Integral laminations on non-orientable surfaces

S. Öykü YURTTAŞ1,∗, Mehmetcik PAMUK1
1 Department of Mathematics, Dicle University, 21280 Diyarbakır, Turkey
saadet.yurttas@dicle.edu.tr
2 Department of Mathematics, Middle East Technical University, Ankara, Turkey
mpamuk@metu.edu.tr

Abstract

We describe triangle coordinates for integral laminations on a non-orientable surface $N_{k,n}$ of genus $k$ with $n$ punctures and one boundary component, and give an explicit bijection from the set of integral laminations on $N_{k,n}$ to $(\mathbb{Z}^{2(n+k-2)} \times \mathbb{Z}^k) \setminus \{0\}$.

Keywords: non-orientable surfaces, triangle coordinates, Dynnikov coordinates

AMS Mathematics Subject classification: 57N05, 57N16, 57M50

1 Introduction

Let $N_{k,n}$ be a non-orientable surface of genus $k$ with $n$ punctures and one boundary component. In this paper we shall describe the generalized Dynnikov coordinate system for the set of integral laminations $\mathcal{L}_{k,n}$, and give an explicit bijection between $\mathcal{L}_{k,n}$ and $(\mathbb{Z}^{2(n+k-2)} \times \mathbb{Z}^k) \setminus \{0\}$. To be more specific, we shall first take a particular collection of $3n + 2k - 4$ arcs and $k$ curves embedded in $N_{k,n}$, and describe each integral lamination by an element of $\mathbb{Z}_0^{3n+2k-4} \times \mathbb{Z}^k$, its geometric intersection numbers with these arcs and curves. Generalized Dynnikov coordinates are certain linear combinations of these integers that provide a one-to-one correspondence between $\mathcal{L}_{k,n}$ and $(\mathbb{Z}^{2(n+k-2)} \times \mathbb{Z}^k) \setminus \{0\}$.

The motivation for this paper comes from a recent work of Papadopoulos and Penner [7] where they provide analogues for non-orientable surfaces of several results from Thurston theory of surfaces which were studied only for orientable surfaces.

∗Correspondence: saadet.yurttas@dicle.edu.tr.
surfaces before [4, 8]. Here we shall give the analogy of the Dynnikov Coordinate System \([2]\) on the finitely punctured disk which has several useful applications such as giving an efficient method for the solution of the word problem of the \(n\)-braid group \([1]\), computing the geometric intersection number of integral laminations \([9]\), and counting the number of components they contain \([11]\).

Throughout the text we shall work on a standard model of \(N_{k,n}\) as illustrated in Figure 1 where a disc with a cross drawn within it represents a crosscap, that is the interior of the disc is removed and the antipodal points on the resulting boundary component are identified (i.e. the boundary component bounds a Möbius band).

The structure of the paper is as follows. In Section 1.1 we give the necessary terminology and background. In Section 2 we describe and study the triangle coordinates for integral laminations on \(N_{k,n}\), and construct the generalized Dynnikov Coordinate System giving the bijection \(\rho: L_{k,n} \rightarrow (\mathbb{Z}^{2(n+k-2)} \times \mathbb{Z}^k) \setminus \{0\}\). An explicit formula for the inverse of this bijection is given in Theorem 2.14.

1.1 Basic terminology and background

A simple closed curve in \(N_{k,n}\) is inessential if it bounds an unpunctured disk, once punctured disk, or an unpunctured annulus. It is called essential otherwise. A simple closed curve is called 2-sided (respectively 1-sided) if a regular neighborhood of the curve is an annulus (respectively Möbius band). We say that a 2-sided curve is non-primitive if it bounds a Möbius band \([7]\), and a 1-sided curve is non-primitive if it is a core curve of a Möbius band. They are called primitive otherwise.

An integral lamination \(L\) on \(N_{k,n}\) is a disjoint union of finitely many essential simple closed curves in \(N_{k,n}\) modulo isotopy. Let \(A_{k,n}\) be the set of arcs \(\alpha_i (1 \leq i \leq 2n-2), \beta_i (1 \leq i \leq n+k-1), \gamma_i (1 \leq i \leq k-1)\) which have each endpoint either on the boundary or at a puncture, and the curves \(c_i (1 \leq i \leq k)\) which are the core curves of Möbius bands in \(N_{k,n}\) as illustrated in Figure 1 the arcs \(\alpha_{2i-3} \text{ and } \alpha_{2i-2}\) for \(2 \leq i \leq n\) join the \(i\)-th puncture to \(\partial N_{k,n}\), the arc \(\beta_i\) has both end points on \(\partial N_{k,n}\) and passes between the \(i\)-th and \((i+1)\)-st punctures for \(1 \leq i \leq n-1\), the \(n\)-th puncture and the first crosscap for \(i = n\), and the \((i-n)\)-th and \((i+1-n)\)-th crosscaps for \(n + 1 \leq i \leq n + k - 1\). The arc \(\gamma_i\) (\(1 \leq i \leq k-1\)) has both endpoints on \(\partial N_{k,n}\) and surrounds the \(i\)-th crosscap.

