CR MANIFOLDS ADMITTING A CR CONTRACTION

KANG-TAE KIM AND JEAN-CHRISTOPHE YOCCOZ

Abstract. We classify the germs of $C^\infty$ CR manifolds that admit a smooth CR contraction. We show that such a CR manifold is embedded into $C^n$ as a real hypersurface defined by a polynomial defining function consisting of monomials whose degrees are completely determined by the extended resonances of the contraction. Furthermore the contraction map extends to a holomorphic contraction that coincides in fact with its polynomial normal form. Consequently, several results concerning Complex and CR geometry are derived.

1. Introduction

1.1. A CR manifold is defined to be a smooth ($C^\infty$) manifold, say $M$, equipped with a CR structure, i.e., a sub-bundle $L$ of its complexified tangent bundle $T_C M$ satisfying two conditions:

(i) $L_p \cap \overline{L}_p = \{0\}$, and

(ii) $[L_1, L_2]$ is a section of $L$ whenever $L_1$ and $L_2$ are.

Such a CR structure is in fact called integrable. But we restrict ourselves in this paper to $C^\infty$ smooth manifolds with an integrable CR structure only.

The CR structure defines a distribution $(p \in M) \mapsto \zeta_p := \text{Re} (L_p \oplus \overline{L}_p)$ with a complex structure that is a smooth assignment $p \in M \mapsto J_p : \zeta_p \to GL(\zeta_p)$ satisfying $J_p \circ J_p = -\text{id}$, where $\text{id}$ represents the identity map. In fact, $J_p$ for each $p$ is the almost complex structure on $\zeta_p$ induced by the action of $C$ on $L$. It is known that the CR structure $L$ is uniquely determined by $J$.

Typical CR manifolds came originally from the smooth boundaries of a domain in $C^n$. In such a case the boundary has real dimension $2n - 1$ and the distribution $\zeta$ consists of $2n - 2$ dimensional hyperplanes. Accordingly a CR manifold is said to be of hypersurface type if $\dim_{\mathbb{R}} M = 2n - 1$ and $\dim_{\mathbb{R}} \zeta_p = 2n - 2$ for every $p \in M$. Again, we restrict ourselves only to the CR manifolds of hypersurface type.

The research of the first named author has been supported in part by the Grant R01-2005-000-10771-0 from the Korea Science and Engineering Foundation.
Thus from here on, by a CR manifold we always mean a CR manifold of hypersurface type.

A CR map is a smooth mapping, say \( f \), of a CR manifold \((M, \zeta, J)\) into another \((N, \xi, \tilde{J})\) satisfying \( df \circ J = \tilde{J} \circ df \).

1.2. The purpose of this article is to study the germ \((M, p)\) of a \( C^{\infty} \) smooth CR manifold \( M \) at a point, say \( p \), that admits a smooth CR contraction \( f : (M, p) \to (M, p) \). Here, a CR map \( f \) is said to be a contraction, if every eigenvalue of \( df_p : T^C_p M \to T^C_p M \) has modulus strictly smaller than 1.

For a germ of a CR manifold, it is quite a strong assumption to admit a CR contraction, and yet there is a reasonably broad collection of examples. Hence it is natural to attempt to classify them; that is exactly the purpose of this article.

The contractions admit a polynomial normal form (up to a conjugation by a smooth diffeomorphism), as is well-known, depending upon the resonances between the eigenvalues. The main results of this paper (cf. Theorem 3.2 for precise presentation) are as follows:

(i) The CR manifold germ admitting a CR contraction is CR equivalent to a CR hypersurface germ in \( \mathbb{C}^n \) (consequently, it is embedded) and the CR contraction extends to a holomorphic contraction.

(ii) The CR equivalence, mentioned in (i), conjugates the CR contraction to a polynomial mapping that is in fact its normal form.

(iii) The defining function of the CR hypersurface germ (the image of the embedding mentioned in (i)) is a weighted homogeneous real-valued polynomial each of whose monomial terms has degrees completely determined by the extended resonances (see Section 2) of the contraction.

1.3. Organization of this paper. We present the contents in the following order.

First we recall several known facts concerning the contractions. Some statements have been modified so that it can fit to the purpose of this paper. However, we point out that there is no novelty by us in this part; we collect them and compose them as an appendix (See Section 5) to this paper for the conveniences of the readers.
Second, we analyze (in Section 2) the real hypersurface germ in \( \mathbb{C}^n \) at the origin that is invariant under a holomorphic contraction.

Third, we show (in Section 3) that any abstract smooth CR manifold germ admitting a CR contraction is CR embedded as a hypersurface in \( \mathbb{C}^n \). In fact we establish the aforementioned main results (i), (ii) and (iii).

Section 4 then lists some of the applications. It may be worth noting that in contrast to the preceding related achievements, this paper takes the existence of a CR contraction as the governing environment. Then we derive the complex and CR geometric consequences. It appeals to us that this change of viewpoint is worthwhile; in particular, our classification of CR manifolds includes even the ones that contain complex lines.

2. **Real hypersurface germ in \( \mathbb{C}^n \) invariant under a holomorphic contraction**

2.1. **A holomorphic contraction.** Let \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0) \) be a local germ of a holomorphic contraction with a fixed point at the origin 0. Denote by \( \lambda_1, \ldots, \lambda_m \) the distinct eigenvalues of the derivative \( Df_0 \) of \( f \) at 0 indexed in such a way that

\[
|\lambda_m| \leq |\lambda_{m-1}| \leq \cdots \leq |\lambda_1| < 1.
\]

Let \( \nu \in \{1, \ldots, m\} \), and let \( E_\nu \) represent the characteristic subspace corresponding to \( \lambda_\nu \). Then it holds obviously that \( \mathbb{C}^n = E_1 \oplus \cdots \oplus E_m \). We shall now use a vector coordinate system \( v_1, \ldots, v_m \) for \( \mathbb{C}^n \) so that each \( v_\nu \) gives a vector coordinate system for \( E_\nu \) for every \( \nu \).

2.2. **Resonances and Extended Resonances.** Let \( \mathbb{N} \) denote the set of positive integers. For each \( \nu \in \{1, \ldots, m\} \) consider the set of resonances

\[
\mathcal{R}_\nu := \{ I = (I_1, \ldots, I_m) \in \mathbb{N}^m \mid \sum I_\ell \geq 2, \lambda_\nu = \lambda^{I_k}, \forall k = 1, \ldots, m \}.
\]

For the purpose of this article, one needs another type of resonances. Let

\[
\mathcal{R}'_\nu = \{(I, I') = (I_1, \ldots, I_m; I'_1, \ldots, I'_m) \in \mathbb{N}^m \times \mathbb{N}^m \mid \sum I_\ell + \sum I'_\ell \geq 2, \lambda_\nu = \lambda^{I+I'} \}
\]

and call it the set of extended resonances.
Remark 2.1. (1) The resonance set $\mathcal{R}_j$ and the generalized resonance set $\mathcal{R}'_j$ are finite.

