ON THE UNIQUENESS OF RICCI FLOW

MAN-CHUN LEE

Abstract. In this note, we study the problem of uniqueness of Ricci flow on complete noncompact manifold. We consider the class of solutions with curvature bounded above by $C/t$ when $t > 0$ and proved uniqueness if initial curvature is of polynomial growth and Ricci curvature of the flow is relatively small.

1. Introduction

Let $(M^n, g_0)$ be a complete Riemannian manifold. In [11], Hamilton introduced the Ricci flow

$$\frac{\partial g}{\partial t} = -2Ric$$

and established short time existence and uniqueness on compact manifolds. Later on, using the idea of DeTurck’s trick [7], Shi [23] and Chen-Zhu [5] generalised the existence and uniqueness result to complete manifolds with bounded curvature. They assert that for any $(M, g_0)$ with bounded curvature, there is a unique solution starting from $g_0$ with bounded curvature. It is natural to ask to what extent that we have short-time existence and uniqueness of Ricci flow on a manifold.

When $n = 2$, Giesen and Topping [9, 26] successfully extended the classical results. In particular, they showed that for any initial surface (including those that are incomplete and with unbounded curvature) can be flowed in a unique way by a smooth and instantaneously complete solution and the maximal existence time can be explicitly calculated. In case of $n = 2$, the Ricci flow is reduced to logarithmic fast diffusion Equation whose study can be reduced to a single scalar PDE. However for $n \geq 3$, the Ricci flow is a system of nonlinear weakly parabolic PDE. It is unclear how much their works can be extended to higher dimension.

There are nevertheless a number of existence results in which the initial metric has potentially unbounded curvature. For example, Cabezas-Riva and Wilking [1] use Cheeger-Gromoll exhaustion to construct a Ricci flow when the initial metric has nonnegative complex sectional curvature. Moreover the curvature is instantaneously smoothed out by $c/t$ if the initial metric is non-collapsing. When $M = \mathbb{C}^n$, Chau-Li-Tam [2] and Yang-Zheng [28] have constructed solutions to Kähler Ricci flow which is $U(n)$-invariant. In related work, Koch-Lamm [16] and Schrör-Schulze-Simon [20, 21, 24] considered flow
from metrics which are possibly nonsmooth and $C^0$ perturbation of some metric with bounded geometry. Xu [27] proved existence of solutions under some integral bound on curvature and He [12] used Xu’s partial flow to consider metric with small Ricci curvature but unbounded curvature. More recently, Hochard [14] proved a short time existence result on dimension three when the initial metric has Ricci lower bound and non-collapsing. He [13] and Lee-Tam [15] constucted solutions if the initial metric is locally Euclidean in appropriate sense.

Comparatively, not much is known about the uniqueness except for some special cases. To the best of our knowledge, it is still unknown whether any of the above examples are unique. There are a very few result in this direction. For example, Fan [8] and Chau-Li-Tam [3] showed the uniqueness of Kähler Ricci flow if the flows are uniformly equivalent to a metric with bounded curvature. Sheng-Wang [22] studied complete solutions with lower bound on complex sectional curvature and proved uniqueness under some extra assumptions on initial metric. When $n = 3$, Chen [4] obtained a local estimates which implies two flows starting from a three dimensional manifold with bounded nonnegative sectional curvature must agree identically. More recently, Kotschwar [18, 17] introduced energy method and extended the classical uniqueness result to the case when two flows are uniformly equivalent and their curvature is bounded above by $C(d_0(x, p)^2+1)/t^\gamma, \gamma < 1/2$ or $C/t^\gamma, \gamma < 1$. Compare to recent examples, most of solutions have curvature bound of $C/t$.

It is natural to ask to what extend that uniqueness holds within the class of solutions with $C/t$ bound. In this note, we discuss the uniqueness problem on the class of solutions with curvature bound $C/t$. Within the Kähler category, we consider flows which are uniformly equivalent. In this work, we will prove:

**Theorem 1.** Let $(M^n, g_0)$ be a complete noncompact Kähler manifold with complex dimension $n$. Suppose $g(t)$ and $\tilde{g}(t)$ are two solution to the Kähler Ricci flow with initial data $g_0$. Assume further that $\exists \lambda > 1, C_1 > 0$ such that

\begin{enumerate}
\item $\lambda^{-1}\tilde{g}(t) \leq g(t) \leq \lambda\tilde{g}(t)$ on $M \times [0, T]$ and
\item $|Rm(g(t))| \leq \frac{C_1}{t}$ on $M \times (0, T]$.
\end{enumerate}

Then $g(t) = \tilde{g}(t)$ on $[0, T]$.

