AN INTEGRODIFFERENTIAL EQUATION DRIVEN BY FRACTIONAL BROWNIAN MOTION

HAKIMA BESSAIH AND CHANDANA WIJERATNE

Abstract. This paper deals with the well posedness of an integrodifferential equation that describes a vortex filament associated to a 3D turbulent fluid flow. This equation is driven by a fractional Brownian motion of Hurst parameter $H > 1/2$. We prove the global existence and uniqueness of a solution in a functional space of Sobolev type.

1. Introduction

The definition of stochastic integrals with respect to the fractional Brownian motions has been investigated intensively by several authors, see for example [1], [19], [7], [15] and for a more comprehensive introduction to this topic see [6] and [9]. There are two approaches to the construction of such integrals, the pathwise approach that is based on the Riemann-Stieltjes construction and is due to Young [21] and the rough path approach. While the pathwise theory is fairly well understood, it is applicable only for Hurst parameters $H > 1/2$, the rough path theory is receiving a lot of interest, see for example [10], [13], [16] and the references therein and is applicable for Hurst parameters $H > 1/4$.

In the present paper, we are using the pathwise argument to solve an integral equation which is an approximation of the line vortex equation. In particular, we will assume that the vorticity field associated to an ideal inviscid incompressible homogeneous fluid in $\mathbb{R}^3$ is described by a fractional Brownian motion with Hurst parameter $H > 1/2$ and we will study its evolution through a pathline equation. Let us denote by $\omega$ this vorticity field then

$$\vec{\omega} := \nabla \times \vec{u},$$

where $\vec{u}$ is the velocity field in $\mathbb{R}^3$. If we denote by $\vec{X}(t, x)$ the position at time $t$ of the fluid particle that at time 0 was at $x \in \mathbb{R}^3$. We have the following path-lines equation

$$\frac{d\vec{X}(t, x)}{dt} = \vec{u}(t, \vec{X}(t, x)). \quad (1.1)$$

Now, let us assume that the vorticity field is concentrated on a fractional Brownian curve $\vec{B}^H$ as follows

$$\vec{\omega}(t, x) = \Gamma \int_0^1 \delta(x - \vec{B}^H(t, \xi))d\vec{B}^H(t, \xi), \quad (1.2)$$

where $\delta$ is the usual “Dirac delta function”, $\Gamma > 0$ is the intensity of vorticity, $\xi \in [0, 1]$ is the arc-length, while the parameter $t$ represents the time. Using the Biot-Savart formula, the equation (1.1) becomes

$$\frac{d\vec{X}(t, \xi)}{dt} = \int_0^1 Q(\vec{X}(t, \xi) - \vec{B}^H(t, \eta))d\vec{B}^H(t, \eta), \quad (1.3)$$
with 
\[ \vec{X}(0, \xi) = \vec{\phi}(\xi). \]

Here \( \vec{\phi} \) is the initial condition and the matrix valued function \( Q \) is the singular matrix 
\[
\begin{pmatrix}
\frac{-1}{4\pi|y|^\gamma} & y_3 & -y_2 \\
-y_3 & 0 & y_1 \\
y_2 & -y_1 & 0
\end{pmatrix}.
\]

For an introduction to this topic we refer to [4], [8], [20], [18] and for a more probabilistic approach to [12]. When \( \vec{B}^H \) is replaced by \( \vec{X} \), the equation (1.3) has been studied by [2] for a smooth closed curve \( \vec{X} \) in the Sobolev space \( W^{1,2} \) and an existence and uniqueness theorem has been proved for local solutions in time. Later, local solutions in some spaces of Hölder continuous functions have been investigated in [3] that have been extended to global solutions in [3].

In the present paper, we will be dealing with an approximation of equation (1.3), the approximation will be on the matrix \( Q \), while the study of equation (1.3) is left for a subsequent paper. Furthermore, in order to make the exposition of our results easier to understand, we will make the assumption that the function \( \vec{X} \) is a real valued function, however, our results will still be true for a vector valued function \( \vec{X} \). More precisely, we are interested in the following integrodifferential equation
\[
\dot{Y}(t, \xi) = \phi(\xi) + \int_0^t \int_0^\xi A(Y(s, \eta))dB^H(s, \eta)ds,
\]
(1.4)

where for all \( t \in [0, T] \), \( B^H(t) = \{B^H(t, \xi), \xi \in [0, 1]\} \) is a real valued fractional Brownian motion of Hurst parameter \( H > \frac{1}{2} \), \( A \) is a bounded and differentiable real valued function with a Lipschitz continuity property, \( \xi \) is a parameter in \([0, 1]\).

Let us describe the content of the paper. In Section 2, we introduce the notions of FBM, our assumptions and the functional setting of our problem. In Section 3, we recall the notions of fractional integrals and some related a priori estimates that would be used later. In Section 4, we introduce the notion of integration wrt to a FBM. Section 5 contains our main results about the existence and uniqueness of a global solution for the equation (1.4) with their proofs.

2. Some preliminaries

2.1. Fractional Brownian motion.

**Definition 2.1.** Let \( B^H = \{B^H, \eta \geq 0\} \) be a stochastic process, and \( H \in (0, 1) \). \( B^H \) is called a Fractional Brownian Motion (FBM) with Hurst parameter \( H \), if it is a centered Gaussian process with the covariance function 
\[
R_H(\gamma, \eta) = \mathbb{E}[B^H(\gamma)B^H(\eta)] = \frac{1}{2}(\eta^{2H} + \gamma^{2H} - |\eta - \gamma|^{2H}).
\]
(2.1)

2.2. The integrodifferential equation. In this paper we study the following equation.
\[
\frac{\partial Y(t, \xi)}{\partial t} = \int_0^\xi A(Y(t, \eta))dB^H(t, \eta), \ \xi \in [0, 1] \text{ and } t \in [0, T]
\]
(2.2)

with
\[
Y(0, \xi) = \phi(\xi)
\]
or alternatively, we can consider the integral form
\[ Y(t, \xi) = \phi(\xi) + \int_0^t \int_0^\xi A(Y(s, \eta))dB^H(s, \eta)ds, \quad (2.3) \]

where \( B^H \) is a fractional Brownian motion defined on a complete probability space \((\Omega, \mathcal{F}, P)\), \( \phi(\xi) \) is the initial condition and \( A : \mathbb{R} \to \mathbb{R} \) is a measurable function that satisfies the assumptions given below.

2.3. Assumptions. Let us assume that:

A1. \( A \) is differentiable.

A2. There exists \( M_1 > 0 \) such that \( |A(x) - A(y)| \leq M_1 |x - y| \) for all \( x, y \in \mathbb{R} \).

A3. There exists \( M_2 > 0 \) such that \( |A(x)| \leq M_2 \) for all \( x \in \mathbb{R} \).

A4. For every \( N \) there exists \( M_N > 0 \), such that \( |A'(x) - A'(y)| \leq M_N |x - y| \) for all \( |x|, |y| \leq N \).

2.4. Functional Setting. Let \( \frac{1}{2} < H < 1 \), \( 1 - H < \alpha < \frac{1}{2} \). We will introduce the following functional spaces.

