Research Article

Reliability Sensitivity Analysis Method for Mechanical Components

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1. Introduction

The responses of mechanical components or engineering structures are often random due to random inputs of them including loads, material properties, and geometry. Many reliability analysis methods have been developed to calculate the failure probabilities of these random structures [1–4]. In reliability analysis, reliability sensitivity is defined as the partial derivative of the failure probability with respect to the distribution parameters (e.g., mean and standard deviation) of fundamental random input variables. Reliability sensitivity provides information about the importance of each input random variable to a structure’s failure probability. Reliability sensitivity analysis methods can be divided into the numerical simulation method and approximate analytic calculation method.

The numerical simulation method can be divided into many different methods due to different sampling methods including importance sampling, direction sampling, line sampling, subset simulation, and low-discrepancy sampling [5–10]. These different simulation methods are all based on Monte Carlo simulation but have different sampling methods. Although the program of these methods is not complicated, lots of samples are needed to get precise results because the structural failure probability is often very small and the computational efficiency is therefore reduced.

Analytic methods for reliability sensitivity analysis are always based on analytic reliability methods. These reliability analysis methods are the mean-value first-order reliability method (MVFORM)/mean-value second-order reliability method (MVSORM) [11, 12], JC method, mean-value first-order saddlepoint approximation (MVFOSA) [13], and moment method. FORM/SORM needs to expand the performance function into first-order or second-order Taylor series at the most likely failure point. The expansion is widely used because of its relatively moderate precision and efficiency. The corresponding reliability sensitivity calculation methods have been derived from these reliability methods [14–23]. Analytic methods for reliability sensitivity analysis always have higher calculation efficiency than MCS.
In this study, a feasible method to compute the sensitivity of failure probability with respect to distribution parameters of basic random input variables is proposed. The method can be applied to estimate failure probability and reliability sensitivity based on Edgeworth series. The proposed reliability sensitivity analysis method includes the following steps. (1) Choose random input variables and determine their distributions and distribution parameters. (2) Calculate the first four response origin moments and their sensitivity with respect to distribution parameters of random input variables by UDRM. (3) Calculate failure probability and reliability sensitivity defined by partial derivatives of failure probability to response origin moments. (4) Calculate partial derivatives of response central moments with respect to response central moments. (5) Calculate partial derivatives of response origin moments with respect to distribution parameters of input random variables by UDRM. (6) Calculate reliability sensitivity defined by partial derivatives of failure probability with respect to distribution parameters of input random variables.

Equation (5) can be expressed using the recursive formula as follows:

\[ m_l = E[Y^l(X)] = \int_{\mathbb{R}^N} Y^l(X)f_X(x)dx, \]

where \( m_l \) is the \( l \)-th order origin moment of \( Y(X) \), \( f_X(x) \) is the joint probability density function of \( X \), and \( E(\cdot) \) is the expectation operator. According to the univariate dimension reduction method proposed by Rahman and Xu [24–26], \( Y(X) \) can be approximately written as follows:

\[ Y(X) = \bar{Y}(X) = \bar{Y}(X_1, X_2, \Lambda, X_N) = \sum_{j=1}^{N} Y(\mu_1, \Lambda, \mu_{j-1}, x_j, \mu_{j+1}, \Lambda, \mu_N) - (N-1)Y(\mu_1, \Lambda, \mu_N), \]

where \( Y(\mu_1, \Lambda, \mu_{j-1}, x_j, \mu_{j+1}, \Lambda, \mu_N) \) is the response which depends on the \( j \)-th random input \( X_j \), \( Y(\mu_1, \Lambda, \mu_{j-1}, x_j, \mu_{j+1}, \Lambda, \mu_N) \) is the response where \( X = [X_1, X_2, \Lambda, X_N]^T \), and \( \mu_j = E(X_j) \) is the mean of \( X_j \).

Then, the \( l \)-th order origin moment \( m_l \) can be written as follows:

\[ m_l = E[\bar{Y}^l(X)] = E\left\{ \left[ \sum_{j=1}^{N} Y(\mu_1, \Lambda, \mu_{j-1}, X_j, \mu_{j+1}, \Lambda, \mu_N) - (N-1)Y(\mu_1, \Lambda, \mu_N) \right]^l \right\}. \]

Applying the binomial formula on the right-hand side of equation (3), the \( l \)-th response origin moment \( m_l \) can be written as follows:

\[ m_l = \sum_{i=0}^{l} C_l^i E\left[ \sum_{j=1}^{N} Y(\mu_1, \Lambda, \mu_{j-1}, X_j, \mu_{j+1}, \Lambda, \mu_N) \right]^i \left[ -(N-1)Y(\mu_1, \Lambda, \mu_N) \right]^{l-i}, \]

where \( C(i, \cdot) \) is the combination operator.

