High-order perturbative expansions of multi-parameter $\Phi^4$ quantum field theories

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Abstract

We present high-order perturbative expansions of multi-parameter $\Phi^4$ quantum field theories with an $N$-component fundamental field, containing up to 4th-order polynomials of the field. Multi-parameter $\Phi^4$ theories generalize the simplest $O(N)$-symmetric $\Phi^4$ theories, and describe more complicated symmetry breaking patterns. These notes collect several high-order perturbative series of physically interesting multi-parameter $\Phi^4$ theories, to five or six loops. We consider the $O(M) \otimes O(N)$-symmetric $\Phi^4$ model, the so-called $MN$ model, and a spin-density-wave $\Phi^4$ model containing five quartic terms. The corresponding Tables of the coefficients are reported in Ref. [1].
I. INTRODUCTION

In the framework of the renormalization-group (RG) approach to critical phenomena, a quantitative description of many continuous phase transitions can be obtained by considering an effective Landau-Ginzburg-Wilson (LGW) $\Phi^4$ field theory, containing up to fourth-order powers of the field components. The simplest example is the $O(N)$-symmetric $\Phi^4$ theory, defined by the Lagrangian density

$$L_{O(N)} = \frac{1}{2} \sum_i (\partial_\mu \Phi_i)^2 + \frac{1}{2} r \sum_i \Phi_i^2 + \frac{1}{4!} u (\sum_i \Phi_i^2)^2$$

(1.1)

where $\Phi$ is an $N$-component real field. These $\Phi^4$ theories describe are characterized by the symmetry breaking $O(N) \rightarrow O(N-1)$. See, e.g., Refs. [2,3] for recent reviews discussing these models. Beside the transitions described by $O(N)$ models, there are also other physically interesting transitions described by more general Landau-Ginzburg-Wilson (LGW) $\Phi^4$ field theories, characterized by more complex symmetries and symmetry breaking patterns. The general LGW $\Phi^4$ theory for an $N$-component field $\Phi_i$ can be written as

$$L = \frac{1}{2} \sum_i (\partial_\mu \Phi_i)^2 + \frac{1}{2} \sum_i r_i \Phi_i^2 + \frac{1}{4!} \sum_{ijkl} u_{ijkl} \Phi_i \Phi_j \Phi_k \Phi_l$$

(1.2)

where the number of independent parameters $r_i$ and $u_{ijkl}$ depends on the symmetry group of the theory. Here, we are only assuming a parity symmetry which forbids third-order terms. An interesting class of models are those in which $\sum_i \Phi_i^2$ is the unique quadratic polynomial invariant under the symmetry group of the theory, corresponding to the case all field components become critical simultaneously. This requires that all $r_i$ are equal, $r_i = r$, and $u_{ijkl}$ must be such not to generate other quadratic invariant terms under RG transformations, for example, it must satisfy the trace condition [2] $\sum_i u_{iikl} \propto \delta_{kl}$. In these models, criticality is driven by tuning the single parameter $r$, which physically may correspond to the reduced temperature.

More general LGW $\Phi^4$ theories, which allow for the presence of independent quadratic parameters $r_i$, must be considered to describe multicritical behaviors where there are independent correlation lengths that diverge simultaneously, which may arise from the competition of distinct types of ordering, see e.g. Ref. [4] and references therein. Note that, like the simplest $O(N)$ models, all multi-parameter $\Phi^4$ field theories are expected to be trivial in four dimensions.

High-order perturbative expansions, to five and six loops, of several multi-parameter $\Phi^4$ theories have been computed in Refs. [3–20]. These notes collect several high-order series of physically interesting multi-parameter $\Phi^4$ theories, to five or six loops. The corresponding Tables of the coefficients are reported in Ref. [1]. We consider two perturbative schemes: the three-dimensional (3D) massive zero-momentum (MZM) scheme in three dimensions and the massless (critical) $\overline{\text{MS}}$ scheme. In the MZM scheme, one expands in powers of the MZM quartic couplings $g_{ijkl}$, defined by

$$\Gamma_{ij}^{(2)} (p) = \delta_{ij} Z_{\phi}^{-1} \left[ m^2 + p^2 + O(p^4) \right], \quad \Gamma_{ijkl}^{(4)} (0) = m Z_{\phi}^{-2} g_{ijkl}$$

