Quantum scissors device of Pegg et al (1998 Phys. Rev. Lett. 81 1604) enables truncation of the Fock-state expansion of an input optical field to qubit and qudit (three-dimensional) states only. Here, a generalized scissors device is proposed using an eight-port optical interferometer. Upon post-selection based on photon counting results, the interferometer implements generation and teleportation of qudit (d-dimensional) states by truncation of an input field at the (d−1)th term of its Fock-state expansion up to d = 6. Examples of selective truncations, which can be interpreted as a Fock-state filtering and hole burning in the Fock space of an input optical field, are discussed. Deterioration of the truncation due to imperfect photon counting is discussed including inefficiency, dark counts and realistic photon-number resolutions of photodetectors.

Keywords: linear optics, quantum state engineering, quantum teleportation, projection synthesis, multiport interferometer, quantum scissors, positive operator valued measure
limited to the discussed truncation of the number-state expansion of a given state. For instance, by considering truncation of a coherent state, defined by the action of the displacement operator on the vacuum state, one can truncate the displacement operator and then apply it to the vacuum state. Such a truncated state is essentially different (for \( d > 2 \)) from that given by \[25\] \[26\] \[27\]. Nevertheless, it is physically realizable, e.g., in a pumped ring cavity with a Kerr nonlinear medium as was demonstrated by Leoński et al \[28\] \[29\] \[30\] \[31\]. The QSD schemes can also be generalized for the truncation of two-mode \[32\] or multi-mode fields.

The original Pegg-Phillips-Barnett QSD enables the truncation of an input state only to qubit and qutrit (3-dimensional qudit) states as was shown by Koniorczyk et al \[8\]. Here, we discuss a generalization of the QSD for qudits and suggest a way to extend the scheme for an arbitrary \( d \). Our approach is essentially different from the other qudit truncation schemes \[11\] \[29\] \[30\] \[33\] \[34\] and we believe that it is easier to be experimentally realized. The proposed scheme is based on an eight-port optical interferometer shown in figure 2. The setup resembles a well-known multiport interferometer of Zeilinger et al \[35\] \[36\] \[37\] \[38\] \[39\], which has been theoretically analyzed \[33\] \[34\] \[35\] \[36\] \[37\] \[38\] \[39\] \[40\] \[41\] \[42\] \[43\] \[46\] \[47\] and experimentally applied \[41\] \[41\] \[42\] for various purposes but, to our knowledge, has not yet been used for optical-state truncation. An important difference between the standard multiport and that proposed here lies in the elimination of the apex BS of the triangle. This elimination is important for the processes of truncation and teleportation.

The paper is organized as follows. In section 2, the generalized quantum scissors device is proposed including a description of the setup (2.1), a short review of multiport unitary transformations (2.2), and explanation of the projection synthesis (2.3), which enables the qudit state truncation. Reductions of the generalized QSD to the Pegg-Phillips-Barnett QSD are demonstrated in section 3. Detailed analyses of the generalized QSD for the truncation up to three, four and five photon-number states are given in sections 4, 6, and 7, respectively. Selective truncations, which can be interpreted as a Fock-state filtering \[13\] or a hole burning in Fock space \[44\] \[47\], are discussed in section 5. How imperfect photon counting deteriorates the truncation processes is discussed in section 8 by including realistic photon-number resolutions, inefficiency, and dark counts of photodetectors. Final conclusions with a discussion of open problems including a generalization of the scheme for an arbitrary qudit state truncations are presented section 9.

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2. GENERALIZED QSD

2.1 The setup

We analyze a generalized quantum scissors device (GQSD) based on an eight-port optical interferometer, also referred to as a multiport mixer or multiport beam splitter, which is assembled in a pyramid-like configuration of ordinary beam splitters (BSs) and phase shifters (PSs) as shown in figure 2. The most general four-port beam-splitter scattering matrix reads as (see, e.g., \[46\] \[47\])

\[
B' = e^{i\theta_0} \begin{bmatrix}
t \exp(i\theta_t) & r \exp(i\theta_r) \\
-r \exp(-i\theta_r) & t \exp(-i\theta_t)
\end{bmatrix}
\]  

where \( T = t^2 \) describes the transmittance, and \( R = r^2 = 1 - T \) is the reflectance of the BS. The associated phase factors \( \theta_t \) and \( \theta_r \) can be realized by the external phase shifters, described by \( P_{\pm} = \text{diag}[\exp(i\theta_t \pm i\theta_r), 1] \), which are placed in front of and behind the beam splitter described by real scattering matrix

\[
B = \begin{bmatrix}
t & r \\
-r & t
\end{bmatrix}
\]

as comes from the decomposition

\[
B' = \exp(i\theta'_0)P_{-}BP_{-}
\]  

where \( \theta'_0 = \theta_0 - \theta_t \). Without loss of generality the global phase factors \( \exp(i\theta'_0) \) and \( \exp(i\theta_0) \) can be omitted. Note that \[8\] describes an asymmetric BS, as marked in all figures by bars with distinct surfaces. We use the same convention as in \[48\] \[49\] that beams reflected from the white surface are \( \pi \) phase shifted so, according to \[8\], the reflection from the black surface and transmissions from any side are without phase shift.
The total scattering matrix \( S \) of the GQSD, shown in figure 2, can be given by

\[
S = P_6 B_5 P_5 B_4 P_4 B_3 P_3 B_2 P_2 B_1 P_1
\]

