Factorization of numbers with Gauss sums: II. Suggestions for implementation with chirped laser pulses

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Abstract. We propose three implementations of the Gauss sum factorization schemes discussed in part I of this series (Wölk et al 2011 New J. Phys. 13 103007): (i) a two-photon transition in a multi-level ladder system induced by a chirped laser pulse, (ii) a chirped one-photon transition in a two-level atom with a periodically modulated excited state and (iii) a linearly chirped one-photon transition driven by a sequence of ultrashort pulses. For each of these quantum systems, we show that the excitation probability amplitude is given by an appropriate Gauss sum. We provide rules on how to encode the number $N$ to be factored in our system and how to identify the factors of $N$ in the fluorescence signal of the excited state.

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1. Introduction

In [1], we have shown that Gauss sums are excellent tools to factor numbers. Throughout that paper we have concentrated on the general principles and the underlying mathematical foundations. However, we have not addressed physical implementations of the Gauss sums. The present article complements part I of this series [1], and proposes three distinct realizations of the Gauss sum factorization.

Several experiments [2–11] have already successfully demonstrated factorization with the help of Gauss sums. However, all of them have implemented the truncated Gauss sum

\[ A_N^{(M)}(\ell) \equiv \frac{1}{M+1} \sum_{m=0}^{M} \exp \left( -2\pi i m^2 \frac{N}{\ell} \right), \]  

but none of the Gauss sums introduced in [1]. Moreover, in all these experiments, except for the recent one using the Michelson interferometer, the ratio N/\ell had to be precalculated. In contrast, the three implementations of Gauss sum factorization proposed in the present paper encode the number N to be factored and the trial factor \ell in two independent variables. As a consequence, the ratio N/\ell need not be precalculated.

The first suggestion utilizes a two-photon transition in an equidistant ladder system driven by a chirped laser pulse. According to [12], in this system the excitation probability amplitude is given by the continuous Gauss sum

\[ S_N(\xi) \equiv \sum_{m=-M}^{M} w_m \exp \left[ 2\pi i \left( \pm m + \frac{m^2}{N} \right) \xi \right], \]  

which can factor numbers.

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In addition, we suggest two alternative approaches based on one-photon transitions in a laser-driven two-level system. Both rely on quantum interference of multiple optical excitation paths, but lead to different quadratic phase factors.

We consider a two-level system with a permanent dipole moment in the excited state. A cw-microwave field interacting with this dipole modulates the energy of the excited state and induces in this way an equidistant set of sidebands. The probability amplitude for a one-photon transition caused by a chirped laser pulse is the sum over all possible excitation channels involving one optical photon as well as multiple quanta of the microwave field. The Gaussian nature of this sum arises from the quadratic chirp of the laser pulse.

The second approach uses a multi-pulse excitation of a two-level system with a linearly chirped resonance frequency. The probability amplitude of excitation is a sum of contributions arising from each pulse. The quadratic phase dependence characteristic of a Gauss sum originates from the linear variation of the resonance frequency of the two-level system.

Throughout this paper, we neglect spontaneous emission since the interaction time with the pulses is much shorter than the decay time of the atomic level. This fact allows us to describe the system by the Schrödinger equation rather than a density matrix.

This paper is organized as follows. Since we rely heavily on the physics of chirped laser pulses, we first summarize in section 2 the basic elements of this branch of optics. In section 3, we then recall the main results of [12] and, in particular, the expression for the excitation probability amplitude corresponding to a chirped two-photon transition in a harmonic ladder system. Starting from this formula we demonstrate that this system allows us to factor numbers and that in principle a single realization of a factorization experiment can be employed to reveal the factors of another number.

Section 4 provides the basic elements of sections 5 and 6 by engineering a one-photon transition. As a second physical system implementing Gauss sums, we investigate in section 5 a two-level system with a permanent dipole moment driven by a microwave field. A chirped laser pulse interacts with the so-called engineered Floquet ladder. The resulting probability amplitude is again given by a Gauss sum. Section 6 is devoted to the analysis of the third system which features a linear variation of the resonance condition of a two-level atom exposed to a pulse train. This system provides an alternative way to factor numbers based on Gauss sums. After a comparison of both factorization schemes in section 7, we conclude in section 8 with a summary of our results and a brief outlook.

2. Chirped pulses: essentials

Throughout this paper we take advantage of the technology of chirped laser pulses. For this reason, we briefly summarize in the present section the key ideas and formulae of this field.

A chirped laser pulse

\[ E(t) \equiv E_0 \left[ e^{-i\omega_L t} f(t) + \text{c.c.} \right] \]

(3)

consists of an amplitude \( E_0 \), a carrier frequency \( \omega_L \) and a pulse shape function

\[ f(t) \equiv f_0 \exp \left[ -\frac{1}{2} (\Delta\omega f_0)^2 t^2 \right]. \]

(4)

Here, we have introduced the complex-valued amplitude

\[ f_0 \equiv \sqrt{\frac{1 + ia}{1 + a^2}}, \]

(5)
and the dimensionless parameter
\[ a \equiv \Delta \omega^2 \phi'' \] (6)
represents the second-order dispersion. Moreover, \( \Delta \omega \) denotes the bandwidth of the pulse and \( \phi'' \equiv d^2\phi(\omega)/d\omega^2 \) is a measure of the quadratic frequency dependence of the phase of the laser pulse.

When we substitute (5) into the exponent of the Gaussian in (4), we find that the pulse shape
\[ f(t) = f_0 \exp \left( -\frac{\Delta \omega^2}{2(1+a^2)} t^2 \right) \exp \left( -i \frac{a \Delta \omega^2}{2(1+a^2)} t^2 \right) \] (7)
of a chirped pulse consists of the product of a real-valued Gaussian and a phase factor whose phase is quadratic in time.

