Dark halo microphysics and massive black hole scaling relations in galaxies

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Accepted 2014 September 22. Received 2014 September 22; in original form 2014 March 23

ABSTRACT

We investigate the black hole (BH) scaling relation in galaxies using a model in which the galaxy halo and central BH are a self-gravitating sphere of dark matter (DM) with an isotropic, adiabatic equation of state. The equipotential where the escape velocity approaches the speed of light defines the horizon of the BH. We find that the BH mass \( m_\bullet \) depends on the DM entropy, when the effective thermal degrees of freedom (\( F \)) are specified. Relations between BH and galaxy properties arise naturally, with the BH mass and DM velocity dispersion following \( m_\bullet \propto \sigma^2/F \) (for global mean density set by external cosmogony). Imposing observationally derived constraints on \( F \) provides insight into the microphysics of DM. Given that DM velocities and stellar velocities are comparable, the empirical correlation between \( m_\bullet \) and stellar velocity dispersions \( \sigma_\star \) implies that \( 7 \lesssim F < 10 \). A link between \( m_\bullet \) and globular cluster properties also arises because the halo potential binds the globular cluster swarm at large radii. Interestingly, for \( F > 6 \) the dense dark envelope surrounding the BH approaches the mean density of the BH itself, while the outer halo can show a nearly uniform kpc-scale core resembling those observed in galaxies.

Key words: black holes — dark matter — galaxies: haloes — galaxies: kinematics and dynamics — galaxies: structure — globular clusters: general

1 INTRODUCTION

Over the last two decades, empirical correlations between different galaxy components — nuclear supermassive black hole (SMBH), stellar bulge and disc, dark matter (DM) halo — have shaped our understanding of galaxy structure evolution and of SMBH/galaxy co-evolution (Kormendy & Ho 2013, for a review). The most significant correlations are the one observed between SMBH masses (\( m_\bullet \)) and velocity dispersions (\( \sigma \)) of their host stellar bulges or spheroids (\( m_\bullet \sigma \) relation: Ferrarese & Merritt 2000; Gebhardt et al. 2000; Tremaine et al. 2002; Graham et al. 2011; Xiao et al. 2011; Aller & Richstone 2007; Graham & Scott 2013; Scott, Graham & Schombert 2013); and the one between SMBH masses and bulge masses (Magorrian et al. 1998; Graham et al. 2001; Graham & Driver 2007; Hernquist et al. 2004; Haring & Rix 2004; Graham & Scott 2013; Scott, Graham & Schombert 2013). The kinetic or potential energy of the bulge also correlates with \( m_\bullet \) (Feoli & Mele 2000a, 2000b; Aller & Richstone 2007; Hopkins et al. 2007a, 2007b; Feoli & Mancini 2008; Mancini & Feoli 2010). Interestingly, for \( F > 6 \) the dense dark envelope surrounding the BH approaches the mean density of the BH itself, while the outer halo can show a nearly uniform kpc-scale core resembling those observed in galaxies.

The kinetic or potential energy of the bulge also correlates with \( m_\bullet \) (Feoli & Mele 2000a, 2000b; Aller & Richstone 2007; Hopkins et al. 2007a, 2007b; Feoli & Mancini 2008; Mancini & Feoli 2010). The correlation may also take other forms, such as a dependence on the Sérsic index of the stellar profile (Graham et al. 2001; Graham & Driver 2007; Savorgnan et al. 2013).

Our investigation follows the dual nature of galaxy structure and dynamics — galaxies: structure — globular clusters: general.
total number of GCs and the SMBH mass also show a correlation with each other.

Taking all these empirical correlations together points to the presence of a general scaling relation between SMBH mass, stellar mass in the host spheroid and the total mass/number of GCs in the galaxy. This scaling appears straightforward to understand, at least qualitatively. Rapid growth of the nuclear black hole of a galaxy (particularly at redshifts $2 < z < 6$) might be fuelled by a massive inflow of cold gas towards the centre of the galaxy. The gas inflow would trigger starbursts and the formation of new GCs. Numerical simulations often show massive gas inflows in mergers of gas-rich galaxies (e.g. Hopkins et al. 2006; Barnes & Hernquist 1991, 1996; Hernquist 1989). This scenario expects a coeval growth of SMBH and stellar components (which includes the spheroid and GCs), regulated by the gas supply that reaches the inner region of the galaxy, and ultimately primed by the merger rate (Volonteri & Natarajan 2009). The parallel growth of the SMBH and stellar component cannot continue indefinitely, and it terminates when the gas supply ceases. The accretion into a black hole at super-Eddington rates will emit copious radiation, which exerts radiative pressure on the inflowing gas, leading to a massive galactic-scale outflows. When the central black hole in a galaxy has grown to a sufficiently large mass (and can therefore attain a sufficiently high Eddington luminosity), the momentum-driven, expanding shell of the swept-up gas will achieve a velocity higher than the escape velocity from the galaxy Silk & Rees 1995; King 2005; Murray, Quataert & Thompson 2005). When most of the gas is expelled, star formation and SMBH accretion are quenched.

However, the reality could be more complicated than described above, as there is evidence that SMBH accretion and star formation do not always trace each other (Zheng et al. 2009). Thus, there could be pathways (or even multiple pathways) of SMBH and spheroid growth without invoking self-regulation (e.g. Anglés-Alcázar, Özel & Davé 2013) that lead to the SMBH scaling relations that we observe today (Zheng 2013). It is worth noting that the duration of SMBH growth in the co-evolution scenario depends on the initial mass of their seed black holes. Some authors (e.g. Shibata & Shapiro 2002; Volonteri & Madau 2003; Begelman 2010) argued that seed black holes may come from direct collapse of supermassive stars, which were formed directly from large-scale gas inflows in the DM halo. As such, the seed black hole mass distribution would be a function of the DM halo virial temperature and the black-hole spin. Also, there would be an angular momentum ceiling for the DM halo, only below which inflows can occur and supermassive stars can form.

The existence and nature of a correlation between GCs, SMBH and dark halo is not free from disputes. GCs have a bimodal colour distribution, probably the signature of two physically distinct populations: younger, metal-rich red and older, metal-poor blue clusters (Brodie & Strader 2006). Co-evolution of stellar populations and SMBH due to major mergers should produce a correlation only between red GCs (formed during the starburst phase and located closer to the nucleus) and SMBH (Kormendy & He 2013). The correlation is indeed tighter for red GCs (Sadoun & Colín 2012), but the fraction of red/blue GCs is similar for most galaxies (Burkert & Tremaine 2010), indicating some residual correlation also with the blue (old) population, or perhaps an initial correlation between blue GCs and seed BH. Intriguingly, it was recently noted (Harris, Harris & Alessi 2013) that the relation between GC mass fraction (i.e., fraction of a galaxy mass that is contained in GCs) and galaxy mass is not a constant but has a characteristic U-shape: both dwarf and giant ellipticals have a larger fraction of baryonic mass located in GCs, than intermediate-mass galaxies. This could be due to different rates of GC formation or subsequent GC destruction. Alternatively, perhaps dwarf and giant galaxies have formed field stars less efficiently, owing to gas losses from super-winds and SMBH activity respectively. Only in intermediate-mass systems is the observed GC mass fraction a true indication of how much gas was initially present in the galaxy potential well. Proponents of collisionless cold DM theories also invoke a scenario of gas blowouts to explain the differences between the simulated halo mass spectrum and the visible baryonic mass function, especially at the low-mass and high-mass ends (e.g. Persic & Salucci 1992; Bell et al. 2003; Read & Tremonti 2003; Papastergis et al. 2012). Either way, it follows that the mass in GCs is determined by the amount of gas initially present in the (DM-dominated) potential well of a galaxy, and therefore there must be some correlation between GC mass and DM halo mass (Harris, Harris & Alessi 2013; Georgiev et al. 2010). In particular, Harris, Harris & Alessi (2013) propose a linear correlation with $M_{\text{GC}} \approx 6 \times 10^{-5} M_{\text{halo}}$.

In summary, there are empirical hints of correlations between SMBHs, DM halos and GCs in galaxies despite the widely different scales of the three types of objects, but it is still not clear to what extent the associations are truly intrinsic or they are mere by-products of other physical processes, such as galaxy mergers. In this work, we search for physical processes that could give rise to such correlations and demonstrate a physical mechanism that naturally links the properties of the SMBH, DM halos and GCs. Motivated by the extent of GC swarms — rounded and far from the direct reach of active galactic nuclei (AGN) in normal galaxies — we seek explanations in which the DM halo is the component controlling the scaling relations.

For galaxies that are large or small; rich or poor in baryons; pristine, star-forming or aged, observations indicate that DM halos feature a kpc-scale central core of nearly uniform density, surrounded by outskirts where the density declines radially till it becomes unmeasurable at $\sim 100$ kpc distances (e.g. Flores & Primack 1994; Moore et al. 1994; Burkert 1995; Salucci & Burkert 2000; Kelson et al. 2002; Kleva et al. 2003; Simon & Geha 2003; Gentile et al. 2004; de Blok 2005; Thomas et al. 2005; Kuzio de Naray et al. 2006; Goerdt et al. 2006; Gilmore et al. 2007; Weijmans et al. 2008; Oh et al. 2008; Inoue et al. 2007; Donato et al. 2007; de Blok 2010; Pu et al. 2010; Murphy, Gebhardt & Adams 2011; Memola, Salucci & Babic 2011; Richtler et al. 2011; Jardel & Gebhardt 2012; Andernach & Evans 2012; Amorisco & Evans 2012; Schuberth et al. 2012; Salucci et al. 2012; Lora et al. 2012; Amorisco, Andernach & Evans 2013; Hague & Wilkinson 2014). However, early theories of collisionless non-interacting DM predicted steep power-law central density cusps and not the observed cores (e.g. Gurevich & Zylinski 2008).
2 MODEL AND FORMULATION

2.1 Halo model properties

We assume a self-gravitating, spherically symmetric and structurally stationary DM halo, in dynamical equilibrium, with the thermodynamics of the DM described in terms of a polytropic equation of state. The DM in the halo is well mixed, without sub-halo clump structures. The SMBH develops from within the DM halo as an integral part of a self-gravitating system, instead of being inserted artificially into the halo centre as a massive external point-like object. Moreover, the black hole has a physically defined horizon directly by attributing the halo structure to the innate microphysics of DM. Studies (see e.g. Milosavljević & Tremaine 2004; MacMillan & Henriksen 2005; Merritt 2006; Zakharov et al. 2007; Ghez et al. 2008; Saxton & Wu 2008; Zakharov et al. 2010) indicate that the central density profile can rise locally in a sharp spike in the sub-pic to pc-scale gravitational sphere of influence around the SMBH. Our paper builds upon this finding, allowing a direct material coupling between the halo and SMBH, with a smooth transition from a DM density spike around the horizon of the SMBH to a cored DM halo at galaxy scales, that in turn binds the swarm of GCs located at larger distances. We consider constraints at both scales, and show how SMBH-GC relations emerge from the SMBH- halo connection.

