FRACTIONAL STATISTIC

by

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Abstract: We improve Haldane’s formula which gives the number of configurations for \(N\) particles on \(d\) states in a fractional statistic defined by the coupling \(g = l/m\). Although nothing is changed in the thermodynamic limit, the new formula makes sense for finite \(N = pm + r\) with \(p\) integer and \(0 < r \leq m\). A geometrical interpretation of fractional statistic is given in terms of ”composite particles”.

I. Introduction

Fractional statistic was proposed by Haldane [1] as a generalization of Fermi–Dirac statistic where Pauli exclusion principle is replaced by a more general exclusion principle: \(g\) states are needed to add one more particle to the system. Clearly, \(g = 0\) or \(1\) corresponds respectively to bosons or fermions statistics; however, this new statistic is not restricted to integer values of \(g\). For instance, if \(g = 1/m\), the exclusion principle tells that \(m\) particles can be added to the system on a single state. More generally, if \(g = l/m\) the fractional statistic means that \(l\) states are needed to add \(m\) particles to the system. Of course, this interpretation makes sense for a large number of particles and states (thermodynamic limit); however, a microscopic interpretation is difficult to realize and for instance, we would like to understand what happens if we add one more particle only to the system since in that case an additional fractional number of states is meaningless. In [2] Polychronakos tried to answer this question by arranging the states on a one dimensional open lattice with the restriction that any two particles be at least \(g\) sites apart (\(g\) integer); although this modifies the combinatoric in the microscopic regime, it gives back Haldane’s statistic in the thermodynamic limit. Unfortunately, his proposition together with the assumption of factorizability of the partition function lead necessarily to negative weights for some configurations (this fact was also observed before by Nayak and Wilczek [3] for \(g = 1/2\)). Later on, Chaturvedi and Srinivasan [4] showed that the fractional statistic was not compatible with the factorizability of the partition function. For \(g = 1/2\) and for an odd number of particles, they calculated the positive fractional weight for any configuration. More recently Murthy and Shankar [5] explained that all configurations were not a priori allowed in the fractional statistic and they determine the constraints which define the possible configurations. For sake of symmetrization over all configurations these constraints can be forgotten at the price of introducing positive fractional weights; they calculated these weights for \(g = 1/2\) with an odd number of particles (in agreement with the results of [4]) and for \(g = 1/3\) with a number of particles \(N = 3p + 1\).
In this publication, we improve Haldane’s formula which gives the number of possible configurations for \( N \) particles over \( d \) states without changing its thermodynamic limit; then, we give a geometrical interpretation of fractional statistic for any number of particles and states. This geometrical interpretation generalizes to all \( g = l/m \), to all number of particles \( N \) and to all number of states \( d \), the constraints of ref [5] over the allowed configurations. Finally, by symmetrization over the configurations, we calculate the fractional weights in full generality.

Given a set of states with energy \( \epsilon_i \) and chemical potential \( \mu \), we define the variables

\[
 x_i = \exp \left( -\beta (\epsilon_i - \mu) \right)
\]

where \( \beta \) is the inverse temperature \( 1/T \). Then, the partition function for the bosonic statistic is

\[
 Z(x_1, ..., x_d) = \prod_{i=1}^{d} \frac{1}{1-x_i} = \sum_{\{p_i\}} x_1^{p_1} ... x_d^{p_d}
\]

where we sum over all integers \( p_i \geq 0 \). If all energies are equal, the partition function becomes

\[
 Z(x) = \frac{1}{(1-x)^d} = \sum_{N=0}^{\infty} C_{d+N-1}^N x^N
\]

where \( C_{d+N-1}^N \) is the number of configurations for \( N \) particles and \( d \) states. Similarly, the partition function for the fermionic statistic is

\[
 Z(x_1, ..., x_d) = \prod_{i=1}^{d} (1 + x_i) = \sum_{\{p_i=0,1\}} x_1^{p_1} ... x_d^{p_d}
\]

and with equal energies we get

\[
 Z(x) = (1 + x)^d = \sum_{N=0}^{\infty} C_d^N x^N
\]