(2) If $J \in \mathcal{R}_j$, then $J_k = 0$ for $k \geq j$. Similarly if $(J, J')$ belongs to $\mathcal{R}'_j$, then $J_k = 0 = J'_k$ for $k \geq j$.

2.3. Normalization of the Contraction $f$. By the basic theorem on normal forms for contractions (see Proposition 5.1 in §5.2.1), after a holomorphic change of variables, one can write:

$$f(v_1, \ldots, v_m) = (\tilde{v}_1, \ldots, \tilde{v}_m)$$

with

$$\tilde{v}_\nu = \Lambda_\nu v_\nu + \sum_{J \in R_\nu} c_J(v),$$

where:

- $\Lambda_\nu : E_\nu \to E_\nu$ is linear with a single eigenvalue $\lambda_\nu$.
- $c_J : \mathbb{C}^n \to E_\nu$ is a polynomial with values in $E_\nu$, homogeneous of degree $J_k$ in the variable $v_k$, for each $k \in \{1, \ldots, m\}$.

2.4. $Df_0$-invariant hyperplanes. Let $(M, 0)$ represent the germ of smooth real hypersurface in $\mathbb{C}^n$ and $f$ the holomorphic contraction above. We work now with the assumption that $f((M, 0)) \subset (M, 0)$ throughout this section.

The following lemma is immediate:

**Lemma 2.2.** Consider the action of the transposed operator $(Df_0)^*$ on the real-dual $(\mathbb{R}^{2n})^* = E_1^* \oplus \cdots \oplus E_m^*$. If a non-zero dual vector $\varphi \in (\mathbb{R}^{2n})^*$ satisfies that $(Df_0)^*(\varphi) = \lambda \varphi$ for some real number $\lambda$, then there exists an index $i \in \{1, \ldots, m\}$ such that $\lambda = \lambda_i$ and $\varphi \in E_i^*$.

Since $(M, 0)$ is the germ of a smooth real hypersurface through the origin that is invariant under $f$, the (extrinsic) tangent space $T_0M$ is invariant under the action of $Df_0$. Let $\varphi$ be a non-vanishing real linear-form whose kernel is $T_0M$. Then we must have $(Df_0)^*(\varphi) = \lambda \varphi$ with the constant $\lambda$ real.

By Lemma 2.2, there exists $i \in \{1, \ldots, m\}$ with $\lambda = \lambda_i$ and a decomposition $E_i = \mathbb{C} e_i \oplus E'_i$ (with coordinates $v_i = (z_i, v'_i)$) of $E_i$ such that

- $T_0M$ is the real hyperplane defined by $\text{Re} \ z_i = 0$.
- $\Lambda_i v_i = (\lambda_i z_i, \Lambda'_i v'_i + z_i c')$ for some $c' \in E'_i$. 
Here, we write as: \( z_i = x_i + \sqrt{-1} y_i \) with \( x_i, y_i \in \mathbb{R} \), and \( c_I = (c_0^I, c'_I) \) with \( I \in R_i \). This special index \( i \) will be kept separately from the other indices here as well as in what follows.

Now we are ready to present the main result of this section.

**Theorem 2.3.** The germ \((M,0)\) of a smooth real hypersurface at 0 invariant under the action of the contraction \( f \) can be written as

\[
x_i = \sum_{\mathcal{R}_i'} a_{I,I'}(v, \overline{v})
\]

where each \( a_{I,I'} : \mathbb{C}^n \to \mathbb{C} \) is a polynomial, homogeneous of degree \( I_k \) in \( v_k \) and of degree \( I'_k \) in \( \overline{v}_k \) for each \( k \), such that \( a_{I,I'} + a_{I',I} \) is real-valued.

**Remark 2.4.**

1. If \((J,J')\) belongs to \( \mathcal{R}_i' \) with real \( \lambda_i \) above(!), then \((J',J)\) also belongs to \( \mathcal{R}_i' \). Consequently, \( a_{I,I'} + a_{I',I} \) is real-valued, as necessary, although \( a_{I,I'} \) and \( a_{I',I} \) are in general complex-valued.

2. Most of the proof takes place at the formal (i.e., algebraic and combinatorial) level; there is a small argument to take care of a possible flat part, that we will present at the end of this section.

### 2.5. Some preliminary observations

We continue to use the special role of the index \( i \) as above. Now write

\[
v' = (v_1, \ldots, v_{i-1}, v'_i, y_i; v_{i+1}, \ldots, v_m)
\]

so that \( v = (x_i, v') \). The Taylor development of the defining function of \( M \) can be written as

\[
x_i = \sum a_{\alpha,I,I'}(v')
\]

where:

- the sum is over multi-indices \((\alpha, I, I')\) such that \( \alpha + |I| + |I'| \geq 2 \), (where \( |I| \) denotes the sum of indices constituting the multi-index \( I \) for instance), and
- \( a_{\alpha,I,I'} \) is a real-valued polynomial homogeneous of degree \( \alpha \) in \( y_i, I_k \) in \( v_k, I'_k \) in \( \overline{v}_k \) \((k \neq i)\), \( I_i \) in \( v'_i \), and \( I'_i \) in \( \overline{v}'_i \).

**Lemma 2.5.** If \( a_{\alpha,I,I'} \neq 0 \), then \( \alpha = 0 \), \( I_k = I'_k = 0 \) for every \( k \geq i \). In other words \( x_i \), on the level of Taylor series, depends only upon the variables \( v_1, \ldots, v_{i-1} \).
Proof. Transforming $M$ by $f$, one obtains:

\[
\hat{x}_i = \lambda_i x_i + \text{Re} \sum_{I \in \mathcal{R}_i} c_I^0(v') = \sum a_{\alpha, I, I'}(\hat{v'}),
\]

together with

\[
\hat{v'} = (\hat{v}_1, \ldots, \hat{v}_{i-1}, \hat{v}_i', \hat{y}_i, \hat{v}_{i+1}, \ldots, \hat{v}_m)
\]

\[
\hat{v}_i = \Lambda_{\alpha} v_i + \sum_{J \in \mathcal{R}_i} c_J(v)
\]

\[
\hat{y}_i = \lambda_i y_i + \text{Im} \sum_{I \in \mathcal{R}_i} c_I^0(v)
\]

\[
\hat{v}'_j = \Lambda_i' v'_i + c' z_i + \sum_{I \in \mathcal{R}_i} c'_I(v) \quad (j > i).
\]