We organise the paper as follows. Section 2 presents the model and the formulation. Section 3 shows the solutions, and in Section 4 we discuss the astrophysical implications.

2.2 Equation of state of the dark matter

The equation of state of the DM takes the form

\[ P = s \rho^2 = \rho \sigma^2 \],

(1)

or equivalently

\[ \rho = Q \sigma^F \],

(2)

where \( P \) is the pressure, \( \rho \) is the density, and \( \sigma \) is the isotropic velocity of the particles. The quantity \( s \) is the (pseudo-)entropy, and \( Q \equiv s^{-F/2} \) is the phase-space density. The adiabatic index \( \gamma \) is determined by the DM microphysics. It is related to the effective thermal degrees of freedom of the dark particles \( F \) via

\[ \gamma = 1 + \frac{2}{F} \].

(3)

Many DM scenarios entail a functionally equivalent equation of state (Section 4). Generally, \( F \) describes the number of modes in which the microscopic energies of DM particles can be equipartitioned. For translational motions in three dimensions, \( F = 3 \). When self-interacting DM particles are composite or have internal structure and modes of rotation, vibration and excitation at a comparable energy scale \( \gamma \) \[ Cline et al. 2013 \], then \( F > 3 \). The specific heat capacity at constant volume is \( c_v \equiv Fk/2 \) (where \( k \) is Boltzmann’s constant) and the energy density is \( F \rho \). If instead DM is a sterile neutrino then \( F \geq 3 \) in the degenerate halo core (e.g. neutrino-ball SMBHs, Viollier, Trautmann & Tupper 1993). If DM is a boson scalar field, then \( F \) derives from the index of the self-coupling potential \( \rho \) \[ Peebles 2001 \]. If DM experiences phase changes, then the equation of state is more complicated, but a polytropic law would remain a fair working approximation in limited ranges of temperature and density.

In principle, \( Q \) and \( s \) vary radially, if the halo is stratified, e.g. due to a history of mergers and accretion, or if dynamically significant energy exchange processes are present (e.g. ‘dark radiation,’ \[ Ackerman et al. 2009 \]). In that case, buoyant stability could appear, when \( ds/dr > 0 \) and \( (dQ/dr) < 0 \). However, we have assumed that the adiabatic DM in the halo does not have sub-structures. Thus, \( Q \) and \( s \) are constant for each galaxy in our calculations.

For \(-2 < F < 10\), the outer radius of the halo and total mass enclosed are finite, safeguarding the existence of realistic solutions for the DM halo-SMBH system. In this paper, we discard models with \( F < 0 \), since they have minimum density at the centre and greatest densities outside (which seems inappropriate for galaxies). Isolated polytropes with \( F > 6 \) are sometimes susceptible to interesting dynamical instabilities \( \[ Ritter 1878 \], \[ Emden 1907 \], \[ Chandrasekhar 1939 \] \). The instability can nonetheless be moderated by interactions with the baryonic matter components \( \[ Saxton 2013 \] \), or by a confining external pressure (e.g. \[ McCrea 1954 \], \[ Bonnor 1954 \], \[ Horelli 1970 \], \[ Umemura & Ikeuchi 1986 \]).

2.3 Halo profile

A realistic halo requires that the density \( \rho \) falls to zero at a certain outer radius \( R \), which defines the size of the halo. The mass enclosed by \( R \) is the total mass \( M \) of the halo. Inside the halo, the mass distribution is the solution to

\[ \frac{d\rho(r)}{dr} = 4\pi r^2 \rho(r) \],

(4)

where \( m(r) \) is mass contained within radius \( r \). The gravitational field strength is given by

\[ g(r) = -\frac{Gm(r)}{r^2} \],

(5)

and the gravitational potential \( \Phi(r) \) by

\[ \frac{d\Phi(r)}{dr} = -g(r) \].

(6)
The escape velocity \( v(r) \) satisfies the relation
\[
\frac{dv(r)^2}{dr} = 2g(r) .
\] (7)

If the pressure in the halo were deficient near a central gravitating mass, adiabatic accretion would proceed (Bondi 1952), which, conceivably, feeds the growth of the SMBH (e.g. Peirani & de Freitas Pacheco 2008; Guzmán & Lora-Clavijo 2011a,b; Pepe, Pellizza & Romero 2012 [Lora-Clavijo, Gracia-Linares & Guzman 2014]). Without losing generality we ignore the complications of accretion inflow and focus on the stationary halo, which is pressure-supported everywhere. Under these conditions the velocity dispersion of DM is then given by
\[
\frac{d\sigma(r)^2}{dr} = \frac{2}{F+2} g(r) .
\] (8)

The DM velocity dispersion \( \sigma^2(r) \) can be considered as a measure of the local thermal ‘temperature’. Within the halo, this thermal temperature is related to the local escape velocity and gravitational potential by
\[
\sigma(r)^2 = \frac{1}{F+2} \left[ v(r)^2 - V^2 \right] ,
\] (9)

\( V \) is the escape velocity at the outer boundary of the halo \( (r = R, \rho = 0, \sigma = 0) \). The above expression can be obtained by carrying out an integration after combining equations (7), (8) and (4). The escape velocity \( V \) depends on whether there is any non-DM material extending beyond the outer halo radius \( R \). Otherwise, it takes the value \( V = \sqrt{2Gm/R} \).

Given either the inner or outer boundary conditions, locating the other boundary is performed by numerical integration (Section 2.3). We define a dimensionless gravitational compactness parameter:
\[
\chi \equiv \left( \frac{V}{c} \right)^2 = \frac{2GM}{c^2R} < 1 .
\] (10)

Empirical values of \( \chi \) could be estimated from a characteristic velocity dispersion or mass-radius relation of self-bound objects. For example, massive galaxy clusters have \( \chi \lesssim 10^{-4} \) (\( V \lesssim 3000 \) km s\(^{-1}\)); giant galaxies have \( \chi \lesssim 10^{-6} \) (\( V \lesssim 300 \) km s\(^{-1}\)); and faint dwarf galaxies have \( \chi \lesssim 10^{-8} \) (\( V \lesssim 30 \) km s\(^{-1}\)).

2.4 Central black hole and horizon surface

Most galaxies are expected to possess a central black hole, but observations indicate that some actually do not. In some cases there might never occur a mass concentration dense enough to collapse gravitationally. Effects such as rotational support might help avert black-hole formation in certain late-type galaxies (Section 1.7). Also, merger events could eject a SMBH from the host galaxy. Here however, we investigate only galaxies that have formed a nuclear SMBH and retain it in equilibrium with its DM surroundings.

The escape velocity of a test mass is \( c \), the speed of light, at the event horizon of a (Schwarzschild) black hole. If it is appropriate to consider the ‘formation’ of a black hole in this newtonian model, then the black hole is defined by the sphere where the escape velocity is \( v = c \) at its surface (i.e. the horizon). This black hole contains a mass \( m_\bullet \), inside a horizon radius, which is given by \( r_\bullet \approx 2GM_\bullet/c^2 \). In a dense DM envelope enclosing the central black hole, the horizon radius is larger than the ideal Schwarzschild value in vacuum. We parameterise the ratio between the horizon radius and the Schwarzschild radius \( r_\bullet \) by \( \eta \equiv r_\bullet/r_s \). In a fully relativistic treatment, \( \eta = 1 \) always. Here, the value of \( \eta \) is generally of the order unity. The mean density of the black hole is then
\[
\bar{\rho}_\bullet \equiv \frac{3m_\bullet}{4\pi r_\bullet^3} = \frac{3\epsilon_6}{32\pi G^3m_\bullet^6\eta^4} .
\] (11)

The velocity dispersion of the DM at the horizon surface is
\[
\sigma^2 = \left( \frac{1 - \chi}{F + 2} \right) c^2 ,
\] (12)

If the halo is adiabatic all the way down to the horizon surface of the central black hole, then from the equation of state (2) we obtain
\[
\rho(r)/\rho_\bullet = \left[ \frac{\sigma(r)^2}{\sigma^2_\bullet} \right]^{F/2} .
\] (13)

Define a parameter \( \psi \equiv \bar{\rho}_\bullet/\rho_\bullet \), which is the density ratio of the BH to DM near its horizon surface. Then, we have
\[
m_\bullet = \sqrt{\frac{3\epsilon_6}{32\pi G^3} \left( \frac{F + 2}{1 - \chi} \right)^{F/4} (\eta^2 \psi p)^{-1/2} \left( \frac{\sigma}{c} \right)^{F/2}} ,
\] or,
\[
= \sqrt{\frac{3\epsilon_6}{32\pi G^3} \left( \frac{F + 2}{1 - \chi} \right)^{F/4} \left( \frac{1}{\sqrt{Q\eta^2 \psi}} \right)^{F/2}} .
\] (14)

Substituting any observed set of \((\rho, \sigma)\) values of the DM from elsewhere in the halo’s adiabatic region yields an estimate of the natural mass of the central compact object. The values of \( \rho \) and \( \sigma \) in the above expression are local. They can be constrained by the observations. The dimensionsless correction factors \( \psi \) and \( \eta \) are, however, obtained by numerical solution of a particular halo model. In Section 3.3 we will show that, for the physically relevant models of polytropic DM haloes, the correction factors are moderate.

Appendix A expresses the mass prediction (14) in absolute physical units.

2.5 Numerical integration scheme

The radial profiles of particular polytropic haloes are obtained from direct numerical integration of equations (1), (4) and (8). This is an initial value problem, with radius \( r \) as the independent variable, starting from either the inner boundary \((r = 0)\) or the outer boundary \((r = R)\). The phase-space density \( Q \), (pseudo-)entropy \( s \) and thermal degrees of freedom \( F \) are mutually consistent constants.

We adopt the embedded eighth-order Runge-Kutta Prince-Dormand method with ninth-order error estimate (Prince & Dormand 1981; Hairer, Nørsett & Wanner 2008) in our integration. When integrating outwards from known inner values, we can express the differential equations in their original form. When integrating inwards, each equation is multiplied by \( -1 \), and \( -r \) is used as the independent variable. Near the inner and outer boundaries, the radius is not known a priori, but the velocity dispersion is known exactly \((\sigma = \sigma_\bullet \) and \( \sigma = 0 \) respectively). In those vicinities,
it is more desirable to adopt $\sigma^2$ as the independent variable in the differential equations, i.e. re-expressing each quantity $y$ the equations in the form of $dy/d\sigma^2$ or $dy/d(-\sigma^2)$.

In the numerical integration we first consider small steps (but much larger than the round-off level) until it is appropriate to switch to another independent variable and then continue in the same integration mode. Doing so we can integrate accurately either from the outer boundary of the halo towards the SMBH horizon, or from the SMBH horizon to the outer boundary of the halo.

