The BV Master Equation for the Wilson Action in general Yang-Mills Gauge Theory

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Abstract

The Wilson effective action for general Yang-Mills gauge theory is shown to satisfy the usual form of Batalin-Vilkovisky (BV) master equation, despite that a momentum cutoff apparently breaks the gauge invariance. In the case of Abelian gauge theory, in particular, it actually deduces the Ward-Takahashi identity for Wilson action recently derived by Sonoda.

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§1. Introduction

Exact renormalization group (ERG)\(^1\) provides us with a powerful tool to reveal the dynamics of various field theories. For the important cases of gauge theories, however, the very notion of the momentum cutoff to define the Wilson action is apparently incompatible with the gauge invariance, and there have been many proposals and trials for circumventing the difficulty.\(^2\)–\(^6\)

In this respect it was truly remarkable that Sonoda\(^7\) has recently found that a simple form of Ward-Takahashi (WT) identity holds for the Wilson action in QED. It implies that the momentum cutoff is compatible with gauge invariance. Moreover the Sonoda’s WT identity for the Wilson action was rederived by a simpler path-integral method by Igarashi, Itoh and Sonoda (IIS) in Ref. 8). Those authors also showed that Sonoda’s equation can be lifted into the form of quantum Batalin-Vilkovisky (BV) master equation,\(^9\) remarkably, exactly the same form of equation as the continuum theory without cutoff.

Their work was, however, restricted to QED. In this short note we show that the Wilson effective action for a general non-Abelian gauge theory satisfies the quantum master equation. We follow the method developed by IIS which is really powerful and makes it remarkably easy to derive the master equation. Again the master equation is written in exactly the same form as the usual one for the continuum theory. This derivation is done in Sects. 2 and 3.

In the special case of QED, we can make it explicit how the Wilson action depends on the BV antifields since the Faddeev-Popov (FP) ghost fields are free there. Then we shall elucidate the relation of our Wilson master action with that derived by IIS and, in particular, show that our BV master equation will really reproduces the Sonoda’s WT identity for the Wilson action in QED. We will perform this task in Sects. 4 and 5.

§2. Wilson action in the presence of antifields

We here follow the method and notation developed by Igarashi-Itoh-Sonoda (IIS) in Ref.8).

Now we consider a general system of Yang-Mills gauge theory and denote the action generically as follows by separating the kinetic terms from the interaction terms:

\[
S[\phi] = \frac{1}{2} \phi \cdot D \cdot \phi + S_I[\phi].
\]

(2.1)

Here and henceforth we use the condensed notation for the fields, \(\phi^A\), with the superfix \(A\) standing for all the field indices and the matrix notation in momentum space like

\[
J \cdot \phi = \int_p J_A(-p)\phi^A(p), \quad \int_p \equiv \int \frac{d^dp}{(2\pi)^d},
\]
The action (2.1) is understood to contain the gauge fixing term and the corresponding Faddeev-Popov ghost term. The gauge invariance of the system is therefore represented by the invariance under the BRS transformation, which we

\[ \delta_B \phi^A = F^A(\phi). \] (2.3)

Note that we adopt in this note the convention for this BRS transformation \( \delta_B \) to be the operation from the right: \( \delta_B(FG) = F \delta_B G + (-)^{e(G)}(\delta_B F)G^\sigma \).

We define as usual the generating functional for this system in the presence of the external sources \( J_A \) as well as the BV antifields \( \phi^*_A \), source functions for the BRS transformation:

\[ Z[J, \phi^*] = \int D\phi \exp \left( -S[\phi] + J \cdot \phi - \phi^* \cdot F(\phi) \right). \] (2.4)

To define the Wilson action, we introduce a momentum cutoff function \( K(p) \) depending only on \( p^2 \) that behaves as

\[ K(p) \rightarrow \begin{cases} 1 & (p^2 < \Lambda^2) \\ 0 & (p^2 \to \infty) \end{cases}. \] (2.5)

We take the function going to 0 sufficiently rapidly as \( p^2 \to \infty \), but \( K(p) \neq 0 \) for any finite \( p \) so as for \( 1/K(p) \) to exist. We can now decompose the original fields \( \phi^A \) into the infrared (IR) fields \( \Phi^A \) and the ultraviolet (UV) fields \( \tilde{\phi}^A \) whose propagators are given by \( K(p)(D_{AB}(p))^{-1} \) and \( (1-K(p))(D_{AB}(p))^{-1} \), respectively:

\[ \phi^A = \Phi^A + \tilde{\phi}^A. \] (2.6)

