Compactness of solutions to the Yamabe problem. III

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Abstract

For a sequence of blow up solutions of the Yamabe equation on non-locally conformally flat compact Riemannian manifolds of dimension 10 or 11, we establish sharp estimates on its asymptotic profile near blow up points as well as sharp decay estimates of the Weyl tensor and its covariant derivatives at blow up points. If the Positive Mass Theorem held in dimensions 10 and 11, these estimates would imply the compactness of the set of solutions of the Yamabe equation on such manifolds.

1 Introduction

Let \((M^n, g)\) be a compact, smooth, connected Riemannian manifold (without boundary) of dimension \(n \geq 3\). The Yamabe conjecture has been proved through the works of Yamabe [19], Trudinger [18], Aubin [1] and Schoen [16]: The conformal class of \(g\) contains a metric of constant scalar curvature. Different proofs of the Yamabe conjecture in the case \(n \leq 5\) and in the case \((M, g)\) is locally conformally flat are given by Bahri and Brezis [5] and Bahri [4].

Consider the Yamabe equation and its sub-critical approximations:

\[-\Delta_g u + c(n)R_g u = n(n-2)u^p, \quad u > 0, \quad \text{on} \quad M,\]

where \(1 < p \leq \frac{n+2}{n-2}\), \(\Delta_g\) is the Laplace-Beltrami operator associated with \(g\), \(R_g\) is the scalar curvature of \(g\), and \(c(n) = \frac{(n-2)}{4(n-1)}\). Let

\[\mathcal{M}_p = \{ u \in C^2(M) \mid u \text{ satisfies (1)} \}.\]

If \((M, g)\) is locally conformally flat and is not conformally diffeomorphic to the standard sphere, Schoen [17] proved that for any \(1 < 1 + \epsilon \leq p \leq \frac{n+2}{n-2}\) and any non-negative integer \(k\),

\[\|u\|_{C^k(M, g)} \leq C, \quad \forall \ u \in \mathcal{M}_p,\]
where $C$ is some constant depending only on $(M, g)$, $\epsilon$ and $k$. The same conclusion has been proved to hold in dimension $n \leq 7$ for $(M^n, g)$ which are not locally conformally flat, see Li and Zhang [13] and Marques [15]. See also the introduction of [13] where works of Li-Zhu [14], Li-Zhang [12] and Druet [9]-[10] for dimensions $n = 3, 4, 5$ are described. Extensive works on the problem and closely related ones can be found in [13] as well.

For $n = 8, 9$ and on $(M^n, g)$ which are not locally conformally flat, [13] also contains sharp estimates on blow up solutions of (1) and sharp decay estimates of the Weyl tensor and its first covariant derivatives at blow up points. If the Positive Mass Theorem held in dimensions 8 and 9, these estimates would yield (2) for $n = 8, 9$. Soon after completing [13], we extended these sharp estimates to dimensions $n = 10, 11$ (see Theorem 1.1); however we have encountered some difficulty in extending such estimates to $n \geq 12$. Very interesting results have subsequently been obtained by Aubin in [2, 3].

To study the compactness of solutions to the Yamabe equation, it is crucial to establish sharp estimates of blow up solutions. An important step is to find out the right asymptotic profile of blow up solutions near a blow up point. Our earlier work [13] strongly suggested such a profile in dimensions $n \geq 10$, which we describe below.

Let $\{u_k\}$ be a sequence of solutions to the Yamabe equation on $(M^n, g)$ satisfying, for some $\bar{P}_k \in M$,

$$u_k(\bar{P}_k) := \max_M u_k \to \infty.$$ 

Assume that $g = g_{ij}(z)dz^idz^j$ is already in conformal normal coordinates centered at $\bar{P}_k$.

For dimension $n = 8, 9$, sharp estimates on blow up solutions and sharp decay estimates of the Weyl tensor and its covariant derivatives at blow up points were established in [13] through an iterative procedure. Due to this procedure, we expect to obtain enough estimates on the decay rates of the Weyl tensor and its covariant derivatives of appropriate order before making the next step in the iterative process, and therefore we can use

$$-\Delta u_k + c(n)R_gu_k = n(n - 2)u_k^{\frac{n+2}{n-2}}, \quad (3)$$

instead of the Yamabe equation which would replace $\Delta$ in (3) by $\Delta_g$, to determine the asymptotic profile of $\{u_k\}$ near blow up points. Note that $\Delta$ is the flat Laplacian in the $z-$coordinates.

The Taylor expansion of $R(z)$ in conformal normal coordinates is, for $\bar{l} \geq 2$,

$$R(z) = \sum_{l=2}^{\bar{l}} \sum_{|\alpha|=l} \frac{\partial^\alpha R}{\alpha!} z^\alpha + O(|z|^{\bar{l}+1}). \quad (4)$$

For convenience, we write, with $M_k := u_k(0)$,

$$v_k(y) := M_k^{-\frac{2}{n+2}}M_k^{-\frac{2}{n-2}y}.$$
The equations satisfied by $v_k$ are

\[ \Delta v_k(y) - \bar{c}(y)v_k(y) + n(n - 2)v_k(y)\frac{n+2}{n-2} = 0 \]  

(5)

where

\[ \bar{c}(y) := c(n)R_g(M_k^{-\frac{2}{n-2}}y)M_k^{-\frac{4}{n-2}} = c(n)\sum_{\gamma=2}^i M_k^{-\frac{4+2\gamma}{n-2}} \sum_{|\alpha|=\gamma} \frac{\partial \alpha R}{\alpha!} y^\alpha \]

and

\[ v_k \text{ converges to } U(y) := \left( \frac{1}{1 + |y|^2} \right)^{\frac{n-2}{2}} \text{ in } C^3_{\text{loc}}(\mathbb{R}^n). \]

For dimension $n = 10, 11$, we only need to consider, in the formal expansion of $v_k$,

\[ \tilde{v}_k = v^{(1)} + \frac{8}{n-2} v^{(2)} + \frac{10}{n-2} v^{(3)}. \]

The equations satisfied by $v^{(1)}$, $v^{(2)}$ and $v^{(3)}$ are, determined by (5),

\[ \Delta v^{(1)} + n(n - 2)[v^{(1)}]\frac{n+2}{n-2} = 0, \]

\[ \Delta v^{(2)} - c(n)\left[ \sum_{|\alpha|=2} \frac{\partial \alpha R}{\alpha!} y^\alpha \right] v^{(1)} + n(n - 2)[v^{(1)}]\frac{4}{n-2} v^{(2)} = 0, \]

and

\[ \Delta v^{(3)} - c(n)\left[ \sum_{|\alpha|=3} \frac{\partial \alpha R}{\alpha!} y^\alpha \right] v^{(1)} + n(n - 2)[v^{(1)}]\frac{4}{n-2} v^{(3)} = 0. \]

Let

\[ \tilde{R}^{(l)} := \frac{1}{|S^{n-1}|} \int_{S^{n-1}} \sum_{|\alpha|=l} \frac{\partial \alpha R}{\alpha!} \theta^\alpha, \]  

(6)

and

\[ \tilde{R}^{(l)}(\theta) := -\tilde{R}^{(l)} + \sum_{|\alpha|=l} \frac{\partial \alpha R}{\alpha!} \theta^\alpha, \quad \theta \in S^{n-1}. \]  

(7)

We have, in polar coordinates,

\[ v^{(1)} := U, \quad v^{(2)}(r, \theta) = -c(n)\tilde{R}^{(2)}(\theta)f_2(r), \quad v^{(3)}(r, \theta) = -c(n)\tilde{R}^{(3)}(\theta)f_3(r), \]  

(8)

where $f_2$ and $f_3$ are respectively the solutions of (73) and (76).

Thus the asymptotic profile of $u_k$ near the blow up point $\bar{P}_k$ should be

\[
\tilde{u}_k(x) &= M_k \tilde{v}_k(M_k^{-\frac{2}{n-2}}x) \\
&= u_k(0)[v^{(1)}(u_k(0)\frac{2}{n-2}x) + u_k(0)\frac{8}{n-2}v^{(2)}(u_k(0)\frac{2}{n-2}x) + u_k(0)\frac{10}{n-2}v^{(3)}(u_k(0)\frac{2}{n-2}x)] \\
&= u_k(0)U(u_k(0)\frac{2}{n-2}x) - u_k(0)\frac{10}{n-2}[c(n)\tilde{R}^{(2)}(\theta)f_2(u_k(0)\frac{2}{n-2}|x|)] \\
&\quad - u_k(0)\frac{10}{n-2}[c(n)\tilde{R}^{(3)}(\theta)f_3(u_k(0)\frac{2}{n-2}|x|)].
\]
In the following, we give the previously mentioned sharp estimates in dimensions $n = 10, 11$. The asymptotic profile of blow-up solutions is exactly the one described above.

For $Q \in M$ and $\mu > 0$, let

$$
\xi_{Q, \mu}(P) := \left(\frac{\mu}{1 + \mu^2 \text{dist}_g(P, Q)^2}\right)^{\frac{n-2}{2}}, \quad P \in M
$$

and, in polar coordinates,

$$
\tilde{\xi}_{Q, \mu}(P) = \xi_{Q, \mu}(P) - c(n)\tilde{R}^{(2)}(\theta)\mu^{\frac{n-10}{4}}
- c(n)\tilde{R}^{(3)}(\theta)\mu^{\frac{n-12}{4}},
$$

We use $W_g$ to denote the Weyl tensor of the metric $g$.

**Theorem 1.1** Let $(M^n, g)$ be a compact, smooth, connected Riemannian manifold of dimension $n = 10, 11$, and let $u$ be a smooth solution of (1) with $1 < 1 + \epsilon \leq p \leq \frac{n+2}{n-2}$. Then for some positive constant $C$ and some positive integer $m$ which depend only on $(M, g)$, there exist some local maximum points of $u$, denoted as $S := \{P_1, \ldots, P_m\}$, such that

$$
\text{dist}_g(P_i, P_j) \geq \frac{1}{C}, \quad \frac{1}{C} u(P_i) \leq u(P_j) \leq C u(P_i), \quad \forall i \neq j,
$$

$$
|W_g(P_i)|_g \leq C u(P_i)^{-\frac{n-6}{n-2}}, \quad |\nabla W_g(P_i)|_g \leq C u(P_i)^{-\frac{n-8}{n-2}},
$$

$$
|\nabla^2 W_g(P_i)|_g \leq \begin{cases} C, & \text{if } n = 10, \\ \frac{\sqrt{\log u(P_i)}}{C u(P_i)^{-\frac{n-10}{n-2}}}, & \text{if } n = 11, \end{cases} \quad \forall i,
$$

$$
\frac{1}{C} \sum_{l=1}^{m} \xi_{P_l, u(P_l)}^{\frac{2}{n-2}} \leq u \leq C \sum_{l=1}^{m} \xi_{P_l, u(P_l)}^{\frac{2}{n-2}}, \quad \text{on } M.
$$

Moreover, for each $l$ and modulo a conformal factor which makes $g$ in conformal normal coordinates at $P_l$,

$$
|\nabla_g^\alpha (u - \tilde{\xi}_{P_l, u(P_l)}^{\frac{2}{n-2}})(P)| \leq C u(P)^{\frac{n-14+2|\alpha|}{n-2}} (1 + u(P)^{\frac{2}{n-2}} \text{dist}_g(P, P_l)^{n-|\alpha|}), \quad \forall \text{dist}_g(P, P_l) < \frac{1}{C}, |\alpha| = 0, 1, 2.
$$

A consequence of Theorem 1.1 is
Corollary 1.1 Let \((M^n, g), n = 10, 11\), be a compact, smooth, connected Riemannian manifold which is not locally conformally flat, and let \(1 < 1 + \epsilon \leq p \leq \frac{n+2}{n-2}\). Then
\[
\|u\|_{H^1(M, g)} \leq C, \quad \forall \ u \in \mathcal{M}_p,
\]
where \(C\) is some constant depending only on \((M^n, g)\) and \(\epsilon\).

Remark 1.1 If the Positive Mass Theorem held in dimensions \(n = 10\) and \(11\), Theorem 1.1 would yield (2) in these dimensions.

In the following we give a result which is more local in nature. Let \(B_1 \subset \mathbb{R}^n\) be the unit ball centered at the origin, and let \((a_{ij}(x))\) be a smooth, \(n \times n\) symmetric positive definite matrix function, defined on \(B_1\), satisfying
\[
\frac{1}{2} |\xi|^2 \leq a_{ij}(x)\xi^i\xi^j \leq 2|\xi|^2, \quad \forall \ x \in B_1, \ \xi \in \mathbb{R}^n,
\]
and, for some \(\bar{a} > 0\),
\[
\|a_{ij}\|_{C^5(B_1)} \leq \bar{a}.
\]
Consider
\[-Lu = n(n-2)u^p, \quad u > 0, \text{ on } B_1,\]
where
\[g := a_{ij}(x)dx^i dx^j.\]
If \(\{x^1, \cdots, x^n\}\) are conformal normal coordinates for \(g\), let
\[v := v^{(1)} + u(0)^{-\frac{8}{n-2}}v^{(2)} + u(0)^{-\frac{10}{n-2}}v^{(3)},\]
where \(v^{(1)}, v^{(2)}\) and \(v^{(3)}\) are defined in (8).

