Abstract

Using Hodge diagram combinatorial data, we study qubit and fermionic Fock spaces from the point of view of type II superstring black holes based on complex compactifications. Concretely, we establish a one-to-one correspondence between qubits, fermionic spaces and extremal black holes in maximally supersymmetric supergravity obtained from type II superstring on complex toroidal and Calabi-Yau compactifications. We interpret the differential forms of the $n$-dimensional complex toroidal compactification as states of $n$-qubits encoding information on extremal black hole charges. We show that there are $2^n$ copies of $n$ qubit systems which can be split as $2^n = 2^{n-1} + 2^{n-1}$. More precisely, $2^{n-1}$ copies are associated with even D-brane charges in type IIA superstring and the other $2^{n-1}$ ones correspond to odd D-brane charges in IIB superstring. This correspondence is generalized to a class of Calabi-Yau manifolds. In connection with black hole charges in type IIA superstring, an $n$-qubit system has been obtained from a canonical line bundle of $n$ factors of one dimensional projective space $\mathbb{CP}^1$.

Keywords: Type II superstrings, black holes, fermionic Fock space, qubit systems and toric Calabi-Yau manifolds.
1 Introduction

Extremal black holes have been extensively investigated in the framework of string theory and related topics including M and F-theories compactified on Calabi-Yau manifolds [1, 2, 3, 4]. These black solutions have been studied using attractor mechanism and topological string theory [5, 6, 7, 8, 9]. In attractor mechanism scenario, the scalar fields can be fixed in terms of the black hole charges by extremising the corresponding potential with respect to stringy moduli obtained from the compactification of higher dimensional theories. Moreover, the corresponding entropy functions have been calculated using the U-duality symmetry acting on the invariant black hole charges of the compactified theories. In this issue, the Calabi-Yau compactifications have been investigated producing various results dealing with black holes in type II superstrings using D-brane physics [10, 11].

Besides these activities, black holes in string theory compactification have been connected with quantum information using the qubit formalism [12-22]. Concretely, a possible link between the $N = 2$ STU black hole obtained from type II superstring and three qubits has been given in [15, 16]. The main recent works, in quantum information theory, are trying to connect quantum entanglement and invariant theory, by classifying orbits of entangled states in multipartite configurations of qubit systems, using multidimensional matrix invariants called hyperdeterminants, which were found by Cayley as extensions for the usual determinants of 2-dimensional matrices of classical algebra. In fact, the notion of hyperdeterminant has been relevant in these called black hole/qubit correspondence.

Some of these connections have been extended to construct superqubits using the compactification on supermanifolds [19, 23]. It has also been noted a relationship between qubits and Segre varieties. In a related work, the Segre embedding has been used to give the connection between geometry and concurrence which describes entanglement of formation for bipartite pure states qubit systems [20]. More recently, qubit systems have been embedded into fermionic Fock space [24] by using the classification of pure states entanglement in the framework of spinors classification, by which a physical interpretation has been found.

The main goal of this work is to contribute to these activities by investigating qubit and fermionic Fock spaces from type II superstring black hole framework. Using combinatorial data provided by Hodge diagrams, we establish a one-to-one correspondence between qubits, fermionic spaces and extremal black holes in maximally supersymmetric supergravity obtained from type II superstring on complex toroidal and Calabi-Yau compactifications. To this aim, we interpret the differential forms of the $n$-dimensional complex toroidal compactification as states of $n$-qubits encoding information on extremal black hole charges.

We claim that $2^n$ copies of $n$ qubit systems can be split as $2^n = 2^{n-1} + 2^{n-1}$. In this decompo-
position, \(2^{n-1}\) copies correspond to even D-brane charges in type IIA superstring and the other \(2^{n-1}\) ones are associated with odd D-brane charges in IIB superstring. This toroidal compactification correspondence can be generalized to a class of Calabi-Yau manifolds. In particular, an \(n\)-qubit system has been associated with a canonical line bundle of \(n\) factors of one dimensional projective space \(\mathbb{CP}^1\) which encodes data of type IIA black hole charges.

The organization of the paper is as follows. In section 2, we reconsider the study of the extremal black holes in type II superstrings. Section 3 concerns the correspondence between qubits and black holes. In section 4, we study qubits and fermionic Fock space from type II superstrings on toroidal complex compactifications. The generalization to a class of Calabi-Yau manifolds is given in section 5. The last section is devoted to conclusions and open questions.

2 Black holes in type II superstrings on complex manifolds

In this section we discuss black holes in type II superstrings on complex compact manifolds. Before going further, we recall that compact complex manifolds play an important role in string theory. We will be concerned with the cohomology structure properties which enclose information on the corresponding black hole charges using D-brane objects.

In the context of differential geometry, an \(n\)-dimensional compact complex manifolds \(M^n\) involves complex and real forms carrying a rich structure. These forms play a crucial role in the string theory compactification and related physics, including the geometric construction of gauge theories[25], and they can be summarized in a nice graphic representation called Hodge diagram. The latter is a relevant piece for the determination of lower dimensional spectrum of string theory and the associated black hole charges. It has been remarked that the studied manifolds are the Kählerian ones such as \(T^n = T^{2n}, \mathbb{CP}^n\) and the Calabi-Yau manifolds \(CY^n\).

