Temperature Renormalization Group
and Resummation

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Abstract

The temperature renormalization group equation (TRGE) is compared with a diagrammatic expansion for the \((\phi^4)_4\)-theory. It is found that the one-loop TRGE resums the leading powers of temperature for the effective mass. A two-loop contribution to TRGE is required to do the leading resummation for the coupling constant. It is also shown that the higher order TRGE resums subleading powers of temperature.

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1 Introduction

The temperature renormalization group equation (TRGE) has been used by several authors to study the high temperature behaviour of different theories. With few exceptions, it has not been clear exactly what kind of improvement the TRGE implies. In some cases, a 1-loop approximation of the effective potential was used to determine the renormalization flow of the mass and coupling constant at zero momentum. This gives the first correction in \( \bar{\hbar} \) but not necessarily the leading correction in \( T/m \) as we shall see. The purpose of this paper is to give a more detailed discussion of the TRGE for the \((\phi^4)_4\)-theory.

When calculating thermal corrections perturbatively in a QFT at finite temperature it is often found that the 1-loop result goes like a power of \( T \) at high temperature. The correction is, therefore, always large for large enough \( T \) which makes the perturbation approach unreliable. The self-energy, for instance, is the correction to the effective (mass)\(^2\) of a particle surrounded by a heat-bath. It gets a correction going like \( T^2 \) at high \( T \) in the \((\phi^4)_4\)-theory. If the quasi-particles in the thermal environment have a mass very different from the zero temperature one, it is clear that it is not a good approximation to expand around the zero temperature excitations. On the other hand, expanding around quasi-particles at one temperature to calculate the self-energy at a temperature that is close has a better chance to succeed. Also, a sequence of many small steps in temperature is intuitively more likely to give a better result than taking only one big step. The situation is familiar from zero temperature calculations where the mass can be renormalized at different scales \( \mu \). Using a mass defined at \( \mu \) to calculate the mass at the scale \( \mu' \) gives rise to \( \log(\mu'/\mu) \) terms which can be large. There the remedy is well-known as the renormalization group which sums up small steps in \( \mu \). The renormalization group equation (RGE) at \( T = 0 \) can be formally derived from the freedom of choosing \( \mu \) in the renormalization prescription. At finite temperature, the mass of the quasi-particles, around which to expand, can be defined at any temperature and there is a freedom of choosing that temperature which should not influence on the final answer. This freedom can be used to derive the TRGE which implements the summation over small temperature steps.

The result from solving the 1-loop TRGE is certainly different from just computing a 1-loop correction but it is not obvious exactly what the difference is. It is known that a resummation of leading \( \log(\mu) \) terms is obtained by solving the RGE at zero temperature. Since there is a similarity between \( \log(\mu) \) and \( T \) it is natural to guess that the
TRGE resums the leading powers of $T$. This is true for the $(\phi^4)_4$-theory and that is the basic result of this paper. It turns out, however, that the TRGE can only resum certain subdiagrams in an effective way. By effective I mean getting a result to the $n$-th order without doing an $n$-th order calculation. The diagrams that are effectively resummed are the ones containing loops that give a leading power of $T^2$ (hard thermal loops). These terms can also be resummed using an effective propagator with a $T$ dependent mass [7]. The relation between the two methods is discussed in Section 3.1.

In Section 2 the thermal $\theta$ and $\beta$ functions are discussed and their relation to the diagrammatic expansion of $m(T)$ and $g(T)$. The TRGE to leading order is solved in Section 2.4.

2 Resummation of leading powers of $T$

2.1 General ideas

Our goal is to calculate the mass and the coupling as functions of the temperature $(m(T), g(T))$. The meaning of $m$ and $g$ depends on the renormalization conditions (RC). To give a physical interpretation of $m(T)$ we define it to be the real part of the pole of the propagator of physical excitations. This assumes that it is meaningful to talk about single stable excitations which may be difficult at finite temperature when the excitations eventually dissipate away, but we take it as an approximation. Adding an imaginary part to the mass would take into account the dissipation. In some cases it may be needed to consider excitations with a spectrum that differs drastically from the free one, but we do not discuss that here (see Section 3.2). The coupling constant is related to experiment through some scattering process and we define it to be the real part of the 4-point function with external momenta corresponding to that scattering. For simplicity we assume that the momentum depends only on the mass and some parameters that are kept fix, such as scattering angles. The renormalization point is then denoted by $p_{iR} = p_i(m_R)$. Thus the renormalization condition is

$$\text{Re}\Gamma^{(2)}(p, T)|_{p^2=m_R^2, \ T\to0} = 0,$$

Perturbative propagators with complex poles do not admit any spectral representation, which the full propagator can be shown non-perturbatively to have (at least at equilibrium). It may therefore be necessary to consider some generalization of the TRGE to the spectral function if dissipation is to be taken into account consistently. I want to thank Dr. Umezawa for pointing out this to me.
In addition, the wave function has to be renormalized and we choose
\[ \frac{\partial}{\partial p_0^2} \text{Re} \Gamma^{(2)}|_{p^2=m_R^2, T=T_R} = -1 , \]
as RC. Other conditions (like the MS scheme) can be imposed if the only purpose is to eliminate the UV divergences. Here we also want to give the expansion parameters \( m_R \) and \( g_R \) a physical meaning.

