Out-of-equilibrium dynamics in systems with long-range interactions: characterizing quasi-stationary states

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Abstract. Systems with long-range interactions (LRI) display unusual thermodynamical and dynamical properties that stem from the non-additive character of the interaction potential. We focus in this work on the lack of relaxation to thermal equilibrium when a LRI system is started out-of-equilibrium. Several attempts have been made at predicting the so-called quasi-stationary state (QSS) reached by the dynamics and at characterizing the resulting transition between magnetized and non-magnetized states. We review in this work recent theories and interpretations about the QSS. Several theories exist but none of them has provided yet a full account of the dynamics found in numerical simulations.

Keywords: Vlasov equation ; long-range interactions

1 Introduction

Systems with long-range interactions (LRI) display unusual thermodynamical and dynamical properties such as ensemble inequivalence, lack of relaxation to equilibrium or broken ergodicity (see Refs. [1,2] for a review of the field). These properties stem from the non-additive character of the interparticle interaction potential. Let us mention a few systems belonging to the LRI class: gravitational systems, non-neutral plasmas, 2D fluid dynamics, etc.

We focus in this work on the lack of relaxation to thermal equilibrium in LRI systems when the system is initiated in an out-of-equilibrium state. This phenomenon leaves the system in an intermediary stage of the dynamics that is called a quasi-stationary state (QSS). This state does not correspond to the equilibrium predicted by statistical mechanics and its lifetime increases algebraically with the number $N$ of interacting particles in the system.

The occurrence of QSS should be taken into account if one is interested in the actual properties of a system. The time needed to reach thermal equilibrium may prevent a proper observation of equilibrium properties in the available experimental or simulational setting.
In this work, we review the generic steps of the out-of-equilibrium dynamics of LRI systems and use the paradigmatic Hamiltonian Mean-Field (HMF) model and its Vlasov formulation to illustrate those steps. Then, several theories attempting to predict or describe the QSS are reviewed.

2 The Hamiltonian Mean-Field model and the Vlasov equation

Let us consider the Hamiltonian Mean-Field (HMF) model introduced by Antoni and Ruffo [4]. This model aims at reproducing the collective behavior of more complex models with ferromagnetic or gravitational interactions, for instance.

The particles in the HMF model lie in a 1-dimensional periodic space with position \( \theta \in [-\pi : \pi] \). The \( N \)-body Hamiltonian is

\[
\mathcal{H} = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{1}{2N} \sum_{i,j=1}^{N} (1 - \cos(\theta_j - \theta_i)) \tag{1}
\]

where \( \theta_i \) is the position (in \([-\pi : \pi]\)) of particle \( i \) and \( p_i \) is its momentum. \( N \) is the total number of particles in the system.

One may also consider the continuum limit of the HMF model. This leads to the Vlasov equation

\[
\frac{\partial f}{\partial t} + \mathbf{p} \frac{\partial f}{\partial \theta} - \frac{\partial V[f](\theta, t)}{\partial \theta} \frac{\partial f}{\partial \mathbf{p}} = 0 , \tag{2}
\]

where

\[
V[f](\theta, t) = \int d\theta' dp' f(\theta', p', t) (1 - \cos(\theta' - \theta)) , \tag{3}
\]

is the interaction potential.

The mean field, or magnetization,

\[
\mathbf{m} = \frac{1}{N} \sum_{i=1}^{N} (\cos \theta_i, \sin \theta_i) = (m_x, m_y) , \tag{4}
\]

is used to follow the dynamical evolution of the HMF model. In the continuum limit,

\[
\mathbf{m} = \int d\theta \ dp \ f(\theta, p)(\cos \theta, \sin \theta) = (m_x, m_y) , \tag{5}
\]

The norm of \( \mathbf{m} \) is denoted \( m \).

Equilibrium statistical mechanics allows one to compute the value of \( m \) for a given energy or temperature. The authors of Ref. [4] observed a discrepancy in the caloric curve between the theory and their simulations, close to the second order transition separating the magnetized phase \( (m > 0) \) from the homogeneous phase \( (m = 0) \). Further investigations revealed the origin of the discrepancy:
the system of particles had not reached thermodynamical equilibrium in the
simulations.

The evolution of many systems with long-range interactions consists in the
following generic steps:

1. An initial condition that does not correspond to equilibrium.
2. Violent relaxation: the observables in the system undergo strong changes.
   The time scale of this step does not depend on the number of particles $N$.
3. Quasi-stationary state (QSS). This state may either be stationary or present
   oscillations. Its lifetime grows algebraically with $N$.
4. Equilibrium: the state that is predicted by equilibrium statistical mechanics.

