Nonconvex Penalization in Sparse Estimation: An Approach Based on the Bernstein Function

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Abstract
In this paper we study nonconvex penalization using Bernstein functions whose first-order derivatives are completely monotone. The Bernstein function can induce a class of nonconvex penalty functions for high-dimensional sparse estimation problems. We derive a thresholding function based on the Bernstein penalty and discuss some important mathematical properties in sparsity modeling. We show that a coordinate descent algorithm is especially appropriate for regression problems penalized by the Bernstein function. We also consider the application of the Bernstein penalty in classification problems and devise a proximal alternating linearized minimization method. Based on theory of the Kurdyka-Lojasiewicz inequality, we conduct convergence analysis of these alternating iteration procedures. We particularly exemplify a family of Bernstein nonconvex penalties based on a generalized Gamma measure and conduct empirical analysis for this family.

Keywords: Bernstein functions, completely monotone functions, coordinate descent algorithms, Kurdyka-Lojasiewicz inequality, nonconvex penalization

1. Introduction
Variable selection plays a fundamental role in statistical modeling for high-dimensional data sets, especially when the underlying model has a sparse representation. The approach based on penalty theory has been widely used for variable selection. A principled approach is due to the lasso of Tibshirani (1996, 2011), which employs the $\ell_1$-norm penalty and performs variable selection via the soft thresholding operator. The lasso enjoys attractive statistical properties (Knight and Fu, 2000, Zhao and Yu, 2006). However, Fan and Li (2001) pointed out that the lasso method produces biased estimates for the large coefficients. Zou (2006) argued that the lasso might not be an oracle procedure under certain scenarios.

Fan and Li (2001) proposed three criteria for evaluating a good penalty function. That is, the resulting estimator should hold sparsity, continuity and unbiasedness. Moreover, Fan and Li (2001) showed that a nonconvex penalty generally admits the oracle properties. Zhang and Zhang (2012) presented a general theoretical analysis for nonconvex regularization. Meanwhile, there exist many nonconvex penalties, including the $\ell_q$ ($q \in (0,1)$) penalty, the smoothly clipped absolute deviation (SCAD) (Fan and Li, 2001), the minimax
concave plus penalty (MCP) (Zhang, 2010a), the kinetic energy plus penalty (KEP) (Zhang et al., 2013), the capped-$\ell_1$ function (Zhang, 2010b, Gong et al., 2013), the nonconvex exponential penalty (EXP) (Bradley and Mangasarian, 1998, Gao et al., 2011), the LOG penalty (Mazumder et al., 2011, Armagan et al., 2013), the smooth integration of counting and absolute deviation (SICA) (Lv and Fan, 2009), the hard thresholding penalty (Zheng et al., 2014), etc. These penalty functions have been demonstrated to have attractive properties theoretically and practically.

On the one hand, nonconvex penalty functions typically yield the tighter approximation to the $\ell_0$-norm and hold some good theoretical properties. On the other hand, they would make computational challenges due to nondifferentiability and nonconvexity that they have. Recently, Mazumder et al. (2011) developed a SparseNet algorithm based on coordinate descent. Specifically, the authors studied the coordinate descent algorithm for the MCP function and conducted convergence analysis (also see Breheny and Huang, 2011). Moreover, Mazumder et al. (2011) proposed some desirable properties for thresholding operators based on nonconvex penalty functions. For example, the thresholding operator is expected to have a nesting property; that is, it should be a strict nesting w.r.t. a sparsity parameter (see Section 4.1). However, the authors claimed that not all nonconvex penalty functions are suitable for use with coordinate descent.

In this paper we study Bernstein functions whose first-order derivatives are completely monotone (Schilling et al., 2010, Feller, 1971). The Bernstein function has the Lévy-Khintchine representation (Schilling et al., 2010). Since there is a one-to-one correspondence between Bernstein penalty functions and Laplace exponents of subordinators, the Bernstein function has been used to develop a nonparametric Bayesian approach to sparse estimation problems (Zhang and Li, 2015).

We introduce Bernstein functions into sparse estimation, giving rise to a unified approach to nonconvex penalization. We particularly exemplify a family of Bernstein nonconvex penalties based on a generalized Gamma measure (Aalen, 1992, Brix, 1999). The special cases include the KEP, nonconvex LOG and EXP as well as a penalty function that we call linear-fractional (LFR) function. Moreover, we find that the MCP function is a capped version of KEP. More specifically, our work offers the following major contributions.

- Applying the notion of regular variation (Feller, 1971), we establish the connection of the Bernstein function with the $\ell_q$-norm ($0 \leq q < 1$) and the $\ell_1$-norm. Using the notion of limiting-subdifferentials (Rockafellar and Wets, 1998), we show that the Bernstein function enjoys the Kurdyka-Lojasiewicz property (Lojasiewicz, 1993, Kurdyka, 1998, Bolte et al., 2007).

- We prove that the Bernstein penalty function admits the oracle properties and can result in an unbiased and continuous sparse estimator. We derive a thresholding function based on the Bernstein penalty. We show that this thresholding operator to some extent has the nesting property pointed out by Mazumder et al. (2011).

- We present a coordinate descent algorithm and a proximal alternating linearized algorithm for solving the regression and classification problems, respectively. Based on theory of the Kurdyka-Lojasiewicz inequality, we prove that these alternating iteration procedures have global convergence properties. Specifically, we show that the algorithms can find a strict local minimizer under certain regularity conditions.
The remainder of this paper is organized as follows. Section 2 reviews some preliminaries that will be used. Section 3 exploits Bernstein functions in the construction of nonconvex penalties. In Section 4 we investigate sparse estimation problems based on the Bernstein function. We devise a coordinate descent algorithm for finding the sparse regression solution and a proximal alternating linearized minimization algorithm for solving the classification problem. In Section 5 we show that these algorithms enjoy a global convergence property by using the novel Kurdyka-Lojasiewicz inequality. In Section 6 we conduct our empirical evaluations. Finally, we conclude our work in Section 7. All proofs are given in the appendix. In the appendix we also present some important properties of Bernstein functions and asymptotic consistencies of our concerned sparse estimation model.

2. Preliminaries

Suppose we are given a set of training data \( \{(x_i, y_i) : i = 1, \ldots, n\} \), where the \( x_i \in \mathbb{R}^p \) are the input vectors and the \( y_i \) are the corresponding outputs. We now consider the following supervised learning model:

\[
y = Xb + \varepsilon,
\]

where \( y = (y_1, \ldots, y_n)^T \) is the \( n \times 1 \) output (or response) vector, \( X = [x_1, \ldots, x_n]^T \) is the \( n \times p \) input (or design) matrix, and \( \varepsilon \) is an error vector. In regression problems \( \varepsilon \sim N(\varepsilon | 0, \sigma^2 I_n) \), and in classification problems it is defined as multivariate Bernoulli. We aim at finding a sparse estimate of regression vector \( b = (b_1, \ldots, b_p)^T \) under the regularization framework.

The classical regularization approach is based on a penalty function of \( b \). That is,

\[
\min_b \left\{ F(b) \triangleq L(b; y, X) + \lambda_n P(b) \right\},
\]

where \( L(\cdot) \) is the loss function, \( P(\cdot) \) is the regularization term penalizing model complexity, and \( \lambda_n (> 0) \) is the tuning parameter of balancing the relative significance of the loss function and the penalty. Specifically, \( L(b; X, y) \triangleq \sum_{i=1}^{n} \ell(b; x_i, y_i) \triangleq \sum_{i=1}^{n} \frac{1}{2} (y_i - b^T x_i)^2 = \frac{1}{2} \|y - Xb\|_2^2 \) in the regression problem, and \( L(b; X, y) \triangleq \sum_{i=1}^{n} \ell(b; x_i, y_i) \triangleq \sum_{i=1}^{n} \log(1 + \exp(-y_i b^T x_i)) \) in the classification problem where \( y_i \in \{-1, 1\} \).

A widely used setting for penalty is \( P(b) = \sum_{j=1}^{p} P(b_j) \), which implies that the penalty function consists of \( p \) separable subpenalties. In order to find a sparse solution of \( b \), one imposes the \( \ell_0 \)-norm penalty \( \|b\|_0 \) to \( b \) (i.e., the number of nonzero elements of \( b \)). However, the resulting optimization problem is usually NP-hard. Alternatively, the \( \ell_1 \)-norm \( \|b\|_1 = \sum_{j=1}^{p} |b_j| \) is an effective convex penalty. Additionally, some nonconvex alternatives, such as the LOG-penalty, SCAD, MCP and KEP, have been studied. Meanwhile, iteratively reweighted \( \ell_q \) \((q = 1 \text{ or } 2)\) minimization and coordination descent methods were developed for finding sparse solutions.

We take the MCP function as an example. Specifically, \( P(b) = M(\alpha|b|) \) where \( \alpha \) is a positive constant and \( M \) is defined as

\[
M(|b|) = \begin{cases} 
\frac{1}{2} & \text{if } |b| \geq 1, \\
|b| - \frac{\alpha^2}{2} & \text{if } |b| < 1.
\end{cases}
\]

Clearly, \( M'(|b|) \) exists w.r.t. \( |b| \). But the second derivative \( M''(|b|) \) does not exist at \( |b| = 1 \).
For a nonconvex penalty function \( P(b) \), it is interesting to exploit its maximum concavity, which is defined as
\[
\zeta(P) = \sup_{s,t \in \mathbb{R}, s < t} \frac{P'(t) - P'(s)}{t - s}.
\]
When \( P \) is twice differentiable on \([0, \infty)\), \( \zeta(P) = \sup_{s \in (0, \infty)} -P''(s) \) (Lv and Fan, 2009). That is, it is the maximum curvature of the curve \( P \).

We now recall the notion of subdifferentials, which can be used to define the subdifferential of a nonconvex function.

**Definition 1 (Subdifferentials)** (Rockafellar and Wets, 1998) Consider a proper and lower semi-continuous function \( f : \mathbb{R}^p \to (-\infty, +\infty] \) and a point \( x \in \text{dom}(f) \).

(i) The Fréchet subdifferential of \( f \) at \( x \), denoted \( \partial f(x) \), is the set of all vectors \( u \in \mathbb{R}^p \) which satisfy
\[
\liminf_{y \to x, y \neq x} \frac{f(y) - f(x) - u^T(y - x)}{\|y - x\|} \geq 0.
\]

(ii) The limiting-subdifferential of \( f \) at \( x \), denoted \( \partial f(x) \), is defined as
\[
\partial f(x) \equiv \left\{ u \in \mathbb{R}^p : \exists x_k \to x, f(x_k) \to f(x) \text{ and } u_k \in \hat{\partial} f(u_k) \to u \text{ as } k \to \infty \right\}.
\]

Notice that when \( x \notin \text{dom} f \), \( \hat{\partial} f(x) = \emptyset \) is the default setting. It is well established that \( \hat{\partial} f(x) \subseteq \partial f(x) \) for each \( x \in \mathbb{R}^p \), and both \( \partial f(x) \) and \( \partial f(x) \) are closed (Rockafellar and Wets, 1998). Moreover, if \( x \) is a critical point of \( f \), then \( 0 \in \partial f(x) \).

Let us see an example in which \( \partial f(x) \subseteq \partial f(x) \) for each \( x \in \mathbb{R}^p \), and both \( \partial f(x) \) and \( \partial f(x) \) are closed (Rockafellar and Wets, 1998). Moreover, if \( x \) is a critical point of \( f \), then \( 0 \in \partial f(x) \).

Next we briefly review the Kurdyka-Łojasiewicz property, which will play an important role in our global convergence analysis. Given \( \eta \in (0, \infty] \), we let \( \Pi_{\eta} \) denote the class of continuous concave functions \( \pi : [0, \eta) \to \mathbb{R}_+ \) which satisfy the following conditions:

(a) \( \pi(0) = 0 \),

(b) \( \pi \) is \( C^1 \) on \((0, \eta)\),

(c) \( \pi'(u) > 0 \) for all \( u \in (0, \eta) \).

Clearly, function \( \pi(u) = u^{1-\gamma} \) for \( \gamma \in (0, 1] \) belongs to \( \Pi_{\eta} \) with \( \eta = +\infty \).

**Definition 2 (Kurdyka-Łojasiewicz property)** Let the function \( f : \mathbb{R}^p \to (-\infty, +\infty] \) be proper and lower semi-continuous. Then \( f \) is said to have the Kurdyka-Łojasiewicz property at \( \bar{x} \in \text{dom} \partial f \) if there exist \( \eta \in (0, +\infty] \), a neighborhood \( U \) of \( \bar{x} \), and a function \( \pi \in \Pi_{\eta} \) such that for all \( x \in U \cap [f(\bar{x}) < f < f(\bar{x}) + \eta] \), the following Kurdyka-Łojasiewicz inequality holds
\[
\pi'(f(x) - f(\bar{x})) \text{dist}(0, \partial f(x)) \geq 1
\] under the notational conventions: \( 0^0 = 1 \) and \( \infty/\infty = 0/0 = 0 \). If \( f \) has the Kurdyka-Łojasiewicz property at every point in \( \text{dom} \partial f \), then \( f \) is said to have the Kurdyka-Łojasiewicz property. Here \( \text{dom}(\partial f) = \{ x : \partial f(x) \neq \emptyset \} \), and \( \text{dist}(v, A) = \inf \{ \|v - u\| : u \in A \} \).
The Kurdyka-Lojasiewicz property was proposed by Lojasiewicz (1993), who proved that for a real analytic function \( f \) and \( \pi(u) = u^{1-\gamma} \) with \( \gamma \in \left[ \frac{1}{2}, 1 \right) \), \( \frac{f(x) - f(y)}{\text{dist}(0, \partial f(x))} \) is bounded around any critical point \( \bar{x} \). Kurdyka (1998) extended this property to a class of functions with the \( \sigma \)-minimal structure. Bolte et al. (2007) extended to nonsmooth subanalytic functions. Recently, the Kurdyka-Lojasiewicz property has been used to establish convergence analysis of proximal alternating minimization for nonconvex problems (Attouch et al., 2010, Bolte et al., 2013, Xu and Yin, 2013).

We again consider the MCP function in Eqn. (1), and define \( f(b) = M(|b|) \). The graph of \( f \) is the closure of the set

\[
\{ (y, b) : y = \frac{1}{2}, b < -1 \} \cup \{ (y, b) : y = -\frac{b^2}{2} - b, -b < 1, b < 0 \} \\
\cup \{ (y, b) : y = -\frac{b^2}{2} + b, -b > 0, b < 1 \} \cup \{ (y, b) : y = \frac{1}{2}, -b < -1 \}.
\]

Thus, the graph is semialgebraic (Bolte et al., 2007). This implies that the MCP function satisfies the Kurdyka-Lojasiewicz property. Analogously, we also obtain that the SCAD function satisfies the Kurdyka-Lojasiewicz property.

The Huber loss function is a classical tool in robust regression. It is

\[
L_\delta(z) = \begin{cases} 
\frac{1}{2} z^2 & \text{if } |z| \leq \delta, \\
\delta |z| - \frac{1}{2} \delta^2 & \text{otherwise.}
\end{cases}
\] (2)

Obviously, the Huber loss function enjoyed the the Kurdyka-Lojasiewicz property.

### 3. Bernstein Penalty Functions

Let \( S \subset [0, \infty) \) and \( f \in C^\infty(S) \) with \( f \geq 0 \). We say \( f \) to be completely monotone if \((-1)^k f^{(k)} \geq 0 \) for all \( k \in \mathbb{N} \) and to be a Bernstein function if \((-1)^k f^{(k)} \leq 0 \) for all \( k \in \mathbb{N} \). It is well known that \( f \) is a Bernstein function if and only if the mapping \( s \mapsto \exp(-tf(s)) \) is completely monotone for all \( t \geq 0 \). Additionally, \( f \) is a Bernstein function if and only if it has the representation

\[
f(s) = a + \beta s + \int_0^{\infty} \left[ 1 - \exp(-su) \right] \nu(du) \quad \text{for all } s > 0,
\]

where \( a \geq 0 \) and \( \beta \geq 0 \), and \( \nu \) is the Lévy measure satisfying additional requirements \( \nu(-\infty, 0) = 0 \) and \( \int_0^{\infty} \min(u, 1) \nu(du) < \infty \). Moreover, this representation is unique. The representation is famous as the Lévy-Khintchine formula.

