Is Temporal Difference Learning Optimal?
An Instance-Dependent Analysis

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Abstract
We address the problem of policy evaluation in discounted Markov decision processes, and
provide instance-dependent guarantees on the $\ell_\infty$-error under a generative model. We estab-
lish both asymptotic and non-asymptotic versions of local minimax lower bounds for policy
evaluation, thereby providing an instance-dependent baseline by which to compare algorithms.
Theory-inspired simulations show that the widely-used temporal difference (TD) algorithm is
strictly suboptimal when evaluated in a non-asymptotic setting, even when combined with
Polyak-Ruppert iterate averaging. We remedy this issue by introducing and analyzing variance-
reduced forms of stochastic approximation, showing that they achieve non-asymptotic, instance-
dependent optimality up to logarithmic factors.

1 Introduction

Reinforcement learning (RL) is a class of methods for the optimal control of dynamical systems [6,
5, 39, 7] that has begun to make inroads in a wide range of applied problem domains. This empirical
research has revealed the limitations of our theoretical understanding of this class of methods—
popular RL algorithms exhibit a variety of behavior across domains and problem instances, and
existing theoretical bounds, which are generally based on worst-case assumptions, fail to capture
this variety. An important theoretical goal is to develop instance-specific analyses that help to
reveal what aspects of a given problem make it “easy” or “hard,” and allow distinctions to be
drawn between ostensibly similar algorithms in terms of their performance profiles. The focus of
this paper is on developing such a theoretical understanding for a class of popular stochastic
approximation algorithms used for policy evaluation.

RL methods are generally formulated in terms of a Markov decision process (MDP). An agent
operates in an environment whose dynamics are described by an MDP but are unknown: at each
step, it observes the current state of the environment, and takes an action that changes the state
according to some stochastic transition function. The eventual goal of the agent is to learn a
policy—a mapping from states to actions—that optimizes the reward accrued over time. In the
typical setting, rewards are assumed to be additive over time, and are also discounted over time.
Within this broad context, a key sub-problem is that of policy evaluation, where the goal is estimate
the long-term expected reward of a fixed policy based on observed state-to-state transitions and one-step rewards. It is often preferable to have $\ell_\infty$-norm guarantees for such an estimate, since these are particularly compatible with policy-iteration methods. In particular, policy iteration can be shown to converge at a geometric rate when combined with policy evaluation methods that are accurate in $\ell_\infty$-norm (see, e.g., [1, 7]).

In this paper, we study a class of stochastic approximation algorithms for this problem under a generative model for the underlying MDP, with a focus on developing instance-dependent bounds. Our results complement an earlier paper by a subset of the authors [32], which studied the least squares temporal difference (LSTD) method through such a lens.

1.1 Related work

We begin with a broad overview of related work, categorizing that work as involving asymptotic analysis, non-asymptotic analysis, or instance-dependent analysis.

Asymptotic theory: Markov reward processes have been the subject of considerable classical study [20, 19]. In the context of reinforcement learning and stochastic control, the policy evaluation problem for such processes has been tackled by various approaches based on stochastic approximation. Here we focus on past work that studies the temporal difference (TD) update and its relatives; see [15] for a comprehensive survey. The TD update was originally proposed by Sutton [38], and is typically used in conjunction with an appropriate parameterization of value functions. Classical results on the algorithm are typically asymptotic, and include both convergence guarantees [22, 10, 11] and examples of divergence [4]; see the seminal work [41] for conditions that guarantee asymptotic convergence.

It is worth noting that the TD algorithm is a form of linear stochastic approximation, and can be fruitfully combined with the iterate-averaging procedure put forth independently by Polyak [33] and Ruppert [35]. In this context, the work of Polyak and Juditsky [33] deserves special mention, since it shows that under fairly mild conditions, the TD algorithm converges when combined with Polyak-Ruppert iterate averaging. To be clear, in the specific context of the policy evaluation problem, the results in the Polyak-Juditsky paper [33] allow noise only in the observations of rewards (i.e., the transition function is assumed to be known). However, the underlying techniques can be extended to derive results in the setting in which we only observe samples of transitions; for instance, see the work of Tadic [40] for results of this type.

Non-asymptotic theory: Recent years have witnessed significant interest in understanding TD-type algorithms from the non-asymptotic standpoint. Bhandari et al. [8] focus on proving $\ell_2$-guarantees for the TD algorithm when combined with Polyak-Ruppert iterate averaging. They consider both the generative model as well as the Markovian noise model, and provide non-asymptotic guarantees on the expected error. Their results also extend to analyses of the popular TD($\lambda$) variant of the algorithm, as well as to Q-learning in specific MDP instances. Also noteworthy is the analysis of Lakshminarayanan and Szepesvari [27], carried out in parallel with Bhandari et al. [8]; it provides similar guarantees on the TD(0) algorithm with constant stepsize and averaging. Note that both of these analyses focus on $\ell_2$-guarantees (equipped with an associated inner product), and thus can directly leverage proof techniques for stochastic optimization [3, 31].
Other related results\footnote{There were some errors in the results of Korda and La \cite{26} that were pointed out by both Lakshminarayanan and Szepesvari \cite{27} and Xu et al. \cite{49}.} include those of Dalal et al. \cite{13}, Doan et al. \cite{16}, Korda and La \cite{26}, and also more contemporary papers \cite{49, 45}. The latter three of these papers introduce a variance-reduced form of temporal difference learning, a variant of which we analyze in this paper.

**Instance-dependent results:** The focus on instance-dependent guarantees for TD algorithms is recent, and results are available both in the $\ell_2$-norm setting \cite{8, 27, 14, 49} and the $\ell_\infty$-norm settings \cite{32}. In general, however, the guarantees provided by work to date are not sharp. For instance, the bounds in \cite{14} scale exponentially in relevant parameters of the problem, whereas the papers \cite{8, 27, 49} do not capture the correct “variance” of the problem instance at hand. A subset of the current authors \cite{32} derived $\ell_\infty$ bounds on policy evaluation for the plug-in estimator. These results were shown to be locally minimax optimal in certain regions of the parameter space. There has also been some recent focus on obtaining instance-dependent guarantees in online reinforcement learning settings \cite{30}. This has resulted in more practically useful algorithms that provide, for instance, horizon-independent regret bounds for certain episodic MDPs \cite{50, 23}, thereby improving upon worst-case bounds \cite{2}. Recent work has also established some instance-dependent bounds, albeit not sharp over the whole parameter space, for the problem of state-action value function estimation in Markov decision processes, for both ordinary $Q$-learning \cite{47} and a variance-reduced improvement \cite{48}.

### 1.2 Contributions

In this paper, we study stochastic approximation algorithms for evaluating the value function of a Markov reward process in the discounted setting. Our goal is to provide a sharp characterization of performance in the $\ell_\infty$-norm, for procedures that are given access to state transitions and reward samples under the generative model. In practice, temporal difference learning is typically applied with an additional layer of (linear) function approximation. In the current paper, so as to bring the instance dependence into sharp focus, we study the algorithms without this function approximation step. In this context, we tell a story with three parts, as detailed below:

**Local minimax lower bounds:** Global minimax analysis provides bounds that hold uniformly over large classes of models. In this paper, we seek to gain a more refined understanding of how the difficulty of policy evaluation varies as a function of the instance. In order to do so, we undertake an analysis of the local minimax risk associated with a problem. We first prove an asymptotic statement (Proposition 1) that characterizes the local minimax risk up to a logarithmic factor; it reveals the relevance of two functionals of the instance that we define. In proving this result, we make use of the classical asymptotic minimax theorem \cite{21, 28, 29}. We then refine this analysis by deriving a non-asymptotic local minimax bound, as stated in Theorem 1 which is derived using the non-asymptotic local minimax framework of Cai and Low \cite{13}, an approach that builds upon the seminal concept of hardest local alternatives that can be traced back to Stein \cite{37}.

**Non-asymptotic suboptimality of iterate averaging:** Our local minimax lower bounds raise a natural question: Do standard procedures for policy evaluation achieve these instance-specific bounds? In Section 3.2 we address this question for the TD(0) algorithm with iterate averaging.
Via a careful simulation study, we show that for many popular stepsize choices, the algorithm fails to achieve the correct instance-dependent rate in the non-asymptotic setting, even when the sample size is quite large. This is true for both the constant stepsize, as well as polynomial stepsizes of various orders. Notably, the algorithm with polynomial stepsizes of certain orders achieves the local risk in the asymptotic setting (see Theorem 1).

Non-asymptotic optimality of variance reduction: In order to remedy this issue with iterate averaging, we propose and analyze a variant of TD learning with variance reduction, showing both through theoretical (see Theorem 2) and numerical results (see Figure 3) that this algorithm achieves the correct instance-dependent rate provided the sample size is larger than an explicit threshold. Thus, this algorithm is provably better than TD(0) with iterate averaging.

1.3 Notation
For a positive integer $n$, let $[n] := \{1, 2, \ldots, n\}$. For a finite set $S$, we use $|S|$ to denote its cardinality. We use $c, C, c_1, c_2, \ldots$ to denote universal constants that may change from line to line. We let $1$ denote the all-ones vector in $\mathbb{R}^D$. Let $e_j$ denote the $j$-th standard basis vector in $\mathbb{R}^D$. We let $v(i)$ denote the $i$-th order statistic of a vector $v$, i.e., the $i$-th largest entry of $v$. For a pair of vectors $(u, v)$ of compatible dimensions, we use the notation $u \preceq v$ to indicate that the difference vector $v - u$ is entrywise non-negative. The relation $u \succeq v$ is defined analogously. We let $|u|$ denote the entrywise absolute value of a vector $u \in \mathbb{R}^D$; squares and square-roots of vectors are, analogously, taken entrywise. Note that for a positive scalar $\lambda$, the statements $|u| \preceq \lambda \cdot 1$ and $\|u\|_\infty \leq \lambda$ are equivalent. Finally, we let $\|M\|_{1,\infty}$ denote the maximum $\ell_1$-norm of the rows of a matrix $M$, and refer to it as the $(1, \infty)$-operator norm of a matrix. More generally, for scalars $q, p \geq 1$, we define $\|M\|_{p,q} := \sup_{\|x\|_p \leq 1} \|Mx\|_q$. We let $M^\dagger$ denote the Moore-Penrose pseudoinverse of a matrix $M$.

2 Background and problem formulation
We begin by introducing the basic mathematical formulation of Markov reward processes (MRPs) and generative observation models.

2.1 Markov reward processes and value functions
We study MRPs defined on a finite set of $D$ states, which we index by the set $[D] \equiv \{1, 2, \ldots, D\}$. The state evolution over time is determined by a set of transition functions, $\{P(\cdot|i), i \in [D]\}$. Note that each such transition function can be naturally associated with a $D$-dimensional vector; denote the $i$-th such vector as $p_i$. We let $P \in [0, 1]^{D \times D}$ denote a row-stochastic (Markov) transition matrix, where row $i$ of this matrix contains the vector $p_i$. Also associated with an MRP is a population reward function, $r : [D] \rightarrow \mathbb{R}$, possessing the semantics that a transition from state $i$ results in the reward $r(i)$. For convenience, we engage in a minor abuse of notation by letting $r$ also denote a vector of length $D$; here $r_i$ corresponds to the reward obtained at state $i$.

We formulate the long-term value of a state in the MRP in terms of the infinite-horizon, discounted reward. This value function (denoted here by the vector $\theta^* \in \mathbb{R}^D$) can be computed as the unique solution of the Bellman fixed-point relation, $\theta^* = r + \gamma P \theta^*$. 

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2.2 Observation model

In the learning setting, the pair \((P, r)\) is unknown, and we accordingly assume access to a black box that generates samples from the transition and reward functions. In this paper, we operate under a setting known as the synchronous or generative setting \([25]\); this setting is also often referred to as the “i.i.d. setting” in the policy evaluation literature. For a given sample index, \(k \in \{1, 2, \ldots, N\}\) and for each state \(j \in [D]\), we observe a random next state \(X_{k,j} \sim P(\cdot|j)\) for \(j \in [D]\). (1a)

We collect these transitions in a matrix \(Z_k\), which by definition contains one 1 in each row: the 1 in the \(j\)-th row corresponds to the index of state \(X_{k,j}\). We also observe a random reward vector \(R_k \in \mathbb{R}^D\), where the rewards are generated independently across states with\(^2\)

\[R_{k,j} \sim \mathcal{N}(r_j, \sigma_r^2).\] (1b)

Given these samples, define the \(k\)-th (noisy) linear operator \(\hat{T}_k : \mathbb{R}^D \mapsto \mathbb{R}^D\) whose evaluation at the point \(\theta\) is given by

\[\hat{T}_k(\theta) = R_k + \gamma Z_k \theta.\] (2)

The construction of these operators is inspired by the fact that we are interested in computing the fixed point of the population operator,

\[\mathcal{T} : \theta \mapsto r + \gamma P \theta,\] (3)

and a classical and natural way to do so is via a form of stochastic approximation known as temporal difference learning, which we describe next.

2.3 Temporal difference learning and its variants

Classical temporal difference (TD) learning algorithms are parametrized by a sequence of stepsizes, \(\{\alpha_k\}_{k \geq 1}\), with \(\alpha_k \in (0, 1]\). Starting with an initial vector \(\theta_1 \in \mathbb{R}^D\), the TD updates take the form

\[\theta_{k+1} = (1 - \alpha_k)\theta_k + \alpha_k \hat{T}_k(\theta_k) \quad \text{for } k = 1, 2, \ldots.\] (4)

In the sequel, we discuss three popular stepsize choices:

- **Constant stepsize:** \(\alpha_k = \alpha\), where \(0 < \alpha \leq \alpha_{\text{max}}\). (5a)
- **Polynomial stepsize:** \(\alpha_k = \frac{1}{k^\omega}\) for some \(\omega \in (0, 1)\). (5b)
- **Recentered-linear stepsize:** \(\alpha_k = \frac{1}{1 + (1 - \gamma)k}\). (5c)

In addition to the TD sequence \((4)\), it is also natural to perform *Polyak-Ruppert averaging*, which produces a parallel sequence of averaged iterates

\[\bar{\theta}_k = \frac{1}{k} \sum_{j=1}^{k} \theta_j \quad \text{for } k = 1, 2, \ldots.\] (6)

\(^2\)All of our upper bounds extend with minor modifications to the sub-Gaussian reward setting.
Such averaging schemes were introduced in the context of general stochastic approximation by Polyak [33] and Ruppert [35]. A large body of theoretical literature demonstrates that such an averaging scheme improves the rates of convergence of stochastic approximation when run with overly “aggressive” stepsizes.

