RANK-TWO MILNOR IDEMPOTENTS FOR THE MULTIPULLBACK
QUANTUM COMPLEX PROJECTIVE PLANE.

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Abstract. The $K_0$-group of the C*-algebra of multipullback quantum complex projective plane is known to be $\mathbb{Z}^3$, with one generator given by the C*-algebra itself, one given by the section module of the noncommutative (dual) tautological line bundle, and one given by the Milnor module associated to a generator of the $K_1$-group of the C*-algebra of Calow-Matthes quantum 3-sphere. Herein we prove that these Milnor modules are isomorphic either to the section module of a noncommutative vector bundle associated to the $SU_q(2)$-prolongation of the Heegaard quantum 5-sphere $S^5_H$ viewed as a $U(1)$-quantum principal bundle, or to a complement of this module in the rank-four free module. Finally, we demonstrate that one of the above Milnor modules always splits into the direct sum of the rank-one free module and a rank-one non-free projective module that is not associated with $S^5_H$.

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1. Introduction

The even \(K\)-theory groups of complex projective spaces were computed by Atiyah and Todd to be \(K^0(\mathbb{C}P^n) = \mathbb{Z}[x]/(x^{n+1})\). Indeed, combining propositions 2.3, 3.1, and 3.3 in [2], and specializing stunted complex projective spaces to usual ones \((P_{n+1,n} = \mathbb{C}P^n)\) yields the result (cf. [1, Theorem 7.2]). Here \(x\) is explicitly given as the difference \([L_1] - [1]\) between the \(K_0\)-class of the dual tautological line bundle \(L_1\) and the \(K_0\)-class of the trivial line bundle. In their proof, arguments are based on the ring structure of \(K\)-theory, the multiplicative Chern character, integral cohomology rings, homotopy classes of classifying maps, and contracting projective hyperplanes to obtain even-dimensional spheres: \(\mathbb{C}P^n/\mathbb{C}P^{n-1} \cong S^{2n}\). None of these tools are available in noncommutative geometry.

However, since \(x := [L_1] - [1]\), the above generators \(1, x, \ldots, x^n\) of can be expressed in terms of line bundles \(L_k := L_{1,k}^\otimes, k \in \mathbb{Z}\), associated with the Hopf principal \(U(1)\)-fibration of \(S^{2n+1}\) over \(\mathbb{C}P^n\), which is a construction available also in the noncommutative setting. In particular, for \(n = 2\) these generators are built from the tautological line bundle \(L_{-1}\) and its dual \(L_1\) as follows:

\[
1 = [1], \quad x = [L_1] - [1], \quad x^2 = [L_1 \oplus L_{-1}] - 2[1].
\]

Furthermore, one can easily check that the direct sum of line bundles \(L_1 \oplus L_{-1}\) is isomorphic with the rank-two vector bundle \(E\) associated via the fundamental representation of \(SU(2)\) to the natural \(SU(2)\)-prolongation \(S^5 \times_{U(1)} SU(2) \to \mathbb{C}P^2\) of the Hopf principal bundle \(S^5 \to \mathbb{C}P^2\). It is precisely this point of view that allows us to see the third generator of \(K^0(\mathbb{C}P^2)\) as the value of the Milnor connecting homomorphism (in the sense of Higson, as explained in [15, Section 0.4]) applied to the \(K^1(S^3)\)-class of the fundamental representation of \(SU(2)\) viewed in a natural way as a continuous function from \(S^3\) to the matrix algebra \(M_2(\mathbb{C})\).

More precisely, the complex projective space \(\mathbb{C}P^n\) (thought of as a topological space) can be obtained by gluing a \(2n\)-dimensional cell along its boundary sphere \(S^{2n-1}\) to the closed tubular neighborhood of a complex projective hyperplane \(\mathbb{C}P^{n-1} \subset \mathbb{C}P^n\). Now it follows from the Mayer-Vietoris six-term exact sequence in \(K\)-theory that the \((n+1)\)-many generators of \(K^0(\mathbb{C}P^n) = \mathbb{Z}^{n+1}\) can be obtained from \(n\) generators of \(K^0(\mathbb{C}P^{n-1})\) and the value of the Milnor connecting homomorphism applied to a generator \(K^1(S^{2n-1})\). The thus obtained \((n+1)\)-generator vanishes when restricted to \(\mathbb{C}P^{n-1}\). What is special for \(n = 2\) is that the Milnor clutching construction takes place over \(S^3\), which is the topological space underlying \(SU(2)\). In this case, we can take the fundamental representation of \(SU(2)\) as a generator of \(K^1(S^3)\), and show that the resulting Milnor clutching yields the aforementioned associated vector bundle \(E\). Unlike tools used by Atiyah and Todd, this approach still makes sense in noncommutative topology, and we will pursue it in this paper.

Multipullback quantum complex projective spaces \(\mathbb{C}P^\mathbb{T}_n\) were constructed in [11] to provide a noncommutative-geometric model of a free distributive lattice. The basic building block of \(\mathbb{C}P^\mathbb{T}_n\) is the Toeplitz algebra \(\mathcal{T}\) viewed as the \(C^*\)-algebra of the quantum unit disk [13]. As a special \(n = 1\) case, the aforementioned construction includes the mirror quantum sphere [13]. The \(K\)-theory groups and a line-bundle generator of the even group of the multipullback quantum complex projective plane \(\mathbb{C}P^\mathbb{T}_2\) were determined in [22] [16].
Theorem 1.1. Let $C$ from $C$-group of $CP^2_\mathbb{T}$ projective plane tautological and dual tautological line bundles over the multipullback quantum complex projective plane $CP^2_\mathbb{T}$. Furthermore, let $U$ be an invertible matrix with entries in $C^*\text{-algebra}$ $C(S^3_H)$ of the Heegaard quantum 3-sphere such that $[U]$ generates $K_1(C(S^3_H)) = \mathbb{Z}$, and let $p_{U^\pm}$ be the associated Milnor idempotents over the $C^*$-algebra $C(CP^2_\mathbb{T})$. Then there exists a projection $p\in C(CP^2_\mathbb{T})$ such that $C(CP^2_\mathbb{T})p$ cannot be realized as a finitely generated projective module associated with the $U(1)$-$C^*$-algebra $C(S^3_H)$ of the Heegaard quantum 3-sphere, and such that one of the following splittings holds:

1. $C(CP^2_\mathbb{T})^2p_U \cong C(CP^2_\mathbb{T}) \oplus C(CP^2_\mathbb{T})p$
2. $C(CP^2_\mathbb{T})^2p_{U^{-1}} \cong C(CP^2_\mathbb{T}) \oplus C(CP^2_\mathbb{T})p$.

As was shown in [14], the $C^*$-algebra $C(CP^m_\mathbb{T})$ of the multipullback quantum complex projective space $CP^m_\mathbb{T}$ can be realized as a fixed-point subalgebra for the diagonal $U(1)$-action on the $C^*$-algebra $C(S^{2n+1}_H)$ of the Heegaard quantum sphere $S^{2n+1}_H$. Recall that, for any integer $n$ and for any unital $C^*$-algebra $A$ equipped with a $U(1)$-action $\alpha : U(1) \to \text{Aut}(A)$, we can define the $n$-th spectral subspace (the section module of the associated line bundle with winding number $n$) as

\[ A_k := \{ a \in A \mid \alpha_\lambda(a) = \lambda^k a \}. \]

In particular, we can take $A = C(S^{2n+1}_H)$, and denote the respective spectral subspaces by $L_k$.

While the first part of the theorem shows that the multipullback noncommutative deformation of the complex projective plane leaves its K-theory unchanged, its second part exhibits a purely quantum phenomenon, as there are no nontrivial projections in the commutative $C^*$-algebra $C(CP^2)$. We obtain the projection $p$ in the following way. First, it follows from the K-theory computation in [22] that $[L_1 \oplus L_{-1} - 2[1]] = \pm \partial_{10}([U])$, where $\partial_{10}$ is the Milnor connecting homomorphism. Next, since $C(S^3_H)$ is a pullback $C^*$-algebra over the $C^*$-algebra $C(\mathbb{T}^2)$ of the two-torus $\mathbb{T}^2$, it is easy to see that for $[U]$ we can take the value of the even-to-odd connecting homomorphism $\partial_{10}$ applied to the nontrivial generator $[\beta]$ of $K_0(C(\mathbb{T}^2)) = \mathbb{Z}[1] \oplus \mathbb{Z}[\beta]$. As in the calculation of a generator of $K_1(C(SU_q(2)))$ in [8], we take $\beta$ to be the Loring projection $[19]$. Thus we obtain

\[ [L_1 \oplus L_{-1} - 2[1]] = \pm (\partial_{10} \circ \partial_{10})([\beta]). \]

Homotoping the idempotent part of $(\partial_{10} \circ \partial_{10})([\beta])$ along the lines of homotopy arguments used in [8], we arrive at an elementary projection $p$ in the $C^*$-algebra $C(CP^2_\mathbb{T})$.

It is at this point that we make contact with recent work of Albert Sheu [23]. Therein, he uses the isomorphism $C(S^{2n+1}_H) \cong \mathcal{T}^{\otimes(n+1)}/\mathcal{K}^{\otimes(n+1)}$ found in [14] to express generators of $K_0(C(CP^2_\mathbb{T}))$ in terms of elementary (i.e. given by polynomials in the generating
isometry and its adjoint) projections in $T^\otimes(n+1)$. This can be understood as transforming topological information into purely operator-algebraic data by means of a quantum deformation. The exact relationship between our and the Sheu presentations of the generators of $K_0(C(CP^2_T))$ is established in [9]. Also in [9], the validity of the Atiyah-Todd presentation given in terms of polynomials in $[L_1]$ is verified.

The above described interplay between topology, operator algebras and group representations, all contributing to unravelling the K-theory of $CP^2_T$, can be imagined as follows:

To prove the first part our theorem, we begin by constructing an auxiliary pullback $U(1)$-$C^*$-algebra $P_5$ and showing that the even K-group of its fixed-point subalgebra is given by

\[ K_0(P_5^{U(1)}) = \mathbb{Z}[1] \oplus \mathbb{Z}([\tilde{L}_1] - [1]) \oplus \mathbb{Z}([\tilde{L}_1 + \tilde{L}_{-1}] - 2[1]). \]

Here $\tilde{L}_k$ is the section module of the associated line bundle given by the winding number $k$. The crux of the argument is the computation of the third generator as the value of the Milnor connecting homomorphism on the $K_1(C(SU_q(2)))$-class of the fundamental representation of $SU_q(2)$. The idempotent part of the Milnor connecting homomorphism turns out to be the section module $\tilde{E}$ of the rank-two noncommutative vector bundle associated via the fundamental representation of $SU_q(2)$ with a natural $SU_q(2)$-prolongation of $P_3$, and $\tilde{E}$ can be easily seen as isomorphic with $\tilde{L}_1 + \tilde{L}_{-1}$. Then we construct a $U(1)$-equivariant *-homomorphism from $P_5$ to $C(S^5_H)$, and prove that its restriction-corestriction $F$ to the fixed-point subalgebras induces an isomorphism on K-theory:

\[ F_* : K_0(P_5^{U(1)}) \xrightarrow{\cong} K_0(C(CP^2_T)). \]

Next, we take advantage of the “pushforwards commute with associations” theorem in [12] to conclude that $F_*([\tilde{L}_k]) = [L_1]$. Finally, since $F_*$ is an isomorphism, we infer that $[L_1 \oplus L_{-1}] - 2[1]$ is the third generator of $K_0(C(CP^2_T))$. As before, $L_1 \oplus L_{-1}$ can be easily identified with the section module $E$ of the rank-two noncommutative vector bundle associated via the fundamental representation of $SU_q(2)$ with a natural $SU_q(2)$-prolongation of $S^5_H$.

