The Donaldson-Witten function
for gauge groups
of rank larger than one

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We study correlation functions in topologically twisted $\mathcal{N} = 2, d = 4$ supersymmetric
Yang-Mills theory for gauge groups of rank larger than one on compact four-manifolds $X$. We find that the topological invariance of the generator of correlation functions of BRST invariant observables is not spoiled by noncompactness of field space. We show how to express the correlators on simply connected manifolds of $b_{2,+}(X) > 0$ in terms of Seiberg-Witten invariants and the classical cohomology ring of $X$. For manifolds $X$ of simple type and gauge group $SU(N)$ we give explicit expressions of the correlators as a sum over $\mathcal{N} = 1$ vacua. We describe two applications of our expressions, one to superconformal field theory and one to large $N$ expansions of $SU(N) \mathcal{N} = 2, d = 4$ supersymmetric Yang-Mills theory.
1. Introduction and conclusion

The Donaldson invariants of 4-manifolds have played an important role in the development of both mathematics and physics during the past fifteen years. Donaldson’s invariants are defined using nonabelian gauge theory for gauge group $G = SU(2)$ or $G = SO(3)$ on a compact oriented Riemannian 4-manifold $X$ [1][2]. They were interpreted by Witten in [3] as correlation functions in an $\mathcal{N} = 2$ supersymmetric Yang-Mills theory (SYM) and as such are best presented as a function on $H_*(X)$ defined by a path integral:

$$Z_{DW}(v \cdot S + pP) = \left\langle \exp[v \cdot I(S) + pO(P)] \right\rangle$$

(1.1)

where $P \in H_0(X; \mathbb{Z})$, $S \in H_2(X; \mathbb{Z})$, $I(S)$ and $O(P)$ are certain operators in the gauge theory, and the right hand side of (1.1) is an expectation value. We refer to (1.1) as the Donaldson-Witten function. Witten’s interpretation has lead to significant progress in the subject [4].

1.1. Questions, and answers.

Since (1.1) is a correlation function in an $SU(2)$ or $SO(3)$ gauge theory it is quite natural to ask about the generalization to compact simple gauge groups $G$ of rank larger than one. The formal definitions, both mathematical and physical, proceed with little essential modification to the higher rank case so we may ask the following three basic questions:

1. Is $Z_{DW}$ an invariant of the diffeomorphism type of $X$?
2. Does $Z_{DW}$ define new 4-manifold invariants that go beyond the classical cohomology ring and the Seiberg-Witten invariants?
3. Can $Z_{DW}$ be evaluated explicitly?

In this paper we answer these questions:

1. Yes, $Z_{DW}$ is a topological invariant for rank$(G) \equiv r > 1$.
2. No, $Z_{DW}$ does not contain any new topological information, at least for 4-manifolds of $b_{2,+} > 0$.
3. Yes, $Z_{DW}$ can be explicitly evaluated in terms of the classical cohomology ring and Seiberg-Witten invariants.

These conclusions require further comment.
The first question is not silly. From the mathematical point of view the instanton moduli spaces are quite singular and it is not obvious that there is a well-defined intersection theory on them. From the physical point of view, although the path integral is formally topologically invariant the expression for $Z_{DW}$ given below is, to say the least, intricate and delicate, and involves integrals over noncompact spaces with singular integrands. Given the phenomenon of wall-crossing and the surprising discovery of continuous metric dependence in a superconformal $SU(2)$ theory the topological invariance of $Z_{DW}$ is not obvious. Among other things one should worry about continuous metric dependence of $Z_{DW}$ arising from integration over the subvarieties in the moduli space of supersymmetric vacua with superconformal symmetry. The main technical work in this paper consists of carefully defining the integrals and checking their metric dependence. Our conclusion, as stated, is that there is no continuous variation. Somewhat surprisingly, in stark contrast to the rank one case, we find that there is no wall-crossing from the measure in the semiclassical regime.

The answer to the second question is, of course, a disappointment. One of the main motivations for this work was the suggestion of E. Witten, made during the investigations of, that wall-crossing phenomena at superconformal points would lead to the discovery of new 4-manifold invariants. We would like to stress that we are not suggesting that $\mathcal{N} = 2$ superconformal theories provide no new topological information (in fact we believe the opposite). However, if there are new invariants, they are inaccessible via the wall-crossing technique used in and described below.

Regarding the third question, the general formula is rather complicated and is only described in full detail for $G = SU(3)$ in equations (9.1) – (9.6) below. An important representative case is that of simply connected manifolds of $b^+ > 1$ and of simple type. The resulting expression for $G = SU(N)$ is given in equation (9.13) below. It is a natural generalization of the expression found by Witten in for the rank one case.

1.2. Method of derivation

Deriving the higher rank Donaldson invariants using the standard mathematical methods of does not work very well. Formal aspects of the problem, like the $\mu$-map generalize straightforwardly but, because of the singularities of instanton moduli space, the intersection theory is difficult to define.

The physical approach to the problem turns out to be much more powerful. By the physical approach we mean the program proposed by Witten in, and brought to
fruition in \[4\]. Some further technical developments described in \[4\] make the derivation of the main result of \[4\] and its higher rank generalizations conceptually straightforward, (although technically challenging in the higher rank case).

The main insight we use from \[9\] is that one can derive the relation between Donaldson and Seiberg-Witten invariants from the phenomenon of wall-crossing. \[1\] This wall-crossing method proceeds as follows. One begins by considering the contribution to \(Z_{DW}\) of the Coulomb branch of the moduli space of supersymmetric vacua on \(\mathbb{R}^4\). This contribution, denoted by \(Z_{\text{Coulomb}}\), is nonzero only for manifolds of \(b_{2,+} = 1\). Nevertheless \(Z_{\text{Coulomb}}\) turns out to contain the essential information for deriving the contributions of the SW branch to \(Z_{DW}\). In particular, cancellation of metric-dependence of \(Z_{\text{Coulomb}}\) at strong coupling singularities in moduli space allows a complete derivation of the universal functions appearing in the Lagrangian of the magnetic dual theory with the light monopole or dyon hypermultiplet fields included in the theory. (See section seven of \[9\].)

The wall-crossing method generalizes to simple gauge groups of rank \(r > 1\). The Coulomb branch \(\mathcal{M}_{\text{Coulomb}}\) is now a quasi-affine variety of complex dimension \(r\). In the weak coupling asymptotic region \(\mathcal{M}_{\text{Coulomb}}\) may be described as \((\mathfrak{k} \otimes \mathfrak{t})/W\) where \(\mathfrak{k}\) is a Cartan subalgebra for \(G\) and \(W\) is the Weyl group. More globally, the Coulomb branch has the form:

\[
\mathcal{M}_{\text{Coulomb}} = (\mathfrak{g}^r - \mathcal{D})
\]

In Seiberg-Witten theory the space of vacuum expectation values \(\langle \text{Tr}\phi^j \rangle\), for \(j\) ranging over the the exponents of \(G\), is identified with \(\mathfrak{g}^r\). The low energy theory is characterized by a family of Abelian varieties over \(\mathbb{C}^r\), and \(\mathcal{D}\) is the singular locus for this family. One can introduce local special coordinates \(a^I\) on (1.2), but these are never global coordinates and together with their duals \(a_{D,I}\) transform in nontrivial ways under the quantum monodromy group \(\Gamma\), determined in principle from the explicit SW curve and differential. For the example of \(SU(N)\), \(\mathcal{D}\) is defined by the vanishing of the “quantum discriminant” of equation (2.10) below. Unfortunately a concise description of the discrete group \(\Gamma \subset Sp(2r; \mathbb{Z})\) does not appear to be available.

The expression \(Z_{\text{Coulomb}}\) turns out to have the general form:

\[
Z_{\text{Coulomb}} = \int_{\mathcal{M}_{\text{Coulomb}}} [d\mathfrak{s}] A(\mathfrak{s}) B(\mathfrak{s}) e^{U + S^2 T^V(\mathfrak{s})} \Psi,
\]
where \( a^I \) are local special coordinates, \( A, B \) are holomorphic automorphic forms for \( \Gamma \) described below, \( U, T \) are forms associated with operator insertions, \( \chi, \sigma \) are the Euler character and signature of \( X \), and \( \Psi \) is a certain Narain-Siegel lattice theta function associated to \( H^2(X; \mathbb{Z}) \otimes \Lambda_w(G) \), where \( \Lambda_w(G) \) is the weight lattice of \( G \). The details of this expression are explained in sections 3 and 4 below. Some aspects in the derivation of the integrand were independently worked out in [12].

It is far from obvious that the integrand of (1.3) is single-valued on \( M_{Coulomb} \). We check this carefully for the case \( G = SU(3) \) and give a less-detailed general argument for single-valuedness for \( G = SU(N) \) (although we prove the invariance under the semiclassical monodromies for any simply-laced group). We do not seriously doubt that the integrand of (1.3) is single-valued for all \( G \) of \( r > 1 \), but our arguments for this leave room for improvement.

The integrand of (1.3) is singular on \( \mathcal{D} \) and in the weak-coupling regime at infinity. Hence, some discussion is required to give rigorous meaning to the integral (1.3). To do this we need to understand the structure of the divisor \( \mathcal{D} \) more thoroughly. This divisor is a stratified space. The maximum dimension stratum is a union of several smooth complex codimension one components \( \mathcal{D}^{(1)}_i \) corresponding physically to moduli for which a single \( u(1) \subset \mathfrak{k} \) becomes strong and for which a single monopole hypermultiplet becomes massless. The strata of \( \mathcal{D} \) of higher codimension correspond to singularities in \( \mathcal{D} \) where successively larger numbers of hypermultiplets become massless. We denote the smooth components of the codimension \( \ell \) strata by \( \mathcal{D}^{(\ell)}_i \). In particular, \( \mathcal{D}^{(r)}_i \) contains \( h(G) \) (the dual Coxeter number) points corresponding to the supersymmetric vacua of the \( \mathcal{N} = 1 \) theory, as well as points corresponding to multicritical superconformal field theories.

Following the discussion in [9] we define the integral by introducing a cutoff in the weak-coupling regime and by introducing tubular neighborhoods of \( \mathcal{D}^{(1)}_i \) and doing a phase integral first over the relevant special coordinates. The definition of the integration near \( \mathcal{D}^{(\ell)}_i \) for \( \ell > 1 \) is more problematic and we only discuss it in full detail in the case \( G = SU(3) \). All this is described in sections 6 and 8. We expect our considerations to generalize to other gauge groups, but again our treatment leaves room for improvement.

We then implement the wall-crossing argument of [9] by postulating that the metric variation of \( Z_{Coulomb} \) from the singularities near \( \mathcal{D}^{(1)}_i \) is cancelled by compensating metric-variation of a mixed Coulomb/monopole theory which describes the low-energy physics
near $D_i^{(1)}$. A consequence of our postulate $2$ is that $Z_{DW}$ must take the form:

$$Z_{DW} = Z_{Coulomb} + \sum_i Z_{D_i^{(1)}} + \sum_i Z_{D_i^{(2)}} + \cdots + \sum_i Z_{D_i^{(r)}}$$

(1.4)

This is the generalization of equation (1.8) of [4]. The integrals $Z_{D_i^{(1)}}$ along the codimension one varieties are derived from the wall-crossing of $Z_{Coulomb}$. This wall-crossing is described in section 6, and the explicit formulae for $G = SU(3)$ are derived in complete detail in equations (9.1) – (9.6) below. The integrals $Z_{D_i^{(1)}}$ themselves have wall-crossing behavior which is compensated by metric dependence of $Z_{D_i^{(2)}}$. This allows a derivation of the integrand of $Z_{D_i^{(2)}}$ and so on. The procedure terminates at the codimension $r$ singularities of $D$.

The central question of metric variation at a superconformal point is addressed in section eight. We analyze the behavior at the Argyres-Douglas points for $G = SU(3)$ in detail and show that there cannot be any continuous metric dependence unless $\sigma < -11$. We also give a general argument that shows there cannot be any continuous metric dependence for any signature. This argument is based on the blow-up and wall-crossing formulae. The blow-up formula for the higher rank case is derived in section seven by a straightforward generalization of the derivation in [9]. Using this formula we can relate the invariants on $X$ to invariants on a blowdown with sufficiently large signature that there can be no continuous metric variation.

In the case $b_{2,+} > 1$ only the last term $\sum_i Z_{D_i^{(r)}}$ of (1.4) is nonvanishing, and indeed only the $\mathcal{N} = 1$ vacua contribute. This allows us to write the generalization of Witten’s formula [4] to $G = SU(N)$ in equation (9.13).

1.3. Applications

Ironically, our work, which was motivated by topology, might find its most interesting applications in physics. In sections 10 and 11 we describe two applications. In section 10 we use the behavior of the integrand of (1.3) at superconformal points to deduce a selection rule for correlators of $\mathcal{N} = 2$, $d = 4$ superconformal theories. In section 11 we use the explicit result (9.13) to study some questions about the large $N$ behavior of certain correlation functions in $SU(N)$ SYM theory.

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2 which may be justified physically from considerations of tunneling between vacua at finite volume
1.4. Directions for future work

There are plenty of opportunities for future work. First, there are technical gaps in our discussion which have been indicated above. We are confident in our conclusions, but it should be possible to give a better treatment of the analysis of $Z_{\text{Coulomb}}$.

As in [9] the discussion is rather easily extended to theories with matter. For example, almost the same formulae hold for $G = SU(N_c)$ with $N_f$ fundamental hypermultiplets, as long as the masses of the hypermultiplets are generic. For special values of the masses some very interesting things should happen and this remains an interesting avenue for future research.

Another generalization worth studying is the case of $SU(N_c)$ with $\mathcal{N} = 4$ supersymmetry perturbed to $\mathcal{N} = 2$ by the addition of a mass perturbation for adjoint hypermultiplets. A discussion of $Z_{\text{Coulomb}}$ for these theories is technically challenging but might find interesting applications in string/M theory.

The generalization of the $w$-plane integrals studied in [9] to the higher rank case is probably only the first of a series of interesting generalizations of similar integrals associated to special Kähler manifolds.

2. Higher rank $\mathcal{N} = 2$ gauge theories

In this section, we review some properties of the low-energy structure of $\mathcal{N} = 2$ gauge theories that we will need in this paper [13][14][15]. We then focus on the case of $SU(N)$ Yang-Mills theory, and in particular on the $SU(3)$ moduli space, which has been explored in some detail [10][14]. We also work out some aspects of the solution near the superconformal, or Argyres-Douglas (AD) points that will be needed in the rest of the paper.

2.1. General structure

The classical moduli space of $\mathcal{N} = 2$ SYM with a rank $r$ gauge group is determined by the vacuum expectation value of the field $\phi$, which can always be rotated into the Cartan subalgebra. Following [14], we will denote these expectation values by a vector $\vec{a}$ in the root lattice, and the components of $\vec{a}$, $a^I$ with $I = 1, \cdots, r$ will correspond to a basis of simple roots. The charges will be specified by vectors $\vec{q}$ expanded in the Dynkin basis (i.e.}
the basis of fundamental weights). The central charges of electric BPS states are then written as

\[ Z_\vec{q} = \vec{q} \cdot \vec{a}, \]  

(2.1)

where the product is given by the usual bilinear form in the weight lattice. One can then introduce the Casimirs \( u_k \) as Weyl-invariant coordinates in the classical moduli space. Singularities in this moduli space are associated semiclassically to massless gauge bosons, and they occur when \( Z_{\vec{a}} = 0 \), where \( \vec{a} \) is a positive root. They are located at the zeroes of the classical discriminant,

\[ \Delta_0(\vec{u}) = \prod_{\vec{\alpha} > 0} Z_{\vec{\alpha}}^2. \]  

(2.2)

The low energy effective action is determined by a prepotential \( \mathcal{F} \) which depends on \( r \) \( \mathcal{N} = 2 \) vector multiplets \( A^I \). The VEVs of the scalar components of these vector superfields are the \( a^I \). The dual variables and gauge couplings are defined as

\[ a_{D,I} = \frac{\partial \mathcal{F}}{\partial a^I}, \quad \tau_{IJ} = \frac{\partial^2 \mathcal{F}}{\partial a^I \partial a^J}. \]  

(2.3)

The moduli space of vacua has a natural Kähler metric given by

\[ (ds)^2 = \text{Im} \tau_{IJ} da^I d\bar{a}^J, \]  

(2.4)

which is invariant under the group \( \text{Sp}(2r, \mathbb{Z}) \) (the restriction to integer valued entries comes from the integrality requirement of the charges, as we will see in a moment). The inverse metric will be denoted by \( (\text{Im} \tau)^{IJ} \). Matrices in \( \text{Sp}(2r, \mathbb{Z}) \) have the structure

[\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \]

(2.5)

where the \( r \times r \) matrices \( A, B, C, D \) satisfy:

\[ A^t D - C^t B = 1, \quad A^t C = C^t A, \quad B^t D = D^t B. \]  

(2.6)

The generators of the symplectic group \( \text{Sp}(2r, \mathbb{Z}) \) are

\[ A = \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix}, \quad A \in \text{GL}(r, \mathbb{Z}), \]

\[ T_\theta = \begin{pmatrix} 1 & \theta \\ 0 & 1 \end{pmatrix}, \quad \theta_{IJ} \in \mathbb{Z}, \quad \theta^t = \theta, \]  

(2.7)

\[ \Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \]
The symplectic group acts on the $a^I, a_{D,I}$ variables as $v \rightarrow \gamma v$, where $v^t = (a_{D,I}, a^I)$. In particular, we have the following transformation properties which will be useful later,

$$\frac{\partial}{\partial a^I} \rightarrow [(C \tau + D)^{-1}]^J_{\alpha} \frac{\partial}{\partial a^J},$$

$$\tau \rightarrow (A \tau + B)(C \tau + D)^{-1},$$

$$\text{Im} \tau \rightarrow [(C \tau + D)^{-1}]^t (\text{Im} \tau)(C \tau + D)^{-1}.$$  \hspace{1cm} (2.8)

2.2. $SU(N)$

In the $SU(N)$ case, the quantum theory is described by the hyperelliptic curve $[17][18]$: 

$$y^2 = P(x)^2 - \Lambda_{2N}^2, \quad P(x) = x^N - \sum_{I=2}^{N} u_I x^{N-I},$$  \hspace{1cm} (2.9)

where the $u_I, I = 2, \ldots, N$, are the elementary symmetric polynomials in the roots of $P(x)$. The quantum discriminant associated to this curve is given by

$$\Delta_A = \Lambda_{2N}^2 \Delta_0(u_2, \ldots, u_{N-1}, u_N + \Lambda^N) \Delta_0(u_2, \ldots, u_{N-1}, u_N - \Lambda^N)$$  \hspace{1cm} (2.10)

The Coulomb branch of the quantum theory is then given by $C^{r - D}$, where $D$ is the vanishing locus of (2.10). To obtain the couplings $\tau_{IJ}$ and the masses of the BPS states, one chooses a symplectic homology basis for the genus $r$ Riemann surface described by (2.9), $\alpha_I, \beta_I, I = 1, \cdots, r$, and the basis of holomorphic differentials $\omega_I = x^{r-1} dx/y$. The periods of the curve are then

$$A^J_I = \int_{\alpha_I} \omega_J = \frac{\partial a_{D,I}}{\partial u_{J+1}},$$

$$B^{IJ} = \int_{\beta_I} \omega_J = \frac{\partial a^I}{\partial u_{J+1}},$$  \hspace{1cm} (2.11)

where $I, J = 1, \cdots, r$. The gauge coupling is then given by

$$\tau_{IJ} = A^K_I (B^{-1})_{KJ}.$$  \hspace{1cm} (2.12)

One then introduces a meromorphic one form on the hyperelliptic curve (usually known as Seiberg-Witten differential) $\lambda_{SW}$ satisfying

$$\frac{\partial \lambda_{SW}}{\partial u_{I+1}} = \omega_I,$$  \hspace{1cm} (2.13)
which has the explicit expression [17]:
\[
\lambda_{SW} = \frac{1}{2\pi i} \frac{\partial P}{\partial x} dx.
\]
(2.14)

The BPS masses are then given by the periods of \(\lambda_{SW}\):
\[
a_{D,I} = \oint_{\alpha_I} \lambda_{SW}, \quad a' = \oint_{\beta'_I} \lambda_{SW}.
\]
(2.15)

Quantum-mechanically, the singularities in the moduli space are associated to massless dyons. Their charges will be denoted by \(\vec{\nu} = (g, q)\), where \(g, q\) are the r-component vectors of magnetic and electric charges, respectively. When one of these dyons becomes massless at a certain submanifold of the moduli space, one of the cycles of the hyperelliptic curve degenerates and there is an associated monodromy given by
\[
M_{\vec{\nu}} = \begin{pmatrix} 1 + \bar{q} \otimes \bar{g} & \bar{q} \otimes \bar{q} \\ -\bar{g} \otimes \bar{g} & 1 - \bar{g} \otimes \bar{q} \end{pmatrix}.
\]
(2.16)

2.3. \(SU(3)\) and the AD points

In the case of \(\mathcal{N} = 2\) SYM theory with gauge group \(SU(3)\), the moduli space is parametrized by the Casimirs \(u = u_2, v = u_3\). There is a discrete, anomaly-free subgroup \(\mathbb{Z}_6\) of the \(R\)-symmetry which acts as \(u \rightarrow e^{2\pi i/3} u, v \rightarrow -v\). The quantum discriminant is given by
\[
\Delta_A = \Lambda^{18}[4u^3 - 27(v + \Lambda^3)^2][4u^3 - 27(v - \Lambda^3)^2].
\]
(2.17)

There are two codimension one submanifolds given by \(\Delta_0(u, v\pm\Lambda^3) = 0\), which intersect in the three \(\mathbb{Z}_2\) vacua \(4u^3 = (3\Lambda^2)^3, v = 0\). At these points there are two mutually local dyons becoming massless, and when \(\mathcal{N} = 2\) is softly broken down to \(\mathcal{N} = 1\) with a superpotential \(\text{Tr}\Phi^2\), they give the three vacua of \(\mathcal{N} = 1\) SYM [17]. The charges \((n^1_m, n^2_m; n^1_e, n^2_e)\) of these states are the following [14]:
\[
\vec{\nu}_1 = (1, 0; 0, 0), \quad \vec{\nu}_2 = (0, 1; 0, 0), \\
\vec{\nu}_3 = (0, 1; -1, 2), \quad \vec{\nu}_4 = (-1, -1; 2, -1), \\
\vec{\nu}_5 = (-1, -1; 1, -2), \quad \vec{\nu}_6 = (1, 0; 2, -1).
\]
(2.18)

The charges in the same row in (2.18) are mutually local. The first row corresponds to the \(\mathbb{Z}_2\) vacuum at \(u_1 = (27/4)^{1/3} \Lambda^2, v = 0\). The second and third rows correspond to the
vacua located at \( u_2 = e^{2\pi i/3}u_1, \) \( v = 0, \) and \( u_3 = e^{4\pi i/3}u_1, \) \( v = 0, \) respectively. In fact one can find a matrix \( U \) which implements the \( \mathbb{Z}_3 \) symmetry in moduli space,

\[
U = \begin{pmatrix}
-1 & -1 & 1 & 2 \\
1 & 0 & -2 & 1 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & -1 \\
\end{pmatrix}.
\]

One can check that, acting on the right on \( \vec{\nu}_1, \vec{\nu}_2, \) we obtain the other two pairs of massless states at the \( \mathcal{N} = 1 \) points, \( i.e., \) \( \vec{\nu}_1U^{-1} = \vec{\nu}_3, \) \( \vec{\nu}_2U^{-1} = \vec{\nu}_4, \) etc.

