RADO’S CONJECTURE AND ITS BAIRE VERSION

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Abstract. Rado’s Conjecture is a compactness/reflection principle that says any nonspecial tree of height \( \omega_1 \) has a nonspecial subtree of size \( \leq \aleph_1 \). Though incompatible with Martin’s Axiom, Rado’s Conjecture turns out to have many interesting consequences that are consequences of forcing axioms. In this paper, we obtain consistency results concerning Rado’s Conjecture and its Baire version. In particular, we show a fragment of PFA, that is the forcing axiom for Baire Indestructibly Proper forcings, is compatible with the Baire Rado’s Conjecture. As a corollary, Baire Rado’s Conjecture does not imply Rado’s Conjecture. Then we discuss the strength and limitations of the Baire Rado’s Conjecture regarding its interaction with simultaneous stationary reflection and some families of weak square principles. Finally we investigate the influence of the Rado’s Conjecture on some polarized partition relations.

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1. Introduction

Definition 1.1. A partial order \((T, \prec)\) is a tree if for each \( t \in T \), \( \{s \in T : s \prec t \} \) is well ordered under the tree order.

We sometimes suppress the tree order when it is clear from the context.

Definition 1.2. For a given tree \( T \), for each \( t \in T \), the height of \( t \) in \( T \) is the order type of its predecessors under the tree order, denoted as \( ht_T(t) \). The height of the tree \( T \) is the least ordinal \( \alpha \) such that for all \( t \in T \), \( ht_T(t) < \alpha \).

Remark 1.3. A tree \( T \)

(1) is non-trivial if each \( t \in T \) has two incompatible extensions;

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(2) does not split on the limit levels if for each limit $\alpha$ and $s, s' \in T$ such that $ht_T(s) = ht_T(s') = \alpha$, if $\{ t \in T : t < s \} = \{ t \in T : t < s' \}$, then $s = s'$.

Restricting ourselves to trees that are non-trivial and do not split on the limit levels does not affect any application of Rado’s Conjecture.

The trees we deal with for the rest of the paper are non-trivial, of height $\omega_1$ and do not split on the limit levels, unless otherwise stated.

Todorcevic studied Rado’s Conjecture and established some of its equivalent forms and showed its consistency by collapsing a supercompact cardinal to $\omega_2$ in [17]. The particular form of the Rado’s conjecture that is the most relevant for us is the tree version.

**Definition 1.4.** A tree $T$ is special if there exists $g : T \to \omega$ such that $g$ is injective on chains.

**Definition 1.5.** $RC$ abbreviates the following: any nonspecial tree has a nonspecial subtree of size $\leq \aleph_1$.

Rado’s conjecture has interesting consequences. To sample a few:

**Theorem 1.1** (Todorcevic [20], [17]). Rado’s Conjecture implies:

1. $2^\omega \leq \omega_2$
2. $\theta^\omega = \theta$ for all regular $\theta \geq \omega_2$
3. the Singular Cardinal Hypothesis
4. $\Box(\kappa)$ fails for all regular $\kappa \geq \omega_2$
5. the Strong Chang’s Conjecture
6. for all regular cardinal $\lambda \geq \omega_2$, any stationary subset of $\lambda \cap \text{cof}(\omega)$ reflects.

**Theorem 1.2** (Feng [7]). Rado’s Conjecture implies the non-stationary ideal on $\omega_1$ is presaturated.

**Theorem 1.3** (Doebler [6]). Rado’s conjecture implies that all stationary set preserving forcings are semiproper.

**Theorem 1.4** (Torres-Pérez and Wu [21]). Rado’s Conjecture along with $\neg CH$ implies $\omega_2$ has the strong tree property. Rado’s Conjecture implies the failure of $\Box(\lambda, \omega)$ for all regular $\lambda \geq \omega_2$.

As remarked in [17], many known consequences of Rado’s conjecture follow from a weaker principle, the Baire version of Rado’s conjecture.

**Definition 1.6.** Let $T$ be a given tree. A subset $A \subset T$ is

1. open if for all $t \leq t' \in T$, if $t \in A$, then $t' \in A$;
2. dense if for all $t \in T$, there exists $t' \in A$ such that $t \leq t'$.

**Definition 1.7.** A non-trivial forcing poset $\mathbb{P}$ is $\omega$-distributive if forcing with $\mathbb{P}$ does not add new $\omega$-sequences of ordinals. If $\mathbb{P}$ is separative, then $\mathbb{P}$ is $\omega$-distributive if and only if for any countable collection of open dense sets $\{ U_n \subset \mathbb{P} : n \in \omega \}, \bigcap_n U_n$ is dense.

**Definition 1.8.** A tree is Baire if it is $\omega$-distributive as a forcing notion.

**Remark 1.9.** It is not always the case that a tree is separative. However, we do have that: for any tree $T$, the following are equivalent:

1. forcing with $T$ adds no new countable sequences of ordinals;
(2) forcing with \( T \) adds no new functions from \( \omega \) to \( V \);

(3) for any countable collection of dense open subsets \( \{ U_n : n \in \omega \} \) of \( T, \bigcap_n U_n \) is dense in \( T \).

The implications (1) \( \leftrightarrow \) (2), (3) \( \rightarrow \) (1) are standard. To see (2) \( \rightarrow \) (3), assume \( \{ U_n : n \in \omega \} \) is a collection of dense open sets such that \( \bigcap_n U_n \) is not dense, say there exists \( t \in T \) such that no extension of \( t \) is in \( \bigcap_n U_n \). Let \( G \subseteq T \) be generic that contains \( t \). In \( V[G] \), define \( f : \omega \to G \subseteq V \) inductively as follows: \( f(0) = t \). Given \( f(i) \), find \( t' \geq f(i) \) in \( U_i \cap G \) such that there exists two incompatible immediate extensions of \( t' \) in \( T \). The reason we can find this is that \( f(i) \in G \) and \( U_i \) is dense above \( f(i) \) so there is \( t'' \in U_i \cap G \) and \( t'' \geq f(i) \). By non-triviality of \( T \), there are \( s, s' \geq t'' \) that are incompatible. Let \( t' \geq t \) be such that it has two incompatible immediate extensions and no \( s \) with \( t \leq s < t' \) has this property. The existence of such \( t' \) follows from the fact that the tree does not split on the limit levels. Note \( t \Vdash t' \in G \) and by openness of \( U_i \), \( t' \in U_i \). Define \( f(i + 1) = t' \). It is now easy to check that \( t \Vdash f \notin V \).

Notice that any Baire tree is nonspecial. Hence the following is a statement weaker than \( RC \).

**Definition 1.10.** \( RC_b \) abbreviates the following: any Baire tree has a nonspecial subtree of size \( \leq \aleph_1 \).

We can also formulate a slightly stronger principle:

**Definition 1.11.** \( sRC_b \) abbreviates the following: any Baire tree has a Baire subtree of size \( \leq \aleph_1 \).

**Definition 1.12.** A proper poset \( P \) is Baire indestructible if for any Baire tree \( T \), \( T \Vdash P \) is proper. We call this class Baire Indestructibly Proper (BIP).

In general, \( MA \) is incompatible with \( RC_b \). To see this, consider the tree \( T(S) \), which is the forcing poset to shoot a club into a stationary co-stationary \( S \subseteq \omega_1 \). \( T(S) \) is easily seen to be Baire. \( MA \) implies any \( \aleph_1 \)-sized subtree of \( T(S) \) is special while \( RC_b \) implies there exists a nonspecial subtree of size \( \aleph_1 \).

One of the motivations of our work is to understand what fragment of the standard forcing axioms is compatible with \( RC_b \). A natural guess is that it should include the “non-specializing” part. Our main result shows that it could even include some “harmless” specializing forcings.

The main result of this paper is:

**Theorem 1.5.** Assume the existence of a supercompact cardinal. There exists a forcing extension where \( sRC_b \) and \( MA_{\omega_1}(BIP) \) both hold.

Since \( MA_{\omega_1}(BIP) \) implies the failure of \( RC \), we have

**Corollary 1.13.** \( sRC_b \) in general does not imply \( RC \).

We are also interested in the influence of Rado’s Conjecture to singular cardinal combinatorics, stationary reflections and polarized partition relations. We recall some notations of partition calculus.

**Definition 1.14.** For ordinals \( \alpha, \beta \), let \( \{ \alpha \}^{\beta} \) denote \( \{ A \subseteq \alpha : \text{otp}(A) = \beta \} \).

**Definition 1.15.** \( \alpha \beta \rightarrow (\gamma \delta \sigma)_{1,1}^{1,1} \) abbreviates: for any \( f : \alpha \times \beta \to \sigma \), there exists \( A \in \{ \alpha \}^\gamma \) and \( B \in \{ \beta \}^\delta \), such that \( f \restriction A \times B \) is constant.
The organization of this paper is the following:

- In Section 2 we sketch the proof that $RC$ holds in the classical Mitchell model, where $RC \vdash \neg CH$ hold and $\omega_2$ does not have the super tree property.
- In Section 3 we present a mixed-support model which is a Mitchell variant where $sRC^b$ holds but $RC$ fails.
- In Section 4 we prove Theorem 1.5.
- In Section 5 we present some streamlined proofs of known consequences of $RC^b$ and we also show that $RC^b$ in general is compatible with failures of simultaneous reflection and some versions of the weak square principles.
- In Section 6 we show that in the classical Mitchell model, where $RC$ is countably capturing, in $\mathcal{P}$-

\[ Q \] holds but $\mathcal{P}[G]$ fails.

We end the introduction by including a simple lemma characterizing forcings that preserve $\omega$-distributivity, which is a variant of the well-known Easton’s Lemma in the context of forcing with products.

**Definition 1.16.** A poset $\mathbb{P}$ is countably capturing if for any $p \in \mathbb{P}$, any $\mathbb{P}$-name of a countable sequence of ordinals $\check{\tau}$, there exists another $\mathbb{P}$-name $\check{\sigma}$ such that $|\check{\sigma}| \leq \aleph_0$, and $q \leq p$ such that $q \Vdash \check{\tau} = \check{\sigma}$.

**Remark 1.17.** Here we think of each $\mathbb{P}$-name $\check{\tau}$ for a countable sequence of ordinals as represented by a function $f_\check{\tau}$ whose domain is $\omega$ such that for each $n \in \omega$, $f_\check{\tau}(n) = \{(\alpha_p, p) : p \in A_n\}$ where $A_n$ is some antichain chain of $\mathbb{P}$ such that for each $p \in A_n$, $p \Vdash \check{\tau} = \check{\sigma}$.

**Remark 1.18.** Any proper forcing is countably capturing. To see this, let $p \in \mathbb{P}$ and a nice name for a countable sequence of ordinals $\check{\tau}$ be given. Let $\lambda$ be a sufficiently large regular cardinal and let $M < H(\lambda)$ contain $\mathbb{P}, p, \check{\tau}$. By properness, find $q \leq p$ that is $(M, \mathbb{P})$-generic. Let $\check{\sigma} = \check{\tau} \cap M$. Then $|\check{\sigma}| \leq \aleph_0$ and $q \Vdash \check{\sigma} = \check{\tau}$.

**Lemma 1.19.** Let $\mathbb{P}$ be countably capturing and $Q$ be $\omega$-distributive. Then TFAE:

1. $\Vdash \check{Q}$ is $\omega$-distributive
2. $\Vdash_{\mathbb{P}} \check{P}$ is countably capturing.

**Proof.**

- 2) implies 1): Let $G \times H$ be generic for $\mathbb{P} \times Q$ and let $\check{\tau}$ be a $(\mathbb{P} \times Q)$-name of a countable sequence of ordinals. We need to show that $\check{\tau}^{G \times H}$ is in $V[G]$. Since $\Vdash_{\mathbb{Q}} \mathbb{P}$ is countably capturing, in $V[H]$, there exists a nice $\mathbb{P}$-name $\check{\sigma}$ with $|\check{\sigma}| \leq \aleph_0$ such that in $V[H][G]$, $\check{\tau}^{H \times G} = (\check{\sigma})^G$. Since $Q$ is $\omega$-distributive, $\check{\sigma} \in V$. But then $\check{\tau}^{H \times G} = (\check{\sigma})^G \in V[G]$.
- 1) implies 2): Let $H$ be $\mathbb{Q}$-generic, we need to show $\mathbb{P}$ is countably capturing in $V[H]$. Let $\check{\tau}$ be a $\mathbb{Q} \times \mathbb{P}$-name for a countable sequence of ordinals. So $\check{\tau}^H$ is a $\mathbb{P}$-name for a countable sequence of ordinals in $V[H]$. Let $p \in \mathbb{P}$ be given. Now let $G$ containing $p$ be generic for $\mathbb{P}$ over $V[H]$. In $V[G \times H]$, by the assumption we know that $\check{\tau}^{G \times H} \in V[G]$, hence there exists a $\mathbb{P}$-name $\check{\sigma} \in V$ such that $\check{\sigma}^G = \check{\tau}^{G \times H}$ in $V[G \times H]$. Find $q \in G, q \leq p$ that forces over $V[H]$ that $\check{\sigma} = \check{\tau}$. In $V[H]$, $q \Vdash_{\mathbb{P}} \check{\sigma}$ is a countable sequence of ordinals. But the same statement is also true in $V$. By the fact that $\mathbb{P}$ is countably capturing in $V$, we can find $q' \leq q$ and $\check{\phi}$ such that $|\check{\phi}| \leq \aleph_0$ and
\(q' \forces \dot{\varphi} = \dot{\sigma}\) (the same forcing relation also holds in \(V[H]\) by the product lemma). Finally in \(V[H]\) we have found \(q'\) such that \(q' \forces \dot{\varphi} = \dot{\tau}\) and \(|\dot{\varphi}| \leq \aleph_0\).

\[\square\]

2. \(RC + \neg CH\) DOES NOT IMPLY THE SUPER TREE PROPERTY

Fix cardinals \(\kappa \leq \lambda\). Recall the following definitions (see [23] for instance).

**Definition 2.1.** \(\langle d_a : a \in P_\kappa \lambda \rangle\) is a \(P_\kappa \lambda\)-list if \(d_a \subset a\) for all \(a \in P_\kappa \lambda\).