Define

\[ S_j^l = E\left[ \left[ \sum_{i=1}^{j} Y(\mu_1, \Lambda, \mu_{j-1}, X_j, \mu_{j+1}, \Lambda, \mu_N) \right]^i \right], \quad j = 1, \ldots N, \quad i = 1, \ldots, l. \]

Equation (5) can be expressed using the recursive formula as follows [23]:

2. Response Origin Moments and Response Sensitivity

2.1. Response Origin Moments by UDRM. Structures subject to random input vector \( X = [X_1, X_2, \Lambda, X_N]^T \in \mathbb{R}^N \), which characterizes uncertainty in loads, material properties, and geometry. Let \( Y(X) \) represent a response of interest that depends on independent random variables \( X = [X_1, X_2, \Lambda, X_N]^T \); then, the \( l \)-th order origin moment of \( Y(X) \) can be written as follows:

\[ m_l = E[Y^l(X)] = \int_{\mathbb{R}^N} Y^l(X)f_X(x)dx, \]

where \( m_l \) is the \( l \)-th order origin moment of \( Y(X) \), \( f_X(x) \) is the joint probability density function of \( X \), and \( E(\cdot) \) is the expectation operator. According to the univariate dimension reduction method proposed by Rahman and Xu [24–26], \( Y(X) \) can be approximately written as follows:

\[ Y(X) = \bar{Y}(X) = \bar{Y}(X_1, X_2, \Lambda, X_N) = \sum_{j=1}^{N} Y(\mu_1, \Lambda, \mu_{j-1}, x_j, \mu_{j+1}, \Lambda, \mu_N) - (N-1)Y(\mu_1, \Lambda, \mu_N), \]

where \( Y(\mu_1, \Lambda, \mu_{j-1}, x_j, \mu_{j+1}, \Lambda, \mu_N) \) is the response which depends on the \( j \)-th random input \( X_j \), \( Y(\mu_1, \Lambda, \mu_{j-1}, x_j, \mu_{j+1}, \Lambda, \mu_N) \) is the response where \( X = [X_1, X_2, \Lambda, X_N]^T \), and \( \mu_j = E(X_j) \) is the mean of \( X_j \).
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Orthogonal polynomial for different probability densities are

\[ S_i^l = E\left[Y_i^l (X_1, \mu_2, \Lambda, \mu_N)\right], \quad (i = 1, \ldots, N), \]

\[ S_2^l = \sum_{k=0}^l C_i^k S_i^k E\left[Y_i^{l-k} (\mu_1, X_2, \Lambda, \mu_N)\right], \quad (i = 1, \ldots, N), \]

\[ S_j^l = \sum_{k=0}^l C_i^k S_{j-1}^l E\left[Y_i^{l-k} (\mu_1, \Lambda, \mu_{j-1}, X_j, \mu_{j+1}, \Lambda, \mu_N)\right], \quad (i = 1, \ldots, N), \]

\[ S_N^l = \sum_{k=0}^l C_i^k S_{N-1}^l E\left[Y_i^{l-k} (\mu_1, \Lambda, \mu_{N-1}, X_N)\right], \quad (i = 1, \ldots, N). \]

Then, the \( l \)th origin moment of \( Y(X) \) can be written as follows:

\[ m_l = E\left[Y_l^l (X)\right] = \sum_{i=0}^l C_i^l S_N^l \left[-(N-1)Y(\mu_1, \Lambda, \mu_{N-1}, \mu_N)\right]^{l-1}. \]  

(7)

From equations (6) and (7), the \( q \)th response origin moment \( m_q^l \) can be calculated out by using the origin moments of univariate functions \( Y(\mu_1, \Lambda, \mu_{j-1}, X_j, \mu_{j+1}, \Lambda, \mu_N) (j = 1, 2, \Lambda, N) \). The \( q \)th origin moment of univariate function \( Y(\mu_1, \Lambda, \mu_{j-1}, X_j, \mu_{j+1}, \Lambda, \mu_N) \) can be written as follows:

\[ m_q^l = E\left[Y_q^l (\mu_1, \Lambda, \mu_{j-1}, X_j, \mu_{j+1}, \Lambda, \mu_N)\right] = \int Y_q^l (\mu_1, \Lambda, \mu_{j-1}, X_j, \mu_{j+1}, \Lambda, \mu_N) f_X(x_j) dx_j, \]  

(8)

where \( f_X(x_j) \) is the probability density function of random variable \( X_j \).

