similar argument shows that the wall \( C^-_{18} \) is a \( \mathbb{P}(\text{Ext}^1((0, F_{m+4}), (1, F_{4m-2}))) \)-fiberation over \( M^\infty(4, -2) \times M^\infty(1, 4) \cong \mathbb{P}^{14} \times \mathbb{P}^2 \). From the exact sequence in Lemma 2.1 again, one can see that

\[ \text{Ext}^1((0, F_{m+4}), (1, F_{4m-2})) \cong \text{Ext}^1(F_{m+4}, F_{4m-2}). \]
The later space is divided into two cases. From the short exact sequence $\mathcal{O}(0) \to \mathcal{O}(2) \to \mathcal{O}(3) \to \mathcal{F}_{m+4} \to 0$,

$\mathcal{O} \to \text{Ext}^1(\mathcal{F}_{m+4}, \mathcal{F}_{4m-2}) \to H^1(\mathcal{F}_{4m-2}(-3)) \to H^1(\mathcal{F}_{4m-2}(-2)) \to \text{Ext}^2(\mathcal{F}_{m+4}, \mathcal{F}_{4m-2}) \to 0$.

By the Serre duality, $\text{Ext}^2(\mathcal{F}_{m+4}, \mathcal{F}_{4m-2}) \cong \text{Ext}^0(\mathcal{F}_{4m-2}, \mathcal{F}_{m+1})$. But the later space is zero if $L \not\subseteq C$ and $C$ otherwise. So we have,

$$\text{Ext}^1((0, \mathcal{F}_{m+4}), (1, \mathcal{F}_{4m-2})) \simeq \begin{cases} \mathbb{C}^4 & \text{if } L \not\subseteq C, \\ \mathbb{C}^5 & \text{if } L \subseteq C. \end{cases}$$

Applying these one in (2.2), we get the result. \hfill \Box

**Lemma 2.4.** (1) The wall crossing locus $C^+_{13}$ (resp. $C^-_{13}$) at $\alpha = 13$ is a $\mathbb{P}^6$ (resp. $\mathbb{P}^3$)-bundle over the product space $\mathbb{P}^2 \times B(4, 1)$.

(2) The locus $C^+_{8}$ (resp. $C^-_{8}$) at $\alpha = 8$ is a $\mathbb{P}^5$ (resp. $\mathbb{P}^3$)-bundle over the product space $\mathbb{P}^2 \times B(4, 2)$.

**Proof.** One can easily check that, if the pair $(1, \mathcal{F}_{4m-1})$ (resp. $(1, \mathcal{F}_{4m})$) is semistable, so is $\mathcal{F}_{4m-1}$ (resp. $\mathcal{F}_{4m}$). Hence the description of the base spaces come from that $\mathcal{M}^a(4, -1) \cong B(4, 1)$ and $\mathcal{M}^a(4, 0) \cong B(4, 2)$ for all $\alpha$. Also, the free resolutions of $\mathcal{F}_{dm+x}$ are given in [10].

$$0 \to 2\mathcal{O}(-2) \to \mathcal{O}(-1) \oplus \mathcal{O} \to \mathcal{F}_{4m-1} \to 0,$$

$$0 \to \mathcal{O}(-2) \oplus \mathcal{O}(-3) \to \mathcal{O} \oplus \mathcal{O}(-1) \to \mathcal{F}_{4m} \to 0.$$

Using these one and Proposition 2.1, we obtain that

- $\text{Ext}^1((1, \mathcal{F}_{4m-1}), (0, \mathcal{F}_{m+3})) \cong \mathbb{C}^7$,
- $\text{Ext}^1((0, \mathcal{F}_{m+3}), (1, \mathcal{F}_{4m-1})) \cong \mathbb{C}^3$,
- $\text{Ext}^1((1, \mathcal{F}_{4m}), (0, \mathcal{F}_{m+2})) \cong \mathbb{C}^6$; and
- $\text{Ext}^1((0, \mathcal{F}_{m+2}), (1, \mathcal{F}_{4m})) \cong \mathbb{C}^5$.

