Reduction of Dilute Ising Spin Glasses

Stefan Boettcher and James Davidheiser

Physics Department, Emory University, Atlanta, Georgia 30322, USA

(Dated: June 30, 2008)

The recently proposed reduction method for diluted spin glasses is investigated in depth. In particular, the Edwards-Anderson model with $\pm J$ and Gaussian bond disorder on hyper-cubic lattices in $d = 2$, 3, and 4 is studied for a range of bond dilutions. The results demonstrate the effectiveness of using bond dilution to elucidate low-temperature properties of Ising spin glasses, and provide a starting point to enhance the methods used in reduction. Based on that, a new greedy heuristic call “Dominant Bond Reduction” is introduced and explored.

PACS numbers: 02.60.Pn, 05.50.+q, 75.10.Nr.

I. INTRODUCTION

Despite more than three decades of intensive research, many properties of spin glasses, especially in finite dimensions, are still not well understood. The most simple model is the Edwards-Anderson model (EA) $^5$,

$$H = - \sum_{<i,j>} J_{i,j} x_i x_j, \quad (x_i = \pm 1), \quad (1)$$

with Ising spins $x_i = \pm 1$ arranged on a finite-dimensional lattice with nearest-neighbor bonds $J_{i,j}$, randomly drawn from a distribution $P(J)$ of zero mean and unit variance.

In Refs. $^6$$^7$$^8$, it was proposed to study the EA in Eq. (1) on bond-diluted lattices at $T = 0$ to obtain more accurate scaling behavior for low-temperature excitations. There, it is shown how to remove iteratively low-connected spins from the lattice and alter the interactions, i.e., to reduce the system, in such a way that the ground-state energy of the reduced system is the same as the original system. In this way often much larger lattice sizes $L$ can be simulated compared to undiluted ones and, in combination with finite-size scaling, enhanced scaling regimes are achieved. With these methods, for instance, we have solved spin glasses exactly at the bond-percolation threshold $p_c$, the edge of the glassy regime, in $d = 2, \ldots , 7$ by reducing a large number of systems with up to $10^8$ spins $^2$.

There is, of course, a long history of studying spin systems on dilute lattices, including spin glasses, going back to Ref. $^9$ itself; see for example Refs. $^{10}$$^{11}$$^{12}$$^{13}$$^{14}$. Coniglio and co-workers have proposed to map the ensemble of critical Ising (or Potts) spin models $^{15}$$^{16}$ onto percolating clusters, based on the ideas of Fortuin and Kasteleyn $^{17}$, to treat ferromagnetic $^{18}$ and spin glass phenomena $^{19}$$^{20}$. Our approach here is based on transformations in the Hamiltonian of an Ising spin system that are exact for each instance. The price paid is that these transformations reducing the Hamiltonian only apply at $T = 0$. Extending our earlier work on the Migdal-Kadanoff approximation $^8$, Ref. $^{21}$ very recently included infinitesimal temperature corrections. Their method is limited to discrete bonds and a subset of the rules we discuss here. As we can merely consider $T = 0$, we are unfortunately not sensitive to the novel transition seen by Ref. $^{21}$ that is said to emerge only at non-zero temperatures.

Our approach, combined with the highly efficient “Extremal Optimization” (EO) heuristic $^{22}$$^{23}$, have lead to a comprehensive characterization of low-temperature excitations in spin glasses in low dimensions (up to $d = 7$) $^{24}$. These results allow for a direct comparison with mean-field predictions $^{22}$, and have recently motivated a reassessment of fundamental scaling relations $^{26}$$^{27}$. This work has also inspired the use of dilution for more effective Monte Carlo simulations of disordered systems $^{21}$$^{28}$$^{29}$$^{30}$$^{31}$.

Here we study the connection between lattice-topology and the reduction method. In particular, we explore the effectiveness of each of the reduction rules as a function of bond dilution. The results provides the reader with an opportunity to evaluate in more detail the conclusions drawn in previous papers $^{6}$$^{7}$$^{24}$$^{31}$$^{32}$, and might suggest possible extensions of these rules for improved effectiveness. As an example of a concrete application, we introduce and evaluate “Dominant Bond Reduction” (DBR), a new heuristic which provides a greedy, almost linear algorithm to obtain approximate spin-glass ground states on average with bounded relative error for increasing system sizes.