(1.3)
The $\overline{\text{MS}}$ scheme is based on a minimal subtraction procedure within the dimensional regularization, and can give rise to an $\epsilon \equiv 4 - d$ expansion, and also 3D expansions in the renormalized $\overline{\text{MS}}$ couplings $g_{ijkl}$ by setting $\epsilon = 1$ after renormalization. The RG flow is determined by the FPs, which are common zeroes $g_{ijkl}^*$ of the $\beta$-functions, $\beta_{ijkl}(g_{abcd}) \equiv m \partial g_{ijkl}/\partial m$ and $\beta_{ijkl}(g_{abcd}) \equiv \mu \partial g_{ijkl}/\partial \mu$ in the MZM and $\overline{\text{MS}}$ schemes respectively. We report series for the $\text{O}(M)\otimes\text{O}(N)$-symmetric $\Phi^4$ model, the so-called $MN$ model, and a spin-density-wave $\Phi^4$ model containing five quartic terms. We also mention that high-order perturbative series for $\text{U}(N) \times \text{U}(N)$, $\text{SU}(4)\otimes\text{SU}(4)$, $\text{U}(N)$ and $\text{SU}(N)$ $\Phi^4$ field theories have been computed in Refs. [13,16,19].

II. THE $\text{O}(M)\otimes\text{O}(N)$-SYMMETRIC MODEL

The $\text{O}(M)\otimes\text{O}(N)$-symmetric $\Phi^4$ model is defined by the Hamiltonian density

$$\frac{1}{2} \sum_{ai} \left[ \sum_{\mu} \left( \partial_{\mu} \Phi_{ai} \right)^2 + r \Phi_{ai}^2 \right] + \frac{u_0}{4!} \left( \sum_{ai} \Phi_{ai}^4 \right) + \frac{v_0}{4!} \left[ \sum_{ij} \left( \sum_a \Phi_{ai} \Phi_{aj} \right)^2 - \left( \sum_a \Phi_{ai}^2 \right)^2 \right], \quad (2.1)$$

where $\Phi_{ai}$ is a real $N \times M$ matrix field ($a = 1, \ldots, N$ and $i = 1, \ldots, M$).

We also consider the four independent quadratic perturbations $Q^{(k)}$ that break the $\text{O}(M)\otimes\text{O}(N)$ symmetry, i.e.

$$Q^{(1)}_{ab} = \Phi_{ai} \Phi_{bj} - \Phi_{aj} \Phi_{bi},$$
$$Q^{(2)}_{ab} = \frac{1}{2} (\Phi_{ai} \Phi_{bj} + \Phi_{aj} \Phi_{bi}) - \frac{1}{M} \delta_{ab} \Phi_{ci} \Phi_{cj} - \frac{1}{N} \delta_{ij} \Phi_{ak} \Phi_{bk} + \frac{1}{MN} \delta_{ab} \delta_{ij} \Phi_{ck} \Phi_{ck},$$
$$Q^{(3)}_{ij} = \Phi_{ci} \Phi_{cj} - \frac{1}{N} \delta_{ij} \Phi_{ck} \Phi_{ck},$$
$$Q^{(4)}_{ab} = \Phi_{ak} \Phi_{bk} - \frac{1}{MN} \delta_{ab} \Phi_{ck} \Phi_{ck}.$$ 

The above four perturbations are related to different representations of the $\text{O}(M)$ and $\text{O}(N)$ groups. Therefore, they do no mix under renormalization-group (RG) transformations.

In the following we report the perturbative expansions in the massive zero-momentum (MZM) scheme and in the minimal-subtraction ($\overline{\text{MS}}$) scheme. For further details see Refs. [3,9–12,14,15,17].

A. The 3D massive zero-momentum perturbative expansion

In the MZM scheme the theory is renormalized by introducing a set of zero-momentum conditions for the one-particle irreducible two-point and four-point correlation functions:

$$\Gamma^{(2)}_{ai,bj}(p) = \delta_{ai,bj} Z_{\phi}^{-1} \left[ m^2 + p^2 + O(p^4) \right], \quad (2.3)$$

where $\delta_{ai,bj} \equiv \delta_{ab} \delta_{ij}$.

$$\Gamma^{(4)}_{ai,bj,ck,dl}(0) = Z_{\phi}^{-2} m (u S_{ai,bj,ck,dl} + v C_{ai,bj,ck,dl}), \quad (2.4)$$
and $S$, $C$ are appropriate tensorial factors associated with the two quartic terms of Hamiltonian (2.1):

$$S_{ai,bj,ck,dl} = \frac{1}{3}(\delta_{ai,bj}\delta_{ck,dl} + \delta_{ai,ck}\delta_{bj,dl} + \delta_{ai,dl}\delta_{bj,ck}),$$

$$C_{ai,bj,ck,dl} = \frac{1}{6}[\delta_{ad}\delta_{cd}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \delta_{ac}\delta_{bd}(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}) + \delta_{ad}\delta_{bc}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl})] - S_{ai,bj,ck,dl}$$ (2.5)

