ULRICH BUNDLES ON K3 SURFACES

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ABSTRACT. We show that any polarized K3 surface supports special Ulrich bundles of rank 2.

Given an n-dimensional closed subvariety \( X \subset \mathbb{P}^N \), a coherent sheaf \( F \) on \( X \) is Ulrich if \( H^i(F(-t)) = 0 \) for \( n \) consecutive values of \( t \). We refer to [Cos17, Bea18] for an introduction. We mention that Ulrich sheaves are related to Chow forms (this was the main motivation for their study in [ESW03]), to determinantal representations and generalized Clifford algebras, to Boij-Söderberg theory (cf. [SE10]) to the minimal resolution conjecture, and to the representation type of varieties (cf. [FP15]).

Conjecturally, Ulrich sheaves exist for any \( X \), see [ESW03]. They are known to exist for several classes of varieties e.g. complete intersections, curves, Veronese, Segre, Grassmann varieties. Low-rank Ulrich bundles on surfaces have been studied intensively, and Ulrich bundles of rank 2 (or sometimes 1) are known in many cases. We refer to [Cas17, Bea18] for a survey and further references. Let us only review some of the cases that are most relevant for us, namely among surfaces with trivial canonical bundle.

In [Bea16], Ulrich bundles of rank 2 are proved to exist on abelian surfaces. In [AFO17], it is proved that K3 surfaces support Ulrich bundles of rank 2, provided that some Noether-Lefschetz open condition is satisfied. The case of quartic surfaces was previously analyzed in detail in [CKM12]. The main techniques used so far are the Serre construction starting from points on \( X \) and Lazarsfeld-Mukai bundles.

In this note, we show that any K3 surface supports an Ulrich bundle \( E \) of rank 2 with \( c_1(E) = 3H \), for any polarization \( H \). So these bundles are special, cf. [ESW03]. We allow singular surfaces with trivial canonical bundle. The main tool is a sort of enhancement of Serre’s construction based on unobstructedness of simple sheaves on a K3 surface.

Let us state the result more precisely. We work over an algebraically closed field \( k \). Let \( X \) be an integral (i.e. reduced and irreducible) projective surface with \( \omega_X \cong \mathcal{O}_X \) and \( H^1(\mathcal{O}_X) = 0 \). We denote by \( X_{\text{sm}} \) the smooth locus of \( X \).

Fix a very ample divisor \( H \) on \( X \). Under the closed embedding given by the complete linear series \( |\mathcal{O}_X(H)| \) we may view \( X \) as a subvariety of some projective space \( \mathbb{P}^g \). A hyperplane section \( C \) of \( X \) is a projective curve of arithmetic genus \( g \) with \( \omega_C \cong \mathcal{O}_C(H) \), where \( H \) also denotes the restriction of \( H \) to \( C \). We may choose \( C \) to be integral too.

A locally Cohen-Macaulay sheaf \( E \) on \( X \) is arithmetically Cohen-Macaulay (ACM) if \( H^i(E(tH)) = 0 \) for all \( t \in \mathbb{Z} \). A special class of ACM sheaves are Ulrich sheaves, which are characterized by the property \( H^i(E(-tH)) = 0 \) for a pair of consecutive integers \( t \). Sometimes these integers are required to be 1 and 2 in which case we speak of an initialized Ulrich sheaf. Of course all these notions depend on the polarization \( H \). We call simple a sheaf whose only endomorphisms are homotheties.

Theorem 1. Let \( X \) and \( H \) be as above. Then there exists a simple Ulrich vector bundle of rank 2 on \( X \) whose determinant is \( \mathcal{O}_X(3H) \).

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The strategy to prove the theorem is the following. First we build an ACM vector bundle $E$ of rank 2 by Serre’s construction applied to a projective coordinate system in $X$. Then we make a sort of elementary modification of $E$ along a single generic point $p \in X$, producing a simple non-reflexive sheaf having the Chern character of an Ulrich bundle. Finally we flatly deform such sheaf and check that this yields generically the desired Ulrich bundle.

Prior to all this, we start by observing that the trivial bundle is a (trivial) example of ACM line bundle. Indeed, using that $H^i(O_X) = 0$ and that $C$ is connected, one checks that $H^i(O_X(-H)) = 0$. In turn, this easily implies $H^i(O_X(-tH)) = 0$ for all $t \geq 2$. Also, Serre duality combined with $\omega_X \cong O_X$ gives $H^i(O_X(tH)) = 0$ for all $t \geq 0$. This way, we see that $O_X$ is an ACM line bundle on $X$ and $X \subset \mathbb{P}^g$ is an ACM surface of degree $2g - 2$.