The surface is divided by these arcs into \(2n + 2k - 2\) regions, \(2n + k - 3\) of these are triangular since each \(\Delta_i (1 \leq i \leq 2n-2)\) and \(\Sigma_i (1 \leq i \leq k-1)\) is bounded by three arcs when the boundary of the surface is identified to a point. The two triangles \(\Delta_{2i-3}\) and \(\Delta_{2i-2}\) on the left and right hand side of the \(i\)-th puncture are defined by the arcs \(\alpha_{2i-3}, \alpha_{2i-2}, \beta_{i-1}\) and \(\alpha_{2i-3}, \alpha_{2i-2}, \beta_i\) respectively. The triangle \(\Sigma_i\) is defined by the arcs \(\gamma_i, \beta_n+i-1, \beta_{n+i}\). Each \(\Delta'_i (1 \leq i \leq k-1)\) is bounded by \(\gamma_i\), and the two end regions \(\Delta_0\) and \(\Delta'_k\) are bounded by \(\beta_1\) and \(\beta_{n+k-1}\) respectively.
Figure 1: The arcs $\alpha_i$, $\beta_i$, $\gamma_i$, the 1-sided curves $c_1, c_2, \ldots, c_k$ and the regions $\Delta_i$ and $\Sigma_i$.

Given $L \in L_{k,n}$, let $L$ be a taut representative of $L$ with respect to the elements of $A_{k,n}$. That is, $L$ intersects each of the arcs and curves in $A_{k,n}$ minimally.

Figure 2: There is 1 left loop component in the first case and 2 right loop components in the second case. There are 2 above and 3 below components in each case.

**Definition 1.1.** Set $S_i = \Delta_{2i-1} \cup \Delta_{2i}$ for each $i$ with $1 \leq i \leq n-1$. A path component of $L$ in $S_i$ is a component of $L \cap S_i$. There are four types of path components in $S_i$ as depicted in Figure 2.
• An above component has end points on $\beta_i$ and $\beta_{i+1}$, passing across $\alpha_{2i-1}$.
• A below component has end points on $\beta_i$ and $\beta_{i+1}$, passing across $\alpha_{2i}$.
• A left loop component has both end points on $\beta_{i+1}$.
• A right loop component has both end points on $\beta_i$.

Figure 3: There is 1 right core loop and 1 straight core component in the first case; 1 left loop and 1 left core loop component in the second case; 1 right non-core loop and 1 right core loop component in the third case and 1 1-sided and 1 2-sided non-primitive curves in the fourth case. There are 2 above and 2 below components in each case.

Definition 1.2. Set $S'_i = \Delta'_i \cup \Sigma_i$ for each $1 \leq i \leq k - 1$. A path component of $L$ in $S'_i$ is a component of $L \cap S'_i$. There are 7 types of path components in $S'_i$ as depicted in Figure 3.

• An above component has end points on $\beta_{n+i-1}$ and $\beta_{n+i}$, and passes across $\gamma_i$ without intersecting $c_i$.
• A below component has end points on $\beta_{n+i-1}$ and $\beta_{n+i}$, and doesn’t pass across $\gamma_i$.
• A left loop component has both end points on $\beta_{n+i}$.
• A right loop component has both end points on $\beta_{n+i-1}$.

If a loop component intersects $c_i$, it is called core loop component otherwise it is called non-core loop component.
• A straight core component has end points on $\beta_{n+i-1}$ and $\beta_{n+i}$, and intersects $c_i$. 
A non-primitive 1-sided curve,
If \( L \) contains a non-primitive 1-sided curve \( c_i \) we depict it with a ring with end points on the \( i \)-th crosscap as shown in the fourth case in Figure 3.

A non-primitive 2-sided curve.

## 2 Triangle coordinates

Let \( L \) be a taut representative of \( \mathcal{L} \). Write \( \alpha_i, \beta_i, \gamma_i \) and \( c_i \) for the geometric intersection number of \( L \) with the arc \( \alpha_i, \beta_i, \gamma_i \) and the core curve \( c_i \) respectively. It will always be clear from the context whether we mean the arc or the geometric intersection number assigned on the arc.

**Definition 2.1.** The triangle coordinate function \( \tau: \mathcal{L}_{k,n} \rightarrow (\mathbb{Z}^{3n+2k-4}_{\geq 0} \times \mathbb{Z}^k) \setminus \{0\} \)

is defined by

\[
\tau(L) = (\alpha_1, \ldots, \alpha_{2n-2}; \beta_1, \ldots, \beta_{n+k-1}; \gamma_1, \ldots, \gamma_{k-1}; c_1, \ldots, c_k)
\]

where \( c_i = -1 \) if \( L \) contains the \( i \)-th core curve; \( c_i = -2m \) if it contains \( m \in \mathbb{Z}^+ \) disjoint copies 2-sided non-primitive curves around the \( i \)-th crosscap, and \( c_i = -2m - 1 \) if it contains \( m \) disjoint copies of 2-sided non-primitive curves around the \( i \)-th crosscap plus the \( i \)-th core curve.