There are also singular points on each of the submanifolds (also called \( \mathbb{Z}_3 \) vacua) at the points \( u = 0, \) \( v = \pm \Lambda^3. \) These are the Argyres-Douglas (AD) points, where three mutually non-local hypermultiplets become massless \( \text{[16]} \). We will be particularly interested in the behaviour of the theory near these points. Let’s focus on the point \( v = \Lambda^3, \) \( u = 0 \) (the behaviour near the other AD point can be obtained using the \( \mathbb{Z}_6 \) symmetry which sends \( v \to -v \)). The states that become massless near this point are \( \vec{\nu}_1, \vec{\nu}_3 \) and \( \vec{\nu}_5. \) The symplectic transformation \( T_\theta \Omega^{-1}A, \) where

\[
A = \begin{pmatrix}
-1 & 0 \\
2 & 1 \\
\end{pmatrix}, \quad \theta = \begin{pmatrix}
-1 & -1 \\
-1 & \theta_{22} \\
\end{pmatrix}, \quad \theta_{22} \in \mathbb{Z},
\]

(2.20)
gives a basis where all the states are charged only with respect to the first \( U(1) \) factor. Applying \( A^{-1}\Omega T_\theta^{-1} \) on the right to the charge vectors \( \nu_i, \) we find that the new charges \( \vec{\nu} \cdot A^{-1}\Omega T_\theta^{-1} \) are (in this new basis) \( (n_e, n_m) = (-1, 0) \) for \( \vec{\nu}_1, \) \((1, -1) \) for \( \vec{\nu}_3 \), and \((0, 1) \) for \( \vec{\nu}_5, \) \( i.e. \) we have one electron, one dyon, and one monopole. In this basis, the hyperelliptic curve degenerates and at leading order in \( u, \) \( v - \Lambda^3, \) it splits into a “small” torus whose periods go to zero (and correspond to \( a^1, a_{D,1} \)) and a “large” torus whose periods \( a^2, a_{D,2} \) are of order \( \Lambda. \)

We introduce now the useful parameters \( \epsilon, \rho \) around the AD point, defined by

\[
u = 3\epsilon^2 \rho, \quad v - \Lambda^3 = 2\epsilon^3.
\]

(2.21)
The variable \( \rho \) parametrizes the direction along which we approach the AD point in the \( u, v \) moduli space. The equation defining the small torus near the AD point is given by

\[
w^2 = z^3 - 3\rho z - 2,
\]

(2.22)
with discriminant

\[
\Delta_\rho = 4 \cdot 27(\rho^3 - 1),
\]

(2.23)
and the Seiberg-Witten differential on the curve (2.22) degenerates to

$$\lambda_{SW} = p \frac{\epsilon^{5/2}}{\Lambda^{3/2}} w dz,$$

(2.24)

where $p$ is some constant that depends on the normalization of $\lambda_{SW}$. The small torus theory gives us the dependence on $\rho$ for the leading terms in $\epsilon$ of $a^1, a_{D,1}$. We can put the curve (2.22) in Weierstrass form and compute $a^1, a_{D,1}$ explicitly (at leading order in $\epsilon$) in terms of the periods of the curve $\omega_\rho, \omega_{\rho,D}$ (with $\text{Im}(\omega_{\rho,D}/\omega_\rho) > 0$):

$$a^1 = \frac{\epsilon^{5/2}}{\Lambda^{3/2}} f(\rho), \quad a_{D,1} = \frac{\epsilon^{5/2}}{\Lambda^{3/2}} f_D(\rho),$$

(2.25)

where

$$f(\rho) = \frac{48p}{5}(\rho \eta - \frac{\omega_\rho}{8}), \quad f_D(\rho) = \frac{48p}{5}(\rho \eta_D - \frac{\omega_{\rho,D}}{8})$$

(2.26)

In this equation, $\eta = \zeta(\omega_\rho/2), \eta_D = \zeta(\omega_{\rho,D}/2)$ are the usual values of the Weierstrass zeta function at the half-periods.

The curve (2.22) has singularities when $\rho^3 = 1$ and also at infinity. At $\rho^3 = 1$ we have $A_0$ singularities and the behaviour of $\tau(\rho)$ is

$$\tau(\rho) = \frac{1}{2\pi i} \log(\rho - \rho_k),$$

(2.27)

where $\rho_k = e^{2\pi i k/3}, k = 0, 1, 2$ are the corresponding singularities. At $\rho \to \infty$, there is an $H_1$ singularity (Kodaira’s type III) and the monodromy is given by $S^{-1}$. The behaviour of $\tau(\rho)$ is given by

$$\tau(\rho) = i + \frac{C}{\rho^{3/2}} + \cdots,$$

(2.28)

where $C$ is a nonzero constant.

The behaviour of $a^2, a_{D,2}$ (the “long periods”) can be found [16] [19] to be

$$a^2 = b \Lambda + c \frac{u}{\Lambda} + d \frac{v - \Lambda^3}{\Lambda^2} + \cdots, \quad a_{D,2} = b_D \Lambda + c_D \frac{u}{\Lambda} + d_D \frac{v - \Lambda^3}{\Lambda^2} + \cdots,$$

(2.29)

where $b, c, d, b_D, c_D$ and $d_D$ are non-zero constants. From these explicit expressions we can compute the matrix of periods of the hyperelliptic curve, $B^{IJ}$, at leading order:

$$\begin{pmatrix}
\frac{\partial a^1}{\partial u} & \frac{\partial a^1}{\partial v} \\
\frac{\partial a^2}{\partial u} & \frac{\partial a^2}{\partial v}
\end{pmatrix} = \begin{pmatrix}
\frac{\epsilon^{1/2}}{3\Lambda^{3/2}} f'(\rho) & \frac{\epsilon^{-1/2}}{3\Lambda^{3/2}} H(\rho) \\
\frac{\epsilon^{1/2}}{\Lambda} & \frac{\epsilon^{-1/2}}{\Lambda}
\end{pmatrix},$$

(2.30)
where the derivatives are with respect to $\rho$ and
\[
f'(\rho) = 12 \rho p, \quad H(\rho) = \frac{5}{4} f(\rho) - \rho f'(\rho) = -\frac{3}{2} \rho \omega_\rho.
\] (2.31)

The matrix $A^J_I$ has a similar expression in terms of $c_D$, $d_D$ and $f_D(\rho)$. We then find that
\[
\det \frac{\partial u_{I+1}}{\partial a^J} = \frac{2 \Lambda^{5/2} \epsilon^{1/2}}{pc} \omega_\rho + O(\epsilon^{3/2}).
\] (2.32)

The gauge couplings can be also computed in a straightforward way and are given by
\[
\tau_{11} = \tau(\rho) = -\frac{e d f_D(\rho)}{c \Lambda H(\rho)} + O(\epsilon^2),
\]
\[
\tau_{12} = -\frac{4\pi i p}{c \Lambda^{1/2}} \epsilon^{1/2} + O(\epsilon^{3/2}),
\]
\[
\tau_{22} = \frac{c_D}{c} - \frac{e d_D}{c \Lambda} \frac{f'(\rho)}{H(\rho)} + O(\epsilon^2),
\] (2.33)

and $c_D/c = e^{\pi i/3}$ is the period of the large torus at the AD point (some aspects of the behaviour of the couplings at the AD point have been investigated in [21]).

Finally, we will need the behaviour of the third derivatives of the prepotential near this point (and in particular their leading behaviour in $\epsilon$). These are given by:
\[
F_{111} = -\frac{\Lambda^{3/2} \epsilon^{-5/2}}{H(\rho)} \rho \frac{d \tau(\rho)}{d \rho} + O(\epsilon^{-3/2}),
\]
\[
F_{112} = \frac{5 \Lambda \epsilon^{-2}}{12c} \frac{f(\rho)}{H(\rho)} \frac{d \tau(\rho)}{d \rho} + O(\epsilon^{-1}),
\]
\[
F_{122} = \frac{\Lambda^{1/2} \epsilon^{-3/2} d_D}{c H(\rho)} \left[ \left( \frac{f'(\rho)}{H(\rho)} \right)^' - \frac{1}{2} \frac{f'(\rho)}{H(\rho)} \right] + O(\epsilon^{-1/2}),
\]
\[
F_{222} = \frac{\epsilon^{-1} d_D}{6c^2} \left[ \left( \frac{f'(\rho)}{H(\rho)} \right)^2 - \frac{5}{2} \frac{f(\rho)}{H(\rho)} \left( \frac{f'(\rho)}{H(\rho)} \right)^' \right] + O(1).
\] (2.34)

This behaviour is consistent with the $R$-charge assignment near the superconformal point, $R(a^1) = 1$, $R(a^2) = R(u) = 4/5$, $R(\mathcal{F}) = 2$.

3. The twisted effective theory on the Coulomb branch

To study the twisted supersymmetric $\mathcal{N} = 2$ SYM theory on a four-manifold $X$, we consider the low-energy description encoded in the solution presented in the last section. The procedure we will follow is a straightforward generalization of the one presented in
The field content of the low-energy theory consists of \( r \) twisted abelian \( \mathcal{N} = 2 \) vector multiplets. The \( \mathcal{Q} \)-transformations are given by

\[
\begin{align*}
[\mathcal{Q}, A^I] &= \psi^I, & [\mathcal{Q}, \psi^I] &= 4\sqrt{2} da^I, \\
[\mathcal{Q}, a^I] &= 0, & [\mathcal{Q}, \bar{a}^I] &= \sqrt{2} i \eta^I, \\
[\mathcal{Q}, \eta^I] &= 0, & [\mathcal{Q}, \chi^I] &= i(F_+^I - D_+^I), \\
[\mathcal{Q}, D^I] &= (d\psi^I)^+. 
\end{align*}
\] (3.1)

We will also need the action of the one-form operator \( G \), which gives a canonical solution to the descent equations (this operator was denoted by \( K \) in [9]). It is given by

\[
\begin{align*}
[G, a^I] &= \frac{1}{4\sqrt{2}} \psi^I, & [G, \psi^I] &= -2(F_+^I + D_+^I), \\
[G, A^I] &= -2i \chi^I, & [G, \bar{a}^I] &= 0, \\
[G, \eta^I] &= -\frac{i\sqrt{2}}{2} da^I, & [G, D^I] &= -\frac{3i}{4} * d\eta^I + \frac{3i}{2} d\chi, \\
[G, \chi^I] &= -\frac{3i\sqrt{2}}{4} * da^I. 
\end{align*}
\] (3.2)

The twisted effective Lagrangian can be written in a manifestly topological way as:

\[
\begin{align*}
\frac{i}{6\pi} G^4 \mathcal{F}(a^I) + \frac{1}{16\pi} \{ \mathcal{Q}, \mathcal{F}_{IJ} \chi^I (D + F_+)^J \} - \frac{i\sqrt{2}}{32\pi} \{ \mathcal{Q}, \mathcal{F}_I d * \psi^I \} \\
&\quad - \frac{\sqrt{2} i}{3 \cdot 2^5 \pi} \{ \mathcal{Q}, \mathcal{F}_{IJK} \chi^I_{\mu\nu} \chi^J_{\nu\lambda} \chi^K_{\mu\lambda} \}, 
\end{align*}
\] (3.3)

which may be expanded out to give:

\[
\begin{align*}
&\frac{i}{16\pi} (\tau_{IJ} F^I_+ \wedge F^J_+ + \tau_{IJ} F^I_- \wedge F^J_-) + \frac{1}{2\pi} (Im \tau_{IJ}) da^I \wedge * da^J - \frac{1}{8\pi} (Im \tau_{IJ}) D^I \wedge * D^J \\
&\quad - \frac{1}{16\pi} \tau_{IJ} \psi^I \wedge * d\eta^J + \frac{1}{16\pi} \tau_{IJ} \eta^I \wedge d * \psi^J + \frac{1}{8\pi} \tau_{IJ} \psi^I \wedge d\chi^J - \frac{1}{8\pi} \tau_{IJ} \chi^I \wedge (d\psi^J)_+ \\
&\quad + \frac{i\sqrt{2}}{16\pi} \mathcal{F}_{IJK} \eta^I \chi^J \wedge (D_+ + F_+)_K - \frac{i\sqrt{2}}{2^7 \pi} \mathcal{F}_{IJK} (\psi^I \wedge \psi^J) \wedge (F_- + D_+)_K \\
&\quad + \frac{i}{3 \cdot 2^{11} \pi} \mathcal{F}_{IJKL} \psi^I \wedge \psi^J \wedge \psi^K \wedge \psi^L - \frac{\sqrt{2} i}{3 \cdot 2^5 \pi} \{ \mathcal{Q}, \mathcal{F}_{IJK} \chi^I_{\mu\nu} \chi^J_{\nu\lambda} \chi^K_{\mu\lambda} \}. 
\end{align*}
\] (3.4)

It is important to notice that the part of the action involving the fourth descendant of the prepotential can be written as a \( \mathcal{Q} \)-exact term plus terms which are topological (i.e.
they do not involve the metric of the four-manifold $X$):

$$
\frac{i}{6\pi} G^4 F(a^I) = 
\left\{ \bar{Q}, \tau_{IJ} \left[ -\frac{\sqrt{2i}}{2^5 \cdot \pi} \psi^I \wedge *d\bar{a}^J - \frac{1}{16\pi} \chi^I \wedge (F^- + D)^J \right] + \frac{\sqrt{2}}{2^7 \cdot \pi} F_{IJK} \psi^I \wedge \psi^J \wedge \chi^K \right\} 
+ \frac{i\tau^{IJ}}{16\pi} F^I \wedge F^J - \frac{i\sqrt{2}}{2^7 \pi} F_{IJK}(\psi^I \wedge \psi^J) \wedge F^K + \frac{i}{3 \cdot 2^{11} \pi} F_{IJKL} \psi^I \wedge \psi^J \wedge \psi^K \wedge \psi^L,
$$

(3.5)

where integration over $X$ is understood.

4. The integrand in the higher rank case

The $u$-plane integral in the higher rank case is given by a general expression of the form:

$$
Z_u(p, S; m_i, \tau_0) = \int_{\mathcal{M}_{\text{Coulomb}}} [d\bar{a}d\bar{a}] A(u) B(u)^\sigma \epsilon^{U+SU^2T_V(u)} \Psi,
$$

(4.1)

where $\Psi$ is a certain lattice theta function. We will explain in some detail the structure of the different terms involved in (4.1). The resulting expression, as we will see, holds for any simply-laced gauge group.

4.1. The observables

In (4.1) the 0-observable is a general invariant function $U$ on the Lie algebra. This generalizes $2pu$ in the rank one case. We will restrict attention to expressions linear in the Casimirs of the group,

$$
U = \sum_{I=2}^{r+1} p^I \text{Tr} \phi^I.
$$

(4.2)

Here we are using the standard notation for the Casimirs of $SU(N)$. The VEVs of these operators can be related to the symmetric polynomials in (2.4) which parametrize the quantum moduli space by standard results on symmetric functions.

The 2-observable is obtained by canonical descent from another general function $V$. When 2-observables are included one has to take into account contact terms, denoted by $T_V(u)$. We will discuss them below. For simplicity, we will restrict attention to 2-observables obtained from the quadratic Casimir, $V = u_2$. Other 2-observables involve new contact terms discussed in [12]. In general, the 2-observable is obtained as in [9] using
the one-form operator $G$, which gives canonical solutions to the descent equations. In the rank $r$ case we have

$$G^2 u_I = \frac{1}{32} \frac{\partial^2 u_I}{\partial a^J \partial a^K} \psi^J \wedge \psi^K - \frac{\sqrt{2}}{4} \frac{\partial u_I}{\partial a^J} (F_- + D_+)^J.$$  \hfill (4.3)

The two-observable associated with a surface $S$ is given by

$$\bar{I}(S) = \frac{i}{\pi\sqrt{2}} \int_S G^2 u_I,$$  \hfill (4.4)

where we use the normalization of \[9\].

The four-observables come again from a general function $W$. Using the canonical solution to the descent equations we see that they merely shift $\tau_{IJ} \rightarrow \tau_{IJ} + W_{IJ}$. This will involve further contact terms, which can be written by a process of covariantizing derivatives.

4.2. The measure factor

The $A, B$ functions in (4.1) are the higher rank generalization of the gravitational factors considered in \[21\]\[9\]. They are given by:

$$A^\chi = \alpha^\chi \left( \det \frac{\partial u_I}{\partial a^J} \right)^{\chi/2}$$  \hfill (4.5)

$$B^\sigma = \beta^\sigma \Delta^\sigma/8$$  \hfill (4.6)

This may be proved by a modification of the argument of \[21\]\[9\]. The twisted theory with gauge group $G$ has a gravitational contribution to the anomaly given by $- (\text{dim} G)(\chi + \sigma)/2$. In the semiclassical regime the effective $U(1)^r$ theory gives the anomaly $- r(\chi + \sigma)/2$. The remaining anomaly should be carried by the measure factor in the semiclassical region. On the other hand, near the divisor where a single hypermultiplet becomes massless, there is an accidental low-energy $R$-symmetry given by $-\sigma/4$ which should also show up in the measure factor in this region.

We first check that the $B^\sigma$ factor gives the needed behaviour for the $\sigma$ dependence. Near the divisor where a single hypermultiplet becomes massless, the quantum discriminant has the structure

$$\Delta^{\sigma/8}_\Lambda \sim Z^{\sigma/8} \tilde{\Delta}_\Lambda,$$  \hfill (4.7)
where $Z$ is the transverse coordinate at the divisor and $\tilde{\Delta}_\Lambda \neq 0$ along it. As $Z$ has $R$-charge two, we see that $B^\sigma$ gives the right behaviour. On the other hand, in the semiclassical region we have that

$$\Delta_\Lambda \sim (\Delta_0)^2. \quad (4.8)$$

The $Z_\alpha$ have $R$-charge 2. As there are $(\dim G - r)/2$ positive roots, the $R$-charge of $\Delta_\Lambda$ in the semiclassical region is given by $4(\dim G - r)$, and again we find the right charge. As $\Delta_\Lambda$ is a modular form of weight zero, $B/\Delta_\Lambda$ has no zeroes and is a constant. This proves (4.6).