From the RC the corresponding \( \theta \), \( \beta \) and \( \gamma \) functions can be defined
\[ \theta(g_R, m_R, T_R) = - \frac{1}{2m_R} \frac{\partial}{\partial T} \text{Re} \Gamma^{(2)}(p_R, T)|_{T=T_R} , \]
\[ \beta(g_R, m_R, T_R) = 4g_R - \frac{\partial}{\partial T} \text{Re} \Gamma^{(4)}(p_R, T)|_{T=T_R} , \]
\[ \gamma(g_R, m_R, T_R) = - \frac{1}{2} \frac{\partial}{\partial T} \frac{\partial}{\partial p_0^2} \text{Re} \Gamma^{(2)}(p, T)|_{p=p_R, T=T_R} . \]
The mass and coupling, defined through Eq.(1), are obtained by integrating \( \theta \) and \( \beta \),
\[ \begin{cases} \frac{dm}{dT} = \theta(g, m, T) \\ \frac{dg}{dT} = \beta(g, m, T) \end{cases} \Rightarrow \begin{cases} m = m(T; g_R, m_R, T_R) \\ g = g(T; g_R, m_R, T_R) \end{cases} , \]
and we get in this way the zero of \( \text{Re} \Gamma^{(2)}(p, T) \) and the value of \( \text{Re} \Gamma^{(4)}(p_R, T) \) as functions of \( g_R, m_R, T_R \) and \( T \).

The same functions could in principle be calculated by the diagrammatic perturbation series as an expansion in \( g_R \). The idea here is to compute \( \theta \) and \( \beta \) perturbatively and then solve the non-linear ordinary differential equations Eq.(1) which are non-perturbative relations. To study the relation between the two approaches it is convenient to use the invariance under a change of the initial conditions to write the linear partial differential equations
\[ \left( \frac{\partial}{\partial T_R} + \theta(g_R, m_R, T_R) \frac{\partial}{\partial m_R} + \beta(g_R, m_R, T_R) \frac{\partial}{\partial g_R} \right) m(T; g_R, m_R, T_R) = 0 , \]
\[ \left( \frac{\partial}{\partial T_R} + \theta(g_R, m_R, T_R) \frac{\partial}{\partial m_R} + \beta(g_R, m_R, T_R) \frac{\partial}{\partial g_R} \right) g(T; g_R, m_R, T_R) = 0 . \]

\[^{3}\text{The definition of } \theta, \beta \text{ and } \gamma \text{ differs from Ref.[1] by factors of } T_R \text{ and } m_R.\]
If now $\theta$ and $\beta$ are computed to finite order in $g_R$ what do the corresponding solutions for $m$ and $g$ mean in terms of perturbation theory? To answer that question we make an Ansatz for the solution as a power series in $g_R$ and determine the coefficients. Before doing that we shall take a closer look at the leading $T$ behaviour of $\theta$ and $\beta$. We must also be aware that, by making the Ansatz above, we assume a power series expansion of $m$ and $g$ around $g_R = 0$.

### 2.2 Power series for $\theta$ and $\beta$

We are interested in the leading $T$ dependence of $\theta$ and $\beta$ at each order in $g_R$, so we have to find the diagrams that contribute most. According to Eq. (3) the $\theta$ and $\beta$ functions are obtained by taking suitable $T$ and $p$ derivatives of $\text{Re}\Gamma^{(2)}$ and $\text{Re}\Gamma^{(4)}$ and putting $p = p_R, \quad T = T_R$ afterwards. The $\text{Re}\Gamma^{(2)}$ and $\text{Re}\Gamma^{(4)}$ should be renormalized with the physical self-consistent RC in Eqs. (1, 2). This turns to be crucial.

The dominant diagram at high $T$ is the tadpole correction to the mass which gives a factor $T^2$. All other loops give a factor $T$. Therefore, the dominant diagrams to each order in $g_R$ are the ones with as many tadpole subdiagrams as possible. This is true for $\Gamma^{(N)}$. However, for each such tadpole there is a corresponding diagram with a counterterm defined at $T_R$ (see Fig. (1)).

