Time-dependent Transport in arbitrary extended driven tunnel junctions

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We develop a very general perturbative theory of time-dependent transport in a weak tunneling junction which is independent of experimental details and on many-body correlated states in the coupled conductors. These can be similar or different, with arbitrary internal or mutual interactions, superconducting correlations, disorder, and coupled to an electromagnetic environment or other quantum systems. The junction can be spatially extended, and is subject, simultaneously, to time-dependent voltage, local magnetic field and modulation of the tunneling amplitudes. All observables at arbitrary frequencies: average current, non-equilibrium admittance and current correlations can be expressed in a universal way through the out-of-equilibrium DC current only, yielding perturbative time-dependent non-equilibrium fluctuation relations. In particular, charge fluctuations are shown to be universally super-poissonian, and to become poissonian if the junction is driven by a series of Lorentzian pulses. We also generalize, for constant voltage and tunneling, the poissonian shot noise and the fluctuation relation between the derivatives of the noise and the conductance. Thus we provide a compact, general and transparent unifying theory at arbitrary dimension, in contrast with involved derivations based explicitly on particular models and profiles of a single time-varying field.

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Introduction. A weak tunneling junction between two conductors is the simplest though one of the most useful mesoscopic systems. Its theoretical study is facilitated by perturbative computations with respect to tunneling amplitudes. Among the huge number of its experimental interests, let us recall that: -it is often a building block of hybrid structures -it serves to probe the density of states -it allows to measure the tunneling charge carrier, using the poissonian non-equilibrium shot noise. Weak tunneling, thus high resistance, is also expected to enhance some physical phenomena. This is the case for dynamical Coulomb blockade\textsuperscript{1} when the current is reduced by inelastic tunneling due to exchange of photons with the electromagnetic environment.\textsuperscript{2,3} Inelasticity can also be induced by coupling to time-dependent (TD) fields, such as the bias voltage \( V(t) \), gate voltage or local magnetic field, or classical noisy sources. Then understanding its interplay with inelasticity due to Coulomb interactions and to an electromagnetic environment is one exciting challenge very little explored so far.\textsuperscript{4} In addition, and to the best of our knowledge, no TD profile other than sine or abrupt switch has been treated so far, nor the simultaneous presence of TD barrier modulation and a TD voltage bias \( V(t) \). There are also realistic and crucial features which are important to include: spatial extension of the junction, capacitive coupling, and coupling to other quantum conductors. The aim of the present letter is to develop a very general framework for TD transport including all these ingredients simultaneously, without knowledge of experimental details, nor the underlying Hamiltonian and many-body states. It pursues along the lines Ref.\textsuperscript{5} focussing on the current in periodically driven tunnel junctions. Two aspects of TD transport can be distinguished and treated here.\textsuperscript{6} On one hand, in the DC bias regime, one deals with spontaneous generation through finite frequency non-equilibrium admittance and shot noise at finite frequency.\textsuperscript{7} On the other hand, TD driving field\textsuperscript{8} generate a TD current, whose DC component yields the rectified current or pumped charge. They also affect current fluctuations, function of two frequencies as well\textsuperscript{9} and of which the (double) DC component, yields the second charge cumulant. Even more, we express for the first time the generalized non-equilibrium admittance depending non-linearly on the TD fields, and whose microscopic exact expression was derived in Ref.\textsuperscript{10}.

We show that all these time-dependent observables can be expressed in a universal manner through the DC tunneling non-equilibrium current. This striking fact is based on a second-order perturbation with respect to a general tunneling operator \( \mathcal{T} \), without recourse to Keldysh technique, but to basic properties of correlation functions. In addition to be weak, the unique restriction on \( \mathcal{T} \) is that its auto-correlations vanish in absence of tunneling. This amounts to require the absence of pairing correlations of the tunneling excitations, thus, when we specify to a thermal distribution and a superconducting junction, to a negligible supercurrent, ensured by a large capacitance, a magnetic field or dissipation. As \( \mathcal{T} \) nor the global Hamiltonian \( \mathcal{H}_0 \) without tunneling are specified, our theory is relevant, more generally, to highly resistive strongly correlated conductors, and should be useful in other contexts of perturbation schemes.

