Generalized quaternionic free rotational Dirac equation and spinor solutions in the electromagnetic field

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Abstract

Starting with the quaternionic Minkowski space-time and its four-vector representation, a rotational analogue of the quaternionic Dirac equation in the electromagnetic field is developed, which includes not only the energy solutions but also the angular momentum solutions for rotating Dirac fermions. The striking feature of the quaternionic approach is that it depicts a unified representation of energy-momentum in a single framework as the space-time. Furthermore, we establish the generalized Schrodinger-Pauli-like equation associated with generalized electric and magnetic dipole energies that corresponding to their dipole moments. As such, the present analysis deals with the invariant behavior of the extended quaternionic rotational Dirac equation under Lorentz-Poincaré, Gauge, duality, and CPT invariance.

Keywords: Minkowski space-time, quaternion, Dirac equation, electromagnetic field, dipole moments, energy-momentum.

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1 Introduction

The predictions of the Classical theory have a substantial impact on the complete description of matter, but its failure to characterize subatomic particle behavior led to the development of quantum mechanics. Quantum mechanics employs the idea of a wave function, which has mathematical construct that provides information in the form of probability amplitudes [1]. With the generalization of de Broglie postulates, Erwin Schrodinger proposed the non relativistic wave equation, however it could not be applied to particles traveling with relativistic velocity [2]. Furthermore, by combining quantum mechanics with special theory of relativity, Dirac and Klein-Gordon discovered the relativistic wave equation, which became the foundation of ‘Relativistic Quantum Mechanics’. The behavior of particles at high energies is described by relativistic wave equations, which are well invariant under Lorentz translation. Dirac suggested his renowned differential equation to correct for the deficiencies of the Klein-Gordon equation, which describes the motion of spin 1/2 particles [3]. The presence of anti-particles was predicted by the Dirac equation. His equation was expressed in terms of Dirac matrices, which were also used to describe the interaction of Dirac particles with an external electromagnetic field. Generally, researchers used a four-vector potential with scalar and

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vector potentials to characterize an external electromagnetic field. In this case, the Dirac equation can be changed using minimal coupling when a particle interacts with an electromagnetic field \[4\]. The hyper-complex algebra is now more commonly used in modern physics to express many physical quantities. The first hyper-complex algebra that is commonly used in physics is the quaternion algebra \[5\]. According to Dickson \[6\], there are four forms of division algebra: real numbers, complex numbers, quaternion, and octonion algebra. Complex numbers (two dimensions), quaternionic algebra (four dimensions), and octonion algebra (eight dimensions) are all extensions of real numbers. Many novel results in modern physics have been explained using hypercomplex algebras \[7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\]. Quaternions have a structure that is extremely similar to that of four vectors, with quaternionic counterparts of four vectors such as four momentum, four velocity, four potential, and so on \[21\]. Apart from that, the quaternionic algebra has been used to describe a variety of physical theories \[22, 23, 24, 25\], including Maxwell’s electromagnetic field equations, Dirac Lagrangian, dual magneto-hydrodynamics (DMHD) equations, and the Bernoulli and Navier–Stokes equations for dyonic plasma, and so on. Davies \[26\] examined the Dirac equation in quaternionic form, and Rawat et al. \[27\] discussed the quaternionic Dirac equation in the bi-spin state, providing positive and negative energy solutions. Rawat and Negi \[28\] have proposed a link between the quaternionic form of the Dirac equation and supersymmetry in presence of electromagnetic field \[29\]. The non-conservation of the photon energy-momentum tensor has been investigated \[30, 31, 32, 33, 34\], which is more in accordance with the terminology used in this paper. Non-conservation of the photon energy-momentum tensor \[34\] is also identified in (i) de Broglie- Proca’s theory (ii) Standard-Model Extension (iii) Non-Linear Electromagnetism. Chanyal \[35, 36\] provided a fresh notion for relativistic quantized electromagnetic fields theory of dyons in terms of quaternion variable. In addition, the quaternionic algebra has been utilized to represent the rotational motion of a rigid body \[37\], where the quaternionic unit elements are typically associated with matrices of the rotational group \[SO(3)\]. Beside, on the \(R \times S^3\) topology, Carmeli \[38, 39, 40\] studied the Klein-Gordon equation, the Weyl equation, and the Dirac equation in rotational form. The rotational analogue of the dynamics of relativistic spin-1/2 fermions and the rotational Dirac equation in terms of quaternionic form have recently been constructed \[41\]. So, in light of the aforementioned literature on rotating Dirac fermions, we have extended the quaternionic rotational Dirac equation in the presence of an external electromagnetic (EM) field.

Starting with the quaternionic Minkowski space-time and its four-vector representation, we write the rotational analogue of the generalized quaternionic Dirac equation for a free fermion spinor where the rotational Dirac matrices are associated with quaternionic basis vectors. We also define the fermionic spinor field for quaternions. The energy-momentum like solutions for Dirac rotating particles are expressed in terms of four-component spinors. Using the quaternionic minimum coupling of four-momentum with the EM-field, a rotational analogy of the quaternionic Dirac equation in the EM-field is developed, which contains not only the rotational energy (corresponding to the quaternionic scalar component) but also the angular momentum (corresponding to the quaternionic vector component) for Dirac fermions associated with electric and magnetic fields. Furthermore, the solutions for these quaternionic rotational energy and momentum equations in the EM-field are obtained in terms of the four component form of the Dirac spinors with spin up and down state. We establish the generalized Schrodinger-Pauli like energy equation, which is associated with unperturbed and perturbed energies due to EM-field interaction. The generalized electric and magnetic dipole moments are constructed corresponding to generalized electric and magnetic dipole energies. Moreover, we also addressed the invariant behavior of the extended quaternionic rotational Dirac equation for Lorentz, gauge, duality, and CPT invariance to check the various symmetries.
2 The quaternions

The quaternions, denoted by $\mathbb{H}$, are a type of hypercomplex algebra which is a four-dimensional norm division algebra over a set of real numbers ($\mathbb{R}$). It is a four-dimensional extension of complex numbers with a scalar and a vector part that is used to explain dynamic behaviour. A quaternion variable $Q_q \in \mathbb{H}$ can be represented as

$$Q_q = e_0 x_0 + (e_1 x_1 + e_2 x_2 + e_3 x_3),$$

where $(x_0, x_1, x_2, x_3) \in \mathbb{R}$ and $(e_0, e_1, e_2, e_3)$ are the fundamental quaternionic units known as the quaternionic basis elements. However, the scalar unit is denoted by $e_0$, and the vector units are denoted by $e_j$ ($j = 1, 2, 3$). The quaternion multiplication can be determined by the following rules:

$$e_0^2 = e_0 = -e_k^2 = 1,$$
$$e_0 e_k = e_k e_0 = e_k,$$
$$e_k e_j = -\delta_{kj} e_0 + \epsilon_{klm} e_m, \quad \forall (k, l, m = 1, 2, 3).$$

Here $\delta_{kl}$ represents delta symbol and $\epsilon_{klm}$ represents the Levi-Civita symbol. Moreover, the quaternionic multiplication \[\text{(2)}\] can also be written as given Table-1. As such, the relation

$$e_k e_l + e_l e_k = -2 \delta_{kl},$$
$$e_k e_l - e_l e_k = 2 \epsilon_{klm} e_m,$$

represent the anti-commutation and commutation property of the quaternionic units. The addition of two quaternions will be expressed as

$$Q_q + R_q = e_0 (x_0 + y_0) + e_1 (x_1 + y_1) + e_2 (x_2 + y_2) + e_3 (x_3 + y_3)$$
$$= e_0 z_0 + (e_1 z_1 + e_2 z_2 + e_3 z_3) \equiv T_q, \quad (Q_q, R_q, T_q \in \mathbb{H}),$$

which satisfies the closure property of addition. As such, the quaternions show the commutative and associative properties of addition, respectively. $Q_q + R_q = R_q + Q_q$ and $(Q_q + R_q) + T_q = Q_q + (R_q + T_q)$. The quaternionic multiplication of any two quaternions can be written by using Table-1 as

$$Q_q \circ R_q = (e_0 x_0 + e_1 x_1 + e_2 x_2 + e_3 x_3) \circ (e_0 y_0 + e_1 y_1 + e_2 y_2 + e_3 y_3)$$
$$= e_0 (x_0 y_0 - \vec{x} \cdot \vec{y}) + e_j (x_0 y_j + x_j y_0 + (\vec{x} \times \vec{y})),$$
$$= \text{Sr}\{Q_q \circ R_q\} + \text{Vr}\{Q_q \circ R_q\}, \quad (\forall j = 1, 2, 3),$$

where ‘$\circ$’ indicates the quaternion product while ‘$\cdot$’ and ‘$\times$’ are the usual scalar and vector products. The scalar

\[
\begin{array}{c|cccc}
  & e_0 & e_1 & e_2 & e_3 \\
\hline
  e_0 & 1 & e_1 & e_2 & e_3 \\
  e_1 & e_1 & -1 & e_3 & -e_2 \\
  e_2 & e_2 & -e_3 & -1 & e_1 \\
  e_3 & e_3 & e_2 & -e_1 & -1 \\
\end{array}
\]

