Affine Stanley symmetric functions

Thomas Lam

American Journal of Mathematics, Volume 128, Number 6, December 2006, pp. 1553-1586 (Article)

Published by Johns Hopkins University Press

DOI: https://doi.org/10.1353/ajm.2006.0045

For additional information about this article
https://muse.jhu.edu/article/206759/summary

For content related to this article
https://muse.jhu.edu/related_content?type=article&id=206759
AFFINE STANLEY SYMMETRIC FUNCTIONS

By THOMAS LAM

Abstract. We define a new family \( \tilde{F}_w(X) \) of generating functions for \( w \in \tilde{S}_n \) which are affine analogues of Stanley symmetric functions. We establish basic properties of these functions including symmetry, dominance and conjugation. We conjecture certain positivity properties in terms of a subfamily of symmetric functions called affine Schur functions. As applications, we show how affine Stanley symmetric functions generalize the (dual of the) \( k \)-Schur functions of Lapointe, Lascoux and Morse as well as the cylindric Schur functions of Postnikov. Conjecturally, affine Stanley symmetric functions should be related to the cohomology of the affine flag variety.

1. Introduction. In [28], Stanley introduced a family \( \{F_w(X)\} \) of symmetric functions now known as Stanley symmetric functions. He used these functions to study the number of reduced decompositions of permutations \( w \in S_n \). Later, the functions \( F_w(X) \) were found to be closely related to the Schubert polynomials of Lascoux and Schützenberger [21], which are well known to be related to the geometry of flag varieties.

The aim of this paper is to define and study an analogue \( \tilde{F}_w(X) \) of Stanley symmetric functions for the affine symmetric group \( \tilde{S}_n \) which we call affine Stanley symmetric functions. Our definition of \( \tilde{F}_w(X) \) is motivated by [7] and [4] and involve an algebra which we call the affine nilCoxeter algebra. This algebra is an affine version of the nilCoxeter algebra used in [7]. When \( w \in S_n \subset \tilde{S}_n \), we have \( \tilde{F}_w(X) = F_w(X) \). Our first main theorem is that these functions \( \tilde{F}_w(X) \) are indeed symmetric functions. Imitating [28], we show basic properties of these functions:

(1) the relation to reduced words:

\[
[x_1, x_2, \ldots, x_{l(w)}] \tilde{F}_w(X) = \# \{ \text{reduced words of } w \},
\]

(2) a skewing formula:

\[
x_1^\perp \cdot \tilde{F}_w = \sum_{w \gg v} \tilde{F}_v
\]

where \( \gg \) denotes the covering relation in weak Bruhat order,

(3) a conjugacy formula:

\[
\tilde{F}_{w^*} = \omega^*(\tilde{F}_w)
\]
where $*: \tilde{S}_n \rightarrow \tilde{S}_n$ and $\omega^+: \Lambda^{(n)} \rightarrow \Lambda^{(n)}$ are involutions ($\Lambda^{(n)} = \mathbb{C}\langle m_{\lambda} \mid \lambda_1 \leq n - 1 \rangle$),

(4) and the existence of a unique dominant monomial term $m_{\mu(w)}$:

$$\tilde{F}_w = m_{\mu(w)} + \sum_{\lambda < \mu(w)} b_{\lambda \mu} m_{\lambda}.$$  

An important special case occurs when $w$ is a Grassmannian permutation. A permutation $w \in \tilde{S}_n$ is Grassmannian if it is a minimal length coset representative of a coset of $S_n \setminus \tilde{S}_n$. In this case we obtain the affine Schur functions $\tilde{F}_\lambda(X) = \tilde{F}_w(X)$ which may be labelled by partitions with no part greater than $n - 1$. We show that the affine Schur functions $\{\tilde{F}_\lambda \mid \lambda_1 \leq n - 1\}$ form a basis of the space $\Lambda^{(n)}$ spanned by $\{m_{\lambda} \mid \lambda_1 \leq n - 1\}$ where the $m_{\lambda}$ are monomial symmetric functions. Edelman and Greene [3] and separately Lascoux and Schützenberger [22] have shown that Stanley symmetric functions $F_w(X)$ expand positively in terms of Schur functions $s_{\lambda}(X)$. We conjecture that affine Stanley symmetric functions expand positively in terms of affine Schur functions. We prove that a unique maximal and minimal "dominant" term exists in such an expansion.

Our definition of affine Stanley symmetric functions is motivated by relations with two other classes of symmetric functions which have received attention lately. Lapointe, Lascoux and Morse [16] initiated the study of $k$-Schur functions, denoted $s_{\lambda}^{(k)}(X)$, in their study of Macdonald polynomial positivity. It is conjectured that $k$-Schur functions form an “intermediate” basis between the Macdonald polynomials $\{H_{\mu}(X; q, t)\}$ [24] and the Schur functions $\{s_{\lambda}(X)\}$ so that the transition coefficients are positive in both intermediate steps. Lapointe and Morse have more recently connected the multiplication of $k$-Schur functions with the Verlinde algebra of $U(m)$.

Our affine Schur functions had earlier been defined using “$k$-tableaux” by Lapointe and Morse, who called these functions dual $k$-Schur functions; see [19]. Work of Lapointe and Morse [18] relating $k$-Schur functions to $n$-cores and to the affine symmetric group show that our definition of affine Schur functions are indeed dual to $k$-Schur functions. In this context the symmetry of affine Schur functions is not obvious, but follows from the symmetry of general affine Stanley symmetric functions. The relation with $k$-Schur functions also suggest the study of skew affine Schur functions $\tilde{F}_{\lambda/\mu}(X)$, another special case of affine Stanley symmetric functions which we study.

Separately, cylindric Schur functions were defined by Postnikov [27] (see also [8]). He showed that certain coefficients of the expansion of toric Schur functions (a special case of cylindric Schur functions) in terms of Schur functions were equal to the 3-point genus 0 Gromov-Witten invariants of the Grassmannian $Gr_{m,n}$ (which are the multiplication constants of the quantum cohomology $QH^*(Gr_{m,n})$ of the Grassmannian). Cylindric Schur functions are defined as generating functions of cylindric semi-standard tableaux, which are tableaux drawn on a cylinder. We show that cylindric Schur functions are special cases of skew affine Schur
functions and that they are exactly equal to affine Stanley symmetric functions $\tilde{F}_w$ labelled by affine permutations $w$ which are “321-avoiding”. These results are affine analogues of some of the results in [1]. However, the affine case is significantly more difficult. For example, any normal Stanley symmetric function $F_w$ is equal to some skew affine Schur function.

We also show that our conjecture that affine Stanley symmetric functions expand positively in terms of affine Schur functions implies both the Schur positivity of toric Schur polynomials and the positivity of the multiplication of $k$-Schur functions. Our work also explains why cylindric Schur functions are not in general Schur positive in infinitely many variables; see [25]. The three families of symmetric functions: affine Stanley, skew affine Schur and cylindric Schur can be thought of as arising from three different representations of the affine nilCoxeter algebra $\mathcal{U}_n$ in a uniform manner. The representation from which affine Stanley symmetric functions arise is the left regular representation of $\mathcal{U}_n$ and so leads to the most general symmetric functions which can arise in this manner.

The connections with $k$-Schur functions and cylindric Schur functions already indicate that affine Stanley symmetric functions are important objects. In fact our work, combined with the known connection between quantum cohomology and the Verlinde algebra, essentially implies that the main results of [27] and [19] are equivalent. However, it seems the most exciting direction to take is to extend our definition of affine Stanley symmetric functions $\tilde{F}_w$ to affine Schubert polynomials $\tilde{S}_w$ and connect them with the affine flag variety $\tilde{G}/B$ of type $\tilde{A}_{n-1}$. Usual Stanley symmetric functions are certain stable “limits” of the Schubert polynomials $\mathcal{S}_w$, which are well known to represent Schubert varieties in the cohomology $H^*(G/B)$ of the flag variety and possess numerous remarkable properties. In the affine case, the cohomology classes $[\Omega_w] \in H^*(\tilde{G}/B)$ representing Schubert varieties are labelled by $w \in \tilde{S}_n$ and should conjecturally be related to affine Stanley symmetric functions.

In the Grassmannian case, Morse and Shimozono [26] have conjectured that affine Schur functions represent the Schubert classes in the cohomology of the affine Grassmannian. Recently, the author has established this conjecture; see [15]. The study of affine Schur functions makes explicit the relationship between the affine Grassmannian $\tilde{G}/\mathcal{P}$, and the Verlinde algebras of $U(m)$ with level $n - m$ or the quantum cohomology $QH^*(Gr_{m,n})$ of the Grassmannian, which are already known to be connected; see for example [30].

Finally, we make the natural generalisation to affine stable Grothendieck polynomials $\tilde{G}_w(x)$ which speculatively should be stable limits of the $K$-theory Schubert classes of the affine flag variety.

Much work has also been done with a version of the Stanley symmetric functions for the hyperoctahedral group; see [11], [12], [6]. We intend to generalize this to the affine case and also investigate Stanley symmetric functions for general Coxeter groups in later work.
Overview. In Sections 2 and 3, we establish some notation for affine permutations and for symmetric functions. In Section 4 we recall the definition of Stanley symmetric functions, give their main properties and explain the relationship with Schubert polynomials. In Section 5, we define the affine nilCoxeter algebra and affine Stanley symmetric functions and prove that the latter are symmetric. In Section 6, we explain how affine Stanley symmetric functions arise from different representations of the affine nilCoxeter algebra. In Section 7, we prove a coproduct formula for affine Stanley symmetric functions. In Section 8, we show that affine Stanley symmetric functions have a unique dominant monomial term. In Section 9, we prove a conjugacy formula, imitating [28]. In Section 10, we define and study affine Schur functions. In Section 11, we study the relationship between \( n \)-cores and the affine symmetric group, following in part [20]. In Sections 12, 13 and 14 we define skew affine Schur functions and relate them to \( k \)-Schur functions. In Section 15, we recall the definition of a cylindric Schur function and connect them with skew affine Schur functions. In Section 16, we show that cylindric Schur functions correspond exactly to 321-avoiding permutations. In Section 17, we make a number of positivity conjectures concerning the expansion of affine Stanley symmetric functions in terms of affine Schur functions. Finally, in Section 18, we discuss some further extensions of our theory and in particular a generalisation to affine stable Grothendieck polynomials.

A condensed preliminary version of this paper appeared as [14].

Acknowledgments. I thank Jennifer Morse for interesting discussions about \( k \)-Schur functions and dual \( k \)-Schur functions. I also thank Mark Shimozono for discussions relating to the affine Grassmannian. I am grateful to Alex Postnikov for introducing cylindric and toric Schur functions in his class. I am also indebted to my advisor, Richard Stanley, for guidance over the last couple of years.

2. Affine symmetric group. A positive integer \( n \geq 3 \) will be fixed throughout the paper. Let \( \tilde{S}_n \) denote the affine symmetric group with simple generators \( s_0, s_1, \ldots, s_{n-1} \) satisfying the relations

\[
\begin{align*}
s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} & \text{for all } i \\
s_i^2 &= 1 & \text{for all } i \\
s_i s_j &= s_j s_i & \text{for } |i - j| \geq 2.
\end{align*}
\]

Here and elsewhere, the indices will be taken modulo \( n \) without further mention. One may realize \( \tilde{S}_n \) as the set of all bijections \( w: \mathbb{Z} \rightarrow \mathbb{Z} \) such that \( w(i+n) = w(i)+n \) for all \( i \) and \( \sum_{i=1}^n w(i) = \sum_{i=1}^n i \). In this realization, to specify an element \( w \in \tilde{S}_n \) it suffices to give the “window” \([w(1), w(2), \ldots, w(n)]\). The product \( w \cdot v \) of two affine permutations is then the composed bijection \( w \circ v: \mathbb{Z} \rightarrow \mathbb{Z} \). Thus \( w_{k} \) is obtained from \( w \) by swapping the values of \( w(i+kn) \) and \( w(i+kn+1) \) for every \( k \in \mathbb{Z} \). See [2] for more details.
The symmetric group $S_n$ embeds in $\tilde{S}_n$ as the subgroup generated by $s_1, s_2, \ldots, s_{n-1}$. Since there are many embeddings of the $S_n$ into $\tilde{S}_n$ we will denote this particular embedding by $S_{\eta}^\circ$.

For an element $w \in \tilde{S}_n$ let $R(w)$ denote the set of reduced words for $w$. A word $\rho = (\rho_1 \rho_2 \cdots \rho_l) \in [0, n-1]^l$ is a reduced word for $w$ if $w = s_{\rho_1}s_{\rho_2} \cdots s_{\rho_l}$ and $l$ is the smallest possible integer for which such a decomposition exists. Abusing notation slightly, we also call $s_{\rho_1}s_{\rho_2} \cdots s_{\rho_l}$ a reduced word for $w$. The integer $l = l(w)$ is called the length of $w$. If $\rho, \pi \in R(w)$ for some $w$, then we write $\rho \sim \pi$. If $\rho$ is an arbitrary word with letters from $[0, n-1]$ then we write $\rho \sim 0$ if it is not a reduced word of any affine permutation. If $w, u \in \tilde{S}_n$ then we say that $w$ covers $u$ if $w = s_i \cdot u$ and $l(w) = l(u) + 1$; and we write $w \triangleright u$. The transitive closure of $\triangleright$ is called the weak Bruhat order and denoted $\trianglerighteq$.

The code $c(w)$ or affine inversion table [2], [20] of an affine permutation $w$ is a vector $c(w) = (c_1, c_2, \ldots, c_n) \in \mathbb{N}^n - \mathbb{P}^n$ of nonnegative entries with at least one 0. The entries are given by $c_i = \# \{j \in \mathbb{Z} \mid j > i \text{ and } w(j) < w(i)\}$. It is shown in [2] that there is a bijection between codes and affine permutations and that $l(w) = |c(w)| = \sum_{i=1}^n c_i$. The right action of the simple generator $s_i$ on the code $c = (c_1, c_2, \ldots, c_n)$ is given by

$$(c_1, \ldots, c_i, c_{i+1}, \ldots, c_n) \cdot s_i = (c_1, \ldots, c_{i+1} + 1, c_i, \ldots, c_n)$$

whenever $c_i > c_{i-1}$. Thus $c(w s_i) = c(w) \cdot s_i$.

3. Symmetric functions. A partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l \geq 0)$ is a weakly decreasing finite sequence of positive integers. We use $\lambda$, $\mu$ and $\nu$ to denote partitions and will always draw them in the English notation (top-left justified). The dominance order $\preceq$ on partitions is given by $\lambda \preceq \mu$ if and only if $\lambda_1 + \lambda_2 + \cdots + \lambda_i \leq \mu_1 + \mu_2 + \cdots + \mu_i$ for every $i$. We will also assume the reader is reasonably familiar with the usual notions of corners, conjugates and semistandard Young tableaux.

