On tangential transversality✩

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Abstract

This is the first of two closely related papers on transversality. Here we introduce the notion of tangential transversality of two closed subsets of a Banach space. It is an intermediate property between transversality and subtransversality. Using it, we obtain a variety of known results and some new ones in a unified way. Our proofs do not use variational principles and we are concentrated mainly on tangential conditions in the primal space.

Keywords: nonseparation of sets, tangential transversality, intersection properties, Lagrange multiplier rule

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1. Introduction

Transversality is a classical concept of mathematical analysis and differential topology. Recently, it has proven to be useful in variational analysis as well. As it is stated in \cite{13}, the transversality-oriented language is extremely natural and convenient in some parts of variational analysis, including subdifferential calculus and nonsmooth optimization.

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The classical definition of transversality at an intersection point of two smooth manifolds in a Euclidean space is that the sum of the corresponding tangent spaces at the intersection point is the whole space (cf. [9], [10]). Equivalently, the intersection of the corresponding normal spaces is the origin.

In order to prove the Pontryagin maximum principle (cf., for example, the bibliography of [26]), Hector Sussmann generalizes the definition of transversality for closed convex cones in \( \mathbb{R}^n \): the cones \( C^A \) and \( C^B \) are transversal if and only if
\[
C^A - C^B = \mathbb{R}^n
\]
and strongly transversal, if they are transversal and \( C^A \cap C^B \neq \{0\} \) (cf. Definitions 3.1 and 3.2 from [26]). In finite-dimensional case, strong transversality of the approximating cones of the same type (either Clarke or Boltyanski) is a sufficient condition for local nonseparation of sets. The sets \( A, B \) containing a point \( x_0 \) are said to be locally separated at \( x_0 \), if there exists a neighborhood \( \Omega \) of \( x_0 \) so that \( \Omega \cap A \cap B = \{x_0\} \) (cf., for example, the introduction of [26]). In infinite-dimensional case, strong transversality of the approximating cones of the same type does not imply local nonseparation of sets – take for example a Hilbert cube \( A := \{(x_n) \in l_2 : |x_n| \leq 1/n\} \subset l_2 \) and a ray \( B := \{\lambda y : \lambda \geq 0\} \), where \( y := (1/n^{3/4})_{n=1}^{\infty} \). We have that the corresponding Clarke tangent cones \( \hat{T}_A(0) = l_2 \) and \( T_B(0) = B \) are strongly transversal, while the sets \( A \) and \( B \) are locally separated at 0.

In the literature there exist many notions generalizing the classical transversality as well as transversality of cones. Some of them are introduced under different names by different authors, but actually coincide. We refer to [20] for a survey of terminology and comparison of the available concepts. The central ones among them are transversality and subtransversality. They are also objects of study in the recent book [14].

These notions are closely connected to some important relations between tangent [normal] cones to two sets and the tangent [normal] cone to their intersection. To be more specific, let \( T_C(x) \) be the tangent cone (in some sense – Bouligand, derivable, Clarke, ...) to the closed subset \( C \) of the Banach space \( X \) at \( x \in C \) and \( N_C(x) \) be the normal cone (in some sense – proximal, limiting, \( G \)-normal, Clarke, ...) to the closed set \( C \subset X \) at \( x \in C \). For the sake of convenience, we introduce the following

**Definition 1.1.** Let \( A \) and \( B \) be closed subsets of the Banach space \( X \) and let \( x_0 \) belong to \( A \cap B \). We say that \( A \) and \( B \) have tangential intersection...
property at $x_0$ with respect to the type(s) of the approximating cones $T_A(x_0)$ and $T_B(x_0)$ if

$$T_{A \cap B}(x_0) \supset T_A(x_0) \cap T_B(x_0).$$

We say that $A$ and $B$ have normal intersection property at $x_0$ with respect to the type(s) of the normal cones $N_A(x_0)$ and $N_B(x_0)$ if

$$N_{A \cap B}(x_0) \subset N_A(x_0) + N_B(x_0)$$

and the right-hand side of the above inclusion is weak$^*$-closed.

The reader is referred to [14] and [23] for the precise definitions of the above mentioned cones.

Equivalent definitions of transversality (see Theorem 2 in [20]) have been around for almost 20 years and are mostly used as sufficient conditions for normal intersection property with respect to the limiting normal cones in Asplund spaces (cf. [22], [23]). The term subtransversality is recently introduced in [7] in relation to proving linear convergence of the alternating projections algorithm. However, this property has been around for more than 20 years as well, but under different names – see Remark 4 in [20] and the references therein. It is a key assumption for two types of results: linear convergence of sequences generated by projection algorithms and a qualification condition for normal intersection property with respect to the limiting normal cones and a sum rule for the limiting subdifferentials. Subtransversality is a weaker condition than transversality, but also implies normal intersection property (cf. Theorem 6.41 in [23] for the limiting normal cones in Asplund spaces and Theorem 7.13 in [14] for the $G$-normal cones in Banach spaces).

We arrived to the study of transversality of sets when investigating Pontryagin’s type maximum principle for optimal control problems with terminal constraints in infinite dimensional state space. In order to prove a nonseparation result (if one can not separate the approximating cones of two closed sets at a common point and, moreover, the cones have nontrivial intersection, then the sets can not be separated as well) we introduced the notion of uniform tangent set (cf. [17]). It happened to be very useful for obtaining necessary conditions for optimal control problems in infinite dimensional state space, because the diffuse variations (which are naturally defined and easy to calculate) form a uniform tangent set to the reachable set of a control system. The present manuscript and [2] are an effort to understand the relation of our study to the established results and methods of nonsmooth
optimization. As we arrived to some known notions and results using our approach, the proofs of the known theorems are completely different from the classical ones and, moreover, we found some new results. Our proofs do not use variational principles and we are concentrated mainly on tangential conditions in the primal space. We were able to obtain a vast variety of results in a unified and economical way.

Here we introduce the notion of *tangential transversality* in Banach spaces. It is an intermediate property between transversality and subtransversality. There are many really useful sufficient conditions for tangential transversality of two sets. One of them – strong tangential transversality – involves uniform tangent sets and is studied in detail in [2]. Moreover, we obtain here a sufficient condition for tangential transversality which is different from the conditions for subtransversality we know about.

The present paper is organized as follows: The second section contains the definition of tangential transversality and the main technical tool allowing us to use this concept. Its proof uses a nontrivial construction which essentially appeared in [18] and [17]. The relation of tangential transversality with the concepts of transversality and subtransversality is obtained. Some consequences of subtransversality are gathered in the third and the fourth sections. These include a nonseparation theorem, an abstract Lagrange multiplier rule and some tangential intersection properties. In the fifth section the notion of an *almost massive* set is introduced and a sufficient condition for tangential transversality is proved when one of the sets involved is almost massive. As corollaries a sum rule for $G$-subdifferentials and a Lagrange multiplier rule are obtained.

Throughout the paper if $Y$ is a Banach space, we will denote by $B_Y$ [respectively $\overline{B_Y}$] its open [closed] unit ball, centered at the origin. The index could be omitted if there is no ambiguity about the space. If $S$ is a closed subset of $Y$ at $y \in S$, we will denote by $T_S(y)$ the Bouligand tangent cone to $S$ at $y$, i.e.

$$T_S(y) := \left\{ v \in Y : \frac{y_k - y}{\tau_k} \to v \quad \text{for some sequences } y_k \in S, y_k \to y \text{ and } \tau_k > 0, \tau_k \to 0 \right\};$$

by $G_S(y)$ the derivable tangent cone to $S$ at $y$, i.e.

$$G_S(y) := \left\{ v \in Y : \frac{\xi(\tau_k) - y}{\tau_k} \to v \quad \text{for some vector-valued function } \xi : [0, \varepsilon] \to S, \xi(0) = y \text{ and for every choice of a sequence } \tau_k > 0, \tau_k \to 0 \right\};$$
and by \( \hat{T}_S(y) \) the Clarke tangent cone to \( S \) at \( y \), i.e.

\[
\hat{T}_S(y) := \left\{ v \in Y : \begin{array}{l}
\text{for each sequence } y_k \in S, y_k \to y \text{ and } \\
\text{for each sequence } \tau_k > 0, \tau_k \to 0 \text{ there } \\
\text{exists a sequence } z_k \in S \text{ with } \frac{z_k - y_k}{\tau_k} \to v
\end{array} \right\}.
\]

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2. Tangential transversality

The definition below is central for our considerations:

**Definition 2.1.** Let \( A \) and \( B \) be closed subsets of the Banach space \( X \). We say that \( A \) and \( B \) are tangentially transversal at \( x_0 \in A \cap B \), if there exist \( M > 0, \delta > 0 \) and \( \eta > 0 \) such that for any two different points \( x^A \in (x_0 + \delta B) \cap A \) and \( x^B \in (x_0 + \delta B) \cap B \), there exists a sequence \( \{t_m\}, t_m \searrow 0 \), such that for every \( m \in \mathbb{N} \) there exist \( w^A_m \in X \) with \( \|w^A_m\| \leq M \) and \( x^A + t_m w^A_m \in A \), and \( w^B_m \in X \) with \( \|w^B_m\| \leq M \), \( x^B + t_m w^B_m \in B \), and the following inequality holds true

\[
\|x^A - x^B + t_m (w^A_m - w^B_m)\| \leq \|x^A - x^B\| - t_m \eta.
\]

The intuition behind it is that the sets can be drawn together uniformly in a neighborhood of the reference point at a linear rate. Different sufficient conditions and examples of applications of this notion can be found in \([2]\). Another way of looking at this property, due Professor Alexander Ioffe, is presented in the following

**Remark 2.2.** It is straightforward to check that the sets \( A \) and \( B \) are tangentially transversal at \( x_0 \in A \cap B \), if there exist \( \delta > 0 \) and \( \zeta > 0 \) such that for any two different points \( x^A \in (x_0 + \delta B) \cap A \) and \( x^B \in (x_0 + \delta B) \cap B \), there exists a sequence \( \{t_m\}, t_m \searrow 0 \), such that for every \( m \in \mathbb{N} \) the following inequality holds true

\[
\text{dist} \left( B_{t_m}(x^A) \cap A, B_{t_m}(x^B) \cap B \right) \leq \text{dist} \left( x^A, x^B \right) - t_m \zeta.
\]
Here $B_m(x)$ is the set $x + t_m B$, dist $(C, D) := \inf \{ \| x - y \| : x \in C, y \in D \}$ and $\zeta := \eta / M$. This notation highlights the metric nature of the concept in question (a comment of Prof. A. Ioffe).

