The anti-diagonal filtration: reduced theory and applications

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Abstract

Seidel and Smith introduced the graded fixed-point symplectic Khovanov cohomology group $\text{Kh}_{\text{symp},\text{inv}}(K)$ for a knot $K \subset S^3$, as well as a spectral sequence converging to the Heegaard Floer homology group $\widehat{HF}(\Sigma(K)\#(S^2 \times S^1))$ with $E_1$-page isomorphic to a factor of $\text{Kh}_{\text{symp},\text{inv}}(K)$. A previous paper showed that the higher pages of this spectral sequence are knot invariants. Here we discuss a reduced version of the spectral sequence which directly computes $\widehat{HF}(\Sigma(K))$. Under some degeneration conditions, one obtains a new absolute Maslov grading on the group $\widehat{HF}(\Sigma(K))$. This occurs when $K$ is a two-bridge knot, and we compute the grading in this case.

1 Introduction

Let $K \subset S^3$ be a knot, and let $\Sigma(K)$ denote the double cover of $S^3$ branched along $K$. This paper is a continuation of a previous one in which we studied the invariance properties of the spectral sequence whose $E_1$ page is isomorphic to a factor of Seidel and Smith’s fixed-point symplectic Khovanov cohomology group and which converges to the Heegaard Floer homology group $\widehat{HF}(\Sigma(K)\#(S^2 \times S^1))$. In the previous paper, we proved that the filtered chain homotopy type of the filtered chain complex inducing this spectral sequence is a knot invariant; this implies that the higher pages of the spectral sequence are knot invariants also.

The present paper will give a definition for a reduced version of the theory in the form of a filtration on the Heegaard Floer chain complex $\widehat{CF}(\Sigma(K))$. The definition will resemble that for the original theory, and the reader should refer to for the prerequisite background material and some lemmas which will be useful here.

Let $b \in B_{2n}$ be a braid whose plat closure is a diagram of the knot $K$. Reviewing Manolescu’s construction in and following the work of Bigelow, we described how to define a fork diagram for $b$ and compute a function $R : \mathcal{G} \to \mathbb{Z} + \frac{1}{2}$, where $\mathcal{G}$ is a set of Bigelow generators in the diagram. One can also define a reduced version of $R$ (denoted by $\tilde{R}$), as suggested in, by restricting to reducible fork diagrams. The reduced grading comes from an analogous version of the holomorphic volume form used in the unreduced theory. One can define a set $\mathcal{G}$ of reduced Bigelow generators which are determined by omitting a pair of arcs from the fork diagram. These reduced Bigelow generators are in one-to-one correspondence with a set of generators for $\widehat{CF}(\Sigma(K))$ (notice that the $S^2 \times S^1$ factor has been removed). Definitions for reduced functions $Q, T, P : \mathcal{G} \to \mathbb{Z}$ are analogous to those of their nonreduced counterparts, and we see that $\tilde{R} : \mathcal{G} \to \mathbb{Z}$ satisfies

$$\tilde{R} = T - Q + P.$$
We acquire $R$ from $\tilde{R}$ via a rational shift $s_R$; let

$$R = \tilde{R} + s_R(b, D), \quad \text{where} \quad s_R(b, D) = \frac{e(b) - w(D) - 2(n - 1)}{4}.$$  

Notice that since $\Sigma(K)$ is a rational homology sphere, every $s \in \text{Spin}^c(\Sigma(K))$ is torsion. Thus we can define a filtration $\rho$ on the entire complex $\hat{CF}(\Sigma(K))$ via

$$\rho = R - \tilde{gr}.$$  

One would hope to relate the reduced and unreduced theories. Observe that the functions $R$ and $\tilde{R}$ also provide filtrations on the Heegaard Floer complexes, and it is easy to see that invariance results hold for them as well. Then we obtain the following:

**Proposition 1.** Let $b \in B_{2n}$ be a braid which induces a reducible fork diagram and whose closure is a diagram for the knot $K$. Let $\mathcal{H}$ (respectively $\mathcal{H}'$) be the Heegaard diagram for $\Sigma(K)$ (respectively $\Sigma(K) \# (S^2 \times S^1)$) provided by Proposition 17 (respectively Proposition 22 from [13]). Let $s \in \text{Spin}^c(\Sigma(K))$ and let $s_0 \in \text{Spin}^c(S^2 \times S^1)$ denote the torsion element. Equip $\hat{CF}(\mathcal{H}, s \# s_0)$ with the $R$-filtration and equip $\hat{CF}(\mathcal{H}', s)$ with the $\tilde{R}$-filtration. Then the filtered complexes

$$\hat{CF}(\mathcal{H}, s \# s_0) \quad \text{and} \quad \hat{CF}(\mathcal{H}, s) \otimes H_{*+1/2}(S^1)$$  

have the same filtered chain homotopy type. Furthermore, equipping $\hat{CF}(\mathcal{H}, s \# s_0)$ with the $\rho$-filtration and $\hat{CF}(\mathcal{H}, s)$ with the $\rho$-filtration,

$$\hat{CF}(\mathcal{H}, s \# s_0) \quad \text{and} \quad \hat{CF}(\mathcal{H}, s) \oplus \hat{CF}(\mathcal{H}, s)$$  

have the same filtered chain homotopy type.

This correspondence, along with Theorem 1 from [13], provides the following:

**Theorem 2.** Let the braids $b \in B_{2n}$ and $b' \in B_{2n}$ have isotopic knot plat closures which are diagrams for the knot $K$, and assume that both induce reducible fork diagrams. Let $\mathcal{H}$ and $\mathcal{H}'$ be the pointed Heegaard diagrams for $\Sigma(K)$ induced by $b$ and $b'$, respectively, in the sense of Proposition 17 below. Then the $\rho$-filtered chain complexes

$$\hat{CF}(\mathcal{H}) \quad \text{and} \quad \hat{CF}(\mathcal{H}')$$  

have the same filtered chain homotopy type.

**Corollary 3.** The $\rho$-filtered chain homotopy type of the complex $\hat{CF}(\Sigma(K))$ is an invariant of the knot $K$.

The filtration $\rho$ induces a reduced version of the spectral sequence with pages $E_{k}$, and Corollary 3 implies the following:

**Corollary 4.** For $k \geq 1$, the page $E_{k}$ is a knot invariant.

Thus one obtains the following relationship between the pages of the spectral sequences induced by $\rho$ and $\tilde{\rho}$, indicating that it suffices to study the reduced theory:
Corollary 5. For \( k \geq 1 \), \( E_k \cong E_k \oplus E_k \).

In Section 3.1.1, we show that the reduced theory enjoys a Künneth-type theorem with respect to connected sums of knots; this result provides a computational tool for composite knots.

Theorem 6. Let \( K_1, K_2 \subset S^3 \) be knots, let \( s_i \in \text{Spin}^c(\Sigma(K_i)), i = 1, 2 \). Then the filtered chain complexes

\[
\hat{CF}(\Sigma(K_1 \# K_2), s_1 \# s_2) \quad \text{and} \quad \hat{CF}(\Sigma(K_1), s_1) \otimes \hat{CF}(\Sigma(K_2), s_2)
\]

have the same filtered chain homotopy type, where \( \hat{CF}(\Sigma(K_1 \# K_2), s_1 \# s_2) \) is equipped with the \( \rho \)-filtration and \( \hat{CF}(\Sigma(K_1), s_1) \otimes \hat{CF}(\Sigma(K_2), s_2) \) is equipped with the tensor product filtration induced by the \( \rho \)-filtrations on the factors.