We will follow mostly [24], [29] for our symmetric function notation. Let $\Lambda$ denote the ring of symmetric functions over $\mathbb{C}$. Usually, our symmetric functions will have an infinite set of variables $x_1, x_2, \ldots$ and will be written as $f(x_1, x_2, \ldots)$ or $f(X)$. If we need to emphasize the variables used, we write $\Lambda_X$.

We will use $m_\lambda$, $p_\lambda$, $e_\lambda$, $h_\lambda$ and $s_\lambda$ to denote the monomial, power sum, elementary, homogeneous and Schur bases of $\Lambda$. It is well known that $\{e_n\}$ and $\{h_n\}$ are algebraically independent generators of $\Lambda$. Let $\langle \cdot, \cdot \rangle$ denote the Hall inner product of $\Lambda$ satisfying $\langle h_\lambda, m_\mu \rangle = \langle s_\lambda, s_\mu \rangle = \delta_{\lambda \mu}$. For $f \in \Lambda$, write $f^\perp : \Lambda \to \Lambda$ for the linear operator adjoint to multiplication by $f$ with respect to $\langle \cdot, \cdot \rangle$. We let $\omega : \Lambda \to \Lambda$ denote the $\mathbb{C}$-algebra involution of $\Lambda$ sending $h_n$ to $e_n$.

If $f(X) \in \Lambda$ then $f(x_1, x_2, \ldots, y_1, y_2, \ldots) = f(X, Y) = \sum_i f_i(X) \otimes g_i(Y) \in \Lambda_X \otimes \Lambda_Y$ for some $f_i$ and $g_i$. This is the coproduct of $f$, written $\Delta f = \sum_i f_i \otimes g_i \in \Lambda \otimes \Lambda$. 


We have the following formula for the coproduct [24]:

\[ \Delta f = \sum_{\lambda} x_\lambda^f \otimes s_\lambda. \]

The ring of symmetric functions \( \Lambda \) is a self dual Hopf-algebra with respect to \( \langle \cdot, \cdot \rangle \), so that

\[ \langle \Delta f, g \otimes h \rangle = \langle f, gh \rangle, \]

where \( \langle f_1 \otimes f_2, g_1 \otimes g_2 \rangle := \langle f_1, g_1 \rangle \langle f_2, g_2 \rangle \).

Let \( \text{Par}^n \) denote the set \( \{ \lambda \mid \lambda_1 \leq n-1 \} \) of partitions with no row longer than \( n-1 \). The following two subspaces of \( \Lambda \) will be important to us:

\[ \Lambda^{(n)} := \mathbb{C} \langle m_\lambda \mid \lambda \in \text{Par}^n \rangle \]
\[ \Lambda(n) := \mathbb{C} \langle h_\lambda \mid \lambda \in \text{Par}^n \rangle = \mathbb{C} \langle e_\lambda \mid \lambda \in \text{Par}^n \rangle = \mathbb{C} \langle p_\lambda \mid \lambda \in \text{Par}^n \rangle. \]

If \( f \in \Lambda^{(n)} \) and \( g \in \Lambda(n) \) then define \( \langle f, g \rangle \) to be their usual Hall inner product within \( \Lambda \). Thus \( \{ h_\lambda \} \) and \( \{ m_\lambda \} \) with \( \lambda \in \text{Par}^n \) form dual bases of \( \Lambda^{(n)} \) and \( \Lambda(n) \). Note that \( \Lambda(n) \) is a subalgebra of \( \Lambda \) but \( \Lambda^{(n)} \) is not closed under multiplication. Instead, \( \Lambda^{(n)} \) is a coalgebra; it is closed under comultiplication.

4. Stanley symmetric functions. Let \( w \in S_n \) with length \( l = l(w) \). Define the generating function \( F_w^{-1}(X) \) by

\[ F_{w^{-1}}(x_1, x_2, \ldots) = \sum_{a_1 a_2 \cdots a_l \in R(w)} \sum_{1 \leq b_1 \leq b_2 \leq \cdots \leq b_l \atop a_i > a_{i+1} \Rightarrow b_i > b_{i+1}} x_{b_1} x_{b_2} \cdots x_{b_l}. \]

We have indexed the \( F_{w^{-1}}(X) \) by the inverse permutation to agree with the definition we shall give later. Note that the length \( l(w) \) is equal to the degree of \( F_w \) and the number \( |R(w)| \) of reduced decompositions of \( w \) is given by the coefficient of \( x_1 x_2 \cdots x_l \) in \( F_w \).

**Theorem 1** [28]. The following properties of the generating function \( F_w \) hold for each \( w \in S_n \):

1. \( F_w(X) \) is a symmetric function in \( (x_1, x_2, \ldots) \).
2. Define \( a_{w,\lambda} \in \mathbb{Z} \) by \( F_w(X) = \sum_\lambda a_{w,\lambda} s_\lambda(X) \). Then there exists partitions \( \lambda(w) \) and \( \mu(w) \) so that \( a_{w,\lambda(w)} = a_{w,\mu(w)} = 1 \) and

\[ F_w(X) = \sum_{\lambda(w) \leq \lambda \leq \mu(w)} a_{w,\lambda} s_\lambda(X). \]
(3) Define an involution $\ast : S_n \to S_n$ by $\ast : w_1 w_2 \cdots w_n \mapsto (n + 1 - w_n)(n + 1 - w_{n-1}) \cdots (n + 1 - w_1)$. Then

$$\omega(F_w) = F_w^\ast.$$ 

(4) We have

$$s_1^\perp \cdot F_w = \sum_{w \triangleright v} F_v.$$

Edelman and Greene and separately Lascoux and Schützenberger showed the following (significantly harder) result concerning the coefficients $a_{w\lambda}$.

**Theorem 2** ([3] and [22]). The coefficients $a_{w\lambda}$ are nonnegative.

We now give a different formulation of the definition in a manner similar to [7]. Let $\mathbb{C}[S_n]$ denote the group algebra of the symmetric group equipped with an inner product $\langle w, v \rangle = \delta_{wv}$. Define linear operators $u_i : \mathbb{C}[S_n] \to \mathbb{C}[S_n]$ for $i \in [1, n - 1]$ by

$$u_{i,w} = \begin{cases} s_{i,w} & \text{if } l(s_{i,w}) > l(w), \\ 0 & \text{otherwise.} \end{cases}$$

The operators satisfy the braid relations $u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1}$ together with $u_i^2 = 0$ and $u_i u_j = u_j u_i$ for $|i - j| \geq 2$. They generate an algebra known as the nilCoxeter algebra. Note that the action on $\mathbb{C}[S_n]$ is a faithful representation of these relations.

Let $A_k(u) = \sum_{b_1 > b_2 > \cdots > b_k} u_{b_1} u_{b_2} \cdots u_{b_k}$. Then the Stanley symmetric functions can be written as

$$(3) \quad F_w(X) = \sum_{\alpha = (a_1, a_2, \ldots, a_k)} \langle A_{a_k}(u) A_{a_{k-1}}(u) \cdots A_{a_1}(u) \cdot 1, w \rangle x_1^{a_1} x_2^{a_2} \cdots x_t^{a_t}$$

where the sum is over all compositions $\alpha$. The symmetry of $F_w(X)$ is then a consequence of the fact that the $A_k(u)$ commute.

For completeness, we explain briefly the relationship between $F_w(X)$ and the Schubert polynomials of Lascoux and Schützenberger (see [1]). For $w \in S_n$, we have a Schubert polynomial $G_w \in \mathbb{C}[x_1, x_2, \ldots, x_n-1]$. If $w \in S_n$, then $w \times 1^s \in S_{n+s}$ denotes the corresponding permutation of $S_{n+s}$ acting on the elements $[1, n]$ of $[1, n+s]$. Similarly, $1^s \times w \in S_{n+s}$ denotes the corresponding permutation acting on the elements $[s+1, n+s]$ of $[1, n+s]$. Schubert polynomials have the important stability property $G_w = G_{1^s \times w}$. Stanley symmetric functions $F_w(1)$ are obtained by taking the other limit: $F_w = \lim_{s \to \infty} G_{1^s \times w}$. The limit is taken by treating both sides as formal power series and taking the limit of each coefficient.
5. Affine Stanley symmetric functions. Our first definition of affine Stanley symmetric functions will imitate the definition (3) above. Let $U_n$ be the affine nilCoxeter algebra generated over $\mathbb{C}$ by generators $u_0, u_1, \ldots, u_{n-1}$ satisfying
\begin{align*}
u_i^2 &= 0 \quad \text{for all } i \in [0, n-1], \\
u_i u_{i+1} u_i &= u_{i+1} u_i u_{i+1} \quad \text{for all } i \in [0, n-1], \\
u_i u_j &= u_j u_i \quad \text{for all } i, j \in [0, n-1] \text{ satisfying } |i - j| \geq 2.
\end{align*}
Here and henceforth the indices are to be taken modulo $n$. A basis of $U_n$ is given by the elements $u_\rho = u_{\rho_1} u_{\rho_2} \cdots u_{\rho_l}$ where $\rho = (\rho_1, \rho_2, \ldots, \rho_l)$ is some reduced word for $w$ (see [10, Chapter 7]). The element $u_\rho \in U_n$ does not depend on the choice of reduced word $\rho$.

Let $a = a_1 a_2 \cdots a_k$ be a word with letters from $[0, n-1]$ so that $a_i \neq a_j$ for $i \neq j$. Let $A = \{a_1, a_2, \ldots, a_k\} \subset [0, n-1]$. The word $a$ is cyclically decreasing if for every $i$ such that $i, i+1 \in A$, the letter $i+1$ precedes $i$ in $a$. We will call an element $u \in U_n$ cyclically decreasing if $u = u_a = u_{a_1} \cdots u_{a_k}$ for some cyclically decreasing word $a$. If $u$ is cyclically decreasing and $u = u_{a_1} \cdots u_{a_k}$ then necessarily $a = a_1 \cdots a_k$ will be cyclically decreasing. The element $u$ is completely determined by the set $A = \{a_1, a_2, \ldots, a_k\} \subset [0, n-1]$ and we write $u = u_A$. Replacing $u_i$ by $s_i$ we make similar definitions of cyclically decreasing affine permutations for the affine symmetric group.

Define $h_k(u) \in U_n$ for $k \in [0, n-1]$ by
\begin{equation*}h_k(u) = \sum_{A \in \binom{[0, n-1]}{k}} u_A\end{equation*}
where the sum is over subsets of $[0, n-1]$ of size $k$. For example if $n = 9$ and $A = \{0, 2, 4, 5, 6, 8\}$ then $u_A = u_0 u_2 u_4 u_5 u_6 u_8 = u_2 u_4 u_5 u_6 u_0 u_8 = \cdots$. A related formula was given in [27], in the context of the affine nil-Temperley-Lieb algebra. The affine nil-Temperley-Lieb algebra is a quotient of the affine nilCoxeter algebra given by the additional relations $u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1} = 0$.

Define a representation of $U_n$ on $\mathbb{C}[S_n]$ by
\begin{equation*}u_i w = \begin{cases} s_i w & \text{if } l(s_i w) > l(w), \\
0 & \text{otherwise.} \end{cases}\end{equation*}
It is easy to see that this is indeed a representation of $U_n$. If we identify $w \in \mathbb{C}[S_n]$ with $u_w \in U_n$ then this is essentially the left regular representation of $U_n$. Equip $\mathbb{C}[S_n]$ with the inner product $\langle w, v \rangle = \delta_{wv}$. The following definition was heavily influenced by [4].
Definition 3. Let \( w \in \tilde{S}_n \). Define the affine Stanley symmetric functions \( \tilde{F}_w(X) \) by

\[
\tilde{F}_w(X) = \sum_{a=(a_1, a_2, \ldots, a_t)} \langle h_{a_1}(u)h_{a_{t-1}}(u) \cdots h_{a_1}(u) \cdot 1, w \rangle x_1^{a_1}x_2^{a_2} \cdots x_t^{a_t},
\]

where the sum is over compositions of \( l(w) \) satisfying \( a_i \in [0, n-1] \).

The seemingly more general “skew” affine Stanley symmetric functions

\[
\tilde{F}_{w/v}(X) = \sum_{a=(a_1, a_2, \ldots, a_t)} \langle h_{a_1}(u)h_{a_{t-1}}(u) \cdots h_{a_1}(u) \cdot v, w \rangle x_1^{a_1}x_2^{a_2} \cdots x_t^{a_t}
\]

are actually equal to the usual affine Stanley symmetric functions \( \tilde{F}_{w^{-1}}(X) \).

Two properties follow straight from the definition.

Proposition 4. Let \( w \in \tilde{S}_n \). Then the coefficient of \( x_1 x_2 \cdots x_{l(w)} \) in \( \tilde{F}_w(X) \) is equal to the number of reduced words of \( w \).

Proposition 5. Suppose \( w \in S_{w_0} \subset \tilde{S}_n \). Then \( \tilde{F}_w(X) = F_w(X) \).

The main theorem of this section is the following.

Theorem 6. The generating functions \( \tilde{F}_w(X) \in \Lambda(n) \) are symmetric.

Theorem 6 follows immediately from Proposition 8. In the following, intervals \([a, b]\) are to be taken in the cyclic fashion within \([0, n-1]\). Also, max and min of a cyclic interval is meant to be taken modulo \( n \) in the obvious manner. So if \( n = 6 \) then \([4, 1] = \{4, 5, 0, 1\}\) and max \(([4, 1]) = 1\) and min \(([4, 1]) = 4\). We will need a technical lemma first.

Lemma 7. We have the following identities for reduced words.

1. Let \( a, b \in [0, n-1] \) with \( a \neq b - 1 \). Then

\[
a(a-1)(a-2) \cdots ba(a-1)(a-2) \cdots b \sim 0.
\]

2. Let \( a, b, c \in [0, n-1] \) satisfying \( a \neq b - 1; c \neq b \) and \( c \in [b, a] \). Then

\[
a(a-1)(a-2) \cdots bc \sim (c-1)a(a-1)(a-2) \cdots b.
\]

Proof. Both results can be calculated by induction. \( \Box \)

So for example, the element \((s_4s_3s_2)(s_4s_3s_2)\) is not reduced and we have \((s_6s_5s_4s_3s_2s_1)s_4 = s_3(s_6s_5s_4s_3s_2s_1)\).

