Abstract. Reverse Mathematics is a program in the foundations of mathematics. Its results give rise to an elegant classification of theorems of ordinary mathematics based on computability. In particular, the majority of theorems fall into only five categories of which the associated logical systems are dubbed ‘the Big Five’. Recently, a lot of effort has been directed towards finding exceptional principles, i.e. which fall outside the Big Five categories. The so-called Reverse Mathematics zoo is a collection of such exceptional principles (and their relations). In this paper, we show that uniform versions of the zoo-principles, i.e. where a functional computes the object stated to exist, do fall in line with the Big Five categorisation. In other words, the zoo seems to disappear at the uniform level. In particular, we shall formulate a general template which establishes equivalence between uniform zoo-principles and arithmetical comprehension, the third Big Five category, inside Kohlenbach’s framework of higher-order Reverse Mathematics. Our template works for theorems whose objects exhibit little structure, a notion we conjecture to be connected to Montalbán’s notion robustness.

1. Introduction: Reverse Mathematics and its zoo

In two words, the subject of this paper is the Reverse Mathematics classification in Kohlenbach’s framework (25) of uniform versions of principles from the Reverse Mathematics zoo (10), namely as equivalent to arithmetical comprehension. We first discuss the italicised notions in more detail.

For an introduction to the foundational program Reverse Mathematics (RM for short), we refer to [39, 40]. One of the main results of RM is that the majority of theorems from ordinary mathematics, i.e. about countable and separable objects, fall into only five categories of which the associated logical systems are dubbed ‘the Big Five’ (See e.g. [29, p. 432]). In the last decade or so, a huge amount of time and effort was invested in identifying principles falling outside of the Big Five categories. All such exceptional principles (and their relations) falling below the third Big Five system, are collected in the so-called RM zoo (See [10]).

In this paper, we shall establish that the exceptional principles inhabiting the RM zoo become non-exceptional at the uniform level, namely that the uniform versions of RM zoo-principles are all equivalent to arithmetical comprehension, the aforementioned third Big Five system of RM. As a first example, consider the principle UDNR, studied in Section 3.1

\[(\exists \Psi^{1→1})(\forall A^1)(\forall e^0)(\Psi(A)(e) \neq \Phi^A_e(e))\]  

(UDNR)
Clearly, UDNR is the uniform version of the zoo principle DNR, defined as:

\[(\forall A^1)(\exists f^1)(\forall e^0)[f(e) \neq \Phi^A_e(e)]\]  

(DNR)

The principle DNR was first formulated in [18] and is even strictly implied by WWKL (See [1]) where the latter principle sports some Reverse Mathematics equivalences ([29,42,43]) but is not a Big Five system. Nonetheless, we shall prove that UDNR \(\leftrightarrow (\exists^2)\), where the second principle is the functional version of arithmetical comprehension, the third Big Five system of RM. In other words, the ‘exceptional’ status of DNR disappears completely if we consider its uniform version.

More generally, in Sections 3, 4, and 6, we show that a number of uniform zoo-principles are equivalent to arithmetical comprehension, based on our results for UDNR. In Section 5, we formulate a general\(^1\) template for classifying (past and future) zoo-principles in the same way. As will become clear, our template provides a uniform and elegant approach to classifying uniform principles originating from the RM zoo; In other words, the RM zoo seems to disappear at the uniform level (but see Remark 5.13). As to a possible explanation for this phenomenon, the axiom of extensionality plays a central role in our template, as discussed in Remark 4.21.

Another key ingredient of the template is the presence of ‘little structure’ (which is e.g. typical of statements from combinatorics) on the objects in RM zoo principles, which gives rise to non-robust theorems in the sense of Montalbán ([29]), as discussed in Section 5.2.

The results in this paper are formulated as theorems of Kohlenbach’s base theory RCA\(\omega\) from [25], but are often proved in the base theory RCA\(\omega^1\), a conservative extension (See Theorem 2.1 below) with certain axioms from Nelson’s internal set theory ([31]) of RCA\(\omega\). The latter system is in turn a conservative extension of RCA\(\omega\), the usual ‘base theory’ of Reverse Mathematics. We shall introduce RCA\(\omega^\Omega\) in Section 2. In other words, Nonstandard Analysis, in the form of Nelson’s internal approach, is used as a tool in this paper.

Finally, as to conceptual considerations, the above-mentioned ‘disappearance’ of the RM zoo suggests that Kohlenbach’s higher-order RM ([25]) is not just ‘RM with higher types’, but a separate field of study giving rise to a completely different classification; In particular, the latter comes equipped with its own notion of exceptionality, notably different from the one present in Friedman-Simpson-style RM. In light of the results in Section 6, one could go even as far as saying that, at the uniform level, weak König’s lemma is more exceptional than e.g. Ramsey’s theorem for pairs, as the latter is more robust than the former, due to the behaviour of their contrapositions (at the uniform level).

In conclusion, the stark contrast in exceptional behaviour between principles from the RM zoo and their uniform counterparts, speaks in favour of the study of higher-order RM. Notwithstanding the foregoing, ‘unconditional’ arguments for the study of higher-order RM are also available, as discussed in Section 6.4.

\(^1\)For instance, as shown in Section 6, our template is certainly not limited to \(\Pi^1_2\)-formulas, and surprisingly even applies to contrapositions of RM zoo principles, including the Ramsey theorems.
2. About and around the base theory RCA$^\Omega_0$

In this section, we introduce the base theory RCA$^\Omega_0$ in which we will work. We discuss some basic results and introduce some notation.

2.1. The system RCA$^\Omega_0$. In two words, RCA$^\Omega_0$ is a conservative extension of Kohlenbach’s base theory RCA$^\omega_0$ from [25] with certain axioms from Nelson’s Internal Set Theory ([31]) based on the approach from [4,5]. This conservation result is proved in [5], while certain partial results are implicit in [4]. In turn, the system RCA$^\omega_0$ is a conservative extension of the ‘usual’ base theory RCA$^0$ for the second-order language by [25, Prop. 3.1].

In Nelson’s syntactic approach to Nonstandard Analysis ([31]), as opposed to Robinson’s semantic one ([33]), a new predicate ‘st(x)’, read as ‘x is standard’ is added to the language of ZFC. The notations (∀_{st}x) and (∃_{st}y) are short for (∀x)(st(x)→...) and (∃y)(st(y)∧...). The three axioms Idealization, Standard Part, and Transfer govern the new predicate ‘st’ and yield a conservative extension of ZFC. We follow Nelson’s notation, repeated in Notation 2.5 below.

Nelson’s approach has been studied in the context of higher-type arithmetic in e.g. [3–5]. Following Nelson’s approach in arithmetic, define RCA$^\Omega_0$ as:

$$\text{E-PRA}^{\omega^*}_0 + \text{QF-AC}^{1,0} + \text{I} + \text{HAC}_{\text{int}} + \text{PF-TP}^\forall$$ (2.1)

from [5, §3.2-3.3]. Nelson’s idealization axiom I is available in RCA$^\Omega_0$, but to guarantee that the latter is a conservative extension of RCA$^\omega_0$, Nelson’s axiom Standard Part must be limited to HAC$_{\text{int}}$, while Nelson’s axiom Transfer has to be limited to universal formulas without parameters, as in PF-TP$^\forall$.

2.1. Theorem. The system RCA$^\Omega_0$ is a conservative extension of RCA$^\omega_0$. The system RCA$^\Omega_0$ is a $\Pi^0_2$-conservative extension of PRA.

Proof. As noted in [5 §3.2], the results in [5 Theorem 5] are also valid for Peano arithmetic replaced by PRA (and the associated reduction of recur sor constants in the language). Since QF-AC$^{1,0}$ is internal, the theorem now follows. □

The conservation result for E-PRA$^{\omega^*}_0 + \text{QF-AC}^{1,0}$ is trivial. Furthermore, omitting PF-TP$^\forall$ from (2.1), the theorem is implicit in [4 Cor. 7.6] as the proof of the latter goes through as long as EFA is available. We now discuss the Transfer principle of RCA$^\Omega_0$ in more detail, as it is essential for our results.

2.2. The Transfer principle of RCA$^\Omega_0$. In this section, we discuss the Transfer principle included in RCA$^\Omega_0$, which is as follows.

2.2. Principle (PF-TP$^\forall$). For any internal formula $\varphi(x^\gamma)$ with all parameters shown, we have $\langle \forall^{st} x^\gamma \rangle \varphi(x) \rightarrow (\forall x) \varphi(x)$.

A special case of the previous axiom can be found in Avigad’s system NPRA$^\omega$ from [3]. The omission of parameters in PF-TP$^\forall$ is essential, as is clear from the following theorem, relating to:

$$(\forall^{st} f^1)(\forall^{st} n)(f(n) = 0 \rightarrow (\forall n)f(n) = 0), \quad (\Pi^0_1\text{-TRANS})$$

$$(\exists\varphi^2)(\forall g^1)([\exists x^0]g(x) = 0 \leftrightarrow \varphi(g) = 0]. \quad (\exists^2)$$

Note that standard parameters are allowed in $f$, and that $(\exists^2)$ is the functional version of ACA$^0$ ([40 III]), i.e. arithmetical comprehension.
2.3. Theorem. The system \( \text{RCA}_0^\Omega \) proves \( \Pi_1^0\)-TRANS \( \leftrightarrow (\exists^2) \)

Proof. By \([5\ Cor. 12]\). \(\square\)

Besides being essential for the proof of the previous theorem, PF-TP\(\gamma\) is also convenient for other reasons. Indeed, as discussed in the next remark, we may assume all functionals defined without parameters are standard, thanks to PF-TP\(\gamma\).

2.4. Remark (Standard functionals). First of all, given the existence of a functional, like e.g. the existence of the fan functional as follows:

\[
(\exists^0\Theta^3)(\forall \varphi^2)(\forall f^1, g^1 \leq 1)[\overline{\{ \Omega(\varphi) = 0 \overline{\{ \Omega(\varphi) \rightarrow \varphi(f) = \varphi(g) \}}\}],
\]

we immediately obtain, via the contraposition of PF-TP\(\gamma\), that

\[
(\exists^0\Theta^3)(\forall \varphi^2)(\forall f^1, g^1 \leq 1)[\overline{\{ \Theta(\varphi) = 0 \overline{\{ \Theta(\varphi) \rightarrow \varphi(f) = \varphi(g) \}}\]]. \tag{2.2}
\]

In other words, we may assume that the fan functional is standard. The same holds for any functional of which the definition does not involve additional parameters.

Secondly, again for the fan functional, we may assume \(\Omega(\varphi)\) is the least number as in \([\text{MUC}]\), which implies that \(\Theta(\varphi)\) from (2.2) can also be assumed to have this property. However, then \(\Theta(\varphi) = 0 \Omega(\varphi)\) for any \(\varphi\), implying \(\Theta = \Omega\), i.e. if it exists, the fan functional is unique and standard. The same again holds for any uniquely-defined functional of which the definition does not involve additional parameters. The previous observations prompted the addition to \(\text{RCA}_0^\Omega\) of axioms reflecting the uniqueness and standardness of certain functionals (See \([5\ §3.3]\)).