Since \( \lim_{s \to 0} f(s) = a \) and \( \lim_{s \to \infty} f(s) = \beta \) (Schilling et al., 2010), we will assume that \( \lim_{s \to 0} f(s) = 0 \) and \( \lim_{s \to \infty} f(s) = 0 \) to make \( a = 0 \) and \( \beta = 0 \). Notice that \( s^q \) for \( q \in (0, 1) \), is a Bernstein function of \( s \) on \((0, \infty)\) satisfying the above assumptions. However, \( f(s) = s \) is Bernstein but does not satisfy the condition \( \lim_{s \to \infty} \frac{f(s)}{s} = 0 \). Indeed, \( f(s) = s \) is an extreme example because \( \beta = 1 \) and \( \nu(du) = \delta_0(u)du \) (the Dirac Delta measure at the origin) in its Lévy-Khintchine formula. In fact, the condition \( \lim_{s \to \infty} \frac{f(s)}{s} = 0 \) aims at excluding this Bernstein function for our concern in this paper.
3.1 Properties

We now define the penalty function $P(b)$ as $\Phi(|b|)$, where the penalty term $\Phi(s)$ is a Bernstein function of $s$ on $[0, \infty)$ such that $\Phi(0) = 0$ and $\lim_{s \to \infty} \frac{\Phi(s)}{s} = 0$. Clearly, $\Phi(s)$ is nonnegative, nondecreasing and concave on $[0, \infty)$, because $\Phi(s) \geq 0$, $\Phi'(s) \geq 0$ and $\Phi''(s) \leq 0$. As a function of $b \in \mathbb{R}$, $\Phi(|b|)$ is of course continuous. Moreover, we have the following theorem.

**Theorem 3** Let $\Phi(s)$ be a nonzero Bernstein function of $s$ on $[0, \infty)$. Assume $\Phi(0) = 0$ and $\lim_{s \to \infty} \frac{\Phi(s)}{s} = 0$. Then

(a) $\Phi(|b|)$ is a nonnegative and nonconvex function of $b$ on $(-\infty, \infty)$, and an increasing function of $|b|$ on $[0, \infty)$.

(b) $\Phi(|b|)$ is continuous w.r.t. $b$ but nondifferentiable at the origin.

(c) Define $P(b) \triangleq \Phi(|b|)$. Then $\hat{\partial}P(0) = \partial P(0) = [-1, 1]$, and $\partial P(b) = \Phi'(|b|) \partial |b|$. Moreover, $P(b)$ satisfies the Kurdyka-Lojasiewicz property.

Recall that under the conditions in Theorem 3, $a$ and $\beta$ in the Lévy-Khintchine formula vanish. Theorem 3 (b) says that $\Phi(|b|)$ is singular at the origin. Thus, $\Phi(|b|)$ can define a class of sparsity-inducing nonconvex penalty functions. Theorem 3 (a) shows that $\Phi(|b|)$ satisfies Condition 1 given in Lv and Fan (2009). Theorem 3 (c) says that the Bernstein function has the same subdifferential with the function $|b|$ at the origin. Moreover, the Bernstein function has the Kurdyka-Lojasiewicz property. As mentioned earlier, however, the subdifferential of the function $|b|^q$ for $0 < q < 1$ at $b = 0$ is $(-\infty, \infty)$.

We can clearly see the connection of the bridge penalty $|b|^q$ with the $\ell_0$-norm and the $\ell_1$-norm, as $q$ goes from 0 to 1. However, the sparse estimator resulted from the bridge penalty is not continuous. This would make numerical computations and model predictions unstable (Fan and Li, 2001). In this paper we consider another class of Bernstein nonconvex penalty functions.

In particular, to explore the relationship of the Bernstein penalty with the $\ell_0$-norm and the $\ell_1$-norm, we further assume that $\Phi'(0) = \lim_{s \to 0} \Phi'(s) < \infty$. Since $\Phi(s)$ is a nonzero Bernstein function of $s$, we can conclude that $\Phi'(0) > 0$. If it were not true, we would have $\Phi'(s) = 0$ due to $\Phi'(s) \leq \Phi'(0)$. This implies that $\Phi(s) = 0$ for any $s \in (0, \infty)$ because $\Phi(0) = 0$. This conflicts with that $\Phi(s)$ is nonzero. Similarly, we can also deduce $\Phi''(0) < 0$. Based on this fact, we can change the assumption $\Phi'(0) < \infty$ as $\Phi'(0) = 1$ without loss of generality. In fact, we can replace $\Phi(s)$ with $\frac{\Phi(s)}{\Phi'(0)}$ to meet this assumption, because the resulting $\Phi$ is still Bernstein and satisfies $\Phi(0) = 0$, $\lim_{s \to \infty} \frac{\Phi(s)}{s} = 0$ and $\Phi'(0) = 1$.

**Theorem 4** Assume that the conditions in Theorem 3 hold. Let $\Phi_\alpha(|b|) = \frac{\Phi(\alpha|b|)}{\Phi'(\alpha)}$ for $\alpha > 0$. If $\Phi'(0) = 1$, then

$$\lim_{\alpha \to 0^+} \Phi_\alpha(|b|) = |b| \quad \text{and} \quad \lim_{\alpha \to 0^+} \zeta(\Phi_\alpha) = 0.$$ 

Furthermore, for $b \neq 0$ we have that

$$\lim_{\alpha \to \infty} \Phi_\alpha(|b|) = |b|^\gamma \quad \text{and} \quad \lim_{\alpha \to \infty} \zeta(\Phi_\alpha) = \infty.$$
Here γ = \lim_{s \to \infty} \frac{s \Phi'(s)}{\Phi(s)} \in [0, 1]. Especially, if γ ∈ (0, 1), we also have that
\lim_{\alpha \to \infty} \frac{\Phi'(|\alpha| b)}{\Phi'(|\alpha|)} = |b|^{-1}.

Remarks 1  It is worth noting that Φ' is completely monotone on [0, ∞). Moreover, Φ' is the Laplace transform of some probability distribution due to Φ'(0) = 1 (Feller, 1971). Additionally, Lemma 14 in the appendix shows that \lim_{s \to \infty} \frac{s \Phi'(s)}{\Phi(s)} = 0 whenever \lim_{s \to \infty} \Phi(s) < ∞. Notice that we cannot ensure that γ < 1. For example, it is known that the function Φ(s) ≜ \frac{s}{\log(e + s)} is a Bernstein function on (0, ∞) (Schilling et al., 2010). It is directly computed that \lim_{s \to 0^+} \Phi(s) = 0, \lim_{s \to 0^+} \Phi'(s) = 1, and \lim_{s \to 0^+} \Phi''(s) = -\frac{2}{e}. However, it is also obtained that \lim_{s \to \infty} \frac{s \Phi'(s)}{\Phi(s)} = 1.

Remarks 2  It follows from Theorem 1 in Chapter VIII.9 of Feller (1971) that \lim_{s \to \infty} \frac{s \Phi'(s)}{\Phi(s)} = γ ∈ (0, 1) if and only if \lim_{\alpha \to \infty} \frac{\Phi'(|\alpha| b)}{\Phi'(|\alpha|)} = |b|^{-1}. However, \lim_{\alpha \to \infty} \frac{\Phi'(|\alpha| b)}{\Phi'(|\alpha|)} = |b|^{-1} (i.e., γ = 0) is only sufficient for \lim_{s \to \infty} \frac{s \Phi'(s)}{\Phi(s)} = 0.

Remarks 3  It is direct to obtain ζ(Φ_α) = -\frac{s^2}{\Phi'(\alpha)} Φ''(0). Clearly, ζ(Φ_α) is increasing in α. Thus, α controls the maximum concavity of Φ_α.

The second part of Theorem 4 shows that the property of regular variation for the Bernstein function Φ(s) and its derivative Φ'(s) (Feller, 1971). That is, Φ and Φ' vary regularly with exponents γ and γ−1, respectively. If γ = 0, then Φ varies slowly. This property implies an important connection of the Bernstein function with the ℓ_0-norm and ℓ_1-norm. With this connection, we see that α plays a role of sparsity parameter because it measures sparseness of Φ(α|b)|/Φ(α). In the following we present a family of Bernstein functions which admit the properties in Theorem 4.

| Bernstein functions | First-order derivatives | Lévy measures |
|---------------------|------------------------|--------------|
| KEP     | Φ_{-1}(s) = \sqrt{2s+1} - 1 | ν(du) = \frac{2}{\sqrt{2\pi}} u^{-\frac{3}{2}} \exp(-\frac{u}{2})du |
| LOG    | Φ_{0}(s) = \log(s+1) | ν(du) = \frac{1}{\sqrt{2\pi}} \exp(-u)du |
| LFR    | Φ_{1/2}(s) = \frac{2s}{s+2} | ν(du) = 4 \exp(-2u)du |
| EXP    | Φ_{1}(s) = 1 - \exp(-s) | ν(du) = \delta_1(u)du |

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| EXP    | Φ_{1}(s) = 1 - \exp(-s) | ν(du) = \delta_1(u)du |

3.2 Examples

We consider a family of Bernstein functions of the form
\[ Φ_ρ(s) = \begin{cases} 
\log(1 + s) & \text{if } ρ = 0, \\
\frac{1}{ρ} \left[ 1 - (1 + (1-ρ)s)^{-\frac{1}{1-ρ}} \right] & \text{if } ρ < 1 \text{ and } ρ \neq 0, \\
1 - \exp(-s) & \text{if } ρ = 1.
\end{cases} \]
It is worth noting that this function is related to the Box-Cox transformation (Box and Cox, 1964). It can be directly verified that $\Phi_0(s) = \lim_{\rho \to 0} \Phi_\rho(s)$ and $\Phi_1(s) = \lim_{\rho \to 1^-} \Phi_\rho(s)$. The corresponding Lévy measure is

$$\nu(du) = \frac{((1-\rho))^{-1/(1-\rho)}}{\Gamma(1/(1-\rho))} u^{\frac{\rho}{1-\rho} - 1} \exp\left(-\frac{u}{(1-\rho)}\right) du. \quad (4)$$

Notice that $u\nu(du)$ forms a Gamma measure for random variable $u$. Thus, this Lévy measure $\nu(du)$ is referred to as a generalized Gamma measure (Brix, 1999). This family of the Bernstein functions was studied by Aalen (1992) for survival analysis. We here show that they can also be used for sparsity modeling.

It is easily seen that the Bernstein functions $\Phi_\rho$ for $\rho \leq 1$ satisfy the conditions: $\Phi(0) = 0$, $\Phi'(0) = 1$ and $(-1)^k \Phi^{(k+1)}(0) < \infty$ for $k \in \mathbb{N}$, in Theorem 4 and Lemma 14 (see the appendix). Thus, $\Phi_\rho$ for $\rho \leq 1$ have the properties given in Theorem 4 and Lemma 14. These properties show that when letting $s = |b|$, the Bernstein functions $\Phi(|b|)$ form nonconvex penalty functions.

The derivative of $\Phi_\rho(s)$ is defined by

$$\Phi'(\rho)(s) = \begin{cases} \frac{1}{1+s} & \text{if } \rho = 0, \\ \frac{1}{1+(1-\rho)s} & \text{if } \rho < 1 \text{ and } \rho \neq 0, \\ \exp(-s) & \text{if } \rho = 1. \end{cases} \quad (5)$$

It is also directly verified that $\Phi_0'(s) = \lim_{\rho \to 0} \Phi_\rho'(s)$ and $\Phi_1'(s) = \lim_{\rho \to 1^-} \Phi_\rho'(s)$. When $\rho \in [0, 1]$, we have $\lim_{s \to \infty} s\Phi_\rho'(s) = 0$ (or $\lim_{s \to \infty} \Phi_\rho(s) \ln(s) < \infty$). When $\rho < 0$, we then have $\lim_{s \to \infty} s\Phi'_\rho(s) = \frac{\rho}{\rho - 1} \in (0, 1)$.

**Proposition 5** Let $\Phi_\rho(s)$ on $[0, \infty)$ be defined in (3) and $\Phi_\rho,\alpha(s) = \frac{\Phi_\rho(\alpha s)}{\Phi_\rho(\alpha)}$ for $\alpha > 0$. Then

(a) If $-\infty < \rho_1 < \rho_2 \leq 1$ then $\Phi_{\rho_1}'(s) \geq \Phi_{\rho_2}'(s)$, $\Phi_{\rho_1}(s) \geq \Phi_{\rho_2}(s)$, and $\zeta(\Phi_{\rho_1},\alpha) \leq \zeta(\Phi_{\rho_2},\alpha)$;

(b) $\lim_{\alpha \to \infty} \frac{\Phi_{\rho}'(\alpha)}{\alpha^{\gamma - 1}} = (1 - \gamma)^{1 - \gamma}$ where $\gamma = 0$ if $\rho \in (0, 1]$ and $\gamma = \frac{\rho}{\rho - 1}$ if $\rho \in (-\infty, 0]$; and

$$\lim_{\alpha \to \infty} \frac{\Phi_{\rho}(\alpha s)}{\Phi_{\rho}(\alpha)} = \begin{cases} 1 & \text{if } \rho \in [0, 1], \\ \frac{s}{s^{\rho - 1}} & \text{if } \rho \in (-\infty, 0). \end{cases}$$

Proposition 5-(b) shows the property of regular variation for $\Phi_\rho$; that is, $\Phi_\rho$ varies slowly when $0 \leq \rho \leq 1$, while it varies regularly with exponent $\rho/(\rho-1)$ when $\rho < 0$. Thus, $\Phi_{\rho,\alpha}(b)$ for $\rho < 0$ approaches to the $\ell_2/(\rho-1)$-norm $\|b\|_{\rho/(\rho-1)}$ as $\alpha \to \infty$.

We list four special Bernstein functions in Table 1 by taking different $\rho$. Specifically, these are the kinetic energy plus (KEP) function, nonconvex log-penalty (LOG), nonconvex exponential-penalty (EXP), and linear-fractional (LFR) function, respectively. Figure 1 depicts these functions and their derivatives. In Table 1 we also give the Lévy measures corresponding to these functions. Clearly, KEP gets a continuum of penalties.
from $\ell_{1/2}$ to the $\ell_1$, as varying $\alpha$ from $\infty$ to 0 (Zhang et al., 2013). But the LOG, EXP and LFR functions get the entire continuum of functions from $\ell_0$ to the $\ell_1$.

The LOG, EXP and LFR functions have been applied in the literature (Bradley and Mangasarian, 1998, Gao et al., 2011, Weston et al., 2003, Geman and Reynolds, 1992, Nikolova, 2005, Lv and Fan, 2009). Lv and Fan (2009) also called LFR a smooth integration of counting and absolute deviation (SICA) penalty. In image processing and computer vision, these functions are usually also called potential functions. However, to the best of our knowledge, there is no work to establish their connection with Bernstein functions.

![Figure 1](a) The Bernstein functions $\Phi_{\rho}(s)$ for $\rho = -1$, $\rho = 0$, $\rho = \frac{1}{2}$ and $\rho = 1$ corresponding to KEP, LOG, LFR and EXP. (b) The corresponding derivatives $\Phi'_{\rho}(s)$.

![Figure 2](a) The Bernstein function $\Phi_2(|b|)$ and the MCP function $M(|b|)$.
Finally, we note that the MCP function can be regarded as a capped version of $\Phi_2$ (i.e., $\rho = 2$). Clearly, $\Phi_2(s)$ is well-defined for $s \geq 0$ but no longer Bernstein, because $\Phi_2(s)$ is negative when $s > 2$. Moreover, it is decreasing when $s \geq 1$ (see Figure 2). To make a concave penalty function from $\Phi_2$, we truncate $\Phi_2(s)$ into $\frac{1}{2}$ whenever $s \geq 1$, yielding the MCP function given in (1).

