Integrability and concentration of the truncated variation for the sample paths of fractional Brownian motions, diffusions and Lévy processes

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For a real càdlàg function \( f \) defined on a compact interval, its truncated variation at the level \( c > 0 \) is the infimum of total variations of functions uniformly approximating \( f \) with accuracy \( c/2 \) and (in opposite to the total variation) is always finite. In this paper, we discuss exponential integrability and concentration properties of the truncated variation of fractional Brownian motions, diffusions and Lévy processes. We develop a special technique based on chaining approach and using it we prove Gaussian concentration of the truncated variation for certain class of diffusions. Further, we give sufficient and necessary condition for the existence of exponential moment of order \( \alpha > 0 \) of truncated variation of Lévy process in terms of its Lévy triplet.

Keywords: diffusions; Gaussian processes; Lévy processes; sample boundedness; truncated variation

1. Introduction

Let \( X = (X(t))_{t \geq 0} \) be a real valued stochastic process with càdlàg trajectories. In general, the total path variation of \( X \) on the compact interval \( [a, b] \subset [0, +\infty) \), defined as

\[
TV(X, [a, b]) = \sup_n \left( \sup_{a \leq t_0 < t_1 < \cdots < t_n \leq b} \sum_{i=1}^n |X(t_i) - X(t_{i-1})| \right),
\]

may be (and in many most important cases is) almost surely infinite. However, in the neighborhood of every càdlàg path we may easily find a function with finite total variation.

Let \( f \) be a càdlàg function \( f : [a, b] \to \mathbb{R} \) and let \( c > 0 \). The natural question arises, what is the smallest possible (or the greatest lower bound for the) total variation of functions from the ball \( B(f, c/2) = \{ g : \|f - g\|_\infty \leq c/2 \} \), where \( \|f - g\|_\infty := \sup_{s \in [a, b]} |f(s) - g(s)| \). Some bound
from below reads as
\[ \text{TV}(g, [a, b]) \geq \text{TV}^c(f, [a, b]), \]
where
\[ \text{TV}^c(f, [a, b]) := \sup_n \sup_{a \leq t_0 < t_1 < \cdots < t_n \leq b} \sum_{i=1}^{n} \max\{|f(t_i) - f(t_{i-1})| - c, 0\} \tag{1.1} \]
and follows immediately from the inequality
\[ |g(t_i) - g(t_{i-1})| \geq \max\{|f(t_i) - f(t_{i-1})| - c, 0\}. \]
It is possible to show (cf. Łochowski [11]) that in fact we have equality
\[ \inf\{\text{TV}(g, [a, b]) : \|f - g\|_{\infty} \leq c/2\} = \text{TV}^c(f, [a, b]) \tag{1.2} \]
attained for some function \( f^c \) from the ball \( B(f, c/2) \).

**Remark 1.** Since we deal with càdlàg functions, a more natural setting of our problem would be the investigation of
\[ \inf\{\text{TV}(g, [a, b]) : g \text{ càdlàg}, d_D(f, g) \leq c/2\}, \]
where \( d_D \) denotes the Skorohod metric. Since the total variation does not depend on the (continuous and strictly increasing) change of argument and the function \( f^c \) minimizing \( \text{TV}(g, [a, b]) \) appears to be a càdlàg one, solutions of both problems coincide.

The quantity (1.1) is called truncated variation and it is finite for any càdlàg function, since every such a function may be uniformly approximated by step functions. Moreover, the truncated variation is a continuous and convex function of the parameter \( c > 0 \) (cf. Łochowski [11]) and it obviously tends to the total variation as \( c \downarrow 0 \). For a process with paths with almost surely infinite total variation may be of interest to assess the rate at which \( \text{TV}^c \) diverges to infinity.

This was done so far for continuous semimartingales and it appears (cf. Łochowski and Miłoś [12]) that for any continuous semimartingale \( X \) we have that
\[ c \cdot \text{TV}^c(X, [a, b]) \rightarrow_{c \downarrow 0} \langle X \rangle_b - \langle X \rangle_a \quad \text{almost surely,} \tag{1.3} \]
where \( \langle \cdot \rangle \) denotes the quadratic variation of \( X \). The truncated variation appears also implicitly in the paper Picard [14] where it corresponds to the double Lebesgue measure \( L^c \) of a trimmed tree at the level \( c \), associated with a càdlàg path. In Picard [14] there were established deep connections of this measure, the variation index and the upper box (or Minkowski) dimension, as well as the counterparts of (1.3) in terms of \( L^c \) for fractional Brownian motions and stable Lévy processes.

For \( t \geq 0 \) denote \( \text{TV}^c(X, t) = \text{TV}^c(X, [0, t]) \). For \( X \) being the unique strong solution of the equation \( X_0 = 0, dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dW_t, t \in [0, S] \), driven by a standard Brownian motion
Integrability of the truncated variation

where $\tilde{W}$ is a standard Brownian motion independent from $W$ and the convergence $\Rightarrow$ is understood as the weak functional convergence in $C([0, S], \mathbb{R})$ topology.

The truncated variation is more informative than $p$-variation, since the latter may be described in terms of the asymptotic properties of $TV_c$ as $c \downarrow 0$ but for any fixed $c > 0$, $TV_c(X, S)$ is a proper random variable and it is possible to consider its distribution. For $X = W$ and fixed $S, c > 0$ convergence result (1.4) seems to indicate very strong concentration of $TV_c(W, S)$ around $S/c$, but it still does not tell anything about the tail probabilities of the functional considered.

These observations motivated us to study the integrability and concentration properties of the truncated variation in greater detail. Some investigation into this direction was already undertaken in Łochowski [10], where the existence of the moment generating function of the truncated variation of Brownian motion with drift on the whole real line was proven. In this paper, we obtain much stronger – Gaussian concentration result, by which we mean the integrability of $\exp(\alpha TV_c(X, S)^2)$ for some positive $\alpha$.

Another incentive for the study of the magnitude of truncated variation for possibly broad class of processes is the pathwise approach to stochastic integration. In Łochowski [13], it was shown that when both – integrand and integrator are semimartingales then it is possible to define the stochastic integral, with some correction term, as an almost sure limit of pathwise Lebesgue–Stieltjes integrals. The construction utilizes uniform approximation of the integrator with finite variation processes. The truncated variation gives the magnitude of such integrals, more precisely

$$\inf \|X-X^c\|_\infty \leq c/2 \sup \|Y\|_\infty \int_0^S Y^- dX^c = \inf \|X-X^c\|_\infty \leq c/2 TV(X^c, S) = TV(X, S),$$

where the supremum is over all càdlàg processes $Y$ with absolute value uniformly bounded by 1 and the infimums are over all pathwise càdlàg approximations $X^c$ of $X$ such that $\|X-X^c\|_\infty := \sup_{t \geq 0} |X(t) - X^c(t)| \leq c/2$.

In this paper, we study the magnitude of the truncated variation for a broad class of stochastic processes, including Gaussian processes, among them fractional Brownian motions, and diffusions. Further we also consider Lévy processes. Our main goal is to describe the tail behavior of $TV_c(X, S)$ assuming that $X$ satisfies some increment condition. We use various techniques depending on the assumption we make.

At the beginning, we use the chaining concept, we assume that $X$ satisfies some exponential integrability condition on increments and deduce the exponential integrability of the truncated variation (e.g., diffusions with bounded covariance and drift coefficients). The chaining approach was first used to study problems of sample boundedness of processes on the general index space Fernique [4,5]. The method was developed to give the full description of classes of processes that are sample bounded, under certain integrability condition Bednorz [1,2], Bednorz [3], Ledoux and Talagrand [8], Talagrand [17], and the small ball probability Li and Shao [9].
For a comprehensive study where many analytical examples are given, see Talagrand [18]. In our study, we need some modification of this idea, since we are interested in bounding the supremum of special sums of increments, not the supremum over increments itself. Therefore, we have to invent a special random variable of exponential integrability that bounds the truncated variation.

Our main guiding example is the class of fractional Brownian motions, that is, centered Gaussian processes $W_H$, $H \in (0, 1)$, starting from 0 and such that $\mathbb{E}|W_H(t) - W_H(s)|^2 = |t - s|^{2H}$. One of the corollaries we get is the following concentration inequality

$$P(\text{TV}_c(W_H, S) \geq c^{(H-1)/H}S(A_H + B_H u)) \leq C_H \exp(-u^{2H}),$$

for $u \geq 0$, where $A_H, B_H, C_H$ are constants; moreover, for $H \geq \frac{1}{2}$ one can set $C_H = 1$. By the homogeneity of increments, we deduce that for $Sc^{-1/H} \geq 2$, $\mathbb{E}\text{TV}^c(W_H, S)$ is comparable with $c^{(H-1)/HS}$ and in this way we prove that for $u \geq 0$,

$$P(\text{TV}^c(W_H, S) \geq \mathbb{E}\text{TV}^c(W_H, S)(\tilde{A}_H + \tilde{B}_Hu)) \leq \tilde{C}_H \exp(-u^{2H}),$$

for some constants $\tilde{A}_H, \tilde{B}_H, \tilde{C}_H$ (again $\tilde{C}_H = 1$ for $H \geq \frac{1}{2}$). In fact, any process with similar properties as the fractional Brownian motion, that is, satisfying some boundedness condition of the increments (inequality (2.2)) may be treated by our method.

Next, we turn to investigate the standard Brownian motion, that is, $W = W_{1/2}$, and diffusions driven by it. Here we can improve our result using the Markov property. It turns out that for Markov processes with moderate growth some local exponential integrability can be extended to the global one. Note that (1.5) implies the existence of the Laplace transform $\mathbb{E}\exp(\alpha \text{TV}_c(W, S))$ for sufficiently small $\alpha > 0$; assuming the Markov property for diffusions with moderate growth we get the estimate for the Laplace transform of their truncated variations on the whole real line. The main result we get this way is Theorem 2, which for a standard Brownian motion and $Sc^{-2} \geq 2$ implies the following concentration inequality

$$P(\text{TV}_c(W, S) \geq \tilde{A}\mathbb{E}\text{TV}^c(W, S) + \tilde{B}\sqrt{Su}) \leq \exp(-u^2),$$

for $u \geq 0$, here $\tilde{A}, \tilde{B}$ are universal constants. Therefore, the Gaussian concentration holds for the truncated variation of the standard Brownian motion. Our result gives better understanding of the already mentioned result (1.4) from which follows that $S^{-1/2}(\text{TV}_c(W, S) - S/c)$ converges in distribution to $\mathcal{N}(0, 1/3)$ as $c \downarrow 0$.

We conclude the paper by proving sufficient and necessary condition for the finiteness of $\mathbb{E}\exp(\alpha \text{TV}^c(X, S))$ for a Lévy process $X$, in terms of its generating triplet. Here we apply the method of level crossing stopping times.