We finish this section with some remarks. First of all, we shall follow Nelson’s notations as in \([5]\), and given as follows.

2.5. Remark (Standardness). As suggested above, we write \((\forall x^0 x^r)\Phi(x^r)\) and also \((\exists x^0 x^r)\Psi(x^r)\) as short for \((\forall x^r) \overline{\{ \text{st}(x^r) \rightarrow \Phi(x^r) \}}\) and \((\exists x^0) \overline{\{ \text{st}(x^r) \land \Psi(x^0) \}}\). We also write \((\forall x^0 \in \Omega)\Phi(x^0)\) and \((\exists x^0 \in \Omega)\Psi(x^0)\) as short for \((\forall x^0) \overline{\{ \text{st}(x^0) \rightarrow \Phi(x^0) \}}\) and \((\exists x^0) \overline{\{ \text{st}(x^0) \land \Psi(x^0) \}}\). Furthermore, if \(\neg \text{st}(x^0)\) (resp. \(\text{st}(x^0)\)), we also say that \(x^0\) is ‘infinite’ (resp. finite) and write \(\text{‘}0 \in \Omega\text{‘}\). Finally, a formula \(A\) is ‘internal’ if it does not involve \text{st}, and \(A^{\text{st}}\) is defined from \(A\) by appending ‘\text{st}’ to all quantifiers (except bounded number quantifiers).

Secondly, we shall use the usual notations for rational and real numbers and functions as introduced in \([25\ p. 288-289]\) and \([40\ I.8.1]\) for the former. Nonetheless, we try to avoid set-theoretic notation involving \(Q\) and \(R\) to avoid confusion.

2.6. Remark (Real number). A (standard) real number \(x\) is a (standard) fast-converging Cauchy sequence \(q^1_{(i)}\), i.e. \((\forall n^0, i^0)(|q_n - q_{n+i}| < 0 \frac{1}{2^r})\). We freely make use of Kohlenbach’s ‘hat function’ from \([25\ p. 289]\) to guarantee that every sequence \(f^1\) can be viewed as a real. Two reals \(x, y\) represented by \(q^1_{(i)}\) and \(r^1_{(i)}\) are equal, denoted \(x = y\), if \((\forall n)(|q_n - r_n| \leq \frac{1}{2^n})\). Inequality \(<\) is defined similarly.

We also write \(x \approx y\) if \((\forall n)(|q_n - r_n| \leq \frac{1}{2^n})\) and \(x \gg y\) if \(x > y \land x \neq y\). Functions \(F\) mapping reals to reals are represented by functionals \(\Phi^{\text{st}}\) such that \((\forall x, y)(x = y \rightarrow \Phi(x) = \Phi(y))\), i.e. equal reals are mapped to equal reals. Finally, sets are denoted \(X^1, Y^1, Z^1\),... and are given by their characteristic functions \(f^1_{X}\), i.e. \((\forall x^0)(x \in X \leftrightarrow f_X(x) = 1)\), where \(f^1_{X}\) is assumed to be binary.

Finally, the notion of equality in \(\text{RCA}_0^\Omega\) is important to our enterprise.
2.7. Remark (Equality). The system RCAω₀ only includes equality between natural numbers ‘=₀’ as a primitive. Equality ‘=ₓ’ for type τ-objects x, y is then defined as follows:

\[ x =_\tau y \equiv (\forall z^\tau)(xz_1 \ldots z^\tau_k)(xz_1 \ldots z^\tau_k =_0 yz_1 \ldots z^\tau_k) \tag{2.3} \]

if the type τ is composed as τ = (τ₁ → \ldots → τₖ → 0). In the spirit of Nonstandard Analysis, we define ‘approximate equality ≈’ as follows:

\[ x \approx_\tau y \equiv (\forall x\in \tau)(\forall y\in \tau)(x_1 \ldots z^\tau_k)(xz_1 \ldots z^\tau_k =_0 yz_1 \ldots z^\tau_k) \tag{2.4} \]

with the type τ as above. Furthermore, the system RCAω₀ includes the axiom of extensionality as follows:

\[ (\forall \varphi^{\text{st}})(\forall x^0)(\forall y^0)(x =_\tau y \to \varphi(x) =_\tau \varphi(y)). \tag{E} \]

However, as noted in [4, p. 1973], the axiom of standard extensionality [E]₀ cannot be included in RCAω₀. Although [E]₀ is not available, certain classes of objects can be proved to be standard extensional (See Theorem 3.5 and Corollary 3.7).

3. UNIFORM DNR

3.1. Classifying UDNR. In this section, we prove that the principle UDNR from the introduction is equivalent to arithmetical comprehension (\(\exists^2\)) from Section 2.2.

As to notation, the formula in square brackets in (DNR) is actually an abbreviation for (\(\forall m, n\))(\(\Phi_{e,n}(e) = m \to f(e) \neq m\)). Using Kleene’s normal form theorem ([11, p. 15]), we can write the formula \(\Phi_{e,n}(e) = m\) in terms of Kleene’s primitive recursive function \(U\) and predicate \(T\), and the characteristic function for \(A\). Thus, the only parameters in the square bracketed formula in (DNR) are \(f, e, A\).

The principle (DNR) and its kin seem weak in that they only provide negative information about the diagonal recursive function \(f\) claimed to exist; i.e., we only know what \(f\) is not, namely \(\Phi_{e,n}\), everywhere the latter exists. Another way in which these principles seem weak is that there is no immediate reason why the functional \(\Psi\) from (UDNR) should satisfy standard extensionality as in (E)₀. Indeed, even if \(A \approx_1 B\), it seems perfectly consistent with (UDNR) that, for all standard \(e^0\), \(\Psi(A)(e) \neq \Phi_{e,n}(e)\), \(\Psi(B)(e) \neq \Phi_{e,n}(e)\), and \(\Psi(A)(e) \neq \Psi(B)(e)\).

We now first prove that (UDNR) does not follow from WKL. To this end, we shall use the so-called fun functional (MUC), introduced in Remark 2.6. In [24, Prop. 3.15], Kohlenbach proves that RCAω₀ + (MUC) is a conservative extension of WKL₀ for the language of second-order arithmetic.

3.1. Theorem. The principles UDNR and (MUC) are inconsistent over RCA₀₀:\n
Proof. First of all, note that the functional \(\Psi\) from UDNR takes sets as input, which are given by their characteristic function, as discussed in Remark 2.6. In other words, \(\Psi\) actually takes binary sequences as input. Furthermore, we may assume \(\Psi\) (and also \(\Omega^3\) from (MUC)) is standard by Remark 2.4.

Secondly, the principles (MUC) and (\(\exists^2\)) are inconsistent as the latter is equivalent to the existence of a discontinuous type 2 functional ([25, Prop. 3.7]), while the former clearly implies that all such functionals are continuous. Furthermore, (\(\exists^2\)) is also equivalent to \(\Pi^0_1\)-TRANS over RCA₀₀ by [3, Cor. 12], as noted in Section 2. Hence, the principle (MUC) implies \(\neg\Pi^0_1\)-TRANS, i.e., there is some standard \(h^1\) such that \((\forall m)(h(m) = 0)\) and \((\exists m)(h(m) \neq 0)\).
Next, we prove that the functional $\Psi$ from UDNR is standard extensional assuming (MUC), i.e. we prove that

$$(\forall^* A^1, B^1)(A \approx_1 B \rightarrow \Psi(A) \approx_1 \Psi(B)).$$

To this end, assume that for some standard sets $A_0^1, B_0^1$ (which are actually binary sequences) and standard $e_0$, we have $A_0 \approx_1 B_0 \wedge \Psi(A_0)(e_0) \neq 0 \neq \Psi(B_0)(e_0)$. Then define the standard functional $\psi^2$ as $\psi^2(\cdot) := \Psi(\cdot)(e_0)$. Now let $\Omega^3$ be the standard functional from (MUC) and note that $\Omega^3$ is standard extensional, assuming (MUC).

Finally, fix a standard pairing function $\pi^1$ and its inverse $\xi^1$. Now let the standard number $e_1$ be the code of the following program: On input $n$, set $k = n$ and check if $k \in A$ and if so, return the second component of $\xi(k)$; If $k \notin A$, repeat for $k + 1$. Intuitively speaking, $e_1$ is such that $\Phi_{e_1}(n)$ outputs $m$ if starting at $n = k$, we eventually find $\pi((l,m)) \in A$, and undefined otherwise. Furthermore, define $C = 0$ (which is the sequence $00\ldots$) and

$$D = \{\pi(e, \Psi(C)(e_1)) : h(e) \neq 0 \wedge (\forall i < e) h(i) = 0\},$$

where $h$ is the exception to $\Pi^0_1\text{-TRANS}$ from the second paragraph of this proof. Note that $C \approx_1 D$ by definition, implying that $\Psi$ satisfies $\Psi(C) \approx_1 \Psi(D)$ due to its standard extensionality (proved in the third paragraph of this proof). However, the latter combined with UDNR gives us:

$$\Psi(C)(e_1) = 0 \neq \Psi(D)(e_1)$$

for large enough (infinite) $m_0$. This contradiction implies that (MUC) and UDNR are inconsistent. Note that we only applied the functional from UDNR to standard sets, i.e. the theorem still follows if ‘$(\exists \Psi^1 \forall A^1)(\forall A^1)$’ in UDNR is replaced by ‘$(\exists \Psi^1 \forall A^1)$’.

Assuming an uncontroversial consistency result, we have:

3.2. Corollary. The system $\text{RCA}_0^\Omega + \text{WKL}$ cannot prove UDNR.

Proof. In [23, p. 293], Kohlenbach notes that (MUC) implies that every real function on $[0,1]$ is uniformly continuous (with a modulus), yielding WKL via [40 IV.2.3]. Hence, if $\text{RCA}_0^\Omega + \text{WKL} \vdash \text{UDNR}$, then also $\text{RCA}_0^\Omega + \text{MUC} \vdash \text{UDNR}$, yielding a contradiction.

We now turn the negative result of the theorem into a positive result. To this end, let UDNR$^+$ be the statement that there is a standard functional $\Psi$ such that for all standard $A^1$ we have $(\forall e^0)[\Psi(A)(e) \neq \Phi^A_e(e)]$ and $\Psi$ is standard extensional.

3.3. Theorem. In $\text{RCA}_0^\Omega$, we have UDNR$^+ \leftrightarrow \Pi^0_1\text{-TRANS} \leftrightarrow (\exists^2)$.