Theorem 6 implies the following fact regarding the reduced spectral sequence:

Corollary 7. If the coefficient ring is a field, then \( E_k(K_1 \# K_2) \cong E_k(K_1) \otimes E_k(K_2) \) for \( k \geq 1 \).

Given a knot \( K \) and some \( s \in \text{Spin}^c(\Sigma(K)) \), we denote by \( \hat{CF}^*(\Sigma(K), s) \) the dual complex to \( \hat{CF}_*(\Sigma(K), s) \). Then \( \hat{CF}^*(\Sigma(K)s) \) is a chain complex, and we define a filtration \( \rho^* \) via

\[
\rho^*(x^*) = -\rho(x) \quad \text{for each} \quad x \in T_\alpha \cap T_\beta.
\]

Theorem 8. Let \(-K\) denote the mirror image of the knot \( K \subset S^3 \), and let \( s \in \text{Spin}^c(\Sigma(K)) \). Then

\[
\hat{CF}_*(\Sigma(-K), s) \quad \text{and} \quad \hat{CF}^*(\Sigma(K), s)
\]

are filtered chain isomorphic, where the complexes carry the filtrations \( \rho \) and \( \rho^* \), respectively.

Section 3.2 will discuss how one can distill a family of knot invariants from the reduced filtration in the form of a function \( r_K : \text{Spin}^c(\Sigma(K)) \to \mathbb{Q} \). Restricting \( r_K \) to the unique Spin structure \( s_0 \) on \( \Sigma(K) \), one obtains the \( \mathbb{Q} \)-valued knot invariant \( r(K) = r_K(s_0) \). Theorems 6 and 8 imply the following:

Corollary 9. Let \( K_1, K_2 \subset S^3 \) be knots, and let \( s_i \in \text{Spin}^c(\Sigma(K_i)) \) for \( i = 1, 2 \). Then

\[
r_{K_1 \# K_2}(s_1 \# s_2) = r_{K_1}(s_1) + r_{K_2}(s_2) \quad \text{and} \quad r(K_1 \# K_2) = r(K_1) + r(K_2).
\]

Corollary 10. Let \( K \subset S^3 \) be a knot and let \( s \in \text{Spin}^c(\Sigma(K)) \). Then

\[
r_{-K}(s) = -r_K(s) \quad \text{and} \quad r(-K) = -r(K).
\]

In [12], Seidel and Smith conjectured the existence of a concordance invariant arising from this theory. Motivated by their suggestion and by Corollaries 9 and 10, we make the following speculation:

Conjecture 11. Let \( \mathcal{C} \) denote the smooth knot concordance group. The knot invariant \( r(K) \) provides a well-defined group homomorphism

\[
r : \mathcal{C} \to \mathbb{Q}.
\]
We defined in [13] the notion of $\rho$-degeneracy of a knot. A consequence of Proposition 1 is that a knot is $\rho$-degenerate if and only if the reduced spectral sequence converges at $E_1$ and the induced filtration $\rho$ is constant on each nontrivial factor $\hat{HF}(\Sigma(K),\mathfrak{s})$. This observation implies the following.

**Proposition 12.** Let $K \subset S^3$ be a knot. Then the following are equivalent:

(i) $K$ is $\rho$-degenerate.

(ii) The filtration $\mathcal{R}$ is a grading and lifts the relative Maslov $\mathbb{Z}$-grading on each nontrivial factor $\hat{HF}(\Sigma(K),\mathfrak{s})$.

Moreover, the grading $\mathcal{R}$ is a knot invariant when the above hold.

One sees this behavior when $K$ is a two-bridge knot.

**Theorem 13.** Let $K \subset S^3$ be a two-bridge knot. Then $K$ is $\rho$-degenerate, and

$$\text{rank}\left(\hat{HF}_{\mathcal{R}=k}(\Sigma(K))\right) = \begin{cases} \text{det}(K) & \text{if } k = \frac{\sigma(K)}{2} \\ 0 & \text{otherwise} \end{cases},$$

where $\sigma(K)$ denotes the classical signature of $K$ and $\text{det}(K)$ denotes the determinant of $K$.

We speculate the following, as suggested by Seidel and Smith in [12]:

**Conjecture 14.** Every knot $K \subset S^3$ is $\rho$-degenerate.

**Remark 15.** Although we restricted ourselves to knots above, all constructions can in fact be extended to oriented links $L \in S^3$ with $b_1(\Sigma(L)) = 0$ (with analogous invariance properties). Furthermore, all two-bridge links have $b_1(\Sigma(L)) = 0$, and thus a version of Theorem 13 holds for oriented links.

More generally, one can partially extend these constructions to all oriented links, with the limitation that the filtration $\rho$ can only be defined on the factors of $\hat{CF}$ associated to torsion Spin$^c$ structures (because the definitions rely on the grading $\tilde{gr}$).

## 2 The reduced theory

We describe how to define a reduced version of the $\rho$-filtration, denoted by $\bar{\rho}$, which is a $\mathbb{Q}$-valued filtration on the chain complex $\hat{CF}(\Sigma(K))$ (a definition first mentioned by Manolescu in [4]). This reduced version is much simpler to compute than the unreduced theory, and Theorem 2 gives an invariance result for the reduced filtration.

The reduced grading $\mathcal{R}$ is computed for a reduced set of Bigelow generators given by omitting a pair of arcs $\alpha_n, \beta_n$ from the fork diagram (with a mild restriction on the diagram used). We’ll see that the set $\mathcal{G}$ is in one-to-one correspondence with a set of generators for $\hat{CF}(\mathcal{H})$, where $\mathcal{H}$ is an admissible Heegaard diagram for the manifold $\Sigma(K)$ obtained from the reduced fork diagram.
2.1 The reduced Bigelow picture

We can define the reduced filtration for fork diagrams of a special type.

**Definition 16.** A reducible fork diagram for a knot $K$ is a fork diagram for $K$ with at least four punctures such that $\mu_{2n} \in \alpha_n \cap \beta_n$.

Notice that a reducible fork diagram exists for any knot $K$, as one can be obtained by performing a Birman stabilization on any braid whose closure is $K$. We’ll define the reduced theory by omitting the pair of arcs $\alpha_n$, $\beta_n$ from a reducible fork diagram for $K$.

Consider a reducible fork diagram for $K$ induced by a braid $b \in B_n$. Denote by $\tilde{Z}$ the set of intersections $\alpha_i \cap \beta_j$, where $i, j \leq n - 1$. Similarly, define $Z$ to be points $\alpha_i \cap bE_j'$, $i, j \leq n - 1$. We then define

\[
\tilde{\mathcal{G}} = (\alpha_1 \times \ldots \times \alpha_{n-1}) \cap (\beta_1 \times \ldots \times \beta_{n-1}) \subset Conf^n(\mathbb{C}), \\
\mathcal{G} = (\alpha_1 \times \ldots \times \alpha_{n-1}) \cap (bE_1' \times \ldots \times E_{n-1}' \times E_0') \subset Conf^n(\mathbb{C}).
\]

Reduced versions $Q, T, P$ are calculated for elelements in $\mathcal{G}$ in the same way as their unreduced counterparts $Q, T, P$.

2.2 A Heegaard diagram for $\widehat{HF}(\Sigma(K))$

Let the polynomial $P_n$, the affine space $\tilde{S}$, and the lifts $\tilde{\alpha}_k$ and $\tilde{\beta}_k$ be defined as in Section ??.

