Erratum: Trichotomous noise induced stochastic resonance in a fractional oscillator with random damping and random frequency (2016 J. Stat. Mech. 023201)

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Several corrections were not applied at the proofreading stage. The Production team regrets the errors, and presents the corrections here. These corrections do not affect the scientific outcomes of the paper.

(1) On page 6, there should be a ‘.’ at the end of the last one of the equation (6).

(2) On page 6, there should be a ‘.’ at the end of the equation (7).

(3) On the 6th row of page 7, there is an extra $\dot{x}$ in the expression of $d_{34}$.

(4) On the 7th row of page 7, ‘ and $x_1(0)$, $x_2(0)$ and $x_3(0)$ are the initial conditions’ should be corrected into ‘and $x_1(0)$, $x_2(0)$, $x_3(0)$, $\dot{x}_1(0)$, $\dot{x}_2(0)$, $\dot{x}_3(0)$ are the initial conditions’.
(5) On the 14th row of page 7, before the sentence ‘Applying the inverse Laplace transformation technique’, there should be another sentence, which is ‘Let $x_{i+3}(0) = x_i(0), i = 1, 2, 3$’.

(6) For equation (16) of page 7, the second term on the right side $\sum_{k=1}^{3} h_{ik}(t)x_k(0)$ should be corrected into $\sum_{k=1}^{6} h_{ik}(t)x_k(0)$.

(7) On page 7, in the line below equation (16), ‘$k = 0, 1, 2, 3$’, should be corrected into ‘$k = 0–6$’.

(8) On page 7, in the line below equation (17), ‘$k = 1, 2, 3$’, should be corrected into ‘$k = 1–6$’.

(9) Figure 4 on page 13 should have appeared on page 12, before section ‘3.2 Stochastic resonance’.

(10) On the 8th row of page 16 (at the end of paragraph 1), ‘(see also figures 8(a), (c) and (d))’ should be read ‘(see also figures 6 (a), (c) and (d))’.
Trichotomous noise induced stochastic resonance in a fractional oscillator with random damping and random frequency

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Abstract. In this study, we investigate the stochastic resonance (SR) of a fractional linear oscillator subjected to multiplicative trichotomous noise and additive fractional Gaussian noise and driven by a periodic signal. Using the Shapiro-Loginov formula and the Laplace transformation technique, we acquire the exact expression of the first-order moment of the system’s steady response. Meanwhile, we discuss the evolutions of the output amplitude with the frequency of the periodic signal and noise parameters. We determine that bona fide SR, SR and reverse-resonance exist in this system. Specifically, the evolution of the output amplitude with the frequency of the periodic signal presents one-peak oscillation, double-peak oscillation and triple-peak oscillation. Moreover, the interplay of the trichotomous noise and memory can induce and diversify the stochastic multi-resonance (SMR) phenomena and give this linear system richer dynamic behavior. A hypersensitive response of the output amplitude to the noise intensity is demonstrated, which is not observed in systems driven by dichotomous noise.

Keywords: stochastic particle dynamics (theory), fluctuations (theory), stochastic processes (theory), Brownian motion
1. Introduction

The term ‘stochastic resonance’ (SR) was first proposed in 1981 by Benzi et al to account for the periodic reoccurrence of ice ages on Earth [1]. Since then, SR has gained considerable attention due to its wide applications in many diverse fields, such as physics, biomedical engineering, and medicine [2], and has been studied extensively in theory and experiment. Various studies have convincingly shown that properly adding noise to a system can sometimes induce new and more orderly behavior of the system [3, 4] and overturned the traditional viewpoint that noise produces nothing but destructive effects.

In early studies, the SR phenomenon was typically found in nonlinear systems driven by periodic signals and noise [3–6]; thus, a nonlinear system, periodic signal and noise were regarded as three essential ingredients to generate SR [7, 8]. Nevertheless, recent studies have shown that SR can also take place in linear systems subjected to periodic signals and multiplicative noise or linear systems driven only by multiplicative noise [9, 10]. In addition, the original understanding of SR is extended. To avoid any misunderstanding, note that, in this paper, we use the term SR in the broad sense [11], which means that the non-monotonic behaviors of the output signal amplitude depend on the changes in the noise characteristic parameters, rather than the usually considered the signal-to-noise ratio (SNR) [12, 13].

As the simplest toy model to describe different phenomena in nature since Chandrasekhar [14] originally considered the problem of noise-driven dynamics of a Brownian harmonic oscillator, the harmonic oscillator with random mass [15, 16], random damping [17, 18], and random frequency [19–22] has attracted continued interest from scientists in different fields. In this paper, we will be mainly interested in
the harmonic oscillator including frequency fluctuation which has been extensively investigated in biology (population dynamics [23]), physics (dye lasers [24], turbulent flows [25]), economics (stock market prices [26]), etc. In most of the above-mentioned studies, the models used to investigate SR are usually based on the classical integral-order oscillator, which is suitable to describe normal diffusion [15–22]. It is a physical phenomenon in ideal fluctuating environment, where the friction term is only dependent on the current velocity. However, in the real world, anomalous diffusion processes can be found in a wide range of areas [27–31]. For example, the motion of individual fluorescently labeled mRNA molecules inside live *Escherichia coli* cells is subdiffusive [32], and the lipid granules typically perform subdiffusive motion in the cytoplasm [33]. Even intrinsic conformational dynamics of protein macromolecules can be subdiffusive [34, 35]. Anomalous diffusion can be modeled as the fractional oscillator (FO) [36, 37] and described by the fractional Langevin equation (FLE) [38, 39]. As a generalization of the classical harmonic oscillator, the fractional oscillator (FO) plays a significant role, and the characteristics of fractional operators make them suitable for describing those systems with long-range dependence and long memory. Therefore, close attention has been paid to exploring SR mechanics in fractional noisy oscillators including frequency fluctuation in more recent years [40–45]. For example, Soika *et al* [42] investigated the resonant behavior of a fractional oscillator with fluctuating frequency, and discussed the necessary and sufficient conditions for the cooperation effects arising as a consequence of the interplay of colored noise and memory. Mankin *et al* [43] studied the long-time limit behavior of the variance and correlation function for the output signal of a fractional oscillator with fluctuating eigenfrequency subjected to a periodic force and discussed the effect of memory-induced energetic stability encountered when the harmonic potential is absent. Zhong *et al* [44] obtained the exact expressions of the amplitude and signal-to-noise (SNR) of the fractional Langevin equation, which is driven by the additive fractional Gaussian noise, and investigated the SR in this type of fractional linear system with random frequency. He *et al* [45] considered the synergy of the fluctuation of damping and spring stiffness, which can influence the dynamics of a system, and studied the SR phenomenon of a fractional oscillator with random damping strength and random spring stiffness.