Let us denote the tadpole contribution by $I(T, m_R)$. The sum of all diagrams with $n$ tadpoles and their counterterms contains a factor $(I(T, m_R) - I(T_R, m_R))^n$. Evidently, after taking the the $T$ derivative and putting $T = T_R$, this factor vanishes for all $n \geq 2$. We conclude that only diagrams with at most one tadpole as subdiagram contributes to $\theta$ and $\beta$ and thus the $n$-th order term in for $\theta$ and $\beta$ goes like $g_R^n T^n$ and $g_R^{n+1} T^n$ respectively.
We write the high $T$ expansion as
\[
\beta\left(\frac{T_R}{m_R}\right) = \frac{g_R}{m_R} \sum_{k=1}^{\infty} g^k_R \left(\frac{T_R}{m_R}\right)^k \sum_{p=0}^{\infty} \left(\frac{T_R}{m_R}\right)^{-p} \beta_{kp},
\]
\[
\theta\left(\frac{T_R}{m_R}\right) = \sum_{k=1}^{\infty} g^k_R \left(\frac{T_R}{m_R}\right)^k \sum_{p=0}^{\infty} \left(\frac{T_R}{m_R}\right)^{-p} \theta_{kp}.
\]
where $k$ is the order to which $\theta$ and $\beta$ are calculated. By looking at low order diagrams we see that $\beta_{10}$ is zero which is important in next section.

In addition to the terms in Eq. (6) there are subleading terms like $T^m \ln^n T$. If they are considered to be of the same order as $T^m$ independently of $n$, they do not affect the ordering of terms with different powers of $T$ that is used in next section. They also do not occur to the leading order. Therefore, I do not write them out since they would only complicate the formulas.

2.3 Power series for $m$ and $g$

The purpose of this section is to relate the solutions of Eq. (5) to the perturbative calculation of $\theta$ and $\beta$. A diagrammatic expansion of $m$ and $g$ yields a power series in $g_R$ so we make the Ansatz
\[
m(T; g_R, m_R, T_R) = m_R \sum_{n=0}^{\infty} g^n_R M_n\left(\frac{T}{m_R}; \frac{T_R}{m_R}\right),
\]
\[
g(T; g_R, m_R, T_R) = g_R \sum_{n=0}^{\infty} g^n_R G_n\left(\frac{T}{m_R}; \frac{T_R}{m_R}\right),
\]
where $M_n$ and $G_n$ only depend on $\frac{T}{m_R}$ and $\frac{T_R}{m_R}$ for dimensional reasons. This is the reason why we related $p_R$ to $m_R$: it reduces the number of dimensionful variables. Of course, we could have related $p_R$ to $T_R$ and get the same simplification here but with an other physical interpretation of $m$ and $g$. The leading behaviour of $\theta$ and $\beta$ in last section was considered for a $p_R$ which does not depend on $T_R$. We could also have treated the renormalization group equation in both $p_R$ and $T_R$ as proposed in [1].

Our primary interest is to see how the $T$ dependence in $\theta$ and $\beta$ gives a $T$ dependence in $m$ and $g$ so we further expand $M_n$ and $G_n$
\[
M_n\left(\frac{T}{m_R}; \frac{T_R}{m_R}\right) = \sum_{q,l} \left(\frac{T_R}{m_R}\right)^q \left(\frac{T}{m_R}\right)^l M_{nq l},
\]
\[
G_n\left(\frac{T}{m_R}; \frac{T_R}{m_R}\right) = \sum_{s,u} \left(\frac{T_R}{m_R}\right)^s \left(\frac{T}{m_R}\right)^u G_{ns u}.
\]
where the summation range for \( q, l, s \) and \( u \) is yet to be determined. The comment at the end of last section regarding possible \( \ln T \) factors apply to this expansion as well.

From Eq.(8) and Eq.(5) we find

\[
qM_{nql} = -\sum_{k=1}^{\infty} \sum_{p=0}^{\infty} M_{n-k,q-1+p-k,l}[\theta_{kp}(2+k-p-q-l) + \beta_{kp}(n-k)] ,
\]

\[
sG_{nsu} = -\sum_{k=1}^{\infty} \sum_{p=0}^{\infty} G_{n-k,s-1+p-k,u}[\theta_{kp}(1+k-p-s-u) + \beta_{kp}(n-k+1)] ,
\]

by identifying the coefficients of the same powers of \( g_R, T_R, \) and \( T \). The Eq.(9) determines \( M_{nql} \) (\( G_{nsu} \)) in terms of \( M_{n'ql} \) (\( G_{n'su} \)) where \( n' < n \), and it can be applied inductively except for \( q = 0 \) (\( s = 0 \)) \(^4\).

We are interested in the \( T \) dependence but Eq.(5) is a differential equation in \( T_R \). The \( T \) dependence enters through the boundary conditions which also determines \( M_{nql} \) (\( G_{nsu} \)) for \( q = 0 \) (\( s = 0 \)). The boundary conditions are

\[
m(T_R; g_R, m_R, T_R) = m_R ,
g(T_R; g_R, m_R, T_R) = g_R ,
\]

or in terms of \( M_n \) and \( G_n \)

\[
M_n\left(\frac{T_R}{m_R};\frac{T_R}{m_R}\right) = 0 \ , \ n \geq 1 \ ; \ \ M_0 = 1 ,
\]

\[
G_n\left(\frac{T_R}{m_R};\frac{T_R}{m_R}\right) = 0 \ , \ n \geq 1 \ ; \ \ G_0 = 1 .
\]

They lead to

\[
M_{n0l} = -\sum'_{r} M_{n,r,l-r} ,
G_{n0u} = -\sum'_{v} G_{n,v,u-v} ,
\]

where \( \sum' \) means that \( r = 0 \) (\( v = 0 \)) is excluded.