However this line bundle is never Ulrich. So generically $X$ will not support Ulrich line bundles. We thus move to rank two and start by constructing a simple ACM bundle.

**Lemma 1.** Let $Z \subset X_{\text{sm}}$ be a set of $g + 2$ points in general linear position. Then there is a unique coherent sheaf $E$ of rank 2 fitting into a non-splitting exact sequence:

\[
0 \to O_X \to E \to I_Z(H) \to 0.
\]

The sheaf $E$ is locally free, simple and ACM. It satisfies:

\[
E \cong E^*(H), \quad h^0(E) = 1, \quad \text{ext}^1_X(E, E) = 2g + 4.
\]

**Proof.** Taking cohomology of the exact sequence

\[
0 \to I_Z(H) \to O_X(H) \to O_Z \to 0,
\]

and using the fact that $Z$ is in general linear position and hence contained in no hyperplane, we get $H^0(I_Z(H)) = 0$ and $h^1(I_Z(H)) = 1$.

By Serre duality we get $\text{ext}^1_X(I_Z(H), O_X) = h^1(I_Z(H)) = 1$ so, up to proportionality, there is a unique non-splitting extension of the desired form. Correspondingly, there exists a unique coherent sheaf $E$ of rank two fitting into a non-splitting exact sequence of the form $\text{(1)}$. The sheaf $E$ we obtain this way satisfies $h^0(E) = 1$ and $H^1(E) = \text{ext}^1_X(E, O_X)^\ast = 0$ because applying $\text{Hom}_X(-, O_X)$ to $\text{(1)}$ we obtain a non-zero map (and thus an isomorphism) $H^0(O_X) \to \text{ext}^1_X(I_Z(H), O_X)$. This is the dual of the map $H^1(I_Z(H)) \to H^1(O_X)$ obtained by taking global sections in $\text{(1)}$. So $H^1(E) = 0$.

If $X$ is smooth we deduce that $E$ is locally free from the Cayley-Bacharach property, cf. for instance [HL97 Theorem 5.1.1]. Indeed, since $Z$ is in general linear position (i.e. $Z$ is a projective frame in $\mathbb{P}^g$), no hyperplane passes through any subset of $g + 1$ points of $Z$. Anyway the statement follows in general by a minor modification of the argument appearing in [PP15 Lemma 7.2]. Indeed by the local-to-global spectral sequence, using $H^0(O_X(-H)) = 0$ and $\text{Hom}_X(I_Z(H), O_X) \cong O_X(-H)$ we get the following exact sequence:

\[
0 \to \text{Ext}^1_X(I_Z(H), O_X) \to H^0(\text{Ext}^1_X(I_Z(H), O_X)) \to H^1(X, O_X(-H)) \to 0.
\]

In turn, using $\text{Ext}^1_X(I_Z(H), O_X) \cong O_Z$ and $H^2(X, O_X(-H)) \cong H^2(X, O_X(H))^\ast$, if we choose $Z$ to be a projective coordinate system of $\mathbb{P}^g$, we rewrite this exact sequence as:

\[
\begin{pmatrix}
1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 1
\end{pmatrix}
\begin{pmatrix}
\vdots \\
\vdots \\
0
\end{pmatrix}
\begin{pmatrix}
\text{Ext}^1_X(I_Z(H), O_X) \\
H^0(O_Z) \\
H^2(X, O_X(H))^\ast
\end{pmatrix} \to 0.
\]

So $\text{Ext}^1_X(I_Z(H), O_X)$ is generated by the vector $(1, \ldots, 1,-1)^t$ and since this vector corresponds to an extension in $\text{Ext}^1_X(I_Z(H), O_X)$ which is non-zero at any point of $Z$ we have that the sequence defining $E$ is locally non-split around each point of $Z$, which in turn implies that $E$ is locally free at each such point (and hence everywhere). From $c_1(E) = H$, since $E$ is locally free of rank 2, we get a canonical isomorphism $E \cong E^*(H)$. 
Let us prove that \( E \) is ACM. We already have \( h^1(E) = 0 \) and thus by Serre duality \( h^1(E(-H)) = h^1(E^*(H)) = h^1(E) = 0 \). Also \( h^0(E(-H)) = 0 \) and \( h^0(E(-H)) = 1 \). Note that, choosing an integral hyperplane section curve \( C \) that avoids \( Z \), (1) becomes:

\[
0 \rightarrow O_C \rightarrow E|_C \rightarrow O_C(H) \rightarrow 0.
\]

