ON THE SUBALGEBRA OF A FOURIER-STIELTJES ALGEBRA
GENERATED BY PURE POSITIVE DEFINITE FUNCTIONS

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Abstract. For a locally compact group $G$, the first-named author considered the closed subspace $a_0(G)$ which is generated by the pure positive definite functions. In many cases $a_0(G)$ is itself an algebra. We illustrate using Heisenberg groups and the $2 \times 2$ real special linear group, that this is not the case in general. We examine the structures of the algebras thereby created and examine properties related to amenability.

For a locally compact group $G$ let $B(G)$ denote its Fourier-Stieltjes algebra and $A(G)$ its Fourier algebra, as defined in [5]. The first named author ([3]) defined $a_0(G)$ to be the closed linear span in $B(G)$ of the pure positive definite functions, and then let $a(G)$ denote the closed subalgebra in $B(G)$ generated by $a_0(G)$. In the case that $G$ is abelian, and $B(G) = M(\hat{G})$ via Fourier-Stieltjes transform, we have that $a_0(G) = a(G) \cong \ell^1(G)$, where the latter is the closed subspace (algebra) generated by Dirac measures. Thus we think of the space $a_0(G)$, and the algebra $a(G)$, as dual analogues of $\ell^1(G)$.

We use the notation and many results from [1]. We let for a continuous unitary representation $\pi: G \to U(H_\pi)$

$$A_\pi = \text{span}\{s \mapsto \langle \pi(s)\xi|\eta \rangle : \xi, \eta \in H_\pi\}^{\| . \|_B}.$$  

We also use the facts that $A_\pi = A_{\pi'}$ if and only if $\pi \simeq \pi'$, i.e. the representations are quasi-equivalent ([1 (3.1)]); if $\pi$ and $\sigma$ are disjoint, i.e. they share no equivalent sub-representations, then $A_{\pi \oplus \sigma} = A_\pi \oplus_{\ell^1} A_\sigma$ ([1 (3.13)]); and $\text{span} A_\pi A_\sigma \| . \|_B = A_{\pi \otimes \sigma}$ ([1 (3.25)]). We define for any family of representations $\Sigma$, $A_\Sigma = \sum_{\sigma \in \Sigma} A_\sigma \| . \|_B$. Thus if this family of representations is pairwise disjoint we have

$$A_\Sigma = \ell^1 \bigoplus_{\sigma \in \Sigma} A_\sigma.$$  

Thus if $\hat{G}$ denotes the set of irreducible representations, i.e. a full set of representatives, one from each unitary equivalence class, we have

$$A_{\hat{G}} = a_0(G).$$

This can be easily seen from the fact that any pure positive definite function belongs to some $A_\pi$, with $\pi$ in $\hat{G}$. This is thanks to the Gelfand-Naimark-Segal construction characterising pure positive definite functions, and the fact that each
\[ A_\pi = \text{span}\{ s \mapsto \langle \pi(s)\xi : \xi \in \mathcal{H}_\pi \} \}. \] If \( \hat{G}_F \) denotes the family of finite dimensional irreducible representations we let

\[ A_F(G) = A_{\hat{G}_F}. \]

If \( \gamma_{ap} : G \to G^\text{ap} \) denotes the almost periodic compactification, we have \( A_F(G) = A(G^\text{ap}) \circ \gamma_{ap} \) ([5] (2.97)).

In [3] several cases were examined in which \( a_0(G) = a(G) \). This holds for Moore groups (i.e. those groups whose irreducible representations are each finite dimensional), in which case \( a_0(G) = a(G) = A_F(G) \). Also, for the \( ax+b \)-group \( G \), we have \( a_0(G) = A(G) \oplus \ell_1 \mathcal{A} = a_0(G) \). Our goal is to investigate \( a(G) \) for some cases where \( a(G) \neq a_0(G) \), and learn about the structures of these algebras. The two examples considered in the sequel exhibit some features in common, though quite different structures in terms of operator amenability theory. See the survey [18] for more context on amenability properties of Fourier and Fourier-Stieltjes algebras.

