REPETITIVE CLUSTER CATEGORIES OF TYPE $D_n$

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Abstract. In this paper, we show that the repetitive cluster category of type $D_n$, defined as the orbit category $\mathcal{D}^b(\text{mod} kD_n)/(\tau^{-1}[1])^p$, is equivalent to a category defined on a subset of tagged edges in a regular punctured polygon. This generalizes the construction of Schiffler, (23), which we recover when $p=1$.

Introduction

Cluster categories were introduced in (6) and, independently, in (8) for type $A_n$, with the aim of better understanding the cluster algebras of Fomin and Zelevinsky (9) (10). Since then cluster categories have been the subject of many investigations.

In the approach of (6), the cluster category $\mathcal{C}_A$ is defined as the quotient of the derived category $\mathcal{D}^b(A)$ of a hereditary algebra $A$ by the endofunctor $F=\tau^{-1}[1]$, where $\tau$ is the Auslander-Reiten translation in $\mathcal{D}^b(A)$ and $[1]$ is the shift. Thus the objects $\mathcal{M}$ of $\mathcal{C}_A$ are the orbits $\mathcal{M}=(F^i M)_{i \in \mathbb{Z}}$ of objects $M \in \mathcal{D}^b(A)$ and $\text{Hom}_A(\mathcal{M}, N) = \oplus_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{D}^b(A)}(M, F^i N)$.

For any positive integer $p$, the repetitive cluster category $\mathcal{C}_{F^p}(\mathcal{H})$ was introduced by Zhu, in (26), for any hereditary abelian category $\mathcal{H}$ with tilting objects. These are orbit categories $\mathcal{D}^b(\mathcal{H})/F^p$ of the derived category $\mathcal{D}^b(\mathcal{H})$ by the endofunctor $F^p=(\tau^{-1}[1])^p$, where $\tau$ is the Auslander-Reiten translation and $[1]$ is the shift. Repetitive cluster categories are triangulated by Keller (17).

The cluster tilting objects in these repetitive cluster categories are shown to correspond one-to-one to those in the classical cluster categories. The endomorphism algebras of cluster tilting objects in $\mathcal{D}^b(\mathcal{H})/F^p$ are the coverings of the cluster tilted algebras (26).

When $\mathcal{H}$ is an hereditary algebra of type $D_n$ we will denote its repetitive cluster category as $\mathcal{C}_{n,p}$.

When $p=1$ we recover the usual cluster category of type $D_n$, which we denote simply by $\mathcal{C}_n$.

The particularity of the repetitive cluster categories is that they are fractionally Calabi-Yau of dimension $\frac{2n}{p}$, this means that $(\tau[1])^p \cong [2p]$ as triangle functors, and the fraction cannot be simplified (15). It is precisely in this point that the category $\mathcal{C}_{n,p}$ differs from $\mathcal{C}_n$.

The association of geometric models to algebraic categories has been studied and developed by many authors, among others we mention: (4) (5) (7) (8) (18) (19) (23) (25). This approach is not only beautiful but also fruitful as it gives new ways to understand the intrinsic combinatorics of the category. In particular repetitive cluster categories of type $A_n$ were studied by Lamberti in (19). She showed an equivalence of categories between the repetitive cluster category of type $A_n$ and a category of diagonals in a regular $p(n+2)$-gon. The model proposed also leads to a geometric interpretation of cluster tilting objects in the repetitive cluster category for type $A_n$.

In this paper, we give a geometric realization of the repetitive cluster categories of type $D_n$ in the spirit of (8) and (19). We adapt the geometric description given by Schiffler in (23).

Our main result is the equivalence of the category of tagged edges and the repetitive cluster category $\mathcal{C}_{n,p}$, see Corollary (7.2).

The article is organized as follows. After a preliminary section, in which we fix the notations and recall some concepts needed later, section 2 is devoted to recall the definition of the repetitive cluster category $\mathcal{C}_{n,p}$ and some basic properties. In section 3, we study the relation between repetitive cluster categories and cluster categories. Section 4 is dedicated to the definition of the category $\mathcal{C}(\mathcal{P}_{np})$ of
tagged edges. We show the equivalence of this category and the repetitive cluster category in section 5. Finally, in section 6, we give a geometric interpretation of cluster tilting objects in $C_{n,p}$.

1. Definitions and Preliminaries

1.1. Notation. Let $k$ be an algebraically closed field. If $Q$ is a quiver, we denote by $Q_0$ the set of vertices and by $Q_1$ the set of arrows of $Q$. The path algebra of $Q$ over $k$ will be denoted by $kQ$. It is of finite representation type if there is only a finite number of isoclasses of indecomposable modules. By Gabriel’s theorem $kQ$ is of finite representation type if and only if $Q$ is a Dynkin quiver, that is, the underlying graph of $Q$ is a Dynkin diagram of type $A_n, D_n$ or $E_n$ [11].

If $A$ is an algebra, we denote by $\text{mod} A$ the category of finitely generated right $A$-modules and by $\text{ind} A$ a full subcategory whose objects are a full set of representatives of the isoclasses of indecomposable $A$-modules. Let $\mathcal{D}^b(\text{mod} A)$ denote the derived category of bounded complexes of finitely generated $A$-modules. For further facts about $\text{mod} (A)$ and $\mathcal{D}^b(\text{mod} A)$ we refer the reader to [1, 2, 12, 22].

Throughout the paper we denote by $\mathcal{D}$ the bounded derived category $\mathcal{D}^b(\text{mod} D_n)$.

1.2. Serre duality and Calabi-Yau categories. Let $k$ be a field and let $\mathcal{K}$ be a $k$-linear triangulated category which is $\text{Hom}$-finite, i.e. for any two objects in $\mathcal{I}$ the space of morphisms is a finite dimensional vector space.