This paper is structured as follows. In the next Section, we will introduce the reduction method and its rules. In Sec. III we display and discuss our numerical investigation of the reduction rules. In Sec. IV we discuss our simulation results for DBR, followed by some concluding remarks in Sec. V. In the Appendix, a generalized description of the reduction method is provided, with some speculations on its applicability.

II. REDUCTION METHOD

To exploit the advantages of spin glasses on a bond-diluted lattice, we can often reduce the number of relevant degrees of freedom substantially before a call to an optimization algorithm becomes necessary. Such a reduction, in particular of low-connected spins, leads to a smaller, compact remainder graph, bare of trivially fluctuating
variables, which is easier to optimize. These reductions are very similar to a procedure known as $k$-core decomposition in graph theory, which is often applied to analyze hard combinatorial or real-world problems.

Furthermore, rules of this sort have also been used to study planar and 3d resistor networks. Here, we focus exclusively on the reduction rules for the ground-state energy (i.e., $T = 0$); a subset of these rules also permit the exact determination of the entropy and overlap $\Delta$ at $T = 0$. These rules apply to general Ising spin glass Hamiltonians as in Eq. (1), with any bond distribution $P(J)$, discrete or continuous, on arbitrary sparse graphs.

The reductions affect both spins and bonds, eliminating recursively all zero-, one-, two-, and three-connected spins. These rules are supplemented here with one that is not topological but concerns bond values directly, which is especially effective for broad, continuous bond distributions, like Gaussian or power-law bonds. The addition of more elaborate rules that depend of suitably chosen bond distributions is conceivable, the universality of the underlying physics permitting. These operations eliminate and add terms to the expression for the Hamiltonian in Eq. (1), but leave it form-invariant. Again, loosening this requirement may lead to an even more efficient procedure for certain problems, although it should be understood that an unrestricted reduction in general leads to an exponential growth in the number and form of the newly created terms, involving all combinations of $p$-spin interactions (see the Appendix).

Offsets in the energy along the way are accounted for by a variable $\Delta$, which is exact for a ground-state configuration. The rules discussed here are as follows:

Rule I: An isolated spin can be ignored entirely.

Rule II: A one-connected spin $i$ can be eliminated, since its state can always be chosen in accordance with its neighboring spin $j$ to satisfy the bond $J_{i,j}$. For its energetically most favorable state we adjust $H_o := H_o - |J_{i,j}|$ and eliminate the term $-J_{i,j} x_i x_j$ from $H$.

Rule III: A double bond, $J_{i,j}^{(1)}$ and $J_{i,j}^{(2)}$, between two spins $i$ and $j$ can be combined to a single bond by setting $J_{i,j} = J_{i,j}^{(1)} + J_{i,j}^{(2)}$ or be eliminated entirely, if the resulting bond vanishes. This operation is very useful to lower the connectivity of $i$ and $j$ at least by one. (For an example, see Fig. 2.)

Rule IV: For a two-connected spin $i$, rewrite the two terms pertaining to $x_i$ in Eq. (1) as

$$x_i(J_{i,1} x_1 + J_{i,2} x_2) \leq |J_{i,1} x_1 + J_{i,2} x_2|$$

$$= J_{i,2} x_1 x_2 + \Delta H,$$

where

$$J_{1,2} = \frac{1}{2}(|J_{i,1} + J_{i,2}| - |J_{i,1} - J_{i,2}|),$$

$$\Delta H = \frac{1}{2}(|J_{i,1} + J_{i,2}| + |J_{i,1} - J_{i,2}|),$$

leaving the graph with a new bond $J_{1,2}$ between spin 1 and 2, and acquiring an offset $H_o := H_o - \Delta H$. In the ground state, the bound in Eq. (2) becomes an equality.

Rule V: A three-connected spin $i$ can be reduced via a “star-triangle” relation, as depicted in Fig. 1. We rewrite the three terms pertaining to $x_i$ in Eq. (1) as:

$$J_{i,1} x_i x_1 + J_{i,2} x_i x_2 + J_{i,3} x_i x_3$$

$$\leq |J_{i,1} x_1 + J_{i,2} x_2 + J_{i,3} x_3|$$

$$= J_{i,2} x_1 x_2 + J_{i,3} x_1 x_3 + J_{2,3} x_2 x_3 + \Delta H,$$

where

$$J_{1,2} = -A - B + C + D, \quad J_{1,3} = A - B + C - D,$$

$$J_{2,3} = -A + B + C - D, \quad \Delta H = A + B + C + D,$$

$$A = \frac{1}{2} |J_{i,1} - J_{i,2} + J_{i,3}|, \quad B = \frac{1}{2} |J_{i,1} - J_{i,2} - J_{i,3}|,$$

$$C = \frac{1}{2} |J_{i,1} + J_{i,2} + J_{i,3}|, \quad D = \frac{1}{2} |J_{i,1} + J_{i,2} - J_{i,3}|.$$

As before, in the ground state, the bound in Eq. (4) becomes an equality.