In order to interpret qubits and fermionic Fock spaces in terms of differential forms, we emphasize the geometric properties of cohomology space. Without loss of generality, consider an \(n\)-dimensional compact complex manifold \(M^n\). The space of \(k\)-forms \(\Omega^k(M)\) is decomposed as

\[
\Omega^k(M) = \bigoplus_{p+q=k} \Omega^{p,q}(M).
\] (2.1)

It is recalled that any element \(\omega\) of \(\Omega^{p,q}(M)\) is an antisymmetric tensor with \(p\) holomorphic and \(q\) anti-holomorphic components. It is recalled that \(\omega\) of \(\Omega^{p,0}(M)\) satisfies \(\bar{\partial}\omega = 0\) where \(\bar{\partial}\) is a Dolbeault operator. A sequence of \(\mathbb{C}\)-linear maps reads as

\[
\Omega^{p,0}(M) \xrightarrow{\bar{\partial}} \Omega^{p,1}(M) \xrightarrow{\bar{\partial}} \ldots \xrightarrow{\bar{\partial}} \Omega^{p,n-1}(M) \xrightarrow{\bar{\partial}} \Omega^{p,n}(M)
\]

which is the Dolbeault complex. From the antisymmetry property of differential forms, \(\bar{\partial}\) verifies the nilpotency property \(\bar{\partial}^2 = 0\) which leads to the construction of the \((p,q)^{th}\) \(\bar{\partial}\) cohomology
group, that forms also a complex vector space. It is defined as

$$H^p,q_\partial(M) \equiv \frac{Z^p,q_\partial(M) \equiv \ker \bar{\partial}}{B^p,q_\partial \equiv \text{im} \bar{\partial}} \quad (2.2)$$

where $Z^p,q_\partial(M)$ is the set of $(p, q)$-cocycles on $M$, and $B^p,q_\partial(M)$ is the set of $(p, q)$-coboundaries. Later, we shall drop the $\partial$ symbol, since we will be concerned only by the cohomology space of Kählerian manifolds.

It is known that $H^{p,q}(M)$ measures the topological non-triviality of complex manifolds, since its dimension is the hodge number $h^{p,q}$, and we have $\dim \mathbb{C} H^{p,q}(M) = h^{p,q}$. An enlightening remark is that the cohomology space $H$ of a complex manifold has an $\mathbb{N}$-graded ring structure which reads as

$$H = \bigoplus_{k \in \mathbb{N}} H^k(M) \quad \text{where} \quad H^k(M) = \bigoplus_{k=p+q} H^{p,q}(M). \quad (2.3)$$

The Hodge numbers summarize the information needed to build the qubit spaces associated with black hole charges in type II superstrings. They spread in a two dimensional diagram commonly called Hodge Diamond. Generally, it takes the following form

$$
\begin{array}{cccccc}
& & h^{0,0} & & \\
& h^{1,0} & & h^{1,0} & & \\
h^{n,0} & & \ldots & & \ldots & h^{0,n} \\
h^{n,n-1} & & \vdots & & h^{n-1,n} \\
& h^{n,n} & & & & \\
\end{array}
$$

Accordingly, the Hodge diamond for Kähler manifolds has vertical and horizontal symmetries, and therefore has the following properties:

- $h^{p,q} = h^{q,p}$
- $h^{p,q} = h^{n-p,n-q}$
- the number of independent Hodge numbers are $(\frac{1}{2}n^2)$ if $n$ even, and $\frac{1}{2}(n+1)(n+3)$ if $n$ is odd.

More constraints can be implemented by imposing other geometric conditions including the Calabi-Yau one.

To make a possible contact with qubit systems, we consider a special compact geometry

$$M^n = T^2 \times T^2 \times \ldots \times T^2 \times T^2 \quad (2.4)$$
where $T^2$ is 2-dimensional torus. It is convenient to use the complex coordinates. Indeed, $T^2$ is defined by the following identifications and constraints

$$ z \equiv z + 1, \quad z \equiv z + i, \quad i^2 = -1 $$

(2.5)

and we will use the complex notation $M^n = T^n_C$.

The cohomology classes associated with the holomorphic and the anti holomorphic $(p, q)$ forms are

$$ 1, \ dz_i, \ \overline{dz}_j, \ dz_i \wedge dz_j, \ dz_i \wedge \overline{dz}_j, \ldots, \ dz_1 \wedge \ldots dz_n \wedge \overline{dz}_1 \wedge \ldots \overline{dz}_n $$

(2.6)

where $i, j, \ldots = 1, \ldots, n$. The corresponding $(n + 1)^2$ Hodge numbers can be listed in the previous Hodge diagram. A close inspection shows that the total number of forms on such a manifold is

$$ \sum_{k=0,1,\ldots,n} \sum_{k=p+q} h^{p,q} = 2^{2n} = 2^n \times 2^n. $$

(2.7)

There are many ways to interpret this number. In connection with quantum information, this indicates that there are $2^n$ copies of $n$ qubit systems. It has been remarked that these systems can be linked with black objects which can be obtained from type II superstring D-branes wrapping non trivial cycles of $T^n_C$. It has been shown that the near horizon of these black objects can be given by the product of Anti-de-Sitter spaces and spheres

$$ Ads_{p+2} \times S^{6-2n-p}, $$

(2.8)

where integers $n$ and $p$ should satisfy the following constraint

$$ 2 \leq 6 - 2n - p. $$

(2.9)

The electric/magnetic duality requires that the $p$-dimensional electric black branes and the $q$-dimensional magnetic black branes are related by

$$ p + q = 6 - 2n. $$

(2.10)

This condition can generate many black solutions which can be classified by the values of $(p, q)$. However, we discuss here only the case associated with $(p, q) = (0, 6 - 2n)$ describing electric charged black holes. Indeed, the compactification on $T^n_C$ may produce the black holes configurations in $10 - 2n$ dimensional maximally supersymmetric supergravity coupled to abelian gauge symmetries associated with the NS-NS and R-R bosonic fields of various ranks in type II superstrings $[25]$. Concretely, the black holes can be constructed using the following type II D-brane configurations

**Type IIA**: D0-branes, D2-branes, D4-branes, D6-branes, D8-branes

**Type IIB**: D1-branes, D3-branes, D5-branes, D7-branes, D9-branes.
Connections have been made recently between entropy of black holes, in string theory, and qubit entanglement in quantum information theory. These connections go under the name of the Black Hole Qubit Correspondence (BHQC) [15, 16] which will be discussed in the following sections.

