Random growth lattice filling model of percolation: a crossover from continuous to discontinuous transition

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Abstract. A random growth lattice filling model of percolation with a touch and stop growth rule is developed and studied numerically on a two dimensional square lattice. Nucleation centers are continuously added one at a time to the empty lattice sites and clusters are grown from these nucleation centers with a growth probability $g$. For a given $g (\in [0, 1])$, the system passes through a critical point during the growth process where the transition from a disconnected to a connected phase occurs. The model is found to exhibit second order continuous percolation transitions as ordinary percolation for $g \leq 0.5$ whereas for $g \geq 0.8$ it exhibits weak first order discontinuous percolation transitions. The continuous transitions are characterized by estimating the values of the critical exponents associated with the order parameter fluctuation and the fractal dimension of the spanning cluster over the whole range of $g$. The discontinuous transitions, however, are characterized by a compact spanning cluster, lattice size independent fluctuation of the order parameter per lattice, departure from power law scaling in the cluster size distribution and weak bimodal distribution of the order parameter. The nature of transitions are further confirmed by studying the Binder cumulant. Instead of a sharp tricritical point, a tricritical region is found to occur for $0.5 < g < 0.8$ within which the values of the critical exponents change continuously until the crossover from continuous to discontinuous transition is completed.

Keywords: classical Monte Carlo simulations, classical phase transitions, finite-size scaling, percolation problems
1. Introduction

A new era in the study of percolation has started recently through the development of a series of new models [1–3] such as percolation on growing networks [4], percolation in the models of contagion [5, 6], k-core percolation [7, 8], explosive percolation [9–11], percolation on interdependent networks [12–14], agglomerative percolation [15, 16], percolation on hierarchical structures [17], drilling percolation [18, 19], two-parameter percolation with preferential growth [20, 21] and many others. In these models, instead of a robust second order (continuous) transition with formal finite size scaling (FSS) as observed in original percolation [22, 23], a variety of new features are noted. Sometimes the transitions are characterized as a discontinuous first-order transition [24–27], sometimes a crossover from second order to first order with a tricritical point (or region) is observed [6, 20, 28–30], sometimes features of both first and second order transitions are simultaneously exhibited in a single model [31–33], sometimes second order transitions with unusual FSS are found to appear [34–37]. Such knowledge not only enriches the understanding of a variety of physical problems, but also leads to the creation of newer models alongside the extension of the existing models for deeper understanding of the existence of such non-universal behavior.

In this article, we propose another novel model of percolation, namely the ‘random growth lattice filling’ (RGLF) model, in which nucleation centers are added continuously to lattice sites as long as a site is available and clusters are grown from these randomly implanted nucleation centers with a fixed, but tunable growth probability \( g \). RGLF can be considered as a discrete version of the continuum space filling model (SFM) [38] with the touch and stop rule in growing the clusters as that of the growing
cluster model (GCM) \cite{39, 40}. However, RGLF displays a crossover from continuous to discontinuous transitions of a weak first order type as the value of $g$ is tuned from 0 to 1 in contrast to both SFM and GCM which display a second order continuous transition. Below we present the model and analyze data that are obtained from extensive numerical computations.

2. The model

The model RGLF is defined on a 2-dimensional (2d) square lattice. Initially, the lattice was empty except for one nucleation center placed randomly onto an empty site. In the next step, besides adding a new nucleation center randomly to another empty site, one layer of perimeter sites of all the existing active clusters (including the nucleation center implanted in the previous time step) are occupied with a constant growth probability $g$ following the Leath algorithm \cite{41}. A cluster is called an active cluster as long as it remains isolated from any other cluster or the nucleation center at least by a layer of empty nearest neighbours (NN). At the end of the growth process, if an active cluster is found separated just by the NN bond from another cluster (active or dead), they are merged into a single cluster and marked as a single dead cluster. The growth of a dead cluster is seized forever as in GCM. Each cluster (active or dead) is marked with a unique label. This constitutes a single Monte Carlo (MC) step and the process is then repeated. If a peripheral site of a cluster is rejected during the growth of an active cluster, it will not be available for occupation by any other growing cluster as in the original percolation (OP). However, such a site can be occupied by a new nucleation center added externally. The growth of a cluster stops either because it becomes a dead cluster by merging with another cluster or all its peripheral sites become forbidden sites for occupation. The process of lattice filling stops when no lattice site is available to add a nucleation center. Time in this model is equal to the number of nucleation centers added. Therefore, for any given value of $g$, there will always be a percolation transition (PT) during the evolution. The model with $g = 0$ naturally corresponds to the Hoshen–Kopelman \cite{42} percolation as the instantaneous area fraction $p(t)$ reaches the OP threshold $p_c(\text{OP}) \approx 0.5927$. It can be noted here that in the SFM with growth rate tending to 0, PT occurs only at unit area fraction \cite{38}. For a given value of $g$, the area fraction $p(t)$ in this model at time $t$ is the total number of occupied sites upto time $t$ per lattice site.