Table 1: Quaternion multiplication table
part of the quaternionic multiplication is ‘Sr’, while the vector part is ‘Vr’. Under multiplication, quaternions are associative but not commutative since $\vec{x} \times \vec{y} \neq \vec{y} \times \vec{x}$. The quaternionic conjugate ($Q^*_q$) of $Q_q$ for which $e_0^* = e_0$, $e_1^* = -e_1$, $e_2^* = -e_2$, $e_3^* = -e_3$ is defined as

$$Q^*_q = e_0 x_0 - (e_1 x_1 + e_2 x_2 + e_3 x_3) \quad (6)$$

Thus, using the conjugation property of quaternion, the real part (i.e. scalar) can be distinguished from its imaginary part (i.e. vector), as

$$Sr = \frac{1}{2} (Q_q + Q^*_q) = x_0$$,

$$Vr = \frac{1}{2} (Q_q - Q^*_q) = e_1 x_1 + e_2 x_2 + e_3 x_3 \quad (7)$$

Notice that, the quaternions are called real quaternions if the vector part is zero, and pure quaternions if the scalar part is zero. As such, the modulus (or norm) of a quaternion is expressed as

$$|Q_q| = \sqrt{Q^*_q \circ Q_q} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2} \quad (8)$$

which satisfy

$$|Q_q R_q| = |Q_q||R_q| \quad (9)$$

The quaternionic inverse becomes

$$Q_q^{-1} = \frac{Q^*_q}{|Q_q|^2}, \quad (|Q_q| \neq 0) \quad (10)$$

A scalar product can also be defined for quaternions as

$$\langle Q_q \cdot R_q \rangle = -\frac{1}{2} (Q_q \circ R^*_q + R_q \circ Q^*_q) = -\frac{1}{2} (Q^*_q \circ R_q + R^*_q \circ Q_q) \quad (11)$$

Even so, the quaternionic basis may also be shown as a 2 × 2 matrix form, such that $e_0 = 1_{2 \times 2}$ and $e_j = -i\sigma_j$ where $\sigma_j$ are the standard Pauli matrices [29]. Furthermore, quaternions show a remarkable resemblance to rotational tau matrices (or isospin matrices) and can be utilized to discuss particle rotational motion [37].

### 3 Quaternionic Minkowski space-time

The quaternions can be utilized as an analogous version of the four vectors because they are regarded comparable to the 4-dimensional Minkowski space-time with structure (−, +, +, +). Thus,

$$X_q = \begin{pmatrix} -ict, \vec{R} \end{pmatrix} = \{ e_0 (-ict) + e_1 X_1 + e_2 X_2 + e_3 X_3 \} \quad (12)$$

$$X^*_q = \begin{pmatrix} -ict, -\vec{R} \end{pmatrix} = \{ e_0 (-ict) - e_1 X_1 - e_2 X_2 - e_3 X_3 \} \quad (13)$$
where \( X_q \) is the quaternionic 4-position. As such, the quaternionic 4-displacement vector, 4-velocity, and 4-gradient will be expressed as, respectively,
\[
ds_q = (\mathbf{ic} \, dt, dx, dy, dz) = e_0 (\mathbf{ic} \, dt) + e_1 dx + e_2 dy + e_3 dz,
\]
\[
U_q = \left( -\mathbf{ic}, \mathbf{u} \right) = e_0 (\mathbf{ic}) + e_1 u_1 + e_2 u_2 + e_3 u_3,
\]
\[
D_q = \left( -\frac{1}{ic} \frac{\partial}{\partial t}, \mathbf{\nabla} \right) = e_0 \left( -\frac{1}{ic} \frac{\partial}{\partial t} \right) + e_1 \frac{\partial}{\partial x} + e_2 \frac{\partial}{\partial y} + e_3 \frac{\partial}{\partial z}.
\]

The quaternionic 4-momentum can be written as
\[
P_q = \left( -i \frac{E}{c}, \mathbf{p} \right) = e_0 \left( -i \frac{E}{c} \right) + e_1 p_1 + e_2 p_2 + e_3 p_3.
\]

Further, the generalized quaternionic four potential \((V_q)\), the electric four potential \((A_q)\), and the magnetic four potential \((B_q)\) can all be expressed in terms of Minkowski structure as
\[
V_q = \left( -i \frac{\Phi}{c}, \mathbf{V} \right) = e_0 \left( -i \frac{\Phi}{c} \right) + e_1 V_1 + e_2 V_2 + e_3 V_3,
\]
\[
A_q = \left( -i \frac{\phi}{c}, \mathbf{A} \right) = e_0 \left( -i \frac{\phi}{c} \right) + e_1 A_1 + e_2 A_2 + e_3 A_3,
\]
\[
B_q = \left( -i \frac{\phi_m}{c}, \mathbf{B} \right) = e_0 \left( -i \frac{\phi_m}{c} \right) + e_1 B_1 + e_2 B_2 + e_3 B_3,
\]

where the generalized vector potential \((\mathbf{V})\) and scalar potential \((\Phi)\) are written as \(\mathbf{V} = \mathbf{A} - ic\mathbf{B}\) and \(\Phi = \phi - ic\phi_m\), respectively. Additionally, the quaternionic mass can be expressed as,
\[
M_q = e_0 m_0 + \left( e_1 m_1 + e_2 m_2 + e_3 m_3 \right),
\]
where \(m_0 = \frac{E_0}{c}\) is defined the rest mass while \(m_j = \left| \frac{p_j}{E_0} \right|, \forall (j = 1, 2, 3)\) is defined the moving mass. The quaternionic moment of inertia (MOI) can thus be written as
\[
I_q = M_q \left( X_q \odot (X_q)^\ast \right) = M_q \left( X_1^2 + X_2^2 + X_3^2 \right) = e_0 I_0 + \left( e_1 I_1 + e_2 I_2 + e_3 I_3 \right),
\]

where \(I_0 \sim m_0|X|^2\) is the MOI about the quaternionic \(e_0\) axis while \(I_j \sim m_j|X_j|^2\) are the MOI about the quaternionic \(e_j\) axes. We can also express the quaternionic rotational 4-momentum as using the above quaternionic quantities, i.e.,
\[
L_q = X_q \odot P_q = e_0 E_0 + \left( e_1 L_1 + e_2 L_2 + e_3 L_3 \right).
\]

Here, equation \((23)\) gives the scalar component \((E_0)\) that can be represented as quaternionic rest mass energy, while the vector components \((L)\) can represented as pure quaternionic angular momentum. So, we have
\[
E_0 = X_0 p_0 - \mathbf{\dot{X}} \cdot \mathbf{p},
\]
\[
L = X_0 \mathbf{\dot{p}} + p_0 \mathbf{\dot{X}} + \left( \mathbf{\dot{X}} \times \mathbf{p} \right). \tag{24}
\]
Interestingly, the quaternionic rest mass energy consisted the scalar terms while the quaternionic angular momen-
tum consisted all the vector terms. We get the pure rotational energy $E_0 \rightarrow -\overrightarrow{X} \cdot \overrightarrow{p}$ and angular momentum $\overrightarrow{L} \rightarrow (\overrightarrow{X} \times \overrightarrow{p})$ for a pure quaternion, while for real quaternionic consideration, we only have the rest mass energy $E_0 \rightarrow X_0 p_0$ with no rotational motion $\overrightarrow{L} \rightarrow 0$.