Theorem 1.2 Let \((B_1, g)\) be as above and let \(u\) be a solution of (11), with \(1 < 1 + \epsilon \leq p \leq \frac{n+2}{n-2}\) and \(n = 10, 11\). Assume, for some constant \(\bar{b} \geq 1\),
\[
\nabla u(0) = 0, \quad 1 \leq \sup_{B_1} u \leq \bar{b}u(0).
\]
Then there exist some positive constants \(\delta\) and \(C\), depending only on \(\bar{b}, \epsilon\) and \(\bar{a}\), such that
\[
u(0)u(x)|x|^{n-2} \leq C, \quad \forall \ 0 < |x| \leq \delta,
\]
and
\[
|W_g(0)|^2_g u(0)^{-\frac{8}{n-2}} + |\nabla_g W_g(0)|^2_g u(0)^{-\frac{12}{n-2}} + |\nabla^2_g W_g(0)|^2_g u(0)^{-\frac{16}{n-2}} \log u(0)
\leq C \bar{u}(0)^{-2}, \quad n = 10,
\]
\[
|W_g(0)|^2 u(0)^{- \frac{8}{n-2}} + |\nabla g W_g(0)|^2 u(0)^{- \frac{12}{n-2}} + |\nabla^2 g W_g(0)|^2 u(0)^{- \frac{16}{n-2}} \leq C u(0)^{-2}, \quad n = 11, \tag{17}
\]
\[
\frac{1}{C} u(0) U(u(0) \frac{2}{n-2} x) \leq u(x) \leq C u(0) U(u(0) \frac{2}{n-2} x), \quad \forall 0 < |x| \leq \delta, \tag{18}
\]
and, if \{x^1, \ldots, x^n\} are conformal normal coordinates for \(g\), we have, with \(v\) given in (13),
\[
|\nabla^\alpha (u - u(0) v(u(0) \frac{2}{n-2} x))| \leq C u(0) \frac{n-14+2|\alpha|}{n-2} (1 + u(0) \frac{2}{n-2} |x|)^{8-n-|\alpha|}, \quad \forall 0 < |x| \leq \delta, \ |\alpha| = 0, 1, 2. \tag{19}
\]

It is not difficult to see that Theorem 1.1 follows from Theorem 1.2. Our proof of Theorem 1.2 follows closely the arguments in [13]. In particular the sharp estimates on blow up solutions and on the decay rates of the Weyl tensor and its derivatives are obtained iteratively with improved estimates after each iteration. The main difference between the arguments in this paper and those of [13] is that some Riemannian tensor inequalities in conformal normal coordinates which we used for dimension \(n = 8, 9\) are not sufficient for higher dimensions. Our proof of Theorem 1.2 requires an estimate from below of some integral quantity associated with \(v(2)\).

We will only prove Theorem 1.2 for \(p = \frac{n+2}{n-2}\) since modifications of the arguments yield the result for \(1 + \epsilon \leq p \leq \frac{n+2}{n-2}\), see section 5 of [13]. We will always assume that \(n = 10, 11\) unless otherwise stated. In Section 2 we prove Theorem 1.2. In the appendices we establish some facts which we use for the proof.

## 2 The proof of Theorem 1.2

In this section we prove Theorem 1.2. In the first four subsections we establish (15) using the method of moving spheres. In the last section we derive (16)-(19) using the Pohozaev type identity.

### 2.1 The set up for proving (15)

Suppose the contrary of (15), then for some \(\bar{a} > 0, \bar{b} \geq 1\), there exist a sequence of Riemannian metrics \{\(g_k\)\} of the form (12) that satisfy (9) and (10), and some solutions \(u_k\) of (11), with \(p = \frac{n+2}{n-2}\) and with \(g\) replaced by \(g_k\), satisfying (14), such that
\[
\max_{|x| \leq \frac{1}{k}} \left( u_k(0) u_k(x) |x|^{n-2} \right) \geq k, \tag{20}
\]
We will simply use \( g \) to denote \( \tilde{g}_k \), and we assume that \( g_{ij}(z)dz^idz^j \) is already in conformal normal coordinates centered at the origin — as in the proof of theorem 2.1 in [13]. As in [13],

\[
M_k := u_k(0) \to \infty.
\]

Write

\[
(g_k)_{ij}(y) = g_{ij}(M_k^{-\frac{2}{n-2}}y)dy^idy^j,
\]

\[
v_k(y) := M_k^{-1}u_k(M_k^{-\frac{2}{n-2}}y),
\]

\[
c(x) = c(n)R_g(x), \quad \text{and} \quad \bar{c}(y) = c(n)R_g(M_k^{-\frac{2}{n-2}}y)M_k^{-\frac{1}{n-2}}.
\]

Then

\[
\begin{aligned}
\Delta g_{k}v_k(y) - \bar{c}v_k(y) + n(n-2)v_k(y)^{\frac{n+2}{n-2}} &= 0, \quad |y| \leq \frac{1}{2}M_k^{\frac{2}{n-2}}, \\
1 = v_k(0) &\geq (\bar{b} - 1 + o(1))v_k(y), \quad |y| \leq \frac{1}{2}M_k^{\frac{2}{n-2}}, \quad \nabla v_k(0) = 0.
\end{aligned}
\] (21)

By the Liouville type theorem of Caffarelli, Gidas and Spruck [6], together with some standard elliptic estimates,

\[v_k\] converges to \( U(y) := \left(\frac{1}{1 + |y|^2}\right)^{\frac{n-2}{2}} \) in \( C^3_{loc}(\mathbb{R}^n) \).

In local coordinates,

\[
g_{pq}(x) = \delta_{pq} + \frac{1}{3}R_{pijq}x^ix^j + \frac{1}{6}R_{pijq,k}x^ix^jx^k + O(r^5).
\]

In conformal normal coordinates, write

\[
\Delta_g = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}\partial_j) = \Delta + b_1\partial_i + d_{ij}\partial_{ij},
\]

where \((g^{ij})\) denotes the inverse matrix of \((g_{ij})\), \(\partial_i = \partial_{x^i}\), \(\partial_{ij} = \partial_{x^i}\partial_{x^j}\), \(\Delta = \sum_{i=1}^n \partial_i^2\),

\[
b_i(x) = \partial_j g^{ij}(x)
\]

\[
= -\frac{1}{6}R_{ia,bk}x^ax^b - \frac{1}{6}R_{ia,bp}x^a x^b - \left(\frac{1}{20}R_{ia,bc} - \frac{1}{15}R_{ipad}R_{pbc}ight) + O(r^4),
\]

and

\[
d_{ij}(x) = g^{ij} - \delta_{ij} = -\frac{1}{3}R_{ijpq}x^p x^q - \frac{1}{6}R_{ijpq,k}x^p x^q x^k - \left(\frac{1}{20}R_{ijpq,kl} - \frac{1}{15}R_{ipqm}R_{jklm}\right) + O(r^5),
\]

\[\]
Thus
\[ \Delta g_k = \Delta + \bar{b}_i \partial_i + \bar{d}_{ij} \partial_{ij}, \]
where
\[ \bar{b}_i(y) = M_k^{-2} b_i(M_k^{n-2} y), \quad \bar{d}_{ij}(y) = d_{ij}(M_k^{n-2} y). \]

For \( \lambda > 0 \) and for any function \( v \), let, as in [13],
\[ v^\lambda(y) := \left( \frac{\lambda}{|y|} \right)^{n-2} v(y^\lambda), \quad y^\lambda := \frac{\lambda^2 y}{|y|^2}, \]
denote the Kelvin transformation of \( v \), and
\[ \Sigma_\lambda := B(0, \frac{1}{\sqrt{k}} M_k^{n-2}) \setminus \overline{B(0, \lambda)} = \{ y \mid |y| < \frac{1}{\sqrt{k}} M_k^{n-2}, \lambda < |y| \}. \]

As in (33)-(35) in [13], the equation for \( w_\lambda = v_k(y) - v^\lambda_k(y), \quad y \in \Sigma_\lambda. \)
As in [13] we apply the method of moving spheres to \( w_\lambda + h_\lambda \), an appropriate perturbation of \( w_\lambda \), for \( \lambda \) in a fixed neighborhood of 1, to reach a contradiction. We recall that the perturbation \( h_\lambda \) is required to satisfy the following properties:

1. \( h_\lambda = 0 \) on \( \partial B_\lambda, \lambda = o(1) |y|^{2-n} \) in \( \Sigma_\lambda. \)
2. For \( O_\lambda := \{ y \in \Sigma_\lambda \mid v_k(y) < 2v^\lambda_k(y) \}, \)
\[ \left( \Delta + \bar{b}_i \partial_i + \bar{d}_{ij} \partial_{ij} - \bar{c} + n(n+2)\xi^\frac{4}{n-2} \right) h_\lambda + E_\lambda \leq 0 \quad \text{in} \quad O_\lambda. \] (23)

Note that by the first property, \( w_\lambda + h_\lambda > 0 \) in \( \Sigma_\lambda \setminus \bar{O}_\lambda \), so we do not need (23) to hold outside \( O_\lambda. \)
2.2 Estimate of $v_k$ in $|y| \leq M_k^{16-\epsilon}/(n-2)^2$

In [13] we established estimates on $v_k - U$ with an error term of the order $M_k^{-8/n-2}$. Now, for dimension $n = 10, 11$, we need to work with terms in the formal expansion of $v_k$ as described in the introduction which are of order $M_k^{-8/n-2}$ and $M_k^{-10/n-2}$. The main result of this subsection is the following estimate of $v_k$ with an error term of the order $M_k^{-12/n-2}$ in the region $|y| \leq M_k^{16-\epsilon}/(n-2)^2$ for any $\epsilon > 0$.

**Proposition 2.1** For $n \geq 10$, and for any $\epsilon > 0$, there exists some positive constant $C(\epsilon)$ such that

$$
\left| \nabla^l \left( v_k - (v^{(1)} + M_k^{-8/n-2} v^{(2)} + M_k^{-10/n-2} v^{(3)}) \right) \right| 
\leq C(\epsilon) M_k^{-12/n-2} (1 + r)^{8-n+\bar{a}-l}, \quad 0 < r \leq M_k^{16-\epsilon/(n-2)^2}, \quad l = 0, 1, 2,
$$

(24)

where $\bar{a} = \frac{3}{4} (n - 10 + \sqrt{\epsilon})$, $v^{(1)}$, $v^{(2)}$ and $v^{(3)}$ are defined in (8).

We first recall some notations in [13]. For $\bar{l} \geq 2$, write the Taylor expansion of $R(x)$ at 0 as (4). Let $\bar{R}^{(l)}$ and $\tilde{R}^{(l)}(\theta)$ be defined as in (6) and (7). We know (see (44) in [13]), with $W$ denoting the Weyl tensor, that

$$
\bar{R}^{(2)} = \frac{1}{2n} \Delta R = -\frac{1}{12n} |W|^2, \quad \text{and } \bar{R}^{(3)} = 0.
$$

(25)

We write

$$
\tilde{R}^{(l)}(\theta) = \sum_{p \geq 1} \tilde{R}_{lp} \epsilon_{lp}(\theta), \quad 2 \leq l \leq \bar{l}.
$$

(26)

where $\epsilon_{lp}$s, depending only on $n$, are non-constant eigenfunctions of $-\Delta_{g^{n-1}}$. The following lemma, whose proof can be found in Appendix A, is used in our arguments.

**Lemma 2.1**

$$
\tilde{R}^{(3)} = \sum_{p=1}^{l_3} \tilde{R}^{(3)}_{p} \epsilon_{3p}(\theta) + O(|W|),
$$

(27)

where $\{\epsilon_{3p}(\theta)\}_{1 \leq p \leq l_3}$ is a set of eigenfunctions of $-\Delta_{g^{n-1}}$ associated with the eigenvalue $3(n + 1)$.

Let $f_2$ and $f_3$ be defined as in Appendix C, set

$$
F^{(2)} := -c(n) \tilde{R}^{(2)}(\theta) f_2(r) M_k^{-8/n-2} = v^{(2)} M_k^{-8/n-2}
$$
To see this, we first recall some known facts. We know from (21), (44) and (123) in [13] that
\[\Delta + n(n + 2)U \frac{4}{n-2} F^{(2)} = c(n) \tilde{R}^{(2)}(\theta) r^2 U M_k \frac{4}{n-2} ,\]
and
\[\Delta + n(n + 2)U \frac{4}{n-2} F^{(3)} = c(n) \sum_{l=2}^{3} \tilde{R}^{(l)}(\theta) r^l U M_k \frac{4+2l}{n-2} + O(|W|) M_k \frac{10}{n-2} (1 + r)^{5-n} .\]

**Proof of Proposition 2.1.** We claim that
\[\Delta g_k - \bar{c}(U + F^{(3)}) + n(n - 2)(U + F^{(3)}) \frac{n+2}{n-2} = O(M_k \frac{10}{n-2} (1 + r)^{6-n} .\] (29)
To see this, we first recall some known facts. We know from (21), (44) and (123) in [13] that
\[v_k(y) \leq CU(y), \quad |y| \leq M_k^{\frac{16-\epsilon}{n-2}} , \quad n \geq 10.\] (30)
\[|\nabla^l R_{abcd}| = O(M_k \frac{2(2-l)}{n-2} + \epsilon), \quad l = 0, 1, \quad n \geq 10.\] (31)
\[\tilde{R}^{(2s+2)} = O(M_k \frac{4(2-s)}{n-2} + \epsilon), \quad s = 0, 1, \quad n \geq 10.\]
These lead to
\[\bar{b}_i(y) = O(M_k \frac{8-i}{n-2}) r^2 + O(M_k \frac{8-i}{n-2}) r^3,\]
\[\bar{d}_{ij}(y) = O(M_k \frac{8-i}{n-2})(1 + r)^3 + O(M_k \frac{8-i}{n-2}) r^4.\] (32)
To derive (29) we use (28) and (31) to obtain
\[
\begin{align*}
\Delta (U + F^{(3)}) + n(n - 2)(U + F^{(3)}) \frac{n+2}{n-2} & \nonumber \\
= \Delta (U + F^{(3)}) + n(n - 2)U \frac{n+2}{n-2} (1 + \frac{F^{(3)}}{U}) \frac{n+2}{n-2} & \\
= \Delta U + n(n - 2)U \frac{n+2}{n-2} + \Delta F^{(3)} + n(n + 2)U \frac{n+2}{n-2} F^{(3)} + O(\frac{F^{(3)}}{U})^2 U \frac{n+2}{n-2} & \\
= c(n) \sum_{l=2}^{3} \tilde{R}^{(l)} r^l U M_k \frac{4+2l}{n-2} + O(|W|) M_k \frac{10}{n-2} (1 + r)^{5-n} + O(M_k \frac{16}{n-2})(1 + r)^{6-n} & \\
= c(n) \sum_{l=2}^{3} \tilde{R}^{(l)} r^l U M_k \frac{4+2l}{n-2} + O(M_k \frac{14-\epsilon}{n-2})(1 + r)^{5-n}.
\end{align*}
\]
Note that we used the estimates of $f_2$ and $f_3$ in Appendix C. To estimate $(\Delta g_k - \Delta)(U + F^{(3)})$, we observe that for any smooth functions $a(\theta)$ and $b(r)$, by the definition of the conformal normal coordinates, $(\Delta g_k - \Delta)b(r) = 0$, consequently

$$(\Delta g_k - \Delta) \left( a(\theta)b(r) \right) = \left( (\Delta g_k - \Delta)a(\theta) \right)b(r). \quad (33)$$