We shall now determine the summation ranges for \( q, l \) and \( s, u \) in Eq.(8) using Eq.(9) and Eq.(12). It is obvious that there is no lower limit in general. We define the upper limit by

\[
Q(n, l) = \max q \text{ for which } M_{nql} \neq 0
\]

\[
L(n, q) = \max l \text{ for which } M_{nql} \neq 0 ,
\]

\(^4\) Again, when including \( \ln T \) terms this exception is modified.
\[ S(n, u) = \max s \text{ for which } G_{nsu} \neq 0 \]
\[ U(n, s) = \max u \text{ for which } G_{nsu} \neq 0 . \]  

(14)

When doing this we must be careful with the last factors in brackets in Eq.(8) which can be equal to zero for certain values of \( k, p, n, q, l, s \) and \( u \) (remember also that \( \beta_{10} = 0 \)). As noted before, their form is different when including possible \( \ln T \) factors. The easiest way to find \( Q, L, S \) and \( U \) is to do an explicit calculation for small \( n \), guess the general form and then prove it by induction. There are some comments about the procedure in Appendix A.

The initial condition for \( n = 0 \) is
\[
Q(0, 0) = 0 \quad ; \quad Q(0, l \neq 0) = -\infty ,
\]
\[
L(0, 0) = 0 \quad ; \quad L(0, q \neq 0) = -\infty ,
\]
\[
S(0, 0) = 0 \quad ; \quad S(0, u \neq 0) = -\infty ,
\]
\[
U(0, 0) = 0 \quad ; \quad U(0, s \neq 0) = -\infty .
\]

(15)

The result for \( n \geq 1 \) is
\[
Q(n, l < 0) = 2n - 2
\]
\[
Q(n, 0 \leq l \leq 2n) = 2(n - \lfloor \frac{l+1}{2} \rfloor)
\]
\[
Q(n, l > 2n) = -\infty ,
\]
\[
L(n, q < 0) = 2n - 2
\]
\[
L(n, 0 \leq q \leq 2n) = 2(n - \lfloor \frac{q+1}{2} \rfloor)
\]
\[
L(n, q > 2n) = -\infty ,
\]
\[
S(n, u < 0) = 2n - 2
\]
\[
S(n, u = 0) = 2n - 1
\]
\[
S(n, 1 \leq u \leq 2n - 1) = 2(n - 1 - \lfloor \frac{u}{2} \rfloor)
\]
\[
S(n, u > 2n - 1) = -\infty ,
\]
\[
U(n, s < 0) = 2n - 3
\]
\[
U(n, s = 0) = 2n - 1
\]
\[
U(n, 1 \leq s \leq 2n - 1) = 2(n - \lfloor \frac{s+1}{2} \rfloor) - 1
\]
\[
U(n, s > 2n - 1) = -\infty .
\]

(16)

(17)

(18)

(19)

Here, \( \lfloor q \rfloor \) means the largest integer less or equal to \( q \). These upper limits show that the leading \( T \) dependence for each order in \( g_R \) goes like
\[
m \propto n_R \sum_{n=0}^{\infty} g_R^n T^{2n}
\]
\[
g \propto g_R \sum_{n=0}^{\infty} g_R^n T^{2n-1}
\]

(20)
which could be seen directly from the diagrammatic expansion. We shall now find out what values of \( k \) that enters in the determination of the \( i \)-th subleading power of \( T \) in \( M_n \) and \( G_n \). Therefore, we look at the terms in Eq. (3) with \( l = 2n - i \) and \( u = 2n - 1 - i \), which are given by

\[
\sum_q \frac{\left(T_R\right)^q}{m_R} M_{n,q,2n-i} = - \sum_{q \leq Q(n,2n-i)} \frac{\left(T_R\right)^q}{m_R} \sum_{k=1}^{\infty} \sum_{p=0}^{q} \frac{1}{q} M_{n-k,q-1+p-k,2n-i}
\]

\[
\left[\theta_{kp}(2 + k - p - q - 2n + i) + \beta_{kp}(n - k)\right]
\]

\[
+ \sum_{r \leq Q(n,2n-i-r)} \sum_{k=1}^{\infty} \sum_{p=0}^{\infty} \frac{1}{r} M_{n-k,r-1+p-k,2n-i-r}\left[\theta_{kp}(2 + k - p - 2n - i) + \beta_{kp}(n - k)\right]
\]