The model is studied at several different values of $g$ ranging from 0 to 1, and the nature of PT at each $g$ value is characterized. At smaller values of $g$ (close to 0), small fractal clusters will grow here and there and by coagulation of these small clusters a large spanning cluster is expected to appear that would lead to a continuous transition. On the other hand for higher values of $g$ (close to 1), a large cluster will appear due to the coagulation of compact finite clusters originated from continuously added nucleation centers that would lead to a discontinuous transition with a discrete jump in the spanning cluster size. A smooth crossover from a continuous to discontinuous transition is expected to occur as $g$ varies from 0 to 1.
3. Results and discussion

An extensive computer simulation of the above model is performed on 2d square lattice of size $L \times L$. The size $L$ of the lattice is varied from $L = 64$ to 1024 in multiples of 2. Clusters are grown applying periodic boundary condition (PBC) in both the horizontal and the vertical directions. All dynamical quantities are stored as the function time $t$, the MC step or the number of nucleation centers added. As there is no control parameter in the system for a given $g$, time evolution of the cluster properties is finally studied as a function of the corresponding instantaneous area fraction $p(t)$ instead of time $t$ directly in order to compare the results with that of the other models. Averages are made over $2 \times 10^5$ to $5 \times 10^3$ ensembles as the system size is varied from $L = 64$ to 1024.

3.1. Cluster morphology and time evolution of the largest cluster

Snapshots of cluster configurations generated on a 2d square lattice of the size $L = 64$ are taken just prior to the appearance of the spanning cluster in the system. The cluster configurations are shown for $g = 0.4$ in figure 1(a) and for $g = 0.8$ in figure 1(b). The largest cluster is shown in red and the other, smaller clusters of different sizes are depicted in different colors. At a lower growth probability $g = 0.4$, clusters of many different sizes exist along with a large finite cluster (figure 1(a)). Several smaller clusters are found to be enclaved inside the larger clusters and the largest cluster becomes fractal. PT occurs in the next step with no significant change in the largest cluster size as the largest cluster in the present step is about to span the lattice. Such continuous change in the size of the largest cluster is usually found to appear in continuous transitions [22, 32, 43]. On the other hand, as the growth probability is taken to be high $g = 0.8$, clusters of smaller sizes are merged with the fast growing other finite clusters. As a result, only a few large compact clusters are found to exist beside newly planted nucleation centers (figure 1(b)). Clusters of intermediate sizes are absent. Enclaves of smaller clusters within the larger clusters have almost disappeared. The clusters in cyan and red in figure 1(b) are merged in the next step and generate a compact spanning cluster with a significant increase in the largest cluster size which might lead discontinuity in the order parameter.

The variations of the largest clusters size $S_{\text{large}}(t)$ of the three different samples are shown against $p(t)$ for $g = 0.4$ in figure 2(a) and for $g = 0.8$ in figure 2(b) for a system of size $L = 512$. The average area fractions corresponding to the thresholds at which PT occur in these systems are marked by the crosses on the respective $p(t)$-axis. Around the respective thresholds, the evolution of $S_{\text{large}}(t)$ against $p(t)$ for $g = 0.4$ and $g = 0.8$ are drastically different. For $g = 0.4$, the value of $S_{\text{large}}(t)$ in different samples is found to increase almost continuously with the instantaneous area fraction $p(t)$ around the threshold whereas for $g = 0.8$, it grows with discontinuous jumps and increases drastically with the largest possible jump of the order of $10^5$ ($\gg L$) at the threshold. It can also be noted that the thresholds of individual samples are in the vicinity of $p_c$ for $g = 0.4$ whereas for $g = 0.8$ they are found to be spread over a wide range of $p(t)$ around $p_c$. However in the thermodynamic limit, such a jump in the order parameter at higher
values of $g \geq 0.8$ will not only be more prominent but also will appear in the near vicinity of the threshold $p_c$.