The paper is organized as follows. First, we recall basic facts and definitions necessary to formulate and prove our theorem. Next, we prove two general results allowing us to make the above described two steps in the proof of the theorem. Then we apply these general results in our concrete setting to conclude the first part of the theorem. We end the paper by proving the second part of the theorem using a homotopy argument.
Throughout the paper, unless indicated otherwise, the unadorned tensor product between C*-algebras stands for the minimal (spatial) complete tensor products, and we use the Heynemann-Sweedler notation (with the summation sign suppressed) for this completed product. Since all C*-algebras that we tensor are nuclear, this completion is unique.

2. Preliminaries

2.1. Pullback diagrams. The name “pullback” is used to denote different things in distinct contexts in mathematics. To avoid confusion, in this paper we use it solely to mean the limit of diagrams of the form \( A \xleftarrow{\pi_1} A_{12} \xrightarrow{\pi_2} A_2 \). These diagrams can be taken in whatever category we need, for instance in the category of unital C*-algebras or in the category of modules. A pullback of a diagram as above is a pair of morphisms \( A \leftarrow p_1 A \rightarrow p_2 A_2 \) such that \( \pi_1 \circ p_1 = \pi_2 \circ p_2 \) and which is universal among such pairs, i.e., for any other pair of maps \( A \xrightarrow{k_1} Z \xleftarrow{k_2} A_2 \) such that \( \pi_1 \circ k_1 = \pi_2 \circ k_2 \), there exists a unique map \( \langle k_1, k_2 \rangle : Z \rightarrow A \) such that \( p_1 \circ \langle k_1, k_2 \rangle = k_1, i = 1, 2 \).

In what follows, we use multiple times the following well-known result, whose proof we include for completeness.

**Theorem 2.1.** Let \( C \) be a category. Assume that we have two pullbacks in \( C \):

- a pullback \( A \xleftarrow{p_1} A \rightarrow p_2 A_2 \) of \( A \xrightarrow{\pi_1} A_{12} \xrightarrow{\pi_2} A_2 \), with \( \pi_2 \) surjective, and
- a pullback \( B \xleftarrow{q_1} B \rightarrow q_2 B_2 \) of \( B \xrightarrow{\rho_1} B_{12} \xrightarrow{\rho_2} B_2 \), with \( \rho_2 \) surjective.

Furthermore, suppose that we have morphisms \( \phi_1 : A_1 \rightarrow B_1 \), \( \phi_2 : A_1 \rightarrow B_2 \) and \( \phi_{12} : A_{12} \rightarrow B_{12} \) such that the solid-line diagram below commutes:

\[
\begin{array}{c}
A \xrightarrow{\phi} B \\
\xleftarrow{p_1} & \xrightarrow{p_2} & \xleftarrow{q_1} & \xrightarrow{q_2} \\
\xleftarrow{\pi_1} & \xrightarrow{\pi_2} & \xleftarrow{\phi_1} & \xrightarrow{\phi_2} \\
\xleftarrow{\phi_{12}} & \xrightarrow{\phi_{12}} \\
A_{12} & B_{12}
\end{array}
\]

Then there exists a unique map \( \tilde{\phi} : A \rightarrow B \) which makes Diagram (2.1) commutative. We usually write the map \( \tilde{\phi} \) as \( A \ni (a_1, a_2) \mapsto (\phi_1(a_1), \phi_2(a_2)) \in B \).

**Proof.** First note that

\[
\rho_1 \circ \phi_1 \circ p_1 = \phi_{12} \circ \pi_1 \circ p_1 = \phi_{12} \circ \pi_2 \circ p_2 = \rho_2 \circ \phi_2 \circ p_2.
\]

Then, by the universal property of the pullback \( B \xleftarrow{q_1} B \rightarrow q_2 B_2 \) of \( B \xrightarrow{\rho_1} B_{12} \xrightarrow{\rho_2} B_2 \), there exist a unique map \( \tilde{\phi} : A \rightarrow B \) such that

\[
q_i \circ \tilde{\phi} = \phi_i \circ p_i, \quad i = 1, 2,
\]

which proves the existence of \( \tilde{\phi} \). \( \Box \)
2.2. Definitions of quantum spaces.

2.2.1. The quantum disk $D_q$. We identify the C*-algebra $C(D_q)$ (see [18]) with the Toeplitz algebra $\mathcal{T}$. The Toeplitz algebra can be viewed as the universal C*-algebra generated by one isometry $S$. There is a natural $U(1)$-action $\alpha$ on $\mathcal{T}$ given by rephasing the isometry $S$: $\alpha(\lambda)(S) := \lambda S$. Viewing the C*-algebra $C(S^1)$ of continuous complex-valued functions on the unit circle as the universal C*-algebra generated by one unitary $u$, and mapping $S$ to $u$, we obtain the following short exact sequence of $U(1)$-C*-algebras:

$$0 \rightarrow K \rightarrow \mathcal{T} \xrightarrow{\sigma} C(S^1) \rightarrow 0.$$  

Here $K$ stands for the ideal of compact operators, and $\sigma$ is called the symbol map. The $U(1)$-action on $C(S^1)$ is the obvious one.

2.2.2. The standard Podleś quantum sphere $S_q^2$. The C*-algebra $C(S_q^2)$ (see [21]) can be presented as a pullback $\mathcal{C} \leftarrow C(S_q^2) \rightarrow \mathcal{T}$ of $C \xrightarrow{1} C(S^1) \leftarrow C(S^1)$:

$$C(S_q^2) \xleftarrow{\sigma} \mathcal{T} \xrightarrow{1} C(S^1) \xrightarrow{\sigma} \mathcal{C}.$$  

2.2.3. The compact quantum group $SU_q(2)$. The C*-algebra $C(SU_q(2))$ (see [24]) can be presented as the pullback $\mathcal{T} \otimes C(S^1) \leftarrow C(SU_q(2)) \rightarrow C(S^1)$ of $\mathcal{T} \otimes C(S^1) \xrightarrow{\sigma \otimes 1} C(S^1) \otimes C(S^1) \xleftarrow{id \otimes 1} C(S^1)$, endowed with the standard diagonal $U(1)$-action [17]:

$$\mathcal{T} \otimes C(S^1) \xleftarrow{\sigma \otimes 1} C(SU_q(2)) \xrightarrow{\sigma \otimes 1} C(S^1) \xleftarrow{id \otimes 1} C(S^1) \otimes C(S^1).$$

2.2.4. The Calow-Matthes quantum sphere $S_H^3$. The C*-algebra $C(S_H^3)$ (see [7]) is defined to be the pullback $\mathcal{T} \otimes C(S^1) \leftarrow C(S_H^3) \rightarrow C(S^1) \otimes \mathcal{T}$. 

$$\mathcal{T} \otimes C(S^1) \xrightarrow{\sigma \otimes 1} C(S_H^3) \xleftarrow{id \otimes 1} C(S^1) \otimes \mathcal{T}.$$
2.2.5. The quantum 4-ball $\mathbb{C}^*$-algebra $B_4$. It is defined as the pullback

$C(SU_q(2)) \xrightarrow{pr_1} B_4 \xrightarrow{pr_2} T \otimes T$

of $C(SU_q(2)) \xrightarrow{\omega} C(S^3_H) \xleftarrow{\nu} T \otimes T$, with the diagonal $U(1)$-action:

Here $\omega$ and $\nu$ are given by:

$\omega : C(SU_q(2)) \to C(S^3_H), \quad (t \otimes u, v) \mapsto (t \otimes u, v \otimes 1_T),$

$\nu : T \otimes T \to C(S^3_H), \quad t_1 \otimes t_2 \mapsto (t_1 \otimes \sigma(t_2), \sigma(t_1) \otimes t_2)$.

2.2.6. The Heegaard quantum sphere $S^5_H$. The $\mathbb{C}^*$-algebra $C(S^5_H)$ (see [22, 16, 14]) is defined as the pullback

$C(S^3_H) \otimes T \xleftarrow{id} C(S^5_H) \xrightarrow{id} T \otimes T \otimes C(S^1)$

of $C(S^3_H) \otimes T \xrightarrow{id \otimes \sigma} C(S^3_H) \otimes C(S^1) \xleftarrow{\nu \otimes id} T \otimes T \otimes C(S^1)$, with the diagonal $U(1)$-action:

of $\mathcal{T} \otimes C(S^1) \xrightarrow{\sigma \otimes id} C(S^1) \otimes C(S^1) \xleftarrow{id \otimes \sigma} C(S^1) \otimes C(S^1)$, with the diagonal $U(1)$-action:
2.2.7. The 5-sphere $C^*$-algebra $P_5$. It is defined as the pullback

$$C(SU_q(2)) \xleftarrow{B_4 \otimes C(S^1)} P_5 \xrightarrow{B_4 \otimes C(S^1)} C(SU_q(2)),$$

of

$$C(SU_q(2)) \xrightarrow{id \otimes 1} C(SU_q(2)) \otimes C(S^1) \xleftarrow{pr_1 \otimes id} B_4 \otimes C(S^1),$$

with the diagonal $U(1)$-action:

$$\begin{align*}
P_5 & \xleftarrow{id \otimes 1_{C(S^1)}} C(SU_q(2)) \otimes C(S^1) \xrightarrow{pr_1 \otimes id} B_4 \otimes C(S^1).
\end{align*}$$

2.2.8. The multipullback quantum complex projective plane $\mathbb{C}P^2_T$. The $C^*$-algebra $C(\mathbb{C}P^2_T)$ (see [11]) can be presented as the iterated pullback [22]:

$$\begin{align*}
P_1 & \xrightarrow{p_1} C(\mathbb{C}P^2_T) \\
\mathcal{T} \otimes \mathcal{T} & \xrightarrow{(\sigma_2, \sigma_2)} T \otimes T \\
\sigma_1 & \xrightarrow{\psi_{01} \circ \sigma_1} C(S^1) \otimes \mathcal{T} \\
\mathcal{T} \otimes C(S^1) & \xrightarrow{\sigma_1} C(S^1) \otimes C(S^1),
\end{align*}$$

$$\begin{align*}
\psi_{01}(v \otimes t) & := S(vt(1)) \otimes t(0), \\
\psi_{02}(v \otimes t) & := t(0) \otimes S(vt(1)), \\
\psi_{12}(t \otimes v) & := t(0) \otimes S(t(1)v), \\
\phi(s \otimes v) & := S(sv(2)) \otimes v(1), \\
\gamma & := (\psi_{02} \circ \sigma_1, \psi_{12} \circ \sigma_2).
\end{align*}$$
The following is a diagrammatic proof that $P_2$ and $C(S^3_H)$ are isomorphic as $U(1)$-$C^*$-algebras:

\[(2.7)\]

\[
\begin{array}{c}
\mathcal{T} \otimes C(S^1) \\
\downarrow \sigma \otimes \text{id} \\
\mathcal{T} \otimes C(S^1) \\
\downarrow \phi \circ (\sigma \otimes \text{id}) \\
C(S^1) \otimes C(S^1) \\
\downarrow \text{id} \otimes \text{id} \\
\end{array}
\]

\[
\begin{array}{c}
P_2 - \text{id} \otimes \text{id} \to \mathcal{T} \otimes C(S^1) \\
\downarrow \phi \otimes \text{id} \\
\mathcal{T} \otimes C(S^1) \\
\downarrow \phi \circ (\sigma \otimes \text{id}) \\
C(S^1) \otimes C(S^1) \\
\downarrow \text{id} \otimes \text{id} \\
\end{array}
\]

\[(2.8)\]

\[
\tilde{\phi}(s \otimes v) = S(sv_{(2)}) \otimes v_{(1)}, \quad \psi(s \otimes v) = S(s)v_{(1)} \otimes v_{(2)}.
\]

Here the antipode map $S : C(S^1) \to C(S^1)$ is given by $S(f)(e^{i\phi}) = f(e^{-i\phi})$.

2.3. **Gauging $U(1)$-$C^*$-algebras and computing fixed-point subalgebras.** Throughout this paper, we denote a right action of a group $G$ on a space $X$ by juxtaposition, that is $(x,g) \mapsto xg$. The general idea for converting between diagonal and rightmost actions of a group $G$ is as follows. We regard $X \times G$ as a right $G$-space in two different ways, which we distinguish notationally as follows.

