COMPARATIVE QUANTIZATIONS OF (2+1)-DIMENSIONAL GRAVITY

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Abstract

We compare three approaches to the quantization of (2+1)-dimensional gravity with a negative cosmological constant: reduced phase space quantization with the York time slicing, quantization of the algebra of holonomies, and quantization of the space of classical solutions. The relationships among these quantum theories allow us to define and interpret time-dependent operators in the “frozen time” holonomy formulation.

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1. Introduction

Over the past few years, there has been a growing interest in (2+1)-dimensional quantum gravity as a simple model for realistic (3+1)-dimensional quantum gravity. As a generally covariant theory of spacetime geometry, general relativity in 2+1 dimensions has the same conceptual foundations as ordinary (3+1)-dimensional gravity. But the reduction in the number of dimensions greatly simplifies the structure of the theory, reducing the infinite number of physical degrees of freedom of ordinary general relativity to a finite number of global degrees of freedom. The model thus allows us to explore the conceptual problems of quantum gravity within the framework of ordinary quantum mechanics, avoiding such issues as nonrenormalizability associated with field degrees of freedom.

A number of different approaches to quantizing (2+1)-dimensional general relativity have been developed recently. These include reduced phase space quantization with ADM variables [1, 2, 3]; quantization of the space of classical solutions of the first-order Chern-Simons theory [4, 5, 6, 7]; and quantization of the holonomy algebra [8, 9, 10, 11, 12, 13, 14, 15]. Each approach has its strengths and weaknesses. ADM quantization, for example, leads to states and operators with clear physical interpretations, but depends on an arbitrary classical choice of time slicing, breaking manifest covariance. Quantization of the space of solutions involves no such choice, but requires a detailed understanding of the classical solutions. Quantization of the holonomy algebra is also manifestly covariant, and reveals important underlying algebraic structures, but the physical interpretation of the resulting operators is not at all clear.

The goal of this paper is to explore the relationships among these three methods of quantization. Such comparisons have been made in the past [6, 16, 17, 18, 19], but the powerful holonomy algebra approach has not generally been considered. We shall see below that quantization of the space of solutions (sometimes called “covariant canonical quantization”) provides a natural bridge between the ADM and holonomy algebra approaches, allowing one to introduce “time”-dependent physical operators into the latter formalism.

The structure of the paper is as follows. In section 2, we discuss the first- and second-order formulations of classical general relativity, solving the constraints and introducing the basic physical variables in each approach. In section 3, we describe the classical solutions for spacetimes with the topology $\mathbb{R} \times T^2$, focusing on the case of a negative cosmological constant but also discussing the $\Lambda \to 0$ limit and briefly considering the $\Lambda > 0$ case. In section 4, we describe the three methods of quantization, and explore their relationships. Our results are summarized in section 5.

A preliminary report on aspects of this work has appeared in [20]. Portions of our discussion of classical solutions and our comparison of ADM and Chern-Simons quantization have been found independently by Ezawa [18, 21], who also discusses the $\Lambda > 0$ case in more detail.
2. Classical Theories

To understand the quantization of (2+1)-dimensional gravity, it is first necessary to understand the classical theory. Classical general relativity has two very different formulations: the second-order form, in which the metric is the only fundamental variable, and the first-order form, in which the metric and the connection (or spin connection) are treated independently. As we shall see, these two formulations lead naturally to two different approaches to quantization.

The fundamental feature of classical general relativity in 2+1 dimensions is that the full Riemann curvature tensor depends linearly on the Ricci tensor. As a result, the empty space field equations

\[ R_{\mu\nu} = 2\Lambda g_{\mu\nu} \]

imply that spacetime has constant curvature, that is, that every point has a neighborhood isometric to a neighborhood of de Sitter, Minkowski, or anti-de Sitter space. For a topologically trivial spacetime, this condition eliminates all degrees of freedom. For a spacetime with nontrivial topology, however, there remain a finite number of degrees of freedom that describe the gluing of constant curvature patches around noncontractible curves (see [22] for a more detailed description). It is these degrees of freedom that we shall eventually quantize.

2.1. ADM Formalism

A most traditional approach to classical gravity in 2+1 dimensions is to begin with the standard second-order form of the Einstein action and perform an ADM-style splitting into spatial and temporal components. This approach has been discussed in some detail by Moncrief [2] and Hosoya and Nakao [1]; in this section, we briefly summarize their results. We assume that spacetime has the topology \( \mathbb{R} \times \Sigma \), where \( \Sigma \) is a closed genus \( g \) surface.

The Einstein action takes the form

\[
I_{\text{Ein}} = \int d^3x \sqrt{-g} \left( R - 2\Lambda \right) = \int dt \int_\Sigma d^2x \left( \pi^{ij} \dot{g}_{ij} - N^i \mathcal{H}_i - N \mathcal{H} \right),
\]

(2.1)

where the metric has been decomposed as

\[
ds^2 = N^2 dt^2 - g_{ij} (dx^i + N^i dt) (dx^j + N^j dt)
\]

(2.2)

and the momenta conjugate to the \( g_{ij} \) are \( \pi^{ij} = \sqrt{g} (K^{ij} - g^{ij} K) \), where \( K^{ij} \) is the extrinsic curvature of the surface \( t = \text{const.} \). The supermomentum and super-Hamiltonian constraints are

\[
\mathcal{H}_i = -2\nabla_j \pi^j_i, \quad \mathcal{H} = \frac{1}{\sqrt{g}} g_{ij} g_{kl} (\pi^{ik} \pi^{jl} - \pi^{ij} \pi^{kl}) - \sqrt{g} (R - 2\Lambda).
\]

(2.3)

A convenient coordinate choice is the York time slicing [23], in which the mean (extrinsic) curvature is used as a time coordinate, \( K = \pi / \sqrt{g} = \tau \). In reference [2], Moncrief shows

*We use standard ADM notation: \( g_{ij} \) and \( R \) refer to the induced metric and scalar curvature of a time slice, while the spacetime metric and curvature are denoted \( g_{\mu\nu} \) and \( R^{\mu\nu} \).
that this is a good global coordinate choice for classical solutions of the field equations. For spacetimes that do not satisfy the field equations, on the other hand, \( \tau \) need not be a good coordinate—there is no \textit{a priori} reason that a spacetime should not admit two different slices with the same value of \( K \).

The next step is to solve the constraints \( \mathcal{H} = 0, \mathcal{H}^i = 0 \) to obtain a theory on a reduced phase space. We begin with a convenient parametrization of the spatial metric and momentum. By the uniformization theorem of Riemann surfaces \([24]\), any two-metric on \( \Sigma \) can be written in the form

\[
\bar{g}_{ij} = e^{2\lambda} \tilde{g}_{ij}, \tag{2.4}
\]

where \( \tilde{g}_{ij} \) is a metric of constant curvature \( k \) on \( \Sigma \), with \( k = 1 \) for the sphere, 0 for the torus, and \(-1\) for any surface of genus \( g \geq 2 \). Moreover, it is a standard result that up to spatial diffeomorphisms, the space of such constant curvature metrics is finite dimensional; that is, any \( \tilde{g}_{ij} \) can be obtained by a suitable diffeomorphism from one of a finite-dimensional family of metrics \( \bar{g}_{ij}(m_\alpha) \). The dimension of this space of “standard” metrics is 0 for the sphere (that is, there is only one diffeomorphism class of constant curvature 1 metrics on \( S^2 \)), 2 for the torus, and \( 6g - 6 \) for a surface of genus \( g \geq 2 \).

For the torus, for instance, we can write \( \bar{g}_{ij} = \tilde{g}_{ij}(m) \), where \( m = m_1 + im_2 \) is a complex number, the modulus, which can be taken without loss of generality to have a positive imaginary part. Concretely, the spatial metric corresponding to a given value of \( m \) is

\[
d\sigma^2 = m_2^{-1} |dx + mdy|^2, \tag{2.5}
\]

where \( x \) and \( y \) each have period 1. Any other flat metric on the torus is diffeomorphic to one of this form, up to a rescaling that can be absorbed in \( \lambda \) in equation (2.4). For higher genus surfaces, we can again write \( \bar{g}_{ij} \) as a function of \( 6g - 6 \) moduli \( m_\alpha \). An explicit representation of the metric similar to (2.5) is much more difficult, however, although it may be possible to handle the genus 2 case by using the theory of hyperelliptic surfaces.

