We propose that the ten-dimensional $E_8 \times E_8$ heterotic string is related to an eleven-dimensional theory on the orbifold $\mathbb{R}^{10} \times S^1/\mathbb{Z}_2$ in the same way that the Type IIA string in ten dimensions is related to $\mathbb{R}^{10} \times S^1$. This in particular determines the strong coupling behavior of the ten-dimensional $E_8 \times E_8$ theory. It also leads to a plausible scenario whereby duality between $SO(32)$ heterotic and Type I superstrings follows from the classical symmetries of the eleven-dimensional world, just as the $SL(2,\mathbb{Z})$ duality of the ten-dimensional Type IIB theory follows from eleven-dimensional diffeomorphism invariance.

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1. Introduction

In the last year, the strong coupling behavior of many supersymmetric string theories (or more exactly of what we now understand to be the one supersymmetric string theory in many of its simplest vacua) has been determined. For instance, the strong coupling behavior of most of the ten-dimensional theories and their toroidal compactifications seems to be under control \([1]\). A notable exception is the \(E_8 \times E_8\) heterotic string theory in ten dimensions; no proposal has yet been made that would determine its low energy excitations and interactions in the strong coupling regime. One purpose of this paper is to fill this gap.

We also wish to further explore the relation of string theory to eleven dimensions. The strong coupling behavior of the Type IIA theory in ten dimensions has turned out \([2,1]\) to involve eleven-dimensional supergravity on \(R^{10} \times S^1\), where the radius of the \(S^1\) grows with the string coupling. An eleven-dimensional interpretation of string theory has had other applications, some of them explained in \([3-5]\). The most ambitious interpretation of these facts is to suppose that there really is a yet-unknown eleven-dimensional quantum theory that underlies many aspects of string theory, and we will formulate this paper as an exploration of that theory. (But our arguments, like some of the others that have been given, could be compatible with interpreting the eleven-dimensional world as a limiting description of the low energy excitations for strong coupling, a view taken in \([1]\).) As it has been proposed that the eleven-dimensional theory is a supermembrane theory but there are some reasons to doubt that interpretation, we will non-committally call it the \(M\)-theory, leaving to the future the relation of \(M\) to membranes.

Our approach to learning more about the \(M\)-theory is to consider its behavior on a certain eleven-dimensional orbifold \(R^{10} \times S^1/\mathbb{Z}_2\). In the process, beyond making a proposal for how the \(E_8 \times E_8\) heterotic string is related to the \(M\)-theory, we will make a proposal for relating the classical symmetries of the \(M\)-theory to the conjectured heterotic - Type I string duality in ten dimensions \([1,5,7]\), much as the classical symmetries of the \(M\)-theory have been related to Type II duality symmetries \([1,5,10]\). These proposals suggest a common eleven-dimensional origin of all ten-dimensional string theories and their dualities.

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1 To get the right spectrum of BPS saturated states after toroidal compactification, the eleven-dimensional theory should support stable macroscopic membranes of some sort, presumably described at long wavelengths by the supermembrane action \([6,2]\). We will indeed make this assumption later. But that the theory can be understood as a theory of fundamental membranes seems doubtful because (i) on the face of it, membranes cannot be quantized; (ii) there is no dilaton or coupling parameter that would justify a classical expansion in membranes.
2. The $M$-Theory On An Orbifold

The $M$-theory has for its low energy limit eleven-dimensional supergravity. On an eleven-manifold, with signature $- + + \ldots +$, we introduce gamma matrices $\Gamma_I$, $I = 1, \ldots, 11$, obeying $\{\Gamma_I, \Gamma_J\} = 2\eta_{IJ}$ and (in an oriented orthonormal basis)

$$\Gamma^1 \Gamma^2 \ldots \Gamma^{11} = 1.$$  \hfill (2.1)

We will assume that the $M$-theory has enough in common with what we know of string theory that it makes sense on a wide class of orbifolds — but possibly, like string theory, with extra massless modes arising at fixed points. We will consider the $M$-theory on the particular orbifold $R^{10} \times S^1 / \mathbb{Z}_2$, where $\mathbb{Z}_2$ acts on $S^1$ by $x^{11} \rightarrow -x^{11}$, reversing the orientation. Note that eleven-dimensional supergravity is invariant under orientation-reversal if accompanied by change in sign of the three-form $A^{(3)}$, so this makes sense at least for the massless modes coming from eleven dimensions.

On $R^{10} \times S^1$, the $M$-theory is invariant under supersymmetry generated by an arbitrary constant spinor $\epsilon$. Dividing by $\mathbb{Z}_2$ kills half the supersymmetry; sign conventions can be chosen so that the unbroken supersymmetries are generated by constant spinors $\epsilon$ with $\Gamma^{11} \epsilon = \epsilon$. Together with (2.1), this condition means that

$$\Gamma^1 \Gamma^2 \ldots \Gamma^{10} \epsilon = \epsilon,$$  \hfill (2.2)

so $\epsilon$ is chiral in the ten-dimensional sense.

The $M$-theory on $R^{10} \times S^1 / \mathbb{Z}_2$ thus reduces at low energies to a ten-dimensional Poincaré-invariant supergravity theory with one chiral supersymmetry. There are three string theories with that low energy structure, namely the $E_8 \times E_8$ heterotic string and the two theories — Type I and heterotic — with $SO(32)$ gauge group. It is natural to wonder whether the $M$-theory on $R^{10} \times S^1 / \mathbb{Z}_2$ reduces, as the radius of the $S^1$ shrinks to zero, to one of these three string theories, just as the $M$-theory on $R^{10} \times S^1$ reduces to the Type IIA superstring in the same limit. We will give three arguments that all show that if the $M$-theory on $R^{10} \times S^1 / \mathbb{Z}_2$ reduces for small radius to one of the three string theories, it must be the $E_8 \times E_8$ heterotic string. The arguments are based respectively on space-time gravitational anomalies, the strong coupling behavior, and world-volume gravitational anomalies.

\[2\] One might wonder whether there is a global anomaly that spoils parity conservation, as described on p. 309 of [11]. This does not occur, as there are no exotic twelve-spheres [12].
(i) Gravitational Anomalies

First we consider the gravitational anomalies of the $M$-theory on $\mathbb{R}^{10} \times S^1 / \mathbb{Z}_2$; these should be computable without detailed knowledge of the $M$-theory because anomalies can be computed from only a knowledge of the low-energy structure. In raising the question, we understand a metric on $\mathbb{R}^{10} \times S^1 / \mathbb{Z}_2$ to be a metric on $\mathbb{R}^{10} \times S^1$ that is invariant under $\mathbb{Z}_2$; a diffeomorphism of $\mathbb{R}^{10} \times S^1 / \mathbb{Z}_2$ is a diffeomorphism of $\mathbb{R}^{10} \times S^1$ that commutes with $\mathbb{Z}_2$. The standard massless fermions of the $M$-theory are the gravitinos; by a gravitino mode on $\mathbb{R}^{10} \times S^1 / \mathbb{Z}_2$ we mean a gravitino mode on $\mathbb{R}^{10} \times S^1$ that is invariant under $\mathbb{Z}_2$. With these specifications, it makes sense to ask whether the effective action obtained by integrating out the gravitinos on $\mathbb{R}^{10} \times S^1 / \mathbb{Z}_2$ is anomaly-free, that is, whether it is invariant under diffeomorphisms.

