Spectral Picture For Rationally Multicyclic Subnormal Operators

Liming Yang  
Department of Mathematics  
Virginia Polytechnic and State University  
Blacksburg, VA 24061  
yliming@vt.edu

Abstract

For a pure bounded rationally cyclic subnormal operator $S$ on a separable complex Hilbert space $H$, Conway and Elias (1993) shows that $\text{clos}(\sigma(S) \setminus \sigma_e(S)) = \text{clos}(\text{Int}(\sigma(S)))$. This paper examines the property for rationally multicyclic (N-cyclic) subnormal operators. We show: (1) There exists a 2-cyclic irreducible subnormal operator $S$ with $\text{clos}(\sigma(S) \setminus \sigma_e(S)) \neq \text{clos}(\text{Int}(\sigma(S)))$. (2) For a pure rationally $N$-cyclic subnormal operator $S$ on $H$ with the minimal normal extension $M$ on $K \supset H$, let $K_m = \text{clos}(\text{span}\{M^kx : x \in H, 0 \leq k \leq m\})$. Suppose $M|_{K^{N-1}}$ is pure, then $\text{clos}(\sigma(S) \setminus \sigma_e(S)) = \text{clos}(\text{Int}(\sigma(S)))$.

1 Introduction

Let $H$ be a separable complex Hilbert space and let $\mathcal{L}(H)$ be the space of bounded linear operators on $H$. An operator $S \in \mathcal{L}(H)$ is subnormal if there exist a separable complex Hilbert space $K$ containing $H$ and a normal operator $M_z \in \mathcal{L}(K)$ such that $M_zH \subset H$ and $S = M_z|_{H}$. By the spectral theorem of normal operators, we assume that $K = \bigoplus_{i=1}^{m} L^2(\mu_i)$ (1-1) where $\mu_1 >> \mu_2 >> ... >> \mu_m$ ($m$ may be $\infty$) are compactly supported finite positive measures on the complex plane $\mathbb{C}$, and $M_z$ is multiplication by $z$ on $K$. For $H = (h_1, ..., h_m) \in K$ and $G = (g_1, ..., g_m) \in K$, we define

$$\langle H(z), G(z) \rangle = \sum_{i=1}^{m} h_i(z)\overline{g_i(z)} \frac{d\mu_i}{d\mu_1} |H(z)|^2 = \langle H(z), H(z) \rangle .$$ (1-2)

The inner product of $H$ and $G$ in $K$ is defined by

$$(H, G) = \int \langle H(z), G(z) \rangle d\mu_1(z).$$ (1-3)

$M_z$ is the minimal normal extension if

$$K = \text{clos}(\text{span}(M_z^{k}x : x \in H, k \geq 0)) ,$$ (1-4)

We will always assume that $M_z$ is the minimal normal extension of $S$ and $K$ satisfies (1-1) to (1-4). For details about the functional model above and basic knowledge of subnormal operators, the reader shall consult Chapter II of the book Conway (1991).

For $T \in \mathcal{L}(H)$, we denote by $\sigma(T)$ the spectrum of $T$, $\sigma_e(T)$ the essential spectrum of $T$, $T^*$ its adjoint, $\ker(T)$ its kernel, and $\text{Ran}(T)$ its range. For a subset $A \subset \mathbb{C}$, we set $\text{Int}(A)$ for its
interior, $\text{cl}(A)$ for its closure, $A^c$ for its complement, and $\bar{A} = \{z : z \in A\}$. For $\lambda \in \mathbb{C}$ and $\delta > 0$, we set $B(\lambda, \delta) = \{z : |z - \lambda| < \delta\}$ and $\mathbb{D} = B(0, 1)$. Let $\mathcal{P}$ denote the set of polynomials in the complex variable $z$. For a compact subset $K \subset \mathbb{C}$, let $\text{Rat}(K)$ be the set of all rational functions with poles off $K$ and let $R(K)$ be the uniform closure of $\text{Rat}(K)$.

A subnormal operator $S$ on $\mathcal{H}$ is pure if for every non-zero invariant subspace $I$ of $S$ ($SI \subset I$), the operator $|S|$ is not normal. For $F_1, F_2, ..., F_N \in \mathcal{H}$, let

$$R^2(S|F_1, F_2, ..., F_N) = \text{cl}\{r_1(S)F_1 + r_2(S)F_2 + ... + r_N(S)F_N\}$$

in $\mathcal{H}$, where $r_1, r_2, ..., r_N \in \text{Rat}(\sigma(S))$ and let

$$P^2(S|F_1, F_2, ..., F_N) = \text{cl}\{p_1(S)F_1 + p_2(S)F_2 + ... + p_N(S)F_N\}$$

in $\mathcal{H}$, where $p_1, p_2, ..., p_N \in \mathcal{P}$. A subnormal operator $S$ on $\mathcal{H}$ is rationally multicyclic ($N$-cyclic) if there are $N$ vectors $F_1, F_2, ..., F_N \in \mathcal{H}$ such that

$$\mathcal{H} = R^2(S|F_1, F_2, ..., F_N)$$

and for any $G_1, ..., G_{N-1} \in \mathcal{H},$

$$\mathcal{H} \neq R^2(S|G_1, G_2, ..., G_{N-1}).$$

We call $N$ is the rationally cyclic multiple of $S$. $S$ is multicyclic ($N$-cyclic) if

$$\mathcal{H} = P^2(S|F_1, F_2, ..., F_N)$$

and for any $G_1, ..., G_{N-1} \in \mathcal{H},$

$$\mathcal{H} \neq P^2(S|G_1, G_2, ..., G_{N-1}).$$

We call $N$ is the cyclic multiple of $S$. In this case, $m \leq N$ where $m$ is as in (1-1).

Let $\mu$ be a compactly supported finite positive measure on the complex plane $\mathbb{C}$ and let $\text{spt}(\mu)$ denote the support of $\mu$. For a compact subset $K$ with $\text{spt}(\mu) \subset K$, let $R^2(K, \mu) = \text{cl}(\text{Rat}(K))$ in $L^2(\mu)$. Let $P^2(\mu)$ denote the closure of $\mathcal{P}$ in $L^2(\mu)$.

If $S$ is rationally cyclic, then $S$ is unitarily equivalent to multiplication by $z$ on $R^2(\sigma(S), \mu_1)$, where $m = 1$ and $F_1 = 1$. We may write $R^2(S|F_1) = R^2(\sigma(S), \mu_1)$. If $S$ is cyclic, then $S$ is unitarily equivalent to multiplication by $z$ on $P^2(\mu_1)$. We may write $P^2(S|F_1) = P^2(\mu_1)$.

For a rationally $N$-cyclic subnormal operator $S$ with cyclic vectors $F_1, F_2, ..., F_N$ and $\lambda \in \sigma(S)$, we denote the map

$$E(\lambda) : \sum_{i=1}^{N} r_i(S)F_i \rightarrow \begin{bmatrix} r_1(\lambda) \\ r_2(\lambda) \\ \vdots \\ r_N(\lambda) \end{bmatrix},$$

(1-5)

where $r_1, r_2, ..., r_N \in \text{Rat}(\sigma(S))$. If $E(\lambda)$ is bounded from $K$ to $(\mathbb{C}^N, \||.||_1, N)$, then every component in the right hand side extends to a bounded linear functional on $\mathcal{H}$ and we will call $\lambda$ a bounded point evaluation for $S$. We use $\text{bpe}(S)$ to denote the set of bounded point evaluations for $S$. The set $\text{bpe}(S)$ does not depend on the choices of cyclic vectors $F_1, F_2, ..., F_N$ (see Corollary 1.1 in [Mbekhta et al. 2014]). A point $\lambda_0 \in \text{int}(\text{bpe}(S))$ is called an analytic bounded point evaluation for $S$ if there is a neighborhood $B(\lambda_0, \delta) \subset \text{bpe}(S)$ of $\lambda_0$ such that $E(\lambda)$ is analytic as a function of $\lambda$ on $B(\lambda_0, \delta)$ (equivalently (1-5) is uniformly bounded for $\lambda \in B(\lambda_0, \delta)$). We use $\text{abpe}(S)$ to denote the set of analytic bounded point evaluations for $S$. The set $\text{abpe}(S)$ does not depend on the choices of cyclic vectors $F_1, F_2, ..., F_N$ (also see Remark 3.1 in [Mbekhta et al. 2010]). Similarly, for an $N$-cyclic subnormal operator $S$, we can define $\text{bpe}(S)$ and $\text{abpe}(S)$ if we replace $r_1, r_2, ..., r_N \in \text{Rat}(\sigma(S))$ in (1-5) by $p_1, p_2, ..., p_N \in \mathcal{P}$.