Remember from [10] that a $k$-triangulated category $\mathcal{K}$ has a Serre functor if it is equipped with an auto-equivalence $\nu : \mathcal{K} \to \mathcal{K}$ together with bifunctorial isomorphisms

$$\text{DHom}_{\mathcal{K}}(X,Y) \cong \text{Hom}_{\mathcal{K}}(Y,\nu X),$$

for each $X, Y \in \mathcal{K}$, where $\text{D}$ indicates the vector space duality $\text{Hom}_k(-, k)$.

We will say that $\mathcal{K}$ has Serre duality if $\mathcal{K}$ admits a Serre functor. If $\mathcal{D}$ denotes the category $\mathcal{D}^b(\text{mod} D_n)$ and we consider the case $\mathcal{K} = \mathcal{D}$ a Serre functor exists ([10] p. 24), it is unique up to isomorphism and $\nu \to \tau[1]$, where $\tau$ is the Auslander-Reiten translate and [1] is the shift functor of $\mathcal{D}$.

For $n, m > 0$, a category $\mathcal{K}$ with Serre functor $\nu$ is said to be fractionally Calabi-Yau of dimension $\frac{m}{n}$ or $\frac{m}{n}$-Calabi-Yau if there is an isomorphism of triangle functors:

$$\nu^n \cong [m]$$

where $[m]$ indicates the composition of the shift functor with itself $m$ times.

1.3. Translation quivers. Following [21], we define a stable translation quiver to be a pair $(\Gamma, \tau)$ where $\Gamma$ is a locally finite quiver and $\tau : \Gamma_0' \to \Gamma_0'$ is an injective map defined on a subset $\Gamma_0'$ of the vertices of $\Gamma$ such that for any $x \in \Gamma_0$, $y \in \Gamma_0'$, the number of arrows from $x$ to $y$ is the same as the number of arrows from $\tau(y)$ to $x$.

If $\Gamma_0' = \Gamma_0$ and $\tau$ is bijective, $(\Gamma, \tau)$ is called a stable translation quiver.

A stable translation quiver is said to be connected if it is not a disjoint union of two non-empty stable subquivers.

Given a stable translation quiver $(\Gamma, \tau)$, a polarization of $\Gamma$ is a bijection $\sigma : \Gamma_1 \to \Gamma_1$ such that $\sigma(\alpha) : \tau x \to y$ for every arrow $\alpha : y \to x \in \Gamma_1$. If $\Gamma$ has no multiple arrows, then there is a unique polarization.

We remark that in all examples of stable translation quivers appearing in this article, the number of arrows between two vertices is always at most 1.

The path category of $(\Gamma, \tau)$ is the category whose objects are the vertices of $\Gamma$, and given $x, y \in \Gamma_0$, the $k$-space of morphisms from $x$ to $y$ is given by the $k$-vector space with basis the set of all paths from $x$ to $y$. The composition of morphisms is induced from the usual composition of paths.

The mesh ideal in the path category of $\Gamma$ is the ideal generated by the mesh relations

$$m_x = \sum_{\alpha : y \to \tau x} \sigma(\alpha)\alpha.$$
The mesh category $\mathcal{M}(\Gamma, \tau)$ of $(\Gamma, \tau)$ is the quotient of the path category of $(\Gamma, \tau)$ by the mesh ideal.

Given a quiver $Q$ one can construct a stable translation quiver $\mathbf{Z}Q$ as follows: $(\mathbf{Z}Q)_0 = \mathbf{Z} \times Q_0$ and the number of arrows in $\mathbf{Z}Q$ from $(i, x)$ to $(j, y)$ equals the number of arrows in $Q$ from $x$ to $y$ if $i = j$, and equals the number of arrows in $Q$ from $y$ to $x$ if $j = i + 1$, and there are no arrows otherwise. The translation $\tau$ is defined by $\tau((i, x)) = (i - 1, x)$.

Important examples of translation quivers are the Auslander-Reiten quivers of the derived categories of hereditary algebras of finite representation type. We shall need the following proposition.

1.1. Proposition. Let $Q$ be a Dynkin quiver. Then

(1) the Auslander-Reiten quiver of $\mathcal{D}^b(\text{mod} Q)$ is $\mathbf{Z}Q^{\text{op}}$.

(2) the category $\text{ind} \mathcal{D}^b(\text{mod} Q)$ is equivalent to the mesh category of $\mathbf{Z}Q^{\text{op}}$.

1.3.1. The stable translation quiver of $\mathcal{D}$. Let $Q$ be a quiver of underlying Dynkin type $D_n$. We denote the vertices of $Q$ with $0, \bar{0}, 1, \cdots, n - 2$ and the arrows are $i - 1 \to i$ ($i = 1, \cdots, n - 2$) together with $\bar{0} \to 1$; see Figure 1.

\[
\begin{array}{c}
(n - 2) \leftarrow (n - 3) \leftarrow \cdots \leftarrow 2 \leftarrow 1 \\
\downarrow \quad \downarrow \quad \downarrow \\
\bar{0} \quad \bar{0} \quad \bar{0}
\end{array}
\]

\textbf{Figure 1.} The quiver of type $D_n$

By proposition 1.1, the Auslander-Reiten quiver of $\mathcal{D}$ is the stable translation quiver $\mathbf{Z}Q^{\text{op}}$. The labels of $Q_0$ induce labels on the vertices of $\mathbf{Z}Q^{\text{op}}$ as usual:

$$(\mathbf{Z}Q^{\text{op}})_0 = \{(i, j) \mid i \in \mathbf{Z}, j \in Q_0\} = \mathbf{Z} \times \{0, \bar{0}, 1, \ldots, n - 2\}.$$

Moreover, we can identify the indecomposable objects of $\mathcal{D}$ with the vertices of $\mathbf{Z}Q$. Let $P_1$ be the indecomposable projective module corresponding to the vertex $1 \in Q_0$. Then, by defining the position of $P_1$ to be $(0, 1)$, we have a bijection

$$\text{pos} : \text{ind} \mathcal{D} \to \mathbf{Z} \times \{0, \bar{0}, 1, \ldots, n - 2\}.$$ 

In other terms, for $M \in \text{ind} \mathcal{D}$, we have $\text{pos} (M) = (i, j)$ if and only if $M = \tau^{-i} P_j$, where $P_j$ is the indecomposable projective $A$-module at vertex $j$. The "integer" $j \in \{0, 1, \ldots, n - 2\}$ is called the level of $M$ and will be denoted by $\text{level} (M)$.