Proof. The second equivalence follows by [5 Cor. 12]; To prove $\Pi^0_1\text{-TRANS} \rightarrow \text{UDNR}^+$, one can also use the latter. Alternatively, define:

$$\Theta(A, M)(e) := \begin{cases} 
\Phi_{e,M}(e) + 1 & (\exists y, s \leq M)(\Phi^A_{e,s}(e) = y) \\
0 & \text{otherwise} 
\end{cases}$$

(3.2)
This functional is called the canonical approximation of the functional from UDNR. Assuming $\Pi^0_1$-TRANS, the functional from (3.2) clearly satisfies:

\[(\forall^* e^0, A^1)(\forall M, N \in \Omega) [\Theta(A, M)(e) = \Theta(A, N)(e)]. \tag{3.3}\]

The formula (3.3) clearly implies

\[(\forall^* e^0, A^1)(\exists k^0)(\forall M, N \geq k) [\Theta(A, M)(e) = \Theta(A, N)(e)]. \tag{3.4}\]

Since RCA$_Ω^0$ proves minimisation for $\Pi^0_1$-formulas, there is a least $k$ as in (3.4), which must be finite by (3.3). Hence, we obtain:

\[(\forall^* e^0, A^1)(\exists^* k^0)(\forall M, N \geq k) [\Theta(A, M)(e) = \Theta(A, N)(e)]. \tag{3.5}\]

Applying HAC$_{\text{int}}$, there is a standard functional $\Psi^2$ such that

\[(\forall^* e^0, A^1)(\exists k^0 \in \Psi(A, e))(\forall M, N \geq k) [\Theta(A, M)(e) = \Theta(A, N)(e)]. \tag{3.6}\]

Now define $\Xi(A)(e)$ as $\Theta(A, \zeta(A, e))(e)$, where $\zeta(A, e)$ is the maximum of $\Psi(A, e)(i)$ for $i < |\Psi(A, e)|$. We then have that:

\[(\forall^* e^0, A^1)(\forall M \in \Omega) [\Theta(A, M)(e) = \Xi(A)(e)]. \tag{3.7}\]

By the definition of $\Theta$ in (3.2), $\Xi$ is standard extensional and satisfies, for standard $A$, the formula $(\forall^* e^0)[\Xi(A)(e) \neq \Phi^A_e(e)]$, where the ‘st’ predicates in the latter formula may be dropped by $\Pi^0_1$-TRANS. Hence, $\Xi$ is as required for UDNR$^+$. Indeed, in the final part of the latter proof, a contradiction is derived from $\neg \Pi^0_1$-TRANS and the standard extensionality of the functional from UDNR. In exactly the same way, we can obtain a contradiction from UDNR$^+ \land \neg \Pi^0_1$-TRANS, implying that UDNR$^+ \rightarrow \Pi^0_1$-TRANS.

The proof of the theorem hinges on the standard extensionality included in UDNR$^+$. We now discuss standard extensionality in more detail.

3.4. Remark. As noted in Remark 2.7, the (standard by Remark 2.4) functional $\varphi$ from (3.2) is standard extensional by Theorem 2.3. To see this, use $\Pi^0_1$-TRANS to obtain $\varphi(f) = 0 \iff (3\forall x^0) f(x) = 0$ for standard $f^1$, immediately implying $(\forall^* f^1, g^1)(f \approx g \rightarrow \varphi(f) = \varphi(g))$, i.e. the functional $\varphi$ from (3.2) is standard extensional. Note that (3.2) provides ‘positive’ information about $\varphi$: We know that $\varphi(f) = 0$ whenever $(3x)f(x) = 0$ and vice versa. Nonetheless, since standard extensionality (3.2) cannot be included in RCA$_Ω^0$ (See [4] p. 1973), not all standard functionals are standard extensional. In particular, there does not seem to be any reason why the (standard by Remark 2.4) functional from UDNR is standard extensional. An argument like for (3.2) does not seem likely, especially since we only have ‘negative’ information about $\Psi$ from UDNR, namely that $\Psi(A)(e) \neq \Phi^A_e(e)$.

Surprisingly, we can prove the following theorem.

3.5. Theorem. In RCA$_Ω^0 +$ UDNR, there is a standard and standard extensional functional $\Psi$ such that $(\forall A^1)(\forall e^0)[\Psi(A)(e) \neq \Phi^A_e(e)]$.

Proof. Let $\Psi$ be as in UDNR and consider the associated axiom of extensionality:

\[(\forall A^1, B^1)(A =_1 B_1 \rightarrow \Psi(A) =_1 \Psi(B)). \tag{3.8}\]

In light of the definition of $\approx =_1$, (3.8) can be brought in the following form:

\[(\forall A^1, B^1, k^0)(\exists N^0)[\overline{A}N =_0 \overline{B}N \rightarrow \Psi(A)(k) =_0 \Psi(B)(k)] \tag{3.9}\]
Hence, UDNR implies the following sentence:

\[(\exists \psi^{1\to 1})(\forall a^1)(\psi(a) \neq \phi^2_a(e))
\land (\forall a^1, b^1, k^0)[\overline{a}N = \overline{b}N \rightarrow \psi(a)(k) = \psi(b)(k)]\].

Applying QF-AC\(^{1,0}\) to the second conjunct in the (big) square brackets, we obtain

\[(\exists \psi^{1\to 1}, \xi^{(1\times 1\times 0)\to 0})(\forall a^1)(\psi(a) \neq \phi^2_e(e))
\land (\forall a^1, b^1, k^0)[\overline{a}\xi(a, b, k) = \overline{b}\xi(a, b, k) \rightarrow \psi(a)(k) = \psi(b)(k)]\].

Since the formula in big square brackets is internal and does not contain parameters besides \(\xi\) and \(\psi\), we may apply PF-TP\(\forall\) to obtain:

\[(\exists \psi^{1\to 1}, \xi^{(1\times 1\times 0)\to 0})(\forall a^1)(\psi(a) \neq \phi^2_e(e))
\land (\forall a^1, b^1, k^0)[\overline{a}\xi(a, b, k) = \overline{b}\xi(a, b, k) \rightarrow \psi(a)(k) = \psi(b)(k)]\].

Now, for standard \(a^1, b^1\) such that \(a \approx_1 b\), we have \(\overline{a}\xi(a, b, k) = \overline{b}\xi(a, b, k)\) for all standard \(k\), as \(\xi(a, b, k)\) is standard for standard input. Hence, \(\psi(b)(k) = \psi(a)(k)\) for all standard \(k\) follows, which is just \(\psi(a) \approx_1 \psi(b)\), i.e. \(\psi\) is also standard extensional, and we are done.

Note that it is essential for the proof that the functional from UDNR has low type, namely \(1 \to 1\), so that we may apply QF-AC\(^{1,0}\). Furthermore, although standard extensionality \((\exists^2)\) cannot be included in RCA\(^\omega\)\(0\) (See [4, p. 1973]), the previous proof shows that any functional ‘of low enough type’ and ‘with a defining axiom without parameters’ can be proved to be standard extensional. Hence, we obtain the following theorem, for which the axiom of extensionality is essential.

3.6. **Corollary.** In RCA\(^\omega\)\(0\), we have UDNR \(\Leftrightarrow (\exists^2)\).

**Proof.** We prove the equivalence UDNR \(\Leftrightarrow (\exists^2)\) in RCA\(^\omega\)\(0\). The latter is a conservative extension of RCA\(^\omega\)\(0\) for the original language. The reverse implication is immediate as \((\exists^2)\) can decide whether a Turing machine halts or not. The forward implication follows from the theorem and Theorem 3.3 (or directly form the proof of Theorem 3.1). The statement in the theorem is stronger than UDNR\(^+\). \(\square\)

3.7. **Corollary.** In RCA\(^\omega\)\(0\), every term \(t^{1\to 1}\) is standard extensional.

**Proof.** As in the proof of the theorem, standard extensionality follows from applying QF-AC\(^{1,0}\) and PF-TP\(\forall\) to \((3.9)\) with \(\psi^{1\to 1}\) replaced by the term \(t^{1\to 1}\). \(\square\)

3.2. **Classifying the strong Tietze extension theorem.** In this section, we study a uniform version of the Tietze (extension) theorem. Non-uniform versions of the Tietze theorem are studied in [40, II.7] and [18]. We are interested in the ‘strong’ Tietze theorem [18, 6.15.(5)] since it implies DNR and is implied by WWKL (See [18, §6]). Furthermore, Montalbán lists the status of the Tietze theorem as an open question in Reverse Mathematics in [29, Question 16]. We will establish an equivalence between \((\exists^2)\) (and hence UWKL) and the uniform strong Tietze theorem. We make essential use of Corollary 3.6.
First of all, since the Tietze theorem from [18] 6.15.(5) is about uniformly continuous functions with a modulus, it does not really matter which definition of continuity is used by [26] Prop. 4.4. Thus, let \( f^1 \in C_{\text{un}}(X) \) mean that \( f \) is continuous in the sense of Reverse Mathematics on \( X \), i.e. as in [40] II.6.1 or [18] Def. 2.7. Furthermore, let \( \mathcal{C}(X) \) be the Banach space used in the Tietze theorem [18] 6.15.(5) as defined in [18] p. 1454. Finally, we use the same definition for closed and separably closed sets as in [18].

3.8. Principle (UTIE). There is a functional \( \Psi^{(1 \times 1) \rightarrow 1} \) such that for closed and separably closed sets \( A \subseteq [0, 1] \) and for \( f \in C_{\text{un}}(A) \) with modulus of uniform continuity \( g \), we have \( \Psi(f, g, A) \in \mathcal{C}([0, 1]) \) and \( f \) equals \( \Psi(f, g, A) \) on \( A \).

We also study the following uniform version of Weierstraß’ (polynomial) approximation theorem. The non-uniform version is equivalent to WKL by [40] IV.2.5.

3.9. Principle (UWA). There is \( \Psi^{1 \rightarrow 1} \) such that
\[
(\forall f \in C_{\text{un}}[0, 1]) (\forall x^1 \in [0, 1], n^0) [\Psi(f)(n) \in \text{POLY} \land |f(x) - \Psi(f)(n)(x)| < \frac{1}{2^n}].
\]

3.10. Theorem. In RCA\(_0^\omega\), we have UWA \( \leftrightarrow \) UTIE \( \leftrightarrow \) (\( \mathcal{C}^2 \)).

Proof. As in the proof of [25] Prop. 3.14], it is straightforward to obtain UWA using (\( \mathcal{C}^2 \)) from the associated non-uniform proof, even when \( f \) is a type \( 1 \rightarrow 1 \) functional which happens to be \( \varepsilon \)-\( \delta \)-continuous. Indeed, it is well-known that \( \lim_{n \to \infty} B_n(f)(x) = f(x) \) uniformly for \( x \in [0, 1] \), if \( f \) is continuous on \( [0, 1] \) and \( B_n(f) \) are the associated Bernstein polynomials ([50] p. 6]). Using (\( \mathcal{C}^2 \)) it is then easy to define \( \Psi(f)(n) \) as the least \( N \) such \( \sup_{x \in [0, 1]} |B_N(f)(x) - f(x)| \leq \frac{1}{2^n} \).