Let \( C([0, T], W^{\alpha, \infty}[0, 1]) \) be the space of measurable functions \( f : [0, T] \times [0, 1] \to \mathbb{R} \) such that

\[ \| f \|_{\alpha, \infty} := \sup_{t \in [0, T]} \sup_{\xi \in [0, 1]} \left( |f(t, \xi)| + \int_0^\xi \frac{|f(t, \xi) - f(t, \eta)|}{(\xi - \eta)^{\alpha + 1}} d\eta \right) < \infty. \quad (2.4) \]

Let \( C([0, T], W_0^{1-\alpha, \infty}[0, 1]) \) be the space of measurable functions \( f : [0, T] \times [0, 1] \to \mathbb{R} \) such that

\[ \| f \|_{1-\alpha, \infty, 0} := \sup_{t \in [0, T]} \sup_{0 < \eta < \xi < 1} \left( \frac{|f(t, \xi) - f(t, \eta)|}{(\xi - \eta)^{1-\alpha}} + \int_\eta^\xi \frac{|f(t, \gamma) - f(t, \eta)|}{(\gamma - \eta)^{2-\alpha}} d\gamma \right) < \infty. \quad (2.5) \]

Let \( W^{1, 1}([0, 1]) \) be the space of measurable functions \( f : [0, 1] \to \mathbb{R} \) such that

\[ \| f \|_{\alpha, 1} := \int_0^1 \frac{|f(\eta)|}{\eta^\alpha} d\eta + \int_0^1 \int_0^\eta \frac{|f(\eta) - f(\delta)|}{(\eta - \delta)^{\alpha + 1}} d\delta d\eta < \infty. \quad (2.6) \]

3. Some a priori estimates

Since FBM with Hurst parameters \( H > 1/2 \) have sample paths that are \( \lambda \)-Hölder continuous for all \( \lambda \in (0, H) \), the construction of the integral with respect to a FBM will be performed using a pathswise argument by means of fractional derivatives and integrals. We refer to [14] and [19] for more details.

3.1. Fractional integrals and derivatives. As usual we denote by \( L^p(a, b) \) the space of all Lebesgue measurable functions \( f : (a, b) \to \mathbb{R} \) such that

\[ \| f \|_{L^p(a, b)} := \left( \int_a^b |f(x)|^p dx \right)^{1/p} < \infty \quad (3.1) \]

for \( a < b \) and \( 1 \leq p < \infty \). Let us recall some definitions on Riemann-Liouville fractional integrals and Weyl derivative.
Definition 3.1. Let \( f \in L^1(a, b) \) and \( \alpha > 0 \). The left-sided and right-sided Riemann-Liouville fractional integrals of \( f \) of order \( \alpha \) are defined for almost all \( x \in (a, b) \) by

\[
I^\alpha_{a+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(y)}{(x-y)^{1-\alpha}} dy
\]

(3.2)

and

\[
I^\alpha_{b-} f(x) = \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{f(y)}{(y-x)^{1-\alpha}} dy
\]

(3.3)

respectively, where \((-1)^{-\alpha} = e^{-i\pi\alpha}\) and \(\Gamma(\alpha) = \int_0^\infty r^{(\alpha-1)} e^{-r} dr\) is the Gamma function or the Euler integral of the second kind.

Definition 3.2. Suppose \( I^\alpha_{a+}(L^p) \) is the image of \( L^p(a, b) \) under the operator \( I^\alpha_{a+} \) and \( I^\alpha_{b-}(L^p) \) is the image of \( L^p(a, b) \) under the operator \( I^\alpha_{b-} \). Let \( 0 < \alpha < 1 \), then we define the Weyl derivative for almost all \( x \in (a, b) \) as

\[
D^\alpha_{a+} f(x) = \frac{1}{\Gamma(1 - \alpha)} \left( \frac{f(x)}{(x-a)^\alpha} + \alpha \int_a^x \frac{f(y) - f(y)}{(y-x)^{\alpha+1}} dy \right) 1_{(a,b)}(x)
\]

(3.4)

when \( f \in I^\alpha_{a+}(L^p) \), and

\[
D^\alpha_{b-} f(x) = \frac{(-1)^\alpha}{\Gamma(1 - \alpha)} \left( \frac{f(x)}{(b-x)^\alpha} + \alpha \int_x^b \frac{f(y) - f(y)}{(y-x)^{\alpha+1}} dy \right) 1_{(a,b)}(x)
\]

(3.5)

when \( f \in I^\alpha_{b-}(L^p) \).

The convergence of the integrals at the singularity \( y = x \) holds pointwise for almost all \( x \in (a, b) \) when \( p = 1 \), and in \( L^p \) sense when \( 1 < p < \infty \).

We now introduce the following notations in order to define the generalized Stieltjes integrals. Assuming the limits exist and are finite, let

\[
f(a+) = \lim_{\epsilon \searrow 0} f(a+\epsilon),\]

\[
g(b-) = \lim_{\epsilon \searrow 0} g(b-\epsilon),\]

\[
f_{a+}(x) = [f(x) - f(a+)]1_{(a,b)}(x),\]

\[
g_{b-}(x) = [g(x) - g(b-)]1_{(a,b)}(x).\]

Definition 3.3. Suppose that \( f \) and \( g \) are functions such that \( f(a+) \), \( g(a+) \) and \( g(b-) \) exist, \( f_{a+} \in I^\alpha_{a+}(L^p) \) and \( g_{b-} \in I^{1-\alpha}_{b-}(L^q) \) for some \( p, q \geq 1 \), \( \frac{1}{p} + \frac{1}{q} \leq 1 \) and \( 0 < \alpha < 1 \). Then the generalized Stieltjes integral of \( f \) with respect to \( g \) is defined as follows, (using (3.4) and (3.5)),

\[
\int_a^b f dg = (-1)^\alpha \int_a^b D^\alpha_{a+} f_{a+}(x) D^{1-\alpha}_{b-} g_{b-}(x) dx + f(a+) [g(b-) - g(a+)].
\]

(3.6)

Remark that if \( \alpha p < 1 \), under the assumptions of the above definition, we have \( f \in I^\alpha_{a+}(L^p) \), and (3.6) can be written as

\[
\int_a^b f dg = (-1)^\alpha \int_a^b D^\alpha_{a+} f_{a+}(x) D^{1-\alpha}_{b-} g_{b-}(x) dx.
\]

(3.7)

For \( a \leq c < d \leq b \), the restriction of \( f \in I^\alpha_{a+}(L^p(a, b)) \) to \((c, d)\) belongs to \( I^\alpha_{c+}(L^p(c, d)) \) and the continuation of \( f \in I^\alpha_{b-}(L^p(c, d)) \) by zero beyond \((c, d)\) belongs to \( I^\alpha_{b+}(L^p(a, b)) \). Thus, if \( f \in I^\alpha_{a+}(L^p) \) and \( g_{b-} \in I^{1-\alpha}_{b-}(L^q) \), then the integral \( \int_a^b 1_{(c,d)} f dg \) in the sense of
(3.7) exists for any \( a \leq c < d \leq b \), and whenever the left-hand side is defined in the sense of (3.7), we have
\[
\int_c^d f \, dg = \int_a^b 1_{(c,d)} f \, dg.
\] (3.8)

3.2. A priori estimates. We have the following basic estimates. Let
\[
\Lambda_\alpha (g) := \frac{1}{\Gamma(1 - \alpha)} \sup_{0 < \eta < \xi < 1} |D^{1 - \alpha}_\xi g_\xi(\eta)|.
\] (3.9)
Then
\[
\Lambda_\alpha (g) \leq \frac{1}{\Gamma(1 - \alpha) \Gamma(\alpha)} \|g\|_{1 - \alpha, \infty, 0 < \infty}.
\] (3.10)

If \( f \in W^{\alpha, 1}(0, 1) \), and \( g \in W^{1 - \alpha, \infty}_0(0, 1) \) then the integral \( \int_0^\xi f \, dg \) exists for all \( \xi \in [0, 1] \).

Also, by (3.8) we have
\[
\int_0^\xi f \, dg = \int_0^1 f_{1(0,\xi)} \, dg.
\]
Using (3.7) we get
\[
\int_0^\xi f \, dg = (-1)^\alpha \int_0^\xi D^\alpha_\xi f(\eta) D^{1 - \alpha}_\xi g_\xi(\eta) \, d\eta.
\]
Then
\[
\left| \int_0^\xi f \, dg \right| \leq \sup_{0 < \eta < \xi} |D^{1 - \alpha}_\xi g_\xi(\eta)| \int_0^\xi |D^\alpha_\xi f(\eta)| \, d\eta.
\]
Hence
\[
\left| \int_0^\xi f \, dg \right| \leq \Lambda_\alpha (g) \|f\|_{\alpha, 1}.
\] (3.11)

4. Stochastic integrals with respect to a FBM

Let us recall the following result, for details and proofs see [19].