Given the recent examples, most of those solutions have non-integrable curvature bound. It is hard to say if two flows are uniformly equivalent to each other. From this point of view, we discuss Ricci flows in which uniform equivalence is not assumed. We use the energy method in [18] to prove that if the curvature of initial metric is polynomially growth and the curvature of flows is bounded by $a/t$ where $a$ is relatively small. Then the solutions must agree.
Theorem 2. For any $m \in \mathbb{N}$, $\exists \epsilon = \epsilon(m) > 0$ such that the following hold: Suppose $(M, g_0)$ is a complete noncompact manifold satisfying

$$|Rm(g_0)| \leq C_0 (d(x, p) + 1)^m$$

for some $C_0 > 0$, $p \in M$. If $g(t)$ and $\tilde{g}(t)$ are two smooth solutions to Ricci flow on $M \times [0, T]$ with $g(0) = \tilde{g}(0) = g_0$ for which

$$|Rm(\tilde{g}(t))|_{\tilde{g}(t)} + |Rm(g(t))|_{g(t)} \leq \frac{\epsilon}{t}$$

on $M \times (0, T]$, then $g(t) = \tilde{g}(t)$ for all $t \in [0, T]$.

The paper is organized as follows: Theorem 1 will be proved in section 2, and Theorem 2 will be proved in section 3.

2. Uniqueness of Kähler Ricci flow

In [17], the author consider the uniqueness problem if the curvature is bounded above by $C t^{-\gamma}$ for $\gamma \in [0, 1)$. In particular, the flows are uniformly equivalent. In this section, we consider the problem within Kähler category and extend the uniqueness to the case of $\gamma = 1$ provided that the flows are uniformly equivalent.

Proof of Theorem 2. By parabolic rescaling, we may assume $T = 1$. Since $\tilde{g} = g(1)$ has bounded curvature. By [25], $\exists \rho \in C^\infty(M)$ such that

$$\begin{cases}
1 + d_{\tilde{g}}(x, p) \leq \rho(x) \leq C_2 (1 + d_\tilde{g}(x, p)) \\
|\nabla \tilde{g} \rho|_{\tilde{g}(T)} \leq C_2 \\
|\nabla_\tilde{g}^2 \rho| \leq C_2
\end{cases}$$

for some $C_2 > 0$. By Ricci flow equation, for all $t \in (0, 1]$,

$$\lambda^{-1} t^\alpha \tilde{g} \leq \tilde{g}(t), \quad g(t) \leq \frac{\lambda}{t^\alpha} \tilde{g}$$

where $\alpha = 2(n - 1)C_1$. We may assume $\alpha = 1 + \beta > 1$.

As in [3], we define $w : M \times [0, 1] \to \mathbb{R}$ to be $w(x, t) = \int_0^t \log \frac{\det g(s)}{\det \tilde{g}(s)} ds$. Let

$$h(x, t, s) = sg(x, t) + (1 - s)\tilde{g}(x, t),$$

then

$$\frac{\partial w}{\partial t} = \log \frac{\det g}{\det \tilde{g}}$$

$$= \int_0^1 \frac{\partial}{\partial s} \log \det h(x, t, s) ds$$

$$= \int_0^1 h_{ij} (g_{ij}(x, t) - \tilde{g}_{ij}(x, t)) ds.$$

On the other hand,

$$w_{ij}(x, t) = \int_0^t -R_{ij}(x, s) + \tilde{R}_{ij}(x, s) ds$$

$$= g_{ij}(x, t) - \tilde{g}_{ij}(x, t).$$
Hence, \[
\frac{\partial w}{\partial t} = \int_0^1 h^{ij} w_{ij}(x, s) \, ds.
\]
In particular, \[
\frac{\partial w^2}{\partial t} - \int_0^1 h^{ij}(w^2)_{ij}(x, s) \, ds
\]
\[
= 2w \left( \partial_t w - \int_0^1 h^{ij} w_{ij}(x, s) \, ds \right) - 2 \int_0^1 h^{ij} w_i(x, s) w_j(x, s) \, ds
\]
\[
\leq 0.
\]
Let \(\eta(x, t) : M \times (0, 1] \to [0, +\infty)\) by \(\eta(x) = f(t) \rho(x)\) where \(f(t) = \exp \left( -\frac{2C \beta}{\lambda t^\beta} \right) \).