Though all these additional extensions are included, specifying some among our universal formulas to either a sine potential at frequency \( \Omega \) or a cosine modulation of the barrier allows to recover their form within the Tien-Gordon theory for photo-assisted tunneling\textsuperscript{9,11,12} provided the electron charge \( e \) is replaced by a general value of the tunneling charge \( q \). Derived for independent particles, this theory is based on the side-band transmission scheme: a tunneling electron can exchange \( n \) "pho-
tons” of frequency $\Omega$ for all integer $n$ with a probability given by a Bessel function squared. Thus different observables can be obtained by weighing their DC values evaluated at $V$ shifted by $n\Omega/eh$. Indeed, even though the one-particle picture is inappropriate, we show that a similar picture can be extended fully in terms of many-body correlated states. We also obtain formulas never obtained so far, i.e. pioneering even in absence of interactions. In addition to works dealing with non-interacting systems the present theory includes, unifies and goes beyond many other works based on specific models and tedious derivations. It applies as well to an arbitrary series of Lorentzian pulses, claimed to generate single-particle excitations free of holes achieved in a pioneering recent experiment of which we revisit the extension to non-linear conductors.

**FIG. 1:** Extended tunneling between arbitrary similar or different conductors with mutual (thus capacitive) or internal Coulomb interactions, disorder, superconducting correlations, and coupling to quantum systems and an electromagnetic environment with impedance $Z(\omega)$. Tunneling amplitudes can depend on many-body states $\alpha$, as well as on time through an arbitrary complex function $\gamma(t)e^{iqVT}$, which is the case for TD local magnetic field and gate voltage

**Model and observables.** Let us consider a tunnel junction between two similar or different interacting/disordered conductors 1 and 2 with mutual interactions, connected to an electromagnetic environment or other quantum systems (see Figure 1). We let $k_B = \hbar = 1$. The whole quantum circuit is described by a Hamiltonian $H_0$ in absence of tunneling, required to obey $[Q, H_0] = 0$ where $Q$ is the charge operator of the junction. Thus only tunneling, described by an operator $T$, transfers a charge $q$, i.e., $[T, \hat{Q}] = qT$, which defines $q$, possibly as an “effective” charge.

We don’t have to specify any form for $T$; it can, for instance, describe tunneling between single or many-body states $\alpha$ with associated TD amplitudes $\Gamma_\alpha(t)$, provided the ratio $\Gamma_\alpha(t)/\Gamma_\alpha(0) = \gamma(t)e^{iqVT}$ is state independent. $\gamma(t)$ is an arbitrary complex function, controlled, for instance, by a local gate voltage or a magnetic field: any linear term in its phase $qV_Tt$ is separated, such that stationary regime corresponds to a finite $V_T$ and $\gamma(t) = 1$. $V_T$ enters implicitly into a total effective DC voltage $V$, thus $V(t) = V + \tilde{V}(t)$ where $\tilde{V}(t)$ is the ac part. After gauge transformation the total Hamiltonian reads:

$$H_T(t) = e^{iqVT}E(t)T + e^{-iqVT}E^*(t)T^\dagger,$$

where $E(t) = \gamma(t)e^{iqVT}$ and $\partial_t \tilde{\phi}(t) = \tilde{V}(t)$. We can show that $\int d\omega' E(\omega') \tilde{E}(\omega' + \omega) = 2\pi |\gamma|^2(\omega)$, i.e. the Fourier transform of $|\gamma(t)|^2$. In particular, when $|\gamma(t)| = 1$, $|E(\omega')|^2$ can be written as the probability to exchange photons with an arbitrary frequency $\omega'$, furnished by the effective ac voltage $V(t) + \partial_t \arg \gamma(t)$. The whole system is at equilibrium at $t = -\infty$, with a density matrix $\hat{\rho}$, thus the average current reads:

$$\bar{I}(qV; t) = \text{Tr} \left[ \hat{\rho} \hat{I}_H(t) \right],$$

where the subscript $H$ denotes the Heisenberg representation with respect to $H(t)$ of the operator:

$$\hat{I}(t) = -iq \left[ e^{iqVT}E(t)T - e^{-iqVT}E^*(t)T^\dagger \right].$$

The non-equilibrium TD admittance, non-local in time, reads:

$$\tilde{G}(qV; t, t') = \frac{\delta \tilde{I}(qV; t)}{\delta \tilde{V}(t')}.$$ (3)