The following theorem is the main technical result to be used later on. The idea of its proof is already present in the proofs of Theorem 3.3 in [18] and Theorem 2.6 in [17].

**Theorem 2.3.** Let the closed sets $A$ and $B$ be tangentially transversal at $x_0 \in A \cap B$ with constants $M > 0$, $\delta > 0$ and $\eta > 0$. Let $x^A \in A$ and $x^B \in B$ be such that

$$\max \{ \| x^A - x_0 \|, \| x^B - x_0 \| \} + \frac{M}{\eta} \| x^A - x^B \| \leq \delta.$$  

Then, there exists $x^{AB} \in A \cap B$ with $\| x^{AB} - x^A \| \leq \frac{M}{\eta} \| x^A - x^B \|$ and $\| x^{AB} - x^B \| \leq \frac{M}{\eta} \| x^A - x^B \|$.

**Proof.** We are going to construct inductively four transfinite sequences indexed by ordinal numbers (cf., for example, § 2 Ordinal numbers of Chapter 1 in [15]). More precisely, we prove that there exist an ordinal number $\alpha_0$ and transfinite sequences $\{ x^A_{\alpha} \}_{1 \leq \alpha \leq \alpha_0} \subset (x_0 + \delta B) \cap A$, $\{ x^B_{\alpha} \}_{1 \leq \alpha \leq \alpha_0} \subset (x_0 + \delta B) \cap B$, $\{ t_{\alpha} \}_{1 \leq \alpha \leq \alpha_0} \subset [0, +\infty)$, $\{ h_{\alpha} \}_{0 \leq \alpha < \alpha_0} \subset (0, +\infty)$ such that $x^A_{\alpha_0} = x^B_{\alpha_0}$ and for each $\alpha \in [1, \alpha_0]$ we have that $t_{\alpha} = \sum_{1 \leq \beta < \alpha} h_{\beta}$ and the following estimates hold true for each $\beta$, $1 \leq \beta \leq \alpha$ and each $\gamma$, $1 \leq \gamma \leq \alpha$:

(S1) $\| x^A_{\beta} - x^B_{\beta} \| \leq \| x^A_{1} - x^B_{1} \| - t_{\beta} \eta$ (and hence $t_{\beta}$ is bounded by $\frac{\| x^A_{1} - x^B_{1} \|}{\eta}$);

(S2) $\| x^A_{\beta} - x_0 \| \leq \| x^A_{1} - x_0 \| + t_{\beta} M$;

(S3) $\| x^B_{\beta} - x_0 \| \leq \| x^B_{1} - x_0 \| + t_{\beta} M$;

(S4) $\| x^A_{\beta} - x^A_{\gamma} \| \leq M \left( t_{\beta} - t_{\gamma} \right)$;

(S5) $\| x^B_{\beta} - x^B_{\gamma} \| \leq M \left( t_{\beta} - t_{\gamma} \right)$.  

6
Note that the first step $h_0 > 0$ is the “big” one and $t_\alpha$ is the sum of all subsequent steps. Till the end of this proof the notation $\sum_{\gamma < \beta} h_\gamma$ means $\sum_{1 \leq \gamma < \beta} h_\gamma$.

We implement our construction using induction on $\alpha$. We start with $x^A_1 := x^A \in (x_0 + \delta B) \cap A$, $x^B_1 := x^B \in (x_0 + \delta B) \cap B$, $t_1 = 0$ and $h_0$ equal to the first element of the sequence $\{t_m\}$ from the definition of tangential transversality. It is straightforward to verify the induction assumptions (S1)-(S5) for $\beta = 1$ and $\gamma = 1$.

Assume that $x^A_\beta \in (x_0 + \delta B) \cap A$, $x^B_\beta \in (x_0 + \delta B) \cap B$, $h_\beta > 0$ and $t_\beta = \sum_{\gamma < \beta} h_\gamma > 0$ are constructed and (S1)-(S5) are true for all ordinals $\beta$ less than $\alpha$ and the process has not been terminated.

Let us first consider the case when $\alpha$ is a non limit ordinal number, i.e. $\alpha = \beta + 1$. As $\beta < \alpha_0$ (the process has not been terminated), we have $\|x^A_\beta - x^B_\beta\| \neq 0$. Then we set $h_\beta \in (0, \|x^A_\beta - x^B_\beta\|)$ to be equal to $t_m$ for some $m$, where the sequence $\{t_m\}$ is from the definition of tangential transversality (it is possible, because $t_m \downarrow 0$). Then, using again the definition of tangential transversality, there exist $w^A_\beta \in X$ with $\|w^A_\beta\| \leq M$ and $w^B_\beta \in X$ with $\|w^B_\beta\| \leq M$ such that

$$x^A_\alpha := x^A_\beta + h_\beta w^A_\beta \in A,$$

$$x^B_\alpha := x^B_\beta + h_\beta w^B_\beta \in B$$

and

$$\|x^A_\alpha - x^B_\alpha\| = \|x^A_\beta - x^B_\beta + h_\beta (w^A_\beta - w^B_\beta)\| \leq \|x^A_\beta - x^B_\beta\| - h_\beta \eta$$

$$\leq \|x^A_1 - x^B_1\| - (t_\beta + h_\beta) \eta.$$  

Setting $t_\alpha := t_\beta + h_\beta$, we have

$$\|x^A_\alpha - x^B_\alpha\| \leq \|x^A_1 - x^B_1\| - t_\alpha \eta.$$  

Therefore, (S1) is verified for $\alpha$.

(S2) yields

$$\|x^A_\alpha - x_0\| \leq \|x^A_1 - x_0\| + h_\beta \|w^A_\beta\|$$

$$\leq \|x^A_1 - x_0\| + t_\beta M + h_\beta M = \|x^A_1 - x_0\| + t_\alpha M.$$  

Analogously, using (S3) instead of (S2), we obtain

$$\|x^B_\alpha - x_0\| < \|x^A_1 - x_0\| + t_\alpha M.$$  

7
Using the estimate for $\|x^A_1 - x_0\|$ from (1) and that $t_\beta \leq \frac{\|x^A_1 - x^B_1\|}{\eta}$, we obtain

$$\|x^A_\alpha - x_0\| \leq \|x^A_1 - x_0\| + t_\alpha M \leq \delta - \frac{M}{\eta} \|x^A_1 - x^B_1\| + \frac{\|x^A_1 - x^B_1\|}{\eta} M = \delta,$$

and similarly

$$\|x^A_\alpha - x_0\| \leq \delta.$$

Now let $\gamma < \alpha$. Then

$$\|x^A_\alpha - x^A_\gamma\| = \|x^A_\alpha - x^A_\gamma + h_\beta v^A_\beta\| \leq \|x^A_\beta - x^A_\gamma\| + h_\beta \|v^A_\beta\| \leq M(t_\beta - t_\gamma) + M(t_\beta - t_\gamma)$$

and in the same way

$$\|x^B_\alpha - x^B_\gamma\| \leq M(t_\alpha - t_\gamma).$$

We have verified the inductive assumptions (S1)-(S5) for the case of a non limit ordinal number $\alpha$.

We next consider the case when $\alpha$ is a limit ordinal number. Let $\beta < \alpha$ be arbitrary. Then $\beta + 1 < \alpha$ too. Since the transfinite process has not stopped at $\beta + 1$, then $\|x^B_\beta - x^A_\beta\| > 0$, and hence taking into account (S1) we obtain that

$$t_\beta < \frac{\|x^A_1 - x^B_1\|}{\eta}.$$

Hence the increasing transfinite sequence $\{t_\beta\}_{1 \leq \beta < \alpha}$ is bounded, and so it is convergent. We denote $t_\alpha := \lim_{\beta \to \alpha} t_\beta = \lim_{\beta \to \alpha} \sum_{\gamma < \beta} h_\gamma = \sum_{\gamma < \alpha} h_\gamma$. Since $\|x^A_\beta - x^A_\gamma\| \leq (t_\beta - t_\gamma)M$, the transfinite sequence $\{x^A_\beta\}_{1 \leq \beta < \alpha}$ is fundamental. Hence there exists $x_\alpha$ so that $\{x^A_\beta\}_{1 \leq \beta < \alpha}$ tends to $x^A_\alpha$ as $\beta$ tends to $\alpha$ with $\beta < \alpha$. In the same way one can prove the existence of $x^B_\alpha$ so that the transfinite sequence $\{x^B_\beta\}_{1 \leq \beta < \alpha}$ tends to $x^B_\alpha$ as $\beta$ tends to $\alpha$. To verify the inductive assumptions for $\alpha$, one can just take a limit for $\beta$ tending to $\alpha$ with $\beta < \alpha$ in the same assumptions written for each $\beta < \alpha$.

