SUFFICIENT CONDITIONS FOR ABSOLUTE CONVERGENCE OF MULTIPLE FOURIER INTEGRALS

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Abstract. Various new sufficient conditions for representation of a function of several variables as an absolutely convergent Fourier integral are obtained in the paper. The results are given in terms of $L^p$ integrability of the function and its partial derivatives, each with the corresponding $p$. These $p$ are subject to certain relations known earlier only for some particular cases. Sharpness and applications of the obtained results are also discussed.

1. Introduction

If

$$f(y) = \int_{\mathbb{R}^d} g(x)e^{i\langle x, y \rangle}dx, \quad g \in L_1(\mathbb{R}^d),$$

we write $f \in A(\mathbb{R}^d)$, with $\|f\|_A = \|g\|_{L_1(\mathbb{R}^d)}$.

The possibility to represent a function as an absolutely convergent Fourier integral has been studied by many mathematicians and is of importance in various problems of analysis. For example, belonging of a function $m(x)$ to $A(\mathbb{R}^d)$ makes it to be an $L_1 \to L_1$ Fourier multiplier (or, equivalently, $L_\infty \to L_\infty$ Fourier multiplier); written $m \in M_1$ ($m \in M_\infty$, respectively). One of such $m$-s attracted much attention in 50-80s (see, e.g., [19], [3], [16, Ch.4, 7.4], [12], and references therein):

$$m(x) := m_{\alpha, \beta}(x) = \theta(x)\frac{e^{\|x\|^\alpha}}{|x|^{\beta}},$$

where $\theta$ is a $C^\infty$ function on $\mathbb{R}^d$, which vanishes near zero, and equals 1 outside a bounded set, and $\alpha, \beta > 0$. In is known that for $d \geq 2$:

2010 Mathematics Subject Classification. Primary 42B10; Secondary 42B15, 42A38, 42A45.

Key words and phrases. Fourier integral, Fourier multiplier, Hardy-Steklov inequality.
I) If \( \frac{\beta}{\alpha} > \frac{d}{2} \), then \( m \in M_1(M_\infty) \).

II) If \( \frac{\beta}{\alpha} \leq \frac{d}{2} \), then \( m \not\in M_1(M_\infty) \).

The first assertion holds true for \( d = 1 \) as well, while the second one only when \( \alpha \neq 1 \); however, the case \( \alpha = d = 1 \) is obvious.

Various sufficient conditions for absolute convergence of Fourier integrals were obtained by Titchmarsh, Beurling, Karleman, Sz.-Nagy, Stein, and many others. One can find more or less comprehensive and very useful survey on this problem in [15]. Let us mention also [13] and a couple of recent papers [1, 4].

New sufficient conditions of belonging to \( A(\mathbb{R}^d) \) are obtained in this paper.

Let us unite certain of the known one-dimensional results closely related to our study in the following theorem. First, it is natural to consider functions \( f \in A(\mathbb{R}) \) that satisfy the condition

\[(N-1) \quad \text{Let } f \in C_0(\mathbb{R}), \text{ that is, } f \in C(\mathbb{R}) \text{ and } \lim_{|t| \to \infty} f(t) = 0, \text{ and let } f \text{ be locally absolutely continuous on } \mathbb{R}. \]

**Theorem A-1.** Let \( f \) satisfy the condition (N-1), \( f \in L^p(\mathbb{R}) \) with \( 1 \leq p \leq 2 \), and \( f' \in L^q(\mathbb{R}) \) with \( 1 < q \leq 2 \). Then \( f \in A(\mathbb{R}) \).

For the multivariate case, we need additional notations. Let \( \eta \) be \( d \)-dimensional vector with the entries either 0 or 1 only. The inequality of vectors is meant coordinate wise. Here and in what follows \( D^\eta f \) for \( \eta = 0 = (0, 0, ..., 0) \) or \( \eta = 1 = (1, 1, ..., 1) \) mean the function itself and the mixed derivative in each variable, respectively, where

\[ D^\eta f(x) = \left( \prod_{j: \eta_j = 1} \frac{\partial}{\partial x_j} \right) f(x). \]

Let us give multidimensional results we are going, in a sense, to generalize (see [11] and [14], respectively).

**Theorem A1-d.** Let \( f \in L^2(\mathbb{R}^d) \). If all the mixed derivatives (in the distributional sense) \( \frac{\partial^{\beta_j}}{\partial x_j^{\beta_j}} f(x) \in L^2(\mathbb{R}^d) \), \( j = 1, 2, ..., d \), where \( \beta_j \) are positive integers such that \( \sum_{j=1}^{d} \frac{1}{\beta_j} < 2 \), then \( f \in A(\mathbb{R}^d) \).

**Theorem A2-d.** Let \( f \in L^1(\mathbb{R}^d) \). If all the mixed derivatives (in the distributional sense) \( D^n f(x) \in L^p(\mathbb{R}^d) \), \( \eta \neq 0 \), where \( 1 < p \leq 2 \), then \( f \in A(\mathbb{R}^d) \).

The outline of the paper is as follows. In the next section we formulate the results. In Section 3 we present the needed auxiliary results.
Then, in Section 4 we concentrate on the one-dimensional version of our main results. In the last section we give multidimensional proofs; one-dimensional arguments from the preceding section will be intensively used.

We shall denote absolute positive constants by $C$, these constants may be different in different occurrences.

2. Main results

It turns out that in several dimension there is a variety of results in terms of different combinations of derivatives. It is still not clear which one is ”better”, not always the sharpness of the obtained results can be proved. We continue to study whether there is a scale of such results, their sharpness and applicability.

Our first main result reads as follows.

**Theorem 2.1.** Let $f \in C_0(\mathbb{R}^d)$ and let $f$ and its partial derivatives $D^\eta f$, for all $\eta, \eta \neq 1$, be locally absolutely continuous on $(\mathbb{R} \setminus \{0\})^d$ in each variable. Let $f \in L_{p_0}$, $1 \leq p_0 < \infty$, and let each partial derivative $D^\eta f$, $\eta \neq 0$, belong to $L_{p_\eta}(\mathbb{R}^d)$, where $1 < p_\eta < \infty$. If for all $\eta, \eta \neq 0$,

$$\frac{1}{p_0} + \frac{1}{p_\eta} > 1,$$

then $f \in A(\mathbb{R}^d)$.