Now, the partition function for Haldane’s fractional statistic when all energies are equal \( (\epsilon_i = \epsilon; x_i = x) \) is defined as

\[
 Z(x) = \sum_{N=0}^{\infty} G_d^N (g) x^N
\]

where \( g \) is a rational \( l/m \) and where the number of configurations for \( N \) particles on \( d \) states is given by

\[
 G_d^N (g) = C_{d+(1-g)(N-1)}^N
\]
The above statistic interpolates between the bosonic statistic \((g = 0)\) and the fermionic statistic \((g = 1)\). Of course, \(G^N_d (g)\) must be an integer number since it represents a number of configurations. However, when \(g = l/m\), the above formula has a meaning only if we restrict the number of particles to be of the form

\[
N = pm + 1
\]

\((p \text{ integer})\) because of the factorial function included in the \(C\) symbol (an analytic continuation using the Euler \(\Gamma\) function is wrong since it gives a fractional number of configurations). Usually, this statistic is understood in the large \(N\) and large \(d\) case (thermodynamic limit) where we define the average number of particles per state as

\[
n = \lim_{d \to \infty} N \frac{d}{d} \quad (9)
\]

where \(\overline{N}\) is defined as the extremum over \(N\) of \([G^N_d (g) x^N]\) (in this calculation, \(\lg(N!) \sim N \lg \left(\frac{N}{e}\right)\) for large \(N\)). It is found that

\[
n = \frac{1}{W(x, g) + g} < \frac{1}{g} \quad (10)
\]

where \(W(x, g)\) is a positive quantity which satisfies the equation [6–9]:

\[
W^g (1 + W)^{1-g} = \frac{1}{x} \quad (11)
\]

At small temperature \((T \to 0)\) and when \(\epsilon < \mu\) the quantity \(W \to 0\) so that the average number of particles per energy level \(n \to \frac{1}{g}\).

One purpose of this contribution is to generalize Haldane’s fractional statistic to all numbers of particles

\[
N = pm + r, \; 0 < r \leq m \quad (12)
\]

In the next section, we give a geometrical interpretation to the fractional statistic for any \(N\) and any \(d\) and we prove that the corresponding number of possible configurations for a given \(g = l/m\) is given by the following function

\[
F^N_d (l, m) = C^N_d + N - 1 - E \left(\frac{N - 1}{m}\right) \quad (13)
\]

where the function ”integer part” is such that \(E \left(\frac{N - 1}{m}\right) = p\).

This fractional statistic gives the bosonic statistic for \(l = 0\), the fermionic statistic for \(l = m = 1\) and Haldane’s fractional statistic for \(N = pm + 1\). Of course, for \(N\) large, the effect of the function ”integer part” disappears and we get back the average number of particles per state \(n\) as given by (10) and Ouvry’s equation (11). It is interesting to note that for \(g = 1\), we obtain the fermionic
statistic only if \( l = m = 1 \), but we obtain different statistics if \( l = m = k \) although they all coincide in the thermodynamic limit.

The general idea for the geometrical interpretation is to take seriously for any finite number of particles \( N \), an organisation containing \( p \) "composite particles" (set of \( m \) particles and at least \( (l-1) \) empty states) plus one uncomplete "composite particle" containing \( r \) particles and any number of empty states. When we symmetrize this picture over all states, we obtain a statistic of \((p+1)\) "composite particles" with fractional weights for each configuration. Let us try to clarify this point:

In the bosonic statistics with states of different energy \( \epsilon_i \), the partition function may be written as (2) where we sum over all possible monomials \( x_1^{p_1} \ldots x_d^{p_d} \) with weight 1 for all monomials. In the fermionic statistic, the monomials are such that \( p_i = 0, 1 \) for all \( i = 1, \ldots, d \) so that the weight for a given monomial is 1 or 0 accordingly. Of course, for these two statistics, the partition function \( Z(x_1, \ldots, x_d) \) is symmetric in the variables \( x_i \), and is factorizable as

\[
Z(x_1, \ldots, x_d) = \prod_{i=1}^{d} Z(x_i) \quad (14)
\]

It is now known [4, 5] that for the fractional statistic, the partition function is not factorizable. After S. Chaturvedi and V. Srinivasan, we write

\[
Z(x_1, \ldots, x_d) = \sum_{\{p_i\}} f(p_1, \ldots, p_d) x_1^{p_1} \ldots x_d^{p_d} \quad (15)
\]

where the weights \( f(p_1, \ldots, p_d) \) are \( \geq 0 \) and symmetric. Since we generate in (15) symmetric polynomials, we may introduce the set of homogeneous symmetric polynomials

\[
M_{\lambda}(x) = \sum x_1^{\lambda_1} \ldots x_d^{\lambda_d} \quad (16)
\]

attached to a given Young tableau \( \lambda \) and where the sum runs over all distinct permutations of \((\lambda_1, \ldots, \lambda_d)\). The surface \( |\lambda| \) of the Young tableau represents the number \( N \) of particles. Then

\[
Z(x_1, \ldots, x_d) = \sum_{\lambda} f_{\lambda} M_{\lambda}(x) \quad (17)
\]