From \( H^k(E(-H)) = 0 \) for \( k = 0, 1 \) we deduce \( h^0(E|_C) = 1 \) so the previous exact sequence does not split. Then \( h^0(E(-H)) = 0 \). This easily implies \( H^1(E(-2H)) = 0 \) and actually \( H^1(E(-tH)) = 0 \) for all \( t \geq 2 \). Serre duality now gives \( H^1(E(tH)) = 0 \) for all \( t \geq 1 \). In other words \( E \) is ACM.

It remains to check that \( E \) is simple. Applying \( \text{Hom}_X(E, -) \) to the exact sequence (2) we get that the non-zero space \( \text{Hom}_X(E, I_Z(H)) \) is contained in \( \text{Hom}_X(E, O_X(H)) \cong H^0(E) \cong k \), so \( \text{hom}_X(E, I_Z(H)) = 1 \). Therefore, applying \( \text{Hom}_X(E, -) \) to the (1), since \( \text{Hom}_X(E, O_X) \cong h^2(E) = 0 \) we get that \( \text{End}_X(E) \) is contained in \( \text{Hom}_X(E, I_Z(H)) \) and is therefore 1-dimensional. This says that \( E \) is simple. By Serre duality \( \text{ext}^2_X(E, E) = 1 \). In turn, we easily deduce \( \text{ext}^1_X(E, E) = 2g + 4 \).

Given a reduced subscheme \( Z \in \text{Hilb}_{g+2}(X_{\text{sm}}) \) consisting of points in general linear position, there is a unique rank-2 bundle associated with \( Z \) according to the previous lemma. We denote it by \( E_Z \). We write \( O_p \) for the skyscraper sheaf of a point \( p \in X \).

**Lemma 2.** Assume \( \eta : E_Z \rightarrow O_p \) is surjective. Then \( E^\eta = \ker(\eta) \) is a simple sheaf with:

\[
c_1(E^\eta) = H, \quad c_2(E^\eta) = g + 3, \quad \text{ext}^1_X(E^\eta, E^\eta) = 2g + 8.
\]

**Proof.** Recall that \( E = E_Z \) is simple and observe that this implies \( \text{Hom}_X(E, E^\eta) = 0 \), as the composition of any non-zero map \( E \rightarrow E^\eta \) with \( E^\eta \rightarrow E \) would provide a self-map of \( E \) which is not a multiple of the identity. Also, since \( E \) is locally free we have \( \text{hom}_X(E, O_p) = 2 \) and \( \text{Ext}^1_X(E, O_p) = 0 \) for \( k > 0 \). Therefore, using Lemma 1 and applying \( \text{Hom}_X(E, -) \) to the exact sequence:

\[
(3) \quad 0 \rightarrow E^\eta \rightarrow E \rightarrow O_p \rightarrow 0.
\]

we obtain \( \text{ext}^1_X(E^\eta, E^\eta) = 2g + 5 \) and \( \text{ext}^2_X(E^\eta, E^\eta) = 1 \).

Next, Serre duality gives \( \text{ext}^1_X(O_p, E) = 2\delta_{2,k} \), while \( \text{ext}^2_X(O_p, O_p) \) is the dimension of the \( k \)-th exterior power of the normal bundle of \( p \) in \( X \) and thus takes value \( \binom{g+1}{k} \). Therefore, applying \( \text{Hom}_X(O_p, -) \) to (3) we find \( \text{ext}^1_X(O_p, E^\eta) = 1 \) and \( \text{ext}^2_X(O_p, E^\eta) = 3 \). Putting these computations together and applying \( \text{Hom}_X(-, E^\eta) \) again to (3) we get:

\[
\text{hom}_X(E^\eta, E^\eta) = \text{ext}^2_X(E^\eta, E^\eta) = 1, \quad \text{ext}^1_X(E^\eta, E^\eta) = 2g + 8.
\]

The computation of Chern classes is straightforward. \( \square \)

**Lemma 3.** Let \( p \in X_{\text{sm}} \setminus Z \). Then, for a generic map \( \eta : E_Z \rightarrow O_p \), the induced map on global sections \( H^0(\eta): H^0(E_Z) \rightarrow H^0(O_p) \) is an isomorphism.

