LIPSCHITZ PROPERTY OF BISTABLE OR COMBUSTION
FRONTS AND ITS APPLICATIONS

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Abstract. For a class of reaction-diffusion equations describing propagation phe-
nomena, we prove that for any entire solution $u$, the level set $\{u = \lambda\}$ is a Lipschitz
graph in the time direction if $\lambda$ is close to 1. Under a further assumption that $u$
connects 0 and 1, it is shown that all level sets are Lipschitz graphs. By a blowing
down analysis, the large scale motion law for these level sets and a characterization
of the minimal speed for travelling waves are also given.

1. Introduction

1.1. Lipschitz property for level sets. Consider a smooth, entire solution $u$ to
the reaction-diffusion equation

$$\partial_t u - \Delta u = f(u), \quad 0 < u < 1 \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R}. \quad (1.1)$$

In this paper we are mainly interested in the Lipschitz property of the level sets
$\{u = \lambda\}$ and their geometric motion law at large scales.

Our main hypothesis on $f$ are

(F1): $f \in \text{Lip}([0, 1])$, $f(0) = f(1) = 0$ and $f \in C^{1,\alpha}([0, \gamma) \cup (1 - \gamma, 1])$ for some
$\alpha, \gamma \in (0, 1)$;

(F2): $f'(1) < 0$;

(F3): $\int_0^1 f(u)du > 0$.

Sometimes we also need

(F4): there exists a $\theta \in (0, 1)$ such that $f > 0$ in $(\theta, 1)$, and either $f < 0$ in
$(0, \theta)$ with $f'(0) < 0$, or $f \equiv 0$ in $(0, \theta)$.

Typical examples are the bistable nonlinearity $f(u) = u(u - \theta)(1 - u)$ with $\theta < 1/2$
and the combustion nonlinearity.

The reaction-diffusion equation (1.1) is used in the modelling of biological prop-
agation phenomena, see Aronson and Weinberger [1]. Entire solutions have been
studied by many people since the work of Hamel and Nadirashvili [19, 20], see
[14, 15, 22, 9] for the homogeneous case. There is also a large literature devoted to
the study of heterogeneous cases. In particular, a very general notion of travelling
fronts, transition fronts, was introduced by Berestycki and Hamel in [6, 7].
The geometry of an entire solution is complicated in general. To study the Lipschitz property of \( \{u = \lambda\} \), we introduce some assumptions on the entire solution \( u \). The first one is

\[ (H1): \text{For any } t \in \mathbb{R}, \sup_{x \in \mathbb{R}^n} u(x, t) = 1. \]

Under this assumption we prove

**Theorem 1.1** (Half Lipschitz property for entire solutions). Suppose \( f \) satisfies \((F1-F3)\). There exists a \( b_0 \in (0, 1) \) such that, if an entire solution \( u \) satisfies \((H1)\), then for any \( \lambda \in [1 - b_0, 1) \), \( \{u = \lambda\} = \{t = h_\lambda(x)\} \) is a globally Lipschitz graph on \( \mathbb{R}^n \).

In general, if \( \lambda \) is close to 0, \( \{u = \lambda\} \) does not satisfy this Lipschitz property, see the example given after Theorem 1.3. In order to establish the full Lipschitz property, we need more assumptions. A natural one is

\[ (H2): u \to 0 \text{ uniformly as } \text{dist}((x, t), \{u \geq 1 - b_0\}) \to +\infty. \]

Here \( \text{dist} \) denotes the standard Euclidean distance on \( \mathbb{R}^n \times \mathbb{R} \).

**Theorem 1.2** (Full Lipschitz property for entire solutions). Suppose \( f \) satisfies \((F1-F4)\), and \( u \) is an entire solution satisfying \((H1-H2)\). Then for any \( \lambda \in (0, 1) \), \( \{u = \lambda\} = \{t = h_\lambda(x)\} \) is a globally Lipschitz graph on \( \mathbb{R}^n \).

The proof of Theorem 1.1 relies on the propagation phenomena (see Aronson and Weinberger [1]) in (1.1). Roughly speaking, by \((F3)\), 1 represents a more stable state than 0, so \( \{u \approx 1\} \) will invade \( \{u \approx 0\} \). This gives us a cone of monotonicity at large scales, see Lemma 2.6 for the precise statement. Although this only implies a Lipschitz property for \( \{u = \lambda\} \) at large scales, it can be propagated to a real Lipschitz property by utilizing some estimates on positive solutions to the linear parabolic equation

\[ \partial_t w - \Delta w = f'(1)w. \]

Here the nondegeneracy condition \( f'(1) < 0 \) will be crucial for this argument.

After establishing the Lipschitz property of \( \{u = \lambda\} \) for \( \lambda \) close to 1, under the hypothesis \((H2)\), we can apply the maximum principle and sliding method (in the time direction, cf. Guo and Hamel [14]) to extend this Lipschitz property backwardly in time, which is Theorem 1.2.

1.2. **Travelling wave solutions.** The solution \( u \) is a travelling wave in the direction \(-e_n\) and with speed \( \kappa > 0 \), if there exists a function \( v \in C^2(\mathbb{R}^n) \) such that

\[ u(x, t) = v(x + \kappa t e_n). \]

Here \( v \) satisfies the elliptic equation

\[ -\Delta v + \kappa \partial_n v = f(v) \quad \text{in } \mathbb{R}^n. \quad (1.2) \]

Among the class of travelling wave solutions, the one dimensional travelling wave is of particular importance. By [1, Theorem 4.1], under the hypothesis \((F1-F4)\), there exists a unique constant \( \kappa_* > 0 \) and a unique (up to a translation) solution to the one dimensional problem

\[ -g''(t) + \kappa_* g'(t) = f(g(t)), \quad g(-\infty) = 0, \quad g(+\infty) = 1. \quad (1.3) \]
Theorem 1.1 applied to $v$ gives

**Theorem 1.3** (Half Lipschitz property for travelling waves). Suppose $f$ satisfies (F1-F3) and $v$ is an entire solution of (1.2), satisfying $\sup_{\mathbb{R}^n} v = 1$. For any $\lambda \in [1 - b_0, 1)$, \{v = \lambda\} = \{x_n = h_{\lambda}(x'), x' \in \mathbb{R}^{n-1}\} is a globally Lipschitz graph on $\mathbb{R}^{n-1}$.

As in the entire solution case, in general, this property does not hold for level sets $\{v = \lambda\}$ with $\lambda$ close to 0. For example, in Hamel and Roquejoffre [21], it is shown that when $n = 2$, there exist solutions $v$ of (1.1), which is monotone in $x_1$ and satisfies

$$v(x_1, x_2) \to 1 \text{ uniformly as } x_1 \to +\infty.$$  

$$v(x_1, x_2) \to \varphi(x_2) \text{ locally uniformly as } x_1 \to -\infty,$$

where $\varphi$ is an $L$-periodic solution (for some $L > 0$) of

$$-\varphi'' = f(\varphi) \text{ in } \mathbb{R}.$$

Hence when $\lambda$ is close to 0, \{v = \lambda\} is the graph of an $L$-periodic function $h_{\lambda}$, satisfying $h_{\lambda}(kL) = -\infty$ for any $k \in \mathbb{Z}$. Clearly it cannot be a globally Lipschitz graph.

As in the entire solution case, in order to get the Lipschitz property for all level sets, we need more assumptions. A natural one is

**Theorem 1.4** (Full Lipschitz property for travelling waves). Suppose $f$ satisfies (F1-F4), $v$ is an entire solution of (1.2), satisfying $\sup_{\mathbb{R}^n} v = 1$ and

$$v(x) \to 0 \text{ uniformly as } \text{dist}(x, \{v \geq 1 - b_0\}) \to +\infty. \quad (1.4)$$

Then for any $\lambda \in (0, 1)$, \{v = \lambda\} = \{x_n = h_{\lambda}(x')\} is a globally Lipschitz graph on $\mathbb{R}^{n-1}$.

This theorem also holds for the monostable case, that is, instead of (F4), we assume

(F4'): $f > 0$ in $(0, 1)$, and $f'(0) > 0$.

The assumption (1.4) holds automatically in the monostable case. Hence we get a small improvement on the same Lipschitz property for all level sets proved in [20], where they require the nonlinearity $f$ to be concave. However, we do not know how to prove the parabolic case, see discussions in Subsection 3.2.

Existence, qualitative properties and classification of solutions to (1.2) with Lipschitz level sets have been studied by many people, see [16, 17, 18, 27, 28, 33, 34, 23, 35, 36].

1.3. **Blowing down limits.** Once we know level sets of $u$ are Lipschitz graphs, we would like to study their large scale structures. Take a $b \in (0, 1)$ such that \{u = b\} = \{t = h(x)\} is a globally Lipschitz graph on $\mathbb{R}^n$. For any $\lambda > 0$, let

$$h_{\lambda}(x) := \frac{1}{\lambda} h(\lambda x).$$
They are uniformly Lipschitz. Therefore for any \( \lambda_i \to \infty \), there exist a subsequence (not relabelling) such that \( h_{\lambda_i} \) converges to \( h_\infty \) in \( C_{\text{loc}}(\mathbb{R}^n) \). (This limit may depend on the choice of subsequences.)

We have the following characterization of \( h_\infty \).

**Theorem 1.5.** Under the assumptions of Theorem 1.2, the blowing down limit \( h_\infty \) is a viscosity solution of

\[
|\nabla h_\infty|^2 - \kappa_*^{-2} = 0 \quad \text{in} \quad \mathbb{R}^n.
\]

**Remark 1.6 (Level set formulation).** Equation (1.5) is the level set formulation of the geometric motion equation for the family of hypersurfaces \( \Sigma(t) := \{ x : h_\infty(x) = t \} \),

\[
V_{\Sigma(t)} = \kappa_* V_{\Sigma(t)}.
\]

Here \( \nu_{\Sigma(t)} = -\nabla h_\infty / |\nabla h_\infty| \) is the unit normal vector of \( \Sigma(t) \). See Fife [13, Chapter 1] for a formal derivation of this equation.

The equation (1.5) also corresponds to the fact that the global mean speed of transition fronts equals \( \kappa_* \), see Hamel [15].

**Remark 1.7.** Because \( h_\infty(0) = 0 \), the following representation formula holds for \( h_\infty \) (see for example Monneau, Roquejoffre and Roussier-Michon [26, Section 2]): there exists a closed set \( \Xi \subset \mathbb{S}^{n-1} \) such that

\[
h_\infty(x) = \inf_{\xi \in \Xi} \xi \cdot x.
\]

As a consequence, \( h_\infty \) is concave and 1-homogeneous.

The connection between reaction-diffusion equations and motion by mean curvatures in the framework of viscosity solutions has been explored by many people in 1980s and 1990s. In particular, the asymptotic behavior of solutions to the Cauchy problem for (1.1) has been studied by Barles, Bronsard, Evans, Soner and Souganidis in [11, 3, 2, 5], in the framework of Hamilton-Jacobi equation and level set motions. We use the same idea, but now for the study of entire solutions of (1.1) (in the spirit of [19, 20]), where we are free to perform scalings to study the large scale structure of entire solutions.

From this blowing down analysis we also get a characterization of the minimal speed.

**Theorem 1.8.** Suppose \( f \) satisfies (F1-F4), \( v \) is an entire solution of (1.2), satisfying \( \sup_{\mathbb{R}^n} v = 1 \) and (1.4). Then \( \kappa \geq \kappa_* \).