For $N = 1$, [Thomson 1991] proves a remarkable structural theorem for $P^2(\mu)$.
Thomson’s Theorem. There is a Borel partition \( \{ \Delta_i \} \) of \( \text{spt} \mu \) such that the space \( P^2(\mu|_{\Delta_i}) \) contains no nontrivial characteristic functions and

\[
P^2(\mu) = L^2(\mu|_{\Delta_0}) \oplus \{ \oplus_{i=1}^\infty P^2(\mu|_{\Delta_i}) \}.
\]

Furthermore, if \( U_i \) is the open set of analytic bounded point evaluations for \( P^2(\mu|_{\Delta_i}) \) for \( i \geq 1 \), then \( U_i \) is a simply connected region and the closure of \( U_i \) contains \( \Delta_i \).

Conway and Elias (1993) extends some results of Thomson’s Theorem to the space \( R^2(K, \mu) \), while Brennan (2008) expresses \( R^2(K, \mu) \) as a direct sum that includes both Thomson’s theorem and results of Conway and Elias (1993). For a compactly supported complex Borel measure \( \nu \) of \( C \), by estimating analytic capacity of the set \( \{ \lambda : |C\nu(\lambda)| > c \} \), where \( C\nu \) is the Cauchy transform of \( \nu \) (see Section 3 for definition), Brennan (2006, English), Aleman et al. (2009), and Aleman et al. (2010) provide interesting alternative proofs of Thomson’s theorem. Both their proofs rely on X. Tolsa’s deep results on analytic capacity. There are other related research papers for \( N = 1 \) in the history. For example, Brennan (1979, Russian), Brennan (2006, English), Aleman et al. (2009), and Yang (2018) extend the result to rationally cyclic subnormal operators.

It is shown in Theorem 2.1 of Conway and Elias (1993) that if \( K \) is a pure rationally cyclic subnormal operator, then

\[
clos(\sigma(S) \setminus \sigma_e(S)) = clos(\text{Int}(\sigma(S))).
\]

This leads us to examine if (1-6) holds for a rationally cyclic subnormal operator.

A Gleason part of \( R(K) \) is a maximal set \( \Omega \) in \( C \) such that for \( x, y \in \Omega \), if \( e_x \) and \( e_y \) denote the functional values at \( x \) and \( y \) respectively, then \( \|e_x - e_y\|_{R(K)^*} < 2 \). Olin and Thomson (1980) shows that a compact set \( K \) can be the spectrum of an irreducible subnormal operator if and only if \( R(K) \) has only one nontrivial Gleason part \( \Omega \) and \( K = clos(\Omega) \). McGuire (1988) and Feldman and McGuire (2003) construct irreducible subnormal operators with a prescribed spectrum, approximate point spectrum, essential spectrum, and the (semi) Fredholm index. Our first result is to construct a (rationally) 2-cyclic irreducible subnormal operator for a prescribed spectrum and essential spectrum. Consequently we show that (1-6) may not hold for a (rationally) \( N \)-cyclic irreducible subnormal operator with cyclic multiple \( N > 1 \).

**Theorem 1.** Let \( K \) and \( K_e \) be two compact subsets of \( C \) such that \( R(K) \) has only one nontrivial Gleason part \( \Omega \), \( K = clos(\Omega) \), and \( \partial K \subset K_e \subset K \). Then there exists a rationally 2-cyclic irreducible subnormal operator \( S \) such that \( \sigma(S) = K \), \( \sigma_e(S) = K_e \), and \( \text{ind}(S - \lambda) = -1 \) for \( \lambda \in K \setminus K_e \). If, in particular, \( C \setminus K \) has only one component, then \( K \) can be constructed as a 2-cyclic irreducible subnormal operator.

Let \( K = clos(D) \) and \( K_e = \partial D \cup clos(\frac{1}{2}D) \). We see that

\[
clos(K \setminus K_e) = \{ z : \frac{1}{2} \leq |z| \leq 1 \} \neq clos(\text{Int}(K)) = clos(D).
\]

From Theorem 1 we get the following result.

**Corollary 1.** There exists a 2-cyclic irreducible subnormal operator \( S \) such that (1-6) does not hold.

In the second part of this paper, we will investigate certain classes of rationally \( N \)-cyclic subnormal operators that have the property (1-6). Let \( S \) be a rationally \( N \)-cyclic subnormal operator on \( \mathcal{H} = R^2(S|F_1, F_2, ..., F_N) \). Let \( \psi \) be a smooth function with compact support. Define

\[
\mathcal{K}^\psi_n = clos \{ \psi^m x : x \in \mathcal{H}, 0 \leq m \leq n \},
\]

then

\[
\mathcal{H} \subset \mathcal{K}^\psi_1 \subset ... \subset \mathcal{K}^\psi_n \subset ... \subset \mathcal{K}
\]

and \( M_\psi|_{\mathcal{K}^\psi_n} \) is a subnormal operator.
Definition 1. A subnormal operator satisfies the property \((N, \psi)\) if the following conditions are met:

1. \(S\) is a pure (rationally) \(N\)-cyclic subnormal operator on \(\mathcal{H} = R^2(S|F_1, F_2, \ldots, F_N)\).

2. \(\psi\) a smooth function with compact support and \(
\text{Area}(\sigma(S) \cap (\partial \psi = 0)) = 0
\). Let \(M_z\) on \(K\) be the minimal normal extension of \(S\) satisfying (1-1) to (1-4), then \(M_z|_{K_{N-1}}\) is also a pure subnormal operator.

Theorem 2. Let \(N > 1\) and let \(S\) be a pure subnormal operator on \(\mathcal{H}\) satisfying the property \((N, \psi)\), then there exist bounded open subsets \(U_i\) for \(1 \leq i \leq N\) such that

\[
\begin{align*}
\sigma_e(S) & = \bigcup_{i=1}^{N} \partial U_i, \\
\sigma(S) & = \bigcup_{i=1}^{N} \text{clos}(U_i),
\end{align*}
\]

and

\[
\text{ind}(S - \lambda) = -i
\]

for \(\lambda \in U_i\) and \(i = 1, 2, \ldots N\). Consequently,

\[
\sigma(S) = \text{clos}(\sigma(S) \setminus \sigma_e(S)) = \text{clos}(\text{Int}(\sigma(S))).
\]

An important special case is that \(\psi = \bar{\xi}\). In section 3, we will provide several examples of subnormal operators that satisfy the property \((N, \psi)\). We prove Theorem 1 in section 2 and Theorem 2 in section 3.

