Abstract. Recently in [23], we have investigated Lie algebras and abelian Lie algebras derived from Lie hyperalgebras using the fundamental relations $L$ and $A$, respectively. In the present paper, continuing this method we obtain solvable Lie algebras from Lie hyperalgebras by $S_n$-relations. We show that $\bigcap_{n\geq 1} S_n^*$ is the smallest equivalence relation on a Lie hyperalgebra such that the quotient structure is a solvable Lie algebra. We also provide some necessary and sufficient conditions for transitivity of the relation $S_n$ using the notion of $S_n$-part.

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1 Introduction

In the classical theory, the quotient of a group by a normal subgroup is a group. In 1934, Marty [18] states that the quotient of a group by any subgroup is a hypergroup. Concretely, if $G$ is a group and $H$ is a subgroup of $G$, then the set of all left cosets of $H$ in $G$ is a hypergroup with the multiplication

$$xH \cdot yH := \{zH | z = xhy, h \in H\}.$$ 

Clearly, if $H$ is normal in $G$, then the set of cosets turns into the quotient group $G/H$.

Following such an approach to algebraic structures, many investigations have been made on hyperstructure theory (hypergroups, hyperrings, hypermodules, hypervector spaces, hyperalgebras, etc.) whose applications nowadays have been known in other sciences such as algebra and geometry as well as automata, cryptography, artificial intelligence and probability, relational algebras, sensor networks, theoretical physics and chemistry (see [17, 9, 24] for more information). An important part of studies on hyperstructures is about strongly regular relations (fundamental relations) that actually makes a bridge between classical structures and hyperstructures. In 1970, this connection was achieved by Koskas [17] using the relation $\beta$ on (semi)hypergroups and its transitive closure to obtain (semi)groups from a quotient of (semi)hypergroups. This relation has been studied mainly by Corsini [6], Vougiouklis [24], Davvaz [8], Davvaz and Leoreanu-Fotea
This quotient set not only links hyperstructures with classical structures, but also enhances the view of hyperstructures as a generalization of the corresponding classical algebraic structures. Hence, studies on this topic have been continued to obtain commutative semigroups, cyclic groups, nilpotent groups, Engel groups, solvable groups, rings, commutative rings, Boolean rings, modules over commutative rings, and also modules over commutative rings [1, 2, 21, 22].

The authors in [23] studied strongly regular relations on Lie hyperalgebras and obtained Lie algebras and abelian Lie algebras from Lie hyperalgebras using the fundamental relations $L$ and $A$, respectively. Now in this paper, we investigate solvable Lie algebras derived from Lie hyperalgebras by $S$-relations ($L \subseteq S_n \subseteq S_1 = A$). We introduce the relations $S_n$ and show that their transitive closures are strongly regular relations on a Lie hyperalgebra $L$ such that the quotient $L/S^*_n$ is a solvable Lie algebra of length at most $n$. We also prove that if the fundamental Lie algebra $L/L^*$ is finite dimensional, then $S = \bigcap_{n \geq 1} S^*_n$ is the smallest equivalence relation on $L$ such that $L/S$ is a (fundamental) solvable Lie algebra. Moreover, it is shown that if $L$ is a simple Lie algebra, then $L/S = 0$ and $L/L \cong L$, which is in coincidence with the concept of simplicity. Finally, we give some necessary and sufficient conditions for transitivity of the relation $S_n$ using its $S_n$-part.

Definition 1.1 Let $(R, +, \cdot)$ be a hypergroup, $(M, +)$ be a hypergroup and $\mathcal{P}^*(M)$ denote the family of all non-empty subsets of $M$, together with an external map $R \times M \to \mathcal{P}^*(M)$, defined by $(r, m) \mapsto rm$ such that for all $a, b \in R$ and $x, y \in M$, we have $a(x + y) = ax + ay$, $(a + b)x = ax + bx$ and $(ab)x = a(bx)$. Then $M$ is called a hypermodule over $R$. If we consider a hyperfield $F$ instead of a hyperring $R$, then $M$ is called a hypervector space.

Throughout the paper, any hypervector space $M$ is assumed to have a scalar identity $0_M$, and any hyperfield $F$ contains a scalar identity $0_F$ and a multiplication unit 1 such that $0_F x = \{0_M\}$ and $1x = \{x\}$ for all $x \in M$.

2 \hspace{1em} $L$-relation and $A$-relation on Lie hyperalgebras

This section is devoted to discuss some preliminary results on Lie algebras obtained from Lie hyperalgebras from [23].

