Thermodynamics of the Double-Layer Quantum Hall Systems

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In this paper we apply the exact solution of the sine-Gordon model to describe thermodynamic properties of the soliton liquid in the incommensurate phase of the double-layer quantum Hall systems. In this way we include thermal fluctuations and extend to finite temperatures the results obtained by C.B. Hanna, A.H. MacDonald and S.M. Girvin [Phys. Rev. B 63, 125305 (2001)]. In addition we calculate the specific heat of the system. While the results obtained for the sine-Gordon model are available in a temperature interval $(0, T_c)$, where $T_c = 8 \pi \rho_s$, $\rho_s$ the pseudospin stiffness, they can be applied in the bilayer system up to temperatures $3 T_{BKT}$, where $T_{BKT} = \pi \rho_s/2$ is the vortex mediated Berezinskii-Kosterlitz-Thouless transition temperature. Above this temperature the operators $\cos \beta \phi$ and $\cos(2 \pi d/\beta)$ are both relevant and the system is in a phase with coexisting order parameters. $\beta$ is the dual field of $\varphi$ and $\beta$ is the sine-Gordon coupling constant. We provide numerical estimates for thermodynamic quantities for the range of parameters relevant for GaAs.

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Strong electron-electron interactions in Quantum Hall Double Layer systems lead to interesting collective effects\textsuperscript{[3]}. Perhaps, the most remarkable among them is the interlayer coherence which is established when the interlayer spacing $d$ is of the order of the magnetic length. In that case the Coulomb interaction between electrons on different layers is comparable to the interaction on the same layer $\textsuperscript{[4]}$. The dynamics of this coherent state is conveniently described in terms of the pseudospin variable $\varphi$ being on the upper or the lower layer $\textsuperscript{[3]}$, whereas real spins are totally frozen.

At finite interlayer distance the symmetry in pseudospin space is lowered down to U(1) such that the corresponding Ginzburg-Landau free energy is of an easy-plane anisotropic ferromagnet. The latter symmetry is further broken by interlayer tunneling. In the limit of strong anisotropy one can treat pseudospins as belonging to a plane such that at low temperatures the free energy is simplified down to the classical sine-Gordon model $\textsuperscript{[2]}$:

\begin{equation}
\frac{F}{T} = \frac{1}{T} \int dxdy \left[ \frac{1}{2} \rho_s (\nabla \varphi - \mathbf{Q})^2 + \frac{t}{2\pi l^2} (1 - \cos \varphi) \right],
\end{equation}

where $(\phi - \mathbf{Q} r)$ is the phase angle describing pseudospins in the $XY$ plane at different positions. $\mathbf{Q} = (2\pi d/\phi_0) \mathbf{B}_\parallel \times \mathbf{z}$ is the parallel magnetic field wave vector, with $\phi_0 = hc/e$ being one flux quantum between the layers. $t$ is the tunneling energy which generally depends on magnetic field $Q$ as well as on $m_z$

\begin{equation}
t = t_0 e^{-Q^2 l^2/4 \sqrt{1 - m_z^2}}.
\end{equation}

$m_z = \nu_1 - \nu_2$ is the layer imbalance and $\rho_s$ is the pseudospin stiffness which depends on $m_z$ as well

\begin{equation}
\rho_s = (1 - m_z^2) \rho_E.
\end{equation}

$\rho_s$ arises from the loss of the Coulomb exchange in the presence of a phase gradient (in what follows we neglect the dependence of $\rho_s$ on $m_z$, considering that the layer imbalance is small or equivalently $\nu_1 = 1 - \nu_2 \approx 1/2$). In the above formulas $l$ is the magnetic length equal to $(hc/eB_L)^{1/2}$, where $B_L$ is the strength of the magnetic field perpendicular to the layers.

At higher temperatures Eq. (1) does not properly describe the pseudospin system as it does not take into account the $2\pi$ periodicity of the field $\varphi$. This free energy should be supplemented by the contribution originating from the vortex configurations of angle variable $\phi$:

\begin{equation}
F_{\text{vortex}}/T = e^{-S_0} \int d^2 x \cos 2\pi \theta, \quad \frac{T}{\rho_s} \partial^2 \phi = \epsilon_{\mu\nu} \partial_\mu \phi, \quad (4)
\end{equation}

where $S_0$ is the thermodynamic action of a vortex and $\theta$ is the dual field of $\varphi$. The scaling dimensions of this term and the tunneling term are respectively