**Remark 2.2.** Let \( b_i = \beta_i - \beta_{i+1} \) for \( 1 \leq i \leq n+k-2 \). Then in each \( S_i \) (\( 1 \leq i \leq n-1 \)) and \( S'_i \) (\( n \leq i \leq n + k - 2 \)) there are \( |b_i| \) loop components. Furthermore, if \( b_i < 0 \) these loop components are left, and if \( b_i > 0 \) they are right.

The proof of the next lemma is obvious from Figure 2.

**Lemma 2.3.** Let \( 1 \leq i \leq n-1 \). The number of above and below components in \( S_i \) are given by \( a_{S_i} = \alpha_{2i-1} - |b_i| \) and \( b_{S_i} = \alpha_{2i} - |b_i| \) respectively.

Let \( \lambda_i \) and \( \lambda_{c_i} \) denote the number of non-core and core loop components, \( \psi_i \) the number of straight core components, and \( a_{S'_i} \) and \( b_{S'_i} \) the number of above and below components in \( S'_i \).

**Lemma 2.4.** Let \( L \) be a taut representative of \( \mathcal{L} \in \mathcal{L}_{k,n} \), and set \( c_i^+ = \max(c_i, 0) \). Then for each \( 1 \leq i \leq k-1 \) we have

\[
\lambda_i = \max(|b_{n+i-1}| - c_i^+, 0), \quad \lambda_{c_i} = \min(|b_{n+i-1}|, c_i^+), \\
\psi_i = \max(c_i^+ - |b_{n+i-1}|, 0).
\]
Proof. Assume that $L$ doesn’t contain any non-primitive curve in $S'_i$. Since $c_i$ gives the sum of straight core and core loop components and $|b_{n+i-1}|$ gives the sum of non-core loop and core loop components in $S'_i$ (see Figure 3) we have

\[ c_i = \psi_i + \lambda_{c_i} \quad \text{and} \quad |b_{n+i-1}| = \lambda_i + \lambda_{c_i}. \tag{1} \]

If $c_i > |b_{n+i-1}|$, then clearly there exists a straight core component in $S'_i$, and hence no non-core loop component in $S'_i$ that is $\lambda_i = 0$. Therefore in this case, $\lambda_{c_i} = |b_{n+i-1}|$ and hence $\psi_i = c_i - |b_{n+i-1}|$ by Equation (1).

If $c_i < |b_{n+i-1}|$, there exists a non-core loop component in $S'_i$, and hence no straight core components in $S'_i$ that is $\lambda_i = 0$. Therefore $c_i = \lambda_{c_i}$ and hence $\lambda_i = |b_{n+i-1}| - c_i$ by Equation (1). We get

\[
\lambda_i = \max(|b_{n+i-1}| - c_i, 0) \\
\psi_i = \max(c_i - |b_{n+i-1}|, 0).
\]

Also if $|b_{n+i-1}| < c_i$, $\lambda_i = 0$ and hence $\lambda_{c_i} = |b_{n+i-1}|$, if $|b_{n+i-1}| > c_i$, $\psi_i = 0$ and hence $\lambda_{c_i} = c_i$ by Equation (1) Therefore we get, $\lambda_{c_i} = \min(|b_{n+i-1}|, c_i)$.

Finally, if $L$ contains a non-primitive curve in $S'_i$, there can be no straight core and core loop component in $S'_i$ that is $\psi_i = \lambda_{c_i} = 0$, hence $\lambda_i = |b_{n+i-1}|$. Since $c_i < 0$ by definition, setting $c_i^+ = \max(c_i, 0)$ we can write

\[
\lambda_i = \max(|b_{n+i-1}| - c_i^+, 0), \quad \lambda_{c_i} = \min(|b_{n+i-1}|, c_i^+), \\
\psi_i = \max(c_i^+ - |b_{n+i-1}|, 0).
\]

\[
\square
\]

Lemma 2.5. Let $L$ be a taut representative of $\mathcal{L} \in \mathcal{L}_{k,n}$. For each $1 \leq i \leq k-1$ we have

\[
a_{S'_i} = \frac{\gamma_i}{2} - |b_{n+i-1}| - \psi_i \\
b_{S'_i} = \max(\beta_{n+i-1}, \beta_{n+i}) - |b_{n+i-1}| - \frac{\gamma_i}{2}
\]

Proof. To compute the number of above and below components in $S'_i$ we observe that each path component other than a below component in $S'_i$ intersects $\gamma_i$ twice, that is $\gamma_i = 2(a_{S'_i} + |b_{n+i-1}| + \psi_i)$. Therefore we get,

\[
a_{S'_i} = \frac{\gamma_i}{2} - |b_{n+i-1}| - \psi_i.
\]

\[ 6 \]
To compute the number of below components, we note that the sum of all path components in $S'_i$ is given by $\beta = \max(\beta_{n+i} - 1, \beta_{n+i})$. Then $b_{S'_i}$ is $\beta$ minus the number of above, straight core components and twice the number loop components in $S'_i$ (each loop component intersects $\beta$ twice). We get

$$b_{S'_i} = \max(\beta_{n+i} - 1, \beta_{n+i}) - a_{S'_i} - 2|b_{n+i-1}| - \psi_i$$

$$= \max(\beta_{n+i-1}, \beta_{n+i}) - |b_{n+i-1}| - \frac{\gamma_i}{2}$$

Another way of expressing $a_{S'_i}$ and $b_{S'_i}$ is given in item P4. in Properties 2.12.