It follows, using the estimates of $\overline{b}_i$ and $\overline{d}_{ij}$ in (32), that

$$\overline{b}_i \overline{\partial}_i + \overline{d}_{ij} \overline{\partial}_{ij} (U + F^{(3)}) = (\overline{b}_i \overline{\partial}_i + \overline{d}_{ij} \overline{\partial}_{ij})F^{(3)}$$

$$= O(M_k^{-\frac{16-\epsilon}{n-2}})(1 + r)^{7-n} + O(M_k^{-\frac{16}{n-2}})(1 + r)^{8-n}.$$ 

Also, by $\tilde{R}^{(2)} = O(M_k^{-\frac{8-\epsilon}{n-2}})$ and the estimates of $f_2$ and $f_3$,

$$\tilde{c}(U + F^{(3)}) = c(n) \sum_{l=2}^{3} \tilde{R}^{(l)} r^l U M_k^{\frac{4+2l}{n-2}} + O(M_k^{-\frac{12}{n-2}})(1 + r)^{6-n}.$$ 

Then (29) is the consequence of the above. By (29) and the equation for $v_k$, we have

$$(\Delta g_k - \tilde{c})(v_k - U - F^{(3)}) + n(n - 2)(v_k^\frac{n+2}{n-2} - (U + F^{(3)})^\frac{n+2}{n-2})$$

$$= O(M_k^{-\frac{12}{n-2}})(1 + r)^{6-n}, \quad |y| \leq M_k^{\frac{16-\epsilon}{(n-2)^2}}. \quad (34)$$

Since $f'_2(0) = f'_3(0) = 0$, $\nabla (U + F^{(3)})(0) = 0$. By these facts and (30) we can prove

$$\Lambda_k := \max_{|y| \leq M_k^{\frac{16-\epsilon}{(n-2)^2}}} |(v_k - U - F^{(3)})(y)| \leq CM_k^{-\frac{12}{n-2}}. \quad (35)$$

Indeed, let

$$w_k := \Lambda_k^{-1}(v_k - U - F^{(3)}).$$

Then we see from (34) and (31) that, for some $\epsilon > 0$ independent of $k$,

$$\left( \Delta + \frac{c(1)}{(1 + |y|)^{1+\epsilon}} + \frac{c(1)}{(1 + |y|)^{2+\epsilon}} + \frac{c(1)}{(1 + |y|)^{3+\epsilon}} \right) w_k(y)$$

$$= O(1) \Lambda_k^{-1} M_k^{-\frac{12}{n-2}}(1 + |y|)^{6-n} + O(1)(1 + |y|)^{-4} w_k$$

$$= O(1) \Lambda_k^{-1} M_k^{-\frac{12}{n-2}}(1 + |y|)^{-2-\epsilon} + O(1)(1 + |y|)^{-2-\epsilon}, \quad |y| \leq M_k^{\frac{16-\epsilon}{(n-2)^2}}.$$ 

If (35) did not hold, then $\Lambda_k^{-1} M_k^{-\frac{12}{n-2}} = c(1)$ along a subsequence, and the argument below (101) in [13] (with $\delta R_k$ replaced by $M_k^{\frac{16-\epsilon}{(n-2)^2}}$) yields a contradiction. See also lemma 3.3 in
[7] for a similar argument. (24) is proved for \( l = 0 \) and \(|y| < R\) for \( R \) being a fixed large constant. Next we use (35) to compare \((v_k - U - F^{(3)})(y)\) with \(QM_k^{-\frac{12}{n-2}}r^{8-n+\bar{a}}\) for some large \( Q \) over \( R < |y| < M_k^{\frac{16-n}{2}}\). By the maximum principle,

\[
|v_k - (U + F^{(3)})| \leq QM_k^{-\frac{12}{n-2}}r^{8-n+\bar{a}}.
\]

The estimates for the first and the second derivatives of \(v_k - (U + F^{(3)})\) follow from this and the equation for \(v_k - (U + F^{(3)})\) by elliptic estimates. Proposition 2.1 is established. \( \square \)

### 2.3 Estimate of \( E_\lambda \)

In this and the next subsections, we assume \( \lambda \in (\frac{1}{2}, 2) \) and we assume \( \lambda \leq |y| \leq \frac{1}{2}M_k^{\frac{2}{n-2}} \) unless otherwise stated. We use \( E_1, \cdots, E_4 \) to denote the following terms:

\[
E_1 = c(n)U^\lambda \sum_{s=0}^{2} \tilde{R}^{(2s+2)} M_k^{-\frac{8+4s}{n-2}} r^{2s+2}(1 - (\frac{\lambda}{r})^{4s+8})
\]

\[
- \frac{c(n)^2}{2n(n+2)} \left( \sum_{i<j} 2(\partial_{ij} R)^2 + \sum_i (\partial_i R)^2 \right) (1 - (\frac{\lambda}{r})^8)r^2 f_2^\lambda M_k^{-\frac{16}{n-2}}
\]

\[
E_2 = \begin{cases} 
  c(n)U^\lambda \sum_{l=2}^{6} \tilde{R}^{(l)} M_k^{-\frac{4+2l}{n-2}} r^l (1 - (\frac{\lambda}{r})^{2l+4}), & n = 10, \\
  c(n)U^\lambda \sum_{l=2}^{7} \tilde{R}^{(l)} M_k^{-\frac{4+2l}{n-2}} r^l (1 - (\frac{\lambda}{r})^{2l+4}), & n \geq 11.
\end{cases}
\]

\[
E_3 = \sum_{s=1}^{J} \tilde{a}_{s,k}(r)e_s, \quad E_4 = \begin{cases} 
  O(M_k^{-\frac{18-\epsilon}{n-2}} r^{9-\frac{\epsilon}{2}-n}), & n = 10, \\
  O(M_k^{-\frac{20-\epsilon}{n-2}} r^{10-\frac{\epsilon}{2}-n}), & n \geq 11.
\end{cases}
\]

(36)

where \( e_s = e_s(\theta) \), independent of \( k \), is a homogeneous spherical harmonic of degree \( s \), \( J \) is a positive integer, \( \tilde{a}_{s,k} \) satisfies

\[
|\tilde{a}_{s,k}(r)| = O(M_k^{-\frac{16-\epsilon}{n-2}} r^{3-n}) + O(M_k^{-\frac{16}{n-2}} r^{4-n}).
\]

From now on we say a term is \( E_3 \) or \( E_4 \) if it is of the form in (36). The main result in this subsection is

**Proposition 2.2** For \( n \geq 10, \frac{1}{2} \leq \lambda \leq 2 \),

\[
E_\lambda = E_1 + E_2 + E_3 + E_4, \quad \lambda \leq |y| \leq \frac{1}{2}M_k^{\frac{2}{n-2}}.
\]
Proof. First by (24) we have
\[ \left| \nabla^l \left[ v_k^\lambda - (U^\lambda + F_\lambda^{(3)}) \right] \right| \leq CM_k^{-\frac{12}{n-2}} |y|^{2-n}, \quad l = 0, 1, 2, \] (37)
where \( F_\lambda^{(3)} \) is the Kelvin transformation of \( F^{(3)} \). Note that (37) holds over the whole \( \Sigma_\lambda \).
Similarly we can define \( F_\lambda^{(2)} \) as the Kelvin transformation of \( F^{(2)} \) and we shall use this expression:
\[ F_\lambda^{(2)} = -c(n)\bar{R}^{(2)}(\theta) f_2^\lambda(r) M_k^{-\frac{8}{n-2}}. \]
where
\[ \bar{R}^{(2)}(\theta) = \sum_{i<j} \partial_{ij} R \theta_i \theta_j + \sum_i \frac{\partial_i R}{2}(\theta_i^2 - \frac{1}{n}) = \sum_{i<j} \partial_{ij} R \theta_i \theta_j + \frac{1}{2} \sum_i (\partial_i R) \theta_i^2 + O(|W|^2). \]
and \( f_2^\lambda(r) = (\lambda^2 \theta) f_2(\lambda^2/r) \) is the Kelvin transformation of \( f_2 \). \( f_3^\lambda \) is understood similarly.
Since \( 0 \leq f_2(r), f_3(r) \leq C r \) for \( 0 \leq r \leq 1 \), we have
\[ |f_2^\lambda(r)| + |f_3^\lambda(r)| \leq C r^{1-n}, \quad F_\lambda^{(3)} = O(M_k^{-\frac{8}{n-2}}) r^{1-n}. \] (38)

Now we consider \( (\bar{b}_i \partial_i + \bar{d}_{ij} \partial_{ij}) v_k^\lambda \). Since \( U^\lambda \) is radially symmetric, we have, using (33), (32), (37),
\[ (\bar{b}_i \partial_i + \bar{d}_{ij} \partial_{ij}) v_k^\lambda = (\bar{b}_i \partial_i + \bar{d}_{ij} \partial_{ij}) F_\lambda^{(3)} + E_4 \]
\[ = -c(n) \left\{ (\bar{b}_i \partial_i + \bar{d}_{ij} \partial_{ij}) \bar{R}^{(2)}(\theta) \right\} f_2^\lambda(r) M_k^{-\frac{8}{n-2}} \]
\[ -c(n)(\bar{b}_i \partial_i + \bar{d}_{ij} \partial_{ij}) \sum_{p=1}^{l_3} \bar{R}_p^{(3)} e_{3p} f_3^\lambda M_k^{-\frac{10}{n-2}} + E_4. \]

For any smooth function \( a(\theta) \),
\[ \int_{S^{n-1}} (\bar{b}_i \partial_i + \bar{d}_{ij} \partial_{ij}) a(\theta) = \int_{S^{n-1}} (\Delta g_k - \Delta) a(\theta) = 0. \] (39)

Expanding \( \bar{b}_i(y) \) and \( \bar{d}_{ij}(y) \) to the fourth and the fifth order respectively and using (31), we have
\[ (\bar{b}_i \partial_i + \bar{d}_{ij} \partial_{ij}) \bar{R}^{(2)}(\theta) = \sum_{l=1}^{7} a_{l,k}(r) P_l(\theta) + O(M_k^{-\frac{12}{n-2}} r^4), \]
\[ (\bar{b}_i \partial_i + \bar{d}_{ij} \partial_{ij}) \sum_{p=1}^{l_3} \bar{R}_p^{(3)} e_{3p} = \sum_{l=1}^{5} b_{l,k}(r) P_l(\theta) + O(M_k^{-\frac{12}{n-2}} r^4), \]
where \( a_{l,k}(r) \) and \( b_{l,k}(r) \) are radial functions satisfying
\[
|a_{l,k}(r)| + |b_{l,k}(r)| = O(M_k^{\frac{8-2l}{n-2}} r) + O(M_k^{\frac{8}{n-2}} r^2),
\]
while \( P_l(\theta) \) is a homogeneous polynomial in \( \theta \) of degree \( l \) and is also independent of \( k \). Consequently, using (39), we have
\[
(\bar{b}_i \partial_i + \bar{d}_{ij} \partial_{ij})\tilde{R}^{(2)}(\theta) = \sum_{l=1}^{7} a_{l,k}(r) e_l(\theta) + O(M_k^{\frac{12}{n-2}} r^4),
\]
\[
(\bar{b}_i \partial_i + \bar{d}_{ij} \partial_{ij}) \sum_{p=1}^{l_3} \tilde{R}^{(3)}_p e_{3p} = \sum_{l=1}^{7} b_{l,k}(r) e_l(\theta) + O(M_k^{\frac{12}{n-2}} r^4),
\]
where \( e_l(\theta) \), independent of \( k \), is a homogeneous spherical harmonic of degree \( l \), and \( a_{l,k}(r) \) and \( b_{l,k}(r) \), independent of \( \theta \), satisfy (40). Consequently, using also (38),
\[
(\bar{b}_i \partial_i + \bar{d}_{ij} \partial_{ij}) v_k^\lambda = (\bar{b}_i \partial_i + \bar{d}_{ij} \partial_{ij}) F^{(3)}_\lambda + E_4 = E_3 + E_4.
\]
Similarly we can show that
\[
\left(\frac{\lambda}{|y|}\right)^{n+2} (\bar{b}_i (y^\lambda) \partial_i v_k(y^\lambda) + \bar{d}_{ij} (y^\lambda) \partial_{ij} v_k(y^\lambda)) = E_3 + E_4.
\]
We have discussed the minor terms in \( E_\lambda \), by (22), the main term in \( E_\lambda \) is
\[
\bar{c}(y) v_k^\lambda - \left(\frac{\lambda}{|y|}\right)^{n+2} \bar{c}(y^\lambda) v_k(y^\lambda).
\]
We shall use the following two expansions of \( \bar{c} \) according to circumstances. First we know
\[
\bar{c} = c(n) \sum_{l=2}^{7} \tilde{R}^{(l)}(\theta) r^l M_k^{\frac{4+2l}{n-2}} + c(n) \sum_{s=0}^{2} \tilde{R}^{(2s+2)} r^{2s+2} M_k^{\frac{8+4s}{n-2}} + O(M_k^{\frac{20}{n-2}} r^8), \quad \lambda < r < \frac{1}{2} M_k^{\frac{2}{n-2}}. \tag{41}
\]
On the other hand, using (25), (27) and the rate of \( |W| \),
\[
\bar{c} = c^{(3)}(3) + O(M_k^{\frac{12}{n-2}} r^4) + O(M_k^{\frac{14-4}{n-2}} r^3), \quad \lambda < r < \frac{1}{2} M_k^{\frac{2}{n-2}}. \tag{42}
\]
where
\[
c^{(3)} = c^{(2)} + c(n) \sum_{p=1}^{l_3} \tilde{R}^{(3)}_p e_{3p} r^3 M_k^{\frac{10}{n-2}}, \tag{43}
\]
Using (37), we have
\[ \hat{c}v_k^\lambda - (\frac{\lambda}{r})^{n+2}\hat{c}(y^\lambda)v_k(y^\lambda) \]
\[ = \hat{c}U^\lambda - (\frac{\lambda}{r})^{n+2}\hat{c}(y^\lambda)U(y^\lambda) + c^{(3)}F^{(3)}(y^\lambda) - (\frac{\lambda}{r})^{n+2}c(y^\lambda)F^{(3)}(y^\lambda) + E_4. \]  