Using (2.1) and (2.2), we have \[
\frac{\partial \eta}{\partial t} - \int_0^1 h^{ij} \eta_{ij} \, ds = f'(t) \rho(x) - f(t) \int_0^1 h^{ij} \rho_{ij} \, ds
\]
\[
\geq f'(t) \rho(x) - \frac{\lambda}{t^{1+\beta}} f(t) |\nabla^2 \rho|_g
\]
\[
\geq f'(t) - \frac{\lambda C_2}{t^{1+\beta}} f(t)
\]
\[
> 0.
\]
For any \(\epsilon > 0\), the function \(F_\epsilon = w^2 - \epsilon \eta - \epsilon t\) satisfies
(2.3) \[
\frac{\partial F}{\partial t} - \int_0^1 h^{ij} F_{ij} \, ds < 0 \quad \text{on} \quad M \times (0, 1].
\]
Since \(g(t)\) and \(\tilde{g}(t)\) are uniformly equivalent, \(w^2 \leq Ct^2\). Hence, \(\exists t_0 \in (0, 1]\) such that for all \(t \in [0, t_0]\), \(w^2 - \epsilon t < 0\). On the other hand, as \(w^2\) is bounded, there exists a compact set \(K\) such that for all \((x, t) \in M \setminus K \times (t_0, 1]\),
\[
F_\epsilon(x, t) < 0.
\]
Suppose \(F_\epsilon > 0\) somewhere, then it achieved positive maximum at \((x', t') \in K \times (t_0, 1]\) which is impossible by (2.3). Therefore, for any \(\epsilon > 0\), \(F_\epsilon(x, t) \leq 0\) on \(M \times [0, T]\). By letting \(\epsilon \to 0\), we conclude that \(w \equiv 0\) which implies \(g(t) = \tilde{g}(t)\) by differentiating \(w\).

\section{Uniqueeness of Ricci flow}

If the curvature is bounded above by \(Ct^{-\gamma}\) where \(\gamma < 1\), then we can integrate from 0 to \(t\) and conclude that two flows are equivalent. However there are examples of solutions with curvature bounded above by \(Ct^{-1}\) which is not integrable, then it is hard to say if two Ricci flow are uniformly equivalent. In this section, we consider Ricci flows with curvature bound \(a/t\) where \(a\) is small but without assuming two flows are equivalent. In addition, we need to assume that the initial metric has mild curvature growth at infinity. More precisely, we prove the following.
Theorem 3. For any \( m \in \mathbb{N} \), \( \exists \epsilon = \epsilon(m) > 0 \) such that the following hold: Suppose \((M, g_0)\) is a complete noncompact manifold satisfying

\[
|Rm(g_0)| \leq C_0(d(x, p) + 1)^m
\]

for some \( C_0 > 0, p \in M \). If \( g(t) \) and \( \tilde{g}(t) \) are two smooth solutions to Ricci flow on \( M \times [0, T] \) with \( g(0) = \tilde{g}(0) = g_0 \) for which

\[
|Rm(\tilde{g}(t))|_{\tilde{g}(t)} + |Rm(g(t))|_{g(t)} \leq \frac{K}{t}
\]

and

\[
|Ric(\tilde{g}(t))|_{\tilde{g}(t)} + |Ric(g(t))|_{g(t)} \leq \frac{\epsilon}{t}
\]
on \( M \times (0, T) \) for some \( K > 0 \), then \( g(t) = \tilde{g}(t) \) for all \( t \in [0, T] \).

In order to apply maximum principle using energy method. We first need to construct a exhaustion function on \( M \).

3.1. Exhaustion function on \( M \).

Lemma 1. Suppose \( g(t) \) is a solution of Ricci flow on \( M \times [0, T] \) satisfying

\[
|Ric(g(t))| \leq \frac{1}{4t} \quad \text{on} \quad M \times (0, T].
\]

Then for any \( L_1, L_2 > 0 \), there exists \( \tau = \tau(L_1, L_2, T) > 0 \) and a smooth function \( \eta : M \times [0, \tau] \rightarrow [0, +\infty) \) such that

\[
\frac{\partial \eta}{\partial t} \geq L_1 |\nabla \eta|^2 \quad \text{and} \quad \eta(x, t) \geq L_2 r(x)^2
\]
on \( M \times (0, T] \) where \( r(x) = d_{g(T)}(x, p) \) for some \( p \in M \).

Proof. By proposition 2.1 in [10], there exists smooth function \( \rho : M \rightarrow [0, +\infty) \) such that \( d_{g(T)}(x, p) \leq \rho(x) \leq d_{g(T)}(x, p) + 1 \) and \( |g(T) \nabla \rho|_{g(T)} \leq 2 \). Due to the curvature assumption, for any \( t \in (0, T] \),

\[
\sqrt{\frac{t}{T}} g(T) \leq g(t) \leq \sqrt{\frac{T}{t}} g(T).
\]

Let \( \eta(x, t) = f(t)\rho^2(x) \), then \( \frac{\partial \eta}{\partial t} = f'(t)\rho^2 \) while

\[
|\nabla \eta|^2 = f^2(t)|\nabla \rho|^2 = f^2 g^{ij} \partial_i \rho \partial_j \rho \leq 2 \sqrt{\frac{T}{t}} f^2.
\]

Hence if we take

\[
f(t) = \frac{1}{a - 4L_1\sqrt{T}\sqrt{t}}
\]

then the first inequality holds. The second inequality holds when we choose \( a \) and \( \tau \) small enough. \( \square \)
3.2. Estimates on curvature and its derivatives. In this section, we use a result in [4] to modify estimate of $|Rm(g(t))|$ in term of $d_t(x,p)$ if $|Rm| < \frac{a}{t}$ and initial curvature is of polynomial growth.

