Full Diversity Codes for MISO Systems Equipped with Linear or ML Detectors

Jing Liu, Student Member, IEEE, Jian-Kang Zhang, Member, IEEE, and Kon Max Wong, Fellow, IEEE

Abstract—In this paper, a general criterion for space time block codes (STBC) to achieve full-diversity with a linear receiver is proposed for a wireless communication system having multiple transmitter and single receiver antennas (MISO). Particularly, the STBC with Toeplitz structure satisfies this criterion and therefore, enables full-diversity. Further examination of this Toeplitz STBC reveals the following important properties: a) The symbol transmission rate can be made to approach unity. b) Applying the Toeplitz code to any signalling scheme having nonzero distance between the nearest constellation points results in a non-vanishing determinant. In addition, if QAM is used as the signalling scheme, then for independent MISO flat fading channels, the Toeplitz codes is proved to approach the optimal diversity-vs-multiplexing tradeoff with a ZF receiver when the number of channel uses is large. This is, so far, the first non-orthogonal STBC shown to achieve the optimal tradeoff for such a receiver. On the other hand, when ML detection is employed in a MISO system, the Toeplitz STBC achieves the maximum coding gain for independent channels. When the channel fading coefficients are correlated, the inherent transmission matrix in the Toeplitz STBC can be designed to minimize the average worst case pair-wise error probability.

Index Terms—Full Diversity, Linear Receiver, MISO, ML detection, Non-vanishing determinant, Optimal diversity-vs-multiplexing tradeoff, STBC, Toeplitz

I. INTRODUCTION

THE recent arrival of the Information Age has created an explosive demand for knowledge and information exchange in our society. This demand has triggered off an enormous expansion in wireless communications in which severe technical challenges, including the need of transmitting speech, data and video at high rates in an environment rich of scattering, have been encountered. A recent development in wireless communication systems is the multi-input multi-output (MIMO) wireless link which, due to its potential in meeting these challenges caused by fading channels together with power and bandwidth limitations, has become a very important area of research. The importance of MIMO communications lies in the fact that they are able to provide a significant increase in capacity over single-input single-output (SISO) channels. Existing MIMO designs employ multiple transmitter antennas and multiple receiver antennas to exploit the high symbol rate provided by the capacity available in the MIMO channels. Full symbol rate is achieved when, on average, one symbol is transmitted by each of the multiple transmitter antennas per time slot (often called a “channel use”). In the case of $M$ transmitter antennas, we will have an average of $M$ symbols per channel use (pcu) at full rate. Furthermore, to combat fading and cross-talk, MIMO systems provide different replicas of transmitted symbols to the receiver by using multiple receiver antennas with sufficient separation between each so that the fading for the receivers are independent of each other. Such diversity can also be achieved at the transmitter by spacing the transmitter antennas sufficiently and introducing a code for the transmitted symbols distributed over transmitter antennas (space) and symbol periods (time), i.e., space-time coding [1]–[4]. Full diversity is achieved when the total degree of freedom available in the multi-antenna system is utilized.

Over the past several years, various space-time coding schemes have been developed to take advantage of the MIMO communication channel. Using a linear processor, orthogonal space-time block codes [2], [3], [5]–[8] can provide maximum diversity achievable by a maximum likelihood detector. However, they have a limited transmission rate [8]–[11] and thus, do not achieve full MIMO channel capacity [12]. Linear dispersion codes have been proposed in [13] for which each transmitted codeword is a linear combination of certain weighted matrices maximizing the ergodic capacity of the system. Unfortunately, good error probability performance for these codes is not strictly guaranteed. To bridge the gap between multiplexing and diversity, a linear dispersion code design has been proposed using frame theory [14] that typically performs well both in terms of ergodic capacity and error performance, but full diversity still cannot be guaranteed. Thus far, with the exception of the orthogonal STBC, all existing STBC are designed such that full diversity can only be achieved when the ML detector is employed. Recent research [15]–[19] based on number theory has shown that employing a ML receiver, it is possible to design linear space-time block codes and dispersion codes which are full rate and full diversity without information loss. The major concern on these designs is that the coding gain vanishes rapidly as the constellation size increases. Therefore, designs of full-rate, full-diversity space-time codes with non-vanishing coding gain have drawn much attention [20]–[32] since such structured space-time codes could achieve the optimal diversity-vs-multiplexing tradeoff developed by Zheng and Tse [33]. However, most available STBC possessing these properties are for ML receivers only.

In this paper, we consider a coherent communication system equipped with multiple transmitter antennas and a single receiver antenna, i.e., a MISO system. These systems are often employed in mobile communications for which the mobile receiver may not be able to support multiple antennas. The highest transmission rate for a MISO system is unity, i.e.,
one symbol pcu. For such a MISO system with ML receivers, rate-1 and full diversity STBC have been proposed by various authors [29], [34]--[36]. In this paper, however, we consider such a MISO system equipped with linear receivers for which we propose a general criterion for the design of a full-diversity STBC. In particular, we introduce the Toeplitz STBC as a member of the family of the full diversity STBC. It should be noted that the Toeplitz structure has already been successfully employed as a special case of the delay diversity code (DDC) [37]–[40] applied to MIMO systems having outer channel coding and ML detection. Here, we extend its application to the construction of STBC in a MISO system by having a Toeplitz coding matrix cascaded with a beamforming matrix. We show that the Toeplitz STBC has several important properties which enable the code, when applied to a MISO system with a linear receiver, to asymptotically achieve unit symbol rate, to possess non-vanishing determinants for signal constellations having non-zero distance between nearest neighbours, and to achieve full diversity [41] accomplishing the optimal tradeoff of diversity and multiplexing gains [33].

On the other hand, we also consider the MISO system in which the channel has zero mean and fixed covariance known to the transmitter. For such MISO systems, sacrificing the transmission rate by repeating the transmitted symbols and employing maximum ratio combining together with orthogonal space-time coding, an optimal precoder can be designed [42], [43] by minimizing the upper bound of the average symbol error probability (SEP). Here in this paper, we apply the Toeplitz STBC to such a MISO system. Maintaining rate one and full diversity, we present a design that minimizes the exact worst case average pair-wise error probability when the ML detector is employed at the receiver.

II. MISO SYSTEM MODEL AND PROPERTIES OF THE CHANNEL MATRIX

Consider a MIMO communication system having \( M \) transmitter antennas and \( M_R \) receiver antennas transmitting the symbols \( \{s_\ell \} \), \( \ell = 1, \ldots , L \) which are selected from a given constellation, i.e., \( s_\ell \in S \). To facilitate the transmission of these \( L \) symbols through the \( M \) antennas in the \( N \) time slots (channel use), each symbol is processed by an \( M \times M \) coding matrix \( A_\ell \), and then summed together, resulting in an \( M \times M \) STBC matrix given by \( X = \sum_{\ell=1}^{L} s_\ell A_\ell \) where the \((m,n)\)th element of \( X \) represents the coded symbol to be transmitted from the \( m \)th antenna at the \( n \)th time slot. These coded symbols are then transmitted to the receiver antennas through flat-fading path coefficients which form the elements of the \( M \times M_R \) channel matrix \( H \). The received space-time signal, denoted by the \( N \times M_R \) matrix \( Y \), can be written as

\[
Y = XH + \Xi
\]

where \( \Xi \) is the \( N \times M_R \) additive white space-time noise matrix whose elements are of complex circular Gaussian distribution \( \mathcal{CN}(0,1) \).

Let us now turn our attention to a MISO wireless communication system which is a special case of the MIMO system having \( M \) transmitter antennas and a single receiver antenna. Just as in the MIMO system, the transmitted symbols \( s_\ell \), \( \ell = 1, \ldots , L \) in the MISO system are coded by linear \( N \times M \) STBC matrices \( A_\ell \) which are then summed together so that

\[
X = \sum_{\ell=1}^{L} A_\ell s_\ell
\]

where \( L \) is the total number of symbols to be transmitted. If \( L = N \), the system is at full-rate (rate-one). At the time slot \( n \), the \( n \)th row of the coding matrix \( X \) feeds the \( M \) coded symbols to the \( M \) antennas for transmission. Each of these transmitter antennas is linked to the receiver antenna through a channel path coefficient \( h_m \), \( m = 1, \ldots , M \). At the receiver of such a system, for every \( N \) time slots \( (n = 1, \ldots , N) \), we receive an \( N \)-dimensional signal vector \( y = [y_1 \ y_2 \ \cdots \ y_N]^T \) which, as a special case of Eq. (1), can then be written as

\[
y = Xh + \xi
\]

where \( h = [h_1, \cdots , h_M]^T \) is an \( M \times 1 \) channel vector assumed to be circularly symmetric complex Gaussian distributed with zero-mean and covariance matrix \( \Sigma \), and \( \xi \) is an \( N \times 1 \) noise vector assumed to be circularly symmetric complex Gaussian with covariance \( \sigma^2 I_N \). Putting Eq. (2) into Eq. (3), writing the symbols to be transmitted as a vector and aligning the code-channel products to form the new channel matrix we can write

\[
H = (A_1 h \ A_2 h \ \cdots \ A_L h) \quad \text{and} \quad s = [s_1 \ s_2 \ \cdots \ s_L]
\]

the received signal vector can now be written as

\[
y = Hs + \xi
\]

In this paper, we emphasize on the application of linear receivers for the MISO system in Eq. (4). In the following, we will derive a condition on the equivalent channel \( H \) that renders full-diversity when the signals are received by a linear receiver. First, we present the following properties of the equivalent channel matrix \( H \):

Property 1: Suppose the equivalent channel \( H \) in Eq. (4) is such that \( H^H H \) is non-singular for any nonzero \( h \). Then we have the following inequality:

\[
C_{\min} \| h \|^2 \leq \det (H^H H) \leq C_{\max} \| h \|^2
\]

where \( C_{\min} \) and \( C_{\max} \) are positive constants independent of \( h \).

Proof: Since \( h \) is nonzero, we normalize the \( L \times L \) matrix \( H^H H \) by dividing each of its elements with \( \| h \|^2 \), i.e., \( H^H H = \| h \|^2 \mathbb{H} \), where \( \mathbb{H} \) is the normalized matrix with the \( i \)th element being equal to

\[
[\mathbb{H}]_{ij} = \frac{H_{ij}}{\| h \|} = \frac{A_j h}{\| h \|} \quad i, j = 1, 2, \cdots , L
\]

The determinant of positive semi-definite (PSD) matrix \( \mathbb{H} \) is continuous in a closed bounded feasible set \( \{ h : \| h \|^2 = 1 \} \) where \( h = \frac{h}{\| h \|} \). It has the maximum and minimum values that are denoted by \( C_{\max} \) and \( C_{\min} \) respectively. Now, since \( H^H H \) is non-singular for any nonzero \( h \), its determinant is positive. Therefore, \( 0 < C_{\min} \leq C_{\max} \) and Eq. (6) holds.
The following example serves to illustrate the above property.  