3 Black Hole/Qubit Correspondence

It is recalled that the qubit is a building block, in quantum information theory, which has been extensively investigated using different physical and mathematical approaches [26, 27, 28]. It is a two level system which can be associated, for instance, with the electron in the hydrogen atom. The general state of a single qubit is usually given by the Dirac notation as follows

$$|\psi\rangle = a_0|0\rangle + a_1|1\rangle \quad (3.1)$$

where $a_i$ are complex coefficients satisfying the normalization condition

$$|a_0|^2 + |a_1|^2 = 1. \quad (3.2)$$

This equation can be interpreted geometrically in terms of the so called Bloch sphere. The two qubits are four states systems. In this case, the most general state reads

$$|\psi\rangle = a_{00}|00\rangle + a_{10}|10\rangle + a_{01}|01\rangle + a_{11}|11\rangle \quad (3.3)$$

where $a_{ij}$ are complex numbers satisfying the normalization condition

$$|a_{00}|^2 + |a_{10}|^2 + |a_{01}|^2 + |a_{11}|^2 = 1, \quad (3.4)$$

describing a three dimensional complex projective space $\mathbb{C}P^3$ generalizing the Bloch sphere. This analysis can be extended to $n$-qubits associated with $2^n$ configuration states using the same binary notation. It is observed that the 3-qubit systems can be represented by STU black hole charges in type IIB superstring on $T^6$ [15, 16]. The connection has been established using the entropy formulae. Indeed, it is recalled that the entropy can be defined in two different physical contexts.

In statistical mechanics, entropy is the measure of the number of available quantum states. This interpretation is well motivated by string theory. In a thermodynamical system, entropy is associated with the measure of order or disorder, which can only increase according to the second law. Hawking showed that one can assign this quantity to a black hole, which can radiate energy due to quantum mechanical effects [29]. The entropy is then linked to the event horizon area by Bekenstein-Hawking formulae

$$S_{BH} = \frac{c^3 A}{4 G \hbar} \propto \frac{1}{A}. \quad (3.5)$$
This shows a first interplay between a thermodynamic quantity of quantum mechanical origin and a geometric quantity from the classical theory of gravity. Moreover, it has been revealed that the entropy of a black hole is related to the entanglement measure of qubits using Cayley’s hyperdeterminant. To see that, one considers the case of a pair of qubits \{A, B\} described by a general state

\[ |\Psi\rangle = a_{AB}|AB\rangle \quad A, B = 0, 1 \]  

(3.6)

where the Einstein summation convention is used. Then, the bipartite entanglement of A and B is given in terms of the 2-tangle \( \tau_{AB} \)

\[ \tau_{AB} = 4|\det a_{AB}|^2 = 4|a_{00}a_{11} - a_{01}a_{10}|^2 = 4|\det \rho_A| = 4|\det \rho_B| \]  

(3.7)

where \( \rho_A \) and \( \rho_B \) are respectively the partial trace of \( \rho_{AB} \) over A and B. The 2-tangle \( \tau_{AB} \) is invariant under \( SL(2) \times SL(2) \) stochastic local operations and classical communication (SLOCC). It is also invariant under the permutations of A and B.

To discuss the case of 3-qubit system \{A, B, C\} described by a general state \( |\Psi\rangle = a_{ABC}|ABC\rangle \), we need to introduce Cayley’s hyperdeterminant considered as a generalization of a determinant from matrices to multidimensional hypermatrices. The simplest version of hypermatrix A of format \( 2 \times 2 \times 2 \) can be visualized as a matrix for which the 8 entries are placed in the corner of a cube. The hypermatrix \( A_{2\times2\times2} \equiv [a_{ABC}] \) where \( A, B, C = 0, 1 \), can be represented in two dimensions by frontal slices of the cube (where \( k \) encodes the two slices)

\[
A = \begin{bmatrix}
a_{00} & a_{01} & a_{01} & a_{01} \\
a_{10} & a_{11} & a_{11} & a_{11}
\end{bmatrix}
\]

Cayley’s hyperdeterminant denoted \( \text{Det } A \) (with capital “D” to make distinction with usual determinant) is defined as follows

\[
\text{Det } A \equiv -\frac{1}{2} \epsilon^{a_{1}A_{1}a_{2}A_{2}a_{3}A_{3}a_{4}A_{4}} \epsilon^{B_{1}B_{2}B_{3}B_{4}} \epsilon^{C_{1}C_{2}C_{3}C_{4}} a_{A_{1}B_{1}C_{1}} a_{A_{2}B_{2}C_{2}} a_{A_{3}B_{3}C_{3}} a_{A_{4}B_{4}C_{4}}
\]

(3.8)

It is a homogeneous quartic polynomial composed by the entries \( a_{ABC} \), which acts as discriminant for trilinear form \( A(X, Y, Z) \). The condition of discriminant \( \text{Det } A = 0 \) is equivalent to the non-trivial vanishing of the trilinear form, which gives the singular points of the latter other than the origin. \( \text{Det } A \) can be written explicitly as

\[
\text{Det } A = a_{000}^2a_{111}^2 + a_{001}^2a_{110}^2 + a_{010}^2a_{101}^2 + a_{011}^2a_{100}^2 \\
-2(a_{000}a_{001}a_{100}a_{111} + a_{000}a_{010}a_{101}a_{111} + a_{000}a_{011}a_{100}a_{111} + a_{000}a_{010}a_{100}a_{111}) \\
+ a_{001}a_{010}a_{101}a_{110} + a_{001}a_{011}a_{100}a_{110} + a_{001}a_{010}a_{100}a_{111} \\
+ 4(a_{000}a_{011}a_{101}a_{110} + a_{001}a_{010}a_{100}).
\]

(3.9)

This polynomial involves an interesting physical interpretation in terms of the STU black hole embedded in type II superstrings. The corresponding theory involves 4 photons producing
8 charges. More precisely, the bosonic sector consists of gravity coupled to 4 photons and 3 complex scalar fields denoted S, T and U playing a role of string dualities. The equations of motion carry the same symmetries occurring in the definition of the hyperdeterminant. Namely, they display a discrete triality that interchanges S, T and U. Moreover, they involve an $SL(2)_S \times SL(2)_T \times SL(2)_U$ symmetry [21], and the solution of an STU black hole in the case of spherical symmetry is given in terms of 8 charges $(q^0, q^1, q^2, q^3, p^0, p^1, p^2, p^3)$.