**Definition 2.2.** A \(P_\kappa \lambda\)-list \(\langle d_a : a \in P_\kappa \lambda \rangle\) is thin if there exists a club \(C \subset P_\kappa \lambda\) such that \(|\{d_a \cap c : c \subset a \in P_\kappa \lambda\}| < \kappa\) for every \(c \in C\).

**Definition 2.3.** Given \(P_\kappa \lambda\)-list \(D = \langle d_a : a \in P_\kappa \lambda \rangle\) and \(d \subset \lambda\), we say \(d\) is an ineffable branch of \(D\) if there exists a stationary set \(S \subset P_\kappa \lambda\) such that \(d \cap a = d_a\) for all \(a \in S\).

**Definition 2.4.** We say \(\kappa\) has the super tree property if for any \(\lambda \geq \kappa\), any thin \(P_\kappa \lambda\)-list \(D\), there exists an ineffable branch of \(D\).

**Remark 2.5.** \(\kappa\) is supercompact iff \(\kappa\) is inaccessible and has the super tree property.

We show in this section that in the classical Mitchell model of the tree property when \(\kappa\) is inaccessible, then if \(\kappa\) (which is the \(\omega_2\) of the forcing extension) has the super tree property in the model, then \(\kappa\) must already be supercompact in the ground model. This heavily relies on Viale and Weiss’ analysis in [23]. This shows that \(RC + \neg CH\) does not imply the super tree property at \(\omega_2\), answering a question of Torres-Pérez and Wu in [21].

2.1. **Proof sketch of RC in the Mitchell’s model.** We give a proof that \(RC\) holds in the Mitchell’s classical model of the tree property. This is due to Todorcevic, who in [20] pointed out the model works. This shows that \(RC\) is compatible with \(2^\omega = \omega_2\). For completeness, we supply a proof here.

Let \(\kappa\) be a strongly compact cardinal. Let \(M_\kappa\) denote the Mitchell forcing. Specifically, the poset consists of pairs \((p, f)\) where \(p \in Add(\omega, \kappa)\) and \(f\) is a function on \(\kappa\) with countable support such that for each \(\alpha \in \kappa\), \(f(\alpha)\) is an \(Add(\omega, \alpha)\)-name for an element in \(Add(\omega_1, 1)^{\kappa(\omega, \omega)}\). \((q, g) \leq (p, f)\) iff \(q \supset p\) and for each \(\alpha \in \text{supp}(f)\), \(q \forces Add(\omega, \alpha) \frown g(\alpha) \leq f(\alpha)\).

Let \(R\) be the term poset. More precisely, conditions in \(R\) are countably supported functions \(f\) with domain \(\kappa\) such that for each \(\alpha \in \text{supp}(f)\), \(f(\alpha)\) is \(Add(\omega_1, 1)\)-name for an element in \(Add(\omega_1, 1)^{\kappa(\omega, \alpha)}\). For \(f, g \in R\), \(f \leq g\) iff \(supp(f) \supset supp(g)\) and for each \(\alpha \in \text{supp}(g)\), \(f(\alpha) \leq g(\alpha)\). \(R\) is countably closed.

We list a few well-known properties of the Mitchell poset:

1. \(M_\kappa\) is \(\kappa\)-c.c. (see Lemma [21]);
2. \(M_\kappa\) is a projection of \(Add(\omega, \kappa) \times R\) which is proper, so in particular \(M_\kappa\) is proper;
3. for each inaccessible \(\delta < \kappa\), we can truncate \(M_\kappa\) at \(\delta\) to get \(M_\delta\). Then for any \(G \subset M_\delta\) that is generic, \(M_\kappa/G\) is susceptible to the same analysis. Namely in \(V[G]\), \(M_\kappa/G\) is equivalent to a projection of \(Add(\omega, \kappa) \times R^*\), where \(R^*\) is countably closed.

We need the following two general facts regarding non-specializing forcings.
Claim 2.6 (Lemma 12). No countably closed forcing can specialize a nonspecial tree.

Claim 2.7. No Cohen forcing can specialize a nonspecial tree.

Proof. Let $T$ be a given nonspecial tree. Let $\lambda$ be a cardinal and $Add(\omega, \lambda)$ be the Cohen forcing of adding $\lambda$ reals. Suppose for the sake of contradiction that there exists a name $\dot{g} : T \rightarrow \omega$ that specializes the tree $T$. For each $t \in T$, find $p_t \in Add(\omega, \lambda)$ such that $p_t \Vdash \dot{g}(t) = n_t$ for some $n_t \in \omega$. Without loss of generality, we might assume $p_t$ is a finite function from $T$ to $2$. Let $F_t = dom(p_t)$ for all $t \in T$. Since $T$ is not special, by going to a nonspecial subtree if necessary, we can assume there exists $m \in \omega$ and $n \in \omega$ such that $n_t = m$ and $|F_t| = n$ for all $t \in T$. Fix some well ordering $\lhd$ on $T$.

We shrink the trees in $n$ rounds. Let $T_0 = T$. At stage $i + 1$, define a regressive function on $T_i$ such that $t \in T_i$ is mapped to its immediate predecessor if it has one, otherwise $t \in T_i$ is mapped to the $\lhd$-least proper initial segment $s$ such that the $i$-th element of $F_i$ is in $F_s$ if it exists, otherwise, just map it to the root. Apply the Pressing Down Lemma for nonspecial trees (Todorčević [13]) and let $T_{i+1}$ be the nonspecial subchain on which the function is a constant, say $s_{i+1}$. Then we have the following property, for each $i < t' \in T_{i+1}$, if the $i$-th element in $F_{i'}$ is in $F_i$, then it is already in $F_{s_{i+1}}$. Let $T' = T_n$. Given $t < t' \in T'$, by the observation, all elements in $F_{i'}$ that are in $F_s$ for some $s < t'$ and $s \in T'$ are already in $D = \bigcup_{s \leq n} F_s$. Thus $F_i \cap F_i' \subset F_s \cap D$. As $2^{[\omega]}$ is finite, we can further shrink $T'$ to $T^*$ and find $r < [\omega]$, $h : r \rightarrow 2$ such that for all $t \in T^*$, $F_i \cap D = r$ and $p_t \Vdash r = h$. This implies that for any $t < t' \in T'$, $p_t$ and $p_{i'}$ are compatible. But this is a contradiction as $p_t \Vdash \dot{g}(t) = m$ and $p_{i'} \Vdash \dot{g}(t') = m$ while $\Vdash \dot{g} : T \rightarrow \omega$ is a specializing function.

Proof that RC holds in $V^{M_\kappa}$. Let $G \subset M_\kappa$ be generic and let $T \in V[G]$ be a nonspecial tree of size $\theta$. Without loss of generality $T = (\theta, \lhd)$ for some tree order $\lhd$. Let $\lambda > \theta$ be a sufficiently large regular cardinal and fix $j : V \rightarrow M$ witnessing $\kappa$ is $\lambda$-compact. By $\kappa$-cc-ness, $j \upharpoonright M_\kappa = id$ is a complete embedding of $M_\kappa$ into $j(M_\kappa)$. Moreover, it is not hard to see that $j(M_\kappa) \upharpoonright \kappa = M_\kappa$. By strong compactness, there exists $Y \in M$ such that $M \models [Y] < j(\kappa)$ and $j^n \theta < Y$. Let $K$ be generic over $V[G]$ for $j(M_\kappa)/G$. Then we can lift $j$ to an elementary embedding from $V[G]$ to $\mathcal{M}[G*K]$. Hence in $\mathcal{M}[G*K]$, $Y \cap j(T) \supset j^n T$ so $(Y \cap j(T), <_{j(T)})$ is a subtree of size $< j(\kappa)$ that contains $(j^n T, <_{j(T)})$. Notice that $(j^n T, <_{j(T)}) \simeq (T, < T)$. We will be done if we can manage to show that $Y \cap j(T)$ is nonspecial. Since $j(\kappa) = (\omega_2)^{\mathcal{M}[G*K]}$, it is clearly sufficient to show $(T, < T)$ remains nonspecial after forcing with $j(M_\kappa)/G$ over $V[G]$. By the properties listed above, we know $j(M_\kappa)/G$ is a projection of $(Add(\omega, j(\kappa)) \times R^*)^{\mathcal{M}[G]}$ where $R^* \in M[G]$ is a countably closed poset in $M[G]$. But $R^*$ is also countably closed in $V[G]$, since $V \models M^\kappa \subset M$ and $M_\kappa$ is $\kappa$-cc so $V[G] \models M[G]^\kappa \subset M[G]$. To summarize, in $V[G]$, $j(M_\kappa)/G$ is a projection of $Add(\omega, j(\kappa)) \times R^*$ where $R^*$ is countably closed in $V[G]$. By Claim 2.6 and Claim 2.7, we know that $(T, < T)$ remains nonspecial in $V[G*K]$.

2.2. Putting it together. The idea is to apply the characterization of Viale and Weiss. Let $M_\kappa$ be the Mitchell forcing with respect to $\kappa$.

Definition 2.8. A forcing poset $\mathbb{P}$ such that $\Vdash_{\mathbb{P}} \kappa$ is regular has
(1) the \( \kappa \)-covering property if for any generic \( G \subset P \) and any subset of ordinals \( A \in V[G] \) such that \( |A| < \kappa \), there exists \( B \in V \) such that \( V | B | \leq \kappa \); and \( V | A | < \kappa \) and \( A \subset V \).

(2) the \( \kappa \)-approximation property if for any generic \( G \subset P \) and any subset of ordinals \( A \in V[G] \), if \( A \cap a \in V \) for all \( a \in V \) with \( V | a | < \kappa \), then \( A \in V \).

**Remark 2.9.** For any poset \( P \) and regular \( \kappa \), if \( P \) is \( \kappa \)-c.c, then \( P \) has the \( \kappa \)-covering property.

We will use the following lemma due to Unger.

**Lemma 2.10** ([22] Lemma 2.4). If a poset \( P \) satisfies that \( P \times P \) has \( \kappa \)-c.c, then \( P \) has \( \kappa \)-approximation property.

**Lemma 2.11** ([1] Lemma 2.4). \( M_{\kappa} \) is \( \kappa \)-Knaster. In particular by Lemma 2.10 it satisfies the \( \kappa \)-approximation property and the \( \kappa \)-covering property.

We use the following result due to Viale and Weiss [23].

**Theorem 2.1** ([23]). Let \( \kappa \) be an inaccessible cardinal and \( P \) be a proper poset with \( \kappa \)-covering and \( \kappa \)-approximation property. If in \( V^P \), the super tree property holds at \( \kappa \), then the super tree property must already hold in \( V \) at \( \kappa \). In particular, \( \kappa \) must be supercompact in \( V \).

**Theorem 2.2.** Let \( \kappa \) be a strongly compact cardinal that is not supercompact. Then there exists a forcing extension in which \( RC \) and \( 2^\omega = \kappa = \omega_2 \) hold but the super tree property at \( \omega_2 \) fails.

**Proof.** Force with \( M_{\kappa} \). In the final model, \( RC + 2^\omega = \kappa = \omega_2 \) hold by the discussion in the subsection 2.2 but the super tree property at \( \omega_2 \) fails by Theorem 2.1.

**Remark 2.12.** If we are more careful about the choice of \( \kappa \) in Theorem 2.2, say for example \( \kappa \) is a strongly compact cardinal but not \( \kappa^{+} \)-supercompact, then in the resulting model \( WRP(\omega_3) \) (see Definition 6.3) fails. The reason is that the model also satisfies \( MA_{\omega_1}(\text{Cohen}) \). Hence by theorems in [15] and [23], we know if \( WRP(\omega_3) + MA_{\omega_1}(\text{Cohen}) \) holds, then \( ITP(\omega_2, \omega_3) \) holds, which in turns implies that \( \kappa \) is \( \kappa^{+} \)-supercompact in the ground model. Separation of \( RC \) from \( WRP(\omega_3) \) first appeared in [14] with a different model.

### 3. Separating sRC\(^b\) from RC

In this section we show \( sRC^{b} \) does not imply \( RC \) in general. Another model separating them will be presented in Section 4. We start off introducing a tree that will be central in the proof.

**Definition 3.1.** Let \( T(\mathbb{R}) \) denote the tree consisting of bounded subsets of \( \mathbb{R} \) well ordered by the natural order on \( \mathbb{R} \). The tree is ordered by end-extension.

We list a few observations about \( T(\mathbb{R}) \).

**Observation 3.2.**

1. \( T(\mathbb{R}) \) is nonspecial (Todorčević [17]);
2. \( T(\mathbb{R}) \) is not Baire;
(3) For any subset \( U \subset T(\mathbb{R}) \), in any outer model, \( U \) has no uncountable branches.

Given a tree \( T \), let \( S(T) \) denote the specializing poset of \( T \). More precisely, it contains finite functions \( s: T \to \omega \) that are injective on chains. The poset is ordered by containment. We need the following characterization of this poset due to Baumgartner.

**Theorem 3.1** ([3], [2]). \( S(T) \) is c.c.c iff \( T \) does not contain an uncountable branch.

Let \( \kappa \) be a supercompact cardinal. Let \( \langle P_\alpha, \check{Q}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle \) denote the finite support iteration such that for each \( \alpha \), \( \models_\alpha \check{Q}_\alpha \) is the specializing poset for \( (T(\mathbb{R}))^{V^{P_\alpha}} \). More specifically, \( \models_\alpha \check{Q}_\alpha \) consists of finite functions from \( (T(\mathbb{R}))^{V^{P_\alpha}} \) to \( \omega \) that are injective on chains, ordered by reverse inclusion. We ensure that \( \models_\kappa \) all \( < \kappa \)-subset of \( T(\mathbb{R}) \) is special. The reason is that: \( P_\kappa \) is c.c.c and each \( < \kappa \)-subset of \( T(\mathbb{R}) \) in \( V^{P_\kappa} \) appears in \( V^{P_\kappa} \) as a subset of \( T(\mathbb{R})^{V^{P_\kappa}} \) for some \( \alpha < \kappa \).