Now denote by $\mathbb{T}_\tilde{\alpha}$ and $\mathbb{T}_\tilde{\beta}$ the totally real tori in $\text{Sym}^{n-1}(S)$ defined by

\[
\mathbb{T}_\tilde{\alpha} = \tilde{\alpha}_1 \times \ldots \times \tilde{\alpha}_{n-1} \text{ and } \mathbb{T}_\tilde{\beta} = \tilde{\beta}_1 \times \ldots \times \tilde{\beta}_{n-1}.
\]

Recall that $\tilde{S}$ can be seen as $\Sigma_{n-1} \setminus \{\pm \infty\};$ because an $\alpha, \beta$ pair was removed, we can have a pointed Heegaard diagram without stabilizing the surface.

**Proposition 17.** The collection of data

\[
\mathcal{H} = (\Sigma_{n-1}; \tilde{\alpha}_1, \ldots, \tilde{\alpha}_{n-1}; \tilde{\beta}_1, \ldots, \tilde{\beta}_{n-1}; +\infty)
\]

is an admissible pointed Heegaard diagram for $\Sigma(K)$.

**Proof.** It is enough to show that

\[
\mathcal{H}' = \left(\Sigma_n = \Sigma_{n-1} \# \Sigma_1; \tilde{\alpha}_1, \ldots, \tilde{\alpha}_{n-1}, \alpha_0; \tilde{\beta}_1, \ldots, \tilde{\beta}_{n-1}, \beta_0; +\infty\right)
\]

gives an admissible pointed Heegaard diagram for $\Sigma(K) \# (S^2 \times S^1)$, where $(\Sigma_1; \alpha_0; \beta_0; +\infty)$ is the standard pointed Heegaard diagram for $S^2 \times S^1$ shown in Figure [4].

We accomplish this by showing that $\mathcal{H}$ can be obtained from $\mathcal{H}'$ by a sequence of handleslides.

Recall the construction in the unreduced context found in [4]. When the surface $\Sigma_{n-1}$ is stabilized to obtain $\Sigma_n$, we obtain the circle $\tilde{\alpha}_n$. Let $\alpha'_n$ denote the $n^{th}$ circle after instead stabilizing the surface via the attachment of the handle near $+\infty$ only. Then we see that $\tilde{\alpha}_n = \alpha'_n \# \alpha_0$, which is illustrated in Figure [2].

Well, $[\alpha'_n] = [\tilde{\alpha}_1] + \ldots + [\tilde{\alpha}_{n-1}] \in H_1(\Sigma_{n-1}; \mathbb{Z})$. Since $[\tilde{\alpha}_n] = [\alpha'_n] + [\alpha_0]$, then the set of attaching circles $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n$ can be obtained via a sequence of handleslides and isotopies from the
Figure 1: A pointed Heegaard diagram for $S^2 \times S^1$

Figure 2: Obtaining $\Sigma_n$ by stabilizing $\Sigma_{n-1}$ in two different ways

(a) Attachment of a handle near $+\infty$ only. The connected sum with $\alpha_0$ is grayed.

(b) Attachment of a handle near $+\infty$ and $-\infty$.

set $\hat{\alpha}_1, \ldots, \hat{\alpha}_{n-1}, \alpha_0$ (where these Heegaard moves avoid $\pm \infty$). As a result, $\hat{\beta}_1, \ldots, \hat{\beta}_n$ can be obtained via an analogous sequence of handleslides and isotopies from $\hat{\beta}_1, \ldots, \hat{\beta}_{n-1}, \beta_0$.

Admissibility can be shown via an argument which is completely analogous to that of Proposition 7.4 from [4].

2.3 Gradings on reduced tori

Let $W = \text{Sym}^{n-1}(\hat{S}) - \nabla$. Via an argument that is identical to that of Proposition 23 from [13], we obtain the following.

**Proposition 18.** There exists a complex volume form $\Theta$ on $W$ which induces Seidel gradings on $\mathbb{T}_{\hat{\alpha}}$ and $\mathbb{T}_{\hat{\beta}}$ in the sense of Section 4.1 of [13]. The resulting absolute Maslov grading on the elements of $\mathbb{T}_{\hat{\alpha}} \cap \mathbb{T}_{\hat{\beta}}$ inside $W$ is exactly $P - Q + T$.

We then define $R$ via a shift which depends on the signed count of braid generators in $b$ and the writhe of the diagram $D$ which is its plat closure.

$$R = \tilde{R} + s_R(b, D), \text{ where } s_R(b, D) = \frac{\epsilon(b) - w(D) - 2(n-1)}{4} \in \mathbb{Q}.$$  

Notice that the reducibility condition guarantees that $+\infty$ and $-\infty$ lie in the same component of $\Sigma_{n-1} \cup (\cup_{i=1}^{n-1} \hat{\alpha}_i) \cup (\cup_{i=1}^{n-1} \hat{\beta}_i)$. Therefore, if $x, y \in \mathbb{T}_{\hat{\alpha}} \cap \mathbb{T}_{\hat{\beta}}$, and $\phi \in \pi_2(x, y)$, then $n_{+\infty}(\phi) = 0$ if and only if $n_{-\infty}(\phi) = 0$.

When viewed as a grading, the function $R$ is also not compatible with Maslov index calculations in the entire symmetric product. By mimicking the argument used in the unreduced case, one can see that the volume form $\Theta$ has an order-one zero along the anti-diagonal $\nabla \subset \text{Sym}^{n-1}(\hat{S})$. Therefore, if $x, y \in \mathbb{T}_{\hat{\alpha}} \cap \mathbb{T}_{\hat{\beta}}$, and $\phi \in \pi_2(x, y)$ with $n_{+\infty}(\phi) = 0$, then

$$R(x) - R(y) = \mu(\phi) + 2[\phi] \cdot [\nabla].$$
Recall that for every knot $K$, the manifold $\Sigma(K)$ is a rational homology sphere. All elements of $\text{Spin}^c(\Sigma(K))$ are then torsion, and thus the absolute grading $\bar{r}$ can be defined for all generators in $H$. One then can obtain a filtration grading $\rho$ on the complex $\widehat{CF}(H)$ via

$$\rho(x) = R(x) - \bar{r}(x).$$

### 2.4 The reduced-unreduced correspondence

Proposition 1 provides the relationship between the filtrations $\rho$ and $\rho$. However, we’ll need Lemma 19, which concerns a braid $b \in B_{2n-2}$ inducing a reducible fork diagram which looks as in Figure 3 in a neighborhood of the two rightmost punctures. Such a braid can be obtained for any knot via some appropriate stabilization.

![Figure 3: The local configuration near $\mu_{2n-3}$ and $\mu_{2n-2}$ in Lemma 19. The dashed arc is $\beta_{n-1}$.](image)

Given a link $L$, one can endow $\widehat{CF}(\Sigma(L)\#(S^2 \times S^1), s\#s_0)$ with the $R$ and $\rho$ filtrations for $s \in \text{Spin}^c(\Sigma(L))$ and $s_0 \in \text{Spin}^c(S^2 \times S^1)$ torsion (see Remark 15).

Notice also that if a fork diagram has some puncture $\mu_k \in \alpha_j \cap \beta_j$ (for arbitrary $j$ and $k = 2j - 1$ or $k = 2j$), then one can define versions of $R$ and $\rho$ in the obvious way by omitting $\alpha_j$, $\beta_j$.

**Lemma 19.** Let $b \in B_{2n-2}$ be a braid inducing a fork diagram of the form shown in Figure 3. Denote by $H_n$ the Heegaard diagram obtained by omitting the pair $\alpha_n$, $\beta_n$ from the reducible fork diagram induced by the braid $b \times (1)^2 \in B_{2n}$, and by $H_{n-1}$ the Heegaard diagram obtained by omitting the pair $\alpha_{n-1}$, $\beta_{n-1}$ from the same fork diagram. Then $\widehat{CF}(H_n)$ and $\widehat{CF}(H_{n-1})$ have the same $R$-filtered chain homotopy type.