3 RESULTS

Before presenting the results of our DM halo-SMBH calculations, we briefly review the general properties of adiabatic self-gravitating polytropic spherically symmetric bodies. This class of spheroids has been investigated previously, but more often in the context of stars instead of larger spheroids such as galaxies or galaxy clusters. There are three sub-classes (Fig. 1) with these characteristics:

(i) **Nonsingular**, with a zero density gradient at the centre. The density declines outward until reaching zero at a large radius $R$. The Lane-Emden spheres are examples of these (Lane 1870; Emden 1907). Observations show that galaxy haloes often have a uniform density core with a profile resembling that of these polytropic spheres.

(ii) **Singular**, with a density spike around a massive nuclear object. Shallower density gradients further out resemble that of a galaxy core. The profile of the outer fringe is similar to that of the nonsingular polytropic haloes.

(iii) **Terraced**, with the radial density profile alternating between power-law slopes and cores, nestled inside each other. The centre is, however, singular. Medvedev & Rybicki (2001) studied terraced polytropes with $F \approx 10$.

Nonsingular polytropic spheroids can be obtained by setting variables according to inner boundary conditions and then integrating the system of profile differential equations outwards. In the context of DM halo-SMBH model considered in our paper this kind of spheroid does not provide a self-consistent description for the circum-nuclear properties explored in our paper this kind of spheroid does not provide a self-consistent description for the circum-nuclear properties.

3.1 Relevant solutions

The polytropes have a nonzero compact central mass surrounded by a density spike, where $\rho \sim r^{-F/2}$ (e.g. Huntley & Saslaw 1973; Quinlan, Hernquist & Sigurdsson 1995; Ullo, Zhao & Kamionkowski 2001). The gravitational potential is keplerian near the origin, with $\Phi \sim r^{-1}$, and the velocity dispersion peaks in the same manner, i.e. $\sigma^2 \sim r^{-1}$. The escape velocity reaches $c$ at some sufficiently small radius.

In computing the radial profile, we set a fiducial outer radius, say $R = 1$, where $\sigma = 0$, and choose trial values of the total mass $M$. This implies a specific value for the compactness parameter $\chi$. Keeping these fixed, we test trial values of the phase-space density $Q$, and integrate the profile differential equations inwards. If the condition (12) is satisfied then we record the conditions of that inner boundary, $(r_*, m_*, \eta, \psi)$. If the origin is reached, or if a condition of $m \leq 0$ is encountered at any $r > 0$, then no horizon for the central gravitating object is obtained, and the trial value of $Q$ is recorded as an unphysical case.

Fig. B depicts the radial profiles of solutions for haloes with $F = 9$ and compactness appropriate for a galaxy ($\chi = 10^{-6}$). The inner tip (left) of each curve locates a horizon $(r_*)$; the outer tip (corresponding to $R = 1$, which is by construction) is where the halo truncates itself. Taking the outer radius to be on the order of $R \sim 300$ kpc, the nearly uniform core has a radius $\sim 1$ kpc. The nonsingular solution (black curve) is uniform at the origin. For a model near the nonsingular limit (e.g. blue curve) the DM velocity dispersion (and density) rise at radii within a parsec. This is the sphere of gravitational influence of the central mass. In this particular solution, the dense DM envelope surrounding the horizon outweighs the influence of central object $(m_*)$ by almost an order of magnitude at radii $r \lesssim 10r_*$.

The central mass fraction is $m_*/M \approx 3.24 \times 10^{-5}$, consistent with the observed ratios between the SMBH and their host galaxy haloes (e.g. under slightly different assumptions, $m_*$ vs $M$ results of equations (4)–(7) and Fig. 5 of Ferrarese 2002). For models with lower $Q$ (green and red curves) the inner dense-hot spike is radially larger, $m_*$ is heavier, and the halo core is more compact. At the opposite extreme (large $Q$) we have $m_* \rightarrow 0$ and obtain the biggest possible halo core. Its maximal mass and radius depend on $F$. When $F$ is smaller, the maximal core is wide and contains much of the halo mass. (For an incompressible fluid, $F = 0$, $m_*$...
the entire halo is a core.) When $F$ is larger, the maximal core is radially smaller and is relatively lightweight.

### 3.2 Configuration space

Particular radial profiles can be obtained for choices of $(\chi, Q)$ across a two-dimensional configuration space at fixed $F$. This task can be wrapped within a root-finding routine or an amoeba-like minimiser, seeking a specific or optimal value of any desired property of the central object (e.g. $m_*/M$ or $\psi$). We explore the $(\chi, Q)$ plane numerically at high resolution. Fig. 3 maps the varying properties of the central object, for $F = 9.5$ halo models. For clarity of presentation, the vertical axis value is a dimensionless adjusted version of the phase-space density,

$$q \equiv QV^F/\bar{\rho}$$

(15)

where $\bar{\rho} = 3M/4\pi R^3$ is the mean density of the system, and $V$ is the surface escape velocity. The four panels show results for: the horizon density contrast ($\psi$); the horizon radius correction ($\eta$); the central object’s fractional mass ($m_*/M$); and its radius ($r_*/R$). Several distinct domains appear. The top-left panel labels these domains:

(i) **forbidden zone**: For sufficiently high $q$ there are no self-consistent solutions. The halo is too dense and cold to reach the assumed outer radius $R$.

(ii) **border zone**: For $q$ slightly below the forbidden zone, there is a thin domain of solutions with extremely high or low $\psi$ values (and steep gradients of $\partial \psi / \partial q$). The upper edge of the border is where the nonsingular solutions occur ($m_*/M \rightarrow 0$ and $r_*/R \rightarrow 0$, which cannot describe a galaxy hosting a SMBH).

(iii) **moderate plateau**: If $6 < F < 10$ then there is a domain of $q$ values below the border, where $m_*/M$ and $r_*/R$ are small but finite (and astronomically significant). The envelope density is non-negligible compared to the mean density of the black hole ($\psi \lesssim 100$). This plateau zone is more extensive in $q$ (or in $Q$) if $\chi$ is small (systems with low escape velocities). Viewed in $(\chi, q)$ or $(\chi, Q)$ planes, the plateau is roughly triangular. Terraced haloes occur here.

(iv) **valleys**: Within the plateau, there are local minima in $\psi$, coinciding with spikes in $\eta$. This implies gradual density continuity between the BH and its immediate dark envelope. Valleys are more numerous for smaller $\chi$.

(v) **hole-dominated**: At $q$ values lower than the plateau zone, the black hole mass becomes dominant, $m_*/M \rightarrow 1$. The halo is relatively tenuous: $r_*/R$ remains small. The black hole is effectively decoupled from the density and pressure of its diffuse surroundings. The density contrast $\psi$ rises by orders of magnitude, and the gradient $\partial \psi / \partial q$ is steep.

The plateau and $\psi$-valleys only exist for haloes with $6 < F < 10$. For $0 \leq F \leq 6$, the transition from nonsingular border to the hole-dominated domain is much narrower than 1 dex in $q$. For $6 < F < 10$, the plateau becomes wider with increasing $F$, until the plateau vanishes suddenly around $F = 10$ (infinite profiles including the classic Plummer model). Fig. 4 depicts the $\psi$ landscape of the plateau in the $(\chi, q)$ plane, for various equations of state ($F = 6, 7, 8$ and 9). The $\psi$-valleys are conspicuous diagonal stripes. The valleys are more numerous for greater $F$. For large-$F$ models, the $\psi$-valleys coincide with steps in the ratio $m_*/M$ (Fig. 5 compare left panels). For lower $F$, the steps are less distinct (gradients $\partial(m_*/M)/\partial q$ and $\partial(m_*/M)/\partial \chi$ are steadier). Across most of the plateau, the $\psi$ contours (such as the valleys) are approximately parallel to contours of $m_*/M$.

As noted below (Section 4.1.3) the $\psi$-valleys are locally energetically favoured states. Haloes in $\psi$-valleys have varied properties:

(i) In the valley at lowest $Q$, the BH is obese ($m_*/M \gtrsim 0.1$) with only a tenuous halo. This is unrealistic for a galaxy.

(ii) Valleys at intermediate $Q$: dark diagonals in Fig. 4. Running along these valleys keeps $m_*/M$ nearly constant, resembling a [Magorrian et al. 1998] relation.

(iii) Valleys near the nonsingular border have branches and irregular $\psi$-topography. Values of $m_*/M$ are lowest here.

For larger $F$, the valleys reach lower values of $m_*/M$ at any given $\chi$. For $\chi \approx 10^{-6}$ the lowest valley haloes with $F = 8$ and $F = 9$ give $m_*/M \sim 10^{-5.5}$ and $\sim 10^{-4.5}$ respectively. This range may be consistent with observed SMBH-bulge relations if the DM halo is $\sim 10^1$ times the baryonic mass. Thus, on the one hand, when $7 \lesssim F < 10$ some of the $\psi$-valleys are consistent with realistic SMBH masses relative to the host galaxy. Conversely, assuming these theories of $F$, we predict that some galaxies host SMBH in low-$\psi$ configurations: the dark envelope is dense near the horizon. At least in the present newtonian model, the edge of such a SMBH is blurry. This deserves further investigation through general relativistic calculations.

### 3.3 Relation of central mass to halo core

Formally the configuration space at fixed $F$ is two-dimensional, with gravitational compactness and phase-space density parameters ($\chi, Q$) or ($\chi, q$). In practical applications, the compactness parameter ($\chi = 2GM/Rc^2$) may be difficult to estimate, since it depends on the total mass contained within the dark halo truncation radius $R$, which is not directly observable. It may be more useful to specify models in terms of quantities pertaining to the measurable DM core. If the local index of the density profile is

$$\alpha \equiv -\frac{\ln \rho}{\ln r},$$

(16)

then we can annotate slope-radii ($R_\alpha < R$) at a standard chosen $\alpha$. The radii $R_\alpha$ can be multi-valued (in terraced haloes) and this is more likely when $F$ is larger. We shall define the DM core to be the region enclosed by the outermost locations where $\alpha = 1, 2, 3$, i.e. $R_1 < R_2 < R_3 < R$. The mass contained within these radii satisfies $m_0 \ll M_1 < M_2 < M_3$. The gravitational compactness of the core can be written as $\chi_\alpha \equiv -2\Phi_\alpha/c^2$. Similarly, we might define the core at the half-mass radius $R_m$ and potential $\Phi_m$ (where $m(R_m) = \frac{1}{2}M$), with core compactness $\chi_m = -2\Phi_m/c^2$. In our discussions below, we can abbreviate the core compactness $\chi_\alpha$ standing for $\chi_1, \chi_2, \chi_3$ or $\chi_m$. The particular choice does not change the qualitative conclusions. These notations disregard the tenuous outskirts of the halo, which in any case are difficult to measure astronomically.

When the $(\chi, q)$ plane transforms to $(\chi_\alpha, q)$, the plateau region straightens from a wedge with diagonal stripes to a rectangular region with vertical stripes (see Fig. 4). Except near the upper $q$ border (the nonsingular limit) the models
of fixed \( \chi_c \) are almost independent of \( q \). The two-dimensional parameter-space is almost (but not quite) reduced to a one-dimensional space in terms of the core compactness.

