Identification of Shallow Neural Networks by Fewest Samples

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Abstract

We address the uniform approximation of sums of ridge functions \( \sum_{i=1}^{m} g_i(a_i \cdot x) \) on \( \mathbb{R}^d \), representing the shallowest form of feed-forward neural network, from a small number of query samples, under mild smoothness assumptions on the functions \( g_i \)'s and near-orthogonality of the ridge directions \( a_i \)'s. The sample points are randomly generated and are universal, in the sense that the sampled queries on those points will allow the proposed recovery algorithms to perform a uniform approximation of any sum of ridge functions with high-probability. Our general approximation strategy is developed as a sequence of algorithms to perform individual sub-tasks. We first approximate the span of the ridge directions. Then we use a straightforward substitution, which reduces the dimensionality of the problem from \( d \) to \( m \). The core of the construction is then the approximation of ridge directions expressed in terms of rank-1 matrices \( a_i \otimes a_i \), realized by formulating their individual identification as a suitable nonlinear program, maximizing the spectral norm of certain competitors constrained over the unit Frobenius sphere. The final step is then to approximate the functions \( g_1, \ldots, g_m \) by \( \hat{g}_1, \ldots, \hat{g}_m \). Higher order differentiation, as used in our construction, of sums of ridge functions or of their compositions, as in deeper neural network, yields a natural connection between neural network weight identification and tensor product decomposition identification. In the case of the shallowest feed-forward neural network, second order differentiation and tensors of order two (i.e., matrices) suffice as we show in this paper. Since our results clarify constructively how many training samples one needs in order to train a shallow feed-forward neural network, they might be useful to shed some light on estimating the minimal number of data needed to train a deeper neural network when the learning is performed by means of layer-by-layer procedures.

Keywords: training shallow neural networks, ridge functions, randomized algorithms, nonlinear programming

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1 Introduction

1.1 Sums of ridge functions and neural networks

A ridge function, in its simplest form, is a function \( f : \mathbb{R}^d \to \mathbb{R} \) of the type

\[ f(x) = g(a \cdot x), \]  

where \( g : \mathbb{R} \to \mathbb{R} \) is a scalar univariate function and \( a \in \mathbb{R}^d \) is the direction of the ridge function. Ridge functions are constant along the hyperplanes \( a \cdot x = \lambda \) for any given level \( \lambda \in \mathbb{R} \) and are among the most simple form of multivariate functions. For this reason they have been extensively studied in the past couple of decades as approximation building blocks for more complicated functions. Nevertheless, ridge functions appeared long before in several contexts. For instance, in multivariate Fourier series, the basis functions are of the form \( e^{i n \cdot x} \) for \( n \in \mathbb{Z}^d \) and \( e^{i a \cdot x} \) for arbitrary directions \( a \in \mathbb{R}^d \) in the Radon transform. Also ridge polynomials \( (a \cdot x)^k \) are used in many settings. The term “ridge function” has been actually coined by Logan and Shepp in 1975 in their work on computer tomography, where they show how ridge functions solve the corresponding \( L_2 \)-minimum norm approximation problem. Ridge function approximation has been as well extensively studied during the 80’s in mathematical statistics under the name of projection pursuit, see for instance [17] [9]. Projection pursuit algorithms approximate a function of \( d \) variables by functions of the form

\[ \sum_{i=1}^{m} g_i(a_i \cdot x), \quad x \in \mathbb{R}^d, \]  

for some functions \( g_i : \mathbb{R} \to \mathbb{R} \) and some non-zero vectors \( a_i \in \mathbb{R}^d \). In the early 90’s there has been an explosion of interest in the field of neural networks. One very popular model
is the multilayer feed-forward neural network with input, hidden (internal), and output layers. The simplest case of such a network (the one with only one internal hidden layer, \( m \) processing units, and one output) is described mathematically by a function of the form

\[
\sum_{i=1}^{m} \alpha_i \sigma \left( \sum_{j=1}^{m} w_{ij} x_j + \theta_i \right),
\]

where \( \sigma : \mathbb{R} \rightarrow \mathbb{R} \) is somehow given and called the activation function and \( w_{ij} \) are suitable weights indicating the amount of contribution of the input layer neurons. One can easily recognize this model to be of type (2). Due to the tremendous attention raised by neural networks in the early 90’s the question of whether one can use sums of ridge functions to approximate well arbitrary functions has been at the center of the attention of the approximation theory community for more than two decades, see for instance [21, 28] for two extensive overviews on the topic, and the references therein. Also the efficiency of such an approximation compared to, e.g., spline type approximation for smoothness classes of functions, has been extensively investigated [3, 27]. The identification of a ridge function has also been thoroughly considered, in particular we mention the work [3], and, for what concerns multilayer neural networks, we refer to the groundbreaking paper [10].

1.2 Identifications of ridge functions from sample queries

Except perhaps the work of Candès on ridglets [4], after these fundamental developments and the rather complete exploration of the subject, there has been less attention on the problem of approximating functions by means of ridge functions, until a more specific issue came into focus again recently, namely the approximation of ridge functions by the minimal amount of sampling queries. In fact the above mentioned results on the identification of such functions were mainly based on disposing of any possible output of the function or even of its derivatives, which might be in certain practical situations very expensive, hazardous or even merely impossible. In the work [5] the authors explored deterministic and adaptive choices of point queries to learn functions of the type (2) for the specific case where \( m = 1 \), under mild regularity assumptions on the function \( g \) as in (1), i.e., \( g \in C^{1,\alpha} \). The approach pursued in this paper was based on suitable finite differences yielding approximations to the gradient of the function \( f \). A similar approach has been considered in [11] where a universal sampling strategy has been derived, i.e., the points on which to evaluate the function \( f \) are not adaptively chosen depending on \( f \), at the price though of randomizing their selection. This approach was inspired by the recent developments of the theory of compressed sensing. The approximation of a function of the type (1) has been actually considered as a learning of a compressible vector \( a \) from the minimal amount of nonlinear measurements \( f(x_i) = g(a \cdot x_i) \), provided by the applications of the function \( f \) on a randomized set of sampling points \( x_i \)’s. In the same paper and similarly, the approximation of functions of the type \( f(x) = g(A^T x) \), for a matrix \( A \in \mathbb{R}^{d \times m} \) and \( g : \mathbb{R}^m \rightarrow \mathbb{R} \), has been explored. The problem of identifying the matrix \( A \) (and consequently the function \( g \)) has been lately popularized under the name of “active subspace” detection [6, 7] with a large number of potential applications. The optimal complexity of such sampling strategies, especially for ridge functions, has been by now completely explored [23].

In this paper, we address the open problem of approximating a function of the type (2) for \( m > 1 \) from a small number of sampling points. We shall assume throughout, that \( m \leq d \) (with the interesting possibility of \( m \ll d \) included), that the vectors \((a_i)_{i=1}^{m}\) are
linearly independent, and \(|a_i|_2 = 1\) for all \(i = 1, \ldots, m\). We reiterate that such functions play a relevant role, for instance, in forming layers of neural networks. The robust and fast calibration of multilayer neural networks, especially in deep learning, is a standing open problem \([10]\). As a matter of fact one can rewrite (2) as follows

\[ f(x) = g(A^T x) = \sum_{i=1}^{m} g_i(a_i \cdot x), \; x \in \mathbb{R}^d, \]

where \(A \in \mathbb{R}^{d \times m}\) is the matrix whose columns are the vectors \(\{a_1, \ldots, a_m\}\), so that \(A^T x = (a_1 \cdot x, a_2 \cdot x, \ldots, a_m \cdot x)\), and \(g(y_1, \ldots, y_m) = g_1(y_1) + \cdots + g_m(y_m)\). Hence the problem of learning \(f\) uniformly can certainly be addressed by using the methods explored, e.g., in \([11]\). However, there are at least two relevant motivations for searching for alternative approaches to the learning of functions of the type (2). First of all, one can hope that the specific structure of being a sum of \(m\) ridge functions can be more precisely identified, in particular, in some applications one may really wish to identify precisely the ridge directions \(a_1, \ldots, a_m\), as it happens in the calibration of a neural network. Unfortunately, this is actually not possible with the methods in \([11]\). Indeed, as clarified in \([11, \text{Lemma 2.1}]\), one can identify the matrix \(A\) only up to an orthogonal transformation, since \(f(x) = g(A^T x) = \tilde{g}(\tilde{A}^T x)\) for \(\tilde{A}^T = O A^T\), for any \(O \in O(m)\) and \(\tilde{g}(y) = g(O^T y)\). Hence, whatever algorithm one uses to approximate \(f(x)\) uniformly with functions of the type \(\tilde{g}(\tilde{A}^T x)\), it is impossible to uniquely identify the ridge directions. The second motivation for searching for a more specific method to approximate (2) from samples is the intrinsic complexity of the problem. In fact, learning a function of the form \(f(x) = g(A^T x)\) requires eventually an exponential number of samples in \(m\), while functions of the type \(f(x) = \sum_{i=1}^{m} g_i(a_i \cdot x)\) are essentially a sum of \(m\) one dimensional ridge functions and one expects that the number of necessary samples should scale at most polynomially in \(m\).

1.3 Main results of the paper: identification of sums of ridge functions from sample queries

For these two relevant reasons we proceed differently in this paper. The approach we are intending to follow is dictated by the following result, whose proof we describe in more detail in Section 2

**Theorem 1** (Reduction to \(m\) dimensions). Let us consider a function

\[ f(x) = \sum_{i=1}^{m} g_i(a_i \cdot x), \; x \in B_1^d = \{x \in \mathbb{R}^d : ||x||_2 \leq 1\}, \]  

for \(m \leq d\) and we denote \(A = \text{span}\{a_1, \ldots, a_m\}\). Let us now fix a \(m\)-dimensional subspace \(\tilde{A} \subset \mathbb{R}^d\), whose we choose an orthonormal basis \(\{b_1, \ldots, b_m\}\), so that \(\tilde{A} = \text{span}\{b_1, \ldots, b_m\}\). We arrange the vectors \(b_i\)'s as columns of a matrix \(B\), and we denote with \(P_A\) and \(P_{\tilde{A}}\) the orthogonal projections onto \(A\) and \(\tilde{A}\) respectively. Then one can construct a function

\[ \tilde{f}(y) = \sum_{i=1}^{m} \tilde{g}_i(a_i \cdot y), \; y \in B_1^m \subset \mathbb{R}^m, \]

\footnote{With a certain abuse of notation, we often use in this paper the symbol \(A\) also to denote the matrix whose columns are the vectors \(\{a_1, \ldots, a_m\}\).}
with \( \alpha_i = B^T a_i \), such that for any other function \( \hat{f} : \mathbb{R}^m \rightarrow \mathbb{R} \) the following estimate holds
\[
\| f - \hat{f}(B^\top \cdot) \|_\infty \leq \| f \|_{\text{Lip}} \| P_A - P_{\tilde{A}} \|_F + \| \hat{f} - \hat{f} \|_\infty. \tag{5}
\]

Moreover, for any other set of vectors \( \{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_m\} \subset \mathbb{R}^m \),
\[
\| a_i - B\tilde{\alpha}_i \|_2 \leq \| P_A - P_{\tilde{A}} \|_F + \| \alpha_i - \tilde{\alpha}_i \|_2. \tag{6}
\]

In view of (4), (5), and (6), the approximation of a sum of \( m \) ridge functions (3) on \( \mathbb{R}^d \) and the identification of its ridge directions can be reduced to the approximation of a sum of \( m \) ridge functions (4) on \( \mathbb{R}^m \) and the identification of its ridge directions, as soon as one can approximate well the subspace \( A \) by means of any other subspace \( \tilde{A} \). Hence, we need to focus on two relevant tasks. The first one is the approximation of the subspace \( A \) and the second is the identification of ridge directions of a sum of ridge functions defined on \( B_1^m \subset \mathbb{R}^m \). For both these tasks, we intend to follow the approach proposed in [11, 3] and we consider approximate higher order differentiation to “extract” from the function \( f \) and “test” its principal directions \( a_i \)‘s against some given vectors \( c_j \)’s:
\[
D_{c_j}^{\alpha_1} \cdots D_{c_j}^{\alpha_k} f(x) = \sum_{i=1}^{m} g_i^{(\alpha_1+\cdots+\alpha_k)}(a_i \cdot x)(a_i \cdot c_1)^{\alpha_1} \cdots (a_i \cdot c_k)^{\alpha_k}, \tag{7}
\]
where \( k \in \mathbb{N}, c_i \in \mathbb{R}^d, \alpha_i \in \mathbb{N} \) for all \( i = 1, \ldots, k \) and \( D_{c_j}^{\alpha} \) is the \( \alpha \)-th derivative in the direction \( c_j \). Since we will employ a finite difference approximation of (7) and we also intend to keep numerical stability in mind, we restrict our attention to first and second order derivatives of \( f \) only, i.e. to \( k \leq 2 \) and \( |\alpha| \leq 2 \). Hence, we need to assume certain regularity of the functions \( g_1, \ldots, g_m \). Namely, we suppose that they are three times continuously differentiable and introduce the quantities
\[
C_j := \max_{i=1,\ldots,m} \max_{1 \leq t \leq 1} |g_i^{(j)}(t)| < \infty, \quad j = 0, 1, 2, 3. \tag{8}
\]

We propose two different algorithms to construct a subspace \( \tilde{A} \) which approximates \( A \) with high accuracy. The first algorithm (Algorithm 1) requires that the matrix
\[
J[f] := \int_{S^{d-1}} \nabla f(x) \nabla f(x)^T d\mu_{S^{d-1}}(x)
\]
has full rank \( m \), being \( \mu_{S^{d-1}} \) the uniform measure on the sphere \( S^{d-1} \). It computes an approximating subspace \( \tilde{A} \) by using \( m\chi(d+1) \) point evaluations of \( f \) with high probability, increasing exponentially to one with the number \( m\chi \) of samples on the sphere. This algorithm computes an approximation space with arbitrary accuracy with high-probability, by using only first order derivatives, and a number of points which is actually linear with respect to the dimension \( d \).

The second algorithm (Algorithm 2) is deterministic and requires that the matrix
\[
H[f](x) = \left( \frac{\partial^2 f(0)}{\partial e_j \partial e_k} \right)^d_{j,k=1} = \sum_{i=1}^{m} g_i^{(2)}(a_i \cdot x)a_i \cdot a_i^T,
\]
has rank \( m \) at \( x = 0 \). It computes an approximating subspace \( \tilde{A} \) with arbitrary accuracy by using \( (d+1)(d+2)/2 \) point evaluations of \( f \). This algorithm computes an accurate
approximation space deterministically, by using second order derivatives, and a number of points which is quadratic with respect to the dimension \(d\).

In case the ridge profiles \(a_i\) exhibit some compressibility, i.e., they can be approximated by sparse vectors, then one can further reduce the number of sampling points by using techniques from compressed sensing. We do not explore this scenario in detail, as it would be a minor modification of the approach given in [11]. As soon as an approximating space \(\tilde{A}\) of \(A\) has been identified, we can then apply Theorem 1 and reduce the dimensionality of the problem to \(m = d\).

It remains to find a way of approximating the ridge directions of a sum of ridge functions in \(\mathbb{R}^m\). We approach this latter issue in Section 3.1 (cf. Algorithm 3) by first finding an approximating matrix space \(\tilde{A} \in \mathbb{R}^{m \times m}\) to \(A = \text{span}\{a_1 \otimes a_1, \ldots, a_m \otimes a_m\}\). The result (Theorem 8) resembles very much the one for the identification of \(A = \text{span}\{a_1, \ldots, a_m\}\) provided by Algorithm 1 and it reads as follows: if the matrix

\[
H_2[f] := \int_{S^{m-1}} H[f](x) \otimes_v H[f](x) d\mu_{S^{m-1}}(x)
\]

has full rank \(m\) (the symbol \(\otimes_v\) represents a suitable tensor product of the vectorization of the matrices \(H[f](x)\), see (31) for its precise definition), then Algorithm 3 based on second differentiation of \(f\) provides an approximating matrix space \(\tilde{A} \in \mathbb{R}^{m \times m}\) to \(A\) with arbitrary accuracy by using \(m_X (m+1)(m+2)/2\) point evaluations of \(f\) with high probability, increasing exponentially to one with the number \(m_X\) of samples on the sphere.

If this approximation is fine enough, then we can assume \(\tilde{A}\) to be spanned by near rank 1 matrices of unit Frobenius norm as well, and those to be good approximations to \(a_1 \otimes a_1, \ldots, a_m \otimes a_m\). For identifying such a basis for \(\tilde{A}\) we need to find in it elements of minimal rank. Let us stress that this problem is strongly related to similar and very relevant ones appearing recently in the literature addressing nonconvex programs to identify sparse vectors and low-rank matrices in linear subspaces, see, e.g., in [25, 29]. We perform such a search by solving a nonlinear program, maximizing the spectral norm among competitors in \(\tilde{A}\) of the Frobenius norm bounded by one, i.e.,

\[
\text{arg max } \|M\|_\infty, \quad \text{s.t. } M \in \tilde{A}, \|M\|_F \leq 1.
\]  

We analyze the optimization algorithm (9) under the assumptions that the ridge profiles \(a_1, \ldots, a_m\) are \(\varepsilon\)-nearly-orthogonal, i.e. that they are close to some orthonormal basis of \(\mathbb{R}^m\) as described in the following definition.

**Definition 2.** Let \(a_1, \ldots, a_m \in \mathbb{R}^m\) be unit vectors. Then we define

\[
S(a_1, \ldots, a_m) = \inf \left\{ \left( \sum_{i=1}^m \|a_i - w_i\|^2_2 \right)^{1/2} : \text{w_1, \ldots, w_m \ orthonormal basis in } \mathbb{R}^m \right\}.
\]  

We say that \(a_1, \ldots, a_m \in \mathbb{R}^m\) are \(\varepsilon\)-nearly-orthogonal, if \(S(a_1, \ldots, a_m) \leq \varepsilon\), for \(\varepsilon > 0\) relatively small.

In Section 3.2 we characterize the solutions to the problem (9) by analyzing its first and second order optimality conditions, and we show that the local maximizers are actually close to \(\{a_1 \otimes a_1, \ldots, a_m \otimes a_m\}\) as soon as \(\tilde{A}\) is a good approximation to \(A\). In Section 3.4 we present an algorithm (Algorithm 5), which is easy to implement and which strives for the solution of this optimization problem, by a sort of iteratively projected gradient ascent,
and we prove some of its convergence properties.

Once we have identified the approximations \( \hat{a}_1, \ldots, \hat{a}_m \) of \( a_1, \ldots, a_m \) by Algorithm 4 or Algorithm 5, the final step addressed in Section 3.3 is then to approximate the functions \( g_1, \ldots, g_m \) by \( \hat{g}_1, \ldots, \hat{g}_m \). The approximation of \( f \) is then given by Algorithm 6 as follows

\[
\hat{f}(x) = \sum_{i=1}^{m} \hat{g}_i(\hat{a}_i \cdot x), \quad x \in B^m_1.
\]

At this point, it is worth to summarize all the construction through the different algorithms in a single higher level result. We use the notations introduced so far.

