The boundary state from open string fields

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Abstract

We construct a class of BRST-invariant closed string states for any classical solution of open string field theory. The closed string state is a nonlinear functional of the open string field and changes by a BRST-exact term under a gauge transformation of the solution. As a result, its contraction with an on-shell closed string state provides a gauge-invariant observable of open string field theory. Unlike previously known observables, however, the contraction with off-shell closed string states in the Fock space is well defined and regular. Moreover, we claim that the BRST-invariant closed string state coincides, up to a possible BRST-exact term, with the boundary state of the boundary conformal field theory which the solution is expected to describe. Our construction requires a choice of a propagator strip. If we choose the Schnabl propagator strip, the BRST-invariant state becomes explicitly calculable. We calculate it for various known analytic solutions of open string field theory and, remarkably, we find that it precisely coincides with the boundary state without any additional BRST-exact term. Our results imply, in particular, that the wildly oscillatory rolling tachyon solution of open string field theory actually describes the regular closed string physics studied by Sen using the boundary state.
1 Introduction

The current formulation of open string field theory \cite{1} requires a choice of a consistent open string background described by a boundary conformal field theory (BCFT)\footnote{See \cite{2,3,4,5} for reviews.}. Other boundary conformal field theories are expected to be described by classical solutions to the equation of motion of the open string field theory based on the original BCFT. There have been remarkable developments in open string field theory since Schnabl’s discovery of an analytic solution for tachyon condensation \cite{6}, and various analytic solutions have been constructed and studied \cite{7}–\cite{36}. It still remains difficult, however, to extract information on the BCFT represented by a solution of open string field theory.

A useful object that contains information on a BCFT is the boundary state $| B \rangle$. The one-point function of a closed string vertex operator $\phi_c$ inserted at the origin of a unit disk can be written using the boundary state $| B \rangle$ as

$$\langle (c_0 - \tilde{c}_0) \phi_c(0) \rangle_{\text{disk}} = \langle B | (c_0 - \tilde{c}_0) | \phi_c \rangle,$$

where $| \phi_c \rangle$ is the state corresponding to $\phi_c$ and the operator $c_0 - \tilde{c}_0$ is associated with a conformal Killing vector on the disk. A classical solution $\Psi$ of open string field theory is expected to describe a consistent open string background and thus a boundary conformal field theory, which we denote by BCFT*. If we can construct the boundary state $| B_* \rangle$ for BCFT* from the solution $\Psi$, we can extract all information contained in bulk one-point functions in the new background. Interesting progress in that direction was recently reported by Ellwood \cite{37}. It was argued that for on-shell closed string vertex operators $\mathcal{V}$, the one-point functions on the disk with BCFT* boundary conditions can be calculated from the gauge-invariant observables $W(\mathcal{V}, \Psi)$ introduced in \cite{38,39} as follows:

$$\langle B_* | (c_0 - \tilde{c}_0) | \mathcal{V} \rangle - \langle B | (c_0 - \tilde{c}_0) | \mathcal{V} \rangle = -4\pi i W(\mathcal{V}, \Psi).$$

\footnote{See \cite{2,3,4,5} for reviews.}
This remarkable observation means that the on-shell part of the information encoded in the BCFT boundary state $|B_\star\rangle$ can be extracted from the corresponding solution of open string field theory.

The restriction to on-shell closed string states arises because the operator $\mathcal{V}$ in $W(\mathcal{V}, \Psi)$ is inserted at a point with a conical singularity on a Riemann surface. Therefore $W(\mathcal{V}, \Psi)$ is not well defined when $\mathcal{V}$ is not a primary field of weight $(0,0)$. Unfortunately, there are few on-shell vertex operators with nonvanishing one-point functions on a disk. On the other hand, the boundary state is well defined when it is contracted with an arbitrary off-shell closed string state and contains more information on the BCFT. If we can relax the on-shell restriction on the closed string state in $W(\mathcal{V}, \Psi)$, we will be able to extract much more information on the BCFT from the solution $\Psi$. This is our motivation.

In this paper we construct, for any open string field theory solution $\Psi$, a class of closed string states $|B_\star(\Psi)\rangle$ of ghost number three. Their contraction with arbitrary off-shell closed string states is regular. The states $|B_\star(\Psi)\rangle$ are BRST invariant, namely,

$$Q|B_\star(\Psi)\rangle = 0. \quad (1.3)$$

Under a gauge transformation $\delta_\chi \Psi$ of the solution $\Psi$, the states $|B_\star(\Psi)\rangle$ change at most by a BRST-exact term:

$$|B_\star(\Psi + \delta_\chi \Psi)\rangle = |B_\star(\Psi)\rangle + (Q - \text{exact}). \quad (1.4)$$

Therefore, a gauge-invariant observable can be constructed from $|B_\star(\Psi)\rangle$ by its contraction with an on-shell closed string state $\mathcal{V}$:

$$\langle \mathcal{V}|(c_0 - \tilde{c}_0)|B_\star(\Psi + \delta_\chi \Psi)\rangle = \langle \mathcal{V}|(c_0 - \tilde{c}_0)|B_\star(\Psi)\rangle. \quad (1.5)$$

The novelty in the construction of these observables is that they admit a perfectly regular off-shell extension and, as we will show, the state $|B_\star(\Psi)\rangle$ is explicitly calculable in certain cases. We claim that the state $|B_\star(\Psi)\rangle$ coincides with the boundary state $|B_\star\rangle$ up to a possible BRST-exact term. In fact, they precisely coincide in all calculable examples that we examined.

Our construction of $|B_\star(\Psi)\rangle$ was inspired by Ellwood’s paper [37] and by recent developments in the calculation of Feynman diagrams in Schnabl gauge [40, 41] and in a class of gauges called linear $b$-gauges [42]. In the construction of $|B_\star(\Psi)\rangle$, we first choose a propagator strip associated with a linear $b$-gauge. The shape of the strip is determined by the operator $\mathcal{B}$ used in the gauge-fixing condition on the open string field of ghost number one. The length of the strip is determined by a choice of a Schwinger parameter $s > 0$. Then the chosen propagator strip can be represented as the surface generated by the operator $e^{-s \mathcal{L}}$, where $\mathcal{L}$, defined by $\mathcal{L} = \{Q, \mathcal{B}\}$, is the BRST transformation of $\mathcal{B}$.

\footnote{For progress related to other gauge choices, see [43, 44, 45].}
Figure 1: (a) An annulus constructed from a half-propagator strip. The boundary conditions of the original BCFT are imposed on the outer boundary. The grey line in the figure represents the identified half-string edges of the half-propagator strip. The path integral over this annulus defines a closed string state at the inner boundary depicted as a dashed line in the figure. (b) An annulus with four slits. The classical solution \( \Psi \) is glued to each slit. The path integral over this annulus after gluing classical solutions defines a closed string state at the inner boundary.

The main ingredient for the construction of \( |B_s(\Psi)\rangle \) is a *half-propagator strip*. We cut the chosen propagator strip in half along the line traced by the open string midpoint. We then take one of the resulting half-propagator strips and form an annulus by identifying its initial and final half-string edges. Imposing the original BCFT boundary conditions at the open string boundary of this annulus, the path integral over the annulus defines a closed string state at the other boundary where we originally cut the propagator strip. See Figure 1(a). It is clear that this closed string state, after an appropriate exponential action of \( L_0 + \bar{L}_0 \), reproduces the boundary state \( |B\rangle \) of the original BCFT. We can thus construct the boundary state \( |B\rangle \) for any choices of \( B \) and \( s \). It should be pointed out, however, that the propagator for non-BPZ-even gauges \( (B^* \neq B) \) is a complicated object, while our construction is based on \( e^{-sL} \) which in these cases is not the full propagator surface.

Let us now repeat the construction with the above half-propagator strip replaced by the one for the background associated with \( \Psi \). The modified half-propagator strip can be constructed by gluing the solution \( \Psi \) to slits which are inserted at various positions along the annulus. See Figure 1(b). The shape of the slits is correlated with the shape of the half-propagator strip before the identification of the half-string edges and is determined by the operator \( B \). The slits are accompanied by appropriate \( b \)-ghost line integrals, and the positions of the slits are integrated over. A closed string state is again defined by the path integral over this annulus. After an appropriate exponential action of \( L_0 + \bar{L}_0 \) and summing over the number of solution
insertions, this defines the state $|B_*(\Psi)\rangle$.

The resulting state $|B_*(\Psi)\rangle$ depends on $B$ and $s$, but the gauge-invariant observables $\langle V | (c_0 - \bar{c}_0) | B_*(\Psi) \rangle$ are independent of $B$ and $s$. Indeed, we can show that as we vary $B$ and $s$, the closed string state $|B_*(\Psi)\rangle$ changes at most by a possible BRST-exact term. While it is difficult to calculate the state $|B_*(\Psi)\rangle$ for generic choices of $B$, it is explicitly calculable for solutions based on the familiar wedge surfaces [16, 17] if we choose Schnabl’s propagator strip. In fact, in this case the methods developed in [41] to map Riemann surfaces for one-loop amplitudes in Schnabl gauge to an annulus can be used to construct the Riemann surfaces which define $|B_*(\Psi)\rangle$. We explicitly calculate $|B_*(\Psi)\rangle$ based on the Schnabl propagator strip of arbitrary length $s$ for various known solutions of string field theory such as Schnabl’s tachyon vacuum solution and the solutions for marginal deformations with regular operator products constructed in [16, 17] and in [25]. We find that $|B_*(\Psi)\rangle$ vanishes identically for the tachyon vacuum solution, which is consistent with Sen’s conjecture that the D-brane disappears at the tachyon vacuum. For the marginal deformations, $|B_*(\Psi)\rangle$ precisely reproduces the BCFT boundary state $|B_s\rangle$. Both results hold independent of the length $s$ of the propagator strip used in the construction. At least for these examples the exact BCFT boundary state can be obtained from the corresponding open string field solution!

Our results imply, in particular, that the boundary state $|B_*(\Psi)\rangle$ calculated from the known rolling tachyon solutions of open string field theory coincides with the BCFT boundary state discussed in [18, 49, 50, 51, 52, 53]. This boundary state describes a regular behavior of D-brane decay in the far future. For example, the pressure decreases monotonically and vanishes in the far future. The rolling tachyon solution $\Psi$, on the other hand, exhibits ever-growing oscillations for the component fields of the open string [16, 17]. It has been a long-standing puzzle whether such wildly oscillatory solutions describe a regular time-dependent process in the far future [54, 55, 56]. Our explicit construction of the boundary state from the rolling tachyon solution confirms that the solution represents the expected regular physics. Our interpretation is that the wild oscillatory behavior is due to the description of the regular physics in the closed string channel in terms of the open string degrees of freedom.

The paper is organized as follows. In section 2 we introduce the half-propagator strips and explain the construction of closed string states using half-propagator strips. In section 3 we define the closed string state $|B_*(\Psi)\rangle$. We show its BRST invariance and prove that it changes at most by a BRST-exact term under a gauge transformation of $\Psi$. We show in section 4 that as we vary $B$ and $s$, the state $|B_*(\Psi)\rangle$ changes at most by a BRST-exact term.

In section 5 we discuss the relation of our work with an earlier approach to boundary states based on open-closed string field theory [57, 58, 59]. Indeed, a set of open-closed vertices can be used to construct an alternative BRST-invariant closed string state $|B_{oc}^s(\Psi)\rangle$ associated with a solution $\Psi$. This state, just like $|B_*(\Psi)\rangle$, changes at most by a BRST-exact term under an
open string gauge transformation. We show that for choices of $\mathcal{B}$ that are invariant under BPZ conjugation, $|B_s(\Psi)\rangle$ encodes a set of consistent open-closed vertices. Curiously, for general choices of $\mathcal{B}$, the open-closed vertices encoded by $|B_s(\Psi)\rangle$ do not satisfy the reality condition. Even so, the state $|B_s(\Psi)\rangle$ can be real for some classical solutions, and we indeed find that it is the case for all the explicit examples we discuss in section 7. If we assume the background independence of a certain version of open-closed string field theory, we can argue that $|B_s(\Psi)\rangle$ and $|B_s\rangle$ coincide up to a BRST-exact term.

In sections 6 and 7 we demonstrate that the state $|B_s(\Psi)\rangle$ is calculable for solutions based on wedge states if we choose Schnabl’s propagator strip. We then explicitly calculate $|B_s(\Psi)\rangle$ for various known solutions and find that it coincides with the BCFT boundary state for arbitrary $s$. In these cases, the state $|B_s(\Psi)\rangle$ factorizes into matter and ghost sectors, and the ghost sector coincides with the boundary state of the $bc$ CFT. It is important to understand when this factorization holds because the state $|B_s(\Psi)\rangle$ factorized in this way can be a consistent BCFT boundary state without any BRST-exact term. In section 7.2 we discuss this factorization and show that for solutions based on wedge states with a certain class of ghost insertions and arbitrary matter insertions, the state $|B_s(\Psi)\rangle$ constructed from the Schnabl propagator always factorizes in this way. In section 8 we end with concluding remarks.

## 2 Half-propagator strips and closed string states

We begin this section by reviewing the construction of propagator strips in linear $b$-gauges [42]. We then introduce half-propagator strips by cutting the full strips along the line traced by the open string midpoint. We further introduce various ingredients to be used in section 3 for the construction of $|B_s(\Psi)\rangle$, such as the star multiplication of half-propagator strips and operator insertions on the strips. Finally, we construct closed string states from half-propagator strips by identifying the half-string edges. The coordinate curve of the closed string is the curve traced by the open string midpoint.

### 2.1 Half-propagator strips for regular linear $b$-gauges

A large class of gauge choices for string perturbation theory was discussed in [42]. These so-called linear $b$-gauges impose a gauge condition

$$\mathcal{B} |\psi_{cl}\rangle = 0$$

(2.1)
on the classical open string field $|\psi_{cl}\rangle$ of ghost number one,^{3} where the operator $B$ is a linear combination of even-moded $b$-ghost oscillators:

$$B = \sum_{j \in \mathbb{Z}} v_{2j} b_{2j} = \oint d\xi v(\xi) b(\xi) \quad \text{with} \quad v(\xi) = \sum_{j \in \mathbb{Z}} v_{2j} \xi^{2j+1}, \quad v_{2j} \in \mathbb{R}.$$  

(2.2)

If the associated vector field $v(\xi)$ is analytic in a neighborhood of the unit circle $|\xi| = 1$ and furthermore satisfies the condition

$$\Re (\bar{\xi} v(\xi)) > 0 \quad \text{for} \quad |\xi| = 1,$$

(2.3)

the gauge choice is called regular. It was shown in \[42\] that regular linear $b$-gauges correctly reproduce open string on-shell amplitudes and that pure-gauge external states decouple.

The propagator of a regular linear $b$-gauge is characterized by the strip surface generated by $e^{-sL}$, where

$$L = \{Q, B\}.$$  

(2.4)

In a certain conformal frame $w$, this strip surface is generated by horizontal translations. See Figure 2(a). The $w$ frame is obtained from the vector field $v(\xi)$ through

$$\frac{dw(\xi)}{d\xi} = \frac{1}{v(\xi)}, \quad w(1) = 0.$$  

(2.5)

Normalizing $v(\xi)$ appropriately, we can impose the additional condition^{4}

$$w(-1) = i\pi.$$  

(2.6)

The horizontal boundaries of the strip are then located at $\Im(w) = 0$ and $\Im(w) = \pi$. The left boundary is the parameterized curve $\gamma(\theta) = w(e^{i\theta})$ for $0 \leq \theta \leq \pi$. It follows from (2.5) and (2.6) that

$$\gamma(0) = 0, \quad \Im(\gamma(\frac{\pi}{2})) = \frac{\pi}{2}, \quad \gamma(\pi) = i\pi,$$

(2.7)

where the middle equation holds because of (2.2).

When $e^{-sL}$ acts on an open string state $A$, the parameterization on $\gamma$ is used to glue the strip associated with $e^{-sL}$ to the coordinate curve of $A$. In the $w$ frame, the right boundary of the strip $e^{-sL}$ is a horizontal translation by $s$ of its left boundary. It is therefore parameterized by the curve $s + \gamma(\theta)$ with $0 \leq \theta \leq \pi$. This fixes the horizontal position of the strip surface $e^{-sL}$ in the $w$ frame. We will now consider the surfaces which arise when we cut the strip $e^{-sL}$ along the line $\Im(w) = \frac{\pi}{2}$. This line is generated by horizontal translations of $\gamma(\frac{\pi}{2})$ and

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^{3} String fields of different ghost numbers are introduced in the process of gauge fixing. See \[42\] for detailed discussions about gauge conditions on such quantum string fields.

^{4} This definition of the $w$ frame differs from the conventions of \[11\]. They are related by $w_{\text{here}} = -w_{\text{there}} + i\pi$. 

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Figure 2: (a) Illustration of the surface associated with $e^{-sL}$. It is generated by horizontal translations in the $w$ frame. The half-propagator strip $\mathcal{P}(0, s)$ is obtained by cutting the surface $e^{-sL}$ along the line $\Im(w) = \frac{\pi}{2}$. (b) The surface $\mathcal{P}(s_a, s_b)$ is a horizontal translation of the surface $\mathcal{P}(0, s_b - s_a)$ by $s_a$.

is thus associated with the open string midpoint. The resulting surfaces from this particular cut are of interest for a number of reasons. In the annulus amplitude, we cut the propagator surface along a closed string curve to read off the boundary state along the boundary generated by the cut. Choosing the open string midpoint for this cut is natural because of the special role of the midpoint in open string field theory. Furthermore, if one chooses this cut for the strip $e^{-sL}$ in Schnabl gauge, the resulting surfaces are the so-called slanted wedges introduced in [41]. The remarkable algebraic properties of these slanted wedges under gluing allowed the explicit map of one-loop Riemann surfaces to an annulus frame, which is expected to facilitate the explicit calculation of off-shell amplitudes. Our analysis will experience a similarly drastic simplification in the Schnabl gauge limit.

The cutting of the strip $e^{-sL}$ along the line $\Im(w) = \frac{\pi}{2}$ yields two surfaces. We will denote the bottom one, located in the region $0 \leq \Im(w) \leq \frac{\pi}{2}$, by $\mathcal{P}(0, s)$. See Figure 2(a). The arguments 0 and $s$ remind us that the open string boundary of $\mathcal{P}(0, s)$ is located on the real axis between $w = 0$ and $w = s$. More generally, we use the notation $\mathcal{P}(s_a, s_b)$ with $s_b \geq s_a$ for the surface $\mathcal{P}(0, s_b - s_a)$ shifted horizontally by $s_a$ in the $w$ frame. See Figure 2(b). The left and right boundaries of $\mathcal{P}(s_a, s_b)$ are parameterized by $s_a + \gamma(\theta)$ and $s_b + \gamma(\theta)$, respectively, where the range of $\theta$ is now restricted to $0 \leq \theta \leq \frac{\pi}{2}$. Finally, $\mathcal{P}(s_a, s_b)$ has a boundary induced by the cut. This boundary is neither an open string boundary nor the coordinate line of an open string state. For reasons that will become apparent later, we will refer to this boundary as the closed string boundary.$^5$

$^5$For the particular case of Schnabl gauge, this boundary is the so-called hidden boundary introduced in [41].
Naively, the surface $\mathcal{P}(0, s)$ is generated by the operator $e^{-sL_R}$, where $L_R$ is the right half of $L$. This notation, however, is misleading. It suggests, incorrectly, that the surface $\mathcal{P}(0, s)$ with $L_R$ inserted at the left edge is the same as $\mathcal{P}(0, s)$ with $L_R$ inserted at the right edge because $[L_R, e^{-sL_R}] = 0$. This is not the case because the line integral $L_R$ has an endpoint on the closed string boundary and this endpoint cannot be moved by contour deformation. Let us denote by $L_R(t)$ the line integral $L_R$ along the contour $t + \gamma(\theta)$ with $0 \leq \theta \leq \frac{\pi}{2}$. As $L_R$ generates translations in the $w$ frame, we have

$$L_R(t) \equiv \int_{t}^{\gamma(\frac{\pi}{2}) + t} \left[ \frac{dw}{2\pi i} T(w) + \frac{d\bar{w}}{2\pi i} \tilde{T}(\bar{w}) \right].$$

(2.8)

The surface $\mathcal{P}(s_a, s_b)$ can then be properly expressed as the path-ordered exponential:

$$\mathcal{P}(s_a, s_b) = P\exp \left[ - \int_{s_a}^{s_b} dt L_R(t) \right].$$

(2.9)

Our convention for the path-ordering is $L_R(t_1) L_R(t_2)$ for $t_1 < t_2$. It is now clear that

$$L_R(s_a) \mathcal{P}(s_a, s_b) - \mathcal{P}(s_a, s_b) L_R(s_b) \neq 0$$

(2.10)

because the left-hand side represents a surface with two disconnected contour integrals. It is therefore natural to introduce an operator that supplements the remaining line integral on $\mathcal{P}(s_a, s_b)$ along the closed string boundary. We thus define

$$\tilde{L} = \int_{\gamma(\frac{\pi}{2}) + s_a}^{\gamma(\frac{\pi}{2}) + s_b} \left[ \frac{dw}{2\pi i} T(w) + \frac{d\bar{w}}{2\pi i} \tilde{T}(\bar{w}) \right],$$

(2.11)

for $\tilde{L}$ acting on $\mathcal{P}(s_a, s_b)$. We then have the identity

$$L_R(s_a) \mathcal{P}(s_a, s_b) - \mathcal{P}(s_a, s_b) L_R(s_b) + \tilde{L} \mathcal{P}(s_a, s_b) = 0,$$

(2.12)

which follows from first connecting the three line integrals in (2.12) and then shrinking the resulting integral contour to zero size. Furthermore, we have

$$\partial_{s_b} \mathcal{P}(s_a, s_b) = -\mathcal{P}(s_a, s_b) L_R(s_b), \quad \partial_{s_a} \mathcal{P}(s_a, s_b) = L_R(s_a) \mathcal{P}(s_a, s_b),$$

(2.13)

which follow from the definition (2.9).

Following the definitions (2.8) and (2.11) of the line integrals of the energy-momentum tensor, we define the corresponding $b$-ghost line integrals as follows:

$$\mathcal{B}_{R}(t) = \int_{t}^{\gamma(\frac{\pi}{2}) + t} \left[ \frac{dw}{2\pi i} b(w) + \frac{d\bar{w}}{2\pi i} \tilde{b}(\bar{w}) \right],$$

$$\mathcal{B} = \int_{\gamma(\frac{\pi}{2}) + s_a}^{\gamma(\frac{\pi}{2}) + s_b} \left[ \frac{dw}{2\pi i} b(w) + \frac{d\bar{w}}{2\pi i} \tilde{b}(\bar{w}) \right].$$

(2.14)
for $\tilde{B}$ acting on $\mathcal{P}(s_a, s_b)$. We also define the corresponding line integrals of the BRST current

$$Q_R(t) = \int_t^{\gamma(\pi)} \left[ \frac{dw}{2\pi i} j_B(w) - \frac{d\bar{w}}{2\pi i} \bar{j}_B(\bar{w}) \right],$$

for $\tilde{Q}$ acting on $\mathcal{P}(s_a, s_b)$. Just as in (2.12), we connect line integrals of the BRST current and the $b$ ghost to find

$$\mathcal{B}_R(s_a) \mathcal{P}(s_a, s_b) - \mathcal{P}(s_a, s_b) \mathcal{B}_R(s_b) + \tilde{B} \mathcal{P}(s_a, s_b) = 0,$$

$$Q_R(s_a) \mathcal{P}(s_a, s_b) - \mathcal{P}(s_a, s_b) Q_R(s_b) + \tilde{Q} \mathcal{P}(s_a, s_b) = 0.$$  

(2.16)

### 2.2 Star multiplication of half-propagator strips

The surface $\mathcal{P}(s_a, s_b)$ is equipped with two parameterized boundaries which can be glued to open string states $A$ or to other surfaces $\mathcal{P}(s'_a, s'_b)$. For the latter, we require that the glued boundaries in the $w$ frame match. As for regular star multiplication of open string states, we use the symbol $*$ to denote this gluing. For the special case of Schnabl gauge, this type of gluing operation was discussed extensively in [11]. From the definition of $\mathcal{P}(s_a, s_b)$ it immediately follows that

$$\mathcal{P}(s_a, s_b) * \mathcal{P}(s_b, s_c) = \mathcal{P}(s_a, s_c).$$

(2.17)

Open string states $A$ do not carry a closed string boundary. Therefore the gluing operation

$$\mathcal{P}(s_a, s_b) * A * \mathcal{P}(s_b, s_c)$$

(2.18)

is well defined and yields a surface with one connected closed string boundary between $\gamma(\pi) + s_a$ and $\gamma(\pi) + s_c$. It can be thought of as the surface $\mathcal{P}(s_a, s_c)$ with a *slit* along the curve $s_b + \gamma(\theta)$, where the open string state $A$ is to be inserted. See Figure 3. It follows from (2.13) that a change in $s_b$ on the surface (2.13) is generated by

$$\partial_{s_b} \left[ \mathcal{P}(s_a, s_b) * A * \mathcal{P}(s_b, s_c) \right]$$

$$= - \mathcal{P}(s_a, s_b) \mathcal{L}_R(s_b) * A * \mathcal{P}(s_b, s_c) + \mathcal{P}(s_a, s_b) * A * \mathcal{L}_R(s_b) \mathcal{P}(s_b, s_c)$$

$$\equiv - \mathcal{P}(s_a, s_b) * \left[ \mathcal{L}_R(s_b), A \right] * \mathcal{P}(s_b, s_c).$$

(2.19)

Note the location of the star symbols in the second line, which is implicit in the commutator defined in the last line. This definition applies to any commutator or anticommutator of a line integral up to the closed string boundary with an open string state. For example,

$$\mathcal{P}(s_a, s_b) * \{ \mathcal{B}_R(s_b), A \} * \mathcal{P}(s_b, s_c)$$

$$\equiv \mathcal{P}(s_a, s_b) \mathcal{B}_R(s_b) * A * \mathcal{P}(s_b, s_c) + \mathcal{P}(s_a, s_b) * A * \mathcal{B}_R(s_b) \mathcal{P}(s_b, s_c).$$

(2.20)

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*We will suppress explicit * symbols in later sections.*
Figure 3: Illustration of the surface $\Sigma(s_a, s_b)$ for $k = 2$. It is obtained from the surface $P(s_a, s_b)$ by inserting $k$ open string states along the parameterized slits $s_i + \gamma(\theta)$.

In the case of $Q_R(t)$, we have

$$P(s_a, s_b) \ast \{ Q_R(s_b), A \} \ast P(s_b, s_c) \equiv P(s_a, s_b) Q_R(s_b) \ast A \ast P(s_b, s_c) + P(s_a, s_b) \ast A \ast Q_R(s_b) P(s_b, s_c)$$

(2.21)

for any Grassmann-odd state $A$ because the BRST current is a primary field of weight one so that its integral in the $w$ frame can be deformed and easily mapped to the frame for $A$. Similar relations do not hold for $L_R(t)$ and $B_R(t)$ because the energy-momentum tensor and the $b$ ghost are not primary fields of weight one:

$$P(s_a, s_b) \ast [L_R(s_b), A] \ast P(s_b, s_c) \neq -P(s_a, s_b) \ast (LA) \ast P(s_b, s_c),$$

$$P(s_a, s_b) \ast \{ B_R(s_b), A \} \ast P(s_b, s_c) \neq -P(s_a, s_b) \ast (BA) \ast P(s_b, s_c).$$

(2.22)

More generally, we will consider surfaces $\Sigma(s_a, s_b)$ resulting from multiple insertions of open string states $A_1, A_2, \ldots, A_k$ into $P(s_a, s_b)$ in the following form:

$$\Sigma(s_a, s_b) = P(s_a, s_1) \ast A_1 \ast P(s_{i-1}, s_i) \ast A_i \ast P(s_i, s_{i+1}) \ast A_k \ast P(s_k, s_b)$$

(2.23)

with $s_a \leq s_1$, $s_i \leq s_{i+1}$, and $s_k \leq s_b$. The surface $\Sigma(s_a, s_b)$ is illustrated in Figure 3 for $k = 2$. The surface $\Sigma(s_a, s_b)$ is $P(s_a, s_b)$ with $k$ parameterized slits along the curves $s_i + \gamma(\theta)$ where the states $A_i$ are to be glued. We denote the Grassmann property of $\Sigma$ by $(-)^\Sigma$:

$$(-)^\Sigma = \prod_{i=1}^{k} (-)^{A_i}.$$  

(2.24)
The operators \( \hat{L} \), \( \hat{B} \), and \( \hat{Q} \) are derivations when acting on products of the form (2.23). For example, we have

\[
\hat{L} [ P(s_a, s_b) * A * P(s_b, s_c)] \\
= [ \hat{L} P(s_a, s_b)] * A * P(s_b, s_c) + P(s_a, s_b) * [ \hat{L} A] * P(s_b, s_c) + P(s_a, s_b) * A * [ \hat{L} P(s_b, s_c)] \\
= [ \hat{L} P(s_a, s_b)] * A * P(s_b, s_c) + P(s_a, s_b) * A * [ \hat{L} P(s_b, s_c)].
\]

(2.25)

Here we used the fact that an open string state \( A \) does not have a closed string boundary and it is therefore annihilated by \( \hat{L} \), \( \hat{B} \), and \( \hat{Q} \):

\[
\hat{L} A = 0, \quad \hat{B} A = 0, \quad \hat{Q} A = 0. \tag{2.26}
\]

We define the BRST operator \( Q \) acting on a surface \( \Sigma(s_a, s_b) \) of the form (2.23) by

\[
Q \Sigma(s_a, s_b) \equiv (-)^{\Sigma} \Sigma(s_a, s_b) Q_R(s_b) - \hat{Q} \Sigma(s_a, s_b) - Q_R(s_a) \Sigma(s_a, s_b). \tag{2.27}
\]

Note that the three integral contours can be connected. We have

\[
\{Q, B_R(t)\} = L_R(t), \quad \{Q, \hat{B}\} = \hat{L}.
\]

(2.28)

From (2.16) we know that \( P(s_a, s_b) \) is annihilated by this operator:

\[
Q P(s_a, s_b) = 0. \tag{2.29}
\]

On an open string state \( A \), inserted along a slit \( s_a + \gamma(\theta) \) in the \( w \) frame, the definition of \( Q \) in (2.27) reduces to the usual BRST transformation \( Q \):

\[
Q A = (-)^{A} A Q_R(s_a) - \hat{Q} A - Q_R(s_a) A = Q A. \tag{2.30}
\]

Here we have used \( \hat{Q} A = 0 \). Combining (2.29), (2.30), and the fact that the BRST current is a primary field of weight one, we find that the BRST transformation of a product \( \Sigma(s_a, s_b) \) of the form (2.23) reduces to BRST transformations of the Fock-space states \( A_i \). We have

\[
Q \Sigma(s_a, s_b) = \sum_{i=1}^{k} (-)^{\Sigma_{i-1}} A_j P(s_a, s_1) * A_1 \ldots P(s_{i-1}, s_i) * (QA_i) * P(s_i, s_{i+1}) \ldots A_k * P(s_k, s_b). \tag{2.31}
\]

Similarly, the properties (2.12), (2.25) and (2.26) can be used to show

\[
\Lambda_R(s_a) \Sigma(s_a, s_b) - \Sigma(s_a, s_b) \Lambda_R(s_b) + \hat{L} \Sigma(s_a, s_b) \\
= \sum_{i=1}^{k} P(s_a, s_1) * A_1 \ldots P(s_{i-1}, s_i) * [\Lambda_R(s_i), A_i] * P(s_i, s_{i+1}) \ldots A_k * P(s_k, s_b) \\
= - \sum_{i=1}^{k} \partial_i \Sigma(s_a, s_b), \tag{2.32}
\]

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where we used (2.19) in the last step. The corresponding identity for $b$-ghost line integrals is
\[
\mathcal{B}_R(s_a) \Sigma(s_a, s_b) - (-)^\Sigma \Sigma(s_a, s_b) \mathcal{B}_R(s_b) + \mathcal{B} \Sigma(s_a, s_b) \\
= \sum_{i=1}^k (-)^{\Sigma_{j=1}^{i-1} A_j} \mathcal{P}(s_a, s_1) * A_1 \ldots (\mathcal{B}_R(s_i) A_i - (-)^{A_i} A_i \mathcal{B}_R(s_i)) \ldots A_k * \mathcal{P}(s_k, s_b).
\]

(2.33)

2.3 Closed string states from half-propagator strips

A surface $\Sigma(s_a, s_b)$ of the form (2.23) can be used to construct a closed string surface state. To do this, we first introduce the identification $w \sim w + (s_b - s_a)$ in the $w$ frame. This identifies the left boundary $s_a + \gamma(\theta)$ with the right boundary $s_b + \gamma(\theta)$ of $\Sigma(s_a, s_b)$. We are left with the closed string boundary at $\Im(w) = \pi 2$, whose name we are now doing justice by gluing it to the coordinate line $0 \leq \sigma \leq 2\pi$ of a closed string coordinate patch. The map from $\sigma$ to the closed string boundary $\Im(w) = \pi 2$ of $\Sigma(s_a, s_b)$ is given by
\[
\sigma \rightarrow w = i \frac{\pi}{2} + (s_b - s_a) \frac{\sigma}{2\pi}.
\]

(2.34)

This map is consistent with the identifications $w \sim w + (s_b - s_a)$ and $\sigma \sim \sigma + 2\pi$. The resulting surface is a closed string surface state with its coordinate line at $\Im(w) = \pi 2$ parameterized by (2.34). We denote this closed string surface state by
\[
\oint_{s_b - s_a} \Sigma(s_a, s_b).
\]

(2.35)

We have represented the operation that turns the surface $\Sigma(s_b, s_a)$ into a closed string surface state by the symbol $\oint_{s_b - s_a}$. This notation is somewhat reminiscent of the notation $\int A$, often used in open string field theory, which glues the left and right parts of the open string state $A$. The subscript $s_b - s_a$ is a reminder that the width $s_b - s_a$ of the strip $\Sigma(s_a, s_b)$ explicitly enters the gluing prescription (2.34).