Non-symmetrized current correlations are defined as:

$$\tilde{S}(qV; t, t') = \text{Tr} \left[ \hat{\rho} \delta \hat{I}_H(t') \delta \hat{I}_H(t) \right],$$ (4)

where $\delta \hat{I}_H(t') = \hat{I}_H(t') - \hat{I}(qV; t')$. $\tilde{G}(qV; t, t')$ was indeed introduced in Ref. (31), providing a microscopic expression still depending on the driving fields, and is related exactly to $\tilde{S}(qV; t, t')$ (see the Supplemental Material).

The double Fourier transforms $\tilde{G}(qV; \omega, \Omega), \tilde{S}(qV; \omega, \Omega)$ are such that $\omega + \Omega/2$ and $\omega - \Omega/2$ are conjugate to $t$ and $t'$. Thus $\omega$ becomes the relevant frequency in the stationary regime, while $\Omega$ allows to construct deviations from stationarity. While the dependence of $\tilde{I}, \tilde{G}$ and $\tilde{S}$ on $qV$ is explicit, their functional dependence on $qV(t)$ and $\gamma(t)$ is implicit, recalled through the tilde. This is dropped in the stationary regime, i.e. when $\tilde{V}(t) = 0$, $\gamma(t) = 1$, for which $\tilde{I}(qV; \Omega) \rightarrow \delta(\Omega)I(qV), \tilde{G}(qV; \omega, \Omega) \rightarrow \delta(\Omega)G(qV; \omega)$ and $\tilde{S}(qV; \omega, \Omega) \rightarrow \delta(\Omega)S(qV; \omega)$.

In order to express perturbatively $\tilde{I}, \tilde{G}, \tilde{S}$ to second order in $T, T^\dagger$, we replace $\hat{\rho}$ by $\hat{\rho}_0$ in Eqs. (2) and apply the standard procedure of expansion of the evolution operator in the interaction representation, a step required only for $\tilde{I}$, as $\tilde{G}, \tilde{S}$ are already of second order in $T, T^\dagger$. This leads to the appearance of correlators of $T, T^\dagger$ which are invariant by time translation, as TD is controlled by $H_0$ only: $\tilde{T}(t) = e^{iH_0t}Te^{-iH_0t}$, and average taken without tunneling, $\langle \cdots \rangle_0 = \text{Tr} [\hat{\rho}_0 \cdots]$. We assume that:

$$\langle T(t)T(0) \rangle_0 = 0.$$ (5)

Thus non-equilibrium dynamics is contained only in $E(t), E^*(t)$, while only two building blocks at equilibrium are needed: $X^>(t) = \langle T^\dagger(t)T(0) \rangle_0$ and $X^<(t) = \langle T(t)T(0) \rangle_0$. 


\( \langle T(0) T^\dagger(t) \rangle_0 \). Let us summarize the results through three main facts. First, \( X^R(\omega) \), the Fourier transform of \( X^R(t) = \theta(t)[X^>(t) - X^<(t)] \), determines the DC out-of-equilibrium current in the stationary regime:

\[
I(qV) = -\frac{q}{\pi} \text{Re} X^R(-qV). \tag{6}
\]