The \( q \)th origin moment \( m_q^l \) in equation (8) can be calculated out by the numerical integration method and can be written as follows:

\[ m_q^l = \int Y_q^l (\mu_1, \Lambda, \mu_{j-1}, x_j, \mu_{j+1}, \Lambda, \mu_N) f(x_j) dx_j = \sum_{k=0}^l W_j^l \left[\int Y_q^l (\mu_1, \Lambda, \mu_{j-1}, x_j, \mu_{j+1}, \Lambda, \mu_N) f(x_j) dx_j\right]. \]  

(9)

2.2. Sensitivity of Response Moment with Respect to Distribution Parameters of Inputs. Calculate the partial derivative on both side of equation (1) with respect to distribution parameters \( T_j^l \) of random input variables \( X = [X_1, X_2, \Lambda, X_N]^T \). The partial derivative can be written as follows:

\[ \frac{\partial m_i}{\partial T_j^l} = \frac{\partial E(Y_i^l (X))}{\partial T_j^l} = \frac{\partial \left(\int_{R^n} \int_{R^n} Y_i^l (X) f(X) dx \right)}{\partial T_j^l} = \int_{R^n} Y_i^l (X) \frac{1}{f_j(x)} \frac{\partial f_j(x)}{\partial T_j^l} f(X) dx \]

\[ = E\left(Y_i^l (X) \frac{1}{f_j(x)} \frac{\partial f_j(x)}{\partial T_j^l}\right) \]

\[ \approx E\left(\sum_{j=1}^N \frac{1}{f_j(x)} (Y_i^l (X) - (N-1)Y(\mu_1, \Lambda, \mu_N)) \right) \frac{1}{f_j(x)} \frac{\partial f_j(x)}{\partial T_j^l}, \quad j = 1, 2, \ldots, N, i = 1, 2, \ldots, l. \]  

(10)
where $T_i^j (j = 1, 2, \ldots, N, i = 1, 2)$, is the $i$th distribution parameter of the $j$th random input variable ($i = 1$, represents the mean; $i = 2$ represents standard deviation), $f_j(x)$ is the probability density function of the $j$th random variable $X_j$, and $\bar{Y}_j(X_j) = Y(\mu_1, \Lambda, \mu_{j-1}, X_j, \mu_{j+1}, \Lambda, \mu_N)$.

Applying the binomial formula on the right-hand side of equation (10), $\partial m_i/\partial T_i^j$ can be written as follows:

\[
\frac{\partial m_i}{\partial T_i^j} \approx \sum_{z=0}^{\frac{1}{2}} C_z^i E \left\{ \sum_{j=1}^{N} \dot{Y}_j(X_j) \right\}^z \left( \frac{1}{f_j(x)} \frac{\partial f_j(x)}{\partial T_i^j} \right) \left[ -(N-1)Y(\mu_1, \Lambda, \mu_N) \right]^{1-z}, \tag{11}
\]

where $\partial m_i/\partial T_i^j$ can be calculated out by equation (11) and recursive formula similar to equations (5) and (6). Suppose

\[
k_i^j = \frac{1}{f_j(x)} \frac{\partial f_j(x)}{\partial T_i^j}, \quad j = 1, 2, \ldots, N, i = 1, 2, \tag{12}
\]

where $k_i^j$ is the kernel function of the $i$th distribution parameter $T_i^j$ of $j$th the random input variable.

The kernel function in equation (12) of $X_j$ with different probability density functions can be obtained as follows: here, $x_j$ is denoted by $x$.

If $X$ is a normal distributed variable, its probability density function can be written as follows:

\[
f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{(x-\mu)^2}{2\sigma^2}\right)}, \tag{13}
\]

where $\mu$ is the mean of $X$, and $\sigma$ is the standard deviation of $X$.