**Lemma 2.** Suppose $(M,g(t))$ is a complete Ricci flow for $t \in [0, T]$ satisfying

$$|Rm|_{g(0)} \leq C_1(d_m^0(x,p) + 1) \quad \text{and} \quad |Rm|_{g(t)} \leq \frac{a}{t} \quad \text{on} \quad M \times (0, T]$$

for some $a, m, C_1 > 0, p \in M$. Then there exists $C_2 = C_2(n, m, C_1, a) > 0$ such that

$$|Rm|(x,t) \leq C_2(d_t(x,p)^m + 1).$$

**Proof.** Let $L > 0$ be a fixed constant first. Let $x \in M$ so that $d_0(x,p) = R \geq L$. On $B_0(p, 2R)$,

$$|Rm|_{g(0)} \leq C_1(2^m R^m + 1) = r^{-2}.$$

We may assume $L = L(m, C_1)$ to be sufficiently large so that $r \leq R$. Then by result in [4],

$$|Rm|(x,t) \leq e^{C_n a} r^{-2} = C_1 e^{C_n a} (2^n R^n + 1), \quad \forall t \in [0, T].$$

But since $Ric \leq a(n - 1)t^{-1}$ on $M \times (0, T]$, by [19],

$$d_t(x,p) \geq d_0(x,p) - \beta_n \sqrt{at}.$$

Hence if we choose $L = L(a, n, C_1, m)$ even larger, then $2d_t(x,p) \geq d_0(x,p)$. Hence, $\exists C_2 = C_2(n, a, C_1, m) > 0$ such that

$$|Rm|(x,t) \leq C_2(d_t(x,p)^m + 1) \quad \text{whenever} \quad d_0(x,p) \geq L.$$

On the other hand, the curvature is bounded on $B_0(p, L)$. Use Chen’s result again, we get the upper bound of $|Rm|_{g(t)}$ on $B_0(p, L)$. So by choosing a larger $C_2$, we conclude that

$$|Rm|(x,t) \leq C_2(d_t(x,p)^m + 1) \quad \forall (x,t) \in M \times [0, T].$$

□

By Shi type estimate, we can also obtain a estimate of $|\nabla Rm|$ in term of spatial information.

**Lemma 3.** Suppose $(M, g(t))$ is a complete Ricci flow for $t \in [0, T]$ satisfying

$$|Rm|_{g(0)} \leq C_1(d_m^0(x,p) + 1) \quad \text{and} \quad |Rm|_{g(t)} \leq \frac{a}{t} \quad \text{on} \quad M \times (0, T]$$

for some $a, m, C_1 > 0, p \in M$. Then there exists $C_3 = C_3(T, n, m, a, C_1) > 0$ such that

$$|\nabla Rm|_{g(t)} \leq \frac{C_3[1 + d_t(x,p)^m]^{3/2}}{\sqrt{t}} \quad \text{on} \quad M \times (0, T].$$
Proof. Since $Ric \leq a(n - 1)t^{-1}$ on $(0, T]$, we know that for all $t \in (0, T]$, $s \in [t/2, t]$,

$$
\lambda^{-1}d_t(x, p) \leq d_s(x, p) \leq \lambda d_t(x, p) \quad \text{where} \quad \lambda = 2^a(n-1).
$$

Now we fix $t_0 > 0$, take $\epsilon = t_0/2$ and $R \geq 1$. For $t \in [\epsilon, 2\epsilon]$, $B_\epsilon(p, 2\lambda R) \subset B_t(p, 2\lambda^2 R)$. Thus, on $B_\epsilon(p, 2R) \times [\epsilon, 2\epsilon]$,

$$
|Rm|(z, t) \leq C_3[1 + (2\lambda^2 R)^m] \leq K = C_3(R^m + 1)
$$

By Shi/Hamilton’s first order estimate [6], on $B_\epsilon(p, \lambda R) \times (\epsilon, 2\epsilon)$,

$$
|\nabla Rm|(z, t) \leq C_nK \left( \frac{1}{\lambda^2 R^2} + \frac{1}{t - \epsilon} + K \right)^{1/2}.
$$