A natural representation of $\oint_{s_b - s_a} \Sigma(s_a, s_b)$ can be obtained in the $\zeta$ frame defined by
\[
\zeta = \exp \left( \frac{2\pi i}{s_b - s_a} w \right).
\]

(2.36)

This maps the surface $\Sigma$ to an annulus with the open string boundary placed at $|\zeta| = 1$ and the closed string coordinate line located at $|\zeta| = e^{-\frac{\pi^2}{s_b - s_a}}$. The surface state $\oint_{s_b - s_a} \Sigma(s_a, s_b)$ is then defined through inner products with arbitrary closed string states $|\phi_c\rangle$ in the Fock space by
\[
\langle \phi_c, \oint_{s_b - s_a} \Sigma(s_a, s_b) \rangle = \langle d_{s_b - s_a} \circ \phi_c(0) [\ldots] \rangle_{\text{disk}},
\]

(2.37)
where the operator \( \phi_c(0) \) corresponding to \(|\phi_c\rangle\) is mapped from its canonical coordinate patch \(|\xi| \leq 1\) to the shrunk coordinate patch \(|\zeta| \leq e^{-\frac{z^2}{s-1}}\) by the retraction

\[
d_{s-b-a}(\xi) = e^{-\frac{z^2}{s-1}} \zeta.
\]  

(2.38)

The dots [\ldots] in (2.37) represent the slits where the open string states \(A_i\) are inserted. They are also mapped from the \(w\) frame to the \(\zeta\) frame via (2.36). For the case of \(k = 4\) slits, the \(\zeta\)-frame representation of the closed string surface state \(\oint_{s-b-a} \Sigma(a, b)\) was illustrated in Figure 1(b).

The identification \(w \sim w + (s_b - s_a)\) allows us to move line integrals cyclically in \(\oint_{s-b-a}\). We have

\[
\oint_{s-b-a} L_R(s_a) \Sigma(s_a, b) = \oint_{s-b-a} \Sigma(s_a, b) L_R(s_b),
\]

\[
\oint_{s-b-a} B_R(s_a) \Sigma(s_a, b) = (-)^s \oint_{s-b-a} \Sigma(s_a, b) B_R(s_b),
\]

\[
\oint_{s-b-a} Q_R(s_a) \Sigma(s_a, b) = (-)^s \oint_{s-b-a} \Sigma(s_a, b) Q_R(s_b).
\]

(2.39)

Let us examine how operators acting on the closed string state \(\oint_{s-b-a} \Sigma(a, b)\) translate into line integrals on \(\Sigma(a, b)\). The BRST operator is invariant under conformal transformations, and we find

\[
Q \oint_{s-b-a} \Sigma(s_a, b) = -\oint_{s-b-a} \bar{Q} \Sigma(s_a, b)
\]

\[
= -\oint_{s-b-a} \left( \bar{Q} \Sigma(s_a, b) + Q_R(s_a) \Sigma(s_a, b) - (-)^s \Sigma(s_a, b) Q_R(s_b) \right)
\]

\[
= \oint_{s-b-a} Q \Sigma(s_a, b),
\]

(2.40)

where we used (2.39) in the second step and (2.27) in the last step. Let us now consider the action of \(L_0 - \bar{L}_0\) on \(\oint_{s-b-a} \Sigma(a, b)\). As \(L_0 - \bar{L}_0\) generates rotations in the \(\zeta\) frame, we expect it to generate horizontal translations in the \(w\) frame. As a first step we note that

\[
L_0 - \bar{L}_0 = \int_{|\zeta| = \exp(-\frac{z^2}{s-1})} \left[ \frac{d\zeta}{2\pi i} \zeta T(\zeta) - \frac{d\bar{\zeta}}{2\pi i} \bar{T}(\bar{\zeta}) \right]
\]

\[
= \frac{s_b - s_a}{2\pi i} \int_{\gamma(\zeta) + s_a} \left[ \frac{dw}{2\pi i} T(w) + \frac{d\bar{w}}{2\pi i} \bar{T}(\bar{w}) \right],
\]

(2.41)

and therefore

\[
(L_0 - \bar{L}_0) \oint_{s-b-a} \Sigma(s_a, b) = \frac{s_b - s_a}{2\pi i} \oint_{s-b-a} \bar{L} \Sigma(s_a, b).
\]

(2.42)
For $\Sigma(s_a, s_b)$ of the form (2.23) we can use (2.39) and (2.32) to find
\[
(L_0 - \tilde{L}_0) \oint_{s_b - s_a} \Sigma(s_a, s_b) = \frac{s_b - s_a}{2\pi i} \oint_{s_b - s_a} \left( [\mathcal{L}_R, \Sigma(s_a, s_b)] + \tilde{\mathcal{L}} \Sigma(s_a, s_b) \right) 
\]
\[
= -\frac{s_b - s_a}{2\pi i} \oint_{s_b - s_a} \left[ \sum_{i=1}^k \partial s_i \Sigma(s_a, s_b) \right].
\]
(2.43)

This result is consistent with the intuition that the generator of rotations in the $\zeta$ frame, $L_0 - \tilde{L}_0$, should generate horizontal translations on the positions $s_i$ of the slits where the open string states $A_i$ are glued. The translation in the $w$ frame is proportional to the total length of the strip $s_b - s_a$, as expected. The corresponding identity for $b_0 - \tilde{b}_0$ reads
\[
(b_0 - \tilde{b}_0) \oint_{s_b - s_a} \Sigma(s_a, s_b) = \frac{s_b - s_a}{2\pi i} \oint_{s_b - s_a} \tilde{B} \Sigma(s_a, s_b)
\]
\[
= \frac{s_b - s_a}{2\pi i} \oint_{s_b - s_a} \sum_{i=1}^k (-)^{i-1} A_i \mathcal{P}(s_a, s_1) * A_1 \ldots
\]
\[
\quad \times (B_R(s_i) A_i - (-)^{A_i} A_i B_R(s_i)) \ldots A_k * \mathcal{P}(s_k, s_b).
\]
(2.44)

We can move an open string state $A$ cyclically within $\oint_{s_b - s_a}$ just as we did for line integrals in (2.39). We have
\[
\oint_{s_b - s_a} A * \Sigma(s_a, s_b) = (-)^{A \Sigma} \oint_{s_b - s_a} \Sigma(s_a, s_b) * A.
\]
(2.45)

Similarly, we can cyclically move half-propagator strips in $\oint_{s_b - s_a}$, but all surfaces must attach to the same segment of the closed string boundary after using the cyclicity. We conclude that
\[
\oint_{s_b - s_a} \mathcal{P}(s_a, s_1) * A_1 * \mathcal{P}(s_1, s_2) \ldots A_k * \mathcal{P}(s_k, s_b)
\]
\[
= \oint_{s_b - s_a} A_1 * \mathcal{P}(s_1, s_2) \ldots A_k * \mathcal{P}(s_k, s_b) * \mathcal{P}(s_b, s_1 + s_b - s_a)
\]
\[
= \oint_{s_b - s_a} A_1 * \mathcal{P}(s_1, s_2) \ldots A_k * \mathcal{P}(s_k, s_1 + s_b - s_a),
\]
(2.46)

where the position of $\mathcal{P}(s_a, s_1)$ was translated by $s_b - s_a$, which is consistent with the periodicity $w \sim w + (s_b - s_a)$ in the $w$ frame.

### 3 Construction of BRST-invariant closed string states

In this section we construct a class of closed string states from a solution of open string field theory using the half-propagator strips we discussed in section 2. We then show that the closed string states are BRST invariant and change by BRST-exact terms under gauge transformations of the classical solution.
3.1 The boundary state from the half-propagator strip

The surface $\mathcal{P}(0, s)$ is closely related to the BCFT boundary state $|B\rangle$. Recall from (1.1) that a one-point function of a closed string vertex operator at the origin on a unit disk can be expressed in terms of $|B\rangle$ as follows:

$$\langle B|(c_0 - \tilde{c}_0)|\phi_c\rangle, \quad (3.1)$$

where $|\phi_c\rangle$ is the closed string state corresponding to the vertex operator. When we cut the unit disk along a circle of radius $e^{-\pi s/2}$, the one-point function can be thought of as an inner product of $\langle B| e^{-\pi s/2(L_0+\tilde{L}_0)}(c_0 - \tilde{c}_0)|\phi_c\rangle$:

$$\langle B|(c_0 - \tilde{c}_0)|\phi_c\rangle = \langle B| e^{-\pi s/2(L_0+\tilde{L}_0)} e^{\pi s/2(L_0+\tilde{L}_0)}(c_0 - \tilde{c}_0)|\phi_c\rangle. \quad (3.2)$$

The half-propagator strip of length $s$ with the initial and final half-string boundaries identified can be mapped to the annulus region on the unit disk bounded by the unit circle and the circle of radius $e^{-\pi s/2}$. We therefore have

$$\oint_s \mathcal{P}(0, s) = e^{-\pi s/2(L_0+\tilde{L}_0)}|B\rangle. \quad (3.3)$$

The boundary state $|B\rangle$ can thus be expressed in terms of the half-propagator strip as follows:

$$|B\rangle = e^{\pi s/2(L_0+\tilde{L}_0)} \oint_s \mathcal{P}(0, s). \quad (3.4)$$

This definition reproduces the BCFT boundary state for any value of $s$. In particular, we conclude

$$\partial_s \left[ e^{\pi s/2(L_0+\tilde{L}_0)} \oint_s \mathcal{P}(0, s) \right] = \partial_s |B\rangle = 0. \quad (3.5)$$

Later we will confirm this explicitly in section 4.2.

3.2 Construction of the closed string state $|B_*(\Psi)\rangle$

We now define a closed string state that is expected to be a generalization of $|B\rangle$ to the background associated with a solution to the equation of motion of open string field theory. In (3.3) the boundary state $|B\rangle$ was expressed in terms of the surface $\mathcal{P}(0, s)$ which is the right half of the propagator strip generated by $e^{-s\mathcal{L}}$. Since $\{Q, \mathcal{B}\} = \mathcal{L}$, we can write

$$e^{-s\mathcal{L}} = e^{-s\{Q, \mathcal{B}\}}. \quad (3.6)$$

We generalize $e^{-s\mathcal{L}}$ by replacing $Q$ in this expression by the BRST operator associated with the new background.
When we expand the open string field theory action around a solution $\Psi$ of the equation of motion
\[ Q\Psi + \Psi^2 = 0, \quad (3.7) \]
the BRST operator $Q_*$ associated with the new background is given by
\[ Q_* A \equiv Q A + \Psi A - (-)^A A \Psi \quad (3.8) \]
for any state $A$. Thus the operator $e^{-sL}$ should be modified as
\[ e^{-sL} \rightarrow e^{-s\{Q_*, B\}}. \quad (3.9) \]

To define a modified half-propagator strip $P_*(0, s)$, we have to extract the right half of the surface associated with $e^{-s\{Q_*, B\}}$. To do this, we first examine the action of $\{Q_*, B\}$ on an arbitrary state $A$. Making use of (3.8), we readily find
\[ \{Q_*, B\} A = L A + \Psi (BA) + (-)^A (BA) \Psi + B (\Psi A) - (-)^A B (A\Psi). \quad (3.10) \]

If we write $L = L_R + L_L$ and $B = B_R + B_L$, we find that the action of $\{Q_*, B\}$ on $A$ decomposes into right and left pieces as
\[ \{Q_*, B\} A = \left[ L_R A + (-)^A (B_R A) \Psi - (-)^A B_R (A\Psi) \right] + \left[ L_L A + \Psi (B_L A) + B_L (\Psi A) \right], \quad (3.11) \]
where terms with a mixed action on both right and left halves of the state $A$ have canceled as follows:
\[ \Psi (B_R A) + B_R (\Psi A) = 0, \quad (-)^A (B_L A) \Psi - (-)^A B_L (A\Psi) = 0. \quad (3.12) \]

Therefore the operator $L_R(t)$ in the half-propagator strip $P(s_a, s_b)$ defined in (2.9) should be modified as
\[ L_R(t) \rightarrow L_R(t) + \{B_R(t), \Psi\}. \quad (3.13) \]

The sign factors of $(-)^A$ in (3.11) have disappeared because of our path-ordering convention stated after (2.9). We thus define the modified half-propagator strip by
\[ P_*(s_a, s_b) \equiv \text{Pexp} \left[ - \int_{s_a}^{s_b} dt \left[ L_R(t) + \{B_R(t), \Psi\} \right] \right]. \quad (3.14) \]

It is useful to explicitly expand $P_*(s_a, s_b)$ in powers of the classical solution. We obtain
\[ P_*(s_a, s_b) = P(s_a, s_b) - \int_{s_a}^{s_b} ds_1 P(s_a, s_1) \{B_R(s_1), \Psi\} P(s_1, s_b) \]
\[ + \int_{s_a}^{s_b} ds_1 \int_{s_1}^{s_b} ds_2 P(s_a, s_1) \{B_R(s_1), \Psi\} P(s_1, s_2) \{B_R(s_2), \Psi\} P(s_2, s_b) + \ldots \]
\[ = \sum_{k=0}^{\infty} (-1)^k \int_{s_a}^{s_b} ds_1 \ldots \int_{s_{i-1}}^{s_b} ds_i \ldots \int_{s_{k-1}}^{s_b} ds_k P(s_a, s_1) \{B_R(s_1), \Psi\} P(s_1, s_2) \ldots \]
\[ \times \ldots P(s_{i-1}, s_i) \{B_R(s_i), \Psi\} P(s_i, s_{i+1}) \ldots P(s_{k-1}, s_k) \{B_R(s_k), \Psi\} P(s_k, s_b) \]. \quad (3.15) \]
The modified half-propagator strip obeys the following relations:

\[
\partial_s \mathcal{P}_s(s_a, s_b) = -\mathcal{P}_s(s_a, s_b) \left( 2 \mathcal{L}_R(s_b) + \{ \mathcal{B}_R(s_b), \Psi \} \right),
\partial_s \mathcal{P}_s(s_a, s_b) = ( 2 \mathcal{L}_R(s_a) + \{ \mathcal{B}_R(s_a), \Psi \} ) \mathcal{P}_s(s_a, s_b).
\] (3.16)

The formula (2.19) is generalized as follows:

\[
\partial_t \left[ \mathcal{P}_s(s_a, t) \mathcal{A} \mathcal{P}_s(t, s_b) \right] = -\mathcal{P}_s(s_a, t) \left[ 2 \mathcal{L}_R(t) + \{ \mathcal{B}_R(t), \Psi \}, \mathcal{A} \right] \mathcal{P}_s(t, s_b).
\] (3.17)

By analogy with the expression (3.4) of the original boundary state \( |B \rangle \), we now introduce the following background-dependent state:

\[
|B_\ast(\Psi) \rangle \equiv e^{\pi \mathcal{B} \left( \mathcal{L}_0 + \tilde{\mathcal{L}}_0 \right)} \int_s \mathcal{P}_s(0, s) \mathcal{P}_s(s, s_b).
\] (3.18)

For future use we expand \( |B_\ast(\Psi) \rangle \) in powers of the solution:

\[
|B_\ast(\Psi) \rangle = \sum_{k=0}^{\infty} |B_\ast^{(k)}(\Psi) \rangle,
\] (3.19)

where

\[
|B_\ast^{(0)}(\Psi) \rangle = |B \rangle,
|B_\ast^{(1)}(\Psi) \rangle = -e^{\pi \mathcal{B} \left( \mathcal{L}_0 + \tilde{\mathcal{L}}_0 \right)} \int_s \mathcal{P}_s(0, s_1) \{ \mathcal{B}_R(s_1), \Psi \} \mathcal{P}(s_1, s),
|B_\ast^{(2)}(\Psi) \rangle = e^{\pi \mathcal{B} \left( \mathcal{L}_0 + \tilde{\mathcal{L}}_0 \right)} \int_s \mathcal{P}_s(0, s_1) \int_s \mathcal{P}_s(0, s_2) \{ \mathcal{B}_R(s_1), \Psi \} \{ \mathcal{B}_R(s_2), \Psi \} \mathcal{P}(s_2, s),
\]

\[
\vdots
\]

\[
|B_\ast^{(k)}(\Psi) \rangle = (-1)^k e^{\pi \mathcal{B} \left( \mathcal{L}_0 + \tilde{\mathcal{L}}_0 \right)} \int_s \mathcal{P}_s(0, s_1) \int_s \mathcal{P}_s(0, s_2) \cdots \int_s \mathcal{P}(s_{k-1}, s_k) \{ \mathcal{B}_R(s_k), \Psi \} \mathcal{P}(s_k, s).
\] (3.20)

We expect that \( |B_\ast(\Psi) \rangle \) is related to the boundary state of the BCFT described by the solution \( \Psi \). In the following we study various properties of \( |B_\ast(\Psi) \rangle \) and in section 7 we explicitly calculate it for analytic solutions.

### 3.3 BRST invariance of \( |B_\ast(\Psi) \rangle \)

We show that the closed string state \( |B_\ast(\Psi) \rangle \) is BRST closed when \( \Psi \) satisfies the equation of motion of open string field theory. The BRST transformation of \( \{ \mathcal{B}_R(t), \Psi \} \) is
\[ Q \{ B_R(t), \Psi \} = Q ( \{ B_R(t), \Psi \} + \{ B_R(t), \Psi \} ) = L_R(t) \Psi + B_R(t) \Psi^2 - \Psi^2 B_R(t) - \Psi L_R(t) \]
\[ = [ L_R(t), \Psi ] + \{ B_R(t), \Psi \} \Psi - \Psi \{ B_R(t), \Psi \} = [ L_R(t) + \{ B_R(t), \Psi \}, \Psi ] , \]
\[ \quad \text{where we have used the equation of motion } Q \Psi + \Psi^2 = 0 . \]
\[ \text{Using (3.17), we conclude} \]
\[ Q \mathcal{P}_s(s_a, s_b) = - \int_{s_a}^{s_b} dt \mathcal{P}_s(s_a, t) \left( Q \{ B_R(t), \Psi \} \right) \mathcal{P}_s(t, s_b) \]
\[ = - \int_{s_a}^{s_b} dt \mathcal{P}_s(s_a, t) [ L_R(t) + \{ B_R(t), \Psi \}, \Psi ] \mathcal{P}_s(t, s_b) \]
\[ = \int_{s_a}^{s_b} dt \partial_t \left[ \mathcal{P}_s(s_a, t) \Psi \mathcal{P}_s(t, s_b) \right] \]
\[ = - \left[ \Psi, \mathcal{P}_s(s_a, s_b) \right] . \]

It is instructive to derive this relation explicitly from the expansion (3.15) of the path-ordered exponential which defines \( \mathcal{P}_s(s_a, s_b) \). It follows from (3.21) that
\[ Q \left[ \mathcal{P}(s_{i-1}, s_i) \{ B_R(s_i), \Psi \} \mathcal{P}(s_{i-1}, s_{i+1}) \right] = \mathcal{P}(s_{i-1}, s_i) \left( Q \{ B_R(s_i), \Psi \} \right) \mathcal{P}(s_{i-1}, s_{i+1}) \]
\[ = - \partial_{s_i} \left[ \mathcal{P}(s_{i-1}, s_i) \Psi \mathcal{P}(s_{i-1}, s_{i+1}) \right] + \mathcal{P}(s_{i-1}, s_i) \left[ \{ B_R(s_i), \Psi \}, \Psi \right] \mathcal{P}(s_{i-1}, s_{i+1}) , \]
\[ \quad \text{where we have used (2.19) and (2.31). Let us consider the term in the expansion of } \mathcal{P}_s(s_a, s_b) \text{ with } k \text{ insertions of the classical solution. We need to calculate its BRST transformation} \]
\[ (-1)^k Q \int_{s_{i-1}}^{s_i} ds_1 \cdots \int_{s_{i-1}}^{s_i} ds_i \cdots \int_{s_{i-1}}^{s_i} ds_k \mathcal{P}(s_a, s_1) \cdots \mathcal{P}(s_{i-1}, s_i) \{ B_R(s_i), \Psi \} \mathcal{P}(s_{i-1}, s_{i+1}) \cdots \mathcal{P}(s_k, s_b) . \]
\[ \quad \text{Using (3.23) and the formula} \]
\[ \int_{s_{i-1}}^{s_i} ds_i \int_{s_i}^{s_{i+1}} ds_i+1 \partial_{s_i} f(s_{i-1}, s_i, s_{i+1}) \]
\[ = \int_{s_{i-1}}^{s_i} ds_i \partial_{s_i} \left[ \int_{s_i}^{s_{i+1}} ds_{i+1} f(s_{i-1}, s_i, s_{i+1}) \right] + \int_{s_{i-1}}^{s_i} ds_i f(s_{i-1}, s_i, s_i) \]
\[ = - \int_{s_{i-1}}^{s_i} ds_{i+1} f(s_{i-1}, s_i, s_{i+1}) \bigg|_{s_i=s_{i-1}} + \int_{s_{i-1}}^{s_i} ds_i f(s_{i-1}, s_i, s_{i+1}) \bigg|_{s_{i+1}=s_i} , \]

\[ 19 \]
we calculate (3.24) as

\[
(-1)^k \sum_{i=1}^k \int_{s_a}^{s_b} ds_1 \ldots \int_{s_{i-1}}^{s_i} ds_i \ldots \int_{s_{k-1}}^{s_k} ds_k \\
\times \mathcal{P}(s_a, s_1) \ldots \mathcal{P}(s_{i-1}, s_i) [\{\mathcal{B}_R(s_i), \Psi\}, \Psi] \mathcal{P}(s_i, s_{i+1}) \ldots \mathcal{P}(s_k, s_b) \\
- (-1)^{k-1} \int_{s_a}^{s_b} ds_1 \ldots \int_{s_{i-1}}^{s_i} ds_i \ldots \int_{s_{k-2}}^{s_k} ds_k \\
\times \left[ \sum_{i=1}^{k-1} \mathcal{P}(s_a, s_1) \ldots \mathcal{P}(s_{i-1}, s_i) [\{\mathcal{B}_R(s_i), \Psi\}, \Psi] \mathcal{P}(s_i, s_{i+1}) \ldots \mathcal{P}(s_{k-1}, s_b) \\
+ [\Psi, \mathcal{P}(s_a, s_1) \ldots \mathcal{P}(s_{i-1}, s_i) \{\mathcal{B}_R(s_i), \Psi\} \mathcal{P}(s_i, s_{i+1}) \ldots \mathcal{P}(s_{k-1}, s_b)] \right].
\]

(3.26)

Here we relabeled the indices in the last two terms so that the \(k-1\) integration variables are \(s_1, \ldots, s_{k-1}\). We see that after summing over all \(k\), the first term at order \(k-1\) cancels the second term at order \(k\), and we are left with contributions from the last term. As in (3.22), we conclude that

\[
Q \mathcal{P}_*(s_a, s_b) = - [\Psi, \mathcal{P}_*(s_a, s_b)].
\]

(3.27)

Recalling the definition of the modified BRST operator \(Q_*\) in (3.8), this is in fact a natural modification of (2.29). We then find

\[
Q \int_{s_b-s_a} \mathcal{P}_*(s_a, s_b) = \int_{s_b-s_a} Q \mathcal{P}_*(s_a, s_b) = - \int_{s_b-s_a} [\Psi, \mathcal{P}_*(s_a, s_b)] = 0,
\]

(3.28)

where we used (2.40) in the first step and (2.45) in the last step. Since the BRST operator commutes with \(L_0 + \tilde{L}_0\), we obtain

\[
Q \vert B_*(\Psi) \rangle = 0.
\]

(3.29)

We have thus constructed a BRST-invariant closed string state for a given solution \(\Psi\).

It is easy to see that the closed string state \(\vert B_*(\Psi) \rangle\) is annihilated by \(b_0 - \tilde{b}_0\). It follows from (2.44) and \([\mathcal{B}_R(t), \{\mathcal{B}_R(t), \Psi\}] = 0\) that

\[
(b_0 - \tilde{b}_0) \int_{s_b-s_a} \mathcal{P}_*(s_a, s_b) = 0.
\]

(3.30)

Since \(b_0 - \tilde{b}_0\) commutes with \(L_0 + \tilde{L}_0\), we conclude that

\[
(b_0 - \tilde{b}_0) \vert B_*(\Psi) \rangle = 0.
\]

(3.31)

The state \(\vert B_*(\Psi) \rangle\) is also annihilated by \(L_0 - \tilde{L}_0\) because

\[
(L_0 - \tilde{L}_0) \vert B_*(\Psi) \rangle = \{Q, b_0 - \tilde{b}_0\} \vert B_*(\Psi) \rangle = 0.
\]

(3.32)
In summary, we have found that the state $|B_*(\Psi)\rangle$ satisfies three consistency requirements for its interpretation as a boundary state, namely,

$$
Q |B_*(\Psi)\rangle = 0, \quad (b_0 - \bar{b}_0) |B_*(\Psi)\rangle = 0, \quad (L_0 - \bar{L}_0) |B_*(\Psi)\rangle = 0.
$$

(3.33)

### 3.4 Variation of $|B_*(\Psi)\rangle$ under open string gauge transformations

Consider an infinitesimal gauge transformation of the solution:

$$
\delta \chi \Psi = Q\chi + [\Psi,\chi].
$$

(3.34)

It follows from the path-ordered expression of $P_*(s_a, s_b)$ that it changes under the gauge transformation as follows:

$$
\delta \chi P_*(s_a, s_b) = -\int_{s_a}^{s_b} dt P_*(s_a, t) \{B_R(t),\delta \chi \Psi\} P_*(t, s_b)
= -\int_{s_a}^{s_b} dt P_*(s_a, t) \{B_R(t),Q\chi\} P_*(t, s_b) - \int_{s_a}^{s_b} dt P_*(s_a, t) \{B_R(t),[\Psi,\chi]\} P_*(t, s_b).
$$

(3.35)

The first term in the second line can be written as

$$
= Q \int_{s_a}^{s_b} dt P_*(s_a, t) [B_R(t),\chi] P_*(t, s_b) - \int_{s_a}^{s_b} dt (Q P_*(s_a, t)) [B_R(t),\chi] P_*(t, s_b)
- \int_{s_a}^{s_b} dt P_*(s_a, t) [L_R(t),\chi] P_*(t, s_b) + \int_{s_a}^{s_b} dt P_*(s_a, t) [B_R(t),\chi] (Q P_*(t, s_b)).
$$

(3.36)

The first term on the right-hand side of (3.36) is BRST exact. The second and fourth terms on the right-hand side can be written using (3.27) as follows:

$$
- \int_{s_a}^{s_b} dt (Q P_*(s_a, t)) [B_R(t),\chi] P_*(t, s_b) + \int_{s_a}^{s_b} dt P_*(s_a, t) [B_R(t),\chi] (Q P_*(t, s_b))
= \left\{ \Psi, \int_{s_a}^{s_b} dt P_*(s_a, t) [B_R(t),\chi] P_*(t, s_b) \right\} - \int_{s_a}^{s_b} dt P_*(s_a, t) \{\Psi,[B_R(t),\chi]\} P_*(t, s_b).
$$

(3.37)

Since

$$
\{B_R(t),[\Psi,\chi]\} + \{\Psi,[B_R(t),\chi]\} = \{\{B_R(t),\Psi\},\chi\},
$$

(3.38)
Therefore, since 

\[ |V\rangle \langle V| \]

using the Jacobi identity, we find any closed string state has to be annihilated by \( b \) operator.

\[ \langle V| (c_0 - \tilde{c}_0) \Omega \rangle = 0 \]

is annihilated by both \( Q \) and \( b_0 - \tilde{b}_0 \). Thus \( |\Omega\rangle \) can be written as

\[ |\Omega\rangle = (b_0 - \tilde{b}_0) |\tilde{\Omega}\rangle . \]

Since \( |\Omega\rangle \) is annihilated by both \( Q \) and \( b_0 - \tilde{b}_0 \), we have

\[ \langle V| (c_0 - \tilde{c}_0) Q |\Omega\rangle = \langle V| (c_0 - \tilde{c}_0) Q (b_0 - \tilde{b}_0) |\tilde{\Omega}\rangle \]

\[ = \langle V| \{c_0 - \tilde{c}_0, Q\} (b_0 - \tilde{b}_0) |\tilde{\Omega}\rangle = \langle V| \{c_0 - \tilde{c}_0, Q\} (b_0 - \tilde{b}_0) |\tilde{\Omega}\rangle . \]

Using the Jacobi identity, we find

\[ \{c_0 - \tilde{c}_0, Q\}, b_0 - \tilde{b}_0\} = - \{Q, b_0 - \tilde{b}_0\}, c_0 - \tilde{c}_0\} - \{b_0 - \tilde{b}_0, c_0 - \tilde{c}_0\}, Q] \]

\[ = - [L_0 - \tilde{L}_0, c_0 - \tilde{c}_0] = 0 . \]

Therefore \( \langle V| (c_0 - \tilde{c}_0) Q |\Omega\rangle \) vanishes for any closed string state \( |\Omega\rangle \) and thus we conclude that \( \langle V| (c_0 - \tilde{c}_0)|B_\ast(\Psi)\rangle \) is gauge-invariant for any closed string state \( |V\rangle \) annihilated by the BRST operator.
4 Dependence on the choice of the propagator strip

The closed string state $|B_*(\Psi)\rangle$ for a given solution $\Psi$ depends on the total strip length $s$ and the operator $B$ for the gauge-fixing condition. In this section we study the dependence of the state $|B_*(\Psi)\rangle$ on $s$ and $B$.