The structure of the paper is as follows. In Section 2, we introduce the chaining approach which will lead us to the main result on the concentration for processes with increments of exponential decay. Then in Section 2.3, we discuss the application of the developed methodology to the fractional Brownian motions and then, in Section 3 its improvement for a standard Wiener process and diffusions with moderate growth. In Section 4 we deal with truncated variation of Lévy processes.

**Remark 2.** In the whole paper, any dependence of a nonnegative constant on some parameters is always indicated by listing them in brackets or in subscripts, for example, $C(n, \varepsilon)$ or $C_{n, \varepsilon}$.


2. The chaining approach

In this section, we prove the fundamental Theorem 1, which will allow us to establish integrability and concentration properties of the truncated variation for a broad class of processes satisfying some increment condition.

For simplicity, we consider processes indexed by a parameter from the metric space \((T, d)\), where \(T\) is the compact interval \([0, S]\), \(S > 0\), equipped with the distance \(d(s, t) = |s - t|^q\), \(s, t \in T\), where \(0 < q < 1\). Further, we introduce an Orlicz function \(\phi : [0, +\infty) \to \mathbb{R}\) convex, even, satisfying \(\phi(0) = 0, \phi(1) = 1\), strictly increasing and such that there exists \(L < +\infty\) such that for any \(x, y \geq 0\),

\[
\phi^{-1}(xy) \leq L(\phi^{-1}(x) + \phi^{-1}(y)).
\]

Moreover, we will require that \(x \mapsto \phi(x^q), x \geq 0\), is also convex.

Remark 3. The convexity assumptions of \(\phi\) may be weakened in such a way that \(\phi\) is convex on some interval \([C_\phi, +\infty)\), where \(C_\phi \geq 0\), and \(\phi(x^q)\) is convex on some interval \([C_{\phi,q}, +\infty)\), where \(C_{\phi,q} \geq 0\).

The standard example of functions with properties mentioned are \(\phi_p(x) = 2^{xp} - 1, p > 0\), for which condition (2.1) holds with \(L_p = \max\{1, 2^{(1-p)/p}\}\). Note that when \(p \geq 1\), \(\phi_p\) is convex on whole interval \([0, +\infty)\) but when \(0 < p < 1\), \(\phi_p\) is convex only on the interval \([C_p; +\infty)\) where \(C_p = (\frac{1}{p} - \frac{1}{q})^{1/p}\). Clearly \(\phi_p(x^q) = \phi_{pq}(x)\) and therefore this function is convex on the whole interval \([0; +\infty)\) if \(pq \geq 1\) and convex on the interval \([C_{pq}; +\infty)\), where \(C_{pq} = C_{pq}\), if \(pq < 1\). We use the notation \(C_p, C_{pq}\) for all \(p > 0, 0 < q < 1\), setting \(C_p = 0\) for \(p \geq 1\) (thus \(C_{pq} = 0\) for \(pq \geq 1\)). Further, we denote \(D_p = \phi_p(C_p), D_{pq} = \phi_p(C_{pq}) = \phi_{pq}(C_{pq})\). Note that \(D_{pq} = 0\) for \(pq \geq 1\). In more general case, we will denote \(D_\phi = \phi(C_\phi)\) and \(D_{\phi,q} = \phi(C_{\phi,q})\).

Let now \(X(t), t \in T\), be a stochastic process with increments controlled by \(\phi\). Namely

\[
\mathbb{E}\phi\left(\frac{|X(s) - X(t)|}{Cd(s, t)}\right) \leq 1
\]

for \(s, t \in T, s \neq t\), where \(0 < C < \infty\) is a universal constant.

Remark 4. In fact, in (2.2) one may consider any distance \(d\) of the form \(d(s, t) = \eta(|s - t|)\), where \(\eta\) is positive, concave, increasing to \(\infty\) and such that \(\eta(0) = 0\). We choose \(\eta(x) = x^q, 0 < q < 1\), for the sake of simplicity, however we stress that our results can be easily extended to a more general \(\eta\).

Condition (2.2) enables us to control the magnitude of the increments of the process \(X\), while the truncated variation takes into account only increments greater than \(c\) (cf. formula (1.1)). Note that as the consequence of (2.2) and the compactness of \(T\) we obtain the existence of a separable modification of \(X(t), t \in T\). Then by the linear order of \(T\) we can define the càdlàg modification of \(X\) which we refer to from now on.

Integrability of the truncated variation 441
The fundamental result of this paper, from which exponential integrability and concentration properties will follow, is the following theorem.

**Theorem 1.** Let $X(t)$, $t \in T$, satisfies (2.2). Then there exist random variables $Z_1, Z_2 \geq 0$ such that $E Z_1, E Z_2 \leq 1$ and for some universal constants $K_1(q), K_2(\varphi, q) < \infty$ the following estimate holds

$$TV^c(X, S) \leq c(q^{-1})^{1/q} S\left[ K_1(\varphi, q)\varphi^{-1}(Z_1 + D_{\varphi}) + K_2(\varphi, q)\left[\varphi^{-1}(Z_2 + D_{\varphi, q})\right]^{1/q}\right].$$

**Remark 5.** The main reason why the result holds is that (2.2) gives an exponential decay of increments with large jumps. Therefore, we can show a global upper bound on increments in the defined set approximation of the truncated variation. Such an idea is used to bound suprema of processes, for example, Bednorz [1], Fernique [4], Kwapieñ and Rosiński [6] and Talagrand [17]. In this paper, the main technical contribution is to invent a common upper bound for an arbitrary sum of truncated increments.

The meaning of the result the that for suitable $\varphi$ and $0 < q < 1$ there holds some concentration inequality. To formulate results in an elegant way, observe that there exists $E_q \in [0; 1]$ such that $E_q + x^{1/q} \geq x$ for $x \geq 0$ and hence we get

$$E_q + \left[\varphi^{-1}(x + \max\{D_{\varphi}, D_{\varphi, q}\})\right]^{1/q} \geq \varphi^{-1}(x + D_{\varphi}) \quad \text{for } x \geq 0. \quad (2.3)$$

As a consequence of Theorem 1, (2.3) and Jensen’s inequality we get the following corollary.

**Corollary 1.** Under the assumptions of Theorem 1 there exist r.v. $Z$ such that $Z \geq 0$, $E Z \leq 1$ and for some constants $A_{\varphi, q}, B_{\varphi, q}$ the following estimate holds

$$TV^c(X, S) \leq c(q^{-1})^{1/q} S\left[ A_{\varphi, q} + B_{\varphi, q}\left[\varphi^{-1}(Z + \max\{D_{\varphi}, D_{\varphi, q}\})\right]^{1/q}\right].$$

For $\varphi = \varphi_p$ let us denote $A_{p, q} = A_{\varphi, q}$ and $B_{p, q} = B_{\varphi, q}$. Applying Corollary 1, the Markov inequality and the fact that $D_{p, q} \geq D_p$ we obtain:

**Corollary 2.** Let $X(t)$, $t \in T$, satisfies (2.2) with $\varphi = \varphi_p$. The following inequality holds

$$P\left(TV^c(X, S) \geq c(q^{-1})^{1/q} S[A_{p, q} + \tilde{B}_{p, q} u]\right) \leq \tilde{D}_{p, q} \exp(-uvpq), \quad \text{for } u > 0,$$

where $\tilde{A}_{p, q}, \tilde{B}_{p, q}$ are universal constants, $\tilde{A}_{p, q} = A_{p, q} + (2/\ln 2)^{1/(pq)} B_{p, q}$, $\tilde{B}_{p, q} = (2/\ln 2)^{1/(pq)} B_{p, q}$ and $\tilde{D}_{p, q} = D_{p, q} + 1$. In particular, $\tilde{D}_{p, q} = 1$ for $pq \geq 1$.

To prove Theorem 1, we start with the construction of finite sets approximating $T$.

### 2.1. Approximating sequence

The first tool we need is a proper geometric approximation of the set $T$. The approximation consists of a sequence of finite sets $(T_n)_{n=0}^\infty$, $T_n \subset T$ constructed in such a way that for each
point \( t \in T \) and \( n = 0, 1, 2, \ldots \), there exists a point \( s \in T_n \), such that \( s \leq t \) and \( d(s, t) \leq r^{-n}S^q \).

Here, we fix \( r \geq 4 \). One of possible constructions is the following

\[
T_n = \left\{ kr^{-n/q}S : k = 0, 1, 2, \ldots \right\} \cap T. \tag{2.4}
\]

For \( T_n \) defined by (2.4) and \( t \in T \), by \( \pi_n(t) \) we denote the unique point \( s \in T_n \) such that \( s \leq t \) and \( d(s, t) \leq r^{-n}S^q \).

This way we define the function \( \pi_n : T \to T_n \). We have \( d(t, \pi_n(t)) < r^{-n}S^q \) for all \( t \in T \) and \( \pi_n(s) \leq \pi_n(t) \) if \( s \leq t \). Note also that for \( s, t \in T_n \), \( s \neq t \), \( d(s, t) \geq r^{-n}S^q \).

Clearly

\[
\frac{r^n}{q} < |T_n| = \left\lfloor \frac{r^n}{q} \right\rfloor + 1 \leq \frac{r^n}{q} + 1. \tag{2.5}
\]

Moreover for any \( m = 1, 2, \ldots \)

\[
\sum_{n=0}^{m} r^{-n}|T_{n+1}| \leq \sum_{n=0}^{m} r^{-n}(r^{(n+1)/q} + 1) \leq A(r, q)\frac{m(1-q)}{q}, \tag{2.6}
\]

where \( A(r, q) := r/(2-q)/r/(1-q)/q - 1 \) (note that \( r \geq 2 \)). For each \( t \in T_{n+1} \) let \( I_{n+1}(t) \) denote the set of the nearest neighbors of \( t \) in \( T_{n+1} \), namely

\[
I_{n+1}(t) = \{ s \in T_{n+1} : d(s, t) \leq 2r^{-n}S^q \} \tag{2.7}
\]

Observe that since \( |s - t| \geq r^{-(n+1)/q}S \) for \( s, t \in T_{n+1}, s \neq t \),

\[
|I_{n+1}(t)| \leq 2^{1/q}r^{-n/q}S + 1 = 2^{1/q}r^{1/q} + 1 =: B(r, q). \tag{2.8}
\]

### 2.2. Proof of the main theorem

The plan of the proof is the following. After having constructed the set approximation of \( T \), we use this approximation to build a type of discretization of any given partition and derive a chaining bound on the truncated variation (Lemma 1). Then we turn to estimate each increment in the partition bound (Lemma 5) and finally apply the bounds as well as some technical observations (Lemmas 3, 4 and 2) to derive the required bounds (Lemmas 6, 7).