**Lemma 3.3.** For any Baire \( T \), \( T \models P_\kappa \) is c.c.c. Hence \( P_\kappa \models T \) is Baire.

**Proof.** We induct on \( \alpha \leq \kappa \). When \( \alpha \) is a limit ordinal, let \( H \subset T \) be generic over \( V \). Then in \( V[H] \), \( P_\alpha \) is the direct limit of \( \langle P_\beta : \beta < \alpha \rangle \) and each \( P_\beta \) is c.c.c by the induction hypothesis. So by usual \( \Delta \)-system argument we know that \( P_\alpha \) is also c.c.c in \( V[H] \). When \( \alpha = \beta + 1 \), let \( H \times G \subset T \times P_\beta \) be generic over \( V \). We examine \( Q_\beta = \langle Q_\beta \rangle^G \) in \( V[H \times G] \). By induction hypothesis, \( \models_T P_\beta \) is c.c.c. By Lemma 1.19 \( \models_{P_\alpha} T \) is Baire. By our definition of the iteration, \( Q_\beta \) lives in \( V[G] \), and is a specializing poset for \( (T(\mathbb{R}))^{V[G]} \). Note that \( (T(\mathbb{R}))^{V[G]} = (T(\mathbb{R}))^{V[G \times H]} \) since \( T \) is Baire in \( V[G] \) and \( (T(\mathbb{R}))^{V[G]} \) does not have any uncountable branch in \( V[G \times H] \). \( Q_\beta \) is the same as the specializing poset defined in \( V[G \times H] \) for \( (T(\mathbb{R}))^{V[G \times H]} \). By Theorem 3.1, \( Q_\beta \) is c.c.c. in \( V[G \times H] \). The last part follows from Lemma 1.19. \( \square \)

**Remark 3.4.** Lemma 3.3 remains valid if we replace the Baire tree \( T \) with any \( \omega \)-distributive forcing \( \mathbb{P} \).

We define our main forcing as a variant of Mitchell’s forcing.

**Definition 3.5.** \( Q \) is a poset consisting of \( (p, f) \) where \( p \in P_\kappa \) and \( f \) is a function on \( \kappa \) with countable support and for each \( \alpha \in \text{dom}(f) \), \( f(\alpha) \) is a \( P_\alpha \)-name for a condition in \( (Add(\omega_1,1))^{V^{P_\alpha}} \). \( (p_1, f_1) \leq (p_2, f_2) \) if and only if \( p_1 \leq p_2 \), \( p_2 \) and \( \text{supp}(f_1) \supset \text{supp}(f_2) \) and for each \( \alpha \in \text{supp}(f_2) \), \( p_1 \vdash f_1(\alpha) \leq f_2(\alpha) \).

**Claim 3.6.** \( Q \) is \( \kappa \)-c.c.

**Proof.** The proof is similar to that of Theorem 2.11. Let \( \langle (p_\alpha, f_\beta) : \alpha < \kappa \rangle \subset Q \) be given. Apply \( \Delta \)-system lemma to get \( A \subset [\kappa]^\omega \) such that \( \{ \text{dom}(f_\alpha) : \alpha \in A \} \) forms a \( \Delta \)-system with root \( h \in [\kappa]^\omega \) and for all \( \alpha, \beta \in A \), \( f_\alpha \upharpoonright h = f_\beta \upharpoonright h \). This is possible since for any \( \omega \leq \beta < \kappa \), the collection of nice \( P_\beta \)-names for \( Add(\omega_1,1) \) is contained in \( V_{(2^{P_\beta})^+} \) and \( h \) is countable.

Since \( P_\kappa \) is c.c.c, we may find \( \alpha < \beta \in A \) such that \( p_\alpha \) and \( p_\beta \) are compatible. Fix some \( r \leq p_\alpha, p_\beta \). Let \( f_\alpha + f_\beta \) be the function such that

\[
\begin{align*}
f_\alpha + f_\beta(\gamma) &= \begin{cases} f_\alpha(\gamma) & \gamma \in h \\ f_\alpha(\gamma) & \gamma \in \text{supp}(f_\alpha) - h \\ f_\beta(\gamma) & \gamma \in \text{supp}(f_\beta) - h \\ \emptyset & \text{otherwise} \end{cases}
\end{align*}
\]
Consider \((r, f_\alpha + f_\beta)\), which is clearly an element in \(Q\) extending both \((p_\alpha, f_\alpha)\) and \((p_\beta, f_\beta)\).

We will recall some standard analysis of this poset. Let \(R\) be the poset consisting of functions \(f\) with domain \(\kappa\) of countable support such that for each \(\alpha \in \kappa\), \(f(\alpha)\) is a \(P_\alpha\)-name for an element in \(Add(\omega_1, 1)^{V_{f_\alpha}}\) and for any \(f, g \in R\), \(f \leq_R g\) iff \(\text{supp}(f) \supset \text{supp}(g)\) and for each \(\beta \in \text{supp}(g)\), \(\Vdash_{P_\beta} f(\beta) \leq g(\beta)\). Notice \(R\) is countably closed.

**Claim 3.7.** \(Q\) projects onto \(P_\kappa\).

*Proof.* The projection onto the first coordinate works.

**Claim 3.8.** \(P_\kappa \times R\) projects onto \(Q\).

*Proof.* Consider the identity map. To see that it is a projection map, for each \((p, f) \in P_\kappa \times R\), and \((q, g) \leq_Q (p, f)\), we need to find \((p', f') \in P_\kappa \times R\) such that \((p', f') \leq_{P_\kappa \times R} (p, f)\) and \((p', f') \leq_Q (q, g)\). Let \(p' = q\). Let \(f'\) be a function with support \(\text{supp}(g)\) and for each \(\beta \in \text{supp}(g)\), \(q \upharpoonright \beta \Vdash f'(\beta) = g(\beta)\) and \(\Vdash_{P_\beta} f'(\beta) \leq f(\beta)\). We can find such a function by the maximality principle of forcing.

**Claim 3.9.** For any countably closed \(\mathbb{D}\), \(\mathbb{D} \times T\) is \(\omega\)-distributive.

*Proof.* It immediately follows from the fact that \(\Vdash_T \mathbb{D}\) is countably closed, as \(T\) is Baire.

We need the similar product analysis on the quotient forcing. Let \(\delta < \kappa\) be inaccessible, then we can truncate \(Q\) to \(Q \upharpoonright \delta\) in the obvious way. Let \(G_\delta\) be generic for \(Q \upharpoonright \delta\). Let \(H_\delta\) be the projection of \(G_\delta\) to the first coordinate, which is \(V\)-generic for \(P_\delta\).

**Claim 3.10.** Let \(T \in V[G_\delta]\) be a Baire tree. Then in \(V[G_\delta]\), \(\Vdash_{Q/G_\delta} T\) is a Baire tree.

*Proof.* In \(V[G_\delta]\), we will show that, similarly as in Claim 3.8, \(Q/G_\delta\) is a projection of \((P_{[\delta, \kappa)} V[H_\delta]) R^*, \) where \(R^*\) is some countably closed poset in \(V[G_\delta]\). Let \(E = \left(P_{[\delta, \kappa)} V[H_\delta] \times R^*\right).\) Let us give a more detailed description of what \(R^*\) is and what the projection is. \(R^*\) consists of countably supported functions \(f\) with domain \([\delta, \kappa)\) such that for each \(\beta \in \text{dom}(f)\), \(f(\beta) \in V[H_\delta]\) is a \((P_{[\delta, \beta)} V[H_\delta])\)-name for an element in \(Add(\omega_1, 1)\). In \(V[G_\delta]\), \(f \leq_{R^*} g\) if \(\text{supp}(f) \supset \text{supp}(g)\) and for each \(\gamma \in \text{supp}(g)\), \(\Vdash_E f(\gamma) \leq \text{Add}(\omega_1, 1) g(\gamma)\).

To see that \(R^*\) is countably closed in \(V[G_\delta]\), it is sufficient to notice that the quotient forcing \(D = (Q \upharpoonright \delta)/H_\delta\) is \(\omega\)-distributive, which is due to our product analysis. Thus in particular, \(V[G_\delta] \models V[H_\delta]^{\omega} \subset V[H_\delta]\) and \(R^*\) is countably closed in \(V[G_\delta]\).

Notice that in \(V[G_\delta]\), \(Q/G_\delta\) is equivalent to the poset \(B\) such that \((s, f) \in B\) iff \(s \in E\) and \(f\) is a countably supported function with domain \([\delta, \kappa)\) and \(\text{range}(f) \subset V[H_\delta]\) such that for any \(\alpha \in \text{supp}(f), \Vdash_{E[\alpha]} f(\alpha) \in \text{Add}(\omega_1, 1)\). The ordering on \(B\) is that \((s', f') \leq (s, f)\) iff \(s' \leq_{E} s\), \(\text{supp}(f') \supset \text{supp}(f)\) and for each \(\alpha \in \text{supp}(f)\), \(\text{range}(f') \subset V[H_\delta]\) such that \((s', f') \leq_{B} (s, f)\). Let \(s'' = s'\) and \(\text{supp}(f'') = \text{supp}(f')\). For each \(\alpha \in \text{supp}(f'')\),
define \( f''(\alpha) \in V[H_\delta] \) such that \( \models_{E/\alpha} f''(\alpha) \leq f'(\alpha) \) and \( s' \upharpoonright \alpha \models_{E/\alpha} f''(\alpha) \leq f'(\alpha) \).

This can be achieved by applying the maximality principle in \( V[H_\delta] \) to \( E \upharpoonright \alpha \), noting that \((s, f), (s', f') \in V[H_\delta] \) by the fact that \( V[G_\delta] = V[H_\delta]^{\omega} \subset V[H_\delta] \) as discussed above.

In \( V[H_\delta] \), let \( \dot{R}^* \) be the \( D \)-name for the countably closed poset as above. Over \( V[H_\delta] \), let \( \dot{T} \) be a \( D \)-name for a Baire tree. Then \( D \ast (\dot{T} \times \dot{R}^*) \) is \( \omega \)-distributive by Claim 3.9. By Lemma 3.3 in \( V[H_\delta] \) we have \( \models_{D\ast(T \times \dot{R}^*)} E \) is c.c.c. This means in \( V[G_\delta] \), \( \models_{T \times \dot{R}^*} E \) is c.c.c. So \( \models_E \dot{R}^* \times \dot{T} \) is \( \omega \)-distributive, hence \( \models_E \dot{R} \times \dot{T} \) is Baire.

Since in \( V[G_\delta] \), \( E \times \dot{R}^* \) projects onto \( Q/G_\delta \), we know \( \models_{Q/G_\delta} \dot{T} \) is Baire.

**Proof of Theorem 1.13.** In fact, we show that in the forcing extension by \( Q \), \( sRC^b \) holds and all \( \aleph_1 \)-subtree of \( T(\mathbb{R}) \) is special. The latter clearly implies the failure of \( RC \) by Observation 3.2. Claim 3.10 implies that \( \omega_1 \) is preserved.

Let \( G \) be generic for \( Q \). As \( Q \) projects onto \( P_\kappa \), we can get \( H \in V[G] \) that is \( V \)-generic for \( P_\kappa \). Let \( T \in V[G] \) be a Baire tree of size \( \theta \). Without loss of generality, we might assume \( T = (\theta, \prec) \) for some tree order \( \prec \). Let \( \dot{T} \) be a \( Q \)-name that names it. Let \( j : V \rightarrow M \) witness that \( \kappa \) is \( \lambda \)-supercompact for some sufficiently large regular cardinal \( \lambda > \theta \). We may choose \( \lambda \) large enough so that it is larger than the cardinality of any nice \( Q \)-name of a subset of \( \theta \). Since \( Q \subset V_\kappa \) is \( \kappa \)-c.c.c., we see that \( j \upharpoonright Q = id \upharpoonright Q \) is a complete embedding. Hence we can view \( Q \) as an initial segment of \( j(Q) \). In fact, \( Q = j(Q) \upharpoonright \kappa \).

By the choice of \( \lambda \), we know \( T \in M \). Hence \( T \in M[G] \). By Claim 3.10 we know that in \( M[G] \), \( \models_{j(Q)/G} T \) is Baire. Let \( K \subset j(Q)/G \) be generic over \( M[G] \), then we can lift \( j \) to an elementary embedding from \( V[G] \) to \( M[G * K] \). In \( M[G * K] \), \( |T| = \theta < j(\kappa) = (\omega_2)^M[G * K] \). Since \( j \upharpoonright \theta, j''T \in M[G * K] \) and \( (T, \prec_t) \) is isomorphic to \( (j''T, \prec_{j(T)}) \), \( M[G * K] \models \) there exists \( A \subset j(T) \) such that \( |A| \leq \aleph_1 \) and \( A \) is Baire. By elementarity, the same statement is true in \( V[G] \). So we show that \( V[G] \models sRC^b \).

Finally we show that any \( \aleph_1 \)-subset of \( T(\mathbb{R}) \) is special. Let \( A \subset T(\mathbb{R}) \) be a \( \aleph_1 \)-subset in \( V[G] \). Note that \( Q/H \) is \( \omega \)-distributive, so \( (T(\mathbb{R}))/V[G] = (T(\mathbb{R}))/V[H] \). \( Q \) is \( \kappa \)-c.c. and so is \( P_\kappa \), therefore, \( Q/H \) is also \( \kappa \)-c.c. Thus there exists \( A' \in V[H] \) and \( A' \subset T(\mathbb{R}) \) of size \( < \kappa \) such that in \( V[H] \), \( \models_{Q/H} \dot{A} \subset A' \), where \( \dot{A} \) is a \( Q/H \)-name for \( A \) in \( V[H] \). In \( V[H] \), \( A' \) is special. In \( V[G] \), \( A \subset A' \) so \( A \) is also special.

\[ \square \]

4. CONSISTENCY OF \( RC^b + MA_{\omega_1}(BIP) \)

**Definition 4.1.** A poset \( P \) is semi-strongly proper if for sufficiently large regular \( \lambda \), for any \( M \prec H(\lambda) \) containing \( P \), and any countable sequence of dense subsets \( \langle D_n : n \in \omega \rangle \) of \( P \cap M \) and any \( p \in P \cap M \), there exists \( q \leq p \), such that for all \( n \in \omega \), \( q \upharpoonright D_n \cap G \neq \emptyset \). We say such \( q \) is semi-strongly generic for \( \langle D_n : n \in \omega \rangle \).