\begin{equation}
d_v = \frac{\pi \rho_s}{T}, \quad d_t = \frac{T}{4\pi \rho_s}, \quad (5)
\end{equation}

such that

\begin{equation}
d_v d_t = 1/4. \quad (6)
\end{equation}

In what follows we shall apply non-perturbative results derived for the sine-Gordon model to the bilayer systems. Strictly speaking, this approach is valid only below the Berezinskii-Kosterlitz-Thouless transition temperature

\begin{equation}
T_{BKT} = \frac{\pi \rho_s}{2}. \quad (7)
\end{equation}

when $d_v > 2$. We shall first discuss the region $T < T_{BKT}$ and then briefly discuss what happens at higher temperatures.

To study two-dimensional classical sine-Gordon (SG) model $\textsuperscript{[1]}$ we use the well known analogy with the quantum (1+1)-dimensional SG model, at $T = 0$. The action
of the quantum SG model corresponding to (1) is:
\[
S = \int \frac{d\tau dx}{2} \left[ \frac{1}{2} \nabla \varphi - h \beta / 2\pi \right]^2 + 2\mu (1 - \cos \beta \varphi) ,
\]
where \( \beta^2 = T / \rho_s \), \( \varphi = \phi / \beta \), \( \mu = t / (4\pi^2 T) \). The field \( \phi / \beta \) coupled to the soliton topological charge density, plays the role of a chemical potential. The two models have the same partition function. The current field theory description is valid only at distances much greater than the magnetic length \( l \) which serves as the ultraviolet cutoff.

The ground state energy \( E_0 \) of the quantum model is related to the free energy of the classical model:
\[
E_0 = \frac{F}{T} .
\]
This is a fundamental formula which we use to establish a link between the exact solution of the quantum sine-Gordon model and the model describing Quantum Hall double layer.

The SG model, given by Eq. (1), contains two competing periodicities: the periodic potential tends to lock the system into a commensurate configuration with the field \( \phi \) being locked to a minima of the cosine potential, whereas the gradient term prefers the field configuration be \( \phi = Q \chi \). This competition takes place only for temperatures smaller than the critical temperature
\[
T_c = 8\pi \rho_s ,
\]
below which the cosine potential is relevant. Notice that \( T_c \) is 16 times larger than \( T_{BKT} \) and therefore the interlayer tunneling is highly relevant in the entire area of validity of the sine-Gordon description. In this area there is a critical value of chemical potential \( Q_c(T) \) such that for \( Q > Q_c \), the competition is resolved by a formation of a liquid of solitons (domain walls). Each soliton interpolates between minima of the cosine potential. The transition into such incommensurate state occurs when the soliton energy equals the chemical potential.

According to [1], numerical values of the parameters for a typical GaAs double layer QH sample are the following. The effective mass is \( m^* \approx 0.07 m_e \), total particle aerial density is \( n_T = 1.0 \times 10^{11} \text{ cm}^{-2} \), layer (midwell to midwell) separation is \( d = 20 \text{ nm} \) and tunneling energy is \( t_0 = 0.1 \text{ meV} \). For such a sample we would have \( l \approx 12.6 \text{ nm} \), \( d / l \approx 1.6 \), \( \hbar / \omega_c \approx 6.9 \text{ meV} \) for \( \nu_T = 1 \), and \( e^2 / 4\pi d l \approx 8.8 \text{ meV} \). This gives \( \rho_G \approx 0.03 \text{ meV} \), which corresponds to \( T_{BKT} \approx 0.5K \).

At \( Q = Q_c \), a single soliton is introduced in the system. The value of \( Q_c \) as determined by the exact solution [3] is given by the equation
\[
Q_c = 4\pi h \approx 8\pi \Gamma \left( \frac{1}{2} - \frac{\tau}{2(1 - \tau)} \right) \left[ \mu_c \Gamma(1 - \tau) / \Gamma(1 + \tau) \right]^{1/[2(1 - \tau)]} ,
\]
where
\[
\tau = \frac{\beta^2}{8\pi} = \frac{T}{T_c} , \quad \mu_c = \frac{t_0 e^{-Q^2 l^2 / 4}}{32 \pi \rho_s} ,
\]
In the limit \( T \to 0 \) this equation coincides with the expression used by Hanna et al. [1]:
\[
Q_c(T) = \frac{4}{\pi} \sqrt{\frac{T}{2 \pi \rho_s}} , \quad (T = 0) .
\]