**Proof.** We compare the filtered chain homotopy types of the complexes obtained from $H_n$ and $H_{n-1}$. This will be accomplished via an intermediate picture: let $\alpha'_n$ be the arc obtained from $\alpha_n$ via the finger-like isotopy shown in Figure 4, and let $\beta'_n = b\alpha'_n$.

![Figure 4: The $\alpha$-arcs in the fork diagram $\mathcal{F}'; \alpha_n$ is dashed, $\alpha'_n$ is dotted, and $\alpha_i$ is solid for $i < n$.](image)

Let $\mathcal{F}$ denote the reduced fork diagram obtained by omitting the pair $\alpha_n$, $\beta_n$, and let $\mathcal{F}'$ denote the diagram obtained from $\mathcal{F}$ by replacing $\alpha_{n-1}$ and $\beta_{n-1}$ with $\alpha'_n$ and $\beta'_n$. We’ll denote by $H'_{n-1}$ the Heegaard diagram for $\Sigma(K)$ covering $\mathcal{F}'$. Furthermore, let

$$\mathcal{Z} = (\alpha_1 \times \ldots \times \alpha_{n-1}) \cap (\beta_1 \times \ldots \times \beta_{n-1}) \quad \text{and} \quad \mathcal{Z}' = (\alpha_1 \times \ldots \times \alpha_{n-2} \times \alpha'_n) \cap (\beta_1 \times \ldots \times \beta_{n-2} \times \beta'_n).$$

The set of attaching circles $\{\hat{\alpha}_1, \ldots, \hat{\alpha}_{n-1}, \hat{\alpha}_{n+1}\}$ can be obtained from $\{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n\}$ via a sequence of handleslides avoiding $\pm \infty$, and the set $\{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_{n-1}, \tilde{\alpha}_{n+1}\}$ is the result of a small
isotopy of \{\hat{\alpha}_1, \ldots, \hat{\alpha}_{n-1}, \hat{\alpha}_{n+1}\}. The set \{\hat{\beta}_1, \ldots, \hat{\beta}_{n-2}, \hat{\beta}_n\} can analogously be obtained from \{\hat{\beta}_1, \ldots, \hat{\beta}_{n-1}\} via isotopies and handleslides avoiding \pm \infty.

It suffices to find triangle injections

\[ g_\alpha : T_\alpha \cap T_\beta \to T_{\alpha'} \cap T_{\beta'} \quad \text{and} \quad g_\beta : T_{\alpha'} \cap T_{\beta'} \to T_{\alpha'} \cap T_{\beta'} \]

such that

\[ R((g_\beta \circ g_\alpha)(x)) = R(x) \quad \text{for all} \quad x \in T_\alpha \cap T_\beta. \]

Such injections exist and will be discussed in Sections 2.4.1 and 2.4.2 below. Case I will deal with generators \(w x \in \mathcal{Z}\) with \(x \in \alpha_{n-1} \cap \beta_{n-1}\), and case II will deal with generators \(z x y\) with \(x \in \alpha_{n-1} \cap \beta_j\) and \(y \in \alpha_k \cap \beta_{n-1}\) (where \(j, k < n - 1\)).

**Proof of Proposition 7** Let \(b \in B_{2n-2}\) be a braid whose plat closure is the knot \(K\) (with \(n > 1\)), and which induces a reducible fork diagram of the form shown in Figure 3. The plat closure of \(b' = b \times (1)^2 \in B_{2n}\) is the two-component link \(L = K \sqcup U\), where \(U\) is the unknot; recall that \(\Sigma(L) \cong \Sigma(K) \# (S^2 \times S^1)\). Let \(\mathcal{H}_n\) and \(\mathcal{H}_{n-1}\) be the Heegaard diagrams for \(\Sigma(L)\) obtained from \(b'\) as in the statement of Lemma 19.

Let \(s_0\) denote the torsion \(\text{Spin}^c\)-structure on \(S^2 \times S^1\), and let \(s \in \text{Spin}^c(\Sigma(K))\). Equipped with their respective \(R\) filtrations, Lemma 19 provides that the complexes \(\widehat{CF}(\mathcal{H}_{n-1}, s \# s_0)\) and \(\widehat{CF}(\mathcal{H}_n, s \# s_0)\) have the same filtered chain homotopy type.

Further, let \(\mathcal{H}'\) denote the genus-(\(n - 1)\) Heegaard diagram for \(\Sigma(K) \# (S^2 \times S^1)\) obtained from \(b\) in the unreduced sense. One can see that this Heegaard diagram is isotopic to \(\mathcal{H}_n\). Also, the fork diagram covered by \(\mathcal{H}_n\) only differs from the fork diagram covered by \(\mathcal{H}'\) by an extra pair of punctures. Identifying Bigelow generators in the obvious way and observing that \(s_R(b') = s_R'(b')\), one obtains that the \(R\)-filtered complex \(\widehat{CF}(\mathcal{H}')\) has the same filtered chain homotopy type as the \(\overline{R}\)-filtered complex \(\widehat{CF}(\mathcal{H})\).

Now denote by \(\mathcal{H}\) the genus-(\(n - 2)\) Heegaard diagram for \(\Sigma(K)\) obtained from \(b\) in the reduced sense by omitting the pair \(\alpha_{n-1}, \beta_{n-1}\). Letting \(\mathcal{H}_0\) denote the standard Heegaard diagram for \(S^2 \times S^1\) of genus 1 (containing only generators from \(s_0\)), one can see that \(\mathcal{H}_{n-1} = \mathcal{H} \# \mathcal{H}_0\) (where the sum region is near \(+ \infty)\). As a result, all generators of \(\widehat{CF}(\mathcal{H}_{n-1})\) lie in \(\text{Spin}^c\) structures of the form \(s \# s_0\). Further, the connected sum provides a natural correspondence between each generator \(x\) for \(\widehat{CF}(\mathcal{H}, s)\) and a pair of generators \(x y\) and \(x z\) for \(\widehat{CF}(\mathcal{H}_{n-1}, s \# s_0)\), where \(y, z \in \hat{\alpha}_n \cap \hat{\beta}_n\).

Now one can verify that

\[
\begin{align*}
P^*(y) &= 2, & P^*(z) &= 0, & Q^*(y) &= 1, & Q^*(z) &= 0, \\
\mathcal{T}(x y) &= \mathcal{T}(x), & \mathcal{T}(x z) &= \mathcal{T}(x), & s_{R}(b') &= s_{R}(b) - \frac{1}{2},
\end{align*}
\]

Figure 5: The \(\alpha\)-circles in the Heegaard diagram \(\mathcal{H}'_{n-1}\)
and so
\[ R(xy) = R(x) + \frac{1}{2} \quad \text{and} \quad R(xz) = R(x) - \frac{1}{2}. \]
As a result, if one equips \( \hat{CF}(H_{n-1}) \) and \( \hat{CF}(H) \) with their respective \( R \)-filtrations, then the filtered complex \( \hat{CF}(H_{n-1}) \) has the same filtered chain homotopy type as the filtered complex \( \hat{CF}(H) \otimes H_{n+1/2}(S^1) \). Furthermore, notice that
\[ g\tilde{r}(xy) = g\tilde{r}(x) + \frac{1}{2} \quad \text{and} \quad g\tilde{r}(z) = g\tilde{r}(x) - \frac{1}{2}, \]
and so \( \hat{CF}(H_{n-1}) \) and \( \hat{CF}(H) \oplus \hat{CF}(H) \) have the same \( \rho \)-filtered chain homotopy types. The results follow.

