Lorentz gauge and Gribov ambiguity
in the compact lattice $U(1)$ theory

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Abstract

The Gribov ambiguity problem is studied for compact $U(1)$ lattice theory within
the Lorentz gauge. In the Coulomb phase, it is shown that apart from double Dirac
sheets all gauge (i.e. Gribov) copies originate mainly from the zero-momentum
modes of the gauge fields. The removal of the zero–momentum modes turns out to
be necessary for reaching the absolute maximum of the gauge functional $F(\theta)$. A
new gauge fixing procedure – zero-momentum Lorentz gauge – is proposed.

1 Introduction

To gain a better understanding of the structure of the lattice theory and to interprete
correctly the numbers obtained in Monte Carlo simulations, it is very instructive to
compare gauge variant quantities such as gauge and fermion field propagators with
the corresponding analytical perturbative results. In this respect, compact $U(1)$
pure gauge theory within the Coulomb phase serves as a very useful ‘test ground’,
because in the weak coupling limit this theory is supposed to describe noninteracting
photons. However, previous lattice studies [1, 2, 3, 4, 5] have revealed some rather
nontrivial effects. It has been shown that the standard Lorentz (or Landau) gauge
fixing procedure leads to a $\tau$–dependence of the transverse gauge field correlator
$\Gamma_T(\tau; \vec{p})$ being inconsistent with the expected zero-mass behavior [1]. Numerical
[2, 3, 6] and analytical [3] studies have shown that there is a connection between
‘bad’ gauge (or Gribov) copies and the appearance of periodically closed double
Dirac sheets (DDS). The removal of DDS restores the correct perturbative behavior
of the transverse photon correlator $\Gamma_T(\tau; \vec{p})$ with momentum $\vec{p} \neq 0$. However, it
does not resolve the Gribov ambiguity problem \[8\] completely. Other Gribov copies connected with zero–momentum modes of the gauge fields still appear, which can ‘damage’ such observables as the zero–momentum gauge field correlator \(\Gamma(\tau; 0)\) or the fermion propagator \(\Gamma_\psi(\tau)\) \[4, 5, 7\].

After having understood, why gauge variant lattice correlators behave unexpectedly from the perturbation theory point of view, one can search for the ‘true’ gauge fixing procedure. This constitutes the main goal of this note. We propose a zero–momentum Lorentz gauge (ZML), which permits to get rid of the lattice artifacts and provides correct values for various correlation functions.

We are going to compare also the standard Lorentz gauge fixing procedure (LG) \[9\] with the axial Lorentz gauge (ALG) proposed in \[10\]. We show that ALG produces just the same problems as LG does and, therefore, cannot resolve the Gribov ambiguity problem.

2 Gauge fixing : Lorentz gauge and axial Lorentz gauge

The standard Wilson action with \(U(1)\) gauge group is \[11\]

\[
S(U) = \beta \sum_x \sum_{\mu > \nu} (1 - \cos \theta_{x,\mu\nu}) ,
\]

where the link variables are \(U_{x\mu} = \exp(i\theta_{x\mu}) \in U(1)\) and \(\theta_{x\mu} \in (-\pi, \pi]\). The plaquette angles are given by \(\theta_{x,\mu\nu} = \theta_{x,\mu} + \theta_{x+\hat{\mu},\nu} - \theta_{x+\hat{\nu},\mu} - \theta_{x,\nu}\). This action makes the part of the full QED action \(S_{QED}\), which is supposed to be compact if we consider QED as arising from a subgroup of a non–abelian (e.g., grand unified) gauge theory \[12\].

The plaquette angle \(\theta_P \equiv \theta_{x;\mu\nu}\) can be split up: \(\theta_P = [\theta_P] + 2\pi n_P\), where \([\theta_P] \in (-\pi; \pi]\) and \(n_P = 0, \pm 1, \pm 2\). The plaquettes with \(n_P \neq 0\) are called Dirac plaquettes. The dual integer valued plaquettes \(m_{x,\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} n_{x,\rho\sigma}\) form Dirac sheets \[13\]. In our calculations we monitored the total number of Dirac plaquettes \(N^{(\mu\nu)}_{DP}\) for every plane \((\mu; \nu) = (1; 1), \ldots, (3; 4)\) and \(N_{DP} \equiv \max_{(\mu\nu)} N^{(\mu\nu)}_{DP}\). The appearance of periodically closed double Dirac sheet means that at least for one out of six planes \((\mu\nu)\) the number of Dirac plaquettes \(N^{(\mu\nu)}_{DP}\) should be

\[
N^{(\mu\nu)}_{DP} \geq 2 \frac{V_4}{N_\mu N_\nu} .
\]

For example, on the lattice \(12 \times 6^3\) the appearance of DDS means \(N_{DP} \geq 72\).