For the proof of UTIE \( \rightarrow \) (\( \mathcal{C}^2 \)), we will make essential use of Corollary 3.6 and [18] [6]. In two words, we will ‘uniformise’ the proof of [18] Lemma 6.17.

First of all, by [18] Lemma 6.17], the strong Tietze theorem [18] 6.15.(5)] implies DNR. In this proof, a function \( f \) defined on a set \( C \) is constructed in RCA\(_0\) (See the proof of [18] Lemma 6.16]). This function satisfies all conditions of the strong Tietze theorem; In particular, it has a modulus of uniform continuity of \( f \). Applying [18] 6.15.(5)], one obtains \( F \in \mathcal{C}[0, 1] \), an extension of \( f \) to \( [0, 1] \).

Secondly, by the definition of \( \mathcal{C}(X) \) from [18] p. 1454], \( F \) is coded by a sequence of polynomials \( p_n \) such that \( \|p_n - F\| < \frac{1}{k} \), and we can define \( h(n) := \sharp(p_n) \). The latter is then such that \( (\forall \varepsilon^0)(h(e) \neq \Phi_\varepsilon(e)) \). The case of DNR where \( A \neq \emptyset \) is then straightforward. Indeed, the initial function \( f \) (from the proof of [18] Lemma 6.16]) is defined using a recursive counterexample to the Heine-Borel lemma. Such a counterexample can be found in [40] I.8.6] and clearly relativizes (uniformly) to any set \( A \). Let us use \( f_A \) to denote the function \( f \) obtained from the previous construction relative to the set \( A \), and let \( C_A \) and \( g_A \) be the relativized domain and modulus. Now let \( \Psi \) be the functional from UTEI and define \( \Xi^{1 \rightarrow 1} \) by
\[
\Xi(A) := \sharp(\Psi(f_A, g_A, C_A)),
\]
where \( f_A, g_A, \) and \( C_A \) are as in the previous paragraph of this proof. In the same way as in the proof of [18] Lemma 6.17], one proves that for any \( A^1 \), we have \( (\forall \varepsilon^0)(\Xi(A)(e) \neq \Phi_\varepsilon^1(e)) \). However, this means that UTIE implies UDNR, which implies (\( \mathcal{C}^3 \)) by Corollary 3.6.

Next, to prove the implication UWA \( \rightarrow \) UTIE, note that Simpson proves an effective version of the Tietze theorem in [40] II.7.5]. Following the proof of the
latter, it is clear that there is a functional $\Phi$ in $\text{RCA}_0^\omega$ such that for closed and separably closed $A$ and $f \in C_{\text{rm}}(A)$, the image $\Phi(f, g, A) \in C_{\text{rm}}[0, 1]$ is the extension of $f$ to $[0, 1]$ provided by [10] II.7.5. For $\Psi$ as in UWA, the functional $\Psi(\Phi(f, g, A))$ is as required by UTIE, and hence $(\exists^2)$ follows by the above. □

Let UTIE’ and UWA’ be the versions of UTIE and UWA with ‘$f \in C[0, 1]$’ instead of the Reverse Mathematics definition of continuity. The following corollary is immediate from the proof of the theorem.

3.11. Corollary. In $\text{RCA}_0^\omega$, we have $\text{UWA}' \leftrightarrow \text{UTIE}' \leftrightarrow (\exists^2)$.

In the proof of the theorem, we established $\text{UTIE} \rightarrow (\exists^2)$ by showing that UTIE $\rightarrow \text{UDNR}$, and then applying Corollary 3.6. The latter implication goes through because of the uniformity of the proof of DNR from the strong Tietze theorem (See [18, Lemma 6.17]). In general, if WKL $\rightarrow T \rightarrow \text{DNR}$ and the proof of second implication is sufficiently uniform, then $UT \leftrightarrow (\exists^2)$. We now list some examples of such theorems $T$ in the following remark.

3.12. Remark (Immediate consequences). First of all, let DNR$_k$ be DNR where the function $f^1$ satisfies $f \leq_k k$, and let UDNR$_k$ be UDNR with the same restriction for $\Psi(A)$. Clearly, for any $k \geq 1$, we have UDNR$_k \leftrightarrow (\exists^2)$.

Secondly, let RKL be the ‘Ramsey type’ version of WKL from [15] and let URKL be its obvious uniform version. In $\text{RCA}_0^\omega$, we have that URKL $\leftrightarrow (\exists^2)$, as it seems the proof of RKL $\rightarrow \text{DNR}$ from [15] Theorem 8 can be uniformized. Indeed, in this proof, RKL is applied to a specific tree $T_0$ from [15] Lemma 7 to obtain a certain set $H$. Then the function $g$ is defined such that $W_{g(e)}$ is the least $e + 3$ elements of $H$. This function $g$ is then shown to be fixed-point free, which means it gives rise to a DNR-function by [41, V.5.8, p. 90]. Noting that the tree $T_0$ has positive measure, we even obtain WRKL $\rightarrow \text{DNR}$ (and the associated uniform equivalence to $(\exists^2)$), where the tree has positive measure in the latter (See [7]).

Thirdly, let SEM be the stable Erdős-Moser theorem from [22]. In [32, Theorem 3.11], the implication SEM $\rightarrow \text{DNR}$ is proved, and the proof is clearly uniform. Hence, for USEM the uniform version of SEM, we have USEM $\leftrightarrow (\exists^2)$. The same obviously holds for EM, the version of SEM without stability conditions.

In the next section, we shall study principles from the zoo for which a ‘uniformising’ proof as in the previous remark is not immediately available. We finish this section with a remark on the Reverse Mathematics zoo ([10]).

3.13. Remark (A higher-order zoo). Since DNR is rather ‘low’ in the zoo, it is to be expected that uniform versions of ‘most’ of the zoo’s principles will behave as UDNR, i.e. turn out equivalent to $(\exists^2)$ (as we will establish below). In particular, since Friedman-Simpson style Reverse Mathematics is limited to second-order arithmetic, the proof of Theorem 3.5 will go through for principles other than UDNR as the associated functionals can only have type $1 \rightarrow 1$ (by the limitation to second-order arithmetic). However, it is conceivable that uniform higher-type principles, to which the proof of Theorem 3.5 does not apply in the absence of the axiom of choice, will populate a ‘higher-order’ RM zoo.
4. Classifying the zoo

In this section, we classify uniform versions of principles from the Reverse Mathematics zoo. After these case studies, we shall formulate in Section 5 a template which seems sufficiently general to apply to virtually any (past or future) principle from the Reverse Mathematics zoo.

4.1. Ascending and descending sequences. In this section, we study the uniform version of the ascending-descending principle ADS (See e.g. [20 Def. 9.1]).

4.1. Definition. For a linear order \( \preceq \), a sequence \( x_n^1 \) is ascending if \( x_0 \prec x_1 \prec \ldots \) and descending if \( x_0 \succ x_1 \succ \ldots \).

4.2. Definition (ADS). Every infinite linear ordering has an ascending or a descending sequence.

Recall that \( \text{LO}(X^1) \) is short for ‘\( X^1 \) is a linear order'; We append ‘\( \infty \)' to ‘\( \text{LO} \)' to stress that \( X \) is an infinite linear order, meaning that its field is not bounded by any number (See [10 V.1.1]). With this in place, uniform ADS is as follows:

4.3. Definition (UADS).

\[
(\exists n)(\forall X)([\text{LO}_\infty(X) \rightarrow (\forall n^0)(\Phi(X)(n) \prec_X (\Phi(X)(n+1))) \\
\quad \lor (\forall m^0)(\Phi(X)(m) \succ_X (\Phi(X)(m+1))].
\]

(4.1)

Note that we can decide which case of the disjunction of UADS holds by testing \( \Phi(X)(0) \prec_X \Phi(X)(1) \). We have the following theorem.

4.4. Theorem. In \( \text{RCA}_0^\omega \), we have UADS \( \leftrightarrow (\exists^2) \).

Proof. We will prove the equivalence in \( \text{RCA}_0^\omega \), a conservative extension of \( \text{RCA}_0^\omega \). For the forward implication, we first of all briefly repeat the proof of Theorem 3.5 for UADS instead of UDNR. Let \( A(\Psi, X) \) be the formula in square brackets in (4.1). Then UADS together with the axiom of extensionality implies:

\[
(\exists n)(\forall X)(A(\Psi, X) \land (\forall Y^1, Z^1)(Z = 1 Y \rightarrow (\Psi(Y) = 1 (\Psi(Z))].
\]

(4.2)

Similar to the proof of Theorem 3.5, resolve \( = 1 \) in (4.2), bring the type 0-quantifiers to the front of the second disjunct, and apply QF-AC\(^{1,0} \) to obtain:

\[
(\exists n, \Xi)[(\forall X^1)\text{A}(\Psi, X) \land (\forall Y^0, Z^0)] \Xi(\Xi(Y, Z, k) = \Xi(\Xi(Y, Z, k) \rightarrow (\Psi(\Xi) = k = (\Xi(Z)\Xi(k)).
\]

Applying PF-TP\( \forall \) implies that \( \Psi \) and \( \Xi \) may be taken to be standard. However, for standard \( Y^1 = Z^1 \), this implies \( \Xi(\Xi(Y, Z, k) = \Xi(\Xi(Y, Z, k) \rightarrow (\Psi(\Xi) = k = \Xi(Z)k) \).

Secondly, assume UADS and suppose \( \Pi_0^1 \)-TRANS is false, i.e. there is a standard \( h^1 \) such that \( (\forall n)(h(n) = 0 \land (\exists m) h(m) \neq 0 \). Now let \( m_0 \) be the least number such that \( h(m_0) \neq 0 \) and define the ordering ‘\( \prec \)' as follows:

\[
\ldots \prec m_0 + 2 \prec m_0 + 1 \prec 0 \prec 1 \prec 2 \prec \ldots \prec m_0.
\]

\( \text{Here, ‘infinite’ should not be confused with the notation ‘} \) (4.3)
It is straightforward to define the *standard* ordering \( < \) using the function \( h \). Now consider the usual strict ordering \( <_0 \) and note that \( (\prec) \approx_1 (\preceq_0) \) (with some abuse of notation in light of [40, V.1.1]). By the standardness of \( \prec \) and standard extensionality for the standard \( \Psi \), we have \( \Psi (\prec) \approx_1 \Psi (\preceq_0) \) (again with some abuse of notation). However, this leads to a contradiction as \( \preceq_0 \) only has ascending infinite sequences, while \( \prec \) only has descending infinite sequences. Indeed, while only the first case in (4.1) can hold for \( \Psi (\prec) \), only the second case can hold for \( \Psi (\preceq_0) \). But then \( \Psi (\prec) \approx_1 \Psi (\preceq_0) \) is impossible. This contradiction guarantees that \( \Pi^0_1 \text{-TRANS} \), and the consequent is equivalent to \( (\exists^2) \) by [1] Cor. 12.