However, most of the previous works have investigated SR in harmonic oscillators driven by dichotomous noise [15, 16, 18–22, 32, 42–45]. In contrast, relatively few literature reports focused on the SR phenomenon driven by trichotomous noise [41, 46–51], which is a type of three-level Markovian noise. Both dichotomous noise and trichotomous noise are called random telegraph noise [49] and are useful to model natural colored fluctuations. However, the latter is more flexible and includes all cases of dichotomous noise [50]. In fact, the dichotomous noise can be regarded as a special case of the trichotomous noise, specifically, the stationary probabilities of dichotomous noise $q_0 = 0.5$. Furthermore, the flatness parameter $\kappa = \frac{1}{2q}$ of the trichotomous noise can vary from 1 to $+\infty$, unlike the flatness for Gaussian colored noise $\kappa = 3$ and symmetric dichotomous noise $\kappa = 1$. The extra degree of freedom is more useful in modeling actual fluctuations [50, 51]. Therefore, the effects of trichotomous noise on lots of dynamic systems were considered [52]. For instance, Brownian particles in a spatially periodic asymmetric potential (ratchet) [47], and in a piecewise linear spatially periodic potential [53] have been investigated.
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Ion channels are complex membrane proteins which provide ion-conducting and nanoscale pores in biological membranes [54]. These proteins undergo spontaneous conformational dynamics resulting in stochastic intermittent events of opening and closing of the pores [55]. Gao [56] proposed a linear fractional random-delayed oscillator with random frequency to investigate this phenomenon, and discussed the cooperative effects of random delay and the fractional memory on the generalized stochastic resonance (GSR). Usually, the damping and spring stiffness are immersed in the same environment, and fluctuations in the damping and spring stiffness can influence the dynamics of systems [45]. Thus motivated, we consider a fractional oscillator similar to the one presented in [45], except for some details of the noise, i.e. a fractional oscillator with random damping and random frequency subjected to an external periodic force and an additive thermal noise. In this work, the fluctuations of the damping and frequency are modeled as trichotomous noises, and this fractional linear oscillator with trichotomous-noise-fluctuated damping and frequency would present richer and more complicated dynamic behaviors. The main purpose of this paper is to investigate the SR phenomenon in this type of fractional oscillator (FO) with random damping and random frequency and discuss how the trichotomous noise parameters influence the resonance phenomenon.

The paper is organized as follows: section 2 presents the model of the fractional linear oscillator with trichotomous-noise-fluctuated damping and frequency and gives an analytical expression of the first-order moment of the system’s steady response. Section 3 presents the simulation results. Section 4 presents the conclusions.

2. System model

We consider an underdamped fractional linear oscillator subjected to multiplicative trichotomous noise $\xi(t)$, an additive internal noise $\eta(t)$, and an external periodic force, which is described by the generalized Langevin equation (GLE) with $m = 1$:

$$\frac{d^2 x(t)}{dt^2} + \gamma [1 + \xi(t)] \int_0^t \beta(t - t') \ddot{x}(t') dt' + \omega^2 [1 + \xi(t)] x(t) = R \cos(\Omega t) + \eta(t),$$

(1)

where $x(t)$ is the displacement of a particle, $\ddot{x}(t)$ is the Newton’s acceleration term, $\gamma$ is the friction coefficient, $\omega$ is the intrinsic frequency of the system, and $R$ and $\Omega$ are the amplitude and frequency of the periodic signal, respectively.

In many physical and biological environments, viscous media usually have power-law memory that represents the dependence of the viscous force on the velocity history of particle [57]. Therefore, the damping kernel function $\beta(t)$ is expressed as $\beta(t) = \frac{1}{\Gamma(1 - \alpha)} |t|^{1-\alpha}$. According to Caputo’s definition of fractional derivative, equation (1) can be written as

$$\frac{d^2 x(t)}{dt^2} + \gamma [1 + \xi(t)] \int_0^t \beta(t - t') \ddot{x}(t') dt' + \omega^2 [1 + \xi(t)] x(t) = R \cos(\Omega t) + \eta(t),$$

(2)

and equation (2) was named as a fractional Langevin equation.
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We investigate this fractional linear oscillator mentioned above, which is described by equation (2). Here, the ‘external noises’ $\xi(t)$ is modeled as symmetric trichotomous noises such that $\xi(t) \in \{-\sigma, 0, \sigma\}$ with stationary probabilities $P(-\sigma) = P(\sigma) = q$, $P(0) = 1 - 2q$, for $0 < q \leq 0.5$. The statistical properties of $\xi(t)$ and noise flatness $\kappa$ are defined as follows:

$$\langle \xi(t) \rangle = 0, \quad \langle \xi(t) \xi(t') \rangle = 2q\sigma^2 \exp(-\lambda|t - t'|), \quad \kappa = \frac{\langle \xi^4(t) \rangle}{\langle \xi^2(t) \rangle^2} = \frac{1}{2q},$$

(3)

where $\sigma^2$ and $\lambda$ are the noise intensity and the correlation rate of $\xi(t)$, respectively. In addition, $\eta(t)$ represents the ‘internal noise’ that drives the GLE and shares the same origin as the damping force of the system [58]. Therefore, the relationship between the damping kernel function $\beta(t)$ and the additive noise $\eta(t)$ can be established via the fluctuation-dissipation theorem [59]:

$$\langle \eta(t) \eta(t') \rangle = \kappa_B T \gamma \beta(t - t') = \kappa_B T \gamma \frac{[t-t']^{\alpha}}{\Gamma(1-\alpha)},$$

(4)

where $\kappa_B$ is the Boltzmann constant and $T$ is the absolute temperature.