When $t = 2\epsilon = t_0$,

$$
|\nabla Rm|(z, 2\epsilon) \leq C(R^m + 1) \left[ \frac{1}{\lambda^2 R^2} + \frac{1}{\epsilon} + (R^m + 1) \right]^{1/2}
$$

$$
\leq \frac{C_3(R^m + 1)^{3/2}}{t_0^{1/2}}.
$$

where $C_3 = C_3(n, m, \alpha, C_1, T) > 0$. As $t_0 > 0$ is arbitrary, we know that if $R \geq 1$, then

$$
|\nabla Rm|(x, t) \leq \frac{C_3(R^m + 1)^{3/2}}{\sqrt{t}} \quad \text{on} \quad B_t(p, R).
$$

Now we wish to get pointwise estimate. If $d_t(x, p) \leq 1$, then

$$
|\nabla Rm|(x, t) \leq \frac{4C_3}{\sqrt{t}}.
$$

If $d_t(x, p) = R \geq 1$, $x \in B_t(p, 2R)$. And hence,

$$
|\nabla Rm|(x, t) \leq \frac{C_3[(2R)^m + 1]^{3/2}}{\sqrt{t}} \leq \frac{C_4(1 + d_t(x, p)^m)^{3/2}}{\sqrt{t}}.
$$

Combines the above result and interpolates with Shi estimate [6], we have the following estimate.

**Corollary 1.** Suppose $(M, g(t))$ is a complete Ricci flow for $t \in [0, T]$ satisfying

$$
|Rm|_{g(0)} \leq C_1(d_{t_0}^m(x, p) + 1) \quad \text{and} \quad |Rm|_{g(t)} \leq \frac{a}{t} \quad \text{on} \quad M \times [0, T]
$$

for some $a, m, C_1 > 0$, $p \in M$. Then there exists $C_3 = C_3(T, n, m, a, C_1) > 0$ such that for any $x \in M$, $t \in (0, T]$, $\delta \in [0, 1]$,

$$
|Rm| \leq C_3(d_T(x, p) + 1)^m \left( \frac{1}{t} \right)^{1-\delta}
$$
and
\[ |\nabla Rm| \leq C_3 \frac{(d_T(x, p) + 1)^{3\delta_m/2}}{t^{\delta/2-\delta}}. \]

Proof. By Lemma 8.3 in [19], we have
\[ d_T(x, p) \geq d_t(x, p) - C_n \sqrt{aT}. \]
If \( d_t(x, p) \geq 2C_n \sqrt{aT} \), then \( d_T(x, p) \geq \frac{1}{2}d_t(x, p) \). Hence we can combine with above Lemmas and interpolate with the assumption to conclude the result. If \( d_t(x, p) < 2C_n \sqrt{aT} \), we can choose a larger constant \( C_3 \) so that the conclusion holds. \( \square \)

3.3. Evolution for energy quantities. Following the argument in [18], we consider the following quantities
\[ h_{ij} = g_{ij} - \tilde{g}_{ij}, \quad A^k_{ij} = \Gamma^k_{ij} - \tilde{\Gamma}^k_{ij}, \quad \text{and} \quad S^l_{ijk} = R^l_{ijk} - \tilde{R}^l_{ijk}. \]
Now let us recall the evolution equation of \( h, A \) and \( S \).
\[ \partial_t h = -2S^l_{ij}, \]
\[ \partial_t A = g^{-1} \ast \nabla S + g^{-1} \ast A \ast \tilde{R} + g^{-1} \ast \tilde{g}^{-1} \ast h \ast \tilde{\nabla} \tilde{R} \]
and
\[ \partial_t S = \Delta S + \nabla_a (g^{ab} \tilde{\nabla}_b \tilde{R} - \tilde{g}^{ab} \tilde{\nabla}_b \tilde{R}) + \tilde{g}^{-1} \ast A \ast \tilde{\nabla} \tilde{R} \]
\[ + \tilde{g}^{-1} \ast \tilde{g}^{-1} \ast h \ast \tilde{R} \ast R + \tilde{g}^{-1} \ast S \ast R + \tilde{g}^{-1} \ast \tilde{R} \ast S. \]
Denote \( \lambda(x, t) \geq 1 \) to be a function on \( M \times [0, T] \) such that
\[ \lambda^{-1} g(t) \leq g(t) \leq \lambda g(t), \quad \text{on} \ M \times [0, T]. \]