The kernel function for normal variable $X$ with respect to its distribution parameters (mean and standard deviation) can be derived from equation (12) directly and can be written as follows:

\[
k_{\mu} = \frac{x-\mu}{\sigma},
\]

\[
k_{\sigma} = \frac{1}{\sigma} \left( \left( \frac{x-\mu}{\sigma} \right)^2 - 1 \right), \tag{14}
\]

where $k_{\mu}$ and $k_{\sigma}$ are the kernel functions with respect to mean and standard deviation, respectively. $\mu$ and $\sigma$ are the mean and standard deviation of normal variable $X$, respectively.

### Table 1: Orthogonal polynomial and Gauss integration.

| Distribution | Orthogonal polynomial | Integral formula |
|--------------|-----------------------|-----------------|
| Normal       | Hermite               | $(1/\sqrt{\pi}) \sum_{k=0}^{n} W_{k}^{i} Y(k \sqrt{2\sigma_{1}^{i} + \mu})$ |
| Lognormal    | Hermite               | $(1/\sqrt{\pi}) \sum_{k=0}^{n} W_{k}^{i} Y(k \sqrt{e^{2\sigma_{1}^{i} + \mu}})$ |
| Uniform      | Legendre             | $(1/2) \sum_{k=0}^{n} W_{k}^{i} Y((b-a)/2) Y^{i} (z)$ |
| Exponential  | Laguerre              | $(1/4) \sum_{k=0}^{n} W_{k}^{i} Y((a+b)/2) Y^{i} (z)$ |
| Weibull      | Laguerre              | $\sum_{k=0}^{n} W_{k}^{i} Y((at+b)/2)$ |

If $X$ is a lognormal distributed variable, its probability density function can be written as follows:

\[
f(x) = \frac{1}{\sqrt{2\pi}\sigma_{1} x} e^{-\left(\frac{(\ln x-\mu_{1})^2}{2\sigma_{1}^2}\right)}, \tag{15}
\]

where $\mu_{1}$ is the mean of $\ln X$, and $\sigma_{1}$ is the standard deviation of $\ln X$.

The kernel function of lognormal variable $X$ with respect to $\mu_1$ and $\sigma_1$ can be derived from equation (12) directly and can be written as follows:

\[
k_{\mu_1} = \frac{\ln(x) - \mu_1}{\sigma_1}, \tag{16}
\]

\[
k_{\sigma_1} = \frac{[\ln(x) - \mu_1]^2 - \sigma_1^2}{\sigma_1^2},
\]

where $k_{\mu_1}$ and $k_{\sigma_1}$ are the kernel functions with respect to $\mu_1$ and $\sigma_1$, respectively.

Distribution parameters $\mu_1$ and $\sigma_1$ of the lognormal distributed variable $x$ can be written as follows:

\[
\mu_1 = \ln \left( \frac{\mu^2}{\sqrt{\sigma^2 + \mu^2}} \right), \tag{17}
\]

\[
\sigma_1 = \ln \left( \frac{\sigma^2}{\mu^2 + 1} \right),
\]

where $\mu$ and $\sigma$ are the mean and standard deviation of the lognormal distributed variable $x$, respectively.

Calculate partial derivative of $\mu_1$ and $\sigma_1$ with respect to $\mu$ and $\sigma$, respectively, according to equation (16); then,
\[
\frac{\partial \mu_1}{\partial \mu} = \frac{2\sigma^2 + \mu^2}{\mu (\sigma^2 + \mu^2)},
\frac{\partial \mu_1}{\partial \sigma} = \frac{2\sigma}{\sigma^2 + \mu^2},
\frac{\partial \sigma_1}{\partial \mu} = \frac{\mu}{\sqrt{\ln(\sigma^2/\mu^2 + 1) \cdot (\mu^2 + 1)}},
\frac{\partial \sigma_1}{\partial \sigma} = \frac{1}{\sigma \sqrt{\ln(\sigma^2/\mu^2 + 1)}}
\] (18)

Jacobi matrix \( J \) can be written as follows:
\[
J = \begin{bmatrix}
\frac{\partial \mu_1}{\partial \mu} & \frac{\partial \sigma_1}{\partial \mu} \\
\frac{\partial \mu_1}{\partial \sigma} & \frac{\partial \sigma_1}{\partial \sigma}
\end{bmatrix}
\] (19)