So we have the results in lemma. \hfill \Box

Recall that the wall types at $\alpha = 3$ are given by

$$(1, (4, 1)) \oplus (0, (1, 1)), (1, (3, 0)) \oplus (0, (2, 2)) \text{ or } (1, (3, 0)) \oplus (0, (1, 1)) \oplus (0, (1, 1)).$$

Since the wall may not be simple, we need more detail calculation. Obviously, the first two types are general case. The third one is the intersection part. Let us denote the general case by $C^+_{3} = A^+ \cup B^+$ and $C^-_{3} = A^- \cup B^-$. Let $D^+ := A^+ \cap B^+$ and $D^- := A^- \cap B^-$. Since the stable pairs in the intersection part may have non-trivial automorphism, we compute the wall crossing separately in Lemma 2.5 and Lemma 2.6.

**Lemma 2.5.** (1) (a) The locus $A^+$ is a $\mathbb{P}^4$-bundle over $\mathbb{P}^2 \times B(4, 3)$. The locus $A^+ \cap D^+$ is a disjoint union of a $\mathbb{P}^3$-bundle over a $\mathbb{P}^3$-bundle over $(\mathbb{P}^2 \times (\mathbb{P}^2 - pt)) \times B(3, 0)$ and a $\mathbb{P}^2$-bundle over $\mathbb{P}^3 \times B(3, 0)$.

(b) The locus $B^+ - D^+$ is a $\mathbb{P}^2$-bundle over $\mathcal{M}^+(3, 0) \times \mathcal{M}^+(2, 2)$. Here the space $\mathcal{M}(2, 2)^s$ consists of the stable sheaves which is isomorphic to $\mathbb{P}^5 - V$ where the $V \cong \text{Sym}^2(\mathbb{P}^2)$ is the space of degenerated conics.
(2) (a) The locus $A$ is a $\mathbb{P}^3$-bundle over $M(1,1) \times M^-(4,1)$. The locus $A \cap D$ is a disjoint union of a $\mathbb{P}^2$-bundle over a $\mathbb{P}^2$-bundle over $(\mathbb{P}^2 \times (\mathbb{P}^2 - pt)) \times B(3,0)$ and a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$-bundle over $\mathbb{P}^2 \times B(3,0)$.

(b) The locus $B - D$ is a $\mathbb{P}^5$-bundle over $M^+(3,0) \times M^+(2,2)$.

Proof. For $\alpha > 3 + \epsilon$, the stable pairs $(s, F)$ in $A^+$ fits into the exact sequence

$$0 \rightarrow (0, F_{m+1}) \rightarrow (s, F) \rightarrow (1, F_{4m+1}) \rightarrow 0.$$  

(2.3)

Also one can easily check that all of the non-split extension in the equation are $\alpha$-stable. Thus $A^+$ is the $\mathbb{P}(\text{Ext}^1((1, F_{4m+1}), (0, F_{m+1})))$-bundle over $M(1,1) \times M^-(4,3)$. Note that $M^-(4,1) \cong B(4,3)$ for $\alpha > 3$. By direct computation, we know that

$$\text{Ext}^1((1, F_{4m+1}), (0, F_{m+1})) \cong \mathbb{C}^5.$$  

If $(s, F) \in A^+ \cap D^+$ in (2.3), the pair $(1, F_{4m+1})$ should fit into the exact sequence

$$0 \rightarrow (0, F'_{m+1}) \rightarrow (1, F_{4m+1}) \rightarrow (1, F_{m+1}) \rightarrow 0.$$  

By the long exact sequence obtained by (2.3), we see

$$\xi : \text{Ext}^1((1, F_{4m+1}), (0, F_{m+1})) \rightarrow \text{Ext}^1((0, F'_{m+1}), (0, F_{m+1})) \rightarrow 0.$$  

Then one can check that the non-split extensions in $A^+$ lie in the intersection part $D^+$ if and only if $\xi = 0$. But, we know that

$$\text{Ext}^1((0, F'_{m+1}), (0, F_{m+1})) = \text{Ext}^1(F'_{m+1}, F_{m+1}) \cong \begin{cases} \mathbb{C} & \text{if } F'_{m+1} \neq F_{m+1}, \\ \mathbb{C}^2 & \text{if } F'_{m+1} = F_{m+1}. \end{cases}$$

Thus the kernel of $\xi$ depends on the choices of $F'_{m+1}$ and $F_{m+1}$. Note that the non-split extension class in (2.3) is parameterized by $\mathbb{P}(\text{Ext}^1((1, F_{3m}), (0, F_{m+1}))) \cong \mathbb{P}^3$. Combining with these one, we get the results.