**Lemma 4.1.** Let \( \{B^H_\eta : \eta \geq 0\} \) be a real-valued FBM of Hurst parameter \( H \in \left( \frac{1}{2}, 1 \right) \). If \( 1 - H < \alpha < \frac{1}{2} \), then
\[
\mathbb{E} \sup_{0 \leq \gamma \leq \eta \leq 1} |D^{1 - \alpha}_{\eta -} B^H_{\eta -}(\gamma)|^p < \infty,
\] (4.1)
for all \( p \in [1, \infty) \).

Let \( \{B^H_\eta : \eta \in [0, 1]\} \) be a real-valued FBM, with the Hurst parameter \( \frac{1}{2} < H < 1 \), defined on a complete probability space \((\Omega, \mathcal{F}, P)\). By (2.1), we have
\[
\mathbb{E}(|B^H_\eta - B^H_\gamma|^2) = |\eta - \gamma|^{2H},
\]
and for any \( p \geq 1 \) we have
\[
\|B^H_\eta - B^H_\gamma\|_p = \mathbb{E}(|B^H_\eta - B^H_\gamma|^p)^{\frac{1}{p}} = c_p |\eta - \gamma|^H.
\] (4.2)

By Lemma (4.1) we know that the random variable
\[
G = \frac{1}{\Gamma(1 - \alpha)} \sup_{0 < \gamma < \eta < 1} |D^{1 - \alpha}_{\eta -} B^H_{\eta -}(\gamma)|
\] (4.3)
has moments of all orders.
As a consequence for $1 - H < \alpha < \frac{1}{2}$, the pathwise integral $\int_0^1 u_\gamma dB_\gamma^H$ exists when $u = \{u_\eta, \eta \in [0, 1]\}$ is a stochastic process whose trajectories belong to the space $W^{\alpha,1}$ and $B_\gamma^H$ is a FBM with $H > \frac{1}{2}$. Moreover, we have the estimate
\[
\left| \int_0^1 u_\gamma dB_\gamma^H \right| \leq G \|u\|_{\alpha,1}.
\] (4.4)

5. Main results with proofs

5.1. Main result. We state the main result of the present paper:

**Theorem 5.1.** Let $\alpha \in (1 - H, \frac{1}{2})$. Assume that $\phi \in W^{\alpha,\infty}[0, 1]$ and the function $A$ satisfies the assumptions A1, A2, A3, and A4. Then for every $T > 0$, there exists a unique stochastic process $Y \in L^0((\Omega, \mathcal{F}, P), C([0, T], W^{\alpha,\infty}[0, 1]))$ solution of the equation (2.2).

As we already defined in Section 4, the stochastic integral wrt to the FBM is well defined pathwise. The above theorem will be proved using a contraction principle that will give us the existence and uniqueness of local solutions. In order to get the global solution, we will need to have an a priori estimate of the solution in some functional spaces. The proof will follow after several steps and will be stated in Section 5.3.

5.2. Fixed point argument. Consider the operator
\[
F : C([0, T], W^{\alpha,\infty}[0, 1]) \rightarrow C([0, T], W^{\alpha,\infty}[0, 1])
\]
defined by
\[
F(Y(t, \xi)) := \phi(\xi) + \int_0^t \int_0^\xi A(Y(s, \gamma)) d\gamma ds,
\] (5.1)
where $g \in C([0, T], W^{-1,\alpha,\infty}[0, 1])$, $\phi \in W^{\alpha,\infty}[0, 1]$ and $A$ satisfies the assumptions A1, A2, A3, and A4.

**Remark 5.2.** Let us remark that the function $g$ in the equation (5.1) is a function of time $t$ and $\gamma$ and the fractional integration is with respect to the parameter $\gamma \in (0, 1)$.

For a given $R > 0$, let us define the ball $B_{R,T}$ in $C([0, T], W^{\alpha,\infty}[0, 1])$ as
\[
B_{R,T} = \{ Y \in C([0, T], W^{\alpha,\infty}[0, 1]) : \|Y\|_{\alpha,\infty} \leq R \}.
\]

**Lemma 5.3.** Given a positive constant $R_1 > \|\phi\|_{\alpha,\infty}$, there exists $T_1 > 0$ such that $F(B_{R_1,T_1}) \subseteq B_{R_1,T_1}$. The time $T_1$ depends on $R_1$, $\alpha$ and the initial condition $\|\phi\|_{\alpha,\infty}$.

**Proof.**
\[
\|F(Y)\|_{\alpha,\infty} = \sup_{t \in [0, T], \xi \in [0, 1]} \left( |F(Y(t, \xi))| + \int_0^\xi \frac{|F(Y(t, \xi)) - F(Y(t, \eta))|}{(\xi - \eta)^{\alpha+1}} d\eta \right).
\] (5.2)

Consider the first term inside the parenthesis in (5.2). We note that
\[
|F(Y(t, \xi))| \leq |\phi(\xi)| + \int_0^t \left| \int_0^\xi A(Y(s, \gamma)) d\gamma \right| ds
\]
\[
\leq |\phi(\xi)| + \int_0^t (\Lambda_\alpha(g) A(Y(s, \cdot)))_{\alpha,1} ds,
\]
where $\Lambda_\alpha(g) = \frac{1}{\Gamma(1-\alpha)} \sup_{0 < \eta \leq \xi} |D_{\xi-}^{1-\alpha} g(\eta, \gamma)|$. 

Further,

\[ ||A(Y(s, \cdot))||_{\alpha,1} \]
\[ = \int_0^1 \frac{|A(Y(s, \gamma))|}{\gamma^{\alpha}} d\gamma + \int_0^1 \int_0^\gamma \frac{|A(Y(s, \gamma)) - A(Y(s, \delta))|}{(\gamma - \delta)^{\alpha+1}} d\delta d\gamma \]
\[ \leq M_2 \int_0^1 \gamma^{-\alpha} d\gamma + M_1 \int_0^1 \int_0^\gamma \frac{|Y(s, \gamma) - Y(s, \delta)|}{(\gamma - \delta)^{\alpha+1}} d\delta d\gamma \]
\[ \leq \frac{M_2}{1 - \alpha} + M_1 \int_0^1 \left( \sup_{\gamma \in [0,1]} \int_0^\gamma \frac{|Y(s, \gamma) - Y(s, \delta)|}{(\gamma - \delta)^{\alpha+1}} d\delta \right) d\gamma \]
\[ \leq \frac{M_2}{1 - \alpha} + M_1 ||Y||_{\alpha,\infty}. \]

Thus, we see that

\[ |F(Y(t, \xi))| \leq |\phi(\xi)| + T \sup_{0 \leq t \leq T} \Lambda_\alpha(g) \left( \frac{M_2}{1 - \alpha} + M_1 ||Y||_{\alpha,\infty} \right). \]

(5.3)

Now, we consider the second term in (5.2).

\[ \int_0^\xi \frac{|F(Y(t, \xi)) - F(Y(t, \eta))|}{(\xi - \eta)^{\alpha+1}} d\eta \]
\[ = \int_0^\xi \frac{1}{(\xi - \eta)^{\alpha+1}} \left| \phi(\xi) - \phi(\eta) + \int_0^t \int_0^\xi A(Y(s, \gamma)) d\gamma d\eta + \int_0^\eta \int_0^\xi A(Y(s, \gamma)) d\gamma d\eta \right| d\eta \]
\[ \leq \int_0^\xi \frac{|\phi(\xi) - \phi(\eta)|}{(\xi - \eta)^{\alpha+1}} + \int_0^\xi \frac{1}{(\xi - \eta)^{\alpha+1}} \left| \int_0^t \int_0^\xi A(Y(s, \gamma)) d\gamma d\eta \right| d\eta \]
\[ = \int_0^\xi \frac{|\phi(\xi) - \phi(\eta)|}{(\xi - \eta)^{\alpha+1}} + \int_0^t \left( \int_0^\xi \frac{1}{(\xi - \eta)^{\alpha+1}} \left| \int_0^\xi A(Y(s, \gamma)) d\gamma \right| d\eta \right) d\eta. \]