First of all, on a smooth eleven-manifold, the effective action obtained by integrating out gravitinos is anomaly-free; purely gravitational anomalies are absent (except possibly for global anomalies in $8k$ or $8k + 1$ dimensions) in any dimension not of the form $4k + 2$ for some integer $k$. But the result on an orbifold is completely different. In the case we are considering, it is immediately apparent that the Rarita-Schwinger field has a gravitational anomaly. In fact, the eleven-dimensional Rarita-Schwinger field reduces in ten dimensions to a sum of infinitely many massive fields (anomaly-free) and the massless chiral ten-dimensional gravitino discussed above – which \[11\] gives an anomaly under ten-dimensional diffeomorphisms. Thus, at least under those diffeomorphisms of the eleven-dimensional orbifold that come from diffeomorphisms of $\mathbb{R}^{10}$ (times the trivial diffeomorphism of $S^1 / \mathbb{Z}_2$), there is an anomaly.

To compute the form of this anomaly, it is not necessary to do anything essentially new; it is enough to know the standard Rarita-Schwinger anomaly on a ten-manifold, as well as the result (zero) on a smooth eleven-manifold. Thus, under a space-time diffeomorphism $\delta x^I = \epsilon v^I$ generated by a vector field $v^I$, the change of the effective action is on general grounds of the form

$$\delta \Gamma = i \epsilon \int_{\mathbb{R}^{10} \times S^1 / \mathbb{Z}_2} d^{11}x \sqrt{g} v^I(x) W_I(x),$$

where $g$ is the eleven-dimensional metric and $W_I(x)$ can be computed \textit{locally} from the data at $x$. The existence of a local expression for $W_I$ reflects the fact that the anomaly can be understood to result entirely from failure of the regulator to preserve the symmetries, and so can be computed from short distances.
Now, consider the possible form of $W_I(x)$ in our problem. If $x$ is a smooth point in $\mathbb{R}^{10} \times S^1/\mathbb{Z}_2$ (not an orbifold fixed point), then lack of anomalies of the eleven-dimensional theory implies that $W_I(x) = 0$. $W_I$ is therefore a sum of delta functions supported on the fixed hyperplanes $x^{11} = 0$ and $x^{11} = \pi$, which we will call $H'$ and $H''$, so (2.3) actually takes the form

$$\delta \Gamma = i \epsilon \int_{H'} d^{10}x \sqrt{g'} v^I W'_I + i \epsilon \int_{H''} d^{10}x \sqrt{g''} v^I W''_I$$

(2.4)

where now $g'$ and $g''$ are the restrictions of $g$ to $H'$ and $H''$ and $W', W''$ are local functionals constructed from the data on those hyperplanes. Obviously, by symmetry, $W''$ is the same as $W'$, but defined from the metric at $H''$ instead of $H'$. The form of $W'$ and $W''$ can be determined as follows without any computation. Let the metric on $\mathbb{R}^{10} \times S^1/\mathbb{Z}_2$ be the product of an arbitrary metric on $\mathbb{R}^{10}$ and a standard metric on $S^1/\mathbb{Z}_2$, and take $v$ to be the pullback of a vector field on $\mathbb{R}^{10}$. In this situation (as we are simply studying a massless chiral gravitino on $\mathbb{R}^{10}$ plus infinitely many massive fields), $\delta \Gamma$ must simply equal the standard ten-dimensional anomaly. The two contributions in (2.4) from $x^{11} = 0$ and $x^{11} = \pi$ must therefore each give one-half of the standard ten-dimensional answer. Though we considered a rather special configuration to arrive at this result, it was general enough to permit an arbitrary metric at $x^{11} = 0$ (or $\pi$) and hence to determine the functionals $W', W''$ completely.

Since the anomaly is not zero, the massless modes we know about cannot be the whole story for the $M$-theory on this orbifold. There will have to be additional massless modes that propagate only on the fixed planes; they will be analogous to the twisted sector modes of string theory orbifolds. The modes will have to be ten-dimensional vector multiplets because the vector multiplet is the only ten-dimensional supermultiplet with all spins $\leq 1$. Let us determine what vector multiplets there may be.

First of all, part of the anomaly can be canceled by a generalized Green-Schwarz mechanism \[13\], with the fields $B'$ and $B''$, defined as the components $A^{(3)}_{ij11}$ of the three-form on $H'$ and $H''$, entering roughly as the usual $B$ field does in the Green-Schwarz mechanism. There will be interactions $\int_{H'} B' \wedge Z'_8$ and $\int_{H''} B'' \wedge Z''_8$ at the fixed planes, with some eight-forms $Z'$ and $Z''$, and in addition the gauge transformation law of $A^{(3)}$ will have terms proportional to delta functions supported on $H'$ and $H''$. In this way – as in the more familiar ten-dimensional case – some of the anomalies can be canceled, but not all.

In fact, recall that the anomaly in ten-dimensional supergravity is constructed from a twelve-form $Y_{12}$ that is a linear combination of $(\text{tr } R^2)^3$, $\text{tr } R^2 \cdot \text{tr } R^4$, and $\text{tr } R^6$ (with $R$
the curvature two-form and \(\text{tr} \text{ the trace in the vector representation})\). The first two terms are “factorizable” and can potentially be canceled by a Green-Schwarz mechanism. The last term is “irreducible” and cannot be so canceled. The irreducible part of the anomaly must be canceled by additional massless modes – necessarily vector multiplets – from the “twisted sectors.”

In ten dimensions, the story is familiar [13]. The irreducible part of the standard ten-dimensional anomaly can be canceled precisely by the addition of 496 vector multiplets, so that the possible gauge groups in ten-dimensional \(N = 1\) supersymmetric string theory have dimension 496. We are in the same situation now except that the standard anomaly is divided equally between the two fixed hyperplanes. We must have therefore precisely 248 vector multiplets propagating on each of the two hyperplanes! 248 is, of course, the dimension of \(E_8\). So if the \(M\)-theory on this orbifold is to be related to one of the three string theories, it must be the \(E_8 \times E_8\) theory, with one \(E_8\) propagating on each hyperplane. \(SO(32)\) is not possible as gauge invariance would force us to put all the vector multiplets on one hyperplane or the other.