Remarkably, IIS achieved this task very concisely by multiplying the generating functional (2.4) by a gaussian integral over new fields \( \theta^A \)

\[ \int D\theta \exp -\frac{1}{2} \theta \cdot \frac{D}{K(1-K)} \cdot \theta = \text{constant} \] (2.7)

and rewriting the \( \theta \) fields as

\[ \theta^A = (1-K)\Phi^A - K\tilde{\phi}^A - (-)^{e_A}(D^{-1})^{AB}(1-K)J_B. \] (2.8)

\(^{\ast} \) We are following this convention by IIS in this note for ease of comparison of our results with theirs, although the conversion to the more natural convention of left-operation is easy.
Indeed, performing the change of integration variables \((\phi^A, \theta^A) \to (\Phi^A, \tilde{\phi}^A)\), we find after a little algebra

\[
Z_{\phi}[J, \phi^*] = N_J \int D\Phi D\tilde{\phi} \exp \left( - \frac{1}{2} \Phi \cdot K^{-1} D \cdot \Phi + \frac{1}{2} \tilde{\phi} \cdot (1 - K)^{-1} D \cdot \tilde{\phi} + S_I[\Phi + \tilde{\phi}] + \phi^* \cdot F(\Phi + \tilde{\phi}) - J \cdot K^{-1} \Phi \right),
\]

where

\[
N_J \equiv \exp \frac{1}{2} (-)^{A} J_A (1 - K^{-1}) (D^{-1})^{AB} J_B.
\]

Note that the external sources \(J_A\) no longer couple to the UV fields \(\tilde{\phi}^A\) and do to the IR fields \(\Phi^A\) with the factor \(K^{-1}\). In this sense \(K^{-1} J_A\) play the role of the low-energy source functions for the IR fields \(\Phi^A\). We will shortly see that it is suitable to define the antifields \(\Phi_A^*\) in the low-energy world as

\[
\Phi_A^* = K^{-1} \phi_A^*.
\]

The Wilson action for the IR fields \(\Phi^A\) is thus defined in the presence of the BV antifields \(\Phi_A^* = K^{-1} \phi_A^*\) by

\[
S[\Phi, \Phi^*] = \frac{1}{2} \Phi \cdot K^{-1} D \cdot \Phi + S_I[\Phi, \Phi^*]
\]

with \(S_I[\Phi, \Phi^*]\) given by

\[
e^{-S_I[\Phi, \Phi^*]} \equiv \int D\tilde{\phi} \exp \left( - \frac{1}{2} \tilde{\phi} \cdot (1 - K)^{-1} D \cdot \tilde{\phi} + S_I[\Phi + \tilde{\phi}] + K \Phi^* \cdot F(\Phi + \tilde{\phi}) \right).
\]

The generating functional in the low-energy world is given in terms of this Wilson action by

\[
Z_{\Phi}[K^{-1} J, \Phi^*] = \int D\Phi \exp \left( - S[\Phi, \Phi^*] + K^{-1} J \cdot \Phi \right),
\]

which is related to the original one for \(\phi\) by

\[
Z_{\phi}[J, \phi^*] = N_J Z_{\Phi}[K^{-1} J, \Phi^*].
\]

§3. BV master equation for the Wilson action

It is now easy to prove that the Wilson action \(S[\Phi, \Phi^*]\) thus defined in (2.12) with (2.13) satisfies the quantum BV master equation.