Since the instantaneous frequency
\[ \nu(t) \equiv \frac{d}{dt} \left( \frac{a \Delta \omega^2}{2(1+a^2)} t^2 \right) = \frac{a \Delta \omega^2}{1+a^2} t \] (8)
of the pulse is the derivative of this phase with respect to time, the frequency changes linearly in time as the pulse switches on and off. For a positive value of \( \phi'' \), we find an increasing frequency, whereas a negative \( \phi'' \) corresponds to a decreasing frequency.

3. Chirping a two-photon transition

In [12], we have considered a two-photon transition in the ladder system of figure 1 driven by a chirped laser pulse. In the weak field limit, the probability amplitude to be in the excited state results from the interference of multiple quantum paths each contributing a quadratic phase factor. Since the population in the excited state has the form of a continuous Gauss sum, we can use this observable to factor numbers as suggested [1] in part I of this series.

We first briefly summarize the essential results of [12]. Then, we turn to a demonstration of the factorization capability and bring out the physical origin of the scaling property derived in [1] from mathematical arguments.

3.1. Brief review of the model

We consider a quantum system with a ground state \( |g\rangle \) and an excited state \( |e\rangle \) separated by an energy \( 2\hbar \omega_0 \). In the neighborhood of the midpoint of the energy difference, we assume to have a manifold of equidistant energy levels as shown in figure 1. Their offset
\[ \delta_m \equiv \delta + m \Delta \] (9)
with respect to the central frequency \( \omega_0 \) is the sum of the off-set \( \delta \) to the central state and integer multiples \( m \) of the separation \( \Delta \) of two neighboring states of the manifold.

At this point, it is important to note that this decomposition of \( \delta_m \) is not unique. Indeed, we could have chosen a ‘central’ level that is different from the one indicated in figure 1. This choice would have changed the integers \( m \). This ambiguity in the labeling of the states is the deeper physical origin of the scaling property of the Gauss sum already mentioned in [1].
Next we drive this ladder system by a chirped laser pulse of the form given by (3). In second-order perturbation theory, the probability amplitude

\[ c_{e(\text{TPT})} = e^{i\gamma} S_N \]  

to be in the excited state after such a two-photon transition is given [12], apart from the phase factor \( \exp(i\gamma) \), by the Gauss sum

\[ S_N(\xi) = \sum_{m=-M'}^{M} w_m \exp \left[ 2\pi i \left( m + \frac{m^2}{N} \right) \xi \right] \]  

discussed in part I of this series. Here the variable

\[ \xi \equiv \frac{\delta \Delta}{\pi} \phi'' \]  

is expressed in terms of the parameters \( \delta \) and \( \Delta \) of the harmonic manifold of intermediate states and the dimensionless chirp \( \phi'' \). Hence, by varying the chirp we can tune \( \xi \). For this reason we call the variable \( \xi \) the dimensionless chirp.

The number

\[ N \equiv \frac{2\delta}{\Delta} \]  

to be factored is represented by the ratio of the two characteristic frequencies of the ladder.

The weight factors

\[ w_m \equiv \bar{\omega}_m \text{erfc} \left( i \frac{\delta_m}{\Delta \omega} \sqrt{1 - i a} \right) \exp \left[ - \left( \frac{\delta_m}{\Delta \omega} \right)^2 \right] \]  

"Figure 1. Model of the ladder system. The ground state \( |g\rangle \) is connected by a two-photon transition to the excited state \( |e\rangle \). We include a harmonic manifold of \( D \equiv M' + M + 1 \) intermediate states \( |m\rangle \) with \( M' \leq m \leq M \), which are shifted by the offset \( \delta_m \equiv \delta + m \Delta \) with respect to the central frequency \( \omega_0 \). The offset of the central state with \( m = 0 \) is \( \delta \), whereas neighboring states in the harmonic manifold are separated by \( \Delta \)."
contain the complementary error function \[ \text{erfc}(z) \equiv \frac{2}{\sqrt{\pi}} \int_{z}^{\infty} du \ e^{-u^2} \quad (15) \]
of the complex argument \( z \), and the abbreviation
\[ \tilde{\omega}_m \equiv -\frac{\pi \Omega_{em} \Omega_{mg}}{2 \Delta \omega^2} \quad (16) \]
involves the Rabi frequencies \( \Omega_{mg} \) and \( \Omega_{em} \) connecting the ground and the excited state with the intermediate states, respectively.

3.2. Factorization

We are now in a position to discuss our factorization scheme. For this purpose, we first address the experimental requirements and limitations and then present an example demonstrating the capability of this system to factor numbers.

3.2.1. Requirements. The probability amplitude \( c_e^{(TPT)} \) of populating the excited state given by (10) is of the form of a continuous Gauss sum discussed in part I of this series. Thus, the dependence of the population \( |c_e^{(TPT)}|^2 \) in the excited state on the dimensionless chirp \( \xi \) can be employed to reveal the factors of an integer \( N \). For this purpose, we encode \( N \) according to (13) in the parameters \( \delta \) and \( \Delta \) of the harmonic manifold of intermediate states. In order to apply the factorization scheme to a broad range of numbers \( N \), we require control over \( \delta \) and \( \Delta \).

We emphasize that the equidistant spacing within the harmonic manifold is essential for obtaining the Gauss sum and for our factorization scheme. Moreover, the dimension \( D \equiv M' + M + 1 \) of the intermediate levels has to be adapted to the number \( N \) to be factored; the larger \( N \) is, the greater the number of intermediate states that are required for a meaningful signal. For the factorization of \( N \), we require \( N \leq D \leq 2N \), where the lowest quantum number is bound by \( M' > N/2 \).