A similar decomposition is possible for the canonical momenta \( \pi^{ij} \); one obtains

\[
\pi^{ij} = e^{-2\lambda} \sqrt{\bar{g}} \left( p^{ij} + \frac{1}{2} \bar{g}^{ij} \pi / \sqrt{\bar{g}} + \nabla^i Y^j + \nabla^j Y_i - \bar{g}^{ij} \nabla_k Y^k \right), \tag{2.6}
\]

where \( \nabla_i \) is the covariant derivative for the connection compatible with \( \bar{g}_{ij} \), indices are now raised and lowered with \( \bar{g}_{ij} \), and \( p^{ij} \) is a transverse traceless tensor with respect to \( \nabla_i \),

\[
\nabla_i p^{ij} = 0. \tag{2.7}
\]

In the language of Riemann surface theory, \( p^{ij} \) is a holomorphic quadratic differential. Roughly speaking, \( p^{ij} \) is canonically conjugate to \( \bar{g}_{ij} \), \( \pi \) to \( \lambda \), and \( Y^i \) to the spatial diffeomorphisms. Standard mathematical results guarantee that our dimensions match—the dimension of the space of quadratic differentials \( p^{ij} \) is equal to the dimension of the space of diffeomorphism classes of constant curvature metrics \( \bar{g}_{ij} \), and, in fact, the quadratic differentials naturally parametrize the cotangent space of the moduli space \([24]\).

\[1\text{In the mathematics literature, the modulus is usually denoted by } \tau. \text{ Following Moncrief } [2], \text{ however, we have already used } \tau \text{ to denote the York time coordinate.}\]
With these parametrizations of $g_{ij}$ and $\pi^{ij}$, the momentum constraints simply imply that $Y^i = 0$, while the Hamiltonian constraint becomes

$$H = -\frac{1}{2}\sqrt{\bar{g}}e^{2\lambda}(\tau^2 - 4\Lambda) + \sqrt{\bar{g}}e^{-2\lambda}p^{ij}p_{ij} + 2\sqrt{\bar{g}}\left[\Delta\lambda - \frac{1}{2}\bar{R}\right] = 0.$$  \hfill (2.8)

Moncrief has shown that this constraint uniquely determines $\lambda$ as a function of the $\bar{g}_{ij}$ and $p^{ij}$. If we define a set of coordinates $p^\alpha$ conjugate to the moduli $m_\alpha$ by

$$p^\alpha = \int_\Sigma e^{2\lambda} \pi^{ij} \frac{\partial \bar{g}_{ij}}{\partial m_\alpha} d^2x,$$  \hfill (2.9)

the action (2.1) reduces to

$$I_{Ein} = \int d\tau \left( p^\alpha \frac{dm_\alpha}{d\tau} - H(m, p, \tau) \right)$$  \hfill (2.10)

with

$$H = \int_\Sigma \sqrt{\bar{g}} \, d^2x = \int_\Sigma e^{2\lambda(m, p, \tau)} \sqrt{\bar{g}} \, d^2x,$$  \hfill (2.11)

where $\lambda(m, p, \tau)$ is determined by (2.8). Three-dimensional gravity is thus reduced to a finite-dimensional mechanical system, albeit one with a complicated and time-dependent Hamiltonian. In particular, the symplectic structure on the reduced phase space $(m_\alpha, p^\alpha)$ can be read off from the action (2.10):

$$\{ m_\alpha, p^\beta \} = \delta^\beta_\alpha.$$  \hfill (2.12)

This system—and in particular the Hamiltonian (2.11)—is generally quite complicated, but it simplifies greatly when $\Sigma$ is a torus. In that case, the super-Hamiltonian constraint (2.8) requires that $\lambda$ be spatially constant, and the Hamiltonian reduces to

$$H = (\tau^2 - 4\Lambda)^{-1/2} \left[ m_2^2 p \bar{p} \right]^{1/2}.$$  \hfill (2.13)

The momentum-dependent term in this expression may be recognized as the square of the momentum $p$ with respect to the Poincaré (constant negative curvature) metric

$$\frac{dmd\bar{m}}{m_2^2}$$  \hfill (2.14)

that is the standard metric on the torus moduli space. The Poisson brackets (2.12) are now

$$\{ m, \bar{p} \} = \{ \bar{m}, p \} = 2, \quad \{ m, p \} = \{ \bar{m}, \bar{p} \} = 0.$$  \hfill (2.15)

By construction, the moduli $m_\alpha$ and momenta $p^\alpha$ are invariant under infinitesimal diffeomorphisms, and therefore under diffeomorphisms that can be obtained by exponentiating such infinitesimal transformations. For a spacetime with the topology $\mathbb{R} \times \Sigma$, however, there are also “large” diffeomorphisms, which cannot be obtained in this fashion. These are generated by Dehn twists, that is, by the operation of cutting open a handle, twisting
one end by $2\pi$, and regluing the cut edges. The set of equivalence classes of such large diffeomorphisms (modulo diffeomorphisms that can be deformed to the identity) is known as the mapping class group of $\Sigma$; for the torus, it is also known as the modular group.

For the torus, in particular, there are two independent Dehn twists, corresponding to the two independent circumferences $\gamma_1$ and $\gamma_2$, which we choose to have intersection number $+1$. These act on the fundamental group $\pi_1(T^2)$ by interchanging or mixing the circumferences $\gamma_1$ and $\gamma_2$:

$$S: \gamma_1 \to \gamma_2^{-1}, \quad \gamma_2 \to \gamma_1$$
$$T: \gamma_1 \to \gamma_1 \cdot \gamma_2, \quad \gamma_2 \to \gamma_2,$$

where the dot in the last line of (2.16) represents composition of curves, or multiplication of homotopy classes. These transformations do not leave the modulus invariant, but instead give rise to the modular transformations

$$S: m \to -\frac{1}{m}, \quad p \to m^2 p$$
$$T: m \to m + 1, \quad p \to p,$$

which may be seen to preserve the Poincaré metric (2.14) and the Poisson brackets (2.15). It may be shown that the transformations (2.16) generate the entire group of large diffeomorphisms of $\mathbb{R} \times T^2$.

Classically, observables should presumably be invariant under all spacetime diffeomorphisms, including those in the mapping class group. Quantum mechanically, this condition may be relaxed, but operators and wave functions should still transform under some unitary representation of the mapping class group. This restriction will be important when we discuss quantization.

### 2.2. First-Order Formalism

Rather than starting with the metric as the fundamental variable, we may instead write the Einstein action in first-order form, treating the triad one-form (or coframe) $e^a = e^a_\mu dx^\mu$ and the spin connection $\omega^{ab} = \omega^{ab}_\mu dx^\mu$ as independent variables. This leads to the first-order, connection approach to (2+1)-dimensional gravity, inspired by Witten [4] (see also [23]) and developed by Nelson, Regge and Zertuche [8,9,10,11,12,13]. The triad $e^a$ is related to the metric of the previous section through

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab},$$

and the action (2.1) becomes

$$I_{Ein} = \int (d\omega^{ab} - \omega^a \wedge \omega^b + \frac{\Lambda}{3} e^a \wedge e^b) \wedge e^c \epsilon_{abc}, \quad a, b, c = 0, 1, 2.$$
For $\Lambda \neq 0$, this action can be written (apart from a total derivative) in the Chern-Simons form

$$I_{CS} = -\frac{\alpha}{4} \int (d\omega^{AB} - \frac{2}{3} \omega^A_E \wedge \omega^E_B) \wedge \omega^{CD} \epsilon_{ABCD}, \quad A, B, C = 0, 1, 2, 3$$ (2.20)

provided the (anti-)de Sitter spin connection $\omega^{AB}$ is identified with the variables $e^a, \omega^{ab}$ in the following way:

Let $k$ denote the sign of $\Lambda$ (i.e., $k = +1$ for de Sitter space and $k = -1$ for anti-de Sitter space), and set

$$\Lambda = k \alpha^{-2}.$$ 

Let $\sqrt{k}$ mean unambiguously $+1$ for $k = 1$ and $+i$ for $k = -1$. Define the tangent space metric as $\eta^{AB} = (-1, 1, 1, k)$ and the Levi-Civita density as $\epsilon_{a b c} = -\epsilon_{a b c}$. Now incorporate the triads by setting $e^a = \alpha \omega^a_3$, that is,

$$\omega^A_B = \begin{pmatrix} \omega^a_b & \frac{k}{\alpha} e^a \\ -\frac{1}{\alpha} e^b & 0 \end{pmatrix}. \quad (2.21)$$

The curvature two-form for the connection $\omega^{AB}$,

$$R^{AB} = d\omega^{AB} - \omega^{AC} \wedge \omega^C_B,$$ (2.22)

has components $R^{ab} + \Lambda e^a \wedge e^b, R^{a3} = \frac{1}{\alpha} R^a$, with

$$R^{ab} = d\omega^{ab} - \omega^{ac} \wedge \omega^c_b$$

$$R^a = de^a - \omega^{ab} \wedge e_b.$$ (2.23)

$R^{ab}$ and $R^a$ may be recognized the ordinary $(2+1)$-dimensional curvature and torsion forms. The field equations derived from the action (2.20) are simply

$$R^{AB} = 0,$$ (2.24)

implying, as in the second-order formalism of the previous section, that the torsion vanishes everywhere and that the curvature $R^{ab}$ is constant.

