Big Free Groups are Almost Free

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Abstract

It is shown that the big free group (the set of countably-long words over a countable alphabet) is almost free, in the sense that any function from the alphabet to a compact topological group factors through a homomorphism. This statement is in fact a simple corollary of the more general result proven below on the extendability of homomorphisms from subgroups (of a certain kind) of the big free group to a compact topological group.

It is an elementary fact that the free group over a set $A$, $F(A)$ can be defined in two equivalent ways:

Definition 1. $F(A)$ is the set of all finite, reduced words over the alphabet $A$.

Definition 2. $F(A)$ is the unique, up to an isomorphism, group such that any function $f : A \rightarrow G$, where $G$ is some group, factors through a homomorphism from $F(A)$ to $G$.

The first definition suggests a natural generalization of the concept of a free group: what happens if the finiteness requirement on the words is dropped? Indeed, the study of such generalizations can be traced as far back as [1]. Recently, such groups have been under intense study as it was realized that, in addition to their intrinsic interest, they play an important role in the study of the fundamental groups of spaces which are not semilocally simply-connected [2, 3]. Such groups also appear in the study of smooth loop groups [4], and thus are relevant for the theory of gauge connections on principal bundles.

Let us give now the precise definitions of the group of transfinite words. We follow [3] closely, to which the reader is referred to for more details if needed.

Definition 3. Let $A$ be the alphabet set and let $A^{-1}$ be the set of formal inverses of elements of $A$. A transfinite word is a map $w$ from a countable, linearly ordered set $S$ into $A \cup A^{-1}$ such that the preimage of any element of $A \cup A^{-1}$ is finite.

Intuitively, a transfinite word is a countable string of letters such that each letter appears at most finitely many times. Two words $w_1 : S_1 \rightarrow A \cup A^{-1}$ and $w_2 : S_2 \rightarrow A \cup A^{-1}$ are considered identical if there is a bijection $f : S_1 \rightarrow S_2$ such that $w_1 = w_2 \circ f$. In this paper we will only deal with words for which both $A$ and $S$ are countable.

Transfinite words can be multiplied in essentially the same way as the finite ones:
Definition 4. If \( w_1 : S_1 \to A \cup A^{-1} \) and \( w_2 : S_2 \to A \cup A^{-1} \) are two words, then \( w_1 w_2 : S_1 S_2 \to A \cup A^{-1} \) is the transfinite word which acts in the obvious way on the domain \( S_1 S_2 \) consisting of the disjoint union of the elements of \( S_1 \) with \( S_2 \) with all elements of \( S_1 \) preceding those of \( S_2 \).

Reduction is a little more involved to formulate in the transfinite case:

Definition 5. Denoting \( \{ s \in S : a \leq s \leq b \} \) by \( [a, b]_S \), we say that the word \( w : S \to A \cup A^{-1} \) admits a cancellation if there is a subset \( T \) of \( S \) and a mapping \( * : T \to T \) satisfying the following four conditions for all \( t \in T \):

- \( * \) is an involution.
- \( [t, t^*]_S = [t, t^*]_T \).
- \( [t, t^*]_T = ([t, t^*]_T)^* \).
- \( w(t^*) = w(t)^{-1} \).

Denoting \( S - T \) by \( S/\ast \) and \( w \) restricted to \( S/\ast \) by \( w/\ast \) we say that \( w/\ast \) arises by a cancellation from \( w \). If a word does not admit cancellations, it will be called reduced.

We shall consider all the words which are related to each other by a cancellation to be equivalent. It can be shown that any such equivalence class contains a unique reduced word and that the set of such equivalence classes becomes a group, which is called the big free group over \( A \), denoted by \( BF(A) \).

The big free group is known to be not free, and there is quite some work on its free subgroups \([2, 3, 5]\). In this paper, we show that the big free group is almost free in the sense that it almost satisfies definition \([2]\). We are going to show that any function from \( A \) to a compact topological group \( G \) can be factored through a homomorphism from \( BF(A) \) to \( G \), with the only difference being that this homomorphism is not unique. In fact, we are going to show more: Any homomorphism from a subgroup of \( BF(A) \), of a special kind, to a compact topological group \( G \) can be extended to a homomorphism from the whole of \( BF(A) \) to \( G \), which will make the factorizability of a function from \( A \) a simple special case. Let us define the special class of subgroups of \( BF(A) \) that we need:

Definition 6. A subgroup \( H \) of \( BF(A) \) is called tame if for any reduced \( w \in H \) we have that every subword of \( w \) is also in \( H \), where a subword is the restriction of the word \( w : S \to A \cup A^{-1} \) to a set of the form \( [a, b]_S \).