Example 1: Consider the following channel matrix \( \mathbf{H} = \begin{pmatrix} h_1 & 0 \\ h_2 & h_1 \\ 0 & h_2 \end{pmatrix} \). The determinant of matrix \( \mathbf{H}^H \mathbf{H} \) can be written as

\[
\det(\mathbf{H}^H \mathbf{H}) = \| \mathbf{h} \|^4 \left( 1 - \frac{|h_1|^2 |h_2|^2}{\| \mathbf{h} \|^2} \right)
\]

(7)

Since \( \frac{|h_1|^2}{\| \mathbf{h} \|^2} + \frac{|h_2|^2}{\| \mathbf{h} \|^2} = 1 \), we can define \( \frac{|h_1|^2}{\| \mathbf{h} \|^2} = \cos \theta \), and \( \| \mathbf{h} \|^2 = \sin \theta \), and Eq. (7) becomes

\[
\det(\mathbf{H}^H \mathbf{H}) = \| \mathbf{h} \|^4 \left( 1 - \sin^2 \theta \cos^2 \theta \right) = \| \mathbf{h} \|^4 \left( 1 - \frac{1}{4} \sin^2(2\theta) \right)
\]

(8)

It is obvious that the function \( f(\theta) = 1 - \frac{1}{4} \sin^2(2\theta) \) is continuous in a closed bounded set. The minimum and maximum of it can be easily obtained as \( C_{\min} = \frac{1}{2} \); \( C_{\max} = 1 \). Both values are constants and are independent of the random channel. Thus, the determinant of the channel matrix is bounded by \( \frac{3}{4} \| \mathbf{h} \|^4 \leq \det \left( \mathbf{H}^H \mathbf{H} \right) \leq \| \mathbf{h} \|^4 \). \( \square \)

Property 2: If \( \mathbf{H}^H \mathbf{H} \) is non-singular for any nonzero \( \mathbf{h} \), then the diagonal elements of \( \left( \mathbf{H}^H \mathbf{H} \right)^{-1} \) satisfies the following inequality

\[
\left( \mathbf{H}^H \mathbf{H} \right)^{-1} \leq C_0 \| \mathbf{h} \|^2
\]

for \( \ell = 1, 2, \cdots, L \) where \( C_0 \) is a constant independent of \( \mathbf{h} \).

Proof: From the matrix inversion algorithm [44], we have

\[
\left( \mathbf{H}^H \mathbf{H} \right)^{-1} = \frac{\det(\mathbf{H}^H \mathbf{H})}{\det(\mathbf{H}^H \mathbf{H})}
\]

(10)

where \( \mathbf{H}^H \mathbf{H} \) is the matrix obtained by deleting the \( \ell \)th column vector, \( \mathbf{A}_\ell \mathbf{h} \), from \( \mathbf{H} \). We notice that the matrix \( \mathbf{H}^H \mathbf{H} \) is still PSD and therefore satisfies the right side inequality of Eq. (9) having an upper bound denoted by \( C_{\ell_{\max}} \| \mathbf{h} \|^2 \). Applying the lower bound of Eq. (9) to the numerator and the upper bound to the denominator of Eq. (10), we obtain

\[
\left( \mathbf{H}^H \mathbf{H} \right)^{-1} \leq C_{\min} \| \mathbf{h} \|^2 \leq C_{\max} \| \mathbf{h} \|^2 \leq \| \mathbf{h} \|^2 \]

(11)

where \( C_{\min} = \frac{C_{\min}}{C_{\ell_{\max}}} \), with \( C_{\ell_{\max}} = \max \{ C_{\ell_{\max}}, \ell = 1, 2, \cdots, L \} \).

Properties 1 and 2 are of fundamental importance to the design of full diversity STBC for a MISO system employing a linear detector. This will be presented in the following section.

### III. Diversity Gain of STBC for a MISO System Employing a Linear Receiver

Let us first review the concept of diversity gain with reference to a MIMO system. Consider the MIMO system in Eq. (1) equipped with a maximum likelihood (ML) detector. It is well-known that an upper bound for the average pair-wise error probability is given by [45]

\[
P(s \rightarrow s') \leq \frac{1}{2} \frac{\det \left( \mathbf{I}_M + \frac{\rho}{8M} \mathbf{X}^H \mathbf{X}(e) \right)^{-M_R}}{M_R} \leq \frac{1}{2} \left( \prod_{m=1}^r \lambda_m \right)^{-M_R} \left( \frac{\rho}{8M} \right)^{-M_R}
\]

(12)

where \( \rho = \mathbb{E} [\mathbb{E}^H \mathbf{X}(e) / \mathbb{E}^H \mathbf{X}(e)] \) is the SNR, \( e = s - s' \) with \( s, s' \in \mathcal{S} \) is the error vector, \( r \leq M \) is the rank, and \( \{ \lambda_m \}, m = 1, \cdots, r \) are the non-zero eigenvalues of the matrix \( \mathbf{X}(e) \mathbf{X}(e) \). The middle part of Eq. (12) is the Chernoff bound, which at high SNR, can be further tightly bounded by the right side. For a given \( M_R \), two factors dictate the minimization of this bound on the right side of Eq. (12):

a) The Rank of \( \mathbf{X}(e) \mathbf{X}(e) \): The exponent \( rM_R \) of the second term governs the behaviour of the upper bound with respect to SNR and is known as the diversity gain. To keep the upper bound as low as possible, we should make the diversity gain as large as possible. Full diversity is achieved when \( r = M \), i.e., \( \mathbf{X}(e) \) is of full column rank. This implies that the diversity gain achieved by an ML detector depends on \( e \), which is decided by the type of signalling.

b) The Determinant of \( \mathbf{X}(e) \mathbf{X}(e) \): The first term consists of the product of the non-zero eigenvalues of \( \mathbf{X}(e) \mathbf{X}(e) \) and is called the coding gain. For \( \mathbf{X}(e) \mathbf{X}(e) \) being full rank, this product is its determinant the minimum value of which (taken over all distinct symbol vector pairs \( \{ s, s' \} \)) must be maximized.

At high SNR, the upper bound in Eq. (12) is dominated by the exponent \( -rM_R \) of \( \rho \). This leads to a more general definition of diversity gain [33] as being the total degrees of freedom offered by a communication system, reflected by the factor involving the negative power of the SNR in the expression of the error probability. Full diversity gain is achieved when the total degrees of freedom (\( = MM_R \)) offered in the multi-antenna system are utilized. We adopt this latter notion of diversity gain when we consider the STBC for the MISO system.

Since \( M_R = 1 \), full diversity for a MISO system is achieved if the exponent of the SNR in the expression of the error probability is equal to \(-M \). Let us now consider the condition on \( \mathbf{H} \) for which full-diversity is achieved by a MISO system employing a linear receiver. We need only to consider the use of a linear zero-forcing (ZF) receiver because the same condition extends to MISO systems using linear minimum mean square (MMSE) receivers or other more sophisticated receivers. Since the diversity gain of a communication system relates the probability of error to SNR, we first analyze the symbol error probability (SEP) of detecting different signal constellations by a linear ZF equalizer and express these in terms of the SNR.

#### A. Symbol Error Probability of Various Signalling Schemes

Here, we examine three commonly used signalling schemes: 1) square QAM, 2) PAM and 3) PSK constellations respec-
tively. Let $\mu$ denote the cardinality. Firstly, we summarize the definition of some common parameters which govern the performance of the ZF linear detectors under these schemes. We use the index $i = 1, 2, 3$ to denote parameters associated with the three signalling schemes as ordered above. Let $E_{s_i}$, $i = 1, 2, 3$, denote the respective average symbol energy in each of the above schemes, and let $\sigma^2$ be the noise variance at the receiver antenna. Therefore, the SNR for each symbol at the receiver is given by

$$P_i(h, s_\ell) = \frac{\rho_\ell}{E_{s_\ell}/\sigma^2}; \quad i = 1, 2, 3$$

Note that $\sigma^2[H H^H]_{\ell\ell}$ is the noise power at the output of the ZF equalizer for the $\ell$th symbol.

1. **Square QAM signals**: The SEP of a ZF receiver for the square QAM signal $s_\ell$ is [46]

$$P_1(h, s_\ell) = 4 \left(1 - \frac{1}{\sqrt{\mu}}\right) Q\left(\sqrt{\frac{3E_{s_1}}{2(\mu - 1)\sigma^2 \left[H H^H\right]_{\ell\ell}}} \right) - 4 \left(1 - \frac{1}{\sqrt{\mu}}\right)^2 Q^2\left(\sqrt{\frac{3E_{s_1}}{2(\mu - 1)\sigma^2 \left[H H^H\right]_{\ell\ell}}} \right)$$

where $Q(z) = \int_0^\infty e^{-x^2/2}dx$. We use the following alternative expressions for the $Q$- and $Q^2$-functions [46]–[49]

$$Q(z) = \frac{1}{\pi} \int_0^{\pi/2} \exp\left(-\frac{z^2}{2\sin^2 \theta}\right) d\theta \quad z \geq 0 \quad (15)$$

$$Q^2(z) = \frac{1}{\pi} \int_0^{\pi/2} \exp\left(-\frac{z^2}{2\sin^2 \theta}\right) d\theta \quad z \geq 0 \quad (16)$$

Substituting Eqs. (15) and (16) into Eq. (14) and after a little manipulation, we obtain

$$P_1(h, s_\ell) = 4 \pi \left(1 - \frac{1}{\sqrt{\mu}}\right) \int_0^{\pi/4} \exp\left(-\frac{3E_{s_1}}{4(\mu - 1)\sigma^2 \left[H H^H\right]_{\ell\ell} \sin^2 \theta}\right) d\theta$$

$$+ \frac{4}{\pi \sqrt{\mu}} \left(1 - \frac{1}{\sqrt{\mu}}\right) \int_{\pi/4}^{\pi/2} \exp\left(-\frac{3E_{s_1}}{4(\mu - 1)\sigma^2 \left[H H^H\right]_{\ell\ell} \sin^2 \theta}\right) d\theta \quad (17)$$

We can obtain an upper bound for Eq. (17) by putting $\sin \theta = 1$ in the two integrals and, writing $\rho_1 = E_{s_1}/\sigma^2$, this easily simplifies to

$$P_1(h, s_\ell) \leq \frac{\mu - 1}{\mu} \exp\left(-\frac{3\rho_1}{4(\mu - 1)\left[H H^H\right]_{\ell\ell} \sin^2 \theta}\right) \quad (18)$$

2. **PAM signals**: The SEP of the ZF receiver for a $\mu$-ary PAM signal $s_\ell$ is given by [46]

$$P_2(h, s_\ell) = \frac{2(\mu - 1)}{\mu} Q\left(\sqrt{\frac{3E_{s_2}}{(\mu^2 - 1)\sigma^2 \left[H H^H\right]_{\ell\ell}}} \right)$$

Now, using Eq. (15) and noting that by putting $\sin \theta = 1$ in the integral, we have $Q(z) \leq \exp(-z^2)$ for $z \geq 0$. Together with $\rho_2 = E_{s_2}/\sigma^2$, we arrive at an upper bound of Eq. (19).

$$P_2(h, s_\ell) \leq \frac{\mu - 1}{\mu} \exp\left(-\frac{3\rho_2}{2(\mu^2 - 1)\left[H H^H\right]_{\ell\ell} \sin^2 \theta}\right) \quad (20)$$

3. **PSK signals**: The SEP of a ZF receiver for the PSK signal $s_{k\ell}$ is given by [46]

$$P_3(h, s_{k\ell}) = \frac{1}{\pi} \int_0^{(\mu - 1)\pi/\mu} \exp\left(-\frac{E_{s_3} \sin^2(\pi/\mu)}{2\sigma^2 \left[H H^H\right]_{\ell\ell} \sin^2 \theta}\right) d\theta \quad (21)$$

Writing $\rho_3 = E_{s_3}/\sigma^2$, similar to the PAM signal, Eq. (21) can be upper bounded by

$$P_3(h, s_{k\ell}) \leq \frac{(\mu - 1)}{\mu} \exp\left(-\frac{\rho_3 \sin^2(\pi/\mu)}{2 \left[H H^H\right]_{\ell\ell} \sin^2 \theta}\right) \quad (22)$$

B. **Design Criterion for Full-Diversity STBC for a MISO System with Linear Receivers**

We now examine the diversity gain achievable by a MISO system.

**Theorem 1**: For a MISO system employing a square QAM, a PAM, or a PSK signalling scheme of cardinality $\mu$ in the transmission, a linear receiver (ZF/MMSE) achieves full diversity for the system if $H H^H$ is non-singular for any nonzero $h$, or equivalently, if $X H(s) X(s)$ is non-singular for any nonzero $s$.