2 Spectral Pictures for Irreducible Rationally 2-Cyclic Subnormal Operators

In this section, we assume that \(K\) is a compact subset of \(\mathbb{C}\), \(\text{Int}(K) \neq \emptyset\), and \(R(K)\) has only one nontrivial Gleason part \(\Omega\) with \(K = \text{clos}(\Omega)\). Theorem 5 and Corollary 6 in McGuire (1988) constructs a representing measure \(\nu\) of \(R(K)\) at \(z_0 \in \text{Int}(K)\) with support on \(\partial K\) such that \(S_0\), on \(R^2(K, \nu)\), is irreducible, \(\sigma(S_0) = K\), \(\sigma_e(S_0) = \partial K\), and \(\text{ind}(S_0 - \lambda) = -1\) for \(\lambda \in \text{Int}(K) = \sigma(S_0) \setminus \sigma_e(S_0)\). From Theorem 6.2 in Gamelin (1969), we get

\[
L^2(\nu) = R^2(K, \nu) \oplus N^2 \oplus R_0^2(K, \nu)
\]

(2-1)

where \(R_0^2(K, \nu) = \{ \bar{r} : r(z_0) = 0 \text{ and } r \in R^2(K, \nu) \}\). The operator \(M_z\), multiplication by \(z\) on \(L^2(\nu)\), can be written as the following matrix with respect to (2-1):

\[
M_z = \begin{bmatrix}
S_0 & A & B \\
0 & C & D \\
0 & 0 & T_v
\end{bmatrix}
\]

where \(T_v\), multiplication by \(\bar{\varepsilon}\) on \(R_0^2(K, \nu)\), is an irreducible rationally cyclic subnormal operator with \(\sigma(T_v) = K\), \(\sigma_e(T_v) = \partial K\), and \(\text{ind}(T_v - \lambda) = -1\) for \(\lambda \in \text{Int}(K)\). Let

\[
S = \begin{bmatrix}
S_0 & A \\
0 & C
\end{bmatrix},
\]

then \(S\) is the dual of \(T_v\). From the properties of dual subnormal operators (see, for example, Theorem 2.4 in Feldman and McGuire (2003)), we see that \(S\) is an irreducible subnormal operator with \(\sigma(S) = K\), \(\sigma_e(S) = \partial K\), and \(\text{ind}(S - \lambda) = -1\) for \(\lambda \in \text{Int}(K)\).

The following lemma, due to Cowen and Douglas (1978) on page 194, allows us to choose eigenvectors for \(S^*\) in a co-analytic manner whenever the Fredholm index function for \(S\) is \(-1\).

Lemma 1. If \(X \in L(H)\) and \(\text{ind}(X - \lambda) = -1\) for all \(\lambda \in G := \sigma(X) \setminus \sigma_e(X)\), then there exists a co-analytic function \(h : G \rightarrow H\) that is not identically zero on any component of \(G\) such that \(h(\lambda) \in \text{ker}(X - \lambda)^*\). In particular, for every \(x \in H\), the function \(\lambda \rightarrow (x, h(\lambda))\) is analytic on \(G\).
Lemma 3. Let $\mu$ be as in (2-2) and let $\mathcal{H}_1$ be as in Lemma 2. Define

$$F(z) = \begin{cases} z - z_0, & z \in \partial K, \\ 0, & z \in \text{Int}(K). \end{cases}$$

and

$$G_n(z) = \begin{cases} k_{\lambda_n}(z), & z \in \partial K, \\ -1/c_n, & z = \lambda_n, \\ 0, & z = \lambda_m, \; m \neq n. \end{cases}$$

Then

$$\mathcal{H}^+ = \overline{\text{span}\{r(z)F, G_j, \; 1 \leq j < \infty, \; r \in \text{Rat}(K)\}}.$$
Proof: It is straightforward to check, from (2-1), (2-2), and (2-3), that $F,G_j \in \mathcal{H}_1^+$. Now let $H(z) \perp \text{clos} \{\text{span}\{r(z)F,G_j, \ 1 \leq j < \infty, \ r \in \text{Rat}(K)\}\}$; then
\[
\int H(z)r(z)F(z)d\mu = \int H(z)r(z)(z - z_0)d\nu = 0
\]
for $r \in \text{Rat}(K)$. From (2-1), we see that the function $H|_{\partial K} \in \mathcal{H}$. It follows from $\int H(z)\tilde{G}_j(z)d\mu = 0$ that $H(\lambda_j) = (H|_{\partial K}, k_{\lambda_j})$. Thus, $H(z) \in \mathcal{H}_1$. The lemma is proved.

**Lemma 4.** Let $\mu, T_1, F,$ and $G_n$ be as in (2-2), (2-4), (2-5) and (2-6), respectively. Then there exists a sequence of positive numbers $\{a_n\}$ satisfying
\[
\sum_{n=1}^{\infty} a_n \|G_n\| < \infty, \quad G = \sum_{n=1}^{\infty} a_n G_n,
\]
and
\[
\mathcal{H}_1^+ = \text{clos} \{\text{span}\{r(z)F(z) + p(z)G(z) : r \in \text{Rat}(K), | \ p \in \mathcal{P}\}\}.
\]
Therefore, $T_1$ is a rationally 2-cyclic irreducible subnormal operator with
\[
\sigma(T_1) = \tilde{K}, \quad \sigma_e(T_1) = \tilde{K}_e, \quad \text{and ind}(T_1 - \lambda) = -1, \ \lambda \in \tilde{K} \setminus \tilde{K}_e.
\]

**Proof:** Notice that
\[
\int f(z)(z - \lambda_n)k_{\lambda_n}(z)d\nu = 0
\]
for $f \in \mathcal{H}$. We conclude, from (2-1), that $(\bar{z} - \tilde{\lambda}_n)k_{\lambda_n}(z) \in \overline{R^2(K, \nu)}$. Hence, there are $\{r_n\} \subset R^2(K, \nu)$ such that
\[
k_{\lambda_n}(z) = \frac{r_n(\bar{z})}{\bar{z} - \tilde{\lambda}_n}(\bar{z} - \bar{z}_0).
\]
We will recursively choose $\{a_n\}$. First choose $a_1 = 1$. Then we assume that $a_1, a_2, \ldots, a_n$ have been chosen. Now we will choose $a_{n+1}$. Let
\[
p_k(z) = \frac{\prod_{j \neq k, 1 \leq j \leq n}(z - \tilde{\lambda}_j)}{a_k \prod_{j \neq k, 1 \leq j \leq n}(\tilde{\lambda}_k - \tilde{\lambda}_j)},
\]
for $k = 1, 2, \ldots, n$. Denote
\[
q_{1k}(z) = p_k(z) \sum_{j \neq k, 1 \leq j \leq n} \frac{a_j}{z - \tilde{\lambda}_j}r_j(z)
\]
and
\[
q_{2k}(z) = \frac{a_k(p_k(z) - p_k(\tilde{\lambda}_k))}{z - \tilde{\lambda}_k}r_k(z).
\]
So $p_k \in \mathcal{P}$ and $q_{1k}, q_{2k} \in R^2(K, \nu)$ for $k = 1, 2, \ldots, n$. Clearly,
\[
p_k(\bar{z}) \sum_{j=1}^{n} a_j G_j(z) - (q_{1k}(\bar{z}) + q_{2k}(\bar{z}))(\bar{z} - \bar{z}_0) = \frac{r_k(\bar{z})(\bar{z} - \bar{z}_0)}{\bar{z} - \tilde{\lambda}_k}, \quad z \in \partial K.
\]
Hence,
\[
p_k(\bar{z}) \sum_{j=1}^{n} a_j G_j(z) - (q_{1k}(\bar{z}) + q_{2k}(\bar{z}))F(z) = G_k(z), \ a.e. \ \mu.
\]
We have the following calculation:
\[
\int \left| p_k(\bar{z}) \sum_{j=1}^{n+1} a_j G_j(z) - (q_{1k}(\bar{z}) + q_{2k}(\bar{z}))F(z) - G_k(z) \right|^2 d\mu
\]
\[
= \int |p_k(\bar{z})a_{n+1}G_{n+1}(z)|^2 d\mu
\]
\[
\leq \left( \frac{a_{n+1}}{a_k} \right)^2 \left( \frac{4D^2}{\prod_{j \neq k, 1 \leq j \leq n} |\tilde{\lambda}_k - \tilde{\lambda}_j|^2} \right) \|G_{n+1}\|^2
\]
where \( D = \max \{|z| : z \in K\} \). Now set
\[
a_{n+1} = \min \left( \frac{1}{2^{n+1}}, \min_{1 \leq k \leq n} \frac{a_k \prod_{j \neq k, 1 \leq j \leq n} \min(1, |\lambda_k - \lambda_j|)}{4^n \max(1, D)^{n+1}} \right) / \max(1, \|G_n\|). \tag{2-8}
\]
So we have chosen all \( \{a_n\} \). From (2-8), we have the following calculation.
\[
\left\| p_k \sum_{i=n+2}^{\infty} a_j G_j \right\| \leq \left( \frac{2D}{\sigma_k \prod_{j \neq k, 1 \leq j \leq n} |\lambda_k - \lambda_j|} \right) \frac{\sum_{i=n+2}^{\infty} a_k \prod_{j \neq k, 1 \leq j \leq i-1} \min(1, |\lambda_k - \lambda_j|)}{4^{i-1} \max(1, D)^{i-2}} \leq \frac{1}{2^{n+2}} \]
Therefore,
\[
\left\| p_k (\bar{z}) - (q_{1k}(\bar{z}) + q_{2k}(\bar{z})) F - G_k(\bar{z}) \right\| \leq \left\| p_k (\bar{z}) \sum_{j=1}^{n+1} a_j G_j - (q_{1k}(\bar{z}) + q_{2k}(\bar{z})) F - G_k(\bar{z}) \right\| + \left\| p_k (\bar{z}) \sum_{j=n+2}^{\infty} a_j G_j \right\| \leq \frac{1}{2^n}.
\]
Hence,
\[
G_k \in \text{clos} (\text{span}(r(\bar{z}) F(z) + p(\bar{z}) G(z) : r \in \text{Rat}(K), p \in \mathcal{P})), \quad k = 1, 2, \ldots
\]
Since \( T_1 \) is the dual of \( S_1 \), we see that \( \sigma(M_1^*) \subset \sigma_e(S_1) \cup \overline{\sigma_e(T_1)} \) (see, for example, Theorem 2.4 in Feldman and McGuirk (2003)), \( \sigma_e(S_1) = \partial K \), and \( \sigma_e(T_1) \supset \partial \bar{K} \). So (2-7) follows. This completes the proof.