Proposition 8. The elements \( h_k(u) \) for \( k \in [0, n-1] \) commute.
Proof. For each \( w \in \tilde{S}_n \) satisfying \( l(w) = x + y \), we calculate the coefficient of \( u_w \) in \( h_x(u)h_y(u) \) and \( h_y(u)h_x(u) \). We assume that \( x \) and \( y \) are both not equal to 0 for otherwise the result is obvious. Let \( u_w = u_Au_B \) where \( |A| = x \) and \( |B| = y \).

We need to exhibit a bijection between reduced decompositions of this form and those of the form \( u_w = u_Cu_D \) with \( |C| = y \) and \( |D| = x \). We assume for simplicity (though it is not crucial to our proof) that \( A \cup B = [0, n - 1] \) for otherwise we are in the nonaffine case and the proposition follows from results of Stanley [28] or Fomin-Greene [4]. Let \( A = \bigcup_i A_i \) and \( B = \bigcup_j B_j \) be minimal decompositions of \( A \) and \( B \) into cyclic intervals. If \( A_i \subset B_j \) for some pair \((i, j)\) then we call \( A_i \) an inner interval and similarly for \( B_k \subset A_l \). Otherwise the interval is called outer.

Using Lemma 7 and our assumption that \( A \cup B = [0, n - 1] \) we can describe the outer intervals in an explicit manner. Each outer interval \( A_i \) touches an outer interval \( m(A_i) = B_k \) called the right neighbor of \( A_i \), for a unique \( k \), so that \( \min(A_i) = \max(B_k) + 1 \). Also \( A_i \) overlaps with an outer interval \( \ln(A_i) = B_l \) for a unique \( l \), so that \( \max(A_i) \geq \min(B_l) - 1 \) called the left neighbor. If \( m(A_i) = B_k \) then we also write \( A_i = \ln(B_k) \) and similarly for \( m(B_k) \). Note that it is possible that \( m(A_i) = \ln(A_i) \) since we are working cyclically.

Our bijection will depend only locally on each pair of an outer interval \( A^* \) and its right neighbor \( B^* = m(A^*) \). We call the interval \( I = [\min(B^*), \min(\ln(A^*)) - 1] \) a critical interval. Critical intervals cover \([0, n - 1]\) in a disjoint manner. For example, suppose \( n = 10 \) and \( A = \{1, 2, 3, 6, 7, 8, 9\} \) and \( B = \{0, 2, 3, 4, 5, 7, 9\} \) (Figure 1), so that \( u_{A\cup B} = u_{0u_8u_7u_6u_5u_3u_2u_1u_0u_9u_7u_5u_4u_2u_0} \). Then \( A_1 = [1, 3] \) and \( A_2 = [6, 9] \) are both outer intervals. Also \( B_1 = [2, 5] \), \( B_2 = [7] \) and \( B_3 = [9, 0] \). Only \( B_2 \) is an inner interval. The left neighbor of \( A_1 \) is \( \ln(A_1) = B_1 \) and the right neighbor is \( m(A_1) = B_3 \). The critical intervals are \([9, 1]\) and \([2, 8]\).

Let \( a = \min(\ln(A^*)) - 1 \) and \( b = \min(B^*) \). Let \( c = |[b, a]| \), \( d = |A \cap [b, a]| \) and \( e = |B \cap [b, a]| \). Renaming for convenience, we let \( S_1, S_2, \ldots, S_r \) be the inner intervals (of \( B \)) contained in \( A^* \) and \( T_1, \ldots, T_t \) be those contained in \( B^* \), arranged so that \( S_k > S_{k+1} \) for all \( k \) within \([b, a]\) and similarly \( T_k > T_{k+1} \). We now define a subset \( U \subset [b, a] \) satisfying \( |U| = d \). The algorithm begins with \( U = [b, a] \) and a changing index \( i \) set to \( i := a \) to begin with. The index \( i \) decreases from \( a \) to \( b \) and at each step the element \( i \) may be removed from \( U \) according to the rule:

1. If \( i \in A^* \) then we remove it from \( U \) unless \( i \in S_k \) for some \( k \in [1, r] \).
2. If \( i \in B^* \) then we remove it from \( U \) unless \( i \in T_k + 1 \) for some \( k \in [1, t] \).
3. Otherwise we do not remove \( i \) from \( U \) and set \( i := i - 1 \). Repeat.

When \( |U| = d \) we stop the algorithm. The algorithm always terminates with \( |U| = d \) since there are at least \( c - d = |[b, a]| - (A \cap [b, a]) \) elements to remove. In fact the algorithm terminates before \( i = b \) since \( \bigcup_i S_i \neq A^* \cap I \). We will denote the result of the algorithm by \( \phi(A^* \cup_i T_i, B^* \cup_i S_i) := U \). Note that \( \min(U) = b \).

The bijection \( u_{A\cup B} \leftrightarrow u_{C\cup D} \) is obtained by letting \( D \subset [0, n - 1] \) be the subset obtained from \( B \) by changing \( B \cap I \) in each critical interval \( I \) to \( U \). By the definition of \( U \) we see that \( |D| = |A| \). We claim that \( u_{A\cup B} = u_{C\cup D} \) or alternatively \( s_{A\cup B}(s_D)^{-1} = s_C \) for some \( C \) satisfying \( |C| = |B| \) (here it is slightly more
It is convenient to calculate within the affine symmetric group, which is legal since our words are all reduced. We can calculate this locally on each critical interval since the $s_{D^r I}$ commute as $I$ varies over critical intervals. Note that $U$ always has the form of a disjoint union $S_1 \cup S_2 \cup \cdots \cup S_r \cup [b, a']$ for some $r' \leq r$ where $a' > \max (B^*)$ or the form $S_1 \cup \cdots \cup S_r \cup \{T_1 + 1\} \cup \{T_2 + 1\} \cup \cdots \cup \{T_{r'} + 1\} \cup [b, a']$ where $a' \leq \max (B^*)$.

Let us assume that $U$ has the first form. Focusing on $I = [b, a] = [\min (B^*), \min (\ln(A^*)) - 1]$ we are interested in

$$\xi = s_{A^* \cap J} S_{T_1} \cdots S_{T_r} S_{S_1} \cdots s_{S_r} s_{B^*(s_{[b, a']})^{-1}} (s_{S_r})^{-1} \cdots (s_{S_1})^{-1}.$$

Then we get

$$\xi = s_{A^* \cap J} S_{T_1} \cdots S_{T_r} S_{S_1} \cdots s_{S_r} (s_{[\max (B^*) + 1, a'])^{-1}$$

$$= s_{S_r}^{-1} \cdots s_{S_1}^{-1} \cdots s_{T_r} S_{A^* \cap J} (s_{[\max (B^*) + 1, a'])^{-1}$$

$$= s_{S_r}^{-1} \cdots s_{S_1}^{-1} \cdots s_{T_r} [a'+1, a] \text{ using } \max (B^*) + 1 = \min (A^*).$$

We used Lemma 7 repeatedly and also the fact that certain intervals do not “touch” and so commute. Let $U'$ be the disjoint union $[a' + 1, a] \cup \{S_{r+1} - 1\} \cup \cdots \cup \{S_s - 1\} \cup T_1 \cup \cdots T_r$. Note that it is always the case that $\max (U') = a$. The other form of $U$ involves a similar calculation. One checks that we can combine this argument for each critical interval showing that $s_A s_B (s_D)^{-1}$ is indeed equal to $s_C$ for some $C$.

Finally, we need to show that this map is a bijection. Again we work locally on a critical interval and assume that $U$ has the first form. If we replace $A^*$ (more precisely $A^* \cap I$) by $U'$ and $B^*$ by $U$, then our internal intervals are $S'_1 = S_1, \ldots , S'_r = S_r$ and $T'_1 = S_{r+1} - 1, \ldots , T'_{r-r'} = S_r - 1, T'_{r-r'+1} = T_1, \ldots , T'_{r-r'+r} = T_r$. We now show that $B^* \cup S_i = \phi(U', U)$ from which the bijectivity will follow. Note that since $\min (U) = b$ and $\max (U') = a$ the critical intervals of $u_{A^*B}$ are the same as those of $u_{A^*B}$. By definition $\phi(U', U)$ keeps $S'_1, S'_2, \ldots$ and keeps $T'_1 + 1, T'_2 - 1, \ldots , T'_{r-r'} + 1$, removing all other values up to this point. At this point the algorithm stops since $\phi(U', U)$ is of the correct size. We see that we obtain $\phi(U', U) = B^* \cup S_i$ back in this way. A similar argument works for the second form of $U$.

Example 1. We illustrate the map $U = \phi(A^* \cup_i T_i, B^* \cup_i S_i)$ of the proof. Suppose $[b, a] = [2, 20]$ and $A^* = [14, 20]$, $B^* = [2, 13]$. Let $S_1 = [16, 18]$ and $T_1 = [8, 11]$ and $T_2 = \{5\}$ be the inner intervals. Then $d = 12$ and $U = \{2, 3, 4, 5, 6, 9, 10, 11, 12, 16, 17, 18\}$. We can compute that

$$s_{A^*} s_{[11, 15]} s_{[9, 15]} s_{[5, 15]}$$

$$s_{B^*} s_{[15, 18]} s_{[15, 17]} s_{[15, 16]} s_{[15, 15]}$$

so that $U' = [7, 20] \cup \{5\}$. Finally one checks that $B^* \cup_i S_i = \phi(U', U)$. 

---

**AFINE STANLEY SYMMETRIC FUNCTIONS 1563**
We end this section by giving two alternative descriptions of the affine Stanley symmetric functions, the first one imitating the original definition of Stanley. Let \( w \in \tilde{S}_n \). Let \( a = (a_1, \ldots, a_l) \in R(w) \) be a reduced word and \( b = (b_1 \geq b_2 \cdots \geq b_l) \) be an positive integer sequence. Then \( (a, b) \) is called a compatible pair for \( w \) if whenever \( b_i = b_{i+1} = \cdots = b_j \) and \( \{k, k+1\} \subset \{a_i, a_{i+1}, \ldots, a_j\} \) then we have that \( k + 1 \) precedes \( k \) (for any \( i, j, k \)). Two compatible pairs \( (a, b) \) and \( (a', b') \) are equivalent if \( b = b' \) and for any maximal interval \([i, j] \subset [1, l] \) satisfying \( b_i = b_{i+1} = \cdots = b_j \) we have that \( a_i a_{i+1} \cdots a_j \) and \( a'_i a'_{i+1} \cdots a'_j \) are reduced words for the same affine permutation. The following proposition is clear from the definitions.

**Proposition 9 (Alternative Definition 1).** The affine Stanley symmetric functions are given by

\[
\tilde{F}_w(X) = \sum_{(a, b)} x_{b_1} x_{b_2} \cdots x_{b_l}
\]

where the sum is over equivalence classes \( (a, b) \) of compatible pairs for \( w \).

Now let \( w \in \tilde{S}_n \) be of length \( l \) and suppose \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r) \) is a composition of \( l \). An \( \alpha \)-decomposition of \( w \) is an ordered \( r \)-tuple of cyclically decreasing affine permutations \( (w^1, w^2, \ldots, w^r) \in \tilde{S}_n^r \) satisfying \( l(w^r) = \alpha_l \) and \( w = w^1 w^2 \cdots w^r \). The following alternative definition is also immediate.

**Proposition 10 (Alternative Definition 2).** The affine Stanley symmetric function \( \tilde{F}_w(X) \) is given by

\[
\tilde{F}_w(X) = \sum_{\alpha} \text{(number of } \alpha \text{-decompositions of } w) \cdot x^\alpha
\]

where the sum is over all compositions \( \alpha \) of \( l \).

6. **Representations of the affine nilCoxeter algebra.** Let \( \mathcal{V} \) be a complex representation of \( \mathcal{U}_n \) with a distinguished basis \( \{v_p \mid p \in P\} \) for some indexing set \( P \). Let \( \langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V} \to \mathbb{C} \) be the inner product defined by \( \langle v_p, v_q \rangle = \delta_{pq} \) for
For any \( p, q \in P \) one can define \( V \)-affine Stanley symmetric functions by
\[
\tilde{F}_{q/p}(X) = \sum_{a=(a_1, a_2, \ldots, a_t)} \langle h_{a_1}(u) h_{a_{t-1}}(u) \cdots h_{a_1}(u) \cdot v_p, v_q \rangle x_1^{a_1} x_2^{a_2} \cdots x_t^{a_t},
\]
where the sum is over compositions of \( l(w) \) satisfying \( a_i \in [0, n-1] \). By Proposition 8 these functions are indeed symmetric functions.

**Proposition 11.** Suppose \( u_w \cdot v_p = v_q \) and \( w \in \tilde{S}_n \) is the only affine permutation such that \( \langle u_w \cdot v_p, v_q \rangle \neq 0 \). Then \( \tilde{F}_{q/p}(X) = \tilde{F}_w(X) \).

**Proof.** For each composition \( a = (a_1, a_2, \ldots, a_t) \) expand \( \hat{u}^{a_1} \hat{v}^{a_{t-1}} \hat{u}^{a_1} \) in the basis \( \{u_v\} \) of \( U_n \). Using the assumption, the proposition follows immediately upon comparison with Definition 3. \( \square \)

More generally, for arbitrary \( v_p \) and \( v_q \) let \( c_w = \langle u_w \cdot v_p, v_q \rangle \). Then \( \tilde{F}_{q/p} = \sum_{w \in \tilde{S}_n} c_w \tilde{F}_w \). We have not found any interesting generating functions of this form.

If in addition \( u_i \) acts on the basis \( \{v_p\}_{p \in P} \) with nonnegative matrix coefficients, then \( \tilde{F}_{p/q} \) will be monomial-positive. This will be the case for all the representations of \( \mathcal{U}_n \) that we will be considering.

**7. Coproduct formula.** We now give the analogue of part (1) of Theorem 1.

**Theorem 12 (Coproduct formula).** The following coproduct expansion holds:
\[
\tilde{F}_w(x_1, x_2, \ldots, y_1, y_2, \ldots) = \sum_{u \uparrow w: l(u) + l(v) = l(w)} \tilde{F}_u(x_1, x_2, \ldots) \tilde{F}_v(y_1, y_2, \ldots).
\]

In particular we have
\[
s_1^{1/2} \tilde{F}_w = \sum_{w \uparrow v} \tilde{F}_v.
\]

**Proof.** The first formula follows immediately from the definition and the fact that \( \tilde{F}_{w/\nu}(Y) = \tilde{F}_{w^{-1}}(Y) \). To obtain the second formula, we first write, using the first formula and (1),
\[
\sum_{u \uparrow w} \tilde{F}_u(X) \otimes \tilde{F}_v(Y) = \sum_\lambda s_\lambda^{1/2}(X) \tilde{F}_w(X) \otimes s_\lambda(Y).
\]

The terms of the formula are to be interpreted within \( \Lambda \), even though the sum is an element of \( \Lambda^{(n)} \). Now take the inner product of both sides with \( s_1(Y) \) to get
\[
s_1^{1/2}(X) \tilde{F}_w(X) = \sum_{u \uparrow w} \tilde{F}_u(X) \langle \tilde{F}_u(Y), s_1(Y) \rangle.
\]
Now \( \langle \tilde{F}_u(Y), s_1(Y) \rangle = 0 \) unless \( u = s_i \) is a simple reflection for some \( i \), in which case \( \tilde{F}_u(Y) = s_1(Y) \). This gives the second formula. \( \blacksquare \)

8. Monomial dominance. We now show that there is a dominant term in the monomial expansion of an affine Stanley symmetric function \( \tilde{F}_w(X) \). Let \( c'(w) = c(w^{-1}) \) denote the code of the inverse \( w^{-1} \) of \( w \), so that \( c'_{w(i)} = \# \{ j : j < i \text{ and } w(j) > w(i) \} \). Let \( \mu(w) \) denote the partition which is conjugate to the decreasing permutation of \( c'(w) \).