**Remark 2.2.** Condition (2.1) is sharp when $\eta = 1$, while for other $\eta$ it is apparently not sharp.

We can also obtain a result in which all the derivatives interplay rather than the pairs $p_0$ and $p_\eta$.

**Theorem 2.3.** Let $f \in C_0(\mathbb{R}^d)$ and let $f$ and its partial derivatives $D^\eta f$, for all $\eta, \eta \neq 1$, be locally absolutely continuous on $(\mathbb{R} \setminus \{0\})^d$ in each variable. Let $f \in L_{p_0}$, $1 \leq p_0 < \infty$, and let each partial derivative $D^\eta f$, $\eta \neq 0$, belong to $L_{p_\eta}(\mathbb{R}^d)$, where $1 < p_\eta < \infty$. If

$$\sum_{0 \leq \eta \leq 1} \frac{1}{p_\eta} > 2^{d-1},$$

and

$$\sum_{\eta \neq 0} \frac{1}{p_\eta} \leq 2^{d-1},$$

then $f \in A(\mathbb{R}^d)$. 
This theorem can be given in the following equivalent form.

**Theorem 2.3.** Let $f \in C_0(\mathbb{R}^d)$ and let $f$ and its partial derivatives $D^n f$, for all $\eta, \eta \neq 1$, be locally absolutely continuous on $(\mathbb{R} \setminus \{0\})^d$ in each variable. Let $f \in L_p, 1 \leq p < p_0 \leq \infty$, and let each partial derivative $D^n f, \eta \neq 0$, belong to $L_{p_\eta}(\mathbb{R}^d)$, where $1 < p_\eta < \infty$. If

$$
\sum_{0 \leq \eta \leq 1} \frac{1}{p_\eta} = 2^{d-1},
$$

then $f \in A(\mathbb{R}^d)$.

**Remark 2.4.** We will see from the proofs of these theorems that when $d = 2$, the assertion holds true if we replace assumption (2.3) by $\frac{1}{p_0} + \frac{1}{p_1} > 1$.

If to assume additionally that any of the $2^{d-1} - 1$ derivatives $D^n f$ are essentially bounded, then condition (2.3) is satisfied. In this case the following statement holds.

**Corollary 2.5.** Let $f \in C_0(\mathbb{R}^d)$ and let $f$ and its partial derivatives $D^n f$, for all $\eta, \eta \neq 1$, be locally absolutely continuous on $(\mathbb{R} \setminus \{0\})^d$ in each variable. Let $f \in L_{p_0}, 1 \leq p_0 < \infty$, and for the derivatives $D^n f \in L_{p_\eta}, 1 < p_\eta < \infty$. Let also $D^n f \in L_{\infty}$ for $|\eta| \leq \frac{d}{2}$. If

$$
(2.4) \quad \sum_{0 \leq \eta \leq 1} \frac{1}{p_\eta} > 2^{d-1},
$$

then $f \in A(\mathbb{R}^d)$.

For $d$ even, we can refine Corollary 2.5 as follows.

**Proposition 2.6.** Let $f \in C_0(\mathbb{R}^d)$, let $d$ be even, and let $f$ and its partial derivatives $D^n f$, for all $\eta, \eta \neq 1$, be locally absolutely continuous on $(\mathbb{R} \setminus \{0\})^d$ in each variable. Let $f \in L_{p_0}, 1 \leq p_0 < \infty$, and for the derivatives $D^n f \in L_{p_\eta}, 1 < p_\eta < \infty$. Let also $D^n f \in L_{\infty}$ for $|\eta| \leq \frac{d}{2} - 1$ and $\frac{1}{p_0} + \frac{1}{p_1} > 1$. If

$$
\sum_{0 \leq \eta \leq 1} \frac{1}{p_\eta} > 2^{d-1},
$$

then $f \in A(\mathbb{R}^d)$.

The next corollary gives conditions on which exponent decay of a function $f$ and its derivatives ensures $f \in A(\mathbb{R}^d)$. 
Corollary 2.7. If

\begin{equation}
|D^\chi f(x)| \leq C \frac{1}{(1 + |x|)^{\gamma_\chi}},
\end{equation}

where \( \gamma_\chi > 0 \) for all \( \chi \), \( 0 \leq \chi \leq 1 \), and

\begin{equation}
\sum_{0 \leq \chi \leq 1} \gamma_\chi > d2^{d-1},
\end{equation}

then \( f \in A(\mathbb{R}^d) \).

It is often naturally to suppose that the derivatives of the same order are of the same growth, for example, when the function is radial, like \( m_{\alpha,\beta} \). The above result then reduces to the next assertion.

Corollary 2.8. Let \( f \in C_0(\mathbb{R}^d) \) be a radial function, that is, \( f(x) = f_0(|x|) \), and let \( f \) and its partial derivatives \( D^\eta f \), for all \( \eta, \eta \neq 1 \), be locally absolutely continuous on \( (\mathbb{R} \setminus \{0\})^d \) in each variable. Let \( f \in L_{p_0}, \ 1 \leq p_0 < \infty \), and for \( j = |\eta| = \eta_1 + \cdots + \eta_d > 0 \) the derivatives \( D^\eta f \in L_{p_j}, \ 1 < p_j < \infty \). Let also

\begin{equation}
f^{(s)}_0 \in C(0, \infty), \quad \lim_{t \to \infty} t^s f^{(s)}_0(t) = 0, \quad 0 \leq s \leq \frac{d-1}{2}.
\end{equation}

If

\[ \sum_{j=0}^{d} \left( \frac{d}{j} \right) \frac{1}{p_j} > 2^{d-1}, \]

and when \( d \) is even

\begin{equation}
\frac{1}{p_0} + \frac{1}{p_d} > 1,
\end{equation}

then \( f \in A(\mathbb{R}^d) \).

Remark 2.9. Note that (2.7) are necessary conditions for belonging to \( A(\mathbb{R}^d) \) (see [17] and [7]).