For any \( a, b > 0, \tilde{h} = t^{-a} h, \tilde{A} = t^{-b} A \) and \( |S|^2 \) satisfies
\[ \partial_t (|\tilde{h}|^2) \leq C_n |R||\tilde{h}|^2 + C_n t^{-a} |S||\tilde{h}| \]  
\[ \partial_t (|\tilde{A}|^2) \leq C_n |R||\tilde{A}|^2 + C_n t^{-b} |\nabla S||\tilde{A}| + C_n \lambda^3 |\tilde{A}|^2 |\tilde{R}|_{\tilde{g}} \]
\[ + C_n t^{-a-b} \lambda^4 |\tilde{A}||\tilde{h}||\tilde{\nabla} \tilde{R}|_{\tilde{g}} \]  
\[ \partial_t |S|^2 \leq 2(\Delta S + div U, S) + C_n \lambda^3 |S|^2 (|R| + |\tilde{R}|_{\tilde{g}}) \]
\[ + C_n \lambda^2 t^{a} |\tilde{h}||S||\tilde{R}|^2 + C_n \lambda^4 t^{b} |\tilde{A}||S||\tilde{\nabla} \tilde{R}|_{\tilde{g}} \]
where \( (\text{div} U)^l_{ijk} = \nabla_a (g^{ab} \nabla_b \tilde{R}^l_{ijk} - \tilde{g}^{ab} \nabla_b \tilde{R}^l_{ijk}). \)

Moreover,
\[ |U| \leq C_n |A||\tilde{R}| + C_n |\tilde{g}^{-1}| |h||\tilde{\nabla} \tilde{R}| \]
\[ \leq C_n \lambda^4 t^{b} |\tilde{A}||\tilde{R}|_{\tilde{g}} + C_n \lambda^4 t^{a} |\tilde{h}||\tilde{\nabla} \tilde{R}|_{\tilde{g}} \]

In [18], the author considered the uniqueness problem where the metrics are uniformly equivalent in which \( \lambda(x, t) \) is uniformly bounded. In the following, we show that \( \lambda \) can be controlled in term of \( d_t(\cdot, p) \) under assumption of theorem 3.
Lemma 4. Under the assumption of theorem \([3]\) there exists \(C = C(\epsilon, n, m, K, C_0, T) > 0\) such that
\[
C^{-1}(d_T(x, p) + 1)^{-2\epsilon}g_0 \leq g(t) \leq C(d_T(x, p) + 1)^{2\epsilon}g_0
\]
for all \((x, t) \in M \times [0, T]\). In particular, we can pick
\[
\lambda(x, t) = C^2(d_T(x, p) + 1)^4\epsilon.
\]

Proof. Due to Corollary \([1]\) when \(d_T(x, p) \leq r\) where \(r \geq 1\), we have
\[
|\text{Ric}(g(t))| \leq \left(\frac{\epsilon}{t}\right)^{1-\delta} (C_1r^n)^{\delta}, \quad \forall \delta \in [0, 1]
\]
for some \(C_1 = C_1(K, n, C_0, m, T) > 0\). Using Ricci flow equation, we therefore conclude that for any \(\delta \in (0, 1]\),
\[
\exp\left[-2\delta^{-1}\epsilon(C_1\epsilon^{-1}r^m)^{\delta}\right] g_0 \leq g(t) \leq \exp\left[2\delta^{-1}\epsilon(C_1\epsilon^{-1}r^m)^{\delta}\right] g_0.
\]
Now we choose \(\delta \in (0, 1]\) so that \(\delta \cdot \log(C_1\epsilon^{-1}r^m) = 1\). Noted that \(\delta \in (0, 1]\) if \(r\) is sufficiently large. Then the above relation becomes
\[
(C_1\epsilon^{-1}r^m)^{-2\epsilon}g_0 \leq g(t) \leq (C_1\epsilon^{-1}r^m)^{2\epsilon}g_0.
\]

3.4. Energy argument.

Proof of theorem \([3]\). We basically modify the curvature estimates so that we can apply the energy argument in \([18]\). We sketch the proof here.

Let \(\phi(x) = \Phi(\rho(x)/r)\) where \(\Phi\) is a smooth non-increasing function defined on \([0, +\infty)\) which is identical to 1 on \([0, 1]\), vanishes outside \([0, 2]\) and \(|\Phi|^2 \leq 100\Phi\) and \(\rho\) is smooth function in which \(|g^{(T)}\nabla \rho|_{g(T)} \leq 2\) and \(|\rho - d_{g(T)}| \leq 1\). Let \(\eta\) to be the function obtained from Lemma \([1]\) with \(L_i\) to be chosen later.