**Proof of Theorem** It follows from Lemma [1]

### 3 Spectral Picture of a Class of Rationally Multicyclic Subnormal Operators

In this section, we will prove Theorem [2] First we provide some examples of subnormal operators that have the property \((N, \psi)\) in Definition [1]

**Example 1.** Every pure subnormal operator \( S \) on \( \mathcal{H} \) with finite rank self-commutator has the property \((N, \psi)\). Notice that the structure of such subnormal operators has been established based on Xia’s model (see Xia (1996) and Yakubovich (1996)).

**Proof:** Assume that \( M_\xi \) on \( K \) is the minimal normal extension satisfying (1-1) to (1-4). Define the self-commutator as the following
\[
D = [S^*, S] = S^* S - SS^*.
\]
The element \( x \in \ker(D) \) if and only if \( \bar{z} x \in \mathcal{H} \). This implies \( \ker(D) \subset \ker(D) \). Therefore,
\[
S^* \ker(D) \subset \ker(D).
\]

Let
\[
\mathcal{H}_0 = \text{clos} (\text{span}(S^n f : f \in \ker(D), n \geq 0)),
\]
then \( S|_{\mathcal{H}_0} \) is \( N \)-cyclic subnormal where \( N = \text{dim}(\ker(D)) \).

On the other hand,
\[
S^* S^n D = SS^* S^{n-1} D + DS^{n-1} D,
\]
hence, we can recursively show that $S^* S^n \text{ Ran}(D) \subset \mathcal{H}_0$ since (3-1). So $S^* \mathcal{H}_0 \subset \mathcal{H}_0$. This implies that
\[ S(\mathcal{H} \ominus \mathcal{H}_0) \subset \mathcal{H} \ominus \mathcal{H}_0 \]
and $S|_{\mathcal{H} \ominus \mathcal{H}_0}$ is normal. Since $S$ is pure, we conclude that $\mathcal{H} = \mathcal{H}_0$ and $S$ is $N$-cyclic. From (3-1), we see that there is a polynomial $p$ such that
\[ p(S|_{\text{Ran}(D)}) = 0. \]
Therefore,
\[ p(S) : \mathcal{H} \to \text{ker}(D). \]
Hence,
\[ \|M^*_p \psi \| = \|M_x \psi \| = \|S \psi \| = \|S^* \psi \| \]
for $\psi \in \mathcal{H}$. This implies $\psi \mathcal{H} \subset \mathcal{H}$. Let $\psi = \psi p$, then $\text{Area} \{ \psi \psi = 0 \} = \text{Area} \{ \psi : p(\psi) = 0 \} = 0$, $\mathcal{K}_{N-1} = \mathcal{H}$, and $S$ satisfies the property $(N, \psi)$ in Definition 4.

**Example 2.** In Lemma 4 if $K = \text{clo}(\mathcal{D})$ and $K_\psi = (\partial \mathcal{D}) \cup (\frac{1}{2} \partial \mathcal{D})$, then the operator $T_1$ is a $2$-cyclic irreducible subnormal operator satisfying the property $(2, \psi)$ where $\psi = |z|^4 - \frac{2}{3} |z|^2$.

**Proof:** For $f \in \mathcal{H}_1$, we get
\[ \psi f = (|z|^2 - 1)(|z|^2 - \frac{1}{4})f + \frac{1}{4} f = \frac{1}{4} f \]
since $\text{spt}(\mu) \subset K_\psi$. Hence, $\mathcal{K}_1 = \mathcal{H}_1$. On the other hand,
\[ \text{Area} \{ \psi \partial \psi = 0 \} \leq \text{Area} \left( \{ 0 \} \cup \{ |z| = \frac{5}{8} \} \right) = 0. \]
Therefore, the operator $T_1$ satisfies the property $(2, \psi)$.

In the remaining section, we assume that $N > 1$ and $S$ is a pure rationally $N$-cyclic subnormal operator on $\mathcal{H} = R^2(S|F_1, F_2, ..., F_N)$ and $M_x$ on $K$, which satisfies (1-1) to (1-4), is the minimal normal extension of $S$. Moreover, $S$ satisfies the property $(N, \psi)$ in Definition 4.

Let $U_k$ be the set of $\lambda \in \text{Int}(\sigma(S))$ such that $\text{Ran}(S - \lambda)$ is closed and $\text{dim} \text{(ker}(S - \lambda)) = k$, where $k = 1, 2, ..., N$.

**Lemma 5.** If $1 \leq k \leq N$, $\delta > 0$, $B(\lambda_0, 2\delta) \subset \text{Int}(\sigma(S))$, $I$ is an index subset of $\{1, 2, ..., N\}$ with size $N - k$, $F = \sum_{i=1}^N r_i F_i$ where $r_i \in \text{Rat}(\sigma(S))$, and $\{ a_i(\lambda) \}_{1 \leq i \leq N-k, 1 \leq s \leq k}$ are analytic on $B(\lambda_0, 2\delta)$ such that
\[ \sup_{1 \leq s \leq k, \lambda \in B(\lambda_0, \delta)} |r_j(\lambda) + \sum_{l=1}^{N-k} a_{i_k}(\lambda^l) r_{i_l}(\lambda)| \leq M \| F \| \tag{3-2} \]
and
\[ F_{ij}(z) = \sum_{s=1}^k a_{i_s}(z) F_j(z), \text{ a.e. } \mu_1|_{B(\lambda_0, \delta)}, \tag{3-3} \]
where $i_j \in I$ and $j_s \notin I$. Then $\lambda_0 \in \bigcup_{k=1}^N U_k$.