In the generating functional \(Z_{\phi}\) in (2.4), we perform the change of integration variables

\[
\phi^A \to \phi^A + F^A(\phi) \lambda
\]

\[\text{(3.1)}\]
where the field shift $\delta \phi^A$ is the same as the BRS transformation $\delta B \phi^A$ multiplied by a Grassmann-odd parameter $\lambda$. Since the original action $S[\phi]$ as well as BV antifield terms $\phi^* \cdot F(\phi)$ are BRS invariant, $\delta B S[\phi] = 0$ and $\delta B (\phi^* \cdot F(\phi)) = \phi^* \cdot \delta B (\delta B \phi) = 0$, the only changes come from the external source terms $J \cdot \phi$ and so we obtain

$$
\int D\phi \left( J \cdot F(\phi) \right) \exp (-S[\phi] + J \cdot \phi - \phi^* \cdot F(\phi)) = 0 \quad \rightarrow \quad J \cdot \frac{\delta^l}{\delta \phi^*} Z_{[J, \phi^*]} = 0,
$$

(3.2)

where the notations $\delta^l$ and $\delta^r$ distinguish the operations from the left and right, respectively, when necessary. Using the relations (2.11), (2.14) and (2.15), we can rewrite this into

$$
J \cdot K^{-1} \frac{\delta^l}{\delta \phi^*} Z_{[K^{-1}J, \Phi^*]} = \left\langle J \cdot K^{-1} \frac{\delta^l}{\delta \phi^*} S \right\rangle_{K^{-1}J, \Phi^*} = 0,
$$

(3.3)

where use has been made of the notation:

$$
\langle \cdots \rangle_{K^{-1}J, \Phi^*} \equiv \int D\Phi \left( \cdots \right) e^{-S[\Phi, \Phi^*] + K^{-1}J \cdot \Phi}.
$$

(3.4)

On the other hand, since integration of any total derivative quantity is vanishing we have

$$
0 = \int D\Phi \ \frac{\delta^r}{\delta \phi^A} \left( \frac{\delta^l S}{\delta \phi^*} e^{(-S[\Phi, \Phi^*] + K^{-1}J \cdot \Phi)} \right)
= \left\langle \frac{\delta^r \delta^l S}{\delta \phi^A \delta \phi^*} - \frac{\delta^r S \delta^l S}{\delta \phi^A \delta \phi^*} + K^{-1}J \cdot \frac{\delta^l S}{\delta \phi^*} \right\rangle_{K^{-1}J, \Phi^*}.
$$

(3.5)

The above identity (3.3), therefore, yields

$$
\left\langle \frac{\delta^r \delta^l S}{\delta \phi^A \delta \phi^*} - \frac{\delta^r S \delta^l S}{\delta \phi^A \delta \phi^*} \right\rangle_{K^{-1}J, \Phi^*} = 0.
$$

(3.6)

This identity holds for any values of the external source functions $J_A$, and so implies that the inside quantity itself vanishes:

$$
\frac{\delta^r S \delta^l S}{\delta \phi^A \delta \phi^*} = \frac{\delta^r \delta^l S}{\delta \phi^A \delta \phi^*}.
$$

(3.7)

This is the quantum BV master equation for the Wilson action $S[\Phi, \Phi^*]$ (4.12) in the presence of antifield background. It is quite remarkable that exactly the same form of BV master equation as that for the continuum theory without UV cutoff holds here for the Wilson effective action for the IR fields. In this sense, the UV momentum cutoff (smooth momentum cutoff, at least) seems compatible with the general non-Abelian gauge-invariance, as claimed by Sonoda in QED case.\(^7\)
§4. The Sonoda’s WT identity for Wilson action in QED case

Up to here we have been considering the general non-Abelian gauge theories. For the rest of this note, however, we consider the Abelian gauge theories. The special circumstance in the Abelian case is that the Faddeev-Popov ghost and antighost are free from interaction in the usual covariant gauges. This fact makes the BV antifield dependence of the master action considerably explicit and enables us to make connections with the previous works in QED case by Sonoda\textsuperscript{7)} and by IIS.\textsuperscript{8)}

We now show that our quantum BV Master equation reproduces the Sonoda’s WT identity for the Wilson action in the case of QED. We also show that our Wilson action in the presence of the antifields indeed deduces the IIS’s BV master action in Ref.8) which they elaborated starting from the Sonoda’s WT identity. The master action constructed by them is not necessarily equal to our master action, although the Wilson action in the absence of the antifields is of course unique. As was already noted by IIS, there is an ambiguity of performing canonical transformations in the field and antifield variable space. In the next section, we will explicitly give the canonical transformation which transforms our variables to theirs, and show that our master action indeed deduces their master action exactly.