For the reverse implication, \( (\exists^2) \) implies ADS and hence every infinite linear order satisfies either \( (\forall n^0)(\exists m^0)(m > X n) \) or \( (\forall k^0)(\exists p^0)(k > X l) \). Since \( (\exists^2) \) can decide which formula holds, the search operator \( (\mu^2) \) (equivalent to \( (\exists^2) \) by [3] Cor. 12] or [27 Cor. 3.5]) now allows us to define the functional \( \Psi \) from UADS.

In [19 Prop. 3.7], it is proved that ADS is equivalent to the principle CCAC. In light of the uniformity of the associated proof, the uniform version of the latter is also equivalent to \( (\exists^2) \). Furthermore, the equivalence in the previous theorem translates into a result in *constructive* Reverse Mathematics (See [23]) as follows.

4.5. Remark (Constructive Reverse Mathematics). The ordering \( \prec \) defined in (4.3) yields a proof that ADS \( \rightarrow \Pi^0_1 \text{-LEM} \) over the (constructive) base theory from [23]. Indeed, for a function \( h^1 \), define the ordering \( \prec_h \) from Footnote 3. By ADS, there is a sequence \( x_n \) which is either ascending or descending in \( \prec_h \). It is now easy to check that if \( x_0 \prec_h x_1 \), then \( (\forall n)h(n) = 0 \), and if \( x_0 \succ_h x_1 \) then \( (\forall n)h(n) = 0 \). Hence, ADS provides a way to decide whether a \( \Pi^0_0 \) -formula holds or not, i.e. the law of excluded middle limited to \( \Pi^0_1 \) -formulas.

Next, we consider a special case of ADS. The notion of *discrete* and *stable* linear orders from [20 Def. 9.15] is defined as follows:

4.6. Definition. [Discrete and stable orders] A linear order is *discrete* if every element has an immediate predecessor, except for the first element of the order if there is one, and every element has an immediate successor, except for the last element of the order if there is one. A linear order is *stable* if it is discrete and has more than one element, and every element has either finitely many predecessors or finitely many successors. (Note that a stable order must be infinite.)

Again, to be absolutely clear, the notion of ‘finite’ and ‘infinite’ in the previous definition constitutes the ‘usual’ *internal* definitions of infinite orders in \( \text{RCA}_0^w \) and have nothing to do with our notation ‘\( M \) is infinite’ for \( \neg \text{st}(M^0) \). In particular, note the type mismatch between orders and numbers.

Now denote by SADS the principle ADS limited to stable linear orderings, and let USADS be its uniform version.

4.7. Corollary. In \( \text{RCA}_0^w \), we have USADS \( \leftrightarrow (\exists^2) \),

Proof. Note that both the orderings \( \preceq_0 \) and \( \prec \) defined in the proof of the theorem are stable and this proof thus also yields USADS \( \rightarrow \Pi^0_1 \text{-TRANS} \).\( \square \)

\footnote{The order \( \prec \) from [43] can be defined as: \( i \prec j \) holds if \( i < j \land (\forall k \leq j - 1)h(k) = 0 \) or \( i > j \land (\exists k \leq j - 1)h(k) \neq 0 \) or \( i > j \land (\forall k \leq i - 1)h(k) \neq 0 \land (\forall k \leq j - 1)h(k) = 0 \).}
Let SRT$^2$ be Ramsey’s theorem for pairs limited to stable colourings (See e.g. [20, Def. 6.28]), and let USRT$^2$ be its uniform version where a functional $\Psi^{1\rightarrow 1}$ takes as input a stable 2-colouring of pairs of natural numbers and outputs an infinite homogeneous set.

4.8. Corollary. In RCA$^0_0$, we have USRT$^2_2 \leftrightarrow (\exists^2)$.

Proof. By [19] Prop. 2.8, we have SRT$^2_2 \rightarrow$ SADS. The proof of the latter is clearly uniform, yielding the forward implication by Corollary 4.7. By [34, Theorem 4.2], the reverse implication follows. □

We can prove similar results for SRAM and related principles from [11], but do not go into details. Our next corollary deals with the chain-antichain principle.

4.9. Definition. [CAC] Every infinite partial order $(P, \leq_P)$ has an infinite subset $S$ that is either a chain, i.e. $(\forall x^0, y^0 \in S)(x \leq_P y \lor y \leq_P x)$, or an antichain, i.e. $(\forall x^0, y^0 \in S)(x \neq y \rightarrow x \not<_P y \land y \not<_P x)$.

Let UCAC be the principle CAC with the addition of a functional $\Psi^{1\rightarrow 1}$ such that $\Psi(P, \leq_P)$ is the infinite subset which is either a chain or antichain. Let USCAC be UCAC limited to stable partial orders (See [19, Def. 3.2]).

4.10. Corollary. In RCA$^0_0$, we have UCAC $\leftrightarrow (\exists^2) \leftrightarrow$ USCAC.

Proof. In [19, Prop. 3.1], CAC $\rightarrow$ ADS is proved. The proof is clearly uniform, implying UCAC $\rightarrow$ UADS. By Theorem 4.4 we obtain the first forward implication in the theorem. The first reverse implication is proved as in the final part of the proof of Theorem 4.4. For the final reverse implication, the implication SCAC $\rightarrow$ SADS is proved in [19, Prop. 3.3]. Since the latter proof is clearly uniform, we have USCAC $\rightarrow (\exists^2)$ by Corollary 4.7. The remaining implication is immediate. □

4.2. Thin and free sets. In this section, we study the so-called thin- and free set theorems from [9]. In the latter, the thin set theorem $TS$ is defined as follows; $TS(k)$ is $TS$ limited to some fixed $k \geq 1$.

4.11. Principle (TS). $(\forall k)(\forall f : [N]^k \rightarrow N)(\exists A)(A \text{ is infinite } \land f([A]^k) \neq N)$.

We define UTS(2) as follows:

$(\exists \Psi^{1\rightarrow 1})(\forall f^1 : [N]^2 \rightarrow N)[(\Psi(f) \text{ is infinite } \land (\exists n^0)[n \not\in f([\Psi(f)]^2))]]$. (UTS(2))

We do not use ‘$N$’ to avoid confusion. Recall that ‘$\Psi(f)$ is infinite’ has nothing to do with infinite numbers $M \in \Omega$; Note in particular the type mismatch.

4.12. Theorem. In RCA$^0_0$, we have $(\exists^2) \leftrightarrow$ UTS(2).

Proof. We will prove the equivalence in RCA$^0_0$, a conservative extension of RCA$^0_0$. The forward implication is immediate from the results in [9, §5]. For the reverse implication, let $\Psi$ be as in UTS(2) and apply QF-AC$^{1,0}$ to $(\forall f^1 : [N]^2 \rightarrow N)(\exists n^0)[n \not\in f([\Psi(f)]^2)]$ to obtain $\Xi^2$ witnessing $n^0$. Hence, UTS(2) becomes

$(\exists \Phi^{1\rightarrow (1\times 0)})(\forall f^1 : [N]^2 \rightarrow N)[(\Phi(f)(1) \text{ is infinite } \land \Phi(f)(2) \not\in f([\Phi(f)(1)]^2))].$

As in the proofs of Theorems 3.3 and 4.4 we can prove that $\Phi$ is standard and standard extensional. Now suppose $h^1$ is a counterexample to $\Pi^0_1$-TRANS, i.e.
\((\forall^* n) h(n) = 0 \land (\exists m) h(m) \neq 0\). Fix standard \(f^1 : [N]^2 \to N\) and define \(g^1 : [N]^2 \to N\) as:

\[
g(k, l) := \begin{cases} 
  f(k, l) & (\forall i \leq \max(k, l)) h(i) = 0 \\
  \Phi(f)(2) & \text{otherwise}
\end{cases}.
\]

(4.4)

By assumption, \(f \approx_1 g\), and we obtain \(\Phi(f) \approx_{(1 \times 0)} \Phi(g)\) by the standard extensionality of \(\Phi\). Note that in particular \(\Phi(f)(2) = \Phi(g)(2)\), and since \(\Phi(g)(1)\) is infinite, there are some \(k'_0 > k_0 > m_0\) such that \(k_0, k'_0 \in \Phi(g)(1)\) where \(m_0\) is such that \(h(m_0) \neq 0\). However, by the definition of \(g\), we obtain \(\Phi(f)(2) \in g(\Phi(g)(1)^2)\), as we are in the second case of (4.4) for \(g(k_0, k'_0)\). Since \(\Phi(f)(2) = \Phi(g)(2)\), the previous yields the contradiction \(\Phi(g)(2) \in g(\Phi(g)(1)^2)\), and hence \(\Pi^0_1\text{-TRANS}\) must hold. By [25 Cor. 12], the latter implies \((\exists^* n)\), and we are done. \(\Box\)

Clearly, the previous proof also goes through for the uniform version of \(\text{STS}(2)\), which is \(\text{TS}(2)\) limited to stable functions, i.e. for functions \(f : [N]^2 \to N\) such that \((\forall x^0)(\exists y^0)(\forall z^0 \geq y)(f(x, y) = f(x, z))\).

Next, we consider the following corollary regarding the free set theorem, where \(\text{UTS}(k)\) and \(\text{UFS}(k)\) have obvious definitions in light of the notations in [9].

4.13. Corollary. In \(\text{RCA}^\omega\), we have \((\exists^* n) \leftrightarrow \text{UTS}(k) \leftrightarrow \text{UFS}(k)\) for \(k \geq 1\).

Proof. The case \(k \geq 2\) is immediate from the theorem, the uniformity of [9] Theorems 3.2 and 3.4, and the fact that \(\text{ACA}_0\) proves \(\text{FS}\) ([9]). To obtain the set \(B\) in the proof of the former theorem, apply \(\text{QF-AC}^{1,0}\) to the fact that the free set is infinite. For the case \(k = 1\), proceed as in the theorem. \(\Box\)

As noted by Kohlenbach in [25 §3], the (necessary) use of the law of excluded middle in the proof of a theorem, gives rise to a discontinuity in the uniform version of this theorem. Now, even the proof of \(\text{FS}(1)\) in [9] Theorem 2.2 uses this law, explaining the equivalence to \((\exists^* n)\) of the associated uniform version.

4.3. Cohesive sets. In this section, we study principles based on cohesiveness (See e.g. [20 Def. 6.30]). We start with the principle \(\text{COH}\).

4.14. Definition. A set \(C\) is cohesive for a collection of sets \(R_0, R_1, \ldots\) if it is infinite and for each \(i\), either \(C \subseteq^* R_i\) or \(C \subseteq^* \overline{R}_i\). Here, \(\overline{A}\) is the complement of \(A\) and \(A \subseteq^* B\) means that \(A \setminus B\) is finite.

