Nonexistence of global solutions of nonlinear wave equations with weak time-dependent damping related to Glassey conjecture

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Abstract

This work is devoted to the nonexistence of global-in-time energy solutions of nonlinear wave equation of derivative type with weak time-dependent damping in the scattering and scale invariant range. By introducing some multipliers to absorb the damping term, we succeed in establishing the same upper bound of the lifespan for the scattering damping as the non-damped case, which is a part of so-called Glassey conjecture on nonlinear wave equations. We also study an upper bound of the lifespan for the scale invariant damping with the same method.

1 Introduction

In this work, we consider the following Cauchy problem for the nonlinear damped wave equations.

\[
\begin{aligned}
  &u_{tt} - \Delta u + \frac{\mu}{(1 + t)^\beta} u_t = |u_t|^p \quad \text{in } \mathbb{R}^n \times [0, \infty), \\
  &u(x, 0) = \varepsilon f(x), \quad u_t(x, 0) = \varepsilon g(x), \quad x \in \mathbb{R}^n,
\end{aligned}
\]

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where $\mu \geq 0$, $n \in \mathbb{N}$ and $\beta \geq 1$. We assume that $\varepsilon > 0$ is a “small” parameter and that $f, g$ are in the energy space with compact support. The restriction on $\beta$ is so-called scattering case ($\beta > 1$) in which the solution of the linear equation scatters to the one of free wave equations, and scale invariant case ($\beta = 1$), in which the linear equation in (1.1) is invariant under the following scaling transform

$$\tilde{u}(x, t) := u(\sigma x, \sigma (1 + t) - 1), \quad \sigma > 0.$$  

We refer the reader to Wirth [14, 15, 16] for the classifications on $\beta$.

First we shall outline the results on (1.1) without damping, i.e. $\mu = 0$. It has been conjectured that there is no global solution for $p > 1$ when $n = 1$, and also that there is a critical power

$$p_c(n) := \frac{n + 1}{n - 1}$$

in the sense that we have global existence for $p > p_c(n)$ while the blow-up in finite time occurs for $1 < p \leq p_c(n)$ when $n \geq 2$. This problem is so-called Glassey conjecture appeared in Glassey [2], and was initiated by John [5] in which he studied more general equations for $n = 3$, and proved that the solution blows-up for $p = 2$. We note that his method works also for $1 < p \leq 2$. After [5], Masuda [9] obtained the blow-up result for $p = 2$ and $n = 1, 2, 3$. Schaeffer [11] established a blow-up result for $n = 2$ and $p = 3$, and conjectured that $p_c(2) = 3$. See also John [6]. Agemi [1] extended the result in [11] to $1 < p \leq 3$. Moreover, Rammaha [10] studied the blow-up result for high dimensional case, $n \geq 4$, under the radially symmetric assumption. Finally, Zhou [18] introduced a simple proof of the blow-up result for all $n \geq 2$ and $1 < p \leq p_c(n)$ as well as $p > 1$ and $n = 1$, and obtained the upper bound of lifespan of the solution. For global existence part, Sideris [12] proved it for $n = 3$ and $p \geq 2$ under the radially symmetric assumption. Hidano and Tsutaya [3], and independently Tzvetkov [13], obtained the global-in-time solution for $n = 2, 3$ and $p > p_c(n)$ without radially symmetric assumption. Finally, Hidano, Wang and Yokoyama [4] generalized the global existence result to high dimensional cases, i.e. $n \geq 4$, under the radially symmetric assumption.

In this work, we are going to study Cauchy problem (1.1) for $\mu > 0$. We focus on the blow-up result and lifespan estimate from above. Without the damping term, the corresponding results has been obtained in Zhou [18], as mentioned above. For our problem we have to overcome the difficulty caused by the damping term. However, due to the scattering and scale invariant coefficients, we may use the multipliers introduced in the authors [7] and the
authors and Wakasa [8] respectively, to absorb the damping term. Then by combining the method used in Zhou and Han [19], we get the blow-up result and the upper bound of lifespan estimate.

2 Main Result

Before showing the main result, we first define the energy and weak solution of the Cauchy problem (1.1).