Fig. 6 shows the variation of \( m_\bullet /M_c \) with respect to core compactness, for equations of state with \( F = 6.5, 7.0, 7.5, 8.0, 8.5, 9.0 \). Each ribbon depicts the entire plateau region, apart from the border strip. The physically uninteresting 'hole-dominated' region hides in the top-right point where \( m_\bullet \approx M \). The thinness of the ribbons in this projection shows how \( q \) becomes inconsequential compared to \( \chi_c \). The nonsingular solutions (not shown) have smaller \( m_\bullet /M_c \) for given \( \chi_c \): down to arbitrarily small values as \( q \) approaches its maximum. In Fig. 6 they occupy the region of the \( (\chi_c, m_\bullet /M_c) \) plot below the ribbon of given \( F \). Thus each ribbon represents the maximum possible \( m_\bullet \) hosted within a DM halo core of given compactness.

Now we can interpret a characteristic velocity dispersion of tracer objects in the core region, \( \sigma_{gc} \propto c \sqrt{\chi_c} \). Observable velocity dispersions of the globular cluster swarm should match this value to within a factor of a few. For an assumed halo \( F \) and known \( \sigma_{gc} \), one can estimate \( \chi_c \) and infer a narrow range of possible \( m_\bullet /M_c \) (if the galaxy is in the 'plateau' regime) or an upper limit on \( m_\bullet /M \) (if it is a nearly nonsingular case). An estimate \( \sigma \propto \sigma_{gc} \) can be substituted in equations (14) and (A3). Estimates should be most robust for haloes where GCS projected \( \sigma_{gc} \) appears nearly constant within the DM core (e.g. Côté et al. 2003; Bridges et al. 2006; Norris et al. 2012; Napolitano et al. 2014).

The shading of the ribbons in Fig. 6 shows the horizon correction factor \( \log_{10} \sqrt{\eta/v_\psi} \) that applies in equation (14) when predicting SMBH masses in physical units. For the \( F > 6 \) plateau halo models, the variation of this term is small. For galaxy compactness \( \chi_c \ll 10^{-4} \) the correction factors vary by less than 0.6 dex. The variation is smaller for low \( F \). Thus equation (14) is robust enough to apply approximately, even when \( Q \) is unobservable.

4 DISCUSSION

4.1 Preferred solutions and \( m_\bullet \)

Although the \( \chi_c \) core representation simplifies the projected configuration space, the present spherical model still has two free parameters: the compactness \( \chi_c \) and some measure of the orderliness (such as \( Q \) or \( q \)). Given the apparent simplicity of the empirical relations between \( m_\bullet \) and host galaxy properties, it is worth seeking a simple causal explanation. Is there any physical principle that constrains \( q \) as a function of \( \chi_c \) or \( \chi^2 \)? A satisfactory model would involve a simple intuitive rule involving instantaneous halo properties, without complexities involving fine-tuning processes, such as those invoked in feedback scenarios, local contingencies or accidents of evolutionary history. Here we shall discuss some conceivable rule-of-thumb explanations.

4.1.1 cosmic density

In the theory and simulations of cosmological collapse of collisionless haloes, galaxy-like objects lack a clearly defined outer boundary surface. Instead they are described

\[
R = \sqrt[3]{\frac{3c^2}{8\pi G \rho_c}}
\]

and the corresponding mass is \( M = \chi c^2 R / 2 G \propto \sqrt[4]{\chi_c / \rho_c} \). Thus, the dimensionless parameter \( \chi \) is linked to empirical properties of each galaxy. Halo star and GC velocity dispersions would scale as \( \sigma \propto \sqrt{\chi} \). Local densities would scale in proportion to the standard \( \rho_c \). The internal mass distribution still depends non-trivially upon \( q \) or \( Q \) however, which determines whether a particular galaxy halo is nonsingular, highly singular, or any condition in between. If \( \eta = \eta(\chi_\bullet, q) \) were weakly dependent on \( q \) then equation (14) would imply a power-law correlation, \( m_\bullet \propto \sigma^{F/2} \). If \( \eta = \eta(\chi_\bullet, q) \) also has non-trivial variations, the assumption of a universal virial density \( \rho_c \) would not predict \( m_\bullet \) tighter than the ribbon relations in Fig. 6. At least one extra principle is needed.

4.1.2 halo entropy

The total entropy of the DM in the halo is

\[
\text{Figure 2. Top: profile of DM + BH mass enclosed within radius } r. \text{ Bottom: the corresponding velocity dispersion } \sigma/c \text{ vs radius. In all models } F = 9, \text{ and the compactness is galaxy-like } (\chi = 10^{-6}) \text{ but different phase-space densities (annotated). The nonsingular solution is black } (q = 0.177; \text{ the dark envelope around the horizon has density contrast } \psi = 3.0, \text{ and radius factor } \eta = 2.92). \text{ Lower values of } q \text{ (higher entropy) give larger } m_\bullet. \text{ The green curve } (q = 7.0) \text{ has a terraced profile. Vertical ticks in the lower panel mark scale radii } R_2 \text{ to show core sizes.}
\]
Figure 3. Characteristics of the central massive object in a halo with $F = 9.5$, in terms of the compactness ($\chi$) and adjusted phase-space density ($q = QV^F/\rho$). The upper panels show the density contrast at the horizon ($\psi = \bar{\rho}/\rho$); and the correction to Schwarzschild radius ($\eta = r_s/r_h$). Lower panels show the mass and radius fractions ($m_*/M$ and $r_*/R$).

$$S = -Nk \ln(Q/Q_0)$$

where the constant $Q_0$ depends on universal particle properties, $N = (M - m_*)/\mu$ is the number of DM particles, and $\mu$ is the particle mass. The event horizon also contributes entropy, $S_* \approx \pi k (r_*/l_P)^2$ where $l_P$ is the Planck length (Bekenstein 1973). When models are normalised (Appendix B) to the same total mass $M$, the total entropy is

$$S = 4\pi k \left( \frac{M}{m_P} \right)^2 \left( m_*/M \right)^2 \frac{1}{m_*/M} \ln \left( \frac{Q}{Q_0} \right)$$

where $m_P$ is the Planck mass. The left (horizon) term of the equation dominates if $\mu \gg m_P^2/M$ and vanishes if $\mu \ll m_P^2/M$. At fixed $\chi$, the right term is monotonic in $Q$, and so is the left term, except subtle wrinkles within 1 dex of the nonsingular border. Maximal entropy prefers a maximally massive BH with only a tenuous dark envelope. Realistic SMBH scaling relations cannot derive from a simple entropic principle.

4.1.3 energetic constraints

For the $F > 6$ scenarios, some mass profiles are energetically more or less favourable, depending non-trivially on the system parameters. For fixed $M$ and $\chi$, the gravitational potential energy $|W|$ and total energy are extremal at the $\psi$-valley where $q$ is lowest. This is the valley where $m_*/M \gg 0.1$, which is excessive. The black hole mass is significant compared to the envelope, but not dominant. This solution is energetically favoured because DM is mostly concentrated
Figure 4. Maps of the contrast between DM envelope and SMBH \((\log_{10} \psi \text{ shaded})\) across the \((\chi, q)\) configuration space. The \(\psi\)-valleys are the dark streaks. Panels depict haloes with \(F = 6, 7, 8, 9\) as annotated.

Deep in the potential well. At lower \(q\), the black hole dominated profiles \(m_*/M \approx 1\) are less energetically favourable because there isn’t much matter in the tenuous halo. In the medium-\(q\) plateau domain, the configurations are less energetically favourable because the mass is less concentrated.

However, the other \(\psi\)-valleys (where the \(m_*/M\) ratios are more astronomically realistic) are subtle local extrema of \(|W|\). In energetic terms, these states may be locally preferred to adjacent configurations in \((\chi, q)\) space.

It isn’t obvious whether or not these energetically favourable states are effective attractors in galaxy halo evolution. An evaluation of realistic evolutionary tracks in \((\chi, q)\)-space might require Monte Carlo simulations that apply hierarchical mergers to an initial population of primordial mini-haloes. As in toy-model studies of SMBH demographics (e.g. Yu & Tremaine 2002) it would be necessary to assume whether adiabatic agglomeration or mass-energy conservation takes priority during mergers, flybys and fission events. Dark shocks and mixing would introduce inelastic and dissipative factors. These issues are non-trivial and deserve a separate investigation.

4.1.4 landscape of \(\psi\)

The density ratio of the central object to its envelope, \(\psi\), is a diagnostic of the halo solutions. One might wonder whether
Figure 5. Mass fraction of the central object (shaded) in relation to the DM core, for various values of $F$ (as annotated). Use of core compactness ($\chi_c = \chi_2, \chi_3$ or $\chi_m$) instead of global compactness $\chi$ reveals a projection in which the model properties are insensitive to $q$ except near the upper border (nonsingular solutions).

A sensible physical condition involving $\psi$ might select the astronomically realistic models. Large values of $\psi$ imply a central object with a high density contrast to its surroundings. The rare cases with $\psi < 1$ are perhaps unnatural as they would imply an overdense inner envelope, and a density inversion in whatever primal object formed the SMBH seed in the first place. Small values of $\psi \gtrsim 1$ are of special interest, as they imply systems where the inner DM envelope is comparable to the mean density of the central object. In some sense, this implies a SMBH that is maximally blended and coupled with the galaxy halo.

An inner condition $\frac{\partial m_*}{\partial r_*} = \frac{dm}{dr}$ would describe a seamless continuity between the SMBH and its envelope. If the horizon occurred at the Schwarzschild radius ($r_* \approx r_s$ and $\eta \approx 1$) the optimal continuity condition would imply $\psi \approx 3$. When the envelope is massive enough that $\eta \neq 1$ by a significant amount, the seamless condition will prefer another value of $\psi$ (depending on the actual non-Schwarzschild $\frac{\partial m_*}{\partial r_*}$ at fixed $\chi$). Ideally that rule should be calculated from the numerical maps of $\frac{\partial m_*}{\partial q}$ and $\frac{\partial r_*}{\partial q}$. We would expect the special value of $\psi$ to be a number of order three or unity: probably near the minima in the $\psi$-valleys, and not in the large-$\psi$ regions outside the plateau.

The seamless envelope condition is theoretically interesting, but what would it imply about SMBH formation and growth? It might be the natural outcome if DM accretion was the main mass supply to the black hole, either through gradual, adiabatic contraction or violent, supernova-like im-
4.1.5 summary

If haloes share a cosmologically determined mean density, then their individual masses are functions of $\chi$, but the variation of internal structure means this ansatz does not provide unique predictions for $m_\bullet$. Entropy maximisation ideally favours high $m_\bullet/M$, which could not describe a realistic galaxy. Gravitational energy also favours models with $m_\bullet$ too large, but there is a subtler preference for moderate-$m_\bullet$ profiles in the $\psi$-valleys of the $(\chi, q)$ space.