### 3.2. Dynamical finite size scaling

The dynamical order parameter $P_{\infty}(t)$, the probability of finding a lattice site in the spanning cluster at time $t$, is defined as

$$P_{\infty}(t) = \frac{S_{\text{max}}(t)}{L^2}$$  \hspace{1cm} (1)

where $S_{\text{max}}(t)$ is the size of the spanning cluster at time $t$. A PT is expected to occur at a critical area fraction $p_c(g)$ for a given growth probability $g$. Following the usual continuous percolation theory [22, 23], the FSS form of $P_{\infty}(t)$ is assumed to be

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**Figure 1.** Snapshots of cluster configurations just before the appearance of a spanning cluster: (a) for $g = 0.4$ at $t = 859$ and (b) for $g = 0.8$ at $t = 75$ on a 2d square lattice of size $L = 64$. The red color shows the largest cluster and other different colors represent the presence of clusters of different sizes. Periodic boundary condition is applied in both horizontal and vertical directions during cluster growth. (a) $g = 0.4$, $t = 859$. (b) $g = 0.8$, $t = 75$.

**Figure 2.** Plot of time evolution of the size of the largest cluster $S_{\text{large}}(t)$ for a few samples against the area fraction $p(t)$ for $g = 0.4$ (a) and for $g = 0.8$ (b) for a system of size $L = 512$. 

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\[
P_\infty(t) = L^{-\beta/\nu} \tilde{P}_\infty \{ p(t) - p_c(g) \} L^{1/\nu}
\]
where \( \beta/\nu = d - d_f \), \( \beta \) is the order parameter exponent, \( \nu \) is the correlation length exponent, \( d \) is the space dimension, \( d_f \) is the fractal dimension of the spanning cluster and \( P_\infty \) is a scaling function. The distribution of \( P_\infty \) is taken as
\[
P[P_\infty(t)] = L^{\beta/\nu} \tilde{P}[P_\infty(t) L^{\beta/\nu}]
\]
where \( \tilde{P} \) is a scaling function following the formalism of thermal critical phenomena [44] as well as recent models of percolation [36, 45, 46]. With such a scaling form of \( P[P_\infty(t)] \), one could easily show that
\[
\langle P^2_\infty(t) \rangle \sim L^{-2\beta/\nu} \quad \text{and} \quad \langle P_\infty(t) \rangle^2 \sim L^{-2\beta/\nu}.
\]
Following Melchert [47], the fluctuation in \( P_\infty(t) \) at an area fraction \( p(t) \) is defined as
\[
\chi_\infty(t) = L^2 \left[ \langle P^2_\infty(t) \rangle - \langle P_\infty(t) \rangle^2 \right] \sim L^{d-2\beta/\nu}
\]
where \( d (=2) \) is the space dimension. The full FSS form of \( \chi_\infty(t) \) is then given by
\[
\chi_\infty(t) = L^{\gamma/\nu} \tilde{\chi} \{ p(t) - p_c(g) \} L^{1/\nu}
\]
where \( \gamma \) is the average cluster size exponent, \( \tilde{\chi} \) is a scaling function and \( \gamma/\nu = d - 2\beta/\nu \), consistent with the hyper-scaling relation. The formalism above describes a continuous transition as \( \nu \) corresponds to the diverging correlation length.

In the case of discontinuous transitions, \( \nu \) does not correspond to the correlation length exponent as is already observed in several explosive percolation models [45, 48]. Since the order parameter exponent goes to zero at the discontinuous transition, the FSS form of \( P_\infty(t) \) reduces to
\[
P_\infty(t) = \tilde{P}_\infty \{ p(t) - p'_c(g) \} L^{1/\nu'}
\]
where \( \nu' \) is a new exponent and \( p'_c(g) \) corresponds to the area fraction at which the first order transition occurs. Accordingly, the fluctuation in the order parameter is expected to scale as \( \chi_\infty(t) \sim L^{d} \), where \( d \) is the space dimension. A new FSS form of \( \chi_\infty(t) \) is then proposed as
\[
\chi_\infty(t) = L^{d} \tilde{\chi} \{ p(t) - p'_c(g) \} L^{1/\nu'}
\]
A thorough verification of the above scaling theory for both continuous and discontinuous transitions will be made by extensive numerical simulations.