which is the sequence of two-mode ‘real’ beam splitters described explicitly by

\[
B_1 = \begin{bmatrix} t_1 & r_1 & 0 & 0 \\ -r_1 & t_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} t_2 & 0 & r_2 & 0 \\ 0 & 1 & 0 & 0 \\ -r_2 & 0 & t_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
\]

\[
B_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & t_3 & r_3 & 0 \\ 0 & -r_3 & t_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & t_4 & 0 & r_4 \\ 0 & 0 & 1 & 0 \\ 0 & -r_4 & 0 & t_4 \end{bmatrix},
\]

\[
B_5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & t_5 & r_5 \\ 0 & 0 & -r_5 & t_5 \end{bmatrix}
\]

and phase shifters represented by

\[
P_k = \text{diag} \{ \exp(\imath \xi_k), 1, 1, 1 \} \quad \text{for} \quad k = 1, 2, 6,
\]

\[
P_k = \text{diag} \{ 1, \exp(\imath \xi_k), 1, 1 \} \quad \text{for} \quad k = 3, 4,
\]

\[
P_5 = \text{diag} \{ 1, 1, \exp(\imath \xi_5), 1 \}. \tag{9}
\]

Similarly, the diagonal matrix

\[
M_k = \text{diag} \{ \exp(\imath \delta_{1k}), \exp(\imath \delta_{2k}), \exp(\imath \delta_{3k}), \exp(\imath \delta_{4k}) \}
\]

\tag{10}

describes the \( k \)th mode reflection phase shift caused by the mirror (see figure 2), where \( \delta_{ik} \) is Kronecker delta. We have not written explicitly \( M_k \) in the sequence \( B \), since the reflection phase shifts \( \zeta \) for modes 1, 2 and 3 can be incorporated in \( \xi_1, \xi_2 \) and \( \xi_3 \), respectively. Besides the reflection phase shift in mode 4 does not affect photodetection in D4, and thus can be neglected.

The described setup resembles a well-known multiport interferometer in triangle configuration of the beam splitters as studied in various contexts, since its theoretical proposal and experimental realizations by Zeilinger et al [33, 34, 35, 40]. However, an important difference between the standard multiport interferometer and that analyzed here is the elimination of the apex BS of the triangle, which is usually placed at the crossing of beams 1 and 4 in figure 2. In particular, the setup for the Pegg-Phillips-Barnett QSD resembles the Zeilinger six-port interferometer with one BS removed. This elimination is essential for the processes of truncation and teleportation of qudits as will be shown in the following.

2.2 Multiport unitary transformation

The annihilation operators \( \hat{a}_i \) at the \( N \) inputs to multiport linear interferometer are related to the annihilation operators \( \hat{b}_i \) at the \( N \) outputs as follows (see, e.g., [47, 48]):

\[
\hat{b}_i = \hat{U}^\dagger \hat{a}_i \hat{U} = \sum_{j=1}^{N} S_{ij} \hat{a}_j \tag{11}
\]

where \( S_{ij} \) are the elements of the unitary scattering matrix \( S \), and \( \hat{U} \) is the unitary operator describing the evolution of the \( N \)-mode input state, say \( |\Psi\rangle \), into the \( N \)-mode output state, say \( |\Phi\rangle \):

\[
|\Phi\rangle = \hat{U} |\Psi\rangle. \tag{12}
\]

By introducing the column vectors \( \hat{a} \equiv [\hat{a}_1; \hat{a}_2; \ldots; \hat{a}_N] \) and \( \hat{b} \equiv [\hat{b}_1; \hat{b}_2; \ldots; \hat{b}_N] \), the set of equations (11) can compactly be rewritten as

\[
\hat{b} = \hat{U}^\dagger \hat{a} \hat{U} = S \hat{a} \tag{13}
\]

from which follow the inverse relations for the creation operators

\[
\hat{a}^\dagger = \hat{U} \hat{b}^\dagger \hat{U}^\dagger = S^T \hat{b}^\dagger. \tag{14}
\]

Then, one can observe that

\[
\hat{U} \hat{a}_i^\dagger \hat{U}^\dagger = \hat{U} \left( \sum_j S_{ji} \hat{b}_j^\dagger \right) \hat{U}^\dagger = \sum_j S_{ji} \hat{U} \hat{b}_j^\dagger \hat{U}^\dagger = \sum_j S_{ji} \hat{a}_j^\dagger. \tag{15}
\]

By applying (15) and noting that neither BSs nor PSs change the vacuum state, \( \hat{U} |\Phi\rangle = |\Phi\rangle \), one can calculate the output state \( |\Phi\rangle \) of the multiport interferometer described by the scattering matrix \( S \) for the input Fock states \( |\Psi\rangle = |n_1, \ldots, n_N\rangle \equiv |\mathbf{n}\rangle \) as follows (for detailed
which is transformed into the output state \( \langle 12 \rangle \). Now, photon-counting of the output modes \( \hat{a}_2 \) stands for multiple sum over \( n \). In the following, we present so-
\[ \Delta(T, \xi) \equiv \sum_{n=1}^{d-1} |c_n^{(d)}(T, \xi) - c_n^{(d)}(T, \xi)| = 0 \] 
for some properly chosen values of the BS transmittances \( T \) and phase shifts \( \xi \). So, the problem is to find such \( T \) and \( \xi \), for which the amplitudes \( c_n^{(d)} \) satisfy condition (20). It is worth noting that, for a possible experimental realization of the scheme, it is essential to have BSs with variable transmittance. A possible solution is to replace each of the BSs by a Mach-Zehnder interferometer composed of two symmetric 50:50 BSs, two mirrors and two PSSs [43, 44, 45].