Remark 2.10. We will see that Corollary 2.8 holds true if we replace condition (2.8) by

\[ \frac{1}{2} \left( \frac{d}{j} \right) \frac{1}{p_j^2} + \sum_{j=\frac{d}{2}+1}^{d} \left( \frac{d}{j} \right) \frac{1}{p_j} \leq 2^{d-1}. \]

Remark 2.11. We can prove that the conditions of the above results are sharp only for certain \( p_0 \). The point is that we make use of \( m_{\alpha,\beta} \) for which intermediate derivatives cannot be arbitrary.
As is mentioned, there is a variety of statements of above type. Let us give one more, it can be proved similarly to those above.

**Theorem 2.12. a)** Let \( f \in C_0(\mathbb{R}^d) \cap L_{p_0}(\mathbb{R}^d), \, 1 \leq p_0 < \infty, \, r > \frac{d}{2}, \, r \in \mathbb{N}, \, \frac{\partial^{r-1}}{\partial x_j} f \) be locally absolutely continuous in \( x_j \), and \( \frac{\partial^r}{\partial x_j} f \in L_{p_j}(\mathbb{R}^d), \, 1 < p_j < \infty, \, j = 1, \ldots, d. \) If

\[
r < \frac{2r - d}{p_0} + \sum_{j=1}^{d} \frac{1}{p_j} \leq \frac{2r - d}{p_0} + r,
\]

then \( f \in A(\mathbb{R}^d). \)

**b)** Let \( 1 \leq p < \infty, \, 1 < q < \infty, \) and

\[
\frac{2r - d}{p} + \frac{d}{q} < r.
\]

Then there is a function \( f \in C_0(\mathbb{R}^d) \cap L_p(\mathbb{R}^d) \) such that \( \frac{\partial^r}{\partial x_j} f \in L_q(\mathbb{R}^d), \) \( j = 1, \ldots, d, \) but \( f \not\in A(\mathbb{R}^d). \)

Theorem 2.12 yields

**Corollary 2.13.** Let \( r > \frac{d}{2}, \, r \in \mathbb{N}, \, \beta, \alpha > 0, \, \alpha \neq 1, \) and \( \beta > r(\alpha - 1). \) If \( \beta > \frac{d \alpha}{2}, \) then \( m \in A(\mathbb{R}^d). \)

Here the point is that using other theorems results in a corollary under more restrictive condition \( \beta > d(\alpha - 1). \)

### 3. Auxiliary Results.

One of the basic tools is the following lemma (see Lemma 4 in [17] or Theorem 3 in [2], in any dimension).

**Lemma B.** Let \( f \in C_0(\mathbb{R}). \) If

\[
\sum_{\nu = -\infty}^{\infty} 2^{\nu/2} \left( \int_{\mathbb{R}} |f(t + h(\nu)) - f(t - h(\nu))|^2 dt \right)^{1/2} < \infty,
\]

where \( h(\nu) = \pi 2^{-\nu}, \, \nu \in \mathbb{Z}, \) then \( f \in A(\mathbb{R}). \)

This lemma is a natural extension of the celebrated Bernstein’s test for the absolute convergence of Fourier series (see [5] Ch.II, §6). In order to formulate the multidimensional version, we denote

\[
\Delta_{\eta}^{n} f(x) = \Delta_{\eta_1, \ldots, \eta_d}^{n} f(x) = \prod_{\eta_j} \Delta_{\eta_j}^{n} f(x),
\]
where \( \eta = (\eta_1, \ldots, \eta_d) \) and \( \Delta_1^{e_j, r} f \) is defined as

\[
\Delta_1^{e_j, r} f(x) = \sum_{k=0}^{r} \binom{r}{k} (-1)^k f(x + (2k - r)u_j e_j), \quad 1 \leq j \leq d.
\]

Here \( e_j \) are basis unit vectors. Denote also \( \Delta_1^{u_1, \ldots, u_d} f(x) = \Delta_{u_1, \ldots, u_d} f(x) \).

**Lemma C.** Let \( f \in C_0(\mathbb{R}^d) \). If

\[
\sum_{s_1 = -\infty}^{\infty} \ldots \sum_{s_d = -\infty}^{\infty} 2^{\frac{1}{2} \sum_{j=1}^{d} s_j} \left\| \Delta_{\frac{x_1}{2}, \ldots, \frac{x_d}{2}} f \right\|_2 < \infty,
\]

where the norm is that in \( L_2(\mathbb{R}^d) \), then \( f \in A(\mathbb{R}^d) \).

We will make use of the following Hardy type inequality (see [6, Cor.3.14]):

For \( F \geq 0 \) and \( 1 < q \leq Q < \infty \)

\[
(\int_{\mathbb{R}} \left[ \int_{-h}^{t+h} F(s) \, ds \right] \, dt)^{1/Q} \leq C h^{1/Q + 1/q} \left( \int_{\mathbb{R}} F^q(t) \, dt \right)^{1/q}.
\]

Here \( \frac{1}{q} + \frac{1}{q'} = 1 \). Similarly \( \frac{1}{p} + \frac{1}{p'} = 1 \).

We need the following direct multivariate generalization of (3.1).

**Lemma 3.1.** For \( F(u) \geq 0 \), \( 1 \leq k < d \), and \( 1 < q \leq Q < \infty \)

\[
\left( \int_{\mathbb{R}^d} \left[ \int_{x_1-h_1}^{x_1+h_1} \ldots \int_{x_k-h_k}^{x_k+h_k} F(u_1, \ldots, u_k, x_{k+1}, \ldots, x_d) \, du_1 \ldots du_k \right] \, dx \right)^{1/Q}
\]

\[
\leq C (h_1 \ldots h_k)^{\frac{1}{2} + \frac{1}{q}} \left( \int_{\mathbb{R}^d} \left[ \int_{\mathbb{R}^k} F^q(x) \, dx_1 \ldots dx_k \right] \, dx_{k+1} \ldots dx_d \right)^{1/Q}.
\]

If \( k = d \),
\[
\left( \int_{\mathbb{R}^d} \left[ \int_{x_1-h_1}^{x_1+h_1} \cdots \int_{x_{d-h_d}}^{x_{d-h_d}} F(u) \, du \right] \, dx \right)^{1/Q} \leq C(h_1 \ldots h_d)^{1/Q+1/q'} \left( \int_{\mathbb{R}^d} F^q(x) \, dx \right)^{1/q}.
\]

(3.3)

Of course, the first \( k \) variables are taken in (3.2) for simplicity, the result is true for any \( k \) variables.

