PFDE WITH NONAUTONOMOUS PAST

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Abstract. We study Cauchy problems associated to partial differential equations with infinite delay where the history function is modified by an evolution family. Using sophisticated tools from semigroup theory such as evolution semigroups, extrapolation spaces, or the critical spectrum, we prove well-posedness and characterize the asymptotic behavior of the solution semigroup by an operator-valued characteristic equation.

1. Introduction. Linear partial differential equations with infinite delay can be written in abstract form as

\[ \dot{x}(t) = Bx(t) + \Phi x_t, \quad t \geq 0, \]  
\[ x_0 = f. \]

Here, the function \( x(\cdot) \) takes values in a Banach space \( X \), \( B \) is a linear operator on \( X \), the history function \( x_t : \mathbb{R} \to X \) is defined by

\[ x_t(s) := x(s + t), \quad s \leq 0, \]

and \( \Phi \), the delay operator, is a linear operator from a space of \( X \)-valued functions on \( \mathbb{R} \) into \( X \). Semigroup methods have been applied to such equations with great success. To that purpose one rewrites (DDE) as an abstract Cauchy problem

\[ \dot{u}(t) = G_{B,\Phi} u(t), \quad t \geq 0, \]  
\[ u(0) = f \]

on the Banach space \( E := C_0([\mathbb{R}, X]) \), where \( G_{B,\Phi} \) is the operator

\[ G_{B,\Phi} f := f', \]

\[ D(G_{B,\Phi}) := \{ f \in C_0([\mathbb{R}, X]) : f(0) \in D(B) \text{ and } f'(0) = Bf(0) + \Phi f \}. \]

We refer to the monographs [5], [8] or [35] for the general theory, and to [6], [30] or [32] for some recent contributions to the theory of delay semigroups.

Our approach in this paper to more general delay equations (see (1) and (2) below) is inspired by Section VI.6 of [7], where it is shown that (DDE) is well-posed if and only

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if (ACP) is well-posed in the semigroup sense. Moreover, if $B$ generates a strongly continuous semigroup $(e^{tB})_{t \geq 0}$ and if $\Phi \in \mathcal{L}(E, X)$, then $G_{B, \Phi}$ generates a strongly continuous semigroup $(T_{B, \Phi}(t))_{t \geq 0}$ on $E$ (cf. [7], Theorem VI.6.1), hence (ACP) is well-posed. This delay semigroup satisfies the so-called translation property

$$(T_{B, \Phi}(t)f)(s) = \begin{cases} f(s + t) & \text{for } s + t \leq 0 \\ (T_{B, \Phi}(s + t)f)(0) & \text{for } s + t \geq 0 \end{cases}$$

for all $f \in E$ (cf. [7], Lemma VI.6.2), which implies that the solutions of (DDE) are given by

$$x(t) = \begin{cases} f(t) & \text{for } t \leq 0 \\ (T_{B, \Phi}(t)f)(0) & \text{for } t \geq 0 \end{cases}$$

(cf. [7], Corollary VI.6.3). Moreover, the translation property says that the delay semigroup $(T_{B, \Phi}(t))_{t \geq 0}$ acts on a history function $f \in E$ by left translation and that, in particular, the value $f(0)$ at time 0 remains unchanged while being translated into the past.

Assume now that, while time is passing, a backward evolution family (cf. [4] p. 8 or [25]) modifies the values of the history function. More precisely, assume that $(U(s, r))_{r \leq s \leq 0}$ is a family of bounded linear operators on $X$ satisfying

$$U(s, r)U(t, s) = U(t, r) \quad \text{for } r \leq s \leq t \leq 0,$$

$$U(t, t) = I \quad \text{for } t \leq 0,$$

and that the semigroup $(T_{B, \Phi}(t))_{t \geq 0}$ solving (DDE) satisfies a modified translation property of the form

$$(T_{B, \Phi}(t)f)(s) = \begin{cases} U(s + t, s)f(s + t) & \text{for } s + t \leq 0 \\ U(0, s)(T_{B, \Phi}(s + t)f)(0) & \text{for } s + t \geq 0 \end{cases}$$

for each $f \in E$. If we put

$$u(t, s) := (T_{B, \Phi}(t)f)(s)$$

and differentiate formally, we obtain

$$\frac{\partial}{\partial t} u(t, s) = \frac{\partial}{\partial s} u(t, s) + A(s)u(t, s)$$

for linear operators

$$A(s) := -\frac{\partial}{\partial r} U(s, r) \bigg|_{r=s}$$

on $X$, while at $s = 0$ the delay differential equation (DDE) takes the form

$$\frac{\partial}{\partial s} u(t, 0) = Bu(t, 0) + \Phi u(t, \cdot).$$

It is our goal in this paper to study the combination of these two equations. For finite dimensional $X$ similar equations occur, e.g., in [22]. For unbounded $A(s)$ however, no systematic study is known to us. However, the population equation with diffusion studied by Nickel, Rhandi and Schnaubelt in [24], [28], [29] can be viewed as an equation of this type.