By placing one \(E_8\) at each end, we cancel the irreducible part of the anomaly, but it may not be immediately apparent that the reducible part of the anomaly can be similarly canceled. To see that this is so, recall some facts about the standard ten-dimensional anomaly. With the gauge fields included, the anomaly is derived from a twelve-form \(\tilde{Y}_{12}\) that is a polynomial in \(\text{tr} F_1^2\) and \(\text{tr} F_2^2\) (\(F_1\) and \(F_2\) are the two \(E_8\) curvatures; the symbol \(\text{tr}\) denotes \(1/30\) of the trace in the adjoint representation) as well as \(\text{tr} R^2\), \(\text{tr} R^4\). (\(\text{tr} R^6\) is absent as that part has been canceled by adding vector multiplets.) \(\tilde{Y}_{12}\) has the properties

\[
\frac{\partial^2 \tilde{Y}_{12}}{\partial \text{tr} F_1^2 \partial \text{tr} F_2^2} = 0
\]

\[
\tilde{Y}_{12} = (\text{tr} F_1^2 + \text{tr} F_2^2 - \text{tr} R^2) \wedge \tilde{Y}_8
\]

where the details of the polynomial \(\tilde{Y}_8\) will not be essential. The factorization in the second equation is the key to anomaly cancellation. The first equation reflects the fact that (as the massless fermions are in the adjoint representation) there is no massless fermion charged under each \(E_8\), so that the anomaly has no “cross-terms” involving both \(E_8\)’s.

Note that if we set \(U_i = \text{tr} F_i^2 - \frac{1}{2}\text{tr} R^2\) for \(i = 1, 2\), then the first equation in (2.5) implies that

\[
U_1 \wedge \left(\tilde{Y}_8(U_1, U_2, \text{tr} R^2, \text{tr} R^4) - \tilde{Y}_8(U_1, 0, \text{tr} R^2, \text{tr} R^4)\right)
+ U_2 \wedge \left(\tilde{Y}_8(U_1, U_2, \text{tr} R^2, \text{tr} R^4) - \tilde{Y}_8(0, U_2, \text{tr} R^2, \text{tr} R^4)\right) = 0.
\]

\[
(2.6)
\]
Hence we can write

\[
\tilde{Y}_{12} = (\text{tr} F_1^2 - \frac{1}{2} \text{tr} R^2) \wedge Z_8(\text{tr} F_1^2, \text{tr} R^2, \text{tr} R^4) + (\text{tr} F_2^2 - \frac{1}{2} \text{tr} R^2) \wedge Z_8(\text{tr} F_2^2, \text{tr} R^2, \text{tr} R^4).
\]  

(2.7)

Here \( Z_8 \) is defined by \( Z_8(\text{tr} F_1^2, \text{tr} R^2, \text{tr} R^4) = \tilde{Y}_8(U_1, 0, \text{tr} R^2, \text{tr} R^4) \). (2.7) is the desired formula showing how the anomalies can be canceled by a variant of the Green-Schwarz mechanism adapted to the eleven-dimensional problem. The first term, involving \( F_1 \) but not \( F_2 \), is the contribution from couplings of what was above called \( B' \), and the second term, involving \( F_2 \) but not \( F_1 \), is the contribution from \( B'' \).

(ii) Strong Coupling Behavior

If it is true that the \( M \)-theory on this orbifold is related to the \( E_8 \times E_8 \) superstring theory, then the relation between the radius \( R \) of the circle (in the eleven-dimensional metric) and the string coupling constant \( \lambda \) can be determined by comparing the predictions of the two theories for the low energy effective action of the supergravity multiplet in ten dimensions. The analysis is precisely as in [1] and will not be repeated here. It gives the same relation

\[ R = \lambda^{2/3} \]  

(2.8)

that one finds between the \( M \)-theory on \( \mathbb{R}^{10} \times S^1 \) and Type IIA superstring theory.

In particular, then, for small \( R \) – where the supergravity cannot be a good description as \( R \) is small compared to the Planck length – the string theory is weakly coupled and can be a good description. On the other hand, we get a candidate for the strong coupling behavior of the \( E_8 \times E_8 \) heterotic string: it corresponds to supergravity on the \( \mathbb{R}^{10} \times S^1 / \mathbb{Z}_2 \) orbifold, which is an effective description (of the low energy interactions of the light modes) for large \( \lambda \) as then \( R \) is much bigger than the Planck length. If our proposal is correct, then what a low energy observer sees in the strongly coupled \( E_8 \times E_8 \) theory depends on where he or she is; a generic observer, far from one of the fixed hyperplanes, sees simply eleven-dimensional supergravity (or the \( M \)-theory), and does not distinguish the strongly coupled \( E_8 \times E_8 \) theory from a strongly coupled Type IIA theory, while an observer near one of the distinguished hyperplanes sees eleven-dimensional supergravity on a half-space, with an \( E_8 \) gauge multiplet propagating on the boundary.

We can also now see another reason that if the \( M \)-theory on \( \mathbb{R}^{10} \times S^1 / \mathbb{Z}_2 \) is related to one of the three ten-dimensional string theories, it must be the \( E_8 \times E_8 \) heterotic string.
Indeed, there is by now convincing evidence [1,7-9] that the strong coupling limit of the Type I superstring in ten dimensions is the weakly coupled $SO(32)$ heterotic string, and vice-versa, so we would not want to relate either of the two $SO(32)$ theories to eleven-dimensional supergravity. We must relate the orbifold to the $E_8 \times E_8$ theory whose strong coupling behavior has been previously unknown.

(iii) Extended Membranes

As our third and last piece of evidence, we want to consider extended membrane states in the $M$-theory after further compactification to $\mathbb{R}^9 \times S^1 \times S^1 / \mathbb{Z}_2$.

Our point of view is not that the $M$-theory “is” a theory of membranes but that it describes, among other things, membrane states. There is a crucial difference. For instance, any spontaneously broken unified gauge theory in four dimensions with an unbroken $U(1)$ describes, among other things, magnetic monopoles. That does not mean that the theory can be recovered by quantizing magnetic monopoles; that is presumably possible only in very special cases. Classical magnetic monopole solutions of gauge theory, because of their topological stability, can be quantized to give quantum states. But topologically unstable monopole-antimonopole configurations, while representing possibly interesting classical solutions, cannot ordinarily be quantized to understand photons and electrons. Likewise, we assume that when the topology is right, the $M$-theory has topologically stable membranes (presumably described if the length scale is large enough by the low energy supermembrane action [2]) that can be quantized to give quantum eigenstates. Even when the topology is wrong – for instance on $\mathbb{R}^{11}$ where there is no two-cycle for the membrane to wrap around – macroscopic membrane solutions (with a scale much bigger than the Planck scale) will make sense, but we do not assume that they can be quantized to recover gravitons.

