Exact rotational space-time transformations, Davies-Jennison experiments and limiting Lorentz-Poincaré invariance

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\textbf{Abstract.} Jennison deduced from the rotational experiments that a rotating radius $r_r$ measured by the rotating observer is contracted by $r_r = r (1 - \omega^2 r^2 / c^2)^{1/2}$, compared with the radius $r$ measured in an inertial frame. This conclusion differs from the result based on Lorentz transformations. Since rotational frames are not equivalent to inertial frames, we analyze the rotational experiments by using the exact rotational space-time transformations rather than the Lorentz transformations. We derive exact rotational transformations on the basis of the principle of limiting Lorentz-Poincaré invariance. The exact rotational transformations form a pseudo-group rather than the usual Lie group. They support Jennison’s contraction of a rotating radius and are consistent with two Davies-Jennison experiments. We also suggest new experimental tests for the exact rotational transformations.

\section{1 Introduction}

It is evident that the space-time transformations between, say, a non-inertial and an inertial frame must simplify to the Lorentz transformations in the limit of zero acceleration. This limiting property appears to be necessary in order for the space-time coordinates of non-inertial frames to have an operational meaning. Furthermore, it will also pave the way to formulate theories such as a unified electroweak theory and gravitational theory and to understand physics in non-inertial frames.

The question is then how one can generalize the Lorentz transformations to non-inertial frames such as frames with linear accelerations or rotations. To accomplish this generalization, one must find a general principle for physical laws in non-inertial frames $F(w, x, y, z)$, similar to the principle of relativity for physical laws in inertial frames $F_I(w, x, y, z)$. In this connection, it is convenient to use $w$ with the dimension of length as the evolution variable for both inertial and non-inertial frames of reference [1], so that one can avoid the complicated properties of the speed of light (which is not constant in non-inertial frames) measured in terms of the usual unit, i.e., meter per second, in non-inertial frames.

It appears natural to postulate that all physical laws, including the laws of space-time coordinates and energy-momentum transformations, in constant-linear-acceleration frames become the corresponding laws in inertial frames in the limit of zero acceleration. This was called the principle of limiting Lorentz-Poincaré invariance [2,3]. For more complicated and general non-inertial frames, we first classify reference frames into different classes. For example, we may have the class of frames with an arbitrary-linear-acceleration (ALA) $\alpha(w)$, the class of frames with a constant-linear-acceleration (CLA) $\alpha_o$ and the class of inertial frames. The general principle states that physical laws in ALA frames reduce to the corresponding laws in CLA frames in the limit $\alpha(w) \to \alpha_o$. We called it the principle of limiting continuation for physical laws [4]. It includes the principle of limiting Lorentz-Poincaré invariance as a special case.

In previous works [2,3,5], we applied the principle of limiting continuation to derive coordinate transformations between an inertial frame and a frame with a velocity and acceleration pointing in the same constant direction. They were consistent with known experiments. We now use the same approach to derive coordinate transformations between an inertial frame and a frame that rotates with a constant angular velocity. As we shall see, our results are not only consistent with the results of rotational experiments [6–9] and high-energy experiments involving unstable particles in a circular storage ring [10], but also support Pellegrini and Swift’s analysis of the Wilson experiment [11], in which they point out that rotational transformations cannot be locally replaced by Lorentz transformations.

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While it is true that the Lorentz transformations have been used to analyze experiments involving rotational or orbital motion and that in some cases, the theoretical predictions are consistent with experimental results (for example, in the case of calculating the lifetime dilation of unstable particles moving in a circular storage ring [10]) the fact remains that, rigorously speaking, such applications are inappropriate.

In this paper, we show first that the rotational space-time transformations can be obtained on the basis of the principle of limiting Lorentz-Poincaré invariance and that the resultant transformations are exact because they reduce to the exact Lorentz transformations in the limit of zero acceleration. The fundamental metric tensors of the rotational frames are derived. The set of exact rotational transformations of space-time forms a “pseudo-group”, which reduces to the usual Lie group only in the limit of zero acceleration. Then we use these exact rotational transformations to discuss Jennison’s contraction of a rotating radius [8] and to analyze the experimental results of Davies-Jennison [9]. Both Jennison’s contraction of a rotating radius and the Davies-Jennison experiment are beyond the special theory of relativity. However, we show that they are consistent with the exact rotational space-time transformations based on the principle of limiting Lorentz-Poincaré invariance. We also discuss new rotational experiments, which can help serve as a test of the proposed rotational space-time transformations and the principle of limiting continuation for physical laws.

2 Exact rotational transformations with limiting Lorentz-Poincaré invariance

Suppose $F_I(w_I, x_I, y_I, z_I)$ is an inertial frame and $F(w, x, y, z)$ (which we subsequently refer to as $F(\Omega)$) is a frame that rotates with a constant angular velocity $\Omega$ (to be defined more precisely below). The origins of both frames coincide at all times and we use a Cartesian coordinate system in both frames. The usual classical transformation equations between $F_I$ and $F$ are

$$
\begin{align*}
  w_I &= w, & x_I &= x \cos(\Omega w) - y \sin(\Omega w), \\
  y_I &= x \sin(\Omega w) + y \cos(\Omega w), & z_I &= z,
\end{align*}
$$

in which the relation, say, $w_I = w$ is incorrect because it is incompatible with the experimental result for the lifetime dilation of unstable particle decay in flight in a circular storage ring [10].

In order to derive a set of exact coordinate transformations between $F_I$ and $F$ that satisfy the requirements of limiting Lorentz-Poincaré invariance (i.e., limiting four-dimensional symmetry), we first consider a slightly more general case. In this more general case, there is an inertial reference frame $F_I$ and a non-inertial frame $F_R(\Omega)$ whose origin orbits the origin of the inertial frame at a constant distance $R$, with a constant angular velocity $\Omega$. A Cartesian coordinate system is used in both frames, set up in such a way that the positive portion of the $y$-axis of the $F_R(\Omega)$ frame always extends through the origin of $F_I$. This is useful because in the limit $R \to \infty$ and $\Omega \to 0$ such that the product $R\Omega = \beta$, is a finite non-zero constant velocity, the two frames become inertial frames with coordinates related by the familiar Lorentz transformations. With that in mind, the classical coordinate transformations between $F_I$ and $F_R(\Omega)$ (the orbiting frame) are

$$
\begin{align*}
  w_I &= w, & x_I &= x \cos(\Omega w) - (y - R) \sin(\Omega w), \\
  y_I &= x \sin(\Omega w) + (y - R) \cos(\Omega w), & z_I &= z.
\end{align*}
$$

According to the principle of limiting continuation, we postulate that the transformations between $F_I$ and $F_R(\Omega)$ that satisfy the requirements of limiting Lorentz-Poincaré invariance (i.e., the transformations satisfy Lorentz and Poincaré group properties in the limit of zero acceleration) take the following form:

$$
\begin{align*}
  w_I &= A w + B \rho \cdot \beta, & x_I &= G x \cos(\Omega w) + E (y - R) \sin(\Omega w), \\
  y_I &= I x \sin(\Omega w) + H (y - R) \cos(\Omega w), & z_I &= z,
\end{align*}
$$

where $\rho = (x, y)$, $S = (x, y - R)$, $\beta = \Omega \times S$, $\Omega = (0, 0, \Omega)$, and the functions $A, B, E, G, H$, and $I$ may, in general, depend on the coordinates $x^\mu$.1

One unusual feature of the transformation eqs. (3) is that both the constant angular velocity $\Omega$ (defined as $\Omega = d\phi/dw$ under a suitable condition) and the orbital radius $R$ are quantities that are measured by observers in the non-inertial frame $F_R(\Omega)$. This is counter to the usual procedure of measuring such parameters from the “lab” or “inertial” reference frame, but will simplify the following discussion since all quantities on the right side of the transformation eqs. (3) are measured with respect to $F_R(\Omega)$ observers. Thus, when $w = w_I = 0$, the $y$ and $y_I$ axes overlap and the origin of $F_I$, i.e., $x_I = y_I = 0$, is found at the coordinates $(x, y) = (0, +R)$. Equivalently, the origin

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1 This is a special case of the principle of limiting continuation of physical laws when the accelerated frame becomes an inertial frame in the limit of zero acceleration.
of the rotating frame $F_R(\Omega)$, i.e., $x = y = 0$, is found at the inertial coordinates $(x_I, y_I) = (0, -HR)$, where $H$ is an unknown function to be determined.