3 Main results

We turn to the statements of our main results and discussion of their consequences. All of our statements involve certain measures of the local complexity of a given problem, which we introduce first. We then turn to the statement of lower bounds on the $\ell_\infty$-norm error in policy evaluation. In Section 3.1, we prove two lower bounds. Our first result, stated as Proposition 1, is asymptotic in nature (holding as the sample size $N \to +\infty$). Our second lower bound, stated as Theorem 1, provides a result that holds for a range of finite sample sizes. Given these lower bounds, it is then natural to wonder about known algorithms that achieve them. Concretely, does the TD(0) algorithm combined with Polyak-Ruppert averaging achieve these instance-dependent bounds? In Section 3.2, we undertake a careful empirical study of this question, and show that in the non-asymptotic setting, this algorithm fails to match the instance-dependent bounds. This finding sets up the analysis in Section 3.3, where we introduce a variance-reduced version of TD(0), and prove that it does achieve the instance-dependent lower bounds from Theorem 1 up to a logarithmic factor in dimension.

Local complexity measures: Recall the generative observation model described in Section 2.2. For a transition matrix $P$, we write $Z \sim P$ to denote a draw of a random matrix with $\{0, 1\}$ entries and a single one in each row (with the position of the one in row $Z_j$ determined by sampling from the multinomial distribution specified by $p_j$). For a fixed vector $\theta \in \mathbb{R}^D$, note that $(Z - P)\theta$ is a random vector in $\mathbb{R}^D$, and define its covariance matrix as follows:

$$\Sigma_P(\theta) = \text{cov}_{Z \sim P} ((Z - P)\theta). \quad (7)$$

We often use $\Sigma(\theta)$ as a shorthand for $\Sigma_P(\theta)$ when the underlying transition matrix $P$ is clear from the context.

With these definitions in hand, define the complexity measures

$$\nu(P, \theta) := \max_{\ell \in [D]} \left( e_{\ell}^T (I - \gamma P)^{-1} \Sigma(\theta) (I - \gamma P)^{-1} e_{\ell} \right)^{1/2}, \quad \text{and} \quad (8a)$$

$$\rho(P, r) := \sigma_r \| (I - \gamma P)^{-1} \|_{2, \infty} \equiv \sigma_r \max_{\|u\|_2 = 1} \| (I - \gamma P)^{-1} u \|_\infty. \quad (8b)$$

Note that $\nu(P, \theta)$ corresponds to the maximal variance of the random vector $(I - \gamma P)^{-1} (Z - P)\theta$. As we demonstrate shortly, the quantities $\nu(P, \theta^*)$ and $\rho(P, r)$ govern the local complexity of estimating the value function $\theta^*$ induced by the instance $(P, r)$ under the observation model (1). A portion of our results also involve the quantity

$$b(\theta) := \frac{\|\theta\|_{\text{span}}}{1 - \gamma}, \quad (8c)$$

where $\|\theta\|_{\text{span}} = \max_{j \in [D]} \theta_j - \min_{j \in [D]} \theta_j$ is the span seminorm.
3.1 Local minimax lower bound

Throughout this section, we use the letter \(P\) to denote an individual problem instance, \(P = (P, r)\), and use \(\theta(P) := \theta^* = (I - \gamma P)^{-1}r\) to denote the target of interest. The aim of this section is to establish instance-specific lower bounds for estimating \(\theta(P)\) under the observation model (1). In order to do so, we adopt a local minimax approach.

The remainder of this section is organized as follows. In Section 3.1.1, we prove an asymptotic local minimax lower bound, valid as the sample size \(N\) tends to infinity. It gives an explicit Gaussian limit for the rescaled error that can be achieved by any procedure. The asymptotic covariance in this limit law depends on the problem instance, and is very closely related to the functionals \(v(P, \theta^*)\) and \(\rho(P, r)\) that we have defined. Moreover, we show that this limit can be achieved—in the asymptotic limit—by the TD algorithm combined with Polyak-Ruppert averaging. While this provides a useful sanity check, in practice we implement estimators using a finite number of samples \(N\), so it is important to obtain non-asymptotic lower bounds for a full understanding. With this motivation, Section 3.1.2 provides a new, non-asymptotic instance-specific lower bound for the policy evaluation problem. We show that the quantities \(v(P, \theta^*)\) and \(\rho(P, r)\) also cover the instance-specific complexity in the finite-sample setting. In proving this non-asymptotic lower bound, we build upon techniques in the statistical literature based on constructing hardest one-dimensional alternatives [37, 9, 17, 18, 12]. As we shall see in later sections, while the TD algorithm with averaging is instance-specific optimal in the asymptotic setting, it fails to achieve our non-asymptotic lower bound.

3.1.1 Asymptotic local minimax lower bound

Our first approach towards an instance-specific lower bound is an asymptotic one, based on classical local asymptotic minimax theory. For regular and parametric families, the Hájek–Le Cam local asymptotic minimax theorem [21, 28, 29] shows that the Fisher information—an instance-specific functional—characterizes a fundamental asymptotic limit. Our model class is both parametric and regular (cf. Eq. (1)), and so this classical theory applies to yield an asymptotic local minimax bound. Some additional work is needed to relate this statement to the more transparent complexity measures \(v(P, \theta^*)\) and \(\rho(P, r)\) that we have defined.

In order to state our result, we require some additional notation. Fix an instance \(P = (P, r)\). For any \(\epsilon > 0\), we define an \(\epsilon\)-neighborhood of problem instances by

\[
\mathcal{N}(P; \epsilon) = \{P' = (P', r') : \|P - P'\|_F + \|r - r'\|_2 \leq \epsilon\}.
\]

Adopting the \(\ell_\infty\)-norm as the loss function, the local asymptotic minimax risk is given by

\[
\mathcal{M}_\infty(P) \equiv \mathcal{M}_\infty(P; \|\cdot\|_\infty) = \lim_{\epsilon \to 0} \lim_{N \to \infty} \inf_{\hat{\theta}_N} \sup_{\mathcal{N}(P; \epsilon)} \mathbb{E}_Q \left[ \sqrt{N} \|\hat{\theta}_N - \theta(Q)\|_\infty \right].
\]

(9)

Here the infimum is taken over all estimators \(\hat{\theta}_N\) that are measurable functions of \(N\) i.i.d. observations drawn according to the observation model (1).

Our first main result characterizes the local asymptotic risk \(\mathcal{M}_\infty(P)\) exactly, and shows that it is attained by stochastic approximation with averaging. Recall the Polyak-Ruppert (PR) sequence \(\{\hat{\theta}_k\}_{k \geq 1}\) defined in Eq. (6), and let \(\{\tilde{\theta}_k\}_{k \geq 1}\) denote this sequence when the underlying SA algorithm is the TD update with the polynomial stepsize sequence (5b) with exponent \(\omega\).
Proposition 1. Let $Z \in \mathbb{R}^D$ be a multivariate Gaussian with zero mean and covariance matrix

$$(I - \gamma P)^{-1}(\gamma^2 \Sigma_P(\theta(P)) + \sigma^2 \gamma I)(I - \gamma P)^{-T}. \tag{10a}$$

Then the local asymptotic minimax risk at problem instance $P$ is given by

$$M_\infty(P) = \mathbb{E}[\|Z\|_\infty]. \tag{10b}$$

Furthermore, for each problem instance $P$ and scalar $\omega \in (1/2, 1)$, this limit is achieved by the TD algorithm with an $\omega$-polynomial stepsize and PR-averaging:

$$\lim_{N \to \infty} \sqrt{N} \cdot \mathbb{E}\left[\|\hat{\theta}_N^\omega - \theta(P)\|_\infty\right] = \mathbb{E}[\|Z\|_\infty]. \tag{10c}$$

With the convention that $\theta^* \equiv \theta(P)$, a short calculation bounding the maximum absolute value of sub-Gaussian random variables (see, e.g., Ex. 2.11 in Wainwright [46]) yields the sandwich relation

$$\nu(P, \theta^*) + \rho(P, r) \leq \mathbb{E}[\|Z\|_\infty] \leq \sqrt{2 \log D} \cdot (\nu(P, \theta^*) + \rho(P, r)), \tag{8}$$

so that Proposition 1 shows that, up to a logarithmic factor in dimension $D$, the local asymptotic minimax risk is entirely characterized by the functional $\nu(P, \theta^*) + \rho(P, r)$.

It should be noted that lower bounds similar to Eq. (10b) have been shown for specific classes of stochastic approximation algorithms [43]. However, to the best of our knowledge, a local minimax lower bound—one applying to any procedure that is a measurable function of the observations—is not available in the existing literature.

Furthermore, Eq. (10c) shows that stochastic approximation with polynomial stepsizes and averaging attains the exact local asymptotic risk. Our proof of this result essentially mirrors that of Polyak and Juditsky [33], and amounts to verifying their assumptions under the policy evaluation setting. Given this result, it is natural to ask if averaging is optimal also in the non-asymptotic setting; answering this question is the focus of the next two sections of the paper.

3.1.2 Non-asymptotic local minimax lower bound

Proposition 1 provides an instance-specific lower bound on $\theta(P)$ that holds asymptotically. In order to obtain a non-asymptotic guarantee, we borrow ideas from the non-asymptotic framework introduced by Cai and Low [12] for nonparametric shape-constrained inference. Adapting their definition of local minimax risk to our problem setting, given the loss function $L(\theta - \theta^*|P) = \|\theta - \theta^*\|_\infty$, the (normalized) local non-asymptotic minimax risk for $\theta(\cdot)$ at instance $P = (P, r)$ is given by

$${\mathcal{M}_N}(P) = \sup_{P'} \inf_{\hat{\theta}_N, Q \in \{P, P'\}} \max_{\theta, \theta^*} \sqrt{N} \cdot \mathbb{E}_Q\left[\|\hat{\theta}_N - \theta(Q)\|_\infty\right]. \tag{11}$$

Here the infimum is taken over all estimators $\hat{\theta}_N$ that are measurable functions of $N$ i.i.d. observations drawn according to the observation model (1), and the normalization by $\sqrt{N}$ is for convenience. The definition (11) is motivated by the notion of the hardest one-dimensional alternative [44, Ch. 25]. Indeed, given an instance $P$, the local non-asymptotic risk $\mathcal{M}_N(P)$ first looks for the hardest alternative $P'$ against $P$ (which should be local around $P$), then measures the worst-case risk over $P$ and its (local) hardest alternative $P'$.

With this definition in hand, we lower bound the local non-asymptotic minimax risk using the complexity measures $\nu(P, \theta^*)$ and $\rho(P, r)$ defined in Eq. (8):
Theorem 1. There exists a universal constant \( c > 0 \) such that for any instance \( \mathcal{P} = (\mathbf{P}, r) \), the local non-asymptotic minimax risk is lower bounded as

\[
\mathfrak{M}_N(\mathcal{P}) \geq c \left( \gamma \nu(\mathbf{P}, \theta^*) + \rho(\mathbf{P}, r) \right).
\]  

This bound is valid for all sample sizes \( N \) that satisfy

\[
N \geq N_0 := \max \left\{ \frac{\gamma^2}{(1-\gamma)^2}, \frac{b^2(\theta^*)}{\nu^2(\mathbf{P}, \theta^*)} \right\}.
\]

A few comments are in order. First, it is natural to wonder about the necessity of condition (13) on the sample size \( N \) in our lower bound. Our past work provides upper bounds on the \( \ell_\infty \)-error of the plugin estimator \([32]\), and these results also require a bound of this type. In fact, when the rewards are observed with noise (i.e., for any \( \sigma > 0 \)), the condition \( N \geq \frac{\gamma^2}{(1-\gamma)^2} \) is natural, since it is necessary in order to obtain an estimate of the value function with \( \mathcal{O}(1) \) error. On the other hand, in the special case of deterministic rewards (\( \sigma = 0 \)), it is interesting to ask how the fundamental limits of the problem behave in the absence of this condition.

Second, note that the non-asymptotic lower bound (12) is closely connected to the asymptotic local minimax bound from Proposition 1. In particular, for any sample size \( N \) satisfying the lower bound (13), our non-asymptotic lower bound (12) coincides with the asymptotic lower bound (10b) up to a constant factor. Thus, it cannot be substantially sharpened. The finite-sample nature of the lower bound (12) is a powerful tool for assessing optimality of procedures: it provides a performance benchmark that holds over a large range of finite sample sizes \( N \). Indeed, in the next section, we study the performance of the TD learning algorithm with Polyak-Ruppert averaging. While this procedure achieves the local minimax lower bound asymptotically, as guaranteed by Eq. (10c) in Proposition 1, it falls short of doing so in natural finite-sample scenarios.

3.2 Suboptimality of averaging

Polyak and Juditsky \([33]\) provide a general set of conditions under which a given stochastic-approximation (SA) algorithm, when combined with Polyak-Ruppert averaging, is guaranteed to have asymptotically optimal behavior. For the current problem, the bound (10c) in Proposition 1, which is proved using the Polyak-Juditsky framework, shows that SA with polynomial stepsizes and averaging have this favorable asymptotic property.

However, asymptotic theory of this type gives no guarantees in the finite-sample setting. In particular, suppose that we are given a sample size \( N \) that scales as \((1 - \gamma)^{-2}\), as specified in our lower bounds. Does the averaged TD(0) algorithm exhibit optimal behavior in this non-asymptotic setting? In this section, we answer this question in the negative. More precisely, we describe a parameterized family of Markov reward processes, and provide careful simulations that reveal the suboptimality of TD without averaging.

3.2.1 A simple construction

The lower bound in Theorem 1 predicts a range of behaviors depending on the pair \( \nu(\mathbf{P}, \theta^*) \) and \( \rho(\mathbf{P}, r) \). In order to observe a large subset of these behaviors, it suffices to consider a very simple
Figure 1. Illustration of the 2-state MRP used in the simulation. The triple of scalars \((p, \nu, \tau)\), along with the discount factor \(\gamma\), are parameters of the construction. The chain remains in state 1 with probability \(p\) and transitions to state 2 with probability \(1 - p\); on the other hand, state 2 is absorbing. The rewards in states 1 and 2 are deterministic, specified by \(\nu\) and \(\nu \tau\), respectively.

In the MRP, \(\mathcal{P} = (\mathbf{P}, r)\) with \(D = 2\) states, as illustrated in Figure 1. In this MRP, the transition matrix \(\mathbf{P} \in \mathbb{R}^{2 \times 2}\) and reward vector \(r \in \mathbb{R}^2\) take the form

\[
\mathbf{P} = \begin{bmatrix} p & 1 - p \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad r = \begin{bmatrix} \nu \\ \nu \tau \end{bmatrix}.
\]

Here the triple \((p, \nu, \tau)\), along with the discount factor \(\gamma \in [0, 1]\), are parameters of the construction.