(44)

To estimate the first two terms we use the expression (41) of \( \hat{c} \).

\[ \hat{c}U^\lambda - (\frac{\lambda}{r})^{n+2}\hat{c}(y^\lambda)U(y^\lambda) \]
\[ = c(n)U^\lambda \left( \sum_{l=2}^{7} \hat{R}^{(l)} r^l M_k \frac{\lambda^{4+2l}}{n-2} + \sum_{s=0}^{2} \hat{R}^{(2s+2)} M_k \frac{\lambda^{8+4s}}{n-2} r^{2s+2} \right) \]
\[ - c(n)\left( \frac{\lambda}{r} \right)^{n+2} U(y^\lambda) \left( \sum_{l=2}^{7} \hat{R}^{(l)} \lambda^2 r^l M_k \frac{\lambda^{4+2l}}{n-2} + \sum_{s=0}^{2} \hat{R}^{(2s+2)} M_k \frac{\lambda^{8+4s}}{n-2} \lambda^2 r^{2s+2} \right) \]
\[ + E_4 \]  
\[ = c(n)U^\lambda \sum_{s=0}^{2} \hat{R}^{(2s+2)} M_k \frac{\lambda^{8+4s}}{n-2} r^{2s+2} (1 - \left( \frac{\lambda}{r} \right)^{4s+8}) + E_2 + E_4. \]

For the third and the fourth terms of (44) we use (42), (43) and (38) to obtain
\[ c^{(3)}F^{(3)} - (\frac{\lambda}{r})^{n+2}c^{(3)}(y^\lambda)F^{(3)}(y^\lambda) \]
\[ = \left( c^{(3)} - \left( \frac{\lambda}{r} \right)^4 c^{(3)}(y^\lambda) \right) F^{(3)}_\lambda \]
\[ = \left( c^{(2)} - \left( \frac{\lambda}{r} \right)^4 c^{(2)}(y^\lambda) \right) F^{(2)}_\lambda + E_3 \]
\[ = -c(n)^2 \left( \sum_{i<j} \partial_{ij} R\theta_i\theta_j + \frac{1}{2} \sum_i \partial_{ii} R\theta_i^2 \right)^2 r^2 (1 - \left( \frac{\lambda}{r} \right)^8) f_2^2 M_k \frac{16}{n-2} \]
\[ + E_3 + E_4. \]

In the expansion of the product in the last equality, we use the fact that homogeneous polynomials of degree 2 and 3 are orthogonal to each other — when a term has average 0 on \( S^{n-1} \) it contributes a term \( E_3 \).

The term that needs to be evaluated is \( \left( \sum_{i<j} \partial_{ij} R\theta_i\theta_j + \frac{1}{2} \sum_i \partial_{ii} R\theta_i^2 \right)^2 \). It is elementary to verify the following identities:
\[ \frac{1}{|S^{n-1}|} \int_{S^{n-1}} \theta_i^2 \theta_j^2 = \frac{1}{n(n+2)} \quad i \neq j, \quad \frac{1}{|S^{n-1}|} \int_{S^{n-1}} (\theta_i)^4 = \frac{3}{n(n+2)}. \]
In the following we often write a polynomial $P(\theta)$ of degree less or equal to 7 as the sum of $\frac{1}{|S_{n-1}|} \int_{S_{n-1}} P(\theta)$ and $\sum_{p=1}^{7} C_p e_p(\theta)$ where $e_p(\theta)$ are homogeneous spherical harmonics of degree $p$.

By the above, we write
\[
\left( \sum_{i<j} \partial_{ij} R \theta_i \theta_j + \frac{1}{2} \sum_i \partial_{ii} R \theta_i^2 \right)^2
\]
\[
= \sum_{i<j} (\partial_{ij} R)^2 \theta_i^2 \theta_j^2 + \frac{1}{4} \left( \sum_i \partial_{ii} R \theta_i^2 \right)^2 + \sum_{p=1}^{7} C_p e_p(\theta)
\]
\[
= \sum_{i<j} (\partial_{ij} R)^2 \theta_i^2 \theta_j^2 + \frac{1}{4} \sum_{i \neq j} \partial_{ii} R \partial_{jj} R \theta_i^2 \theta_j^2 + \frac{1}{4} \sum_i (\partial_{ii} R)^2 \theta_i^4 + \sum_{p=1}^{7} C_p e_p(\theta)
\]
\[
= \frac{1}{n(n+2)} \sum_{i<j} (\partial_{ij} R)^2 + \frac{1}{4n(n+2)} \sum_{i \neq j} \partial_{ii} R \partial_{jj} R + \frac{3}{4n(n+2)} \sum_i (\partial_{ii} R)^2 + \sum_{p=1}^{7} C_p e_p.
\]

Note that
\[
\sum_{i \neq j} \partial_{ii} R \partial_{jj} R + \sum_{i=1}^{n} (\partial_{ii} R)^2 = (\sum_i \partial_{ii} R)^2 = O(|W|^4) = O(M_k^{-\frac{16}{n-2}}).
\]

We have
\[
\left( \sum_{i<j} \partial_{ij} R \theta_i \theta_j + \frac{1}{2} \sum_i \partial_{ii} R \theta_i^2 \right)^2
\]
\[
= \frac{1}{2n(n+2)} \left[ \sum_{i<j} 2(\partial_{ij} R)^2 + \sum_i (\partial_{ii} R)^2 \right] + O(M_k^{-\frac{16}{n-2}}) + \sum_{p=1}^{7} C_p e_p.
\]  

Thus
\[
c^{(3)} \hat{E}^{(3)} - \left( \frac{\lambda}{r} \right)^{n+2} c^{(3)} (y^\lambda) E^{(3)} (y^\lambda)
\]
\[
= -\frac{c(n)^2}{2n(n+2)} \left[ \sum_{i<j} 2(\partial_{ij} R)^2 + \sum_i (\partial_{ii} R)^2 \right] r^2 \left[ 1 - \left( \frac{\lambda}{r} \right)^8 f_2^2 M_k^{-\frac{16}{n-2}} + E_3 + E_4. \right.
\]

Proposition 2.2 follows from the above.  

2.4 Construction of auxiliary functions and proof of (15)

The goal of this subsection is to finish the proof of (15) by finding a contradiction to (20). Before we construct the auxiliary functions, we discuss two relatively minor terms in $E_1$. 
Recall that $\bar{R}^{(2)} = -\frac{W^2}{12n}$. The following properties of conformal normal coordinates are established in [11]: If $W = 0$, then $R_{abcd} = 0$ and, for some constant $c_1(n) > 0$, $\bar{R}^{(4)} = -c_1(n)|R_{abcd,e}|^2$; if $W = 0$ and $\nabla W = 0$, then $R_{abcd,e} = 0$. Examining the proofs there, we arrive at

$$|R_{abcd}| = O(1)|W|, \quad |R_{abcd,e}| = O(|W|) + O(\nabla g W),$$

and

$$\bar{R}^{(4)} = -c_1(n)|\nabla R_{abcd}|^2 + O(|W|)O(|\nabla R_{abcd}| + O(|W|^2)).$$

Thus

$$\bar{R}^{(4)} \leq -\frac{1}{2}c_1(n)|\nabla R_{abcd}|^2 + O(|W|^2) = -\frac{1}{2}c_1(n)|\nabla R_{abcd}|^2 + O(M_k^{-\frac{2n}{n-2}}). \quad (46)$$

So

$$E_1 \leq c(n)U^\lambda \bar{R}^{(6)}r^6 M_k^{-\frac{16}{n-2}}(1 - \left(\frac{\lambda}{r}\right)^{16})$$

$$-\frac{c(n)^2}{2n(n+2)} \left( \sum_{i<j} 2(\partial_{ij} R)^2 + \sum_i (\partial_i R)^2 \right) (1 - (\frac{\lambda}{r})^8)r^2 f_2^\lambda M_k^{-\frac{16}{n-2}}. \quad (47)$$

Recall that the most important requirement for the test function $h_\lambda$ is (23), for which we construct $h_\lambda$ as the sum of four test functions $h_1, \ldots, h_4$. Each of the first three functions is constructed with respect to $\Delta$ and $V_\lambda$ (a radial function, to be defined later), rather than $\Delta_{g_k} - \bar{c}$ and $\xi$. So even though each of them cancels a major part of $E_\lambda$, they also create some minor extra errors because of the difference between $\Delta, V_\lambda$ and $\Delta_{g_k} - \bar{c}, \xi$. Eventually all these new error terms will be put together and be controlled by $h_4$.

For the convenience of our discussion, we define

$$\bar{l} = 6 \text{ if } \quad n = 10, \quad \bar{l} = 7 \text{ if } \quad n = 11.$$  

To cancel the term $c(n)U^\lambda \bar{R}^{(6)} \sum_{l=2}^{\bar{l}} M_k^{-\frac{4+2l}{n-2}} r^l (1 - (\frac{\lambda}{r})^{2l+4})$ in $E_\lambda$ we use (26). The $f_{2,\lambda}$ defined in Appendix C is to deal with $\bar{R}^{(2)}$.

Let

$$V_\lambda(r) := n(n+2) \int_0^1 (tU + (1-t)U^\lambda)^{\frac{1}{n-2}} dt.$$  

For $3 \leq l \leq \bar{l}$, let $\lambda_p > 0$ be the eigenvalue corresponding to $e_p$, we consider

$$\left\{ \begin{array}{l}
\frac{f''_{p\lambda}(r)}{r} + \frac{n+2}{r^2} f'_{p\lambda}(r) + (V_\lambda - \frac{\lambda_p}{r^2}) f_{p\lambda}(r) = -r^l U^\lambda(r) (1 - (\frac{\lambda}{r})^{2l+4}), \\
\lambda < r < 2M_k^{\frac{2}{n-2}}, \quad 3 \leq l \leq \bar{l}.
\end{array} \right.$$  

$$f_{p\lambda}(\lambda) = f_{p\lambda}(2M_k^{\frac{2}{n-2}}) = 0. \quad (48)$$
By Proposition 6.1 in Appendix A of [13], there exists some small $\epsilon_4 = \epsilon_4(n) > 0$ such that for $\lambda \in [1 - \epsilon_4, 1 + \epsilon_4]$, equation (48) has a unique classical solution satisfying
\[0 \leq f_{p\lambda}(r) \leq Cr^{l+4-n}, \quad 3 \leq l \leq \bar{l}, \quad \lambda \leq r \leq 2M_k^{-\frac{2}{n-2}}.\]

Let
\[h_1 = c(n)\bar{R}^{(2)}(\theta)f_{2,\lambda}M_k^{-\frac{8}{n-2}}.\]
This is a major part of $h_\lambda$. Let
\[h_2 = c(n)\sum_{l=3}^{\bar{l}}\sum_{p=1}^{\tilde{l}}\bar{R}_{lp}e_p(\theta)f_{p\lambda}(r)M_k^{-\frac{\tilde{l} + 8}{n-2}},\]
where $\bar{R}_{lp}$ and $e_p(\theta)$ are the ones in (26). By the definitions of $h_1$ and $h_2$, we have
\[\Delta(h_1 + h_2) + V_\lambda(h_1 + h_2) = -E_2.\]

The extra error terms created by $h_1$ are
\[(b_1\partial_l + \tilde{d}_{ij}\partial_{ij} - \bar{c} + (n(n + 2)\xi^{-\frac{4}{n-2}} - V_\lambda))h_1.\]

We need to estimate the above in $O_\lambda$, note that by definition $h_1, h_2 = O(1)|y|^{2-n}$ in $\Sigma_\lambda$. For $(b_1\partial_l + \tilde{d}_{ij}\partial_{ij})h_1$, we just analyze it the same way as analyzing $(b_1\partial_l + \tilde{d}_{ij}\partial_{ij})F^{(3)}_\lambda$ to obtain
\[(b_1\partial_l + \tilde{d}_{ij}\partial_{ij})h_1 = \bar{E}_3 + E_4.\]

Here and in the following, $\bar{E}_3$ denotes a term of the form
\[\bar{E}_3 = \sum_{s=1}^{J}\tilde{c}_{s,k}(r)e_s\]
with $\tilde{c}_{s,k}(r)$ depending only on $r$ and satisfying
\[\tilde{c}_{s,k}(r) = O(M_k^{\frac{16-4}{n-2}r^{7-n}} + O(M_k^{-\frac{16}{n-2}r^{8-n}}).\] (49)