4.15. Definition. [COH] Every countable collection of sets has a cohesive set.

It is important to note that \(\text{COH}\) involves multiple significant existential quantifiers: The ‘\((\exists C^1)\)’ quantifier, but also the existential type 0-quantifiers in \(C \subseteq^* R_i \lor C \subseteq^* \overline{R}_i\). As we will see, it is important that the functional from the uniform version of \(\text{COH}\) outputs both the set \(C\) and an upper bound to \(C \setminus R_i\) or \(C \setminus \overline{R}_i\).

4.16. Definition. [UCOH] There is \(\Phi^{(0 \to 1)} \to (1 \times 1)\) such that for all \(R^{0 \to 1}\)

\[
(\forall h)(\exists l \in \Phi(R)(1)) \land (\forall i^0)[(\forall n \in \Phi(R)(1)) (n \geq \Phi(R)(2) \implies n \in R(i)) \lor (\forall m \in \Phi(R)(1)) (m \geq \Phi(R)(2) \implies m \in \overline{R}(i))].
\]

(4.5)

Note that we may treat the collection \(R^{0 \to 1}\) as a type 1-object, namely as a double sequence (See for instance [40 p. 13]), and the same holds for \(\Phi(R)\).
4.17. Theorem. In RCA\textsubscript{0}^\omega, we have UCOH ↔ (∃\exists).

Proof. For the reverse implication, since cohesiveness is an arithmetical property, it is easy to build the functional Φ from UCOH assuming (∃\exists).

For the forward implication, consider UCOH and apply QF-AC\textsuperscript{1,0} to the first conjunct of (4.5) to obtain Ξ\textsuperscript{2} such that (\forall R^{\omega+1},k^0)[Ξ(R,k) > k ∧ Ξ(R,k) ∈ Φ(R)(1)]. The resulting formula (starting with (∃Ψ,Ξ)) qualifies for PF-TP\textsubscript{\forall}, i.e. we may assume that Φ and Ξ are standard. Following the proof of Theorem 3.5, we may assume Φ and Ξ are also standard extensional. Note that we can decide which disjunct holds (for given i) in the second conjunct of (4.5) by checking if Ξ(R,Φ(R)(2)(i)) ∈ R(i). For standard R, i, the latter only involves standard objects.

Now assume UCOH and suppose Π\textsubscript{0}^1\text{-TRANS} is false, i.e. there is standard h\textsuperscript{1} such that (\forall n)h(n) = 0 ∧ (\exists m)h(m) ≠ 0. Suppose for some fixed standard R, there is standard i\textsubscript{0} such that the first disjunct holds in the second conjunct of (4.5).

Now define R’ as follows: k ∈ R’(j) ↔ [k ∈ R(j) ∧ (\forall n ≤ max(j,k))h(n) = 0]. Clearly, R’ is standard and we have R ≈\textsubscript{0→1} R’, implying Φ(R) ≈\textsubscript{1\times1} Φ(R’).

In particular, Φ(R)(2)(i\textsubscript{0}) = Φ(R’)(2)(i\textsubscript{0}), and Φ(R)(1) ≈\textsubscript{1} Φ(R’)(1). However, then the first disjunct holds in the second conjunct of (4.5) for R’, i\textsubscript{0} too, since Ξ(R’, Φ(R’)(2)(i\textsubscript{0})) ∈ R(i\textsubscript{0}) is equivalent to Ξ(R, Φ(R)(2)(i\textsubscript{0})) ∈ R(i\textsubscript{0}). However, now let m\textsubscript{0} be such that h(m\textsubscript{0}) ≠ 0 and take m\textsubscript{0} < l\textsubscript{0} ∈ Φ(R’)(1). Clearly, l\textsubscript{0} > Φ(R’)(2)(i\textsubscript{0}) as the first number is infinite and the second finite. But then l\textsubscript{0} ∈ R’(i\textsubscript{0}) by UCOH, which is impossible by the definition of R’. A similar procedure leads to a contradiction in case the second disjunct holds in the second conjunct of (4.5) for some standard i\textsubscript{0}. In light of these contradictions, we must have Π\textsubscript{0}^1\text{-TRANS}, and (∃\exists) follows from UCOH by [5, Cor. 12].

While Ramsey’s theorem for pairs RT\textsubscript{2} does not imply WKL (See e.g. [20,28]), the uniform versions are equivalent.

4.18. Corollary. In RCA\textsubscript{0}^\omega, we have URT\textsubscript{2} ↔ UWKL.

Proof. The implication RT\textsubscript{2} → COH is proved in [20, 6.32]. This proof is clearly uniform (as also noted at the end of [20 p. 85]), yielding URT\textsubscript{2} → UCOH, and the theorem implies the forward implication, since (∃\exists) ↔ UWKL (24). By [34, Theorem 4.2], the reverse implication follows.  

Next, we study the cohesive version of ADS. Recall the definition of a stable order from Definition 4.6. Denote by CADS the statement that every infinite linear order has a stable suborder. The connection between CADS and cohesiveness is discussed between [20, 9.17-9.18]. Now let UCADS be the ‘fully’ uniform version of CADS as follows.

4.19. Definition. [UCADS] There is Φ\textsuperscript{1→(1\times1)} such that for infinite linear orders X\textsuperscript{1}, Y ≡ Φ(X)(1) is a stable suborder of X and Φ(X)(2) witnesses this, i.e. for y\textsuperscript{0} ∈ Y:

\[(∀w\textsuperscript{0})(y ≤_Y w → w ≤_Y Φ(X)(2)(y)) ∨ (∀v\textsuperscript{0})(y ≥_Y v → v ≥_Y Φ(X)(2)(y)). \quad (4.6)\]

4.20. Theorem. In RCA\textsubscript{0}^\omega, we have UCADS ↔ (∃\exists).
Proof: The reverse implication is immediate in light of Theorem 4.17 and the uniformity of the proofs of [19 Prop. 1.4 and 2.9]. For the forward implication, we proceed as in the proof of Theorem 4.17. Consider UCADS and apply QF-AC to the formula expressing that $\Phi(X)(1)$ is infinite to obtain $\Xi^2$ such that $(\forall X^1, k^0)[\Xi(X, k) > 0 \wedge \Xi(X, k) \in \Phi(X)(1)]$. The resulting formula (starting with $(\exists \Phi, \Xi)$) qualifies for PF-TP, i.e., we may assume that $\Phi$ and $\Xi$ are standard. Following the proof of Theorem 5.5, $\Phi$ and $\Xi$ are also standard extensional.

Now suppose $\Pi^0_1$-TRANS is false and $h$ is a counterexample to the latter, and consider again the orders $<_0$ and $\prec$ from the proof of Theorem 4.4. Since $<_0 \approx_{1 \times 1} \Phi(\prec)$. Now take standard $n_0 \in \Phi(<_0)(1)$ (which exist by the standardness of $\Xi$ and also satisfies $n_0 \in \Phi(\prec)(1)$ by standard extensionality) and consider the standard number $\Phi(<_0)(2)(n_0) =_0 \Phi(\prec)(2)(n_0)$, the latter equality again by standard extensionality. However, by the infinitude of $\Phi(<_0)(1)$ (resp. of $\Phi(\prec)(1)$) only the second (resp. first) disjunct of (4.10) can hold for $<_0$ (resp. for $\prec$). Then, the second (resp. first) disjunct of (4.10) for $<_0$ (resp. $\prec$) implies $n_0 \geq_0 \Phi(<_0)(2)(n_0)$ (resp. $n_0 \leq \Phi(\prec)(2)(n_0)$). Since all objects are standard, we obtain $n_0 =_0 \Phi(<_0)(2)(n_0) =_0 \Phi(\prec)(2)(n_0)$. However, then $\Phi(<_0)(1) \approx_1 \Phi(\prec)(1)$ is impossible as the ‘overlap’ between the latter two orders is a singleton, namely $\{n_0\}$.

In [19 Prop. 2.9], a uniform proof of CADS from $\text{CRT}^2_2$, a cohesive version of $\text{RT}^2_2$, is presented. Hence, it follows that the (fully) uniform version of $\text{CRT}^2_2$ is also equivalent to $(\exists^2)$.

4.21. Remark (The role of extensionality). At the risk of stating the obvious, the axiom of extensionality is central in proving all above equivalences; In particular, the fact that functionals with a defining sentence (like those originating from the RM zoo) are standard extensional, is essential for the equivalence to $(\exists^2)$ of uniform versions of RM zoo principles. Hence, an approach to uniform computability not involving the axiom of extensionality will yield different results. It is a matter of opinion whether in the latter such ‘non-extensional framework’, the glass is half-full (finer distinctions) or half-empty (more complicated picture). In our opinion, it is remarkable how uniform our uniform classification has turned out.

5. Taming the Future Reverse Mathematics Zoo

In this section, we formulate a general template for obtaining equivalences between $(\exists^2)$ and uniform versions of principles from the RM zoo.

5.1. General template. Our template is defined as follows.

Template. Let $T \equiv (\forall X^1)(\exists Y^1)\varphi(X, Y)$ be a RM zoo principle and let $UT$ be $(\exists^2)(\forall X^1)\varphi(X, \Phi(X))$. The proof of $(\exists^2) \rightarrow UT$ is usually straightforward; To prove $UT \rightarrow (\exists^2)$, proceed as follows:

(i) Following Theorem 5.5, prove that the functional in $UT$ is standard and standard extensional in $\text{RCA}_0 + UT$.

(ii) Suppose the standard function $h^1$ is such that $(\forall n^0)h(n) = 0$ and $(\exists m^0)h(m) \neq 0$, i.e. $h$ is a counterexample to $\Pi^0_1$-TRANS.

(iii) For standard $V^1$, use $h$ to define standard $W^1 \approx_1 V$ such that $\Phi(W) \neq_1 \Phi(V)$, i.e. $W$ is $V$ with the nonstandard elements changed sufficiently to yield a different image under $\Phi$. 


(iv) The previous contradiction implies $\Pi^0_1$-TRANS and $(\exists^2)$ by [5, Cor. 12].

In Section 5.2, we speculate why uniform principles $UT$ originating from RM zoo-principles are equivalent to $(\exists^2)$ en masse. We conjecture a connection to Montalbán’s notion of robustness from [29].

Finally, the above template treats zoo-principles in a kind of ‘$\Pi^1_2$-normal form’, for the simple reason that most zoo-principles are formulated in such a way. Nonetheless, it is a natural question, discussed in Section 6, whether principles not formulated in this normal form gives rise to uniform principles not equivalent to $(\exists^2)$. Surprisingly, the answer to this question turns out to be negative.

5.2. Robustness and structure. In this section, we try to explain why our template works so well for RM zoo principles. We conjecture a connection to Montalbán’s notion of robustness from [29].

First of all, standard computable functions are determined by their behaviour on the standard numbers (by the Use principle from [41, p. 50]), while e.g. a standard Turing machine may well halt at some infinite number (given e.g. MUC and hence $-\Pi^0_1$-TRANS), i.e. non-computable problems, like the Halting problem for standard Turing machines, are not necessarily determined by the standard numbers.