To achieve our goal we use tools from semigroup theory such as evolution semigroups ([3] or [7], Section VI.9), extrapolation spaces (see [18] or [7], Section II.5) or the critical spectrum from [17].
In a first step, we show that, under appropriate assumptions, a strongly continuous semigroup \((T_B,\Phi(t))_{t \geq 0}\) exists on \(E\) such that \(u(t,s) := (T_B,\Phi(t)f)(s)\) solves (1) and (2) in a mild sense for an initial value \(f \in E\). In a next step, we characterize the spectrum of the generator \(G_{B,\Phi}\) of \((T_B,\Phi(t))_{t \geq 0}\) by a characteristic operator equation in \(X\). Finally, we estimate the critical growth bound of \((T_B,\Phi(t))_{t \geq 0}\) and obtain as a consequence that the spectrum of \(G_{B,\Phi}\) determines the growth bound of \((T_B,\Phi(t))_{t \geq 0}\), hence of the solutions of (1) and (2). By adding appropriate positivity assumptions, the stability criteria become much simpler and lead, in the case of the example discussed in Section 5, to quite explicit results.

2. Well-Posedness. We start by stating the assumptions which we will keep throughout the paper.

**Assumption 2.1.** Let \(B\) be the generator of a strongly continuous semigroup \((e^{tB})_{t \geq 0}\) on a Banach space \(X\). On \(E := C_0(\mathbb{R}_-,X)\) we consider a bounded linear operator \(\Phi\) from \(E\) into \(X\). Finally, let \(U = (U(s,r))_{r \leq s \leq 0}\) be a strongly continuous backward evolution family, i.e., a family of bounded linear operators on \(X\) satisfying

\[
U(s,r)U(t,s) = U(t,r) \quad \text{for } r \leq s \leq t \leq 0,
\]

\[
U(t,t) = I \quad \text{for } t \leq 0,
\]

and such that the mapping \((s,r) \mapsto U(s,r)\) is strongly continuous. Moreover, we assume that \((U(s,r))_{r \leq s \leq 0}\) is exponentially bounded, i.e., there exist constants \(M\) and \(w \in \mathbb{R}\) such that

\[
\|U(s,r)\| \leq Me^{w(s-r)}
\]

for all \(r \leq s \leq 0\). We then define the growth bound of \((U(s,r))_{r \leq s \leq 0}\) by

\[
\omega_0(U) := \inf \{w \in \mathbb{R} : \exists M \text{ such that } \|U(s,r)\| \leq Me^{w(s-r)} \text{ for all } r \leq s \leq 0\}.
\]

In a first step we extend \((U(s,r))_{r \leq s \leq 0}\) to a backward evolution family \((\tilde{U}(s,r))_{r \leq s}\) on all of \(\mathbb{R}\) and then define a corresponding evolution semigroup. Such evolution semigroups have been studied systematically in the monograph [3] or in the survey article by Schnaubelt in [7], Section VI.9, and have been applied with great success to characterize qualitative properties of solutions to nonautonomous evolution equations (cf. [19], [20], [31]).

**Definition 2.2.** (i) The backward evolution family \(U = (U(s,r))_{r \leq s \leq 0}\) on \(X\) is extended to a backward evolution family \(\tilde{U} = (\tilde{U}(s,r))_{r \leq s}\) by

\[
\tilde{U}(s,r) := \begin{cases} 
U(s,r) & \text{for } r \leq s \leq 0 \\
U(0,r) & \text{for } r \leq s \\
U(0,0) & \text{for } 0 \leq r \leq s.
\end{cases}
\]

(ii) On \(\tilde{E} := C_0(\mathbb{R},X)\) we then define the corresponding evolution semigroup \(\tilde{T} = (\tilde{T}(t))_{t \geq 0}\) by

\[
(\tilde{T}(t)f)(s) := \tilde{U}(s+t,s)f(s+t)
\]

for all \(s \in \mathbb{R}, \ t \geq 0,\) and \(f \in \tilde{E}\).

Then one easily proves the following property of \((\tilde{T}(t))_{t \geq 0}\).
Proposition 2.3. With the above definitions, \((\bar{T}(t))_{t \geq 0}\) is a strongly continuous semigroup on \(\bar{E}\), and its generator \(\bar{G}\) is a local operator in the sense that \(\bar{f} \in D(\bar{G})\), \(\bar{f}(s) = 0\) for all \(a < s < b\) implies \(\bar{G}\bar{f}(s) = 0\) for all \(a < s < b\).

Remark that we did not assume \((\bar{U}(t,r))_{r \leq s}\) to be differentiable, and hence the precise description of \(G\) and its domain \(D(\bar{G})\) is difficult. However, assuming differentiability, the generator of an evolution semigroup has been described more explicitly in [23], Proposition 2.7. We adapt this result to our situation.

Proposition 2.4. There exist operators \(\bar{A}(s), s \in \mathbb{R}\), on \(X\) with the following property:

If \(\bar{f} \in \bar{E}\) satisfies \(\bar{f}'(\cdot) \in \bar{E}\) and \(\bar{A}(\cdot)\bar{f}(\cdot) \in \bar{E}\), then \(\bar{f}\) belongs to \(D(\bar{G})\) and

\[
(\bar{G}\bar{f})(s) = \bar{f}'(s) + \bar{A}(s)\bar{f}(s)
\]

for \(s \in \mathbb{R}\). Moreover, we have \(\bar{A}(s) = 0\) for \(s > 0\).

We remark that, in general, the operators \(\bar{A}(s)\) may have trivial domain. On the other hand, if the evolution family corresponds to a well-posed Cauchy problem, then there is a core \(D\) of \(D(\bar{G})\) such that \(\bar{G}\bar{f} = \bar{f}' + \bar{A}(\cdot)\bar{f}\) for \(\bar{f} \in D\) (cf. [3], Theorem 3.12 or [25], Theorem 2.3).