Definition 2.1 As in [7] and [8], we say that $u$ is an energy solution of (1.1) on $[0, T]$ if

$$u \in C([0, T), H^1(\mathbb{R}^n)) \cap C^1([0, T), L^2(\mathbb{R}^n)) \cap C^1([0, T), L^p(\mathbb{R}^n)),$$

satisfies

$$\int_{\mathbb{R}^n} u_t(x, t) \phi(x, t) dx - \int_{\mathbb{R}^n} u_t(x, 0) \phi(x, 0) dx + \int_0^t ds \int_{\mathbb{R}^n} \{ -u_t(x, s) \phi_t(x, s) + \nabla u(x, s) \cdot \nabla \phi(x, s) \} dx + \int_0^t ds \int_{\mathbb{R}^n} \mu u_t(x, s) \phi(x, s) dx = \int_0^t ds \int_{\mathbb{R}^n} |u_t(x, s)|^p \phi(x, s) dx$$

with any $\phi \in C_0^\infty(\mathbb{R}^n \times [0, T))$ and any $t \in [0, T)$.

Employing the integration by parts in (2.1) and letting $t \to T$, we have that

$$\int_{\mathbb{R}^n \times [0, T]} u(x, s) \left\{ \phi_u(x, s) - \Delta \phi(x, s) - \left( \frac{\mu \phi(x, s)}{(1 + s)^\beta} \right)_s \right\} dx ds$$

$$= \int_{\mathbb{R}^n} \mu u(x, 0) \phi(x, 0) dx - \int_{\mathbb{R}^n} u(x, 0) \phi_t(x, 0) dx + \int_{\mathbb{R}^n} u_t(x, 0) \phi(x, 0) dx + \int_{\mathbb{R}^n \times [0, T]} |u_t(x, s)|^p \phi(x, s) dx ds.$$

This is exactly the definition of the weak solution of (1.1).

Our main results are stated in the following two theorems.

Theorem 2.1 Let $\mu > 0$ and $\beta > 1$. Assume that both $f \in H^1(\mathbb{R}^n)$ and $g \in L^2(\mathbb{R}^n)$ are non-negative, and $g$ does not vanish identically. Suppose that an energy solution $u$ of (1.1) on $[0, T]$ satisfies

$$\text{supp } u \subset \{(x, t) \in \mathbb{R}^n \times [0, \infty) : |x| \leq t + R\} \quad (2.2)$$
with some $R \geq 1$. Then, there exists a constant $\varepsilon_0 = \varepsilon_0(f, g, n, p, \mu, \beta, R) > 0$ such that $T$ has to satisfy

$$T \leq \begin{cases} \varepsilon_0 - (p-1)/(1-(n-1)(p-1)/2) & \text{for } 1 < p < p_c(n) \quad \text{when } n \geq 2, \\ \exp(C\varepsilon_0^{-(p-1)}) & \text{for } p = p_c(n) \text{ and } n \geq 2 \end{cases}$$

with $0 < \varepsilon \leq \varepsilon_0$, where $C$ is a positive constant independent of $\varepsilon$.

Remark 2.1 This estimate provides us the same upper bound of the lifespan as the case of $\mu = 0$ in Zhou [18].

Theorem 2.2 Let $\mu > 0$ and $\beta = 1$. Assume the same condition on $f, g$ and $\text{supp } u$ to Theorem 2.1. Then, for $n \geq 1$, there exists a constant $\varepsilon_0 = \varepsilon_0(f, g, n, p, \mu, \beta, R) > 0$ such that $T$ has to satisfy

$$T \leq \begin{cases} \varepsilon_0 - (p-1)/(1-(n+2\mu-1)(p-1)/2) & \text{for } 1 < p < p_c(n+2\mu), \\ \exp(C\varepsilon_0^{-(p-1)}) & \text{for } p = p_c(n+2\mu) \end{cases}$$

with $0 < \varepsilon \leq \varepsilon_0$, where $C$ is a positive constant independent of $\varepsilon$.

Remark 2.2 Along with the definition of the scattering case by Wirth [14, 15, 16], Theorem 2.1 can be established for generalized damping for which $\mu/(1+t)^{\beta}u_t$ in (1.1) is replaced by positive function $b(t)u_t$ satisfying $b \in L^1([0, \infty))$. It is easy to prove this fact by our proof below if one substitutes the definition of the multiplier $m$ in (3.1) by

$$m(t) = \exp \left( -\int_t^\infty b(s)ds \right)$$

due to the fact that we only need a boundedness of $m$. But such a generalization can not be available in Theorem 2.2 due to the unboundedness of the multiplier $m_1(t)$ in (4.1) below.