4 Quaternionic Rotational Dirac (QRD) equation

A relativistic equation that explains the dynamic behaviour of spin 1/2 particles is the Dirac equation. Let us start with the rotational analogue of the Dirac equation in order to write the generalized quaternionic Dirac equation for a free fermion spinor field \(41\), as

\[
(H_q \circ L_q - \mathfrak{B} \lambda^2 I_q) \circ \Psi = 0 , \tag{26}
\]

where $H_q, L_q, \mathfrak{B}, \lambda, I_q$ and $\Psi$ are rotational analogous to the standard Dirac variables $\overrightarrow{d}, \overrightarrow{p}, \beta, c, m$ and $\psi$. The rotational Dirac matrix elements $H_q$ and $\mathfrak{B}$ can be represented in terms of $2 \times 2$ matrix as

\[
H_q = \{D^0(H), D^j(H)\}, \quad \forall D^0(H) = \left( \begin{array}{cc} e_0 & 0 \\ 0 & e_0 \end{array} \right), \quad D^j(H) = \left( \begin{array}{cc} 0 & ie_j \\ ie_j & 0 \end{array} \right) , \tag{27}
\]

\[
\mathfrak{B} = \left( \begin{array}{cc} e_0 & 0 \\ 0 & -e_0 \end{array} \right) , \tag{28}
\]

where $D^0(H)$ and $D^j(H)$ for $j = 1, 2, 3$ are the quaternionic $D$-matrices. Here the speed of rotating fermions is denoted by ‘$\lambda$’ i.e., $\lambda = c \sqrt{\frac{\overrightarrow{X}}{e_0}} = \frac{\overrightarrow{X}}{|\overrightarrow{X}|}$ (see Ref. \[40\]). Therefore, the generalized QRD equation becomes

\[
\left[ e_0 \left( D^0(H) E_0 - \lambda \left( \overrightarrow{D} (H) \cdot \overrightarrow{L} \right) - \mathfrak{B} \lambda^2 I_0 \right) + e_j \left( \lambda D^0(H) L_j + D^j(H) E_0 + \lambda \left( \overrightarrow{D} (H) \times \overrightarrow{L} \right) j - \mathfrak{B} \lambda^2 I_j \right) \right] \circ \Psi = 0 , \tag{29}
\]

The advantage of QRD equation is that it describing not only the rotational energy but also the rotating momentum for the quaternionic scalar and vector parts. This is the most significant aspect of quaternionic analysis, as it allows for dual representation of four vectors in a single framework. Thus, the energy-momentum solutions of the generalized QRD equation \[29\] can be obtained with the quaternion fermionic spinor field $(\Psi)$ as

\[
\Psi = e_0 \Psi_0 + (e_1 \Psi_1 + e_2 \Psi_2 + e_3 \Psi_3) ,
\]

\[
\simeq \left( \begin{array}{c} \Psi_0 \\ \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{array} \right) . \tag{30}
\]

Equation \[30\] indicates the four components representation of quaternionic Dirac spinor, and it can be simplified as $\Psi = (\Psi_K + e_3 \Psi_L)$, where $\Psi_K = (\Psi_0 + e_1 \Psi_1)$ and $\Psi_L = (\Psi_2 - e_1 \Psi_3)$. So, the energy solutions of equation \[29\] can be examined by equating the scalar coefficient $(e_0)$:

\[
\left[ D^0(H) E_0 - \lambda \left( \overrightarrow{D} (H) \cdot \overrightarrow{L} \right) - \mathfrak{B} \lambda^2 I_0 \right] \Psi = 0 , \tag{31}
\]
which gives

\[ \Psi_0(E_0, \vec{L}) = \frac{i\lambda (\vec{e} \cdot \vec{L})}{E_0 - \lambda^2 I_0} \Psi_2(E_0, \vec{L}) , \]  
\(32\)

\[ \Psi_1(E_0, \vec{L}) = \frac{i\lambda (\vec{e} \cdot \vec{L})}{E_0 - \lambda^2 I_0} \Psi_3(E_0, \vec{L}) , \]  
\(33\)

\[ \Psi_2(E_0, \vec{L}) = \frac{i\lambda (\vec{e} \cdot \vec{L})}{E_0 + \lambda^2 I_0} \Psi_0(E_0, \vec{L}) , \]  
\(34\)

\[ \Psi_3(E_0, \vec{L}) = \frac{i\lambda (\vec{e} \cdot \vec{L})}{E_0 + \lambda^2 I_0} \Psi_1(E_0, \vec{L}) . \]  
\(35\)

Equations \(32\) and \(33\) give the negative energy solutions for the anti-fermions while equations \(34\) and \(35\) give the positive energy solutions for fermions. Accordingly, by equating vector coefficient \((e_j)\) in equation \(29\) the momentum like solutions for Dirac rotating particles can be expressed as

\[ \Psi_0(E_0, \vec{L}) = \left[ e_j E_0 + \lambda (\vec{e} \times \vec{L})_j \right] \frac{i\lambda}{(L_j - \lambda I_j)} \Psi_2(E_0, \vec{L}) , \]  
\(36\)

\[ \Psi_1(E_0, \vec{L}) = \left[ e_j E_0 + \lambda (\vec{e} \times \vec{L})_j \right] \frac{i\lambda}{(L_j - \lambda I_j)} \Psi_3(E_0, \vec{L}) , \]  
\(37\)

\[ \Psi_2(E_0, \vec{L}) = \left[ e_j E_0 + \lambda (\vec{e} \times \vec{L})_j \right] \frac{i\lambda}{(L_j - \lambda I_j)} \Psi_0(E_0, \vec{L}) , \]  
\(38\)

\[ \Psi_3(E_0, \vec{L}) = \left[ e_j E_0 + \lambda (\vec{e} \times \vec{L})_j \right] \frac{i\lambda}{(L_j - \lambda I_j)} \Psi_1(E_0, \vec{L}) , \]  
\(39\)

where the equations \(36\) and \(37\) represent the momentum solution in fermionic spinor field while equations \(38\) and \(39\) represent the momentum solution in anti fermionic spinor field.

5 Energy-momentum description of rotational Dirac spinors in presence of electromagnetic field

Let us start with the quaternionic four-momentum minimum coupling on the interaction of a charged particle with an electromagnetic (EM) field as

\[ p^\mu \to p^\mu - \frac{Q}{c} v^\mu \equiv \Pi^\mu , \quad (\mu = 0, 1, 2, 3) , \]  
\(40\)

where \(p^\mu\) is the canonical momentum, \(Q\) is the generalized charge of dyons \((Q = e - ic(m))\) where \(e\) is the electric charge and \(m\) is the magnetic charge and \(\Pi^\mu\) is the kinetic momentum. Now, considering the rotating Dirac fermions, as an analogous, the four angular momentum can be transferred as 

\[ L^\mu \to L^\mu - \frac{Q}{\lambda} (X^\mu v^\mu) \equiv \Pi_L^\mu , \]  
where \(\Pi_L^\mu\) is the kinetic angular momentum. Thus we can express the quaternionic form of rotational four momentum in presence of
Consider the scalar functions \( \Phi \) as rotational EM-energy, i.e.,

\[ E \equiv \Phi \cdot X = \Phi \cdot \mathbf{X} + \left( \mathbf{X} \times \mathbf{V} \right) \phi . \]

On substituting the values of \( V \) given in (19), we have

\[ \mathcal{L} = \frac{q}{\lambda} \left( e_0 \left( X_0 \Phi - \mathbf{X} \cdot \mathbf{V} \right) + e_j \left( X_0 \mathbf{V} + \Phi \mathbf{X} + \left( \mathbf{X} \times \mathbf{V} \right) \right) \right) . \]

The quaternionic rotational four-momentum can be obtained in presence of EM-field, i.e.,

\[ L_{qem} = e_0 \left( X_0 p_0 - \mathbf{X} \cdot \mathbf{p} \right) - \frac{q}{\lambda} \left( X_0 V_0 - \mathbf{X} \cdot \mathbf{V} \right) \]

\[ + e_j \left( X_0 \mathbf{p} + p_0 \mathbf{X} + \left( \mathbf{X} \times \mathbf{p} \right) \right) - \frac{q}{\lambda} \left( X_0 \mathbf{V} + \Phi \mathbf{X} + \left( \mathbf{X} \times \mathbf{V} \right) \right) . \]