The stable pairs in the complement of $D^+$ in $B^+$ should be supported on a quintic curve with smooth conic as a component. Hence, the locus $B^+ - D^+$ is a $\mathbb{P}(\text{Ext}^1((1, F_{3m}), (0, F_{2m+2})))$-bundle over $M^+(3,0) \times M^+(2,2) \cong \mathbb{P}^1 \times (\mathbb{P}^5 - V)$. By using the resolution of the sheaves, we see that Ext$^1((1, F_{3m}), (0, F_{2m+2})) \cong \mathbb{C}^8$ and so we get (1)-(b).

The proof of the case $\alpha < 3 - \epsilon$ is the same as that of $\alpha > 3 + \epsilon$ except that

- $\text{Ext}^1((0, F_{m+1}), (1, F_{4m+1})) \cong \mathbb{C}^4$,
- $\text{Ext}^1((0, F_{m+1}), (1, F_{3m})) \cong \mathbb{C}^3$, and
- $\text{Ext}^1((0, F_{2m+2}), (1, F_{3m})) \cong \mathbb{C}^6$.

Plug these one in above, one can finish the proof of lemma. \hfill $\square$

Lemma 2.6. (1) The locus $D^+$ is the disjoint union of a $\mathbb{P}^3 \times \mathbb{P}^3$-bundle over $\mathbb{P}^9 \times (V - D)$ and a $\text{Gr}(2,4)$-bundle over $\mathbb{P}^9 \times D$.

(2) The intersection locus $D^-$ is the disjoint union of a $\mathbb{P}^2 \times \mathbb{P}^2$-bundle over $\mathbb{P}^9 \times (V - D)$ and $\text{Gr}(2,3)$-bundle over $\mathbb{P}^9 \times D$.

Proof. The stable pairs $(s, F) \in D^+$ fits into the exact sequence

$$0 \rightarrow (0, F_{2m+2}) \rightarrow (s, F) \rightarrow (1, F_{3m}) \rightarrow 0.$$  

(2.5)

Note that a non-split extension fitting in (2.5) may not be $\alpha$-stable. Also, the automorphism of the pair $(0, F_{2m+2}) = (0, F'_{m+1}) \oplus (0, F_{m+1})$ varies depending on the choice of $F'_{m+1}$ and $F_{m+1}$. Thus we treat the case by case.
If $F'_{m+1} \neq F_{m+1}$, then one can easily check that the pair $(s, F)$ is $\alpha$-stable if and only if it is contained in
\[ \text{Ext}^1((1, F_{m+1}), (0, F_{m+1})) \cup \text{Ext}^1((1, F_{m+1}), (0, F'_{m+1})). \]
By quotient out the $\text{Aut}((0, F_{m+1}))$ we see that it is isomorphic to the product space
\[ \mathbb{P}(\text{Ext}^1((1, F_{m+1}), (0, F_{m+1}))) \times \mathbb{P}(\text{Ext}^1((1, F_{m+1}), (0, F'_{m+1}))). \]
If $F'_{m+1} = F_{m+1}$, then $\text{Ext}^1((1, F_{m+1}), (0, F_{m+1})) \cong \mathbb{C}^2 \otimes \text{Ext}^1((1, F_{m+1}), (0, F_{m+1}))$ and $\text{Aut}((0, F_{m+1}))$ acts on this $\mathbb{C}^2$ in the standard fashion. Hence we have
\[ \text{Ext}^1((1, F_{m+1}), (0, F_{m+1})) \cong \mathbb{P}^3. \]
Since $\text{Ext}^1((1, F_{m+1}), (0, F_{m+1})) \cong \mathbb{C}^4$, we have proved the first part of the item (1).

**Remark 2.7.** We remark that the locus satisfying the condition $F'_{m+1} \neq F_{m+1}$ in (1) (similarly in (2)) is not a Zariski locally trivial fibration. It can be described as an way to compute the virtual Poincaré polynomial (13). For convenience of the reader, we address the detail. Note that $V - D \cong (\mathbb{P}^2 \times \mathbb{P}^2 - D)/\mathbb{Z}_2$. Let $Z$ be the projective bundle over $\mathbb{P}^2 \times \mathbb{P}^2$ with the fiber $\mathbb{P}(\text{Ext}^1((1, F_{m+1}), (0, F_{m+1}))) \cong \mathbb{P}^3$. The bundle $Z$ can be constructed from the tautological pair of the extensions (23). Let $p : Z \times \mathbb{P}^2 \rightarrow \mathbb{P}^2 \times \mathbb{P}^2$ be the canonical projection. Then the group $\mathbb{Z}_2$ equivariantly acts on the both spaces. Let us denote the descent map by
\[ p : Z \times \mathbb{P}^2 Z/\mathbb{Z}_2 \rightarrow \mathbb{P}^9 \times \mathbb{P}^2 \times \mathbb{P}^2/\mathbb{Z}_2 \cong \mathbb{P}^9 \times V. \]
Then one can easily see that the inverse image $p^{-1}(\mathbb{P}^9 \times (V - D))$ is exactly the $\mathbb{P}^3 \times \mathbb{P}^3$-fibration over $\mathbb{P}^9 \times (V - D)$, which is isomorphic to the quotient space $(Z \times \mathbb{P}^2) - (p \times \mathbb{P}^2 p)^{-1}(D)/\mathbb{Z}_2$. As applying the formula in [20, Lemma 2.6], one can get the virtual Poincaré polynomial of the later space.