4.1 Variation of the propagator

Let us consider an infinitesimal change of the gauge-fixing condition (2.1) for the propagator. The corresponding changes of $B_R$ and $L_R$ are

$$B_R(t) \rightarrow B_R(t) + \delta B_R(t), \quad L_R(t) \rightarrow L_R(t) + \{Q_R(t), \delta B_R(t)\}.$$  \hspace{1cm} (4.1)

Thus the modified half-propagator strip $\mathcal{P}_*(s_a, s_b)$ changes as follows:

$$\delta \mathcal{P}_*(s_a, s_b) = - \int_{s_a}^{s_b} dt \mathcal{P}_*(s_a, t) \delta[\mathcal{L}_R(t) + \{B_R(t), \Psi\}] \mathcal{P}_*(t, s_b)$$

$$= - \int_{s_a}^{s_b} dt \mathcal{P}_*(s_a, t) \{Q_R(t), \delta B_R(t)\} \mathcal{P}_*(t, s_b) - \int_{s_a}^{s_b} dt \mathcal{P}_*(s_a, t) \{\delta B_R(t), \Psi\} \mathcal{P}_*(t, s_b) \quad (4.2)$$

$$= - Q \int_{s_a}^{s_b} dt \mathcal{P}_*(s_a, t) \delta B_R(t) \mathcal{P}_*(t, s_b) - \left\{ \Psi, \int_{s_a}^{s_b} dt \mathcal{P}_*(s_a, t) \delta B_R(t) \mathcal{P}_*(t, s_b) \right\},$$

where we used (3.27) in the last step. We therefore find that

$$\delta \oint_{s_b - s_a} \mathcal{P}_*(s_a, s_b) = - Q \oint_{s_b - s_a} \int_{s_a}^{s_b} dt \mathcal{P}_*(s_a, t) \delta B_R(t) \mathcal{P}_*(t, s_b). \quad (4.3)$$

We conclude that the closed string state $|B_*(\Psi)\rangle$ changes by a BRST-exact term under a variation (4.1) of the gauge-fixing condition:

$$\delta |B_*(\Psi)\rangle = Q - \text{exact}. \quad (4.4)$$

4.2 Change of the strip length and the action of $L_0 + \tilde{L}_0$

To understand the $s$ dependence of $|B_*(\Psi)\rangle$, let us begin by relating closed string states of the type $\oint_{s_b - s_a} \Sigma(s_a, s_b)$ defined in (2.23) with different values for $s_b$. Consider an infinitesimal change in $s_b$. A change in the strip length $s_b - s_a$ affects the gluing of the strip to the closed string coordinate, as can be seen from (2.31). We thus need to reparameterize the closed string boundary. Infinitesimally, we account for this change by inserting a line integral of the energy-momentum tensor along the closed string boundary. From (2.31) it follows that the vector field
$u$ which adjusts the parameterization of the closed string boundary to an infinitesimal change in $s_b$ is given by

$$u(w) = \frac{w - (\gamma(\frac{\pi}{2}) + s_a)}{s_b - s_a}. \quad (4.5)$$

This vector field is tangential to the closed string boundary $\Im(w) = \frac{\pi}{2}$, vanishes at $w = \gamma(\frac{\pi}{2}) + s_a$, and satisfies

$$u \left( \gamma(\frac{\pi}{2}) + s_b \right) = 1. \quad (4.6)$$

It thus follows that the corresponding line integral of the energy-momentum tensor,

$$\tilde{\mathcal{L}}^\text{rep} = \oint_{\gamma(\frac{\pi}{2})+s_b} \left[ \frac{dw}{2\pi i} \frac{w - \gamma(\frac{\pi}{2}) - s_a}{s_b - s_a} T(w) + \frac{d\bar{w}}{2\pi i} \frac{\bar{w} - \gamma(\frac{\pi}{2}) - s_a}{s_b - s_a} \tilde{T}(\bar{w}) \right], \quad (4.7)$$

generates the desired linear stretching of the closed string boundary:

$$\partial_{s_b} \oint_{s_b-s_a} \Sigma(s_a, s_b) = \oint_{s_b-s_a} \partial_s \Sigma(s_a, s_b) + \oint_{s_b-s_a} \tilde{\mathcal{L}}^\text{rep} \Sigma(s_a, s_b). \quad (4.8)$$

Note that the constant part of the vector field $u(w)$ has an imaginary contribution $\frac{-i}{2(s_b-s_a)}$ arising from $\gamma(\frac{\pi}{2})$ and thus we cannot immediately derive a useful identity analogous to (2.12).

This contribution to the operator $\tilde{\mathcal{L}}^\text{rep}$ is proportional to $L_0 + L_0$, which can be written in the $w$ frame as

$$L_0 + \tilde{L}_0 = \int_{|\zeta|=\exp(-\frac{\pi s_b^2}{s_a^2 s_b s_a})} \frac{d\zeta}{2\pi i} \left[ \zeta T(\zeta) + \frac{d\bar{\zeta}}{2\pi i} \bar{\zeta} \tilde{T}(\bar{\zeta}) \right]$$

$$= \frac{s_b - s_a}{2\pi i} \oint_{\gamma(\frac{\pi}{2})+s_b} \left[ \frac{dw}{2\pi i} T(w) - \frac{d\bar{w}}{2\pi i} \tilde{T}(\bar{w}) \right]. \quad (4.9)$$

Therefore, if we define

$$\tilde{\mathcal{L}}^\text{rep}' = \oint_{\gamma(\frac{\pi}{2})+s_b} \left[ \frac{dw}{2\pi i} \frac{w - \Re(\gamma(\frac{\pi}{2})) - s_a}{s_b - s_a} T(w) + \frac{d\bar{w}}{2\pi i} \frac{\bar{w} - \Re(\gamma(\frac{\pi}{2})) - s_a}{s_b - s_a} \tilde{T}(\bar{w}) \right] \quad (4.10)$$

for $\tilde{\mathcal{L}}^\text{rep}'$ acting on $\Sigma(s_a, s_b)$, we have

$$\partial_{s_b} \oint_{s_b-s_a} \Sigma(s_a, s_b) = \oint_{s_b-s_a} \partial_s \Sigma(s_a, s_b) + \oint_{s_b-s_a} \tilde{\mathcal{L}}^\text{rep}' \Sigma(s_a, s_b) + \frac{\pi^2}{(s_b-s_a)^2} (L_0 + \tilde{L}_0) \oint_{s_b-s_a} \Sigma(s_a, s_b). \quad (4.11)$$

We introduce $\mathcal{L}_R^\text{rep}'(t)$ with the same integrand as $\tilde{\mathcal{L}}^\text{rep}'$ by

$$\mathcal{L}_R^\text{rep}'(t) = \int_t^{\gamma(\frac{\pi}{2})+t} \left[ \frac{dw}{2\pi i} \frac{w - \Re(\gamma(\frac{\pi}{2})) - s_a}{s_b - s_a} T(w) + \frac{d\bar{w}}{2\pi i} \frac{\bar{w} - \Re(\gamma(\frac{\pi}{2})) - s_a}{s_b - s_a} \tilde{T}(\bar{w}) \right], \quad (4.12)$$
and we have the following identity analogous to (2.12):

\[ L_R^{\text{rep}'}(s_a) P(s_a, s_b) - P(s_a, s_b) L_R^{\text{rep}'}(s_b) + \tilde{L}_R^{\text{rep}'} P(s_a, s_b) = 0. \]  

(4.13)

Note that unlike the line integrals \( L_R, B_R \) and \( Q_R \), the integrand of \( L_R^{\text{rep}'} \) is not invariant under the identification \( w \sim w + (s_b - s_a) \). We instead have

\[
\oint_{s_b - s_a} L_R^{\text{rep}'}(s_a) \Sigma(s_a, s_b) - \oint_{s_b - s_a} \Sigma(s_a, s_b) L_R^{\text{rep}'}(s_b) + \oint_{s_b - s_a} \Sigma(s_a, s_b) L_R(s_b) = 0.
\]  

(4.14)

Adding the left-hand side of (4.14) to (4.11) allows us to localize the integration contour around the slits where the open strings are inserted. We obtain

\[
\partial_s \oint_{s_b - s_a} \Sigma(s_a, s_b) = \oint_{s_b - s_a} \left[ \partial_s \Sigma(s_a, s_b) + \Sigma(s_a, s_b) L_R(s_b) \right] + \frac{\pi^2}{(s_b - s_a)^2} (L_0 + \tilde{L}_0) \oint_{s_b - s_a} \Sigma(s_a, s_b) \\
+ \sum_{i=1}^k \oint_{s_b - s_a} P(s_a, s_1) * A_1 \cdots [L_R^{\text{rep}'}(s_i), A_i] \cdots A_k * P(s_k, s_b).
\]

(4.15)

Let us apply this result to the expression on the right-hand side of (3.4) for the boundary state \( |B\rangle \). The last term in (4.15) vanishes for this case as \( P(0, s) \) does not contain any open string state insertions \( A_i \). Furthermore, the first term in (4.15) also vanishes because of (2.13). We find

\[ \partial_s |B\rangle = \partial_s \left[ e^{\frac{\pi^2}{s^2} (L_0 + \tilde{L}_0)} \oint_s P(0, s) \right] = -\frac{\pi^2}{s^2} (L_0 + \tilde{L}_0) |B\rangle + e^{\frac{\pi^2}{s^2} (L_0 + \tilde{L}_0)} \partial_s \oint_s P(0, s) = 0. \]  

(4.16)

We have thus confirmed (3.5), and the right-hand side of (3.4) reproduces the BCFT boundary state of the original theory independent of \( s \).

### 4.3 Variation of \( s \)

We now use the results of the previous subsection to study the \( s \) dependence of the closed string state \( |B_*(\Psi)\rangle \). Recalling that

\[ \partial_s \mathcal{P}_*(0, s) = -\mathcal{P}_*(0, s) (L_R(s) + \{ B_R(s), \Psi \}) , \]

(4.17)

we find

\[
\partial_s |B_*(\Psi)\rangle = -\frac{\pi^2}{s^2} (L_0 + \tilde{L}_0) e^{\frac{\pi^2}{s^2} (L_0 + \tilde{L}_0)} \oint_s \mathcal{P}_*(0, s) + e^{\frac{\pi^2}{s^2} (L_0 + \tilde{L}_0)} \partial_s \oint_s \mathcal{P}_*(0, s) \\
= e^{\frac{\pi^2}{s^2} (L_0 + \tilde{L}_0)} \oint_s \left[ \tilde{L}_R^{\text{rep}'} \mathcal{P}_*(0, s) - \mathcal{P}_*(0, s) (L_R(s) + \{ B_R(s), \Psi \}) \right] ,
\]

(4.18)
where we have used \((4.11)\). We define the \(b\)-ghost line integral \(\vec{B}_{\text{rep}}'\) as \(\vec{L}_{\text{rep}}'\) in \((4.10)\) with \(T(z)\) and \(\vec{T}(\bar{z})\) replaced by \(b(z)\) and \(\bar{b}(\bar{z})\), respectively. It follows that \(\{Q, \vec{B}_{\text{rep}}'\} = \vec{L}_{\text{rep}}'\). Consider now the first term on the above right-hand side. We have

\[
\oint_s \vec{L}_{\text{rep}}' \mathcal{P}_s(0, s) = \oint_s \{Q, \vec{B}_{\text{rep}}'\} \mathcal{P}_s(0, s) = Q \oint_s \vec{B}_{\text{rep}}' \mathcal{P}_s(0, s) + \oint_s \vec{B}_{\text{rep}}'[\Psi, \mathcal{P}_s(0, s)] .
\]

(4.19)

The commutator term vanishes using the cyclicity property \((2.45)\) and we conclude that the first term in \((4.18)\) is BRST exact:

\[
e^{-\frac{\pi}{2} s (L_0 + \tilde{L}_0)} \oint_s \vec{L}_{\text{rep}}' \mathcal{P}_s(0, s) = Q \left[ e^{-\frac{\pi}{2} s (L_0 + \tilde{L}_0)} \oint_s \vec{B}_{\text{rep}}' \mathcal{P}_s(0, s) \right] .
\]

(4.20)

The second term in \((4.18)\) is also BRST exact. In fact, we again use the cyclicity property to find

\[
\oint_s \mathcal{P}_s(0, s) \left( \mathcal{L}_R(s) + \{\mathcal{B}_R(s), \Psi\} \right) = \oint_s \left( \mathcal{P}_s(0, s) \{Q_R(s), \mathcal{B}_R(s)\} - [\Psi, \mathcal{P}_s(0, s)] \mathcal{B}_R(s) \right)
= Q \oint_s \mathcal{P}_s(0, s) \mathcal{B}_R(s) .
\]

(4.21)

As both terms in \((4.18)\) are BRST exact, we conclude that \(|B_s(\Psi)\rangle\) changes by a BRST exact piece under a variation of \(s\):

\[
\partial_s |B_s(\Psi)\rangle = Q - \text{exact} .
\]

(4.22)

Using the formula

\[
\oint_s \mathcal{B}_{\text{rep}}(0) \mathcal{P}_s(0, s) - \oint_s \mathcal{P}_s(0, s) \mathcal{B}_{\text{rep}}'(s) + \oint_s \mathcal{P}_s(0, s) \mathcal{B}_R(s) = 0 ,
\]

(4.23)

which can be derived in the same way as \((4.14)\), the total BRST-exact term in \((4.22)\) can be written as

\[
\partial_s |B_s(\Psi)\rangle = Q \left[ e^{-\frac{\pi}{2} (L_0 + \tilde{L}_0)} \oint_s \left( \mathcal{B}_{\text{rep}}' \mathcal{P}_s(0, s) + \mathcal{B}_{\text{rep}}'(0) \mathcal{P}_s(0, s) - \mathcal{P}_s(0, s) \mathcal{B}_{\text{rep}}'(s) \right) \right] .
\]

(4.24)

so that the three \(b\)-ghost contours can be connected.

---

7 The cyclicity property is unaffected by the presence of the line integral \(\vec{B}_{\text{rep}}'\) along the closed string boundary.
4.4 The $s \to 0$ limit

Let us now consider the limit $s \to 0$ of the state $|B_s(\Psi)\rangle$. The first term in the expansion (3.19) is the original boundary state $|B\rangle$. It is independent of $s$ and thus the limit $s \to 0$ is trivial. The next term $|B^{(1)}_s(\Psi)\rangle$ can be written as

$$
|B^{(1)}_s(\Psi)\rangle = -e^{\frac{x^2}{s^2} (L_0 + \tilde{L}_0)} \int_0^s ds_1 \oint_s \mathcal{P}(0, s_1) \{\mathcal{B}_R(s_1), \Psi\} \mathcal{P}(s_1, s)
$$

where we used (2.44). We can further transform it using an integrated version of (2.43) as follows:

$$
|B^{(1)}_s(\Psi)\rangle = -e^{\frac{x^2}{s^2} (L_0 + \tilde{L}_0)} \frac{2\pi i}{s} \int_0^s ds_1 (b_0 - \tilde{b}_0) e^{-s_1 \frac{2\pi i x}{s^2} (L_0 - \tilde{L}_0)} \oint_s \Psi \mathcal{P}(0, s)
$$

where in the last step we defined $\theta = 2\pi s_1/s$. One might have suspected that the state

$$
\lim_{s \to 0} \int_0^s ds_1 \oint_s \mathcal{P}(0, s_1) \{\mathcal{B}_R(s_1), \Psi\} \mathcal{P}(s_1, s)
$$

vanishes because the integration region of $s_1$ collapses in the limit $s \to 0$. We see from (4.26), however, that the integration over $s_1$ effectively rotates the surface state once around the closed string coordinate. The vanishing integration region over $s_1$ was only a coordinate effect of our parameterization of the integral over the rotational modulus, and the final expression in (4.26) is clearly nonvanishing in the limit $s \to 0$ for generic $\Psi$.

Let us next consider inner products $\langle \mathcal{V} | (c_0 - \tilde{c}_0) | B^{(k)}_s(\Psi)\rangle$ with $k \geq 2$, where $\mathcal{V}$ is an arbitrary on-shell closed string state:

$$
\langle \mathcal{V} | (c_0 - \tilde{c}_0) | B^{(k)}_s(\Psi)\rangle = (-1)^k \langle \mathcal{V} | (c_0 - \tilde{c}_0) | \oint_s \int_0^s ds_1 \oint_0^s ds_2 \cdots \oint_0^s ds_k \mathcal{P}(0, s_1) \{\mathcal{B}_R(s_1), \Psi\} \mathcal{P}(s_1, s_2) \cdots
$$

$$
\times \mathcal{P}(s_{k-1}, s_k) \{\mathcal{B}_R(s_k), \Psi\} \mathcal{P}(s_k, s)\rangle .
$$

The factor $e^{\frac{x^2}{s^2} (L_0 + \tilde{L}_0)}$ did not contribute because $\mathcal{V}$ is a primary field of weight $(0, 0)$. The limit $s \to 0$ of these inner products were essentially discussed in section 4 of [37], where it was argued that the terms with $k \geq 2$ vanish for a certain regular class of solutions. Let us review the argument in [37].

---

8 The analysis in [37] was based on the Siegel propagator strip, but this choice does not enter the following argument in an essential way.
As we did in (4.25), one can extract a factor of $b_0 - \tilde{b}_0$ from the expression in (4.28). It is accompanied by a factor of $2\pi i/s$, thus the integrand in (4.28) is singular in the limit $s \to 0$. However, the $k$ dimensional integral should be transmuted into one integral for the overall rotation and a $k-1$ dimensional integral, and the Jacobian should cancel the singularity of the factor $2\pi i/s$ as in (4.26). It was argued in [37] that the resulting integrand is finite in the limit $s \to 0$, while the $k-1$ dimensional integration region vanishes. Then the inner products in (4.28) vanish in the limit $s \to 0$ for $k \geq 2$.

As was remarked in one of the footnotes of [37], however, it is difficult to identify necessary regularity conditions on the solution for the finiteness of the resulting integrand in the limit $s \to 0$ and thus difficult to prove rigorously that the inner products in (4.28) vanish in the limit $s \to 0$ for $k \geq 2$. For example, the open string midpoint of the solution approaches the closed string vertex operator in the limit $s \to 0$, and we may find singular operator products. In fact, the analytic solutions in Schnabl gauge constructed in [6, 16, 17] contain $b$-ghost integrals extending up to the open string midpoint, and their operator products with $\mathcal{V}$ can potentially be singular. Furthermore, we have to be careful when we judge whether expressions are finite using the $w$ frame because conformal factors associated with the map to a disk coordinate can potentially be singular in the limit $s \to 0$. In fact, we have seen that the singular factor in (4.25) arose from such a conformal factor. Similarly, we have

\[
\oint_s \mathcal{P}(0, s_1) \mathcal{P}(s_1, s_2) \mathcal{P}(s_2, s) = \frac{2\pi i}{s} (b_0 - \tilde{b}_0) \oint_s \mathcal{P}(0, s_1) \Psi \mathcal{P}(s_1, s_2) \mathcal{P}(s_2, s).
\]

This is also singular in the limit $s \to 0$, but we expect that

\[
\oint_s \mathcal{P}(0, s_1) \Psi \mathcal{P}(s_1, s_2) \mathcal{P}(s_2, s)
\]

is finite in the limit $s \to 0$ if the solution $\Psi$ is regular. The difference in the behavior as $s \to 0$ between (4.29) and (4.30) is not so obvious.

We therefore do not make a general claim that the inner products in (4.28) vanish in the limit $s \to 0$ for $k \geq 2$. On the other hand, an advantage of our approach is that the state $|B_\ast(\Psi)\rangle$ is explicitly calculable for solutions based on wedge states if we choose the propagator strip of Schnabl gauge, as we demonstrate in section 6. We revisit the limit $s \to 0$ in section 6.2 and we examine this suppression of higher-order terms explicitly in section 7 for various known analytic solutions. We indeed find that the states $|B_\ast^{(k)}(\Psi)\rangle$ with $k \geq 2$ vanish in the limit $s \to 0$ for all explicit examples that we consider.

If the inner products $\langle \mathcal{V} | (c_0 - \tilde{c}_0) | B_\ast^{(k)}(\Psi) \rangle$ with $k \geq 2$ vanish in the limit $s \to 0$ for a given
regular solution $\Psi$, we find
\[
\lim_{s \to 0} \langle \mathcal{V}|(c_0 - \tilde{c}_0)|B_s(\Psi) \rangle - \langle \mathcal{V}|(c_0 - \tilde{c}_0)|B \rangle = \lim_{s \to 0} \langle \mathcal{V}|(c_0 - \tilde{c}_0)|B_s^{(1)}(\Psi) \rangle
\]
\[
= -i \lim_{s \to 0} \int_{0}^{2\pi} d\theta \langle \mathcal{V}|(c_0 - \tilde{c}_0)(b_0 - \tilde{b}_0)| \oint_s \Psi \mathcal{P}(0, s) \rangle
\]
\[
= -4\pi i \lim_{s \to 0} \langle \mathcal{V}| \oint_s \Psi \mathcal{P}(0, s) \rangle .
\] (4.31)

Here we used that $|\mathcal{V}\rangle$ is annihilated by $L_0 \pm \tilde{L}_0$ and $b_0 - \tilde{b}_0$. The parameter $s$ in (4.31) is simply a regularization of a contraction of $\Psi$ with the identity state. In the limit $s \to 0$, the closed string vertex operator is inserted at the open string midpoint of $\Psi$, and we recover the familiar string field theory observables $W(\mathcal{V}, \Psi)$:
\[
\lim_{s \to 0} \langle \mathcal{V}|(c_0 - \tilde{c}_0)|B_s(\Psi) \rangle - \langle \mathcal{V}|(c_0 - \tilde{c}_0)|B \rangle = -4\pi i W(\mathcal{V}, \Psi) ,
\] (4.32)
where $W(\mathcal{V}, \phi)$ for a generic open string state $|\phi\rangle$ in the Fock space is defined by
\[
W(\mathcal{V}, \phi) = \langle \mathcal{V}(i) f_I \circ \phi(0) \rangle_{\text{UHP}} .
\] (4.33)

We denote by $f_I \circ \phi(0)$ the conformal transformation of the operator $\phi(0)$ corresponding to the state $|\phi\rangle$ under the identity map
\[
f_I(\xi) = \frac{2\xi}{1 - \xi^2} .
\] (4.34)

As is well known, the observables $W(\mathcal{V}, \Psi)$ are gauge-invariant. The vanishing of terms with two or more solution insertions in the limit $s \to 0$ is thus consistent because $|B_s^{(0)}(\Psi)\rangle + |B_s^{(1)}(\Psi)\rangle$ is gauge-invariant in this limit. Furthermore, when we vary $s$, the state $|B_s(\Psi)\rangle$ only changes by a BRST-exact term which has a vanishing inner product with $\langle \mathcal{V}|(c_0 - \tilde{c}_0)$, and thus we conclude that (4.32) holds even for finite $s$:
\[
\langle \mathcal{V}|(c_0 - \tilde{c}_0)|B_s(\Psi) \rangle - \langle \mathcal{V}|(c_0 - \tilde{c}_0)|B \rangle = -4\pi i W(\mathcal{V}, \Psi) \text{ for any } s .
\] (4.35)

It was argued in [37] that the observables $W(\mathcal{V}, \Psi)$ represent the difference between the original boundary state $|B\rangle$ and the boundary state $|B_s\rangle$ contracted with $\langle \mathcal{V}|(c_0 - \tilde{c}_0)$
\[
\langle \mathcal{V}|(c_0 - \tilde{c}_0)|B_s \rangle - \langle \mathcal{V}|(c_0 - \tilde{c}_0)|B \rangle = -4\pi i W(\mathcal{V}, \Psi) .
\] (4.36)

This implies that
\[
\langle \mathcal{V}|(c_0 - \tilde{c}_0)|B_s(\Psi) \rangle = \langle \mathcal{V}|(c_0 - \tilde{c}_0)|B_s \rangle
\] (4.37)

---

9 The relation in the notation of [37] is $W(\mathcal{V}, \Psi) = \mathcal{A}_\Psi^{\text{disk}}(\mathcal{V}) - \mathcal{A}_0^{\text{disk}}(\mathcal{V})$, where $\mathcal{A}_\Psi^{\text{disk}}(\mathcal{V})$ and $\mathcal{A}_0^{\text{disk}}(\mathcal{V})$ are related to the inner products as $\langle \mathcal{V}|(c_0 - \tilde{c}_0)|B_s \rangle = -4\pi i \mathcal{A}_\Psi^{\text{disk}}(\mathcal{V})$ and $\langle \mathcal{V}|(c_0 - \tilde{c}_0)|B \rangle = -4\pi i \mathcal{A}_0^{\text{disk}}(\mathcal{V})$. 

29
and thus
\[ |B_\ast(\Psi)\rangle = |B_\ast\rangle + (Q - \text{exact}) \, . \] (4.38)

To summarize, if the inner products \( \langle V | (c_0 - \tilde{c}_0)|B_\ast(k)(\Psi)\rangle \) with \( k \geq 2 \) vanish for a given regular solution \( \Psi \), the relation (4.36) which was argued in [37] to hold in general implies that the closed string state \( |B_\ast(\Psi)\rangle \) coincides with the BCFT boundary state \( |B_\ast\rangle \) up to a possible BRST-exact term. Again, instead of attempting to prove this relation in general, we explicitly calculate \( |B_\ast(\Psi)\rangle \) for various known analytic solutions in section 7. We find, surprisingly, that the possible BRST-exact term vanishes for arbitrary \( s \), and we precisely obtain the BCFT boundary state for all the solutions we consider in section 7.

5 The boundary state and open-closed vertices

In this section we explain that \( |B_\ast(\Psi)\rangle \) encodes a set of open-closed vertices. They are generically “complex” open-closed vertices and the reality condition is satisfied only for certain choices of the propagator strip. As we will review below, open-closed interactions feature prominently in the construction of open-closed string field theory [57, 59]. Using a natural classical sector of this theory and assuming its physical background independence, we will do the following:

- Give a brief proof that the observable \( W(V, \Psi) \) encodes the change in one-point functions of on-shell closed strings on a disk upon change of open string background.
- Give a brief proof that the closed string state \( |B_\ast(\Psi)\rangle \) has the correct on-shell content, \( i.e., \) agrees on-shell with \( |B_\ast\rangle \).

5.1 Open-closed string field theory and background independence

We consider the following open-closed string field theory action:

\[
S_{\text{oc}}(\Psi, V) = S_\text{o}(\Psi) + \left( \langle B | + \sum_{k=1}^\infty \langle B_k | \Psi \rangle \ldots | \Psi \rangle \right) (c_0 - \tilde{c}_0) |V\rangle \\
+ \sum_{p=2}^\infty \sum_{k=0}^\infty \langle B_{k,p} | \Psi \rangle \ldots | \Psi \rangle |V\rangle \ldots |V\rangle \, .
\] (5.1)

Here \( S_\text{o}(\Psi) \) is the familiar classical open string field theory action of Witten, which is well known to reproduce correctly \( all \) amplitudes that involve only external open string states. The action \( S_{\text{oc}}(\Psi, V) \) also includes, in the first line, a series of terms that are linear in the closed string field \( |V\rangle \) but include any number of open string fields. These terms are particularly important for us. The state \( \langle B | \) is the boundary state of the original BCFT represented by a vanishing
open string field. The state $\langle B_1 \rangle$ arises from an open-closed vertex that couples one open string to one closed string. More generally, the state $\langle B_k \rangle$ couples $k$ open strings to one closed string. In the second line in (5.1) we include the terms that couple, via a disk, two or more closed strings to arbitrary numbers of open strings. The above action gives the correct amplitudes for processes that include any numbers of closed and open strings scattering through a disk and all states can be off-shell. Note that the action $S_{oc}(\Psi, \mathcal{V})$ includes neither a closed string kinetic term nor the classical, genus-zero, closed string interactions. The closed string field should not be treated as dynamical.