Now, we examine at the quaternionic scalar portion known as rotational EM-energy \( (E_{em}) \) that corresponds to the basis element \( e_0 \) as

\[ E_{em} = \left( X_0 p_0 - \mathbf{X} \cdot \mathbf{p} \right) - \frac{q}{\lambda} \left\{ X_0 \phi_e - \mathbf{X} \cdot \mathbf{A} \right\} - i \lambda \left( X_0 \phi_m - \mathbf{X} \cdot \mathbf{B} \right) \]

\[ = E_0 - (E_c - i \lambda E_m) , \]  \( (45) \)

where the first term \( E_0 \) is the quaternionic rotational energy given in equation (44), the second term indicates the rotational electric energy \( (E_c) \) and the third term indicates the rotational magnetic energy \( (E_m) \), so that

\[ E_c = \frac{q}{\lambda} \left\{ X_0 \phi_e - \mathbf{X} \cdot \mathbf{A} \right\} , \]  \( (46) \)

\[ E_m = \frac{q}{\lambda} \left\{ X_0 \phi_m - \mathbf{X} \cdot \mathbf{B} \right\} . \]  \( (47) \)

Consider the scalar functions \( \phi_e \) and \( \phi_m \) as, respectively, the generalized quaternionic rotational electric and magnetic potentials. Then, the equation (45) can be written as \( E_{em} = E_0 - \frac{q}{\lambda} \Phi^X \), where \( \Phi^X = \phi_e - i \lambda \phi_m \) is the quaternionic rotational generalized scalar potential. Similarly, in equation (44) the quaternionic vector portion denoted the EM-angular momentum \( (L_{em}) \) corresponding to basis element \( e_j \), i.e.,

\[ \mathbf{L}_{em} = \left( X_0 \mathbf{p} + p_0 \mathbf{X} + \left( \mathbf{X} \times \mathbf{p} \right) - \frac{q}{\lambda} \left\{ X_0 \mathbf{A} + \phi_e \mathbf{X} + \left( \mathbf{X} \times \mathbf{A} \right) \right\} - i \lambda \left( X_0 \mathbf{B} + \phi_m \mathbf{X} + \left( \mathbf{X} \times \mathbf{B} \right) \right) \]

\[ = \mathbf{L} - (\mathbf{L}_c - i \lambda \mathbf{L}_m) . \]  \( (48) \)
In equation 48, $L_e$ and $L_m$ are the electric and magnetic angular momentum, respectively. As such,

$$\overrightarrow{L}_e = \frac{Q}{\hbar} \overrightarrow{A}^x,$$

$$\overrightarrow{L}_m = \frac{Q}{\hbar} \overrightarrow{B}^x,$$

where $\overrightarrow{A}^x \mapsto (\mathbf{e}_0 \overrightarrow{A} + \phi \mathbf{X} + \left(\mathbf{X} \times \mathbf{e}_0 \overrightarrow{A}\right))$ can be considered as quaternionic generalized rotational electric vector potential and $\overrightarrow{B}^x \mapsto (\mathbf{e}_0 \overrightarrow{B} + \phi \mathbf{X} + \left(\mathbf{X} \times \mathbf{e}_0 \overrightarrow{B}\right))$ is considered as quaternionic rotational generalized magnetic vector potential. Thus, the generalized quaternion angular momentum in presence of EM-field becomes $\overrightarrow{L}_{em} = \overrightarrow{L} - \frac{Q}{\hbar} \overrightarrow{\nabla}^x$ along with vector potential $\overrightarrow{\nabla}^x = \overrightarrow{A}^x - i\lambda \overrightarrow{B}^x$.

6 Generalized QRD solutions in presence of EM-field

Let us rewrite the generalized Dirac equation in the presence of an EM-field associated with four-momentum minimum coupling 41 for rotating fermions as

$$[H_q \circ (L_q - \mathcal{L}) - \mathfrak{B} \lambda^2 I_q] \circ \Psi = 0.$$  \hspace{1cm} (51)

It is worth noting that equation 51 is rotational analog of the standard Dirac equation in presence of external EM-field as $\left(\overrightarrow{d} \cdot (\overrightarrow{p} - \frac{Q}{\hbar} \overrightarrow{\nabla}) - \beta mc^2\right) \Psi = 0$. As a result, using the equations 44, the equation 51 gives the generalized QRD equation in EM-field,

$$e_0 \left[ D^0 (H) \left( E_0 - \frac{Q}{\hbar} \Phi^x \right) - \lambda \left( \overrightarrow{D} (H) \cdot \left( \overrightarrow{L} - \frac{Q}{\hbar} \overrightarrow{\nabla}^x \right) \right) - \mathfrak{B} \lambda^2 I_0 \right] + e_j \left[ \lambda D^j (H) \left( \overrightarrow{L} - \frac{Q}{\hbar} \overrightarrow{\nabla}^x \right) + D^j (H) \left( E_0 - \frac{Q}{\hbar} \Phi^x \right) + \lambda \left( \overrightarrow{D} (H) \times \left( \overrightarrow{L} - \frac{Q}{\hbar} \overrightarrow{\nabla}^x \right) \right) - \mathfrak{B} \lambda^2 I_j \right] \circ \Psi = 0,$$  \hspace{1cm} (52)

where the components of quaternionic electromagnetic coupling term $H_q \circ (L_q - \mathcal{L}) = H_q \circ L_{em}$ can be expressed as

$$H_q \circ L_{em} = e_0 \left[ D^0 (H) \left( E_0 - \frac{Q}{\hbar} \Phi^x \right) - \lambda D^1 (H) \left( L_1 - \frac{Q}{\hbar} \Phi^1 \right) - \lambda D^2 (H) \left( L_2 - \frac{Q}{\hbar} \Phi^2 \right) - \lambda D^3 (H) \left( L_3 - \frac{Q}{\hbar} \Phi^3 \right) \right] + e_1 \left[ \lambda D^0 (H) \left( L_1 - \frac{Q}{\hbar} \Phi^1 \right) + D^1 (H) \left( E_0 - \frac{Q}{\hbar} \Phi^x \right) + \lambda D^2 (H) \left( L_2 - \frac{Q}{\hbar} \Phi^2 \right) + \lambda D^3 (H) \left( L_3 - \frac{Q}{\hbar} \Phi^3 \right) \right] + e_2 \left[ \lambda D^0 (H) \left( L_2 - \frac{Q}{\hbar} \Phi^2 \right) + D^2 (H) \left( E_0 - \frac{Q}{\hbar} \Phi^x \right) + \lambda D^1 (H) \left( L_1 - \frac{Q}{\hbar} \Phi^1 \right) + \lambda D^3 (H) \left( L_3 - \frac{Q}{\hbar} \Phi^3 \right) \right] + e_3 \left[ \lambda D^0 (H) \left( L_3 - \frac{Q}{\hbar} \Phi^3 \right) + D^3 (H) \left( E_0 - \frac{Q}{\hbar} \Phi^x \right) + \lambda D^1 (H) \left( L_1 - \frac{Q}{\hbar} \Phi^1 \right) + \lambda D^2 (H) \left( L_2 - \frac{Q}{\hbar} \Phi^2 \right) \right],$$

(53)

It is important to note that in the presence of an electromagnetic field, the generalized QRD equation has a dual structure that includes both the EM-rotational energy (scalar component) and the EM-rotational momentum (vector component) of the Dirac fermions. Now, with adding the two and four-component spinors field, we will construct the energy and momentum like solutions of generalized QRD equation 52 in the following subsections.
6.1 Energy solutions corresponding to real quaternion

In this case, we must equate the scalar coefficient of \( e_0 \) as to calculate the energy solution of the generalized QRD equation as

\[
[D^0 (H) \left( E_0 - \frac{Q}{\lambda} \Phi^x \right) - \lambda \left( \vec{D} \cdot (\vec{L} - \frac{Q}{\lambda} \vec{V}^x) \right) - 2\lambda^2 I_0] \Psi = 0. \tag{54}
\]

On substituting the values of quaternionic \( D \)-matrices, we obtain

\[
\begin{pmatrix}
\left( \left( E_0 - \frac{Q}{\lambda} \Phi^x \right) - \lambda^2 I_0 \right) & -i\lambda \left[ \vec{\sigma} \cdot \left( \vec{L} - \frac{Q}{\lambda} \vec{V}^x \right) \right] \\
-\lambda \left[ \vec{\sigma} \cdot \left( \vec{L} - \frac{Q}{\lambda} \vec{V}^x \right) \right] & \left( \left( E_0 - \frac{Q}{\lambda} \Phi^x \right) + \lambda^2 I_0 \right)
\end{pmatrix}
\begin{pmatrix}
\Psi_K \\
\Psi_L
\end{pmatrix} = 0. \tag{55}
\]