**Remark 2.6.** Observe that the loop components in $\Delta_0$ are always left and the number of them is given by $\frac{\beta_n}{2}$. Similarly, the loop components in $\Delta'_k$ are always right and the number of core and non-core loop components in $\Delta'_k$ are given by $c_k$ and $\lambda_k = \frac{\beta_n + k}{2} - c_k$.

Lemma 2.7 and Lemma 2.8 are obvious from Figure 2 and Figure 3.

**Lemma 2.7.** There are equalities for each $S_i$:

- **When there are left loop components** ($b_i < 0$),
  $$\alpha_{2i} + \alpha_{2i-1} = \beta_{i+1}$$
  $$\alpha_{2i} + \alpha_{2i-1} - \beta_i = 2|b_i|,$$

- **When there are right loop components** ($b_i > 0$),
  $$\alpha_{2i} + \alpha_{2i-1} = \beta_i$$
  $$\alpha_{2i} + \alpha_{2i-1} - \beta_{i+1} = 2|b_i|,$$

- **When there are no loop components** ($b_i = 0$),
  $$\alpha_{2i} + \alpha_{2i-1} = \beta_i = \beta_{i+1}.$$

**Lemma 2.8.** There are equalities for each $S'_i$:

- **When there are left loop components** ($b_{n+i-1} < 0$),
  $$a_{S'_i} + b_{S'_i} + \psi_i + 2|b_{n+i-1}| = \beta_{n+i}$$
  $$a_{S'_i} + b_{S'_i} + \psi_i = \beta_{n+i-1}$$
• **When there are right loop components** \((b_{n+i-1} > 0)\)

\[
a_{S'_i} + b_{S'_i} + \psi_i + 2|b_{n+i-1}| = \beta_{n+i-1}.
\]

\[
a_{S'_i} + b_{S'_i} + \psi_i = \beta_{n+i}.
\]

• **When there are no loop components** \(b_{n+i-1} = 0\)

\[
a_{S'_i} + b_{S'_i} + \psi_i = \beta_{n+i} = \beta_{n+i-1}.
\]

**Example 2.9.** Let \(\tau(L) = (4, 2, 2, 6; 2, 6, 8, 4; 8, 1, 1)\) be the triangle coordinates of an integral lamination \(L \in \mathcal{L}_{2,3}\). We shall show how we draw \(L\) from its given triangle coordinates. First, we compute the loop components in the two end regions \(\Delta_0\) and \(\Delta'_2\) using Remark 2.6. Since \(\beta_1 = 2\) there is one loop component in \(\Delta_0\). Similarly, since \(\beta_4 = 4\) and \(c_2 = 1\), we get \(\lambda_2 = \frac{\beta_4}{2} - c_2 = 1\).

Next, we compute loop components in \(S_1\), \(S_2\) and \(S'_1\). Since \(b_i = \frac{\beta_i - \beta_{i+1}}{2}\) for each \(1 \leq i \leq 3\) we have \(b_1 = -2, b_2 = -1\). Hence there are two left loop components in \(S_1\), and one left component in \(S_2\). Similarly since \(b_3 = 2\) there are 2 right loop components in \(S'_1\), and by Lemma 2.3 \(\lambda_3 = \max(|b_3| - c_1, 0) = 1\) (hence \(\psi_1 = 0\)) and \(\lambda_{c_1} = \min(|b_3|, c_1) = 1\). Using Lemma 2.3 and Lemma 2.5 we compute the number of above and below components. We get \(a_{S_1} = \alpha_1 - |b_1| = 2, b_{S_1} = \alpha_2 - |b_1| = 0, a_{S_2} = \alpha_3 - |b_2| = 1, b_{S_2} = \alpha_4 - |b_2| = 5,\) and

\[
a_{S'_1} = \frac{\gamma_1}{2} - |b_3| - \psi_1 = 2
\]

\[
b_{S'_1} = \max(\beta_3, \beta_4) - |b_3| - \frac{\gamma_1}{2} = 2.
\]

Connecting the path components in each \(\Delta_0, \Delta'_2, S_1, S_2\) and \(S'_1\) we draw the integral lamination as shown in Figure 4.

Figure 4: \(\tau(L) = (4, 2, 2, 6; 2, 6, 8, 4; 8, 1, 1)\)
Lemma 2.10. The triangle coordinate function $\tau: L_{k,n} \to (\mathbb{Z}_{\geq 0}^{3n+2k-4} \times \mathbb{Z}^k) \setminus \{0\}$ is injective.

Proof. We can determine the number of loop, above and below components in each $S_i$ by Remark 2.2 and Lemma 2.3, core and non-core loop, straight core, above and below components in each $S'_i$ by Lemma 2.4 and Lemma 2.5 as illustrated in Example 2.9. The components in each $S_i$ and $S'_i$ are glued together in a unique way up to isotopy, and hence $L$ is constructed uniquely. \qed

Remark 2.11. The triangle coordinate function $\tau: L_{k,n} \to (\mathbb{Z}_{\geq 0}^{3n+2k-4} \times \mathbb{Z}^k) \setminus \{0\}$ is not surjective: an integral lamination must satisfy the triangle inequality in each $S_i$ and $S'_i$, and some additional conditions such as the equalities in Lemma 2.7 and Lemma 2.8.