- We write $(X \times G)^R$ for the product $X \times G$ with $G$-action $(x,g) \cdot h := (x,gh)$.
- We write $X \times G$ for the same space with diagonal $G$-action $(x,g)h := (xh,gh)$.

There is a $G$-equivariant homeomorphism $\kappa : (X \times G)^R \to X \times G$ determined by $\kappa(x,g) := (xg,g)$, with inverse given by $\kappa^{-1}(x,g) = (xg^{-1},g)$. In general, given any Cartesian product of $G$-spaces, we will regard it as a $G$-space with the diagonal action, except for those of the form $(X \times G)^R$ just described.

We often identify the unit circle $\mathbb{T}$ with the unitary group $U(1)$, and take advantage of the induced quantum-group structure (coproduct, counit, antipode) on $C(U(1))$. Even though we only use the classical compact Hausdorff group $U(1)$, we are forced to use the quantum-group language of coactions, etc., to write explicit formulas, and carry out computations.

Let $G$ be a compact Hausdorff group, and let $H := C(G)$. Then $\text{S} : H \to H$, given by $\text{S}(h)(g) := h(g^{-1})$, is the antipode map, $\varepsilon(h) := h(e)$ defines the counit ($e$ is the neutral element of $G$), and

\[
\Delta : H \to H \otimes H \cong C(G \times G),
\]

\[
\Delta(h)(g_1, g_2) := h(g_1, g_2) = (h_{(1)} \otimes h_{(2)})(g_1, g_2) = h_{(1)}(g_1)h_{(2)}(g_2),
\]

is a coproduct. If $\alpha : G \to \text{Aut}(A)$ is a $G$-action on a unital $C^*$-algebra $A$, then there is a coaction $\delta : A \to A \otimes H \cong C(G, A)$ given by

\[
\delta(a)(g) := \alpha_g(a) =: (a_{(0)} \otimes a_{(1)})(g) = a_{(0)}a_{(1)}(g).
\]

Consider $A \otimes H$ as a $C^*$-algebra with the diagonal coaction $p \otimes h \mapsto p_{(0)} \otimes h_{(1)} \otimes p_{(1)}h_{(2)}$, and denote by $(A \otimes H)^R$ the same $C^*$-algebra with the coaction on the rightmost factor:
\[ p \otimes h \mapsto p \otimes h_{(1)} \otimes h_{(2)}. \]

Then the following map is a $G$-equivariant (i.e. intertwining the coactions) isomorphism of $C^*$-algebras:

\[
g_A : (A \otimes H) \to (A \otimes H)^R, \quad a \otimes h \mapsto a_{(0)} \otimes a_{(1)} h.
\]

Its inverse is explicitly given by

\[
g_A^{-1} : (A \otimes H)^R \to (A \otimes H), \quad a \otimes h \mapsto a_{(0)} \otimes S(a_{(1)}) h.
\]

2.4. The Milnor construction. We will now recall the clutching construction of a non-commutative vector bundle \[20, 3, 8\]. It is given in terms of the pullback of $C^*$-algebras and *-homomorphisms $\pi_1$ and $\pi_2$ such that at least one of them is surjective:

\[
\begin{array}{ccc}
B & \xleftarrow{pr_1} & B_1 \\
& \searrow_{\pi_1} & \downarrow \pi_2 \\
& & B_2 \\
& \swarrow_{pr_2} & \\
H & \xleftarrow{\pi_2} & \\
& \downarrow & \\
& H \otimes V & \xrightarrow{\chi} & H \otimes V.
\end{array}
\]

Furthermore, assume that $H$ is the algebra of continuous functions on a fixed compact quantum group. Given a finite-dimensional complex vector space $V$ and an isomorphism of left $H$-modules $\chi : H \otimes V \to H \otimes V$, we construct a finitely generated projective left $B_1$-module $B_1 \otimes V$ and the free left $B_2$-module $B_2 \otimes V$:

\[
\begin{array}{ccc}
M(B_1 \otimes V, B_2 \otimes V, \chi) & \xleftarrow{\pi_1 \otimes \text{id}} & B_1 \otimes V \\
& \xrightarrow{\chi} & H \otimes V \\
& \xleftarrow{\pi_2 \otimes \text{id}} & B_2 \otimes V
\end{array}
\]

**Theorem 2.2.** Let $B$ be the pullback $C^*$-algebra of Diagram (2.11), $V$ be an $N$-dimensional vector space, with $N \in \mathbb{N}$, and let $E := M(B_1 \otimes V, B_2 \otimes V, \chi)$ be the pullback left $B$-module from Diagram (2.12). Define $E_i = B_i \otimes V$, and take $a := (a_{\alpha \beta}) \in GL(N, H)$ to be the matrix implementing an isomorphism $\chi : \pi_1^*(E_1) \to \pi_2^*(E_2)$. Then, if $\pi_2$ is surjective, $E \cong B^{2N} p$ as a left module, where

\[
p := \begin{pmatrix}
(1, c(2 - dc)d) & (0, c(2 - dc)(1 - dc)) \\
(0, (1 - dc)d) & (0, (1 - dc)^2)
\end{pmatrix}
\]

is an idempotent matrix in $M_{2N}(B)$ such that all entries of the sub-matrices $c, d \in M_N(B_2)$ satisfy $\pi_2(c_{\alpha \beta}) = a_{\alpha \beta}$, and $\pi_2(d_{\alpha \beta}) = (a_{\alpha \beta})^{-1}$, for all $\alpha, \beta$.

3. General results

3.1. Comparing $K$-groups of pullback $C^*$-algebras. We will now derive from Theorem 2.1 some functoriality properties for the $K$-theory of pullbacks.
Theorem 3.1. Assume that we have two pullbacks in the category of $C^*$-algebras:

- Let $A_1 \xrightarrow{p_1} A \xrightarrow{p_2} A_2$ be a pullback of $A_1 \xrightarrow{\pi_1} A_{12} \xrightarrow{\pi_2} A_2$, with $\pi_2$ surjective, and
- Let $B_1 \xleftarrow{q_1} B \xleftarrow{q_2} B_2$ be a pullback of $B_1 \xrightarrow{p_1} B_{12} \xrightarrow{p_2} B_2$, with $p_2$ surjective.

Let $\phi_1 : A_1 \to B_1$, $\phi_2 : A_1 \to B_2$, $\phi_{12} : A_{12} \to B_{12}$, $\tilde{\phi} : A \to B$ be as in Theorem 2.7 so that the diagram below is commutative:

\begin{equation}
\begin{array}{ccc}
A & \xrightarrow{\phi} & B \\
\downarrow{\pi_1} & & \downarrow{\pi_2} \\
A_{12} & \xrightarrow{\phi_{12}} & B_{12}
\end{array}
\end{equation}

Assume that the morphisms $\phi_i$ and $\phi_{12}$ induce isomorphisms on $K$-groups:

\begin{equation}
\phi_{i*} : K_*(A_i) \xrightarrow{\cong} K_*(B_i), \quad \phi_{12*} : K_*(A_{12}) \xrightarrow{\cong} K_*(B_{12}).
\end{equation}

Then the morphism $\tilde{\phi}$ also induces an isomorphism on $K$-theory:

\begin{equation}
\tilde{\phi}_* : K_*(A) \xrightarrow{\cong} K_*(B).
\end{equation}

Proof. Observe that, by Theorem 2.1, we have:

\begin{equation}
\tilde{\phi}((a_1, a_2)) = (\phi_1(a_1), \phi_2(a_2)) \in B.
\end{equation}

Now recall that we have the following $K$-theory six-term exact sequence (see e.g. [9]) induced by the pullback $A_1 \xleftarrow{p_1} A \xrightarrow{p_2} A_2$ of $A_1 \xrightarrow{\pi_1} A_{12} \xrightarrow{\pi_2} A_2$:

\begin{equation}
\begin{array}{c}
K_0(A) \xrightarrow{(p_1, p_2)} K_0(A_1) \oplus K_0(A_2) \xrightarrow{\pi_2* - \pi_1*} K_0(A_{12}) \\
\downarrow{\delta_{10}} & & \downarrow{\delta_{11}} \\
K_1(A_{12}) \xrightarrow{\pi_2* - \pi_1*} K_1(A_1) \oplus K_1(A_2) \xrightarrow{(p_1, p_2)} K_1(A).
\end{array}
\end{equation}

Similarly, we have the following $K$-theory 6-term exact sequence induced by the pullback $B_1 \xleftarrow{q_1} B \xrightarrow{q_2} B_2$ of $B_1 \xrightarrow{p_1} B_{12} \xrightarrow{p_2} B_2$:

\begin{equation}
\begin{array}{c}
K_0(B) \xrightarrow{(q_1, q_2)} K_0(B_1) \oplus K_0(B_2) \xrightarrow{\rho_2* - \rho_1*} K_0(B_{12}) \\
\downarrow{\delta_{10}} & & \downarrow{\delta_{11}} \\
K_1(B_{12}) \xrightarrow{\rho_2* - \rho_1*} K_1(B_1) \oplus K_1(B_2) \xrightarrow{(q_1, q_2)} K_1(B).
\end{array}
\end{equation}

To prove that $\tilde{\phi}$ induces isomorphisms on $K$-groups, we will use functoriality of the $K$-theory, and the 5-lemma. We will divide the proof into two parts. In the first part, we will prove that $\tilde{\phi}$ induces an isomorphism between $K_0(A)$ and $K_0(B)$. In the second part, we will prove that $\tilde{\phi}$ induces an isomorphism between $K_1(A)$ and $K_1(B)$.
Part 1: The map $\tilde{\phi}_*: K_0(A) \to K_0(B)$ is an isomorphism. By using the 6-term exact sequences associated to pullbacks, and taking advantage of the functoriality of $K$-theory, we obtain:

$$K_1(A_1) \oplus K_1(A_2)^{\pi_2-\pi_1} K_1(A_{12}) \xrightarrow{\partial_{10}} K_0(A)^{(\rho_{11} \circ \rho_{22})} K_0(A_1) \oplus K_0(A_2)^{\pi_2-\pi_1} K_0(A_{12})$$

$$K_1(B_1) \oplus K_1(B_2)^{\pi_2-\pi_1} K_1(B_{12}) \xrightarrow{\partial_{10}} K_0(B)^{(\rho_{11} \circ \rho_{22})} K_0(B_1) \oplus K_0(B_2)^{\pi_2-\pi_1} K_0(B_{12}).$$

Here the top and bottom rows are exact because they come from 6-term exact sequences (see e.g. [3]). Also, note that all vertical arrows apart from the middle one are isomorphisms by hypothesis. Hence, if we can prove that the above diagram is commutative, we can conclude by the 5-Lemma that the middle vertical arrow is also an isomorphism, which is our claim.

Next, observe that the commutativity of all the squares above, apart from the one containing connecting homomorphisms follows immediately from the commutativity of Diagram (3.1). Hence it only remains to show the commutativity of the square in Diagram (3.7) containing connecting homomorphisms.