To determine the unknown functions $A, B, E, G, H,$ and $I$, we consider the following limiting cases. First, when $R = 0$, transformations (3) must have $x/y$ symmetry (i.e., be symmetric under an exchange of $x$ and $y$). This implies that

$$-E = G = H = I, \quad \text{when} \quad R = 0.$$  \hspace{1cm} (4)

Second, in the limit of small $\Omega$ (or $|\Omega \times S| \ll 1$) with $R \rightarrow 0$, transformations (3) should reduce to the classical rotational transformations (1). Thus,

$$-E \approx G \approx H \approx I \approx 1, \quad \text{for small} \quad \Omega \quad \text{with} \quad R = 0.$$  \hspace{1cm} (5)

Finally, and perhaps most importantly, in the limit where $R \rightarrow \infty$ and $\Omega \rightarrow 0$ such that their product $R\Omega = \beta_0$ is a non-zero constant velocity parallel to the $x_I$-axis, the finite and differential forms of (3) must reduce to the Lorentz transformations, in both its finite form,

$$w_I = \gamma_0 (w + \beta_0 x), \quad x_I = \gamma_0 (x + \beta_0 y), \quad y_I = y, \quad z_I = z,$$

$$\gamma_0 = \frac{1}{\sqrt{1 - \beta_0^2}},$$  \hspace{1cm} (6)

and its differential form,

$$dw_I = \gamma_0 (dw + \beta_0 dx), \quad dx_I = \gamma_0 (dx + \beta_0 dw), \quad dy_I = dy, \quad dz_I = dz.$$  \hspace{1cm} (7)

(Strictly speaking, $y_I = y$ should be replaced by $y_I = -\infty$ in (6), because $R \rightarrow \infty$ in this limit. Nevertheless, one may shift the $y$-axis so that $y_I = y$.) This is necessary for (3) to satisfy the limiting Lorentz-Poincaré invariance, as specified by the principle of limiting continuation of physical laws. The time $w$ in (2) reduces to the usual relativistic time (with the dimension of length) in this limit $[5,1]$. Thus, we have

$$A = B = G = -E = \gamma_0, \quad H = 1,$$  \hspace{1cm} (8)

when $R \rightarrow \infty$ and $\Omega \rightarrow 0$ such that $\beta_0 = R\Omega$.

The requirements put on $A, B, E, G, H,$ and $I$ by considering these three limiting cases do not lead to a unique solution for the unknown functions. This is analogous to the case in which gauge symmetry does not uniquely determine the electromagnetic action [12] and one must also postulate a minimal electromagnetic coupling. Here, as we did in the previous papers [2,3] in deriving the accelerated Wu transformations, we postulate a minimal generalization of the classical rotational transformations (2).

Based on the limiting cases considered above, it is not unreasonable for transformations (3) to have the following two properties: i) For non-zero $\Omega$ and finite $R$, the functions $A, B, G, I$ and $-E$ are simply generalized from $\gamma_0 = (1 - \beta^2)^{-1/2}$ to $\gamma = (1 - \beta^2)^{-1/2}$, where $\beta = S\Omega$, and ii) only $H$ depends on $R$ and it is required to be the simplest function involving only the first power of $\gamma$. Combining (4), (5) and (8) with the two properties just named leads to the following solutions:

$$A = B = G = I = \gamma = (1 - \beta^2)^{-1/2}, \quad E = -\gamma,$$

$$H = (\gamma + R/R_0)(1 + R/R_0), \quad \gamma = (1 - \beta^2)^{-1/2}.$$  \hspace{1cm} (9)

The quantity $R_0$ in (9) is an undetermined length parameter that seems to be necessary for the transformations to satisfy the limiting criteria specified in (4), (5) and (8), as well as the limiting Lorentz-Poincaré invariance. From a comparison of rotational transformations and their classical approximation, the value of $R_0$ should be very large. However, since it is unsatisfactory to have such undetermined parameters in a theory and there are no known experimental results that can help to determine a value for $R_0$, we shall take the limit $R_0 \rightarrow \infty$ for simplicity. This is consistent with the minimal generalization of the classical rotational transformations (2).

Thus, a simple rotational transformation corresponding to the classical transformations (2) is

$$w_I = \gamma (w + \rho \cdot \beta), \quad x_I = \gamma [x \cos(\Omega w) - (y - R) \sin(\Omega w)],$$

$$y_I = \gamma [x \sin(\Omega w) + (y - R) \cos(\Omega w)], \quad z_I = z,$$

$$\beta = |\Omega \times S| = \Omega \sqrt{x^2 + (y - R)^2} = \Omega S < 1, \quad \rho \cdot \beta = xR\Omega.$$  \hspace{1cm} (10)

Since the transformations (10) include the exact Lorentz transformations as a special limiting case, as shown in (6), the transformations (10) are also exact. We shall call (10) the “rotational taiji transformations” to distinguish it from other rotational transformations in the literature\(^2\).

\(^2\) In ancient Chinese thought, the word taiji, or taichi, denotes “The Absolute”, i.e., the ultimate principle or the condition that existed before the creation of the world.
Because all known experiments can be analyzed in the $R = 0$ case, for the rest of this paper, we set $R = 0$ and concentrate solely on the implications of the transformations under that condition. With $R = 0$ the coordinate transformations between an inertial frame $F_I$ and a non-inertial frame $F(\Omega)$ whose origin coincides with that of $F_I$ and that rotates with a constant angular velocity $\Omega$ about the origin are
\[
\begin{align*}
    w_I &= \gamma(w + \rho \cdot \beta) = \gamma w, \quad x_I = \gamma[x \cos(\Omega w) - y \sin(\Omega w)], \\
    y_I &= \gamma[x \sin(\Omega w) + y \cos(\Omega w)], \quad z_I = z, \quad \gamma = 1/\sqrt{1 - \rho^2 \Omega^2},
\end{align*}
\]
where $\rho \cdot \beta = \Omega R x = 0$ for $R = 0$.