In a $(2+1)$-dimensional splitting of spacetime, the action (2.20) decomposes as

$$I_{CS} = \frac{\alpha}{4} \int dt \int d^2 x \epsilon^{ij} \epsilon_{ABCD} \left( \omega^{CD}_j \dot{\omega}^{AB}_i - \omega^{AB}_0 R^{CD}_{ij} \right)$$ (2.25)

(with $\epsilon^{0ij} = -\epsilon^{ij}$), from which the constraints are

$$R^{AB}_{ij} = 0.$$ (2.26)

These are equivalent to the conditions $\mathcal{H} = 0, \mathcal{H}' = 0$ of equation (2.3) of the last section. The Poisson brackets can be read off from (2.25):

$$\{ \omega^{AB}_i(x), \omega^{CD}_j(y) \} = \frac{k}{2\alpha} \epsilon^{ij} \epsilon^{ABCD} \delta^2(x - y),$$ (2.27)
or equivalently

\[ \{ e^a_i(x), \omega^{bc}_{\; j}(y) \} = -\frac{1}{2} \epsilon_{ij} \epsilon^{abc} \delta^2(x - y) \]

\[ \{ e^a_i(x), e^b_j(y) \} = \{ \omega^{ab}_{\; i}(x), \omega^{cd}_{\; j}(y) \} = 0 \]  \hspace{1cm} (2.28)

where \( x, y \in \Sigma \) are generic points on the \( t = \text{const.} \) surface \( \Sigma \) and \( \epsilon_{12} = 1 \).

The constraints (2.26) imply that the (anti-)de Sitter connection \( \omega^{AB}_{\; i} \) is flat. It can therefore be written locally in terms of an \( \text{SO}(3, 1) \)- or \( \text{SO}(2, 2) \)-valued zero-form \( \psi^{AB} \) as

\[ d\psi^{AB} = \omega^{AC}_{\;} \psi_C^B. \]  \hspace{1cm} (2.29)

Of course, \( \omega^{AB} \) may have nontrivial holonomies around closed loops in \( \Sigma \), so \( \psi^{AB} \) will not necessarily be single-valued. It is actually more convenient to use the spinor groups, where the spinor group of \( \text{SO}(2, 2) \) is \( \text{SL}(2, \mathbb{R}) \otimes \text{SL}(2, \mathbb{R}) \) and that of \( \text{SO}(3, 1) \) is \( \text{SL}(2, \mathbb{C}) \). Define the one-form

\[ \Delta(x) = \Delta_i(x) dx^i = \frac{1}{4} \omega^{AB}_{\;}(x) \gamma_{AB} \]  \hspace{1cm} (2.30)

(for \( \gamma \)-matrix conventions and identities see the Appendix), for which (2.26) implies that \( d\Delta - \Delta \wedge \Delta = 0 \). This form of the constraints can be integrated using multivalued \( \text{SL}(2, \mathbb{R}) \) or \( \text{SL}(2, \mathbb{C}) \) matrices \( S \), which satisfy

\[ dS(x) = \Delta(x) S(x). \]  \hspace{1cm} (2.31)

These are related to the matrices \( \psi \) of (2.29) by

\[ \psi^{AB}_{\;} \gamma_B = S^{-1} \gamma^A S. \]  \hspace{1cm} (2.32)

From (2.27), these spinor representations satisfy the Poisson brackets

\[ \{ \Delta^\pm_i(x), \Delta^\pm_j(y) \} = \pm \frac{i}{2 \alpha \sqrt{k}} \epsilon_{ij} \sigma^m \otimes \sigma^m \delta^2(x - y) \]

\[ \{ \Delta^+_i(x), \Delta^-_j(y) \} = 0, \]  \hspace{1cm} (2.33)

where the \( \sigma^m \) are Pauli matrices and the \( \pm \) refer to the decomposition of the \( 4 \times 4 \) representations of \( \Delta(x), S(x) \) into \( 2 \times 2 \) irreducible parts by using the projectors \( p_\pm \) (see Appendix),

\[ \Delta(x) = \Delta^+(x) \otimes p_+ + \Delta^-(x) \otimes p_- \]

\[ S(x) = S^+ \otimes p_+ + S^- \otimes p_-, \]  \hspace{1cm} (2.34)

so

\[ dS^\pm(x) = \Delta^\pm(x) S^\pm(x). \]  \hspace{1cm} (2.35)

In this approach to \( (2+1) \)-dimensional gravity, the existence of nontrivial classical solutions arises from the fact that the \( S^\pm \) may be multivalued—that is, the connections \( \omega^{AB} \)
and $\Delta$ may have nontrivial holonomies. In particular, let $\gamma : [0, 1] \to \Sigma$ be a noncontractible closed curve based at $\gamma(0) = x_0$, and take $S^\pm(\gamma(0)) = 1$ as an initial condition for the differential equation (2.31). Then (2.31) can be integrated to obtain a nontrivial value for $S^\pm(\gamma(1)) = S^\pm[\gamma]$, the $\text{SL}(2, \mathbb{R})$ or $\text{SL}(2, \mathbb{C})$ holonomy of $\Delta$. In general, this holonomy will depend on the curve $\gamma$, but for our system, the flatness of the connection $\Delta$ implies that $S^\pm[\gamma]$ depends only on the homotopy class of $\gamma$.

The Poisson brackets (2.27) now induce corresponding brackets between $S^\pm[\sigma]$ and $S^\pm[\gamma]$, where $\sigma, \gamma \in \pi_1(\Sigma, x_0)$. The matrices $S^\pm[\gamma]$ therefore furnish a representation of $\pi_1(\Sigma, x_0)$ in $\text{SL}(2, \mathbb{R})$ or $\text{SL}(2, \mathbb{C})$. Under a gauge transformation or a change of base point, the $S^\pm$ transform by conjugation, so their traces provide an (overcomplete) set of gauge-invariant variables.

The classical Poisson brackets for these variables were calculated by hand for the genus 1 and genus 2 cases, and then generalized and quantized in [3]. For the genus 1 case that is the focus of this paper, the Poisson algebra is

$$\{ R^\pm_1, R^\pm_2 \} = \mp \frac{i}{4\alpha \sqrt{k}} (R^\pm_{12} - R^\pm_1 R^\pm_2) \quad \text{and cyclic permutations,} \quad (2.36)$$

where $R^\pm = \frac{1}{2} \text{Tr} S^\pm$. Here the subscripts 1 and 2 refer to the two independent intersecting circumferences $\gamma_1, \gamma_2$ on $\Sigma$ with intersection number $+1$, while the third holonomy, $R^\pm_{12}$, corresponds to the path $\gamma_1 \cdot \gamma_2$, which has intersection number $-1$ with $\gamma_1$ and $+1$ with $\gamma_2$.

Classically, the six holonomies $R^\pm_{1,2,12}$ provide an overcomplete description of the space-time geometry of $\mathbb{R} \times T^2$, which, as we saw in the last section, is completely characterized by four real or two complex parameters $m$ and $p$. To understand this overcompleteness, consider the cubic polynomials

$$F^\pm = 1 - (R^\pm_1)^2 - (R^\pm_2)^2 - (R^\pm_{12})^2 + 2 R^\pm_1 R^\pm_2 R^\pm_{12}. \quad (2.37)$$

These polynomials have vanishing Poisson brackets with all of the traces $R^\pm_a$, and are cyclically symmetric in the $R^\pm_a$. Moreover, the classical algebra (2.36) and the central elements (2.37) are invariant under the modular transformations (2.16).