Note that we don’t require \( D_n = D \cap M \) for some \( D \in M \).

**Remark 4.2.** We abbreviate the above as \( P \) is semi-strongly proper for \( M \) and \( \langle D_n : n \in \omega \rangle \). In the following, when the model \( M \) is clear from the context, we will just say \( P \) is semi-strongly proper for \( \langle D_n : n \in \omega \rangle \).

**Remark 4.3.** Note that the class of semi-strongly proper forcings here properly contains the class of strongly proper forcings in the sense of Mitchell. Strongly
proper forcing always adds Cohen reals while any countable closed forcing will be semi-strongly proper. Shelah in [16] used the name “strongly proper forcings” to refer to what we call “semi-strongly proper forcings”. We do this to avoid confusion.

In general, Baire trees are preserved when forcing with semi-strongly proper posets.

**Lemma 4.4.** Let $T$ be a Baire tree and $P$ be a semi-strongly proper poset. Then $\Vdash_T P$ is semi-strongly proper. In particular, $\Vdash_P T$ is Baire.

*Proof.* Let $H \subset T$ be generic over $V$. Let $\lambda$ be large enough regular and let $M \prec H(\lambda)$ such that $M \cap V \subseteq (H(\lambda))^V$ containing $P$. Let $p \in M \cap P$ and a sequence of dense subsets of $M \cap P$, say $\bar{D} = \langle D_n : n \in \omega \rangle$ be given. Since $D_n \subset M \cap P = (M \cap V) \cap P$, by countable closure, $\langle D_n : n \in \omega \rangle \in V$. Since in $V$, $P$ is semi-strongly proper for $M \cap V$. There exists $q \leq p$ that is semi-strongly generic for $M \cap V$ and $\bar{D}$, namely $q \Vdash_P D_n \cap \bar{G} \neq \emptyset$ for all $n \in \omega$. But this property persists to $V[H]$ by the absoluteness of the definability of forcing.

Notice what we have shown is that in $V[H]$, for sufficiently large $\lambda$, there exists a club subset of $[H(\lambda)]^\omega$ witnessing the strong properness of $P$. By the standard trick, we can eliminate the club in the statement.

\[\square\]

**Remark 4.5.** The reader may notice that we restrict our attention to proper forcings. This is natural since any forcing that preserves all Baire trees is necessarily proper.

**Remark 4.6.** It is a theorem of Shelah (see [15]) that countable support iteration of semi-strongly proper forcings is semi-strongly proper. In light of Lemma 4.4 we can get the consistency of $RC^b$ and $MA_{\omega_1}$ (semi-strongly proper) rather easily. However, this forcing axiom is not strong enough to ensure the failure of $RC$. For example Baumgartner specializing forcing for $T(\mathbb{R})$ is not semi-strongly proper. Also there are many natural examples of BIP forcings that are not semi-strongly proper, like the Laver forcing.

**Remark 4.7.** It may be tempting to show that for a Baire tree $T$, countable support iteration of forcings that preserve the Baireness of $T$ preserves the Baireness of $T$. However, in general this is false. In fact, it is consistent that there exists a Baire tree $T$ and a countable support iteration of proper forcings $\langle p_i, \dot{Q}_j : i \leq \omega, j < \omega \rangle$ such that $\Vdash_{p_i} T$ is Baire for all $i < \omega$, but $\Vdash_{p_\omega} T$ is special.

In light of Remark 4.7 we need to consider stronger property that implies Baire-preserving so that the property is also preserved under countable support iteration. This class should also include semi-strongly proper forcings. The class $BIP$ (see Definition 4.12) turns out to be as desired.

**Definition 4.8.** Fix $M \prec H(\lambda)$ containing relevant objects, including a countable support iteration of proper forcings $\langle P_i, \dot{Q}_j : i \leq \alpha, j < \alpha \rangle$. Let $C$ be a countable collection of dense subsets of $P_i \cap M$ for $i \in M \cap \alpha + 1$. We say $C$ is closed under operations with respect to $M$ (if $M$ is clear from the context we will just say $C$ is closed under operations) if for any $D \in C$, $\gamma < \gamma' \in M \cap \alpha + 1$ such that $D$ is a dense subset of $P_i \cap M$, $(p, \dot{q}) \in M \cap (P_\gamma \ast P_{\langle \gamma' \rangle})$, $A_{D, \gamma}(p, \dot{q}) = \{r \in P_\gamma \cap M : r \perp p \lor \exists \gamma'' \leq D \cap M, r' =_{def} (r, \dot{q}) \in D \cap M, r' \leq (p, \dot{q})\}$ is also in the collection. We let $C_\gamma$ to denote the collection of $D \in C, D$ is a dense subset of $P_\gamma \cap M$.  


Remark 4.9. In order for the definition above to make sense, we need to verify \( A_{D,\gamma_j}(p,\dot{\beta}) \) as defined is dense in \( P_\gamma \cap M \). But this is clear.

Claim 4.10. Let \( M \) and \( \langle P_i, \dot{Q}_i : i \leq \alpha, j < \alpha \rangle \) be as in Definition 4.8. For any \( C \) closed under operations, and \( \gamma \in M \cap \alpha + 1 \), suppose \( G \subseteq P_\gamma \) such that \( G \) meets all the dense sets in \( C_\gamma \), then in \( V[G] \), any \( D \in C_{\gamma + 1} \), \( (D)^G := \{ \langle \dot{q} \rangle^G : \exists p \in G \ (p, \dot{q}) \in D \} \) is dense in \( M[G] \cap Q_\gamma \).

Proof. In \( V[G] \), let \( t \in M[G] \cap Q_\gamma \). Let \( \dot{t} \in M \) be its name. Since \( M[G] \prec H(\lambda)^{V[G]} \), we know that there exists \( q \in G \cap M[G] \cap P_\gamma \subseteq M \cap P_\gamma \) such that \( p \Vdash \dot{t} \in \dot{Q}_\gamma \). Consider \( A_{D,\gamma_j}(p,\dot{\beta}) \), which is a set in \( C_\gamma \) by the closure assumption. Since \( G \cap A_{D,\gamma_j}(p,\dot{\beta}) \neq \emptyset \), we can pick \( r \) in the intersection. By the definition of \( A_{D,\gamma_j}(p,\dot{\beta}) \) and the fact that \( r, p \in G \) which implies they are compatible, there exists \( \dot{r} \in M \) such that \( (r, \dot{r}) \in D \) and is below \( (p, \dot{t}) \). Then since \( r \in G \), \( (\dot{r})^G \in (D)^G \subseteq M[G] \cap Q_\gamma \) and is stronger than \( t \). \( \square \)

Before we proceed with our iteration lemma, we need an extension lemma essentially due to Shelah about iteration of semi-strongly proper posets.

Lemma 4.11 (Shelah). Let \( \langle P_i, \dot{Q}_i : i \leq \alpha, j < \alpha \rangle \) be a countable support iteration of proper forcings and \( M \prec H(\lambda) \) contain relevant objects including \( P_\alpha \). Fix \( \alpha_0 \in M \cap \alpha + 1 \). Suppose \( C \) is a countable collection of dense subsets of \( P_\gamma \cap M \) for \( \gamma \in M \cap (\alpha + 1) \) closed under operations.

Suppose for each \( \gamma \in M \cap \alpha \) and \( q \in P_\gamma \) that \( q \) is semi-strongly generic for \( M \) and \( C_\gamma \), \( q \Vdash_{P_\gamma} \dot{Q}_\gamma \) is strongly proper for \( M[G_\gamma] \) and \( (C_{\gamma + 1})^G_\gamma := \{ (D)^G_\gamma : D \in C_{\gamma + 1} \} \).

If \( q \in P_{\alpha_0} \) is semi-strongly generic for \( C_{\alpha_0} \) and \( \dot{p} \in P_{\alpha_0} \) is a \( P_{\alpha_0} \)-name such that

\[
q \Vdash_{P_\gamma} \dot{p} \in P_\gamma \cap M, \dot{p} \upharpoonright \alpha_0 \in \dot{G}_{\alpha_0},
\]

then there exists \( q' \in P_\alpha, q' \upharpoonright \alpha_0 = q \) and \( q' \) is semi-strongly generic for \( C_\alpha \) and

\[
q' \Vdash_{P_\gamma} \dot{p} \in \dot{G}_\alpha.
\]

Proof. We proceed by the induction on \( \alpha \). If \( \alpha = \beta + 1 \), fix such \( M \prec H(\lambda) \) containing the iteration and \( \gamma_0 \in M \cap \alpha \). Note that \( \beta \in M \). Let \( q \in P_{\alpha_0} \), and \( \dot{p} \in V^{P_{\alpha_0}} \) be as given. Apply the induction hypothesis, we get \( q' \leq q, q' \upharpoonright \alpha_0 = q \) such that \( q' \) is semi-strongly generic for \( C_\beta \) and \( q' \Vdash_{P_\gamma} \dot{p} \upharpoonright \beta \in \dot{G}_\beta \) and \( \dot{p} \in P_\alpha \cap M \). By the hypothesis and Claim 4.10, we have \( q' \Vdash_{P_\gamma} \dot{Q}_\beta \) is semi-strongly generic for \( \dot{M}[\dot{G}_\beta] \) and \( \dot{Q}_\beta \) is semi-strongly proper for \( \dot{M}[\dot{G}_\beta] \) and \( (C_{\beta + 1})^{\dot{G}_\beta} \). Let \( G_\beta \subseteq P_\beta \) be generic over \( V \) containing \( q' \), then in \( V[G_\beta], (\dot{p})^{G_\beta} = p \in P_\alpha \cap M \). Since in \( V[G_\beta], Q_\beta \) is semi-strongly proper for \( M[G_\beta] \) and \( (C_{\beta + 1})^{G_\beta} \), there exists \( t \leq Q_\beta(p(\beta))^{G_\beta} \) that is semi-strongly generic for \( (C_{\beta + 1})^{G_\beta} \). Let \( \dot{t} \) be a \( P_\beta \)-name for \( t \) such that \( q' \) forces it satisfies all the properties above, which exists by the maximality principle of forcing. Hence \( (q', \dot{t}) \) is the desired extension. Indeed, \( (q', \dot{t}) \Vdash_{P_\gamma} \dot{p} \in \dot{G}_\alpha, (q', \dot{t}) \upharpoonright \alpha_0 = q' \upharpoonright \alpha_0 = q \) and \( (q', \dot{t}) \) is semi-strongly generic for \( C_\alpha \) (which is easily implied by the fact that \( p \) is semi-strongly generic for \( C_\beta \) and \( p \Vdash_{\dot{t}} \dot{p} \) is semi-strongly generic for \( (C_{\beta + 1})^{G_\beta} \)).

When \( \alpha \) is a limit, list \( \{ D_n : n \in \omega \} \) in \( C_\alpha \) and fix \( \alpha_0 \in M \cap \alpha, q \in P_{\alpha_0}, \dot{p} \in V^{P_\beta} \) as in the statement. Fix an increasing \( \langle \alpha_i : \alpha_i \in M \cap \alpha : i \in \omega \rangle \) cofinal in \( \sup M \cap \alpha \).

We build the following sequences: \( \langle q_i : i < \omega \rangle, \langle \dot{p}_i : i < \omega \rangle \) such that
• \( \dot{p}_0 = \dot{q}_0 = q \)
• \( q_i \in P_\alpha \) is semi-strongly generic for \( C_\alpha \)
• \( p_i \) is a \( P_\alpha \)-name
• \( q_{i+1} \Vdash_{P_{\alpha_{i+1}}} p_{i+1} \in P_\alpha \cap M, \dot{p}_{i+1} \mid \alpha_{i+1} \in \dot{G}_{\alpha_{i+1}}, \dot{p}_i \leq \dot{p}_{i+1} \in D_i \)
• \( q_{i+1} \mid \alpha_i = q_i \)

If the construction is successful, then by the standard argument as in the properness preservation theorem, we will be done.

Now we demonstrate given \( q_i, p_i \), how to find \( q_{i+1}, \dot{p}_{i+1} \) satisfying the requirements. Let \( G \subseteq P_\alpha \) be generic over \( V \) containing \( q_i \). In \( V[G] \), we have \( p_i = (\dot{p}_i)^G \in P_\alpha \cap M \) and \( p_i \mid \alpha_i \in G \). Consider the set \( A_{D_i, \alpha_i, p_i} \subseteq C_\alpha \). By the strong genericity of \( q_i \) with respect to \( \alpha_i \), \( G \cap A_{D_i, \alpha_i, p_i} \neq \emptyset \). Let \( r \in G \cap A_{D_i, \alpha_i, p_i} \), then there exists \( (r, \dot{q}) \in D_i \) and \( (r, \dot{q}) \leq p_i \). By the maximality principle of forcing we can find a \( P_\alpha \)-name \( \dot{p}_{i+1} \) such that \( q_i \Vdash_{P_\alpha} \dot{p}_{i+1} \leq \dot{p}_i, \dot{p}_{i+1} \in D_i \subseteq M \cap P_\alpha, \dot{p}_{i+1} \mid \alpha_i \in \dot{G}_{\alpha_i} \).

Apply the induction hypothesis, we can get \( q_{i+1} \in P_{\alpha_{i+1}} \) with \( q_{i+1} \mid \alpha_i = q_i, q_{i+1} \) is semi-strongly generic for \( C_{\alpha_{i+1}} \) and \( q_{i+1} \Vdash_{P_{\alpha_{i+1}}} p_{i+1} \mid \alpha_{i+1} \in \dot{G}_{\alpha_{i+1}} \).