If $\text{pos} (M) = (i, j)$ with $j \in \{0, \bar{0}\}$ then let $M^-$ be the indecomposable object such that $\text{pos} (M^-) = (i, j')$, where $j'$ is the unique element in $\{0, \bar{0}\} \setminus \{j\}$.

The structure of the derived category $\mathcal{D}$ is well known. In particular, we have the following result.

1.2. Lemma. Let $M \in \text{ind} \mathcal{D}$.

(1) If $n$ is even, then $M[1] = \tau_D^{-n+1} M$.

(2) If $n$ is odd, then

$$M[1] = \begin{cases} 
\tau_D^{-n+1} M & \text{if } \text{level} (M) \notin \{0, \bar{0}\} \\
\tau_D^{-n+1} M^- & \text{if } \text{level} (M) \in \{0, \bar{0}\}
\end{cases}$$
2. Repetitive cluster categories of type $D_n$

In the following we give the algebraic description of the repetitive cluster category of type $D_n$. This is the orbit category of the bounded derived category $D$ of $\text{mod} D_n$ under the action of the cyclic group generated by the auto-equivalence $(\tau^{-1}[1])^p = \tau^{-p}[p]$ for $p > 0$, where $\tau$ is the AR-translation in $D$ and $[1]$ is the shift functor. Repetitive cluster categories were defined in [26] as orbit categories of $D^b(\mathcal{K})$, for $\mathcal{K}$ a hereditary abelian category with tilting objects.

2.1. Definition. The repetitive cluster category

$$\mathcal{C}_{n,p} := D/ < \tau^{-p}[p] >$$

of type $D_n$, has as class of objects the same as in $D$. The class of morphism is given by:

$$\text{Hom}_{\mathcal{C}_{n,p}}(X,Y) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_D(X,(\tau^{-p}[p])^iY)$$

Observe that when $p = 1$, one gets back the usual cluster category which we simply denote by $\mathcal{C}_n$.

Furthermore, one can define the projection functor $\eta_p : \mathcal{C}_{n,p} \to \mathcal{C}_n$ which sends an object $X$ in $\mathcal{C}_{n,p}$ to an object $X$ in $\mathcal{C}_n$, and $\phi : X \to Y$ in $\mathcal{C}_{n,p}$ to the morphism $\phi : X \to Y$ in $\mathcal{C}_n$, [26]. Then one has that $\pi_1 = \pi_p \circ \eta_p$, where $\pi_p : D \to \mathcal{C}_{n,p}$ and $\pi_1 : D \to \mathcal{C}_n$ are the projections.

In the following let $\mathcal{F} = \mathcal{F}_1$ be the fundamental domain for the functor $F := \tau^{-1}[1]$ in $D$ given by the isoclasses of indecomposables objects in $\text{mod} D_n$ together with the $[1]$-shift of the projective indecomposable modules. After [4, Proposition 1.6] one can identify the subcategory of isomorphism classes of indecomposable objects of $\mathcal{C}_n$, denoted by $\text{ind}(\mathcal{C}_n)$, with the objects in $\mathcal{F}$. Let $F^k := F \circ \cdots \circ F$, $k$-times, then denote by $\mathcal{F}_k$ the $F^k$-shift of $\mathcal{F}$ and we can draw the fundamental domain for the functor $\tau^{-p}[p]$ as in Figure 2.

![Figure 2. Partition of the fundamental domain of $\tau^{-p}[p]$](image)

As next we summarize some basic properties of $\mathcal{C}_{n,p}$ proven in [26 Proposition 3.3].

2.2. Lemma. Let $\mathcal{C}_{n,p}$ be the repetitive cluster category of type $D_n$ and $\mathcal{C}_n$ be the cluster category of type $D_n$. Then

(1) $\mathcal{C}_{n,p}$ is a triangulated category with AR-triangles and Serre functor $\nu := \tau[1]$.
(2) The projections $\pi_p : D \to \mathcal{C}_{n,p}$ and $\eta_p : \mathcal{C}_{n,p} \to \mathcal{C}_n$ are triangle functors.
(3) $\mathcal{C}_{n,p}$ is fractionally CY of dimension $\frac{2p}{p}$.
(4) $\mathcal{C}_{n,p}$ is a Krull-Schmidt category.
(5) $\text{ind}(\mathcal{C}_{n,p}) = \bigcup_{i=1}^{p} \text{ind}(\mathcal{F}_i)$.

Repetitive cluster categories were studied by Zhu in [26] from a purely algebraic point of view and for type $A_n$ by Lamberti in [19] in a geometrical-combinatorial way. We will do a similar study for type $D_n$. 
2.1. The Auslander-Reiten quiver of $\mathcal{C}_{n,p}$. Since the Auslander-Reiten quiver of the cluster category $\mathcal{C}_n$, denoted by $\Gamma(D_n,1)$, is the stable translation quiver built from $n$ copies of $Q^\text{op}$ (see [9 §1], [13]), after Lemma 2.2 we can see the Auslander-Reiten quiver of the repetitive cluster category $\mathcal{C}_{n,p}$, denoted by $\Gamma_{n,p}$, as the stable translation quiver built from $np$ copies of $Q^\text{op}$. The vertices of $\Gamma_{n,p}$ are:

$$(\Gamma_{n,p})_0 = \{(i,j) \mid i \in \mathbb{Z}_{np}, j \in Q_0\} = \mathbb{Z}_{np} \times \{0, \bar{0}, 1, \ldots, n - 2\};$$

and there is an arrow $(i,j) \rightarrow (i,k)$ and an arrow $(i,k) \rightarrow (i+1,k)$ whenever there is an arrow $j \rightarrow k$ in $Q^\text{op}$. Finally, the translation $\tau$ is given by

$$\tau(i,j) = \begin{cases} (i-1,j), & \text{if } i = 0, j \in \{0, \bar{0}\} \text{ and } np \text{ is odd;} \\ (i-1,j), & \text{otherwise.} \end{cases}$$

We use the convention that $\bar{0} = 0$. Note that the switch described here only occurs for odd $np$.