Our first step is to analyze a given partition \( \Pi_n = \{ t_0, t_1, \ldots, t_n \} \), where \( 0 \leq t_0 < t_1 < \cdots < t_n \leq S \). We decompose the set \( \{1, \ldots, n\} \) into subsets \( J_m, m = 0, 1, 2, \ldots \), defined in the following way

\[
J_m = \{ i \in \{1, \ldots, n\} : r^{-m-1}S^q < d(t_{i-1}, t_i) \leq r^{-m}S^q \}.
\]

Let \( M_0 := 12CL \), where \( L \) and \( C \) are constants appearing in (2.1) and (2.2). The level \( m_0 \in \{0, 1, 2, \ldots\} \) such that

\[
r^{-m_0-1}S^q < c/M_0 \leq r^{-m_0}S^q
\]

will be of particular meaning in the proof. Since \( \Pi_n \) is finite, \( J_m = \emptyset \) for \( m \) large enough, say \( m \geq N_0 \geq m_0 \). We will use different bounds for \( i \in J_m \) with \( m > m_0 \) and for \( i \in J_m \) with \( m \leq m_0 \).
Therefore, let us make the trivial separation

\[
\sum_{i=1}^{n} \left( |X(t_i) - X(t_{i-1})| - c \right)_{+} \leq \sum_{m=0}^{m_0} \sum_{i \in J_m} \left( |X(t_i) - X(t_{i-1})| - c \right)_{+} \leq \sum_{m=m_0+1}^{\infty} \sum_{i \in J_m} \left( |X(t_i) - X(t_{i-1})| - c \right)_{+}. \tag{2.9}
\]

Now we turn to describe the chaining method which is the main tool in the proof. First, we fix \( N \geq N_0 \) and define \( t^{N+1}_l = \pi_{N+1}(t_l) \), then for \( l \in \{0, 1, \ldots, N\} \) we put by the reverse induction \( t^l_i = \pi_l(t^{l+1}_i) \). Note that by the construction of \( \pi_l \) we preserve the order of the projections, namely \( t^l_i \leq t^l_{i+1} \leq \cdots \leq t^l_n \) for any \( 0 \leq l \leq N + 1 \). Moreover since \( N \geq N_0 \) points \( \{t^{N+1}_0, t^{N+1}_1, \ldots, t^{N+1}_n\} \) are separated, that is, \( t^{N+1}_i \neq t^{N+1}_{i+1}, i \in \{1, \ldots, n\} \). Let us denote \( \bar{m} = \max\{m, m_0\} \). For \( i \in J_m \) with \( m > m_0 \), we estimate

\[
\left( |X(t_i) - X(t_{i-1})| - c \right)_{+} \leq \left( |X(t^{m+1}_i) - X(t^{m+1}_{i-1})| - c \right)_{+} + \sum_{s \in [i-1, i]} |X(t^{N+1}_s) - X(t_s)| \tag{2.10}
\]

\[
+ \sum_{l=m+1}^{N} \sum_{s \in [i-1, i]} \left( |X(t^{l+1}_s) - X(t^{l+1}_{s+1})| - 2^{-l+m} \frac{c}{3} \right)_{+}
\]

and for \( i \in J_m \) with \( m \leq m_0 \) we have

\[
\left( |X(t_i) - X(t_{i-1})| - c \right)_{+} \leq |X(t^{m+1}_i) - X(t^{m+1}_{i-1})| \tag{2.11}
\]

\[
+ \sum_{s \in [i-1, i]} |X(t^{N+1}_s) - X(t_s)| + \sum_{l=m+1}^{m_0} \sum_{s \in [i-1, i]} |X(t^{l+1}_s) - X(t^{l+1}_{s+1})| \]

\[
+ \sum_{l=m_0+1}^{N} \sum_{s \in [i-1, i]} \left( |X(t^{l+1}_s) - X(t^{l+1}_{s+1})| - 2^{-l+m} \frac{c}{3} \right)_{+}.
\]

Putting together estimates (2.9), (2.10) and (2.11), we obtain the following decomposition lemma.

**Lemma 1.** For any partition \( \Pi_n = \{t_0, \ldots, t_n\} \), where \( n \geq 0, 0 \leq t_0 < t_1 < \cdots < t_n \leq S \) and \( N > m_0 \) the following estimate holds

\[
\sum_{i=1}^{n} \left( |X(t_i) - X(t_{i-1})| - c \right)_{+} \leq V_1 + V_2 + W_1 + W_2 + \sum_{i=1}^{n} \sum_{s \in [i-1, i]} |X(t_s) - X(t^{N+1}_s)|,
\]
where

\[ V_1 := \sum_{m=0}^{m_0} \sum_{i \in J_m} \sum_{l=m+1}^{m_0} \sum_{s \in \{i-1, i\}} |X(t_l^s) - X(t_{l+1}^s)|; \]

\[ W_1 := \sum_{m=0}^{m_0} \sum_{i \in J_m} |X(t_i^{m+1}) - X(t_{i-1}^{m+1})|; \]

\[ V_2 := \sum_{m=0}^{\infty} \sum_{i \in J_m} \sum_{l=\bar{m}+1}^{N} \sum_{s \in \{i-1, i\}} \left( |X(t_l^s) - X(t_{l+1}^s)| - 2^{-l+\bar{m}} \frac{c}{3} \right) + \]

\[ W_2 := \sum_{m=m_0+1}^{\infty} \sum_{i \in J_m} \left( |X(t_i^{m+1}) - X(t_{i-1}^{m+1})| - \frac{c}{3} \right) + . \]

For each \( i \in J_m, m \geq 0 \) we say that \( t_i^{m+1}, t_i^{m+2}, \ldots, t_i^{N+1}, s = i - 1, i \) are path approximations of \( t_{i-1} \) and \( t_i \), respectively (see Figure 1). Note that for \( i \in \{1, \ldots, n-1\} \) there are two path approximations of \( t_i \), one from the pair \( t_{i-1}, t_i \) and the second from the pair \( t_i, t_{i+1}, \) that coincide, starting from some point, yet may differ on the length since \( i \in J_m, i + 1 \in J_m' \) and numbers \( m \) and \( m' \) may be different. The fundamental property of the path approximation is that for

![Figure 1](image-url). Path approximations.
a given \( u \in T_{i+1} \) the step \( \pi_i(u), u \) may occur in at most two path approximations of some \( t_i, i \in \{0, 1, \ldots, n\} \).

**Lemma 2.** Consider \( u \in T_{i+1}, l \in \{0, 1, \ldots, n\} \). The step \( \pi_i(u), u \) may occur in at most two path approximations of some \( t_i, i \in \{0, 1, \ldots, n\} \), that is, there exits no more than one \( i \in \{0, 1, \ldots, n\} \) such that \( i \in J_m, m + 1 \leq l \) and \( t_i = \pi_i(u), t_i^{l+1} = u \) or \( i + 1 \in J_{m'}, m' + 1 \leq l \) and \( t_i = \pi_i(u), t_i^{l+1} = u \) for some \( m, m' = 0, 1, 2, \ldots, N \).

**Proof.** Recall that \( r \geq 4 \). It suffices to prove that for a given \( i \in J_m, l \geq m + 1 \) points \( t_i^{l+1} \) and \( t_i^{l+1} \) are different. Indeed since \( t_0^{l+1} \leq t_1^{l+1} \leq \cdots \leq t_n^{l+1} \) the property implies that there can be at most one \( i \in \{0, 1, \ldots, n\} \) such that \( t_i^{l+1} = u \). To prove the assertion, we use \( d(t_i, t_{i-1}) > r^{-m-1} S^q \) which implies that for \( l \geq m + 1 \)

\[
d(t_i^{l+1}, t_{i-1}^{l+1}) \geq r^{-m-1} S^q - d(t_i^{l+1}, t_{i-1}) - d(t_i^{l+1}, t_i) \geq r^{-m-1} S^q - 2 \sum_{j=l+1}^{\infty} r^{-j} S^q \geq r^{-m-1} S^q - 2 \frac{r^{-m-2} S^q}{1 - r^{-1}} > 0.
\]

In the sequel, we will use two simple observations concerning increasing function \( \psi \) that is convex starting from some \( C_0 \geq 0 \), that is, convex for \( x \geq C_0 \).

**Lemma 3.** Let \( \psi : [0; +\infty) \to [0; +\infty) \) be a strictly increasing function. Assume that \( \psi \) is convex on the interval \( [C_0; +\infty) \) where \( C_0 \geq 0 \), then for any nonnegative \( x_1, \ldots, x_k \) and positive \( \alpha_1, \ldots, \alpha_k \) such that \( \sum_{i=1}^{k} \alpha_i \leq M \) we have

\[
\sum_{i=1}^{k} \alpha_i x_i \leq M \psi^{-1} \left( M^{-1} \sum_{i=1}^{k} \alpha_i \psi(x_i) + \psi(C_0) \right).
\]

**Proof.** Observe that the function \( \tilde{\psi}(x) = \psi(x + C_0) - \psi(C_0) \) for \( x \geq 0 \) is convex, strictly increasing and such that \( \tilde{\psi}(0) = 0 \). Consequently, \( \tilde{\psi}^{-1}(y) = \psi^{-1}(y + \psi(C_0)) - C_0 \) is concave with \( \tilde{\psi}^{-1}(0) = 0 \) and we have

\[
\sum_{i=1}^{k} \alpha_i x_i \leq \sum_{i=1}^{k} \alpha_i \psi^{-1}(\psi(x_i) + \psi(C_0)) = \sum_{i=1}^{k} \alpha_i (\tilde{\psi}^{-1}(\psi(x_i)) + C_0) \leq MC_0 + M \sum_{i=1}^{k} \frac{\alpha_i}{M} \tilde{\psi}^{-1}(\psi(x_i)) \leq MC_0 + M \tilde{\psi}^{-1} \left( \sum_{i=1}^{k} \frac{\alpha_i}{M} \psi(x_i) \right) = M \psi^{-1} \left( M^{-1} \sum_{i=1}^{k} \alpha_i \psi(x_i) + \psi(C_0) \right).
\]
where the last inequality follows from Jensen’s inequality
\[
\left(1 - \sum_{i=1}^{k} \frac{\alpha_i}{M}\right)\bar{\psi}^{-1}(0) + \sum_{i=1}^{k} \frac{\alpha_i}{M} \bar{\psi}^{-1}(\psi(x_i)) \leq \bar{\psi}^{-1}\left(\sum_{i=1}^{k} \frac{\alpha_i}{M} \psi(x_i)\right).
\]

Further, we also have the following lemma.