(5.4)

Again we consider the second term in (5.3) and get

\[ \int_0^\xi \frac{1}{(\xi - \eta)^{\alpha+1}} \left| \int_0^\xi A(Y(s, \gamma)) d\gamma d\eta \right| d\eta \]
\[ \leq \Lambda_\alpha(g) \int_0^\xi \frac{1}{(\xi - \eta)^{\alpha+1}} \left( \int_\eta^\xi \frac{|A(Y(s, \gamma))|}{(\gamma - \eta)^{\alpha}} d\gamma \right) d\eta \]
\[ + \int_\eta^\xi \frac{|A(Y(s, \gamma)) - A(Y(s, \delta))|}{(\gamma - \delta)^{\alpha+1}} d\delta d\gamma d\eta \]
\[ \leq \Lambda_\alpha(g) \int_0^\xi \frac{1}{(\xi - \eta)^{\alpha+1}} \int_\eta^\xi \frac{|A(Y(s, \gamma))|}{(\xi - \eta)^{\alpha}} d\gamma d\eta \]
\[ + \Lambda_\alpha(g) M_1 \int_0^\xi \frac{1}{(\xi - \eta)^{\alpha+1}} \int_\eta^\xi \frac{|Y(s, \gamma) - Y(s, \delta)|}{(\gamma - \delta)^{\alpha+1}} d\delta d\gamma d\eta. \]

(5.5)

First, we use the Fubini theorem on the first term in (5.5) and then the substitution \( \eta = \gamma - (\xi - \gamma)x \). Thus,

\[ \int_0^\xi \frac{1}{(\xi - \eta)^{\alpha+1}} \int_\eta^\xi \frac{|A(Y(s, \gamma))|}{(\gamma - \eta)^{\alpha}} d\gamma d\eta \]
\[
\int_0^\xi \int_0^\xi (\xi - \eta)^{-\alpha - 1}(\gamma - \eta)^{-\alpha} |A(Y(s, \gamma))| d\gamma d\eta \\
= \int_0^\xi \int_0^\gamma (\xi - \eta)^{-\alpha - 1}(\gamma - \eta)^{-\alpha} |A(Y(s, \gamma))| d\eta d\gamma \\
= \int_0^\xi \int_0^\gamma |A(Y(s, \gamma))| \left( \int_0^\gamma (\xi - \eta)^{-\alpha - 1}(\gamma - \eta)^{-\alpha} d\eta \right) d\gamma \\
= \int_0^\xi |A(Y(s, \gamma))| \left( (\xi - \gamma)^{-2\alpha} \int_0^{\gamma - \alpha} (1 + x)^{-\alpha - 1} x^{-\alpha} dx \right) d\gamma \\
\leq \int_0^\xi |A(Y(s, \gamma))| \left( (\xi - \gamma)^{-2\alpha} \int_0^{\infty} (1 + x)^{-\alpha - 1} x^{-\alpha} dx \right) d\gamma \\
\leq \frac{M_2 b_1^{(1)}}{1 - 2\alpha} \xi^{1 - 2\alpha}, \tag{5.6}
\]

where \(b_1^{(1)} = B(2\alpha, 1 - \alpha)\) and \(B(p, q)\) is the beta function given by

\[
B(p, q) = \int_0^1 \frac{x^{q-1}}{(1 + x)^{p+q}} dx.
\]

Now consider the second part in (5.5). We see that

\[
\int_0^\xi \frac{1}{(\xi - \eta)^{\alpha + 1}} \int_0^\gamma \frac{|Y(s, \gamma) - Y(s, \delta)|}{(\gamma - \delta)^{\alpha + 1}} d\delta d\gamma d\eta \\
= \int_0^\xi \int_0^\gamma \int_0^\gamma \frac{|Y(s, \gamma) - Y(s, \delta)|}{(\xi - \eta)^{\alpha + 1}(\gamma - \delta)^{\alpha + 1}} d\delta d\gamma d\eta \\
= \int_0^\xi \int_0^\gamma \int_0^\gamma \frac{|Y(s, \gamma) - Y(s, \delta)|}{(\xi - \eta)^{\alpha + 1}(\gamma - \delta)^{\alpha + 1}} d\delta d\eta d\gamma \\
= \int_0^\xi \int_0^\gamma |Y(s, \gamma) - Y(s, \delta)| \left( \int_0^\gamma \frac{1}{(\xi - \eta)^{-\alpha - 1} d\eta} \right) d\delta d\gamma \\
\leq \alpha^{-1} \int_0^\xi \int_0^\gamma |Y(s, \gamma) - Y(s, \delta)| (\gamma - \delta)^{\alpha + 1} (\gamma - \eta)^{-\alpha} d\delta d\gamma \\
\leq \alpha^{-1} \int_0^\xi (\xi - \gamma)^{-\alpha} \int_0^\gamma |Y(s, \gamma) - Y(s, \delta)| (\gamma - \delta)^{\alpha + 1} d\delta d\gamma \\
\leq \alpha^{-1} \int_0^\xi (\xi - \gamma)^{-\alpha} \sup_{\gamma \in [0, 1], \delta \in [0, T]} \int_0^\gamma \frac{|Y(s, \gamma) - Y(s, \delta)|}{(\gamma - \delta)^{\alpha + 1}} d\delta d\gamma \\
\leq \frac{||Y||_{0, \infty}}{\alpha(1 - \alpha)} \xi^{1 - \alpha}. \tag{5.7}
\]

Now using the estimates \((5.3), (5.4), (5.5), (5.6)\) and \((5.7)\) in \((5.2)\), we get

\[
||F(Y)||_{0, \infty} \\
\leq \sup_{t \in [0, T], \xi \in [0, 1]} \left[ |\phi(\xi)| + \Lambda_\alpha(g) t \left( \frac{M_2}{1 - \alpha} + M_1 ||Y||_{0, \infty} \right) \right]
\]
+ \int_0^\xi \frac{\phi(\xi) - \phi(\eta)}{(\xi - \eta)^{\alpha+1}} d\eta + \int_0^t \left( \frac{\Lambda_\alpha(g)M_2b_\alpha^{(1)}}{1 - 2\alpha}\xi^{1-2\alpha} + \frac{\Lambda_\alpha(g)M_1||Y||_{\alpha,\infty}\xi^{1-\alpha}}{\alpha(1 - \alpha)} \right) ds \right]
\leq \sup_{t \in [0,T], \xi \in [0,1]} \sup_{\xi \in [0,1]} \left( \frac{\phi(\xi)}{\xi^{\alpha+1}} + \int_0^\xi \frac{\phi(\xi) - \phi(\eta)}{(\xi - \eta)^{\alpha+1}} d\eta \right)
+ \Lambda_\alpha(g)t \left( \frac{M_2}{1 - \alpha} + M_1||Y||_{\alpha,\infty} \right) + \frac{\Lambda_\alpha(g)M_2b_\alpha^{(1)}}{1 - 2\alpha}\xi^{1-2\alpha} + \frac{\Lambda_\alpha(g)M_1||Y||_{\alpha,\infty}}{\alpha(1 - \alpha)}t\xi^{1-\alpha}
\leq ||\phi||_{\alpha,\infty} + T\sup_{0 \leq t \leq T} \Lambda_\alpha(g) \left[ M_2 \left( \frac{1}{1 - \alpha} + \frac{b_\alpha^{(1)}}{1 - 2\alpha} \right) + M_1 \left( 1 + \frac{1}{\alpha(1 - \alpha)} \right) ||Y||_{\alpha,\infty} \right]
\leq ||\phi||_{\alpha,\infty} + Tb_\alpha^{(2)} \left[ 1 + ||Y||_{\alpha,\infty} \right],

where $b_\alpha^{(2)} = \left[ M_2 \left( \frac{1}{1 - \alpha} + \frac{b_\alpha^{(1)}}{1 - 2\alpha} \right) + M_1 \left( 1 + \frac{1}{\alpha(1 - \alpha)} \right) \right] \sup_{0 \leq t \leq T} \Lambda_\alpha(g)$.