Suppose, expecting a contradiction, that there exists $(\alpha, I, I')$ with $a_{\alpha, I, I'} \neq 0$ and $\alpha + \sum_{k \geq i} I_k + \sum_{k \geq i'} I'_k > 0$. Amongst such terms $a_{\alpha, I, I'}$, choose one which minimizes the total degree of $a_{\alpha, I, I'} = \alpha + \sum_{k \geq i} I_k + \sum_{\ell} I'_\ell$. Then the invariance of $M$ under the action by $f$ would imply that

\[
\lambda_i = \lambda_i^0 \lambda_{I'}.
\]

But this last identity cannot hold, because we have

\[
|\lambda_j| \leq |\lambda_i| \quad (\forall j \geq i),
\]

\[
\alpha + \sum_{k \geq i} I_k + \sum_{k \geq i'} I'_k \geq 1,
\]

and

\[
\alpha + \sum_{\ell} I_\ell + \sum_{\ell} I'_\ell \geq 2.
\]

This contradiction yields the proof. \(\square\)

We may now express $M$ (on the Taylor series level) by the expression

\[
x_i = \sum a_{I, I'}(v_1, \ldots, v_{i-1}, v_i, \ldots, v_{i-1})
\]
with \( I_\ell = I'_\ell = 0 \) for \( \ell \geq i \). Notice that the invariance under the action of \( f \) yields

\[
\sum a_{I,I'}(v_1, \ldots, v_{i-1}, \overline{v}_1, \ldots, \overline{v}_{i-1}) = \frac{1}{\lambda_i} \left( \sum a_{I,I'}(\hat{v}_1, \ldots, \hat{v}_{i-1}, \overline{v}_1, \ldots, \overline{v}_{i-1}) \right) - \sum \text{Re } c_i^0 \tag{2.1}
\]

The main observation is the following

**Lemma 2.6.** If \( a_{I,I'}(v_1, \ldots, v_{i-1}, \overline{v}_1, \ldots, \overline{v}_{i-1}) \) is homogeneous of degree \((I,I') \in \mathcal{R}_i'\) and if we set

\[
\hat{a}_{I,I'} = a_{I,I'}(\hat{v}_1, \ldots, \hat{v}_{i-1}, \overline{v}_1, \ldots, \overline{v}_{i-1})
\]

(with \( \hat{v}_j = \Lambda_j v_j + \sum_{\mathcal{R}_j} c_j(v) \) as above), then the degree of each non-zero homogeneous component of \( \hat{a}_{I,I'} \) also belongs to \( \mathcal{R}_i' \).

**Proof.** Start with the relation

\[
\lambda_i = \lambda' \lambda'.
\]

When one develops \( \hat{a}_{I,I'} \) into homogeneous components, each \( \lambda_j \) or \( \overline{\lambda}_j \) in the right hand term either remains the same or is replaced by \( \lambda^J \) (respectively \( \overline{\lambda}^J \)) for some \( J \in \mathcal{R}_j \). As \( \lambda_j = \lambda^J \) for \( J \in \mathcal{R}_j \) we still have equality. Thus the corresponding degree belongs to \( \mathcal{R}_i' \). \( \square \)

2.6. **Proof of Theorem 2.3** We first prove that the Taylor series only contains components with degree in \( \mathcal{R}_i' \). If not, take the non-zero component with degree not in \( \mathcal{R}_i' \) and with the smallest total degree. Considering the corresponding components on both sides of (2.1) gives a contradiction.

It is now established that \( M \) can be expressed by

\[
x_i = \sum_{\mathcal{R}_i'} a_{I,I'}(v,v) + \varphi(v')
\]

where \( \varphi \) is \( C^\infty \) and vanishes to infinite order at 0 (which is often called flat at 0).
Assume that there exists $v'_0$ with $\varphi(v'_0) \neq 0$. Observe that, since $M$ is $f$-invariant, the hypersurface

$$M_0 : x_i = \sum_{R_i'} a_{I_i} v_i$$

is also invariant. Let $(x_{i,0}^0, v'_0)$ (resp., $(x_{i,0}, v'_0)$) be the point of $M_0$ (resp., $M$) above $v'_0$. Write, for $\nu \geq 0$,

$$f^\nu(x_{i,0}^0, v'_0) = (x_{i,\nu}, v'_\nu).$$

The coordinate $x_{i,\nu}$ of $M$ that is above $v'_\nu$ is represented by

$$x_{i,\nu} = x_{i,0}^0 + \varphi(v'_\nu).$$

Denote its image under $f$ by $(\tilde{x}_{i,\nu+1}, \tilde{v}'_{(\nu+1)})$; see the drawing below:

Here, the vector joining $(x_{i,\nu+1}, v'_{(\nu+1)})$ to $(\tilde{x}_{i,\nu+1}, \tilde{v}'_{(\nu+1)})$ is (when $\nu$ is sufficiently large) nearly vertical (i.e., nearly parallel to the $x_i$-axis) while the vector joining $(\tilde{x}_{i,\nu+1}, \tilde{v}'_{(\nu+1)})$ to $(x_{i,\nu+1}, v'_{(\nu+1)})$ is nearly horizontal. Therefore we have

$$\lim_{\nu \to +\infty} \frac{\varphi(v'_{(\nu+1)})}{\varphi(v'_\nu)} = \lambda_i.$$  

On the other hand, the size of $v'_\nu$ approaches zero exponentially fast; this is not compatible with the flatness of $\varphi$ at 0. Therefore, $\varphi$ has to be identically zero. This completes the proof. $\square$
2.7. An Example. Consider a holomorphic contraction \( z \mapsto \hat{z} : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0) \) with its linear part

\[
\begin{pmatrix}
\lambda \\
\lambda^2 \\
\lambda^4
\end{pmatrix},
\]

for some real number \( \lambda \) with \( 0 < \lambda < 1 \). The resonance sets are

\[
R_2 = \{(2, 0, 1)\} \quad \text{and} \quad R_3 = \{(4, 0, 1), (2, 1, 1), (0, 2, 1)\}.
\]

Hence in an appropriate coordinate system, \( f \) is represented by

\[
(z_1, z_2, z_3) \mapsto (\hat{z}_1, \hat{z}_2, \hat{z}_3)
\]

with

\[
\begin{align*}
\hat{z}_1 &= \lambda z_1 \\
\hat{z}_2 &= \lambda^2 z_2 + Dz_1^2 \\
\hat{z}_3 &= \lambda^4 z_3 + Az_2^2 + Bz_1^2 z_2 + Cz_1^4.
\end{align*}
\]

The extended resonance set \( R'_3 \) is described as follows (Note here that the extended resonances is written as \((a, a'; b, b'; \ldots)\) instead of the presentation \((a, b, \ldots; a', b', \ldots)\).):

\[
R'_3 = \{(4, 0; 0, 0; 0, 0), (0, 4; 0, 0; 0, 0), (3, 1; 0, 0; 0, 0), (1, 3; 0, 0; 0, 0), (2, 2; 0, 0; 0, 0), (2, 0; 1, 0; 0, 0), (0, 2; 0, 1; 0, 0), (2, 0; 0, 1; 0, 0), (0, 2; 0, 1; 0, 0), (1, 1; 0, 0; 0, 0), (0, 0; 2, 0; 0, 0), (0, 0; 0, 2; 0, 0), (0, 0; 1, 1; 0, 0)\}\]