4. Sparse Estimation Based on Bernstein Penalty Functions

We now study mathematical properties of the sparse estimators based on Bernstein penalty functions. These properties show that Bernstein penalty functions are suitable for use of a coordinate descent algorithm.

4.1 Thresholding Operators

Let $\Phi$ be a Bernstein penalty function. We define a univariate penalized least squares problem as follows.

$$
J_1(b) \triangleq \frac{1}{2}(z - b)^2 + \lambda \Phi(|b|),
$$

(6)

where $z = x^Ty$. It has been established by Fan and Li (2001) that a good penalty should result in an estimator with three properties. (a) “Unbiasedness:” it is nearly unbiased when the true unknown parameter takes a large value in magnitude; (b) “Sparsity:” there is a thresholding operator, which automatically sets small estimated coefficients to zero; (c) “Continuity:” it is continuous in $z$, which can avoid instability in model computation and prediction.

It suffices for the estimator obtained from (6) to be nearly unbiased that $\Phi'(|b|) \to 0$ as $|b| \to \infty$. The Bernstein penalty function satisfies the conditions $\Phi(0+) = 0$ and $\lim_{s \to \infty} \Phi'(s) = 0$, so it can result in an unbiased sparse estimator.

**Theorem 6** Let $\Phi$ be a nonzero Bernstein function on $[0, \infty)$ such that $\Phi(0) = 0$ and $\lim_{s \to \infty} \frac{\Phi(s)}{s} = \lim_{s \to \infty} \Phi'(s) = 0$. Consider the penalized least squares problem in (6).

(i) If $\lambda \leq -\frac{1}{\Phi''(0)}$, then the resulting estimator is defined as

$$
\hat{b} = S(z, \lambda) \triangleq \begin{cases} 
\operatorname{sgn}(z)\kappa(|z|) & \text{if } |z| > \lambda \Phi'(0), \\
0 & \text{if } |z| \leq \lambda \Phi'(0),
\end{cases}
$$

where $\kappa(|z|) \in (0, |z|)$ is the unique positive root of $b + \lambda \Phi'(b) - |z| = 0$ in $b$.

(ii) If $\lambda > -\frac{1}{\Phi''(0)}$, then the resulting estimator is defined as

$$
\hat{b} = S(z, \lambda) \triangleq \begin{cases} 
\operatorname{sgn}(z)\kappa(|z|) & \text{if } |z| > s^* + \lambda \Phi'(s^*), \\
0 & \text{if } |z| \leq s^* + \lambda \Phi'(s^*),
\end{cases}
$$

where $s^* > 0$ is the unique root of $1 + \lambda \Phi''(s) = 0$ and $\kappa(|z|)$ is the unique root of $b + \lambda \Phi'(b) - |z| = 0$ on $(s^*, |z|)$. 

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As we see earlier, we always have $\Phi'(0) > 0$ and $\Phi''(0) < 0$. It is worth noting that when $\lambda \leq -\frac{1}{\Phi''(0)}$, the function $h(b) = b + \lambda \Phi'(b) - |z|$ is increasing on $(0, |z|)$ and that when $\lambda > -\frac{1}{\Phi''(0)}$, it is also increasing on $(s^*, |z|)$. Thus, we can employ the bisection method to find the corresponding root $\kappa(|z|)$. We will see that an analytic solution for $\kappa(|z|)$ is available when $\Phi(s)$ is either of KEP, LOG and LFR. Therefore, a coordinate descent algorithm is especially appropriate for Bernstein penalty functions, which will be presented in Section 4.2.

It suffices for the resulting estimator to be sparse that the minimum of the function $|b| + \lambda \Phi'(|b|)$ is positive. Moreover, a sufficient and necessary condition for “continuity” is that the minimum of $|b| + \lambda \Phi'(|b|)$ is attained at 0. In our case, it follows from the proof of Theorem 6 that when $\lambda \leq -\frac{1}{\Phi''(0)}$, $|b| + \lambda \Phi'(|b|)$ attains its minimum value $\lambda \Phi''(0)$ at $s^* = 0$. Thus, the resulting estimator is sparse and continuous when $\lambda \leq -\frac{1}{\Phi''(0)}$. In fact, the continuity can be also concluded directly from Theorem 6-(i). Specifically, when $\lambda \leq -\frac{1}{\Phi''(0)}$, we have $\kappa(\lambda \Phi'(0)) = 0$ because 0 is the unique root of equation $b + \lambda \Phi'(b) - \lambda \Phi''(0) = 0$.

Recall that if $\Phi(s) = s^q$ with $q \in (0, 1)$, we have $\lim_{s \to 0} \Phi'(s) = +\infty$ and $\lim_{s \to 0} \Phi''(s) = -\infty$. This implies that $\lambda \leq -\frac{1}{\Phi''(0)}$ does not hold. In other words, this penalty cannot result in a continuous solution.

In this paper we are especially concerned with the Bernstein penalty functions which satisfy the conditions in Theorem 4. In this case, since $-\infty < \Phi''(0) < 0$ and $0 < \Phi'(0) < \infty$, such Bernstein penalties are able to result in a continuous sparse solution. Consider the regular variation property of $\Phi(s)$ given in Theorem 4. We let $P(b) = \Phi(\alpha|b|)$ and $\lambda = \frac{\eta}{\Phi'(0)}$, where $\eta$ and $\alpha$ are positive constants. We now denote the thresholding operator $S(z, \lambda)$ in Theorem 6 by $S_{\alpha}(z, \eta)$. As a direct corollary of Theorem 6, we particularly have the following results.

**Corollary 7** Assume $\Phi'(0) = 1$ and $\Phi''(0) > -\infty$. Let $P(b) = \Phi(\alpha|b|)$ and $\lambda = \frac{\eta}{\Phi'(0)}$ where $\alpha > 0$ and $\eta > 0$, and let $S_{\alpha}(z, \eta)$ be the thresholding operator defined in Theorem 6.

(i) If $\eta \leq -\frac{\Phi'(0)}{\alpha \Phi''(0)}$, then the resulting estimator is defined as

$$
\hat{b} = S_{\alpha}(z, \eta) \triangleq \begin{cases} 
\text{sgn}(z)\kappa(|z|) & \text{if } |z| > \frac{\alpha}{\Phi'(0)} \eta, \\
0 & \text{if } |z| \leq \frac{\alpha}{\Phi'(0)} \eta,
\end{cases}
$$

where $\kappa(|z|) \in (0, |z|)$ is the unique positive root of $b + \frac{\alpha^2}{\Phi'(0)} \Phi'(ab) - |z| = 0$ w.r.t. $b$.

(ii) If $\eta > -\frac{\Phi'(0)}{\alpha \Phi''(0)}$, then the resulting estimator is defined as

$$
\hat{b} = S_{\alpha}(z, \eta) \triangleq \begin{cases} 
\text{sgn}(z)\kappa(|z|) & \text{if } |z| > s^* + \frac{\alpha \Phi'(\alpha s^*)}{\Phi'(0)} \eta, \\
0 & \text{if } |z| \leq s^* + \frac{\alpha \Phi'(\alpha s^*)}{\Phi'(0)} \eta,
\end{cases}
$$

where $s^* > 0$ is the unique root of $1 + \frac{\alpha^2}{\Phi'(0)} \Phi''(\alpha s^*) = 0$ and $\kappa(|z|)$ is the unique root of the equation $b + \frac{\alpha}{\Phi'(0)} \Phi'(ab) - |z| = 0$ on $(s^*, |z|)$.

**Proposition 8** Assume $\Phi'(0) = 1$ and $\Phi''(0) > -\infty$. Then
(a) $\Phi'(\alpha)$ is increasing and $\frac{1}{\Phi(\alpha)}$ is decreasing, both in $\alpha$ on $(0, \infty)$. Moreover, $\lim_{\alpha \to 0^+} \frac{\alpha}{\Phi(\alpha)} = 1$ and $\lim_{\alpha \to \infty} \frac{\alpha}{\Phi(\alpha)} = \infty$.

(b) The root $\kappa(|z|)$ is strictly increasing w.r.t. $|z|$.

(c) When $\lambda < -1/\Phi''(0)$, the function $\frac{1}{2}b^2 + \lambda \Phi(|b|)$ is strictly convex in $b \in \mathbb{R}$.

The Bernstein function $\Phi_\rho$ given in (3) satisfies the conditions in Corollary 7 and Proposition 8. Proposition 8-(a) and (c) implies that $\Phi(|b|)$ satisfies Assumption 1 made in Loh and Wainwright (2013). Recall that $\alpha$ controls sparseness of $\Phi(\alpha|b|)/\Phi(\alpha)$ as varying $\alpha$ from 0 to $\infty$. It follows from Proposition 8 that $|z| \geq \eta$ due to $|z| \geq \frac{\eta}{\Phi(\alpha)}$. This implies that the Bernstein function $\Phi(\alpha|b|)/\Phi(\alpha)$ has stronger sparseness than the $\ell_1$-norm when $\eta \leq \frac{\Phi(\alpha)}{\Phi'(\alpha)}$. Moreover, for a fixed $\eta$, there is a strict nesting of the shrinkage thresholding $\frac{\eta}{\Phi(\alpha)}$ as $\alpha$ increases. Thus, the Bernstein penalty to some extent satisfies the nesting property, a desirable property for thresholding functions pointed out by Mazumder et al. (2011).

As we stated earlier, when $\rho \in [0, 1]$ $\Phi_\rho$ gives a smooth homotopy between the $\ell_0$-norm and the $\ell_1$-norm. We now explore a connection of the thresholding operator $S_\alpha(z, \eta)$ with the soft thresholding operator based on the lasso and the hard thresholding operator based on the $\ell_0$-norm.

**Theorem 9** Let $S_\alpha(z, \eta)$ be the thresholding operator defined in Corollary 7. Then

$$\lim_{\alpha \to 0^+} S_\alpha(z, \eta) = \begin{cases} \text{sgn}(z)(|z| - \eta) & \text{if } |z| > \eta, \\
0 & \text{if } |z| \leq \eta. \end{cases}$$

Furthermore, if $\lim_{\alpha \to \infty} \frac{\alpha \Phi'(\alpha)}{\Phi(\alpha)} = 0$ or $\lim_{\alpha \to \infty} \frac{\Phi(\alpha)}{\log(\alpha)} < \infty$, then

$$\lim_{\alpha \to \infty} S_\alpha(z, \eta) = \begin{cases} |z| & \text{if } |z| > 0, \\
0 & \text{if } |z| \leq 0. \end{cases}$$

In the limiting case of $\alpha \to 0$, Theorem 9 shows that the thresholding function $S_\alpha(z, \eta)$ approaches the soft thresholding function $\text{sgn}(z)(|z| - \eta)_+$. However, as $\alpha \to \infty$, the limiting solution does not fully agree with the hard thresholding function, which is defined as $z I(|z| \geq \sqrt{2\eta})$.

Let us return the concrete Bernstein functions in Table 1. We are especially interested in the KEP, LOG and LFR functions, because there are analytic solutions for $\kappa(|z|)$ based on them. Corresponding to LOG and LFR, $\kappa(|z|)$ are respectively

$$\kappa(|z|) = \frac{\alpha|z| - 1 + \sqrt{(1 + \alpha|z|^2)^2 - 4\alpha^2}}{2\alpha} \tag{7}$$

and

$$\kappa(|z|) = \frac{2(\alpha|z| + 2)}{3\alpha} \cos \left[\frac{1}{3} \arccos \left(1 - \frac{3}{\alpha|z| + 2} \right)^3\right] + \frac{\alpha|z| + 2}{3\alpha} - \frac{2}{\alpha} \tag{8}.$$

The derivation can be obtained by using direct algebraic computations. We here omit the derivation details. As for KEP, $\kappa(|z|)$ was derived by Zhang et al. (2013). That is,

$$\kappa(|z|) = \frac{4(2\alpha|z| + 1)}{3} \cos^2 \left[\frac{1}{3} \arccos \left(-\lambda^2 \left(\frac{3}{2\alpha|z| + 1} \right)^3\right)\right] - \frac{1}{\alpha}.$$
4.2 Coordinate Descent Algorithm for the Penalized Regression Problem

Based on the discussion in the previous subsection, the Bernstein penalty function is suitable for the coordinate descent algorithm under the regression setting where \( L(b; X, y) = \frac{1}{2} ||y - Xb||_2^2 \). Specifically, we give the coordinate descent procedure in Algorithm 1. If the LOG and LFR functions are used, the corresponding thresholding operators have the analytic forms in (7) and (8). Otherwise, we employ the bisection method for finding the root \( \kappa(|z|) \). The method is also very efficient.

When \( \lambda \leq -\frac{1}{\alpha^2 \Phi''(0)} \) (or \( \lambda > -\frac{1}{\alpha^2 \Phi''(0)} \)), we can obtain that \( |\hat{b}| \leq |z| \) always holds. The objective function \( J_1(b) \) in (6) is strictly convex in \( b \) whenever \( \lambda \leq -\frac{1}{\alpha^2 \Phi''(0)} \). Moreover, according to Theorem 8, the estimator \( \hat{b} \) in both the cases is strictly increasing w.r.t. \( |z| \). As we see, \( P(b) \triangleq \Phi(\alpha|b|) \) satisfies \( P(b) = P(-b) \). Moreover, \( P''(b) \) is positive and uniformly bounded on \([0, \infty)\), and \( \inf_b \lambda P''(b) > -1 \) on \([0, \infty)\) when \( \lambda < -\frac{1}{\alpha^2 \Phi''(0)} \). Thus, the algorithm shares the same convergence property as in Mazumder et al. (2011) (see Theorem 4 therein). It is worth noting that Theorem 4 of Mazumder et al. (2011) requires the second-order derivative \( P''(b) \) on \([0, \infty)\) to exist. However, for the MCP function \( M \) defined in (1) its second-order derivative at \( |b| = 1 \) does not exist. In contrast, our used Bernstein penalty function meets this requirement. In Section 5 we will give a global convergence analysis based on the Kurdyka-Łojasiewicz property.