1. **Heisenberg groups**

We let

\[ \mathbb{H}_n = \{ (p, q, t) : p, q \in \mathbb{R}^n, t \in \mathbb{R} \} \]

be the Heisenberg group of dimension \( 2n + 1 \) with usual matricial group law

\[ (p, q, t)(p', q', t') = (p + p', q + q', t + p \cdot q' + t') \]

where \( p \cdot q \) is the usual dot product of \( p \) and \( q \). This is also called the “polarized form” in [7]. Notice that the centre \( Z \) of \( \mathbb{H}_n \) is given by

\[ Z = \{(0, 0, t) : t \in \mathbb{R} \} \]

and there is a natural isomorphism

\[ \mathbb{H}_n/Z \cong \mathbb{R}^2. \]

Following [6] we have that the unitary dual is given by

\[ \hat{\mathbb{H}}_n = \{ \rho_h, \chi_{\xi, \eta} : h \in \mathbb{R}^{\mathbb{Z}_0}, \xi, \eta \in \mathbb{R}^n \} \]

where the Schrödinger representations are given by

\[ \rho_h(p, q, t)f(x) = e^{i(ht + pq \cdot x)}f(x + hp), \ f \in L^2(\mathbb{R}^2) \]

(at least up to unitary equivalence) and the finite dimensional irreducible representations are simply the characters

\[ \chi_{\xi, \eta}(p, q, t) = e^{i[p \xi + q \eta]} . \]

**Proposition 1.1.** We have the following tensor product equivalences

\[ \chi_{\xi, \eta} \otimes \rho_h \cong \rho_h \]

\[ \rho_h \otimes \rho_{h'} \simeq \rho_{h + h'}, \text{ if } h + h' \neq 0 \]

\[ \rho_h \otimes \rho_{-h} \cong \lambda_{\mathbb{H}_n/Z} \]

where \( \cong \) is the relation of unitary equivalence and \( \simeq \) that of quasi-equivalence, \( \lambda_{\mathbb{H}_n/Z} \) is the left regular representation of \( \mathbb{H}_n/Z \), and \( q : \mathbb{H}_n \to \mathbb{H}_n/Z \) is the quotient map.
**Proof.** The first two follow, in part, from the Stone-von Neumann theorem:

\[(1.2) \quad \pi \simeq \rho_h \iff \pi(0,0,t) = e^{iht}I.\]

See [6, 7], for example. Thus unitary equivalence in the first tensor product follows from the following: for any group \(G\) if we have \(\chi \otimes \pi \simeq \pi\) for a character \(\chi\) and a representation \(\pi\), then \(\chi \otimes \pi \cong \pi\). Indeed \(\{\chi(g)\pi(g) : g \in G\}\) can admit only those operators which commute with each \(\pi(g)\) as commutators, and hence by Schur’s lemma is irreducible. Two irreducible representations are quasi-equivalent only when they admit non-trivial intertwiners and thus are unitarily equivalent.

Now, let us consider \(\rho_h \otimes \rho_{-h}\). We have for \(f \in L^2(\mathbb{R}^n \times \mathbb{R}^n) \cong L^2(\mathbb{R}^n) \otimes_2 L^2(\mathbb{R}^n)\) that

\[\rho_h \otimes \rho_{-h}(p, q, t)f(x, y) = e^{iqt(x+y)}f(x + hp, y - hp).\]

We let \(W : L^2(\mathbb{R}^n \times \mathbb{R}^n) \to L^2(\mathbb{R}^n \times \mathbb{R}^n)\) be implemented by the orthogonal transformation \((x, y) \mapsto \frac{1}{\sqrt{2}}(x - y, x + y)\) so

\[Wf(x, y) = f\left(\frac{1}{\sqrt{2}}(x - y), \frac{1}{\sqrt{2}}(x + y)\right)\text{ and } W^*f(x, y) = f\left(\frac{1}{\sqrt{2}}(x + y), \frac{1}{\sqrt{2}}(-x + y)\right)\]

We thus have

\[W\rho_h \otimes \rho_{-h}(p, q, t)W^*f(x, y) = [\rho_h \otimes \rho_{-h}(p, q, t)W^*f]\left(\frac{1}{\sqrt{2}}(x - y), \frac{1}{\sqrt{2}}(x + y)\right)\]

\[= e^{i\sqrt{2}t \cdot x}W^*f\left((\frac{1}{\sqrt{2}}(x - y) + hp, \frac{1}{\sqrt{2}}(x + y) - hp\right)\]

\[= e^{i\sqrt{2}t \cdot x}f(x, y - \sqrt{2}hp)\]

\[= (V \otimes I)\lambda_{\mathbb{R}^n \times \mathbb{R}^n}(\sqrt{2q}, \sqrt{2hp})(V^* \otimes I)f(x, y)\]

where \(V : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)\) is the Fourier-Plancherel transform. Now the map \((p, q, t) \mapsto \sqrt{2}(q, hp) : \mathbb{H}_n \to \mathbb{R}^n \times \mathbb{R}^n\) is an open surjective homomorphism with kernel \(\mathbb{Z}\). The unitary equivalence \(\rho_h \otimes \rho_{-h} \cong \lambda_{\mathbb{H}_n/Z}q\) follows. \(\square\)