A Lie algebra $(L, \oplus, \odot)$ is a vector space over a field $\mathbb{F}$ equipped with a bilinear map $[-, -]: L \times L \to L$, usually called the Lie bracket of $L$, satisfying the following conditions:

(i) $[x, x] = 0$, for all $x \in L$,

(ii) $[x, [y, z]] \oplus [y, [z, x]] \oplus [z, [x, y]] = 0$, for all $x, y, z \in L$ (Jacobi identity).
Definition 2.1. Let \((L, +, \cdot)\) be a hypervector space over a hyperfield \((F, +, \cdot)\) and \([-,-] : L \times L \to \mathcal{P}^*(L)\) given by \((x, y) \mapsto [x, y]\) be a bilinear map which we would call the hyperbracket of \(L\). Then \(L\) is said to be a Lie hyperalgebra if the following axioms hold:

(i) \(0 \in [x, x]\), for all \(x \in L\),

(ii) \(0 \in ([x, [y, z]] + [y, [z, x]] + [z, [x, y]])\), for all \(x, y, z \in L\).

We use the operations "\(\oplus, \circ\)" and "\([-, -]\)" as the bracket in Lie algebras, and "\(+, \cdot\)" and "\([-,-]\)" as the hyperbracket in Lie hyperalgebras. Also, the bilinearity of the hyperbracket means:

\[\lambda_1 x_1 + \lambda_2 x_2, y] = \lambda_1 [x_1, y] + \lambda_2 [x_2, y],\]

\[[x, \lambda_1 y_1 + \lambda_2 y_2] = \lambda_1 [x, y_1] + \lambda_2 [x, y_2],\]

for all \(x, x_1, x_2, y, y_1, y_2 \in L\) and \(\lambda_1, \lambda_2 \in F\) (see [10, 23] for details and examples).

Let \(L\) be a Lie hyperalgebra on \(F\) and \(\rho \subseteq L \times L\) be an equivalence relation. For non-empty subsets \(A\) and \(B\) of \(L\), define

\[\bar{A} \bar{\rho} \bar{B} \iff a \rho b, \ \forall a \in A, \forall b \in B.\]

The relation \(\rho\) is said to be strongly regular on the left (on the right) if \(x \rho y\) implies \(a + x \bar{\rho} a + y, \ \lambda \cdot x \bar{\rho} \lambda \cdot y\) and \([a, x] \bar{\rho} [a, y]\) (if \(x \rho y\) implies \(x + a \bar{\rho} y + a, \ x \cdot \lambda \bar{\rho} y \cdot \lambda\) and \([x, a] \bar{\rho} [y, a]\)), for all \(x, y, a \in L\) and \(\lambda \in F\). In addition, \(\rho\) is called strongly regular if it is strongly regular on the both left and right.

Lemma 2.2 [23, Proposition 2.5] Let \(L\) be a Lie hyperalgebra over a hyperfield \(F\), and \(\rho\) and \(\delta\) be equivalence relations on \(L\) and \(F\), respectively. If \(\rho\) and \(\delta\) are strongly regular, then the quotient \(L/\rho\) is a Lie algebra over the field \(F/\delta\), under the following operations:

\[\bar{x} \oplus \bar{y} = \bar{z}, \ \text{for all} \ z \in x + y\]

\[\bar{\lambda} \odot \bar{x} = \bar{z}, \ \text{for all} \ z \in \lambda \cdot x\]

\[[\bar{x}, \bar{y}] = \bar{z}, \ \text{for all} \ z \in [x, y],\]

for \(\bar{x}, \bar{y} \in L/\rho\) and \(\bar{\lambda} \in F/\delta\). Conversely, if \(L/\rho\) together with the above operations is a Lie algebra over the field \(F/\delta\), then \(\rho\) is strongly regular.

In [23], the relation \(L\) on a Lie hyperalgebra \(L\) (over a hyperfield \(F\)) is defined as follows:

\[x \mathcal{L} y \iff \{x, y\} \subseteq \sum_{i=1}^{n} \ell_i \text{ where } \ell_i = f_i(g_{i1}, g_{i2}, \ldots, g_{im_i})\]

with \(g_{ij} = \left(\sum_{k=1}^{p_{ij}} \prod_{r=1}^{q_{ijk}} (\lambda_{ijkr})\right) \cdot h_{ij},\)
such that $1 \leq j \leq m_i$, $h_{ij} \in L$, $\lambda_{ijk} \in F$, and $f_i$ is an arbitrary hyperbracket function, i.e. $f_i(g_{i1}, g_{i2}, \ldots, g_{im_i})$ is an arbitrary composition of the elements $g_{i1}, g_{i2}, \ldots, g_{im_i}$ only by hyperbracket. Moreover, it is shown that $\mathcal{L}^*$ is a strongly regular relation on $L$, which is also the smallest equivalence relation on $L$ such that $L/\mathcal{L}^*$ is a (fundamental) Lie algebra (see Theorem 3.3 and Corollary 3.4 in [23]).