In order to parameterize this MRP in a scalarized manner, we vary the triple \((p, \nu, \tau)\) in the following way. First, we fix a scalar \(\lambda \geq 0\), and then we set

\[
p = \frac{4\gamma - 1}{\beta \gamma}, \quad \nu = 1 \quad \text{and} \quad \tau = 1 - (1 - \gamma)^{\lambda}.
\]

Note that this sub-family of MRPs is fully parametrized by the pair \((\gamma, \lambda)\). Let us clarify why this particular scalarization is interesting. It can be shown via simple calculations that the underlying MRP satisfies

\[
\nu(\mathbf{P}, \theta^*) \sim \left(\frac{1}{1 - \gamma}\right)^{1.5 - \lambda}, \quad \rho(\mathbf{P}, r) = 0 \quad \text{and} \quad b(\theta^*) \sim \left(\frac{1}{1 - \gamma}\right)^{2 - \lambda},
\]

where \(\sim\) denotes equality that holds up to a constant pre-factor. Consequently, by Theorem 1 the minimax risk, measured in terms of the \(\ell_\infty\)-norm, satisfies

\[
\mathcal{M}_N(\mathcal{P}) \geq c \cdot \left(\frac{1}{1 - \gamma}\right)^{1.5 - \lambda}. \quad (14)
\]

Thus, it is natural to study whether the TD(0) algorithm with PR averaging achieves this error.

3.2.2 A simulation study

In order to compare the behavior of averaged TD with the lower bound (14), we performed a series of experiments of the following type. For a fixed parameter \(\lambda\) in the range \([0, 1.5]\), we generated a range of MRPs with different values of the discount factor \(\gamma\). For each value of the discount parameter \(\gamma\), we consider the problem of estimating \(\theta^*\) using a sample size \(N\) set to be one of two possible values: namely, \(N \in \left\{\left\lceil \frac{8}{(1 - \gamma)^2} \right\rceil, \left\lfloor \frac{8}{(1 - \gamma)^2} \right\rfloor\right\}\).
Figure 2. Log-log plots of the $\ell_\infty$-error versus the discount complexity parameter $1/(1 - \gamma)$ for various algorithms. Each point represents an average over 1000 trials, with each trial simulations are for the 2-state MRP depicted in Figure 1 with the parameter choices $p = \frac{4\gamma - 1}{(1 - \gamma)^3}$, $\nu = 1$ and $\tau = 1 - (1 - \gamma)^\lambda$. We have also plotted the least-squares fits through these points, and the slopes of these lines are provided in the legend. In particular, the legend contains the stepsize choice for averaged SA (denoted as $\alpha_k$), the slope $\hat{\beta}$ of the least-squares line, and the ideal value $\beta^*$ of the slope computed in equation 15. We also include the lower bound predicted by Theorem 1 for these examples as a dotted line for comparison purposes. Logarithms are to the natural base.

In Figure 2, we plot the $\ell_\infty$-error of the averaged SA, for constant stepsize (5a), polynomial-decay stepsize (5b) and recentered linear stepsize (5c), as a function of $\gamma$. The plots show the behavior for $\lambda \in \{0.5, 1.5\}$. Each point on each curve is obtained by averaging 1000 Monte Carlo trials of the experiment. Note that from our lower bound calculations above (14), the log $\ell_\infty$-error is related to the complexity $\log(\frac{1}{1 - \gamma})$ in a linear fashion; we use $\beta^*$ to denote the slope of this idealized line. Simple algebra yields

$$\beta^* = \frac{1}{2} - \lambda \quad \text{for} \quad N = \frac{1}{(1 - \gamma)^2}, \quad \text{and} \quad \beta^* = -\lambda \quad \text{for} \quad N = \frac{1}{(1 - \gamma)^3}. \quad (15)$$

In other words, for an algorithm which achieves the lower bound predicted by our theory, we expect a linear relationship between the log $\ell_\infty$-error and log discount complexity $\log(\frac{1}{1 - \gamma})$, with the slope $\beta^*$.

Accordingly, for the averaged SA estimators with the stepsize choices in (5a)-(5c), we performed a linear regression to estimate the slopes between the log $\ell_\infty$-error and the log discount-complexity $\log(\frac{1}{1 - \gamma})$. The plot legend reports the stepsize choices $\alpha_k$ and the slope $\hat{\beta}$ of the fitted regression line. We also include the lower bound in the plots, as a dotted line along with its slope, for a visual comparison. We see that the slopes corresponding to the averaged SA algorithm are higher compared to the ideal slopes of the dotted lines. Stated differently, this means that the averaged SA algorithm does not achieve the lower bound with either the constant step or the polynomial-decay step. Overall, the simulations provided in this section demonstrate that the averaged SA algorithm,
although guaranteed to be asymptotically optimal by Eq. (10c) in Proposition 1, does not yield the ideal non-asymptotic behavior.

3.3 Variance-reduced policy evaluation

In this section, we propose and analyze a variance-reduced version of the TD learning algorithm. As in standard variance-reduction schemes, such as SVRG [24], our algorithm proceeds in epochs. In each epoch, we run a standard stochastic approximation scheme, but we recenter our updates in order to reduce their variance. The recentering uses an empirical approximation to the population Bellman operator $\mathcal{T}$.

We describe the behavior of the algorithm over epochs by a sequence of operators, $\{V_m\}_{m \geq 1}$, which we define as follows. At epoch $m$, the method uses a vector $\hat{\theta}_m$ in order to recenter the update, where the vector $\hat{\theta}_m$ should be understood as the best current approximation to the unknown vector $\theta^*$. In the ideal scenario, such a recentering would involve the quantity $T(\theta_m)$, where $T$ denotes the population operator previously defined in Eq. (3). Since we lack direct access to the population operator $T$, however, we use the Monte Carlo approximation

$$\tilde{T}_{N_m}(\theta_m) := \frac{1}{N_m} \sum_{i \in D_m} \tilde{T}_i(\theta_m),$$

where the empirical operator $\tilde{T}_i$ is defined in Eq. (2). Here the set $D_m$ is a collection of $N_m$ i.i.d. samples, independent of all other randomness.

Given the pair $((\hat{\theta}_m, \tilde{T}_{N_m}(\hat{\theta}_m))$ and a stepsize $\alpha \in (0, 1)$, we define the operator $V_k$ on $\mathbb{R}^D$ as follows:

$$\theta \mapsto V_k(\theta; \alpha, \theta_m, \tilde{T}_{N_m}) := (1 - \alpha)\theta + \alpha \left\{ \tilde{T}_k(\theta) - \tilde{T}_k(\hat{\theta}_m) + \tilde{T}_{N_m}(\hat{\theta}_m) \right\}.$$  

As defined in Eq. (2), the quantity $\tilde{T}_k$ is a stochastic operator, where the randomness is independent of the set of samples $D_m$ used to define $\tilde{T}_{N_m}$. Consequently, the stochastic operator $\tilde{T}_k$ is independent of the recentering vector $\tilde{T}_{N_m}(\hat{\theta}_m)$. Moreover, by construction, for each $\theta \in \mathbb{R}^D$, we have

$$\mathbb{E} \left[ \tilde{T}_k(\theta) - \tilde{T}_k(\hat{\theta}_m) + \tilde{T}_{N_m}(\hat{\theta}_m) \right] = T(\theta).$$

Thus, we see that $V_k$ can be seen as an unbiased stochastic approximation of the population-level Bellman operator. As will be clarified in the analysis, the key effect of the recentering steps is to reduce its associated variance.

3.3.1 A single epoch

Based on the variance-reduced policy evaluation update defined in Eq. (17), we are now ready to define a single epoch of the overall algorithm. We index epochs using the integers $m = 1, 2, \ldots, M$, where $M$ corresponds to the total number of epochs to be run. Epoch $m$ requires as inputs the following quantities:

- a vector $\tilde{\theta}$, which is chosen to be the output of the previous epoch,
- a positive integer $K$ denoting the number of steps within the given epoch,
• a positive integer $N_m$ denoting the number of samples used to calculate the Monte Carlo update (16),

• a sequence of stepsizes $\{\alpha_k\}_{k \geq 1}$ with $\alpha_k \in (0, 1)$, and

• a set of fresh samples $\{\hat{T}_i\}_{i \in \mathcal{E}_m}$, with $|\mathcal{E}_m| = N_m + K$. The first $N_m$ samples are used to define the dataset $\mathcal{D}_m$ that underlies the Monte Carlo update (16), whereas the remaining $K$ samples are used in the $K$ steps within each epoch.

We summarize the operations within a single epoch in Algorithm 1.

Algorithm 1

\begin{algorithm}
\begin{algorithmic}[1]
\State Given (a) Epoch length $K$, (b) Recentering vector $\tilde{\theta}$, (c) Recentering sample size $N_m$, (d) Stepsize sequence $\{\alpha_k\}_{k \geq 1}$, (e) Samples $\{\hat{T}_i\}_{i \in \mathcal{E}_m}$
\State Compute the recentering quantity $\hat{T}_{N_m}(\tilde{\theta}) = \frac{1}{N_m} \sum_{i \in \mathcal{D}_m} \hat{T}_i(\tilde{\theta})$
\State Initialize $\theta_1 = \tilde{\theta}$
\For{$k = 1, 2, \ldots, K$}
\State Compute the variance-reduced update:
\State $\theta_{k+1} = V_k(\theta_k; \alpha_k, \tilde{\theta}, \hat{T}_{N_m})$
\EndFor
\end{algorithmic}
\end{algorithm}

The choice of the stepsize sequence $\{\alpha_k\}_{k \geq 1}$ is crucial, and it also determines the epoch length $K$. Roughly speaking, it is sufficient to choose a large enough epoch length to ensure that the error is reduced by a constant factor in each epoch. In Section 3.3.3 to follow, we study three popular stepsize choices—the constant stepsize (5a), the polynomial stepsize (5b) and the recentered linear stepsize (5c)—and provide lower bounds on the requisite epoch length in each case.

3.3.2 Overall algorithm

We are now ready to specify our variance-reduced policy-evaluation (VRPE) algorithm. The overall algorithm has five inputs: (a) an integer $M$, denoting the number of epochs to be run, (b) an integer $K$, denoting the length of each epoch, (c) a sequence of sample sizes $\{N_m\}_{m=1}^M$ denoting the number of samples used for recentering, (d) Sample batches $\{\{\hat{T}_i\}_{i \in \mathcal{E}_m}\}_{m=1}^M$ to be used in $m$ epochs, and (e) a sequence of stepsize $\{\alpha_k\}_{k \geq 1}$ to be used in each epoch. Given these five inputs, we summarize the overall procedure in Algorithm 2.

In the next section, we provide a detailed description on how to choose these input parameters for three popular stepsize choices (5a)–(5c). Finally, we reiterate that at epoch $m$, the algorithm uses $N_m + K$ new samples, and the samples used in the epochs are independent of each other. Accordingly, the total number of samples used in $M$ epochs is given by $KM + \sum_{m=1}^M N_m$. 

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Algorithm 2  
Variance-reduced policy evaluation (VRPE)

1: Given (a) Number of epochs $M$, (b) Epoch length $K$, (c) Recentering sample sizes $\{N_m\}_{m=1}^M$, (d) Sample batches $\{\mathcal{T}_i\}_{i \in \mathcal{E}_m}$, for $m = 1, \ldots, M$, (e) Stepsize $\{\alpha_k\}_{k=1}^K$

2: Initialize at $\theta_1$

3: for $m = 1, 2, \ldots, M$ do

4:    $\theta_{m+1} = \text{RunEpoch}(\theta_m; K, N_m, \{\alpha_k\}_{k=1}^K, \{\mathcal{T}_i\}_{i \in \mathcal{E}_m})$

5: end for

6: Return $\theta_{M+1}$ as the final estimate

3.3.3 Instance-dependent guarantees

Given a desired failure probability, $\delta \in (0, 1)$, and a total sample size $N$, we specify the following choices of parameters in Algorithm 2:

Number of epochs :  $M := \log_2 \left( \frac{N(1-\gamma)^2}{8 \log((8D/\delta) \cdot \log N)} \right)$ (18a)

Recentering sample sizes :  $N_m := 2^m \frac{4^2 \cdot 9^2 \cdot \log(8MD/\delta)}{(1-\gamma)^2}$ for $m = 1, \ldots, M$ (18b)

Sample batches: Partition the $N$ samples to obtain $\{\mathcal{T}_i\}_{i \in \mathcal{E}_m}$ for $m = 1, \ldots, M$ (18c)

Epoch length:  $K = \frac{N}{2M}$ (18d)

In the following theorem statement, we use $(c_1, c_2, c_3, c_4)$ to denote universal constants.

Theorem 2. (a) Suppose that the input parameters of Algorithm 2 are chosen according to Eq. (18). Furthermore, suppose that the sample size $N$ satisfies one of the following three stepsize-dependent lower bounds:

(a) $\frac{N}{M} \geq c_1 \frac{\log(8ND/\delta)}{(1-\gamma)^2}$ for recentered linear stepsize $\alpha_k = \frac{1}{1+(1-\gamma)k}$,

(b) $\frac{N}{M} \geq c_2 \log(8ND/\delta) \cdot \left(\frac{1}{1-\gamma}\right)^{\left(\frac{1}{1-\gamma} \sqrt{2}\right)}$ for polynomial stepsize $\alpha_k = \frac{1}{k^\omega}$ with $0 < \omega < 1$,

(c) $\frac{N}{M} \geq \frac{c_3}{\log(\frac{1}{1-\gamma})}$ for constant stepsize $\alpha_k = \alpha \leq \frac{1}{5} \cdot \frac{\sqrt{2}}{32} \cdot \frac{(1-\gamma)^2}{\log(8ND/\delta)}$.

Then for any initialization $\tilde{\theta}_1$, the output $\tilde{\theta}_{M+1}$ satisfies

$$\|\tilde{\theta}_{M+1} - \theta^*\|_{\infty} \leq c_4 \cdot \left\|\tilde{\theta}_1 - \theta^*\|_{\infty} + \frac{\log^2((8D/\delta) \cdot \log N)}{N^2(1-\gamma)^4} \right. \left. + c_4 \cdot \left\{ \sqrt{\frac{\log(8DM/\delta)}{N}} \cdot \left( \gamma \cdot \nu(P, \theta^*) + \rho(P, r) \right) + \frac{\log(8DM/\delta)}{N} \cdot b(\theta^*) \right\}, \right.$$ (19)

with probability exceeding $1 - \delta$. 

See Section 4.3 for the proof of this theorem.

A few comments on the upper bound provided in Theorem 2 are in order. In order to facilitate a transparent discussion in this section, we use the notation \( \gg \) in order to denote a relation that holds up to logarithmic factors in the tuple \((N, D, (1-\gamma)^{-1})\).

**Initialization dependence:** The first term on the right-hand side of the upper bound (19) depends on the initialization \( \tilde{\theta}_1 \). It should be noted that when viewed as a function of the sample size \( N \), this initialization-dependent term decays at a faster rate compared to the other two terms. This indicates that the performance of Algorithm 2 does not depend on the initialization \( \theta_1 \) in a significant way. A careful look at the proof (cf. Section 4.3) reveals that the coefficient of \( \|\tilde{\theta}_1 - \theta^*\|_\infty \) in the bound (19) can be made significantly smaller. In particular, for any \( p \geq 1 \) the first term in the right-hand side of bound (19) can be replaced by

\[
\frac{c_4 \cdot \|\tilde{\theta}_1 - \theta^*\|_\infty}{N^p} \cdot \frac{\log^p((8D/\delta) \cdot \log N)}{(1-\gamma)^{2p}},
\]

by increasing the recentering sample size (18b) by a constant factor and changing the values of the absolute constants \((c_1, c_2, c_3, c_4)\), with these values depending only on the value of \( p \). We have stated and proved a version for \( p = 2 \). Assuming the number of samples \( N \) satisfies \( N \geq (1-\gamma)^{-(2+\Delta)} \) for some \( \Delta > 0 \), the first term on the right-hand side of bound (19) can always be made smaller than the other two terms. In the sequel we show that each of the lower bound conditions (a)-(c) in the statement of Theorem 2 requires a lower bound condition \( N \gg (1-\gamma)^{-3} \).