The fields, antifields and external sources in the QED case are given by

\[
\phi^A = \{a_\mu, b, c, \bar{c}, \psi, \bar{\psi}\},
\phi^*_A = \{a^*_\mu, b^*, c^*, \bar{c}^*, \psi^*, \bar{\psi}^*\},
J_A = \{J_\mu, J_b, J_c, J_{\bar{c}}, J_\psi, J_{\bar{\psi}}\}. \tag{4.1}
\]

But the antifields \(b^*, c^*\) actually do not appear in QED case since \(\delta_B b = \delta_B c = 0\).\textsuperscript{\textasteriskcentered} We denote the IR fields \(\Phi^A\) by the corresponding upper case letters:

\[
\Phi^A = \{A_\mu, B, C, \bar{C}, \Psi, \bar{\Psi}\}. \tag{4.2}
\]

The action (2.1) in the covariant \(\xi\) gauge reads explicitly

\[
\frac{1}{2} \phi \cdot D \cdot \phi = \int_k \left[ \frac{1}{2} a_\mu (-k) \left( k^2 \delta_{\mu \nu} - k_\mu k_\nu \right) a_\nu (k) + \bar{c}(-k) i k^2 c(k) 
- b(-k) \left( i k^\mu a_\mu (k) + \frac{\xi}{2} b(k) \right) \right] + \int_p \bar{\psi} (-p) \left( \gamma^\mu \psi(p) \right) a_\mu (k),
\]

\[
S_I[\phi] = \int_{p,k} -e \bar{\psi} (-p - k) \gamma^\mu \psi(p) a_\mu (k). \tag{4.3}
\]

\textsuperscript{\textasteriskcentered} Since the antifield variables \(b^*, c^*\) are missing, the fields \(b\) and \(c\) are no longer the canonical field variables but becomes mere ‘parameters’ in the BV field-antifield formalism in the next section.
The antifield terms $\phi^A F^A(\phi)$ are given by

$$
\phi^A F^A(\phi) = i \int_k \left[ a^*_{ik} (-k) \left(-ik^\mu c(k)\right) + \bar{c}^* (-k) ib(k) \right]
- ie \int_{p,k} \left[ \psi^* (-p) \psi (p-k) c(k) + c(k) \bar{\psi} (p-k) \bar{\psi}^* (-p) \right].
$$

(4.4)

To make explicit the antifield dependence of the Wilson action, we proceed as follows. First, the antifield dependence solely comes from the antifield term $K\Phi^* \cdot F(\Phi + \tilde{\phi})$ which is contained in the exponent of the integrand of the $\tilde{\phi}^A$ integration defining $S_I[\Phi, \Phi^*]$ in (2.13). That term is expanded in powers of the integration variables $\tilde{\phi}^A$.

$$
K\Phi^* \cdot F(\Phi + \tilde{\phi}) = K\Phi^* \cdot \left[ F(\Phi) + F'(\Phi) \tilde{\phi} + \frac{1}{2} F''(\Phi) \tilde{\phi}^2 \right].
$$

(4.5)

Among the integration variables $\tilde{\phi}^A$, the FP ghost variable $\bar{c}$ can be set equal to zero in QED case. This is because FP ghosts are free and the antighost variable $\bar{c}$ appears only in the free kinetic term $\bar{c}(-k)(1 - K)^{-1} ik^2 \bar{c}(k)$ so that any terms proportional to $\bar{c}$ can be absorbed by shifting the antighost variable $\bar{c}$. By this procedure, all the quadratic terms and a part of the linear terms in $\bar{c}$ are eliminated here in Eq. (4.5). The remaining linear terms in $\tilde{\phi}^A$ are now for $\tilde{\phi}^A = \{ \bar{b}, \tilde{\psi}, \tilde{\tilde{\psi}} \}$. They can be absorbed into the kinetic terms $(1/2) \bar{\phi} \cdot (1 - K)^{-1} D \cdot \tilde{\phi}$ by shifting the integration variables

$$
\tilde{\phi}^A = \tilde{\phi}^A + f^{AB}(\Phi) \Phi_B^*
$$

(4.6)

with the shift $(f(\Phi) \cdot \Phi^*)^A$ determined by the condition

$$
K\Phi^* \cdot F'(\Phi) \tilde{\phi} = (f(\Phi) \cdot \Phi^*) \cdot (1 - K)^{-1} D \cdot \tilde{\phi}.
$$