We now consider the $A^\chi$ factor. The $R$-charge at the semiclassical regime is easily computed to give $\chi(\dim G - r)/2$, again in agreement with the behaviour we need. On the other hand, we have to check that (in the appropriate local variables) this factor does not have zeros or singularities on the moduli space. Notice that, at a generic point in the moduli space, the $A^\chi$ factor can be written as

$$A^\chi = \alpha^\chi(\det B^{IJ})^{-\chi/2}, \quad (4.9)$$

where $\det B^{IJ}$ is the first minor of the period matrix of the hyperelliptic curve (2.9), and is given in (2.11). It follows from the Riemann bilinear relations that this minor is nonsingular. On a divisor where a hypermultiplet becomes massless, there are good coordinates $a^I$ around it in the sense that the Jacobian of the change of variables from $u_I$ to $a^I$ is nonsingular, and again we see that in the appropriate variables this factor has no zeros or singularities. Since $\det \partial u_{I+1}/\partial a^J$ is a modular form of weight $(-1,0)$, we have proved (4.5). We will comment on the constant $\alpha$ below.

4.3. The lattice $\Gamma$ and generalized Stiefel-Whitney classes

The function $\Psi$ in (1.1), as we will see, involves the evaluation of the photon partition function for the effective $U(1)^r$ theory. Therefore, it includes a sum over electric line bundles [21]. We will consider theories with a non-abelian magnetic flux. This is possible, for instance, in the case of an $SU(N)$ theory, because the gauge group is actually $SU(N)/\mathbb{Z}_N$ (provided all fields are in the adjoint representation of the group). A bundle $E$ with this gauge group is characterized up to isomorphism by two topological invariants: the instanton number and the generalized Stiefel-Whitney class (or non-abelian magnetic flux) $\tilde{w}_2(E) \in H^2(X, \mathbb{Z}_N)$. For a gauge group $G$, the non-abelian magnetic flux $\tilde{w}_2(E)$ takes values in $H^2(X, \pi_1(G))$. Equivalently [22], for any simply-laced gauge group, the
magnetic fluxes are cohomology classes in $H^2(X, \Lambda_w/\Lambda_r)$, where $\Lambda_w(r)$ are the weight and root lattices of the group, respectively. For every weight lattice, there is a set of weights called minimal weights which are in one-to-one correspondence with the cosets $\Lambda_w/\Lambda_r$ ([23], p. 72). There are in general $c - 1$ minimal weights, where $c = \det C$ is the “index of connection”, that is, the determinant of the Cartan matrix (notice that $c$ is precisely the order of $\Lambda_r$ in $\Lambda_w$). The set of minimal weights is in general a subset of the set of fundamental weights. We will denote these weights by $\tilde{m}_I$, $I = 1, \cdots, c - 1$. In the case of $SU(N)$, they are just the fundamental weights $\tilde{w}_I$, $I = 1, \cdots, N - 1$. The electric line bundles are then classified by vectors of the form:

$$\lambda = \lambda_Z + \vec{v}, \quad \lambda_Z = \sum_{I=1}^{r} \lambda_I^Z \alpha_I, \quad \vec{v} = \sum_{I=1}^{c-1} \pi_I^J \tilde{m}_I,$$

where $\alpha_I$ is a set of simple roots. In this expression, $\lambda_I^Z, \pi_I^J$ are all integer classes in $H^2(X; \mathbb{Z})$. The $\pi_I^J$ are fixed, and represent a choice of $\tilde{w}_2(E) \in H^2(X, \Lambda_w/\Lambda_r)$ lifted to $H^2(X, \Lambda_r)$. Notice that we can always expand the minimal weights in the basis of simple roots:

$$\tilde{m}_I = \sum_{J=1}^{r} m_I^J \tilde{a}_J, \quad I = 1, \cdots, c - 1, \quad m_I^J \in \frac{1}{c} \mathbb{Z},$$

therefore we can write

$$\lambda = \sum_{I=1}^{r} \lambda_I^I \tilde{a}_I, \quad \lambda_I^I = \lambda_I^Z + \sum_{J=1}^{c-1} m_I^J \pi_I^J \in \frac{1}{c} H^2(X, \mathbb{Z}).$$

For $SU(N)$, one has $m_I^J = (C^{-1})_I^J$, where $C_I^J$ is the Cartan matrix. Finally, later we will need the result that the instanton number of the original bundle $E$ satisfies

$$c_2(E) = -\frac{\vec{v} \cdot \vec{v}}{2} \mod 1.$$ 

### 4.4. The lattice sum

The lattice sum $\Psi$ appearing in (4.1) is obtained after integrating over the zero modes of the fields, integrating out the auxiliary fields (after including the 2-observable (4.4)) and taking into account the photon partition function. The procedure is entirely analogous to the one presented in [4] for $SU(2)$. The only difference is that the photon partition function includes now a factor $(\det \text{Im}\tau)^{-1/2}$ (for simply-connected manifolds). We also have $r$ zero modes for $\eta^I$ as well as for $\chi^I$ (when $b_2^+ = 1$). Because of the argument based on the
scaling of the metric of \[9\], the contribution of the Coulomb branch vanishes if \(b_j^+ > 1\). We also write \(F^I = 4\pi \lambda^I\), which is the appropriate normalization for the line bundles involved in the sum. The lattice \(\Gamma\) of \(\lambda^I\) has been already specified for the general case in which we have non-abelian magnetic fluxes. After taking all this into account, we finally obtain a formula for the factor \(\Psi\) in (4.1):

\[
\Psi = (\det \text{Im} \tau)^{-1/2} \exp \left[ \frac{1}{8\pi} V_J (\text{Im} \tau)^{JK} V_K S^2 \right] \sum_{\lambda \in \Gamma} 
\exp \left[ -i\pi \tau_{IJ}(\lambda^I_+, \lambda^I_-) - i\pi \tau_{IJ}(\lambda_-^I, \lambda^I_+) - i\pi((\lambda - \lambda_0) \cdot \vec{\rho}, w_2(X)) - iV_I(S, \lambda^I_-) \right] 
\int \prod_{I=1}^r \eta^I d\chi^I \exp \left\{ - \frac{i\sqrt{2}}{16\pi} F_{JK} \eta^J \chi^J [4\pi (\lambda_+^K, \omega) + i(\text{Im} \tau)^{KL} V_L(S, \omega)] 
+ \frac{1}{64\pi} F_{KL} (\text{Im} \tau)^{IJ} F_{PQ} \eta^K \chi^L \eta^P \chi^Q \right\} 
\] (4.14)

Here \(V_I = \frac{\partial V}{\partial a^I}\). The phase factor \(\exp[-i\pi((\lambda - \lambda_0) \cdot \vec{\rho}, w_2(X))]\) can be derived by a generalization of Witten’s analysis in [21] (see [12] for a derivation along these lines). We found it (independently) from invariance of the Coulomb integral under the semiclassical monodromy. \(\lambda_0\) is an element in \(\Gamma\) such that \(\lambda - \lambda_0 \in H^2(X, \Lambda_r)\), and corresponds to a choice of orientation of the higher rank instanton moduli spaces. Notice that its inclusion is necessary in order for the phase factor to be defined independently of the integral lift we choose for \(\lambda\). One should also include in the lattice sum a global phase factor depending on the generalized Stiefel-Whitney class \(\vec{v}\), in order to obtain invariants that are real. In the \(SU(2)\) case, this factor turns out to be \(e^{i\pi \vec{v} \cdot \vec{\sigma}} = e^{i\pi w_2(E)^2 / 2}\) [9]. We will find the appropriate factor for \(SU(N)\) after computing the resulting invariants in section 9.

In the rank one case, this lattice sum is related to the sum \(\Psi_{r=1}\) introduced in [9] as follows:

\[
\Psi = - \frac{i\sqrt{2}}{4} \frac{1}{y^{1/2}} \frac{d\tau}{da} \sum_{\lambda \in \Gamma} (-1)^{(\lambda - \lambda_0) \cdot w_2(X)} \left[ (\lambda, \omega) + \frac{i}{4\pi y} \frac{du}{da}(S, \omega) \right] 
\cdot \exp \left[ -i\pi \tau_+(\lambda^+)^2 - i\pi \tau_-(\lambda_-)^2 - i\frac{du}{da}(S, \lambda_-) \right] 
\] (4.15)

where \(\tau = x + iy\).
We can explicitly evaluate the Grassmann integral in the rank two case, with the result:

\[
\int \prod_{I=1,2} d\eta^I d\chi^I \exp \left\{ -\frac{i\sqrt{2}}{16\pi} F_{IJK} \eta^I \chi^J [4\pi (\lambda_+^K, \omega) + i(\Im \tau)^{KL} V_L(S, \omega)] \\
+ \frac{1}{64\pi} F_{KLI} (\Im \tau_{IJ})^{-1} F_{JPQ} \eta^K \chi^L \eta^P \chi^Q \right\}
\]

\[
= -\frac{1}{\pi^2} (F_{11}^I F_{22}^J - F_{12}^I F_{12}^J) \left\{ -4\pi (\Im \tau)^{IJ} \\
+ [4\pi (\lambda_+^I, \omega) + i(\Im \tau)^{IK} V_K(S, \omega)][4\pi (\lambda_+^J, \omega) + i(\Im \tau)^{JL} V_L(S, \omega)] \right\}
\]

(4.16)

In general, the integration over the Grassmann variables will give a factor of the form

\[
\det_{IJ} (F_{IJ} \lambda_+^K) + \cdots,
\]

where the remaining terms should be regarded as contact terms.

A more compact expression for (4.14) can be found if we introduce \( r \) bosonic auxiliary variables \( b^I \):

\[
\Psi = \sum_{\lambda \in \Gamma} \int \prod_{I=1}^r d\eta^I d\chi^I \int_{-\infty}^{+\infty} \prod_{I=1}^r db^I \exp \left[ -i\pi \tau_{IJ}(\lambda_+^I, \lambda_+^J) - i\pi \tau_{IJ}(\lambda_+^I, \lambda_+^J) \\
+ \frac{1}{8\pi} b^I (\Im \tau)_{IJ} b^J - iV_I(S, \lambda_+^I) - \frac{i}{4\pi} V_I(S, \omega) b^I \\
- \frac{i\sqrt{2}}{16\pi} F_{IJK} \eta^I \chi^J (b^K + 4\pi \lambda_+^K) - i\pi ((\bar{\lambda} - \bar{\lambda}_0) \cdot \bar{\rho}, w_2(X)) \right].
\]

(4.17)

We emphasize that the integral in (4.17) is finite-dimensional, and not a path integral. This expression can be formally considered as the partition function of a finite-dimensional topological “field” theory, obtained from the original one after restriction to the sector of harmonic forms. The topological invariance is obtained from (3.1) and reads:

\[
[\bar{\mathcal{Q}}, \lambda] = 0, \quad [\bar{\mathcal{Q}}, a^I] = 0, \quad [\bar{\mathcal{Q}}, \bar{a}^I] = \sqrt{2} i\eta^I, \quad [\bar{\mathcal{Q}}, b^I] = 0, \\
[\bar{\mathcal{Q}}, \eta^I] = 0, \quad [\bar{\mathcal{Q}}, \chi^I] = i(4\pi \lambda_+^I - b^I).
\]

(4.18)

We can consider minus the exponent in (4.17) as the (euclidean) action \( S_E \) of this topological field theory. It is \( \bar{\mathcal{Q}} \)-closed.
As explained in [9], when 2-observables are taken into account there are possible contact terms in the low-energy description. As will become clear in the next section, the contact term \( T_V \) must be such that

\[
\hat{T}_V(\vec{u}) = T_V(\vec{u}) + \frac{1}{8\pi} V_J(\text{Im} \tau)^{JK} V_K
\]  

(4.19)

is duality invariant. We give its form for \( V = u_2 \) and a general \( SU(N) \) theory with \( N_f \) matter hypermultiplets, \( N_f \leq 2N \), following the approach of [9][24]. Introduce the parameter \( \tau_0 \) as \( \Lambda_{N,N_f}^{2N-N_f} = e^{i\pi \tau_0} \) for the asymptotically free theories, and as the microscopic gauge coupling for the theories with \( N_f = 2N \). The prepotential verifies the relation \[ \frac{\partial F}{\partial \tau_0} = \frac{1}{4} u_2, \]  

(4.20)

and under a symplectic transformation we have the following behaviour,

\[
\frac{\partial^2 F}{\partial \tau_0^2} \rightarrow \frac{\partial^2 F}{\partial \tau_0^2} - \frac{\partial^2 F}{\partial \tau_0 \partial a_I} [(C\tau + D)^{-1}]^{IJ} C^{JK} \frac{\partial^2 F}{\partial \tau_0 \partial a^K}. \]  

(4.21)

If we take into account that \( V_I = 4(\partial^2 F / \partial a^I \partial \tau_0) \), we see that the shift of (4.21) has the same structure as the shift of the second term in (4.19) under symplectic transformations.

It follows that the contact term can be written as

\[
T(\vec{u}) = 4 \frac{\partial^2 F}{\pi i \partial \tau_0^2},
\]  

(4.22)

In some cases we can use the homogeneity properties of \( u_2 \) to write more explicit expressions for \( T(\vec{u}) \). In the case of \( N_f < 2N \) massless hypermultiplets we have

\[
T(\vec{u}) = \frac{1}{2N-N_f} \left( 2u_2 - \sum_I a^I \frac{\partial u_2}{\partial a^I} \right),
\]  

(4.23)

Notice from this expression that \( T(\vec{u}) \) vanishes in the semiclassical regime, as required by asymptotic freedom. This coincides with [9][12] in the appropriate cases.

Using the relation between higher rank \( SU(N) \) Yang-Mills theory and the Toda-Whitham hierarchy [28][29][30][31][32], one can introduce a set of “times” in the prepotential which can be seen to be dual to the higher order Casimirs. This makes possible the computation of contact terms for the two-observables coming from these Casimirs using the same arguments we have given here, and generalizing the expression (4.22) to include the rest of the time variables [33]. These variables were also considered in [12] in the context of the twisted theory, and the contact terms for the higher Casimirs were derived using a blow-up argument.
4.6. Remark on the normalization

The overall normalization of the integral (4.1) has a meaning and can in principle be fixed by physical computations or by comparison to topological invariants. In particular the constants $\alpha, \beta$ in (4.5) are functions of the group and for $G = SU(N)$ are functions of $N$. Some constraints on these constants can be obtained from the factorization of the measure in certain regions of $\mathcal{M}_{\text{Coulomb}}$ expected on physical grounds.

Let us focus on $G = SU(N)$ and consider a semiclassical region of moduli space with scalar vevs:

$$\phi = M \begin{pmatrix} N_2 & 1 & N_1 \\ 0 & -N_1 & 1 \end{pmatrix} + \begin{pmatrix} \phi_1 \\ 0 \\ \phi_2 \end{pmatrix}$$

(4.24)

where $\phi_1, \phi_2$ are traceless. Since it is important to keep track of quantum scales to understand the behavior of the measure we introduce the quantum scale $\Lambda_N$ and require that $Z_{DW}$ be dimensionless. We work in the semiclassical region

$$|M| \gg |\phi_1^a - \phi_1^b|, |\phi_2^i - \phi_2^j| \gg |\Lambda_N|$$

(4.25)

for $1 \leq a < b \leq N_1$, $N_1 + 1 \leq i < j \leq N_1 + N_2$. The physics of this region is that we have a hierarchy of symmetry breakings:

$$SU(N) \xrightarrow{M} SU(N_1) \times SU(N_2) \times U(1) \rightarrow U(1)^{N_1-1} \times U(1)^{N_2-1} \times U(1)$$

(4.26)

with $N = N_1 + N_2$. At the large scale $M$ we integrate out $N_1 N_2$ vectormultiplets corresponding to the off-diagonal blocks. It is not difficult to show that, up to relative corrections of order $\mathcal{O}(\phi/M)$, the semiclassical prepotential reduces to:

$$F = \frac{i}{4\pi} \sum_{a < b} (\phi_{1a} - \phi_{1b})^2 \log \frac{(\phi_{1a} - \phi_{1b})^2}{\Lambda_N^{2N_1}} + \frac{i}{4\pi} \sum_{i < j} (\phi_{2i} - \phi_{2j})^2 \log \frac{(\phi_{2i} - \phi_{2j})^2}{\Lambda_N^{2N_2}} + \frac{i}{4\pi} N_1 N_2 (NM)^2 \log \frac{(NM)^2}{\Lambda_N^4}$$

(4.27)

with renormalization group matching conditions:

$$\frac{\Lambda_{N_1}}{\Lambda_N} = \left( \frac{\Lambda_N}{NM} \right)^{N_2}, \quad \frac{\Lambda_{N_2}}{\Lambda_N} = \left( \frac{\Lambda_N}{NM} \right)^{N_1}$$

(4.28)

One can then show that the $SU(N)$ $\Psi$-function with scale $\Lambda_N$, denoted $\Psi_{SU(N),\Lambda_N}$ factorizes in the region (4.25) as:

$$\Psi_{SU(N),\Lambda_N} \rightarrow \Psi_{SU(N_1),\Lambda_{N_1}} \Psi_{SU(N_2),\Lambda_{N_2}} \Psi_{U(1)} \left( 1 + \mathcal{O}(\phi/M) \right)$$

(4.29)
Moreover, in the semiclassical region we have:

\[ A^x B^\sigma \rightarrow \alpha(N)^x \beta(N)^\sigma \left( \prod_{\beta > 0} \left( \frac{\tilde{\alpha} \cdot \tilde{\phi}}{\Lambda_N} \right)^{(\chi + \sigma)/2} \right) \]  

(4.30)

with the product over the positive roots of \( SU(N) \). From this we easily find the factorization in the region (4.25):

\[ \prod_{\tilde{\alpha} > 0} (\tilde{\alpha} \cdot \tilde{\phi}) \rightarrow (NM)^{N_1 N_2} \prod_{\tilde{\alpha}_1 > 0} (\tilde{\alpha}_1 \cdot \tilde{\phi}_1) \prod_{\tilde{\alpha}_2 > 0} (\tilde{\alpha}_2 \cdot \tilde{\phi}_2) \]  

(4.31)

where \( \tilde{\alpha}_1, \tilde{\alpha}_2 \) are positive roots of \( SU(N_1), SU(N_2) \), respectively.

Thus, the nontrivial functions in the measure factorize in the region (4.25) as expected on physical grounds. Factorization of the entire measure implies that the measure \( \frac{dM dM}{|\Lambda_U(1)|^2} \) picks up nontrivial dependence on \( |M|^2 \) which we have not predicted on a priori grounds. However, the holomorphic part of the measure, \( (NM)^{N_1 N_2} \) can be expected on a priori grounds since it accounts for the \( R \)-charge anomaly of the vectormultiplets integrated out at scale \( M \). Combining this insight with Seiberg’s trick of regarding constants in an effective Lagrangian as vev’s in some theory at a higher scale to determine holomorphic dependence, we can give an heuristic argument for the \( N \)-dependence of \( \alpha(N), \beta(N) \). We regard the constants \( \alpha(N), \beta(N) \) as well as \( \Lambda_N \) as carrying \( R \)-charge. Thus, as in (4.31) we expect the factorization

\[ \alpha(N) = \alpha(N_1) \alpha(N_2) (\alpha_{U(1)})^{N_1 N_2} \]  

(4.32)

for some constant \( \alpha_{U(1)} \). Consequently, there should be \( N \)-independent constants \( \kappa_1, \kappa_2 \) such that

\[ \alpha(N) = e^{\kappa_1 N + \kappa_2 N^2} \]  

(4.33)

Similar formulae hold for \( \beta \). Unfortunately we can only fix one linear combination using the known constants for the \( SU(2) \) case, which have been found in [4] by comparing to explicit results for Donaldson invariants. As remarked in [34][4], to compare the results of the physical theory to mathematical results, one has to multiply the Donaldson-Witten function by the order of the center of the gauge group.
5. Single-valuedness of the integrand

The generalized $u$-plane integral (4.1) derived in the previous section is not manifestly well-defined because of monodromy around divisors where the SW curve $\Sigma$ (or the abelian variety $J(\Sigma)$ in the integrable system) degenerates. In this section we perform a careful check of the monodromy invariance of the integral in the case of simply-connected manifolds. The semiclassical analysis applies to any simply-laced gauge group. The strong-coupling analysis is only complete for $G = SU(3)$.