We can now state the result:

Theorem. Let \( H \) be a tame subgroup of \( BF(A) \) and \( f : H \to G \) a homomorphism where \( G \) is a compact topological group, then \( f \) extends to a homomorphism from \( BF(A) \) to \( G \).

Proof. Let \( I \) be an ordered set and let \( G \) be the set of all functions from \( I \) to \( G \), i.e. the set of all nets in \( G \) indexed by \( I \). Note that \( G \) is naturally a group under pointwise multiplication. Let \( G_0 \) be the subgroup of nets which are

\[1\]As is customary, we assume that \( G \) is Hausdorff.
eventually constant. There is an obvious homomorphism \( \pi : \mathcal{G}_0 \to G \), given by \( \pi(g_0) = \lim g_0 \). The crucial fact that we shall use repeatedly in what follows is that there is an extension of \( \pi \) to all of \( \mathcal{G} \). To see this, choose an ultrafilter on \( \mathcal{I} \) and let \( *G \) stand for the set of equivalence classes of \( G \)-valued nets where we consider two nets equivalent if they are identical on an element of the ultrafilter. Proceeding as is customary in nonstandard analysis [6], it is easy to see that \( *G \) is a group. Since it is compact, any element of \( *G \) is near standard, which allows us to define the standard part map from \( *G \to G \). This map is a homomorphism and is an extension of \( \pi \).

Using this, we can show that, without loss of generality, \( H \) can be taken to contain all the letters of the alphabet \( A \). For suppose that we have demonstrated the theorem in this case. If \( H \) now is a tame subgroup which contains only a subset \( A' \) of the letters, then we know that we can extend \( f \) to \( BF(A') \). We will be done if we can show that any homomorphism from \( BF(A') \) can be extended to \( BF(A) \). This is indeed the case. To see this, consider the set of all finite collections of words in \( BF(A') \). Order this set by inclusion making it into a directed set \( \mathcal{J} \). Suppose \( w \in BF(A) \). By deleting the letters not appearing in \( A' \), this word splits into a string of elements in \( BF(A') \). Let us denote this string by \( w' \). Now pick any element \( j \in \mathcal{J} \) and associate to it the group element \( f(w_1)f(w_2)\ldots \) where \( w_1, w_2, \ldots \) are the elements of \( j \) appearing in the string \( w' \). Note that since any given letter can appear only finitely many times, the product is a finite one. If no elements of \( j \) appear in the string \( w' \), associate to \( j \) the identity element. We thus get a \( G \)-valued net indexed by \( \mathcal{J} \) which we shall denote by \( \{w_j\}_{j \in \mathcal{J}} \). Note that if \( w \in BF(A) \) happens to be in \( BF(A') \), then this net is eventually constant and is equal to \( f(w) \). It is easy to check that if \( w_1 \) and \( w_2 \) are any two elements in \( BF(A) \) then eventually \( (w_1 \cdot w_2)_j = (w_1)_j(w_2)_j \). It follows at once that the map \( w \mapsto \{w_j\}_{j \in \mathcal{J}} \overset{\pi}{\to} G \) is a homomorphism which extends \( f \). In view of the above, we shall assume below that \( H \) includes all the letters of \( A \).

Let now \( \mathcal{I} \) be the following set

\[
\mathcal{I} = \left\{ \{l_n\}_{n=1}^\infty : l_{n+1} \leq l_n , \quad l_n > 0 , \quad \sum_{n=1}^\infty l_n < \infty \right\},
\]

i.e. the set of all monotone decreasing, strictly positive sequence whose sum is convergent. Order this set by stipulating that \( \{l_n\}_{n=1}^\infty \prec \{l'_n\}_{n=1}^\infty \iff l_n \leq l'_n , \forall n \). It is obvious that \( (\mathcal{I},\prec) \) becomes now a directed set.

Let \( w \in BF(A) \) be a reduced word. Denote by \( k_i \) the total number of times the letter \( a_i \) appears in \( w \), where we count both the letter and its inverse. Suppose that \( \iota \in \mathcal{I} \) is such that \( \sum_{n=1}^\infty k_n l_n = L_w < \infty \). We decompose \([0,L_w] \) into two sets, a countable collection of disjoint open intervals and its complement, where the intervals correspond to the letters of the word \( w \). This is done in the following way: For every letter in the word, associate an open interval of length \( l_n \), if the letter is \( a_n \) or \( a^{-1}_n \), whose starting point is equal to the sum of the lengths of the intervals corresponding to all the letters preceding the given letter. Thus, if for example our word is \( a_2a_1^{-1}a_2^{-1} \), we get the following intervals \( \{(0,l_2),(l_2,l_2+l_1),(l_2+l_1,l_2+2l_1),(l_2+2l_1,2l_2+2l_1)\} \). It is obvious in
this example and, as is easy to check, true generally, that any two such obtained intervals are disjoint, that they are all a subset of $[0, L_w]$ and that the complement of their union in $[0, L_w]$, which shall be denoted by $C$, is a closed set.