### 3 Proof of Theorem 2.1

In the proof of the main theorem, we make use of two key tools. The first one is a multiplier

$$m(t) := \exp \left( \mu \frac{(1+t)^{1-\beta}}{1-\beta} \right), \quad (3.1)$$

which was first introduced in Lai and Takamura [7] and has a property

$$\frac{m'(t)}{m(t)} = \frac{\mu}{(1+t)^\beta}.$$
This multiplier is specially useful for the study of nonlinear damped wave equation with $\beta > 1$ due to its boundedness from above and below as

$$1 \geq m(t) \geq m(0) > 0 \quad \text{for } t \geq 0. \quad (3.2)$$

The other one is defined as

$$\psi(x, t) := e^{-t} \phi_1(x), \quad \phi_1(x) := \begin{cases} \int_{S^{n-1}} e^{x \omega} dS_{\omega} & \text{for } n \geq 2, \\ e^x + e^{-x} & \text{for } n = 1, \end{cases} \quad (3.3)$$

which was introduced in Yordanov and Zhang [17] and admits the following good properties:

$$\psi_t = -\psi, \quad \psi_{tt} = \Delta \psi = \psi. \quad (3.4)$$

We note that there exists a constant $C_1 = C_1(n, R) > 0$ such that

$$\int_{|x| \leq t+R} \psi(x, t) dx \leq C_1(t + 1)^{(n-1)/2} \quad \text{for } t \geq 0 \quad (3.5)$$

with any constant $R > 0$.

Setting

$$F_1(t) := \int_{\mathbb{R}^n} u(x, t) \psi(x, t) dx, \quad (3.6)$$

we have the following lemma.

**Lemma 3.1** Under the same assumption of Theorem 2.1, it holds that

$$F_1(t) \geq \frac{m(0) \varepsilon}{2} \int_{\mathbb{R}^n} f(x) \phi_1(x) dx \geq 0 \quad \text{for } t \geq 0. \quad (3.7)$$

**Proof.** The proof of Lemma 3.1 is parallel to that of (3.9) in [7]. For convenience, we write down the details. By the definition (2.1), we get

$$\frac{d}{dt} \int_{\mathbb{R}^n} u_t(x, t) \phi(x, t) dx + \int_{\mathbb{R}^n} \frac{\mu u_t(x, t)}{(1 + t)^{\beta}} \phi(x, t) dx$$

$$+ \int_{\mathbb{R}^n} \{ -u_t(x, t) \phi_t(x, t) - u(x, t) \Delta \phi(x, t) \} dx$$

$$= \int_{\mathbb{R}^n} |u_t(x, t)|^p \phi(x, t) dx.$$

Multiplying the both sides of the above equality by $m(t)$ we have

$$\frac{d}{dt} \left\{ m(t) \int_{\mathbb{R}^n} u_t(x, t) \phi(x, t) dx \right\}$$

$$+ m(t) \int_{\mathbb{R}^n} \{ -u_t(x, t) \phi_t(x, t) - u(x, t) \Delta \phi(x, t) \} dx$$

$$= m(t) \int_{\mathbb{R}^n} |u_t(x, t)|^p \phi(x, t) dx.$$
Integration this equality over $[0, t]$ implies that

\[
m(t) \int_{\mathbb{R}^n} u_t(x, t) \phi(x, t) dx - m(0) \epsilon \int_{\mathbb{R}^n} g(x) \phi(x, 0) dx
- \int_0^t m(s) ds \int_{\mathbb{R}^n} \{u_t(x, s) \phi_t(x, s) + u(x, s) \Delta \phi(x, s)\} dx
= \int_0^t m(s) ds \int_{\mathbb{R}^n} |u_t(x, s)|^p \phi(x, s) dx.
\]