3.3. Fluctuation in the order parameter

The values of \( \chi_\infty(t) \) are obtained for several values of \( g \) on the lattices of different sizes. In figure 3, \( \chi_\infty(t)/L^2 \) is plotted against \( p(t) \) for different lattice sizes at two extreme values of \( g \), for \( g = 0.1 \) in figure 3(a) and for \( g = 0.8 \) in figure 3(b). Each plot has a maximum at a certain value of \( p(t) \) for a given \( g \) and \( L \) and the value of the maximum decreases with \( L \) to the power \( \gamma/\nu - 2 \) as observed in [49]. The locations of the peaks correspond to the critical thresholds \( p_c(L) \) (marked by crosses on the \( p(t) \)-axis) at which a spanning cluster appears for the first time in the system of size \( L \). The critical area fraction \( p_c(L) \) is expected to scale with the system size \( L \) as

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In the case of continuous PT as in OP [22]. In the limit $L \to \infty$, the value of $p_c(L)$ should correspond to $p_c(g)$, the percolation threshold of the model for a given $g$. In figure 4(a), $p_c(L)$ is plotted against $L^{-1/\nu}$ taking $1/\nu = 0.75$, that of the OP, for $g \leq 0.5$. The scaling form given in equation (9) is found to be well satisfied for $g \leq 0.5$ with $1/\nu = 0.75$ and the plots intersect the $p_c(L)$-axis at different $p_c(g)$ values as seen in figure 4(a). For discontinuous phase transitions, such a difference in the local and global thresholds scales with the system size as $1/L^d$ where $d$ is the space dimension [50, 51]. However, it is rarely observed in non-equilibrium discontinuous PTs, instead a new exponent $1/\nu'$ of various different values less than the space dimension in 2d is reported [52–54].

For $g \geq 0.8$, $p_c(L)$ is plotted against $L^{-2}$ in figure 4(b) and it can be seen that the plots are not linear. The numerical value of $1/\nu'$ will be determined later by performing FSS analysis of $\chi_\infty(t)$ corresponding to the best possible collapse in this regime.

The exponent $\gamma/\nu$ that describes the singularity in association with $\chi_\infty(t)$ can be determined from the variation of the peak values with the system size $L$ in figure 3. As $p(t) \to p_c(L)$, the local threshold, following equation (9) the scaling function in equation (6) becomes $\tilde{\chi}(1)$ and it is assumed to be a constant. Hence, for the continuous PT the fluctuation $\chi_\infty$ would scale with the lattice size $L$ as

$$\chi_\infty(\max) \sim L^{\gamma/\nu}.$$  

In order to extract the exponent $\gamma/\nu$ at a given value of $g (< 0.5)$, the peak values of the fluctuation $\chi_\infty(\max)$ are plotted against $L$ for $g \leq 0.5$ in figure 5(a) in double logarithmic scale. The magnitudes of $\chi_\infty(\max)$ are found to be independent of $g$ at a given $L$ in this regime of $g$. The exponent $\gamma/\nu$ is determined by a linear least square fit through the data points. For $g \leq 0.5$, it is found to be $\gamma/\nu = 1.79 \pm 0.01$ as that of the OP (43/24) corresponding to continuous transition. On the other hand, for $g \geq 0.8$ no global percolation threshold exists for this model and hence, the local percolation threshold $p_c(L)$ is used in place of $p'(g)$ in equation (8) for FSS analysis. As $p(t) \to p_c(L)$, the scaling function in equation (8) then reduces to $\tilde{\chi}(0)$. Assuming $\tilde{\chi}(0)$ to be a constant, $\chi_\infty$ at the threshold would scale with lattice size $L$ as

$$\chi_\infty(\max) \sim L^{\gamma/\nu}.$$
\( \chi_{\infty}(\text{max}) \sim L^d \) \hspace{1cm} (11)