The most familiar example of a situation in which there are topologically stable membrane states is that of compactification of the $M$-theory on $\mathbb{R}^9 \times S^1 \times S^1$. With $x^1$ understood as the time and $x^{10}, x^{11}$ as the two periodic variables, the classical membrane equations have a solution described by $x^2 = \ldots = x^9 = 0$. This solution is certainly topologically stable so (if the radii of the circles are big in Planck units) it can be reliably quantized to obtain quantum states. The solution is invariant under half of the supersymmetries, namely those obeying

$$\Gamma^1 \Gamma^{10} \Gamma^{11} \epsilon = \epsilon,$$

(2.9)
so these will be BPS-saturated states. This latter fact gives the quantization of this particular membrane solution a robustness that enables one (even if the membrane in question can not for other purposes be usefully treated as elementary) to extrapolate to a regime in which one of the $S^1$’s is small and one can compare to weakly coupled string theory.

Let us recall the result of this comparison, which goes under the name of double dimensional reduction of the supermembrane [14]. The membrane solution described above breaks the eleven-dimensional Lorentz group to $SO(1,2) \times SO(8)$. The massless modes on the membrane world-volume are the oscillations of $x^2, \ldots, x^9$, which transform as $(1,8)$, and fermions that transform as $(2,8’’)$). Here 2 is the spinor of $SO(1,2)$ and 8, 8’, and 8’’ are the vector and the two spinors of $SO(8)$. To interpret this in string theory terms, one considers only the zero modes in the $x^{11}$ direction, and decomposes the spinor of $SO(1,2)$ into positive and negative chirality modes of $SO(1,1)$; one thus obtains the world-sheet structure of the Type IIA superstring. So those membrane excitations that are low-lying when the second circle is small will match up properly with Type IIA states. Since many of these Type IIA states are BPS saturated and can be followed from weak to strong coupling, the membrane we started with was really needed to reproduce this part of the spectrum.

Now we move on the the $\mathbb{R}^{10} \times S^1 / \mathbb{Z}_2$ orbifold. We assume that the classical membrane solution $x^2 = \ldots = x^9$ is still allowed on the orbifold; this amounts to assuming that the membranes of the $M$-theory can have boundaries that lie on the fixed hyperplanes. In the orbifold, unbroken supersymmetries (as we discussed at the outset of section two) correspond to spinors $\epsilon$ with $\Gamma^{11} \epsilon = \epsilon$; these transform as the 16 of $SO(1,9)$, or as $8’_+ \oplus 8’’_-$ of $SO(1,1) \times SO(8)$. The spinors unbroken in the field of the membrane solution also obey (2.4), or equivalently $\Gamma^1 \Gamma^{10} \epsilon = \epsilon$. Thus, looking at the situation in string terms (for an observer who does not know about the eleventh dimension), the unbroken supersymmetries have positive chirality on the string world sheet and transform as $8’_+$ (where the + is the $SO(1,1)$ chirality) under $SO(1,1) \times SO(8)$. The massless world-sheet bosons, oscillations in $x^2, \ldots, x^9$, survive the orbifolding, but half of the fermions are projected out. The survivors transform as $8’’_-$; one can think of them [15] as Goldstone fermions for the $8’’_-$ supersymmetries that are broken by the classical membrane solution. The – chirality means that they are right-moving.

So the massless modes we know about transform like the world-sheet modes of the heterotic string that carry space-time quantum numbers: left and right-moving bosons
transforming in the $\mathbf{8}$ and right-moving fermions in the $\mathbf{8}''$. Recovering much of the world-sheet structure of the heterotic string does not imply that the string theory (if any) related to the $M$-theory orbifold is a heterotic rather than Type I string; the Type I theory also describes among other things an object with the world-sheet structure of the heterotic string $\mathbb{P}$. It is by considerations of anomalies on the membrane world-volume that we will reach an interesting conclusion.

The Dirac operator on the membrane three-volume is free of world-volume gravitational anomalies as long as the world-volume is a smooth manifold. (Recall that except possibly for discrete anomalies in $8k$ or $8k + 1$ dimensions, gravitational anomalies occur only in dimensions $4k + 2$.) In the present case, the world-volume is not a smooth manifold, but has orbifold singularities (possibly better thought of as boundaries) at $x^{11} = 0$ and at $x^{11} = \pi$. These singularities give rise to three-dimensional gravitational anomalies; this is obvious from the fact that the massless world-sheet fermions in the two-dimensional sense are the fermions $\mathbf{8}''$ of definite chirality. By analysis just like we gave in the eleven-dimensional case, the gravitational anomaly on the membrane world-volume vanishes at smooth points and is a sum of delta functions supported at $x^{11} = 0$ and $x^{11} = \pi$. As in the eleven-dimensional case, these delta functions each represent one half of the usual two-dimensional anomaly of the effective massless two-dimensional $\mathbf{8}''$ field.

As is perhaps obvious intuitively and we will argue below, the gravitational anomaly of the $\mathbf{8}''$ field is the usual gravitational anomaly of right-moving RNS fermions and superconformal ghosts in the heterotic string. So far we have only considered the modes that propagate in bulk on the membrane world-volume. If the membrane theory makes sense in the situation we are considering, the gravitational anomaly of the $\mathbf{8}''$ field must be canceled by additional world-volume “twisted sector” modes, supported at the orbifold fixed points $x^{11} = 0$ and $x^{11} = \pi$. If we are to recover one of the known string theories, these “twisted sector modes” should be left-moving current algebra modes with $c = 16$. (In any event there is practically no other way to maintain space-time supersymmetry.) Usually both $SO(32)$ and $E_8 \times E_8$ are possible, but in the present context the anomaly that must be cancelled is supported one half at $x^{11} = 0$ and one half at $x^{11} = \pi$, so the only possibility is to have $E_8 \times E_8$ with one $E_8$ supported at each end. This then is our third reason that if the $M$-theory on the given orbifold has a known string theory as its weak coupling limit, it must be the $E_8 \times E_8$ heterotic string.

It remains to discuss somewhat more carefully the gravitational anomalies that have just been exploited. Some care is needed here as there is an important distinction between
objects that are quantized as elementary strings and objects that are only known as macroscopic strings embedded in space-time. (See, for instance, [16,17].) For elementary strings, one usually considers separately both right-moving and left-moving conformal anomalies. The sum of the two is the total conformal anomaly (which generalizes to the conformal anomaly in dimensions above two), while the difference is the world-sheet gravitational anomaly. For objects that are only known as macroscopic strings embedded in space-time, the total conformal anomaly is not a natural concept, since the string world-sheet has a natural metric (and not just a conformal class of metrics) coming from the embedding. But the gravitational anomaly, which was exploited above, still makes sense even in this situation, as the world-sheet is still not endowed with a natural coordinate system.