In lattice calculations the usual choice of the Lorentz (or Landau) gauge is

\[
\sum_{\mu=1}^{4} \bar{\partial}_\mu \sin \theta_{x\mu} = 0 ,
\]
which is equivalent to finding an extremum of the functional $F(\theta)$

$$F(\theta) = \frac{1}{V_4} \sum_x F_x(\theta) ; \quad F_x(\theta) = \frac{1}{8} \sum_{\mu=1}^4 \left[ \cos \theta_{x\mu} + \cos \theta_{x-\hat{\mu},\mu} \right]$$

with respect to the (local) gauge transformations

$$U_{x\mu} \rightarrow \Lambda_x U_{x\mu} \Lambda_x^* ; \quad \Lambda_x = \exp\{i\Omega_x\} \in U(1) . \quad (2.5)$$

The standard gauge fixing procedure (referred in what follows as LG) consists of the maximization of the value $F_x(\theta)$ at some site $x$ under the local gauge transformations $\Lambda_x$, then at another site and so on. After a certain number of gauge fixing sweeps a local maximum $F_{\text{max}}$ of the functional $F(\theta^A)$ is reached. In order to improve the convergence of the iterative gauge fixing procedure one can use the optimized overrelaxation procedure [14] with some parameter $\alpha$ which depends on the volume and $\beta$. For convergence criteria we use

$$\max \left| \sum_{\mu} \bar{\partial}_\mu \sin \theta_{x\mu} \right| < 10^{-5} \quad \text{and} \quad \frac{1}{V} \sum_x \left| \sum_{\mu} \bar{\partial}_\mu \sin \theta_{x\mu} \right| < 10^{-6} .$$

In Figure 1a we show a typical Monte Carlo time history of $F_{\text{max}}$ obtained using the standard Lorentz gauge fixing procedure. Partially, the comparatively big dispersion of $F_{\text{max}}$ is due to periodically closed double Dirac sheets [2, 3]. In Figure 1b we show the corresponding time history of $N_{DP} = \max_{\mu \nu} N_{(\mu \nu)}^{DP}$. For the given lattice size and $\beta$–value $\sim 20\%$ of configurations turn out to possess DDS. Exactly because of the configurations containing DDS the transverse photon correlator

$$\Gamma_T(\tau; \vec{p}) = \langle \Phi(\tau; \vec{p}) \Phi^*(0; \vec{p}) \rangle , \quad \Phi(\tau; \vec{p}) = \sum_{\vec{x}} \exp(i\vec{p}\vec{x} + i\tau \vec{p}_\mu) \sin \theta_{\tau \vec{x}, \mu}$$

($\mu = 1, 3$, $\vec{p} = (0, p, 0)$) exhibits an unphysical ‘tachion–like’ behavior [2, 3, 4]. In Figure 2 we show the normalized photon correlator $\Gamma_T(\tau; \vec{p})/\Gamma_T(0; \vec{p})$ for lowest non-vanishing momentum and LG together with results obtained with other Lorentz gauge fixing procedures to be discussed lateron. We clearly see for LG the deviation from the expected zero-mass behavior. All the observations described above have not been seen changing, when $\beta$ and/or the lattice size were increased considerably [4].

It is a long–standing believe [15] that the ‘true’ gauge copy corresponds to the absolute maximum of $F(\theta)$. Thus it would be highly desirable to find a Lorentz gauge fixing procedure which produces a (unique) gauge copy of the given configuration with the absolute maximum of $F(\theta)$. However, if one repeatedly subjects a configuration $\{\theta_{x\mu}\}$ to a random gauge transformation as in Eq.(2.5) and then subsequently applies to it the LG procedure, one usually obtains gauge (Gribov) copies with different values of $F_{\text{max}}$. 


An attempt to resolve this problem has been made in [10] where a modified Lorentz gauge fixing procedure has been proposed. This procedure (which we refer to as the axial Lorentz gauge fixing or ALG) consists of the two steps:

i) first transform every configuration to satisfy a maximal tree temporal gauge condition (‘axial’ gauge) [16];

ii) then apply the Lorentz gauge fixing procedure.

An axial gauge with a chosen maximal tree is unique by definition. In practice, this is easily checked by random gauge transformations applied first. Consecutive gauge fixing steps – e.g. the Lorentz gauge iterations – will lead always to the same result as long as we do not change the detailed prescription for these steps. The question is, whether this ‘unique’ Lorentz gauge obtainable for each gauge field configuration resolves the problems mentioned above. The answer is ‘no’. We do not find the absolute maximum of the gauge functional in the majority of the cases. There is a quite high percentage of Gribov copies left containing DDS (around 10% for \(\beta = 1.1\) and a \(12 \times 6^3\) lattice). As a consequence the transverse non-zero momentum photon propagator does not come out correctly again. The corresponding data are shown in Figure 2, too. Doubling of the linear lattice size and enlarging \(\beta\) (we checked \(\beta = 2.0\)) do not improve the behavior.