If galaxies tend to evolve to minimise $\psi$ then this implies that a relativistic dark envelope surrounds the SMBH horizon, where local densities of DM could be significant compared to the SMBH mean density. In that case the preferred configurations include: tracks where $m_\bullet/M$ is almost independent of $\chi$ (resembling Magorrian relations); a track of minimum $q$ with unrealistic $m_\bullet/M$; and low $m_\bullet/M$ cases near the maximum $q$ (nonsingular border, minimal entropy). Detailed scaling relations depend on $F$ sensitively.

When the halo is described in terms of properties of its DM core, the plateau region of the parameter space simplifies considerably. In these terms, the $m_\bullet/M_c$ ratio falls within a narrow ribbon that depends on $F$ and $\chi_c$ but only weakly on $q$. The valley and inter-valley solutions shrink together into the same projected region. The ribbons are thinner than the present scatter in observational $m_\bullet$ values, so it is practically almost a one-dimensional $m_\bullet$-$\sigma$ scaling relation. When nearly nonsingular models are allowed, the ribbons in Fig. 6 become strict upper limits on $m_\bullet/M_c$. If we can link the velocity dispersion of halo tracers — such as globular clusters — to the core compactness ($\chi_c = \chi_1$, $\chi_2$, $\chi_3$ or $\chi_m$) then realistic $m_\bullet$ vs $\sigma$ relations emerge. These relations may have some intrinsic scatter, due to the model dependencies of $\sqrt{\eta^3 \psi}$ envelope correction factor. This factor reintroduces some $Q$-dependence, though it is subtle: for $6.5 \leq F \leq 9.5$ and $\chi_c > 10^{-8}$ we find $0.5 < \frac{1}{2} \log_{10}(\eta^3 \psi) < 1.81$ across the plateau region. (ter-
raced haloes, as coloured in Fig. 3. For galaxy-like compactness \( \chi_c \ll 10^4 \), 0.5 \( \lesssim \frac{1}{2} \log_{10}(\eta^3) \psi \) \( < 1.1 \)

4.2 Comparison to observed SMBH

Empirically, the heaviest known ultramassive black holes amount to a few times \( 10^9 m_\odot \) (McConnell et al. 2011, 2012; van den Bosch et al. 2012). The smallest confirmed SMBH are a few times \( 10^6 m_\odot \), residing in bulgeless discs and dwarf galaxies (Filippenko & Ho 2003; Barth et al. 2004; Peterson et al. 2004; Shields et al. 2008; Seth et al. 2010; Reines, Greene & Geha 2013). For the central black hole of a massive elliptical galaxy, \( m_\bullet = 10^9 m_\odot \) and the Schwarzschild radius \( r_s \approx 2.95 \times 10^{14} \) cm \( \approx 7 \times 10^6 \) kpc. For a realistic galaxy-sized halo, this implies \( r_s/R \lesssim 10^{-5} \); and \( m_\bullet/M \lesssim 10^{-3} \) or \( \lesssim 10^{-4} \) assuming a large galaxy with \( \chi \approx 10^{-6} \). We adopt these values as a benchmark.

For haloes with \( 0 \leq F \leq 6 \), astronomically realistic ratios of \( m_\bullet/M \) only exist very close to the border of nonsingular models. A small relative amount of heating (e.g. dissipative effects of a tidal flyby) could induce a significant jump in \( m_\bullet \), unless the compactness \( \chi \) also changes. In the thin band of \( (\chi, q) \) states where \( 6 \leq F \) is compatible with SMBH scaling trends, the density contrast between the hole and its dark envelope tends to be immense (\( \psi \gg 10^{10} \)). This means that when \( F \leq 6 \), an astronomically realistic BH is much denser than its surroundings, and effectively decoupled from the ambient halo pressure. It would be necessary to invoke elaborate, mundane non-DM physics to explain the observed correlations. The \( 6 < F < 10 \) regime however enables observationally plausible \( m_\bullet/M \) values throughout the small-\( \psi \) plateau and valleys, as well as near the nonsingular border. Many orders of magnitude are available in \( q \) and pseudo-entropy \( s \). Incrementally heating a galaxy halo need not be catastrophic for SMBH growth.

Observational constraints are met in the \( \psi \)-plateau when \( 6 < F < 10 \) (for \( r_s/R \) and \( 7.5 \lesssim F < 10 \) (for \( m_\bullet/M \)), according to our Fig. 4). This then is an observationally favoured range of DM microphysics. We typically find \( \psi < 100 \) in the best models. This entails a dark envelope with gravitationally significant density near the event horizon. This envelope declines radially as a nuclear ‘spike’, though more steeply than the low-\( F \) spikes of previous modelling (Gondolo & Silk 1999; Mouawad et al. 2004; Hall & Gondolo 2006; Zakharov et al. 2010). For large \( F \), the spike’s steepness makes the combined DM envelope plus BH appear (from afar) as if it were a more massive black hole.

To an accuracy comparable to the present observational scatter, it is useful to represent the mass trend as a power-law, \( m_\bullet/M_c \sim M_c^{-\beta} \). If we assume that galaxy haloes share a nearly universal cosmic mean density (Section 4.1) \( M_c \sim \sqrt{\chi_c^2/\rho_c} \) then the expectation is \( m_\bullet/M_c \sim \chi_c^{3(\beta-1)/2} \). In our numerical results, the domain \( 6 < F < 10 \) ensures \( 1 < \beta < 2 \); while \( F < 6 \) only gives solutions near the nonsingular limit (\( \beta = 1 \)). Observations of \( m_\bullet \) in local galaxies and AGN (Laor 2001) show \( \beta = 1.54 \pm 0.15 \). In our Fig. 4 this would correspond to slopes of \( \partial \ln(m_\bullet/M_c)/\partial \ln \chi_c = 0.81 \pm 0.23 \), which graphically is consistent with the ribbons of higher \( F \) cases. Bandara, Crampton & Simard (2009) modelled the strong gravitational lensing effects of a set of elliptical galaxies that also have \( m_\bullet \) estimates. They found a correlation that implies \( \beta = 1.55 \pm 0.31 \) or \( \beta = 1.57 \pm 0.39 \) depending on their fitting methods. Our equivalent ribbon slopes would be \( \partial \ln(m_\bullet/M_c)/\partial \ln \chi_c = 0.82 \pm 0.47 \) or \( 0.86 \pm 0.59 \). These constraints are lax, but would seem to prefer \( F \gtrsim 7 \).

In our model, the predicted ratios \( m_\bullet/M \) refer to \( M \) as the halo mass, not the stellar bulge (\( M_\bullet \)). Peculiar galaxies observed with high \( m_\bullet/M_\star \) (Bogdán et al. 2012; van den Bosch et al. 2012) could be normal products of the SMBH-halo relationship, but impoverished in stars and gas for some other reason. Alternatively, if they are genuinely overweighed in \( m_\bullet/M \) terms, they might be high-enteropy outliers: low \( q \) or high \( \chi \) due to an unlucky history of tidal buffeting or other halo heating processes.

Our model also has implications for the presence of intermediate mass black holes (IMBH; \( 10^3 \lesssim m_\bullet/m_\odot \lesssim 10^5 \)) in the least massive systems. Based on velocity dispersions, escape velocities and tidal radii, ultra-compact dwarf and faint dwarf galaxies could have compactness parameters \( \chi \ll 10^{-7} \). If they bind substantial amounts of DM then the \( \psi \)-plateaus of \( F > 6 \) haloes set upper limits on \( m_\bullet/M \) that are rather low (left extreme of Fig. 4). In a system amassing \( M = 10^6 m_\odot \), the ‘plateau’ configurations of \( F = 7, 8, 9 \) haloes with \( \chi \approx 10^{-8} \) predict a maximum central object of \( m_\bullet \approx 10^{4.2}, 10^{2.6} \) and \( 10^{1.0} m_\odot \) respectively. For objects with \( \chi \approx 10^{-7} \), these \( F = 7, 8, 9 \) models give \( m_\bullet \approx 10^{5.2}, 10^{2.5} \) and \( 10^{1.0} m_\odot \) respectively. This object could be a stellar black hole, rather than an IMBH. If there is a non-trivial central stellar density, then the predicted central mass is lost amidst stellar granularity, and the model breaks down. Even if an IMBH were formed, there are plausible processes that might remove it: the ‘gravitational rocket’ effect during high-spin black hole mergers; random walks due to scattering in dense stellar environments; random walks due to momentary imbalances between the thrusts of two jets during a gas accretion episode. The rarity or non-observation of IMBH in dwarf galaxies and GC is unsurprising.

4.3 Possible observational tests of the dark matter envelope

The presence of a DM envelope around a black hole contributes to the gravitational potential, which will produce observational consequences. Here we list a few examples.

(i) The gravitational potential of the DM envelope will cause stellar orbits to deviate from the Keplerian orbits that are expected for motion around a bare spherically symmetric gravitating object (Rubilar & Eckart 2001; Hall & Gondolo 2006; Mouawad et al. 2003; Zakharov et al. 2007; Ghez et al. 2008; Will 2008; Zakharov et al. 2011). A possible means to detect this deviation is timing observations of pulsars, if present, around the central black holes in nearby galaxies (Wex & Kopeikin 1999; Pfahl & Loeb 2004; Kramer et al. 2004; Liu et al. 2012; Singh, Wu & Sarty 2014).

(ii) Stars with non-circular orbits traversing the DM envelope around a black hole would experience a gentler tidal-force gradient than around a bare BH of equal total mass. In a stellar tidal disruption process (see Rees 1988; Komossa 2002; Bloom et al. 2011; Saxton et al. 2012) the stellar debris tracks would have morphologies different to those resulting from a rapid change in the tidal force field.
(iii) AGN are powered by accretion of gas into a massive black hole. The inner accretion disc region unleashes most of the accretion power, in the form of radiation and outflows. For objects orbiting around a black hole, there is a well-defined innermost stable circular orbit (ISCO). This orbit is assumed to be the inner boundary of the accretion disc, because beyond that, the infall matter plummets towards the horizon without having time to dissipate and radiate energy. X-ray emission line profiles are often used as a diagnostic of space-time properties and conditions near the inner-disc radius (see e.g. Fabian et al. 1989; Stella 1990; Laor 1991; Fabian et al. 2000; Fuerst & Wu 2004; Younsi, Wu & Fuerst 2011). However, the ISCO location does not have a simple analytic solution when a massive DM envelope is present. The gravity of the DM envelope modifies the accretion flow dynamics, and hence the thermodynamics and radiative properties of the inner disc. Accretion discs around black holes in the presence and in the absence of a massive DM envelope would show different spectral profiles (cf. Joshi, Malafarina & Narayan 2011, 2014; Bambi & Malafarina 2013). Thus, black hole parameter estimates derived without accounting for DM could give incorrect results.