(21)

\[
\sum_s \frac{\left(T_R\right)^s}{m_R} G_{n,s,2n-1-i} = - \sum_{s \leq S(n,2n-1-i)} \frac{\left(T_R\right)^s}{m_R} \sum_{k=1}^{\infty} \sum_{p=0}^{s} \frac{1}{s} G_{n-k,s-1+p-k,2n-1-i}
\]

\[
\left[\theta_{kp}(2 + k - p - s - 2n + i) + \beta_{kp}(n - k + 1)\right]
\]

\[
+ \sum_{r \leq S(n,2n-1-i-r)} \sum_{k=1}^{\infty} \sum_{p=0}^{\infty} \frac{1}{r} G_{n-k,r-1+p-k,2n-1-i-r}\left[\theta_{kp}(2 + k - p - 2n - i) + \beta_{kp}(n - k + 1)\right]
\]

(22)

The first term on the RHS is for \( q \neq 0 \) (\( s \neq 0 \)) and the second is for \( q = 0 \) (\( s = 0 \)). The general solution to the problem of finding what values of \( k \) contribute for a given value of \( i \) is rather complicated because of the many special cases that have to be considered. The cases of \( i = 0,1 \) are treated in next section. The way of solving the problem for general \( i \) is discussed in Appendix B, here I just state the result. To resum the \( i \)-th subleading powers of \( T \) using the TRGE, the \( \theta \) and \( \beta \) functions have to be calculated to the \( k \)-th order where \( k \) satisfies

\[
m: \quad k \leq i/2, \text{ all } p
\]

or \( p + k \leq i + 1 \) (23)
\[
g : \quad k \leq i/2, \text{ all } p \text{ \ or } p + k \leq i + 2
\]  
(24)

Remember that the \(i\) is defined as \(T^{2n-i}\) for \(m\) and as \(T^{2n-1-i}\) for \(g\).

A discussion of the result is given in Section 3 and here we just notice that the solution of the TRGE, using a perturbatively calculated \(\theta\) and \(\beta\) (i.e. finite \(k\)), indeed results in a resummation of leading and subleading factors of \(\frac{T}{m_R}\).

### 2.4 Special cases when \(i = 0, 1\)

The most important cases are, of course, the leading and first subleading contributions. We start with the leading terms for the mass which go like in Eq.(20). The analysis in Appendix E gives that only \(k = 1\), \(p = 0\) contribute, i.e. \(\theta_{10}\). That can also be seen from a diagrammatic expansion of \(m^2\); only the 1-loop tadpole give a \(g_R T^2\) dependence, all other diagrams give at most \(g_R n T^{2n-1}\). Therefore, the leading \(T\) dependence of \(m(T)\) is given by

\[
m(T) = \sqrt{m_R + g_R \theta_{10} (T^2 - T_R^2)}.
\]  
(25)

The next-to-leading order \((i = 1)\) requires the knowledge of \(\theta_{10}, \theta_{11}, \theta_{20}, \beta_{11}\) and \(\beta_{20}\) (from \(p + k \leq 2\), Eq.(23)) which gives a much more complicated TRGE to solve.

Let us continue with the expansion of \(g(T)\) which gives a more interesting result. It turns out that to leading order \((i = 0)\) it is necessary to calculate

\[\theta_{10}, \beta_{11} \text{ and } \beta_{20}\]

That is, to get the leading \(T\) dependence we must actually do a 2-loop calculation! Since the leading order TRGE can be solved explicitly we shall take a closer look at it to see what the actual values of \(\theta, \beta, m(T)\) and \(g(T)\) are. What we need to know is \(\theta_{10}, \beta_{11}\) and \(\beta_{20}\), and they are obtained from the diagrams in Fig.(2), where we use the notation

\[
F_q^p(T, m) = \int_0^\infty dk \frac{k^p}{\omega^q} \frac{1}{e^{\omega/T} - 1}, \quad \omega = \sqrt{k^2 + m^2}.
\]  
(26)

From these diagrams we find the TRGE

\[
\frac{dm}{dT} = \frac{g_T}{24m},
\]  
(27)

\(^5\) From Eq.(24) we have \(p + k \leq 2\), but \(\theta_{11}\) and \(\theta_{20}\) do not contribute since the bracket \([2+k-p-s-2n+i]\) vanishes, see Eq.(22). If we had included the \(\ln T\) terms the bracket would in general not be of the same form. However, to highest order there are no \(\ln T\) terms so the conclusion is correct.
\[
\frac{dg}{dT} = -\frac{3g^2}{16\pi m} + \frac{g^3T^2}{128\pi m^3}.
\]

The last equation can be written as an exact differential and the solution when \(T_R = 0\) is

\[
g(T) = \frac{g_R}{1 + \frac{3g_R}{16\pi m(T)}}.
\]