\( \Box \)

**Corollary 4.12.** Let \( \langle P_i, \dot{Q}_i : i \leq \alpha, j < \alpha \rangle \) be a countable support iteration of proper forcing. Let \( M \prec H(\lambda) \) contain \( P_\alpha \). Let \( C \) be a countable collection of dense subsets of \( P_\gamma \cap M \) for \( \gamma \in M \cap (\alpha + 1) \) closed under operations.

Suppose for each \( \gamma \in M \cap \alpha \) and \( q \in P_\gamma \) that is semi-strongly generic for \( M \) and \( C_\gamma, q \Vdash_{P_\gamma} \dot{Q}_\gamma \) is strongly proper for \( M[\dot{G}_\gamma] \) and \( (C_{\gamma+1})^{\dot{G}_\gamma} \).

Then for each \( p \in M \cap P_\alpha \), there exists \( q \leq p \) that is semi-strongly generic for \( C_\alpha \).

**Proof.** Apply Lemma 4.11 by setting \( \alpha_0 = 0 \). \( \Box \)

Our main idea is that in order to prove properness of a poset in a countably distributive extension, it suffices to prove strong properness in the ground model with respect to the relevant collection of dense sets.

The following iteration lemma is key to the proof of the main theorem.

**Lemma 4.13** (Key Lemma). Let \( T \) be a Baire tree and \( \langle P_i, \dot{Q}_i : i \leq \alpha, j < \alpha \rangle \) be a countable support iteration of proper forcings such that for each \( i < \alpha, \Vdash_{T \times P_i} \dot{Q}_i \) is proper. Then \( \Vdash_T P_\alpha \) is proper.

**Remark 4.14.** Notice in Lemma 4.13 there is some abuse of notation, in that \( \dot{Q}_i \) is actually a \( P_i \)-name, but it can be canonically identified as a \( (T \times P_i) \)-name, say \( \dot{Q}'_i \). So here we really mean \( \Vdash_{T \times P_i} \dot{Q}'_i \) is proper. But in general, there is only one way of interpretation based on the context, so we confuse \( \dot{Q}_i \) with \( \dot{Q}'_i \). We extend this practice to other similar situations.

**Proof.** We proceed by induction on \( \alpha \). If \( \alpha = \beta + 1 \), then by the hypothesis, \( \Vdash_{T \times P_\beta} \dot{Q}_\beta \) is proper. Let \( H \subset T \) be generic over \( V \). We need to show that \( V[H] \models P_\alpha \) is proper. We will be done once we realize that in \( V[H], P_\alpha \) is a dense subset of \( P_\beta \ast \dot{Q}_\beta \). The difference between these two sets is \( P_\alpha \) is the two-step iteration defined in \( V \), so \( (p, \dot{q}) \in P_\alpha \) \( \rightarrow (p, \dot{q}) \in V \) while \( P_\beta \ast \dot{Q}_\beta \) is the iteration defined in \( V[H] \) so it may contain \( (p, \dot{q}) \) where \( \dot{q} \notin V \). Given \( (p, \dot{q}) \in P_\beta \ast \dot{Q}_\beta \), \( p \Vdash \dot{q} \in \dot{Q}_\beta \). Since \( \dot{Q}_\beta \) is a \( P_\beta \)-name living in \( V \), we know \( p \Vdash \exists \hat{t} \in V \hat{t} = \dot{q} \). Let \( p' \leq p \) and \( \hat{t} \in V \) be such that \( p' \Vdash \hat{t} = \dot{q} \). Then \( (p', \hat{t}) \leq (p, \dot{q}) \) and \( (p', \hat{t}) \in P_\alpha \). By the hypothesis, \( V[H] \models P_\beta \ast \dot{Q}_\beta \) is proper, so we are done.
When $\alpha$ is a limit ordinal, let $H \subseteq T$ be generic over $V$. In $V[H]$, let $\lambda$ be a large enough regular cardinal, and $M \prec H(\lambda)$ contain relevant objects including $P_\gamma$ such that $M \cap V \prec H(\lambda)V$. Note since $V[H]$ is a countably distributive extension of $V$, we have that $M' = M \cap V \in V$.

For each $\gamma \in M \cap (\alpha + 1)$, enumerate the dense subsets of $P_\gamma$ in $M$ as $D^\gamma = \{D^\gamma_n : n \in \omega\}$. Let $\bar{D}^\gamma \uparrow M = \{D^\gamma_n \cap M : n \in \omega\}$. As each $D^\gamma_n \cap M \subseteq P_\gamma \cap M = P_\gamma \cap M' \subset V$, so by countable distributivity, $\bar{D}^\gamma \uparrow M \in V$.

**Claim 4.15.** $\bigcup_{\gamma \text{M}(\alpha+1)} \bar{D}^\gamma \uparrow M \in V$ is closed under operations.

**Proof.** For any $\gamma' \leq \alpha$ and $\gamma < \gamma' \in M$ and $p \in P_{\gamma'} \cap M$, dense $D \subset P_{\gamma'}$ with $D \in M$, we need to show that $A_{D \cap M, P_{\gamma}, p} \in \bar{D}^\gamma \uparrow M$. But this is immediate from the fact that $A' = \{r \in P_{\gamma'} : r \perp p | \gamma \vee \exists q' r' = r \in D, r' < p\}$ is a dense subset of $P_\gamma$ living in $M$ and $A_{D \cap M, P_{\gamma}, p} = A' \cap M \in \bar{D}^\gamma \uparrow M$ by elementarity. $\square$

**Claim 4.16.** In $V$, for each $\gamma \in M \cap \alpha$ and any $p \in P_\gamma$ that is semi-strongly generic for $\bar{D}^\gamma \uparrow M$, $q \models P_\gamma[\dot{G}_{\gamma}]$ is semi-strongly proper for $M'[\dot{G}_\gamma]$ and $\bar{D}^{\gamma+1} \uparrow M[\dot{G}_\gamma]$.

**Proof of the claim.** Fix $\bar{r} \in M$ a $P_{\gamma}$-name for a condition in $\dot{Q}_\gamma$. Let $G_{\gamma} \subset P_{\gamma}$ be generic over $V[H]$ containing $\bar{q}$, then $V[H \times G_{\gamma}] \models Q_{\gamma} = (\dot{Q}_{\gamma})^{G_{\gamma}}$ is proper with respect to $M[G_{\gamma}]$. Let $r \leq (\dot{r})^{G_{\gamma}}$ be a $(M[G_{\gamma}], Q_{\gamma})$-generic condition. We claim that in $V[G_{\gamma}]$, $r$ is semi-strongly generic for $(\bar{D}^{\gamma+1} \uparrow M)\dot{G}_{\gamma}$. Then the claim follows immediately.

The fact that in $V$, $q$ that is semi-strongly generic for $\bar{D}^\gamma \uparrow M$ implies that in $V[H]$, $q$ is $(M, P_{\gamma})$-generic. Therefore, for each $D^\gamma_n = \bar{D}^{\gamma} \uparrow M$, $(D^\gamma_n)^{G_{\gamma}}$ is a dense subset of $Q_{\gamma} = (\dot{Q}_{\gamma})^{G_{\gamma}}$ living in $M[G_{\gamma}]$. Fix $n \in \omega$. Since in $V[H][G_{\gamma}]$, $r$ is $(M[G_{\gamma}], Q_{\gamma})$-generic, we know $r \models Q_{\gamma} (D^\gamma_{n+1})^{G_{\gamma}} \cap M[G_{\gamma}] \cap W \neq \emptyset$, where $W$ is the canonical name for the generic filter. To see that this implies $r \models Q_{\gamma} (D^\gamma_{n+1})^{G_{\gamma}} \cap G \neq \emptyset$, let $W \subset Q_{\gamma}$ be generic over $V[H][G_{\gamma}]$ containing $r$. In $V[H][G_{\gamma}][W]$, there exists $\bar{r} \in M$ such that $(\dot{r})^{G_{\gamma}} = (D^\gamma_n)^{G_{\gamma}}$ and $(\dot{r})^{G_{\gamma}} \in W$. Since both $(\dot{r})^{G_{\gamma}}$ and $(D^\gamma_{n+1})^{G_{\gamma}}$ are in $M[G_{\gamma}] \prec (H(\lambda))^{V[H][G_{\gamma}]}$, $M[G_{\gamma}] \models \exists p \in G_{\gamma} \exists (p, \bar{r}) \in D^\gamma_{n+1}$ and $(\dot{r})^{G_{\gamma}} = (\dot{\bar{r}})^{G_{\gamma}}$. Find $p, \bar{r} \in D^\gamma_{n+1} \cap M[G_{\gamma}]$ witnessing the statement above. Since $q$ is $(M, P_{\gamma})$-generic and $G_{\gamma}$ contains $q$, we know $M[G_{\gamma}] \cap V[H] = M$. So $(p, \bar{r}) \in M$. Hence in $V[H][G_{\gamma}][W]$, we have found $(\dot{r})^{G_{\gamma}} \in (D^\gamma_{n+1} \cap M)^{G_{\gamma}} \cap W$.

We claim that $V[G_{\gamma}]$ models the same statement, namely $r \models Q_{\gamma} (D^\gamma_{n+1} \cap M)^{G_{\gamma}} \cap W \neq \emptyset$.

Suppose not for the sake of contradiction, in $V[G_{\gamma}]$ we can extend $r$ to $r'$ to force the negation of the statement. Let $W \subset Q_{\gamma}$ containing $r'$ be generic over $V[H][G_{\gamma}]$. Then $V[G_{\gamma} \ast W] \models (D^\gamma_n \cap M)^{G_{\gamma}} \cap W = \emptyset$ but $V[H][G_{\gamma} \ast W] \models (D^\gamma_{n+1} \cap M)^{G_{\gamma}} \cap W \neq \emptyset$. By the product lemma, $V[G_{\gamma} \ast W]$ is a submodel of $V[H][G_{\gamma}][W]$ so the statement is absolute between these two models. We thus get a contradiction. $\square$

Work in $V$. By Claim 4.13 and Corollary 4.12 we can conclude that $P_\alpha$ is semi-strongly proper for $M'$ and $\bar{D}^\alpha \uparrow M$. Using the same argument as in Lemma 4.2, this implies that in $V[H], P_\alpha$ is proper with respect to $M$.

$\square$

**Theorem 4.1.** Countable support iteration of BIP forcings is BIP.
Proof. Let \( \langle P_i, \dot{Q}_i : i \leq \alpha, j < \alpha \rangle \) be the iteration and \( T \) be a given Baire tree. We show this by induction. If \( \alpha = \beta + 1 \), then by the induction hypothesis, \( \forces_T P_\beta \) is proper. In particular by Lemma \[1.19\] \( \forces_T P_\beta \) is Baire indestructible, we have \( \forces_T P_\beta \forces_T \dot{Q}_\beta \) is proper. Hence by the product lemma, \( \forces_T (P_\beta \times \dot{Q}_\beta) \) is proper and \( \forces_T \dot{Q}_\beta \) is proper. Since \( \forces_T (P_\beta \times \dot{Q}_\beta) \) is a dense subset of \( P_\beta \times \dot{Q}_\beta \), we see that \( \forces_T P_{\beta+1} \) is proper. If \( \alpha \) is a limit, then we check the hypotheses in the Key Lemma \[4.13\] are satisfied. For each \( i < \alpha \), \( \forces_T P_i \) is proper. By the same argument as above, we have \( \forces_T \times P_i \dot{Q}_\iota \) is proper. Hence the hypothesis is satisfied, so \( \forces_T P_\alpha \) is proper.

\( \square \)

Corollary 4.17. Countable support iteration of BIP forcings preserve Baire trees.

Proof. Immediately from Lemma \[1.19\] and Theorem \[4.1\] \( \square \)

Proof of Theorem \[1.8\] Let \( \kappa \) be a supercompact cardinal. Let \( \langle P_\alpha, \dot{Q}_\alpha : \alpha < \kappa \rangle \) be the countable support iteration of BIP forcings guided by the Laver function (see \[4\] for more details) of length \( \kappa \). In this model \( MA_{\omega_1}(BIP) \) holds. We claim that \( RC^b \) holds. Let \( G \) be a generic for \( P_\kappa \). Let \( T \in V[G] \) be a Baire tree of height \( \omega_1 \). Let the size of \( T \) be \( \theta \). Furthermore, we may assume \( T \) is of the form \( (\theta, <) \) for some tree order \( < \). Let \( \dot{T} \) be a \( P_\kappa \)-name for \( T \). Let \( \lambda >> \max\{|\dot{T}|, \kappa\} \) be a sufficiently large regular cardinal.

Fix an elementary embedding \( j : V \to M \) such that \( M^\lambda \subset M \). Let \( H \subset j(P_\kappa)/G \) be generic over \( V[G] \). Then we can lift \( j \) to an elementary embedding from \( V[G] \) to \( M[G][H] \). We will slight abuse the notation by still using \( j \) to refer to the lifted embedding in \( V[G][H] \). Notice by the closure assumption on \( j \), we have \( j''T, T \in M[G][H] \).

By Corollary \[4.17\] \( T \) remains Baire in \( M[G][H] \). Since \( M[G][H] \models (T, <_T) \simeq (j''T, <_{j(T)}) \). By elementarity, in \( V[G] \), \( T \) has a Baire subtree of size \( < \kappa = \omega_2 \). \( \square \)

The following lemma gives yet another model separating \( RC^b \) from \( RC \).

Lemma 4.18. \( MA_{\omega_1}(BIP) \) implies all \( \aleph_1 \) subtrees of \( T(\mathbb{R}) \) are special.

Proof. This just follows from the observation that for any \( \aleph_1 \) subset \( T' \) of \( T(\mathbb{R}) \), the Baumgartner specializing forcing for \( T' \) is BIP (see Lemma \[3.3\]) \( \square \)

5. The Strength and Limitations of the Baire Rado’s Conjecture

Definition 5.1. For any regular cardinal \( \lambda \) and stationary \( S \subset \lambda \), we say \( S \) reflects if there exists \( \beta \in \lambda \cap cof(\omega) \) such that \( S \cap \beta \) is stationary in \( \beta \). Given a family \( S \) of stationary subsets of \( \lambda \), we say \( S \) reflects simultaneously if there exists \( \beta \in \lambda \cap cof(\omega) \) such that for each \( S \in \mathcal{S} \), \( S \cap \beta \) is stationary.