\( Q_c(T) \) is a monotonously decreasing function of \( T \), as shown on Fig. 1. The maximal value of \( Q_c \) is achieved for \( T = 0 \); substituting the numerical values used for GaAs, we obtain the estimate for a critical magnetic field necessary to observe the soliton liquid:
\[
Q_c(T = 0) l \approx 0.3 , \quad B_c(T = 0) \sim 0.1 \phi_0 / l d .
\]

From this estimate we also see that since the value of \( Q_c l / 2 < 0.15 \) is always small, it is not always necessary to take into account the field dependence of the tunneling integral in Eq. (11). (This term is however important in cases of strong magnetic fields and mistakenly it is not accounted in [1] in all asymptotic cases \( Q \to +\infty \) they considered. The correction terms to quantities like \( M(\nu) \), \( \chi_{sol} \) etc., resulting from the \( Q \) dependence of \( t \), should decrease exponentially when \( Q \to +\infty \) instead of the power low decay of [1].

The authors of [1] used \( T \to 0 \) limit to describe the soliton state. To establish whether such description can be extended to finite \( T \), we have to recall some fundamental facts about the sine-Gordon model.

The particle spectrum of the quantum SG model consists of solitons and antisolitons of mass \( M_s \) and for \( \tau < 1/2 \) also of their bound states
\[
M_n = 2M_s \sin \left( \frac{n \tau}{2(1 - \tau)} \right) , \quad n = 1, \cdots , \left( \frac{1}{\tau} - 1 \right) ,
\]
where
\[
M_d = \frac{2}{\sqrt{\pi}} \Gamma \left( \frac{1}{2} + \frac{\tau}{2(1 - \tau)} \right) \left[ \mu(Q) \Gamma(1 - \tau) / \Gamma(1 + \tau) \right]^{1/[2(1 - \tau)]} ,
\]
where
\[
\mu(Q) = \frac{t_0 e^{-Q^2 l^2 / 4}}{32 \pi \rho_s} .
\]
Recall that the sine-Gordon description is valid below \( T_{BKT} \) which corresponds to \( \tau < 1/16 \). In the case the system will be exposed to an external "magnetic field" only the solitons’ and antisolitons’ energy will be affected. The breathers have zero charge and do not interact with external magnetic fields. The solitons acquire additional energy \( -h \) whereas the antisolitons \( h \), and in the ground state only solitons can appear. The soliton’s excitation
FIG. 1: Plot of $4\tau M_{\text{sol}}$ on temperature ($\tau = T/T_c$) and magnetic field $Q$ dependence is shown. $t_o$ is the tunneling energy and $T_c$ is the critical temperature above which the soliton generating operator becomes irrelevant. The line on the surface shows the critical magnetic field, above which the system crosses to the incommensurate phase. (On the vertical axis we show $\tau M_{\text{sol}}$ instead of the diverging soliton mass $M_{\text{sol}}$.)

As can be seen, for fixed $\tau$, the mass of the solitons decreases exponentially as $Q$ increases. The $Q$ and $M_s$ axes have units $t/l$, (the inverse magnetic length, $t = 12.6$ nm).

The parameter range relevant for GaAs we have $\mu(0) \approx 0.03$.

In Fig. 1 we plot $\tau M_s$ as function of $\tau$ and $Q$. The soliton mass $M_s$ diverges as $1/\tau$ as $\tau \to 0$. On the surface of the three dimensional plot $(\tau, Q, 4\tau M_s)$ we show also the critical field $Q_c(\tau)$. Its dependence on temperature is shown also in the inset of Fig. 1. The vertical dotted line in it separates the temperature intervals $0 < T < T_{\text{BKT}}$ and $T_{\text{BKT}} < T < T_c$.

At $Q \neq 0$ only the solitons’ and antisolitons’ energies are affected. The breathers have zero topological charge and do not interact with external magnetic fields.

As we have said, the sine-Gordon model description is rigorously valid only below $T_{\text{BKT}}$; however one can use it as a good approximation even above $T_{\text{BKT}}$ provided the correlation length generated by the interlayer tunneling is shorter than the correlation length generated by vortices:

$$M_1(T) \gg [\exp(-S_0)]^{1/(2-d_s)},$$

where $M_1$ is the mass of the first breather determined by Eq.(15). Assuming that $S_0 \approx 2\pi \rho_s/T$ which is the thermodynamic action of a skyrmion in the O(3) nonlinear sigma model, we estimate that this inequality is reasonably fulfilled up to $T \approx 3T_{\text{BKT}}$.