It is enough to choose $T_1 = \frac{R_1 - ||\phi||_{\alpha,\infty}}{b_\alpha^{(2)}(1 + R_1)}$ with $R_1 > ||\phi||_{\alpha,\infty}$ to get that $F(B_{R_1}) \subseteq B_{R_1}$.

This completes the proof. \qed

Now in order to prove that the operator $F$ is a contraction, we will need the following two propositions.

**Proposition 5.4.** Let $f \in C([0,T],W^\alpha_{0,\infty}[0,1])$ and $g \in C([0,T],W^{1-\alpha,\infty}_{0,1}[0,1])$. Then, for all $\xi \in [0,1]$ and $t \in [0,T]$,

\[
\left| \int_0^t \int_0^\xi f(s,\gamma)dg_\gamma ds \right| + \int_0^\xi (\xi - \eta)^{-\alpha-1} \left| \int_0^t \int_0^\xi f(s,\gamma)dg_\gamma ds - \int_0^t \int_0^{\eta} f(s,\gamma)dg_\gamma ds \right| d\eta
\leq \sup_{0 \leq t \leq T} \Lambda_\alpha(g)b_\alpha^{(3)} \int_0^t \int_0^\xi [(\xi - \gamma)^{-2\alpha} + \gamma^{-\alpha}] \left| f(s,\gamma) \right|
+ \int_0^\gamma \frac{|f(s,\gamma) - f(s,\delta)|}{(\gamma - \delta)^{\alpha+1}} d\delta \right) d\gamma ds. \tag{5.8}
\]

where $b_\alpha^{(3)}$ is a constant which depends on $\alpha$, its explicit expression is given below.

**Proof.** Let $0 \leq \eta < \xi \leq 1$. Then,

\[
\left| \int_0^t \int_0^\xi f(s,\gamma)dg_\gamma ds - \int_0^t \int_0^{\eta} f(s,\gamma)dg_\gamma ds \right|
= \left| \int_0^t \int_0^\xi f(s,\gamma)dg_\gamma ds \right|
= \int_0^t \left| \int_0^\xi f(s,\gamma)dg_\gamma \right| ds.
\]
\[ = \int_0^t |D_\eta^\alpha f(s, \gamma) D_\xi^{1-\alpha} g_\xi(s, \gamma) d\gamma| \]
\[ \leq \sup_{0 \leq \eta \leq T} \Lambda_\alpha(g) \int_0^t \left( \int_\eta^\xi \frac{|f(s, \gamma)|}{(\gamma - \eta)^\alpha} d\gamma + \alpha \int_\eta^\xi \int_\eta^{\gamma} \frac{|f(s, \gamma) - f(s, \delta)|}{(\gamma - \delta)^{\alpha+1}} d\delta d\gamma \right) ds. \tag{5.9} \]

We multiply (5.9) by \((\xi - \eta)^{-\alpha}\), integrate from 0 to \(\xi\), use Fubini’s theorem and we get
\[ \int_0^\xi \frac{1}{(\xi - \eta)^{\alpha+1}} \left| \int_0^t \int_\eta^\xi f(s, \gamma) dg ds - \int_0^t \int_\eta^\xi f(s, \gamma) dg ds \right| d\eta \]
\[ \leq \sup_{0 \leq \eta \leq T} \Lambda_\alpha(g) \left( \int_0^t \int_\eta^\xi \frac{1}{(\xi - \eta)^{\alpha+1}} \left( \int_\eta^\xi \frac{|f(s, \gamma)|}{(\gamma - \eta)^\alpha} d\gamma \right) + \alpha \int_\eta^\xi \int_\eta^{\gamma} \frac{|f(s, \gamma) - f(s, \delta)|}{(\gamma - \delta)^{\alpha+1}} d\delta d\gamma \right) d\eta ds \]
\[ \leq \sup_{0 \leq \eta \leq T} \Lambda_\alpha(g) \left( \int_0^t \int_\eta^\xi \int_\eta^\gamma \frac{|f(s, \gamma)|}{(\xi - \eta)^{\alpha+1}(\gamma - \eta)^\alpha} d\gamma d\eta ds \right) + \alpha \int_0^t \int_\eta^\xi \int_\eta^\gamma \int_\eta^{\gamma} \frac{|f(s, \gamma) - f(s, \delta)|}{(\xi - \eta)^{\alpha+1}(\gamma - \delta)^{\alpha+1}} d\delta d\gamma d\eta ds. \tag{5.10} \]

Consider the first term on the right side of (5.10):
\[ \int_0^\xi \int_\eta^\gamma (\xi - \eta)^{-\alpha-1}(\gamma - \eta)^{-\alpha} |f(s, \gamma)| d\gamma d\eta \]
\[ = \int_0^\xi \int_\eta^\gamma (\xi - \eta)^{-\alpha-1}(\gamma - \eta)^{-\alpha} |f(s, \gamma)| d\eta d\gamma \]
\[ = \int_0^\xi |f(s, \gamma)| \left( \int_\eta^\gamma (\xi - \eta)^{-\alpha-1}(\gamma - \eta)^{-\alpha} d\eta \right) d\gamma \]
using again the same substitution used in (5.5), we get
\[ \leq b_\alpha^{(1)} \int_0^\xi |f(s, \gamma)|(\xi - \gamma)^{-2\alpha} d\gamma. \tag{5.11} \]