To do so, take $[u] \in K_1(A_{12})$, with representative $u \in GL_n(A_{12})$. Recall that the odd-to-even connecting homomorphism $\partial_{10}$ is defined as follows. Let $c, d \in M_n(A_2)$, be such that

$$\text{(id} \otimes \pi_2)(c) = u, \quad (\text{id} \otimes \pi_2)(d) = u^{-1}. $$

Then $\partial_{10}([u]) = [p] - [I_n]$, where

$$p = \begin{pmatrix} (1, c(2 - dc)d) & (0, c(2 - dc)(1 - dc)) \\ (0, (1 - dc)d) & (0, (1 - dc)^2) \end{pmatrix}. $$

Now, because $\text{id} \otimes \phi_2$ is a $C^*$-homomorphism on $M_\infty(A_2)$, it follows that:

$$\tilde{\phi}_*(\partial_{10}([u])) = \tilde{\phi}_*([p]) - [I_n] = \begin{pmatrix} (1, c'(2 - dc')d') & (0, c'(2 - dc')(1 - dc')) \\ (0, (1 - dc')d') & (0, (1 - dc')^2) \end{pmatrix} - [I_n].$$

Here $c' := (\text{id} \otimes \phi_2)(c)$ and $d' := (\text{id} \otimes \phi_2)(d)$. Also, note that $c'$ and $d'$ satisfy

$$\text{(id} \otimes \rho_2)(c') = (\text{id} \otimes (\rho_2 \circ \phi_2))(c) = (\text{id} \otimes (\phi_{12} \circ \pi_2))(c) = (\text{id} \otimes \phi_{12})(u),$$

together with:

$$\text{(id} \otimes \rho_2)(d') = (\text{id} \otimes (\rho_2 \circ \phi_2))(d) = (\text{id} \otimes (\phi_{12} \circ \pi_2))(d) = (\text{id} \otimes \phi_{12})(u^{-1}) = ((\text{id} \otimes \phi_{12})(u))^{-1}. $$

Moreover, by using the definition of $\partial'_{10}$ (which is analogous to the definition of $\partial_{10}$), we can also write

$$\partial'_{10}(\phi_{12*}([u])) = \partial'_{10}([(\text{id} \otimes \phi_{12})(u)]) = \tilde{\phi}_*(\partial_{10}([u]),$$

thus proving the desired commutativity.
Part 2: The map $\tilde{\phi}_*: K_1(A) \to K_1(B)$ is an isomorphism. As before, we have the following diagram with exact rows:

\[
\begin{array}{ccc}
K_0(A_1) \oplus K_0(A_2)^{\pi_2 - \pi_1} K_0(A_{12}) & \xrightarrow{(\phi_{1*}, \phi_{2*})} & K_1(A) \\
\downarrow \phi_{12*} & & \downarrow \phi_* \\
K_0(B_1) \oplus K_0(B_2)^{\rho_2 - \rho_1} K_0(B_{12}) & \xrightarrow{\tilde{\phi}_{12*}} & K_1(B) \end{array}
\]

Again, the vertical maps apart from $\tilde{\phi}_*$ are isomorphisms by assumption, and the commutativity of all squares apart from the one containing connecting homomorphisms is an easy consequence of the commutativity of Diagram (3.1). So, by the 5-lemma, it only remains to prove that

\[
(3.15) \quad \tilde{\phi}_* \circ \partial_{01} = \partial_{01} \circ \phi_{12*}.
\]

To this end, we need to recall an explicit form of even-to-odd connecting homomorphisms. Let $[p] \in K_0(A_{12})$, where $p$ is a projection in $M_n(A_{12})$ for some $n \in \mathbb{N}$. Consider any self-adjoint lifting $\tilde{p}$ of $p$ to $M_n(A_2)$ such that $(id \otimes \pi_2)(\tilde{p}) = p$. Then it is known (e.g., see [5]) that

\[
(3.16) \quad \partial_{01}([p]) = [(I_n, e^{2\pi i\tilde{p}})].
\]

Now observe that $(id \otimes \pi_2)(e^{2\pi i\tilde{p}}) = I_n$, and so $(I_n, e^{2\pi i\tilde{p}}) \in K_0(A)$. Thus we obtain

\[
(3.17) \quad \tilde{\phi}_*(\partial_{01}([p])) = [\tilde{\phi}((I_n, e^{2\pi i\tilde{p}}))] = [(I_n, (id \otimes \phi_2)(e^{2\pi i\tilde{p}}))] = [(I_n, e^{2\pi i(id \otimes \phi_2)(\tilde{p})})].
\]

Furthermore, we have:

\[
(3.18) \quad \phi_{12*}([p]) = [(id \otimes \phi_{12})(p)] \in K_0(B_{12}).
\]

To compute $\partial_{01}([(id \otimes \phi_{12})(p)])$, we will again need a self-adjoint lifting of $(id \otimes \phi_{12})(p)$, which we can then choose to be $(id \otimes \phi_2)(\tilde{p})$. Indeed, $(id \otimes \phi_2)(\tilde{p})$ is self-adjoint because $\phi_2$ is a $*$-homomorphism, and it is a lifting of $(id \otimes \phi_{12})(p)$ because:

\[
(3.19) \quad (id \otimes \rho_2)((id \otimes \phi_2)(\tilde{p})) = (id \otimes (\rho_2 \circ \phi_2))(\tilde{p}) = (id \otimes (\phi_{12} \circ \pi_2))(\tilde{p}) = (id \otimes \phi_{12})(p).
\]

This proves the desired commutativity (3.15). \hfill \Box

3.2. Modules associated to piecewise cleft coactions. In this section, we will unravel the structure of Milnor’s module of Section 2.4 in the setting of noncommutative vector bundles associated to compact quantum principal bundles obtained by glueing two trivial (or more generally, two cleft) parts.

Let $(H, \Delta)$ be a compact quantum group acting on unital C*-algebras $P_1$, $P_2$, and $P_{12}$, and let

\[
\begin{array}{ccc}
P & \xleftarrow{\phi} & P_1 \\
\downarrow & & \downarrow \\
& P_{12} & \xleftarrow{\phi} & P_2 \\
\end{array}
\]
be a pullback diagram in the category of $H$-$\ast$-algebras. Next, denote by $\mathcal{H}$ Woronowicz’s Peter-Weyl $\ast$-Hopf algebra generated by the matrix elements of finite-dimensional representations of the compact quantum group $(H, \Delta)$, and by $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_{12}$ denote the respective Peter-Weyl $\mathcal{H}$-comodule algebras of $H$-$\ast$-algebras $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_{12}$. Recall that, for any unital $\ast$-algebra $A$ endowed with a coaction $\delta : A \to A \otimes H$, the Peter-Weyl $\mathcal{H}$-comodule algebra \cite[Equation (0.4)]{4} consists of all $a \in A$ such that $\delta(a) \in A \otimes_{\text{alg}} H$.

Moreover, recall that assigning Peter-Weyl $\mathcal{H}$-comodule algebras to $H$-$\ast$-algebras defines a functor (called the Peter-Weyl functor) transforming pullback diagrams into pullback diagrams \cite[Lemma 2]{5}. Hence applying the Peter-Weyl functor to Diagram 3.20 yields the following pullback diagram of $\mathcal{H}$-comodule algebras:

\[
\begin{array}{ccc}
\mathcal{P} & \rightarrow & \mathcal{P} \\
\downarrow \phi_1 & \swarrow \phi_2 & \swarrow \phi_2 \\
\mathcal{P}_1 & \leftarrow & \mathcal{P}_2
\end{array}
\]

Next, assume that $\tilde{\pi}_2$ is surjective and that both $\mathcal{P}_1$ and $\mathcal{P}_2$ are cleft, i.e. there exist $\mathcal{H}$-colinear, convolution-invertible linear morphisms

\[
\gamma_i : \mathcal{H} \to \mathcal{P}_i, \quad i = 1, 2,
\]

called cleaving maps. Furthermore, suppose that $\mathcal{P}^{\mathcal{H}}_{12} = H$, where $\mathcal{P}^{\mathcal{H}}_{12}$ is the coaction-invariant subalgebra. Let us denote by $\pi_i : \mathcal{P}^{\mathcal{H}}_i \to H$ the restriction-corestriction of $\tilde{\pi}_i$ to the respective $\mathcal{H}$-coaction-invariant subalgebras, $i = 1, 2$. Let $V$ be a finite-dimensional complex vector space that is also a left $\mathcal{H}$-comodule. Consider the following isomorphisms of $\mathcal{P}^{\mathcal{H}}_i$-modules

\[
\Lambda_i : \mathcal{P}^{\mathcal{H}}_i \otimes V \to \mathcal{P}_i \otimes H V, \quad b_i \otimes v \mapsto b_i \gamma_i(v(-1)) \otimes v(0).
\]

They give rise to the following diagram:

\[
\begin{array}{cccc}
\mathcal{P}^{\mathcal{H}}_1 \otimes V & \xrightarrow{\Lambda_1} & \mathcal{P}_1 \otimes H V \\
\downarrow \pi_1 \otimes \text{id} & \xrightarrow{\pi_2 \otimes \text{id}} & \downarrow \tilde{\pi}_1 \otimes \text{id} \\
H \otimes V & \overset{\chi}{\longrightarrow} & H \otimes V \\
\end{array}
\quad
\begin{array}{cccc}
\mathcal{P}^{\mathcal{H}}_2 \otimes V & \xrightarrow{\Lambda_2} & \mathcal{P}_2 \otimes H V \\
\downarrow \pi_2 \otimes \text{id} & \xrightarrow{\tilde{\pi}_2 \otimes \text{id}} & \downarrow \text{id} \\
\mathcal{P}_{12} \otimes H V & \longrightarrow & \mathcal{P}_{12} \otimes H V
\end{array}
\]

**Theorem 3.2.** In the situation depicted in Diagram (3.24), the following equality holds:

\[
\chi(b \otimes v) = b \gamma_1(v(-2)) \gamma_2^{-1}(v(-1)) \otimes v(0), \quad \gamma_i := \tilde{\pi}_i \circ \gamma_i, \quad i = 1, 2.
\]

Observe that the $(\mathcal{P}^{\mathcal{H}}_1 \oplus \mathcal{P}^{\mathcal{H}}_2)$-module isomorphism $\Lambda_1 + \Lambda_2$ restricts to the $\mathcal{P}^{\mathcal{H}}$-module isomorphism $\Upsilon$:

\[
\Upsilon : M(\mathcal{P}^{\mathcal{H}}_1 \otimes V, \mathcal{P}^{\mathcal{H}}_2 \otimes V, \chi) \to \mathcal{P}
\]

between respective pullbacks if and only if, for any

\[
c := (b_1 \otimes v_1, b_2 \otimes v_2) \in \mathcal{P}^{\mathcal{H}}_1 \otimes V, \mathcal{P}^{\mathcal{H}}_2 \otimes V,
\]
we have
\[ (3.27) \quad c \in M(\mathcal{P}_1^{co H} \otimes V, \mathcal{P}_2^{co H} \otimes V, \chi) \iff (\Lambda_1 \oplus \Lambda_2)(c) \in \mathcal{P} \square^H V, \]
that is
\[ (3.28) \quad ((\tilde{\pi}_1 \otimes \text{id}) \circ \Lambda_1)(b_1 \otimes v_1) = ((\tilde{\pi}_2 \otimes \text{id}) \circ \Lambda_2)(b_2 \otimes v_2) \]
\[ (3.29) \quad \chi(\pi_1(b_1) \otimes v_1) = (\pi_2(b_2) \otimes v_2). \]

**Proof.** Note first that \( \tilde{\gamma}_i \) is a cleaving map into \( \mathcal{P}_{12} \). Indeed, it is obviously a right \( H \)-comodule map and has convolution inverse \( \tilde{\gamma}_i^{-1} = \tilde{\pi}_i \circ \gamma_i^{-1} \). Note next that Equation \( (3.28) \) after substituting the definitions of \( \Lambda_i \)'s and \( \tilde{\gamma}_i \)'s can be written as:
\[ (3.30) \quad \pi_1(b_1)\tilde{\gamma}_1(v_{1(-1)}) \otimes v_{1(0)} = \pi_2(b_2)\tilde{\gamma}_2(v_{2(-1)}) \otimes v_{2(0)}. \]
Using the convolution inverse of \( \tilde{\gamma}_2 \) leads to an equivalent equation:
\[ (3.31) \quad \pi_1(b_1)\tilde{\gamma}_1(v_{1(-2)})\tilde{\gamma}_2^{-1}(v_{1(-1)}) \otimes v_{1(0)} = (\pi_2(b_2) \otimes v_2). \]
Hence
\[ (3.32) \quad \chi(b \otimes v) = b\tilde{\gamma}_1(v_{(-2)})\tilde{\gamma}_2^{-1}(v_{(-1)}) \otimes v_{(0)}, \]
as claimed. \( \Box \)

It follows from the above theorem that we can use the image under \( \gamma_1 \ast \tilde{\gamma}_2^{-1} \) of a representation matrix of the left \( H \)-comodule \( V \) to construct an idempotent representing \( \mathcal{P} \square^H V \) by taking advantage of Theorem 2.2.