Furthermore, if \( \kappa = \kappa_* \), there exists a constant \( t \in \mathbb{R} \) such that

\[
v(x) \equiv g(x_n + t) \quad \text{in} \quad \mathbb{R}^n.
\]

1.4. **Further problems.** To put our results in a wide perspective, here we mention some further problems about (1.1) and (1.2). Some of these problems are well known to experts in this field.

**Problem 1.** Extend results in this paper to the monostable case.

**Problem 2.** Theorem 1.5 gives only the main order term of the front motion law. The next order term has been formally derived in Fife [13]. Using the language
of viscosity solutions, the family of hypersurfaces \( \Sigma(t) := \{ u(t) = 1/2 \} \) should be an approximate viscosity solution at large scales (in the sense of Savin \cite{Sa1, Sa2}) of the forced mean curvature flow
\[
V_{\Sigma(t)} = [\kappa_* - H_{\Sigma(t)}] \nu_{\Sigma(t)}.
\] (1.7)
Here \( H_{\Sigma(t)} \) denotes the mean curvature of \( \Sigma(t) \).

**Problem 3.** In \cite{HN}, Hamel and Nadirashvili proposed a conjecture about the classification of entire solutions. For travelling wave solutions in the bistable and combustion case, this conjecture may be broken into two steps:

1. There exists a one to one correspondence between solutions of (1.2) and solutions of
\[
\text{div} \left( \frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right) = \kappa_* - \frac{\kappa}{\sqrt{1 + |\nabla h|^2}}.
\] (1.8)

   This is the travelling wave equation of (1.7), see \cite{KMS} for a discussion on this equation.

2. There exists a one to one correspondence between solutions of (1.8) and nonnegative Borel measures on \( S^{n-1} \).

**Problem 4.** In view of the above discussion and Taniguchi’s theorem in \cite{T}, a not so ambitious question is if the reverse of Theorem 1.5 is true, that is, given a homogeneous viscosity solution \( h_\infty \) of (1.5), does there exist an entire solution of (1.1) so that its level set \( \{ u = 1/2 \} \) is asymptotic to \( \{ t = h_\infty(x) \} \)?

1.5. **Notations and organization of the paper.** Throughout the paper we keep the following conventions.

- We use \( C \) (large) and \( c \) (small) to denote various universal constants, which could be different from line to line.
- The parabolic boundary of a domain \( \Omega \subset \mathbb{R}^n \times \mathbb{R} \) is denoted by \( \partial^p \Omega \).
- A function \( u \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}) \) if it is \( C^2 \) in \( x \)-variables and \( C^1 \) in \( t \)-variable.

The remaining part of this paper is organized as follows. In Section 2 we study the propagation phenomena in (1.1) and use this to prove Theorem 1.1. In Section 3 we prove Theorem 1.2 by the sliding method. An elliptic Harnack inequality is established in Section 4. In Section 5 we perform the blowing down analysis. In Section 6 we prove Theorem 1.5. In Section 7 we give a representation formula for the blowing down limits. With these knowledge on blowing down limits, we prove Theorem 1.8 in Section 8 by using the sliding method again.

2. **Propagation phenomena**

2.1. **Cone of monotonicity at large scales.** Standard parabolic regularity theory implies that \( u, \nabla u, \nabla^2 u \) and \( \partial_t u \) are all bounded in \( \mathbb{R}^n \times \mathbb{R} \). By the Lipschitz property of \( u \) in \( t \), \( \sup_{x \in \mathbb{R}^n} u(x, t) \) is a Lipschitz function of \( t \).

We start with the following simple observation, which is related to the hypothesis (H1).

**Proposition 2.1.** Either \( \sup_{x \in \mathbb{R}^n} u(x, t) \equiv 1 \) or \( \sup_{x \in \mathbb{R}^n} u(x, t) < 1 \) in \((-\infty, +\infty)\).
Proof. Denote  
\[ \mathcal{I} := \left\{ t : \sup_{x \in \mathbb{R}^n} u(x, t) = 1 \right\}. \]
By continuity, \( \mathcal{I} \) is a closed subset of \( \mathbb{R} \).

We claim that \( \mathcal{I} \) is also open. Therefore it is either empty or the entire real line. Indeed, if \( \sup_{x \in \mathbb{R}^n} u(x, t_0) = 1 \), then there exist a sequence of points \( x_j \in \mathbb{R}^n \) such that \( u(x_j, t_0) \to 1 \). Let 
\[ u_j(x, t) := u(x_j + x, t_0 + t). \]
By standard parabolic regularity theory and Arzela-Ascoli theorem, \( u_j \to u_\infty \) in \( C^{2,1}_{loc}(\mathbb{R}^n \times \mathbb{R}) \), where \( u_\infty \) is an entire solution of (1.1). Since \( 0 \leq u_\infty \leq 1 \) and \( u_\infty(0, 0) = 1 \), by (F1) and the strong maximum principle, \( u_\infty \equiv 1 \). As a consequence, for any \( \varepsilon > 0 \) and \( t \in (-\varepsilon, \varepsilon) \),
\[ \lim_{j \to \infty} u(x_j, t_0 + t) = 1. \]
Hence \( \sup_{x \in \mathbb{R}^n} u(x, t) = 1 \) in \( (t_0 - \varepsilon, t_0 + \varepsilon) \) and the claim follows. \( \square \)

From now on it is always assumed that (H1) holds, i.e. \( \sup_{x \in \mathbb{R}^n} u(x, t) \equiv 1 \) for any \( t \in \mathbb{R} \).

Lemma 2.2. For any \( b > 0 \) and \( R > 0 \), there exists a constant \( \varepsilon := \varepsilon(b, R) > 0 \) such that for any \( (x, t) \in \mathbb{R}^n \times \mathbb{R} \), if \( u(x, t) \geq 1 - \varepsilon \), then \( u \geq 1 - b \) in \( B_R(x) \times (t - R, t + R) \).

Proof. This follows from a contradiction argument similar to the proof of Proposition 2.1, by applying the strong maximum principle to the limiting solution. \( \square \)

The following result is essentially [1, Lemma 5.1] (see also [30, Lemma 3.5]). We will use the notations of forward and backward light cones in space-time:
\[
\begin{align*}
\mathcal{C}_+^\lambda(x, t) &:= \{ (y, s) : s > t, |y - x| < \lambda(s - t) \}, \\
\mathcal{C}_-^\lambda(x, t) &:= \{ (y, s) : s < t, |y - x| < \lambda(t - s) \}.
\end{align*}
\]

Lemma 2.3 (Propagation to state 1). There exists a constant \( b_1 \in (0, 1) \) such that for any \( b \in [0, b_1) \) and \( \delta > 0 \), there exists an \( R := R(b, \delta) \) so that the following holds. If \( w \) is the solution to the Cauchy problem
\[
\begin{align*}
\partial_t w - \Delta w &= f(w) \quad \text{in} \quad \mathbb{R}^n \times (0, +\infty), \\
w(0) &= (1 - b) \chi_{B_R},
\end{align*}
\]
where \( R \geq R(b, \delta) \), then
\[ w(x, t) > 1 - b \quad \text{in} \quad \mathcal{C}_+^{\kappa, \delta}(0, 0). \]

By decreasing \( b_1 \) further, we may assume \( f' \leq f'(1)/2 \) in \( [1 - b_1, 1] \).

For applications below, we need an a priori estimates for a linear parabolic equation.

Lemma 2.4. Given a constant \( M > 0 \), if \( w \) satisfies
\[
\begin{align*}
\partial_t w - \Delta w &\leq -Mw \quad \text{in} \ B_1 \times (-1, 0), \\
0 &\leq w \leq 1 \quad \text{in} \ B_1 \times (-1, 0),
\end{align*}
\]
then

\[ w \leq \frac{C}{M} \text{ in } B_{1/2} \times (-1/2, 0). \]

This can be proved, for example, by constructing a suitable sup-solution. The first application of this lemma is

**Lemma 2.5.** For any entire solution \( u \), if it is not exactly 1, then

\[ \inf_{\mathbb{R}^n \times \mathbb{R}} u < 1 - b_1. \]

**Proof.** Assume by the contrary, \( u \geq 1 - b_1 \) everywhere. By our choice of \( b_1 \), we get

\[ \partial_t (1 - u) - \Delta (1 - u) \leq \frac{f'(1)}{2} (1 - u) \text{ in } \mathbb{R}^n \times \mathbb{R}. \]  

(2.2)

An iteration of Lemma 2.4 gives \( u \equiv 1 \). \( \square \)

The next lemma is our main technical tool for the proof of Lipschitz property.

**Lemma 2.6.** There exist two constants \( D > 0, 0 < b_2 < b_1 \) so that the following holds. For any \((x, t) \in \{u = 1 - b_2\}\),

\[ u > 1 - b_2 \text{ in } C_{\kappa* - \delta}(x, t + D). \]

**Proof.** Take \( R := R(b_1, \delta) \) according to Lemma 2.3, \( b_2 := \varepsilon(b_1, R) \) according to Lemma 2.2. Then \( u_x(t) = 1 - b_2 \) implies \( u_y(y, t) \geq 1 - b_1 \) for any \( y \in B_R(x) \).

Combining Lemma 2.3 and comparison principle, we deduce that \( u > 1 - b_1 \) in \( C_{\kappa* - \delta}(x, t) \).

Now \( 1 - u \) satisfies the differential inequality (2.2) in \( C_{\kappa* - \delta}(x, t) \). By Lemma 2.4, we find a \( D > 0 \), which depends only on \( b_1, b_2 \) and \( f'(1) \), such that \( u > 1 - b_2 \) in \( C_{\kappa* - \delta}(x, t + D) \). \( \square \)

Three corollaries follow from this lemma. The first two of them are rather direct consequences of this lemma.

**Corollary 2.7.** For any \((x, t) \in \{u = 1 - b_2\}\),

\[ u < 1 - b_2 \text{ in } C_{\kappa* - \delta}(x, t - D). \]

**Corollary 2.8.** For any \((x, t) \in \{u = 1 - b_2\}, \{u = 1 - b_2\}\) lies between \( \partial C_{\kappa* - \delta}(x, t - D) \) and \( \partial C_{\kappa* - \delta}(x, t - D) \).

**Corollary 2.9.** If \( u \) is an entire solution of (1.1) satisfying (H1), then it cannot be independent of \( t \), unless \( u \equiv 1 \).

**Proof.** Assume by the contrary, \( \partial_t u \equiv 0 \) in \( \mathbb{R}^n \times \mathbb{R} \). By (H1), there exists a point \((x, 0) \in \{u = 1 - b_2\}\). By Lemma 2.6, \( u > 1 - b_2 \) in \( C_{\kappa* - \delta}(x, D) \). Then we get \( u \geq 1 - b_2 \) everywhere. By Lemma 2.5, \( u \equiv 1 \). \( \square \)

**Proposition 2.10.** The level set \( \{u = 1 - b_2\} \) belongs to the \( D \)-neighborhood of a globally Lipschitz graph \( \{t = h_*(x)\} \).
Proof. Let
\[ h^*_*(x) := \inf_{(y, s) \in \{u = 1 - b\}^2} \left[ s + D + \frac{|x - y|}{\kappa_* - \delta} \right]. \]
It is a globally Lipschitz function on \( \mathbb{R}^n \), with its Lipschitz constant at most \((\kappa_* - \delta)^{-1}\). To check this, we need only to show that \( h^*_* > -\infty \) at one point. (This then implies that it is finite everywhere.) In fact, take an arbitrary point \((x_0, t_0) \in \{u = 1 - b\}\). (The existence of such a point is guaranteed by (H1) and Lemma 2.5.) By Corollary 2.7, we see for any \((y, s) \in \{u = 1 - b\}^2\),
\[ |y - x_0| > (\kappa_* - \delta) (t_0 - D - s). \]
In other words, \((x_0, t_0) \notin C^{+}_{\kappa_0 - \delta}(y, s + D)\). Then by definition, we get
\[ h^*_*(x_0) \geq t_0. \]
□

We modify \( h^*_* \) into a smooth function. Take a standard cut-off function \( \eta \in C^\infty_0(\mathbb{R}^n), \eta \geq 0 \) and \( \int_{\mathbb{R}^n} \eta = 1 \). Define
\[ h^*(x) := \int_{\mathbb{R}^n} \eta(x - y) [h^*_*(y) + 1] \, dy. \]
It is directly verified that \( h^* \) has the same Lipschitz constant with \( h^*_* \). Moreover, by choosing \( \eta \) suitably, we have
\[ h_* \leq h^* \leq h_* + 2. \quad (2.3) \]
Denote
\[ \Omega^* := \{(x, t) : t > h^*(x)\}. \]

Lemma 2.11. There exists a universal constant \( c < 1 \) such that
\[ cb_2 \leq 1 - u \leq b_2 \quad \text{on} \{t = h^*(x)\}. \]

Proof. The second inequality is a direct consequence of the fact that \( u > 1 - b_2 \) in \( \Omega^* \), thanks to (2.3) and the construction of \( h^*_* \) in the proof of Proposition 2.10.