Rule VI: A spin $i$ (of any connectivity) for which the absolute weight $|J_{i,j'}|$ of one bond to a spin $j'$ is larger than the absolute sum of all its other bond-weights to neighboring spins $j \neq j'$, i.e.

$$|J_{i,j'}| > \sum_{j \neq j'} |J_{i,j}|,$$
bond $J_{i,j'}$, must be satisfied in any ground state. Then, spin $i$ is determined in the ground state by spin $j'$ and it, as well as this “super-bond” $J_{i,j'}$, can be eliminated accordingly, as depicted in Fig. 2. Here, we obtain $H_0 := H_0 - |J_{i,j'}|$. All other bonds connected to $i$ are simply reconnected with $j'$, but with reversed sign, if $J_{i,j'} < 0$.

This procedure is costly, and hence best applied after the other rules are exhausted. But it can be highly effective for very widely distributed bonds. In particular, since neighboring spins may reduce in connectivity and become susceptible to the previous rules again, an avalanche of further reductions may ensue, see Fig. 2.

After a recursive application of these rules, the original lattice or graph is either completely reduced (which is almost always the case below or near $p_c$), in which case $H_o$ provides the exact ground state energy already, or we are left with a reduced, compact graph in which no spin has less than four connections, from which one could obtain the ground state with some optimization procedure, as described in Refs. 6,31. Reducing even higher-connected spins would lead to new (hyper-)bonds between more than two spins, unlike Eq. (1), as discussed in the Appendix.

III. NUMERICAL SIMULATIONS

In our simulations we have studied EA spin glasses on hyper-cubic lattices over a range of sizes $L$ in dimensions $d = 2, 3$, and $4$ at various bond fractions $p$ for $\pm J$ bonds and Gaussian bonds. Similar studies could as well have been done on an arbitrary family of sparse graphs, with-
TABLE I: List of the range of system sizes \( L \) chosen for each dimension \( d \) in Figs. 3–6.

| \( d \) | \( L \) |
|-------|-------|
| 2     | 10, 20, \ldots, 100 |
| 3     | 5, 10, \ldots, 20 |
| 4     | 3, 4, \ldots, 10, 15 |

out restriction. We have applied the rules described in Sec. II recursively, until no further reductions were possible. We have calculated a number of aspects of this reduction, such as the number of spins in the remainder graph relative to the original lattice, the average connectivity in the remainder graph, and the fraction of systems that is completely reducible (without remainder), all as a function of bond-density \( p \). Similarly, we have counted along the way how many times each of the reduction rules has been applied for graphs of a certain size \( L \) and bond fraction \( p \). The system sizes used in each figure of this section are listed in Tab. I for each dimension.

In Figs. 3–6 we have plotted the efficiency of the reduction step for one-connected spins (Rule II above, labeled \( \text{R1} \) here), two-connected spins (Rule IV, \( \text{R2} \) here), three-connected spins (Rule V, \( \text{R3} \)), double bond elimination (Rule III, \( \text{Rd} \)), and super-bonds (Rule VI, \( \text{SB} \)) as a function of \( p \) for dimensions \( d = 2, 3, \) and 4. Efficiency is defined here as the number of calls to that step in a run relative to the original number of spins \( N = L^d \) in the original lattice. We observe that each of the reduction rules reaches a peak for increasing bond densities, in order of \( \text{R1}, \text{R2}, \text{R3}, \) and \( \text{SB} \). Thus, the further to the right a rule peaks, the more powerful it is, and its decline signals significant changes in the structure of the graph. The peak of \( \text{R1} \) near \( p_c \) (marked by a vertical line in each plot) is a consequence of the well-known fact that a percolating graph is predominantly one-connected, i.e., \( p_c \approx 1/(2d) \) such that the connectivity is \( \alpha_c \approx 2dp_c \approx 1 \) for \( d \to \infty \). The values for the bond-percolation thresholds on hyper-cubic lattices are \( p_c = 1/2 \) in \( d = 2 \), \( p_c \approx 0.2488 \) in \( d = 3 \), and \( p_c \approx 0.1601 \) in \( d = 4 \). These thresholds are indicated by vertical lines in each plot.