In \[17\], Todorcevic shows that \( RC \) implies any stationary subset of \( \lambda \cap cof(\omega) \) reflects for regular \( \lambda \geq \omega_2 \). The proof there uses some equivalent characterizations of \( RC \). We include a short argument here (from \( RC^b \) actually) using directly the tree formulation of \( RC \) as in Definition \[1.6\] and \[1.11\]. It is worth noting that Sakai derives the same conclusion from the Semistationary Reflection Principle (see \[14\]), which is a consequence of \( RC^b \) by Theorem \[5.2\].

Theorem 5.1 (Todorcevic). \( RC^b \) implies any stationary subset of \( \lambda \cap cof(\omega) \) reflects for regular cardinal \( \lambda \geq \omega_2 \).
Proof. Let \( S \subset \lambda \cap \text{cof}(\omega) \) be given. Let \( T(S) \) be the tree consisting functions \( t : \gamma + 1 \rightarrow S \), such that \( t \) is increasing and continuous, ordered by end extension. The stationarity of \( S \) implies \( T(S) \) is Baire by the standard argument. For any subtree \( T \leq T(S) \), let \( \sup T \) be \( \sup \{ \text{image}(t) : t \in T \} \). Apply \( RC^b \) to \( T(S) \) and pick some \( \aleph_1 \)-sized nonspecial subtree \( T \leq T(S) \) with the least supremum, say \( \delta \). \( \delta \) must have cofinality \( \omega_1 \) by the minimality of \( \delta \) and the fact that any two nodes of the same top element are incompatible. We claim that \( S \cap \delta \) is stationary. Suppose not for the sake of contradiction, then there exists a club \( C \subset \delta \) that is disjoint from \( S \cap \delta \). Define a pressing down function \( f : T \rightarrow T \) where for each \( t \in T \) of limit height, \( f(t) = s \) if \( s \) is the \( <_T \)-least predecessor of \( t \) such that \( (\max \text{image}(s), \max \text{image}(t)) \cap C = \emptyset \). By the Pressing Down Lemma for trees, there exists a nonstationary subtree \( T' \leq T \) such that \( f \) gets constant value \( s \) with \( \gamma = \max \text{image}(s) \). But then for each \( t \in T' \), \( \max \text{image}(t) \leq \min C \setminus \gamma < \delta \). Hence \( \sup T' < \delta \). By the minimality of \( \delta \), \( T' \) is special. This is a contradiction. \( \square \)

Remark 5.2. The theorem above shows that \( RC^b \) implies the ordinary stationary reflection at ordinals of cofinality \( \omega_1 \). In general, \( RC \) does not imply stationary reflection at ordinals of cofinality \( \omega_1 \). In fact \( RC \) is compatible with the fact that \( \aleph_{\omega_1 + 1} \cap \text{cof}(\geq \omega_2) \) carrying a partial square (see section 5 of [10]), which in turn implies the failure of the ordinal stationary reflection at ordinals of cofinality \( \geq \omega_2 \).

Similar to the argument above, we are able to present an alternative argument that \( RC^b \) implies Semi-stationary Reflection (SSR) due to Doebler [6].

Theorem 5.2 (Doebler [6]). \( RC^b \) implies SSR, where the latter means that for any regular cardinal \( \lambda \geq \omega_2 \) and any \( S \subset [\lambda]^{\omega} \) closed under \( \subset \) (\( x \subset y \) iff \( x \subset y \) and \( x \cap \omega_1 = y \cap \omega_1 \)), there exists \( W \in [\lambda]^{\omega_1} \) containing \( \omega_1 \) such that \( S \cap [W]^{\omega} \) is stationary.

Proof. Fix \( \lambda, S \) as above. We may assume for any \( x \in S \), \( x \cap \omega_1 \in \omega_1 \).

Build the tree \( T(S) \) consisting of continuous strongly \( \subset \)-increasing sequences of elements in \( S \), namely for any \( t \in T(S) \) and \( a \neq b \in t \) if \( a \subset b \) then \( a \cap \omega_1 < b \cap \omega_1 \) and \( \sup a < \sup b \). Hence by design, each element \( t \) in \( T(S) \) has a \( \subset \)-maximum element. Let \( \max t \) denote this element. This tree is clearly Baire by the fact that \( S \) is stationary.

Apply \( RC^b \), we can find a subtree \( T' \leq T \) such that \( T' \) is nonspecial. Let \( W = \bigcup_{t \in T'} \max t \). We can choose such a \( T' \) such that \( \sup W \) is the least. Notice that \( W \supset \omega_1 \) and \( \text{cf}(W) > \omega \). This follows from the fact that for any \( t, t' \in T(S) \) with \( \sup \max t = \sup \max t' \), then they are incompatible.

We claim that \( S \cap [W]^{\omega} \) is stationary. Suppose not, there exists a function \( F : W^{<\omega} \rightarrow W \) such that \( \text{cl}_F \cap S = \emptyset \). We may redefine \( F \) such that for any \( y \in [W]^{\omega} \), \( F^y : y^{<\omega} \rightarrow \text{cl}_F(y) \). For each \( t \in T' \), \( \max t \in S \), so by the assumption \( t \) is not closed under \( F \). In particular, \( F^{\max t} \max t^{<\omega} \notin S \). Since \( S \) is closed under \( \subset \), we know that \( F^{\max t} \max t^{<\omega} \cap \omega_1 > \max t \cap \omega_1 \). Hence there exists \( \bar{a} \in \max t \) such that \( F(\bar{a}) \geq \max t \cap \omega_1 \). Now we can use this fact to define a regressive function so by Pressing Down Lemma for nonspecial trees, there exists \( T'' \leq T' \) nonspecial and \( \bar{a} \in W^{<\omega} \) such that for each \( t \in T'' \), \( F(\bar{a}) = \max t \cap \omega_1 \). But this means that the height of \( T'' \) is bounded above by \( F(\bar{a}) \), which is a contradiction to the fact that \( T'' \) is nonspecial. \( \square \)
From the aspect of simultaneous reflection, Theorem 5.1 is optimal.

**Theorem 5.3.** $RC^b$ does not imply any two stationary subsets of $\omega_2 \cap \text{cof}(\omega)$ reflects simultaneously.

**Proof.** We first prepare the ground model $V$ such that it satisfies that $RC^b$ is indestructible under $\omega_2$-directed closed forcings. We can do this by Lévy collapsing $\kappa$ to $\omega_2$ where $\kappa$ is a supercompact cardinal that is Laver indestructible (i.e. indestructible under $\kappa$-directed closed forcings).

Let $\mathbb{P}$ be the standard poset that adds two stationary subsets of $\omega_2 \cap \text{cof}(\omega)$ that do not reflect simultaneously. More precisely, $p \in \mathbb{P}$ if $p = (p_0, p_1)$ where

- $p_0 \cap p_1 = \emptyset$
- $p_0, p_1 \subset \omega_2 \cap \text{cof}(\omega)$ and there is $\gamma = \sup p_0 = \sup p_1 < \omega_2$
- for all $\beta \in \omega_2 \cap \text{cof}(\omega_1)$, there exists a club $C \subset \beta$ such that there is $i < 2$, $p_i \cap C = \emptyset$.

Let $\dot{S}_0, \dot{S}_1$ be the $\mathbb{P}$-name for the two stationary sets that are added by $\mathbb{P}$. Define in $V^\mathbb{P}$ forcings $\dot{t}_i$ for shooting a club with initial segments through the complement of $\dot{S}_i$ for $i < 2$. It is a standard fact that (for example see [4]) that $\mathbb{P} * \dot{Q}_i$ has a dense $\omega_2$-directed closed subset for $i < 2$. Let $G \subseteq \mathbb{P}$ be generic over $V$. We show that $V[G]$ is the desired model. To this end, fix a Baire tree $T \in V[G]$.

**Claim 5.3.** In $V[G]$, it is forced by $T$ that there exists $i < 2$, such that $\omega_2^V \cap \text{cof}^V(\omega) - S_i$ remains stationary.

**Proof.** Otherwise, we can find $H \subset T$ be generic over $V[G]$ such that in $V[G*H]$, for each $i < 2$, there is a club $C_i \subset \dot{S}_i \cup (\omega_2^V \cap \text{cof}^V(\omega_1))$. As $T$ is Baire, $\text{cf}^{V[G*H]}(\omega_2^V) > \omega$. Hence in $V[G*H]$, the fact that $S_0 \cap S_1 = \emptyset$ implies that $C_0 \cap C_1 \subset \omega_2^V \cap \text{cof}^V(\omega_1)$ is a club. But this implies that there is an ordinal whose cofinality is $\omega_1$ in $V$ now has cofinality $\omega$ in $V[G*H]$. This contradicts with the $\omega_1$-distributivity of $\mathbb{P}$ in $V$ and the Baireness of $T$ in $V[G]$.

**Claim 5.4.** In $V[G]$, it is forced by $T$ that there exists $i < 2$, $\dot{Q}_i$ is $\omega$-distributive.

**Proof.** Work in $V[G]$. For any given $t' \in T$, use Claim 5.3 to find $t \leq t'$ and $i < 2$ such that $t \Vdash \omega_2^V \cap \text{cof}^V(\omega) - S_i$ is stationary. Let $H \subset T$ be generic over $V[G]$ containing $t$. We show $V[G*H] \models Q_i = (\dot{Q}_i)^{V[G]}$ is $\omega$-distributive. Let $\dot{\tau}$ be a $Q_i$-name for a $\omega$-sequence of ordinals. Let $\lambda < H(\lambda)$ contain relevant objects such that $\gamma = \sup M \cap \omega_2^V \notin S_i$ where $\lambda$ is some large enough regular cardinal. Fix a generic sequence $\langle p_i : i \in \omega \rangle$ for $M$ so in particular for any $j < \omega$ there exists some $i < \omega$ such that $p_i$ decides $\dot{\tau}(j)$ and $\sup_{i \in \omega} p_i = \gamma$. As $T$ is Baire in $V[G]$, $\langle p_i : i \in \omega \rangle \in V[G]$. Since $\gamma \notin S_i$, we see that $\bigcup_{i \in \omega} p_i \cup \{\gamma\} \in Q_i$, which decides $\dot{\tau}$.

By Claim 5.4 in $V[G]$, find $t \in T$ and $i < 2$ such that $t \Vdash T Q_i$ is $\omega$-distributive. By the Product Lemma, $\Vdash_{Q_i} T \upharpoonright t$ is Baire. Let $R$ be generic for $Q_i$ over $V[G]$. Since $\mathbb{P} * \dot{Q}_i$ has a $\omega_2$-directed closed dense subset, it follows that $V[G*R] \models RC^b$. Hence there exists a nonspecial $T' \leq T \upharpoonright t$ of size $\aleph_1$. Since over $V[G]$, $Q_i$ is $\omega_1$-distributive, we know that $T' \in V[G]$ and it remains nonspecial in $V[G]$.

**Definition 5.5.** Let $\mu$ be a cardinal. $\langle A_i \subset \mu : i < \mu^+ \rangle$ is said to be an almost disjoint sequence if for each $i < \mu^+$, $A_i$ is unbounded in $\mu$ and for each $\beta < \mu^+$,
there exists $F : \beta \to \mu$ such that $\langle A_i \setminus F(i) : i < \beta \rangle$ is pairwise disjoint. $ADS_\mu$ abbreviates the assertion that there exists such a sequence.

The interesting case of the above principle is when $\mu$ is singular. It is known (see [3]) $ADS_\mu$ follows from the existence of a PCF-theoretic object, a better scale at $\mu$, which in turn is a consequence of $\square^*_\mu$. It is a theorem of Shelah that if $SCH$ fails, then the least ordinal where it fails (whose cofinality is necessarily $\omega$ by a theorem of Silver) carries a better scale. On the other hand, Sakai and Velickovic in [15] show that the Semistationary Reflection principle implies there is no better scale, extending the theorem of Todorcevic [20] that $RC$ implies $SCH$. We present a proof of $RC^b \to \neg ADS_\mu$ using the same ideas as those in Theorem 5.1 and 5.2.

**Proposition 5.6.** $RC^b$ implies $\neg ADS_\mu$ for all singular cardinal $\mu$ with $cf(\mu) = \omega$.

**Proof.** Suppose for the sake of contradiction that there exists $\langle A_i : i < \mu^+ \rangle$ that witnesses $ADS_\mu$. We may assume that each set in the sequence has order type $\omega$. $S = \{x \in [\mu^+]^\omega : A_{sup}x \subset x\}$ is stationary since for any Skolem function $F : (\mu^+)^\omega \to \mu^+$, we can find a $F$-closure point $\alpha \in \mu^+ \cap cof(\omega) \cap \mu$ and a cofinal sequence $\langle \alpha_i : i < \omega \rangle$ in $\alpha$ such that $cl_F(A_\alpha \cup \{\alpha_i : i \in \omega\}) \in S$. Define $T(S)$ for shooting a continuous $\omega_1$-sequence through $S$. More precisely, $t \in T(S)$ iff there exists $\gamma < \omega_1$ and $t : \gamma + 1 \to S$ that is continuous increasing and for any $\alpha < \beta \leq \gamma$, $sup(t(\alpha)) < sup(t(\beta))$. $T(S)$ is ordered by end extension. For each $t \in T(S)$ with domain $\gamma + 1$, let $max t$ be $t(\gamma)$.