**Lemma 4.** For any strictly increasing function \(\psi : [0; +\infty) \to [0; +\infty)\) such that \(\psi\) is convex on the interval \([C_0; +\infty)\) where \(C_0 \geq 0\) and for any \(M > 0\) and \(y \geq 0\), we have
\[
\psi^{-1}(y + \psi(C_0)) \leq \max\{M, 1\}\psi^{-1}\left(y/M + \psi(C_0)\right).\tag{2.13}
\]

**Proof.** Again, we consider the function \(\bar{\psi}^{-1}\). If \(M < 1\), then (2.13) follows from the monotonicity of \(\bar{\psi}^{-1}\). Now assume that \(M \geq 1\). By concavity and \(\bar{\psi}^{-1}(0) = 0\), for \(y \geq 0\) and \(M \geq 1\), we get
\[
M\bar{\psi}^{-1}(y/M) \geq \bar{\psi}^{-1}(y),
\]
which reads as
\[
M\left(\psi^{-1}(y/M + \psi(C_0)) - C_0\right) \geq \psi^{-1}(y + \psi(C_0)) - C_0,
\]
\[
M\psi^{-1}(y/M + \psi(C_0)) \geq \psi^{-1}(y + \psi(C_0)) + (M - 1)C_0
\]
and which gives
\[
\psi^{-1}(y + \psi(C_0)) \leq M\psi^{-1}(y/M + \psi(C_0)).\tag{\Box}
\]

Now we formulate some basic bounds on increments in the chaining argument. For simplicity, we use the following notation
\[
\Delta(u, v) = \varphi\left(\frac{|X(u) - X(v)|}{Cd(u, v)}\right), \quad \text{for all } u, v \in T.
\]

Recall that \(\bar{m} = \max\{m, m_0\}\).

**Lemma 5.** Suppose that \(i \in J_m, m \geq 0\) then

1. for any \(m \leq m_0, l \in \{m + 1, \ldots, m_0\}\)
\[
|X(t_i^l) - X(t_i^{l+1})| \leq Cr^{-l}S^q\varphi^{-1}\left(\Delta(t_i^l, t_i^{l+1})\right);
\]
2. for any \(m \geq 0, l \in \{\bar{m} + 1, \ldots, N\}\)
\[
\left|\frac{|X(t_i^l) - X(t_i^{l+1})| - 2^{-l+\bar{m}+\bar{c}}}{3}\right| \leq b_{\bar{m}, \bar{m}}\left[\varphi^{-1}(a_{\bar{m}}\Delta(t_i^l, t_i^{l+1}))\right]^{1/q},
\]
where

\[ a_{l,m} = \varphi \left( \frac{2^{-l+m} r^l c}{6CL S^q} \right), \quad b_{l,m} = \left( \frac{c}{6} 2^{-l+m} \right)^{(q-1)/q} (CL r^{-l} S^q)^{1/q}; \]

3. for \( m \leq m_0 \)

\[ |X(t_i^{m+1}) - X(t_{i-1}^{m+1})| \leq 2Cr^{-m} S^q \varphi^{-1} (\Delta(t_i^{m+1}, t_{i-1}^{m+1})); \]

4. and for \( m > m_0 \)

\[ \left( |X(t_i^{m+1}) - X(t_{i-1}^{m+1})| - \frac{c}{3} \right) + \leq b_m \left[ \varphi^{-1} (a_{m-1} \Delta(t_i^{m+1}, t_{i-1}^{m+1})) \right]^{1/q}, \]

where

\[ a_m = \varphi \left( \frac{r^m c}{12CL S^q} \right), \quad b_m = \left( \frac{c}{6} \right)^{(q-1)/q} (2CL r^{-m} S^q)^{1/q}. \]

**Remark 6.** Note that the choice of \( M_0 \) in the definition of \( m_0 \) guarantees that \( a_{l,m} \geq 1 \) and \( a_m \geq 1 \) for \( m > m_0 \). Moreover, for \( \varphi \) convex on the whole real line, that is, \( C_\varphi = 0 \) one can deduce \( a_{l,m}^{-1} \leq (r/2)^{-l+m} \) and \( a_m^{-1} \leq r^{-m+m_0+1} \).

**Proof of Lemma 5.** Let us denote \( u = t_i^l \), then the construction of approximation paths implies that \( t_i^{l+1} = \pi_l(u) \). Clearly \( d(u, \pi_l(u)) \leq r^{-l} S^q \) and hence

\[ |X(\pi_l(u)) - X(u)| \leq Cr^{-l} S^q \varphi^{-1} (\Delta(\pi_l(u), u)). \] (2.14)

To prove the second assertion, we use (2.14) to get

\[ \left( |X(\pi_l(u)) - X(u)| - \frac{2^{-l+m} c}{3} \right) \leq \left( Cr^{-l} S^q \varphi^{-1} (\Delta(\pi_l(u), u)) - \frac{2^{-l+m} c}{3} \right). \] (2.15)

Now we rewrite (2.15) using \( a_{l,m} \)

\[ \left( |X(\pi_l(u)) - X(u)| - \frac{2^{-l+m} c}{3} \right) \]

\[ \leq \left[ Cr^{-l} S^q (\varphi^{-1} (\Delta(\pi_l(u), u)) - L\varphi^{-1} (a_{l,m})) - \frac{2^{-l+m} c}{3} \right]. \]

Therefore, we can apply (2.1) and see

\[ \left( |X(\pi_l(u)) - X(u)| - \frac{2^{-l+m} c}{3} \right) \]

\[ \leq \left( CL r^{-l} S^q \varphi^{-1} (a_{l,m}^{-1} \Delta(\pi_l(u), u)) - \frac{2^{-l+m} c}{6} \right). \] (2.16)
Using the inequality \((x - 1)_+ \leq x^{1/q}\) valid for \(x \geq 0\), we get
\[
\left( |X(\pi_l(u)) - X(u)| - 2^{-l+m} \frac{c}{3} \right)_+ \\
\leq \frac{c}{6} 2^{-l+m} (6c^{-1} 2^{l-m} C L r^{-1} S^q \varphi^{-1}(a_{l,m}^{-1} \Delta(\pi_l(u), u)) - 1)_+ \\
\leq \frac{c}{6} 2^{-l+m} (6c^{-1} 2^{l-m} C L r^{-1} S^q)^{1/q} \left[ \varphi^{-1}(a_{l,m}^{-1} \Delta(\pi_l(u), u)) \right]^{1/q} \\
= b_{l,m} \left[ \varphi^{-1}(a_{l,m}^{-1} \Delta(\pi_l(u), u)) \right]^{1/q}.
\]

To prove the third assertion, we first observe that \(d(t_i, t_i - 1) \leq r^{-m} S^q\) for \(i \in J_m\) and hence
\[
d(t_{i,m}^{m+1}, t_{i-1,m}^{m+1}) \leq d(t_i, t_i - 1) + d(t_i^{N+1}, t_i) + d(t_i^{N+1}, t_i - 1) + \sum_{l=m+1}^{N} \left[ d(t_{i+1}^{l+1}, t_{i}^{l}) + d(t_{i-1}^{l+1}, t_{i-1}^{l}) \right] \\
\leq r^{-m} S^q + 2 \sum_{l=m+1}^{\infty} r^{-l} S^q,
\]
so
\[
d(t_{i,m}^{m+1}, t_{i-1,m}^{m+1}) \leq 2r^{-m} S^q. \tag{2.17}
\]

Denoting \(u = t_{i,m}^{m+1}\) and \(v = t_{i-1,m}^{m+1}\) we get in the same way as (2.14) that
\[
|X(u) - X(v)| \leq 2Cr^{-m} S^q \varphi^{-1}(\Delta(u, v)).
\]

Then using the same idea as for the second assertion we deduce the remaining inequality. \(\square\)

We turn to apply the above lemmas to bound increments in the chaining bound formulated in Lemma 1. First, we consider a bound on \(V_1 + W_1\).

**Lemma 6.** There exists a universal constant \(K_1(\varphi, r, q) < \infty\) and a random variable \(Z_1 \geq 0\) independent from the partition \(\Pi_n\), such that \(\mathbf{E} Z_1 \leq 1\) and for \(V_1\) and \(W_1\) defined in Lemma 1 one has
\[
V_1 + W_1 \leq K_1(\varphi, r, q)c^{(q-1)/q} S \varphi^{-1}(Z_1 + D_\varphi).
\]

**Proof.** By Lemma 2 and the first bound in Lemma 5, we get
\[
V_1 \leq 2 \sum_{l=0}^{m_0} \sum_{u \in T_{l+1}} |X(u) - X(\pi_l(u))| \\
\leq \mathcal{V}_1 := 2C \sum_{l=0}^{m_0} r^{-l} S^q \sum_{u \in T_{l+1}} \varphi^{-1}(\Delta(\pi_l(u), u)). \tag{2.18}
\]
To bound $W_1$ we use (2.17), that is, that $d(i_m^{m+1}, i_{m-1}^{m+1}) \leq 2r^{-m}S^q$ for $i \in I_m$. Using the already defined sets $I_m(u) = \{ v \in T_m^{m+1} : d(u, v) \leq 2r^{-m}S^q \}$, and the third bound in Lemma 5

$$W_1 \leq \sum_{l=0}^{m_0} \sum_{u \in T_{l+1}} \sum_{v \in I_{l+1}(u)} \left| X(u) - X(v) \right|$$

$$\leq \mathcal{W}_1 := C \sum_{l=0}^{m_0} r^{-l}S^q \sum_{u \in T_{l+1}} \sum_{v \in I_{l+1}} \varphi^{-1}(\Delta(u, v)).$$

We calculate the sum of all weights appearing in (2.18) and (2.19). By (2.8) for each $u \in T_m$ we have $|I_{l+1}(u)| \leq B(r, q)$ and hence, using also (2.6)

$$M_1 := \sum_{l=0}^{m_0} r^{-l}S^q \left[ |T_{l+1}| + \sum_{u \in T_{l+1}} |I_{l+1}(u)| \right]$$

$$\leq \left[ 1 + B(r, q) \right]S^q \sum_{l=0}^{m_0} r^{-l}|T_{l+1}|$$

$$\leq A(r, q) \left[ 1 + B(r, q) \right] r^{-m_0(1-q)/q} S^q.$$

Therefore by $c \leq M_0 r^{-m_0} S^q$ we get $M_0 r^{-m_0(1-q)/q} S^q \leq M_0^{1/q} c(q^{-1})/q S$ and hence

$$M_1 \leq M_0^{1/q} A(r, q) \left[ 1 + B(r, q) \right] c(q^{-1})/q S.$$

Using Lemma 3 for $\varphi$ which is convex above $C_\varphi$ we get

$$\mathcal{V}_1 + \mathcal{W}_1 \leq 2CM_1 \varphi^{-1}(Z_1 + \varphi(C_\varphi)) \leq K_1(r, q) c(q^{-1})/q S \varphi^{-1}(Z_1 + \varphi(C_\varphi)),$$

where $K_1(\varphi, r, q) := 2CM_0^{1/q} A(r, q)[1 + B(r, q)]$ (the dependence on $\varphi$ is through $C$ and $L$) and

$$Z_1 = M_1^{-1} \sum_{l=0}^{m_0} r^{-l}S^q \sum_{u \in T_{l+1}} \left( \Delta(\pi_l(u), u) + \sum_{v \in I_{l+1}(u)} \Delta(u, v) \right).$$

Obviously $Z_1 \geq 0$ and $EZ_1 \leq 1$ by (2.2) and the definition of $M_1$. Combining (2.18), (2.19) and (2.20) we get the result. \hfill \Box

Our second goal is to prove a bound for $V_2 + W_2$ in Lemma 1 above the level $m_0$.