We are grateful to H.H. Lee for pointing out the role of the Stone-von Neumann theorem, [12] above. K.F. Taylor kindly informs us that the formula for \(\rho_h \otimes \rho_{-h}\) may also be deduced from results of Mackey; details of will be available in his forthcoming book with E. Kaniuth. Our procedure has the benefit of being direct and elementary.

We let \(R = \{\rho_h : h \neq 0\}\). From \([11]\) we see that

\[a_0(\mathbb{H}_n) = A_R \oplus_{\ell^1} A_F(\mathbb{H}_n).\]

The conclusions of the proposition above may be reinterpreted as follows:

\[\chi_{\xi, \eta}A_{\rho_h} = A_{\rho_h}\]

\[(1.3) \quad \text{span}_{\|\cdot\|_B}A_{\rho_h}A_{\rho_{h'}}^{\|\cdot\|_B} = A_{\rho_{h+h'}} \quad \text{if } h + h' \neq 0\]

\[\text{(1.4)} \quad \text{span}_{\|\cdot\|_B}A_{\rho_{-h}}A_{\rho_{-h}}^{\|\cdot\|_B} = A(\mathbb{H}_n/Z)q.\]

In particular \(a_0(\mathbb{H}_n)\) is not an algebra.

**Proposition 1.2.** *The closed algebra generated by \(a_0(\mathbb{H}_n)\) is given by*

\[a(\mathbb{H}_n) = A_R \oplus_{\ell^1} A(\mathbb{H}_n/Z)q \oplus_{\ell^1} A(\mathbb{H}_n/Z^p)q_{ap}.\]
Proof. The multiplication relation (1.2) shows that \( A(\mathbb{H}_n/Z) \circ q \subset a(\mathbb{H}_n) \). Each character \( \chi_{\ell,n} \) clearly multiplies elements of \( A(\mathbb{H}_n/Z) \circ q \), respectively \( A_{\rho_n} \), into the same space. The multiplication relation (1.3) shows that \( A_{R^+} = \ell^1-\bigoplus_{h>0} A_{\rho_h} \) and \( A_{R^-} = \ell^1-\bigoplus_{h>0} A_{\rho_h} \) are subalgebras of \( a(\mathbb{H}_n) \). Finally, we see that \( [A(\mathbb{H}_n/Z) \circ q] A_{\rho_n} \subset A_{\rho_n} \). Indeed, it is immediate from (1.2), applied to \( \rho \), that this representation is quasi-equivalent to \( \rho_h \). Hence \( \text{span} A_{R^+} A_{R^-} \) \( \oplus \) \( \mathbb{H}_n \) \( \oplus \) \( A(\mathbb{H}_n/Z) \circ q \) is an ideal in \( a(\mathbb{H}_n) \).

Our goal is to now understand the ideal \( A_R \oplus \ell^1 \) \( A(\mathbb{H}_n/Z) \circ q \). To this end, let us consider a partial compactification of \( \mathbb{H}_n \). Let

\[
\mathbb{H}_n = \{(p, q, z) : p, q \in \mathbb{R}^n, z \in \mathbb{R}^{ap}\}.
\]

This group has group law

\[
(p, q, z)(p', q', z') = (p + p', q + q', z + q'(p'q')^{-1})z
\]

where \( \gamma : \mathbb{R} \to \mathbb{R}^{ap} \) is the compactification map, and we write the group law on \( \mathbb{R}^{ap} \) multiplicatively. Let \( \tilde{\gamma} : \mathbb{H}_n \to \mathbb{H}_n \) be the homomorphism given by

\[
\tilde{\gamma}(p, q, t) = (p, q, \gamma(t)).
\]

Theorem 1.3. We have

\[
A_R \oplus \ell^1 A(\mathbb{H}_n/Z) \circ q = A(\mathbb{H}_n) \circ \tilde{\gamma}.
\]

In particular, this algebra has Gelfand spectrum isomorphic to \( \mathbb{H}_n \).