(4.7)

Then the expression for the $S_I[\Phi, \Phi^*]$ in (2.13) now becomes

$$
\exp(-S_I[\Phi, \Phi^*]) = \exp \left( - \left( S_I'[\Phi'] + K\Phi^* \cdot F(\Phi) - \frac{1}{2} (f(\Phi) \cdot \Phi^*) \cdot (1 - K)^{-1} D \cdot (f(\Phi) \cdot \Phi^*) \right) \right)
\exp(-S_I'[\Phi']) = \int D\tilde{\phi}' \exp \left( \frac{1}{2} \tilde{\phi}' \cdot (1 - K)^{-1} D \cdot \tilde{\phi}' + S_I[\Phi' + \tilde{\phi}'] \right).
$$

(4.8)

Here in the argument of the interaction term $S_I[\Phi' + \tilde{\phi}']$, we have also shifted the IR fields

$$
\Phi'^A = \Phi^A - (f(\Phi) \cdot \Phi^*)^A,
$$

(4.9)

\footnote{In the usual covariant gauges, all the quadratic terms in $\tilde{\phi}$ contain the FP ghost $\bar{c}$ as one of the two $\tilde{\phi}$’s. The fact that all the quadratic terms can be eliminated in QED gives the reason why all the antifield dependences are represented by the shift of the fields in the Wilson action. The linear terms in $\tilde{\phi}$ can be eliminated by the shift of the UV fields which turns to give the shift of the IR fields in the Wilson action, as we will show shortly. In non-Abelian cases, however, the FP ghosts are interacting fields and so the quadratic terms in $\tilde{\phi}$ cannot be eliminated, implying that the antifield dependences exit which cannot be represented by the shift of the fields.}
such that the arguments remain intact, $\Phi + \tilde{\phi} = \Phi' + \tilde{\phi}'$. At this stage $S'_1[\Phi']$ depends on the antifields only through the shifted IR variables $\Phi'^A$. Explicitly these field shifts are given by

$$A'_\mu(k) = A_\mu(k) + \frac{k_\mu}{k^2} (1 - K(k)) K(k) \tilde{C}^*(k),$$

$$\Psi'(p) = \Psi(p) - ie \frac{1 - K(p)}{\tilde{p} + m} \int_k K(p-k) \bar{\Psi}^*(p-k) C(k),$$

$$\bar{\Psi}'(-p) = \bar{\Psi}(-p) - ie \int_k K(p+k) \bar{\Psi}^*(-p-k) C(k) \frac{1 - K(p)}{\tilde{p} + m}. \quad (4.10)$$

The other fields $B$, $C$ and $\tilde{C}$ remain intact.

Now, we should also do the rewriting $\Phi \to \Phi' + (f(\Phi) \cdot \Phi^*)$ in all the other terms in the Wilson action

$$S[\Phi, \Phi^*] = \frac{1}{2} \Phi \cdot K^{-1} D \cdot \Phi + K \Phi^* \cdot F(\Phi) - \frac{1}{2} (f(\Phi), \Phi^*) (1 - K)^{-1} D \cdot (f(\Phi) \cdot \Phi^*) + S'_1[\Phi'] . \quad (4.11)$$

We need to rewrite the kinetic terms $(1/2) \Phi \cdot K^{-1} D \cdot \Phi$ and the antifield terms $K \Phi^* \cdot F(\Phi)$, but $f(\Phi)$ need not be rewritten since it contains only $C$. We have

$$S[\Phi, \Phi^*] = \frac{1}{2} \Phi' \cdot K^{-1} D \cdot \Phi' + S'_1[\Phi']$$

$$+ \text{(linear terms in } \Phi^*) + \text{(quadratic terms in } \Phi^*), \quad (4.12)$$

where the (linear terms in $\Phi^*$) is given after a short computation by

$$\text{(linear terms in } \Phi^*)$$

$$= K \Phi^* \cdot F(\Phi') + \Phi' \cdot K^{-1} D \cdot (f(\Phi) \cdot \Phi^*)$$

$$= \int_k \left( K(k) A^*_\mu(-k) (-ik^\mu C(k)) + \tilde{C}^*(-k) iB(k) \right)$$