There are two experimentally measurable quantities one can extract from the sine-Gordon thermodynamics: the specific heat and the magnetization. Both quantities are related to the free energy; at $\tau < 1/2$ the latter one contains two contributions:

$$F = F_1 + F_2,$$

$$F_1 = \frac{1}{4} T_c (M_s l)^2 \frac{\pi}{2(1 - \tau)} \frac{\rho_s}{2} \frac{(Ql)^2}{2},$$

$$F_2 = \frac{\pi}{2(1 - \tau)} \int_{-B}^B d\theta \cosh \theta [\epsilon(\theta)]^2,$$

$F_2$ being related to the ground state energy of the solitonic Fermi sea. (The solitons have a relativistic dispersion relation and usually their spectrum is parametrized in terms of the rapidity $\theta$, in the form $p_{\text{sol}} = M_s \sinh \theta$, $\epsilon_{\text{sol}}(\theta) = M_s \cosh \theta$, for the bare values of energy and momentum. $B$ and the renormalized value of $\epsilon(\theta)$ can be found in [3]). Therefore $F_2 \neq 0$ only at $Q > Q_c$. In the commensurate phase there will be also a contribution from the the first part of $F_1$ due to the $Q$-dependence of $M_s$:

$$\left. \frac{\phi_0}{2\pi l d} M_{||} \right|_{Q < Q_c} = - \frac{\rho_s}{Q_c} [Q - Q_c(0)] + \frac{T_c}{4} \left\{ n_{\text{sol}} + \left( Q/2 \right) \frac{\pi}{2(1 - \tau)} (M_l)^2 \right\}$$

$$n_{\text{sol}} = \frac{\partial \mathcal{E}_0}{\partial h},$$

with $n_{\text{sol}}$ being the soliton density. In the limiting case $T \to 0$ the subcritical contribution to the magnetization is given by

$$\left. \frac{\phi_0}{2\pi l d} M_{||} \right|_{Q < Q_c; T = 0} = \frac{\pi T_c}{256} \frac{Q}{Q_c} (l Q_c)^3,$$

where $Q_c$ is given by Eq.(13).
amplitude is field dependent. The last term is a result of the fact that the tunneling of solitons (or the soliton lattice wave number \(z\)) changes from 1 to 0 at the critical line, in the I phase the thermal fluctuations of the soliton condensate leads to a decrease in the specific heat, except close to the critical line where it diverges. As \(Q\) changes, the critical point goes towards \(T = 0\) point and the divergence changes form from \(1/\sqrt{T}\) to \(1/|\epsilon\ln(1/\epsilon)|\). Here the tunneling amplitude is \(t = 0.1\) meV.

Similarly along the fixed \(Q\) line, \(Q_s\), to first order in \((\epsilon_\tau)^{1/2}\), has square root behavior in \(T\)

\[
Q_s = \frac{8\sqrt{2} \tau M_s^2}{Q c} \sqrt{-\frac{Q e}{Q c} \left(\frac{\tau}{\tau_c} - 1\right)}^{1/2} + \ldots .
\]

For smaller temperatures (fixed \(\tau\)) the square root dependence holds only in a decreasingly small interval \(Q\) above \(Q_c\). At these temperatures for larger \(Q\) the dependence of \(1/L_s\) on \(Q\) changes using the expression for the soliton free energy \([6]\) we find

\[
\frac{Q_s}{Q} = -\frac{8\pi^2 \tau_0^2 \bar{M}_s^2 e^{-Q(\tau/2)^2}}{Q Q_c} \left[\frac{(\ln \epsilon - 1)}{\ln^2 \epsilon} + \frac{\epsilon}{\ln(1/\epsilon)}\right],
\]

where \(\bar{M}_s\) would be the soliton mass in the case the tunneling amplitude is independent of the magnetic field \(Q\).

On Fig. 3 we give the plot of the specific heat for \(\mu = 0.03\). At small \(T\) in the C phase the specific heat per unit area is linear in temperature:

\[
C_V/T = 1/2 \gamma R \frac{\pi}{8} \mu \ln^2 \mu .
\]

For \(\mu = 0.03\) we have \(\gamma^2 \approx 0.14R\).

In the I phase the soliton condensate leads to a decrease of the specific heat, except close to the critical line where it diverges. It is worth mentioning that the leading order term on the condensate contribution to the specific heat changes form from \(1/|\epsilon\ln(1/\epsilon)|\) when \(T \to 0\) to \(1/\sqrt{\epsilon}\) at finite temperatures.

