FURTHER RESULTS FOR STARLIKE FUNCTIONS RELATED WITH BOOTH LEMNISCATE

ABSTRACT. In this paper we investigate an interesting subclass BS(α) (0 ≤ α < 1) of starlike functions in the unit disk Δ. The class BS(α) was introduced by Kargar et al. [R. Kargar, A. Ebadian and J. Sokol, On Booth lemniscate and starlike functions, Anal. Math. Phys. (2017) DOI: 10.1007/s13324-017-0187-3] which is strongly related to the Booth lemniscate. Some geometric properties of this class of analytic functions including, radius of starlikeness of order γ (0 ≤ γ < 1), the image of f({z : |z| < r}) when f ∈ BS(α), an special example and estimate of bounds for Re(f(z)/z) are studied.

1. Introduction

Let H denote the class of analytic functions in the open unit disk Δ = {z ∈ C : |z| < 1} on the complex plane C. Also let A denote the subclass of H including of functions normalized by f(0) = f′(0) − 1 = 0. The subclass of A consists of all univalent functions f(z) in Δ is denoted by S. We denote by B the class of functions w(z) analytic in Δ with w(0) = 0 and |w(z)| < 1, (z ∈ Δ). For two analytic and normalized functions f and g, we say that f is subordinate to g, written f ≺ g in Δ, if there exists a function w ∈ B such that f(z) = g(w(z)) for all z ∈ Δ. In special case, if the function g is univalent in Δ, then f(z) < g(z) ⇔ (f(0) = g(0) and f(Δ) ⊂ g(Δ)). It is easy to see that for any complex numbers λ ≠ 0 and μ, we have:

\[ f(z) < g(z) \Rightarrow λf(z) + μ < λg(z) + μ. \]

The set of all functions f ∈ A that are starlike univalent in Δ will be denoted by S∗ and the set of all functions f ∈ A that are convex univalent in Δ will be denoted by K. Robertson (see [5]) introduced and studied the class S∗(γ) of starlike functions of order γ ≤ 1 as follows

\[ S∗(γ) := \left\{ f ∈ A : \text{ Re} \left\{ \frac{zf′(z)}{f(z)} \right\} > γ, \ z ∈ Δ \right\}. \]

We note that if γ ∈ [0, 1), then a function in S∗(γ) is univalent. Also we say that f ∈ K(γ) (the class of convex functions of order γ) if and only if zf′(z) ∈ S∗(γ). In particular we put S∗(0) = S∗ and K(0) = K.

Recently, Kargar et al. [3] introduced and studied a class functions related to the Booth lemniscate as follows.

Definition 1.1. (see [3]) The function f ∈ A belongs to the class BS(α), 0 ≤ α < 1, if it satisfies the condition

\[ \left( \frac{zf′(z)}{f(z)} - 1 \right) < \frac{z}{1 - αz^2} \quad (z ∈ Δ). \]

Recall that [4], a one-parameter family of functions given by

\[ F_α(z) := \frac{z}{1 - αz^2} = \sum_{n=1}^{∞} α^{n-1} z^{2n-1} \quad (z ∈ Δ, \ 0 ≤ α ≤ 1), \]

are starlike univalent when 0 ≤ α ≤ 1 and are convex for 0 ≤ α ≤ 3−2√2 ≈ 0.1715. We have also F_α(Δ) = D(α), where

\[ D(α) = \left\{ x + iy ∈ C : (x^2 + y^2)^2 - \frac{x^2}{(1-α)^2} - \frac{y^2}{(1+α)^2} < 0, (0 ≤ α < 1) \right\} \]

and

\[ D(1) = \{ x + iy ∈ C : (\forall t ∈ (−∞, −i/2) ∪ [i/2, ∞)) [x + iy ≠ it] \}. \]

It is clear that the curve

\[ (x^2 + y^2)^2 - \frac{x^2}{(1-α)^2} - \frac{y^2}{(1+α)^2} = 0 \quad (x, y) ≠ (0, 0), \]

is the Booth lemniscate of elliptic type (see Fig. 1, for α = 1/3). For more details, see [3].