Now, consider the second term on the right side of (5.10):
\[ \int_0^\xi \int_\eta^\gamma \int_\eta^{\gamma} \frac{|f(s, \gamma) - f(s, \delta)|}{(\xi - \eta)^{\alpha+1}(\gamma - \delta)^{\alpha+1}} d\delta d\gamma d\eta \]
\[ = \int_0^\xi \int_\eta^\gamma \int_\eta^{\gamma} \frac{|f(s, \gamma) - f(s, \delta)|}{(\xi - \eta)^{\alpha+1}(\gamma - \delta)^{\alpha+1}} d\delta d\eta d\gamma \]
\[ = \int_0^\xi \int_\eta^\gamma \int_\eta^{\delta} \frac{|f(s, \gamma) - f(s, \delta)|}{(\xi - \eta)^{\alpha+1}(\gamma - \delta)^{\alpha+1}} d\eta d\delta d\gamma \]
\[ = \int_0^\xi \int_\eta^\gamma \int_\eta^{\delta} \frac{|f(s, \gamma) - f(s, \delta)|}{(\xi - \eta)^{\alpha+1}(\gamma - \delta)^{\alpha+1}} \left( \int_\eta^{\delta} (\xi - \eta)^{-\alpha-1} d\eta \right) d\delta d\gamma \]
\[ \leq \alpha^{-1} \int_0^\xi \int_\eta^\gamma \int_\eta^{\delta} \frac{|f(s, \gamma) - f(s, \delta)|}{(\gamma - \delta)^{\alpha+1}(\xi - \gamma)^{-\alpha}} d\delta d\gamma \]
\[ \leq \alpha^{-1} \int_0^\xi (\xi - \gamma)^{-\alpha} \int_\eta^{\gamma} \frac{|f(s, \gamma) - f(s, \delta)|}{(\gamma - \delta)^{\alpha+1}} d\delta d\gamma. \tag{5.12} \]
Using (5.10) (5.11) and (5.12), we obtain
\[
\int_0^\xi \frac{1}{(\xi - \eta)^{\alpha+1}} \left| \int_0^t \int_0^\xi f(s, \gamma)dg_\gamma ds - \int_0^t \int_0^\eta f(s, \gamma)dg_\gamma ds \right| d\eta \\
\leq \sup_{0 \leq t \leq T} \Lambda_\alpha(g) \int_0^t \left( b^{(1)}_\alpha \int_0^\xi \left| f(s, \gamma) \right| (\xi - \eta)^{-2\alpha} d\eta \right) + \int_0^\xi (\xi - \gamma)^{-\alpha} \int_0^\gamma \left| \frac{f(s, \gamma) - f(s, \delta)}{(\gamma - \delta)^{\alpha+1}} \right| d\delta d\gamma ds.
\]
By (5.9) we have
\[
\int_0^t \int_0^\xi f(s, \gamma)dg_\gamma ds \leq \sup_{0 \leq t \leq T} \Lambda_\alpha(g) \int_0^t \left( \int_0^\xi \frac{|f(s, \gamma)|}{\gamma^\alpha} d\gamma + \int_0^\xi \frac{|f(s, \gamma) - f(s, \delta)|}{(\gamma - \delta)^{\alpha+1}} d\delta d\gamma \right) ds.
\]
From (5.13) and (5.14), we have
\[
\left| \int_0^t \int_0^\xi f(s, \gamma)dg_\gamma ds \right| + \sup_{0 \leq t \leq T} \Lambda_\alpha(g) \int_0^t \left( b^{(1)}_\alpha \int_0^\xi |f(s, \gamma)| (\xi - \eta)^{-2\alpha} + \int_0^\xi |f(s, \gamma)| (\xi - \eta)^{-\alpha} \right) d\gamma ds \\
\leq \sup_{0 \leq t \leq T} \Lambda_\alpha(g) \int_0^t \left( b^{(1)}_\alpha (\xi - \eta)^{-2\alpha} + \int_0^\xi |f(s, \gamma) - f(s, \delta)| (\gamma - \delta)^{-\alpha} d\delta \right) d\gamma ds
\]
using the fact that $\alpha < \gamma^{-\alpha}$ and $\frac{1}{(\xi - \eta)^{\alpha+1}} < \frac{1}{(\xi - \gamma)^{2\alpha}}$ we get
\[
\leq \sup_{0 \leq t \leq T} \Lambda_\alpha(g) \int_0^t \int_0^\xi \left[ (\xi - \eta)^{-2\alpha} + \gamma^{-\alpha} \right] |f(s, \gamma)| + \int_0^\gamma \left| \frac{f(s, \gamma) - f(s, \delta)}{(\gamma - \delta)^{\alpha+1}} \right| d\delta d\gamma ds
\]
and
\[
\leq \sup_{0 \leq t \leq T} \Lambda_\alpha(g) b^{(3)}_\alpha \int_0^t \int_0^\xi \left[ (\xi - \eta)^{-2\alpha} + \gamma^{-\alpha} \right] |f(s, \gamma)| + \int_0^\gamma \left| \frac{f(s, \gamma) - f(s, \delta)}{(\gamma - \delta)^{\alpha+1}} \right| d\delta d\gamma ds,
\]
where $b^{(3)}_\alpha = \max\{1, b^{(1)}_\alpha\}$. This completes the proof. 

**Proposition 5.5.** Let $h : \mathbb{R} \to \mathbb{R}$ be a function satisfying the assumptions A3 and A4. Then for all $N > 0$ and $|X_1|, |X_2|, |X_3|, |X_4| \leq N$ for all $X_1, X_2, X_3, X_4 \in \mathbb{R}$,
\[
|h(X_1) - h(X_2) - h(X_3) + h(X_4)| \\
\leq M_1|X_1 - X_2 - X_3 + X_4| + M_N |X_1 - X_3||X_1 - X_2| + |X_3 - X_4|.
\]
Using the expression (5.2) and Proposition 5.4, we have that

\[ \frac{h(X_1) - h(X_3)}{X_1 - X_3} = h'(\theta X_1 + (1 - \theta)X_3) \quad (0 < \theta < 1) \]

\[ = h'(\beta) \quad \text{where} \quad \beta = \theta X_1 + (1 - \theta)X_3, \quad X_1 < \beta < X_3 \]

\[ = \frac{1}{X_3 - X_1} \int_{X_1}^{X_3} h'(\gamma)d\gamma. \]

Using the substitution \( \gamma = \theta X_1 + (1 - \theta)X_3 \), we get

\[ h(X_1) - h(X_3) = (X_1 - X_3) \int_0^1 h'(\theta X_1 + (1 - \theta)X_3)d\theta. \quad (5.17) \]

Similarly we get

\[ h(X_2) - h(X_4) = (X_2 - X_4) \int_0^1 h'(\theta X_2 + (1 - \theta)X_4)d\theta. \quad (5.18) \]

By (5.17) and (5.18) we obtain

\[ |h(X_1) - h(X_2) - h(X_3) + h(X_4)| \]

\[ = |(X_1 - X_3) \int_0^1 h'(\theta X_1 + (1 - \theta)X_3)d\theta - (X_2 - X_4) \int_0^1 h'(\theta X_2 + (1 - \theta)X_4)d\theta| \]

\[ = \int_0^1 (X_1 - X_2 - X_3 + X_4)h'(\theta X_2 + (1 - \theta)X_4)d\theta \]

\[ + \int_0^1 (X_1 - X_3)[h'(\theta X_1 + (1 - \theta)X_3) - h'(\theta X_2 + (1 - \theta)X_4)]d\theta \]

\[ \leq M_1|X_1 - X_2 - X_3 + X_4| \]

\[ + |X_1 - X_3| \int_0^1 [|h'(\theta X_1 + (1 - \theta)X_3) - h'(\theta X_2 + (1 - \theta)X_3)]d\theta \]

\[ + |X_1 - X_3| \int_0^1 [|h'(\theta X_1 + (1 - \theta)X_3) - h'(\theta X_2 + (1 - \theta)X_4)]d\theta \]

\[ \leq M_1|X_1 - X_2 - X_3 + X_4| + M_N|X_1 - X_3| \int_0^1 \theta d\theta \]

\[ + |X_3 - X_4| \int_0^1 \theta d\theta \]

\[ \leq M_1|X_1 - X_2 - X_3 + X_4| + M_N|X_1 - X_3|(|X_1 - X_2| + |X_3 - X_4|). \]

This completes the proof. \( \square \)

**Lemma 5.6.** Given a positive constant \( R_2 > \| \phi \|_{\alpha, \infty} \), there exists \( T_2 > 0 \) and a constant \( 0 < C < 1 \) such that

\[ \| F(Y_1) - F(Y_2) \|_{\alpha, \infty} \leq C \| Y_1 - Y_2 \|_{\alpha, \infty} \]

for all \( Y_1, Y_2 \in B_{R_2, T_2} \).