In this paper, $\eta(t)$ is modeled as the fractional Gaussian noise (fGn) [60] and satisfies

$$\langle \eta(t) \rangle = 0, \quad \langle \eta(t) \eta(t') \rangle = 2DH(2H - 1)|t - t'|^{2H-2},$$

(5)

where $D$ is the noise intensity of $\eta(t)$ and $H$ is the Hurst parameter. By comparing equations (4) with (5), we obtain

$$H = (2 - \alpha)/2, \quad D = \kappa_B T \gamma / \Gamma(3 - \alpha)$$

(6)

We assume that the ‘external noises’ $\xi(t)$ and the ‘internal noise’ $\eta(t)$ are uncorrelated because they have different origins. That is, the two noises satisfy

$$\langle \xi(t) \eta(t') \rangle = 0$$

(7)

In the next part, we will obtain the exact expression of the first-order moment of the system’s steady response.

2.1. First-order moment of the system stationary state response

Average equation (2), then we will obtain

$$\frac{d^2 \langle x \rangle}{dt^2} + \gamma_0 D^\alpha_0 \langle x \rangle + \gamma e^{-\lambda t} D^\alpha_0 \langle (\xi x) e^{\lambda t} \rangle + \omega^2 \langle x \rangle + \omega^2 \langle \xi x \rangle = R \cos(\Omega t).$$

(8)

Multiplying both sides of equation (2) by $\xi(t)$ and averaging all terms, we have

$$\langle \xi \dot{x} \rangle + 2q\sigma^2 \gamma_0 D^\alpha_0 \langle x \rangle + \gamma e^{-\lambda t} D^\alpha_0 [\langle (\xi x) + \langle \xi^2 x \rangle - 2q\sigma^2 \langle x \rangle \rangle e^{\lambda t}]$$

$$+ \omega^2 \langle \xi x \rangle + \omega^2 \langle \xi^2 x \rangle = 0$$

(9)

Using a similar method, we multiply both sides of equation (2) by $\xi^2(t)$ and average all terms. Based on the properties of trichotomous noise $\xi(t)$, that is, $\xi^3(t) = \sigma^2 \xi(t)$ and $\langle \xi^2(t) \rangle = 2q\sigma^2$, we obtain the following equation:

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\[
\langle \xi^2(x) \rangle + 2q\sigma^2 C_0^\alpha D_t^\alpha (x) + \gamma e^{-\lambda t} C_0^\alpha D_t^\alpha [((\sigma^2 + 2q\sigma^2) (x) - 2q\sigma^2 (x)) e^{\lambda t}] + \omega^2 \sigma^2 \langle \xi^2(x) \rangle + \omega^2 \langle \xi^2(x) \rangle = 2q\sigma^2 R \cos(\Omega t).
\]

To perform the splitting of correlators, we use the well-known Shapiro-Loginov formula [61], which reads as

\[
\left\langle \frac{d^n}{dt^n} \xi^2(x) \right\rangle = \left( \frac{d}{dt} + \lambda \right)^n \langle \xi(x) \rangle.
\]

Furthermore, to address the correlation factor \( \langle \xi^2(t) \hat{x}(t) \rangle \), we use the Shapiro-Loginov formula again and obtain

\[
\langle \xi^2(x) \rangle = \left( \frac{d}{dt} + \lambda \right)^2 \langle \xi^2(x) \rangle - 2q\sigma^2 \lambda \left( \frac{d^2}{dt^2} + \lambda \right) \langle \xi(x) \rangle.
\]

Inserting equations (11) and (12) into equations (9) and (10), we obtain the fractional differential equations as follows:

\[
\begin{align*}
\left( \frac{d^2}{dt^2} + \gamma_0^\alpha D_t^\alpha + \omega^2 \right) \langle x(t) \rangle + \omega^2 \langle \xi(x) \rangle + \gamma e^{-\lambda t} C_0^\alpha D_t^\alpha [ (\langle \xi(x) \rangle e^{\lambda t}) ] = R \cos \Omega t, \\
2q\sigma^2 \gamma_0^\alpha D_t^\alpha \langle x(t) \rangle + \left[ \left( \frac{d}{dt} + \lambda \right)^2 + \omega^2 \right] \langle \xi(x) \rangle + \omega^2 \langle \xi^2(x) \rangle \\
+ \gamma e^{-\lambda t} C_0^\alpha D_t^\alpha [ (\langle \xi(x) \rangle + \langle \xi^2(x) \rangle - 2q\sigma^2 (x)) e^{\lambda t} ] = 0,
\end{align*}
\]

\[
2q\sigma^2 \left[ -2\lambda \frac{d}{dt} + \gamma_0^\alpha D_t^\alpha - \lambda^2 \right] \langle x(t) \rangle + \omega^2 \sigma^2 \langle \xi(x) \rangle + \left[ \left( \frac{d}{dt} + \lambda \right)^2 + \omega^2 \right] \langle \xi^2(x) \rangle \\
+ \gamma e^{-\lambda t} C_0^\alpha D_t^\alpha [ (\sigma^2 + \langle \xi^2(x) \rangle - 2q\sigma^2 (x)) e^{\lambda t} ] = 2q\sigma^2 R \cos \Omega t.
\]