To derive the inverse transformations of (11), we must first find a way to express $\Omega w$ and $\gamma = 1/\sqrt{1 - \rho^2 \Omega^2}$ in terms of quantities measured in the inertial frame $F_I$. While the coordinate transformations (10) for the non-zero $R$ case can only be written in Cartesian coordinates (so that they satisfy the limiting Lorentz-Poincaré invariance), the transformations in the $R = 0$ case (11) can be written in terms of cylindrical coordinates. We introduce the relations
\[
    x_I = \rho_I \cos(\phi_I), \quad y_I = \rho_I \sin(\phi_I), \quad x = \rho \cos \phi, \quad \text{and} \quad y = \rho \sin \phi
\]
so that (11) can be written as
\[
    w_I = \gamma w, \quad \rho_I = \sqrt{x_I^2 + y_I^2} = \gamma \rho, \quad \phi_I = \phi + \Omega w, \quad z_I = z,
\]
with
\[
    \Omega_I \equiv \frac{d\phi_I}{dw_I} = \frac{d\phi_I}{dw} \frac{dw}{dw_I} = \Omega \frac{1}{\gamma} \quad (\rho \text{ and } \phi \text{ fixed}).
\]
The conditions that $\rho$ and $\phi$ are fixed mean that an object is at rest in the rotating frame, so that an observer in $F_I$ can measure its angular velocity $\Omega_I$ and identify it with the angular velocity of the rotating frame. These equations give the operational definitions of $\Omega_I$ and $\Omega$. Thus, we have
\[
    w_I \Omega_I = w \Omega, \quad \rho_I \Omega_I = \rho \Omega, \quad \gamma = \frac{1}{\sqrt{1 - \rho^2 \Omega^2}} = \frac{1}{\sqrt{1 - \rho^2_I \Omega^2}}.
\]
From (11) and (12), we can then derive the inverse rotational transformations
\[
\begin{align*}
    w &= \frac{w_I}{\gamma}, \quad x = \frac{1}{\gamma}[x_I \cos(\Omega_I w_I) + y_I \sin(\Omega_I w_I)], \\
    y &= \frac{1}{\gamma}[-x_I \sin(\Omega_I w_I) + y_I \cos(\Omega_I w_I)], \quad z = z_I,
\end{align*}
\]
where
\[
    \gamma = \frac{1}{\sqrt{1 - \Omega_I^2(x_I^2 + y_I^2)}}.
\]

If one imagines the frame $F(\Omega)$ as a carousel, then for any given angular velocity, objects at rest relative to $F(\Omega)$ that are sufficiently far from the origin would have a classical linear velocity that exceeds the speed of light in the inertial frame $F_I$. As expected, the rotational transformations display some unusual behavior near that region. From the relationship between $\Omega$ and $\Omega_I$, i.e., $\Omega_I = \Omega / \gamma = \Omega \sqrt{1 - \rho^2 \Omega^2}$, one can see that for a given constant angular velocity $\Omega$, the corresponding angular velocity $\Omega_I$ depends on $\rho$ as well as $\Omega$. In particular, when $\rho = 1/\Omega$, the value of $\Omega_I$ becomes zero. Furthermore, the reading of $F(\Omega)$ clocks at that radius, as viewed by $F_I$ observers, stops changing, much like the case in special relativity where the rate of ticking of clocks slows down and stops as the speed of those clocks approaches the speed of light. These effects are not unexpected because the rotational transformations (11) map only a portion of the space in the rotating frame $F(\Omega)$ ($\rho < 1/\Omega$) to the entire inertial frame $F_I$. Similar to the singular walls found in frames with a linear acceleration [2], there is a cylindrical singular wall at $\rho = 1/\Omega$ in the rotating frame $F(\Omega)$.

The rotational transformations (11) have the inverse transformations (13), but when $R \neq 0$ in (10), it appears that there is no algebraic expression for the inverse of the transformations (10). However, mathematically, (10) defines a function from one neighborhood of the origin in space-time to another. The implicit function theorem then guarantees the existence of an inverse in a neighborhood of any point where the Jacobian is non-zero\(^3\).

\(^3\) We would like to thank Dana Fine for helpful discussions.
3 Metric tensors for the space-time of rotating frames

We note that the transformations of the contravariant 4-vectors \( dx_\mu = (dw_I, dx_I, dy_I, dz_I) \) and \( dx^\mu = (dw, dx, dy, dz) \) can be derived from (11). We have

\[
dw_I = \gamma [dw + (\gamma^2 \Omega^2 w_x) dx + (\gamma^2 \Omega^2 w_y) dy],
\]

\[
dx_I = \gamma \{[\cos(\Omega w) + \gamma^2 \Omega^2 x^2 \cos(\Omega w) - \gamma^2 \Omega^2 x y \sin(\Omega w)] dx
\]

\[-[\sin(\Omega w) + \gamma^2 \Omega^2 y^2 \sin(\Omega w) - \gamma^2 \Omega^2 x y \cos(\Omega w)] dy
\]

\[-[\Omega x \sin(\Omega w) + \Omega y \cos(\Omega w)] dw],
\]

\[
dy_I = \gamma \{[\sin(\Omega w) + \gamma^2 \Omega^2 x^2 \sin(\Omega w) + \gamma^2 \Omega^2 x y \cos(\Omega w)] dx
\]

\[+[\cos(\Omega w) + \gamma^2 \Omega^2 y^2 \cos(\Omega w) + \gamma^2 \Omega^2 x y \sin(\Omega w)] dy
\]

\[+[\Omega x \cos(\Omega w) - \Omega y \sin(\Omega w)] dw\},
\]

\[
dz_I = dz.
\]

(14)

To find the metric tensors \( P_{\mu\nu} \), which will be called the Poincaré metric tensors for the rotating frame \( F(\Omega) \), it is convenient to use (11) to write \( ds^2 = \eta_{\mu\nu} dx_\mu dx_\nu = dw_I^2 - dx_I^2 - dy_I^2 - dz_I^2 \), where \( \eta_{\mu\nu} = (1, -1, -1, -1) \), as

\[
ds^2 = d(\gamma w)^2 - (x^2 + y^2) \gamma^2 \Omega^2 dw^2 - d(\gamma x)^2 - d(\gamma y)^2 - dz^2 + 2\gamma \Omega y d(x \gamma x) dw - 2\Omega \gamma x d(\gamma y) dw.
\]

(15)

Then, with the help of the relation \( d\gamma = \gamma^3 \Omega^2 (xdz + ydy) \), (15) can be written as

\[
ds^2 = P_{\mu\nu} dx^\mu dx^\nu, \quad \mu, \nu = 0, 1, 2, 3.
\]

(16)

It follows from (14), (15) and (16) that the non-vanishing components of \( P_{\mu\nu} \) are given by

\[
P_{00} = 1,
\]

\[
P_{11} = -\gamma^2 \left[1 + 2\gamma^2 \Omega^2 x^2 - \gamma^4 \Omega^4 x^2 (w^2 - x^2 - y^2)\right],
\]

\[
P_{22} = -\gamma^2 \left[1 + 2\gamma^2 \Omega^2 y^2 - \gamma^4 \Omega^4 y^2 (w^2 - x^2 - y^2)\right],
\]

\[
P_{33} = -1,
\]

\[
P_{01} = \gamma^3 [\Omega y + \gamma^2 \Omega^2 wx],
\]

\[
P_{02} = \gamma^2 [-\Omega x + \gamma^2 \Omega^2 wy],
\]

\[
P_{12} = -\gamma^4 \Omega^2 xy \left[2 - \gamma^2 \Omega^2 (w^2 - x^2 - y^2)\right].
\]

(17)