The overcompleteness of our description arises because the polynomials $F^\pm$ vanish classically by the $\text{SL}(2, \mathbb{R})$ or $\text{SL}(2, \mathbb{C})$ Mandelstam identities. Indeed, $F^\pm$ may be expressed as

$$F^\pm = \frac{1}{2} \text{Tr} \left( I - S^\pm[\gamma_1] S^\pm[\gamma_2] S^\pm[\gamma_1^{-1}] S^\pm[\gamma_2^{-1}] \right), \quad (2.38)$$

where $S^\pm[\gamma_i^{-1}] = (S^\pm[\gamma_i])^{-1}$; this expression reduces to (2.37) by virtue of the identities

$$A + A^{-1} = I \text{Tr} A$$

for $2 \times 2$ unimodular matrices $A$. The classical vanishing of $F^\pm$ can thus be thought of as the application of the group identity

$$\gamma_1 \cdot \gamma_2 \cdot \gamma_1^{-1} \cdot \gamma_2^{-1} = I$$

8Paths with intersection number 0, ± 1 are sufficient to characterize the holonomy algebra for genus 1. For $g > 1$, one must in general consider paths with two or more intersections, for which the brackets (2.38) are more complicated; see [11,12].
to the representations \( S^\pm \).

In this first-order approach, the constraints have now been solved exactly. There is no Hamiltonian, however, and no time development. One can think of this formalism as initial data for some (unspecified) choice of time, or alternatively as giving a time-independent description of the entire spacetime geometry.

### 3. Classical Solutions

Before turning to quantization, it is useful to explore the structure of the classical solutions of \((2+1)\)-dimensional gravity in more detail. We shall concentrate on spacetimes with the topology \( \mathbb{R} \times T^2 \), for which the space of classical solutions is completely understood.

It would, of course, be valuable to extend these results to more complicated topologies. Some progress towards this goal has been made in the holonomy formulation \([8]\), but a full geometric understanding is still lacking. We shall also specialize to the case \( \Lambda < 0 \), briefly discussing the corresponding picture for \( \Lambda \geq 0 \) at the end of this section. Many of the results presented in this section have been discovered independently by Ezawa \([18,21]\), and related solutions were found by Fujiwara and Soda \([3]\).

An obvious place to begin is with the ADM formalism of section 2.1: the spatial metric on a constant \( \tau \) slice is given by equation \((2.3)\), and the results of Moncrief \([2]\) permit a straightforward computation of the lapse and shift functions in the full ADM metric \((2.2)\). Rather than using the York time slicing from the start, it is actually somewhat easier to begin in a slightly different gauge, “time gauge,” in which \( N = 1 \) and \( N^i = 0 \). Equivalently, in the first order form, we choose \( e_i^0 = 0 \) and \( e_t^0 = 1 \). We shall see below that for the topology \( \mathbb{R} \times T^2 \), this choice is equivalent to the York gauge, although this is no longer the case for spaces of genus \( g > 1 \).

With this gauge choice, it is easy to check that the first-order field equations \((2.24)\) are solved by

\[
\begin{align*}
e^0 &= dt \\
e^1 &= \frac{\alpha}{2} \left[ (r_1^+ - r_1^-) dx + (r_2^+ - r_2^-) dy \right] \sin \frac{t}{\alpha} \\
e^2 &= \frac{\alpha}{2} \left[ (r_1^+ + r_1^-) dx + (r_2^+ + r_2^-) dy \right] \cos \frac{t}{\alpha}
\end{align*}
\]

\[\omega^{12} = 0\]

\[\omega^{01} = -\frac{1}{2} \left[ (r_1^+ - r_1^-) dx + (r_2^+ - r_2^-) dy \right] \cos \frac{t}{\alpha}\]

\[\omega^{02} = \frac{1}{2} \left[ (r_1^+ + r_1^-) dx + (r_2^+ + r_2^-) dy \right] \sin \frac{t}{\alpha},\]

where \( r_1^\pm \) and \( r_2^\pm \) are four arbitrary parameters and the coordinates \( x \) and \( y \) have period one. The triad \((3.1)\) determines a spacetime metric \( g_{\mu\nu} = e^\alpha_\mu e_\alpha^\nu \), which can be used to
compute the moduli and momenta of section 2.1. In particular, it is not hard to show that the
spatial metric $g_{ij}$ on a slice of constant $t$ describes a torus with modulus

$$m = \left( r_1^- e^{it/\alpha} + r_1^+ e^{-it/\alpha} \right) \left( r_2^- e^{it/\alpha} + r_2^+ e^{-it/\alpha} \right)^{-1}.$$  \hspace{1cm} (3.3)

The conjugate momentum $p$ of equation (2.9) can be similarly computed from the $g_{\mu\nu}$ in
terms of the extrinsic curvature of a slice of constant $t$; it takes the form

$$p = -\frac{i\alpha}{2 \sin \frac{\alpha}{2}} \left( r_2^+ e^{it/\alpha} + r_2^- e^{-it/\alpha} \right)^2.$$  \hspace{1cm} (3.4)

Finally, the York time is

$$\tau = -\frac{d}{dt} \ln \sqrt{g} = -\frac{2}{\alpha} \cot \frac{2t}{\alpha},$$  \hspace{1cm} (3.5)

which ranges from $-\infty$ to $\infty$ as $t$ varies from 0 to $\pi\alpha/2$. Clearly, $\tau$ is a monotonic function
of $t$ in this range, so a slicing by surfaces of constant $t$ is equivalent to the York slicing by
surfaces of constant $K$, as claimed.

To check the generality of the solution (3.1)–(3.2), observe first that the four parameters
$r_{\alpha}^\pm$ can be chosen arbitrarily, which in turn implies that the modulus $m$ and momentum $p$ of
equations (3.3) and (3.4) can take arbitrary values at an initial surface $t = t_0$. This means
that we can specify arbitrary initial data $(m(t_0), p(t_0))$ in the ADM formalism. Results
of Moncrief [2] and Mess [26] then guarantee that such data determine a unique maximal
spacetime—technically, a maximal domain of dependence of the initial surface—and that
any such spacetime can be obtained from suitable initial data.\*

We can obtain additional information about this solution by calculating the $\text{SL}(2, \mathbb{R})$
holonomies of equation (2.31), using the decomposition of the spinor group of $\text{SO}(2, 2)$
described in Section 2.2. The computation is again straightforward, and gives traces

$$R_1^\pm = \frac{1}{2} \text{Tr} S^\pm [\gamma_1] = \cosh \frac{r_{\alpha}^\pm}{2},$$

$$R_2^\pm = \frac{1}{2} \text{Tr} S^\pm [\gamma_2] = \cosh \frac{r_{\alpha}^\pm}{2},$$

$$R_{12}^\pm = \frac{1}{2} \text{Tr} S^\pm [\gamma_1 \cdot \gamma_2] = \cosh \frac{(r_{\alpha}^1 + r_{\alpha}^2)}{2}. \hspace{1cm} (3.6)$$

Conversely, the metric $g_{\mu\nu}$ can be obtained directly from the holonomies by a quotient space
construction. Three-dimensional anti-de Sitter space is naturally isometric to the group
manifold of $\text{SL}(2, \mathbb{R})$. Indeed, anti-de Sitter space can be represented as the submanifold
of flat $\mathbb{R}^{2,2}$ (with coordinates $(X_1, X_2, T_1, T_2)$ and metric $dS^2 = dX_1^2 + dX_2^2 - dT_1^2 - dT_2^2$) on which

$$\det|X| = 1, \hspace{1cm} X = \frac{1}{\alpha} \begin{pmatrix} X_1 + T_1 & X_2 + T_2 \\ -X_2 + T_2 & X_1 - T_1 \end{pmatrix},$$  \hspace{1cm} (3.7)$$

*This is no longer true in the case $\Lambda > 0$ [24]; the resulting ambiguity is discussed briefly in [27] and [28]. Moreover, as Louko and Marolf have observed [29], if one starts with a solution that is a domain of
dependence, it may be possible to find further extensions to regions containing closed timelike curves.
i.e., $X \in \text{SL}(2, \mathbb{R})$. If one allows the $S_a^+$ to act on $X$ by left multiplication and the $S_a^-$ to act by right multiplication, it may be shown that the triad (3.1) represents the geometry of the quotient space $(S_1^+, S_2^+) \setminus \text{AdS}/(S_1^-, S_2^-)$.