Proof. We begin by noting that the Haar measure on \( \mathbb{H}_n \) is the product measure \( m_n \times m_n \times \mu \) where \( m_n \) is the Lebesgue measure and \( \mu \) is the Haar measure on \( \mathbb{R}^{ap} \); indeed

\[
\int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{ap}} \varphi(p + p', q + q', z\gamma(p'q')z') \, dp \, dq \, dz = \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{ap}} \varphi(p, q, z) \, dp \, dq \, dz
\]

for compactly supported continuous \( \varphi : \mathbb{H}_n \to \mathbb{C} \). Thus we obtain a decomposition

\[
L^2(\mathbb{H}_n) \cong L^2(\mathbb{R}^{2n}) \otimes_2 \left( \ell^2 \bigoplus_{h \in \mathbb{R}} \mathbb{C} \chi_h \right)
\]

where \( \chi_h \) is the character on \( \mathbb{R}^{ap} \) given by \( h \) in \( \mathbb{R} \) \( \cong \mathbb{R}^{ap} \). We let

\[
\mathcal{H}_h = L^2(\mathbb{R}^{2n}) \otimes_2 \mathbb{C} \chi_h \cong L^2(\mathbb{R}^{2n})
\]

so for \( f \) in \( \mathcal{H}_h \), \( f(x, y, zz') = \chi_h(z)f(x, y, z') \). We compute for such \( f \) the left regular representation

\[
\lambda_{\mathbb{H}_n}(p, q, z)f(x, y, z') = f((-p, -q, \gamma(p\cdot q)z^{-1})(x, y, z'))
\]

\[
= f(x - p, -q, \gamma(p\cdot (q - y))z^{-1}z')
\]

\[
= e^{ip\cdot (q - y)\gamma(t)} \chi_{-h}(z)f(x - p, q - y, z')
\]

We immediately observe that

\[
\lambda_{\mathbb{H}_n}(\cdot)|_{\mathcal{H}_h} \circ \tilde{\gamma}(0, 0, t) = \chi_{-h} \circ \gamma(t)I = e^{-it}I.
\]

Thus it follows (1.2) that \( \lambda_{\mathbb{H}_n}(\cdot)|_{\mathcal{H}_h} \circ \tilde{\gamma} = \rho_{-h} \) for \( h \neq 0 \). We also immediately see that \( \lambda_{\mathbb{H}_n}|_{\mathcal{H}_h} \circ \gamma \cong \lambda_{\mathbb{H}_n/Z} \circ q \). (Observe that \( \mathcal{H}_h \) is the only component of \( L^2(\mathbb{H}_n) \) which is not annihilated by averaging over the centre \( Z \) of \( \mathbb{H}_n \).)
The identification of the algebras is immediate. The identification of the spectrum follows \cite{5} (3.34)].

Any irreducible representation of $\mathbb{H}_n$ must also be an irreducible representation of $\mathbb{H}_n$. Hence it is immediate that $a(\mathbb{H}_n) = a(\mathbb{H}_n) \circ \gamma \cong a(\mathbb{H}_n)$. This is despite that $\mathbb{H}_n \not\cong \mathbb{H}_n$. Of course, a similar phenomenon may be observed with any non-compact Moore group $G$: $a(G) = A(G^{ap}) \circ \gamma_{ap} \cong a(G^{ap})$.

The spine $A^*(G)$ for a locally compact group $G$ is defined in \cite{11}. It may be given as

$$A^*(G) = \sum_{(\gamma,H)} A(H) \circ \eta$$

where the sum runs over all pairs where $\eta: G \to H$ is a continuous homomorphism into a locally compact group with dense range. Details as to why this sum can be determined as a sum over an index set are given in \cite{11}.

**Corollary 1.4.** (i) We have $a(\mathbb{H}_n) \subset A^*(\mathbb{H}_n)$.

(ii) The algebra $a(\mathbb{H}_n)$ is operator amenable, but not amenable.