Now, we are looking only at the invariant hypersurfaces tangent to the real hyperplane represented by the equation \( x_3 = 0 \). The candidates for the invariant hypersurface \( M \) are consequently given by

\[
\begin{align*}
z_3 + \bar{z}_3 &= az_1^4 + \bar{a} \bar{z}_1^4 + bz_1^3 \bar{z}_1 + \bar{b} \bar{z}_1^3 z_1 + cz_1^2 \bar{z}_1^2 \\
&\quad + dz_1^2 z_2 + \bar{d} \bar{z}_1^2 \bar{z}_2 + ez_1^2 \bar{z}_2 + \bar{e} \bar{z}_1^2 z_2 + fz_1 \bar{z}_1 z_2 + \bar{f} z_1 \bar{z}_1 \bar{z}_2 \\
&\quad + gz_2^2 + \bar{g} \bar{z}_2^2 + h z_2 \bar{z}_2
\end{align*}
\]

where \( c, h \) are real-valued.

A direct calculation shows the following:

- If such an example exists, then \( A = 0 \).
• If $D = 0$, then $A = B = C = 0$. In particular, $f$ is linear, and in this case any choices for $a, b, c, d, e, f, g, h$ will define $M$ invariant under the action by the contraction.

• If $D \neq 0$, then the following relations should hold:
  \[ f = h = 0, \quad \text{Re} \, e\bar{D} = 0, \quad B = 2g\lambda^2 D, \quad C = d\lambda^2 D + D^2. \]
  In other words, $d, g, \lambda$ and $D$ determines $B$ and $C$. $e$ just need to satisfy $\text{Re} \, e\bar{D} = 0$.

• If $D \neq 0$, then $M$ contains a non-trivial complex analytic set. For instance with choices $\lambda = 1/2, D = d = f = 1, e = \sqrt{-1}$, it follows that $B = 1/2, C = 5/4$. Then $M$ contains the variety defined by $z_1 = 0, z_3 = z_2^2$. (Compare this with the main theorem of [6].)

3. CR structures admitting a CR contraction

Recall that a CR structure (of hypersurface type) on a real $2n - 1$ dimensional smooth manifold $M$ consists of a smooth hyperplane distribution $\zeta = (\zeta_x)_{x \in M}$ (each $\zeta_x$ is of dimension $2n - 2$) in the tangent bundle $TM$ and an integrable smooth almost complex structure $\tilde{J} = (\tilde{J}_x)_{x \in M}$ on $\zeta$.

Let $(\zeta, \tilde{J})$ be a CR structure in a neighborhood of 0 in $\mathbb{R}^{2n-1}$. Assume that there exists a smooth contraction $\tilde{f}$ at 0 which preserves the CR structure. We will see that after an appropriate smooth change of coordinates, $\zeta, \tilde{J}$ and $\tilde{f}$ can be written in a very simple form, so that in particular $M$ admits a CR embedding into $\mathbb{C}^n$.

The first step is to use a theorem of Catlin ([3]): taking $\mathbb{R}^{2n-1}$ as the hyperplane $\{\text{Re} \, z_n = 0\}$ in $\mathbb{C}^n$, there exists a smooth integrable almost complex structure $J$ on $(\mathbb{C}^n, 0)$ such that the CR structure on $(\mathbb{R}^{2n-1})$ induced by $J$ coincides with $(\zeta, \tilde{J})$ up to a flat error at 0 (i.e., an error vanishing to infinite order at 0).

We actually need slightly more: we want that the contraction $\tilde{f}$ on $(\mathbb{R}^{2n-1}, 0)$ can be extended to a contraction $f$ of $(\mathbb{C}^n, 0)$ which preserves $J$ up to an error which is flat at 0.

The next step is to rectify $J$: there exists a smooth diffeomorphism $\Phi_0$ of $(\mathbb{C}^n, 0)$ such that $\Phi_0^*J$ is the standard complex structure $J_{st}$ on $(\mathbb{C}^n, 0)$. Let $M_0 := \Phi_0^{-1}(\{\text{Re} \, z_n = 0\})$; it is equipped with a CR structure $(\zeta_0, \tilde{J}_0)$ that is the image of $(\zeta, \tilde{J})$ under $\Phi_0^*$, which coincides, up to a flat term, with the structure $(\zeta_{M_0}, \tilde{J}_{M_0})$ induced by $J_{st}$.
Let \( f_0 := \Phi_0^{-1} \circ f \circ \Phi_0 \). This is a local smooth contraction of \((\mathbb{C}^n, 0)\) that preserves \( M_0 \) and \((\zeta_0, \tilde{J}_0)\); it also preserves \( J_{st} \), up to a flat error at 0.

We now use the normal form theorem (see §5.2.1) for smooth contractions (on the finite jet level only): given any \( k \geq 1 \), there exists a local holomorphic diffeomorphism \( \Phi_1 \) of \((\mathbb{C}^n, 0)\) such that the local contraction \( f_1 := \Phi_1^{-1} \circ f \circ \Phi_1 \) has Taylor expansion of order \( k \) consisting only of resonant terms. Take \( k \) so large that all resonant terms have order strictly smaller than \( k - 1 \).

Let \( \tilde{f}_1 \) be the polynomial diffeomorphism of \( \mathbb{C}^n \) (of degree \( \ll k \)) given by the Taylor expansion of \( f_1 \) of order \( k \). Let \( M_1 = \Phi_1^{-1}(M_0) \) and let it be equipped with \((\zeta_1, \tilde{J}_1) := \Phi_1^*(\zeta_0, \tilde{J}_0)\). The hypersurface \( M_1 \) is invariant under \( f_1 \), and the restriction of \( f_1 \) to \( M_1 \) preserves \((\zeta_1, \tilde{J}_1)\).

Observe that the \( k \)-jet of \( M_1 \) at 0 is therefore invariant under \( \tilde{f}_1 \). By the arguments on the hypersurfaces invariant under holomorphic contractions in the proof of Theorem 2.3, the \( k \)-jet of \( M_1 \) only contains the resonant terms (of degree \( \ll k \)). This \( k \)-jet defines a polynomial hypersurface \( \tilde{M}_1 \) which is invariant under \( \tilde{f}_1 \), and consequently also invariant under \( f_1 \) up to a finite order \( \geq k \).

Let \((\zeta_1, \tilde{J}_1)\) be the canonical CR structure on \( \tilde{M}_1 \); let \( \Phi_2 \) be a smooth diffeomorphism of \((\mathbb{C}^n, 0)\) which satisfies \( \Phi_2(M_1) = \tilde{M}_1 \) and coincides with the identity up to an order \( \geq k \) at 0. Then \( \Phi_2^*(\zeta_1, \tilde{J}_1) = (\zeta_2, \tilde{J}_2) \) is a CR structure on \( M_1 \) which coincides with \((\zeta_1, \tilde{J}_1)\) up to an order \( \geq k \).