**Theorem 3.** Let \( f \) be a real-valued function defined on the neighborhood of \( B^d_1 \), which takes the form

\[
f(x) = \sum_{i=1}^{m} g_i(a_i \cdot x),
\]

for \( m \leq d \). Let \( g_i \) be three times continuously differentiable on a neighborhood of \([-1,1]\) for all \( i = 1, \ldots, m \), and let \( \{a_1, \ldots, a_m\} \) be \( \varepsilon \)-nearly-orthogonal, for \( \varepsilon > 0 \) small enough. We additionally assume both \( J[f] \) and \( H_2[f] \) of maximal rank \( m \). Then using at most \( mX[(d+1)+(m+1)(m+2)/2] \) random point evaluations of \( f \), Algorithms 4-5 construct approximations \( \{\hat{a}_1, \ldots, \hat{a}_m\} \) of the ridge directions \( \{a_1, \ldots, a_m\} \) up to a sign change for which

\[
\left( \sum_{i=1}^{m} \|\hat{a}_i - a_i\|_2^2 \right)^{1/2} \lesssim \varepsilon,
\]

with probability at least \( 1 - m \exp \left( -\frac{mXc}{2\max\{\varepsilon_1, \varepsilon_2\}^2m^2} \right) \), for a suitable constant \( c > 0 \) intervening (together with some fixed power of \( m \)) in the asymptotical constant of the approximation (11). Moreover, Algorithm 6 constructs an approximating function \( \hat{f} : B^d_1 \to \mathbb{R} \) of the form

\[
\hat{f}(x) = \sum_{i=1}^{m} \hat{g}_i(\hat{a}_i \cdot x),
\]

such that

\[
\|f - \hat{f}\|_{L_{\infty}(B^d_1)} \lesssim \varepsilon.
\]

Let us mention that this constructive result would not hold if \( \varepsilon \), measuring the deviation of \( \{a_1, \ldots, a_m\} \) from orthonormality, gets too large. In fact, in case \( \{a_1, \ldots, a_m\} \) deviates significantly from being an orthonormal system, it is unclear whether they can be again uniquely identified by any algorithm (see Example 1 below). Moreover, in absence of noise on the point evaluations of \( f \) as in Theorem 3, the usage of more point evaluations does not improve the accuracy in (11) and (12). The result would need to be significantly modified in case of noise on the point evaluations of \( f \) in order to deal with stability issues determined by employing finite differences in order to approximate the gradient and the Hessian of \( f \).

The identifiability of the parameters of (deep) neural networks has been addressed recently [19, 31]. Also the connection between fitting of neural networks and the search for the decomposition of tensors in the sense of [12] was observed in the recent literature. Using decompositions of tensors of the third order, [15] proposed and analyzed the algorithm NN-LIFT, which learns a two-layer feed-forward neural network, where the second
layer has a linear activation function. Our use of first and second order of differentiation (cf. Algorithms 1-3) of functions \( f \) of the form (2) yields a natural connection between neural network weight identification and decompositions of second order tensors (i.e. non-orthogonal decompositions of matrices). Interestingly, [24] shows that learning the weights of a simple neural network (which essentially coincides with (2)) is as hard as the problem of decomposition of a tensor built up from these weights. As it is known that many problems involving tensors are NP-hard [13, 14], this shows that also the exact recovery of parameters of a neural network is a difficult task, at least without some additional assumptions on its structure.

Let us conclude with a glimpse on future developments. In the case of the shallowest feed-forward neural network (2), second order differentiation and tensors of order two (i.e., matrices) suffice as we show in this paper. Since our results clarify constructively how many training samples one needs in order to train a shallow feed-forward neural network, they might be extended also to shed some light on estimating the minimal number of data needed to train a deeper neural network when the learning is performed by means of layer-by-layer procedures [2, 15].

The notation used throughout the paper is rather standard. For \( 0 < p < \infty \), we denote by \( \|x\|_p = \left( \sum_{j=1}^d |x_j|^p \right)^{1/p} \) the \( p \)-(quasi)-norm of a vector \( x \in \mathbb{R}^d \). This notation is complemented by setting \( \|x\|_\infty = \max_{j=1,\ldots,d} |x_j| \). If \( M \in \mathbb{R}^{m \times d} \) is an \( m \times d \) matrix, we denote by \( \|M\|_F \) its Frobenius norm and by \( \|M\| = \|M\|_\infty \) its spectral norm. The inner product of two vectors \( x, y \in \mathbb{R}^d \) is denoted by \( \langle x, y \rangle = x \cdot y = x^T y \). Their tensor product is a rank-1 matrix denoted by \( xy^T = x \otimes y \). More specific notation is introduced along the way, when needed.

2 Dimensionality reduction

The aim of this paper is the uniform approximation of sums of ridge functions

\[
f(x) = \sum_{i=1}^m g_i(a_i \cdot x), \quad x \in B_1^d. \tag{13}
\]

We assume throughout that the vectors \( a_1, \ldots, a_m \in \mathbb{R}^d \) are linearly independent and, therefore, \( m \leq d \). Nevertheless, the typical setting we have in mind is that the number \( d \gg 1 \) of variables is very large and the number \( m \) of summands in (13) is significantly smaller than \( d \), i.e. \( m \ll d \).

The main aim of this section is the proof of Theorem 1 which allows to reduce the general case \( m \leq d \) to \( m = d \). Due to the typical range of parameters we have in mind, this step is crucial in reducing the complexity of the approximation of (13).

2.1 Reduction to dimension \( d = m \)

**Proof of Theorem 1.** Let us assume that the unknown function \( f : B_1^d \to \mathbb{R} \) takes the form of a sum of ridge functions (13) with unknown univariate functions \( g_i : [-1,1] \to \mathbb{R} \) and unknown ridge profiles \( a_1, \ldots, a_m \in \mathbb{R}^d \). We denote \( A = \text{span}\{a_1, \ldots, a_m\} \).

We assume (and we shall discuss this point later in this section) that we were able to find a subspace \( \tilde{A} \subset \mathbb{R}^d \), which approximates \( A \). We select an (arbitrary) orthonormal basis \( (b_i)_{i=1}^m \) of \( \tilde{A} \), and consider the \( d \times m \) matrix \( B \) with columns \( b_1, \ldots, b_m \). Finally, we set \( \alpha_i = B^T a_i \in \mathbb{R}^m \) with \( \|\alpha_i\|_2 = \|P_{\tilde{A}} a_i\|_2 \leq 1 \).
We observe that the function
\[
\hat{f}(y) := f(By) = \sum_{i=1}^{m} g_i(a_i \cdot By) = \sum_{i=1}^{m} g_i(\alpha_i \cdot y), \quad y \in B_1^m,
\]
is a sum of \(m\) ridge functions on \(B_1^m \subset \mathbb{R}^m\). Furthermore, sampling of \(\hat{f}\) can be easily transferred to sampling of \(f\) by \(\hat{f}(y) = f(By)\). Let us assume that \(\hat{f}\) is a uniform approximation of \(f\) on \(B_1^m\). Then the function \(\hat{f}(B^T x)\) is a uniform approximation of \(f\) on \(B_1^d\). Indeed, let \(x \in B_1^d\). We have
\[
|f(x) - \hat{f}(B^T x)| \leq |f(x) - \hat{f}(B^T x)| + |\hat{f}(B^T x) - \hat{f}(B^T x)|
\leq |f(x) - f(BB^T x)| + \|\hat{f} - \hat{f}\|_{\infty} = |f(P_A x) - f(P_A x)| + \|\hat{f} - \hat{f}\|_{\infty}
\leq \|f\|_{\text{Lip}} \cdot \|P_A x - P_A x\|_2 + \|\hat{f} - \hat{f}\|_{\infty}.
\]
If we take the supremum over \(x \in B_1^d\), we get
\[
\|f - \hat{f}(B^T \cdot)\|_{\infty} \leq \|f\|_{\text{Lip}} \cdot \|P_A - P_A\|_{\infty} + \|\hat{f} - \hat{f}\|_{\infty}.
\]
A crucial step in the construction of the uniform approximation \(\hat{f}\) of \(f\) on \(B_1^m\) will be the identification of the ridge profiles \(\alpha_1, \ldots, \alpha_m\). Naturally, we will not be able to recover them exactly and we will only obtain some good approximation \(\{\hat{\alpha}_1, \ldots, \hat{\alpha}_m\} \subset \mathbb{R}^m\). Then the vectors \(B\hat{\alpha}_i\) approximate well the original ridge profiles \(\alpha_i\) as can be observed by using \(B\alpha_i = BB^T a_i = P_A a_i\) and
\[
\|a_i - B\hat{\alpha}_i\|_2 \leq \|a_i - B\alpha_i\|_2 + \|B(\alpha_i - \hat{\alpha}_i)\|_2
= \|(P_A - P_A)a_i\|_2 + \|B(\alpha_i - \hat{\alpha}_i)\|_2
\leq \|P_A - P_A\|_{\infty} + \|\alpha_i - \hat{\alpha}_i\|_2,
\]
which finishes the proof.

\[\square\]

**Remark 1.** Let \(\{a_1, \ldots, a_m\}\) be \(\varepsilon\)-nearly orthogonal and let \(\{w_1, \ldots, w_m\} \subset \mathbb{R}^d\) be such that
\[
S(a_1, \ldots, a_m) = \left(\sum_{j=1}^{m} \|a_j - w_j\|_2^2\right)^{1/2} = \varepsilon.
\]
By Theorem \ref{thm:identification} (and its proof) we can assume that \(\{w_1, \ldots, w_m\} \subset A\). Then
\[
S(\alpha_1, \ldots, \alpha_m) = S(B^T a_1, \ldots, B^T a_m) = S(BB^T a_1, \ldots, BB^T a_m)
= S(P_A a_1, \ldots, P_A a_m) \leq \left(\sum_{j=1}^{m} \|P_A a_j - w_j\|_2^2\right)^{1/2}
\leq \left(\sum_{j=1}^{m} \|P_A a_j - P_A w_j\|_2^2\right)^{1/2} + \left(\sum_{j=1}^{m} \|P_A w_j - P_A w_j\|_2^2\right)^{1/2}
\leq \varepsilon + \|P_A - P_A\|_F.
\]
Hence, if the vectors \(a_1, \ldots, a_m\) are orthogonal, or nearly-orthogonal in the sense of Definition \ref{def:orthogonal}, the vectors \(\alpha_1, \ldots, \alpha_m\) behave similarly.
2.2 Approximation of the span of ridge profiles

As previously shown, as soon as we can produce a subspace \( \tilde{A} \subset \mathbb{R}^d \) approximating \( A = \text{span}\{a_1, \ldots, a_m\} \), we can eventually reduce the problem of approximating a sum of ridge functions in \( \mathbb{R}^d \) to the same problem in \( \mathbb{R}^m \), preserving even the quasi-orthogonality, cf. Remark 1. In this section we describe two different methods of identification of \( A \). The first one is inspired by the results in [11] and makes use of first order differences. The second one works with second order differences.

2.2.1 Identifying \( A \) using first order derivatives

We observe that the vector

\[
\nabla f(x) = \sum_{i=1}^{m} g_i'(a_i \cdot x) a_i
\]

(14)

lies in \( A \) for every \( x \in \mathbb{R}^d \). We consider (14) for different \( x_1, \ldots, x_m \in \mathbb{R}^d \), where \( m_X \geq m \). In a generic situation for the points \( x_i \)'s, \( A \) is likely given as the span of \( \{\nabla f(x_1), \ldots, \nabla f(x_m)\} \).

As we would like to use only function values of \( f \) in our algorithms, we use for every \( j = 1, \ldots, d \) and every \( k = 1, \ldots, m_X \) the Taylor’s expansion

\[
\frac{\partial}{\partial \epsilon_j} f(x_k) = \frac{f(x_k + \epsilon e_j) - f(x_k)}{\epsilon} - \left[ \frac{\partial}{\partial \epsilon_j} f(x_k + \eta_{j,k} \epsilon e_j) - \frac{\partial}{\partial \epsilon_j} f(x_k) \right]
\]

(15)

for some \( \eta_{j,k} \in [0, \epsilon] \). We recast the \( d \times m_X \) instances of (15) into the matrix notation

\[
X = Y - \mathcal{E},
\]

(16)

where

\[
X_{j,k} = \frac{\partial}{\partial \epsilon_j} f(x_k), \quad Y_{j,k} = \frac{f(x_k + \epsilon e_j) - f(x_k)}{\epsilon},
\]

(17)

and

\[
\mathcal{E}_{j,k} = \frac{\partial}{\partial \epsilon_j} f(x_k + \eta_{j,k} \epsilon e_j) - \frac{\partial}{\partial \epsilon_j} f(x_k)
\]

for \( j = 1, \ldots, d \) and \( k = 1, \ldots, m_X \). It follows from (14), that \( A \) is the linear span of columns of \( X \). Naturally, we define \( \tilde{A} \) using the linear span of the singular vectors of \( Y \) corresponding to its \( m \) largest singular values. This is formalized in the following algorithm.
Algorithm 1.

- For a natural number $m, X \geq m$, choose $x_1, \ldots, x_m \in S^{d-1}$ uniformly at random.
- Construct $Y$ according to (17).
- Compute the singular value decomposition of $Y^T = (\tilde{U}_1 \tilde{U}_2) \left( \tilde{\Sigma}_1 0 \tilde{\Sigma}_2 \right) \left( \tilde{V}_1^T \tilde{V}_2^T \right)$, where $\tilde{\Sigma}_1$ contains the $m$ largest singular values.
- Set $\tilde{A}$ to be the row space of $\tilde{V}_1^T$.

The aim of the rest of this section is to show that $\tilde{A}$ constructed in Algorithm 1 is in some sense close to $A$. To be more specific, we need to bound $\|P_A - P_{\tilde{A}}\|$, i.e. the operator or the Frobenius norm of the difference between the orthogonal projections onto $A$ and $\tilde{A}$, respectively. For this first approximation method we need to introduce a new quantity

$$J[f] := \int_{S^{d-1}} \nabla f(x) \nabla f(x)^T d\mu_{S^{d-1}}(x).$$ (19)

**Lemma 4.** Assume the vectors $(a_i)_{i=1}^m$ linearly independent, and $\|a_i\|_2 = 1$ for all $i = 1, \ldots, m$. Additionally assume

$$C_1 := \max_{i=1, \ldots, m} \max_{-1 \leq t \leq 1} |g_i(t)| < \infty.$$ (20)

Suppose that $\sigma_m(J[f]) \geq \alpha > 0$, i.e., the $m$th singular value of the matrix $J[f]$ is bounded away from zero. Then for any $s \in (0, 1)$ we have that

$$\sigma_m(X) \geq \sqrt{m \chi \alpha (1 - s)}$$ (21)

with probability at least $1 - m \exp\left( \frac{m \chi \alpha s^2}{2C_1^2 m^2} \right)$, where $X$ is constructed as in (16) for $x_1, \ldots, x_m \in S^{d-1}$ drawn uniformly at random.

**Proof.** The result will follow by a suitable application of Theorem 24 in the Appendix. Let $w_1, \ldots, w_m$ be an orthonormal basis of $A$. We denote $P^A x = ((w_1 \cdot x), \ldots, (w_m \cdot x))^T \in \mathbb{R}^m$. We identify $P^A$ with the corresponding $m \times d$ matrix, i.e. the matrix with rows $w_1^T, \ldots, w_m^T$.

We observe that $\sigma_j(X) = \sigma_j(P^A X) = \sqrt{\sigma_j((P^A X)^T (P^A)^T)}$,

$$XX^T = \sum_{l=1}^{m \chi} \nabla f(x_l) \nabla f(x_l)^T$$

and

$$P^A XX^T (P^A)^T = \sum_{l=1}^{m \chi} P^A \nabla f(x_l) \nabla f(x_l)^T (P^A)^T.$$
Furthermore, we obtain for every $x \in \mathbb{R}^d$

$$\sigma_1(P^A \nabla f(x) \nabla f(x)^T (P^A)^T) = \sigma_1(\nabla f(x) \nabla f(x)^T) = \|\nabla f(x) \nabla f(x)^T\|_F$$

$$= \|\nabla f(x)\|_2^2 = \sum_{i=1}^{m} \left( \sum_{i=1}^{d} g'_i(a_i \cdot x) a_i \right)^2$$

$$\leq C_1^2 \sum_{i=1}^{d} \left( \sum_{i=1}^{m} |a_{i,l}| \right)^2$$

$$\leq C_1^2 \left( \sum_{i=1}^{d} \sum_{i=1}^{m} |a_{i,l}|^2 \right)^2 = C_1^2 m^2. \tag{22}$$

Hence $X_j = P^A \nabla f(x_j) \nabla f(x_j)^T (P^A)^T$ is a random $m \times m$ positive-semidefinite matrix, that is almost surely bounded. Moreover,

$$\mathbb{E} X_j = P^A \int_{S^{d-1}} \nabla f(x) \nabla f(x)^T d\mu_{S^{d-1}}(x)(P^A)^T = P^A J[f](P^A)^T.$$

We conclude that $\mu_{\min} = \mu_{\min} \left( \sum_{j=1}^{m} \mathbb{E} X_j \right) \geq m \chi \alpha$, and by Theorem 24 in the Appendix

$$\sigma_m(X) = \sqrt{\sigma_m(P^A X X^T (P^A)^T)} \geq \sqrt{\mu_{\min}(1 - s)} \geq \sqrt{m \chi \alpha(1 - s)}$$

with probability at least

$$1 - m \exp \left( - \frac{\mu_{\min} s^2}{2 C_1^2 m^2} \right) \geq 1 - m \exp \left( - \frac{m \chi \alpha s^2}{2 C_1^2 m^2} \right).$$

\[\square\]

Remark 2. If we further assume that $a_1, \ldots, a_m$ are $\varepsilon$-nearly-orthonormal and $w_1, \ldots, w_m$ are orthonormal vectors with

$$S(a_1, \ldots, a_m) = \left( \sum_{i=1}^{m} \|a_i - w_i\|_2^2 \right)^{1/2} \leq \varepsilon,$$

we can improve \[\[24\] to

$$\sigma_1(P^A \nabla f(x) \nabla f(x)^T (P^A)^T) \leq \|\nabla f(x)\|_2^2 = \left\| \sum_{i=1}^{m} g'_i(a_i \cdot x) a_i \right\|_2^2$$

$$\leq \left( \left\| \sum_{i=1}^{m} g'_i(a_i \cdot x) w_i \right\|_2 + \left\| \sum_{i=1}^{m} g'_i(a_i \cdot x)(a_i - w_i) \right\|_2 \right)^2$$

$$\leq \left( \left( \sum_{i=1}^{m} g'_i(a_i \cdot x)^2 \right)^{1/2} + \sum_{i=1}^{m} |g'_i(a_i \cdot x)| \|a_i - w_i\|_2 \right)^2$$

$$\leq (1 + \varepsilon)^2 \sum_{i=1}^{m} |g'_i(a_i \cdot x)| \leq C_1^2 (1 + \varepsilon)^2 m.$$
The rest of the proof then follows in the same manner, only the probability changes to

$$1 - m \exp\left( -\frac{mX\alpha s^2}{2C_1^2(1 + \varepsilon)^2m} \right).$$

The same remark applies also to Theorem 8.

Following theorem quantifies the distance between the subspace \(\tilde{A}\) constructed in Algorithm 1 and \(A\).

**Theorem 5.** Assume the vectors \((a_i)_{i=1}^m\) linearly independent, and \(\|a_i\|_2 = 1\) for all \(i = 1, \ldots, m\). Additionally assume that

$$C_1 := \max_{i=1}^m \max_{-1 \leq t \leq 1} |g'_i(t)| < \infty$$

and that the Lipschitz constants of all \(g'_j, j = 1, \ldots, m\), are bounded by \(C_2 < \infty\).