For bosonic statistic \( f_{\lambda} = 1 \) for all \( \lambda \) and for fermionic statistic \( f_{\lambda} = 1 \) for \( \lambda = \{1^N\} \) and 0 otherwise. For fractional statistic, the weights \( f_{\lambda} \) are fractional.

These weights have been determined for \( g = 1/2 \) and \( N = 2m + 1 \) by Chaturvedi and Srinivasan; they were also obtained for \( g = 1/2, 1/3 \) and \( N = 2p+1, 3p+1 \) respectively by Murthy and Shankar who mentioned the existence of an algorithm to calculate \( f_{\lambda} \) for arbitrary \( g = 1/m \) and \( N = pm + 1 \).

In section II, we give a geometrical interpretation of the fractional statistic which leads to an intermediate non-symmetric partition function and we calculate \( F^N_{\lambda}(g) \) for this construction. In section III, we symmetrize the result of section II and calculate the corresponding weights \( f_{\lambda} \).
II. Geometrical construction of the intermediate unsymmetric partition function.

Let us give ourself \( d \) states which can be drawn from the top first level to the bottom \( d^{th} \) level. Let us give ourself \( N = pm + r \) \((0 < r \leq m)\) particles to be placed on the \( d \) states in the following way:

We define a ”composite particle” as a set of \( m \) particles and \( \delta \geq l \) states with the constraint that the \((l - 1)\) bottom states are empty and the \( l^{th} \) state (from the bottom) has at least one particle. We define an uncomplete ”composite particle” as a set of \( 0 < r \leq m \) particles with no constraint on the empty states. The situation is described in fig.1 with \((p + 1) \)” composite particles”, the uncomplete one being placed at the bottom. Clearly, the main constraint in this construction is that a non empty state is entirely included inside a ”composite particle” and cannot be splitted over several ”composite particles” (as showed in fig.2).

We now calculate the number of possible configurations for a given choice of \( N, d, l, m \). We first consider the uncomplete ”composite particle” with \( r \) particles on \( \delta_{p+1} \) states. Since \( r \leq m \) and since the empty states have no constraint, the uncomplete ”composite particle” satisfies a bosonic statistic and the number of possible configurations is

\[ C^r_{\delta_{p+1} + r - 1} \]  

(18)

We now consider a ”composite particle”; once the \((l - 1)\) empty bottom states are fixed, we have a bosonic statistic for \( m \) particles on \( \delta - (l - 1) \) states but we have to subtract the number of configurations where there is no particle on the \( l^{th} \) state. The number of possible configurations for a ”composite particle” is

\[ C^m_{\delta - l + m} - C^m_{\delta - l + m - 1} = C^{m-1}_{\delta - l + m - 1} \]  

(19)

Consequently, given \( p \)” composite particles” with \( \delta_i \) states satisfying

\[ \delta_i \geq l, \ i = 1, \ldots, p \]  

(20)

and given one uncomplete ”composite particle” with \( \delta_{p+1} \) states satisfying

\[ \delta_{p+1} \geq 1 \]  

(21)

such that the total number of states is

\[ \sum_{i=1}^{p+1} \delta_i = d, \]  

(22)

then, the total number of possible configurations is

\[ F^N_d (l, m) = \sum_{\{\delta_i\}} C^r_{\delta_{p+1} + r - 1} \prod_{i=1}^{p} C^{m-1}_{\delta_i - l + m - 1} \]  

(23)
where we sum over all $\delta_i$'s defined above. To perform this sum, we proceed as usual: we define the functional

$$F(z_1, \ldots, z_{p+1}) = \sum_{\{\delta_i\}} C_{\delta_{p+1}+r+1}^{r+1} \prod_{i=1}^{p} C_{\delta_{i-l+m-1}}^{m-1} z_{\delta_{i-l}}$$ (24)

where the sums over the variables $\delta_i$ satisfy (20,21) and run to $\infty$. We get

$$F(z_1, \ldots, z_{p+1}) = \frac{1}{(1-z_{p+1})^{r+1}} \prod_{i=1}^{p} \frac{1}{(1-z_i)^m}$$ (25)