Lemma 1.1. (see [3]) Let F_α(z) be given by (1.3). Then for 0 ≤ α < 1, we have

\[ \frac{1}{α - 1} < \text{Re} \{ F_α(z) \} < \frac{1}{1 - α} \quad (z ∈ Δ). \]

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Therefore by definition of subordination and by the Lemma 1.1 \( F \in \mathcal{A} \) belongs to the class \( BS(\alpha) \), if it satisfies the condition
\[
\frac{\alpha}{\alpha - 1} < \Re \left\{ \frac{zf'(z)}{f(z)} \right\} < \frac{2 - \alpha}{1 - \alpha} \quad (z \in \Delta).
\]

The following lemma will be useful.

**Lemma 1.2.** (see [6]) Let \( F, G \in \mathcal{H} \) be any convex univalent functions in \( \Delta \). If \( f \prec F \) and \( g \prec G \), then
\[
f * g \prec F * G \quad \text{in} \quad \Delta.
\]

In this work, some geometric properties of the class \( BS(\alpha) \) are investigated.

## 2. Main results

We start with the following lemma that gives the structural formula for the function of the considered class.

**Lemma 2.1.** The function \( f \in \mathcal{A} \) belongs to the class \( BS(\alpha) \), \( 0 \leq \alpha < 1 \), if and only if there exists an analytic function \( q \), \( q(0) = 0 \) and \( q \prec F_\alpha \) such that
\[
f(z) = z \exp \left( \int_0^z \frac{q(t)}{t} \, dt \right).
\]

The proof is easy. Putting \( q = F_\alpha \) in Lemma 2.1 we obtain the function
\[
f(z) = z \left( 1 + \frac{z\sqrt{\alpha}}{1 - z\sqrt{\alpha}} \right)^{\frac{1}{\sqrt{\alpha}}},
\]
which is extremal for several problems in the class \( BS(\alpha) \). Moreover, we consider
\[
F(z) := \frac{f(z)}{z} = \left( 1 + \frac{z\sqrt{\alpha}}{1 - z\sqrt{\alpha}} \right)^{1/\sqrt{\alpha}} = 1 + z + \frac{1}{2}z^2 + \frac{1}{3} \left( \alpha + \frac{1}{2} \right) z^3 \ldots.
\]

From [1,7] we conclude that \( f \in BS(\alpha) \) is starlike of order \( \frac{\alpha}{\alpha - 1} < 0 \), hence \( f \) may not be univalent in \( \Delta \). It may therefore be interesting to consider a problem to find the radius of starlikeness of order \( \gamma \), \( \gamma \in [0,1) \) (hence univalence) of the class \( BS(\alpha) \), i.e., the largest radius \( r_s(\alpha, \gamma) \) such that each function \( f \in BS(\alpha) \) is starlike of order \( \gamma \) in the disc \( |z| < r_s(\alpha, \gamma) \). For this purpose we recall the following property of the class \( \mathfrak{B} \).

**Lemma 2.2.** (Schwarz lemma) (see [11]) Let \( w \) be analytic in the unit disc \( \Delta \), with \( w(0) = 0 \) and \( |w(z)| < 1 \) in \( \Delta \). Then \( |w'(0)| \leq 1 \) and \( |w(z)| \leq |z| \) in \( \Delta \). Strict inequality holds in both estimates unless \( w \) is a rotation of the disc: \( w(z) = e^{i\theta} z \).

**Theorem 2.1.** Let \( \alpha \in (0,1) \) and \( \gamma \in [0,1) \) be given numbers. If \( f \in BS(\alpha) \), then \( f \) is starlike of order \( \gamma \) in the disc \( |z| < r_s(\alpha, \gamma) = \frac{\sqrt{1 + 4\alpha(1-\gamma)^{-1}} - 1}{2\alpha(1-\gamma)} \). The result is sharp.

**Proof.** Let \( f \in BS(\alpha), \alpha \in (0,1) \). Then through [1,2] we have \( \left| \frac{zf'(z)}{f(z)} - 1 \right| = \frac{z}{1 - \alpha z^2} \) so there exists \( w \in \mathfrak{B} \) such that
\[
\Re \left\{ \frac{zf'(z)}{f(z)} \right\} = \Re \left\{ \frac{1 + w(z) - \alpha w^2(z)}{1 - \alpha w^2(z)} \right\}
\]
for all \( z \in \Delta \). Applying the Schwarz lemma we obtain
\[
\Re \left\{ \frac{zf'(z)}{f(z)} \right\} \geq 1 - \frac{|w(z)|}{1 - \alpha |w(z)|^2} \geq 1 - \frac{|z|}{1 - \alpha |z|^2} = 1 - \frac{r}{1 - \alpha r^2}.
\]
where $r = |z| < 1$. Let us consider a function $h(r) = 1 - \frac{r}{1 + \alpha r^2}$, $r \in (0, 1)$. Note that $h'(r) = -\frac{1 + \alpha r^2}{(1 + \alpha r^2)^2} < 0$ for all $r \in (0, 1)$ hence $h$ is a strictly decreasing function and it decreases from 1 to $\frac{\alpha}{\alpha - 1} < 0$. Therefore the equation $h(r) = \gamma$ has for given $\alpha$ and $\gamma$ the smallest positive root $r_s(\alpha, \gamma)$ in $(0, 1)$. Therefore $f$ is starlike of order $\gamma$ in $|z| < r \leq r_s(\alpha, \gamma)$. Note that for the function $f$ given in (2.2) we obtain