Let \(\tilde{A}\) be constructed as described in Algorithm 1 by sampling \(mX(d + 1)\) values of \(f\). Let \(0 < s < 1\), and assume \(\sigma_m(J[f]) \geq \alpha > 0\). Then

$$\|P_A - P_{\tilde{A}}\|_F \leq 2C_2\varepsilon m \sqrt{\alpha(1 - s)} - C_2\varepsilon m$$

with probability at least \(1 - m \exp\left( -\frac{mX\alpha s^2}{2mC_1^2} \right)\).

**Proof.** We intend to apply the so-called Wedin’s bound, as recalled in Theorem 23 in the Appendix, to estimate the distance between \(A\) and \(\tilde{A}\). If we choose \(B = X^T\) and \(\tilde{B} = Y^T\), we get \(\Sigma_2 = 0\) and we observe that (88) and (89) are satisfied with \(\tilde{\alpha} = \sigma_m(Y^T)\). Therefore, Theorem 23 implies

$$\|P_A - P_{\tilde{A}}\|_F = \|V_1V_1^T - \tilde{V}_1\tilde{V}_1^T\|_F \leq \frac{2\|X - Y\|_F}{\sigma_m(Y^T)}$$

$$\leq \frac{2\|X - Y\|_F}{\sigma_m(X^T) - \|X - Y\|_F},$$

where we have used Weyl’s inequality \(|\sigma_m(X^T) - \sigma_m(Y^T)| \leq \|X - Y\|_F\) in the last step.

To continue in (21), we have to estimate \(\|X - Y\|_F\) and \(\sigma_m(X^T)\).

We use the relation

$$\left| \frac{\partial}{\partial e_j} f(x_k + \eta_{j,k}e_j) - \frac{\partial}{\partial e_j} f(x_k) \right| = \left| \sum_{i=1}^m \left[ g'_i(a_i \cdot (x_k + \eta_{j,k}e_j)) - g'_i(a_i \cdot x_k) \right] a_{i,j} \right| \leq C_2\varepsilon \sum_{i=1}^m a_{i,j}^2$$

to obtain the estimate

$$\|X - Y\|_F = \|E\|_F \leq C_2\varepsilon \left( \sum_{k=1}^{mX} \sum_{j=1}^d \left( \sum_{i=1}^m a_{i,j}^2 \right)^2 \right)^{1/2} \leq C_2\varepsilon \sqrt{mX} \sum_{j=1}^d \sum_{i=1}^m a_{i,j}^2 = C_2\varepsilon \sqrt{mXm}.$$

The statement now follows by a combination of (24) with (25) and (21).
Remark 3. The same argument as in the proof of Theorem 5 allows to show that
\[ \sigma_m(Y) - \sigma_{m+1}(Y) \geq \sqrt{m} \chi(\sqrt{\alpha(1-s)} - 2C_2 \epsilon) \]
with the same probability as before. Hence, for \( \epsilon \) small enough and \( m \chi \) large, there is (with high probability) a gap in the spectrum of \( Y \) between \( \sigma_m(Y) \) and \( \sigma_{m+1}(Y) \). This can be used to detect \( m \) if it is unknown. The same argument applies to Algorithm 2 and Theorem 6 with virtually no modifications.

2.3 Identifying \( A \) using second order derivatives

If we collect the second order derivatives of \( f \) at \( x \), we get the corresponding Hessian matrix
\[ H[f](x) = \left( \frac{\partial^2 f(x)}{\partial e_j \partial e_k} \right)_{j,k=1}^{d} = \sum_{i=1}^{m} g_i''(a_i \cdot x) a_i a_i^T, \] (26)
which is a symmetric \( d \times d \) matrix with rank \( H[f](x) \leq m \) and the span of its columns (or rows) is a subspace of \( A \).

For simplicity, we fix \( x = 0 \) and approximate \( H[f](0) \) by finite order differences of second order, i.e. for \( \epsilon > 0 \) we consider a matrix \( \Delta[f](0) \) with entries
\[ (\Delta[f](0))_{j,k} = \frac{f(\epsilon(e_j + e_k)) - f(\epsilon e_j) - f(\epsilon e_k) + f(0)}{\epsilon^2}, \quad j, k = 1, \ldots, d. \] (27)
The subspace \( \tilde{A} \) is then given as the linear span of the singular vectors corresponding to \( m \) largest singular values of \( \Delta[f](0) \). We summarize this procedure in the following Algorithm.

Algorithm 2.

- Construct \( \Delta[f](0) \) according to (27).
- Compute the singular value decomposition of
  \[ (\Delta[f](0))^T = \begin{pmatrix} \tilde{U}_1 & \tilde{U}_2 \end{pmatrix} \begin{pmatrix} \tilde{\Sigma}_1 & 0 \\ 0 & \tilde{\Sigma}_2 \end{pmatrix} \begin{pmatrix} \tilde{V}_1^T \\ \tilde{V}_2^T \end{pmatrix}, \]
  where \( \tilde{\Sigma}_1 \) contains the \( m \) largest singular values.
- Set \( \tilde{A} \) to be the row space of \( \tilde{V}_1^T \).

The distance between \( A \) and \( \tilde{A} \) is quantified in the following Theorem.

Theorem 6. Assume the vectors \( (a_i)_{i=1}^{m} \) linearly independent, and \( ||a_i||_2 = 1 \) for all \( i = 1, \ldots, m \). Additionally assume that \( g_j, j = 1, \ldots, m \), are two times differentiable with the Lipschitz constant of all \( g_j'' \), \( j = 1, \ldots, m \), bounded from above by \( C_3 > 0 \). Let \( H[f](0) \) and \( \Delta[f](0) \) be defined by (26) and (27), respectively, the latter by using \((d+1)(d+2)/2\) sampling values of \( f \). Additionally assume that
\[ 0 < \epsilon < \frac{\sigma_m(H[f](0))}{2C_3 m}. \]
Then
\[ ||P_A - \tilde{P}_{\tilde{A}}||_F \leq \frac{4C_3 \epsilon m}{\sigma_m(H[f](0)) - 2C_3 \epsilon m}. \]
Proof. Let \( g(t) = f(te_j + \epsilon e_k) - f(te_j) \), where \( 0 \leq t \leq \epsilon \). Then by the mean value theorem

\[
(\Delta[f](0))_{j,k} = \frac{g(\epsilon) - g(0)}{\epsilon^2} = \frac{g'(\xi_1)}{\epsilon} = \frac{\frac{\partial f}{\partial x_j}(\xi_1 e_j + \epsilon e_k) - \frac{\partial f}{\partial x_j}(\xi_1 e_j)}{\epsilon}
\]

where \( 0 < \xi_1, \xi_2 < \epsilon \). Therefore

\[
\|(H[f](0))_{j,k} - (\Delta[f](0))_{j,k}\| \leq \sum_{l=1}^{m} \left| \frac{\partial^2 f}{\partial x_k \partial x_j}(0) - \frac{\partial^2 f}{\partial x_k \partial x_j}(\xi_1 e_j + \xi_2 e_k) \right| \cdot |a_{l,j}| \cdot |a_{l,k}|
\]

and by triangle inequality and \( \|a_i\|_4 \leq \|a_i\|_2 = 1 \), we obtain

\[
\|H[f](0) - (\Delta[f](0))\|_F \leq 2C_3 \epsilon \left[ \sum_{j,k=1}^{d} \left( \sum_{l=1}^{m} \left| a_{l,j}^2 \right| \right) \right]^{1/2} \leq 2C_3 \epsilon \left( \sum_{l=1}^{m} \left( \sum_{j,k=1}^{d} a_{l,j}^2 a_{l,k}^2 \right) \right)^{1/2} \leq 2C_3 \epsilon m.
\]

The rest follows again by Wedin’s bound (Theorem 2.3) and Weyl’s inequality

\[
\|P_A - P_A^\dagger\|_F \leq 2\frac{\|H[f](0) - \Delta[f](0)\|_F}{\sigma_m(\Delta[f](0))} \leq 2\frac{\|H[f](0) - \Delta[f](0)\|_F}{\sigma_m(H[f](0)) - \|H[f](0) - \Delta[f](0)\|_F}
\]

and

\[
\leq \frac{4C_3 \epsilon m}{\sigma_m(H[f](0)) - 2C_3 \epsilon m}.
\]

\[\square\]

2.4 Sparse profiles

If the dimension \( d \) is very large, it might be reasonable to assume, that not all the coordinates of the profiles \( a_1, \ldots, a_m \) are nonzero. To be more specific, let us assume that all of them are \( s \)-sparse, i.e., they are vectors with at most \( s \) nonzero entries. Then \( \nabla f(x) \) is \( ms \) sparse, cf. [14] and we can then use the methods of compressed sensing to identify \( \nabla f(x_1), \ldots, \nabla f(x_{m\chi}) \) with only very few sampling points. We just replace (15) by

\[
\frac{\partial}{\partial \phi_j} f(x_k) = \nabla f(x_k) \cdot \phi_j = \frac{f(x_k + \epsilon \phi_j) - f(x_k)}{\epsilon} - \frac{\partial^2 f}{\partial \phi_j^2} f(x_k)
\]

for \( j = 1, \ldots, m_\phi \) and \( k = 1, \ldots, m_\chi \). We choose \( m_\phi \geq Csm \log(d) \) and the directions \( \phi_1, \ldots, \phi_{m_\phi} \) at random

\[
\phi_{j,l} = \frac{1}{\sqrt{m_\phi}} \begin{cases} +1 & \text{with probability } 1/2, \\ -1 & \text{with probability } 1/2 \end{cases}, \quad j = 1, \ldots, m_\phi, \quad l = 1, \ldots, d.
\]
Then we may use the methods of sparse recovery, to approximate (with high probability) \( \nabla f(x_k) \) from the linear measurements
\[
\nabla f(x_k) \cdot \varphi_j
\]
or their noisy counterparts
\[
\frac{f(x_k + \epsilon \varphi_j) - f(x_k)}{\epsilon}
\]
Let us observe, that this reduces the number of sampling points to \( m_X(m + 1) \approx C m_X s m \log(d) \).

As the treatment of sparse profiles is not crucial for the rest of our work, we refer to \([11]\) for more details, where a similar problem was discussed extensively.

3 Identification of ridge profiles

We explained how to recover an approximation of \( A = \text{span}\{a_1, \ldots, a_m\} \) and how to use it to reduce the dimensionality of the problem from \( d \) to \( m \). We therefore concentrate on the case \( m = d \) in the rest of the paper.

We will also assume throughout this section that vectors \( a_1, \ldots, a_m \in \mathbb{R}^m \) are \( \varepsilon \)-nearly-orthonormal in the sense of Definition 2, i.e., that \( S(a_1, \ldots, a_m) \leq \varepsilon \) for \( \varepsilon > 0 \) small. (Some basic properties of this notion are collected in the Appendix for reader’s convenience and we refer to them below quite often.)

We start to build the approximation scheme by approximating the ridge profiles. As we will show below, a crucial step in order to be able to identify/approximate the vectors \( a_1, \ldots, a_m \in \mathbb{R}^m \) is first to identify/approximate the span of their tensor products. Accordingly, we denote by
\[
\mathcal{A} = \text{span}\{a_i \otimes a_i, i = 1, \ldots, m\} \subset \mathbb{R}^{m \times m}
\]
the subspace of symmetric matrices generated by their tensor products \( a_i \otimes a_i = a_i a_i^T \).

For the actual recovery of \( a_i \)'s we proceed according to the following strategy: we first recover an approximating subspace \( \tilde{\mathcal{A}} \) of \( \mathcal{A} \), which is (in some sense) close to \( \mathcal{A} \). Then, we consider the following nonlinear program
\[
\arg \max \|M\|_{\infty}, \quad \text{s.t.} \quad M \in \tilde{\mathcal{A}}, \|M\|_F \leq 1
\] (28)
to recover the \( a_i \)'s - or, more precisely, their approximations \( \tilde{a}_i \) (which is of course possible only up to the sign).

The optimization problem (28) is not convex and may in general have a large number of local maximas. Nevertheless, we shall prove that to every local maximizer of (28), there is one of the matrices \( a_i \otimes a_i \), which lies very close to it. In particular, for \( \tilde{\mathcal{A}} = \mathcal{A} \) we obtain the exact recovery of the \( a_i \otimes a_i \)'s.

Let us clarify now that the \( \varepsilon \)-near-orthogonality of the ridge profiles \( a_1, \ldots, a_m \in \mathbb{R}^m \) is actually necessary for their identification by any algorithm relying on an approximating subspace \( \tilde{\mathcal{A}} \) of \( \mathcal{A} \).

Example 1. In this instructive example we clarify that the near-orthonormality of \( a_i \)'s is actually necessary. Let \( a_1 = (1, 0)^T, a_2 = (\sqrt{2}/2, \sqrt{2}/2)^T \) and \( a_* = (a_1 + a_2)/\|a_1 + a_2\|_2 \). We assume that \( \mathcal{A} = \text{span}\{a_1 a_1^T, a_2 a_2^T\} \) and that
\[
\tilde{\mathcal{A}} = \text{span}\left\{\begin{pmatrix} 1 & \varepsilon \\ \varepsilon & -\varepsilon \end{pmatrix}, \begin{pmatrix} 0.5 & 0.5 + \varepsilon \\ 0.5 + \varepsilon & 0.5 - \varepsilon \end{pmatrix}\right\}
\]
When choosing \( \epsilon = 0.05 \), we find out that
\[
\{ \text{dist}(a_1 a_1^T, \tilde{A}), \text{dist}(a_2 a_2^T, \tilde{A}), \text{dist}(a_* a_*^T, \tilde{A}) \} \subset [0.07, 0.08].
\]
Hence, looking at \( \tilde{A} \) alone, every algorithm will have difficulties to decide, which two of the three rank-1 matrices above are the generators of the true \( A \). Nevertheless, \( \|a_* - a_1\|_2 = \|a_* - a_2\|_2 \geq 0.39 \). We see that although the level of noise is rather small in this case, we cannot distinguish between well-separated vectors.

### 3.1 Approximation of \( A \)

First of all we construct here an approximation \( \tilde{A} \) to the space \( \mathcal{A} = \text{span}\{a_i \otimes a_i, i = 1, \ldots, m\} \) by generating again \( m_X \in \mathbb{N} \) points \( x_l \sim \mu_{S^{m-1}}, l = 1, \ldots, m_X \) uniformly at random on the \( m - 1 \) dimensional sphere (remind that now we assume \( m = d \)), and we define
\[
(\Delta [f](x_l))_{j,k} = \frac{f(x_l + \epsilon e_j + \epsilon e_k) - f(x_l + \epsilon e_j) - f(x_l + \epsilon e_k) + f(x_l)}{\epsilon^2}, \quad j, k = 1, \ldots, m.
\]
(29)

As \( \Delta [f](x) \sim H[f](x) = \sum_{i=1}^{m} g''(a_i \cdot x) a_i a_i^T \in \mathcal{A} \), we define \( \tilde{A} \) as the \( m \)-dimensional subspace approximating the points \( (\Delta [f](x_l))_{m_X}^{m} \) in the least-square sense. We show below that \( \tilde{A} \) is indeed a good approximation to \( A \) by showing that the difference of the respective orthogonal projections \( \|P_A - P_{\tilde{A}}\|_{F \rightarrow F} \) in the operator norm associated to the Frobenius norm of matrices is small with high probability, as soon as \( m_X \) is large enough.

We need now to introduce some notations to facilitate the presentation. We define the vectorization of a matrix \( A = (a_{i,j})_{ij} \in \mathbb{R}^{m \times m} \) as the column vector in \( \mathbb{R}^{m^2} \)
\[
\text{vec}(A)_k := a_{\lfloor \frac{k-1}{m} \rfloor +1, (k-1 \mod m)+1}, \quad k = 1, \ldots, m^2.
\]

For two matrices \( A, B \in \mathbb{R}^{m \times m} \) we define their vectorized tensor product by
\[
A \otimes_v B := \text{vec}(A) \otimes \text{vec}(B) = \text{vec}(A) \text{vec}(B)^T.
\]
(30)

(Note that such a product of matrices does coincide neither with the Hadamard product nor with the Kronecker product.) Thanks to these definitions and notations we can introduce the matrix
\[
H_2[f] := \int_{S^{m-1}} H[f](x) \otimes_v H[f](x) d\mu_{S^{m-1}}(x).
\]
This \( m^2 \times m^2 \) matrix plays exactly the same role as \( J[f] \) in Section 2.2.1.
Algorithm 3.  
• Generate \( m_X \) points \( x_l \sim \mu_{\mathbb{S}^{m-1}} \), \( l = 1, \ldots, m_X \).
• Compute \( \Delta[f](x_l) \) for \( l = 1, \ldots, m_X \) by (29).
• Construct \( Y \) as a \( m^2 \times m_X \) matrix with columns \( \text{vec}(\Delta[f](x_l)) \).
• Compute the singular value decomposition of 
\[
Y^T = \left( \begin{array}{cc} \tilde{U}_1 & \tilde{U}_2 \end{array} \right) \left( \begin{array}{cc} \tilde{\Sigma}_1 & 0 \\ 0 & \tilde{\Sigma}_2 \end{array} \right) \left( \begin{array}{c} \bar{V}_1^T \\ \bar{V}_2^T \end{array} \right),
\]
where \( \tilde{\Sigma}_1 \) contains the \( m \) largest singular values.
• Set \( \bar{A} \) to be the space of matrices, whose vectorization lies in the row space of \( \bar{V}_1^T \).

As we follow the same strategy as the one used in Section 2.2.1 to approximate the space \( A = \text{span}\{a_i, \, i = 1, \ldots, m\} \), we limit ourselves to reformulate it in the context of the vector space of matrices \( A \). We start with a technical estimate, which can be shown with the same proof as in Theorem 6.

Lemma 7. Assume the vectors \( (a_i)_{i=1}^m \) satisfy \( \|a_i\|_2 = 1 \) for all \( i = 1, \ldots, m \) and assume that \( g_j, j = 1, \ldots, m, \) are two times differentiable with the Lipschitz constant of all \( g_j' \) bounded from above by \( C_3 > 0 \). Then, for all \( x \in \mathbb{S}^{m-1} \),
\[
\|H[f](x) - \Delta[f](x)\|_F \leq 2C_3m\epsilon.
\]

Proof. By the same argument as in the proof of Theorem 6
\[
|(H[f](x))_{j,k} - (\Delta[f](x))_{j,k}| \leq C_3\epsilon \sum_{i=1}^m (a_{i,j}^2 |a_{i,k}| + |a_{i,j}| a_{i,k}^2).
\]
Using triangle inequality and \( \|a_j\|_4 \leq \|a_j\|_2 = 1 \), we estimate
\[
\|H[f](x) - \Delta[f](x)\|_F \leq 2C_3\epsilon \left[ \sum_{i=1}^m \left( \sum_{j,k=1}^m a_{i,j}^2 |a_{i,k}| \right)^2 \right]^{1/2} \leq 2C_3\epsilon \sum_{i=1}^m \left( \sum_{j,k=1}^m a_{i,j}^4 a_{i,k}^2 \right)^{1/2} \leq 2C_3\epsilon m.
\]

Theorem 8. Assume the vectors \( (a_i)_{i=1}^m \) linearly independent, and \( \|a_i\|_2 = 1 \) for all \( i = 1, \ldots, m \). Additionally assume
\[
C_j := \max_{i=1, \ldots, m-1} \max_{1 \leq t \leq 1} |g_{t,j}^{(j)}(t)| < \infty, \quad j = 0, 1, 2.
\]
Let \( \bar{A} \) be constructed as described in Algorithm 3 by sampling \( m_X \) \( [(m+1)(m+2)/2] \) values of \( f \). Let \( 0 < s < 1 \), and assume \( \sigma_m(H_2[f]) \geq \alpha_2 > 0 \), i.e., the \( m \)th singular value of the matrix \( H_2[f] \) is bounded away from zero. Then
\[
\|P_{\bar{A}} - P_{\bar{A}}\|_{F \to F} \leq \frac{4C_3m\epsilon}{\sqrt{\alpha_2(1-s)} - 2C_3m\epsilon}.
\]
with probability at least \(1 - m \exp\left(- \frac{m \alpha^2 \sigma^2}{2m^2 C_2^2}\right)\). In particular \(\dim(A) = \dim(\tilde{A}) = m\).