Our numerical minimization of \( \Delta \) reveals that perfect truncation using the generalized QSD can be realized for various values of the BS transmittances and phase shifts, which however, correspond to different probabilities of successful truncation. Here, we focus on very simple analytical solutions rather than numerically optimized approximate ones. In the following, we will analyze in detail truncations up to six-dimensional qudits.

The described truncation process via the GQSD can be considered a kind of the form-limited quantum teleportation of the first \( d \) terms of the Fock-state expansion of the incident light in analogy to the qutrit-limited teleportation via the Pegg-Phillips-Barnett QSD discussed in [3, 4, 5, 6, 11, 12]. Even a brief analysis of figure 2, shows that no light from the input port 4 can reach the output port 1. In fact, this transformation is based on the same principles of the quantum entanglement and the Bell-state measurement (the projection postulate) as the original Bennett et al teleportation scheme [22]. The multi-

 photon-entangled state is created by the beam splitters and, as required, the original state \( |\psi \rangle \) is destroyed by the Bell-state measurement implemented by the BSs 3-5 and detectors 2-4. Thus, by assuming that an incident light is already prepared in a \( d \)-dimensional (up to \( d=6 \)) qudit state \( |\psi^{(d)} \rangle \) and the conditional measurement is successfully performed, then the state is teleported from mode \( \hat{a}_4 \) to \( |\phi^{(d)} \rangle = |\psi^{(d)} \rangle \) in mode \( b_1 \).

3. REDUCTIONS TO THE PEGG-PHILLIPS-BARNETT QSD

First, we show how the eight-port QSD can truncate the input state \( |\psi \rangle \), given by (11), to qubit state, given by (9), as expected by the original Pegg-Phillips-Barnett scissors device [3, 4]. The system shown in figure 1 is a special case of that shown in figure 2 by assuming, e.g., that BSs 1, 3 and 5 are perfectly reflecting, thus a complete set of transmittances is \( T = [0, t_2^0, 0, t_4^0, 0] \).
and we can also set that the only nonzero phase shift is \( \xi_4 \).

Note that, in this configuration, the input port 1 and the output port 4 are unimportant, so can be neglected. Another way to generate the truncated state \( |3\rangle \) is to remove BSs 2, 3 and 5 as shown in figure 3(a). Thus, the simplified version of the GQSD is described by the transmittances

\[
T = [t_1^2, 1, 1, t_4^2, 1],
\]

which can be obtained in our setup by assuming the single-photon Fock states in the input modes \( \hat{a}_1 \) and \( \hat{a}_2 \), together with the single-photon counts in detectors D2 and D4. Under these assumptions and by denoting \( f'_i \equiv r_i^2 - t_i^2 \), the output state in mode \( \hat{b}_1 \) becomes

\[
|\phi_{11}^{\text{trun}}\rangle \sim 2r_1t_1t_4^4(e^{2\xi_4}r_1^2|0\rangle + \gamma_2|2\rangle) + e^{i\xi_1}f'_1f'_4f_1^2|1\rangle
\]

and the phase shift \( \xi_4 \) is equal to \( \pi \) or zero, respectively. By inspection of (24) one readily finds that the optimum solutions occur for \( t_4^2 \) equal either to \((3 - \sqrt{3})/6 \approx 0.21\) or to \((3 + \sqrt{3})/6 \approx 0.79\) and \( t_4^2 = t_1^2 \) if \( \xi_4 = 0 \), in agreement with the results of (24), but also for \( t_4^2 = 1 - t_1^2 \) if \( \xi_4 = \pi \).

\section{4. Truncation to Quartit States}

In this section, we demonstrate how to realize truncation of an input state \( |\psi\rangle \), given by (4), to the four-dimensional qudit

\[
|\phi_{\text{trun}}^{(4)}\rangle \sim |\gamma_00\rangle + |\gamma_11\rangle + |\gamma_22\rangle + |\gamma_33\rangle
\]

referred to as the quartit. We apply the GQSD shown in figure 2, assuming that modes \( \hat{a}_1, \hat{a}_2, \) and \( \hat{a}_3 \) are the in single-photon Fock states, and single photons have been measured in all detectors, \( N_2 = N_3 = N_4 = 1 \). If the light to be truncated enters the interferometer in mode \( \hat{a}_4 \), then the output state \( |\phi\rangle = |\phi_{1111}\rangle \) obtained via the projection synthesis is given by (17) for \( d = 4 \) and the amplitudes

\[
c_n^{(4)} \equiv \langle n111|\hat{U}|111n\rangle
\]

dependent on the BS and PS parameters. By applying the procedure described in section 2, we find that the simplest solution is for \( n = d - 1 \) and reads as

\[
c_3^{(4)} = 6e^{2i\xi_2}r_1t_1r_2t_2^2r_4 t_3^2 r_5 t_5.
\]