**Lemma 7.** There exists a universal constant $K_2(\varphi, r, q) < \infty$ and a random variable $Z_2 \geq 0$ independent from the partition $\Pi_n$ such that $EZ_2 \leq 1$ and for $V_2$ and $W_2$ defined in Lemma 1 the following inequality holds

$$V_2 + W_2 \leq K_2(\varphi, r, q) \left[ \varphi^{-1}(Z_2 + D_\varphi, q) \right]^{1/q}.$$
**Proof.** First, we prove a bound for $V_2$. We analyze the increment

$$
\left( |X(t_{i+1}^l) - X(t_i^l)| - 2^{l-\tilde{m}} \frac{c}{3} \right)_+, \quad l > \tilde{m}, i \in J_m, m \geq 0.
$$

Using the second inequality in Lemma 5, we obtain that

$$
\left( |X(t_{i+1}^l) - X(t_i^l)| - 2^{-l+\tilde{m}} \frac{c}{3} \right)_+ \leq b_{l, \tilde{m}} [\varphi^{-1}(a_{l, \tilde{m}} \Delta(t_{i+1}^l, t_i^l))]^{1/q}.
$$

Now observe that $|t_i - t_{i-1}| \geq r^{-(m+1)/q} S$ for $i \in J_m, m \geq 0$. Therefore,

$$
\sum_{l=\tilde{m}+1}^N b_{l, \tilde{m}} = \sum_{l=\tilde{m}+1}^N \left( \frac{c}{6} \right)^{l-\tilde{m}} (CLr^{-l} S^q)^{1/q}
$$

$$
\leq (6^{-q} CL)^{1/q} \sum_{l=\tilde{m}+1}^\infty (2^{(q-1)/q} r^{-1/q})^{l-\tilde{m}} c^{(q-1)/q} r^{-\tilde{m}/q} S
$$

$$
\leq M_2 c^{(q-1)/q} |t_i - t_{i-1}|^{1/q},
$$

where $M_2 = M_2(\varphi, r, q)$ is defined by

$$
(6^{-q} CL)^{1/q} \sum_{l=\tilde{m}+1}^\infty (2^{(q-1)/q} r^{-1/q})^{l-\tilde{m}} \leq ((12)^{1-q} CL)^{1/q} \sum_{l'=0}^\infty (2^{1-q} r^{-1/q})^{l'}
$$

$$
= ((12)^{1-q} CL)^{1/q} (1 - 2^{1-q} r^{-1/q})^{-1} =: M_2.
$$

Consequently

$$
\sum_{m=0}^\infty \sum_{i \in J_m} \sum_{l=\tilde{m}+1}^N \sum_{s \in \{i-1, i\}} b_{l, \tilde{m}} \leq 2M_2 c^{(q-1)/q} \sum_{i=1}^n |t_i - t_{i-1}| = 2M_2 c^{(q-1)/q} S.
$$

Thus we can apply Lemma 3 for $\varphi(x^q)$ which is convex above $C_{\varphi, q}$ and get

$$
V_2 \leq 2M_2 c^{(q-1)/q} S [\varphi^{-1}(\tilde{V}_2 + D_{\varphi, q})]^{-1},
$$

(2.22)

where

$$
\tilde{V}_2 := (2M_2)^{-1} \sum_{m=0}^\infty \sum_{i \in J_m} \sum_{l=\tilde{m}+1}^N \sum_{s \in \{i-1, i\}} \frac{\tilde{b}_{l, \tilde{m}}}{d_{l, \tilde{m}}} \Delta(t_i^l, t_i^{l+1})
$$

and

$$
\tilde{b}_{l, \tilde{m}} = (c^{(q-1)/q} S)^{-1} b_{l, \tilde{m}} = (6^{-1} 2^{l-\tilde{m}})^{(q-1)/q} (CLr^{-l})^{1/q}.
$$
Observe that
\[ \bar{b}_{l, \bar{m}} \leq \tilde{b}_{l,m_0} \leq a_{l,m_0}, \]
which implies \( \bar{b}_{l, \bar{m}} / a_{l, \bar{m}} \leq \tilde{b}_{l,m_0} / a_{l,m_0} \). Hence by Lemma 2, we have
\[ \tilde{V}_2 \leq V_2 := M_2^{-1} \sum_{l=m_0+1}^{\infty} \frac{\tilde{b}_{l,m_0}}{a_{l,m_0}} \sum_{u \in \mathcal{N}_l} \Delta(\pi_l(u), u). \] (2.23)

By the construction, \( a_{l,m_0} \geq 1 \), furthermore by (2.5)
\[ (6^{-12-l+m_0})(q-1)/q (C \varepsilon r^{-l})^{1/q} \left( \psi((2^{-l+m_0} r^l) / (6CL^2q)) \right) \left( \psi((2^{-l+m_0} r^l)) / \psi((2^{-l} r^l)) \right) \leq (1 - 4r^{-1})^{-1}. \] (2.24)

For \( \psi \) which is convex for \( x \geq C_\psi \) basically the same argument works but \( l' \) must be large enough to apply the convexity. Indeed, using that \( \psi(x) = \psi(x + C_\psi) - \psi(C_\psi) \) is convex and \( \psi(0) = 0 \) we deduce \( \psi(2^{-l'} r^{l'} x) \geq 2^{-l'} r^{l'} \psi(x) \) for \( x \geq 0 \) and thus for all \( x \geq 0 \),
\[ \psi(2^{-l'} r^{l'} x + C_\psi) \geq 2^{-l'} r^{l'} (\psi(x + C_\psi) - \psi(C_\psi)) + \psi(C_\psi). \] (2.25)

Now choosing a suitable \( x \) one can get a bound similar to (2.24) yet for general \( \psi \). Note that in this case the bounding constant may depend on \( \psi \). It proves that \( M_3 < \infty \). Finally, by (2.22), (2.23) and Lemma 4 we get
\[ V_2 \leq 2 M_2 \max\{M_3, 1\} c^{(q-1)/q} S[\psi^{-1}(\varphi_{2/M_3 + D_{\varphi,q}})]^{1/q}. \] (2.26)

Clearly, by (2.2) and the definition of \( M_3 \) we have \( E \nu_{2/M_3} \leq 1 \).

A similar argument can be used to bound increments in \( W_2 \). Namely using the forth inequality in Lemma 5 we get that for \( m > m_0 \) and \( i \in J_m \)
\[ \left( |X(t_i^{m+1}) - X(t_i^{m+1})| - \frac{c}{3} \right)_+ \leq b_m \left[ \psi^{-1}(a_m^{-1} \Delta(t_i^{m+1}, t_i^{m+1})) \right]^{1/q}. \]
Using that \( r^{-(m+1)/q} S \leq |t_i - t_{i-1}| \leq r^{-m/q} S \) we get
\[
b_m = \left( \frac{c}{6} \right)^{(q-1)/q} \left( 2CLr^{-m} S^q \right)^{1/q} \leq M_4 c^{(q-1)/q} |t_i - t_{i-1}| ,
\]
where \( M_4 = (2 \cdot 6^{1-q} CLr^{-1})^{1/q} \). Therefore,
\[
\sum_{m=m_0+1}^{\infty} \sum_{i \in J_m} b_m \leq M_4 c^{(q-1)/q} \sum_{i=1}^{n} |t_i - t_{i-1}| = M_4 c^{(q-1)/q} S ,
\]
and thus using Lemma 3 for \( \varphi(x^q) \) we get
\[
W_2 \leq M_4 c^{(q-1)/q} S \left[ \varphi^{-1} (\bar{W}_2 + D\varphi, q) \right]^{1/q} , \tag{2.27}
\]
where
\[
\bar{W}_2 := M_4^{-1} \sum_{m=m_0+1}^{\infty} \sum_{i \in J_m} \bar{b}_m \Delta(t_i^{m+1}, t_i^{m+1})
\]
and \( \bar{b}_m = (c^{(q-1)/q} S)^{-1} b_m = (2 \cdot 6^{1-q} CLr^{-m})^{1/q} \). By (2.17) we have \( d(t_i^{m+1}, t_i^{m+1}) \leq 2r - mS^q \) and thus using the definition of the set \( I_{m+1}(u) \) for each \( m > m_0 \) and \( u \in T_{m+1} \)
\[
\bar{W}_2 \leq W_2 = M_4^{-1} \sum_{m=m_0+1}^{\infty} \bar{b}_m \sum_{u \in T_{m+1}} \Delta(u, v) .
\]
Note that by (2.5), (2.8)
\[
\sum_{u \in T_{m+1}} |I_{m+1}(u)| \leq 2^{-1} B(r, q) (r^{(m+1)/q} + 1) \leq B(r, q) r^{(m+1)/q} .
\]
Hence
\[
M_4^{-1} B(r, q) \sum_{m=m_0+1}^{\infty} \bar{b}_m \sum_{u \in T_{m+1}} \Delta(u, v) \leq M_4^{-1} B(r, q) (2 \cdot 6^{1-q} CLr)^{1/q} \sum_{m'=0}^{\infty} (\varphi(r^{m'}))^{-1} =: M_5 ,
\]
where in the last line we used that \( r^{-m_0-1} S^q < c/M_0 , \) \( M_0 = 12CL \). The same argument as for \( M_3 \) proves that \( M_5 < \infty \). Note that in the case of convex \( \varphi \) we can easily bound \( \sum_{m'=0}^{\infty} (\varphi(r^{m'}))^{-1} \) by \( (1 - r^{-1})^{-1} \). By (2.27) and Lemma 4, we get
\[
W_2 \leq M_4 \max\{ M_5, 1 \} c^{(q-1)/q} S \left[ \varphi^{-1} (\bar{W}_2 / M_5 + D\varphi, q) \right]^{1/q} . \tag{2.28}
\]
Obviously $E|W_2/M_5| \leq 1$, consequently by (2.26), (2.28) and Jensen’s inequality we obtain the desired result. \hfill \Box

Now we are ready to finish the proof of Theorem 1.

**Proof of Theorem 1.** Note that for fixed $q$ and $\varphi$ we may minimize constants $K_1(\varphi, r, q)$ and $K_2(\varphi, r, q)$ appearing Lemmas 6, 7 with respect to $r \geq 4$. It is clear from our discussion about the finiteness of $M_3, M_5$ that one can set $r = 4$ in the case of convex $\varphi$. If $C_\varphi > 0$ the choice of $r \geq 4$ may be of meaning as we have explained in (2.25). Such minimal constants depend only on $\varphi$ and $q$, and we will denote them by $K_1(\varphi, q)$ and $K_2(\varphi, q)$ respectively.