Now in step (iii), the assumption $-\Pi^0_1$-TRANS allows us to change the nonstandard part of a standard set $V^1$, resulting in standard $W^1 \approx V$. Since $\Phi(V)$ (resp. $\Phi(W)$) is not computable from $V$ (resp. $W$), the former depends on the nonstandard numbers in $V$ (resp. $W$). However, making the nonstandard parts of $V$ and $W$ different enough, we can guarantee $\Phi(W) \not\approx_1 \Phi(V)$, and obtain a contradiction with standard extensionality. Hence, $\Pi^0_1$-TRANS follows and so does $(\exists^2)$.

Secondly, note that step (iii) crucially depends on the fact that we can modify the nonstandard numbers in the set $V$ without changing the standard numbers, i.e. while guaranteeing $V \approx_1 W$. Such a modification is only possible for structures which are not closed downwards: For instance, our template will fail for the fan theorem (See Section 4), as the latter deals with (finite) binary trees, which are closed downwards. Of course, many of the zoo-principles have a distinct combinatorial flavour, which implies that the objects at hand exhibit little structure.

Thirdly, in light of this absence of structure in principles of the RM zoo, we conjecture that robust theorems (in the sense of [29, p. 432]) are (exactly) those which deal with mathematical objects with lots of structure like trees, continuous functions, metric spaces, et cetera. In particular, the presence of this structure ‘almost guarantees’ a place in one of the Big Five categories. The non-robust theorems, by contrast, deal with objects which exhibit little structure and for this reason have the potential to fall outside the Big Five and in the RM zoo. However, as we observed in the previous paragraph, the absence of structure in RM zoo principles, is exactly what makes our template from Section 5.1 work.

In conclusion, what makes the principles in the RM zoo exceptional (namely the presence of little structure on the objects at hand) guarantees that the uniform versions of the RM zoo principles are non-exceptional (due to the fact that the above template works form them).
6. Converse Mathematics

In this section, we classify the uniform versions of the contrapositions of zoo-principles. This study is motivated by the question whether the template from Section 5 "always" works, i.e. perhaps we can find counterexamples to this template by studying contra-posed zoo-principles (which are not necessarily in $\Pi^1_2$-normal form)? We first discuss this motivation in detail.

First of all, the weak König’s lemma (WKL) is rejected in all varieties of constructive mathematics, while the (classical logic) contraposition of WKL, called the fan theorem is accepted in Brouwer’s intuitionistic mathematics (See e.g. [8]). This difference in constructive content is also visible at the uniform level: The uniform version of WKL satisfies the template from the previous section, and is indeed equivalent to arithmetical comprehension, while the uniform version of the fan theorem is not stronger than WKL itself. (See [25, 37]). Hence, we observe that, from the constructive and uniform point of view, a principle can behave rather differently compared to its contraposition.

Secondly, the template from Section 5 would seem to work for any zoo principle in ‘$\Pi^1_2$-normal form’ $T \equiv (\forall X^1)(\exists Y^1) \varphi(X,Y)$ and the associated uniform version $UT \equiv (\exists \Phi^1 \rightarrow 1)(\forall X^1) \varphi(X, \Phi(X))$. Nonetheless, while $UT$ is the most natural uniform version of $T$ in our opinion, there sometimes exists an alternative uniform version of $T$, similar to the uniform version of the fan theorem. With regard to examples, the principle ADS from Section 6.1 is perhaps the most obvious candidate, while various Ramsey theorems can also be recognised as suitable candidates.

In conclusion, it seems worthwhile investigating the uniform versions of contra-posed zoo-principles, inspired by the difference in behaviour of the fan theorem and weak König’s lemma. However, somewhat surprisingly, we shall only obtain principles equivalent to arithmetical comprehension, i.e. our study will not yield exceptions to our observation that the RM zoo disappears at the uniform level.

6.1. The contraposition of ADS. In this section, we study the uniform version of the contraposition of ADS. Recall that ADS states that every infinite linear order either has an ascending or a descending chain. Hence, the contraposition of ADS is the statement that if a linear order has no ascending and descending sequences, then it must be finite, as follows:

$$(\forall X^1)[LO(X) \land (\forall x^1) \in \text{Seq}(X)(\exists n^0, k^0)(x_n \leq_X x_{n+1} \land x_k \geq_X x_{k+1})$$

$$(\rightarrow (\exists n^0, k^0 \in \text{field}(X))(\forall m^0 \in \text{field}(X))(1 \leq_X m \leq_X 1)).$$ (6.1)

By removing all existential quantifiers, we obtain the following alternative uniform version of ADS.

6.1. Principle (UADS$_2$). There is $\Phi^3$ such that for all linear orders $X^1$ and $g^2$

$$(\forall x^1) \in \text{Seq}(X)(\exists n^0, k^0 \leq g(x^1))(x_n \leq_X x_{n+1} \land x_k \geq_X x_{k+1})$$

$$\rightarrow (\forall m^0 \in \text{field}(X))(\Phi(X, g)(1) \leq_X m \leq_X \Phi(X, g)(2)).$$ (6.2)

For the following theorem, we recall that in [36,37] certain equivalences were only proved over the base theory extended by the principle QF-AC$^{2,0}$. This situation is similar to the observation that some RM equivalences are proved over RCA$_0 + I\Sigma^2_1$. Hunter notes in [21, §2.1.2] that any QF-AC$^{\sigma,0}$ still results in a conservative extension of RCA$_0$. 


6.2. Theorem. In RCA\textsuperscript{0} + QF-AC\textsuperscript{2,0}, we have UADS\textsubscript{2} ↔ (3\textsuperscript{2}).

Proof. The reverse direction is immediate since (3\textsuperscript{2}) implies ADS and the upper and lower bounds to ≤\textsubscript{X} in the consequent of (6.1) can be found using the search operator (μ\textsuperscript{2}), and the latter is equivalent to (3\textsuperscript{2}) by [23, §3]. For the forward direction, we shall work in RCA\textsuperscript{0} which is a conservative extension of RCA\textsubscript{0} for the internal language. Thus, assume UADS\textsubscript{2}, let \(\Phi\) be as in the latter and consider the following formula expressing extensionality for \(\Phi\):

\[(\forall g^2, X^1, Y^1) [X =_1 Y \rightarrow \Phi(X, g) = 0 \cdot \Phi(Y, g)].\]

Resolve ‘=\textsubscript{1}’ and bring the type 0-quantifier outside the square brackets as follows:

\[(\forall g^2, X^1, Y^1)(\exists N^0)(\exists N = 0 \rightarrow \Phi(X, g) = 0 \cdot \Phi(Y, g)).\]

Now apply QF-AC\textsuperscript{2,0} to obtain \(\Xi\) witnessing \(N\) in the previous formula:

\[(\exists N^0)(\forall g^2, X^1, Y^1)(\forall g, X, Y) (\Phi(X, g) = 0 \cdot \Phi(Y, g)).\] (6.3)

Hence, UADS\textsubscript{2} implies a formula of the form (\(\exists^3, \Xi^3\)) \(A(\Phi) \wedge B(\Xi, \Phi)\), where \(B\) is the formula in square brackets in (6.3) and \(A\) is (6.2) quantified over all linear orders \(X\) and all \(g^2\). Applying PF-TP\textsubscript{v}, we observe that \(\Xi\) and \(\Phi\) may be taken to be standard. This yields the standard extensionality of \(\Phi\) as follows:

\[(\forall^\ast g^2)(\forall^\ast X^1, Y^1)(X \approx_1 Y \rightarrow \Phi(X, g) = 0 \cdot \Phi(Y, g)),\]

since the standard functional \(\Xi\) is standard for standard input. Now fix standard \(X_0 \neq \emptyset\) and \(g_0\) such that the antecedent of UADS\textsubscript{2} holds. Then \(\Phi(X_0, g_0)\) is standard and consider the standard function \(h_0^2\) which is constant and always outputs \(\Phi(X_0, g_0)(1) + \Phi(X_0, g_0)(2) + 4\). Clearly, we have:

\[(\forall x^1_{(i)} \in \text{Seq}(X_0))(\exists n^0, k^0 \leq h_0(x_{(i)})) (x_n \leq_{X_0} x_{n+1} \land x_k \geq_{X_0} x_{k+1}),\] (6.4)

as there are less than \(\Phi(X_0, g_0)(1) + \Phi(X_0, g_0)(2) + 2\) distinct elements in the finite linear order induced by \(X_0\), by the consequent of UADS\textsubscript{2}. Indeed, by the definition of linear order ([[10] V.1.1]), if \(x \leq X y \land x \geq X y\), then \(x =_0 y\), i.e. equality in the sense of \(X\) is equality on the natural numbers. By (6.3), the associated consequent of UADS\textsubscript{2} also follows for \(\Phi(X_0, h_0)\).

Now suppose that \(\Pi^0_1\)-TRANS is false, i.e. there is standard function \(h_1^1\) such that

\[(\forall^\ast n)(h_1(n) = 0) \land (\exists n)(h_1(n) \neq 0).\]

Following [[10] V.1.1], define the standard set \(Y_0\) by adding to \(X_0\) the pairs \((x, m_0)\) for \(x \in \text{field}(X_0)\) and where \(m_0\) is such that \((\forall i < m_0) h_1(i) = 0 \land h_1(m_0) \neq 0\). Intuitively speaking, the standard set \(Y_0\) represents the linear order \(X_0\) with a ‘point at infinity’ \(m_0\) added (in a standard way, thanks to \(h_1\)). Since the order induced by \(Y_0\) is only a one-element extension of the order induced by \(X_0\), we also have

\[(\forall x^1_{(i)} \in \text{Seq}(Y_0))(\exists n^0, k^0 \leq h_0(x_{(i)})) (x_n \leq_{Y_0} x_{n+1} \land x_k \geq_{Y_0} x_{k+1}),\]

i.e. the antecedent of UADS\textsubscript{2} holds for \(Y_0\) and \(h_0\). Hence, the order induced by \(Y_0\) is bounded by \(\Phi(Y_0, h_0)\) as in the consequent of UADS\textsubscript{2}. However, by definition, we have \(X_0 \approx_1 Y_0\), implying \(\Phi(X_0, h_0) = 0 \cdot \Phi(Y_0, h_0)\). By the latter, we cannot have \(m \leq Y_0, \Phi(Y_0, h_0)\) for the unique (and necessarily infinite) element \(m_0\) in \(\text{field}(Y_0) \setminus \text{field}(X_0)\), i.e. a contradiction. Hence \(\Pi^0_1\)-TRANS holds and (3\textsuperscript{2}) also follows. \(\square\)
In the previous proof, we added the ‘point at infinity’ \( m_0 \) to the finite linear order induced by \( X_0 \); such a modification is only possible for structures which are not closed downwards. In particular, the above approach clearly does not work for theorems concerned with trees, like e.g. the fan theorem. On the other hand, we can easily obtain a version of the previous theorem for e.g. the chain-antichain principle CAC, and of course for stable versions of the latter and of ADS.