**Proof.** The proof is inductive. For \( d = 1 \), the result holds true: (3.1). Supposing that it is true for \( d-1 \), \( d = 2, 3, \ldots \), let us prove (3.2) with \( k = d \). Applying inductive assumption for the first \( d-1 \) variables, we obtain

\[
\left( \int_{\mathbb{R}^d} \left[ \int_{x_1-h_1}^{x_1+h_1} \cdots \int_{x_{d-h_d}}^{x_{d-h_d}} F(u_1, \ldots, u_d) \, du_1 \ldots du_d \right] \, dx_1 \ldots dx_d \right)^{1/Q}
\]

\[
= \left( \int_{\mathbb{R}^{d-1}} \left[ \int_{x_1-h_{d-1}}^{x_1+h_{d-1}} \cdots \int_{x_{d-1}-h_{d-1}}^{x_{d-1}+h_{d-1}} \left[ \int_{x_{d-h_d}}^{x_{d+h_d}} F(u_1 \ldots, u_d) \, du_1 \ldots du_d \right] \, dx_1 \ldots dx_{d-1} \right] \, dx_d \right)^{1/Q}
\]

\[
\leq C(h_1 \ldots h_{d-1})^{1/Q+1/q'} \left( \int_{\mathbb{R}^{d-1}} \left[ \int_{x_{d-h_d}}^{x_{d+h_d}} F(x_1, \ldots, x_{d-1}, u_d) \, du_d \right] \, dx_{d-1} \ldots dx_d \right)^{q/Q}.
\]

Applying now the generalized Minkowski inequality with exponent \( Q/q \geq 1 \), we bound the right-hand side by, times a constant,

\[
(h_1 \ldots h_{d-1})^{1/Q+1/q'} \left( \int_{\mathbb{R}^{d-1}} \left[ \int_{x_{d-h_d}}^{x_{d+h_d}} F(x_1, \ldots, x_{d-1}, u_d) \, du_d \right] \, dx_{d-1} \ldots dx_d \right)^{q/Q} \left( \int_{\mathbb{R}^d} F^q(x) \, dx \right)^{1/q}.
\]
To obtain (3.2), it remains again to make use of (3.1) for the $d$-th variable.

If $k < d$, we just represent the considered integral as

\[
\left( \int_{\mathbb{R}^{d-k}} \left( \int_{\mathbb{R}^k} \left[ \int_{x_1-h_1}^{x_1+h_1} \ldots \int_{x_d-h_d}^{x_d+h_d} F(u_1, \ldots, u_k, x_{k+1}, \ldots, x_d) \, du_1 \ldots du_k \right] \, dx_{k+1} \ldots dx_d \right)^{(1/Q)Q} \right)^{1/Q} \left( \int_{\mathbb{R}^{d-k}} \right)^Q \]

and apply the proved version to the inner integral. The proof is complete. \qed

We will also apply the following simple result.

**Lemma 3.2.** Let $f \in C_0(\mathbb{R}^d)$, $D^1 f \in L_q(\mathbb{R}^d)$, $1 < q < \infty$, and partial derivatives $D^\eta f$, $\eta \neq 1$, are locally absolutely continuous on $(\mathbb{R} \setminus \{0\})^d$ in each variable. Then

\[
\| \Delta_{h_1, \ldots, h_d} f \|_\infty \leq 2^{d/q'} (h_1 \ldots h_d)^\frac{1}{q} \| D^1 f \|_q.
\]

**Proof.** By Hölder’s inequality,

\[
\| \Delta_{h_1, \ldots, h_d} f \|_\infty \leq \int_{x_1-h_1}^{x_1+h_1} \ldots \int_{x_d-h_d}^{x_d+h_d} \left| D^1 f(u_1, \ldots, u_d) \right| \, du_1 \ldots du_d \leq \left( \int_{x_1-h_1}^{x_1+h_1} \ldots \int_{x_d-h_d}^{x_d+h_d} \, du_1 \ldots du_d \right)^{\frac{1}{q'}} \| D^1 f \|_q,
\]

as required. \qed

## 4. One-dimensional result

Our main result in dimension one reads as follows (see [K]), here we present a proof of the sufficiency different from that in [K].

**Theorem 4.1.** Suppose a function $f$ satisfies condition (N-1).

a) Let $f(t) \in L_p(\mathbb{R})$, $1 \leq p < \infty$, and $f'(t) \in L_q(\mathbb{R})$, $1 < q < \infty$. If $\frac{1}{p} + \frac{1}{q} > 1$, then $f \in A(\mathbb{R})$.

b) If $\frac{1}{p} + \frac{1}{q} < 1$, then there exists a function $f$ satisfying (N-1) such that $f(t) \in L_p(\mathbb{R})$ and $f'(t) \in L_q(\mathbb{R})$ but $f \not\in A(\mathbb{R})$. 
Proof. To prove b) of the theorem, let us consider the function \( m \) from the introduction. Suppose that \( p\beta > 1 \) and \( q(\beta - \alpha + 1) > 1 \), with \( \alpha \neq 1 \).
Simple calculations show that \( m \in L^p(\mathbb{R}) \) and \( m' \in L^q(\mathbb{R}) \). If \( \frac{\beta}{\alpha} < \frac{1}{2} \), then \( m \not\in A(\mathbb{R}) \). The last inequality is equivalent to \( 2\beta - \alpha + 1 < 1 \).
Therefore,
\[
\frac{1}{p} + \frac{1}{q} < 2\beta - \alpha + 1 < 1,
\]
and the considered \( m \) delivers the required counterexample.