**Remark 3.3.** Theorem 3.2 can be proven in the same way for pullback diagrams of arbitrary comodule algebras, not necessarily those in the image of the Peter-Weyl functor. In particular, we do not need the cosimplicity of a structure Hopf algebra, which is a key feature of all Peter-Weyl Hopf algebras. Such an algebraic version of Theorem 3.2 would perfectly fit with the original Milnor’s version of Theorem 2.2, which is also purely algebraic.

### 4. Associated-module construction of a Milnor module

It was proven in [22] that
\[ (4.1) \quad K_0(C(\mathbb{C}P_7^2)) = s(K_0(P_1)) \oplus \partial_{10}(K_1(C(S^3_H))). \]
Here \( P_1 := (C(S^3_H) \otimes \mathcal{T})^U(1), K_0(P_1) = \mathbb{Z} \oplus \mathbb{Z}, \partial_{10}(K_1(C(S^3_H))) = \mathbb{Z}, \) and \( s \) is a splitting of
\[ (4.2) \quad (p_1)_* : K_0(C(\mathbb{C}P_7^2)) \longrightarrow K_0(P_1). \]
Moreover, in [10], generators of \( s(K_0(P_1)) \) were identified as [1] and \([L_1] - [1]. \) The goal of this section is to identify a generator of \( \partial_{10}(K_1(C(S^3_H))) \), which must be of the form \( \partial_{10}([U]) \) for some invertible matrix \( U \).

To this end, using again the Mayer-Vietoris six-term exact sequence in K-theory, first we show that \( K_0(P_5^{U(1)}) \) enjoys an analogous decomposition as \( K_0(C(\mathbb{C}P_7^2)) \):
\[ (4.3) \quad K_0(P_5^{U(1)}) = t(K_0(C(S^2_q))) \oplus \partial_{10}(K_1(C(SU_q(2)))). \]
Here \( t \) is a splitting of
\[
(q_1)_*: K_0(P_5^{U(1)}(1))) \rightarrow K_0(C(S^2_q)),
\]
with \( q_1 \) defined in Diagram (4.24). Next, using [10] and [12], we can easily see that [1] and \([\hat{L}_1] - [1]\) generate \( t(K_0(C(S^2_q))) \). Moreover, taking advantage of Theorem 3.2 we compute that \([\hat{L}_1 \oplus \hat{L}_{-1}] - 2[1]\) generates \( \partial_{10}(K_1(C(SU_q(2)))) \).

Finally, we construct a \( U(1) \)-equivariant *-homomorphism \( f: P_5 \rightarrow C(SU_q^0(2)) \). This allows us, using [14, Theorem 5.1] (cf. [12, Theorem 0.1]), to conclude that \( \tilde{f}_* \) (where \( \tilde{f} \) is the restriction-corestriction of \( f \) to the fixed-point subalgebras) maps \([\tilde{L}_n]\) to \([L_n]\) for any \( n \in \mathbb{Z} \). Then we prove that \( \tilde{f}_* \) is an isomorphism of \( K \)-groups. This allows us to infer that \([L_1 \oplus L_{-1}] - 2[1]\) generates \( \partial_{10}(K_1(C(SU_q(2)))) \).

### 4.1. Associated-module construction in the auxiliary case.
Recall first that the cotensor product of a right comodule \( M \) with a coaction \( \rho : M \rightarrow M \otimes_{alg} C \) with a left comodule algebra via the coaction \( L \) that \( f \) is the restriction-corestriction of \( \alpha \) to the Peter-Weyl Hopf subalgebras by \( \pi \).

The assignment \( \alpha \mapsto \pi(\alpha) := u, \gamma \mapsto \pi(\gamma) = 0 \), defines a *-homomorphism
\[
C(SU_q(2)) \rightarrow C(U(1)).
\]

We denote its restriction-corestriction
\[
O(SU_q(2)) \rightarrow O(U(1))
\]

to the Peter-Weyl Hopf subalgebras by \( \pi \). We can view \( O(SU_q(2)) \) as a left \( O(U(1)) \)-comodule algebra via the coaction
\[
\Phi_L := (\pi \otimes id) \circ \Delta: O(SU_q(2)) \rightarrow O(U(1)) \otimes_{alg} O(SU_q(2)),
\]
where \( \Delta \) is the coproduct.

Let \( A \) be a unital \( U(1) \)-C*-algebra. Then \( \mathcal{P}_{U(1)}(A) \) stands for its \( O(U(1)) \)-Peter-Weyl-comodule algebra. Consider the following pullback presentation of \( \mathcal{P}_{U(1)}(P_5) \otimes O(U(1)) O(SU_q(2)) \):

\[
\mathcal{P}_{U(1)}(P_5) \otimes O(U(1)) O(SU_q(2))
\]

\[
\mathcal{P}_{U(1)}(SU_q(2)) \otimes O(U(1)) O(SU_q(2)) \rightarrow \mathcal{P}_{U(1)}(B_5 \otimes C(S^1)) \otimes O(U(1)) O(SU_q(2))
\]

\[
(C(SU_q(2)) \otimes_{alg} O(S^1))^{\otimes R} \otimes O(U(1)) O(SU_q(2)).
\]
Here \( g : (C(SU_q(2)) \otimes C(S^1)) \rightarrow (C(SU_q(2)) \otimes C(S^1))^R \) is a gauging map (see Equation (2.9)). It is easy to see that we have the following cleaving maps:

\[
\begin{align*}
\gamma_1 : & \mathcal{O}(SU_q(2)) \rightarrow \mathcal{P}_{U(1)}(SU_q(2)) \square_{\mathcal{O}(SU_q(2))} \mathcal{O}(SU_q(2)), \quad s \mapsto s_{(1)} \otimes s_{(2)}, \\
\gamma_2 : & \mathcal{O}(SU_q(2)) \rightarrow \mathcal{P}_{U(1)}(B_4 \otimes C(S^1)) \square_{\mathcal{O}(SU_q(2))} \mathcal{O}(SU_q(2)), \quad s \mapsto 1 \otimes \pi(s_{(1)}) \otimes s_{(2)}. 
\end{align*}
\]

It follows that

\[
\begin{align}
\gamma_1(s_{(1)})\gamma_2^{-1}(s_{(2)}) &= (s_{(1)} \otimes \pi(s_{(2)}) \otimes s_{(3)})(1 \otimes \pi(S(s_{(5)}) \otimes S(s_{(4)})) \\
&= s_{(1)} \otimes \pi(s_{(2)})S(s_{(3)}) \otimes 1 \\
&= s \otimes 1 \otimes 1. 
\end{align}
\]

**Lemma 4.1.** The element \([\tilde{L}_1 \oplus \tilde{L}_{-1}] - 2[1]\) generates \(\partial_{10}(C(SU_q(2)))\).

**Proof.** It suffices to show that \([\tilde{L}_1 \oplus \tilde{L}_{-1}] - 2[1] = \pm \partial_{10}([U])\). We define a left \(\mathcal{O}(SU_q(2))-\)coaction on \(\mathbb{C}^2\) via the fundamental representation matrix \(U\). Then

\[
\tilde{E} := \mathcal{P}_{U(1)}(P_3) \square_{\mathcal{O}(SU_q(2))} \mathcal{O}(SU_q(2)) \square_{\mathcal{O}(SU_q(2))} \mathbb{C}^2 \\
= \mathcal{P}_{U(1)}(P_3) \square_{\mathcal{O}(SU_q(2))} \mathbb{C}^2 \\
= (\mathcal{P}_{U(1)}(P_3) \square_{\mathcal{O}(SU_q(2))} \mathbb{C}^+) \oplus (\mathcal{P}_{U(1)}(P_3) \square_{\mathcal{O}(SU_q(2))} \mathbb{C}^-) \\
= \tilde{L}_1 \oplus \tilde{L}_{-1}.
\]

Here \(\mathbb{C}^\pm\) stands for the left comodule \(\mathbb{C}\) with its coaction given by \(1 \mapsto u^{\pm 1} \otimes 1\).

On the other hand, using Equation (3.9) Theorem 2.2 Theorem 3.2 and Equation (4.12), we compute

\[
\partial_{10}([U]) = [\tilde{E}] - 2[1].
\]

Finally, combining (4.13) with (4.14) completes the proof. \(\square\)

### 4.2. Reducing the multipullback case to the auxiliary case.

**Lemma 4.2.** There is a \(U(1)\)-equivariant \(C^*-\)morphism \(f : P_3 \rightarrow C(S_H^3)\) which makes the diagram below commutative:

\[
\begin{array}{ccccccccc}
P_3 & \xrightarrow{\omega \otimes 1_T} & f & \xleftarrow{\gamma} & C(S_H^3) \\
C(SU_q(2)) & \xrightarrow{id \otimes 1_{C(S^1)}} & B_4 \otimes C(S^1) & \xrightarrow{pr_1 \otimes id} & C(S_H^3) \otimes T & \xrightarrow{id \otimes \sigma} & T^2 \otimes C(S^1) \\
\end{array}
\]

\[
\begin{array}{ccccccccc}
\omega \otimes id & \xrightarrow{id \otimes \sigma} & \omega \otimes id & \xrightarrow{pr_2 \otimes id} & \omega \otimes id & \xrightarrow{\nu \otimes id} & \omega \otimes id \\
C(SU_q(2)) \otimes C(S^1) & \xrightarrow{id \otimes pr_2} & C(S_H^3) \otimes C(S^1). \\
\end{array}
\]
Proof. We will use the universal property of a pullback to show the existence of the map $f$. To this end, we need to prove the commutativity of the part of Diagram $\text{(4.15)}$ drawn with solid lines. Explicitly, we need to show that

$$
(\omega \otimes \text{id}) \circ (\text{pr}_1 \otimes \text{id}) = (\nu \otimes \text{id}) \circ (\text{pr}_2 \otimes \text{id}),
$$

(4.16)

$$(\text{id} \otimes \sigma) \circ (\omega \otimes 1_\tau) = (\omega \otimes \text{id}) \circ (\text{id} \otimes 1_{C(S^1)}).$$

The first equality follows from Diagram $\text{(2.4)}$, while the second one follows from the calculation $(\text{id} \otimes \sigma) \circ (\omega \otimes 1_\tau) = \omega \otimes 1_{C(S^1)} = (\omega \otimes \text{id}) \circ (\text{id} \otimes 1_{C(S^1)})$. □

Diagram $\text{(4.15)}$ assumes that all $U(1)$-actions are diagonal. In order to obtain the C*-algebra $C(CP^2_\tau)$ of the multipullback quantum complex projective plane as a fixed-point subalgebra, we first need to gauge the diagonal actions to actions of $U(1)$ on the rightmost $C(S^1)$-component. This is applied to the $U(1)$-algebras $A$ in Diagram $\text{(4.15)}$, and uses the natural gauge isomorphisms $g_A$ defined in Equations $\text{(2.9)}$ and $\text{(2.10)}$. A slight complication is that some of components of Diagram $\text{(4.15)}$ are not trivial. However, they are all pullbacks of trivial components. We extend the gauge isomorphisms to pullbacks in a natural way (componentwise). For instance, we define $g_{C(SU_q(2))} : C(SU_q(2)) \rightarrow C(SU_q(2))^R$ and its inverse by

$$
g_{C(SU_q(2))} : (t \otimes u,v) \mapsto (t(0) \otimes t(1)u,v),
$$

(4.17)

$$
g_{C(SU_q(2))}^{-1} : (t \otimes u,v) \mapsto (t(0) \otimes S(t(1))u,v).
$$

In what follows, we will omit the subscript denoting an algebra, and write $g$ instead of $g_A$. We will also implicitly include in the gauge isomorphisms the permutations that make the $C(S^1)$-factor the rightmost component. By applying gauge isomorphisms to each component of Diagram $\text{(4.15)}$ and conjugating the original maps by appropriate gauge isomorphisms, we derive the diagram:

$$
(\text{id} \otimes \sigma) \circ (\omega \otimes 1_\tau) = (\omega \otimes \text{id}) \circ (\text{id} \otimes 1_{C(S^1)}).
$$