Secondly, the average current \( \bar{I}(qV;\Omega) \) for the driven junction can be expressed in terms of \( X^R(\Omega) \), thus related to \( I(\omega = qV) \) using Eq. (6) and Kramers-Kronig relations. Thirdly, \( \bar{S}(qV;\omega;\Omega) \) can be expressed in terms of \( X^R(\omega), X^<(\omega) \). If we specify to \( \rho_0 = e^{-\beta H_0}/\text{Tr}(e^{-\beta H_0}) \), \( X^>(\omega) \) and \( X^<(\omega) \) become related to \( \text{Re} X^R(\omega) = -\pi I(-\omega = qV)/q \) by equilibrium FDTs: \( X^>(\omega) = -\pi[N(\omega)+1]I(-\omega)/q \), \( X^<(\omega) = -\pi N(\omega)I(-\omega)/q \), with \( N(\omega) = [e^{\beta\omega} - 1]^{-1} \). Therefore, \( \bar{S}(qV;\omega;\Omega) \) can be expressed merely in terms of \( I(\omega = qV) \). Now we will give directly these universal expressions, referring to the Supplemental Material for more details.

**Average current and admittance** To lowest order/ tunneling, Eq. (2) is expressed universally as:

\[
\bar{I}(qV;\Omega) = i \int \frac{d\omega d\omega' \tilde{E}^*(\omega' - \Omega/2)\tilde{E}(\omega + \Omega/2)}{(\Omega^2 + \omega^2 + 2i\delta)I(qV + \omega' - \omega)} - \frac{\delta^2}{\Omega^2}, \tag{7}
\]

where the limit of vanishing \( \delta \) has to be taken. The rectified current, the easiest to measure experimentally (or transferred charge), obeys a universal relation:

\[
\bar{I}(qV;0) = \int d\omega' |\tilde{E}(\omega')|^2 I(qV - \omega'). \tag{8}
\]

Even though simple and compact, the r. h. s. depends non-trivially on \( V(t) \) and \( \gamma(t) \). This unique formula contains the results of numerous previous works based explicitly on a specific model within much more restrictive framework, for instance the series of works dealing with the Tomonaga-Luttinger model. Interestingly, one can interpret it by extending the side-band transmission picture to complicated global many-body states. A tunneling charge \( q \) has a probability \( |\tilde{E}(\omega')|^2 \) to absorb (emit) an energy \( \omega' < 0(> 0) \), inducing a transition between the many-body eigenstates of \( H_0 \) spaced by \( \omega' \), thus sees an effective voltage \( V - \omega'/q \). The total current is given by the superposition of DC currents at \( V - \omega'/q \) weighted by \( |\tilde{E}(\omega')|^2 \). An interesting application to a periodic series of pulses of area \( \phi_0 \), and \( \gamma(t) = 1 \), led to a Josephson type term, \( \sin^2(q\phi_0/2) \) in \( \bar{I}(qV;0) \). Another one consists into series of Lorentzian pulses; here we focus on a single one, centered around time \( t_1 \) with width \( \tau_1 \): \( qV(t) = -2\tau_1/(t - t_1)^2 + \tau_1^2 \), the DC voltage is given by the surface of the pulse, \( V = V_1 = 2\tau_1/q \). Then \( \tilde{E}(qV_1 - \omega) = \delta(\omega - 2\tau_1\gamma^{-1}(\tau_1 + t_1))\tilde{\theta}(\omega) \) (the Heaviside function). In case one has \( I(V = 0) = 0 \), Eq. (5) does not depend on \( t_1 \), and has a very general expression: \( \bar{I}(qV_1;0) = 4\tau_1^2 J_0^\infty d\omega e^{-2\omega\tau_1}I(\omega) \). For instance, for tunneling of fractional charge \( q = ne \) between edge states in the FQHE at filling factor \( \nu = 1/(2n + 1) \), and for a thermal distribution such that \( 2eT\tau_1 << 1 \), \( \bar{I}(qV;0) \propto \tau_1^{2(1-\nu)} \).

Now, we focus on the integrated value of the non-equilibrium TD admittance over fast modes, \( \bar{G}(qV;\omega;0) = \delta\bar{I}(qV;\omega)/\delta\Omega(\omega) \), which obeys:

\[
\text{Re}\bar{G}(qV;\omega,0) = \frac{q}{2\omega} \left[ \bar{I}(qV + \omega;0) - \bar{I}(qV - \omega;0) \right] \tag{9}
\]

where the rectified current on the r.h.s are given by Eq. (5), thus \( \bar{G} \) depends non-trivially on \( V(t) \) and \( \gamma(t) \) (complex). In the stationary regime, Eq. (9) reduces to that we found previously\(^{12} \) \( \text{Re}\bar{G}(qV;\omega) = q[I(qV + \omega) - I(qV - \omega)]/2\omega \).