5.1. Semiclassical monodromy

The classical monodromy group is isomorphic to the Weyl group of the gauge group, and it is generated by the Weyl reflections $r_i$ associated to the root basis, $i = 1, \cdots, r$. Semiclassically this monodromy has a quantum correction due to the one-loop contribution to the prepotential.

The general form of the semiclassical monodromy has been presented in [14][15] for any gauge group. The action of the $r_i$ monodromy on $\vec{a}$ is given by the matrix

$$r_i = 1 - \vec{\alpha}_i \otimes \vec{w}_i,$$  

(5.1)

where the simple roots $\vec{\alpha}_i$ are expanded in the Dynkin basis, and $\vec{w}_i$ are the fundamental weights. The classical monodromy acting on $(\vec{a}_D, \vec{a})$ is given then by

$$P(r_i) = \begin{pmatrix} (r_i^{-1})^t & 0 \\ 0 & r_i \end{pmatrix}.$$  

(5.2)

The one-loop correction to the prepotential

$$F_{\text{one-loop}} = \frac{i}{4\pi} \sum_{\vec{\alpha} > 0} Z_{\vec{\alpha}}^2 \log \left( \frac{Z_{\vec{\alpha}}^2}{\Lambda^2} \right),$$  

(5.3)

(where the sum is over the positive roots) gives, in addition to the Weyl reflection, a theta-shift in the coupling constant of the form

$$\tau \rightarrow (r_i^{-1})^t [\tau - \vec{\alpha}_i \otimes \vec{\alpha}_i] r_i^{-1}.$$  

(5.4)

The semiclassical monodromy matrix is then given by

$$M(r_i) = \begin{pmatrix} (r_i^{-1})^t & -(r_i^{-1})^t(\vec{\alpha}_i \otimes \vec{\alpha}_i) \\ 0 & r_i \end{pmatrix}.$$  

(5.5)
The invariance of the Coulomb path integral under these monodromies is a non-trivial check of our expression. First we analyze the lattice sum, then the measure factor including the gravitational contributions. To analyze the lattice sum, it is convenient to redefine the variables \( \lambda, D, \eta \) and \( \chi \) by performing the Weyl transformation \( r_i^{-1} \). Notice that the lattice \( \Gamma \) is invariant under this redefinition, as the fundamental weights are shifted by roots. The measure for the variables \( D, \eta, \chi \) is invariant under this transformation, since it is an orthogonal transformation. Also, the two-observables \( V_I \) are derivatives with respect to \( a_I \) of duality-invariant quantities, so they transform as \( \partial/\partial a_I \), therefore the terms involving the two-observables remain invariant after the Weyl transformation of \( \lambda \) and \( D \). For the phase factor depending on \( w_2(X) \), we can use the properties of the Cartan-Killing form and the vector \( \vec{\rho} \) and see that it gives the additional term \( \pi i (w_2(X), \vec{\alpha}_i \cdot \vec{\lambda}) \), where the dot denotes the Cartan-Killing form on the weight space and \((\cdot, \cdot)\) denotes the usual product in integer cohomology. We then have an additional phase factor in the lattice sum:

\[
\exp \left( i\pi (\vec{\lambda} \cdot \vec{\alpha}_i, \vec{\alpha}_i) + i\pi (w_2(X), \vec{\alpha}_i \cdot \vec{\lambda}) \right). \tag{5.6}
\]

The simple roots are expanded in the Dynkin basis. To see that this phase factor is one, we take into account the decomposition in \((110)\). The term \((5.6)\) then reads

\[
i\pi \left( (\vec{\lambda}_Z \cdot \vec{\alpha}_i, \vec{\alpha}_i) + (w_2(X), \vec{\lambda}_Z \cdot \vec{\alpha}_i) + \sum_{I,J=1}^{c-1} (\vec{m}_I \cdot \vec{\alpha}_i)(\vec{m}_J \cdot \vec{\alpha}_i)(\pi^I, \pi^J) + \sum_{I=1}^{c-1} (\vec{m}_I \cdot \vec{\alpha}_i)(w_2(X), \pi^I) + 2 \sum_{I=1}^{c-1} (\vec{m}_I \cdot \vec{\alpha}_i)(\pi^I, \vec{\lambda}_Z \cdot \vec{\alpha}_i) \right). \tag{5.7}
\]

The last term is an even integer, and the other terms can be combined into even integers using the Wu formula

\[
(w_2(X), z) = (z, z) \mod 2, \quad z \in H^2(X, Z), \tag{5.8}
\]

Therefore, the lattice sum is invariant under the semiclassical monodromy.

Next we examine the measure factor in the Coulomb path integral. The measure \([d\bar{a}d\bar{a}]\) is invariant under the monodromy, and for the gravitational factor involving \( \chi \) we can use the symplectic transformation properties and the fact that \( \det r_i = -1 \) to derive

\[
\left( \det \frac{\partial u_I}{\partial a^J} \right)^{\chi/2} \rightarrow \exp \left[ \frac{\pi i \chi}{2} \right] \left( \det \frac{\partial u_I}{\partial a^J} \right)^{\chi/2}. \tag{5.9}
\]

24
Finally, we analyze the factor involving the discriminant. In the semiclassical regime we can use the expression (4.8). The Weyl reflection acts as follows on the $Z_\alpha$, with $\alpha > 0$: the basic root $\alpha_i$ changes its sign, therefore

$$Z_{\alpha_i} \to -Z_{\alpha_i},$$  \hspace{1cm} (5.10)$$
and the rest of the positive roots are permuted, so the product of the rest of the roots $Z_\alpha$ in the classical discriminant is invariant. The only change in the discriminant comes from this minus sign, and we finally obtain

$$\Delta_\Lambda^{\sigma/8} \to \exp[i\pi \sigma/2] \Delta_\Lambda^{\sigma/8}.$$  \hspace{1cm} (5.11)$$
For a four-manifold with $b_1 = 0$ and $b_2^+ = 1$, $\chi + \sigma = 4$, and the measure factor does not change under the monodromy. Therefore, the Coulomb integral is invariant under the semiclassical monodromies.

5.2. Duality transformations

To analyze the quantum monodromy, we have to consider the duality transformations of the Coulomb integral in the appropriate descriptions. To do this, we introduce the generalization of the lattice theta function of [9] to the higher rank case

$$\Theta_\Gamma(\tau_{IJ}, \alpha_I, \beta_I; P, \xi_I) = \exp \left[ -i\pi (\alpha_I, \beta_I) + \frac{\pi}{2} \left( \xi_{IJ} + (\text{Im}\tau)^{IJ} \xi_{IJ} \right) \right]$$

$$\times \sum_{\lambda \in \Gamma} \left[ -i\pi \tau_{IJ}(\hat{\lambda}_I^+, \hat{\lambda}_I^-) \tau_{IJ}(\hat{\lambda}_I^-, \hat{\lambda}_I^+) - 2\pi i(\hat{\lambda}_I^+, \xi_I) + 2\pi i(\hat{\lambda}_I^-, \alpha_I) \right],$$  \hspace{1cm} (5.12)$$
where $\hat{\lambda}_I^+ = \lambda_I^+ + \beta_I$. Notice that in the rank one case we recover the complex conjugate of the theta function introduced in [9].

If we take

$$\xi_I = \frac{1}{2\pi} V_I S_- + \frac{\sqrt{2}}{16\pi} F_{IJ} \eta^J \chi^K \omega,$$

$$\beta_I = \sum_{J=1}^{c-1} m_J \pi^J, \hspace{0.5cm} \alpha_I = \frac{1}{2} w_2(X), \hspace{0.5cm} I = 1, \cdots, r,$$

and consider $\lambda_I^+$ as the integer class $\lambda_I^+ \in \mathbb{Z}$ introduced in (4.10), we see that the lattice sum (4.14) can be written as

$$\Psi = \exp \left[ \frac{S^2}{8\pi} V_I (\text{Im}\tau)^{IJ} V_J \right] \exp[i\pi (\alpha_I, \beta_I)] (\text{det} \text{Im}\tau_{IJ})^{-1/2}$$

$$\times \int \prod d\eta d\chi \exp \left[ \frac{\sqrt{2}}{16\pi} F_{IJ} \eta^J \chi^K (\text{Im}\tau)^{KL} V_L(S, \omega) \right] \Theta_\Gamma(\tau_{IJ}, \alpha_I, \beta_I; P, \xi_I),$$  \hspace{1cm} (5.14)$$
The overall factor involving $S^2$ combines with $T(\vec{u})$ to give the duality-invariant quantity $\hat{T}(\vec{u})$ introduced in (4.19).

We will now consider the transformation properties of this theta function under the group $Sp(2r, \mathbb{Z})$. The generators of the symplectic group are given in (2.7). The transformation properties are the following:

Under $\Omega$ we have:

$$\Theta_{\Omega}(-\tau^{-1})^{I J} \alpha_I, \beta^I ; P, -(\tau^{-1})^{I J} \xi_{J,+} - (\tau^{-1})^{I J} \xi_{J,-}$$

$$= \sqrt{\left| \frac{\Gamma}{\Gamma'} \right|} (\det i\tau^{I J})^{b_+ / 2} (\det -i\tau^{I J})^{b_- / 2} \Theta_{\Omega'}(\tau_{I J}, \beta^I, -\alpha_I ; P, \xi_I).$$

(5.15)

where $\Gamma'$ is the dual lattice. To derive the transformation law for $\xi_I$ in (5.15), one has to use that

$$(\text{Im} \tau)^{I J} - 2i(\tau^{-1})^{I J} = (\text{Im} \tau)^{I K} \tau_{K L} (\tau^{-1})^{L J}.$$

(5.16)

If there is a characteristic element $w_2$ such that $(\lambda^I, w_2) = (\lambda^I, \lambda^I) \mod 2$, the transformation law of (5.12) under $T_\theta$ is:

$$\Theta_{T}(\tau_{I J} + \theta_{I J}, \alpha_I, \beta^I ; P, \xi_I) = \exp\left[\frac{i}{2} \sum_I (w_2, \theta_{II} \beta^I)\right] \Theta_{T}(\tau_{I J}, \alpha_I - \frac{1}{2} \theta_{II} w_2 - \theta_{I J} \beta^I, \beta^I ; P, \xi_I).$$

(5.17)

Finally, under the transformation $A$ we have

$$\Theta_{\Gamma}(A \tau A^t, \alpha_I, \beta^I ; P, \xi_I) = \Theta_{\Gamma}(\tau_{I J}, A^{-1} \alpha, A^t \beta ; P, A^{-1} \xi).$$

(5.18)

Using these transformations, it is easy to check that the lattice sum (5.14)(except for the exponential involving $S^2$ and the phase) is then a modular form of weights ($(b_- + 1) / 2, (b_+ - 3) / 2$). To derive this result, one formally considers the Grassmann variables $\eta, \chi$ as modular forms of weight $(0, 1)$, and takes into account the change induced in the Grassmann measure. The modular factors then combine with the measure $[d\alpha d\beta]$ and the gravitational factors to give the Coulomb integral for the dual variables.

5.3. Explicit check of quantum monodromy invariance for $SU(3)$

Using the above transformation properties, we can analyze the quantum monodromy in the $SU(3)$ case, as we know the explicit strong coupling spectrum in this case [14]. To obtain the appropriate form of the integral, we will make a symplectic transformation for each pair of mutually local charges in such a way that in the resulting theory there are
two electrically charged particles with charges $q^I_i = \delta^I_i$, $i, I = 1, 2$. For the two massless states $\vec{\nu}_1, \vec{\nu}_2$ in (2.18), we have to perform the transformation $\Omega^{-1}$. For the states $\vec{\nu}_3, \vec{\nu}_4$, the appropriate symplectic transformation is given by $\Omega^{-1}\mathcal{A}^{-1}T_{-\theta}$, where

$$A = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \quad \theta = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}. \quad (5.19)$$

Finally, for $\vec{\nu}_5, \vec{\nu}_6$, the symplectic transformation has again the structure $\Omega^{-1}\mathcal{A}^{-1}T_{-\theta}$ with

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad \theta = \begin{pmatrix} 2 & -1 \\ -1 & -1 \end{pmatrix}. \quad (5.20)$$

Therefore, in this basis, the monodromies associated to the two mutually local massless states are given by:

$$M_i = \begin{pmatrix} 1 & e^{ii} \\ 0 & 1 \end{pmatrix}, \quad i = 1, 2, \quad (5.21)$$

where $(e^{ii})_{IJ} = \delta^I_i \delta^j_I$, and this holds for each of the three pairs of mutually local dyons. An important aspect of these transformations is in that in all cases we are left with dual theories where the shifts in the $\Gamma$ lattice are given by

$$\beta^I = \frac{1}{2} w_2(X), \quad I = 1, \cdots, r, \quad (5.22)$$

i.e. they are Spin$^c$ structures. This result is obtained using the higher rank theta function transformations (5.15), (5.17) and (5.18). It is important to notice that the shifts in the $\Gamma$ lattice are defined modulo integer cohomology classes.

Now the monodromy invariance of the integral can be easily checked. The strong coupling monodromies are just theta-angle shifts in the dual coupling constants, and they are given by

$$\theta^{(i)}_{I,J} = \delta^I_i \delta^j_J. \quad (5.23)$$

We can now use (5.17) to see that the only change in the higher rank theta function is given by the phase $\exp[-\pi iw_2(X)^2/4]$. There is also a change in the measure associated to the factor involving the discriminant. Near a singular locus this factor has the structure given in (4.7), and the monodromy acts on $Z$ as $Z \to e^{2\pi i} Z$. We then obtain a factor $\exp[i\pi \sigma /4]$. But the second Stiefel-Whitney class verifies that

$$w_2(X)^2 = \sigma \mod 8, \quad (5.24)$$

therefore both phases combine to 1 and the integral is invariant under the strong coupling monodromies.
5.4. General case

In the general case, the verification of quantum monodromy invariance requires a precise knowledge of the strong coupling spectrum (or, equivalently, of the monodromy subgroup of the symplectic group associated to the relevant hyperelliptic curve). On the other hand, one can always write the monodromy associated to a monopole divisor in the form (5.21) with a submatrix $q^2 e^{i\theta}$, where $q = \gcd(\nu_k)$, through an appropriate symplectic transformation [16]. If this symplectic transformation is such that the $U(1)^i$ factor has a shift like (5.22), then the above argument goes through. Notice that, near the divisor where a hypermultiplet of charge $q$ becomes massless, the discriminant of the curve has a zero of order $q^2$. Thus a general proof of quantum monodromy invariance follows if the symplectic transformation taking the monodromy to (5.21) also makes the dual line bundle $\lambda^i$ a Spin$^c$ structure. Unfortunately, the monodromy group $\Gamma$ has not been studied in sufficient detail to make the explicit check, although we fully expect it to work.

6. Definition of the integral and wall-crossing

The generalized $u$-plane integral (4.1) above is a formal (albeit monodromy-invariant) expression. In order to give meaning to the integral and, ultimately, derive topological invariants for four-manifolds, we must define it carefully and examine its metric dependence properties.

6.1. Defining the integral

The integrand of (4.1) has bad behaviour at the singularities on the moduli space. Therefore, we should regularize it appropriately near the codimension one submanifolds where dyons become massless and also near infinity.

The first step in the regularization is to choose appropriate coordinates along these submanifolds. A divisor where a hypermultiplet becomes massless can be given locally by the equation $a^i = 0$, and this gives a preferred coordinate along this locus. The other coordinates should be chosen according to the region we are considering along the locus. At the $\mathcal{N} = 1$ points, for instance, one should choose dyonic coordinates for all the dyonic $U(1)$ factors, while at the region where a monopole locus goes to infinity, one should choose electric coordinates for the remaining variables. At a generic point at infinity, we choose electric variables for all the $U(1)$ factors. Finally, near a point where mutually non-local dyons become massless, there are no truly appropriate coordinates, and the divergences
of the quantities involved in the integral are quite different from the ones associated to the monopole loci and to infinity. In the case of the AD points of the $SU(3)$ theory, we will check that the integral is well-behaved near these points by using the $\epsilon, \rho$ coordinates introduced in $[2.21]$. 

The second step in the regularization is to introduce appropriate cutoffs at the singularities. In the case of the monopole loci, we choose tubular neighbourhoods defined by $|a^i| > r$, where $r$ is some radius that we will take to zero at the end. Near a generic point at infinity, the prepotential is naturally expressed in terms of the combinations $Z_{\vec{\alpha}}$, and the cutoff is given by $|Z_{\vec{\alpha}}| < R$, where $R \to \infty$, for some positive roots $\vec{\alpha}$ (different choices of these roots give different directions at infinity). Near the AD points of the $SU(3)$ theory, we will introduce an IR cutoff in the $\epsilon$ plane, $|\epsilon| > r$, and analyze the behaviour of the theory when we take $r \to 0$ (recall that $\epsilon$ is a coordinate near this point). Notice that this regularization can be interpreted as the substitution of the original moduli space $\mathcal{M}_{\text{Coulomb}}$ by a “regularized” moduli space $\mathcal{M}_{\text{Coulomb}}^{\text{reg}}$, which is a manifold with a non-connected boundary.

Finally, we perform the integrals over the corresponding variables. The procedure is now similar to the one in $[9]$. We first perform the integral over the phase of the complex coordinates chosen for each region, and this procedure gives a projection of the terms of the form $a^i \tau^i$ onto terms with $\nu = \mu$. As we will see in the next sections, in the $SU(3)$ case, the resulting integrals converge, although their metric dependence can be discontinuous, resulting in wall-crossing.

6.2. Metric dependence of the integral

To study the possible metric dependence of the integral, we follow the strategy in $[3]$. We consider the variation of the Coulomb integral with respect to a first order variation $\delta \omega$ in the period point. All the dependence on $\omega$ in the Coulomb integral appears in the lattice sum $\Psi$. The variation is most easily expressed using the representation $[1.17]$, and reads

$$
\delta \Psi = \sum_{\lambda \in \Gamma} \int \prod_{l=1}^r d\eta^I d\chi^I \int_{-\infty}^{+\infty} \prod_{l=1}^r db^I e^{-SE} \left[ -4\pi (\text{Im} \tau_{IJ}) \lambda^I_+ (\lambda^J, \delta \omega) - \frac{i}{4\pi} V_I(S, \delta \omega) b^I 
- \frac{i}{4} \sqrt{2} \int_{ijkl} \eta^I \eta^J (\lambda^K, \delta \omega) + iV_I ((S, \delta \omega) \lambda^I_+ + (\lambda^I, \delta \omega) S_+) \right],
$$

(6.1)

\footnote{We trust there will be no confusion with $r$ as the rank of the gauge group, nor $r_i$ as a Weyl reflection.}
where

\[
S_E = i\pi \tau_{IJ}(\lambda^I_+, \lambda^I_-) + i\pi \tau_{IJ}(\lambda^I_-, \lambda^I_+) - \frac{1}{8\pi} b^I (\text{Im}\tau)_{JJ} b^J + i V_I(S, \lambda^I_-) \\
+ \frac{i}{4\pi} V_I(S, \omega)b^I + \frac{i\sqrt{2}}{16\pi} F_{IJ} \eta^I \chi^J (b^K + 4\pi \lambda^K_+) + i\pi((\bar{\lambda} - \bar{\lambda}_0) \cdot \bar{\rho}, w_2(X)).
\] (6.2)

Topological field theory promises us that \(\delta \Psi\) is the integral of a total derivative. In fact, the metric variation in (6.1) can be written as

\[
\delta \Psi = \sum_{\lambda \in \Gamma} \int \prod_{I=1}^r d\eta^I d\chi^I \int_{-\infty}^{+\infty} \prod_{I=1}^r db^I \left[ \{ \bar{Q}, \Phi e^{-S_E} \} - 4\pi \frac{\partial}{\partial b^I} e^{-S_E}(\lambda^I, \delta \omega) \right],
\] (6.3)

where

\[
\Phi = i(\text{Im}\tau)_{IJ}(\lambda^I, \delta \omega) + \frac{1}{4\pi} (S, \delta \omega) V_I \chi^I.
\] (6.4)

Now we can use the fact that, according to the transformations (4.18) the \(\bar{Q}\) operator is given by

\[
\bar{Q} = i\sqrt{2}\eta^I \frac{\partial}{\partial a^I} + i(4\pi \lambda^I_+ - b^I) \frac{\partial}{\partial \chi^I}
\] (6.5)

and write the metric variation as a total derivative in field space with respect to the antiholomorphic coordinates:

\[
\delta \Psi = i\sqrt{2} \frac{\partial}{\partial a^I} \Upsilon^I,
\] (6.6)

where

\[
\Upsilon^I = \sum_{\lambda \in \Gamma} \int \prod_{J=1}^r d\eta^J d\chi^J \int_{-\infty}^{+\infty} \prod_{K=1}^r db^K \eta^I \Phi e^{-S_E}.
\] (6.7)

We introduce now an \((r, r-1)\) form on the Coulomb moduli space as

\[
\Omega = i\sqrt{2} \sum_{l=1}^r (-1)^{l+r-1} \Upsilon^I da^1 \wedge \cdots da^r \wedge d\bar{a}^1 \wedge \cdots \wedge d\bar{a}^r.
\] (6.8)

which satisfies

\[
d\Omega = \bar{\Omega} = i\sqrt{2} \sum_{l=1}^r \frac{\partial}{\partial a^I} \Upsilon^I da^1 \wedge \cdots da^r \wedge d\bar{a}^1 \wedge \cdots \wedge d\bar{a}^r.
\] (6.9)

Taking into account that the measure and the observables in the Coulomb integral are holomorphic, we can use Stokes theorem to write the metric dependence as an integral over the boundary of the regularized Coulomb branch:

\[
\delta Z^{\text{reg}}_{\text{Coulomb}} = \int_{\partial \mathcal{M}^{\text{reg}}_{\text{Coulomb}}} A^I B^\sigma e^{U+S E} T^V \Omega.
\] (6.10)
6.3. Wall-crossing formulae along the monopole loci

We will analyze now the generic wall-crossing at a monopole locus. For simplicity and concreteness, we will focus on the case of the pure $SU(3)$ theory. The monopole locus is defined by the equation $a^2 = 0$. As explained in section 4.3, there is a symplectic transformation to a basis in which $\lambda^2 \in H^2(X, \mathbb{Z}) + \frac{1}{2} w_2(X)$. For the other $U(1)$ factor, according to our remarks in section 6.1, we should choose the appropriate coordinates depending on the region of the monopole locus, i.e. we are allowed to perform symplectic transformations that leave $a^2$ fixed but change $a^1$.