**Claim 5.7.** $T(S)$ is Baire.

**Proof.** Fix a countable collection of dense open subsets of $T(S)$, say $\{D_i : i \in \omega\}$. Let $\lambda$ be a large enough regular cardinal and $M \prec H(\lambda)$ contain relevant objects such that $M \cap \mu^+ \in S$. Now build a sequence $p_i \in T(S) \cap M$ such that

- $p_{2i} \cap D_i$ for all $i \in \omega$
- $p_i \subset p_{i+1}$ for all $i \in \omega$
- $\bigcup_{i \in \omega} p_i = M \cap \mu^+$

It is easy to see that $\bigcup_{i \in \omega} p_i \cup \{\langle M \cap \omega_1, M \cap \mu^+ \rangle\}$ is in $\bigcap_{i \in \omega} D_i$. \hfill $\square$

For each $T \leq T(S)$, define $sup T = sup\{sup(max t) : t \in T\}$. Apply $RC^b$ to $T(S)$ and pick a nonspecial subtree $T' \leq T(S)$ of size $\omega_1$ with the least supremum, say $sup T' = \delta$. The cofinality of $\delta$ must be $\omega_1$ by the minimality of $\delta$ and the fact that any $t, t' \in T(S)$ with $sup max t = sup max t'$ are incompatible. Let $W = \bigcup_{t \in T'} max t$. By the almost-disjointness, there exists $F : W \to \mu$ such that $\langle A_i \setminus F(i) : i < \mu^+ \rangle$ is pairwise disjoint. Define a pressing down function $f : T' \to T''$ such that for each $t$ of limit height $f(t)$ is the $<_{T'}$-least $s$ such that there exists $\alpha \in (A_{sup}max t \setminus F(sup max t)) \cap max s$. By the Pressing Down Lemma and countable completeness of nonspecial trees, we can find nonspecial subtree $T'' \leq T'$ and $\alpha \in \mu^+$ such that for each $t \in T''$, $\alpha \in (A_{sup}max t \setminus F(sup max t))$. Therefore by the fact that $\langle A_i \setminus F(i) : i < \mu^+ \rangle$ is pairwise disjoint, there exists $\delta' < \delta$, for all $t \in T''$, $sup max t = \delta'$. By the minimality of $\delta$, $T''$ is special. This is a contradiction. \hfill $\square$

**Remark 5.8.** In [5], Cummings, Foreman and Magidor showed that $WRP^*([\mu^+]^\omega)$ implies the failure of $ADS_\mu$, where $WRP^*([\mu^+]^\omega)$ is the same $WRP([\mu^+]^\omega)$ (see Definition 6.3) except that in addition we require that the reflected set $W \in [\lambda]^{\omega_1}$ to satisfy that $cf(W) = \omega_1$. It is also known that in general $RC$ does not imply $WRP$ (see [14]).
Remark 5.9. It follows from the work of Foreman and Magidor in \cite{FoMa98} that \( RC \) is compatible with the Approachability Property at \( \mu \) where \( \mu \) is a singular cardinal of countable cofinality.

The last part of this section is dedicated to showing that in general \( RC \) is compatible with \( \square(\lambda, \omega_2) \) for all regular \( \lambda \geq \omega_2 \).

Definition 5.10. Let \( \lambda \) be a regular cardinal and \( \kappa \) be a cardinal. \( \square(\lambda, \kappa) \) asserts the existence of a sequence \( \langle C_\alpha : \alpha \in \text{lim} \lambda \rangle \) such that

- \( C_\alpha \) is a \( \leq \kappa \)-collection of clubs in \( \alpha \) for each \( \alpha \in \text{lim} \lambda \)
- for each \( \alpha < \beta \in \text{lim} \lambda \) and \( C \in C_\beta \), if \( \alpha \in \text{lim} C \), then \( C \cap \alpha \in C_\alpha \)
- there does not exist a thread, namely there is not club \( D \subseteq \lambda \) such that for any \( \alpha \in \text{lim} D \), \( D \cap \alpha \in C_\alpha \).

Fix a Laver indestructible supercompact cardinal \( \kappa \) and a regular cardinal \( \lambda > \kappa^+ \). We define a forcing poset that is \( \kappa \)-directed closed which adds a \( \square(\lambda, \kappa) \)-sequence.

Definition 5.11. Let \( \mathbb{P} = \mathbb{P}_{\square(\lambda, \kappa)} \) be the poset consisting of functions \( t \) where \( t \) is a function of domain \( (\gamma + 1) \times \kappa \) for some \( \gamma = \gamma_1 < \lambda \) such that for all \( \beta \leq \gamma \) and \( i < \kappa \), \( t(\beta, i) \) is a club in \( \beta \) and for any \( \alpha < \beta \in \text{lim} \lambda \cap \gamma + 1 \), \( i < \kappa \) and \( \alpha \in t(\beta, i) \), if \( \alpha \in \text{lim} a \), then \( a \cap \alpha \in \bigcup_{j < \kappa} t(\alpha, j) \). \( t \) extends \( t' \) iff \( t' \) end extends \( t \) and there exists \( \eta < \kappa \) such that for all \( \nu > \eta \), \( t(\gamma_\nu, \nu) = t'(\gamma_\nu, \nu) \cap \gamma_\nu \).

Let us collect some standard facts about this forcing.

Fact 5.12. \hspace{1em} \begin{itemize}
\item \( \mathbb{P} \) is \( \kappa \)-directed closed
\item \( \mathbb{P} \) is \( \lambda \)-strategically closed
\item Forcing with \( \mathbb{P} \) adds a \( \square(\lambda, \kappa) \)-sequence.
\end{itemize}

Proof. The second and third items are standard. To see the poset is \( \kappa \)-directed closed, for any directed collection \( \langle p_\beta \in \mathbb{P} : i < \beta < \kappa \rangle \), since the ordering is by end extension, the collection must be linearly ordered. Let \( p' = \bigcup_{i < \beta} p_i \). If this is a condition, then we are done. Otherwise, \( \text{dom}(p') = \gamma = \sup_{i < \beta} \gamma_i \), and we need to extend \( p' \) to \( p \) whose domain is \( (\gamma + 1) \times \kappa \). For each \( i < \beta \), there exists \( j_i \in \kappa \) such that for all \( k > j_i \), for all \( l < i \), \( p_l(\gamma_i, k) = p_i(\gamma_i, k) \cap \gamma_i \). Let \( \nu = \sup_{i < \beta} j_i \). For each \( \nu' > \nu \), let \( p(\gamma, \nu') = \bigcup_{i < \beta} p_i(\gamma_i, \nu') \). Finally let \( p(\gamma, \mu) = p(\gamma, \nu + 1) \) for all \( \mu \leq \nu \). It is easy to see that \( p \) as defined is a desired lower bound. \( \square \)

Since the forcing is \( \kappa \)-directed closed, it preserves the supercompactness of \( \kappa \). If we follow by the Lévy Collapse \( \text{Coll}(\omega_1, \kappa) \), then we will have \( RC \) along with \( \square(\lambda, \omega_2) \) in the model. One final point is that the \( \square(\lambda, \kappa) \)-sequence remains so after forcing with the Lévy Collapse. This follows from Corollary 2.22 in \cite{Levy81}.

Remark 5.13. As mentioned in the introduction, \( RC \) is known to refute \( \square(\lambda, \omega) \) for regular \( \lambda \geq \omega_2 \). By a theorem of Weiss that \( \square(\lambda, \omega_1) \) refutes the \( (\lambda, \omega_2) \)-strong tree property and of Torres–Wu that \( RC + \neg CH \) implies the \( (\lambda, \omega_2) \)-strong tree property for all \( \lambda \geq \omega_2 \), it follows that \( RC + \neg CH \) implies the failure of \( \neg \square(\lambda, \omega_1) \) for all regular \( \lambda \geq \omega_2 \). \( \square(\omega_2, \omega_1) \) is just a consequence of \( CH \). It is not known for regular \( \lambda > \omega_2 \), whether \( RC + CH \) is compatible with \( \square(\lambda, \omega_1) \).

The following observation imposes some restriction on the model of \( RC + CH + \square(\lambda, \omega_1) \) (if there is one at all).
Observation 5.14. If $\omega_2$ is generically strongly compact via proper forcings, then $\square(\lambda, \omega_1)$ fails for all regular $\lambda > \omega_2$.

Proof. Let $\kappa = \omega_2$. Suppose otherwise for the sake of contradiction. Let $C = \langle C_{\alpha,i} : i < \omega_1, \alpha \in \text{lim } \lambda \rangle$ be the $\square(\lambda, \omega_1)$-sequence in $V$. Let $P$ be some semi-proper forcing such that whenever $G \subset P$ is generic, in $V[G]$ there exists an elementary embedding $j : V \rightarrow M$ such that $\text{crit}(j) = \kappa$ and there exists $Y \in M$ such that $j''\lambda \subset Y$ and $M \models |Y| < j(\kappa)$. Let $\gamma = \sup j''\lambda$. We may assume $Y \subset \gamma$. In $M$, let $C = j(C)(\gamma)$. So $C \subset \gamma$ is a club. Consider $A = j^{-1}((\text{lim } C) \cap Y)$. $A$ is unbounded in $\lambda$ since $(\text{lim } C) \cap Y \supset (\text{lim } C) \cap j''\lambda$ and the latter is a $\omega$-club. For each $\alpha \in A$, we know $j(\alpha) \in (\text{lim } C) \cap Y$, so by coherence there exists $i_\alpha < \omega_1$ such that $C \cap j(\alpha) = j(C_{\alpha,i_\alpha})$. Find $A' \subset A$ unbounded in $\lambda$ and $i < \omega_1$ such that for all $\alpha \in A'$, $i_\alpha = i$. Then $\bigcup_{\alpha \in A'} C_{\alpha,i}$ threads $C$ in $V[G]$, since for any $\alpha < \beta \in A'$, $j(C_{\beta,i}) \cap j(\alpha) = (C \cap j(\beta)) \cap j(\alpha) = C \cap j(\alpha) = j(C_{\alpha,i})$ so by elementarity $C_{\beta,i} \cap \alpha = C_{\alpha,i}$. However, by Corollary 2.22 in [12], no proper forcing can introduce a thread to $C$. This is a contradiction. $\square$

6. Rado’s conjecture and polarized partition relations

Garti, Horowitz and Magidor recently show in [11] that under MM, a certain saturation property of $NS_{\omega_1}$ stronger than the usual saturation is true. There are certain combinatorial consequences, for example, polarized partition relations involving $\omega_1$ and $\omega_2$. It is natural to ask if $RC$ has any influence on those polarized partition relations.

Definition 6.1. Let $I$ be an ideal on $X$. Then $I$ is $(\alpha, \beta, \gamma)$-saturated if for any $\{X_i : i < \alpha\} \subset I^+$, there exists $A \in [\alpha]^\beta$ such that for any $B \in [A]^\gamma$, $\bigcap_{j \in B} X_j \in I^+$. We can define $(\alpha, \beta, < \gamma)$-saturation in a similar way.

Given an ideal $I$ on $X$, we can consider equivalence relation: $A \sim B$ iff $A \Delta B \in I$.

Then $P(X)/I$ is the poset that consists all $\sim$-equivalent classes, such that $p \leq q$ iff there exists $A \in p, B \in q$ such that $A \subseteq I B$.

Definition 6.2. Let $I$ be an ideal on $\omega_1$. Then

- $I$ is precipitous if whenever $U \subset P(\omega_1)/I$ is a generic ultrafilter, then $Ult(V, U)$ is well-founded in $V[U]$.
- $I$ is presaturated if $I$ is precipitous and forcing with $P(\omega_1)/I$ preserves $\omega_{2}^{V}$ as a cardinal.

Definition 6.3 ([8]). The Weak Reflection Principle ($WRP$) refers to the following principle: for any regular $\lambda \geq \omega_2$ and any stationary $S \subset [\lambda]^\omega$, there exists $W \subset \lambda$ such that $|W| = \aleph_1$, $\omega_1 \subset W$ and $S \cap [W]^\omega$ is stationary in $[W]^\omega$. $WRP(\lambda)$ is the localized version regarding stationary subsets of $[\lambda]^\omega$.

Theorem 6.1 ([11]). Under MM, the non-stationary ideal on $\omega_1$ is $(\omega_2, \omega_1, < \omega)$-saturated. In fact, the conclusion follows from $WRP$ and the $(\omega_2, 2, 2)$-saturation of $NS_{\omega_1}$.

Theorem 6.2 ([11]). If there exists a $(\omega_2, \omega_1, < \omega)$-saturated ideal, then $(\omega_2, \omega_1) \rightarrow (\omega, \omega_1)^{1,1}$ and $(\omega_2, \omega_1) \rightarrow (k, \omega_1)^{1,1}$ for any $k \in \omega$. 
It turns out that all we need is a much weaker assumption.

**Lemma 6.4.** If there exists a presaturated ideal I on \( \omega_1 \), then \( \left( \frac{\omega_2}{\omega_1} \right) \rightarrow \left( \omega \right)_{\omega}^{1,1} \).

**Proof.** Let \( f : \omega_2 \times \omega_1 \rightarrow \omega \) be the given coloring. Let \( G \) be the generic ultrafilter for \( P(\omega_1) / I \) over \( V \). Let \( j : V \rightarrow M \) be the generic elementary embedding in \( V[G] \). We know by the assumption that \( M^\omega \cap V[G] \subset M \) and \( j(f(\omega_1^\omega)) > \omega \) (see \[9\] for basic properties of presaturated ideals). In \( V[G] \), \( A = j''\omega_2 \cap \omega_1^\prime \subset M \) is of size \( \omega_2 = (\omega_1)^{V[G]} \). Therefore in \( V[G] \) it is possible to find \( k \in \omega \) and uncountable \( A_0 \subset A \) such that for all \( i \in A_0 \), \( j(f)(i, \omega_1^i) = k \). Pick \( \beta_0 \in \omega_2^i \) such that \( j(\beta_0) = \min A_0 \). For each \( j(\eta) \in A_0 \), in \( M_i \), \( \exists \xi < j(\omega_1^i) \ j(f)(j(\beta_0), \xi) = j(f)(j(\eta), \xi) = k \).