It is seen that \( c_3^{(4)} \) is independent of the BS3 parameters, so for simplicity let us assume that BS3 is removed (\( t_3 = 1, \xi_3 = 0 \)). Thus, instead of a general setup, shown in figure 2, we first analyze its simplified version shown in figure 3(b). Under the assumption, the solutions for the other amplitudes are found to be

\[
c_0^{(4)} = -2e^{i(\xi_1 + \xi_3)}t_3 t_4(e^{i(\xi_2 + \xi_3)}f_1^2 r_2 r_3 t_5 + e^{i\xi_1}f'_1 r_1 t_4 r_4),
\]

\[
c_1^{(4)} = -2r_1t_1r_2 r_3 t_5(e^{i(\xi_2 + \xi_3)}f'_2 + e^{i\xi_1}f'_1) + e^{i(\xi_2 + \xi_3)}f_1^2 f_4 f'_5,
\]

\[
c_2^{(4)} = 2e^{i\xi_2}t_2 t_4(e^{i(\xi_2 + \xi_3)}r_1 t_1 g_2 r_4 f'_5 + e^{i\xi_1}f'_1 r_2 g_4 r_5 t_5)
\]

with the results of (24), but also for \( t_4^2 = 1 - t_1^2 \) if \( \xi_4 = \pi \).
where, for brevity, the $n$-primed $f$ denotes $r_k^2 - n_l^2_x$, $g_k = 2r_k^2 - t_k^2$, and the global phase factor $\exp(i \xi_1)$, the same for all $c^{(4)}_n$, was omitted as it cancels out during the renormalization with $N$. Our multiport interferometer can act as a good quantum scissors device if there exist parameters $T$ and $\xi$ such that condition (29) is satisfied. As explained in subsection 2.3, we focus on the simplicity of the solutions, although we realize that they are not optimal as implied by the results of our numerical experiments. For example, a simple solution is found for the transmittances equal to

$$
T = \left[ \frac{1}{13} \frac{1}{3} \frac{1}{4}, \frac{1}{13} \frac{1}{3} \frac{1}{4} \right]
$$

(30)

and zero phase shifts $\xi$ except $\xi_5 = \pi/2$. By applying these values to (29), one readily finds that $c^{(4)}_n = 1/12$ for $n = 0, 1, 2, 3$. Another solution of (29) is found for the transmittances

$$
T = \left[ \frac{1}{26} \frac{1}{3} \frac{1}{4}, \frac{1}{26} \frac{1}{3} \frac{1}{4} \right]
$$

(31)

and $\xi = 0$, which results in a constant amplitude $c^{(4)}_n = 1/(4 \sqrt{39})$, although much lower than that found for the first solution. Numerically it is easy to find solutions, which correspond to higher $c^{(4)}_n$ than those given by our analytical solutions. E.g., by choosing $T = [0.78494, 0.69001, 1.087451, 0.70185]$ and zero phase shifts $\xi$ except $\xi_5 = \pi$, then the amplitudes $c^{(4)}_n$ for the output state $|\psi\rangle = |\phi_1^{111}\rangle$ are constant at the value of 0.134.

A closer look at the amplitudes (29) reveals that they remain unchanged by some permutations of transmittances accompanied by a proper change in the phase shifts, or by replacement of $t_k^2$ by $1 - t_k^2$. In particular, we find the following relations:

$$
c^{(4)}_n(T, \xi) = c^{(4)}_n(t_k^2, t_k^2, 1, 1, 1, 1),
$$

(32)

if $\xi_1 = 0, \xi_4 = \xi_2 + \xi_5$, and

$$
c^{(4)}_n(T, \xi) = c^{(4)}_n(1 - t_k^2, t_k^2, 1, 1, 1, 1),
$$

(33)

if $\xi = \xi' = \xi''$ except for $\xi_2 = \xi_2 + \pi$ or, equivalently, $\xi'_1 = \xi_1 + \pi$, and $\xi''_1 = \xi_5 + \pi$. Thus, by transforming solutions (30) and (31) according to (32) and (33) one can find new solutions.

5. SELECTIVE TRUNCATIONS

Here, we focus on generalized truncations of the input state $|\psi\rangle$ into a finite superposition of the form $|\phi^{(d)}_{\text{run}}\rangle$ albeit with some states (say $|k_1\rangle$, $|k_2\rangle$, ...$)$ removed, i.e.,

$$
|\psi\rangle \rightarrow |\phi^{(d)}_{\text{holes \; k_1, k_2, \ldots}}\rangle = N \sum_{n=0}^{d-1} \sum_{n \neq k_1, k_2, \ldots} \gamma_n |n\rangle
$$

(34)

corresponding to the case when the amplitudes $c_{n}^{(d)}$ vanish for $n = k_1, k_2, \ldots$ and are constant but nonzero for the other $n$. This kind of quantum state engineering can be interpreted as the truncation with hole burning. In general, the hole burning in the Fock space of a given state of light, according to Baseia et al. (see [14] and references therein) and Gerry and Beunoussa [15], means selective removal of one or more specific Fock states from the field. Although, originally, the hole burning was applied to infinite-dimensional states, given by (11), this concept can also be used in the case of finite-dimensional states $|\phi^{(d)}_{\text{run}}\rangle$. Alternatively, following the interpretation of D’Ariano et al [13], one can refer to the above quantum state engineering, especially when the number of holes exceeds $(d - 1)/2$, as a kind of Fock-state filtering, which enables selection of some Fock states (say $|j_1\rangle$, $|j_2\rangle$, ...) from a given input state $|\psi\rangle$, i.e.,

$$
|\psi\rangle \rightarrow |\phi^{(d)}_{\text{filter \; j_1, j_2, \ldots}}\rangle = N (\gamma_1 |j_1\rangle + \gamma_2 |j_2\rangle + \cdots ).
$$

(35)

Here, we show how the GQSD can be used for the Fock-state filtering and the hole burning in the case of $d = 4$. As the first example, we analyze the truncation to $|\phi^{(4)}_{\text{run}}\rangle$ with the two-photon Fock state removed which results in

$$
|\phi^{(4)}_{\text{hole \; 2}}\rangle \sim \gamma_0 |0\rangle + \gamma_1 |1\rangle + \gamma_3 |3\rangle.
$$

(36)