Furthermore, the $A$-relation on $L$ is defined as:

$$x A y \iff \exists n \in \mathbb{N}, \exists \sigma \in S_n, \exists f_1, \ldots, f_n, \exists m_1, \ldots, m_n \in \mathbb{N}, \exists \sigma_i \in S_{m_i}, \exists h_{i1}, \ldots, h_{im_i} \in L, \exists p_{i1}, \ldots, p_{im_i} \in \mathbb{N}, \exists \sigma_{ij} \in S_{p_{ij}}, \exists q_{ij1}, \ldots, q_{ijp_{ij}} \in \mathbb{N}, \exists \lambda_{ijk1}, \ldots, \lambda_{ijkq_{ijk}} \in F, \exists \sigma_{ij} \in S_{q_{ijk}}$$

such that

$$x \in \sum_{i=1}^{n} \ell_i \quad \text{where} \quad \ell_i = f_i(g_{i1}, g_{i2}, \ldots, g_{im_i}) \quad \text{with} \quad g_{ij} = \left( \sum_{k=1}^{p_{ij}} \prod_{r=1}^{q_{ijk}} \lambda_{ijk} \right) \cdot h_{ij},$$

for $1 \leq j \leq m_i$, and

$$y \in \sum_{i=1}^{n} \ell_i' \quad \text{where} \quad \ell_i' = f_{\sigma(i)}(g_{i1}', g_{i2}', \ldots, g'_{im_{\sigma(i)}}) \quad \text{with} \quad g'_{ij} = A_{\sigma(i)\sigma(i)(j)} \cdot h_{\sigma(i)\sigma(i)(j)},$$

in which $1 \leq j \leq m_{\sigma(i)}$ and $A_{ij} = \sum_{k=1}^{p_{ij}} B_{ij\sigma_{ij}(k)}$ \text{with} $B_{ijk} = \prod_{r=1}^{q_{ijk}} \lambda_{ijk}\sigma_{ijk}(r)$.

It is proved that $A^*$ is a strongly regular relation, and also the smallest equivalence relation on $L$ such that $L/A^*$ is an abelian Lie algebra ([23 Theorem 4.2, Corollary 4.3]).

## 3 Main results

In this section, we introduce and study the smallest equivalence relation on a Lie hyperalgebra such that the quotient structure is a solvable Lie algebra. A Lie algebra $L$ is called solvable of length $n$, if $L^{(n)} = 0$ and $L^{(n-1)} \neq 0$, where $L^{(i)}$ denotes the $(i+1)$st term of the derived series of $L$, defined inductively by $L^{(0)} = L$ and $L^{(i)} = [L^{(i-1)}, L^{(i-1)}] = \langle [x, y] | x, y \in L^{(i-1)} \rangle$ for $i \geq 1$.

Now, let $L$ be a Lie hyperalgebra. For non-empty subsets $A, B$ of $L$, define

$$[A, B] = \bigcup_{a \in A, b \in B} [a, b],$$

and put $L^{[0]} = L$ and $L^{[i]} = [L^{[i-1]}, L^{[i-1]}]$ for $i \geq 1$. Clearly, $L^{[i]} \subseteq L^{[i-1]}$ for every $i \geq 1$.

Also, for every Lie algebra $L$ which may be considered as a trivial Lie hyperalgebra (see Example 3.4 below), we have $L^{[i]} \subseteq L^{[i]}$ for every $i \geq 0$. 