**Proof.** Put \( E = E_Z \). It suffices to check that there exists \( \eta \) such that the induced map \( H^0(\eta): k = H^0(E) \rightarrow H^0(O_p) = k \) is an isomorphism, for this is an open condition. To do it, we apply \( \text{Hom}_X(I_Z(H), -) \) to the exact sequence:

\[
0 \rightarrow I_p \rightarrow O_X \rightarrow O_p \rightarrow 0.
\]

This gives an exact sequence:

\[
\text{Ext}^1_X(I_Z(H), I_p) \rightarrow \text{Ext}^1_X(I_Z(H), O_X) \rightarrow \text{Ext}^1_X(I_Z(H), O_p).
\]

Observe that \( \text{Hom}_X(I_Z(H), O_p) \cong O_p \) and \( \text{Ext}^1_X(I_Z(H), O_p) = 0 \) as these sheaves are computed locally on \( X \) and, since \( p \cap Z = \emptyset \), we may choose an open cover of \( X \) consisting of subsets where \( I_Z \) is trivial or \( O_p \) vanishes. Then the local-to-global spectral sequence
gives Ext^1_X(I_Z(H), O_p) = 0 so the extension corresponding to \( I \) admits a lifting to \( I_p \). In other words, we get the commutative exact diagram:

\[
\begin{array}{c}
0 \\
\downarrow \quad \quad \downarrow \\
I_p \\
\downarrow \quad \quad \downarrow \\
E^0 \\
\downarrow \quad \quad \downarrow \\
I_Z(H) \\
\downarrow \quad \quad \downarrow \\
O_p \\
\downarrow \quad \quad \downarrow \\
E \\
\downarrow \quad \quad \downarrow \\
I_Z(H) \\
\downarrow \quad \quad \downarrow \\
O_p \\
\downarrow \quad \quad \downarrow \\
0 \\
\end{array}
\]

where \( \eta \) and \( E^0 \) are defined by the diagram. For this choice of \( \eta \) we get, by the top row of the diagram, \( H^0(E^0) = 0 \), which implies that \( H^0(\eta) \) is an isomorphism. 

By the previous lemma, we may choose \( E_Z \) as in Lemma 1 a point \( p \in X_{\text{sm}} \setminus Z \), some \( \eta : E_Z \rightarrow O_p \) and consider the sheaf \( E^0 \). The goal is to deform \( E^0 \) to an Ulrich bundle.

**Lemma 4.** There exist a smooth connected variety \( S_0 \) of dimension \( 2g + 8 \) and a flat family of simple sheaves \( \mathcal{F} \) on \( X \times S_0 \) such \( \mathcal{F}_s \) is an Ulrich bundle for \( s \) generic in \( S_0 \) and \( \mathcal{F}_s \simeq E^0 \) for some distinguished point \( s_0 \) of \( S_0 \).

**Proof.** We proved in Lemma 2 that \( E^0 \) is simple. Since the non-locally free locus of \( E^0 \) is disjoint from the singular locus of \( X \), we may apply the arguments of [Art90, Theorem 0.1] and [Art90]. In particular, the moduli functor of simple sheaves on \( X \) is pro-represented by a moduli space \( \text{Spl}_X \) which can be constructed in the \( \acute{e}tale \) topology and which is smooth of dimension \( 2g + 8 \) at \( E^0 \). Therefore there exists an open piece of \( \text{Spl}_X \) which is a quasi-projective variety \( S \) equipped with a flat family \( \mathcal{F} \) of simple sheaves on \( X \), such that the induced map \( S \rightarrow \text{Spl}_X \) is a local isomorphism around the point corresponding to \( E^0 \). We denote by \( s_0 \) this point, so that \( \mathcal{F}_{s_0} \simeq E^0 \).

We may assume that \( S \) is smooth and connected of dimension \( 2g + 8 \). Since the reflexive hull \( E \) of \( E^0 \) is locally free and satisfies the assumption of [Art90, Corollary 1.5], we get that \( \mathcal{F}_s \) is locally free for all \( s \) in an open dense subset \( S_0 \) of \( S \).