**Comparing the upper and lower bounds:** The second and the third terms in (19) show the instance-dependent nature of the upper bound, and they are the dominating terms. Furthermore, assuming that the minimum sample size requirements from Theorems 1 and 2 are met, we find that the upper bound (19) matches the lower bound (12) up to logarithmic terms.

It is worthwhile to explicitly compute the minimum sample size requirements in Theorems 1 and 2. Ignoring the logarithmic terms and constant factors for the moment, unwrapping the lower bound conditions (a)-(c) in Theorem 2, we see that for both the constant stepsize and the recentered linear stepsize the sample size needs to satisfy \( N \gg (1-\gamma)^{-3} \). For the polynomial stepsize \( \lambda_k = \frac{1}{k^\omega} \), the sample size has to be at least \( (1-\gamma)^{-(1+\omega)} \). Minimizing the last bound for different values of \( \omega \in (0,1) \), we see that the minimum value is attained at \( \omega = 2/3 \), and in that case the bound (19) is valid when \( N \gg (1-\gamma)^{-3} \). Overall, for all the three stepsize choices discussed in Theorem 2 we require \( N \gg (1-\gamma)^{-3} \) in order to certify the upper bound. Returning to Theorem 1, from assumption (13) we see that in the best case scenario, Theorem 1 is valid as soon as \( N \gg (1-\gamma)^{-2} \). Putting together the pieces we find that the sample size requirement for Theorem 2 is more stringent than that of Theorem 1. Currently we do not know whether the minimum sample size requirements in Theorems 1 and 2 are necessary; answering this question is an interesting future research direction.

**Simulation study:** It is interesting to demonstrate the sharpness of our bounds via a simulation study, using the same scheme as our previous study of TD(0) with averaging. In Figure 3 we report the results of this study; see the figure caption for further details. At a high level, we see
that the VRPE algorithm, with either the recentered linear stepsize (panel (a)) or the polynomial stepsize $t^{-2/3}$, produces errors that decay with the exponents predicted by our instance-dependent theory for $\lambda \in \{0.5, 1.0, 2.0\}$. See the figure caption for further details.

4 Proofs

We now turn to the proofs of our main results.

4.1 Proof of Proposition 1

Recall the definition of the matrix $\Sigma_P(\theta)$ from Eq. (7), and define the covariance matrix

$$V_P = (I - \gamma P)^{-1}(\gamma^2 \Sigma_P(\theta) + \sigma_r^2 I)(I - \gamma P)^{-T}. \quad (20)$$

Recall that we use $Z$ to denote a multivariate Gaussian random vector $Z \sim \mathcal{N}(0, V_P)$, and that the sequence $\{\theta_k^\omega\}_{k \geq 1}$ is generated by averaging the iterates of stochastic approximation with polynomial stepsizes (5b) with exponent $\omega$. With this notation, the two claims of the theorem are:
\[ M_\infty(\mathcal{P}) = \mathbb{E}[\|Z\|_\infty], \quad \text{and} \]
\[ \lim_{N \to \infty} \mathbb{E} \left[ \sqrt{N} \cdot \|\hat{\theta}_N - \theta^*\|_\infty \right] = \mathbb{E}[\|Z\|_\infty]. \quad (21a) \]

We now prove each of these claims separately.

### 4.1.1 Proof of Eq. (21a)

For the reader's convenience, let us state a version of the Hájek–Le Cam local asymptotic minimax theorem:

**Theorem 3.** Let \( \{P_\varphi\}_{\varphi \in \Theta} \) be a family of parametric models, quadratically mean differentiable with Fisher information matrices \( J_\varphi \). Fix some parameter \( \varphi \in \Theta \), and consider a function \( \psi : \Theta \to \mathbb{R}^D \) that is differentiable at \( \varphi \). Then for any quasi-convex loss \( L : \mathbb{R}^D \to \mathbb{R} \), we have:

\[ \lim_{c \to \infty} \lim_{N \to \infty} \inf_{\hat{\varphi}_N} \sup_{\varphi'} \mathbb{E}_{\varphi'} \left[ L \left( \sqrt{N} \cdot (\hat{\varphi}_N - \varphi') \right) \right] = \mathbb{E}[L(Z)], \quad (22) \]

where the infimum is taken over all estimators \( \hat{\varphi}_N \) that are measurable functions of \( N \) i.i.d. data points drawn from \( P_\varphi \), and the expectation is taken over a multivariate Gaussian \( Z \sim \mathcal{N}(0, \nabla \psi(\varphi)^T J_\varphi \nabla \psi(\varphi)) \).

Returning to the problem at hand, let \( \varphi = (P, r) \) denote the unknown parameters of the model and let \( \psi(\varphi) = \theta(P) = (I - \gamma P)^{-1} r \) denote the target vector. A direct application of Theorem 3 shows that

\[ M_\infty(\mathcal{P}) = \mathbb{E}[\|Z\|_\infty], \quad \text{where} \ Z = \mathcal{N}(0, \nabla \psi(\varphi)^T J_\varphi \nabla \psi(\varphi)), \quad (23) \]

where \( J_\varphi \) is the Fisher information at \( \varphi \). The following result provides a more explicit form of the covariance of \( Z \):

**Lemma 1.** We have the identity

\[ \nabla \psi(\varphi)^T J_\varphi \nabla \psi(\varphi) = (I - \gamma P)^{-1} (\gamma^2 \Sigma P(\theta) + \sigma^2 I)(I - \gamma P)^{-T}. \quad (24) \]

Although the proof of this claim is relatively straightforward, it involves some lengthy and somewhat tedious calculations; we refer the reader to Appendix A.1 for the proof.

Given the result from Lemma 1, the claim (21a) follows by substituting the relation (24) into (23).

### 4.1.2 Proof of Eq. (21b)

The proof of this claim follows from the results of Polyak and Juditsky [33, Theorem 1], once their assumptions are verified for TD(0) with polynomial stepsizes. Recall that the TD iterates in Eq. (4) are given by the sequence \( \{\theta_k\}_{k \geq 1} \), and that \( \hat{\theta}_k \) denotes the \( k \)-th iterate generated by averaging.

For each \( k \geq 1 \), note the following equivalence between the notation of our paper and that of Polyak and Juditsky [33], or PJ for short:

\[ x_k \equiv \theta_k, \quad \gamma_k \equiv \alpha_k, \quad A \equiv I - \gamma P, \quad \text{and} \quad \xi_k = (R_k - r) + (Z_k - P)\theta_k. \]
Let us now verify the various assumptions in the PJ paper. Assumption 2.1 in the PJ paper holds by definition, since the matrix $\mathbf{I} - \gamma \mathbf{P}$ is Hurwitz. Assumption 2.2 in the PJ paper is also satisfied by the polynomial stepsize sequence for any exponent $\omega \in (0, 1)$.

It remains to verify the assumptions that must be satisfied by the noise sequence $\{\xi_k\}_{k \geq 1}$. In order to do so, write the $k$-th such iterate as

$$\xi_k = (R_k - r) + (Z_k - \mathbf{P})\theta^* + (Z_k - \mathbf{P})(\theta_k - \theta^*) .$$

Since $Z_k$ is independent of the sequence $\{\theta_i\}_{i=1}^k$, it follows that the condition

$$\lim_{N \to \infty} \mathbb{E} \left[ \|\theta_N - \theta^*\|^2 \right]$$

suffices to guarantee that Assumptions 2.3–2.5 in the PJ paper are satisfied. We now claim that for each $\omega \in (1/2, 1]$, condition (25) is satisfied by the TD iterates. Taking this claim as given for the moment, note that applying Theorem 1 of Polyak and Juditsky \cite{33} establishes claim (21b), for any exponent $\omega \in (1/2, 1)$.

It remains to establish condition (25). For any $\omega \in (1/2, 1]$, the sequence of stepsizes $\{\alpha_k\}_{k \geq 1}$ satisfies the conditions

$$\sum_{k=1}^{\infty} \alpha_k = \infty \quad \text{and} \quad \sum_{k=1}^{\infty} \alpha_k^2 < \infty .$$

Consequently, classical results due to Robbins and Munro \cite{34} Theorem 2] guarantee $\ell^2$-convergence of $\theta_N$ to $\theta^*$.

### 4.2 Proof of Theorem 1

Throughout the proof, we use the notation $\mathcal{P} = (\mathbf{P}, r)$ and $\mathcal{P}' = (\mathbf{P}', r')$ to denote, respectively, the problem instance at hand and its alternative. Moreover, we use $\theta^* \equiv \theta(\mathcal{P})$ and $\theta(\mathcal{P}')$ to denote the associated target parameters for each of the two problems $\mathcal{P}$ and $\mathcal{P}'$. We use $\Delta_{\mathcal{P}} = \mathbf{P} - \mathbf{P}'$ and $\Delta_r = r - r'$ to denote the differences of the parameters. For probability distributions, we use $\mathcal{P}$ and $\mathcal{P}'$ to denote the marginal distribution of a single observation under $\mathcal{P}$ and $\mathcal{P}'$, and use $\mathcal{P}^N$ and $(\mathcal{P}')^N$ to denote the distribution of $N$ i.i.d observations drawn from $\mathcal{P}$ or $\mathcal{P}'$, respectively.

#### 4.2.1 Proof structure

We introduce two special classes of alternatives of interest, denoted as $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively:

$$\mathcal{S}_1 = \{ \mathcal{P}' = (\mathbf{P}', r') \mid r' = r \} , \quad \text{and} \quad \mathcal{S}_2 = \{ \mathcal{P}' = (\mathbf{P}', r') \mid \mathbf{P}' = \mathbf{P} \} .$$

In words, the class $\mathcal{S}_1$ consists of alternatives $\mathcal{P}'$ that have the same reward vector $r$ as $\mathcal{P}$, but a different transition matrix $\mathbf{P}'$. Similarly, the class $\mathcal{S}_2$ consists of alternatives $\mathcal{P}'$ with the same transition matrix $\mathbf{P}$, but a different reward vector. By restricting the alternative $\mathcal{P}'$ within class $\mathcal{S}_1$ and $\mathcal{S}_2$, we can define restricted versions of the local minimax risk, namely

$$\mathcal{M}_N(\mathcal{P}; \mathcal{S}_1) \equiv \sup_{\mathcal{P}' \in \mathcal{S}_1} \inf_{\theta_N} \max_{\mathcal{P} \in \{\mathcal{P}, \mathcal{P}'\}} \mathbb{E}_{\mathcal{P}} \left[ \sqrt{N} \cdot \|\hat{\theta}_N - \theta(\mathcal{P})\|_{\infty} \right] , \quad \text{and} \quad (26a)$$

$$\mathcal{M}_N(\mathcal{P}; \mathcal{S}_2) \equiv \sup_{\mathcal{P}' \in \mathcal{S}_2} \inf_{\theta_N} \max_{\mathcal{P} \in \{\mathcal{P}, \mathcal{P}'\}} \mathbb{E}_{\mathcal{P}} \left[ \sqrt{N} \cdot \|\hat{\theta}_N - \theta(\mathcal{P})\|_{\infty} \right] . \quad (26b)$$
The main part of the proof involves showing that there is a universal constant \( c > 0 \) such that the lower bounds

\[
\mathcal{M}_N(\mathcal{P}; S_1) \geq c \cdot \gamma(\mathcal{P}, \theta^*) ,
\]

and

\[
\mathcal{M}_N(\mathcal{P}; S_2) \geq c \cdot \rho(\mathcal{P}, r)
\]

both hold (assuming that the sample size \( N \) is sufficiently large to satisfy the condition \([13]\)). Since we have \( \mathcal{M}_N(\mathcal{P}) \geq \max \{ \mathcal{M}_N(\mathcal{P}; S_1), \mathcal{M}_N(\mathcal{P}; S_2) \} \), these lower bounds in conjunction imply the claim Theorem 1. The next section shows how to prove these two bounds.

4.2.2 Proof of the lower bounds (27a) and (27b):

Our first step is to lower bound the local minimax risk for each problem class in terms of a modulus of continuity between the Hellinger distance and the \( \ell_\infty \)-norm.

Lemma 2. For each \( S \in \{ S_1, S_2 \} \), we have the lower bound

\[
\mathcal{M}_N(\mathcal{P}; S) \geq \frac{1}{8} \cdot \mathcal{M}_N(\mathcal{P}; S),
\]

where we define

\[
\mathcal{M}_N(\mathcal{P}; S) := \sup_{\mathcal{P}' \in S} \left\{ \sqrt{N} \cdot \| \theta(\mathcal{P}) - \theta(\mathcal{P}') \|_{\ell_\infty} \mid d_{hel}(\mathcal{P}, \mathcal{P}') \leq \frac{1}{2}\sqrt{N} \right\}.
\]

The proof of Lemma 2 follows a relatively standard argument, one which reduces estimation to testing; see Appendix B.1 for details.

This lemma allows us to focus our remaining attention on lower bounding the quantity \( \mathcal{M}_N(\mathcal{P}; S) \). In order to do so, we need both a lower bound on the \( \ell_\infty \)-norm \( \| \theta(\mathcal{P}) - \theta(\mathcal{P}') \|_{\ell_\infty} \) and an upper bound on the Hellinger distance \( d_{hel}(\mathcal{P}, \mathcal{P}') \). These two types of bounds are provided in the following two lemmas. We begin with lower bounds on the \( \ell_\infty \)-norm:

Lemma 3. (a) For any \( \mathcal{P} \) and for all \( \mathcal{P}' \in S_1 \), we have

\[
\| \theta(\mathcal{P}) - \theta(\mathcal{P}') \|_{\ell_\infty} \geq \left( 1 - \gamma \frac{1}{1 - \gamma} \| \Delta \mathcal{P} \|_{\ell_\infty} \right) \cdot \| (\mathbf{I} - \gamma \mathcal{P})^{-1} \Delta \mathcal{P} \theta^* \|_{\ell_\infty}.
\]

(b) For any \( \mathcal{P} \) and for all \( \mathcal{P}' \in S_2 \), we have

\[
\| \theta(\mathcal{P}) - \theta(\mathcal{P}') \|_{\ell_\infty} \geq \| (\mathbf{I} - \gamma \mathcal{P})^{-1} \Delta \|_{\ell_\infty}.
\]

See Appendix B.2 for the proof of this claim.