**Current correlations**. Here we present other main results of the paper: universal TD non-equilibrium FDR for the non-symmetrised current correlations. For that, we need to specify \( \rho_0 \) to be thermal (see Supplemental Material). Let’s start first by the non-equilibrium stationary regime. Then the finite frequency non-symmetrised noise \( S(qV;\omega) \) obeys the FDR:

\[
S(qV;\omega)/\pi q = \sum_{\pm} N(\omega \pm qV)I(qV \pm \omega), \tag{10}
\]

generalizing that based explicitly on the Tomonaga-Luttinger liquid in Ref. (10). It leads to a universal FDR for the symmetrized noise: \( \pi \sum_{\pm} I(qV \pm \omega) \coth \beta(\omega \pm \omega)/2 \), generalizing its derivation without interactions\(^{10} \) and with coupling to an electromagnetic environment\(^{13} \), provided one replaces \( e \) by the arbitrary value of the tunneling charge \( q \). Let us now specify Eq. (10) to zero-frequency: \( S(qV) = S(qV,0) = \pi q \coth (V/2T) I(qV) \). This allows us to derive simultaneously, in the most economical and unifying scheme, two important results. First, for \( T << qV \), the poissonian shot noise result:

\[
S(qV) = qI(qV). \tag{11}
\]

Secondly, the FDR between the derivatives of the noise and the differential conductance at finite temperatures, taking the zero-bias limit after expansion/\( V \):

\[
\frac{dS}{d\Omega}(V = 0) = 2T \frac{dG}{dV}(V = 0). \tag{12}
\]

The quest for such an FDR in nonlinear conductors has been the subject of intensive theoretical and experimental activities recently. Indeed, a lengthy proof of Eq. (12) was given in a specific Tunnel junction, without capacitive coupling neither an environment\(^{13} \) while we have generalized it here straightforwardly, as a direct consequence of Eq. (12). Thus Eq. (12) is shown to unify in a transparent and a single shot all these FDRs, which are different simply due to different regimes of temperatures, voltages and frequencies.

Let us now turn to a TD driven junction by arbitrary \( V(t) \) and complex \( \gamma(t) \). The current fluctuations can be
expressed fully in terms of the noise generated by a constant voltage, Eq. (10), thus in terms of $I$ too:

$$
\tilde{S}(qV; \omega, \Omega) = \int \frac{d\omega'}{2\pi} \tilde{E}(\omega') \tilde{E}^*(\omega' + \Omega) \times S(qV - \omega'; \omega - \Omega/2).
$$

(13)

Symmetrized correlations can be deduced too, as $\sum_{\pm} \tilde{S}(qV; \pm \omega, \Omega)$. Let’s infer charge fluctuations, obtained by double integration of $\tilde{S}(qV; t, t')$, thus:

$$
\tilde{S}(qV; 0, 0) = q \int \frac{d\omega'}{2\pi} |\tilde{E}(qV - \omega')|^2 \coth \left( \frac{\omega'}{2T} \right) I(\omega').
$$

(14)

As the two last terms combine into the zero-frequency noise $S(qV = \omega')$, the same interpretation as that of Eq. (8) holds here. Eq. (14) reduces to the result derived for noninteracting conductors, if we replace $I(\omega')$ by $T \omega'$, with $T$ a weak transmission coefficient, showing that this substitution holds in much more general setups whenever the DC current is linear. Now let’s compare Eq. (14), specified to zero temperature, to Eq. (8). This leads to a universal inequality:

$$
\tilde{S}(qV; 0, 0) \geq qI(0; qV),
$$

(15)

thus charge fluctuations are super-poissonian (for a thermal density matrix). In particular, they are not bounded by their value in the stationary regime, $S(qV)$: the inequality by Levitov et al. can be recovered only for a linear junction, and for $\gamma(t) = 1$, such that the r. h. s. reduces to the DC current, $qI(qV; 0) = qI(qV) = GV = S$ (see Eq. (11)). Interestingly, for $\gamma(t) = 1$, the equality can be reached when $V(t)$ is a series of Lorentzian pulses, as $\tilde{E}(qV - \omega')$ vanishes for either negative or positive $\omega'$, leading to poissonian charge fluctuations: $\tilde{S}(qV; 0, 0) = qI(0; qV)$.