Now observe that \( H^i(\mathcal{F}_s) = 0 \) by Lemmas 1 and 3. Then, semicontinuity ensures that \( H^i(\mathcal{F}_s) = 0 \) for all \( s \) in an open dense subset \( S_0 \) of \( S_1 \). Therefore, the isomorphism \( \mathcal{F}_s = \mathcal{F}_s(-H) \) and Serre duality give \( H^i(\mathcal{F}_s(-H)) \simeq H^{2-i}(\mathcal{F}_s(0)) \simeq H^{2-i}(\mathcal{F}_s)^* \simeq H^{2-i}(\mathcal{F}_s)^* = 0 \). This says that \( \mathcal{F}_s \) is an initialized special Ulrich bundle, for all \( s \in S_0 \).

Recall the notation \( M_X(v) \) for the moduli space of \( H \)-semistable sheaves \( \mathcal{F} \) on \( X \) whose Mukai vector \( v = (c_0, c_1, c_2) \) satisfies \( c_0 = \text{rk}(\mathcal{F}), c_1 = c_1(\mathcal{F}) \) and \( c_2 = \chi(\mathcal{F}) - \text{rk}(\mathcal{F}) \). From [Qin93, Lemma 2.1] we obtain the following strong version of Theorem 1.

**Corollary 1.** If \( X \) is smooth, \( M_X(2, H, -2) \) is of dimension \( 2g + 8 \) and a general point of it corresponds to a sheaf \( \mathcal{E} \) with is stable (with respect to all polarizations) and such that \( \mathcal{E}(H) \) is an initialized special Ulrich bundle.

It follows from Theorem 1 that \( X \) is strictly Ulrich wild in the sense of [FP15]. The next result refines this fact in terms of moduli spaces. It was proved when \( \text{Pic}(X) \) is generated by \( H \) in [AFO17, Theorem 2.7]. A modification of that argument allows to prove the result in general.

**Theorem 2.** Let \( X \) be a K3 surface and \( H \) be a very ample line bundle on \( X \). Then, for any positive integer \( r \), the moduli space \( M_X(2r, rH, -2r) \) is of dimension \( 2(r^2(g + 5) + 1) \) and its generic point is a stable Ulrich bundle.
Proof. Given a coherent sheaf $E$ or rank $r > 0$ on $X$ we write $P(E) \in \mathbb{Q}[t]$ for the Hilbert polynomial of $E$ and $p(E)$ for its reduced version, namely $P(E) = \chi(E(tH))$ and $p(E) = P(E)/r$. We put $p_0 = (g-1)(r+1)t$ so that, if $E$ is an initialized Ulrich sheaf, then $p(E(-H)) = p_0$. Note that, if $E_1$ and $E_2$ are non-isomorphic stable sheaves with $p(E_1) = p(E_2)$, then $\text{Ext}^1_{\mathcal{O}_X}(E_1, E_2) = 0$ for $k = 0, 2$ and $i \neq j$.

The proof goes by induction on $r$, the case $r = 1$ being given by Corollary 1. For $r \geq 1$, we select a stable Ulrich bundle $E_2$ in $M_X(2r, rH, -2r)$ given by the induction hypothesis and a stable Ulrich bundle $E_1$ in $M_X(2, H, -2)$, taking care that $E_1$ is not isomorphic to $E_2$ for $r = 1$, which is of course possible since $\dim M_X(2, H, -2) > 0$. This way we have:

$$\text{(4)} \quad \text{Ext}^k_{\mathcal{O}_X}(E_1, E_2) = 0, \quad \text{for } k = 0, 2 \text{ and } i \neq j,$$

$$\text{(5)} \quad \text{ext}^1_{\mathcal{O}_X}(E_1, E_2) = 2(g+3) \quad \text{for } i \neq j.$$

It follows plainly by [FP15 Theorem 1, iii] that, for any choice of $\xi \in \mathbb{P}(\text{Ext}^1_{\mathcal{O}_X}(E_2, E_1))$, the sheaf $E^\xi$ fitting as middle term of the associated extension is a simply locally free Ulrich sheaf lying in $M_X(2(r+1), (r+1)H, -2(r+1))$. Of course this sheaf is not stable, as $E_1$ is a sub-sheaf of $E^\xi$ with quotient $E_2$ and the reduced Hilbert polynomial of all these sheaves is $p_0$. We record the defining sequence:

$$\text{(6)} \quad 0 \to E_1 \to E^\xi \to E_2 \to 0.$$

In the same spirit as in Lemma 2 we take a deformation of $E^\xi$ in the space of simple sheaves, which is unobstructed of dimension $2((r+1)(g+3)+1)$ at $E^\xi$. We consider thus an integral quasi-projective variety $S$ as base of an $S$-flat family of simple Ulrich sheaves $\mathcal{T}_s$ with $\mathcal{T}_s \cong E^\xi$ for some $s_0 \in S$ and $S$ locally isomorphic to the moduli space of simple sheaves around the point $s_0$. We may assume that $\mathcal{T}_s$ is locally free for all $s \in S$.