Replacing $\phi(x, t)$ with $\psi(x, t)$ on $\text{supp } u$ in the above inequality, making use of (3.4) and integration by parts in $t$-integral in the second line, we come to

\[
m(t)\{F'_1(t) + 2F_1(t)\} - m(0) \epsilon \int_{\mathbb{R}^n} \{f(x) + g(x)\} \phi_1(x) dx
= \int_0^t m'(s) F_1(s) ds + \int_0^t m(s) ds \int_{\mathbb{R}^n} |u_t(x, s)|^p \psi(x, s) dx.
\]

which yields

\[
F'_1(t) + 2F_1(t) \geq \frac{m(0)}{m(t)} C_{f,g} \epsilon + \frac{1}{m(t)} \int_0^t m(s) \frac{\mu}{(1 + s)^\beta} F_1(s) ds
\geq m(0) C_{f,g} \epsilon + \frac{1}{m(t)} \int_0^t m(s) \frac{\mu}{(1 + s)^\beta} F_1(s) ds,
\]

where

\[
C_{f,g} := \int_{\mathbb{R}^n} \{f(x) + g(x)\} \phi_1(x) dx.
\]

Here we have used the boundedness of $m$ in (3.2). Hence it is easy to get from (3.8) that

\[
e^{2t} F_1(t) \geq F_1(0) + m(0) C_{f,g} \epsilon \int_0^t e^{2s} ds
+ \int_0^t \frac{e^{2s}}{m(s)} ds \int_0^s m(r) \frac{\mu}{(1 + r)^\beta} F_1(r) dr,
\]

which leads, by comparison argument, to

\[
e^{2t} F_1(t) \geq m(0) C_{f,g} \epsilon + \frac{m(0)}{2} C_{f,0} \epsilon (e^{2t} - 1),
\]

and finally to

\[
F_1(t) \geq \frac{1}{2} m(0) C_{f,0} \epsilon \quad \text{for } t \geq 0
\]

which is exactly the inequality (3.7) we need.

\[\square\]
Now we are in a position to prove Theorem 2.1. First we have

\[
\frac{d}{dt} \left[ m(t) \int_{\mathbb{R}^n} \{u_t(x,t) + u(x,t)\} \psi(x,t)dx \right]
= \frac{\mu}{(1 + t)^\beta} m(t) \int_{\mathbb{R}^n} \{u_t(x,t) + u(x,t)\} \psi(x,t)dx
+ m(t) \frac{d}{dt} \int_{\mathbb{R}^n} \{u_t(x,t) + u(x,t)\} \psi(x,t)dx.
\] (3.10)

Replacing the test function \( \phi \) in the definition (2.1) by \( \psi \) and taking derivative to both sides with respect to \( t \), we have that

\[
\frac{d}{dt} \int_{\mathbb{R}^n} u_t(x,t) \psi(x,t)dx - \int_{\mathbb{R}^n} u_t(x,t) \psi_t(x,t)dx
+ \int_{\mathbb{R}^n} \nabla u(x,t) \cdot \nabla \psi(x,t)dx + \frac{\mu}{(1 + t)^\beta} \int_{\mathbb{R}^n} u_t(x,t) \psi(x,t)dx
= \int_{\mathbb{R}^n} |u_t(x,t)|^p \psi(x,t)dx.
\] (3.11)

Hence the integration by parts in the first term in the second line of (3.11) with (3.4) yields that

\[
\frac{d}{dt} \int_{\mathbb{R}^n} \{u_t(x,t) + u(x,t)\} \psi(x,t)dx
= \int_{\mathbb{R}^n} |u_t(x,t)|^p \psi(x,t)dx - \frac{\mu}{(1 + t)^\beta} \int_{\mathbb{R}^n} u_t(x,t) \psi(x,t)dx.
\]

Plugging this equality into (3.10) we have

\[
\frac{d}{dt} \left[ m(t) \int_{\mathbb{R}^n} \{u_t(x,t) + u(x,t)\} \psi(x,t)dx \right]
= m(t) \int_{\mathbb{R}^n} |u_t(x,t)|^p \psi(x,t)dx + \frac{\mu}{(1 + t)^\beta} m(t) F_1(t)
\] (3.12)

for \( t \geq 0 \). Then (3.12) and the positivity of \( F_1 \) by Lemma 3.1 yield

\[
m(t) \int_{\mathbb{R}^n} \{u_t(x,t) + u(x,t)\} \psi(x,t)dx
\geq m(0) \varepsilon \int_{\mathbb{R}^n} \{f(x) + g(x)\} \phi_1(x)dx
+ \int_0^t ds \int_{\mathbb{R}^n} m(s)|u_t(x,s)|^p \psi(x,s)dx.
\] (3.13)
On the other hand, (3.11) also yields that
\[
\frac{d}{dt} \int_{\mathbb{R}^n} u_t(x, t) \psi(x, t) dx + \frac{n'(t)}{m(t)} \int_{\mathbb{R}^n} u_t(x, t) \psi(x, t) dx \\
+ \int_{\mathbb{R}^n} \{u_t(x, t) - u(x, t)\} \psi(x, t) dx \\
= \int_{\mathbb{R}^n} |u_t(x, t)|^p \psi(x, t) dx.
\]

