A NEW DISCRETE ANALYSIS OF FOURTH ORDER ELLIPTIC VARIATIONAL INEQUALITIES

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ABSTRACT. This paper applies the gradient discretisation method (GDM) for fourth order elliptic variational inequalities. The GDM provides a new formulation of error estimates and a complete discrete analysis of different approximating schemes. We show that the convergence is unconditional. Classical assumptions on data are only sufficient to establish the convergence results. These results are applicable to all schemes that fall in the framework of GDM.

1. INTRODUCTION

Fourth order variational inequalities are used to model numerous problems arising in mechanics and physics [16, 18, 21]. Mathematical results corresponding to the well-posedness, stability, and regularity of the solutions to obstacle problems can be found in [3, 5, 11, 13, 14, 16, 19, 20].

Numerically, fourth order variational inequalities have been approximated by several schemes. [10] establishes a generic convergence rate of conforming methods applied to variational inequalities. With smooth data, [4] drives the best error estimate for quadratic and linear finite element methods. Error estimates for discontinuous Galerkin methods have been provided in [6]. Without studying the convergence rates, [12] solves variational inequalities by mixed finite element methods. [17, 23] generates a convergence order for the non conforming finite element method.

Although different studies apply finite volume methods (as in [1, 2, 9, 15, 22]) to second order variational inequalities, it seems that such these schemes have not yet been designed for the fourth order variational inequalities.

The purpose of this work is to extend the gradient discretisation methods [7] to elliptic variational inequalities with a fourth order operator to obtain general error estimates and convergence analysis that hold for different conforming and non conforming numerical methods.

The outlines of this paper are as follows: Section 2 states the formulation of continuous problems and its full discrete scheme. Section 3 introduces and proves the main theoretical results, error estimates, and convergences.
2. Continuous and discrete setting

Let Ω ⊂ Rd (d > 1) be a bounded connected domain with boundary ∂Ω, f ∈ L²(Ω), the barrier ψ ∈ C²(Ω) ∩ C(Ω) and ψ ≥ 0 on ∂Ω. In this paper, we study here the following fourth order variational inequality: Seek ¯c ∈ K satisfying

\[ \int_{\Omega} \Delta \bar{c}(x) \Delta (\bar{c}(x) - \varphi(x)) \, dx \leq \int_{\Omega} f(x)(\bar{c}(x) - \varphi(x)) \, dx, \quad \forall \varphi \in \mathcal{K}, \quad (2.1) \]

where the non empty closed convex set K is defined by

\[ K := \{ \varphi \in H^2_0(\Omega) : \varphi \leq \psi \text{ in } \Omega \}. \]

Note that the standard theory founded in [16] shows that the above problem is well-posed. We see that the model can be formulated by the following equivalent energy minimization equation:

Find \( \bar{c} \in K \) such that, for all \( \varphi \in \mathcal{K} \),

\[ \frac{1}{2} \int_{\Omega} \Delta \bar{c}(x) \Delta \varphi(x) \, dx - \int_{\Omega} f(x) \varphi(x) \, dx. \]

Here, we define the discrete elements (called gradient discretisation) to construct the approximation scheme for our problem.

**Definition 2.1.** Let Ω be an open domain of Rd (d > 1). A gradient discretisation \( D \) for fourth order obstacle problems is given by a family \( D = (X_{D,0}, \Pi_D, \nabla_D, \Delta_D) \), where:

- The discrete set \( X_{D,0} \) is a finite-dimensional vector space on \( \mathbb{R} \), dealing with the unknowns of the method.
- The linear operator \( \Pi_D : X_{D,0} \to L^2(\Omega) \) is the reconstruction of the approximate function.
- The linear operator \( \nabla_D : X_{D,0} \to L^2(\Omega)^d \) is the reconstruction of the gradient of the function.
- \( \Delta_D : X_{D,0} \to L^2(\Omega) \) is a linear mapping to construct a discrete of the biharmonic ∆ form, and must be defined so that \( \| \Delta_D \cdot \|_{L^2(\Omega)} \) is a norm on \( X_{D,0} \).