Next, we give a list of properties an integral lamination $L \in L_{k,n}$ satisfies in terms of its triangle coordinates as in [9], and then construct a new coordinate system from the triangle coordinates which describes integral laminations in a unique way. In particular, we shall generalize the Dynnikov coordinate system [1–3, 5, 9–11] for $N_{k,n}$.

Properties 2.12. Let $L$ be a taut representative of $L \in L_{k,n}$.

P1. Every component of $L$ intersects each $\beta_i$ an even number of times. Recall from Remark 2.2 that the number of loop components is given by $|b_i|$ where $b_i = \frac{\beta_i - \beta_{i+1}}{2}$.

P2. Set $x_i = |\alpha_{2i} - \alpha_{2i-1}|$ and $t_i = |a_{S'_i} - b_{S'_i}|$. Then $x_i$ and $t_i$ gives the difference between the number of above and below components in $S_i$ and $S'_i$ respectively. Set $m_i = \min\{\alpha_{2i} - |b_i|, \alpha_{2i-1} - |b_i|\}; 1 \leq i \leq n-1$ and $n_i = \min\{a_{S'_i}, b_{S'_i}\}; 1 \leq i \leq k-1$. See Figure 5. Note that $x_i$ is even since $L$ intersects $\alpha_{2i} \cup \alpha_{2i-1}$ an even number of times. Clearly, this may not hold for $t_i$ since when $\psi_i$ is odd the sum of above and below components (and hence their difference) is odd. See Lemma 2.8.

P3. Set $2a_i = \alpha_{2i} - \alpha_{2i-1} (|a_i| = x_i/2)$. Then, by Lemma 2.7 we get

- If $b_i \geq 0$, then $\beta_i = \alpha_{2i} + \alpha_{2i-1}$ and hence
  $$\alpha_{2i} = a_i + \frac{\beta_i}{2} \quad \text{and} \quad \alpha_{2i-1} = -a_i + \frac{\beta_i}{2}.$$  

- If $b_i \leq 0$, then $\beta_{i+1} = \alpha_{2i} + \alpha_{2i-1}$ and hence
  $$\alpha_{2i} = a_i + \frac{\beta_{i+1}}{2} \quad \text{and} \quad \alpha_{2i-1} = -a_i + \frac{\beta_{i+1}}{2}.$$  

9
Figure 5: $m_i$ and $n_i$ denote the smaller of above and below components in $S_i$ and $S'_i$ respectively.

That is,

$$\alpha_i = \begin{cases} 
(-1)^i a_{\lfloor i/2 \rfloor} + \frac{b_{\lfloor i/2 \rfloor}}{2} & \text{if } b_{\lfloor i/2 \rfloor} \geq 0, \\
(-1)^i a_{\lfloor i/2 \rfloor} + \frac{b_{1+\lfloor i/2 \rfloor}}{2} & \text{if } b_{\lfloor i/2 \rfloor} \leq 0.
\end{cases}$$

where $\lfloor i/2 \rfloor$ denotes the smallest integer that is not less than $i/2$.

P4. Since $t_i = a_{S'_i} - b_{S'_i}$ for $1 \leq i \leq k - 1$, from Lemma 2.8 we get

- If $b_{n+i-1} \geq 0$ then $a_{S'_i} + b_{S'_i} + \psi_i + 2b_{n+i-1} = \beta_{n+i-1}$, and
  $$a_{S'_i} = \frac{t_i - \psi_i + \beta_{n+i-1} - 2b_{n+i-1}}{2}$$
• If $b_{n+i-1} \leq 0$ then $a_{S'_i} + b_{S'_i} + \psi_i - 2b_{n+i-1} = \beta_{n+i}$, and
\[
a_{S'_i} = \frac{t_i - \psi_i + \beta_{n+i} + 2b_{n+i-1}}{2}
\]
And hence
\[
a_{S'_i} = \frac{t_i - \psi_i + \max(\beta_{n+i}, \beta_{n+i-1}) - 2|b_{n+i-1}|}{2}
\]
Similarly we compute
\[
b_{S'_i} = \frac{-t_i - \psi_i + \max(\beta_{n+i}, \beta_{n+i-1}) - 2|b_{n+i-1}|}{2}
\]

P5. It is easy to observe from Figure 5 that
\[
\beta_i = 2[|a_i| + \max(b_i, 0) + m_i] \quad \text{for} \quad 1 \leq i \leq n - 1
\]
\[
\beta_{n+i} = |t_i| + 2 \max(b_{n+i-1}, 0) + \psi_i + 2n_i \quad \text{for} \quad 1 \leq i \leq k - 1.
\]
Therefore, since $b_i = \frac{\beta_i - \beta_{i+1}}{2} \quad 1 \leq i \leq n + k - 2$ we can compute $\beta_1$ using one of the two equations below:
\[
\beta_1 = 2 \left[ |a_i| + \max(b_i, 0) + m_i + \sum_{j=1}^{i-1} b_j \right] \quad \text{for} \quad 1 \leq i \leq n - 1,
\]
\[
\beta_1 = |t_i| + 2 \max(b_{n+i-1}, 0) + \psi_i + 2n_i + 2 \sum_{j=1}^{n+i-2} b_j \quad \text{for} \quad 1 \leq i \leq k - 1.
\]