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Definition 3.1 Let $L$ be a Lie hyperalgebra over a hyperfield $F$. For $n \in \mathbb{N}$, we define the relation $S_n$ on $L$ as follows:

$$x S_n y \iff \exists t \in \mathbb{N}, \exists \sigma \in S_t, \exists f_1, \ldots, f_t, \exists m_1, \ldots, m_t \in \mathbb{N}, \exists h_{i_1}, \ldots, h_{i_m} \in L,$$

$$\exists p_1, \ldots, p_{im} \in \mathbb{N}, \text{ and for all } 1 \leq i \leq t \text{ and } 1 \leq j \leq m_i :$$

$$(\exists \sigma_i \in S_m \text{ such that } \sigma_i(j) = j \text{ if } h_{ij} \notin L^{[n-1]} \text{), and also}$$

$$\exists \sigma_{ij} \in S_{p_{ij}}, \exists q_{ij1}, \ldots, q_{ijp_{ij}} \in \mathbb{N}, \exists \lambda_{ij1}, \ldots, \lambda_{ijk_{ij}k_{ij}} \in F, \exists \sigma_{ijk} \in S_{q_{ijk}}$$

such that

$$x \in \sum_{i=1}^t \ell_i \text{ where } \ell_i = f_i(g_{i1}, g_{i2}, \ldots, g_{im_i}) \text{ with } g_{ij} = \left( \sum_{k=1}^{p_{ij}} q_{ijk} \prod_{r=1}^{q_{ijk}} \lambda_{ijk} \right) \cdot h_{ij},$$

for $1 \leq j \leq m_i$, and

$$y \in \sum_{i=1}^t \ell_i' \text{ where } \ell_i' = f_{\sigma(i)}(g'_{i1}, g'_{i2}, \ldots, g'_{im_{\sigma(i)}}) \text{ with } g_{ij}' = A_{i\sigma(i)\sigma(i)} \cdot h_{i\sigma(i)\sigma(i)}$$

in which $1 \leq j \leq m_{\sigma(i)}$ and $A_{ij} = \sum_{k=1}^{p_{ij}} B_{ij\sigma_{ij}(k)}$ with $B_{ijk} = \prod_{r=1}^{q_{ijk}} \lambda_{ijk}$.

Clearly for every $n \geq 1$, $S_n$ is a reflexive and symmetric relation and $L \subseteq S_{n+1} \subseteq S_n \subseteq S_1 = \mathcal{A}$. Consider $S_n^*$ as the transitive closure of $S_n$ ([23] Example 3.2) shows that $S_n$ is not necessarily transitive, for all $n \geq 1$). Also, $S_n^*(x)$ denotes the equivalence class of $x \in L$.

Proposition 3.2 For every $n \geq 1$, $S_n^*$ is a strongly regular relation on a Lie hyperalgebra $L$.

Proof. We use an argument similar to the proof of [23] Theorem 4.2] on the $A$-relation. We show that $x S_n y$ implies $[x, a] \subseteq S_n^* [y, a]$ for every $a \in L$. Using above notations, $x \in \sum_{i=1}^t \ell_i$ and $y \in \sum_{i=1}^t \ell_i'$, and hence

$$[x, a] \subseteq \left[ \sum_{i=1}^t \ell_i, a \right] = \sum_{i=1}^t \left[ f_i(g_{i1}, g_{i2}, \ldots, g_{im_i}), a \right],$$

$$[y, a] \subseteq \left[ \sum_{i=1}^t \ell_i', a \right] = \sum_{i=1}^t \left[ f_{\sigma(i)}(g'_{i1}, g'_{i2}, \ldots, g'_{im_{\sigma(i)}}), a \right].$$

For all $1 \leq i \leq t$, consider $\tau_i \in S_{m_i+1}$ given by $\tau_i(j) = \sigma_i(j)$ for all $1 \leq j \leq m_i$ and $\tau_i(m_i + 1) = m_i + 1$, and put $g_i(m_i+1) = h_i(m_{i+1}) = a$ and $f_i'(-, a) = [f_i(-, a)]$. Thus

$$[x, a] \subseteq \sum_{i=1}^t f_i'(g_{i1}, g_{i2}, \ldots, g_{im_i}, g_i(m_i+1)) \text{ with } g_{ij} = \left( \sum_{k=1}^{p_{ij}} q_{ijk} \prod_{r=1}^{q_{ijk}} \lambda_{ijk} \right) \cdot h_{ij}$$
for $1 \leq j \leq m_i + 1$, and

$$[y, a] \subseteq \sum_{i=1}^{t} f_{\sigma(i)}^*(g'_{i1}, g'_{i2}, \ldots, g'_{i(m_{\sigma(i)})})$$

with $g'_{ij} = A_{\sigma(i)}\tau_{\sigma(i)}(j) \cdot h_{\sigma(i)}\tau_{\sigma(i)}(j)$

in which $1 \leq j \leq m_{\sigma(i)} + 1$ and $A_{ij} = \sum_{k=1}^{p_{ij}} B_{ij\sigma(i)(k)}$ with $B_{ijk} = \prod_{r=1}^{q_{ijk}} \lambda_{ijk}\sigma(i)(r)$,