Next we consider $-\bar{c}h_1$. By the definition of $h_1$, (42) and (43) we have
\[-\bar{c}h_1 = -c^{(3)}h_1 + O(M_k^{-\frac{20}{n-2}r^{10-n}}).\]

The major part to contribute to $E_1$ is, using (45),
\[-c^{(2)}h_1 = -c(n)^2\left(\sum_{i<j} \partial_{ij}R\theta_i\theta_j + \frac{1}{2} \sum_i \partial_{ii}R\theta_i^2\right) r^2f_{2,\lambda}M_k^{-\frac{16}{n-2}} + E_4 = -c(n)^2\left(\sum_{i<j} 2(\partial_{ij}R)^2 + \sum_i (\partial_{ii}R)^2\right) r^2f_{2,\lambda}M_k^{-\frac{16}{n-2}} + \bar{E}_3 + E_4.\]
Since $\tilde{R}^{(2)}(\theta)$ is orthogonal to $e_{3p}(\theta)$

$$(c^{(3)} - c^{(2)})h_1 = \tilde{E}_3. \quad (50)$$

For $-\tilde{c}h_1 + (n(n + 2)\xi_{\frac{4}{n-2}} - V_\lambda)h_1$ we use (24) over $|y| \leq M_k^{(n-2)^2}$ and

$$n(n + 2)\xi_{\frac{4}{n-2}} - V_\lambda = O(r^{-4}), \quad r \in (M_k^{\frac{16-n}{(n-2)^2}}, \frac{1}{\sqrt{k}}M_k^{\frac{2}{(n-2)^2}}).$$

First for $r \leq M_k^{\frac{16-n}{(n-2)^2}}$, using (24) and (37),

$$n(n + 2)\xi_{\frac{4}{n-2}} = n(n + 2)\int_0^1 (t(U + a) + (1 - t)(U^\lambda + b))^{\frac{4}{n-2}} dt$$

$$= V_\lambda + \frac{4n(n + 2)}{n - 2} \int_0^1 (tF^{(3)} + (1 - t)F^{(3)}_\lambda)(tU + (1 - t)U^\lambda)^{\frac{6-n}{2}} dt$$

$$+ O(M_k^{\frac{-12}{n-2} - r^{2-\alpha}}), \quad \lambda \leq r \leq M_k^{\frac{16-n}{(n-2)^2}}, \quad (51)$$

where $a := v_k - U = F^{(3)} + O(M_k^{\frac{-12}{n-2} - r^{2-n}})$ and $b := v_k^\lambda - U^\lambda = F^{(3)}_\lambda + O(M_k^{\frac{-12}{n-2} - r^{2-n}})$.

Since $\tilde{R}^{(2)}(\theta)$ is orthogonal to $e_{3p}(\theta)$

$$\left( \int_0^1 (tF^{(3)} + (1 - t)F^{(3)}_\lambda - tF^{(2)} - (1 - t)F^{(2)}_\lambda)(tU + (1 - t)U^\lambda)^{\frac{6-n}{2}} dt \right)h_1$$

$$= \tilde{E}_3.$$

Clearly,

$$(tF^{(2)} + (1 - t)F^{(2)}_\lambda)(tU + (1 - t)U^\lambda)^{\frac{6-n}{2}}h_1 \leq 0.$$ 

So

$$\left( n(n + 2)\xi_{\frac{4}{n-2}} - V_\lambda \right)h_1 \leq \tilde{E}_3 + E_4, \quad \lambda \leq M_k^{\frac{16-n}{(n-2)^2}}.$$

We use

$$|(n(n + 2)\xi_{\frac{4}{n-2}} - V_\lambda)h_1| \leq O(M_k^{\frac{-8}{n-2} - r^{2-n}}) = E_4, \quad M_k^{\frac{16-n}{(n-2)^2}} \leq r \leq \frac{1}{\sqrt{k}}M_k^{\frac{2}{(n-2)^2}}.$$ 

Now we estimate

$$\tilde{b}_i \partial_i + \tilde{d}_{ij} \partial_{ij} - \tilde{c} + n(n + 2)\xi_{\frac{4}{n-2}} - V_\lambda)h_2.$$

As usual,

$$(\tilde{b}_i \partial_i + \tilde{d}_{ij} \partial_{ij})h_2 = \tilde{E}_3,$$
\[-c h_2 = -c^{(2)} h_2 + E_4 = -c^{(2)} c(n) \left( \sum_{p=1}^{l_1} \tilde{R}_{ip} e_p(\theta) f_{p\lambda}(r) M_k^{-\frac{4+2i}{n-2}} \right) \bigg|_{l=3} + E_4.\]

Compare (27) and (26), we deduce from above, using the orthogonality of \( \tilde{R}^{(2)} \) and \( e_{3p}(\theta) \) and the decay of \( W \),

\[-c h_2 = \bar{E}_3 + E_4. \quad (52)\]

By (51)

\[
n(n+2)\xi^{\frac{4}{n-2}} - V_\lambda = \frac{4n(n+2)}{n-2} \int_0^1 (t F^{(2)} + (1-t) F^{(2)}_\lambda)(t U + (1-t) U^\lambda)^{\frac{6-n}{2}} \, dt + O(M_k^{-\frac{10}{n-2}} r) + O(M_k^{-\frac{12}{n-2}} r^{2+\eta}), \quad \lambda < r < M_k^{\frac{16-c}{(n-2)^2}}.
\]

Therefore, as in the derivation of (52),

\[
(n(n+2)\xi^{\frac{4}{n-2}} - V_\lambda)h_2 = \frac{4n(n+2)}{n-2} h_2 \int_0^1 \left( t F^{(2)} + (1-t) F^{(2)}_\lambda \right) (t U + (1-t) U^\lambda)^{\frac{6-n}{2}} \, dt + E_4
\]

\[
= \frac{4n(n+2)}{n-2} \left( c(n) \sum_{p=1}^{l_1} \tilde{R}_{ip} e_p(\theta) f_{p\lambda}(r) M_k^{-\frac{4+2i}{n-2}} \right) \bigg|_{l=3}
\]

\[
\cdot \int_0^1 \left( t F^{(2)} + (1-t) F^{(2)}_\lambda \right) (t U + (1-t) U^\lambda)^{\frac{6-n}{2}} \, dt + E_4
\]

\[
= \bar{E}_3 + E_4, \quad \lambda < r < M_k^{\frac{16-c}{(n-2)^2}}.
\]

The first term on the right is of the form \( \sum s=1^J \tilde{c}_{s,k}(r) e_s \) and \( \tilde{c}_{s,k}(r) = O(M_k^{-\frac{18}{n-2}} r^7-n) \). So after this term is extended to \( r \leq \frac{1}{\sqrt{k}} M_k^{\frac{2}{n-2}} \), it can be combined with \( E_3 \). The extended part has a good decay.

For \( M_k^{\frac{16-c}{(n-2)^2}} \leq r \leq \frac{1}{\sqrt{k}} M_k^{\frac{2}{n-2}} \), we have

\[
|n(n+2)\xi^{\frac{4}{n-2}} - V_\lambda| h_2 | \leq O(M_k^{-\frac{10}{n-2}} r^{3-n}) = E_4.
\]

Recall that our purpose is to obtain (23). By putting \( h_1 \) and \( h_2 \) together and using (47), we have

\[
(\Delta + \bar{b}_i \partial_i + \bar{d}_{ij} \partial_{ij} - \bar{c} + n(n+2)\xi^{\frac{4}{n-2}}) (h_1 + h_2) + E_1 + E_2 \leq c(n) U^\lambda \tilde{R}^{(6)} M_k^{-\frac{16}{n-2}} \, r^6 (1 - (\frac{\lambda}{r})^{16})
\]
\[
- \frac{c(n)^2}{2n(n+2)} \sum_{i,j} (\partial_{ij} R)^2 r^2 \left( (1 - (\frac{\lambda}{r})^8) f_2^\lambda + f_{2,\lambda} \right) M_k^{-\frac{16}{r^2}} + \bar{E}_3 + E_4. \tag{53}
\]

For \( \bar{R}(6) \) we use (63), we also know the lower bounds for \( f_2 \) and \( f_{2,\lambda} \), respectively (see (74) and (75)). These three estimates are in the appendix and are sufficient for \( h_1 + h_2 \) to cancel the major part of \( E_\lambda \). In fact, first by (74) and (75)

\[
\frac{c(n)}{2n(n+2)} (1 - (\frac{\lambda}{r})^8) f_2^\lambda + f_{2,\lambda} \leq U \lambda^4 (1 - (\frac{\lambda}{r})^{16} \frac{1}{8(n+4)(n+2)n} (\frac{n-8}{n-2} - \frac{49}{20n^2} + \epsilon)
\]

where we have used the following inequality that holds only for \( n = 10, 11 \).

\[
\frac{1}{8(n+4)(n+2)n} (\frac{n-8}{n-2} - \frac{49}{20n^2} + \epsilon) \leq \frac{c(n)}{2n(n+2)} \frac{1}{6(n-4)},
\]

Thus, by using (63), we deduce from (53) that

\[
(\Delta + V_\lambda) h_3 = - \bar{E}_3.
\]

Next we construct test functions to control \( \bar{E}_3 \) and \( E_4 \). The \( h_3 \) to be constructed later will create much minor error terms than before. Then eventually all the minors terms will be controlled by \( h_4 \). Let \( \tilde{f}_{s,\lambda} \) be the solution of

\[
\begin{cases}
\tilde{f}''_{s,\lambda}(r) + \frac{n-1}{r} \tilde{f}_{s,\lambda}(r) + (V_\lambda - \frac{\lambda}{r^8}) \tilde{f}_{s,\lambda}(r) = -\tilde{c}_{s,k}(r), \quad \lambda < r < M_k^{-\frac{2}{r^2}} , \\
\tilde{f}_{s,\lambda}(\lambda) = \tilde{f}_{s,\lambda}(M_k^{-\frac{2}{r^2}}) = 0,
\end{cases}
\]

and let

\[
h_3 := \sum_{s=1}^f \tilde{f}_{s,\lambda}(r) e_s.
\]

By (49), \( \tilde{c}_{s,k}(r) = O(M_k^{-\frac{14}{r^2}} r^{7-n}) \). Consequently \( |\tilde{f}_{s,\lambda}(r)| \leq C M_k^{-\frac{14}{r^2}} r^{9-n} \). Therefore

\[
(\Delta + V_\lambda) h_3 = - \bar{E}_3.
\]

By the estimates of \( \bar{b}_i, \bar{d}_{ij} \) and \( h_3 \), etc we obtain

\[
(\bar{b}_i \partial_i + \bar{d}_{ij} \partial_{ij} - \bar{c} + n(n+2) \xi^{\frac{4}{n-2}} - V_\lambda) h_3 = E_4.
\]
Finally we define, for $Q >> 1$ that

$$h_4(r) = \begin{cases} QM_k^{-\frac{18-2\sqrt{\epsilon}}{n-2}} f_{n,n-9+\sqrt{\epsilon}}(\frac{r}{k}), & n = 10, \\ QM_k^{-\frac{20-2\sqrt{\epsilon}}{n-2}} f_{n,n-10+\sqrt{\epsilon}}(\frac{r}{k}), & n = 11. \end{cases}$$

where $f_{n,\alpha}$ is defined in [13]. Let $h_{\lambda} := h_1 + h_2 + h_3 + h_4$, then (23) is obtained. This $h_{\lambda}$ satisfies all the requirements for the test function to make the method of moving spheres work. Then the standard moving sphere argument leads to the following conclusion:

$$\min_{|y| \leq r} v_k \leq (1 + \epsilon)U(r), \quad 0 \leq r \leq \frac{1}{\sqrt{k}}M_k^{\frac{n-2}{n-2}},$$

where $\epsilon$ is an arbitrary small positive constant. Then following the argument in [13] one gets a contradiction to (20). (15) is established.

### 2.5 Vanishing rates of the Weyl tensor and the completion of the proof of Theorem 1.2

In this subsection we use (15) to prove (16) and (17), the vanishing rates of the Weyl tensor and its covariant derivatives at the blow up point. By (15),

$$v_k(y) \leq CU(y), \quad |y| \leq \delta M_k^{\frac{n-2}{n-2}}.$$  \hfill (54)

This estimate leads to an improved estimate of $v_k$ than that in Proposition 2.2.