We can read out the population of the excited state by its fluorescence. In [1], we have formulated a rule for determining the factors of \( N \): if the signal shows a distinct maximum around the integer \( \xi = \ell \), then \( \ell \) is a factor of \( N \). In order to resolve the signal in the vicinity of candidate prime numbers, we require sufficient stability of and accuracy in the chirp \( \phi'' \).

According to [1], we also need to impose a restriction on the weight factor \( w_m \) (equation (14)) of the contribution arising from the quantum path through the \( m \)th intermediate state: indeed, \( w_m \) must be slowly varying as a function of \( m \) in order to ensure that no specific excitation path is favored or discriminated. As a result, the dipole matrix elements associated with all possible transitions should be of the same order of magnitude.

3.2.2. Example. Next we present numerical results for an artificial ladder system consisting of \( D \) intermediate states. Here, we have made the idealized assumption that the Rabi frequencies associated with the sequential path \( |g\rangle \rightarrow |m\rangle \rightarrow |e\rangle \) are identical, that is, \( \Omega_{em} \Omega_{mg} = \text{const} \).

In figure 2, we display the population \( |c_e^{(TPT)}|^2 \) of the excited state for the number \( N = 15 = 3 \times 5 \) as a function of the dimensionless chirp \( \xi \). For a single intermediate state, there exist [12] several interfering quantum paths only if \( \xi < 0 \). However, for an equidistant manifold, the population is symmetric with respect to \( \xi \), that is, \( |c_e^{(TPT)}(\xi)|^2 = |c_e^{(TPT)}(-\xi)|^2 \).

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Figure 2. Factorization of $N = 15 = 3 \times 5$ with the help of the population $|c_{\xi}^{(TPT)}|^2$ of the excited state given by (10) and (11) after a chirped two-photon transition through the intermediate levels of the ladder system of figure 1. In the center, we provide an overview of the complete signal as a function of the dimensionless chirp $\xi$. The insets magnify the signal in the vicinity of candidate prime factors. Pronounced maxima at the prime factors $\xi = 3$ and 5 are clearly visible. In contrast, at non-factors $\xi = 2$ and 7 the signal does not exhibit any peculiarities. Parameters are $\delta = 0.0225 \text{ fs}^{-1}$, $\Delta = 0.003 \text{ fs}^{-1}$, $\Delta w = 0.1525 \text{ fs}^{-1}$, $a = -10824$ and $D = 23$.

The insets magnify the signal in the vicinity of trial factors $\xi = 2, 3, 5$ and 7. Following our criterion of a dominant maximum at an integer indicating a factor, we clearly identify from the distinct maxima at $\xi = 3$ and 5 the factors of $N = 15$. In contrast, the signal at non-factors such as $\xi = 2$ or 7 does not show any characteristic features.

3.3. Factorization by rescaling

In part I of this series [1], we have already shown that the realization of the Gauss sum $S_N$ for $N$ is sufficient to reveal the factors of another number $N'$. Whereas the proof presented in [1] relied on mathematical arguments, we now use the freedom in labeling the levels of the equidistant ladder system to verify this surprising feature.

For this purpose, we recall from (13) that $N$ is encoded in the ratio $2\delta/\Delta$ described by the offset $\delta$ of the reference level $|m = 0\rangle$ and the level spacing $\Delta$. However, when we choose a different reference level $|k\rangle$ with the associated offset

$$\delta' \equiv \delta + k\Delta,$$

(17)
the number to be factored is
\[ N' \equiv \frac{2\delta'}{\Delta} = N + 2k. \] (18)

With the help of definition (13) of the number \( N \) to be factored in terms of the offset \( \delta \) and the detuning \( \Delta \), we find from (12) that the dimensionless chirp
\[ \xi = N \frac{\Delta^2}{2\pi} \phi'' \] (19)
is proportional to \( N \).

As a result, the dimensionless chirp
\[ \xi' \equiv N' \frac{\Delta^2}{2\pi} \phi'' \] (20)
corresponding to \( N' \) is related to \( \xi \) and \( N \) by the scaling transformation
\[ \xi' = \frac{N'}{N} \xi. \] (21)

Therefore, we can factor the number \( N' \) by analyzing the signal \( c_e^{(\text{TPT})} \), which was recorded in its dependence on \( \xi \) for \( N \), using the new scale \( \xi' \equiv (N'/N) \cdot \xi \), in complete agreement with [1].

4. Engineering a one-photon transition

In the preceding section, we have used a chirped two-photon transition going through an equidistant ladder system to factor numbers. Unfortunately, the requirements on the system are rather stringent and it is hard to identify quantum systems with such an arrangement of levels. For these reasons, we now study more elementary models based on a two-level system with a ground state \( |g\rangle \) and an excited state \( |e\rangle \) separated by an energy \( \hbar \omega_0 \).

We assume the excited state to have a permanent dipole moment \( \varphi_{ee} \) which interacts with a time-dependent modulating field \( E_m = E_m(t) \). In the following sections, we consider two cases: (i) a sinusoidal time dependence manifesting itself in a periodic modulation of the excited state and (ii) a quadratic chirp reflecting itself in a linear shift.

In addition to \( E_m \) we have a time-dependent weak driving field \( E_d = E_d(t) \) causing transitions between the ground and the excited state. Depending on the two cases, \( E_d \) is either a single chirped laser pulse or a sequence of pulses.

This arrangement corresponds to the interaction Hamiltonian
\[ V \equiv -\varphi_{ee} E_m(t) |e\rangle \langle e| - \left( \varphi_{ge} |e\rangle \langle g| + \text{c.c.} \right) E_d(t), \] (22)
where \( \varphi_{ge} \) denotes the dipole moment of the two-level transition. Here, we have assumed that the frequencies of \( E_m \) and \( E_d \) are clearly separated. Hence, \( E_m \) only acts on the excited state and \( E_d \) only on the transition.