For the study of our delay differential equation, we have to restrict the evolution semigroup \((\bar{T}(t))_{t \geq 0}\) to the space \(E = C_0(\mathbb{R}_-, X)\), and therefore need a boundary condition for the left shift. Such evolution semigroups on the half line have been considered recently in, e.g., [11] or [12] in order to study the asymptotic behavior of the corresponding nonautonomous problems. Here, we use the semigroup \((e^{tB})_{t \geq 0}\) generated by \(B\) to define the appropriate evolution semigroup.

Definition 2.5. For \(t \geq 0\) and \(f \in E\) we define

\[
(T_{B,0}(t)f)(s) := \begin{cases} 
U(s + t, s)f(s + t) & \text{for } s + t \leq 0 \\
U(0, s)e^{(s+t)B}f(0) & \text{for } s + t \geq 0.
\end{cases}
\]

Then one can again prove the semigroup property of \((T_{B,0}(t))_{t \geq 0}\).

Proposition 2.6. With the above definition, \(T_{B,0} = (T_{B,0}(t))_{t \geq 0}\) is a strongly continuous semigroup on \(E\) having growth bound \(\omega_0(T_{B,0}) = \max\{\omega_0(B), \omega_0(\mathcal{U})\}\).

We denote its generator by \(G_{B,0}\) and try to express it in terms of the generator \(\bar{G}\) of \((\bar{T}(t))_{t \geq 0}\). To this purpose we first restrict \(\bar{G}\) to the space \(E\).

Definition 2.7. Take

\[D(\bar{G}) := \{\bar{f}|_{\mathbb{R}_-} : \bar{f} \in D(\bar{G})\}\]

and define

\[Gf := \bar{G}\bar{f}|_{\mathbb{R}_-} \quad \text{for } f = \bar{f}|_{\mathbb{R}_-}.
\]

Since \(\bar{G}\) is a local operator (cf. Proposition 2.3), \(G\) is well-defined, and we will obtain the generator \(G_{B,0}\) of \((T_{B,0}(t))_{t \geq 0}\) as a restriction of \(G\).
Proposition 2.8. The generator \( G_{B,0} \) of \( (T_{B,0}(t))_{t \geq 0} \) is given by
\[
G_{B,0} f = Gf
\]
with domain
\[
D(G_{B,0}) = \{ f \in D(G) : f(0) \in D(B) \text{ and } (Gf)(0) = Bf(0) \}.
\]

Proof. Take a function \( f \in D(G) \) with \( f(0) \in D(B) \) and \( (Gf)(0) = Bf(0) \). Since \( f \in D(G) \), we can extend \( f \) to a function \( \tilde{f} \in D(G) \). We then have
\[
(T_{B,0}(t) f)(s) - f(s) - t(Gf)(s) = (\tilde{T}(t) \tilde{f})(s) - \tilde{f}(s) - t(\tilde{G}\tilde{f})(s)
\]
for \( s + t \leq 0 \) and
\[
(T_{B,0}(t) f)(s) - f(s) - t(Gf)(s) = (\tilde{T}(-s) \tilde{f})(s) - \tilde{f}(s) - (-s)(\tilde{G}\tilde{f})(s)
\]
\[
+ U(0,s)[e^{(s+t)B} f(0) - f(0) - (s+t)Bf(0)]
\]
\[
- (s+t)[(\tilde{G}\tilde{f})(s) - U(0,s)Bf(0)]
\]
for \( s + t \geq 0 \). Using \( \tilde{f} \in D(G) \) and \( (Gf)(0) = Bf(0) \), we obtain
\[
\lim_{t \downarrow 0} \sup_{s + t \leq 0} \frac{1}{t} \| (T_{B,0}(t) f)(s) - f(s) - t(Gf)(s) \| = 0
\]
and
\[
\lim_{t \downarrow 0} \sup_{s + t \geq 0} \frac{1}{t} \| (T_{B,0}(t) f)(s) - f(s) - t(Gf)(s) \| = 0.
\]
This implies
\[
\lim_{t \downarrow 0} \frac{1}{t} \| T_{B,0}(t) f - f - tGf \| = 0,
\]
hence \( f \in D(G_{B,0}) \) and \( G_{B,0} f = Gf \).

Conversely, consider a function \( f \in D(G_{B,0}) \). Since \( (T_{B,0}(t) f)(0) = e^{tB} f(0) \) for all \( t \geq 0 \), we obtain \( f(0) \in D(B) \). Choose now a continuously differentiable function \( g : \mathbb{R}_+ \to X \) with compact support such that \( g(0) = f(0) \) and \( g'(0) = Bf(0) \). Then, the function \( \tilde{f} \) defined by
\[
\tilde{f}(s) := \begin{cases} f(s) & \text{for } s \leq 0 \\ g(s) & \text{for } s \geq 0 \end{cases}
\]
is an extension of \( f \) belonging to \( D(\tilde{G}) \). Therefore, \( f \in D(G) \), \( f(0) \in D(B) \) and \( (Gf)(0) = Bf(0) \). \( \square \)

We now use the delay operator \( \Phi \in \mathcal{L}(E, X) \) to define the operator
\[
G_{B,\Phi} f := Gf
\]
with domain
\[
D(G_{B,\Phi}) := \{ f \in D(G) : f(0) \in D(B) \text{ and } (Gf)(0) = Bf(0) + \Phi f \}.
\]
In order to show that this operator is again a generator we use a technique introduced by Rhandi [27] and extend \( G_{B,\Phi} \) to an operator on \( \mathcal{E} := X \times E \).
Lemma 2.9. The operator
\[ G_{B,0} := \begin{pmatrix} 0 & -\delta_0 G + B\delta_0 \\ 0 & G \end{pmatrix} \]
with domain
\[ D(G_{B,0}) := \{0\} \times \{f \in D(G) : f(0) \in D(B)\} \]
is a Hille-Yosida operator on the Banach space \( E := X \times E \).