where \( d \) is the space dimension as expected for a discontinuous PT [25, 55]. In order to verify the above scaling form at a given value of \( g (>0.8) \), the peak values of the fluctuation \( \chi_{\infty}(\text{max}) \) are plotted against \( L \) for \( g \geq 0.8 \) in figure 5(b) in double logarithmic scale. The magnitudes of \( \chi_{\infty}(\text{max}) \) are found to increase with \( g \) at a given \( L \) in this regime of \( g \). The exponent is determined by linear least square fit through the data points and it is found to be \( 2.0 \pm 0.01 \), the space dimension as it appears in discontinuous transitions [25, 52]. Such a result is also consistent with the constant peak values in the plots of \( \chi_{\infty}(t)/L^2 \) in figure 3. The solid straight lines with desire slopes in figures 5(a) and (b) respectively are a guide for the eye. For \( 0.5 < g < 0.8 \), the exponent \( \gamma/\nu \) is found to change continuously from 1.79 to 2.0 that defines a region of crossover. The values of \( \gamma/\nu \) for different values of \( g \) are also confirmed by estimating the average cluster size of all the finite clusters (excluding the spanning cluster) at their respective percolation thresholds. Though there is evidence in some other percolation models such as \( k \)-core percolation [56] that the scaling behavior of order parameter fluctuation and that of the average cluster size are not identical.
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A phase diagram separating the percolating and non-percolating regions is obtained by plotting the variation of $p_c(L)$ against $g$ for a system of size $L = 1024$ in figure 6. The phase line starts at $p_c(L) \approx 0.59$ and $g = 0$, has a dip at $g = 0.65$ then increases almost linearly with $g$ up to $g = 1$. It is obvious that the area fraction would be $\approx 0.59$ at the criticality for $g = 0$ as this corresponds to OP. If $g$ is finite but small, the growth of small clusters will stop. Mainly because of a lesser growth probability besides rarely merging with another small cluster or a newly added nucleation center. A large number of small finite clusters will be in the system before transition and the merging of such a small cluster will lead to a spanning cluster which will have many voids in it. As a result, the area fraction will be less. Such an effect will be more predominant when $g$ is around the percolation threshold of OP as at this growth probability large fractal clusters will be grown. PT occurs due to the merging of such large fractal clusters which contain the maximum void space. Hence, the area fraction is expected to be the lowest around $p_c$(OP). It is interesting to note that the critical area fraction has a minimum, as low as $\approx 0.45$, at a growth probability of $g = 0.65$, a little above the critical threshold of OP (0.59). Beyond such a growth probability, compact clusters start appearing which occupy most of the space at the time of transition. Area fraction is found to increase almost linearly with $g$ in this regime. Such variation of $p_c$ is also observed in a percolation model with a repulsive or attractive rule in site occupation [57]. The phase space is further divided into three regimes by two vertical dashed lines. The value of $\gamma/\nu$ in the region left of the vertical dashed line at $g = 0.5$ is always 1.79 corresponds to the usual continuous PT, whereas it is 2 in the region right of the vertical dashed line at $g = 0.8$ corresponding to a discontinuous PT. In the intermediate region, though the transitions are continuous they do not belong to the same universality class of OP. Such a region may be termed as a tricritical region [58, 59] with no definite tricritical point. The existence of such a tricritical should be verified in the $L \to \infty$ limit.

For continuous PT, the FSS form of $\chi(t)$ is verified plotting the scaled fluctuation $\chi(t)/L^{\gamma/\nu}$ against the scaled variable $[p(t) - p_c(g)]L^{1/\nu}$ for $g = 0.1$ in figure 7(a). For

Figure 6. Plot of $p_c(L)$ against $g$ for $L = 1024$. The line joining the data points separates the percolating region from the non-percolating region. The dotted lines with an arrow towards the left and towards the right denote the regions of continuous and discontinuous transitions respectively.
$g = 0.1$, a good collapse of data is obtained with $\gamma/\nu = 1.79$ and $1/\nu = 0.75$ as those of OP. Since no $p_c(g)$ can be identified for $g \geq 0.8$, FSS of $\chi_\infty(t)$ is performed against the scaled variable $[p(t) - p_c(L)]L^{1/\nu}$ replacing $p_c$ by the size dependent threshold $p_c(L)$. For discontinuous PT at $g = 0.8$, $\chi_\infty(t)/L^2$ is plotted against $[p(t) - p_c(L)]L^{1/\nu}$ in figure 7(b) where $1/\nu'$ is a tunable parameter. The best possible collapse is obtained by tuning the value of $1/\nu'$ to 0.5 as in the Bohman–Frieze–Wormald model of EP [53] which represents a first-order transition. Such collapse for all other values of $g > 0.8$ is also found to occur for $d = 2.0$ and $1/\nu' = 0.5$. The collapse of the peak values confirms the values of the scaling exponent $\gamma/\nu$ as 1.79 for $g \leq 0.5$ corresponding to continuous transition and 2.0 for $g \geq 0.8$ corresponding to discontinuous transition.

3.4. Dimension of spanning cluster

For the system size $L \ll \xi$, the size of the spanning cluster $S_{\text{max}}$ at the criticality varies with the system size $L$ as