If we use a four-component quaternionic spinors field, then we get

\[
\begin{align*}
\Psi_0(E_{em}, L_{em}) &= \frac{i\lambda \left[ \vec{\sigma} \cdot \left( \vec{L} - \frac{Q}{\lambda} \vec{V}^x \right) \right]}{(E_0 - \frac{Q}{\lambda} \Phi^x) - \lambda^2 I_0} \Psi_2(E_{em}, L_{em}), \\
\Psi_1(E_{em}, L_{em}) &= \frac{i\lambda \left[ \vec{\sigma} \cdot \left( \vec{L} - \frac{Q}{\lambda} \vec{V}^x \right) \right]}{(E_0 - \frac{Q}{\lambda} \Phi^x) - \lambda^2 I_0} \Psi_3(E_{em}, L_{em}), \\
\Psi_2(E_{em}, L_{em}) &= \frac{i\lambda \left[ \vec{\sigma} \cdot \left( \vec{L} - \frac{Q}{\lambda} \vec{V}^x \right) \right]}{(E_0 - \frac{Q}{\lambda} \Phi^x) + \lambda^2 I_0} \Psi_0(E_{em}, L_{em}), \\
\Psi_3(E_{em}, L_{em}) &= \frac{i\lambda \left[ \vec{\sigma} \cdot \left( \vec{L} - \frac{Q}{\lambda} \vec{V}^x \right) \right]}{(E_0 - \frac{Q}{\lambda} \Phi^x) + \lambda^2 I_0} \Psi_1(E_{em}, L_{em}).
\end{align*}
\]

Now, the entire spinor solutions for the QRD equation may now be calculated by introducing the plane wavefunction \( \psi \) as \( \Psi = \chi \psi \) where \( \chi \) is the Dirac spinor in presence of EM-field. To calculate the rotational energy solutions of generalized quaternionic spinors with spin up state \( \chi \), we put \( \Psi_0 = 1, \Psi_1 = 0 \) for positive energy and \( \Psi_2 = 1, \Psi_3 = 0 \) for negative energy Dirac spinors as

\[
\Psi : \rightarrow \Psi^\dagger_+ = N_+ \begin{pmatrix}
1 \\
0 \\
0 \\
0
\end{pmatrix} \begin{pmatrix}
\frac{i\lambda \left[ \vec{\sigma} \cdot \left( \vec{L} - \frac{Q}{\lambda} \vec{V}^x \right) \right]}{(E_0 - \frac{Q}{\lambda} \Phi^x) + \lambda^2 I_0} e^{i \left( \vec{k} \cdot \vec{r} - \omega \right)}, \\
\frac{-i\lambda \left[ \vec{\sigma} \cdot \left( \vec{L} - \frac{Q}{\lambda} \vec{V}^x \right) \right]}{(E_0 - \frac{Q}{\lambda} \Phi^x) - \lambda^2 I_0} e^{i \left( \vec{k} \cdot \vec{r} - \omega \right)}, \\
0 \\
1
\end{pmatrix},
\]

\[
\Psi : \rightarrow \Psi^\dagger_- = N_- \begin{pmatrix}
1 \\
0 \\
0 \\
0
\end{pmatrix} \begin{pmatrix}
\frac{i\lambda \left[ \vec{\sigma} \cdot \left( \vec{L} - \frac{Q}{\lambda} \vec{V}^x \right) \right]}{(E_0 - \frac{Q}{\lambda} \Phi^x) - \lambda^2 I_0} e^{i \left( \vec{k} \cdot \vec{r} - \omega \right)}, \\
\frac{-i\lambda \left[ \vec{\sigma} \cdot \left( \vec{L} - \frac{Q}{\lambda} \vec{V}^x \right) \right]}{(E_0 - \frac{Q}{\lambda} \Phi^x) + \lambda^2 I_0} e^{i \left( \vec{k} \cdot \vec{r} - \omega \right)}, \\
0 \\
1
\end{pmatrix},
\]

where \( \Psi = (\Psi_+, \Psi_-) \).
where \( N_+ = \frac{(E_0 - \frac{Q}{\lambda} \Phi^x) + \lambda^2 l_0}{\sqrt{[(E_0 - \frac{Q}{\lambda} \Phi^x)^2 + \lambda^2 (T - \frac{Q}{\lambda} \nabla^x)^2]}} \) and \( N_- = \frac{(E_0 - \frac{Q}{\lambda} \Phi^x) - \lambda^2 l_0}{\sqrt{[(E_0 - \frac{Q}{\lambda} \Phi^x)^2 + \lambda^2 (T - \frac{Q}{\lambda} \nabla^x)^2]}} \) are the normalization constants. Similarly, for spin down state we put \( \Psi_0 = 0, \Psi_1 = 1 \) for positive energy and \( \Psi_2 = 0, \Psi_3 = 1 \) for negative energy as

\[
\Psi :\rightarrow \Psi^\dagger_+ = N_+ \begin{pmatrix}
0 \\
1 \\
0 \\
i \lambda \left[ \bar{\sigma} \left( T - \frac{Q}{\lambda} \nabla^x \right) \right] / (E_0 - \frac{Q}{\lambda} \Phi^x + \lambda^2 l_0) \\
i \lambda \left[ \bar{\sigma} \left( T - \frac{Q}{\lambda} \nabla^x \right) \right] / (E_0 - \frac{Q}{\lambda} \Phi^x - \lambda^2 l_0) \\
0 \\
1
\end{pmatrix} e^{i \left( \bar{k} \cdot \bar{r} - \omega t \right)} ,
\]  
\( \text{(62)} \)

\[
\Psi :\rightarrow \Psi^\dagger_- = N_- \begin{pmatrix}
0 \\
1 \\
0 \\
i \lambda \left[ \bar{\sigma} \left( T - \frac{Q}{\lambda} \nabla^x \right) \right] / (E_0 - \frac{Q}{\lambda} \Phi^x + \lambda^2 l_0) \\
i \lambda \left[ \bar{\sigma} \left( T - \frac{Q}{\lambda} \nabla^x \right) \right] / (E_0 - \frac{Q}{\lambda} \Phi^x - \lambda^2 l_0) \\
0 \\
1
\end{pmatrix} e^{i \left( \bar{k} \cdot \bar{r} - \omega t \right)} .
\]  
\( \text{(63)} \)

### 6.2 Momentum solutions corresponding to pure quaternion

In order to calculate the rotational momentum solutions of generalized QRD equation in EM-field, we equate the coefficient of \( e_j \) in equation (62) as

\[
\left[ \lambda D^0 (H) \left( \bar{T} - \frac{Q}{\lambda} \nabla^x \right) + D^j (H) \left( E_0 - \frac{Q}{\lambda} \Phi^x \right) + \lambda \left( \bar{T} (H) \times \left( \bar{T} - \frac{Q}{\lambda} \nabla^x \right) \right) _j - 2 \lambda^2 I_j \right] \Psi = 0 ,
\]  
\( \text{(64)} \)

which gives

\[
\begin{pmatrix}
\lambda \left[ \left( \bar{T} - \frac{Q}{\lambda} \nabla^x \right) - \lambda I_j \right] \\
i \lambda \left[ \left( E_0 - \frac{Q}{\lambda} \Phi^x \right) \times \left( \bar{T} - \frac{Q}{\lambda} \nabla^x \right) \right] _j \\
i \lambda \left[ \left( E_0 - \frac{Q}{\lambda} \Phi^x \right) \times \left( \bar{T} - \frac{Q}{\lambda} \nabla^x \right) \right] _j \\
\lambda \left[ \left( \bar{T} - \frac{Q}{\lambda} \nabla^x \right) + \lambda I_j \right]
\end{pmatrix} \begin{pmatrix}
\Psi_K \\
\Psi_L
\end{pmatrix} = 0 .
\]  
\( \text{(65)} \)