Now it suffices to use Lemma 1, then universal bounds given in Lemmas 6, 7 and finally let $N \to \infty$. Recall that by the construction variables $Z_1$ and $Z_2$ of Lemmas 6, 7 do not depend on $N$ and $\lim_{N \to \infty} d(t, \pi_{N+1}(t)) = 0$ for any $t \in T$. From condition (2.2), for a given partition $\Pi_n = \{t_0, t_1, \ldots, t_n\}$ we get $R_N := \sum_{i=1}^n |X(t_i) - X(t_i^{N+1})| \to 0$ in probability as $N \uparrow +\infty$. Taking subsequence $N_k$ such that $R_{N_k} \to 0$ almost surely, we get the universal bound for the sum $\sum_{i=1}^n (|X(t_i) - X(t_{i-1})| - c)_+$. Since $\Pi_n$ was arbitrary we get the result for $TV^c(X, S)$. \hfill \Box

**2.3. Application to the fractional Brownian motion**

Let $W_H(t), t \geq 0$, be a fractional Brownian motion of the Hurst parameter $H \in (0, 1)$, that is, a centered Gaussian process which has the following covariance function

\[ E(W_H(s)W_H(t)) = \frac{1}{2} (s^{2H} + t^{2H} - |s - t|^{2H}). \]  

(2.29)

Let us consider $T = [0, S]$ with distance $d(s, t) = |t - s|^H$. From (2.29), it follows that $W_H(t) - W_H(s) \sim \mathcal{N}(0, |t - s|^{2H})$ and thus, for some constant $C(H)$,

\[ E\varphi_2\left(\frac{|W_H(t) - W_H(s)|}{C(H)|t - s|^H}\right) \leq 1, \quad \text{for } s, t \in T, s \neq t. \]

Consequently, all assumptions of Corollary 2 are satisfied with $p = 2$, $q = H$ and we get the following corollary.

**Corollary 3.** For any fractional Brownian motion $W_H(t), t \in T$, the following inequality holds

\[ P(TV^c(W_H, S) \geq c^{(H-1)/H} S(A_H + B_H u)) \leq C_H \exp(-u^{2H}), \quad \text{for } u > 0, \]

where $A_H, B_H, C_H$ are universal constants and $C_H = 1$ for $H \geq 1/2$.

Note that Corollary 3 implies that $E TV^c(W_H, S) \leq K_H c^{(H-1)/H} S$, where $K_H < \infty$. On the other hand $c^{(H-1)/H} S$ is also the proper lower bound for $E TV^c(W_H, S)$ when $Sc^{-1/H}$ is not too small. Indeed, let us consider the partition $\Pi = \{0 \leq t_0 < t_1 < \cdots < t_N \leq S\}$ given by $t_i = ic^{1/H}, \ i = 0, 1, 2, \ldots, N = \lfloor Sc^{-1/H} \rfloor$. We have

\[ TV^c(W_H, S) \geq \sum_{i=1}^N (|W_H(t_i) - W_H(t_{i-1})| - c)_+. \]
Clearly, for $Sc^{-1/H} \geq 2$, $N > Sc^{-1/H} - 1 \geq Sc^{-1/H}/2$ and $\mathbf{E}(|W_H(t_i) - W_H(t_{i-1})| - c) + \geq k_H c$ for some positive constant $k_H$. It proves that when $Sc^{-1/H} \geq 2$, $c^{(H-1)/H}S$ is comparable with $\mathbf{E} \text{TV}^c(W_H, S)$ up to a constant depending only on $H$. Therefore, we have another formulation of Corollary 3.

**Corollary 4.** Assume that $Sc^{-1/H} \geq 2$. For any fractional Brownian motion $W_H(t), t \in T$, the following inequality holds

$$
P(\text{TV}^c(W_H, S) \geq \mathbf{E} \text{TV}^c(W_H, S)(\bar{A}_H + \bar{B}_H u)) \leq \tilde{C}_H \exp(-u^{2H}), \quad \text{for } u > 0,$$

where $\bar{A}_H, \bar{B}_H, \tilde{C}_H < \infty$ are universal constants. Moreover $\tilde{C}_H = 1$ for $H \geq 1/2$.

### 3. Application to the standard Brownian motion and diffusions

For a standard Brownian motion $W = W_{1/2}$, which is the only fractional Brownian motion with independent increments one may, using this property, strengthen the results obtained for general fBm and obtain Gaussian concentration of $\text{TV}^c(W, S)$. The generalization of this result for diffusions with moderate growth, driven by $W$, is also possible.

Let us assume that $X_t$, $t \geq 0$, is a one-dimensional diffusion satisfying

$$X(t) = x_0 + \int_0^t \mu(s, X(s)) \, ds + \int_0^t \sigma(s, X(s)) \, dW(s). \quad (3.1)$$

We assume that $\sigma : [0; +\infty) \times \mathbb{R} \to [-R; R]$ is measurable and bounded (i.e., $0 < R < +\infty$) and $\mu : [0; +\infty) \times \mathbb{R} \to \mathbb{R}$ is measurable and satisfying the following linear growth condition: there exists $C, D \geq 0$ such that for all $t \geq 0$

$$|\mu(t, x)| \leq C + D|x|.
\quad (3.2)$$

We will also need the natural assumption that $X$ is a Markov process. With this assumption, we have the following theorem.

**Theorem 2.** For $X$ being a Markov process satisfying (3.1) with $\mu$ and $\sigma$ as above and $\lambda \geq 0$ one has

$$\mathbf{E} \exp(\lambda \text{TV}^c(X, S)) \leq 2 \exp(\lambda^2 S\alpha_R + \lambda Sc^{-1}\beta_R + \lambda \gamma_{x_0, C, D, S}) \times (1 + 8\lambda \eta_{D, R, S} \exp(\lambda^2 \eta_{D, R, S}^2)),
\quad$$

where $\gamma_{x_0, C, D, S} = (C + D|x_0|) Se^{DS}$, $\delta_{D, S} = DSe^{DS}$ and $\eta_{D, R, S} = \delta_{D, S} R^{1/\sqrt{2}}$. In particular, when $D = 0$ we get

$$\mathbf{E} \exp(\lambda \text{TV}^c(X, S)) \leq 2 \exp(\lambda^2 S\alpha_R + \lambda S(c^{-1}\beta_R + C))$$

and for the standard Brownian motion $X = W$ we get

$$\mathbf{E} \exp(\lambda \text{TV}^c(W, S)) \leq 2 \exp(\lambda^2 S\alpha + \lambda Sc^{-1}\beta),
\quad (3.3)$$
where $\alpha, \beta$ are universal constants.

**Proof.** Let us define

$$M(t) := \int_0^t \mu(s, X(s)) \, ds, \quad Y(t) := \int_0^t \sigma(s, X(s)) \, dW(s)$$

and $Y^* = \sup_{0 \leq s \leq t} |Y(s)|$. We have $X(t) = x_0 + M(t) + Y(t)$, and due to (3.2) we estimate

$$|M(t)| \leq \int_0^t |\mu(s, X(s))| \, ds \leq \int_0^t C + D|X(s)| \, ds$$

$$\leq \int_0^t C + D|x_0| + D|Y(s)| + D|Y^*| \, ds$$

$$\leq (C + D|x_0| + D|Y^*|)S + D\int_0^t |M(s)| \, ds. \quad (3.4)$$

Hence, from Gronwall’s lemma (cf. Revuz and Yor [15], Appendix §1), we get

$$|M(t)| \leq (C + D|x_0| + D|Y^*|)Se^{Dt}. \quad (3.5)$$

Notice that due to (3.5) $M$ is adapted, absolute continuous process with locally bounded total variation. Indeed, repeating estimates (3.4) and using (3.5) we get

$$\text{TV}(M, S) \leq \int_0^S |\mu(s, X(s))| \, ds$$

$$\leq (C + D|x_0| + D|Y^*|)S + D\int_0^S |M(t)| \, ds$$

$$\leq (C + D|x_0| + D|Y^*|)S + D(C + D|x_0| + D|Y^*|)S \int_0^S e^{Dt} \, ds$$

$$= (C + D|x_0|)Se^{DS} + DSe^{DS}Y^*. \quad (3.6)$$

(TV = TV$^0$ denotes here the total variation.)

By Łochowski and Miłoś [12], Fact 17, we have

$$\text{TV}^c(X, S) \leq \text{TV}(M, S) + \text{TV}^c(Y, S). \quad (3.7)$$

Now we will investigate $\text{TV}^c(Y, S)$.

First, let us prove that $Y$ satisfies condition (2.1) with $\varphi_1 = \varphi_2$ and $d(s, t) = |s - t|^{1/2}$. Indeed, let us fix $0 \leq s < t \leq S$ and consider the following martingale $Z(u) := Y(s + u) - Y(s), u \in [0; t - s]$. We have

$$Z(u) = \int_s^{s+u} \sigma(\tau, X(\tau)) \, dW(\tau)$$
and
\[
\langle Z \rangle(u) = \int_s^{s+u} \sigma(\tau, X(\tau))^2 \, d\tau \leq R^2(t-s).
\]

Hence, by Bernstein's inequality (cf. Revuz and Yor [15], Chapter IV, Exercise 3.16), we have
\[
P\left( |Y(t) - Y(s)| \geq x \right) \leq 2P\left( \sup_{u \in [0; t-s]} Z(u) \geq x \right)
\]
\[
= 2P\left( \sup_{u \in [0; t-s]} Z(u) \geq x, \langle Z \rangle(t-s) \leq R^2(t-s) \right)
\]
\[
\leq 2 \exp(-x^2/(2R^2(t-s))).
\]

From (3.8), we immediately get that \( Y \) satisfies condition (2.1) for \( \varphi = \varphi_2 \) and \( d(s, t) = |s-t|^{1/2} \).