**Proof**: From Eqs. (18), (20), and (22), we can arrive at a generalized upper bound on the symbol error probability for the $\mu$-ary QAM, PAM, and PSK signals such that

$$P_i(h, s_\ell) \leq \frac{\mu - 1}{\mu} \exp\left(-\frac{a_i\rho_i}{\left[H H^H\right]_{\ell\ell} \sin^2 \theta}\right)$$

where $\rho_i = E_{s_i}/\sigma^2$, and

$$a_1 = 3/[4(\mu - 1)], a_2 = 3/[2(\mu^2 - 1)], \text{ and } a_3 = \sin^2(\pi/\mu)/2$$

Since $H H^H$ is non-singular for any nonzero $h$, we can apply Eq. (9) on Eq. (23). Here, we see that the arithmetic mean
of the SEP of all the three signalling schemes have a general upper bound given by

\[ P_i(h) \leq \frac{\mu - 1}{\mu} \exp\left(-a_i\rho_i C_0 \|h\|^2\right) \]

= \frac{\mu - 1}{\mu} \exp\left(-a_i\rho_i C_0 h^H h\right) \quad (25)

Now, \( h \) is assumed to be Gaussian with zero mean and covariance matrix \( \Sigma \). Therefore, averaging the exponential part of the right side of Eq. (25) over the density function of \( h \) yields

\[
\frac{1}{\pi^M \det \Sigma} \int \exp \left(-a_i\rho_i C_0 h^H h\right) \exp \left(-h^H \Sigma^{-1} h\right) dh \]

= \left[ \det \left((a_i\rho_i C_0 + \Sigma^{-1})^{-1}\right) \right] = \det(I + a_i\rho_i C_0 \Sigma)^{-1} \quad (26)

Substituting Eq. (26) into Eq. (25), we establish the following inequalities:

\[
E[P_i(h)] \leq \frac{\mu - 1}{\mu} \det(I + a_i\rho_i C_0 \Sigma)^{-1}
\]

\[
\leq \left(\frac{\mu - 1}{\mu} \det(C_0 \Sigma)^{-1} a_i^{-M}\right) \rho_i^{-M}, \quad i = 1, 2, 3 \quad (27)
\]

The exponent of \( \rho_i \) in Eq. (27) indicates that the upper bound of the SEP using a ZF receiver in a MISO system to detect signals from the three schemes indeed achieves full diversity for non-singular \( H^H H \).

We now show the equivalence of the following two statements:
1) \( H^H H \) is non-singular for any nonzero \( h \); and 2) \( X^H(s)X(s) \) is non-singular for any nonzero \( s \).

We will show 1) \( \Rightarrow \) 2), and the reverse can be similarly proved. From the development of Eq. (5), we have \( X(s)h = Hs \).

Now, if \( H^H H \) is non-singular for any nonzero \( h \), then \( H \) has full column rank, and hence \( Hs \neq 0 \) for all \( s \neq 0 \). Therefore \( X(s)h \neq 0 \) for any \( s \neq 0 \), \( h \neq 0 \). This implies full column rank of matrix \( X(s) \), and hence \( X^H(s)X(s) \) is non-singular for any nonzero \( s \). \( \square \)

**Remarks on Theorem 1:**

a) Although the proof provided here is for square QAM, PAM and PSK signalling schemes, Theorem 1 can be shown to be valid for any signal constellation.

b) Since the condition provided here is sufficient for a linear receiver to achieve full diversity, the same condition naturally yields full diversity for more sophisticated receivers such as MMSE/ZF-DFE or ML receivers.

c) For a MIMO system using an ML detector, the requirement for full diversity as indicated by Eq. (12) is that the coding matrix \( X^H(e)X(e) \) is maintained at full-rank for the signals \( s, s' \in S \). However, for a MISO system employing a linear receiver, Theorem 1 shows us that full diversity is achieved if the coding matrix \( X^H(s)X(s) \) is of full rank for any signal \( s \), a much stronger condition than that required by systems using an ML detector.

In the following section, we present the Toeplitz STBC which has a simple structure satisfying the full-rank condition in Theorem 1 and is therefore a full-diversity STBC for a MISO system employing a linear receiver.

**IV. TOEPLITZ SPACE-TIME BLOCK CODES AND THEIR PROPERTIES**

**A. Toeplitz STBC for a MISO System [41]**

To examine the structure of the Toeplitz space-time block code, we let \( \alpha = [\alpha_1, \alpha_2, \ldots, \alpha_L] \). A \((K + L - 1) \times K\) Toeplitz matrix generated by \( \alpha \) and a positive integer \( K \), denoted by \( T(\alpha, L, K) \), is defined as

\[
[T(\alpha, L, K)]_{ij} = \begin{cases} 
\alpha_{i-j+1}, & \text{if } i \geq j \text{ and } i-j < L \\
0, & \text{otherwise}
\end{cases}
\]

which can be explicitly written as

\[
T(\alpha, L, K) = \begin{pmatrix} 
\alpha_1 & 0 & \cdots & 0 \\
\alpha_2 & \alpha_1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & \alpha_L \\
\end{pmatrix}
\]

\[(K+L-1)\times K \quad (29)\]

If we replace \( \alpha \) by \( \beta \), the information symbols to be transmitted, then a Toeplitz STBC matrix \( X_B(s) \) is defined as

\[
X_B(s) = T(s, L, K) \cdot B
\]

where, for \( K \leq M \), \( B \) is a \( K \times M \) matrix of rank \( K \) placed in the coding matrix to facilitate the transmitter antennas with beamforming capability. At time slot \( n \), the \( n \)th row of the \( N \times M \) matrix \( X_B(s) \) is fed to the \( M \) transmitter antennas for transmission. Apply the Toeplitz space-time block coding matrix to the MISO system described in Eq. (3), and we have

\[
y = T(\tilde{h}, K, L) s + \xi
\]

(31)

where \( \tilde{h} = Bh \), and \( L = N - K + 1 \). Thus, \( T(\tilde{h}, K, L) \) can be viewed as the overall channel matrix of the MISO system.

**Example 2.** For \( K = M = L = 2, N = K + L - 1 = 3 \), and \( B = I_2 \), the codeword matrix and channel matrix are, respectively,

\[
X_B(s) = \begin{pmatrix} 
s_1 & 0 \\
s_2 & s_1 \\
0 & s_2 \\
\end{pmatrix}, \quad T(\tilde{h}, 2, 2) = \begin{pmatrix} 
h_1 & 0 \\
h_2 & h_1 \\
0 & h_2 \\
\end{pmatrix}
\]

For this code, there are \( L = 2 \) symbols to be transmitted in \( N = 3 \) channel uses. Therefore, the symbol transmission rate of this system is \( R_s = \frac{2}{3} \) symbols per channel use.

**Remarks on Toeplitz STBC:**

a) Eq. (31) is identical in form to that describing a MIMO intersymbol interference channel for zero-paddling block data transmission (e.g. [50]). It can thus be interpreted that the original MISO channel is transformed into a Toeplitz virtual MIMO channel. In other words, the space diversity has been exchanged for delay (time) diversity. This is realized by transforming the flat fading channel.
into a frequency selective channel with zero-padding. This technique is parallel to that employed in [40].

b) For such a system, we can utilize the efficient Viterbi algorithm [51] to detect the signal \( s \) if perfect channel knowledge is available at the receiver. On the other hand, when channel coefficients are not known at the receiver, we can make use of the second order statistics of the received signal to blindly identify the channel [50], [52].

c) Toeplitz STBC is a non-orthogonal STBC whose coding matrix \( XH \) possesses non-vanishing determinant for any signalling scheme. Hence according to Theorem 1 the code achieves full diversity even with the use of a linear receiver. On the other hand, since full diversity STBC designed for ML receivers (e.g., [34]–[36], [53]) maintain non-vanishing determinant only for certain types of signalling, full diversity gain is not guaranteed when a linear receiver is used.

d) When \( B = I \), Toeplitz STBC becomes a special delay diversity code (DDC) [37], [38] with padded zeroes. In general, DDC is applied with the use of outer channel coding and ML detectors to achieve the full diversity gain. However, here we show that the Toeplitz STBC possesses special properties which enable full space diversity to be achieved even with the use of the simplest linear receiver and the signals can be of any type.

B. Properties of Toeplitz STBC

We now examine some important properties of the Toeplitz space-time block codes introduced in the previous subsection. These properties will be useful in performance analysis and code designs in the ensuing sections.

Property 3: The definition of the Toeplitz space-time code shows that the symbol transmission rate is \( R_s = \frac{K}{N} = \frac{N-K+1}{N} \) symbols per channel use when \( K \leq M \). Therefore, for a fixed \( M \), the transmission rate \( R \) can approach unity if the number of channel uses is sufficiently large.

Property 4: For any nonzero vector \( \alpha \), there exists \( 0 < C_{T_{\min}} \leq C_{T_{\max}} \leq 1 \), and the matrix \( (TH(\alpha, L, K)T(\alpha, L, K)) \) satisfies the following inequality,

\[
C_{T_{\min}} ||\alpha||_{2K}^2 \leq \det (TH(\alpha, L, K)T(\alpha, L, K)) \leq C_{T_{\max}} ||\alpha||_{2K}^2.
\] (32)

Proof: By letting \( \alpha = h \) in Eq. (4) and choosing

\[
A_\ell = P^{\ell-1} A_0, \quad \ell = 1, \ldots, L
\]
where

\[
P = \begin{pmatrix}
0_{(L-1) \times 1} & 1 \\
I_{L-1} & 0_{1 \times (L-1)}
\end{pmatrix}, \quad A_0 = \begin{pmatrix}
I_L \\
0_{(K-1) \times L}
\end{pmatrix}
\]
we obtain an equivalent channel \( H \) of the same structure as \( TH(\alpha, L, K) \). Hence, \( T(\alpha, L, K) \) is a special case of \( H \). Thus, from Property 4 there exist \( C_{T_{\min}} \) and \( C_{T_{\max}} \) for which Eq. (32) holds. Now, we note that the diagonal entries of the matrix \( TH(\alpha, L, K)T(\alpha, L, K) \) are all the same and are equal

\[
[TH(\alpha, L, K)T(\alpha, L, K)]_{kk} = ||\alpha||_2, \quad k = 1, \ldots, K.
\]

By applying Hadamard’s inequality [44], we arrive at:

\[
\det (TH(\alpha, L, K)T(\alpha, L, K)) \leq ||\alpha||_{2K}^2 \\
\Rightarrow C_{T_{\max}} \leq 1
\] (33)

Furthermore, since \( \alpha \) is nonzero, we can assume, without loss of generality, that the first element \( \alpha_1 \neq 0 \). (Otherwise, we can always permute the nonzero element to the first position.) The \( N \times K \) “all” matrix, \( T(\alpha, L, K) \), can be partitioned into a top \( K \times K \) matrix \( \Omega_1 \), and a bottom matrix \( \Omega_2 \) containing the rest of \( T(\alpha, L, K) \), i.e.,

\[
T(\alpha, L, K) = \begin{pmatrix}
\alpha_1 & 0 & \cdots & 0 \\
\alpha_2 & \alpha_1 & \cdots & \vdots \\
& \ddots & \ddots & \ddots \\
\alpha_K & \alpha_{K-1} & \cdots & \alpha_1 \\
\alpha_{K+1} & \alpha_K & \cdots & \alpha_2 \\
& \vdots & \ddots & \ddots \\
\alpha_L & \alpha_{L-1} & \cdots & \alpha_{L-K-1} \\
0 & \alpha_L & \cdots & \alpha_{L-K} \\
& & \vdots & \ddots \\
& & & \alpha_L
\end{pmatrix}
\]

\( \Omega_1 \)

\( \Omega_2 \)

We note that \( \Omega_1 \) is a lower triangular matrix having equal diagonal elements \( \alpha_1 \neq 0 \). (Here, we assume that \( K \leq L \). The proof is equally valid if \( L \leq K \) by exchanging the roles of \( K \) and \( L \).) Since \( T(\alpha, L, K) = \Omega_1 H_1 + \Omega_2 H_2 \), using a standard result [44, p. 484] on the determinant of the sum of a positive definite and a positive semi-definite matrix we have

\[
\det (TH(\alpha, L, K)T(\alpha, L, K)) \geq \det(\Omega_1 H_1 + \Omega_2 H_2) \geq \det(\Omega_1 H_1) = |\alpha_1|_{2K}^2.
\] (34)

Since \( \alpha_1 \neq 0 \), \( \det (TH(\alpha, L, K)T(\alpha, L, K)) \) is nonzero for any nonzero \( \alpha \) and thus we have

\[
C_{T_{\min}} > 0
\] (35)

The conclusions in Eqs. (33) and (35) complete the proof of the property.