**Proof.** Take \( \lambda \) with \( \text{Re} \lambda > \omega_0(T_{B,0}) \). Since \( \text{Re} \lambda > \omega_0(\mathcal{U}) \), we can define
\[ (e_\lambda x)(s) := e^{\lambda s}U(0,s)x \]
for \( s \leq 0 \) and \( x \in X \). This yields an operator \( e_\lambda \in \mathcal{L}(X,E) \) satisfying \( \delta_0 e_\lambda = I \). We show that \( Ge_\lambda = \lambda e_\lambda \). To this end, we choose for every \( x \in X \) a function \( \tilde{f}_\lambda \in \tilde{E} \) such that
\[ \tilde{f}_\lambda(s) = e^{\lambda s}U(0,s)x \]
for \( s \leq 0 \),
\[ \tilde{f}_\lambda(s) = e^{\lambda s}x \]
for \( 0 \leq s \leq 1 \), and such that \( \tilde{f}_\lambda \) is continuously differentiable for \( s \geq 0 \). These conditions imply \( \tilde{f}_\lambda \in D(G) \). Moreover, we have
\[ (\tilde{T}(t)\tilde{f}_\lambda)(s) = e^{\lambda t}\tilde{f}_\lambda(s) \]
for \( s + t \leq 1 \), hence
\[ (\tilde{G}\tilde{f}_\lambda)(s) = \lambda\tilde{f}_\lambda(s) \]
for \( s < 1 \). By definition of \( G \), we conclude that \( e_\lambda x \in D(G) \) and
\[ Ge_\lambda x = \lambda e_\lambda x \].

Using the operator \( e_\lambda \), the resolvent of \( G_{B,0} \) can be expressed as
\[ R(\lambda, G_{B,0}) = \begin{pmatrix} 0 & 0 \\ e_\lambda R(\lambda, B) & R(\lambda, G_{B,0}) \end{pmatrix} \]
for \( \text{Re} \lambda > \omega_0(T_{B,0}) \). In order to show this we take \( (\tilde{x}, \tilde{f}) \in \tilde{E} \) and put
\[ g := e_\lambda R(\lambda, B)x + R(\lambda, G_{B,0})f. \]
Since \( g(0) \in D(B) \), we have \( g \in D(G_{B,0}) \) and
\[ \begin{pmatrix} 0 & 0 \\ e_\lambda R(\lambda, B) & R(\lambda, G_{B,0}) \end{pmatrix} \begin{pmatrix} x \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix} \in D(G_{B,0}). \]
If we apply \( \lambda - G_{B,0} \) to \( \begin{pmatrix} x \\ f \end{pmatrix} \), we obtain \( \begin{pmatrix} x \\ f \end{pmatrix} \), hence
\[ \begin{pmatrix} 0 & 0 \\ e_\lambda R(\lambda, B) & R(\lambda, G_{B,0}) \end{pmatrix} \]
is a right inverse of \( \lambda - G_{B,0} \). Moreover,
\[ (\lambda - G_{B,0})\begin{pmatrix} x \\ f \end{pmatrix} = 0 \]
implies \( \lambda f - G_{B,0}f = 0 \), hence \( f = 0 \), and it follows that the above operator matrix is the resolvent \( R(\lambda, G_{B,0}) \). For the powers of \( R(\lambda, G_{B,0}) \) we obtain
\[ R(\lambda, G_{B,0})^k = \begin{pmatrix} 0 & 0 \\ R(\lambda, G_{B,0})^k e_\lambda R(\lambda, B) & R(\lambda, G_{B,0})^k \end{pmatrix} \]
for $\text{Re} \lambda > \omega_0(T_{B,0})$ and $k \in \mathbb{N}$. Since $G_{B,0}$ is a generator on $E$, $G_{B,0}$ is a Hille-Yosida operator on $E$.

**Lemma 2.10.** The operator

$$G_{B,\Phi} := \begin{pmatrix} 0 & -\delta_0 G + B\delta_0 + \Phi \\ G \\ 0 \end{pmatrix}$$

with domain

$$D(G_{B,\Phi}) := \{0\} \times \{f \in D(G) : f(0) \in D(B)\}$$

is a Hille-Yosida operator on the Banach space $E := X \times E$.

**Proof.** This follows from the previous lemma together with the Bounded Perturbation Theorem for Hille-Yosida operators (see e.g. [21], Lemma 4.1.1).

**Theorem 2.11.** The operator $G_{B,\Phi}$ generates a strongly continuous semigroup on $E$.

**Proof.** Since $G_{B,\Phi}$ is a Hille-Yosida operator on $E$, the part of $G_{B,\Phi}$ in $\{0\} \times E$ generates a strongly continuous semigroup on $\{0\} \times E$. We now identify $E$ with $\{0\} \times E$. Under this identification, the operator $G_{B,\Phi}$ on $E$ corresponds to the part of $G_{B,\Phi}$ in $\{0\} \times E$.

We now prove that the semigroup $T_{B,\Phi} = (T_{B,\Phi}(t))_{t \geq 0}$ generated by $G_{B,\Phi}$ satisfies a modified translation property. This will be deduced from the following property of the operator $G$.