**Proof.** We define the matrices \(X, Y\) whose columns are given by \(\text{vec}(H[f](x_j)), j = 1, \ldots, m_X\) and \(\text{vec}(\Delta[f](x_j)), j = 1, \ldots, m_X\) respectively, namely

\[X = (\text{vec}(H[f](x_1)) \ldots | \text{vec}(H[f](x_{m_X}))), \quad Y = (\text{vec}(\Delta[f](x_1)) \ldots | \text{vec}(\Delta[f](x_{m_X}))).\]

Notice that these matrices have dimension \(m^2 \times m_X\). As done in (24) and by assuming for the moment that \(\sigma_m(X) \neq 0\) (but obviously \(\sigma_{m+1}(X) = 0\) because the \(H[f](x_i)\)'s lie all in the \(m\)-dimensional space \(A\)), we deduce the estimate

\[
\|P_A - P_{\tilde{A}}\|_{F \rightarrow F} \leq \frac{2\|X - Y\|_F}{\sigma_m(X) - \|X - Y\|_F},
\]

as an application of Wedin’s bound, Theorem 23 in the Appendix. From Lemma 7 we easily deduce

\[
\|X - Y\|_F = \left(\sum_{j=1}^{m_X} \|H[f](x_j) - \Delta[f](x_j)\|_F^2\right)^{1/2} \leq 2C_3 \epsilon m \sqrt{m_X}.
\]

In order to apply (32) we need finally to estimate \(\sigma_m(X)\) from below and we shall do it by using again the Chernoff’s bound for matrices Theorem 24.

Given an orthonormal basis \(\{B_1, \ldots, B_m\}\) for \(A\) we define the projector from \(\mathbb{R}^{m^2} \rightarrow \mathbb{R}^m\) given by \(P_A v = (\text{vec}(B_1)^T v, \ldots, (\text{vec}(B_m)^T v)\) for any \(v \in \mathbb{R}^{m^2}\). We additionally define with some abuse of notation

\[
P_A X := (P_A (\text{vec}(H[f](x_1))) \ldots | (P_A \text{vec}(H[f](x_{m_X}))))).
\]

Notice that now this matrix has dimension \(m \times m_X\). Thanks to the fact that \(P_A\) is an orthogonal transformation, we obtain the following equivalences

\[
\sigma_m(X) = \sqrt{\sigma_m((P_A X)(P_A X)^T)}.
\]

Hence to estimate \(\sigma_m(X)\) is sufficient to do it for \(\sigma_m((P_A X)(P_A X)^T)\), whose argument is explicitly expressed as a sum

\[
(P_A X)(P_A X)^T = \sum_{j=1}^{m_X} X_j,
\]

where

\[
X_j = P_A \text{vec}(H[f](x_j)) \otimes \text{vec}(H[f](x_j))(P_A)^T.
\]

We wish to apply Theorem 24 for the sequence of positive semidefinite matrices \(X_1, \ldots, X_{m_X}\). We notice first that

\[
\mathbb{E}X_j = P_A H^T \frac{1}{2} (P_A)^T,
\]

and therefore

\[
\mu_{\min}(\mathbb{E}X_j) \geq m_X \alpha_2.
\]

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Additionally, for every \( x \in \mathbb{S}^{m-1} \)

\[
\sigma_1 (P^A \text{vec}(H[f](x)) \otimes \text{vec}(H[f](x))(P^A)^T) = \sigma_1 (\text{vec}(H[f](x)) \otimes \text{vec}(H[f](x))) (35)
\]

\[
\leq \|H[f](x_j)\|_F^2
\]

\[
= \| \sum_{i=1}^m g''(a_i \cdot x)a_i \otimes a_i \|_F^2
\]

\[
\leq C_2^2 m^2 .
\]

An application of Theorem 24 under conditions (34) and (36) yields

\[
\sigma_m (X) \geq \sqrt{m \alpha^2 (1 - s)},
\]

(36)

with probability

\[
1 - m \exp \left( - \frac{m \alpha^2 s^2}{2m^2 C_2^2} \right).
\]

We conclude from (33) and (36) that, with the same probability

\[
\| P_A - \tilde{P}_A \|_{F \rightarrow F} \leq \frac{4C_3m \epsilon}{\sqrt{\alpha^2 (1 - s) - 2C_3m \epsilon}}.
\]

(37)

### 3.2 Properties of local maximizers

In this section we return to the analysis of the optimization program (28), i.e.

\[
\text{arg max} \| M \|_\infty, \quad \text{s.t.} \quad M \in \tilde{A}, \quad \| M \|_F \leq 1,
\]

(38)

and we derive a characterization of its local maximal solutions. First of all let us observe that every local maximizer of (38) will be always found on the sphere \( \mathbb{S} \tilde{A} = \{ M \in \tilde{A} : \| M \|_F = 1 \} \). The set \( \mathbb{S} \tilde{A} \) is a unit sphere in a Hilbert space of (symmetric) matrices, intersected with a linear subspace, and therefore everywhere differentiable. Despite the nonsmoothness of the objective function, i.e. \( M \rightarrow \| M \|_\infty \), the solution of the nonconvex program (38) will be tackled by means of differential methods.

Our general approach is the following. Let us assume, that \( M \) is a local maximizer of (38), then there is a neighborhood \( U \subset \mathbb{S} \tilde{A} \) of \( M \) on the sphere \( \mathbb{S} \tilde{A} \), such that \( \| X \|_\infty \leq \| M \|_\infty \) for every \( X \in U \). Hence for every \( X \in \tilde{A} \), the function

\[
f_X : \gamma \rightarrow \frac{\| M + \gamma X \|_\infty}{\| M + \gamma X \|_F}
\]

(39)

has a local maximum in \( \gamma = 0 \). Furthermore, without loss of generality it is enough to restrict ourselves to matrices \( X \in \tilde{A} \) with \( \| X \|_F = 1 \) and \( X \perp M \) (in the scalar product \( \langle \cdot, \cdot \rangle_F \) induced by the Frobenius norm). In fact, by considering any \( X = \alpha M + X_\perp \) with \( X_\perp \perp M \) we have

\[
f_X (\gamma) = \frac{\| M + \gamma (\alpha M + X_\perp) \|_\infty}{\| M + \gamma (\alpha M + X_\perp) \|_F} = \frac{\| M + \gamma/(1 + \gamma \alpha)X_\perp \|_\infty}{\| M + \gamma/(1 + \gamma \alpha)X_\perp \|_F} = f_{X_\perp} (\gamma/(1 + \gamma \alpha)).
\]

Let now

\[
(M + \gamma X)u_j (\gamma) = \lambda_j (\gamma) u_j (\gamma), \quad j = 1, \ldots, m,
\]

(40)
be the spectral decomposition of \( M + \gamma X \) with eigenvalues \( \lambda_j(\gamma) \) and eigenvectors \( u_j(\gamma) \). For the sake of a simple introduction to the characterization result (we provide of it below a more formal proof), let us assume just for now that \( \lambda_j(\gamma) \) and \( u_j(\gamma) \) depend smoothly on \( \gamma \). For \( \gamma = 0 \), we denote briefly \( u_j = u_j(0) \) and \( \lambda_j = \lambda_j(0) \) and \( (41) \) becomes \( Mu_j = \lambda_j u_j \).

Due to \( \|M + \gamma X\|_F^{-1} = (1 + \gamma^2)^{-1/2} = 1 - \gamma^2/2 + o(\gamma^2) \), we obtain asymptotically for \( \gamma \to 0 \)

\[
f_X(\gamma) = (1 - \gamma^2/2) \cdot \max_{j:|\lambda_j(0)| = \|M\|_\infty} |\lambda_j(0) + \lambda_j'(0)\gamma + \lambda_j''(0)\gamma^2/2| + o(\gamma^2).
\]

(41)

We conclude, that if \( f_X \) has a local maximum in \( \gamma = 0 \), then, by a simple asymptotic argument for \( \gamma \to 0 \), we conclude

\[
\lambda_j'(0) = 0,
\]

(42)

for all \( j \in \{1, \ldots, m\} \) with \( |\lambda_j(0)| = \|M\|_\infty \).

In order to determine \( \lambda_j'(0) \), we differentiate \( (40) \)

\[
Mu_j'(\gamma) + Xu_j(\gamma) + \gamma Xu_j'(\gamma) = \lambda_j'(\gamma)u_j(\gamma) + \lambda_j(\gamma)u_j'(\gamma),
\]

(43)

evaluate \( (43) \) in \( \gamma = 0 \) and multiply it with \( u_j \). We obtain

\[
(u_j')^TMu_j + u_j^TXu_j = \lambda_j'(0) + \lambda_ju_j^Tu_j.
\]

We now plug in the relation \( Mu_j = \lambda_j u_j \) together with \( (u_j')^Tu_j = 0 \), which follows by differentiating the orthogonality relation \( \langle u_i(\gamma), u_j(\gamma) \rangle = \delta_{i,j} \), and obtain

\[
\lambda_j'(0) = u_j^TXu_j.
\]

In view of \( (42) \), we have in particular

\[
0 = u_j^TXu_j.
\]

for all \( j \in \{1, \ldots, m\} \) with \( |\lambda_j(0)| = \|M\|_\infty \). The latter equations are actually the first order optimality conditions for \( M \) being an extremal point for \( (38) \).

To distinguish between local minimizers and local maximizers, we study also the second derivatives. Again for the sake of simplicity, we assume now that the largest eigenvalue of \( M \) is simple (below we actually prove this property). As we can always exchange \( M \) with \(-M\), we shall assume in the sequel that \( \lambda_1 = \|M\|_\infty > \max\{\lambda_2, \ldots, \lambda_m\} \). In this case we reformulate \( (41) \) using \( \lambda_1'(0) = 0 \) and \( (41) \) becomes

\[
f_X(\gamma) = (1 - \gamma^2/2)(\lambda_1(0) + \lambda_1''(0)\gamma^2/2) + o(\gamma^2) = \lambda_1(0) + \frac{\lambda_1''(0) - \lambda_1(0)}{2}\gamma^2 + o(\gamma^2).
\]

(41)

If \( f_X \) has a local maximum at \( \gamma = 0 \), again by a simple asymptotic argument for \( \gamma \to 0 \), we conclude that

\[
\lambda_1''(0) \leq \lambda_1(0).
\]

(44)

We differentiate \( (43) \) with \( j = 1 \) to obtain

\[
Mu_1''(\gamma) + 2Xu_1'(\gamma) + \gamma Xu_1'(\gamma) = \lambda_1''(\gamma)u_1(\gamma) + 2\lambda_1'(\gamma)u_1'(\gamma) + \lambda_1(\gamma)u_1''(\gamma).
\]

We evaluate this equation at \( \gamma = 0 \) and take again the inner product with \( u_1 \), yielding

\[
u_1^TMu_1'' + 2u_1^TXu_1' = \lambda_1''(0) + 2\lambda_1'(0)u_1^Tu_1' + \lambda_1u_1^Tu_1''.
\]
Using $u_1^T M u_1'' = \lambda_1 u_1^T u_1''$ and $u_1^T u_1' = 0$, the equation becomes

$$\lambda_1''(0) = 2 u_1^T X u_1'.$$

For eliminating $u_1'$, we multiply (43) for $j = 1$ with $u_k, k \neq 1$ at $\gamma = 0$. This gives

$$u_k^T M u_1' + u_k^T X u_1 = \lambda_1'(0) u_k^T u_1 + \lambda_1 u_k^T u_1'.$$

Using $u_k^T M u_1' = \lambda_k u_k^T u_1'$ and $u_k^T u_1 = 0$, this can be reformulated as $u_k^T u_1' = (u_k^T X u_1)/(\lambda_1 - \lambda_k)$ for $\lambda_1 \neq \lambda_k$. Hence

$$u_1' = \sum_{k=2}^{m} \langle u_1', u_k \rangle u_k = \sum_{k=2}^{m} \frac{(u_k^T X u_1)}{\lambda_1 - \lambda_k} u_k$$

and (44) becomes

$$2 u_1^T X \left( \sum_{k=2}^{m} \frac{u_k^T X u_k}{\lambda_1 - \lambda_k} \right) = 2 \sum_{k=2}^{m} \frac{(u_1^T X u_k)^2}{\lambda_1 - \lambda_k} \leq \lambda_1$$

for all $X \in \mathcal{S}_{\widetilde{A}}$ with $X \perp M$. The equation (45) corresponds to the second order optimality condition for $M$ being a local maximizer for (38).

In the argument above we made heavy use of the additional requirement of smooth dependence of the spectral decomposition of $M + \gamma X$ on the parameter $\gamma$. We will show now, that the same is true even without such an assumption.

**Theorem 9.** Let $M$ be any local maximizer of

$$\arg \max \|M\|_\infty, \quad \text{s.t.} \quad M \in \widetilde{A}, \quad \|M\|_F \leq 1.$$  (46)

Then

$$u_j^T X u_j = 0 \quad \text{for all} \quad X \in \mathcal{S}_{\widetilde{A}} \quad \text{with} \quad X \perp M$$

and all $j \in \{1, \ldots, m\}$ with $|\lambda_j(0)| = \|M\|_\infty$.  (47)

If furthermore

$$S(a_1, \ldots, a_m) \leq \varepsilon \quad \text{and} \quad 3m \|P_A - P_{\widetilde{A}}\| < (1 - \varepsilon)^2,$$  (48)

then $|\lambda_1| = \|M\|_\infty$, $\lambda_1 \notin \{\lambda_2, \ldots, \lambda_m\}$ and

$$2 \sum_{k=2}^{m} \frac{(u_1^T X u_k)^2}{|\lambda_1 - \lambda_k|} \leq |\lambda_1| \quad \text{for all} \quad X \in \mathcal{S}_{\widetilde{A}} \quad \text{with} \quad X \perp M.$$  (49)

Before presenting the proof of this result let us add some comments on the orthogonality condition $X \perp M$.

**Remark 4.** (i) If $X \in \widetilde{A}$ is not orthogonal to $M$, we consider the matrix

$$\frac{X - \langle X, M \rangle M}{\|X - \langle X, M \rangle M\|_F}$$

for every $X \in \widetilde{A}$ not co-linear with $M$, and can rewrite (47) as

$$u_j^T X u_j = \langle X, M \rangle \|M\|_\infty \quad \text{for every} \quad X \in \widetilde{A},$$  (50)
Proof of Theorem 9.

(i) The formulas (47) and (49) resemble very much the so-called first and second Hadamard variation formula, cf. [33, Chapter 1.3]. At least in the case when the spectrum of $M$ contains only simple eigenvalues, the proof we give resembles very much the one in [33].

Proof of Theorem 9

Step 1. Proof of (47)

Let us assume, that $M \in \tilde{A}$ is fixed and that $f_X$ has local maximum at $\gamma = 0$ for $X \in \tilde{A}$ with $\|X\|_F = 1$ and $X \perp M$. Hence, for $|\gamma|$ small, we have

$$\|M\|_\infty \geq \frac{\|M + \gamma X\|_\infty}{\|M + \gamma X\|_F} \geq \left(1 - \gamma^2/2 + o(\gamma^2)\right) \cdot \max_{j=1, \ldots, m} |u_j^T(M + \gamma X)u_j|$$

$$= \max_{j=1, \ldots, m} \left|\lambda_j(0) + \gamma u_j^T X u_j\right| + O(\gamma^2).$$

Considering $j \in \{1, \ldots, m\}$ with $|\lambda_j(0)| = \|M\|_\infty$ and $|\gamma|$ small, we arrive to (47).

Step 2. Proof of (49)

We derive (49) under the assumption that $\lambda_1 = \|M\|_\infty$ and $\lambda_1 \notin \{\lambda_2, \ldots, \lambda_m\}$. If $\lambda_1 = -\|M\|_\infty$, the result follows by considering $-M$ instead of $M$. Let again $Mu_j = \lambda_j u_j$ be the singular value decomposition of $M$. Then

$$\|M + \gamma X\|_\infty = \sup_{\|\sigma\| \leq 1} \left(\sum_{i=1}^m \sigma_i u_i\right)^T (M + \gamma X) \left(\sum_{j=1}^m \sigma_j u_j\right)$$

$$= \sup_{\|\sigma\| \leq 1} \left(\sum_{i,j=1}^m \sigma_i \sigma_j u_i^T M u_j + \gamma \sum_{i,j=1}^m \sigma_i \sigma_j u_i^T X u_j\right)$$

$$= \sup_{\|\sigma\| \leq 1} \left(\sum_{i=1}^m \sigma_i \lambda_i + \gamma \sum_{i,j=1}^m \sigma_i \sigma_j A_{ij}\right) =: \sup_{\|\sigma\| \leq 1} f(\sigma),$$

where $A_{ij} = u_i^T X u_j = A_{ji}$. We will use an approximate solution of the Lagrange’s multiplier equations to estimate $\|M + \gamma X\|_\infty$ from below.