**Discussion and conclusion** We have developed a very general framework to deal with TD transport in a tunnel junction independent on experimental details, with a potential validity for other perturbative schemes. Thus many complications, often discarded, can be included here: spatial extension, capacitive coupling, strong correlations, coupling to other quantum conductors or an electromagnetic environment. Simultaneous TD bias voltage, gate voltage and local magnetic field are shown to affect in a nontrivial way the finite frequency average current, the non-equilibrium TD admittance, and the asymmetric part of current correlations. Though all are given by compact universal expressions, using only the out-of-equilibrium DC current $I(\omega = qV)$, through which intervene the unspecified global Hamiltonian and density matrix. If the latter is thermal, we derive universal a TD non-equilibrium FDR for current correlations at arbitrary frequencies. This allowed us to show that charge fluctuations are super-poissonian, and become poissonian when $V(t)$ is a series of Lorentzian pulses and constant tunneling amplitudes. We have shown that it is possible to extend the side-band transmission picture in terms of many-body correlated eigenstates. This explains why some of the relations here obtained reduce to the same form as those derived within the Tien-Gordon Theory for non-interacting electrons, once we specify to a single driving sine field and to $q = e$. Others have not been derived before, even in absence of interactions, such as the finite frequency current and the non-equilibrium TD admittance.

When specified to the stationary regime, universal non-equilibrium FDRs are obtained for the non-symmetrized noise, leading as well to an FDR for the symmetrized noise. Both can be exploited, by comparison to $I(qV)$, to check experimentally whether symmetrized or non-symmetrised noise is measured, as performed by the group of F. Portier. Taking the zero-frequency limit, we have shown the universality of the poissonian DC noise, Eq. (11), as well as the non-equilibrium FDRs between higher derivatives, Eq. (12). This gives a unifying framework for all these FDRs which goes well beyond related works performed independently of each other, as we don’t decouple $H_0$ neither specify $T$, we can include capacitive coupling, extended or energy-dependent tunneling, an electromagnetic environment and other quantum conductors.

Let us finally discuss the relevance of this theory, derived at arbitrary dimension, to one-dimensional strongly correlated systems. First, it provides, in a compact, general and transparent way, finite frequency observables, in contrast with much more involved derivations, which moreover, are: -using explicitly the Tomonaga Luttinger Liquid- specifying to a unique driving field -computing only either the rectified current or current correlations not compared to the DC current. The present theory accomplishes as well other remarkable achievements as: -as it does not require a specific Hamiltonian, it deals with arbitrary series of fractional filling factors in the FQHE and edge reconstruction. It is formally valid for both a tunneling or weak backscattering barrier. It takes systematically into account the realistic extension of tunneling between edge states and their mutual Coulomb interactions, which are unavoidable. It shows the universal non-equilibrium FDR in the stationary regime (Eq. (10)), extending fully Ref. (16), as well as the DC shot noise, Eq. (11). It provides various complementary methods to measure $q$ in addition to those proposed in Ref. (15), which will be discussed separately.

Finally, our theory is promising to address the interplay between inelasticity, non-linearities and decoherence due to Coulomb interactions, and the exchange of photons with TD driving fields and the electromagnetic environment or other conductors. It provides powerful general tests which have to be obeyed, more generally, when the highly resistive limit is taken, within one or two impurity problems, quantum dots, or any structure described by an effective energy-dependent
transmission. It has many potential applications such as: -pumping-mixing, choosing different periods for the voltages on different sides of the junction, the modulus and phase of $\gamma(t)$ - classical sources of noise, with arbitrary distributions of these driving fields.