As an example, we draw the quivers $\Gamma(D_n,1)$ for $n = 3$ and $n = 4$; see Figures 3 and 4. The translation $\tau$ is indicated by dotted lines (it is directed to the left).

![Figure 3. The quiver $\Gamma(D_3,1)$](image)

![Figure 4. The quiver $\Gamma(D_4,1)$](image)

3. $\mathcal{C}_{n,p}$ and the link to the cluster category

One of our goals is to realise the Auslander-Reiten quiver (or AR-quiver for short) for the repetitive cluster category in terms of the AR-quiver of a cluster category of type $D_t$ for certain value of $t$.

In the following denote by $(\Gamma_{n,p}, \tau_{n,p})$ the AR-quiver of the repetitive cluster category $\mathcal{C}_{n,p}$ and by $(\Gamma(D_t,1), \tau_t)$ the AR-quiver of the cluster category $\mathcal{C}_t$.

3.1. Lemma. Let $t = np$. Then $(\Gamma_{n,p}, \tau_{n,p})$ is a subquiver of $(\Gamma(D_t,1), \tau_t)$.

Proof. To prove the claim we establish an isomorphism of stable translation quivers between $(\Gamma_{n,p}, \tau_{n,p})$ and a stable translation subquiver of $(\Gamma(D_{np},1), \tau_{np})$.

We observe that the union of the $\tau_t$-orbits of the bottom $n$ rows of the quiver $\Gamma(D_t,1)$, illustrated in the darkest strip in figure 5 defines a subquiver $\tilde{\Gamma}_t$ of $\Gamma(D_t,1)$. 
As $\Gamma(D_{1}, 1)$ is a stable translation quiver, the same remains true for $\tilde{\Gamma}_{t}$. By construction, the two quivers have the same number of rows (namely $n$).

Remember that $\Gamma(D_{r}, 1)$ identifies the vertices $(0, j)$ and $(r, j)$ for every $j \notin \{0, 0\}$. For $r$ even, also identifies the vertices $(0, 0)$ and $(0, 0)$ with the vertices $(r, 0)$ and $(r, 0)$ respectively. However, for $r$ odd it identifies $(0, 0)$ with $(r, 0)$ and $(0, 0)$ with $(r, 0)$.

To see the isomorphism just on the level of quivers we compare the induced action of the autoequivalence $(\tau^{-1})_{n,p}^{}[1]$ on $\Gamma(n,p) \subset \Gamma(D_{np}, 1)$. It is easy to check that the actions coincide because $np$ is odd if and only if $n$ and $p$ are odd. Furthermore, because the meshes of the quivers $\tilde{\Gamma}_{t}$ and $\Gamma(n,p)$ coincide we deduce that this gives an isomorphism of stable translation quivers.

Now, we describe the other component arising in the translation quiver $(\Gamma(D_{np}, 1), \tau)$.

3.2. Proposition. The quiver $\Gamma(D_{np}, 1)$ has 1 connected component isomorphic to the Auslander-Reiten quiver of $D^{b}(A_{n(p-1)})/\tau_{np}$.

Proof. We consider the following subset $Z$ of vertices of the quiver $\Gamma(D_{np}, 1)$:

$$Z := \{(i, j) \mid i \in \mathbb{Z}_{np}, j \in \{n-1, \ldots, np-2\}\}$$

Such a set $Z$ is the union of the first $n(p-1)$ orbits at the top of the quiver $\Gamma(D_{np}, 1)$. It is clear that the translation quiver generated by $Z$ (i.e. the full subquiver induced by $Z$, together with $\tau$) is a connected component of $\Gamma(D_{np}, 1)$. Each row is of length $np$. Then $Z$ is isomorphic to the Auslander-Reiten quiver of $D^{b}(A_{n(p-1)})/\tau_{np}$.

Since, by lemma 3.1, $\Gamma_{n,p}$ is a connected component of $\Gamma(D_{np}, 1)$ and the vertices of $\Gamma(D_{np}, 1)$ are exhausted by the vertices of $\Gamma_{n,p}$ and $Z$, we obtain the following corollary.

3.3. Corollary. The quiver $(\Gamma(D_{np}, 1), \tau)$ is the union of the following connected components:

$$(\Gamma(D_{np}, 1), \tau_{np}) = \Gamma_{n,p} \cup \Gamma(D^{b}(A_{n(p-1)})/\tau_{np}),$$

where $\Gamma(D^{b}(A_{n(p-1)})/\tau_{np})$ denotes the Auslander-Reiten quiver of $D^{b}(A_{n(p-1)})/\tau_{np}$.

We illustrate on the following example the results presented.

3.4. Example. Let $n = 4$ and $p = 2$. 

\[\text{Figure 5. Inclusion } \Gamma_{n,p} \subset \Gamma(D_{np}, 1).\]
3.1. **Triangulated equivalence for \( \mathcal{C}_{n,p} \).** Here we desire to compare the category \( \mathcal{C}_{n,p} \) with the cluster category of type \( D_t \) for \( t = np \).

Since the inclusion of the AR-quiver of \( \mathcal{C}_{n,p} \) in the AR-quiver of the cluster category \( \mathcal{C}_t \) for \( t = np \) does not give rise to an inclusion at the level of full subcategories, \( \mathcal{C}_{n,p} \) is in particular not a triangulated subcategory of \( \mathcal{C}_{np} \). However, it is possible to prove that it is triangulated equivalent to a quotient category of \( \mathcal{C}_{np} \) in the sense of Jørgensen [14].