The additional use of \( \text{SB} \) does not seem to affect the other rules much (except for \( \text{Rd} \)). While it does not seem to trigger avalanches of activity for lower rules (except just above \( p_c \)), it in itself often leads to nearly 10\% further reduction at larger \( p \).

In Figs. 3–6 we have plotted, as a function of bond density \( p \), the fraction of instances that result in a remainder graph after a complete exhaustion of the reduction rules. We note that below and near the bond-percolation threshold \( p_c \) in each dimension, almost all graphs are completely reducible. This implies that the optimization of their energy can be done in polynomial time. Physically, this means that there can not be an ordered, glassy state even at \( T = 0 \) below \( p_c \), of course. For increasing system size, a sharp transition emerges such that almost every graph has some non-empty remainder (of unspecified size) above that transition. In case of the discrete \( \pm J \) bonds this transition appears to be related with presumed onset of spin-glass order at \( p = p^* > p_c \), as discussed in Refs. 6, 7. For Gaussian bonds, \( p^* = p_c \), and the transition appears to be centered close to that. Asymptotically, the use of \( \text{SB} \) seems to push the transition just above \( p_c \), whereas it seems to locate somewhat below without \( \text{SB} \).

In Figs. 3–6 we have plotted the average size of the remainder graph (empty or not) as a function of bond density \( p \). Including empty remainder graphs in the weight of the average is important, of course, and explains the values below unity at low \( p \). The pivot point indicates a well-defined transition point closely related to a 3-core percolation transition\(^{33,35,36} \) above \( p_c \), as our rules reduce at least all vertices of degree less than 4. The correspondence is not exact, as cooperative effects between bond weights (such as Rule III) or superbonds (Rule VI) distort the pure case. Predictably, for \( p \to 1 \) the graphs remains unaltered, except maybe for a few spins reducible by \( \text{SB} \) at lower \( d \).

In Figs. 3–6 we have plotted the average connectivity \( \langle \alpha \rangle \) of any non-empty remainder graph as a function of bond density \( p \). By virtue of the reduction rules, it is \( \langle \alpha \rangle \geq 4 \). The data is very noisy below \( p_c \), since almost all remainders are empty there. These connectivities will eventually approach \( 2d \) for \( p \to 1 \), except when \( \text{SB} \) is included. There is a strong effect due to \( \text{SB} \) also right above the threshold \( p_c \), where Rule VI leads to an increasingly sharper maximum with size and dimension, as Eq. 5 is more likely satisfied there.

IV. DOMINANT BOND REDUCTION HEURISTIC

The Rule VI in Sec. II is based on the following observation: If the absolute weight \( |J_{i,j'}| \) of one bond incident on spin \( x_i \) from a neighboring spin \( x_{j'} \) exceeds the absolute sum of all its other \( \alpha_i - 1 \) bond-weights with adjacent spins, i.e., if by Eq. 5

\[
|J_{i,j'}| > \sum_{j \neq j', j \neq i} |J_{i,j}| < 1,
\]

the bond \( J_{i,j'} \) must be satisfied in any ground state. In the exact reduction procedure, as applied in Sec. III, such a dominant bond is used to eliminate it and the spin \( x_i \) from the problem.
FIG. 4: Plot of the probability to obtain any non-empty remainder graph after a complete exhaustion of the reduction rules as a function of bond density $p$ in $d = 2$ (left column), $d = 3$ (middle column), and $d = 4$ (right column) for $\pm J$ bonds (top row), Gaussian bonds without super-bond reduction $\text{SB}$ according to Rule VI (middle row), and Gaussian bonds with $\text{SB}$ (bottom row). The sequence of graphs in each plot steepen for increasing system size $L$ from right to left. There is a strong dependence on $L$, and it appears that the probabilities converge to a $0 - 1$ step function at or near the bond percolation threshold (indicated by a vertical line). With super-bond reduction, Rule VI, graphs are far more reducible even significantly above the threshold, at least at any finite size.