Next, we require upper bounds on the Hellinger distance:

Lemma 4. (a) For each \( \mathcal{P} \) and for all \( \mathcal{P}' \in S_1 \), we have

\[
d_{hel}(\mathcal{P}, \mathcal{P}')^2 \leq \frac{1}{2} \sum_{i,j} \frac{((\Delta \mathcal{P})_{i,j})^2}{\mathcal{P}_{i,j}}.
\]

(b) For each \( \mathcal{P} \) and for all \( \mathcal{P}' \in S_2 \), we have

\[
d_{hel}(\mathcal{P}, \mathcal{P}')^2 \leq \frac{1}{2\sigma^2} \| r_1 - r_2 \|_2^2.
\]
Using Lemmas 3 and 4, we can derive two different lower bounds. First, we have the lower bound \( \mathcal{M}_N(P; S_1) \geq \mathcal{M}_N(P; S_1) \), where

\[
\mathcal{M}_N(P; S_1) \equiv \sup_{P' \in S_1} \left\{ \sqrt{N} \cdot \left( 1 - \frac{\gamma \| \Delta P \|_\infty}{1 - \gamma} \right) \cdot \left\| \gamma(I - \gamma P)^{-1} \Delta P \theta^* \right\|_\infty \right\}
\]

\[
\text{where the last inequality follows by the assumed lower bound } N \geq \frac{4\gamma^2}{(1-\gamma)^2}. \quad \text{(31a)}
\]

Second, we have the lower bound \( \mathcal{M}_N(P; S_2) \geq \mathcal{M}_N(P; S_2) \), where

\[
\mathcal{M}_N(P; S_2) \equiv \sup_{P' \in S_2} \left\{ \sqrt{N} \cdot \| (I - \gamma P)^{-1} \Delta r \|_\infty \right\}
\]

In order to complete the proofs of the two lower bounds (27a) and (27b), it suffices to show that

\[
\mathcal{M}_N(P; S_1) \geq \frac{1}{\sqrt{2}} \cdot \rho(P, r), \quad \text{and} \quad \mathcal{M}_N(P; S_1) \geq \frac{1}{2\sqrt{2}} \cdot \gamma \nu(P, \theta^*). \quad \text{(32a)}
\]

**Proof of the bound (32a):** This lower bound is easy to show—it follows from the definition:

\[
\mathcal{M}_N(P; S_2) = \frac{\sigma_r}{\sqrt{2}} \| (I - \gamma P)^{-1} \Delta r \|_\infty = \frac{1}{\sqrt{2}} \rho(P, r).
\]

**Proof of the bound (32b):** The proof of this claim is much more delicate. Our strategy is to construct a special “hard” alternatif, \( \bar{P} \in S_1 \), that leads to a good lower bound on \( \mathcal{M}_N(P; S_1) \). Lemma 5 below is the main technical result that we require:

**Lemma 5.** There exists some probability transition matrix \( \bar{P} \) with the following properties:

(a) It satisfies the constraint \( \sum_{i,j} \left( \frac{P_{i,j} - \bar{P}_{i,j}}{\bar{P}_{i,j}} \right)^2 \leq \frac{1}{2N} \).

(b) It satisfies the inequalities

\[
\| \bar{P} - P \|_\infty \leq \frac{1}{\sqrt{2N}}, \quad \text{and} \quad \left\| \gamma(I - \gamma P)^{-1}(\bar{P} - P) \theta^* \right\|_\infty \geq \frac{\gamma}{\sqrt{2N}} \cdot \nu(P, \theta^*).
\]

See Appendix [B.3] for the proof of this claim.

Given the matrix \( \bar{P} \) guaranteed by this lemma, we consider the “hard” problem \( \bar{P} := (\bar{P}, r) \in S_1 \). From the definition of \( \mathcal{M}_N(P; S_1) \) in Eq. (31a), we have that

\[
\mathcal{M}_N(P; S_1) \geq \sqrt{N} \cdot \left( 1 - \frac{\gamma}{1 - \gamma} \| \bar{P} - P \|_\infty \right) \cdot \left\| \gamma(I - \gamma P)^{-1}(\bar{P} - P) \theta^* \right\|_\infty
\]

\[
\geq \sqrt{N} \cdot \left( 1 - \frac{\gamma}{1 - \gamma} \cdot \frac{1}{\sqrt{2N}} \right) \cdot \frac{\gamma}{\sqrt{2N}} \cdot \nu(P, \theta^*) \geq \frac{1}{2\sqrt{2}} \cdot \nu(P, \theta^*),
\]

where the last inequality follows by the assumed lower bound \( N \geq \frac{4\gamma^2}{(1-\gamma)^2} \). This completes the proof of the lower bound (32b).
4.3 Proof of Theorem 2

This section is devoted to the proof of Theorem 2, which provides the achievability results for variance-reduced policy evaluation.

4.3.1 Proof of part (a):

We begin with a lemma that characterizes the progress of Algorithm 2 over epochs:

**Lemma 1.** Under the assumptions of Theorem 2 (a), there is an absolute constant $c$ such that for each epoch $m = 1, \ldots, M$, we have:

$$
\left\| \hat{\theta}_{m+1} - \theta^* \right\|_\infty \leq \frac{\left\| \hat{\theta}_m - \theta^* \right\|_\infty}{4} + c \left\{ \frac{\log(8DM/\delta)}{N_m} \left( \gamma \cdot \nu(P, \theta^*) + \rho(P, r) \right) + \frac{\log(8DM/\delta)}{N_m} \cdot b(\theta^*) \right\},
$$

(33)

with probability exceeding $1 - \frac{\delta}{M}$.

Taking this lemma as given for the moment, let us complete the proof. We use the shorthand

$$
\tau_m := \sqrt{\frac{\log(8DM/\delta)}{N_m}} \left( \gamma \cdot \nu(P, \theta^*) \rho(P, r) \right) \quad \text{and} \quad \eta_m := \frac{\log(8DM/\delta)}{N_m} \cdot b(\theta^*)
$$

(34)

to ease notation, and note that $\frac{\tau_m}{\sqrt{2}} \leq \tau_{m+1}$ and $\frac{\eta_m}{2} \leq \eta_{m+1}$, for each $m \geq 1$. Using this notation and unwrapping the recursion relation from Lemma 1, we have

$$
\left\| \hat{\theta}_{M+1} - \theta^* \right\|_\infty \leq \frac{\left\| \hat{\theta}_M - \theta^* \right\|_\infty}{4} + c(\tau_M + \eta_M)
$$

\[ (i) \]

$$
\leq \frac{\left\| \hat{\theta}_{M-1} - \theta^* \right\|_\infty}{4^2} + \frac{c}{2} (\tau_M + \eta_M) + c(\tau_M + \eta_M)
$$

\[ (ii) \]

$$
\leq \frac{\left\| \hat{\theta}_{1} - \theta^* \right\|_\infty}{4^M} + 2c(\tau_M + \eta_M).
$$

Here, step (i) follows by applying the one-step application of the recursion (33), and by using the upper bounds $\frac{\tau_m}{\sqrt{2}} \leq \tau_{m+1}$ and $\frac{\eta_m}{2} \leq \eta_{m+1}$. Step (ii) follows by repeated application of the recursion (33). The last inequality holds with probability at least $1 - \frac{\delta}{M}$ by a union bound over the $M$ epochs.

It remains to express the quantities $4^M$, $\tau_M$ and $\eta_M$—all of which are controlled by the centering sample size $N_M$—in terms of the total number of available samples $N$. Towards this end, observe that the total number of samples used for centering at $M$ epochs is given by

$$
\sum_{m=1}^{M} N_m \approx 2^M \cdot \frac{\log(8MD/\delta)}{(1 - \gamma)^2}.
$$
Substituting the value of $M = \log_2 \left( \frac{N(1-\gamma)^2}{8\log((8D/\delta) \log N)} \right)$ we have

$$c_1 N \leq N_M \approx \sum_{m=1}^{M} N_m \leq \frac{N}{2},$$

where $c_1$ is a universal constant. Consequently, the total number of samples used by Algorithm 2 is given by

$$MK + \sum_{m=1}^{M} N_m \leq \frac{N}{2} + \frac{N}{2} = N,$$

where in the last equation we have used the fact that $MK = \frac{N}{2}$. Finally, using $M = \log_2 \left( \frac{N(1-\gamma)^2}{8\log((8D/\delta) \log N)} \right)$ we have the following relation for some universal constant $c$:

$$4^M = c \cdot \frac{N^2(1-\gamma)^4}{\log^2((8D/\delta) \cdot \log N)}$$

Putting together the pieces, we conclude that

$$\|\tilde{\theta}_{M+1} - \theta^*\|_\infty \leq c_2 \|\tilde{\theta}_1 - \theta^*\|_\infty \cdot \frac{\log^2((8D/\delta) \cdot \log N)}{N^2(1-\gamma)^4}
+ c_2 \left\{ \sqrt{\frac{\log(8DM/\delta)}{N}} \left( \gamma \cdot \nu(P, \theta^*) + \rho(P, r) \right) + \frac{\log(8DM/\delta)}{N} \cdot b(\theta^*) \right\},$$

for a suitable universal constant $c_2$. The last bound is valid with probability exceeding $1 - \delta$ via the union bound. In order to complete the proof, it remains to prove Lemma 1, which we do in the following subsection.

### 4.3.2 Proof of Lemma 1

We now turn to the proof of the key lemma within the argument. We begin with a high-level overview in order to provide intuition. In the $m$-th epoch that updates the estimate from $\tilde{\theta}_m$ to $\tilde{\theta}_{m+1}$, the vector $\tilde{\theta} \equiv \tilde{\theta}_m$ is used to recenter the updates. Our analysis of the $m$-th epoch is based on a sequence of recentered operators $\{J^m_k\}_{k \geq 1}$ and their population analogs $J^m(\theta)$. The action of these operators on a point $\theta$ is given by the relations

$$J^m_k(\theta) := \tilde{T}_k(\theta) - \tilde{T}_k(\tilde{\theta}_m) + \tilde{T}_N(\tilde{\theta}_m), \quad \text{and} \quad J^m(\theta) := T(\theta) - T(\tilde{\theta}_m) + \tilde{T}_N(\tilde{\theta}_m). \quad (35a)$$

By definition, the updates within epoch $m$ can be written as

$$\theta_{k+1} = (1 - \alpha_k) \theta_k + \alpha_k J^m_k(\theta_k). \quad (35b)$$

Note that the operator $J^m$ is $\gamma$-contractive in $\| \cdot \|_\infty$-norm, and as a result it has a unique fixed point, which we denote by $\tilde{\theta}_m$. Since $J^m(\theta) = \mathbb{E} [J^m_k(\theta)]$ by construction, when studying epoch $m$, it is natural to analyze the convergence of the sequence $\{\theta_k\}_{k \geq 1}$ to $\tilde{\theta}_m$. 

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Suppose that we have taken \( K \) steps within epoch \( m \). Applying the triangle inequality yields the bound
\[
\|\theta_{m+1} - \theta^*\|_\infty = \|\theta_{K+1} - \theta^*\|_\infty \leq \|\theta_{K+1} - \hat{\theta}_m\|_\infty + \|\hat{\theta}_m - \theta^*\|_\infty.
\]
(35c)
With this decomposition, our proof of Lemma 1 is based on two auxiliary lemmas that provide high-probability upper bounds on the two terms on the right-hand side of inequality (35c). 

**Lemma 2.** Let \((c_1, c_2, c_3)\) be positive numerical constants, and suppose that the epoch length \( K \) satisfies one of the following three stepsize-dependent lower bounds:
\[(a)\] \( K \geq c_1 \frac{\log(8KMD/\delta)}{(1-\gamma)^2} \) for recentered linear stepsize \( \alpha_k = \frac{1}{1+(1-\gamma)k} \),
\[(b)\] \( K \geq c_2 \log(8KMD/\delta) \cdot \left( \frac{1}{1-\omega} \right)^{\frac{1}{2}} \) for polynomial stepsize \( \alpha_k = \frac{1}{k^\omega} \) with \( 0 < \omega < 1 \),
\[(c)\] \( K \geq \frac{c_3}{\log \left( \frac{1}{1-\alpha(1-\gamma)} \right)} \) for constant stepsize \( \alpha_k = \alpha \leq \frac{(1-\gamma)^2}{\log(8KMD/\delta)} \cdot \frac{1}{2^8\cdot 32^2} \).

Then after \( K \) update steps with epoch \( m \), the iterate \( \theta_{K+1} \) satisfies the bound
\[
\|\theta_{K+1} - \hat{\theta}_m\|_\infty \leq \frac{1}{8}\|\hat{\theta}_m - \theta^*\|_\infty + \frac{1}{8}\|\hat{\theta}_m - \theta^*\|_\infty \quad \text{with probability at least} \quad 1 - \frac{\delta}{2^{12M}}.
\]
(36)
See Appendix C.1 for the proof of this claim.

Our next auxiliary result provides a high-probability bound on the difference \( \|\hat{\theta}_m - \theta^*\|_\infty \).

**Lemma 3.** There is an absolute constant \( c_4 \) such that for any recentering sample size satisfying \( N_m \geq 4^2 \cdot 9^2 \cdot \frac{\log(MD/\delta)}{(1-\gamma)^2} \), we have
\[
\|\hat{\theta}_m - \theta^*\|_\infty \leq \frac{1}{8}\|\hat{\theta}_m - \theta^*\|_\infty + c_4 \left\{ \sqrt{\frac{\log(8DM/\delta)}{N_m}} \left( \gamma \cdot \nu(P, \theta^*) + \rho(P, r) \right) + \frac{\log(8DM/\delta)}{N_m} \cdot b(\theta^*) \right\},
\]
with probability exceeding \( 1 - \frac{\delta}{2^{12M}} \).

See Appendix C.2 for the proof of this claim.

With Lemmas 2 and 3 in hand, the remainder of the proof is straightforward. Recall from Eq. (34) the shorthand notation \( \tau_m \) and \( \eta_m \). Using our earlier bound (35c), we have that at the end of epoch \( m \) (which is also the starting point of epoch \( m + 1 \)),
\[
\|\theta_{m+1} - \theta^*\|_\infty \leq \|\theta_{K+1} - \hat{\theta}_m\|_\infty + \|\hat{\theta}_m - \theta^*\|_\infty
\]
\[
\leq \left\{ \frac{\|\theta_m - \theta^*\|_\infty}{8} + \frac{9}{8}\|\hat{\theta}_m - \theta^*\|_\infty \right\} + \|\hat{\theta}_m - \theta^*\|_\infty
\]
\[
= \frac{\|\theta_m - \theta^*\|_\infty}{8} + \frac{9}{8}\|\hat{\theta}_m - \theta^*\|_\infty
\]
\[
\leq \frac{\|\theta_m - \theta^*\|_\infty}{8} + \frac{1}{8}\left\{ \|\hat{\theta}_m - \theta^*\|_\infty + c_4(\tau_m + \eta_m) \right\}
\]
\[
\leq \frac{\|\hat{\theta}_m - \theta^*\|_\infty}{4} + c_4(\tau_m + \eta_m),
\]
where inequality (i) follows from Lemma 2(a), and inequality (ii) from Lemma 3. Finally, the sequence of inequalities above holds with probability at least $1 - \frac{\delta}{M}$ via a union bound. This completes the proof of Lemma 1.