**FIG. 2:** Plot of the magnetization on both the C and I phases of the double-layer QH system as a function of the magnetic field \(Q\) and temperature \((\tau = T/T_c)\) is shown. In the C phase the magnetization increases with \(Q\) until it reaches the C-I transition critical point. The presence of the soliton condensate in the I phase results in a fast decrease of the magnetization.

**FIG. 3:** The total specific heat of the double layer QH system for different values of the magnetic field \(Q\). The presence of the soliton condensate leads to a decrease in the specific heat, except close to the critical line where it diverges. As \(Q\) increases the critical point goes towards \(T = 0\) point and the divergence changes form from \(1/\sqrt{T}\) to \(1/|\epsilon\ln(1/\epsilon)|\). Here the tunneling amplitude is \(t = 0.1\) meV.

On Fig. 2 we give the dependence of the magnetization on the magnetic field and temperature. As can be also observed in the figure the presence of the solitons leads to a decrease of the magnetization of the system. It takes biggest value at small temperatures and close to the critical line. For high fields the magnetization vanishes. Close to the critical line, in the I phase the thermal fluctuations of the solitons renormalize the magnetization decrease by changing its behavior from \(1/\ln(1/\epsilon)\) as \(T \to 0\) to \(\sqrt{\epsilon}\) at finite temperatures, where \(\epsilon = |\tau/\tau_c(Q) - 1|\).

In the following we give also the asymptotic value of the density of solitons (or the soliton lattice wave number \(Q_s = 2\pi/L_s = 2\pi n_{sol}, L_s\) is the distance of solitons from each other). This is related to the soliton lattice magnetization by \(M_l = (4\pi^2 \rho_\phi/\phi_0) n_{sol}\), where \(n_{sol}\) is the density of the solitons in the system in the IC phase and \(\phi_0\) is the magnetic flux quantum. The density of solitons can be found as:

\[
n_{sol} = \frac{1}{L_s} = -\frac{\partial F_2}{\partial h} = \frac{h}{2\pi} \frac{\partial F_2}{\partial Q} = -\frac{\partial F_0}{\partial h} .
\]

\(h = 2\pi Q/\beta^2\) is the field coupled to the soliton charge, or the number of particles, and \(\beta^2 = 1/\rho_s\), [see Eq. (11)]. At finite temperature we find

\[
\frac{Q \to Q_c, \ (T \neq 0)}{Q} = \frac{8\sqrt{2} \tau M_s^2 e^{-Q(\tau/2)^2}}{Q Q_c} \left[\frac{(\ln \epsilon - 1)}{\ln^2 \epsilon} + \frac{\epsilon}{\ln(1/\epsilon)}\right],
\]

The last term is a result of the fact that the tunneling amplitude is field dependent.
In the following we calculate the compressional stiffness element $K_{11}$ of the stiffness tensor $K_{ij}$. The plot which is given below applies in the case of the bilayers up to temperature $T \approx 3T_{KT}$, and for other systems analogues to SG, in the whole temperature region. We calculate $K_{11}$ as the change of the free energy by varying the spacing between solitons. This is achieved by calculating the second derivative with respect to the component $Q_{s1}$ for fixed $Q$.

\[
K_{11}(Q, \tau) = \frac{\partial^2 f_{sol}}{\partial Q_{s1}^2}
\]

where $f_{sol}$ is the soliton contribution to the free energy and is equal to $F_2$. Substituting

\[
Q_{s1} = -\frac{1}{\rho_s} \frac{\partial f_{sol}}{\partial Q}
\]

one finds for $K_{11}$ the following expression

\[
K_{11} = -\rho_s^2 \left[ \frac{1}{f_{solQQ}} - \frac{f_{solQ}f_{solQQQ}}{f_{solQQ}^3} \right]
\]

\[
= \rho_s \left( \frac{\partial^2 Q_s/\partial Q}{\partial Q_s/\partial Q} \right) \left[ 1 - \frac{\rho_s (\partial^2 Q_s/\partial Q^2)}{(\partial Q_s/\partial Q)^2} \right]
\]

Note that in Ref. [1] the second term is not taken into account.

We calculate $K_{11}$ numerically and give in Fig. 4 its dependence as a function of temperature and magnetic field.

In the following we give the asymptotic behavior of $K_{11}$, keeping only the leading order term.