Using \( P_{\mu\lambda} P^{\lambda\nu} = \delta^\nu_\mu \), the contravariant metric tensor \( P^{\mu\nu} \) is found to be

\[
P^{00} = \gamma^{-2} \left[1 - \Omega^4 w^2 (x^2 + y^2)\right],
\]

\[
P^{33} = -1,
\]

\[
P^{11} = -\gamma^{-2} \left[\gamma^{-2} (1 - \Omega^2 x^2) - 2\gamma^{-2} \Omega^3 w x y + \Omega^6 w^2 y^2 (x^2 + y^2)\right],
\]

\[
P^{22} = -\gamma^{-2} \left[\gamma^{-2} (1 - \Omega^2 y^2) + 2\gamma^{-2} \Omega^3 w x y + \Omega^6 w^2 x^2 (x^2 + y^2)\right],
\]

\[
P^{01} = -\gamma^{-2} \left[-\Omega y - \gamma^{-2} \Omega^2 w x + \Omega^5 w^2 y (x^2 + y^2)\right],
\]

\[
P^{02} = -\gamma^{-2} \left[\Omega x - \gamma^{-2} \Omega^2 wy - \Omega^5 w^2 x (x^2 + y^2)\right],
\]

\[
P^{12} = \gamma^{-2} \left[\gamma^{-2} \Omega^2 xy - \gamma^{-2} \Omega^4 w (x^2 - y^2) + \Omega^6 w^2 xy (x^2 + y^2)\right].
\]

(18)

All other components in (17) and (18) are zero. One can verify that these Poincaré metric tensors \( P_{\mu\nu} \) for rotating frames satisfy vanishing Riemann-Christoffel curvature tensors.

In the special case \( R = 0 \), one can easily express the exact rotational transformations (11) in cylindrical coordinates \( x^\mu = (w, \rho, \phi, z) \) and obtain the metric tensors for the rotating frame,

\[
P_{00} = 1, \quad P_{11} = -\gamma^6 (1 - \Omega^4 \rho^2 w^2), \quad P_{22} = -\gamma^2 \rho^2,
\]

\[
P_{33} = -1, \quad P_{01} = \gamma^4 \Omega^2 \rho w, \quad P_{02} = -\gamma^2 \Omega \rho^2.
\]
4 The rotational pseudo-group

As mentioned previously, the rotational transformations (11) imply the existence of a cylindrical singular wall at \( \rho = \sqrt{x^2 + y^2} = 1/\Omega \equiv \rho_o \), which depends on the angular velocity \( \Omega \). Like the accelerated Wu transformations, the rotational transformations map only the portion of space-time of the rotating frame within the singular wall to the entire space-time of an inertial frame and the “group” of rotating transformations is a pseudo-group [13]. In general, the rotational transformations map only a portion of the space in a rotating frame \( F(\Omega) \) to a portion of the space in another rotating frame \( F(\Omega') \). To deal with these types of transformations, mathematicians O. Veblen and J.H.C. Whitehead developed the concept of a pseudo-group [13]. A set of transformations forms a pseudo-group if i) the resultant of two transformations in the set is also in the set, and ii) the set contains the inverse of every transformation in the set. The complete set of rotational transformations forms a pseudo-group, which may be called the rotational pseudo-group.

Two of the group properties of the rotational transformations are straightforward to verify. One can easily see that in the limit \( \Omega \to 0 \), the rotational transformations (11) reduce to the identity transformation. Also, the inverse transformations of (11) are given by (13).

In order to see other group properties of the rotational transformations, let us consider two other rotating frames, \( F'(\Omega') \) and \( F''(\Omega'') \), which are characterized by two different constant angular velocities \( \Omega' \) and \( \Omega'' \), respectively. With the help of (11), we can derive the rotational transformations between \( F(0) = F_1, F(\Omega), F'(\Omega') \) and \( F''(\Omega'') \),

\[
\begin{align*}
\omega_1 &= \omega' = \gamma' \omega', \\
\omega_2 &= \omega'' = \gamma'' \omega'', \\
x_1 &= \gamma [x' \cos(\Omega w') - y' \sin(\Omega w')] = \gamma' [x' \cos(\Omega' w') - y' \sin(\Omega' w')]
= \gamma'' [x'' \cos(\Omega'' w'') - y'' \sin(\Omega'' w'')], \\
y_1 &= \gamma [x' \sin(\Omega w') + y' \cos(\Omega w')] = \gamma' [x' \sin(\Omega' w') + y' \cos(\Omega' w')]
= \gamma'' [x'' \sin(\Omega'' w'') + y'' \cos(\Omega'' w'')], \\
\gamma' &= \frac{1}{\sqrt{1 - \beta'^2}}, \quad \beta' = \Omega' \rho', \quad \gamma'' = \frac{1}{\sqrt{1 - \beta''^2}}, \quad \beta'' = \Omega'' \rho'',
\end{align*}
\]

where we have neglected to write the trivial transformation equations for the \( z \)-direction. The rotational transformations between \( F(\Omega) \) and \( F'(\Omega') \) can be obtained from (19)

\[
\begin{align*}
w &= \frac{\gamma'}{\gamma} \omega', \\
x &= \gamma' \left\{ [x' \cos(\Omega' w') - y' \sin(\Omega' w')] \frac{\cos(\Omega w)}{\gamma} 
+ [x' \sin(\Omega' w') + y' \cos(\Omega' w')] \frac{\sin(\Omega w)}{\gamma} \right\}, \\
y &= \gamma' \left\{ [x' \sin(\Omega' w') + y' \cos(\Omega' w')] \frac{\cos(\Omega w)}{\gamma} 
- [x' \cos(\Omega' w') - y' \sin(\Omega' w')] \frac{\sin(\Omega w)}{\gamma} \right\}.
\end{align*}
\]

If one compares these transformations with those in eqs. (11) through (13), one can see that the transformations between two rotating frames are more algebraically complicated than those between an inertial frame and a rotating frame. In general, it does not seem possible to write the inverse of the transformations (20) in analytic form. However, the implicit function theorem guarantees the existence of an inverse in the neighborhood of any point where the Jacobian is non-zero (see footnote).  

Moreover, the set obtained from generators of the rotational transformations using the Lie bracket, and the brackets of brackets, and so on, never closes in general. That is, the rotational transformations define a group with infinitely many generators (see footnote). This difference is likely related to the distortion of space-time coordinates in non-inertial frames relative to an inertial frame, as shown by the Poincaré metric tensors \( P_{\mu \nu} \) in (17).

The space-time transformations for frames involving constant linear accelerations or arbitrary linear accelerations also form pseudo-groups rather than Lie groups [5]. Thus, regarding the group nature of coordinate transformations in flat space-time between general physical frames of reference, we conjecture that only in the limits of zero acceleration do we have Lie groups such as the Lorentz and Poincaré groups associated with the space-time transformations of inertial frames.

\footnote{Here, “set” is defined as an infinite-dimensional set, rather than the usual Lie set.}
5 Physical implications

Some physical implications of the rotational transformations for rotating frames worthy of note are the following.

5.1 Operational definitions of space-time coordinates in rotating frames

Since the speed of light in rotating frames is not constant, as shown by \( ds = 0 \) in (15), it is very complicated to use light signals to synchronize clocks in a rotating frame \( F(\Omega) = F(w, x, y, z) \). However, as discussed previously [3] in reference to linearly accelerated Wu transformations, one can synchronize a set of clocks and setup a coordinate system in the rotating frame \( F(\Omega) \) by using a grid of “computerized space-time clocks” [4] that are programmed to accept information concerning their positions \((x_I, y_I)\) relative to the \( F_I \) frame, obtain \( w_I \) from the nearest \( F_I \) clock\(^5\), and then compute and display \( w, x \) and \( y \) using the inverse transformation (13) with the given parameter \( \Omega_I \). In a rotating frame \( F(\Omega) \), the values the physical time \( w \) can take on are restricted by the condition \( \Omega^2(x^2 + y^2) < 1 \) and the coordinates of physical space in \( F(\Omega) \) are limited by \( \rho = \sqrt{x^2 + y^2} < 1/\Omega \).