**Proposition 2.3** There exists $\delta' > 0$, independent of $k$, such that

$$\left| \nabla^l \left( v_k - (v^{(1)} + M_k^{-\frac{8}{n-2}}v^{(2)} + M_k^{-\frac{10}{n-2}}v^{(3)}) \right) \right| = O(M_k^{-\frac{12}{n-2}})(1 + |y|)^{8-n-l}, \quad |y| \leq \delta' M_k^{\frac{n-2}{n-2}}, \quad l = 0, 1, 2. \hfill (55)$$

**Proof.** Write

$$E^{(3)} := v_k - (U + F^{(3)}).$$

We only need to prove that

$$|\nabla^l E^{(3)}| = O(M_k^{-\frac{12}{n-2}})(1 + |y|)^{8-n-l}, \quad |y| \leq \delta' M_k^{\frac{n-2}{n-2}}, \quad l = 0, 1, 2. \hfill (55)$$

It follows from (21) and (29) that

$$(\Delta_{g_k} - \tilde{\epsilon})E^{(3)} + n(n+2)\xi^{\frac{1}{n-2}}E^{(3)} = O(M_k^{-\frac{12}{n-2}})(1 + r)^{6-n}, \quad 0 < r < M_k^{\frac{n-2}{n-2}}, \hfill (56)$$
where
\[ \xi \frac{n-2}{n} (y) = \int_0^1 (tv_k + (1-t)(U + F^{(3)}))^\frac{4}{n-2} dt. \]

Arguing as on page 212 of [13], we see that the operator \( \Delta g_k - \tilde{c} + n(n+2) \xi \frac{n-2}{n} \) satisfies the maximum principle over \( R_1 < |y| < \delta' M_k^{-\frac{2}{n-2}} \) for some constants \( R_1, \delta' > 0 \) which are independent of \( k \). For \( C_{10} \) large, but independent of \( k \), we see, using (56), (24) and (54), that
\[
\left( \Delta g_k - \tilde{c} + n(n+2) \xi \frac{n-2}{n} \right) \left( E^{(3)} - f \right) \geq 0, \quad R_1 < |y| < \delta' M_k^{-\frac{2}{n-2}},
\]
\[
\left( E^{(3)} - f \right) (y) < 0 \quad \text{on} \quad \{ r = R_1 \} \cup \{ r = \delta' M_k^{-\frac{2}{n-2}} \},
\]
where \( f(r) := C_{10}M_k^{-\frac{n-2}{2}}r^{8-n} \). Thus, in view of (24), estimate (55) for \( l = 0 \) follows from the maximum principle. The estimate for \( l = 1,2 \) can then be deduced from the equation satisfied by \( E^{(3)} \) using elliptic estimates. \( \square \)

Recall the Pohozaev type identity (102) in [13], with \( R'_k = \delta' M_k^{-\frac{2}{n-2}} \),
\[
I_1[v_k] + I_2[v_k] + I_3[v_k] + I_4[v_k] = I_5[v_k],
\]
\[
I_1[v_k] = \int_{|y| \leq R'_k} \left( -\tilde{b}_i \partial_i v_k - \tilde{d}_{ij} \partial_i v_k \right) \left( \nabla v_k \cdot y + \frac{n-2}{2} v_k \right),
\]
\[
I_2[v_k] = -\frac{c(n)}{2} M_k^{-\frac{n-2}{2}} \int_{|y| \leq R'_k} \left( (M_k^{-\frac{2}{n-2}} y) \cdot \nabla R(M_k^{-\frac{2}{n-2}} y) + 2R(M_k^{-\frac{2}{n-2}} y) \right) v_k^2(y),
\]
\[
I_3[v_k] = \frac{c(n)}{2} M_k^{-\frac{n-2}{2}} R'_k \int_{|y| = R'_k} R(M_k^{-\frac{2}{n-2}} y) v_k^2(y),
\]
\[
I_4[v_k] = -\frac{(n-2)^2}{2} R'_k \int_{|y| = R'_k} v_k(y) \frac{2n}{n-2},
\]
\[
I_5[v_k] = \int_{|y| = R'_k} \left( \left| \frac{\partial v_k}{\partial \nu} \right|^2 - \frac{1}{2} |\nabla v_k|^2 \right) R'_k + \frac{n-2}{2} v_k \frac{\partial v_k}{\partial \nu} \right) = O(M_k^{-2}).
\]

Write
\[
\nabla v_k \cdot y + \frac{n-2}{2} v_k = \tilde{U} + \tilde{F}^{(3)} + \tilde{E}^{(3)}
\]
where
\[
\tilde{U} = \nabla U \cdot y + \frac{n-2}{2} U, \quad \tilde{F}^{(3)} = \nabla F^{(3)} \cdot y + \frac{n-2}{2} F^{(3)}.
\]

Clearly,
\[
|\nabla^l \tilde{F}^{(3)}| = O(M_k^{-\frac{2}{n-2}})(1 + r)^{6-n-l}, \quad l = 0,1,2,
\]
(58)
and
\[ |\nabla^l \tilde{E}^{(3)}| = O(M_k^{-\frac{12}{n-2}} |y|^{8-n-l}), \quad l = 0, 1, 2. \] (59)

With these and (32),(55),(58),(59), we have
\[
I_1[v_k] = \int_{B(0,M_k^{-\frac{2}{n-2}})} (-\tilde{b}_i \partial_i - \tilde{d}_{ij} \partial_{ij})(v_k - U)(\nabla v_k \cdot y + \frac{n-2}{2} v_k) dy
= \int_{B(0,M_k^{-\frac{2}{n-2}})} (\Delta - \Delta_{g_k})(F^{(3)} + E^{(3)}) \cdot (\tilde{U} + \tilde{F}^{(3)} + \tilde{E}^{(3)})
= \int_{B(0,M_k^{-\frac{2}{n-2}})} (\Delta - \Delta_{g_k})F^{(3)} \tilde{U} + O(M_k^{-2}).
\]

Since \( \tilde{U} \) is radially symmetric, \( \int_{B(0,M_k^{-\frac{2}{n-2}})} (\Delta - \Delta_{g_k})F^{(3)} \tilde{U} = 0 \). Thus \( I_1[v_k] = O(M_k^{-2}) \), and with notation in (26),
\[
I_2[v_k] = -\frac{c(n)}{2} \int_{|y| \leq R_k} \left\{ \sum_{l=2}^{\infty} \sum_{|\alpha| = l} \int_{|y| \leq R_k} \left\{ \left( \frac{l+2}{\alpha!} \partial_\alpha R g^\alpha M_k^{-\frac{4+4l}{n-2}} \right) v_k^2 + O(M_k^{-2}) \right\} \right\} \cdot \left( U^2 + 2U(F^{(3)} + E^{(3)}) + (F^{(3)} + E^{(3)})^2 \right) + O(M_k^{-2}).
\]

Using Lemma 3.2 and (46), and for small \( \epsilon' \), we have
\[
-\frac{c(n)}{2} \int_{|y| \leq R_k} \left\{ \sum_{s=0}^{2} (2s + 4) \tilde{R}^{(2s+2)} M_k^{-\frac{4+4s}{n-2}} r^{2s+2} \right\} \left( U^2 + 2U(F^{(3)} + E^{(3)}) + (F^{(3)} + E^{(3)})^2 \right) \geq c_1(n) |W|^2 M_k^{-\frac{8}{n-2}} + c_2(n) |\nabla R_{abcd}|^2 M_k^{-\frac{12}{n-2}}
+ 4c(n) \left( \epsilon |\nabla^2 R_{abcd}|^2 - \frac{1}{8(n+4)(n+2)n} (\frac{n-8}{n-2} - \frac{49}{20n^2} + \epsilon') |\nabla^2 R|^2 \right) \left( \frac{r^6 U^2 dy M_k^{-\frac{12}{n-2}}} + \frac{1}{\epsilon} \left( |W|^2 M_k^{-\frac{4}{n-2}} + |\nabla R_{abcd}|^2 M_k^{-\frac{12}{n-2}} \right) \right). \] (60)

where \( c_1(n), c_2(n), \epsilon \) and \( \epsilon' \) are some positive constants depending only on \( n \). \( \epsilon \) and \( \epsilon' \) are sufficiently small. Also we observe that
\[
-\frac{c(n)}{2} \int_{|y| \leq R_k} \left( \sum_{l=2}^{\infty} \sum_{|\alpha| = l} \int_{|y| \leq R_k} (l+2) \tilde{R}_{ip} e^p \| M_k^{-\frac{4+4l}{n-2}} \right) U^2 = 0.
\]
We only need to deal with

\[-\frac{c(n)}{2} \int_{|y| \leq R'_k} \left( \sum_{l=1}^{7} \sum_{p=1}^{l} (l + 2) \tilde{R}_{lp} e_p r^l M_k^{\frac{4+2l}{n-2}} + \sum_{s=0}^{2} (2s + 4) \tilde{R}^{(2s+2)} M_k^{-\frac{8+4s}{n-2}} r^{2s+2} \right) \cdot \left( 2U(F^{(3)} + E^{(3)}) + (F^{(3)} + E^{(3)})^2 \right).\]

By previous estimates

\[-\int_{|y| \leq R'_k} \left( \sum_{l=1}^{7} \sum_{p=1}^{l} (l + 2) \tilde{R}_{lp} e_p r^l M_k^{\frac{4+2l}{n-2}} \right) (2U(F^{(3)} + E^{(3)}) + (F^{(3)} + E^{(3)})^2) = O(M_k^{-2}).\]

\[-\int_{|y| \leq R'_k} \left( \sum_{p=1}^{l_2} \tilde{R}_{2p} e_p r^2 M_k^{-\frac{2}{n-2}} \right) (2UE^{(3)} + (F^{(3)} + E^{(3)})^2) = O(M_k^{-2}).\]

Finally \(-\frac{c(n)}{2} \int_{|y| \leq R'_k} \sum_{l=2}^{7} \sum_{p=1}^{l} (l + 2) \tilde{R}_{lp} e_p r^l M_k^{\frac{4+2l}{n-2}} 2UF^{(3)}\) contributes another important term.

\[-\frac{c(n)}{2} \int_{|y| \leq R'_k} \sum_{l=1}^{7} \sum_{p=1}^{l} (l + 2) \tilde{R}_{lp} e_p r^l M_k^{\frac{4+2l}{n-2}} \cdot 2UF^{(3)}\]

\[= -c(n) \int_{|y| \leq R'_k} \left( \sum_{p=1}^{l_2} 4\tilde{R}_{2p} e_p r^2 M_k^{-\frac{2}{n-2}} + \sum_{l=3}^{7} \sum_{p=1}^{l} (l + 2) \tilde{R}_{lp} e_p r^l M_k^{\frac{4+2l}{n-2}} \right) U(F^{(2)} + F^{(3)} - F^{(2)}).\]

Using the fact the eigenfunction corresponding to \(l = 2\) are orthogonal to those corresponding to \(l = 3\), we have

\[-\frac{c(n)}{2} \int_{|y| \leq R'_k} \sum_{l=1}^{7} \sum_{p=1}^{l} (l + 2) \tilde{R}_{lp} e_p r^l M_k^{\frac{4+2l}{n-2}} 2UF^{(3)}\]

\[= -\frac{4c(n)}{n(n+2)} \int_{|y| \leq R'_k} \tilde{R}^{(2)}(\theta) r^2 UF^{(2)} M_k^{-\frac{8}{n-2}} + O(M_k^{-2}) + o(1)M_k^{-\frac{8}{n-2}} |W|^2\]

\[= \frac{2c(n)^2}{n(n+2)} \left( \sum_{i<j} (\partial_{ij} R)^2 + \sum_{i} (\partial_i R)^2 \right) \int_{|y| \leq R'_k} r^2 Uf_2 dy M_k^{-\frac{16}{n-2}} + O(M_k^{-2})\]

\[+ o(1)M_k^{-\frac{8}{n-2}} |W|^2\]

\[\geq \frac{2c(n)^2}{n(n+2)} \int_{|y| \leq R'_k} r^2 Uf_2 dy M_k^{-\frac{16}{n-2}} + O(M_k^{-2}) + o(1)M_k^{-\frac{8}{n-2}} |W|^2\]  \(\text{(61)}\)
The Pohozaev type identity (57) yields, in view of (60), (61) and the lower bound of $f_2$ in (74), that

$$|W|^2 M_k^{-\frac{n-2}{n-4}} + |\nabla R_{abcd}|^2 M_k^{-\frac{n-2}{n-4}} + |\nabla^2 R_{abcd}|^2 M_k^{-\frac{n-2}{n-4}} \log M_k = O(M_k^{-2}), \quad n = 10.$$  

$$|W|^2 M_k^{-\frac{n-2}{n-4}} + |\nabla R_{abcd}|^2 M_k^{-\frac{n-2}{n-4}} + |\nabla^2 R_{abcd}|^2 M_k^{-\frac{n-2}{n-4}} = O(M_k^{-2}), \quad n = 11.$$  

Thus we have proved (16) and (17). Estimates (18) and (19) follow from (55). Theorem 1.2 is established. □

3 Appendix A: Some curvature inequalities in conformal normal coordinates

3.1 The inequality for $\hat{R}^{(6)}$

In this subsection we prove the following two lemmas.

**Lemma 3.1** If $|W(0)| = |\nabla W(0)| = 0$, then we have, in conformal normal coordinates centered at 0,

$$\hat{R}^{(6)} = -\frac{R_{p1p2p3p4p5p6}R_{p1p2p3p4,p5p6}}{40(n+4)(n+2)n} - \frac{R_{p1p2,p3p4}(R_{p1p2,p3p4} + R_{p3p4,p1p2})}{8(n+4)(n+2)n}$$

$$+ \frac{\sum_{p1p2}(\partial_{p1p2}R)^2}{8(n+4)(n+2)n}, \text{ at } 0,$$

where repeated indices mean summation, and $R_{ijkl,pq}$ denotes covariant derivatives of $R_{ijkl}$.

**Lemma 3.2** For some small $\epsilon = \epsilon(n) > 0$, we have, in conformal normal coordinates centered at 0,

$$\hat{R}^{(6)} < -\frac{\epsilon R_{p1p2p3p4,p5p6}R_{p1p2p3p4,p5p6}}{1} + \frac{1}{8(n+4)(n+2)n} \left( \frac{n-8}{n-2} - \frac{49}{20n^2} + \epsilon \right) \sum_{p1p2}(\partial_{p1p2}R)^2 + O(|\nabla R_{abcd}|^2 + |W|^2).$$

We first assume Lemma 3.1 and give the proof of Lemma 3.2.