The kernel function of the lognormal distributed variable \( X \) with respect to its mean and standard deviation can be written as follows:
\[
\begin{bmatrix} k_{\mu} \\ k_{\sigma} \end{bmatrix} = \begin{bmatrix}
\frac{\partial \mu_1}{\partial \mu} k_{\mu_1} + \frac{\partial \sigma_1}{\partial \mu} k_{\sigma_1} \\
\frac{\partial \mu_1}{\partial \sigma} k_{\mu_1} + \frac{\partial \sigma_1}{\partial \sigma} k_{\sigma_1}
\end{bmatrix} = \begin{bmatrix} \frac{\partial \mu_1}{\partial \mu} \mu_
\frac{\partial \sigma_1}{\partial \mu} \sigma_
\end{bmatrix} \cdot \begin{bmatrix} k_{\mu_1} \\ k_{\sigma_1} \end{bmatrix} = J \cdot \begin{bmatrix} k_{\mu_1} \\ k_{\sigma_1} \end{bmatrix}
\] (20)

where \( k_{\mu} \) and \( k_{\sigma} \) are the kernel functions with respect to mean and standard deviation of the lognormal distributed variable \( x \), respectively.

If \( x \) is two parameter Weibull distributed variable, its probability density function can be written as follows:
\[
f(x) = \left( \frac{\beta}{\alpha} \right) x^{\beta - 1} e^{- (x/\alpha)^{\beta}}, \quad (21)
\]
where \( \beta \) is the shape parameter, and \( \alpha \) is the scale parameter.

The kernel function of the Weibull distributed variable \( X \) with respect to scale parameter \( \alpha \) and shape parameter \( \beta \) can be derived from equation (12) directly, which can be written as follows:
\[
k_{\alpha} = \frac{\beta}{\alpha} \left[ \left( \frac{x}{\alpha} \right)^{\beta} - 1 \right],
\] (22)

\[
k_{\beta} = \frac{1}{\beta} \left[ 1 - \left( \frac{x}{\beta} \right)^{\beta} \right] \ln \left( \frac{x}{\alpha} \right).
\]

The mean and standard deviation \( \mu \) and \( \sigma \) of \( X \) can be written as follows:
\[
\mu = \alpha \cdot \Gamma \left( \frac{1}{\beta} + 1 \right),
\sigma = \alpha \cdot \sqrt{\left( \frac{2}{\beta} + 1 \right) - \left( \Gamma \left( \frac{1}{\beta} + 1 \right) \right)^2},
\] (23)

where \( \mu \) and \( \sigma \) are the mean and standard deviation of \( X \), respectively, and \( \Gamma (\cdot) \) is the gamma function.

Calculate partial derivative of \( \mu \) and \( \sigma \) with respect to \( \alpha \) and \( \beta \), respectively, from equation (21); then,
\[
\frac{\partial \mu}{\partial \alpha} = \Gamma \left( \frac{1}{\beta} + 1 \right),
\frac{\partial \sigma}{\partial \alpha} = \frac{\sigma}{\alpha}
\] (24)

\[
\frac{\partial \mu}{\partial \beta} = -a \Gamma \left( \frac{1}{\beta} + 1 \right) \Psi \left( \frac{1}{\beta} + 1 \right),
\frac{\partial \sigma}{\partial \beta} = \frac{\sigma}{\beta^2 \sigma} \left[ \Gamma^2 \left( \frac{1}{\beta} + 1 \right) \Psi \left( \frac{1}{\beta} + 1 \right) - \Gamma \left( \frac{2}{\beta} + 1 \right) \Psi \left( \frac{2}{\beta} + 1 \right) + \right]
\] (25)

where \( \Psi(\cdot) \) is the psi function.

Jacobi matrix \( J \) can be written as follows:
\[
J = \begin{bmatrix}
\frac{\partial \mu_1}{\partial \mu} & \frac{\partial \sigma_1}{\partial \mu} \\
\frac{\partial \mu_1}{\partial \beta} & \frac{\partial \sigma_1}{\partial \beta}
\end{bmatrix}
\]