On substituting the values of \( \Psi_K \) and \( \Psi_L \), we obtain four component spinors fields as

\[
\Psi_0 \left( E_{cm}, \bar{T}_{cm} \right) = \frac{-i \left[ e_j \left( E_0 - \frac{Q}{\lambda} \Phi^x \right) + \lambda \left( \bar{T} \times \left( \bar{T} - \frac{Q}{\lambda} \nabla^x \right) \right) _j \right]}{\lambda \left( \left( \bar{T} - \frac{Q}{\lambda} \nabla^x \right) - \lambda I_j \right)} \Psi_2 \left( E_{cm}, \bar{T}_{cm} \right) ,
\]  
\( \text{(66)} \)

\[
\Psi_1 \left( E_{cm}, \bar{T}_{cm} \right) = \frac{-i \left[ e_j \left( E_0 - \frac{Q}{\lambda} \Phi^x \right) + \lambda \left( \bar{T} \times \left( \bar{T} - \frac{Q}{\lambda} \nabla^x \right) \right) _j \right]}{\lambda \left( \left( \bar{T} - \frac{Q}{\lambda} \nabla^x \right) - \lambda I_j \right)} \Psi_3 \left( E_{cm}, \bar{T}_{cm} \right) ,
\]  
\( \text{(67)} \)

\[
\Psi_2 \left( E_{cm}, \bar{T}_{cm} \right) = \frac{-i \left[ e_j \left( E_0 - \frac{Q}{\lambda} \Phi^x \right) + \lambda \left( \bar{T} \times \left( \bar{T} - \frac{Q}{\lambda} \nabla^x \right) \right) _j \right]}{\lambda \left( \left( \bar{T} - \frac{Q}{\lambda} \nabla^x \right) + \lambda I_j \right)} \Psi_0 \left( E_{cm}, \bar{T}_{cm} \right) ,
\]  
\( \text{(68)} \)
\[ \Psi_3(E_{em}, \vec{L}_{em}) = \frac{-i\left[e_j\left(E_0 - \frac{\Phi}{\lambda} \Phi^x\right) + \lambda\left(\vec{\sigma} \times \left(\vec{L} - \frac{\Phi}{\lambda} \vec{\nabla} x\right)\right)\right]}{\lambda \left(\vec{L} - \frac{\Phi}{\lambda} \vec{\nabla} x + \lambda I_j\right)} \Psi_1(E_{em}, \vec{L}_{em}), \]  

(69)

where equations (66) and (67) yield the angular momentum solution for the anti-particle whereas from equations (68) and (69) yield the momentum solutions for the particle. The term \( e_jE_{em} \) is due to the electromagnetic energy of spin \( \frac{1}{2} \) particles and \( \lambda\left(\vec{L}_{em} - \lambda I_j\right), \lambda\left(\vec{L}_{em} + \lambda I_j\right) \) are the energy terms for the anti-particles whereas \( \vec{\sigma} \times \vec{L}_{em} \) can be interpreted as the directional interaction governed by spin and the orbital angular momentum. As a result, for angular momentum solutions with rotating fermions in the spin up state, we put \( \Psi_0 = 1, \Psi_1 = 0 \) for fermionic angular momentum and \( \Psi_2 = 1, \Psi_3 = 0 \) for anti-fermionic angular momentum as

\[
\Psi :\rightarrow \Psi^+ = M_+ \begin{pmatrix} 1 \\ 0 \\ -i\left[e_j\left(E_0 - \frac{\Phi}{\lambda} \Phi^x\right) + \lambda\left(\vec{\sigma} \times \left(\vec{L} - \frac{\Phi}{\lambda} \vec{\nabla} x\right)\right)\right] \\ \lambda \left(\vec{L} - \frac{\Phi}{\lambda} \vec{\nabla} x + \lambda I_j\right) \end{pmatrix} e^{\frac{i}{\hbar} \left(\vec{k} \cdot \vec{r} - \omega t\right)}, \]

(70)

\[
\Psi :\rightarrow \Psi^- = M_- \begin{pmatrix} 1 \\ 0 \\ -i\left[e_j\left(E_0 - \frac{\Phi}{\lambda} \Phi^x\right) + \lambda\left(\vec{\sigma} \times \left(\vec{L} - \frac{\Phi}{\lambda} \vec{\nabla} x\right)\right)\right] \\ \lambda \left(\vec{L} - \frac{\Phi}{\lambda} \vec{\nabla} x - \lambda I_j\right) \end{pmatrix} e^{\frac{i}{\hbar} \left(\vec{k} \cdot \vec{r} - \omega t\right)}. \]

(71)

where the normalization constants are

\[
M_+ = \frac{\lambda \left(\vec{L} - \frac{\Phi}{\lambda} \vec{\nabla} x + \lambda I_j\right)}{\sqrt{\left[\lambda \left(\vec{L} - \frac{\Phi}{\lambda} \vec{\nabla} x + \lambda I_j\right)\right]^2 - \left[e_j\left(E_0 - \frac{\Phi}{\lambda} \Phi^x\right) + \lambda\left(\vec{\sigma} \times \left(\vec{L} - \frac{\Phi}{\lambda} \vec{\nabla} x\right)\right)\right]^2}},
\]

\[
M_- = \frac{\lambda \left(\vec{L} - \frac{\Phi}{\lambda} \vec{\nabla} x - \lambda I_j\right)}{\sqrt{\left[\lambda \left(\vec{L} - \frac{\Phi}{\lambda} \vec{\nabla} x - \lambda I_j\right)\right]^2 - \left[e_j\left(E_0 - \frac{\Phi}{\lambda} \Phi^x\right) + \lambda\left(\vec{\sigma} \times \left(\vec{L} - \frac{\Phi}{\lambda} \vec{\nabla} x\right)\right)\right]^2}}.
\]

Correspondingly, to write the angular momentum solutions of rotating fermions in the spin down state, we put \( \Psi_0 = 0, \Psi_1 = 1 \) for fermionic angular momentum and \( \Psi_2 = 0, \Psi_3 = 1 \) for anti-fermionic angular momentum as

\[
\Psi :\rightarrow \Psi^\dagger = M_+ \begin{pmatrix} 0 \\ 1 \\ 0 \\ -i\left[e_j\left(E_0 - \frac{\Phi}{\lambda} \Phi^x\right) + \lambda\left(\vec{\sigma} \times \left(\vec{L} - \frac{\Phi}{\lambda} \vec{\nabla} x\right)\right)\right] \\ \lambda \left(\vec{L} - \frac{\Phi}{\lambda} \vec{\nabla} x + \lambda I_j\right) \end{pmatrix} e^{\frac{i}{\hbar} \left(\vec{k} \cdot \vec{r} - \omega t\right)}, \]

(72)
\[ \Psi : \rightarrow \Psi^\dagger = M \begin{pmatrix} 0 & -\frac{e_i (E_0 - \frac{2}{\lambda} \Phi \times (\vec{E} - \frac{2}{\lambda} \vec{V}^x))}{\lambda (\vec{E} - \frac{2}{\lambda} \vec{V}^x)} \cr \frac{1}{\lambda} \xi (\vec{E} - \frac{2}{\lambda} \vec{V}^x) - \lambda L_j \cr 0 \cr 1 \end{pmatrix} e^{\frac{i}{\hbar} (\vec{E} \cdot \vec{r} - \omega t)} . \]  

(73)

Now, using fundamental symmetries and conservation rules, we will examine the validity of the foregoing QRD equations and their solutions in the EM field.

### 6.2.1 Lorentz-Poincaré Invariance

The Lorentz transformation can be expressed using a 4 \times 4 transformation matrix \( \Lambda \) as

\[ (X^\mu)' = \sum_{\nu=0}^{3} (\Lambda^\mu_{\nu}) X^\nu, \ (\mu, \nu = 0, 1, 2, 3), \]  

(74)

where

\[ \Lambda^\mu_{\nu} = \begin{pmatrix} \cosh \omega & -i \sinh \omega & 0 & 0 \\
 i \sinh \omega & \cosh \omega & 0 & 0 \\
 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 1 \end{pmatrix} . \]  

(75)

Here \( \omega \) denotes the boost parameter. To check the Lorentz covariance, we can use the covariant form of the QRD equation as follows:

\[ \left[ \gamma^\mu \left( L^\mu_q - \frac{Q}{\lambda} (X^\mu_q V_{q}^\mu) \right) - \lambda I_q \right] \Psi = 0 . \]  

(76)

Here \( \gamma^\mu \) are the Dirac gamma matrices which can be represented in terms of quaternionic D-matrices as

\[ \gamma^0 = \mathfrak{B} = \begin{pmatrix} 1 & 0 \\
 0 & -1 \end{pmatrix} \quad \text{and} \quad \gamma^i = \mathfrak{B} D^i (H) = \begin{pmatrix} 0 & ie_j \\
 -ie_j & 0 \end{pmatrix} . \]  

(77)

Now, we have

\[ \left[ \gamma^\mu \left( L^\mu_q - \frac{Q}{\lambda} (X^\mu_q V_{q}^\mu) \right) - \lambda I_q \right] \Psi' = 0 , \]  

(78)

with

\[ \Psi' = S \Psi , \]  

(79)

where \( S \) is a 4 \times 4 matrix. Therefore, we may write

\[ S^{-1} \left[ \gamma^\mu S \Lambda^\mu_{\nu} \left( L^\mu_q - \frac{Q}{\lambda} (X^\mu_q V_{q}^\mu) \right) - \lambda I_q S \right] \Psi = 0 , \]  

(80)
which shows
\[ S^{-1} \gamma^\mu \left( L_q^\mu - \frac{Q}{\lambda} (X_q^\mu V_q^\mu) \right) = \Lambda' \gamma'^\nu \left( L_{q'}^\mu - \frac{Q}{\lambda} (X_{q'}^\mu V_{q'}^\mu) \right). \] (81)

where the corresponding term \((\lambda L_q)\) is treated as a constant during the transformation in order to keep the covariant form of the Dirac equation. Furthermore, the Lorentz boost can define \(S\) for a Lorentz transformation along the \(e_1\)-axis as
\[ S = \cosh \frac{\omega}{2} + iD^1(H) \sinh \frac{\omega}{2}, \] (82)
where \(D^1(H) \sim \begin{pmatrix} 0 & ie_1 \\ ie_1 & 0 \end{pmatrix}\). As a result, under Lorentz transformation, the generalized QRD equation in the EM-field is well invariant.