**Proof.** We have $a(\mathbb{H}_n) = A(\mathbb{H}) \circ \gamma \oplus A(H_n^{ap}) \circ \gamma_{ap}$, which clearly gives (i). That $a(\mathbb{H}_n)$ is operator amenable is an immediate consequence of \cite{16} Prop. 3.1. We observe that $a(\mathbb{H}_n)$ admits $a(\mathbb{H}) \circ \gamma \cong a(\mathbb{H}_n)$ as a complemented ideal. This ideal is not amenable thanks to \cite{8} Thm. 2.3] of \cite{15} Cor. 3.3. Thus by \cite{13} 2.3.7, $a(\mathbb{H}_n)$ is not amenable. \hfill \Box

Motivated by all of the examples we have thus far, we are emboldened to suggest the following. We let $\nu n(G) = L^\infty - \bigoplus_{\pi \in \hat{G}} \mathcal{B}(\mathcal{H}_\pi)$, which is a von Neumann algebra and the dual of $a(G)$. We also refer to \cite{2} \cite{11} \cite{19} for information on topological Clifford semigroups.

**Conjecture 1.5.** (i) For any locally compact group $G$, there is an injective continuous homomorphism into a locally compact group with dense range, $\gamma: G \to H$, such that $A(H) \circ \gamma$ is contained in $a(G)$ and is an ideal in $a(G)$.

(ii) The Gelfand spectrum $\Phi_{a(G)} \subset \nu n(G)$ is a Clifford semigroup with a dense open subgroup isomorphic to $H$.

Indeed, for Moore groups we use the almost periodic compactification $\gamma_{ap}: G \to G^{ap}$, and $\Phi_{a(G)} \cong G^{ap}$. For $G$ being either of the $ax+b$-group, $SL_2(\mathbb{R})$ (see below), or $\mathbb{H}_n$, we use $id: G \to G$; and $\Phi_{a(G)} = G \sqcup G^{ap}$. For $G = \mathbb{H}_n$ we use $\tilde{\gamma}: \mathbb{H}_n \to \mathbb{H}_n$, and $\Phi_{a(H_n)} = \mathbb{H}_n \sqcup \mathbb{H}_n \sqcup \mathbb{H}_n^{ap}$. The truth of (i), above, would verify a conjecture in \cite{3}, that the invertible part of $\Phi_{a(G)}$ consists of unitaries.

Of course for a discrete non-Moore group, i.e. not Type I (see \cite{12} 12.6.37), we will not be able to calculate $a_0(G)$, nor $a(G)$, in the elementary manner presented here.

2. $SL_2(\mathbb{R})$

We show how results of Repka \cite{13}, on the tensor products of irreducible representations on $SL_2(\mathbb{R})$, give a structure theory for $a(SL_2(\mathbb{R}))$.

We begin by listing all of the families of irreducible unitary representations. Our notation is similar to that of \cite{6} p. 247, with the exception of our parametrisation
of the complementary series. We shall, not need, and thus will not indicate, any of
the actual formulas for the representations themselves.
principal continuous series \[ \Pi^+ = \{ \pi^+_t : t \in [0, \infty) \}, \quad \Pi^- = \{ \pi^-_t : t \in (0, \infty) \} \]
complementary series \[ K = \{ \kappa_s : s \in (-1, 0) \} \]
discrete series \[ \Delta^\pm = \{ \delta^\pm_n : n = 2, 3, 4, \ldots \} \]
mock discrete series \[ M = \{ \delta_1^+, \delta_1^- \}. \]
There is also the trivial representation 1. We will consider two direct integral
of indices \[ k > l \]

\[ \text{Let} \quad u \in a(\text{SL}_2(\mathbb{R})) \]

\[ \text{We record a crude summary of [13], Theorems 4.6, 5.9, 6.4, 7.1, 7.3 and 8.1.} \]

**Lemma 2.1.** Let \( \sigma, \tau \) be any two non-trivial irreducible unitary representations of
\( \text{SL}_2(\mathbb{R}) \). Then we have quasi-containments
\[ \sigma \otimes \tau \prec \begin{cases} 
\pi^+ \otimes \pi^- + \delta \otimes \kappa_{s+t+1} & \text{if } \{ \sigma, \tau \} = \{ \kappa_s, \kappa_t \} \text{ and } s + t < -1 \\
\pi^+ \otimes \pi^- + \delta & \text{otherwise.} 
\end{cases} \]