In light of this relationship, one can now obtain an invariance result for the reduced theory.

Proof of Theorem 2. Let \( H_1 \) and \( H_2 \) be two Heegaard diagrams for \( \Sigma(K) \) obtained from reducible fork diagrams, and let \( H_1 \) and \( H_2 \) be corresponding diagrams for \( \Sigma(K) \#(S^2 \times S^1) \). For \( s \in \text{Spin}^c(\Sigma(K)) \) and \( s_0 \in \text{Spin}^c(S^2 \times S^1) \) torsion, equip \( \hat{CF}(H_i, s) \) with \( \rho \) filtrations and equip \( \hat{CF}(H_i, s \# s_0) \) with \( \rho \) filtrations. Then the result follows via Proposition 1 and Theorem 1 from [13].

### 2.4.1 Case I for Lemma 19

Let \( wx \in Z \), where \( x \in \alpha_{n-1} \cap \beta_{n-1} \). Local pictures of \( F \) and \( F' \) can be seen in Figures 6 and 7, respectively.

**Figure 6:** The fork diagram \( F \). The solid arc is \( \alpha_{n-1} \) and the dashed arc is \( \beta_{n-1} \). The omitted arcs \( \alpha_n \) and \( \beta_n \) are grayed.

**Figure 7:** The fork diagram \( F' \). The solid arc is \( \alpha'_n \) and the dotted arc is \( \beta'_n \). The omitted arcs \( \alpha_{n-1} \) and \( \beta_{n-1} \) are grayed.

Let \( \tilde{H} \) denote the Heegaard diagram obtained from \( H_n \) via handleslides among the \( \alpha \) circles such that \( H'_{n-1} \) can be obtained from \( \tilde{H} \) via handleslides among the \( \beta \) circles. Figure 8 shows local regions of the Heegaard diagrams \( H_n, H \) and \( H'_{n-1} \) (covering a neighborhood of the puncture \( \mu_{2n-2} \) in the respective fork diagrams).

The local components of 3-gons in Figure 9 allow one to define \( g_\alpha \) and \( g_\beta \) such that \( g_\alpha(wx) = wy \) and \( g_\beta(wy) = wz \).
However, we should justify that $R(wz) = R(wx)$. Strictly speaking, we should calculate the gradings for $wz$ after performing an isotopy on $F'$ so that $\alpha'_n$ is a horizontal joining $\mu_{2n-1}$ and $\mu_{2n}$; the result is shown in Figure 10.

One can verify that indeed $Q(wz) = Q(wx) + 1$, $P(wz) = P(wx) + 1$, and $T(wz) = T(wx)$. So, $(g_\beta \circ g_{ba})$ preserves $R$ in this case.

2.4.2 Case II for Lemma 19

Now instead let $zxy \in \mathcal{Z}$ with $x \in \alpha_{n-1} \cap \beta_j$ and $y \in \alpha_k \cap \beta_{n-1}$ (where $j, k < n-1$). Local regions of the fork diagram $\mathcal{F}$ containing possible candidates for $x$ and $y$ are shown in Figure 11. Local regions of $\mathcal{F}'$ are shown in Figure 12.

Figures 13 and 14 exhibit local pictures of $\mathcal{H}_n$ and $\mathcal{H}'_{n-1}$.

The local components of 3-gon domains in Figures 15-18 allow us to let $g_\alpha(zx_iy_j) = zu_iv_j$ and $g_\beta(zu_iv_j) = zu_iv_j$ for $i, j = 1, 2$.

Figure 19 shows the result of the isotopy making $\alpha'_n$ horizontal. One can verify that $T(zu_iv_j) = T(zx_iy_j)$ and that for $i, j = 1, 2$,

$$(P^* - Q^*) (u_i) = (P^* - Q^*) (x_i) + 1 \quad (P^* - Q^*) (v_j) = (P^* - Q^*) (y_i) - 1.$$  

Furthermore, $R(zu_3v_j) = R(zu_2v_j)$ for $j = 1, 2$. So, $(g_\beta \circ g_{ba})$ preserves $R$.  

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Figure 10: Result of an isotopy on Figure 7

Figure 11: Local pictures of $\mathcal{F}$ showing possible components of generators

Figure 12: Local pictures of $\mathcal{F}'$

Figure 13: Local pictures of the Heegaard diagram $\mathcal{H}_n$
3 Applications and computations

3.1 Operations on knots

Theorems 6 and 8 determine the behavior of the filtration \( \rho \) under the processes of taking connected sums and mirrors of knots.

3.1.1 Connected sums

Recall that Ozsváth and Szabó showed that the relatively-graded Heegaard Floer chain complexes satisfy a Künneth-type relationship under connected sums of 3-manifolds. We state the result for \( \widehat{CF} \), but it also holds for \( CF^+ \):
Figure 17: Local components of 3-gon domains for $\psi_+^\beta$ in case II

Figure 18: Local components of 3-gon domains for $\psi_-^\beta$ in case II

Figure 19: The result of an isotopy on Figure 12a

**Theorem 20** (Proposition 6.1 from [8]). Let $M_1$ and $M_2$ be oriented 3-manifolds with $\text{Spin}^c$ structures $s_i \in \text{Spin}^c(M_i)$ for $i = 1, 2$. Then

$$\widehat{CF}(M_1 \# M_2, s_1 \# s_2) \cong \widehat{CF}(M_1, s_1) \otimes_{\mathbb{Z}} \widehat{CF}(M_2, s_2).$$

Recall also that for $M$ a rational homology 3-sphere with $s \in \text{Spin}^c(M)$, Ozsváth and Szabó defined in [6] the correction term $d(M, s) \in \mathbb{Q}$ to be the minimal absolute grading $\tilde{\text{gr}}$ of any non-torsion element in the image of $HF^\infty(M, s)$ in $HF^+(M, s)$. It was shown in [6] that

$$d(M_1 \# M_2, s_1 \# s_2) = d(M_1, s_1) + d(M_2, s_2).$$

It follows that when $M_1$ and $M_2$ are rational homology 3-spheres, the $\widehat{CF}$ complexes satisfy a Künneth formula as absolutely-graded chain complexes with grading $\tilde{\text{gr}}$, i.e.

$$\widehat{CF}_{\tilde{\text{gr}}=k}(M_1 \# M_2, s_1 \# s_2) \cong \bigoplus_{i+j=k} \left( \widehat{CF}_{\tilde{\text{gr}}=i}(M_1, s_1) \otimes_{\mathbb{Z}} \widehat{CF}_{\tilde{\text{gr}}=j}(M_2, s_2) \right). \quad (3.1)$$

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As stated in Theorem 6 when $M_i = \Sigma(K_i)$ for $i = 1, 2$, then the complexes for $M_1, M_2,$ and $\Sigma(K_1 \# K_2) \cong M_1 \# M_2$ satisfy a Künneth-type relationship as filtered complexes upto filtered chain homotopy type.

**Proof of Theorem 6.** Let $K_1$ and $K_2$ be knots in $S^3$. Then we can find braids $b_1 \in B_{2m-1}$ and $b_2 \in B_{2n-2}$ such that the plat closure of $b_1' = (b_1 \times \sigma_1^{-1} \times 1) \in B_{2m+2}$ is a diagram $D_1$ for $K_1$ and the plat closure of $b_2' = (1 \times b_2 \times 1) \in B_{2n}$ is a diagram $D_2$ for $K_2$. Then the plat closure of $b = (b_1 \times \sigma_1^{-1} \times b_2 \times 1) \in B_{2n+2m}$ is a diagram $D$ for $K = K_1 \# K_2$ (where we view $\sigma_1 \in B_2$). These plat closures are illustrated in Figure 20.