Along the monopole locus, then, the behaviour of the $\tau_{22}$ coupling is given by

$$\tau_{22} = \frac{1}{2\pi i} \log a^2 + \cdots,$$

$$\mathcal{F}_{222} = -\frac{1}{2\pi i} \frac{1}{a^2} + \cdots,$$  \hspace{1cm} (6.11)

while $\tau_{11}$, $\tau_{12}$ and the other $\mathcal{F}_{IJK}$ are smooth (except when we are at an $\mathcal{N} = 1$ point or at infinity, where we will have “wall-crossing for wall-crossing”. This will be analyzed in a moment). Denote $y \equiv \text{Im} \tau_{22}$. An analysis similar to the one performed in [1] shows that the possible discontinuities in the integral are associated to terms involving only $1/(y^{1/2} \pi^2)$, and they occur when $(\lambda^2, \lambda^2) < 0$, $\lambda^2 = 0$. These are the usual conditions for SW wall-crossing for $\lambda^2$.

Taking this into account, we can easily find the terms that contribute to wall-crossing in the integral (4.1), using the explicit expression in (4.16). First of all, the factor $\det \text{Im} \tau_{IJ}$ appearing in the photon partition function has the structure

$$\det \text{Im} \tau_{IJ} = y (\text{Im} \tau_{11} + O(1/y)),$$  \hspace{1cm} (6.12)

and similarly

$$\left(\text{Im} \tau\right)^{-1} = \begin{pmatrix} (\text{Im} \tau_{11})^{-1} + O(1/y) & O(1/y) \\ O(1/y) & O(1/y) \end{pmatrix}. \hspace{1cm} (6.13)$$

Therefore, in the term written in (4.16) the only surviving contribution is given by

$$-\frac{1}{32\pi} \lambda_+^2 \mathcal{F}_{111} \mathcal{F}_{222} \left[ 4\pi (\lambda^1, \omega) + i (\text{Im} \tau_{11})^{-1} V_1(S, \omega) \right]. \hspace{1cm} (6.14)$$

The term in $S_+^2$ (involving $(\text{Im} \tau)^{-1}$) in (4.14) can be analyzed in the same way, with the result that the only contribution comes from $V_1^2 (\text{Im} \tau_{11})^{-1} (S_+^2/8\pi)$. On the monopole locus, we also have the following expansion in powers of $a^2$:

$$\tau_{11}(a^1, a^2) = \tau_{11}^{(0)} + a^2 \tau_{11}^{(1)} + \cdots,$$ \hspace{1cm} (6.15)
where $\tau_{11}^{(0)}(\bar{a}^1, \bar{a}^2 = 0)$ is the first term in an expansion in $\bar{a}^2$ and is different from zero. As we have already noticed, the positive powers of $\bar{a}^2$ do not contribute to the discontinuity of the integral, therefore we can put $\tau_{11} = \tau_{11}^{(0)}$ in the wall-crossing formula and write it as the integral of a residue:

$$ WC(\lambda^2) = \frac{\sqrt{2}}{8} e^{i\phi(\lambda^2)} \int_D da^1 da^2 \text{Res}_{a^2=0} \left\{ A^x B^\sigma \exp[pU + S^2 T_V - iV_2(S, \lambda^2)] \cdot \sum_{\lambda_1 \in \Gamma_1} q_{22}^{-(\lambda^2, \lambda^2)/2} q_{12}^{-(\lambda^1, \lambda^2)} \Psi(\lambda^1) \right\}. $$

In this expression, $D$ denotes the monopole divisor, $\phi(\lambda^2)$ is a global phase which depends on $\lambda^2$ and is obtained through the appropriate symplectic transformation to the monopole locus, $q_{IJ} = \exp(2\pi i \tau_{IJ})$, and

$$ \Psi(\lambda^1) = \frac{1}{\sqrt{(\text{Im}\tau_{11})_{(0)}}} \frac{\partial \tau_{11}^{(0)}}{\partial \bar{a}^1} \exp \left\{ \frac{1}{8\pi} V_1^2(\text{Im}\tau_{11})_{(0)}^{-1} S_+^2 \right\} \cdot \exp \left[ -i\pi \tau_{11}^{(0)} (\lambda^1_+)^2 - i\pi \tau_{11} (\lambda^1_-)^2 - iV_1(S, \lambda^1_+) - iV_1(S, \lambda^1_-) - i\pi(\lambda^1_+, \alpha_1) \right] \cdot [4\pi(\lambda^1, \omega) + i(\text{Im}\tau_{11})_{(0)}^{-1} V_1(S, \omega)], $$

where we have denoted $(\text{Im}\tau_{11})_{(0)} = (1/(2i)) (\tau_{11} - \tau_{11}^{(0)})$ and $\alpha_1$ is the phase we have for the $a^1$ theory. This phase, as well as the shift in the lattice $\Gamma_1$, depends on our choice of symplectic basis (we will make definite choices when we consider the “wall-crossing for wall crossing,” because in this case there is also a preferred $a^1$ variable).

Notice that the wall-crossing formula involves an integral which is very similar to a rank one $u$-plane integral depending on a “background field” $a^2$, and where the antiholomorphic part of the theory involves a restriction to $\bar{a}^2 = 0$.

### 6.4. Wall-crossing for wall-crossing

An important aspect of the integral (6.16) is that it has wall-crossing by itself. Along the monopole locus, there are three distinguished points where the $a^1$ theory has singularities. These are the $N = 1$ points where the divisors intersect, the region at infinity, and the AD points. The behaviour near the AD points will be analyzed later. In this section we will focus on the wall-crossing for wall-crossing near the $N = 1$ points and at infinity.

Near an $N = 1$ point the appropriate variable for the $a^1$ theory is also a “magnetic” one, therefore near this point we have $a^1 \to 0$ and the behaviour of $\tau_{11}$ is similar to (6.11).
For this choice of the variable, the shift in the lattice $\Gamma_1$ is also given by $w_2(x)/2$, i.e. the $\lambda^1$ are also Spin$^c$ structures. The wall-crossing behaviour of the integral for the $a^1$ variable is very similar to the usual SW wall-crossing analyzed in [9]. Again, we have wall-crossing for $(\lambda^1, \lambda^1) < 0, \lambda^1_+ = 0$. The discontinuity is now a double residue and is given by

\[
WC(WC(\lambda^1, \lambda^2)) = -4 \pi^2 e^{2\pi i (\lambda^1, \lambda^0_0)} \text{Res}_{a^1, a^2=0} \left\{ A^B B^\sigma \exp[ pU + S^2TV - iV_1(S, \lambda') ] \right. \\
\left. \cdot \prod_{l, j=1}^2 q_{l,j}^{-\frac{1}{2}(\lambda^l, \lambda^j)} \right\}.
\]

We have chosen the $\mathcal{N} = 1$ point at $u = (27/4)^{1/3} \Lambda$. The phase factor involving $\tilde{\lambda}_0$ is the generalization to the higher rank case of a similar factor considered in [9]. It gives the dependence of the SW contribution on the generalized Stiefel-Whitney class, and can be obtained from (5.15). The wall-crossing for wall-crossing at the other $\mathcal{N} = 1$ points can be obtained in a similar way (they will have different global phases, according to the $\mathbb{Z}_3$ symmetry).

At infinity along the monopole locus, the physics is that of an $SU(2)$ theory embedded in $SU(3)$, i.e. we have the quantum-corrected gauge symmetry breaking pattern $SU(3) \to U(1) \times SU(2)$, where the $U(1)$ (corresponding to the $a^1$ theory) is weakly coupled in electric variables, and the $SU(2) \to U(1)$ is weakly coupled in magnetic variables. There is a duality frame, therefore, where the behaviour of $\tau_{11}$ is given by

\[
\tau_{11} = \frac{i}{2\pi} \log a^1 + \cdots
\]

and corresponds to electric variables, i.e. the shift in the lattice $\Gamma_1$ is given by $\beta^1 = (C^{-1})^{1/2} \pi^J$. The wall-crossing of the integral on $a^1$ will then be a Donaldson wall-crossing, exactly like the one analyzed in [9]. The expression we get is formally identical to the one in (6.18), although the conditions for wall-crossing in $\lambda^1$ are the ones for Donaldson wall-crossing, and one must use the appropriate duality frame.

6.5. Wall-crossing at infinity

The relevant information to analyze the wall-crossing at infinity is encoded in the semiclassical one-loop correction to the prepotential (5.3). In the $SU(3)$ case it is given by:

\[
\mathcal{F}_{\text{one-loop}} = \frac{i}{4\pi} \sum_{i=1}^3 Z_i^2 \log \left( \frac{Z_i^2}{\Lambda^2} \right),
\]
where we denote $Z_i = Z_{\tilde\alpha_i}$, corresponding to the three positive roots of $SU(3)$, $\tilde\alpha_i$, $i = 1, 2, 3$. The explicit expressions are $Z_1 = 2a^1 - a^2$, $Z_2 = -a^1 + 2a^2$, $Z_3 = Z_1 + Z_2 = a^1 + a^2$. To analyze the conditions for wall-crossing, we focus on the photon partition function of the lattice sum (4.14):

$$\exp\left[-i\pi \tau_{IJ}(\lambda_1^I + \lambda_2^J) - i\pi \tau_{IJ}(\lambda_1^I - \lambda_2^J)\right] \sim \prod_{\tilde\alpha > 0} \left(\frac{Z_{\tilde\alpha}}{\Lambda}\right)^{-(\tilde\lambda_+ \cdot \tilde\alpha)^2} \left(\frac{Z_{\tilde\alpha}}{\Lambda}\right)^{(\tilde\lambda_- \cdot \tilde\alpha)^2}. \quad (6.21)$$

We can approach the region at infinity in moduli space in many ways, keeping one of the $Z_i$, $i = 1, 2, 3$ to be finite and the other two, $Z_j$, $j \neq i$, going to infinity (notice that we cannot keep two of the $Z_i$ finite, as $Z_3 = Z_1 + Z_2$). The conditions for a possible wall-crossing in $\tilde\lambda$ are then given by $\tilde\lambda_+ \cdot \tilde\alpha_j = 0$, $j \neq i$, as one can easily check from (6.21). As any two positive roots are linearly independent, we find $\tilde\lambda_+ = 0$. Therefore, there is no wall-crossing at infinity for $SU(3)$ (across codimension one walls): the integral is not discontinuous when $\tilde\lambda_+ = 0$. This is in contrast with the case of the non-simple rank two group $SU(2) \times SU(2)$, where there are only two positive roots and therefore there are directions at infinity where one finds wall-crossing (namely, the Donaldson wall-crossing associated to each of the $SU(2)$ factors).

One can also check this behaviour for $SU(3)$ using the $u$, $v$ variables, going to infinity along the $u$ or the $v$ planes, and using the explicit expressions for the behaviour of the prepotential given in [14]. Again, one finds that the condition for a possible wall-crossing along these directions is $\tilde\lambda_+ = 0$ and there is no discontinuity in the integral.

7. The blowup formula

The blowup formula generalizing [35] can be easily derived following the method used in [9]. Since there are manifolds with vanishing SW contributions it suffices to derive the formula for $Z_{\text{Coulomb}}$. The latter is easily derived by studying the change of the measure in (4.1). One then applies a universality argument.

Let $\tilde{X} = Bl_P(X)$ be the blowup at a smooth point. Then $\tilde{\sigma} = \sigma - 1$, $\tilde{\chi} = \chi + 1$. The change in the measure under $X \to \tilde{X}$ is just:

$$\mu_{\tilde{X}} = \frac{\alpha}{\beta} \left(\det \frac{\partial u_{I,J}}{\partial a^I}\right)^{1/2} \Delta^{-1/8} \mu_X \quad (7.1)$$

Now let $B$ denote the class of the exceptional divisor, with $B^2 = -1$. In the chamber $B_+ = 0$ (or more properly, for a fixed correlation function, where $B_+ < \epsilon$ for some
sufficiently small $\epsilon$) the $\Psi$ function factorizes to a $\Psi$-function for $X$ times a holomorphic $\Phi$-function involving a sum over the root lattice. Indeed we may write:

$$\tilde{\lambda}_+^I = \lambda_+^I$$

$$\tilde{\lambda}_-^I = \lambda_-^I + n^I B$$

where $n^I$ is in $\mathbb{Z} + m_j e^J$. The shift $e^J$ depends on the generalized Steifel-Whitney class of the gauge bundle $\tilde{E} \to \tilde{X}$. In the chamber $B_+ = 0$ the $\Phi$-function factorizes as:

$$\tilde{\Psi}_\tilde{X} = \sum_{n^I} e^{i\pi\tau_I n^I + i t V_I n^I - i \sum_I n^I \Phi_X} \equiv \Theta_{m^I \vec{e}, \Delta}(t \tilde{V} | \tau) \Phi_X$$  \hspace{1cm} (7.3)

where we have written $\tilde{S} = S + t B$ and $\tilde{\Delta} = (1, \ldots, 1)$. Thus, accounting for the contact term, the integrand for the blown-up manifold $\tilde{X}$ is related to that for $X$ by the replacement of zero-observables:

$$e^U \to e^U \frac{\alpha}{\beta} \left( \det \frac{\partial u_J}{\partial a^I} \right)^{1/2} \Delta^{-1/8} e^{-t^2 T_V} \Theta_{m^I \vec{e}, \Delta}(t \tilde{V} | \tau)$$  \hspace{1cm} (7.4)

Note that the expression must be monodromy invariant. Indeed, it has modular weight zero. This observation can be used to derive the required contact terms $T_V$ for $V$ other than the quadratic Casimir [12]. Moreover, since it is invariant, it is a function of $t$ and the Casimirs $u_2, \ldots, u_r$.

Physically, we expect the defect $B$ creating the blown-up manifold can be represented by an infinite number of local observables. The ring of local BRST invariant observables is generated by the Casimirs $u_2, \ldots, u_r$. Thus there must be polynomials $B_{\vec{e}, k}(u_2, \ldots, u_r)$ such that

$$\frac{\alpha}{\beta} \left( \det \frac{\partial u_J}{\partial a^I} \right)^{1/2} \Delta^{-1/8} e^{-t^2 T_V} \Theta_{m^I \vec{e}, \Delta}(t \tilde{V} | \tau) = \sum_{k \geq 0} t^k B_{\vec{e}, k}(u_2, \ldots, u_r+1)$$  \hspace{1cm} (7.5)

The fact that $B_{\vec{e}, k}(u_2, \ldots, u_r)$ are polynomials can be proven as follows: the blowup expression (7.4) is monodromy invariant, in particular of weight zero, so it must be a function of $u_I$, $I = 2, \ldots, r+1$, and $t$. Using the $R$-symmetry, we see that $t$ has to be of charge $-2$, hence the polynomial $B_{\vec{e}, k}$ has charge $2k$. On the other hand, the expression (7.3) has no singularities in the moduli space. This is because the theta function involved in the expression never has singularities, and the only possible singularities come from $\Delta^{-1/8}$.  

35
But these must be cancelled by zeros of the theta function, as follows from monodromy invariance.

In the case of the $SU(2)$ theory, the explicit expression for these polynomials was obtained in [9] using the expansion of the theta functions in terms of Eisenstein series, but in the higher rank case these expansions are not available. However, these expressions can probably be obtained using the relation between Seiberg-Witten theory and integrable systems. For $A_r$, the integrable system relevant to the Seiberg-Witten solution is the periodic Toda lattice [28][29][30]. The solutions to both models are straightline motions in the Jacobian of a hyperelliptic curve. Indeed, we recognize that (7.3) is essentially the $\tau$ function for the Toda hierarchy. Solutions to the Toda equations can be obtained from the Baker-Akhiezer function, and comparing the $t$ expansion of these solutions should determine the polynomials $B_{\vec{e},k}(u_2,\ldots,u_{r+1})$. We have not carried out the details of this procedure.

In any case, the blowup formula at higher rank is:

$$
\left\langle \exp[I(S) + tI(B) + pO] \right\rangle_{\tilde{X}} = \left\langle \exp[I(S) + pO] \tau(t,O_2,\ldots,O_{r+1}) \right\rangle_X
$$

$$
= \sum_{k \geq 0} t^k \left\langle \exp[I(S) + pO] B_{\vec{e},k}(O_2,\ldots,O_{r+1}) \right\rangle_X
$$

(7.6)

8. Behaviour at the Argyres-Douglas points

The Coulomb integral (4.1) depends on the metric of the four-manifold $X$. Its variation with respect to the metric can be written in terms of an integral over the boundary of the regularized Coulomb branch, as in (6.10). In general, this integral over the boundary will vanish, due to the damping factors associated to the behaviour of the couplings near the singularities or in the semiclassical region. However, at the AD points of the $SU(3)$ theory, there is an $\mathcal{N} = 2$ superconformal field theory with a finite value of the gauge coupling. The situation is reminiscent of the behaviour of the $N_f = 4$ theory analyzed in [4], where it was found that generic correlation functions have a continuous dependence on the metric. Therefore, one should analyze the possible continuous metric dependence associated to these superconformal points.
8.1. A general argument

The blowup formula derived in section 7 severely constrains the possibility of continuous metric dependence. This is because the blow-up formula relates the Donaldson-Witten function of manifolds with different signatures. As we will show below, for sufficiently large signature (e.g., $\sigma > -11$ for $G = SU(3)$) the measure near the superconformal points is sufficiently smooth that the metric variation vanishes. Now, the blow-up formula relates the invariants on $\tilde{X}$ to invariants on a manifold with $\sigma(X) = \sigma(\tilde{X}) + 1$. If there is no continuous variation in the latter correlators there cannot be any such variation in the former. Care should be taken with this argument since the blowup formula only applies for $\omega$ in certain chambers of the forward light cone in $H^2(X; \mathbb{R})$. For any given correlator, the formula applies in a chamber with $B_+ < \epsilon$ for some sufficiently small $\epsilon$, where $B$ is the exceptional divisor of the blow-up. If there is no continuous metric variation in this chamber then, given metric-independent wall-crossing formulae, there cannot be any continuous variation in any other chamber. (In fact, as we have seen there is no wall-crossing from infinity on codimension one walls, so there is really only one chamber.)

One could ask why an argument like this doesn’t rule out continuous metric dependence in the $N_f = 4$ theory considered in [9]. The reason is that, in this case, the inequality involving the signature also includes the ghost number $Q$ of the correlators, and the condition not to have any metric dependence has the form of an upper bound on $2\sigma + Q$. The above argument does not apply in this case, and one can easily check that the blowup formula is perfectly compatible with continuous metric dependence for the $N_f = 4$ theory. The reason for the different behaviours (and for the different bounds on the signature) has to do with the fact that, in the $N_f = 4$ theory, the continuous metric dependence comes from the behaviour at infinity, while the superconformal points in $SU(N)$ super Yang-Mills theories are in an “interior” region of the moduli space.