In paper [2] a Lorentz gauge fixing prescription with a preconditioning step based on a non-periodic gauge transformation was proposed. The latter has been chosen in such a way that the spatial Polyakov loop averages were transformed into real numbers as a first step. We convinced ourselves that the Lorentz gauge fixed configurations with very high probability did not contain DDS. As a consequence, the photon correlator becomes correct. However, this gauge fixing procedure does not provide the absolute maximum of the Lorentz gauge functional, too. Thus, the lattice Gribov problem for QED in the Coulomb phase has not been solved.

3 Zero–momentum Lorentz gauge (ZML)

As is well known, apart from the local symmetry Wilson action \(S\) has an additional (global) symmetry with respect to non–periodic transformations

\[
U_{x\mu} \rightarrow \bar{\Lambda}_x U_{x\mu} \bar{\Lambda}^*_{x+\bar{\mu}} = U_{x\mu} \cdot e^{-i\bar{c}_\mu} ; \quad \bar{\Lambda}_x = e^{i\sum_{\mu} c_\mu x_\mu} ,
\]

or equivalently

\[
\theta_{x\mu} \rightarrow \theta_{x\mu} - c_\mu .
\]

\(^1\)In [10] an analytically known local solution of the Lorentz gauge condition has been applied, instead of an iteratively obtained one. But, as we have convinced ourselves, this solution does not resolve the problems we are discussing here.
Note that these transformations do not spoil the periodicity of the gauge fields $U_{x\mu}$ and $\theta_{x\mu}$, respectively. The transformation in Eq. (3.2) changes the zero–momentum mode $\phi_\mu$ of the link angle $\theta_{x\mu} \equiv \phi_\mu + \delta\theta_{x\mu}$, where $\sum_x \delta\theta_{x\mu} = 0$, and a proper choice of the parameters $c_\mu$ can make $\phi_\mu$ equal to zero.

It is rather evident that for the infinitesimal fluctuations $\delta\theta_{x\mu}$ about the zero–momentum mode $\phi_\mu$ the absolute maximum of the functional $F(\theta)$ corresponds to the case $\phi_\mu = 0$.

Our main statement (to be proved in this section) is that in most of the cases ($\sim 99.99\%$) gauge copies of the given configuration $\theta_{x\mu}$ are due to $a$) double Dirac sheets; $b$) the zero–momentum modes of this field. It is the exclusion of DDS and of the zero–momentum modes that permits to obtain a gauge copy of the given configuration with the absolute maximum of the functional $F(\theta)$.

We define our gauge fixing prescription (which we refer to as the zero–momentum Lorentz gauge or ZML gauge) as follows. Every iteration consists of one sweep with (global) transformations as in Eq. (3.2) and one sweep with local gauge transformations as in the standard Lorentz gauge fixing procedure.

In Figures 3a,b we show time histories of $F_{\text{max}}$ and of $N_{DP}$ for the gauge fixing procedure with the minimization of the zero–momentum modes of the gauge field. Both the time histories should be compared with the corresponding ones for the standard Lorentz gauge shown in Figures 1a,b. One can see that after suppression of the zero–momentum modes the average value of $F_{\text{max}}$ became essentially larger as compared with the LG case. At the same time the number of DDS (to be precise, the number of configurations with $N_{DP} \geq 72$) has drastically decreased ($\sim 1\%$). Typically, those configurations with DDS have smaller values of $F_{\text{max}}$ than configurations without DDS have.

It is a very easy task to remove the remaining DDS. If a DDS in a ZML gauge has really appeared, then perform a random gauge transformation to the same field configuration and repeat the ZML procedure again. As a result gauge field configurations do not contain DDS.

To convince ourselves that the ZML gauge fixing procedure provides an absolute maximum of the functional $F(\theta)$ we generated many random gauge copies for every thermalized configuration $\{\theta_{x\mu}\}$. The number of these random gauge copies $N_{RC}$ varied between 10 and 1000 for different $\beta$’s and lattices. Let $\{\theta_{x\mu}^{(j)}\}$ be the $j^{th}$ gauge copy of the configuration $\{\theta_{x\mu}\}$ obtained with the random gauge transformation as in Eq. (2.5), $j = 1, \ldots, N_{RC}$. For any configuration $\{\theta_{x\mu}\}$ we define a ‘variance’ $\delta F_{\text{max}}(\theta)$ of $F_{\text{max}}(\theta)$

$$\delta F_{\text{max}}(\theta) = \max_{ij} \left( F_{\text{max}}(\theta^{(i)}) - F_{\text{max}}(\theta^{(j)}) \right) ; \ i, j = 1, \ldots, N_{RC} . \quad (3.3)$$

Different gauge copies can, in principle, have different values of $F_{\text{max}}(\theta)$. Therefore, its ‘variance’ $\delta F_{\text{max}}(\theta)$ can be non–zero. For example, the standard gauge fixing procedure (LG) described above gives typical values $\delta F_{\text{max}}(\theta) \sim 0.06 \div 0.07$ for
\(N_{RC} = 10\). As an example we show in Figure 4a a time history of the ‘variance’ \(\delta F_{\text{max}}(\theta)\) for the standard LG and for ZML. In the case of ZML this ‘variance’ is zero \(^2\).