As explained in [57, 59], one can view $S_{oc}(\Psi, \mathcal{V})$ as an action for open strings propagating in a nontrivial closed string background. Indeed, the open string gauge invariance of $S_0(\Psi)$ can be extended to a gauge invariance of (5.1) when the closed string configuration $|\mathcal{V}\rangle$ satisfies the equation of motion of pure closed string field theory [60]. It is a natural generalization of the pure open string field theory, and the action $S_{oc}(\Psi, \mathcal{V})$ defines a consistent classical theory.

The action (5.1) gives the correct amplitudes mentioned above when the open-closed vertices satisfy sets of recursion relations. In particular, for the vertices $\langle B_k \rangle$ with $k = 1, 2, \ldots \infty$ that couple to a single closed string, the recursion relations express the BRST action on $\langle B_k \rangle$ as a series of terms in which $\langle B_{k-1} \rangle$ is glued to the three-open-string vertex:

$$
\langle B_k; 1, \ldots, k \rangle \left( Q_c + \sum_{p=1}^{k} Q^{(p)} \right) \sim \langle V_3; 1, 2, x \rangle \langle B_{k-1}; x, 3, 4, \ldots, k \rangle | R; x, x' \rangle \\
+ \langle V_3; 2, 3, x \rangle \langle B_{k-1}; x, 4, \ldots, k, 1 \rangle | R; x, x' \rangle \\
+ \langle V_3; k, 1, x \rangle \langle B_{k-1}; x, 2, \ldots, k-1 \rangle | R; x, x' \rangle.
$$

The right-hand side is a sum of $k$ terms in which $\langle B_{k-1} \rangle$ is glued to the three-open-string vertex $\langle V_3 \rangle$ by the reflector $| R \rangle$. The labels for the external open string states have been written out while the closed string label remains implicit. The various terms arise from the possible cyclic ordering of the external state labels. Open-closed vertices are said to be consistent if they satisfy the above recursion relations.

We also note that in (5.1) the state that couples linearly to the closed string field and couples to no open string field is the boundary state $\langle B \rangle$. It encodes the on-shell one-point functions of closed string states on a disk. Let $\bar{\Psi}$ denote a fixed string field solution of $S_0$ (or of $S_{oc}(\Psi, \mathcal{V} = 0)$). We explore the new background by shifting the open string field by the classical solution, namely, letting $\Psi \rightarrow \bar{\Psi} + \Psi$ in the action. In the background defined by $\bar{\Psi}$, the role of boundary state is played by the terms in $S_{oc}(\bar{\Psi} + \Psi, \mathcal{V})$ that couple to $\mathcal{V}$ and to no
open string field $\Psi$. This boundary state $\langle B^\text{oc}_\bullet(\Psi) \rangle$ is therefore defined by

$$\langle B^\text{oc}_\bullet(\Psi) \rangle \equiv \langle B \rangle + \sum_{k=1}^\infty \langle B_k \rangle (\langle |\Psi\rangle \rangle)^k.$$  (5.3)

Without possible confusion, we revert to our earlier notation where $\Psi$ denotes a classical solution and thus simply write

$$\langle B^\text{oc}_\bullet(\Psi) \rangle \equiv \langle B \rangle + \sum_{k=1}^\infty \langle B_k \rangle (\langle |\Psi\rangle \rangle)^k.$$  (5.4)

By construction, $\langle B^\text{oc}_\bullet(\Psi) \rangle$ gives one-point functions of closed string states in the background defined by $\Psi$. In fact, the role of the sum in (5.4) as a “energy-momentum tensor” associated with the open string field was suggested long ago in [57]. The statement of physical background independence for the open-closed string field theory (5.1) includes the claim that the physical one-point functions determined by $\langle B^\text{oc}_\bullet(\Psi) \rangle$ agree with those of the BCFT boundary state $\langle B_\bullet \rangle$ associated with the new background:

$$\text{Background independence} \quad \implies \quad |B^\text{oc}_\bullet(\Psi) \rangle = |B_\bullet \rangle + (Q - \text{exact}).$$  (5.5)

The background independence of (5.1) has not been proven, but it is motivated by the fact that (5.1) is a consistent, gauge-invariant extension of the familiar open string field theory.

In a nutshell, the claim in (5.5) is that the state built as in (5.4) using any consistent set of open-closed vertices, i.e., vertices satisfying (5.2), agrees on-shell with the boundary state.

A few consistency checks can be readily performed. The recursion relations (5.2) guarantee that $\langle B^\text{oc}_\bullet(\Psi) \rangle$ is BRST closed whenever $\Psi$ is an open string field theory solution. Moreover, under a gauge transformation of the solution $\Psi$ the state $\langle B^\text{oc}_\bullet(\Psi) \rangle$ changes by a BRST-exact term. Under geometric changes of the open-closed vertices, the state $\langle B^\text{oc}_\bullet(\Psi) \rangle$ also changes by a BRST-exact term. For regular open-closed vertices the local coordinate for the closed string insertion is non-singular and the inner product $\langle B^\text{oc}_\bullet(\Psi) \rangle \langle c_0 - \tilde{c}_0 | \phi_c \rangle$ can be evaluated for any off-shell closed string state $|\phi_c \rangle$.

While $|B^\text{oc}_\bullet(\Psi) \rangle$ provides a solution to the problem of constructing a boundary state associated with the background represented by the solution $\Psi$, it is difficult to evaluate it explicitly for any previously known set of open-closed string vertices. There was, in addition, no evidence that there is a choice of open-closed vertices and a representative of the solution for which the resulting $|B^\text{oc}_\bullet(\Psi) \rangle$ coincides with $|B_\bullet \rangle$ without any BRST-exact term.

We will show that, up to the action of $e^{\frac{2\pi i}{L_0 + \tilde{L}_0}}$, the state $|B_\bullet(\Psi) \rangle$ can be viewed as the state $|B^\text{oc}_\bullet(\Psi) \rangle$ associated with a set of “complex” open-closed vertices. For the complex vertices derived from the Schnabl propagator strip the state becomes calculable.

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10The existing proofs of background independence apply to classical open string field theory and to quantum closed string field theory and have been established for infinitesimal marginal deformations only.
5.2 Recovering the gauge-invariant observables $W(\Psi, \mathcal{V})$

The open-closed vertices that define $\langle B_{oc}^{\mathcal{V}}(\Psi) \rangle$ are not unique. A one-parameter family of vertices was introduced and discussed in [58]. The vertices in [58] arise from the minimal area metrics subject to the constraint that all nontrivial open curves be longer than or equal to $\pi$ while all nontrivial closed curves be longer than or equal to a parameter $\ell_c$. For a vertex which couples one open string to one closed string, for example, one considers a disk with one closed string puncture (in the interior) and one open string puncture (on the boundary) and searches for the appropriate minimal area metric. In this metric the neighborhood of the open string puncture is isometric to a semi-infinite flat strip of width $\pi$ and the neighborhood of the closed string puncture is isometric to a semi-infinite flat cylinder of circumference $\ell_c$. The coordinate curves that define the vertex are the natural boundaries of the strip and the cylinder. For any value of $\ell_c > 0$, one finds a set of consistent open-closed vertices $\langle B_k \rangle$ and $\langle B_{k,p} \rangle$. This family of open-closed vertices is parameterized by $\ell_c$.

It was noted in [58] that the construction simplifies dramatically as $\ell_c \to 0$. In this limit, the open-closed vertex $\langle B_1 \rangle$ alone constructs, through the Feynman rules, a full cover of the moduli space of all disks that involve any number of closed strings and any number of open strings. No higher vertices are thus needed and we can set them to zero. The open-closed string field theory (5.1) thus becomes

$$S_{oc}(\Psi, \mathcal{V}) = S_o(\Psi) + \left( \langle B \rangle + \langle B_1 | \Psi \rangle \right) (c_0 - \tilde{c}_0) |\mathcal{V}\rangle \quad \text{for } \ell_c \to 0.$$ (5.6)

The open-closed vertex $\langle B_1 \rangle$ obtained for $\ell_c \to 0$ is the vertex used to define $W(\mathcal{V}, \Psi)$. Indeed, in this limit the closed string cylinder disappears and the open string semi-infinite strip terminates by having its left and right half-string edges identified, an operation implemented by the open string identity field. The closed string local coordinate is singular in this vertex. As a result the vertex can only be used for on-shell closed string states.

Not only is the simplification dramatic, but the action (5.6) has one important advantage over (5.1). While it can only be used for on-shell closed string states, it reproduces all amplitudes involving any number of external open string and closed string states! It does so for surfaces of any topology that have at least one boundary. In other words, it produces all amplitudes except those of pure closed string theory [58]. The fact that (5.6) correctly reproduces all on-shell amplitudes shows that the on-shell contributions of all the higher open-closed vertices vanish as $\ell_c \to 0$.

For the action (5.6) the associated boundary state $\langle B_{oc}^{\mathcal{V}}(\Psi) \rangle$ in (5.4) becomes

$$\langle B_{oc}^{\mathcal{V}}(\Psi) \rangle = \langle B \rangle + \langle B_1 | \Psi \rangle \quad \text{for } \ell_c \to 0.$$ (5.7)

It is a boundary state for the background described by $\Psi$. As we noted above, the open-closed
vertex $\langle B_1 \rangle$ defines the gauge-invariant observable $W(V, \Psi)$. Therefore,

$$\langle B_1 | \Psi (c_0 - \tilde{c}_0) | V \rangle = -4\pi i W(V, \Psi)$$  (5.8)

for on-shell $V$. It follows from the last two equations that

$$\langle B^{oc}_s (\Psi) | (c_0 - \tilde{c}_0) | V \rangle - \langle B | (c_0 - \tilde{c}_0) | V \rangle = -4\pi i W(V, \Psi) .$$  (5.9)

Using the background independence statement \([5.5]\) and the on-shell property of $V$, we can replace $\langle B^{oc}_s (\Psi) \rangle$ by $\langle B_s \rangle$ in the above expression. The result is

$$\langle B_s | (c_0 - \tilde{c}_0) | V \rangle - \langle B | (c_o - \tilde{c}_0) | V \rangle = -4\pi i W(V, \Psi) .$$  (5.10)

This is the claim \([1.2]\) that was discussed in \([37]\).

### 5.3 Generalized open-closed vertices from $|B_s (\Psi)\rangle$

By expanding the closed string state $|B_s (\Psi)\rangle$ in powers of $\Psi$, we can deduce a series of couplings of open string states to a single closed string state. Indeed, its expansion \([3.19]\) looks like the set of terms \([5.1]\) in the open-closed string field theory. To elucidate this statement and examine its limitations we will first consider the coupling encoded in $|B_s^{(1)} (\Psi)\rangle$, a coupling of one open string to one closed string.

The closed string local coordinate in $|B_s (\Psi)\rangle$ is obtained by acting with the scaling operator $e^{\frac{i}{2} (L_0 + \tilde{L}_0)}$ on the local coordinate defined by $\oint_s \mathcal{P}_s (0, s)$, as can be seen in the definition \([3.18]\). Since the scale of the closed string local coordinate patch will not be relevant to our discussion, we focus on $\oint_s \mathcal{P}_s (0, s)$, for which the closed string coordinate curve is the one traced by the open string midpoint.\(^{11}\)

For the term in $\oint_s \mathcal{P}_s (0, s)$ with one open string field, let us look at \([4.26]\). Up to rotation of the closed string local coordinate and the $b$-ghost integral insertion, the interaction of one open string with one closed string is encapsulated by

$$\oint_s \Psi \mathcal{P}(0, s) .$$  (5.11)

The picture of the open-closed interaction described by \([5.11]\) is shown in Figure 4(a). We can deduce from Figure 4 the limit of this interaction as $s \to 0$. When the circumference of the cylinder becomes smaller than the horizontal span of the slit it is convenient to redraw

\(^{11}\)After the action of $e^{\frac{i}{2} (L_0 + \tilde{L}_0)}$ one cannot generally define the closed string coordinate curve. Naively, it seems to be at the boundary of the disk, but this only holds if there are no open string insertions, as in the case of $|B\rangle$. In the presence of the open string slits a closed string local coordinate that extends to the boundary would fail to be continuous at the slits. The expanded local coordinate, however, can be safely used to insert closed string states in the Fock space.
Figure 4: (a) The open-closed vertex constructed from $\oint P(0,s)$. The closed string local coordinate is mapped from the unit disk of \( \eta \) to the semi-infinite cylinder of circumference \( s \) bounded by the geodesic at a distance $\pi/2$ from the open string boundary. The open string local coordinate is mapped from the half disk of \( \xi \) to the strip that is attached at the slit on the cylinder. (b) The $s \to 0$ limit of the open-closed vertex constructed from $\oint P(0,s)$. The two sides of the slit are identified in the limit and the semi-infinite closed string cylinder disappears.

the vertex as in Figure 4(b). It is then clear that as $s \to 0$ the upper part of the cylinder disappears and the two sides of the slit are identified. The closed string insertion is now at the open string midpoint. Since points at equal heights are identified, this implies the identification \( \xi \to -1/\xi \) on the boundary $|\xi| = 1$ with \( \Im(\xi) \geq 0 \) of the open string local coordinate frame. This is precisely the identification that defines the contraction of an open string field with the identity. We have thus recovered the singular ($\ell_c \to 0$) open-closed vertex that defines the gauge-invariant observables $W(\Psi, \mathcal{V})$:

$$
\lim_{s \to 0} \langle \mathcal{V} | \oint_s P(0,s) = W(\Psi, \mathcal{V}) .
$$

(5.12)

This relation holds for an arbitrary choice of the shape of the propagator strip. In section 6.2 we will use the example of the Schnabl propagator strip to calculate the open string local coordinate of the open-closed vertex for finite $s$. This can be carried out explicitly and one manifestly recovers the geometry of the open string identity field.

For a general number of solution insertions, the construction of $|B_s(\Psi)\rangle$ suggests the follow-
ing definition of vertices \( \langle B_k \rangle \):

\[
(\langle \Psi \rangle |^k | B_k \rangle = (-1)^k \oint_{s_0}^s ds_1 \ldots \oint_{s_{i-1}}^s ds_i \ldots \oint_{s_{k-1}}^s ds_k \mathcal{P}(0, s_1) \{ \mathcal{B}_R(s_1), \Psi \} \mathcal{P}(s_1, s_2) \ldots \times \ldots \mathcal{P}(s_{i-1}, s_i) \{ \mathcal{B}_R(s_i), \Psi \} \mathcal{P}(s_i, s_{i+1}) \ldots \mathcal{P}(s_{k-1}, s_k) \{ \mathcal{B}_R(s_k), \Psi \} \mathcal{P}(s_k, s) .
\]

(5.13)

It is important to note that these vertices satisfy the recursion relations (5.2). This is geometrically clear. Consider \( | B_k \rangle \) in (5.13) for any \( k \). The boundaries in the integration region consist of configurations where \( s_i = s_{i+1} \) for some \( i \). These configurations represent the collision of two slits, or more precisely, two insertions of \( \Psi \), which then couple through the three-string vertex. The number of slits is effectively reduced by one, and they are still integrated over the cylinder as in \( \langle B_{k-1} \rangle \). This is clearly what we see on the right-hand side of (5.2). The property that the vertices satisfy the recursion relations provides an alternative way to understand why \( | B_\ast(\Psi) \rangle \) is BRST closed and why it changes by a BRST-exact term under a gauge transformation of the open string field.

There is an interesting complication that must be addressed. The open-closed vertices (5.13) typically fail to satisfy a reality condition. For each vertex the reality condition guarantees that the associated term in the action is real when the open and closed string fields are real (as they must be). For the vertex that couples one open string and one closed string, reality requires an antiholomorphic automorphism of the surface and the local coordinates that define the vertex. For vertices coupling more than one open string the vertex includes a sum over surfaces. Reality then requires the overall invariance of the set of surfaces under the action of an antiholomorphic automorphism on each surface.

For the vertex that couples one open string to a closed string, the automorphism can be described easily after a map that takes the disk to the upper-half plane of \( u \), with the open string puncture at \( u = 0 \) and the closed string puncture at \( u = i \). Since the real axis of the open string local frame \( \xi \) is mapped to the real axis of the upper-half plane, we have \( u(\bar{\xi}) = u(\xi) \). Reality requires that, in addition, \( u(-\xi) = -u(\xi) \). As a result one has \( u(-\bar{\xi}) = u(\bar{\xi}) \). This implies that the surface and the coordinate curve is invariant under \( u \to -\bar{u} \). The same condition must hold for the closed string coordinate curve.\(^\text{12}\) This is the antiholomorphic automorphism required by reality. This reality condition is not satisfied for open-closed vertices based on non-BPZ-even gauge choices. We will explicitly demonstrate it in section 6.2 for the open-closed vertex \( | B_1 \rangle \) associated with the Schnabl propagator strip.

On the other hand, we claim that for propagators that arise from BPZ-even gauge-fixing conditions, the vertices (5.13) are not only consistent but also real. These vertices can thus be

\(^\text{12}\) In the \( \zeta \) frame this condition is invariance under a reflection about the real axis. Our closed string coordinate curve is a circle in the \( \zeta \) frame so the condition is clearly satisfied.
used in the action (5.1) and the corresponding open-closed boundary state $|B^\ast_{oc}(\Psi)\rangle$ coincides with $|B^\ast_{*}(\Psi)\rangle$ up to the action of $\exp\left[\frac{\pi i}{s}(L_0 + \tilde{L}_0)\right]$. It is simplest to see the reality of the vertex coupling one closed string to one open string. A BPZ-even gauge condition implies that the curve $\gamma(\theta)$ is vertical \cite{42}, i.e., $\Re(\gamma(\theta)) = 0$. Thus the slit on the half-propagator strip is vertical, as shown in Figure 5. Imagine now gluing a semi-infinite open string strip to the slit. The surface has an automorphism under a combination of reflection across the line traced by the open string midpoint on the glued strip and reflection across the slit and its extension up the cylinder. If the surface is mapped to the upper-half plane of $u$ with the open string puncture at $u = 0$ and the closed string puncture at $u = i$, this automorphism can be presented as the transformation $u \rightarrow -\bar{u}$ that leaves the open string coordinate curve invariant. This is the automorphism that guarantees reality.

If we choose Siegel gauge, which is a BPZ-even gauge choice, the associated open-closed vertices (5.13) take a simple form, shown in Figure 6. One can visualize each vertex as a semi-infinite cylinder of circumference $s$ that has vertical slits of height $\pi/2$ cut from the bottom edge. \footnote{This Siegel-gauge geometry was discussed in the context of the factorization of the closed string two-point function in \cite{37}.} Semi-infinite strips of width $\pi$ can be folded and glued isometrically to the slits. The separation between the slits is integrated over. It should be noted that the vertex coupling one open string to one closed string in this construction is not the same as that of the open-closed string field theory built with $\ell_c = 2\pi$. (See Figure 1 in \cite{57}.)

We have seen that for propagator strips associated with BPZ-even gauges, the vertices (5.13)
are consistent and real. Up to scaling of the closed string coordinate (which does not change the on-shell content), the state $|B_\ast(\Psi)\rangle$ can be thought of as the state $|B_\ast^{oc}(\Psi)\rangle$ built with the open-closed vertices (5.13). These vertices can be supplemented with vertices coupling multiple closed strings to construct a complete action in the form of (5.1). If we then assume background independence, as stated in (5.5), we conclude that $|B_\ast(\Psi)\rangle$ has the on-shell content of $|B_\ast\rangle$ for BPZ-even gauges. Since $|B_\ast(\Psi)\rangle$ changes by a BRST-exact term when the choice of propagator strip is modified, this implies that $|B_\ast(\Psi)\rangle$ always has the correct on-shell content.

In summary, propagators corresponding to non-BPZ-even gauge conditions result in states $|B_\ast(\Psi)\rangle$ that do not give real open-closed vertices. In those cases $|B_\ast(\Psi)\rangle$ cannot be viewed as a boundary state $|B_\ast^{oc}(\Psi)\rangle$ of open-closed string field theory. Indeed, we use the propagator strip of Schnabl gauge, which is a non-BPZ-even gauge choice, for the explicit calculations of $|B_\ast(\Psi)\rangle$ in the following sections. In this sense our proposal of $|B_\ast(\Psi)\rangle$ goes beyond the framework of open-closed string field theory. Since it is based on complex open-closed vertices, one may wonder if the state $|B_\ast(\Psi)\rangle$ based on Schnabl gauge is real. In fact, our arguments only guarantee that its contraction with on-shell closed string states is real, but the contraction with off-shell closed string states could be complex. Of course, it is also possible that a state $|B_\ast(\Psi)\rangle$ that is not real for arbitrary real open string states turns out to be real for open string states satisfying the equation of motion. We will find that the state $|B_\ast(\Psi)\rangle$ based on Schnabl gauge is indeed real for the solutions we consider in section 7.
6 Regular and calculable boundary states

6.1 Simplifications for the Schnabl propagator and wedge-based solutions

For any choice of parameter $s$, propagator gauge-fixing condition $B$, and classical solution $\Psi$, we can construct the closed string state $|B_s(\Psi)\rangle$. In general, however, it is difficult to calculate $|B_s(\Psi)\rangle$ explicitly because the gluing of insertions of classical solutions to the slits in the $w$ frame generically requires calculations of correlation functions on a complicated Riemann surface. A drastic simplification occurs if we choose Schnabl’s gauge condition $B = B$ with\(^{14}\)

$$B = \int \frac{d\xi}{2\pi i} v_S(\xi) b(\xi), \quad (6.1)$$

where

$$v_S(\xi) = \frac{f(\xi)}{f'(\xi)}, \quad f(\xi) = \frac{2}{\pi} \arctan \xi. \quad (6.2)$$

We use the doubling trick in this section and in the next section. An explicit mode expansion of $B$ is

$$B = b_0 + 2 \sum_{j=1}^{\infty} (-1)^{j+1} b_{2j}. \quad (6.3)$$

If we choose $B = B$ and a classical solution $\Psi$ based on wedge states, we can use the results of [41] to map the resulting surface to an annulus and calculate the state $|B_s(\Psi)\rangle$ explicitly. In this section we assemble the main ingredients necessary for this calculation.

We define the wedge region $W_\alpha$ by the semi-infinite strip on the upper-half plane of $z$ between the vertical lines $\Re(z) = -\frac{1}{2}$ and $\Re(z) = \frac{1}{2} + \alpha$ with these lines identified by translation. The wedge state $W_\alpha$ is defined by

$$\langle \phi, W_\alpha \rangle = \langle f \circ \phi(0) \rangle W_\alpha. \quad (6.4)$$

Here and in what follows we denote a generic open string state in the Fock space by $|\phi\rangle$ and its corresponding operator by $\phi(0)$. When a solution is made of wedge states with operator insertions, we call it a wedge-based solution.

We now map the $w$-frame geometry with its parameterized slits to a frame which is convenient for the propagator choice $B = B$. It is related to the $w$ frame via

$$z = \frac{1}{2} e^w. \quad (6.5)$$

The $z$ frame is closely related to the familiar sliver frame. In fact, the image of the curve $\gamma(\theta)$ in the $z$ frame coincides with the sliver-frame coordinate line $f(e^{i\theta})$. More generally, $s_i + \gamma(\theta)$,

\(^{14}\) The simplification also occurs for $B \propto (B + \alpha B^*)$ with $\alpha \neq 1$. 

39
which parameterizes a slit in a half-propagator strip or its boundary, becomes vertical in this frame and is located at \( \Re(z) = \frac{1}{2} e^{s_i} \). The parameterization in the \( z \) frame is given by

\[
s_i + \gamma(\theta) \to z = e^{s_i} f(e^{i\theta}) = \frac{1}{2} e^{s_i} + i e^{s_i} \left[ \frac{2}{\pi} \arctanh \left( \tan \frac{\theta}{2} \right) \right]. \tag{6.6}
\]

Since the slits are infinite, the closed string boundary is hidden at \( z \to i \infty \). This property can be traced back to the fact that Schnabl gauge is not a regular linear \( b \) gauge. In fact, the vector field \( v_S(\xi) \) vanishes at \( \xi = i \) and thus violates the condition (2.3) at the open string midpoint. This simplifies the analysis in Schnabl gauge, but all manipulations of surfaces must be justified by regularizing the propagator and taking the Schnabl limit. Fortunately, all manipulations that our analysis requires were already justified in [11], so we can simply apply the prescriptions developed there. In particular, it is important to understand the contour of the integrals \( B_R(t) \) and \( L_R(t) \) in the \( z \) frame as a limit of a regulated curve. We denote the contour after using the doubling trick by \( C(t) \):

\[
B_R(t) \to \int_{C(t)} \frac{dz}{2\pi i} z b(z), \quad L_R(t) \to \int_{C(t)} \frac{dz}{2\pi i} z T(z). \tag{6.7}
\]

In the regularized analysis, the closed string boundary is a finite segment on the imaginary \( z \) axis, and \( t \) parameterizes the endpoint of the contour \( C(t) \) on that line segment. In the Schnabl limit, the location of the closed string boundary diverges to \( i \infty \). The contour \( C(t) \) in this limit naively runs from \(-i \infty \) to \( i \infty \) along the vertical line \( \Re(z) = \frac{e^t}{2} \) and the \( t \) dependence of its endpoints on the imaginary axis is hidden. As we discussed in section 2, however, it does depend on \( t \) even in the limit, namely,

\[
\int_{C(t)} \frac{dz}{2\pi i} z b(z) - \int_{C(t')} \frac{dz}{2\pi i} z b(z) \neq 0 \tag{6.8}
\]

for \( t' > t \) even when there are no operator insertions between the two contours. This is the \( z \)-frame representation of

\[
B_R(t) \mathcal{P}(t, t') - \mathcal{P}(t', t) B_R(t') \neq 0 \tag{6.9}
\]

which follows from the inequality (2.10). Let us now consider a calculation of

\[
(-1)^k \prod_{i=1}^{k} \int_{s_i}^{s_{i-1}} ds_i \mathcal{P}(s_{i-1}, s_i) \{ B_R(s_i), A_{\alpha_i} \} \mathcal{P}(s_k, s), \tag{6.10}
\]

where \( A_{\alpha_i} \) is a Grassmann-odd state made of the wedge state \( W_{\alpha_i} \) with operator insertions, and \( s_0 = 0 \). Before gluing the states \( A_{\alpha_i} \) to the parameterized slits, the total surface is located in the region

\[
\frac{1}{2} \leq \Re(z) \leq \frac{1}{2} e^s, \tag{6.11}
\]

15 In the notation of [11], the endpoints of the contour \( C(t) \) are \( \pm ie^t \Lambda \). In the Schnabl limit we have \( \Lambda \to \infty \).
and the parameterizations of its vertical boundaries are given by $f(e^{i\theta})$ for the left boundary and $e^{s}f(e^{i\theta})$ for the right boundary. These boundaries are glued by the operation $\oint_{s}$ which forms a closed string state from the surface, and this gluing is compatible with the identification $z \sim e^{s}z$. See Figure 7. The map to the annulus frame $\zeta$ is
\[
\zeta = \exp\left(\frac{2\pi i}{s} \ln 2z\right),
\]
which is compatible with the identification $z \sim e^{s}z$.

From the parameterization (6.6) it is obvious how to insert the wedge-based state $A_{\alpha_{1}}$ to the slit at $\Re(z) = \frac{e^{s_{1}}}{2}$: simply translate the remaining surface in the region $\Re(z) > \frac{e^{s_{1}}}{2}$ horizontally to the right by $e^{s_{1}}\alpha_{1}$ and map $A_{\alpha_{1}}$ from its sliver frame $z^{(1)}$ used in (6.4) to the resulting gap in the $z$ frame via
\[
z = e^{s_{1}}z^{(1)}.
\]
The next slit is now located at $\Re(z) = e^{s_{1}}\alpha_{1} + \frac{1}{2}e^{s_{2}}$. We translate the remaining surface in the region $\Re(z) > e^{s_{1}}\alpha_{1} + \frac{1}{2}e^{s_{2}}$ by $e^{s_{2}}\alpha_{2}$, and map the state $A_{\alpha_{2}}$ into the resulting gap via
\[
z = e^{s_{1}}\alpha_{1} + e^{s_{2}}z^{(2)}.
\]
See Figure 8. The construction iterates. For the insertion of $A_{\alpha_{i}}$ after having inserted the previous $i - 1$ states, the slit is located at
\[
\Re(z) = \sum_{j=1}^{i-1} e^{s_{j}}\alpha_{j} + \frac{1}{2} e^{s_{i}},
\]
Figure 8: Illustration of (6.10) for $k = 2$ with gaps of width $e^{s_1}\alpha_1$ and of width $e^{s_2}\alpha_2$ inserted at the two slits. Compare this with Figure 7. The states $A_{\alpha_1}$ and $A_{\alpha_2}$ are then mapped into these gaps.

and we translate the remaining surface to the right of the slit by $e^{s_i}\alpha_i$. Then we map the state $A_{\alpha_i}$ into the resulting gap via

$$z = \sum_{j=1}^{i-1} e^{s_j} \alpha_j + e^{s_i} z^{(i)}.$$ (6.16)

At the end of the process, the whole surface corresponding to (6.10), which we denote by $\Sigma$, is located in the $z$ frame in the region

$$\frac{1}{2} \leq \Re(z) \leq \sum_{j=1}^{k} e^{s_j} \alpha_j + \frac{1}{2} e^s.$$ (6.17)

The parameterization of the identified left and right boundaries of this resulting surface $\Sigma$ are not related by scaling $z \sim e^sz$, so we cannot map $\Sigma$ directly to the annulus frame via (6.12). To restore this relation, we use the prescription given in section 6.1 of [41]. We shift the entire surface $\Sigma$ horizontally by

$$a_0 = \frac{1}{e^s - 1} \sum_{j=1}^{k} e^{s_j} \alpha_j.$$ (6.18)

With this value of the shift, $\Sigma$ is then located in the region

$$\frac{1}{2} + a_0 \leq \Re(z) \leq e^s \left(\frac{1}{2} + a_0\right),$$ (6.19)

and the gluing of the left to the right boundary of $\Sigma$ is now compatible with the identification $z \sim e^sz$. This translated frame is called the natural $z$ frame. The total map from the coordinate $z^{(i)}$ of the wedge surface on which $A_{\alpha_i}$ is defined to the natural $z$ frame is a combination of the
map \((6.16)\) and a horizontal translation by \(a_0\). It is thus given by

\[
z = \ell_i + e^{s_i}z^{(i)},
\]

(6.20)

where

\[
\ell_i = \sum_{j=1}^{i-1} \alpha_j e^{s_j} + a_0, \quad \ell_1 = a_0.
\]

(6.21)

It is consistent with the identification \(z \sim e^{s}z\) in the natural \(z\) frame to map \(\Sigma\) to the annulus frame \(\zeta\) via \((6.12)\). The gluing to the closed string coordinate patch is unaffected by our manipulations in the \(z\) frame, as was shown in \([41]\). One may worry that the horizontal translations of the surfaces may have resulted in a relative rotation in the \(\zeta\) frame. This is not the case, as can be easily seen from the regularized analysis of \([41]\). We can thus analytically map the surface that defines the closed string state \((6.10)\) to an annulus that has exactly the same modulus as the annulus that defines \(\oint_s \mathcal{P}(0,s)\). In particular, all closed string surface states contributing to \(|B^\ast(\Psi)\rangle\) for wedge-based solutions are represented on exactly the same Riemann surface. This remarkable property of the Schnabl propagator strip will be crucial for our explicit calculations in section 7.