Proof of a). This is apparently the shortest possible proof. Denoting
\[
\Delta(h) = \left( \int_{\mathbb{R}} |\Delta_h f(t)|^2 dt \right)^{1/2},
\]
we are going to prove the positive part by showing that
\[
\sum_{\nu=1}^{\infty} 2^{-\nu/2} \Delta(h(-\nu)) + \sum_{\nu=0}^{\infty} 2^{\nu/2} \Delta(h(\nu)) < \infty.
\]
It is obvious that for \( h > 0 \)
\[
|f(t + h) - f(t - h)| = | \int_{t-h}^{t+h} f'(s) ds |.
\]
Let start with the first sum in (4.2) which is
\[
\sum_{\nu=1}^{\infty} 2^{-\nu/2} \left( \int_{\mathbb{R}} |\Delta_h(-\nu) f(t)|^2 dt \right)^{1/2}.
\]
Using (4.3), we represent the integral as
\[
\left( \int_{\mathbb{R}} |\Delta_h(-\nu) f(t)| \left| \int_{t-h}^{t+h} f'(s) ds \right| dt \right)^{1/2}.
\]
By Hölder’s inequality, it is estimated via
\[
\left( \int_{\mathbb{R}} |\Delta h(-\nu) f(t)|^p dt \right)^{\frac{1}{2p}} \left( \int_{\mathbb{R}} \left[ \int_{t-h(-\nu)}^{t+h(-\nu)} |f'(s)| ds \right]^{p'} dt \right)^{\frac{1}{2p'}}.
\]

Since \( p' > q \), we use (3.1) with \( F(s) = |f'(s)| \) and \( Q = p' \). Therefore, the first sum in (4.2) is controlled by

\[
\|f\|_p^{1/2} \|f'\|_{q'}^{1/2} \sum_{\nu=1}^{\infty} 2^{-\frac{p}{q} \left(1 - \frac{1}{p'} - \frac{1}{q'}\right)},
\]
and is bounded since

\[
1 - \frac{1}{p'} - \frac{1}{q'} = \frac{1}{p} + \frac{1}{q} - 1 > 0.
\]

To handle the second sum, we represent it as (see (4.3))

\[
(\int_{\mathbb{R}} |\Delta h(\nu) f(t)|^{q'} dt)^{\frac{1}{2q'}} \left( \int_{\mathbb{R}} \left[ \int_{t-h(\nu)}^{t+h(\nu)} |f'(s)| ds \right]^q dt \right)^{\frac{1}{2q'}}.
\]

Applying Hölder’s inequality with the exponents \( q' > 1 \) and \( q \), we estimate (4.5) via

\[
(\int_{\mathbb{R}} |\Delta h(\nu) f(t)|^{q} dt)^{\frac{1}{2q'}} \left( \int_{\mathbb{R}} \left[ \int_{t-h(\nu)}^{t+h(\nu)} |f'(s)| ds \right]^{q'} dt \right)^{\frac{1}{2q'}}.
\]

By (4.3) and Lemma 3.2 in dimension one, the first integral in (4.6) is controlled by

\[
\left( \int_{\mathbb{R}} |\Delta h(\nu) f(t)|^p dt \right)^{\frac{1}{2q'}} \left( \int_{\mathbb{R}} \left[ \int_{t-h(\nu)}^{t+h(\nu)} |f'(s)| ds \right]^{q'} dt \right)^{\frac{1}{2q'}} \leq C h(\nu)^{\frac{q'-p}{q'} \frac{1}{2q'}} \|f\|_{p}^{\frac{2p}{2q'}} \|f'\|_{q'}^{\frac{2q}{2q'}}.
\]

To estimate the second one, we use (3.1) with \( F(s) = |f'(s)| \) and \( Q = q \). We thus estimate the second factor in (4.6) via

\[
C[h(\nu)]^{1/2} \|f'\|_{q}^{1/2}.
\]

Since \( q' > p \), the series

\[
\sum_{\nu=1}^{\infty} 2^{-\frac{p}{q} \left(1 - \frac{1}{p'} - \frac{1}{q'}\right)},
\]

is bounded since

\[
1 - \frac{1}{p'} - \frac{1}{q'} = \frac{1}{p} + \frac{1}{q} - 1 > 0.
\]
\begin{align*}
\sum_{\nu=1}^{\infty} 2^{d+\nu \frac{1}{2\nu}}
\end{align*}
converges, which ensures the finiteness of (4.2). □

5. **Proofs of multidimensional results**

We give, step by step, proofs of the results formulated in Introduction.

5.1. **Proof of Theorem 2.1** The proof is surprisingly very similar to that in dimension one. When we deal with the part of the sum from Lemma C with

\begin{align*}
\sum_{k_1=1}^{\infty} \ldots \sum_{k_d=1}^{\infty} 2^{-\frac{1}{2} \sum_{j=1}^{d} k_j},
\end{align*}

we represent this sum as

\begin{align*}
\sum_{k_1=1}^{\infty} \ldots \sum_{k_d=1}^{\infty} 2^{-\frac{1}{2} \sum_{j=1}^{d} k_j} \left( \int_{\mathbb{R}^d} |\Delta_{h(-k_1),\ldots,h(-k_d)} f(x)| \right) \\
\times \left| \begin{array}{ccc}
x_1+h(-k_1) & x_1+h(-k_1) & \ldots \\
x_1 & x_1+h(-k_1) & \ldots \\
x_d+h(-k_d) & x_d & \ldots \end{array} \right| D^1 f(u) du \ dx \right)^{1/2}
\end{align*}

and manage it exactly as in the either proof of the first sum in dimension one.

Further, when we deal with the part of the sum from Lemma C with

\begin{align*}
\sum_{k_1=0}^{\infty} \ldots \sum_{k_d=0}^{\infty} 2^{\frac{1}{2} \sum_{j=1}^{d} k_j},
\end{align*}

we proceed as in the (1st) proof of the second sum in dimension one.

In both cases (3.3) from Lemma 3.1 is applied.

Finally, when we deal with the parts of the sum from Lemma C with

\begin{align*}
\sum_{i: \eta_i=0} 2^{-\frac{1}{2} \sum_{j=1}^{d} k_j} \ldots \sum_{i: \eta_i=1} 2^{\frac{1}{2} \sum_{j=1}^{d} k_j},
\end{align*}
where $\eta \neq 0$ and $\eta \neq 1$, the point is that we do not need to treat the first sum at all: when the rest is bounded, the series corresponding to $\eta_i = 0$ converge automatically then. As for the second sum, we proceed to it as in the proof of the second sum in dimension one. Since we always apply Lemma 3.1 with $Q = q$, we get that (3.2) is reduced to usual $L_{p_\eta}$ spaces.