5.2 The invariant action for electrodynamics in rotating frames

We are now able to write the invariant action \( S_{em} \) in natural units in a rotating frame for a charged particle with mass \( m \) and charge \( e \) moving in the 4-potential \( A_\mu \),

\[
S_{em} = \int \left[ -mds - eA_\mu dx^\mu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}\sqrt{-P} d^4x \right],
\]

where \( ds \) is given by (16) and the Poincaré metric tensors \( P_{\mu\nu} \) for rotating frames are given in (17). Also, \( D_\nu \) denotes the covariant partial derivative associated with the metric tensors \( P_{\mu\nu} \).

The Lagrange equation of motion of a charged particle can be derived from (21). We obtain

\[
m\frac{Du_\mu}{ds} = eF_{\mu\nu}u^\nu,
\]

\[
Du_\mu = D_\nu u_\mu dx^\nu, \quad u^\nu = \frac{dx^\nu}{ds}, \quad u_\mu = P_{\mu\nu}u^\nu.
\]

Starting with the invariant action (21) and replacing the second term \(-\int eA_\mu dx^\mu\) with

\[
-\int A_\mu j^\mu\sqrt{-P} d^4x,
\]

for a continuous charge distribution in space, we obtain covariant Maxwell’s equations in a rotating frame,

\[
D_\nu F^{\mu\nu} = j^\mu, \quad \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0.
\]

Based on gauge invariance and the rotational invariance of the action (21), the term \( eA_\mu dx^\mu \) implies that the electromagnetic potential \( A_\mu \) must be a covariant vector in non-inertial frames because the coordinate differential \( dx^\mu \) is, by definition, a contravariant vector. Since the force \( F \) and the fields \( E \) and \( B \) are related to a change of the potential \( A_\mu \) with respect to a change of coordinates \( x^\nu \), by definition, the electromagnetic fields \( E \) and \( B \) are naturally identified with components of the covariant tensor \( F_{\mu\nu} \) as given by (22) in non-inertial frames. The metric tensor \( P_{\mu\nu} \) behaves like a constant under covariant differentiation, \( D_\lambda P_{\mu\nu} = 0 \).

Similarly, one can also formulate classical Yang-Mills gravity in rotating frames and other non-inertial frames [14,15]. Nevertheless, the quantization of fields and the derivation of Feynman rules for QED and Yang-Mills gravity in rotating frames are difficult due to the complicated metric tensors in (17) and (18)\(^6\).

\(^5\) A system of clocks in an inertial frame \( F_I \) can be synchronized in the usual way.

\(^6\) For a discussion of quantization of fields and Feynman rules in a simpler non-inertial frames, say, those with constant-linear accelerations, see ref. [5], appendix C.
5.3 Absolute contraction of a rotating radius and absolute slow-down of a rotating clock

In contrast to the classical rotational transformations (1), the exact rotational transformations (11) predict that the length of a rotating radius \( \sqrt{x^2 + y^2} \) is contracted by a factor of \( \gamma \),

\[
\sqrt{x'^2 + y'^2} = \gamma \sqrt{x^2 + y^2}.
\]

This contraction is absolute, meaning that both observers in the inertial frame \( F_I \) and in the rotating frame \( F(\Omega) \) agree that the radius, as measured in the rotating frame \( F(\Omega) \) is shorter, because there is no relativity between an inertial frame and a rotating (non-inertial) frame.

In some discussions of phenomena involving circular motion at high speeds, such as the lifetime dilation of unstable particles traveling in a circular storage ring, the argument is made that during a very short time interval, one can approximate the true rotational transformations using the Lorentz transformations \([10,11]\). However, making this approximation leads to a completely different conclusion regarding the radius, namely, that it does not contract because it is always perpendicular to the direction of motion.

Furthermore, for a given \( \rho \), eq. (11) implies \( \Delta w_I = \gamma \Delta w_r \), which is independent of the spatial distance between two events\(^7\). In other words, clocks at rest relative to a rotating frame and located at a distance \( \rho = \sqrt{x^2 + y^2} \) from the center of rotation slow down by a factor of \( \gamma = \sqrt{1 - \beta^2} \) in comparison with clocks in the inertial frame \( F_I \).

Analogous to the absolute contraction of radial distances as shown in (27), this time dilation is also an absolute effect in that observers in both \( F_I \) and \( F(\Omega) \) agree that it is the accelerated clocks that are slowed.

Both the contraction of radial distances and the slowing down of clocks are consequences of requiring the rotational transformations to satisfy the limiting Lorentz-Poincaré invariance.

It is interesting to note that, in 1964, Jennison \([8]\) showed by deductions from the results of an experiment by Champeney and Moon that “the radius of a rotating system, measured from a single domain rotating with the system, is contracted by a factor of \( \sqrt{2} \), \( \sqrt{3} \), or \( \sqrt{1} \) as the evolution variable in the Lagrangian formalism. From the invariant “free” action \( S_I = \int \sqrt{d\tau} \) for a “non-interacting particle” with mass \( m \) in the rotating frame \( F(\Omega) \), the spatial components of the physical momentum are

\[
p_i = -\frac{\partial L}{\partial v_i} = mP_{\nu i} \frac{dx^\nu}{ds} = P_{\nu i} p^\nu, \quad L = -m\sqrt{P_{\mu \nu} v^\mu v^\nu},
\]

where \( i = 1, 2, 3 \). Both \( L \) and \( p_i \) have the dimension of mass and \( v^\mu \equiv dx^\mu/d\tau = (1, v^i) \). The zeroth component \( p_0 \) (or the Hamiltonian) with the dimension of mass is defined as usual,

\[
p_0 = \nu^i \frac{\partial L}{\partial v^i} - L = mP_{\nu 0} \frac{dx^\nu}{ds} = P_{\nu 0} p^\nu.
\]

Thus, the covariant momentum \( p_\mu \) defined in (28) and (29) is the physical momentum of a particle in the rotating frame.

The rotational transformations of the differential operators \( \partial/\partial x_0^\mu \) and \( \partial/\partial x^\mu \) can be calculated from (13). Just as in quantum mechanics, the covariant momentum \( p_\mu \) has the same transformation properties as the covariant differential

\(^7\) For comparison with the usual time \( t \), if one were to define \( w = ct \) and \( w_I = ct_I \), then one would have the usual relation \( \Delta t_I = \gamma \Delta t \). However, the constant “speed of light” \( c \) in a non-inertial frame is not well defined because the speed of a light signal is no longer isotropic or constant.
operator $\partial/\partial x^\mu$. From (13), we obtain

$$
p_{10} = \gamma^{-1}(p_0 + \Omega y p_1 - \Omega x p_2),
p_{11} = [-\gamma^{-2} \Omega^2 x_2] p_0 + \gamma^{-2} \left[ \gamma \cos(\Omega w) - \Omega^2 x_1 x - \Omega^3 x_1 y \right] p_1
+ \gamma^{-2} \left[ -\gamma \sin(\Omega w) - \Omega^2 x_1 y + \Omega^3 x_1 x \right] p_2,
p_{12} = [-\gamma^{-2} \Omega^2 y_2] p_0 + \gamma^{-2} \left[ \gamma \sin(\Omega w) - \Omega^2 y_1 x - \Omega^3 y_1 y \right] p_1
+ \gamma^{-2} \left[ -\gamma \cos(\Omega w) - \Omega^2 y_1 y + \Omega^3 y_1 x \right] p_2,
p_{13} = p_3, \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}},
$$

(30)

where $x_I$ and $y_I$ can be expressed in terms of $x$, $y$, and $w$ of the rotating frame $F(\Omega)$ using (11).