It is invariant under \( f_2 := \Phi_2^{-1} \circ \tilde{f}_1 \circ \Phi_2 \) (because \((\zeta_1, \tilde{J}_1)\) is invariant under \( \tilde{f}_1 \)). Moreover, \( f_1 \) and \( f_2 \) coincide up to an order \( \geq k \).

The following commutative diagram summarizes what we have done so far:

\[
\begin{array}{ccccccc}
\mathbb{R}^{2n-1} & \Phi_0 & \mathbb{C}^{n-1} & \Phi_1 & \mathbb{C}^{n-1} & \Phi_2 & \mathbb{C}^{n-1} \\
(\zeta, \tilde{J}) & \circlearrowleft & (\zeta_1, \tilde{J}_1) & \circlearrowleft & (\zeta_2, \tilde{J}_2) & \circlearrowleft & (\zeta_3, \tilde{J}_3) \\
\downarrow \tilde{f} & \downarrow f_0 & \downarrow f_1 & \downarrow f_2 & \downarrow \tilde{f}_1 \\
\mathbb{R}^{2n-1} & \Phi_0 & \mathbb{C}^{n-1} & \Phi_1 & \mathbb{C}^{n-1} & \Phi_2 & \mathbb{C}^{n-1} \\
(\zeta, \tilde{J}) & \circlearrowleft & (\zeta_1, \tilde{J}_1) & \circlearrowleft & (\zeta_2, \tilde{J}_2) & \circlearrowleft & (\zeta_3, \tilde{J}_3) \\
\end{array}
\]

Here, \((\tilde{M}_1, \zeta_1, \tilde{J}_1)\) is a polynomial model. We now present:
Lemma 3.1. There exists a local diffeomorphism $\Phi$ of $(M_1, 0)$ which coincides with the identity up to terms of order $\geq k$ and satisfies

$$\Phi^*(\zeta_1, \tilde{J}_1) = (\zeta_2, \tilde{J}_2)$$

and

$$f_2 = \Phi^{-1} \circ f_1 \circ \Phi.$$

Proof. The normal form theorem (cf. Proposition 5.1, §5.2.1) for smooth contractions guarantees that there exists a unique local diffeomorphism $\Phi$ of $(M_1, 0)$ which coincides with the identity up to terms of order $\geq k$, and satisfies $f_2 = \Phi^{-1} \circ f_1 \circ \Phi$. Therefore in the lemma we can assume that $f_2 = f_1$, and that $f_1$ (or, equivalently $f_2$) preserves both $(\zeta_1, \tilde{J}_1)$ and $(\zeta_2, \tilde{J}_2)$; these two CR structures coincide up to order $\geq k$ and we have to show that they are in fact equal.

Let us first check that $\zeta_1 = \zeta_2$. Let $x$ be any point close to 0. We have

$$\zeta_\ell(x) = [Df_1^N(x)]^{-1}(\zeta_\ell(f_1^N(x)))$$

for $\ell = 1, 2$. Let $\lambda < 1$ such that $\|f_1(y)\| \leq \lambda \|y\|$ for $y$ close to 0. Then we have

$$\|\zeta_1(f_1^N(x)) - \zeta_2(f_1^N(x))\| \leq C \lambda^{Nk}.$$ 

As $k$ can be arbitrarily large, one can immediately conclude that $\zeta_1(x) = \zeta_2(x)$. The proof that $\tilde{J}_1 = \tilde{J}_2$ is similar: if $v \in \zeta_1(x) = \zeta_2(x)$, then one has

$$\tilde{J}_{\ell,x}(v) = [Df_1^N(x)]^{-1}(\tilde{J}_{1,f_1^N x}(Df_1^N(x)v))$$

with

$$\|\tilde{J}_{1,f_1^N x} - \tilde{J}_{2,f_1^N x}\| \leq C \lambda^{Nk}.$$ 

Hence the assertion of lemma follows. $\square$

As a consequence of the arguments by far, one obtains:

Theorem 3.2. Let $(M, 0)$ be a germ of abstract smooth CR manifold of real dimension $2n - 1$ with the CR dimension $(2n - 2)$. If there exists a smooth CR contraction $f$ at 0, then there exists a smooth CR embedding $\psi : (M, 0) \rightarrow (\mathbb{C}^n, 0)$ such that:

1. the image $\psi(M)$ is the hypersurface defined by a real-valued weighted homogeneous polynomial;
2. $\psi \circ f \circ \psi^{-1}$ coincides with the polynomial normal form for the contraction $f$ at 0 holomorphic in a neighborhood of 0 in $\mathbb{C}^n$; and
(3) the degree of each monomial term in the expression of the polynomial defining function for \( \psi(M) \) is determined completely by the extended resonance set of \( f \).

4. Applications and Remarks

4.1. Strongly pseudoconvex case. Assume that \((M,0)\) is a smooth, strongly pseudoconvex CR manifold of hypersurface type, and that it admits a smooth CR contraction \( f \). Then Theorem \ref{thm:contraction} implies that our \((M,0)\) is CR equivalent to an embedded real hypersurface germ in \( \mathbb{C}^n \) that is invariant under a holomorphic contraction, and furthermore that \((M,0)\) has to be CR equivalent to the strongly pseudoconvex real hypersurface germ represented by

\[
\text{Re } z_n = Q(z_1, \ldots, z_{n-1})
\]

where \( Q \) is a real-valued quadratic polynomial that is positive definite (or, negative definite). Therefore, \((M,0)\) is CR equivalent to the hypersurface represented by

\[
\text{Re } z_n = |z_1|^2 + \cdots + |z_n|^2.
\]

Notice that this yields an alternative proof of the following theorem:

**Theorem 4.1** (Schoen \[12\]). Any smooth strongly pseudoconvex smooth CR manifold of hypersurface type that admits a CR automorphism contracting at a point is locally CR equivalent to the sphere.