8.2. An explicit check

The above argument is rather general and should be checked by explicit computation. We now give a detailed analysis of the behavior near the AD points for $G = SU(3)$. In particular we explicitly show the absence of continuous variation for $\sigma > -11$. 

37
8.2.1. Convergence near the AD points

First of all, we must analyze the convergence of the Coulomb integral itself, as the
divergences of the integrand of \((1.1)\) near the superconformal point are rather different from
the ones we have considered. We have to introduce a cutoff \(r\) for the variable \(\epsilon\) introduced
in \((2.21)\), and study the behaviour of the integral as \(r\) goes to zero, as we have indicated in
section 6.1. To do that, we first consider the antiholomorphic terms with \(\tau^{-n}\) behaviour.
These come from the terms in \(\mathcal{F}_{IJK}\), in \((4.16)\), and whose structure near the AD point
was presented in \((2.34)\). The most divergent term corresponds to \(\mathcal{F}_{111}\mathcal{F}_{221} \sim \tau^{-4}\). We
have to write the measure of the integral in terms of \(\epsilon, \rho\) variables. The jacobian of the
change of variables from \(a^I\) to \(x^J = \epsilon, \rho\) can be computed at leading order from \((2.25),
(2.29)\):

\[
\det \frac{\partial a^I}{\partial x^J} = \frac{6c\epsilon^{7/2}}{\Lambda^{5/2}} H(\rho) + \cdots
\]  

(8.1)

The measure is then

\[
[dad\overline{a}] = \left( \frac{36c^2}{\Lambda^5} |H(\rho)|^2 |\epsilon|^7 + \cdots \right) d\epsilon d\overline{\epsilon} d\rho d\overline{\rho}.
\]  

(8.2)

Because of the factor \(|\epsilon|^7\) in the measure, we see that the leading behaviour of the integral
is smooth, so it converges. Notice that that the rest of the terms involved in the integrand
\((V_I, \tau_{IJ}, T_V, u, v)\) are smooth as \(\epsilon\) goes to zero, as one can see from \((2.21), (2.30),\) and
\((2.33)\). Thus we conclude that the integral is well-defined in the limit \(r \to 0\).

8.2.2. Explicit formulae for the metric variation

Now we want to study the possible metric dependence of the integral. The first step
in doing that is to write explicit expressions for the \(\Upsilon^T\) quantities defined in \((6.7)\). After
doing the Grassmann integrals in the rank two case, one obtains

\[
\begin{align*}
\Upsilon^T &= - \frac{i \sqrt{2}}{16 \pi} (\det \text{Im} \tau)^{-1/2} \exp \left[ \frac{1}{8 \pi} V_J (\text{Im} \tau)^{JK} V_K S_2^T \right] \\
& \quad \cdot \sum_{\lambda \in \Gamma} \exp \left[ -i \pi \tau_{IJ} (\lambda_+^J, \lambda_-^I) - i \pi \tau_{IJ} (\lambda_-^J, \lambda_+^I) - i \pi \left( (\tilde{\lambda} - \tilde{\lambda}_0) \cdot \tilde{\rho}, w_2(X) \right) - i V_I (S, \lambda_-^I) \right] \\
& \quad \cdot [4 \pi (\lambda_+^J, \omega) + i (\text{Im} \tau)^{KL} V_L (S, \omega)] \\
& \quad \cdot \left\{ F_{22K} (i (\text{Im} \tau)_{1J} (\lambda_-^J, \omega) + \frac{1}{4 \pi} (S, \omega) V_1) - F_{21K} (i (\text{Im} \tau)_{2J} (\lambda_-^J, \omega) + \frac{1}{4 \pi} (S, \omega) V_2) \right\}, \\
\Upsilon^T &= - \frac{i \sqrt{2}}{16 \pi} (\det \text{Im} \tau)^{-1/2} \exp \left[ \frac{1}{8 \pi} V_J (\text{Im} \tau)^{JK} V_K S_2^T \right] \\
& \quad \cdot \sum_{\lambda \in \Gamma} \exp \left[ -i \pi \tau_{IJ} (\lambda_+^J, \lambda_-^I) - i \pi \tau_{IJ} (\lambda_-^J, \lambda_+^I) - i \pi \left( (\tilde{\lambda} - \tilde{\lambda}_0) \cdot \tilde{\rho}, w_2(X) \right) - i V_I (S, \lambda_-^I) \right] \\
& \quad \cdot [4 \pi (\lambda_+^J, \omega) + i (\text{Im} \tau)^{KL} V_L (S, \omega)] \\
& \quad \cdot \left\{ F_{11K} (i (\text{Im} \tau)_{2J} (\lambda_-^J, \omega) + \frac{1}{4 \pi} (S, \omega) V_2) - F_{12K} (i (\text{Im} \tau)_{1J} (\lambda_-^J, \omega) + \frac{1}{4 \pi} (S, \omega) V_1) \right\}. 
\end{align*}
\]

To analyze the behaviour near the superconformal point, we use the duality frame specified by the symplectic transformation \( \text{(2.20)} \), in order to use the “small torus” \( \text{(2.22)} \) and the explicit solutions in section 2. The differential form \( \Omega \) of \( \text{(6.8)} \) can be written now in terms of the \( \epsilon, \rho \) variables. The explicit expression follows from:

\[
\begin{align*}
\Omega &= i \sqrt{2} \left( \det \frac{\partial a^J}{\partial x^J} \right) \left\{ \left( \frac{\partial a^2}{\partial \tau^1} \Upsilon^T - \frac{\partial a^1}{\partial \tau^2} \Upsilon^T \right) d \epsilon \wedge d \rho \wedge d \tau \right. \\
& \quad \left. + \left( \frac{\partial a^2}{\partial \tau^2} \Upsilon^T - \frac{\partial a^1}{\partial \tau^1} \Upsilon^T \right) d \epsilon \wedge d \rho \wedge d \tilde{\rho} \right\}.
\end{align*}
\]

There are two terms in \( \text{(8.4)} \) which can lead to variation \( \delta Z_{\text{Coulomb}} \). In the first term in \( \text{(8.4)} \) we take the integral over the \( \rho \) boundary, which will be a set of three tubular neighbourhoods of the monopole divisors \( \rho^3 = 1 \). The contributions of these boundaries leads to discontinuous, wall-crossing type, metric dependence. This is just the monopole wall-crossing analyzed in section 6.3.

The second term in \( \text{(8.4)} \) is more interesting and it gives the possible metric dependence associated to the AD points. We regularize the integral by cutting a small disk of radius \( r \) around \( \epsilon = 0 \). The boundary integral in \( \epsilon \) will then be along the circle of radius \( r, S_r, \) with center at \( \epsilon = 0 \). We want to know if there are surviving contributions as \( r \to 0 \).

To analyze the integral over \( S_r \) it is important to take into account monodromy invariance under \( \epsilon \to e^{2 \pi i \epsilon} \). This invariance can be verified explicitly using the fact, crucial
to the entire argument we are giving, that after the symplectic transformation (2.20), the line bundles $\lambda^1$ define Spin$^c$ structures. First, one can easily check, using the behaviour of $F_{IJK}$ in (2.34), that all the powers of $\epsilon$ appearing in the expression are positive or zero. Actually the only contribution one can have when $r \to 0$ comes from the terms with no powers of $\epsilon$. These involve $F_{111}, F_{112}$. We can write the metric dependence then as the integral of a residue, in the same way that we have written the wall-crossing formulae:

$$
\delta Z_{\text{Coulomb}}(\omega) = -\frac{1}{8\pi} \int d\epsilon \int d\rho d\bar{\rho} A^\chi B^\sigma \left( \det \frac{\partial a^I}{\partial x^J} \right) e^{U + S^2 T_V} 
\cdot (\det \text{Im}\tau(0))^{-1/2} \exp \left[ \frac{1}{8\pi} V_J(\text{Im}\tau)^{(0)}_{JK} V_K S^2_+ \right] \sum_{\lambda \in \Gamma} \left[ i(\text{Im}\tau)^{(0)}_{2J}(\lambda^J, \delta \omega) + \frac{1}{4\pi}(S, \delta \omega)V_2 \right] 
\cdot \exp \left[ -i\pi \tau(\bar{\rho})(\lambda^1_+)^2 - i\pi \tau^{(0)}_{22}(\lambda^2_+)^2 - i\pi \tau_{IJ}(\lambda^I_-, \lambda^J_+) - iV_I(S, \lambda^I_-) \right] 
\cdot \frac{d\tau(\bar{\rho})}{d\bar{\rho}} [4\pi(\lambda^1_+, \omega) + i(\text{Im}\tau)^{1L}_{(0)} V_L(S, \omega)],
$$

where the (0) (sub)superscript means that in the antiholomorphic quantities we take $\epsilon = 0$. In (8.5) we have omitted a global phase depending on the non-abelian magnetic fluxes.

The expression (8.5) is not zero in general. We conclude that $Z_{\text{Coulomb}}$ has continuous metric dependence from the AD points.

8.2.3. The $\rho$-plane theory

We now examine the metric dependence we have discovered in more detail. One of the interesting things about (8.5) is that it involves, essentially, a rank one integral associated to the elliptic curve (2.22). We will refer to this curve as the "$\rho$-curve." To see this, let us study the leading behaviour for $\epsilon \to 0$ of the measure appearing in (8.5). Up to a constant that can be computed from (2.17), (2.32), and (8.1), together with (2.31), we find the behaviour

$$
A^\chi B^\sigma \left( \det \frac{\partial a^I}{\partial x^J} \right) \sim \epsilon^{\frac{3\sigma + 14}{4}} \Delta^\sigma_{\rho} \omega_\rho^{1-\frac{1}{\sigma}},
$$

where $\omega_\rho$ is the period of the curve (2.22). Similarly, using (2.30) we also have for the 2-observable

$$
V_1 \sim \frac{\epsilon^{1/2}}{\omega_\rho},
$$

which again behaves as the 2-observable of the rank one case (involving the period of the $\rho$-curve). Comparing the factors (8.6)(8.7) to the general expressions for the rank one $u$-plane integrals we see that the leading behavior for $\epsilon \to 0$ is governed by a family of
effective supersymmetric theories described by the $\rho$-curve and which we will refer to as the “$\rho$-plane theory.”

8.2.4. Monopoles to the rescue

The nonvanishing continuous metric variation (8.5) of $Z_{\text{Coulomb}}$ appears to spell doom for the topological invariance of $Z_{\text{DW}}$. Before jumping to this conclusion we must consider the possible continuous metric variation of the other terms in (1.4). In particular, we must examine the continuous dependence of the mixed SW/Coulomb integrals along the monopole divisors $Z_{D_i(1)}$. In the present case the relevant divisors are $D_i^{(1)}$, $i = 1, 2, 3$ defined by the roots of $\rho^3 = 1$. The integrals $Z_{D_i(1)}$ will be analyzed in some detail in the next section, but their continuous metric dependence is easy to analyze here. The Seiberg-Witten contributions are obtained by cancellation of wall-crossing of $Z_{\text{Coulomb}}$ along the monopole divisors, and they are integrals along these subvarieties involving the Seiberg-Witten invariants at the singularities of the $\rho$-plane (corresponding to the dyons becoming massless at $\rho^3 = 1$). They are obtained from the behaviour of $Z_{\text{Coulomb}}$ near $\rho^3 = 1$ in such a way that wall-crossings cancel:

$$WC_{\rho_i}(Z_{\text{Coulomb}}) + WC(Z_{D_i^{(1)}}) = 0$$

(8.8)

where $\rho_i$, $i = 1, 2, 3$ are the roots of $\rho^3 = 1$ and label the three monopole divisors near the AD point. We want to know the continuous metric dependence of these Seiberg-Witten contributions. That is we want to compute $\frac{\delta}{\delta \omega} Z_{D_i^{(1)}}$ for generic $\omega$, not just at walls. The continuous variation comes from the region $\epsilon = 0$, and we will denote this variation by $\delta_{\epsilon=0} Z_{D_i^{(1)}}$. Since the continuous metric dependence and the discontinuous metric dependence involve the behaviour with respect to different variables, we see that the wall-crossing of the integral over $\rho$, $\bar{\tau}$ in (8.5) near the $\rho^3 = 1$ divisors has to match the Seiberg-Witten wall-crossing of $\delta_{\epsilon=0} Z_{D_i^{(1)}}$ at these singularities. We then have

$$\delta_{\epsilon=0} Z_{D_i^{(1)}} = \int d\epsilon \epsilon^{\frac{3\sigma + \chi + 14}{4}} \sum_{\lambda \in \Gamma} SW(\lambda^1) \text{Res}_{\rho = \rho_i} F(\rho, \epsilon, \lambda^I, \delta \omega),$$

(8.9)

One must exercise caution when expressing the behavior of the integral at the AD points in terms of the $\rho$-plane theory since the matrix $(\text{Im} \tau)^{IJ}$ and $(\text{det Im} \tau)^{1/2}$ does lead to subleading terms in $1/(\text{Im} \tau(\rho))$.4
where $F(\rho, \epsilon, \lambda^I, \delta \omega)$ is a holomorphic function of $\epsilon, \rho$ which depends also on $\lambda^I$ and $\delta \omega$. It can be obtained, as we have indicated, by computing the wall-crossing of the $\rho, \overline{\rho}$ integral in (8.5) and matching it to the wall-crossing of a Seiberg-Witten contribution with the appropriate insertion of observables, as in the following section. Now we note that in the $\rho$-plane integral in (8.5), all the terms that do not correspond to a rank one integral for the curve (2.22) do not contribute to wall-crossing, as they involve subleading powers in $1/(\text{Im} \tau(\rho))$. Thus, the continuous metric dependence of $Z^{(1)}_D$ is expressed in terms of the $\rho$-plane theory.

Taking this into account, the metric dependence of $Z_{DW}$ near the AD point is a sum of two terms: one from the integral in (8.5) and one from the Seiberg-Witten contributions near the singularities at $\epsilon = 0, \rho^3 = 1$, and can therefore be written schematically as

$$
\delta Z_{DW} = \oint d\epsilon \overline{\epsilon} \sigma + 14 \{ \int d\rho d\overline{\rho} \left[ \cdots \right] + \sum_{\lambda \in \Gamma} \sum_{i=1}^{3} \text{SW}^{(1)}(\lambda^I) \text{Res}_{\rho = \rho_i} F(\rho, \epsilon, \lambda^I, \delta \omega) \},
$$

(8.10)

where $[\cdots]$ denotes the integrand of (8.5) up to the global power of $\epsilon$ that we have factored out.

8.2.5. Vanishing of $\delta Z_{DW}$ for $G = SU(3), \sigma > -11$.

We are finally ready to justify the assertion that $\delta Z_{DW} = 0$ for sufficiently large signature. This is a simple consequence of (8.10). From the scaling behaviour of the terms in (8.5) we see that all of the terms in the $\epsilon$ expansion of (8.5) have positive powers. Therefore, (8.10) will vanish if the power of $\epsilon$ in the measure is bigger than $-1$, i.e., if

$$
\sigma > -11,
$$

(8.11)

where we have taken into account that $\chi + \sigma = 4$. Notice that we can always make insertions of 2-observables which have no leading powers of $\epsilon$ (for example, $V_2 = \Lambda/c + O(\epsilon)$). Therefore, we can not write a general selection rule involving the ghost number of a given correlator, as in the $N_f = 4$ case analyzed in [9]. Rather we have a condition on the signature of the manifold, given by the bound (8.11). This bound is particular to the gauge group $SU(3)$. For other superconformal points associated to other gauge groups and/or matter content [36][37], we expect other explicit bounds depending on the $R$-charge spectrum near these points.

We can now complete the argument for topological invariance of $Z_{DW}$ by invoking the general argument at the beginning of this section since the blow-up formula holds for $Z_{DW}$ and the measure factor depending on $\epsilon$ is common to both contributions in (8.10).

5 Notice once more that the consistency of this procedure requires that the $\lambda^I$ bundles, which are the “line bundles” that couple to the $\rho$ theory, define Spin$^c$ structures.
9. The Seiberg-Witten contributions

9.1. The SW contribution along the monopole loci

As in [9], we expect that the higher rank Donaldson-Witten functional is given by the Coulomb integral (4.1) plus the contributions coming from the monopole divisors, as we have indicated in (1.4). Generically, along these divisors a dyon becomes massless and the low-energy effective theory contains one hypermultiplet coupled to one of the $U(1)$ factors. The twisted theory is now a “mixed” theory where one of the variables (the one that we have called $a^2$) is a distinguished coordinate but we can still perform duality transformations which leave $a^2$ fixed. We expect, however, that the twisted theory will localize to supersymmetric configurations for the $A^2$ vector multiplet coupled to the hypermultiplet. These simply give the Seiberg-Witten monopole equations for the $A^2$ variables. At the $\mathcal{N} = 1$ points, there are distinguished coordinates for both $a^1, a^2$, the effective theory contains two mutually local hypermultiplets (each of them coupled to each of the vector multiplets), and the twisted theory will localize to supersymmetric configurations for both vector multiplets coupled to the hypermultiplets. At these points, the contribution will be given then by Seiberg-Witten invariants $SW(\lambda^1), SW(\lambda^2)$.

On general grounds, the Donaldson-Witten functional for $SU(3)$ will be given by

$$ Z_{DW} = Z_{\text{Coulomb}} + \sum_i \int_{D_i^{(1)}} da^1_i d\overline{a}^1_i \sum_{\lambda^1, \lambda^2} \int_{\mathcal{M}_{SW}(\lambda^2)} \mu^i_{\lambda^1, \lambda^2}(a^1_i, \overline{a}^1_i, a^2) + \sum_{i=1}^3 \sum_{\lambda^1, \lambda^2} \int_{\mathcal{M}_{SW}(\lambda^1) \times \mathcal{M}_{SW}(\lambda^2)} \Phi_i(a^1, a^2), \quad (9.1) $$

where we have included a sum over the components of the codimension one divisor $D_i^{(1)}$, and also the contribution of the three $\mathcal{N} = 1$ points. $\mathcal{M}_{SW}(\lambda)$ is the Seiberg-Witten moduli space for the Spin$^c$ structure $\lambda$. The structure of the functions $\mu^i_{\lambda^1, \lambda^2}(a^1_i, \overline{a}^1_i, a^2)$, $\Phi_i(a^1, a^2)$ can be obtained by cancellation of wall-crossing, as in [9], and comparing to the formulae derived in section 6. We find that the Seiberg-Witten contribution along a monopole divisor is given by the function

$$ \mu^i_{\lambda^1, \lambda^2}(a^1_i, \overline{a}^1_i, a^2) = e^{\phi_i(\lambda^2)} \exp[pU + S^2 T_V - iV_2(S, \lambda^2)] C_{22}(a^1, a^2)^{(\lambda^2)^2/2} C_{12}(a^1, a^2)^{(\lambda^1, \lambda^2)} \cdot P(a^1, a^2)^{\sigma/8} L(a^1, a^2)^{\lambda^1/4} \Psi(\lambda^1), \quad (9.2) $$

43
where $\Psi(\lambda^1)$ is given in (6.17), $\phi_i(\lambda^2)$ is the appropriate global phase depending on the divisor and the corresponding symplectic transformation, and the functions $C_{22}, C_{12}, P(a^1, a^2), L(a^1, a^2)$ are given by

$$
\begin{align*}
C_{22}(a^1, a^2) &= \frac{a^2}{q_{22}}, \\
C_{12}(a^1, a^2) &= q_{12}^{-1}, \\
L(a^1, a^2) &= -\frac{\sqrt{2}}{8} \alpha^2 (\det \frac{\partial u_I}{\partial a_J})^2, \\
P(a^1, a^2) &= \frac{1}{32} \beta^8 \frac{\Delta^\Lambda}{a^2}. 
\end{align*}
$$

(9.3)

As we explained in section 8, the Seiberg-Witten contributions along the monopole divisors have continuous metric dependence near the AD point, which can be obtained by matching the wall-crossing of the $\rho, \bar{\rho}$ integral in (8.5) near $\rho^3 = 1$ to the wall-crossing coming from the Seiberg-Witten contributions at these singularities. This can be verified using the computations above, with the only difference that instead of having an integral over $a_i^1, \bar{a}_i^1$ (the coordinate which parametrizes the monopole divisors) we have a contour integral in $\epsilon$.