We see that \( T \) is non-singular by Property 4 then using the fact that \( T \) as a special case of \( H \) in Property 2 we have:

Property 5:

\[
[TH(\alpha, L, K)T(\alpha, L, K)]^{-1} \geq C_{T_{\min}} ||\alpha||^2
\]

for \( k = 1, 2, \ldots, K \) (36)

where the existence of \( C_{T_{\min}} > 0 \) is shown in Property 4 holds.

We now introduce the following definition related to the measure in a signal constellation \( S \):

Definition 1: For \( s, s' \in S \), the minimum distance of the signal constellation is defined as

\[
d_{\min}(S) = \min_{s \neq s'} \|s - s'\|
\] (37)
If \( \|s - s'\| = d_{\text{min}}(S) \), we say that \( s \) and \( s' \) are *neighbours*.

From the definition of the coding matrix \( \mathbf{X} \) in Eq. (30) with \( \mathbf{B} = \mathbf{I}_M \), we can now establish a lower bound for a metric between \( \mathbf{X}_{\Theta_M}(s) \) and \( \mathbf{X}_{\Theta_M}(s') \) in \( d_{\text{min}}(S) \). Let \( s = (s - s') \). For notational convenience, we let \( \mathbf{X}_{\Theta_M}(e, \{i_1, i_2, \ldots, i_m\}) \) denote the matrix consisting of \( m \) of the columns of \( \mathbf{X}_{\Theta_M}(e) \) indexed by \( \{i_1, i_2, \ldots, i_m\} \) where \( i_1 < i_2 < \cdots < i_m \) and these columns are not necessarily consecutively chosen. Then, we have

**Property 6:** For \( s \neq s' \in S^L \), where \( S^L = S \times S \times \cdots \times S \), and any nonzero vector \( e = (s - s') \), we have

\[
\det [\mathbf{X}_{\Theta_M}^H(e, \{i_1, i_2, \ldots, i_m\}) \mathbf{X}_{\Theta_M}(e, \{i_1, i_2, \ldots, i_m\})] \geq d_{\text{min}}^2(S)
\] (38)

for \( m = 0, 1, \ldots, M - 1 \), where the equality holds if and only if \( s \) and \( s' \) are neighbours, i.e., iff \( \|e\| = d_{\text{min}}(S) \).

**Proof:** The proof of this property is similar to that of Property 4. As in Property 4 without loss of generality, we can always assume that \( e_1 \neq 0 \) with \( e_1 \) being the first element of \( e \). Thus, \( \mathbf{X}_{\Theta_M}(e) \) can be written as

\[
\mathbf{X}_{\Theta_M}(e) = \begin{pmatrix}
e_1 & 0 & \ldots & 0 \\
e_2 & e_1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
e_M & \cdots & \cdots & e_1 \\
e_L & \cdots & \cdots & e_M \\
0 & \cdots & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & e_L \
\end{pmatrix}_{N \times M}
\] (39)

An important observation in Eq. (39) is that the top submatrix consisting of the first \( m \) rows of \( \mathbf{X}_{\Theta_M}(e) \) is a \( M \times M \) lower triangular matrix with nonzero diagonal entries and therefore, nonzero determinant. We can also see that the submatrix \( \mathbf{X}_{\Theta_M}(e, \{i_1, i_2, \ldots, i_m\}) \) preserves the same property because by permuting its rows and columns, an \( m \times m \) lower triangular matrix can always be formed as its top part, i.e., \( \mathbf{X}_{\Theta_M}(e, \{i_1, i_2, \ldots, i_m\}) \) can be expressed as

\[
\Pi_1 \mathbf{X}_{\Theta_M}(e, \{i_1, i_2, \ldots, i_m\}) \Pi_2 = \begin{pmatrix}
\mathbf{H}_1 \\
\mathbf{H}_2 
\end{pmatrix}
\]

where \( \Pi_1 \) and \( \Pi_2 \) denote the \( N \times N \) and \( m \times m \) permutation matrices, respectively. \( \Omega_1 \) contains the first \( m \) rows of \( \mathbf{X}_{\Theta_M}(e, \{i_1, i_2, \ldots, i_m\}) \) and hence is lower triangular, and \( \Omega_2 \) denotes the remaining submatrix of \( \mathbf{X}_{\Theta_M}(e, \{i_1, i_2, \ldots, i_m\}) \). Since the permutation of the rows and columns of \( \mathbf{X}_{\Theta_M}(e, \{i_1, i_2, \ldots, i_m\}) \) does not change the determinant of its autocorrelation matrix \( \mathbf{X}_{\Theta_M}^H(e, \{i_1, i_2, \ldots, i_m\}) \mathbf{X}_{\Theta_M}(e, \{i_1, i_2, \ldots, i_m\}) \), therefore, as in Property 4 we arrive at

\[
\det [\mathbf{X}_{\Theta_M}^H(e, \{i_1, i_2, \ldots, i_m\}) \mathbf{X}_{\Theta_M}(e, \{i_1, i_2, \ldots, i_m\})] \geq \det(\Omega_1^H \Omega_1) + \det(\Omega_2^H \Omega_2) \geq d_{\text{min}}^2(S)
\] (40)

where the equality holds iff \( \Omega_2 \) is a zero matrix, i.e., iff \( s \) and \( s' \) are neighbouring points.

We can establish another useful property on the metric between \( \mathbf{X}_{\Theta_M}(s) \) and \( \mathbf{X}_{\Theta_M}(s') \) by first recalling an important property in matrix algebra [54]:

The characteristic polynomial of an \( M \times M \) matrix \( \mathbf{A} \) is the polynomial whose roots are the eigenvalues of \( \mathbf{A} \). Mathematically, it can be re-written as

\[
h(\nu) = \det (\mathbf{I} + \nu \mathbf{A}) = \nu^M + c_1 \nu^{M-1} + \cdots + c_{M-1} \nu + c_M \]
(40)

such that

\[
c_m = \sum_{\varnothing} \det(\mathbf{A})_{i_1, \ldots, i_m}
\]
(41)

where \( \mathbf{A}_{i_1, \ldots, i_m} \) denotes the principal submatrix obtained by deleting the rows and columns of \( \mathbf{A} \) except the \( i_1 \)th, the \( i_2 \)th, \ldots, and the \( i_m \)th ones, and \( \varnothing \) denotes the combination set of \( i_1, \ldots, i_m \). We note, in particular, \( c_1 = \text{tr} (\mathbf{A}) \) and \( c_M = \det(\mathbf{A}) \). Now, the following property provides us with another metric on the matrix between \( \mathbf{X}_{\Theta_M}(s) \) and \( \mathbf{X}_{\Theta_M}(s') \) in relation to \( d_{\text{min}}(S) \).

**Property 7:** Let \( \Delta = \text{diag}(\delta_1, \delta_2, \ldots, \delta_M) \) with \( \delta_m > 0 \) for \( m = 1, 2, \ldots, M \). Then, for any nonzero vector \( e \), the following inequality holds

\[
\det (\Delta + \mathbf{X}_{\Theta_M}^H(e) \mathbf{X}_{\Theta_M}(e)) \geq \prod_{m=1}^M (\delta_m + d_{\text{min}}^2(S))
\]
(42)

with equality holding if and only if \( s \) and \( s' \) are neighbours.

**Proof:** Let us first rewrite the left side of Eq. (42) as

\[
\det (\Delta + \mathbf{X}_{\Theta_M}^H(e) \mathbf{X}_{\Theta_M}(e)) = 
\]

\[
\det(\Delta) \det (\mathbf{I} + (\mathbf{X}_{\Theta_M}(e) \Delta^{-1/2})^H \mathbf{X}_{\Theta_M}(e) \Delta^{-1/2})
\]
(43)

Now, let \( \mathbf{A} = (\mathbf{X}_{\Theta_M}(e) \Delta^{-1/2})^H \mathbf{X}_{\Theta_M}(e) \Delta^{-1/2} \), then Eqs. (40) and (41) becomes

\[
\det \left[ \mathbf{I} + (\mathbf{X}_{\Theta_M}(e) \Delta^{-1/2})^H \mathbf{X}_{\Theta_M}(e) \Delta^{-1/2} \right] = 1 + \sum_{m=1}^M c_m.
\]
(44)

where

\[
c_m = \sum_{\varnothing} \det [\mathbf{X}_{\Theta_M}^H(e, \{i_1, i_2, \ldots, i_m\})] \Pi_1 \mathbf{X}_{\Theta_M}(e, \{i_1, i_2, \ldots, i_m\}) \Pi_2
\]
(45)

Using Eq. (38) in Property 6 on the right side of Eq. (45), we have

\[
c_m \geq d_{\text{min}}^2(S) \sum_{\varnothing} \Pi_1 \mathbf{X}_{\Theta_M}(e, \{i_1, i_2, \ldots, i_m\}) \Pi_2
\]
(46)

where \( \sum_{\varnothing} \Pi_1 \mathbf{X}_{\Theta_M}(e, \{i_1, i_2, \ldots, i_m\}) \Pi_2 \) denotes the sum of the combination of the product of \( \delta_i \) taken \( m \) at a time. Equality in Eq. (46) holds if and only if \( s \) and \( s' \) are neighbours. Combining Eqs. (44) and (46) results in

\[
\det \left[ \mathbf{I} + (\mathbf{X}_{\Theta_M}(e) \Delta^{-1/2})^H \mathbf{X}_{\Theta_M}(e) \Delta^{-1/2} \right]
\]
The multiplexing gain for diversity gain and

\[ \delta_m^{-1} d_{\min}^2(\mathbf{S}) \]

where in the second step, we recognize that the right side of Eq. (47a) is the eigen polynomial of the matrix \( d_{\min}^2(\Delta) \), which, in turn, equals \( \text{det}(1 + d_{\min}^2(\Delta)) \) and hence Eq. (47b). Combining Eqs. (47a) and (49), we complete the proof of Property 7.

V. TOEPLITZ STBC APPLIED TO A MISO SYSTEM WITH A LINEAR RECEIVER

We now apply the Toeplitz STBC to the MISO communication system using the properties presented in Section IV-B. From Property 6 and 7 in Section IV-B and Theorem 1 in Section III-B, we can see that the Toeplitz STBC can approach \( \text{unit-rate} \) as well as \( \text{full diversity} \) even if only a linear ZF or linear MMSE receiver is used in a MISO system. In the following, we examine the optimal tradeoff between diversity gain and multiplexing gain [33] when the Toeplitz STBC is employed in a MISO system equipped with a linear receiver. We first make the assumption that the channel coefficients are independent, i.e., \( \Sigma = \mathbf{I} \).

Now, our MISO system has \( M \) transmitter antennas transmitting a signal vector \( \mathbf{s} \) of length \( L \) in \( N = L + M - 1 \) time slots. Also, all the above three signalling schemes have constellation cardinality \( \mu \). Thus, employing any of the three schemes described in Section III, in our MISO system will result in a transmission data rate \( r \) given by

\[ r = \frac{L}{N} \log_2 \mu \]  

(48)

Note that \( r \) is the bit rate of transmission. The multiplexing gain \( g \), on the other hand, is dependent on the scheme and in general, is defined as [33]

\[ g_i = \frac{r}{\log_2 \text{SNR}_i} \]  

(49)

where “SNR” refers to the general SNR in the received data. Here in our analysis, we use the SNR of the received data block for the SNR, and denote this by \( \rho_{\text{bl1}} \). \( i = 1, 2, 3 \) when the \( i \)th signalling scheme is employed. Notice that in the MISO system, we always have \( 0 \leq g_i \leq 1 \) \( \forall i \), since the system has only one receiver antenna. Hence, we can write \( \mu = \rho_{\text{bl1}}^{Ng/L} \).