$$S_{\text{max}} \approx L^{d_f}$$

where $d_f$ is the fractal dimension of the spanning cluster. Since the clusters are grown here applying PBC, the horizontal and vertical extensions of the largest cluster are stored. If either the horizontal or the vertical extension of the largest cluster is found to be greater than or equal to $L$, it is identified as a spanning cluster. The value of $S_{\text{max}}$ are noted at the respective thresholds for several lattice sizes $L$ for a given $g$. For a continuous PT, the spanning cluster is a random object with all possible sizes of holes in it and is expected to be fractal whereas in the case of a discontinuous transition it becomes a compact cluster whose size is expected to scale with the system size as $L^d$ where $d$ is the space dimension [20, 45]. The values of $S_{\text{max}}$ are plotted against $L$ in double logarithmic scale for the different values of $g \leq 0.5$ in figure 8(a) and for $g \geq 0.8$ in figure 8(b). For $g \leq 0.5$, $S_{\text{max}}$ scales with $L$ as a power law with $d_f = 1.895 \pm 0.002$ almost that of OP (91/48). On the other hand, for $g \geq 0.8$, $S_{\text{max}}$ scales with $L$ as a power law with $d_f = 2.0 \pm 0.01$ as that of the space dimension $d$. The solid lines with desired slopes 1.896 and 2.0 in figures 8(a) and (b) respectively are a guide for the eye. Thus for $g \leq 0.5$, the spanning cluster is found to be fractal as in OP whereas for $g \geq 0.8$ it is
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compact as expected in a discontinuous transition. As a result, there would be enclaves in spanning clusters for $g \leq 0.5$ whereas such enclaves would be absent in the spanning clusters for $g \geq 0.8$ as is evident in the cluster morphology shown in figure 1. Such a presence or absence of enclaves in the spanning cluster determines whether it would be fractal or compact which essentially determines the nature of the transition as continuous or discontinuous [32, 60]. In the regime $0.5 < g < 0.8$ the dimension of spanning cluster $d_f$ changes continuously from $d_f = 1.895$ to $d_f = 2.0$.

3.5. FSS study of $P_\infty(t)$

Following the scaling relations $\gamma/\nu = d - 2\beta/\nu$ and $d_f = d - \beta/\nu$, the exponent $\beta/\nu$ is found to be $\approx 0.105$ as that of continuous PT in OP (5/48) for $g \leq 0.5$ and 0 as that of a discontinuous PT for $g \geq 0.8$. Instead of a direct measurement of the exponent $\beta/\nu$ or $\beta$, the values of the exponent will be verified by studying the FSS of $P_\infty(t)$ in both the regimes of $g$.

The variation of $P_\infty(t)$ is plotted against $p(t)$ for different lattice sizes for $g = 0.1$ in figure 9(a) and for $g = 0.8$ in figure 9(b). As the system size $L$ increases, $P_\infty(t)$ becomes sharper and sharper for both $g = 0.1$ and $g = 0.8$. Moreover, for $g = 0.1$, the plots of $P_\infty(t)L^{\beta/\nu}$ cross at a particular value of $p(t)$ corresponding to the critical threshold $p_c(g)$ taking $\beta/\nu = 0.105$ as shown in inset-I of figure 9(a). As $\beta = 0$ for $g = 0.8$, by no means they could make a cross at a definite $p(t)$. For $g = 0.1$, a complete collapse of the plots is obtained by re-scaling the $P_\infty(t)$ axis by $P_\infty(t)L^{\beta/\nu}$ and the $p(t)$ axis by $[p(t) - p_c(g)]L^{1/\nu}$ taking $\beta/\nu = 0.105$ and $1/\nu = 0.75$ as shown in inset-II of figure 9(a). However, the collapse of $P_\infty(t)$ plots for $g = 0.8$ is obtained by re-scaling the $p(t)$ axis only by $[p(t) - p_c(L)]L^{1/\nu}$ taking $1/\nu = 0.5$ as shown in the inset of figure 9(b). Such a collapse of data not only confirms the validity of the scaling forms assumed, but also confirms the values of the scaling exponents obtained. The observations at $g = 0.1$ are found to be the same for all $g \leq 0.5$ and those that are at $g = 0.8$ are the same for $g \geq 0.8$. For $0.5 < g < 0.8$, a collapse of data is observed for continuously varied exponents $\beta/\nu$ that depend on $g$ as also seen in other EP models [61, 62].

Though a discontinuous jump in the order parameter of a single sample of finite size is observed in SFM, the PT is characterized as continuous [38]. On the other hand, in GCM, a discontinuous transition is found to occur only in the vanishingly small fixed

![Figure 8](https://doi.org/10.1088/1742-5468/aab50d)
initial seed concentration [39], but for intermediate seed concentrations the transitions are found to be continuous which belong to OP universality class [40, 47].

The variations of the critical exponents $\gamma/\nu$, $\beta/\nu$ and fractal dimension $d_f$ with the growth probability $g$ are presented in figure 10. The values of the critical exponents clearly distinguish the discontinuous transitions for $g \geq 0.8$ from the continuous transitions for $g \leq 0.5$.