We have constructed inductively the transfinite sequences

$$\{x^A_\beta\}_{\beta \leq \alpha} \subset A, \{x^B_\beta\}_{\beta \leq \alpha} \subset B$$

and $\{t_\beta\}_{\beta \leq \alpha} \subset [0, +\infty)$. The process will terminate when $x^A_\alpha = x^B_\alpha$ for some $\alpha$. Since

$$\|x^A_\alpha - x^B_\alpha\| \leq \|x^A_1 - x^B_1\| - t_\beta \eta$$

8
and the transfinite sequence \( t_\alpha \) is strictly increasing, the equality \( x_\alpha^A = x_\alpha^B \) will be satisfied for some \( \alpha = \alpha_0 \) strictly preceding the first uncountable ordinal number. Indeed, the successor ordinals indexing the so constructed transfinite sequences form a countable set (because to every successor ordinal \( \alpha + 1 \) corresponds the open interval \( (t_\alpha, t_\alpha + h_\alpha) \subset \mathbb{R} \), these intervals are disjoint and the rational numbers are countably many and dense in \( \mathbb{R} \)). Therefore, \( \alpha_0 \) is countable accessible. On the other hand, according to the Corollary after Lemma 5.1 on page 40 of [15], \( \aleph_{\gamma+1} \) is a regular cardinal (under the assumption of the Axiom of choice) for every \( \gamma \), in particular the first uncountable cardinal \( \aleph_1 \) is not countably accessible. Thus \( \omega_1 \) is not countably accessible (as \( \omega_1 \) is the first ordinal with \( |\omega_1| = \aleph_1 \)). Hence our inductive process must terminate before \( \omega_1 \).

If we assume that the process has not been terminated before \( \omega_1 \) (\( \omega_1 \) is the first ordinal with \( |\omega_1| = \aleph_1 \)), then this contradicts the fact that \( \omega_1 \) is not countably accessible. Hence our inductive process must terminate before \( \omega_1 \).

Then \( x^{AB} := x_{\alpha_0}^A = x_{\alpha_0}^B \in A \cap B \) and because of (S1) we have that

\[
t_{\alpha_0} \leq \frac{\|x_1^A - x_1^B\|}{\eta}.
\]

Applying (S4) we obtain

\[
\|x^{AB} - x_1^A\| \leq M(t_{\alpha_0} - t_1) \leq \frac{M}{\eta}\|x^A - x^B\|.
\]

Analogously, due to (S5),

\[
\|x^{AB} - x_1^B\| \leq \frac{M}{\eta}\|x^A - x^B\|.
\]

This completes the proof.

\begin{proof}
Indeed,

\[
\max \left\{ \|x^A - x_0\|, \|x^B - x_0\| \right\} + \frac{M}{\eta}\|x^A - x^B\| \\
\leq \zeta + \frac{M}{\eta} \left( \|x^A - x_0\| + \|x_0 - x^B\| \right) \leq \zeta + \frac{M}{\eta} 2\zeta = \delta.
\]

\end{proof}

\textbf{Proposition 2.4.} If the sets \( A \) and \( B \) are tangentially transversal at \( x_0 \in A \cap B \) with constants \( M > 0 \), \( \delta > 0 \) and \( \eta > 0 \), then we have that (1) holds true for all \( x^A \in A \), \( \|x^A - x_0\| \leq \zeta \) and \( x^B \in B \), \( \|x^B - x_0\| \leq \zeta \), where

\[
\zeta := \frac{\delta}{1 + 2\frac{M}{\eta}}.
\]
We are going to show that transversality implies tangential transversality, which implies subtransversality due to the above theorem. The definitions below are taken from the recent book [14].

**Definition 2.5.** Let $A$ and $B$ be closed subsets of the Banach space $X$. $A$ and $B$ are said to be transversal at $x_0 \in A \cap B$, if there exist $\delta > 0$ and $K > 0$, such that

$$d(x, (A - a) \cap (B - b)) \leq K(d(x, A - a) + d(x, B - b))$$

for all $x \in x_0 + \delta B$ and $a$ and $b$ close enough to the origin.

**Definition 2.6.** Let $A$ and $B$ be closed subsets of the Banach space $X$. $A$ and $B$ are said to be subtransversal at $x_0 \in A \cap B$, if there exist $\delta > 0$ and $K > 0$, such that

$$d(x, A \cap B) \leq K(d(x, A) + d(x, B))$$

for all $x \in x_0 + \delta B$.

**Proposition 2.7.** Let the closed sets $A$ and $B$ be transversal at $x_0 \in A \cap B$. Then, $A$ and $B$ are tangentially transversal at $x_0$.

**Proof.** In the proof we are going to use the equivalent definition of transversality given in [19] (cf. Definition 3.1 (iii) and Theorem 3.1 (iii) in [19]): $A$ and $B$ are transversal at $x_0 \in A \cap B$, if and only if there exist $\alpha > 0$ and $\delta > 0$ such that

$$(A - x^A - \rho w_1) \cap (B - x^B - \rho w_2) \cap \rho \bar{B} \neq \emptyset$$

for all $\rho \in (0, \delta)$, $w_i \in \alpha \bar{B}$, $i = 1, 2$, $x^A \in (x_0 + \delta \bar{B}) \cap A$ and $x^B \in (x_0 + \delta \bar{B}) \cap B$.

We will show that $A$ and $B$ are tangentially transversal at $x_0$ with constants $M := \alpha + 1$, $\delta$ and $\eta := \alpha$.

Let us fix $x^A \in A \cap (x_0 + \delta \bar{B})$, $x^B \in B \cap (x_0 + \delta \bar{B})$ and $t_m \in (0, \min \{\delta, \frac{\|B - A\|}{\alpha}\})$. We put $w_1 := \alpha \frac{x^B - x^A}{\|x^B - x^A\|} \in \alpha \bar{B}$ and $w_2 := 0$. Then, (2) (with $\rho := t_m$) is equivalent to the existence of $u \in \bar{B}$ such that

$$t_m u \in (A - x^A - t_m w_1) \cap (B - x^B - t_m w_2).$$
The last inclusion implies that

\[ x^A + t_m w^A_m \in A \text{ and } x^B + t_m w^B_m \in B , \]

where \( w^A_m := w_1 + u \) and \( w^B_m := w_2 + u \). We also have that \( \|w^A_m\| \leq \alpha + 1 = M \) and \( \|w^B_m\| \leq 1 \leq M \).

We estimate

\[
\|x^A - x^B + t_m (w^A_m - w^B_m)\| = \left\|x^A - x^B + t_m \alpha \frac{x^B - x^A}{\|x^B - x^A\|}\right\|
\]

\[
= \left\|x^A - x^B\right\| \left| 1 - \frac{t_m \alpha}{\|x^B - x^A\|}\right| = \left\|x^A - x^B\right\| - t_m \eta .
\]

This proves the tangential transversality.  

\[\Box\]

**Proposition 2.8.** Let the closed sets \( A \) and \( B \) be tangentially transversal at \( x_0 \in A \cap B \). Then, \( A \) and \( B \) are subtransversal at \( x_0 \).

**Proof.** Let the constants \( M > 0, \delta > 0 \) and \( \eta > 0 \) be from the definition of tangential transversality.

Let us set \( \zeta := \frac{\delta}{2 \left(1 + 2 \frac{M}{\eta}\right)} \in (0, \delta) \). Let \( x \) be an arbitrary element of \( x_0 + \zeta B \). Let us fix an arbitrary \( \varepsilon \in (0, \zeta - \|x - x_0\|) \). We have that there exist \( x^A \in A \) and \( x^B \in B \) such that

\[
\|x^A - x\| < d(x, A) + \varepsilon \text{ and } \|x^B - x\| < d(x, B) + \varepsilon . \tag{3}
\]

Since \( d(x, A) \leq \|x - x_0\| \), we obtain that

\[
\|x^A - x\| < \|x_0 - x\| + \varepsilon < \zeta
\]

and therefore \( x^A \in (x_0 + 2\zeta B) \cap A \). Analogously, \( x^B \in (x_0 + 2\zeta B) \cap B \).

We have that

\[
\max \left\{ \|x^A - x_0\|, \|x^B - x_0\| \right\} + \frac{M}{\eta} \|x^A - x^B\| < 2\zeta + \frac{4M}{\eta} \zeta = \delta .
\]

We can apply Theorem 2.3 and obtain \( x^{AB} \in A \cap B \) with

\[
\|x^{AB} - x^A\| \leq \frac{M}{\eta} \|x^A - x^B\| \text{ and } \|x^{AB} - x^B\| \leq \frac{M}{\eta} \|x^A - x^B\|. \tag{4}
\]
Applying (3) and (4), we obtain
\[
d(x, A \cap B) \leq \|x - x^{AB}\| \leq \|x - x^A\| + \|x^A - x^{AB}\|
\]
\[
< d(x, A) + \varepsilon + \frac{M}{\eta} \|x^A - x^B\| \leq d(x, A) + \varepsilon + \frac{M}{\eta}(\|x^A - x\| + \|x - x^B\|) \leq \left(1 + \frac{M}{\eta}\right) (d(x, A) + d(x, B)) + \varepsilon \left(1 + 2\frac{M}{\eta}\right).
\]

Letting \(\varepsilon\) go to 0 proves the subtransversality with constants \(\zeta > 0\) and \(K := 1 + \frac{M}{\eta} > 0\).