Eqs. (2.3) and (2.4) relate the mass scale (inverse correlation length) $m$, and the zero-momentum quartic couplings $u$ and $v$ to the corresponding Hamiltonian parameters $r$, $u_0$, and $v_0$. In addition, the function $Z_t$ is defined by the relation

$$\Gamma_{ai,bj}^{(1,2)}(0) = \delta_{ai,bj}Z_t^{-1},$$

(2.6)

where $\Gamma^{(1,2)}$ is the one-particle irreducible two-point function with an insertion of $\frac{1}{2}\Phi^2$. The RG flow in the quartic-coupling $u, v$ plane is determined by the $\beta$-functions

$$\beta_u(u,v) = m \frac{\partial u}{\partial m} \bigg|_{u_0,v_0}, \quad \beta_v(u,v) = m \frac{\partial v}{\partial m} \bigg|_{u_0,v_0}.$$ (2.7)

Their common zeroes are the fixed points (FP’s) of the RG flow. The RG functions $\eta_\phi$ and $\eta_\nu$ associated with the standard critical exponents are

$$\eta_\phi(t,u,v) = \frac{\partial \ln Z_{\Phi,t}}{\partial \ln m} = \beta_u \frac{\partial \ln Z_{\Phi,t}}{\partial u} + \beta_v \frac{\partial \ln Z_{\Phi,t}}{\partial v},$$ (2.8)

In order to compute the RG dimensions of the quadratic operators (2.2), one computes the four functions $Z_{Qk}$ defined through the relation $\Gamma_{ai,bj}^{(Qk,2)}(0) = q_{k,ai,bj}Z_{Qk}^{-1}$, where $\Gamma_{ai,bj}^{(Qk,2)}$ is the one-particle irreducible two-point function with an insertion of $Q^{(k)} (q_{k,ai,bj}$ are the appropriate tensorial factors). One then derives the corresponding RG functions $\eta_{Qk}$ through the equation

$$\eta_{Qk}(u,v) = \frac{\partial \ln Z_{Qk}}{\partial \ln m}$$ (2.9)

The critical exponents $\eta$ and $\nu$ and the RG dimensions $y_{Qk}$ of the quadratic operators $Q^{(k)}$ are obtained by evaluating the RG functions at the fixed point $u^*, v^*$, as

$$\eta = \eta_\phi(u^*, v^*),$$

$$\nu = [2 - \eta_\phi(u^*, v^*) + \eta_\nu(u^*, v^*)]^{-1},$$

$$y_{Qk} = 2 - \eta_\phi(u^*, v^*) + \eta_{Qk}(u^*, v^*).$$ (2.10)

The 3D expansions of the RG functions in powers of the zero-momentum couplings $u, v$ are [we set $A = 3/(16\pi)$]

$$\beta_u = \sum_{ijkl} b^{(a)}_{ijkl} M^k N_i u^i v^j = -u + \frac{8+MN}{9} Au^2 - \frac{2(1-M)(1-N)}{9} Av^2 + \frac{(1-M)(1-N)}{9} Av^2$$

$$- \frac{760+164MN}{2187} A^2 u^3 + \frac{300(1-M)(1-N)}{2187} A^2 u^2 v - \frac{118(1-M)(1-N)}{729} A^2 Av^2$$

$$+ \frac{40(1-M)(1-N)}{2187} A^2 v^3 + ...$$
\[
\beta_v = \sum_{ijkl} b^{(v)}_{ijkl} M^k N^l u^i v^j = -v - \frac{8-M-N}{9} Av^2 + \frac{4}{3} Au - \frac{1480 + 92MN}{2187} A^2 u^2 v
\]

\[
\eta_\phi = \sum_{ijkl} e^{(\phi)}_{ijkl} M^k N^l u^i v^j = \frac{1912 - 400M - 400N + 184MN}{2187} A^2 u^2 v^2 - \frac{298 - 106M - 106N + 40MN}{729} A^2 u^2 v + \ldots
\]

\[
\eta_t = \sum_{ijkl} e^{(t)}_{ijkl} M^k N^l u^i v^j = -\frac{2 + MN}{9} Au + \frac{(1-M)(1-N)}{9} Av + \frac{4 + 2MN}{81} A^2 u^2 - \frac{4(1-M)(1-N)}{81} A^2 uv
\]

\[
\eta_{Q1} = \sum_{ijkl} e^{(Q1)}_{ijkl} M^k N^l u^i v^j = -\frac{2}{9} Au + \frac{1}{9} Av + \frac{12 + 2MN}{243} A^2 u^2 - \frac{28 - 4M - 4N + 4MN}{243} A^2 uv
\]