To solve the equation (13) with three variables \( x_1 = \langle x \rangle, x_2 = \langle \xi(x) \rangle \) and \( x_3 = \langle \xi^2(x) \rangle \), we use the Laplace transformation technique and obtain [62]:

\[
\begin{align*}
d_{11}X_1(s) + d_{12}X_2(s) + d_{13}X_3(s) &= \frac{Rs}{s^2 + \Omega^2} + d_{14}, \\
d_{21}X_1(s) + d_{22}X_2(s) + d_{23}X_3(s) &= d_{24}, \\
d_{31}X_1(s) + d_{32}X_2(s) + d_{33}X_3(s) &= \frac{2q\sigma^2 Rs}{s^2 + \Omega^2} + d_{34},
\end{align*}
\]

where \( X_i(s) = \mathcal{L} \{ x_i(t) \} = \int_0^{\infty} x_i(t) e^{-st} dt \), \( i = 1, 2, 3, \)

\[
\begin{align*}
d_{11} &= s^2 + 2\gamma \sigma^\alpha + \omega^2, \\
d_{12} &= \gamma(s + \lambda)^\alpha + \omega^2, \\
d_{13} &= 0, \\
d_{14} &= (s + \gamma s^{\alpha - 1}) x_1(0) + \dot{x}_1(0) + \gamma(s + \lambda)^{\alpha - 1} x_2(0), \\
d_{21} &= 2q\sigma^2 [s^\alpha - (s + \lambda)^\alpha], \\
d_{22} &= (s + \gamma) + \gamma(s + \lambda)^\alpha + \omega^2, \\
d_{23} &= \gamma(s + \lambda)^\alpha + \omega^2,
\end{align*}
\]
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\[ d_{24} = 2q\sigma^2\gamma[(s^{\alpha-1} - (s + \lambda)^{\alpha-1})x_1(0) + [s + 2\lambda + \gamma(s + \lambda)^{\alpha-1}]]x_2(0) + \dot{x}_2(0) \]
\[ + \gamma(s + \lambda)^{\alpha-1}x_3(0), \]
\[ d_{31} = 2q\sigma^2[-2\lambda s + \gamma s^\alpha - \gamma(s + \lambda)^\alpha - \lambda^2], \quad d_{32} = \sigma^2[\gamma(s + \lambda)^\alpha + \omega^2], \]
\[ d_{33} = (s + \lambda)^2 + \gamma(s + \lambda)^\alpha + \omega^2, \]
\[ d_{34} = 2q\sigma^2[\gamma s^{\alpha-1} - \gamma(s + \lambda)^{\alpha-1} - 2\lambda]x_1(0) + \sigma^2\gamma(s + \lambda)^{\alpha-1}x_2(0) \]
\[ + [s + 2\lambda + \gamma(s + \lambda)^{\alpha-1}]x_3(0) + \dot{x}_3(0), \]

and \( x_1(0), x_2(0) \) and \( x_3(0) \) are the initial conditions.

The solutions of equation (14) can be represented as

\[
\begin{bmatrix}
X_1(s) \\
X_2(s) \\
X_3(s)
\end{bmatrix} = \frac{1}{H(s)}\begin{bmatrix}
(d_{22}d_{33} - d_{23}d_{32} + 2q\sigma^2d_{12}d_{23})Rs \\
\frac{(d_{23}d_{31} - d_{21}d_{33} - 2q\sigma^2d_{12}d_{23})}{s^2 + \Omega^2} + d_{14}(d_{22}d_{33} - d_{23}d_{32}) - d_{12}(d_{21}d_{33} - d_{23}d_{31}) \\
\frac{(d_{21}d_{32} - d_{22}d_{31} + 2q\sigma^2(d_{11}d_{22} - d_{12}d_{21}))}{s^2 + \Omega^2} + \frac{1}{H(s)}[d_{14}(d_{21}d_{32} - d_{22}d_{31}) - d_{12}(d_{11}d_{22} - d_{12}d_{21}) + d_{34}(d_{11}d_{22} - d_{12}d_{21})]
\end{bmatrix},
\]

where \( H(s) = d_{11}d_{22}d_{33} - d_{12}d_{21}d_{33} - d_{13}d_{22}d_{33} + d_{12}d_{23}d_{31} \).

Applying the inverse Laplace transformation technique, we obtain

\[ x_i(t) = R \int_0^t h_{10}(t - t')\cos(\Omega t')dt' + \sum_{k=1}^{3} h_{ik}(t)x_k(0), \quad i = 1, 2, 3, \]

where \( H_{ik}(s) \) are the Laplace transforms of \( h_{ik}(t) \), \( k = 0, 1, 2, 3 \), and can be determined by equation (15). Here, the expressions of \( x_i(t), i = 1, 2, 3 \) are the Wiener forms adherent to Stratonovich description \([11, 15, 18, 63]\). Specifically, \( H_{10}(s) \) is the transfer function of system, which is written as

\[
H_{10}(s) = \frac{2q\sigma^2d_{12}d_{23} + d_{22}d_{33} - d_{23}d_{32}}{d_{11}d_{22}d_{33} - d_{12}d_{21}d_{33} - d_{13}d_{22}d_{33} + d_{12}d_{23}d_{31}}.
\]

In the long-time regime \( t \to \infty \), the functions \( h_{ik}(t), k = 1, 2, 3 \) tend to zero only if

\[ \sigma^2 < (\sigma^2)_{cr} = \frac{(\lambda^2 + \gamma\lambda^\alpha + \omega^2)^2}{(2q\lambda^2 + \gamma\lambda^\alpha + \omega^2)(\gamma\lambda^\alpha + \omega^2)}, \quad \lambda > 0. \]