Let $F_1, F_2, \text{and } F$ denote the reducible fork diagrams induced by $b_1', b_2,' \text{and } b, \text{respectively.}$ We choose the braids $b_i$ in such a way that the fork diagrams $F_i$ have the local behavior indicated in Figure 21; this can always be achieved through stabilization. The fork diagram $F$ can be seen in Figure 22.

Let’s compute the reduced gradings $R$ by omitting the pair $\alpha_m, \beta_m$ from $F_1$, the pair $\alpha_n, \beta_n$ from $F_2$, and the pair $\alpha_{m+n}, \beta_{m+n}$ from $F$.

![Figure 20: Plat closures of $b_1' \in B_{2m+2}, b_2' \in B_{2n}, \text{and } b \in B_{2m+2n}$](image)

![Figure 21: Local pictures of fork diagrams for summand knots $K_i$](image)

![Figure 22: The fork diagram $F$ for $K_1 \# K_2 \text{with the arcs } \alpha_{m+n}, \beta_{m+n} \text{ grayed.}$](image)
Figure 23: Heegaard diagrams depicting local connected sum behavior

Figure 23 shows the Heegaard diagrams $H_1$, $H_2$, and $H$ covering $F_1$, $F_2$, and $F$, respectively. Let the sets of attaching circles for these diagrams be denoted by $\{\alpha^1, \beta^1\}$, $\{\alpha^2, \beta^2\}$, and $\{\alpha, \beta\}$, respectively.

Recall that if the connected sum region in each Heegaard diagram is near the basepoint, then $H_1$, $H_2$, and $H_\# = H_1 \# H_2$ are Heegaard diagrams exhibiting the correspondences in Theorem 20 and Equation 3.1. The sets of attaching circles for $H_\#$ are $\{\alpha^\#, \beta^\#\} = \{\alpha^1 \cup \alpha^2, \beta^1 \cup \beta^2\}$.

Let $xy$ be a generator in $H_1$ and let $wz$ be a generator in $H_2$ (where $x$ is a $(m-1)$-tuple, $w$ is a $(n-2)$-tuple, $y \in \alpha^1_m$, and $z \in \alpha^2_1$). The generators in $H_\#$ are exactly of the form $xwyz$, and we set $R(xwyz) = R(xy) + R(wz)$.

Now notice that $\alpha$ can be obtained from $\alpha^\#$ by a sequence of handleslides avoiding $\pm \infty$, and analogous handleslides transform $\beta$ to $\beta^\#$. The local components of domains shown in Figures 24 through 27 indicate that we can define triangle injections $g_\alpha : T_{\alpha^\#} \cap T_{\beta^\#} \to T_\alpha \cap T_\beta$ and $g_\beta : T_\alpha \cap T_{\beta^\#} \to T_\alpha \cap T_\beta$ by $g_\alpha(xwyz) = xwuy_i$ and $g_\beta(xwyz) = xwvy_i$ for $i = 1, 2$. It remains to show that $(g_\beta \circ g_\alpha)$ preserves $R$. 

Figure 24: Local components of 3-gon domains for $\psi^\#_{\alpha}$
One can verify that indeed \( \tilde{R}(xwuv_i) = \tilde{R}(xy) + \tilde{R}(wz_i) \) for \( i = 1, 2 \). Further, notice that \( e(b) = e(b'_1) + e(b'_2) \) and \( w(D) = w(D_1) + w(D_2) \). Therefore,

\[
s_R(b, D) = \frac{e(b) - w(D) - 2(m + n - 2)}{4} = \frac{e(b'_1) - w(D_1) - 2(m - 1)}{4} + \frac{e(b'_2) - w(D_2) - 2(n - 1)}{4} = s_R(b_1, D_1) + s_R(b_2, D_2).
\]

\[\square\]

### 3.1.2 Knot mirrors

Given a braid word \( b = \sigma_{i_1}^{k_1} \cdots \sigma_{i_m}^{k_m} \in B_{2n} \), let \( -b \) denote the braid word

\[
-b = \sigma_{-i_1}^{-k_1} \cdots \sigma_{-i_m}^{-k_m} \in B_{2n}.
\]

Notice that if the plat closure of \( b \) is \( K \), then the plat closure of \( -b \) is \( -K \), the mirror of \( K \).

**Proof of Theorem** \[\s\] Let \( b \in B_{2n} \) be a braid whose closure is the knot \( K \). Then recall that the closure of the braid \( -b \) is the knot \( -K \), the mirror image of \( K \). Also, let \( \mathcal{H}_\pm \) denote the admissible Heegaard diagram for \( \Sigma(\pm K) \) induced by \( \pm b \). Let \( s \in \text{Spin}^c(\Sigma(K)) \). Then by an argument
and gives rise to another more concise knot invariant. Let \( t \) of genus 1; this diagram can be seen in Figure 28. Let the generators be labelled from left to right ready found in [13]). Thus \( L \) was studied in [13], and the fork diagram shown there is reducible. We can omit the pair \( \alpha \) and \( \beta \).

3.3 The left-handed trefoil and the lens space \( L(3,1) \)

Let \( K \) be the left-handed trefoil, viewed as the plat closure of the braid \( \sigma_2^3 \in B_4 \). This braid was studied in [13], and the fork diagram shown there is reducible. We can omit the pair \( \alpha_2, \beta_2 \) from the fork diagram and obtain via Proposition [17] an admissible Heegaard diagram for \( L(3,1) \) of genus 1; this diagram can be seen in Figure 28. Let the generators be labelled from left to right as \( t', t, \) and \( x_1 \).

The set of reduced Bigelow generators is \( G = \{ t, t', x_1 \} \), and one can verify that all elements occupy the level \( R = 1 \). In this case \( \tilde{\partial} \equiv 0 \), and indeed we see evidence of \( \rho \)-degeneracy (which was already found in [13]). Thus \( \tilde{R} \) provides an absolute Maslov grading on the group \( \tilde{HF}(L(3,1); \mathbb{Z}/2\mathbb{Z}) \), and

\[
\tilde{HF}(L(3,1); \mathbb{Z}/2\mathbb{Z}) = \left[ (\mathbb{Z}/2\mathbb{Z})^{\oplus 3} \right]_{R=1}.
\]

One should observe that \( \tilde{HF}(L(3,1)) \) is supported entirely in the grading level \( R = 1 \) (and 1 is half of the classical signature \( \sigma(K) \)). Theorem [13] states that all two-bridge knot behave this way.

3.4 Two-bridge knots

Let us first give some background on two-bridge knots and links.
3.4.1 The Conway form of a two-bridge knot or link

Recall that a two-bridge knot is a knot which has a projection on which the natural height function has exactly two maxima and two minima. A two-bridge link is defined similarly, with exactly one maximum and one minimum of the height function lying on each of the two components.

For each two-bridge knot or link $L$, there are nonzero integers $b_1, b_2, \ldots, b_k$ (either all positive or all negative) such that one of the two diagrams in Figure 29 is a projection of $L$. Note that if $b_i < 0$, then the $i^{th}$ bunch of half-twists is reversed from those shown in the Figure 29.

This diagram for $L$ is referred to as the Conway form, with Conway notation given by the continued fraction

$$[b_1, \ldots, b_k] = b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cfrac{1}{b_4 + \ldots}}} = \frac{p}{q}.$$  

Remark 21. If $L$ is a knot or link with Conway notation $[b_1, \ldots, b_k] = \frac{p}{q}$, $p$ and $q$ coprime, then the two-fold cover of $S^3$ branched along $L$ is the lens space $L(p, q)$.