Consider the case of a particle traveling at high speed in a circular storage ring. Such a particle can be considered to be at rest in a rotating frame $F(\Omega)$, so that $d x^i = 0$ and hence, $d s = d w$. Based on $p^\mu = m dx^\mu/ds$ in (28), the contravariant momenta are $p^i = 0$, $i = 1, 2, 3$, and $p^0 = m$. In this case, the covariant momenta of this particle in $F$ are

$$
p_0 = m, \quad p_1 = m \gamma^2 \Omega y, \quad p_2 = -m \gamma^2 \Omega x, \quad p_3 = 0,
$$

(31)

where we have used the relations $p_\mu = P_{\mu\nu} P^\nu$, $dx^i = 0$, $p^i = (x, y, 0)$ is fixed, $\rho^2 = \text{const}$. Since both $\rho$ and $\gamma$ are constant in the rotational transformations (11) and (13), eq. (14) will be greatly simplified and, hence, (17), (18), and (30) will be simplified, too. As a result, we also have very simple relations in (31). This difference between $p_\mu$ and $p^\mu$ is mainly due to the non-vanishing metric tensor components $P_{10}$ and $P_{02}$ in the rotating frame. The physical momenta of the particle, as measured in an inertial frame $F_1$, are given by (30) and (31),

$$
p_{10} = \gamma m, \quad p_{11} = m \gamma [\Omega x \sin(\Omega w) + \Omega y \cos(\Omega w)],
p_{12} = -m \gamma [\Omega x \cos(\Omega w) - \Omega y \sin(\Omega w)],
p_{13} = 0.
$$

(32)

Thus, we have seen that the expression for the energy of a rotating particle $p_{10} = \gamma m$ agrees with the well-established results of high-energy experiments performed in an inertial laboratory frame $F_1$.

The transformation laws of the covariant wave vector $k_\mu$ are also given by (30) because a photon’s momentum $p_\mu$ and wave vector $k_\mu$ are related by $p_\mu = \kappa_\mu$ (in natural units).

6 Experimental tests of the rotational taiji transformations

Because particles moving in a straight line at relativistic speeds travel large distances in very short times, designing experiments to test relativistic effects with linear motion, where the Lorentz transformations are most directly applicable, can be challenging. Experiments in which objects move in a circular path at relativistic speeds can be performed with much more compact apparatus. However, as was discussed briefly in the previous section, applying the Lorentz transformations to analyzing such experiments is problematic because of the non-relativity between inertial frames and rotating frames, as shown in the transformations (10), (11) and (13). In this section, we discuss three experiments that can test the usefulness of the rotational taiji transformations (10) or (11).

6.1 Absolute dilation of decay length for particle decay in circular motion

Let us first consider the lifetime dilation of unstable particles traveling in a circular storage ring with a constant radius [10]. If the particle’s rest lifetime in the rotating frame $F(\Omega) = F(w, x, y, z)$ is denoted by $\Delta w_{\text{(rest)}}$, then the rotational taiji transformation (11) gives

$$
\Delta w_1 = \gamma \Delta w_{\text{(rest)}}, \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}}, \quad \beta = \Omega \rho.
$$

(33)

But the rest lifetime of the particle $\Delta w_{\text{(rest)}}$ in the rotating frame $F$ cannot be directly measured in the inertial laboratory. However, according to the weak equivalence of non-inertial frames (see eq. (A.2) in the appendix) [2, 4], we have the relation

$$
\Delta w_{\text{(rest)}} \sqrt{P_{00}} = \Delta w_1_{\text{(rest)}} \sqrt{\eta_{00}}.
$$

(34)
Since the lifetime is typically very short, \( \Delta \nu_{\text{rest}} \) can be considered as the zeroth component of the contravariant differential coordinate vector \( dx^0; \Delta \nu = d\nu = dx^0 \). It follows from (33) and (34) that

\[
\Delta \nu_I = \gamma \Delta \nu_{\text{rest}},
\]

where we have used \( P_{\mu \nu} \) in (17). Result (35) is consistent with well-established experimental results of the decay lifetime (or decay length) dilation of muons in flight in a circular storage ring [5,10].

6.2 Davies-Jennison experiment 1 – null transverse frequency shift

An experiment that can test the usefulness of the taiji rotational transformations is the Davies-Jennison experiment 1 [8,16], in which the shift in frequency of transversely emitted radiation from an orbiting source was measured. The arrangement of the apparatus for this experiment is roughly as follows: A laser beam is directed straight downward onto a mirror at the center of a horizontal rotating table. The mirror sends the beam radially outward, parallel to the surface of the table, where it strikes a second mirror mounted at the edge of the table. This second mirror reflects the beam in a direction perpendicular to its velocity, thus acting as a source emitting radiation in the transverse direction. This resultant beam is then mixed with light from the original laser in order to produce interference fringes and measure any shift in its frequency compared to the original laser light. In the Davies-Jennison experiment 1, no such shift was observed. Let us analyze this experiment using the rotational taiji transformations.

Consider a radiation source at rest in a rotating frame \( F(\Omega) \), located at \( \rho_s = (x_s, y_s) \), that emits a wave with a frequency \( k_0 = k_0(\text{rest}) \), as measured in \( F(\Omega) \). If this wave, travels along the \( -\rho \) direction to the origin, then \( k_1/k_2 = k_2/k_2 = x/y \), because the two vectors \( -\rho \) and \( k \) are on the \( x-y \) plane and parallel to each other. Because the rotational taiji transformations for the covariant wave vector \( k \), are the same as those for the covariant momentum \( p_\mu = \hbar k_\mu \), the frequency shift of transversely emitted radiation as measured by observers in an inertial laboratory frame \( F_I \) is given by (30). To detect the transverse effect, the observer is assumed to be at rest at the origin \( x_I = x_I = x = y = 0 \). Equation (30) with \( p_\mu = \hbar k_\mu \) gives the relation for frequencies,

\[
k_{10} = \gamma^{-1}(k_0 + \Omega[yk_1 - xk_2]) = \gamma^{-1}k_0,
\]

\[
xk_2 = yk_1, \quad \gamma = \frac{1}{\sqrt{1 - \rho_s^2 \Omega^2}}, \quad \rho_s^2 = x_s^2 + y_s^2 = \text{const},
\]

where \( k_0 = k_0(\text{rest}) \) is the frequency of the source at rest in the rotating frame \( F(\Omega) \), as measured by observers in \( F(\Omega) \). Just as in our discussion of the lifetime dilation of unstable particles traveling in a circle, it is extremely difficult to measure the frequency \( k_0(\text{rest}) \) directly because we cannot move our measuring apparatus to the rotating frame \( F(\Omega) \). Furthermore, because of the non-equivalence of inertial and rotating frames, we cannot assume that \( k_0(\text{rest}) \) is equal to \( k_{10}(\text{rest}) \), the frequency of the radiation from the same source at rest in an inertial frame \( F_I \) as measured by observers in \( F_I \). However, we can again use the weak equivalence of non-inertial frames (cf. eq. (A.2) in the appendix) [4,15] to write

\[
k_{00}(\text{rest})\sqrt{P_{00}} = k_{10}(\text{rest})\sqrt{\eta_{00}}, \quad P_{00} = \gamma^{-2}, \quad \eta_{00} = 1.
\]

