The Nonlinear Filtering Feature Identification Methods for the Duffing Chaotic Fractal State

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Abstract. With the wide applications of weak signal detection techniques for Duffing system, it is important to study the fast and accurate identification methods for fractal state, such as in the field of all kinds of signals detections, by fractal state discriminant of Duffing, the target detections can be achieved under the conditions of extremely low signal-to-noise ratios. Aiming at the existing problems of the algorithm, a new idea is put forward, the Duffing oscillator is converted to the state equation of nonlinear filters, respectively by CKF(Cubature Kalman Filter,) PF(Particle Filter) and EKF(Extended Kalman Filter) three nonlinear filtering algorithm for recursive estimations, the inherent rules whose filtering parameters change with the system state changing will be researched, and a discrimination method of oscillator Fractal state is proposed; the experimental results show that the three kinds of filtering methods can all achieve the Duffing oscillator state tracking, and with the help of the filtering characteristics, the identification of the oscillator state can be distinguished and feature extraction can both realize the qualitative and quantitative detections.

1. Introduction
In recent years, the use of chaotic systems for weak signal detection has become a hot spot of research, and the core of detection is the feature differentiation of the chaotic fractal state after the weak signal input, and it can be called phase change discrimination. In other words, only the exact discrimination of the state of the chaotic system can be realized, and the signal sensitivity can be fully applied to the chaotic oscillator. The characteristic of the noise immunity realizes the weak signal detection [1~3].

For the Duffing oscillator, an important basis for the response to its morphological change is to judge the phase transition of its oscillator, but the present decision means have a great improvement space in the precision and efficiency. An important research direction, for example, in the field of digital communication with high bit rate, it is urgent to guarantee the discriminant means of real-time communication under the condition of very low signal to noise ratio. For the state decision, the existing detection process is mainly based on the change of the system phase diagram, that is to judge the transfer of the state of the system , The quantitative detection methods, such as the Lyapunov characteristic index (Lyapunov Characteristic Exponents) method, and K-entropy method, the fractal dimension method and other analytical methods[4~6], have the disadvantages of complex algorithm and low precision, so the phase change decision means need to be improved, and it is urgent to be able to distinguish the system state quickly and accurately. As a special nonlinear system, Duffing oscillator can be written as the form of state equation by Euler equation. This also creates a prerequisite for the recursive estimation of the Duffing oscillator by using the nonlinear filtering technique. In this paper, the state equation of nonlinear filtering is set, and three nonlinear filtering algorithms are used respectively. It is estimated that the filter parameters change with the change of the state. The
characteristic quantity in the filtering process reflects the details of the chaotic motion, which can be used for the rapid and accurate discrimination of the state of the oscillator.

2. State Modeling of Chaotic Oscillator
The traditional Duffing oscillator can be expressed as a form of two order differential equations. In the Kalman recursive filtering process of a Duffing two-dimensional nonlinear system, the filter model of the system should be written as the form of two element equation of the state and measurement equation [7], as shown below:

State equation:
\[
\begin{align*}
\dot{x}(n+1) &= x(n) + \omega \cdot h \cdot y(n) + v(n) \\
y(n+1) &= (1 - 0.5 \cdot \omega \cdot h) \cdot y(n) + \omega \cdot h \cdot [x(n) - x'(n)] \\
&\quad + \gamma \cdot \omega \cdot h \cdot \cos(\omega \cdot (h \cdot n))
\end{align*}
\] (1)

Measurement equation:
\[
z(n+1) = h[x(n), y(n)] + w(k)
\] (2)

It is assumed that the process noise and measurement noise are additive zero mean Gauss white noise.

The system state equation is usually obtained by \(a = b = 1, k = 0.5\), and the minimum value should be selected for the simulation step, so as to ensure a more accurate approximation of the Duffing system model, and to represent the Gauss white noise with a mean of 0 and variance. The following three different nonlinear filtering algorithms are applied to the recursive estimation of the system, and the state discrimination method for the Duffing system is found.