The first inequality follows by applying Harnack inequality to the linear parabolic equation
\[ \partial_t (1 - u) - \Delta (1 - u) = V (1 - u) \]
in the parabolic cylinder \( B_{3\sqrt{D}}(x) \times (t - 9D, t + 9D) \). In the above \( V := -f(u)/(1 - u) \) is an \( L^\infty \) function. □

2.2. Proof of Theorem 1.1. Before proving Theorem 1.1, first we need to construct a comparison function. Consider the problem
\[
\begin{cases}
\partial_t w^* - \Delta w^* = f'(1)w^*, & \text{in } \Omega^*, \\
w^* = 1 & \text{on } \partial\Omega^*.
\end{cases}
\]

Proposition 2.12. \( C^\infty(\overline{\Omega}^*) \).

(1) There exists a unique solution of \((2.4)\) in \( L^\infty(\Omega^*) \cap C^\infty(\overline{\Omega}^*) \).

(2) There exists a universal constant \( C \) such that for any \((x, t) \in \Omega^*\),
\[ \frac{1}{C} e^{C[1 - h^*(x)]} \leq w^*(x, t) \leq C e^{t - h^*(x)}. \]

(2.5)
There exists a universal constant \( C \) such that
\[
\frac{|\nabla w^*|}{w^*} + \frac{\partial tw^*}{w^*} \leq C \quad \text{in} \quad \Omega^*.
\] (2.6)

There exists a universal constant \( c \) such that
\[
\partial tw^* \leq -c \quad \text{in} \quad \Omega^*.
\] (2.7)

There exists a universal constant \( C \) such that
\[
\frac{|\nabla w^*|}{|\partial tw^*|} \leq C \quad \text{in} \quad \Omega^*.
\] (2.8)

As a consequence, all level sets of \( w^* \) are Lipschitz graphs in the \( t \)-direction.

**Proof.**

(i) Existence, uniqueness and regularity of the solution can be proved as in Oleinin and Radkevic [29, Chapter 1], see respectively Section 5, Section 6 and Section 8 therein.

(ii) The exponential upper bound follows from iteratively applying the estimate in Lemma 2.4. The lower bound follows from iteratively applying the standard Harnack inequality.

(iii) By the regularity theory in [29], there exists a universal constant \( C \) such that
\[
|\nabla w^*| + |\partial tw^*| \leq C \quad \text{in} \quad \Omega^*.
\] Hence for any constant vector \((\xi, s) \in \mathbb{R}^{n+1}, \xi \cdot \nabla w^* + s\partial tw^*\) is a bounded solution of (2.4). As in (ii), it converges to 0 as \( t - h^*(x) \to +\infty \). Then (2.6) follows from an application of the comparison principle.

(iv) For any \( \rho > 0 \), in the half ball
\[
\mathcal{B}_\rho(0, 0) := \{(x, t) : |x|^2 + t^2 < \rho^2, \; t < 0\}\,
\] the function
\[
w_\rho(x, t) := e^{\alpha(|x|^2 + t^2 - \rho^2)}
\] is a sup-solution of (2.4), provided that \( \alpha \) is small enough (depending only on \( \rho \), the space dimension \( n \) and \( f'(1) \)).

On \( \{|x|^2 + t^2 = \rho^2, \; t < - (\kappa_* - \delta)^{-1} |x|\} \), there exists a constant \( c(\rho) > 0 \) such that
\[
\partial_\rho w_\rho(x, t) = 2\alpha t w_\rho(x, t) \leq -c(\rho).
\]
Since \( \sup_{\mathbb{R}^n} |\nabla^2 h^*| \leq C \), for any \((x, h^*(x))\), there exists an half ball \( \mathcal{B}_{1/C}(y, s) \) tangent to \( \partial \Omega^* \) at this point. Moreover, because \( |\nabla h^*(x)| \leq (\kappa_* - \delta)^{-1} \),
\[
t - s \leq - (\kappa_* - \delta)^{-1} |x - y|.
\]

By the comparison principle in \( \mathcal{B}_{1/C}(y, s), w_{1/C}(\cdot - (y, s)) \geq w^* \) in \( \mathcal{B}_{1/C}(y, s) \). Therefore
\[
\partial_tw^*(x, h^*(x)) \leq -c.
\]

Then (2.7) follows from an application of the comparison principle as in (iii).

(v) This is a direct consequence of (2.6) and (2.7). \( \square \)
Lemma 2.13. There exists a universal constant $C$ such that
\[
\frac{b_2}{C} \leq \frac{1 - u}{w^*} \leq Cb_2 \quad \text{in } \Omega^*.
\] (2.9)

Proof. For each $k \geq 1$, let
\[
\Omega_k^* := \{(x, t) : k - 1 < t - h^*(x) < k\}.
\]

Similar to (2.5), for any $(x, t) \in \Omega^*$ we have
\[
\frac{1}{C} e^{-C[t-h^*(x)]} \leq 1 - u(x, t) \leq Ce^{-C[t-h^*(x)]}.
\] (2.10)

Hence there exists a $\sigma \in (0, 1)$ such that
\[
\sigma k f'(1) (1 - u) \leq \left[ \partial_t - \Delta - f'(1) \right] (1 - u) \leq -\sigma k f'(1) (1 - u) \quad \text{in } \Omega_k^*.
\]

For each $k$, define $w_k^*$ inductively as the unique solution of
\[
\begin{aligned}
\partial_t w_k^* - \Delta w_k^* &= f'(1) (1 - \sigma^k) w_k^* \quad \text{in } \Omega_k^*, \\
w_k^* &= w_{k-1}^* \quad \text{on } \{t = h^*(x) + k - 1\}.
\end{aligned}
\] (2.11)

Here $w_0^* := b_2$. As before, the existence and uniqueness of $w_k^*$ follows from Oleinik and Radkevič [29, Chapter 1].

By inductively applying the comparison principle, we get
\[
1 - u \leq w_k^* \quad \text{in } \Omega_k^*.
\] (2.12)

Next for each $k$, denote
\[
M_k := \sup_{\Omega_k^*} \frac{w_k^*}{w^*}.
\]

A direct calculation shows that $(M_k w^*)^{1-\sigma^k}$ is a sup-solution of (2.11) in $\Omega_{k+1}^*$. (Here we also need an inductive assumption that $M_k w^* < 1$ on $\{t = h^*(x) + k\}$.) From this we deduce that
\[
M_{k+1} \leq M_k^{1-\sigma^k} \sup_{\Omega_{k+1}^*} (w^*)^{-\sigma^k} \leq M_k^{1-\sigma^k} e^{C k \sigma^k},
\]

where the last inequality follows by substituting (2.5) to estimate $\inf_{\Omega_{k+1}^*} w^*$.

From this inequality it is readily deduced that there exists a universal, finite upper bound on $M_k$ as $k \to \infty$. Combining this fact with (2.12) we obtain the upper bound in (2.9).

The lower bound in (2.9) follows in the same way by considering
\[
\begin{aligned}
\partial_t w_{k,*} - \Delta w_{k,*} &= f'(1) (1 + \sigma^k) w_{k,*} \quad \text{in } \Omega_k^*, \\
w_{k,*} &= w_{k-1,*} \quad \text{on } \{t = h^*(x) + k - 1\}.
\end{aligned}
\]

\[\square\]

Corollary 2.14. There exists a universal constant $C > 0$ such that
\[
\frac{|\nabla u| + |\partial_t u|}{1 - u} \leq C \quad \text{in } \Omega^*. 
\] (2.13)
Proof. For any \((x, t) \in \Omega^*\), standard gradient estimate gives
\[
|\nabla u(x, t)| + |\partial_t u(x, t)| \leq C \sup_{B_1(x) \times (t-1, t)} (1 - u). \tag{2.14}
\]
By the previous lemma, for any \((y, s) \in B_1(x) \times (t - 1, t)\),
\[
1 - u(y, s) \leq Cw^*(y, s) \leq C^2w^*(x, t) \leq C^3[1 - u(x, t)]. \tag{2.15}
\]
Here the second inequality follows by integrating (2.6) along the segment from \((y, s)\) to \((x, t)\).
Substituting (2.15) into (2.14) we get (2.13). \qed

**Proposition 2.15.** There exists a \(b_0 \in (0, b_2)\) such that in \(\{u > 1 - b_0\}\),
\[
\frac{\partial_t u}{1 - u} \geq c. \tag{2.16}
\]

Proof. Assume by the contrary, there exists a sequence of points \((x_k, t_k)\) such that
\[
u(x_k, t_k) \to 1 \quad \text{but} \quad \frac{\partial_t u(x_k, t_k)}{1 - u(x_k, t_k)} \to 0. \tag{2.17}
\]
Denote \(R_k := \text{dist}((x_k, t_k), \partial \Omega^*)\). Because \(u(x_k, t_k) \to 1\), by (2.10), \(R_k\) goes to infinity as \(k \to \infty\).

Let
\[
u_k(x, t) := \frac{1 - u(x_k + x, t_k + t)}{1 - u(x_k, t_k)}, \quad w_k(x, t) := \frac{w^*(x_k + x, t_k + t)}{1 - u(x_k, t_k)}.
\]
By definition, \(u_k(0, 0) = 1\), while by (2.9), we have
\[
\frac{b_2}{C} \leq \frac{u_k}{w_k} \leq Cb_2 \quad \text{in} \quad B_{cR_k}(0) \times (-cR_k, cR_k).
\]
Furthermore, (2.6) and (2.13) are transformed into
\[
\frac{|\nabla u_k| + |\partial_t u_k|}{u_k} \leq C, \quad \frac{|\nabla w_k| + |\partial_t w_k|}{w_k} \leq C \quad \text{in} \quad B_{cR_k}(0) \times (-cR_k, cR_k).
\]
Then by standard parabolic regularity theory, \(u_k\) and \(w_k\) are uniformly bounded in \(C_{loc}^{2+\alpha, 1+\alpha/2}(\mathbb{R}^n \times \mathbb{R})\). After passing to a subsequence, \(u_k \to u_\infty, \ w_k \to w_\infty\) in \(C_{loc}^2(\mathbb{R}^n \times \mathbb{R})\). Both of them are solutions of
\[
\partial_t w - \Delta w = f'(1)w \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R}.
\]
By [37] or [24], there exists a Borel measure supported on \(\{\lambda = |\xi|^2\} \subset \mathbb{R}^{n+1}\) such that
\[
w_\infty(x, t) = \int_{\{\lambda = |\xi|^2\}} e^{[f'(1)+\lambda]t+\xi \cdot x} d\mu(\xi, \lambda).
\]
Because
\[
\frac{b_2}{C} \leq \frac{u_\infty}{w_\infty} \leq Cb_2 \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R}, \tag{2.18}
\]
there exists a function $\Theta$ on $\{\lambda = |\xi|^2\}$ with $b_2/C \leq \Theta \leq Cb_2$ such that

$$u_\infty(x, t) = \int_{\{\lambda = |\xi|^2\}} e^{[f'(1) + \lambda]t + \xi \cdot x} \Theta(\xi, \lambda) d\mu(\xi, \lambda).$$

This follows by writing $u_\infty$ as the same integral representation with another measure $\tilde{\mu}$, applying Radon-Nikodym theorem to these two measures, and then use (2.18) to estimate the differential $\frac{d\tilde{\mu}}{d\mu}$.