Here, we consider relaxing that constraint to assess the quality of approximate results that can be obtained with a heuristic approach. We assume that even if $r_i \geq 1$ in Eq. (6), any almost-dominant bond on a spin $x_i$ is more likely satisfied in a ground state. This suggest a fast, greedy heuristic: Find the spin $x_i$ with $r_i = r_{\text{min}} = \min_{1 \leq j \leq N} \{r_j\}$ in an instance and eliminate it and its strongest bond as in Rule VI. This step can be repeated until any number of the heaviest bonds have been removed to treat the remainder with an optimization heuristic like EO, or even until the entire lattice is reduced. The latter heuristic we call “dominant bond reduction” (DBR).

Like Rule VI itself, DBR is not useful for homogeneous bond distributions like $\pm J$, where all bonds have the same absolute weight $|J| \equiv 1$ (at least initially). But it may be very effective for continuous bond distributions on dilute lattices with low average connectivity $\langle \alpha \rangle$, as we have shown in Sec. III. We can further exploit the universality of bond distributions and utilize broadly distributed bonds, such as a power law, $P(J) \sim |J|^{-\gamma}$ for $|J| \to \infty$, as long as $P(J)$ has zero mean and finite width. In fact, such a greedy procedure has already been described for extremely widely separated bonds (each bond is larger in weight than the sum of all smaller ones), where it becomes exact: the problem is no longer NP-hard. For the power-law bonds, we expect that there is a transition in the behavior of this procedure at some finite value of $\gamma_c$, where it becomes exact, the problem is no longer NP-hard. Hence, we may find a “sweet spot”: a sufficiently broad distribution on a sufficiently dilute lat-
FIG. 5: Plot of the average (fractional) size of the remainder graph (empty or not) as a function of bond density \( p \) in \( d = 2 \) (left column), \( d = 3 \) (middle column), and \( d = 4 \) (right column) for \( \pm J \) bonds (top row), Gaussian bonds without super-bond reduction \( \text{SB} \) according to Rule VI (middle row), and Gaussian bonds with \( \text{SB} \) (bottom row). Super-bond reduction with Rule VI lowers the remainder size near the threshold by about an order of magnitude, but with diminishing effects for larger \( p \). The sequence of graphs in each plot steepen for increasing system size \( L \).

Frustration leaves many bonds violated in the ground state, obviously, and our recursive elimination procedure accounts for that through compounding bonds in Rule III, as described also in Fig. 2. In a simple benchmark, shown in Fig. 7 we found that DBR obtains an approximate ground state energy density of \( \langle e \rangle_N = -1.308(1) \) for the undiluted EA, in \( d = 2 \) at \( N = 100^2 \), only 0.5\% above \( \langle e \rangle_\infty = -1.31479(2) \), the best-known result\(^{39} \), and \( \langle e \rangle_N = -1.631(1) \) in \( d = 3 \) at \( N = 20^3 \), 4\% above \( \langle e \rangle_\infty = -1.700(1) \).\(^{44} \) Computational costs are trivial, \( O(dN \ln N) \), but our implementation is limited to \( N < 10^4 \) by a data structure problem: repeated application of Rule VI leads to a few highly connected spins with hundreds of neighbors. Of course, to calculate properties of low-\( T \) excitations, even a systematic error of 4\% would be unacceptably large, and our naive DBR algorithm will have to be developed into a meta-heuristic, again combining reduction and EO.

To explore the effect of broadly distributed bonds, we have compared DBR for one undiluted cubic lattice of size \( N = 20^3 \) with Gaussian and power-law bonds at \( \gamma = 1.5 \). In Fig. 8 we show the value of the (smallest) \( r_i = r_{\text{min}} \) in Eq. (6) of the currently reduced spin \( x_i \) during one run of DBR. Initially, for all \( r_{\text{min}} < 1 \), DBR is exact, which persists much longer for power-law bonds. Even when \( r_{\text{min}} \geq 1 \), the size of \( r_{\text{min}} \) is typically smaller for power-law bonds. Considering that DBR’s systematic error is only 4\% for Gaussian bonds, we project that power-law bonds should be even more successful. We
FIG. 6: Plot of the average connectivity $\langle \alpha \rangle$ of any non-empty remainder graph as a function of bond density $p$ in $d = 2$ (left column), $d = 3$ (middle column), and $d = 4$ (right column) for $\pm J$ bonds (top row), Gaussian bonds without super-bond reduction $SB$ according to Rule VI (middle row), and Gaussian bonds with $SB$ (bottom row). The dramatic effect of implementing Rule VI becomes apparent near the percolation threshold, especially for increasing $d$. The sequence of graphs in each plot steepen for increasing system size $L$.

expect to conduct more extensive tests for varying $\gamma$ and increasing $N$, which will require a significantly revised data structure compared to the one used in these studies.