Let us justify the claim that the world-sheet gravitational anomaly of the $8''$ fermions encountered above equals the usual gravitational anomaly from right-moving RNS fermions and superconformal ghosts. A detailed calculation is not necessary, as this can be established by the following simple means. First let us state the problem (as it appears after double dimensional reduction to the Green-Schwarz formulation of the heterotic string) in generality. The problem really involves, in general, a two-dimensional world-sheet $\Sigma$ embedded in a ten-manifold $M$. The normal bundle $N$ to the world-sheet is a vector bundle with structure group $SO(8)$. If $S_-$ is the bundle of negative chirality spinors on $\Sigma$ and $N''$ is the bundle associated to $N$ in the $8'$ representation of $SO(8)$, then the $8''$ fermions that we want are sections of $L_- \otimes N''$. By making a triality transformation in $SO(8)$, we can replace the fermions with sections of $L_- \otimes N$ without changing the anomalies. Now using the fact that the tangent bundle of $M$ is the sum of $N$ and the tangent bundle of $\Sigma - TM = N \oplus T\Sigma$ – we can replace $L_- \otimes N$ by $L_- \otimes TM$ if we also subtract the contribution of fermions that take values in $L_- \otimes T\Sigma$. The $L_- \otimes TM$-valued fermions are the usual right-moving RNS fermions, and (as superconformal ghosts take values in $L_- \otimes T\Sigma$) subtracting the contribution of fermions valued in $L_- \otimes T\Sigma$ has the same effect as including the superconformal ghosts.

3. Heterotic - Type I Duality from the $M$-Theory

In this section, we will try to relate the eleven-dimensional picture to another interesting phenomenon, which is the conjectured duality between the heterotic and Type I $SO(32)$ superstrings.
So far we have presented arguments indicating that the $E_8 \times E_8$ heterotic string theory is related to the $M$-theory on $\mathbb{R}^{10} \times S^1/\mathbb{Z}_2$, just as the Type IIA theory is related to the $M$-theory compactified on $\mathbb{R}^{10} \times S^1$.

We can follow this analogy one step further, and compactify the tenth dimension of the $M$-theory on $S^1$. Schwarz [5] and Aspinwall [10] explained how the $SL(2,\mathbb{Z})$ duality of the ten-dimensional Type IIB string theory follows from space-time diffeomorphism symmetry of the $M$-theory on $\mathbb{R}^9 \times T^2$. (For some earlier results in that direction see also [18].) Here we will argue that the $SO(32)$ heterotic-Type I duality similarly follows from classical symmetries of the $M$-theory on $\mathbb{R}^9 \times S^1 \times S^1/\mathbb{Z}_2$.

First we need several facts about $T$-duality of open-string models.

**$T$-Duality in Type I Superstring Theory**

The Type I theory in ten dimensions can be interpreted as a generalized $\mathbb{Z}_2$ orbifold of the Type IIB theory [19,23]. The orbifold in question acts by reversing world-sheet parity, and acts trivially on the space-time. Projection of the Type IIB spectrum to $\mathbb{Z}_2$-invariant states makes the Type IIB strings unoriented; this creates an anomaly in the path integral over world-sheets with crosscaps, which must be compensated for by introducing boundaries. The open strings, which are usually introduced to cancel the anomaly, are naturally interpreted as the twisted states of the parameter-space orbifold.

More generally, one can combine the reversal of world-sheet orientation with a space-time symmetry, getting a variant of the Type I theory [20,22]. A special case of this will be important here. Upon compactification to $\mathbb{R}^9 \times S^1$, Type IIB theory is $T$-dual to Type IIA theory. Analogously, the Type I theory – which is a $\mathbb{Z}_2$ orbifold of Type IIB theory – is $T$-dual to a certain $\mathbb{Z}_2$ orbifold of Type IIA theory. This orbifold is constructed by dividing the Type IIA theory by a $\mathbb{Z}_2$ that reverses the world-sheet orientation and acts on the circle by $x^{10} \rightarrow -x^{10}$. Note that the Type IIA theory is invariant under combined reversal of world-sheet and space-time orientations (but not under either one separately), so the combined operation is a symmetry. This theory has been called the Type $I'$ or Type IA theory. In this theory, the twisted states are open strings that have their endpoints at the fixed points $x^{10} = 0$ and $x^{10} = \pi$. To cancel anomalies, these open strings must carry

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3 More generally still, one can divide by a group containing some elements that act only on space-time and some that also reverse the world-sheet orientation; the construction of the twisted states then has certain subtleties that were discussed in [24].
Chan-Paton factors. If we want to treat the two fixed points symmetrically – as in natural
in an orbifold – while canceling the anomalies, there must be $SO(16)$ Chan-Paton factors
at each fixed point, so the gauge group is $SO(16) \times SO(16)$.

In fact, it has been shown \([20,22]\) that the $SO(16) \times SO(16)$ theory just described is the
$T$-dual of the vacuum of the standard Type I theory on $R^9 \times S^1$ in which $SO(32)$ is broken
to $SO(16) \times SO(16)$ by a Wilson line. This is roughly because $T$-duality exchanges the
usual Neumann boundary conditions of open strings with Dirichlet boundary conditions,
and gives a theory in which the open strings terminate at the fixed points.

Of course, the Type I theory on $R^9 \times S^1$ has moduli corresponding to Wilson lines;
by adjusting them one can change the unbroken gauge group or restore the full $SO(32)$. In
the $T$-dual description, turning on these moduli causes the positions at which the open
strings terminate to vary in a way that depends upon their Chan-Paton charges \([25]\). The
vacuum with unbroken $SO(32)$ has all open strings terminating at the same fixed point.

**Heterotic - Type I Duality**

We are now ready to try to relate the eleven-dimensional picture to the conjectured
heterotic - Type I duality of ten-dimensional theories with gauge group $SO(32)$.

What suggests a connection is the following. Consider the Type I superstring on
$R^9 \times S^1$. Its $T$-dual is related, as we have discussed, to the Type IIA theory on an
$R^9 \times S^1/\mathbb{Z}_2$ orbifold. We can hope to identify the Type IIA theory on $R^9 \times S^1/\mathbb{Z}_2$ with
the $M$-theory on $R^9 \times S^1/\mathbb{Z}_2 \times S^1$, since in general one hopes to associate Type IIA theory
on any space $X$ with $M$-theory on $X \times S^1$.

On the other hand, we have interpreted the $E_8 \times E_8$ heterotic string as $M$-theory
on $R^{10} \times S^1/\mathbb{Z}_2$, so the $M$-theory on $R^9 \times S^1 \times S^1/\mathbb{Z}_2$ should be the $E_8 \times E_8$ theory on
$R^9 \times S^1$.

So we now have two ways to look at the $M$-theory on $X = R^9 \times S^1/\mathbb{Z}_2 \times S^1$. (1) It
is the Type IIA theory on $R^9 \times S^1/\mathbb{Z}_2$ which is also the $T$-dual of the Type I theory on
$R^9 \times S^1$. (2) After exchanging the last two factors so as to write $X$ as $R^9 \times S^1 \times S^1/\mathbb{Z}_2$,
the same theory should be the $E_8 \times E_8$ heterotic string on $R^9 \times S^1$. So it looks like we
can predict a relation between the Type I and heterotic string theories!