We set the constraint condition $g(\sigma) = \|\sigma\|^2_H = 1$ and use Lagrange’s multiplier theorem on

$$\theta(\sigma, \nu) := f(\sigma) + \nu (g(\sigma) - 1).$$

This leads to equations

$$\frac{\partial \theta}{\partial \nu} = g(\sigma) - 1 = 0,$$

$$\frac{\partial \theta}{\partial \sigma_j} = 2\sigma_j \lambda_j + 2\gamma \sum_{i=1}^m \sigma_i A_{ij} + \nu \cdot 2\sigma_j = 0, \quad j = 1, \ldots, m.$$ (53)
For \( j = 1 \) we use \( A_{11} = u_1^T X u_1 = 0 \) and (53) becomes

\[
\sigma_1(\lambda_1 + \nu) = -\gamma \sum_{j=2}^{m} \sigma_j A_{1j}.
\] (54)

If \( j \geq 2 \), we reduce (53) by the following observation. The optimal value of \( \sigma \) in (52) for \( \gamma = 0 \) is \( \sigma = e_1 = (1, 0, \ldots, 0)^T \). We therefore expect that for \( |\gamma| \) small, the optimal value of \( \sigma \) in (52) will be close to \( e_1 \), i.e. \( \sigma_2, \ldots, \sigma_m \) are expected to be of order \( \gamma \). The values \( A_{ij} \) with \( i, j \geq 2 \) therefore come into the value of \( f(\sigma) \) only in the third order in \( \gamma \) and may be neglected. Then (53) becomes

\[
\sigma_j(\lambda_j + \nu) = -\gamma \sigma_1 A_{1j}.
\]

Finally, (54) shows that \( \nu \) is close to \( -\lambda_1 \). We are then naturally led to chose \( \sigma \) according to the equations

\[
\sum_{j=1}^{m} \sigma_j^2 = 1, \quad \sigma_1 \neq 0 \quad \text{and} \quad \frac{\sigma_k}{\sigma_1} = \gamma \cdot \frac{u_1^T X u_k}{\lambda_1 - \lambda_k}, \quad k = 2, \ldots, m.
\] (55)

Up to the sign of \( \sigma \), there is exactly one solution to (55), which we plug into (52). This leads to

\[
\|M + \gamma X\|_\infty \geq f(\sigma) = \sum_{j=1}^{m} \sigma_j^2 \lambda_j + \gamma \sum_{i,j=1}^{m} \sigma_i \sigma_j A_{ij}
\]

\[
= \sigma_1^2 \lambda_1 + \sum_{k=2}^{m} \sigma_k^2 \lambda_k + 2\gamma \sum_{j=2}^{m} \sigma_1 \sigma_j (u_1^T X u_j) + \gamma \sum_{i,j=2}^{m} \sigma_i \sigma_j (u_i^T X u_j)
\]

\[
= \sigma_1^2 \lambda_1 + \sum_{k=2}^{m} \lambda_k \gamma^2 \frac{\sigma_1^2 (u_1^T X u_k)^2}{(\lambda_1 - \lambda_k)^2} + 2\gamma^2 \sigma_1^2 \sum_{k=2}^{m} \frac{(u_1^T X u_k)^2}{\lambda_1 - \lambda_k} + o(\gamma^2)
\] (56)

\[
= \sigma_1^2 \lambda_1 + \gamma^2 \frac{\sigma_1^2}{\lambda_1 - \lambda_k} \left\{ \sum_{k=2}^{m} \lambda_k \right\} + \gamma^2 \frac{\sigma_1^2}{\lambda_1 - \lambda_k} \left\{ \sum_{k=2}^{m} \lambda_k \right\} + o(\gamma^2).
\]

Furthermore, from \( \|\sigma\|_2^2 = 1 \), we derive

\[
\sigma_1^2 + \sum_{k=2}^{m} \lambda_k \gamma^2 (\frac{u_1^T X u_k)^2}{(\lambda_1 - \lambda_k)^2} = \sigma_1^2 \left( 1 + \gamma^2 \sum_{k=2}^{m} (u_1^T X u_k)^2 \right) = 1,
\]

which, by the Taylor theorem, leads to

\[
\sigma_1^2 = \left( 1 + \gamma^2 \sum_{k=2}^{m} (u_1^T X u_k)^2 \right)^{-1} = 1 - \gamma^2 \sum_{k=2}^{m} \frac{(u_1^T X u_k)^2}{(\lambda_1 - \lambda_k)^2} + o(\gamma^2).
\]

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We plug this estimate into (56) and get
\[
f(\sigma) = \left(1 - \gamma^2 \sum_{k=2}^{m} \frac{(u_k^T X u_k)^2}{(\lambda_1 - \lambda_k)^2}\right) \cdot \left\{\lambda_1 + \gamma^2 \sum_{k=2}^{m} \frac{(u_k^T X u_k)^2}{(\lambda_1 - \lambda_k)^2}(2\lambda_1 - \lambda_k)\right\} + o(\gamma^2)
\]
\[
= \lambda_1 + \gamma^2 \sum_{k=2}^{m} \frac{(u_k^T X u_k)^2}{(\lambda_1 - \lambda_k)^2} \left\{-\frac{\lambda_1}{(\lambda_1 - \lambda_k)^2} + \frac{2\lambda_1 - \lambda_k}{(\lambda_1 - \lambda_k)^2}\right\} + o(\gamma^2)
\]
\[
= \lambda_1 + \gamma^2 \sum_{k=2}^{m} \frac{(u_k^T X u_k)^2}{(\lambda_1 - \lambda_k)^2} + o(\gamma^2).
\]
This allows to conclude that
\[
f_X(\gamma) = \frac{\|M + \gamma X\|_\infty}{\sqrt{1 + \gamma^2}} \geq f(\sigma)(1 - \gamma^2/2) = \lambda_1 + \gamma^2 \left(\sum_{k=2}^{m} \frac{(u_k^T X u_k)^2}{(\lambda_1 - \lambda_k)^2} - \frac{\lambda_1}{2}\right) + o(\gamma^2).
\]
If \(f_X\) has local maximum at \(\gamma = 0\), the coefficient at \(\gamma^2\) has to be smaller or equal to zero, giving (55).

**Step 3. Uniqueness of the largest eigenvalue**
We proceed by contradiction. Let (58) be fulfilled and let \(M \in \bar{A}\) with \(\|M\|_F = 1\) be a local maximizer of (58) with \(\lambda_1 = \lambda_2 = \|M\|_\infty\). The case \(\lambda_1 = \lambda_2 = -\|M\|_\infty\) follows in the same manner. Taking \(X \in S_{\bar{A}}\) with \(X \perp M\) and considering again the function \(f_X\) from (59), we can write
\[
f_X(\gamma) = \frac{\|M + \gamma X\|_\infty}{\|M + \gamma X\|_F} \geq \sup_{(\sigma_1, \sigma_2):\sigma_1^2 + \sigma_2^2 = 1} (\sigma_1 u_1 + \sigma_2 u_2)^T (M + \gamma X)(\sigma_1 u_1 + \sigma_2 u_2) + o(\gamma)
\]
\[
= \|M\|_\infty + \gamma \sup_{(\sigma_1, \sigma_2):\sigma_1^2 + \sigma_2^2 = 1} \{\sigma_1^2 u_1^T X u_1 + \sigma_2^2 u_2^T X u_2 + 2\sigma_1 \sigma_2 u_1^T X u_2\} + o(\gamma)
\]
If \(f_X\) has a local maximum at \(\gamma = 0\), we choose \((\sigma_1, \sigma_2)\) equal to \((1, 0), (0, 1),\) or \((1, 1)/\sqrt{2}\), respectively. We conclude that
\[
u_k^T X u_1 = u_k^T X u_2 = u_k^T X u_2 = 0.
\]
(57)
If \(X \in \bar{A}\) is not orthogonal to \(M\), we apply (57) to \(\frac{X - (X, M)M}{\|X - (X, M)M\|_2}\), cf. Remark 4 and obtain
\[
u_k^T X u_1 = (X, M) \cdot \|M\|_\infty = u_k^T X u_2,
\]
\[
u_k^T X u_2 = 0.
\]
We set \(X_j = P_{\bar{A}}(a_j \otimes a_j) - X_j = a_j \otimes a_j\). Then \(X_j \in \bar{A}\) and we derive from these conditions
\[
\langle u_1, a_j \rangle^2 + u_k^T \mathcal{E}_j u_1 = \langle u_2, a_j \rangle^2 + u_k^T \mathcal{E}_j u_2, \quad j = 1, \ldots, m,
\]
\[
\langle u_1, a_j \rangle \cdot \langle u_2, a_j \rangle = -u_k^T \mathcal{E}_j u_2, \quad j = 1, \ldots, m.
\]
Solving these equations for \(\langle u_1, a_j \rangle^2\), we arrive at
\[
\langle u_1, a_j \rangle^2 \leq \left|\frac{\|u_k^T \mathcal{E}_j u_2 - u_k^T \mathcal{E}_j u_1\|}{2}\right| + \sqrt{\|u_k^T \mathcal{E}_j u_2 - u_k^T \mathcal{E}_j u_1\|^2 + 4u_k^T \mathcal{E}_j u_2^2}
\]
\[
\leq \frac{2|u_k^T \mathcal{E}_j u_2 - u_k^T \mathcal{E}_j u_1| + 2|u_k^T \mathcal{E}_j u_2|}{2}
\]
\[
= |u_k^T \mathcal{E}_j u_2 - u_k^T \mathcal{E}_j u_1| + |u_k^T \mathcal{E}_j u_2| \leq 3\|\mathcal{E}_j\|_\infty.
\]
(58)
By assumption \( S(a_1, \ldots, a_m) \leq \varepsilon \), Lemma 21 and (58), we then obtain

\[
(1 - \varepsilon)^2 = (1 - \varepsilon)^2 \|u_1\|_2^2 \leq \|A^T u_1\|_2^2 = \sum_{j=1}^m \langle u_1, a_j \rangle^2 \leq 3 \sum_{j=1}^m \|E_j\|_\infty
\]

\[
\leq 3 \sum_{j=1}^m \|(P_{\tilde{A}} - P_A)(a_j \otimes a_j)\|_F \leq 3m \|P_{\tilde{A}} - P_A\|_{F \to F},
\]

which leads to a contradiction. This finishes the proof of Theorem 9.

3.3 Approximation of ridge profiles

We show how to use Theorem 9 to develop approximation schemes for sums of ridge functions. We proceed in two steps. In the first step we identify vectors \( \hat{a}_1, \ldots, \hat{a}_m \in \mathbb{R}^m \), which approximate the true ridge profiles \( a_1, \ldots, a_m \). In the second step we define with their help a function \( \hat{f} \), which is the uniform approximation of \( f \).

We show, how to use the conditions (50) and (51) to analyze the minimization problem (46). First, we summarize the notation and assumptions used throughout this section. We assume that

- \( a_1, \ldots, a_m \in \mathbb{R}^m \) are the unknown ridge profiles,
- \( \mathcal{A} = \text{span}\{a_j \otimes a_j, j = 1, \ldots, m\} \subset \mathbb{R}^{m \times m} \),
- the vectors \( a_1, \ldots, a_m \) are \( \varepsilon \)-nearly-ornthonormal, i.e., there is an orthonormal basis \( w_1, \ldots, w_m \), such that \( \left( \sum_{j=1}^m \|a_j - w_j\|_2^2 \right)^{1/2} = \varepsilon > 0 \),
- \( \hat{\mathcal{A}} = \text{span}\{w_j \otimes w_j, j = 1, \ldots, m\} \),
- \( \tilde{\mathcal{A}} \) is the approximation of \( \mathcal{A} \) available after the first step with \( \|P_{\hat{\mathcal{A}}} - P_{\tilde{\mathcal{A}}}\|_{F \to F} \leq \eta \) (Algorithm 3 and Theorem 8),
- by Lemma 22 in the Appendix we then have \( \|P_{\hat{\mathcal{A}}} - P_{\tilde{\mathcal{A}}}\|_{F \to F} \leq \|P_{\hat{\mathcal{A}}} - P_{\tilde{\mathcal{A}}}\|_{F \to F} + \|P_{\tilde{\mathcal{A}}} - P_{\mathcal{A}}\|_{F \to F} \leq 4\varepsilon + \eta =: \nu \).

We start with several lemmas needed later on. We will use throughout the notation just introduced.

**Lemma 10.** Let \( Z \in \hat{\mathcal{A}} \) and \( \nu < 1 \). Then

\[
\|P_{\tilde{\mathcal{A}}}(Z)\|_F \leq \|Z\|_F \leq \frac{1}{1 - \nu} \cdot \|P_{\tilde{\mathcal{A}}}(Z)\|_F.
\]

In particular, \( P_{\tilde{\mathcal{A}}} \) is bijective as a map from \( \hat{\mathcal{A}} \) to \( \tilde{\mathcal{A}} \).

**Proof.** Let \( Z \in \hat{\mathcal{A}} \). Then \( \|P_{\tilde{\mathcal{A}}}Z\|_F \leq \|Z\|_F \) and

\[
\|P_{\tilde{\mathcal{A}}}Z\|_F \geq \|P_{\mathcal{A}}(Z)\|_F - \|P_{\tilde{\mathcal{A}}} - P_{\mathcal{A}}\|(Z)\|_F \geq \|Z\|_F - \nu\|Z\|_F = (1 - \nu)\|Z\|_F.
\]

The inequality implies the injectivity of \( P_{\tilde{\mathcal{A}}} \) on \( \hat{\mathcal{A}} \) and from Theorem 8 we know that \( \dim(\tilde{\mathcal{A}}) = \dim(\mathcal{A}) = m \), hence \( P_{\tilde{\mathcal{A}}} \) is also surjective. 

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Lemma 10 ensures that for any $M \in \hat{A}$ with $\|M\|_F = 1$ there exists $Z = \sum_k \sigma_k w_k \otimes w_k \in \hat{A}$ such that $M = P_{\hat{A}}(Z)$ and

$$1 \leq \left\| \sum_{k=1}^m \sigma_k w_k \otimes w_k \right\|_F = \|\sigma\|_2 \leq \frac{1}{1-\nu}. \quad (59)$$

We will use this property repetitively below, especially for $M$ being a local maximizer of (38).

If $X = w_j \otimes w_j$ and $\|w_j\|_2 = \|u\|_2 = 1$, then

$$\|Xu\|_2^2 = \langle (w_j, u)w_j \rangle_2^2 = |\langle w_j, u \rangle|^2 = (u^T w_j)(w_j^T u) = u^T Xu.$$

If $X = P_{\hat{A}}(w_j \otimes w_j)$ instead, we expect the difference between $\|Xu\|_2^2$ and $u^T Xu$ to be small. This statement is made precise in the following lemma.

**Lemma 11.** Let $W_j = w_j \otimes w_j$, $X = P_{\hat{A}}(W_j)$ and $\|u\|_2 = 1$. Then

$$\left| \|Xu\|_2^2 - u^T Xu \right| \leq 2\nu.$$

**Proof.** Indeed, using $W_j = P_{\hat{A}}(W_j) = W_j^2$ we obtain

$$\left| \|Xu\|_2^2 - u^T Xu \right| = \left| \langle P_{\hat{A}}(W_j)u, P_{\hat{A}}(W_j)u \rangle - u^T (P_{\hat{A}}W_j)u \right|$$

$$= \left| u^T [(P_{\hat{A}}W_j)(P_{\hat{A}}W_j) - (P_{\hat{A}}W_j)]u \right| \leq \| (P_{\hat{A}}W_j)^2 - (P_{\hat{A}}W_j) \|_\infty$$

$$= \| [(P_{\hat{A}} - P_{\hat{A}})(W_j) + P_{\hat{A}}(W_j)]^2 - (P_{\hat{A}}W_j) + (P_{\hat{A}} - P_{\hat{A}})(W_j) \|_\infty$$

$$= \| [(P_{\hat{A}} - P_{\hat{A}})(W_j)][(P_{\hat{A}} - P_{\hat{A}})(W_j)] + P_{\hat{A}}(W_j) - (P_{\hat{A}} - P_{\hat{A}})(W_j) \|_\infty$$

$$= \| (P_{\hat{A}}W_j)(P_{\hat{A}} - P_{\hat{A}})(W_j)) + (P_{\hat{A}} - P_{\hat{A}})(W_j)(\text{Id} - W_j) \|_\infty \leq 2\nu. \quad \square$$

We show that the local maximizers $M$ of (38) are (possibly after replacing $M$ by $-M$) nearly positive semi-definite.

**Lemma 12.** Let $\nu \leq 1/4$ and let $M$ be a local maximizer of (10) with $\|M\|_\infty = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$ being its eigenvalues. Then

$$\lambda_m \geq \frac{2\nu}{\lambda_1} - 4\nu. \quad (60)$$

**Proof.** We plug $X = P_{\hat{A}}(w_j \otimes w_j)$ into (60) and obtain

$$u_1^T P_{\hat{A}}(w_j \otimes w_j)u_1 = (P_{\hat{A}}(w_j \otimes w_j), M)\lambda_1.$$ 

We observe now that from $\|P_{\hat{A}} - P_{\hat{A}}\|_F \leq \nu$, we have $|u_1^T [(P_{\hat{A}} - P_{\hat{A}})(w_j \otimes w_j)]u_1| \leq \nu$, implying $u_1^T [(P_{\hat{A}} - P_{\hat{A}})(w_j \otimes w_j)]u_1 \geq -\nu$ and

$$-\nu \leq \langle w_j, u_1 \rangle^2 + u_1^T [(P_{\hat{A}} - P_{\hat{A}})(w_j \otimes w_j)]u_1 = u_1^T (w_j \otimes w_j)u_1 + u_1^T [(P_{\hat{A}} - P_{\hat{A}})(w_j \otimes w_j)]u_1$$

$$= u_1^T P_{\hat{A}}(w_j \otimes w_j)u_1 = \lambda_1 (P_{\hat{A}}(w_j \otimes w_j), M) \geq \lambda_1 \sum_{k=1}^m \sigma_k (P_{\hat{A}}(w_j \otimes w_j), P_{\hat{A}}(w_k \otimes w_k)).$$

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We conclude that \( \maximizers \) is actually close to one of the ridge profiles. We show that the eigenvector corresponding to the largest eigenvalue of any of its local maximizers is
\[
\nu \leq \sigma_j \| P_{\hat{\mathcal{A}}} (w_j \otimes w_j) - P_{\hat{\mathcal{A}}} \| + \sum_{k \neq j} \sigma_k \| P_{\hat{\mathcal{A}}} (w_j \otimes w_j), P_{\hat{\mathcal{A}}} (w_k \otimes w_k) \|
\leq \sigma_j (1 - \nu)^2 + \left( \sum_{k \neq j} \sigma_k \right)^{1/2} \left( \sum_{k \neq j} \| (P_{\hat{\mathcal{A}}} - P_{\hat{\mathcal{A}}})(w_j \otimes w_j), w_k \otimes w_k \| \right)^{1/2}
\leq \sigma_j (1 - \nu)^2 + \| \sigma \|_2 \| (P_{\hat{\mathcal{A}}} - P_{\hat{\mathcal{A}}})(w_j \otimes w_j) \| \leq \sigma_j (1 - \nu)^2 + \frac{\nu}{1 - \nu}.
\]
We conclude that
\[
\min_{j=1,\ldots,m} \sigma_j \geq \left( \frac{\nu}{\lambda_1} + \frac{\nu}{1 - \nu} \right) \frac{1}{(1 - \nu)^2}.
\]
Finally,
\[
\lambda_m = u_m^T M u_m = \sum_{j=1}^m \sigma_j \langle P_{\hat{\mathcal{A}}} (w_j \otimes w_j), u_m \otimes u_m \rangle
= \sum_{j=1}^m \sigma_j \langle w_j \otimes w_j, u_m \otimes u_m \rangle + \sum_{j=1}^m \sigma_j \langle (P_{\hat{\mathcal{A}}} - P_{\hat{\mathcal{A}}})(w_j \otimes w_j), u_m \otimes u_m \rangle
\geq \left( \min_{j} \sigma_j \right) \sum_{j=1}^m \langle w_j, u_m \rangle^2 + \langle (P_{\hat{\mathcal{A}}} - P_{\hat{\mathcal{A}}}) \left( \sum_{j=1}^m \sigma_j w_j \otimes w_j \right), u_m \otimes u_m \rangle
\geq \left( \min_{j} \sigma_j \right) - \| P_{\hat{\mathcal{A}}} - P_{\hat{\mathcal{A}}} \| \left\| \sum_{j=1}^m \sigma_j w_j \otimes w_j \right\|_F
\geq \left( \min_{j} \sigma_j \right) - \frac{\nu}{1 - \nu} \geq - \left( \frac{\nu}{\lambda_1} + \frac{\nu}{1 - \nu} \right) \frac{1}{(1 - \nu)^2} - \frac{\nu}{1 - \nu},
\]
where in the second last inequality we used (69), and the result follows now by simple algebraic computations for \( \nu \leq 1/4 \).

The recovery algorithm based on the optimization problem (46) is quite straightforward. We show that the eigenvector corresponding to the largest eigenvalue of any of its local maximizers is actually close to one of the ridge profiles.

**Algorithm 4.**

- **Let** \( M \) **be a local maximizer of** (46)
  - **If** \( \| M \|_\infty \) **is not an eigenvalue of** \( M \), **replace** \( M \) **by** \( -M \)
  - **Denote** by \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \) **the eigenvalues of** \( M \) **arranged in decreasing order**
  - **Take** the eigenvalue decomposition of \( M \), i.e. \( M = \sum_{j=1}^m \lambda_j u_j \otimes u_j \)
  - **Put** \( \hat{\mathcal{A}} := u_1 \)

The performance of Algorithm 4 is guaranteed by the following theorem.
Theorem 13. If $0 < \nu < 1/(6m)$, then there is $j_0 \in \{1, \ldots, m\}$, such that the vector $\hat{a}$ found by Algorithm 4 satisfies $\|\hat{a} - a_{j_0}\|_2 \leq 5\nu$.