Thus, if we want to maintain nonzero multiplexing gain at high SNR, then \( \mu \), the cardinality of the constellation must be large. It has been shown [33] that at high SNR, we can trade-off the multiplexing gain for diversity gain and vice versa, and the optimum trade-off for our MISO system with \( M \) transmitter antennas is given by

\[ D_{\text{op}} = M(1 - g) \]  

(50)

where \( D_{\text{op}} \) is the optimal diversity gain. Let us examine trade-off of the multiplexing gain for diversity gain in the three signalling schemes:

1. QAM signals: The averaged symbol energy \( E_s \) for square QAM signal is [55]

\[ E_{s1} = \frac{2}{3}(\mu - 1) \]  

(51)

We note that \( E_{s1} \) increases with the constellation cardinality \( \mu \). From Eq. (51), the averaged transmission energy per block can be calculated as \( E_{\text{bl1}} = \frac{2}{3}(\mu - 1)ML \). Given \( \sigma^2 \) being the noise variance at the receiver antenna, the averaged noise power per block is \( \sigma^2_{\text{bl1}} = \sigma^2N \). Therefore, the block SNR is

\[ \rho_{\text{bl1}} = \frac{2(\mu - 1)ML}{3N\sigma^2} \]

leading to

\[ \sigma^2 \approx \frac{2\mu ML}{3N\rho_{\text{bl1}}} = \frac{2ML}{3N\rho_{\text{bl1}}} \frac{N\rho_{\text{bl1}}}{N\rho_{\text{bl1}}^\Delta} \]  

(52)

where the approximation is under the assumption of large \( \mu \), and Eq. (49) has been used. Therefore, from Eqs. (13), (24), (51) and (52), we obtain

\[ a_1\rho_1 = \frac{3E_s}{4(\mu - 1)\sigma^2} = \frac{1}{2\sigma^2} = \frac{3N}{4ML} \rho_{\text{bl1}}^{1 - \frac{N\rho_{\text{bl1}}}{N\rho_{\text{bl1}}^\Delta}} \]  

(53)

Now, consider Eq. (27) on the upper bound of the SEP for a ZF receiver, i.e.,

\[ E[P_1(h)] \leq C_{T_{\text{min}}}(a_1\rho_1)^{-M} \]  

(54)

Substituting Eq. (53) into Eq. (54), we obtain

\[ E[P_1(h)] \leq C_{T_{\text{min}}}(3NML)^{-M} \rho_{\text{bl1}}^{\frac{N\rho_{\text{bl1}}}{N\rho_{\text{bl1}}^\Delta}} g^{-M} \]  

(55)

Hence, the diversity gain for this scheme, \( D_1 \), is given by

\[ D_1(g) = M \left( 1 - \frac{N}{L}g \right) \]  

(56)

where \( \varepsilon = \frac{M-1}{L} \geq 0 \). From Eq. (56), we can see that \( D_1(g) \leq D_{\text{op}}(g) \). However, we can make \( \varepsilon \) small by choosing \( L \) sufficiently large so that \( D_1(g) \approx D_{\text{op}}(g) \). Hence, \( D_1(g) \) is the \( \varepsilon \)-approximation of \( D_{\text{op}}(g) \). We can always choose \( L = \lceil \frac{N-1}{\mu - 1} \rceil + 1 \), where \( \lceil - \rceil \) denotes the integer part of a quantity, and therefore, we can say that the ZF receiver is able to approach the optimal diversity-multiplexing tradeoff if the proposed Toeplitz code is used with a square QAM signalling scheme of large cardinality. If a MMSE receiver is employed, utilizing an expression parallel to Eq. (55) for the MMSE receiver, we can also show that the optimal diversity-multiplexing tradeoff can be asymptotically achieved if the Toeplitz STBC is applied to the MISO system in which a square QAM signalling scheme of large cardinality is used for transmission.

2. PAM signals: We note that the averaged transmission energy \( E_s \) for \( \mu \)-ary PAM signal is given by [55] \( E_s = \frac{1}{4}(\mu^2 - 1) \). Hence the averaged transmission energy per block is \( E_{\text{bl2}} = \frac{1}{4}(\mu^2 - 1)ML \). Following similar arguments resulting in Eq. (52), for PAM signals we have,

\[ \sigma^2 \approx \frac{\mu^2 ML}{6N\rho_{\text{bl2}}} = \frac{ML}{6N\rho_{\text{bl2}}} \rho_{\text{bl2}}^{\frac{N\rho_{\text{bl2}}}{N\rho_{\text{bl2}}^\Delta}} \]  

Also, similarly to Eq. (53), we have

\[ a_2\rho_2 = \frac{3E_s}{2(\mu^2 - 1)\sigma^2} = \frac{1}{4\sigma^2} = \frac{3N}{2ML} \rho_{\text{bl2}}^{1 - \frac{N\rho_{\text{bl2}}}{N\rho_{\text{bl2}}^\Delta}} \]
Therefore, from Eq. (27), the upper bound on SEP for PAM signal is

\[
E[P_2(h)] \leq C_{T}^{-M} \left( \frac{a_2 \rho_2}{2N_{\min}} \right)^{-M}
\]

\[
= \left( \frac{3NC_{\min}}{2ML} \right)^{-M} \rho_{\text{bl}} 2 \frac{N_{\text{bl}}}{N_{\text{bl}}} g^{-M}
\]

Hence, the diversity order is

\[
D_2(g) = M \left( 1 - \frac{2N}{L} g \right) = M(1 - 2g) - 2\varepsilon M g
\]

\[
\leq M(1 - 2g) \leq M(1 - g) = D_{\text{opt}}(g)
\]

Equality in (57) holds iff \( g = 0 \). Therefore, for finite multiplexing gain, \( D_2(g) \) cannot approach the optimal tradeoff \( D_{\text{opt}}(g) \).

3. **PSK signals**: The averaged transmission energy \( E_s \) for \( \mu \)-ary PSK signal is \([55]\) \( E_s = 1 \). Hence the averaged transmission energy per block is \( E_{\text{abl}} = ML \). Therefore, we have

\[
\sigma^2 = \frac{ML}{N_{\text{bl}}}
\]

We also have

\[
\alpha_3 \rho_3 = \frac{E_s \sin^2(\pi/\mu)}{2 \sigma^2} \approx \frac{\pi^2}{2 \sigma^2 \mu^2} = \frac{N \pi^2}{2ML} \rho_{\text{bl}} 1 - 2\varepsilon \frac{N_{\text{bl}}}{N_{\text{bl}}}
\]

where, the second step comes from the assumption of large \( \mu \). Following similar arguments as PAM scheme, it can be shown that PSK signalling cannot achieve the optimal tradeoff of diversity-multiplexing gains.

VI. **OPTIMAL TOEPLITZ STBC DESIGN FOR MISO SYSTEM WITH ML DETECTOR**

The previous section shows what could be achieved when the Toeplitz STBC is applied to a MISO system equipped with a linear receiver. In this section, we will examine the application of the Toeplitz STBC to a MISO system equipped with a ML detector. In particular, we seek for the optimal design of the matrix \( B \) inherent in a Toeplitz space-time block code Eq. (30) such that the worst case pair-wise error probability is minimized when a maximum likelihood detector is employed.

Given a channel realization \( h \) and a transmission matrix \( B \), the probability \( P(s \rightarrow s'|h, B) \) of transmitting \( s \) and deciding in favor of \( s' \neq s \) with the ML detector is given by [56]

\[
P(s \rightarrow s'|h, B) = Q \left( \frac{d(s, s')}{2\sigma} \right)
\]

where \( d(s, s') \) is the Euclidean distance between the Toeplitz coded signals \( s \) and \( s' \) after being transmitted through the channel, i.e., it is the Euclidean distance between \( X_B(s)h \) and \( X_B(s')h \). Because of the relation of Eq. (30), we can write:

\[
d^2(s, s') = (s - s')^H T^H \left( h, M, N \right) T \left( h, M, N \right) (s - s')
\]

\[
= h^H X_B^H(e) X_B(e) h
\]

(60)

where \( e = s - s' \). By employing the alternative expression of the \( Q \)-function in Eq. (15) and taking the average of Eq. (59) over the Gaussian random vector \( h \), the average pair-wise error probability can be written as

\[
P(s \rightarrow s'|B) = \frac{1}{\pi} \int_0^{\pi/2} d\theta \det \left( I + (8\sigma^2 \sin^2 \theta)^{-1} \Sigma B^H(e) B(e) \right)
\]

(61)

with \( \Sigma \) being the covariance matrix of \( h \). Our design problem can now be stated as:

**Design Problem**: For a fixed number \( M \) of transmitter antennas, find a \( K \times M \) matrix \( B \) such that the worst-case average pair-wise error probability \( P(s \rightarrow s'|B) \) is minimized, subject to the transmission power constraint, \( \text{tr}(B^H B) \leq 1 \), i.e.,

\[
B_{\text{opt}} = \arg \min_{\text{tr}(B^H B) \leq 1} \max_{s, s' \in S^L} P(s \rightarrow s'|B)
\]

(62)

where \( S^L = S \times S \cdot \cdot \cdot \times S \) with \( S \) denoting the signal constellation of each element of \( s \).

To solve the above design problem, we not only have to find the optimum \( B \), but also have to determine its dimension \( K \). Let us first examine the \( M \times M \) covariance matrix, \( \Sigma = E[hh^H] \), of the transmission channels \( h \). Suppose we perform an eigen decomposition such that \( \Sigma = VAV^H \) where \( V \) is an \( M \times M \) unitary matrix and \( \Lambda = \text{diag}(\gamma_1, \gamma_2, \ldots, \gamma_K) \). \( K \leq M \), be the singular values of \( B \) and let \( G(\Lambda_K \Gamma, \varepsilon) \) denote the integral

\[
G(\Lambda_K \Gamma, \varepsilon) = \frac{1}{\pi} \int_0^{\pi/2} \prod_{k=1}^K \left( 1 + \frac{\varepsilon \lambda_k \gamma_k^2}{\sin^2 \theta} \right)^{-1} d\theta \quad \text{for} \quad \varepsilon > 0
\]

(63)

We obtain an optimal \( \Gamma \) by solving the following convex optimization problem

\[
\Gamma_{\text{opt}} = \arg \min_{\text{tr}(\Gamma) \leq 1} G \left( \Lambda_K \Gamma, \frac{d_{\text{min}}^2(S)}{8\sigma^2} \right)
\]

(64)

where \( \Gamma_{\text{opt}} \) is a \( K \times K \) diagonal matrix given by \( \Gamma_{\text{opt}} = \text{diag}(\gamma_{op1}, \gamma_{op2}, \ldots, \gamma_{opK}) \) with \( K \) being the highest integer for which \( [\Gamma_{op}]_{kk} = \gamma_{opk} > 0, k = 1, 2, \ldots, K \). Then the optimum transmission matrix is given by

\[
B_{\text{opt}} = \Gamma_{\text{opt}} V_K^H
\]

(65)

where \( V_K \) is the \( M \times K \) matrix containing the \( K \) eigenvectors corresponding to the \( K \) largest eigenvalues in the eigen decomposition of \( \Sigma \). Furthermore, the worst case pair-wise error probability is lower bounded by

\[
\max_{s, s' \in S^L} P(s \rightarrow s'|B) \geq G \left( \Lambda_K \Gamma_{\text{opt}}, \frac{d_{\text{min}}^2(S)}{8\sigma^2} \right)
\]

(66)

Note that the work presented here is different from that in [57] in which a precoder matrix is designed for a frequency-selective fading channel even though both involve Toeplitz structured matrices. Here, the Toeplitz matrix containing \( B \) and \( h \) is separated from the signal vector \( s \). This, by the properties of the Toeplitz STBC shown, transforms the design of \( B \) into a convex optimization problem. In [57], however, the design parameter and the signal vector are all parts of the Toeplitz structure resulting in a non-convex design problem that can only be solved by numerical method with no guarantee for global optimality.
Equality in Eq. (66) holds if and only if
i) \(|s - s'| = d_{\text{min}}(S)|

ii) \(B = B_{\text{op}}\).