### 3.6. Binder cumulant

The evidence presented above shows a continuous transition for $g \leq 0.5$ and a discontinuous transition for $g \geq 0.8$. In order to confirm the order of transition in different regimes of the growth probability $g$, a dynamical Binder cumulant $B_L(t)$ [55, 63], the fourth moment of $S_{\text{max}}(t)$, is studied as function of area fraction $p(t)$. The dynamical Binder cumulant $B_L(t)$ is defined as

$$B_L(t) = \frac{3}{2} \left[ 1 - \frac{\langle S_{\text{max}}^4(t) \rangle}{3 \langle S_{\text{max}}^2(t) \rangle^2} \right].$$

The cumulants when plotted against the area fraction $p(t)$ for different system sizes $L$ are expected to cross each other at a definite $p(t)$ corresponding to the critical threshold of the system for a continuous transition, whereas no such crossing is expected to occur in the case of a discontinuous transition [20]. Though the cumulant has some unusual behavior [64, 65], it is rarely used in the study of recent models of percolation except in a few references [47, 66]. The values of $B_L(t)$ are plotted against $p(t)$ for different system sizes $L$ in figure 11(a) for $g = 0.1$ and in figure 11(b) for $g = 0.8$. For $g = 0.1$, the plots of $B_L(t)$ cross at a particular $p(t)$ corresponding to $p_c(g)$, marked by a cross on the $p(t)$-axis whereas for $g = 0.8$ no such crossing of $B_L(t)$ is observed for different values of $L$. The value of the Binder cumulant at the critical threshold $B_L(p_c)$ is found to be 0.945 as shown by a dotted line in figure 11(a) for $g = 0.1$ and remains close to this for other values of $g \leq 0.5$. It is verified that the value of $B_L(p_c)$ is the same as that of ordinary site percolation, though a lesser value is reported for the bond percolation [67].

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The FSS form of $B_L(t)$ for continuous PT is given by
\[ B_L(t) = \tilde{B} \left[ \{ p(t) - p_c(g) \} L^{1/\nu} \right] \]
whereas that for the discontinuous PT is given by
\[ B_L(t) = \tilde{B} \left[ \{ p(t) - p_c(L) \} L^{1/\nu'} \right] \]
where $\tilde{B}$ is a universal scaling function. The FSS form is verified by obtaining a collapse of the plots of $B_L(t)$ against the scaled variable $[p(t) - p_c(g)] L^{1/\nu}$ taking $1/\nu = 0.75$ for $g = 0.1$. For $g = 0.8$, however, such a collapse is obtained by plotting the cumulants against $[p(t) - p_c(L)] L^{1/\nu'}$ taking $1/\nu' = 0.5$. The data collapse is shown in the insets of the respective plots. Such scaling behavior of $B_L(t)$ for $g = 0.1$ is found to occur for the whole range of $g \leq 0.5$ and that of $g = 0.8$ is found to occur for $g \geq 0.8$. Once again, the Binder cumulant provides a strong evidence that the dynamical transition is continuous for $g \leq 0.5$ whereas it is discontinuous for $g \geq 0.8$. For $0.5 < g < 0.8$, a region of crossover, the cumulants do not cross at a particular value of $p(t)$, rather they cross each other over a range of $p(t)$ values.
3.7. Cluster size and order parameter distributions

Power law distribution of cluster sizes at the critical threshold is an essential criterion in a second-order continuous phase transition. Following OP, a dynamical cluster size distribution \( n_s(t) \), the number of clusters of size \( s \) per lattice site at time \( t \), is assumed to be

\[
n_s(t) = s^{-\tau} f[(p(t) - p_c)s^\sigma]
\]

(16)