(iv) Interferometric imaging of SMBH in nearby galaxies will be possible with the development of the Event Horizon Telescope (EHT 2 and the Greenland Telescope (GLT) 3. A SMBH that is heavily enveloped by DM might show an ‘event horizon’ shadow smaller than that expected from stellar-kinematic mass deductions (cf. Falcke, Melia & Agol 2000; Nusser & Broadhurst 2004; Doeleman et al. 2008).

4.4 Dark matter physics and microphysics

There are many theories of DM physics that can viably describe the gravitational fields of galaxy haloes. At galaxian scales, the only essential requirements are that the unknown material is electromagnetically invisible and has no discernible effect on nucleosynthesis or the stability of normal stars. Since halo shapes are spheroidal, the DM seems unable to lose energy as readily as the radiatively cooling gas in classic astrophysical discs (although see Fan et al. 2013).

Often DM is simply assumed to be collisionless: practically an invisible self-gravitating dust. This provides an easy prescription for cosmological simulations employing N-body methods. However, observations do not confirm the predicted density cusps (see Section 1). Simulations also over-predict numbers of dwarf galaxies (Klypin et al. 1999; Moore et al. 1999; D’Onghia & Lake 2004; Tikhonov & Klypin 2006; Zwaan, Meyer & Staveley-Smith 2010; Klypin et al. 2014), and dense large satellites that are unseen in reality (Boylan-Kolchin, Bullock & Kaplinghat 2011, 2012; Miller et al. 2014; Tollerud, Boylan-Kolchin & Bullock 2014; Kirby et al. 2014; Garrison-Kimmel et al. 2014). The question then is: what variety of modification or alternative theory is necessary? Suppose that some process drives the DM phase-space distribution function to become locally isotropic and proportional to a power of the single-particle energy, $f \propto \left[ -E \right]^{(P-3)/2}$. Then a polytropic relation (1) emerges $f \propto \left[ -E \right]^{(P-3)/2}$. In collisionless DM simulations, the cuspy haloes have $Q$ following a power of $r$ when assuming $F = 3$ (Taylor & Navarro 2001; Ludlow et al. 2011), which implies a constant-$Q$ singular polytrope for some non-integer $F$ value. For those models, the severest challenge is to explain why kpc cores occur in real haloes. Driving processes might involve shaking by an elaborate baryonic feedback (e.g. Peirani, Kay & Silk 2008), or collective phenomena similar to bar-mode instabilities. This needs fine-tuning to achieve a realistic core radius. When simulations invoke ad hoc feedback recipes, these can be decisive or ineffectual, depending on numerical implementation (e.g. Governaro et al. 2010; Vogelsberger et al. 2014).

The polytropic condition is claimed to be a natural equilibrium for self-gravitating systems, according to the Tsallis (1988) conjecture of extended thermostatistics. Collisionless spheres may settle as ‘stellar polytropes’ (Plastino & Plastino 1993; Vignat, Plastino & Plastino 2011). Our parameter $F$ is linearly related to Tsallis’ extensivity parameter, which is a non-integer. Férón & Hiorth (2008) find that stellar polytropes are a poor representation of cuspy haloes that emerge in numerical simulations. However, this is not a fatal criticism of the thermostatistical models, since real observed galaxies have cored (not cuspy) profiles.

Another possibility is that DM is adiabatic and self-interacting (SIDM). The polytropic equation of state $F$ arises from basic thermodynamics, in the absence of complications such as phase changes. SIDM interactions may consist of direct interparticle scattering, short-ranged Yukawa interactions or long-ranged dark forces analogous to magnetism (e.g. Spergel & Steinhardt 2000; Ahn & Shapiro 2001; Backley & Fox 2010; Ackerman et al. 2002; Loeb & Weiner 2011). If the fluid consists of point-like particles with only translational motions then $F = 3$. This is a common, unquestioned assumption, algorithmically built into many simulations of weakly interacting SIDM (Moore et al. 2000; Yoshida et al. 2000; Davé et al. 2001; Vogelsberger, Zavala & Loeb 2012; Rocha et al. 2013; Peter et al. 2013; Vogelsberger et al. 2014). However, if DM has additional internal energy then $F > 3$, e.g. $\frac{1}{2} k T$ for each degree of freedom of rotational kinetic energy of ‘dark molecules’. For diatomic dark molecules, $F = 5$. The $F$ value increases if DM particles have more composite complexity. Independently, some efforts to reconcile direct detection experiments invoke composite or inelastic DM (e.g. Smith et al. 2001; Chang et al. 2009; Alves et al. 2010; Kaplan et al. 2010, 2011). The astroparticle physics implications are increasingly recognised (e.g. Cline et al. 2013, 2014; Boddy et al. 2014). This possibility is inherently beyond the scope of N-body codes. The quantity $F$ might effectively vary in some DM theories that yield pressure anisotropies and/or a more complicated equation of state (e.g. Sobott, Hasani Zonoozi & Haghi 2009; Harko & Lobo 2011, 2012). For now, we assume constant $F$.

The nature of SIDM is still under debate. It is possible that SIDM is not a gas but a scalar field or boson condensate (e.g. Sin 1994; Ji & Sin 1994; Lee & Koh 1996; Hu, Barkana & Gruzinov 2000). A polytropic equation of state can be obtained from some boson models, with

2 http://www.eventhorizontelescope.org/
3 http://www.cfa.harvard.edu/greenland12m/
the value of $F$ depending on the self-coupling potential in the lagrangian. Many works assume $F = 2$ with $s$ and $Q$ fixed universally by particle properties (e.g., Goodman 2000; Arbey et al. 2003, 2004; B"ohmer & Harko 2007; Harko 2011; Chavanis & Delfini 2011), but other $F$ values are possible (Peebles 2000). It was also suggested that phase changes can occur in bosonic DM and this would alter the spatial variations in the properties of large astrophysical objects (see e.g., Arbey 2006; Blobig & Goodman 2012).

Alternatively, DM may consist of neutral fermions (Dodelson & Widrow 1994). Warm DM made of sterile neutrinos ($\sim 1$–7 keV mass range, depending on the primordial particle distribution) might decay, producing X-ray emission lines (Boevsky et al. 2014; Bulbul et al. 2014). In this case, a degenerate phase can act as a cored polytropy with $F = 3$ (e.g., Mynaveza & Biermann 2004; Richter, Tupper & Violliet 2006; Chan & Chu 2005, 2006; Richter, Tupper & Violliet 2006; Arbey, Lesgourgues & Salati 2003; B"ohmer & Harko 2007; Harko 2011; Chavanis & Delfini 2011), but other $F$ values are possible (Peebles 2000). It was also suggested that phase changes can occur in bosonic DM and this would alter the spatial variations in the properties of large astrophysical objects (see e.g., Arbey 2006; Blobig & Goodman 2012).

4.5 Globular clusters as a tracer

Globular clusters (GC) inhabit the host galaxy’s halo and provide a useful physical probes where other visible tracers are rare. The GC swarm diminishes with distance from the core, but can also develop central deficits (Capuzzo-Dolcetta & Mastrobuono-Battisti 2000). GC consist of uniformly old and metal-poor stellar populations. They appear to lack DM of their own; stellar mass suffices to explain the internal kinematics (e.g., Heggie & Hut 1993; Baumgardt et al. 2003; Sollima et al. 2009; Lane et al. 2010; Conroy, Loeb & Spergel 2011; Hanelk & Ce1 2011; Bradford et al. 2011; Sollima, Bellazzini & Lee 2012; Ibata et al. 2013).

GC formation was either a purely baryonic process, or else their miniature DM haloes were ablated later. The oldest GC apparently formed in brief single starbursts comparable to a dynamical time of the proto-galaxy, perhaps caused by thermal instabilities or shock compressions of clouds in the halo (e.g., Searle & Zinn 1978; Fall & Rees 1983). Newer (metal rich) GC may form from shocked gas in wet mergers (Ashman & Zepf 1992; Zepf & Ashman 1993; Whitmore & Schweizer 1993; Hancock et al. 2009; Whitmore et al. 2009; Smith et al. 2014). Dry mergers of galaxies combine preexisting GCs and preserve the ratios of SMBH, stellar and GC masses. GC on radial orbits traversing the inner galaxy can be destroyed by tidal shocking (e.g., Ostrikov, Sperit & Chevalier 1972; Fall & Rees 1977; Gnedin & Ostriker 1997; Gnedin, Lee & Ostriker 1999; Fall & Zhang 2001). Compared to ellipticals, disc galaxies seem more efficient at GC destructions or less efficient GC formers (e.g., Harris 1988; Georgiev et al. 2010). The surviving GC population depends on: the primordial baryonic mass endowment; the subsequent formation and destruction processes; and the breath and depth of the halo potential binding GC to the galaxy. By the virial theorem or Jeans modelling, the radial velocity dispersion of the GC system is proportional to the depth of the halo potential.

Since the GC swarm traces aspects and properties of the whole galaxy halo, it is significant that GC observables correlate with the SMBH ($m_*$) (Spitzer & Forbes 2004; Burkert & Tremaine 2014; Harris & Harris 2011; Harris, Harris & ALESSI 2013; Snyder, Hopkins & Hernquist 2011) interpret the SMBH-GC correlations as consequences of the depth of the galaxy bulge’s gravitational potential. Sadoun & Colin (2012) relate the velocity dispersion of the GC system, $m_* \sim \sigma_{\text{GC}}^3$. 
with $\beta = 3.78 \pm 0.53$. \text{Pota et al.} \ (2013) also linked $m_*$ with $\sigma_{gc}$ ($3 \lesssim \beta \lesssim 6$ or $\beta \approx 4.45$ on average); and \text{Rhode} (2012) found $\beta \approx 5.3$ or 5.9. These $\sigma_{gc}$ relations have great implications. This correlation could be evidence of a link between SMBH formation and the halo properties, not merely the properties of the stellar bulge. The stronger the $m_*-\sigma_{gc}$ relation is, the less likely that these components are controlled by BH feedback, and the more likely that it depends somehow on the underlying DM potential.

\text{Burkert \\ & Tremaine} (2010) and \text{Rhode} (2012) have a different interpretation: attributing the correlation to the effect of mergers later on (more mergers produce more GCs and a bigger SMBH). We suggest that the correlation would not be so tight if the individual merging blocks did not already have a correlation on their own. Furthermore, mergers cannot have been the controlling process in bulgeless thin-disc galaxies that host a SMBH but have never experienced a major merger (Section 4.7). Mergers cannot be the universal explanation. Instead we propose that the halo controls the SMBH origin and the GC properties separately. In each large galaxy, there will be a fraction of large GCs produced \textit{in situ} during the initial collapse, and a fraction coming later from the disruption of nucleated satellite galaxies. Stellar populations and orbital kinematics are usually clues to which is which (e.g. M54 and $\omega$ Cen may be satellite accretions). It would be interesting to predict the implications if local GCs are those formed without a DM potential well, and those coming from accreted galaxies are formed at the bottom of that galaxy’s DM potential, perhaps even with their own nuclear black holes.