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Supplemental Material

Exact expressions for the non-equilibrium TD admittance and FDR.

Considering an arbitrary conductor described by a TD Hamiltonian and connected to many terminals at TD voltages, we have provided a microscopic formulae for the generalized non-equilibrium TD admittance, depending on the all TD driving fields (it is a matrix indeed)\cite{45,46} This general formulae was adapted to arbitrary driven Tunnel junction, without requiring weak tunneling, thus

$$\hat{G}(qV; t, t') = \frac{\delta \tilde{I}(qV; t)}{\delta \tilde{V}(t')}, \quad (16)$$

is given exactly by:

$$\partial_t \tilde{G}(qV; t, t') = i\theta(t - t') \langle [\hat{I}_H(t), \hat{I}_H(t')] \rangle - q^2 \delta(t - t') \langle H_T(t) \rangle. \quad (17)$$

The consequence of Eq. (17) is an exact FDR for the asymmetric part of the non-symmetrized current correlations, Eq. (16), which verify:

$$S^{-}(qV; t, t') = \tilde{S}(qV; t, t') - \tilde{S}(qV; t', t) = h[c_0 \partial_t G(qV; t, t') - \partial_t G(qV; t, t')]. \quad (18)$$

Once Fourier transformed, this yields

$$\tilde{S}(qV; \omega, \Omega) - \tilde{S}(qV; -\omega, \Omega) = (\Omega/2 - \omega) \tilde{G}(qV; \omega, \Omega) - (\Omega/2 + \omega) \tilde{G}(qV; -\omega, \Omega). \quad (19)$$

Here the double Fourier transform of $F(qV; t, t')$ is defined as: $F(qV; \omega, \Omega) = \int dx \oint dx e^{i\omega x + \Omega x} F(qV; x, s)$, where $F(qV; t, t') = F(qV; x, s)$, with $x = t - t'$ and $s = (t + t')/2$.

Nevertheless, it is only in the stationary regime that this theory has been used so far, leading to a non-equilibrium admittance which depends on the DC voltages and a single frequency, computed for the first time in a three-terminal geometry within the Tomonaga-Luttinger Liquid model with leads\cite{10} or exactly (without leads) at the specific interaction parameter $1/2$\cite{12}

Other works have followed as well, mainly concerning the Kondo problem in a two-terminal geometry\cite{13,14,15}. The advantage of the present theory, restricted to weak tunneling regime, is the opportunity to compute the generalized TD admittance and current fluctuations for the driven junction.

Perturbative expressions for the average current, non-equilibrium TD admittance and current fluctuations.

Let’s first compute the average current to second order with respect to $\mathcal{T}$.

$$\tilde{I}(qV; \Omega) = -q \int \frac{d\omega'}{2\pi} \tilde{E}^*(\omega') \tilde{E}(\omega' + \Omega)$$

$$[\chi^R(\omega' + \Omega - qV) + \chi^R(\omega' - qV)] \quad (20)$$

We recall that $X^R(t) = \theta(t) \langle [\mathcal{T}[t^+(t)], \mathcal{T}[0]] \rangle$.

In the stationary regime, $\tilde{I}(qV; \Omega) \to \delta(\Omega) \tilde{I}(qV)$ with:

$$I(qV) = -\frac{q}{\pi} \text{Re} \chi^R(-qV). \quad (21)$$

If we specify first to a vanishing frequency, thus express the rectified current, one has directly a compact equation (given in the main text):

$$\tilde{I}(qV; 0) = \int d\omega' |\tilde{E}(qV - \omega')|^2 I(\omega'). \quad (22)$$