By elementarity, we know in \( V \), \( \exists \xi < \omega_1 \), \( f(\beta_0, \xi) = j(\eta, \xi) = k \). Hence in \( V[G] \) we can map \( j(\xi) \in A_0 \) to \( \alpha_\xi \in \omega_1^i \) satisfying the above. However, since \( \omega_1^i \) is countable in \( V[G] \), by the Pigeon Hole Principle, we can find \( \alpha_0 \in \omega_1^i \) and an uncountable \( A_1 \subset A_0 \) such that for all \( j(\xi) \in A_1 \), \( \alpha_\xi = \alpha_0 \). In general, suppose we have \( \alpha_0 < \cdots < \alpha_n \in \omega_1^i \), \( \beta_0 < \cdots < \beta_n \in \omega_2^i \), uncountable \( A_{n+1} \subset \cdots \subset A_n \). Let \( \beta_{n+1} \) be such that \( j(\beta_{n+1}) = \min A_{n+1} \). For each \( j(\eta) \in A_{n+1} \), in \( M_i \), we have \( \exists \xi < j(\omega_1^i), \xi > \alpha_n \) \& \( j(f)(j(\beta_0), \xi) = k = j(f)(j(\eta), \xi) \) for all \( j \leq n + 1 \). As usual by elementarity, we can define the map \( A_{n+1} \) to \( \omega_1^i \), sending \( j(\eta) \) to \( \alpha_n \). Since \( \omega_1^i \) is countable in \( V[G] \), we can find a popular value above \( \alpha_n \), say \( \alpha_{n+1} \), and shrink \( A_{n+1} \) to some uncountable subset \( A_{n+2} \) such that for all \( j(\xi) \in A_{n+2} \), \( \alpha_\xi = \alpha_{n+1} \).

In this process we ensure that in \( V[G] \), \( j(f) \upharpoonright \{ \beta_n : n \in \omega \} \times \{ \alpha_n : n \in \omega \} \equiv k \). However, \( M^\omega \cap V[G] \subset M \), so the same statement is actually true in \( M \). Thus \( M \models \text{there exist } C \in [j(\omega_1^\omega)]^\omega, D \in [j(\omega_1^\omega)]^\omega \text{ such that } j(f) \upharpoonright C \times D \equiv k \). By elementarity, we know \( V \models C \in [\omega_2^\omega]_\omega, D \in [\omega_1^\omega]_\omega \text{ such that } f \upharpoonright C \times D \equiv k \).

\[ \square \]

**Remark 6.5.** The same proof as in Lemma 6.4 shows that the existence of a presaturated ideal on \( \omega_1 \) implies that \( \left( \frac{\omega_2}{\omega_1} \right) \rightarrow \left( k \right)_{\omega}^{1,1} \) for any \( k \in \omega \).

**Remark 6.6.** The same polarized partition relations follow from Chang’s Conjecture, as shown by Todorcevic in \[19\]. In fact he obtains a characterization of Chang’s Conjecture in terms of the c.c.c.-indestructibility of certain polarized partition relations. The proof presented above is an adaptation of his method.

**Corollary 6.7.** \( RC \) implies \( \left( \frac{\omega_2}{\omega_1} \right) \rightarrow \left( \omega \right)_{\omega}^{1,1} \) and \( \left( \frac{\omega_2}{\omega_1} \right) \rightarrow \left( k \right)_{\omega}^{1,1} \) for any \( k \in \omega \).

**Proof.** There are two ways of seeing this. One on hand, \( RC \) implies Chang’s Conjecture, as shown by Todorcevic in \[20\]. Then apply Remark 6.6. On the other hand, \( RC \) implies the non-stationary ideal on \( \omega_1 \) is presaturated, as shown by Feng in \[17\]. Then apply Lemma 6.4 and Remark 6.5.

\[ \square \]

It is now natural to ask if we can prove anything stronger, in particular, the next natural partition relation to consider is \( \left( \frac{\omega_2}{\omega_1} \right) \rightarrow \left( \omega \right)_{\omega}^{1,1} \). It turns out that \( RC \) does not decide the truth of this statement.

We use and modify a little the idea of Prikry (\[13\]) to add an \( \omega_2 \)-sequence of \( \omega_1 \)-partitions of \( \omega_1 \): \( \langle A_{\alpha, \beta} \subset \omega_1 : \alpha < \omega_2, \beta < \omega_1 \rangle \) such that
• for each $\alpha < \omega_2$, $\{A_{\alpha, \beta} : \beta \in \omega_1\}$ is a partition of $\omega_1$
• for any distinct $\langle \alpha_n \in \omega_2 : n \in \omega \rangle$, not necessarily distinct $\langle \xi_n \in \omega_1 : n \in \omega \rangle$,
  $|\omega_1 - \bigcup_{n \in \omega} A_{\alpha_n, \xi_n}| \leq \aleph_0$.

Notation 6.8. Given a countable function $S$ with its domain a subset of $\omega_2 \times \omega_1$, let $S_0 \in [\omega_2]^{\leq \omega}$ denote the projection of $\text{dom}(S)$ to its first coordinate and $S_1 \in [\omega_1]^{\leq \omega}$ denote the projection of $\text{dom}(S)$ to its second coordinate.

$\mathbb{P}_{\omega_2, \omega_1}$ consists of pairs $(S, \mathcal{A})$ where $S : \omega_2 \times \omega_1 \rightarrow \omega_1$ is a countable partial function such that $S_1 \in \omega_1$ and $\mathcal{A}$ is a countable collection of countably infinite partial functions from $\omega_2$ to $\omega_1$ closed under co-finite restrictions, namely for each $f \in \mathcal{A}$, if $A =^{CH} \text{dom}(f)$, $A \subset \text{dom}(f)$, then $f \upharpoonright A \in \mathcal{A}$.

We say $(S', \mathcal{A}') \leq (S, \mathcal{A})$ iff $S' \supseteq S$ and $\mathcal{A}' \supseteq \mathcal{A}$ and for all $\beta \in S_1' - S_1$ and for all $f \in \mathcal{A}$, there exists $\alpha \in \text{dom}(f)$ such that $(\alpha, \beta) \in \text{dom}(S')$ and $S'(\alpha, \beta) = f(\alpha)$.

Remark 6.9. In general, for a regular cardinal $\kappa$, we can define $\mathbb{P}_{\kappa, \omega_1}$ analogously by simply replacing $\omega_2$ with $\kappa$.

It is easy to see that the ordering is transitive.

Claim 6.10. $\mathbb{P}_{\omega_2, \omega_1}$ is $\aleph_2$-c.c. and countably closed assuming CH.

Proof. Countable closure is immediate. To see $\aleph_2$-c.c., given a collection of conditions $p_i = (S_{i1}, \mathcal{A}_i)$ for $i < \omega_2$, we apply $\Delta$-system lemma to get $A \in [\omega_2]^{\leq \omega}$ such that

- $(S_{i1})_1$ is the same for all $i \in A$.  
- $\text{dom}(S_i)$ forms a $\Delta$-system with root $r$ and $S_i \upharpoonright r$ are the same for all $i \in A$.

Now any two conditions with indices in the set are easily seen to be compatible.  

Claim 6.11. For any $\alpha \in \omega_2$ and any $\beta \in \omega_1$ and any $p = (S_p, \mathcal{A}_p) \in \mathbb{P}$, there exists $p' \leq p$ such that $(\alpha, \beta) \in \text{dom}(S_{p'})$.

Proof. If $(\alpha, \beta) \in \text{dom}(S_p)$, then we take $p'$ to be $p$. Otherwise, if $\beta \in (S_p)_1$, then just add $(\alpha, \beta, 0)$ to $S_p$. If $\beta \notin S_p$, then for each $f_n \in \mathcal{A}$, find distinct $\alpha_n \in \text{dom}(f_n)$ and different from $\alpha$, then add $(\alpha, \beta', 0)$ and $(\alpha_n, \beta', f(\alpha_n))$ for $n \in \omega$ to $S_p$ for each $\beta' \leq \beta$ and $\beta' \notin S_p$.

For the following claim, simply prove by adding a new function and close the collection under co-finite restrictions.

Claim 6.12. For any $f : \omega_2 \rightarrow \omega_1$ with countable support, and $p \in \mathbb{P}$, there exists $p' \leq p$ such that $f \in \mathcal{A}_{p'}$.

Lemma 6.13 (Prikry [13]). Assume $V \models \text{CH}$. Then in $V^{\mathbb{P}_{\omega_2, \omega_1}}$, $(\omega_2) \not\in (\omega_1)^{\omega_1}$.  
Namely, there exists $f : \omega_2 \times \omega_1 \rightarrow \omega_1$ such that for any $A \in [\omega_2]^{\omega}$ and $B \in [\omega_1]^{\omega_1}$, $f'' A \times B = \omega_1$. This clearly implies $(\omega_2) \not\in (\omega_1)^{\omega_1}$.

Lemma 6.14. Let $\kappa$ be a supercompact cardinal. In $V[G][H]$ where $G$ is generic for $\text{Coll}(\omega_1, < \kappa)$ and $H$ is generic for $\langle \mathbb{P}_{\omega_2, \omega_1} \rangle^{V[G]}$ over $V[G]$, WRP and RC both hold.
Proof: We only show $RC$ holds in the model. The proof for $WRP$ is almost the same.

First notice that $\mathbb{P}_{\omega_2, \omega_1}$ defined in $V[G]$ is the same as $\mathbb{P}_{\kappa, \omega_1}$ defined in $V$ since $Coll(\omega_1, < \kappa)$ is countably closed.

Fix $T \in V[G][H]$ of size $\theta$ that is nonspecial. Let $\lambda > \theta$ be some sufficiently large regular cardinal, specifically larger than the cardinality of any nice $Coll(\omega_1, < \kappa) \times \mathbb{P}_{\kappa, \omega_1}$-name of a subset of $\theta$.

Fix $j : V \rightarrow M$, an embedding witnessing $\kappa$ is $\lambda$-supercompact, namely $j(\kappa) > \lambda$ and $M^\lambda \subset M$. $j(Coll(\omega_1, < \kappa) \times \mathbb{P}_{\kappa, \omega_1}) = Coll(\omega_1, < j(\kappa)) \times \mathbb{P}_{j(\kappa), \omega_1}$. Let $R \subset Coll(\omega_1, [\kappa, j(\kappa)])$ be generic over $V[G][H]$.

Then we can lift $j$ to $j : V[G] \rightarrow M[G][R]$ (we slightly abuse the notation by using the same $j$). Notice $j \upharpoonright \mathbb{P}_{\omega_2, \omega_1}$ in $V[G]$ is a complete embedding since $V[G] \models CH$ so by Lemma 6.11, $\mathbb{P}_{\omega_2, \omega_1}$ is $\aleph_2$-c.c. in $V[G]$. So each maximal antichain of $\mathbb{P}_{\omega_2, \omega_1}$ in $V[G]$ is mapped to its pointwise image since $crit(j) = \kappa = (\omega_2)^{V[G]}$. Notice also that the complete embedding is just the identity.

Claim 6.15. $j(\mathbb{P}_{\omega_2, \omega_1})/H$ is countably closed in $V[G][R][H]$.

Proof of the Claim. Let $\mathbb{P} = (\mathbb{P}_{\kappa, \omega_1})^V = (\mathbb{P}_{\omega_2, \omega_1})^{V[G]}$. Suppose $p_0 \geq p_1 \geq \ldots$, then we claim that $q = \bigcup_n p_n$ is $\mathbb{P}$-generic over $H$, i.e. we show for all $h \in H$, $q$ is compatible with $h$. Note that for each $n \in \omega$, $S_n$ must agree with $S_H = \bigcup \{S_t : t \in H\}$ which is a total function on $\kappa \times \omega_1 \rightarrow \omega_1$. Fix $h \in H$. By the observation above, we know $S_h$ is consistent with $S_q$. By extending $h$ and by Lemma 6.11 we might assume $(S_h)_1 > (S_q)_1$.

Fix $\beta \in (S_h)_1 - (S_q)_1$. For each $f \in A_q$, there exists $n \in \omega$ such that $f \in A_{p_n}$. By the fact that $h$ and $p_n$ are compatible, there exists $\alpha \in dom(f)$ such that $(\alpha, \beta, f(\alpha))$ can be added to $S_h$. By the closure under co-finite restrictions of $A_{p_n}$, we know in fact there are infinitely many such candidates. What is left to do is to recursively build a condition $h'$ extending both $h$ and $q$ such that $(S_{h'})_1 = (S_h)_1$ and $A_{h'} = A_h \cup A_q$.

Enumerate $(S_h)_1 - (S_q)_1$ as $\{\beta_i : i \in \omega\}$. Define $\langle l_i : i \in \omega\rangle$ where each $l_i$ is a countable function from $j(\kappa) \times \omega_1 \rightarrow \omega_1$. For each $i \in \omega$, for each $f \in A_q$, find $\alpha = \alpha_f \in dom(f)$ such that $\alpha_f \neq \alpha_0$ for any $f \neq g \in A_q$ and $l_i = \{ (\alpha_f, \beta_i, f(\alpha_f)) : f \in A_q \}$ is compatible with $S_q$. Also $l_i$ is necessarily compatible with $S_h$ since $\beta_i \notin (S_q)_1$. Notice for $i \neq j \in \omega$, $l_i$ is compatible with $l_j$. Let $l_\omega = \bigcup_{i \in \omega} l_i \cup S_h \cup S_q$. It is easy to verify that $h' = (l_\omega, A_q \cup A_h)$ works.

To see the claim implies the conclusion, let $L$ be generic for $j(\mathbb{P}_{\omega_2, \omega_1})/H$ over $V[G][R][H]$, then we can further lift $j$ to $j : V[G][H] \rightarrow M[G][R][H][L]$. By the choice of $\lambda$, $T \in M[G][H]$. By Claim 2.6 $T$ remains nonspecial in $M[G][H][R][L]$. Since $M[G][H][R][L] \models [T] = \theta < j(\kappa) = \omega_2$, by elementarity, $V[G][H] \models \exists T$ of size $\leq \aleph_1$.

Combining Lemma 6.13 and Lemma 6.14 we have the following.

Theorem 6.3. Let $\kappa$ be a supercompact cardinal. Then there exists a forcing extension in which $\kappa = \omega_2$, RC and WRP both hold, and $\left(\omega_2 \mathcal{W}_{\omega_1} \mathcal{W} \right) \nRightarrow \left(\omega_1 \mathcal{W}_{\omega} \mathcal{W} \right)^{1,1}_{\omega}$. 

□
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