Let $x \in C$. Note that any such point naturally splits $w$ into the part ‘before $x$’ and the part ‘after $x$’. We shall say that $x$ is regular on the right/left if there is an initial/final segment of the word after/before $x$ which is contained in $H$. If a point is both regular on the left and on the right, then we shall simply say that it is regular. Points which are not regular will be called singular. It follows from the fact that $H$ is a tame subgroup that the set of singular points, denoted by $C'$, is closed.

Fix now $m \in \mathbb{N}$. Suppose $x$ is not regular on the right. Let $\alpha_x = \sup\{x \in C' \cap [x, x + \frac{1}{m}]\}$. If $\alpha_x = x$ or $\alpha_x = x + \frac{1}{m}$, then associate to $x$ the interval $[x, x + \frac{1}{m}] \cap [0, L_w]$. Alternatively, associate to $x$ the interval $[x, x + \alpha_x]$. Perform the procedure with obvious changes if $x$ is not regular on the left. Consider the union of all such intervals. First, it follows from the fact that $C'$ is closed that any connected component of this union is a closed interval. Second, we shall classify these connected components into two classes. The first class are those whose length is greater or equal to $\frac{1}{m}$, while the second one are those whose length is strictly less. The second possibility happens when the left endpoint is regular on the left while the right endpoint is regular on the right and there are no elements of $C'$ within $\frac{1}{m}$ of the endpoints outside the interval. It follows that the number of intervals of the first class is finite (as the sum of their lengths has to be finite) with the same being true of the intervals of the second class (as such intervals have to be at least $\frac{1}{m}$ apart from each other).

It could happen that an interval of the first class has its two endpoints not in $C$. In this case replace it by the smallest closed interval containing it whose endpoints are in $C$. Now, note that if we delete all subwords of $w$ whose letters are contained in these intervals, we will be left with a finite string of subwords $w_1 w_2 \ldots w_n$. We claim that each one of these subwords is in $H$. To see this, pick any letter in the subword $w_i$ and consider the set of all words in $H$ which are subwords of $w_i$ containing this particular letter. Every one of such subwords corresponds to a subinterval of the interval corresponding to $w_i$. Let $\alpha$ be the infimum of the left endpoints of these intervals. Similarly, let $\beta$ be the supremum of the right endpoints. Suppose $\alpha$ was not the left endpoint of the interval corresponding to $w_i$. It is easy to see that since $\alpha$ is regular on the right and $\beta$ is regular on the left then the subword corresponding to $[\alpha, \beta]$ is in fact in $H$. If this word was not in fact equal to $w_i$, i.e. if e.g. $\alpha$ was not the left endpoint of $w_i$, it would follow that $\alpha$ is regular and there would be a strictly longer subword than $[\alpha, \beta]$ which would still be in $H$. This would contradict the way $\alpha$ was defined.

To sum up the above, for any element $i \in I$ and any $m \in \mathbb{N}$ we have managed to associate to $w$ a finite string of words in $H$. Note that if the original word was in $H$, then this construction simply gives the word back. Keeping $i$ fixed for now, and using $f$, we can associate to $w$ the sequence of elements in $G$, $m \rightarrow g_m = f(w_1)f(w_2)\ldots f(w_n)$. It is not difficult, albeit a little tedious, to show by considering the different cases that if $w$ and $\tilde{w}$ are two words, such
that $L_w, L_{\bar{w}} < \infty$, whose corresponding sequences are $\{g_m\}_{m=1}^{\infty}$ and $\{h_m\}_{m=1}^{\infty}$, then the sequence that corresponds to $w \cdot \bar{w}$ is eventually equal to $\{g_m h_m\}_{m=1}^{\infty}$.

Using $\pi$ (nets are simply sequences here), we can thus map the sequence to a single group element $g_i$. Let $g_i$ be equal to the identity element of $G$ if $L_w = \infty$.

We thus get for any word a net of group elements $\{g_i\}_{i \in I}$ such that the net corresponding to the product of two words is equal eventually to the product of the two individual nets. Using $\pi$ again (the nets here are indexed by $I$ of course), we see that the map $w \to \{g_i\}_{i \in I} \mapsto G$ is the homomorphism extension that we seek.

We have the following immediate

**Corollary.** $BF(A)$ satisfies definition 2, if $G$ is a compact, topological group.

**Proof.** This follows at once from the fact that $F(A)$ is a tame subgroup of $BF(A)$.

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