**Definition 2.2.** Let \( D \) be a gradient discretisation. The discrete version of the problem (2.1) is given by

\[ \int_{\Omega} \Delta_D \varphi(x) \Delta_D (\varphi(x) - \varphi(x)) \, dx \leq \int_{\Omega} f(x) \varphi(x) \, dx. \quad (2.2) \]

The accuracy of this approximation can be measured by the following three indicators; the first one is the constant \( C_D \), which measures the coercivity and is defined by

\[ C_D = \max_{\omega \in X_{D,0} \setminus \{0\}} \left( \frac{\| \Pi_D \omega \|_{L^2(\Omega)} \| \nabla_D \omega \|_{L^2(\Omega)^d}}{\| \Delta_D \omega \|_{L^2(\Omega)} \| \Delta_D \omega \|_{L^2(\Omega)}} \right). \quad (2.3) \]

It implies the discrete Poincaré inequalities, for any \( w \in X_{D,0} \),

\[ \| \Pi_D \omega \|_{L^2(\Omega)} \leq C_D \| \Delta_D \omega \|_{L^2(\Omega)} \quad \text{and} \quad \| \nabla_D \omega \|_{L^2(\Omega)^d} \leq C_D \| \Delta_D \omega \|_{L^2(\Omega)}. \quad (2.4) \]
The second quantity is the function $S_D : K \to K_D$, which measures the interpolation error and it is given by: For all $v \in K$,

$$S_D(v) = \min_{\omega \in K_D} \left( \| \Pi_D \omega - v \|_{L^2(\Omega)} + \| \nabla D \omega - \nabla v \|_{L^2(\Omega)^d} + \| \Delta_D \omega - \Delta v \|_{L^2(\Omega)} \right). \quad (2.5)$$

The last one is the function $W_D : H_{\Delta}(\Omega) \to \mathbb{R}$, which refers to the limit–conformity and it is defined by: For $v \in H_{\Delta}(\Omega)$,

$$W_D(v) = \max_{\omega \in X_{D,0}\setminus\{0\}} \frac{1}{\| \Delta_D \omega \|_{L^2(\Omega)}} \int_{\Omega} (\Delta v \Pi_D \omega - \int_{\Omega} v \Delta_D \omega) \, dx, \quad (2.6)$$

where $H_{\Delta}(\Omega) := \{ v \in L^2(\Omega) : \Delta v \in L^2(\Omega) \}$.

**Definition 2.3.** Let $(\mathcal{D}_m)_{m \in \mathbb{N}}$ be a sequence of a gradient discretisation in the sense of definition 2.1. We say that

- $(\mathcal{D}_m)_{m \in \mathbb{N}}$ is coercive if there exists $C_P \in \mathbb{R}^+$ such that $C_{\mathcal{D}_m} \leq C_P$ for all $m \in \mathbb{N}$.
- $(\mathcal{D}_m)_{m \in \mathbb{N}}$ is consistent if

$$\text{for all } v \in K, \quad \lim_{m \to \infty} S_{\mathcal{D}}(v) = 0.$$

- $(\mathcal{D}_m)_{m \in \mathbb{N}}$ is limit–conforming if

$$\text{for all } v \in H_{\Delta}(\Omega), \quad \lim_{m \to \infty} W_{\mathcal{D}}(v) = 0.$$

- $(\mathcal{D}_m)_{m \in \mathbb{N}}$ is compact if for any sequence $(\omega_m)_{m \in \mathbb{N}}$ in $X_{\mathcal{D}_m,0}$, such that $\| \Delta_{\mathcal{D}_m} \omega_m \|_{L^2(\Omega)}$ is a bounded, then there exists $\varphi \in H_{0}^1(\Omega)$, such that the sequence $(\Pi_{\mathcal{D}_m} \omega_m)_{m \in \mathbb{N}}$ converges strongly to $\varphi$ in $L^2(\Omega)$, as $m \to \infty$.