In this paper, we assume that the condition is satisfied. Thus, in the case of the long-time limit \( t \to \infty \), the influence of the initial conditions will vanish, and the asymptotic expression of \( \langle x(t) \rangle \) is written in the following form:

\[ \langle x(t) \rangle_{as} = \langle x(t) \rangle_{t \to \infty} = R \int_0^t h_{10}(t - t')\cos(\Omega t')dt'. \]
Using the linear response theory, equation (19) can be further expressed as [62]

\[ \langle x(t) \rangle_{\text{as}} = \langle x(t) \rangle_{t \to \infty} = A \cos(\Omega t + \phi), \] (20)

where \( A \) and \( \phi \) are the amplitude and the phase shift of the system stationary state response \( \langle x(t) \rangle_{\text{as}} \) respectively, and they satisfy

\[ A = R|H_{10}(j\Omega)|, \quad \phi = \arg(H_{10}(j\Omega)). \] (21)

Using equations (17) and (21), one can find that

\[ A = R \sqrt{\frac{f_1^2 + f_2^2}{f_3^2 + f_4^2}}, \] (22)

\[ \phi = \arctan \left( \frac{f_2 f_3 - f_1 f_4}{f_1 f_3 + f_2 f_4} \right), \] (23)

where

\[ f_1 = b^4 \cos(4\theta) + 2\gamma b^{2+\alpha}\cos[(2 + \alpha)\theta] + 2\omega^2 b^2 \cos(2\theta) + M\gamma^2 b^{2\alpha} \cos(2\alpha\theta) + 2M\omega^2 b^\alpha \cos(\alpha\theta) + M\omega^4, \]

\[ f_2 = b^4 \sin(4\theta) + 2\gamma b^{2+\alpha}\sin[(2 + \alpha)\theta] + 2\omega^2 b^2 \sin(2\theta) + M\gamma^2 b^{2\alpha} \sin(2\alpha\theta) + 2M\omega^2 b^\alpha \sin(\alpha\theta), \]

\[ f_3 = N b^4 \cos(4\theta) + 2q\sigma^2 \gamma b^{2+2\alpha}\cos[(2 + \alpha)\theta] + 2(N + q\sigma^2 \omega^2)\gamma b^{2+\alpha} \cos[(2 + \alpha)\theta]
+ 2N\omega^2 b^2 \cos(2\theta) + [N(1 - \sigma^2) - 2q\sigma^2 \lambda^2] \gamma b^{2\alpha} \cos(2\alpha\theta)
+ 2[N(1 - \sigma^2) - 2q\sigma^2 \lambda^2] \gamma b^\alpha \omega^2 \cos(\alpha\theta) + (1 - \sigma^2) \gamma \omega^4 b^{\alpha} \cos \left( \frac{\pi}{2} \right)
+ \omega^4 [N(1 - \sigma^2) - 2q\sigma^2 \lambda^2] + \gamma b^4 \Omega^\alpha \cos \left( 4\theta + \frac{\pi}{2} \right)
+ 2(1 - q\sigma^2)\gamma b^{2+\alpha} \Omega^\alpha \cos \left( 2\theta + \frac{\pi}{2} \right)
+ 2(1 - q\sigma^2)\gamma \omega^2 b^{\alpha} \Omega^\alpha \cos \left( \alpha\theta + \frac{\pi}{2} \right)
+ (1 - \sigma^2) \gamma b^{2\alpha} \Omega^\alpha \cos \left( 2\alpha\theta + \frac{\pi}{2} \right)
+ 2(1 - q\sigma^2)\gamma \omega^2 b^{\alpha} \Omega^\alpha \cos \left( \alpha\theta + \frac{\pi}{2} \right)
- 4q\sigma^2 \lambda\gamma b^{2\alpha} \Omega^\alpha \cos \left( 2\alpha\theta + \frac{\pi}{2} \right) - 8q\sigma^2 \lambda\gamma \omega^2 b^{\alpha} \Omega^\alpha \cos \left( \alpha\theta + \frac{\pi}{2} \right). \]
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\[ f_1 = N b^4 \sin (4\theta) + 2q\sigma^2\omega^2b^2\sin [(2 + 2\alpha)\theta] + 2(N + q\sigma^2\omega^2)\gamma b^{2+\alpha} \sin [(2 + \alpha)\theta] \\
+ 2N\omega^2b^2\sin (2\theta) + [N(1 - \sigma^2) - 2q\sigma^2\lambda^2]\gamma^2b^{2\alpha} \sin (2\alpha\theta) \\
+ 2[N(1 - \sigma^2) - 2q\sigma^2\lambda^2]b^\alpha\omega^2 \sin (\alpha\theta) + (1 - \sigma^2)\gamma\omega^4\Omega^\alpha \sin \left(\frac{\pi}{2}\alpha\right) \\
+ \gamma b^4\Omega^\alpha \sin \left(4\theta + \frac{\pi}{2}\alpha\right) + 2(1 - q\sigma^2)\gamma^2b^{2+\alpha}\Omega^\alpha \sin \left[(2 + \alpha)\theta + \frac{\pi}{2}\alpha\right] \\
+ 2(1 - q\sigma^2)\gamma\omega^2b^2\Omega^\alpha \sin \left(2\theta + \frac{\pi}{2}\alpha\right) + (1 - \sigma^2)\gamma^3b^{2\alpha}\Omega^\alpha \sin \left(2\alpha\theta + \frac{\pi}{2}\alpha\right) \\
+ 2(1 - \sigma^2)\gamma\omega^2b^2\Omega^\alpha \sin \left(\alpha\theta + \frac{\pi}{2}\alpha\right) - 4q\sigma^2\lambda^2\gamma^2b^{2\alpha}\Omega^\alpha \sin \left(2\alpha\theta + \frac{\pi}{2}\alpha\right) \\
- 8q\sigma^2\lambda\gamma\omega^2b^\alpha\Omega^\alpha \sin \left(\alpha\theta + \frac{\pi}{2}\alpha\right) - 4q\sigma^2\lambda\omega^4\Omega^\alpha, \]

\[ b = \sqrt{\Omega^2 + \lambda^2}, \quad \theta = \arctan \left(\frac{\Omega}{\lambda}\right), \]

\[ M = 1 - (1 - 2q)\sigma^2, \quad N = \omega^2 - \Omega^2. \]