Then the equation for \(m(T)\) reads

\[
\frac{dm}{dT} = \frac{2\pi g_R T}{48\pi m + 9g_R T},
\]

which, after a change of variables \(m(T) = v(T) T\), turns into a separable equations and can be solved by integration. The implicit solution is

\[
m^2 + \frac{3g_R}{16\pi} mT - \frac{g_R T^2}{24} = m_R^2 \left( \frac{96\pi m + T\left(\sqrt{81g_R^2 + 384\pi^2 g_R^2 + 9g_R}\right) + \sqrt{81g_R^2 + 384\pi^2 g_R^2 - 9g_R}}{96\pi m - T\left(\sqrt{81g_R^2 + 384\pi^2 g_R^2 - 9g_R}\right)} \right)^{\frac{9g_R}{g_R}}
\]

In the \(T \to \infty\) limit we find that

\[
\frac{m(T)}{T} \to \sqrt{\frac{g_R}{24} + \left(\frac{3g_R}{32\pi}\right)^2} - \frac{3g_R}{32\pi},
\]
and
\[ g(T) \to g_R \frac{\sqrt{81g_R^2 + 384\pi^2g_R} - 9g_R}{\sqrt{81g_R^2 + 384\pi^2g_R} + 9g_R}. \]  
(33)

In order to determine the subleading behaviour of \( g(T) \) we would need to calculate \( \theta_{10}, \theta_{11}, \theta_{20}, \beta_{11}, \beta_{12}, \beta_{20}, \beta_{21} \) and \( \beta_{30} \). Note that the subleading terms can become dominant since the leading terms sum up to a constant when \( T \to \infty \) (see Section 3.1).

3 Conclusion and outlook

The main conclusion of this paper is that the TRGE resums the leading and subleading powers of \( T \) in much the same way as the \( T = 0 \) RGE resums leading \( \ln \mu \) terms. It is also interesting to notice that a 2-loop calculation is required to get the leading behaviour for the coupling constant. This was observed by Funakubo and Sakamoto in Ref.[3] who studied the \( O(N) \) model in the limit \( N \to \infty \) where it is exactly solvable.

It is, of course, not a miracle that some higher order diagrams are determined in terms of lower order diagrams. At zero temperature, the leading \( \ln \mu \) contribution comes from diagrams with many divergent subdiagrams. It is those subdiagrams that are controlled by the \( \beta \) function. Higher order diagrams with leading \( \ln \mu \) terms are constructed by multiple insertion of those subdiagrams. At finite temperature there is one type of loop that differs from others: the quadratically divergent tadpole which gives a factor \( T^2 \) \[7, 9\]. All other loops give a factor \( T \) (in the high \( T \) limit). At the \( n \)-th order, if we want the \( n \)-th subleading contribution to \( m(T) \) (\( \propto g_R^nT^n \)) all diagrams to that order contribute. It is, therefore, natural that we have to calculate the \( \theta \) and \( \beta \) functions to \( n \)-th order (actually, even if we want the \( (n-1) \)-th subleading term we get a \( k = n \) contribution from the condition \( p + k \leq i + 1 \) (Eq.(23)) when \( p = 0 \) and \( i = n - 1 \)). We can briefly say that the TRGE is a way of resumming the subdiagrams with a \( T^2 \) dependence of which there is only the 1-loop tadpole in the \( (\phi^4)_4 \)-theory.

A difference between the zero and finite \( T \) RGE is that the \( \beta \) function at zero \( T \) does not depend on \( \mu \) in the MS scheme, while the \( \beta \) function at finite \( T \) depends on \( T \). That makes it harder to prove a leading \( T \) resummation and it works only because the \( T \) dependence is not too strong, only \( g_R^nT^n \) and not \( g_R^nT^{2n} \). The reason for that is, as we saw in Section 2.2, that the highest power of \( T \) is subtracted off by the counterterms. Then we are back to the physical starting point, namely to use the physical \( T \) dependent mass already in the propagator. In that way the highest \( T \) dependence does not contribute to the \( \beta \) function.
3.1 Effective propagators

The method of using an effective propagator with a mass term \( m_R^2 + g_R T^2 / 24 \) also resums the leading powers of \( T \) to each order in \( g_R \). By definition of the effective mass we have the same expression for \( m(T) \) when \( T_R = 0 \) (Eq. (25)) and from the 1-loop diagram in Fig. (2) we get for the coupling constant

\[
g(T) = g_R + \frac{3g_R^2}{32\pi^2} \ln(1 + \frac{g_R T^2}{24m_R^2}) - \frac{3g_R^2}{16\pi} \frac{T}{\sqrt{m_R^2 + g_R T^2 / 24}}
\]

which agrees with Eq. (29) to leading order.

Still they predict quantitatively different behaviours at very high temperature. The logarithmic term in Eq. (34), that comes from the renormalization at \( T = 0 \), is formally of lower order in \( T/m_R \) and should be neglected. Then Eq. (34) gives a negative effective 4-point coupling at sufficiently high \( T \) if \( g_R > 32\pi^2 / 27 \), which indicates an instability even though it can be stabilized by higher n-point functions (such as the 6-point function). On the other hand the logarithmic term is actually dominant at high \( T \) and gives an increasing \( g(T) \). The TRGE result in Eq. (33) does not show any of these behaviours. The difference between the effective propagator approach and the TRGE is in the subleading terms, and it is clear that if the leading terms sum up to something finite when \( T \to \infty \) the subleading terms can become dominant. That is also the reason why \( m(T) \) and \( g(T) \) are found to diverge at a critical temperature in Ref. [3] but not here.