9.2. Contributions from the $\mathcal{N} = 1$ points

Now we follow the same approach to compute the functions involved at the $\mathcal{N} = 1$ points. By comparison with wall-crossing, their structure is

$$
\Phi_i(a^1, a^2) = e^{i\phi_i} \exp[2\pi i(\lambda^I, \lambda^I_0)] \exp[pU + S^2 T_V - iV_I(S, \lambda^I)] \\
\cdot \prod_{I, J=1}^{2} (\tilde{C}_{IJ}(a^1, a^2))^{\frac{1}{4}(\lambda^I, \lambda^J)} \tilde{P}(a^1, a^2)^{\sigma/8} \tilde{L}(a^1, a^2)^{\chi/4},
$$

(9.4)

where $\phi_i$ is a global phase depending on the generalized Stiefel-Whitney class and on the $\mathcal{N} = 1$ point. The functions $\tilde{C}_{IJ}(a^1, a^2), I, J = 1, 2, \tilde{P}(a^1, a^2), \tilde{L}(a^1, a^2)$ are given by

$$
\begin{align*}
\tilde{C}_{II}(a^1, a^2) &= \frac{a^I}{q_{II}}, \ I = 1, 2 \\
\tilde{C}_{IJ}(a^1, a^2) &= q^{-1}_{IJ}, \ I, J = 1, 2, \ I \neq J, \\
\tilde{L}(a^1, a^2) &= -4\pi^2 \alpha^2 (\det \frac{\partial u_I}{\partial a_J})^2, \\
\tilde{P}(a^1, a^2) &= 16\pi^4 \beta^8 \frac{\Delta^\Lambda}{a^1 a^2}.
\end{align*}
$$

(9.5)
We can thus write the SW contribution at the $N = 1$ points for $SU(N)$, which is a straightforward generalization of the above procedure,

$$\langle e^{U+I_2(S)} \rangle^{(i)}_{\lambda^1, \ldots, \lambda^r} = \alpha^\chi \beta^\sigma e^{i\phi_1} e^{2\pi i (\lambda^I, \lambda^J)} \left( \prod_{I=1}^r SW(\lambda^I) \right)$$

$$\cdot \text{Res}_{a^1 = \ldots = a^r = 0} \left\{ \left( \prod_{I=1}^r (a^I)^{2x^a + 3} - \frac{(\lambda^I)^2}{2} - \tilde{q}_{II} (\lambda^I)^2/2 \right) \prod_{1 \leq I < J \leq r} \left( q_{IJ}^{(\lambda^I, \lambda^J)} \right) \right\} \cdot \left( \frac{\Delta_{\Lambda}}{\prod_{I=1}^r a^I} \right)^{\sigma/8} \left( \frac{\det \partial a^I}{\partial a^J} \right)^{\chi/2} \exp[U + S^2 V_I - iv_I(S, \lambda^I)] \right\},$$

(9.6)

where $\tilde{q}_{II} = q_{II}/a^I$, and we have included in $\alpha$, $\beta$ the numerical factors that are obtained, as in (9.5), from matching to wall-crossing. It is important to notice that the quantities $\tilde{q}_{II}$ as well as the factor involving $\Delta_{\Lambda}/\prod_{I=1}^r a^I$ in (9.6) are regular at $a^I = 0$.

9.3. $SU(N)$ Donaldson invariants for manifolds of simple type

In this section we generalize the gauge group to $G = SU(N)$ for all $N$, but specialize the class of manifolds to those of simple type.

In the simple type case, we can evaluate the contribution at the $N = 1$ points using the explicit expressions given in [38][39] for the $N = 1$ point where $N-1$ monopoles become massless, together with the discrete $\mathbb{Z}_4N$ symmetry relating the $N = 1$ vacua. The eigenvalues for $\phi$ are [38]

$$\phi_n = 2 \cos \frac{\pi (n - \frac{1}{2})}{N}, \quad n = 1, \ldots, N.$$  

(9.7)

and from this expression we can easily compute the VEVs of the Casimirs,

$$c_{2s} \equiv \langle \text{Tr} \phi^{2s} \rangle = \left( \frac{2s}{s} \right) N, \quad c_{2s+1} \equiv \langle \text{Tr} \phi^{2s+1} \rangle = 0,$$

(9.8)

One also finds a relation

$$\sum_{I=1}^r \frac{\partial a^I}{\partial \phi_i} \sin \frac{\pi k I}{N} = i \cos \frac{\pi (i - \frac{1}{2})}{N},$$

(9.9)

and from this we can obtain,

$$\frac{\partial u_2}{\partial a^I} = -2i \sin \frac{\pi I}{N}.$$

(9.10)

This gives the value of $V_I$. 

45
Finally, we need the value of the off-diagonal couplings at the $N = 1$ point. These can be obtained using the scaling trajectory for the eigenvalues $\phi_n$ obtained in \cite{38}. This trajectory depends on a parameter $s$, and $s = 0$ corresponds to the $N = 1$ point. The value of the magnetic gauge coupling along this trajectory is given by:

$$
\tau_{IJ}(s) = \frac{2}{N} \sum_{K=1}^{N-1} \tau_K(s) \sin \frac{\pi IK}{N} \sin \frac{\pi JK}{N},
$$

(9.11)

and the eigenvalues of this matrix, $\tau_K(s)$, can be explicitly written in terms of some integrals. In particular, the leading behaviour of $\tau_K(s)$ as $s \to 0$ is given by \cite{38}

$$
\tau_K(s) = \frac{i}{2\pi \sin \frac{2\kappa}{2}} \int_{-b}^{b} d\theta \frac{\cos(1 - \kappa)\theta}{\sqrt{e^{-2s} - \sin^2 \theta}},
$$

(9.12)

where $\kappa = K/N$ and $b = \arcsin e^{-s}$. This integral has a divergent part $\frac{1}{2\pi i} \log s$ as $s \to 0$, and this produces the usual logarithmic divergence, diagonal in $I$ and $J$, in $\tau_{IJ}$. But the off-diagonal terms of $\tau_{IJ}$ are finite at $s = 0$ (i.e. at the $N = 1$ point), and they can be obtained from the finite part of the integral in (9.12). For $SU(3)$ we have for instance \cite{14},

$$
\tau_{12}(0) = \frac{i}{\pi} \log 2,
$$

(9.13)

as one can check from (9.12) and (9.11). Although we have not found an explicit expression for the finite part of (9.12), there are two important properties of the couplings $\tau_{IJ}(0)$, $I \neq J$, that one can deduce from (9.12) and (9.11): they are imaginary, and they satisfy the following symmetry property

$$
\tau_{IJ}(0) = \tau_{N-I \ N-J}(0).
$$

(9.14)

We can already write the contribution from the $N = 1$ points to the $SU(N)$ invariants, using the fact that these points are related by the $\mathbb{Z}_{4N} \subset U(1)_R$ symmetry. We must take into account the $R$-charges of the different operators in the correlation function, as well as the gravitational contribution to the anomaly that appears on a curved four-manifold. This anomaly can be computed from the microscopic theory (as in \cite{34}) or directly from the expression given in (9.6). The two computations must give the same result because
the factors in the measure were actually determined from an \( R \)-charge argument. In fact, as the \( R \)-charge of \( q_{II} \) is zero, we have

\[
R\left( \frac{\Delta_\Lambda}{\prod_{I=1}^r a^I} \right) \sigma^8 = \frac{\sigma}{2} N(N-1) - \frac{\sigma}{4} (N-1),
\]

\[
R(\det \frac{\partial u_I}{\partial a^J}) \chi^2 = \chi^2 N(N-1),
\]

\[
R\left( \prod_{I=1}^r \tilde{q}_{II}^{-(\lambda^I)^2/2} \right) = (N-1) \frac{2\chi + 3\sigma}{4},
\]

and the \( R \)-charges of these terms give the right \( R \)-charge coming from the underlying twisted SYM theory, namely

\[
\frac{N^2 - 1}{2} (\chi + \sigma).
\]

We conclude that the contribution of the \( \mathcal{N} = 1 \) vacua is

\[
\langle e^{U + U_2(S)} \rangle_{SU(N)} = \tilde{\alpha} \tilde{\beta} \sum_{k=0}^{N-1} \omega^k [N^2-1)\delta + N\tilde{\lambda}_0 \cdot \tilde{\lambda}_0] \sum_{\lambda^I} e^{2\pi i (\lambda^I, \lambda^I_0)} \left( \prod_{I=1}^{N-1} SW(\lambda^I) \right) \cdot \prod_{1 \leq I < J \leq r} \left( \tilde{q}_{IJ}^{-(\lambda^I, \lambda^J)} \right) \exp \left[ \sum_{s=1}^{N-1} p_{2s} \omega^{2k_s} c_{2s} + 2\omega^{2k_s} S^2 + 2\omega^k \sum_{I=1}^{N-1} (S, \lambda^I) \sin \frac{\pi I}{N} \right],
\]

(9.17)

where \( \omega = \exp[i\pi/N] \) and \( \delta = (\chi + \sigma)/4 \). In \( \tilde{\alpha}, \tilde{\beta} \) we have reabsorbed numerical factors that come from the evaluation of \( L, P \) and \( \tilde{q}_{II} \) at the point where \( N - 1 \) monopoles become massless (the values at the other points are obtained using \( R \)-symmetry.) The \( q_{IJ}, I < J \), can be obtained from (9.11) and (9.12) when \( s = 0 \), as we have discussed. We have also included in the phase factor labeling the \( \mathcal{N} = 1 \) vacua in (9.17) an additional term depending on the generalized Stiefel-Whitney class, which generalizes the \( SO(3) \) case considered in [34][4]. This term can be obtained if we take into account that the instanton number of the bundle, once non-abelian fluxes are included, satisfies (4.13). Equivalently, one can take into account the transformations (5.15), (5.17) and (5.18) to find this extra factor when we go through the different \( \mathcal{N} = 1 \) vacua and see that they generate precisely this extra phase.

We will now find a natural generalization of the phase factor \( e^{i\pi w_2^2(E)/2} \) obtained in [9] to guarantee that the resulting expression is real. Notice that this is also a consistency check of the above answer, as this factor must be a global phase depending on the generalized Stiefel-Whitney class. We then have to consider the properties of (9.17) under complex
conjugation (assuming that the overall factor $\tilde{\alpha}_N^N \beta_N^N$ is real). First notice that the first term in the sum, $k = 0$, changes by conjugation of the global phase: $\exp[-2\pi i(\lambda^I, \lambda_0^I)]$ (the factors $q_{IJ}$, $I < J$, are real). Now we take into account that, due to (9.14) and the expression (9.10), the right hand side of (9.17) has the symmetry $\lambda^I \to \lambda^{N-I}$, hence we can write the resulting phase as

$$e^{2\pi i(\lambda^I, \lambda_0^I)} e^{-2\pi i[(\lambda^I, \lambda_0^I) + (\lambda^{N-I}, \lambda_0^I)]}.$$  \hspace{1cm} (9.18)

Using the fact that $(C^{-1})^J_J + (C^{-1})^N_N^I$ is an integer, for all $I, J = 1, \ldots, N - 1$, we can write the second factor as

$$\exp\left\{-\pi i \sum_{J=1}^{N-1} [J(J + N - 2)(\pi^J, w_2(X))]\right\} = \exp[\pi i N\vec{v} \cdot \vec{v}],$$  \hspace{1cm} (9.19)

where we have taken into account the Wu formula (5.8) and the explicit form of the inverse Cartan matrix for $SU(N)$. For the rest of the terms in the sum, $k = 1, \ldots, N - 1$, we take into account that, under conjugation, $\omega^k \to -\omega^{N-k}$, and we change $\lambda^I \to -\lambda^I$. Using the transformation [4]:

$$SW(-\lambda) = (-1)^{\delta} SW(\lambda),$$  \hspace{1cm} (9.20)

one easily checks that the sum of the terms $k = 1, \ldots, N - 1$ changes by an overall sign of the form $(-1)^{N\vec{v} \cdot \vec{v}}$ (notice that, for manifolds of simple type, $\delta$ is an integer). Comparing with (9.13) we see that, under conjugation, (9.17) picks a global sign depending on the generalized Stiefel-Whitney class $\vec{v}$. Notice that $N\vec{v} \cdot \vec{v}$ is always an integer. Moreover, for $N$ odd it is an even integer, because in this case $NC^{-1}$ is an even form. Therefore, for $N$ odd, (9.17) is real. For $N$ even, it is then natural to include a phase factor of the form $e^{i\pi N\vec{v} \cdot \vec{v}/2}$ to make the above expression real. This factor is independent of the lifting of $\vec{v}$ as long as $N$ is even, and in the special case of $SU(2)$ we recover the factor introduced in [3].

10. **Application 1: Twisted $\mathcal{N} = 2$ superconformal field theories**

At the AD points there is an $\mathcal{N} = 2$ superconformal field theory, and as we are studying the twisted version of $\mathcal{N} = 2$ super Yang-Mills theory, the relevant spacetime symmetry algebra describing the model there is the twisted version of the $\mathcal{N} = 2$ extended superconformal algebra in four dimensions. Recall that this algebra includes extra bosonic
generators $K_\mu$ and $D$, corresponding to the special conformal transformations and the dilatations, respectively, as well as two Weyl spinors $S_{AI}$, $\overline{S}^A_I$, where $A$, $\hat{A}$ are spinor indices of $SU(2)_- \times SU(2)_+$, and $I$ is the $SU(2)_R$ index. There is also an $R$ generator corresponding to the non-anomalous $U(1)_R$-current. The topological twist changes the coupling to gravity of all the fields charged with respect to the internal symmetry group $SU(2)_R$, in the usual way. This means that in the twisted theory we consider the diagonal subgroup $SU(2)$ of $SU(2)_+ \times SU(2)_R$, and the internal symmetry index $I$ is promoted to a spinor index, $I \rightarrow \hat{A}$. We then obtain two scalar supercharges,

$$\overline{Q} = \epsilon^{\hat{A}\hat{B}} Q_{\hat{A}\hat{B}}, \quad \overline{S} = \epsilon^{\hat{A}\hat{B}} S_{\hat{A}\hat{B}},$$

(10.1)

where $\overline{Q}_{AI}$ is the usual supersymmetry charge. We can also define descent operators,

$$G_\mu = \frac{i}{4} \sigma^\mu_{\hat{A}\hat{B}} Q_{\hat{A}\hat{B}}, \quad T_\mu = \frac{i}{4} \sigma^\mu_{\hat{A}\hat{B}} S_{\hat{A}\hat{B}},$$

(10.2)

The twisted $\mathcal{N} = 2$ superconformal algebra includes the relations:

$$[\overline{Q}, D] = \frac{1}{2} \overline{Q}, \quad [\overline{S}, D] = -\frac{1}{2} \overline{S},$$

$$[\overline{Q}, R] = -\overline{Q}, \quad [\overline{S}, R] = \overline{S},$$

$$[G_\mu, D] = \frac{1}{2} G_\mu, \quad [T_\mu, D] = -\frac{1}{2} T_\mu,$$

$$[G_\mu, R] = G_\mu, \quad [T_\mu, R] = -T_\mu,$$

$$\{\overline{Q}, G_\mu\} = iP_\mu, \quad \{\overline{S}, T_\mu\} = iK_\mu,$$

$$\{\overline{Q}, \overline{Q}\} = 0, \quad \{\overline{S}, \overline{S}\} = 0,$$

$$\{\overline{Q}, \overline{S}\} = 2R - 4D.$$

(10.3)

One can define in a natural way (topological) chiral primary fields as those fields satisfying

$$\{\overline{Q}, \Phi\} = \{\overline{S}, \Phi\} = 0.$$ 

(10.4)

From the last relation in (10.3), we recall the well-known fact that for such fields $R(\Phi) = 2D(\Phi)$. The topological descendants of a chiral primary field are $n$-forms with the structure,

$$\Phi^{(n)}_{\mu_1 \cdots \mu_n} = \{G_{\mu_1}, \cdots, \{G_{\mu_n}, \Phi\} \cdots\},$$

(10.5)

and we see from (10.3) that they are also annihilated by $\overline{S}$. After integrating them on $n$-cycles, we find new chiral primary fields. Notice that the $R$-charge of the topological descendant $\Phi^{(n)}$ satisfies $R(\Phi^{(n)}) = R(\Phi) - n$. 

49
Now we can try to extract some information about the twisted superconformal field theory at the AD point from the results we have obtained above. An important new feature arising near the AD points is that the gravitational factors $A^\chi, B^\sigma$ have the following behaviour:

$$A^\chi B^\sigma \sim \epsilon^{\chi/4} \epsilon^{3\sigma/4}. \quad (10.6)$$

Recall that this factor measures the gravitational contribution to the anomaly of the $R$-current. The factor involving $\sigma$ is naturally interpreted as the $U(1)_R$ anomaly of the three mutually nonlocal hypermultiplets becoming massless at the AD point. We interpret the factor involving $\chi$ as a signal of the $R$-charge of the superconformal vacuum, leading to an anomaly $-\chi(X)/4$ in units where the anomaly of a single hypermultiplet is $-\sigma(X)/4$. We conclude that in the twisted superconformal theory on a manifold $X$ there is a selection rule for correlation functions:

$$\langle \Phi_1 \cdots \Phi_n \rangle_X \neq 0 \quad (10.7)$$

only for

$$\sum_i R[\Phi_i] = \frac{1}{10} \chi(X), \quad (10.8)$$

where we have taken into account that $R(\epsilon) = 2/5$. This has a striking resemblance to the selection rule for correlators in a twisted $d=2 \mathcal{N}=(2,2)$ sigma model on a Riemann surface $\Sigma$, where the $R$-charge of the vacuum is given by $-\hat{c}\chi(\Sigma)/2$.

The generalization of (10.7)(10.8) to $SU(N)$ can be determined by examining the order of vanishing of $\det \frac{\partial a}{\partial u}$. For simplicity, assume $N$ is odd. Since $u_j \sim \epsilon^j$, and there are $\frac{N-1}{2}$ vanishing $\beta$-periods, $a_i \sim \epsilon^{N+2}$, we expect that $\det \frac{\partial a}{\partial u} \sim \epsilon^{-(N-1)^2/8}$ and hence the RHS of (10.8) becomes $\frac{1}{8} \frac{(N-1)^2}{N+2} \chi(X)$ for the $\mathbb{Z}_N$ multicritical superconformal theories. \footnote{In deriving this result we have simply counted factors of $\epsilon$ in the determinant $\det \frac{\partial a}{\partial u}$. In principle the $O(1)$ factors in this determinant, which we have not computed, could lead to a cancellation of the coefficient of the leading divergence $\epsilon^{-(N-1)^2/8}$. We are assuming this does not happen.}

It is interesting to compare this result with some similar recent results for $\mathcal{N}=1$ theories \cite{10}. We believe that further information about the behavior of superconformal theories can be extracted from the above results, in particular from the $\rho$-plane theory of section 8 \cite{11}.
11. Application 2: Large $N$ limits of $SU(N) \mathcal{N} = 2$ SYM

As an application of (9.17) we now sketch the large $N$ asymptotics of the Donaldson invariants for $SU(N)$. While this has no obvious interest for topology, explicit results for correlators in supersymmetric Yang-Mills theory are rarities. It is even rarer that one can explicitly study a large $N$ limit using exact results. Since on hyperkähler manifolds the correlators of $\overline{\mathcal{Q}}$-invariant operators are the same in the topologically twisted theory as the “physical correlators” [22] we may hope to understand something of the physics of large $N \mathcal{N} = 2$ $SU(N)$ SYM theory. We know from [38] that this limit presents some unusual features.

The following identities will be useful to evaluate the correlators from (9.17):

\[
\sum_{k=0}^{N-1} \omega^{k\ell} = \begin{cases} 
N & \ell = 0 \mod 2N, \\
0 & \ell = 0 \mod 2, \quad \ell \neq 0 \mod 2N \\
\frac{2}{1 - \omega} & \ell = 1 \mod 2. 
\end{cases}
\] (11.1)

The last case, $\ell$ odd, leads to a nontrivial $1/N$ series:

\[
\sum_{k=1}^{N-1} \omega^{k\ell} = \frac{2i}{\pi \ell} N + 1 + 2i \sum_{t=1}^{\infty} \frac{B_{2t}(-1)^t}{(2t)!} \left( \frac{\pi \ell}{N} \right)^{2t-1} \] (11.2)

where the $B_{2t}$ are Bernoulli numbers.