Algorithm 1 Coordinate descent algorithm for the penalized regression problem

\begin{itemize}
  \item \textbf{Input:} \( \{x_i, y_i\}_{i=1}^n \) where each column of \( X = [x_1, \ldots, x_n]^T \) is standardized to have mean 0 and length 1, a grid of increasing values \( \Lambda = \{\eta_1, \ldots, \eta_L\} \), a grid of decreasing values \( \Gamma = \{\alpha_1, \ldots, \alpha_K\} \) where \( \alpha_K \) indexes the Lasso penalty. Set \( \hat{b}_{\alpha_K, \eta_{L+1}} = 0. \)
  \item for each value of \( l \in \{L, L-1, \ldots, 1\} \) do
    \begin{itemize}
      \item Initialize \( \tilde{b} = \hat{b}_{\alpha_K, \eta_{L+1}} \);
      \item for each value of \( k \in \{K, K-1, \ldots, 1\} \) do
        \begin{itemize}
          \item if \( \eta_l \leq -\frac{\Phi(\alpha_k)}{\alpha_k^2 \Phi''(0)} \) then
            \begin{itemize}
              \item Cycle through the following one-at-a-time updates
              \[ \tilde{b}_j = S_{\alpha_k} \left( \sum_{i=1}^n (y_i - z_i^j) x_{ij}, \eta_l \right), \quad j = 1, \ldots, p \]
              \end{itemize}
          \end{itemize}
        \end{itemize}
      \end{itemize}
    \end{itemize}
  \item Output: Return the two-dimensional solution \( \hat{b}_{\alpha, \eta} \) for \((\alpha, \eta) \in \Lambda \times \Gamma.\)
\end{itemize}
### 4.3 Extension to Classification and Robust Regression Problems

We now consider the classification problem in which the loss function is defined as $L(b; X, y) = \sum_{i=1}^{n} \log(1 + \exp(-y_i b^T x_i))$ and the penalty function is still defined as Bernstein function $\Phi(\alpha|b|)$. Breheny and Huang (2011) suggested that the corresponding minimization problem is approached by first obtaining a quadratic approximation to the loss function $L(b; X, y)$ based on a Taylor series expansion about the current iterative value of $b$. That is,

$$
\mathbf{b}^{(t+1)} = \arg\min_{\mathbf{b}} \left\{ L(b^{(t)}; X, y) + \langle \nabla L(b^{(t)}; X, y), \mathbf{b} - b^{(t)} \rangle + \frac{1}{2}(\mathbf{b} - b^{(t)})^T \nabla^2 L(b^{(t)}; X, y)(\mathbf{b} - b^{(t)}) + \lambda_n \sum_{j=1}^{p} \Phi(\alpha|b_j|) \right\}.
$$

Alternatively, we resort to a proximal alternating linearized minimization (PALM) procedure to solve the minimization problem. Specifically, the procedure chooses variables $b_1, \ldots, b_p$ in cyclic order at each time. Let $L_j^{(t)}(b_j) = L(b_j^{(t)}; X, y)$ where $b_j^{(t)} = (b_1^{(t)}, \ldots, b_j^{(t)}, b_{j+1}^{(t)}, \ldots, b_p^{(t)})^T$. When optimizing variable $b_j$ with the rest variables fixed, we use a linear approximation of $L_j$ with a proximal regularization term. That is,

$$
b_j^{(t+1)} = \arg\min_{b_j} \left\{ \frac{1}{2} L_j^{(t)}(b_j^{(t)}) + \nabla_j L_j^{(t)}(b_j^{(t)})(b_j - b_j^{(t)}) + \frac{\nu_j^{(t)}}{2}(b_j - b_j^{(t)})^2 + \lambda_n \Phi(\alpha|b_j|) \right\}. \quad (9)
$$

Typically, the optimal solution can be represented as $b_j^{(t+1)} = \text{Prox}_{\nu_j}(b_j^{(t)} - \frac{\nu_j^{(t)}}{\nu_j^{(t)}} \nabla_j L_j^{(t)}(b_j^{(t)}))$.

The proximal operator is defined as

$$
\text{Prox}_{\nu}(u) \triangleq \arg\min_{x} \left\{ \frac{\nu}{2} \|x - u\|^2 + g(x) \right\}.
$$

We summary the whole procedure in Algorithm 2.

Algorithm 2 can be also used to solve a robust regression problem regularized by the Bernstein function. The loss function is then defined by

$$
L(b; X, y) = \sum_{i=1}^{n} = L_\delta(y_i - b^T x_i),
$$

where $L_\delta(y_i - b^T x_i)$ is the Huber loss as in (2).

### 5. Convergence Analysis

In this section we present the global convergence analysis of the previous coordinate descent and PALM procedures in Algorithms 1 and 2. In particular, we consider the following optimization problem

$$
\min_{\mathbf{b}} F(b) \triangleq L(b; X, y) + \lambda_n \sum_{j=1}^{p} P(b_j),
$$

where $P(b_j) = \Phi(\alpha|b_j|)$. We further make the following assumptions:
Algorithm 2 PALM for the penalized classification problem

Initialization: set the initial value $b^{(0)}$.

for $t = 0, 1, \ldots$ do
  for $j = 1, 2, \ldots, p$ do
    $b^{(t+1)}_j = \text{Prox}_{\nu^{(t)}_j} (b^{(t)}_j - \frac{1}{\nu^{(t)}_j} \nabla L^{(t)}_j (b^{(t)}_j))$.
  end for
  if stopping criterion is met then
    Return $b^{(t)}$.
  end if
end for

Assumption 1 In Algorithm 2, assume that $0 < m_0 < \nu^{(t)}_j < M_0 < \infty$ for every $j$ and $t$.

Assumption 2 $L^{(t)}_j$ is strongly convex with modulus $0 < -\lambda_n n^2 \Phi''(0) < \gamma^{(t)}_j < \gamma_M < \infty$, namely,

$$L^{(t)}_j (u) - L^{(t)}_j (v) \geq \langle \nabla L^{(t)}_j (v), u - v \rangle + \frac{\gamma^{(t)}_j}{2} \| u - v \|^2;$$

and $\nabla L^{(t)}_j$ is Lipschitz continuous.

Notice that both the logistic loss function and the least squares loss function meet Assumption 2 (Xu and Yin, 2013). The Huber loss function also satisfies Assumption 2. Thus, the following convergence analysis applies to the case that the loss function is defined as the Huber loss function in (2).

Our convergence analysis mainly includes three steps. First, we show the sequences \{$F(b^{(t)}): t \in \mathbb{N}$\} generated by the algorithms have the sufficient decrease property, and hence establish the square summable result $\sum_{t=0}^{\infty} \| b^{(t+1)} - b^{(t)} \|^2 < +\infty$ in Theorem 10. Second, based on the fact that the Bernstein penalty function has the Kurdyka-Lojasiewicz property, in Theorem 12 we improve the result to $\sum_{t=0}^{\infty} \| b^{(t+1)} - b^{(t)} \| < +\infty$, which implies that the generated sequence \{b^{(t)}: t \in \mathbb{N}\} is a Cauchy sequence. Consequently, it converges to a critical point of the objective function $F$. Third, noting that the Bernstein penalty function satisfies Condition 1 of Lv and Fan (2009), as a direct corollary of Theorem 1 of Lv and Fan (2009), Fan and Lv (2011), we prove that \{b^{(t)}: t \in \mathbb{N}\} converges to a strict local minimizer of $F$ under certain regularity conditions.

**Theorem 10** Suppose Assumption 1 holds for Algorithm 2, or Assumption 2 holds for Algorithm 1. Let the sequence \{b^{(t)}: t \in \mathbb{N}\} be generated by Algorithm 1 or Algorithm 2. Then we have the following properties:

(i) [Sufficient decrease property] The generated sequence \{F(b^{(t)}): t \in \mathbb{N}\} is nonincreasing; particularly,

$$F(b^{(t)}) - F(b^{(t+1)}) \geq \frac{C_0}{2} \| b^{(t)} - b^{(t+1)} \|^2, \quad \forall t \geq 0,$$

where $C_0$ is some positive constant.
(ii) [Square summable property]
\[ \sum_{t=0}^{\infty} \| b^{(t+1)} - b^{(t)} \|^2 < +\infty, \]
which implies \( \lim_{t \to \infty} \| b^{(t+1)} - b^{(t)} \| = 0. \)

(iii) [Subgradient lower bound for the iterative gap] There exists a positive constant \( C_1 \) such that for \( w^{(t+1)} \in \partial F(b^{(t+1)}), \)
\[ \| w^{(t+1)} \| \leq C_1 \| b^{(t+1)} - b^{(t)} \|, \quad \forall \ t = 0, 1, \ldots \quad (10) \]
Notice that the function \( F(b) \) is coercive, which means that \( F(b) \to \infty \) iff \( \| b \| \to \infty. \)
Then by Theorem 10-(i), with a bounded initial \( b^{(0)} \) the sequence \( \{ b^{(t)} : t \in \mathbb{N} \} \) is bounded. Hence, there exists a convergent subsequence \( \{ b^{(t_k)} \} \) that converges to \( b^* \). The set of all stationary points which are started with a bounded \( b^{(0)} \) is denoted by \( \mathcal{M}(b^{(0)}) \). That is,
\[ \mathcal{M}(b^{(0)}) \triangleq \left\{ b \in \mathbb{R}^p : \exists t_k, \{ t_k \}_{k \in \mathbb{N}}, \text{ such that } b^{(t_k)} \to b \text{ as } k \to \infty \right\}. \]

Lemma 11 (property of the limit points) Let \( \{ b^{(t)} : t \in \mathbb{N} \} \) be generated by Algorithm 1 or Algorithm 2. Then we have

(i) \( \mathcal{M}(b^{(0)}) \) is not empty and \( \mathcal{M}(b^{(0)}) \subseteq \text{crit} F; \)

(ii)
\[ \lim_{t \to \infty} \text{dist}(b^{(t)}, \mathcal{M}(b^{(0)})) = 0. \quad (11) \]

Lemma 11 implies that \( \mathcal{M}(b^{(0)}) \) is a subset of stationary or critical points of \( F \) and \( \{ b^{(t)} \}_{t \in \mathbb{N}} \) approaches to one point of \( \mathcal{M}(b^{(0)}) \). Our current concern is to prove \( \lim_{t \to \infty} b^{(t)} = b^* \). As in Bolte et al. (2013), we know that \( \mathcal{M}(b^{(0)}) \) is compact and connected. Moreover, the objective function \( F \) is finite and constant on \( \mathcal{M}(w^{(0)}) \).

As we have mentioned previously, as a function of \( b \), \( \Phi(|b|) \) is sub-analytic, which satisfies the Kurdyka-Lojasiewicz property. Moreover, both the least squares loss and the logistic loss function are real analytic. This implies that \( F(b) \) also satisfies the Kurdyka-Lojasiewicz property. Accordingly, combining Theorem 10 and Lemma 11, we have the global convergence property of Algorithm 1 and of Algorithm 2 as follows.

Theorem 12 Assume that \( \Phi \) is a nonzero Bernstein function on \( [0, \infty) \) such that \( \Phi(0) = 0 \) and \( \Phi'(\infty) = 0. \) Let the sequence \( \{ b^{(t)} : t \in \mathbb{N} \} \) be generated by Algorithm 1 or Algorithm 2. Under the conditions in Theorem 10, then the following assertions hold.

(i) The sequence \( \{ b^{(t)} : t \in \mathbb{N} \} \) has finite length,
\[ \sum_{t=0}^{\infty} \| b^{(t+1)} - b^{(t)} \| < \infty. \quad (12) \]
(ii) The sequence \( \{b^{(t)} : t \in \mathbb{N}\} \) converges to a critical point \( b^* = (b_1^*, \ldots, b_p^*)^T \) of \( F \).

The convergence property of Algorithm 2 is a direct corollary of the results studied by Bolte et al. (2013). Notice that in the proof for Algorithm 2, it is not necessarily required that \( P^*_{\alpha}(b) \) on \((0, \infty)\) exists. Thus, when we use the MCP or SCAD function in Algorithm 2, the resulting procedure also has the convergence results in Theorems 10 and 12 because both MCP and SCAD satisfy the Kurdyka-Lojasiewicz property (see Section 3.2).

It is worth pointing out that the convergence analysis of Mazumder et al. (2011) (see Theorem 4 therein) has only established the square summable result \( \sum_{t=0}^{\infty} \|b^{(t+1)} - b^{(t)}\|^2 < +\infty \). However, by the square summable result, it can not be directly obtained that \( \{b^{(t)}\}_{t \in \mathbb{N}} \) is a Cauchy sequence. The theory of the Kurdyka-Lojasiewicz inequality is an essential tool to obtain this result.

Theorem 12 says that \( b^* \) is a critical point of \( F \). Let \( S = \text{supp}(b^*) \) and \( S^c \) denote the complement of \( S \) in \( \{1, \ldots, p\} \). Hence, when taking Algorithm 1 for regression, we have

\[
0 \in X^TXb^* - X^Ty + \lambda_n z,
\]

where \( z = (z_1, \ldots, z_p)^T \) and \( z_j = \alpha \Phi'(\alpha|b_j^*|)\|b_j^*\| \). Let \( z_S \) be the sub-vector of \( z \) with entries in \( S \) and \( X_S \) is the submatrix of \( X \) with columns indexed by \( S \). Then the Karush-Kuhn-Tucker (KKT) condition in (13) is equivalent to

\[
X_S^TX_Sb_S^* - X_S^Ty + \lambda_n z_S = 0
\]

and \( \|X_S^TX_Sb_{Sc} - X_S^Ty\|_{\text{max}} \leq \lambda_n \alpha \Phi'(0) \).

When \( L = \sum_{i=1}^{n} \log(1 + \exp(-y_i b^T x_i)) \) is defined in the classification problem, we have

\[
0 \in \lambda_n z - \sum_{i=1}^{n} \omega_i y_i x_i = \lambda_n z - X^T D y,
\]

where \( \omega_i = \frac{\exp(-y_i b^T x_i)}{1 + \exp(-y_i b^T x_i)} \) and \( D = \text{diag}(\omega_1, \ldots, \omega_n) \). The current KKT condition is equivalent to

\[
\lambda_n z_S - X_S^T D y = 0
\]

and \( \|X_S^T D y\|_{\text{max}} \leq \lambda_n \alpha \Phi'(0) \). Notice that \( \frac{\partial^2 L}{\partial b \partial b^T} = X^T(I_n - D)DX \). The following theorem is a direct corollary of Theorem 1 of Lv and Fan (2009), Fan and Lv (2011). It shows that \( b^* \) is a strict local minimizer of \( F \) under certain regularity conditions.

**Theorem 13** Assume that the conditions in Theorem 12 are satisfied. If \( \lambda_{\min}(X_S^T X_S) + \frac{\lambda_n \alpha^2}{\Phi''(\alpha)} \Phi''(0) > 0 \) and \( \|X_S^T X_S b_{Sc} - X_S^T y\|_{\text{max}} < \lambda_n \alpha \Phi'(0) \) in Algorithm 1 or \( \lambda_{\min}(X_S^T (I_n - D)DX_S) + \frac{\lambda_n \alpha^2}{\Phi''(\alpha)} \Phi''(0) > 0 \) and \( \|X_S^T D y\|_{\text{max}} < \lambda_n \alpha \Phi'(0) \) in Algorithm 2, then \( b^* \) is a strict local minimizer of \( F \). Here \( \lambda_{\min}(A) \) denotes the smallest eigenvalue of the positive semidefinite matrix \( A \).

Notice that without the condition \( \|X_S^T X_S b_{Sc} - X_S^T y\|_{\text{max}} < \lambda_n \alpha \Phi'(0) \) or \( \|X_S^T D y\|_{\text{max}} < \lambda_n \alpha \Phi'(0) \), we only can ensure that \( b^* \) is a local minimizer of \( F \).
6. Experimental Analysis

In this paper our principal focus has been to explore the theoretical properties of the Bernstein function in nonconvex sparse modeling. However, we have also developed the coordinate descent algorithm and the PALM algorithm for the supervised learning problems with the Bernstein penalty. Thus, it is interesting to conduct empirical analysis of the estimation algorithms with different Bernstein penalty functions. We particularly study the nonconvex LOG, EXP and LFR functions because they bridge the $\ell_0$-norm and the $\ell_1$-norm. The MATLAB code will be available at the homepage of the author.

6.1 Regression Analysis on Simulated Datasets

First we conduct experiments on the regression problem with the coordinate descent algorithm based on LOG, EXP and LFR, respectively. We also implement the lasso, and the $\ell_{1/2}$-norm and MCP based methods (Mazumder et al., 2011). All these methods are also solved by using coordinate descent. Moreover, the hyperparameters involved in all the methods are selected via cross validation.

Our empirical analysis is based on a simulated data, which was used by Mazumder et al. (2011). In particular, we generate data from the following model:

$$y = x^T b + \sigma e$$

where $e \sim N(0, 1)$. We choose $\sigma$ such that the Signal-to-Noise Ratio (SNR), which is $\text{SNR} = \frac{\sqrt{b^T \Sigma b}}{\sigma}$, is a specified value. Following the setting in Mazumder et al. (2011), we use SNR = 3.0 in all the experiments.

We generate three different datasets with different $p$ and $n$ to implement the experiments.

That is,

**Data 1**: $n = 100$, $p = 200$, $b_1$ has 10 non-zeros such that $b_{1,20i+1} = 1$ for $i = 0, 1, \ldots, 9$, and $\Sigma_1 = \left\{0.7^{i-j}\right\}_{1 \leq i,j \leq p}$.