Figure 6: $L^*$ is a simple closed curve on the right but it is not on the left.
P6. Some integral laminations contain R-components: an R-component of $L$ has geometric intersection numbers $i(R, \alpha_j) = 1$ for each $1 \leq j \leq 2n - 2$, $i(R, \beta_j) = 2$ for each $1 \leq j \leq n + k - 1$ and $i(R, \gamma_j) = 2$ for each $1 \leq j \leq k - 1$, which has its end points on the $k$-th crosscap (denoted red in Figure 6). Set $L^* = L \setminus R$. Note that $L^*$ is a component of $L$ which isn’t necessarily a simple closed curve (the two possible cases are depicted in Figure 6). Let $\alpha^*_i, \beta^*_i$ and $\gamma^*_i$ denote the number of intersections of $L^*$ with the arcs $\alpha_i, \beta_i$ and $\gamma_i$ respectively. Define $a^*_i, b^*_i, t^*_i$ and $\lambda^*_i, \lambda^*_i, a^*_S, b^*_S$ and $\psi^*_i$ similarly as above. We therefore have

$$\beta^*_i = 2 \left[ |a^*_i| + \max(b^*_i, 0) + m^*_i + \sum_{j=1}^{i-1} b^*_j \right] \text{ for } 1 \leq i \leq n - 1,$$

$$\beta^*_i = |t^*_i| + 2 \max(b^*_n+i-1, 0) + \psi^*_i + 2n^*_i + 2 \sum_{j=1}^{n+i-2} b^*_j \text{ for } 1 \leq i \leq k - 1.$$

where $m^*_i = \min \{ \alpha^*_2i - |b^*_1|, \alpha^*_{2i-1} - |b^*_i| \}; 1 \leq i \leq n - 1$ and $n^*_i = \min \{ a^*_S, b^*_S \}; 1 \leq i \leq k - 1$. Furthermore, there is some $m^*_i = 0$, or some $n^*_i = 0$ since otherwise $L^*$ would have above and below components in each $S_i$ and $S^*_i$ which would yield curves parallel to $\partial N_{k,n}$, or $L^*$ would contain R-components which is impossible by definition. Write $a^*_i = a_i, b^*_i = b_i, t^*_i = t_i$ since deleting R-components doesn’t change the $a, b, t$ values. Set

$$X_i = 2 \left[ |a_i| + \max(b_i, 0) + \sum_{j=1}^{i-1} b_j \right] \text{ for } 1 \leq i \leq n - 1,$$

$$Y_i = |t_i| + 2 \max(b_{n+i-1}, 0) + \psi_i + 2 \sum_{j=1}^{n+i-2} b_j \text{ for } 1 \leq i \leq k - 1.$$

Then one of the three following cases hold for $L^*$:

I. If $m^*_i > 0$ for all $1 \leq i \leq n - 1$, then there is some $j$ with $1 \leq j \leq k - 1$ such that $n^*_j = 0$. Therefore, $\beta^*_i > X_i$ and $\beta^*_i = Y_j$.

II. If $n^*_i > 0$ for all $1 \leq i \leq k - 1$, then there is some $j$ with $1 \leq j \leq n - 1$ such that $m^*_j = 0$. Therefore, $\beta^*_i > Y_i$ and $\beta^*_i = X_j$.
III. There is some $i$ with $1 \leq i \leq n-1$ such that $m^*_i = 0$ and some $j$ with $1 \leq j \leq k-1$ such that $n^*_j = 0$. Therefore, $\beta^*_1 = X_i = Y_j$.

We therefore have

$$\beta^*_i = \max(X, Y) - 2 \sum_{j=1}^{i-1} b_j$$

where

$$X = 2 \max_{1 \leq r \leq n-1} \left\{ |a_r| + \max(b_r, 0) + \sum_{j=1}^{r-1} b_j \right\}$$

and

$$Y = \max_{1 \leq s \leq k-1} \left\{ |t_s| + 2 \max(b_{n+s-1}, 0) + \psi_s + 2 \sum_{j=1}^{n+s-2} b_j \right\}.$$