The proof is complete. □

5.2. **Proof of Theorem 2.3.** When we deal with the sum

\[
\sum_{k_1=1}^{\infty} \cdots \sum_{k_d=1}^{\infty} 2^{-\frac{1}{2}(k_1+\cdots+k_d)} \| \Delta_{h(-k_1),\ldots,h(-k_d)}f \|_2
\]

we only need to use condition (2.2). Choosing $p^*_\eta > p_\eta$, $0 \leq \eta \leq 1$, such that

\[
\sum_{\eta \leq 1, \eta_j=1} \frac{1}{p^*_\eta} = 2^{d-1}.
\]

Applying Hölder inequality and Lemma 3.1 we obtain

\[
\| \Delta_{h(-k_1),\ldots,h(-k_d)}f \|_2 \leq C \left( \prod_{0 \leq \eta \leq 1} \| \Delta^\eta_{h(-k_1),\ldots,h(-k_d)}f \|_{p_\eta} \right)^{\frac{1}{p}}
\]

\[
\leq C \left( \prod_{0 \leq \eta \leq 1} \| f \|_{\infty}^{1-\frac{p_\eta}{p^*_\eta}} (h(-k_1)^{\eta_1} \cdots h(-k_d)^{\eta_d})^{\frac{p_\eta}{p^*_\eta}} \| D^n f \|_{p_\eta}^{\frac{p_\eta}{p^*_\eta}} \right)^{\frac{1}{p}}
\]

\[
= C \prod_{j=1}^{d} 2^{\frac{1}{2d} \left( \sum_{0 \leq \eta \leq 1, \eta_j=1} \frac{p_\eta}{p^*_\eta} \right) k_j} \left( \prod_{0 \leq \eta \leq 1} \| f \|_{\infty}^{1-\frac{p_\eta}{p^*_\eta}} \| D^n f \|_{p_\eta}^{\frac{p_\eta}{p^*_\eta}} \right)^{\frac{1}{p}}.
\]

The last inequality together with the following inequality

\[
\sum_{0 \leq \eta \leq 1, \eta_j=1} \frac{p_\eta}{p^*_\eta} < 2^{d-1}
\]

yield the convergence of the sum in (5.1).

In what follows, we denote for simplicity $p_0 = p_0$, $p_d = p_1$.

Let us show that

\[
\sum_{k_1=0}^{\infty} \cdots \sum_{k_d=0}^{\infty} 2^{\frac{1}{2}(k_1+\cdots+k_d)} \| \Delta_{h(k_1),\ldots,h(k_d)}f \|_2 < \infty
\]

Assuming that condition (2.3) holds with strong inequality, we can choose $p^*_0 > p_0$ such that

\[
\frac{1}{p_0} + \sum_{\eta \neq 0} \frac{1}{p_\eta} = 2^{d-1}.
\]
Applying then Hölder’s inequality, Lemma 3.2 and Lemma 3.1, we obtain

$$
\|\Delta_{h(k_1),...,h(k_d)} f\|_2 \\
\leq C \left( \|\Delta_{h(k_1),...,h(k_d)} f\|_{\infty} \|f\|_{p_0} \prod_{\eta \neq 0} \|\Delta_{h(k_1),...,h(k_d)} f\|_{p_\eta}\right)^{\frac{1}{2d}}
$$

Thus, choosing

$$
p^* \to p_0 \quad \text{and} \quad p^* \to p_{e_1}
$$

such that

$$
1 - \frac{p^*}{p_0} = \sum_{\eta \neq 0, \eta \neq e_1} \frac{1}{p_\eta} = 2^{d-1}.
$$

Note that we can choose $p^*_0$ to be sufficiently large and $p^*_1$ to be sufficiently close to $p_{e_1}$.

Applying then Hölder’s inequality, we obtain

$$
(5.3) \quad \|\Delta_{h(k_1),...,h(k_d)} f\|_2 \leq C \left( \|\Delta_{h(k_1),...,h(k_d)} f\|_{\infty} \|f\|_{p_0} \prod_{\eta \neq 0} \|\Delta_{h(k_1),...,h(k_d)} f\|_{p_\eta}\right)^{\frac{1}{2d}}
$$

From (5.3), Lemma 3.1 and Lemma 3.2 we get

$$
\|\Delta_{h(k_1),...,h(k_d)} f\|_2 = O\left(2^{-\left(\frac{1}{2} + \frac{1}{2d}\right)\left(-1 + \frac{p^*_{e_1}}{p_{e_1}} + \frac{1}{p_d}\left(2 - \frac{p^*_{e_1}}{p_{e_1}}\right)\right)k_1} \cdot \frac{1}{2} \right).
$$

Thus, choosing $p^*_0$ and $p^*_1$ such that

$$
\frac{1}{p_d}\left(2 - \frac{p_0}{p_0} - \frac{p_{e_1}}{p^*_{e_1}}\right) > \frac{p_{e_1}}{p^*_{e_1}},
$$

we obtain the convergence of (5.2).

To complete the proof of the theorem, it remains to show the convergence of the series of type

$$
(5.4) \quad \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{l_{j+1}=1}^{\infty} \sum_{l_d=1}^{\infty} \frac{2^{\frac{1}{2}(k_1+\cdots+k_d)}}{2^{\frac{1}{2}(l_{j+1}+\cdots+l_d)}} \|\Delta_{h} f\|_2,
$$

From (5.3), Lemma 3.1 and Lemma 3.2 we get

$$
\|\Delta_{h(k_1),...,h(k_d)} f\|_2 \leq C \left( \|\Delta_{h(k_1),...,h(k_d)} f\|_{\infty} \|f\|_{p_0} \prod_{\eta \neq 0} \|\Delta_{h(k_1),...,h(k_d)} f\|_{p_\eta}\right)^{\frac{1}{2d}}
$$

Thus, choosing $p^*_0$ and $p^*_1$ such that

$$
\frac{1}{p_d}\left(2 - \frac{p_0}{p_0} - \frac{p_{e_1}}{p^*_{e_1}}\right) > \frac{p_{e_1}}{p^*_{e_1}},
$$

we obtain the convergence of (5.2).
where $1 \leq j \leq d - 1$ and $h = (h(k_1), \ldots, h(k_j), h(-l_{j+1}), \ldots, h(-l_d))$.