The only maps changed by conjugation are $\beta$, $\chi$ and $\Omega$. In the above diagram, the rightmost coaction is supported on the sole explicit tensor factor $C(S^1)$ except in two cases: $C(SU_q(2))^R$ and $(C(S^3_H) \otimes \mathcal{T})^R$. Here the $C(S^1)$ tensor factor supporting the rightmost $U(1)$-action is hidden inside $C(SU_q(2))^R$ and $C(S^3_H)$. The case $C(SU_q(2))^R$ was discussed earlier in Equation $\text{(4.17)}$. Now we define $(C(S^3_H) \otimes \mathcal{T})^R$ as a pullback obtained by gauging componentwise the pullback presentation of $C(S^3_H) \otimes \mathcal{T}$:

$$
(\mathcal{T} \otimes \mathcal{T} \otimes C(S^1))^R \longrightarrow (C(S^3_H) \otimes \mathcal{T})^R \longrightarrow (\mathcal{T} \otimes \mathcal{T} \otimes C(S^1))^R
$$

$$
(C(S^1) \otimes \mathcal{T} \otimes C(S^1))^R.
$$
Here $\alpha$ is given to be the conjugation of $\text{id} \otimes \sigma \otimes \text{id}$ with gauging isomorphisms:

\begin{equation}
(4.19) \quad (T \otimes T \otimes C(S^1))^R \xrightarrow{g^{-1}} C(S^1) \otimes T \otimes T \xrightarrow{\text{id} \otimes \sigma \otimes \text{id}} (C(S^1) \otimes T \otimes C(S^1))^R \xrightarrow{g} C(S^1) \otimes C(S^1) \otimes T.
\end{equation}

In the bottom line of the above diagram, there are two $C(S^1)$ factors in the source of $g$. Note that it is the second one (the one where symbol map was applied) which $g$ moves to the front to support the rightmost $C(U(1))$-coaction. Explicitly, we have:

\begin{equation}
(4.20) \quad \alpha(t \otimes t' \otimes u) = u(1)S(\sigma(t)t'(1)) \otimes t'(0) \otimes u(2).
\end{equation}

The $U(1)$-equivariant gauge isomorphism $g : C(S^3_H) \otimes T \longrightarrow (C(S^3_H) \otimes T)^R$ is obtained by applying appropriate gauge isomorphisms to respective pullback components. Other gauge isomorphisms are defined in equations (2.9) and (4.17). Now we are ready to compute functions $\Omega$, $\beta$ and $\chi$ by conjugation with gauge isomorphisms. The map $\beta$ is defined to be the following composition:

\[ C(S^3_q)^R \xrightarrow{g^{-1}} C(S^3_q) \xrightarrow{\text{id} \otimes \sigma} C(S^3_q) \otimes C(S^1) \xrightarrow{g} (C(S^3_q) \otimes C(S^1))^R. \]

Explicitly,

\begin{equation}
(4.21) \quad \beta(s) = (g \circ (\text{id} \otimes 1) \circ g^{-1})(s) = g(g^{-1}(s) \otimes 1) = g^{-1}(s(0)) \otimes s(1).
\end{equation}

The map $\Omega$ is the following composition:

\[ C(S^3_q)^R \xrightarrow{g^{-1}} C(S^3_q) \xrightarrow{\omega \otimes 1} C(S^3_q) \otimes T \xrightarrow{g} (C(S^3_q) \otimes T)^R. \]

Explicitly,

\begin{equation}
(4.22) \quad \Omega((t \otimes u, v)) = (g \circ (\omega \otimes 1) \circ g^{-1}((t \otimes u, v))
= (g \circ (\omega \otimes 1))((t(0) \otimes S(t(1))u, v))
= g((t(0) \otimes S(t(1))u \otimes 1_T, v \otimes 1_T \otimes 1_T))
= (t \otimes 1_T \otimes u, 1_T \otimes 1_T \otimes v).
\end{equation}

Finally, $\chi$ is defined to be the composition:

\[ (C(S^3_H) \otimes T)^R \xrightarrow{g^{-1}} C(S^3_H) \otimes T \xrightarrow{\text{id} \otimes \sigma} C(S^3_H) \otimes C(S^1) \xrightarrow{g} (C(S^3_H) \otimes C(S^1))^R. \]

Explicitly, we have

\begin{equation}
(4.23) \quad \chi((s \otimes s' \otimes u, t \otimes t' \otimes v))
= (g \circ (\text{id} \otimes \sigma) \circ g^{-1}((s \otimes s' \otimes u, t \otimes t' \otimes v))
= (g \circ (\text{id} \otimes \sigma))((s(0) \otimes uS(s(1)s'(1)) \otimes s'(0), vS(t(1)t'(1)) \otimes t(0) \otimes t'(0))
= g((s(0) \otimes uS(s(1)s'(2)) \otimes \sigma(s'(1)), vS(t(1)\sigma(t')(2)) \otimes t(0) \otimes \sigma(t'(1)))
= (s(0) \otimes u(1)S(s(1)\sigma(s')) \otimes u(2), t(0) \otimes v(1)S(t(1)\sigma(t')) \otimes v(2)).
\end{equation}
Taking the $U(1)$-invariant subalgebras in all the nodes in Diagram (4.18), and appropriately restricting and corestricting the diagram morphisms, we get explicitly describe above maps as:

\[
\begin{align*}
\tilde{\beta} : C(S^2_q) & \longrightarrow C(SU_q(2)), \quad (t, c) \longmapsto (t(0) \otimes S(t(1)), c_1C(S^1)) \\
\omega : C(SU_q(2)) & \longrightarrow C(S^3_H), \quad (t \otimes u, v) \longmapsto (t \otimes u, v \otimes 1_\mathcal{T}), \\
\nu : \mathcal{T} \otimes \mathcal{T} & \longrightarrow C(S^3_H), \quad t_1 \otimes t_2 \longmapsto (t_1 \otimes \sigma(t_2), \sigma(t_1) \otimes t_2), \\
\tilde{\Omega} : C(S^2_q) & \longrightarrow P_1, \quad (t, c) \longmapsto (t \otimes 1_\mathcal{T}, c_1T \otimes 1_\mathcal{T}).
\end{align*}
\]

(4.25)

Theorem 4.3. The map $f^*_* : K_*((P^R_3)^{(1)}) \rightarrow K_* (C(\mathbb{CP}^2))$ is an isomorphism.

Proof. By Theorem 3.1 it suffices to show that the maps $\tilde{\Omega}$, $pr_2$ and $\omega$ in Diagram (4.24) induce isomorphisms between the respective $K$-groups. We extract the needed isomorphism claims into the subsequent three lemmas.

Lemma 4.4. The map $pr_2$ in Diagram (4.24) induces isomorphisms of $K$-groups. Furthermore, $K_0(B_4) = \mathbb{Z}[1]$.

Proof. The pullback definition (2.4) yields the 6-term exact sequence:

\[
\begin{align*}
K_0(B_4) & \xrightarrow{(pr_1, pr_2)} K_0(C(SU_q(2))) \oplus K_0(\mathcal{T} \otimes \mathcal{T}) \xrightarrow{\nu_\ast - \omega_\ast} K_0(C(S^3_H)) \\
K_1(C(S^3_H)) & \xleftarrow{\nu_\ast - \omega_\ast} K_1(C(SU_q(2))) \oplus K_1(\mathcal{T} \otimes \mathcal{T}) \xrightarrow{(pr_1, pr_2)} K_1(B_4).
\end{align*}
\]

(4.26)

Note that $K_0(C(SU_q(2))) = K_0(C(S^3_H)) = \mathbb{Z}[1]$, and $K_1(C(SU_q(2)))$ is isomorphic with $\mathbb{Z}$ and is generated by the fundamental representation. Therefore, using the Künneth formula, we get

\[
\begin{align*}
K_0(B_4) & \xrightarrow{(pr_1, pr_2)} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\nu_\ast - \omega_\ast} \mathbb{Z} \\
\mathbb{Z} & \xleftarrow{\nu_\ast - \omega_\ast} \mathbb{Z} \oplus 0 \xrightarrow{(pr_1, pr_2)} K_1(B_4).
\end{align*}
\]

(4.27)

Furthermore, for the $K_0$-groups, the map $(\nu_\ast - \omega_\ast)$ is onto because it is given by the formula $(\nu_\ast - \omega_\ast)(m, n) = m - n$. Therefore, $\partial_{01} = 0$ due to the exactness of the
above diagram. Next, since the generator of $K_1(C(SU_q(2))) \cong \mathbb{Z}$ goes to the generator of $K_1(C(S^3_{H})) \cong \mathbb{Z}$, and $\nu_* = 0$, we conclude that $\nu_* - \omega_*$ is an isomorphism of $K_1$ groups. This leads to the following exact sequence

$$0 \to K_1(B_4) \xrightarrow{(pr_1, pr_2)} \mathbb{Z} \oplus 0 \xrightarrow{(\nu_* - \omega_*)} \mathbb{Z} \to 0,$$  

which implies that $(pr_1, pr_2)$ is the zero map. It follows that $K_1(B_4) = 0$.

For the $K_0(B_4)$, we get

$$0 \to K_0(B_4) \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{(m,n) \mapsto m-n} \mathbb{Z} \to 0.$$  

Hence $K_0(B_4) = \mathbb{Z}$. This also shows that $pr_2$ is an isomorphism. Finally, since both $K_0(C(SU_q(2)))$ and $K_0(T \otimes T)$ are generated by $[1]$, we infer that also $K_0(B_4)$ is generated by $[1]$.

**Lemma 4.5.** The map $\tilde{\Omega}$ in Diagram (4.24) induces isomorphisms of $K$-groups.

**Proof.** By the pullback presentations of $C(S^2_q)$ and $P_1$, as well as the definition of $\tilde{\Omega}$, we immediately obtain:

![Diagram](image)

Here $\tilde{a}$ is given by (cf. Equation (4.20)):

$$\tilde{a}(t \otimes t') = S(\sigma(t)\sigma'(1)) \otimes t'(0).$$  

The components of $\tilde{\Omega}$ are easily shown to induce isomorphism on $K$-groups by naturality of the Künneth’s formula. Then, by Theorem 3.1 also $\tilde{\Omega}$ induces isomorphisms on $K$-groups.

**Lemma 4.6.** The map $\omega$ in Diagram (4.24) induces isomorphisms on $K$-groups.

**Proof.** The pullback presentations of $C(SU_q(2))$ and $C(S^3_{H})$ yield the commutative diagram

![Diagram](image)
The naturality of the Künneth’s formula easily implies that the maps id_{T ⊗ C(S^1)}, id_{C(S^1) ⊗ C(S^1)} and id_{C(S^1)} ⊗ 1_T induce isomorphisms on K-groups. Hence also Ω induces isomorphisms of K-groups.

As a corollary of Theorem 4.3, we obtain the following result which ends the proof of the first part of the main theorem:

**Corollary 4.7.** The element $[L_1 ⊕ L_{-1}] − 2[1]$ generates $∂_0(K_1(C(S^3_H)))$.

**Proof.** Note first that [14, Theorem 5.1] (cf. [12, Theorem 0.1]) implies that $\tilde{f}^R$ (where $\tilde{f}^R$ is the restriction and corestriction of $f^R$ to fixed-point subalgebras) maps $[L_n]$ to $[L_n]$ for any $n ∈ \mathbb{Z}$.