**Proof.** Recall \( F(Y(t, \xi)) = \phi(\xi) + \int_t^\xi \int_0^s A(Y(s, \gamma))dg_\gamma ds \). Then,

\[ F(Y_1(t, \xi)) - F(Y_2(t, \xi)) = \int_t^\xi \int_0^s [A(Y_1(s, \gamma)) - A(Y_2(s, \gamma))]dg_\gamma ds. \]

Using the expression (5.2) and Proposition 5.1 we have that
\[ \|F(Y_1) - F(Y_2)\|_{\alpha,\infty} \]

\[
\leq \sup_{0 \leq t \leq T} \Lambda_\alpha(g) b_\alpha^{(1)} \sup_{t \in [0,T], \xi \in [0,1]} \int_0^t \int_0^\xi \left( \frac{[\xi - \gamma]^{-2\alpha} + \gamma^{-\alpha}}{(\gamma - \eta)^{\alpha+1}} \right) A(Y_1(s,\gamma)) - A(Y_2(s,\gamma)) \, d\gamma d\eta 
\]

\[
+ \int_0^\gamma \frac{|A(Y_1(s,\gamma)) - A(Y_2(s,\gamma)) - \{A(Y_1(s,\eta)) - A(Y_2(s,\eta))\}|}{(\gamma - \eta)^{\alpha+1}} \, d\gamma ds 
\]

By Proposition 5.5 we get

\[
\|F(Y_1) - F(Y_2)\|_{\alpha,\infty} \leq \sup_{0 \leq t \leq T} \Lambda_\alpha(g) b_\alpha^{(1)} \sup_{t \in [0,T], \xi \in [0,1]} \int_0^t \int_0^\xi \left( \frac{[\xi - \gamma]^{-2\alpha} + \gamma^{-\alpha}}{(\gamma - \eta)^{\alpha+1}} \right) M_1|Y_1(s,\gamma) - Y_2(s,\gamma)| 
\]

\[
+ M_1 \int_0^\gamma \frac{|Y_1(s,\gamma) - Y_2(s,\gamma) - Y_1(s,\eta) + Y_2(s,\eta)|}{(\gamma - \eta)^{\alpha+1}} \, d\eta 
\]

\[
+ M_N |Y_1(s,\gamma) - Y_2(s,\gamma)| \int_0^\gamma \frac{|Y_1(s,\gamma) - Y_2(s,\gamma)|}{(\gamma - \eta)^{\alpha+1}} \, d\eta \] 

Using the notation

\[
\Delta(Y_j) = \sup_{s \in [0,T], \gamma \in [0,1]} \int_0^\gamma \frac{|Y_j(s,\gamma) - Y_j(s,\eta)|}{(\gamma - \eta)^{\alpha+1}} \, d\eta, \quad j \in \{1, 2\},
\]

we get

\[
\|F(Y_1) - F(Y_2)\|_{\alpha,\infty} \leq \sup_{0 \leq t \leq T} \Lambda_\alpha(g) b_\alpha^{(1)} \sup_{t \in [0,T], \xi \in [0,1]} \int_0^t \int_0^\xi \left( \frac{[\xi - \gamma]^{-2\alpha} + \gamma^{-\alpha}}{(\gamma - \eta)^{\alpha+1}} \right) M_1|Y_1(s,\gamma) - Y_2(s,\gamma)| 
\]

\[
+ M_1 \int_0^\gamma \frac{|Y_1(s,\gamma) - Y_2(s,\gamma) - Y_1(s,\eta) + Y_2(s,\eta)|}{(\gamma - \eta)^{\alpha+1}} \, d\eta 
\]

\[
+ M_N |Y_1(s,\gamma) - Y_2(s,\gamma)|(\Delta(Y_1) + \Delta(Y_2)) \] 

\[
\leq \sup_{0 \leq t \leq T} \Lambda_\alpha(g) b_\alpha^{(1)} b^{(4)} \sup_{t \in [0,T], \xi \in [0,1]} \int_0^t \int_0^\xi \left( \frac{[\xi - \gamma]^{-2\alpha} + \gamma^{-\alpha}}{(\gamma - \eta)^{\alpha+1}} \right) |Y_1(s,\gamma) - Y_2(s,\gamma)| 
\]

\[
+ \int_0^\gamma \frac{|Y_1(s,\gamma) - Y_2(s,\gamma) - Y_1(s,\eta) + Y_2(s,\eta)|}{(\gamma - \eta)^{\alpha+1}} \, d\eta \] 

where \( b^{(4)} = (M_1 + M_{R_1})(1 + 2R_1) \). Hence,

\[
\|F(Y_1) - F(Y_2)\|_{\alpha,\infty}
\]
Lemma 5.6 the operator

Choose

Choose

T

for all

solution

Theorem 5.8. □

proof.

φ

T

t

where

Now, we are able to state the following theorem:

Let

Consider the first term on the right side of the above equality,

\[
\sup_{0 \leq t \leq T} \Lambda_{\alpha}(g) b^{(1)}_{\alpha} b^{(4)} \sup_{t \in [0,T], \xi \in [0,1]} \int_{0}^{t} \int_{0}^{\xi} \left[ (\xi - \gamma)^{-2\alpha} + \gamma^{-\alpha} \right]
\]

\times \sup_{t \in [0,T], \xi \in [0,1]} \left( |Y_1(s, \gamma) - Y_2(s, \gamma)| + \int_{0}^{\gamma} |Y_1(s, \gamma) - Y_2(s, \gamma) - \{Y_1(s, \eta) - Y_2(s, \eta)\}| d\eta \right) d\gamma ds
\]

\leq \sup_{0 \leq t \leq T} \Lambda_{\alpha}(g) b^{(1)}_{\alpha} b^{(4)} \sup_{t \in [0,T], \xi \in [0,1]} \int_{0}^{t} \int_{0}^{\xi} \left[ (\xi - \gamma)^{-2\alpha} + \gamma^{-\alpha} \right] |Y_1(s, \gamma) - Y_2(s, \gamma)|_{a,\infty} d\gamma ds
\]

\leq \sup_{0 \leq t \leq T} \Lambda_{\alpha}(g) b^{(1)}_{\alpha} b^{(4)} \frac{2 - 3\alpha}{(1 - 2\alpha)(1 - \alpha)} T |Y_1 - Y_2|_{a,\infty}
\]

Choose \(T_2\) such that \(b^{(5)}T_2 = C < 1\). This completes the proof. □

Now, we are able to state the following theorem:

Theorem 5.7. Let \(0 < \alpha < \frac{1}{2}\), \(g \in C([0, T], W^{1-a,\infty}[0, 1])\). Consider the integrodifferential equation

\[
Y(t, \xi) = \phi(\xi) + \int_{0}^{t} \int_{0}^{\xi} A(Y(s, \eta)) dg(s, \eta) d\xi ds
\]

(5.19)

where \(t \in [0, T], \xi \in [0, 1]\). Assume that \(A\) satisfies assumptions \(A1, A2, A3,\) and \(A4\) and that \(\phi \in W^{a,\infty}([0, 1])\). Then, there exists \(T_0 > 0\) such that the above equation has a unique solution

\[
Y \in C([0, T], W^{a,\infty}[0, 1]),
\]

for all \(T \leq T_0\).

Proof. Choose \(T_0 = \min\{T_1, T_2\}\), and \(R = \min\{R_1, R_2\}\). Then, using Lemma 5.3 and Lemma 5.6 the operator \(F\) is a contraction on \(B_{R,T}\) for all \(T < T_0\) and this completes the proof. □

Now, we can show that the solution of (5.19) is global in time.

Theorem 5.8. Let \(1 - H < \alpha < \frac{1}{2}\), \(g \in C([0, T], W^{1-a,\infty,0}[0, 1])\). Assume that \(A\) satisfies assumptions \(A1, A2, A3,\) and \(A4\) and that \(\phi \in W^{a,\infty}([0, 1])\). Then for all \(T > 0\), there exists a unique \(Y \in C([0, T], W^{a,\infty}[0, 1])\) solution of (5.19).