**Proof:** From (3-3), we get
\[ \int_{B(\lambda_0, \delta)} |F|^2 d \mu_1 = \int_{B(\lambda_0, \delta)} \left| \sum_{s=1}^k \left( r_{i_s}(z) + \sum_{l=1}^{N-k} a_{i_s}(z) r_{i_l}(z) \right) F_{j_s}(z) \right|^2 d \mu_1. \]
Using (3-2) and the maximal modulus principle,
\[ \sup_{1 \leq s \leq k, \lambda \in B(\lambda_0, \delta)} \left| r_j(\lambda) + \sum_{l=1}^{N-k} a_{i_s}(\lambda) r_{i_l}(\lambda) \right| \leq \frac{M}{\delta} \| (S - \lambda_0) F \|. \]
Hence,
\[
\int |F|^2 \,d\mu_1 \leq \int_{B(\lambda_0, \delta)^c} |F|^2 \,d\mu_1 + \left( \sum_{j \notin I} \|F_j\|^2 \right)^2 \sup_{1 \leq s \leq k, \lambda \in B(\lambda_0, \delta)} \left| r_{j_s}(\lambda) + \sum_{i=1}^{N-k} a_{is}(\lambda) r_i(\lambda) \right|^2 .
\]

Therefore,
\[
\|F\| \leq M_1 \|(S - \lambda_0)F\|,
\]
where
\[
M_1^2 = \left( 1 + \left( \sum_{j \notin I} \|F_j\|^2 \right)^{\frac{1}{2}} \right)^2.
\]

So \( \text{Ran}(S - \lambda_0) \) is closed. On the other hand, there are \( k \) linearly independent \( k_j \in \mathcal{H} \) such that
\[
r_{j_s}(\lambda) + \sum_{i=1}^{N-k} a_{is}(\lambda) r_i(\lambda) = \int \langle F(z), k_j(z) \rangle \,d\mu_1(z)
\]
where \( j_s \notin I \) and \( \lambda \in B(\lambda_0, \delta) \). This implies
\[
\dim(\ker (S - \lambda_0)^\ast) \geq k.
\]

Therefore, \( \lambda_0 \in \bigcup_{i=1}^{N} U_i \).

Let \( \nu \) be a compactly supported finite measure on \( \mathbb{C} \). The transform
\[
C_\psi \nu(z) = \int \frac{(\psi(w) - \psi(z))^i}{w - z} \,d\nu(w)
\]
is continuous at each point \( z \) with \( |\nu(\{z\})| = 0 \) and \( i > 0 \). For \( i = 0 \), the transformation
\[
C^0_\psi(\nu) = C(\nu) = \int \frac{1}{w - z} \,d\nu(w)
\]
is the Cauchy transform of \( \nu \). Let \( M^G(z) \) be the following \( N \) by \( N \) matrix,
\[
M^G(z) = \left[ C_{\psi}^{i-1}(\langle F_j, G \rangle \mu_1) \right]_{N \times N}
\]
where we assume that \( G \perp K^\psi_{N-1} \) or equivalently \( G \) satisfies the following conditions
\[
\tilde{\psi}^n G \perp \mathcal{H}, \; n = 0, 1, 2, ..., N - 1.
\]
(3-4)

The set \( W^G \subset \mathbb{C} \) is defined by:
\[
W^G = \{ \lambda : \int \frac{1}{|z - \lambda|} |\langle F_i(z), G(z) \rangle| \,d\mu_1(z) < \infty, \; 1 \leq i \leq N \}.
\]

Let
\[
\Omega^G = \text{Int}(\sigma(S)) \cap W^G \cap \{ \lambda : |\det(M^G(\lambda))| > 0 \}.
\]
(3-5)

Then for \( \lambda \in \Omega^G \), the matrix
\[
\left[ C(\langle F_j \psi^{i-1}, G \rangle \mu_1) \right]_{N \times N}
\]
(3-6)
is invertible. By Construction, we see that
\[
\det(M^G(z)) = 0 \; \text{a.e. Area}_{\text{close}(\Omega^G)^c}.
\]

**Lemma 6.** Using above notations, we conclude that
\[
\Omega^G \subset \text{abpe}(S).
\]

**Hence, by Lemma 6** we get \( \Omega^G \subset U_N \).
Therefore, by construction of $\Omega^G$, we see that the lemma is a direct application of Theorem 2 in Yang (2018).

Let $A = \{ \lambda_n : \mu_1(\{ \lambda_n \}) > 0 \}$ be the set of atoms for $\mu_1$. Now let us define the matrix $M^G_j(z)$ to be a submatrix of $M^G(z)$ by eliminating the first row and $j$ column. Let $B^G_j(z)$ be the $j$ column of the matrix $M^G(z)$ by eliminating the first row. Define

$$\Omega^G_j = \left( \text{Int}(\sigma(S)) \cap A^c \cap \{ z : |\det(M^G_j(z))| > 0 \} \right) \setminus \text{clo}(\Omega^G).$$

(3-7)

Notice that $M^G_j(\lambda)$ is continuous at each $\lambda \in \Omega^G_j$. On $\Omega^G_j$, we can define the following function vector

$$a_j(z) = [a_{ij}(z)]_{(N-1) \times 1} = (M^G_j(z))^{-1}B^G_j(z).$$

(3-8)

**Lemma 7.** Let $G, \Omega^G, \Omega^G_j$, and $a_j(z)$ be as in (3-4), (3-5), (3-7), and (3-8), respectively. Then for $\lambda_0 \in \Omega^G_j$, there exists $\delta > 0$ such that $a_j(z)$ equals an analytic function on $B(\lambda_0, \delta) \subset \text{Int}(\sigma(S))$ almost everywhere with respect to the area measure. Moreover,

$$C((F_j, G)\mu)(z) = \sum_{k=1}^{j-1} a_{kj}(z)C((F_k, G)\mu)(z) + \sum_{k=j+1}^{N} a_{k-1,j}(z)C((F_k, G)\mu)(z), \ a.e. \ Area|_{B(\lambda_0, \delta)},$$

and

$$\langle F_j, G \rangle = \sum_{k=1}^{j-1} a_{kj}(z)\langle F_k, G \rangle + \sum_{k=j+1}^{N} a_{k-1,j}(z)\langle F_k, G \rangle, \ a.e. \mu|_{B(\lambda_0, \delta)}.$$  

(3-9)

(3-10)

Proof: Without loss of generality, we assume that $j = N$. For $z \in \text{Int}(\sigma(S)) \cap W^G \cap \Omega^G_N$, write

$$M^G(z) = \begin{bmatrix} A^G_N(z) & c^G_N(z) \\ M^G_N(z) & B^G_N(z) \end{bmatrix}$$

where

$$A^G_N(z) = [C((F_1, G)\mu_1)(z), C((F_2, G)\mu_1)(z), ..., C((F_{N-1}, G)\mu_1)(z)]$$

and

$$c^G_N(z) = C((F_N, G)\mu_1)(z).$$

By construction of $\Omega^G_N$, we conclude that

$$\det(M^G(z)) = (A^G_N(z)(M^G_N(z))^{-1}B^G_N(z) - c^G_N(z))\det(M^G_N(z)) = 0 \ a.e. \ Area|_{\Omega^G_N}.$$  

Therefore,

$$c^G_N(z) = A^G_N(z)(M^G_N(z))^{-1}B^G_N(z) \ a.e. \ Area|_{\Omega^G_N}.$$  

(3-11)

Let $\nu_1 = (F_1, G)\mu_1$ and $H_{i,m}(z) = \frac{m^2}{\pi} \nu_1(B(z, \frac{1}{m}))$, then the functions $H_{i,m}(z)$ are bounded with compact supports. We have

$$C(H_{i,m}dA)(w) = \int_{|\lambda-w|\geq \frac{1}{m}} \frac{1}{\lambda-w} d\nu_1(\lambda) + \int_{|\lambda-w|< \frac{1}{m}} \frac{m^2|\lambda-w|^2}{\lambda-w} d\nu_1(\lambda).$$