If we choose all variables $z_i = z$, we get

$$\sum_{d=pl+1}^{\infty} F_d^N (l, m) z^{d-pl-1} = \frac{1}{(1-z)^{N+1}} = \sum_{n=0}^{\infty} C_N^N z^n$$ (26)

By identification in $z$ and because $p = E(\frac{N-1}{m})$ we proved that the above organization of the particles in "composite particles" leads to a statistic defined by

$$F_d^N (l, m) = C_N^N d N + l E(\frac{N-1}{m})$$ (27)

We now construct the partition function for $N$ particles. To each state $i$ we associate a variable $x_i$ ($i = 1, \ldots, d$). Then, each allowed configuration defines a monomial $x_1^{p_1} \ldots x_d^{p_d}$ where $p_i$ is the occupation number of the state $i$. The above geometrical construction means that the partition function is

$$Z_N (x_1, \ldots, x_d) = \sum_{\{p_i\} \in \Lambda} x_1^{p_1} \ldots x_d^{p_d}$$ (28)

The set $\Lambda$ of possible $p_i$'s is defined by three constraints:

$1^o$)

$$\sum_{i=1}^{d} p_i = N$$ (29)

$2^o$) let

$$\pi_i = \sum_{j \leq i} p_j \quad i = 1, \ldots, d$$ (30)

then,

$$\{m, 2m, \ldots, pm\} \subseteq \{\pi_i\}$$ (31)
where \( N = pm + r \) \((0 < r \leq m)\).

3°) we consider the indices \((i_1, ..., i_p)\) such that \(\pi_{i_q} =qm\), then

\[
p_{i_q+1} = p_{i_q+2} = ... = p_{i_q+l-1} = 0, \quad q = 1, ..., p
\]

(32)

We proved above that the number of monomials defined by \(\Lambda\) is \(F_d^N(l, m)\).

The partition function in (28) is an intermediate partition function which is non-symmetric in the variables \(x_1, ..., x_d\). A non-symmetric partition function does not seem to contradict any principle of thermodynamics; however, we may have to deal with physical systems where symmetrization is needed without changing the total number of configurations. Then, fractional weights is a consequence of the symmetrization procedure described in next section. Before closing this section, let us make two remarks:

1°) we are now in position to answer the question: how many states do we need if we add one particle to the system? Let us consider a system with \(N = pm + 1\) particles and \((l-1)p\) empty states. Adding one more particle does not necessarily change the number of states as it completes partly the uncomplete "composite particle" (from \(r = 1\) to \(r = 2\)). Adding \((m-1)\) particles does not necessarily change the number of states for the same reason. Now, if \(N = pm + m\), adding one more particle takes \(l\) extra states: \((l-1)\) empty states to form the new "composite particle" and one more state to receive the added particle. Altogether, adding \(m\) particles to \(N\) particles takes at least \(l\) extra states; as a consequence, the average occupation number per state is \(n = N/d < m/l\).

2°) in the thermodynamic limit when \(N\) and \(d \to \infty\) and in the zero temperature limit \(T \to 0\), the average occupation number per state \(n \to 1/g = m/l\). This situation occurs only when all "composite particles" have \((l-1)\) empty states and one state with \(m\) particles.

III Fractional weights.

We wish to transform the partition function (28) into the symmetric form

\[
Z_N (x_1, ..., x_d) = \sum_{\{\lambda, |\lambda| = N\}} f_{\lambda} M_{\lambda} (x_1, ..., x_d) \quad (33)
\]

where the symmetric polynomial \(M_{\lambda} (x_1, ..., x_d)\) is defined in (16) and in such a way that the total number of configurations for \(N\) particles on \(d\) states is unchanged

\[
Z_N (1, ..., 1) = F_d^N (l, m) \quad (34)
\]

Clearly, the weights \(f_{\lambda}\) are the ratio between the number of monomials of \(M_{\lambda}\) which belong to \(\Lambda\) (defined in section II) and the total number of monomials in \(M_{\lambda}\). This last number is known to be

\[
M_{\lambda} (1, ..., 1) = \frac{d!}{q_0!q_1!q_2!...q_m!} = C_d^{q_0} [q_1, ..., q_m] \quad (35)
\]