For more about the Conway form and Conway notation, see [11].
3.4.2 The Goeritz matrix and the classical knot signature

Gordon and Litherland give a formula in [3] for calculating the signature $\sigma(K)$ of a knot $K$. Here we review their construction briefly, but one can find more details in [3].

Given a regular projection $D$ of a knot $K$ in $\mathbb{R}^2$, color the components of $\mathbb{R}^2 - D$ black and white in a checkerboard fashion, denoting the white regions by $X_0, \ldots, X_n$. Denote by $c(X_i, X_j)$ the set of crossings of $D$ which are incident to $X_i$ and $X_j$. Then assign an incidence number $\eta(C) = \pm 1$ to each crossing $C$ in the projection, following the convention in Figure 30.

![Figure 30: Conventions for $\eta(C)$, where $C$ is a crossing in a colored diagram $D$](image)

**Definition 22.** Let $D$ be a regular projection of a knot $K$ equipped with a checkerboard coloring and white regions $X_0, \ldots, X_n$. Then let $G'(D)$ be the $(n + 1) \times (n + 1)$ matrix with

$$G'(D) = (g_{ij})_{i,j=0}^{n}, \quad \text{where} \quad g_{ij} = \begin{cases} -\sum_{c(X_i, X_j)} \eta(C) & \text{if } i \neq j \\ -\sum_{k \neq i} g_{ik} & \text{if } i = j. \end{cases}$$

Then define $G(D)$, the **Goeritz matrix of $D$**, to be the $n \times n$ symmetric integer matrix obtained from $G'(D)$ by deleting the 0th row and 0th column.

![Figure 31: Types of double point orientations counted by $\mu_I$ and $\mu_{II}$](image)

Now fixing an orientation on $K$, we separate the double points of $D$ into types I and II, according to Figure 31. Note that when classifying a crossing in this way, we ignore which strand is passing over the other.

Now we define two integers $\mu_I$ and $\mu_{II}$ by

$$\mu_I(D) = \sum_{C \text{ of type I}} \eta(C) \quad \text{and} \quad \mu_{II}(D) = \sum_{C \text{ of type II}} \eta(C).$$

It is shown in [3] that the quantity $\text{sign}(G(D)) - \mu_{II}(D)$ is independent of the projection $D$, and thus an invariant of the knot $K$. We’ll make use of the following theorem:

**Theorem 23** (Theorem 6 from [3]). Let $K$ be a knot with regular projection $D$. Then

$$\sigma(K) = \text{sign}(G(D)) - \mu_{II}(D).$$
We’ll need the following fact, the proof of which will be left as an exercise.

**Lemma 24.** Let $K$ be a knot with oriented regular projection $D$. Then

$$w(D) = \mu_{II}(D) - \mu_I(D).$$

### 3.4.3 Computations for two-bridge knots

We’ll perform our calculation for a two-bridge knot $K$ using the braid whose closure is the Conway form of $K$. Further, we assume that the number of Bigelow generators in cannot be reduced by isotopy of fork diagram. Because Conway forms never involve the rightmost strand in the braiding, the induced fork diagram will always be reducible. After reducing the diagram, $G$ will be the set of 1-tuples $\alpha_1 \cap bE_i'$. First we show that the function $R$ is very simple for such a reduced fork diagram:

**Proposition 25.** Let $K$ be a two-bridge knot with Conway notation $[b_1, \ldots, b_k]$. Then for any reduced Bigelow generator $x \in G$ in the special reduced fork diagram above,

$$R(x) = \begin{cases} 
  e - \frac{w - 2}{4} & \text{if } b_1 > 0 \\
  e - \frac{w + 2}{4} & \text{if } b_1 < 0.
\end{cases}$$

where $e$ is the signed count of braid generators in the Conway form and $w$ is the writhe of that diagram.

Since our entire set of Bigelow generators (and thus a set of generators for $\widehat{CF}(\Sigma(K))$ lies in the same $R$ level. As a result, we have that $\widehat{\partial} = 0$ and

$$|G| = rk(\widehat{HF}(\Sigma(K)) = det(K).$$

Before proving Proposition 25, we’ll have to define some new terminology.

**Definition 26.** Let $K$ be an oriented two-bridge knot, and consider the special fork diagram acquired from the Conway form of $K$. Augment the diagram by adding an extra horizontal tine edge $\alpha$ connecting $\mu_2$ and $\mu_3$, and give it a vertical handle $h$. Then we’ll call $x \in (\alpha - \mu_2) \cap \beta_1$ a **central intersection**.

We then extend the $\widetilde{R}$ grading to central intersections in the natural way. Of course our reduced fork diagrams don’t include $\alpha$ or its handle; we will simply use these central intersections as an inductive tool for proving Proposition 25.

**Proof of Proposition 25.** We first prove the proposition for the case where $b_i > 0$ for $i = 1, \ldots, k$. The fork diagram corresponding to only $\sigma_2^b$ includes $b_1$ Bigelow generators with $\widetilde{R} = 0$. Notice that in such a diagram, when $\beta_1$ is incident to $\mu_i$, it approaches from below when $i = 1$ or $i = 3$ and from below when $i = 2$. Furthermore, the actions of $\sigma_2$ and $\sigma_1^{-1}$ don’t change these incidence trajectories. Let us develop inductive steps for applying $\sigma_1^{-1}$ and $\sigma_2$, the building blocks for diagrams of this type.

For the first inductive step, we examine the application of $\sigma_1^{-1}$ to an existing braid $b$. Each existing element $g \in G$ on the interior of $\alpha_1$ spawns one new central intersection $c$ and is itself replaced by another interior $g' \in G$, with $\widetilde{R}(g') = \widetilde{R}(g)$ and $\widetilde{R}(c) = \widetilde{R}(g) + 1$. 

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We should also examine the effects on an arc terminating at $\mu_1$ (case I) or $\mu_2$ (case II), as shown in Figure 32.

In both cases, the element $g \in G$ is replaced by $g' \in G$. We see that $\overline{R}(g') = \overline{R}(g)$. In case II, the application of $\sigma_1^{-1}$ also spawns a new interior central intersection point, $c$. We see that $\overline{R}(c) = \overline{R}(g) + 1$.

The loops used for grading calculations above can be seen in Figure 33.

For the second inductive step, we examine the application of $\sigma_2$ to an existing braid $b$. Each existing central intersection $c$ spawns a new element $g \in G$ on the interior of $\alpha_1$, and is itself replaced by a new central intersection $c'$. We see that $\overline{R}(c') = \overline{R}(c)$ and $\overline{R}(g) = \overline{R}(c) - 1$.

Some new elements of $\tilde{G}$ can also result from twisting a strand that originally terminated at $\mu_2$ (case I) or $\mu_3$ (case II), as shown in Figure 34.

In case I, we gain a central intersection $c$, and the element $g \in G$ is replaced by $h \in G$. In case II, the central intersection $c$ is replaced by $g \in G$. In either case, we have that $\overline{R}(c) = \overline{R}(g) + 1$ and $\overline{R}(h) = \overline{R}(g)$.

The grading comparisons calculated above are demonstrated by the loops in Figure 35.

Using the results of these two inductive steps, we see that all elements of $\tilde{G}$ for the $b_1 > 0$ case
lie in the 0 level of the $\tilde{R}$ grading.

The proof for the $b_i < 0$ case contains analogous inductive steps corresponding to applying the braid generators $\sigma_1$ and $\sigma_1^{-1}$. The fork diagram corresponding to only $\sigma_2^{b_i}$ includes $-b_1$ Bigelow generators with $\tilde{R} = 1$. So, all elements of $\tilde{G}$ lie in level 1 of the $\tilde{R}$ grading when $b_1 < 0$.