The kernel function of the Weibull distributed variable \( X \) with respect to its mean and standard deviation can be written as follows:
\[
\begin{bmatrix} k_{\mu} \\ k_{\sigma} \end{bmatrix} = \begin{bmatrix}
\frac{\partial \mu_1}{\partial \mu} k_{\mu_1} + \frac{\partial \sigma_1}{\partial \mu} k_{\sigma_1} \\
\frac{\partial \mu_1}{\partial \beta} k_{\mu_1} + \frac{\partial \sigma_1}{\partial \beta} k_{\sigma_1}
\end{bmatrix} = \begin{bmatrix} \frac{\partial \mu_1}{\partial \mu} \mu_
\frac{\partial \sigma_1}{\partial \mu} \sigma_
\end{bmatrix} \cdot \begin{bmatrix} k_{\mu_1} \\ k_{\sigma_1} \end{bmatrix} = J \cdot \begin{bmatrix} k_{\mu_1} \\ k_{\sigma_1} \end{bmatrix}
\] (26)

where \( k_{\mu} \) and \( k_{\sigma} \) are the kernel functions with respect to scale parameter and shape parameter, respectively. \( \alpha \) and \( \beta \) are the scale parameter and shape parameter of the two parameter Weibull distributed \( X \), respectively.

3. Reliability and Reliability Sensitivity Based on Edgeworth Series

3.1. Reliability Based on Edgeworth Series. The failure probability of a mechanical component or a random structure can be calculated out by the following multidimensional integral:
\[
P_f = P[g(X) < 0] = \int_{g(X) < 0} f_X(x) dx,
\] (27)

where \( P_f \) is the failure probability of a random structure, \( X = [X_1, X_2, \ldots, X_N]^T \in \mathbb{R}^N \) represents the \( N \)-dimensional random input variables of the random structure, \( f_X(x) \) is the joint probability density function of random input variable \( X \), \( g(X) \) is the performance function, \( g(X) < 0 \) represents the failure domain, \( g(X) > 0 \) represents the safety domain, and \( g(X) = 0 \) represents the limit state.
In practice, it is difficult to obtain the analytical solution of the multidimensional integral in equation (25) because of the nonlinear integration boundary \( g(X) = 0 \) and high dimension of \( X \). There are two classes of approaches available for estimating the failure probability \( P_f \) in equation (25), the analytical method and numerical simulation method. The numerical simulation method is the Monte Carlo simulation as known. Analytical methods (e.g., first-order reliability method (FORM)/second-order reliability method (SORM)), higher-order moment method, and saddlepoint approximation) can always be applied to calculate failure probability \( P_f \) because of high computational efficiency and accuracy of them.

Suppose the first four central moments of performance function \( g(X) \) are \( \mu_g, \nu_g, \theta_g, \) and \( \eta_g \). The failure probability \( P_f \) in equation (27) can be approximately estimated by Edgeworth series which can be written as follows:

\[
P_f(\zeta) = \Phi(\zeta) - \phi(\zeta) \left[ \frac{1}{6} \frac{\theta_g}{\sigma_g^2} H_2(\zeta) + \frac{1}{24} \left( \frac{\eta_g}{\sigma_g} - 3 \right) H_3(\zeta) + \frac{1}{72} \left( \frac{\theta_g}{\sigma_g^3} \right)^2 H_5(\zeta) \right],
\]

where \( \Phi(\cdot) \) is the standard normal distribution function, \( \phi(\cdot) \) is the standard normal probability density, \( \mu_g, \nu_g, \theta_g, \) and \( \eta_g \) are the mean, deviation, the third central moment, and the fourth central moment of performance function \( g(X) \), \( \sigma_g \) is the standard deviation of \( g(X) \), and \( H_j(\cdot) \) is the \( j^{th} \) Hermite polynomial and can be written as follows:

\[
H_{j+1}(\zeta) = 2\zeta H_j(\zeta) - 2jH_{j-1}(\zeta),
\]

\[
H_0(\zeta) = 1,
\]

\[
H_1(\zeta) = 2\zeta.
\]

3.2. Reliability Sensitivity Based on Edgeworth Series.

Partial derivative of failure probability \( P_f \) in equation (28) with respect to the first four central moments \( \mu_g, \nu_g, \theta_g, \) and \( \eta_g \) of performance function \( g(X) \) can be written as follows:

\[
\frac{\partial P_f}{\partial \zeta} = \phi(\zeta) + \left\{ \phi(\zeta) \left[ \frac{1}{3} \frac{\theta_g}{\sigma_g^2} H_2(\zeta) + \frac{1}{8} \left( \frac{\eta_g}{\sigma_g} - 3 \right) H_3(\zeta) + \frac{5}{72} \left( \frac{\theta_g}{\sigma_g^3} \right)^2 H_4(\zeta) \right] \right\}.
\]