### 3. Main results

Let us now introduce the main results; the general error estimates and the convergence results. We begin with defining the continuous interpolant $I_D : K \to K_D$ by

$$I_D h = \text{argmin}_{\omega \in K_D} \left( \| \Pi_D \omega - h \|_{L^2(\Omega)} + \| \nabla D \omega - \nabla h \|_{L^2(\Omega)^d} + \| \Delta_D \omega - \Delta h \|_{L^2(\Omega)} \right). \quad (3.1)$$

Thus, from the definition of $S_D$, we have, for any $v \in K$,

$$\| \Pi_D I_D v - v \|_{L^2(\Omega)} + \| \nabla D I_D v - \nabla v \|_{L^2(\Omega)^d} + \| \Delta_D I_D v - \Delta v \|_{L^2(\Omega)} \leq S_D(v). \quad (3.2)$$

**Theorem 3.1** (Convergence rates). Assume that $\Omega \subset \mathbb{R}^d$ ($d > 1$) is a bounded domain with the boundary $\partial \Omega$, $f \in L^2(\Omega)$, $\psi \in C^2(\Omega) \cap C(\overline{\Omega})$, $\psi \geq 0$ on $\partial \Omega$, and $\bar{c}$ be the solution to (2.1). Let $D$ be a gradient discretisation such that $K_D \neq \emptyset$, then the discrete scheme (2.2) has a unique solution $c \in K_D$. Moreover, if it is assumed that $\Delta^2 \bar{c} \in L^2(\Omega)$, then:

$$\| \Pi_D c - \bar{c} \|_{L^2(\Omega)} \leq C_D \frac{\sqrt{2}}{2} W_D(\Delta \bar{c}) + (C_D \frac{\sqrt{2}}{2} + 1) S_D(\bar{c}) + C_D R_D(\bar{c}) \frac{1}{2}, \quad (3.3)$$
\[ \| \nabla Dc - \nabla \tilde{c} \|_{L^2(\Omega)} \]
\[ \leq C_D \frac{\sqrt{\gamma}}{2} W_D(\Delta \tilde{c}) + (C_D \frac{\sqrt{\gamma} + 2}{2} + 1) S_D(\tilde{c}) + C_D R_D(\tilde{c}), \quad (3.4) \]
\[ \| \Delta Dc - \Delta \tilde{c} \|_{L^2(\Omega)} \leq \frac{\sqrt{\gamma}}{2} W_D(\Delta \tilde{c}) + \frac{\sqrt{\gamma} + 2}{2} S_D(\tilde{c}) + R_D(\tilde{c}), \quad (3.5) \]

where \( R_D(\tilde{c}) := \int_{\Omega} (\Delta^2 \tilde{c} + f) (\psi - \Pi_D I_D \tilde{c}) \, dx \).

**Proof.** The existence and uniqueness of the discrete solution follow from Stampacchia’s theorem [16], thanks to the assumption that \( K_D \) is a non empty set. Under the regularity \( \Delta^2 \tilde{c} \in L^2(\Omega) \), we can apply (2.6) to \( v := \Delta \tilde{c} \in H_\Delta(\Omega) \) with taking \( w := I_D \tilde{c} - c \in X_{D,0} \) to deduce

\[ \int_{\Omega} \Delta \tilde{c}(x) \Delta D(I_D \tilde{c} - c)(x) \, dx + \int_{\Omega} \Delta^2 \tilde{c}(x) \Pi_D(I_D \tilde{c} - c)(x) \, dx \]
\[ \leq \| \Delta D(I_D \tilde{c} - c) \|_{L^2(\Omega)} W_D(\Delta \tilde{c}), \]

where \( I_D \) is the interpolant defined by (3.1). Since \( c \) is the solution to (2.2), the above inequality implies

\[ \int_{\Omega} \Delta D(I_D \tilde{c} - c)(x)(\Delta \tilde{c}(x) - \Delta Dc(x)) \, dx + \int_{\Omega} \Delta^2 \tilde{c}(x) \Pi_D(I_D \tilde{c} - c)(x) \, dx \]
\[ \leq \| \Delta D(I_D \tilde{c} - c) \|_{L^2(\Omega)} W_D(\Delta \tilde{c}) - \int_{\Omega} f(x) \Pi_D(I_D \tilde{c} - c)(x) \, dx. \quad (3.6) \]