Considering an extremal STU black hole, with vanishing surface gravity and minimal mass compatible with the given charges, the square of the entropy is proportional to a quartic polynomial of $q_0, q_1, q_2, q_3, p_0, p_1, p_2, p_3$ [15, 16]

$$S^2 = \pi^2 \left\{ - (p^0 q_0 + p^1 q_1 + p^2 q_2 + p^3 q_3)^2 
+ 4((p^1 q_1)(p^2 q_2) + (p^1 q_1)(p^3 q_3) + (p^2 q_2)(p^3 q_3) + q_0 p_0 p^2 p^3 - p^0 q_1 q_2 q_3) \right\}.$$ (3.10)

Under a suitable correspondence, the 8 charges of the STU black hole are linked to the components $a_{ABC}$ of 3-qubit system \{A, B, C\} through the entropy formulae. By taking the following identification

$$\begin{pmatrix}
q_0 \\
q_1 \\
q_2 \\
q_3 \\
p^0 \\
p^1 \\
p^2 \\
p^3 
\end{pmatrix} \leftrightarrow 
\begin{pmatrix}
a_{000} \\
-a_{001} \\
-a_{010} \\
-a_{100} \\
a_{111} \\
a_{110} \\
a_{101} \\
a_{011} 
\end{pmatrix},$$

the correspondence between the 3-tangle $\tau_{ABC}$ and the black hole entropy can be established through

$$S = \pi \sqrt{|\text{Det}a_{ABC}|} = \frac{\pi}{2} \sqrt{\tau_{ABC}}.$$ (3.11)

In type IIB superstring, the black hole charges can be obtained from D3-branes wrapping cycles on internal geometry. These charges are obtained from the reduction of the real 5-form $F_5$, which is the gauge invariant field strength associated with the $RR$ gauge field 4-form coupled to D3-branes. The corresponding gauge invariant field strengths are

$$F^\alpha_2 = dA^\alpha, \quad \alpha = 0, 1, 2, 3.$$ (3.12)

The gauge fields $A^\alpha_\mu$ are obtained from the compactification of the gauge 4-form on 3-cycles. The integration of the field strength $F^\alpha_2$ and its dual Hodge form $F^{\star \alpha}_2$ throughout the sphere $S^2_\infty$ gives the electric $q^\alpha$ and magnetic charges $p^\alpha$ which are given respectively by

$$q^\alpha = \int_{S^2_\infty} F^{\star \alpha}_2, \quad p^\alpha = \int_{S^2_\infty} F^\alpha_2, \quad \alpha = 0, 1, 2, 3.$$ (3.13)
These charges can be also obtained using T-duality, transforming D3-branes to even D-branes in type IIA superstring. Geometrically, it transforms 3-cycles to even cycles in the internal geometry encoding black hole charges.

The dictionary between black hole charges and qubits is still growing to establish a complete picture. In this sense, we intend to propose additional elements to enrich such a dictionary.

4 Qubit and Fermionic spaces from type II superstring compactifications

In this section, we show that qubits and fermionic Fock spaces can be embedded in type II superstrings on the compact complex manifold $T^C_n$ using black hole charges associated with $(p,q)$-forms. In fact, the structure of the space $H$ can be associated with a vector space on which rely D-brane charges in type II superstrings. This can be explored to approach qubits from string Hodge diagrams. A close inspection shows that this structure can be associated with the fermionic Fock space in which $2^n$ copies of $n$-qubit systems are embedded. Before giving the connection, it is worth noting that the Fock space is a Hilbert space completion, on which all possible states of $n$-identical particles are represented. It reads as

$$\mathcal{F}(\mathcal{H}) = \bigoplus_{n \in \mathbb{N}} S_{\epsilon} \mathcal{H}^{\otimes n} \quad \text{where} \quad \epsilon = \pm$$

(4.1)

where $S_{\epsilon}$ corresponds to the symmetrizer tensor associated with bosonic statistics if $\epsilon = +$, or to the antisymmetrizer tensor associated with fermions if $\epsilon = -$.

The fermionic case is of interest due to the antisymmetry property, and moreover the associated Fock space is finite whenever $\mathcal{H}$ is finite.

Let $\mathcal{H}$ be a complex $n$-dimensional Hilbert space, with canonical basis $\{e_i, \ i = 1, \ldots, n\}$ describing single particle states. Its dual space is $\mathcal{H}^*$ with basis $\{e^j, \ j = 1, \ldots, n\}$.

The fermionic Fock space is given in terms of the Grassmann algebra based on $\mathcal{H}^*$

$$\mathcal{F}(\mathcal{H}^*) = \bigoplus_{n \in \mathbb{N}} S_{\epsilon} (\mathcal{H}^*)^{\otimes n} = \mathbb{C} \oplus \mathcal{H}^* \oplus \wedge^2 \mathcal{H}^* \oplus \ldots \oplus \wedge^n \mathcal{H}^*$$

(4.2)

where $\mathbb{C}$ is the ray of vacuum state, $\mathcal{H}^*$ are representing one-particle subspace, $\wedge^2 \mathcal{H}^*$ are representing 2-particle subspace, etc. The dimensionality is given by

$$\dim \wedge^k \mathcal{H}^* = \binom{n}{k} \quad \text{and} \quad \dim \mathcal{F}(\mathcal{H}^*) = 2^n.$$  

(4.3)

Moreover, we can form a $2n$-dimensional complex space $E = \mathcal{H} \oplus \mathcal{H}^*$ with basis $\{e_I\} \equiv \{e_i, e^j\}$ where $I = \{1, \ldots, n, n + 1, \ldots, 2n\}$. 