3. Numerical discussion

Now, we perform the numerical work on the above analytical expression in equation (22), which shows the behaviors of the output amplitude \( A \) for any combination of the parameters \( \alpha, \gamma, \Omega, \sigma^2, \lambda, \) and \( q. \)

3.1. Bona fide stochastic resonance

In figure 1, we plot the curves of the output amplitude \( A \) as a function of the driving frequency \( \Omega \) with different values of the trichotomous noise correlation rate \( \lambda. \)

As shown in figure 1, all of the curves show that \( A \) attains a maximum value with increasing \( \Omega, \) indicating that the bona fide SR takes place. Furthermore, for different values of \( \lambda, \) the picture of the resonance behavior of \( A(\Omega) \) is quite different. Obviously, there are three different types of the bona fide SR which are represented as depending on \( \lambda: \) (1) when \( \lambda = 0.01, \) there are three maximum values of \( A(\Omega) \) (triple-peak SR phenomenon); (2) when \( \lambda = 0.3, \) there are two maximum values of \( A(\Omega) \) (double-peak SR phenomenon); (3) when \( \lambda = 1, \) there is only one maximum value of \( A(\Omega) \) (one-peak SR phenomenon). In the former two situations, there is more than one peak in each curve of \( A(\Omega), \) i.e. stochastic multi-resonance (SMR) [64, 65] phenomenon occurs, which is not observed in conventional linear system. Specifically, in the case of the system parameters applied in figure 1, when \( \lambda = 0.01, \) triple-peak SR takes place, and the

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three resonance peaks occur at the frequency $\Omega_1 \approx \sqrt{\omega^2 - \sigma} \approx 0.475$, $\Omega_2 \approx \omega = 1$, and $\Omega_3 \approx \sqrt{\omega^2 + \sigma} \approx 1.332$, which correspond to the noise states $\xi_1 = -\sigma$, $\xi_2 = 0$, and $\xi_3 = \sigma$, respectively. In addition, the positions of the three peaks merge as $\lambda$ increases \cite{Note1}, and the resonance peak occurs at the frequency $\Omega \approx -\gamma$ when $\lambda \to 1$.

In figure 2, we plot the phase diagrams in the $\alpha - \gamma$ plane for the emergence of the bona fide SR of $A$ versus $\Omega$ under different values of $\lambda$ at $R = 1$, $\omega = 1$, $q = 0.2$ and $\sigma^2 = 0.6$. In the unshaded regions, bona fide SR is impossible. The shaded regions in the figure correspond to the regions where the bona fide SR of $A$ versus $\Omega$ is possible. Three phases can be discerned in the resonance domain: the light gray region, where there is only one resonance peak, the dark gray region, where double resonance peaks appear, and the black region, where a triple-peak phenomenon is observed.

As shown in figure 2, with the increase of $\lambda$, the triple-peak SR region gets smaller, and finally disappears, and the one-peak SR region becomes bigger; however, the double-peak SR region and non-resonance region non-monotonously vary. In addition, with all other parameters fixed and increasing $\lambda$, the type of bona fide SR transfers from stochastic multi-resonance (triple-peak SR and double-peak SR) to one-peak SR.

In figure 3, we depict the phase diagrams in the $\alpha - \gamma$ plane for the emergence of the bona fide SR of $A$ versus $\Omega$ under different values of the stationary probability $q$ at $R = 1$, $\omega = 1$, $\lambda = 0.1$ and $\sigma^2 = 0.6$. In the unshaded regions, bona fide SR is impossible. The shaded regions in the figure correspond to those regions where bona fide SR is possible. Three phases can be discerned in the resonance domain: the light gray region (one-peak SR region), the dark gray region (double-peak SR region) and the black region (triple-peak SR region). As shown in figure 3, with the increase of $q$, the non-resonance region is getting smaller and smaller, and the other three types of regions vary non-monotonously. Specifically, when $q = 0.5$, the trichotomous noise degenerates.
into dichotomous noise, such that the triple-peak SR phenomenon disappear (see also in figure 3(d)).

Therefore, it is noteworthy that the stationary probability $q$ diversifies the stochastic multi-resonance phenomena and gives this linear system richer dynamic behavior.

In figure 4, we depict the curves of the output amplitude $A$ as a function of $\Omega$ with different values of $q$. As shown in figure 4(a), all of the curves show that $A$ attains a maximum value with increasing $\Omega$, indicating that the bona fide SR takes place. Furthermore, for different values of $q$, the picture of the resonance behavior of $A(\Omega)$ is quite different. When $q = 0.01$, there is only one maximum value of $A(\Omega)$ (one-peak SR phenomenon takes place). Since the stationary probability for jumps to the noise states $\xi_1 = -\sigma$ and $\xi_2 = \sigma$ is very small, the secondary resonance peaks are absent. When $q = 0.1$, there are two maximum values of $A(\Omega)$ (double-peak SR phenomenon). When $q = 0.25$, there are three maximum values of $A(\Omega)$ (triple-peak SR phenomenon). With further increases in $q$, the triple-peak SR region disappear (see also in figure 3(d)).