Now, the holonomies (3.6) are not the most general possible: an $\text{SL}(2, \mathbb{R})$ matrix can have an arbitrary trace, while our solution requires the holonomies to be hyperbolic. The solution (3.1)–(3.2) thus represents only one sector in the space of holonomies, the “hyperbolic-hyperbolic” sector, out of nine possibilities [21]. On the other hand, we argued above that (3.1)–(3.2) gave the most general solution to the problem of evolution of initial data on a spacelike surface with the topology $T^2$. These two statements are not, in fact, inconsistent: for solutions with elliptic or parabolic holonomies, spacetime still has the topology $\mathbb{R} \times T^2$, but the toroidal slices are not spacelike [21]. In particular, the choice of time gauge is only possible when all holonomies are hyperbolic. A similar phenomenon has been investigated in detail by Louko and Marolf [29] in the case $\Lambda=0$.

### 3.1. Classical Time Evolution

By construction, we know that the traces (3.6) must satisfy the nonlinear classical Poisson bracket algebra (2.36). One may easily verify from (3.6) that the central elements (2.37) are identically zero. The algebra (2.36) implies that the holonomy parameters $r^\pm$ satisfy

$$\{r^\pm_1, r^\pm_2\} = \mp \frac{1}{\alpha}, \quad \{r^+_a, r^-_b\} = 0.$$  (3.8)

It is easily checked that the brackets (3.8) induce the correct brackets (2.15) for the modulus $m$ and momentum $p$ defined by (3.3)–(3.4), confirming the consistency of the classical first- and second-order descriptions.

From (3.3), it may be shown that the time-dependent moduli $m_1, m_2$ lie on a semicircle of radius $R$,

$$(m_1 - c)^2 + m_2^2 = R^2,$$  (3.9)

where

$$c = \frac{r^+_1 r^+_2 - r^-_1 r^-_2}{(r^+_2)^2 - (r^-_2)^2}, \quad R^2 = \frac{(r^+_1 r^-_2 - r^-_1 r^+_2)^2}{(r^+_2)^2 - (r^-_2)^2}.$$  (3.10)

This agrees with the results of Fujiwara and Soda [3]. The imaginary part $m_2$ of the modulus ranges from zero to $R$ and back to zero, while $m_1$ ranges from $c + R$ to $c$ to $c - R$. The momenta (3.4) are similarly constrained through

$$p(\bar{m} - c) + \bar{p}(m - c) = 0.$$  (3.11)

In this range $p^1$ is constant while $p^2$ varies from $-\infty$ to zero to $+\infty$. This behavior clearly illustrates the nontrivial dynamics of the system, even though the full (2+1)-dimensional curvature tensor is everywhere constant. This is not a “gauge” effect, but rather reflects the nontrivial, time-dependent identifications needed to construct a torus from patches of anti-de Sitter space.

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† An $\text{SL}(2, \mathbb{R})$ matrix $R$ is called hyperbolic if $|\text{Tr}R| > 2$, parabolic if $|\text{Tr}R| = 2$, and elliptic if $|\text{Tr}R| < 2$. 

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The Hamiltonian $H$ of equation (2.13) generates this classical $\tau$ development of the moduli and their momenta through Hamilton’s equations,

\[
\frac{dp}{d\tau} = \{p, H\}, \quad \frac{dm}{d\tau} = \{m, H\}.
\]  

(3.12)

$H$ may be calculated in terms of the holonomy parameters $r^\pm_a$ as

\[
H = g^{1/2} = \frac{\alpha^2}{4} \sin \frac{2t}{\alpha} (r_1^- r_2^+ - r_1^+ r_2^-) \\
= \frac{\alpha}{2\sqrt{\tau^2 - 4\Lambda}} (r_1^- r_2^+ - r_1^+ r_2^-),
\]  

(3.13)

where we assume that $r_1^- r_2^+ - r_1^+ r_2^- > 0$ so that $g^{1/2} > 0$ in the range $t \in (0, \pi\alpha/2)$. This guarantees that the imaginary part $m^2$ of the modulus is always positive, as it is in the standard description of torus geometry. The area of the torus expands from zero to a maximum at $t = \pi\alpha/4$, $\tau = 0$ and then recollapses back to zero at $t = \pi\alpha/2$, $\tau = \infty$. Alternatively, the Hamiltonian

\[
H' = \frac{d\tau}{dt} H = \frac{4}{\alpha^2} \csc^2 \frac{2t}{\alpha} H
\]  

(3.14)

generates evolution in coordinate time $t$ by

\[
\frac{dp}{dt} = \{p, H'\}, \quad \frac{dm}{dt} = \{m, H'\}.
\]  

(3.15)

The standard action of the modular group on the traces (3.6) suggests that the holonomy parameters transform as

\[
S: r_1^\pm \to r_2^\pm, \quad r_2^\pm \to -r_1^\pm \\
T: r_1^\pm \to r_1^\pm + r_2^\pm, \quad r_2^\pm \to r_2^\pm
\]  

(3.16)

It may be checked that these transformations do indeed leave the brackets (3.8) and the Hamiltonians (3.13) and (3.14) invariant, and that they induce the correct modular transformations (2.17) on the moduli and momenta.

The relationships among the moduli and the holonomy parameters $r^\pm$ allows us to write the reduced phase space action of (2+1)-dimensional gravity in several equivalent forms:

\[
I_{E_{in}} = \int dt \int d^2x \pi^{ij} \dot{g}_{ij} = \int dt \int d^2x \, 2\epsilon^{ij} \epsilon_{abc} e^c_j \dot{\omega}^{ab}_i \\
= \frac{1}{2} (\bar{p} dm + p d\bar{m}) + H d\tau - d(p^1 m_1 + p^2 m_2) \\
= \alpha \int (r_1^- dr_2^- - r_1^+ dr_2^+),
\]  

(3.17)

showing that the holonomy parameters $r_{1,2}^\pm$ are related to the modulus $m$ and momentum $p$ through a (time-dependent) canonical transformation.
3.2. \( \Lambda \to 0^- \)

Much of the classical behavior discussed above was studied previously in [4,5] for the case of vanishing cosmological constant. The \( \Lambda = 0 \) theory is easy to describe in ADM variables, but the holonomies analogous to \( R^\pm_a \) are considerably more complicated, since the relevant gauge group is the non-semisimple Lie group \( \text{ISO}(2,1) \), the \((2+1)\)-dimensional Poincaré group. It is therefore useful to describe the relationship between the \( \Lambda < 0 \) and \( \Lambda = 0 \) theories. (The corresponding limit for \( \Lambda \to 0^+ \) has also been studied by Ezawa [18].)

The \( \Lambda \to 0 \) limit is most easily seen by rescaling the holonomy parameters as follows. Define

\[
\begin{align*}
  w_a &= \alpha \left( r_a^+ + r_a^- \right)/2 \\
  u_a &= \left( r_a^+ - r_a^- \right)/2, \quad a = 1, 2
\end{align*}
\]

where the (time-independent) \( w_a \) and \( u_a \) remain finite as \( \Lambda \to 0, \alpha \to \infty \). In this limit, the York time (3.5) becomes

\[
\tau = -\frac{1}{t},
\]

and the solution (3.1)–(3.2) reduces to

\[
\begin{align*}
  e^0 &= dt \\
  e^1 &= t [u_1 dx + u_2 dy] \\
  e^2 &= [w_1 dx + w_2 dy] \\
  \omega^{12} &= 0 \\
  \omega^{01} &= -[u_1 dx + u_2 dy] \\
  \omega^{02} &= 0.
\end{align*}
\]

The moduli and momenta of (3.3)–(3.4) are now

\[
\begin{align*}
  m &= (w_1 - itu_1)(w_2 - itu_2)^{-1} \\
  p &= -\frac{i}{t}(w_2 - itu_2)^2,
\end{align*}
\]

and the Hamiltonian (3.13) is

\[
H = g^{1/2} = t(w_1 u_2 - u_1 w_2).
\]