such that $\tau_i(j) = j$ if $h_{ij} \notin L^{[n-1]}$, for $1 \leq i \leq t$ and $1 \leq j \leq m_i + 1$. Note that due to the definition of $\tau_i$, it does not matter whether the element $h_{i(m_i+1)} = a$ belongs to $L^{[n-1]}$ or not. It follows that $u \mathcal{S}_n^* v$ for all $u \in [x, a]$ and $v \in [y, a]$, and so $[x, a] \mathcal{S}_n^* [y, a]$. Similarly, we may prove that $[a, x] \mathcal{S}_n^* [a, y]$. Also similar to [5, Lemma 2.2], one can show that $\mathcal{S}_n^*$ is strongly regular on $(L, +, \cdot)$, which completes the proof. \(\square\)

Let $(R, +, \cdot)$ be a hyperring and $x, y \in R$. In [11], the relation $\alpha$ is defined on $R$ as follows:

$$x \alpha y \iff \exists t \in \mathbb{N}, \exists \sigma \in \mathcal{S}_t, \exists k_1, \ldots, k_t \in \mathbb{N}, \exists \sigma_i \in \mathcal{S}_{k_i}, \exists x_{i1}, \ldots, x_{ik_i} \in R$$

such that

$$x \in \sum_{i=1}^{t} \left( \prod_{j=1}^{k_i} x_{ij} \right) \text{ and } y \in \sum_{i=1}^{t} A_{\sigma(i)} \text{ where } A_i = \prod_{j=1}^{k_i} x_{i\sigma_i(j)}.$$ 

Also, $\alpha^*$ is the transitive closure of $\alpha$ (see also [9, Definition 7.1.1]).

A Lie algebra $L$, over a field $\mathbb{F}$, is said to be abelian, if $L^{(1)} = 0$. It is easy to see that if the characteristic of $\mathbb{F}$ is not 2 and $[x, y] = [y, x]$ for all $x, y \in L$, then $L$ is abelian (see also [23]). In the following results, $F$ is a hyperfield such that the characteristic of the field $F/\alpha^*$ is not 2.

**Theorem 3.3** Let $L$ be a Lie hyperalgebra over $F$. Then $L/\mathcal{S}_n^*$ is a solvable Lie algebra of length at most $n$ over $F/\alpha^*$.

**Proof.** By Lemma 2.2 and Proposition 3.2 $L/\mathcal{S}_n^*$ is a Lie algebra over $F/\alpha^*$ (note that $(L/\mathcal{S}_n^*, \oplus)$ is also an abelian group (see [5, 23])). We show that $L/\mathcal{S}_n^*$ is solvable of length at most $n$. One may inductively prove that

$$\left( L/\mathcal{S}_n^* \right)^{(i)} = \left[ \mathcal{S}_n^*(h) \right] h \in L^{[i]},$$

for all $i \geq 0$. In Definition 3.1 let $t = 1, m_1 = 2, g_{11} = h_{11} \in L^{[n-1]}$, $g_{12} = h_{12} \in L^{[n-1]}$, $h_{1}(g_{11}, g_{12}) = [g_{11}, g_{12}]$, and $\sigma_1 = \mathcal{S}_2$ such that $\sigma_1(1) = 2$ and $\sigma_1(2) = 1$. For every $x \in [g_{11}, g_{12}]$ and $y \in [g_{1\sigma_1(1)}, g_{1\sigma_1(2)}] = [g_{12}, g_{11}]$, we have $x \mathcal{S}_n y$. Hence $x \mathcal{S}_n^* y$ and thus

$$[\mathcal{S}_n^*(g_{11}), \mathcal{S}_n^*(g_{12})] = \mathcal{S}_n^*(x) = \mathcal{S}_n^*(y) = [\mathcal{S}_n^*(g_{12}), \mathcal{S}_n^*(g_{11})].$$

Now since $g_{11}, g_{12} \in L^{[n-1]}$, equalities (3.1) and (3.2) imply that $(L/\mathcal{S}_n^*)^{(n-1)}$ is an abelian Lie algebra, which means $(L/\mathcal{S}_n^*)^{(n)} = 0_{L/\mathcal{S}_n^*}$. Therefore, $L/\mathcal{S}_n^*$ is solvable of length at most $n$. \(\square\)
The following trivial Lie hyperalgebra illustrates the difference between the relations and also the quotients of a Lie hyperalgebra by them.