When $q = 0.5$, the curve presents double-peak SR phenomenon. From figure 4(b), we find that as $q$ changes, the types of bona fide SR can transfer among the three types of SRs mentioned above. Furthermore, with the increase of $q$, the peak value decreases and the second resonance peak appear in the left of the original one, that is, the type of bona fide SR transfers from one-peak SR to double-peak SR. Here, the left peak value is smaller than the original one. With the further increase of $q$, the left peak value increase, the original peak value still decrease, and the third resonance peak appear in the right of the original peak. That is, the type of bona fide SR transfers from double-peak SR to triple-peak SR. When $0.25 \leq q \leq 0.47$, the left peak value is bigger than

Figure 2. The phase diagrams for bona fide stochastic resonance versus $\Omega$ in the $\alpha - \gamma$ plane at $R = 1$, $\omega = 1$, $q = 0.2$, $\sigma^2 = 0.6$. Other parameter values: (a) $\lambda = 0.01$; (b) $\lambda = 0.15$; (c) $\lambda = 0.5$; (d) $\lambda = 1$. 

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the original one, that is, the left peak becomes the main peak. When $0.47 < q \leq 0.5$, the type of bona fide SR transfers from triple-peak SR to double-peak SR again. It is worth emphasizing that the peak in the one-peak SR ($q = 0.01$) and the valley in the two-peak SR ($q = 0.5$) appear at the same position. In addition, with the decrease of $q$, the position of the main peak (where the peak value is maximum) shifts from $0.561\Omega \approx 1.232\Omega$.

3.2. Stochastic resonance

In this part, we aim to examine the dependence of the output amplitude $A$ on the noise parameters (including the noise intensity $\sigma^2$ and the stationary probability $q$). Here, because $\kappa = \frac{1}{2q}$, we do discuss the dependence of the output amplitude $A$ on the noise flatness $\kappa$.

In figure 5, we present the curves of $A$ as functions of $\sigma^2$ with different values of $\lambda$ and $q$. In figure 5(a), the curve shows that $A$ attains a resonance-like minimum value with increasing $\sigma^2$, indicating that reverse-resonance [49] occurs. In figures 5(b) and (c), these two curves show that $A$ attains a maximum value with increasing $\sigma^2$, indicating that the SR phenomenon occurs. Under the synergy between external trichotomous noise, internal fractional Gaussian noise and periodic signal, the power of multiplicative noise transforms into the power of the periodic signal, therefore, it enhances the output amplitude $A$. In addition, figure 5(c) also demonstrates that $A$ sharp decreases

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**Figure 3.** The phase diagrams for bona fide stochastic resonance versus $\Omega$ in the $\alpha - \gamma$ plane at $R = 1, \omega = 1, \sigma^2 = 0.6, \lambda = 0.1$. Other parameter values: (a) $q = 0.01$; (b) $q = 0.1$; (c) $q = 0.25$; (d) $q = 0.5$. 

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from peak value to valley value with increasing $\sigma^2$, which is called the hypersensitive response [41] phenomenon. This interesting SR phenomenon reveals that the characteristics of stochastic resonance are extremely sensitive to tiny variations in the noise intensity $\sigma^2$ [51].

Figure 6 shows the phase diagrams in the $\alpha - \gamma$ plane for the emergence of stochastic resonance of $A$ versus $\sigma^2$ under different values of $\lambda$ and $q$ at $R = 1, \omega = 1$, and $\Omega = 1$. In the unshaded regions, the resonance phenomenon is impossible. The shaded regions in the figure correspond to those regions where stochastic resonance and reverse-resonance of $A$ versus $\sigma^2$ is possible. Three phases can be discerned in the resonance domain:

Figure 4. Bona fide stochastic resonance for the response function $A$ versus the driving frequency $\Omega$. Other parameter values: $R = 1, \omega = 1, \gamma = 0.5, \sigma^2 = 0.6, \lambda = 0.1, \alpha = 0.2$. 
Figure 5. Stochastic resonance for the response function $A$ versus $\sigma^2$ at $R = 1, \omega = \Omega = 1, \gamma = 0.1, \alpha = 0.1$. Other parameter values: (a) $\lambda = 0.01, q = 0.1$; (b) $\lambda = 0.1, q = 0.1$; (c) $\lambda = 0.01, q = 0.4$. 

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Figure 6. The phase diagrams for stochastic resonance versus $\sigma^2$ in the $\alpha - \gamma$ plane at $R = 1, \omega = \Omega = 1$. Other parameter values: (a) $\lambda = 0.01, q = 0.1$; (b) $\lambda = 0.01 q = 0.4$; (c) $\lambda = 0.1, q = 0.1$; (d) $\lambda = 1, q = 0.1$.

(1) the light gray region (i), where there is only one valley; (2) the light gray region (ii), where only one resonance peak is observed; (3) the dark gray region, where one peak and one valley appear. In addition, when $\lambda$ and $q$ are given different values, there are three types of resonance phenomena: one-valley reverse-resonance, one-peak SR, and one-peak and one-valley SR (see also figure 5). Furthermore, when $q$ is fixed, with the increase in $\lambda$, one can observe that the one-peak SR region becomes larger while the one-valley reverse-resonance region and one-peak and one-valley SR region become smaller monotonously and disappear eventually (see also figures 8(a), (c) and (d)).

Figure 7 shows curves of the output amplitude $A$ versus the noise intensity $\sigma^2$ with different values of parameters (including $\lambda$ and $q$).

As shown in figure 7, each curve shows that $A$ attains a maximum value by increasing $\sigma^2$, indicating that the SR appears. Moreover, from the equation

$$\frac{d(A)}{d(\sigma^2)} = 0$$

the position of the peak of the curve $A(\sigma^2)$ is determined. When other parameters was fixed, $\sigma^2(\lambda)$ determine the position of the resonance peaks. In figure 7(a), with the increase of $\lambda$, the type of SR of $A$ versus $\sigma^2$ changes from one-peak and one-valley SR to one-peak SR, the resonance peak becomes flat and the position of the peak shifts toward the right. Furthermore, the peak value of $A(\sigma^2)$ varies non-monotonously with changes in $\lambda$, decreasing to a minimum and then increasing. In figure 7(b), with the increase of $q$, the maximum of $A$ increases, the resonance peak becomes sharper and the position of the peak shifts to the right.
Figure 8 shows curves of the output amplitude $A$ versus the noise intensity $\sigma^2$ with different values of the parameters (including the fractional order $\alpha$ and the friction coefficient $\gamma$).