\[
\eta_{Q2} = \sum_{ijkl} e^{(Q2)}_{ijkl} M^k N^l u^i v^j = -\frac{2}{9} Au + \frac{1}{9} Av + \frac{12 + 2MN}{243} A^2 u^2 - \frac{12 - 4M - 4N + 4MN}{243} A^2 uv
\]

\[
\eta_{Q3} = \sum_{ijkl} e^{(Q3)}_{ijkl} M^k N^l u^i v^j = -\frac{2}{9} Au + \frac{1}{9} Av + \frac{12 + 2MN}{243} A^2 u^2 - \frac{12 - 12M - 4N + 4MN}{243} A^2 uv
\]

\[
\eta_{Q4} = \sum_{ijkl} e^{(Q4)}_{ijkl} M^k N^l u^i v^j = -\frac{2}{9} Au + \frac{1}{9} Av + \frac{12 + 2MN}{243} A^2 u^2 - \frac{12 - 12M - 4N + 4MN}{243} A^2 uv
\]

The values of the coefficients \( b^{(u)}_{ijkl}, b^{(v)}_{ijkl}, e^{(\phi)}_{ijkl}, e^{(t)}_{ijkl}, \) and \( e^{(Qk)}_{ijkl} \) up to six loops are reported in the file `OMN-MZM.TXT` attached to Ref. [1]. Each line of this file contains 7 numbers. The first one indicates the quantity at hand: 1, 2, 3, 4, 5, 6, 7 correspond respectively to \( \beta_u, \beta_v, \eta_\phi, \eta_t, \eta_{Q1}, \eta_{Q2}, \eta_{Q3} \) (the expansion of \( \eta_{Q4} \) can be obtained from that of \( \eta_{Q3} \) by interchanging \( M \) with \( N \)); the second integer number gives the number of loops; the subsequent four integer numbers are the indices \( i, j, k, l \); finally, the last real number is the value of the coefficient.

**B. The minimal-subtraction perturbative expansion**

In the \( \overline{\text{MS}} \) scheme one sets

\[
\Phi = [Z_\phi(u, v)]^{1/2} \Phi_R,
\]

\[
u_0 = A_d \mu^d Z_u(u, v),
\]

\[
\nu_0 = A_d \mu^d Z_v(u, v),
\]

where the renormalization functions \( Z_\phi, Z_u, \) and \( Z_v \) are determined from the divergent part of the two- and four-point one-particle irreducible correlation functions computed in dimensional regularization. They are normalized so that \( Z_\phi(u, v) \approx 1, Z_u(u, v) \approx u, \) and \( Z_v(u, v) \approx v \) at tree level. Here \( A_d \) is a \( d \)-dependent constant given by \( A_d \equiv 2^{d-1} \pi^{d/2} \Gamma(d/2). \) Moreover, one defines a mass renormalization constant \( Z_t(u, v) \) by requiring \( Z_t \Gamma^{(1,2)} \) to be
finite when expressed in terms of \( u \) and \( v \). Here \( \Gamma^{(1,2)} \) is the one-particle irreducible two-point function with an insertion of \( \frac{1}{2} \Phi^2 \). The \( \beta \) functions,

\[
\beta_u(u, v) = \mu \frac{\partial u}{\partial \mu}
\left|_{u_0,v_0}\right., \quad \beta_v(u, v) = \mu \frac{\partial v}{\partial \mu}
\left|_{u_0,v_0}\right., \tag{2.13}
\]

have a simple dependence on \( d \):

\[
\beta_u = (d - 4)u + B_u(u, v), \quad \beta_v = (d - 4)v + B_v(u, v), \tag{2.14}
\]

where the functions \( B_u(u, v) \) and \( B_v(u, v) \) are independent of \( d \). The RG dimensions of the quadratic operators \( Q^{(k)} \) are obtained by computing the renormalization functions \( Z_{Qk}(u, v) \). These functions are determined by requiring \( Z_{Qk} \Gamma^{(Qk,2)} \) to be finite when expressed in terms of \( u \) and \( v \). Here \( \Gamma^{(Qk,2)} \) is the one-particle irreducible two-point function with an insertion of \( Q^{(k)} \). The RG functions \( \eta_\phi \) and \( \eta_t \), associated with the critical exponents, are obtained from

\[
\eta_{\phi,t}(u, v) = \frac{\partial \log Z_{\phi,t}}{\partial \log \mu}
\left|_{u_0,v_0}\right.. \tag{2.15}
\]

The same equation allows the determination of \( \eta_{Qk} \) from \( Z_{Qk} \). The RG functions \( \eta_\phi \), \( \eta_t \), and \( \eta_{Qk} \) are independent of \( d \). The standard critical exponents \( \eta \) and \( \nu \), and the RG dimensions \( y_{Qk} \) of the quadratic perturbations \( Q_k \) are obtained by using Eq. (2.10).