Multiplying this equality by \(m(t)\), we have
\[
\frac{d}{dt} \left[ m(t) \int_{\mathbb{R}^n} u_t(x, t) \psi(x, t) dx \right] \\
+ m(t) \int_{\mathbb{R}^n} \{u_t(x, t) - u(x, t)\} \psi(x, t) dx \\
= m(t) \int_{\mathbb{R}^n} |u_t(x, t)|^p \psi(x, t) dx.
\]

Adding (3.13) and (3.14) together, we obtain that
\[
\frac{d}{dt} \left[ m(t) \int_{\mathbb{R}^n} u_t(x, t) \psi(x, t) dx \right] + 2m(t) \int_{\mathbb{R}^n} u_t(x, t) \psi(x, t) dx \\
\geq m(0) \varepsilon \int_{\mathbb{R}^n} \{f(x) + g(x)\} \phi_1(x) dx + m(t) \int_{\mathbb{R}^n} |u_t(x, t)|^p \psi(x, t) dx \\
+ \int_0^t m(s) ds \int_{\mathbb{R}^n} |u_t(x, s)|^p \psi(x, s) dx.
\]

Setting
\[
G(t) := m(t) \int_{\mathbb{R}^n} u_t(x, t) \psi(x, t) dx - \frac{m(0) \varepsilon}{2} \int_{\mathbb{R}^n} g(x) \phi_1(x) dx \\
- \frac{1}{2} \int_0^t m(s) ds \int_{\mathbb{R}^n} |u_t(x, s)|^p \psi(x, s) dx,
\]
we have
\[
G(0) = \frac{m(0) \varepsilon}{2} \int_{\mathbb{R}^n} g(x) \phi_1(x) dx > 0.
\]

Then it follows from (3.15) and direct computation that
\[
G'(t) + 2G(t) \geq \frac{m(t)}{2} \int_{\mathbb{R}^n} |u_t(x, t)|^p \psi(x, t) dx + m(0) \varepsilon \int_{\mathbb{R}^n} \phi_1(x) f(x) dx \geq 0
\]

which implies
\[
G(t) \geq e^{-2t} G(0) > 0 \quad \text{for} \ t \geq 0.
\]
Hence, by the definition (3.16), it holds that
\[
m(t) \int_{\mathbb{R}^n} u_t(x,t) \psi(x,t) dx \\
\geq \frac{1}{2} \int_0^t m(s) ds \int_{\mathbb{R}^n} |u_t(x,s)|^p \psi(x,s) dx \\
+ \frac{m(0)\varepsilon}{2} \int_{\mathbb{R}^n} g(x) \phi_1(x) dx.
\]  
(3.17)

Denoting
\[
H(t) := \frac{1}{2} \int_0^t m(s) ds \int_{\mathbb{R}^n} |u_t(x,s)|^p \psi(x,s) dx + \frac{m(0)\varepsilon}{2} \int_{\mathbb{R}^n} g(x) \phi_1(x) dx,
\]
then, by (3.17), we have
\[
m(t) \int_{\mathbb{R}^n} u_t(x,t) \psi(x,t) dx \geq H(t) \quad \text{for } t \geq 0.
\]  
(3.18)

On the other hand, Hölder inequality and (3.5) yield that
\[
2H'(t) \geq C_1^{1-p}(t + 1)^{-(n-1)(p-1)/2} \left( m(t) \int_{\mathbb{R}^n} u_t(x,t) \psi(x,t) dx \right)^p.
\]  
(3.19)

Here we have used the boundedness of \( m \) in (3.2) as \( m(t)^{1-p} \geq 1 \). We then conclude by (3.18) and (3.19) that
\[
H'(t) \geq \frac{C_1^{1-p}}{2(1 + t)^{(n-1)(p-1)/2}} H^p(t) \quad \text{for } t \geq 0,
\]
from which with the initial data \( H(0) = (m(0)C_{0,\theta}/2)\varepsilon > 0 \) we can easily get the upper bound of lifespan estimate in Theorem 2.1.

