A REFLEXIVE BANACH SPACE WHOSE ALGEBRA OF OPERATORS IS NOT A GROTHENDIECK SPACE

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Abstract. By a result of Johnson, the Banach space \( F = (\bigoplus_{n=1}^{\infty} \ell_1^n)_{\ell_\infty} \) contains a complemented copy of \( \ell_1 \). We identify \( F \) with a complemented subspace of the space of (bounded, linear) operators on the reflexive space \( (\bigoplus_{n=1}^{\infty} \ell_1^n)_{\ell_p} \) \((p \in (1, \infty))\), thus solving negatively the problem posed in the monograph of Diestel and Uhl which asks whether the space of operators on a reflexive Banach space is Grothendieck.

1. Introduction

A Banach space \( E \) is Grothendieck if weak* convergent sequences in \( E^* \) converge weakly. Certainly, every reflexive Banach space is Grothendieck. Notable examples of non-reflexive Grothendieck spaces are \( C(K) \)-spaces for extremally disconnected compact spaces \( K \) ([4]) and the Hardy space \( H^\infty \) of bounded holomorphic functions on the unit disc ([1]). Diestel and Uhl wrote in their famous monograph [3, p. 180]:

Finally, there is some evidence (Akemann [1967], [1968]) that the space \( \mathcal{L}(H;H) \) of bounded linear operators on a Hilbert space is a Grothendieck space and that more generally the space \( \mathcal{L}(X;X) \) is a Grothendieck space for any reflexive Banach space \( X \).

The question of whether the space of (bounded, linear) operators on a reflexive Banach space is Grothendieck was raised also by Soybaş ([4]). Pfitzner proved in [6] that C*-algebras have the so-called Pelczyński’s property (V) which for dual Banach spaces is equivalent to being a Grothendieck space (cf. [2, Exercise 12, p. 116]). In particular, von Neumann algebras are Grothendieck spaces which confirms that the space of operators on a Hilbert space is Grothendieck. It is known that duals of spaces with property (V) are weakly sequentially complete. We shall present an example of a reflexive Banach space \( E \) such that \( \mathcal{B}(E) \) fails to be Grothendieck, giving thus a negative answer to the above-mentioned problem. To do this, we require a result of Johnson which asserts that the Banach space \( F = (\bigoplus_{n=1}^{\infty} \ell_1^n)_{\ell_\infty} \) contains a complemented copy of \( \ell_1 \) (cf. the Remark after Theorem 1 in [5]), so it is not a Grothendieck space.

By an operator we understand a bounded, linear operator acting between Banach spaces. The space \( \mathcal{B}(E_1, E_2) \) of operators acting between spaces \( E_1 \) and \( E_2 \) is a Banach space when endowed with the operator norm. We write \( \mathcal{B}(E) \) for \( \mathcal{B}(E, E) \). Let \( p \in [1, \infty] \). We denote by \( (\bigoplus_{n=1}^{\infty} E_n)_p \) the \( \ell_p \)-sum of a sequence \( (E_n)_{n=1}^{\infty} \) of Banach spaces. We identify elements of \( \mathcal{B}((\bigoplus_{n=1}^{\infty} E_n)_p) \) with matrices \( (T_{ij})_{i,j\in \mathbb{N}} \), where \( T_{ij} \in \mathcal{B}(E_j, E_i) \) \((i, j \in \mathbb{N})\). Let \( (e_n)_{n=1}^{\infty} \) be the canonical basis of \( \ell_1 \). For each \( n \in \mathbb{N} \) we define \( \ell_1^n = \text{span}\{e_1, \ldots, e_n\} \).
2. The result

The main result. Let $p \in (1, \infty)$ and consider the reflexive Banach space $E = \left( \bigoplus_{n=1}^{\infty} \ell_1^n \right)_{\ell_p}$. Then $\mathcal{B}(E)$ is not a Grothendieck space.