This cannot be right in the form stated, since the model in (1) has gauge group
$SO(16) \times SO(16)$, while that in (2) has gauge group $E_8 \times E_8$. Without really understanding
the $M$-theory, we cannot properly explain what to do, but pragmatically the most sensible
course is to turn on a Wilson line in theory (2), breaking $E_8 \times E_8$ to $SO(16) \times SO(16)$.
At this point, it is possible that (1) and (2) are equivalent (under exchanging the last two factors in $\mathbb{R}^9 \times S^1/\mathbb{Z}_2 \times S^1$). The equivalence does not appear, at first sight, to be a known equivalence between string theories. We can relate it to a known equivalence by making a $T$-duality transformation on each side. In (1), a $T$-duality transformation will convert to the Type I theory on $\mathbb{R}^9 \times S^1$ (in its $SO(16) \times SO(16)$ vacuum). In (2), a $T$-duality transformation will convert to an $SO(32)$ heterotic string with $SO(32)$ spontaneously broken to $SO(16) \times SO(16)$.

At this point, theories (1) and (2) are Type I and heterotic $SO(32)$ theories (in their respective $SO(16) \times SO(16)$ vacua), so we can try to compare them using the conjectured heterotic - Type I duality. It turns out that this acts in the expected way, exchanging the last two factors in $X = \mathbb{R}^9 \times S^1/\mathbb{Z}_2 \times S^1$.

Since the logic may seem convoluted, let us recapitulate. On side (1), we start with the $M$ theory on $\mathbb{R}^9 \times S^1/\mathbb{Z}_2 \times S^1$, and interpret it as the $T$-dual of the Type I theory on $\mathbb{R}^9 \times S^1$. On side (2), we start with the $M$-theory on $\mathbb{R}^9 \times S^1 \times S^1/\mathbb{Z}_2$, interpret it as the $E_8 \times E_8$ heterotic string on $\mathbb{R}^9 \times S^1$ and (after turning on a Wilson line) make a $T$-duality transformation to convert the gauge group to $SO(32)$. Then we compare (1) and (2) using heterotic - Type I duality, which gives the same relation that was expected from the eleven-dimensional point of view. One is still left wishing that one understood better the meaning of $T$-duality in the $M$-theory.

We hope that this introduction will make the computation below easier to follow.

**Side (1)**

We start with the $M$-theory on $\mathbb{R}^9 \times S^1/\mathbb{Z}_2 \times S^1$, with radii $R_{10}$ and $R_{11}$ for the two circles. We interpret this as the Type IA theory, which is the $\mathbb{Z}_2$ orbifold of the Type II theory on $\mathbb{R}^9 \times S^1$, a $T$-dual of the Type I theory in a vacuum with unbroken $SO(16) \times SO(16)$.

The relation between $R_{10}$ and $R_{11}$ and the Type IA parameters (the ten-dimensional string coupling $\lambda_{IA}$ and radius $R_{1A}$ of the $S^1$) can be computed by comparing the low

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4 $R \to 1/R$ symmetry, with $R$ the radius of the circle in $\mathbb{R}^9 \times S^1$, maps the heterotic string vacuum with unbroken $SO(32)$ to itself, and maps the heterotic string vacuum with unbroken $E_8 \times E_8$ to itself, but maps a heterotic string vacuum with $E_8 \times E_8$ broken to $SO(16) \times SO(16)$ to a heterotic string vacuum with $SO(32)$ broken to $SO(16) \times SO(16)$. This follows from facts such as those explained in [26,27].
energy actions for the supergravity multiplet. The computation and result are as in [1]:

\[
R_{11} = \frac{\lambda_{IA}^{2/3}}{R_{10}} = \frac{1}{\lambda_{IA}^{2/3}}
\]

Now we make a T-duality transformation to an ordinary Type I theory (with unbroken gauge group $SO(16) \times SO(16)$), by the standard formulas $R_1 = 1/R_{IA}$, $\lambda_1 = \lambda_{IA}/R_{IA}$. So

\[
R_{11} = \frac{\lambda_1^{2/3}}{R_1^{2/3}}
\]

\[
R_{10} = \frac{\lambda_1^{1/3}}{R_1^{2/3}}.
\]

Side (2)

Now we start with the $M$-theory on $\mathbb{R}^9 \times S^1 \times S^1/\mathbb{Z}^2$, with the radii of the last two factors denoted as $R'_{10}$ and $R'_{11}$. This is hopefully related to the $E_8 \times E_8$ heterotic string on $\mathbb{R}^9 \times S^1$, with the $M$-theory parameters being related to the heterotic string coupling $\lambda_{E_8}$ and radius $R_{E_8}$ by formulas

\[
R'_{11} = \frac{\lambda_{E_8}^{2/3}}{R_{10}} = \frac{1}{\lambda_{E_8}^{2/3}}
\]

\[
R'_{10} = \frac{\lambda_{E_8}^{1/3}}{R_{11}^{2/3}}.
\]

The second equation is equivalent to the Weyl rescaling $g_{10,M} = \lambda_{IA}^{-2/3} g_{10,IIA}$ obtained in [1] between the ten-dimensional metrics as measured in the $M$-theory or Type IIA. Also, by $\lambda_{IA}$ we refer to the ten-dimensional string coupling constant; similar convention is also used for all other string theories below.

\[5\]
Comparison

Now we compare the two sides via the conjectured $SO(32)$ heterotic - Type I duality according to which these theories coincide with

$$\lambda_h = \frac{1}{\lambda_I},$$

$$R_h = \frac{R_I}{\lambda_I^{1/2}}.$$  \hfill (3.5)

A comparison of (3.2) and (3.4) now reveals that the relation of $R_{10}, R_{11}$ to $R'_{10}, R'_{11}$ is simply

$$R_{10} = R'_{11},$$

$$R_{11} = R'_{10}.\hfill (3.6)$$

So – as promised – under this sequence of operations, the natural symmetry in eleven dimensions becomes standard heterotic - Type I duality.

4. Comparison to Type II Dualities

We have seen a close analogy between the dualities that involve heterotic and Type I string theories and relate them to the $M$-theory, and the corresponding Type II dualities that relate the Type IIA theory to eleven dimensions and the Type IIB theory to itself. The reason for this analogy is of course that while the Type II dualities are all related to the compactification of the $M$-theory on $\mathbb{R}^9 \times \mathbb{T}^2$, the heterotic and Type I dualities are related to the compactification of the $M$-theory on a $\mathbb{Z}_2$ orbifold of $\mathbb{R}^9 \times \mathbb{T}^2$. It is the purpose of this section to make this analogy more explicit.