$$\text{Re} \frac{zf'(z)}{f(z)} = \text{Re} \left\{ 1 + \frac{z}{1 - \alpha z^2} \right\} =: A(z)$$

and $A(-r_s(\alpha, \gamma)) = \gamma$. \hfill $\square$

Putting $\gamma = 0$ in Theorem 2.1 we obtain:

**Corollary 2.1.** Let $\alpha \in (0, 1)$. If $f \in \mathcal{B}S(\alpha)$ then $f$ is starlike univalent in the disc $|z| < r_s(\alpha) = \frac{\sqrt{1 + 16\alpha - 1}}{2\alpha}$. The result is sharp.

**Remark 2.1.** Note that $\lim_{\alpha \to 0^+} r_s(\alpha) = \lim_{\alpha \to 0^+} \frac{2}{\sqrt{1 + 4\alpha + 1}} = 1$. Moreover, it is worth mentioning that $\lim_{\alpha \to 1^-} r_s(\alpha) = \frac{\sqrt{15} - 1}{2} = 0.618 \ldots$, i.e. this limit is a reciprocal of the golden ratio $\frac{\sqrt{5} + 1}{2}$.

Now we consider the following question: For a given number $r \in (0, 1]$ find $\alpha(r)$ such that for each function $f \in \mathcal{B}S(\alpha(r))$ the image $f(\{z : |z| < r\})$ is a starlike domain.

**Theorem 2.2.** Let $r \in (0, 1]$ be the given number. If $0 \leq \alpha < \frac{1 - r^2}{\sqrt{r^2 - 1}}$, then each function $f \in \mathcal{B}S(\alpha)$ maps a disc $|z| < r$ onto a starlike domain.

**Proof.** After using the same argument as in the proof of Theorem 2.1 we conclude that $f \in \mathcal{B}S(\alpha)$ satisfies the equality

$$\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} = \text{Re} \left\{ 1 + \frac{w(z)\alpha - \alpha w(z)}{1 - \alpha w(z)} \right\}$$

for all $z \in \Delta$ with some $w \in \mathfrak{B}$. Then we have by Schwarz’s lemma that

$$\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} \geq 1 - \frac{|w(z)|^2}{1 - \alpha |w(z)|^2} = 1 - \frac{|z|}{1 - \alpha |z|^2}.$$  

Consequently, for $|z| < r$, we obtain $\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 1 - \frac{r}{1 - \alpha r^2}$. Let us note that a function $g(\alpha) = 1 - \frac{r}{1 - \alpha r^2}$, $\alpha \in (0, 1)$, has positive values for $0 \leq \alpha < \frac{1 - r^2}{\sqrt{r^2 - 1}}$. Therefore the image of the disc $|z| < r$ is a starlike domain. \hfill $\square$

**Theorem 2.3.** Let $n \geq 2$ be integer. If one of the following conditions holds

(i) $\frac{1}{\alpha + n(1 - \alpha)} < |c| < 1,$
(ii) $n > \frac{1}{\alpha - 2n(1 - \alpha)} < |c| < 1,$
(iii) $n > \frac{2}{\alpha} \alpha$ and $|c| > 1,$
(iv) $n < \frac{2}{\alpha - 2n} \alpha$ and $1 < |c| < \frac{1}{\alpha + n(\alpha - 1)}$,

then the function $g_n(z) = z + cz^n$ does not belong to the class $\mathcal{B}S(\alpha)$.