In the interaction picture, the equations of motion for the probability amplitudes \( c_e = c_e(t) \) and \( c_g = c_g(t) \) to be in the excited and ground states read [14]
\[ i \frac{d}{dt} c_e(t) = -\Omega_{ee}(t) c_e(t) - \Omega_{ge}(t) e^{i\omega_0 t} c_g(t) \] (23)
and
\[ i \frac{d}{dt} c_g(t) = -\Omega_{eg}(t) e^{-i\omega_0 t} c_e(t). \] (24)
Here we have defined the time-dependent Rabi frequencies
\[ \Omega_{ee}(t) \equiv \frac{g_{ee} E_m(t)}{\hbar} \quad \text{and} \quad \Omega_{eg}(t) \equiv \frac{g_{eg} E_d(t)}{\hbar} \quad (25) \]
associated with the two electric fields \( E_m \) and \( E_d \), respectively.

To solve (23) and (24), we first recall that the strong field \( E_m \) that causes a modulation of the excited state appears through \( \Omega_{ee} \) in (23) and multiplies \( c_e \). Only the weak driving field \( E_d \) that enters (23) via \( \Omega_{ge} \) and multiplies \( c_g \) induces transitions. For this reason, it suffices to describe this process by perturbation theory of first order.

At time \( t_0 \) the two-level system occupies the ground level, that is, \( c_g(t_0) = 1 \) and \( c_e(t_0) = 0 \).

In the weak field limit, the probability amplitude for the excited state does not change significantly under the action of a weak chirped pulse that yields \( c_g(t) \approx 1 \). Hence, equation (23) reduces to the inhomogeneous differential equation
\[ i \frac{d}{dt} c_e(t) \approx -\Omega_{ee}(t) c_e(t) - \Omega_{ge}(t) e^{i\omega_0 t}, \quad (26) \]
where the interaction with the chirped laser pulse acts as an inhomogeneity.

It is easy to verify that the solution of (26) reads
\[ c_e(t) = i e^{i\beta(t)} \int_{t_0}^{t} dt' \exp[-i \beta(t')] \exp(i \omega_0 t') \Omega_{ge}(t'), \quad (27) \]
where we have introduced the phase
\[ \beta(t) \equiv \int_{t_0}^{t} dt' \Omega_{ee}(t') = \frac{g_{ee}}{\hbar} \int_{t_0}^{t} dt' E_m(t'). \quad (28) \]

When we substitute the electric field
\[ E_d(t) \equiv \mathcal{E}_d e^{-\i \omega_0 t} h(t) + \text{c.c.} \quad (29) \]
of amplitude \( \mathcal{E}_d \), carrier frequency \( \omega_0 \) and envelope \( h = h(t) \) into the solution (27) for the probability amplitude \( c_e \), we find in rotating wave approximation
\[ c_e(t) = i \Omega_{ge} e^{i\beta(t)} \int_{t_0}^{t} dt' \exp[-i \beta(t')] \exp(i \delta t') h(t'). \quad (30) \]
Here we have defined the time-independent Rabi frequency
\[ \Omega_{ge} \equiv \frac{g_{ge}}{\hbar} \mathcal{E}_d \quad (31) \]
associated with the electric field \( \mathcal{E}_d \) of the transfer pulse and the detuning \( \delta \equiv \omega_0 - \omega_0 \) between the atomic and the carrier frequency.

5. Floquet ladder

So far we have specified neither the modulating nor the driving field. In this section, we consider a sinusoidal modulation of the excited state by a strong cw field creating a set of equidistant sidebands, very much in the spirit of the harmonic manifold of figure 1. Moreover, we include a weak chirped laser pulse driving the transition; that is, the envelope \( h = h(t) \) of \( E_d \) is given by \( f(t) \) of (4). Figure 3 summarizes this engineering of the Floquet ladder.
Figure 3. Engineering the Floquet ladder. We consider a two-level system with the ground state $|g\rangle$ and the excited state $|e\rangle$ separated by the energy $\hbar \omega_0$. The excited state is modulated by a strong sinusoidal field $E_m = E_m(t)$ (equation (32)), giving rise to equidistant sidebands separated by $\hbar \Delta$. The one-photon transition is driven by a chirped laser pulse $E_d = E_d(t)$ (equation (29)), characterized by a linear variation of the instantaneous frequency $\omega = \omega(t)$.

5.1. Excitation probability in the weak field limit

We now evaluate the probability amplitude $c_e^{(FL)}$ given by (28) and (30) for the excitation of the Floquet ladder. Here we proceed in two steps: we first include the modulation and then calculate the remaining integral for the case of a chirped pulse.

5.1.1. Sinusoidal modulation. In the case of the modulation field

$$E_m(t) \equiv F_0 \cos(\Delta t + \varphi),$$

with period $2\pi/\Delta$, amplitude $F_0$ and phase $\varphi$, the time-dependent phase $\beta = \beta(t)$ defined by (28) takes the form

$$\beta(t) \equiv \kappa \sin(\Delta t + \varphi).$$

Here, we have chosen the lower integration limit $t_0 \equiv (n_0 \pi - \varphi)/\Delta$ and have introduced the dimensionless ratio

$$\kappa \equiv \frac{\Omega_{ee}}{\Delta}$$

of the time-independent Rabi frequency $\Omega_{ee} \equiv \varphi_{ee} F_0/\hbar$ and $\Delta$.