In Figure 4b we compare values of \(F_{\text{max}}\) for these two gauges. The lower broken line for the LG corresponds to the first gauge copy which is just a thermalized configuration produced by our updating subroutine. Applying random gauge transformations we produced different random gauge copies \((N_{RC} = 10\) in this case\) and then chose the gauge copy with the maximal value of \(F_{\text{max}}\). The corresponding values are represented by the solid LG line in Figure 4b. One can see that even this is far below the solid line which corresponds to the ZML gauge. Increasing the number \(N_{RC}\) up to 1000 does not change this conclusion.

One reason for the non–zero value of the ‘variance’ is the appearance of gauge copies containing double Dirac sheets \([2, 3, 5]\). Gauge copies with double Dirac sheets have typically much lower values of \(F_{\text{max}}(\theta)\) as compared to \(F_{\text{max}}\) for gauge copies without DDS. Moreover, a contribution of configurations with DDS ‘spoils’ the photon correlator \(\Gamma_{\tau}(\tau; \vec{p})\) with \(\vec{p} \neq 0\) which leads to a wrong dispersion relation inconsistent with the dispersion relation for the massless photon. Double Dirac sheets represent a spectacular example of lattice artifacts which can lead to a misleading interpretation of the results of numerical calculations (see also \([17]\)).

But, as we have seen, the mere exclusion of gauge copies with DDS in the standard Lorentz gauge fixing procedure does not yet provide a zero value of the ‘variance’ \(\delta F_{\text{max}}(\theta)\). Different gauge copies (without DDS) still have different values of \(F_{\text{max}}(\theta)\). The exclusion of the zero-momentum modes turns out to be sufficient to remove the ambiguity.

4 Conclusions

Now let us summarize our findings.

Our main result is that \(\sim 99.99\%\) (if not all) gauge copies for the given gauge field configuration are due to two reasons:

a) periodically closed double Dirac sheets;

2) the zero–momentum mode of the gauge field \(\theta_{x\mu}\).

We didn’t find any other reason for the appearance of the gauge copies. The minimization of the zero–momentum mode can be performed sweep by sweep using a global transformation as in Eq. (3.2). We proposed a modified gauge fixing procedure consisting of local gauge transformations in Eq. (2.3) and global transformations in Eq. (3.2) (ZML gauge). The exclusion of the DDS (which appear

\(^2\)In rather rare cases \((\lesssim 0.01\%)\) in ZML gauge \(\delta F_{\text{max}}(\theta) \neq 0\). So far we have no explanation for it.
rather rarely in the ZML gauge) can be easily performed on the algorithmic level as described in the text.

The application of the ZML gauge fixing procedure provides us with absolute maximum of the functional \( F(\theta) \).

We have shown that the gauge fixing procedure with axial gauge preconditioning (ALG) cannot solve the problem of the Gribov ambiguity in this theory. The axial gauge preconditioning cannot exclude the appearance of DDS as well as of nonzero values of the zero–momentum mode.

In this paper we present our results only for the bosonic sector of the theory. However, we believe that this study solves ultimatively the Gribov ambiguity puzzle in the case of quenched compact QED within the gauge as well as the fermionic sector. Work on the fermion case is in progress [7].

The inclusion of dynamical fermions changes somewhat the symmetry group of the full action, i.e. the parameters \( c_\mu \) in Eq.'s (3.1), (3.2) can have only discrete values. This case needs some additional study.

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**Figure captions**

Figure 1. Time history of $F_{\text{max}}$ (a) and $N_{DP}$ (b) at $\beta = 1.1$ on the $12 \cdot 6^3$ lattice in the standard Lorentz gauge.

Figure 2. Transverse propagator at $\beta = 1.1$ on the $12 \cdot 6^3$ lattice in three different gauges.

Figure 3. Time history of $F_{\text{max}}$ (a) and $N_{DP}$ (b) at $\beta = 1.1$ on the $12 \cdot 6^3$ lattice in ZML gauge.

Figure 4. Time history of the ‘variance’ $\delta F_{\text{max}}$ (a) and $F_{\text{max}}$ (b) at $\beta = 1.1$ on the $12 \cdot 6^3$ lattice. LG solid line corresponds to $N_{RC} = 10$. Gauge copies with DDS are excluded.
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