![Fig. 1: A section of the moduli space of compactifications of the $M$-theory on $\mathbb{R}^9 \times \mathbb{T}^2$. By virtue of the $\mathbb{Z}_2$ symmetry between the two compact dimensions, only the shaded half of the diagram is relevant.](image)
The moduli space of compactifications of the $M$-theory on a rectangular torus is shown in Figure 1, following [10,5]. Let us first recall how one can see the $SL(2, \mathbb{Z})$ duality of the ten-dimensional Type IIB theory in the moduli space of the $M$-theory on $\mathbb{R}^9 \times T^2$. The variables of the Type IIB theory on $\mathbb{R}^9 \times S^1$ are related to the compactification radii of the $M$-theory on $\mathbb{R}^9 \times T^2$ by

$$
\lambda_{\text{IIB}} = \frac{R_{11}}{R_{10}},
$$

$$
R_{\text{IIB}} = \frac{1}{R_{10}R_{11}^{1/2}}.
$$

The string coupling constant $\lambda_{\text{IIB}}$ depends on the shape of the two-torus of the $M$-theory, but not on its area. As we send the radius $R_{\text{IIB}}$ to infinity to make the Type IIB theory ten-dimensional at fixed $\lambda_{\text{IIB}}$, the radii $R_{10}$ and $R_{11}$ go to zero. So, the ten-dimensional Type IIB theory at arbitrary string coupling corresponds to the origin of the moduli space of the $M$-theory on $\mathbb{R}^9 \times T^2$ as shown in Figure 1. The $SL(2, \mathbb{Z})$ duality group of the ten-dimensional Type IIB theory can be identified with the modular group acting on the $T^2$ [3,10].

Similarly, the region of the moduli space where only one of the radii $R_{10}$ and $R_{11}$ is small corresponds to the weakly coupled Type IIA string theory. When both radii become large simultaneously, the Type IIA string theory becomes strongly coupled, and the low energy physics of the theory is described by eleven-dimensional supergravity [1].

Now we can repeat the discussion for the heterotic and Type I theories. The moduli space of the $M$-theory compactified on $\mathbb{R}^9 \times S^1/\mathbb{Z}_2 \times S^1$ is sketched in Figure 2.

![Fig. 2: A section of the moduli space of compactifications of the $M$-theory on $\mathbb{R}^9 \times S^1 \times S^1/\mathbb{Z}_2$. Here $R_{10}$ is the radius of $S^1/\mathbb{Z}_2$.](image-url)
In the previous sections we discussed the relations between the Type IA theory on $\mathbb{R}^9 \times S^1/\mathbb{Z}_2$, the Type I theory on $\mathbb{R}^9 \times S^1$, and the $M$-theory on $\mathbb{R}^9 \times S^1/\mathbb{Z}_2 \times S^1$. This relation leads to the following expression for the Type I variables in terms of the variables of the $M$-theory,

$$\lambda_I = \frac{R_{11}}{R_{10}},$$
$$R_I = \frac{1}{R_{10} R_{11}^{1/2}}. \quad (4.2)$$

The string coupling constant $\lambda_I$ depends only on the shape of the two-torus of the $M$-theory. By the same reasoning as in the Type IIB case, the ten-dimensional Type I theory at arbitrary string coupling $\lambda_I$ is represented by the origin of the moduli space, which should be – in both cases – more rigorously treated as a blow-up.

We have related the $SO(32)$ heterotic string is related to variables of the $M$-theory by

$$\lambda_h = \frac{R'_{11}}{R'_{10}},$$
$$R_h = \frac{1}{R'_{10} R'_{11}^{1/2}}. \quad (4.3)$$

Using heterotic - Type I duality, which simply exchanges the two radii,

$$R'_{10} = R_{11},$$
$$R'_{11} = R_{10}, \quad (4.4)$$
we can express this relation in terms of $R_{10}$ and $R_{11},$

$$\lambda_h = \frac{R_{10}}{R_{11}},$$
$$R_h = \frac{1}{R_{11} R_{10}^{1/2}}. \quad (4.5)$$

Just like the ten-dimensional Type I theory, the ten-dimensional $SO(32)$ heterotic theory at all couplings corresponds to the origin of the moduli space. The heterotic - Type I duality maps one of these theories at strong coupling to the other theory at weak coupling, and vice-versa.

Now we would like to understand the regions of the moduli space where at least one of the radii $R_{10}, R_{11}$ is large. Recall that $R_{10}$ and $R_{11}$ – as measured in the $M$-theory – are related to the string coupling constants by

$$R_{10} = \lambda_{E_8}^{2/3},$$
$$R_{11} = \lambda_{IA}^{2/3}. \quad (4.6)$$
If one of the radii is large and the other one is small, the natural description of the physics at low energies is in terms of the weakly-coupled Type IA or $E_8 \times E_8$ heterotic string theory. Conversely, as we go to the limit where both $R_{10}$ and $R_{11}$ are large, both string theories are strongly coupled, and the low energy physics is effectively described by the $M$-theory.

Comparison of the Spectra

We can gain some more insight into the picture by looking at some physical states of the $M$-theory and interpreting them as states in different weakly-coupled string theories.

A particularly natural set of states in the $M$-theory on $\mathbb{R}^9 \times T^2$ is given by the Kaluza-Klein (KK) states of the supergravity multiplet, that is the states carrying momentum in the tenth and eleventh dimension, along with the wrapping modes of the membrane. As measured in the $M$-theory, these states have masses

$$M^2 = \frac{\ell^2}{R_{10}^2} + \frac{m^2}{R_{11}^2} + n^2 R_{10}^2 R_{11}^2$$

for certain values of $m, n, \ell$.

We are of course interested in states of the $M$-theory on $\mathbb{R}^9 \times S^1/\mathbb{Z}_2 \times S^1$. In order to get the states that survive on the orbifold, we must project (4.7) to the $\mathbb{Z}_2$ invariant sector. Schematically, the orbifold group acts on the states with the quantum numbers of (4.7) as follows:

$$|\ell, m, n\rangle \rightarrow \pm | - \ell, m, n \rangle.$$  \hspace{1cm} (4.8)

The action of the orbifold group on the KK modes follows directly from its action on the space-time coordinates. The action on the membrane wrapping modes indicates that the orbifold changes simultaneously the space-time orientation as well as the world-volume orientation of the membrane.

While $m$ and $n$ are conserved quantum numbers even in the orbifold, $\ell$ is not. Nevertheless, we include it in the discussion since $\ell$ is approximately conserved in some limits.

If our prediction about the relation of the $M$-theory on $\mathbb{R}^9 \times S^1/\mathbb{Z}_2 \times S^1$ to the heterotic and Type I string theories is correct, the stable states of the $M$-theory must have an interpretation in each of these string theories. The string masses of these states as measured by the Type I observer are

$$M_I^2 = \ell^2 R_i^2 + \frac{m^2 R_i^2}{\lambda_i} + \frac{n^2}{R_i^2}.$$ \hspace{1cm} (4.9)
The membrane wrapping modes can be identified with the KK modes of the Type I string, while the unstable states correspond to unstable winding modes of the elementary Type I string. The membrane KK modes along the eleventh dimension are non-perturbative states in the Type I theory. These states can be identified with winding modes of non-perturbative strings with tension $T_2 \propto \lambda_\gamma^{-1}$. We will see below that this is simply the solitonic heterotic string of the Type I theory [7,8,9].