**Data 2**: $n = 500$, $p = 1000$, $b_2 = (b_1, \ldots, b_1)$, and $\Sigma_2 = \text{diag}(\Sigma_1, \cdots, \Sigma_1)$ (five blocks).

**Data 3**: $n = 500$, $p = 2000$, $b_3 = (b_1, \ldots, b_1)$, and $\Sigma_3 = \text{diag}(\Sigma_1, \cdots, \Sigma_1)$ (ten blocks).

Our experimental analysis is performed on the above training datasets, and the corresponding test datasets of $m = 10000$ samples. Let $\hat{b}$ denote the solution obtained from each algorithm. We use a standardized prediction error (SPE) and a feature selection error (FSE) as measure metrics. SPE is defined as $\text{SPE} = \frac{\sum_{i=1}^{m} (y_i - x_i^T \hat{b})^2}{m \sigma^2}$ and FSE is proportion of coefficients in $\hat{b}$ which is incorrectly set to zero or nonzero based on the true $b$.

Table 2 reports the average results over 25 repeats. From them, we can see that all the methods are competitive in both prediction accuracy and feature selection accuracy. But nonconvex penalization outperforms convex penalization in sparsity. Although there does not exist a closed-form thresholding operator in the coordinate descent method with EXP, it is still efficient in computational times. This agrees with our analysis in Section 4.2. We find that the $\ell_{1/2}$ penalty indeed suffers from the numerically unstable problem during the computation and prediction. It is worth pointing out that SparseNet is based on a calibrated
version of MCP (Mazumder et al., 2011). Our experiments show that the performance of the conventional MCP is less effective. Thus, the calibration technique is necessary for MCP. However, we do not apply any calibration techniques to LOG, EXP and LFR in their implementations. Thus, the Bernstein penalty function is effective and efficient in sparse modeling.

Comparing further the several nonconvex penalty functions, we see that the performance of LFR is slightly better than those of the remainders. Recall that for any fixed $\alpha > 0,$

$$1 - \exp(-\alpha|b|) \leq \frac{2\alpha|b|}{\alpha|b|+2} \leq \log(\alpha|b|+1) \leq \alpha|b|,$$

with equality only when $b = 0.$ However, we have seen that related to EXP, LFR has the closed-form thresholding operator. This would be an important reason why LFR has the best performance. In summary, the experimental results show that LFR is an especially good choice for nonconvex penalization in finding sparse solutions.

Table 2: Results of the coordinate descent algorithms with different penalty functions on the simulated data sets.

|      | SPE(±STD) | “FSE” | SPE(±STD) | “FSE” | SPE(±STD) | “FSE” |
|------|-----------|-------|-----------|-------|-----------|-------|
|      | p = 200 and n = 100 |       | p = 1000 and n = 500 |       | p = 2000 and n = 500 |       |
| LOG  | 1.2307 (±0.0131) 0.0146 |       | 1.2192 (±0.0028) 0.0121 |       | 3.5200 (±1.1410) 0.0739 |       |
| EXP  | 1.1452 (±0.0074) **0.0037** |       | 1.1407 (±0.0013) 0.0024 |       | 3.6499 (±0.1800) 0.0785 |       |
| LFR  | 1.1145 (±0.0093) 0.0050 |       | **1.1205 (±0.0018) 0.0018** |       | **3.4109 (±0.1590) 0.0610** |       |
| $\ell_{1/2}$ | 1.2480 (±0.0230) 0.0277 |       | 1.2689 (±0.0071) 0.0262 |       | 3.7468 (±0.2041) 0.0896 |       |
| MCP  | 1.1195 (±0.0041) 0.0051 |       | 1.2736 (±0.0509) 0.0430 |       | 3.6853 (±0.1580) 0.0821 |       |
| Lasso| 1.6678 (±0.0654) .01555 |       | 1.6588 (±0.0184) 0.1520 |       | 4.0433 (±0.1607) 0.1470 |       |

6.2 Classification Analysis on Real Datasets

We now conduct empirical analysis of the classification problem with the PALM algorithm based on the Bernstein penalty functions. More specifically, our experiments are performed on four real datasets. The heart data (270 samples and 13 features), the Australian data (690 samples and 14 features), and the German number data (1000 samples and 24 features) come from Statlog. The splice dataset (1000 samples and 60 features) is from Delve. The datasets are used for the binary classification problem.

For comparison, we also implement the conventional SVM and the penalized logistic regression with the $\ell_1$ penalty. We use the 70% of the data for training and the rest 30% for testing. Table 3 reports the average results over 30 repeats. We see that the methods based on the Bernstein penalty functions slightly outperform the two convex methods. Figure 3 illustrates the convergence results of the PALM procedures with EXP, LOG, and LFR, respectively. As we see, the PALM method for the nonconvex penalization problem admits the convergence property.
It is worth pointing out that in the PALM algorithm the input samples are not necessarily standardized such that $\sum_{i=1}^{n} x_{ij} = 0$ and $\sum_{i=1}^{n} x_{ij}^2 = 1$. However, the coordinate descent methods studied by Mazumder et al. (2011) and Breheny and Huang (2011) typically require such standardization.

Table 3: Classification accuracies

|                  | Heart $n = 270, p = 13$ | Australian $n = 690, p = 14$ | German number $n = 1000, p = 24$ | Splice $n = 3175, p = 60$ |
|------------------|-------------------------|-------------------------------|---------------------------------|-------------------------|
| LOG              | 0.8463 (±0.0340)        | 0.8589 (±0.0196)              | 0.7603 (±0.0236)                | 0.7997 (±0.0150)        |
| EXP              | **0.8639** (±0.0296)    | **0.8638** (±0.0221)          | 0.7602 (±0.0237)                | **0.8037** (±0.0241)    |
| LFR              | 0.8620 (±0.0297)        | 0.8638 (±0.0222)              | **0.7622** (±0.0248)           | 0.7999 (±0.0201)        |
| $\ell_1$-norm    | 0.8231 (±0.0212)        | 0.8595 (±0.0194)              | 0.7356 (±0.0230)                | 0.7875 (±0.0077)        |
| SVM              | 0.8417 (±0.0455)        | 0.8522 (±0.0223)              | 0.7600 (±0.0226)                | 0.7890 (±0.0235)        |

Finally, we apply the classification method based on the PALM algorithm on the microarray gene expression data of leukemia patients (Golub et al., 1999). This data involves 7129 genes for 72 patients. Following the treatment in Breheny and Huang (2011), we use 38 patients for training and the other 34 for testing. We implement the PALM algorithm with LOG, EXP, LFR, and MCP, respectively. For comparison, we also implement the method of Breheny and Huang (2011), which is based on the second order Taylor approximation at the current estimate value of the regression vector. We report the misclassification error evaluated on the leukemia dataset. We point out that there can be 33/34 accuracy adopted by the PALM with the either of LOG, EXP, FLR, and MCP, while 31/34 used by the method of Breheny and Huang (2011).

7. Conclusion

In this paper we have exploited Bernstein functions in the definition of nonconvex penalty functions. To the best of our knowledge, it is the first time that we apply theory of Bernstein functions to systematically study nonconvex penalization problems. We have illustrated the KE, LOG, EXP and LFR functions, which have wide applications in many scenarios but sparse modeling. We have conducted empirical analysis with LOG, EXP and LFR, which shows they are good choices.

The Bernstein function has attractive ability in sparsity modeling. Geometrically, the Bernstein function holds the property of regular variation (Feller, 1971). In other words, the Bernstein function bridges the $\ell_q$-norm ($0 \leq q < 1$) and the $\ell_1$-norm. Computationally, the resulting estimation problems can be efficiently solved by using coordinate descent algorithms. The Bernstein function enjoys the Kurdyka-Lojasiewicz property (Lojasiewicz, 1993, Kurdyka, 1998, Bolte et al., 2007), which makes the coordinate descent procedure have global convergence properties. Theoretically, the Bernstein function admits the oracle properties (more details given in the appendix) and can result in an unbiased and continuous sparse estimator.
Appendix A. Several Important Results on Bernstein functions

In this section we present several lemmas that are useful for Bernstein functions.

**Lemma 14** Let $\Phi(s)$ be a nonzero Bernstein function of $s$ on $(0, \infty)$. Assume $\lim_{s \to 0} \Phi(s) = 0$ and $\lim_{s \to \infty} \frac{\Phi(s)}{s} = 0$. Then

(a) $\lim_{s \to +\infty} \Phi^{(k)}(s) = 0$ and $\lim_{s \to 0^+} s^k \Phi^{(k)}(s) = 0$ for any $k \in \mathbb{N}$. Additionally, if $\lim_{s \to \infty} \Phi(s) < \infty$, then $\lim_{s \to \infty} s^k \Phi^{(k)}(s) = 0$ for $k \in \mathbb{N}$.

(b) If $\lim_{s \to \infty} s\Phi'(s)$ exists (possibly infinite), then $\lim_{s \to 0^+} s(-1)^{k-1}(k-1)! s^k \Phi^{(k)}(s)$ for all $k \in \mathbb{N}$ exist and are identical. In fact, if $\Phi'(0) = \lim_{s \to 0^+} s\Phi'(s) = 1$, then $\lim_{u \to 0^+} \frac{\Phi'(u)}{u}$
where $F(u)$ is the probability distribution which concentrated on $(0, \infty)$ and whose Laplace transform is $\Phi'(s)$.

**Proof** First, it follows from the Lévy-Khintchine representation that

$$\Phi(s) = \int_0^\infty [1 - e^{-su}] \nu(du)$$

due to $\Phi(0) = 0$ and $\lim_{s \to \infty} \frac{\Phi(s)}{s} = 0$. Thus, we have

$$\Phi^{(k)}(s) = (-1)^{k-1} \int_0^\infty e^{-su} u^k \nu(du).$$

When $s \geq k$ for any $k \in \mathbb{N}$, it is easily verified that $e^{-su} u^k \leq \frac{u^k}{1 + u^k}$ for $u > 0$. Notice that

$$\int_0^\infty \min(u^k, 1) \nu(du) \leq \int_0^\infty \min(u, 1) \nu(du) < \infty$$

and

$$\frac{u^k}{1 + u^k} \leq \min(u^k, 1) \leq \frac{2u^k}{1 + u^k}, \quad u \geq 0.$$ 

This implies that $\int_0^\infty \min(u^k, 1) \nu(du) < \infty$ is equivalent to that $\int_0^\infty \frac{u^k}{1 + u^k} \nu(du) < \infty$. As a result, we have that when $s \geq k$,

$$\int_0^\infty e^{-su} u^k \nu(du) = \int_0^\infty e^{-su} u^k \nu(du) \leq \int_0^\infty \frac{u^k}{1 + u^k} \nu(du) < \infty.$$

Thus,

$$\lim_{s \to \infty} \Phi^{(k)}(s) = (-1)^{k-1} \lim_{s \to \infty} \int_0^\infty e^{-su} u^k \nu(du) = (-1)^{k-1} \int_0^\infty \lim_{s \to \infty} e^{-su} u^k \nu(du) = 0.$$ 

Additionally, since $e^{-su} (su)^k \leq k^k e^{-k}$ for $s \geq 0$ and $u \geq 0$, we have

$$\int_0^\infty e^{-su} (su)^k \nu(du) = \int_0^1 e^{-su} (su)^k \nu(du) + \int_1^\infty e^{-su} (su)^k \nu(du) \leq \int_0^1 e^{-su} (su)^k \nu(du) + \int_1^\infty k^k e^{-k} \nu(du). \quad (14)$$

Hence, for any $s \leq 1$,

$$\int_0^\infty e^{-su} (su)^k \nu(du) \leq \int_0^1 u \nu(du) + \int_1^\infty k^k e^{-k} \nu(du) \leq \max(1, k^k e^{-k}) \int_0^\infty \min(1, u) \nu(du) < \infty.$$ 

As a result, we obtain

$$\lim_{s \to 0} s^k \Phi^{(k)}(s) = (-1)^{k-1} \lim_{s \to 0} \int_0^\infty e^{-su} (su)^k \nu(du) = (-1)^{k-1} \int_0^\infty \lim_{s \to 0} e^{-su} (su)^k \nu(du) = 0.$$ 

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Furthermore, \( \lim_{s \to \infty} \Phi(s) = M_0 < \infty \) implies that \( \int_0^\infty \nu(du) < \infty \), so we always have
\[
\int_0^\infty e^{-su} (su)^k \nu(du) \leq k^k e^{-k} \int_0^\infty \nu(du) < \infty,
\]
which leads us to \( \lim_{s \to \infty} s^k \Phi(k)(s) = 0 \) for any \( k \in \mathbb{N} \).

We now prove Part (b). Consider that
\[
\frac{(-1)^{k-1} s^k \Phi(k)(s)}{(k-1)!} = \int_0^\infty \frac{s^k}{(k-1)!} e^{-su} u^{k-1} \nu(du)
\]
and that \( \frac{s^k}{(k-1)!} e^{-su} u^{k-1} \) is the p.d.f. of gamma random variable \( u \) with shape parameter \( k \) and scale parameter \( 1/s \). Such a gamma random variable converges to the Dirac Delta measure \( \delta_0(u) \) in distribution as \( s \to \infty \). For a fixed \( u > 0 \), \( \frac{s^k}{(k-1)!} e^{-su} u^{k-1} \) is monotone w.r.t. sufficiently large \( s \). Accordingly, using monotone convergence, we have
\[
\lim_{s \to \infty} \frac{(-1)^{k-1} s^k \Phi(k)(s)}{(k-1)!} = 0 \nu(\{0\}) + \lim_{s \to \infty} \int_0^\infty \frac{s^k}{(k-1)!} e^{-su} u^{k-1} \nu(du)
\]
\[
= 0 \nu(\{0\}) = \int_0^\infty \delta_0(u) \nu(du) = \lim_{s \to \infty} s \Phi'(s).
\]

When \( \Phi'(0) = \lim_{s \to 0^+} \Phi'(s) = 1 \), it is a well-known result that \( \Phi'(s) \) is the Laplace transform of some probability distribution (say, \( F(u) \)). That is,
\[
\Phi'(s) = \int_0^\infty \exp(-su)dF(u) = \int_0^\infty s \exp(-su) F(u)du.
\]
Here we use \( F(0) = 0 \) because \( F \) is concentrated on \((0, \infty)\). Recall that \( s^2u \exp(-su) \to \delta_0(u) \) in distribution as \( s \to +\infty \). We thus have
\[
\lim_{s \to \infty} s \Phi'(s) = \lim_{u \to 0^+} \frac{F(u)}{u}.
\]
This result can be also obtained from Tauberian Theorem (Widder, 1946). Furthermore, if \( F(u) \) is the probability distribution of some continuous nonnegative random variable \( U \), we have \( \lim_{s \to \infty} s \Phi'(s) = F'(0^+) \). If \( U \) is discrete, we see two cases. In the first case, \( \Pr(U = 0) > 0 \). This implies that \( F(u) \geq \Pr(U = 0^+) > 0 \) for any \( u > 0 \). Thus, we have \( \lim_{s \to \infty} s \Phi'(s) = \lim_{u \to 0^+} \frac{F(u)}{u} = \infty \). In the second case, \( \Pr(U = 0) = 0 \). Then there exists a small positive number \( \delta \) such that \( F(u) = 0 \) for \( u > \delta \). As a result, we obtain that \( \lim_{s \to \infty} s \Phi'(s) = \lim_{u \to 0^+} \frac{F(u)}{u} = 0 \).