P7. If $L$ doesn’t have an $R$-component, that is if $L^* = L$ then $2c_k \leq \beta^*_{n+k-1} = \beta_{n+k-1}$ since $\beta_{n+k-1} = 2c_k + 2\lambda_k$. If $L$ has an $R$-component then $2c_k > \beta^*_{n+k-1}$ and $\lambda_k = 0$. See Figure 6. Hence the number of $R$-components of $L$ is given by

$$R = \max(0, 2c_k - \beta^*_{n+k-1})/2.$$ 

For example, the integral laminations in Figure 6 (from left to right) has $c_1 = 2, \beta^*_5 = 2$, and hence $R = 1$; and $c_1 = 1, \beta^*_5 = 0$, and hence $R = 1$. Then $L$ is constructed by identifying the two end points of an $R$ component with the pieces of $L^*$ on the $k$-th crosscap. Since $R$-components intersect each $\beta_i$ twice we get

$$\beta_i = \beta^*_i + 2R; 1 \leq i \leq n + k - 1.$$ 

Then

$$\beta_i = \max(X, Y) - 2 \sum_{j=1}^{i-1} b_j + 2R$$ 

Also, from item P3. we have
\[\alpha_i = \begin{cases} 
(-1)^i a_{\lceil i/2 \rceil} + \frac{\beta_{\lceil i/2 \rceil}}{2} & \text{if } b_{\lceil i/2 \rceil} \geq 0, \\
(-1)^i a_{\lceil i/2 \rceil} + \frac{\beta_{\lceil i/2 \rceil}}{2} & \text{if } b_{\lceil i/2 \rceil} \leq 0, 
\end{cases}\]

Finally, it is easy to observe from Figure 3 that

\[\gamma_i = 2(a_{S'_i} + |b_{n+i-1}| + \psi_i)\]

Making use of the properties above, we shall define the generalized Dynnikov coordinate system which coordinatizes \(\mathcal{L}_{k,n}\) bijectively and with the least number of coordinates.

**Definition 2.13.** The generalized Dynnikov coordinate function

\[\rho: \mathcal{L}_{k,n} \rightarrow (\mathbb{Z}^{2(n+k-2)} \times \mathbb{Z}^k) \setminus \{0\}\]

is defined by

\[\rho(\mathcal{L}) = (a; b; t; c) := (a_1, \ldots, a_{n-1}; b_1, \ldots, b_{n+k-2}; t_1, \ldots, t_{k-1}; c_1, \ldots, c_k)\]

where

\[a_i = \frac{\alpha_{2i} - \alpha_{2i-1}}{2} \quad \text{for } 1 \leq i \leq n - 1,\]
\[b_i = \frac{\beta_i - \beta_{i+1}}{2} \quad \text{for } 1 \leq i \leq n + k - 2,\]
\[t_i = a_{S'_i} - b_{S'_i} \quad \text{for } 1 \leq i \leq k - 1,\]

where \(a_{S'_i}\) and \(b_{S'_i}\) are as given in Lemma 2.5.

**Theorem 2.14.** Let \((a; b; t; c) \in (\mathbb{Z}^{2(n+k-2)} \times \mathbb{Z}^k) \setminus \{0\}\). Set

\[X = 2 \max_{1 \leq r \leq n-1} \left\{|a_r| + \max(b_r, 0) + \sum_{j=1}^{r-1} b_j\right\}\]
\[Y = \max_{1 \leq s \leq k-1} \left\{|t_s| + 2 \max(b_{n+s-1}, 0) + \psi_s + 2 \sum_{j=1}^{n+s-2} b_j\right\}.\]
Then \((a; b; t; c)\) is the Dynnikov coordinate of exactly one element \(L \in \mathcal{L}_{k,n}\) which has

\[
\beta_i = \max(X, Y) - 2 \sum_{j=1}^{i-1} b_j + 2R, \quad (2)
\]

\[
\alpha_i = \begin{cases} 
(-1)^i a_{\lceil i/2 \rceil} + \frac{\beta_{i+1/2}}{2} & \text{if } b_{\lceil i/2 \rceil} \geq 0, \\
(-1)^i a_{\lceil i/2 \rceil} + \frac{\beta_{i+1/2}}{2} & \text{if } b_{\lceil i/2 \rceil} \leq 0,
\end{cases} \quad (3)
\]

\[
\gamma_i = 2(a_{S_i'} + |b_{n+i-1}| + \psi_i) \quad (4)
\]

where \(a_{S_i'}\) is defined as in item P4. in Properties 2.12.

**Proof.** Given \(L \in \mathcal{L}_{k,n}\) with \(\tau(L) = (\alpha, \beta, \gamma, c)\) and \(\rho(L) = (a, b, t, c)\), Properties 2.12 show that \(\alpha, \beta\) and \(\gamma\) must be given by (2), (3) and (4) respectively, and hence \(L\) is unique by Lemma 2.10. Therefore \(\rho\) is injective. By Properties 2.12 we can draw non-intersecting path components in each \(S_i (1 \leq i \leq n-1)\), \(S_i' (1 \leq i \leq k-1)\), \(\Delta_0\) and \(\Delta_k'\) which intersect each element of \(\mathcal{A}_{k,n}\) the number of times given by \((\alpha, \beta, \gamma, c)\). Gluing together these path components gives a disjoint union of simple closed curves in \(N_{k,n}\). There are no curves that bound a puncture or parallel to the boundary by construction, and hence \((\alpha, \beta, \gamma, c)\) where \(\alpha, \beta\) and \(\gamma\) are defined by (2), (3) and (4) respectively, correspond to some \(L\) with \(\rho(L) = (a, b, t, c)\). Therefore, \(\rho\) is surjective. \(\square\)