We find that state $|\phi^{(4)}_{111}\rangle$, with amplitudes given by (29), is reduced to (30) for various transmittances $T$ and phase shifts $\xi$, including the following:

$$
T = \left[ \frac{1}{14} \frac{1}{3} \frac{1}{2}, \frac{1}{14} \frac{1}{3} \frac{1}{2} \right]
$$

(37)

and $\xi = 0$. Similarly, the other truncated states with a single hole:

$$
|\phi^{(4)}_{\text{hole \; 0}}\rangle \sim \gamma_1 |1\rangle + \gamma_2 |2\rangle + \gamma_3 |3\rangle,
$$

(38a)

$$
|\phi^{(4)}_{\text{hole \; 1}}\rangle \sim \gamma_0 |0\rangle + \gamma_2 |2\rangle + \gamma_3 |3\rangle,
$$

(38b)

can be generated by the GQSD from $|\phi^{(4)}_{111}\rangle$ if, e.g., $\xi = 0$ and the transmittances are as follows:

$$
T = \left[ \frac{1}{2} \frac{1}{2} \frac{1}{2}, \frac{1}{2} \frac{1}{2} \frac{1}{2} \right]
$$

(39a)

respectively. One can check that superpositions of any two Fock states $|k\rangle$ and $|l\rangle$ for $k, l = 0, \ldots, 3$, i.e.,

$$
|\phi^{(4)}_{\text{filter \; k}, \text{filter \; l}}\rangle \sim \gamma_k |k\rangle + \gamma_l |l\rangle,
$$

(40)

can be obtained as special cases of $|\phi^{(4)}_{111}\rangle$, e.g., for $\xi = 0$ and the transmittances given by

$$
|\phi^{(4)}_{\text{filter \; 02}}\rangle : T = \left[ \frac{1}{2} \frac{1}{2} \frac{1}{2}, \frac{1}{2} \frac{1}{2} \frac{1}{2} \right]
$$

(41a)

$$
|\phi^{(4)}_{\text{filter \; 03}}\rangle : T = \left[ \frac{1}{2} \frac{1}{2} \frac{1}{2}, \frac{1}{2} \frac{1}{2} \frac{1}{2} \right]
$$

(41b)

$$
|\phi^{(4)}_{\text{filter \; 13}}\rangle : T = \left[ \frac{1}{2} \frac{1}{2} \frac{1}{2}, \frac{1}{2} \frac{1}{2} \frac{1}{2} \right]
$$

(41c)

$$
|\phi^{(4)}_{\text{filter \; 23}}\rangle : T = \left[ \frac{1}{2} \frac{1}{2} \frac{1}{2}, \frac{1}{2} \frac{1}{2} \frac{1}{2} \right]
$$

(41d)
Note that our exemplary state $|\phi_{\text{filter02}}\rangle$, given in (11), can be realized in the Pegg-Phillips-Barnett scheme, since the chosen transmittances are a special case of (21). All the above examples were found for $t_3 = 1$. But we have not found solutions $|\phi_{\text{filter12}}\rangle$ by assuming $t_3 = 1$, together with the single-photon states in the input modes $a_1, a_2, a_3$ and the single-photon counts in detectors D2, D3, and D4. While keeping the latter two requirements, one can change only the transmittance of BS3. Then, we find a solution

$$
|\phi_{\text{filter12}}\rangle : \quad T = \left[\frac{1}{2} + \frac{1}{\sqrt{5}} \frac{8}{9} \frac{1}{2} \frac{1}{3} + \frac{1}{\sqrt{5}} \frac{1}{3} \right]
$$

and $\xi = 0$. On the other hand, the state $|\phi_{\text{filter12}}\rangle$ can be realized in the QSD with $t_3 = 1$, e.g., as a special case of the output state $|\phi_{110}\rangle$ for

$$
|\phi_{\text{filter12}}\rangle : \quad T = \left[\frac{1}{10} (5 - \sqrt{15}) \frac{2}{3} \frac{1}{3} \frac{1}{2} \frac{1}{2} \right]
$$

and $\xi = 0$. The scheme enables also a synthesis of Fock states via teleportation. From $|\phi_{111}\rangle$, one can synthesize the two and three photon Fock states for the following transmittances:

$$
|2\rangle : \quad T = \left[1, \frac{1}{2}, \frac{1}{3}, \frac{1}{2}, 1, 1\right]
$$

$$
|3\rangle : \quad T = \left[1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{2}, 1\right]
$$

and, e.g., $\xi = 0$ except $\xi_5 = \pi/2$ in the latter case. Actually, with the choice (12a), the state $|2\rangle$ is generated for arbitrary phase shifts. The two-photon Fock state cannot be obtained from $|\phi_{111}\rangle$ assuming $t_3 = 1$, which can be shown analytically. However, the state $|2\rangle$ can easily be obtained even for $t_3 = 1$ but from other states, e.g. $|\phi_{110}\rangle$. It is worth noting that by applying the transformations given by (22) and (23) to the above solutions for T, one can easily obtain new analytical solutions for the generation of states $|\phi_{\text{hole} k}\rangle$ and $|\phi_{\text{filter} k}\rangle$. We have given only some specific examples of T, which guarantee the desired truncation. Although, it is out of the main goal of this paper, it is possible to give more general conditions for T, e.g.: (i) for any $T = [t_1', t_2', 1, 1, 1/2]$ with $t_1' \neq 1/2$ and $t_2 \neq 0, 1$, the output state is $|\phi_{\text{filter02}}\rangle$, (ii) for any $T = [1/2, t_2', 1, 1, 1/2]$ with $t_2 \in (0, 2/3)$ the output state is $|\phi_{\text{filter13}}\rangle$, assuming in both cases $\xi = 0$. As already emphasized, the solutions presented here are usually not optimized, but simple enough to show analytically that the specific truncations can be realized by the GQSD.