Claim 1. There is an open dense subset $S_0$ of $S$ such that, for any stable sheaf $K$ with $\text{rk}(K) < 2(r+1)$, $\text{rk}(K) \neq 2$ and $p(K) = p_0$, we have $\text{Hom}_X(K, \mathcal{T}_s) = 0$, for all $s \in S_0$.

Proof of the claim. Clearly it suffices to find such open subset for a fixed rank $u$ of $K$ and take the intersection of the corresponding open subsets for all $u < 2(r+1)$, $u \neq 2$.

So let $N$ be the moduli space of stable sheaves $E$ on $X$ with Hilbert polynomial $P(E) = u p_0$. Let $\mathcal{U}$ be a quasi-universal family over $X \times N$, cf. [HL97 Proposition 4.6.2] and denote by $\sigma$ and $\pi$ the projection maps $X \times N \to N$ and $X \times N \to X$, respectively.

For $y \in N$ let $\mathcal{U}_y$ be the corresponding sheaf over $X$. We observe that, applying $\text{Hom}_X(\mathcal{U}_y, -)$ to (6), using the definition of $N$ and $\xi$ and the fact that the $E_i$’s are stable with $p(E_i) = p(\mathcal{U}_y)$ we get $\text{Hom}_X(\mathcal{U}_y, E^\xi) = 0$. Indeed, the only case to check is for $u = 2r$ when $y$ corresponds to the sheaf $E_2$, but $\text{Hom}_X(E_2, E^\xi) = 0$, for otherwise by stability of $E_2$, the exact sequence (6) would split, contradicting our assumption on $\xi$.

Then, Serre duality gives, for all $y \in N$:

$$\text{(7)} \quad H^2((E^\xi)^* \otimes \mathcal{U}_y) \cong \text{Ext}^2_{\mathcal{O}_X}(E^\xi, \mathcal{U}_y) = 0.$$

Now consider $X \times N \times S$, put $r$ for the projection $X \times S \to S$ and denote by $\tilde{\sigma}, \tilde{\pi}, \tilde{\tau}$ the projection maps from $X \times N \times S$ onto $X \times S, N \times S$ and $X \times N$, respectively. Let $\mathcal{V} = \tilde{\pi}^*(F^*) \otimes \tilde{\tau}^*(\mathcal{U})$. Since $\mathcal{V}$ is flat over the integral base $X \times S$ and $\tilde{\sigma}$ has relative dimension 2, base-change gives, for all $(y, s) \in N \times S$:

$$\text{(8)} \quad R^2\tilde{\sigma}_*(\mathcal{V})_{(y, s)} \cong H^2(F^* \otimes \mathcal{U}_y).$$

Let $W$ be the support of $R^2\sigma_*(\mathcal{V})$, i.e. the closed subset of points $(y, s) \in N \times S$ such that $R^2\sigma_*(\mathcal{V})_{(y, s)} \neq 0$. By (7) and (8), we have $W \cap N \times \{s_0\} = \emptyset$, i.e. $s_0$ does not lie in $r(W)$. Then there is an open neighbourhood $S_0 \subset S$ of $s_0$ which is disjoint from $r(W)$. Again by (8), we get $H^2(F^* \otimes \mathcal{U}_y) = 0$ for all $(y, s) \in N \times S_0$, which proves the claim.

Let us now conclude the proof of the theorem. In view of the claim, we have two alternatives for $s$ generic in $S_0$: either $\text{Hom}(K, \mathcal{T}_s) = 0$ for any stable sheaf $K$ with $\text{rk}(K) < 2(r+1)$, or $\text{Hom}(K, \mathcal{T}_s) \neq 0$ for some $s \in S_0$. By [HL97 Theorem 4.6.4] the former case cannot happen since $\mathcal{T}_s$ is locally free for all $s \in S_0$.
2(r + 1) and p(K) = p_u, or otherwise this happens for all such K except for rk(K) = 2 and there actually exists a stable K in N such that Hom(K, F_i) ≠ 0.