For later use, let us put the variation of the virtual Poincaré polynomial at the wall $\alpha = 3$.

**Corollary 2.8.**

(26)
\[ P(C_+^3 - C_3^-) = p^4 + 4p^5 + 13p^6 + 27p^7 + 44p^8 + 57p^9 + 66p^{10} + 70p^{11} + 72p^{12} + 72p^4 + 72p^{15} + 70p^{16} + 66p^{17} + 57p^{18} + 44p^{19} + 27p^{20} + 13p^{21} + 4p^{22} + 2^{23}. \]

**Proof.** The wall crossing terms are a disjoint union of the locally closed subsets. Thus,
\[ P(C_3^+ - C_3^-) = [P(A^+ - A^+ \cap D^+) - P(A^- - A^- \cap D^-)] + [P(B^+ - D^+) - P(B^- - D^-)] + P(D^+ - D^-). \]
By the descriptions in Lemma 2.5, Lemma 2.6, and Remark 2.7 we obtain the result. □

Since the proof of the below lemma is very similar with that of Lemma 2.4. We omit the proof.
Lemma 2.9. The flipping locus $C^+_\frac{1}{2}$ (resp. $C^-_{\frac{1}{2}}$) at $\alpha = \frac{1}{2}$ is a $\mathbb{P}^6$ (resp. $\mathbb{P}^5$)-bundle over the product space $\mathbb{P}^5 \times \mathcal{B}(3,1)$.

In a summary, through Lemma 2.3, 2.4, 2.5, 2.6, and Lemma 2.9, the first part of Theorem 1.1 has been proved.

Remark 2.10. The wall crossing loci $C^+_\alpha$ for each $\alpha$ can be described by a geometric way (cf. [6]). The wall-crossing loci are the locus of pairs of seven points, six points, five, four points on a line with a quartic curve at the wall $\alpha = 18, 13, 8, 3$, respectively, and lastly six points on a conic curve with a cubic curve at $\alpha = \frac{1}{2}$.

2.2. Stratification of the moduli space $M(5, 2)$. In this subsection, we will study the forgetting map $M^+(5, 2) \to M(5, 2)$, $(s, F) \mapsto F$ by using the stratification of stable sheaves in $M(5, 2)$ ([5]).

Proof of (2) in Theorem 1.1. By [5, Theorem 1.1], we know that $h^0(F) \leq 3$ for stable sheaves $F \in M(5, 2)$. On the other hand, since $(5, 2) = 1$, there exists a universal family of sheaves $\mathcal{F}$ on $M(5, 2) \times \mathbb{P}^2$ ([15]). Therefore the Proj of the direct image sheaf $p_*\mathcal{F}$ is isomorphic to $M^+(5, 2)$ and thus the moduli space $M^+(5, 2)$ is decomposed into locally closed subsets:

1. the $\mathbb{P}^1$-bundle over $M(5, 2)_2$
2. the $\mathbb{P}^2$-bundle over $M(5, 2)_3$

where $M(5, 2)_k := \{F \in M(5, 2)|h^0(F) = k\}$.

For later use, we compute the Poincaré polynomial of the exceptional locus $M(5, 2)_3$.