6. TRUNCATION TO FIVE-DIMENSIONAL QUdit STATES

A question arises whether the GQSD, shown in figure 3(b) with the removed BS3, can be used for truncation of the input state up to more than quartits. So, first we analyze possibilities of the truncation of an input state $|\psi\rangle$, given by (11), to the qudit state $|\phi_{\text{ran}}(5)\rangle$ being a special case of (2) for $d = 5$. As usual, we assume that light to be truncated is in mode $a_4$, and the input modes $a_1$ and $a_3$ are in the single-photon states, but, by contrast to the former sections, we choose mode $a_2$ to be in the two-photon state. So, the total input state is $|\Psi\rangle = |121\psi\rangle$. We assume that the conditional measurement yields the single-photon counts in detectors D2 and D4, but the two-photon count in D3, thus the resulting output state $|\phi\rangle = |\phi_{\text{ran}}(5)\rangle$ is given by (15) for $d = 5$ and the amplitudes

$$
c^{(5)}_n = \langle n | \Psi | 121 \rangle
$$

equal to

$$
c^{(5)}_0 = e^{i(\xi_4 + \xi_5)} t_2 t_4 (3e^{2i(\xi_3 + \xi_5)} f''_1 r_1 f_2 r_3 f_5^2 + 2e^{i(\xi_3 + \xi_4 + \xi_5)} g_1 t_2 r_2 r_3 g_5 t_5 + 3e^{2i(\xi_4 + \xi_5)} t_1' f_1'' f_4'' f_5'),
$$

$$
c^{(5)}_1 = 3 r_1 t_1 r_2 r_4 r_5 g_5 (e^{i(\xi_3 + \xi_5)} f''_1 r_2 r_3 f_5^4 + e^{i(\xi_3 + \xi_4 + \xi_5)} e^{i(\xi_4 + \xi_5)} f''_1 f_1'' f_4'' f_5'),
$$

$$
c^{(5)}_2 = e^{i\xi_2} t_2 t_4 (2t_2 t_4 r_5 g_5 e^{2i(\xi_3 + \xi_5)} r_2 f_2^2 g_5^2 + 2t_2 r_4^2 g_5^2 f_5') - 2e^{i(\xi_3 + \xi_4 + \xi_5)} e^{i(\xi_4 + \xi_5)} f''_2 r_5 f_4^2 f_5' - 2t_2' (f_2^2 + r_2^2) g_5 f_5' - r_1 g_2 (t_2 r_4 f_5' + 2r_2^2 f_5'),
$$

$$
c^{(5)}_3 = 3e^{2i\xi_2} r_1 t_2 f_2^2 r_5 g_5 e^{i(\xi_3 + \xi_5)} r_1 (3r_2^2 - r_2^2) r_4 f_5' + e^{i\xi_2} r_2 (3r_2^2 - r_2^2) r_5 t_5',
$$

$$
c^{(5)}_4 = 12e^{3i\xi_2} r_1 t_1 t_2 t_3 r_4 r_4 r_5 r_5 r_5'.
$$

As in (24), the irrelevant global phase factor $\exp(i\xi_1)$ was canceled out from all $c^{(5)}_n$ in (46). We have not found analytical solutions for the BS and PS parameters satisfying condition (21) with $d = 5$ for the amplitudes given by (15), as the problem requires finding roots of 6th order equations. Thus, we have applied numerical procedure for finding the BS parameters for which $\Delta = 0$ with precision of the order of $10^{-18}$. We have found various solutions including that for the transmittances equal to $T = [0.30464, 0.38775, 1, 0.81740, 0.18438]$, $\xi_4 = \pi$ and the other phase shifts set to zero, which results in the constant amplitude $c^{(5)}_4$ for $n = 0, ..., 4$. By placing BS3 in the setup with $t_3 \neq 1$ one can find the other solutions with larger constant $c^{(5)}_4$. However, since we are not interested in the optimization but rather simplicity of the scheme, we do not present these solutions here.
7. TRUNCATION TO SIX-DIMENSIONAL QUDIT STATES

The eight-port QSD, shown in figure 2, enables truncation of the incident light $|\psi\rangle$ even to the six-dimensional qudit state $|\psi^{(6)}\rangle$ as a special case of (2) for $d = 6$. To achieve the perfect truncation, we assume a single-photon Fock state in mode $\hat{c}$, two-photon states in modes $\hat{a}_1, \hat{a}_3$, and that the conditional measurement yields $N_2 = N_4 = 2$ and $N_3 = 1$ photon counts. Then, the output state $|\phi\rangle = |\psi^{(12)}\rangle$ is given by (15) for $d = 6$ and the amplitudes

$$c_n^{(6)} = \langle n212|\hat{U}|212n\rangle.$$  \hspace{1cm} (47)

The simplest-form amplitude is for $n = d-1$, which reads as

$$c_n^{(6)} = 30e^{-2\xi_1+3i\xi_2}r_1^2r_2^2r_4^2t_5\{ (3r_2^2-2t_5^2) \times e^{i\xi_2}e^{i\xi_2}r_1(3r_2^2-2t_5^2)r_3 + e^{i\xi_2}g_1r_2t_3)r_5t_5 + e^{i\xi_2}g_2r_1t_1(3r_2^2-2t_5^2)t_3 - e^{i\xi_2}g_1r_2t_3)r_4g_3 \}.$$  \hspace{1cm} (48)