The expansions of the RG functions in powers of the \( \overline{\text{MS}} \) couplings \( u, v \) are

\[
B_u = \sum_{ijkl} b_{ijkl}^{(u)} M^k N^l u^i v^j = \frac{8+MN}{6} u^2 - \frac{(1-M)(1-N)}{3} uv + \frac{(1-M)(1-N)}{6} v^2 - \frac{14+3MN}{12} u^3
\]

\[
+ \frac{11(1-M)(1-N)}{18} u^2 v - \frac{13(1-M)(1-N)}{24} uv^2 + \frac{5(1-M)(1-N)}{36} v^3 + ...
\]

\[
B_v = \sum_{ijkl} b_{ijkl}^{(v)} M^k N^l u^i v^j = 2uv - \frac{8-M-N}{6} v^2 - \frac{82+5MN}{36} u^2 v + \frac{53-11M-11N+5MN}{18} uv^2
\]

\[
- \frac{99-35M-35N+13MN}{72} v^3 + ...
\]

\[
\eta_\phi = \sum_{ijkl} e_{ijkl}^{(\phi)} M^k N^l u^i v^j = \frac{2+MN}{72} u^2 - \frac{(1-M)(1-N)}{36} uv + \frac{(1-M)(1-N)}{48} v^2 + ...
\]

\[
\eta_t = \sum_{ijkl} e_{ijkl}^{(t)} M^k N^l u^i v^j = - \frac{2+MN}{6} u + \frac{(1-M)(1-N)}{6} v
\]

\[
+ \frac{2+MN}{12} u^2 - \frac{(1-M)(1-N)}{6} uv + \frac{(1-M)(1-N)}{8} v^2 + ...
\]

\[
\eta_{Q1} = \sum_{ijkl} e_{ijkl}^{(Q1)} M^k N^l u^i v^j = - \frac{1}{3} u + \frac{1}{72} v + \frac{6+MN}{36} u^2 - \frac{7-M-N+MN}{18} uv + \frac{17-5M-5N+2MN}{72} v^2 + ...
\]

\[
\eta_{Q2} = \sum_{ijkl} e_{ijkl}^{(Q2)} M^k N^l u^i v^j = - \frac{1}{3} u + \frac{1}{6} v + \frac{6+MN}{36} u^2 - \frac{3-M-N+MN}{18} uv + \frac{9-3M-3N+2MN}{72} v^2 + ...
\]

\[
\eta_{Q3} = \sum_{ijkl} e_{ijkl}^{(Q3)} M^k N^l u^i v^j = - \frac{1}{3} u + \frac{1-N}{6} v + \frac{6+MN}{36} u^2 - \frac{3-M-N+MN}{18} uv + \frac{3-M-3N+MN}{24} v^2 + ...
\]

\[
\eta_{Q4} = \sum_{ijkl} e_{ijkl}^{(Q4)} M^k N^l u^i v^j = - \frac{1}{3} u + \frac{1-M}{6} v + \frac{6+MN}{36} u^2 - \frac{3-3M-N+MN}{18} uv + \frac{3-M-N+MN}{24} v^2 + ...
\]
The values of the coefficients \( b^{(u)}_{ijkl}, b^{(v)}_{ijkl}, e^{(\phi)}_{ijkl}, e^{(t)}_{ijkl} \) and \( e^{(Qk)}_{ijkl} \) up to five loops are reported in the file \texttt{OMN-MS.TXT} attached to Ref. [1]. The meaning of the numbers reported in this file is the same as that of the numbers appearing in file \texttt{OMN-MZM.TXT}, see the end of Sec. II A. In file \texttt{OMN-MS.TXT} the coefficients are given numerically for simplicity, although we computed them exactly in terms of fractions and \( \zeta \) functions.

### III. THE \( MN \) MODEL

The so-called \( mn \) model is defined by the Hamiltonian density

\[
\mathcal{H} = \frac{1}{2} \sum_{ai} \left( \sum_{\mu} (\partial_\mu \Phi_{ai})^2 + r \Phi_{ai}^2 \right) + \frac{u_0}{4!} \left( \sum_{ai} \Phi_{ai}^2 \right)^2 + \frac{v_0}{4!} \sum_{abi} \Phi_{ai} \Phi_{bi}^2,
\]

(3.1)

where \( \Phi_{ai} \) is a real \( m \times n \) matrix, i.e., \( a = 1, \ldots, m \) and \( i = 1, \ldots, n \).