6.2.2 Gauge Invariance

Let us consider the analogous gauge transformation of fermionic spinor field as
\[ \Psi' = \exp \left( -i \frac{Q}{\hbar \lambda} \xi(\mathbf{r},t) \right) \Psi, \] (83)
where \(\xi(\mathbf{r},t)\) is an arbitrary differentiable gauge function in Minkowski space-time. The gauge transformation of the two four-potentials is also introduced as
\[ \mathbf{A}' = \mathbf{A} - \nabla \xi \quad \text{and} \quad \phi_e' = \phi_e + \left( \frac{\partial \xi}{\partial t} \right), \] (84)
\[ \mathbf{B}' = \mathbf{B} - \nabla \xi \quad \text{and} \quad \phi_m' = \phi_m + \left( \frac{\partial \xi}{\partial t} \right), \] (85)
along with
\[ \left[ H_q \circ \left( L_q - \frac{Q}{\lambda} (X_q \circ V_q') \right) - \mathfrak{H} \lambda^2 I_q \right] \circ \Psi' = 0, \] (86)
where \(V_q'\) is the generalized two four-potential transforming under the gauge transformation and can be defined as
\[ \mathbf{V}' = \mathbf{V} - \nabla \xi \quad \text{and} \quad \Phi' = \Phi + \left( \frac{\partial \xi}{\partial t} \right), \] (87)
where \(\mathbf{V}\) and \(\Phi\) are already defined in equations (18) and (19).
6.2.3 Duality Invariance

To check the duality invariance of generalized QRD equation in EM-field, let us define the transformation of two field tensors $F_{\mu\nu}$ and $F'_{\mu\nu}$ as

$$
F'_{\mu\nu} \rightarrow F_{\mu\nu} \cos \alpha - \mathcal{F}_{\mu\nu} \sin \alpha \quad (88)
$$

$$
F'_{\mu\nu} \rightarrow \mathcal{F}_{\mu\nu} \sin \alpha + F_{\mu\nu} \cos \alpha \quad (89)
$$

The electric and magnetic field vectors can be transformed by replacing $F_{\mu\nu} \rightarrow E$ and $F_{\mu\nu} \rightarrow H$ where $E$ and $H$ are the electric and magnetic field vectors. Then, the transformation becomes

$$
\begin{pmatrix}
E \\
H
\end{pmatrix} := \begin{pmatrix}
0 & -i\lambda \\
i\lambda & 0
\end{pmatrix}
\begin{pmatrix}
E \\
H
\end{pmatrix} .
$$

(90)

Here we took the general case of $\alpha = \frac{\pi}{2}$. The matrix $\begin{pmatrix}
0 & -i\lambda \\
i\lambda & 0
\end{pmatrix}$ is the duality transformation matrix. Accordingly, dual scalar potentials, dual vector potentials and dual charge can also transform as

$$
\begin{pmatrix}
\phi_e \\
\phi_m
\end{pmatrix} := \begin{pmatrix}
0 & -i\lambda \\
i\lambda & 0
\end{pmatrix}
\begin{pmatrix}
\phi_e \\
\phi_m
\end{pmatrix} .
$$

(91)

$$
\begin{pmatrix}
A \\
B
\end{pmatrix} := \begin{pmatrix}
0 & -i\lambda \\
i\lambda & 0
\end{pmatrix}
\begin{pmatrix}
A \\
B
\end{pmatrix} ,
$$

(92)

$$
\begin{pmatrix}
e \\
m
\end{pmatrix} := \begin{pmatrix}
0 & -i\lambda \\
i\lambda & 0
\end{pmatrix}
\begin{pmatrix}
e \\
m
\end{pmatrix} .
$$

(93)

The generalized QRD equation in the presence of an EM-field is thus well invariant under duality transformation when applying the above transformation equations.

6.2.4 CPT Invariance

In order to check the CPT invariance of the generalized QRD equation in the EM-field, Table 2 summarizes the transformations of various physical parameters under parity, time reversal, and charge conjugation [44, 45]. Apart from the physical quantities, the transformation of the fermionic spinor fields can be represented as

$$
P \Psi(\vec{r}, t) P^{-1} = \gamma_0 \Psi(-\vec{r}, t) ,
$$

(94)

$$
C \Psi(\vec{r}, t) C^{-1} = i\gamma_2 \Psi^\ast(\vec{r}, t) ,
$$

(95)

$$
T \Psi(\vec{r}, t) T^{-1} = i\gamma_1 \gamma_3 \Psi(\vec{r}, -t) ,
$$

(96)

where $P$, $C$ and $T$ represent parity, charge conjugation and time reversal operators while $\gamma_0 = \mathbb{B}$, $\gamma_j = \mathbb{B} D^j(H)$ are equivalent to the quaternionic matrices. From equations (94) to (96), we get

$$
\text{CPT} \left[ (H_q \circ L_{qem} - \mathbb{B} \lambda^2 I_q) \circ \Psi \right] T^{-1} P^{-1} C^{-1} = 0 ,
$$

(97)

Therefore, under CPT transformation, [51] yields an invariant QRD equation in the EM-field.
Table 2: CPT transformation of various quaternionic physical quantities

| Physical quantities | P-symmetry | C-symmetry | T-symmetry |
|---------------------|------------|------------|------------|
| ∇                   | −∇        | ∇         | ∇         |
| ∂t                  | ∂t        | ∂t        | −∂t       |
| λ                   | −λ        | λ         | −λ        |
| X₀                  | X₀        | X₀        | −X₀       |
| X                   | −X        | X         | −X        |
| p₀                  | p₀        | p₀        | p₀        |
| p                   | −p        | p         | −p        |
| ϕₑ                  | −ϕₑ       | ϕₑ        | ϕₑ        |
| ϕₘ                  | −ϕₘ       | −ϕₘ       | −ϕₘ       |
| A                   | −A        | −A        | A         |
| B                   | B         | −B        | B         |
| E                   | −E        | −E        | E         |
| H                   | −H        | −H        | H         |