We now recall the definition of quotient categories and some properties. Let \( \mathcal{C} \) be an additive category and \( \mathcal{X} \) a class of objects of \( \mathcal{C} \). Then the *quotient category* \( \mathcal{C}_\mathcal{X} \) has by definition the same objects as \( \mathcal{C} \), but the morphism spaces are taken modulo all the morphisms factoring through an object of \( \mathcal{X} \).

Observe that if \( \mathcal{C} \) is a triangulated category, then \( \mathcal{C}_\mathcal{X} \) needs not to be triangulated for all choices of \( \mathcal{X} \). However, \( \mathcal{C}_\mathcal{X} \) is always pre-triangulated ([14] Theorem 2.2) and taking a particular choice of the class \( \mathcal{X} \), \( \mathcal{C}_\mathcal{X} \) becomes a triangulated category ([14] Theorem 3.3]).
3.5. Proposition. The repetitive cluster category $\mathcal{C}_{n,p}$ is triangulated equivalent to a quotient of the cluster category $\mathcal{C}_t$ for $t = np$.

Proof. Denote by $X$ the additive full subcategory generated by the indecomposable objects in the first $n(p - 1)$ orbits at the top of the quiver $\Gamma(D_{np},1)$. Then, we have that $\tau_t X = X$. So, by [14, Theorem 4.2], the AR-quiver of the quotient category $(\mathcal{C}_t)_X$ is obtained by deleting the vertices corresponding to the objects of $X$ and the arrows linked with them. Then, $(\mathcal{C}_t)_X$ is connected, and has finitely many indecomposable objects up to isomorphism. Furthermore, again by [14, Theorem 4.2] $(\mathcal{C}_t)_X$ is standard and of algebraic origin. Proceeding as in [14, Theorem 5.2] we conclude that $(\mathcal{C}_t)_X$ is triangulated equivalent to a quotient of a cluster category of type $D_n$.

It remains to see that $(\mathcal{C}_t)_X$ and $\mathcal{C}_{n,p}$ are equivalent as triangulated categories. For this we observe that $\mathcal{C}_{n,p}$ is of algebraic origin by results of [16, Section 9.3] and standard by [3, Proposition 6.1.1.]. Furthermore, it is straightforward to see that the AR-quivers of $(\mathcal{C}_t)_X$ and $\mathcal{C}_{n,p}$ are isomorphic as translation quivers. Thus we are in the conditions of Amiot’s Theorem [14, Theorem 5.1] applied to $(\mathcal{C}_t)_X$ and $\mathcal{C}_{n,p}$. Hence we deduce that these categories are equivalent as triangulated categories, and so the claim follows. □

4. Geometric model of $\mathcal{C}_{n,p}$

In this section we present the geometric model for $\mathcal{C}_{n,p}$. It is a simple modification of the model constructed by Schiffler in [23] for the cluster category of type $D_n$.

4.1. Tagged edges. Let $n \geq 3$ and $p \geq 1$. Consider a regular polygon $P_{np}$ with $np$ vertices and one puncture in its center. We label the vertices of $P_{np}$ counterclockwise $1, 2, \ldots, np$.

If $a \neq b$ are any two vertices on the boundary then let $\delta_{a,b}$ denote a path along the boundary from $a$ to $b$ in counterclockwise direction which does not run through the same point twice. Let $\delta_{a,a}$ denote a path along the boundary from $a$ to $a$ in counterclockwise direction which goes around the polygon exactly once, and such that $a$ is the only point through which $\delta_{a,a}$ runs twice. For $a \neq b$, let $|\delta_{a,b}|$ be the number of vertices on the path $\delta_{a,b}$ (including $a$ and $b$), and let $|\delta_{a,a}| = n + 1$.

Figure 9. The path $\delta_{a,b}$ on the punctured polygon $P_{np}$.

An edge is a triple $(a, \alpha, b)$ where $a$ and $b$ are vertices of the punctured polygon and $\alpha$ is a path from $a$ to $b$ such that

(E1) $\alpha$ is homotopic to $\delta_{a,b}$,
(E2) except for its starting point $a$ and its endpoint $b$, the path $\alpha$ lies in the interior of the punctured polygon,
(E3) $\alpha$ does not cross itself, that is, there is no point in the interior of the punctured polygon through which $\alpha$ runs twice,
(E4) $|\delta_{a,b}| \geq np - n + 3$.

Two edges $(a, \alpha, b), (c, \beta, d)$ are equivalent if $a = c$, $b = d$ and $\alpha$ is homotopic to $\beta$. Let $E$ be the set of equivalence classes of edges. Then, an element of $E$ is uniquely determined by the ordered pair of vertices $(a, b)$. We will therefore use the notation $M_{a,b}$ for the equivalence class of edges $(a, \alpha, b)$ in $E$. 
Define the set of tagged edges $E'$ as follows:

$$E' = \{ M^\epsilon_{a,b} \mid M_{a,b} \in E, \epsilon = \pm 1 \text{ and } \epsilon = 1 \text{ if } a \neq b \}$$

If $a \neq b$, we will often drop the exponent and write $M_{a,b}$ instead of $M^1_{a,b}$.

A simple count shows that there are $pnr^2$ elements in $E'$. These tagged edges will correspond to the indecomposable objects in the repetitive cluster category.

4.2. **Elementary moves.** Adapting a concept from [23], we will define elementary moves, which will correspond to irreducible morphisms in the repetitive cluster category.

An elementary move sends a tagged edge $M_{a,b} \in E'$ to another tagged edge $M'^{\epsilon'}_{a',b'} \in E'$ in the following way:

1. If $\delta_{a,b} = np - n + 3$, then there is precisely one elementary move $M_{a,b} \mapsto M_{a,b+1}$.
2. If $np - n + 4 \leq |\delta_{a,b}| \leq np - 1$, then there are precisely two elementary moves $M_{a,b} \mapsto M_{a+1,b}$ and $M_{a,b} \mapsto M_{a,b+1}$.
3. If $|\delta_{a,b}| = np$, then there are precisely three elementary moves $M_{a,b} \mapsto M_{a+1,b}$, $M_{a,b} \mapsto M^1_{a,a}$ and $M_{a,b} \mapsto M^{-1}_{a,a}$.
4. If $|\delta_{a,b}| = np + 1$, then $a = b$ and there is precisely one elementary move $M^1_{a,a} \mapsto M_{a+1,a}$.