We refer to Refs. [3,7,8,18] for further details on the perturbative expansions in the \( mn \) model.

#### A. The 3D massive zero-momentum perturbative expansion

The basic relations in the MZM scheme are the same as those reported in Sec. II A. Beside Eq. (2.3), we have

\[
\Gamma^{(4)}_{ai,bj,ck,dl}(0) = Z^{-2} m \left( uS_{ai,bj,ck,dl} + vC_{ai,bj,ck,dl} \right) \]

(3.2)

where \( S \) and \( C \) are the tensorial factors corresponding to the quartic terms of the \( mn \) Hamiltonian, i.e.

\[
S_{ai,bj,ck,dl} = \frac{1}{3} \left( \delta_{ai,bj} \delta_{ck,dl} + \delta_{ai,ck} \delta_{bj,dl} + \delta_{ai,dl} \delta_{bj,ck} \right),
\]

\[
C_{ai,bj,ck,dl} = \frac{1}{3} \left( \delta_{ij} \delta_{ik} \delta_{il} \right) \left( \delta_{ab} \delta_{cd} + \delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc} \right).
\]

The function \( Z_t \) is defined as in Eq. (2.6). The \( \beta \)-functions \( \beta_u, \beta_v \) and RG functions \( \eta_{\phi,t} \) are defined as in Eqs. (2.7) and (2.8). Their expansions are [we set \( A = 3/(16\pi) \)]:

\[
\beta_u = \sum_{ijkl} b^{(u)}_{ijkl} m^k n^l u^i v^j = -u + \frac{8+mn}{9} A u^2 + \frac{4+2m}{9} A u v - \frac{760+164m}{2187} A^2 u^3 + \frac{-800+400m}{2187} A u^2 v - \frac{184+92m}{2187} A^2 u^2 v^2 + \ldots
\]

(3.3)

\[
\beta_v = \sum_{ijkl} b^{(v)}_{ijkl} m^k n^l u^i v^j = -v + \frac{8+m}{9} A v^2 + \frac{1}{3} A u v - \frac{8+2+m}{9} A^2 u^2 v - \frac{1480+92m}{2187} A^2 u^2 v^2
\]

\[
- \frac{2096+400m}{2187} A^2 u^2 v^2 + \frac{-760+164m}{2187} A^2 v^3 + \ldots
\]

\[
\eta_{\phi} = \sum_{ijkl} e^{(\phi)}_{ijkl} m^k n^l u^i v^j = \frac{16+8m}{2187} A^2 u^2 + \frac{32+16m}{2187} A^2 u v + \frac{16+8m}{2187} A^2 v^2 + \ldots
\]

\[
\eta_{t} = \sum_{ijkl} e^{(t)}_{ijkl} m^k n^l u^i v^j = -\frac{2+mn}{9} A u - \frac{2+m}{9} A v + \frac{4+2m}{81} A^2 u^2 + \frac{8+4m}{81} A^2 u v + \frac{4+2m}{81} A^2 v^2 + \ldots
\]
The values of the coefficients $b_{ijkl}^{(u)}$, $b_{ijkl}^{(v)}$, $e_{ijkl}^{(\phi)}$, and $e_{ijkl}^{(t)}$ up to six loops are reported in the file \texttt{MN-MZM.TXT} attached to Ref. [1]. Each line of this file contains 7 numbers. The first one indicates the quantity at hand: 1, 2, 3, 4 correspond respectively to $\beta_u$, $\beta_v$, $\eta_\phi$, and $\eta_t$; the second integer number gives the number of loops; the subsequent four integer numbers are the indices $i, j, k, l$; finally, the last real number is the value of the coefficient.

B. The minimal-subtraction perturbative expansion

The RG functions in the $\overline{MS}$ scheme are

$$
B_u = \sum_{ijkl} b_{ijkl}^{(u)} n^i u^j v^j = \frac{8+mn}{6} u^2 + \frac{2+m}{3} u v - \frac{14+3mn}{12} u^3 - \frac{22+11m}{18} u^2 v - \frac{10+5m}{36} u v^2 + \ldots \quad (3.5)
$$

$$
B_v = \sum_{ijkl} b_{ijkl}^{(v)} n^i u^j v^j = 2uv + \frac{8+m}{6} v^2 - \frac{82+5mn}{36} u^2 v - \frac{58+11m}{18} u v^2 - \frac{14+3m}{12} v^3 + \ldots
$$

$$
\eta_\phi = \sum_{ijkl} e_{ijkl}^{(\phi)} n^i u^j v^j = \frac{2+mn}{72} u^2 + \frac{2+m}{36} u v + \frac{2+m}{72} v^2 + \ldots
$$

$$
\eta_t = \sum_{ijkl} e_{ijkl}^{(t)} n^i u^j v^j = -\frac{2+mn}{6} u - \frac{2+m}{6} v + \frac{2+mn}{12} u^2 + \frac{2+m}{6} u v + \frac{2+m}{12} v^2 + \ldots
$$