The proof of this theorem, which we report below, is fundamentally based on proving the following bound

$$\|M\|_\infty = \lambda_1 \geq 1 - \epsilon' \nu,$$  \hfill (62)

for some $\epsilon' > 0$ and for any local maximizers $M$ of (46). This will allow to ensure a sufficient spectral gap to apply the Wedin’s bound (Theorem 23) for showing good approximation properties of $\hat{a}$ as in Algorithm 4 to one of the ridge directions $a_1, \ldots, a_m$. We shall obtain (62) by a bootstrap argument: first we need to establish a weaker bound

$$\lambda_1 > \frac{\nu}{1 - \nu},$$

and use it for deducing (62).

Lemma 14. Assume $0 < \nu < 1/(6m)$. Then the following inequalities hold:

$$\|Xu_1\|_2^2 \leq \lambda_1^2 \cdot \frac{1 + \langle X, M \rangle^2}{2} + 3\nu,$$  \hfill (63)

for any $X \in \tilde{\mathbb{A}}$ such that $\|X\|_F \leq 1$ and

$$\lambda_1 > \frac{\nu}{1 - \nu}.$$  \hfill (64)

Proof. Let $M$ be any of the local maximizers of (46) with $\lambda_1 = \|M\|_\infty$. Further let $X \in \tilde{\mathbb{A}}$. We estimate the left-hand side of (51) using (50)

$$\begin{align*}
2 \sum_{k=2}^m \frac{(u_1^T Xu_k)^2}{\lambda_1 - \lambda_k} &\geq 2 \min \left( \frac{1}{\lambda_1 - \lambda_k}, \sum_{k=2}^m (u_1^T Xu_k)^2 \right) = \frac{2}{\lambda_1 - \lambda_m} \sum_{k=2}^m \langle Xu_1, u_k \rangle^2 \\
&= \frac{2}{\lambda_1 - \lambda_m} \left( \|Xu_1\|_2^2 - \langle Xu_1, u_1 \rangle^2 \right) \\
&= \frac{2}{\lambda_1 - \lambda_m} \left( \|Xu_1\|_2^2 - \lambda_1^2 \langle X, M \rangle^2 \right).
\end{align*}

Together with (51), this leads to

$$\frac{2}{\lambda_1 - \lambda_m} \left( \|Xu_1\|_2^2 - \lambda_1^2 \langle X, M \rangle^2 \right) \leq \lambda_1(\|X\|_F^2 - \langle X, M \rangle^2).$$

If moreover $\|X\|_F \leq 1$, $\nu \leq \frac{1}{4}$, and using (60) we conclude that

$$\begin{align*}
\|Xu_1\|_2^2 &\leq \frac{\lambda_1(1 - \lambda_m)}{2} (\|X\|_F^2 - \langle X, M \rangle_F^2) + \lambda_1^2 \langle X, M \rangle_F^2 \\
&\leq \lambda_1^2 \left( \frac{1 + \langle X, M \rangle^2}{2} - \lambda_1 \lambda_m \cdot \frac{1 - \langle X, M \rangle^2}{2} \right) \\
&\leq \lambda_1^2 \cdot \frac{1 + \langle X, M \rangle^2}{2} + 2\lambda_1 \left( \frac{\nu}{\lambda_1} + 2\nu \right) \cdot \frac{1 - \langle X, M \rangle^2}{2} \\
&\leq \lambda_1^2 \cdot \frac{1 + \langle X, M \rangle^2}{2} + 3\nu.
\end{align*}$$  \hfill (65)
Let us choose $X = \sum_{k=1}^{m} x_k P \tilde{A} (w_k \otimes w_k)$ with $\|x\|_2 \leq 1$, a generic element in $\tilde{A}$ such that $\|X\|_F \leq 1$. We compute

$$\|X u_1\|_2^2 = \left\| \sum_{k=1}^{m} x_k P \tilde{A} (w_k \otimes w_k) u_1 \right\|_2^2 \geq \left\| \sum_{k=1}^{m} x_k (w_k \otimes w_k) u_1 \right\|_2^2 - \left\| \sum_{k=1}^{m} x_k (P \tilde{A} - P \hat{A}) (w_k \otimes w_k) u_1 \right\|_2^2 \geq (1 - \nu) \left\| \sum_{k=1}^{m} x_k \langle u_1, w_k \rangle w_k \right\|_2 = (1 - \nu) \left( \sum_{k=1}^{m} x_k^2 \langle u_1, w_k \rangle^2 \right)^{1/2}.$$ 

By choosing $x_k = 1/\sqrt{m}$ for all $k = 1, \ldots, m$, we obtain

$$\|X u_1\|_2^2 \geq \frac{(1 - \nu)^2}{m} \sum_{k=1}^{m} \langle u_1, w_k \rangle^2 = \frac{(1 - \nu)^2}{m}.$$ 

Combining with (65) yields

$$\frac{(1 - \nu)^2}{m} \leq \|X u_1\|_2^2 \leq \lambda_1^2 \cdot \frac{1 + \langle X, M \rangle^2}{2} + 3\nu \leq \lambda_1^2 + 3\nu,$$ 

or

$$\lambda_1^2 \geq \frac{(1 - \nu)^2}{m} - 3\nu.$$ 

In order to show that $\lambda_1 > \frac{\nu}{1 - \nu}$, it is in fact sufficient to show that for $\nu \leq 1/(6m)$

$$\frac{(1 - \nu)^2}{m} - 3\nu > \frac{16}{9} \nu^2.$$ (66)

Indeed one would have then

$$\lambda_1^2 \geq \frac{(1 - \nu)^2}{m} - 3\nu > \frac{16}{9} \nu^2 = \frac{(1 - 1/4)^2}{(1 - \nu)^2} \geq \frac{\nu^2}{(1 - \nu)^2}.$$ 

The quadratic inequality (66) is in fact fulfilled for $0 < \nu < 1/(6m)$.

**Proof of Theorem 13.** Let $M$ be any of the local maximizers of (46) with $\lambda_1 = \|M\|_\infty$. Further let $X \in \hat{A}$. If $X = P\tilde{A} (\omega_j \otimes \omega_j)$ by Lemma 11,

$$\|X u_1\|_2^2 \geq u_1^T X u_1 - 2\nu = \lambda_1 \langle X, M \rangle_F - 2\nu.$$ 

Using (63), we then arrive at

$$\lambda_1 \langle X, M \rangle_F - 2\nu \leq \lambda_1^2 \cdot \frac{1 + \langle X, M \rangle^2}{2} + 3\nu.$$ 

This can be further rewritten as

$$0 \leq \lambda_1^2 - 1 + (1 - \lambda_1 \langle X, M \rangle_F)^2 + 10\nu.$$ (67)
Assume now that there exists $Z = \sum_{k=1}^m \sigma_k (w_k \otimes w_k) \in \hat{A}$ such that $M = P_{\hat{A}} Z$ and using \eqref{eq:59}, we can estimate $\lambda_1$ from above

$$
\lambda_1 = u_1^T M u_1 = u_1^T \left( \sum_{k=1}^m \sigma_k P_{\hat{A}} (w_k \otimes w_k) \right) u_1
$$

$$= u_1^T (P_{\hat{A}} - P_{\hat{A}}) \left( \sum_{k=1}^m \sigma_k (w_k \otimes w_k) \right) u_1 + u_1^T \left( \sum_{k=1}^m \sigma_k (w_k \otimes w_k) \right) u_1
$$

$$\leq \nu \left\| \sum_{k=1}^m \sigma_k (w_k \otimes w_k) \right\|_F^2 + \sum_{k=1}^m \sigma_k \langle w_k, u_1 \rangle^2 \leq \frac{\nu}{1-\nu} + \max_{j=1, \ldots, m} \sigma_j.
$$

From Lemma \ref{lem:4} and in particular by \eqref{eq:63} we deduce that

$$
\max_{j=1, \ldots, m} \sigma_j \geq \lambda_1 - \frac{\nu}{1-\nu} > 0.
$$

Hence there exists certainly some $j$ for which $\sigma_j > 0$. If $X = P_{\hat{A}} (w_j \otimes w_j)$ we get for $\sigma_j > 0$

$$
\langle X, M \rangle = \langle P_{\hat{A}} (w_j \otimes w_j), M \rangle = \left\langle P_{\hat{A}} (w_j \otimes w_j), \sum_{k=1}^m \sigma_k P_{\hat{A}} (w_k \otimes w_k) \right\rangle
$$

$$= \sigma_j \langle P_{\hat{A}} (w_j \otimes w_j), P_{\hat{A}} (w_j \otimes w_j) \rangle + \langle w_j \otimes w_j, \sum_{k \neq j} \sigma_k (P_{\hat{A}} - P_{\hat{A}}) (w_k \otimes w_k) \rangle
$$

$$= \sigma_j \langle P_{\hat{A}} (w_j \otimes w_j), w_j \otimes w_j \rangle + \left\langle (P_{\hat{A}} - P_{\hat{A}}) \left( \sum_{k \neq j} \sigma_k w_j \otimes w_j \right), w_k \otimes w_k \right\rangle
$$

$$\geq \sigma_j \cdot (1-\nu) - \nu \|\sigma\|_2 \geq \sigma_j \cdot (1-\nu) - \frac{\nu}{1-\nu}.
$$

We conclude, that there is $j_0 \in \{1, \ldots, m\}$ with

$$
\langle P_{\hat{A}} (w_{j_0} \otimes w_{j_0}), M \rangle \geq (1-\nu) \max_{j=1, \ldots, m} \sigma_j - \frac{\nu}{1-\nu}.
$$

Combining \eqref{eq:69} with \eqref{eq:68}, we obtain for $\nu \leq 1/4$

$$
\langle P_{\hat{A}} (w_{j_0} \otimes w_{j_0}), M \rangle \geq \left( \lambda_1 - \frac{\nu}{1-\nu} \right) \cdot (1-\nu) - \frac{\nu}{1-\nu}
$$

$$= \lambda_1 (1-\nu) - \frac{2\nu}{1-\nu} \geq \lambda_1 (1-\nu) - \frac{5\nu}{2}
$$

and

$$0 \leq 1 - \lambda_1 \langle P_{\hat{A}} (w_{j_0} \otimes w_{j_0}), M \rangle \leq 1 - \lambda_1^2 (1-\nu) + \frac{5\lambda_1 \nu}{2}.
$$

Finally, \eqref{eq:67} with \eqref{eq:70} give

$$0 \leq \lambda_1^2 - 1 + \left( 1 - \lambda_1^2 (1-\nu) + \frac{5\lambda_1 \nu}{2} \right)^2 + 10\nu
$$

$$= \lambda_1^2 (\lambda_1^2 - 1) + \nu \left\{ -2\lambda_1^4 + \nu \lambda_1^4 - 2\lambda_1^2 + \frac{25}{4} \lambda_1^2 \nu + 5\lambda_1 - 5\lambda_1^3 (1-\nu) + 10 \right\}
$$

$$\leq \lambda_1^2 (\lambda_1^2 - 1) + c\nu \leq (\lambda_1^2 - 1) + c\nu,$$

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We conclude, that not only $\lambda_1 > \frac{\nu}{1-\nu}$, but we indeed have the significantly better lower bound

$$\lambda_1 \geq 1 - c' \nu,$$  \hfill (71)

for some $c' > 0$. Finally, we apply the Wedin’s bound, Theorem 23 in the Appendix, to

$$\tilde{B} = M = \sum_{j=1}^{m} |\lambda_j| \text{sign} \lambda_j u_j \otimes u_j \quad \text{and} \quad B = \sum_{k=1}^{m} |\sigma_k| \text{sign} \sigma_k w_k \otimes w_k.$$  

We assume without loss of generality that $\sigma_1 = \max_{k=1,\ldots,m} |\sigma_k|$. We observe that

$$\|\tilde{B} - B\|_F = \|P_{\tilde{A}}(B) - B\|_F = \|(P_{\tilde{A}} - P_A)(B)\|_F \leq \nu\|B\|_F = \nu\|\sigma\|_2 \leq \frac{\nu}{1-\nu}$$  

Moreover

$$\left(\sum_{j=1}^{m} |\lambda_j|^2\right)^{1/2} \leq 1 \quad \text{and} \quad \lambda_1 \geq 1 - c' \nu$$

imply $|\lambda_k| \leq c' \nu$ for all $k = 2,\ldots,m$. Moreover,

$$\frac{1}{1-\nu} \geq \sigma_1 \geq \lambda_1 - \frac{\nu}{1-\nu} \geq 1 - \left(\frac{1}{1-\nu} + c'\right) \nu$$

imply $|\sigma_k| \leq \frac{2 + c'(1-\nu)}{1-\nu} \nu$ for all $k = 2,\ldots,m$. We deduce that we can choose $\tilde{\alpha} \geq \frac{1}{2}$ as in Theorem 23 for $0 < \nu < \nu_0$ small enough, i.e.,

$$\min_k |\lambda_1 - \sigma_k| \geq \left| 1 - c' \nu - \frac{2 + c'(1-\nu)}{1-\nu} \nu \right| \geq \tilde{\alpha} \geq 1/2,$$  \hfill (72)

and

$$|\lambda_1| \geq |1 - c' \nu| \geq \tilde{\alpha} \geq 1/2,$$  \hfill (73)

are verified for $0 < \nu < \nu_0$ small enough. We do not specify $\nu_0 > 0$ but its existence follows by a simple continuity argument. We therefore obtain $\|u_1 \otimes u_1 - w_1 \otimes w_1\|_F \leq 4\nu$. After a possible sign change of $w_1$, we can assume that $\langle u_1, w_1 \rangle \geq 0$ and obtain

$$16\nu^2 \geq \|u_1 \otimes u_1 - w_1 \otimes w_1\|_F^2 = 2(1 - \langle u_1, w_1 \rangle)^2 \geq 2(1 - \langle u_1, w_1 \rangle) = \|u_1 - w_1\|_2^2$$

and, finally,

$$\|u_1 - a_1\|_2 \leq \|u_1 - w_1\|_2 + \|w_1 - a_1\|_2 \leq 4\nu + \varepsilon \leq 5\nu.$$

\[\square\]

### 3.4 A simple algorithm to approach the nonlinear program \([28]\)

Let us describe in this section how to approach practically the solution of the nonlinear program \([28]\). Let us introduce first for a given parameter $\gamma > 1$ an operator acting on the singular values of a matrix $X = U\Sigma V^T$ as follows. If $\Sigma \in \mathbb{R}^{m \times m}$ is a diagonal matrix with the singular values of $X$ denoted by $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_m$ on the diagonal, we set

$$\Pi_\gamma(X) = \frac{1}{\sqrt{\gamma^2 \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_m^2}} U \begin{pmatrix} \gamma \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & 0 & \cdots \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \sigma_m \end{pmatrix} V^T.$$  

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Notice that $\Pi_\gamma$ maps any matrix $X$ onto a matrix of unit Frobenius norm, simply exalting the first singular value and damping the others. It is not a linear operator. Furthermore, the definition of $\Pi_\gamma(X)$ is not well-posed if $\sigma_1 = \sigma_2 \geq \sigma_3 \geq \cdots \geq \sigma_m$ and in this case it is assumed that a choice of ordering is made for just this one application of $\Pi_\gamma(\cdot)$. Notice that if $\sigma_1 > 1/\sqrt{2}$ and $\|X\|_F \leq 1$, then $\sigma_1 > \sigma_2$ and $\Pi_\gamma(X)$ is well-defined. This is the case for example under the conditions of Lemma 16.

We propose the following algorithm

**Algorithm 5.**

- Fix a suitable parameter $\gamma > 1$
- Generate an initial guess $X^0 = \sum_{j=1}^m \zeta_j \Delta |f|(0)(x_j)$, e.g., by choosing at random $\zeta_j \geq 0$, so that $X^0 \in \tilde{A}$ and $\|X^0\|_F = 1$;
- For $\ell \geq 0$:
  \[ X^{\ell+1} := P_{\tilde{A}} \Pi_\gamma(X^\ell). \]

This algorithm performs essentially an iteratively projected gradient ascent as the two operations executed within the loop are respectively a gradient ascent step towards the maximization of the spectral norm by means of $\Pi_\gamma$, and a projection back onto $\tilde{A}$ by $P_{\tilde{A}}$.

In the following we analyze some of the convergence properties of this algorithm and its relationship to (28). Assume for a moment now that $\tilde{A} = A$ and that $a_1, \ldots, a_m$ are orthonormal. In this case the algorithm can be rather trivially analyzed and performs a straightforward computation of one of the maximizers of (28). As we shall see later, such maximizer in this case coincides (up to the sign) with one of the matrices $a_j \otimes a_j$.

**Proposition 15.** Assume that $\tilde{A} = A$ and that $a_1, \ldots, a_m$ are orthonormal. Let $\gamma > \sqrt{2}$ and let $\|X^0\|_\infty > 1/\sqrt{\gamma^2 - 1}$. Then there exists $\mu_0 < 1$ such that

\[
1 - \|X^{\ell+1}\|_\infty \leq \mu_0 \left( 1 - \|X^\ell\|_\infty \right), \quad \text{for all } \ell \geq 0. \tag{74}
\]

Being the sequence $(X^\ell)_\ell$ made of matrices with Frobenius norm bounded by 1, we conclude that any accumulation point of it has both unit Frobenius and spectral norm and therefore it has to coincide with one maximizer of (28).

**Proof.** We can assume now that $X^0$ can already be expressed in terms of its singular value decomposition $X^0 = \sum_{j=1}^m \sigma_j(X^0) a_j \otimes a_j$. Since at each iteration $\|X^\ell\|_F \leq 1$ or $1 \geq \sum_{j=1}^m \sigma_j(X^\ell)^2$, it is a straightforward observation that

\[
\|X^{\ell+1}\|_\infty = \sigma_1(X^{\ell+1}) = \frac{\gamma \sigma_1(X^\ell)}{\sqrt{\gamma^2 \sigma_1(X^\ell)^2 + \sigma_2(X^\ell)^2 + \cdots + \sigma_m(X^\ell)^2}} \geq \frac{\gamma \sigma_1(X^\ell)}{\sqrt{(\gamma^2 - 1) \sigma_1(X^\ell)^2 + 1}}. \tag{75}
\]
Using elementary calculations we further estimate
\[
1 - \sigma_1(X^{\ell+1}) \leq \frac{\sqrt{(\gamma^2 - 1)\sigma_1(X^\ell)^2} + 1 - \gamma \sigma_1(X^\ell)}{\sqrt{(\gamma^2 - 1)\sigma_1(X^\ell)^2} + 1} = \frac{1 - \sigma_1(X^\ell)^2}{[\sqrt{(\gamma^2 - 1)\sigma_1(X^\ell)^2} + 1 + \gamma \sigma_1(X^\ell)]\sqrt{(\gamma^2 - 1)\sigma_1(X^\ell)^2} + 1} \leq \frac{2(1 - \sigma_1(X^\ell))}{(\gamma^2 - 1)\sigma_1(X^\ell)^2 + 1}
\]
and we get (74) with
\[
\mu_0 := \frac{2}{(\gamma^2 - 1)\|X^0\|_{\infty}^2 + 1} < 1.
\]

Let us now move away from the ideal case of the \( \tilde{A} = A \) and assume that \( \tilde{A} \) is only a good approximation to \( A \), in the sense that \( \|P_A - P_{\tilde{A}}\| \leq \varepsilon \).