Because $w_\infty$ still satisfies the inequality (2.7), the support of $\mu$ is contained in $\{\lambda \leq -f'(1) - c\}$. Hence we also have

$$\partial_t u_\infty = \int_{\{\lambda = |\xi|^2\}} [f'(1) + \lambda] e^{[f'(1) + \lambda]t + \xi \cdot x} \Theta(\xi, \lambda) d\mu(\xi, \lambda) \leq -cu_\infty.$$

This is a contradiction with (2.17). \qed

Theorem 1.1 follows by combining (2.13) and (2.16).

3. Proof of Theorem 1.2

For simplicity, denote $h(x) := h_{1-b_0}(x)$.

3.1. The combustion and bistable case. In these two cases we need the assumption (H2), that is, $u(x, t) \to 0$ uniformly as $\text{dist}((x, t), \{t = h(x)\}) \to +\infty$.

First we use the sliding method to prove

**Proposition 3.1.** $u$ is increasing in $t$.

**Proof.** The fact that $\partial_t u > 0$ in $\{u > 1 - b_0\}$ has been established in Proposition 2.15. Now we use the sliding method to show the remaining case.

For any $\lambda \in \mathbb{R}$, let

$$u^\lambda(x, t) := u(x, t + \lambda).$$

We want to show that for any $\lambda > 0$, $u^\lambda \geq u$ in $\mathbb{R}^n \times \mathbb{R}$.

**Step 1.** If $\lambda$ is large enough, $u^\lambda \geq u$ in $\mathbb{R}^n \times \mathbb{R}$.

By (H2), there exists a constant $L > 0$ such that

$$\sup_{\{t < h(x) - L\}} u \ll 1. \quad (3.1)$$

If $\lambda > L$, we have

$$u^\lambda \geq 1 - b_0 \geq u \quad \text{in } \{h(x) - L \leq t \leq h(x)\}.$$ 

In $\{t < h(x) - L\}$, we have

$$\partial_t \left( u - u^\lambda \right)_+ - \Delta \left( u - u^\lambda \right)_+ \leq V \left( u - u^\lambda \right)_+,$$

where

$$V := \begin{cases} \frac{f(u) - f(u^\lambda)}{u - u^\lambda}, & \text{if } u > u^\lambda \\ f'(u), & \text{otherwise.} \end{cases}$$
By (3.1) and (F4), $V \leq 0$ in $\{ t < h(x) - L \}$. Because $(u - u^\lambda)_+ = 0$ on $\{ t = h(x) - L \}$ and $(u - u^\lambda)_+ \to 0$ as $\text{dist}((x,t), \{ t = h(x) - L \}) \to +\infty$, by the maximum principle we obtain

$$u \leq u^\lambda \quad \text{in} \quad \{ t < h(x) - L \}.$$ 

**Step 2.** Now

$$\lambda_* := \inf \left\{ \lambda : \forall \lambda' > \lambda, \ u^{\lambda'} \geq u \ \text{in} \ \mathbb{R}^n \times \mathbb{R} \right\}$$

is well defined. We claim that $\lambda_* = 0$.

By continuity, $u^{\lambda_*} \geq u$ in $\mathbb{R}^n \times \mathbb{R}$. By the strong maximum principle, either $u^{\lambda_*} > u$ strictly or $u^{\lambda_*} \equiv u$. The later is excluded if $\lambda_* > 0$, because in this case $u^{\lambda_*} > u$ in $\{ t > h(x) \}$.

**Claim.** If $\lambda_* > 0$, for any $L > 0$, there exists a constant $\epsilon_1 := \epsilon(\lambda_*, L) > 0$ such that

$$u^{\lambda_*} - u \geq \epsilon_1 \quad \text{in} \quad \{ h(x) - L \leq t \leq h(x) \}.$$

We prove this claim by contradiction. Assume there exists a sequence of points $(x_i, t_i) \in \{ h(x) - L \leq t \leq h(x) \}$ such that $u^{\lambda_*}(x_i, t_i) - u(x_i, t_i) \to 0$. Set

$$u_i(x, t) := u(x_i + x, t_i + t).$$

They satisfy the following conditions:

- there exists a constant $b(L) \in (0, 1)$ such that $b(L) \leq u_i(0, 0) \leq 1 - b(L)$;
- $u^{\lambda_*}_i \geq u_i$ in $\mathbb{R}^n \times \mathbb{R}$;
- $u^{\lambda_*}_i - u_i \geq c\lambda_*(1 - u_i)$ in $\{ u_i \geq 1 - b_0 \}$ (thanks to Proposition 2.15);
- $u^{\lambda_*}_i(0, 0) - u_i(0, 0) \to 0$.

These lead to a contradiction after letting $i \to +\infty$, and the proof of this claim is complete.

By this claim and Proposition 2.15, for any $L > 0$, we find another constant $\epsilon_2 := \epsilon(\lambda_*, L) > 0$ such that, if $\lambda \geq \lambda_* - \epsilon_2$,

$$u^\lambda \geq u \quad \text{in} \quad \{ t \geq h(x) - L \}. \quad (3.2)$$

Then as in Step 1, by (3.2) and the comparison principle, for these $\lambda$, $u^\lambda \geq u$ in $\{ t < h(x) - L \}$ (hence everywhere in $\mathbb{R}^n \times \mathbb{R}$). This is a contradiction with the definition of $\lambda_*$. Therefore we must have $\lambda_* = 0$. \( \square \)

Combining this proposition with Corollary 2.9 and strong maximum principle, we obtain

**Corollary 3.2.** In $\mathbb{R}^n \times \mathbb{R}$, $\partial_t u > 0$ strictly.

**Proposition 3.3.** There exists a universal constant $c > 0$ such that

$$\partial_t u \geq c|\nabla u| \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R}.$$ 

**Proof.** In $\{ u > 1 - b_0 \}$, this inequality follows by combining Corollary 2.14 and Proposition 2.15.
In view of Corollary 3.2, following the argument in the second step of the proof of Proposition 3.1, for any $L > 0$, we find a positive lower bound for $\partial_t u$ in $\{h(x) - L \leq t \leq h(x)\}$. Hence trivially we have

$$|\nabla u| \leq C \leq \frac{C}{c} \partial_t u \quad \text{in} \quad \{h(x) - L \leq t \leq h(x)\}.$$  

Finally, we apply the maximum principle to the linearized equation

$$(\partial_t - \Delta) (\partial_t u - c \xi \cdot \nabla u) = f'(u) (\partial_t u - c \xi \cdot \nabla u)$$

to show that $\partial_t u - c \xi \cdot \nabla u \geq 0$ in $\{t < h(x) - L\}$, where $\xi$ is an arbitrary unit vector in $\mathbb{R}^n$ and $c > 0$ is a small constant. \[\square\]

Theorem 1.2 is a direct consequence of this proposition.

3.2. A remark on the monostable case. In this subsection we give a remark on the monostable case.

For this case, we note the following important “hair trigger” phenomena (see [1, Theorem 3.1]).

**Lemma 3.4.** For any $\lambda \in (0, 1)$, $\delta > 0$ and $(x, t) \in \mathbb{R}^n \times \mathbb{R}$, there exists a constant $D := D(x, t, \lambda) > 0$ such that

$$u > \lambda \quad \text{in} \quad C^{+}_{\kappa, \delta}(x, t + D).$$

**Lemma 3.5.** If $f$ is monostable, then $u \to 0$ uniformly as $\text{dist}((x, t), \{u \geq 1 - b_0\}) \to +\infty$.

**Proof.** This follows from the following Liouville type result: suppose $u$ is an entire solution of (1.1) satisfying $0 \leq u \leq 1 - b_0$, then $u \equiv 0$. This Liouville theorem is a direct consequence of Lemma 3.4. \[\square\]

Unfortunately, in this case we need to assume the following assumption:

$$|\partial_t u| + |\nabla u| \leq C \quad \text{in} \quad \{t < h(x)\}. \quad (3.3)$$

Of course, if $u$ is a travelling wave solution, this assumption holds by applying standard elliptic Harnack inequality and interior gradient estimates (see Lemma 8.1 below), but we do not know how to prove the parabolic case.

**Lemma 3.6.** Given $\kappa > 0$, assume $u$ is an entire positive solution of

$$\partial_t u - \Delta u = \kappa u.$$  

Then

$$\frac{|\nabla u|}{\partial_t u} \leq \frac{1}{2\sqrt{\kappa}} \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R}.$$  

As a consequence, all level sets of $u$ are Lipschitz graphs in the $t$ direction, with their Lipschitz constants at most $2\sqrt{\kappa}$.  

Proof. By [37] or [24], there exists a Borel measure $\mu$ supported on \( \{ \lambda = |\xi|^2 \} \subset \mathbb{R}^{n+1} \) such that
\[
\begin{align*}
u(x,t) &= \int_{\{ \lambda = |\xi|^2 \}} e^{[\kappa + \lambda]t + |\xi|^2 x^2} d\mu(\xi, \lambda).
\end{align*}
\]
Then we have
\[
\begin{align*}
\partial_t \nu(x,t) &= \int_{\{ \lambda = |\xi|^2 \}} \left[ \kappa + \lambda \right] e^{[\kappa + \lambda]t + |\xi|^2 x^2} d\mu(\xi, \lambda) \\
&= \int_{\{ \lambda = |\xi|^2 \}} \left[ \kappa + |\xi|^2 \right] e^{[\kappa + \lambda]t + |\xi|^2 x^2} d\mu(\xi, \lambda) \\
&\geq 2\sqrt{\kappa} \int_{\{ \lambda = |\xi|^2 \}} |\xi| e^{[\kappa + \lambda]t + |\xi|^2 x^2} d\mu(\xi, \lambda) \\
&\geq 2\sqrt{\kappa} |\nabla \nu(x,t)|.
\end{align*}
\]
□

Corollary 3.7. There exists a constant $L > 0$ such that
\[
|\nabla \nu| \leq \frac{1}{4\sqrt{f'(0)}} \quad \text{in} \quad \{ t < h(x) - L \}.
\]

Proof. For any $(x_i, t_i) \in \{ t < h(x) \}$ with $t_i - h(x_i) \to -\infty$, by Lemma 3.5, $u(x_i, t_i) \to 0$. Set
\[
u_i(x,t) := \frac{u(x_i + x, t_i + t)}{u(x_i, t_i)}.
\]
By definition, $u_i > 0$ and $u_i(0,0) = 1$. Integrating (3.3), we see $u_i$ are uniformly bounded in any compact set of $\mathbb{R}^n \times \mathbb{R}$. Then by standard parabolic regularity theory, we can take a subsequence $u_i \to u_\infty$ in $C^{2,1}_{\text{loc}}(\mathbb{R}^n \times \mathbb{R})$. Here $u_\infty$ is an entire solution of
\[
\partial_t u_\infty - \Delta u_\infty = f'(0)u_\infty.
\]
The claim then follows from Lemma 3.6. □

Theorem 1.2 in the monostable case (under the hypothesis (3.3)) follows from the same sliding method as in the previous subsection.