The values of the coefficients $b_{ijkl}^{(u)}$, $b_{ijkl}^{(v)}$, $e_{ijkl}^{(\phi)}$, and $e_{ijkl}^{(t)}$, up to five loops are reported in the file \texttt{MN-MS.TXT} attached to Ref. [1]. The meaning of the numbers reported in this file is the same as that of the numbers appearing in file \texttt{MN-MZM.TXT}, see the end of Sec. III A.

IV. THE SPIN-DENSITY-WAVE $\Phi^4$ MODEL

The spin-density-wave (SDW) $\Phi^4$ model is defined by the Hamiltonian density

$$
\mathcal{H} = |\partial_\mu \Phi_1|^2 + |\partial_\mu \Phi_2|^2 + r(|\Phi_1|^2 + |\Phi_2|^2) + \frac{u_{1,0}}{2}(|\Phi_1|^4 + |\Phi_2|^4) + \frac{u_2,0}{2}(|\Phi_1|^2 + |\Phi_2|^2) + w_{1,0} |\Phi_1|^2 |\Phi_2|^2 + w_{2,0} |\Phi_1 \Phi_2|^2 + w_{3,0} |\Phi_1^* \Phi_2|^2
$$

$$
(4.1)
$$

where $\Phi_{ai}$ is a complex $2 \times N$ matrix field ($a = 1, 2$ and $i = 1, \ldots N$).

We refer to Refs. [20] for further details on the perturbative expansions in this model.

A. The 3D massive zero-momentum perturbative expansion

The RG functions of the SDW model in the MZM scheme are defined following the same steps as in the cases considered in the preceding sections, see Sec. II A.

The $\beta$-functions of the renormalized quartic couplings $u_i$, $w_i$ corresponding to the quartic Hamiltonian parameters $u_{i,0}$, $w_{i,0}$ can be written as

$$
\beta_\# = \sum_{ijklmp} b_{ijklmp}^{(\#)} N^p u_1^i u_2^j u_3^k u_4^l w_3^m
$$

$$
(4.2)
$$
where the symbol # indicates \( u_1, u_2, w_1, w_2, w_3 \). At one loop we have \(( A \equiv 1/(8\pi) )\)

\[
\begin{align*}
\beta_{u_1} &= -u_1 + A \left[ (N+4)u_1^2 + 4u_1u_2 + 4u_2^2 + Nu_1^2 + w_1^2 + w_2^2 + 2w_1w_2 + 2w_1w_3 \right] + \ldots \\
\beta_{u_2} &= -u_2 + A \left[ 6w_1u_2 + Nu_2^2 + 2w_2w_3 \right] + \ldots \\
\beta_{w_1} &= -w_1 + A \left[ 2w_1^2 + w_2^2 + w_3^2 + 2(N + 1)u_1w_1 + 4u_2w_1 + 2u_1w_2 + 2u_1w_3 \right] + \ldots \\
\beta_{w_2} &= -w_2 + A \left[ Nu_2^2 + 2u_1w_2 + 4u_2w_3 + 4w_1w_2 + 2w_2w_3 \right] + \ldots \\
\beta_{w_3} &= -w_3 + A \left[ Nu_3^2 + 2u_1w_3 + 4u_2w_2 + 4w_1w_3 + 2w_2w_3 \right] + \ldots 
\end{align*}
\]

Analogous expansions apply to \( e^{(\phi)} \) and \( e^{(t)} \) with coefficients \( e^{(\phi)}_{ijklmp} \), and \( e^{(t)}_{ijklmp} \). The values of the coefficients \( b^{(#)}_{ijklmp}, e^{(\phi)}_{ijklmp}, \) and \( e^{(t)}_{ijklmp} \) up to six loops are reported in the file `SDW-MZM.TXT` attached to Ref. [1]. The file is formatted as before. Nine numbers appear in each line. The first one indicates the quantity one is considering: \( 1, 2, 3, 4, 5 \) correspond to the \( \beta \) functions \( \beta_{u_1}, \beta_{u_2}, \beta_{w_1}, \beta_{w_2}, \) and \( \beta_{w_3} \); \( 6 \) and \( 7 \) to \( e^{(\phi)} \) and \( e^{(t)} \). The second number gives the number of loops. The subsequent six integer numbers correspond to \( i, j, k, l, m, p \). Finally, the last number gives the coefficient.