Proof: We first establish a lower bound on the worst case average pair-wise error probability. Let \(s_{\ell}\) and \(s'_{\ell}\) be neighbour symbol vectors differing in only the \(\ell\)th symbol, i.e., \(s_{\ell} - s'_{\ell} = e_{\ell} = [0 \cdots 0 \; \epsilon_{\ell} \; 0 \cdots 0]^T\) where \(|\epsilon_{\ell}| = d_{\text{min}}(S)\). Then, we can write

\[
\det\left(I_M + \frac{1}{8\sigma^2 \sin^2 \theta} \sum \lambda^H_{B}(e_{\ell}) \lambda_B(e_{\ell}) \right) = \det\left(I_M + \frac{d_{\text{min}}^2(S)}{8\sigma^2 \sin^2 \theta} \Sigma B^H B \right) \\
\leq \prod_{k=1}^{M} \left(1 + \frac{d_{\text{min}}^2(S) \gamma_k^2 \lambda_k}{8\sigma^2 \sin^2 \theta} \right) (72)
\]

where the first step is a result of the structure of \(e_{\ell}\) on the Toeplitz code, and the second step is the result of an inequality for the determinant of a matrix. Equality in Eq. (67) holds if and only if \(B = B_o = \Gamma V H\), i.e., the singular vectors of \(B\) are the eigenvectors of \(\Sigma\). Substituting the inequality of (67) in Eq. (61), we have \(P(\delta \rightarrow s'/B) \geq G(\Lambda_K \Gamma, \frac{d_{\text{min}}^2(S)}{8\sigma^2})\). Since \(\max_{s \in S, s \neq s'} P(\delta \rightarrow s'/B) \geq P(\delta \rightarrow s'/B)\), the worst case average pair-wise error probability is lower bounded by

\[
\max_{s \neq s'} P(\delta \rightarrow s'/B) \geq G(\Lambda_K \Gamma, \frac{d_{\text{min}}^2(S)}{8\sigma^2}) (68)
\]

If we minimize both sides of Eq. (68), we can write

\[
\min_{B} \left(\max_{s \neq s'} P(\delta \rightarrow s'/B)\right) \geq G(\Lambda_K \Gamma, \frac{d_{\text{min}}^2(S)}{8\sigma^2}) (69)
\]

where \(\Gamma_{\text{op}}\) is obtained according to Eq. (64).

Let us now establish an upper bound for the worst case average pair-wise error probability for the specially structured transmission matrix \(B_o\) above. For any error vector \(e\), we have

\[
\det\left(I_M + \frac{1}{8\sigma^2 \sin^2 \theta} \Sigma \lambda_{B}(e) \lambda_B(e) \right) = \left(\frac{1}{8\sigma^2 \sin^2 \theta}\right)^M \det(\Lambda_K \Gamma^2) \det(\Delta + \lambda^H_{I_M}(e) \lambda_{I_M}(e)) (70)
\]

where the special structure of \(B_o\) has been utilized, \(\Lambda_K\) denotes the diagonal matrix containing the largest \(K\) positive eigenvalues of \(\Sigma\) and \(\Delta = (8\sigma^2 \sin^2 \theta) \Lambda_K^{-1} \Gamma^{-2}\). Using Eq. (70) in Property 2, for any nonzero vector \(e\) and nonzero \(\theta\) in the interval \([0, \pi/2]\), we have

\[
\det(\Delta + \lambda^H_{I_M}(e) \lambda_{I_M}(e)) \geq \prod_{k=1}^{M} \left(\frac{8\sin^2 \theta}{\gamma_k^2 \lambda_k} + \frac{d_{\text{min}}^2(S)}{\gamma_k^2 \lambda_k} \right) (71)
\]

where, according to Property 2, equality holds if and only if \(s\) and \(s'\) are neighbour vectors. Eq. (70) and Eq. (71) together yield

\[
\det\left(I_M + \frac{1}{8\sigma^2 \sin^2 \theta} \Sigma \lambda_{B}(e) \lambda_B(e) \right) \geq \prod_{k=1}^{M} \left(1 + \frac{d_{\text{min}}^2(S) \gamma_k^2 \lambda_k}{8\sigma^2 \sin^2 \theta} \right) (72)
\]

Again, substituting Eq. (72) in Eq. (61) and using the optimum \(\Gamma_{\text{op}}\) yields

\[
\max_{s \neq s'} P(s \rightarrow s'/B_{\text{op}}) \leq G(\Lambda_K \Gamma_{\text{op}}, \frac{d_{\text{min}}^2(S)}{8\sigma^2})
\]

where equality holds if and only if \(|s - s'| = d_{\text{min}}(S)|\). Thus, the proof of Theorem 2 is complete. 

Remarks on Theorem 2

a) Theorem 2 shows us that the lower bound of the worst case pair-wise error probability can be reached by having \(B = \Gamma_{\text{op}} V H\). Thus, the design problem in Eq. (62) becomes finding the optimum \(\Gamma_{\text{op}}\) of Eq. (64). 

b) The original non-convex optimization problem has been transformed into the convex problem in Eq. (64) and can be solved efficiently by interior point methods. The convexity of the objective function can be verified by re-writing \(G(\Lambda_K \Gamma, \varepsilon)\) in Eq. (65) as

\[
G(\Lambda_K \Gamma, \varepsilon) = \frac{1}{\pi} \int_0^{\pi/2} \exp \left(-\sum_{k=1}^{K} \ln \left(1 + \frac{\varepsilon \gamma_k^2 \lambda_k}{\sin^2 \theta}\right) \right) d\theta,
\]

\(\varepsilon > 0 (75)\)

We notice that \(-\ln(\cdot)\) is a convex function over \(\gamma_k^2 \lambda_k\) and hence their sum is also convex over \(\gamma_1^2, \cdots, \gamma_K^2\). Now, \(\exp(x)\) is monotonically increasing with \(x\). By composition rule [60] (Page 84), the integrand in Eq. (75) is a convex function implying that \(G(\Lambda_K \Gamma, \varepsilon)\) is convex.

c) The solution of Eq. (64) yields the values of the diagonal elements \({\gamma_{o1}, \gamma_{o2}, \cdots, \gamma_{oM}}\). Some of these values may not be positive. We choose all the \(K\) positive ones to form the singular values of \(B_{\text{op}}\).

Theorem 2 provides us with an efficient scheme to obtain the optimal matrix \(B_{\text{op}}\) by numerically minimizing \(G(\Lambda_K \Gamma, \varepsilon)\). However, if the Chernoff bound [46] of the pairwise error probability is employed as the objective function for minimization instead, a closed-form optimal \(B\) can be obtained. This can be shown by setting \(\sin^2 \theta = 1\) in the pairwise error probability of Eq. (61), so that we obtain the Chernoff bound as

\[
P(s \rightarrow s'/B) \leq \frac{1}{2\det \left(1 + (8\sigma^2)^{-1} \Sigma \lambda^H_{B}(e) \lambda_B(e) \right)} (76)
\]
Seeking to minimize the worst case Chernoff bound, and following similar arguments which establish the optimization problem in Eq. (64), we arrive at the following problem,

$$\hat{\Gamma}_{\text{op}} = \arg\min_{\mathbf{T} \in \mathbb{C}^{K \times K}} \frac{1}{2} \sum_{k=1}^{K} \left( 1 + \frac{d_{\text{min}}^2(S)}{8\sigma^2} \lambda_k \gamma_k \right)^{-1}$$

(77)

where $\hat{\Gamma}_{\text{op}}$ is a diagonal matrix with diagonal elements $\hat{\gamma}_{opk}$.

This problem is a relaxed form of that in Eq. (64) and its solution is provided by the following corollary:

**Corollary 1:** The solution, $\hat{\Gamma}_{\text{op}}$, for the optimization problem of Eq. (77) can be obtained by employing the water-filling strategy [59]. The diagonal elements of $\hat{\Gamma}_{\text{op}}$ are given by

$$\hat{\gamma}_{opk} = \sqrt{\frac{1}{M_0} \left( 1 + \frac{8\sigma^2}{d_{\text{min}}^2(S)} \sum_{\ell=1}^{M_0} \frac{1}{\lambda_\ell} \right) - \frac{1}{\lambda_m}}$$

(78)

where notation $[x]_+$ denotes $\max(x, 0)$. The optimal choice of $K$ is $K = M_0$, where $M_0$ is the maximum positive integer satisfying

$$\frac{1}{M_0} \left( 1 + \frac{8\sigma^2}{d_{\text{min}}^2(S)} \sum_{\ell=1}^{M_0} \frac{1}{\lambda_\ell} \right) - \frac{1}{\lambda_m} > 0, \quad m = 1, 2, \cdots, M_0$$

The optimum transmission matrix is thus $\hat{\mathbf{B}}_{\text{op}} = \hat{\Gamma}_{\text{op}} \mathbf{V}_K^H$.

**Proof:** The minimization of the type of problems of Eq. (77) has been studied by several researchers and the water-filling solution is shown in [59].

Remarks on Corollary 1:

a) For the particular case in which the channel coefficients are mutually independent, i.e., $\Sigma = \mathbf{I}_M$, then any $M \times M$ unitary matrix scaled by a factor $1/\sqrt{M}$ is a suitable choice for $\hat{\mathbf{B}}_{\text{op}}$.

b) The optimal design $\hat{\mathbf{B}}_{\text{op}}$ maximizes the coding gain [45] which is defined to be the normalized minimum determinant of $\mathbf{A}(\mathbf{e})^H \mathbf{A}(\mathbf{e})$ for all nonzero $\mathbf{e}$. The criterion of coding gain maximization is derived from minimizing the Chernoff bound on pairwise error probability for a system equipped with a ML detector. Since $\hat{\mathbf{B}}_{\text{op}}$ minimizes the worst case Chernoff bound on PEP, we conclude that it achieves optimal coding gain.

Remarks on the Optimum Transmission Matrix Design:

- The derivations of Theorem 2 and Corollary 1 are based on the consideration of using a ML receiver in the MISO system. For the case when the system is equipped with a linear ZF (or MMSE) receiver under the environment of correlated channels, the problem of obtaining an optimum $\mathbf{B}$ becomes very complicated. This is because we seek for a matrix $\mathbf{B}$ to minimize the respective average error probability obtained by averaging the expressions of error probability in Eqs. (14), (19) and (21) for the respective signalling schemes over the random channel matrix. This requires the knowledge of the PDF of the equivalent channel matrix. However, the equivalent channel matrix in these cases is of a Toeplitz structure for which the PDF is unavailable, and therefore, the expressions for the average error probability cannot be obtained. (For the case of linear MMSE receivers, a similar problem exists).

- An alternative way to attack the problem in the case when a linear receiver is employed is to consider the upper bound on the averaged error probability given in Eq. (27). We can minimize this upper bound with respect to $\mathbf{B}$. However, this necessitates the knowledge of the value of $C_0$. From Property 4 $C_0$ is the minimum value of the determinant of the Toeplitz matrix having its column vector belonging to unit ball and this renders the solving of $C_0$ difficult. Thus, in this paper, we are unable to come up with any true optimal $\mathbf{B}$ for MISO systems equipped with linear receivers.