A problem with the effective propagator is that it leads to \( T \) dependent infinities in the tadpole diagram in Fig. (1) when the mass is explicitly \( T \) dependent. It occurs in subleading terms and is subtracted away if all diagrams to that order are included [8]. Also, in [9] this was pointed out as a defect of not treating subleading terms consistently while in [10] it was considered to be legitimate with such \( T \) dependent infinities. In my point of view the bare parameters of the theory has nothing to do with the thermal state of the system and must, therefore, be temperature independent. The TRGE is, however, derived from the \( T \) independence of the bare parameters and satisfy this criterion by definition. It is a subject for further study to understand significance of subleading terms in the TRGE approach.

3.2 Other theories

As we now understand, the method of resummation using the TRGE works because the leading temperature corrections can be accurately approximated by a \( T \) dependent mass term. In the \((\phi^4)_4\)-theory it is particularly simple since the tadpole does not depend on
the external momentum. The possibility of extending the result of this paper to other theories is related to the possibility of including the leading $T$ dependence in the propagator, i.e. to expand about the physical excitations at finite temperature. For instance, the $(\phi^3)_6$-theory has a momentum dependent 1-loop correction to the mass. It turns out, however, that the leading $T^2$ part of that correction is momentum independent and the TRGE should work even for that theory.

In QCD, the gluon polarization tensor has a $T^2$ term which is momentum dependent and cannot be approximated by a constant \[11, 12\]. This indicates that the TRGE, as described here, does not yield a consistent resummation. It may be possible to generalize the TRGE to include a momentum dependent mass term. The choice of RC determines the propagator and we have only considered finite changes in a momentum independent mass by renormalizing at a fixed momentum. In principle, one can allow for a momentum dependent mass term. The reason for not doing so is that it makes calculations more complicated, but if the physical excitations cannot be well approximated by free particles with $T$ dependent mass, one has to use something better. A perturbation method using effective propagators and vertices to include the $T^2$ contributions was developed in \[13\]. The analysis in this paper indicates that a 2-loop calculation would be necessary also in QCD to get a TRGE which resums the leading powers of $T$ for the coupling constant.

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A Appendix

We outline the determination of the functions $Q, L, S$, and $U$ defined in Section 2.3. From Eq.(9) we see that

$$Q(n, l) = \max_{k, p} [Q(n - k, l) + 1 - p + k]$$  \hspace{1cm} (35)

where $1 \leq k \leq n$ and $p \geq 0$. If the result is less or equal to zero we must also check Eq.(12) which gives a lower limit

$$Q(n, l) \geq 0 \text{ if } l \leq L(n, r) + r \ ; \ r \neq 0 \ .$$  \hspace{1cm} (36)
Since the determination of $Q(n, l)$ only involves $Q(n', l)$ with $n' < n$ we can solve the problem iteratively in $n$ with initial condition

$$Q(0, 0) = 0, \ Q(0, l \neq 0) = -\infty.$$  

(37)

Each time we determine $k, p$ in Eq.(35) we must be sure that the factor in brackets in Eq.(9) does not vanish for these values. It does not pose any problem here but it does when calculating $S$ and $U$.

Similarly $L$ is then determined by

$$L(n, q \neq 0) = \max_{k,p}[L(n - k, q - 1 + p - k)]$$
$$L(n, 0) = \max_{r \neq 0}[L(n, r) + r],$$

(38)

with the same initial condition as for $Q$.

The procedure is the same for $S$ and $U$ but we find that

$$S(1, 0) = \max_p[S(0, 0) + 1 - p + 1] = 2,$$

(39)

occurs for $k = 1, p = 0$. This term is, however, absent in Eq.(4) since $\beta_{10} = 0$ and $1 + k - p - s - u = 0$. Therefore, $S(1, 0) = 1$ for $k = 1, p = 1$. That is the major difference between $Q, L$ and $S, U$. As we can see from Eq.(4) $S(n, 0)$ and $U(n, 0)$ has to be treated separately from $S(n, 1 \leq u \leq 2n - 1)$ and $U(n, 1 \leq s \leq 2n - 1)$.

**B Appendix**

In order to determine which values of $k$ and $p$ that occurs in the summations in Eqs.(21, 22) we look at the first and second term separately.

In the first term in Eq.(21) the summation over $q$ runs through all values satisfying

$$q \leq Q(n, 2n - i)$$
$$2n - i \leq L(n, q).$$

(40)

For each such $q$ the terms that contribute to the $k$ and $p$ summations satisfy

$$q - 1 + p - k \leq Q(n - k, 2n - i)$$
$$2n - i \leq L(n - k, q - 1 + p - k).$$

(41)

Finally, the factors in brackets in Eqs.(21, 22) must be non-zero. This analysis gives that for $i > 2n$ there is no restriction on $k$ and $p$ (other than $1 \leq k \leq n$ and $p \geq 0$, of course)
and for $2 \leq i \leq 2n$ we have $k \leq i/2$. There is no contribution from the first term when $i = 0, 1$.