3. A Lagrange multiplier rule

It is our understanding that the following result is crucial for obtaining necessary optimality conditions.

**Proposition 3.1** (Nonseparation result). Let \(A\) and \(B\) be closed subsets of the Banach space \(X\). Let \(A\) and \(B\) be subtransversal at \(x_0 \in A \cap B\) with constants \(\delta > 0\) and \(K > 0\). Let there exist \(v^A\) with unit norm which belongs to the Bouligand tangent cone to \(A\) at \(x_0\), \(v^B\) with unit norm which belongs to the derivable tangent cone to \(B\) at \(x_0\) and let \(\|v^A - v^B\| < \frac{1}{K}\). Then \(A\) and \(B\) cannot be locally separated at \(x_0\).

**Proof.** Since \(v^A\) belongs to the Bouligand tangent cone to \(A\) at \(x_0\), we have that there exist sequences \(t_m \downarrow 0\) and \(v_m^A \to v^A\) such that
\[
x_m^A := x_0 + t_m v_m^A \in A.
\]

Since \(v^B\) belongs to the derivable tangent cone to \(B\) at \(x_0\), we have that for all small enough \(t > 0\) there exists \(v_t^B \in X\), such that \(x_0 + tv_t^B \in B\) and \(v_t^B \to v^B\) as \(t \downarrow 0\). Let us set
\[
x_m^B := x_0 + t_m v_m^B \in B
\]
for \(m\) – large enough. We wrote \(v_m^B\) instead of \(v_t^B\) for the sake of simplicity.

From the triangle inequality we obtain that \(\|v_m^A\| \geq \|v^A\| - \|v_m^A - v^A\| = 1 - \|v_m^A - v^A\|\) and therefore for \(m\) large enough we have
\[
t_m = \frac{\|x_m^A - x_0\|}{\|v_m^A\|} \leq \frac{\|x_m^A - x_0\|}{1 - \|v_m^A - v^A\|}.
\]
and
\[
\|x^A_m - x^B_m\| = t_m \|v^A_m - v^B_m\| \leq \frac{\|x^A_m - x_0\|}{1 - \|v^A_m - v^B_m\|} \left( \|v^A_m - v^A\| + \|v^A - v^B\| + \|v^B - v^B_m\| \right).
\]

We have that
\[
\frac{\|v^A_m - v^A\| + \|v^A - v^B\| + \|v^B - v^B_m\|}{1 - \|v^A_m - v^A\|} \to_{m \to +\infty} \|v^A - v^B\|
\]
and \(\|v^A - v^B\| < \frac{1}{K + \varepsilon}\) for some small enough \(\varepsilon > 0\). Therefore there exists \(m_0 \in \mathbb{N}\) such that
\[
\|x^A_m - x^B_m\| \leq \|x^A_m - x_0\| \cdot \frac{1}{K + \varepsilon}
\]
for all \(m \geq m_0\).

Let \(m_1 \geq m_0\) be such that \(t_m \|v^A_m\| \leq \delta\) and \(t_m \|v^B_m\| \leq \delta\) whenever \(m \geq m_1\). Then, for \(m \geq m_1\) we have
\[
d(x^A_m, A \cap B) \leq K \left( d(x^A_m, A) + d(x^A_m, B) \right) \leq K \cdot d(x^A_m, x^B_m) = K \|x^A_m - x^B_m\|.
\]
From the definition of a distance from a point to a set there exists \(x^{AB}_m \in A \cap B\) with
\[
\|x^{AB}_m - x^A_m\| \leq d(x^A_m, A \cap B) + \frac{\varepsilon}{2} \|x^A_m - x^B_m\|.
\]
Note that if \(x^A_m = x^B_m\) we just put \(x^{AB}_m\) to coincide with these points and all addends are zero. Then
\[
\|x^{AB}_m - x^A_m\| \leq K \|x^A_m - x^B_m\| + \frac{\varepsilon}{2} \|x^A_m - x^B_m\| \leq \|x^A_m - x_0\| \frac{K + \varepsilon/2}{K + \varepsilon} < \|x^A_m - x_0\|
\]
using (5). Therefore \(x^{AB}_m \neq x_0\). Moreover,
\[
\|x^{AB}_m - x_0\| \leq \|x^A_m - x_0\| + \|x^A_m - x^{AB}_m\| \leq 2\|x^A_m - x_0\| \leq 2t_m \|v^A_m\| \to_{m \to +\infty} 0.
\]
Thus, \(x^{AB}_m \to x_0\) and \(A\) and \(B\) cannot be locally separated at \(x_0\). \qed

In the first version of this paper the nonseparation result has been proved under the assumption of tangential transversality instead of subtransversality. We include below the respective formulation because of some small differences in the constants involved.
Proposition 3.2. Let $A$ and $B$ be closed subsets of the Banach space $X$. Let $A$ and $B$ be tangentially transversal at $x_0 \in A \cap B$ with constants $M > 0$, $\delta > 0$ and $\eta > 0$. Let there exist $v^A$ with unit norm which belongs to the Bouligand tangent cone to $A$ at $x_0$, $v^B$ with unit norm which belongs to the derivable tangent cone to $B$ at $x_0$ and let $\|v^A - v^B\| < \frac{\eta}{M}$. Then $A$ and $B$ cannot be locally separated at $x_0$.

We will apply the above nonseparation result to obtain an abstract Lagrange multiplier rule. Let $X$ be a Banach space. We consider $X \times \mathbb{R}$ equipped with the uniform norm $\|(x, r)\| := \max\{\|x\|, |r|\}$. We will need Lemma 3.3 below. It is a natural generalisation of the fact that in finite dimensions if two cones are transversal and one of them is not a subspace, then they are strongly transversal.

Lemma 3.3. Let $\tilde{C}_1$ and $\tilde{C}_2 := C_2 \times (-\infty, 0]$ be closed convex cones in $X \times \mathbb{R}$ (hence $C_2$ is a closed convex cone in $X$). Let $\tilde{C}_1 - \tilde{C}_2$ be dense in $X \times \mathbb{R}$. Then, for each $\varepsilon > 0$ there exist $\tilde{w}_1 \in \tilde{C}_1$ and $\tilde{w}_2 \in \tilde{C}_2$ with unit norm such that $\|\tilde{w}_1 - \tilde{w}_2\| < \varepsilon$.

Proof. Let us fix an arbitrary $\varepsilon \in (0, 1)$. We consider the vector $\tilde{v} := (0, -1) \in X \times \mathbb{R}$. Now the density of $\tilde{C}_1 - \tilde{C}_2$ yields the existence of two vectors $\tilde{v}_i = (v_i, r_i) \in \tilde{C}_i$, $i = 1, 2$, such that

$$\|\tilde{v} - (\tilde{v}_1 - \tilde{v}_2)\| < \frac{\varepsilon}{2},$$

hence

$$\|v_1 - v_2\| < \frac{\varepsilon}{2} \text{ and } |1 - (r_1 - r_2)| = |r_1 - (r_2 - 1)| < \frac{\varepsilon}{2}. \quad (6)$$

Due to the definition of $\tilde{C}_2$ and that $(v_2, r_2) \in \tilde{C}_2$, we have $(v_2, r_2 - 1) \in \tilde{C}_2$. Also,

$$\|(v_2, r_2 - 1)\| \geq |r_2 - 1| \geq 1 \quad (7)$$

since $r_2 \leq 0$. Moreover, $|r_1| \geq |r_2 - 1| - \varepsilon/2 > 1/2$.

Let us set

$$\tilde{w}_1 := \frac{(v_1, r_1)}{\|(v_1, r_1)\|} \in \tilde{C}_1 \text{ and } \tilde{w}_2 := \frac{(v_2, r_2 - 1)}{\|(v_2, r_2 - 1)\|} \in \tilde{C}_2.$$
Apparently, $\|\tilde{w}_1\| = 1$ and $\|\tilde{w}_2\| = 1$. Using (6) and (7), we estimate

\[
\|\tilde{w}_1 - \tilde{w}_2\| = \left\| \frac{(v_1, r_1)}{\|(v_1, r_1)\|} - \frac{(v_2, r_2 - 1)}{\|(v_2, r_2 - 1)\|} \right\|
\leq \left\| \frac{(v_1, r_1)}{\|(v_1, r_1)\|} - \frac{(v_1, r_1)}{\|(v_2, r_2 - 1)\|} \right\| + \left\| \frac{(v_2, r_2 - 1)}{\|(v_2, r_2 - 1)\|} - \frac{(v_2, r_2 - 1)}{\|(v_2, r_2 - 1)\|} \right\|
= \frac{1}{\|(v_2, r_2 - 1)\|} \left( \left\| \frac{(v_1, r_1)}{\|(v_1, r_1)\|} \right\| - \left\| \frac{(v_1, r_1)}{\|(v_2, r_2 - 1)\|} \right\| \right) + \frac{1}{\|(v_2, r_2 - 1)\|} \left( \left\| \frac{(v_2, r_2 - 1)}{\|(v_2, r_2 - 1)\|} \right\| - \left\| \frac{(v_2, r_2 - 1)}{\|(v_2, r_2 - 1)\|} \right\| \right)
\leq 2 \frac{\|v_1 - v_2\|}{\|(v_2, r_2 - 1)\|} \leq 2 \max\{\|v_1 - v_2\|, |r_1 - (r_2 - 1)|\} < \varepsilon.
\]

The proof is complete. \(\square\)

**Theorem 3.4** (Lagrange multiplier rule). Let us consider the optimization problem

\[
f(x) \to \min \text{ subject to } x \in S,
\]

where $f : X \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and proper and $S$ is a closed subset of the Banach space $X$. Let $x_0$ be a solution of the above problem. Let $C_{epif}(x_0, f(x_0))$ and $C_S(x_0)$ be closed convex cones, contained in the corresponding Bouligand approximating cones $T_{epif}(x_0, f(x_0))$ and $T_S(x_0)$. Let at least one of them consist of derivable tangent vectors.