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where $q_i$ is the number of rows of length $i$ in $\lambda$ (that is the number of states with $i$ particles), and where the symbol $[q_1, \ldots, q_m]$ is

$$[q_1, \ldots, q_m] = \frac{(\sum_{i=1}^m q_i)!}{\prod_{i=1}^m q_i!}$$ (36)

The numbers $q_i$ satisfy two relations

$$\sum_{i=0}^m q_i = d$$ (37)

$$\sum_{i=1}^m i q_i = N$$ (38)

The problem of finding how many monomials of $M_\lambda$ are in $\Lambda$ is much more elaborate and is treated in the appendix. Let us simply describe the result:

$$f_\lambda = \frac{[t_1, \ldots, t_{K(m)}] \prod_{i=1}^p \Psi_{\mu_i} \Phi_{\nu} f_0}{[q_1, \ldots, q_m]}$$ (39)

where $f_0$ is related to the statistic of the empty states

$$f_0 = \frac{C_{q_0}^{(l-1)p}}{C_d^{(l-1)p}} = \frac{C_{d-(l-1)p}}{C_d^{(l-1)p}}$$ (40)

In (39), the integers $t_i$, $\Psi_{\mu_i}$ and $\Phi_{\nu}$ are independent of the empty states so that we may define them in a system where $l = 1$ and $q_0 = 0$. In that case, each "composite particle" $i$ defines a Young tableau $\mu_i$; for each Young tableau, we define the set of integers $r_1, \ldots, r_m$ corresponding to the number of rows with length 1, ..., $m$ (that is the number of states with 1, ..., $m$ particles) and satisfying

$$\sum_{i=1}^m i r_i = m$$ (41)

Then,

$$\Psi_{\mu_i} = [r_1, \ldots, r_m]$$ (42)

Similarly, the incomplete "composite particle" defines a Young tableau $\nu$ such that the corresponding multiplicities $s_1, \ldots, s_r$ satisfy

$$\sum_{i=1}^r i s_i = r$$ (43)
Then,
\[ \Phi_{\nu} = [s_1, \ldots, s_r] \quad (44) \]

Finally, we denote by \( K(m) \) the number of Young tableaux of surface \( m \) and by \( t_1, \ldots, t_{K(m)} \) the multiplicities in the chosen set \( \{\mu_1, \ldots, \mu_r (\nu \text{ if } r = m)\} \). These multiplicities satisfy
\[ \sum_{i=1}^{K(m)} t_i = E \left( \frac{N}{m} \right) \quad (45) \]

Then, a combinatorial factor \( [t_1, \ldots, t_{K(m)}] \) is generated in (39) when we symmetrise over the "composite particles" (and eventually over the uncomplete one if \( r = m \)).

We now calculate the weights \( f_\lambda \) by application of the formula (39) to four simple examples:

1\(^o\) the case \( N \leq m \): in this case, \( p = 0 \) so that \( f_0 = 1 \). Also, we have \( E \left( \frac{N}{m} \right) = 0 \) or 1 so that \( [t_1, \ldots, t_{K(m)}] = 1 \). Finally, \( s_i = q_i (q_{r+1} = \ldots = q_m = 0) \) so that \( [s_1, \ldots, s_r] = [q_1, \ldots, q_m] \). Consequently, all \( f_\lambda = 1 \) and we are in a bosonic situation.

2\(^o\) the case \( m = 1 \): in this case, there exists only one possible Young tableau so that \( [r_1] = [s_1] = [q_1] = [t_1] = 1 \). In that case, \( f_\lambda = f_0 \) depends only of the statistic of the empty states.

3\(^o\) the case \( m = 2 \): in this case, we have two Young tableaux satisfying \( [0, 1] = [2, 0] = 1 \); clearly, \( [s_1, s_2] = 1 \) whether \( r = 1 \) or 2. The total number of Young tableaux \( [0, 1] \) is given by \( t_2 = q_2 \) so that \( [t_1, t_2] = C_{E \left( \frac{q_2}{q_1 + q_2} \right)}^{q_2} \). Consequently,
\[ f_\lambda = \frac{C_{E \left( \frac{q_2}{q_1 + q_2} \right)}^{q_2} f_0}{C_{q_1 + q_2}^{q_2}} \quad (46) \]

If \( l = r = 1 \), we obtain the results of ref.[4, 5].