When we apply the generating function [13]

$$\exp(i \kappa \sin \theta) = \sum_{n=-\infty}^{\infty} J_n(\kappa) e^{in\theta}$$

of the Bessel function $J_n$ to evaluate the phase factor in the integrand of (30), we find the probability amplitude

$$c_e^{(FL)}(t) = i \hbar \Omega_{gs} e^{i \beta(t)} \sum_n J_n(\kappa) e^{-in\varphi} h_n(t)$$

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to be in the excited state. Here, we have interchanged the order of integration and summation
and have introduced the integral

$$h_n(t) \equiv \int_{t_0}^{t} dt' \exp(i \delta_n t') h(t')$$

(37)

with the offset

$$\delta_n = \delta - n \Delta$$

(38)
of the \(n\)th satellite of the excited state in the split manifold.

Hence, the modulation of the excited state causes equidistant sidebands and \(J_n(\kappa)\)
determines the weight of the \(n\)th sideband. In the language of Floquet theory, we have replaced
the time-dependent Hamiltonian by an infinite-dimensional Floquet matrix.

5.1.2. Chirped pulse. So far our calculation is valid for an arbitrary pulse shape \(h = h(t)\). We
now perform the integration for the case of a chirped pulse where the envelope \(h\) is given by the
complex-valued Gaussian \(f\) defined by (4).

Since we are interested in times after the pulse has interacted, that is, for \(\sqrt{1 + a^2/\Delta \omega} \ll t\),
we extend the upper and lower limits of the integration in (37) to \(+\infty\) and \(-\infty\), respectively,
which yields

$$h_n(t) \approx h_n \equiv f_0 \int_{-\infty}^{\infty} dt \ e^{-\frac{1}{2}(\Delta \omega f_0 t)^2 + i \delta_n t}$$

(39)
or

$$h_n = \frac{\sqrt{2\pi}}{\Delta \omega} \exp \left[ -\frac{1}{2} f_0^2 \left( \frac{\delta_n}{\Delta \omega} \right)^2 \right].$$

(40)

When we recall definition (5) of \(f_0\), we can decompose this Gaussian into a real-valued one and
into a quadratic phase factor, that is,

$$h_n = \frac{\sqrt{2\pi}}{\Delta \omega} \exp \left[ -\frac{1}{2} \left( \frac{\delta_n}{\Delta \omega} \right)^2 \right] \exp \left[ i \frac{\delta_n^2 \phi''}{2} \right].$$

(41)

Here, we have also made use of (6).

5.2. Emergence of the continuous Gauss sum

In the expression for \(h_n\) given by (41), the offset \(\delta_n\) that depends linearly on \(n\) enters
quadratically. Hence, \(h_n\) contains quadratic phase factors. In this section, we cast the probability
amplitude \(c_e^{(FL)}\) given by (36) into the form of a Gauss sum which allows us to factor numbers.

With the help of (41), we find from (36) the formula

$$c_e^{(FL)} = i \Omega g e^{i \beta(t)} \frac{\sqrt{2\pi}}{\Delta w} \sum_n \exp \left[ -\frac{1}{2} \left( \frac{\delta_n}{\Delta w} \right)^2 \right] J_n(\kappa) e^{-i n \phi} \exp \left[ i \frac{\delta_n^2 \phi''}{2} \right].$$

(42)

Next, we recall definition (38) of \(\delta_n\) and express it by

$$\delta_n = -\frac{n - \delta}{\Delta w} \Delta w / \Delta \equiv -\frac{n - N}{\Delta n}.$$

(43)
where in the last step we have introduced the abbreviations
\[ N \equiv \frac{\delta}{\Delta} \quad \text{and} \quad \Delta n \equiv \frac{\Delta w}{\Delta}. \] (44)

Likewise, we obtain from (38) the identity
\[ \delta^2_n = \delta^2 - \left( n - n^2 \frac{\Delta}{2\delta} \right) 2\delta \Delta. \] (45)

When we substitute (43) and (45) into (42), define the dimensionless chirp
\[ \xi \equiv \frac{\delta}{\pi \phi''} \] (46)
and recall definition (44) of \( N \), we arrive at the probability amplitude
\[ c_e^{(FL)}(t) = \mathcal{N}(t) \sum_n \tilde{w}_n \exp\left[ -i\pi \left( n - \frac{n^2}{2N} \right) \xi \right], \] (47)
to be in the excited state in the Floquet-ladder scheme. Here, we have used the abbreviation
\[ \mathcal{N}(t) = 2\pi i \frac{\Omega_{ee} e^{i\beta(t)}}{\Delta} e^{i\delta \phi'/2} \] (48)


together with the weight factor
\[ \tilde{w}_n \equiv \frac{1}{\sqrt{2\pi \Delta n}} \exp\left[ - \frac{1}{2} \left( n - \frac{N}{\Delta n} \right)^2 \right] J_n(\kappa)e^{-i\pi \phi}. \] (49)

In order to reduce the influence of the Bessel function \( J_n \) in \( \tilde{w}_n \), we adjust the modulation index
\[ \kappa \equiv \frac{\Omega_{ee}}{\Delta} = \frac{F_0 S_{ee}}{\hbar \Delta} \] (50)
determined by the microwave field, equation (32), such that \( \tilde{w}_n \) is slowly varying as a function of \( n \). For this purpose, we recall [13] the asymptotic expansion
\[ J_n(z) \approx \sqrt{\frac{2}{\pi z}} \cos \left( z - n \frac{\pi}{2} - \frac{\pi}{4} \right) \] (51)
of the Bessel function in the limit of large arguments \( n \ll z \). Indeed, we find for the choice
\[ \kappa \equiv 2\pi s + \frac{\pi}{4}, \] (52)
where \( s \) is a large integer, the approximation
\[ J_n(\kappa) \approx \sqrt{\frac{2}{\pi \kappa}} \left\{ \begin{array}{ll} (-1)^m & \text{for } n = 2m, \\ 0 & \text{for } n = 2m + 1. \end{array} \right. \] (53)