**Remark 5.** As we learned already in the previous section, we can retain without loss of generality the assumption of \( a_1, \ldots, a_m \) being orthonormal to a certain extent. Indeed, were \( \{a_1, \ldots, a_m\} \) just \( \varepsilon \)-near-orthonormal and \( \{w_1, \ldots, w_m\} \) its approximating orthonormal basis, then we could denote \( a_i = a_i a_i^T \in \mathbb{R}^{m \times m} \), \( w_i = w_i w_i^T \in \mathbb{R}^{m \times m} \), \( A = \text{span}\{a_1, \ldots, a_m\} \), and \( \tilde{A} = \text{span}\{w_1, \ldots, w_m\} \). It is shown in Lemma 22 (iv), that
\[
(\sum_{i=1}^m \|a_i - w_i\|_F^2)^{1/2} \leq 2\varepsilon.
\]
Combining this result with Lemma 22 we would obtain \( \|P_A - P_{\tilde{A}}\|_{F \to F} \leq 8\varepsilon \). Hence, at the price of changing slightly the reference orthonormal basis and accepting some additional approximation error of order \( \varepsilon \), also in the case of a \( \varepsilon \)-near-orthonormal system of vectors we can reduce the arguments to the case of an orthonormal system.

Unfortunately, in the perturbed case \( \tilde{A} \neq A \), there is no direct way of estimating \( \|X^{\ell+1}\|_{\infty} \) by some function of \( \|X^{\ell}\|_{\infty} \) as it is done in (75) as the singular value decompositions of the matrices \( X^{\ell+1} \) and \( X^{\ell} \) are in principle different. However, the singular vectors of both these matrices can be approximated by \( \{a_1, \ldots, a_m\} \) (we reiterate that here we assume them orthonormal) and we need to take advantage of this reference orthonormal system. First, we need to show a certain continuity property of the operator \( \Pi_{\gamma} \).

**Lemma 16.** Assume \( X, \tilde{X} \) to be two matrices in \( \mathbb{R}^{m \times m} \) with respective singular value decompositions \( X = U\Sigma V^T \) and \( \tilde{X} = \tilde{U}\tilde{\Sigma}\tilde{V}^T \). Let us also assume that \( \|X - \tilde{X}\|_F \leq \varepsilon \) for some \( 0 < \varepsilon < 1 \). Assume additionally that \( \max \{\|\tilde{X}\|_F, \|X\|_F\} \leq 1 \) and \( \sigma_1(X) \geq t_0 := \frac{1}{\sqrt{2}} + \varepsilon + \xi, \) for \( \xi > 0 \). Then, for \( \gamma > 1 \)
\[
\|\Pi_{\gamma}(X) - \Pi_{\gamma}(\tilde{X})\|_F \leq 2^{3/2}\varepsilon + \frac{4\varepsilon}{\xi} + 2\sqrt{1 - (t_0 - \varepsilon)} := \mu_1(\gamma, t_0, \varepsilon)
\]
Notice in particular that \( \mu_1(\gamma, t_0, \varepsilon) \to 0 \) for \( (t_0, \varepsilon) \to (1, 0) \).

**Proof.** As \( \sigma_1 := \sigma_1(X) \geq t_0 \) and \( \sigma_1^2 + \cdots + \sigma_m^2 \leq 1 \), we have also \( \sum_{j=2}^m \sigma_j^2 \leq 1 - t_0^2 \).
By the assumption \( \|X - \tilde{X}\|_F \leq \varepsilon \) and by the well known Mirsky’s bound we have that \( \|\Sigma - \tilde{\Sigma}\|_F \leq \varepsilon \).

Hence, \( \tilde{\sigma}_1 := \sigma_1(\tilde{X}) \geq t_0 - \varepsilon, \tilde{\sigma}_j := \sigma_j(\tilde{X}) \leq \sqrt{1 - t_0^2} + \varepsilon \) and
\[
|\tilde{\sigma}_1 - \sigma_j| \geq t_0 - \varepsilon - \sqrt{1 - t_0^2} := \tilde{\alpha} > 0,
\]

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for all \( j = 2, \ldots, m \). The positivity of \( \hat{\alpha} > 0 \) comes from the assumption that \( t_0 = \frac{1}{\sqrt{2}} + \epsilon + \xi \). Hence, by applying Wedin’s bound, Theorem 24 in Appendix, we easily obtain

\[
\max\{\|u_1v_1^T - \bar{u}_1\bar{v}_1^T\|_F, \|v_1v_1^T - \bar{v}_1\bar{v}_1^T\|_F\} \leq \frac{2}{t_0 - \epsilon - \sqrt{1 - t_0^2}} \leq \frac{2\epsilon}{\xi}. \tag{78}
\]

The last inequality comes from \( 1 < 2(t_0 - \epsilon - \xi)^2 < (t_0 - \epsilon - \xi)^2 + t_0^2 \). For later use we notice already that for any unit-norm vectors \( x, \tilde{x} \in \mathbb{R}^m \)

\[
\|xx^T - \tilde{x}\tilde{x}^T\|_F^2 = \|xx^T\|_F^2 + \|\tilde{x}\tilde{x}^T\|_F^2 - 2\langle xx^T, \tilde{x}\tilde{x}^T \rangle = 2(1 - \langle x, \tilde{x} \rangle)
\]

and

\[
\|\langle x, \tilde{x} \rangle\|_2 = \sqrt{1 - \frac{\|xx^T - \tilde{x}\tilde{x}^T\|_F^2}{2}} \geq 1 - \frac{\|xx^T - \tilde{x}\tilde{x}^T\|_F^2}{2}.
\]

If moreover \( \langle x, \tilde{x} \rangle \geq 0 \), we get

\[
\|x - \tilde{x}\|_2^2 = 2(1 - \langle x, \tilde{x} \rangle) \leq \|xx^T - \tilde{x}\tilde{x}^T\|_F^2. \tag{79}
\]

We now address (77) by considering the estimates of different components of the singular value decompositions. We start by comparing the first singular value components. To simplify the notation, we set for \( s = (s_1, \ldots, s_m) \)

\[
\pi_\gamma(s) = \pi_\gamma(s_1, \ldots, s_m) = \frac{\gamma s_1}{\sqrt{\gamma^2 s_1^2 + s_2^2 + \cdots + s_m^2}}.
\]

We first derive a bound for \( \|u_1\pi_\gamma(\sigma)u_1^T - \bar{u}_1\pi_\gamma(\bar{\sigma})\bar{u}_1^T\|_F \), where \( \sigma = (\sigma_1, \ldots, \sigma_m) \) and similarly for \( \bar{\sigma} = (\bar{\sigma}_1, \ldots, \bar{\sigma}_m) \).

For that, we need first to show the Lipschitz continuity of the function \( s \to \pi_\gamma(s) \). From

\[
|\partial_{s_1} \pi_\gamma(s_1, \ldots, s_m)| = \frac{\gamma s_1}{|\sqrt{\gamma^2 s_1^2 + s_2^2 + \cdots + s_m^2}|^{3/2}}
\]

and

\[
|\partial_{s_j} \pi_\gamma(s_1, \ldots, s_m)| = \frac{\gamma s_1 s_j}{|\sqrt{\gamma^2 s_1^2 + s_2^2 + \cdots + s_m^2}|^{3/2}}
\]

we obtain for \( 1 \leq s_1 > t_0 - \epsilon \) and \( s_2^2 + \cdots + s_m^2 \leq 1 - t_0^2 \leq 1/2 \)

\[
\|\nabla_\pi_\gamma(s)\|_2^2 \leq \frac{\gamma^2 (s_1^2 + \cdots + s_m^2)^2 + \gamma^2 s_1^2 (s_2^2 + \cdots + s_m^2)}{(\gamma^2 s_1^2 + s_2^2 + \cdots + s_m^2)^3} \leq \frac{\gamma^2}{(\gamma^2 s_1^2 + s_2^2 + \cdots + s_m^2)^3} \leq \frac{1}{2\gamma^4 s_1^6}.
\]

Therefore

\[
|\pi_\gamma(\sigma) - \pi_\gamma(\bar{\sigma})| \leq \|\nabla_\pi_\gamma\|_2 \cdot \|\sigma - \bar{\sigma}\|_2 \leq \frac{\epsilon}{\sqrt{2s_1^2}} \leq 2\epsilon. \tag{80}
\]

As the signs of the singular vectors can be chosen arbitrarily, we can assume without loss of generality that \( \langle u_1, \bar{u}_1 \rangle \geq 0 \). Together with (78) and (79) we obtain \( \|u_1 - \bar{u}_1\|_2 \leq \frac{2\epsilon}{\xi} \)
and the same holds also for \(\|v_1 - \tilde{v}_1\|_2\). Therefore, we may estimate the difference of the first singular value components by
\[
\|u_1 \pi_1(\sigma) v_1^T - \tilde{u}_1 \pi_1(\tilde{\sigma}) \tilde{v}_1^T\|_F \leq \|(u_1 - \tilde{u}_1) \pi_1(\sigma) v_1^T\|_F + \|\tilde{u}_1 (\pi_1(\sigma) - \pi_1(\tilde{\sigma})) v_1^T\|_F + \|\tilde{u}_1 \pi_1(\tilde{\sigma})(v_1^T - \tilde{v}_1^T)\|_F \\
\leq \|u_1 - \tilde{u}_1\|_2 + |\pi_1(\sigma) - \pi_1(\tilde{\sigma})| + \|v_1 - \tilde{v}_1\|_2 \\
\leq \frac{2\epsilon}{\xi} + \max \left\{ \frac{1}{(t_0 - \epsilon)}, \frac{1}{\gamma^2(t_0 - \epsilon)^2} \right\} \epsilon + \frac{2\epsilon}{\xi} \\
\leq 2^{3/2}\epsilon + \frac{4\epsilon}{\xi}.
\] (81)

We now need to estimate the difference of the other components of the singular value decomposition. Now notice that
\[
\left\| \sum_{j=1}^{k} y_j \tilde{z}_j^T \right\|_F^2 = \sum_{j=1}^{k} \|y_j\|_2^2
\] (82)

for arbitrary vectors \(\{y_1, \ldots, y_k\} \subset \mathbb{R}^m\) and orthonormal vectors \(\{z_1, \ldots, z_k\} \subset \mathbb{R}^m\). By applying the triangle inequality and (82)
\[
\left\| \sum_{j=2}^{m} u_j \frac{\sigma_j}{\sqrt{\gamma^2 \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_m^2}} v_j^T - \sum_{j=2}^{m} \tilde{u}_j \frac{\tilde{\sigma}_j}{\sqrt{\gamma^2 \tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 + \cdots + \tilde{\sigma}_m^2}} \tilde{v}_j^T \right\|_F \\
\leq \left\| \sum_{j=2}^{m} u_j \frac{\sigma_j}{\sqrt{\gamma^2 \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_m^2}} v_j^T \right\|_F + \left\| \sum_{j=2}^{m} \tilde{u}_j \frac{\tilde{\sigma}_j}{\sqrt{\gamma^2 \tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 + \cdots + \tilde{\sigma}_m^2}} \tilde{v}_j^T \right\|_F \\
= \left( \frac{1 - t_0^2}{\gamma \tilde{u}_0} \right)^{1/2} + \left( \frac{1 - (t_0 - \epsilon)^2}{\gamma^2(t_0 - \epsilon)^2} \right)^{1/2} \leq \frac{\sqrt{1 - (t_0 - \epsilon)^2}}{\gamma(t_0 - \epsilon)} \leq \sqrt{1 - (t_0 - \epsilon)},
\] (83)
as \(t_0 - \epsilon > 1/\sqrt{2}\) and \(\frac{1-u}{u} \leq 2\sqrt{1-u}\) for \(1 > u > 1/\sqrt{2}\). The statement now follows by adding (81) and (83).

**Proposition 17.** Assume for that \(\|P_A - P_\mathcal{A}\|_{F \to F} < \epsilon < 1\) and that \(a_1, \ldots, a_m\) are orthonormal. Let \(\|X^0\|_\infty = \max \{ \frac{1}{\sqrt{\gamma - 1}}, \sqrt{2} + \epsilon + \xi \}\) and \(\sqrt{2} < \gamma\). Then for the iterations \((X^\ell)_{\ell \in \mathbb{N}}\) produced by Algorithm 8, there exists \(\mu_0 < 1\) such that
\[
\limsup_{\ell \to \infty} \|1 - \|X^\ell\|_\infty\| \leq \frac{\mu_1(\gamma, t_0, \epsilon) + 2\epsilon}{1 - \mu_0},
\] (84)

where \(\mu_1(\gamma, t_0, \epsilon)\) is as in Lemma 16. The sequence \((X^\ell)_{\ell \in \mathbb{N}}\) is bounded and its accumulation points \(\bar{X}\) satisfy simultaneously the following properties
\[
\|\bar{X}\|_F \leq 1 \text{ and } \|\bar{X}\|_\infty \geq 1 - \frac{\mu_1(\gamma, t_0, \epsilon) + 2\epsilon}{1 - \mu_0} - \epsilon,
\]
and
\[
\|P_A \bar{X}\|_F \leq 1 \text{ and } \|P_A \bar{X}\|_\infty \geq 1 - \frac{\mu_1(\gamma, t_0, \epsilon) + 2\epsilon}{1 - \mu_0} - 2\epsilon.
\]

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Proof. We denote the singular value decomposition of \( X^\ell \) by \( X^\ell = \sum_{j=1}^m \sigma_j u_j \otimes v_j \) and the one of \( P_A X^\ell \) by \( P_A X^\ell = \sum_{j=1}^m \sigma_j a_{ij} \otimes a_{ij} \), where \( ij \) is a suitable rearrangement of the index set \( \{1, \ldots, m\} \). By Lemma \( \text{(16)} \) we can further develop the following estimates:

\[
\epsilon + \sigma_1^{\ell+1} \geq \|X^{\ell+1}\|_\infty = \|P_A \Pi_{\gamma}(X^\ell)\|_\infty \\
\geq \|P_A \Pi_{\gamma}(X^\ell)\|_\infty - \|\Pi_{\gamma}(P_A - P_A) \Pi_{\gamma}(X^\ell)\|_F \\
\geq \|P_A \Pi_{\gamma}(X^\ell)\|_\infty - \epsilon \\
= \|P_A \Pi_{\gamma}(P_A X^\ell) + P_A \Pi_{\gamma}(P_A X^\ell) - P_A \Pi_{\gamma}(P_A X^\ell)\|_\infty - \epsilon \\
\geq \|P_A \Pi_{\gamma}(P_A X^\ell)\|_\infty - \Pi_{\gamma}(P_A X^\ell) - \Pi_{\gamma}(P_A X^\ell)\|_F - \epsilon \\
\geq \frac{\gamma \sigma_1^{\ell}}{\sqrt{\gamma^2 - 1}} - \mu_1(\gamma, t_0, \epsilon) - \epsilon.
\]

Hence, we obtain

\[
1 - \sigma_1^{\ell+1} \leq 1 - \frac{\gamma \sigma_1^{\ell}}{\sqrt{\gamma^2 - 1}} + \mu_1(\gamma, t_0, \epsilon) + 2 \epsilon.
\]

By an estimate similar to \( \text{(76)} \) and following the arguments given before, we conclude that

\[
1 - \sigma_1^{\ell+1} \leq \mu_0(1 - \sigma_1^{\ell}) + \eta_0,
\]

where \( \eta_0 = \mu_1(\gamma, t_0, \epsilon) + 2 \epsilon. \) As \( X^\ell = P_A \Pi_{\gamma}(X^{\ell-1}) \) and \( P_A \) and \( P_A \) are orthogonal projections we have that

\[
\sigma_1^{\ell} \leq \|P_A X^\ell\|_F \leq \|X^\ell\|_F \leq \|\Pi_{\gamma}(X^{\ell-1})\|_F = 1.
\]

Hence, actually, the recursion \( \text{(85)} \) can be rewritten as

\[
|1 - \sigma_1^{\ell+1}| \leq \mu_0 \left| 1 - \sigma_1^{\ell} \right| + \eta_0 \\
\leq \mu_0^{\ell+1} \left| 1 - \sigma_1^{0} \right| + \eta_0 \sum_{k=0}^{\ell} \mu_0^k.
\]

This implies

\[
\limsup_{\ell \to \infty} |1 - \|X^\ell\|_\infty| = \limsup_{\ell \to \infty} |1 - \sigma_1^{\ell}| \leq \limsup_{\ell \to \infty} |1 - \sigma_1^{\ell}| + \epsilon \leq \frac{\eta_0}{1 - \mu_0} + \epsilon.
\]

(86)

Since the sequence \( (X^\ell)_\ell \) is bounded, it has accumulation points \( \bar{X} \), and as a consequence of \( \text{(86)} \) we obtain that \( \bar{X} \) has simultaneously the following properties

\[
\|\bar{X}\|_F \leq 1 \text{ and } \|\bar{X}\|_\infty \geq 1 - \frac{\eta_0}{1 - \mu_0} - \epsilon,
\]

and

\[
\|P_A \bar{X}\|_F \leq 1 \text{ and } \|P_A \bar{X}\|_\infty \geq 1 - \frac{\eta_0}{1 - \mu_0} - 2 \epsilon.
\]

\[
\square
\]

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Remark 6. Given the singular value decompositions $\bar{X} = \sum_{j=1}^{m} \tilde{\sigma}_j \tilde{u}_j \otimes \tilde{v}_j$ and $P_A \bar{X} = \sum_{j=1}^{m} \sigma_j^\infty a_i \otimes a_{ij}$, by applying again Wedin’s bound we obtain that, for instance

$$\|\tilde{v}_1 \otimes \tilde{v}_1 - a_{i1} \otimes a_{i1}\|_F \leq \frac{2}{1 - \frac{m}{1-\mu_0} + \epsilon - \sqrt{1 - (1 - \frac{m}{1-\mu_0})^2}} \epsilon.$$  

Notice that for $\epsilon \to 0$ we obtain $\eta_0 = \mu_1(\gamma, t_0, \epsilon) + 2\epsilon \to 2\sqrt{1-t_0}$.

4 Approximating the sum of ridge functions

By running Algorithm 4 with different initial values, we can approximate the ridge directions $(a_j)_{j=1}^{m}$ by unit-norm vectors $(\hat{a}_j)_{j=1}^{m}$. We then sample $f$ along the vectors in the dual basis $(\hat{b}_j)_{j=1}^{m}$ to obtain an approximation of the univariate ridge profiles $g_1, \ldots, g_m$. The approximation $\hat{f}$ of $f$ is then obtained by putting all these ingredients together. Obviously, the representation (2) is not unique also due to constant factors, which may be spread among the profiles $(g_j)_{j=1}^{m}$ arbitrarily. By simply sampling $f$ at zero and subtracting this value from $f$, we may assume that $f(0) = 0$ and, consequently, that also $g_1(0) = \cdots = g_m(0) = 0$.