**Example 3.4** Let \((L, \oplus, \odot)\) be a 4-dimensional vector space over \(\mathbb{R}\) with a basis \(\{a, b, c, d\}\). It is easy to check that \(L\) is a Lie hyperalgebra (over the trivial hyperfield \(\mathbb{R}\)) with the hyperoperations: \(x + y := \{x \oplus y\}\), \(r \cdot x := \{r \odot x\}\) for all \(x, y \in L\) and \(r \in \mathbb{R}\), and the bilinear hyperbracket:

\[
\begin{array}{c|cccc}
[-, -] & a & b & c & d \\
\hline
a & \{0\} & \{0\} & \{0\} & \{0\} \\
b & \{0\} & \{0\} & \{a\} & \{b\} \\
c & \{0\} & \{-1 \odot a\} & \{0\} & \{-1 \odot c\} \\
d & \{0\} & \{-1 \odot b\} & \{c\} & \{0\} \\
\end{array}
\]

Clearly, \(L/L^*\) is the 4-dimensional Lie algebra (over \(\mathbb{R}/\gamma^* \cong \mathbb{R}\)) with the basis \(\{\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}\}\) and brackets \([\tilde{b}, \tilde{c}] = \tilde{a}\), \([\tilde{b}, \tilde{d}] = \tilde{b}\) and \([\tilde{d}, \tilde{c}] = \tilde{c}\) where \(\tilde{x} = L^*(x)\) for every \(x \in L\) (see [23, 25] for the definition of the \(\gamma\)-relation).

Also, \(a \in [b, c]\) and \((-1 \odot a) \in [c, b]\), which imply that \(a A (-1 \odot a)\) and hence \(\tilde{a} = -1 \odot a\), or equivalently \(\tilde{a} = \tilde{0}\) where \(\tilde{x} = A^*(x)\). By using a similar manner, we get \(\tilde{b} = \tilde{c} = \tilde{0}\). Therefore, \(L/A^*\) is the abelian 1-dimensional Lie algebra (over \(\mathbb{R}/\alpha^* \cong \mathbb{R}\)) with the basis \(\{\tilde{a}\}\).

Moreover, \(b \in [b, d]\) and \(c \in [d, c]\) which means \(b, c \in L[1]\). Now since \(a \in [b, c]\) and \((-1 \odot a) \in [c, b]\), we get \(a S_2 (-1 \odot a)\) and hence \(\tilde{a} = \tilde{0}\) where \(\tilde{x} = S_2^*(x)\). Observe that \(d \not\in L[1]\), so \(\tilde{b}, \tilde{c}\) and \(\tilde{d}\) are non-zero. Thus, \(L/S_2^*\) is the 3-dimensional Lie algebra over \(\mathbb{R}\) with the basis \(\{\tilde{b}, \tilde{c}, \tilde{d}\}\) and brackets \([\tilde{b}, \tilde{d}] = \tilde{b}\) and \([\tilde{d}, \tilde{c}] = \tilde{c}\). It is easy to see that \(L/S_2^*\) is solvable of length 2.

Finally, it is not difficult to show that \(S_n = L\) for all \(n \geq 3\) (note that \(L/L^*\) is solvable of length 3). In this example, we have \(L \supseteq S_2 \supseteq A\) and \(\dim(L/A^*) \leq \dim(L/S_2^*) \leq \dim(L/L^*)\).

Let \(L\) be a Lie hyperalgebra such that the fundamental Lie algebra \(L/L^*\) is finite dimensional, and define

\[S = \bigcap_{n \geq 1} S_n^*\]

Since

\[\dim \frac{L}{A^*} \leq \dim \frac{L}{S_2^*} \leq \cdots \leq \dim \frac{L}{S_n^*} \leq \dim \frac{L}{S_{n+1}^*} \leq \cdots \leq \dim \frac{L}{L^*}\]

for all \(n \geq 2\), there exists \(m \in \mathbb{N}\) such that \(\dim(L/S_m^*) = \dim(L/S_{m+i}^*)\) for all \(i \geq 1\). Thus, \(L/S_m^* \cong L/S_{m+i}^*\) (as vector spaces) and \(S_m^* = S_{m+i}^*\) for all \(i \geq 1\), which implies that \(S = S_m^*\) (for instance, in the above example \(m = 3\)). Therefore by Proposition 3.2 and Theorem 3.3 \(S\) is a strongly regular relation on \(L\) and \(L/S\) is a solvable Lie algebra.
Theorem 3.5 Let $L$ be a Lie hyperalgebra such that $L/L^*$ is finite dimensional. Then $S$ is the smallest equivalence relation on $L$ such that $L/S$ is a solvable Lie algebra.