\[
\frac{\partial P_f}{\partial \mu_g} = \frac{1}{\sigma_g}.
\]

\[
\frac{\partial \zeta}{\partial \sigma_g} = \frac{\mu_g}{\sigma_g}
\]

\[
\frac{\partial \sigma_g}{\partial \nu_g} = \frac{1}{2 \sigma_g},
\]

\[
\frac{\partial \sigma_g}{\partial \nu_g} = \frac{1}{2 \sigma_g},
\]

\[
\frac{\partial P_f}{\partial \sigma_g} = \phi(\zeta) \left[ \frac{1}{2} \frac{\theta_g}{\sigma_g^2} H_2(\zeta) + \frac{1}{6} \frac{\eta_g}{\sigma_g} H_3(\zeta) + \frac{1}{12} \left( \frac{\theta_g}{\sigma_g^3} \right)^2 H_5(\zeta) \right].
\]
The first four central moments \(\mu_g, \nu_g, \theta_g,\) and \(\eta_g\) of performance function \(g(X)\) can be written as follows:

\[
\begin{align*}
\mu_g &= m_1, \\
\nu_g &= m_2 - m_1^2, \\
\theta_g &= m_3 - 3m_1m_2 + 2m_1^3, \\
\eta_g &= m_4 - 4m_1m_3 + 6m_2m_2 - 3m_1^4,
\end{align*}
\]

(34)

where \((\partial m_i/\partial T_j^i)(i = 1, 2, 3, 4)\) can be calculated out by equation (11).

The failure probability sensitivity defined as partial derivative of failure probability \(P_f\) with respect to distribution parameters \(T^i_j\) of random inputs \(X = [X_1, X_2, \ldots, X_N]^T\) can be written as follows:

\[
\frac{\partial P_f}{\partial T^i_j} = \frac{\partial P_f}{\partial \mu_g} \frac{\partial \mu_g}{\partial T^i_j} + \frac{\partial P_f}{\partial \nu_g} \frac{\partial \nu_g}{\partial T^i_j} + \frac{\partial P_f}{\partial \theta_g} \frac{\partial \theta_g}{\partial T^i_j} + \frac{\partial P_f}{\partial \eta_g} \frac{\partial \eta_g}{\partial T^i_j},
\]

(i = 1, 2, \(j = 1, 2, \ldots, N),\)

(35)

4. Numerical Examples

4.1. Example 1: Nonlinear Performance Function with Normal Variables. This example considers a nonlinear performance function which is the stress limit state function of a multileaf spring written as follows:

\[
g(X) = r - \frac{3P_l}{7bh},
\]

(37)

where basic random variables \(X = [r, p, l, b, h]\) are the independent and identically distributed Gaussian random variables, \(r\) is the material strength of the leaf spring, \(p\) is the load, and \(b, h,\) and \(l\) are the geometric dimension width, thickness, and span of the leaf spring, respectively. The mean and standard deviation of random variables \(X = [r, p, l, b, h]\) are listed in Table 2 [22].

The proposed method and MCS are applied to analyze the failure probability and reliability sensitivity, respectively, and their results are listed in Table 3 for comparison. The MCS results can be considered as accurate results and \(N\) is the number of samples.

In Table 3, the results obtained by the proposed method have small errors compared with those obtained by MCS. Example 1 indicates that the proposed method is suitable for problems with nonlinear performance functions.

4.2. Example 2: Linear Performance Function with Six Lognormal Variables. This example considers a linear performance function, a plastic collapse mechanism of a one-bay frame which has been used as example 5 by Der et al. [28] and Zhao and Ono [29], written as follows:
\[ g(X) = x_1 + 2x_2 + 2x_3 + x_4 - 5x_5 - 5x_6, \]  
\[ (38) \]

where basic random variables \( X = [x_1, x_2, x_3, x_4, x_5, x_6] \) are the independent and identically distributed lognormal random variables. The mean and standard deviation of random variables \( X = [x_1, x_2, x_3, x_4, x_5, x_6] \) are listed in Table 4.

The proposed method and MCS are applied to analyze the failure probability and reliability sensitivity, respectively, and their results are listed in Table 5 for comparison.

In Table 5, the results obtained by the proposed method have small errors compared with those obtained by MCS. Example 2 indicates that the proposed method is suitable for problems with nonnormal random input variables.