$$+ ie \int_{p,k} \left( K(p) \Psi^*(-p) C(k) \frac{\Psi'(p-k)}{K(p-k)} + \frac{\bar{\Psi}'(p-k)}{K(p-k)} K(p) \bar{\Psi}^*(-p) C(k) \right) \quad (4.13)$$

and the (quadratic terms in $\Phi^*$) is

$$\text{(quadratic terms in } \Phi^*)$$

$$= K \Phi^* \cdot F'(\Phi')(f(\Phi) \cdot \Phi^*) + \frac{1}{2} (f(\Phi) \cdot \Phi^*) \left[ -\frac{1}{1 - K} + \frac{1}{K} \right] D \cdot (f(\Phi) \cdot \Phi^*). \quad (4.14)$$

Here we can set $F'(\Phi') = F'(\Phi)$ since only fermion terms are relevant and $\Phi = C$ appears there. Therefore, using the relation (4.7), we find

$$K \Phi^* \cdot F'(\Phi)(f(\Phi) \cdot \Phi^*) = (f(\Phi) \cdot \Phi^*) \cdot (1 - K)^{-1} D \cdot (f(\Phi) \cdot \Phi^*), \quad (4.15)$$
so that the quadratic terms become
\[
\text{(quadratic terms in } \Phi^*) = \frac{1}{2} (f(\Phi) \cdot \Phi^*) \cdot \left[ \frac{1}{1 - K} + \frac{1}{K} \right] D \cdot (f(\Phi) \cdot \Phi^*). \quad (4.16)
\]

The shift \((f(\Phi) \cdot \Phi^*)\) for the gauge field \(A_\mu\) yields no quadratic term in \(\Phi^*\) here since it is proportional to \(ik_\mu\) while the kinetic operator \(D\) is transversal \(\propto (\delta_{\mu\nu}k^2 - k_\mu k_\nu)\). Thus only the fermion kinetic term is relevant and we find the explicit form for the (quadratic terms in \(\Phi^*)\):
\[
= (f(\Phi) \cdot \Phi^*) \bar{\Psi}(-p) \frac{\dot{p} + m}{K(p)(1 - K(p))} (f(\Phi) \cdot \Phi^*) \Psi(p)
= (ie)^2 \int_{p,k,l} K(p + k)\Psi^*(-p - k)C(k) \frac{1 - K(p)}{K(p)(\dot{p} + m)} K(p - l)\bar{\Psi}^*(p - l)C(l). \quad (4.17)
\]

Now that we have made explicit the antifield dependence of the Wilson action, we can easily see that our BV master equation \((3.7)\) really reproduces the Sonoda’s Ward identity for the Wilson action by putting all the antifield equal to zero after evaluating the antifield derivatives.

§5. Quantum BV master equation by IIS

Our master action \(S[\Phi, \Phi^*]\), however, still does not coincide with that derived by IIS, although both reproduce the same Sonoda equation when the antifields are set equal to zero. This is due to the fact that there is a freedom of doing canonical transformation in the field and antifield space.

The clear differences of the two actions are that the implicit antifield dependence in our case exists through three primed variables \(A'_\mu, \Psi', \bar{\Psi}'\) in \((4.10)\), while it is only through \(\bar{\Psi}'\) in the IIS case, and also the shifts in \(\Psi'\) do not coincides with each other. We are thus lead to performing the canonical transformation which identify our first two primed variables \(A'_\mu, \Psi'\) with IIS’s variables \(A_{\mu \text{IIS}}, \Psi_{\text{IIS}}\). The generating function(al) of the desired canonical transformation is easily found to be
\[
W[\Phi, \Phi^*_{\text{IIS}}]
= \int_k \left[ A^\mu_{\text{IIS}}(-k) \left( A_{\mu}(k) + \frac{k_\mu}{k^2} K(k)(1 - K(k))\bar{C}_{\text{IIS}}(k) \right) + \bar{C}_{\text{IIS}}(-k)C(k) \right]
  + \int_p \left[ \Psi_{\text{IIS}}^*(-p) \left( \Psi(p) - ie(1 - K(p)) \int_k (\dot{p} + m)^{-1} K(p - k)\bar{\Psi}_{\text{IIS}}^*(p - k)C(k) \right) \right.
  + \bar{\Psi}(p)\bar{\Psi}^*_{\text{IIS}}(-p) \right]. \quad (5.1)
\]
Then, actually, the canonical variable relation \( \Phi^*_{A}(k) = \delta^t W[\Phi, \Phi_{IIS}^*] / \delta \Phi_A^*(-k) \) says first that all the antifield variables coincide