(a) If $C_{epif}(x_0, f(x_0)) - C_S(x_0) \times (-\infty, 0]$ is not dense in $X \times \mathbb{R}$, then there exists a pair $(\xi, \eta) \in X^* \times \mathbb{R}$ such that

(i) $(\xi, \eta) \neq (0, 0)$;

(ii) $\eta \in \{0, 1\}$;

(iii) $(\xi, v) \leq 0$ for every $v \in C_S(x_0)$;

(iv) $(\xi, w) + \eta s \geq 0$ for every $(w, s) \in C_{epif}(x_0, f(x_0))$.

(b) If $C_{epif}(x_0, f(x_0)) - C_S(x_0) \times (-\infty, 0]$ is dense in $X \times \mathbb{R}$, then $epif$ and $S \times (-\infty, f(x_0)]$ are not subtransversal at $(x_0, f(x_0))$.

**Proof.** (a) If $C_{epif}(x_0, f(x_0)) - C_S(x_0) \times (-\infty, 0]$ is not dense in $X \times \mathbb{R}$, then there exist $(\bar{x}, \bar{r}) \in X \times \mathbb{R}$ and $d > 0$ such that

\[
(C_{epif}(x_0, f(x_0)) - C_S(x_0) \times (-\infty, 0]) \cap ((\bar{x}, \bar{r}) + dB_{X \times \mathbb{R}}) = \emptyset.
\]
Then, \[ \tilde{C} \cap \tilde{D} = \emptyset, \]

where \[ \tilde{C} := \tilde{C}_{epif}(x_0, f(x_0)) - C_S(x_0) \times (-\infty, 0] \]

is a closed convex cone and \[ \tilde{D} := \{(x, r) \in X \times \mathbb{R} \mid (x, r) = \alpha((\tilde{x}, \tilde{r}) + d(x_1, r_1)), \quad \alpha > 0, \quad (x_1, r_1) \in B_{X \times \mathbb{R}} \} \]

is an open convex cone (non-empty). We can separate \( \tilde{C} \) and \( \tilde{D} \) and find a non-zero pair \( (\xi, \eta) \in X^* \times \mathbb{R} \) and a real \( \alpha \) such that

\[ \langle \xi, v_1 \rangle + \eta r_1 \geq \alpha > \langle \xi, v_2 \rangle + \eta r_2 \]

for all \((v_1, r_1) \in \tilde{C} \) and \((v_2, r_2) \in \tilde{D} \). Since \((0, 0)\) lies in \( \tilde{C} \) and on the boundary of \( \tilde{D} \), we have that \( \alpha = 0 \). Hence,

\[ \langle \xi, v_1 \rangle + \eta r_1 \geq 0 \]

for all \((v_1, r_1) \in \tilde{C} \), which is

\[ \langle \xi, v' - v'' \rangle + \eta(r' - r'') \geq 0 \]

for all \((v', r') \in \tilde{C}_{epif}(x_0, f(x_0)) \) and \((v'', r'') \in C_S(x_0) \times (-\infty, 0] \). By taking \( v' = v'' = 0, r' = 0 \) and \( r'' < 0 \) we obtain that \( \eta \geq 0 \). Hence without loss of generality we may assume that \( \eta \in \{0, 1\} \). By taking \( v' = 0, v'' = v \in C_S(x_0) \) and \( r' = r'' = 0 \) we obtain that \( \langle \xi, v \rangle \leq 0 \). By taking \( (v', r') = (w, s) \in \tilde{C}_{epif}(x_0, f(x_0)) \) and \((v'', r'') = (0, 0)\), we obtain that \( \langle \xi, w \rangle + \eta s \geq 0 \). (b) Let \( \tilde{C}_{epif}(x_0, f(x_0)) - C_S(x_0) \times (-\infty, 0] \) be dense in \( X \times \mathbb{R} \). Without loss of generality we may assume that \( x_0 \) is a strong minimum of \( f \) on \( S \). This is due to the fact that if \( g : X \to \mathbb{R} \cup \{+\infty\} \) is strictly Fréchet differentiable at \( x_0, g(x_0) = 0 \) and \( g'(x_0) = 0 \), then

\[ T_{epif}(x_0, f(x_0)) = T_{epif(g+g)}(x_0, f(x_0)) \]

and \( G_{epif}(x_0, f(x_0)) = G_{epif(g+g)}(x_0, f(x_0)) \).

Indeed, \((v_0, r_0) \in T_{epif}(x_0, f(x_0))\) if and only if there exist sequences \((v_m, r_m) \to (v_0, r_0)\) and \( t_m \searrow 0 \) such that

\[ (x_0, f(x_0)) + t_m(v_m, r_m) \in \text{epi} f \]

which is equivalent to

\[ \frac{f(x_0 + t_m v_m) - f(x_0)}{t_m} \leq r_m. \]
Let us denote
\[ r_m' := \frac{g(x_0 + t_m v_m) - g(x_0)}{t_m} = \frac{g(x_0) + \langle g'(x_0), t_m v_m \rangle + o(\|t_m v_m\|) - g(x_0)}{t_m} \to 0. \]

Then,
\[ \frac{(f + g)(x_0 + t_m v_m) - (f + g)(x_0)}{t_m} \leq r_m + r_m' \]
which is equivalent to
\[ (x_0, (f + g)(x_0)) + t_m(v_m, r_m + r_m') \in \text{epi}(f + g) \]
for the sequences \((v_m, r_m + r_m') \to (v_0, r_0)\) and \(t_m \searrow 0\). This verifies that \(T_{\text{epi}}(x_0, f(x_0)) \subset T_{\text{epi}}(f + g)(x_0, f(x_0))\). As \(-g\) satisfies the same assumptions as \(g\), the reverse inclusion is verified as well. The proof for derivable tangent cones is analogous. By putting \(g(x) := \|x - x_0\|^2\), we obtain that
\[ \tilde{C}_{\text{epi}}(x_0, f(x_0)) = C_{\text{epi}}(f + g)(x_0, f(x_0)) \]
and \(x_0\) is a strong minimum of \(f + g\) on \(S\).

Let us assume that \(\text{epi} f\) and \(\tilde{S} := S \times (-\infty, f(x_0)]\) are subtransversal at \((x_0, f(x_0))\) with constant \(K > 0\). By applying Lemma 3.3 for \(\varepsilon := \frac{1}{K}\) and then Proposition 3.1, we obtain that the sets \(\text{epi} f\) and \(\tilde{S}\) can not be separated. That is, there exists a sequence \((x_m, r_m) \in \text{epi} f \cap \tilde{S}\) converging to \((x_0, f(x_0))\) such that \((x_m, r_m) \neq (x_0, f(x_0))\) for every positive integer \(m\). But \((x_m, r_m) \in \text{epi} f \cap \tilde{S}\) implies that \(r_m \geq f(x_m)\) and \(r_m \leq f(x_0)\). Because \(x_0\) is a strong local minimum of \(f\) on \(S\), for each sufficiently large \(m\) the following inequalities hold true \(r_m \geq f(x_m) > f(x_0) \geq r_m\), a contradiction.

Therefore \(\text{epi} f\) and \(\tilde{S} := S \times (-\infty, f(x_0)]\) are not tangentially transversal at \((x_0, f(x_0))\), which completes the proof. \(\square\)

4. Intersection properties

Let \(A\) and \(B\) be two smooth manifolds and \(x_0 \in A \cap B\). The classical meaning of transversality in this case is that the tangent space to the manifold \(A \cap B\) at the point \(x_0\) equals the intersection of the tangent spaces to \(A\) and \(B\), respectively, at \(x_0\). Next we obtain some tangential intersection properties as corollaries of subtransversality.
Proposition 4.1 (Intersection property with respect to Bouligand and derivable tangent cones). Let $A$ and $B$ be closed subsets of the Banach space $X$ and let $A$ and $B$ be subtransversal at $x_0 \in A \cap B$. Then,

$$T_A(x_0) \cap G_B(x_0) \subset T_{A \cap B}(x_0),$$

where $T_A(x_0)$ ($T_{A \cap B}(x_0)$) is the Bouligand tangent cone to $A$ ($A \cap B$) at $x_0$ and $G_B(x_0)$ is the derivable tangent cone to $B$ at $x_0$. Moreover,

$$G_A(x_0) \cap G_B(x_0) = G_{A \cap B}(x_0).$$

Proof. Let $v_0$ be in $T_A(x_0) \cap G_B(x_0)$. Without loss of generality, we may assume that $\|v_0\| = 1$. Since $v_0$ belongs to the Bouligand tangent cone to $A$ at $x_0$, we have that there exist sequences $t_m \searrow 0$ and $v^A_m \rightarrow v_0$ such that $x_0 + t_m v^A_m \in A$. Since $v_0$ belongs to the derivable tangent cone to $B$ at $x_0$, we have that for all small enough $t > 0$ there exists $v^B_t \in X$, such that $x_0 + tv^B_t \in B$ and $v^B_t \rightarrow v_0$ as $t \searrow 0$. Therefore, $x_0 + t_m v^B_m \in B$ for $m \in \mathbb{N}$ large enough and $v^B_m \rightarrow v_0$ (here $v^B_m := v^B_{t_m}$).