Choosing $p_0^* > p_0$ and $p_{e_i}^* > p_{e_i}$, $i = j + 1, \ldots, d$ such that

$$\frac{1}{p_0^*} + \sum_{i=1}^j \frac{1}{p_{e_i}} + \sum_{i=j+1}^d \frac{1}{p_{e_i}^*} + \sum_{|\eta| > 1} \frac{1}{p_\eta} = 2^{d-1}.$$ 

Applying Hölder’s inequality, we obtain

(5.5) $\|\Delta_h f\|_2 \leq C(S_1 S_2 S_3 S_4)^{\frac{1}{2d}},$

where

$S_1 = \|\Delta_h f\|_{p_0^*},$

$S_2 = \prod_{i=1}^j \|\Delta_{h(k_i)} f\|_{p_{e_i}},$

$S_3 = \prod_{i=j+1}^d \|\Delta_{h(-l_i)} f\|_{p_{e_i}^*},$

and

$S_4 = \prod_{|\eta| > 1} \|\Delta_\eta f\|_{p_\eta}.$

Applying Lemma 3.2 we get

(5.6) $S_1 \leq C \frac{p_0}{p_0^*} \frac{p_0}{p_0} \frac{\Delta_h f}{\Delta_h f} \|f\|_{\infty} \frac{p_0}{p_0} \frac{\|\Delta_h f\|_{\infty}}{\frac{p_0}{p_0^*} \|\Delta_h f\|_{\infty}} \leq C 2^{1-\frac{\varepsilon}{p_0}} \frac{p_0}{p_0^*} \frac{p_0}{p_0} \frac{\|\Delta_h f\|_{\infty}}{\frac{p_0}{p_0^*} \|\Delta_h f\|_{\infty}} \frac{1}{p_0^*} \frac{1}{p_0^*},$

where $\varepsilon \in (0, 1 - \frac{p_0}{p_0^*}).$

Further, applying Lemma 3.1 we obtain

(5.7) $S_2 \leq C \prod_{i=1}^j 2^{-k_i} \|D^{p_{e_i}} f\|_{p_{e_i}},$

(5.8) $S_3 \leq C \prod_{i=j+1}^d \|\Delta_{h(-l_i)} f\|_{\infty} \frac{p_{e_i}}{p_{e_i}} \frac{\|\Delta_{h(-l_i)} f\|_{p_{e_i}}}{\frac{p_{e_i}}{p_{e_i}}},$

$\leq C \prod_{i=j+1}^d \|f\|_{\infty} \frac{p_{e_i}}{p_{e_i}} \frac{p_{e_i}}{p_{e_i}} \|D^{p_{e_i}} f\|_{p_{e_i}}.$
and

\[(5.9) \quad S_4 \leq C \prod_{|\eta| > 1} 2^{n_k_1 + \cdots + n_j k_j} \eta_{j+1} \cdots \eta_{d} \|D^n f\|_{p_0}.
\]

Combining then (5.5) and (5.6)-(5.9), we arrive at

\[
\|\Delta_h f\|_2 = O\left(\prod_{i=1}^j 2^{-\left(\frac{1}{2} + \frac{\epsilon}{2p_0 p_i}\right) k_i} \prod_{i=j+1}^d 2^{-\left(\frac{1}{2} + \frac{\epsilon}{2p_0 p_i}\right) k_i}\right).
\]

Choosing \(\epsilon \in (0, 1 - \frac{p_0}{p_i})\) such that

\[
\frac{\epsilon}{p_i} - 1 + \frac{p_0}{p_i} < 0, \quad i = j + 1, \ldots, d,
\]

we obtain that (5.4) is finite.

This completes the proof. \(\square\)

5.3. Proof of Corollary 2.7 Let us rewrite (2.6) as

\[
\sum_{0 \leq \chi \leq 1} \gamma_\chi = d 2^{d-1} + \epsilon.
\]

For each \(\chi\), let us choose \(p_\chi\) so that \(\gamma_\chi p_\chi = d + \frac{\epsilon}{2^d}\). Then

\[
\sum_{0 \leq \chi \leq 1} \frac{d + \epsilon/2^d}{p_\chi} = d 2^{d-1} + \epsilon.
\]

Since

\[
\sum_{0 \leq \chi \leq 1} \frac{\epsilon/2^d}{p_\chi} < \epsilon,
\]

there holds

\[
\sum_{0 \leq \chi \leq 1} \frac{d}{p_\chi} > d 2^{d-1}.
\]

This is equivalent to (2.4), and hence \(f \in A(\mathbb{R}^d)\). \(\square\)

Proof of Proposition 2.6 Convergence of the series like (5.1) and (5.2) is proved as in Theorems 2.1 and 2.3. Thus, in order to complete the proof of the proposition, it suffices to demonstrate the convergence of series of type (5.4). We will restrict ourselves to the case \(j = d - 1\), that is, to the series
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\begin{equation}
\sum_{k_1=0}^{\infty} \cdots \sum_{k_{d-1}=0}^{\infty} \sum_{l_d=1}^{\infty} \frac{2^{\frac{1}{2}(k_1+\cdots+k_d)}}{2^{\frac{1}{2}l_d}} \|\Delta_{h(k_1),\ldots,h(k_d),h(-l_d)}f\|_2.
\end{equation}

For \( j < d - 1 \), the proof goes along the same lines as in the following arguments.

Let
\begin{equation}
\sum_{|\eta| \geq \frac{d}{2}} \frac{1}{p_\eta} \geq 2^{d-1},
\end{equation}
otherwise the proof is obvious.

Assume that there is a collection \( \{p_\eta^*\} \) such that \( p_\eta^* > p_\eta \) when \( |\eta| \leq \frac{d}{2} - 1 \), \( p_{\eta(1)}^* > p_{\eta(1)} \), where \( \eta(1) = (\eta_1^{(1)}, \ldots, \eta_{d-1}^{(1)}, 1), |\eta(1)| = \frac{d}{2} \), and
\begin{equation}
\sum_{|\eta| \leq \frac{d}{2} - 1} \frac{1}{p_\eta^*} + \frac{1}{p_{\eta(1)}^*} + \sum_{|\eta| \geq \frac{d}{2}, \eta \neq \eta(1)} \frac{1}{p_\eta} = 2^{d-1}.
\end{equation}

Observe that \( p_\eta^* \) can be chosen arbitrary large when \( |\eta| \leq \frac{d}{2} - 1 \).