**Example 2.15.** Let \(\rho(\mathcal{L}) = (a_1; b_1, b_2; t_1; c_1, c_2) = (-1; 2, 0; 1; 1, 0)\) be the generalized Dynnikov coordinates of an integral lamination \(\mathcal{L}\) on \(N_{2,2}\). We shall use Theorem 2.14 to compute the triangle coordinates of \(\mathcal{L}\) from which we determine the number of path components in \(S_1\) and \(S_1'\), and hence draw \(\mathcal{L}\) as illustrated in Example 2.9. By Lemma 2.4 \(\psi_1 = \max(c_1^+ - |b_2|, 0) = 1\) so we have

\[
X = 2(|a_1| + \max(b_1, 0)) = 6 \quad \text{and} \quad Y = |t_1| + 2 \max(b_2, 0) + \psi_1 + 2b_1 = 6.
\]

Therefore

\[
\beta_1 = \max(6, 6) = 6, \quad \beta_2 = \max(6, 6) - 2b_1 = 2, \quad \beta_3 = \max(6, 6) - 2(b_1 + b_2) = 2,
\]

\[
\alpha_1 = -a_1 + \frac{\beta_1}{2} = 4, \quad \alpha_2 = a_1 + \frac{\beta_1}{2} = 2.
\]

Since \(0 = 2c_2 < \beta_3^* = 2\), there are no \(R\)-components by item P8. of Properties 2.12. Since \(\beta_1 = 6\) there are 3 loop components in \(\Delta_0\), and since \(\beta_3 = 2\) and \(c_2 = 0\), there is one non-core loop component in \(\Delta_k'\) that is \(\lambda_2 = 1\). By Remarks 2.2 \(b_1 = 2\) and \(b_2 = 0\), and hence there are 2 right loop components in \(S_1\) and no
loop components in $S'_1$. By Lemma 2.3 we compute that $a_{S_1} = \alpha_1 - |b_1| = 2$ and $b_{S_1} = \alpha_2 - |b_1| = 0$. Finally by item P4 of Properties 2.12

$$a_{S'_1} = \frac{t_1 - \psi_1 + \max(\beta_2, \beta_3) - 2|b_2|}{2} = 1$$

$$b_{S'_1} = \frac{-t_1 - \psi_1 + \max(\beta_2, \beta_3) - 2|b_2|}{2} = 0$$

Gluing together the path components in $S_1$ and $S'_1$ we construct the integral lamination depicted in Figure 7

![Figure 7: $\rho(L) = (-1; 2, 0; 1, 1, 0)$](image)

**Remark 2.16.** Generalized Dynnikov coordinates for integral laminations can be extended in a natural way to generalized Dynnikov coordinates of measured foliations [5]; the transverse measure on the foliation [4, 7, 8] assigns to each element in $A_{k,n}$ a non-negative real number, and hence each measured foliation is described by an element of $(\mathbb{R}^{3n+2k-4} \times \mathbb{R}^k) \setminus \{0\}$, the associated measures of the arcs and curves of $A_{k,n}$. Therefore, the Generalized Dynnikov coordinate system for measured foliations is defined similarly (see Definition 2.13), and provides a one-to-one correspondence between the set of measured foliations (up to isotopy and Whitehead equivalence) on $N_{k,n}$ and $(\mathbb{R}^{2(n+k-2)} \times \mathbb{R}^k) \setminus \{0\}$.

**References**

[1] Dehornoy, Patrick and Dynnikov, Ivan and Rolfsen, Dale and Wiest, Bert, *Ordering braids*, volume 148 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2008.

[2] I. Dynnikov, *On a Yang-Baxter mapping and the Dehornoy ordering*, Uspekhi Mat. Nauk, 57(3(345)): 151–152, 2002.

[3] Dynnikov, I. and Wiest, B., *On the complexity of braids*, J. Eur. Math. Soc. (JEMS), 9(4):801–840, 2007
[4] Fathi, A. and Laudenbach, F. and Poenaru, V. *Travaux de Thurston sur les surfaces*, Société Mathématique de France, (66) 284, 1979.

[5] Hall, Toby and Yurtta¸s, S. Oyku, *On the topological entropy of families of braids*, Topology Appl., 156(8):1554–1564, 2009.

[6] Moussafir, J-O., *On computing the entropy of braids*, Funct. Anal. Other Math., 1(1):37–46, 2006.

[7] Papadopoulos, A. and Penner, R. C., *Hyperbolic metrics, measured foliations and pants decompositions for non-orientable surfaces*, Asian J. Math., 20(1):157–182, 2016.

[8] Thurston, W. P., *On the geometry and dynamics of diffeomorphisms of surfaces*, Bull. Amer. Math. Soc. (N.S.), 19(2):417–431, 1988.

[9] Yurtta¸s, S. ¨Oyk¨ u, *Geometric intersection of curves on punctured disks*, Journal of the Mathematical Society of Japan, 65(4):1554–1564, 2013.

[10] Yurtta¸s, S. ¨Oyk¨ u, *Dynnikov and train track transition matrices of pseudo-Anosov braids*, Discrete Contin. Dyn. Syst., 36(1):541–570, 2016.

[11] Yurtta¸s, S. ¨Oyk¨ u and Hall, Toby, *Counting components of an integral lamination*, manuscripta math. doi:10.1007/s00229-016-0885-4, 2016.