For a finite frequency, we need to use Eq. (21) together with the Kramers-Kronig relation such that we express fully $\chi^R$ in terms of $I$:

$$X^R(\omega) = -\frac{i\pi}{q} \int d\omega' \frac{I(\omega')}{\omega' + \Omega + i\delta}$$

$$= -\frac{q}{\pi} \left[ I(-\omega) + iP \int d\omega' \frac{I(\omega')}{\omega' + \omega} \right], \quad (23)$$

where the limit of vanishing $\delta$ has to be undertaken in the first equation, which is more convenient to adopt here. Substitution into Eq. (20) and additional algebraic steps yield the universal expression for $\tilde{I}(qV, \Omega)$ (given in the text):

$$\tilde{I}(qV; \Omega) = i \int \frac{d\omega' d\omega''}{2\pi} \chi^*(\omega' - \Omega/2) \tilde{E}(\omega' + \Omega/2)$$

$$\frac{\langle\Omega/2 + i\delta\rangle I(qV + \omega'' - \omega')}{\omega''^2 - \langle\Omega/2 + i\delta\rangle^2}, \quad (24)$$

thus is determined fully and universally by $I(\omega = qV)$ and $E(t)$. Now we can derive the generalized non-equilibrium TD admittance in two ways. Either we use the exact formulae in Eq. (17) with the average of the commutator taken now without tunneling, i. e. related to $X^R$:

$$\partial_t \tilde{G}(qV; t, t') = -2q^3 \text{Im} \int d\omega'' e^{-i\omega''(t+t')} \times$$

$$\tilde{E}^*(t) \tilde{E}(t'') X^R(t-t'') [\delta(t-t') - \delta(t' - t')]. \quad (25)$$

Alternatively, we can use the perturbative computation of the TD current, Eq. (20), and take its differential with
which obeys the exact FDR given by Eq.(19), with \( \tilde{\rho}_0 = e^{-\beta \hbar \omega} / Tr(e^{-\beta \hbar \omega}) \). This allows us to benefit from the detailed balance relation: \( X^>(\omega) = e^{-\beta \omega} X^<(\omega) \), and express both \( X^> \) through equilibrium FDTs: \( X^>(\omega) = [N(\omega)+1] X^<(\omega), X^<(\omega) = N(\omega) X^>(\omega) \), where \( N(\omega) = [e^{\beta \omega} - 1]^{-1} \). This allows to recover the general expression for \( \tilde{S}(qV; \omega, \Omega) \):

\[
\tilde{S}(qV; \omega, \Omega) = \int \frac{d\omega'}{2\pi} \tilde{E}(\omega') \tilde{E}^*(\omega' + \Omega) \times S(qV - \omega'; \omega - \Omega/2),
\]

while the symmetrized part can be deduced through by \( \sum_\pm \tilde{S}(qV; \pm \omega, \Omega) \). Here \( S(qV; \omega) \) is shown to be determined by \( I \),

\[
S(qV; \omega) / \pi q = \sum_\pm \pm N(\omega \pm qV) I(qV \pm \omega),
\]

One can check that this finite-frequency noise under a DC voltage is obtained from Eq.(25) in the stationary regime, as one takes the limit \( \tilde{E}(\omega) = \delta(\omega') \).

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Apart from simple filling factors $\nu = 1/(2n + 1)$, for which $q = \nu e$, the value of $q$ for different series of $\nu$ is not a well-settled issue. For an edge state at simple filling factor $\nu = 1/(2n + 1)$, $q = e$ or respectively $\nu e$. If the edge state is coupled to a Fermi metal, $q = 2\nu e/(1 + \nu)$ ($q = e$) in the strong (weak) coupling regime. In quantum wires or carbon nanotubes with arbitrary range and profile of interactions, one has, in all regimes, $q = e$, due to the connection to reservoirs. For two superconductors, $T$ describes tunneling of Cooper pairs with an effective charge $q = 2e$ at energies below the gap, and that of quasi-particles with charge $q = e$ at energies above the gap.

It is only for a linear DC current and for $\gamma(t) = 1$ that the transferred charge, as claimed often, is given by the DC component only, $I(qV; 0) = GV$. Linearity of the DC current can be valid within some regimes in an arbitrary highly resistive correlated system, even when the scattering approach is not appropriate.

Notice that a vanishing current at zero voltage, $I(0) = 0$, is not generally true for an arbitrary $\rho_0$, but can be shown for the thermal distribution.