4.2. Levi non-degenerate case. In the case when the hypersurface germ \((M,0)\) is such that its Levi form is not strongly pseudoconvex but only non-degenerate, Schoen’s theorem does not apply even when \( M \) is assumed further to be real analytic. For this particular case, Kim and Schmalz have presented the following theorem:

**Theorem 4.2** (\[7\]). If a real analytic, Levi non-degenerate hypersurface germ \((M,0)\) in \( \mathbb{C}^{n+1} \) \( (n \geq 2) \) is invariant under the action of a 1-1 holomorphic mapping that is repelling along the normal direction to \( M \) while contracting in some direction, then \((M,0)\) is biholomorphic to the hyperquadric, defined by

\[
\text{Re } z_0 = |z_1|^2 + \cdots + |z_k|^2 - |z_{k+1}|^2 - \cdots - |z_n|^2.
\]

Then they give an example \( \text{Re } z_3 = \text{Re } z_1 z_2 + |z_1|^4 \) that defines a real analytic, Levi non-degenerate hypersurface that is not biholomorphic to any hyperquadrics, while it is invariant under the contraction \((z_1, z_2, z_3) \mapsto (\lambda, \lambda^3, \lambda^4)\), in order to demonstrate that the contraction is not relevant for their purposes.
On the other hand, if one considers smooth \((C^\infty)\) Levi non-degenerate CR manifold of hypersurface type with a CR contraction, notice that this example of Kim-Schmalz does belong to the realm of Theorems \[2.3\] and \[3.2\]. Notice that, in the case that Levi form is indefinite (even when it is non-degenerate) it is just that the Levi form alone cannot determine the eigenvalues of the contracting CR automorphisms. But if one starts from the given CR contraction, instead of Levi geometric assumptions, Theorem \[3.2\] reduces the case to the embedded smooth CR hypersurface with a holomorphic contraction, denoted by \(f\) again, by an abuse of notation. Then Theorem \[3.2\] implies that the hypersurface germ invariant under the action of \(f\) is equivalent to the hypersurface germ defined by an explicit polynomial defining function with multi-degrees determined completely by the extended resonance set for \(f\). (See the example in \[2.7\] for instance.)

4.3. **On the automorphism groups of bounded domains.** Let us now take a bounded domain \(\Omega\) in \(\mathbb{C}^n\) with a smooth \((C^\infty)\) boundary \(\partial \Omega\). Assume further the following:

1. \(f\) is a holomorphic automorphism of \(\Omega\).
2. There exist a point \(p \in \partial \Omega\) and an open neighborhood \(U\) of \(p\) such that \(f\) extends to a smooth non-singular map of \(U\) satisfying \(f(p) = p\) and contracting at \(p\).

It follows that the iteration of \(f\) in \(U\) forms a sequence that converges uniformly to the constant map with value at \(p\). Then Montel’s theorem yields that the iteration of \(f\) converges uniformly on compact subsets of \(\Omega\) to the same constant map.

Choose a smooth extension of \(f\) in a neighborhood \(U\) of the closure of \(\Omega\). There is a smooth local diffeomorphism \(\psi\) from a neighborhood \(V\), say, of \(p\) onto a neighborhood of 0 such that \(P := \psi \circ f \circ \psi^{-1}\) is a polynomial normal form of \(f\) at 0. Moreover, from Section 5.2.3, \(\psi\) is holomorphic in \(V \cap \Omega\). Let \(U'\) be the basin of attraction of \(p\) for the extension of \(f\); it is an open set which contains \(V\) and \(\Omega\). One extends \(\psi\) to a smooth diffeomorphism (which we still denote by \(\psi\)) from \(U'\) onto its image by using the maps \(P^{-k} \circ \psi \circ f^k\), \(k = 1, 2, \ldots\). Then \(\psi\) still conjugates \(f\) and \(P\), and is holomorphic in \(\Omega\). The image under \(\psi\) of the intersection \(U' \cap \partial \Omega\) is invariant under \(P\), and hence is a hypersurface of the form described in Theorem 2.3. Consequently \(\psi\) gives rise to a biholomorphism between \(\Omega\) and one side of this model hypersurface.
Since $\Omega$ is bounded, one must have $i = n$. On the other hand, this implies immediately that $\text{Aut}(\widehat{\Omega})$ admits free translation along $\text{Im } z_n$-direction, as well as the linear contractions. This yields the following:

**Corollary 4.3.** Let $\Omega$ be a bounded domain with a smooth boundary. If there exists $f \in \text{Aut}(\Omega) \cap \text{Diff}(\Omega)$ that is contracting at a boundary point, then $\Omega$ is biholomorphic to the domain $\widehat{\Omega}$ defined by a weighted homogeneous polynomial defining function. In particular, the holomorphic automorphism group contains two dimensional group of complex affine maps that are dilations followed by translations.

Notice that this is more general than the main results of [6], in the sense that we do not put any restrictions on the boundary except the smoothness. The first named author was recently informed that a similar results as above (but on the domains with D’Angelo finite type boundary) has been obtained by S.-Y. Kim[8] from a different argument, generalizing the main result of [6].

On the other hand our Theorem 3.2 also yields the following corollary:

**Corollary 4.4.** Suppose that $\Omega$ is a bounded domain in $\mathbb{C}^n$ with a smooth boundary. If there exists $f \in \text{Aut}(\Omega) \cap \text{Diff}(\Omega)$ that is contracting at a boundary point $p$, then $\partial \Omega$ at $p$ is of finite type in the sense of D’Angelo ([4]).

**Proof.** Note that it suffices to show that the normal form $\partial \widehat{\Omega}$ at the origin is of finite type in the sense of D’Angelo. Since this boundary surface is defined by a real-valued polynomial, notice further that it suffices to show that there is no non-trivial complex analytic variety passing through the origin. (See [5].)

Recall that $\partial \widehat{\Omega}$ is defined by the equation

$$\text{Re } z_n = \rho(z_1, \ldots, z_{n-1})$$

where $\rho$ is the real-valued polynomial consisting of its monomial terms with degrees completely controlled by the set of extended resonances of $f$. Consequently there exist positive integers $1 \leq r_2 \leq \cdots \leq r_n$ such that $\partial \widehat{\Omega}$ is invariant under the action of the contraction

$$P_\lambda := \begin{pmatrix} \lambda^{r_2} & & \\ & \ddots & \\ & & \lambda^{r_n} \end{pmatrix}$$

for every $\lambda$ with $0 < \lambda < 1$. 

Were there a non-trivial variety in $\partial \hat{\Omega}$ passing through 0, then one has a non-constant holomorphic map $\varphi : D \rightarrow \partial \hat{\Omega}$ of the open unit disc $D$ such that $\varphi(0) = 0$.

Since $\partial \hat{\Omega} \subset \mathbb{C}^n$, we may write $\varphi(\zeta) = (\varphi_1(\zeta), \ldots, \varphi_n(\zeta))$. For each $\mu \in \{1, \ldots, n\}$, consider the Taylor developments of $\varphi_\mu(\zeta)$ at the origin. Let $d_\mu$ be the lowest degree term that does not vanish. (Set $d_\mu = +\infty$ if $\varphi_\mu$ is identically zero.) Since $\varphi$ is not identically zero, not every $d_\mu$ can be infinite.