**Theorem 13.** Let \( w \in \tilde{S}_n \). Then

1. If \([m_\lambda] \tilde{F}_w \neq 0\) then \( \lambda \preceq \mu(w) \).
2. We have \([m_\mu(w)] \tilde{F}_w = 1\).

**Proof.** Left multiplication of \( w \) by \( s_i \) acts on \( c'(w) \) by

\[
s_i : (c'_1, \ldots, c'_{i-1}, c'_i, c'_{i+1}, \ldots, c'_n) \mapsto (c'_1, \ldots, c'_{i-1}, c'_i + 1, c'_i, \ldots, c'_n)
\]

whenever \( l(s_iw) > l(w) \). Applying a term of \( h_k(u) \) to \( w \) will increase \( k \) different entries of \( c'(w) \) by 1 and also permute the entries (assuming the result is nonzero), since \( u_i \) never acts after \( u_{i+1} \). Using this repeatedly we see that if \( m_\lambda \) occurs in \( \tilde{F}_w \), we must have \( \mu_1(w) \geq \lambda_1 \) and then \( \mu_1(w) + \mu_2(w) \geq \lambda_1 + \lambda_2 \) and so on. So \( \lambda \preceq \mu(w) \).

Now we check that the coefficient of \( X^{\mu(w)} \) in \( \tilde{F}_w(X) \) is 1. To see this, we work by going down in the Bruhat order or equivalently, acting on \( w \) by \( h_k(u)^\perp \) (the adjoint with respect to \( \langle \cdot, \cdot \rangle \) of \( h_k(u) \)). Multiplying \( w \) by a term of \( h_{\mu_1(w)}(u)^\perp \) means decreasing \( \mu_1(w) \) different entries of \( c'(w) \) by 1 each (and also permuting the entries in some way). But \( c'(w) \) only has \( \mu_1(w) \) nonzero entries, and so there is only one possible resulting code \( c'(v) \): it is obtained from \( c'(w) \) by taking all nonzero entries \( c'_i \) and shifting them each to the right (cyclically) one entry. This is because entries can only decrease (by 1) by shifting to the right, and once such an entry is shifted we are forbidding it from moving again. Now the conjugate of the decreasing permutation of \( c'(v) \) is exactly \( (\mu_2(w), \mu_3(w), \ldots) \) so our result follows from induction. \( \blacksquare \)

**Corollary 14.** The subalgebra \( \Lambda_{(n)}(u) \) generated of \( U_n \) by \( \{ h_k(u) \}_{k=1}^{n-1} \) is isomorphic to \( \Lambda_{(n)} \) with isomorphism given by \( h_i \mapsto h_i(u) \) for \( 1 \leq i \leq n - 1 \).

**Proof.** Suppose to the contrary that the \( h_k(u) \) are not algebraically independent. Then there is some relation \( h_\lambda(u) = \sum_\nu a_\nu h_\nu(u) \) where we may pick \( \lambda \) so that no \( \nu \) appearing on the right hand side satisfies \( \nu \prec \lambda \). Now pick \( w \) so that \( \mu(w) = \lambda \). Then by Theorem 13, \( 1 = \langle h_\lambda(u) \cdot 1, w \rangle = \sum_\nu a_\nu \langle h_\nu(u) \cdot 1, w \rangle = 0 \), a contradiction. \( \blacksquare \)

We denote by \( f(u) \) the image of \( f \in \Lambda_{(n)} \) under the isomorphism \( \Lambda_{(n)} \cong \Lambda_{(n)}(u) \).
9. **Conjugacy.** Define $\omega : \Lambda_{(n)} \to \Lambda_{(n)}$ as usual by $\omega : h_i \mapsto e_j$. Define $\omega^+ : \Lambda^{(n)} \to \Lambda^{(n)}$ by requiring that $\langle \omega(f), \omega^+(g) \rangle = \langle f, g \rangle$ where $f \in \Lambda_{(n)}$ and $g \in \Lambda^{(n)}$. Alternatively, we require that the sets $\{e_{\lambda} | \lambda \in \text{Par}^n\}$ and $\{\omega^+(m_\lambda) | \lambda \in \text{Par}^n\}$ form dual bases of $\Lambda_{(n)}$ and $\Lambda^{(n)}$. The map $\omega^+$ is clearly an involution but it does not agree with $\omega$ (see for example [29, Chapter 7, Ex. 9]).

Denote by $w \mapsto w^*$ the involution of $\tilde{S}_n$ given by $s_i \mapsto s_{n-i}$ (with $s_0 \mapsto s_0$). In terms of the window realization of $\tilde{S}_n$, we have $[w(1), w(2), \ldots, w(n)]^* = [n+1-w(n), n+1-w(n-1), \ldots, n+1-w(1)]$. Similarly, $u_i \mapsto u_{n-i}$ defines an algebra involution (also denoted $^*$) of $U_n$.

**Theorem 15 (Conjugacy formula).** Let $w \in \tilde{S}_n$. Then $\omega^+(\tilde{F}_w) = \tilde{F}_w^*$.

We shall prove Theorem 15 by calculating within the subalgebra $\Lambda_{(n)}(u)$ of Corollary 14. The following result says that $e_k(u) = (h_k(u))^*$.

**Proposition 16.** The elements $e_k(u) \in U_n$ are given by

$$e_k(u) = \sum_{A \in \binom{[0,n-1]}{k}} \tilde{u}_A,$$

where for a $k$-subset $A = \{a_1, a_2, \ldots, a_k\} \subset [0, n-1]$ the element $\tilde{u}_A \in U_n$ is defined as any expression $u_{a_1}u_{a_2}\cdots u_{a_k}$ where if $i$ and $i+1$ (modulo $n$) are both in $A$ then $u_i$ must precede $u_{i+1}$ within $\tilde{u}_A$.

**Proof.** We verify this using the relation

$$(4) \quad e_k(u) = h_k(u) - h_{k-1}(u)e_1(u) + \cdots \pm h_1(u)e_{k-1}(u).$$

First, we restrict our attention to the monomials which only involve the set of generators $\{u_1, u_2, \ldots, u_{n-1}\}$. Then one may write

$$h_k(u) = \sum_{n-1 \geq \{a_1 > a_2 > \cdots > a_k\} \geq 1} u_{a_1}u_{a_2}\cdots u_{a_k}$$

and we assume that

$$e_l(u) = \sum_{n-1 \geq \{a_1 < a_2 < \cdots < a_l\} \geq 1} u_{a_1}u_{a_2}\cdots u_{a_l}$$

is known for $l < k$. (The base case $k = 1$ is clear.) Now for $k > l \geq 1$, $h_{k-l}(u)e_l(u)$ can be written as $A_l + B_l$ where

$$A_l = \sum_{n-1 \geq \{a_1 > a_2 > \cdots > a_{k-l} < a_{k-l+1} < \cdots < a_k\} \geq 1} u_{a_1}u_{a_2}\cdots u_{a_k}$$

and

$$B_l = \sum_{n-1 \geq \{a_1 > a_2 > \cdots > a_{k-l} < a_{k-l+1} \geq 1 < \cdots < a_k\} \geq 1} u_{a_1}u_{a_2}\cdots u_{a_k}.$$
and
\[ B_l = \sum_{n-1 \geq \{a_1, a_2, \ldots, a_k-1 \} > a_k \geq \cdots > a_{k-l+1} \geq \cdots \geq a_k} \prod_{i} u_{a_i}. \]

Note that \( h_k(u) = B_1 \) and for \( k > l \geq 1 \), we have \( A_l = B_{l+1} \) so all but one of the terms on the right hand side of (4) cancel to give \( e_k(u) = A_{k-1} \), which is the desired formula. This proves the theorem when the monomials are restricted to \( \{u_1, u_2, \ldots, u_{n-1}\} \). But since \( k \leq n-1 \), any monomial \( u_w \) in (4) only involves a proper subset of the generators \( \{u_0, u_1, u_2, \ldots, u_{n-1}\} \), so we can calculate the coefficient of that monomial in \( e_k(u) \) by setting \( u_i = 0 \) for some \( i \). The theorem follows.

More generally, when \( \lambda = (a, 1^b) \) is a hook shape satisfying \( s_\lambda \in \Lambda(n) \) then \( s_\lambda(u) \) can be written as a sum over the reading words of certain tableaux (see [13]). We shall not need this generality; however, see Proposition 42.

Proof of Theorem 15. Write the affine noncommutative Cauchy kernel
\[ \Omega^{(n)}(x, u) := \sum\lambda \in \text{Par}^n h_\lambda(u)m_\lambda(X) = \sum\lambda \in \text{Par}^n e_\lambda(u)\omega^+(m_\lambda(X)) \]

where the second equality follows from the definition of \( e_\lambda(u) \) and an argument similar to [29, Lemma 7.9.2].

By definition \( \tilde{F}_w(X) = \langle \Omega^{(n)}(x, u) \cdot 1, w \rangle = \sum\lambda \in \text{Par}^n \langle e_\lambda(u) \cdot 1, w \rangle \omega^+(m_\lambda(X)). \) By Theorem 16, \( e_\lambda(u) \) is obtained from \( h_\lambda(u) \) by the involution \( u_i \mapsto u_{n-1-i} \), so \( \langle e_\lambda(u) \cdot 1, w \rangle = \langle h_\lambda(u) \cdot 1, w^* \rangle. \) This completes the proof of the theorem.

For later use, we have the following proposition.

**Proposition 17.** Let \( w \in \tilde{S}_n \). Then \( \tilde{F}_{w^{-1}} = \tilde{F}_w \).

**Proof.** The reduced words of \( w^{-1} \) are obtained by reversing the reduced words of \( w \). But each term of \( e_\lambda(u) \) is also obtained from a term of \( h_\lambda(u) \) by reversing the order of the generators. This shows that \( \langle h_\lambda(u) \cdot 1, w^{-1} \rangle = \langle e_\lambda(u) \cdot 1, w \rangle = \langle h_\lambda(u) \cdot 1, w^* \rangle. \)

Let \( \mathbb{Z}/n\mathbb{Z} \) act on \( \tilde{S}_n \) by the action \( p.s_i = s_{i+p} \) for \( p \in \mathbb{Z}/n\mathbb{Z} \) on the simple generators. Since the definition of \( h_\lambda(u) \) is invariant under the analogous transformations of \( U_n \), we have the following symmetry of affine Stanley symmetric functions:

**Proposition 18.** Let \( w \in \tilde{S}_n \) and \( p \in \mathbb{Z}/n\mathbb{Z} \). Then \( \tilde{F}_w = \tilde{F}_{p.w} \).

**10. Affine Schur functions.** A permutation \( w \in \tilde{S}_n \) is Grassmannian (or more precisely left-Grassmannian) if it is a minimal length coset representative
for a coset of \( S_n \setminus \hat{S}_n \) where \( S_n \cong S_n \) is the maximal parabolic subgroup generated by the \( n - 1 \) generators \( s_1, \ldots, s_{n-1} \). By general facts concerning parabolic subgroups of Coxeter groups [10], the minimal length coset representative \( \bar{w} \) of a coset \(( S_n \cong S_n )\ w \) is unique and satisfies \( l(u\bar{w}) = l(u) + l(\bar{w}) \) for any \( u \in S_n \). There is a natural correspondence between the minimal length coset representatives corresponding to another embedding of \( S_n \) into \( \hat{S}_n \), and the ones we have called Grassmannian. In particular the associated affine Stanley symmetric functions are equal under this correspondence so we will only consider the Grassmannian permutations.

A permutation \( w \) is Grassmannian if left multiplication by \( s_i \) always increases the length \( l(w) \). This is equivalent to \( c'(w) \) being a weakly increasing sequence, or equivalently, that the window \( [w^{-1}(1), w^{-1}(2), \ldots, w^{-1}(n)] \) of \( w^{-1} \) is increasing. In fact, the correspondence \( w \leftrightarrow \mu(w) \) is a bijection between Grassmannian permutations and \( \text{Par}^n \) (see [2]).

**Definition 19.** An affine Stanley symmetric function \( \tilde{F}_w(X) \) is called an affine Schur function if \( w \) is a Grassmannian permutation. If \( \mu = \mu(w) \), we write \( \tilde{F}_\mu(X) := \tilde{F}_w(X) \).

Affine Schur functions had earlier been defined by Lapointe and Morse in a different manner, and were called dual \( k \)-Schur functions. We will see the origin of this name later.

**Theorem 20.** The affine Schur functions \( \{ \tilde{F}_\mu : \mu \in \text{Par}^n \} \) form a basis of \( \Lambda^{(n)} \).

**Proof.** By Theorem 13, \( \tilde{F}_\mu = \sum_{\lambda \preceq \mu} b_{\mu \lambda} m_\lambda \) for some coefficients \( b_{\mu \lambda} \in \mathbb{Z} \) satisfying \( b_{\mu \mu} = 1 \). Since the transition matrix between \( \{ \tilde{F}_\mu \} \) and \( \{ m_\lambda \} \) is uni-triangular, the theorem follows.

Now define \( a_{w\lambda} \in \mathbb{Z} \) by

\[
\tilde{F}_w(X) = \sum_{\lambda \in \text{Par}^n} a_{w\lambda} \tilde{F}_\lambda(X).
\]

The fact that the coefficients \( a_{w\lambda} \) are integers follows from the fact that the transition matrix between \( \{ \tilde{F}_\mu \} \) and \( \{ m_\lambda \} \) is uni-triangular with integer coefficients, together with the fact that the monomial expansion of \( \tilde{F}_w \) has integer coefficients. Let \( \tilde{f}_\lambda = [x_1,x_2, \ldots, x_{\ell(\mu)}] \tilde{F}_\lambda(X) \) be the number of reduced decompositions of the Grassmannian permutation \( u \) satisfying \( \mu(u) = \lambda \). Thus for any \( w \in \hat{S}_n \) we have

\[
\# R(w) = \sum_{\lambda} a_{w\lambda} \tilde{f}_\lambda.
\]

In fact we conjecture that \( a_{w\lambda} \geq 0 \); see Section 17. In the nonaffine case, the numbers \( \tilde{f}_\lambda \) are dimensions of irreducible representations of the symmetric group and are given by the well-known hook length formula; see [29]. It is unknown
whether a closed formula for $\tilde{f}_\lambda$ exists in the affine case, though $\tilde{f}_\lambda$ does count the number of certain tableaux, known as $k$-tableaux; see Section 13.