Similarly, we can try to interpret the states in the $T$-dual, Type IA theory. A Type IA observer will measure the following masses of the states:

$$M_{1A}^2 = \frac{\ell^2}{R_{1A}^2} + \frac{m^2}{\lambda_{1A}^2} + n^2 R_{1A}^2.$$  \hspace{1cm} (4.10)

As required by $T$-duality, the membrane wrapping modes correspond to the string winding modes. The unstable states correspond to the KK modes of the Type IA closed string; they are unstable because the tenth component of the momentum is not conserved in the Type IA theory. The stable KK states of the $M$-theory correspond to non-perturbative Type IA states. These Type IA states can be identified with the zero-branes (alias extremal black holes) of the Type IIA theory in ten dimensions. Notice that under the $\mathbb{Z}_2$ orbifold action, the quantum number that corresponds to the extremal black hole states is conserved, and the zero-brane states survive the orbifold projection.

In the $E_8 \times E_8$ heterotic theory, the masses of our states are given by

$$M_{E8}^2 = \frac{m^2}{R_{E8}^2} + n^2 R_{E8}^2.$$ \hspace{1cm} (4.11)

(We here omit $\ell$, as the unstable states it labels have no clear interpretation for the weakly coupled heterotic string.) The stable KK modes along the eleventh dimension in the $M$-theory can be interpreted as the KK modes along the tenth dimension in the heterotic theory, while the membrane wrapping modes are the winding modes of the heterotic string.

In the $SO(32)$ heterotic string, the masses are

$$M_h^2 = m^2 R_h^2 + n^2 R_h^2.$$ \hspace{1cm} (4.12)

Again, these are the usual momentum and winding states of the heterotic string. The formulas also make it clear that – as expected from the heterotic - Type I duality – the $m = 1$ non-perturbative Type I string state corresponds to the elementary heterotic string.
We already pointed out an analogy between the heterotic - Type I duality and the $SL(2,\mathbb{Z})$ duality of the Type IIB theory; now we actually see remnants of the $SL(2,\mathbb{Z})$ multiplet of Type IIB string states in the $SO(32)$ heterotic and Type I theories. This can be best demonstrated when we consider the weakly-coupled Type IIB theory, and look at the behavior of its spectrum under the $\mathbb{Z}_2$ orbifold group that leads to the Type I theory. The perturbative Type IIB string of the $SL(2,\mathbb{Z})$ multiplet is odd under the $\mathbb{Z}_2$ orbifold action, and so does not give rise to a stable string. But a linear combination of strings winding in opposite directions survives the projection and corresponds to the elementary Type I closed string, which is unstable but long-lived for weak coupling. The $SL(2,\mathbb{Z})$ Type IIB string multiplet also contains a non-perturbative state that is even under $\mathbb{Z}_2$, and we have just identified it with the elementary heterotic string. Upon orbifolding, the original $SL(2,\mathbb{Z})$ multiplet of Type IIB strings thus gives rise to both Type I and the heterotic string.

**Twisted Membrane States**

The states we have discussed so far are analogous to untwisted states of string theory on orbifolds. The membrane world-volume is without boundary, but the membrane Hilbert space is projected onto $\mathbb{Z}_2$-invariant states; the $\mathbb{Z}_2$ simultaneously reverses the sign of $x^{11}$ and the membrane orientation.

We must also add the twisted membrane states, which are analogous to open strings in parameter space orbifolds of Type II string theory reviewed above. Just like the open strings of the Type IA theory, the twisted membrane states have world-volumes with two boundary components, restricted to lie at one of the orbifold fixed points, $x^{10} = 0$ and $x^{10} = \pi$. Such a state might simply be localized near one of the fixed points (in which case the description as a membrane state might not really be valid), or it might wrap around $S^1/\mathbb{Z}_2 \times S^1$ a certain number of times (in which case the membrane description does make sense at least if the radii are large). The former states might be called twisted KK states, and should include the non-abelian gauge bosons discussed in section two. The latter states will be called twisted wrapping modes. The twisted states carry no momentum in the orbifold direction. Both the momentum $\tilde{m}$ in the $S^1$ direction and the wrapping number $\tilde{n}$ are conserved.

States of these twisted sectors have masses – as measured in the $M$-theory – given by

$$M^2 = \frac{\tilde{m}^2}{R_{11}^2} + \tilde{n}^2 R_{10}^2 R_{11}^2.$$  (4.13)
Just as in the untwisted sector, one has to project out the twisted states that are not invariant under the orbifold group action.

Again, the $\mathbb{Z}_2$-invariant twisted states should have a natural interpretation in the corresponding string theories. In the Type I theory, the twisted states of the $M$-theory have masses

$$M_I^2 = \frac{\tilde{m}^2 R_I^2}{\lambda_I^2} + \frac{\tilde{n}^2}{R_I^2}. \quad (4.14)$$

At generic points of our moduli space, the twisted states carry non-trivial representations of $SO(16) \times SO(16)$. The twisted wrapping modes of the membrane correspond to the KK modes of the open Type I string. The twisted KK modes of the $M$-theory are non-perturbative string states of the Type I theory, with masses $\propto R_I/\lambda_I$; they are charged under the gauge group, and should be identified with the charged heterotic soliton strings of the Type I theory.

In the Type IIA theory, we obtain the following mass formula:

$$M_{IA}^2 = \frac{\tilde{m}^2}{\lambda_{IA}} + \frac{\tilde{n}^2}{R_{IA}^2}. \quad (4.15)$$

While the twisted wrapping modes of the $M$-theory correspond to the perturbative winding modes of the Type IIA open string, the twisted KK modes show up in the Type IIA theory as additional non-perturbative black-hole states, charged under $SO(16) \times SO(16)$.

In the $E_8 \times E_8$ heterotic theory,

$$M_{E_8}^2 = \frac{\tilde{m}^2}{R_{E_8}^2} + \frac{\tilde{n}^2}{R_{E_8}^2}. \quad (4.16)$$

Both sectors are perturbative heterotic string states in non-trivial representations of $SO(16) \times SO(16)$. In the $SO(32)$ heterotic theory, the corresponding masses are

$$M_h^2 = \frac{\tilde{m}^2 R_h^2}{R_h^2} + \frac{\tilde{n}^2}{R_h^2}, \quad (4.17)$$

which is of course in accord with $T$-duality.

One can go on and analyze spectra of other $p$-branes. Let us only notice here that the space-time orbifold singularities of the $M$-theory on $\mathbb{R}^9 \times S^1/\mathbb{Z}_2 \times S^1$ are intriguing $M$-theoretical analogs of Dirichlet-branes of string theory.

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