### 4.3. Example 3: Nonlinear Performance Function with Non-normal Variables

This example considers a nonlinear performance function which describes the displacement response of a \( l \) section cantilever beam shown in Figure 2 written as follows:

\[
g(X) = d - \frac{4Fl^3}{E'\left(b(h^3 - b(h - 2t)^3 + a(h - 2t)^3)\right)}, \tag{39}
\]

where basic random variables \( X = [a, t, b, h, l, F, d, E] \) are the independent random variables. Here, \( a \) is the thickness of web, \( t \) is the thickness of flange, \( b \) is the width of \( I \)-beam section, \( h \) is the height of \( I \)-beam section, \( l \) is the length of cantilever beam, \( f \) is the load force, \( d \) is the allowable maximum deformation, and \( E \) is the elastic modulus. The
distributions and distribution parameters (mean and standard deviation) of these input random variables are listed in Table 6.

The scale parameter and shape parameter of random variable $F$ are $\alpha = 5.197 \times 10^4$ and $\beta = 310.36$, respectively.

The proposed method and MCS are applied to analyze the failure probability and reliability sensitivity, respectively, and their results are listed in Table 7 for comparison.

In Table 7, the results obtained by the proposed method have small errors compared with those obtained by MCS. Example 3 indicates that the proposed method is suitable for problems with nonlinear performance functions with non-normal random input variables. The nonnormal variables need not to be transformed into equivalent normal ones. But the coefficient of variation of the input random variables should be small ($\leq 0.1$ for Weibull distribution variables).
Table 7: Failure probability and sensitivity results of Example 3.

|                      | Proposed method     | MCS (N = 5 × 10^6) |
|----------------------|---------------------|---------------------|
| P_r                 | 9.7363 × 10^{-4}    | 2.8000 × 10^{-4}    |
| ∂P_r/∂μa            | -8.2470 × 10^{-4}   | -5.0299 × 10^{-4}   |
| ∂P_r/∂σa            | 1.0579 × 10^{-4}    | 1.1908 × 10^{-4}    |
| ∂P_r/∂μb            | -2.2000 × 10^{-3}   | -2.0500 × 10^{-3}   |
| ∂P_r/∂σb            | 1.1000 × 10^{-3}    | 1.4000 × 10^{-3}    |
| ∂P_r/∂μh            | -4.4905 × 10^{-4}   | -5.8711 × 10^{-4}   |
| ∂P_r/∂σh            | 3.2129 × 10^{-4}    | 1.8541 × 10^{-4}    |
| ∂P_r/∂μh            | -5.5812 × 10^{-4}   | -6.4358 × 10^{-4}   |
| ∂P_r/∂σh            | 1.1000 × 10^{-3}    | 1.5000 × 10^{-3}    |
| ∂P_r/∂μl            | 9.3914 × 10^{-5}    | 8.9505 × 10^{-5}    |
| ∂P_r/∂σl            | 1.9045 × 10^{-4}    | 1.9436 × 10^{-4}    |
| ∂P_r/∂μf            | 5.5696 × 10^{-6}    | 1.5504 × 10^{-6}    |
| ∂P_r/∂σf            | 8.4887 × 10^{-5}    | 8.6620 × 10^{-7}    |
| ∂P_r/∂μd            | -1.9000 × 10^{-3}   | -2.1000 × 10^{-3}   |
| ∂P_r/∂σd            | 1.6000 × 10^{-3}    | 1.5000 × 10^{-3}    |
| ∂P_r/∂μE            | -2.2514 × 10^{-7}   | -3.6546 × 10^{-7}   |
| ∂P_r/∂σE            | 1.8949 × 10^{-7}    | 2.5002 × 10^{-7}    |

5. Conclusions

The univariate dimension-reduction method, reliability analysis based on Edgeworth series, and reliability sensitivity analysis are employed to present a computational procedure for estimating failure probability sensitivity of random engineering structures. Three numerical examples are employed to illustrate the performance of the proposed method. To illustrate the feasibility of the proposed method, failure probability and failure probability sensitivity results obtained by the new method are compared with those obtained by direct MCS. It is concluded as follows: The proposed method is suitable for models of which the input random variables have small coefficient of variation (≤0.1). The nonnormal input variables need not to be transformed into equivalent normal ones, so the new method can be conveniently applied to solve problems with nonnormal input variables.

Data Availability

The data and models used to support the findings of the study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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