4.1. **Notation.** Observe that when we write $M_{a,b}$ the indices $a, b$ have to be taken modulo $np$.

4.3. **Translation.** We define the translation $\tau$ to be the following bijection $\tau : E' \to E'$:

1. If $a \neq b$ then $\tau M_{a,b} = M_{a-1,b-1}$.
2. If $a = b$ then $\tau M^\epsilon_{a,a} = M^{-\epsilon}_{a-1,a-1}$, for $\epsilon = \pm 1$.

The next lemma follows immediately from the definition of $\tau$.

4.2. **Lemma.** Let $\tau$ be the translation defined above. Then:

1. If $np$ is even then $\tau^{np} = id$.
2. If $np$ is odd then $\tau^{np} M^\epsilon_{a,b} = \begin{cases} M^\epsilon_{a,b} & \text{if } a \neq b, \\ M^{-\epsilon}_{a,a} & \text{if } a = b. \end{cases}$

Observing that our tagged edges are a subset of the set of tagged edges defined in [23]. Precisely, the ones with $|\delta_{a,b}| \geq np - n + 3$ instead of $|\delta_{a,b}| \geq 3$. For $p = 1$ we have exactly the definition of [23]. In particular, we have the following lemma.

4.3. **Lemma.** [23 Lemma 3.6] Let $M^\lambda_{a,b}$, $M^\epsilon_{c,d}$ be two tagged edges. Then there is an elementary move $M^\lambda_{a,b} \mapsto M^\epsilon_{c,d}$ if and only if there is an elementary move $\tau M^\epsilon_{c,d} \mapsto M^\lambda_{a,b}$.
4.4. Quiver of tagged edges of \( P_{np} \). As next we associate a translation quiver \( \Gamma_\odot \) to the tagged edges of \( P_{np} \) with the intention of modelling the AR-quiver of the category \( \mathcal{E}_{n,p} \).

4.4. Definition. Let \( \Gamma_\odot \) be the quiver whose vertices are the tagged edges \( M \in E' \) on the punctured polygon \( P_{np} \). Given \( M, N \in E' \) there is an arrow \( M \to N \) in \( \Gamma_\odot \) whenever there is an elementary move \( M \mapsto N \).

Note that \( \Gamma_\odot \) has no loops and no multiple arrows.

4.5. Lemma. The pair \((\Gamma_\odot, \tau)\) is a stable translation quiver.

Proof. Clearly the map \( \tau \) is bijective. As \( \Gamma_\odot \) is finite, we only need to persuade us that the number of arrows from a tagged edge \( M \) to a tagged edge \( N \) is equal to the number of arrows from \( \tau N \) to \( M \). As there is at most one arrow between any two tagged edges, we only have to check that there is an arrow from \( M \) to \( N \) if and only if there is an arrow from \( \tau N \) to \( M \). It follows directly from Lemma 4.3 \( \square \)

4.5. The category of tagged edges \( \mathcal{E}(P_{np}) \). We will now define a \( k \)-linear additive category of tagged edges \( \mathcal{E}(P_{np}) \) as the mesh category \( \mathcal{M}(\Gamma_\odot, \tau) \) of \((\Gamma_\odot, \tau)\) (as in section 1.3). More specifically, the objects are direct sums of tagged edges in \( E' \). The set of morphisms from a tagged edge \( Y \) to a tagged edge \( X \) is the quotient of the vector space over \( k \) spanned by sequences of elementary moves from \( Y \) to \( X \) by the subspace generated by the mesh relations

\[
m_X = \sum_{Y \to X} \tau Y \xrightarrow{\sigma(\alpha)} Y \xrightarrow{\alpha} X.
\]

5. Equivalence of categories

In this section, we will prove the equivalence between the category \( \mathcal{E}(P_{np}) \) and the repetitive cluster category \( \mathcal{E}_{n,p} \).

5.1. Theorem. The quiver \((\Gamma_\odot, \tau)\) is a translation quiver isomorphic to the Auslander-Reiten quiver \( \Gamma_{n,p} \) of \( \mathcal{E}_{n,p} \).

Proof. Consider the morphism \( \phi_p : \Gamma_{n,p} \to \Gamma_\odot \) such that

\[
\phi_p(i, j) = \begin{cases} 
M_{i+1,i+n+p+1-j}, & \text{if } j \notin \{0,0\}; \\
M_{i+1,i+1}, & \text{if } j = 0 \text{ and } i \text{ is even}; \\
M_{i+1,i+1}, & \text{if } j = 0 \text{ and } i \text{ is odd}; \\
M^{-1}_{i+1,i+1}, & \text{otherwise}.
\end{cases}
\]

It is clear that \( \phi_p \) is a bijection between the vertices of both quivers that sends the \( \tau \)-orbit of the vertex \((0, j)\) to the \( \tau \)-orbit of the vertex \( M_{1,n+p+1-j} \); the \( \tau \)-orbit of the vertex \((0,0)\) to the \( \tau \)-orbit of the vertex \( M_{1,1} \); and the \( \tau \) orbit of the vertex \((0,0)\) to the \( \tau \)-orbit of the vertex \( M^{-1}_{1,1} \). Moreover, the arrows \((i, j) \to (i', j')\) agrees with the elementary moves \( \phi_p(i, j) \to \phi_p(i', j') \).

\( \square \)

Since \( \mathcal{E}(P_{np}) \) is the mesh category \( \mathcal{M}(\Gamma_\odot, \tau) \) of \((\Gamma_\odot, \tau)\) we obtain the following corollary.

5.2. Corollary. The repetitive cluster category of type \( D_n \) is equivalent to the category of tagged edges \( \mathcal{E}(P_{np}) \).