4. AN ELLIPTIC HARNACK INEQUALITY

From now on, unless otherwise stated, it is always assumed that (F1 – F4) and (H1 – H2) hold. In this section we prove an elliptic Harnack inequality for $u$. This will be used in the blowing down analysis in the next section.

In $\{ t > h(x) \}$, what we want has been given in Corollary 2.14, so here we consider the other part $\{ t < h(x) \}$.

Proposition 4.1. There exists a universal constant $C > 0$ such that
\[
\frac{|\partial_t u| + |\nabla u|}{u} \leq C \quad \text{in} \quad \{ t < h(x) \}.
\]

Before proving this proposition, we first notice the following exponential decay of $u$ in $\{ t < h(x) \}$ for later use.
Proposition 4.2. Under the hypothesis (H2),
\[ u(x, t) \leq C e^{c(t - h(x))} \quad \text{in} \quad \{t < h(x)\}. \tag{4.2} \]

Proof. Because \( f \) is of combustion type or bistable, by choosing \( L \) large enough, we have
\[ \partial_t u - \Delta u \leq 0 \quad \text{in} \quad \{t < h(x) - L\}. \]
Take a radially symmetric function \( \varphi \in C^2(\mathbb{R}^n) \) such that \( \varphi \leq 0, \varphi(x) \equiv -2\kappa_s|x| \) outside a large ball, \( |\Delta \varphi| \ll 1 \) and \( |\nabla \varphi| \leq C \) in \( \mathbb{R}^n \). By taking a small \( \mu > 0 \), the function
\[ w(x, t) := e^{\mu|t - \varphi(x)|} \]
is a super-solution of the heat equation in \( \mathcal{D} := \{(y, t) : t < \varphi(x)\} \). Moreover, \( w = 1 \) on \( \partial \mathcal{D} \).

For each \((x, t) \in \{t < h(x)\}\), by enlarging \( L \) further (depending on the Lipschitz constant of \( h \), but independent of \( x \)), the domain
\[ \mathcal{D}_x := \{(y, s) : s < h(x) - 2L - \varphi(y - x)\} \subset \{s < h(y) - L\}. \]
A comparison with a suitable translation of \( w \) leads to (4.2).

Take a large \( L > 0 \) so that \( u \ll 1 \) in \( \{t < h(x) - L\} \). This is possible by (H2). In \( \{h(x) - L \leq t \leq h(x)\} \), (4.1) is a direct consequence of the facts that \( u \) has a positive lower bound here while both \( \partial_t u \) and \( \nabla u \) are bounded. It thus remains to show that (4.1) holds in \( \{t < h(x) - L\} \).

We first prove the combustion case.

Proof of Proposition 4.1 in the combustion case. If \( f \) is of combustion type, \( u, \partial_t u \) and \( \partial_x u \) all satisfy the heat equation in \( \{t < h(x) - L\} \). The estimate (4.1) then follows from the comparison principle. For example, because both \( \partial_t u \) and \( u \) converge to 0 uniformly as \( \text{dist}((x, t), \{t = h(x)\}) \rightarrow +\infty \), if \( \partial_t u \leq Mu \) on \( \{t = h(x) - L\} \) for some constant \( M > 0 \), then
\[ \partial_t u \leq Mu \quad \text{in} \quad \{t < h(x) - L\}. \]

Next, we prove the bistable case.

Proof of Proposition 4.1 in the bistable case. Take a \( b \in (0, 1) \) sufficiently small so that \( f \in C^{1, \alpha}([0, b]) \),
\[ f'(u) \leq f'(0)/2 \quad \text{and} \quad |f'(u) - f'(0)| \leq Cu^\alpha, \quad \text{for any} \ u \in [0, b]. \tag{4.3} \]

For any \( \lambda \in (0, b) \), denote \( \Omega_\lambda := \{u < \lambda\} \). A direct calculation using (4.3) shows that for some universal constant \( C > 1 \) (independent of \( \lambda \)),
\[ \partial_t u^{1-C\lambda^\alpha} - \Delta u^{1-C\lambda^\alpha} \geq f'(u)u^{1-C\lambda^\alpha}. \]
On the other hand, \( \partial_t u \) is a solution of this linearized equation.

Therefore, if we denote
\[ M(\lambda) := \sup_{\partial \Omega_\lambda} \frac{\partial_t u}{u^{1-C\lambda^\alpha}} = \sup_{\partial \Omega_\lambda} \frac{\partial_t u}{\lambda^{1-C\lambda^\alpha}}, \]
applying the comparison principle as in the proof of the combustion case, we obtain
\[ \partial_t u \leq M(\lambda)u^{1-C\lambda^\alpha} \quad \text{in} \quad \Omega_\lambda. \quad (4.4) \]

From this inequality and the fact that \( \partial \Omega_{\lambda/2} \subset \Omega_\lambda \), we deduce that
\[ M \left( \frac{\lambda}{2} \right) \leq M(\lambda) \left( \frac{\lambda}{2} \right)^{-C(1-2^{-\alpha})\lambda^\alpha}. \]

This inequality implies that
\[ \limsup_{\lambda \to 0} M(\lambda) < +\infty. \]

Substituting this estimate into (4.4), we find a constant \( C \) such that for any \( \lambda \in (0, b) \), in \( \Omega_\lambda \setminus \Omega_{\lambda/2} \),
\[ \partial_t u \leq Cu^{1-C(2u)\alpha} \leq 2Cu. \quad (4.5) \]

Here to deduce the last inequality, we have used the inequality (perhaps after choosing a smaller \( b \))
\[ u^{-C2u\alpha} \leq 2, \quad \text{if} \quad u \leq b. \]

Finally, the estimate for \( |\nabla u|/u \) follows by combining (4.5) and Proposition 3.3. \( \square \)

5. Blowing down analysis

Recall that the one dimensional travelling wave \( g \) (see (1.3)) is strictly increasing, and it converges to 1 and 0 exponentially as \( t \to \pm \infty \). In fact, by (F1) and (F4), there exist four positive constants \( \alpha_\pm \) and \( \beta_\pm \) such that
\[ g(t) = 1 - \alpha_+e^{-\beta_+t} + O \left( e^{-(1+\alpha)\beta_+t} \right) \quad \text{as} \quad t \to +\infty, \]
\[ g(t) = \alpha_-e^{\beta_-t} + O \left( e^{(1+\alpha)\beta_-t} \right) \quad \text{as} \quad t \to -\infty, \]
where
\[ \beta_+ := -\lim_{t \to +\infty} \frac{g''(t)}{g'(t)} = \frac{-\kappa_* + \sqrt{\kappa_*^2 - 4f'(1)}}{2}, \]
\[ \beta_- := \lim_{t \to -\infty} \frac{g''(t)}{g'(t)} = \frac{\kappa_* + \sqrt{\kappa_*^2 - 4f'(0)}}{2}. \]

Because \( f''(0) \leq 0, \beta_- \geq \kappa_* \).

Following [2], set \( \Phi := g^{-1} \circ u \). It satisfies
\[ \partial_t \Phi - \Delta \Phi = \kappa_* + \frac{g''(\Phi)}{g'(\Phi)} \left( |\nabla \Phi|^2 - 1 \right). \quad (5.1) \]

**Lemma 5.1.** There exists a universal constant \( C > 0 \) such that
\[ |\partial_t \Phi| + |\nabla \Phi| \leq C \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R}. \]
Proof. By Proposition 4.1,
\[ |\partial_t \Phi| \leq C \frac{|\partial_t u|}{u} \leq C, \quad |\nabla \Phi| \leq C \frac{|\nabla u|}{u} \leq C, \quad \text{in } \{ u \leq 1 - b_0 \}. \]
By Corollary 2.14,
\[ |\partial_t \Phi| \leq C |\partial_t u| \leq C, \quad |\nabla \Phi| \leq C |\nabla u| \leq C, \quad \text{in } \{ u \geq 1 - b_0 \}. \]

Lemma 5.2 (Semi-concavity). There exists a universal constant \( C \) such that for any \( (x, t) \in \{ \Phi > 0 \} \),
\[ \nabla^2 \Phi(x, t) \leq \frac{C}{\Phi(x, t)}, \]
and for any \( (x, t) \in \{ \Phi < 0 \} \),
\[ \nabla^2 \Phi(x, t) \geq \frac{C}{\Phi(x, t)}. \]
This follows from a standard maximum principle argument applied to \( \eta \nabla^2 \Phi(\xi, \xi) \),
for any \( \xi \in \mathbb{R}^n \) and a suitable cut-off function \( \eta \).

For each \( \varepsilon > 0 \), let \( \Phi_\varepsilon(x, t) := \varepsilon \Phi(\varepsilon^{-1}x, \varepsilon^{-1}t) \). It satisfies
\[ \partial_t \Phi_\varepsilon - \varepsilon \Delta \Phi_\varepsilon = \kappa_+ + \frac{g'(\varepsilon^{-1} \Phi_\varepsilon)}{g'(\varepsilon^{-1} \Phi_\varepsilon)} \left( |\nabla \Phi_\varepsilon| - 1 \right). \quad (5.2) \]
By the uniform Lipschitz bound on \( \Phi_\varepsilon \) from Lemma 5.1, there exists a subsequence of \( \varepsilon \to 0 \) such that \( \Phi_\varepsilon \to \Phi_\infty \) in \( C_{\text{loc}}(\mathbb{R}^n \times \mathbb{R}) \).

The limit \( \Phi_\infty \) may depend on the choice of subsequences. But for notational simplicity, we will always write \( \varepsilon \to 0 \) instead of \( \varepsilon_i \to 0 \).

By standard vanishing viscosity method, we get

Lemma 5.3. In the open set \( \{ \Phi_\infty > 0 \} \), \( \Phi_\infty \) is a viscosity solution of
\[ \partial_t \Phi_\infty + \beta_+ |\nabla \Phi_\infty|^2 - \kappa_+ - \beta_+ = 0. \quad (5.3) \]

In the open set \( \{ \Phi_\infty < 0 \} \) (if non-empty), \( \Phi_\infty \) is a viscosity solution of
\[ \partial_t \Phi_\infty - \beta_- |\nabla \Phi_\infty|^2 - \kappa_- + \beta_- = 0. \quad (5.4) \]
Since \( h(x) \) is globally Lipschitz on \( \mathbb{R}^n \), by taking a further subsequence, we may also assume
\[ \varepsilon_i h(\varepsilon_i^{-1}x) \to h_\infty(x) \quad \text{in } C_{\text{loc}}(\mathbb{R}^n). \]

Lemma 5.4. \( \{ \Phi_\infty > 0 \} = \{ t > h_\infty(x) \} \).