4\(^o\) the case \( m = 3 \): in this case, we have three Young tableaux satisfying \( [0, 0, 1] = [3, 0, 0] = 1 \) and \( [1, 1, 0] = 2 \). On the other hand, \( [s_1, s_2, s_3] = 1 \) for \( r = 1 \) or 2. The total number of tableaux \( [0, 0, 1] \) is \( t_3 = q_3 \), the total number of tableaux \( [1, 1, 0] \) is \( t_2 = q_2 \) if \( r = 1 \) or 3, while it is \( t_2 = q_2 - 1 \) if \( r = 2 \) and \( [s_1, s_2] = [0, 1] \). Consequently,
\[ f_\lambda = \frac{[E \left( \frac{q_2}{q_1 + q_2} \right) - q_2 - q_3 + \eta, q_2 - \eta, q_3] 2^{q_2 - 0} f_0}{[q_1, q_2, q_3]} \quad (47) \]

where \( \eta = 1 \) if \( r = 2 \) and \( [s_1, s_2] = [0, 1] \) and 0 otherwise. Again, for \( l = r = 1 \), we obtain the result of ref.[5].

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Acknowledgments

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Appendix

We now calculate the number of monomials of \( M_\lambda(x_1, ..., x_d) \) which are generated by the geometrical construction of section II, that is the number of monomials which belong to \( \Lambda \) defined in (29-32).

The generating functional for at most \( m \) particles on one state can be written as

\[
F(\alpha, x) = \alpha_0 + \alpha_1 x + ... + \alpha_m x^m
\]

(48)

More generally, the generating functional for at most \( m \) particles per state over \( \delta \) states is

\[
\prod_{i=1}^{\delta} F(\alpha, x_i) = \sum_{\lambda \in [\delta*m]} \alpha_{\lambda_1}...\alpha_{\lambda_{\delta}} M_\lambda(x_1, ..., x_\delta)
\]

(49)

where we sum over all Young tableaux \( \lambda \) inside the rectangle \([\delta \times m]\).

At this stage, we find useful to introduce the multiplicities \( s_i \) which is the number of rows of length \( i \) in \( \lambda \); these multiplicities satisfy

\[
\sum_{i=0}^{m} s_i = \delta
\]

(50)

\[
\sum_{i=1}^{m} i s_i = |\lambda|
\]

(51)

Then

\[
\prod_{i=1}^{\delta} F(\alpha, x_i) = \sum_{\lambda \in [\delta*m]} \prod_{i=0}^{m} \alpha_i^{s_i} M_\lambda(x_1, ..., x_\delta)
\]

(52)

The generating functional which gives the number of monomials is

\[
[F(\alpha, x)]^\delta = \sum_{\lambda \in [\delta*m]} [s_0, ..., s_m] \prod_{i=0}^{m} \alpha_i^{s_i} x^{|\lambda|}
\]

(53)

where the symbol \([s_0, ..., s_m] = M_\lambda(1^\delta)\) is given in (36).

In the following, we introduce a generating functional for each "composite particle" and also for the uncomplete "composite particle". To simplify the
writing, we define the generating functional \( \Phi(\alpha, x) \) which takes into account any number of states

\[
\Phi(\alpha, x) = \sum_{\delta=0}^{\infty} [F(\alpha, x)]^{\delta} y^{\delta} = \frac{1}{1 - F(\alpha, x) y}
\] (54)

and we find convenient to expand it under the form

\[
\Phi(\alpha, x) = \sum_{n=0}^{\infty} \left( \sum_{s_i} \frac{(\alpha_1 x + \ldots + \alpha_m x^m)^n}{(1 - \alpha_0 y)^{n+1}} \right) y^n
\] (55)

so that the empty states get separated from the others. In order to describe the \( r \) particles of the uncomplete "composite particle", we must collect in (55) all terms in \( x^r \). We get

\[
\Phi_r(\alpha) = \sum_{\{\nu, |\nu|=r\}} [s_1, \ldots, s_r] \prod_{i=1}^{m} \alpha_{s_i}^{r_i} \frac{y^{n(\nu)}}{(1 - \alpha_0 y)^{n(\nu)+1}}
\] (56)

where we sum over the Young tableaux \( \nu \) with multiplicities \( s_i \) satisfying

\[
\sum_{i=1}^{r} i s_i = r
\] (57)

\[
\sum_{i=1}^{r} s_i = n(\nu)
\] (58)