**Proof of Lemma 3.2:**

It was proved by Hebey and Vaugon in [11] that, if $|W(0)| = |\nabla W(0)| = 0$, then, in conformal normal coordinates centered at 0,

$$R_{p1p2,p3p4}(R_{p1p2,p3p4} + R_{p3p4,p1p2}) \geq \frac{6}{n-2} \sum_{p1p2}(\partial_{p1p2}R)^2, \text{ at } 0.$$
We also need the following inequality under the same assumption:

\[
R_{p_1p_2p_3p_4p_5p_6}R_{p_1p_2p_3p_4p_5p_6} \geq \frac{49}{4n^2} \sum_{p_1p_2} (\partial_{p_1p_2} R)^2, \quad \text{at} \quad 0. \tag{65}
\]

Note that (65) with \(\frac{49}{4n^2}\) replaced by \(\frac{9}{n^2}\) was established in [11]. This weaker version leads to an inequality weaker than (3.2), which is nevertheless enough for applications in this paper. To prove (65), we consider

\[
||R_{ikmjq} - \alpha R_{ij}\delta_{kp}\delta_{mq}||^2 > 0.
\]

Namely,

\[
|\nabla_{pq} R_{ikm}|^2 - 2\alpha R_{ikm,km} R_{ij} + \alpha^2 n^2 R_{ij} R_{ij} \geq 0.
\]

By the second Bianchi identity, \( R_{ikm,k} = R_{im,j} - R_{ij,m} \). So

\[
R_{ikm,km} = R_{im,jm} - R_{ij,mm} = \frac{1}{2} R_{ij} + 3R_{ij} = \frac{7}{2} R_{ij}
\]

It follows that

\[
|\nabla_{pq} R_{ikm}|^2 + (\alpha^2 n^2 - 7\alpha) R_{ij} R_{ij} \geq 0.
\]

Inequality (65) follows from the above by taking \(\alpha = \frac{7}{2n^2}\).

By (62), (64) and (65) we have

\[
\bar{R}^{(6)} < \frac{1}{8(n+4)(n+2)n} \left( \frac{n-8}{n-2} - \frac{49}{20n^2} \right) \sum_{p_1p_2} (\partial_{p_1p_2} R)^2, \quad \text{at} \quad 0.
\]

Then (28) holds for some small \(\epsilon(n) > 0\) under the assumption \(|W(0)| = |\nabla W(0)| = 0\). In general if we do not assume \(|W(0)| = |\nabla W(0)| = 0\), all the extra terms can be estimated by Cauchy’s inequality, and we obtain (63). Lemma 3.2 is established. □

**Proof of Lemma 3.1:**

It was proved in [11] that if \(|W(0)| = |\nabla W(0)| = 0\), then, in conformal normal coordinates centered at 0,

\[
C(2, 2) \text{Sym}_{p_1...p_6} R_{p_1...p_6} + 864R_{p_1p_2p_3p_4p_5p_6} R_{p_1p_2p_3p_4p_5p_6} + 4320R_{p_1p_2p_3p_4}(R_{p_1p_2p_3p_4} + R_{p_3p_4p_1p_2}) - 4320 \sum_{p_1p_2} (\partial_{p_1p_2} R)^2 = 0,
\]

where \(C(2, 2)\) is the complete contraction:

\[
C(2, 2) = \sum_{p_1=p_2=1}^{n} \sum_{p_3=p_4=1}^{n} \sum_{p_5=p_6=1}^{n}.
\]
Since we work in conformal normal coordinates and since $|W(0)| = |\nabla W(0)| = 0$,

$$\text{Sym}_{p_1 \ldots p_6} R_{p_1 \ldots p_6} = \text{Sym}_{p_1 \ldots p_6} \partial_{p_1 \ldots p_6} R - \frac{36}{5} \text{Sym}_{p_1 \ldots p_6} R_{p_p p_2 p_3 p_4 p_5 p_6} = \text{Sym}_{p_1 \ldots p_6} \partial_{p_1 \ldots p_6} R,$$

where, for the second equality, we have used the skew-symmetry of $R_{abcd}$.

Thus we have

$$C(2, 2) \text{Sym}_{p_1 \ldots p_6} R_{p_1 \ldots p_6} = 720 \Delta^3 R(0),$$

where $\Delta$ denotes the flat Laplacian.

Therefore

$$\Delta^3 R(0) + \frac{6}{5} R_{p_1 p_2 p_3 p_4 p_5 p_6} R_{p_1 p_2 p_3 p_4 p_5 p_6} + 6 R_{p_1 p_2 p_3 p_4} (R_{p_1 p_2 p_3 p_4} + R_{p_3 p_4 p_1 p_2})$$

$$- 6 \sum_{p_1 p_2} (\partial_{p_1 p_2} R)^2 = 0.$$  \hspace{1cm} (66)

By some standard computations,

$$\tilde{R}^{(6)} = \frac{1}{|S^{n-1}|} \int_{S^{n-1}} \sum_{|\alpha| = 6} \frac{\partial_\alpha R g^\alpha}{\alpha!} = \frac{\Delta^3 R(0)}{48(n + 4)(n + 2)n}.  \hspace{1cm} (67)$$

It follows from (66) and (67) that

$$\tilde{R}^{(6)} = - \frac{R_{p_1 p_2 p_3 p_4 p_5 p_6} R_{p_1 p_2 p_3 p_4 p_5 p_6}}{40(n + 4)(n + 2)n} - \frac{R_{p_1 p_2 p_3 p_4} (R_{p_1 p_2 p_3 p_4} + R_{p_3 p_4 p_1 p_2})}{8(n + 4)(n + 2)n}$$

$$- \sum_{p_1 p_2} (\partial_{p_1 p_2} R)^2$$

$$8(n + 4)(n + 2)n$$

at 0 where $|W(0)| = |\nabla W(0)| = 0$ is assumed. Lemma 3.1 is established.  \hspace{1cm} □

### 3.2 Proof of Lemma 2.1

The following fact is elementary: Let $k \geq 1$ be an integer, $j \in [1, n]$ be a fixed integer, then

$$\int_{S^{n-1}} \partial_{p_1 \ldots p_{2k+1}} R(0) x^{p_1} \ldots x^{p_{2k+1}} \cdot x^j dx = C(n, k) \partial_j (\Delta^k R)(0)$$  \hspace{1cm} (68)

where

$$C(n, k) = \frac{(2k + 1)! |S^{n-1}|}{(2k + n) 2^k k! \prod_{i=0}^{k-1} (n + 2i)}.$$  

**Proof of Lemma 2.1:** In conformal normal coordinates,

$$\text{Sym}_{p_1 \ldots p_{2k+3}} R_{p_1 p_2 p_3 \ldots p_{2k+3}} = 0, \quad \omega + 2 \leq 2k + 3 \leq 2\omega + 3,$$
if $|\nabla^i R_{abcd}(0)| = 0$ for $0 \leq i \leq \omega - 1$. See [11]. After contraction this implies
\[ \partial_j(\Delta^k)R(0) = 0 \quad j = 1, \ldots, n, \quad \text{if} \quad |\nabla^i R_{abcd}(0)| = 0 \quad \text{for} \quad 0 \leq i \leq \omega - 1. \]

For $n = 10, 11$, we only need to discuss $k = 1$, i.e., we have $\partial_j(\Delta R)(0) = 0$ if $|W(0)| = 0$. In general we have
\[ \partial_j(\Delta R)(0) = O(|W|) \]
This and (68) imply
\[ \int_{S^{n-1}} \tilde{R}^{(3)}(\theta)\theta^j = O(|W|), \quad 1 \leq j \leq n. \]

On the other hand, it is clear that
\[ \int_{S^{n-1}} \tilde{R}^{(3)}(\theta) = 0, \quad \int_{S^{n-1}} \tilde{R}^{(3)}(\theta)\theta^i\theta^j = 0 \quad 1 \leq i, j \leq n. \]

Lemma 2.1 follows from the above. \(\square\)

4 Appendix B: Some estimates on an ODE

**Proposition 4.1** Let $n \geq 3$ be an integer, $\delta_0 \geq n$ be a constant, let $\tilde{H}(r) \in C^0(0, \infty)$ satisfy, for some positive constants $C, \beta$ and $\alpha > 2$, $\delta_0 + (\alpha - 2)(n - \alpha) > 0$,
\[ 0 \leq \tilde{H}(r) \leq Cr^\beta(1 + r)^{-\beta - \alpha}, \quad 0 < r < \infty. \]

Then for any constant $p$ satisfying
\[ 0 < p \leq \beta + 2, \quad p(p + n - 2) < \delta_0, \]
there exists a unique $a(r) \in C^2(0, \infty)$ verifying
\[ \left\{ \begin{array}{l}
Ta(r) := a''(r) + \frac{n-1}{r}a'(r) + (n(n+2)U(r))^{\frac{4}{n-2}} - \frac{\delta_0}{p^2}a(r) = -\tilde{H}(r), \quad 0 < r < \infty, \\
\lim_{r \to 0} a(r) = \lim_{r \to \infty} a(r) = 0.
\end{array} \right. \] (69)

Moreover, for some positive constant $C_0$ depending only on $n$, $\delta_0$, $\alpha$, $\beta$, $p$ and $C$,
\[ 0 \leq a(r) \leq C_0r^p(1 + r)^{-p+2-\alpha}, \quad 0 < r < \infty. \]

**Lemma 4.1** Let $n \geq 3$ be an integer, $\delta_0 \geq n$ be a constant, and let $\tilde{H}(r)$ be a non-negative function in $C^0(0, \infty)$. Then for any $0 < \epsilon < R$, there exists a unique solution $a_{\epsilon,R} \in C^2[\epsilon, R]$ to
\[ \left\{ \begin{array}{l}
a''_{\epsilon,R}(r) + \frac{n-1}{r}a'_{\epsilon,R}(r) + (n(n+2)U(r))^{\frac{4}{n-2}} - \frac{\delta_0}{p^2}a_{\epsilon,R}(r) = -\tilde{H}(r), \quad \epsilon < r < R, \\
a_{\epsilon,R}(\epsilon) = a_{\epsilon,R}(R) = 0.
\end{array} \right. \] (70)

Moreover, $a_{\epsilon,R} \geq 0$ on $[\epsilon, R]$. 
Proof. Let \((S^n, g_0)\) be the standard sphere. It is known that in the stereographic projection coordinates

\[
g_0 = \sum_{i=1}^{n+1} dx_i^2 = u_1(y)^{-\frac{1}{n-2}} dy^2
\]

where

\[
u_1(y) = \left(\frac{2}{1 + |y|^2}\right)^{\frac{n-2}{2}} = 2^{\frac{n-2}{2}} U.
\]

Also we know that

\[
L_{g_0}(\phi) = (\Delta_{g_0} - \frac{n(n-2)}{4})\phi = u_1^{-\frac{n+2}{n-2}} \Delta(u_1\phi).
\]

If we let \(\phi = a_{\epsilon,R}/u_1\), we can rewrite (70) as

\[
(\Delta_{g_0}\phi - \frac{n(n-2)}{4}\phi)u_1^{\frac{n+2}{n-2}} = -(n(n+2)U^{-\frac{1}{n-2}} - \frac{\delta_0}{r^2})a_{\epsilon,R} - \dot{H}(r).
\]

After simplification, we have

\[
\Delta_{g_0}\phi + (n - \frac{\delta_0(1 + r^2)^2}{4r^2})\phi = -u_1^{-\frac{n+2}{n-2}} H(r).
\]

Since \(\delta_0 \geq n\)

\[
n < \frac{\delta_0(1 + r^2)^2}{4r^2}, \quad \text{for } \epsilon \leq r \leq R, \quad r \neq 1.
\]

(71)

So the existence and the uniqueness of \(\phi\) as well as \(a_{\epsilon,R}\) are proved. \(\square\)

Proof of Proposition 4.1. Clearly for some \(R_1 > 1,\)

\[
\frac{\delta_0}{r^2} \geq \frac{8n}{r^4}, \quad \text{for } r \geq R_1/2, \quad \text{and} \quad \frac{\delta_0}{r^2} \geq 8n, \quad \text{for } 0 < r \leq \frac{2}{R_1}.
\]

Fix a \(W \in C^0(0, \infty)\) satisfying

\[
W(r) = \frac{\delta_0}{r^2}, \quad \frac{2}{R_1} < r < \frac{R_1}{2},
\]

\[
\frac{\delta_0}{r^2} \geq W(r) \geq W(r) \geq 4n, \quad \frac{1}{R_1} < r < \frac{2}{R_1},
\]

\[
W(r) = 4n, \quad 0 < r \leq \frac{1}{R_1},
\]

\[
\frac{\delta_0}{r^2} \geq W(r) \geq 4n, \quad \frac{R_1}{2} < r \leq R_1,
\]