**Proof.** Let us put $G(z) = \frac{g_n'(z)}{g_n(z)} - 1 = \frac{1 + \alpha c z^{n-1}}{1 + \alpha c z^n} - 1$. To prove our assertion it suffices to show that the function $G$ is not subordinate to $F_{\alpha}$ or equivalently, because of the univalence of the dominant function $F_{\alpha}$, that the set $G(\Delta)$ is not included in $F_{\alpha}(\Delta) = D(\alpha)$. Upon performing simple calculation we find that the set $G(\Delta)$ is the disc with the diameter from the point $x_1 = \frac{|c|^{(n-1)}}{|c|-1}$ to the point $x_2 = \frac{|c|^{(n-1)}}{|c|+1}$. The set $D(\alpha)$ is bounded by the curve

$$(x^2 + y^2)^2 - \frac{x^2}{(1 - \alpha)^2} - \frac{y^2}{(1 + \alpha)^2} = 0, \quad (x, y) \neq (0, 0).$$

We have $\min_{z=1} \text{Re} \{F_{\alpha}(z)\} = F_{\alpha}(-1) = \frac{1}{\alpha}$ and $\max_{z=1} \text{Re} \{F_{\alpha}(z)\} = F_{\alpha}(1) = \frac{1}{\alpha^2}$. If one of the conditions (i) – (iv) is satisfied then $\min\{x_1, x_2\} < \frac{1}{\alpha - 1}$ or $\max\{x_1, x_2\} > \frac{1}{\alpha - 1}$, and then $G(\Delta)$ is not included in $D(\alpha)$.

The proof of theorem is completed. \hfill $\square$

Recently, one of the interesting problems for mathematician is to find bounds for $\text{Re}\{f(z)/z\}$ (see [2, 4]). In the sequel, we obtain lower and upper bounds for $\text{Re}\{f(z)/z\}$. We first get the following result for the function $F(z)$ given by (2.3).

**Theorem 2.4.** The function $F(z) - 1$ is convex univalent in $\Delta$.

**Proof.** Let us define

$$p(z) := F(z) - 1 = \left( \frac{1 + z\sqrt{\alpha}}{1 - z\sqrt{\alpha}} \right)^{\frac{1}{\alpha}} - 1 = z + \frac{1}{2}z^2 + \frac{1}{3} \left( \alpha + \frac{1}{2} \right) z^3 \ldots.$$
Then we see that \( p(z) \in \mathcal{A} \). A simple calculation gives us
\[
(2.5) \quad 1 + \frac{zp''(z)}{p'(z)} = 1 + \left( \frac{1}{2\sqrt{\alpha}} - 1 \right) \left( \frac{2\sqrt{\alpha}z}{1 - \alpha^2} \right) + \frac{2\sqrt{\alpha}z}{1 - \sqrt{\alpha}z}.
\]
It is sufficient to show that (2.5) has positive real part in the unit disc. From Lemma 1.1 we obtain
\[
\text{Re} \left\{ 1 + \frac{zp''(z)}{p'(z)} \right\} = 1 + 2\sqrt{\alpha} \left( \frac{1}{2\sqrt{\alpha}} - 1 \right) \text{Re} \left\{ \frac{z}{1 - \alpha^2} \right\} + 2\sqrt{\alpha} \text{Re} \left\{ \frac{z}{1 - \sqrt{\alpha}z} \right\}
\]
\[
> 1 + (1 - 2\sqrt{\alpha}) \left( \frac{1}{\alpha - 1} \right) - \frac{2\sqrt{\alpha}}{1 + \sqrt{\alpha}} =: K(\alpha) \quad (0 \leq \alpha < 1).
\]
It is easily seen that \( K'(\alpha) = \frac{1}{(\alpha - 1)^2} > 0 \). Thus \( K(\alpha) \geq K(0) = 0 \), and hence \( F(z) - 1 \) is convex univalent function.

In the proof of the next theorem we will use the following result concerning the convexity of the boundary of \( D(\alpha) \).

Lemma 2.3. (see [4]) Suppose that \( F_\alpha \) is given by (2.3). If \( 0 \leq \alpha \leq 3 - 2\sqrt{2} \approx 0.1715 \), then the curve \( F_\alpha(e^{i\varphi}) \), \( \varphi \in [0, 2\pi) \), is convex. If \( \alpha \in (3 - \sqrt{2}, 1) \), then the curve \( F_\alpha(e^{i\varphi}) \), \( \varphi \in [0, 2\pi) \), is concave. Moreover, in both cases this curve is symmetric with respect to both axes.