Let $w$ be a Grassmannian permutation. Then since the involution $*: \tilde{S}_n \to \tilde{S}_n$ sends $S_n^{\circ}$ to $S_n^{\circ}$, the permutation $w^*$ is also a Grassmannian permutation. We thus obtain an involution $*: \text{Par}_n \to \text{Par}_n$ given by requiring that $\mu(w^*) = \mu(w^*)$ for Grassmannian permutations $w$. Combining this with Theorem 15 we obtain

\begin{equation}
\omega^*(\tilde{F}_\lambda) = \tilde{F}_{\lambda^*}.
\end{equation}

Let $v$ be a minimal coset representative of a right coset in $\tilde{S}_n / S_n^{\circ}$ (a right-Grassmannian permutation). Since $v$ is the inverse of some Grassmannian permutation, by Proposition 17, the associated affine Stanley symmetric function $\tilde{F}_v$ is equal to an affine Schur function so in fact we have lost no generality considering the left-Grassmannian permutations instead of the right-Grassmannian permutations.

The involution $*$ on $\text{Par}_n$ has been studied in a different form in [16] where it is called $k$-conjugation. Define the partial order $\prec^*$ on $\text{Par}_n$ by $\lambda \prec^* \mu$ if and only if $\mu^* \prec \lambda^*$. The partial order $\prec^*$ is not the same as $\prec$. For example $(2, 2)$ and $(2, 1, 1)$ are both fixed points of $*$ for $n = 3$ (the author thanks J. Morse for this example).

Let $\lambda(w) = \mu(w^{-1})^*$ (note that $c(w^{-1})$ and $c(w^*)$ are rearrangements of each other so that $\mu(w^{-1}) = \mu(w^*)$).

**Theorem 21 (Dominant Terms).** Let $w \in \tilde{S}_n$. Then

1. If $a_{w\lambda} \neq 0$ then $\lambda(w) \preceq^* \lambda \preceq \mu(w)$.
2. We have $a_{w\mu(w)} = a_{w\lambda(w)} = 1$.

**Proof.** The statements involving $\mu(w)$ follow from Theorem 13 and the comments earlier. Applying this to $w^{-1}$, we have

$$\tilde{F}_{w^{-1}}(X) = \tilde{F}_{\mu(w^{-1})} + \sum_{\lambda \prec \mu(w^{-1})} a_{w^{-1}\lambda} \tilde{F}_{\lambda}(X).$$

Applying $\omega^*$ to both sides and using Theorem 15, Proposition 17 and (5) we get

$$\tilde{F}_{w}(X) = \tilde{F}_{\lambda(w)} + \sum_{\lambda(w) \prec^* \lambda^*} a_{w^{-1}\lambda} \tilde{F}_{\lambda^*}(X)$$

which implies the other statements of the theorem.

We end this section with a question: for which $w \in \tilde{S}_n$ is $\mu(w^*) = \mu(w^*)$? Is it the same as the class of permutations $w \in \tilde{S}_n$ such that $\tilde{F}_w$ is equal to an affine Schur function? See also Problem 1.
11. Affine symmetric group and \( n \)-cores. We now describe an action of the affine symmetric group on partitions. Further details for the material of this section can be found in [23], [20].

A \( n \)-ribbon is a connected skew shape \( \lambda/\mu \) of size \( n \) which contains no \( 2 \times 2 \) square. A partition \( \lambda \) is an \( n \)-core if no \( n \)-ribbon \( \lambda/\mu \) can be removed from it to obtain another partition \( \mu \). Let \( \mathcal{P}^n \) denote the set of \( n \)-cores.

If \( \lambda \) is a partition, we let \( p(\lambda) \) denote the edge sequence of \( \lambda \). The edge sequence \( p(\lambda) = (\ldots, 1, 1, 0, 1, 0, 0, 1, 0, 0, \ldots) \) is the doubly infinite bit sequence obtained by drawing the partition in the English notation and reading the “edge” of the partition from bottom left to top right – writing a 1 if you go up and writing a 0 if you go to the right (see Figure 2). We shall normalize our notation for edge sequences by requiring that the empty partition \( \emptyset \) has edge sequence \( p(\emptyset)_{i} = 1 \) for \( i \leq 0 \) and \( p(\emptyset)_{i} = 0 \) for \( i \geq 1 \). Adding a box to a partition corresponds to changing two adjacent entries of the edge sequence \( p_{i}, p_{i+1} \) from \( (0, 1) \) to \( (1, 0) \).

Adding a \( n \)-ribbon to a partition \( \lambda \) corresponds to finding an index \( i \in \mathbb{Z} \) such that \( p_i(\lambda) = 1 \) and \( p_{i+n}(\lambda) = 0 \), then changing those two bits to \( p_i(\lambda) = 0 \) and \( p_{i+n}(\lambda) = 1 \).

Let \( \lambda \) be an \( n \)-core with edge sequence \( p(\lambda) = (\ldots, p_{-2}, p_{-1}, p_0, p_1, p_2, \ldots) \). Then there is no index \( i \) so that \( p_i(\lambda) = 0 \) and \( p_{i+n}(\lambda) = 1 \). Equivalently, the subsequences

\[
p^{(i)}(\lambda) = (\ldots, p_{i-2n}, p_{i-n}, p_i, p_{i+n}, p_{i+2n}, \ldots)
\]

all look like \((\ldots, 1, 1, 1, 0, 0, 0, 0, \ldots)\) with a suitable shift. Define the offsets \( \{d_i = d_i(\lambda) \mid i \in \mathbb{Z} \} \) by requiring that \( p_{i+nd_i} = 0 \) and \( p_{i+nd_{i-1}} = 1 \). The offsets satisfy \( d_{i-n} = d_{i} + 1 \) and \( d_{i} + d_{i+1} + \cdots + d_{n} = 0 \) and completely determine the \( n \)-core.

Now let \( \mathcal{P} \) denote the set of doubly infinite \((0,1)\)-sequences \( p=(\ldots, p_{-2}, p_{-1}, p_0, p_1, p_2, \ldots) \) and let \( \mathbb{C}[\mathcal{P}] \) denote the space of formal \( \mathbb{C} \)-linear combinations of such sequences. Let \( \tilde{S}_n \) act on \( \mathcal{P} \) by letting \( s_i \) act on \( p = (\ldots, p_{-2}, p_{-1}, p_0, p_1, p_2, \ldots) \) by swapping \( p_{kn+i} \) and \( p_{kn+i+1} \) for each \( k \in \mathbb{Z} \). One can check directly that this defines a representation of \( \tilde{S}_n \) on \( \mathbb{C}[\mathcal{P}] \).
A sub-representation $\mathbb{C}[\mathcal{P}^*]$ of $\mathbb{C}[\mathcal{P}]$ is given by taking only those bit sequences $p \in \mathcal{P}^*$ satisfying $p_N = 1$ for sufficiently small $0 \gg N$ and $p_N = 0$ for $N \gg 0$. These sequences correspond to possibly shifted edge sequences of partitions. It is easy to see that $\mathbb{C}[\mathcal{P}^*]$ is indeed a sub-representation, but it is by no means irreducible. We let $\tilde{S}_n$ act on partitions by the corresponding action on the edge sequences. The action of $s_i \in \tilde{S}_n$ acts by adding and or removing boxes along certain diagonals.

The proof of the following proposition is straightforward.

**Proposition 22.** The orbit $\tilde{S}_n \cdot \emptyset$ is equal to the set of $n$-cores. Let $\lambda$ be an $n$-core with offsets $d_i(\lambda)$. Then $\mu = s_i \cdot \lambda$ is an $n$-core with offsets $d_i(\mu) = d_i(\lambda)$ for $j \neq i, i+1$ and $d_{i+1}(\mu) = d_{i+1}(\lambda)$ and $d_i(\mu) = d_{i+1}(\lambda)$. If $d_i(\lambda) > d_{i+1}(\lambda)$ then boxes are added; if $d_i(\lambda) < d_{i+1}(\lambda)$ then boxes are removed and if $d_i(\lambda) = d_{i+1}(\lambda)$ then $\lambda = \mu$.

One can see (for example using Proposition 22) that the stabilizer of the empty partition is $\tilde{S}_n^\emptyset \subset \tilde{S}_n$, so the set $\mathcal{P}^n$ of $n$-cores is naturally isomorphic to $\tilde{S}_n^\emptyset / S_n^\emptyset$. We may thus identify $n$-cores with right-Grassmannian permutations – the set $S_n^\emptyset$ of minimal length coset representatives of $\tilde{S}_n / S_n^\emptyset$. If $w \in S_n^\emptyset$ satisfies $w \cdot \emptyset = \lambda \in \mathcal{P}^n$ then we write $w = w(\lambda)$.

The following relation between the $n$-cores and the affine symmetric group is known (see [20]).

**Proposition 23.** Let $\lambda, \mu \in \mathcal{P}^n$ be $n$-cores. Then $\lambda \subset \mu$ if and only if $w(\lambda)$ is less than $w(\mu)$ in (strong) Bruhat order.

The action of $\tilde{S}_n$ on $\mathcal{P}^n$ corresponds to the left action of $\tilde{S}_n$ on $\tilde{S}_n^\emptyset / S_n^\emptyset$. We will need the following general fact for Coxeter groups.

**Lemma 24.** Let $W$ be a Coxeter group, $W_I$ a parabolic subgroup and $W_I^I$ a the set of minimal length coset representatives of $W / W_I$. Let $w \in W_I^I$ and $s_i$ be a simple generator. Then either $s_i w \in W_I^I$ or $s_i w \in w W_I$.

**Proof.** Let $l(w) = l$. Suppose that $s_i w = uv$ for $v \in W_I^I$ and $u \in W_I$. By [10, Proposition 1.10], we have $l(s_i w) = l(uv) = l(v) + l(u)$. But we also have $wu^{-1} = s_i v$ so that $l(s_i v) = l(w) + l(u)$. Suppose first that $l(s_i w) = l - 1$. Then $l(v) = l - 1 - l(u)$ and $l + l(u) = l(s_i v) \leq l - l(u)$ which implies that $u = 1$ so $s_i w \in W_I^I$. Now suppose that $l(s_i w) = l + 1$. Then we have $l - l(u) \leq l + l(u) \leq l + 2 - l(u)$ with equality holding for exactly one inequality. If $l - l(u) = l + l(u)$ then again we have $u = 1$. Otherwise, $l(u) = 1$.

In the last case we have $l(v) = l(w)$ and $l(s_i w) = l(s_i v) = l + 1$. Let $u = s_r$ for some simple generator $s_r$. By the Strong Exchange Condition ([10, Theorem 5.8]), $v = (s_i w) s_r$ is obtained from $s_i w$ by taking a reduced word $s_i s_{a_1} \cdots s_{a_l}$ of $s_i w$ and omitting one generator. If that simple generator is the first $s_i$ then $v = w$ and we are done. Otherwise $v = s_i s_{a_1} \cdots \hat{s}_{a_j} \cdots s_{a_l}$ where $\hat{s}_{a_j}$ denotes omission. But then it is clear that $l(s_i v) = l - 1$, a contradiction. \qed
12. Skew affine Schur functions. The action of \( \hat{S}_n \) on the set of \( n \)-cores induces another representation of \( \mathcal{U}_n \). Let \( \mathcal{U}_n \) act on \( \mathbb{C}[P^n] \) by

\[
 u_i \cdot \nu = \begin{cases} 
 s_i \cdot \nu & \text{if } s_i \cdot \nu \text{ is obtained from } \nu \text{ by adding boxes.} \\
 0 & \text{otherwise.}
\end{cases}
\]

The fact that this defines an action of \( \mathcal{U}_n \) is easy to verify. In fact we have

**Proposition 25.** The above action of \( \mathcal{U}_n \) on \( \mathbb{C}[P^n] \) is isomorphic to the action of \( \mathcal{U}_n \) on \( \mathbb{C}[S_n] \) where for \( w \in S_n \) we define

\[
 u_i \cdot w = \begin{cases} 
 s_i \cdot w & \text{if } s_i w \in S_n^{\phi} \text{ and } l(s_i w) > l(w) \\
 0 & \text{otherwise.}
\end{cases}
\]

Thus the action of \( \mathcal{U}_n \) on \( \mathbb{C}[S_n^{\phi}] \) is obtained from the action on \( \mathbb{C}[\hat{S}_n] \) by setting to 0 all elements \( w \notin S_n^{\phi} \). The isomorphism is given by identifying \( \lambda \in P^n \) and \( w(\lambda) \in P^n \).

More generally, one can define an action of \( \mathcal{U}_n \) on \( \mathbb{C}[S_I] \) for other parabolic subgroups \( S_I \) of \( \hat{S}_n \).

**Proof.** It is straightforward to check that the formulae of the proposition do define a representation of \( \mathcal{U}_n \) on \( \mathbb{C}[S_n^{\phi}] \). By Proposition 22, for \( \nu \in P_n \) the \( n \)-core \( s_i \cdot \nu \) is always obtained from \( \nu \) by either adding boxes or removing boxes or doing nothing. Let \( w = w(\nu) \). Then by Lemma 24 applied to \( W = \hat{S}_n \) and \( W^I = S_n^{\phi} \), we have either \( s_i w = w(\mu) \) for some \( \mu = s_i \cdot \nu \) or \( s_i w \in w S_n^{\phi} \). In the latter case, \( s_i \cdot \nu = \nu \). In the former case, using Proposition 23, adding boxes corresponds to the case that \( l(s_i w) > l(w) \).

Equip \( \mathbb{C}[P^n] \) with the inner product \( \langle \nu, \mu \rangle = \delta_{\nu \mu} \).