Proof. Let $\xi$ be an equivalence relation on $L$ such that $L/\xi$ is a solvable Lie algebra of length $n$ say, and also $(L/\xi, \oplus)$ is an abelian group. It is easy to see that if $h \in L^{[n]}$, then $\xi(h) \in (L/\xi)^{(n)} = 0_{L/\xi}$. Now if $x \in S_{n+1}$, then using the notations of Definition 3.1, we have $x = \sum_{i=1}^{t} \ell_i$ and $y = \sum_{i=1}^{t} \ell_{\sigma(i)}$. Since $(L/\xi, \oplus)$ is abelian, we get $\xi(x) = \oplus_{i \in I} \xi(\ell_i) = \xi(y)$ where $I \subseteq \{1, \ldots, t\}$ such that none of the elements $h_{i_1}, h_{i_2}, \ldots, h_{i_m}$ belong to $L^{[n]}$ for every $i \in I$. Therefore, $x \in S_{n+1}$ and so $S_{n+1} \subseteq \xi$ which implies that $S \subseteq \xi$. This completes the proof. $\Box$

A non-abelian Lie algebra $L$ is called simple, if it contains no ideals other than 0 and $L$. Also, a Lie algebra $L$ is said to be perfect, if $L = L^{(1)}$. It is easy to see that every simple Lie algebra is perfect (see [12]).

Proposition 3.6 Let $L$ be a perfect Lie algebra. Then $L/S = 0$, $L/L \cong L$ and

$$\mathcal{L} = \{(x, x) | x \in L\} \subseteq S = L \times L.$$ 

Proof. Clearly, for every Lie algebra (trivial Lie hyperalgebra) $L$, we have $\mathcal{L} = \mathcal{L}^* = \{(x, x) | x \in L\}$ (the diagonal relation on $L$) and $L/\mathcal{L} \cong L$. Now, let $a \in L$. Since $L$ is perfect, we have $L = L^{(n)}$ for all $n \geq 1$. Without loss of generality, one may suppose that $a = [x, y]$ where $x, y \in L^{[n-1]}$. Hence $a = [x, y]$ and $(-1 \circ a) = [y, x] \in [y, x]$. Thus $a \in S_n$ (where $\mathcal{L}$) and $[x, y] = 0$ where $\bar{a} = S_n^*(a)$. Therefore $L/S_n^* = 0$ and $S_n = L \times L$ for all $n \geq 1$, which completes the proof. $\Box$

Example 3.7 Let $L$ be the 3-dimensional Lie algebra with the basis $\{a, b, c\}$ and non-zero Lie brackets $[a, b] = c$, $[b, c] = a$ and $[c, a] = b$. Clearly, $L$ is simple (perfect) and $S_n = A$ for all $n \geq 1$. One can easily check that $L/S = 0$ and $L/\mathcal{L} \cong L$.

Remark 3.8 Using a similar argument as in Example 3.4, one may show that for a finite dimensional Lie algebra $L$, we have

$$\dim(L/S_n^*) = \dim L - \dim L^{(n)}.$$ 

Now, let $m$ be the smallest positive integer such that $L^{(m)} = L^{(m+i)} \forall i \geq 1$. Then $\dim(L/S) = \dim L - \dim L^{(m)}$. In particular if $L$ is solvable, then $L/S \cong L$ (in this case $S = \mathcal{L}$).

In what follows, we discuss the transitivity conditions of $S_n$ (for a fixed $n \geq 1$). Under the notations of Definition 3.1, a non-empty subset $K$ of a Lie hyperalgebra $L$ (over an arbitrary hyperfield $F$) is said to be an $S_n$-part of $L$ if for every $t \in \mathbb{N}$, every $\sigma \in S_t$, $\bar{K}$
every $\ell_1, \ldots, \ell_t$ such that $\ell_i = f_i(g_{i1}, g_{i2}, \ldots, g_{imi})$ with $g_{ij} = \left( \sum_{k=1}^{p_{ij}} \left( \prod_{r=1}^{n} \lambda_{ijk}^r \right) \right) \cdot h_{ij}$, and every $\sigma_i \in S_m, n$ such that $\sigma_i(j) = j$ if $h_{ij} \not\in L^{[n-1]}$, we have

$$\sum_{i=1}^{t} \ell_i \cap K \neq \emptyset \implies \sum_{i=1}^{t} \ell'_i \subseteq K,$$

where $\ell'_i$ is that given in Definition 3.1

**Example 3.9** In Example 3.4 it is easy to see that the vector subspace $\langle a \rangle$ is an $S_n$-part of $L$ for all $n \geq 1$. Moreover, the singleton $\{a\}$ is an $S_n$-part ($L$-part) of $L$ only for $n \geq 3$, since $[b, c] \cap \{a\} \neq \emptyset$ and $b, c \in L^{[1]}$ but $[c, b] \not\subseteq \{a\}$. Also, $\{b\}$ is an $S_n$-part of $L$ for all $n \geq 2$, since $d \not\in L^{[1]}$.