Let us fix an arbitrary positive $\varepsilon$. Let $K$ and $\delta$ be the constants from the definition of subtransversality. Then, there exists $m_0 \in \mathbb{N}$ such that

$$\|v^A_m - v_0\| \leq \frac{\varepsilon}{2K + 3} \text{ and } \|v^B_m - v_0\| \leq \frac{\varepsilon}{2K + 3}$$

for all $m \geq m_0$. Let $m_1 \geq m_0$ be such that

$$t_m \leq \frac{\delta}{1 + \varepsilon} \text{ for all } m \geq m_1$$

and let us denote

$$x^A_m := x_0 + t_m v^A_m \in A \text{ and } x^B_m := x_0 + t_m v^B_m \in B \text{ for all } m \geq m_1.$$

It is straightforward that

$$\|x^A_m - x^B_m\| = t_m \|v^A_m - v^B_m\| \leq t_m \left(\|v^A_m - v_0\| + \|v_0 - v^B_m\|\right) \leq \frac{2\varepsilon}{2K + 3} t_m.$$

Since

$$\|x^A_m - x_0\| = t_m \|v^A_m\| \leq t_m \left(\|v_0\| + \frac{\varepsilon}{2K + 3}\right) \leq \frac{\delta}{1 + \varepsilon} \left(1 + \frac{\varepsilon}{2K + 3}\right) < \delta$$

18
and analogously $\|x^B_m - x_0\| < \delta$, for $m \geq m_1$ we have
\[d(x^A_m, A \cap B) \leq K \left(d(x^A_m, A) + d(x^A_m, B)\right) \leq K \cdot d(x^A_m, x^B_m) = K \|x^A_m - x^B_m\|.
\]

From the definition of a distance from a point to a set there exists $x_{AB}^m \in A \cap B$ with
\[\|x_{AB}^m - x_m^A\| \leq d(x^A_m, A \cap B) + \|x^A_m - x^B_m\| \leq (K + 1) \|x^A_m - x^B_m\|.
\]

Note that if $x^A_m = x^B_m$ we just put $x_{AB}^m$ to coincide with these points and all addends are zero. We estimate
\[
\|x_{AB}^m - x_0\| = \|x_{AB}^m - (x_0 + t_m v_0)\| = \|x_{AB}^m - (x_0 + t_m v^A_m) - t_m (v_0 - v^A_m)\|
\]
\[
\leq \|x_{AB}^m - x^A_m\| + t_m \|v_0 - v^A_m\| \leq (K + 1) \|x^A_m - x^B_m\| + t_m \frac{\varepsilon}{2K + 3}
\]
\[
\leq (K + 1) \cdot \frac{2\varepsilon}{2K + 3} t_m + \frac{\varepsilon}{2K + 3} t_m = \varepsilon t_m.
\]

Hence, for $m \geq m_1$, the following is true
\[x_{AB}^m \in x_0 + t_m (v_0 + \varepsilon B).
\]

We have obtained that for every $v_0 \in T_A(x_0) \cap G_B(x_0)$, $\|v_0\| = 1$ and for every $\varepsilon > 0$ there exists $m_1 \in \mathbb{N}$ such that
\[(A \cap B) \cap (x_0 + t_m (v_0 + \varepsilon B)) \neq \emptyset
\]
for all $m \geq m_1$. From this, it follows that $T_A(x_0) \cap G_B(x_0)$ is a subset of $T_{A \cap B}(x_0)$.

The inclusion $G_A(x_0) \cap G_B(x_0) \subset G_{A \cap B}(x_0)$ can be proved in the same way. Then, the equality for the derivable cones follows from their monotonicity.

**Proposition 4.2** (Intersection property with respect to Clarke tangent cones). Let $A$ and $B$ be closed subsets of the Banach space $X$ and let $A$ and $B$ be subtransversal at $x_0 \in A \cap B$. Then,
\[
\hat{T}_A(x_0) \cap \hat{T}_B(x_0) \subset \hat{T}_{A \cap B}(x_0),
\]
where $\hat{T}_S(x_0)$ is the Clarke tangent cone to $S$ at $x_0$. 

19
Proof. Let us fix a positive $\varepsilon$ and an arbitrary $v_0 \in \hat{T}_A(x_0) \cap \hat{T}_B(x_0)$. Let $\delta > 0$ and $K > 0$ be the constants from the definition of subtransversality of $A$ and $B$ at $x_0$.

From the definition of Clarke tangent cone for $v_0$, we have that for $\eta := \frac{\varepsilon}{2K+3} > 0$ there exists $\delta_A > 0$ such that for all $x \in (x_0 + \delta_A \bar{B}) \cap A$ and for all $t \in (0, \delta_A)$ it holds true that
\[(x + t(v_0 + \eta \bar{B})) \cap A \neq \emptyset\]
and correspondingly, there exists $\delta_B > 0$ such that for all $x \in (x_0 + \delta_B \bar{B}) \cap B$ and for all $t \in (0, \delta_B)$ it holds true that
\[(x + t(v_0 + \eta \bar{B})) \cap B \neq \emptyset.

Let us fix $\bar{\delta} := \min \{\frac{\varepsilon}{2}, \delta_A, \delta_B\} > 0$ and an arbitrary $\bar{x} \in (x_0 + \bar{\delta} \bar{B}) \cap (A \cap B)$. Let $h_0$ be an arbitrary positive real satisfying
\[h_0 \leq \bar{h} := \min \left\{\bar{\delta}, \frac{\delta}{2(\eta + \|v_0\|)}\right\}.
\]
We obtain that there exist vectors $v_0^A \in X$ and $v_0^B \in X$ such that
\[\|v_0^A - v_0\| \leq \eta, \quad \|v_0^B - v_0\| \leq \eta\]
and
\[x^A := \bar{x} + h_0 v_0^A \in A, \quad x^B := \bar{x} + h_0 v_0^B \in B.
\]
Taking into account (9), we can verify directly that
\[\|x^A - x^B\| = h_0 \|v_0^A - v_0^B\| = h_0 \|v_0^A - v_0 + v_0 - v_0^B\| \leq 2\eta h_0\]
and
\[\|x^A - \bar{x}\| = h_0 \|v_0^A\| = h_0 \|v_0^A - v_0 + v_0\| \leq h_0(\eta + \|v_0\|).
\]
Analogously, we obtain that
\[\|x^B - \bar{x}\| \leq h_0(\eta + \|v_0\|).
\]
We have that
\[\|x^A - x_0\| \leq \|x^A - \bar{x}\| + \|\bar{x} - x_0\| \leq h_0(\eta + \|v_0\|) + \bar{\delta} \leq \frac{\varepsilon}{2} + \bar{\delta} \leq \delta\]
using the estimate of \( h_0 \) and the definition of \( \bar{\delta} \). Analogously, \( \|x^B - x_0\| \leq \bar{\delta} \).

Therefore,

\[
d (x^A, A \cap B) \leq K (d (x^A, A) + d (x^A, B)) \leq K \cdot d (x^A, x^B) = K \|x^A - x^B\|.
\]

From the definition of a distance from a point to a set there exists \( x^{AB} \in A \cap B \) with

\[
\|x^{AB} - x^A\| \leq d (x^A, A \cap B) + \|x^A - x^B\| \leq (K + 1) \|x^A - x^B\|.
\]

Note that if \( x^A = x^B \) we just put \( x^{AB} \) to coincide with these points and all addends are zero.

We estimate

\[
\|x^{AB} - (\bar{x} + h_0 v_0)\| = \|x^{AB} - (\bar{x} + h_0 v^A_0) - h_0 (v_0 - v^A_0)\|
\]

\[
\leq \|x^{AB} - x^A\| + h_0 \|v_0 - v^A_0\| \leq (K + 1) \|x^A - x^B\| + h_0 \eta
\]

\[
\leq (K + 1) 2 \eta h_0 + h_0 \eta = h_0 \eta (2K + 3) = \varepsilon h_0.
\]

Hence,

\[
x^{AB} \in (A \cap B) \cap (\bar{x} + h_0 (v_0 + \varepsilon B)).
\]

We have obtained that for every \( v_0 \in \hat{T}_A(x_0) \cap \hat{T}_B(x_0) \) and for every \( \varepsilon > 0 \) there exists \( \bar{\delta} > 0 \) such that for each point \( \bar{x} \in (A \cap B) \cap (x_0 + \bar{\delta} B) \), there exists \( h \) such that

\[
(A \cap B) \cap (\bar{x} + h_0 (v_0 + \varepsilon B)) \neq \emptyset
\]

for each \( h_0 \in [0, \tilde{h}] \). Therefore, \( v_0 \in \hat{T}_{A \cap B}(x_0) \). This completes the proof. \( \square \)

**Remark 4.3.** It is remarkable that the same intersection properties (cf. Proposition 4.4 and Proposition 4.2) appear in [1] back in 1990. Proposition 4.4 shows that the “local transversality condition” assumed in [1] is a sufficient condition for tangential transversality. Hence, because tangential transversality implies subtransversality, Corollary 4.3.5 in [1] follows from our Proposition 4.4 and Proposition 4.2.