\[
A^*_\mu(k) = A^*_{\mu, IIS}(k), \quad \bar{C}^*(k) = \bar{C}^*_{IIS}(k), \quad \Psi^*(p) = \Psi_{IIS}^*(p), \quad \bar{\Psi}^*(p) = \bar{\Psi}_{IIS}^*(p),
\]

and \( \Phi_{IIS}^A(k) = \delta^t W[\Phi, \Phi_{IIS}^*] / \delta \Phi_{A IIS}^*(-k) \) gives the relation of the field variables:

\[
A_{\mu, IIS}(k) = A_{\mu}(k) + \frac{k_\mu}{k^2} K(k)(1 - K(k)) \bar{C}_{IIS}^*(k) = A'_\mu(k), \tag{5.3}
\]

\[
\bar{C}_{IIS}(k) = \bar{C}(k) - K(k)(1 - K(k)) \frac{k_\mu}{k^2} A^*_{\mu, IIS}(k), \tag{5.4}
\]

\[
\Psi_{IIS}(p) = \Psi(p) - i e (1 - K(p)) \int_k (\bar{p} + m)^{-1} K(p - k) \bar{\Psi}_{IIS}^*(p - k) C(k) = \Psi'(p), \tag{5.5}
\]

\[
\bar{\Psi}_{IIS}(-p) = \bar{\Psi}(-p) - i e (1 - K(p + k)) \int_k \Psi_{IIS}^*(-p - k) (\bar{p} + \bar{k} + m)^{-1} K(p) C(k)
\]

\[
= \bar{\Psi}'(-p) + i e \int_k \Psi_{IIS}^*(-p - k) C(k) U(-p - k, p), \tag{5.6}
\]

with

\[
U(-p - k, p) = \frac{1 - K(p)}{\bar{p} + m} K(p + k) - \frac{1 - K(p + k)}{\bar{p} + \bar{k} + m} K(p). \tag{5.7}
\]

The first and third equations, (5.3) and (5.5), give the desired transformation for the photon and electron variables. The second equation (5.4) gives the ‘unexpected’ transformation for antighost \( \bar{C} \) associated with the canonical transformation of the photon field. This shift of \( \bar{C} \) in the FP ghost kinetic term \( K^{-1}(k) \bar{C}(-k) i k^2 C(k) \) in the Wilson action, yields an additional term

\[
A^*_\mu(-k)(1 - K(k)) (-i k^\mu C(k)), \tag{5.8}
\]

which transforms the non-canonical weight \( K(k) \) in the already existing antifield term \( K(k) A^*_{\mu}(-k)(-i k^\mu C(k)) \) in (4.13) into the canonical weight 1.

Eq. (5.6) indicates that our primed field \( \bar{\Psi}' \) agrees with IIS’s primed field \( \bar{\Psi}' \) since the function \( U(-p - k, p) \) here coincides with theirs. So we have to rewrite the \( \bar{\Psi}' \) variable in the \( \bar{\Psi}' \bar{\Psi} C \) term in (4.13) in terms of \( \Psi_{IIS} \). The shift proportional to \( U \) there yields an additional contribution to the (quadratic terms in \( \Phi^* \)):

\[
e^2 \int_{p,k,l} \frac{K(-p + l)}{K(p)} \Psi^*(-p - k) C(k) U(-p - k, p) \bar{\Psi}^*(p - l) C(l). \tag{5.9}
\]

As shown in Eq. (5.7), the function \( U(-p - k, p) \) contains two terms. The contribution from the first term \( \propto (\bar{p} + m)^{-1} \) to this exactly cancels the (quadratic terms in \( \Phi^* \)) in (4.17), thus leaving the contribution from the second term

\[
-e^2 \int_{p,k,l} \Psi^*(-p - k) C(k) \frac{1 - K(p + k)}{\bar{p} + \bar{k} + m} K(p - l) \bar{\Psi}^*(p - l) C(l). \tag{5.10}
\]
This term, however, vanishes by itself. Indeed, by shifting the momentum $p \rightarrow p - k$, it is rewritten into the form
\[
-e^2 \int_{p,k,l} \bar{\Psi}^*(-p) \frac{1 - K(p)}{\hat{p} + m} K(p - k - l) \bar{\Psi}^*(p - k - l) C(k) C(l),
\] (5.11)
which clearly vanishes since $C(k) C(l)$ is antisymmetric under the exchange of $k$ and $l$ while they appear symmetrically in the form $p - k - l$ in the arguments of the other functions. Thus the quadratic terms in $\Phi^*$ vanish completely.