Now, consider the index $\hat{\ell} \in \{1, \ldots, n\}$ such that
\[ \frac{d_{\hat{\ell}}}{r_{\hat{\ell}}} = \min_{1 \leq \ell \leq n} \frac{d_\ell}{r_\ell}. \]

Let $R > 1$ be arbitrarily given. Then let $\lambda = R^{-d_{\hat{\ell}}/r_{\hat{\ell}}}$. Then we consider the sequence of maps
\[ h_R(\zeta) := P_\lambda^{-1} \circ \varphi \left( \frac{\zeta}{R} \right). \]

It is a simple matter to observe that $\lim_{R \to \infty} h_R(\zeta)$ defines a non-constant entire curve with its image contained in $\partial \hat{\Omega}$.

But then, because the surface $\partial \hat{\Omega}$ is defined by the equation
\[ \Re z_n = \rho(z_1, \ldots, z_{n-1}), \]

one may add a real number to the last coordinate of the entire curve generated above to obtain a non-constant entire curve contained in $\hat{\Omega}$. But this is again impossible because $\hat{\Omega}$ is biholomorphic to the bounded domain $\Omega$. Thus the assertion follows.

Notice that this gives an affirmative answer to a special case of the Greene-Krantz conjecture in several complex variables, which says that, for a bounded pseudoconvex domain with smooth boundary, any boundary point at which an automorphism orbit accumulates is of finite type in the sense of D’Angelo.

As a passing remark, we would like to mention a naturally arising question: when is the domain defined by such special real-valued polynomial defining function biholomorphic to a bounded domain? Considering that it is relatively rare for a basin of attraction to be bounded, this question may be of an interest.
4.4. **A trivial remark.** It seems reasonable to add an obvious remark concerning the point-wise attraction versus contraction. Even though the domain $\Omega$ admits an automorphism $\varphi$ and a boundary point $p \in \partial \Omega$ such that $\lim_{\nu \to \infty} \varphi^\nu(q) = p$ for every $q \in \Omega$ (point-wise attraction property) with the extra property that it has a smooth extension to the boundary near $q$ fixing $q$, this extension may not in general be a contraction at $p$. For each $t \in \mathbb{R}$, the holomorphic automorphism $z \mapsto t + z$ of the upper half plane in $\mathbb{C}$ with the unique fixed point at $\infty$. This gives rise to an automorphism of the unit open ball in $\mathbb{C}^2$ with expression

$$(z, w) \mapsto \left( c_t \cdot \frac{z - \alpha_t}{1 - \alpha_t z}, \frac{\sqrt{1 - |\alpha_t|^2}}{1 - \alpha_t z} w \right)$$

where $c_t = (2i - t)/(2i + t)$ and $\alpha_t = t/(t - 2i)$. Notice that both eigenvalues of its derivative at the fixed point are 1. (This is a typical example of a parabolic map in complex dimension 2.)

5. **Appendix: Normal forms for smooth Contractions**

We recall some standard facts about smooth contractions, which have been well-known for around one century. The exposition here follows that of [11]. A simple fact about the holomorphicity of the conjugacy to normal form is also proved at the end.

5.1. **Jets.**

5.1.1. Let $E$ be a real finite dimensional vector space. Denote by $D(E, 0)$ the group of germs of $C^\infty$ diffeomorphisms of $E$ at 0 which fix 0. For $n \geq 0$, denote by $D_n$ the normal subgroup of $D(E, 0)$ whose elements have contact of order $> n$ with the identity map $id_E$. Namely we have

$$f(x) = x + o(\|x\|^n),$$

for every $f \in D_n$.

Let $D_\infty := \bigcup_{n \geq 0} D_n$. For $0 \leq n \leq +\infty$, the group of $n$-jets $J_n(E)$ is the quotient group $D(E, 0)/D_n$. It is obvious that $J_0(E) = \{1\}$, $J_1(E) = GL(E)$. For $n \geq 1$, $J_{n+1}(E)$ is obtained from $J_n(E)$ by an abelian extension

$$1 \to D_n/D_{n+1} \to J_{n+1}(E) \to J_n(E) \to 1,$$

where $D_n/D_{n+1}$ is canonically identified with the polynomial maps from $E$ to $E$ homogeneous of degree $n+1$. The groups $J_n(E)$, $n < +\infty$, are Lie groups, and $J_\infty(E)$ is the projective limit of the sequence $(J_n(E))_n$. 
5.1.2. A jet \( j \) in the Lie group \( J_n(E) \) is semi-simple if and only if it is conjugated to the \( n \)-jet of a semi-simple linear automorphism of \( E \); it is unipotent if and only if its image \( J_1(E) = GL(E) \) is a unipotent linear automorphism of \( E \). Writing an arbitrarily given jet as the product of its semi-simple and unipotent components and going to the projective limit, we obtain the following:

For any \( F \in D(E,0) \), one can find \( H \in D(E,0), F_1 \in D(E,0), A \in GL(E) \) such that

(i) \( H \circ F \circ H^{-1} = F_1 \circ A \);
(ii) \( A \) is semi-simple;
(iii) \( DF_1(0) \) is unipotent;
(iv) \( F_1 \circ A - A \circ F_1 \) is flat at 0.

5.1.3. Let us describe the centralizer \( Z(A) \) of a semi-simple linear automorphism \( A \) in the group \( J_{\infty}(E) \), in the case where \( A \) is a contraction.

We first complexify the objects: let \( E_c, A_c, J_{\infty}(E_c) \) be these complexifications and let \( Z_c(A) \) be the centralizer of \( A_c \) in \( J_{\infty}(E_c) \). Let \( \lambda_1, \ldots, \lambda_m \) be the distinct eigenvalues of \( A_c \), and let \( E_1, \ldots, E_m \) be the corresponding eigenspaces. As \( A_c \) is semi-simple, it follows that \( E_c = \bigoplus E_\nu \). For \( \nu \in \{1, \ldots, m\} \) let

\[
R_\nu = \{ I = (I_1, \ldots, I_m) \in \mathbb{C}^m \mid \sum I_\ell \geq 2, \lambda_\nu = \lambda^I \}
\]

be the set of resonances. A jet \( g \in J_{\infty}(E_c) \) with \( g(v_1, \ldots, v_m) = (g_1(v_1, \ldots, v_m), \ldots, g_m(v_1, \ldots, v_m)) \) belongs to \( Z_c(A) \) if and only if it can be written as

\[
g_\nu(v_1, \ldots, v_m) = B_\nu(v_\nu) + \sum_{I \in R_\nu} g_{\nu, I}(v_1, \ldots, v_m)
\]

where:

- \( B_\nu \in GL(E_\nu) \), and
- \( g_{\nu, I} \) is a polynomial map from \( E \) to \( E_\nu \) homogeneous of degree \( I_\ell \) with respect to the vector variable \( v_\ell \) for \( E_\ell \).

Observe that, because \( A \) is a contraction, the sets \( R_\nu \) are finite. It follows that \( Z_c(A) \) is a finite dimensional Lie group of polynomial diffeomorphisms of \( E_c \).