For any fixed constant \(r > 10\). By Corollary \([1]\) there exists \(C_1 = C_1(n, C_0, T, m, K) > 0\) such that for any \(x \in \text{supp}(\phi), t \in (0, T], \delta \in [0, 1]\),
\[
|\text{R}|_g + |\tilde{\text{R}}|_{\tilde{g}} \leq C_1^{\frac{r^{2m\delta}}{1-\delta}}
\]
and
\[
|\nabla \text{R}|_g + |\nabla \tilde{\text{R}}|_{\tilde{g}} \leq C_1^{\frac{r^{3m\delta/2}}{1-\delta}}.
\]
On the other hand, by Lemma \([1]\)
\[
|\tilde{g}^{-1}|_g + |\tilde{g}|_g \leq C_1^{r^{4m\epsilon}}.
\]
Abuse of notation, we will denote \(N\) to be any constant depending on \(n, C_0, T, m, K\) for our convenience.
Using the above estimates and cauchy inequalities on equation (3.1), (3.2) and (3.3), we have on supp(\(\phi\)) \(\subset B_T(p, 3r)\)

\[
\frac{\partial}{\partial t}|\tilde{h}|^2 \leq N \frac{r^{\delta \delta}}{t^{1-\delta}} |\tilde{h}|^2 + \frac{N}{t^a} |S|^2 + \frac{N}{t^a} |\tilde{h}|^2,
\]

\[
\frac{\partial}{\partial t}|\tilde{A}|^2 \leq N \frac{r^{m+2m}}{t^{3-\delta+b-a}} |\tilde{h}|^2 + N |\tilde{A}|^2 \left[ \frac{r^{m+1}}{t^{1-\delta}} + \frac{r^{m+1}}{t^{1-\delta}} + \frac{r^{m+1}}{t^{1-\delta}} \right] + |\nabla S|^2,
\]

\[
\frac{\partial}{\partial t}|S|^2 \leq 2(\Delta S + \text{div} U, S) + N \frac{r^{16m+2b}}{t^{2-\delta-b}} |\tilde{A}|^2 + N \frac{r^{8m+m+2\delta}}{t^{1-\delta}} |\tilde{h}|^2
\]

\[
+ N |S|^2 \left[ \frac{r^{12m+2m}}{t^{1-\delta}} + \frac{r^{16m+2\delta}}{t^{2-\delta-b}} \right],
\]

and

\[
|U|^2 \leq N \frac{r^{9m+2m\delta}}{t^{2-2\delta-2b}} |\tilde{A}|^2 + N \frac{r^{32m+4\delta m}}{t^{3-2\delta-2a}} |\tilde{h}|^2
\]

where \(\delta\) can be any number in \([0, 1]\). Now we specify our choice of \(a, b, \delta\) and \(\epsilon\). Choose \(\delta = \frac{1}{4m} > 1 - \delta > 0\), \(\frac{1}{2} > b > \frac{1}{2} - \delta\) and \(\epsilon = \frac{1}{40m}\). Therefore, if we define \(E_r(t)\) to be

\[
E_r(t) = \int_M \mathcal{E} e^{-\eta \phi} d\mu = \int_M \left[ |\tilde{h}|^2 + |\tilde{A}|^2 + |S|^2 \right] e^{-\eta \phi} d\mu_g(t).
\]

Differentiate with respect to \(t\). Using above estimates and integration by parts, we therefore conclude a similar differential equation as in [18] except that we are using cutoff and exhaustion function with respect to \(g(T)\) instead of \(g(0)\).

\[
\partial_t E_r \leq \frac{N r^2}{t^3} E_r + N \int_{A_g(T)(r, 2r)} \frac{|\nabla \phi|^2}{\phi} |S|^2 e^{-\eta \phi} d\mu
\]

\[
+ \int_M \left( N |\nabla \eta|^2 - \frac{\partial \eta}{\partial t} \right) \mathcal{E} e^{-\eta \phi} d\mu
\]

on \((0, T]\), for some \(N = N(n, m, K, C_0, \epsilon, T) > 0\), \(\gamma = \gamma(m) \in (0, 1)\). So the last term vanishes if we choose \(L_1 = N\) in Lemma 1 and restrict \(t \in (0, \tau]\).

Noted that

\[
|\nabla \phi|^2 = g^{ij} \partial_i \phi \cdot \partial_j \phi \leq \frac{N \phi}{r^2 \sqrt{t}}.
\]

Together with Corollary 1 and Lemma 4, the last term is bounded above by

\[
N \int_{A_g(T)(r, 2r)} \frac{|\nabla \phi|^2}{\phi} |S|^2 e^{-\eta \phi} d\mu \leq N \frac{e^{-L \sqrt{t}} V_T(B_T(p, 2r))}{\sqrt{t}} \leq N \frac{e^{-L \sqrt{t}}}{\sqrt{t}}
\]

where we have used volume comparison theorem on \(g(T)\). As in [18], by choosing \(L_2\) sufficiently large depending on \(N\), we can conclude that on \(M \times [0, \tau]\),

\[
\mathcal{E} \equiv 0
\]
by solving the ode and letting $r \to \infty$. Finally we Iterate the argument to conclude $g = \tilde{g}$ on $[0, T]$.

\[\square\]

**References**

[1] Cabezas-Rivas E, Wilking B. How to produce a Ricci flow via Cheeger-Gromoll exhaustion[J]. Journal of the European Mathematical Society, 2015, 17(12): 3153-3194.