Recall that \( k_{00}(\text{rest}) \) is the frequency of radiation emitted from the source at rest in the rotating frame \( F(\Omega) \), located at the constant radius, \( \rho = \text{fixed} \) on the \( x-y \) plane. When this condition is used in the derivation of (14), then we have \( d\nu_I = \gamma d\nu, \text{etc., so that the contravariant component} \quad \eta_{00} \quad \text{reduces to} \quad P_{00} = \gamma^{-2} = \text{const \ in} \quad (37)^8 \). It follows from (36) and (37), that the relationship between the frequency of the transversely emitted radiation emitted from the orbiting source as measured in the inertial laboratory frame \( k_{10} \) and the frequency of the radiation emitted from the same source at rest in the inertial laboratory frame as measured in the inertial laboratory frame \( k_{10}(\text{rest}) \) is given by

\[
k_{10} = \gamma^{-1}k_{00}(\text{rest}) = k_{10}(\text{rest}).
\]

Thus, the rotational taiji transformation predicts the result (38), which implies that no frequency shift for transverse radiation should be observed for an orbiting radiation source, as measured by observers in inertial frames. This result (38) is consistent with Davies-Jennison experiment 1 [8,16].

6.3 Davies-Jennison experiment 2 – longitudinal frequency shifts

Another experiment that can test the taiji rotational transformations is the longitudinal frequency shift of an orbiting source as observed in inertial laboratory. Consider a radiation source at rest in a rotating frame \( F(\Omega) \), located at

\[\text{Footnote 8}: \text{For details, see equations in (39) below.}\]
\( \rho^i = (x, y, 0) = \text{const}, \) that emits a wave with a frequency \( k_0 = k_0(\text{rest}) \), as measured in \( F(\Omega) \). Let us consider the rotational transformations with the condition \( \rho^2 = x^2 + y^2 = \text{const} \), one can show that the rotational transformations for the coordinate differentials \( dz^\mu \) and the results for the metric tensors \( P_{\mu\nu} \) and \( P^{\mu\nu} \) in (17) and (18) will be simplified. For our discussions, it suffices to note that \( P^{\mu\nu} \) in (18) becomes

\[
\begin{align*}
P^{00} &= \gamma^{-2}, & P^{11} &= -\gamma^{-2}(1 - \Omega^2 y^2), & P^{22} &= -\gamma^{-2}(1 - \Omega^2 x^2), \\
P^{01} &= \gamma^{-2} \Omega y, & P^{02} &= -\gamma^{-2} \Omega x, & P^{12} &= \gamma^{-4} \Omega^2 xy,
\end{align*}
\]

for constant \( \rho \). Similarly, the transformations of wave vector \( k_\mu(= p_\mu/\hbar) \) will be simpler than (30),

\[
\begin{align*}
k_{10} &= \gamma^{-1}(k_0 + \Omega y k_1 - \Omega x k_2), \\
k_{11} &= \gamma^{-1} (k_1 [\cos(\Omega w) + \Omega^4 x y \rho^2 \sin(\Omega w)] \\
&- k_2 [\sin(\Omega w) + \Omega^4 x y \rho^2 \cos(\Omega w)]), \\
k_{12} &= \gamma^{-1} (k_1 [\sin(\Omega w) - \Omega^4 x y \rho^2 \cos(\Omega w)] \\
&+ k_2 [\cos(\Omega w) - \Omega^4 x y \rho^2 \sin(\Omega w)]).
\end{align*}
\]

We now concentrate on the beams in the \( \pm x_1 \) directions (i.e., \( k_{12} = k_{13} = 0 \)), as observed in the laboratory. In this case, we also have \( k_2 = k_3 = 0 \). For the light beam emitted at time \( w = 0 \) from the position \( (x, y) = (0, 0) \), transformation (40) leads to the longitudinal frequency shift,

\[
k_{10} = \gamma^{-1}(k_0 + \Omega y k_1),
\]

where \( k_0 = k_0(\text{rest}) \) is the frequency of the source at rest in the rotating frame \( F(\Omega) \), as measured by observers in \( F(\Omega) \).

Because of the non-equivalence of inertial and rotating frames, we cannot assume that \( k_0(\text{rest}) \) is equal to \( k_{10}(\text{rest}) \), the frequency of the radiation from the same source at rest in an inertial frame \( F_i \) as measured by observers in \( F_i \). However, we can again use the weak equivalence of non-inertial frames (37) to obtain the relation

\[
k_0(\text{rest}) \gamma^{-1} = k_{10}(\text{rest}).
\]

The law of propagation of light in \( F(\Omega) \), \( k_0 k_\mu F^{\mu\nu} = 0 \), with the source at \( (x, y) = (0, 0) \) with \( \rho^2 = x^2 + y^2 = \text{const} \), leads to

\[
k_0^2 - k_0^2 \gamma^{-2} + 2\Omega k_0 k_1 = 0, \quad k_2 = k_3 = 0,
\]

where we have used the metric tensor \( F^{\mu\nu} \) in (39).

From eq. (43), we solve for \( k_1 > 0 \),

\[
k_1 = \gamma^2 k_0 (1 + \Omega y), \quad \gamma^2 = \frac{1}{1 - \Omega^2 \rho^2}, \quad \rho^2 = y^2.
\]

It follows from (41), (42) and (44) that the exact rotational space-time transformations predict the orbiting frequency shift to be

\[
k_{10} = k_{10}(\text{rest}) \left[ 1 + y \Omega \gamma^2 (1 + \Omega y) \right] \approx k_{10}(\text{rest}) \left[ 1 + \rho_1 \Omega t + \rho_1^2 \Omega t^2 \ldots \right], \quad \rho \Omega = \rho_1 \Omega t,
\]

where we have used \( k_0 = k_0(\text{rest}) \) for a source at rest in \( F(\Omega) \) and express \( y \Omega \) in terms of quantities measured in the inertial laboratory frame \( F_i \), \( y \Omega = y_i \Omega t_i \), according to (12) with \( \rho = y \) and \( \rho_1 = y_i \). Thus, the exact rotational \( \text{taiji} \) transformations predict a new frequency shift (45) for an orbiting source emitting light in the longitudinal direction, as measured by observers in inertial frames. The result (45) is in sharp contrast to the usual relativistic Doppler shift in special relativity for radiation source moving with \textit{constant velocities}.