Hence,

$$|C(H_{i,m}dA)(w) - C\nu_1(w)| \leq 2 \int_{|w-z|<1/m} \frac{1}{|w-z|} d\nu_1(\lambda) \ a.e. \ Area$$

and

$$\lim_{m \to \infty} C(H_{i,m}dA)(w) = C\nu_1(w), \ a.e. \ Area.$$ 

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Let $C_0 > 0$ be a constant such that $|\psi(z) - \psi(w)| \leq C_0 |z - w|$. We estimate $C_1^\psi(\nu_\epsilon)$ as the following,

$$\left| C_1^\psi(H_{1,m}dA)(w) - C_1^\psi\nu_\epsilon(w) \right|$$

$$\leq \frac{m^2}{\pi} \int_{\lambda - \lambda_\epsilon < \frac{1}{m}} \int_{\lambda - \lambda_\epsilon < \frac{1}{m}} \frac{\psi(z) - \psi(w)}{z - w} dA(z) d\nu_\epsilon(\lambda) - C_1^\psi\nu_\epsilon(w)$$

$$\leq \frac{m^2}{\pi} \int_{w - \lambda < \frac{1}{m}} \int_{w - \lambda < \frac{1}{m}} \frac{\psi(z) - \psi(w)}{z - w} dA(z) d\nu_\epsilon(\lambda) - \int_{w - \lambda < \frac{1}{m}} \frac{\psi(\lambda) - \psi(w)}{\lambda - w} d\nu_\epsilon(\lambda)$$

$$+ \frac{m^2}{\pi} \int_{w - \lambda < \frac{1}{m}} \int_{w - \lambda < \frac{1}{m}} \frac{\psi(z) - \psi(w)}{z - w} dA(z) d\nu_\epsilon(\lambda) + \int_{w - \lambda < \frac{1}{m}} \frac{\psi(\lambda) - \psi(w)}{\lambda - w} d\nu_\epsilon(\lambda).$$

Notice that

$$\frac{m^2}{\pi} \int_{w - \lambda < \frac{1}{m}} \int_{w - \lambda < \frac{1}{m}} \frac{1}{z - w} dA(z) d\nu_\epsilon(\lambda) = \int_{w - \lambda < \frac{1}{m}} \frac{1}{\lambda - w} d\nu_\epsilon(\lambda).$$

We get

$$\left| C_1^\psi(H_{1,m}dA)(w) - C_1^\psi\nu_\epsilon(w) \right|$$

$$\leq \frac{m^2}{\pi} \int_{w - \lambda < \frac{1}{m}} \int_{w - \lambda < \frac{1}{m}} \frac{\psi(z) - \psi(\lambda)}{z - w} dA(z) d\nu_\epsilon(\lambda) + 2C_0|\nu_\epsilon|(B(w, \frac{1}{\sqrt{m}}))$$

$$\leq \frac{m^2}{\pi} \int_{w - \lambda < \frac{1}{m}} \int_{w - \lambda < \frac{1}{m}} C_0|z - \lambda| z - w dA(z) d\nu_\epsilon(\lambda) + 2C_0|\nu_\epsilon|(B(w, \frac{1}{\sqrt{m}}))$$

$$\leq C_0 \frac{1}{\sqrt{m}} \int_{w - \lambda_\epsilon < \frac{1}{m}} |\nu_\epsilon|(B(w, \frac{1}{\sqrt{m}})) + 2C_0|\nu_\epsilon|(B(w, \frac{1}{\sqrt{m}}))$$

$$\leq \frac{C_0}{\sqrt{m} - 1} |\nu_\epsilon| + 2C_0|\nu_\epsilon|(B(w, \frac{1}{\sqrt{m}})).$$

Therefore,

$$\lim_{m \to \infty} C_1^\psi(H_{1,m}dA)(w) = C_1^\psi\nu_\epsilon(w)$$

for $w \notin A$. For $\lambda_0 \in \Omega^\epsilon_N$ and $\epsilon > 0$, we can choose a $\delta > 0$ and $m_0$ such that

$$\left| C_1^\psi(H_{1,m}dA)(w) - C_1^\psi\nu_\epsilon(w) \right|$$

$$\leq 2C_0|\nu_\epsilon|(B(w, \frac{1}{\sqrt{m}})) + \frac{C_0}{\sqrt{m} - 1} |\nu_\epsilon|$$

$$\leq 2C_0|\nu_\epsilon|(B(\lambda_0, \delta + \frac{1}{\sqrt{m}})) + \frac{C_0}{\sqrt{m} - 1} |\nu_\epsilon|$$

$$< \epsilon$$

where $w \in B(\lambda_0, \delta) \setminus A$ and $m \geq m_0$. Since $C_1^\psi\nu_\epsilon(w)$ is continuous at $\lambda_0$, $\delta$ can be chosen so that

$$\left| C_1^\psi\nu_\epsilon(w) - C_1^\psi\nu_\epsilon(\lambda_0) \right| < \epsilon$$

where $w \in B(\lambda_0, \delta) \setminus A$. It is easy to verify that $C_1^\psi(H_{1,m}dA)$ is a smooth function. For $k > 1$, clearly $C_k^\psi\nu_\epsilon(w)$ is a smooth function. Define

$$M_k^G(z) = \frac{[C_1^\psi(H_{1,m}dA), C_1^\psi(H_{2,m}dA), ..., C_1^\psi(H_{N-1,m}dA)]}{C_1^\psi(\nu_1), C_1^\psi(\nu_2), ..., C_1^\psi(\nu_{N-1})}. \frac{[C_{N-1}^\psi(\nu_1), C_{N-1}^\psi(\nu_2), ..., C_{N-1}^\psi(\nu_{N-1})]}.$$
are invertible for \( w \in B(\lambda_0, \delta) \setminus A \) and \( m > m_0 \). Define

\[
B_N^{G_m}(z) = \begin{bmatrix}
C^1_\psi(H_{N,m}dA) \\
C^2_\psi(\nu_N) \\
\ldots \\
C^{N-1}_\psi(\nu_N)
\end{bmatrix},
\]

\[
A_N^{G_m}(z) = \langle C(H_{1,m}dA), C(H_{2,m}dA), \ldots, C(H_{N-1,m}dA) \rangle
\]

and

\[
c_N^{G_m}(z) = C(H_{N,m}dA)(z).
\]

For a smooth function \( \phi \) with compact support in \( B(\lambda_0, \delta) \), using the definition (3-8) and Lebesgue’s Dominated Convergence Theorem, we get the following calculation,

\[
\int \partial \phi(z) a_N(z) dA(z)
= \lim_{m \to \infty} \int \partial \phi(z) \left( (M_N^{G_m}(z))^{-1} B_N^{G_m}(z) \right) dA(z)
= - \lim_{m \to \infty} \int \phi(z) \partial \left( (M_N^{G_m}(z))^{-1} B_N^{G_m}(z) \right) dA(z)
\]

(3-12)

On the other hand,

\[
\partial M_N^{G_m}(z) = \partial \psi(z)
\begin{bmatrix}
-C(H_{1,m}dA), & -C(H_{2,m}dA), & \ldots, & -C(H_{N-1,m}dA) \\
-2C^1_\psi(\nu_1), & -2C^1_\psi(\nu_2), & \ldots, & -2C^1_\psi(\nu_{N-1}) \\
\ldots, & \ldots, & \ldots, & \ldots \\
-(N-1)C^{N-2}(\nu_1), & -(N-1)C^{N-2}(\nu_2), & \ldots, & -(N-1)C^{N-2}(\nu_{N-1})
\end{bmatrix}.
\]

Therefore,

\[
(\partial M_N^{G_m}(z))(M_N^{G_m}(z))^{-1} = -\partial \psi(z)
\begin{bmatrix}
A_N^{G_m}(z) & (M_N^{G_m}(z))^{-1} \\
2, & 0, & \ldots, & 0, \ldots, & 0 \\
\ldots, & \ldots, & \ldots, & \ldots, & \ldots \\
0, & 0, & \ldots, & N-1, & 0
\end{bmatrix}.
\]

Hence,

\[
(\partial M_N^{G_m}(z))(M_N^{G_m}(z))^{-1} B_N^{G_m}(z) - \partial B_N^{G_m}(z) = -\partial \psi(z)
\begin{bmatrix}
A_N^{G_m}(z)(M_N^{G_m}(z))^{-1} B_N^{G_m}(z) - c_N^{G_m} \\
0, \ldots, & \ldots, & \ldots, & \ldots \\
0 
\end{bmatrix}.
\]