[2] Chau A, Li K F, Tam L F. Deforming complete Hermitian metrics with unbounded curvature[J]. arXiv preprint [arXiv:1402.0722] 2014.

[3] Chau A, Li K F, Tam L F. Longtime existence of the Khler-Ricci flow on Cn[J]. arXiv preprint [arXiv:1409.1906] 2014.

[4] B. L. Chen, Strong uniqueness of the Ricci flow, J. Differential Geom. 82 (2009), 363382.

[5] Chen, Bing-Long, and Xi-Ping Zhu. "Uniqueness of the Ricci flow on complete non-compact manifolds." Journal of Differential Geometry 74.1 (2006): 119-154.

[6] Chow B, Lu P, Ni L. Hamilton’s Ricci flow[M]. American Mathematical Soc., 2006.

[7] DeTurck D M. Deforming metrics in the direction of their Ricci tensors, improved version[J]. Collected papers on Ricci flow, 2003, 37: 163-165.

[8] Fan X Q. A uniqueness result of Khler Ricci flow with an application[J]. Proceedings of the American Mathematical Society, 2007, 135(1): 289-298.

[9] Giesen G, Topping P M. Existence of Ricci flows of incomplete surfaces[J]. Communications in Partial Differential Equations, 2011, 36(10): 1860-1880.

[10] Greene, Robert E., and H. Wu. "$C^\infty$ approximations of convex, subharmonic, and plurisubharmonic functions." Annales Scientifiques de l’ cole Normale Suprieure. Vol. 12. No. 1. 1979.

[11] Hamilton R S. Three-manifolds with positive Ricci curvature[J]. Journal of Differential Geometry, 1982, 17(2): 255-306.

[12] He F. Existence, Lifespan and Transfer Rate of Ricci Flows on Manifolds with Small Ricci Curvature[J]. arXiv preprint [arXiv:1604.07923] 2016.

[13] He F. Existence and applications of Ricci flows via pseudolocality[J]. arXiv preprint [arXiv:1610.01738] 2016.

[14] Hochard R. Short-time existence of the Ricci flow on complete, non-collapsed 3-manifolds with Ricci curvature bounded from below[J]. arXiv preprint [arXiv:1603.08726] 2016.

[15] Lee M C, Tam L F. A note on existence of Ricci flow[J]. arXiv preprint [arXiv:1702.02767] 2017.

[16] Koch H, Lamm T. Geometric flows with rough initial data[J]. Asian Journal of Mathematics, 2012, 16(2): 209-235.

[17] Kotschwar, Brett. "Short-time persistence of bounded curvature under the Ricci flow." arXiv preprint [arXiv:1507.08246] (2015).

[18] Kotschwar B. An energy approach to the problem of uniqueness for the Ricci flow[J]. arXiv preprint [arXiv:1206.3225] 2012.

[19] Perelman, G., The entropy formula for the Ricci flow and its geometric applications, [arXiv:math.DG/0211159]

[20] Schrner O C, Schulze F, Simon M. Stability of Euclidean space under Ricci flow[J]. arXiv preprint [arXiv:0706.0421] 2007.

[21] Schrner O C, Schulze F, Simon M. Stability of hyperbolic space under Ricci flow[J]. arXiv preprint [arXiv:1003.2107] 2010.

[22] Sheng L, Wang X. On Uniqueness of Complete Ricci Flow Solution with Curvature Bounded from Below[J]. arXiv preprint [arXiv:1310.1611] 2013.
[23] Shi W X. Deforming the metric on complete Riemannian manifolds[J]. Journal of Differential Geometry, 1989, 30(1): 223-301.
[24] Simon M. Deformation of C0 Riemannian Metrics in the Direction of Their Ricci Curvature[M]. Univ., Math. Fak., 2001.
[25] Tam, Luen-fai. "Exhaustion functions on complete manifolds." Recent advances in geometric analysis 11 (2008): 211-215.
[26] Topping P M. Uniqueness of instantaneously complete Ricci flows[J]. Geometry & Topology, 2015, 19(3): 1477-1492.
[27] Xu G. Short-time existence of the Ricci flow on noncompact Riemannian manifolds[J]. Transactions of the American Mathematical Society, 2013, 365(11): 5605-5654.
[28] Yang B, Zheng F. U (n)-invariant Kahler-Ricci flow with non-negative curvature[J]. Comm. Anal. Geom, 2013, 21: 251-294.

(Man-Chun Lee) DEPARTMENT OF MATHEMATICS, THE CHINESE UNIVERSITY OF HONG KONG, SHATIN, HONG KONG, CHINA.
E-mail address: mclee@math.cuhk.edu.hk