**Lemma 2.12.** For $f \in D(G)$ we have that $s \mapsto U(s, \tau)f(s)$ is differentiable and

$$U(s, r)(Gf)(s) = \frac{\partial}{\partial s} U(s, r)f(s)$$

for $r \leq s \leq 0$.

**Proof.** Since $f \in D(G)$, we can extend $f$ to a function $\tilde{f}$ on $\mathbb{R}_-$ contained in $D(\tilde{G})$. By the definition (see Definition 2.2) of $(\tilde{T}(t))_{t \geq 0}$, we have

$$\tilde{U}(s, r)(\tilde{T}(t)\tilde{f})(s) = \tilde{U}(s + t, r)\tilde{f}(s + t)$$

for $r \leq s$. From this it follows that

$$\frac{\partial}{\partial t} \tilde{U}(s, r)(\tilde{T}(t)\tilde{f})(s) = \frac{\partial}{\partial t} \tilde{U}(s + t, r)\tilde{f}(s + t) = \frac{\partial}{\partial s} \tilde{U}(s + t, r)\tilde{f}(s + t) = \frac{\partial}{\partial s} \tilde{U}(s, r)(\tilde{T}(t)\tilde{f})(s),$$

hence

$$\tilde{U}(s, r)(\tilde{G}\tilde{T}(t)\tilde{f})(s) = \frac{\partial}{\partial s} \tilde{U}(s, r)(\tilde{T}(t)\tilde{f})(s)$$

for $r \leq s$. Putting $t = 0$, we obtain

$$\tilde{U}(s, r)(\tilde{G}\tilde{f})(s) = \frac{\partial}{\partial s} \tilde{U}(s, r)\tilde{f}(s)$$

for $r \leq s$. This finally implies that $U(\cdot, r)f(\cdot)$ is differentiable and

$$U(s, r)(Gf)(s) = \frac{\partial}{\partial s} U(s, r)f(s)$$
for $r \leq s \leq 0$.

**Lemma 2.13.** The semigroup $(T_{B,\Phi}(t))_{t \geq 0}$ satisfies

$$ (T_{B,\Phi}(t)f)(s) = \begin{cases} U(s+t,s)f(s+t) & \text{for } s+t \leq 0 \\ U(0,s)(T_{B,\Phi}(s+t)f)(0) & \text{for } s+t \geq 0 \end{cases} $$

for all $f \in E$.

**Proof.** It suffices to prove the assertion for $f \in D(G_{B,\Phi})$. Since $G_{B,\Phi}$ is a restriction of $G$, we know from Lemma 2.12 that

$$ U(s,r)(G_{B,\Phi}T_{B,\Phi}(t)f)(s) = \frac{\partial}{\partial s} U(s,r)(T_{B,\Phi}(t)f)(s), $$

hence

$$ \frac{\partial}{\partial r} U(s,r)(T_{B,\Phi}(t)f)(s) = \frac{\partial}{\partial s} U(s,r)(T_{B,\Phi}(t)f)(s) $$

for $r \leq s \leq 0$. Consequently, the expression

$$ U(s,r)(T_{B,\Phi}(t)f)(s) $$

can be written as a function of $r$ and $s+t$. From this it follows that

$$ U(s,r)(T_{B,\Phi}(t)f)(s) = \begin{cases} U(s+t,s)f(s+t) & \text{for } s+t \leq 0 \\ U(0,s)(T_{B,\Phi}(s+t)f)(0) & \text{for } s+t \geq 0 \end{cases} $$

for $r \leq s \leq 0$. Putting $r = s$, we obtain

$$ (T_{B,\Phi}(t)f)(s) = \begin{cases} U(s+t,s)f(s+t) & \text{for } s+t \leq 0 \\ U(0,s)(T_{B,\Phi}(s+t)f)(0) & \text{for } s+t \geq 0 \end{cases} $$

\[ \square \]

**Proposition 2.14.** The semigroup $(T_{B,\Phi}(t))_{t \geq 0}$ satisfies

$$ (T_{B,\Phi}(t)f)(s) = \begin{cases} U(s+t,s)f(s+t) & \text{for } s+t \leq 0 \\ U(0,s)e^{(s+t)B}f(0) + \int_0^{s+t} U(0,s)e^{(s+t-\tau)B}\Phi T_{B,\Phi}(\tau)f d\tau & \text{for } s+t \geq 0 \end{cases} $$

for all $f \in E$.

**Proof.** It suffices to prove the assertion for $f \in D(G_{B,\Phi})$. In this case, we have

$$ \frac{d}{dt}(T_{B,\Phi}(t)f)(0) = (G_{B,\Phi}T_{B,\Phi}(t)f)(0) = B(T_{B,\Phi}(t)f)(0) + \Phi T_{B,\Phi}(t)f. $$

Therefore, we obtain

$$ (T_{B,\Phi}(t)f)(0) = e^{tB}f(0) + \int_0^t e^{(t-\tau)B}\Phi T_{B,\Phi}(\tau)f d\tau. $$

Using the translation property from Lemma 2.13, the assertion follows. \[ \square \]
Remark 2.15. If the terms in the above formula for \((T_{\mathcal{B}, \Phi}(t)f)(s)\) are all differentiable and belong to the domains of the relevant operators, we obtain a (classical) solution of the equations (1) and (2) (see [9] for details). Therefore, for each \(f \in E\), the function \(u(t, s) := (T_{\mathcal{B}, \Phi}(t)f)(s)\) can be regarded as a mild solution of (1) and (2).