Using (3-11), we see that

\[
\lim_{m \to \infty} \left( A_N^{G_m}(z)(M_N^{G_m}(z))^{-1} B_N^{G_m}(z) - c_N^{G_m} \right) = 0 \text{ a.e. } Area|_{B(\lambda_0, \delta)}.
\]

Since each component of the above vector function is less than

\[
M \int \frac{1}{|w-z|} d|\nu_1|(z) \text{ a.e. } Area|_{B(\lambda_0, \delta)},
\]

applying Lebesgue’s Dominated Convergence Theorem to the last step of (3-12), we conclude

\[
\int \partial \phi(z) a_N(z) dA(z) = 0.
\]

By Weyl’s lemma, we see that \( a_N(z) \) is analytic on \( B(\lambda_0, \delta) \). From equation (3-8), we get

\[
C^1_\psi(F_N, G)(\mu_1)(z) = \sum_{k=1}^{N-1} a_{k,j}(z)C^1_\psi(F_k, G)(\mu_1)(z), \text{ a.e. } Area|_{B(\lambda_0, \delta)},
\]

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The above equation implies (3-9) since
\[ \partial \mathcal{C}^k_v(s) = -C((s)) \text{ a.e. Area.} \]

For equation (3-10), let \( \phi \) be a smooth function with compact support in \( B(\lambda_0, \delta) \) and let \( \nu \) be a compactly supported finite measure, we get
\[ \int \partial \phi(z) \mathcal{C}v(z) dA(z) = \pi \int \phi(z) d\nu(z). \]

Apply the above equation to the both sides of the equation (3-9) for \( j = N \) and using
\[ \partial \phi(z) a_{kj}(z) = \partial (\phi(z) a_{kj}(z)), \quad z \in B(\lambda_0, \delta), \]
we conclude
\[ \int \phi(F_N, G) d\mu_1 = \int \phi \sum_{k=1}^{N-1} a_{kj}(F_k, G) d\mu_1. \]

Hence the equation (3-10) follows. This completes the proof of the lemma.

**Corollary 2.** Let \( G, \Omega^G, \) and \( \Omega^G_0 \) be as in Lemma 7. Suppose \( G \perp \mathcal{K}^\psi_{N+1} \) (satisfies (3-4)). Then \( \Omega^G_0 \subset U_{N-1} \cup U_N. \)

**Proof:** Without loss of generality, we assume that \( j = N \). From Lemma 7 for \( \lambda_0 \in \Omega^G_N \), there exists \( \delta > 0 \) such that \( B(\lambda_0, \delta) \subset \text{Int}(\sigma(S)) \) and the equations (3-9) and (3-10) hold, which imply (3-3). For \( r_1, r_2, ..., r_N \in \text{Rat}(\sigma(S)) \), let
\[ F = \sum_{i=1}^N r_i F_i. \]

Notice that
\[ r_i(\lambda) \mathcal{C}^k_v(F_i, G) \mu_1) = \mathcal{C}^k_v(\langle F, F_i, G \rangle \mu_1) \]
since \( G \perp \mathcal{K}^\psi_{N+1} \). Then
\[ \sum_{i=1}^N r_i(\lambda) \mathcal{C}^k_v(\langle F_i, G \rangle \mu_1)(\lambda) = \mathcal{C}^k_v(\langle F, G \rangle \mu_1)(\lambda), \]
for \( k = 1, 2, ..., N - 1 \). Now using the equation (3-9) for \( \lambda \in B(\lambda_0, \delta) \setminus A \), we get
\[ \sum_{i=1}^{N-1} (r_i(\lambda) + a_{N1}(\lambda) r_N(\lambda)) \mathcal{C}^k_v(\langle F_i, G \rangle \mu_1)(\lambda) = \mathcal{C}^k_v(\langle F, G \rangle \mu_1)(\lambda), \]
equivalently,
\[ M^G_N(\lambda) \begin{bmatrix} r_1(\lambda) + a_{N1}(\lambda) r_N(\lambda) \\ r_2(\lambda) + a_{N2}(\lambda) r_N(\lambda) \\ \vdots \\ r_{N-1}(\lambda) + a_{N,N-1}(\lambda) r_N(\lambda) \end{bmatrix} = \begin{bmatrix} \mathcal{C}^k_v(\langle F, G \rangle \mu_1)(\lambda) \\ \mathcal{C}^k_v(\langle F, G \rangle \mu_1)(\lambda) \\ \vdots \\ C^{N-1}_\psi(\langle F, G \rangle \mu_1)(\lambda) \end{bmatrix}. \]

where the inverse of \( M^G_N(\lambda) \) is bounded on \( B(\lambda_0, \delta) \setminus A \) and \( a_{N1} \) are analytic on \( B(\lambda_0, \delta) \). Therefore, there exists a positive constant \( M \) such that
\[ \sup_{1 \leq k \leq N-1, \lambda \in B(\lambda_0, \delta)} |r_k(\lambda) + a_{Nk}(\lambda) r_N(\lambda)| \leq M \|F\|, \]
which implies (3-2). Hence, Lemma 3.1 implies \( \Omega^G_N \subset U_{N-1} \cup U_N. \)

Now let us recursively construct other sets such as \( \Omega^G_i \) for a given \( G \perp \mathcal{K}^\psi_{N-1} \). We will only describe the algorithm for \( k = N - 2 \) and the other cases will follow recursively. Let \( E^G_0 = \Omega^G \). 

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and \( E_{N-1}^G = \bigcup_{i=1}^N \Omega_i^G \). Let \( M_{ij}^G \) be an \( N-2 \) by \( N-2 \) submatrix of \( M^G \) by eliminating the first two rows and the \( i \) and \( j \) columns. Define

\[
\Omega_i^G = \left( \text{Int}(\sigma(S)) \cap \mathcal{A} \cap \{ z : |\text{det}(M_{ij}^G(z))| > 0 \} \right) \setminus \text{cl}(E_{N-1}^G \cup E_{N-2}^G).
\]

Without loss of generality, let us assume that \( i = N-1 \) and \( j = N \). Similar to Lemma \[\text{Lemma} \] one can prove that for \( \lambda_0 \in \Omega_{N-1,N}^G \), there exist \( \delta > 0 \), analytic functions \( a_i(z) \) and \( b_i(z) \) on \( B(\lambda_0, \delta) \subset \text{Int}(\sigma(S)) \) such that

\[
F_{N-1} = \sum_{i=1}^{N-2} a_i(z) F_i(z), \quad F_N = \sum_{i=1}^{N-2} b_i(z) F_i(z), \quad \text{a.e.} \mu_1|_{B(\lambda_0, \delta)},
\]

and there exists a constant \( M > 0 \) such that

\[
\sup_{1 \leq k \leq N-2, \lambda \in B(\lambda_0, \frac{\delta}{2})} |r_k(\lambda) + a_k(\lambda)r_{N-1}(\lambda) + b_k(\lambda)r_N(\lambda)| \leq M\|F\|,
\]

where \( r_1, r_2, \ldots, r_N \in \text{Rat}(\sigma(S)) \) and \( F = \sum_{i=1}^N r_i F_i \). (3-13) and (3-14) are the same as (3-2) and (3-3) for the case \( k = N - 2 \). Let

\[
E_{N-2}^G = \bigcup_{i<j} \Omega_i^G.
\]

**Corollary 3.** Let \( E_{N-2}^G \) be as in (3-15). Suppose \( G \perp \mathcal{K}_{N-1}^\psi \) (satisfies (3-4)). Then

\[
E_{N-2}^G \subset U_{N-2} \cup U_{N-1} \cup U_N.
\]

The proof is the same as Corollary \[\text{Corollary} \] Therefore we can recursively construct \( E_k^G \) for \( k = 1, 2, \ldots, N \) such that

\[
E_k^G \subset \bigcup_{i=k}^N U_i
\]

where the proof for \( k = N \) is from Lemma \[\text{Lemma} \] \( k = N - 1 \) is from Corollary \[\text{Corollary} \] and \( k = N - 2 \) is from Corollary \[\text{Corollary} \].