**Definition 26.** Let \( \mu \subset \nu \) be two \( n \)-cores such that there is some \( w \in \hat{S}_n \) satisfying \( u_w \cdot \mu = \nu \). The skew affine Schur function \( \hat{F}_{\nu/\mu}(X) \) is given by

\[
 \hat{F}_{\nu/\mu}(X) = \sum_{a=(a_1,a_2,\ldots,a_t)} \langle h_{a_1}(u)h_{a_2}(u)\cdots h_{a_t}(u) \cdot \mu, \nu \rangle x_1^{a_1} x_2^{a_2} \cdots x_t^{a_t}.
\]

Suppose \( \mu, \nu \in P^n \). Then using Proposition 25 there is at most one permutation \( w \) satisfying \( u_w \cdot \mu = \nu \). If \( \nu \) and \( u \) are right-Grassmannian permutations corresponding to \( \mu \) and \( \nu \) then \( w \) is given by \( w = u \nu^{-1} \) assuming that \( l(w) + l(\nu) = l(u) \). By Proposition 11, we have \( \hat{F}_{\nu/\mu} = \hat{F}_w \) so that skew affine Schur functions are special cases of affine Stanley symmetric functions. We write \( w = w(\nu/\mu) \). It is not true that \( l(w) \) is equal to the number of boxes in \( \nu/\mu \), since the action of \( u_i \) may add more than one box. It is also not true that some \( w \) exists satisfying \( u_w \cdot \mu = \nu \) for every pair of \( n \)-cores \( \mu \subset \nu \) where containment is as subsets of the
Figure 3. A $k$-tableau with shape $(8, 5, 2)(1)$ and weight $(3, 2, 2)$. Here $n = 4$.

plane. For example, $(2, 1, 1) \subset (5, 3, 1)$ and both are 3-cores but such a $w$ does not exist.

When $\mu = \emptyset$, the permutation $w(\nu) = w(\nu/\emptyset)$ is right-Grassmannian as defined earlier. In this case, the skew affine Schur function $\tilde{F}_{\nu/\emptyset}$ is an affine Schur function. We write $\psi$: $\text{Par}^n \to P^n$ for the bijection satisfying $\tilde{F}_{\lambda} = \tilde{F}_{\psi(\lambda)/\emptyset}$.

13. Cores and $k$-tableaux. One can view the skew affine Schur function $\tilde{F}_{\nu/\mu}$ as the generating function for certain semistandard tableaux built on $n$-cores. These tableaux are called $k$-tableaux (with $k = n - 1$) by Lapointe and Morse [18]. A (semistandard) $k$-tableau of shape $\nu/\mu$ and weight $w(T) = (a_1, a_2, a_3, \ldots, a_l)$ is a chain of partitions $\mu = \nu^{(0)} \subset \nu^{(1)} \subset \cdots \subset \nu^{(l)} = \nu$ such that

1. Each partition $\nu^{(i)}$ is an $n$-core.
2. The successive differences $\nu^{(i)}/\nu^{(i-1)}$ contain at most one box in each column. That is they are horizontal strips.
3. The contents $c(i, j) = j - i$ of the boxes $(i, j) \in \nu^{(i)}/\nu^{(i-1)}$ involve exactly $a_i$ different residues $\{r_1, r_2, \ldots, r_{n-1}\}$ modulo $n$ and $\nu^{(i)}$ has no addable-corners with content congruent to one of these residues $r_j$.

When a $k$-tableau is drawn, the boxes of $\nu^{(i)}/\nu^{(i-1)}$ are filled with the number $i$. We have (see also [18])

**Proposition 27.** Let $\mu \subset \nu$ be two $n$-cores such that there is some $w \in \tilde{S}_n$ satisfying $u_w \cdot \mu = \nu$. Then

$$\tilde{F}_{\nu/\mu}(X) = \sum_T X^{w(T)}$$

where the sum is over all $k$-tableaux of shape $\nu/\mu$.

**Proof.** The $n$-cores $\nu^{(i)}$ of a $k$-tableau are obtained by successive applications of terms of $h_{a_i}(u)$. Thus $\nu^{(i)} = u_{A_i} \cdot \nu^{(i-1)}$ for some term $u_{A_i}$ in $h_{a_i}(u)$. This is equivalent to the description of $k$-tableaux given above. Condition (2) in the definition comes from the fact that $u_{i+1}$ always precedes $u_i$ in the definition of $h_i(u)$ so that a box on a diagonal congruent to $i$ modulo $n$ is never added after a box on a diagonal congruent to $i+1$ modulo $n$. Condition (3) follows from the description (in Proposition 22) of the action of $u_i$ on a $n$-core, which adds all possible boxes along diagonals with residue $i$. The set $A_i$ is exactly the set of residues $\{r_1, r_2, \ldots, r_{n-1}\}$. \qed
If \( \lambda \) is a partition fitting inside a \( m \times (n - m) \) box for some \( m \) then at most \( n - 1 \) diagonals are involved in \( \lambda \) and necessarily \( \lambda \in \mathcal{P}^n \). Within the \( m \times (n - m) \) box, only at most one box is added by the action of \( s_i \). In this case the definition of a \( k \)-tableau reduces to a usual semistandard Young tableau. The following is then immediate.

**Proposition 28.** Let \( \lambda \subseteq ((n - m)^m) \) for some \( 1 \leq m \leq n - 1 \). Then \( \lambda \in \text{Par}^n \cap \mathcal{P}^n \) and \( \tilde{F}_\lambda/\emptyset = \tilde{F}_\lambda = s_\lambda \).

**14. Affine Schur and \( k \)-Schur functions.** We now describe the relationship between affine Schur functions and the \( k \)-Schur functions \( \{s^{(k)}(X; t)\} \) (with \( k = n - 1 \)). The \( k \)-Schur functions \( \{s^{(k)}(X; t)\} \) were originally used to investigate Macdonald polynomial positivity. Let \( H_\mu(X; q, t) \) be given by the plethystic substitution \( H_\mu(X; q, t) = J_\mu(X/(1 - t); q, t) \) where \( J_\mu(X; q, t) \) is the integral form of Macdonald polynomials [24]. Let \( K^{(k)}_\mu(q, t) \) and \( \pi^{(k)}_\mu(t) \) be given by

\[
H_\mu(X; q, t) = \sum_\nu K^{(k)}_\mu(q, t)s^{(k)}_\nu(X; t) \quad ; \quad s^{(k)}_\nu(X; t) = \sum_\lambda \pi^{(k)}_\lambda(t)s_\lambda(X).
\]

Then it is conjectured that \( K^{(k)}_\mu(q, t) \in \mathbb{N}[q, t] \) and \( \pi^{(k)}_\mu(t) \in \mathbb{N}[t] \) which would refine the (proven) “Macdonald positivity conjecture” that the Schur expansion of \( H_\mu(X; q, t) \) has coefficients in \( \mathbb{N}[q, t] \); see [9].

There are a number of different definitions of \( k \)-Schur functions [16], [17] which conjecturally agree. The definition of the \( k \)-Schur functions that we will use is from [18] and is (conjecturally) the \( t = 1 \) specialisations of the original definitions but are usually still called \( k \)-Schur functions. Suppose \( \tilde{F}_\lambda(X) = \sum_\mu K^{(n)}_{\lambda\mu}m_\mu \) where \( \lambda \in \text{Par}^n \) and the sum is over \( \mu \in \text{Par}^n \). Then using Proposition 27 and the results of [18], the \( k \)-Schur functions \( s^{(k)}_\lambda(X) \in \Lambda(n) \) are given by requiring that

\[
h_\mu(X) = \sum_\lambda K^{(n)}_{\lambda\mu}s^{(k)}_\lambda(X).
\]

This definition is called the \( k \)-Pieri rule. The following result is also established in [19].

**Proposition 29.** Affine Schur functions and \( k \)-Schur functions are dual bases of \( \Lambda(n) \) and \( \Lambda^{(n)} \), so that \( \langle s^{(k)}_\mu, \tilde{F}_\nu \rangle = \delta_{\mu\nu} \).

**Proof.** Write the affine Cauchy kernel

\[
\Omega^{(n)}(X, Y) = \sum_{\mu: \mu \in \text{Par}^n} h_\mu(X)m_\mu(Y) = \sum_{\mu: \mu \in \text{Par}^n} \left( \sum_{\lambda: \lambda \in \text{Par}^n} K^{(n)}_{\lambda\mu}s^{(k)}_\lambda(X) \right) m_\mu(Y)
\]
\[
\sum_{\lambda: \lambda \in \text{Par}^n} s^{(k)}_{\lambda}(X) \left( \sum_{\mu: \mu \in \text{Par}^n} K^{(n)}_{\lambda \mu} m_\mu(Y) \right) = \sum_{\lambda: \lambda \in \text{Par}^n} s^{(k)}_{\lambda}(X) \tilde{F}_\lambda(Y),
\]
which is equivalent to duality. \qed

15. Cylindric Schur functions. In [27], Postnikov introduced and studied cylindric Schur functions, which he showed were symmetric functions; see also closely related work of Gessel and Krattenthaler [8]. Postnikov studied a special subset of the cylindric Schur functions in finitely many variables which he called toric Schur polynomials. He showed that the expansion coefficients of toric Schur polynomials in the basis of Schur polynomials were equal to 3-point genus 0 Gromov-Witten invariants \( C^d_{\lambda \mu \nu} \) of the Grasmannian \( \text{Gr}_{m,n} \), where \( \lambda, \mu \) and \( \nu \) are partitions contained in a \( m \times (n-m) \) box and \( |\lambda| + |\mu| + |\nu| = dn + m(n-m) \). The Gromov-Witten invariant \( C^d_{\lambda \mu \nu} \) counts the number of maps \( f: \mathbb{P}^1 \to \text{Gr}_{m,n} \) whose image has degree \( d \) and meets generic translates of the Schubert varieties \( \Omega_\lambda, \Omega_\mu \) and \( \Omega_\nu \) at three marked points \( p_1, p_2, p_3 \in \mathbb{P}^1 \). In particular, these coefficients are positive. They are the multiplicative constants of the (small) quantum cohomology ring \( \text{QH}^*(\text{Gr}_{m,n}) \) of the Grassmannian.

In general cylindric Schur functions do not expand positively in terms of Schur functions. See [25] for a detailed discussion of this.

A cylindric shape \( \lambda \) is an infinite lattice path in \( \mathbb{Z}^2 \), consisting only of moves upwards and to the right, invariant under the translation by a vector \( (n-m, -m) \) for some \( m \in [1, n-1] \). We denote the set of such cylindric shapes by \( C^{n,m} \). If \( \lambda, \mu \in C^{n,m} \) are cylindric shapes so that \( \mu \) always lies weakly to the left of \( \lambda \), then \( \lambda/\mu \) is a cylindric skew shape. We write \( \mu \subset \lambda \).

Definition 30. A cylindric semi-standard tableau of shape \( \lambda/\mu \) and weight \( a = (a_1, a_2, \ldots, a_l) \) is a chain \( \mu = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(l)} = \lambda \) of cylindric shapes in \( C^{n,k} \) such that each \( \lambda^{(i)}/\lambda^{(i-1)} \) is a cylindric skew shape with at most one box in each column and \( a_i \) boxes in any \( n-m \) consecutive columns.

When we draw a cylindric semi-standard tableau, we place the number \( i \) into the boxes of \( \lambda^{(i)}/\lambda^{(i-1)} \). The columns will then be strictly increasing and the rows weakly increasing (see Figure 4).

Definition 31. Let \( \lambda/\mu \) be a cylindric skew shape. Then the cylindric Schur function \( s^C_{\lambda/\mu} \) is given by
\[
s^C_{\lambda/\mu}(X) = \sum_T X^T
\]
where the sum is over all cylindric tableau \( T \) of shape \( \lambda/\mu \).

One can alternatively define cylindric Schur functions in the same way as skew affine Schur functions by letting \( U_n \) act on infinite bit sequences \( p = \)
(\ldots, p_{-2}, p_{-1}, p_0, p_1, p_2, \ldots) satisfying the periodicity condition $p_i = p_{i+n}$. It is clear that periodic bit sequences are closed under the action of $\tilde{S}_n$ and in fact form $n+1$ finite orbits depending on the value of $m = p_1 + p_2 + \cdots + p_n \in [0, n]$.

If $\lambda \in C^{n,m}$ is a cylindric shape then $s_i \cdot \lambda$ is the cylindric shape obtained from $\lambda$ by either adding boxes at all corners along diagonals congruent to $i \mod n$, or removing such boxes, or doing nothing. Define $u_i : C[C^{n,m}] \rightarrow C[C^{n,m}]$ by

$$u_i \cdot \lambda = \begin{cases} s_i \cdot \lambda & \text{if } s_i \cdot \lambda \text{ is obtained from } \lambda \text{ by adding boxes.} \\ 0 & \text{otherwise.} \end{cases}$$

This defines a representation of $U_n$ on $C[C^{n,m}]$, and equipping $C[C^{n,m}]$ with the natural inner product one can check directly using the definition of cylindric semistandard tableaux that for $\mu \subset \lambda \in C^{n,m}$ the function $\tilde{F}_\lambda^\mu$ given by

$$\tilde{F}_\lambda^\mu(X) = \sum_{\alpha = (a_1, a_2, \ldots, a_t)} \langle h_{a_1}(u)h_{a_{i-1}}(u) \cdots h_{a_1}(u) \cdot \mu, \lambda \rangle x_1^{a_1}x_2^{a_2} \cdots x_t^{a_t}$$

is equal to the cylindric Schur function $s_\lambda^\mu(X)$.

**Lemma 32.** Suppose $\lambda$ and $\mu$ are cylindric shapes. Then there is at most one $w \in \tilde{S}_n$ satisfying $u_w \cdot \mu = \lambda$.

**Proof.** Suppose $u$ and $w$ satisfy $u_w \cdot \mu = \lambda$ and $u_v \cdot \mu = \lambda$. Let $u_w = u_{a_1}u_{a_2} \cdots u_{a_t}$ and $u_v = u_{b_1}u_{b_2} \cdots u_{b_t}$. We may assume that $a_1 \neq b_1$, otherwise we can reduce to a smaller case by letting $\mu := u_{a_1} \cdot \mu$. So let the rightmost occurrence of $a_i = u_{a_i}$ in $u_{b_1}u_{b_2} \cdots u_{b_t}$ be $u_{b_i}$. The cylindric shape $\mu$ must have an addable corner along the $i$-th diagonal so in particular none of $u_{b_{i+1}}, u_{b_{i-2}}, \ldots, u_{b_i}$ is equal to $u_{i+1}$ or $u_i$ and we can move $u_{b_i}$ to the right most position to get another reduced word for $u_v$, and then reduce to a smaller case.\]

By Proposition 11, cylindric Schur functions are thus also special cases of affine Stanley symmetric functions. In fact more is true.
Proposition 33. Every cylindric Schur function $\tilde{F}_{\lambda/\mu}^c$ is a skew affine Schur function.