3. Spectral Theory. We are now trying to determine the spectrum \(\sigma(\mathcal{G}_{\mathcal{B}, \Phi})\) of the generator \(\mathcal{G}_{\mathcal{B}, \Phi}\) by a condition in the space \(X\) instead of \(E = C_0(\mathbb{R}_-, X)\). To that purpose, we recall that, for \(\Re \lambda > \omega_0(T_{\mathcal{B}, 0})\), we defined in the proof of Lemma 2.9 operators \(e_\lambda \in L(X, E)\) by

\[
(e_\lambda x)(s) := e^{\lambda s} U(0, s)x
\]

for \(s \leq 0\) and \(x \in X\). Since \(\Phi \in L(E, X)\), we obtain that \(\Phi e_\lambda \in L(X, X)\).

Theorem 3.1. For \(\Re \lambda > \omega_0(T_{\mathcal{B}, 0})\), we have

\[
\lambda \in \sigma(\mathcal{G}_{\mathcal{B}, \Phi}) \iff \lambda \in \sigma(B + \Phi e_\lambda).
\]

Proof. It follows from [7], Theorem II.5.15 that the spectra of \(\mathcal{G}_{\mathcal{B}, \Phi}\) and \(\mathcal{G}_{\mathcal{B}, \Phi}\) coincide. Hence, it remains to show that

\[
\lambda \in \sigma(\mathcal{G}_{\mathcal{B}, \Phi}) \iff \lambda \in \sigma(B + \Phi e_\lambda).
\]

To this end, we write

\[
(\lambda - \mathcal{G}_{\mathcal{B}, \Phi}) R(\lambda, \mathcal{G}_{\mathcal{B}, 0}) = \begin{pmatrix}
\lambda & \delta_0 G - B \delta_0 - \Phi \\
0 & \lambda - G
\end{pmatrix}
\begin{pmatrix}
0 & 0 \\
e_\lambda R(\lambda, B) & R(\lambda, \mathcal{G}_{\mathcal{B}, 0})
\end{pmatrix}
\]

\[
= \begin{pmatrix}
(\lambda - B - \Phi e_\lambda) R(\lambda, B) - \Phi R(\lambda, \mathcal{G}_{\mathcal{B}, 0}) \\
0 & 1d
\end{pmatrix}.
\]

Now, \(\lambda\) belongs to the resolvent set of \(\mathcal{G}_{\mathcal{B}, \Phi}\) if and only if \((\lambda - \mathcal{G}_{\mathcal{B}, \Phi}) R(\lambda, \mathcal{G}_{\mathcal{B}, 0})\) is invertible. This holds if and only if \(\lambda - B - \Phi e_\lambda\) is invertible, i.e. \(\lambda\) belongs to the resolvent set of \(B + \Phi e_\lambda\).

The above condition is the appropriate analogue of the characteristic equation for delay differential equations in finite dimensions (see e.g. [8], Lemma 7.2.1) and is a characteristic operator equation as studied in [14], [15]. It is challenging to use it for the qualitative study of nonlinear problems ([10] or [26]). However, the determination of all \(\lambda\) such that \(\lambda \in \sigma(B + \Phi e_\lambda)\) remains a very difficult task. In the remaining part of this section we show how positivity arguments in combination with Theorem 3.1 yield a useful estimate for the spectral bound \(s(\mathcal{G}_{\mathcal{B}, \Phi})\) of the generator \(\mathcal{G}_{\mathcal{B}, \Phi}\).

To this end, we assume \(X\) to be a Banach lattice. Then \(E\) becomes a Banach lattice as well. Furthermore, we assume that the semigroup \((e^{tB})_{t \geq 0}\) generated by \(B\) and the delay operator \(\Phi\) are both positive. Finally, we assume that the backward evolution family \((U(s, r))_{r \leq s \leq 0}\) consists of positive operators. For the general theory of positive semigroups we refer to [13], Chap VI.1.b in [7] or [34].

Theorem 3.2. Under the above positivity assumptions and for \(\lambda > \omega_0(T_{\mathcal{B}, 0})\), we have

\[
s(\mathcal{G}_{\mathcal{B}, \Phi}) < \lambda \iff s(B + \Phi e_\lambda) < \lambda.
\]
**Proof.** We imitate the proof of [7], Lemma VI.6.15. In the first part of the proof, we show that \( s(B + \Phi e_\lambda) < \lambda \) implies \( s(G_{B, \Phi}) < \lambda \). In the second part, we show that \( s(B + \Phi e_\lambda) \geq \lambda \) implies \( s(G_{B, \Phi}) \geq \lambda \).

(i) Assume that \( s(B + \Phi e_\lambda) < \lambda \). From the monotonicity of the spectral bound (see [7], Proposition VI.6.13) it follows that \( s(B + \Phi e_\mu) \leq s(B + \Phi e_\lambda) < \lambda \leq \mu \) for all \( \mu \geq \lambda \). This implies \( \mu \in \rho(B + \Phi e_\mu) \), hence, by Theorem 3.1, \( \mu \in \rho(G_{B, \Phi}) \) for all \( \mu \geq \lambda \). Since \( s(G_{B, \Phi}) \in \sigma(G_{B, \Phi}) \) (see [7], Theorem VI.1.10), we conclude \( s(G_{B, \Phi}) < \lambda \).