**Proposition 4.4.** Let \( A \) and \( B \) be closed subsets of the Banach space \( X \) and let \( x_0 \in A \cap B \). If there exist constants \( \bar{\delta} > 0, \alpha \in [0, 1) \) and \( M > 0 \) such that for each \( x^A \in (x_0 + \bar{\delta} B) \cap A \) and for each \( x^B \in (x_0 + \bar{\delta} B) \cap B \) it is true that \( B \subset (G_A(x^A) \cap M B) - T_B(x^B) + \alpha B \), then the sets \( A \) and \( B \) are tangentially transversal at \( x_0 \).
Proof. Let us fix an arbitrary positive real \( \eta < 1 - \alpha \) and check that \( A \) and \( B \) are tangentially transversal at \( x_0 \) with constants \( \delta, M + 3 \) and \( \eta \).

Let us choose arbitrary \( x^A \in (x_0 + \delta \bar{B}) \cap A \) and \( x^B \in (x_0 + \delta \bar{B}) \cap B \). Then the vector
\[
v := \frac{x^B - x^A}{\|x^B - x^A\|}
\]
is of norm one, and therefore there exist vectors \( w^A \in G_A(x^A), \|w^A\| \leq M \) and \( w^B \in T_B(x^B) \) such that \( \|v - (w^A - w^B)\| \leq \alpha \). Then \( w^B \in T_B(x^B) \) implies the existence of sequences \( t_m \downarrow 0 \) and \( w^B_m \to w^B \) such that \( x^B + t_m w^B_m \in B \). Since \( w^A \) belongs to the derivable tangent cone to \( A \) at \( x^A \), we have that for all small enough \( t > 0 \) there exists \( w^A_t \in X \), such that \( x^A + tw^A_t \in A \) and \( w^A_t \to w^A \) as \( t \to 0 \). Therefore, \( x^A + t_m w^A_m \in A \) for \( m \in \mathbb{N} \) large enough and \( w^A_m \to w^A \) (here \( w^A_m := w^A_{t_m} \)). Moreover,
\[
\|x^A - x^B + t_m(w^A_m - w^B_m)\| \leq \|x^A - x^B + t_m v\| + t_m \|w^A - w^B - v\| + t_m \|w^A_m - w^A\| + t_m \|w^B_m - w^B\| \leq \\
\|x^A - x^B\| - t_m \|w^A - w^B\| + t_m \|w^A_m - w^A\| + t_m \|w^B_m - w^B\| .
\]
Then for all \( m \) big enough (for which \( \|w^A_m - w^A\| \leq (1 - \alpha - \eta)/2, \|w^B_m - w^B\| \leq (1 - \alpha - \eta)/2 \) the estimate
\[
\|x^A - x^B + t_m(w^A_m - w^B_m)\| \leq \|x^A - x^B\| - t_m \eta
\]
holds true. It remains to note that
\[
\|w^A_m\| \leq \|w^A\| + \frac{1 - \alpha - \eta}{2} \leq M + 1 \quad \text{and}
\]
\[
\|w^B_m\| \leq \|w^B\| + \frac{1 - \alpha - \eta}{2} \leq \alpha + \|w^A\| + \|v\| + 1 \leq M + 3 .
\]

\[\square\]

Remark 4.5. It is also remarkable that in 1982 subtransversality is proven to be a sufficient condition for a tangential intersection property for Dubovitzki-Milyutin tangent cones (even with equality) by Dolecki in \cite{Dolecki82}. The word “subtransversality” is not mentioned, but the distance inequality from its definition is used instead.
5. Almost massive sets

We are going to consider a formal generalization of the classical concept of compactly epi-Lipschitz sets in Banach spaces. It was introduced by J.M. Borwein and H.M. Strojwas in 1985 in [4] as appropriate for investigating tangential approximations of the Clarke tangent cone in Banach spaces. Since then, it has been an important notion in nonsmooth analysis and has been frequently used in qualification conditions for obtaining normal intersection properties and calculus rules concerning limiting Fréchet cones and subdifferentials (in Asplund spaces, cf. [22] and [23]) and $G$-cones and $G$-subdifferentials (in general Banach spaces, cf. [13]). Compactly epi-Lipschitz sets are called massive in [14]. Here is the corresponding definition:

**Definition 5.1.** Let $A$ be a closed subset of the Banach space $X$ and $x_0 \in A$. We say that $A$ is compactly epi-Lipschitz (massive) at $x_0$, if there exist $\varepsilon > 0$, $\delta > 0$ and a compact set $K \subset X$, such that for all $x \in A \cap (x_0 + \delta B)$, for all $v \in X$, $\|v\| \leq \varepsilon$ and for all $t \in [0, \delta]$, there exists $k \in K$, for which $x + t(v - k) \in A$.

In our definition of almost massive sets, the “correction” set $K$ is assumed to have a finite net with fixed radius, instead of being compact:

**Definition 5.2.** Let $A$ be a closed subset of the Banach space $X$ and $x_0 \in A$. We say that $A$ is almost massive at $x_0$, if there exist $\varepsilon > 0$, $\delta > 0$ and a set $K \subset X$ with a finite $\varepsilon q$-net for some $q \in (0, 1)$, such that for all $x \in A \cap (x_0 + \delta B)$, for all $v \in X$, $\|v\| \leq \varepsilon$ and for all $t \in [0, \delta]$, there exists $k \in K$, for which $x + t(v - k) \in A$.

During the preparation of this paper we didn’t know the exact relation between almost massive sets and massive sets. After the submission of the manuscript we proved that the class of almost massive sets coincides with the class of massive sets (cf. [24]).

Using the concept of almost massive sets, we are able to prove the following sufficient condition for tangential transversality.

**Theorem 5.3.** Let $A$ and $B$ be closed subsets of the Banach space $X$ and let $x_0 \in A \cap B$. Let $A$ be almost massive and $\hat{T}_A(x_0) - \hat{T}_B(x_0)$ be dense in $X$. Then $A$ and $B$ are tangentially transversal at $x_0$. 

23
Proof. Let $\varepsilon > 0$, $\delta > 0$, the set $K$ and $q \in (0, 1)$ be those from the definition of $A$ – almost massive at $x_0$. Let the set $F := \{k_1, k_2, \ldots, k_n\}$ be an $\varepsilon q$–net for $K$. Let us set $\eta := \frac{\varepsilon(1-q)}{4}$.

Due to the density of $\hat{T}_A(x_0) - \hat{T}_B(x_0)$ in $X$, we obtain that for all $s \in \{1, \ldots, n\}$, there exist $w^A_s \in \hat{T}_A(x_0)$ and $w^B_s \in \hat{T}_B(x_0)$ such that
\[ \|k_s - (w^A_s - w^B_s)\| \leq \eta. \]

From the definition of Clarke tangent cone for $w^A_s \in \hat{T}_A(x_0)$, we have that there exists $\delta^A_s > 0$ such that for all $x \in (x_0 + \delta^A_sB) \cap A$ and for all $t \in (0, \delta^A_s)$ it holds true that
\[ (x + t(w^A_s + \eta B)) \cap A \neq \emptyset. \]

Analogously, for $w^B_s \in \hat{T}_B(x_0)$, we have that there exists $\delta^B_s > 0$ such that for all $x \in (x_0 + \delta^B_sB) \cap B$ and for all $t \in (0, \delta^B_s)$ it holds true that
\[ (x + t(w^B_s + \eta B)) \cap B \neq \emptyset. \]

We set $N := \max\{\|k_s\| : s = 1, \ldots, n\}$, $M := \max\{\|w^A_s\|, \|w^B_s\| : s = 1, \ldots, n\}$, $\tilde{\varepsilon} := N + \varepsilon(1 + q) + \eta$ and
\[ \tilde{\delta} := \min\{\varepsilon, \delta, \frac{\delta^A_s}{1 + N + 2\varepsilon} : s = 1, \ldots, n\}. \]

Let $x^A \in (x_0 + \tilde{\delta}B) \cap A$ and $x^B \in (x_0 + \tilde{\delta}B) \cap B$ and $t \in (0, \min\{\tilde{\delta}, \frac{\|x^A - x^B\|}{\varepsilon}\})$ be arbitrary. Let us set
\[ v := -\frac{x^A - x^B}{\|x^A - x^B\|}. \]

Then, $\|\varepsilon v\| = \varepsilon$, $0 < t < \tilde{\delta} \leq \varepsilon$, $x^A \in (x_0 + \tilde{\delta}B) \cap A \subset (x_0 + \tilde{\delta}B) \cap A$ and therefore there exists $k \in K$ such that
\[ \tilde{x}^A := x^A + t(\varepsilon v - k) \in A. \]

Since $k \in K$, then $\|k - k_s\| \leq \varepsilon q$ for some $s \in \{1, \ldots, n\}$. We estimate
\[ \|\tilde{x}^A - x_0\| \leq \|x^A - x_0\| + t\|\varepsilon v - k\| \leq \tilde{\delta} + \tilde{\delta}(\|\varepsilon v\| + \|k_s\| + \|k - k_s\|) \]
\[ \leq \tilde{\delta}(1 + \varepsilon + N + \varepsilon q) < \tilde{\delta}(1 + N + 2\varepsilon) \leq \delta^A \]
and therefore
\[ (\tilde{x}^A + t(w^A_s + \eta B)) \cap A \neq \emptyset. \]
Then, there exists \( w^A \in X \), \( \| w^A - w^A_s \| \leq \eta \), such that \( \tilde{x}^A + tw^A \in A \) and we obtain
\[
\tilde{x}^A + tw^A = x^A + t(\varepsilon v - k) + tw^A = x^A + t(\varepsilon v - k + w^A) \in A
\]
and
\[
\| \varepsilon v - k + w^A \| = \| \varepsilon v - k_s + (k_s - k) + (w^A - w^A_s) + w^A_s \| \leq \varepsilon + N + \varepsilon q + \| w^A_s \| \leq M.
\]
For \( x^B \in (x_0 + \delta \bar{B}) \cap B \subset (x_0 + \delta_B \bar{B}) \cap B \) we have that
\[
(x^B + t(w^B_s + \eta \bar{B})) \cap B \neq \emptyset,
\]
which implies that there exists \( w^B \in X \), \( \| w^B - w^B_s \| \leq \eta \), such that
\[
x^B + tw^B \in B.
\]
Obviously \( \| w^B \| \leq \| w^B_s \| + \eta < M \).