The summation range for the second term is determined by

$$\begin{align*}
r & \leq Q(n, 2n - i - r) \\
2n - i & \leq L(n, r) + r ,
\end{align*}$$

(42)

and the terms in the $k, p$ sums satisfy

$$\begin{align*}
r - 1 + p - k & \leq Q(n - k, 2n - i - r) \\
2n - i - r & \leq L(n - k, r - 1 + p - k) .
\end{align*}$$

(43)

We find that this term contributes whenever $p + k \leq i + 1$.

In a similar way we check the summation in Eq.(22) where in the first term the summation runs over all $s$ satisfying

$$\begin{align*}
s & \leq S(n, 2n - 1 - i) \\
2n - 1 - i & \leq U(n, s) ,
\end{align*}$$

(44)

and the $k, p$ summation ranges over

$$\begin{align*}
s - 1 + p - k & \leq S(n - k, 2n - 1 - i) \\
2n - 1 - i & \leq U(n - k, s - 1 + p - k) .
\end{align*}$$

(45)

We find, using Eqs.(18, 19), almost the same conditions as for $M$, i.e. $k \leq i/2$ and $p + k \leq i + 1$. However, the expression for $S(0, u)$ and $U(0, s)$ do not follow the general form and give exceptions for $n = k$ in the RHS of Eq.(45). This gives an extra term when $k = n$ and $p \leq i + 2 - n$. The highest $n$ (=k) for which this term can contribute is when $p = 0$, i.e. $n = k = i + 2$.

In summary, we have found that for each iteration of the solution of Eqs.(21, 22) (if we imagine an iterative solution in $n$) the terms contributing are the ones satisfying

$$\begin{align*}
k & \leq i/2, \text{ all } p \text{ (also } 1 \leq k \leq n) \\
or \quad p + k & \leq i + 1 .
\end{align*}$$

(46)

In addition, in the $(i + 2)$-th order in Eq.(22) there is an extra term with

$$p + k \leq i + 2 .$$

(47)
We conclude that, in order to get the $i$-th subleading term of the $T_R/m_R$-expansion of $m(T)$ we need to compute the $\theta$ and $\beta$ functions to the $(i+1)$-th order (respectively the $(i+2)$-th order for $g(T)$). Actually, we only need the leading $T$-dependent part of the highest order diagram when calculating $\theta$ and $\beta$. The Eqs. (46, 47) tell more precisely what is needed. An explicit example for $i = 0$ is given in Section 2.4.

References

[1] H. Matsumoto, Y. Nakano and H. Umezawa,  
“Renormalization group at finite temperature”, Phys. Rev. D 29 (1984) 1116

[2] Y. Fujimoto, K. Ideura, Y. Nakano and H. Yoneyama, “The finite temperature renormalization group equation in $\lambda \phi^4$ theory”, Phys. Lett. B167 (1986) 406

[3] K. Funakubo and M. Sakamoto, “Higher order contribution in finite temperature renormalization group”, Phys. Lett. B186 (1987) 205

[4] Y. Fujimoto and H. Yamada, “Finite-temperature renormalization group equation in QCD. II”, Phys. Lett. B200 (1988) 167

[5] H. Nakkagawa, A. Niégawa and H. Yokota, “Non-Abelian gauge couplings at finite temperature in the general covariant gauge”, Phys. Rev. D 38 (1988) 2566

[6] P. Elmfors, “Finite temperature renormalization for the $(\phi^3)_6$- and $(\phi^4)_4$-models at zero momentum”, NORDITA 91/38 P, to be published in Z. Phys. C

[7] L. Dolan and R. Jackiw,  
“Symmetry behavior at finite temperature”, Phys. Rev. D 9 (1974) 3320

[8] M. B. Kislinger and P. D. Morley, “Collective phenomena in gauge theories. I. The plasmon effect for Yang-Mills fields”, Phys. Rev. D 13 (1976) 2765

[9] P. Fendley, “The effective potential and the coupling constant at high temperature”, Phys. Lett. B196 (1987) 175

[10] R. R. Parwani, “Resummation in a hot scalar field theory”, ITP-SB-91-64, Stony Brook.
[11] O. K. Kalshnikov and V. V. Klimov, “Polarization operator in QCD at finite temperature and density”, Sov. J. Nucl. Phys. 31 (1980) 699

[12] H. A. Weldon, 

“Covariant calculations at finite temperature: The relativistic plasma”, Phys. Rev. D26 (1982) 1394

[13] E. Braaten and R. D. Pisarski, “Soft amplitudes in hot gauge theories: A general analysis”, Nucl. Phys. B337 (1990) 569, and references therein