Thus, all weight factors \( \tilde{w}_n \) with odd index \( n \) vanish and the probability amplitude \( c_e^{(FL)} \) given by (47) reduces to
\[ c_e^{(FL)} \approx \mathcal{N} \sum_m w_m \exp\left[ -2\pi i \left( m - \frac{m^2}{N} \right) \xi \right]. \] (54)
where
\[ w_m = \frac{1}{\pi \Delta n} \frac{1}{\sqrt{\kappa}} \exp \left[ -2 \left( \frac{m - N/2}{\Delta n} \right)^2 \right] e^{im(\pi - 2\varphi)}. \] (55)

We note that for the choice of |\varphi| = \pi/2 the phase factor in \( w_m \) is unity and the only \( m \)-dependence left results from the Gaussian. For an appropriate choice of \( \Delta n \), which according to (44) is determined by the bandwidth \( \Delta \omega \) of the chirped pulse, this Gaussian is slowly varying.

When we compare the probability amplitude \( c_e^{(FL)} \) to be in the excited state given by (54) to the generic representation
\[ S(\xi; A, B) \equiv \sum_m w_m \exp \left[ 2\pi i \left( \frac{m}{A} + \frac{m^2}{B} \right) \xi \right] \] (56)
of the continuous Gauss sum discussed in part I of this series [1], we find that
\[ c_e^{(FL)} = N S(\xi; -1, N). \] (57)
Since the fluorescence signal of the excited state is proportional to the population \( |c_e^{(FL)}|^2 \) in this state, it is proportional to the Gauss sum \( |S(\xi; -1, N)|^2 \).

5.3. Factorization

In order to gain information on the factors of an appropriately encoded number \( N \), we now present two schemes: the first one requires a continuous measurement of the fluorescence signal as a function of the dimensionless chirp \( \xi \). For the second one, it suffices to acquire the fluorescence signal at integer values \( \xi = \ell \).

5.3.1. Continuous tuning of chirp. According to [1], \( \ell \) is a factor, or a multiple of a factor, of \( N \) if \( S(\xi; -1, N) \) given by (56) shows a pronounced maximum at \( \xi = \ell \). In figure 4, we present numerical results for the factorization of \( N = 21 = 3 \times 7 \) employing a truncated Floquet ladder covering \( 2M + 1 = 78 \) harmonics for two choices of the relative phase \( \varphi \). Here, we display \( |c_e^{(FL)}|^2 \) based on (47). Again we indicate candidate prime factors by vertical lines.

For \( \varphi = \pi/2 \), we clearly identify from the insets on the top the prime factors \( \ell = 3 \) and 7. In contrast, the signal does not show any peculiarities at non-factors such as \( \ell = 2 \) and 5 as demonstrated by the insets at the bottom.

For \( \varphi = 0 \), the phase factor \( e^{im\pi} = (-1)^m \) leads to oscillatory weight factors and our criterion of finding pronounced maxima at factors of \( N \) is not applicable here. Nevertheless, the signal \( |c_e^{(FL)}|^2 \) still contains information on the factors of \( N \). Indeed, the corresponding signal shown at the bottom of figure 4 vanishes at the factors \( \ell = 3 \) and 7 as indicated by the insets on the top, but displays no peculiarities at non-factors such as \( \ell = 2 \) and \( \ell = 5 \) depicted at the bottom.

5.3.2. Discrete values of chirp. Next we present another approach to factorization with the help of the Gauss sum, equation (56). For this technique, we assume that we have sufficient control over the dimensionless chirp \( \xi \) to tune it precisely to an integer \( \xi = \ell \). As a consequence, the
Figure 4. Factorization of $N = 21 = 3 \times 7$ with 78 satellites in the Floquet ladder of the excited state. Here, we show the fluorescence signal $|c_{(FL)}|^2$ (equation (47)) as a function of the continuous dimensionless chirp $\xi$ for the phase $|\varphi| = \pi/2$ (top) and $\varphi = 0$ (bottom) of the cw field. The electric field parameters are chosen to yield the width of the weight factor distribution $\Delta n = 12.71$ and the modulation index $\kappa = 100 \times 2\pi + \pi/4$. The positions of candidate prime factors are indicated by vertical lines. The insets demonstrate that the signal exhibits pronounced maxima at the prime factors $\ell = 3$ and 7 (top) but not at $\ell = 2$ and 5 (bottom). For the phase $\varphi = 0$, the factorization criterion of observing distinct maxima at factors of $N$ does not apply. Here, the factors are identified by zeros (top) rather than maxima. The non-factors have a non-vanishing signal (bottom).
Figure 5. Factorization of $N = 105 = 3 \times 5 \times 7$ in the Floquet-ladder approach using the signal $|c_e^{(FL)}|^2$ (equation (47)) for integer values $\xi = \ell$ of the dimensionless chirp. At the prime factors $\ell = 3, 5$ and 7 and products $\ell = 15, 21$ and 35, the signal displays maxima. At non-factors, the signal is suppressed. Data points corresponding to factors of $N$ are situated on a line through the origin. Integer multiples of a factor are characterized by the same value of the signal as illustrated by the two horizontal lines. Since $c_e^{(FL)}$ is only an approximation of $S_N(\ell)$, there are small deviations of this behavior. In order to satisfy the criterion of slowly varying weight factors, we have chosen the parameters $\Delta n = 90$ and $\kappa = 10^5 \times 2\pi + \pi/4$.

term linear in the summation index in the phase factor drops out and the probability amplitude $c_e^{(FL)}$, approximately given by (57), is proportional to the Gauss sum

$$S_N(\ell) \equiv \sum_m w_m \exp \left[ 2\pi i m^2 \frac{\ell}{N} \right].$$

Hence, we deal with a Gauss sum over purely quadratic phases$^6$.