**Theorem 2.2.** (i) We have
\[ a(\text{SL}_2(\mathbb{R})) = a_0(\text{SL}_2(\mathbb{R})) \oplus_{\ell_1} A_{\pi^+} \oplus_{\ell_1} A_{\pi^-} \]
\[ = A_{\Pi^+} \oplus_{\ell_1} A_{\Pi^-} \oplus_{\ell_1} A_K \oplus_{\ell_1} A_M \oplus_{\ell_1} C1 \oplus_{\ell_1} A(G). \]
(ii) We have
\[ \text{span}_{\ell_1}(\text{SL}_2(\mathbb{R}))^2 \subset A_K \oplus_{\ell_1} C1 \oplus_{\ell_1} A(G). \]

**Proof.** Clearly \( a_0(\text{SL}_2(\mathbb{R})) = A_{\Pi^+} \oplus_{\ell_1} A_{\Pi^-} \oplus_{\ell_1} A_K \oplus_{\ell_1} A_M \oplus_{\ell_1} C1 \oplus_{\ell_1} A_{\Delta^+} \oplus_{\ell_1} A_{\Delta^-} \).
For \( \Sigma, T \) being any of \( \Pi^\pm \), \( M \), \( \Delta^\pm \), Lemma 2.1 tells us that
\[ A_{\Sigma} A_K, \quad A_{\Sigma} A_T \subset A_{\pi^+} \oplus_{\ell_1} A_{\pi^-} \oplus_{\ell_1} A_{\delta} = A(G). \]
whereas
\[ A_K^2 \subset A_{\pi^+} \oplus_{\ell_1} A_{\pi^-} \oplus_{\ell_1} A_{\delta} \oplus_{\ell_1} A_K = A(G) \oplus_{\ell_1} A_K. \]
Hence both (i) and (ii) are immediate. \( \Box \)

**Corollary 2.3.** The Gelfand spectrum of \( a(\text{SL}_2(\mathbb{R})) \) is the one-point compactification, \( \text{SL}_2(\mathbb{R})_\infty \).

**Proof.** We first observe that \( \text{SL}_2(\mathbb{R})_\infty \) is the spectrum of \( A(\text{SL}_2(\mathbb{R})) \oplus C1 \). It can be easily derived from Lemma 2.1 that if \( u = \sum_{k=1}^n u_k \) where \( u_k \in A_{\pi_k} \) for \( \pi_k \in \Pi^+ \cup \Pi^- \cup M \cup K \) then \( u^n \in A(\text{SL}_2(\mathbb{R})) \) for some \( n \). Indeed, if, up to reordering of indices \( k \), we have that \( \pi_k = \kappa_{s_k}, s_1 < s_2 \cdots < s_l \) for some \( l \leq m \), \( \pi_k \notin K \) for \( k > l \), then \( n \leq \log_2(\frac{1}{\pi_{s_1}}) - 1 \). We have \( n = 2 \), otherwise. Hence it follows that for \( u \in a(\text{SL}_2(\mathbb{R})) \) and \( \varepsilon > 0 \) that there is \( v \in a(\text{SL}_2(\mathbb{R})) \) for which \( \| u - v \| < \varepsilon \) and \( v^n \in A(G) \oplus C1 \). \( \Box \)
The corollary above can also be deduced from the main result of [4], and has a similar method of proof.

We observe that \( a(\text{SL}_2(\mathbb{R})) \) admits much weaker amenability properties than does \( a(\mathbb{H}_n) \).

**Corollary 2.4.** The algebra \( a(\text{SL}_2(\mathbb{R})) \) admits no non-zero point derivations, but is not operator weakly amenable.

**Proof.** Since \( A(\text{SL}_2(\mathbb{R})) \) is operator weakly amenable ([17]), it admits no non-zero point derivations. For \( u \in a(\text{SL}_2(\mathbb{R})) \) and \( \varepsilon > 0 \), the proof of the corollary above provides \( \alpha \in \mathbb{C} \) and \( v, w \) in \( a(\text{SL}_2(\mathbb{R})) \) for which \( u = v + w + \alpha 1 \) where \( \|w\| \leq \varepsilon \) and \( u^n \in A(\text{SL}_2(\mathbb{R})) \) for some \( n \). It follows that any point derivation is zero. Theorem 2.2 (ii) shows that \( \text{span}(a(\text{SL}_2(\mathbb{R})))^2 \) is not dense in \( a(\text{SL}_2(\mathbb{R})) \), hence \( a(\text{SL}_2(\mathbb{R})) \) is not operator weakly amenable by [9, Lem. 3.1]. □