\( \square \)

Observe that the fundamental domain \( \mathcal{F} \) of the cluster category \( \mathcal{E}_n \) is in correspondence (via \( \phi_p \)) with the tagged edges \( M_{n,b}^a \) for \( a \in \{1, \cdots, n\} \); and in general \( \mathcal{F}_k \), the \( F^k \)-shift of \( F \), is in correspondence (via \( \phi_p \)) with the tagged edges \( M_{n,b}^a \) for \( a \in \{(k-1)n + 1, \cdots, kn\} \). Then the action of \( F \) on \( \mathcal{E}_{n,p} \) can be see as a counterclockwise rotation \( \rho \) through \( \frac{2\pi}{p} \) around the center of \( P_{np} \).
Given a tagged edge $M_{a,b}^p \in \mathcal{P}_{np}$, we can identify the vertices $a + 1$ and $a + 1 + n(p - 1)$ and delete all the edges between. This gives us a new tagged edge $\mu_p(M_{a,b}^p) \in \mathcal{P}_n$. The indices $a, b$ have to be taken modulo $np$ on the punctured polygon $\mathcal{P}_{np}$ and modulo $n$ on the punctured polygon $\mathcal{P}_n$.

![Figure 11. Example of $\mu_2 : \mathcal{P}_8 \rightarrow \mathcal{P}_4$.](image)

It follows that this projection $\mu_p : \mathcal{P}_{np} \rightarrow \mathcal{P}_n$ corresponds to the projection functor $\eta_p : \mathcal{C}_{n,p} \rightarrow \mathcal{C}_n$. Moreover we have the following commutative diagram

\[
\begin{array}{ccc}
\mathcal{C}_{n,p} & \xrightarrow{\eta_p} & \mathcal{C}_n \\
\downarrow{\phi_1} & & \downarrow{\phi_1} \\
\mathcal{C}(\mathcal{P}_{np}) & \xrightarrow{\mu_p} & \mathcal{C}(\mathcal{P}_n)
\end{array}
\]

5.3. **Example.** We illustrate the Auslander-Reiten quiver of the category $\mathcal{C}(\mathcal{P}_{np})$ for $p = 2$ and $n = 3$. The translation $\tau$ is indicated by dotted lines (it is directed to the left).

![Figure 12. The quiver $\Gamma_\phi$ for $p = 2$ and $n = 3$.](image)

6. **Cluster tilting theory for $\mathcal{C}_{n,p}$**

In this section we are interested in understanding the cluster tilting objects of $\mathcal{C}_{n,p}$, and compare them with configurations of tagged edges in the punctured polygon $\mathcal{P}_{np}$. When $p = 1$, it is known that cluster tilting objects of $\mathcal{C}_n$ correspond to triangulations of a regular polygon with $n$ vertices and one puncture; i.e. a maximal collection of pairwise non crossing tagged edges.

Cluster tilting objects in $\mathcal{C}_{n,p}$ have been studied from an algebraic point of view in [26].

In the following definition let $\text{add}(T)$ be the full subcategory consisting of direct summands of direct sum of finitely many copies of $T$.

6.1. **Definition.** [26] An object $T \in \mathcal{C}_{n,p}$ is called a **cluster tilting object** if for any object $X \in \mathcal{C}_{n,p}$ we have that
of Proposition. we have:

Suppose that 

Proof. 

If \( T = T_1 \oplus T_2 \oplus \cdots \oplus T_k \) is an object in \( \mathcal{C}_{n,p} \) denote by \( \mathcal{X}_T \) the set of tagged edges on the punctured polygon \( P_{np} \) via the isomorphism \( \phi_p \)

\[
T = T_1 \oplus T_2 \oplus \cdots \oplus T_k \overset{\phi_p}{\mapsto} \mathcal{X}_T = \{ \phi_p(T_1), \ldots, \phi_p(T_k) \}
\]

Taking \( p = 1 \) we have the next result that follows from [23].

6.2. Lemma. \( T \) is a cluster tilting object in \( \mathcal{C}_n \) if and only if \( \mathcal{X}_T \) is a triangulation of the regular punctured polygon \( P_n \). The cardinality of \( \mathcal{X}_T \) is \( n \).

Now we are going to state a similar result for cluster tilting objects in \( \mathcal{C}_{n,p} \).

If \( \mathcal{X}_T \) is a set of tagged edges \( M'_{a,b} \), with \( a, b \in \{1, \cdots, n\} \), of \( P_n \), by abuse of notation we also denote by \( \mathcal{X}_T \) the set of tagged edges \( M'_{a,b} \) with \( a, b \in \{1, \cdots, n\} \) of \( P_{np} \). Recall that \( \rho \) is the counterclockwise rotation through \( \frac{2\pi}{p} \) around the center of \( P_{np} \) (which corresponds with the action of \( F \) on \( \mathcal{C}_{n,p} \)). Then we have:

6.3. Proposition. \( T \) is a cluster tilting object of \( \mathcal{C}_{n,p} \) if and only if there is a cluster tilting object \( T' \) of \( \mathcal{C}_n \) such that

\[
\mathcal{X}_T = \mathcal{X}_{T'} \cup \rho(\mathcal{X}_{T'}) \cup \cdots \cup \rho^{p-1}(\mathcal{X}_{T'})
\]

Proof. Suppose that \( T \) is a cluster tilting object in \( \mathcal{C}_{n,p} \). Then by [26] Theorem 3.5 \( T' = \eta_p(T) \) is a cluster tilting object in \( \mathcal{C}_n \). By Lemma 6.2 \( \mathcal{X}_{T'} \) is a triangulation of the regular punctured polygon \( P_n \) with \( n \) elements. Let \( \mathcal{X} := \mu_p^{-1}(\mathcal{X}_{T'}) \) the corresponding set of tagged edges in \( P_{np} \). Then \( \mathcal{X}_T := \phi_p(T) = \mu_p^{-1} \phi_1 \eta_p(T) = \mathcal{X} \) and \( \mathcal{X} = \mu_p^{-1}(\mathcal{X}_{T'}) = \mathcal{X}_{T'} \cup \rho(\mathcal{X}_{T'}) \cup \cdots \cup \rho^{p-1}(\mathcal{X}_{T'}) \) by definition of \( \mu_p \).