**Lemma 3.10** Let $K$ be a non-empty subset of a Lie hyperalgebra $L$ and $x, y \in L$. Then

(i) $K$ is an $S_n$-part of $L$, if and only if (ii) $x \in K$ and $x S_n y$ implies $y \in K$, if and only if (iii) $x \in K$ and $x S_n^* y$ implies $y \in K$.

**Proof.** (i) $\Rightarrow$ (ii) Let $x \in K$ and $x S_n y$. Then there exist $\ell_1, \ldots, \ell_t$ such that $x \in \sum_{i=1}^{t} \ell_i$ and $y \in \sum_{i=1}^{t} \ell'_i$. Thus $x \in \sum_{i=1}^{n} \ell_i \cap K$ and by (i) we obtain $y \in K$.

(ii) $\Rightarrow$ (iii) Let $x \in K$ and $x S_n^* y$. Then there exist $m \in \mathbb{N}$ and $x = z_1, z_2, \ldots, z_m = y \in L$ such that $x = z_1 S_n z_2 S_n \ldots S_n z_m = y$. Since $x \in K$, by (ii) we have $z_2 \in K$. Repeating this process ($m - 1$ times) yields $y \in K$.

(iii) $\Rightarrow$ (i) Assume that $\sum_{i=1}^{t} \ell_i \cap K \neq \emptyset$. Then there exists $x \in \sum_{i=1}^{t} \ell_i \cap K$. Suppose that $y \in \sum_{i=1}^{t} \ell'_i$. Thus $x S_n^* y$ and by (iii), $y \in K$ which implies that $\sum_{i=1}^{t} \ell'_i \subseteq K$.

Under the notations of Definition 3.1 consider the sets $T_t(x) = \{ (\ell_1, \ldots, \ell_t) | x \in \sum_{i=1}^{t} \ell_i \}$ such that $\ell_i = f_i(g_{i1}, g_{i2}, \ldots, g_{imi})$ with $g_{ij} = \left( \sum_{k=1}^{p_{ij}} \left( \prod_{r=1}^{n} \lambda_{ijk}^r \right) \right) \cdot h_{ij}$, $P_t(x) = \bigcup_{t \geq 1} \left\{ \sum_{i=1}^{t} \ell_i | (\ell_1, \ldots, \ell_t) \in T_t(x) \right\}$ and $P(x) = \bigcup_{t \geq 1} P_t(x)$.

**Remark 3.11** It is easy to check that $P(x) = \{ y \in L | x S_n y \}$ for every $x \in L$.

In the following theorem, we give the transitivity conditions of $S_n$.

**Theorem 3.12** Let $L$ be a Lie hyperalgebra over a hyperfield $F$. Then (i) $S_n$ is transitive, if and only if (ii) $S_n^*(x) = P(x)$ for every $x \in L$, if and only if (iii) $P(x)$ is an $S_n$-part of $L$ for every $x \in L$.

**Proof.** (i) $\Rightarrow$ (ii) By Remark 3.11 and (i), we trivially have $S_n^*(x) = P(x)$.

(ii) $\Rightarrow$ (iii) Clearly $x \in P(x)$. If $x S_n^* y$, then $y \in S_n^*(x)$ and by (ii), we get $y \in P(x)$.

Now in Lemma 3.10 (iii) $\Rightarrow$ (i), put $K = P(x)$. It implies that $P(x)$ is an $S_n$-part of $L$.

(iii) $\Rightarrow$ (i) Let $x S_n y$ and $y S_n z$. By Remark 3.11 we have $y \in P(x)$ and since $y S_n z$, there exist $t \in \mathbb{N}$ and $\ell_1, \ldots, \ell_t$ such that $y \in \sum_{i=1}^{t} \ell_i \cap P(x)$ and $z \in \sum_{i=1}^{t} \ell'_i$. Now by (iii), we have $\sum_{i=1}^{t} \ell'_i \subseteq P(x)$ and hence $z \in P(x)$. Then by Remark 3.11 we get $x S_n z$ which shows that $S_n$ is transitive. □
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