Proof. Because \( u \geq 1 - b_0 \) in \( \{ t > h(x) \} \), by Lemma 2.4 we get
\[ 1 - u(x) \leq Ce^{-c(t-h(x))} \quad \text{in } \{ t > h(x) \}. \]
Using the expansion of \( g \) near infinity, this is rewritten as
\[ \Phi(x) \geq c[t - h(x)] - C \quad \text{in } \{ t > h(x) \}. \quad (5.5) \]
Taking the scaling \( \Psi_\varepsilon \) and letting \( \varepsilon \to 0 \), we obtain
\[ \Phi_\infty(x) \geq c[t - h_\infty(x)] > 0 \quad \text{in } \{ t > h_\infty(x) \}. \quad (5.6) \]
Finally, because $\Phi \leq g^{-1}(1 - b_0)$ in $\{t < h(x)\}$, $\Phi_\infty \leq 0$ in $\{t < h_\infty(x)\}$. □

**Lemma 5.5.** $h_\infty$ is unbounded from below.

*Proof.* This is a direct consequence of Proposition 2.1. □

**Lemma 5.6.** The Lipschitz constant of $h_\infty$ is at most $\kappa_*^{-1}$.

*Proof.* For any $(x_0, t_0) \in \{\Phi_\infty > 0\}$, for all $\varepsilon$ small, $\Phi_\varepsilon(x_0, t_0) \geq \Phi_\infty(x_0, t_0)/2$. By definition, $u(\varepsilon^{-1}x_0, \varepsilon^{-1}t_0)$ is very close to 1. By Lemma 2.6, for any $\delta > 0$, there exits a $D(\delta)$ such that $C_{\kappa_*-\delta}^+(\varepsilon^{-1}x_0, \varepsilon^{-1}t_0 + D(\delta)) \subset \{u > 1 - b_2\}$. A scaling of this gives $C_{\kappa_*-\delta}^+(x_0, t_0 + D(\delta)\varepsilon) \subset \{\Phi_\varepsilon > 0\}$. Letting $\varepsilon \to 0$ and then $\delta \to 0$, with the help of (5.6), we deduce that $C_{\kappa_*}^+(x_0, t_0) \subset \{\Phi_\infty > 0\}$. This implies that the Lipschitz constant of $h_\infty$ is at most $\kappa_*^{-1}$. □

Finally, under the assumption of Theorem 1.2, the following non-degeneracy condition in $\Omega_\infty^-$ holds.

**Proposition 5.7.** In $\Omega_\infty^-$,

$$\Phi_\infty(x, t) \leq c \left[ t - h_\infty(x) \right] < 0.$$  \hspace{1cm} (5.7)

This can be proved by scaling Proposition 4.2.

6. Geometric motion: Proof of Theorem 1.5

In this section we prove Theorem 1.5. This theorem does not follow directly from the result on front motion law established in [4], although it can be reduced to that one by constructing a suitable comparison function. The main reason is, for entire solutions of (1.1), it is not clear whether $|\nabla \Phi| \leq 1$ everywhere or not. (We believe this is not true in general.)

*Proof of Theorem 1.5.* We divide the proof into two steps, verifying the sub- and sup-solution property respectively.

**Step 1.** For any $\varphi \in C^1(\mathbb{R}^n)$ satisfying $\varphi \geq h_\infty$ and $\varphi = h_\infty$ at one point, say the origin 0, we want to show that $|\nabla \varphi(0)| \leq \kappa_*^{-1}$.

Assume by the contrary, there exists $\delta > 0$ such that

$$|\nabla \varphi(0)| = (\kappa_* - 3\delta)^{-1}. \hspace{1cm} (6.1)$$

The tangent plane of $\{t = \varphi(x)\}$ at $(0, 0)$ is $\{(\kappa_* - 3\delta)t = -x \cdot \xi\}$, where $\xi := \nabla \varphi(0)/|\nabla \varphi(0)|$. Since $h_\infty \leq \varphi$, we find three small constants $\rho > 0$, $t_0 < 0$ and $\sigma > 0$ such that

$$\Phi_\infty(x, t_0) \geq \sigma \quad \text{in} \quad D := \{x \cdot \xi \geq -(\kappa_* - 2\delta)t_0\} \cap B_\rho(0).$$

For all $\varepsilon$ small, consider the Cauchy problem

$$\begin{cases}
\partial_tw - \Delta w = f(w) & \text{in } \mathbb{R}^n \times (t_0/\varepsilon, +\infty), \\
w(x, t_0/\varepsilon) = \left(1 - e^{-t_0\varepsilon^{-1}\text{dist}(x, \partial D/\varepsilon)}\right) \chi_{D/\varepsilon}(x),
\end{cases}$$

where $\chi_{D/\varepsilon}$ denotes the characteristic function of $D/\varepsilon$. 

As in Lemma 2.3 (or by the motion law for front propagation starting from \( \partial(D \cap B_\rho) \)), see [2, Main Theorem] or [4, Theorem 9.1], we get

\[ w(x, 0) \geq 1 - b_0 \quad \text{in} \quad \left\{ x \cdot \xi \geq \frac{\delta t_0}{\varepsilon} \right\} \cap B_{\rho + (\kappa_\ast - \delta)t_0}(0). \]

By comparison principle, \( u \geq w \) in \( \mathbb{R}^n \times [t_0/\varepsilon, 0] \). After a scaling, with the help of (5.6), we get

\[ \Phi_\infty > 0 \quad \text{in} \quad \left\{ x \cdot \xi \geq \delta t_0 \right\} \cap B_{\rho + (\kappa_\ast - \delta)t_0}(0). \]

In particular, \( (0, 0) \) is an interior point of \( \{ \Phi_\infty > 0 \} \). This is a contradiction.

**Step 2.** In the same way, we can show that for any \( \varphi \in C^1(\mathbb{R}^n) \) satisfying \( \varphi \leq h_\infty \) and \( \varphi(0) = h_\infty(0) \), \( |\nabla \varphi(0)| \geq \kappa_\ast^{-1} \).

Assume this is not true, that is, there exists a \( \delta > 0 \) such that \( |\nabla \varphi(0)| = (\kappa_\ast + 3\delta)^{-1} \). The only difference with Step 1 is the construction of the comparison function. Now we need to consider, for all \( \varepsilon > 0 \), the Cauchy problem

\[
\begin{aligned}
\partial_t w - \Delta w &= f(w) & &\text{in} \ \mathbb{R}^n \times (t_0/\varepsilon, +\infty), \\
w(x, t_0) &= 1 - \left(1 - e^{-c_\ast \varepsilon^{-1}\text{dist}(x, \partial D)}\right) \chi_{D/\varepsilon}(x),
\end{aligned}
\]

where \( D = \{ x : \xi \leq - (\kappa_\ast + 2\delta)t_0 \} \cap B_\rho(0) \).

As in Step 1, by the motion law for front propagation starting from \( \partial(D \cap B_\rho) \) given in [2, Main Theorem] or [4, Theorem 9.1], we deduce that

\[ w(x, 0) \leq b_0 \quad \text{in} \quad \left\{ x \cdot \xi \leq - \frac{\delta t_0}{\varepsilon} \right\} \cap B_{\rho + (\kappa_\ast - \delta)t_0}(0). \]

This implies that \( (0, 0) \) is an interior point of \( \{ \Phi_\infty < 0 \} \), which is a contradiction. \( \square \)

7. Representation formula for the blowing down limit

In this section, we give an explicit representation formula for \( \Phi_\infty \). We first consider the forward problem (5.3) in \( \Omega_\infty^+ := \{ t > h_\infty(x) \} \), and then the backward problem (5.4) in \( \Omega_\infty^- := \{ t < h_\infty(x) \} \). The main tool used in this section is the generalized characteristics associated to \( \Phi_\infty \). We will follow closely the treatment in Cannarsa, Mazzola and Sinestrari [10].

7.1. Forward problem. This subsection is devoted to the forward problem (5.3) in \( \Omega_\infty^+ \). We first notice the following pointwise monotonicity relation.

**Lemma 7.1.** For any \( (x, t), (y, s) \in \Omega_\infty^+ \) with \( t > s \), if the segment connecting \( (y, s) \) and \( (x, t) \) is contained in \( \Omega_\infty^+ \), then

\[ \Phi_\infty(x, t) \leq \Phi_\infty(y, s) + (\kappa_\ast + \beta_+) (t - s) + \frac{|x - y|^2}{4\beta_+ (t - s)}. \] (7.1)

**Proof.** Since \( \Phi_\infty \) is Lipschitz, it is differentiable a.e. and satisfies (5.3) a.e. in \( \Omega_\infty^+ \). By avoiding a zero measure set, we may assume for a.e. \( \tau \in [0, 1] \), \( \Phi_\infty \) is differentiable at the point \( X(\tau) := ((1 - \tau)y + \tau x, (1 - \tau)s + \tau t) \). (The general case follows by an approximation using the continuity of \( \Phi_\infty \).)
Then we have
\[
\frac{d}{d \tau} \Phi_\infty(X(\tau)) = \partial_t \Phi(X(\tau))(t-s) + \nabla \Phi(X(\tau)) \cdot (x-y)
\]
\[
= (\kappa_+ + \beta_+) (t-s) - \beta_+ |\nabla \Phi(X(\tau))|^2 (t-s) + \nabla \Phi(X(\tau)) \cdot (x-y)
\]
\[
\leq (\kappa_+ + \beta_+) (t-s) + \frac{|x-y|^2}{4\beta_+ (t-s)}.
\]
Integrating this inequality in \( \tau \), we obtain (7.1). \( \square \)

Next we establish a localized Hopf-Lax formula for \( \Phi_\infty \).

**Lemma 7.2** (Localized Hopf-Lax formula I). There exists a constant \( K \) depending only on the Lipschitz constant of \( \Phi_\infty \) so that the following holds. For any \((x,t) \in \Omega_\infty^+ \), there exists an \( \varepsilon > 0 \) such that \( B_{K\varepsilon}(x) \times (t-\varepsilon,t) \subset \Omega_\infty^+ \), and

\[
\Phi_\infty(x,t) = \min_{y \in B_{K\varepsilon}(x)} \left[ \Phi_\infty(y,t-\varepsilon) + (\kappa_+ + \beta_+) \varepsilon + \frac{|x-y|^2}{4\beta_+ (t-s)} \right].
\]

(7.2)

**Proof.** Denote \( Q := B_{K\varepsilon}(x) \times (t-\varepsilon,t) \). By Lemma 7.1, we can apply Lions [25, Theorem 10.1 and Theorem 11.1] to deduce that

\[
\Phi_\infty(x,t) = \inf_{(y,s) \in \partial Q} \left[ \Phi_\infty(y,s) + (\kappa_+ + \beta_+) (t-s) + \frac{|x-y|^2}{4\beta_+ (t-s)} \right].
\]

(7.3)

If \( K \) is large enough (compared to the Lipschitz constant of \( \Phi_\infty \)), for any \((y,s) \in \partial B_{K\varepsilon}(x) \times [t-\varepsilon,t) \), we have

\[
\Phi_\infty(y,s) + (\kappa_+ + \beta_+) (t-s) + \frac{|x-y|^2}{4\beta_+ (t-s)} > \Phi_\infty(x,t-\varepsilon) + (\kappa_+ + \beta_+) \varepsilon.
\]

Therefore the infimum in (7.3) is attained in the interior of \( B_{K\varepsilon}(x) \times \{t-\varepsilon\} \), and it must be a minimum. \( \square \)

Now we use this localized Hopf-Lax formula to study backward characteristic curves of \( \Phi_\infty \). We will restrict our attention to differentiable points. Take a point \((x_0,t_0) \in \Omega_\infty^+ \) so that \( \Phi_\infty(\cdot,t_0) \) is differentiable at \( x_0 \). Denote \( p_0 := \nabla \Phi_\infty(x_0,t_0) \).