The resulting algorithm and the analysis of its performance are as described below.

\[\text{Algorithm 6.}\]

- Let $\hat{a}_j$ be the normalized approximations of $a_j$, $j = 1, \ldots, m$
- Let $(\hat{b}_j)_{j=1}^{m}$ be the dual basis to $(\hat{a}_j)_{j=1}^{m}$
- Assume, that $f(0) = g_1(0) = \cdots = g_m(0)$
- Put $\hat{g}_j(t) := f(t\hat{b}_j)$, $t \in (-1/\|\hat{b}_j\|_2, 1/\|\hat{b}_j\|_2)$
- Put $\hat{f}(x) := \sum_{j=1}^{m} \hat{g}_j(\hat{a}_j \cdot x)$, $\|x\|_2 \leq 1$

Theorem 18. Let $S(a_1, \ldots, a_m) \leq \epsilon$, $S(\hat{a}_1, \ldots, \hat{a}_m) \leq \epsilon'$, and $(\sum_{j=1}^{m} \|a_j - \hat{a}_j\|^2_2)^{1/2} \leq \eta$. Then $f$ constructed by Algorithm 4 satisfies

$$\|f - \hat{f}\|_\infty \leq 5C_2(1 + \xi(\epsilon, \epsilon')) \max(\eta, \eta^2),$$

where $\xi(\epsilon, \epsilon') \to 0$ if $(\epsilon, \epsilon') \to (0,0)$.

Proof. We use that $g_i(0) = 0$ for $i = 1, \ldots, m$, $a_i \cdot x = \sum_j (\hat{a}_j \cdot x) \cdot (\hat{b}_j \cdot a_i)$, Taylor’s formula,
and estimate for $x \in \mathbb{R}^m$ with $\|x\|_2 \leq 1$

\[
|f(x) - \hat{f}(x)| = \left| \sum_{i=1}^{m} g_i(a_i \cdot x) - \sum_{j=1}^{m} \hat{g}_j(\hat{a}_j \cdot x) \right| = \left| \sum_{i=1}^{m} g_i(a_i \cdot x) - \sum_{j=1}^{m} f((\hat{a}_j \cdot x)\hat{b}_j) \right|
\]

\[
= \left| \sum_{i=1}^{m} g_i(a_i \cdot x) - \sum_{j=1}^{m} \sum_{i=1}^{m} g_i((\hat{a}_j \cdot x) \cdot (\hat{b}_j \cdot a_i)) \right|
\]

\[
\leq \sum_{i=1}^{m} \left| g_i(a_i \cdot x) - g_i((\hat{a}_j \cdot x) \cdot (\hat{b}_j \cdot a_i)) \right|
\]

\[
= \sum_{i=1}^{m} \left| g_i(0) a_i \cdot x - \sum_{j=1}^{m} g_i(0)((\hat{a}_j \cdot x) \cdot (\hat{b}_j \cdot a_i)) \right|
\]

\[
+ \int_0^{\hat{a}_j \cdot x} (a_i \cdot x - u) g'_i(u) du - \sum_{j=1}^{m} \int_0^{(\hat{a}_j \cdot x) - (\hat{b}_j \cdot a_i)} ((\hat{a}_j \cdot x) \cdot (\hat{b}_j \cdot a_i) - u) g''(u) du
\]

\[
\leq \sum_{i=1}^{m} \int_0^{\hat{a}_j \cdot x} (a_i \cdot x - u) g'_i(u) du - \int_0^{(\hat{a}_j \cdot x) - (\hat{b}_j \cdot a_i)} ((\hat{a}_j \cdot x) \cdot (\hat{b}_j \cdot a_i) - u) g''(u) du
\]

\[
+ \sum_{j \neq i} \sum_{j \neq i} \int_0^{(\hat{a}_j \cdot x) - (\hat{b}_j \cdot a_i)} ((\hat{a}_j \cdot x) \cdot (\hat{b}_j \cdot (a_i - \hat{a}_i)) - u) g''(u) du
\]

\[
= I + II.
\]

We use Lemma 19 and Lemma 21 to bound the first term by

\[
I \leq C_2 \sum_{i=1}^{m} \left\{ |a_i \cdot x| \cdot |a_i \cdot x - (\hat{a}_i \cdot x) \cdot (\hat{b}_i \cdot a_i)| + |a_i \cdot x - (\hat{a}_i \cdot x) \cdot (\hat{b}_i \cdot a_i)|^2 / 2 \right\}
\]

\[
\leq C_2 \left( \sum_{i=1}^{m} |a_i \cdot x|^2 \right)^{1/2} \cdot \left( \sum_{i=1}^{m} |a_i \cdot x - (\hat{a}_i \cdot x) \cdot (\hat{b}_i \cdot a_i)|^2 \right)^{1/2}
\]

\[
+ \frac{C_2}{2} \sum_{i=1}^{m} |a_i \cdot x - (\hat{a}_i \cdot x) \cdot (\hat{b}_i \cdot a_i)|^2 = I' + I'',
\]

where

\[
I' \leq C_2(1 + \varepsilon) \cdot \left[ \left( \sum_{i=1}^{m} |(a_i - \hat{a}_i) \cdot x|^2 \right)^{1/2} + \left( \sum_{i=1}^{m} |(\hat{a}_i \cdot x) \cdot (\hat{b}_i \cdot (\hat{a}_i - a_i))|^2 \right)^{1/2} \right]
\]

\[
\leq C_2(1 + \varepsilon) \eta + C_2(1 + \varepsilon) \max_j \|\hat{b}_j\|_2 \eta
\]

and

\[
I'' \leq C_2 \sum_{i=1}^{m} \left( |(a_i - \hat{a}_i) \cdot x|^2 + |(\hat{a}_i \cdot x) \cdot (\hat{b}_i \cdot (\hat{a}_i - a_i))|^2 \right)
\]

\[
\leq C_2 \eta^2 + C_2 \max_j \|\hat{b}_j\|_2^2 \eta^2.
\]
Next, we estimate the second term by
\[
II \leq C_2 \sum_{i=1}^{m} \sum_{j=1}^{m} |(\hat{a}_j \cdot x) \cdot (\hat{b}_j \cdot (a_i - \hat{a}_i))|^2 \leq C_2 \sum_{i,j=1}^{m} (\hat{a}_j \cdot x)^2 \cdot \|\hat{b}_j\|_2^2 \cdot \|a_i - \hat{a}_i\|_2^2
\]
\[
\leq C_2 \max_j \|\hat{b}_j\|_2^2 \cdot \sum_{j=1}^{m} (\hat{a}_j \cdot x)^2 \cdot \sum_{i=1}^{m} \|a_i - \hat{a}_i\|_2^2 \leq C_2 \max_j \|\hat{b}_j\|_2^2 \cdot (1 + \varepsilon')^2 \eta^2.
\]

Using Lemma 21 (vi) and summing up these estimates we get
\[
\|f - \hat{f}\|_\infty \leq 5C_2 (1 + \xi(\varepsilon, \varepsilon')) \max(\eta, \eta^2),
\]
where \(\xi(\varepsilon, \varepsilon') \to 0\) if \((\varepsilon, \varepsilon') \to (0, 0)\).

5 Appendix

We collect several technical tools needed throughout the paper. We start with an auxiliary lemma, which can be easily verified by a straightforward calculation.

Lemma 19. Let \(I \subset \mathbb{R}\) be an interval containing zero and let \(G : I \to \mathbb{R}\) be measurable. Then for any \(x, y \in I\)
\[
\left| \int_0^x (x-u)G(u)du - \int_0^y (y-u)G(u)du \right| \leq \max_{u \in I} |G(u)| \cdot \left( |x| \cdot |y-x| + |y-x|^2/2 \right).
\]

5.1 Nearly orthogonal vectors

In this section we discuss some basic properties of the notion of \(\varepsilon\)-nearly-orthogonal vectors, cf. Definition 2.

Theorem 20. Let \(a_1, \ldots, a_m \in \mathbb{R}^m\) and let \(A \in \mathbb{R}^{m \times m}\) be a matrix with columns \(a_1, \ldots, a_m\). Then
\[
S(a_1, \ldots, a_m) = \left( \sum_{i=1}^{m} (\sigma_i - 1)^2 \right)^{1/2},
\]
where \(\sigma_1 \geq \sigma_2 \geq \cdots \geq 0\) are the singular values of \(A\).

Proof. The result is very well known and the proof follows easily by singular value decomposition of \(A = UAV^T\). The closest orthogonal basis \(w_1, \ldots, w_m\) is given as the columns of the matrix \(W = UV^T\). \(\square\)

Lemma 21. Let \(\varepsilon > 0\) and let \(a_1, \ldots, a_m \in \mathbb{R}^m\) with \(S(a_1, \ldots, a_m) \leq \varepsilon\) and \(\|a_i\|_2 = 1\) for all \(i = 1, \ldots, m\) and let \(A \in \mathbb{R}^{m \times m}\) be a matrix with columns \(a_1, \ldots, a_m\).

(i) Then
\[
(1 - \varepsilon)\|y\|_2 \leq \|Ay\|_2 \leq (1 + \varepsilon)\|y\|_2
\]
for all \(y \in \mathbb{R}^m\). The result holds with identical proof also for \(A^T\) instead of \(A\) substituted in the inequality.

(ii) Let \(M = \sum_{j=1}^{m} \sigma_j a_j \otimes a_j\), then \(\|M\|_\infty \leq (1 + \varepsilon)^2 \|\sigma\|_\infty\).
(iii) \( \sum_{k \neq j} |\langle a_k, a_j \rangle|^2 \leq 2 \varepsilon^2 \) for all \( j = 1, \ldots, m \).

(iv) \( S(a_1 \otimes a_1, \ldots, a_m \otimes a_m) \leq 2 \varepsilon \).

(v) \( (1 - 2 \varepsilon) \| \sigma \|_2 \leq \| \sum_{j=1}^m \sigma_j a_j \otimes a_j \|_F \leq (1 + 2 \varepsilon) \| \sigma \|_2 \).

(vi) Let \( b_j, j = 1, \ldots, m \) be the dual basis of \( a_j, j = 1, \ldots, m \) (i.e. \( \langle b_i, a_j \rangle = \delta_{i,j} \)). Then \( \| b_j \|_2 \leq 1/(1 - \varepsilon) \) for all \( j = 1, \ldots, m \).

**Proof.**

(i) Let \( W \) be the optimal orthonormal matrix for \( A \). Then

\[
\| Ay \|_2 = \| (A - W)y + Wy \|_2 \leq \| A - W \|_\infty \cdot \| y \|_2 + \| Wy \|_2 \leq (1 + \varepsilon) \| y \|_2.
\]

The other estimate from below follows by applying the inverse triangle inequality. The proof can be used similarly also for obtaining the bounds for \( A^T \) instead of \( A \).

(ii) We estimate by (i)

\[
\| M \|_\infty = \sup_{x: \| x \|_2 \leq 1} |\langle x, Mx \rangle| \leq \sup_{x: \| x \|_2 \leq 1} \sum_{j=1}^m |\sigma_j| \cdot |\langle a_j, x \rangle|^2
\]

\[
\leq \| \sigma \|_\infty \cdot \sup_{x: \| x \|_2 \leq 1} \sum_{j=1}^m |\langle a_j, x \rangle|^2 \leq (1 + \varepsilon)^2 \| \sigma \|_\infty.
\]

(iii) We get just by triangle inequality and the inequality between the arithmetic and quadratic mean

\[
\left( \sum_{k \neq j} |\langle a_k, a_j \rangle|^2 \right)^{1/2} = \left( \sum_{k \neq j} (|\langle a_k, w_j \rangle|^2 + \langle w_k, a_j \rangle^2) \right)^{1/2}
\]

\[
\leq \left( \sum_{k \neq j} |\langle a_k - w_k, a_j \rangle|^2 \right)^{1/2} + \left( \sum_{k \neq j} (|\langle a_j, a_j - w_j \rangle|^2) \right)^{1/2}
\]

\[
\leq \left( \sum_{k \neq j} \| a_k - w_k \|^2 \cdot \| a_j \|^2 \right)^{1/2} + \| a_j - w_j \|_2
\]

\[
\leq \sqrt{2} \left( \| a_j - w_j \|^2 + \sum_{k \neq j} \| a_k - w_k \|^2 \right)^{1/2} \leq \sqrt{2} \varepsilon.
\]

(iv) The result follows by

\[
\left( \sum_{j=1}^m \| a_j \otimes a_j - w_j \otimes w_j \|_F^2 \right)^{1/2} = \left( \sum_{j=1}^m \| a_j \otimes (a_j - w_j) + (a_j - w_j) \otimes w_j \|_F^2 \right)^{1/2}
\]

\[
\leq \left( \sum_{j=1}^m \| a_j \otimes (a_j - w_j) \|_F^2 \right)^{1/2} + \left( \sum_{j=1}^m \| (a_j - w_j) \otimes w_j \|_F^2 \right)^{1/2}
\]

\[
= \left( \sum_{j=1}^m \| a_j - w_j \|^2 \cdot \| a_j \|^2 \right)^{1/2} + \left( \sum_{j=1}^m \| a_j - w_j \|^2 \cdot \| w_j \|^2 \right)^{1/2} \leq 2 \varepsilon.
\]

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(v) Using (iv) we obtain
\[
\left\| \sum_{j=1}^{m} \sigma_j a_j \otimes a_j \right\|_F \leq \left\| \sum_{j=1}^{m} \sigma_j (a_j \otimes a_j - w_j \otimes w_j) \right\|_F + \left\| \sum_{j=1}^{m} \sigma_j w_j \otimes w_j \right\|_F \\
\leq \| \sigma \|_2 + \sum_{j=1}^{m} |\sigma_j| \cdot \| a_j \otimes a_j - w_j \otimes w_j \|_F \\
\leq \| \sigma \|_2 + \left( \sum_{j=1}^{m} \sigma_j^2 \right)^{1/2} \cdot \left( \sum_{j=1}^{m} \| a_j \otimes a_j - w_j \otimes w_j \|_F^2 \right)^{1/2} \\
\leq (1 + 2\epsilon)\| \sigma \|_2
\]
and similarly for the other side.

(vi) Using the definition of \( S(a_1, \ldots, a_m) \), we obtain for the optimal orthonormal basis \( w_1, \ldots, w_m \)
\[
\| b_j \|_2 = \left( \sum_{k=1}^{m} (b_j, w_k^T)^2 \right)^{1/2} = \left( \sum_{k=1}^{m} [(b_j, a_k) + (b_j, w_k - a_k)]^2 \right)^{1/2} \\
\leq \left( \sum_{k=1}^{m} (b_j, a_k)^2 \right)^{1/2} + \left( \sum_{k=1}^{m} (b_j, w_k - a_k)^2 \right)^{1/2} \\
\leq 1 + \left( \sum_{k=1}^{m} \| b_j \|_2^2 \cdot \| w_k - a_k \|_2^2 \right)^{1/2} \leq 1 + \epsilon \| b_j \|_2.
\]

\[
\text{Lemma 22. Let } \{\alpha_1, \ldots, \alpha_m\} \subset \mathbb{R}^n \text{ be arbitrary linearly independent vectors with unit Euclidean norm and let } \{\omega_1, \ldots, \omega_m\} \subset \mathbb{R}^n \text{ be orthonormal. Let } A = \text{span}\{\alpha_1, \ldots, \alpha_m\}, \quad \hat{A} = \text{span}\{\omega_1, \ldots, \omega_m\}, \text{ and} \\
\left( \sum_{i=1}^{m} \| \alpha_i - \omega_i \|_2^2 \right)^{1/2} \leq \epsilon < 1.
\]

Then
\[
\| P_A - \hat{P}_A \|_\infty \leq 5\epsilon,
\]
where \( P_A \) and \( \hat{P}_A \) are the orthogonal projections on \( A \) and \( \hat{A} \) respectively, and the norm \( \| \cdot \|_\infty \) represents the operator norm for operators on the Euclidean space \( \mathbb{R}^n \).

\textbf{Proof.} Let \( A \in \mathbb{R}^{n \times m} \) have columns \( \alpha_1, \ldots, \alpha_m \) and let \( W \in \mathbb{R}^{n \times m} \) have columns \( \omega_1, \ldots, \omega_m \). Then \( \hat{P}_A = WW^T \). If \( A = U\Sigma V^T \) is the singular value decomposition of \( A \) with \( U \in \mathbb{R}^{n \times m}, \Sigma \in \mathbb{R}^{m \times m} \) and \( V \in \mathbb{R}^{m \times m} \), then \( P_A = UU^T \). Using \( U\Sigma = AV \), we then obtain for \( \| \cdot \| = \| \cdot \|_\infty \)
\[
\| P_A - \hat{P}_A \| = \| UU^T - WW^T \| = \| UU^T - (U\Sigma)(U\Sigma)^T + (U\Sigma)(U\Sigma)^T - WW^T \| \\
\leq \| UU^T - U\Sigma^2U^T \| + \| AVV^T A^T - WW^T \| \\
= \| U(I - \Sigma^2)U^T \| + \| AA^T - WW^T \| \\
\leq \| I - \Sigma^2 \| + \| A(A^T - W^T) \| + \| (A - W)W^T \| \\
\leq \max_i \| 1 - \sigma_i^2 \| + \| A \| \cdot \| A^T - W^T \|_F + \| A - W \|_F \\
\leq \epsilon^2 + \epsilon(1 + \epsilon) + \epsilon \leq 4\epsilon.
\]
5.2 Stability of the singular value decomposition

Given two matrices \( B \) and \( \tilde{B} \) with corresponding singular value decompositions

\[
B = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix} \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix}
\]

and

\[
\tilde{B} = \begin{pmatrix} \tilde{U}_1 & \tilde{U}_2 \end{pmatrix} \begin{pmatrix} \tilde{\Sigma}_1 & 0 \\ 0 & \tilde{\Sigma}_2 \end{pmatrix} \begin{pmatrix} \tilde{V}_1^T \\ \tilde{V}_2^T \end{pmatrix},
\]

where it is understood that two corresponding submatrices, e.g., \( U_1, \tilde{U}_1 \), have the same size, we would like to bound the difference between \( V_1 \) and \( \tilde{V}_1 \) by the error \( \| B - \tilde{B} \|_F \). As a consequence of Wedin’s perturbation bound \[35\], see also \[32, Section 7\], we have the following useful result.

**Theorem 23** (Stability of subspaces - Wedin’s bound). If there is an \( \bar{\alpha} > 0 \) such that

\[
\min_{\ell, \ell'} |\sigma_{\ell}(\tilde{\Sigma}_1) - \sigma_{\ell}(\Sigma_2)| \geq \bar{\alpha},
\]

(88)

and

\[
\min_{\ell} |\sigma_{\ell}(\tilde{\Sigma}_1)| \geq \bar{\alpha},
\]

(89)

then

\[
\| V_1 V_1^T - \tilde{V}_1 \tilde{V}_1^T \|_F \leq \frac{2}{\bar{\alpha}} \| B - \tilde{B} \|_F.
\]

(90)

5.3 Spectral estimates and sums of random semidefinite matrices

The value of \( \sigma_m(X^T) \) can be estimated by certain matrix Chernoff bounds. The following theorem generalizes Hoeffding’s inequality to sums of random semidefinite matrices and was recently presented by Tropp in [34, Corollary 5.2 and Remark 5.3], improving over results in [1], and using techniques from [30] and [26].

**Theorem 24** (Matrix Chernoff). Let \( X_1, \ldots, X_n \) be independent random, positive-semidefinite matrices of dimension \( m \times m \). Moreover suppose that

\[
\sigma_1(X_j) \leq C
\]

(91)

almost surely for all \( j = 1, \ldots, n \). Let

\[
\mu_{\text{min}} = \sigma_m \left( \sum_{j=1}^n E X_j \right)
\]

(92)

be the smallest singular value of the sum of the expectations. Then

\[
P \left\{ \sigma_m \left( \sum_{j=1}^n X_j \right) - \mu_{\text{min}} \leq -s\mu_{\text{min}} \right\} \leq m \exp \left( -\frac{\mu_{\text{min}} s^2}{2C} \right),
\]

(93)

for all \( s \in (0, 1) \).
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