The \(b\)-ghost line integrals \(B^R_s(s_i)\) in \((6.10)\) have a simple representation in the natural \(z\) frame. Indeed, mapping the line integrals \(B^R_s(s_i)\) from their initial \(z\)-frame representation \((6.7)\) in the presence of slits to their final location in the natural \(z\) frame, we find

\[
- \{B^R_s(s_i), A_{\alpha_i}\} \rightarrow - \int_{C(s_i)} \frac{dz}{2\pi i} (z - \ell_i) b(z) \ldots - \ldots \int_{C(s_i)} \frac{dz}{2\pi i} (z - \ell_{i+1}) b(z).
\]

(6.22)

The dots \([\ldots]\) in \((6.22)\) represent the operator insertions for \(A_{\alpha_i}\). Note that the difference in the endpoints of the contour \(C(s_i)\) generated by translation of the contour in the direction of the real axis vanishes in the Schnabl limit as discussed in \([41]\). Since both contours in \((6.22)\) have the same endpoint on the closed string boundary, the contours can be connected. From the relation \(\ell_{i+1} = \ell_i + e^{s_i} \alpha_i\) we find

\[
- \{B^R_s(s_i), A_{\alpha_i}\} \rightarrow \oint \frac{dz}{2\pi i} (z - \ell_i) b(z) \ldots + e^{s_i} \alpha_i \ldots B^+_R,
\]

(6.23)

where the contour encircles the operators \([\ldots]\) counterclockwise and

\[
B^+_R = \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} b(z).
\]

(6.24)

We do not write the \(t\) dependence on the endpoints of the integration contour of \(B^+_R\) because the integral does not depend on the choice of \(t\). To see this, note that

\[
\mathcal{L}^+_R \equiv \{Q, B^+_R\}
\]

(6.25)
generates horizontal translations in the $z$ frame. It was shown in [11] that the closed string boundary is unaffected by such translations in the Schnabl limit. It follows that the integrand in (6.24) vanishes along the closed string boundary. Thus the operator $B_{R}$ does not depend on the choice of $t$. The position of this insertion is given implicitly by the operator ordering in the correlator.

We have assembled all the ingredients required to explicitly calculate $|B_{s}(\Psi)\rangle$ with $B = B$ for wedge-based solutions. In fact, the map from the wedge surfaces on which the solutions are defined to the natural $z$ frame are explicitly given in (6.20) and the positions and conformal factors of operator insertions on the surface $\Sigma$ can be explicitly calculated. The $b$-ghost line integrals $B_{R}(s_{i})$ have the simple representation (6.23) in the natural $z$ frame. Finally, the map from the natural $z$ frame to the annulus frame is explicitly given in (6.12). We conclude that the boundary state $|B_{s}(\Psi)\rangle$ is explicitly calculable for wedge-based solutions if we use the half-propagator strips associated with Schnabl’s $B$. We will now use this knowledge to illustrate the vanishing of terms with more than one solution insertion in the limit $s \to 0$.

### 6.2 The $s \to 0$ limit revisited

In this subsection we first examine the open-closed vertex encoded in $|B_{s}^{(1)}(\Psi)\rangle$ constructed using the Schnabl propagator strip. The open string local coordinate can be calculated explicitly and one can confirm that this open-closed vertex is not real for finite $s$. We can also confirm that the reality condition is recovered in the $s \to 0$ limit, where the vertex becomes the familiar singular one used in the definition of $W(V, \Psi)$ discussed in section 5.3. We then consider $|B_{s}^{(k)}(\Psi)\rangle$ with $k \geq 2$ constructed using the Schnabl propagator strip and argue that they vanish in the $s \to 0$ limit.

As explained earlier in (5.11), the geometrical configuration of $|B_{s}^{(1)}(\Psi)\rangle$ can be reduced to that of $\oint s \Psi P(0, s)$. To examine the reality of this vertex, it is sufficient to consider the case where the open string field is a generic state $|\phi\rangle$ in the Fock space. Since a state in the Fock space is a wedge state of unit width with a local operator insertion, the surface associated with $\phi P(0, s)$ has total width $\frac{1}{2}(e^{s} + 1)$ in the $z$ frame. The necessary shift $a_{0}$ for the natural $z$ frame from the formula (6.18) with $k = 1$, $\alpha_{1} = 1$, and $s_{1} = 0$ is

$$a_{0} = \frac{1}{e^{s} - 1}. \quad (6.26)$$

For small $s$, it can be expanded as

$$a_{0} \simeq \frac{1}{s} - \frac{1}{2} + \mathcal{O}(s). \quad (6.27)$$

---

16 This can also be explicitly confirmed by mapping (6.24) to the annulus frame $\zeta$ and evaluating the integrand at $|\zeta| = e^{-\frac{\pi}{2}}$. 44
The position $z_p$ of the operator insertion associated with $|\phi\rangle$ in the natural $z$ frame is

$$z_p = 1 + a_0 = \frac{e^s}{e^s - 1}, \quad (6.28)$$

and the local coordinate for the open string is given by

$$z(\xi) = z_p + f(\xi) \quad (6.29)$$

with $f(\xi)$ defined in (6.2). Since the natural $z$ frame with the identification $z \sim e^s z$ is singular in the limit $s \to 0$, it is convenient to map this to the annulus $\zeta$ frame. We then map it further to the upper-half plane to compare it with $W(V, \Psi)$. The local coordinate for the open string in the $\zeta$ frame is

$$\zeta(\xi) = \exp \left[ \frac{2\pi i}{s} \left( \ln 2z(\xi) - \ln 2z_p \right) \right] = \exp \left( \frac{2\pi i}{s} \ln \left[ 1 + \frac{f(\xi)}{1 + a_0} \right] \right), \quad (6.30)$$

where we rotated the $\zeta$ frame to satisfy $\zeta(0) = 1$ for convenience. The surface associated with the state $\oint_s \phi P(0,s)$ in the $\zeta$ frame is an annulus with inner radius $r = e^{-\frac{\pi}{2}}$. We further map this to the upper-half plane of $u$ by the following conformal transformation:

$$u = i \frac{1 - \zeta}{1 + \zeta}. \quad (6.31)$$

The puncture is located at $u = 0$ and the outer boundary of unit radius in the $\zeta$ frame is mapped to the real axis. For small $s$, the inner boundary of the annulus is mapped to a small, almost circular closed curve around $u = i$ in this frame. The local coordinate now takes the form

$$u(\xi) = \tan \left( \frac{\pi}{s} \ln \left[ 1 + \frac{f(\xi)}{1 + a_0} \right] \right). \quad (6.32)$$

When $s$ is small and as $\theta \to \pi/2$, $u(e^{i\theta})$ is a curve that rotates along a small circle around the point $u = i$. As discussed in section 5.3, the reality of the open-closed vertex requires $u(-\xi) = -u(\xi)$. While $f(-\xi) = -f(\xi)$, the coordinate $u(\xi)$ in (6.32) does not satisfy $u(-\xi) = -u(\xi)$. Thus the open-closed vertex is not real for finite $s$.

Let us now consider the limit $s \to 0$. We can make use of the expansion

$$\ln \left[ 1 + \frac{f}{1 + a_0} \right] \simeq sf - \frac{s^2}{2}(f + f^2) + O(s^3), \quad (6.33)$$

which is valid for any $|\xi| \leq 1$ satisfying $|\xi - i| > \epsilon$ with fixed $\epsilon > 0$ because $|f(\xi)|$ is bounded in this region. We then find

$$u(\xi) = \tan \left( \pi f - \frac{\pi}{2} s(f + f^2) + O(s^3) \right), \quad (6.34)$$

$$= \tan(\pi f) - \frac{\pi}{2} s(f + f^2) \sec^2(\pi f) + O(s^3).$$
The limit \( s \to 0 \) is perfectly well defined and we find

\[
\lim_{s \to 0} u(\xi) = \tan(\pi f) = \tan(2 \arctan(\xi)) = \frac{2\xi}{1 - \xi^2}.
\]

This result is valid for the region \( |\xi - i| > \epsilon \) on the unit disk, as stated above. We can see that (6.35) is, in fact, the local coordinate map (4.34) for the identity state. The closed string coordinate curve becomes a vanishingly small curve around \( u = i \) in this limit, and we have recovered, as expected, the open-closed vertex used to define the observable \( W(\mathcal{V}, \Psi) \). In this limit, of course, the open-closed vertex is real.

Let us now examine the limit \( s \to 0 \) of the state \( |B(k)(\Psi)\rangle \) in (3.20) for arbitrary \( k \). Naively, one might expect that only the leading term \( k = 0 \), the boundary state \( |B\rangle \), survives in this limit because the integration region for every Schwinger parameter collapses as \( s \to 0 \). We have seen in section 4.4 that this reasoning fails for \( |B_1(k)(\Psi)\rangle \). This is the open-closed vertex examined above. In this case, the integral over \( s_1 \) covers the entire range of the rotational modulus of the associated surface even in the limit \( s \to 0 \). See (4.26). For general choices of propagator strips, the Schwinger parameters \( s_i \) represent moduli of an annulus with insertions of the classical solution. The \( b \)-ghost line integrals \( B_R(s_i) \) provide the measure for this integration over moduli. For the Schnabl propagator strip and wedge-based solutions, the modulus of the annulus depends only on \( s \), and the Schwinger parameters \( s_i \) represent position moduli for the insertions. It is instructive to calculate explicitly the position moduli by examining the maps of solution insertions to the annulus. We will show that in the limit \( s \to 0 \), the integration over all Schwinger parameters \( s_1, \ldots, s_k \) only covers a one-dimensional subspace of the position moduli. We will argue that this is consistent with the vanishing of \( |B_1(k)(\Psi)\rangle \) with \( k \geq 2 \) in the limit \( s \to 0 \) for regular solutions.

Consider any point \( t^{(i)} \) on the boundary of the wedge surface of width \( \alpha_i \) for the state \( A_{\alpha_i} \) in (6.10). We have \( \frac{1}{2} \leq t^{(i)} \leq \frac{1}{2} + \alpha_i \). By (6.12) and (6.20), this term is mapped on the annulus frame to the point

\[
\zeta_i = e^{i\theta_i}, \quad \text{with} \quad \theta_i = \frac{2\pi}{s} \ln \left[ a_0 + \sum_{j=1}^{i-1} e^{s_j} \alpha_j + e^{s_i} t^{(i)} \right].
\]

(6.36)

It will be useful to consider the angular separation between a point \( t^{(p)} \) on \( A_{\alpha_p} \) and a point \( t^{(q)} \)
on $A_{a_q}$ in the limit $s \to 0$. We choose $q > p$ for definiteness. Recalling $a_0 = \mathcal{O}(s^{-1})$, we obtain

$$\theta_q - \theta_p = \frac{2\pi}{s} \ln \left[ \frac{a_0 + \sum_{j=1}^{q-1} e^{s_j} \alpha_j + e^{s_q} t(q)}{a_0 + \sum_{j=1}^{p-1} e^{s_j} \alpha_j + e^{s_p} t(p)} \right]$$

$$= \frac{2\pi}{s} \ln \left[ 1 + \frac{1}{a_0} \left( t(q) - t(p) + \sum_{j=p}^{q-1} \alpha_j \right) + \mathcal{O}(s) \right]$$

$$= \frac{2\pi}{s} \frac{t(q) - t(p) + \sum_{j=p}^{q-1} \alpha_j}{\sum_{j=1}^{k} \alpha_j} + \mathcal{O}(s),$$

where we used the explicit expression for $a_0$ from (6.18) in the last step and, since $0 \leq s_i \leq s$ for all $i$, we simply treated all $\mathcal{O}(s_i)$ terms as $\mathcal{O}(s)$ terms. We also assumed that the sum of $\alpha_i$’s in the denominator above is greater than zero. This is clearly satisfied if $\alpha_i > 0$ for all $i$. We conclude from (6.37) that the angular separation between $t(p)$ and $t(q)$ on the unit circle $|\zeta| = 1$ freezes up in the limit $s \to 0$: it becomes independent of all Schwinger parameters $s_i$. Thus the relative positions of all points mapped to the open string boundary of the annulus are fixed in this limit. As the closed string coordinate patch in the annulus frame is also independent of all $s_i$ and is only a function of $s$, the only remaining modulus that can vary as we integrate over the $s_i$ is the relative angle of all operator insertions with respect to the closed string coordinate patch. We conclude that a $k$ dimensional integral over the Schwinger parameters $s_i$ is confined to a one-dimensional subspace of moduli in the limit $s \to 0$. As the line integrals $B_{R}(s_i)$ supply the correct measure for the $k$-dimensional integration and the integration region becomes degenerate, this is consistent with the vanishing of the integrals with $k \geq 2$ in the limit $s \to 0$.

We can apply this analysis to the state $|B_s^{(k)}(\Psi)\rangle$ with $k \geq 2$ for solutions $\Psi$ which are defined on wedge surfaces of nonvanishing width. Again, as in section 4.4 we assume that the solution $\Psi$ satisfies some regularity condition that ensures a nonsingular behavior in the limit $s \to 0$. Then we conclude from the above argument that all $|B_s^{(k)}(\Psi)\rangle$ with $k \geq 2$ vanish in this limit. This explicit analysis of the $s_i$ dependence of the maps to the annulus for the Schnabl gauge propagator thus illustrates the more general argument of section 4.4.

### 7 The BCFT boundary state from analytic solutions

In this section we explicitly calculate $|B_s(\Psi)\rangle$ constructed using the Schnabl propagator strips for various known wedge-based solutions. We analyze Schnabl’s tachyon vacuum solution and the two known analytic solutions for marginal deformations with regular operator products. We find that

$$|B_s(\Psi)\rangle = 0 \quad \text{for all } s$$

(7.1)
in the case of the tachyon vacuum solution and
\[ |B_s(\Psi)\rangle = |B_s\rangle \quad \text{for all } s \] (7.2)
with no additional BRST-exact term in the case of marginal deformations.

### 7.1 Schnabl’s solution for tachyon condensation

Schnabl’s solution \( \Psi_S \) for tachyon condensation is given by [6]
\begin{equation}
\Psi_S = \lim_{N \to \infty} \left[ \sum_{n=0}^{N} \psi'_n - \psi_N \right], \quad \psi'_n \equiv \frac{d}{dn} \psi_n ,
\end{equation}
(7.3)
where
\begin{equation}
\langle \phi, \psi_{n-1} \rangle = -\langle f \circ \phi(0) \mathcal{B}_R^+ c(n) \rangle_{\mathcal{W}_n} \quad \text{for } n > 1
\end{equation}
(7.4)
and
\begin{equation}
\langle \phi, \psi_0 \rangle \equiv \lim_{n \to 1} \langle \phi, \psi_{n-1} \rangle = \langle f \circ \phi(0) \rangle_{\mathcal{W}_1} .
\end{equation}
(7.5)
The goal of this subsection is the calculation of \( |B_s(\Psi_S)\rangle \). As a warm-up exercise, it is instructive to calculate
\begin{equation}
|B_s^{(1)}(\psi_0)\rangle = -e^{\frac{\pi^2}{2}(\hat{L}_0 + \hat{\hat{L}}_0)} \oint ds_1 \mathcal{P}(0, s_1) \left\{ \mathcal{B}_R(s_1), \psi_0 \right\} \mathcal{P}(s_1, s) .
\end{equation}
(7.6)
The required shift \( a_0 \) from (6.18) is given by
\begin{equation}
a_0 = \frac{e^{s_1}}{e^{s} - 1} ,
\end{equation}
(7.7)
and the operator \( c(1) \) in (7.5) is mapped to \( e^{-s_1}c(e^{s_1} + a_0) \). The operator insertions in the natural \( z \) frame are
\begin{align}
-e^{-s_1} \int_{C(s_1)} \frac{dz}{2\pi i} (z - a_0) b(z) c(e^{s_1} + a_0) - e^{-s_1}c(e^{s_1} + a_0) \int_{C(s_1)} \frac{dz}{2\pi i} (z - a_0 - e^{s_1}) b(z) \\
= e^{-s_1} \int \frac{dz}{2\pi i} (z - a_0) b(z) c(e^{s_1} + a_0) + c(e^{s_1} + a_0) \mathcal{B}_R^+ \\
= 1 + c(e^{s_1} + a_0) \mathcal{B}_R^+ = -\mathcal{B}_R^+ c(e^{s_1} + a_0) ,
\end{align}
(7.8)
where the first line corresponds to (6.22), the second line corresponds to (6.23), and we used the anticommutation relation \( \{ \mathcal{B}_R^+, c(t) \} = -1 \) in the last step. From the identification \( z_+ = e^s z_- \) in the natural \( z \) frame, we find
\begin{equation}
\int \frac{dz_+}{2\pi i} b(z_+) = e^{-s} \int \frac{dz_-}{2\pi i} b(z_-) .
\end{equation}
(7.9)
and therefore
\[
\int_{C(t+s)} \frac{dz_+}{2\pi i} b(z_+) - \int_{C(t)} \frac{dz_-}{2\pi i} b(z_-) = (e^{-s} - 1) \int_{C(t)} \frac{dz_-}{2\pi i} b(z_-). \tag{7.10}
\]
We thus obtain the following formula:
\[
\mathcal{B}_R^+ \ldots = -\frac{e^s}{e^s - 1} \oint \frac{dz}{2\pi i} b(z) \ldots , \tag{7.11}
\]
where the dots \(\ldots\) represent arbitrary insertions of local operators and the contour encircles all these operators counterclockwise. Using this formula, the operator insertions (7.8) in the natural \(z\) frame can be calculated as
\[
-\mathcal{B}_R^+ c(e^{s_1} + a_0) = \frac{e^s}{e^s - 1} \oint \frac{dz}{2\pi i} b(z) c(e^{s_1} + a_0) = \frac{e^s}{e^s - 1} . \tag{7.12}
\]
For (7.6) we thus obtain
\[
|\mathcal{B}_S^{(1)}(\psi_0)\rangle = - e^{\frac{x^2}{2}(L_0 + \tilde{L}_0)} \oint_s \int_0^s ds_1 \mathcal{P}(0, s_1) \{\mathcal{B}_R(s_1), \psi_0\} \mathcal{P}(s_1, s)
\]
\[
= \int_0^s ds_1 \frac{e^s}{e^s - 1} |B\rangle = \frac{s e^s}{e^s - 1} |B\rangle . \tag{7.13}
\]
Let us now generalize this calculation to closed string states of the following form:
\[
|\Phi(n_1, \ldots, n_k)\rangle \\
\equiv (-1)^k e^{\frac{x^2}{2}(L_0 + \tilde{L}_0)} \oint_s \mathcal{P}(0, s_1) \{\mathcal{B}_R(s_1), \psi_{n_1-1}\} \mathcal{P}(s_1, s_2) \{\mathcal{B}_R(s_2), \psi_{n_2-1}\}
\]
\[
\times \mathcal{P}(s_2, s_3) \{\mathcal{B}_R(s_3), \psi_{n_3-1}\} \cdots \mathcal{P}(s_{k-1}, s_k) \{\mathcal{B}_R(s_k), \psi_{n_k-1}\} \mathcal{P}(s_k, s) , \tag{7.14}
\]
where \(n_i \geq 1\). Note that \(|\mathcal{B}_S^{(k)}(\Psi_S)\rangle\) with \(\Psi_S\) given in (7.3) can be expressed in terms of states of the form \(|\Phi(n_1, \ldots, n_k)\rangle\). For example,
\[
|\mathcal{B}_S^{(2)}(\Psi_S)\rangle = \lim_{N_1, N_2 \to \infty} \int_0^s ds_1 \int_0^s ds_2 \left[ \sum_{n_1=1}^{N_1} \sum_{n_2=1}^{N_2} \frac{\partial}{\partial n_1} \frac{\partial}{\partial n_2} |\Phi(n_1, n_2)\rangle - \sum_{n_1=1}^{N_1} \frac{\partial}{\partial n_1} |\Phi(n_1, N_2)\rangle \right. \]
\[
- \sum_{n_2=1}^{N_2} \frac{\partial}{\partial n_2} |\Phi(N_1, n_2)\rangle \left. + |\Phi(N_1, N_2)\rangle \right] . \tag{7.15}
\]
\footnote{One might worry that the \(\mathcal{B}_R^+\) integral cannot be closed without taking into account a contribution from the hidden boundary similar to the contribution \(\mathcal{B}\) in (2.16) for \(\mathcal{B}_R\). Fortunately, this is not the case because the integrand of \(\mathcal{B}_R^+\) \emph{vanishes} along the closed string boundary in the Schnabl limit, as we discussed in section 6.}
In the calculation of $|\Phi (n_1, \ldots, n_k)\rangle$, the operators we insert for $\psi_{n_i-1}$ in the natural $z$ frame are

$$- e^{-s_i} c(e^{s_i} + \ell_i) \mathcal{B}_R^+ c(n_i e^{s_i} + \ell_i) ,$$

where

$$\ell_i = \sum_{j=1}^{i-1} n_j e^{s_j} + a_0 , \quad \ell_1 = a_0 .$$

All the operators for the state $|\Phi (n_1, \ldots, n_k)\rangle$ can be written using the formula (6.23) as

$$\prod_{i=1}^{k} \left[ - e^{-s_i} \oint \frac{dz}{2\pi i} (z - \ell_i) B(z) c(e^{s_i} + \ell_i) \mathcal{B}_R^+ c(n_i e^{s_i} + \ell_i) - n_i c(e^{s_i} + \ell_i) \mathcal{B}_R^+ c(n_i e^{s_i} + \ell_i) \mathcal{B}_R^+ \right]$$

$$= \prod_{i=1}^{k} \left[ - \mathcal{B}_R^+ c(n_i e^{s_i} + \ell_i) - n_i c(e^{s_i} + \ell_i) \mathcal{B}_R^+ + n_i c(e^{s_i} + \ell_i) \mathcal{B}_R^+ \right]$$

$$= \prod_{i=1}^{k} \left[ - \mathcal{B}_R^+ c(n_i e^{s_i} + \ell_i) \right].$$

(7.18)

Using the anticommutation relation $\{ B_R^+, c(t) \} = -1$ and $(\mathcal{B}_R^+)^2 = 0$ repeatedly, we find

$$\mathcal{B}_R^+ c(t_1) \mathcal{B}_R^+ c(t_2) \cdots \mathcal{B}_R^+ c(t_k) = (-1)^{k-1} \mathcal{B}_R^+ c(t_k) = \frac{(-1)^k e^s}{e^s - 1} ,$$

(7.19)

where we used (7.11) in the last step. Therefore we have

$$\prod_{i=1}^{k} \left[ - \mathcal{B}_R^+ c(n_i e^{s_i} + \ell_i) \right] = \frac{e^s}{e^s - 1}$$

(7.20)

and thus

$$|\Phi (n_1, \ldots, n_k)\rangle = \frac{e^s}{e^s - 1} |B\rangle .$$

(7.21)

Note that this is independent of $n_i$. This means that the $\psi'_n$ piece of the solution does not contribute to $|B_s(\Psi_S)\rangle$ because all derivatives of $|\Phi (n_1, \ldots, n_k)\rangle$ with respect to $n_i$ vanish. In particular, mixed terms that involve $\psi'_n$ and $\psi_N$ do not contribute to $|B_s(\Psi_S)\rangle$. Therefore the whole contribution to $|B_s(\Psi_S)\rangle$ comes entirely from the ‘phantom’ term $-\psi_N$ of the solution, namely,

$$|B_s(\Psi_S)\rangle = (-1)^k \lim_{N_1, \ldots, N_k \to \infty} \int_0^s ds_1 \int_{s_1}^s ds_2 \cdots \int_{s_{k-1}}^s ds_k |\Phi (N_1, N_2, \ldots, N_k)\rangle .$$

(7.22)

Since $|\Phi (N_1, \ldots, N_k)\rangle$ is independent of $N_i$, the limit $N_1, \ldots, N_k \to \infty$ is trivial.\footnote{In particular, the limit $N_1, \ldots, N_k \to \infty$ is independent of the order in which we take $N_i \to \infty$.} The result (7.21) is also independent of $s_i$. Thus the integrals over $s_i$ in (7.22) simply gives the

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following factor:
\[ \int_0^s ds_1 \int_{s_1}^s ds_2 \int_{s_2}^s ds_3 \cdots \int_{s_{k-1}}^s ds_k = \frac{s^k}{k!}. \] (7.23)

We therefore conclude that
\[ |B_*(\Psi_S)\rangle = \left[ 1 + \sum_{k=1}^{\infty} \frac{s^k}{k!} (-1)^k \frac{e^s}{e^s-1} \right] |B\rangle = \left[ 1 + (e^{-s} - 1) \frac{e^s}{e^s-1} \right] |B\rangle = 0. \] (7.24)

In [37, 33] it was shown that the gauge-invariant observables \( W(\mathcal{V}, \Psi) \) vanish for the tachyon vacuum solution \( \Psi_S \). Their result can be reproduced by calculating the on-shell part of the \( k = 1 \) term \( |B_1^s(\Psi_S)\rangle \) and taking the limit \( s \to 0 \). The result (7.24) can thus be viewed as the generalization of the calculation in [37] to the off-shell part and to finite \( s \). Indeed, the terms with \( k \geq 2 \) in (7.24) are suppressed for small \( s \), consistent with our analysis in section 6.2. In the limit \( s \to 0 \) the \( k = 1 \) term by itself cancels the original boundary state \( |B\rangle \). In summary, we conclude that \( |B_*(\Psi_S)\rangle \) for Schnabl’s tachyon vacuum solution \( \Psi_S \) vanishes for any finite \( s \):

\[ |B_*(\Psi_S)\rangle = 0 \quad \text{for} \quad \mathcal{B} = B \quad \text{and any finite} \quad s. \] (7.25)

This is consistent with Sen’s conjecture that the D-brane disappears at the tachyon vacuum.

### 7.2 Factorization of \( |B_*(\Psi)\rangle \) into matter and ghost sectors

We have seen that \( |B_*(^k(\Psi_S))\rangle \) is proportional to \( |B\rangle \) for any \( k \). In particular, this means that the ghost sector of \( |B_*(^k(\Psi_S))\rangle \) is the same as the ghost sector of \( |B\rangle \), namely, the boundary state \( |B^{(bc)}\rangle \) of the \( bc \) CFT. This boundary state satisfies the relations
\[ (b_n - \tilde{b}_{-n}) |B^{(bc)}\rangle = 0, \quad (c_n + \tilde{c}_{-n}) |B^{(bc)}\rangle = 0 \] (7.26)
for all \( n \in \mathbb{Z} \). If the state \( |B_*(\Psi)\rangle \) factorizes into matter and ghost sectors as
\[ |B_*(\Psi)\rangle = |B_*(\text{matter})\rangle \otimes |B^{(bc)}\rangle, \] (7.27)
it follows from \( Q |B_*(\Psi)\rangle = 0 \) and (7.26) that the matter part \( |B_*(\text{matter})\rangle \) satisfies the relation for conformal boundary conditions
\[ (L_n^{(\text{matter})} - \tilde{L}_{-n}^{(\text{matter})}) |B_*(\text{matter})\rangle = 0. \] (7.28)

While our claim is that the state \( |B_*(\Psi)\rangle \) coincides with the BCFT boundary state \( |B_*\rangle \) up to a possible BRST-exact term, the state \( |B_*(\Psi)\rangle \) factorized as (7.27) can be a consistent BCFT boundary state without any BRST-exact term. It is therefore important to examine for what solutions the ghost part of \( |B_*(\Psi)\rangle \) becomes the boundary state \( |B^{(bc)}\rangle \) of the \( bc \) CFT.

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We now claim that the ghost part of a closed string state of the form
\begin{equation}
\frac{e^{s^2(L_0 + \tilde{L}_0)}}{s} \mathcal{P}(0, s_1) \{\mathcal{B}_R(s_1), A_1\} \mathcal{P}(s_1, s_2) \{\mathcal{B}_R(s_2), A_2\} \mathcal{P}(s_2, s_3) \{\mathcal{B}_R(s_3), A_3\} \cdots \times \mathcal{P}(s_{k-1}, s_k) \{\mathcal{B}_R(s_k), A_k\} \mathcal{P}(s_k, s) \tag{7.29}
\end{equation}

coincides with the boundary state of the bc CFT if open string fields \(A_1, A_2, \ldots, A_k\) of ghost number one are made of wedge states with

- local operator insertions of the \(c\) ghost and its derivatives,
- \(b\)-ghost line integrals \(\mathcal{B}_R^+\),
- arbitrary insertions of matter operators, and
- line integrals \(\mathcal{L}_R^+ = \{Q, \mathcal{B}_R^+\}\) of the energy-momentum tensor. We demand that there are no other operators on the contour of each \(\mathcal{L}_R^+\).