Simulations illustrating this dynamical evolution may be found in Refs. [3,5],
for instance. An important consequence of the occurrence of QSS is that in
addition to the equilibrium phase transitions that the system may experience,
one has to consider an “out-of-equilibrium phase diagram” that is based on the
magnetization found in the QSS. In the thermodynamic limit $N \to \infty$, this
“out-of-equilibrium phase diagram” is the relevant one.

3 Theories for the quasi-stationary states

Several attempts have been made at predicting the quasi-stationary state reached
by the dynamics. We review several of those attempts, namely: Lynden-Bell’s
theory that is based on an entropy maximization principle [6,7], the exact station-
ary regime theory proposed by de Buyl, Mukamel and Ruffo [8] and a dynamical
reduction proposed by Levin [9].

None of the aforementioned theories is able to take into account the existence
of states whose observables are not constant in time, i.e. they predict a time
independent distribution $f(\theta, p)$. It is known that dynamical resonances lead to
oscillating regimes [4,10] and those cannot be predicted.

3.1 The theory of Lynden-Bell

In 1967, Lynden-Bell [6] devised a theory to compute the relaxed state of grav-
itational systems obeying a Vlasov equation. His theory is based on the max-
imization of an entropy functional that takes into account the incompressible
character of the distribution function in Vlasov dynamics. The computation is
based on a coarse graining of phase space but leads to a continuous prediction
for the distribution function.

Lynden-Bell’s theory has been applied with success to the prediction of the
intensity of the Colson-Bonifacio model for the free-electron laser [11] and to
provide an out-of-equilibrium phase diagram for the HMF model [6,7].
3.2 BGK like theory

Based on the fact that a distribution function that only depends on the energy is stationary in Vlasov dynamics, one may try to construct stationary states. This approach is well known in plasma physics as Bernstein-Greene-Kruskal modes [12]. The authors of Ref. [8] develop this idea while proposing an approximate correspondence between the initial condition and the state that is reached by the system.

The distribution \( f(\theta, p) \) is expressed directly as a function of the energy distribution function of the initial condition. For low values of the initial magnetization, the theory fails to predict the final magnetization. Else, it provides good results and predicts a second-order phase transition for \( \langle m \rangle \) and \( \langle m_x \rangle \). This theory is based purely on dynamical consideration and as such provides interesting complementary information with respect to Lynden-Bell’s theory.

3.3 Core-halo and envelope

The authors of Ref. [9] propose an ansatz for the distribution function \( f \) that reproduces the core-halo structure found in the phase space of the HMF model

\[
 f_S(\theta, p) = \eta_0 \left[ \Theta(\epsilon_F - \epsilon) + \chi \Theta(\epsilon_h - \epsilon) \Theta(\epsilon - \epsilon_F) \right],
\]

where \( \epsilon(\theta, p) = p^2/2 + (1 - M_S \cos \theta) \), \( M_S \) is the value of the magnetization and \( \Theta \) is the Heaviside function.

This ansatz requires the determination of the energy levels (\( \epsilon_F \) for the core and \( \epsilon_h \) for the halo) and of the magnetization \( M_S \). Those values are provided by a reduced dynamical equation. This theory is tested on the transition between magnetized and homogeneous regimes in the HMF model and predicts a first-order like transition for \( \langle m \rangle \) and \( \langle m_x \rangle \). The order of the transition is confirmed by simulation data for \( \langle m_x \rangle \). Simulation data for \( \langle m \rangle \) is not given however.

4 Discussion and conclusion

Out of the existing theories aimed at predicting the quasi-stationary states (QSS) that have been applied to the Hamiltonian Mean-Field model, none is able to predict the regimes in which oscillations are found. As is pointed out in Ref. [13], one may relate the time averages of the squared norm of the magnetization to the one of the \( x \) component of the magnetization by the following relation:

\[
 \langle m^2 \rangle = \langle m_x^2 \rangle = \langle m_x \rangle^2 + \sigma_{m_x}^2,
\]

where \( \sigma_{m_x} \) is the time-wise standard deviation of \( m_x \). The choice of an observable thus impacts the results that is found in simulations for non-steady QSS, explaining the different results between Ref. [7] and Ref. [9]. As soon as the QSS displays oscillations in the magnetization, \( \sigma_{m_x}^2 > 0 \) and \( \langle m^2 \rangle \neq \langle m_x \rangle^2 \). The

\[\text{[1] Here, } m_y \text{ can be set equal to zero without loss of generality.}\]
phase diagram provided by Lynden-Bell's theory [7] still represents the most ensemble view of the QSS for the HMF model as well as an actual interpretation in terms of phase transitions.

We have reviewed in this work recent advances in the understanding of the out-of-equilibrium dynamics in systems with long-range interactions. Several theories exist but none of them has provided yet a full account of the dynamics found in numerical simulations. Progress in this direction has been made by the construction of counter-rotation BGK clusters by Yamaguchi [14]. This construction is however not predictive.

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