At finite temperatures, near the critical line, $K_{11}$ increases as a square root function of the magnetic field.

\[
K_{11} = \rho_s \frac{Q^2}{4\sqrt{2\pi M_s^2}} e^{1/2} + \ldots
\]

As it can be seen on Fig. 4, the compressional elastic constant characterizing the soliton lattice, reaches quicker its asymptotic value at higher temperatures than at lower ones. At temperatures close to $T_c$, $K_{11}$ has values very close to $\rho_s$ and changes very little for variations of the magnetic field.

For temperatures towards zero the square root holds in a decreasingly small interval $Q$ above $Q_c$. For larger $Q$, outside this interval, the dependence of $K_{11}$ changes. At zero temperatures the $Q$ interval of the square root behavior vanishes and the $Q$ dependence of $K_{11}$ is

\[
K_{11} = \rho_s \frac{Q^2}{8\pi^2 \tau^2 M_s^2} \epsilon \ln^3(1/\epsilon) + \ldots
\]

The large $Q$ asymptotics at zero temperature is given by

\[
K_{11} = \rho_s \left[ 1 + \frac{1}{64} \left( \frac{\pi Q_c}{2Q} \right)^4 \left( 18 + 9Q^2l^2 + 2Q^4l^4 + Q^6l^6 \right) e^{(Ql)^2/2} + \ldots \right]
\]

The asymptotics for the case of the tunneling amplitude independent of the magnetic field is obtained by substituting $l = 0$.

The large $Q$ asymptotics for finite temperatures and field independent tunneling is given by

\[
K_{11} = \rho_s \left[ 1 + \tilde{\alpha}(\tau) \frac{Q^2}{(1 + 84\tau + 128\tau^2 - 64\tau^3)} \right]
\]

where

\[
\tilde{\alpha}(\tau) = \frac{\Gamma(\tau)}{\Gamma(-\tau)} \frac{\Gamma(5/2 - \tau)}{\Gamma(1/2 + \tau)} \frac{m^*_s}{(3 - 2\tau)(2\tau - 1)}
\]

It is not the soliton mass itself, but the combination $m^*_s = 2\sqrt{\pi} \left( \frac{1}{\Gamma[2(1 - \tau)]} \right) \Gamma(\tau) \left( \frac{\pi}{\Gamma[2(1 - \tau)]} \right) \frac{1}{\Gamma[2(1 - \tau)]}$ that is shown. Close to $Q_c$ the compressional stiffness tensor element $K_{11}$ approaches the constant value $\rho_s$. As $T \to T_{KT}$, $K_{11}$ reaches quickly values close to the asymptotic value, $\rho_s$, and does not vary much with the magnetic field.
which enters as a scale in the expansion above.

The case with the tunneling dependent on the magnetic field \( K_{11} \) has the following asymptotic behavior

\[
[Q \rightarrow +\infty, T \neq 0]
\]

\[
K_{11} = \rho_s \left[ 1 + \alpha(\tau) \frac{Q^{2+4\tau}}{e^{\left(0Q\right)^2/2}} + \ldots \right]. \tag{40}
\]

This is consistent with (37), if in (40) \( \tau = 0 \) is substituted. In (40) only the highest order term is kept.

In this work we made use of the exact solution of the sine-Gordon model to extend the previous results for the bilayer QH system \([1]\) to finite temperatures by including the thermal fluctuations of the soliton lattice. We calculated the magnetization and its temperature dependence in both C and I phases. In addition we calculated the specific heat of the bilayer system which could not be achieved with the method of the previous work \([1]\). The presence of the condensate results in a decrease of both of the specific heat of the system (except close to the critical line where it diverges) and its magnetization.

We observe however that this method cannot be applied in the whole temperature interval where the results for the sine-Gordon model are available. Above the Berezinskii-Kosterlitz-Thouless transition temperature, in the interval \( (T_{\text{BKT}}, T_c) \) where \( \cos \beta \varphi \) and \( \cos(2\pi \theta / \beta) \) are both relevant, the system is in a new phase with coexisting order parameters. (As we argued we could extend the interval of SG applicability, with decreasing accuracy, up to \( 3T_{\text{BKT}} \)). In this phase this system can no longer be described by Eq. (11). Furthermore, as pointed out by \([8]\) at higher temperatures the role of the spin degree of freedom becomes important and the model used would need to be modified.

We notice finally that different authors have found, instead of \( T_C \), the Berezinskii-Kosterlitz-Thouless transition temperature as the point where the soliton lattice melts down.

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