In [1], we have analyzed the properties of the function $S_N = S_N(\ell)$. In particular, we have shown that $S_N(\ell)$ allows us to factor numbers in a rather straightforward way. Since $S_N(\ell)$ approximates the excitation probability amplitude $c_e^{(FL)}$ of a Floquet ladder given by (47), the occupation probability $|c_e^{(FL)}|^2$ at integer values $\ell$ of the dimensionless chirp $\xi$ should yield information about the factors of an appropriately encoded number $N$.

In figure 5, we verify this statement by presenting numerical results for the factorization of $N = 105 = 3 \times 5 \times 7$ based on (47). In contrast to the previous scheme, the signal $|c_e^{(FL)}|^2$ is depicted only for integer values of the dimensionless chirp $\xi = \ell$. Moreover, data points with $\ell$ being a factor of $N$ and their products arrange themselves on a straight line through the origin. Data points corresponding to integer multiples of a factor are characterized by identical values. On the other hand, the signal is suppressed at non-factors of $N$ in complete agreement with the predictions of [1].

$^6$ Another quantity where purely quadratic phase factors occur is the autocorrelation function of the two-dimensional quantum rotor. See, for example, [15].
6. Pulse train

In this section, we turn to yet another realization of a Gauss sum in a physical system. In contrast to the method of the preceding section, now the quadratic phase factors are not due to a chirped laser pulse, but arise from the combination of a linear time variation of the resonance condition and a pulse train as shown by figure 6. The probability amplitude $c_e^{(PT)}$ of excitation after a sequence of laser pulses follows from the sum over the contributions from the individual pulses and is of the form of a Gauss sum. Again the system is capable of factoring numbers. However, the roles of the trial factor and the number to be factored are interchanged.

6.1. Excitation probability in the weak field limit

We modulate the energy of the excited state by an electric field

$$E_m(t) = F_0 \frac{t}{T},$$

(59)

which increases linearly in time. Here $F_0$ denotes the amplitude of the field and $T$ is a timescale.

When we substitute this field into the definition (28) of the phase $\beta$, we find the expression

$$\beta(t) = \frac{1}{2} \frac{\Omega_{ee}}{T} t^2 - \beta_0 = \alpha(t) - \beta_0,$$

(60)

which contains the time-independent Rabi frequency $\Omega_{ee} = g_{ee} F_0 / \hbar$ and $\beta_0 \equiv \Omega_{ee} t_0^2 / (2T)$.

Moreover, we drive the one-photon transition with the electric field $E_d$, given by (29) and consisting of a train

$$h(t) = \frac{1}{2M + 1} \sum_{n=-M}^{M} \delta(t - nT)$$

(61)
of $2M + 1$ delta-shaped pulses separated by $T$. Here, we have chosen a normalization
\[
\int_{-\infty}^{\infty} dt \, h(t) = 1. \tag{62}
\]
The approximation of the pulse by a delta function reflects the fact that the temporal width of the individual pulses has to be small compared to $T$.

When we substitute the pulse train $h = h(t)$ (equation (61)) into (30) and perform the integration, we arrive at
\[
c_e^{(PT)}(t) = i \Omega_{ee} e^{i\omega(t)} \frac{1}{2M + 1} \sum_{n=-M}^{M} \exp \left[ i \left( \delta T n - \frac{\Omega_{ee} T n^2}{2} \right) \right]. \tag{63}
\]
Here, we have assumed that the range of integration in (30) is large enough to cover the whole pulse train.

Again the probability amplitude $c_e^{(PT)}$ of excitation involves the sum over quadratic phase factors and is therefore a Gauss sum.

6.2. Emergence of the reciprocate Gauss sum

Although in principle we could apply the same factorization scheme as in section 5, we propose here a more powerful technique. Indeed, by a proper choice of parameters, we eliminate in (63) the phase linear in $n$. For this purpose, we relate the detuning $\delta$ and the pulse separation $T$ to the number $N$ to be factored by
\[
N \equiv \frac{\delta T}{2\pi}. \tag{64}
\]
With this choice we find for the quadratic phase
\[
\frac{1}{2} \Omega_{ee} T n^2 = 2\pi n^2 \frac{N}{\xi}, \tag{65}
\]
where we have introduced the dimensionless variable
\[
\xi \equiv \frac{2\delta}{\Omega_{ee}}. \tag{66}
\]
As a consequence, the probability amplitude $c_e^{(PT)}$ for the pulse train given by (63) reduces to
\[
c_e^{(PT)}(t) = i \Omega_{ee} e^{i\omega(t)} A_N(\xi) \tag{67}
\]
and is governed by the reciprocate Gauss sum
\[
A_N(\xi) \equiv \frac{1}{2M + 1} \sum_{n=-M}^{M} \exp \left[ -2\pi i n^2 \frac{N}{\xi} \right]. \tag{68}
\]
discussed in part I of this series [1]. In contrast to the Gauss sums of the preceding sections, the roles of $N$ and $\xi$ are interchanged. Indeed, now the variable $\xi$ is in the denominator and the number $N$ to be factored appears in the numerator.
6.3. Factorization

In section 5.3, we have shown that the continuous Gauss sum arising in the excitation of the Floquet ladder reveals the factors of $N$ for a continuous tuning of the chirp $\xi$ as well as for integer values $\xi = \ell$. Likewise, the Gauss sum $A_N = A_N(\xi)$ defined by (68) provides us with the factors for continuous as well as integer values of $\xi$. However, the analysis for continuous $\xi$ is more complicated and has been presented in [16]. For this reason, we focus in the present section only on the discrete case.