### 4.3.3 Proof of Theorem 2 parts (b) and (c)

The proofs of Theorem 2 parts (b) and (c) require versions of Lemma 1 for the polynomial stepsize (5b) and constant stepsize (5a), respectively. These two versions of Lemma 1 can be obtained by simply replacing Lemma 2, part (a), by Lemma 2, parts (b) and (c), respectively, in the proof of Lemma 1.

### 5 Discussion

We have discussed the problem of policy evaluation in discounted Markov decision processes. Our contribution is three-fold. First, we provided a non-asymptotic instance-dependent local-minimax bound on the $\ell_\infty$-error for the policy evaluation problem under the generative model. Next, via careful simulations, we showed that the standard TD-learning algorithm—even when combined with Polyak-Rupert iterate averaging—does not yield ideal non-asymptotic behavior as captured by our lower bound. In order to remedy this difficulty, we introduced and analyzed a variance-reduced version of the standard TD-learning algorithm which achieves our non-asymptotic instance-dependent lower bound up to logarithmic factors. Both the upper and lower bounds discussed in this paper hold when the sample size is bigger than an explicit threshold; relaxing this minimum sample size requirement is an interesting future research direction. Finally, we point out that although we have focused on the tabular policy evaluation problem, the variance-reduced algorithm discussed in this paper can be applied in more generality, and it would be interesting to explore applications of this algorithm to non-tabular settings.

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A Proofs of auxiliary lemmas for Proposition

In this section, we provide proofs of the auxiliary lemmas that underlie the proof of Proposition.

A.1 Proof of Lemma

The proof is basically a lengthy computation. For clarity, let us decompose the procedure into three steps. In the first step, we compute an explicit form for the inverse information matrix $J^\dagger_\vartheta$. In the second step, we evaluate the gradient $\nabla\psi(\vartheta)$. In the third and final step, we use the result in the previous two steps to prove the claim of the lemma.
Step 1: In the first step, we evaluate $J_\vartheta^\dagger$. Recall that our data $(Z, R)$ is generated as follows. We generate the matrix $Z$ and the vector $R$ independently. Each row of $Z$ is generated independently. Its $j$th row, denoted by $z_j$, is sampled from a multinomial distribution with parameter $p_j$, where $p_j$ denotes the $j$th row of $P$. The vector $R$ is sampled from $\mathcal{N}(r, \sigma^2_r I)$. Because of this independence structure, the Fisher information $J_\vartheta$ is a block diagonal matrix of the form

$$
J_\vartheta = \begin{bmatrix}
J_{p_1} & 0 & 0 \cdots & 0 & 0 \\
0 & J_{p_2} & 0 \cdots & 0 & 0 \\
0 & 0 & \ddots & \cdots & 0 \\
0 & 0 & 0 \cdots & J_{p_D} & 0 \\
0 & 0 & \cdots & 0 & J_r
\end{bmatrix}.
$$

Here each sub-block matrix $J_{p_j}$ is the Fisher information corresponding to the model where a single data $Z_j$ is sampled from the multinomial distribution with parameter $p_j$, and $J_r$ is the Fisher information corresponding to the model in which a single data point $R$ is sampled from $\mathcal{N}(r, \sigma^2_r I)$. Thus, the inverse Fisher information $J_\vartheta^\dagger$ is also a block diagonal matrix of the form

$$
J_\vartheta^\dagger = \begin{bmatrix}
J_{p_1}^\dagger & 0 & 0 \cdots & 0 & 0 \\
0 & J_{p_2}^\dagger & 0 \cdots & 0 & 0 \\
0 & 0 & \ddots & \cdots & 0 \\
0 & 0 & 0 \cdots & J_{p_D}^\dagger & 0 \\
0 & 0 & \cdots & 0 & J_r^\dagger
\end{bmatrix}. \quad (37)
$$

It is easy to compute $J_{p_j}^\dagger$ and $J_r^\dagger$:

$$
J_{p_j}^\dagger = \text{diag}(p_j) - p_j p_j^T = \text{cov}(Z_j - p_j) \quad \text{for } j \in [D], \quad \text{and} \quad (38a)
$$

$$
J_r^\dagger = J_r^{-1} = \sigma^2_r I. \quad (38b)
$$

For a vector $q \in \mathbb{R}^D$, we use $\text{diag}(q) \in \mathbb{R}^{D \times D}$ to denote the diagonal matrix with diagonal entries $q_j$.

Step 2: In the second step, we evaluate $\nabla \psi(\vartheta)$. Recall that $\psi(\vartheta) = (I - \gamma P)^{-1} r$. It is straightforward to see that

$$
\nabla_r \psi(\vartheta) = (I - \gamma P)^{-1}.
$$

Below we evaluate $\nabla_{p_j} \psi(\vartheta)$ for $j \in [D]$, where $p_j$ is the $j$th row of the matrix $P$. We show that

$$
\nabla_{p_j} \psi(\vartheta) = \gamma (I - \gamma P)^{-1} e_j \theta^T. \quad (40)
$$

Here we recall $\theta = \psi(\vartheta) = (I - \gamma P)^{-1} r$.

To prove Eq. (40), we start with the following elementary fact: for the matrix inverse mapping $A \rightarrow A^{-1}$, we have $\frac{\partial A^{-1}}{\partial A_{jk}} = -A^{-1} e_j e_k^T A^{-1}$ for all $j, k \in [D]$. Combining this fact with chain rule, we find that

$$
\frac{\partial \psi(\vartheta)}{\partial P_{jk}} = \gamma (I - \gamma P)^{-1} e_j e_k^T (I - \gamma P)^{-1} r = \gamma (I - \gamma P)^{-1} e_j \theta^T e_k,
$$

valid for all $j, k \in [D]$. This immediately implies Eq. (40) since $p_j$ is the vector with coordinates $P_{jk}$. 

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In the third step, we evaluate $\nabla \psi(\theta)^T J_\theta^T \nabla \psi(\theta)$. From Eq. (37), we observe that the inverse Fisher information $J_\theta^T$ has a block structure. Consequently, we can write

$$\nabla \psi(\theta)^T J_\theta^T \nabla \psi(\theta) = \sum_{j \in [D]} \nabla_{p_j} \psi(\theta)^T J^T_{p_j} \nabla_{p_j} \psi(\theta) + \nabla_R \psi(\theta)^T J^T_R \nabla_R \psi(\theta). \tag{41}$$

Combining Eqs. (38b) and (39) yields

$$\nabla_R \psi(\theta)^T J^T_R \nabla_R \psi(\theta) = \sigma^2 (I - \gamma P)^{-1} (I - \gamma P)^{-T}. \tag{42}$$

Combining Eqs. (38a) and (40) yields

$$\nabla_{p_j} \psi(\theta)^T J^T_{p_j} \nabla_{p_j} \psi(\theta) = \gamma^2 (I - \gamma P)^{-1} e_j \theta^T \text{cov}(Z_j - p_j) \theta e_j^T (I - \gamma P)^{-T} \tag{43}$$

valid for each $j \in [D]$. Summing over $j \in [D]$ then leads to

$$\sum_{j \in [D]} \nabla_{p_j} \psi(\theta)^T J^T_{p_j} \nabla_{p_j} \psi(\theta) = \gamma^2 (I - \gamma P)^{-1} \left( \sum_{j \in [D]} e_j \text{cov}((Z_j - p_j)^T \theta) e_j^T \right) (I - \gamma P)^{-T} \tag{44}$$

where the last line uses the definition of $\Sigma_P(\theta)$ in Eq. (7). Finally, substituting Eq. (42) and Eq. (43) into Eq. (41) yields the claim (24), which completes the proof of Lemma 1.

### B Proofs of auxiliary lemmas for Theorem 1

In this appendix, we detailed proofs of the auxiliary lemmas that underlie the proof of the non-asymptotic local minimax lower bound stated in Theorem 1.

#### B.1 Proof of Lemma 2

The proof uses the standard device of reducing estimation to testing (see, e.g., [9, 12, 46]). The first step is to lower bound the minimax risk over $\mathcal{P}$ and $\mathcal{P}'$ by its averaged risk:

$$\inf_{\theta_N} \max_{P \in \{\mathcal{P}, \mathcal{P}'\}} \mathbb{E}_P \left[ \|\hat{\theta}_N - \theta\|_\infty \right] \geq \frac{1}{2} \left( \mathbb{E}_{P_N} \left[ \|\hat{\theta}_N - \theta\|_\infty \right] + \mathbb{E}_{P_N} \left[ \|\hat{\theta}_N - \theta'\|_\infty \right] \right). \tag{44}$$

By Markov’s inequality, for any $\delta \geq 0$, we have

$$\mathbb{E}_{P_N} \left[ \|\hat{\theta}_N - \theta\|_\infty \right] + \mathbb{E}_{P_N} \left[ \|\hat{\theta}_N - \theta'\|_\infty \right] \geq \delta \left[ P_N \left( \|\hat{\theta}_N - \theta\|_\infty \geq \delta \right) + P_N \left( \|\hat{\theta}_N - \theta'\|_\infty \geq \delta \right) \right].$$

If we define $\delta_0 := \frac{1}{2} \|\theta - \theta'\|_\infty$, then we have the implication

$$\|\theta - \theta\|_\infty < \delta_0 \implies \|\theta - \theta'\|_\infty > \delta_0,$$  \tag{45}$$

from which it follows that

$$\mathbb{E}_{P_n} \left[ \|\hat{\theta}_n - \theta\|_\infty \right] + \mathbb{E}_{P_n} \left[ \|\hat{\theta}_n - \theta'\|_\infty \right] \geq \delta_0 \left[ 1 - P_n (\|\hat{\theta}_n - \theta\|_\infty < \delta_0) + P_n (\|\hat{\theta}_n - \theta'\|_\infty \geq \delta_0) \right] \tag{46}$$

$$\geq \delta_0 \left[ 1 - P_n (\|\hat{\theta}_n - \theta'\|_\infty \geq \delta_0) + P_n (\|\hat{\theta}_n - \theta'\|_\infty \geq \delta_0) \right] \tag{47}$$

$$\geq \delta_0 \left[ 1 - \|P_n - P_n\|_{TV} \right] \geq \delta_0 \left[ 1 - \sqrt{2 \delta_0} \left( \|P_n - P_n\|_2^2 \right) \right].$$

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The tensorization property of Hellinger distance (cf. Section 15.1 in [46]) guarantees that
\[ d_{\text{hel}}(P^N, P'^N)^2 \leq N \left( 1 - d_{\text{hel}}(P, P')^2 \right)^N. \]
Thus, we have proved that
\[ \inf_{\theta_N} \max_{\mathcal{Q} \in \mathcal{S}(P, P')} \mathbb{E}_\mathcal{Q} \left[ \| \theta - \theta(\mathcal{Q}) \|_\infty \right] \geq \frac{1}{4} \| \theta(P) - \theta(P') \|_\infty \cdot \left( 1 - \sqrt{2} N \cdot d_{\text{hel}}(P, P')^2 \right)_+. \]
Taking the supremum over all the possible alternatives \( P' \in \mathcal{S} \) yields
\[ \mathcal{M}_N(P; \mathcal{S}) \geq \sup_{P' \in \mathcal{S}} \frac{1}{4} \cdot \sqrt{N} \| \theta(P) - \theta(P') \|_\infty \cdot \left( 1 - \sqrt{2} N \cdot d_{\text{hel}}(P, P')^2 \right)_+. \] (46)
A calculation shows that this bound implies the claim in Lemma 2.

B.2 Proof of Lemma 3
Recall the shorthand \( \Delta_P = P - P' \) and \( \Delta_r = r - r' \), and let \( \theta^* \equiv \theta(\mathcal{P}) \). We prove that \( \| \theta(\mathcal{P}) - \theta(\mathcal{P}') \|_\infty \) is lower bounded by
\[ \left\| \gamma (\mathbf{I} - \gamma P)^{-1} \Delta_P \theta^* + (\mathbf{I} - \gamma P)^{-1} \Delta_r \right\|_\infty \geq \left( \frac{\gamma \| \Delta_P \|_\infty}{(1 - \gamma)} \right) \left\| \gamma (\mathbf{I} - \gamma P)^{-1} \Delta_P \theta^* \right\|_\infty + \frac{\gamma \| \Delta_P \|_\infty \| \Delta_r \|_\infty}{(1 - \gamma)^2}. \] (47)
Since \( \theta(\mathcal{P}) = (\mathbf{I} - \gamma P)^{-1} r \) and \( \theta(\mathcal{P}') = (\mathbf{I} - \gamma P')^{-1} r' \) by definition, if we introduce the shorthand \( M_P = (\mathbf{I} - \gamma P)^{-1} - (\mathbf{I} - \gamma P')^{-1} \), some elementary calculation gives the identity
\[ \theta(\mathcal{P}) - \theta(\mathcal{P}') = M_P r + (\mathbf{I} - \gamma P)^{-1} \Delta_r + M_P \Delta_r. \] (48)
Now we find a new expression for \( M_P = (\mathbf{I} - \gamma P)^{-1} - (\mathbf{I} - \gamma P')^{-1} \) that is easy to control. Recall the elementary identity \( A_{1}^{-1} - A_{0}^{-1} + A_{1}^{-1}(A_{0} - A_{1})A_{0}^{-1} \) for any matrices \( A_{0}, A_{1} \). Thus,
\[ M_P = (\mathbf{I} - \gamma P)^{-1} - (\mathbf{I} - \gamma P')^{-1} \]
\[ = \gamma (\mathbf{I} - \gamma P')^{-1} (\mathbf{P} - \mathbf{P}') (\mathbf{I} - \gamma P)^{-1} \]
\[ = \gamma (\mathbf{I} - \gamma P)^{-1} (\mathbf{P} - \mathbf{P}') (\mathbf{I} - \gamma P)^{-1} + \gamma^2 (\mathbf{I} - \gamma P')^{-1} (\mathbf{P} - \mathbf{P}') (\mathbf{I} - \gamma P)^{-1} \]
\[ = \gamma (\mathbf{I} - \gamma P)^{-1} \Delta_P (\mathbf{I} - \gamma P)^{-1} + \gamma^2 (\mathbf{I} - \gamma P')^{-1} \Delta_P (\mathbf{I} - \gamma P)^{-1} \Delta_P (\mathbf{I} - \gamma P)^{-1}. \]
Substituting this identity into Eq. (48), we obtain
\[ \theta(\mathcal{P}) - \theta(\mathcal{P}') = \gamma (\mathbf{I} - \gamma P)^{-1} \Delta_P \theta^* + (\mathbf{I} - \gamma P)^{-1} \Delta_r + R_{01}, \] (49)
where the remainder term \( R_{01} \) takes the form
\[ R_{01} = \gamma^2 (\mathbf{I} - \gamma P')^{-1} \Delta_P (\mathbf{I} - \gamma P)^{-1} \Delta_P \theta^* + M_P \Delta_r. \]
Since \((1 - \gamma)(\mathbf{I} - \gamma P')^{-1}\) is a probability transition matrix, it follows that \( \| (1 - \gamma)(\mathbf{I} - \gamma P')^{-1} \|_\infty \leq 1 \). Thus, the remainder term \( R_{01} \) satisfies the bound
\[ \| R_{01} \|_\infty \leq \frac{\gamma}{(1 - \gamma)} \| \Delta_P \|_\infty \| \gamma (\mathbf{I} - \gamma P)^{-1} \Delta_P \theta^* \|_\infty + \frac{\gamma}{(1 - \gamma)^2} \| \Delta_P \|_\infty \| \Delta_r \|_\infty. \]
The claimed lower bound (47) now follows from Eq. (49) and the triangle inequality. It is clear that Eq. (47) implies the claim in the lemma statement once we restrict \( P' \in \mathcal{S}_1 \) and \( P' \in \mathcal{S}_2 \).
B.3 Proof of Lemma 4