As in the former cases for the truncation with $d < 6$, the amplitude $c_n^{(d)}$ is independent of the BS3 parameters. Unfortunately, contrary to the former cases, we have not found numerically any solutions satisfying $c_n^{(6)} = \text{const}$ for $n = 0, ..., d-1$ for the simplified scheme with removed BS3, shown in figure 3(b). Thus, we analyze again the general scheme shown in figure 2 with $t_3 \neq 0, 1$. As an example, we give a solution for $c_4^{(6)}$ to be as follows:

$$c_4^{(6)} = 6e^{2i(\xi_1+\xi_2)}r_1^2r_2^2r_4^2t_5\{ (3r_2^2-2t_5^2) \times e^{i\xi_2}e^{i\xi_2}r_1(3r_2^2-2t_5^2)r_3 + e^{i\xi_2}g_1r_2t_3)r_5t_5 + e^{i\xi_2}g_2r_1t_1(3r_2^2-2t_5^2)t_3 - e^{i\xi_2}g_1r_2t_3)r_4g_3 \}.$$  \hspace{1cm} (49)

Solutions for $c_n^{(6)}$ with $n = 0, ..., 3$ are quite lengthy so, we do not present them explicitly here. Nevertheless, they have been used in our numerical search of the BS and PS parameters satisfying condition (21) for $d = 6$. Thus, we have found various solutions for which $\Delta \sim 10^{-16}$. We just mention a solution for the BS transmissions equal to $T = [0.75572, 0.41783, 0.32503, 0.83274, 0.50338]$, $\xi_4 = \pi$ and the other phase shifts equal to zero, which results in the constant nonzero amplitude $c_n^{(6)}$ for $n = 0, ..., 5$, and obviously vanishing for $n > 5$. Another solution is found for the same phase shifts but transmissions equal to $T = [0.58154, 0.28519, 0.46753, 0.68558, 0.49836]$, which results also in a constant $c_n^{(6)}$ but one that slightly lower than that in the former case.

8. IMPERFECT PHOTON COUNTING

Now, we address an important problem from an experimental point of view that concerns the deterioration effects of system imperfections on the fidelity of truncation and teleportation. To analyze such effects one can follow the approaches of, for example, [6, 12, 13, 14, 50] applied to the Pegg-Phillips-Barnett QSD. Here, we focus on imperfect photodetection.

Photon counting in mode $\hat{b}_1$ by an imperfect detector with a finite efficiency $\eta$ and a mean dark count rate $\nu$ can be described by a positive-operator-valued measure (POVM) $|51\rangle$ with the following elements

$$\hat{\Pi}^{(b_1)}_{N_i} = \sum_{n=0}^{N_i} \sum_{m=-n}^{n} e^{-\nu_m}g^{|n-m|}\eta_m(1-\eta_m)^{|n-m|}C_m|n\rangle_i\langle m|.$$  \hspace{1cm} (50)

summing up to the identity operator $\hat{I}$. In (50), $N_i$ is the number of registered photocounts in detector $D_i$, $n$ is the actual number of photons entering detector, and $N_i - n$ is the number of dark counts, $C_m$ are binomial coefficients. The mean dark count rate $\nu$ in (50) is related to the standard dark count rate $R_{\text{dark}}$ by the relation $\nu = \tau_{\text{res}}R_{\text{dark}}$, where $\tau_{\text{res}}$ is the detector resolution time. In analogy to the analysis of the Pegg-Phillips-Barnett QSD given in [12], we compare the fidelities for the states truncated by the generalized scissors in relation to three types of applied detectors: (i) Conventional photodetectors (e.g., avalanche photo-diodes, APDs) providing only a binary answer to the question whether any photons have been registered or not, thus described by the POVM with the two elements:

$$\hat{\Pi}^{(b_1)}_{\text{det}} = \hat{\Pi}^{(b_1)}_0, \quad \hat{\Pi}^{(b_1)}_{\text{bin}} = \hat{I} - \hat{\Pi}^{(b_1)}_0.$$  \hspace{1cm} (51)

(ii) Single-photon resolving photodetectors (e.g., visual light photon counters, VLPCs [52]) providing a trinary answer to the question whether zero, one or more than one photons have been registered, so given by the POVM with the following three elements:

$$\hat{\Pi}^{(b_1)}_{\text{det}} = \hat{\Pi}^{(b_1)}_0, \quad \hat{\Pi}^{(b_1)}_{\text{bin}} = \hat{I} - \hat{\Pi}^{(b_1)}_0 - \hat{\Pi}^{(b_1)}_1.$$  \hspace{1cm} (52)

where $\hat{\Pi}^{(b_1)}_{\text{det}}$ in (51) and (52) are given by (50). (iii) And unrealistic detectors (labeled by $\eta$) resolving any number of simultaneously absorbed photons described by the POVM elements $\hat{\Pi}^{(b_1)}_{N_i} \equiv \hat{\Pi}^{(b_i)}_{N_i}$, given by (50) for any $N_i$. Such detectors are not available, although some methods (including the so-called photon chopping [53]) have been proposed to measure photon statistics with conventional devices. By including the imperfect photon counting by detectors $D_2, D_3$ and $D_4$, the output state at mode $\hat{b}_1$ can be described by the following density matrix

$$\hat{\rho}_x = \mathcal{N} \text{Tr}_{(b_2,b_3,b_4)} \left( \hat{\Pi}^{(b_2)}_{x\hat{N}_2} \hat{\Pi}^{(b_3)}_{x\hat{N}_3} \hat{\Pi}^{(b_4)}_{x\hat{N}_4} |\Phi\rangle \langle \Phi| \right),$$  \hspace{1cm} (53)