**B. The minimal-subtraction perturbative expansion**

The RG functions of the SDW model in the \( \overline{\text{MS}} \) scheme are defined following the same steps as in the cases considered in the preceding sections, see Sec. II B.

The \( \beta \)-functions of the renormalized quartic couplings \( u_i, w_i \) corresponding to the quartic Hamiltonian parameters \( u_{i,0}, w_{i,0} \) can be written as

\[
\beta_{\#} = (d - 4)\# + B_{\#}, \quad B_{\#} = \sum_{ijklmp} b^{(#)}_{ijklmp} N^p u_i^i u_j^j w_l^l w_m^m \tag{4.3}
\]

where the symbol \# represents \( u_1, u_2, w_1, w_2, w_3 \). At one loop we have

\[
\begin{align*}
B_{u_1} &= (N + 4)u_1^2 + 4u_1u_2 + 4u_2^2 + Nu_1^2 + w_1^2 + w_2^2 + 2w_1w_2 + 2w_1w_3 \tag{4.4} \\
B_{u_2} &= 6w_1u_2 + Nu_2^2 + 2w_2w_3 \\
B_{w_1} &= 2w_1^2 + w_2^2 + w_3^2 + 2(N + 1)u_1w_1 + 4u_2w_1 + 2u_1w_2 + 2u_1w_3 \\
B_{w_2} &= Nu_2^2 + 2u_1w_2 + 4u_2w_3 + 4w_1w_2 + 2w_2w_3 \\
B_{w_3} &= Nu_3^2 + 2u_1w_3 + 4u_2w_2 + 4w_1w_3 + 2w_2w_3 
\end{align*}
\]

The functions \( e^{(\phi)} \) and \( e^{(t)} \) have analogous expansions with coefficients \( e^{(\phi)}_{ijklmp}, \) and \( e^{(t)}_{ijklmp} \). The values of the coefficients \( b^{(#)}_{ijklmp}, e^{(\phi)}_{ijklmp}, \) and \( e^{(t)}_{ijklmp} \) up to five loops are reported in the file `SDW-MS.TXT` attached to Ref. [1]. The meaning of the numbers reported in this file is the same as that of the numbers appearing in file `SDW-MZM.TXT`, see the end of Sec. IV A.

**C. Perturbations of the \( O(4) \otimes O(N) \) fixed points**

We report the perturbative series of the RG eigenvalues \( \Omega_1 \) and \( \Omega_2 \) defined in App. B of the paper. We write
\[ \Omega_\# = \sum_{ijk} \Omega_{ijk}^# N^k u_1^i u_2^j. \] (4.5)

The values of the coefficients \( \Omega_{ijk}^# \) are reported in the file Perturbations-O4ON.TXT attached to Ref. [1]. In each line we report 6 numbers. The first one specifies the quantity one is referring to: 1 corresponds to \( \Omega_1 \) in the MZM scheme; 2 corresponds to \( \Omega_2 \) in the MZM scheme; 3 corresponds to \( \Omega_1 \) in the MS scheme; 4 corresponds to \( \Omega_2 \) in the MS scheme. The second number gives the number of loops, the third, fourth, and fifth number refer to the indices \( i, j, \) and \( k \) appearing in Eq. (4.5). Finally, the last real number is the coefficient \( \Omega_{ijk}^# \). For \( N = 3 \) these series are reported in App. B.

**D. Perturbations of the mn fixed points**

We report the perturbative series of the RG eigenvalues \( \Omega_1 \) and \( \Omega_2 \) defined in App. C of the paper. We write

\[ \Omega_\# = \sum_{ijk} \Omega_{ijk}^# N^k u_1^i u_2^j. \] (4.6)

The values of the coefficients \( \Omega_{ijk}^# \) are reported in the file Perturbations-MN.TXT attached to Ref. [1]. In each line we report 6 numbers. The first one specifies the quantity one is referring to: 1 corresponds to \( \Omega_1 \) in the MZM scheme; 2 corresponds to \( \Omega_2 \) in the MZM scheme; 3 corresponds to \( \Omega_1 \) in the MS scheme; 4 corresponds to \( \Omega_2 \) in the MS scheme. The second number gives the number of loops, the third, fourth, and fifth number refer to the indices \( i, j, \) and \( k \) appearing in Eq. 4.6. Finally, the last number is the coefficient \( \Omega_{ijk}^# \).
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