The real centralizer \( Z(A) \) is then the real part of \( Z_c(A) \), i.e., formed by those jets which commute with complex conjugation.
5.1.4. There is a special case which we are interested in: assume that $E$ is even-dimensional and equipped with a complex structure, i.e., a linear automorphism $J$ such that $J^2 = -1_E$. Assume also that $A$ is $J$-linear, meaning that $AJ = JA$. Then the subgroup $Z_J(A)$ formed by those jets $g$ in $Z(A)$ which are $J$-holomorphic has the form described above (where now $\lambda_1, \ldots, \lambda_m$ are the eigenvalues of $A$ understood as a complex automorphism of $(E, J)$, $E_1, \ldots, E_m$ being the $J$-invariant eigenspaces).

5.2. The Normalization Theorem for Contractions.

5.2.1. Let $f \in D(E, 0)$ be the germ of a smooth local contraction of $E$ at 0. Write $A \in GL(E)$ for the semi-simple part of $Df(0)$. According to §5.1.2 above, there exists $h \in D(E, 0)$ such that the $\infty$-jet of $h^{-1} \circ f \circ h$ belongs to $Z(A)$. On the other hand, by §5.1.3, $Z(A)$ is canonically identified to a finite dimensional Lie group of polynomial diffeomorphisms of $E$.

**Proposition 5.1.** The following statements hold:

1. There exists $h \in D(E, 0)$ such that $h^{-1} \circ f \circ h$ belongs to $Z(A)$ as an element of $D(E, 0)$.

2. Let $f_0, f_1 \in Z(A) \subset D(E, 0)$. Assume that both $f_0, f_1$ have semi-simple part equal to $A$. Then, any $h \in D(E, 0)$ such that $h \circ f_0 \circ h^{-1} = f_1$ belongs to $Z(A)$.

For the proof, see [11].

5.2.2. We now make two simple observations in relation to Proposition 5.1, one in this section and then the other in the next.

First, assume that $f_0, f_1$ are two local contractions in $D(E, 0)$ with the same $k$-th jet, where $k$ is large enough so that $Z(A)$ injects into $J_k(E)$. (Here, $A$ is the semi-simple part of $Df_0(0) = Df_1(0)$.) Then, there exists $h \in D(E, 0)$ with trivial $k$-th jet such that $h \circ f_0 \circ h^{-1} = f_1$.

5.2.3. The second observation, which we now give, is more closely and explicitly related to the situation under consideration of this paper.

Assume as in §5.1.4 that $E$ is even dimensional, and is equipped with a complex structure $J$. Let $f \in D(E, 0)$ be a local contraction, and let $A$ the semi-simple part of $Df(0)$. Assume also that the $\infty$-jet of $f$ is $J$-holomorphic (but it is not necessarily a convergent series!). Then there exists $h \in D(E, 0)$ with a $J$-holomorphic $\infty$-jet, such that $h \circ f \circ h^{-1} \in Z_J(A)$.
Assume even further that there exists an open set $U$ (with $0 \in \overline{U}$) such that $f(U) \subset U$ and $f$ is $J$-holomorphic on $U$. Then the conjugating map $h$ above is also $J$-holomorphic on $U$. This can be seen as follows: first conjugate $f$ by a truncation $h_1$ of $h$ at a very high order which is locally a biholomorphism at $0$. We get a new diffeomorphism $f_1 := h_1 \circ f \circ h_1^{-1}$ of the form

$$f_1 = f_0 + o(\|x\|^k), \ k \gg 1$$

where $f_0 \in Z_J(A)$. By the remark made in §5.2.2 one has that

$$f_0 = h_0 \circ f_1 \circ h_0^{-1},$$

$$h_0(x) = x + o(\|x\|^k).$$

Now we are to check that $h_0$ is $J$-holomorphic on the open set $U_1 := h_1(U)$. Notice that $U_1$ is $f_1$-invariant, and on $U_1$ the map $f_1$ is $J$-holomorphic. But for any $x \in U_1$ and $n \geq 0$, it holds that

$$Dh_0(x) = Df_0^n(h_0f_1^n(x)) Dh_0(f_1^n x) Df_1^n(x).$$

In this formula, both $Df_1^n(x)$ and $Df_0^{-n}(h_0f_1^n x)$ are $J$-linear and exponential in $n$. On the other hand, one has

$$Dh_0(f_1^n x) = 1_E + O(\|f_1^n x\|^k).$$

Taking $k$ sufficiently large and letting $n$ go to infinity, one deduces that $Dh_0(x)$ is $J$-linear.

References

[1] M.S. Baouendi, C.H. Chang, F. Trèves, Microlocal hypo-analyticity and extension of CR functions, J. Diff. Geom. 18 (1983), no. 3, 331–391.

[2] D. Catlin, Boundary invariants of pseudoconvex domains, Ann. Math. 120 (1984), 529-586.

[3] D. Catlin, Sufficient conditions for the extension of CR structures, J. Geom. Anal., (4) 4 (1994), 467-538.

[4] J. D’Angelo, Real hypersurfaces, orders of contacts, and applications, Ann. Math. 115 (1982), 615-637.

[5] ______, Several complex variables and the geometry of real hypersurfaces, Studies in Adv. Math., CRC Press, 1993.

[6] K.T. Kim and S.Y. Kim, CR hypersurfaces with a contracting automorphism, J. Geom. Anal. (To appear in 2008)

[7] K.T. Kim and G. Schmalz, Dynamics of local automorphisms of embedded CR manifolds, Mat. Zametki 76 (2004), 477-480; Transl. in Math. Notes 76 (2004), 443-446.

[8] S.Y. Kim, Preprint, 2008.

[9] S.G. Krantz, Convexity in complex analysis, Several Complex Variables and Complex Geometry, Part 1, Proc. Sympos. Pure Math., 52-1, Amer. Math. Soc. (1991), 119-137.
[10] N.G. Kruzhilin and A.V. Loboda, Linearization of local automorphisms of the pseudoconvex surfaces (Russian), Dokl. Acad. Nauk. USSR (Ser. Math.) 271 (1983), 280-282.

[11] J. Palis and J.-C. Yoccoz, Differentiable conjugacies of Morse-Smale diffeomorphisms, Bol. Soc. Bras. Mat., 20-2 (1990), 25-48.

[12] R. Schoen, On the conformal and CR automorphism groups, Geom. Funct. Anal., 5 (1995), 464-481.

[13] A.F. Spiro, Smooth real hypersurfaces in $\mathbb{C}^n$ with a non-compact isotropy subgroup of CR transformations, Geom. Dedicata, 67 (1997), 199-221.

(Kim) Department of Mathematics, Pohang University of Science and Technology, Pohang 790-784 The Republic of Korea
E-mail address: kimkt@postech.ac.kr

(Yoccoz) Collège de France, 3 rue d’Ulm, 75005 Paris, France
E-mail address: jean-c.yoccoz@college-de-france.fr