Next, we show that $K_0$-group morphisms induced by the maps

$$P_1 \leftarrow C(\mathbb{C}P^2_5) \rightarrow \mathcal{T} ⊗ \mathcal{T}$$

send $[L_1 ⊕ L_{-1}] − 2[1]$ to 0. This, by exactness, will imply that

$$[L_1 ⊕ L_{-1}] − 2[1] ∈ ∂_0(K_1(C(S^3_H))).$$

To this end, we first use [10, Theorem 2.1] which implies that the map $K_0(\mathcal{T} ⊗ \mathcal{T}) → K_0(P_1)$ sends $[L_1 ⊕ L_{-1}] − 2[1]$ to 0.

To show that also the induced map $K_0(C(\mathbb{C}P^2_5)) \rightarrow K_0(\mathcal{T} ⊗ \mathcal{T})$ sends $[L_1 ⊕ L_{-1}] − 2[1]$ to 0, we again use an argument based on rank. Note first that it follows from [14, Theorem 5.1] that the induced morphism of $K$-groups $K_0(C(\mathbb{C}P^2_5)) \rightarrow K_0(\mathcal{T} ⊗ \mathcal{T})$ sends $[L_n]$ to $[L_n]$ for any $n ∈ \mathbb{Z}$, where $L_n$ is the spectral subspace of $\mathcal{T} ⊗ \mathcal{T} ⊗ C(S^1)$ corresponding to the winding number $n$. Such spectral subspaces are always free modules of rank one:

$$[(\mathcal{T} ⊗ \mathcal{T} ⊗ \mathcal{O}(U(1))) ⊗ \mathcal{O}(U(1))] ≃ \mathcal{T} ⊗ \mathcal{T} ⊗ \mathcal{C}.$$

Consequently, $L_1$ maps to the class of a rank-one module. It follows now that $[L_1 ⊕ L_{-1}] − 2[1]$ maps to a rank-zero element of $K_0(\mathcal{T} ⊗ \mathcal{T}) = \mathbb{Z}[1]$, and the only such element in $\mathbb{Z}[1]$ is 0.

All this proves that $[L_1 ⊕ L_{-1}] − 2[1] ∈ ∂_0(K_1(C(S^3_H)))$. Next, since by Lemma 4.1 $[\tilde{L}_1 ⊕ \tilde{L}_{-1}] − 2[1]$ generates $∂_0(C(SU_q(2)))$, it follows that

$$[L_1 ⊕ L_{-1}] − 2[1] = \tilde{f}^R([\tilde{L}_1 ⊕ \tilde{L}_{-1}] − 2[1])$$

generates $∂_0(K_1(C(S^3_H)))$. Indeed, we know that both $K_0((\mathbb{F}_5^2)^U(1))$ and $K_0(C(\mathbb{C}P^2_5))$ are isomorphic to $\mathbb{Z}^2 ⊕ \mathbb{Z}$, where the last component corresponds to $∂_0(K_1(C(SU_q(2))))$ and $∂_0(K_1(C(S^3_H)))$, respectively. Since $[\tilde{L}_1 ⊕ \tilde{L}_{-1}] − 2[1]$ generates $∂_0(K_1(C(SU_q(2))))$ and is mapped by an isomorphism to $[L_1 ⊕ L_{-1}] − 2[1] ∈ ∂_0(K_1(C(S^3_H)))$, we conclude that $[L_1 ⊕ L_{-1}] − 2[1]$ generates $∂_0(K_1(C(S^3_H)))$.  

5. Milnor idempotents and elementary projections in $\mathcal{T}$

5.1. Computing Milnor idempotents from $K^0(\mathbb{T}^2)$. To begin with, we need to compute a generator of $K_1(C(S^3_H))$. We will do it much in the same way as in [8], where a generator of $K_1(C(SU_q(2))$ was computed from the Loring projection $\beta \in M_2(C(\mathbb{T}^2))$ (see Theorem 3.2 in [8]). Recall that $K^0(\mathbb{T}^2) = K_0(C(\mathbb{T}^2)) = \mathbb{Z}[1] \oplus \mathbb{Z}[\beta]$ and, according to Equation (3.70) in [8], we can write $\beta$ as follows:

\begin{equation}
\beta := \begin{pmatrix} 1 \otimes f & 1 \otimes g + u \otimes h \\ 1 \otimes g + u^* \otimes h & 1 \otimes (1 - f) \end{pmatrix}.
\end{equation}

Here $u$ is the standard unitary generator of $C(S^1)$.

Next, as in [8], we denote by $S$ the isometry generating the Toeplitz algebra $\mathcal{T}$, and lift $\beta$ to a selfadjoint element $Q$ in $M_2(\mathcal{T} \otimes C(S^3))$:

\begin{equation}
Q = \begin{pmatrix} 1 \otimes f & 1 \otimes g + S \otimes h \\ 1 \otimes g + S^* \otimes h & 1 \otimes (1 - f) \end{pmatrix}.
\end{equation}

Now, by [8, Theorem 2.3], we see that

\begin{equation}
\partial_{01}([\beta]) = [(e^{2\pi i Q}, I_2)] \in K_1(C(S^3_H)).
\end{equation}

Here $I_2$ denotes the $2 \times 2$ identity matrix in $M_2(C(S^3) \otimes \mathcal{T})$. Note that Equation (3.84) in [8] gives an explicit formula for $e^{2\pi i Q}$. Putting $\nu := \exp(2\pi i \chi_{[0,1/2]}(f)) \in C(S^1)$, we obtain

\begin{equation}
e^{2\pi i Q} = \begin{pmatrix} 1 \otimes 1 + (1 - SS^*) \otimes (\nu - 1) & 0 \otimes 0 \\ 0 \otimes 0 & 1 \otimes 1 \end{pmatrix}.
\end{equation}

Furthermore, to compute the generator $[p_U] = 2[1]$ of $K_0(C(\mathbb{CP}^2_H))$ associated to $[U] := \partial_{01}([\beta])$, we can use the formula given in Theorem 2.2 (and also in [16, Theorem 1.1]) to construct the projection $p_U$ associated to $U := (e^{2\pi i Q}, I_2)$ using the diagram

\begin{equation}
\begin{array}{ccc}
P_1 & \longrightarrow & \mathcal{T} \otimes \mathcal{T} \\
\downarrow \gamma \circ (\sigma_2, \sigma_2) & & \downarrow \gamma \\
C(S^3_H) & \longrightarrow & \mathcal{T} \otimes \mathcal{T}
\end{array}
\end{equation}

obtained from Diagram 2.7 and the big diagram above it. Recall that the surjective (see [14, Lemma 3.4]) *-homomorphism $\gamma : \mathcal{T} \otimes \mathcal{T} \rightarrow C(S^3_H)$ is given on simple tensors by

\begin{equation}
\gamma(t \otimes t') = (t_{(0)} \otimes t'_{(1)}, \sigma(t), t_{(0)} \otimes t'_{(1)} \sigma(t')).
\end{equation}

Although $e^{2\pi i Q}$ is a $2 \times 2$ matrix, it has only one nontrivial entry, so we focus on finding lifts of

\begin{align*}
X := 1 \otimes 1 + (1 - SS^*) \otimes (\nu - 1) \\
&= SS^* \otimes 1 + (1 - SS^*) \otimes \nu
\end{align*}

and its inverse

\begin{align*}
X^{-1} = SS^* \otimes 1 + (1 - SS^*) \otimes \nu^*.
\end{align*}

These will give us $d$ and $c$ respectively.
Let \( \tilde{\nu} \) denote a lift of \( \nu \in C(S^1) \) to \( T \). That is, we require that \( \sigma(\tilde{\nu}) = \nu \), where \( \sigma \) denotes the symbol map. Since \( \sigma( SS^* ) = 1 \), if we take
\[
d = \begin{pmatrix} SS^* \otimes 1 + (1 - SS^*) \otimes \tilde{\nu} & 0 \\ 0 & 1 \otimes 1 \end{pmatrix}, \quad c = \begin{pmatrix} SS^* \otimes 1 + (1 - SS^*) \otimes \tilde{\nu}^* & 0 \\ 0 & 1 \otimes 1 \end{pmatrix},
\]
then \( (\text{id} \otimes \tilde{\gamma})(d) = (I_2, e^{2\pi i Q}) \) and \( (\text{id} \otimes \tilde{\gamma})(c) = (I_2, (e^{2\pi i Q})^{-1}) \).

We can now compute the Milnor idempotent associated to \( U \) by inserting
\[
dc = \begin{pmatrix} SS^* \otimes 1 + (1 - SS^*) \otimes \tilde{\nu}\tilde{\nu}^* & 0 \\ 0 & 1 \otimes 1 \end{pmatrix},
\]
\[
2 - dc = \begin{pmatrix} SS^* \otimes 1 + (1 - SS^*) \otimes (2 - \tilde{\nu}\tilde{\nu}^*) & 0 \\ 0 & 1 \otimes 1 \end{pmatrix},
\]
\[
1 - dc = \begin{pmatrix} (1 - SS^*) \otimes (1 - \tilde{\nu}\tilde{\nu}^*) & 0 \\ 0 & 0 \end{pmatrix},
\]
\[
c(2 - dc) = \begin{pmatrix} SS^* \otimes 1 + (1 - SS^*) \otimes (2\tilde{\nu}^* - \tilde{\nu}\tilde{\nu}^*) & 0 \\ 0 & 1 \otimes 1 \end{pmatrix},
\]
into
\[
p_U = \begin{pmatrix} (I_2, c(2 - dc)d) & (0, c(2 - dc)(1 - dc)) \\ (0, (1 - dc)d) & (0, (1 - dc)^2) \end{pmatrix}.
\]
Note that completing this computation will yield a complicated formula. However, we will be able to greatly simplify it by using an appropriate homotopy.

Our next step is to study Milnor’s idempotent \( p_U \) in terms of conjugation by a matrix \( V \):
\[
p_U = \begin{pmatrix} (I_2, c(2 - dc)d) & (0, c(2 - dc)(1 - dc)) \\ (0, (1 - dc)d) & (0, (1 - dc)^2) \end{pmatrix} = \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c(2 - dc)d & c(2 - dc)(1 - dc) \\ (1 - dc)d & (1 - dc)^2 \end{pmatrix} = \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix} V \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix} V^{-1}.
\]

Here
\[
V := \begin{pmatrix} c(2 - dc) & cd - 1 \\ 1 - dc & d \end{pmatrix}, \quad V^{-1} = \begin{pmatrix} d & 1 - dc \\ cd - 1 & c(2 - dc) \end{pmatrix}.
\]

For the sake of brevity, we adopt the following notation:
\[
c = \begin{pmatrix} \tilde{c} & 0 \\ 0 & 1 \end{pmatrix}, \quad d = \begin{pmatrix} \tilde{d} & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{with}
\]
\[
\tilde{c} := SS^* \otimes 1 + (1 - SS^*) \otimes y, \quad \tilde{d} := SS^* \otimes 1 + (1 - SS^*) \otimes x, \quad \text{with} \quad x := \tilde{\nu}, \quad y := x^*.
\]
Also, recall that a function \( f \in C(S^1) \) used to define the Loring projection \( \beta \) and the function \( \nu \) is given explicitly in terms of a parameter \( s \in [0, 1] \) (counted modulo 1) as
follows:

\[
\begin{align*}
  f(s) &= \begin{cases} 
    1 - 2s & \text{for } s \in [0, \frac{1}{2}] \mod 1, \\
    2s - 1 & \text{for } s \in [\frac{1}{2}, 1] \mod 1.
  \end{cases}
\end{align*}
\]

Plugging it into \( \nu \) yields

\[
\nu(s) = \exp(2\pi i \chi_{[0,1/2]}(s)(1 - 2s)) = \exp(-4\pi i \chi_{[0,1/2]}(s)s).
\]