VII. NUMERICAL EXPERIMENTS

In this section, we examine the performance of the Toeplitz code in a MISO system. We first evaluate the performance of the system equipped with a linear receiver under the condition that different parameters are varied. We then evaluate the performance of the system employing different beamformers as well as a linear or a ML receiver in an environment in which the channels are correlated. We note that for a linear receiver, the major computation occurs in the inversion of the Toeplitz matrix for which the complexity is of order $O(LM)$ [61], where $L$ is the length of signal vector and $M$ is the number of transmit antennas for the MISO system. On the other hand, the complexity using a ML detector for this MISO system transmitting the Toeplitz code is of order $\mu^M$ where $\mu$ is the constellation cardinality. Thus, for a reasonably large constellation and/or a comparatively large number of transmitter antennas, the ML receiver is substantially more complex than a linear receiver. Finally, we compare the performance of the Toeplitz code with some other known efficient codes.

**Example 1:** In this example, we examine the performance of Toeplitz STBC for a MISO communication system with independent channel fading, i.e., $\Sigma = \mathbf{I}$. The system is equipped with a linear ZF detector at the receiver end. For the Toeplitz STBC, we choose $\mathbf{B} = \mathbf{I}$ in Eq. (30). The following three experiments are performed:

1) We fix the number of transmitter antennas to be $M = 4$ and the length of signal vector $s$ to be $L = 8$, and the
symbol transmission data rate is therefore $R_s = L/N = 0.7273$ symbols per channel use. The symbols are randomly selected from the different constellations of 4-QAM, 16-QAM and 64-QAM. The symbols are transmitted through the MISO system having zero-mean unit-variance i.i.d. Gaussian channels and additive white Gaussian noise as described in Section II and the BER curves are plotted in Fig. 1 in which the three different curves correspond to the performance of the system with different number of transmitting antennas.

In this example, we test the performance of Toeplitz STBC for correlated channels in a MISO system equipped with four transmitter antennas in a linear array and the one receiver antenna on the normal to the axis of the transmitter antenna array (“broadside”). For small angle spread, the correlation coefficient between the $m_1$th and $m_2$th transmitting antennas is $\rho_{m_1 m_2}$. For the different choices of $L=8, 16, 32$ antennas with $M=4, 16$, and $8$ antennas with $L=8$ in the Toeplitz codes and the signals selected from a 16-QAM constellation.

**Example 2:** In this example, we test the performance of Toeplitz STBC for correlated channels in a MISO system equipped with four transmitter antennas in a linear array and the one receiver antenna on the normal to the axis of the transmitter antenna array (“broadside”). For small angle spread, the correlation coefficient between the $m_1$th and $m_2$th transmitting antennas is $\rho_{m_1 m_2}$. For the different choices of $L=8, 16, 32$ antennas with $M=4, 16$, and $8$ antennas with $L=8$ in the Toeplitz codes and the signals selected from a 16-QAM constellation.

Fig. 3. The average BER performance of the proposed Toeplitz STBC for the MISO system with different number of transmitter antennas.

2) In this experiment, we fix the signal constellation to be 16-QAM for a four transmitter antenna MISO system and perform simulations for different signal lengths, $L=4, 8, 16, 32$. For the different choices of $L$, the system has different transmission symbol rates, which are $R_s = 0.5714, 0.7273, 0.8421, 0.9143$ symbols per channel use. The channel and noise assumptions are the same as those in the previous experiment. The BER curves at different SNR are plotted in Fig. 2. It can be observed that the longer is the transmitted signal, the worse is the system BER performance. Again, this is due to the fact that larger $L$ corresponds to higher transmission data rate resulting in worse performance.

3) In this experiment, we vary the number of the transmitter antennas $M$. An increase in $M$ increases the diversity and decreases the transmission symbol rate. Therefore, it is expected that the performance of the system will be enhanced with the increase of the number of transmitter antennas. This is indeed the case as illustrated in Fig. 3 where we compare the performance of the MISO systems having $M=2, 4, 8$ antennas with $L=8$ in the Toeplitz codes and the signals selected from a 16-QAM constellation.

Fig. 2. The average BER performance of the proposed Toeplitz STBC for different $L$. 

Fig. 3. The average BER performance of the proposed Toeplitz STBC for the MISO system with different number of transmitter antennas. 

$[\Sigma]_{m_1 m_2} \approx \frac{1}{2\pi} \int_0^{2\pi} \exp \left(-j2\pi(m_1 - m_2)\Delta d_t \sin \theta\right) d\theta \tag{79}$

where $d_t$ is the antenna spacing, $\varsigma$ is the wavelength of the (narrowband) signal, and $\Delta$ is the angle spread. Here in our simulations, we choose $d_t = 0.5\varsigma$ and $\Delta = 5^\circ$. We examine the performance of the MISO system transmitting 4-QAM signals in the following three cases using Toeplitz STBC having the structures:

i) $\mathbf{B} = \frac{1}{\sqrt{M}} \mathbf{I}$. This is an approximately optimal transmission matrix at sufficiently high SNR in the minimization of the Chernoff bound under the assumption that $\mathbf{B}$ is a square matrix. The approximate optimality can be shown as follows: For Eq. (76), under high SNR, we ignore the identity matrix $\mathbf{I}$ in the denominator and obtain

$$P(s \rightarrow s'/\mathbf{X}) < \frac{(8\sigma^2)^M}{2 \det (\Sigma X_B^H(\mathbf{e})X_B(\mathbf{e}))} \left(\frac{8\sigma^2)^M}{2 \det(\Sigma) \det(\mathbf{B}^H \Sigma^{-1} \mathbf{B})}\right) \tag{80}$$

To minimize the right side of Eq. (80), we maximize the second determinant in the denominator. We note that

$$\det (\mathbf{B}^H \Sigma^{-1} \mathbf{B}) \leq \det (\Sigma) \prod_{m=1}^M [\mathbf{B}^H]_{mm}$$

$$\leq \det (\Sigma) \left(\frac{1}{M} \text{tr} (\mathbf{B}^H \mathbf{B})\right)$$

$$= \frac{1}{M^M} \det (\Sigma) \tag{81}$$
The first inequality is due to Hadamard Inequality [44] and equality holds iff $BB^H$ is diagonal. The second inequality is due to geometric mean being no larger than arithmetic mean, with equality holding iff $BB^H$ has equal diagonal elements. Finally, the trace of $BB^H$ is equal to unity due to the power constraint. Hence, the condition for maximum in Eq. (51) is that $B$ is a scaled unitary matrix of which $B = I$ is one choice:

ii) $B = B_{op}$. This is the optimal solution to minimize the worst case PEP derived in Theorem 2 and can be obtained numerically by solving the convex optimization problem of Eq. (64) and then using the result in Eq. (65).

iii) $B = \tilde{B}_{op}$. This minimizes Chernoff bound on PEP as described in Corollary 1.

In our simulations here, the transmitted signal vector is of length $L = 10$ and each of the symbols is randomly selected from the 4-QAM constellation. At the receiver, the signals in each of the three cases are detected separately by a linear ZF detector and a ML detector and the respective performances of the two detectors are examined. (As mentioned in the end of the last section, we cannot obtain an exact optimum $B$ for the ZF receiver. Nonetheless, we will employ the two optimum transmission matrices derived for the ML receiver $B_{op}$ and $\tilde{B}_{op}$ to the case of ZF receiver to see how the performance is improved). We note that due to the variation of the channel fading, the dimension $K$ of the optimum transmission matrices in Cases ii) and iii) change with SNR.

For the specific correlated channel described in Eq. (79), for Case ii), we found that $K = 1$ when $SNR \leq 8$ dB and $K = 2$ at higher SNR, whereas for Case iii), we found that $K = 1$ when $SNR \leq 10$ dB and $K = 2$ at higher SNR. Therefore, the transmission data rate for these two cases is $R_1 = \frac{K}{L+K-1}$ symbols per channel use, which is lower than $R_2 = \frac{M}{L+M-1}$ for the case of $B = I_M$. For a fair comparison, we choose two different structures for $B$ in Case i) in the following experiments:

1) We maintain the transmission data rate in Case i) the same as that in case iii). This is realized by setting $B = \left[ \frac{1}{\sqrt{K}} I_K . 0_{(K,M-K)} \right]$ in Case i) with $K$ being the dimension of $\tilde{B}_{op} B_{op}$. We evaluated the error performances of the systems equipped with different $B$ in all three cases and the results are shown in Fig. 4 from which the following observations can be made:

- For the system employing a ML detector, performance of Case ii) and iii) are superior to that of Case i), confirming the theoretical analyses in Theorem 2 and Corollary 1.
- For the system employing a ML detector, the BER performance for Cases ii) and iii) employing $B_{op}$ and $\tilde{B}_{op}$ respectively are very close. This shows that Chernoff bound is tight for this system. Close performance in the two cases is also true for the system using a ZF detector.
- Although $B_{op}$ and $\tilde{B}_{op}$ are optimal transmission matrices developed for the ML detector, they are equally effective in providing substantial performance improvement for the same system employing a linear ZF detector.

At lower SNR, we have $K = 1$, i.e., only one transmitter antenna is effective. Therefore, given a coded system, linear ZF and ML detectors provide the same performance.

2) In the second part of the experiment, we put $B = I_M$ in Case i) and examine its performance. The error performance for such a choice is shown in Fig. 5 from which the performance curves from Fig. 4 of Case iii) corresponding to the uses of $B_{op}$ as a transmission matrix. (Since the performance of Cases ii) is almost the same as that of Case iii), we have omitted here the performance curves corresponding to the use of $B_{op}$). It should be noted that when the signals are detected by a ML detector, the system coded with $B = I_M$ has higher transmission data rate than those in Cases ii) and iii). For the sake of comparison, we have re-plotted in Fig. 5 the performance curves of Case iii) of Fig. 4 of Case iii) corresponding to the uses of $B_{op}$ as a transmission matrix. (Since the performance of Cases ii) is almost the same as that of Case iii), we have omitted here the performance curves corresponding to the use of $B_{op}$). It should be noted that when the signals are detected by a ML detector, the system coded with $B = I_M$ has higher diversity gain over the system with $B_{op}$. This is due to the fact that water-filling strategy may not employ all the available transmitter antennas for correlated channels. Specifically in this example, the effective number of antennas for $B_{op}$ is $K \leq 2 < 4$. However, the optimal coding gain achieved by $B_{op}$ with ML detectors ensures a better performance. It is also im-

![Fig. 4. The average bit error rate comparison of the proposed Toeplitz STBC with i) $B = [I_K, 0_{M-K}]$, ii) $B_{op}$ and iii) $\tilde{B}_{op}$. The performances are shown for both linear ZF detectors and ML detectors.](image1)

![Fig. 5. The average bit error rate comparison of the proposed Toeplitz STBC with $B = I_M$ and $B_{op}$. The performances are shown for both linear ZF detectors and ML detectors.](image2)
important to note that the employments of $B = I_M$ and $\overline{B}_{op}$ result in a relatively large difference in performances, revealing that the upper bound on PEP given in Eq. (80) is not tight. Thus, even though this bound is quite commonly employed in STBC designs for independent channels, the results here show that this relaxed bound is a poor design criterion for an environment of highly correlated channel coefficients.