By Lemma 7.2, for any \( s < t_0 \) sufficiently close to \( t_0 \), there exists a point \((x(s),s) \in C_K(x_0,t_0) \cap \Omega_\infty^+ \) such that

\[
\Phi_\infty(x_0,t_0) = \Phi_\infty(x(s),s) + (\kappa_+ + \beta_+) (t_0-s) + \frac{|x_0-x(s)|^2}{4\beta_+(t_0-s)}.
\]

(7.4)

**Lemma 7.3.** Under the above setting, we have

\[
x(s) = x_0 - 2\beta_+(t_0-s)p_0.
\]

(7.5)

**Proof.** By Lemma 7.2, for any \( x \) close to \( x_0 \),

\[
\Phi_\infty(x,t_0) \geq \Phi_\infty(x(s),s) + (\kappa_+ + \beta_+) (t_0-s) + \frac{|x_0-x(s)|^2}{4\beta_+(t_0-s)}.
\]

(7.6)
Subtracting (7.4) from (7.6) leads to
\[ \Phi_\infty(x, t) - \Phi_\infty(x_0, t_0) \geq \frac{x + x_0 - 2x(s)}{4\beta_+(t_0 - s)} \cdot (x - x_0). \]

On the other hand, because \( \Phi_\infty(\cdot, t) \) is differentiable at \( x_0 \), we have
\[ \Phi_\infty(x, t) - \Phi_\infty(x_0, t_0) = p \cdot (x - x_0) + o(|x - x_0|). \]

These two relations hold for any \( x \) sufficiently close to \( x_0 \), so
\[ p_0 = \frac{x_0 - x(s)}{2\beta_+(t_0 - s)}. \]

\[ \square \]

**Corollary 7.4.** The minimum in (7.2) is attained at a unique point.

The curve
\[ \{(x(s), s) : x(s) = x_0 - 2\beta_+(t_0 - s)p_0, s \leq t_0\} \]
is the backward characteristic curve of \( \Phi_\infty \) starting from \( (x_0, t_0) \).

**Lemma 7.5.** Under the above settings, \( \Phi_\infty(\cdot, s) \) is differentiable at \( x(s) \). Moreover,
\[ \nabla \Phi_\infty(x(s), s) = p_0. \] (7.7)

**Proof.** Because \( x(s) \) attains the minimum in (7.3), for any \( z \) sufficiently close to \( x(s) \), we have
\[ \Phi_\infty(x(s), s) + (\kappa_* + \beta_+) (t_0 - s) + \frac{|x_0 - x(s)|^2}{4\beta_+(t_0 - s)} \]
\[ \quad \leq \Phi_\infty(z, s) + (\kappa_* + \beta_+) (t_0 - s) + \frac{|x_0 - z|^2}{4\beta_+(t_0 - s)}. \]

After simplification, this is
\[ \Phi_\infty(z, s) \geq \Phi_\infty(x(s), s) + \frac{x_0 - x(s)}{2\beta_+(t_0 - s)} \cdot [z - x(s)] + O(|z - x(s)|^2). \]

Since \( \Phi_\infty(\cdot, s) \) is semi-concave, this inequality implies that \( \Phi_\infty(\cdot, s) \) is differentiable at \( x(s) \), and its gradient is given by (7.7).

By Lemma 7.2 and Lemma 7.5, the characteristic curve can be extended indefinitely in the backward direction, unless it hits the boundary \( \partial \Omega_\infty^+ \) in finite time. Now we show that the later case must happen.

**Lemma 7.6.** For any \( (x_0, t_0) \in \Omega_\infty^+ \) with \( \Phi(\cdot, t_0) \) differentiable at \( x_0 \), there exists an \( s_0 < t_0 \) such that
\[ (x_0 - 2\beta_+(t_0 - s_0)p_0, s_0) \in \partial \Omega_\infty^+. \]

**Proof.** If \( (x(s), s) = (x_0 - 2\beta_+(t_0 - s)p_0, s) \in \Omega_\infty^+ \), by (7.5), (7.4) can be rewritten as
\[ \Phi_\infty(x_0 - 2\beta_+(t_0 - s)p_0, s) = \Phi_\infty(x_0, t_0) - (\kappa_* + \beta_+ + \beta_+|p_0|^2) (t_0 - s). \] (7.8)

Hence there exists an \( s_0 \) such that \( \Phi_\infty(x_0 - 2\beta_+(t_0 - s_0)p_0, s_0) = 0 \) and \( \Phi_\infty(x_0 - 2\beta_+(t_0 - s)p_0, s) > 0 \) for any \( s \in (s_0, t_0] \). Because \( \Phi_\infty > 0 \) in \( \Omega_\infty^+ \) and \( \Phi_\infty = 0 \) on \( \partial \Omega_\infty^+ \), \( (x_0 - 2\beta_+(t_0 - s_0)p_0, s_0) \in \partial \Omega_\infty^+. \) \( \square \)
Lemma 7.7. For any \((x, t) \in \Omega^+_\infty\),
\[
\Phi_\infty(x, t) = \inf_{y \in \Omega^+_\infty} \left\{ (\kappa_+ + \beta_+)[t - h(y)] + \frac{|x - y|^2}{4\beta_+[t - h(y)]} \right\}.
\]

Proof. Choosing \((y, s) = (y, h_\infty(y))\) with \(h_\infty(y) < t\) in (7.1) (here we may assume the segment connecting this point and \((x, t)\) is contained in \(\Omega^+_\infty\), and then taking infimum over \(y\), we obtain
\[
\Phi_\infty(x, t) \leq \inf_{y \in \Omega^+_\infty} \left\{ (\kappa_+ + \beta_+)[t - h(y)] + \frac{|x - y|^2}{4\beta_+[t - h(y)]} \right\}.
\] (7.9)

To show that this is an equality, we assume without loss of generality that \(x\) is a differentiable point of \(\Phi_\infty(\cdot, t)\). Then by Lemma 7.6, in particular, (7.8), we find that \(y = x_0 - 2\beta_+(t_0 - s_0)p_0\) attains the equality in (7.9). □

7.2. Backward problem. For the backward problem (5.4), we still use backward characteristics to determine the form of \(\Phi_\infty\). The proof is similar to the forward problem, so most results in this subsection will be stated without proof.

Lemma 7.8. For any \((x, t), (y, s) \in \Omega^-_\infty\) with \(t > s\),
\[
\Phi_\infty(x, t) \geq \Phi_\infty(y, s) + (\kappa_+ - \beta_-)(t - s) - \frac{|x - y|^2}{4\beta_-(t - s)}.
\] (7.10)

Because \(\Omega^-_\infty\) is convex (see Remark 1.7), the segment connecting \((y, s)\) and \((x, t)\) is always contained in \(\Omega^-_\infty\).

In \(\Omega^-_\infty\), \(\Phi^-_\infty := -\Phi_\infty\) is a viscosity solution of
\[
\partial_t\Phi^-_\infty + \beta_-|\nabla\Phi^-_\infty|^2 + \kappa_+ - \beta_- = 0.
\] (7.11)

Hence we have the following localized Hopf-Lax formula.

Lemma 7.9 (Localized Hopf-Lax formula II). There exists a constant \(K\) depending only on the Lipschitz constant of \(\Phi_\infty\) so that the following holds. For any \((x, t) \in \Omega^-_\infty\), there exists an \(\varepsilon > 0\) such that \(B_{K\varepsilon}(x) \times (t - \varepsilon, t) \subset \Omega^-_\infty\), and
\[
\Phi_\infty(x, t) = \max_{y \in B_{K\varepsilon}(x)} \left[ \Phi_\infty(y, t - \varepsilon) + (\kappa_+ - \beta_-)\varepsilon - \frac{|x - y|^2}{4\beta_-\varepsilon} \right].
\] (7.12)

Take a point \((x_0, t_0) \in \Omega^+_\infty\) so that \(\Phi_\infty(\cdot, t_0)\) is differentiable at \(x_0\). Denote \(p_0 := \nabla\Phi_\infty(x_0, t_0)\).

By Lemma 7.9, for any \(s < t_0\) sufficiently close to \(t_0\), there exists a point \((x(s), s) \in C_K^-(x_0, t_0) \cap \Omega^-_\infty\) such that
\[
\Phi_\infty(x_0, t_0) = \Phi_\infty(x(s), s) + (\kappa_+ - \beta_-)(t_0 - s) - \frac{|x_0 - x(s)|^2}{4\beta_-(t_0 - s)}.
\] (7.13)

Lemma 7.10. Under the above setting, we have
\[
x(s) = x_0 + 2\beta_-(t_0 - s)p_0.
\] (7.14)

Lemma 7.11. Under the above setting, \(\Phi_\infty(\cdot, s)\) is differentiable at \(x(s)\). Moreover,
\[
\nabla\Phi_\infty(x(s), s) = p_0.
\] (7.15)
By Lemma 7.9 and Lemma 7.11, the characteristic curve can be extended indefinitely in the backward direction, unless it hits the boundary \( \partial \Omega_\infty \) in finite time. Now we show that the latter case must happen.

**Lemma 7.12.** For any \((x_0, t_0) \in \Omega_\infty^-\) with \(\Phi(\cdot, t_0)\) differentiable at \(x_0\), there exists an \(s_0 < t_0\) such that 
\[
(x_0 + 2\beta_+(t_0 - s_0)p_0, s_0) \in \partial \Omega_\infty^-.
\]

**Proof.** If \((x(s), s) = (x_0 + 2\beta_-(t_0 - s)p_0, s) \in \Omega_\infty^-\), by (7.14), (7.13) can be rewritten as
\[
\Phi_\infty(x_0 + 2\beta_-(t_0 - s)p_0, s) = \Phi_\infty(x_0, t_0) - (\kappa_* - \beta_- - \beta_- |p_0|^2) (t_0 - s). \tag{7.16}
\]
If \(\beta_- > \kappa_*\), there exists an \(s_0\) such that \(\Phi_\infty(x_0 - 2\beta_+(t_0 - s_0)p_0, s_0) = 0\) and \(\Phi_\infty(x_0 - 2\beta_+(t_0 - s)p_0, s) < 0\) for any \(s \in (s_0, t_0]\). Because \(\Phi_\infty < 0\) in \(\Omega_\infty^-\) and \(\Phi_\infty = 0\) on \(\partial \Omega_\infty^-\), \((x_0 + 2\beta_-(t_0 - s_0)p_0, s_0) \in \partial \Omega_\infty^-\).

If \(\beta_- = \kappa_*\), this is still the case, unless \(p_0 = 0\). However, if \(p_0 = 0\), the characteristic curve is \((x_0, s)\), and (7.16) reads as
\[
\Phi_\infty(x_0, s) \equiv \Phi_\infty(x_0, t_0), \text{ for any } s < t_0.
\]
This cannot happen by (5.7). \(\square\)

With these lemmas in hand, similar to Lemma 7.7, we get

**Lemma 7.13.** For any \((x, t) \in \Omega_\infty^-\),
\[
\Phi_\infty(x, t) = \sup_{y \in (h_\infty < t)} \left\{ (\kappa_* - \beta_-) [t - h_\infty(y)] - \frac{|x - y|^2}{4\beta_- [t - h_\infty(y)]} \right\}.
\]

**Remark 7.14.**
- In the above, we use only backward characteristic curves starting from differentiable points. For a non-differentiable point, there could exist many backward characteristic curves emanating from it, see [10].
- In the monostable case, where \(\k_* / 2 \leq \beta_- < \k_*\), the backward characteristic curves could always stay in the domain and do not hit the boundary. We expect the above representation formula still holds in this case, but do not know how to prove it.
- The existence of a nontrivial viscosity solution to (5.4) imposes some restrictions on the domain \(\{t < h_\infty(x)\}\). The following question seems to be interesting, and as far as the author knows, has not been explored in the literature: under what conditions on the domain \(\{t < h_\infty(x)\}\), can we prove the nonexistence of viscosity solution of an Hamilton-Jacobi equation? We may ask the same question for the implication of the existence of globally Lipschitz viscosity solutions.