Hence, from Corollary 2 we obtain the following bound on the tails of \( TV^c(Y, S) \):
\[
P\left( TV^c(Y, S) \geq c^{-1} S(A + Bu) \right) \leq e^{-u},
\]
where \( A = A(R) \) and \( B = B(R) \) depend on \( R \) only. Notice that for \( \delta > 0 \) applying Bernstein's inequality to \( Y^* \) we get
\[
P(Y^* \geq x) \leq 2 \exp(-x^2/(2R^2S))
\]
and using integration by parts we have
\[
E \exp(\delta Y^*) \leq 1 + 2\delta \int_0^\infty e^{\delta y} e^{-y^2/(2R^2S)} \, dy \leq 1 + 8\delta R \sqrt{S/2} e^{\delta^2 R^2 S/2}.
\]

Now, we will strengthen estimate (3.9) using the Markov property of \( X \). First, using (3.9) and integration by parts we have
\[
E \exp(\lambda TV^c(Y, S) - c^{-1} SA) \leq \frac{1}{1 - \lambda SB/c}
\]
for \( \lambda < c(SB)^{-1} \). Let now \( S = S_1 + S_2 \), where \( S_1, S_2 > 0 \). Using the inequality \( TV^c(Y, S) \leq TV^c(Y, S_1) + c + TV^c(Y, [S_1, S]) \), which follows easily from the estimate:
\[
\left( |Y(t) - Y(u)| - c \right)_+ \leq \left( |Y(t) - Y(S_1)| - c \right)_+ + \left( |Y(S_1) - Y(u)| - c \right)_+ + c
\]
for \( 0 \leq t < S_1 < u \leq S \), and then the Markov property of \( X \) we get
\[
E \exp(\lambda TV^c(Y, S) - c^{-1} SA)
\]
\[
\leq E \exp(\lambda TV^c(Y, S_1) + \lambda c + \lambda TV^c(Y, [S_1, S]) - \lambda c^{-1} SA)
\]
\[
= e^{\lambda c} E\left[e^{\lambda TV^c(Y, S_1) - \lambda c^{-1} S_1 A} E[e^{\lambda TV^c(Y, [S_1, S]) - \lambda c^{-1} S_2 A} | X(S_1)]] \right]
\]
\[
\leq e^{\lambda c} \frac{1}{1 - \lambda S_1 B/c} \frac{1}{1 - \lambda S_2 B/c}.
\]
The last inequality follows by (3.11), since the right-hand side of (3.11) does not depend on \( x_0 \), and using the Markov property in similar way we have the universal estimate for the conditional
expectation

\[ E \left( \exp \left( \lambda TV^c(Y, [S_1, S]) - \lambda c^{-1} S_2 A \right) \right) | X(S_1) = x_1 \leq \frac{1}{1 - \lambda S_2 B/c} \]

(note that the length of interval \([S_1, S]\) is \(S_2\)). Notice now that from (3.12) it follows that \(E \left( \exp \left( \lambda [TV^c(Y, S) - c^{-1} S A] \right) \right) < +\infty \) for \(\lambda < \min \{c(S_1 B)^{-1}, c(S_2 B)^{-1}\}\). Let us fix integer \(n \geq 1\). Iterating (3.12) we obtain

\[ E \left( \exp \left( \lambda TV^c(Y, S) - c^{-1} S A \right) \right) \leq e^{\lambda c(n-1)} \left( \frac{1}{1 - \lambda S B(cn)^{-1}} \right)^n \]  

(3.13)

for \(\lambda < c n (SB)^{-1}\), which gives that \(E \left( \exp \left( \lambda TV^c(Y, S) - c^{-1} S A \right) \right) < +\infty \) for any \(\lambda \in \mathbb{R}\). Now, let us fix \(\lambda > 0\) and set \(n = \lceil 2 \lambda S B c^{-1} \rceil\). Using (3.13), we get

\[ E \left( \exp \left( \lambda TV^c(Y, S) - c^{-1} S A \right) \right) \leq 2 \exp \left( 2 \lambda^2 S B + 2 (\ln 2) \lambda S B c^{-1} \right) \]

and thus

\[ E \left( \exp \left( \lambda TV^c(Y, S) \right) \right) \leq 2 \exp \left( 2 \lambda^2 S B + \lambda S c^{-1} (A + 2 (\ln 2) B) \right) \]

\[ = 2 \exp \left( \lambda^2 S \alpha_R + \lambda S c^{-1} \beta_R \right), \]  

(3.14)

where \(\alpha_R = 2 B = 2B(R)\) and \(\beta_R = A + 2 (\ln 2) B = A(R) + 2 (\ln 2) B(R)\). Now, from (3.7), (3.6) and (3.14) we get

\[ E \left( \exp \left( \lambda TV^c(X, S) \right) \right) \leq E \left( \exp \left( \lambda TV(M, S) + \lambda TV^c(Y, S) \right) \right) \]

\[ \leq 2 \exp \left( \lambda^2 S \alpha_R + \lambda S c^{-1} \beta_R + \lambda \gamma_{x_0, C, D, S} \right) E \left( \exp \left( \lambda \delta_{D, S} Y^* \right) \right), \]

where \(\gamma_{x_0, C, D, S} = (C + D|x_0|) S e^{DS}, \delta_{D, S} = D S e^{DS}\). Finally, using (3.10) with \(\delta = \lambda \delta_{D, S}\) we get

\[ E \left( \exp \left( \lambda TV^c(X, S) \right) \right) \leq 2 \exp \left( \lambda^2 S \alpha_R + \lambda S c^{-1} \beta_R + \lambda \gamma_{x_0, C, D, S} \right) \]

\[ \times \left( 1 + 8 \lambda \eta_{D, R, S} \right) \exp \left( \lambda^2 \eta_{D, R, S}^2 \right), \]

where \(\eta_{D, R, S} = \delta_{D, S} R \sqrt{S/2}\). \(\square\)

**Remark 7.** Let us notice that the condition that \(\sigma\) is bounded is essential for obtaining the Gaussian concentration of \(TV^c(X, S)\). To see this it is enough to consider the equation \(dX(t) = 2^{-1} X(t) \, dt + X(t) \, dW(t)\) with the starting condition \(X(0) = 1\). Notice that \(TV^c(X, S) \geq X(S) - X(0) - c\) and that \((X(S) - X(0) - c) = (\exp W(S) - 1 - c)\) does not reveal the Gaussian concentration.
Remark 8. Notice that for the standard Brownian motion \( \mathcal{X} = \mathcal{W} \) and \( S c^{-2} \geq 2, S c^{-1} \) is comparable up to a universal constant with \( \mathbb{E} TV^c \mathcal{W}(S) \). Hence, from (3.3) we obtain that for \( c > 0 \) such that \( S c^{-2} \geq 2 \), there exist universal constants \( \bar{A}, \bar{B} < +\infty \) such that the Gaussian concentration holds

\[
\mathbb{P}(TV^c \mathcal{W}(S) \geq \bar{A} \mathbb{E} TV^c \mathcal{W}(S) + \bar{B} \sqrt{S} u) \leq \exp(-u^2), \quad \text{for } u \geq 0.
\]

4. Existence of moment-generating functions of the truncated variation of \( \text{Lévy processes} \)

In this section, we will deal with the existence of finite exponential moments of the truncated variation of a \( \text{Lévy process} \) \( \mathcal{X} \). We will state the necessary and sufficient condition for the finiteness of \( \mathbb{E} \exp(\alpha TV^c \mathcal{X}(S)) \) in terms of the generating triplet of the process \( \mathcal{X} \) (cf. Sato [16], Chapter 2, Section 11). The methodology used here is very similar to the methodology used in Łochowski [10] for a Wiener process \( \mathcal{W} \), where the existence of \( \mathbb{E} \exp(\alpha TV^c \mathcal{W}(S)) \) for any complex \( \alpha \) was proved.

We start with the following lemma.

Lemma 8. Let \( \mathcal{X} \) be a \( \text{Lévy process} \). For any \( c > 0 \) and \( \alpha > 0 \) one has \( \mathbb{E} \exp(\alpha TV^c \mathcal{X}(S)) < +\infty \) if and only if

\[
\mathbb{E} \left( \alpha \sup_{0 \leq s \leq S} |X(s)| \right) < +\infty.
\]

Proof. The ‘only if’ part follows from the inequality

\[
TV^c \mathcal{X}(S) \geq \sup_{0 \leq s \leq S} \max \left\{ |X(s) - X(0)| - c, 0 \right\}
\]

\[
= \max \left\{ \sup_{0 \leq s \leq S} |X(s)| - c, 0 \right\} \geq \sup_{0 \leq s \leq S} |X(s)| - c.
\]

To prove the opposite implication let us define \( T^c_0 = 0 \) and for \( i = 1, 2, \ldots \)

\[
T^c_i = \inf \{ t > T^c_{i-1} : |X(t) - X(T^c_{i-1})| > c/2 \} \wedge (S + T^c_{i-1}).
\]

Observe that \( T^c_i = \inf \{ t > 0 : |X(t)| > c/2 \} \wedge S \leq S \) and that \( (X(t))_{t \geq 0} \overset{d}{=} (X(t) - X(T^c_i))_{t \geq T^c_i} \)

where “\( \overset{d}{=} \)” denotes the equality of distributions. Now let us define

\[
X^c_t = \sum_{i=0}^{\infty} X(T^c_i) I_{(T^c_i, T^c_{i+1})}(t).
\]

Since \( \|X^c - X\|_{\infty} \leq c/2 \), we have

\[
TV^c \mathcal{X}(S) \leq TV(X^c, S)
\]  \hspace{1cm} (4.1)
and since $X^c$ is piecewise constant with the first jump at $T_1^c \leq S$, denoting $\Delta X^c(T_1^c) = X^c(T_1^c) - X^c(T_1^c -)$ we have

$$TV(X^c, S) = |\Delta X^c(T_1^c)| + TV(X^c, [T_1^c, S]) \leq \sup_{0 \leq s \leq T_1^c} |X(s)| + TV(X^c, [T_1^c, S]). \tag{4.2}$$

Let now $\delta \in (0; S)$ be such a small number that

$$E[\exp(\alpha \sup_{0 \leq s \leq S} |X(s)|); T_1^c \leq \delta] \leq \sup_{0 \leq s \leq T_1^c} |X(s)| + TV(X^c, [T_1^c, S]). \tag{4.3}$$

Note that such a number exists, since we assume that $E[\exp(\alpha \sup_{0 \leq s \leq S} |X(s)|) < +\infty$ and from the càdlàg property and stochastic continuity of $X$ it follows that $P(T_1^c \leq \delta) = P(\sup_{0 \leq s \leq \delta} |X(s)| > c/2) \downarrow 0$ as $\delta \downarrow 0$.