(ii) Assume that \( s(B + \Phi e_\lambda) \geq \lambda \). Let \( \nu = \sup\{\mu : s(B + \Phi e_\mu) \geq \mu \} \). Since the mapping \( \mu \mapsto s(B + \Phi e_\mu) \) is continuous from the left (see [7], Proposition VI.6.13), we must have \( s(B + \Phi e_\nu) \geq \nu \). We claim that \( \nu \in \sigma(B + \Phi e_\nu) \). If \( \nu \in \rho(B + \Phi e_\nu) \), then \( \nu + \varepsilon \in \rho(B + \Phi e_{\nu + \varepsilon}) \) for \( \varepsilon > 0 \) sufficiently small. Moreover, \( R(\mu, B + \Phi e_\nu) = \lim_{\varepsilon \downarrow 0} R(\nu + \varepsilon, B + \Phi e_{\nu + \varepsilon}) \). Since \( s(B + \Phi e_{\nu + \varepsilon}) < \nu + \varepsilon \), we have \( R(\nu + \varepsilon, B + \Phi e_{\nu + \varepsilon}) \geq 0 \). This implies \( R(\nu, B + \Phi e_\nu) \geq 0 \) by [7], Lemma VI.1.9. Therefore, we must have \( s(B + \Phi e_\nu) < \nu \), which is a contradiction. Thus, we conclude \( \nu \in \sigma(B + \Phi e_\nu) \), hence \( \nu \in \sigma(G_{B, \Phi}) \). This finally implies \( s(G_{B, \Phi}) \geq \nu \geq \lambda \).

\[ \square \]

4. **Asymptotic Behavior.** By Theorem 3.1 and Theorem 3.2 we reduced the task of determining \( s(G_{B, \Phi}) \) to a problem in the space \( X \) involving the given data \( B \) and \( \Phi \). However, in our infinite dimensional situation the spectrum of the generator may not be sufficient to determine the asymptotic behavior of the semigroup \( (T_{B, \Phi}(t))_{t \geq 0} \). In order to overcome this difficulty we use the critical spectrum of a semigroup as introduced in [17]. We briefly recall its definition and main properties.

For a Banach space \( E \) and a strongly continuous semigroup \( T = (T(t))_{t \geq 0} \) on \( E \), we consider the Banach space \( \hat{E} := \ell^\infty(E) \) of all bounded sequences in \( E \). We extend the semigroup \( (T(t))_{t \geq 0} \) to this space and obtain a new semigroup \( \hat{T} = (\hat{T}(t))_{t \geq 0} \) by

\[ \hat{T}(t)\hat{f} := (T(t)f_n)_{n \in \mathbb{N}} \quad \text{for} \quad \hat{f} = (f_n)_{n \in \mathbb{N}}. \]

For this extended semigroup we consider its space of strong continuity

\[ \hat{E}_T := \{ \hat{f} \in \hat{E} : \lim_{k \downarrow 0} \| \hat{T}(h)\hat{f} - \hat{f} \| = 0 \}. \]

This subspace is closed and \( (\hat{T}(t))_{t \geq 0} \)-invariant. Therefore, the semigroup \( (\hat{T}(t))_{t \geq 0} \) induces a quotient semigroup \( \tilde{T} = (\tilde{T}(t))_{t \geq 0} \) on the quotient space \( \tilde{E} := \hat{E}/\hat{E}_T \), which is given by

\[ \tilde{T}(t)\tilde{f} := \hat{T}(t)\hat{f} + \hat{E}_T \quad \text{for} \quad \tilde{f} = \hat{f} + \hat{E}_T. \]

The critical spectrum of \( T(t) \) is then defined as

\[ \sigma_{crit}(T(t)) := \sigma(\tilde{T}(t)), \]

while the critical growth bound of \((T(t))_{t \geq 0}\) is defined as

\[ \omega_{crit}(T) := \inf \{ w \in \mathbb{R} : \exists M \text{ such that } \| \tilde{T}(t) \| \leq Me^{wt} \text{ for all } t \geq 0 \}. \]

In [17], Proposition 4.6, it is shown that the critical growth bound \( \omega_{crit}(T) \) coincides with the constant \( \delta(T) \) introduced by M. Blake [1] (see also [33]).
**Proposition 4.1.** For a strongly continuous semigroup $T = (T(t))_{t \geq 0}$, the critical growth bound is given by

$$\omega_{\text{crit}}(T) = \inf \{ w \in \mathbb{R} : \exists M \text{ such that } \lim_{h \downarrow 0} \| T(t + h) - T(t) \| \leq Me^{wt} \text{ for all } t \geq 0 \}. $$

As the main property of the critical spectrum, we state the following partial spectral mapping theorem and, as a consequence, a characterization of the growth bound $\omega_0(T)$ (see [17], Theorem 3.2).

**Theorem 4.2.** For a strongly continuous semigroup $T = (T(t))_{t \geq 0}$ with generator $G$ the following statements hold.

(i) $\sigma(T(t)) \setminus \{0\} = e^{G(t)} \cup \sigma_{\text{crit}}(T(t)) \setminus \{0\}$.

(ii) $\omega_0(T) = \max\{s(G), \omega_{\text{crit}}(T)\}$.