### 4.6 SMBH formation and accretion

Our equilibrium configurations do not distinguish how the central object originated. We simply have a non-evolutionary description of the endpoint after the inner halo attains approximate pressure balance. Our model $m_*$ limits do not apply while a system is dynamically disturbed, asymmetric and evolving into another state. However the most realistic equilibrium solutions tend to have small $\psi$ values, meaning that a dark envelope is a significant presence around the horizon. This suggests that DM accretion may be relevant to SMBH seeding and growth. We are aware of at least three scenarios. Steady growth is possible via \text{Bondi} (1952) accretion of fluid (e.g. \text{Munyanzea \\ & Biernann} 2006; \text{Richter, Tupper \\ & Voilier} 2006; \text{Peirani \\ & de Freitas Pacheco} 2008; \text{Guzmán \\ & Lora-Clavijo} 2011a,b; \text{Pepe, Pellizza \\ & Romero} 2012; \text{Lora-Clavijo, Gracia-Linares \\ & Guzman} 2014) or gradual capture of collisionless orbiting particles accompanied by core-loss refilling (e.g. \text{Peebles} 1972; \text{Ullo, Zhao \\ & Kamionkowski} 2003; \text{Vasiliev \\ & Zelhnikov} 2008). If the dark matter self-interactions are weak (with a kpc-sized mean-free-path) but heat conduction is significant then gravothermal instability could form a SMBH (\text{Ostriker} 2001; \text{Hennawi \\ & Ostriker} 2002; \text{Balberg \\ & Shapirol} 2002; \text{Balberg, Shapiro \\ & Inagaki} 2002). If SIDM is a fluid with $F > 6$ then collapse may proceed via a localised gravitational instability in a discrete ‘dark gulp’ lasting a dynamical timescale of the nucleus (\text{Saxton \\ & Wu} 2008, 2014). The gulped dark mass could be an appreciable fraction of the SMBH total.

Initiating this process may require a steep central density gradient. BH seeding is probably helped if there is already a steep spike of stars or accumulation of inflowing gas. It may be necessary for baryons to become denser than some threshold, in order to pinch the DM (via adiabatic contraction. \text{Blumenthal et al.} 1986) and enable collapse of the innermost DM. Perhaps this pinching can partly explain the observed correlations between SMBH and the Sersic index of the stellar surface brightness profile (\text{Graham et al.} 2001; \text{Graham \\ & Driver} 2007; \text{Savorgnan et al.} 2013). Evaluating the collapse thresholds needs multi-component stability analyses, like \text{Saxton} (2013) but with a density spike.

Some comparisons of the mass function of the local SMBH population with the AGN and quasar luminosity distribution were consistent with most of the current SMBH mass coming from radiatively efficient gas accretion (\text{Soltan} 1982; \text{Salucci et al.} 1991; \text{Yu \\ & Tremaine} 2002; \text{Shankar et al.} 2004; \text{Shankar, Weinberg \\ & Miralda-Escudé} 2000). This does not invalidate our proposed scenario. If these audits of light and mass are complete, they are still consistent with an initial relation between bulge mass and seed BH mass, in which the latter could have been $\lesssim 10^{-4}$ bulge mass and much less than the final SMBH mass. That situation corresponds to $\chi \lesssim 10^{-6}$ in the $F \gtrsim 8.5$ halo models. Spatially, $R \sim 10^{12} r_*$ would be a plausible size for a seed black hole, as $r_* \lesssim 10^3$ cm for large galaxies with $R \sim 10$s of kpc ($\sim 10^{23}$ cm). These seeds could have condensed according to our predicted scaling index, $m_* \sim \sigma^{F/2}$, and then grown through (\text{Eddington} 1918) limited luminous accretion of gas. The final observed black-hole mass would be $10^8$ times the seed mass, after $\approx \pi$ Salpeter timescales.

The scaling relations would rise in normalisation but retain the original slope: $m_*^* = 10^n m_* \sim \sigma^{1.5}$ (if $F \approx 9$). The index of SMBH scaling is preserved from our simplistic gasless halo model.

Note that there are always uncertainties and complications in the accounting of total SMBH mass and radiative efficiency of their growth. For example, recalling SMBHs can escape their galaxies after a merger (e.g. \text{Redmount \\ & Rees} 1989; \text{Menou, Haiman \\ & Narayanan} 2001; \text{Haiman} 2004; \text{Madau \\ & Quataert} 2004; \text{Baker et al.} 2004; \text{Gonzalez, Zaritsky \\ & Zabludoff} 2007; \text{Campanelli et al.} 2007a,b; \text{Schnittman \\ & Buonanno} 2007; \text{Lousto \\ & Zlochower} 2011, 2013), and end up dormant in intergalactic space: in that case, simple counts of nuclear SMBHs would underestimate the total cosmic BH mass. The local SMBH density may also have been underestimated if there is a previously unrecognized population of SMBHs in ultracompact dwarf galaxies (\text{Seth et al.} 2014). The presence of ultramassive BHs may require more accretion (via radiatively inefficient modes) than reckoned before (\text{McConnell et al.} 2011, 2012; \text{van den Bosch et al.} 2012; \text{Fabian et al.} 2013). The discovery of modern-sized quasars at high redshift is likely to require a faster early growth than allowed by Eddington-limited luminous accretion (\text{Fan et al.} 2004; \text{Shapiro} 2005; \text{Mortlock et al.} 2011; \text{Venemans et al.} 2013); on the other hand, the X-ray background from high-redshift AGN is dimmer than expected, contradicting rapid radiatively-efficient gas accretion in the $z > 5$ era (\text{Willott} 2011; \text{Salvaterra et al.} 2012; \text{Treister et al.} 2013). The radiative efficiency of quasar accretion may be lower than the standard $\eta \approx 0.1$ disk efficiency during.
super-critical gas accretion phases (Novak 2013) or due to DM accretion; however, accretion can instead appear more radiatively efficient than $\eta \sim 0.1$, for example if it taps into the BH spin (Narayan, Igumenshchev & Abramowicz 2002; Igumenshchev 2008; Tchekhovskoy et al. 2014; Lasota et al. 2014) or if DM envelope dominates the inner potential. Finally, SMBHs might grow via BH-BH coalescence without any radiative emission; however, constraints set by the cosmic gravitational-wave background imply that steady accretion (of gas or DM) dominates (Shannon et al. 2013). Thus, the issues of how early SMBHs were seeded, the role of DM in setting the seed mass (e.g. Mack, Ostriker & Ricotti 2007; Dotan, Rossi & Shaviv 2011; Lora-Clavijo, Gracia-Linares & Guzman 2014) and the DM mass contribution are far from settled.

Our halo model has some similarities to the supermassive star scenario that aims to explain the early SMBH seeding. The proposal is that a $\gtrsim 10^4 m_\odot$ polytropic sphere of gas (e.g. Hoyle & Fowler 1963; Iben 1963; Fowler 1964; Shibata & Shapiro 2002) burns and collapses to produce a seed BH that is born supermassive, thereby reducing the feeding time needed to reach observed SMBH scales (e.g. Begelman, Volonteri & Rees 2006; Begelman 2010; Johnson et al. 2013). The main doubt about this scenario is that the gas may not collapse into a single supermassive object, and may instead fragment into clumps and star clusters because of its angular momentum. Even if a single supermassive star were formed, it may not survive long enough to develop a core and collapse into a single black hole, due to mass losses in intense winds. Our model would create MBH seeds from polytropic DM instead. Eddington limits and winds do not apply to SIDM seeding. Whichever way real SMBH originated, we expect a scaling like $m_\bullet \sim \sigma^2/2$ to emerge from the direct or indirect coupling of the SMBH and halo in equilibrium, since the equilibrium state is independent of what fed the SMBH previously.

4.7 Late-type galaxies

It has long been a puzzle to explain why ellipticals, lenticulars and early-type spirals have a nuclear SMBH, while many late-type spirals have a nuclear star cluster but no SMBH. M33 and NGC205 are local examples of the latter (Gebhardt et al. 2001; Merritt, Ferrarese & Joseph 2001; Valluri et al. 2005). Even more puzzling is the fact that the nuclear star cluster mass versus $\sigma$ relation runs parallel to the $m_\bullet$ scaling relations (Graham & Spitler 2009; Graham 2012c; Ferrarese et al. 2006). On the other hand, some bulgeless galaxies do possess a nuclear SMBH (e.g. Filippenko & Ho 2003; Peterson et al. 2005; Shields et al. 2008; Araya Salvo et al. 2012; Simmons et al. 2013; Reines, Greene & Geha 2013).

Salucci et al. (2000) observed that late-type galaxies have SMBH that are undersized compared to the usual trend with bulge mass ($M_\bullet$). It is arguable those galaxies only have pseudo-bulges (evolved quiescently from the disc via secular processes), whereas SMBH correlate with classical bulges (Kormendy & Bender 2011). Alternatively, perhaps the $m_\bullet$ relation bends downwards in the low-$M_\bullet$ domain (Graham 2012c; Scott, Graham & Schombert 2013) and the SMBH relation to $\sigma$ is straighter. This hints that $\Phi$ plays the fundamental role, consistent with our thesis linking the SMBH to the halo. Either way, the hints of some dependence on luminous morphology (besides the DM halo) deserve an explanation within our theory.

It is worth noting some exemptions from the SMBH mass prediction of equations (14) and Appendix A: If the velocity dispersion $\sigma$ is non-relativistic everywhere in the profile, then there need not be an event horizon at the centre. A nonsingular halo does not grow any central compact mass. This is the lowest entropy condition available. We propose that protogalaxies condensed in this initial state, and some would grow quiescently (without major mergers or gas expulsions) till the present epoch. Those are tranquil disc galaxies, near the nonsingular border, lacking classical bulges, and having undersized SMBH or none at all. For other galaxies, tidal harassment or minor mergers would raise the entropy (lowering $q$), inducing a more centrally peaked density profile. Perhaps if the central DM becomes concentrated enough, a seed BH forms. Subsequent large-scale gas inflows accrete onto the SMBH in a quasar phase. These galaxy haloes enter the ‘plateau region’; they follow the maximum $m_\bullet/M$ scaling relation. For those that suffer more major mergers, the luminous disc converts partially into a classical bulge, or totally into an elliptical. In contrast, for the undisturbed, high-$q$, nonsingular galaxies, if the inner halo never became dense enough, it does not form the initial black hole, and the same large-scale gas inflows produce a nuclear star cluster. The mass in this nuclear star cluster is comparable to the baryonic mass that would have fed the SMBH. We speculate that the knee in the $M_\bullet$ correlation (Graham 2012b) or the underweight SMBH of late-type galaxies (Salucci et al. 2000) may occur:

(i) because the latest-type galaxies are near the high-$q$ nonsingular border and their $m_\bullet/M$ is below the relations in Fig. 6 or
(ii) because these galaxies are near one of the knees in a ribbon relation such as those in Fig. 6 or
(iii) the bulge is incidental and the halo determines $m_\bullet$

Though it is beyond the scope of our spherical modelling, we speculate that the angular momentum of the halo and gas may also affect the outcome. If the inner halo possesses too much angular momentum (and cannot shed it via large-scale dark turbulence) then rotational support inhibits collapse. If the baryons have effective rotational support, then they may not achieve the central densities needed to trigger the inner halo to condense a seed. The result is a pure disc galaxy without a central black hole.