In [11] it is computed that \( A^*(\text{SL}_2(\mathbb{R})) = A(\text{SL}_2(\mathbb{R})) \oplus_\ell^1 \mathbb{C} \). We let, for any locally compact group \( G \), the Rajchman algebra \( B_0(G) \) be the subalgebra of \( B(G) \) consisting of those elements vanishing at \( \infty \). It is observed in [4] that \( B(\text{SL}_2(\mathbb{R})) = B_0(\text{SL}_2(\mathbb{R})) \oplus_\ell^1 \mathbb{C} \). For any continuous singular Borel measure \( \mu \) on \((0, \infty)\) we have that \( \pi_\pm^\mu = \int_{(0, \infty)} \pi_\pm^t \, d\mu(t) \) satisfies \( A_{\pm}^+ \cap a(\text{SL}_2(\mathbb{R})) = \{0\} \); indeed see [11, (3.12) & (3.55)]. Hence

\[
A^*(\text{SL}_2(\mathbb{R})) \subsetneq a(\text{SL}_2(\mathbb{R})) \subsetneq B_0(\text{SL}_2(\mathbb{R})) \oplus_\ell^1 \mathbb{C} = B(\text{SL}_2(\mathbb{R})).
\]

**References**

[1] G. Arsac. Sur l’espace de Banach engendré par les coefficients d’une représentation unitaire. *Publ. Dép. Math. (Lyon)* 13 (1976), 1–101.

[2] J.F. Berglund. Compact semitopological inverse Clifford semigroups. *Semigroup Forum* 5:191–215, 1972/73.

[3] Y.-H. Cheng. Subalgebras generated by extreme points in Fourier-Stieltjes algebras of locally compact groups *Studia Math.* 202 (2011), 289–302

[4] M. Cowling. The Fourier-Stieltjes algebra of a semisimple group. *Colloq. Math.* 41 (1970), 89–94.

[5] P. Eymard. L’algèbre de Fourier d’un groupe localement compact. *Bull. Soc. Math. France* 92 (1964), 181–236.

[6] G.B. Folland *A Course in Abstract Harmonic Analysis*. CRC Press, New York, 1995.

[7] G.B. Folland *Harmonic Analysis in Phase Space*. Princeton University Press, Princeton, 1989.

[8] B.E. Forrest and V. Runde. Amenability and weak amenability of the Fourier algebra. *Math. Z.* 250 (2005), 731–744.

[9] B.E. Forrest and P.J. Wood. Cohomology and the operator space structure of the Fourier algebra and its second dual. *Indiana Univ. Math. J.* 50 (2001), 1217–1240.

[10] Harish-Chandra. Plancherel formula for the 2 × 2 real unimodular group. *Proc. Nat. Acad. Sci. U. S. A.*, 38:337–342, 1952

[11] M. Ilić and N. Spronk. The spine of a Fourier-Stieltjes algebra *Proc. Lond. Math. Soc.* (3), 94 (2007), 273–301.

[12] T.W. Palmer. *Banach algebras and the general theory of \( \hat{\mathbb{A}} \)-algebras*. Encyclopedia of Mathematics and its Applications, 79. Cambridge University Press, Cambridge, 2001

[13] J. Repka. Tensor products of unitary representations of \( SL_2(\mathbb{R}) \). *Amer. J. Math.*, 100 (1978), 747–774.

[14] V. Runde. *Lectures on amenability*. Lecture Notes in Mathematics, 1774. Springer-Verlag, Berlin, 2002.

[15] V. Runde. The amenability constant of the Fourier algebra. *Proc. Amer. Math. Soc.* 134 (2006), 1479–1481.
[16] V. Runde and N. Spronk. Operator amenability of Fourier-Stieltjes algebras. II. Bull. Lond. Math. Soc. 39 (2007), 194–202.

[17] N. Spronk. Operator weak amenability of the Fourier algebra. Proc. Amer. Math. Soc. 130 (2002), 3609–3617

[18] N. Spronk. Amenability properties of Fourier algebras and Fourier-Stieltjes algebras: a survey. Banach algebras 2009, 365–383, Banach Center Publ., 91, Polish Acad. Sci. Inst. Math., Warsaw, 2010.

[19] N. Spronk and R. Stokke. Matrix coefficients of unitary representations and associated compactifications. To appear in Indiana Univ. Math. J., accepted in 2012.