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Accordingly, one may associate an operator to every element of $E$ to act on $\mathcal{F}(\mathcal{H}^*)$, by the following correspondence

$$e^i \mapsto \hat{e}^i \equiv e^i \wedge, \quad e_j \mapsto \hat{e}_j \equiv \iota e_j.$$ (4.4)

where the mappings are respectively the exterior and interior products seen as ladder operators on $\mathcal{F}(\mathcal{H}^*)$. Hence, they naturally obey the usual anticommutator relations

$$\{\hat{e}_i, \hat{e}_j\} = \delta_{ij}, \quad \{\hat{e}^i, \hat{e}^j\} = \{\hat{e}_i, \hat{e}_j\} = 0.$$ (4.5)

In this way, one may see $\hat{e}^i$ and $\hat{e}_j$ as creation and annihilation operators respectively. Thus, one may redefine them as $\hat{e}^i \equiv \hat{p}^i$ and $\hat{e}_j \equiv \hat{n}_j$ where $\hat{p}$ is the momentum operator, and $\hat{n}$ is the number operator.

Taking $|0\rangle$ as a notation for the vacuum state, we have $\hat{n}_j|0\rangle = 0$ which is the defining property for vacuum. $\hat{p}^i|0\rangle$, $\hat{p}^i\hat{p}^j|0\rangle$, $\hat{p}^i\hat{p}^j\hat{p}^k|0\rangle$, ..., $\hat{p}^1\hat{p}^2...\hat{p}^n|0\rangle$ with $(i \leq j \leq k)$ represent respectively a single particle in the $i^{th}$ mode, and similarly two, three ..., $n$-particle states.

We note that a $k$-particle subspace is spanned by the basis vectors $\hat{p}^{i_1}\hat{p}^{i_2}...\hat{p}^{i_k}|0\rangle$ with $(1 \leq i_1 \leq i_2 \leq ... \leq i_k \leq n)$. Hence for a general state $\psi \in \mathcal{F}(\mathcal{H}^*)$ we have

$$\psi = \hat{\Psi}|0\rangle \quad \text{where} \quad \hat{\Psi} = \sum_{k=0}^{n} \left( \sum_{i_1 \leq ... \leq i_k} \frac{1}{k!} \psi^{(k)}_{i_1i_2...i_k} \hat{p}^{i_1}\hat{p}^{i_2}...\hat{p}^{i_k} \right)$$ (4.6)

where $\psi^{(k)}_{i_1i_2...i_k}$ is an antisymmetric tensor of order $k$ that captures the complex amplitudes of the $k$-particle subspace.

In particular, the fermionic Fock space can be decomposed as follows

$$\mathcal{F} = \mathcal{F}_+ \oplus \mathcal{F}_-$$ (4.7)

where the subspaces $\mathcal{F}_-$ and $\mathcal{F}_+$ are associated, respectively, with positive and negative chiralities. This space has been investigated in connection with qubit systems [21]. A close inspection shows that the fermionic Fock space can be associated with space of forms $H$ structure. The latter contains $2^n \times 2^n$ $(p, q)$-forms. These forms can belong to two classes called odd and even forms according to the parity of the number $p + q$. Indeed, we have the following sectors

$$H^+ = \{(p, q) \text{ - forms where } p + q \text{ is an even number}\}$$ (4.8)

$$H^- = \{(p, q) \text{ - forms where } p + q \text{ is an odd number}\}.$$ (4.9)

This factorization can be also understood in terms of a $\mathbb{Z}_2$ symmetry acting on the corresponding complex variables $z$ as follows

$$z_i \rightarrow -z_i, \quad i = 1, \ldots, n.$$ (4.10)
The calculation shows that $H$ splits into positive and negative eigenspaces of the $\mathbb{Z}_2$ operator

$$H = H_+ \oplus H_-$$  \hspace{1cm} (4.11)$$

The splitting of $H$ leads to equal dimensions for positive and negative eigenspaces. Inspired by the BHQC, the total cohomology space $H$ can be identified with the fermionic Fock space

$$H \equiv \mathcal{F}.$$  \hspace{1cm} (4.12)$$

In this way, an arbitrary state of $\mathcal{F}$ can be related to an element of $H$. In this interplay, the one dimensional subspace associated with the vacuum state corresponds to the cohomology subspace $H^{0,0}$. The other states can be associated with the cohomology subspaces $H^{p,q}$ where $(p, q) \neq (0,0)$. We expect that this link could be explored to discuss qubits in terms of black hole charges in type II superstrings on toroidal complex manifolds. This produces the following identifications

$$\mathcal{F}_- \equiv H_- \quad \mathcal{F}_+ \equiv H_+.$$  \hspace{1cm} (4.13)$$

These identifications can be supported by mapping the $(p, q)$-forms of the $H$ space, carrying information on black hole charges, to states, considered as basis vectors, of the fermionic Fock space.

For the cohomology subspaces $H^{p,q}$ where $(p, q) \neq (0,0)$, the mapping reads as

$$(p, q)\text{-form} \to |i_1, \ldots, i_p, j_1, \ldots, j_q\rangle$$ \hspace{1cm} (4.14)$$

with antisymmetric properties. Inspired by the closed string theory spectrum, these states can be obtained by operators acting on the vacuum state associated with the cohomology subspace $H^{0,0}$. Using a useful notation

$$i = (i_1, \ldots, i_p) \quad j = (j_1, \ldots, j_q),$$  \hspace{1cm} (4.15)$$

these states can be associated with the following creation operators

$$| (i, j) \rangle = \prod_{a=1}^{p} \alpha_i^a \prod_{b=1}^{q} \tilde{\alpha}_j^b |0\rangle,$$  \hspace{1cm} (4.16)$$

where $\alpha_0^0 \tilde{\alpha}_0^0$ is considered as the identity operator. $i_k$ is a binary number taking either 1 or 0 according to whether the corresponding variable is present or not.