Thanks to the regularity assumption \( \Delta^2 \tilde{c} \in L^2(\Omega) \), taking \( \varphi := \tilde{c} - v \) (with a non negative \( v \in C_0^\infty(\Omega) \)) as a generic function in (2.1) shows that \( (f + \Delta^2 \tilde{c}) \geq 0 \) for a.e. in \( \Omega \). Thus, we have

\[ \int_{\Omega} \Delta^2 \tilde{c}(x) \Pi_D(I_D \tilde{c} - c)(x) \, dx \]
\[ = \int_{\Omega} (\Delta^2 \tilde{c}(x) + f(x)) \Pi_D(I_D \tilde{c} - c)(x) \, dx - \int_{\Omega} f(x) \Pi_D(I_D \tilde{c} - c)(x) \, dx \]
\[ = \int_{\Omega} (\Delta^2 \tilde{c}(x) + f(x))(\Pi_D I_D \tilde{c}(x) - \psi(x)) \, dx + \int_{\Omega} (\Delta^2 \tilde{c}(x) + f(x))(\psi(x) - \Pi_D I_D \tilde{c}(x)) \]
\[ \quad - \int_{\Omega} f(x) \Pi_D(I_D \tilde{c} - c)(x) \, dx \]
\[ \geq \int_{\Omega} (\Delta^2 \tilde{c}(x) + f(x))(\Pi_D I_D \tilde{c}(x) - \psi(x)) \, dx - \int_{\Omega} f(x) \Pi_D(I_D \tilde{c}(x) - c(x)) \, dx, \]

since the quantity \( \int_{\Omega} (\Delta^2 \tilde{c}(x) + f(x))(\psi(x) - \Pi_D I_D \tilde{c}(x)) \, dx \geq 0 \). Substituting the above relation in (3.6) yields

\[ \int_{\Omega} \Delta D(I_D \tilde{c}(x) - c(x))(\Delta \tilde{c}(x) - \Delta Dc(x)) \, dx \leq \| \Delta D(I_D \tilde{c} - c) \|_{L^2(\Omega)} W_D(\Delta \tilde{c}) + R_D(\tilde{c}). \]
Introduce the term $\Delta D (I_D \bar{c})$ and apply the Cauchy–Schwarz’s inequality to obtain

$$\left\| \Delta D (I_D \bar{c} - c) \right\|^2 \leq \int_{\Omega} \Delta D (I_D \bar{c}(x) - c(x)) (\Delta D I_D \bar{c}(x) - \Delta \bar{c}(x)) \, dx$$

which leads to, thanks to (3.2)

$$\left\| \Delta D (I_D \bar{c} - c) \right\|_{L^2(\Omega)} \leq \left( \frac{1}{2} \right) \left( \left\| \Delta D (\Delta \bar{c}) \right\| + S_D (\bar{c}) \right)^2 + R_D (\bar{c})^+.$$  (3.7)

Applying Young’s inequality to this relation gives

$$\left\| \Delta D I_D \bar{c} - \Delta D c \right\|_{L^2(\Omega)} \leq \left( \frac{1}{2} \right) \left( \left\| \Delta D (\Delta \bar{c}) \right\| + S_D (\bar{c}) \right)^2 + R_D (\bar{c})^+.$$  (3.8)

This inequality with the use of triangle inequality, the fact that $\forall a, b \in \mathbb{R}^+$, $(a + b)^{1/2} \leq a^{1/2} + b^{1/2}$ and (3.2) yield Estimate (3.3). Using the definition of $C_D$, and (3.8), and (3.2), one has

$$\left\| \Pi_D I_D \bar{c} - c \right\|_{L^2(\Omega)} \leq C_D \left( \frac{1}{2} \left( \left\| \Delta D (\Delta \bar{c}) \right\| + S_D (\bar{c}) \right) \right)^2 + R_D (\bar{c})^+ + S_D (\bar{c}).$$

Applying $(a + b)^{1/2} \leq a^{1/2} + b^{1/2}$ again establishes Estimate (3.5). Estimate (3.4) follows in a similar way.