3. Nonlinear Filtering Recursive Estimation of Chaotic Oscillator

3.1. CKF Recursive Estimation and Simulation of Duffing System
Firstly, a new local non-linear filtering algorithm, Cubature Kalman Filter (CKF) algorithm [8, 9], is applied to the recurrence of Duffing system. It uses a set of Equal-Weight volume points to approximate the integral terms of Bayesian filter, which provides a new way to realize the non-linear estimation. In this case, it has higher accuracy than the Unscent Kalman Filter (UKF) algorithm.

I. Time Update:
1) Assuming a posterior probability density function at the time \(k - 1\):
Assumptions \(p(x_k|D_{k-1}) = N(x_{k-1|k-1}, P_{x_{k-1|k-1}})\) are known. The decomposition results are:

\[
P_{x_{k-1|k-1}} = S_{x_{k-1|k-1}} S_{x_{k-1|k-1}}^T
\] (3)

2) Estimating the Value of cubature Points \((i = 1, 2, \cdots, m)\):

\[
X_{i|x_{k-1}} = S_{x_{k-1|k-1}} \xi_i + x_{k-1|k-1}
\] (4)

There \(m = 2n, \xi_i = \sqrt{n} \begin{pmatrix} 1 \\ 0 \\ \cdots \\ 0 \\ \sqrt{n} \end{pmatrix}, \xi_i = \sqrt{n} \begin{pmatrix} 1 \\ 0 \\ \cdots \\ 0 \\ \sqrt{n} \end{pmatrix}\)

3) Estimating the cubature point of propagation \((i = 1, 2, \cdots, m)\):

\[
X_{i|x_{k-1}}^* = f(X_{i|x_{k-1}}, u_{k-1})
\] (5)

4) Estimate the predicted state:

\[
x_{k|x_{k-1}} = \frac{1}{m} \sum_{i=1}^{m} X_{i|x_{k-1}}^*
\] (6)
5) Estimate the square root factor of the prediction error covariance:

\[ S_{x,k|k-1} = \text{Triad}\left[ \begin{bmatrix} \zeta_{x,k|k-1} & S_{x,Q,k-1} \end{bmatrix} \right] \] (7)

The delegates \( \text{Triad}(\cdot) \) take down the triangular matrix by QR decomposition, which \( S_{Q,k-1} \) represents a square root factor of \( Q_{k-1} \).

II. Measurement update:

1) Estimation of propagation Cubature points \((i = 1, 2, \cdots, m)\),

\[ Z_{x,i,k|k-1} = h\left( x_{i,k|k-1}, u_k \right) \] (8)

2) Estimate the square root form of the updated covariance matrix:

\[ S_{x,z,k|k-1} = \text{Triad}\left[ \begin{bmatrix} \Gamma_{x,k|k-1} & S_{R,k} \end{bmatrix} \right] \] (9)

A square root factor of \( R_k \) represented here \( S_{R,k} \), that is \( R_k = S_{R,k}S_{R,k}^T \), and the weight center matrix is:

\[ \Gamma_{x,k|k-1} = \frac{1}{\sqrt{m}} \left[ Z_{x,1,k|k-1} - Z_{x,1,k-1} \cdots Z_{x,m,k|k-1} - Z_{x,m,k-1} \right] \] (10)

3) Estimate the cross-covariance matrix:

\[ P_{x,z,k|k-1} = \zeta_{x,k|k-1} \Gamma_{x,k|k-1}^T \] (11)

4) Estimate update status

\[ x_{4,k|k} = x_{4,k|k-1} + K \left( z_{x,k} - z_{x,k|k-1} \right) \] (12)

5) Estimate the square root factor of the corresponding error covariance:

\[ S_{x,k|k} = \text{Triad}\left[ \begin{bmatrix} \zeta_{x,k|k-1} - K \Gamma_{x,k|k-1} & K_{x,k} S_{R,k} \end{bmatrix} \right] \] (13)

2. Simulation verification
The recursive equation of Duffing oscillator is obtained by Euler algorithm, such as the equation of state (1), which is used to estimate the whole system. The CKF estimation of the Duffing oscillator is shown in Fig. 1. The tracking results for the chaotic state and large-scale periodic state of the system are shown here. The parameters are set as follows: the periodic driving force of the chaotic state is $\gamma = 0.80$, and the periodic driving force of the large-scale periodic state is $\gamma = 0.84$.