The arrangement of the apparatus for the Davies-Jennison experiment 2 is similar to their experiment 1 discussed in sect. 6.2. A small modification is that the second mirror (mounted at the edge of the rotating table) reflects the beam in a direction parallel to its velocity, thus acting as a source emitting radiation in the longitudinal direction. The Davies-Jennison experiment 2 does not measure the frequency shift (45) directly [9,16]. Rather, the experiment sends the reference beam of the interferometer directly to the detector while the other beam is reflected in turn from a mirror moving towards the beam and then from a mirror moving away from the beam, where both beams are parallel to the \( x_j \). Suppose the mirror moving away from the beam is at \( (x, y) = (0, -R_s) \) and the mirror moving towards the beam is at \( (x, y) = (0, +R_s) \). The frequency of this longitudinal beam is then compared to the reference beam in the laboratory.
The shifts of frequencies at these two locations of orbiting mirrors are, respectively, given by eq. (41) with \((x, y) = (0, -R_s)\) and \((x, y) = (0, +R_s)\),

\[
k_{I0} = \gamma^{-1}(k_0 + \Omega y k_1) = k_{I0}(\text{rest}) \left(1 + \Omega y \frac{k_1}{k_0}\right), \quad y = -R_s, \quad \text{(46)}
\]

\[
k_{I0} = \gamma^{-1}(k_0 + \Omega y k_1) = k_{I0}(\text{rest}) \left(1 + \Omega y \frac{k_1}{k_0}\right), \quad y = +R_s. \quad \text{(47)}
\]

Here, the ratio \(k_1/k_0\) is determined by the law \(k_\mu k_\nu P^{\mu\nu} = 0\) for light propagation in the rotating frame, where \(P^{\mu\nu}\) is given by (39). Since \(k_1\) is given by (44), the results in (46) and (47) show that the blue-shift is completely cancelled by the red-shift, as observed in the inertial laboratory frame \(F_I\), so that the rotational transformations predict a null frequency shift, \(k_{I0} = k_{I0}(\text{rest})\), consistent with the null result of the Davies-Jennison experiment 2 [16]. A main physical reason for this null frequency shift is the weak equivalence of non-inertial frames, as expressed by the relation (42).

It would be an interesting challenge to measure the predicted frequencies in (46) and (47) separately to test further the exact rotational taiji transformations (11) and the weak equivalence of non-inertial frames.

Therefore, rather than serving as a test of special relativity, the rotational Davies-Jennison experiments can reveal new principles of physics for non-inertial frames, which are beyond the realm of special relativity\(^9\). They can also help to develop a new and deeper understanding of physics in non-inertial frames based on the principle of limiting Lorentz-Poincaré invariance and the weak equivalence of non-inertial frames.

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**Appendix A. Weak equivalence of non-inertial frames**

Suppose a particle is at rest in an inertial frame \(F_I'\) and has a lifetime \(\tau_I'(\text{rest})\), as measured by observers in \(F_I'\). From the viewpoint of observers in another inertial (laboratory) frame \(F_I\), this particle will decay in flight and has a different lifetime \(\tau_I\), as measured by observers in \(F_I\). The Lorentz transformations give the usual lifetime dilation \(\tau_I = \gamma_\sigma \tau_I'(\text{rest})\). Experimentally, it is difficult to measure \(\tau_I'(\text{rest})\) because it is almost impossible to arrange for an observer to be in a frame that is co-moving with unstable particles in high-energy laboratory. Fortunately, according to the principle of relativity, all inertial frames are equivalent. Therefore, the lifetime \(\tau_I(\text{rest})\) of a particle at rest in \(F_I\) and measured by observers in \(F_I\) is the same as the lifetime \(\tau_I'(\text{rest})\) of the same kind of particle at rest in \(F_I'\) and measured by observers in \(F_I'\),

\[
\tau_I(\text{rest}) = \tau_I'(\text{rest}). \quad \text{(A.1)}
\]

Thus, the lifetime dilation (3.34) in special relativity can be expressed in terms of observable quantities in an inertial frame \(F_I\), \(\tau_I = \gamma_\sigma \tau_I'(\text{rest})\), which has been confirmed by high-energy experiments in the laboratory frame \(F_I\).

Now, we consider the situation involving non-inertial frames \(F\) and \(F'\), which are not equivalent to an inertial frame \(F_I\). In order to calculate the lifetime of an accelerated particle to compare it with experiment, we must generalize the relation \(\tau_I(\text{rest}) = \tau_I'(\text{rest})\) in (A.1) for inertial frames to one for non-inertial frames.

Let us consider observable quantities which transform as vectors. These physically observable vector quantities (e.g., the energy momentum of a particle with mass, or the wave vector of radiation emitted from a source, etc.) can be denoted by contravariant or covariant vectors, \(V^\mu(\text{rest})\) and \(V_\mu(\text{rest})\), where (rest) denotes that the particle or the source of radiation is “at rest” in a non-inertial frame \(F\) and these quantities are measured by observers in the same

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9 If the cylindrical coordinates is used for the rotating frame \(F(\Omega)\), the law for light propagation, \(k_\mu k_\nu P^{\mu\nu} = 0\), implies explicitly that the ratio \(k_1/k_0\) depends on the constant radius \(r > 0\) in this case (where \(k_1\) is understood as proportional to the tangential wave vector).

10 It appears that general relativity will not be able to give a clearcut analysis of rotational experiments. The reason is that the coordinates in general relativity do not have an operational meaning. See, for example, E.P. Wigner, *Symmetries and Reflections, Scientific Essays* (Cambridge, The MIT Press, 1967) pp. 52–53. Wigner explained: “Evidently, the usual statements about future positions of particles, as specified by their coordinates, are not meaningful statements in general relativity. This is a point which cannot be emphasized strongly enough and is the basis of a much deeper dilemma than the more technical question of the Lorentz invariance of the quantum field equations. It pervades all the general theory, and to some degree we mislead both our students and ourselves when we calculate, for instance, the mercury perihelion motion without explaining how our coordinate system is fixed in space... Expressing our results in terms of the values of coordinates became a habit with us to such a degree that we adhere to this habit also in general relativity, where values of coordinates are not per se meaningful.”
non-inertial frame $F$. Following the principle of limiting continuation for physical laws, it is natural to postulate the 
following generalized relations for two different non-inertial frames, $F$ and $F'$,
\[ V^0(\text{rest}) \sqrt{P_{00}} = V'^0(\text{rest}) \sqrt{P'_{00}}, \]
(A.2)
for contravariant vector such as differential coordinate vectors, or
\[ V^0(\text{rest}) \sqrt{P_{00}} = V'^0(\text{rest}) \sqrt{P'_{00}}, \]
(A.3)
for covariant vectors such as momentum and wave vectors. These postulates may be termed the “weak equivalence of 
non-inertial frames”. These relationships (A.2) and (A.3) cannot be derived from space-time transformations for 
non-inertial frames because they refer to two different vectors (or physical quantities) which are not connected by 
the space-time coordinate transformations. However, they are consistent with the principle of limiting continuation of 
physical laws and the relations can be tested experimentally.

It appears that the Davies-Jennison experiments 1 and 2 discussed previously in sect. 6 and Thim’s rotational 
experiment [17] support the weak equivalence of non-inertial frames, as postulated in (A.2) and (A.3). We remark 
that for the special case, where $P_{\mu\nu} = 0$ for $\mu \neq \nu$, (A.2) is the same as (A.3). This property of the metric tensor 
$P_{\mu\nu}$ shows up if the non-inertial frames are constant-linear-acceleration frames, whose space-time transformations are 
given by the accelerated Wu transformations [4,5]. Also, (A.2) and (A.3) are consistent with (A.1) in the limit of zero 
acceleration, where all Poincaré metric tensors $P_{\mu\nu}$ for non-inertial frames reduce to the Minkowski metric tensor, 
$\eta_{\mu\nu} = (1, -1, -1, -1)$ for inertial frames.

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