Proposition 2.11. The (virtual) Poincaré polynomial of the space $M(5, 2)_3$ is given by

$$1 + 3p + 8p^2 + 14p^3 + 19p^4 + 21p^5 + 22p^6 + 22p^7 + 22p^8 + 22p^9 + 22p^{10} + 22p^{11} + 22p^{12} + 22p^{13} + 22p^{14} + 22p^{15} + 22p^{16} + 22p^{17} + 21p^{18} + 19p^{19} + 14p^{20} + 8p^{21} + 3p^{22} + p^{23}.$$

Proof. The locus $M(5, 2)_3$ is isomorphic to the moduli space $M^+(5, -2)$ by [4] Proposition 4.2.8. Let us compute the polynomial of the moduli space $M^+(5, -2)$ by using the wall crossings. Among the moduli spaces $M^+(5, -2)$, one can easily see that there is a single wall-crossing at $(1, (5, -2)) = (1, (4, -2)) = (0, (1, 0))$. Also, by [6, Lemma 2.3], the space $M^+(5, -2)$ is a projective bundle over $\text{Hilb}^2(\mathbb{P}^2)$ with the fiber $\mathbb{P}^{17}$. Since $\text{Ext}^1((1, F_{4m-2}), (0, F_m)) = \text{Ext}^1((0, F_m), (1, F_{4m-2})) = \mathbb{C}^4$, we get

$$P(M(5, 2)_3) = P(M^+(5, -2)) = P(M^+(5, -2)) + (\mathbb{P}^3 - \mathbb{P}^3) \cdot P(\mathbb{P}^2) \times P(\mathbb{P}^{14}).$$

Since $P(\text{Hilb}^2(\mathbb{P}^2)) = 1 + 2q + 5q^2 + 6q^3 + 5q^4 + 2q^5 + q^6$, so the claim is proved.

3. Bridgeland wall crossing of the moduli space $M(5, 2)$

In this section, we study the wall crossing of the space $M(5, 2)$ in the sense of Bridgeland. The wall crossing of $M(5, 2)$ can be similarly done as the case $M(6, 1)$ ([7]). So we omit the detail about the computation of the wall crossing. The remarkable one is to find the final birational model of $M(5, 2)$. To end up, we describe the ray generator of effective cone of the moduli space $M(5, 2)$. As a set, the divisor $D$ is defined as the locus of stable sheaves which is not orthogonal to the
vector bundle $E$. (For detail, see [24].) The existence of such a vector bundle $E$ has been proved in [24] Theorem 4.3. Let $A := \phi^* \mathcal{O}(1)$ where the map $\phi: \mathcal{M}(5,2) \to |\mathcal{O}_{\mathbb{P}^2}(5)|$ is defined by the Fitting ideal ([15]). Obviously, the divisor $A$ is the nef divisor of the moduli space $\mathcal{M}(5,2)$.

**Lemma 3.1.** The effective cone of $\mathcal{M}(5,2)$ are generated by the two geometric divisors $A$ and $D = \overline{X}_{01}$. Here the locus $X_{01}$ consists of the stable sheaves of the forms $\mathcal{O}_C(2)(-Z_4 + Z_1)$ such that $C$ is a smooth quintic curve and $Z_1$ is the subscheme of $C$ with length $1$ in a general position.

**Proof.** By [24] Theorem 5.3], the divisors $A$ and $D$ generate the rays of the effective cone of the $\mathcal{M}(5,2)$. On the other hand, the general free resolution of the stable sheaf $F \in \mathcal{M}(5,2)$ has two types depending on the some algebraic condition ([17] §2.3]). One can easily check that the sheaf $F$ is orthogonal to $E$ if and only if $F$ fits into the exact sequence $0 \to \mathcal{T}_{\mathbb{P}^2}(-4) \to 2\mathcal{O}_{\mathbb{P}^2} \to F \to 0$. Hence the complement $X_{01}$ consisting of the stable sheaves $\mathcal{O}_C(2)(-Z_4 + Z_1)$ ([17] Proposition 2.3]) is exactly the support of the divisor $D$. □

**Proposition 3.2.** The final birational model of the moduli space $\mathcal{M}(5,2)$ is isomorphic to the Grassmannian variety $\text{Gr}(2,15)$.