This can be shown in the following way. First consider the case where there are no line integrals of \(\mathcal{L}_R^+\). The \(b\)-ghost integral \(\mathcal{B}_R(s_i)\) in \(\{\mathcal{B}_R(s_i), A_i\}\) can be written in the form (6.23):
\begin{equation}
\{\mathcal{B}_R(s_i), A_{\alpha_i}\} \rightarrow - \oint \frac{dz}{2\pi i} (z - \ell_i) b(z) [\ldots] - e^{s_i} \alpha_i [\ldots] \mathcal{B}_R^+, \tag{7.30}
\end{equation}

where \(\alpha_i\) is the length of the wedge state associated with \(A_i\) and \(\ell_i\) is defined in (6.21). Nonvanishing contributions to the first term come only from local operator insertions of the \(c\) ghost and its derivatives in \([\ldots]\). Therefore, after performing the integrals of the form (7.30), remaining operator insertions in the ghost sector are insertions of \(\mathcal{B}_R^+\) and insertions of the \(c\) ghost and its derivatives. It then follows from \((\mathcal{B}_R^+)^2 = 0\) and the transformation property (7.29) that there must be at least one insertion of the \(c\) ghost or its derivatives between two insertions of \(\mathcal{B}_R^+\) for the result to be nonvanishing. However, since the total ghost number vanishes, there must be only one insertion of the \(c\) ghost or its derivatives between two insertions of \(\mathcal{B}_R^+\). Then such contributions can be calculated using the formula (7.19). In fact, terms which contain derivatives of the \(c\) ghost vanish because the right-hand side of (7.19) is independent of \(t_i\)'s. Nonvanishing contributions are of the form (7.19) and no ghost operators remain in the end. We have thus shown that the ghost part of (7.29) coincides with \(|\mathcal{B}_{bc}\rangle\).

Let us next consider the case where there is one line integral of \(\mathcal{L}_R^+\). The insertion of \(\mathcal{L}_R^+\) in the definition of a state \(A_i\) appears in the form
\begin{equation}
\langle \phi, A_i \rangle = \langle f \circ \phi(0) [\ldots]_1 \mathcal{L}_R^+ [\ldots]_2 \rangle_w_{\alpha_i}, \tag{7.31}
\end{equation}

where we denoted all the operator insertions to the left of the \(\mathcal{L}_R^+\) line integral by \([\ldots]_1\) and those to the right by \([\ldots]_2\). Such separation is possible because of our assumption that there
are no operators on the contour of $L^+_R$. The state $A_i$ can be written as

$$A_i = -\partial_t A_i(t) \bigg|_{t=0},$$

(7.32)

where the state $A_i(t)$ for $t > 0$ is defined by\(^{19}\)

$$\langle \phi, A_i(t) \rangle = \langle f \circ \phi(0) \ldots g \circ \ldots \rangle_{W_{0+t}} \text{ with } g(z) = z + t. \quad (7.33)$$

Since $A_i(t)$ belongs to the class of states considered before, the ghost part of (7.29) with $A_i$ replaced by $A_i(t)$ for any positive $t$ coincides with $|B^{(bc)}\rangle$. Therefore, the ghost part of (7.29) with $A_i$ replaced by $\partial_t A_i(t)$ also coincides with $|B^{(bc)}\rangle$. It thus follows that the ghost part of (7.29) with $A_i$ given by (7.32) also coincides with $|B^{(bc)}\rangle$. It is straightforward to generalize the proof to the case with an arbitrary number of insertions of $L^+_R$. We conclude that the state $|B_s(\Psi)\rangle$ takes the form (7.27) if the solution $\Psi$ consists of wedge states with local operator insertions of the $c$ ghost and its derivatives, line integrals $B^+_R$ and $L^+_R$, and arbitrary insertions of matter operators\(^{20}\).

This condition on the solution for which $|B_s(\Psi)\rangle$ satisfies (7.27) is a sufficient condition and is not a necessary condition. However, this class of states covers all known wedge-based analytic solutions such as Schnabl’s tachyon vacuum solution\(^{6}\) as analyzed in section 7.1 and the solutions associated with marginal deformations for both regular and singular operator products constructed in \[16, 17, 20, 25\].

In the case of marginal deformations with regular operator products, the solutions in Schnabl gauge constructed in \[16, 17\] and those in \[25\] are expected to be gauge-equivalent. We just argued that these different solutions give the same state $|B_s(\Psi)\rangle$ in the form (7.27). We thus expect that the form (7.27) is preserved for a certain class of gauge transformations. We have shown in section 3.4 that the state $|B_s(\Psi)\rangle$ changes under a gauge transformation $\delta_\chi \Psi = Q\chi + [\Psi, \chi]$ by the following BRST-exact term:

$$\delta_\chi |B_s(\Psi)\rangle = Q \left[ e^{\frac{2}{\alpha} (L_0 + L_0)} \int_0^s dt \mathcal{P}_s(0,t) [B_R(t), \chi] \mathcal{P}_s(t,s) \right]. \quad (7.34)$$

Consider a closed string state of the form

$$e^{\frac{2}{\alpha} (L_0 + L_0)} \int_s \mathcal{P}(0, s_1) \{B_R(s_1), A_1\} \mathcal{P}(s_1, s_2) \{B_R(s_2), A_2\} \mathcal{P}(s_2, s_3) \{B_R(s_3), A_3\} \ldots \times \mathcal{P}(s_{i-1}, s_i) \{B_R(s_i), A_i\} \mathcal{P}(s_i, s_{i+1}) \ldots \mathcal{P}(s_{k-1}, s_k) \{B_R(s_k), A_k\} \mathcal{P}(s_k, s). \quad (7.35)$$

\(^{19}\) The definition of $A_i(t)$ can be extended to $t < 0$ until operator insertions in $[\ldots]_1$ and $g \circ [\ldots]_2$ collide. The derivative of $A_i(t)$ at $t = 0$ is therefore well defined.

\(^{20}\) In general, if we have a one-parameter family of closed string states of the form (7.29) with their ghost sectors being $|B^{(bc)}\rangle$, ghost sectors of closed string states obtained by taking derivatives with respect to the parameter are also given by $|B^{(bc)}\rangle$. We considered the case with line integrals of $L^+_R$ as a particular example of this generalization because $\psi^*_{\text{tach}}$ in the tachyon vacuum solution contains a line integral of $L^+_R$, but various other generalizations will be possible. Derivatives of the $c$ ghost can also be treated in this way.
where $A_i$ carries ghost number zero and all other states $A_1, A_2, \ldots, A_k$ carry ghost number one. They are made of wedge states with local operator insertions of the $c$ ghost and its derivatives, line integrals $B_R^+$ and $L_R^+$, and arbitrary insertions of matter operators. The $b$-ghost integral $B_R(s_i)\left[B_R(s_i), A_i\right]$ for the Grassmann-even state $A_i$ can be written as
\[
\left[B_R(s_i), A\right] \rightarrow -\oint \frac{dz}{2\pi i} \left( z - \ell_i \right) b(z) \left[ \ldots \right] + c s_i \alpha_i \left[ \ldots \right] B_R^+,
\] (7.36)
which follows from the same manipulations that led to (6.23) for Grassmann-odd states. Line integrals of $L_R^+$ can be treated in the same way as before so that it is sufficient to consider the case when they are absent. After performing the integrals of the form (7.30) and (7.36), remaining operator insertions in the ghost sector are again insertions of $B_R^+$ and insertions of the $c$ ghost and its derivatives. In this case, however, the total ghost number is $-1$ and thus we have one more insertions of $B_R^+$ than insertions of the $c$ ghost and its derivatives. Any term with more than one insertion of $B_R^+$ immediately vanishes because of $(B_R^+)^2 = 0$ and the transformation property (7.9). In the case of one insertion of $B_R^+$, we do not have any other insertions of ghost operators, and it follows from the formula (7.11) that the contribution vanishes. We thus conclude that $\delta \chi |B_\ast(\Psi)\rangle$ vanishes if the solution $\Psi$ and the gauge parameter $\chi$ consist of wedge states with local operator insertions of the $c$ ghost and its derivatives, line integrals $B_R^+$ and $L_R^+$, and arbitrary insertions of matter operators. In particular, gauge transformations generated by gauge parameters made of wedge states with only matter operator insertions do not change the state $|B_\ast(\Psi)\rangle$.

7.3 Marginal deformations with regular operator products

Deformations of BCFT generated by a matter primary field $V$ of weight one are exactly marginal when operator products $V(t_1) V(t_2) \ldots V(t_n)$ are regular. In this case we expect to have a one-parameter family of solutions to the equation of motion of open string field theory. Analytic solutions for such marginal deformations with regular operator products were constructed in [16, 17, 25]. In [37, 36], the gauge-invariant observables $W(V, \Psi)$ were calculated for these solutions, and the results confirmed the relation (1.2). These calculations essentially extracted the on-shell part of $|B^{(s)}(\Psi)\rangle$ in the limit $s \rightarrow 0$. In this section we calculate the full state $|B_\ast(\Psi)\rangle$ constructed using the Schnabl propagator strips for these analytic solutions to see if it coincides with the BCFT boundary state
\[
|B_\ast\rangle = \exp \left[ \lambda \int_0^{2\pi} d\theta V(\theta) \right] |B\rangle,
\] (7.37)
where $\theta$ with $0 \leq \theta \leq 2\pi$ parameterizes the boundary of the disk.
7.3.1 The leading term

The leading term $\Psi^{(1)}$ defined by

$$\langle \phi, \Psi^{(1)} \rangle = \langle f \circ \phi(0) \ cV(1) \rangle_{W_1}$$

is identical for all solutions \[16, 17, 25\] associated with a marginal operator $V$. Let us calculate

$$|B_s^{(1)}(\Psi^{(1)})| = - e^{s^2(L_0 + \tilde{L}_0)} \oint_s \int_0^s ds_1 \mathcal{P}(0, s_1) \{B_R(s_1), \Psi^{(1)}\} \mathcal{P}(s_1, s).$$

The required shift $a_0$ from (6.18) is given by

$$a_0 = \frac{e^{s_1}}{e^s - 1}.$$  (7.40)

Using (6.23), the operator insertions in the natural $z$ frame are

$$\oint dz \frac{dz}{2\pi i} (z - a_0) b(z) cV(e^{s_1} + a_0) + e^{s_1} cV(e^{s_1} + a_0) B_R^+$$

$$= e^{s_1} V(e^{s_1} + a_0) + e^{s_1} cV(e^{s_1} + a_0) B_R^+$$

$$= - e^{s_1} B_R^+ cV(e^{s_1} + a_0).$$

The ghost sector of these operator insertions is identical to the one calculated in (7.12) for the tachyon vacuum solution. We obtain

$$- e^{s_1} B_R^+ cV(e^{s_1} + a_0) = \frac{e^s e^{s_1}}{e^s - 1} V(e^{s_1} + a_0).$$

If we define

$$u_1 = e^{s_1} + a_0 = \frac{e^s e^{s_1}}{e^s - 1},$$

we observe that

$$\int_0^s ds_1 \frac{e^s e^{s_1}}{e^s - 1} V(e^{s_1} + a_0) = \int_{\frac{e^s}{e^s - 1}}^{\frac{e^{2s}}{e^s - 1}} du_1 V(u_1).$$

Namely, the measure factor

$$\frac{\partial u_1}{\partial s_1} = \frac{e^s e^{s_1}}{e^s - 1}$$

has been correctly provided through the calculation. Since the point $\frac{e^s}{e^{s_1}}$ and the point $\frac{e^{2s}}{e^{s_1}}$ are identified in the natural $z$ frame, the operator corresponds to

$$\oint_0^{2\pi} d\theta V(\theta)$$

in the $\zeta$ frame, where $\theta$ parameterizes the boundary $|\zeta| = 1$ as $\zeta = e^{i\theta}$. Therefore we find

$$|B_s^{(1)}(\Psi^{(1)})| = - e^{s^2(L_0 + \tilde{L}_0)} \oint_s \int_0^s ds_1 \mathcal{P}(0, s_1) \{B_R(s_1), \Psi^{(1)}\} \mathcal{P}(s_1, s) = \int_0^{2\pi} d\theta V(\theta) |B\rangle.$$

This is indeed the $O(\lambda)$ term in the path-ordered exponential (7.37) that we expected.
7.3.2 Regular marginal deformations in Schnabl gauge

Let us next calculate $|B_*(\Psi)|$ with $\Psi$ being the Schnabl-gauge solutions for marginal deformations constructed in [16, 17]. The solution $\Psi$ is given by

$$\Psi = \sum_{n=1}^{\infty} \lambda^n \Psi^{(n)},$$

where

$$\langle \phi, \Psi^{(n)} \rangle = - \int_0^1 dt_1 \int_0^1 dt_2 \ldots \int_0^1 dt_{n-1} \langle f \circ \phi(0) cV(1) \times V(1 + t_1) V(1 + t_1 + t_2) \ldots V(1 + t_1 + t_2 + \ldots + t_{n-2}) \times B_R^+ cV(1 + t_1 + t_2 + \ldots + t_{n-1}) \rangle_{W_{1+t_1+t_2+\ldots+t_{n-1}}}.$$  

The calculation of the ghost sector has been done in section 7.1 so that we only need to calculate the matter sector.

There are two terms which contribute to $|B_*(\Psi)|$ at $\mathcal{O}(\lambda^2)$. The first one is

$$|B_*^{(2)}(\Psi^{(1)})| = e^{\frac{s^2}{2}(L_0 + \bar{L}_0)} \oint_s \int_s^s ds_1 \int_s^s ds_2 \mathcal{P}(0, s_1) \{B_R(s_1), \Psi^{(1)}\} \times \mathcal{P}(s_1, s_2) \{B_R(s_2), \Psi^{(1)}\} \mathcal{P}(s_1, s).$$

The operator insertion in the natural $z$ frame is given by

$$\frac{e^s}{e^s - 1} \int_s^s ds_1 \int_s^s ds_2 e^{s_1} V \left( \frac{e^s e^{s_1} + e^{s_2}}{e^s - 1} \right) e^{s_2} V \left( \frac{e^s e^{s_1} + e^s e^{s_2}}{e^s - 1} \right),$$

where we used (7.20) to calculate the ghost sector. The second one is

$$|B_*^{(1)}(\Psi^{(2)})| = - e^{\frac{s^2}{2}(L_0 + \bar{L}_0)} \oint_s \int_s^s ds_1 \mathcal{P}(0, s_1) \{B_R(s_1), \Psi^{(2)}\} \mathcal{P}(s_1, s).$$

The operator insertion in the natural $z$ frame is given by

$$\frac{e^s}{e^s - 1} \int_s^s ds_1 \int_0^1 dt_1 e^{s_1} V \left( \frac{e^s e^{s_1} + t_1 e^{s_1}}{e^s - 1} \right) e^{s_1} V \left( \frac{e^s e^{s_1} + t_1 e^{s_1}}{e^s - 1} \right).$$

If we define

$$s'_1 = s_1, \quad s'_2 = s_1 + \ln t_1,$$
this can be written as follows:

\[ \frac{e^s}{e^s - 1} \int_0^s ds_1 \int_{-\infty}^{s_1} ds_2' e^{s_1} V \left( \frac{e^s e^{s_1} + e^{s_2'}}{e^s - 1} \right) e^{s_2} V \left( \frac{e^s e^{s_1} + e^{s_2}}{e^s - 1} \right). \] (7.55)

Note that the second factor of \( e^{s_1} \) in (7.53) has been changed to \( e^{s_2} \) because of the Jacobian

\[ \frac{\partial (s_1', s_2')}{\partial (s_1, t_1)} = \frac{1}{t_1}. \] (7.56)

The two expressions (7.51) and (7.55) are combined to give

\[ \frac{e^s}{e^s - 1} \int_0^s ds_1 \int_{-\infty}^{s_2} ds_2 e^{s_1} V \left( \frac{e^s e^{s_1} + e^{s_2}}{e^s - 1} \right) e^{s_2} V \left( \frac{e^s e^{s_1} + e^{s_2}}{e^s - 1} \right). \] (7.57)

If we define

\[ u_1 = \frac{e^s e^{s_1} + e^{s_2}}{e^s - 1}, \quad u_2 = \frac{e^s e^{s_1} + e^s e^{s_2}}{e^s - 1}, \] (7.58)

this expression can be written in the following form:

\[ \int_{\Gamma^{(2)'}} du_1 du_2 V(u_1) V(u_2), \] (7.59)

where we have used

\[ \frac{\partial (u_1, u_2)}{\partial (s_1, s_2)} = \frac{e^s}{e^s - 1} e^{s_1} e^{s_2}. \] (7.60)

Since

\[ e^{s_1} = u_1 - e^{-s} u_2, \quad e^{s_2} = u_2 - u_1, \] (7.61)

the integration region \( \Gamma^{(2)'} \) can be characterized by

\[ 1 \leq u_1 - e^{-s} u_2 \leq e^s, \quad 0 \leq u_2 - u_1 \leq e^s. \] (7.62)

Using the identification \( z \sim e^s z \) in the natural \( z \) frame, this integral can also be written as

\[ \int_{\Gamma^{(2)}} du_1 du_2 V(u_1) V(u_2) \] (7.63)

with \( \Gamma^{(2)} \) given by

\[ 1 \leq e^s u_1 - u_2 \leq e^s, \quad 0 \leq u_2 - u_1 \leq 1. \] (7.64)

It is straightforward to generalize the calculations to the case of \( \mathcal{O}(\lambda^n) \). The details are presented in appendix A.1. The matter sector of the boundary state \( |B_\ast(\Psi)\rangle \) can be written in the \( z \) frame with the identification \( z \sim e^s z \) as follows:

\[ \sum_{n=0}^{\infty} \lambda^n \int_{\Gamma^{(n)}} du_1 du_2 \ldots du_n V(u_1) V(u_2) \ldots V(u_n), \] (7.65)

57
where the region $\Gamma^{(n)}$ is given by

$$0 \leq u_2 - u_1 \leq 1, \quad 0 \leq u_3 - u_2 \leq 1, \quad \ldots \quad 0 \leq u_n - u_{n-1} \leq 1, \quad 1 \leq e^s u_1 - u_n \leq e^s. \quad (7.66)$$

In appendix A.2 we use the identification $z \sim e^s z$ in the $z$ frame repeatedly to show that

$$\int_{\Gamma^{(n)}} du_1 du_2 \ldots du_n V(u_1) V(u_2) \ldots V(u_n) = \int_1^{e^s} du_1 \int_1^{e^s} du_2 \ldots \int_1^{e^s} du_n V(u_1) V(u_2) \ldots V(u_n). \quad (7.67)$$

It follows that

$$\sum_{n=0}^{\infty} \lambda^n \int_{\Gamma^{(n)}} du_1 du_2 \ldots du_n V(u_1) V(u_2) \ldots V(u_n) = \exp \left[ \lambda \int_1^{e^s} du V(u) \right]. \quad (7.68)$$

Therefore we conclude that

$$|B_*(\Psi)\rangle = \exp \left[ \lambda \int_0^{2\pi} d\theta V(\theta) \right] |B\rangle. \quad (7.69)$$

### 7.3.3 Other solutions for regular marginal deformations

We have so far considered solutions in Schnabl gauge, and thus the choice of the propagator strip $B = B$ and the gauge condition on the solution $B\Psi = 0$ are correlated. However, neither the explicit calculability of $|B_*(\Psi)\rangle$ nor the factorizability (7.27) depends on the Schnabl-gauge condition $B\Psi = 0$. We next consider the analytic solutions for regular marginal deformations constructed in [25], which do not satisfy $B\Psi = 0$. While there is an obstruction in the construction of solutions in Schnabl gauge for marginal deformations with singular operator products, the solutions in [25] can be generalized to such singular cases and they still belong to the class of solutions discussed in section 7.2.

The solution $\Psi_L$ constructed in [25] is given by

$$\Psi_L = \sum_{n=1}^{\infty} \Psi_L^{(n)}, \quad (7.70)$$

where

$$\langle \phi, \Psi_L^{(n)} \rangle = \lambda^n \int_1^{2} dt_2 \int_2^{3} dt_3 \int_3^{4} dt_4 \ldots \int_{t_{n-1}}^{n} dt_n \langle f \circ \phi(0) cV(1) V(t_2) V(t_3) \ldots V(t_n) \rangle_{\mathcal{W}_n}. \quad (7.71)$$

While $\Psi_L$ solves the equation of motion, it does not satisfy the reality condition

$$\Psi = \text{hc}^{-1}(\Psi^*) \quad (7.72)$$
on the string field, where \(hc\) denotes hermitian conjugation. A real solution \(\Psi_{\text{real}}\) that satisfies (7.72) was constructed from \(\Psi_L\) using a gauge transformation [25]. As the gauge parameter \(\chi\) of the required gauge transformation is of the type described in section 7.2, we conclude from the discussion in that section that

\[
|B_s(\Psi_L)\rangle = |B_s(\Psi_{\text{real}})\rangle.
\]  

(7.73)

It is thus sufficient to calculate \(|B_s(\Psi_L)\rangle\).

Consider the term with \(k\) insertions \(\Psi_L^{(n_1)}, \ldots, \Psi_L^{(n_k)}\) in \(|B_s(\Psi_L)\rangle\):

\[
(-1)^k e^{\frac{\pi}{2}(L_0+\bar{L}_0)} \oint_s \int_0^s ds_1 \int_{s_1}^s ds_2 \ldots \int_{s_{k-1}}^s ds_k \mathcal{P}(0, s_1) \{B_R(s_1), \Psi_L^{(n_1)}\} \times \mathcal{P}(s_1, s_2) \{B_R(s_2), \Psi_L^{(n_2)}\} \mathcal{P}(s_2, s_3) \{B_R(s_3), \Psi_L^{(n_3)}\} \ldots \times \mathcal{P}(s_{k-1}, s_k) \{B_R(s_k), \Psi_L^{(n_k)}\} \mathcal{P}(s_k, s).
\]  

(7.74)

At \(O(\lambda^n)\), all partitions \(\vec{n} = \{n_1, \ldots, n_k\}\) with

\[
\sum_{i=1}^k n_i = n
\]  

(7.75)

contribute. The solution \(\Psi_L\) has the structure discussed in section 7.2 and we can thus eliminate all ghost-sector operators in the natural \(z\) frame as described in that section. We show in appendix B.1 that the numerical factor which remains after this manipulation is given by

\[
\Delta_k \prod_{i=1}^k e^{s_i} \quad \text{with} \quad \Delta_k = 1 + \frac{1}{e^s - 1} - \frac{1}{e^s - 1} \prod_{i=1}^k (1 - n_i).
\]  

(7.76)

Let us denote the marginal operators from the insertion \(\Psi_L^{(n_i)}\) in the natural \(z\)-frame picture as \(V(t_1^{(i)}), V(t_2^{(i)}), \ldots, V(t_{n_i}^{(i)})\) with \(t_1^{(i)} \leq t_2^{(i)} \leq \ldots \leq t_{n_i}^{(i)}\). Note that \(cV(t_1^{(i)})\) is the only unintegrated vertex operator from \(\Psi_L^{(n_i)}\) and the others \(V(t_2^{(i)}), \ldots, V(t_{n_i}^{(i)})\) are integrated vertex operators. After the calculation of the ghost sector in the natural \(z\) frame, we expect the factor (7.76) to provide the correct measure for the integration over positions of the operators \(V(t_i^{(1)})\) with \(i = 1 \ldots k\). This is indeed the case:

\[
\frac{\partial(t_1^{(1)}, t_2^{(2)}, \ldots, t_k^{(k)})}{\partial(s_1, s_2, \ldots, s_k)} = \Delta_k \prod_{i=1}^k e^{s_i},
\]  

(7.77)

as we show in appendix B.2.

---

23 The infinitesimal gauge parameter \(\chi\) for this transformation is proportional to \(\ln \sqrt{U}\) with the state \(U\) defined in [25]. As \(U\) is a wedge-based state with only matter operator insertions and \(U = 1 + O(\lambda)\), we conclude that \(\chi\) is well defined perturbatively in \(\lambda\) and of the form described in section 7.2.
The operator insertions in the natural \(z\) frame of any term (7.74) with partition \(\vec{n}\) thus take the form

\[
\int_{\Gamma(\vec{n})} dt_1(t_1^{(1)}) dt_2(t_2^{(1)}) \cdots dt_{n_k}(t_k^{(1)}) V(t_1^{(1)}) V(t_2^{(1)}) \cdots V(t_{n_k}) \tag{7.78}
\]

for some integration region \(\Gamma(\vec{n})\) associated with the particular partition \(\vec{n}\). The integration regions are complicated, and we were not able to explicitly combine the regions \(\Gamma(\vec{n})\) of all partitions into the expected form. We instead choose a different approach to show that the BCFT boundary state is indeed recovered. Consider any point in the integration region \(\Gamma(\vec{n})\) of any partition \(\vec{n}\) which contributes to \(|B_\epsilon(\Psi)|\) at \(O(\lambda^n)\). The associated positions \(\{t_1^{(1)}, \ldots, t_{n_k}^{(1)}\}\) in the \(z\) frame are mapped to a set of angles \(\{\theta_1, \ldots, \theta_n\}\) on the unit circle in the \(\zeta\) frame:

\[
\{t_1^{(1)}, \ldots, t_{n_k}^{(1)}\} \rightarrow \{e^{i\theta_1}, \ldots, e^{i\theta_n}\} \quad \text{with} \quad 0 \leq \theta_1 \leq \cdots \leq \theta_q \leq \cdots \leq \theta_n \leq 2\pi . \tag{7.79}
\]

Note that this is a map between sets, and the order of insertion points \(t_j^{(i)}\) in the \(z\)-frame will in general be a cyclic permutation of the ordering of the angles \(\theta_q\). In particular, we do not expect that \(t_1^{(1)}\) is necessarily mapped to \(\theta_1\).

We pick any one of the angles in the set \(\{\theta_1, \ldots, \theta_n\}\), and denote its index by \(\tilde{q}\). We now vary this angle, keeping all other angles fixed to their original values. In appendix B.3 we show that for any \(\theta_{\tilde{q}-1} \leq \tilde{\theta} \leq \theta_{\tilde{q}+1}\), we can find a partition \(\vec{n}\) such that some point in its integration region \(\Gamma(\vec{n})\) maps to the positions

\[
\{e^{i\theta_1}, \ldots, e^{i\theta_{\tilde{q}-1}}, e^{i\tilde{\theta}}, e^{i\theta_{\tilde{q}+1}}, \ldots, e^{i\theta_n}\} \tag{7.80}
\]

in the \(\zeta\) frame. As \(\tilde{\theta}\) is varied, one generically reaches the boundary of the integration region \(\Gamma(\vec{n})\) of the current partition \(\vec{n}\). We show that such points can always be smoothly matched to the boundary of the integration region \(\Gamma(\vec{n})\) of a different partition \(\tilde{n}\). The variation of \(\tilde{\theta}\) can thus continue until it either coincides with \(\theta_{\tilde{q}-1}\) or \(\theta_{\tilde{q}+1}\) \[24\] Denoting the image of the region \(\Gamma(\vec{n})\) in the \(\zeta\) frame by \(\zeta(\Gamma(\vec{n}))\), we conclude that

\[
\{e^{i\theta_1}, \ldots, e^{i\theta_{\tilde{q}-1}}, e^{i\tilde{\theta}}, e^{i\theta_{\tilde{q}+1}}, \ldots, e^{i\theta_n}\} \subset \bigcup_{\vec{n}} \zeta(\Gamma(\vec{n})) \quad \text{for all} \quad \theta_{\tilde{q}-1} \leq \tilde{\theta} \leq \theta_{\tilde{q}+1} . \tag{7.81}
\]

Note that the angles \(\theta_q\) with \(q \neq \tilde{q}\) are not arbitrary because they are determined by the point in the integration \(\vec{n}\) that we picked originally as the starting point for the argument. It is obvious, however, that we can use the above argument iteratively for all \(1 \leq \tilde{q} \leq n\) and complete the integration region to all \(\{\theta_1, \ldots, \theta_n\}\) satisfying

\[
0 \leq \theta_1 \leq \cdots \leq \theta_n \leq 2\pi . \tag{7.82}
\]

\[24\] For the special case of \(\tilde{q} = 1\), the lower boundary of the variation is \(\theta_1 = 0\). Similarly, the upper boundary of the variation for \(\tilde{q} = n\) is \(\theta_n = 2\pi\).
This is the integration region expected from (7.37) at $\mathcal{O}(\lambda^n)$. As the above argument requires a choice of a starting point, it does not rule out multiple covering and we conclude that

$$|B_s(\Psi_L)\rangle = \sum_{n=0}^{\infty} \frac{C_n}{n!} \left[ \lambda \int_0^{2\pi} d\theta V(\theta) \right]^n |B\rangle \quad \text{for some } C_n \in \mathbb{N}. \quad (7.83)$$

It now remains to show that $C_n = 1$ for all $n$. Consider any partition $\vec{n} = \{n_1, \ldots, n_k\}$ with $k \geq 2$. In the natural $z$ frame, the last operator insertion of $\Psi^{(n_1)}$ and the first operator insertion of $\Psi^{(n_2)}$ are located at

$$t^{(1)}_{n_1} \leq e^{s_1}n_1 + a_0, \quad t^{(2)}_{1} = e^{s_1}n_1 + e^{s_2} + a_0 \quad (7.84)$$

with

$$a_0 = \frac{1}{e^s - 1} \sum_{i=1}^{k} n_i e^{s_i} \leq \frac{e^s n}{e^s - 1}. \quad (7.85)$$

In the $\zeta$ frame, their angular separation $\Delta \theta$ is thus bounded from below:

$$\Delta \theta \geq \Delta \theta_{\text{min}} \quad \text{with} \quad \Delta \theta_{\text{min}} = \frac{2\pi}{s} \log \left[ \frac{e^s n + (e^s - 1)(n_1 + 1)}{e^s n + (e^s - 1)n_1} \right]. \quad (7.86)$$

The lower bound $\Delta \theta_{\text{min}} > 0$ is independent of the Schwinger parameters $s_i$. Using cyclicity, we similarly conclude that

$$\Delta \theta \leq 2\pi - \Delta \theta_{\text{min}}. \quad (7.87)$$

Now consider the subset of the integration region (7.82) where all operator insertions are separated by less than the angle $\Delta \theta_{\text{min}}$, namely,

$$|\theta_i - \theta_j| < \Delta \theta_{\text{min}} \quad \text{for all } 1 \leq i, j \leq n. \quad (7.88)$$

This is a finite region for finite $s$ which can only be covered by $\zeta(\Gamma(\vec{n}))$ with the partition $\vec{n} = \{n\}$ of $k = 1$. On the other hand, we have shown in (4.26) that the region from $|B_s^{(1)}(\Psi^{(n)}_L)\rangle$ covers the rotational modulus exactly once, so the subset (7.88) of the integration region is covered precisely once. Therefore, there cannot be multiple coverings of the integration region (7.82) and we obtain

$$C_n = 1 \quad \text{for all } n. \quad (7.89)$$

Recalling the relation (7.73), we conclude that

$$|B_s(\Psi_L)\rangle = |B_s(\Psi_{\text{real}})\rangle = \exp \left[ \lambda \int_0^{2\pi} d\theta V(\theta) \right] |B\rangle. \quad (7.90)$$
8 Discussion

In this paper we have constructed a class of BRST-invariant closed string states $|B_*(\Psi)\rangle$ for any open string field solution $\Psi$. The construction depends on a choice of a propagator strip. Modifying the propagator strip or performing a gauge transformation on the classical solution generically changes $|B_*(\Psi)\rangle$ by a BRST-exact term. We calculated $|B_*(\Psi)\rangle$ for various known analytic solutions choosing the Schnabl propagator strip and found that $|B_*(\Psi)\rangle$ precisely coincides with the BCFT boundary state $|B_\epsilon\rangle$ of the background that the solutions are expected to describe. This is the first construction of the full BCFT boundary state from solutions of open string field theory.