In type II superstrings, the states $| (i, j) \rangle$ are associated with $D(i + j)$-brane charges where

$$i + j = \sum_{a=1}^{p} i_a + \sum_{b=1}^{q} j_b.$$  \hspace{1cm} (4.17)$$
The corresponding black hole charges can be obtained from the gauge invariant \((i + j + 2)\) form field strength of \((i + j + 1)\) forms \(D_{i+j+1}\) coupled to \(D(i + j)\)-branes. The gauge fields \(A^\alpha\) can be obtained from the decomposition

\[
D^\alpha_{i+j+1} \rightarrow A^\alpha \wedge \omega_{ij} \quad (4.18)
\]

\(\omega_{ij}\) are \((i + j)\)-forms on the associated toroidal compactification. The integration of the field strength \(dD^\alpha_{i+j+1}\) gives the black hole charges

\[
\rho^\alpha = \int_{S^2_\infty \times C_{ij}} F^\alpha_2 \wedge \omega_{ij} \quad (4.19)
\]

where \(C_{ij}\) are \((i + j)\)-cycles on which \(D(i + j)\)-branes are wrapped to produce black holes in lower dimensions.

**4.1 One-qubit**

The 1-qubit system associated with \(n = 1\) can be represented by \((p, q)\)-forms on \(T^2 = T^2_3\). This compactification produces an \(N = 2\) supergravity in eight dimensions. In the case of type IIA superstring, the relevant objects are \(D0\) and \(D2\)-branes producing black hole charges in such a compactification. However, the situation in type IIB is somewhat different. The corresponding black hole charges should be obtained from \(D1\)-branes. Indeed, we obtain two copies: One of them corresponds to the positive eigenspace represented by \(H^+\) associated with type IIA superstring, while the second copy corresponds to negative eigenspace \(H^-\) associated with type IIB superstring.

In this way, for the first case the \(D0\) and \(D2\)-brane charges require that the 1-qubit in type IIA superstring reads as

\[
|\psi_+\rangle = a_1 \alpha_0 \bar{\alpha}_0 |0\rangle + a_2 \alpha_1 \bar{\alpha}_1 |0\rangle \quad (4.20)
\]

associated with \((0,0)\) and \((1,1)\) even forms given by

\[
1, \quad dz_1 \wedge \overline{dz_1}. \quad (4.21)
\]

The corresponding black hole charges can be obtained from the gauge invariant 2-form and 4-from field strengths of 1-form and 3-from \(R-R\) gauge fields respectively. They are coupled to \(D0\) and \(D2\)-branes. The integration of these field strengths gives the black hole charges

\[
p^1 = \int_{S^2_3} F_2, \quad p^2 = \int_{S^2_3 \times C_{11}} F_2 \wedge \omega_{11} \quad (4.22)
\]

where \(w_{11}\) is the 2-form associated with \(dz_1 \wedge \overline{dz_1}\). While the second copy of 1-qubit associated with \(D1\)-brane charges in type IIB superstring reads as

\[
|\psi_-\rangle = b_1 \alpha_1 \bar{\alpha}_1 |0\rangle + b_2 \bar{\alpha}_1 |0\rangle \quad (4.23)
\]
associated with odd forms on $T^2 = T_C^1$, namely with

$$dz_1, \quad \overline{dz}_1.$$  \hfill (4.24)

It is worth noting that this copy can be identified with the one found in [16] associated with the complex structure of the elliptic curve $T_C^1$. However, the type IIA copy is intimately related to its Kähler deformation. We expect that these copies could be related by mirror symmetry duality. This connection deserves to be discussed in higher dimensional compactification.

### 4.2 Two-qubit systems

The 2-qubit case, which is of relevance for entanglement, appears in the compactification of type II superstring on $T_C^2$. To identify the corresponding differential forms and D-branes, we use the Hodge diagram properties.

In type IIA superstring, the Hodge duality allows to select the two copies of 2-qubit system states associated with $H_+$. These can be obtained from two different brane systems. The first copy can be obtained from black hole charges associated with a system of \{D0, D2, D4-branes\}. The corresponding state reads as

$$|\psi_1^+\rangle = a_1\alpha_0^0\alpha_0^0|0\rangle + a_2\alpha_1^1\tilde{\alpha}_1^1|0\rangle + a_2\alpha_2^2\tilde{\alpha}_2^2|0\rangle + a_4\alpha_1^1\alpha_2^2\tilde{\alpha}_1^2\tilde{\alpha}_2^2|0\rangle$$  \hfill (4.25)

which is associated with the following even forms

$$1, \quad dz_1 \wedge \overline{dz}_1, \quad dz_2 \wedge \overline{dz}_2, \quad dz_1 \wedge dz_2 \wedge d\overline{z}_1 \wedge d\overline{z}_2.$$  \hfill (4.26)

It is worth noting that the double occupancy embedding of this 2-qubits can be discussed in terms of the complex geometry of $T_C^2$ and the associated D-branes. In fact, each state will be represented by two boxes associated with 2 factors of $T_C^1$.

![Figure 1: Geometric interpretation of double occupancy embedding of 2-qubits.](image)
The second copy can be obtained from black hole charges associated with a system having only D2-branes. The corresponding state reads as

$$|\psi_+^2\rangle = b_1\alpha_1^1\alpha_2^1|0\rangle + b_2\alpha_1^1\tilde{\alpha}_2^1|0\rangle + b_3\alpha_2^1\tilde{\alpha}_1^1|0\rangle + b_4\tilde{\alpha}_1^1\tilde{\alpha}_2^1|0\rangle,$$

which is associated with the following even forms

$$dz_1 \wedge dz_2, \ dz_1 \wedge \overline{dz}_2, \ dz_2 \wedge \overline{dz}_1, \ \overline{dz}_1 \wedge \overline{dz}_2.$$

The negative eigenspace $H_-$ can be associated with type IIB superstring in the presence of D1 and D3-branes. In fact, we obtain two dual D-brane system linked by the complex conjugate operation:

The first copy is

$$|\psi_-^1\rangle = c_1\alpha_1^1|0\rangle + c_2\alpha_2^1|0\rangle + c_3\alpha_2^1\tilde{\alpha}_1^1\tilde{\alpha}_2^1|0\rangle + c_4\tilde{\alpha}_1^1\tilde{\alpha}_2^1|0\rangle$$

and the corresponding odd forms are given by

$$dz_1, \ dz_2, \ dz_1 \wedge \overline{dz}_1 \wedge \overline{dz}_2, \ dz_1 \wedge \overline{dz}_1 \wedge \overline{dz}_2.$$

The complex conjugate operation produces the second copy of 2-qubit

$$|\psi_-^2\rangle = d_1\tilde{\alpha}_1^1|0\rangle + d_2\tilde{\alpha}_2^1|0\rangle + d_3\alpha_1^1\tilde{\alpha}_2^1|0\rangle + d_4\alpha_1^1\tilde{\alpha}_1^1|0\rangle$$

where the corresponding forms are

$$\overline{dz}_1, \ \overline{dz}_2, \ dz_1 \wedge dz_2 \wedge \overline{dz}_2, \ dz_1 \wedge dz_2 \wedge \overline{dz}_1.$$

The corresponding boxes (single and mixed occupations) can be easily represented by forms on $T^3_C$.

### 4.3 Higher dimensional qubit systems

Higher dimensional qubits can be approached using the same dictionary of even and odd D-branes of type II superstrings. Accordingly, three qubit systems have been extensively investigated in connection with STU black holes in type IIB superstring. This case involves more assumptions and calculation based on the Hodge duality.

In fact, we can identify four copies of 3-qubit systems obtained from type IIA superstring associated with positive eigenspace $H_+$. One copy is related to a type IIA superstring black hole obtained from a system of D0, D2, D4 and D6-branes. The Hodge diagram shows that one could have three systems of type IIA superstring black holes obtained only from D2 and D4-branes.

In type IIB superstring, there are also four copies corresponding to negative eigenspace $H_-$. As in type IIA superstring, one copy can be obtained only from D3-branes. In fact, this copy
can be related to the one involving D0, D2, D4, D6-branes using idea of T-duality and mirror symmetry. This system could recover the STU black hole. The other three involve system of D1, D3 and D5-branes.

For the compactification on $T^n_C$, calculations show that there are $2^n$ copies of $n$ qubit system that can be split as

$$2^n = 2^{n-1} + 2^{n-1}. \tag{4.33}$$

In this way, $2^{n-1}$ copies are associated with positive eigenspace $H_+$ in type IIA geometry with even D-branes. The other $2^{n-1}$ copies correspond to negative eigenspace $H_-$ associated with type IIB geometry with odd D-branes.

It is worth noting that the integer $n$ could be fixed by string theory compactification. However, the complex geometry could be explored to approach higher dimensional qubits by introducing Hodge combinatorial numbers.

## 5 Qubits from black holes on local Calabi-Yau manifolds

In this section, we make contact with local Calabi-Yau compactifications in type IIA superstring. Concretely, we consider a special class of toric manifolds satisfying the Calabi-Yau condition. It is recalled that a toric manifold can be expressed in the following form,

$$X^{n+1} = \frac{\mathbb{C}^{n+r+1}}{\mathbb{C}^r}, \tag{5.1}$$

where the $r$ $\mathbb{C}^*$ actions are given by

$$\mathbb{C}^* : x_i \rightarrow \lambda^{q^a_i} x_i, \quad i = 1, \ldots, n + r + 1; \quad a = 1, \ldots, r. \tag{5.2}$$

In these expressions, the exponents $q^a_i$ can be interpreted as physical charges and are assumed to be integers $[31]$. The latter have been extensively studied in the context of geometric engineering of quantum field theories embedded in string theory and related models including F-theory. From a physical point of view, a clever way to approach such manifolds is possible by using a two-dimensional $\mathcal{N} = 2$ supersymmetric linear sigma model involving $n + 1 + r$ chiral superfields $\Phi_i$ with charges $q^a_i$, $i = 1, \ldots, n + 1 + r$; $a = 1, \ldots, r$ under $U(1)^{\otimes r}$ gauge symmetry $[31]$. The corresponding local geometry can be obtained by solving the D-term potential ($D^a = 0$) of such $\mathcal{N} = 2$ linear sigma model. These equations read as

$$\sum_{i=1}^{n+r+1} q^a_i |\Phi_i|^2 = R^a, \quad a = 1, \ldots, r, \tag{5.3}$$

where the $R^a$’s are FI coupling parameters and where the $\Phi_i$’s are the associated scalar fields of the chiral superfield $\Phi_i$. The quotient operation by the $U(1)^{\otimes r}$ gauge symmetry produces $n + 1$-dimensional toric variety

$$X^{n+1} = \frac{\mathbb{C}^{n+r+1}}{\mathbb{C}^r} \setminus S, \tag{5.4}$$
where the $r$ copies of $\mathbb{C}^*$ actions indexed by $a = 1, \ldots, r$ are given by

$$
\mathbb{C}^r : \Phi_i \to \lambda^a \Phi_i
$$

and $\lambda$ is a non-zero complex number. The local Calabi-Yau condition is ensured by the constraint

$$
\sum_{i=1}^{n+1+r} q_i^a = 0.
$$

In fact, there are many examples used in string theory compactification. An interesting one is the so-called canonical line bundle over projective spaces.