**Lemma 15** Let \( \Phi \) be a nonzero Bernstein function on \([0, \infty)\) such that \( \lim_{s \to \infty} s \Phi'(s) \) is finite. Then we have \( \lim_{s \to \infty} \frac{\Phi(s)}{s \Phi'(s)} = \lim_{s \to \infty} s \Phi'(s) < \infty \). Furthermore, we have
\[
\lim_{s \to \infty} \frac{s \Phi'(s)}{\Phi(s)} = 0.
\]
**Proof** It follows from the condition \( \lim s\Phi'(s) < \infty \) that \( \lim_{s \to \infty} \frac{\Phi(s)}{\log(1+s)} = \lim_{s \to \infty} \frac{\Phi(s)}{\log s} = \lim_{s \to \infty} s\Phi'(s) < \infty \). Thus, when \( \lim \Phi(s) = \infty \), we have \( \lim_{s \to \infty} s\Phi'(s) = 0 \). Otherwise \( \lim_{s \to \infty} \Phi(s) = M \in (0, \infty) \), we always have that \( \lim_{s \to \infty} \frac{\Phi(s)}{\log(1+s)} = \lim_{s \to \infty} \frac{\Phi(s)}{\log s} = \lim_{s \to \infty} s\Phi'(s) = 0 \). Thus, we have \( \lim_{s \to \infty} \frac{s\Phi'(s)}{\Phi(s)} = 0 \) in any case.

**Lemma 16** Let \( \Phi \) be a nonzero Bernstein function on \([0, \infty)\). Assume \( \Phi(0) = 0, \Phi'(0) = 1, \) and \( \Phi'(\infty) = 0 \). Then \( \lim_{s \to \infty} \frac{s\Phi'(s)}{\Phi(s)} \in [0, 1] \).

**Proof** Lemma 14 shows that \( \lim_{s \to \infty} \frac{s\Phi'(s)}{\Phi(s)} = 0 \) whenever \( \lim \Phi(s) < \infty \). If \( \lim \Phi(s) = \infty \), we take \( \Psi(s) \triangleq \log(1 + \Phi(s)) \), which is also Bernstein and holds the conditions \( \Psi(0) = 0, \Psi'(0) = 1 \) and \( \Psi'(\infty) = 0 \). In this case, \( \lim_{s \to \infty} \frac{s\Phi'(s)}{\Phi(s)} = \lim_{s \to \infty} s\Psi'(s) \) due to \( \Psi'(s) = \frac{\Phi'(s)}{1 + \Phi(s)} \).

Thus, Lemma 14-(b) directly applies the Bernstein function \( \Psi(s) \). Thus, \( \lim_{s \to \infty} \frac{s\Phi'(s)}{\Phi(s)} \) exists.

Now consider that \( s\Phi'(s) - \Phi(s) \) is a decreasing function on \((0, \infty)\) because its first-order derivative is non-positive; i.e., \( s\Phi''(s) \leq 0 \). As a result, we have \( 0 \leq \frac{s\Phi'(s)}{\Phi(s)} \leq 1 \). Subsequently, \( \gamma = \lim_{s \to \infty} \frac{s\Phi'(s)}{\Phi(s)} \in [0, 1] \).

**Appendix B. The Proofs**

In this section we present the proofs of the results given in the paper.

**B.1 The Proof of Theorem 4**

**Proof** It is directly verified that

\[
\lim_{\alpha \to 0} \frac{\Phi(\alpha |b|)}{\Phi(\alpha)} = \lim_{\alpha \to 0} \frac{|b|\Phi'(\alpha |b|)}{\Phi'(\alpha)} = \frac{|b|\Phi'(0)}{\Phi'(0)} = |b|
\]

due to \( \Phi'(0) = 1 \in (0, \infty) \). Clearly, we have that \( \lim_{\alpha \to +\infty} \frac{\Phi(\alpha s)}{\Phi(\alpha)} = 0 \) when \( s = 0 \) and that \( \lim_{\alpha \to +\infty} \frac{\Phi(\alpha s)}{\Phi(\alpha)} = 1 \) when \( s = 1 \).

Lemma 16 shows that \( \gamma = \lim_{s \to \infty} \frac{s\Phi'(s)}{\Phi(s)} \in [0, 1] \). When \( \lim_{s \to \infty} \frac{\Phi(s)}{\log(1+s)} < \infty \), Lemma 15 implies that \( \gamma = 0 \). According to Theorem 1 in Chapter VIII.9 of Feller (1971), we have the second part of the theorem.

**B.2 The Proof of Proposition 5**

**Proof** Let \( \omega = \frac{1}{1+\theta} \). For \(-\infty < \rho \leq 1\), we have \( \omega (0, \infty] \). We now write \( \Phi'_\rho(s) \) for a fixed \( s > 0 \) as \( 1/g(\omega) \) where

\[
g(\omega) = (1 + \frac{s}{\omega})^\omega.
\]
It is a well-known result that for a fixed \( s > 0 \) \( g(\omega) \) is increasing in \( \omega \) on \( (0, \infty) \). Moreover, \( \lim_{\omega \to \infty} g(\omega) = \exp(s) \). Accordingly, \( \Phi'_{\rho}(s) \) is decreasing in \( \rho \) on \( (-\infty, 1] \). Moreover, we obtain

\[
\Phi_{\rho_1}(s) = \int_0^s \Phi'(t)dt \geq \int_0^s \Phi'(t)dt = \Phi_{\rho_2}(s)
\]

whenever \( \rho_1 \leq \rho_2 \leq 1 \).

The proof of Part-(b) is immediately. We here omit the details. \( \blacksquare \)

B.3 The Proof of Theorem 6

Proof The first-order derivative of \((6)\) w.r.t. \( b \) is

\[
\text{sgn}(b)(|b| + \lambda \Phi'(|b|)) - z.
\]

Let \( g(|b|) = |b| + \lambda \Phi'(|b|) \). It is clear that if \( |z| < \min_{b \neq 0}\{g(|b|)\} \), the resulting estimator is 0; namely, \( \hat{b} = 0 \). We now check the minimum value of \( g(s) = s + \lambda \Phi'(s) \) for \( s \geq 0 \).

Taking the first-order derivative of \( g(s) \) w.r.t. \( s \), we have

\[
g'(s) = 1 + \lambda \Phi''(s).
\]

Notice that \( \Phi''(s) \) is non-positive and increasing in \( s \). As a result, we have

\[
g'(s) \geq 1 + \lambda \Phi''(0).
\]

Thus, if \( \lambda \leq -\frac{1}{\Phi''(0)} \), \( g(s) \) attains its minimum value \( \lambda \Phi'(0) \) at \( s^* = 0 \). Otherwise, \( g(s) \) attains its minimum value when \( s^* \) is the solution of \( 1 + \lambda \Phi''(s) = 0 \).

First, we consider the case that \( \lambda \leq -\frac{1}{\Phi''(0)} \). In this case, the resulting estimator is 0 when \( |z| \leq \lambda \Phi'(0) \). If \( z > \lambda \Phi'(0) \), then the resulting estimator should be a positive root of the equation \( b + \lambda \Phi'(b) - z = 0 \) in \( b \). Letting \( h(b) = b + \lambda \Phi'(b) - z \), we study the roots of \( h(b) = 0 \). Notice that \( h(z) = \lambda \Phi'(z) > 0 \) and \( h(0) = \lambda \Phi'(0) - z < 0 \). In this case, moreover, we have that \( h(b) \) is increasing on \( [0, \infty) \). This implies that \( h(b) = 0 \) has one and only one positive root. Furthermore, the resulting estimator \( 0 < b < z \) when \( z > \lambda \Phi'(0) \). Similarly, we can obtain that \( z < \hat{b} < 0 \) when \( z < -\lambda \Phi'(0) \). As stated in Fan and Li (2001), a sufficient and necessary condition for “continuity” is the minimum of \( |b| + \lambda \Phi'(|b|) \) is attained at 0. This implies that the resulting estimator is continuous.

Next, we prove the case that \( \lambda > -\frac{1}{\Phi''(0)} \). In this case, \( g(s) \) attains its minimum value \( g(s^*) = s^* + \lambda \Phi'(s^*) \) when \( s^* \) is the solution of equation \( 1 + \lambda \Phi''(s) = 0 \). Notice that \( \Phi''(s) \) is non-positive and increasing in \( s \). Thus, the solution \( s^* \) exists and is unique. Moreover, since \( \Phi''(s^*) = \frac{1}{\lambda} > \Phi''(0) \), we have \( s^* > 0 \). In this case, the resulting estimator is 0 when \( |z| \leq s^* + \lambda \Phi'(s^*) \). We just make attention on the case that \( |z| > s^* + \lambda \Phi'(s^*) \). Subsequently, the resulting estimator is \( \hat{b} = \text{sgn}(z)\kappa(|z|) \) where \( \kappa(|z|) \) should be a positive root of equation \( b + \lambda \Phi'(b) - |z| = 0 \). We now need to prove that \( \kappa(|z|) \) exists and is unique on \( (s^*, |z|) \). We have that \( h(b) = b + \lambda \Phi'(b) - |z| \) is a convex function of \( b \) on \( [0, \infty) \) due to \( h''(b) = \lambda \Phi''(b) \geq 0 \). This implies that \( h(b) \) is increasing on \( [s^*, \infty) \) and decreasing on \( (0, s^*) \). Thus, the equation \( h(b) = 0 \) has at most two positive roots, which are on \( (0, s^*) \)
or \(|s^*, \infty)\). Since \(h(s^*) = s^* + \lambda \Phi'(s^*) - |z| < 0\) and \(h(|z|) = \lambda \Phi'(|z|) \geq 0\), the equation \(h(b) = 0\) has one unique root on \((s^*, |z|)\). Thus, \(\kappa(|z|)\) exists and is unique on \((s^*, |z|)\). It is worth pointing out that if the equation \(h(b) = 0\) has a root on \((0, s^*)\), the objective function \(J_1(b)\) attains its maximum value at this root. Thus, we can exclude this root.

\[\text{B.4 The Proof of Proposition 8}\]

Observe that \(1 = \Phi'(0) = \int_0^\infty \nu(u)du\) and \(\Phi(\alpha) = \int_0^\infty (1 - \exp(-\alpha u))\nu(du)\). Since \(\alpha u > 1 - \exp(-\alpha u)\) for \(u > 0\), we obtain \(\Phi(\alpha) < \alpha\). Additionally, \(\left[\frac{\alpha}{\Phi(\alpha)}\right]' = \frac{\Phi(\alpha) - \alpha \Phi'(\alpha)}{\Phi^2(\alpha)} \geq 0\) due to \(|\Phi(\alpha) - \alpha \Phi'(\alpha)| = -\Phi''(\alpha) \geq 0\). Also, \(\left[\frac{1}{\Phi(\alpha)}\right]' \leq 0\). We thus obtain that \(\frac{\alpha}{\Phi(\alpha)}\) is increasing, while \(\frac{1}{\Phi(\alpha)}\) is decreasing. Furthermore, we can see that \(\lim_{\alpha \to 0+} \frac{\alpha}{\Phi(\alpha)} = \lim_{\alpha \to 0-} \frac{1}{\Phi'(\alpha)} = 1\) and \(\lim_{\alpha \to \infty} \frac{\alpha}{\Phi(\alpha)} = \lim_{\alpha \to \infty} \frac{1}{\Phi'(\alpha)} = \infty\).

\[\text{B.5 The Proof of Theorem 9}\]

\[\text{Proof}\] First, it is easily obtained that \(\lim_{\alpha \to 0} \frac{\alpha}{\Phi(\alpha)} = \frac{1}{\Phi(0)}\) and \(\lim_{\alpha \to 0} \frac{\Phi(\alpha)}{\alpha^2} = \infty\). This implies that in the limiting case the condition \(\eta \leq -\frac{\Phi(\alpha)}{\alpha^2 \Phi''(0)}\) is always met (i.e., Case (i) in Theorem 6). Moreover, \(|z| > \frac{\eta}{\Phi(\alpha)} \Phi'(0)\) degenerates to \(|z| > \eta\). In addition, we have

\[\lim_{\alpha \to 0} \frac{\alpha \Phi'(ab)}{\Phi(\alpha)} = \lim_{\alpha \to 0} \frac{\Phi'(ab) + ab \Phi''(ab)}{\Phi'(\alpha)} = 1.\]

This implies that \(\kappa(|z|)\) converges to the nonnegative solution of equation of the form

\[b + \eta - |z| = 0.\]

That is, \(\kappa(|z|) = |z| - \eta\) when \(|z| > \eta\).

Second, it is easily obtained that \(\lim_{\alpha \to \infty} \frac{\alpha}{\Phi(\alpha)} = \infty\) and \(\lim_{\alpha \to \infty} \frac{\Phi(\alpha)}{\alpha^2} = 0\). This implies that in the limiting case the condition \(\eta > -\frac{\Phi(\alpha)}{\alpha^2 \Phi''(0)}\) is always held.

Recall that \(s^* > 0\) is the unique root of \(1 + \lambda \Phi''(s) = 0\) and \(\Phi''(s)\) is monotone increasing, so we can express \(s^*\) as \(s^* = \frac{1}{\alpha} (\Phi''(s))^1(-\Phi(\alpha)/(\eta \alpha^2))\). Since \(\lim_{\alpha \to \infty} \Phi(\alpha)/(\eta \alpha^2) = 0\), we can deduce that \(\lim_{\alpha \to \infty} (\Phi''(s))^1(-\Phi(\alpha)/(\eta \alpha^2)) = \infty\). Subsequently,

\[\lim_{\alpha \to \infty} s^* = \lim_{\alpha \to \infty} \frac{1}{\alpha} (\Phi''(s))^1(-\Phi(\alpha)/(\eta \alpha^2)) = \lim_{\alpha \to \infty} ((\Phi''(s))^1(-\Phi(\alpha)/(\eta \alpha^2)))' \leq |z|.\]

Additionally,

\[\lim_{\alpha \to \infty} \frac{\eta \alpha}{\Phi(\alpha)} \Phi'((\Phi''(s))^1(-\Phi(\alpha)/(\eta \alpha^2))) \ldots = \lim_{\alpha \to \infty} \frac{[\Phi(\alpha)/\alpha^2]}{\frac{\Phi'(\alpha)}{\Phi(\alpha)} - \frac{\Phi''(\alpha)}{\alpha^2}} \ldots = \lim_{\alpha \to \infty} \ldots = \lim_{\alpha \to \infty} s^*.\]
Assume \( \lim_{\alpha \to \infty} s^* = c \in (0, |z|) \). Then for sufficiently large \( \alpha \), we have \((\Phi')^{-1}(-\Phi(\alpha)/(\eta \alpha^2)) \simeq \alpha \); that is,

\[
\Phi(\alpha) \simeq -\eta \alpha^2 \Phi''(\alpha).
\]

However, if \( \lim_{\alpha \to \infty} \Phi(\alpha) < \infty \) then \(- \lim_{\alpha \to \infty} \alpha^2 \Phi''(\alpha) = \lim_{\alpha \to \infty} \frac{\Phi(\alpha)}{\log(\alpha)} = 0 \); while \( \lim_{\alpha \to \infty} \Phi(\alpha) = \infty \) then \(- \lim_{\alpha \to \infty} \alpha^2 \Phi''(\alpha) = \lim_{\alpha \to \infty} \alpha \Phi'(\alpha) = \lim_{\alpha \to \infty} \frac{\Phi(\alpha)}{\log(\alpha)} < \infty \). This makes the contradiction due to the assumption \( \lim_{\alpha \to \infty} s^* = c \in (0, |z|) \). Thus, we have \( \lim_{\alpha \to \infty} s^* = 0 \). Hence,

\[
\lim_{\alpha \to \infty} s^* + \frac{\eta \alpha}{\Phi(\alpha)} \Phi'(\alpha s^*) = 0.
\]

Finally, we have

\[
\lim_{\alpha \to \infty} \kappa(b) - \frac{\eta \alpha}{\Phi(\alpha)} \Phi'(\alpha \kappa(b)) = |z|,
\]

which implies \( \lim_{\alpha \to \infty} \kappa(|z|) = |z| \). The second part now follows. \( \blacksquare \)

### B.6 The proof of Theorem 10

**Proof** The case for Algorithm 2 has been proved by Bolte et al. (2013). We now only consider the case for Algorithm 1. Without loss of generality, we assume that \( \alpha = 1 \). The proof is obtained by making slight changes to that for Theorem 4 of Mazumder et al. (2011). Specifically, let

\[
g(u) \triangleq F(b_1, \ldots, b_{i-1}, u, b_{i+1}, \ldots, b_p) = L(b_1, \ldots, b_{i-1}, u, b_{i+1}, \ldots, b_p) + \lambda_n \Phi(|u|).
\]

Then the limiting-subdifferential of \( g \) at \( u \) is given as

\[
\partial g(u) = \nabla_i L(b_1, \ldots, b_{i-1}, u, b_{i+1}, \ldots, b_p) + \lambda_n \Phi'(|u|) \partial |u|.
\]

Using the strong convexity of \( L \) and Taylor’s series expansion of \( \Phi(|z|) \) w.r.t. \( |z| \), we have

\[
g(u + \delta) - g(u) = L(b_1, \ldots, b_{i-1}, u + \delta, b_{i+1}, \ldots, b_p) - L(b_1, \ldots, b_{i-1}, u, b_{i+1}, \ldots, b_p) + \lambda_n \Phi'(|u + \delta|) - \Phi(|u|) \\
\geq \nabla_i L(b_1, \ldots, b_{i-1}, u, b_{i+1}, \ldots, b_p) \delta + \frac{\gamma_i}{2} \delta^2 \\
+ \lambda_n \left\{ \Phi'(|u|)(|u + \delta| - |u|) + \frac{1}{2} \Phi''(|u^*|)(|u + \delta| - |u|)^2 \right\},
\]

where \( |u^*| \) is some number between \( |u + \delta| \) and \( |u| \).