In the first alternative F_i is stable, so we assume that the second one takes place and look for a contradiction. We go back to Claim [1] and carry out the same argument for u = 2, with y_0 being the point corresponding to E_2. Observe that K must lie in M_X(2, H, −2) as the proof of Claim [1] applies verbatim on any other component of N.

We note that W ∩ N × {s_0} = {(y_0, s_0)}, as clearly Hom_X(K, E_2) = 0 for all K in N \ {y_0}. So W is properly contained in N × S. Moreover, we easily have hom_X(E_2, E_2) = 1. Recall by construction of the quasi-universal family that there is u_0 such that rk(U) = 2u_0 and that, for y ∈ N, the sheaf U_y is a direct sum of u_0 copies of the stable sheaf of rank 2 in M_X(2, H, −2) corresponding to y. Therefore, the sheaf R^2σ_y(V) has rank at least u_0 at any (y, s) ∈ W, and rank precisely u_0 at (y_0, s_0). So there is an open dense subset W_0 of W where R^2σ_y(V) is free of rank u_0. For any (y, s) ∈ W_0, the stable sheaf K corresponding to y satisfies hom_X(K, F_i) = 1; up to proportionality we have thus a unique non-zero map η_{y,s} : K → F_i. Stability easily impies that η_{y,s} is injective, so there is an exact sequence:

\[
0 → K → F_i → K' → 0,
\]

for a well-defined sheaf K' = coker(η_{y,s}), for all (y, s) ∈ W_0.

For s = s_0 the sheaf K' is just E_2 so, by openness of stability, up to shrinking W_0 we may assume that K' is stable for all (y, s) ∈ W_0. Note that K' lies in M(2r, H, −2r).

Under our assumption, such sequence should exist for any s in an open neighbourhood of s_0. Then the family of sheaves F should be dominated by the family of extensions of K by K' as s varies around s_0. We see that the dimension of this family of extensions is:

\[
\dim(M_X(2, H, −2)) + \dim(M_X(2r, H, −2r)) + \dim(\mathcal{E}xt}_X^1(K', K)),
\]

which equals 2r(r + 1) + 1(q + 3) + 3, as it follows by formulas 4, 5 applied to K and K' instead of E_1 and E_2. On the other hand, the dimension of S is 2(r + 1)^2(q + 3) + 1. The difference of these dimensions is 2r(g + 3) − 1 and since this is always positive for r ≥ 1, g ≥ 3, we get that the family of simples sheaves appearing as extensions cannot be dense in S_0. This contradiction concludes the proof.

The previous result is in some sense optimal as general K3 surfaces do not support Ulrich bundles of odd rank, cf. [AFO17 Corollary 2.2].

Remark 1. An argument similar to the one of Theorem 1 has been used to construct ACM and Ulrich bundles on Fano threefolds of index 1. Indeed, it follows from the main result of [BF11] that any smooth Fano threefold of Picard number 1 and index 1, containing a line L with normal bundle O_L ⊕ O_L(−1) (such a threefold was called "ordinary" in that paper) admits an Ulrich bundle of rank 2. Initialized Ulrich sheaves of rank 2 are precisely ACM sheaves E with c_1(E(−H)) = H and c_2(E(−H)) = (g + 3)L, where L ⊂ X is a line. We do not know if the same result holds for non-ordinary threefolds.

Remark 2. Theorem 1 implies for instance that any integral quartic surface supports an Ulrich bundle of rank 2. If X is not integral, then X must the union of (possibly multiple) surfaces of degree ≥ 3. For each component it is possible to find a rank-2 Ulrich bundle, we refer to [FP15 Lemma 7.2] for the slightly delicate case of singular cubic surfaces. This yields existence of an Ulrich sheaf of rank 2 on an arbitrary quartic surface.

However the resulting sheaf will fail to be locally free over the intersection of the components. Finding locally free Ulrich sheaves of rank 2 seems more tricky when X is not irreducible and might be impossible when X is not reduced. To justify this let us mention that, for instance if X the union of two distinct double planes, the rank of any locally free Ulrich sheaf on X must be a multiple of 4 by [BHMP16 Proposition 4.14].

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