Now collecting the terms altogether and writing the Wilson action in terms of the IIS field variables $\Phi_{\text{IIS}}$ but omitting the index IIS for simplicity, we find
\[
S[\Phi, \Phi^*] = \frac{1}{2} \Phi^\prime \cdot K^{-1} \Phi^\prime + S'_{\text{I}}[\Phi^\prime]
\]
\[
+ \int_k \left( A_\mu^*(-k)(-ik^\mu C(k)) + C^*(-k)iB(k) \right)
\]
\[
+ ie \int_{p,k} \left( K(p) \bar{\Psi}^*(-p) \frac{\Psi(p - k)}{K(p - k)} + \frac{\bar{\Psi}(p - k)}{K(p - k)} K(p) \bar{\Psi}^*(-p) C(k) \right)
\] (5.12)
with $S'_{\text{I}}[\Phi^\prime]$ defined in Eq. (4.8). Here the primed field $\Phi^\prime^A$ denotes that only the $\bar{\Psi}$ is shifted:
\[
\Phi^\prime^A = \{ A_\mu, \ B, \ C, \ \bar{C}, \ \Psi, \ \bar{\Psi}^\prime \},
\]
\[
\bar{\Psi}^\prime(-p) = \bar{\Psi}(-p) - ie \int_k \bar{\Psi}_{\text{IIS}}^*(-p - k) C(k) U(-p - k, p).
\] (5.13)
with $U(-p - k, p)$ given in Eq. (5.7). This expression for the master action exactly agrees with that derived by IIS. [See Eqs.(41) and (42) in Ref. 8.]

§6. Summary and Discussion

In this paper, we have derived the BV master equation for the Wilson action in the general non-Abelian gauge theory following IIS’s work.\(^8\) We have introduced the antifields as the sources for the BRS transformation of the fields from the starting action. In QED case, we have made explicit the antifield dependence of our master action, and have shown that our master action deduces the IIS ‘s master action and the two expressions exactly agree with each other via a canonical transformation in the field and antifield variable space, whose generating function was found explicitly. In particular, our BV master equation reproduced the Sonoda’s WT identity when the antifields are set equal to zero.

In this approach, the BRS transformation for the IR field $\Phi$ is defined as follows by the master action $S[\Phi, \Phi^*]$ itself:
\[
\delta_Q \Phi = (\Phi, S[\Phi, \Phi^*]) - \Delta \Phi,
\] (6.1)
where \((X, Y)\) and \(\Delta\) are:

\[
(X, Y) \equiv \frac{\partial X}{\partial \Phi^A} \frac{\partial Y}{\partial \Phi^*_A} - \frac{\partial X}{\partial \Phi^*_A} \frac{\partial Y}{\partial \Phi^A},
\]

\[
\Delta \equiv (-)^{\epsilon_A+1} \frac{\partial^r}{\partial \Phi^A} \frac{\partial^r}{\partial \Phi^*_A}.
\]

The antifield dependence of the master action is non-trivial in the general non-Abelian gauge theories, and so is this BRS transformation. The explicit form of the BRS transformation is thus determined simultaneously as that of the Wilson action.

The Polchinski equation (exact renormalization group equation) for the master action is invariant under this “quantum” BRS transformation, and we can obtain the BRS invariant renormalization group flows for the Wilson effective action of the gauge theories.

**Note added**

After submitting this note to the arXive, we became aware of the work by Igarashi, Itoh and So, Ref.10, in which the authors already derived the exact quantum BV master equation for the Wilson action in the presence of antifields. Although their IR fields are defined to be “average fields” differently from ours, the BV master equation holds very similar to ours.

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