7 Generalized Quaternionic Electromagnetic Moment

7.1 Corresponding to the real quaternion

The Dirac fermions has an electromagnetic moment or electromagnetic energy that may be determined using the two component form of Pauli’s spinors. To find the electromagnetic moment corresponding to the scalar coefficient in the generalized QRD equation, we may start with the given equation \((55)\) with condition \(E_{em} = E'_{em} + \lambda^2 I_0\) where \(E_{em}\) is rotational EM-energy, \(E'_{em}\) is the rotational kinetic energy and \(\lambda^2 I_0\) is rotational analogous of rest mass energy in EM-field, then one can write
\[
(E'_{em} + 2\lambda^2 I_0) \Psi_L - i\lambda \left( \overrightarrow{e} \cdot \overrightarrow{L}_{em} \right) \Psi_K = 0 ,
\]
(98)
Since, in the non-relativistic limit \(E'_{em} << \lambda^2 I_0\), we have
\[
\Psi_L \simeq \left( i \overrightarrow{e} \cdot \overrightarrow{L}_{em} \right) \frac{1}{2\lambda I_0} \Psi_K .
\]
(99)
As a result, by substituting equation \((99)\) in second component solution of equation \((55)\), we obtain
\[
(E_{em} - \lambda^2 I_0) \Psi_K + \left( \frac{1}{2\lambda I_0} (L_{em})^2 - \frac{Q\hbar}{2\lambda I_0} \left( \overrightarrow{e} \cdot \overrightarrow{H} \right) - \frac{iQ\hbar}{2\lambda I_0} \left( \overrightarrow{e} \cdot \overrightarrow{E} \right) \right) \Psi_K = 0 .
\]
(100)
This equation can be referred as the generalized Schrodinger-Pauli like equation where the unperturbed energy term is \(\frac{1}{2\lambda I_0} (L_{em})^2 + \lambda^2 I_0\), while the perturbed energy terms are \(\frac{Q\hbar}{2\lambda I_0} \left( \overrightarrow{e} \cdot \overrightarrow{H} \right)\) and \(\frac{iQ\hbar}{2\lambda I_0} \left( \overrightarrow{e} \cdot \overrightarrow{E} \right)\) corresponding to magnetic and electric dipole moments \(\frac{Q\hbar}{2\lambda I_0}\) and \(\frac{iQ\hbar}{2\lambda I_0}\), respectively. Furthermore, we have found dual perturbed energies, respectively, the generalized electric dipole energy \((46)\) i.e. \(\kappa \left( \overrightarrow{e} \cdot \overrightarrow{H} \right)\) and the generalized magnetic dipole energy i.e. \(i\kappa \left( \overrightarrow{e} \cdot \overrightarrow{E} \right)\) due to EM‐interaction in fermionic field, where \(\kappa = \frac{Q\hbar}{2\lambda I_0}\) is a constant.
7.2 Corresponding to the pure quaternion

In this case, we choose vector coefficient in the generalized QRD equation and applying the condition \( \overrightarrow{L}_{em} = \overrightarrow{L}'_{em} + \lambda I_j \), and obtain

\[
\lambda \left( L'_{em} + 2\lambda I_j \right) \Psi_L + i \left[ e_j E_{em} + \lambda \left( \overrightarrow{e} \times \overrightarrow{L}_{em} \right)_j \right] \Psi_K = 0 ,
\]

(101)

For the non-relativistic limit \( \lambda L'_{em} \ll 2\lambda^2 I_j \), then equation (101) gives

\[
\Psi_L \approx \frac{-i \left[ e_j E_{em} + \left( \overrightarrow{e} \times \overrightarrow{L}_{em} \right)_j \right]}{2\lambda^2 I_j} \Psi_K .
\]

(102)

Substituting equation (101) in second component solution of equation (65), and obtain

\[
\left[ \lambda \overrightarrow{L}'_{em} - \frac{1}{2\lambda^2 I_j} \left( E_{em}^2 \right)_j + \frac{1}{2I_j} \left\{ -2 \left( \overrightarrow{L}_{em}^2 \right)_j + \frac{Qe}{2I_j} \left( \overrightarrow{e} \cdot \overrightarrow{H} \right) + \frac{Qhi}{2I_j} \left( \overrightarrow{e} \cdot \overrightarrow{E} \right) \right\} + \frac{2e_j E_{em} \left( \overrightarrow{e} \times \overrightarrow{L}_{em} \right)_j}{2\lambda I_j} \right] \Psi_K = 0 .
\]

(103)

As equation (100), the equation (103) can describe the electromagnetic energy in quaternionic rotational momentum space. The last term denotes the energy corresponding to the directional spin orbit coupling in the quaternionic angular momentum-space.

On the other hand, if we consider ‘rotating dyons’ in the generalized fermionic spinor dual-fields that includes both an electric field due to electrons and a magnetic field due to magnetic monopoles, then the following field constituents for generalized Dirac spinors may be used for unified dyonic fields [14, 35, 36]:

\[
Q_{Dyon} = (e - i\lambda m) , \quad \text{(Dyonic Charge)}
\]

\[
\Phi^X_{Dyon} = \left( \phi^e + i\lambda \phi^m \right) , \quad \text{(Dyonic Scalar Potential)}
\]

\[
\overrightarrow{V}^X_{Dyon} = \left( \overrightarrow{A}^e - i\lambda \overrightarrow{B}^m \right) , \quad \text{(Dyonic Vector Potential)}
\]

\[
E_{Dyon} = (E_e - i\lambda E_m) , \quad \text{(Dyonic Rotational EM-Energy)}
\]

\[
\overrightarrow{L}_{Dyon} = \left( \overrightarrow{L}_e - i\lambda \overrightarrow{L}_m \right) , \quad \text{(Dyonic EM-Angular Momentum)}
\]

\[
E_{Dyon}^{di} = \frac{Qe}{2I_0} \left( \overrightarrow{e} \cdot \overrightarrow{\Psi}_{Dyon} \right) , \quad \text{(Dyonic Dipole Energy)}
\]

\[
\overrightarrow{\Psi}_{Dyon} = \left( \overrightarrow{E} - i\lambda \overrightarrow{H} \right) , \quad \text{(Dyonic EM-Field)}
\]

Hence, we examined a symmetrical form of the generalized QRD equation in the presence of an external EM-field, in which two four-potentials are recognized as the gauge potentials associated with a particle (or antiparticle) containing the simultaneous existence of electric and magnetic charge (monopole) of dyons.
### Table 3: Comparison of some physical quantities

| Physical quantities                                      | General Case                                                                 | Quaternionic-Rotational Case                                                                 |
|----------------------------------------------------------|------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------|
| Dirac equation without EM-field:                         | $(\alpha \cdot p - \beta mc^2) \psi = 0$                                  | $(H_q \circ L_q - 2\mathbf{B}\lambda^2 I_q) \circ \Psi = 0$                                      |
| Minimal Substitution:                                    | $\tilde{p} \to \tilde{p} - \frac{e}{c} A, E \to E - e\phi$                  | $L^\mu :\to L^\mu - \frac{Q}{c} (X^\mu \circ V^\mu) \equiv \Pi^\mu_{\mu} (\mu = 0, 1, 2, 3)$  |
| Dirac equation with EM-field:                            | $(\tilde{\alpha} \cdot (\tilde{p} - \frac{e}{c} \hat{A}) - \beta mc^2) \psi = 0$ | $(H_q \circ (L_q - \frac{Q}{c} (X_q \circ V_q)) - 2\mathbf{B}\lambda^2 I_q) \circ \Psi = 0$ |
| Spinor solutions:                                        | $\psi^+_{\pm} = N_+ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \psi^+ = N_+ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ | $\psi^+_+ = N_+ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \psi^+_+ = N_+ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ |
| Spinor solutions with EM-field:                          | $\psi^+_+ = N_+ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$               | $\psi^+_+ = N_+ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\psi^+_+ = N_+ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ |
|                                                          |                                                                              |                                                                                               |
|                                                          |                                                                              |                                                                                               |
8 Conclusion

Hypercomplex division algebra (viz. quaternion) is regarded as a necessity to study an additional dimension of any system or theory in a single framework. In this case, quaternionic representation of four-potential, four-momentum, four-position etc. have been discussed in view of Minkowski structure. The generalized quaternionic rotational Dirac equation and its solutions for fermions without EM-field has been discussed in equation (29). The generalized QRD equation has the benefit of representing both aspects as rotational energy and rotating momentum corresponding to quaternionic scalar and vector components. The most important feature of quaternionic analysis is that it allows for the dual representation of four vectors in a unified structure. As a result, the EM-field interaction in the extended QRD equation is stated in equation (44), where equations (45) and (48) define the resulting quaternionic rotational EM-energy corresponding to the scalar component and quaternionic rotational angular momentum corresponding to the vector component, respectively. Accordingly, the electric and magnetic angular momentum are established in terms of quaternionic generalized rotational electric and magnetic vector potentials. Furthermore, the four component Dirac spinors with spin up and down states have been used to examine the solutions for quaternionic rotational energy and momentum equations in the EM-field. The generalized Schrödinger-Pauli-like energy equation, which is related with unperturbed and perturbed energies as a result of EM-field interaction, has been constructed in equation (100). Due to EM-interaction in a fermionic field, we have interpreted the dual perturbed energies, respectively, the generalized electric dipole energy and the generalized magnetic dipole energy in view of quaternionic analysis. Table-3 shows a comparison of the linear and rotational behavior of several quaternionic physical quantities. The beauty of present analysis, to examine the different symmetries the extended quaternionic rotational Dirac equation in presence of EM-field has been shown to be well invariant under the Lorentz, gauge, duality, and CPT invariances. With the interaction of EM-field, the extended QRD equation possesses symmetrical structure in electric and magnetic fields, with two four-potentials associated with Dirac fermions (or anti-fermions) including the simultaneous existence of electric and magnetic charge of rotating dyons.

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