where \( \tau \) and \( \sigma \) are two exponents and \( f \) is a universal scaling function. For OP, an equilibrium percolation model, the exponents are \( \tau = \frac{187}{91} \) and \( \sigma = \frac{36}{91} \) [22]. The distribution at the percolation threshold \( n_s(p_c) \) is expected to scale as \( s^{-\tau} \). The cluster size distributions \( n_s(p_c) \) are determined taking \( p_c(g) \) as a threshold for \( g \leq 0.5 \) and taking \( p_c(L) \) as threshold for \( g \geq 0.8 \) for a system of size \( L = 1024 \). The data obtained are binned in varying widths and finally normalized by the respective bin widths. In figure 12(a), the distributions \( n_s(p_c) \) are plotted against \( s \) in a double logarithmic scale for \( g = 0.4 \) (green), 0.5 (magenta), 0.9 (orange) and 1.0 (blue) for \( L = 1024 \). It is clearly evident that the distributions for \( g \leq 0.5 \) describes a power law behavior whereas for \( g \geq 0.8 \) the distributions develop a curvature and deviate from power law scaling. The black solid line with slope \( \tau = \frac{187}{91} \) as that of OP drawn over the data points for \( g \leq 0.5 \) is a guide to eye. The exponent \( \tau \) measured by linear least square fit to the data points for \( g \leq 0.5 \) is found to be 2.05 ± 0.01 as that of OP. Whereas no definite value of \( \tau \) can be obtained for \( g > 0.5 \). Thus, a crossover from continuous transition of OP type with power law distribution of cluster sizes to a discontinuous PT without power law distribution of cluster sizes in the different regimes of the growth probability \( g \) is found to occur in this model. This is in contrary to the observations in SFM [38] or cluster merging model [33] where a power law distribution of clusters size is found to occur beside discontinuous transition.

Besides the cluster size distribution, the distribution of the order parameter is also studied for different values of \( g \), as usually it is studied in thermal phase transitions [44]. A bimodal distribution of the order parameter is expected in a discontinuous transition corresponding to coexisting phases, whereas a single peaked distribution is
expected in a continuous transition. An ensemble of largest clusters, both spanning and non-spanning, on different configurations are collected at the percolation threshold of a given $g$. As $S_{\text{max}}$ represents the size of a spanning cluster only, the size of the largest clusters are denoted by $S_{\text{large}}$ and the values of the order parameter are estimated as $P_{\text{large}} = S_{\text{large}}/L^2$. A probability distribution $P(P_{\text{large}})$ is then defined as

$$P(P_{\text{large}}) \sim L^{3/\nu} \tilde{P}[P_{\text{large}}L^{3/\nu}]$$

(17)

where $\tilde{P}$ is a scaling function. Bimodal nature of $\tilde{P}$ is found to be a powerful tool to distinguish discontinuous transitions from continuous transitions in some of the recent percolation models [20, 36, 45, 46]. The distributions of $P(P_{\text{large}})$s are plotted in figure 12(b) for different values of $g$. For $g \geq 0.8$ though, a broad, not sharp, distribution with two weak peaks for $\tilde{P}$s is obtained. Such broad bimodal distributions of the order parameter are also found in the study of mechanical yield in amorphous solids that represents a first-order transition [68]. No FSS of the distributions is found as given equation (17) but the width of the distribution $\Delta = 2[\langle P_{\text{large}}^2 \rangle - \langle P_{\text{large}} \rangle^2]^{1/2}$ for a given $g$ is found to increase with the system size $L$, shown in the inset of figure 12(b), as a signature of discontinuous transition. For a given $L$, the width of the distributions $\Delta$ is also found to increase with $g$. However, the distributions $P(P_{\text{large}})$ for $g \leq 0.5$ are found to be single humped and follow the scaling form given in equation (17) as shown in the other inset. The width of the distributions for a given $g \leq 0.5$ are found to decrease with $L$. The model, thus, exhibits characteristic properties of discontinuous transition for $g \geq 0.8$ and those of continuous transition for $g \leq 0.5$. Since no sharp double humped distribution is obtained for $g \geq 0.8$, discontinuous transitions occurring can be termed as a weak first-order transition.

### 4. Conclusion

A random cluster growth lattice filling percolation model with a touch and stop rule is developed. As the growth probability $g$ is tuned from 0 to 1, a crossover from continuous to discontinuous PT is observed in this model. For $g \leq 0.5$, the order parameter continuously goes to zero and the geometrical quantities follow the usual FSS at the critical threshold with the critical exponents that of OP. The cluster size distribution is found to be scale-free and a single-humped distribution of order parameter is found to occur in this regime of $g$. On the other hand, for $g \geq 0.8$, the PT occurs with a discontinuous jump at the threshold, the order parameter fluctuation per lattice site becomes independent of the system size, the spanning cluster becomes compact with fractal dimension $d_f = 2$ as that of discontinuous transitions. No scale-free distribution is found for the cluster sizes and the order parameter distribution is given by a broad double humped distribution of increasing width with the system size. The discontinuous transition observed here can be termed as a weak first-order transition. The occurrence of the true first-order transition should be verified in the thermodynamic limit. The order of transitions in different regimes of $g$ is further confirmed by the estimates of Binder cumulant. The intermediate regime of growth probability $0.5 < g < 0.8$ remains a region of crossover without a definite tricritical point.
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