Throughout the proof, we use \( (Z, R) \) (respectively \( (Z', R') \)) to denote a sample drawn from the distribution \( P \) (respectively from the distribution \( P' \)). We use \( P_Z, P_R \) (respectively \( P'_Z, P'_R \)) to denote the marginal distribution of \( Z, R \) (respectively \( Z', R' \)). By the independence of \( Z \) and \( R \) (and similarly for \( (Z', R') \)), the joint distributions have the product form

\[
P = P_Z \otimes P_R, \quad \text{and} \quad P' = P'_Z \otimes P'_R.
\]

**Proof of part (a):** Let \( \mathcal{P}' = (P', R') \in S_1 \) (so \( r' = r \)). Because of the independence between \( Z \) and \( R \) (see Eq. (50)) and \( r = r' \), we have that

\[
d_{hel}(P, P')^2 = d_{hel}(P_Z, P_{Z'})^2.
\]

Note that the rows of \( Z \) and \( Z' \) are independent. Thus, if we let \( Z_i, Z'_i \) denote the \( i \)-th rows of \( Z \) and \( Z' \), we have

\[
d_{hel}(P_Z, P_{Z'})^2 = 1 - \prod_i \left( 1 - d_{hel}(P_{Z_i}, P_{Z'_i})^2 \right) \leq \sum_i d_{hel}(P_{Z_i}, P_{Z'_i})^2.
\]

Now, note that \( Z_i \) and \( Z'_i \) have multinomial distribution with parameters \( P_i \) and \( P'_i \), where \( P_{0,i}, P_{1,i} \) are the \( i \)-th row of \( P_0 \) and \( P_1 \). Thus, we have

\[
d_{hel}(P_{Z_i}, P_{Z'_i})^2 \leq \frac{1}{2} \left(D_{\chi^2}(P_{Z_i} \| P_{Z'_i}) = \frac{1}{2} \sum_j \frac{\left(P_{i,j} - P'_{i,j}\right)^2}{P_{i,j}} \right).
\]

Putting together the pieces yields the desired upper bound (30a).

**Proof of part (b):** Let \( \mathcal{P}' = (P', R') \in S_2 \) (so \( P' = P \)). Given the independence between \( Z \) and \( R \) (see Eq. (50)) and \( \mathcal{P} = P' \), we have the relation \( d_{hel}(P, P')^2 = d_{hel}(P_R, P'_R)^2 \). Note that \( R \sim N(r, I) \) and \( R' \sim N(r', I) \). Thus, we have

\[
d_{hel}(P_R, P'_R)^2 \leq D_{kl}(P_R \| P'_R) = \frac{1}{2\sigma_r^2} \|r - r'\|_2^2,
\]

as claimed.

B.4 Proof of Lemma 5

We now specify how to construct the probability matrix \( \tilde{P} \) that satisfies the desired properties stated in Lemma 5. We introduce the shorthand notation \( \tilde{\theta} = \mathbf{P}\theta^* \), and \( U = (I - \gamma \mathbf{P})^{-1} \). Let \( \tilde{\ell} \in [D] \) be an index such that

\[
\tilde{\ell} \in \arg\max_{\ell \in [D]} \left( e_{\ell}^\top (I - \gamma \mathbf{P})^{-1} \Sigma(\theta)(I - \gamma \mathbf{P})^{-\top} e_{\ell} \right)^{1/2} = \arg\max_{\ell \in [D]} \left( \sum_i U^2_{\ell,i} \sigma_i^2(\theta^*) \right)^{1/2}
\]

We construct the matrix \( \tilde{\mathbf{P}} \) entrywise as follows:

\[
\tilde{P}_{i,j} = P_{i,j} + \frac{1}{\sqrt{2N}} \cdot P_{i,j} U_{\tilde{\ell},i}(\theta^* - \tilde{\theta}_i)
\]

for \( \nu \equiv \nu(P, \theta^*) = \left( \sum_i U^2_{\ell,i} \sigma_i^2(\theta_i) \right)^{1/2} \). Now we show that \( \tilde{\mathbf{P}} \) satisfy the following properties:
(P1) The matrix $\bar{P}$ is a probability transition matrix.

(P2) It satisfies the constraint $\sum_{i,j} \left( \frac{(P-P)_{i,j}}{P_{i,j}} \right)^2 \leq \frac{1}{2N}$.

(P3) It satisfies the inequalities

$$\|P - \bar{P}\|_{\infty} \leq \frac{1}{\sqrt{2N}}, \quad \text{and} \quad \|\gamma(I - \gamma P)^{-1}(P - \bar{P})\theta^*\|_{\infty} \geq \frac{\gamma}{\sqrt{2N}} \nu(P, \theta^*).$$

We prove each of these properties in turn.

**Proof of (P1):** For each row $i \in [D]$, we have

$$\sum_j P_{i,j} = \sum_j P_{i,j} + \frac{1}{\nu \sqrt{2N}} U_{\bar{i},i} \sum_j P_{i,j} (\theta^*_j - \bar{\theta}_i) = \sum_j P_{i,j} = 1,$$

thus showing that $\bar{P} 1 = 1$ as desired. Moreover, since $(1 - \gamma)U = (1 - \gamma)(I - \gamma P)^{-1}$ is a probability transition matrix, we have the bound $|U_{\bar{i},i}| \leq \frac{1}{1 - \gamma}$. By the triangle inequality, we have

$$2 \|	heta^*\|_{\text{span}} \geq |\theta^*_j - \bar{\theta}_i|.$$ 

Thus, our assumption on the sample size $N$ implies that $\nu \sqrt{N} \geq \frac{2}{1 - \gamma} \|	heta^*\|_{\text{span}} \geq |U_{\bar{i},i}(\theta^*_j - \bar{\theta}_j)|$, which further implies that

$$\bar{P}_{i,j} = P_{i,j} \left(1 + \frac{1}{\nu \sqrt{2N}} U_{\bar{i},i} (\theta^*_j - \bar{\theta}_i) \right) \geq 0.$$ 

In conjunction with the property $\bar{P} 1 = 1$, we conclude that $\bar{P}$ is a probability transition matrix, as claimed.

**Proof of (P2):** We begin by observing that $(\Delta P)_{i,j} = \frac{1}{\nu \sqrt{2N}} P_{i,j} U_{\bar{i},i} (\theta^*_j - \bar{\theta}_i)$. Now it is simple to check that

$$\sum_{i,j} \left( \frac{(\Delta P)_{i,j}}{P_{i,j}} \right)^2 = \frac{1}{2N\nu^2} \sum_{i,j} P_{i,j} U_{\bar{i},i}^2 (\theta^*_j - \bar{\theta}_i)^2 = \frac{1}{2N\nu^2} \sum_i U_{\bar{i},i}^2 \sigma^2_i(\theta^*) = \frac{1}{2N}, \quad (53)$$

where in step (i), we use $\sigma^2_i(\theta^*) = \sum_j P_{i,j} (\theta^*_j - \bar{\theta}_i)^2$ for each $i$, as the $i$th row of our observation $Z$ is a multinomial distribution with mean specified by the $i$th row of $P$. This proves that $\bar{P}$ satisfies the constraint, as desired.

**Proof of (P3):** In order to verify the first inequality, we note that for any row $i$,

$$\sum_j |(\Delta P)_{i,j}| \leq \left( \sum_j (\Delta P)^2_{i,j} \right)^{1/2} \leq \left( \sum_{i,j} \frac{\Delta P_{i,j}^2}{P_{i,j}} \right)^{1/2} = \frac{1}{\sqrt{2N}},$$
where step (i) follows from the Cauchy-Schwartz inequality, and step (ii) follows by the previously established Property 2. Taking the maximum over row $i$ yields

$$\|\Delta P\|_\infty = \max_i \sum_j |(\Delta P)_{i,j}| \leq \frac{1}{\sqrt{2N}},$$

thus establishing the first claimed inequality in Eq. (51).

In order to establish the second inequality in Eq. (51), our starting point is the lower bound

$$\|\gamma(I - \gamma P)^{-1}\Delta P\|_\infty \geq \|e^T \gamma(I - \gamma P)^{-1}\Delta P\|_\infty = \gamma \cdot \sum_{i,j} U_{\ell,i} (\Delta P)_{i,j} \theta^*_j.$$ 

It is straightforward to check that

$$\sum_{i,j} U_{\ell,i} (\Delta P)_{i,j} \theta^*_j = \sum_{i,j} U_{\ell,i} (\Delta P)_{i,j} (\theta^*_j - \bar{\theta}_i) = \frac{1}{\nu \sqrt{2N}} \sum_{i,j} P_{i,j} U_{\ell,i}^2 (\theta^*_j - \bar{\theta}_i)^2 = \nu^2.$$ 

Here step (i) follows from the fact that $\sum_j (\Delta P)_{i,j} = 0$ for all $i$ (as $\Delta P \bar{1} = \bar{P} - P = 0$); whereas step (ii) follows from our previous calculation (see Eq. (53)) showing that

$$\sum_{i,j} P_{i,j} U_{\ell,i}^2 (\theta^*_j - \bar{\theta}_i)^2 = \nu^2.$$ 

Thus, we have verified the second inequality in Eq. (51).

C Proofs of auxiliary lemmas for Theorem 2

This appendix is devoted to the proofs of auxiliary lemmas involved in the proof of Theorem 2.

C.1 Proof of Lemma 2

In this section, we prove all three parts of Lemma 2, which provides high-probability upper bounds on the suboptimality gap at the end of each epoch. Parts (a), (b) and (c), respectively, of Lemma 2 provides guarantees for the recentered linear stepsize, polynomially-decaying stepsizes and constant stepsizes. In order to de-clutter the notation, we omit the dependence on the epoch $m$ in the operators and epoch initialization $\bar{\theta}_m$. In order to distinguish between the total sample size $N$ and the recentering sample size at epoch $m$, we retain the notation $N_m$ for the recentering sample size.

C.1.1 Proof of part (a)

We begin by rewriting the update Eq. (35b) in a form suitable for application of general results from [47]. Subtracting off the fixed point $\theta$ of the operator $J$, we find that

$$\theta_{k+1} - \bar{\theta} = (1 - \alpha_k) \left( \theta_k - \bar{\theta} \right) + \alpha_k \left( J_k(\theta_k) - \bar{\theta} \right).$$

Note that the operator $\theta \mapsto \hat{J}_k(\theta)$ is $\gamma$-contractive in the $\ell_\infty$-norm and monotonic with respect to the orthant ordering; consequently, Corollary 1 from [47] can be applied. In applying this corollary, the effective noise term is given by

$$W_k := J_k(\bar{\theta}) - J(\bar{\theta}) = \left\{ \hat{T}_k(\bar{\theta}) - \hat{T}_k(\bar{\theta}) \right\} - \left\{ T(\bar{\theta}) - T(\bar{\theta}) \right\}.$$
With this setup, by adapting Corollary 1 from [47] we have

\[ \| \theta_{K+1} - \bar{\theta} \|_\infty \leq \frac{2}{1 + (1 - \gamma)K} \left\{ \| \bar{\theta} - \tilde{\theta} \|_\infty + \sum_{k=1}^{K} \| V_k \|_\infty \right\} + \| V_{K+1} \|_\infty, \]  

(54a)

where the auxiliary stochastic process \( \{ V_k \} \) evolves according to the recursion

\[ V_{k+1} = (1 - \alpha_k) V_k + \alpha_k W_k. \]  

(54b)

We claim that the \( \ell_\infty \)-norm of this process can be bounded with high probability as follows:

**Lemma 4.** Consider any sequence of stepsizes \( \{ \alpha_k \} \) in \((0, 1)\) such that

\[ (1 - \alpha_{k+1}) \alpha_k \leq \alpha_{k+1}. \]  

(55)

Then for any tolerance level \( \delta > 0 \), we have

\[ \mathbb{P} \left[ \| V_{K+1} \|_\infty \geq 4 \| \bar{\theta} - \tilde{\theta} \|_\infty \sqrt{\alpha_k} \sqrt{\log(8KMD/\delta)} \right] \leq \frac{\delta}{2KM}. \]  

(56)

See Appendix C.3 for a proof of this claim. For future reference, note that all three stepsize
choices \((5a)-(5c)\) satisfy the condition \((55)\).

Substituting the bound \((56)\) into the relation \((54a)\) yields

\[ \| \theta_{K+1} - \bar{\theta} \|_\infty \leq c \left\{ \frac{\| \bar{\theta} - \tilde{\theta} \|_\infty}{1 + (1 - \gamma)K} + \frac{\| \tilde{\theta} - \bar{\theta} \|_\infty}{(1 - \gamma)\delta/\sqrt{K}} \right\} \sqrt{\log(8KMD/\delta)} \]

\[ \leq c \| \bar{\theta} - \tilde{\theta} \|_\infty \left\{ \frac{\sqrt{\log(8KMD/\delta)}}{1 + (1 - \gamma)K} + \frac{\sqrt{\log(8KMD/\delta)}}{(1 - \gamma)\delta/\sqrt{K}} \right\}, \]

with probability at least \(1 - \frac{\delta}{2KM}\). Combining the last bound with the fact that \(KM = \frac{N}{2}\) we find that for all \(K \geq c_1 \frac{\log(8ND/\delta)}{(1 - \gamma)^3}\), we have

\[ \| \theta_{K+1} - \bar{\theta} \|_\infty \leq \frac{3}{2} \| \bar{\theta} - \tilde{\theta} \|_\infty \leq \frac{1}{8} \| \tilde{\theta} - \theta^* \|_\infty + \frac{1}{8} \| \tilde{\theta} - \theta^* \|_\infty, \]

which completes the proof of part (a).