11.1. The torus $X = T^4$.

Although our considerations in this paper have been mostly on simply-connected manifolds, the extension to the non-simply connected case can be done along the same lines (some interesting issues arising in this case are addressed in [42]). For the four-torus, however, the situation is very simple because the only basic class is $\lambda = 0$, and the Donaldson invariants are still given by (9.17) (with $\lambda = 0$). The reason for this is the following: since $T^4$ is flat, the monopole field in the SW equations must vanish and the SW moduli space is simply the space of harmonic 1-forms on $T^4$. However, this is a nongeneric situation and the complications of the bundle of antighost zeromodes are most easily handled, as is standard, by perturbing the equations. Then, since $T^4$ is hyperkahler the only basic class is $\lambda = 0$, with $SW(\lambda) = 1$. This is in accord with the physical argument using a nowhere-vanishing mass perturbation [34].
Let us consider the operators in the theory. Since the torus is not simply connected we could also introduce $Q$-closed 1-cycle and 3-cycle operators. These only contribute through their contact terms and do not change the following results in any essential way, so we omit these operators. It is convenient to rescale the $0$-observables to

$$A_j \equiv \frac{1}{c_{2j}} \text{Tr} \theta^2 j$$

for $j = 1, 2, \ldots$. Then we have, simply,

$$\exp \left( \sum t_j A_j + I(S) \right) = \sum_{k=0}^{N-1} \exp \left[ \sum_{j \geq 1} t_j \omega^{kj} + 2S^2 \omega^2 k \right].$$

(11.3)

Now we must decide how to define the large $N$ limit. We consider the $N \to \infty$ limit of finite polynomials in $t_j$ and $S$. By (11.1) we see that all correlators at fixed ghost number vanish identically for sufficiently large $N$. The large $N$ limit exists, but it is utterly trivial.

We can obtain some more interesting correlators in two ways. The first is to add $SU(N)/Z_N$ fluxes to the theory. These produce a factor $\omega^{-kf}$ in the sum over $N = 1$ vacua, where $f = -N \bar{v}^2$, $\bar{v} = \sum_{I=1}^{N-1} \pi^I \bar{w}_I$ with $\bar{w}_I$ the fundamental weights and $\pi^I$ integral classes. A second way to get interesting correlators is to introduce a new “conjugate” set of operators:

$$\bar{A}_j \equiv \frac{1}{c_{2L+2-2j}} \text{Tr} \theta^{2L+2-2j}.$$

(11.4)

where $L \equiv [(N-1)/2]$ and $j = 1, 2, \ldots$. Note that these operators do not have well-defined ghost number in the $N \to \infty$ limit.

With these modifications (11.4) becomes a little more intricate:

$$\left\langle \exp \left( \sum t_j A_j + \sum \bar{t}_j \bar{A}_j + I(S) \right) \right\rangle_f = \sum_{k=0}^{N-1} \omega^{-kf} \exp \left[ \sum t_j \omega^{2kj} + \sum \bar{t}_j \omega^{(2L+2-2j)k} + 2S^2 \omega^2 k \right].$$

(11.5)

Consider now a term $\sim \prod t_j^{\ell_j} \bar{t}_j^{\bar{\ell}_j} (S^2)^r$ where $\ell_j = \bar{\ell}_j = 0$ for all but finitely many $j$. We now apply (11.4) with the exponent

$$-f + 2 \sum j \ell_j + 2 \sum (L + 1 - j) \bar{\ell}_j + 2r.$$

(11.6)

We now must divide the problem into cases.

52
First suppose \( N \) is odd so \( N = 2L + 1 \). In this case \( N(C^{-1})^{IJ} \) is an even integral quadratic form, and hence the squared-fluxes \( f \) are always even. Since \( f, \ell_j, \) etc. are held fixed while \( N \to \infty \) we must have:

\[
\sum \bar{\ell}_j = 0 \mod 2,
\]

\[
-f + 2 \sum j \ell_j - 2 \sum (j-1/2) \bar{\ell}_j + 2r = 0.
\]

Thus, the large \( N \) theory is summarized by:

\[
\left< \exp \left( \sum t_j A_j + \sum \bar{t}_j \bar{A}_j + I(S) \right) \right> =
\]

\[
\frac{N}{2} \oint \frac{dz}{z} z^{-f/2} e^{2Sz} \exp \left[ \sum t_j z^j \right] \cdot \left[ \exp \left[ \sum \bar{t}_j z^{-j} \right] + \exp \left[ - \sum \bar{t}_j z^{-j+1/2} \right] \right].
\]

(11.8)

An interesting point is that the above \( 1/N \) “expansion” is exact.

Now we consider \( N \) even. The integral form \( N(C^{-1})^{IJ} \) is odd, and hence there are two subcases depending on whether the flux-squared \( f \) is even or odd. If \( f \) is even then the evaluation proceeds as before and

\[
\left< \exp \left( \sum t_j A_j + \sum \bar{t}_j \bar{A}_j + I(S) \right) \right> =
\]

\[
\frac{N}{2} \oint \frac{dz}{z} z^{-f/2} e^{2Sz} \exp \left[ \sum t_j z^j \right] \cdot \left[ \exp \left[ \sum \bar{t}_j z^{-j} \right] + \exp \left[ - \sum \bar{t}_j z^{-j+1/2} \right] \right].
\]

(11.9)

If \( f \) is odd then we evaluate the sums in (11.6) using (11.2). Now the \( 1/N \) expansions become nontrivial and do not terminate.

11.2. \( X = K3 \)

The situation for \( X = K3 \) is very similar. \( K3 \) is hyperkahler so the only basic class is \( \lambda = 0 \). Now \( \delta = 2 \) so \( \omega^{(N^2-1)/k} = \omega^{-2k} \). Thus the results (11.9)(11.10) continue to hold with the simple modification:

\[
\oint \frac{dz}{z} (\cdots) \to \tilde{\alpha}(N)^{24} \tilde{\beta}(N)^{-16} \oint \frac{dz}{z^2} (\cdots).
\]

(11.11)

11.3. Other 4-manifolds

For other four-manifolds with nonzero basic classes, the evaluation of the correlators is more complicated due to the off-diagonal couplings in (9.17), which mix the different
We will give here some indications on the structure of these correlators in the simple case of minimal surfaces of general type, and sketch a possible strategy to perform a systematic large $N$ expansion. In the case of minimal surfaces of general type, the only basic classes are $\pm K$, where $K$ is the canonical line bundle of the manifold, and moreover:

$$SW(-K) = 1,$$
$$SW(+K) = (-1)^\delta.$$  \hfill (11.12)

We can then introduce the variables $s^I$, $I = 1, \ldots, N - 1$, taking the values $\pm 1$, and define $2\lambda^I = s^I K$. The sum over the basic classes for the $k$ vacua now takes the form

$$\sum_{s^I = \pm 1} (-s^I)^\delta \exp \left\{ -\frac{\pi i}{2} K^2 \sum_{I < J} \tau_{IJ} s^I s^J + \omega^k(S, K) \sum_{I=1}^{N-1} \sin \frac{\pi I}{N} s^I \right\}.$$  \hfill (11.13)

This is a correlation function for a one-dimensional spin chain with $N - 1$ sites, with long range interactions given by the off-diagonal couplings $\tau_{IJ}$, and in the presence of a space dependent “magnetic field” proportional to $\sin(\pi I/N)$. The large $N$ limit of this expression corresponds then to the thermodynamic limit of the system. This suggests that one can study the large $N$ limit using standard techniques in statistical mechanics. One possible strategy is to rewrite (11.13) by introducing auxiliary variables $x^I$, $I = 1, \ldots, N - 1$, and reexpress the quadratic term in the $s^I$ spin variables using a gaussian integral. To do this, it is useful to consider the invertible matrix of couplings $\tilde{\tau}_{IJ}$, $I, J = 1, \ldots, N - 1$, where the diagonal couplings come from the regular part at the $N = 1$ point of the couplings defined in (9.11) (i.e. after subtracting the logarithmic divergence), and the off-diagonal terms are the ones in (11.13). This is actually the matrix of couplings that naturally appears in the Seiberg-Witten contribution in (9.6). The term involving the diagonal part of $\tilde{\tau}_{IJ}$ is just an overall factor depending on $N$ and $K^2$. After introducing the auxiliary variables $x^I$, the sum over the spin variables can be easily worked out, and the correlation function (11.13) becomes, for $\delta$ even:

$$2^{N-1} C(N) \int_{-\infty}^{+\infty} \prod_{I=1}^{N-1} dx^I \exp \left\{ -\frac{i}{\pi K^2} \sum_{I,J} \tilde{\tau}_{IJ}^{-1} x^I x^J + \sum_{I=1}^{N-1} \log \cosh \left[ \omega^k(S, K) \sin \frac{\pi I}{N} + x^I \right] \right\},$$  \hfill (11.14)

where $C(N)$ is an overall constant depending on $N$ and $K^2$. An analogous expression involving sinh can be obtained for $\delta$ odd. Notice that, for minimal surfaces of general type.
type one has $K^2 > 0$. One could evaluate this integral using a saddle-point approximation (which should be a good one in the large $N$ limit) and then systematically computing the corrections to the saddle-point.

For other four-manifolds, the set of basic classes is more complicated, but in the case of algebraic surfaces one has a complete description of this set and the Seiberg-Witten invariants have been explicitly computed \cite{4,13}. The structure of (9.17) indicates that the contribution of an $\mathcal{N} = 1$ vacuum will be given again by a correlation function in some statistical mechanical system. As we have suggested, this analogy could prove useful in studying the properties of the Donaldson-Witten function in the large $N$ limit, and one may likely find interesting phenomena (like phase transitions.) Another reason to study the large $N$ limit of Donaldson-Witten theory is a possible relation to topological strings, as was conjectured in \cite{45} (a relation between the large $N$ limit of Chern-Simons topological gauge theory and topological strings has been recently found in \cite{46}.)

11.4. Possible applications to Dbranes and matrix theory

There are several ways in which $\mathcal{N} = 2$ SYM is realized in the context of string theory, D-branes and M-theory. See \cite{47} and \cite{48} for recent reviews. We limit ourselves here to a few brief and superficial remarks.

Perhaps the most direct applications are to matrix theory. As noted above the physical theory and twisted theory correlators coincide for the hyperkahler 4-manifolds $X = T^4, K3$. The correlators on the 4-torus can be interpreted as a kind of finite temperature partition function: $Z_{DW} = \text{Tr} e^{-\beta H} (-1)^F \exp[\sum t_i A_i]$. 

In \cite{49} a matrix theory approach to studying Schwarzschild black holes was proposed. This approach requires the existence of a singularity in the equation of state, which should translate into a singularity in $\text{Tr} e^{-\beta H} \prod \mathcal{O}_i$ at large $N$ as a function of $\beta$. Although we are studying theories with 8 rather than 16 supercharges, one would expect the phenomenon required by \cite{49} to be rather generic. Unfortunately we find no evidence of the discontinuity in $N$ posited in \cite{49}. This might be due to the insertion of $(-1)^F$, and indeed that is consistent with the discussion in section 3.2 of \cite{51}.

In the realization of Seiberg-Witten theory via $M$-theory 5-branes described in \cite{51,48} the correlators of $\text{Tr} \phi^{2n}$ carry information about the quantum distribution of positions of $(D4)$ branes. The tendency of large $N$ correlators to vanish as described above would seem to suggest that if the 5branes are wrapped in the $x^{1,2,3}$ directions (to use the standard
choice of coordinates in \([51 \, [48])\) at finite temperature then the D4 branes - or tubes between NS5 branes - are very uniformly distributed.

When the 4-manifold \(X\) is not hyperkahler then the twisted and physical correlators can differ. However, topological correlators can very well be relevant in the theory of D-branes \([52]\) so the above results might also find applications in the theory of branes in more complicated compactifications of string/M theory.

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References

[1] S.K. Donaldson and P.B. Kronheimer, *The Geometry of Four-Manifolds*, Clarendon Press, Oxford, 1990.

[2] R. Friedman and J.W. Morgan, *Smooth Four-Manifolds and Complex Surfaces*, Springer Verlag, 1991.

[3] E. Witten, “Topological Quantum Field Theory,” Commun. Math. Phys. **117** (1988) 353.

[4] E. Witten, “Monopoles and four-manifolds,” hep-th/9411102; Math. Res. Letters **1** (1994) 769.

[5] S. Donaldson, “Irrationality and the $h$-cobordism conjecture,” J. Diff. Geom. **26** (1987) 141.

[6] G. Ellingsrud and L. Göttsche, “Wall-crossing formulas, Bott residue formula and the Donaldson invariants of rational surfaces,” alg-geom/9506019.

[7] L. Göttsche, “Modular forms and Donaldson invariants for 4-manifolds with $b_+ = 1$,” alg-geom/9506013; J. Am. Math. Soc. **9** (1996) 827.

[8] L. Göttsche and D. Zagier, “Jacobi forms and the structure of Donaldson invariants for 4-manifolds with $b_+ = 1$,” alg-geom/9612020.

[9] G. Moore and E. Witten, “Integration over the $u$-plane in Donaldson theory,” hep-th/9709193.

[10] V. Pidstrigach and A. Tyurin, “Localization of the Donaldson invariants along Seiberg-Witten classes,” dg-ga/9507004.

[11] P.M.N. Feehan and T.G. Leness, “$PU(2)$ monopoles and relations between four-manifold invariants,” dg-ga/9709022; “$PU(2)$ monopoles I: Regularity, Uhlenbeck compactness, and transversality,” dg-ga/9710032; “$PU(2)$ monopoles II: Highest-level singularities and realtions between four-manifold invariants,” dg-ga/9712005.

[12] A. Losev, N. Nekrasov, and S. Shatashvili, “Issues in topological gauge theory,” hep-th/9711108; “Testing Seiberg-Witten solution,” hep-th/9801061.

[13] N. Seiberg and E. Witten, “Electric-magnetic duality, monopole condensation, and confinement in $\mathcal{N} = 2$ supersymmetric Yang-Mills Theory,” hep-th/9407087; Nucl. Phys. **B426** (1994) 19

[14] A. Klemm, W. Lerche and S. Theisen, “Nonperturbative effective actions of $\mathcal{N} = 2$ supersymmetric gauge theories,” hep-th/9506150; Int. J. Mod. Phys. **A10** (1996) 1929.

[15] U.H. Danielsson and B. Sundborg, “The moduli space and monodromies of $\mathcal{N} = 2$ supersymmetric gauge theories,” hep-th/9504102; Phys. Lett. **B358** (1995) 273.

[16] P.C. Argyres and M.R. Douglas, “New phenomena in $SU(3)$ supersymmetric gauge theory,” hep-th/9505062; Nucl. Phys. **B448** (1995) 93.

[17] P.C. Argyres and A.E. Faraggi, “Vacuum structure and spectrum of $\mathcal{N} = 2$ supersymmetric $SU(N)$ gauge theory,” hep-th/9411057; Phys. Rev. Lett. **74** (1995) 3931.
[18] A. Klemm, W. Lerche, S. Theisen and S. Yankielowicz, “Simple singularities and $\mathcal{N} = 2$ supersymmetric Yang-Mills theory,” [hep-th/9411048]. Phys. Lett. B344 (1995) 169.

[19] T. Masuda and H. Suzuki, “On explicit evaluations around the conformal point in $\mathcal{N} = 2$ supersymmetric Yang-Mills theories,” [hep-th/9612240]. Nucl. Phys. B495 (1997) 149.

[20] T. Kubota and N. Yokoi, “Renormalization group flow near the superconformal points in $\mathcal{N} = 2$ supersymmetric gauge theories,” [hep-th/9712054].

[21] E. Witten, “On S-duality in abelian gauge theory,” [hep-th/9505180]. Selecta Mathematica 1 383

[22] C. Vafa and E. Witten, “A strong coupling test of S-duality,” [hep-th/9408074]. Nucl. Phys. B431 (1994) 3.

[23] J.E. Humphreys, Introduction to Lie algebras and representation theory, Springer-Verlag, 1972.

[24] M. Mariño and G. Moore, “Integrating over the Coulomb branch in $\mathcal{N} = 2$ gauge theory,” [hep-th/9712062].

[25] M. Matone, “Instantons and recursion relations in $\mathcal{N} = 2$ supersymmetric gauge theory,” [hep-th/9506102]. Phys. Lett. B357 (1995) 342.

[26] T. Eguchi and S.K. Yang, “Prepotentials of $\mathcal{N} = 2$ supersymmetric gauge theories and soliton equations,” [hep-th/9510183]. Mod. Phys. Lett. A11 (1996) 131.

[27] J. Sonnenschein, S. Theisen, and S. Yankielowicz, “On the relation between the holomorphic prepotential and the quantum moduli in supersymmetric gauge theories,” [hep-th/9510129]. Phys. Lett. B367 (1996) 145.

[28] A. Gorsky, I.M. Krichever, A. Marshakov, A. Mironov and A.Morozov, “Integrability and Seiberg-Witten exact solution,” [hep-th/9505035]. Phys. Lett. B355 (1995) 466.

[29] E. Martinec and N.P. Warner, “Integrable systems and supersymmetric gauge theory,” [hep-th/9509161]. Nucl. Phys. B459 (1996) 97.

[30] T. Nakatsu and K. Takasaki, “Whitham-Toda hierarchy and $\mathcal{N} = 2$ supersymmetric Yang-Mills theory,” [hep-th/9509162]. Mod. Phys. Lett. A11 (1996) 157.

[31] R. Donagi and E. Witten, “Supersymmetric Yang-Mills theory and integrable systems”, [hep-th/9510101]. Nucl. Phys. B460 (1996) 299.

[32] R. Donagi, [alg-geom/9705010].

[33] A. Gorsky, A. Marshakov, A. Mironov and A.Morozov, “RG equations from Whitham hierarchy,” [hep-th/9802007].

[34] E. Witten, “Supersymmetric Yang-Mills theory on a four-manifold,” [hep-th/9403193]. J. Math. Phys. 35 (1994) 5101.

[35] R. Fintushel and R.J. Stern, “The blowup formula for Donaldson invariants,” [alg-geom/9405002]. Ann. Math. 143 (1996) 529.
[36] P.C. Argyres, M.R. Plesser, N. Seiberg, and E. Witten, “New $\mathcal{N} = 2$ superconformal field theories in four dimensions,” hep-th/9511154; Nucl. Phys. B461 (1996) 71.
[37] T. Eguchi, K. Hori, K. Ito, and S.K. Yang, “Study of $\mathcal{N} = 2$ superconformal field theories in four dimensions,” hep-th/9603002; Nucl. Phys. B471 (1996) 430.
[38] M.R. Douglas and S.H. Shenker, “Dynamics of $SU(N)$ supersymmetric gauge theory,” hep-th/9503163; Nucl. Phys. B447 (1995) 271.
[39] E. D’Hoker and D.H. Phong, “Strong coupling expansions of $SU(N)$ Seiberg-Witten theory,” hep-th/9701053; Phys. Lett. B397 (1997) 94.
[40] D. Anselmi, J. Erlich, D.Z. Freedman, and A.A. Johansen, “Nonperturbative formulas for central functions of supersymmetric gauge theories,” hep-th/9708042; “Positivity constraints on anomalies in supersymmetric gauge theories,” hep-th/9711035
[41] M. Mariño and G. Moore, work in progress.
[42] M. Mariño and G. Moore, “Donaldson invariants for nonsimply connected manifolds,” hep-th/9804104.
[43] R. Friedman and J.W. Morgan, “Algebraic surfaces and Seiberg-Witten invariants,” alg-geom/9502026; J. Alg. Geom. 6 (1997) 445. “Obstruction bundles, semiregularity, and Seiberg-Witten invariants”, alg-geom/9509007.
[44] J.W. Morgan, The Seiberg-Witten equations and applications to the topology of smooth four-manifolds, Princeton University Press, 1996.
[45] G. Moore, “2D Yang-Mills Theory and Topological Field Theory,” hep-th/9409044; and in Proceedings of the International Congress of Mathematicians 1994, Birkhäuser 1995
[46] R. Gopakumar and C. Vafa, “Topological gravity from large $N$ gauge theory,” hep-th/9802016.
[47] A. Klemm, “On the geometry behind $\mathcal{N} = 2$ supersymmetric effective actions in four dimensions,” hep-th/9705131
[48] A. Giveon and D. Kutasov, “Brane Dynamics and Gauge Theory,” hep-th/9802067
[49] T.Banks, W.Fischler , I.R. Klebanov, L.Susskind, “Schwarzschild black holes from Matrix theory,” hep-th/9709108
[50] E. Witten, “Anti de Sitter space and holography”, hep-th/9802150.
[51] E. Witten, “Solutions of four-dimensional field theories via M-theory,” hep-th/9703066; Nucl. Phys. B500 (1997) 3.
[52] M. Bershadsky, V. Sadov, and C. Vafa, “D-Branes and Topological Field Theories,” Nucl. Phys. B463 (1996) 420; hep-th/9511222.