\[
W(r) = \frac{4n}{r^4}, \quad r > R_1.
\]
Clearly $W(r) \leq \delta_0/r^2$ and
\[
\frac{(1+r^2)^2}{4} W(r) > n, \quad \forall r > 0, \quad r \neq 1.
\]
With this fact, the first eigenvalue of $-\Delta g_0 + \left( \frac{(1+r^2)^2}{4} W(r) - n \right)$ is positive on $S^n$ and the potential $\left( \frac{(1+r^2)^2}{4} W(r) - n \right)$ is in $C^0(S^n)$. Since $\alpha > 2$, $u_1^{-\frac{n+2}{2}} H(r) \in L^q(S^n)$ for some $q > 1$.
Let $\phi_1 \in W^{2,q}(S^n)$ be the solution of
\[
\Delta g_0 \phi_1 + \left( n - \frac{(1+r^2)^2}{4} W(r) \right) \phi_1 = -u_1^{-\frac{n+2}{2}} H(r).
\]
By the symmetry of the data, the uniqueness of the solution, $\phi_1$ depends only on $r$. Since both $W$ and $H$ are continuous for $0 < r < \infty$, $\phi_1$ is $C^2$ in $0 < r < \infty$. By the maximum principle, $\phi_1 \geq 0$. Let
\[
a_1(r) = \phi_1(r) u_1(r), \quad 0 < r < \infty.
\]
Then $a_1 \in C^2(0,\infty)$, $a_1(0) \geq 0$, and
\[
T a_1(r) := a_1''(r) + \frac{n-1}{r} a_1'(r) + [n(n+2)U(r) \frac{4}{n-2} - \frac{\delta_0}{r^2}] a_1(r)
\leq a_1''(r) + \frac{n-1}{r} a_1'(r) + [n(n+2)U(r) \frac{4}{n-2} - W(r)] a_1(r)
= (\Delta g_0 \phi_1 - \frac{n(n-2)}{4} \phi_1) u_1^{\frac{n+2}{2}} - [n(n+2)U(r) \frac{4}{n-2} - W(r)] \phi_1 u_1
= (\Delta g_0 \phi_1 + (n - \frac{(1+r^2)^2}{4} W(r)) \phi_1) u_1^{\frac{n+2}{2}} = -H_1(r).
\]
So $a_1(r)$ is a supersolution. A calculation gives
\[
T(r^p) = -\left( \delta_0 - p(p+n-2) + O(r) \right) r^{p-2}, \quad \text{as} \quad r \to 0,
\]
\[
T(r^{2-\alpha}) = -\left( \delta_0 + (\alpha-2)(n-\alpha) + O(1/r) \right) r^{-\alpha}, \quad \text{as} \quad r \to \infty.
\]
Since both $\delta_0 - p(p+n-2)$ and $\delta_0 + (\alpha-2)(n-\alpha)$ are positive, there exists $R_2 > 1$ such that
\[
T(\gamma r^p) \leq -H(r), \quad \text{for} \quad 0 < r \leq \frac{1}{R_2},
\]
\[
T(\gamma r^{2-\alpha}) \leq -H(r), \quad \text{for} \quad r \geq R_2
\]
for some $\gamma > 1$. Choose $\gamma$ larger if necessary such that
\[
\gamma(\frac{1}{R_2})^p > a_1(\frac{1}{R_2}), \quad \gamma(R_2)^{2-\alpha} > a_1(R_2).
\]
Define
\[
\bar{a}(r) = \begin{cases} 
\min\{\gamma r^p, a_1(r)\}, & 0 < r < \frac{1}{R_2}, \\
a_1(r), & \frac{1}{R_2} \leq r \leq R_2, \\
\min\{\gamma^{2-a}, a_1(r)\}, & r > R_2.
\end{cases}
\]
Then \(\bar{a}(r)\) is a continuous supersolution to \(T\bar{a}(r) = -\hat{H}(r)\) in \((0, \infty)\). Therefore for any \(0 < \epsilon < R < \infty\), then solution of (70) satisfies
\[
0 \leq a_{\epsilon,R}(r) \leq \bar{a}(r), \quad \forall 0 < r < \infty.
\]
Let \(\epsilon \to 0\) and \(R \to \infty\) along a subsequence, \(a_{\epsilon,R}(r)\) tends to \(a(r)\) in \(C^{1,\lambda}_{\text{loc}}(0, \infty)\) for \(0 < \lambda < 1\), which satisfies (69) in the weak sense. Since \(\hat{H} \in C^0(0, \infty)\), we know that \(a \in C^2(0, \infty)\).

Now we prove the uniqueness of the solution of (69). Let \(a(r)\) and \(b(r)\) be two solutions of (69), then their difference verifies the homogeneous equation
\[
T(a - b) \equiv 0, \quad \text{in} \quad (0, \infty).
\]
By Theorem 8.1 in [8], the homogeneous equation has two linearly independent solutions \(a_+(r)\) and \(a_-(r)\) with the asymptotic behavior
\[
\lim_{r \to \infty} \frac{a_+(r)}{r^{\lambda_1}} = \lim_{r \to \infty} \frac{a_-(r)}{r^{\lambda_2}} = 1,
\]
where \(\lambda_1\) and \(\lambda_2\) are the two solutions of \(\lambda^2 + (n - 2)\lambda - \delta_0 = 0\) such that \(\lambda_1 > 0\) and \(\lambda_2 < 2 - n\). Since \((a - b)(r) = C_1a_+ + C_2a_-\) for some constants \(C_1\) and \(C_2\), and since \((a - b)(r) \to 0\) as \(r \to \infty\), we must have \(C_1 = 0\) and therefore
\[
\lim_{r \to \infty} r^{n-2}(a - b)(r) = 0. \tag{72}
\]
Since \(T(a - b) = 0\) corresponds to
\[
(\Delta_{g_0} + [n - \frac{\delta_0(1 + r^2)^2}{4r^2}])\frac{(a - b)(r)}{u_1} = 0, \quad \text{on} \quad S^n \setminus \{P, N\},
\]
where \(N, P\) are the south pole and the north pole, respectively. We know from (72) that \(\frac{(a-b)}{u_1}(p) \to 0\) as \(p \to \{N, P\}\), and in view of (71), we can apply the maximum principle to conclude \(a - b \equiv 0\). Proposition 4.1 is established. \(\square\)

5 Appendix C: Two useful lower bounds

**Proposition 5.1** For \(n \geq 10\), there exists a unique \(f_2 \in C^\infty((0, \infty))\) satisfying
\[
\begin{align*}
\left\{ f_2''(r) + \frac{n-1}{r}f_2'(r) + \left(n(n+2)\frac{1}{4r^2} - \frac{2n}{r^2}\right)f_2 = -r^2U, & \quad 0 < r < \infty, \\
\lim_{r \to 0} f_2(r) = \lim_{r \to \infty} f_2(r) = 0. \tag{73}
\end{align*}
\]
Moreover, for some universal positive constant $C$,

\[
\frac{U}{6(n-4)}(r^4 + \frac{3n-4}{n-2}r^2) \leq f_2(r) \leq Cr^\frac{2}{n}(1+r)^\frac{2}{n-2}, \quad 0 < r < \infty, \quad (74)
\]

**Proposition 5.2** Let $V_\lambda = n(n+2)f_0^1(tU + (1-t)U^\lambda)\frac{1}{n-2}dt$. Then there exists a unique $f_{2,\lambda} \in C^\infty(0,\infty)$ satisfying

\[
\begin{cases}
  f''_{2,\lambda}(r) + \frac{n-1}{r}f'_{2,\lambda}(r) + (V_\lambda - \frac{2n}{r^2})f_{2,\lambda}(r) = -r^2U^\lambda(r)(1 - (\frac{\lambda}{r})^4), \quad r \in (\lambda,\infty) \\
  f_{2,\lambda}(\lambda) = 0, \quad \lim_{r \to \infty} f_{2,\lambda}(r) = 0.
\end{cases}
\]

Moreover for any $\epsilon > 0$, there exist $\delta(\epsilon)$ satisfying $\delta(\epsilon) \to 0$ as $\epsilon \to 0$, and a universal constant $C$ such that for $|\lambda - 1| \leq \delta(\epsilon)$,

\[
\frac{1-\epsilon}{6(n-4)}U^\lambda \left(r^4\left(1 - \left(\frac{\lambda}{r}\right)^4\right) + \frac{3n-4}{n-2}r^2\left(1 - \left(\frac{\lambda}{r}\right)^4\right)\right) \leq f_{2,\lambda}(r) \leq Cr^{6-n} \quad (75)
\]

for $\lambda < r < \infty$.

**Proposition 5.3** For $n \geq 8$, there exists a unique $f_3 \in C^\infty(0,\infty)$ satisfying

\[
\begin{cases}
  f''_3(r) + \frac{n-1}{r}f'_3(r) + (n(n+2)U^\frac{4}{n-2} - \frac{3(n+1)}{r^2})f_3 = -r^2U, \quad 0 < r < \infty, \\
  \lim_{r \to 0} f_3(r) = \lim_{r \to \infty} f_3(r) = 0.
\end{cases}
\]

and, for a universal constant $C$

\[
0 \leq f_3(r) \leq Cr^\frac{5}{n}(1+r)^\frac{3}{n-2}, \quad r > 0.
\]

The existence, uniqueness, and the upper bounds of $f_2$, $f_{2,\lambda}$ and $f_3$ follow from Proposition 4.1. So we only prove the lower bound of $f_2$ and $f_{2,\lambda}$ in this section.

Let $\phi_1(r) = r^4U/(6(n-4))$. Then by elementary computation

\[
\Delta \phi_1 + \left(n(n+2)U^\frac{4}{n-2} - \frac{2n}{r^2}\right)\phi_1 > -r^2U.
\]

Consequently

\[
\begin{cases}
  T(f_2 - \phi_1) := \Delta(f_2 - \phi_1) + (n(n+2)U^\frac{4}{n-2} - \frac{2n}{r^2})(f_2 - \phi_1) = -g \leq 0, \quad 0 < r < \infty, \\
  \lim_{r \to 0}(f_2 - \phi_1)(r) = \lim_{r \to \infty}(f_2 - \phi_1)(r) = 0
\end{cases}
\]

where

\[
g(r) = r^2U\left(\frac{4(n-2)}{3(n-4)} \frac{1}{1+r^2} + \frac{2n}{3(n-4)}r^2U^\frac{4}{n-2}\right).\]
By Proposition 4.1, there exists a positive solution \( a_1(r) \) of

\[
\begin{align*}
T a_1(r) &= -g(r), \quad 0 < r < \infty, \\
\lim_{r \to 0} a_1(r) &= \lim_{r \to \infty} a_1(r) = 0.
\end{align*}
\]

Since

\[
\begin{align*}
T(f_2 - \phi_1 - a_1) &= 0, \quad 0 < r < \infty, \\
\lim_{r \to 0}(f_2 - \phi_1 - a_1)(r) &= \lim_{r \to \infty}(f_2 - \phi_1 - a_1)(r) = 0,
\end{align*}
\]

we know from the proof of Proposition 4.1 that \( f_2 - \phi_1 - a_1 \equiv 0 \). So we only need to obtain a lower bound for \( a_1 \). Let

\[
\phi_2(r) = \frac{3n - 4}{6(n - 4)(n - 2)} r^2 U,
\]

then direct computation gives

\[
\Delta \phi_2 + (n(n + 2)U^{\frac{4}{n-2}} - \frac{2n}{r^2}) \phi_2 = -\frac{2(3n - 4)}{3(n - 4)} U\left(\frac{r^2}{1 + r^2} - \frac{n}{n - 2} r^2 U^{\frac{n-2}{n-4}}\right).
\]

Then one verifies immediately

\[
g(r) > \frac{2(3n - 4)}{3(n - 4)} U\left(\frac{r^2}{1 + r^2} - \frac{n}{n - 2} r^2 U^{\frac{n-2}{n-4}}\right).
\]

This means

\[
f_2(r) - \frac{1}{6(n - 4)} r^4 U > \frac{3n - 4}{6(n - 4)(n - 2)} r^2 U.
\]

(74) is established.

To prove (75), we still use maximum principle as before, but instead of comparing \( f_{2,\lambda} \) directly with the right-hand-side in (75), we compare \( f_{2,\lambda} \) with \((1 - 2\epsilon)(\phi_3 + \phi_4)\) where

\[
\phi_3 := \phi_1 - \phi_1^\lambda = \frac{1}{6(n - 4)} \left( r^4 U^\lambda(1 - \left(\frac{\lambda}{r}\right)) + r^4(U - U^\lambda) \right),
\]

\[
\phi_4 := \phi_2 - \phi_2^\lambda = \frac{3n - 4}{6(n - 4)(n - 2)} \left( r^2 U^\lambda(1 - \left(\frac{\lambda}{r}\right))^4 + r^2(U - U^\lambda) \right).
\]

Our purpose is to show

\[
f_{2,\lambda} \geq (1 - 2\epsilon)(\phi_3 + \phi_4) \tag{77}
\]

where \( \epsilon \) is any fixed small positive constant and \( \lambda \) is close to 1 depending on \( \epsilon \). Once we have (77), (75) follows from (77) and the following well known fact:
\[ U(r) - U^\lambda(r) = (1 - \lambda)(1 - \frac{\lambda}{r})O(r^{2-n}). \]

Let \( T := \Delta + V - \frac{2n}{r^2} \), then by elementary computation,

\[
T \phi_3 = -r^2U^\lambda(1 - \left(\frac{\lambda}{r}\right)^4) + \frac{4(n-2)}{3(n-4)}U^\lambda\left(\frac{r^2}{1+r^2} - \frac{\lambda^8}{r^6(1+\lambda^4/r^2)}\right)
+ \frac{2n}{3(n-4)}r^4(U^\lambda)^{n+2}(1 - \left(\frac{\lambda}{r}\right)^8) + \delta(\lambda)(1 - \frac{\lambda}{r})r^{4-n}.
\]

\[
T \phi_4 = -\frac{2(3n-4)}{3(n-4)}U^\lambda(1 - \left(\frac{\lambda}{r}\right)^4) + \frac{2(3n-4)}{3(n-4)}U^\lambda\left(1 + \frac{1}{1+r^2} - \left(\frac{\lambda}{r}\right)^4\frac{1}{1+\lambda^4/r^2}\right)
+ \frac{2n(3n-4)}{3(n-4)(n-2)}r^2(U^\lambda)^{n+2}(1 - \left(\frac{\lambda}{r}\right)^4) + \delta(\lambda)(1 - \frac{\lambda}{r})r^{2-n}.
\]

where we use \( \delta(\lambda) \) to indicate a function of \( \lambda \) which tends to 0 as \( \lambda \to 1 \).

Then one verifies that

\[ T\left(f_{2,\lambda} - (1 - 2\epsilon)(\phi_3 + \phi_4)\right) < 0, \quad \lambda < r < \infty \]

if \( \lambda \) is close to 1 enough. Since we also have \( f_{2,\lambda}(\lambda) = \phi_3(\lambda) = \phi_4(\lambda) = 0 \) and \( \lim_{r \to \infty}(f_{2,\lambda} - (1 - 2\epsilon)(\phi_3 + \phi_4)) = 0 \), we have proved (77) by maximum principle. Proposition 5.1 is established. \( \Box \)

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