Proof. A generalized $n$-core is an infinite bit sequence obtained from the edge sequence of a $n$-core by shifting the indexes. So if $p(\theta) = (\ldots, p_{-2}, p_{-1}, p_0, p_1, \ldots)$ is the edge sequence of a $n$-core $\theta$ then the sequence $q = (\ldots, q_{-2}, q_{-1}, q_0, q_1, \ldots)$ given by $q_i := p_{i+k}$ defines a generalized $n$-core. Equivalently, generalized $n$-cores are in bijection with offset sequences $\{d_i \mid i \in \mathbb{Z}\}$ satisfying $d_{i-n} = d_i + 1$.

Let $w \in S_n$ satisfy $u_w \cdot \nu = \lambda$. We show first that there are generalized $n$-cores $\nu, \rho$ such that $u_w \cdot \nu = \rho$, which immediately implies $\tilde{F}_{\lambda/\mu}^c = \tilde{F}_{\rho/\nu}$ (the definition of $\tilde{F}_{\rho/\nu}$ for generalized $n$-cores is the obvious one). The edge sequence $p(\nu)$ is obtained from $p(\mu)$ by setting $p_N = 0$ for $N \geq n \cdot (l(w) + 1)$ and $p_N = 1$ for $N \leq -(n \cdot (l(w) + 1))$. Since it is clear that $\nu$ is a generalized $n$-core, $\rho = u_w \cdot \nu$ is also a generalized $n$-core as long as it is nonzero.

So the “central” part of $p(\nu)$ looks the same as $p(\mu)$ and the action of $U_n$ on the central part is identical. An entry of the bit sequence is moved no more than one step for each action by a simple generator, so in total it is moved no further than $l(w)$ from its initial position. The alteration of $p(\nu)$ is thus sufficiently far away from the center that the altered bits cannot affect whether a box is added at each step of the action of the simple generators of $w$ on $\nu$. For the action of some $u_i$ to be nonzero we need only ensure that $s_i$ adds a box somewhere to the shape.

Finally, if $\nu$ and $\rho$ are two generalized $n$-cores with the same “shift” given by $d_1(\nu) + d_2(\nu) + \cdots + d_n(\nu) = d_1(\rho) + d_2(\rho) + \cdots + d_n(\rho)$ then one can shift again to find genuine $n$-cores $\nu^*$ and $\rho^*$ so that $\tilde{F}_{\rho/\nu} = \tilde{F}_{\rho^*/\nu^*}$.

16. 321-avoiding permutations.

Definition 34. An affine permutation $w \in \tilde{S}_n$ is 321-avoiding if no reduced word for $w$ contains a subsequence of the form $i(i+1)i$.

When $w \in S_n$, this definition is the same as $w$ “avoiding” the pattern 321, as shown in [1]. We can extend this naturally to the affine case.

Proposition 35. An affine permutation $w \in \tilde{S}_n$ is 321-avoiding if and only if there do not exist indices $x < y < z \in \mathbb{Z}$ such that $w(x) > w(y) > w(z)$.

Proof. Suppose first that some reduced word for $w$ contains a subsequence of the form $i(i+1)i$, so that $w = u_{i;i+1;i}$. Recall that $w_{i} > w$ if and only if $w(i) < w(i+1)$. Let $v' = u_{i;i+1;i}$. Since the word is reduced, we must have $v(i) < v(i+1) < v(i+2)$ and $v'(i) > v'(i+1) > v'(i+2)$. But since multiplying by each simple generator in $u$ increases the length of the permutation, the 3 integers $a = v'(i), b = v'(i+1)$ and $c = v'(i+2)$ will never be swapped past each other again. So there are indices $x < y < z$ such that $w(x) = a, w(y) = b$ and $w(z) = c$.

Conversely, suppose $w$ has three indices $x < y < z$ so that $w(x) > w(y) > w(z)$. We may assume that there is no index $t$ in the open interval $(x, y)$ such that
$w(t) > w(y)$ for otherwise we can replace $x$ by $t$. Similarly, there is no $r$ in $(y, z)$ so that $w(r) < w(y)$. Now if $x < y - 1$, we multiply $w$ by $s_k$ on the right where $x$ is to be taken modulo $n$ as usual. Let $w' = ws_x$. Since $w(x) > w(x + 1)$, we have $l(w') = l(w) - 1$. Also note that if $w'(z) \neq w(z)$ then we have $w'(z - 1) = w(z)$. This is because $w(z) < w(x)$ and $z > x$ so it is not possible that $z = x + kn$ for some $k \in \mathbb{Z}$. Similarly, $w'(y)$ can only have been moved to the left compared to $w(y)$, so that it is never moved past $w(z)$. So there are indices $x + 1 = x' < y' < z'$ so that $w'(x') > w'(y') > w'(z')$. Furthermore $z' - x' < z - x$. Repeating this (also with the roles of $z$ and $x$ swapped) we eventually obtain $w'' \in S_n$ and $y'' \in Z$ so that $w''(y'' - 1) > w''(y'') > w''(y'' + 1)$. Clearly, $w''$ is not 321-avoiding and since at each step going from $w$ to $w''$ the length is reduced, some reduced word for $w$ contains a reduced word for $w''$ as a subword. This shows that $w$ is not 321-avoiding.

**Theorem 36.** Let $w \in S_n$ be 321-avoiding. Then $\tilde{F}_w$ is equal to a cylindric Schur function (and thus by Proposition 33 also a skew affine Schur function).

**Proof.** We proceed by induction on $l = l(w)$, the case $l(w) = 1$ being trivial. So assume $w = s_1 \cdot v$ with $l(v) = l(w) - 1$ and that $u_v \cdot \lambda = \nu$ for cylindrical shapes $\lambda, \nu$. Pick a reduced word $p = \rho_1 \rho_2 \cdots \rho_l - 1$ for $v$. Pick $k$ minimal so that $\rho_k = i$, if such a $k$ exists. Then since $w$ is 321-avoiding, we must have unique $x, y < k$ satisfying $\rho_x = i + 1$ and $\rho_y = i - 1$. We claim that $s_i \cdot \nu$ is obtained from $\nu$ by adding boxes. This is clear since after applying $s_{i+1}$ and $s_{i-1}$, the shape $\nu$ must have edge sequence satisfying $p_i(\nu) = 0$ and $p_{i+1}(\nu) = 1$.

If no such $k$ exists and $p_i(\nu) = 1$ or $p_{i+1}(\nu) = 0$ then in the first case $i - 1$ does not occur in $\rho$ and $p_i(\nu) = p_i(\mu)$. In the second case $i + 1$ does not occur in $\rho$ and $p_{i+1}(\nu) = p_{i+1}(\mu)$. In either or both cases, we let $\lambda$ be the cylindrical skew shape obtained from $\mu$ by setting $p_i(\nu) = 0$ and $p_{i+1}(\nu) = 1$ (and keeping the rest of the edge sequence the same). Then it is clear that $u_w \cdot \lambda \neq 0$ so that $\tilde{F}_w = \tilde{F}_{(u_w \cdot \lambda)/\lambda}$.

If $\lambda$ and $\mu$ are cylindrical shapes satisfying $u_w \cdot \mu = \lambda$ then $w$ is necessarily 321-avoiding. In fact, the action of $U_n$ on cylindrical shapes always satisfies the additional relation $u_i u_i u_i = u_i u_i u_i = 0$. However, this is not true for $n$-cores. For example, let $n = 3$ and $\mu = (1)$. Let $w = s_1 s_2 s_3 = s_2 s_1 s_2$. Then $w \cdot \mu = (3, 1, 1)$. This shows that skew affine Schur functions are considerably more complicated than cylindrical Schur functions. In fact more is true:

**Proposition 37.** There exists $\mu \in \mathcal{P}_n$ so that for each $w \in S_n$, there is a $n$-core $\lambda$ so that $F_w = \tilde{F}_{\lambda/\mu}$.

**Proof.** We can pick $\mu$ to be any $n$-core with offsets satisfying $d_1(\mu) < d_2(\mu) < \cdots < d_n(\mu)$. Then by Proposition 22, $u_w \cdot \mu \neq 0$ so that $F_w = \tilde{F}_{w, \mu/\mu}$.

**17. Positivity.** We conjecture that affine Schur functions generalize Schur functions for Stanley symmetric function positivity (Theorem 2).
Conjecture 38. The affine Stanley symmetric functions $\tilde{F}_w(X)$ expand positively in terms of the affine Schur functions $\tilde{F}_\lambda(X)$.

This conjecture seems to be consistent with all the known behavior of $k$-Schur functions and cylindric Schur functions.

It has been conjectured [16], [17] that the multiplicative constants $d_{\nu\mu}^\lambda$ for $k$-Schur functions given by

$$s_k^{(k)}(\nu) s_k^{(k)}(\mu) = \sum_{\lambda \in \text{Par}_n} d_{\nu\mu}^\lambda s_{\lambda}^{(k)}$$

are nonnegative. In [19], it is shown that the coefficients $d_{\nu\mu}^\lambda$ include the multiplicative constants of the Verlinde algebra of $U(m)$ at level $n-m$.

Proposition 39. Conjecture 38 implies $d_{\nu\mu}^\lambda \geq 0$.

Proof. Using Proposition 29, together with equation (2), we have

$$d_{\nu\mu}^\lambda = \langle s_k^{(k)}(\nu), \tilde{F}_\lambda \rangle = \langle s_k^{(k)}(\mu) \otimes s_k^{(k)}, \Delta \tilde{F}_\lambda \rangle.$$  

But $\Delta \tilde{F}_\lambda = \sum_{\rho \subset \lambda} \tilde{F}_{\lambda/\rho} \tilde{F}_\rho$ where the sum is over $\rho \in \text{Par}_n$ such $\psi(\rho) \subset \psi(\lambda)$ as $n$-cores, and such that $\tilde{F}_{\lambda/\mu} := \tilde{F}_{\psi(\lambda)/\psi(\mu)}$ is defined (the bijection $\psi$: $\text{Par}_n \to \mathcal{P}_n$ was defined in Section 12). Using Proposition 29 again, we have $d_{\nu\mu}^\lambda = \langle s_k^{(k)}, \tilde{F}_{\lambda/\mu} \rangle$ which would be positive if Conjecture 38 is true. \qed

Call a cylindric skew shape $\nu/\mu$ where $\nu, \mu \in C_{n,m}$ toric if the toric Schur polynomial $\tilde{F}_{\nu/\mu}(x_1, x_2, \ldots, x_m)$ is nonzero [27]. Let

$$\tilde{F}_{\nu/\mu}(x_1, \ldots, x_m) = \sum_{\lambda} C_{\nu/\mu}^\lambda s_{\lambda}(x_1, \ldots, x_m)$$

where the sum is over all partitions $\lambda \subset ((n-m)^m)$. Postnikov showed that the coefficients $C_{\nu/\mu}^\lambda$ were certain Gromov-Witten invariants of $G_{n,m}$. Informally, by restricting the cylindric shapes $\nu, \mu \in C_{n,m}$ to certain $m$ consecutive rows, one obtains partitions $\tilde{\nu}, \tilde{\mu} \subset ((n-m)^m)$. In the notation of Section 15, we have $C_{\nu/\mu}^\lambda = C_{\tilde{\nu}/\tilde{\mu}}^\lambda$ where $\lambda^\nu \subset ((n-m)^m)$ obtained from $\lambda$ by taking the complement of $\lambda$ within $((n-m)^m)$ and rotating the resulting shape and $d$ can be calculated from $\tilde{\nu}, \tilde{\mu}$ and $\lambda^\nu$. These coefficients are known to be nonnegative from their geometric definition, but a combinatorial proof is still lacking.

Proposition 40. Conjecture 38 implies that $C_{\nu/\mu}^\lambda \geq 0$.

Proof. Postnikov showed that the only Schur polynomials $s_{\lambda}(x_1, \ldots, x_m)$ which appear in the Schur expansion of $\tilde{F}_{\nu/\mu}(x_1, \ldots, x_m)$ satisfy $\lambda \subset ((n-m)^m)$. By Proposition 28, these must be exactly the affine Schur functions which occur in the affine Schur expansion of $\tilde{F}_{\nu/\mu}(x_1, \ldots, x_m)$. \qed
See also McNamara’s work on cylindric Schur positivity [25].

**Remark 41.** By Proposition 33 and the proof of Proposition 39, the coefficients \( C_{\nu/\mu}^{\lambda} \) are special cases of multiplication coefficients for \( k \)-Schur functions. It is known [30] that the Verlinde algebra of \( U(m) \) at level \( n - m \) agrees with quantum cohomology of \( Gr_{m,n} \) at \( q = 1 \). Thus our work shows that on the one hand the connection between toric Schur functions and quantum cohomology and on the other hand the connection between \( k \)-Schur functions and the Verlinde algebra are equivalent.

Since \( s^{(k)}_{\lambda}(X) \in \Lambda_{(n)} \) we have an element \( s^{(k)}_{\lambda}(u) \in U_n \) (as before \( k = n - 1 \)). The following proposition is inspired by the paper of Fomin-Greene [4].

**Proposition 42.** Let \( c_{w,\lambda} \in \mathbb{Z} \) be given by

\[
s^{(k)}_{\lambda}(u) = \sum_{w \in S_n} c_{w,\lambda} u_w.
\]

Then \( c_{w,\lambda} = a_{w,\lambda} \) where \( a_{w,\lambda} \) is the coefficient of \( \tilde{F}_{\lambda} \) in \( \tilde{F}_w \).

**Proof.** We compute using the (noncommutative) affine Cauchy kernel that

\[
\tilde{F}_w(X) = \sum_{\lambda \in \text{Par}^n} \langle h_{\lambda}(u) \cdot 1, w \rangle m_{\lambda}(X) = \langle \Omega^{(n)}(x,u) \cdot 1, w \rangle
\]

\[
= \sum_{\lambda \in \text{Par}^n} \langle s^{(k)}_{\lambda}(u) \cdot 1, w \rangle \tilde{F}_{\lambda}(X).
\]

Thus the coefficient of \( \tilde{F}_{\lambda} \) in \( \tilde{F}_w \) is equal to \( c_{w,\lambda} \).

Thus Conjecture 38 is equivalent to \( c_{w,\lambda} \geq 0 \): every noncommutative \( k \)-Schur function can be expressed as a nonnegative sum of monomials in \( \{u_0, u_1, \ldots, u_{n-1}\} \). When \( \lambda \) is contained in some \((n - m) \times m\) box, then the \( k \)-Schur function \( s^{(k)}_{\lambda} \) is actually the Schur function \( s_{\lambda} \) [17]. If in fact \( |\lambda| \leq n - 1 \), then by restricting to proper subsets of the generators \( \{u_0, u_1, \ldots, u_{n-1}\} \) (like in Proposition 16) one can give a positive monomial formula for \( s_{\lambda}(u) \) in terms of reading words of tableaux using the results of [4] on noncommutative Schur functions. This for example gives combinatorial interpretations of some Gromov-Witten invariants corresponding to very small shapes. However, it is likely that such combinatorial interpretations are easily obtained from existing results.