The following theorem proves, under the conditions \( S \) satisfies the property \((N, \psi)\), the set \( \bigcup_{k=1}^N E_k^G \) is big.

**Theorem 3.** Let \( E_i^G \) be constructed for \( i = 1, 2, \ldots, N \) above. Suppose \( \{G_i\} \subset (\mathcal{K}_{N-1}^\psi)^+ \) is a dense subset, then

\[
\text{spt} \mu_1 \subset \text{cl}(\bigcup_{i=1}^N \bigcup_{j=1}^\infty E_i^{G_j}).
\]

**Proof:** First we prove

\[
\mu_1\left( \text{Int}(\sigma(S)) \setminus \text{cl}\left( \bigcup_{i=1}^N \bigcup_{j=1}^\infty E_i^{G_j} \right) \right) = 0.
\]

Suppose that \( B(\lambda_0, \delta) \subset \text{Int}(\sigma(S)) \) and \( B(\lambda_0, \delta) \cap \text{cl}(\bigcup_{i=1}^N \bigcup_{j=1}^\infty E_i^{G_j}) = \emptyset \), then by construction of \( E_i^{G_j} \), we conclude that

\[
C_{\psi}^{-1}(\langle F_i, G_j \rangle_{\mu_1})(z) = 0
\]

on \( B(\lambda_0, \delta) \), where \( i = 1, 2, \ldots, N \). By taking \( \tilde{\partial} \) in the sense of distribution, we see that

\[
C(\langle F_i, G_j \rangle_{\mu_1})(z) = 0
\]

a.e. Area on \( B(\lambda_0, \delta) \) since \( \text{Area}(\{ \tilde{\partial} \psi = 0 \} \cap \sigma(S)) = 0 \), where \( i = 1, 2, \ldots, N \). For a smooth function \( \phi \) with compact support in \( B(\lambda_0, \delta) \),

\[
\int \phi(z) (F_i, G_j) d\mu_1 = \frac{1}{\pi} \int \tilde{\partial} \phi(z) C(\langle F_i, G_j \rangle_{\mu_1})(z) dA(z) = 0.
\]
Therefore, 
\[ \langle F_i(z), G_j(z) \rangle = 0, \text{ a.e. } \mu_1|_{B(\lambda_0, \delta)} \]  
(3-17)
where \( i = 1, 2, \ldots, N \). From (1-4), we see that for \( P \in \oplus_{k=1}^m L^2(\mu_k|_{B(\lambda_0, \delta)}) \), (3-17) implies \( (P, G_j) = 0 \). Therefore, 
\[ \oplus_{k=1}^m L^2(\mu_k|_{B(\lambda_0, \delta)}) \subset \mathcal{K}_N^\psi. \]
Hence, \( \mu_1|_{B(\lambda_0, \delta)} = 0 \) since \( M_\psi|_{\mathcal{K}_N^\psi} \) is pure.

Now assume \( B(\lambda_0, \delta) \cap \text{clos}(\text{Int}(\sigma(S))) = \emptyset \). For \( N > 1 \), the function \( \mathcal{C}_\psi^{N-1}((F_i, G_j)\mu_1)(z) \) is continuous on \( \mathbb{C} \setminus A \) and is zero on \( \mathbb{C} \setminus \sigma(S) \). Hence,
\[ \mathcal{C}_\psi^{N-1}((F_i, G_j)\mu_1)(z) = 0 \]
on \( B(\lambda_0, \delta) \setminus A \), where \( i = 1, 2, \ldots, N \). Using the same proof as above, we see that \( \mu_1|_{B(\lambda_0, \delta)} = 0 \). This implies \( \text{spt}\mu_1 \subset \text{clos}(\text{Int}(\sigma(S))) \). The theorem is proved.

**Proof of Theorem 2** From (3-16) and Theorem 3, we get 
\[ \bigcup_{i=1}^N \partial U_i \subset \sigma_\varepsilon(S) \subset \text{spt}(\mu_1) \subset \text{clos} \left( \bigcup_{i=1}^N U_i \right). \]
This implies 
\[ \sigma_\varepsilon(S) = \bigcup_{i=1}^N \partial U_i \]
since \( \sigma_\varepsilon(S) \cap U_i = \emptyset \). This completes the proof.

For a positive finite measure \( \mu \) with compact support on \( \mathbb{C} \), define 
\[ P^2(\mu|1, \bar{z}, z^{N-1}) = \text{clos} \{ p_1(z) + p_2(z)\bar{z} + \ldots + p_N(z)z^{N-1} : p_1, p_2, \ldots, p_N \in \mathcal{P} \} \]
and \( S_{N,\mu} \) as the multiplication by \( z \) on \( P^2(\mu|1, \bar{z}, z^{N-1}) \). Then \( S_{N,\mu} \) is a multicyclic subnormal operator with the minimal normal extension \( M_\mu \), the multiplication by \( z \), on \( L^2(\mu) \).

**Corollary 4.** Suppose that \( S_{2,\mu} \) on \( P^2(\mu|1, \bar{z}, z^2) \) is pure, then the operator \( S_{1,\mu} \) on \( P^2(\mu|1, \bar{z}, z) \) satisfies 
\[ \sigma(S_{1,\mu}) = \text{clos}(\sigma(S_{1,\mu}) \setminus \sigma_\varepsilon(S_{1,\mu})). \]

**Proof:** Since 
\[ \mathcal{K}_1^\varepsilon = \text{clos}(\text{span}(z^k P^2(\mu|1, \bar{z}) : 0 \leq k \leq 1)) = P^2(\mu|1, \bar{z}, z^2) \]
and \( S_{2,\mu} \) on \( P^2(\mu|1, \bar{z}, z^2) \) is pure. Therefore, the result follows from Theorem 2.

It seems strong to assume that \( S_{2,\mu} \) on \( P^2(\mu|1, \bar{z}, z^2) \) is pure in the corollary. We believe that the condition can be reduced to assume that \( S_{1,\mu} \) on \( P^2(\mu|1, \bar{z}, z) \) is pure. However, we are not able to prove the result under the weaker conditions. We will leave it as an open problem for further research.

**Problem 1.** Does Corollary 4 hold under the weaker assumption that \( S_{1,\mu} \) on \( P^2(\mu|1, \bar{z}, z) \) is pure?

**Corollary 5.** Let \( S \) on \( H \) be a pure rationally \( N \)-cyclic subnormal operator with \( H = R^2(S|F_1, F_2, \ldots, F_N) \) and let \( M_\varepsilon \) be its minimal normal extension on \( K \) satisfying (1-1) to (1-4). Suppose that there exists a smooth function \( \psi \) on \( \mathbb{C} \) such that \( \text{Area}(\{ \partial \psi = 0 \} \cap \sigma(S)) = 0 \) and \( \psi(M_\varepsilon)H \subset H \). Then there exist bounded open subsets \( U_i \) for \( 1 \leq i \leq N \) such that 
\[ \sigma_\varepsilon(S) = \bigcup_{i=1}^N \partial U_i, \quad \sigma(S) \setminus \sigma_\varepsilon(S) = \bigcup_{i=1}^N U_i, \]
and
\[ \text{dimker}(S - \lambda)^* = i. \]
for \( \lambda \in U_i \).
Notice that Example 1 and 2 are special cases of Corollary 5. It seems that further results could be obtained for the special cases where $S$ satisfies the conditions of Corollary 5. Moreover, we might be able to combine the methodology in [McCarthy and Yang (1997)] to obtain the structural models for the class of subnormal operators, which might extend Xia’s model for subnormal operators with finite rank self-commutators.

**Problem 2.** Can the structure of subnormal operators in Corollary 5 be characterized?

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