According to the experimental results of state tracking and parameter evaluation of Duffing system using CKF in Fig. 1, the number of simulated cyclic sampling points is $10^5$. According to the tracking results of $10^5$ points, the state trajectories of the two points almost coincide. Here, only the system is selected to highlight the details of the tracking results. Fig. 1 (a) and (b) are chaotic and periodic states of Duffing system, respectively. The simulation results of tracking the X-direction state of Duffing system using CKF show that CKF can estimate the effective state of Duffing system. Fig. 1 (c) and (d) are the covariances of the states of Duffing system in chaotic and periodic states, respectively. Similarly, as shown in the graph, the state covariance in the X-direction of the periodic state of the system is in a relatively stable periodic state after a short period of fluctuation, while the state covariance in the X-direction of the chaotic state will always be in a fluctuating state.

In a word, through the comparison of the recursive states in Fig. 1, it can be seen that the Kalman gain and covariance in the chaotic and periodic states show different characteristics in the process of system state estimation using CKF, which provides a CKF criterion for the qualitative judgment of the two states. At the same time, there are significant differences in the accumulation of quantities between the two, which creates a premise for quantitative judgment.

3.2. Recursive Estimation and Simulation of Particle Filter for Duffing System

Particle filter is essentially derived from Bayesian theory. Even if particle swarm optimization (PSO) [10-12] is used to approximate the probability of system state occurrence, its core idea is to represent its
distribution by random state particles extracted from posterior probability, which is a Sequential Importance Sampling method. In the filtering process, enough particles can be selected to obtain a detailed description of the posterior probability distribution. The core part of the filter is importance sampling. By sampling, the proposed probability distribution is close to the distribution probability of the real system.

Although particle filter algorithm can be used as an effective means of recursion for non-linear models, it has some problems. The most important problem is that the process needs a large number of samples to better approximate the posterior probability density of the system. The more samples needed to describe the posterior probability distribution, the higher the complexity of the algorithm. In addition, the resampling stage will cause the loss of sample validity and diversity, resulting in samples dilution.

![Diagram](image1)

![Diagram](image2)

(a) Estimation of chaotic state by particle filter  
(b) Particle filter estimation of periodic states

![Diagram](image3)

(c) Sampling weights of chaotic importance  
(d) Sampling weights of periodic importance

**Figure 2.** Comparison of Recursive State of Particle Filtering in Duffing System

1) Initialization
Take \( k = 0 \) and extract \( N \) sample points \( \{x_0^{(i)}, y_0^{(i)}\}, \ i=1\ldots N \) according to the initial value \( p(x_0) \) of Duffing system

2) Importance sampling \( \tilde{x}_k^{(i)} \sim q(x_k | x_{0,k-1}, z_{1:k}) \), \( \tilde{x}_0^{(i)} = (x_0^{(i)}, \tilde{x}_k^{(i)}) \), \( i=1\ldots N \).

3) Calculating weights \( \omega_{i,k}^{(i)} = \omega_{i,k-1}^{(i)} \frac{p(z_{1:k} | x_k^{(i)}, y_k^{(i)})p(y_k^{(i)} | y_{k-1}^{(i)})}{q(x_k^{(i)} | x_{0,k-1}, z_{1:k})} \)

4) Resampling
According to the size of the normalized weights \( \tilde{\omega}_{i,k}^{(i)}, \tilde{\omega}_{i,k}^{(i)} \), \( N \) samples \( \{\tilde{x}_0^{(i)}, \tilde{y}_0^{(i)}\} \) were duplicated and discarded, which were approximately distributed \( \{p(x_0^{(i)} | z_{1:k}), p(y_0^{(i)} | x_{1:k})\} \), \( N \) samples \( \{x_0^{(i)}, y_0^{(i)}\} \) of approximate distribution are obtained.
5) Output results
The output of the algorithm is the particle set \( \{ x_{0,i}^{(i)}, y_{0,i}^{(i)} : i = 1, \ldots, N \} \), which can approximate the posterior probability and the expectation of the function \( f(x_{0,i}) g(y_{0,i}) \).