V. CONCLUSIONS

Our results validate the effectiveness of the recently proposed reduction scheme to determine the stiffness exponent $\gamma$, leaving remainder graphs that are a small fraction (typically $\approx 1 - 10\%$) of the size of the original problem in the interesting regime just above $p_c$. Note that the fact that reduction works well in two-dimensions, where $T_c = 0$ holds, does not imply definitively that it should work for $d > 2$. Nevertheless, since the local interconnections between spins (i.e., graph vertices) near and just above $p_c$ in $d = 2$ and higher dimensions is similar, justifies that the reduction scheme is applicable also for higher dimensions, or even sparse random graphs, where no exact ground-state algorithms are available. As the general discussion and the speculative inferences in the Appendix suggest, it may be possible to extend the methods discussed here for any particular graph topology or bond distribution at hand. We have also presented evidence that a heuristic approach, based purely on the bond reductions introduced in Sec. III provides a fast algorithm to obtain approximate ground states, with the potential to handle even large or undiluted systems within bounded error.
over 10

tic (circles) discussed in the text extrapolate to within 0.5%

22,23

are also results obtained with the EO heuristic (squares)
of the best-known prediction by Campbell, Hartmann, and

LBR also applies

which reproduces CHK within errors, and a simple alterna-

avoids the creation of highly-connected spins.

faster than EO. LBR is marginally simpler than DBR and

total bond-weight, i. e., where the sum of all absolute weigh ts

γ

/L

2.

FIG. 8: Plot of $\langle e \rangle = \langle E \rangle / L^2$ for the two-dimensional EA in

Eq. (1) as a function of $1/L^2$. Each data point is the average

over $10^5$ random instances. The results for the DBR heuristic

circles) discussed in the text extrapolate to within 0.5%
of the best-known prediction by Campbell, Hartmann, and

Katzgraber (CHK)\textsuperscript{20}, marked by the horizontal line. Shown

are also results obtained with the EO heuristic (squares)\textsuperscript{22,23},

which reproduces CHK within errors, and a simple alternative
to DBR we call Least Bond Reduction (LBR, triangles). LBR also applies Rule VI, but to the spin with the weakest

total bond-weight, i. e., where the sum of all absolute weights

of incident bonds is minimal. Both, DBR and LBR, are much

faster than EO. LBR is marginally simpler than DBR and

avoids the creation of highly-connected spins.

SB thanks P. Duxbury for inspiring discussions. This

work was supported by grant 0312510 from the Division

of Materials Research at the National Science Foundation

and a grant from the Emory University Research Council.

Appendix

In general, a spin glass Hamiltonian $H$, such as the

one in Eq. (1), consists of the sum (in the negative) of

a number of terms $J_{i_1 \ldots i_q} x_{i_1} \ldots x_{i_q}$, each representing a

(hyper-)bond of weight $J$ between $p$ spins $x_i \in \{-1, +1\}$. The number of connected spins $p$ may vary between
terms, although $p = 2$ for all terms in Eq. (1). The bonds

$J$ are quenched variables drawn from an arbitrary distribution, discrete or continuous. A particular instance of

the spin glass Hamiltonian is specified by the values of

these quenched variables.

A ground state minimizes $H$, thus we want to maxi-
mize as many terms as possible in $-H$. Each spin, say

$x_0$, appears in a number of such terms, connecting it to

a total of $q$ other spins. In general, there are $2^q$ possible
terms, and we can eliminate $x_0 = \pm 1$ by

$$
J_0 x_0 + J_1 x_1 x_0 + \ldots + J_q x_q x_0 + J_{12} x_1 x_2 x_0 +
\ldots + J_{1 \ldots q} x_1 \ldots x_q
$$

$$
= x_0 (J_0 + J_1 x_1 + \ldots + J_q x_q + J_{12} x_1 x_2 +
\ldots + J_{1 \ldots q} x_1 \ldots x_q)
\leq |J_0 + J_1 x_1 + \ldots + J_q x_q + J_{12} x_1 x_2 +
\ldots + J_{1 \ldots q} x_1 \ldots x_q|
$$

$$
= a_0 + a_1 x_1 + \ldots + a_q x_q + a_{12} x_1 x_2 +
\ldots + a_{1 \ldots q} x_1 \ldots x_q,
$$

where the bound again becomes an equality for the ground

state energy. A new Hamiltonian is obtained which is reduced by one variable. Notice that the last

two lines provide a unique system of $2^q$ linear equations,

one for each assignment of the $x_i = \pm 1$, that determine

the new bonds $a$ in terms of the old bonds $J$.