Assume that \( g(u) \) achieves the minimum value at \( u_0 \). Then \( 0 \in \partial g(u_0) \). Hence,

\[
\nabla_i L(b_1, \ldots, b_{i-1}, u_0, b_{i+1}, \ldots, b_p) + \lambda_n \Phi'(|u_0|) \text{sgn}(u_0) = 0.
\]

Notice that if \( u_0 = 0 \), then the above equation holds true for some value in \([-1, 1]\). For notational convenience, we here still write such a value by \( \text{sgn}(u_0) \). Additionally, we have

\[
\Phi'(|u|)(|u + \delta| - |u|) - \Phi'(|u|) \text{sgn}(u) \delta = \Phi'(|u|)(|u + \delta| - |u| - \text{sgn}(u) \delta) \geq 0
\]
because $\Phi'(|u|) \geq 0$. We now obtain

$$g(u_0 + \delta) - g(u_0) \geq \frac{\gamma_1}{2} \delta^2 + \frac{\lambda_n}{2} \Phi''(|u_0|)(|u_0 + \delta| - |u_0|)^2 \geq \frac{\gamma_1 + \lambda_n \Phi''(0)}{2} \delta^2.$$  

Here we use the fact that $\Phi''(0) \leq \Phi''(|u|) \leq 0$ and $(|u_0 + \delta| - |u_0|)^2 \leq \delta^2$. Let $\rho = \min_i \frac{\gamma_i + \lambda_i \Phi''(0)}{2}$. Then

$$g(u_0 + \delta) - g(u_0) \geq \rho \delta^2.$$  

Applying the above inequality leads us to

$$F_i^{(t)}(b_i^{(t)}) - F_i^{(t)}(b_{i-1}^{(t)}) \geq \rho \|b_i^{(t)} - b_{i-1}^{(t)}\|^2,$$

where $b_i^{(t)} = (b_1^{(t+1)}, \ldots, b_i^{(t+1)}, b_{i+1}^{(t)}, \ldots, b_p^{(t)})^T$. Hence, $F(b^{(t)}) - F(b^{(t+1)}) \geq \rho \|b^{(t)} - b^{(t+1)}\|^2$. This also implies that

$$\sum_{t=0}^{T} \|b^{(t)} - b^{(t+1)}\|^2 \leq \frac{1}{\rho} \sum_{t=0}^{T} (F(b^{(t)}) - F(b^{(t+1)})) = \frac{1}{\rho} (F(b^{(0)}) - F(b^{(T+1)})) < \infty.$$

Thus, $\sum_{t=0}^\infty \|b^{(t)} - b^{(t+1)}\|^2 < \infty$.

Consider that

$$\nabla_i L(b_1^{(t+1)}, \ldots, b_{i-1}^{(t+1)}, b_i^{(t+1)}, b_{i+1}^{(t)}, \ldots, b_p^{(t)}) + \lambda_i \partial P(b_i^{(t+1)}) = 0.$$  

That is,

$$\nabla_i L(b_i^{(t)}) - \nabla_i L(b^{(t+1)}) + \nabla_i L(b^{(t+1)}) + \lambda_i \partial P(b_i^{(t+1)}) = 0.$$  

This implies that

$$w_i^{(t+1)} \in \nabla_i L(b^{(t+1)}) + \lambda_i \partial P(b_i^{(t+1)}) = \partial F_i(b^{(t+1)}).$$  

where $w_i^{(t+1)} = \nabla_i L(b^{(t+1)}) - \nabla_i L(b_i^{(t)})$. Let $w^{(t+1)} = (w_1^{(t+1)}, \ldots, w_p^{(t+1)})^T$. Then $w^{(t+1)} \in \partial F(b^{(t+1)})$. Notice that

$$\left| \nabla_i L(b^{(t+1)}) - \nabla_i L(b_i^{(t+1)}) \right| = \sum_{j=i+1}^n \left| a_{ij} (b_j^{(t+1)} - b_j^{(t)}) \right| \leq \sum_{j=i+1}^n |a_{ij}| \left| b_j^{(t+1)} - b_j^{(t)} \right| \leq \sum_{j=i+1}^n \left| b_j^{(t+1)} - b_j^{(t)} \right| \leq \|b^{(t+1)} - b^{(t)}\| \leq \sqrt{p} \|b^{(t+1)} - b^{(t)}\|,$$

where $X^T X = A = [a_{ij}]$ and $B > 0$ is some constant. Here the second inequality is due to $|a_{ij}| = |\sum_{i=1}^n x_i x_{ij}| \leq 1$. Hence,

$$\|w^{(t+1)}\| = \sqrt{\sum_{i=1}^p \left| \nabla_i L(b^{(t+1)}) - \nabla_i L(b_i^{(t+1)}) \right|^2} \leq C_1 \|b^{(t+1)} - b^{(t)}\|,$$

where $C_1 = p$.  

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This implies that $b^\nabla w$. In terms of the proof of 10-(iii), we know that $F$ verges to $M$. Because the sequence $N > N$ it is obvious that for any integer $K$, we have $t F$ cause the sequence $F$ achieve the convergent sequence. Otherwise, we consider $by (10), we have $L$, we have $F$. As is known, there exists an increasing sequence $\{t_k\}_{k \in \mathbb{N}}$ such that $\{b^{(t_k)}\}$ converges to $F(b^\star)$. Suppose there exists an integer $N_0$ such that $F(b^{(N_0)}) = F(b^\star)$. Then it is obvious that for any integer $N > N_0$, $F(b^{(N)}) = F(b^\star)$ holds. Then it is trivial to achieve the convergent sequence. Otherwise, we consider $F(b^{(t)}) > F(b^\star), \forall t \in \mathbb{N}$. Because the sequence $F(b^{(t)})$ is convergent, it is clear that for any $\eta > 0$, there exists a positive integer $M_1$ such that $F(b^{(t)}) < F(b^\star) + \eta$ for all $t > M_1$. By using (11), we have $\lim dist(b^{(t)}, \mathcal{M}(b^{(0)})) = 0$ which implies that for any $\epsilon > 0$ there exists a positive integer $M_2$ such that $dist(b^{(t)}, \mathcal{M}(b^{(0)})) < \epsilon$ for all $t > M_2$. Let $l = \max \{M_1, M_2\}$. Then by the Kurdyka-Lojasiewicz inequality in Definition 2, we have

$$\pi'(F(b^{(t)}) - F(b^\star)) \det(0, \partial F(b^{(t)})) \geq 1 \text{ for any } t > l.$$ 

By (10), we have

$$\pi'(F(b^{(t)}) - F(b^\star)) \geq \frac{1}{C_1} \|b^{(t)} - b^{(t-1)}\|^{-1}.$$ 

Let $\Delta_{t,t+1} \triangleq \pi(F(b^{(t)}) - F(b^\star)) - \pi(F(b^{(t+1)}) - F(b^\star))$. With the property of concave functions, we have

$$\Delta_{t,t+1} \geq \pi'(F(b^{(t)}) - F(b^\star))(F(b^{(t)}) - F(b^{(t+1)}))$$

$$\geq C_0 \pi'(F(b^{(t)}) - F(b^\star))\|b^{(t+1)} - b^{(t)}\|^2$$

$$\geq \frac{C_0}{2C_1}\|b^{(t)} - b^{(t-1)}\|^{-1}\|b^{(t+1)} - b^{(t)}\|^2.$$
That is,
\[ C \Delta_{t,t+1} \| b^{(t)} - b^{(t-1)} \| \geq \| b^{(t+1)} - b^{(t)} \|^2, \]
where \( C = \frac{2 \Delta}{C_0} \). Notice that
\[ C \Delta_{t,t+1} \| b^{(t+1)} - b^{(t)} \| \leq \left( \frac{C \Delta_{t,t+1} + \| b^{(t+1)} - b^{(t)} \|}{2} \right)^2. \]
We thus have
\[ 2\| b^{(t+1)} - b^{(t)} \| \leq C \Delta_{t,t+1} + \| b^{(t+1)} - b^{(t-1)} \|. \]
Then
\[ \sum_{t=t+1}^{\infty} \| b^{(t+1)} - b^{(t)} \| \leq C \sum_{t=t+1}^{\infty} \Delta_{t,t+1} + \sum_{t=t+1}^{\infty} \left( \| b^{(t)} - b^{(t-1)} \| - \| b^{(t+1)} - b^{(t)} \| \right) \]
\[ \leq C \Delta_{t+1,\infty} + \| b^{(t+1)} - b^{(t)} \| \]
\[ \leq C \pi \left( F(\| b^{(t+1)} \|) - F(b^*) \right) + \| b^{(t+1)} - b^{(t)} \|. \]
Since \( \lim_{l \to \infty} \| b^{(l+1)} - b^{(l)} \| = 0 \) and \( \lim_{l \to \infty} F(b^{(l+1)}) = F(b^*) \), it is clearly seen that
\[ \lim_{l \to \infty} \sum_{t=t+1}^{\infty} \| b^{(t+1)} - b^{(t)} \| = 0. \]
Thus we obtain
\[ \sum_{t=0}^{\infty} \| b^{(t+1)} - b^{(t)} \| < \infty. \]
This implies that \( \{ b^{(t)} \}_{t \in \mathbb{N}} \) is a Cauchy sequence, and hence, it is a convergent sequence that converges to \( b^* \).

**Appendix C. Asymptotic Properties**

We discuss asymptotic properties of the sparse estimator under the regression setting. Following the setup of Zou and Li (2008) and Armagan et al. (2013), we assume two conditions: (i) \( y_i = x_i^T b^* + \epsilon_i \) where \( \epsilon_1, \ldots, \epsilon_n \) are i.i.d. errors with mean 0 and variance \( \sigma^2 \); (ii) \( X^T X / n \to C \) where \( C \) is a positive definite matrix. Let \( A = \{ j : b_j^* \neq 0 \} \). Without loss of generality, we assume that \( A = \{1, 2, \ldots, r\} \) with \( r < p \). Thus, partition \( C \) as
\[ C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}, \]
where \( C_{11} \) is \( r \times r \). Additionally, let \( b_1^* = \{ b_j^* : j \in A \} \) and \( b_2^* = \{ b_j^* : j \notin A \} \).

We are now interested in the asymptotic behavior of the sparse estimator based on the penalty function \( \Phi(\alpha | b) \). That is,
\[ \tilde{b}_n = \arg\min_b \| y - Xb \|^2_2 + \lambda_n \sum_{j=1}^{p} \Phi(\alpha_n | b_j |). \] (15)
Furthermore, we let \( \lambda_n = \frac{\eta_n}{\Phi(\alpha_n)} \) based on Theorem 4. For this estimator, we have the following oracle property.

**Theorem 17** Let \( \tilde{b}_{nj} = \{ \tilde{b}_{nj} : j \in A \} \) and \( \tilde{A}_n = \{ j : \tilde{b}_{nj} \neq 0 \} \). Suppose \( \Phi \) is a Bernstein function on \([0, \infty)\) such that \( \Phi(0) = 0 \) and \( \Phi'(0) = 1 \), and there exists a constant \( \gamma \in [0, 1) \) such that \( \lim_{\alpha \to \infty} \frac{\Phi'(\alpha)}{\alpha^{\gamma}} = c_0 \) where \( c_0 \in (0, \infty) \) when \( \gamma \in (0, 1) \) and \( c_0 \in [0, \infty) \) when \( \gamma = 0 \). If \( \eta_n/n^2 \to c_1 \in (0, \infty) \) and \( \alpha_n/n^2 = c_2 \in (0, \infty) \) where \( \gamma_1 \in (0, 1) \) for \( \gamma = 0 \) or \( \gamma_1 \in (0, 1) \) for \( \gamma > 0 \) and \( \gamma_2 \in (0, 1) \) such that \( \gamma_1 + \gamma_2 > 1 + \gamma \gamma_2 \), then \( \tilde{b}_n \) satisfies the following properties:

1. **Consistency in variable selection:** \( \lim_{n \to \infty} P(\tilde{A}_n = A) = 1 \).

2. **Asymptotic normality:** \( \sqrt{n}(\tilde{b}_{n1} - b^*) \xrightarrow{d} N(0, \sigma^2 C_1^{-1}) \).

Notice that \( \Phi' \) is the Laplace transform of some distribution function (say \( F \)). Based on the Tauberian Theorem (Widder, 1946), the condition \( \lim_{\alpha \to \infty} \frac{\Phi'(\alpha)}{\alpha^{\gamma}} = c_0 \) is equivalent to that \( \lim_{t \to 0^+} \frac{F(t)}{t^{2-\gamma}} = \frac{c_0}{(2-\gamma)^2} \).

Obviously, the function \( \Phi_{\rho} \) in (3) satisfies the conditions in Theorem 17; that is, we see \( \gamma = -\frac{\rho}{\rho_0} \) when \( \rho \leq 0 \) and \( \gamma = 0 \) when \( 0 < \rho \leq 1 \) (see Proposition 5). It follows from the condition \( \lim_{\alpha \to \infty} \frac{\Phi'(\alpha)}{\alpha^{\gamma}} = c_0 \) that \( \lim_{\alpha \to \infty} \frac{\Phi'(\alpha)}{\alpha^{\gamma}} = \frac{c_0}{\gamma} \) for \( \gamma \neq 0 \). As a result, we obtain 
\[
\lim_{\alpha \to \infty} \frac{\alpha \Phi'(\alpha)}{\Phi(\alpha)} = \gamma.
\]

The condition \( \alpha_n/n^{\gamma_2/2} = c_2 \) implies that \( \alpha_n \to \infty \). Subsequently, we have 
\[
\lim_{n \to \infty} \sum_{j=1}^p \Phi(\alpha_n|b_{nj}|) = \sum_{j=1}^p |b_j|^\gamma \quad \text{(see Theorem 4)}.
\]

On the other hand, as stated earlier, 
\[
\lim_{\alpha_n \to 0^+} \sum_{j=1}^p \frac{\Phi(\alpha_n|b_{nj}|)}{\Phi(\alpha_n)} = \lim_{\alpha_n \to 0^+} \sum_{j=1}^p \frac{\Phi(\alpha_n|b_{nj}|)}{\Phi(\alpha_n)} = \|b\|_1.
\]

Thus, we are also interested in the corresponding asymptotic behavior of the sparse estimator. In particular, we have the following theorem.