We now determine the critical growth bound for all $f \in E$. Since the semigroup $(e^{tB})_{t \geq 0}$ is immediately norm continuous, it follows that the operator family $(T_{B,\Phi}(t) - T_{B,\Phi}(0))_{t \geq 0}$ is norm continuous. By Proposition 4.1, the critical growth bounds of $(T_{B,\Phi}(t))_{t \geq 0}$ and $(T_{B,0}(t))_{t \geq 0}$ coincide, i.e.

$$\omega_{\text{crit}}(T_{B,\Phi}) = \omega_{\text{crit}}(T_{B,0}).$$

We now determine the critical growth bound of $(T_{B,0}(t))_{t \geq 0}$. From the definition of $(T_{B,0}(t))_{t \geq 0}$ we obtain the estimates

$$\lim_{h \downarrow 0} \| T_{B,0}(t + h) - T_{B,0}(t) \| \geq \sup_{s + t \leq 0} \| U(s + t, s) \|$$

and

$$\lim_{h \downarrow 0} \| T_{B,0}(t + h) - T_{B,0}(t) \| \leq C \sup_{s + \tau \leq 0, t \leq \tau \leq t + 1} \| U(s + \tau, s) \|$$

for some constant $C$. This implies

$$\omega_{\text{crit}}(T_{B,0}) = \omega_0(U).$$

Therefore, we conclude that

$$\omega_{\text{crit}}(T_{B,\Phi}) = \omega_0(U),$$

and, by Theorem 4.2, the growth bound of $(T_{B,\Phi}(t))_{t \geq 0}$ becomes

$$\omega_0(T_{B,\Phi}) = \max\{s(G_{B,\Phi}), \omega_0(U)\}.$$
5. Example. Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^n$. The Dirichlet Laplacian generates an analytic semigroup $(e^{t\Delta})_{t \geq 0}$ on $X := L^2(\Omega)$. We then define operators $A(s)$ as
\[ A(s) := a(s)\Delta, \]
where the function $0 \leq a(\cdot) \in L^1_{loc}(\mathbb{R}_-)$.

These operators generate a backward evolution family $(U(s, r))_{r \leq s \leq 0}$ given by
\[ U(s, r) = e^{(\int_r^s a(\sigma) \, d\sigma) \Delta} \]
for $r \leq s \leq 0$. We then have
\[ \|U(s, r)\| = e^{(\int_r^s a(\sigma) \, d\sigma) \lambda_0}, \]
where $\lambda_0$ denotes the largest eigenvalue of $\Delta$. This identity allows us to compute directly the growth bound of $(U(s, r))_{r \leq s \leq 0}$.

**Proposition 5.1.** The growth bound of $(U(s, r))_{r \leq s \leq 0}$ is given by
\[ \omega_0(U) = \inf_{t > 0} \sup_{s + t \leq 0} \left( \frac{1}{t} \int_s^{s+t} a(\sigma) \, d\sigma \right) \lambda_0. \]

We now take the operator $B$ as
\[ B := \Delta \]
and define the delay operator $\Phi$ by
\[ \Phi f := \int_{-\infty}^0 \varphi(s) f(s) \, ds \]
for $f \in E$, where $0 \leq \varphi(\cdot) \in L^1(\mathbb{R}_-)$. The semigroup $(T_{B, \Phi}(t))_{t \geq 0}$ from Section 2 is then well defined and solves the equations (1) and (2) corresponding to $\Phi$ and $A(\cdot)$. In order to compute the growth bound of $(T_{B, \Phi}(t))_{t \geq 0}$, it suffices to compute the spectral bound of its generator.

**Proposition 5.2.** The spectral bound $s(G_{B, \Phi})$ of the generator $G_{B, \Phi}$ is the unique solution of the equation
\[ \lambda_0 + \int_{-\infty}^0 \varphi(s) e^{\lambda s} e^{(\int_0^s a(\sigma) \, d\sigma) \lambda_0} \, ds = \lambda. \]

**Proof.** We will apply Theorem 3.2. By definition, we have
\[ B + \Phi \varepsilon \lambda = \Delta + \int_{-\infty}^0 \varphi(s) e^{\lambda s} e^{(\int_0^s a(\sigma) \, d\sigma) \Delta} \, ds. \]
Using the spectral theorem for selfadjoint operators, this implies
\[ s(B + \Phi \varepsilon \lambda) = \lambda_0 + \int_{-\infty}^0 \varphi(s) e^{\lambda s} e^{(\int_0^s a(\sigma) \, d\sigma) \lambda_0} \, ds, \]
where $\lambda_0$ denotes the largest eigenvalue of $\Delta$. By Theorem 3.2, we have
\[ s(G_{B, \Phi}) < \lambda \iff \lambda_0 + \int_{-\infty}^0 \varphi(s) e^{\lambda s} e^{(\int_0^s a(\sigma) \, d\sigma) \lambda_0} \, ds < \lambda. \]
Since the function
\[ \lambda \mapsto \lambda_0 + \int_{-\infty}^0 \varphi(s) e^{\lambda s} e^{(\int_0^s a(\sigma) \, d\sigma) \lambda_0} \, ds \]
is continuous and strictly decreasing, the spectral bound \( s(G_{B,\Phi}) \) of the generator \( G_{B,\Phi} \) is the unique solution of the equation

\[
\lambda_0 + \int_{-\infty}^{0} \phi(s)e^{\lambda s}e^{\left( \int_{0}^{s} a(\sigma) \, d\sigma \right)} \, ds = \lambda.
\]

By Theorem 4.3, we then obtain the growth bound \( \omega_0(T_{B,\Phi}) \) of the semigroup \( (T_{B,\Phi}(t))_{t \geq 0} \) as

\[
\omega_0(T_{B,\Phi}) = \max\{ s(G_{B,\Phi}), \omega_0(U) \}.
\]

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