To solve the linear system, we define

$$
g(x) = |J_0 + J_1 x_1 + \ldots + J_q x_q + J_{12} x_1 x_2 +
\ldots + J_{1 \ldots q} x_1 \ldots x_q|,
$$

and note that for any function $g(x)$, $x \in \{-1\}^q$, it is true that

$$
g(x) = \sum_{\langle e \rangle} g(\epsilon) \prod_{i=1}^q \delta_{x_i, \epsilon_i}
$$

$$
= \sum_{\langle e \rangle} g(\epsilon) \prod_{i=1}^q \frac{1 + \epsilon_i x_i}{2}
$$

$$
= \sum_{\langle e \rangle} \frac{g(\epsilon)}{2^q} (1 + \epsilon_1 x_1 + \ldots + \epsilon_q x_q + \epsilon_1 \epsilon_2 x_1 x_2 +
\ldots + \epsilon_{q-1} x_{q-1} x_q + \ldots + \epsilon_1 \ldots \epsilon_q x_1 \ldots x_q),
$$
where the sum extends over all \(2^n\) permutations of \(\epsilon \in \{\pm 1\}^q\) and the relation \(\delta_{a,b} = (1 + ab)/2\) for \(a, b \in \{\pm 1\}\) was used to represent the Kronecker symbol. Comparison of Eq. (9) with the last two lines in Eq. (7) yields

\[
 a_{i_1, \ldots, i_p} = 2^{-q} \sum_{\epsilon} g(\epsilon) \epsilon_1 \ldots \epsilon_p
\]

(10)

for the new bonds connecting the remaining variables.

In general, it is not useful to reduce the Hamiltonian in this way; after all, if all \(n\) spins are connected to each other, as for the Sherrington-Kirkpatrick model\(^5\), just the elimination of one spin-variable requires \(O(2^n)\) operations. Yet, there are common situations, in particular for lattices of finite dimensionality and sparse graphs, where the application of the previous results can be very useful: either in itself, in combination with heuristic techniques, or as the basis of approximation schemes. Of course, we may also be interested in the system’s entropy, the magnetization, overlaps, etc, which can be considered simultaneously.

Although the combinatorial effort in the preceding expressions seems daunting in general, they possess a rich structure that relate them to other, well-studied subjects. For instance, we can rewrite Eq. (10) as

\[
 a_{i_1, \ldots, i_p} = \frac{1}{2^q} \sum_{j=0}^{2^q-1} g((j)) W_{k,2^q}(j)
\]

(11)

with

\[
 k = 1 + \sum_{\mu=1}^{\frac{p}{2}} 2^{i_{\mu}-1},
\]

(12)

where \((j)\) is the binary encoding (on \(\pm 1\)) of the integer \(j\). In particular, \(W_{k,L}(x)\) is the \(L\)th Walsh function\(^6\) of support \(L\) familiar from wavelet analysis and signal filtering. The orthogonality properties of Walsh functions provide a powerful means to analyze the preceding reduction equations for particular choices of initial bond distributions. For instance, there may be types of graphs with a nontrivial bond distribution for which the reductions could be simple. Also, a transformation may be found that maps the distribution of the \(J^\prime\)'s into that of the \(a^\prime\)'s. Finally, existing, highly optimized wavelet algorithms\(^7\) may in fact produce efficient spin glass solvers based on these reduction equations.

\* URL: \texttt{http://www.physics.emory.edu/faculty/boettcher/}
\† URL: \texttt{http://www.physics.emory.edu/students/davidheiser/}