An expansion in \( y \) of (56) gives for the term in \( y^\delta \) the same result as (53) restricted to \( |\lambda| = r \). We now describe the "composite particle" from the generating functional

\[
\alpha_0^{l-1} (\alpha_1 x + \ldots + \alpha_m x^m) [F(\alpha, x)]^{\delta-l} y^{\delta}
\] (59)

which takes into account the specific structure of the "composite particle". After summation over \( \delta \) from \( l \) to \( \infty \), we obtain the corresponding functional

\[
\Psi(\alpha, x) = \frac{\alpha_0^{l-1} (\alpha_1 x + \ldots + \alpha_m x^m)}{1 - F(\alpha, x) y} y^l
\] (60)

which may also be expanded as

\[
\Psi(\alpha, x) = \sum_{n=0}^{\infty} \frac{\alpha_0^{l-1} (\alpha_1 x + \ldots + \alpha_m x^m)^{n+1}}{(1 - \alpha_0 y)^{n+1}} y^{n+l}
\] (61)

Consequently, the "composite particle" with \( m \) particles has for generating functional

\[
\Psi_m(\alpha) = \alpha_0^{l-1} \sum_{\{\mu, |\mu|=m\}} [r_1, \ldots, r_m] \prod_{i=1}^{m} \alpha_{r_i}^{r_i} \frac{y^{n(\mu)+1-l}}{(1 - \alpha_0 y)^{n(\mu)}}
\] (62)
where we sum over the Young tableaux $\mu$ with multiplicities $r_i$ satisfying

$$\sum_{i=1}^{m} i r_i = m$$

(63)

$$\sum_{i=1}^{m} r_i = n(\mu)$$

(64)

The generating functional for $p$ "composite particles" and one uncomplete "composite particle" with $r$ particles is given by

$$\Psi_m^p(\alpha) \Phi_r(\alpha) = \alpha_0^{(l-1)p} \sum_{\{\mu_1, \ldots, \mu_p, \nu\}} \frac{\Phi_r \prod_{i=1}^{m} \Psi_{\mu_i} \prod_{i=1}^{m} \alpha_i^{q_i}}{(1 - \alpha_0 y)^{d - q_0 + 1}}$$

(65)

where $\Psi_{\mu_i} = [r_1, \ldots, r_m]$ for the Young tableau $\mu_i$, $\Phi_\nu = [s_1, \ldots, s_r]$ for the Young tableau $\nu$ and where the $q_i$'s are defined in (37,38). If we develop (65) in powers of $y$ to get the term in $y^d$ where $d$ is the total number of states, we obtain for the total number of monomials corresponding to the chosen set of Young tableaux $\{\mu_1, \ldots, \mu_p, \nu\}$

$$C_{d-(l-1)p}^{q_0} \sum_{\{\mu_1, \ldots, \mu_p, \nu\}} \Phi_\nu \prod_{i=1}^{m} \Psi_{\mu_i}$$

(66)

Finally, any permutation of the tableaux $\mu_i$ (and eventually $\nu$ if $r = m$) contributes as well to the monomials of $M_\lambda(x_1, \ldots, x_d)$ which belong to $\Lambda$; these permutations generate a combinatoric factor equal to $[t_1, \ldots, t_{K(m)}]$ where $K(m)$ is the number of Young Tableaux with surface $m$ and the integers $t_i$ are their multiplicities (satisfying (45)) in the chosen set $\{\mu_1, \ldots, \mu_p, \nu(\nu \text{ if } r = m)\}$. This ends the calculation of $f_\lambda$ as given in (39).

[1] F.D.M. Haldane, Phys. Rev. Lett. 67, 937 (1991).
[2] A. P. Polychronakos, Phys. Lett. B365, 202 (1996).
[3] C. Nayak and F. Wilczek, Phys. Rev. Lett. 73, 2740 (1994).
[4] S. Chaturvedi and V. Srinivasan, Phys. Rev. Lett. 78, 4316 (1997).
[5] M. V. N. Murthy and R. Shankar, IMSc/99/01/02, [cond-mat/9903278].
[6] B. Sutherland, J. Math. Phys. 12, 251 (1971).
[7] Y.S. Wu, Phys. Rev. Lett. 73, 922 (1994).
[8] D. Bernard, Les Houches, Session LXII (1994).
[9] A. Dasnière de Veigy and S. Ouvry, Phys. Rev. Lett. 72, 600 (1994).
Fig. 1: \( p \) "composite particles" for \( m=3, l=2. \)

Fig. 2: Forbidden configuration for \( m=3. \)