18. Final comments.

18.1. Which affine Stanley symmetric functions are Schur, skew Schur or cylindric? In [1], the question of which Stanley symmetric functions equalled a skew Schur function was studied. As Proposition 37 indicates, the corresponding
problem for affine Stanley symmetric functions may well be more difficult. We call an affine permutation \( w \) affine vexillary (respectively skew affine vexillary or cylindric vexillary) if \( \tilde{F}_w \) is equal to some affine Schur function (respectively some skew affine Schur function or cylindric Schur function).

**Problem 1.** Which affine permutations are affine vexillary, skew affine vexillary and cylindric vexillary?

For example, Theorem 36 shows that all 321-avoiding permutations are cylindric vexillary. It is not clear whether \( \mu(w) = \lambda(w) \) implies that \( w \) is vexillary, in the notation of Section 10. The corresponding statement is true for usual permutations and follows from part (2) of Theorem 1.

Cylindric Schur and affine skew Schur functions arise from representations of \( U_n \) on different sets of infinite bit sequences. It would be interesting to find other sets of infinite bit sequences which are closed under the action of \( \tilde{S}_n \) and to define actions of \( U_n \) on them.

### 18.2. The affine flag variety, quantum cohomology and fusion ring

The connections with \( k \)-Schur functions and with cylindric Schur functions indicate that affine Stanley symmetric functions are important objects.

Our results show directly that \( k \)-Schur functions and cylindric Schur functions are related. In some cases, this was already known if we combine Postnikov’s work on cylindric Schur functions and Gromov-Witten invariants of the Grassmannian with Lapointe and Morse’s work showing that multiplication \( k \)-Schur functions calculate the multiplication in the fusion ring. Finally it is known that the fusion ring agrees with the quantum cohomology \( QH^*(Gr_{m,n}) \) of the Grassmannian at \( q = 1 \) ([30]). These connections suggest that there may be an interesting \( q \)-analogue of our theory. It is not clear whether the \( q \)-analogue in quantum cohomology is related to the \( t \)-analogue of the original \( k \)-Schur functions \( s^{(k)}(X; t) \) arising from Macdonald polynomial theory.

However, the most interesting direction to take seems to be the connections with the affine flag variety (type \( A \)). Shimozono has conjectured that the multiplication of \( k \)-Schur functions calculate the homology multiplication of the affine Grassmannian. The dual conjecture is that affine Schur functions represent the Schubert classes in the cohomology of the affine Grassmannian [26], also discussed in [19]. Recently, these conjectures have been established by the author [15]. Extending these conjectures from the affine Grassmannian to the affine flag variety would involve defining afford Schubert polynomials which should in some sense be “unstable” versions of affine Stanley symmetric functions.

### 18.3. A dual version of \( \tilde{F}_w \)

We have shown that affine Schur functions \( \tilde{F}_\lambda \) are dual to the \( k \)-Schur functions \( s^{(k)}(X) \). The \( k \)-Schur functions are conjectured to be Schur positive [17] (in [16] the Schur positivity is part of the definition).
Define the dual affine Stanley symmetric function $\tilde{F}_w^d$ by

$$
\tilde{F}_w^d(X) = \sum_{\lambda} a_{w,\lambda} \tilde{F}_{\lambda}(X)
$$

where as before $a_{w,\lambda}$ is given by $\tilde{F}_w = \sum_{\lambda} a_{w,\lambda} \tilde{F}_{\lambda}$. If Conjecture 38 is true as well as the Schur positivity of $k$-Schur functions, then $\tilde{F}_w^d$ would be Schur positive. If so, is it the character of a natural $S_m$ or $GL(N)$ module?

### 18.4. Affine stable Grothendieck polynomials.

Whereas Schubert polynomials are representatives for the cohomology of the flag variety, Grothendieck polynomials are representatives for the K-theory of the flag variety. In the same way that Stanley symmetric functions are stable Schubert polynomials, one can define stable Grothendieck polynomials. Our definition of affine Stanley symmetric functions naturally generalizes to a definition of affine stable Grothendieck polynomials (see [4] or [5]).

Let $\tilde{U}_n$ be the algebra obtained from $U_n$ by replacing the relation $u_i^2 = 0$ with $u_i^2 = u_i$. Define $\tilde{h}_k(u) \in \tilde{U}_n$ for $k \in [0, n-1]$ with the same formula as for $h_k(u)$.

**Definition 43.** Let $w \in \tilde{S}_n$. The affine stable Grothendieck polynomial $\tilde{G}_w(X) \in \Lambda^{(n)}$ is

$$
\tilde{G}_w(X) = \sum_{a=(a_1, a_2, \ldots, a_t)} \langle \tilde{h}_{a_t}(u)\tilde{h}_{a_{t-1}}(u)\cdots \tilde{h}_{a_1}(u) \cdot 1, w \rangle x_1^{a_1} x_2^{a_2} \cdots x_t^{a_t},
$$

where the sum is over compositions of $l(w)$ satisfying $a_i \in [0, n-1]$.

The functions $\tilde{G}_w$ are not homogeneous. The lowest degree part of $\tilde{G}_w$ is equal to $\tilde{F}_w$.

**Theorem 44.** The affine stable Grothendieck polynomial $\tilde{G}_w(X)$ is a symmetric function.

We will first need the following lemma.

**Lemma 45.** Let $a, b \in [0, n-1]$ satisfy $a \neq b + 1$. Then in $\tilde{U}_n$ we have

$$
u_b u_{b-1} \cdots u_a u_b u_{b-1} \cdots u_a = u_{b-1} \cdots u_a u_b u_{b-1} \cdots u_a = u_{b} u_{b-1} \cdots u_a u_b u_{b-1} \cdots u_{a+1}.
$$

**Proof:** The result follows easily by induction, the base case being the defining identity $u_b^2 = u_b$.

**Proof of Theorem 44.** We show that $\tilde{h}_k(u)\tilde{h}_l(u) = \tilde{h}_l(u)\tilde{h}_k(u)$, as in Proposition 8. Our approach will be the same as in Proposition 8, but since not just reduced words are involved, the proof is slightly more difficult. We indicate the modifications of the proof of Proposition 8 which are needed—the global structure of the
proof is completely identical, but the calculation within each critical interval is more delicate. The main difference is that an outer interval \( A_i \) may overlap with its right neighbor \( B_k \). Let \( A^* \) and \( B^* \) be an outer interval and its right neighbor as before. We may no longer assume that \( \min (A^*) = \max (B^*) + 1 \), but nevertheless we define \( U = \phi(A^* \cup_i T_i, B^* \cup_j S_j) \subset [b, a] \) with a small modification. So we begin with \( U = [b, a] \) and a changing index \( i \) set to \( i := a \) to begin with. The index \( i \) decreases from \( a \) to \( b \) and at each step the element \( i \) may be removed from \( U \) according to the rule:

1. If \( i \in A^* \) then we remove it from \( U \) unless \( i \in S_k \) or \( i \in (A^* \cap B^*) + 1 \) for some \( k \in [1, s] \).
2. If \( i \in B^* \) and \( i \not\in A^* \) then we remove it from \( U \) unless \( i \in T_k + 1 \) for some \( k \in [1, s] \).
3. Otherwise we do not remove \( i \) from \( U \) and set \( i := i - 1 \). Repeat.

When \( |U| = d \) we stop the algorithm. The proof follows essentially as in Proposition 8 but in addition we need the following types of manipulations in \( \mathcal{U}_U \) for \( i > p > k > j > m > l \) (cyclically):

\[
\begin{align*}
(6) \quad & (u_i u_{i-1} \cdots u_{j+1} u_j) (u_k u_{k-1} \cdots u_{l+1} u_l) \\
& = (u_i u_{i-1} \cdots u_m) (u_k u_{k-1} \cdots u_{j+1} u_m u_{m-1} \cdots u_l),
\end{align*}
\]

and

\[
\begin{align*}
(7) \quad & (u_i u_{i-1} \cdots u_{j+1} u_j) (u_k u_{k-1} \cdots u_{l+1} u_l) \\
& = (u_i u_{i-1} \cdots u_p u_{k-2} u_{k-1} u_k) (u_p u_{p-1} \cdots u_l),
\end{align*}
\]

which follow from Lemma 45. So for example we have \( u_4 u_3 u_2 u_3 u_2 u_1 u_0 = u_4 u_3 u_2 u_1 u_2 u_3 u_0 \). One checks that (6) and (7) are exactly the relations needed at the “overlap” between \( A^* \) and \( B^* \) and show that the definition of \( U = \phi(A^* \cup_i T_i, B^* \cup_j S_j) \) induces the desired bijection. Unlike in Proposition 8 we cannot perform our calculations within the affine symmetric group since some of our words are not reduced. However, the arguments required are nearly identical, as the next example should show.

Example 2. We illustrate the map \( U = \phi(A^* \cup_i T_i, B^* \cup_j S_j) \). Suppose \( [b, a] = [2, 20] \) and \( A^* = [13, 20], B^* = [2, 14] \). Let \( S_1 = [16, 18] \) and \( T_1 = [8, 11] \) and \( T_2 = [5] \) be the inner intervals. Then \( d = 13 \) and \( U = \{2, 3, 4, 5, 6, 9, 10, 11, 12, 14, 16, 17, 18\} \). We can compute that

\[
\begin{align*}
u_{T_1} u_{T_2} u_{A^*} u_{B^*} u_{S_1} &= u_{T_1} u_{T_2} u_{[13, 20]} u_{14} u_{[2, 12]} u_{S_1} \\
&= u_{T_1} u_{[13, 20]} u_{14} u_{[2, 12]} u_{S_1} u_{T_1+1} = u_{T_2} u_{[6, 20]} u_{14} u_{[2, 6]} u_{T_1+1} u_{S_1},
\end{align*}
\]

so that \( U' = [6, 20] \cup \{5\} \). Finally one checks that \( B^* \cup_j S_j = \phi(U', U) \).
When $w$ is 321-avoiding, then we obtain cylindric stable Grothendieck polynomials which should be related to the quantum $K$-theory of the Grassmannian.

DEPARTMENT OF MATHEMATICS, M.I.T., CAMBRIDGE, MA 02139
E-mail: thomasl@math.mit.edu

REFERENCES

[1] S. Billey, W. Jockusch and R. Stanley, Some combinatorial properties of Schubert polynomials, J. Algebraic Combin. 2 (1993), 345–374.
[2] A. Björner and F. Brenti, Affine permutations of type A, Electron. J. Combin. 3(2) (1996), Research paper 18.
[3] P. Edelman and C. Greene, Balanced tableaux, Adv. Math. 631 (1987), 42–99.
[4] S. Fomin and C. Greene, Noncommutative Schur functions and their applications, Discrete Math. 193 (1998), 179–200.
[5] S. Fomin and A. Kirillov, Grothendieck polynomials and the Yang-Baxter equation, Proc. 6th Internat. Conf. on Formal Power Series and Algebraic Combinatorics, DIMACS, 1994, pp. 183–190.
[6] ———, Combinatorial Bn-analogues of Schubert polynomials, Trans. Amer. Math. Soc. 348 (1996), 3591–3620.
[7] S. Fomin and R. Stanley, Schubert polynomials and the nilCoxeter algebra, Adv. Math. 103 (1994), 196–207.
[8] I. Gessel and C. Krattenthaler, Cylindric partitions, Trans. Amer. Math. Soc. 349 (1997), 429–479.
[9] M. Haiman, Hilbert schemes, polygraphs, and the Macdonald positivity conjecture, J. Amer. Math. Soc. 14 (2001), 941–1006.
[10] J. Humphreys, Reflection groups and Coxeter groups, Cambridge Stud. Adv. Math., vol. 29, Cambridge University Press, Cambridge, 1990.
[11] W. Kraskiewicz, Reduced decompositions in hyperoctahedral group, C.R. Acad. Sci. Paris Ser. I 309 (1989), 903–904.
[12] T-K. Lam, $B_n$ Stanley symmetric functions, Discrete Math. 157 (1996), 241–270.
[13] T. Lam, Ribbon Schur operators, Europ. J. Comb. (to appear); math.CO/0409463.
[14] ———, Affine Stanley symmetric functions (extended abstract), Proceedings of FPSAC, 2004, Taormina.
[15] ———, Schubert polynomials for the affine Grassmannian, preprint, 2006; math.CO/0603125.
[16] L. Lapointe, A. Lascoux and J. Morse, Tableau atoms and a new Macdonald positivity conjecture, Duke Math J. 116 (2003), 103–146.
[17] L. Lapointe and J. Morse, Schur function analogs and a filtration for the symmetric function space, J. Combin. Theory Ser. A 101/2 (2003), 191–224.
[18] ———, Tableaux on $k+1$-cores, reduced words for affine permutations, and $k$-Schur expansions, J. Combin. Theory Ser. A 112 (2005), 44–81.
[19] ———, Quantum cohomology and the $k$-Schur basis, preprint, 2005; math.CO/0501529.
[20] A. Lascoux, Ordering the affine symmetric group, Algebraic Combinatorics and Applications (Gosswinstein, 1999), Springer, Berlin, 2001, pp. 218–231.
[21] A. Lascoux and M. Schützenberger, Polynômes de Schubert, C.R. Acad. Sci. Paris 294 (1982), 447–450.
[22] ———, Schubert polynomials and the Littlewood-Richardson rule, Lett. Math. Phys. 10(2–3) (1985), 111–124.
[23] M. van Leeuwen, Edge sequences, ribbon tableaux, and an action of affine permutations, Europ. J. Combinator. 20 (1999), 179–195.
[24] I. Macdonald, Symmetric Functions and Hall Polynomials, Oxford University Press, 1995.
[25] P. McNamara, Cylindric Skew Schur Functions, Adv. Math. 205(1) (2006), 275–312.
[26] J. Morse and M. Shimozono, talk at Park City Mathematical Institute, July, 2004.
[27] A. Postnikov, Affine approach to quantum Schubert calculus, *Duke Math. J.* 128 (2005), 473–509.
[28] R. Stanley, On the number of reduced decompositions of elements of Coxeter groups, *European J. Combinator.* 5 (1984), 359–372.
[29] ________, *Enumerative Combinatorics,* vol. 2, Cambridge University Press, 1999.
[30] E. Witten, The Verlinde algebra and the cohomology of the Grassmannian, *Geometry, Topology and Physics, Conference Proceedings and Lecture Notes in Geometric Topology,* vol. IV, International Press, Cambridge, MA, 1995, pp. 357–422.