8. Characterization of minimal speed: Proof of Theorem 1.8

In this section we consider travelling wave equation (1.2).

Denote the constants
\[
K_+ := \sqrt{1 + \frac{\k_*}{\beta_+} + \frac{\k^2}{4\beta_+^2}}, \quad K_- := \sqrt{1 - \frac{\k_*}{\beta_-} + \frac{\k^2}{4\beta_-^2}}.
\]
By abusing notations, we will use the following notations about cones in $\mathbb{R}^n$:

$$C^+(x) := \{ y : y_n - x_n > \lambda |y' - x'| \}, \quad C^-(x) := \{ y : y_n - x_n < -\lambda |y' - x'| \}. $$

As in Section 5, set $\Psi := g^{-1} \circ u$. It satisfies

$$-\Delta \Psi + \kappa \partial_n \Psi = \kappa_* + \frac{g''(\Psi)}{g'(\Psi)} (|\nabla \Psi|^2 - 1). \quad (8.1)$$

Since this is an elliptic equation, we have the following unconditional, global Lipschitz bound on $\Psi$. This lemma holds once the nonlinearity satisfies $f(0) = f(1) = 0$, no matter whether it is monostable, combustion or bistable.

**Lemma 8.1.** There exists a universal constant $C$ such that $|\nabla \Psi| \leq C$ on $\mathbb{R}^n$.

**Proof.** By definition,

$$\nabla \Psi = \frac{\nabla u}{g'(\Psi)}. $$

Since $g'$ has a positive lower bound on any compact set of $\mathbb{R}$, by the gradient bound on $u$, $|\nabla \Psi|$ is bounded in $\{1/4 < u < 3/4\}$.

In $\{u < 1/4\}$,

$$g'(\Psi) \geq cg(\Psi) = cu.$$ 

Hence here we have

$$|\nabla \Psi| \leq C \frac{|\nabla u|}{u} \leq C,$$

where the last inequality follows from Harnack inequality and interior gradient estimates applied to (1.2).

Similarly, in $\{u > 3/4\}$,

$$|\nabla \Psi| \leq C \frac{|\nabla u|}{1 - u} \leq C. \quad \Box$$

As before, $\Psi$ is still semi-concave.

**Lemma 8.2 (Semi-concavity).** There exists a universal constant $C$ such that for any $x \in \{\Psi > 0\}$,

$$\nabla^2 \Psi(x) \leq \frac{C}{\Psi(x)},$$

and for any $x \in \{\Psi < 0\}$,

$$\nabla^2 \Psi(x) \geq \frac{C}{\Psi(x)}.$$ 

For each $\varepsilon > 0$, let $\Psi_\varepsilon(x) := \varepsilon \Psi(\varepsilon^{-1} x)$, which satisfies

$$-\varepsilon \Delta \Psi_\varepsilon + \kappa \partial_n \Psi_\varepsilon = \kappa_* + \frac{g''(\varepsilon^{-1} \Psi_\varepsilon)}{g'(\varepsilon^{-1} \Psi_\varepsilon)} (|\nabla \Psi_\varepsilon|^2 - 1). \quad (8.2)$$

By the uniform Lipschitz bound on $\Psi_\varepsilon$ from Lemma 8.1, for any sequence $\varepsilon_i \to 0$, there exists a subsequence such that $\Psi_{\varepsilon_i} \to \Psi_\infty$ in $C_{loc}(\mathbb{R}^n)$. Then standard vanishing viscosity method gives
Lemma 8.3. In the open set \( \{\Psi_{\infty} > 0\} \), \( \Psi_{\infty} \) is a viscosity solution of
\[
\kappa \partial_n \Psi_{\infty} - \kappa_* + \beta_+ \left( |\nabla \Psi_{\infty}|^2 - 1 \right) = 0. \tag{8.3}
\]

In the open set \( \{\Psi_{\infty} < 0\} \) (if non-empty), \( \Psi_{\infty} \) is a viscosity solution of
\[
\kappa \partial_n \Psi_{\infty} - \kappa_* - \beta_+ \left( |\nabla \Psi_{\infty}|^2 - 1 \right) = 0. \tag{8.4}
\]

Remark 8.4. Equations (8.3) and (8.4) are the corresponding travelling wave equations for the time-dependent Hamilton-Jacobi equations (5.3) and (5.4).

Recall that
\[
\{v = 1 - b_0\} = \{x_n = h(x')\}. 
\]
As before, we define the blowing down limit \( h_\infty \) from \( h \). By Lemma 5.4, we still have
\[
\{\Psi_{\infty} > 0\} = \{x_n > h_\infty(x')\}. 
\]

Proposition 8.5. The Lipschitz constant of \( h_\infty \) is at most \( \sqrt{\kappa/\kappa_* - 1} \). In particular, we must have \( \kappa \geq \kappa_* \).

Remark 8.6. Under the assumptions of Theorem 1.4, the blowing down limit \( h_\infty \) is a viscosity solution of
\[
|\nabla h_\infty|^2 - \frac{\kappa^2}{\kappa_*} + 1 = 0 \quad \text{in} \quad \mathbb{R}^{n-1}. \tag{8.5}
\]
This follows from a reduction of Theorem 1.5.

Proof of Proposition 8.5. The blowing down limit of the level set for the entire solution \( v(x + \kappa t e_n) \) is the graph
\[
t = \frac{h_\infty(x') - x_n}{\kappa}. 
\]
By Lemma 5.6, its Lipschitz constant is at most \( \kappa_*^{-1} \). \qed

By Lemma 5.2, \( \Psi_\varepsilon \) are uniformly semi-concave in any compact set of \( \{\Psi_{\infty} > 0\} \). As a consequence, \( \Psi_{\infty} \) is locally semi-concave in this open set. The sup-differential of \( \Psi_{\infty} \) is then well defined at every point in \( \{\Psi_{\infty} > 0\} \). Recall that
\[
\partial \Psi_{\infty}(x) := \left\{ \xi \in \mathbb{R}^n : \limsup_{y \to x} \frac{\Psi_{\infty}(y) - \Psi_{\infty}(x) - \xi \cdot (y - x)}{|y - x|} \leq 0 \right\}
\]
is a compact convex subset of \( \mathbb{R}^n \). Because \( \Psi_\varepsilon \to \Psi_{\infty} \) uniformly on any compact set of \( \mathbb{R}^n \), by the uniform semi-concavity of \( \Psi_\varepsilon \), we deduce that for any \( x_\varepsilon \to x_\infty \in \{\Psi_{\infty} > 0\} \),
\[
each \text{limit point of } \nabla \Psi_\varepsilon(x_\varepsilon) \text{ as } \varepsilon \to 0 \in \partial \Psi_{\infty}(x_\infty). \tag{8.6}
\]
If \( \Psi_{\infty} < 0 \) in \( \{x_n < h_\infty(x')\} \), the same result holds for the negative part of \( \Psi_{\infty} \), with sup-differentials replaced by sub-differentials.

A reduction of Lemma 7.7 and Lemma 7.13 gives

Proposition 8.7. \quad \bullet \quad \text{For any } x = (x', x_n) \in \{\Psi_{\infty} > 0\},
\[
\Psi_{\infty}(x) = \inf_{y' \in \mathbb{R}^{n-1}} \left[ K_+ \sqrt{|x' - y'|^2 + (x_n - h_\infty(y'))^2} - \frac{\kappa}{2\beta_+} (x_n - h_\infty(y')) \right]. \tag{8.7}
\]
• Assume $\Psi_\infty < 0$ in $\{x_n < h_\infty(x')\}$. Then for any $x = (x', x_n) \in \{x_n < h_\infty(x')\}$,

$$
\Psi_\infty(x) = -\inf_{y' \in \mathbb{R}^{n-1}} \left[ K_- \sqrt{|x' - y'|^2 + (x_n - h_\infty(y'))^2} - \frac{\kappa}{2\beta_-}(x_n - h_\infty(y')) \right]. \quad (8.8)
$$

**Remark 8.8.** The representation formula (8.7) and (8.8) (when $\beta_- > \kappa_*$), can be proved directly by rewriting (8.3) and (8.4) as eikonal equations. For example, in the case of (8.3), we can define a norm on $\mathbb{R}^n$, $\| \cdot \|$ so that the corresponding unit ball is $B_{K_+} \left(0', -\frac{\kappa}{2\beta_+}\right)$. (This is because this ball contains the origin as an interior point.) The Hamilton-Jacobi equation (8.3) is equivalent to the eikonal type equation

$$
\| \nabla \Psi(x) \|^2 - 1 = 0. \quad (8.9)
$$

Then we can prove that

$$
\Psi_\infty(x) = \inf_{y \in \partial \{\Phi_\infty > 0\}} \|x - y\|^*.
$$

Here $\| \cdot \|^*$ denotes the dual norm of $\| \cdot \|$.

Now we come to

**Proof of Theorem 1.8.** We have shown that $\kappa \geq \kappa_*$ in Proposition 8.5. It remains to characterize the $\kappa = \kappa_*$ case.

From Proposition 8.5, it is seen that, if $\kappa = \kappa_*$, we must have $\nabla h_\infty = 0$ a.e. in $\mathbb{R}^{n-1}$. Since $h_\infty(0) = 0$, we get

$$
h_\infty \equiv 0 \quad \text{in} \quad \mathbb{R}^{n-1}. \quad (8.10)
$$

This holds for any blowing down limit $h_\infty$, so the blowing down limit is unique.

Substituting (8.10) into (8.7) and (8.8), by noting that $\kappa = \kappa_*$ implies

$$
K_+ = 1 + \frac{\kappa_*}{2\beta_+}, \quad K_- = 1 - \frac{\kappa_*}{2\beta_-},
$$

we deduce that

$$
\Psi_\infty(x) \equiv x_n \quad \text{in} \quad \mathbb{R}^n. \quad (8.11)
$$

**Claim.** For any $\varepsilon > 0$, there exists an $L(\varepsilon) > 0$ such that

$$
|\nabla' \Psi| \leq \varepsilon \partial_n \Psi \quad \text{in} \quad \{ |\Psi| \geq L(\varepsilon) \}.
$$

By this claim, similar to the proof of Theorem 1.4 (or as in the proof of Gibbons conjecture in [12, 8]), applying the sliding method we deduce that

$$
|\nabla' \Psi| \leq \varepsilon \partial_n \Psi \quad \text{in} \quad \mathbb{R}^n.
$$

Letting $\varepsilon \to 0$, we deduce that $\nabla' \Psi \equiv 0$, or equivalently, $v$ is a function of $x_n$ only. Because we have (1.4) and $\sup_{\mathbb{R}^n} v = 1$, by the uniqueness of $g$, we find a constant $t \in \mathbb{R}$ such that

$$
v(x) \equiv g(x_n + t) \quad \text{in} \quad \mathbb{R}^n.
$$

**Proof of the claim.** Assume by the contrary, there exists a sequence of points $x_i$ with

$$
\varepsilon_i^{-1} := |\Psi(x_i)| \to +\infty,
$$
but
\[ |\nabla' \Psi(x_i)| \geq \varepsilon \partial_n \Psi(x_i). \] (8.12)

Let \( \Psi_i(x) := \varepsilon_i \Psi(\varepsilon_i^{-1} x) \). Combining (8.6) and (8.12) together leads to a contradiction with (8.11).

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