For convenience, we set \( p_\eta^* = p_\eta \) when \( |\eta| \geq \frac{d}{2} \) \( \eta \neq \eta(1) \), while \( \eta_1^{(1)} = \cdots = \eta_{\frac{d}{2}-1}^{(1)} = \eta_{d-1}^{(1)} = 1 \) and \( \eta_1^{(1)} = \cdots = \eta_{d-1}^{(1)} = 0 \). Let also \( h = (h(k_1), \ldots, h(k_{d-1}), h(-l_d)) \).

Applying Hölder’s inequality, we obtain
\begin{equation}
\|\Delta_hf\|_2 \leq C(S_1S_2S_3S_4)^{\frac{1}{2d}},
\end{equation}
where
\begin{align*}
S_1 &= \|\Delta_hf\|_{p_\eta^*}, \\
S_2 &= \prod_{|\eta| = 1} \|\Delta_hf\|_{p_\eta^*}, \\
S_3 &= \|\Delta_{\eta(1)}f\|_{p_\eta^*}, \\
S_4 &= \prod_{|\eta| > 1, \eta \neq \eta(1)} \|\Delta_{\eta}f\|_{p_\eta^*}.
\end{align*}

As is shown above (see (5.6))
\begin{equation}
S_1 = O \left( 2^{-\frac{d}{p_\eta^*(k_1+\cdots+k_{d-1}-l_d)}} \right).
\end{equation}

Further, applying Lemma 3.1, we get
\[ S_2 \leq C \| \Delta_h \eta_1^{(1)} - e_d \|_{p_{\eta_1}^{(1)} - e_d}^{\frac{d}{2}} \prod_{j=\frac{d}{2}}^{d-1} \| \Delta_h f \|_{p_{\eta_j}^{(1)} - e_d} \]
\[ \leq C 2^{-\frac{d}{2}(k_1 + \cdots + k_{d-1})} \| D \eta_1^{(1)} - e_d \|_{p_{\eta_1}^{(1)} - e_d}^{\frac{d}{2}} \prod_{j=\frac{d}{2}}^{d-1} 2^{-k_j} \| D^{(1)} f \|_{p_{\eta_j}^{(1)}}. \]

\[ S_3 \leq C \| \Delta_h \eta_1^{(1)} \|_{\infty} \frac{p_{\eta_1}^{(1)}}{p_{\eta_1}^{(1)}} \| \Delta_h \eta_1^{(1)} \|_{p_{\eta_1}^{(1)}} \]
\[ \leq C \| f \|_{\infty} \frac{1}{p_{\eta_1}^{(1)}} \left( \prod_{j=1}^{d-1} 2^{-k_j} 2^{d} \| D \eta_1^{(1)} f \|_{p_{\eta_1}^{(1)}} \right) \]
and

\[ S_4 = O \left( \prod_{j=1}^{d-1} 2^{-(2d-1)-2k_j} \prod_{j=\frac{d}{2}}^{d-1} 2^{-(2d-1)-1} k_j 2^{(2d-1)-2k_j} \right) \]

Combining (5.13) and (5.14)-(5.17), we obtain

\[ \| \Delta_h f \|_2 = O \left( \prod_{j=1}^{d-1} 2^{-\left(\frac{1}{2} + \frac{1}{2d} + \frac{p_{\eta_1}^{(1)}}{p_{\eta_1}^{(1)}} + \frac{d}{2d} - 1\right)} k_j \right) \]
\[ \times \prod_{j=\frac{d}{2}}^{d-1} 2^{-\left(\frac{1}{2} + \frac{1}{2d} + \frac{p_{\eta_1}^{(1)}}{p_{\eta_1}^{(1)}} + \frac{d}{2d} - 1\right)} k_j \left( 2^{\frac{d}{2}} \frac{p_{\eta_1}^{(1)}}{p_{\eta_1}^{(1)}} 2^{\frac{d}{2d} - 1} \right) \]

Hence, choosing \( \varepsilon \) to be small enough, we readily get the convergence of the series in question.

Now, if (5.12) holds for no collection \( \{p_{\eta}^{*}\} \) such that \( p_{\eta}^{*} > p_{\eta} \) for \( |\eta| \leq \frac{d}{2} - 1 \), we suppose that there is a collection \( \{p_{\eta}^{*}\} \) such that \( p_{\eta}^{*} > p_{\eta} \) for \( |\eta| \leq \frac{d}{2} - 1 \), \( p_{\eta_1}^{*}^{(1)} \), and \( p_{\eta_2}^{*}^{(2)} \) such that \( p_{\eta_1}^{*} > p_{\eta} \)
for \( |\eta| \leq \frac{d}{2} - 1 \), \( p_{\eta_1}^{(1)} \), \( \eta_1^{(j)} \), \( \eta_1^{(j-1)} \), \( \eta_1^{(1)} \), \( |\eta^{(j)}| = \frac{d}{2} \), \( j = 1, 2 \), and

\[ \sum_{|\eta| \leq \frac{d}{2} - 1} \frac{1}{p_{\eta}^{*}} + \frac{1}{p_{\eta_1}^{(1)}} + \frac{1}{p_{\eta_2}^{(2)}} + \sum_{|\eta| \geq \frac{d}{2}, \eta \neq \eta_1^{(1)}, \eta_2^{(2)}} \frac{1}{p_{\eta}^{*}} = 2^{d-1}. \]

Note that we can now choose \( p_{\eta_1}^{*}^{(1)} \) and \( p_{\eta}^{*} \) arbitrary large when \( |\eta| \leq \frac{d}{2} - 1 \).
We then repeat the above way of reasoning replacing \( \|\Delta_h^{(1)}\|_{p_{\eta(1)}} \) with \( \|\Delta_h^{(1)} - \epsilon_d f\|_{p_{\eta(1) - \epsilon_d}} \). This is always possible, since the numbers \( p_{\eta(1)}^* \) and \( p_{\eta(1) - \epsilon_d}^* \) can be chosen arbitrary large.

The proof can be completed then by repeating this procedure the needed number of times. \( \square \)

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