On the other hand we assume that \( \mathcal{X}_T = \mathcal{X}_{T'} \cup \rho(\mathcal{X}_{T'}) \cup \cdots \cup \rho^{p-1}(\mathcal{X}_{T'}) \) with \( T' \) a cluster tilting object in \( \mathcal{C}_n \). Then \( T = \phi_p^{-1}(\mathcal{X}_T) = T' \oplus F(T') \oplus \cdots \oplus F^{p-1}(T') \) is a cluster tilting object in \( \mathcal{C}_{n,p} \), again by [26] Theorem 3.5.

\[\square\]

6.4. Definition. A set of tagged edges \( \mathcal{X} \) of \( P_{np} \) is said to be a \( p \)-triangulation if there is a triangulation \( \mathcal{Y} \) of \( P_n \) (in the sense of [23]) such that \( \mathcal{X} = \mu_p^{-1}(\mathcal{Y}) \).

Then we can rewrite Proposition 6.3 as follows:

6.5. Proposition. \( T \) is a cluster tilting object of \( \mathcal{C}_{n,p} \) if and only if \( \mathcal{X}_T \) is a \( p \)-triangulation of \( P_{np} \).

Since the \( p \)-triangulations have \( np \) different tagged edges we have the following Corollary.

6.6. Corollary. Any cluster tilting object of \( \mathcal{C}_{n,p} \) has \( np \) pairwise non isomorphic summands.

\[\square\]

References

[1] I. Assem, D. Simson and A. Skowroński, *Elements of the Representation Theory of Associative Algebras, 1: Techniques of Representation Theory*, London Mathematical Society Student Texts 65, (Cambridge University Press, 2006).

[2] M. Auslander, I. Reiten and S.O. Smalø, *Representation Theory of Artin Algebras*, Cambridge Studies in Advanced Math. 36, (Cambridge University Press, 1995).

[3] C. Amiot, *On the structure of triangulated categories with finitely many indecomposables*, Bull. Soc. Math. France 135(3):435–474, 2007.
[4] K. Baur and R. Marsh, *A geometric description of the m-cluster categories of type \( D_n \)*, Int. Math. Res. Not. IMRN, (4): Art. ID rmn011, 19, 2007.
[5] K. Baur and R. Marsh, *A geometric description of \( m \)-cluster categories*, Trans. Amer. Math. Soc. 360(11):5789–5803, 2008.
[6] A. Buan, R. Marsh, M. Reineke, I. Reiten and G. Todorov, *Tilting theory and cluster combinatorics*, Adv. Math., 204(2):572–618, 2006.
[7] A. Buan and H. Torkildsen, *The number of elements in the mutation class of a quiver of type \( D_n \)*, Electron. J. Combin. 16(1): Research Paper 23, 2009.
[8] P. Caldero, F. Chapoton, and R. Schiffler. *Quivers with relations arising from clusters (A_\( n \) case)*, Trans. Amer. Math. Soc. 358(3):1347–1364, 2006.
[9] S. Fomin and A. Zelevinsky, *Cluster algebras I. Foundations*, J. Amer. Math. Soc. 15(2), 497-529 (electronic), 2002.
[10] S. Fomin and A. Zelevinsky, *Cluster algebras II. Finite type classification*, Inventiones Mathematicae, 154(1), 63-121, 2003.
[11] P. Gabriel, *Unzerlegbare Darstellungen*, Manuscripta Math 6 (1972), 71-103.
[12] P. Gabriel, *Auslander-Reiten sequences and representation-finite algebras*. In Representation theory, I (Proc. Workshop, Carleton Univ., Ottawa, Ont., 1979), Lecture Notes in Math. 831, 1-71. (Springer Verlag, 1980).
[13] D. Happel, *Triangulated Categories in the Representation Theory of Finite Dimensional Algebras*, Lecture Notes Series 119, (Cambridge University Press, 1988).
[14] P. Jørgensen. *Quotients of cluster categories*, Proc. Roy. Soc. Edinburgh Sect. A, 140(1):65–81, 2010.
[15] B. Keller, *Calabi-Yau triangulated categories*, Trends in representation theory of algebras and related topics, EMS Ser. Congr. Rep. pages 467–489. Eur. Math. Soc., Zürich, 2008.
[16] B. Keller, *Cluster algebras, quiver representations and triangulated categories*, Triangulated categories, volume 375 of London Math. Soc. Lecture Note Ser. pages 76–160. Cambridge Univ. Press, Cambridge, 2010.
[17] B. Keller, *Triangulated orbit categories*. Document Math., 10: 551-581, 2005.
[18] L. Lamberti, *A geometric interpretation of the triangulated structure of \( m \)-cluster categories*, Communications in Algebra, 42: 962-983, 2014.
[19] L. Lamberti, *Repetitive higher cluster categories of type \( A_n \)*, Journal of Algebra and Its Applications, 1350091.
[20] J. Przytycki and A. Sikora. *Polygon dissections and Euler, Fuss, Kirkman, and Cayley numbers*, J. Combin. Theory Ser. A 92(1):68–76, 2000.
[21] C. Riedtmann. *Algebren, Darstellungsköcher, Überlagerungen und zurück*, Comment. Math. Helv. 55(2):199–224, 1980.
[22] C.M. Ringel, *Tame algebras and integral quadratic forms* Lecture Notes in Math. 1099, (Springer Verlag, 1984).
[23] R. Schiffler. *A geometric model for cluster categories of type \( D_n \)*, J. Algebraic Combin. 27(1):1–21, 2008.
[24] H. Thomas. *Defining an \( m \)-cluster category*, J. Algebra 318(1):37–46, 2007.
[25] H. Torkildsen. *Finite mutation classes of coloured quivers*, Colloq. Math. 122(1):53–58, 2011.
[26] B. Zhu, *Cluster-tilted algebras and their intermediate coverings*, Comm. Algebra 39(7):2437–2448, 2011.

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