Proof. Recall that $F = \left( \bigoplus_{n=1}^{\infty} \ell_1^n \right)_{\ell_\infty}$ contains a complemented copy of $\ell_1$. To complete the proof it is enough to embed $F$ as a complemented subspace of $\mathcal{B}(E)$.

One may identify $\ell_1^1$ with a 1-complemented subspace of $\mathcal{B}(\ell_1^n)$ via the mapping

$$e_k \mapsto e_k \otimes e_1^*(k \leq n, n \in \mathbb{N}),$$

where $e_1^*$ stands for the coordinate functional associated with $e_1$. Consequently, the space $D = \left( \bigoplus_{n=1}^{\infty} \mathcal{B}(\ell_1^n) \right)_{\ell_\infty}$ contains a complemented subspace isomorphic to $F$. Let $\Delta: D \to \mathcal{B}(E)$ be the diagonal embedding, that is, $\Delta((T_n)_{n=1}^{\infty}) = \text{diag}(T_1, T_2, \ldots)$ ($\mathcal{B}(T_n)_{n=1}^{\infty} \in D$); this map is well-defined since the decomposition of $E$ into the subspaces $\ell_1^1, \ell_1^2, \ldots$ is unconditional.

It is enough to notice that $\Delta$ has a left-inverse $\Xi: \mathcal{B}(E) \to D$ given by

$$\Xi(T_{ij})_{i,j \in \mathbb{N}} = (T_{ij})_{i,j = 1}^{\infty} ((T_{ij})_{i,j \in \mathbb{N}} \in \mathcal{B}(E)),$$

which is bounded. To this end, we shall perform a construction inspired by a trick of Tong (cf. [8] Theorem 2.3 and its proof). With each operator $T = (T_{ij})_{i,j \in \mathbb{N}} \in \mathcal{B}(E)$ we shall associate a sequence $(S^{(n)})_{n=1}^{\infty}$ of finite-rank perturbations of $T$ such that for each $n \in \mathbb{N}$ we have $\|S^{(n)}\| \leq \|T\|$ and the matrix of $S^{(n)}$ agrees with the matrix of the diagonal operator $\text{diag}(-T_{11}, \ldots, -T_{nn}, 0, 0, \ldots)$ at entries $(i,j)$ with $i \leq n$ or $j \leq n$. This will immediately yield that

$$\|\Xi(T)\| = \sup_{n \in \mathbb{N}} \|T_{nn}\| = \sup_{n \in \mathbb{N}} \|S^{(n)}_{nn}\| \leq \sup_{n \in \mathbb{N}} \|S^{(n)}\| \leq \|T\|.$$

Define operators $T_k, T_r$ which have the same columns and rows as $T$ respectively, except the first ones, where we instead set $(T_k)_{i1} = -T_{i1}$ and $(T_r)_{1j} = -T_{1j}$ for $i,j \in \mathbb{N}$ (these are indeed elements of $\mathcal{B}(E)$ as rank-1 perturbations of $T$). Certainly, $\|T\| = \|T_k\| = \|T_r\|$ and the norm of $S = (T_k + T_r)/2$ does not exceed the norm of $T$. Arguing similarly, we observe that $\|(S_k^{(n)} + S_r^{(n)})/2\| \leq \|T\|$, where $S^{(1)} = S$ and $S^{(n+1)} = (S_k^{(n)} + S_r^{(n)})/2$ ($n \in \mathbb{N}$). Consequently, $(S^{(n)})_{n=1}^{\infty}$ is the desired sequence. \hfill \Box

Remark. The space $\mathcal{B}(E)$ shares with the space of operators on a Hilbert space a number of common properties. For instance, since $E$ has a Schauder basis, $\mathcal{B}(E)$ can be identified with the bidual of $\mathcal{K}(E)$, the space of compact operators on $E$. Nonetheless, $E$ is plainly not superreflexive and $\mathcal{B}(E)$ fails to have weakly sequentially dual for the obvious reason that $\ell_\infty$ embeds into $\mathcal{B}(E)^*$. We conjecture that the space of operators on a superreflexive space is Grothendieck (or at least it has weakly sequentially complete dual).

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