We then have that for any $x \in \tilde{G}$,

$$\tilde{R}(x) = \begin{cases} 0 & \text{if } b_1 > 0 \\ 1 & \text{if } b_1 < 0. \end{cases}$$

and the result for $R$ follows.

**Lemma 27.** Let $K$ be a knot with projection $D$ given by the closure of $\sigma_2^{b_1} \sigma_1^{-b_2} \ldots \sigma_1^{-b_{k-1}} \sigma_2^{b_k}$. Then

$$-(\mu_I(D) + \mu_{II}(D)) = \sum_{i=1}^{k} b_i.$$ 

**Proof of Lemma 27.** Again consider the four types of colored oriented crossings in Figure ???. The crossings $\sigma_2$ and $\sigma_1^{-1}$ are of type C or D, and crossings $\sigma_2^{-1}$ and $\sigma_1$ are of type A or B. Now,

$$-(\mu_I(D) + \mu_{II}(D)) = -((\#B - \#D) + (\#A - \#C))
= (\#D + \#C) - (\#B + \#A)
= \sum_{i \text{ odd}} b_i + \sum_{i \text{ even}} b_i = \sum_{i=1}^{k} b_i.$$ 

**Lemma 28.** Let $K$ be a knot with projection $D$ given by the closure of $\sigma_2^{b_1} \sigma_1^{-b_2} \ldots \sigma_1^{-b_{k-1}} \sigma_2^{b_k}$. Then

$$\text{sign}(G(D)) = -\text{sign}(b_1) \cdot (\sum_{i \text{ even}} |b_i| + 1) = \begin{cases} -\left(\sum_{i \text{ even}} b_i\right) - 1 & \text{if } b_1 > 0 \\ -\left(\sum_{i \text{ even}} b_i\right) + 1 & \text{if } b_1 < 0. \end{cases}$$

**Proof of Lemma 28.** Consider the case where $b_1 > 0$. First color the components of $\mathbb{R}^2 - K$ such that the exterior region is black. The Goeritz matrix $G(D)$ has dimensions

$$\left(\sum_{i \text{ even}} b_i + 1\right) \times \left(\sum_{i \text{ even}} b_i + 1\right)$$
and is given by:
\[
\begin{pmatrix}
  a_1 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
 1 & -2 & \ddots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 1 & \ddots & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & \ddots & -2 & 1 & \ddots & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & \ddots & 1 & a_3 & \ddots & 0 & 0 & 0 & 0 & 0 & 0 \\
 \vdots & \vdots & \ddots & 0 & 1 & \ddots & 1 & 0 & 0 & 0 & \vdots & \vdots \\
 0 & 0 & \cdots & 0 & 0 & \ddots & -2 & 1 & 0 & 0 & 0 & 0 \\
 0 & 0 & \cdots & 0 & 0 & \ddots & 0 & a_{k-2} & 1 & \ddots & 0 & 0 \\
 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & -2 & \ddots & 0 & 0 \\
 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 1 & \ddots & 1 & 0 \\
 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \ddots & -2 & 1 \\
 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & a_k \\
\end{pmatrix},
\]

where \( a_j = \begin{cases} -(b_j + 2) & \text{if } j = 2, \ldots, > 0 \\ -(b_j + 1) & \text{if } j = 1, k. \end{cases} \)

One can see inductively that all of the upper left minors of the matrix \(-G(D)\) have positive determinant, and so the matrix \(G(D)\) is negative-definite. We then have that

\[
\text{sign}(G(D)) = -\left( \sum_{i \text{ even}} b_i \right) - 1.
\]

When \(b_1 < 0\), the Goeritz matrix is positive-definite and has signature

\[
\text{sign}(G(D)) = -\left( \sum_{i \text{ even}} b_i \right) + 1.
\]

\[\square\]

**Proof of Theorem 13.** By Proposition 25, the filtration \(\rho\) is already constant on \(\widetilde{CF}(\Sigma(K), s)\) for each \(s \in \text{Spin}^c(\Sigma(K))\) and therefore \(K\) is both \(\rho\)-degenerate and \(R\)-thin. It remains to be shown that \(R(K) = \sigma(K)/2\). For \(b_1 > 0\), we have that

\[
R(K) = \frac{e - w - 2}{4} = \frac{1}{4} \left( \sum_{i=1}^{k} (-1)^{i+1} b_i + \mu_I - \mu_{II} - 2 \right),
\]

and

\[
\sigma(K) = \text{sign}(G(D)) - \mu_{II}(D) = -\left( \sum_{i \text{ even}} b_i \right) - 1 - \mu_{II}(D).
\]
So, we then have that
\[
2R(K) - \sigma(K) = \frac{1}{2} \left( \sum_{i=1}^{k} (-1)^{i+1} b_i \right) + 2 \left( \sum_{i \text{ even}} b_i \right) + \mu_I + \mu_{II}
\]
\[
= \frac{1}{2} \left( \sum_{i=1}^{k} b_i \right) + \mu_I + \mu_{II} \right) = \frac{1}{2}(0) = 0.
\]

The calculation for the \( b_1 < 0 \) case is similar. \( \square \)

4 Future directions

4.1 \( \rho \)-degeneracy

We have seen that when the knot \( K \) is \( \rho \)-degenerate, then the function \( R \) provides an absolute Maslov grading on the group \( \widehat{HF}(\Sigma(K)) \). It was established above that this occurs when \( K \) is a two-bridge knot. It is natural to ask whether a larger class of knots is \( \rho \)-degenerate, such as alternating knots or quasi-alternating knots. Following Seidel and Smith in [12], we have conjectured above that all knots are \( \rho \)-degenerate.

4.2 A possible concordance invariant

Such theories have been known to produce invariants of the knot concordance class. Examples include Rasmussen’s \( s \)-invariant in Khovanov homology [10], Manolescu and Owens’s \( \delta \)-invariant in Heegaard Floer homology [5], and Ozsváth and Szabó’s \( \tau \)-invariant in Knot Floer homology [7]. Conjecture [11] speculates that \( r \) also provides one.

In [5], Manolescu and Owens defined a concordance invariant via \( \delta(K) = 2d(\Sigma(K), s_0) \), where \( d \) denotes the correction term defined by Ozsváth and Szabó in [6] and \( s_0 \in \text{Spin}^c(\Sigma(K)) \) is the unique Spin structure. Furthermore, it was shown in [5] that whenever \( K \) is an alternating knot, \( \delta(K) = -\sigma(K)/2 \). Together with Theorem [13] above, this implies that if \( K \) is a two-bridge knot, then
\[
r(K) = \frac{\sigma(K)}{2} - d(\Sigma(K), s_0) = \frac{\sigma(K)}{2} - \frac{\delta(K)}{2} = \frac{3\sigma(K)}{4}.
\]

Recall that the concordance invariants \( s \), \( \delta \), and \( \tau \) have all been shown to be equal to a constant multiple of \( \sigma \) when restricted to the set of alternating knots.

4.3 Knot mutation

Bloom showed in [2] that odd Khovanov homology is invariant under Conway mutation, while Ozsváth and Szabó proved in [9] that knot Floer homology can distinguish a Kinoshita-Terasaka knot from one of its mutants. Viro noted in [14] that mutants links have homeomorphic double branched covers, and thus can’t be distinguished by the Heegaard Floer homology groups discussed here. However, one could ask whether this extra filtration structure on the chain complex can distinguish mutant knots.
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