Since the Rabi frequency $\Omega_{ee}$ is a free parameter, we can adjust $\xi$ to be an integer $\ell$. As a result we arrive at the sum

$$A_N(\ell) \equiv \frac{1}{2M+1} \sum_{n=-M}^{M} \exp \left[ -2\pi i \frac{n^2 N}{\ell} \right].$$  

(69)

We now demonstrate that $A_N$ is even more suited to factor numbers than the two Gauss sums $S$ or $S_N$ given by (56) and (58), respectively. Whenever the integer argument $\ell$ is a factor $q$ of $N$ the phase of each phase factor of $A_N$ is an integer multiple of $2\pi$. As a consequence, each term in the sum is unity. Since the sum contains $2M+1$ terms, the signal at a factor $q$ of $N$ takes on the maximum value of

$$A_N(q) = 1.$$  

(70)

In figure 7, we illustrate the power of this readout mechanism of factors using the example $N = 1911 = 3 \times 7^2 \times 13$. We find that already 21 pulses allow us to decide whether $\ell$ is a factor of $N$ or not.

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\footnote{In [1], we have shown that the Gauss reciprocity relation establishes the connection between the two types of Gauss sums $S_N$ and $A_N$ of (58) and (69), respectively.}
7. Comparison of factorization schemes

We devote this section to a brief comparison of the factorization schemes based on the Floquet ladder and the pulse train discussed in sections 5 and 6. Here, we first concentrate on the methods of readout and then briefly address experimental requirements and the necessary resources.

In the Floquet-ladder approach, two techniques to analyze the fluorescence signal determined by the population $|c_e^{(FL)}|^2$ in the excited state and given by (47) offer themselves: (i) we measure $|c_e^{(FL)}|^2$ as a function of the continuous chirp $\xi$. In this case pronounced maxima at trial factors indicate factors of $N$. (ii) An alternative readout relies on the measurement of the signal at integer values $\xi = \ell$. Here, we find that the signals at factors of $N$ form a straight line through the origin.

For the pulse-train approach, the proposed readout scheme is based on a measurement of the signal $|c_e^{(PT)}|^2$, equation (67), at integer values of the argument $\ell$. Factors of $N$ are characterized by the same maximal value, whereas the signal at non-factors is suppressed.

Next we address the experimental requirements for these schemes to work. To reveal the factors of a given number $N$, it is sufficient to analyze the fluorescence signal for values of the dimensionless chirp $\xi$ in the interval $[0, \sqrt{N}]$. For the continuous version, the resolution in $\xi$ has to be sufficiently high to resolve the shape of the signal in the vicinity of candidate primes. For the discrete scheme, the signal has to be acquired only for integer arguments $\ell$. Nevertheless, we require precise control of $\xi$. When we compare the number of measurements necessary in both schemes to obtain enough information for a decision on the factors, the discrete factorization schemes are favorable since fewer data points are required.

It is also interesting to compare the number of terms in the Gauss sums $S_N$ and $A_N$ necessary to find factors. In the approach based on the Floquet ladder, this number is determined by the width $\Delta n$ of the weight factor distribution $w_m$, equation (55). Indeed, this distribution has to be sufficiently broad in order to achieve a signal with an appropriate contrast. For the pulse-train approach, the number of terms contributing to $A_N$ is determined by the number of pulses in the train. Already with a few terms the signal has enough contrast to bring out the factors.

One may wonder whether the elimination of the phase linear in $n$ in the pulse-train approach is also possible in the Floquet-ladder system. The basic idea was to choose the number $N$ to be factored such that the term linear in the summation index drops out of the phase factor. Here, we had three parameters at our disposal: two are required to encode the number $N$ and one parameter is free to vary the argument $\ell$.

In the Floquet-ladder approach, we also have control over the three parameters $\delta$, $\Delta$ and $\phi''$ for encoding both the number to be factored $N$ and the dimensionless argument $\ell$. However, if all three were used to encode $N$, we would not have a parameter left for controlling $\ell$.

8. Conclusions

In this paper, we have proposed three physical systems to implement three types of Gauss sums. Our ultimate goal was to construct an analogue computer that would calculate these Gauss sums. We have then analyzed the signal to deduce from it the factors of an appropriately encoded integer $N$. 

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Our first system is based on a two-photon transition in a ladder system driven by a chirped laser pulse. Although this factorization scheme performs well for small numbers, its performance for larger numbers is questionable since the required dimension $D$ of the harmonic ladder needs to be of the order of $N$. Moreover, it is rather difficult to find such an equidistant ladder system in nature.

For this reason, we have investigated two other systems. In the approach of the Floquet ladder, a cw field modulates the excited state of a two-level atom giving rise to a manifold of equidistant sidebands. When driven by a chirped laser pulse, the resulting excitation probability amplitude is a Gauss sum. The second technique is based on a linear chirp of the excited state energy. A pulse train of delta-shaped pulses ensures that the excitation probability amplitude is of the form of a Gauss sum. The origin of the quadratic phase factors is different in these two realizations of Gauss sums. In the first one they are due to the chirped laser pulse, whereas in the ladder approach they originate from a linear chirp of the resonance condition.

In all three examples, the excited state probability is experimentally accessible via a detection of the fluorescence signal. Moreover, for each system, we have developed rules for determining the factors of an appropriately encoded number.

Our factorization scheme rests solely on interference, which implies that the required resources grow exponentially with the number of digits of $N$. This feature is in contrast to Shor’s algorithm that achieves an exponential speed-up due to entanglement. The next challenge is to combine these ideas with entanglement and create a Shor algorithm with Gauss sums. Indeed, there are already proposals [17, 18] on how to achieve this task. However, this task goes beyond the scope of the present paper and has to await a future publication.

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