The recontraction of the spatial slices now disappears; the tori instead expand linearly in the range \( t \in (0, \infty) \). The new variables \( u \) and \( w \) may be easily shown to satisfy the classical Poisson brackets

\[
\{u_1, w_2\} = \{w_1, u_2\} = -\frac{1}{2},
\]
derivables and refer to the action
\[ I = -2 \int (u_1 dw_2 + w_1 du_2). \]  
(3.24)

They are related to the parameters of reference [5] by
\[ u_1 = -\lambda, \; u_2 = -\mu, \; w_1 = a, \; w_1 = b. \]  
(3.25)

For the traces (3.6) the limit is more complicated. However, the Poincaré variables and their algebra described in reference [10] may be retrieved by using the holonomy parameters
\[ r_{a}^{\pm} = u_{a} \pm \frac{w_{a}}{\alpha} \]  
(3.26)
to expand the traces (3.6) to first order in \( 1/\alpha \),
\[ R_{a}^{\pm} = \cosh \frac{r_{a}^{\pm}}{2} = \cosh \frac{u_{a}}{2} \pm \frac{w_{a}}{2\alpha} \sinh \frac{u_{a}}{2} \]
\[ = q_{a} \pm \frac{\nu_{a}}{\alpha} \quad (a = 1, 2). \]  
(3.27)

The Poisson brackets (3.23) then imply that to order \( 1/\alpha \),
\[ \{q_{1}, q_{2}\} = -\frac{1}{16\alpha^2} (\nu_{12} - \nu_{12}^{-1}) \]
\[ \{q_{1}, \nu_{2}\} = -\frac{1}{16} (q_{12} - q_{12}^{-1}) + \frac{1}{8\alpha^2} \nu_{1} \nu_{2} \]
\[ \{\nu_{1}, \nu_{2}\} = -\frac{1}{16} (\nu_{12} - \nu_{12}^{-1}). \]  
(3.28)

Here
\[ q_{12} = \cosh \frac{u_{1} + u_{2}}{2}, \quad q_{12}^{-1} = \cosh \frac{u_{1} - u_{2}}{2} \]
\[ \nu_{12} = \left(\frac{w_{1} + w_{2}}{2}\right) \sinh \frac{u_{1} + u_{2}}{2}, \quad \nu_{12}^{-1} = \left(\frac{w_{1} - w_{2}}{2}\right) \sinh \frac{u_{1} - u_{2}}{2}, \]  
(3.29)

which satisfy the identities
\[ \nu_{12}^{-1} + \nu_{12} = 2(\nu_{1} q_{2} + \nu_{2} q_{1}), \quad q_{12}^{-1} + q_{12} = 2 q_{1} q_{2}. \]  
(3.30)

With these identifications, the algebra (3.28) reproduces that of [10] in the limit \( \Lambda \to 0, \alpha \to \infty \).

\[ ^{\dagger} \text{Note that the } \nu \text{ of [10] is } 8 \nu^{\text{here}}. \]
3.3. $\Lambda > 0$

The case of positive cosmological constant has been studied in detail by Ezawa [18]. For completeness, we point out that the classical solutions for $\Lambda > 0$ can easily be derived from our solutions for $\Lambda < 0$ by substituting hyperbolic sines and cosines for sines and cosines. One finds that the range of

$$\tau = -\frac{2}{\alpha} \coth \frac{2t}{\alpha}$$

is now $-\infty$ to $-\frac{2}{\alpha}$ for $t \in (0, \infty)$, and that the area of the torus expands exponentially from zero to $\infty$. The $R_a^\pm$ are now traces of SL(2, $\mathbb{C}$) holonomies, expressed as the complex conjugates

$$R_a^\pm(r_a^+, r_a^-) = \cos \left( \frac{r_a^+ + r_a^-}{4} \right) \pm i \frac{r_a^+ - r_a^-}{4},$$

which can be written in terms of the parameters $u, w$ of equation (3.18) of the previous section as

$$R_a^\pm = \cos \frac{w_a}{2\alpha} \pm i \frac{w_a}{2}$$

$$= \cos \left( \frac{w_a}{2\alpha} \right) \cosh \left( \frac{u_a}{2} \right) \mp i \sin \left( \frac{w_a}{2\alpha} \right) \sinh \left( \frac{u_a}{2} \right)$$

To first order in $1/\alpha$, we have

$$R_a^\pm = q_a \mp i \frac{\nu_a}{\alpha} \quad (a = 1, 2)$$

(compare with (3.27) for $\Lambda < 0$), so the limit $\Lambda \to 0$ is again easy to understand. In particular, the $q_a$ and $\nu_a$ again satisfy the algebra (3.28) in the limit $\Lambda \to 0^+$.

Note that from (3.31),

$$R_a^\pm(r_a^+, r_a^-) = R_a^\pm(r_a^+ + 4\pi n_a, r_a^- + 4\pi n_a)$$

for any integers $n_a$. Hence the parameters $r_a^\pm$—and therefore the moduli—are not uniquely determined by the SL(2, $\mathbb{C}$) traces, in contrast to the $\Lambda < 0$ case. Such a change corresponds to adding the total derivative

$$4\pi n_1 d(r_2^2 - r_2^+)$$

to the action (3.17). This ambiguity was first noted by Mess [26], and was discussed by Witten in reference [27]. It suggests that in addition to the traces $R_a^\pm$, a new discrete quantum number related to the direct quantization of the parameters $r_a^\pm$ may be necessary to describe (2+1)-dimensional gravity with positive cosmological constant.

4. Quantum Theories

We now turn to the quantization of the system described above. As we shall see, the different classical descriptions naturally lead to very different approaches to the quantum theory, whose relationship can give us further information about the structure of (2+1)-dimensional quantum gravity.
4.1. ADM Quantization

Let us begin with the second-order formalism of section 2.1. We saw above that the reduced phase space action (2.10)—the action written in terms of the physical variables \( m_\alpha \) and \( p^\alpha \)—is equivalent to that of a finite-dimensional mechanical system with a complicated Hamiltonian. We know, at least in principle, how to quantize such a system: we simply replace the Poisson brackets (2.12) with commutators,

\[
\hat{m}_\alpha \hat{p}^\beta = i\hbar \delta^\beta_\alpha ,
\]

(4.1)

represent the momenta as derivatives,

\[
p^\alpha = \frac{\hbar}{i} \frac{\partial}{\partial m_\alpha} ,
\]

(4.2)

and impose the Schrödinger equation,

\[
i\hbar \frac{\partial \psi(m, \tau)}{\partial \tau} = \hat{H} \psi(m, \tau) ,
\]

(4.3)

where the Hamiltonian \( \hat{H} \) is obtained from (2.11) by some suitable operator ordering.

One fundamental problem, of course, is hidden in this last step: it is not at all obvious how one should define \( \hat{H} \) as a self-adjoint operator on an appropriate Hilbert space. The ambiguity is already evident for the genus 1 Hamiltonian (2.13): \( \hat{m}_2 \) and \( \hat{p} \) do not commute, so the operator ordering is not unique. The simplest choice of ordering is that of equation (2.13), for which the Hamiltonian becomes

\[
\hat{H} = \frac{\hbar}{\sqrt{\tau^2 - 4\Lambda}} \Delta_0^{1/2} ,
\]

(4.4)

where \( \Delta_0 \) is the ordinary scalar Laplacian for the constant negative curvature moduli space characterized by the metric (2.14). This Laplacian is invariant under the modular transformations (2.17); its invariant eigenfunctions, the weight zero Maass forms, are discussed in considerable detail in the mathematical literature [30].

While this choice of ordering is not unique, the number of possible alternatives is smaller than one might fear. The key restriction is diffeomorphism invariance: the eigenfunctions of the Hamiltonian should transform under a one-dimensional unitary representation of the mapping class group. The representation theory of the modular group (2.17) has been studied extensively [31]; one finds that the possible inequivalent Hamiltonians are all of the form (4.4), but with \( \Delta_0 \) replaced by

\[
\Delta_n = -m_2^2 \left( \frac{\partial^2}{\partial m_1^2} + \frac{\partial^2}{\partial m_2^2} \right) + 2inm_2 \frac{\partial}{\partial m_1} + n(n + 1), \quad 2n \in \mathbb{Z} ,
\]

(4.5)

the Maass Laplacian acting on automorphic forms of weight \( n \). (See [7] for details of the required operator orderings.) Note that when written in terms of the momentum \( p \), the

*It is argued in [6] that the natural choice of ordering in first-order quantization corresponds to \( n = 1/2 \).
operators $\Delta_n$ differ from each other by terms of order $\hbar$, as expected for operator ordering ambiguities. Nevertheless, the various choices of ordering can have drastic effects on the physics: the spectra of the various Maass Laplacians are very different.