As shown in figure 8, each curve shows that $A$ attains a maximum value by increasing $\sigma^2$, indicating that SR appears. In figures 8(a) and (b), one-peak SR occurs, and in figures 8(c) and (d), one-peak and one-valley SR occurs. This is consistent with figure 6. Figure 8(a) shows that the maximum of $A$ decreases, the resonance peak becomes flat and the position of the peak shifts towards the left with increasing $\alpha$. Moreover, there exists a critical noise intensity $(\sigma^2)_c$. When $\sigma^2 < (\sigma^2)_c$, $A$ gradually decreases with increasing $\alpha$. When $\sigma^2 > (\sigma^2)_c$, $A$ gradually increases with increasing $\alpha$. Figure 8(b) shows that the maximum of $A$ decreases, the resonance peak becomes flat and the
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Figure 8. Stochastic resonance for the response function $A$ versus $\sigma^2$ at $R = 1, \omega = \Omega = 1$. Other parameter values: (a) $\gamma = 1, \lambda = 0.01, q = 0.1$; (b) $\lambda = 0.01, q = 0.4, \alpha = 0.6$; (c) $\gamma = 1, \lambda = 0.01, q = 0.1$; (d) $\lambda = 0.01, q = 0.1, \alpha = 0.3$.

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position of the peak shifts towards the right with increasing $\gamma$. Moreover, there are two critical noise intensities $(\sigma^2)_1$ and $(\sigma^2)_2$. When $(\sigma^2)_1 < \sigma^2 < (\sigma^2)_2$, $A$ gradually increases with increasing $\gamma$. When $\sigma^2 > (\sigma^2)_2$ and $\sigma^2 < (\sigma^2)_1$, $A$ gradually decreases with increasing $\gamma$. In figure 8(c), with increasing $\alpha$, the maximum of $A$ decreases, the resonance peak becomes flat and the position of the peak shifts to the left; however, the minimum of $A$ increases, and the position of the valley shifts towards the right. In figure 8(d), with the increase of $\gamma$, the maximum and the minimum of $A$ decrease, the positions of the peak and the valley shift to the right at the same time.

Figure 9 shows the phase diagrams in the $\alpha - \gamma$ plane for the emergence of stochastic resonance of $A$ versus $q$ under different values of $\lambda$ and $\sigma^2$ at $R = 1$, $\omega = \Omega = 1$. In the unshaded regions, the resonance phenomenon is impossible. The shaded regions in the figure correspond to those regions where SR and reverse-resonance of $A$ versus $q$ is possible. Two phases can be discerned in the resonance domain: (1) the light gray region (i), where only one valley appears; (2) the light gray region (ii), where only one resonance peak is observed. As shown in figure 9, one can deduce that the type of the resonance of $A$ versus $q$ is determined by $\lambda$, and the resonance region expands with increasing $\sigma^2$.

Figure 10 shows curves of the output amplitude $A$ versus the stationary probability $q$ with different values of $\gamma$.

In figure 10(a), each curve shows that $A$ attains a minimum value by increasing $\gamma$, indicating that the one-valley reverse-resonance appears. In other words, the stationary probability $q$ has an optimized inhibiting effect on the output amplitude $A$ in this situation. Moreover, with increasing $\gamma$, the valley value of $A(q)$ varies non-monotonously as $\gamma$ changes, increasing to a maximum and then decreasing, and the position of the valley

Figure 9. The phase diagrams for reverse-resonance and stochastic resonance versus $q$ in the $\alpha - \gamma$ plane at $R = 1, \omega = \Omega = 1$. Other parameter values: (a) $\lambda = 0.01, \sigma^2 = 0.1$; (b) $\lambda = 0.01, \sigma^2 = 0.6$; (c) $\lambda = 1, \sigma^2 = 0.1$; (d) $\lambda = 1, \sigma^2 = 0.6$. 
shifts to the left. However, in figure 10(b), each curve shows that $A$ attains a maximum value with increasing $q$, indicating that SR appears. Furthermore, with increasing $\gamma$, the maximum of $A$ decreases, the resonance peak becomes flat and the position of the peak shifts to the right. The resonance peak vanishes for $\gamma > \sim 1.2$ (see also figure 9).

4. Conclusions

In this study, we investigate the phenomenon of stochastic resonance in a fractional linear system subjected to two trichotomous noises and a fractional Gaussian noise
Trichotomous noise induced stochastic resonance in a fractional oscillator with random damping and random frequency and driven by a periodic signal, as random damping and random frequency affect the dynamics of the particles at the same time. We detect bona fide SR, SR and reverse-resonance in this linear system. Furthermore, we determine the regions in plane where resonance phenomena are possible. A major advantage of this investigated model is that the interplay of the multiplied trichotomous noise, the external periodic force, and memory effect in a fractional noisy oscillator can generate a rich variety of nonequilibrium cooperation phenomena, that is, the interplay can induce and diversify the SMR phenomena and give this linear system richer dynamic behavior. Specifically, the evolution of the output amplitude \( A \) with \( \Omega \) presents one-peak oscillation, double-peak oscillation, and triple-peak oscillation.

Finally, by properly adjusting the parameters of trichotomous noise mentioned above, we can effectively control the resonance behavior of this fractional linear system within a certain range. In addition, we expect that the model of a fractional oscillator with random damping and random frequency will find many applications in modern science.

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