where the partial trace is taken over the detected modes $\hat{b}_2, \hat{b}_3,$ and $\hat{b}_4; \hat{\Pi}^{(b_i)}_{x\hat{N}_i}$ are the POVM elements for a given type of detectors $x = c, s, r; |\Phi\rangle$ is the four-mode output state, given by (12), and $\mathcal{N}$ is the normalization. For simplicity, we can assume identical detectors with $\eta \equiv \eta_2 = \eta_3 = \eta_4$ and $\nu \equiv \nu_2 = \nu_3 = \nu_4$. Deviation of the realistically truncated state $\hat{\rho}_x$ from the ideally truncated state $|\phi\rangle$ is usually described by the fidelity

$$F_x = \langle \phi | \hat{\rho}_x | \phi \rangle.$$  \hspace{1cm} (54)
In our numerical analysis we assume: (i) $\eta = 0.7$ and $R_{\text{dark}} \sim 100 \text{s}^{-1}$ for conventional detectors (see e.g., [12]), (ii) $\eta = 0.88$ and $R_{\text{dark}} \sim 10^4 \text{s}^{-1}$ for single-photon detectors (VLPCs) as experimentally achieved by Takeuchi et al. [52], (iii) for theoretic photon-number resolving detectors we choose the same $\eta$ and $R_{\text{dark}}$ as in (ii). Moreover, we put $\tau_{\text{res}} \sim 10 \text{ ns}$. We observe that the truncation fidelity in the system with imperfect photodetection depends on the chosen transmittances. In particular, different solutions described in section 4 for perfect truncation (with $F = 1$) in the lossless system correspond not only to different probabilities of success but also to different fidelities of the truncation in the lossy system. For $\alpha = 0.4$, we find that the fidelity for the truncation up to quartit states in the system described by the transmittances given below equation (54) drop from one to $F_c \approx 0.91$ for the conventional detectors and to $F_c = F_r \approx 0.98$ for the VLPCs and the photon-number resolving detectors. The fidelities of the truncation up to five-dimensional states in the system described in section 6 are estimated to be $F_c \approx 0.67$ for the conventional detectors, $F_c \approx 0.95$ for the VLPCs, and $F_c \approx 0.96$ for the photon-number resolving detectors if $\alpha = 0.4$. These estimations show that the conventional photodetectors can effectively be used for the low-intensity-field truncations described in sections 3–5, where at most single-photon detections are required. In the schemes described in sections 6 and 7, where detections of two-photons are important, one has to apply at least single-photon resolving detectors even in the low-intensity limit. It is worth noting that by increasing the number of detectors and beam splitters in the discussed pyramid configuration, one can achieve truncations to higher-dimensional states by detecting no or single photons only [54]. In our estimation we have assumed, based on [52], relatively high values of the dark count rates. However, it has recently been experimentally demonstrated by Babichev et al. by rigorously synchronizing the photon count events, that the dark counts can be reduced to a negligible level [12]. As multiport optical interferometers have already been experimentally realized [10, 11, 52], thus it seems that the proposed GQSD for the truncation and teleportation of at least quartit states is accessible to experiments with present-day technology for low-intensity incident fields.

9. DISCUSSION AND CONCLUSIONS

We are aware that our analysis of the quantum state truncation via a GQSD is not yet complete. Among open problems to be analyzed to a greater detail we should mention: (i) A generalization of the scheme for the truncation of an arbitrary $d$-dimensional qudit states based on the $2N$-port interferometer in the triangle configuration with the top BS removed. By symmetry of the scheme, shown in figure 2, such generalization is straightforward, e.g., along the lines of [52]. However, it would be desirable to find the minimum number of detections in the multiport QSD, which enables the state truncation to a qudit state of a given dimensionality. (ii) An analysis of other kinds of losses (including mode mismatch in addition to imperfect photon-counting) in the generalized optical state truncation. (iii) A detailed experimental proposal of the scheme for the truncation at least to qutrit and quartit states. (iv) A hard problem is the optimization of solutions for the BS parameters to obtain the highest probability of the truncation for $d \geq 4$. These problems are being under our current investigation [54].

In conclusion, we have proposed a generalization of the Pegg-Phillips-Barnett six-port QSD (shown in figure 1) to the eight-port optical interferometer, depicted in figure 2. The analyzed system enables, upon post-selection based on photon counting results, generation and teleportation of qudit states (for $d = 2, ..., 6$) by truncation of an input optical field at the $(d - 1)$th term of its Fock-state expansion. We have discussed examples of selective truncations, which enable Fock-state filtering and hole burning in the Fock space of an input optical field. We have also analyzed deterioration of the truncation fidelity due to realistic photon counting including finite photon-number resolution, inefficiency and dark counts of photodetectors. Our estimations suggest that the scheme is experimentally feasible at least for the generation and teleportation of qudit states of low-intensity incident fields.

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[1] 1997 Quantum State Preparation and Measurement, J. Mod. Opt. 44 (11/12) (Special issue)
[2] Knill E, Laflamme R, and Milburn G J 2001 Nature 409 46
[3] O’Brien J L, Pryde G J, White A G, Ralph T C, and Branning D A 2003 Nature 426 264
[4] Yamamoto T, Koashi M, Özdemir S K, and Imoto N 2003 Nature 421, 343
[5] Pegg D T, Phillips L S and Barnett S M 1998 Phys. Rev. Lett. 81 1604