### 6.2. The contraposition of Ramsey theorems

In this section, we study the well-known Ramsey’s theorem for pairs \( RT_2^2 \). The latter is the statement that every colouring with two colours of all two-element sets of natural numbers must have an infinite homogenous subset, i.e. of the same colour. Now, \( RT_2 \) has an equivalent version (See [20, §6]) of which the contraposition has the ‘right’ syntactic structure, namely similar to the fan theorem. Thus, consider the following principle.

#### 6.3. Principle (Contraposition of \( RT_2^2 \)).

\[
(\forall X^1, c^1 : |X|^2 \to 2) \left( (\forall H^1 \subseteq X) (\forall i < 2) \left( (\forall s^0 \in |H|^2) (c(s) = i) \to H \text{ is finite} \right) \to (\forall i < 2) \left( (\forall s^0 \in |H|^2) (c(s) = i) \to X \text{ is finite} \right) \right. \]

(6.5)

Here, ‘\( Z^1 \) is finite’ is short for \((\exists n^0)(\forall \sigma^0 \in |\sigma|)(\forall i < |\sigma|)(\sigma(i) \in Z^1) \to |\sigma| \leq n \). We also abbreviate the previous formula by \((\exists n^0)(|Z^1| \leq n)\), where obviously \(|Z^1| \leq n\) is a \( \Pi^1_1 \)-formula. Note that we used the usual notation \(|H|^n\) for the set of \( n \)-element subsets of \( H \), which of course has nothing to do with the typing of variables.

Based on the previous principle, define URTP\(_2\) as the following principle.

#### 6.4. Principle. There is \( \Phi^3 \) such that for all \( g^2, X^1, c^1 : |X|^2 \to 2 \), we have

\[
(\forall H \subseteq X)(\forall i < 2) \left( (\forall s \in |H|^2) (c(s) = i) \to |H| \leq g(H) \right) \to |X| \leq \Phi(X, g, c)
\]

(6.6)

Note that \( g \) does not depend on \( i \), as the quantifier \((\forall i < 2)\) can be brought inside the square brackets to obtain \((\forall s \in |H|^2) (c(s) = 0) \lor (\forall t \in |H|^2) (c(t) = 1)\).

#### 6.5. Theorem. In RCA\(_0^2\) + QF-AC\(_2^0\), we have \((\exists^2) \leftrightarrow \text{URTP}_2\).

**Proof.** The forward direction is immediate as \((\exists^2)\) implies \( RT_2^2 \) and the upper bound to \(|X|\) in the former’s contraposition can be found using \((\mu^2)\). For the reverse direction, we work in RCA\(_0^2\). Thus, assume URTP\(_2\) and establish as in the proof of Theorem 6.2 that \( \Phi \) is standard and (partially) standard extensional as follows:

\[ X \approx_1 Y \land c \approx_1 d \to \Phi(X, g, c) =_0 \Phi(Y, g, d), \]

for all standard \( g^2, X^1, Y^1, c^1 : |X|^2 \to 2, d^1 : |Y|^2 \to 2 \). Now let \( g^0_0, X^1_0, c^1_0 : |X_0|^2 \to 2 \) be standard objects such that the antecedent of (6.6) holds and hence \(|X_0| \leq \Phi(X_0, g_0, c_0)\), where \( X_0 \neq \emptyset \). Now define \( h_0^2 \) to be the functional which is constantly \( \Phi(X_0, g_0, c_0) + 1 \), and note that we have

\[
(\forall H \subseteq X_0)(\forall i < 2) \left( (\forall s \in |H|^2) (c_0(s) = i) \to |H| \leq h_0(H) \right),
\]

as \( H \subseteq X_0 \) implies that \(|H| \leq |X_0|\). By UADS\(_2\), we also have \(|X_0| \leq \Phi(X_0, h_0, c_0)\).

Now suppose \( \Pi^0_1\)-TRANS is false, i.e. there is some standard \( h_1 \) such that \((\forall^*n)h_1(n) = 0 \lor (\exists m_0)h_1(m_0)\), and define the standard set \( Y_0 \) as \( X_0 \cup \{ m_0, m_0 + 1, \ldots, m_0 + \Phi(X_0, h_0, c_0) \} \), where \( m_0 \) is the least number \( k \) such that \( h_1(k) \neq 0 \). Now define the standard colouring \( d_0^1 \) as follows: \( d_0(s) = 0 \) if both elements of \( s \) are
at least $m_0$, 1 if one element of $s$ is at least $m_0$ and the other one is not, and $c_0(s)$ otherwise. By the definition of $Y_0$ and $d_0$, we have

$$(\forall H \subseteq Y_0)(\forall i < 2)[(\forall s \in [H]^2)(d_0(s) = i) \rightarrow |H| \leq h_0(H)],$$

(6.7)
as for $H \subseteq Y_0$ with more than $\Phi(X_0, g_0, c_0) + 1$ elements, the set $H$ is not homogenous for $d_0$. By UADS$_2$, we obtain $|Y_0| \leq \Phi(Y_0, h_0, d_0)$, but we also have $\Phi(Y_0, h_0, d_0) = \Phi(X_0, h_0, c_0)$ by standard extensionality since $X_0 \approx_1 Y_0$ and $c_0 \approx_1 d_0$. However, $Y_0$ by definition has more elements than $\Phi(X_0, h_0, c_0)$, a contradiction. Hence, we must have $\Pi^0_1$-TRANS and $(\exists^2)$ follows. □

6.3. Contraposition of thin and free set theorems. In this section, we study the so-called thin- and free set theorems from [9]. These results are similar to those in the previous two sections, hence our treatment will be brief. Notations are as in [9], except that we write $f : [X]^k \rightarrow \mathbb{N}$ instead of $f : [X]^k \rightarrow \mathbb{N}$; In general, we shall not use the symbol ‘$\mathbb{N}$’ to avoid confusion (with notation from Robinson’s approach to Nonstandard Analysis).

Recall the equivalent version of Ramsey’s theorem from [20, §6] in Principle 6.3. Because of the extra set parameter $X^1$ in the latter, (6.5) is amenable to our treatment as in Theorem 6.5. As it turns out, the free and this set theorems also have such equivalent versions by [9, Lemma 2.4 and Corollary 3.6].

For instance, by the aforementioned lemma, $FS(k)$, the free set theorem for index $k$, is equivalent to the statement that for every infinite set $X^1$ and $f^1 : [X]^k \rightarrow N$, there is infinite $A^1 \subset X$ which is free for $f$. The contraposition of the latter is:

$$(\forall X^1, f^1 : [X]^k \rightarrow N)[(\forall A^1 \subseteq X)[(\forall s^0 \in [A]^k)(f(s) \notin A \lor f(s) \in s)] \rightarrow H \text{ is finite}] \rightarrow X \text{ is finite},$$

which is neigh identical to Principle 6.3 for $k = 2$. Now let UFSP$_k$ be the uniform version of (6.8) similar to URTP$_2$. Similar to Theorem 6.5, one proves the following.

6.6. Theorem. In RCA$_0^\omega + QF\text{-AC}^{2,0}$, we have $(\exists^2) \leftrightarrow UFSP_2$.

The version of the thin set theorem from [9, Corollary 3.6] is not so elegant, hence we do not consider it. We finish this section with some concluding remarks.

6.7. Remark. First of all, Kohlenbach claims in [25, §1] that $(\exists^2)$ sports a rich and very robust class of equivalent principles, which seems to be ‘more than’ confirmed by the above results, especially those in this section.

Secondly, if one were to categorise principles according to robustness at the uniform level, ADS and other principles studied in this section would rank very high, as even their contrapositions give rise to uniform principles equivalent to $(\exists^2)$. By contrast, WKL would rank lower, as the uniform version of the contraposition of WKL is not stronger than WKL, as discussed in the first part of this section. In other words, ADS is exceptional in Friedman-Simpson-style RM, while it is not in the aforementioned ‘uniform’ categorisation.
6.4. Motivation for higher-order Reverse Mathematics. The reader unaccustomed to higher-order arithmetic may deem higher-order principles like UDNR unnatural, compared to e.g. second-order arithmetic. We now argue that, at least from the point of view of second-order RM, higher-order RM is also natural. It should first be mentioned that Montalbán includes higher-order RM among the ‘new avenues for RM’ in [29].

First of all, Fujiwara and Kohlenbach have established the connection (and even equivalence in some cases) between (classical) uniform existence as in $UT$ and intuitionistic provability ([16][17]). Hence, the investigation of uniform principles like UDNR may be viewed as the (second-order) study of intuitionistic provability.

Secondly, the author shows in [38] that higher-order statements are implicit in (second-order) RM-theorems concerning continuity, due to the special nature of the RM-definition of continuity. In particular, consider the statement

\[ \text{All continuous functions on Cantor space are uniformly continuous}. \]

Let (H) be the previous statement with continuity as in the RM-definition. One can then proves (H)$\iff$(UH), where:

There is a functional which witnesses the uniform RM-continuity on Cantor space of any RM-continuous function. 

(UH)

From the treatment in [38], it is clear that the functional in (UH) can only be obtained because the RM-definition of continuity greatly reduces quantifier complexity. In conclusion, higher-order RM is already implicit in second-order RM due to the RM-definition of continuity involving codes. Similar results are in [36][37].

Thirdly, RM can be viewed as a classification based on computability: Theorems provable in RCA$_0$ are part of ‘computable mathematics’; An equivalence between a theorem and a Big Five system classifies the computational strength of the theorem, as the Big Five have natural formulations in terms of computability. Furthermore, as noted by Simpson in [40] I.8.9 and IV.2.8, theorems are analysed in RM ‘as they stand’, in contrast to constructive mathematics, where extra conditions are added to enforce a constructive solution. In other words, the goal of RM is not to enforce computability onto theorems, but to classify how ‘non-computable’ the latter are.

In light of the previous, it is a natural question whether there are other natural ways of classifying theorems of ordinary mathematics. As noted in [36][37], the study of uniform versions of theorems constitutes a classification based on the central tenet of Feferman’s Explicit Mathematics (See [12][14]), which is:

\[ \text{A proof of existence of an object yields a procedure to compute said object}. \]

Indeed, in the same way as the RM-classification is based on the question which axioms (and hence ‘how much’ non-computability) are necessary to prove a theorem, the study of uniform versions of theorems is motivated by the following question:

\[ \text{For a given theorem } T, \text{ what extra axioms are needed to compute the objects claimed to exist by } T? \]

Similar to RM, we do not enforce the central tenet of Explicit Mathematics in higher-order RM: We measure ‘how much extra’ is needed to obtain UT, the uniform version of $T$ where a functional witnesses the existential quantifiers.

\[ \text{4The proof takes place in } RCA_0^\omega + QF-AC^{2,0}, \text{ a conservative extension of RCA}_0 \text{ by [21] §2.1.2}. \]
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