6) \( k = k + 1 \), repeat 2) to 6) steps.
Finally, the state estimation of Duffing oscillator can be achieved by 1) - 6) steps. According to the algorithm process, the following simulation experiments are done.

Figure 2 is the experimental results of state tracking and parameter evaluation of Duffing system using particle filter. It can be seen that particle filter can also realize the effective state estimation. Figure 2 (c) and (d) are recursive values of particle sampling weights of Duffing system in chaotic and periodic states, respectively. From the graph, it can be seen that the sampling weights of Duffing system can be obtained in the process of recursive filtering. The value has obvious difference in numerical value, so it can be used for quantitative detection. The disadvantage of this filtering process is its high computational complexity.

3.3. Space Considerations
Extended Kalman Filter [13, 14] EKF(Extended Kalman Filter) is the non-filterable, the the extend Kalman Filter is the non-filterof the non-lines, and the non-lines state of the non-lines and the the the lines of the example; It's can can can can's the state of, is the.to the extension Kalman Filter, and the Duffing include the process in the Extended Kalman Filter.

![Figure 3. Comparison of EKF Recursive State of Duffing System](image)

1) One-step prediction of state
\[
\begin{bmatrix}
\hat{X}(k+1|k) \\
\hat{Y}(k+1|k)
\end{bmatrix} = f_{X,Y}(k|\hat{X}(k|k), \hat{Y}(k|k)) 
\] (14)
2) Calculate the Jacobian matrix of the system
3) Covariance prediction

\[ P_X(k+1|k) = F_X(k)P_X(k|k)F_X^*(k) + Q(k) \]  \hspace{1cm} (15)

4) Measurements and predictions

\[ \hat{Z}(k+1) = h(k+1, \hat{X}(k+1|k), \hat{Y}(k+1|k)) \]  \hspace{1cm} (16)

5) Covariance

\[ S_X(k+1) = h_X(k)P_X(k+1|k)h_X^*(k) + R(k) \]  \hspace{1cm} (17)

6) Calculating Kalman Gain

\[ K_X(k+1) = P_X(k+1|k)h_X^*(k+1)S_X^{-1}(k+1) \]  \hspace{1cm} (18)

7) State renewal equation

\[ \hat{X}(k+1|k+1) = \hat{X}(k+1|k) + K_X(z_X(k+1) - \hat{z}_X(k+1|k)) \]  \hspace{1cm} (19)

8) Covariance updating equation

\[ P_X(k+1|k+1) = P_X(k+1|k) - K_XS_X(k+1)K_X^* \]  \hspace{1cm} (20)

Fig. 3 is the experimental results of Duffing state tracking using EKF. It represents the results of state tracking of the oscillator at different stages, such as homoclinic orbit, period doubling bifurcation orbit, chaotic state orbit and large-scale periodic state orbit. It can be seen that EKF tracking of the state of the Duffing oscillator not only shows the effectiveness of tracking, but also has good tracking gain. The properties of matching stage states.

4. Summary
For Duffing chaotic detection, an important basis reflecting its detection performance is the result of phase change judgment. There is still much room to improve the accuracy and efficiency of current judgment methods. In this paper, the Euler form of Duffing oscillator is set as the state equation of non-linear filtering, and a new fractal discriminant method of chaotic state eigenvalue is proposed, that is, using volume Kalman filter separately. Three non-linear filtering algorithms, particle filter and extended Kalman filter, are used to estimate the Duffing chaotic oscillator recursively, and the intrinsic law of the change of filtering parameters with the change of system state is fully studied, and a method based on these three filtering methods is proposed. The experimental results show that the three filtering methods can track the Duffing oscillator state and use them to filter. Wave characteristic quantities can distinguish and distinguish the state of oscillator, and feature extraction quantities can simultaneously detect the state qualitatively and quantitatively, which can provide theoretical basis for the development of chaotic fractal state feature detection algorithm technology.

5. References
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