1. K. Binder and A. P. Young, Rev. Mod. Phys. 58, 801 (1986).
2. M. Mézard, G. Parisi, and M. A. Virasoro, Spin glass theory and beyond (World Scientific, Singapore, 1987).
3. K. H. Fischer and J. A. Hertz, Spin Glasses (Cambridge University Press, Cambridge, 1991).
4. A. P. Young, ed., Spin glasses and random fields (World Scientific, Singapore, 1998).
5. S. F. Edwards and P. W. Anderson, J. Phys. F: Metal Phys. 5, 965 (1975).
6. S. Boettcher, Europhys. Lett. 67, 453 (2004).
7. S. Boettcher, Europhys. Lett. 78, 83 (2007).
8. S. Boettcher, Europhys. Lett. 73, 430 (2006).
9. S. Boettcher and E. Marchetti, Phys. Rev. B 77, 100405(R) (2008).
10. M. J. Stephen and G. S. Grest, Phys. Rev. Lett. 38, 567 (1977).
11. B. W. Southern, A. P. Young, and P. Pfeuty, J. Phys. C: Solid State Phys. 12, 683 (1979).
12. A. J. Kolan and R. G. Palmer, J. Appl. Phys. 53, 2198 (1982).
13. J. R. Banavar, A. J. Bray, and S. Feng, Phys. Rev. Lett. 58, 1463 (1987).
14. A. J. Bray and S. Feng, Phys. Rev. B 36, 8456 (1987).
15. A. Coniglio, Phys. Rev. Lett. 62, 3054 (1989).
16. A. Coniglio, Physica A 266, 379 (1999).
17. C. M. Fortuin and P. W. Kasteleyn, Physica 57, 536 (1972).
18. V. Cataudella, G. Franzese, M. Nicodemi, A. Scala, and A. Coniglio, Phys. Rev. E 54, 175 (1996).
19. A. Fierro, G. Franzese, A. de Candia, and A. Coniglio, Phys. Rev. E 59, 60 (1999).
20. G. Franzese and A. Coniglio, Phys. Rev. E 59, 6409 (1999).
21. T. Jörg and F. Ricci-Tersenghi, Phys. Rev. Lett. 100, 177203 (2008).
22. S. Boettcher and A. G. Percus, Phys. Rev. Lett. 86, 5211 (2001).
23. S. Boettcher and A. G. Percus, Artificial Intelligence 119, 275 (2000).
24. S. Boettcher, Phys. Rev. Lett. 95, 197205 (2005).
25. T. Aspelmeier, M. A. Moore, and A. P. Young, Phys. Rev. Lett. 90, 127202 (2003).
26. G. Parisi and T. Rizzo, cond-mat/0706.1180 (2007).
27. T. Aspelmeier, A. Billoire, E. Marinari, and M. A. Moore, J. Phys. A (Special Issue) (condmat/0711.3445).
28. T. Jörg, in Proceedings of "Statistical Physics of Disordered Systems and Its Applications" (Hayama, Japan), 2004.
29. T. Jörg, Phys. Rev. B 73, 224431 (2006).
30. M. Hasenbusch, A. Pelissetto, and E. Vicari, J. Stat. Mech. p. L02001 (2008).
31. S. Boettcher, Europhys. Lett. 31, 29 (2003).
32. S. Boettcher, Phys. Rev. B 67, R060403 (2003).
33. C. Farrow, C. F. Moukarzel, and P. M. Duxbury, Phys. Rev. E 72, 66109 (2005).
34. J. I. Alvarez-Hamelin, L. Dall’Asta, A. Barrat, and A. Vespignani, Advances in Neural Information Processing Systems 18, 41 (2006).
35. C. L. Farrow, P. Shukla, and P. M. Duxbury, J. Phys. A: Math. Theor. 40, F581 (2007).
36. D. J. Frank and C. J. Lobb, Phys. Rev. B 37, 302 (1988).
37. D. B. Gingold and C. J. Lobb, Phys. Rev. B 42, 8220 (1990).
38. J. Adler, Physica A 171, 453470 (1991).
39. I. A. Campbell, A. K. Hartmann, and H. G. Katzgraber,
Phys. Rev. B 70, 054429 (2004).

40 P. Cizeau and J. P. Bouchaud, Journal of Physics A: Mathematical and General 26, L187 (1993).

41 K. Janzen, A. K. Hartmann, and A. Engel, J. Stat. Mech., P04006 (2008).

42 C. M. Newman and D. L. Stein, Phys. Rev. Lett. 72, 2286 (1994).

43 M. Cieplak, A. Maritan, and J. R. Banavar, Phys. Rev. Lett. 72, 2320 (1994).

44 K. F. Pal, Physica A 233, 60 (1996).

45 D. Sherrington and S. Kirkpatrick, Phys. Rev. Lett. 35, 1792 (1975).

46 M. Thuillard, *Wavelets in Soft Computing* (World Scientific, 2001).

47 G. Beylkin, R. Coifman, and V. Rokhlin, Communications on Pure and Applied Mathematics 44, 141 (1991).