To keep the connection with qubit systems of previous sections, we consider the canonical line bundle over $n$ complex dimensional compact toric manifold $M^n$ given by

$$
M^n = \mathbb{C}P^1 \times \mathbb{C}P^1 \times \ldots \times \mathbb{C}P^1
$$

In this case, each $q_i^a$ produces a projective space $\mathbb{C}P^1$. To illustrate this model, we initially consider the leading example of $n = 1$. The corresponding local Calabi-Yau $X^2$ is known by

$$
O(-2) \to \mathbb{C}P^1
$$

In sigma model, this geometry can be obtained from a supersymmetric gauge theory with a $U(1)$ gauge symmetry and three chiral fields $\Phi_i$ with charge $(1, -2, 1)$. The D-term constraint (equation of motion of $V$) reads as

$$
|\Phi_1|^2 + |\Phi_2|^2 - 2|\Phi_3|^2 = Re(t).
$$

This geometry describes the Kähler deformation of the $A_1$ singularity of the ALE spaces

$$
u v = z^2,
$$

where $u, v$ and $z$ are the generators of gauge invariants. They are realized in terms of the scalar fields as follows

$$u = \Phi_1^2 \Phi_2, \quad v = \Phi_3^2 \Phi_2, \quad z = \Phi_1 \Phi_2 \Phi_3.
$$

The compact part is $\mathbb{C}P^1$ which can be obtained by setting $\Phi_3 = 0$ and identifying

$$
|\Phi_1|^2 + |\Phi_2|^2 = Re(t).
$$

The remaining coordinate can be identified with the non-compact direction ensuing the local Calabi-Yau. A similar analysis is a priori possible for higher dimensional cases. In this case, the corresponding local Calabi-Yau is known by

$$
O(-2, \ldots, -2) \to \mathbb{C}P^1 \times \mathbb{C}P^1 \times \ldots \times \mathbb{C}P
$$
The compactification of type II superstrings on such manifolds gives supergravity models with $2^{5-n}$ supercharges interacting with an abelian gauge symmetry. The corresponding abelian gauge theory can be obtained from type II D-branes wrapping the appropriate cycles of such local Calabi-Yau manifolds. These manifolds have been investigated in connection with black holes in topological string theory [32].

A close inspection shows that the Hodge diagram of $V^n$ can be related to the one $T^n_C$ up to $\mathbb{Z}_2$ orbifold actions. Concretely, each $\mathbb{Z}_2$ symmetry affects only one $T^2$ factor. To get the corresponding Hodge numbers one can use the corresponding trivial fibration. Indeed, if $M$ and $N$ are complex manifolds, then the trivial fibration $M \times N$ has Hodge numbers obtained by the following identity [33]

$$h^{(p,q)}(M \times N) = \sum_{u+r=p \atop v+s=q} h^{(u,v)}(M)h^{(r,s)}(N).$$  \hfill (5.14)

In what follows, we will be concerned only with the untwisted sector. For illustration, we list the Hodge diagrams for $n = 1, 2, 3$ corresponding to $\mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, respectively, where the identity (5.14) has been used

| $n = 1$ | $h^{0,0}$ | $h^{1,0}$ | $h^{0,1}$ | $h^{1,1}$ | $1$ | $0$ | $0$ |
|---------|------------|------------|------------|------------|-----|-----|-----|
| $n = 2$ | $h^{0,0}$ | $h^{1,0}$ | $h^{0,1}$ | $h^{1,1}$ | $h^{0,2}$ | $h^{1,2}$ | $h^{2,2}$ | $1$ | $0$ | $0$ | $0$ | $0$ | $1$ |
| $n = 3$ | $h^{0,0}$ | $h^{1,0}$ | $h^{0,1}$ | $h^{1,1}$ | $h^{0,2}$ | $h^{1,2}$ | $h^{2,2}$ | $h^{1,3}$ | $h^{2,3}$ | $h^{3,3}$ | $1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $1$ |

It has been shown that the invariant forms belong to $H^{k,k}_+$ formed by $\prod_{i=0}^{k} d z_i \wedge d \overline{z}_i$. It has been calculated that the corresponding Hodge numbers are

$$\dim H^{k,k}_+ = h^{k,k}_+ = \frac{n!}{k!(n-k)!}. \hfill (5.15)$$
It is clear that one has the following relation

$$dim \ H(T^n_C) = \sum_{k=0}^{n} h_{++}^{k,k} = 2^n$$

(5.16)

associated with $n$ qubit systems. In this way, the relation (4.33) can be replaced by

$$2^n = 1 + (2^n - 1)$$

(5.17)

showing that there is only one copy of $n$-qubit systems in type IIA superstring. This system is associated with the principal vertical line of the Hodge diagram. The black hole charges can be obtained from a class of even D-branes. In this way, the states given in (4.16) reduce to

$$| (i, i) \rangle = \prod_{a=1}^{p} \alpha_a^{i_a} \tilde{\alpha}_a^{i_a} | 0 \rangle.$$  

(5.18)

These states are associated with double occupancy embedding of the $n$-qubit Hilbert space in type IIA superstring. This can be illustrated in the following figure.

![Figure 2: Geometric interpretation of double occupancy embedding of the $n$-qubit Hilbert space in type IIA superstring, where $z_i$ parameterizes the associated $\mathbb{CP}^1$.](image)

6 Conclusion and discussions

In this paper, we have investigated a link between black holes, quantum information and fermionic Fock space using combinatorial data of the internal space Hodge diagram. Concretely, we have elaborated a one-to-one correspondence between qubit systems, Fock space
and extremal black holes embedded in maximally supergravity obtained from II superstrings compactified on complex manifolds. The physical states of $n$-qubit systems can be associated with differential forms of the internal manifolds. In the $n$-dimensional toroidal compactification, we have shown that there are $2^{n-1}$ copies of $n$-qubits associated with even D-branes charges in type IIA superstring. Similar copies can be present in type IIB superstring using odd D-brane charges. Then, we proposed a possible generalization to Calabi-Yau manifolds. More precisely, we have shown that an $n$-qubit system can be associated with a canonical line bundle of $n$ factors of one dimensional projective space $\mathbb{CP}^1$. This qubit system is associated with type IIA D-brane charges. Our paper comes up with many open questions related to quantum information geometry. In fact, it should be of relevance to study quantum information using complex geometry. It should be interesting to make contact with mirror symmetry in toric complex manifolds. This matter will be addressed elsewhere.

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