\[\square\]

## 4 Proof of Theorem 2.2

The proof of Theorem 2.2 can be proceeded along almost the same way as that of Theorem 2.1. The only essential difference is that we replace a multiplier \( m \) defined in (3.1) by
\[
m_1(t) := (1 + t)^\mu,
\]  
(4.1)

which was first introduced in Lai, Takamura and Wakasa [8] and has a property
\[
\frac{m'_1(t)}{m_1(t)} = \frac{\mu}{1 + t}.
\]
Hence the differences in this section from the previous one should appear only in points where the boundedness of $m$ in (3.2) is employed. They are (3.8) and (3.19).

Keeping this fact in our mind, we immediately obtain

**Lemma 4.1** Under the same assumption of Theorem 2.2, it holds that

$$F_1(t) \geq \frac{C_{f,0}\varepsilon}{2m_1(t)} \geq 0 \quad \text{for } t \geq 0,$$

where $F_1$ is defined in (3.6).

**Proof.** The proof is parallel to that of Lemma 3.1. Due to the unboundedness of $m_1$, instead of (3.8), we have

$$F_1'(t) + 2F_1(t) \geq \frac{C_{f,g}\varepsilon}{m_1(t)} + \frac{1}{m_1(t)} \int_0^t \mu(1 + s)^{\mu-1}F_1(s)ds$$

by substituting $m$ with $m_1$ simply. Integrating this inequality over $[0, t]$ with a multiplication $e^{2t}$, we get

$$e^{2t}F_1(t) \geq F_1(0) + C_{f,g}\varepsilon \int_0^t \frac{e^{2s}}{m_1(s)}ds$$

$$+ \int_0^t \frac{e^{2s}}{m_1(s)}ds \int_0^s \mu(1 + r)^{\mu-1}F_1(r)dr.$$

Therefore the comparison argument again yields that

$$F_1(t) > \frac{C_{f,g}\varepsilon}{2m_1(t)}(1 - e^{-2t}) + e^{-2t}F_1(0) \geq \frac{C_{f,0}\varepsilon}{2m_1(t)} \geq 0 \quad \text{for } t \geq 0$$

as desired. \hfill \Box

In this way, we get the positivity of $F_1$ also for the case of $\beta = 1$. Due to this fact, we can proceed the proof of Theorem 2.2 by simple replacement of $m$ by $m_1$ in the one of Theorem 2.1 till making use of the boundedness of $m$ once more at (3.19). Hence, setting

$$H_1(t) := \frac{1}{2} \int_0^t m_1(s)ds \int_{\mathbb{R}^n} |u_t(x, s)|^2 \psi(x, s)dx + \frac{m_1(0)\varepsilon}{2} \int_{\mathbb{R}^n} g(x)\phi_1(x)dx,$$

we have

$$m_1(t) \int_{\mathbb{R}^n} u_t(x, t)\psi(x, t)dx \geq H_1(t) \quad \text{for } t \geq 0. \quad (4.3)$$
This is almost the same as (3.18). On the other hand, Hölder inequality and (3.5) as well as the concrete expression of $m_1$ yield that

$$2H'_1(t) \geq C_1^{1-p}(t+1)^{-(n+2\mu-1)(p-1)/2}\left(m_1(t) \int_{\mathbb{R}^n} u_t(x,t)\psi(x,t)dx\right)^p.$$  (4.4)

We then conclude from (4.3) and (4.4) that

$$H'_1(t) \geq \frac{C_1^{1-p}}{2(1+t)^{(n+2\mu-1)(p-1)/2}}H_1^p(t) \quad \text{for } t \geq 0,$$

from which with the initial data $H_1(0) = (C_0, g/2) \varepsilon > 0$ we can easily get the upper bound of lifespan estimate in Theorem 2.2. \hfill \Box

**Remark 4.1** In the scale invariant damping case, we get an upper bound of the lifespan estimate depending on $\mu$, since we have use the multiplier $m_1(t) = (1+t)^\mu$, which is not bounded from above again, comparing to $m(t)$ used in the scattering case.

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