From Theorem 3.1, we can obtain an optimal convergence rate for the approximation of fourth order variational inequalities in terms of the mesh size. [7, Remark 2.24] provides the link between the mesh size and the parameters $C_D$, $S_D$, and $\mathbb{W}_D$ that appear in Estimates (3.3)–(3.5). Under the assumptions stated in the previous theorem, it is also clear to establish an estimate on the term $R_D (\bar{c})$ since it can be rewritten as

$$R_D (\bar{c}) = \int_{\Omega} (\Delta^2 \bar{c} + f)(\psi - \Pi D I_D \bar{c}) \, dx$$

$$= \int_{\Omega} (\Delta^2 \bar{c} + f) \psi \, dx + \int_{\Omega} (\Delta^2 \bar{c} + f) (\bar{c} - \Pi D I_D \bar{c}) \, dx$$

$$\leq \left\| \Delta^2 \bar{c} + f \right\|_{L^2(\Omega)} S_D (\bar{c}).$$

Note that the regularity assumption that $\Delta^2 \bar{c} \in L^2(\Omega)$ in the previous theorem is only used to establish the convergence rate whereas establishing the convergence of the discrete scheme (2.2) can be obtained under the standard hypothesis on the continuous solution as in the following theorem.

**Theorem 3.2** (Convergence). Assume that $\Omega \subset \mathbb{R}^d$ ($d > 1$) is a bounded domain with the boundary $\partial \Omega$, $f \in L^2(\Omega)$, $\psi \in C^2(\Omega) \cap C(\overline{\Omega})$, $\psi \geq 0$ on $\partial \Omega$, and let $(D_m)_{m \in \mathbb{N}}$ be a sequence of a gradient discretisation, that is coercive, consistent, limit–conforming and compact. Let $\bar{c}$ be the solution to (2.1). If $K_{D_m}$ is a non empty set, then there exists a unique solution $u_m \in K_{D_m}$ to the discrete problem (2.2) (with $D = D_m$), and, as $m \to \infty$,

- $\Pi_{D_m} c_m$ converges strongly to $\bar{c}$ in $L^2(\Omega)$,
- $\nabla D_m c_m$ converges strongly to $\nabla \bar{c}$ in $L^2(\Omega)^d$, and
- $\Delta D_m c_m$ converges to $\Delta \bar{c}$ in $L^2(\Omega)^d$. 


• $\Delta D_m c_m$ converges strongly to $\Delta \tilde{c}$ in $L^2(\Omega)$.

Proof. Let $c := c_m$ and $\varphi := I_{D_m} \tilde{c} \in K_{D_m}$ in (2.2), where $I_D$ is defined by (3.1) with $D = D_m$. Use the discrete relation (2.4) to obtain

$$
\|\Delta D_m c_m\|_{L^2(\Omega)}^2 \leq \|f\|_{L^2(\Omega)} \|\Pi_{D_m} (c_m - I_{D_m} \tilde{c})\|_{L^2(\Omega)} + \|\Delta D_m c_m\|_{L^2(\Omega)} \|\Delta D_m I_{D_m} \tilde{c}\|_{L^2(\Omega)}
$$

$$
\leq C_P \|f\|_{L^2(\Omega)} \|\Delta D_m (c_m - I_{D_m} \tilde{c})\|_{L^2(\Omega)} + \|\Delta D_m c_m\|_{L^2(\Omega)} \|\Delta D_m I_{D_m} \tilde{c}\|_{L^2(\Omega)},
$$

where $C_P$ does not depend on $m$. Thanks to Young’s inequality, the above inequality gives

$$
\|\Delta D_m u_m\|_{L^2(\Omega)}^2 \leq C \left(\|f\|_{L^2(\Omega)} + \|\Delta D_m I_{D_m} \tilde{c}\|_{L^2(\Omega)}\right), \tag{3.9}
$$

where $C$ is also independent on $m$. By the triangle inequality and (3.2), one writes

$$
\|\Delta D_m I_{D_m} \tilde{c}\|_{L^2(\Omega)} \leq \|\Delta D_m I_{D_m} \tilde{c} - \Delta \tilde{c}\|_{L^2(\Omega)} + \|\Delta \tilde{c}\|_{L^2(\Omega)}
$$

$$
\leq S_{D_m} (\tilde{c}) + \|\Delta \tilde{c}\|_{L^2(\Omega)}.
$$

The consistency of $D_m$ and the standard regularity $\Delta \tilde{c} \in L^2(\Omega)$ show that the quantity $\|\Delta D_m I_{D_m} \tilde{c}\|_{L^2(\Omega)}$ is bounded and thus Estimate (3.9) proves that $\|\Delta D_m c_m\|_{L^2(\Omega)}$ remains bounded.