Theorem 2.5. If a function \( f \) belongs to the class \( \mathcal{BS}(\alpha) \), \( 0 \leq \alpha \leq 3 - 2\sqrt{2} \), then
\[
(2.6) \quad \frac{f(z)}{z} \prec F(z) \quad (z \in \Delta),
\]
where \( F(z) \) is given by (2.3).

Proof. Let \( 0 \leq \alpha \leq 3 - 2\sqrt{2} \) and let \( f \) be in the class \( \mathcal{BS}(\alpha) \). Then we have
\[
(2.7) \quad \phi(z) := \frac{zf'(z)}{f(z)} - 1 \prec F_\alpha(z) \quad (z \in \Delta),
\]
where \( F_\alpha \) is given by (2.3). It is well known that the normalized function
\[
l(z) = \log \frac{1}{1 - z} = \sum_{n=1}^{\infty} \frac{z^n}{n} \quad (z \in \Delta),
\]
belongs to the class \( \mathcal{K} \) and for \( f \in \mathcal{A} \) we get
\[
(2.8) \quad \phi(z) * l(z) = \int_0^z \frac{\phi(t)}{t} \, dt \quad \text{and} \quad F_\alpha(z) * l(z) = \int_0^z \frac{F_\alpha(t)}{t} \, dt.
\]
By Lemma 2.3 we deduce that the function \( F_\alpha \) is convex. Thus applying Lemma 1.2 in (2.7) we obtain
\[
(2.9) \quad \phi(z) * l(z) \prec F_\alpha(z) * l(z) \quad (z \in \Delta).
\]
Now from (2.8) and (2.9), we can obtain
\[
\int_0^z \phi(t) \, dt < \int_0^z \frac{F_\alpha(t)}{t} \, dt \quad (z \in \Delta).
\]
Thus
\[
\frac{f(z)}{z} = \exp \int_0^z \frac{\phi(t)}{t} \, dt < \int_0^z \frac{F_\alpha(t)}{t} \, dt = \frac{f(z)}{z}.
\]
This completes the proof of theorem.

Here by combining Theorem 2.4, Theorem 2.5 and 1.1, we get:

Theorem 2.6. Let \( f \in \mathcal{BS}(\alpha) \), \( 0 \leq \alpha \leq 3 - 2\sqrt{2} \) and \( |z| = r < 1 \). Then
\[
(2.10) \quad \left( \frac{1 - r\sqrt{\alpha}}{1 + r\sqrt{\alpha}} \right)^{\frac{1}{1 - \sqrt{\alpha}}} \leq \text{Re} \left( \frac{f(z)}{z} \right) \leq \left( \frac{1 + r\sqrt{\alpha}}{1 - r\sqrt{\alpha}} \right)^{\frac{1}{1 + \sqrt{\alpha}}} \quad (z \in \Delta).
\]
The result is sharp.

Proof. By the subordination principle, we have:
\[
f(z) \prec g(z) \Rightarrow f(|z| < r) \subset g(|z| < r) \quad (0 \leq r < 1).
\]
From Theorem 2.4 since \( F(z) - 1 \) is convex univalent in \( \Delta \), and it is real for real \( z \), thus it maps the disc \( |z| = r < 1 \) onto a convex set symmetric which respect to the real axis laying between \( F(-r) - 1 \) and \( F(r) - 1 \). Now the assertion is obtained from Theorem 2.5.
References

[1] P.L. Duren, *Univalent functions*, Springer-Verlag, 1983.

[2] R. Kargar, A. Ebadian and J. Sokół, *On subordination of some analytic functions*, Sib. Math. J. (2016) 57: 599–604.

[3] R. Kargar, A. Ebadian and J. Sokół, *On Booth lemniscate and starlike functions*, Anal. Math. Phys. DOI: 10.1007/s13324-017-0187-3

[4] K. Piejko and J. Sokół, *Hadamard product of analytic functions and some special regions and curves*, J. Inequal. Appl. 2013, 2013:420.

[5] M.S. Robertson, *Certain classes of starlike functions*, Michigan Math. J. 32 (1985) 135–140.

[6] St. Ruscheweyh and J. Stankiewicz, *Subordination under convex univalent function*, Bull. Pol. Acad. Sci. Math. 33 (1985) 499–502.

[7] Y.J. Sim and O.S. Kwon, *Notes on analytic functions with a bounded positive real part*, J. Inequal. Appl. (2013) 2013: 370. doi:10.1186/1029-242X-2013-370