**Proof of part (b):**

The proof of part (b) is similar to that of part (a). In particular, adapting Corollary 2 from the paper [47] for polynomial steps, we have

\[ \| \theta_{k+1} - \bar{\theta} \|_\infty \leq e^{-\frac{1}{1-\omega}(k^{1-\omega} - 1)} \| \bar{\theta} - \tilde{\theta} \|_\infty + e^{-\frac{1}{1-\omega}k^{1-\omega}} \sum_{\ell=1}^{K} e^{\frac{1-\omega}{\ell\omega}} \frac{k^{1-\omega}}{\ell^{3/2}} \| V_{\ell} \|_\infty + \| V_{K+1} \|_\infty. \]  

(57)

Recall that polynomial stepsize \((5b)\) satisfies the conditions of Lemma 4. Consequently, applying the bound from Lemma 4 we find that

\[ \| \theta_{k+1} - \tilde{\theta} \|_\infty \leq \| \bar{\theta} - \tilde{\theta} \|_\infty \left\{ e^{-\frac{1}{1-\omega}(k^{1-\omega} - 1)} + 4 \sqrt{\log(8KMD/\delta)} \left( e^{-\frac{1}{1-\omega}k^{1-\omega}} \sum_{\ell=1}^{K} e^{\frac{1-\omega}{\ell\omega}} \frac{k^{1-\omega}}{\ell^{3/2}} + \frac{1}{k^{3/2}} \right) \right\}. \]  

(58)
It remains to bound the coefficient of $\|\hat{\theta} - \tilde{\theta}\|_\infty$ in the last equation, and we do so by using the following lemma from [47]:

**Lemma 5** (Bounds on exponential-weighted sums). There is a universal constant $c$ such that for all $\omega \in (0, 1)$ and for all $k \geq \left(\frac{3\omega}{2(1-\gamma)}\right)^{\frac{1}{1-\gamma}}$, we have

$$e^{-\frac{1-\gamma}{1-\omega}k^{1-\omega}} \sum_{\ell=1}^{k} \frac{e^{\frac{1-\gamma}{1-\omega}\ell^{1-\omega}}}{\ell^{3\omega/2}} \leq c \left\{ e^{-\frac{1-\gamma}{1-\omega}(k^{1-\omega}-1)} \left(1 - \frac{1}{1-\gamma}\frac{1}{k^{\omega/2}}\right) + \frac{1}{(1-\gamma)^{\frac{1}{1-\omega}}} + \frac{1}{k^{\omega/2}} \right\}.$$  

Substituting the last bound in Eq. (57) yields

$$\|\theta_{k+1} - \tilde{\theta}\|_\infty \leq c\|\tilde{\theta} - \tilde{\theta}\|_\infty \left\{ e^{-\frac{1-\gamma}{1-\omega}(k^{1-\omega}-1)} \left(1 - \frac{1}{1-\gamma}\frac{1}{k^{\omega/2}}\right) + \frac{1}{(1-\gamma)^{\frac{1}{1-\omega}}} + \frac{1}{k^{\omega/2}} \right\} \leq c\|\tilde{\theta} - \tilde{\theta}\|_\infty \cdot \sqrt{\log(8KMD/\delta)} \left\{ 5 \cdot e^{-\frac{1-\gamma}{1-\omega}(k^{1-\omega}-1)} \left(1 - \frac{1}{1-\gamma}\frac{1}{k^{\omega/2}}\right) + \frac{2}{(1-\gamma)^{\frac{1}{1-\omega}}} \right\}.$$  

Finally, doing some algebra and using the fact that $KM = \frac{N}{2}$ we find that there is an absolute constant $c$ such that for all $K$ lower bounded as $K \geq c\log(4ND/\delta) \cdot \left(\frac{1}{1-\gamma}\right)^{\frac{1}{1-\omega}}$, we have

$$\|\theta_{K+1} - \tilde{\theta}\|_\infty \leq \frac{\|\tilde{\theta} - \tilde{\theta}\|_\infty}{8} \leq \frac{1}{8}\|\tilde{\theta} - \theta^*\|_\infty + \frac{1}{8}\|\tilde{\theta} - \theta^*\|_\infty.$$  

The completes the proof of part (b).

**Proof of part (c):**

Invoking Theorem 1 from [47], we have $\|\theta_{K} - \hat{\theta}\|_\infty \leq a_{K} + b_{K} + \|V_{K}\|_\infty$. For a constant stepsize $\alpha_{K} = \alpha$, the pair $(a_{K}, b_{K})$ is given by

$$b_{K} = \|\hat{\theta} - \tilde{\theta}\|_\infty \cdot (1 - \alpha(1 - \gamma))^{K-1},$$

$$a_{K} = \gamma\alpha \|V_{k}\|_\infty + \gamma\alpha \|V_{l}\|_\infty \sum_{k=1}^{K-1} (1 - (1 - \gamma)\alpha)^{K-k} \leq \|\hat{\theta} - \tilde{\theta}\|_\infty \cdot \left(2\gamma\alpha^2 \sqrt{\log(8KMD/\delta)} + \frac{2\gamma\alpha}{1 - \gamma} \sqrt{\log(8KMD/\delta)}\right),$$

where inequality (i) follows by substituting $\alpha_{K} = \alpha$, and using the bound on $\|V_{l}\|_\infty$ from Lemma 4.

It remains to choose the pair $(\alpha, K)$ such that $\|\theta_{K+1} - \hat{\theta}\|_\infty \leq \frac{1}{8}\|\tilde{\theta} - \tilde{\theta}\|_\infty$. Doing some simple algebra and using the fact that $KM = \frac{N}{2}$ we find that it is sufficient to choose the pair $(\alpha, K)$ satisfying the conditions

$$0 < \alpha \leq \frac{(1 - \gamma)^2}{\log(4ND/\delta)} \cdot \frac{1}{5^2 \cdot 32^2}, \quad \text{and} \quad K \geq 1 + \frac{2\log 16}{\log \left(\frac{1}{1-\alpha(1-\gamma)}\right)}.$$  

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With this choice, we have

\[ \| \theta_{K+1} - \hat{\theta} \|_\infty \leq \frac{\| \hat{\theta} - \theta^* \|_\infty}{8} \leq \frac{1}{8} \| \theta - \theta^* \|_\infty + \frac{1}{8} \| \hat{\theta} - \theta^* \|_\infty, \]

which completes the proof of part (c).

C.2 Proof of Lemma 3

Recall our shorthand notation for the local complexities (8). The following lemma characterizes the behavior of various random variables as a function of these complexities. In stating the lemma, we let \( \hat{P}_n \) be a sample transition matrix constructed as the average of \( n \) i.i.d. samples, and let \( \hat{r}_n \) denote the reward vector constructed as the average of \( n \) i.i.d. samples.

**Lemma 6.** Each of the following statements holds with probability exceeding \( 1 - \frac{\delta}{M} \):

\[
\| (I - \gamma P)^{-1} (\hat{P}_n - P) \theta^* \|_\infty \leq 2 \nu(P, \theta^*) \sqrt{\frac{\log(4DM/\delta)}{n}} + 4 \cdot b(\theta^*) \cdot \frac{\log(4DM/\delta)}{n}, \quad \text{and}
\]

\[
\| (I - \gamma P)^{-1} (\hat{r}_n - r) \|_\infty \leq 2 \rho(P, r) \sqrt{\frac{\log(4DM/\delta)}{n}}.
\]

**Proof.** Entry \( \ell \) of the vector \((I - \gamma P)^{-1}(\hat{P}_n - P) \theta^*\) is zero mean with variance given by the \( \ell \)-th diagonal entry of the matrix \((I - \gamma P)^{-1} \Sigma(\theta^*) (I - \gamma P)^{-T}\), and is bounded by \( b(\theta^*) \) almost surely. Consequently, applying the Bernstein bound in conjunction with the union bound completes the proof of the first claim. In order to establish the second claim, note that the vector \((I - \gamma P)^{-1}(\hat{r}_n - r)\) has sub-Gaussian entries, and apply the Hoeffding bound in conjunction with the union bound. \( \square \)

In light of Lemma 6, note that it suffices to establish the inequality

\[
\Pr \left\{ \| \hat{\theta}_m - \theta^* \|_\infty \geq \frac{1}{9} \| \theta - \theta^* \|_\infty + \| (I - \gamma P)^{-1} (\hat{P}_{N_m} - P) \theta^* \|_\infty + \| (I - \gamma P)^{-1} (\hat{r}_{N_m} - r) \|_\infty \right\} \leq \frac{\delta}{2M},
\]

where we have let \( \hat{P}_{N_m} \) and \( \hat{r}_{N_m} \) denote the empirical mean of the observed transitions and rewards in epoch \( m \), respectively. The proof of Lemma 3 follows from Eq. (59) by a union bound.

**Establishing the bound (59):** Since the epoch number \( m \) should be clear from context, let us adopt the shorthand \( \hat{\theta} \equiv \hat{\theta}_m \), along with the shorthand \( \hat{r} \equiv \hat{r}_{N_m} \) and \( \hat{P} \equiv \hat{P}_{N_m} \). Note that \( \hat{\theta} \) is the fixed point of the following operator:

\[
J(\theta) := \mathcal{T}(\theta) - \mathcal{T}(\hat{\theta}) + \hat{r}_{N_m}(\hat{\theta}) = \hat{r} + \gamma \underbrace{\hat{P} - P}_{\hat{r}} \hat{\theta} + \gamma P \theta,
\]

where we have used the fact that \( \hat{T}_{N_m}(\theta) = \hat{r} + \gamma \hat{P} \theta \).

Thus, we have \( \hat{\theta} = (I - \gamma P)^{-1}(\hat{r} - r) \), so that \( \hat{\theta} - \theta^* = (I - \gamma P)^{-1}(\hat{r} - r) \). Also note that we have

\[
\hat{r} - r = \hat{r} + \gamma (\hat{P} - P) \hat{\theta} - r = \hat{r} - r + \gamma (\hat{P} - P) \theta^* + \gamma (\hat{P} - P) (\hat{\theta} - \theta^*),
\]

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so that putting together the pieces and using the triangle inequality yields the bound
\[ \|\hat{\theta} - \theta^*\|_\infty \leq \|(I - \gamma P)^{-1}(\bar{r} - r)\|_\infty + \gamma\|(I - \gamma P)^{-1}(\hat{P} - P)\theta^*\|_\infty + \gamma\|(I - \gamma P)^{-1}(\hat{P} - P)(\hat{\theta} - \theta^*)\|_\infty \]
\[ \leq \|(I - \gamma P)^{-1}(\bar{r} - r)\|_\infty + \gamma\|(I - \gamma P)^{-1}(\hat{P} - P)\theta^*\|_\infty + \frac{\gamma}{1 - \gamma}\|(\hat{P} - P)(\hat{\theta} - \theta^*)\|_\infty. \]

Note that the random vector \((\hat{P} - P)(\hat{\theta} - \theta^*)\) is the empirical average of \(N_m\) i.i.d. random vectors, each of which is bounded entrywise by \(2\|\hat{\theta} - \theta^*\|_\infty\). Consequently, by a combination of Hoeffding’s inequality and the union bound, we find that
\[ \left\| (\hat{P} - P)(\hat{\theta} - \theta^*) \right\|_\infty \leq 4\|\hat{\theta} - \theta^*\|_\infty \sqrt{\frac{\log(8DM/\delta)}{N_m}}, \]
with probability at least \(1 - \frac{\delta}{4M}\). Thus, provided \(N_m \geq 4^2 \cdot 9^2 \cdot \frac{\gamma^2}{(1 - \gamma)^2} \log(8DM/\delta)\) for a large enough constant \(c_1\), we have
\[ \frac{\gamma}{1 - \gamma}\left\| (\hat{P} - P)(\hat{\theta} - \theta^*) \right\|_\infty \leq \frac{\|\hat{\theta} - \theta^*\|_\infty}{9}. \]

This completes the proof.

C.3 Proof of Lemma 4

Recall that by definition, the stochastic process \(\{V_k\}_{k \geq 1}\) evolves according to the linear recursion \(V_k = (1 - \alpha_k)V_{k-1} + \alpha_k W_{k-1}\), where the effective noise sequence \(\{W_k\}_{k \geq 0}\) satisfies the uniform bound
\[ \|W_k\|_\infty \leq \left\| \hat{T}_k(\hat{\theta}) - \hat{T}_k(\bar{\theta}) \right\|_\infty + \|T(\hat{\theta}) - T(\bar{\theta})\|_\infty \leq 2\|\hat{\theta} - \bar{\theta}\|_\infty \quad \text{for all } k \geq 0. \]

Moreover, we have \(E[W_k] = 0\) by construction so that each entry of the random vector \(W_k\) is a zero-mean sub-Gaussian random variable with sub-Gaussian parameter at most \(2\|\hat{\theta} - \theta^*\|_\infty\). Consequently, by known properties of sub-Gaussian random variables (cf. Chapter 2 in [45]), we have
\[ \log E\left[e^{sW_k(x)}\right] \leq \frac{s^2 b^2}{8} \quad \text{for all scalars } s \in \mathbb{R}, \text{ and states } x. \quad (60) \]

We complete the proof by using an inductive argument to upper bound the moment generating function of the random variable \(V_k\); given this inequality, we can then apply the Chernoff bound to obtain the stated tail bounds. Beginning with the bound on the moment generating function, we claim that
\[ \log E\left[e^{sV_k(x)}\right] \leq \frac{s^2 \alpha_k b^2}{8} \quad \text{for all scalars } s \in \mathbb{R} \text{ and states } x. \quad (61) \]
We prove this claim via induction on \(k\).

**Base case:** For \(k = 1\), we have
\[ \log E\left[e^{sV_1(x)}\right] = \log E\left[e^{s\alpha_1 W_0(x)}\right] \leq \frac{s^2 \alpha_1^2 b^2}{8}, \]
where the first equality follows from the definition of \(V_1\), and the second inequality follows by applying the bound (60).
**Inductive step:** We now assume that the bound (61) holds for some iteration $k \geq 1$ and prove that it holds for iteration $k + 1$. Recalling the definition of $V_k$, and the independence of the random variables $V_k$ and $W_k$, we have

$$
\mathbb{E} \left[ e^{sV_{k+1}(x)} \right] = \log \mathbb{E} \left[ e^{s(1-\alpha_k)V_k(x)} \right] + \log \mathbb{E} \left[ e^{s\alpha_kW_k(x)} \right]
$$

$$
\leq \frac{s^2(1-\alpha_k)^2 b^2}{8} + \frac{s^2 \alpha_k^2 b^2}{8}
$$

$$
\leq \frac{s^2(1-\alpha_k) \alpha_k b^2}{8} + \frac{s^2 \alpha_k^2 b^2}{8}
$$

$$
= \frac{s^2 \alpha_k b^2}{8},
$$

where inequality (i) follows from the assumed condition (55) on the stepsizes.

Simple algebra yields that all the stepsize choices (5a)-(5c) satisfy the condition (55). Finally, combining the bound (61) with the Chernoff bounding technique along with a union bound over iterations $k = 1, \ldots, K$ yields

$$
\mathbb{P} \left[ \|V_\ell\|_\infty \geq 2b \sqrt{\alpha_{\ell-1}} \sqrt{\log(8KMD/\delta)} \right] \leq \frac{\delta}{8KM},
$$

as claimed.