This ambiguity can be viewed as a consequence of the structure of the classical phase space. The torus moduli space is not a manifold, but rather has orbifold singularities, and quantization on an orbifold is generally not unique. Since the space of solutions of the Einstein equations in 3+1 dimensions has a similar orbifold structure [32], we might expect a similar ambiguity in realistic (3+1)-dimensional quantum gravity.

There is another, potentially more serious, ambiguity in this approach to quantization, coming from the classical treatment of the time slicing. The choice of $K$ as a time variable is rather arbitrary—it greatly simplifies the constraints (2.3), but is otherwise no better than any other classical gauge-fixing technique—and it is not at all clear that a different choice would lead to the same quantum theory. The danger of making a “wrong” choice is illustrated by the classical solution (3.1)–(3.2): another standard gauge choice is $\sqrt{g} = t$, but it is evident that when $\Lambda < 0$, $\sqrt{g}$ is not even a single-valued function of $\tau$.

A possible resolution of this problem is to treat the holonomy approach, in which no choice of time slicing is needed, as fundamental. If we can establish a relationship between the \((\hat{m}, \hat{p})\) and suitable operators in the first-order formulation, we can convert the problem of time slicing into one of defining the appropriate physical operators. Different choices of slicing would then merely require different operators to represent moduli, and not different quantum theories.

4.2. Quantizing Traces of Holonomies

We next consider an alternative approach to quantization, starting from the first-order formulation of the classical theory. Without assuming \textit{ab initio} any classical relationship between moduli and holonomies, the algebra of the traces $R^\pm$ can be quantized directly for any value of the cosmological constant $\Lambda$ and any genus $g$ of $\Sigma$. For arbitrary genus, one obtains an abstract quantum algebra, the subject of intense study [8, 9]. In principle, a representation in terms of some finite set parameters, analogous to the $R^\pm$ of section 2.2, would determine a quantization of those parameters. For arbitrary $g$, it is not yet clear exactly how to find such a representation, although for $g = 2$ there has been some recent progress [8].

For the remainder of this section, we shall restrict our attention to the relatively well-understood case of $g = 1$, in order to make contact with the torus moduli quantization of Section 4.1. We can quantize the classical algebra (2.36) as follows:

1. We replace the classical Poisson brackets $\{,\}$ by commutators $[,]$, with the rule
   \[ [x, y] = xy - yx = i\hbar \{x, y\}; \]  
   (4.6)

2. On the right hand side of (2.36), we replace the product by the symmetrized product,
   \[ xy \rightarrow \frac{1}{2}(xy + yx). \]  
   (4.7)
The resulting operator algebra is given by
\[
\hat{R}_1^\pm \hat{R}_2^\pm e^{\pm i\theta} - \hat{R}_2^\pm \hat{R}_1^\pm e^{\mp i\theta} = \pm \frac{2i \sin \theta}{\sqrt{\hbar}} \hat{R}_{12}^\pm \quad \text{and cyclical permutations}
\] (4.8)
with
\[
\tan \theta = \frac{i\sqrt{\hbar}}{8\alpha}.
\] (4.9)
Note that for \(\Lambda < 0\), \(k = -1\), and \(\theta\) is real, while for \(\Lambda > 0\), \(k = 1\), and \(\theta\) is pure imaginary.

The algebra (4.8) is not a Lie algebra, but it is related to the Lie algebra of the quantum group \(SU(2)_q\) [13, 33], where \(q = \exp 4i\theta\), and where the cyclically invariant \(q\)-Casimir is the quantum analog of the cubic polynomial (2.37),
\[
\hat{F}^\pm(\theta) = \cos^2 \theta - e^{\pm 2i\theta} \left( (\hat{R}_1^\pm)^2 + (\hat{R}_{12}^\pm)^2 \right) - e^{\mp 2i\theta} (\hat{R}_2^\pm)^2 + 2e^{\pm i\theta} \cos \theta \hat{R}_1^\pm \hat{R}_2^\pm \hat{R}_{12}^\pm.
\] (4.10)

The operator algebra (4.8) can be represented by
\[
\hat{R}_a^\pm = \sec \theta \cosh \frac{\hat{r}_a^\pm}{2} \quad (a = 1, 2, 12),
\] (4.11)
with
\[
[\hat{r}_1^+, \hat{r}_2^+] = \pm 8i\theta, \quad [\hat{r}_a^+, \hat{r}_b^-] = 0,
\] (4.12)
which differ (for \(\Lambda\) small and negative) from the naive expectation
\[
[\hat{r}_1^+, \hat{r}_2^+] = \mp \frac{i\hbar}{\alpha}.
\] (4.13)
by terms of order \(\hbar^3\).

We must next try to implement the action of the modular group (3.16) on the operators \(\hat{R}_a^\pm\). The action that preserves the commutators (4.8) is (note the factor ordering)
\[
S: \hat{R}_1^\pm \to \hat{R}_2^\pm, \quad \hat{R}_2^\pm \to \hat{R}_1^\pm, \quad \hat{R}_{12}^\pm \to \hat{R}_1^\pm \hat{R}_2^\pm + \hat{R}_2^\pm \hat{R}_1^\pm - \hat{R}_{12}^\pm
\]
\[
T: \hat{R}_1^\pm \to \hat{R}_{12}^\pm, \quad \hat{R}_2^\pm \to \hat{R}_{12}^\pm, \quad \hat{R}_{12}^\pm \to \hat{R}_1^\pm \hat{R}_2^\pm + \hat{R}_2^\pm \hat{R}_1^\pm - \hat{R}_1^\pm.
\] (4.14)
The second of these can be generated by the unitary operators
\[
G^\pm = \exp(\pm \frac{i(\hat{r}_2^\pm)^2}{16\theta})
\] (4.15)
as
\[
y \to G^\pm y(G^\pm)^{-1}
\]
where \(y\) is any function of the \(\hat{r}_a^\pm\).

It is amusing to note that, from (4.12), the operators \(\exp\{n\hat{r}_1^\pm\}\) and \(\exp\{\hat{r}_2^\pm\}\) commute when
\[
\theta = \pi p/4n
\]
with \(n, p \in \mathbb{Z}\) and \(\Lambda < 0\). This occurs when the parameter \(q\) of the quantum group associated with the algebra (4.8) is a root of unity. We see from (4.9) that there are \(2n - 1\) solutions \(\alpha\) of this equation for for any given \(n\). By contrast, in the direct quantization given by equation (4.13), this simplification of the algebra would occur for an infinite number of values of \(\alpha\).
4.3. Quantizing the Space of Solutions

A third method of quantization starts with the parameters \( r_a^\pm \) of the classical solution \( r^1 e^{i\alpha t} + \hat r^2 e^{-i\alpha t} \). This approach can be viewed as a version of covariant canonical quantization, i.e., “quantizing the space of classical solutions” [34, 35]. It has the obvious disadvantage of requiring detailed knowledge of the classical solutions, which are completely understood at present only for the simplest topology, \( \mathbb{R} \times \mathbb{T}^2 \). On the other hand, this approach to quantization provides a natural bridge between the ADM and holonomy approaches discussed above, and in particular allows us to define a natural set of time-dependent physical operators in the latter theory.

Our starting point is now the set of Poisson brackets (3.8). The natural guess is that these should simply become the commutators (4.13). This leads to a legitimate quantum theory, but we know from the previous section that the commutators (4.8) of traces of holonomies will not be reproduced. To obtain these traces, we must instead impose the commutators (4.12). With these definitions, the results of the previous section are all preserved. In particular, it is not hard to show that the quantum modular group action (4.14) is induced by the transformations (3.16) of the \( r_a^\pm \).