**Proof.** From [11] §9.2] and [17] §2.3], the blown-up space $\widetilde{G}$ of $\text{Gr}(2,15)$ along a $\mathbb{P}^2 \times \text{Gr}(2,6)$ is isomorphic to $\mathcal{M}(5,2)$ up to codimension one because the exceptional divisor $E$ is supported on the strict transformation of $X_{01}$ in $\mathcal{M}(5,2)$. Hence $\text{Eff}(\mathcal{M}(5,2)) = \text{Eff}(\widetilde{G})$. Furthermore, by the computation of the wall in the sense of Bridgeland, there exists the corresponding wall in the the effective cone, saying $A+D$ (cf. [7] Remark 2.12)). The divisor $A+D$ is the last one before the collapsing wall $D$. On the other hand, on $\widetilde{G}$,

$$-15A = K_{\mathcal{M}(5,2)} = \pi^* K_{\text{Gr}(2,15)} + 15E.$$ 

The first equality comes from [24] Lemma 3.1] and the second one comes from [12] Exercise 8.5, II]. Hence $\pi^*(-K_{\text{Gr}(2,15)}) = 15A + 15D$ is a nef (but not ample) divisor on $\widetilde{G}$ because $K_{\text{Gr}(2,15)}$ is anti-ample and $D = E = \overline{X}_{01}$. Thus the corresponding birational model of the divisors in $[A + D, D]$ is the space $\text{Gr}(2,15)$.

**Proposition 3.3.** The moduli space $\mathcal{M}(5,2)$ can be obtained from the space $\text{Gr}(2,15)$ by the Bridgeland wall crossings.

**Proof.** Proposition 3.2 and [3] Theorem 1.1] imply the statement. □

**Corollary 3.4.** The Poincaré polynomial of the space $\mathcal{M}(5,2)$ is given by

$$1 + 2q + 6q^2 + 13q^3 + 26q^4 + 45q^5 + 68q^6 + 87q^7 + 100q^8 + 107q^9 + 111q^{10} + 112q^{11} + 113q^{12} + 113q^{13} + 113q^{14} + 112q^{15} + 111q^{16} + 107q^{17} + 100q^{18} + 87q^{19} + 68q^{20} + 45q^{21} + 26q^{22} + 13q^{23} + 6q^{24} + 2q^{25} + q^{26}.$$ 

**Proof.** The computation of the wall crossings is similar with that of $\mathcal{M}(6,1)$ in [7]. So we omit the detail. □

**Remark 3.5.** This result is compatible with [25] Theorem 6.1].
4. Computation of the virtual Poincaré polynomial of $\mathcal{M}^\infty(5,2)$

As cooking up the results in the previous sections, we obtain the virtual Poincaré polynomial of $\mathcal{M}^\infty(5,2)$.

**Corollary 4.1.** The virtual Poincaré polynomial $P(\mathcal{M}^\infty(5,2))$ is given by

\[
1 + 3p + 9p^2 + 22p^3 + 50p^4 + 99p^5 + 173p^6 + 256p^7 + 330p^8 + 379p^9 + 407p^{10} \\
+ 420p^{11} + 426p^{12} + 428p^{13} + 429p^{14} + 428p^{15} + 423p^{16} + 410p^{17} + 382p^{18} \\
+ 333p^{19} + 259p^{20} + 176p^{21} + 101p^{22} + 51p^{23} + 22p^{24} + 9p^{25} + 3p^{26} + p^{27}.
\]

**Proof.** From the part (2) in Theorem 1.1 and Corollary 3.3 we obtain

\[
P(\mathcal{M}^+(5,2)) = (P(M(5,2)) - P(M(5,2), P[\mathcal{P}^1] + P(M(5,2), P[\mathcal{P}^2]) \\
= 1 + 3p + 9p^2 + 22p^3 + 47p^4 + 85p^5 + 132p^6 + 176p^7 + 209p^8 + 229p^9 \\
+ 240p^{10} + 245p^{11} + 247p^{12} + 248p^{13} + 248p^{14} + 247p^{15} + 245p^{16} + 240p^{17} \\
+ 229p^{18} + 209p^{19} + 176p^{20} + 132p^{21} + 85p^{22} + 22p^{23} + 9p^{24} + 3p^{25} + p^{27}.
\]

Let us add the wall crossing terms in Lemma \[2.3\] \[2.4\] \[2.5\] \[2.6\] and Lemma \[2.9\] Let $P(C_\alpha) = P(C_\alpha - C_\alpha)$. Then,

\[
P(\mathcal{M}^\infty(5,2)) = P(\mathcal{M}^+(5,2)) + P(C_{18}) + P(C_{13}) + P(C_8) + P(C_3) + P(C_2)
\]

\[
= P(\mathcal{M}^+(5,2)) + P(\mathcal{P}^2)P(\mathcal{P}^2)P(\mathcal{P}^3)P(\mathcal{P}^4)P(\mathcal{P}^3)P(\mathcal{P}^9)
\]

With the help of a computer program, the claim is proved. \[\square\]

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