While we claim that the state $|B_*(\Psi)\rangle$ in general coincides with the BCFT boundary state $|B_\epsilon\rangle$ up to a possible BRST-exact term, such a term can be absent if the state $|B_*(\Psi)\rangle$ factorizes into the matter and ghost sectors as (7.27). We presented a sufficient condition on the solution $\Psi$ such that $|B_*(\Psi)\rangle$, constructed with the choice of the Schnabl propagator strip, factorizes in this way. It would be useful to understand better when the factorization (7.27) holds. Our analysis indicates that the remarkable simplifications associated with the choice of Schnabl propagator strip play an important role for the factorization (7.27). It follows from the analysis in [41] that all Riemann surfaces associated with closed string states of the form $\oint_s \Sigma(0,s)$ coincide in the Schnabl limit when $A_i$ in the definition (2.23) of $\Sigma(0,s)$ are wedge-based states. In fact, the resulting Riemann surface is an annulus whose modulus only depends on $s$. This outstanding feature was crucially important in this paper. We expect from the analysis in [13] that the simplifications associated with Schnabl’s propagator carry over into other projector-based propagators.

Because the boundary state is a basic object in BCFT, we believe that our construction of $|B_\epsilon\rangle$ from a solution $\Psi$ provides an important step towards establishing the map from classical solutions of open string field theory to BCFT’s. Partial success in the reverse map from BCFT’s to classical solutions was achieved in [20, 25, 27] for backgrounds connected by arbitrary marginal deformations. A systematic procedure to construct solutions from the BCFT operator that implements a change of boundary conditions along a segment on the boundary was presented for the bosonic string in [25] and for the superstring in [27]. Our ambitious goal is a complete understanding of the relation between BCFT’s and classical solutions of open string field theory.

The construction of the closed string state $|B_*(\Psi)\rangle$ is based on the representation (3.4) of the original BCFT boundary state $|B\rangle$ in terms of the half-propagator strip $P(0,s)$. The state $|B_*(\Psi)\rangle$ is obtained by replacing $P(0,s)$ in (3.4) with the half-propagator strip $P_*(0,s)$ associated with the background $\Psi$. This construction of $|B_*(\Psi)\rangle$ is the primary reason for our claim that the state $|B_*(\Psi)\rangle$ coincides with the BCFT boundary state $|B_\epsilon\rangle$ up to a possible
BRST-exact term, but we have not provided a proof for this claim. We have in addition examined two lines of argumentation. The first one is based on the limit \( s \to 0 \) discussed in section 4.4. Since \(|B_*(\Psi)\rangle\) changes only by a BRST-exact term as we vary the parameter \( s \), our claim follows if the relation (4.32) holds and if the claim (1.2) discussed in [37] is proven. The second argument is that our claim is a consequence of the expected, but not yet proven, background independence of a version of open-closed string field theory. We may obtain new insights from further efforts towards a rigorous proof for the claim.

It would be interesting to study the generalization of our construction to open superstring field theory in the WZW formulation by Berkovits [65, 66]. We can construct BRST-invariant closed string states in the superstring by replacing \( \Psi \) in \(|B_*(\Psi)\rangle\) with \( e^{-\Phi}Qe^{\Phi} \), where \( \Phi \) is the open superstring field and \( Q \) is the BRST operator in the superstring. We expect that the resulting states \(|B_*(e^{-\Phi}Qe^{\Phi})\rangle\) are related to the BCFT boundary state. It would be interesting to study such closed string states extending the discussion in [37] on gauge-invariant observables [67] for the superstring. The construction of the BCFT boundary state in the superstring from open superstring fields can then be used for consistency checks on solutions of open superstring field theory. For example, a solution for the tachyon vacuum in this theory has been proposed in [35], and it would be useful to examine if the state \(|B_*(e^{-\Phi}Qe^{\Phi})\rangle\) vanishes for this solution.

We hope that our construction of \(|B_*(\Psi)\rangle\) will pave the way to the study of closed string physics within string field theory. It may lead us to a novel formulation of open-closed string field theory, and the set of open-closed vertices encoded in \(|B_*(\Psi)\rangle\) is the first step in this direction. If we choose a propagator strip associated with a non-BPZ-even gauge condition, we obtain complex open-closed vertices. To obtain real vertices suitable for open-closed string field theory, it may be useful to examine if our construction can be generalized to the full propagator surface of regular linear \( b \)-gauges [42].

Since gravity is contained in the closed string sector, a consistent coupling of open strings to an off-shell closed string in the framework of string field theory can be thought of as a string theory generalization of the energy-momentum tensor. Apart from the reality issue we mentioned earlier, the state \(|B_*(\Psi)\rangle\) can thus be regarded as giving such a generalized energy-momentum tensor. Its expression in terms of the path-ordered exponential (3.14) is reminiscent of the energy-momentum tensor of noncommutative gauge theory in terms of open Wilson lines derived in [68]. While the on-shell part of \(|B_*(\Psi)\rangle\) is gauge-invariant, the off-shell part is not. We believe that information contained in its off-shell part, especially when \(|B_*(\Psi)\rangle\) coincides with the BCFT boundary state \(|B_*\rangle\), is useful in understanding the map from solutions to BCFT’s, but it is an important open problem whether physical observables are contained in the off-shell part in the context of string theory or of string field theory. For example, off-shell information in the BCFT boundary state was used in the study of the rolling tachyon by
Finally, the coupling of open string fields to closed string modes plays an important role in the AdS/CFT correspondence. The study of open string field theory with such open-closed vertices reviewed in section indicates that a large amount of the closed string physics can in principle be reproduced, and the results in this paper further provide a prospect that they might actually be calculable. We hope that exciting developments await us in this direction.

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A Marginal deformations in Schnabl gauge

A.1 The matter sector to all orders

In this appendix we derive (7.65). We generalize the calculations in the subsection to the following case:

\[ (-1)^k e^{\frac{\pi^2}{16}(L_0 + \tilde{L}_0)} \int_s \int_0^s ds_1 \int_{s_1}^s ds_2 \ldots \int_{s_{k-1}}^s ds_k P(0, s_1) \{ B_R(s_1), \Psi^{(n_1)} \} \]
\[ \times P(s_1, s_2) \{ B_R(s_2), \Psi^{(n_2)} \} P(s_2, s_3) \{ B_R(s_3), \Psi^{(n_3)} \} \ldots \]
\[ \times P(s_{k-1}, s_k) \{ B_R(s_k), \Psi^{(n_k)} \} P(s_k, s), \]

where

\[ \sum_{i=1}^k n_i = n. \]
The operator insertion in the natural $z$ frame is given by

$$
\frac{e^s}{e^s - 1} \int_0^s ds_1 \int_{s_1}^s ds_2 \ldots \int_{s_{k-1}}^s ds_k \prod_{i=1}^k \left[ \int_0^{t_1^{(i)}} dt_1^{(i)} \int_0^{t_2^{(i)}} dt_2^{(i)} \ldots \int_0^{t_{n-1}^{(i)} (e^{s_i})^{n_i}} dt_{n-1}^{(i)} \right] 
\times \prod_{a=1}^n V \left( \frac{e^{s'_a} + \ldots + e^{s'_a + e^{s_{a+1}} + \ldots + e^{s_n}}}{e^s - 1} \right),
$$

(A.3)

where

$$
s'_a = s_i + \ln t_{j-1}^{(i)} \quad \text{with} \quad t_0^{(i)} = 1
$$

(A.4)

for

$$
a = n_1 + n_2 + \ldots + n_{i-1} + j, \quad \text{with} \quad j = 1, 2, \ldots, n_i.
$$

(A.5)

The integral can be written in terms of $s'_a$ as follows:

$$
\frac{e^s}{e^s - 1} \int_{\Gamma(n_1, n_2, \ldots, n_k)} ds'_1 ds'_2 \ldots ds'_n \prod_{a=1}^n e^{s_a} V \left( \frac{e^{s'_1 + \ldots + e^{s'_a + e^{s_{a+1}} + \ldots + e^{s_n}}}}{e^s - 1} \right),
$$

(A.6)

where

$$
\int_{\Gamma(n_1, n_2, \ldots, n_k)} ds'_1 ds'_2 \ldots ds'_n = \prod_{i=1}^k \int_{s'_{d_i-1} + 1}^{s'_{d_i}} ds'_1 \int_{s'_{d_i-1}}^{s'_{d_i+1}} ds'_2 \int_{s'_{d_i}}^{s'_{d_i+2}} ds'_3 \int_{s'_{d_i+1}}^{s'_{d_i+3}} ds'_4 \int_{s'_{d_i+2}}^{s'_{d_i+4}} ds'_5 \int_{s'_{d_i+3}}^{s'_{d_i+6}} ds'_6
$$

(A.7)

with

$$
d_1 = 0, \quad d_i = \sum_{j=1}^{i-1} n_j \quad \text{for} \quad i > 1, \quad s'_{d_0 + 1} = 0.
$$

(A.8)

For example, when $(n_1, n_2, n_3) = (3, 2, 1)$, we have

$$
\int_{\Gamma(3, 2, 1)} ds'_1 ds'_2 \ldots ds'_6 = \int_0^s ds'_1 \int_{-\infty}^{s'_1} ds'_2 \int_{s'_1}^{s'_2} ds'_3 \int_{s'_2}^{s'_3} ds'_4 \int_{s'_3}^{s'_4} ds'_5 \int_{s'_4}^{s'_5} ds'_6.
$$

(A.9)

Note that $\Gamma(n_1, n_2, \ldots, n_k)$ satisfies the following relation:

$$
\int_{\Gamma(n_1, n_2, \ldots, n_k)} ds'_1 ds'_2 \ldots ds'_n \int_{-\infty}^{s'_n} ds'_{n+1}
\int_{\Gamma(n_1, n_2, \ldots, n_k)} ds'_1 ds'_2 \ldots ds'_n \int_{-\infty}^{s'_{n+1}} ds'_{n+1} + \int_{\Gamma(n_1, n_2, \ldots, n_k)} ds'_1 ds'_2 \ldots ds'_n \int_{s'_{n+1}}^{s'_{n+1} + 1} ds'_{n+1} = \int_{\Gamma(n_1, n_2, \ldots, n_{k-1}, n_{k+1})} ds'_1 ds'_2 \ldots ds'_{n+1} + \int_{\Gamma(n_1, n_2, \ldots, n_{k-1}, n_{k+1})} ds'_1 ds'_2 \ldots ds'_{n+1}.
$$

(A.10)

Using this relation recursively, the integration regions from all partitions are combined to give

$$
\sum_{n_1+n_2+\ldots+n_k=n} \int_{\Gamma(n_1, n_2, \ldots, n_k)} ds'_1 ds'_2 \ldots ds'_n = \int_0^s ds'_1 \int_{-\infty}^{s'_1} ds'_2 \int_{s'_1}^{s'_2} ds'_3 \int_{s'_2}^{s'_3} ds'_4 \int_{s'_3}^{s'_4} ds'_5 \int_{s'_4}^{s'_5} ds'_6.
$$

(A.11)
If we define
\[ u_a = \frac{e^{s} e^{s_1} + \ldots + e^{s} e^{s_a} + e^{s_{a+1}} + \ldots + e^{s_n}}{e^{s} - 1} \quad \text{for} \quad 1 \leq a \leq n, \quad (A.12) \]
the contributions from all partitions can be written in the following form:
\[
\frac{e^{s}}{e^{s} - 1} \int_{-\infty}^{s} ds'_{1} \int_{-\infty}^{s} ds'_{2} \int_{-\infty}^{s} ds'_{3} \ldots \int_{-\infty}^{s} ds'_{n} \prod_{a=1}^{n} e^{s_{a}} V \left( \frac{e^{s} e^{s_{1}} + \ldots + e^{s} e^{s_{a}} + e^{s_{a+1}} + \ldots + e^{s_{n}}}{e^{s} - 1} \right) 
= \int_{\Gamma(n)^{\prime}} du_{1} du_{2} \ldots du_{n} V(u_{1}) V(u_{2}) \ldots V(u_{n}). \quad (A.13)
\]
To see this, note that
\[
\frac{\partial(u_{1}, u_{2}, \ldots, u_{n})}{\partial(s'_{1}, s'_{2}, \ldots, s'_{n})} = \frac{1}{(e^{s} - 1)^{n}} \prod_{a=1}^{n} e^{s_{a}} = \frac{e^{s} (e^{s} - 1)^{n-1}}{e^{s} - 1} \prod_{a=1}^{n} e^{s_{a}}. \quad (A.14)
\]
Since
\[
e^{s_{1}} = u_{1} - e^{-s} u_{n}, \quad e^{s_{2}} = u_{2} - u_{1}, \quad e^{s_{3}} = u_{3} - u_{2}, \quad \ldots \quad e^{s_{n}} = u_{n} - u_{n-1}, \quad (A.15)
\]
the integration region \(\Gamma(n)^{\prime}\) can be characterized by
\[
1 \leq u_{1} - e^{-s} u_{n} \leq e^{s}, \quad 0 \leq u_{2} - u_{1} \leq e^{s}, \quad 0 \leq u_{3} - u_{2} \leq e^{s}, \quad \ldots \quad 0 \leq u_{n} - u_{n-1} \leq e^{s}. \quad (A.16)
\]
Using the identification \(z \sim e^{s} z\) in the natural \(z\) frame, this integral can also be written as
\[
\int_{\Gamma(n)} du_{1} du_{2} \ldots du_{n} V(u_{1}) V(u_{2}) \ldots V(u_{n}) \quad (A.17)
\]
with \(\Gamma(n)^{\prime}\) given by
\[
1 \leq e^{s} u_{1} - u_{n} \leq e^{s}, \quad 0 \leq u_{2} - u_{1} \leq 1, \quad 0 \leq u_{3} - u_{2} \leq 1, \quad \ldots \quad 0 \leq u_{n} - u_{n-1} \leq 1. \quad (A.18)
\]
This completes the derivation of (7.65).
A.2 Recovering the BCFT boundary state

The matter sector of the boundary state $|B_z(\Psi)\rangle$ with $\Psi$ being solutions for marginal deformations can be written in the $z$ frame with the identification $z \sim e^sz$ as follows:

$$\sum_{n=0}^{\infty} \lambda^n \int_{\Gamma(n)} du_1 du_2 \cdots du_n V(u_1) V(u_2) \cdots V(u_n) ,$$  \hspace{1cm} (A.19)

with $\Gamma(n)$ given in (A.18). In this appendix we show that

$$\int_{\Gamma(n)} du_1 du_2 \cdots du_n V(u_1) V(u_2) \cdots V(u_n) = \int_{1}^{e^s} du_1 \int_{u_1}^{e^s} du_2 \cdots \int_{u_{n-1}}^{e^s} du_n V(u_1) V(u_2) \cdots V(u_n) .$$  \hspace{1cm} (A.20)

Let us first consider the case with $n = 2$. We define a region $U(2)$ in the $(u_1, u_2)$ plane by

$$U(2) \equiv \{ (u_1, u_2) \mid u_1 \leq u_2 , u_2 \leq e^s u_1 \} .$$  \hspace{1cm} (A.21)

The region $\Gamma(2)$ is the subset of $U(2)$ given by

$$\Gamma(2) \equiv \{ (u_1, u_2) \mid 0 \leq u_2 - u_1 \leq 1 , 1 \leq e^s u_1 - u_2 \leq e^s \} .$$  \hspace{1cm} (A.22)

If we define $\Gamma(a_1, a_2) \subset U(2)$ by

$$\Gamma(a_1, a_2) \equiv \{ (u_1, u_2) \mid 0 \leq u_2 - u_1 \leq a_1 , 0 \leq e^s u_1 - u_2 \leq a_2 \} ,$$  \hspace{1cm} (A.23)

the region $\Gamma(2)$ can be written as

$$\Gamma(2) = \Gamma(1, e^s) - \Gamma(1, 1) .$$  \hspace{1cm} (A.24)

Consider the map $g : U(2) \rightarrow U(2)$ given by

$$g((u_1, u_2)) = (u_2, e^s u_1) .$$  \hspace{1cm} (A.25)

Note that the set of angles $\{\theta_1, \theta_2\}$ in the $\zeta$ frame that the points $(u_1, u_2)$ are mapped to via (6.12) is invariant under $g$. The region $\Gamma(a_1, a_2)$ is mapped under $g$ as

$$\Gamma(a_1, a_2) \xrightarrow{g} \Gamma(a_2, e^s a_1) .$$  \hspace{1cm} (A.26)

Thus the region $\Gamma(2)$ is mapped by a sequence of maps $g$ as follows:

$$\Gamma(2) = \Gamma(1, e^s) - \Gamma(1, 1) \xrightarrow{g} \Gamma(e^s, e^s) - \Gamma(1, e^s) \xrightarrow{g} \Gamma(e^s, e^{2s}) - \Gamma(e^s, e^s) \xrightarrow{g} \ldots .$$  \hspace{1cm} (A.27)

The map $g$ is invertible and its inverse is given by

$$g^{-1}((u_1, u_2)) = (e^{-s} u_2, u_1) .$$  \hspace{1cm} (A.28)
Under $g^{-1}$, the region $\Gamma(a_1, a_2)$ is mapped as

$$\Gamma(a_1, a_2) \xrightarrow{g^{-1}} \Gamma(e^{-s}a_2, a_1).$$  \hspace{1cm} (A.29)$$

Thus the region $\Gamma^{(2)}$ is mapped by a sequence of maps $g^{-1}$ as follows:

$$\begin{align*}
\Gamma^{(2)} &= \Gamma(1, e^s) - \Gamma(1, 1) \xrightarrow{g^{-1}} \Gamma(1, 1) - \Gamma(e^{-s}, 1) \\
&\quad \xrightarrow{g^{-1}} \Gamma(e^{-s}, 1) - \Gamma(e^{-s}, e^{-s}) \xrightarrow{g^{-1}} \Gamma(e^{-s}, e^{-s}) - \Gamma(e^{-2s}, e^{-s}) \xrightarrow{g^{-1}} \ldots .
\end{align*}$$  \hspace{1cm} (A.30)$$

Therefore, any point $(u_1, u_2) \in U^{(2)}$ can be mapped to $\Gamma^{(2)}$ either by a sequence of the map $g$ or by a sequence of the map $g^{-1}$. Let us denote this map by $G$. This map

$$G : U^{(2)} \to \Gamma^{(2)}$$  \hspace{1cm} (A.31)$$

is uniquely defined because the images of $\Gamma^{(2)}$ under different sequences of either $g$ or $g^{-1}$ do not intersect. Furthermore, $G$ is onto because it is the identity map when restricted to $(u_1, u_2) \in \Gamma^{(2)}$.

The region $\tilde{\Gamma}^{(2)}$ for the path-ordered exponential in (A.20) at $O(\lambda^2)$ is the subset of $U^{(2)}$ given by

$$\tilde{\Gamma}^{(2)} = \{ (u_1, u_2) \mid 1 \leq u_1 \leq u_2 \leq e^s \}.$$  \hspace{1cm} (A.32)$$

If we define $\Gamma(u_i \leq a) \subset U^{(2)}$ by

$$\Gamma(u_i \leq a) = \{ (u_1, u_2) \mid u_1 \leq u_2, u_2 \leq e^s u_1, u_i \leq a \},$$  \hspace{1cm} (A.33)$$

the region $\tilde{\Gamma}^{(2)}$ can be written as

$$\tilde{\Gamma}^{(2)} = \Gamma(u_2 \leq e^s) - \Gamma(u_1 \leq 1).$$  \hspace{1cm} (A.34)$$

Under the map $g$, the region $\Gamma(u_i \leq a)$ is mapped as

$$\begin{align*}
\Gamma(u_1 \leq a) &\xrightarrow{g} \Gamma(u_2 \leq e^s a), \quad \Gamma(u_2 \leq a) \xrightarrow{g} \Gamma(u_1 \leq a).
\end{align*}$$  \hspace{1cm} (A.35)$$

Thus the region $\tilde{\Gamma}^{(2)}$ is mapped by a sequence of maps $g$ as follows:

$$\begin{align*}
\tilde{\Gamma}^{(2)} &= \Gamma(u_2 \leq e^s) - \Gamma(u_1 \leq 1) \xrightarrow{g} \Gamma(u_1 \leq e^s) - \Gamma(u_2 \leq e^s) \\
&\quad \xrightarrow{g} \Gamma(u_2 \leq e^{2s}) - \Gamma(u_1 \leq e^s) \xrightarrow{g} \Gamma(u_1 \leq e^{2s}) - \Gamma(u_2 \leq e^{2s}) \xrightarrow{g} \ldots .
\end{align*}$$  \hspace{1cm} (A.36)$$

Under the inverse map $g^{-1}$, the region $\Gamma(u_i \leq a)$ is mapped as

$$\begin{align*}
\Gamma(u_1 \leq a) &\xrightarrow{g^{-1}} \Gamma(u_2 \leq a), \quad \Gamma(u_2 \leq a) \xrightarrow{g^{-1}} \Gamma(u_1 \leq e^{-s} a).
\end{align*}$$  \hspace{1cm} (A.37)$$
Thus the region $\tilde{\Gamma}^{(2)}$ is mapped by a sequence of maps $g^{-1}$ as follows:

\[
\tilde{\Gamma}^{(2)} = \Gamma(u_2 \leq e^s) - \Gamma(u_1 \leq 1) \xrightarrow{g^{-1}} \Gamma(u_1 \leq 1) - \Gamma(u_2 \leq 1) \\
\xrightarrow{g^{-1}} \Gamma(u_2 \leq 1) - \Gamma(u_1 \leq e^{-s}) \xrightarrow{g^{-1}} \Gamma(u_1 \leq e^{-s}) - \Gamma(u_2 \leq e^{-s}) \xrightarrow{g^{-1}} \ldots .
\] (A.38)

Therefore, any point $(u_1, u_2) \in U^{(2)}$ can be mapped to $\tilde{\Gamma}^{(2)}$ either by a sequence of maps $g$ or by a sequence of maps $g^{-1}$. Let us denote this map by $\tilde{G}$. This map

\[
\tilde{G} : U^{(2)} \to \tilde{\Gamma}^{(2)}
\] (A.39)

is uniquely defined because the images of $\tilde{\Gamma}^{(2)}$ under different sequences of either $g$ or $g^{-1}$ do not intersect. Furthermore, $\tilde{G}$ is onto because it is the identity map when restricted to $(u_1, u_2) \in \tilde{\Gamma}^{(2)}$.

We have thus constructed a map $G : U^{(2)} \to \Gamma^{(2)}$ and a map $\tilde{G} : U^{(2)} \to \tilde{\Gamma}^{(2)}$. Both maps are onto. We now define a map $H : \tilde{\Gamma}^{(2)} \to \Gamma^{(2)}$ that is the restriction of $G$ to $(u_1, u_2) \in \tilde{\Gamma}^{(2)}$:

\[
H = G \big|_{\tilde{\Gamma}^{(2)}}.
\] (A.40)

Similarly, we define $\tilde{H} : \Gamma^{(2)} \to \tilde{\Gamma}^{(2)}$ as the restriction of $\tilde{G}$ to $(u_1, u_2) \in \Gamma^{(2)}$:

\[
\tilde{H} = \tilde{G} \big|_{\Gamma^{(2)}}.
\] (A.41)

The composition of these two maps, $\tilde{H} \circ H$, is the identity map on $\tilde{\Gamma}^{(2)}$. To show this, assume the contrary, i.e., assume $(u'_1, u'_2) \neq (u_1, u_2)$ with

\[
(u'_1, u'_2) = \tilde{H} \circ H((u_1, u_2)).
\] (A.42)

As $\tilde{H} \circ H$ is built from sequences of $g$ and $g^{-1}$, the points $(u_1, u_2) \in \tilde{\Gamma}^{(2)}$ and $(u'_1, u'_2) \in \tilde{\Gamma}^{(2)}$ are related by some sequence of maps $g$ or $g^{-1}$. However, since

\[
g \circ \ldots \circ g^{(i\text{\ times})} \cap \tilde{\Gamma}^{(2)} = \emptyset \quad \text{for} \quad i \neq 0, \\
g^{-1} \circ \ldots \circ g^{-1}^{(j\text{\ times})} \cap \tilde{\Gamma}^{(2)} = \emptyset \quad \text{for} \quad j \neq 0, 
\] (A.43)

we conclude that they cannot be related by a nontrivial sequence and thus

\[
(u'_1, u'_2) = (u_1, u_2),
\] (A.44)

in contradiction with the assumption. Thus $\tilde{H} \circ H$ is indeed the identity map on $\tilde{\Gamma}^{(2)}$. Similarly, one can show that $H \circ \tilde{H}$ is the identity map on $\Gamma^{(2)}$. The maps $H$ and $\tilde{H}$ are therefore inverses.
of each other, and in particular they must be one-to-one and onto. We have thus constructed a map between the integration region \( \Gamma^{(2)} \) and the integration region \( \tilde{\Gamma}^{(2)} \),

\[
\tilde{H} : \Gamma^{(2)} \rightarrow \tilde{\Gamma}^{(2)}, \text{ one-to-one and onto.} \tag{A.45}
\]

Note that for each fixed \((u_1, u_2)\) this map is either a finite\(^{25}\) sequence of maps \(g\) or a finite sequence of maps \(g^{-1}\). Furthermore, recall that the set of angles \(\{\theta_1, \theta_2\}\) in the \(\zeta\) frame that the points \((u_1, u_2)\) map to, is invariant under the maps \(g\) and \(g^{-1}\). We can thus decompose \(\Gamma^{(2)}\) appropriately and map each piece of the region to reconstruct \(\tilde{\Gamma}^{(2)}\). We can explicitly perform this procedure for a given finite \(s\). We thus conclude that the integration regions \(\Gamma^{(2)}\) and \(\tilde{\Gamma}^{(2)}\) are identical:

\[
\int_{\Gamma^{(2)}} du_1 du_2 V(u_1) V(u_2) = \int_{\tilde{\Gamma}^{(2)}} du_1 du_2 V(u_1) V(u_2). \tag{A.46}
\]

This proves \(A.20\) for \(n = 2\).

This proof can be easily generalized to arbitrary \(n > 2\). We define a region \(U^{(n)}\) by

\[
U^{(n)} \equiv \left\{ (u_1, u_2, \ldots, u_n) \mid u_1 \leq u_2, u_2 \leq u_3, \ldots, u_{n-1} \leq u_n, u_n \leq e^s u_1 \right\}. \tag{A.47}
\]

The region \(\Gamma^{(n)}\) is the subset of \(U^{(n)}\) given by

\[
\Gamma^{(n)} = \left\{ (u_1, u_2, \ldots, u_n) \mid 0 \leq u_2 - u_1 \leq 1, 0 \leq u_3 - u_2 \leq 1, \ldots, 1 \leq e^s u_1 - u_n \leq e^s \right\}. \tag{A.48}
\]

If we define \(\Gamma(a_1, a_2, \ldots, a_n) \subset U^{(n)}\) by

\[
\Gamma(a_1, a_2, \ldots, a_n) \equiv \left\{ (u_1, u_2, \ldots, u_n) \mid 0 \leq u_2 - u_1 \leq a_1, 0 \leq u_3 - u_2 \leq a_2, \ldots, 0 \leq u_n - u_{n-1} \leq a_{n-1}, 0 \leq e^s u_1 - u_n \leq a_n \right\}, \tag{A.49}
\]

the region \(\Gamma^{(n)}\) can be written as

\[
\Gamma^{(n)} = \Gamma(1, 1, \ldots, 1) - \Gamma(1, 1, \ldots, 1). \tag{A.50}
\]

Consider the map \(g : U^{(n)} \rightarrow U^{(n)}\) given by

\[
g\left((u_1, u_2, \ldots, u_n)\right) = (u_2, u_3, \ldots, u_n, e^s u_1). \tag{A.51}
\]

Note that the set of angles \(\{\theta_1, \theta_2, \ldots, \theta_n\}\) in the \(\zeta\) frame that the points \((u_1, u_2, \ldots, u_n)\) are mapped to via \((6.12)\) is invariant under \(g\). The region \(\Gamma(a_1, a_2, \ldots, a_n)\) is mapped under \(g\) as

\[
\Gamma(a_1, a_2, \ldots, a_n) \xrightarrow{g} \Gamma(a_2, a_3, \ldots, a_n, e^s a_1). \tag{A.52}
\]

\(^{25}\)The maps \(g\) and \(g^{-1}\) have a fixed-point at the origin in the \((u_1, u_2)\) plane. However, as neither \(\Gamma^{(2)}\) nor \(\tilde{\Gamma}^{(2)}\) contains the origin, the map \(\tilde{H}\) is perfectly well defined and unaffected by this singularity.