Proof. It is enough to get an estimate in \(C([0, T], W^{a,\infty}[0, 1])\). We can write

\[
|Y(t)|_{a,\infty} = \sup_{\xi \in [0, 1]} \left( Y(t, \xi) + \int_{0}^{\xi} \frac{|Y(t, \xi) - Y(t, \eta)|}{(\xi - \eta)^{\alpha+1}} d\eta \right).
\]

(5.20)

Consider the first term on the right side of the above equality,

\[
|Y(t, \xi)| \leq |\phi(\xi)| + \int_{0}^{t} \left| \int_{0}^{\xi} A(Y(s, \eta)) dg(s, \eta) \right| d\xi.
\]
From (5.3) we obtain
\[ |Y(t, \xi)| \leq |\phi(\xi)| + \int_0^t \Lambda_\alpha(g)t \left( \frac{M_2}{1-\alpha} + M_1 ||Y(s)||_{\alpha, \infty} ds \right). \]  
(5.21)

Consider the second term. We have
\[ \left| \int_0^\xi \frac{Y(t, \xi) - Y(t, \eta)}{(\xi - \eta)^{\alpha+1}} d\eta \right| \]
\[ = \int_0^\xi \frac{1}{(\xi - \eta)^{\alpha+1}} \left| \left( \phi(\xi) - \phi(\eta) \right) + \int_0^\xi A(Y(s, \gamma)) d\gamma ds \right| \]
\[ \leq \int_0^\xi \frac{1}{(\xi - \eta)^{\alpha+1}} \left( |\phi(\xi) - \phi(\eta)| + \int_\eta^\xi A(Y(s, \gamma)) d\gamma ds \right) \]
\[ \leq \int_0^\xi \frac{|\phi(\xi) - \phi(\eta)|}{(\xi - \eta)^{\alpha+1}} d\eta + \int_0^\xi \int_\eta^\xi \frac{1}{(\xi - \eta)^{\alpha+1}} \left| \int_\eta^\xi A(Y(s, \gamma)) d\gamma \right| d\eta ds. \]

From (5.6) and (5.7) we obtain
\[ \left| \int_0^\xi \frac{Y(t, \xi) - Y(t, \eta)}{(\xi - \eta)^{\alpha+1}} d\eta \right| \]
\[ \leq \int_0^\xi \frac{\phi(\xi) - \phi(\eta)}{(\xi - \eta)^{\alpha+1}} d\eta + \Lambda_\alpha(g) \int_0^\xi \left( \frac{M_2 b^{(1)}_\alpha}{1 - 2\alpha} + \frac{M_1}{\alpha(1 - \alpha)} ||Y(s)||_{\alpha, \infty} \right) \xi^{1-\alpha} \right) ds. \]  
(5.22)

By (5.21) and (5.22) we get,
\[ ||Y(t)||_{\alpha, \infty} \]
\[ \leq ||\phi||_{\alpha, \infty} + \Lambda_\alpha(g) \int_0^t \left( M_2 \left( \frac{1}{1-\alpha} + \frac{b^{(1)}_\alpha}{1 - 2\alpha} \right) + M_1 \left( 1 + \frac{1}{\alpha(1 - \alpha)} \right) \right) ||Y(s)||_{\alpha, \infty} ds. \]

By Gronwall’s inequality we obtain,
\[ ||Y||_{\alpha, \infty} \leq ||\phi||_{\alpha, \infty} \exp(KT), \]
where \( K = \sup_{0 \leq t \leq T} \Lambda_\alpha(g) \left[ M_2 \left( \frac{1}{1-\alpha} + \frac{b^{(1)}_\alpha}{1 - 2\alpha} \right) + M_1 \left( 1 + \frac{1}{\alpha(1 - \alpha)} \right) \right]. \)

Hence, the local solution is global in time. \( \square \)

5.3. Proof of Theorem 5.1. For every \( t \in [0, T] \), the random variable \( G = \frac{1}{(1-\alpha)} \sup_{0 \leq \eta < \xi < 1} ||(D_{\xi}^{1-\alpha} B_{\xi -})(t) \) has moments of all orders by Proposition 4.1. \( \square \) Hence, the pathwise integral \( \int_0^1 A(Y(t, \eta)) dB_\eta(t) \) exists for \( 1 - H < \alpha < \frac{1}{2} \) and the existence and uniqueness of solutions follows from Theorem 5.8 which completes the proof.

Acknowledgments: This work was partially supported by NSF grant No. DMS 0608494.

References
[1] Alós, E., Mazet, O. and Nualart, D.: Stochastic calculus with respect to Gaussian processes, Ann. Probab., 29, (2001), no. 2, 766–801.
[2] Berselli, L.C., Bessaih, H.: Some Results for the Line Vortex Equation, Nonlinearity, 15 (2002), 1729–1746.
[3] Berselli, L. C.; Gubinelli, M.: On the global evolution of vortex filaments, blobs, and small loops in 3D ideal flows, Comm. Math. Phys., 269 (2007), no. 3, 693713.
[4] Bertozzi, A.L., Majda, A.J.: Vorticity and Incompressible Flow, Cambridge University Press, Cambridge, 2002.

[5] Bessaih, H.; Gubinelli, M.; Russo, F.: The Evolution of a Random Vortex Filament, The Annals of Probability, 33 (2005), 1825–1855.

[6] Biagini, F.; Hu, Y.; Øksendal, B.; Zhang, T.: Stochastic Calculus for Fractional Brownian Motion and Applications, Springer, 2008.

[7] Carmona, P.; Coutin, L.; Montseny, G.: Stochastic integration with respect to fractional Brownian motion, Ann. Inst. H. Poincar Probab. Statist., 39 (2003), no. 1, 27–68.

[8] Chorin , A. J.: Vorticity and Turbulence, Springer-Verlag, New York, 1994.

[9] Coutin, L.: An introduction to (stochastic) calculus with respect to fractional Brownian motion. Séminaire de Probabilités XL, 365, Lecture Notes in Math., 1899, Springer, Berlin, 2007.

[10] Coutin, L.; Lejay, A.: Semi-martingales and rough paths theory. Electron. J. Probab. 10 (2005), no. 23, 761–785.

[11] Decreusefond, L. and Üstünel, A. S.: Stochastic analysis of the fractional Brownian motion. Potential Anal. 10, 1998, 177–214.

[12] Flandoli, F.: On a probabilistic description of small scale structures in 3D fluids, Ann. Inst. H. Poincar Probab. Statist., 38 (2002), no. 2, 207–228.

[13] Gubinelli, M.; Lejay, A.; Tindel, S.: Young integrals and SPDEs. Potential Anal. 25 (2006), no. 4, 307–326.

[14] Kilbas, A.A., Marichev, O.I. and Samko, S.G.: Fractional Integrals and Derivatives: Theory and Applications, Princton University Press, Gordon and Breach, 1993.

[15] Klingenhöfer, F. and Zähle, M.: Ordinary Differential Equations with Fractal Noise, Proceedings of the American Mathematical Society, 127(1999), 1021–1028.

[16] Lyons, T. J.: Differential equations driven by rough signals, Rev. Mat. Iberoamericana 14 (1998), no. 2, 215–310.

[17] Mandelbrot, B. and Van Ness, J.: Fractional Brownian motions, fractional noises and applications, SIAM Review, 10(1968),422–437.

[18] Marchioro, C., Pulvirenti, M.: Mathematical Theory of Incompressible Nonviscous Fluids, Springer-Verlag, New York, 1994.

[19] Nualart, D. and Răşcanu, A.: Differential Equations driven by Fractional Brownian Motion, Collectanea Mathematica, 53(2001), 55–81.

[20] Saffman, P.G.: Vortex Dynamics, Cambridge University Press, New York, 1992.

[21] Young, L.C.: An inequality of the Hölder type connected with Stieltjes integration, Acta Math, 67(1936),251–282.

H. Bessaih, University of Wyoming, Department of Mathematics, Dept. 3036, 1000 East University Avenue, Laramie WY 82071, United States
E-mail address: Bessaih@uwyo.edu

C. Wijeratne, University of Wyoming, Department of Mathematics, Dept. 3036, 1000 East University Avenue, Laramie WY 82071, United States
E-mail address: cwijerat@uwyo.edu