**Example 3:** In this example, we compare the BER performance of Toeplitz STBC with other STBC for independent MISO channels. Here again, we choose $B = I$ for Toeplitz STBC. The experiments are performed for the two cases in which the number of transmitter antennas in the communication system are $M = 4$ and $M = 8$ respectively:

1) $M = 4$ transmitter antennas and a single receiver antenna:

   We compare BER performance of Toeplitz STBC with other rate one STBC [29], [34], [36], [63]:
   - Quasi-orthogonal STBC. The code for four transmitter antennas was presented in [34], and the maximization of its coding gain was subsequently shown in [36].
   - Dense full-diversity STBC [29]
   - Multi-group decodable STBC [63]

   For the Toeplitz STBC, we choose $L = 9$ for which the symbol transmission rate is $R_s = L/N = 3/4$ symbols pcu. To achieve a fair comparison, the same transmission bit rate is imposed on all the codes such that signals are selected from 256-QAM constellation for Toeplitz STBC and from 64-QAM for the other full-rate STBC. Therefore, the same transmission bit rate, $R_b = 6$ bits pcu, is employed for all the systems. At the receiver, the Toeplitz STBC is processed by a linear ZF equalizer followed by a symbol-by-symbol detector. For the other full-rate STBC, we examine the two cases in which the signals are processed by a) a ML detector and b) a linear ZF receiver. The BER curves are plotted in Fig. 6. When a linear ZF equalizer and a symbol-by-symbol detector is applied at the receiver, it can be observed that Toeplitz STBC outperforms “quasi-orthogonal” STBC and “dense” STBC, and at higher SNR, its performance is superior to multi-group code. It is also interesting to observe that at higher SNR, for Toeplitz STBC with linear ZF receivers, the performance is also superior to that of the Multi-group STBC using a ML receiver. In fact, for the range of SNR tested, the slope of its BER curve is the same as those of the “dense” STBC and the “quasi-orthogonal” STBC processed by ML detectors, indicating they have the same diversity gain.

2) We now consider the system having $M = 8$ transmitter antennas. For the Toeplitz code, we choose $L = 35$ and therefore, the symbol transmission rate is $L/N = 5/6$ symbols pcu. We compare the bit error rate performance of our Toeplitz code with that of the orthogonal STBC having symbol transmission rate of:

   i) $1/2$ symbols pcu [2], [3], [7] and
   ii) $5/8$ symbols pcu [64] (this the highest symbol rate achievable by the orthogonal STBC applied to an eight transmitter antenna system).

   To achieve a fair comparison, the transmitted signals are selected from a 64-QAM constellation for our Toeplitz code, a 256-QAM constellation for the $5/8$ rate orthogonal code and a 1024-QAM constellation for the $1/2$ rate orthogonal code. Hence, all of the codes have the same transmission data rate in bits, i.e., $R_b = 5$ bits pcu. At the receiver end, the orthogonal STBC is decoded by a linear ZF detector for which, because of the orthogonality, the performance is the same as that of a ML detector. For Toeplitz STBC, the signals are decoded separately by a linear ZF receiver and a ZF-DFE receiver. The average bit error rate for these codes are plotted Fig. 7. It can be observed that the performance of the Toeplitz code detected with a linear ZF receiver is superior to that of the $1/2$-rate orthogonal STBC when the SNR is less than or equal to 25 dB. When the Toeplitz STBC is received by a ZF-DFE receiver, due to the higher coding gain, its performance is significantly better than that of the orthogonal STBC. In Fig. 7 at $10^{-5}$, the Toeplitz code with a ZF-DFE receiver outperforms the orthogonal code by about 4 dB.

   It should be noted that for the Toeplitz code, both lin-
When employed in a MISO system equipped with a ML detector, for both independent and correlated channel coefficients, we can design the transmission matrix inherent in the proposed Toeplitz STBC to minimize the exact worst case average pair-wise error probability resulting in full diversity and optimal coding gain being achieved. In particular, when the design criterion of the worst case average pair-wise error probability is approximated by the Chernoff bound, we obtain a closed-form optimal solution.

The use of the Toeplitz STBC (having an identity transmission matrix) in a MISO system fitted with a ZF receiver has been shown by simulations to have the same slope of the BER curves to other full rate STBC employing a ML detector, whereas even better performance can be achieved by using receivers (such as ZF-DFE) more sophisticated than the linear ones to detect the Toeplitz code. For correlated channels, employing the optimum transmission matrices in the Toeplitz code results in substantial additional improvements in performance to using the identity transmission matrix. This substantial improvement of performance is observed in either case for which a ML or a ZF receiver is used.

**REFERENCES**

[1] J.-C. Guey, M. P. Fitz, M. R. Bell, and W.-Y. Kuo, “Signal design for high data rate wireless communication systems over Rayleigh fading channels,” in *Proc. IEEE Vehicular Technology Conf.*, pp. 136–140, 1996.

[2] S. M. Alamouti, “A simple transmitter diversity scheme for wireless communications,” *IEEE J. Select. Areas Commun.*, vol. 16, pp. 1451–1458, Oct. 1998.

[3] V. Tarokh, H. Jafarkhani, and A. R. Calderbank, “Space-time block codes from orthogonal designs,” *IEEE Trans. Inform. Theory*, vol. 45, pp. 1456–1467, July, 1999.

[4] M. O. Damen, A. Tewfik, and J. C. Belfiore, “A construction of a space-time code based on number theory,” *IEEE Trans. Inform. Theory*, vol. 48, pp. 658–661, Mar. 2002.

[5] A. V. Geramita and J. Seberry, “Orthogonal design, quadratic forms and hadamard matrices,” in *Lecture Notes in Pure and Applied Mathematics*, (New York: Marcel Dekker INC), 1979.

[6] G. Ganesan and P. Stoica, “Space-time block codes: a maximum SNR approach,” *IEEE Trans. Inform. Theory*, vol. 47, pp. 1650–1656, May, 2001.

[7] O. Tirkkonen and A. Hottinen, “Square-matrix embeddable space-time codes for complex signal constellations,” *IEEE Trans. Inform. Theory*, vol. 48, pp. 1122–1126, Feb. 2002.

[8] X.-B. Liang, “Orthogonal designs with maximal rates,” *IEEE Trans. Inform. Theory*, vol. 49, pp. 2468–2503, Oct. 2003.

[9] X.-B. Liang and X.-G. Xia, “Nonexistence of rate one space-time blocks from generalized complex linear processing orthogonal designs for more than two transmit antennas,” in *International Symposium on Inform. Theory*, (Washington DC), June. 2001.

[10] W. Su and X.-G. Xia, “Two generalized complex orthogonal space-time block codes of rates 7/11 and 3/5 for 5 and 6 transmit antennas,” in *International Symposium on Inform. Theory*, (Washington DC), June. 2001.

[11] H. Wang and X.-G. Xia, “Upper bounds of rates of complex orthogonal space-time block codes,” in *International Symposium on Inform. Theory*, (Lausanne, Switzerland), June. 2002.

[12] S. Sandhu and A. J. Paulraj, “Space-time block coding: A capacity perspective,” *IEEE Commun. Letters*, vol. 4, pp. 384–386, Dec. 2000.

[13] B. Hassibi and B. M. Hochwald, “High-rate codes that are linear in space and time,” *IEEE Trans. Inform. Theory*, vol. 50, pp. 1804–1824, Jul. 2002.
R. W. Heath and A. J. Paulraj, “Linear dispersion codes for MIMO systems based on frame theory,” IEEE Trans Signal Processing, vol. 50, pp. 2429–2441, Oct. 2002.

H. E. Gamal and M. O. Damen, “Universal space-time coding,” IEEE Trans. Inf. Theory, vol. 49, pp. 1097–1119, May, 2003.

X.-Ma and G. B. Giannakis, “Full-diversity full rate complex-field space-time coding,” IEEE Trans. Signal Processing, vol. 51, pp. 2917–2930, Nov. 2003.

B. A. Sethuraman, B. S. Rajan, and V. Shashidhar, “Full-diversity, high rate space-time block codes from division algebras,” IEEE Trans. Inform. Theory, vol. 49, pp. 2596–2616, Oct. 2003.

J.-K. Zhang, K. M. Wong, and T. N. Davidson, “Information lossless full rate full diversity cyclicotomic linear dispersion codes,” in Int. Conf. Acoust., Speech, Signal Process., (Montreal, Canada), May 2003.

J.-K. Zhang, J. Liu, and K. M. Wong, “Trace-orthonormal full diversity cyclicotomic linear dispersion codes,” in Proceedings IEEE International Symposium on Information Theory, (Chicago), June 2004.

H. Yao and G. W. Wornell, “Achieving the full MIMO diversity-vs-multiplexing frontier with rotation-based space-time codes,” in 41th Annual Allerton Conf. on Comm., Control, and Comput., (Monticello, IL), Oct. 2003.

P. Dayal and M. K. Varanasi, “An optimal two transmit antenna space-time code and its stacked extensions,” in Asilomar Conf. on Signals, Systems and Computers, (Monterey, CA), Nov. 2003.

J. C. Belfiore and G. Rekaya, “Quaternionic lattices for space-time coding,” in Proceedings IEEE of ITW2003, (Paris, France), March 2003.

J. C. Belfiore, G. R. Rekaya, and E. Viterbo, “The Golden code: A 2 × 2 rate space-time code with non-vanishing determinants,” in Proceedings of the IEEE International Symposium on Information Theory, (Chicago), June 2004.

G. Rekaya, J. C. Belfiore, and E. Viterbo, “Algebraic 3 × 3, 4 × 4 and 6 × 6 space-time codes with non-vanishing determinants,” in ISITA, (Parma, Italy), Oct. 2004.

G. Wang and X.-G. Xia, “Optimal multi-layer cyclicotomic space-time code designs,” in Proceedings IEEE International Symposium on Information Theory, (Chicago), June 2004.

J.-K. Zhang, G.-Y. Wang, and K. M. Wong, “Optimal norm form integer space-time codes for two antenna MIMO systems,” in Int. Conf. Acoust., Speech, Signal Process., (Philadelphia, USA), March 2005.

G.-Y. Wang, J.-K. Zhang, Y. Zhang, and K. M. Wong, “Space-time code designs with non-vanishing determinants based on cyclic field extension families,” in Int. Conf. Acoust., Speech, Signal Process., (Philadelphia, USA), March 2005.

H. E. Gamal, G. Caire, and M. O. Damen, “Lattice coding and decoding achieve the optimal diversity–multiplexing tradeoff of MIMO channels,” IEEE Trans. Inform. Theory, vol. 50, pp. 968–985, June 2004.

J. Hiltunen, C. Hollanti, J. Lahtonen, “Dense full-diversity matrix lattices for four transmit antenna MISO channel,” in Proceedings IEEE International Symposium on Information Theory, (Adelaide, Australia), pp. 1290–1294, Sept. 2005.

T. Kiran and B. S. Rajan, “STBC-schemes with nonvanishing determinant for certain number of transmit antennas,” IEEE Trans. Inform. Theory, vol. 51, pp. 2984–2992, Aug. 2005.

P. Eliia, K. Kumar, S. Pawar, P. V. Kumar, and H. Lu, “Explicit space-time codes achieving the diversity-multiplexing gain tradeoff,” IEEE Trans. Inform. Theory, vol. 52, pp. 3869–3884, Sept. 2006.

F. Oggier, G. Rekaya, J.-C. Belfiore, and E. Viterbo, “Perfect space time block codes,” IEEE Trans. Inform. Theory, vol. 52, pp. 3885–3902, Sept. 2006.

L. Zheng and D. N. C. Tse, “Diversity and multiplexing: A fundamental tradeoff in multiple-antenna channels,” IEEE Trans. Inform. Theory, vol. 49, pp. 1073–1096, May 2003.

W. Su and X.-G. Xia, “Signal constellations for quasi-orthogonal space-time block codes with full diversity,” IEEE Trans. Inform. Theory, vol. 50, pp. 2331–2347, Oct. 2004.

N. Sharma and C. B. Papadias, “Full rate full-diversity quasi-orthogonal space-time code for any number of transmit antennas,” EURASIP J. Applied Signal Processing, pp. 1246–1256, 2004.

D. N. Dao and C. Tellambura, “Optimal rotations for quasi-orthogonal STBC with two-dimensional constellations,” in Proceedings IEEE Global Communications Conference, (St. Louis, U.S.A.), pp. 1290–1295, Nov. 2003.

D. Gore, S. Sandhu, and A. Paulraj, “Delay diversity code for frequency selective channels,” Electronics Letters, vol. 37, pp. 1230–1231, Sept. 2001.