Let us fix $M > 0$. Note that on the set $\{T_1^c > \delta\}$ we have $TV(X^c, \delta) = 0$, hence

$$E[\exp(\alpha TV(X^c, \delta) \wedge M); T_1^c \leq \delta] \leq E[\exp(\alpha TV(X^c, \delta) \wedge M); T_1^c \leq \delta] + E[\exp(0 \wedge M); T_1^c > \delta] = E[\exp(\alpha TV(X^c, \delta) \wedge M); T_1^c \leq \delta] + P(T_1^c > \delta). \tag{4.4}$$

Now, applying (4.2), the independence of the process $X(t) - X(T_1^c)$, $t \geq T_1^c$, and the two-dimensional r.v. $(\sup_{0 \leq s \leq T_1^c} |X(s)|, T_1^c)$ (to see this notice that $T_1^c$ is a stopping time and use the strong Markov property of Lévy processes) and the equality of distributions of $TV(X^c, s)$ and $TV(X^c, [T_1^c; T_1^c + s])$ for any $s \geq 0$, we have

$$E[\exp(\alpha TV(X^c, \delta) \wedge M); T_1^c \leq \delta] \leq E[\exp(\alpha \left( \sup_{0 \leq s \leq T_1^c} |X(s)| + TV(X^c, [T_1^c; \delta]) \right) \wedge M); T_1^c \leq \delta]$$

$$\leq E[\exp(\alpha \sup_{0 \leq s \leq T_1^c} |X(s)| + \alpha TV(X^c; [T_1^c; \delta + T_1^c]) \wedge M); T_1^c \leq \delta]$$

$$= E[\exp(\alpha \sup_{0 \leq s \leq T_1^c} |X(s)|); T_1^c \leq \delta]E[\exp(\alpha TV(X^c, \delta) \wedge M); T_1^c \leq \delta] + P(T_1^c > \delta)$$

$$\leq E[\exp(\alpha \sup_{0 \leq s \leq S} |X(s)|); T_1^c \leq \delta]E[\exp(\alpha TV(X^c, \delta) \wedge M).$$
By this and by (4.4), (4.3) we have

$$E \exp(\alpha TV(X^c, \delta) \wedge M) \leq \frac{P(T^c_1 > \delta)}{1 - E[\exp(\alpha \sup_{0 \leq s \leq S} |X(s)|); T^c_1 \leq \delta]}.$$  (4.5)

Using similar arguments as before (i.e., (4.2), independence of $X(t) - X(T^c_1), t \geq T^c_1$, and $(\sup_{0 \leq s \leq T^c_1} |X(s)|, T^c_1)$ and the equality of distributions of $TV(X^c, s)$ and $TV(X^c, [T^c_1; T^c_1 + s])$ for $s \geq 0$) we obtain

$$E \exp(\alpha TV(X^c, S) \wedge M)
\leq E\left[\exp\left(\alpha \sup_{0 \leq s \leq T^c_1} |X(s)| + \alpha TV(X^c; [T^c_1; S + T^c_1]) \wedge M; T^c_1 \leq \delta\right)\right]$$
$$+ E\left[\exp\left(\alpha \sup_{0 \leq s \leq T^c_1} |X(s)| + \alpha TV(X^c; [T^c_1; S + T^c_1 - \delta]) \wedge M; T^c_1 > \delta\right)\right]$$
$$= E\left[\exp\left(\alpha \sup_{0 \leq s \leq T^c_1} |X(s)|\right); T^c_1 \leq \delta\right]E[\exp(\alpha TV(X^c, S) \wedge M)]$$
$$+ E\left[\exp\left(\alpha \sup_{0 \leq s \leq T^c_1} |X(s)|\right); T^c_1 > \delta\right]E[\exp(\alpha TV(X^c, S - \delta) \wedge M)]$$
$$\leq E\left[\exp\left(\alpha \sup_{0 \leq s \leq S} |X(s)|\right); T^c_1 \leq \delta\right]E[\exp(\alpha TV(X^c, S) \wedge M)]$$
$$+ E\exp\left(\alpha \sup_{0 \leq s \leq S} |X(s)|\right)E[\exp(\alpha TV(X^c, S - \delta) \wedge M)].$$

From this, we have

$$E \exp(\alpha TV(X^c, S) \wedge M)$$
$$\leq \frac{E \exp(\alpha \sup_{0 \leq s \leq S} |X(s)|)}{1 - E[\exp(\alpha \sup_{0 \leq s \leq S} |X(s)|); T^c_1 \leq \delta]} E \exp(\alpha TV(X^c, S - \delta) \wedge M).$$

Similarly, if $S - 2\delta > 0$

$$E \exp(\alpha TV(X^c, S - \delta) \wedge M)$$
$$\leq \frac{E \exp(\alpha \sup_{0 \leq s \leq S - \delta} |X(s)|)}{1 - E[\exp(\alpha \sup_{0 \leq s \leq S - \delta} |X(s)|); T^c_1 \leq \delta]} E \exp(\alpha TV(X^c, S - 2\delta) \wedge M)$$
$$\leq \frac{E \exp(\alpha \sup_{0 \leq s \leq S} |X(s)|)}{1 - E[\exp(\alpha \sup_{0 \leq s \leq S} |X(s)|); T^c_1 \leq \delta]} E \exp(\alpha TV(X^c, S - 2\delta) \wedge M).$$
Iterating and putting together the above inequalities, we finally obtain
\[
\mathbb{E}\exp(\alpha TV(X^c, S) \wedge M) \leq \left( \frac{\mathbb{E}\exp(\alpha \sup_{0 \leq s \leq S} |X(s)|)}{1 - \mathbb{E}[\exp(\alpha \sup_{0 \leq s \leq S} |X(s)|); T^c_1 \leq \delta]} \right)^{\lceil S/\delta \rceil} \times \mathbb{E}\exp(\alpha TV(X^c, \delta) \wedge M).
\] (4.6)

By (4.5) and (4.6), and letting \( M \to \infty \) we get \( \mathbb{E}\exp(\alpha TV(X^c, S)) < +\infty \). Finally, from (4.1) we get
\[
\mathbb{E}\exp(\alpha TV^c(X, S)) < +\infty. \quad \square
\]

Now let \((A, \nu, \gamma)\) be the generating triplet of the process \(X\). By Sato [16], Theorem 28.15, we have
\[
\mathbb{E}\exp(\alpha \sup_{0 \leq s \leq S} |X(s)|) < +\infty
\]
if and only if
\[
\mathbb{E}\exp(\alpha |X(1)|) < +\infty
\]
which, by Sato [16], Corollary 25.8, is equivalent with
\[
\int_{|x| > 1} e^{\alpha |x|} \nu(dx) < +\infty. \quad (4.7)
\]
From equivalence of these conditions and Lemma 8 we obtain the following theorem.

**Theorem 3.** Let \((A, \nu, \gamma)\) be the generating triplet of the Lévy process \(X\). For any \(\alpha > 0\) we have
\[
\mathbb{E}\exp(\alpha TV^c(X, S)) < +\infty
\]
if and only if
\[
\int_{|x| > 1} e^{\alpha |x|} \nu(dx) < +\infty.
\]

Theorem 3 may be applied in situations, when the process \(X\) satisfies condition (4.7) with some \(\alpha > 0\) but it is neither Brownian motion nor finite variation process. This holds, for example, for tempered stable process, that is, processes with the Lévy measure given by
\[
\nu(dx) = \frac{c_p}{x^{1+\alpha_p}} e^{-\lambda_p x} 1_{x > 0} dx + \frac{c_n}{(-x)^{1+\alpha_n}} e^{\lambda_n x} 1_{x < 0} dx,
\]
where \(\alpha_p, \alpha_n < 2, \lambda_p, \lambda_n, c_p, c_n > 0\). They satisfy (4.7) for any \(\alpha < \min(\lambda_p, \lambda_n)\) and have infinite variation when \(\alpha_p, \alpha_n \geq 1\). Another example are Meixner processes, used in financial modeling (cf. Kyprianou et al. [7], Chapter I), with Lévy measure given by
\[
\nu(dx) = \delta \frac{\exp(\beta x/\eta)}{x \sinh(\pi x/\eta)} dx,
\]
Integrability of the truncated variation

where $\delta, \eta > 0$, $|\beta| < \pi$. They satisfy (4.7) for $\alpha < (\pi - |\beta|)/\eta$.

Acknowledgements

The authors would like to thank the anonymous referee and the Associate Editor for many valuable comments and suggestions, which considerably improved the presentation of the results. The research of the first and the second author was supported by the National Science Center in Poland under decision no. DEC-2011/01/B/ST1/05089.

References

[1] Bednorz, W. (2006). A theorem on majorizing measures. *Ann. Probab.* **34** 1771–1781. MR2271481
[2] Bednorz, W. (2006). On a Sobolev type inequality and its applications. *Studia Math.* **176** 113–137. MR2264359
[3] Bednorz, W. (2010). Majorizing measures on metric spaces. *C. R. Math. Acad. Sci. Paris* **348** 75–78. MR2586748
[4] Fernique, X. (1978). Caractérisation de processus à trajectoires majorées ou continues. In *Séminaire de Probabilités XII (Univ. Strasbourg, Strasbourg, 1976/1977).* Lecture Notes in Math. **649** 691–706. Berlin: Springer. MR0520032
[5] Fernique, X. (1983). Regularité de fonctions aléatoires non gaussiennes. In *Eleventh Saint Flour Probability Summer School – 1981 (Saint Flour, 1981).* Lecture Notes in Math. **976** 1–74. Berlin: Springer. MR0722982
[6] Kwapień, S. and Rosiński, J. (2004). Sample Hölder continuity of stochastic processes and majorizing measures. In *Seminar on Stochastic Analysis, Random Fields and Applications IV.* Progress in Probability **58** 155–163. Basel: Birkhäuser. MR2096287
[7] Kyprianou, A. E., Schoutens, W. and Willmott, P. (2005). *Exotic Option Pricing and Advanced Lévy Models.* Chichester: Wiley. MR2343205
[8] Ledoux, M. and Talagrand, M. (1991). *Probability in Banach Spaces. Isoperimetry and Processes.* Ergebnisse der Mathematik und Ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)] **23**. Berlin: Springer. MR1102015
[9] Li, W.V. and Shao, Q.M. (2001). Gaussian processes: Inequalities, small ball probabilities and applications. In *Stochastic Processes: Theory and Methods 533–597.* Handbook of Statistics **19.** Amsterdam: North-Holland. MR1861734
[10] Łochowski, R. (2011). Truncated variation, upward truncated variation and downward truncated variation of Brownian motion with drift – their characteristics and applications. *Stochastic Process. Appl.* **121** 378–393. MR2746181
[11] Łochowski, R. (2013). On a generalisation of the Hahn–Jordan decomposition for real càdlàg functions. *Colloq. Math.* **132** 121–138. MR3106093
[12] Łochowski, R. and Miłoś, P. (2013). On truncated variation, upward truncated variation and downward truncated variation for diffusions. *Stochastic Process. Appl.* **123** 446–474. MR3003359
[13] Łochowski, R.M. (2014). Pathwise stochastic integration with respect to semimartingales. *Probab. Math. Statist.* To appear.
[14] Picard, J. (2008). A tree approach to $p$-variation and to integration. *Ann. Probab.* **36** 2235–2279. MR2478682
[15] Revuz, D. and Yor, M. (2005). Continuous Martingales and Brownian Motion. Grundlehren der Mathematischen Wissenschaften 293 [Fundamental Principles of Mathematical Sciences]. Berlin: Springer. MR1725357

[16] Sato, K.-i. (1999). Lévy Processes and Infinitely Divisible Distributions. Cambridge Studies in Advanced Mathematics 68. Cambridge: Cambridge Univ. Press. MR1739520

[17] Talagrand, M. (1990). Sample boundedness of stochastic processes under increment conditions. Ann. Probab. 18 1–49. MR1043935

[18] Talagrand, M. (2005). The Generic Chaining. Upper and Lower Bounds of Stochastic Processes. Springer Monographs in Mathematics. Berlin: Springer. MR2133757

Received November 2012 and revised September 2013