**Theorem 18** Let \( \Phi \) be a Bernstein function such that \( \Phi(0) = 0 \) and \( \Phi'(0) = 1 \). Assume \( \lim_{n \to \infty} \alpha_n = 0 \). If \( \lim_{n \to \infty} \frac{\eta_n}{\sqrt{n}} = 2c_3 \in [0, \infty) \), then \( \tilde{b}_n \xrightarrow{p} b^* \). Furthermore, if \( \lim_{n \to \infty} \frac{\eta_n}{\sqrt{n}} = 0 \), then \( \sqrt{n}(\tilde{b}_n - b^*) \xrightarrow{d} N(0, \sigma^2 C^{-1}) \).

In the previous discussion, \( p \) is fixed. It would be also interested in the asymptotic properties when \( r \) and \( p \) rely on \( n \) (Zhao and Yu, 2006). That is, \( r \uparrow r_n \) and \( p \uparrow p_n \) are allowed to grow as \( n \) increases. Consider that \( \tilde{b}_n \) is the solution of the problem in (15). Thus,
\[
0 \in (X\tilde{b}_n - y)^T x_j + \frac{\eta_n \alpha_n \Phi'(\alpha_n|\tilde{b}_{nj}|)}{\Phi(\alpha_n)} \partial|\tilde{b}_{nj}|, \quad j = 1, \ldots, p.
\]

Under the condition \( \alpha_n \to 0 \), we have
\[
0 \in \lim_{n \to \infty} \left\{ (X\tilde{b}_n - y)^T x_j + \frac{\eta_n \alpha_n \Phi'(\alpha_n|\tilde{b}_{nj}|)}{\Phi(\alpha_n)} \partial|\tilde{b}_{nj}| \right\} = \lim_{n \to \infty} \left\{ (X\tilde{b}_n - y)^T x_j + \eta_n \partial|\tilde{b}_{nj}| \right\}
\]
for \( j = 1, \ldots, p \). Since the minimizer of the conventional lasso exists and unique (denote \( \hat{b}_0 \)), the above relationship implies that \( \lim_{n \to \infty} \tilde{b}_n = \lim_{n \to \infty} \hat{b}_0 \). Accordingly, we can obtain the same result as in Theorem 4 of Zhao and Yu (2006).
Recently, Zhang and Zhang (2012) presented a general theory of nonconvex regularization for sparse learning problems. Their work is built on the following four conditions on the penalty function $P(b)$: (i) $P(0) = 0$; (ii) $P(-b) = P(b)$; (iii) $P(b)$ is increasing in $b$ on $[0, \infty)$; (iv) $P(b)$ is subadditive w.r.t. $b \geq 0$, i.e., $P(s + t) \leq P(s) + P(t)$ for any $s \geq 0$ and $t \geq 0$. It is easily seen that the Bernstein function $\Phi(|b|)$ as a function of $b$ satisfies the first three conditions. As for the fourth condition, it is also obtained via the fact that

$$
\Phi(s + t) = \int_0^\infty [1 - \exp(-(s + t)u)]\nu(du)
\leq \int_0^\infty [1 - \exp(-su) + 1 - \exp(-tu)]\nu(du) = \Phi(s) + \Phi(t), \quad \text{for } s, t > 0.
$$

Thus, we can directly apply the theoretical analysis of Zhang and Zhang (2012) to the Bernstein nonconvex penalty function.

The Bernstein function $\Phi(|b|)$ studied in this paper also satisfies Assumption 1 made in Loh and Wainwright (2013) (see Proposition 8-(a) and (c)). This implies that the theoretical analysis of Loh and Wainwright (2013) applies to the Bernstein penalty function.

### C.1 The Proof of Theorems 17 and 18

The proof is similar to that of Theorem 1 in Armagan et al. (2013). Let $\tilde{b}_n = b^* + \frac{u}{\sqrt{n}}$ and

$$
\hat{u} = \arg\min_u \left\{ G_n(u) \triangleq \|y - X(b^* + \frac{u}{\sqrt{n}})\|_2^2 + \eta_n \sum_{j=1}^p \frac{\Phi(\alpha_n|b_j^* + \frac{u}{\sqrt{n}}|)}{\Phi(\alpha_n)} \right\}.
$$

Then $\hat{u} = \sqrt{n}(\tilde{b}_n - b^*)$. Consider that

$$
G_n(u) - G_n(0) = u^T(X^TX/n)u - 2u^T\frac{X^T\epsilon}{\sqrt{n}} + \eta_n \sum_{j=1}^p \frac{\Phi(\alpha_n|b_j^* + \frac{u}{\sqrt{n}}|)}{\Phi(\alpha_n)} - \Phi(\alpha_n|b_j^*|).
$$

Clearly, $X^TX/n \to C$ and $\frac{X^T\epsilon}{\sqrt{n}} \overset{d}{\to} N(0, \sigma^2 C)$. We now discuss the limiting behavior of the third term of the right-hand side.

We partition $z$ into $z^T = (z_1^T, z_2^T)$ where $z_1 = \{z_j : j \in A\}$ and $z_2 = \{z_j : j \notin A\}$. First, assume $b_j^* = 0$. The previous results imply

$$
\eta_n \frac{\Phi(|u_j| \frac{\alpha_n}{\sqrt{n}})}{\Phi(\alpha_n)} \leq n \frac{n^{\gamma + \frac{1}{2}}}{n^{\frac{\gamma}{2}}} \cdot \frac{\eta_n}{\alpha_n} \frac{n^{\frac{\gamma}{2}}}{n^{\frac{\gamma}{2}}} \cdot \frac{\alpha_n}{\alpha_n} \frac{\log(\alpha_n)}{\Phi(\alpha_n)} \frac{\Phi(|u_j| \frac{\alpha_n}{\sqrt{n}})}{\Phi(\alpha_n)} \alpha_n \alpha_n \rightarrow +\infty
$$

whenever $\gamma = 0$, due to $\lim_{\alpha \to \infty} \frac{\log(\alpha)}{\Phi(\alpha)} = \lim_{\alpha \to \infty} \frac{1}{\alpha \Phi(\alpha)} = \frac{1}{c_0} > 0$. Here we take $\rho$ as a positive constant such that $\rho \leq \frac{2n^{\gamma + \frac{1}{2}} - 1}{n^{\gamma}}$. If $\gamma \in (0, 1)$, we also have

$$
\eta_n \frac{\Phi(|u_j| \frac{\alpha_n}{\sqrt{n}})}{\Phi(\alpha_n)} \leq n \frac{n^{\gamma + \frac{1}{2}}}{n^{\frac{\gamma}{2}}} \cdot \frac{\eta_n}{\alpha_n} \frac{n^{\frac{\gamma}{2}}}{n^{\frac{\gamma}{2}}} \cdot \frac{\alpha_n}{\alpha_n} \frac{\Phi(|u_j| \frac{\alpha_n}{\sqrt{n}})}{\Phi(\alpha_n)} \alpha_n \alpha_n \rightarrow +\infty,
$$

because $\lim_{\alpha \to \infty} \frac{\alpha_n}{\Phi(\alpha)} = \lim_{\alpha \to \infty} \frac{\alpha_n}{\Phi(\alpha)} = \frac{\gamma}{c_0} > 0$. 

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Next, we assume that $b_j^* \neq 0$. Subsequently, for sufficiently large $n$,

\[
\Phi(\alpha_n | b_j^* + \frac{u_j}{\sqrt{n}}|) - \Phi(\alpha_n | b_j^*) \\
\eta_n \frac{\Phi(\alpha_n)}{\Phi(\alpha_n)} \\
= \eta_n \frac{\Phi(\alpha_n(b_j^* + \frac{u_j}{\sqrt{n}})\text{sgn}(b_j^*)) - \Phi(\alpha_n b_j^* \text{sgn}(b_j^*))}{\Phi(\alpha_n)} \\
= \frac{u_j}{b_j^* + \frac{\theta u_j}{\sqrt{n}}} \frac{\Phi'(\alpha_n(b_j^* + \frac{\theta u_j}{\sqrt{n}})\text{sgn}(b_j^*))}{\Phi(\alpha_n)} \\
\times \Phi(\alpha_n(b_j^* + \frac{\theta u_j}{\sqrt{n}})\text{sgn}(b_j^*)) \\
\times \Phi(\alpha_n(b_j^* + \frac{\theta u_j}{\sqrt{n}})\text{sgn}(b_j^*)) \\
\times \{\text{for some } \theta \in (0, 1)\} \quad (16)
\]

\[
\to 0.
\]

Here we use the fact that $\lim_{z \to \infty} \frac{z \Phi'(z)}{\Phi(z)} = \gamma \in [0, 1]$.

By Slutsky’s theorem, we have

\[
G_n(u) - G_n(0) \overset{d}{\to} \begin{cases} 
    u_1^T C_{11} u_1 - 2u_1^T z \quad &\text{if } u_j = 0 \forall j \notin A, \\
    \infty &\text{otherwise.}
\end{cases}
\]

This implies that $G_n(u) - G_n(0)$ converges in distribution to a convex function, whose unique minimum is $(C_{11}^{-1} z_1, 0)^T$. It then follows from epiconvergence (Knight and Fu, 2000) that

\[
\hat{u}_1 \overset{d}{\to} C_{11}^{-1} z_1 \quad \text{and} \quad \hat{u}_2 \overset{d}{\to} 0. \quad (17)
\]

This proves asymptotic normality due to $z_1 \overset{d}{=} N(0, \sigma^2 C_{11})$.

Recall that $\tilde{b}_{nj} \overset{p}{\to} b_j^*$ for any $j \in A$, which implies that $\Pr(j \in A_n) \to 1$. Thus, for consistency in Part (1), it suffices to obtain $\Pr(l \in A_n) \to 0$ for any $l \notin A$. For such an event “$l \in A_n$,” it follows from the KKT optimality conditions that $2x_l^T (y - X\hat{b}_n) = \frac{\eta_n \alpha_n \Phi'(\alpha_n | \tilde{b}_{nj}|)}{\Phi(\alpha_n)}$. Notice that

\[
\frac{2x_l^T (y - X\tilde{b}_n)}{\sqrt{n}} = \frac{2x_l^T X \sqrt{n} (b^* - \tilde{b}_n)}{\sqrt{n}} + \frac{2x_l^T \epsilon}{\sqrt{n}},
\]

and

\[
\lim_{n \to \infty} \frac{\eta_n \alpha_n \Phi'(\alpha_n | \tilde{b}_{nj}|)}{\sqrt{n} \Phi(\alpha_n)} = \lim_{n \to \infty} \frac{\eta_n \alpha_n \Phi'(\sqrt{n} \tilde{b}_{nj} | \alpha_n / \sqrt{n})}{\sqrt{n} \Phi(\alpha_n)} \leq \lim_{n \to \infty} \frac{n^{\gamma_1 + 1/2 - \frac{1}{2} \log(\alpha_n)}}{n^{\gamma_2 / 2} \Phi(\alpha_n)} \to \infty \quad \text{for } \gamma = 0
\]

or

\[
\lim_{n \to \infty} \frac{\eta_n \alpha_n \Phi'(\alpha_n | \tilde{b}_{nj}|)}{\sqrt{n} \Phi(\alpha_n)} = \lim_{n \to \infty} \frac{\eta_n \alpha_n \Phi'(\sqrt{n} \tilde{b}_{nj} | \alpha_n / \sqrt{n})}{\sqrt{n} \Phi(\alpha_n)} \leq \lim_{n \to \infty} \frac{n^{\gamma_1 + 1/2 - \frac{1}{2} \log(\alpha_n)}}{n^{\gamma_2 / 2} \Phi(\alpha_n)} \to \infty \quad \text{for } \gamma > 0
\]

due to $\sqrt{n} \tilde{b}_{nj} | \overset{p}{\to} 0$ by (17) and Slutsky’s theorem. Accordingly, we have

\[
\Pr(l \in A_n) \leq \Pr \left[ 2x_l^T (y - X\tilde{b}_n) = \frac{\eta_n \alpha_n \Phi'(\alpha_n | \tilde{b}_{nj}|)}{\Phi(\alpha_n)} \right] \to 0.
\]

As for the proof of Theorem 18, we consider the case that $\lim \alpha_n = 0$. In this case, we have

\[
\lim_{n \to \infty} \frac{\Phi(\alpha_n / \sqrt{n})}{\alpha_n / \sqrt{n}} = 1 \quad \text{and} \quad \lim_{n \to \infty} \frac{\Phi(\alpha_n)}{\alpha_n} = 1.
\]
Assume that \( \lim_{n \to \infty} \eta_n / \sqrt{n} = 2c_3 \in [0, \infty] \). Then
\[
\phi_n \frac{\Phi(|u_j|^{ \alpha_n} / \sqrt{n})}{\Phi(\alpha_n)} = |u_j| \frac{\eta_n \Phi(|u_j|^{ \alpha_n} / \sqrt{n})}{\Phi(\alpha_n)} \to 2c_3 |u_j|
\]
when \( u_j \neq 0 \). If \( b_j^* \neq 0 \), then
\[
\phi_n \frac{\Phi(\alpha_n |b_j^* + u_j / \sqrt{n}|) - \Phi(\alpha_n |b_j^*|)}{\Phi(\alpha_n)} = \phi_n \frac{\Phi(\alpha_n (b_j^* + u_j / \sqrt{n}) \text{sgn}(b_j^*)) - \Phi(\alpha_n b_j^* \text{sgn}(b_j^*))}{\Phi(\alpha_n)}
\]
\[
= \phi_n \frac{\alpha_n (b_j^* + \theta u_j / \sqrt{n}) \text{sgn}(b_j^*)) - \phi_n \alpha_n (b_j^* + \theta u_j / \sqrt{n}) \text{sgn}(b_j^*))}{\Phi(\alpha_n)}
\]
\[
\to 2c_3 u_j \text{sgn}(b_j^*).
\]

We now first consider the case that \( c_3 = 0 \). In this case, we have
\[
G_n(u) - G_n(0) \xrightarrow{d} u^T Cu - 2u^T z,
\]
which is convex w.r.t. \( u \). Then the minimizer of \( u^T Cu - 2u^T z \) is \( u^* \) if and only if \( Cu^* - z = 0 \). Since \( \hat{u} \xrightarrow{d} u^* \) (by epiconvergence), we obtain \( \sqrt{n}(\hat{b}_n - b^*) = \hat{u} \xrightarrow{d} N(0, \sigma^2 C^{-1}) \).

We then consider the case that \( c_3 \in (0, \infty) \). Right now we have
\[
G_n(u) - G_n(0) \xrightarrow{d} u^T Cu - 2u^T z + 2c_3 \sum_{j \in A} u_j \text{sgn}(b_j^*) + 2c_3 \sum_{j \notin A} |u_j| \xrightarrow{d} H_2(u).
\]

\( H_2(u) \) is convex in \( u \). Let the minimizer of \( H_2(u) \) be \( u^* \). Then
\[
Cu^* - z + c_3 s = 0
\]
where \( s^T = (\text{sgn}(b_j^*)^T, v^T) \) and \( v \in \mathbb{R}^{p_2} \) with \( \max_j |v_j| \leq 1 \). Thus, we have \( u^* \xrightarrow{d} N(t, \sigma^2 \Theta) \) where \( t = (t_1, \ldots, t_p)^T = -c_3 C^{-1}s \) and \( \Theta = [\theta_{ij}] = C^{-1} \). For any \( \epsilon > 0 \), when \( n \) is significantly large and using Chebyshev’s inequality, we have that
\[
\Pr \left[ |u_j^*| / \sqrt{n} \geq \epsilon \right] = \Pr \left[ |u_j^*| \geq \sqrt{n} \epsilon \right] \leq \Pr \left[ |u_j^* - t_j| \geq \sqrt{n} \epsilon - |t_j| \right] \leq \frac{\sigma^2 \theta_{jj}}{\sqrt{n} \epsilon - |t_j|} \to 0
\]
for \( j = 1, \ldots, p \). Consequently, \( |u_j^*| / \sqrt{n} \xrightarrow{p} 0 \); that is, \( \hat{b}_n \xrightarrow{p} b^* \).

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