Hence

\[
V = \begin{pmatrix} 
  \tilde{c}(2 - \tilde{c}d) & 0 \\
  0 & \tilde{d}c \\
  \tilde{d}c - 1 & 0 \\
  0 & \tilde{d}
\end{pmatrix}
\]

\[
V^{-1} = \begin{pmatrix} 
  \tilde{d} & 0 \\
  0 & 1 \\
  1 - \tilde{d}c & 0 \\
  0 & \tilde{c}(2 - \tilde{c}d)
\end{pmatrix}.
\]

We now use a suitable permutation of rows and columns of a \( 4 \times 4 \)-matrix, to conclude that \( V \) and \( V^{-1} \) are similar to

\[
V \sim \begin{pmatrix} 
  \tilde{V} & 0 \\
  0 & I
\end{pmatrix}, \quad V^{-1} \sim \begin{pmatrix} 
  \tilde{V}^{-1} & 0 \\
  0 & I
\end{pmatrix},
\]

where

\[
\tilde{V} = \begin{pmatrix} 
  \tilde{c}(2 - \tilde{c}d) & \tilde{c}d - 1 \\
  1 - \tilde{c}d & \tilde{d}
\end{pmatrix}, \quad \tilde{V}^{-1} = \begin{pmatrix} 
  \tilde{d} & 0 \\
  0 & \tilde{c}(2 - \tilde{c}d)
\end{pmatrix}.
\]

Moreover,

\[
\tilde{V} = SS^* \otimes \begin{pmatrix} 
  1 & 0 \\
  0 & 1
\end{pmatrix} + (1 - SS^*) \otimes \begin{pmatrix} 
  y(2 - xy) & yx - 1 \\
  1 - xy & x
\end{pmatrix},
\]

\[
\tilde{V}^{-1} = SS^* \otimes \begin{pmatrix} 
  1 & 0 \\
  0 & 1
\end{pmatrix} + (1 - SS^*) \otimes \begin{pmatrix} 
  x & 1 - xy \\
  yx - 1 & c(2 - xy)
\end{pmatrix}.
\]

Also notice that the same permutation of rows and columns we used for \( V \) and \( V^{-1} \) can be applied to the idempotent \( \begin{pmatrix} I_2 & 0 \\
  0 & 0 \end{pmatrix} \) to get

\[
\begin{pmatrix} 
  I_2 & 0 \\
  0 & 0
\end{pmatrix} \sim \begin{pmatrix} 
  1 & 0 \\
  0 & 1 \\
  0 & 0 \\
  0 & 1
\end{pmatrix}
\]

and

\[
V \begin{pmatrix} I_2 & 0 \\
  0 & 0 \end{pmatrix} V^{-1} \sim \begin{pmatrix} 
  \tilde{V} & 0 \\
  0 & \tilde{V}^{-1}
\end{pmatrix}.
\]
Next, if we apply the same permutation of rows and columns to $p_U$, we get

$$
(5.21) \quad p_U \sim \begin{pmatrix}
\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & 0 \\
0 & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\end{pmatrix}, \quad \begin{pmatrix} \tilde{V} & \begin{pmatrix} 0 & 0 \\ 0 & \tilde{V}^{-1} \end{pmatrix} \end{pmatrix}.
$$

Finally, note that the pair of the right-lower-blocks in Equation (5.21) describes a rank-one free direct summand in the rank-two free module over $C(CP^2_T)$. Therefore we can rewrite the class $[p_U] \in K_0((C(CP^2_T)))$ as

$$
(5.22) \quad [p_U] = [\tilde{p}_U] + [1], \quad \text{where} \quad \tilde{p}_U := \begin{pmatrix} 0 & \tilde{V} \\ 0 & 0 \end{pmatrix}.
$$

5.2. Homotoping Milnor’s idempotent into an elementary form. In this section, we will homotopy the Milnor idempotent $p_U$ in the simplified form $\tilde{p}_U$ given in Equation (5.22). Let

$$
(5.23) \quad \chi_t(s) := \begin{cases} 1 & \text{for } s \in [0, \frac{1}{t+1}] \text{ mod } 1, \\
0 & \text{otherwise},
\end{cases}
$$

$$
(5.24) \quad \nu_t(s) = \exp \left( -\frac{4\pi i \chi_t(s)s}{1 + t} \right).
$$

Note that $\nu_0 = \nu$.

By using Fourier expansions, we get:

$$
(5.25) \quad \nu_t(s) = c_0(t) + \sum_{k \in \mathbb{N}_+} c_k(t)(e^{2\pi is})^k + \sum_{k \in \mathbb{N}_+} c_{-k}(t)(e^{-2\pi is})^k,
$$

where

$$
(5.26) \quad c_k(t) := \int_0^1 e^{-2\pi i ks} \nu_t(s) ds = \int_0^{1+t} e^{-2\pi i ks} \nu_t(s) ds + \int_0^1 e^{-2\pi i ks} \nu_t(s) ds
$$

$$
= \int_0^{1+t} e^{-2\pi i (k+\frac{2}{t+1})s} ds + \int_0^1 e^{-2\pi i ks} ds.
$$

For $k \neq -2, -1, 0$, we obtain

$$
(5.28) \quad c_k(t) = \frac{i \cdot 1 - e^{-\pi i k(1+t)}}{2\pi k(\frac{k}{t+1} + 1)}.
$$

Moreover:

$$
(5.29) \quad c_{-2}(t) = \frac{1}{2} \begin{cases} 1 & \text{for } t = 0, \\
\frac{e^{2\pi it}}{2\pi it} & \text{for } t \neq 0,
\end{cases}
$$

$$
(5.30) \quad c_{-1}(t) = \begin{cases} \frac{e^{\pi i(t-1)}}{\pi i(t-1)} & \text{for } t \neq 1, \\
1 & \text{for } t = 1,
\end{cases}
$$

$$
(5.31) \quad c_0(t) = \frac{1 - t}{2}.
$$
In particular,
\begin{equation}
    c_k(1) = \begin{cases} 
        0 & \text{for } t \neq -1, \\
        1 & \text{for } t = -1.
    \end{cases}
\end{equation}

Observe that, since for \( k \neq -2, -1, 0 \) and \( t \in [0, 1] \)
\begin{equation}
    |c_k(t)| \leq \frac{6}{\pi k^2},
\end{equation}
we can lift the Fourier series of the homotopy \( t \mapsto \nu_t \) to a homotopy \( t \mapsto x_t \) in the Toeplitz algebra by defining:
\begin{equation}
    x_t := c_0(t) + \sum_{k \in \mathbb{N}_+} c_k(t) S^k + \sum_{k \in \mathbb{N}_+} c_{-k}(t)(S^*)^k, \quad y_t := x_t^*.
\end{equation}

Hence, for \( t = 1 \), we get
\begin{equation}
    x_1 = S^*, \quad y_1 = S.
\end{equation}

Therefore, by replacing \((x, y)\) by \((x_t, y_t)\) in Equation (5.17), we obtain a homotopy \( t \mapsto \tilde{V}_t \) of invertible \( 2 \times 2 \) matrices with entries in \( \mathcal{T} \otimes \mathcal{T} \), with \( \tilde{V}_0 = \tilde{V} \) and
\begin{equation}
    \tilde{V}_1 = SS^* \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (1 - SS^*) \otimes \begin{pmatrix} S & SS^* - 1 \\ 0 & S^* \end{pmatrix},
\end{equation}
\begin{equation}
    \tilde{V}_1^{-1} = SS^* \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (1 - SS^*) \otimes \begin{pmatrix} S^* & 0 \\ SS^* - 1 & S \end{pmatrix}.
\end{equation}

Consequently, after substituting \( \tilde{V}_t \) into Equation (5.10), we obtain a homotopy of idempotents \( t \mapsto \tilde{p}_{U_1} \) with \( \tilde{p}_{U_0} = \tilde{p}_U \), and
\begin{equation}
    \tilde{p}_{U_1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} SS^* \otimes 1 + (1 - SS^*) \otimes SS^* - 1 \\ 0 & 0 \end{pmatrix}.
\end{equation}

Finally, we observe that we can remove all entries except the top left ones in the above formula for \( \tilde{p}_{U_1} \) without changing its class in \( K_0((C(\mathbb{C}P^2)) \), thus getting
\begin{equation}
    [p_U] - 2[1] = [\tilde{p}_{U_1}] - [1] = [p] - [1] \in K_0((C(\mathbb{C}P^2)),
\end{equation}
where
\begin{equation}
    p := (1, SS^* \otimes 1 + (1 - SS^*) \otimes SS^*) = (1, 1) - (0, (1 - SS^*) \otimes (1 - SS^*))
\end{equation}
is an idempotent in \( C(\mathbb{C}P^2) \). Notice also that
\begin{equation}
    [p] - [1] = -[(0, (1 - SS^*) \otimes (1 - SS^*))],
\end{equation}
which by (5.39) expresses the generator \([p_U] - 2[1] \) of \( \partial_{10}(K_1(C(S^3_H))) \subset K_0(C(\mathbb{C}P^2)) \) in terms of the minimal elementary projection \( 1 - SS^* \) in the Toeplitz algebra.

On the other hand, by Corollary 4.7, also \([L_1 + L_{-1}] - 2[1] \) generates \( \partial_{10}(K_1(C(S^3_H))) \) (equal to \( \mathbb{Z} \) by [22]), so
\begin{equation}
    [L_1 + L_{-1}] - 2[1] = \pm ([p] - [1]).
\end{equation}
Suppose that \( C(\mathbb{C}P^2)p \) is a module associated with the \( U(1) \)-C*-algebra \( C(S^3_H) \). Then taking the \( U(1) \)-equivariant *-homomorphism \( f : C(S^3_H) \to \mathcal{T} \otimes \mathcal{T} \otimes C(S^1) \) from Diagram 4.18 to pushforward \( C(\mathbb{C}P^2)p \) to \( (\mathcal{T} \otimes \mathcal{T})f(p) \), by [12], we obtain a positive-rank free module over \((\mathcal{T} \otimes \mathcal{T})\). It follows from (5.41) that the rank of \( C(\mathbb{C}P^2)p \) is one, so it is a spectral subspace \( L_n \) for some integer \( n \). Now we use another \( U(1) \)-equivariant
*-homomorphism $g : C(S^5_H) \to C(S^3_H) \otimes T$ from Diagram 4.18 to pushforward all spectral subspaces in (5.41) to $P_1$:

\[(5.42) \quad [(C(S^3_H) \otimes T)_1 \oplus (C(S^3_H) \otimes T)_{-1}] - 2[1] = \pm([C(S^3_H) \otimes T]_n) - [1]).\]

Then, taking again advantage of [12], we use the equivariant map $\Omega$ in Diagram 4.18 to view any spectral subspace $(C(S^3_H) \otimes T)_k$ as the image of $C(SU_q(2))_k$. Now, since (by Lemma 4.5) $\tilde{\Omega}^*$ is an isomorphism, we apply $\tilde{\Omega}^{-1}$ to (5.43) to obtain

\[(5.43) \quad [C(SU_q(2))_1 \oplus C(SU_q(2))_{-1}] - 2[1] = \pm([C(SU_q(2))_n] - [1]).\]

Finally, we compute an index pairing as in [10, Theorem 2.1] to conclude that $n = 0$. This gives the desired contradiction as $[p] \neq [1]$.

Furthermore, it follows from (4.14) that $[L_1 \oplus L_{-1}] = [C(CP^2_T)^2 p_U]$, so plugging it in (5.41) yields

\[(5.44) \quad [C(CP^2_T)^2 p_U] - 2[1] = \pm([p] - [1]).\]

If we take the above equality with the plus sign, we immediately get Theorem 1.1(1). Next, choosing the minus sign, and remembering that the Milnor map is a group homomorphism, we obtain

\[(5.45) \quad [p] - [1] = -([C(CP^2_T)^2 p_U] - 2[1]) = [C(CP^2_T)^2 p_U - 1] - 2[1].\]

Thus we obtain Theorem 1.1(2) and conclude the proof of our main result.

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