The regularity results of the limit for the second order gradient discretisation stated in [8, Lemma 2.15] can easily be extended to the fourth order gradient discretisation. These results and the boundedness of $\|\Delta D_m c_m\|_{L^2(\Omega)}$ established above show that there exists $\tilde{c} \in H^2_0(\Omega)$, such that $\Pi_{D_m} c_m \rightarrow \tilde{c}$ in $L^2(\Omega)$, $\nabla D_m c_m \rightarrow \nabla \tilde{c}$ in $L^2(\Omega)^d$ and $\Delta D_m c_m \rightarrow \Delta \tilde{c}$ in $L^2(\Omega)$. From the compactness property of $D_m$, we see that $\Pi_{D_m} c_m$ converges strongly to $\tilde{c}$ in $L^2(\Omega)$. Now, since $c_m \in K_{D_m}$, we obtain $\Pi_{D_m} c_m \leq \psi$, which yields $\tilde{c} \in K$.

Let us now show that $\tilde{c}$ satisfies (2.1). For any $v \in K$, the interpolant defined in (3.1) and the consistency property of $(D_m)_{m \in \mathbb{N}}$ imply $\Pi_{D_m} I_{D_m} v \rightarrow v$ in $L^2(\Omega)$, $\nabla D_m I_{D_m} v \rightarrow \nabla v$ in $L^2(\Omega)^d$, and $\Delta D_m I_{D_m} v \rightarrow \Delta v$ in $L^2(\Omega)$. Choosing $c := c_m$ and $\varphi := I_{D_m} \tilde{c} \in K_{D_m}$ in (2.2) and pass to the limit show that (2.1) is satisfied for any $\varphi \in K$, which concludes that $c$ is the continuous solution, thanks to the strong–weak convergences of sequences.

In order to obtain the strong convergence of $\Delta D_m c_m$ and $\nabla D_m c_m$, we take $\varphi := I_{D_m} \tilde{c}$ as a generic function in (2.2) for $D_m = D$. It implies, due to $c$ is the solution to (2.1)

$$
0 \leq \|\Delta D_m c_m - \Delta \tilde{c}\|_{L^2(\Omega)}^2
$$

$$
= \int_{\Omega} (\Delta D_m c_m(x) - \Delta \tilde{c}(x))^2 \, dx
$$

$$
\leq \int_{\Omega} f(x) \Pi_{D_m} (c_m - I_{D_m} \tilde{c})(x) \, dx + \int_{\Omega} \Delta \tilde{c}(x)^2 \, dx.
$$

Pass to the limit in this inequality and use again the strong–weak convergence of sequences to conclude the strong convergence of $\Delta D_m c_m$. Finally, introduce $\Delta I_{D_m} \tilde{c}$ and use the triangle inequality to attain, thanks to (2.3) and (3.2)

$$
\|\nabla D_m c_m - \nabla \tilde{c}\|_{L^2(\Omega)^d}
$$

$$
\leq \|\nabla D_m (c_m - I_{D_m} \tilde{c})\|_{L^2(\Omega)^d} + \|\nabla D_m I_{D_m} \tilde{c} - \nabla \tilde{c}\|_{L^2(\Omega)^d}
$$

$$
\leq C_{D_m} \|\Delta D_m (c_m - I_{D_m} \tilde{c})\|_{L^2(\Omega)} + S_{D_m} (\tilde{c}).
$$
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Since \((D_m)_{m \in \mathbb{N}}\) is coercive and consistent, the strong convergence of \(\Delta_{D_m} c_m\) established above therefore leads to \(C_{D_m} \|\Delta_{D_m} (c_m - I_{D_m} \bar{c})\|_{L^2(\Omega)} + S_{D_m}(\bar{c}) \to 0\), as \(m \to \infty\), which completes the proof. □

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