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Thus the region $\Gamma^{(n)}$ is mapped by a sequence of maps $g$ as follows:

$$
\Gamma^{(n)} = \Gamma(1, 1, \ldots, 1, e^s) - \Gamma(1, 1, \ldots, 1)
\quad \xrightarrow{g} \quad \Gamma(1, 1, \ldots, 1, e^s) - \Gamma(1, 1, \ldots, 1, e^s)
\quad \xrightarrow{g} \quad \Gamma(1, 1, \ldots, 1, e^s, e^s) - \Gamma(1, 1, \ldots, 1, e^s, e^s)
\quad \xrightarrow{g} \quad \ldots
$$

(A.53)

The map $g$ is invertible and its inverse is given by

$$
g^{-1}\left((u_1, u_2, \ldots, u_n)\right) = (e^{-s}u_n, u_1, u_2, \ldots, u_{n-1})
$$

(A.54)

Under $g^{-1}$, the region $\Gamma(a_1, a_2, \ldots, a_n)$ is mapped as

$$
\Gamma(a_1, a_2, \ldots, a_n) \xrightarrow{g^{-1}} \Gamma(e^{-s}a_n, a_1, a_2, \ldots, a_{n-1})
$$

(A.55)

Thus the region $\Gamma^{(n)}$ is mapped by a sequence of maps $g^{-1}$ as follows:

$$
\Gamma^{(n)} = \Gamma(1, 1, \ldots, 1, e^s) - \Gamma(1, 1, \ldots, 1)
\quad \xrightarrow{g^{-1}} \quad \Gamma(1, 1, \ldots, 1) - \Gamma(e^{-s}, 1, 1, \ldots, 1)
\quad \xrightarrow{g^{-1}} \quad \Gamma(e^{-s}, 1, 1, \ldots, 1) - \Gamma(e^{-s}, e^{-s}, 1, 1, \ldots, 1)
\quad \xrightarrow{g^{-1}} \quad \ldots
$$

(A.56)

Therefore, any point $(u_1, u_2, \ldots, u_n) \in U^{(n)}$ can be mapped to $\Gamma^{(n)}$ either by a sequence of the map $g$ or by a sequence of the map $g^{-1}$. Let us denote this map by $G$. This map

$$
G : U^{(n)} \rightarrow \Gamma^{(n)}
$$

(A.57)

is uniquely defined because the images of $\Gamma^{(n)}$ under different sequences of either $g$ or $g^{-1}$ do not intersect. Furthermore, $G$ is onto because it is the identity map when restricted to $(u_1, u_2, \ldots, u_n) \in \Gamma^{(n)}$.

The region $\tilde{\Gamma}^{(n)}$ for the path-ordered exponential at $O(\lambda^n)$ is the subset of $U^{(n)}$ given by

$$
\tilde{\Gamma}^{(n)} \equiv \left\{(u_1, u_2, \ldots, u_n) \mid 1 \leq u_1 \leq u_2 \leq \ldots \leq u_n \leq e^s\right\}
$$

(A.58)

If we define $\Gamma(u_i \leq a) \subset U^{(n)}$ by

$$
\Gamma(u_i \leq a) \equiv \left\{(u_1, u_2, \ldots, u_n) \mid u_1 \leq u_2, u_2 \leq u_3, \ldots, u_{n-1} \leq u_n, u_n \leq e^su_1, u_i \leq a\right\},
$$

(A.59)

the region $\tilde{\Gamma}^{(n)}$ can be written as

$$
\tilde{\Gamma}^{(n)} = \Gamma(u_n \leq e^s) - \Gamma(u_1 \leq 1).
$$

(A.60)

Under the map $g$, the region $\Gamma(u_i \leq a)$ is mapped as

$$
\Gamma(u_1 \leq a) \xrightarrow{g} \Gamma(u_n \leq e^sa),
\Gamma(u_2 \leq a) \xrightarrow{g} \Gamma(u_1 \leq a), \quad \Gamma(u_3 \leq a) \xrightarrow{g} \Gamma(u_2 \leq a), \quad \ldots \quad \Gamma(u_n \leq a) \xrightarrow{g} \Gamma(u_{n-1} \leq a).
$$

(A.61)
Thus the region $\tilde{\Gamma}^{(n)}$ is mapped by a sequence of maps $g$ as follows:

$$
\tilde{\Gamma}^{(n)} = \Gamma(u_n \leq e^s) - \Gamma(u_1 \leq 1) \xrightarrow{g} \Gamma(u_{n-1} \leq e^s) - \Gamma(u_n \leq e^s) \\
\xrightarrow{g} \Gamma(u_{n-2} \leq e^s) - \Gamma(u_{n-1} \leq e^s) \xrightarrow{g} \ldots \\
\xrightarrow{g} \Gamma(u_1 \leq e^s) - \Gamma(u_2 \leq e^s) \xrightarrow{g} \Gamma(u_n \leq e^{2s}) - \Gamma(u_1 \leq e^s) \\
\xrightarrow{g} \Gamma(u_{n-1} \leq e^{2s}) - \Gamma(u_n \leq e^{2s}) \xrightarrow{g} \ldots
$$

(A.62)

Under the inverse map $g^{-1}$, the region $\Gamma(u_i \leq a)$ is mapped as

$$
\Gamma(u_1 \leq a) \xrightarrow{g^{-1}} \Gamma(u_2 \leq a), \quad \Gamma(u_2 \leq a) \xrightarrow{g^{-1}} \Gamma(u_3 \leq a), \quad \ldots \quad \Gamma(u_{n-1} \leq a) \xrightarrow{g^{-1}} \Gamma(u_n \leq a), \\
\Gamma(u_n \leq a) \xrightarrow{g^{-1}} \Gamma(u_1 \leq -s \cdot a).
$$

(A.63)

Thus the region $\tilde{\Gamma}^{(n)}$ is mapped by a sequence of maps $g^{-1}$ as follows:

$$
\tilde{\Gamma}^{(n)} = \Gamma(u_n \leq e^s) - \Gamma(u_1 \leq 1) \xrightarrow{g^{-1}} \Gamma(u_1 \leq 1) - \Gamma(u_2 \leq 1) \\
\xrightarrow{g^{-1}} \Gamma(u_2 \leq 1) - \Gamma(u_3 \leq 1) \xrightarrow{g^{-1}} \ldots \\
\xrightarrow{g^{-1}} \Gamma(u_{n-1} \leq 1) - \Gamma(u_n \leq 1) \xrightarrow{g^{-1}} \Gamma(u_n \leq 1) - \Gamma(u_1 \leq e^s) \\
\xrightarrow{g^{-1}} \Gamma(u_1 \leq e^{-s}) - \Gamma(u_2 \leq e^{-s}) \xrightarrow{g^{-1}} \ldots.
$$

(A.64)

Therefore, any point $(u_1, u_2, \ldots, u_n) \in U^{(n)}$ can be mapped to $\tilde{\Gamma}^{(n)}$ either by a sequence of maps $g$ or by a sequence of maps $g^{-1}$. Let us denote this map by $\tilde{G}$. This map

$$
\tilde{G} : U^{(n)} \to \tilde{\Gamma}^{(n)}
$$

(A.65)

is uniquely defined because the images of $\tilde{\Gamma}^{(n)}$ under different sequences of either $g$ or $g^{-1}$ do not intersect. Furthermore, $\tilde{G}$ is onto because it is the identity map when restricted to $(u_1, u_2, \ldots, u_n) \in \tilde{\Gamma}^{(n)}$.

We have thus constructed a map $G : U^{(n)} \to \Gamma^{(n)}$ and a map $\tilde{G} : U^{(n)} \to \tilde{\Gamma}^{(n)}$. Both maps are onto. We now define a map $H : \tilde{\Gamma}^{(n)} \to \Gamma^{(n)}$ that is the restriction of $G$ to $(u_1, u_2, \ldots, u_n) \in \tilde{\Gamma}^{(n)}$:

$$
H = G \big|_{\tilde{\Gamma}^{(n)}}.
$$

(A.66)

Similarly, we define $\tilde{H} : \Gamma^{(n)} \to \tilde{\Gamma}^{(n)}$ as the restriction of $\tilde{G}$ to $(u_1, u_2, \ldots, u_n) \in \Gamma^{(n)}$:

$$
\tilde{H} = \tilde{G} \big|_{\Gamma^{(n)}}.
$$

(A.67)
The composition of these two maps, $\tilde{H} \circ H$, is the identity map on $\tilde{\Gamma}^{(n)}$. To show this, assume the contrary, i.e., assume $(u'_1, u'_2, \ldots, u'_n) \neq (u_1, u_2, \ldots, u_n)$ with
\[
(u'_1, u'_2, \ldots, u'_n) = \tilde{H} \circ H \left( (u_1, u_2, \ldots, u_n) \right).
\]
As $\tilde{H} \circ H$ is built from sequences of $g$ and $g^{-1}$, the points $(u_1, u_2, \ldots, u_n) \in \tilde{\Gamma}^{(n)}$ and $(u'_1, u'_2, \ldots, u'_n) \in \tilde{\Gamma}^{(n)}$ are related by some sequence of maps $g$ or $g^{-1}$. However, since $g \circ \ldots \circ g \cap \tilde{\Gamma}^{(n)} = \emptyset$ for $i \neq 0$, $g^{-1} \circ \ldots \circ g^{-1} \cap \tilde{\Gamma}^{(n)} = \emptyset$ for $j \neq 0$,
\[
(A.68)
\]
we conclude that they cannot be related by a nontrivial sequence and thus
\[
(u'_1, u'_2, \ldots, u'_n) = (u_1, u_2, \ldots, u_n),
\]
in contradiction with the assumption. Thus $\tilde{H} \circ H$ is indeed the identity map on $\tilde{\Gamma}^{(n)}$. Similarly, one can show that $H \circ \tilde{H}$ is the identity map on $\Gamma^{(n)}$. The maps $H$ and $\tilde{H}$ are therefore inverses of each other, and in particular they must be one-to-one and onto. We have thus constructed a map between the integration region $\Gamma^{(n)}$ and the integration region $\tilde{\Gamma}^{(n)}$,
\[
\tilde{H} : \Gamma^{(n)} \rightarrow \tilde{\Gamma}^{(n)}, \text{ one-to-one and onto.}
\]
\[
(A.71)
\]
Note that for each fixed $(u_1, u_2, \ldots, u_n)$ this map is either a finite sequence of maps $g$ or a finite sequence of maps $g^{-1}$. Furthermore, recall that the set of angles $\{\theta_1, \theta_2, \ldots, \theta_n\}$ in the $\zeta$ frame that the points $(u_1, u_2, \ldots, u_n)$ map to, is invariant under the maps $g$ and $g^{-1}$. We can thus decompose $\Gamma^{(n)}$ appropriately and map each piece of the region to reconstruct $\tilde{\Gamma}^{(n)}$. We can explicitly perform this procedure for a given finite $s$. We thus conclude that the integration regions $\Gamma^{(n)}$ and $\tilde{\Gamma}^{(n)}$ are identical:
\[
\int_{\Gamma^{(n)}} du_1du_2\ldots du_n V(u_1)V(u_2)\ldots V(u_n) = \int_{\tilde{\Gamma}^{(n)}} du_1du_2\ldots du_n V(u_1)V(u_2)\ldots V(u_n).
\]
\[
(A.72)
\]
This completes our proof of the claim (A.20).

### B The solution $\Psi_L$

#### B.1 Ghost sector

Let us consider the ghost sector of $|B_z^{(k)}(\Psi_L)\rangle$ in the natural $z$ frame. The value of $a_0$ is
\[
a_0 = \frac{1}{e^s - 1} \sum_{i=1}^{k} n_i e^{s_i}.
\]
\[
(B.1)
\]
The ghost sector of the term \[(7.4)\] in the natural \(z\) frame can be written as

\[
\prod_{i=1}^{k} \left[ -\int_{C(s_i)} \frac{dz}{2\pi i} (z - \ell_i) b(z) c(e^{s_i} + \ell_i) - c(e^{s_i} + \ell_i) \int_{C(s_i)} \frac{dz}{2\pi i} (z - \ell_{i+1}) b(z) \right], \quad (B.2)
\]

where

\[
\ell_i = \sum_{j=1}^{i-1} n_j e^{s_j} + a_0, \quad \ell_1 = a_0. \quad (B.3)
\]

This can be calculated as

\[
\prod_{i=1}^{k} \left[ -\int_{C(s_i)} \frac{dz}{2\pi i} (z - \ell_i) b(z) c(e^{s_i} + \ell_i) - c(e^{s_i} + \ell_i) \int_{C(s_i)} \frac{dz}{2\pi i} (z - \ell_{i+1}) b(z) \right]
\]

\[
= \prod_{i=1}^{k} \left[ \oint \frac{dz}{2\pi i} (z - \ell_i) b(z) c(e^{s_i} + \ell_i) + n_i e^{s_i} c(e^{s_i} + \ell_i) B^+_R \right] \quad (B.4)
\]

\[
= \prod_{i=1}^{k} e^{s_i} \left[ 1 + n_i c(e^{s_i} + \ell_i) B^+_R \right].
\]

Using the same manipulations as in \((7.19)\) we find

\[
c(t_1) B^+_R c(t_2) B^+_R \ldots c(t_m) B^+_R = (-1)^{m-1} c(t_1) B^+_R = \frac{(-1)^{m-1}}{e^s - 1}. \quad (B.5)
\]

Therefore, we have

\[
\prod_{i=1}^{k} e^{s_i} \left[ 1 + n_i c(e^{s_i} + \ell_i) B^+_R \right] = \Delta_k \prod_{i=1}^{k} e^{s_i} \quad (B.6)
\]

with

\[
\Delta_k = 1 + \frac{1}{e^s - 1} - \frac{1}{e^s - 1} \prod_{i=1}^{k} (1 - n_i). \quad (B.7)
\]

**B.2 Measure**

In this appendix we calculate the Jacobian

\[
\prod_{i=1}^{k} \left. \frac{\partial(t_1^{(1)}, t_2^{(2)}, \ldots, t_k^{(k)})}{\partial(s_1, s_2, \ldots, s_k)} \right|_{t_i^{(i)}}\quad (B.8)
\]

where

\[
t_i^{(i)} = \sum_{j=1}^{i-1} n_j e^{s_j} + e^{s_i} + a_0 \quad (B.9)
\]
with $a_0$ given in (B.1). The derivative of $a_0$ with respect to $s_j$ is given by

$$
\frac{\partial a_0}{\partial s_j} = \frac{n_j}{e^s - 1} e^{s_j} = b_j e^{s_j}, \quad (B.10)
$$

where

$$
b_j \equiv \frac{n_j}{e^s - 1}. \quad (B.11)
$$

We define $\tilde{\Delta}_k$ by

$$
\frac{\partial (t_1^{(1)}, t_1^{(2)}, \ldots, t_1^{(k)})}{\partial (s_1, s_2, \ldots, s_k)} = \tilde{\Delta}_k \prod_{i=1}^{k} e^{s_i}. \quad (B.12)
$$

It follows from

$$
\frac{\partial t_i^{(i)}}{\partial s_j} = \begin{cases} 
(n_j + b_j) e^{s_j} & \text{for } j < i, \\
(1 + b_j) e^{s_j} & \text{for } j = i, \\
b_j e^{s_j} & \text{for } j > i
\end{cases} \quad (B.13)
$$

that

$$
\tilde{\Delta}_k = \begin{vmatrix}
1 + b_1 & b_2 & b_3 & \cdots & b_{k-1} & b_k \\
1 + b_1 & 1 + b_2 & b_3 & \cdots & b_{k-1} & b_k \\
n_1 + b_1 & n_2 + b_2 & 1 + b_3 & \cdots & b_{k-1} & b_k \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
n_1 + b_1 & n_2 + b_2 & n_3 + b_3 & \cdots & 1 + b_{k-1} & b_k \\
n_1 + b_1 & n_2 + b_2 & n_3 + b_3 & \cdots & n_{k-1} + b_{k-1} & 1 + b_k
\end{vmatrix}. \quad (B.14)
$$

Let us prove that

$$
\tilde{\Delta}_k = \Delta_k = 1 + \frac{1}{e^s - 1} - \frac{1}{e^s - 1} \prod_{i=1}^{k} (1 - n_i). \quad (B.15)
$$

For $k = 1$, we have

$$
\tilde{\Delta}_1 = 1 + b_1 = 1 + \frac{n_1}{e^s - 1} = \Delta_1. \quad (B.16)
$$

For $k > 1$, we find that

$$
\tilde{\Delta}_k = \begin{vmatrix}
1 + b_1 & \cdots & b_{k-1} & 0 \\
\vdots & \ddots & \vdots & \vdots \\
n_1 + b_1 & \cdots & 1 + b_{k-1} & 0 \\
n_1 + b_1 & \cdots & n_{k-1} + b_{k-1} & 1
\end{vmatrix} + b_k \begin{vmatrix}
1 + b_1 & \cdots & b_{k-1} & 1 \\
\vdots & \ddots & \vdots & \vdots \\
n_1 + b_1 & \cdots & 1 + b_{k-1} & 1 \\
n_1 + b_1 & \cdots & n_{k-1} + b_{k-1} & 1
\end{vmatrix}. \quad (B.17)
$$
The first term on the right-hand side is $\tilde{\Delta}_{k-1}$. The determinant in the second term can be calculated as follows:

$$
\begin{vmatrix}
1 + b_1 & b_2 & b_3 & \ldots & b_{k-1} & 1 \\
1 + b_1 & 1 + b_2 & b_3 & \ldots & b_{k-1} & 1 \\
1 + b_1 & n_2 + b_2 & 1 + b_3 & \ldots & b_{k-1} & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
n_1 + b_1 & n_2 + b_2 & n_3 + b_3 & \ldots & 1 + b_{k-1} & 1 \\
n_1 + b_1 & n_2 + b_2 & n_3 + b_3 & \ldots & n_{k-1} + b_{k-1} & 1
\end{vmatrix}
$$

(B.18)

$$
\begin{vmatrix}
1 - n_1 & -n_2 & -n_3 & \ldots & -n_{k-1} & 0 \\
0 & 1 - n_2 & -n_3 & \ldots & -n_{k-1} & 0 \\
0 & 0 & 1 - n_3 & \ldots & -n_{k-1} & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 1 - n_{k-1} & 0 \\
n_1 + b_1 & n_2 + b_2 & n_3 + b_3 & \ldots & n_{k-1} + b_{k-1} & 1
\end{vmatrix}
$$

$$
= \prod_{i=1}^{k-1} (1 - n_i).
$$

We therefore have

$$
\tilde{\Delta}_k = \tilde{\Delta}_{k-1} + b_1 \prod_{i=1}^{k-1} (1 - n_i) = \tilde{\Delta}_{k-1} + \frac{n_k}{e^\sigma - 1} \prod_{i=1}^{k-1} (1 - n_i).
$$

(B.19)

On the other hand, it is easy to see that

$$
\Delta_k - \Delta_{k-1} = \frac{n_k}{e^\sigma - 1} \prod_{i=1}^{k-1} (1 - n_i)
$$

(B.20)

for $k > 1$. We thus conclude that

$$
\tilde{\Delta}_k = \Delta_k.
$$

(B.21)

### B.3 Proof of (7.81)

In this appendix we prove the claim (7.81). We consider an arbitrary point $\{t_1^{(1)}, \ldots, t_n^{(k)}\}$ in the integration region $\Gamma(\vec{n})$ of a partition $\vec{n} = (n_1, \ldots, n_k)$ contributing to $|B_k(\Psi_L)|$ at $O(\lambda^n)$. The insertion points $\{t_1^{(1)}, \ldots, t_n^{(k)}\}$ in the $z$ frame are mapped to the unit circle in the $\zeta$ frame as

$$
\{t_1^{(1)}, \ldots, t_n^{(k)}\} \rightarrow \{e^{i\theta_1}, \ldots, e^{i\theta_{q-1}}, e^{i\theta_q}, e^{i\theta_{q+1}}, \ldots, e^{i\theta_n}\}.
$$

(B.22)

We will show that for any

$$
\theta_{q-1} \leq \tilde{\theta} \leq \theta_{q+1},
$$

(B.23)

we can find a partition $\vec{n}$ such that a point in its integration region $\Gamma(\vec{n})$ is mapped to the positions

$$
\{e^{i\theta_1}, \ldots, e^{i\theta_{q-1}}, e^{i\theta}, e^{i\theta_{q+1}}, \ldots, e^{i\theta_n}\}
$$

(B.24)
in the $\zeta$ frame. We will prove this claim by showing that we can determine the required partition $\vec{n}$ starting from the original partition $\vec{n}$ as we continuously vary $\tilde{\theta}$ away from $\theta_q$. For the following analysis it is convenient to recall that the integration region of an arbitrary partition $\vec{n}$. The position of the first $V$ insertion associated with $\Psi^{(n_i)}$ in the natural $z$ frame is given by
\[
t_1^{(i)} = \sum_{j=1}^{i-1} n_j e^{s_j} + e^{s_i} + a_0,
\]
while the integration region for the remaining $n_i - 1$ insertions, which we will denote as internal insertions in the following, are given by
\[
t_j^{(i)} - t_{j-1}^{(i)} \leq t_j^{(i)} \leq t_1^{(i)} + e^{s_i}(j-1) \quad \text{for } j \geq 2.
\]
Let us denote the insertion position in the original partition $\vec{n}$ which is mapped to the angle $\theta_q$ by $t_j^{(i)}$. We need to distinguish several cases.

$V(t_j^{(i)})$ is an internal insertion ($j \geq 2$)

Consider first a variation which decreases $\tilde{\theta}$, i.e., $\tilde{\theta} \leq \theta_q$. If $\theta_{q-1} = \theta_q$ then we are already at the lower end of the interval (B.23) and we are done. If $\theta_{q-1} < \theta_q$ then $t_j^{(i)} > t_{j-1}^{(i)}$, so the internal insertion is not yet at the lower boundary of its integration region (B.26). Varying $t_j^{(i)}$ does not affect any other positions, so we conclude that we can find a configuration within the exact same partition $\vec{n}$ for any
\[
\theta_{q-1} \leq \tilde{\theta} \leq \theta_q.
\]
Eventually, the internal insertion $V(t_j^{(i)})$ collides with the previous insertion $V(t_{j-1}^{(i)})$, which is precisely one of the boundaries in the interval (B.23) that we expected to find.

Now consider increasing $\tilde{\theta}$ into the range $\tilde{\theta} > \theta_q$. When increasing $\tilde{\theta}$ by increasing $t_j^{(i)}$, eventually two things can happen: if $j < n_i$ and $t_{j+1}^{(i)} \leq t_j^{(i)} + e^{s_i}(j-1)$, it follows from (B.26) that $V(t_j^{(i)})$ can collide with the next $V$ insertion at $t_{j+1}^{(i)}$ which corresponds to the upper boundary of the interval (B.23) and we are done. Otherwise, $t_j^{(i)}$ eventually hits its upper limit of integration at $t_j^{(i)} = t_1^{(i)} + e^{s_i}(j-1)$. Within this partition $\vec{n}$, the integration region stops although we have not encountered an operator collision. So this integration region must smoothly match with another integration region in a different partition $\vec{n}$. This is indeed the case. Intuitively, this can be understood as a “breaking” of the solution insertion $\Psi^{(n_i)}$ into two pieces when $t_j^{(i)}$ becomes too large. We now have two solution insertions, $\Psi^{(j-1)}$ and $\Psi^{(n_i-j+1)}$, with a new half-propagator strip opening up between them. The relevant partition is thus obtained by replacing
\[
\cdots \mathcal{P}(s_{i-1}, s_i) \Psi^{(n_i)} \mathcal{P}(s_i, s_{i+1}) \cdots \rightarrow \cdots \mathcal{P}(s_{i-1}, s_i) \Psi^{(j-1)} \mathcal{P}(s_i, \tilde{s}) \Psi^{(n_i-j+1)} \mathcal{P}(\tilde{s}, s_{i+1}) \cdots.
\]

(B.28)

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We thus have
\[
\vec{n} = \{\ldots, n_{i-1}, n_i, n_{i+1}, \ldots\} \rightarrow \vec{n} = \{\ldots, n_{i-1}, (j-1), (n_{i-1} - j + 1), n_{i+1}, \ldots\}. \quad (B.29)
\]
It is easy to see that for \( t_j^{(i)} = t_1^{(i)} + e^{s_i}(j - 1) \) in the first partition, the operator insertions match smoothly to \( \tilde{s} = s_i \) in the second partition. The boundary of the integration region \( \Gamma(\vec{n}) \) at \( t_j^{(i)} = t_1^{(i)} + e^{s_i}(j - 1) \) thus matches smoothly to the boundary of the integration region \( \Gamma(\vec{n}) \).

In the new partition \( \vec{n} \), the \( \mathcal{V} \) insertion at angle \( \tilde{\theta} \) originates from the first \( \mathcal{V} \) insertion in the solution \( \Psi^{(n_i - j + 1)} \). To see that we can then still continue varying \( \tilde{\theta} \), we turn to the second case.

**V(\( t_j^{(i)} \)) is the first insertion in a solution (\( j = 1 \))**

The insertion position \( t_1^{(i)} \) is not independently integrated over in our parameterization using integrals over \( s_i \). In fact, to move \( t_1^{(i)} \) while keeping all positions fixed, we generically need to vary all \( s_m \). Recalling (B.13), we find
\[
\frac{\partial s_m}{\partial t_1^{(i)}} > 0 \quad \text{for all} \quad 1 \leq m, i \leq k. \quad (B.30)
\]
Furthermore, we note that the integral regions (B.26) of *internal* insertions depend on the \( s_m \).

As we vary \( \tilde{\theta} \) and thus \( t_1^{(i)} \), the following things can happen.

1. **\( V(t_1^{(i)}) \) can collide with \( V(t_2^{(i)}) \)**
   This can happen if \( n_i \geq 2 \) and constitutes the upper boundary of the interval (B.23).

2. **An internal insertions hits the upper boundary of its integration**
   As mentioned above, this is caused by the change in integration regions for internal insertions as we vary \( t_1^{(i)} \). We have encountered such a situation before in the previous subsection. Denote the position of the insertion which reaches its upper limit of integration by \( t_j^{(m)} \) with \( j \geq 2 \). Just as above we can match this configuration smoothly by breaking the affected solution into two pieces:
\[
\mathcal{P}(s_{m-1}, s_m)\Psi^{(n_m)}\mathcal{P}(s_m, s_{m+1}) \rightarrow \mathcal{P}(s_{m-1}, s_m)\Psi^{(j-1)}\mathcal{P}(s_m, s)\Psi^{(n_m - j + 1)}\mathcal{P}(s, s_{m+1}). \quad (B.31)
\]
This corresponds to the change of partition
\[
\vec{n} = \{\ldots, n_{m-1}, n_m, n_{m+1}, \ldots\} \rightarrow \vec{n} = \{\ldots, n_{m-1}, (j-1), (n_{m-j+1}), n_{m+1}, \ldots\}. \quad (B.32)
\]
For \( t_j^{(m)} = t_1^{(m)} + e^{s_m}(j - 1) \) in the partition \( \vec{n} \), the operator insertions match smoothly to \( \tilde{s} = s_m \) in the partition \( \vec{n} \).
• $s_m \rightarrow s_{m+1}$ for some $1 \leq m < k$

This is in a sense the opposite case to the ones we have encountered so far. Instead of breaking solutions, in this case solutions merge. We match smoothly to a new configuration

$$\ldots \mathcal{P}(s_{m-1}, s_m) \Psi^{(n_m)} \mathcal{P}(s_m, s_{m+1}) \Psi^{(n_{m+1})} \ldots \rightarrow \ldots \mathcal{P}(s_{m-1}, s_{m+1}) \Psi^{(n_m+n_{m+1})} \ldots .$$

(B.33)

This corresponds to the change of partition

$$\vec{n} = \{ \ldots, n_{m-1}, n_m, n_{m+1}, n_{m+2}, \ldots \} \rightarrow \tilde{\vec{n}} = \{ \ldots, n_{m-1}, (n_m + n_{m+1}), n_{m+2}, \ldots \} .$$

(B.34)

Thus there is no longer an integral over $s_m$ present – instead the solution insertion $\Psi^{(n_m+n_{m+1})}$ carries one more internal integral than the previous solution insertions $\Psi^{(n_m)}$ and $\Psi^{(n_{m+1})}$ combined. Note that if $m+1 = i$, then the insertion $V(t^{(i)}_1)$ which we are varying corresponds to an internal insertion in the new partition $\tilde{\vec{n}}$ and we have to keep track of the variation of its position as we did in the previous subsection. If not, we continue the analysis of this subsection with the new partition $\tilde{\vec{n}}$.

• $s_k \rightarrow s$

As we increase $\tilde{\theta}$ and thus increase $t^{(i)}_1$, $s_k$ also increases because

$$\frac{\partial s_k}{\partial t^{(i)}_1} > 0 .$$

(B.35)

Thus eventually we can hit its upper limit of integration $s_k = s$, and the last half-propagator $\mathcal{P}(s_k, s)$ in the partition (7.74) collapses. As we generically do not have an operator collision in this configuration, we need to match it smoothly to a different partition. But in fact, using the cyclic property of $\oint_s$, we can rewrite this configuration as

$$\left[ \ldots \oint_s \mathcal{P}(0, s_1) \ldots \mathcal{P}(s_{k-1}, s_k) \Psi^{(n_k)} \mathcal{P}(s_k, s) \right]_{s_k = s} \rightarrow \left[ \ldots \oint_s \mathcal{P}(0, s_0) \Psi^{(n_k)} \mathcal{P}(s_0, s_1) \ldots \mathcal{P}(s_{k-1}, s) \right]_{s_0 = 0} .$$

(B.36)

After cyclic index relabeling $i \rightarrow i + 1$, we again obtain a partition of the form (7.74) with $s_1 = 0$. This corresponds to the change of partition

$$\vec{n} = \{ n_1, \ldots, n_{k-1}, n_k \} \rightarrow \tilde{\vec{n}} = \{ n_k, n_1, \ldots, n_{k-1} \} .$$

(B.37)

We can now continue to increase $\tilde{\theta}$, which is now represented by the position $t^{(i+1)}_1$ in $\tilde{\vec{n}}$.

As we increase $\tilde{\theta}$, the new Schwinger parameter $s_1$ leaves its lower boundary $s_1 = 0$ and
enters its allowed region $s_1 > 0$, because
\[
\frac{\partial s_1}{\partial t_{1+1}} > 0. \tag{B.38}
\]

- $s_1 \to 0$
  
  We can hit $s_1 = 0$ when decreasing $\tilde{\theta}$. This is a reversed situation of the collision $s_k \to s$ and can be dealt with in the exact same way.

We conclude that we can continue to vary $\tilde{\theta}$ throughout the interval $\left[B.23\right]$ while keeping all other insertion angles fixed. This completes the proof of $\left[7.81\right]$.

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