4.8 Stellar components

Our gasless and starless model is a simplification. In principle, a galaxy’s stellar mass distribution affects the SMBH/halo equilibrium to some extent. In galaxy clusters, Saxton & Wu (2008, 2014) found that the continuity requirements of gas inflows impose lower limits on $m_\bullet$, however inserting a central galaxy’s stellar profile did not alter these constraints greatly. An isolated elliptical galaxy’s stellar spheroid compresses the dark core slightly (Saxton & Ferreras 2010; Saxton 2013). Nonetheless DM always dominates in the outer halo. DM should also dominate baryons at the centre: within the innermost stellar orbit,
and perhaps throughout the SMBH sphere of influence. Visible matter is most influential at medium radii (kpc for an elliptical galaxy).

Our present models omit stellar profiles, as we are most interested in the link between the DM halo and the SMBH. Because observations already show that these properties correlate, we suspect that the stellar mass does not dominate SMBH scaling relations outright. This motivates our comprehensive exploration of baryon-free configurations. Our model has two components and three key parameters: thermal degrees of freedom ($F$), compactness ($\chi$) and entropy ($s$, via $Q$ and $F$). Adding one more density component will increase the complexity of the formulation, if we want a self-consistent treatment. This topic is worth a separate study, and we intend to resume it elsewhere. However, we would also like to comment qualitatively here. The addition of a stellar spheroid entails three more free variables: total stellar mass ($M_\star$), a half-light radius ($R_e$) and Sersic shape index ($n$), vastly increasing the system’s dimensionality. We ran restricted tests of $F = 9$ models where the stars comprise 10% of the mass. In a preliminary way, we note:

(i) If the stellar component is compact ($R_e \ll R$) it exerts little influence on the scaling relations. This is understandable since this bulge behaves somewhat like a central concentrated point, which is effectively the same as the SMBH.

(ii) For any terraced or singular model, the DM dominates at a sufficiently small radius. The stellar component is also sub-dominant at the radius of halo core and outskirts beyond $r \gg$ kpc. The stellar potential only perturbs the DM density profile locally at intermediate radii.

(iii) Theoretically, the worst scenario is when the DM and the stellar component have similar compactness. Even then, we find that the basic conclusion holds, except that $\psi$ and $m_\star/M$ values shift across the parameter plane. This shift is only significant near the nonsingular border. We will leave the detailed discussion for our next paper.

The robustness of SMBH vs halo scaling relations, in spite of a stellar contribution and medium radii, might be foreseeable on qualitative grounds. The halo core depends on heat capacity and entropy. The location of the event horizon (which sets $m_\star$) coincides with an effectively universal maximum of $\sigma^2$. Both defining structures depend straightforwardly on the gravitational potential and velocity dispersion, which are linearly related. Their correlation arises naturally. Essentially and generally, the empirical SMBH scaling relations reveal how density $\rho$ is stratified with respect to potential $\Phi$ in galaxies.

5 CONCLUSIONS

We investigate the properties of spherical, adiabatic self-gravitating systems with the DM microphysics prescribed by an equation of state. These systems form a halo of DM and a central compact object. We have found that the halo profile is determined by two necessary parameters. One possible combination is the gravitational compactness $\chi$ (equation 10) and a measure of (pseudo-)entropy $s$ (or equivalently the phase-space density $Q$). Characterisation of such halo profile in terms of a single parameter (e.g. asymptotic or peak circular velocity: see Ferrarese 2002).

ACKNOWLEDGMENTS

We thank the referee, P. Salucci, for helpful criticisms and suggestions that improved the scope and focus of our results and commentary. We thank A.W. Graham for discussions of SMBH scaling, and M. Cropper for discussions of GC populations. This work has made use of NASA’s Astrophysics Data System. Our calculations employed mathematical routines from the GNU SCIENTIFIC LIBRARY. This publication has made use of code written by James R. A. Davenport. Specifically, the figures’ colour scheme was developed by Green (2011).
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APPENDIX A: SMBH PREDICTION IN
ABSOLUTE TERMS

The equation (13) for the SMBH mass can be written in
various absolute units for practical applications. The choice
of units depends on context. For example, in the vicinity of
the dark envelope and the circumnuclear region, velocity
dispersions are almost relativistic. DM densities could become
comparable to that of baryonic matter on Earth. In units
suiting that environment, the SMBH mass in solar units is

\[ \frac{m_\bullet}{m_\odot} \approx \frac{4.2919 \times 10^9}{\sqrt{\eta \psi}} \left( \frac{F + 2}{1 - \chi} - \sigma \frac{\rho}{100 \text{ km s}^{-1}} \right) \]  

(A1)

Further out, in the kpc-scale core of the galaxy’s halo,
typical velocities drop to the order of 100 km s\(^{-1}\),
DM core densities are multiples or fractions of 1 m\(_\odot\) pc\(^{-3}\). In these
terms, the predicted central mass (solar units) is

\[ \frac{m_\bullet}{m_\odot} \approx \frac{1.6495 \times 10^9}{\sqrt{\eta \psi}} - 0.018264 \left( \frac{F + 2}{1 - \chi} \right) \left( \frac{1 \text{ m}_\odot \text{ pc}^{-3}}{\rho} \right) \left( \frac{\sigma}{100 \text{ km s}^{-1}} \right)^{\frac{1}{2}} \]  

(A2)

An equivalent logarithmic form says

\[ \log_{10} \left( \frac{m_\bullet}{m_\odot} \right) \approx 19.217 - 1.7384 F + \frac{F}{4} \log_{10} \left( \frac{F + 2}{1 - \chi} \right) - \frac{1}{2} \log_{10} \left( \frac{\rho}{1 \text{ m}_\odot \text{ pc}^{-3}} \right) - \frac{1}{2} \log_{10} \left( \frac{\sigma}{100 \text{ km s}^{-1}} \right) \]  

\[ + \frac{F}{2} \log_{10} \left( \frac{\sigma}{100 \text{ km s}^{-1}} \right) \]  

(A3)

The third term on the right side is < 2.7 when \( \chi \ll 1 \). The
astronomical mass range \( m_\bullet \lesssim 10^{10} m_\odot \) implies that either
\( F > 6 \) (in the second term of the right side) or there is a
large correction factor \( \eta \psi \) (in the fourth term on the right).

APPENDIX B: MODEL HOMOLOGIES AND
SCALE-IN Variant PARAMETERISATION

Given a particular polytropic halo model, a family of
homologous models can be formed by multiplying each quanti-
yty \( y \) by a scale factor \( X_y \). Since we take the speed of
light as an absolute reference scale for velocity dispersions,
escape velocities and gravitational potentials, we necessar-
ily have \( X_\sigma = 1 \), \( X_V = X_m/\chi_\tau = 1 \) and \( X_\psi = 1 \). It
follows that model masses and distances must rescale by
the same factor, \( X_m = X_c \equiv X \), and densities rescale as
\( X_\rho = X_m/X_c^3 = X^{-2} \). The phase-space density rescales as
\( X_\rho = X_\rho/X_c^3 = X^{-2} \). For example, if we choose to stan-
dardise a set of models so that they have the same total
mass \( M \), we could transform the phase-space densities as
\( Q \to Q/R^2 \).

We prefer to classify and compare models in terms
of their dimensionless properties that remain constant un-
der the homology transformations. Dimensionless quanti-
ties such as \( \chi, \eta \) and \( \psi \) remain constant under the homol-
gy transformations. If the outer boundary conditions are
known, then it is possible to define a dimensionless variable
related to \( Q \), for instance \( q \equiv Q V/F/\rho \) for which \( X_q = 1 \).
Similarly, \( \varpi \equiv M^2 Q \) for which \( X_\varpi = X_\tau = 1 \). The properties
of the central object are best described in terms of invariant
fractional quantities such as the \( m_\bullet/M \) and \( r_\bullet/R \).

APPENDIX C: ENTROPY CALCULATION

The Bekenstein (1973) entropy of an event horizon is
\( S_\bullet = kA/(4\rho)^2 \) where \( A \) is the area surface, \( k \) is Boltzmann’s con-
stant, \( \rho \) is the Planck length, and \( m_\bullet \) is the
Planck mass. Substituting the area of the inner boundary
of our model, \( A \approx 4\pi r_\bullet \), we have \( S_\bullet = \pi k (r_\bullet/r_\Psi)^2 \),
which simplifies:

\[ S_\bullet = \pi k \left( \frac{c^2 r_\bullet}{G m_\bullet} \right)^2 = \pi k \left( \frac{c^2 2GM m_\bullet}{G m_\bullet^2} \right)^2 = 4\pi k \left( \frac{m_\bullet}{m_\rho} \right)^2 \]  

(C1)

Since the total mass of the system is \( M \), the mass of
DM outside the SMBH is \( M - m_\bullet \), and the number of dark
particles is \( N = (M - m_\bullet)/\mu \). The DM halo entropy is
\( S_d = -Nk\ln(Q/\rho_0) \). For the total entropy,

\[ S = 4\pi k \left( \frac{M}{m_\rho} \right)^2 \left( \frac{m_\bullet}{M} \right)^2 \eta^2 - \frac{M}{\mu} \left( \frac{1 - m_\bullet}{M} \right) \ln \left( \frac{Q}{\rho_0} \right) \]

\[ = \frac{M k}{4\pi} \left( \frac{M}{m_\rho} \right) \left( \frac{m_\bullet}{M} \right)^2 \eta^2 - \left( \frac{1 - m_\bullet}{M} \right) \ln \left( \frac{Q}{\rho_0} \right) \]  

(C2)

The first term (entropy of the horizon) dominates if \( \mu \gg m_\rho^2/M \), and the second term (entropy of the DM halo) domi-
nates if \( \mu \ll m_\rho^2/M \). Also note the trivial algebraic identity,

\[ \frac{m_\bullet}{M} \eta = \frac{m_\bullet}{M} \frac{r_\bullet}{2GM} = \frac{c^2 m_\bullet r_\bullet}{2GM} = \frac{c^2 r_\bullet}{2GM} = \frac{r_\bullet}{2X} \]  

(C3)

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For fixed $\chi$, the ratio $m_*/M$ is a monotonic function of $Q$, and $\eta$ remains on the order of 1. Also for fixed $\chi$, the ratio $r_*/R$ is monotonic in $Q$ except for wrinkles within one dex of the non-singular border. Therefore, if the right term of (C2) dominates then $S$ is monotonic in $Q$; and if the left term dominates then $S$ is also monotonic in $Q$ (except for subtle features near the nonsingular boundary).