We estimate
\[
\begin{align*}
\| (x^A + t(\varepsilon v - k + w^A)) - (x^B + tw^B) \| \\
&= \| x^A - x^B + t\varepsilon v + t(w^A - k - w^B) \| \leq \| x^A - x^B + t\varepsilon v \| + \| t(w^A - k - w^B) \| \\
&\leq \| x^A - x^B \| 1 - \frac{t\varepsilon}{\| x^A - x^B \|} + \\
&\quad + t \| w^A - w^A_s + w^B_s - w^B + k_s - k + (w^A_s - w^B_s - k_s) \| \\
&\leq \| x^A - x^B \| - t\varepsilon + t(3\eta + \varepsilon q) = \| x^A - x^B \| - t\left( \varepsilon - \varepsilon q - \frac{3\varepsilon(1-q)}{4} \right) \\
&= \| x^A - x^B \| - t\eta,
\end{align*}
\]
where \( \eta := \frac{\varepsilon(1-q)}{4} > 0 \).

This verifies the definition of \( A \) and \( B \) - tangentially transversal at \( x_0 \) with constants \( M > 0 \), \( \delta > 0 \) and \( \eta > 0 \).

**Corollary 5.4.** Let \( A \) and \( B \) be closed subsets of the Banach space \( X \) and let \( x_0 \in A \cap B \). Let \( A \) be almost massive and \( \hat{T}_A(x_0) - \hat{T}_B(x_0) \) be dense in \( X \). Then,
\[
N_{A \cap B}(x_0) \subset N_A(x_0) + N_B(x_0),
\]
where \( N_S(x) \) is the \( G \)-normal cone to the set \( S \) at the point \( x \).
Proof. Due to Theorem 5.3, $A$ and $B$ are tangentially transversal. Tangential transversality implies subtransversality due to Proposition 2.8 and subtransversality implies (10) due to Theorem 7.13 in [14].

Let $f_1 : X \to \mathbb{R}\cup\{+\infty\}$ and $f_2 : X \to \mathbb{R}\cup\{+\infty\}$ be lower semicontinuous and proper and $x_0 \in X$ be in $\text{dom} f_1 \cap \text{dom} f_2$. We are going to apply the results from this section to the closed sets

$$C_1 := \{(x, r_1, r_2) \in X \times \mathbb{R} \times \mathbb{R} \mid r_1 \geq f_1(x)\}$$

and

$$C_2 := \{(x, r_1, r_2) \in X \times \mathbb{R} \times \mathbb{R} \mid r_2 \geq f_2(x)\}$$

in order to obtain a sum rule for the $G$-subdifferential. This is the approach introduced by Ioffe in [11]. We will need the following technical lemma.

Lemma 5.5. The following are equivalent

(i) $\hat{T}_{\text{epi} f_1}(x_0, f_1(x_0)) - \hat{T}_{\text{epi} f_2}(x_0, f_2(x_0))$ is dense in $X \times \mathbb{R}$

(ii) $\hat{T}_{C_1}(x_0, f_1(x_0), f_2(x_0)) - \hat{T}_{C_2}(x_0, f_1(x_0), f_2(x_0))$ is dense in $X \times \mathbb{R} \times \mathbb{R}$

(iii) $\{N^C_{C_1}(x_0, f_1(x_0), f_2(x_0))\} \cap \{-N^C_{C_2}(x_0, f_1(x_0), f_2(x_0))\} = \{(0, 0, 0)\}$, where $N^S_\infty(x)$ is the Clarke normal cone to the set $S$ at the point $x$

(iv) $\{\partial^C f_1(x_0)\} \cap \{-\partial^C f_2(x_0)\} = \{0\}$, where $\partial^C$ is the Clarke singular subdifferential.

Proof. We have that

$N^C_{C_1}(x_0, f_1(x_0), f_2(x_0)) = \{(x^*, s_1, 0) \in X^* \times \mathbb{R} \times \mathbb{R} \mid (x^*, s_1) \in N^C_{\text{epi} f_1}(x_0, f_1(x_0))\}$ and

$N^C_{C_2}(x_0, f_1(x_0), f_2(x_0)) = \{(x^*, s_2, 0) \in X^* \times \mathbb{R} \times \mathbb{R} \mid (x^*, s_2) \in N^C_{\text{epi} f_2}(x_0, f_2(x_0))\}.$

The reals $s_1$ and $s_2$ in the expressions above are non-positive by polarity, since the vector $(0, 1)$ is always contained in a tangent cone to the epigraph of a function. Using this and the definition of singular subdifferential, we obtain that (iv) is equivalent to (iii), which is equivalent to (ii) by polarity.

Using again that (iv) holds if and only if

$$\{N^C_{\text{epi} f_1}(x_0, f_1(x_0))\} \cap \{-N^C_{\text{epi} f_2}(x_0, f_2(x_0))\} = \{(0, 0, 0)\},$$

we obtain that it is equivalent to (i) by polarity. \qed
**Corollary 5.6.** Let \( f_1 : X \to \mathbb{R} \cup \{+\infty\} \) and \( f_2 : X \to \mathbb{R} \cup \{+\infty\} \) be lower semicontinuous and proper and \( x_0 \in X \) be in \( \text{dom} f_1 \cap \text{dom} f_2 \). Let \( \text{epi} f_1 \) be almost massive and
\[
\partial_C^\infty f_1(x_0) \cap \{-\partial_C^\infty f_2(x_0)\} = \{0\}.
\] (11)

Then,
\[
\partial_G(f_1 + f_2)(x_0) \subset \partial_G f_1(x_0) + \partial_G f_2(x_0),
\]
where \( \partial_G \) is the \( G \)-subdifferential.

**Proof.** Let us set
\[
C_i := \{(x, r_1, r_2) \in X \times \mathbb{R} \times \mathbb{R} \mid r_i \geq f_i(x)\} \quad \text{for} \quad i = 1, 2.
\]

We have that the qualification condition (11) is equivalent to
\[
\hat{T}_{C_1}(x_0, f_1(x_0), f_2(x_0)) - \hat{T}_{C_2}(x_0, f_1(x_0), f_2(x_0)) = X \times \mathbb{R} \times \mathbb{R}.
\] (12)
due to Lemma 5.5.

Since \( C_1 \) is almost massive, we can apply Corollary 5.4 and obtain that
\[
N_{C_1 \cap C_2}(x_0, f_1(x_0), f_2(x_0)) \subset N_{C_1}(x_0, f_1(x_0), f_2(x_0)) + N_{C_2}(x_0, f_1(x_0), f_2(x_0)).
\]
It is direct that
\[
C_1 \cap C_2 \subset C := \{(x, r_1, r_2) \in X \times \mathbb{R} \times \mathbb{R} \mid r_1 + r_2 \geq f_1(x) + f_2(x)\}
\]
and by Lemma 5.5 in [12]
\[
N_C(x_0, f_1(x_0), f_2(x_0)) \subset N_{C_1 \cap C_2}(x_0, f_1(x_0), f_2(x_0))
\subset N_{C_1}(x_0, f_1(x_0), f_2(x_0)) + N_{C_2}(x_0, f_1(x_0), f_2(x_0)).
\]
Since \( x^* \in \partial_G(f_1 + f_2)(x_0) \iff (x^*, -1, -1) \in N_C(x_0, f_1(x_0), f_2(x_0)) \), the proof is complete.

The following statement is an abstract Lagrange multiplier rule.

**Corollary 5.7.** Let us consider the optimization problem
\[
f(x) \to \min \quad \text{subject to} \quad x \in S,
\]
where \( f : X \to \mathbb{R} \cup \{+\infty\} \) is lower semicontinuous and proper and \( S \) is a closed subset of the Banach space \( X \). Let \( x_0 \) be a solution of the above problem. If \( \text{epi} f \) is almost massive at \((x_0, f(x_0))\), then there exists a pair \((\xi, \eta) \in X^* \times \mathbb{R}\) such that
(i) \((\xi, \eta) \neq (0, 0)\);

(ii) \(\eta \in \{0, 1\}\);

(iii) \(\langle \xi, v \rangle \leq 0 \) for every \(v \in \hat{T}_S(x_0)\);

(iv) \(\langle \xi, w \rangle + \eta s \geq 0 \) for every \((w, s) \in \hat{T}_{epif}(x_0, f(x_0))\).

**Proof.** The corollary follows from Theorem 3.4 and Theorem 5.3.

Having in mind that almost massive and massive sets coincide, Corollaries 5.4 and 5.6 are not improvements of known results, just these assertions are obtained in a different way (cf. [12], [16])). To our knowledge, Theorem 5.3 (even replacing tangential transversality by subtransversality in the conclusion) and Corollary 5.7 are new.

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