We can now make the connection with the ADM quantization of section 4.1. The basic idea is to treat wave functions \( \psi(r_a) \) as Heisenberg picture states and to define suitable time-dependent operators acting on these states. Now, the classical modulus and momentum on a surface \( K = \tau \) have already been determined in terms of the \( r_a^\pm \), and are given by equations (3.3)–(3.4). Carrying these definitions over to the quantum theory, we obtain a family of operators \( \hat m(\tau) \) and \( \hat p(\tau) \), whose eigenvalues may be interpreted as the ADM modulus and momentum in the York time slicing. Similarly, the operator analog of (3.13) may be interpreted as a Hamiltonian generating the evolution of \( \hat m \) and \( \hat p \). Indeed, if we keep the orderings of (3.3)–(3.4) and (3.13), defining

\[
\hat m = \left( \hat r_1^+ e^{it/\alpha} + \hat r_1^- e^{-it/\alpha} \right) \left( \hat r_2^- e^{-it/\alpha} + \hat r_2^+ e^{it/\alpha} \right)^{-1},
\]

\[
\hat p = -\frac{i\alpha}{2\sin\frac{2\alpha}{\alpha}} \left( \hat r_2^+ e^{it/\alpha} + \hat r_2^- e^{-it/\alpha} \right)^2,
\]

\[
\hat H = \frac{\alpha^2}{4} \sin\frac{2t}{\alpha} (\hat r_1^- \hat r_2^+ - \hat r_1^+ \hat r_2^-),
\]

it follows from the commutators (1.12) that

\[
[\hat m^\dagger, \hat p] = [\hat m, \hat p^\dagger] = 16i\alpha \theta, \quad [\hat m, \hat p] = [\hat m^\dagger, \hat p^\dagger] = 0
\]

\[
[\hat p, \hat H'] = -8i\alpha \theta \frac{d\hat p}{dt}, \quad [\hat m, \hat H'] = -8i\alpha \theta \frac{d\hat m}{dt},
\]

which differ from the corresponding equations in ADM quantization by terms of order \( O(\hbar^3) \), small when \( |\Lambda| = 1/\alpha^2 \) is small. These results depend on operator ordering in \( \hat m, \hat p, \) and \( \hat H \), of course, but the orderings of (4.16) are a bit less arbitrary than they might seem: they were chosen to ensure that the modular transformations (3.16) of the \( r_a^\pm \) induce the correct transformations (2.17) of \( \hat m \) without any \( O(\hbar) \) corrections.
Note that (4.11) can be inverted to give the operators \( \hat{r}_a^\pm \) in terms of the traces \( \hat{R}_a^\pm \). Equation (4.16) can therefore be viewed as a definition of modulus and momentum operators in the holonomy algebra quantization. These operators are, of course, quite complicated—they involve logarithms of the traces \( \hat{R} \)—but given a representation of the holonomy algebra, they provide the first known instance of physical observables with clear geometric interpretations.

To further investigate the connection to ADM quantization, we can examine the properties of wave functions that are eigenfunctions of \( \hat{m}(r_a^\pm, \tau) \) and its adjoint—that is, we can transform to a “Schrödinger picture.” As in the \( \Lambda = 0 \) case [6], Ezawa has shown that these wave functions transform as Maass forms of weight \( 1/2 \), corresponding to an ordering (4.5) of the ADM Hamiltonian with \( n = 1/2 \) [18]. Also as in the \( \Lambda = 0 \) case [7], however, this Hamiltonian can be changed by reordering the operators \( \hat{m}^\dagger(r_a^\pm, \tau) \) and \( \hat{p}^\dagger(r_a^\pm, \tau) \), or equivalently by redefining the inner product.

We can now return to the question of the choice of time slicing raised at the end of section 4.1. In the holonomy quantization of section 4.2 or the approach of this section, no choice of a time coordinate is ever made. A particular time slicing is instead reflected in a choice of “time”-dependent operators \( \hat{m}(r_a^\pm, \tau) \) and \( \hat{p}(r_a^\pm, \tau) \) that describe the geometry of the chosen slice. Other choices of classical time coordinate would presumably lead to other operators, which would be used to answer genuinely different physical questions. In some sense, we have thus succeeded in evading the “problem of time” in quantum gravity.

5. Conclusion

In most quantum field theories, it is fairly clear from the start what the “right” variables to quantize are. Moreover, we have general theorems that guarantee that local field redefinitions will not change the \( S \) matrix. Consequently, we are not used to worrying about how to determine the “right” quantization of, say, electrodynamics.

Quantum gravity is different. Here, the physical observables are necessarily nonlocal, and there is no reason to believe that quantizations based on different variables should be equivalent. In (3+1)-dimensional gravity, of course, the question is rather premature, since we do not yet have even one complete, consistent quantization. In 2+1 dimensions, though, the problem becomes unavoidable.

It might be hoped, however, that this problem can be turned to our advantage. The various approaches to quantizing (2+1)-dimensional gravity have different strengths, and if their relationship can be understood clearly, we might be able to combine these strengths. In ADM quantization, for instance, the fundamental variables—the moduli and momenta \((m, p)\)—have simple geometric interpretations. In the quantization of the traces \( R^\pm \), the physical meaning of the observables is much less clear, but the algebraic structures can be directly generalized to arbitrary spatial topologies. A primary goal of this paper has been to demonstrate the relationships between these approaches, thus allowing us to introduce clearly defined physical observables into the algebraic structure of holonomy quantization.

As we have seen, this goal can be achieved for spacetimes of the form \( \mathbb{R} \times T^2 \). Equations (4.11) and (4.16) give explicit “time”-dependent operators in the holonomy formalism that
represent the moduli and momenta on surfaces of constant York time. These results depend on our knowledge of the exact solutions of the equations of motion, but it may be possible to extend them at least to genus 2: reference [8] has developed the description of the quantum algebra of traces of holonomies, while the hyperelliptic nature of genus 2 surfaces is likely to simplify the ADM analysis.

We have also seen hints of a solution of the “problem of time” in quantum gravity. In ADM quantization, one must choose a classical time-slicing, and it is by no means clear that different choices will lead to equivalent theories. In quantization of the holonomy algebra, on the other hand, no such choice need be made; different choices of time show up only as different families of operators describing the spatial geometry of the corresponding slices. The definition of such operators is difficult, of course, and it would be very useful to find a perturbative approach that did not require complete knowledge of the classical solutions. But in principle, we have found a way to implement Rovelli’s approach to “evolving constants of motion” [36] in a theory of quantum gravity.

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6. Appendix

In this appendix, we briefly describe some conventions of this paper. The γ-matrices are defined by

\[ \gamma_A \gamma_B + \gamma_B \gamma_A = 2g_{AB} \]
\[ \gamma_{AB} = \frac{1}{2}[\gamma_A, \gamma_B] \tag{6.1} \]

and \( \gamma_5 = k \gamma_0 \gamma_1 \gamma_2 \gamma_3 \) satisfies:

\[ \gamma_5^2 = -k, \quad -\frac{1}{2} \epsilon^{ABCD} \gamma_A \gamma_B = \gamma_5 \gamma^C \gamma^D. \tag{6.2} \]

In terms of the Pauli matrices, our representation is

\[ \gamma_0 = i \sigma_2 \otimes \sigma_3, \quad \gamma_1 = \sigma_3 \otimes \sigma_3, \quad \gamma_2 = -1 \otimes \sigma_1, \quad \gamma_3 = -\sqrt{k} \sigma_1 \otimes \sigma_3, \quad \gamma_5 = i \sqrt{k} \quad 1 \otimes \sigma_2. \]

The projectors

\[ P_\pm = \pm (\gamma_5 \pm i \sqrt{k} 1)/(2i \sqrt{k}) \tag{6.3} \]

satisfy \( P_\pm^2 = P_\pm, \quad P_+ P_- = P_- P_+ = 0. \) In this representation,

\[ P_\pm = 1 \otimes p_\pm, \quad p_\pm = \frac{1}{2}(1 \pm \sigma_2). \tag{6.4} \]
It follows that
\[ \gamma^{AB} \otimes \gamma_{AB} = -4(\sigma^i \otimes p_+) \otimes (\sigma^i \otimes p_+) - 4(\sigma^i \otimes p_-) \otimes (\sigma^i \otimes p_-) \] (6.5)

where
\[ (\sigma^i)^{\beta}_{\alpha} (\sigma^i)^\gamma_{\tau} = -\delta^\beta_{\alpha} \delta^\gamma_{\tau} + 2\delta^\beta_{\tau} \delta^\gamma_{\alpha}. \] (6.6)

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