On groupoid graded von Neumann regular rings and a Brandt groupoid graded Leavitt path algebras

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Abstract

Let \( S \) be a partial groupoid, that is, a set with a partial binary operation. An \( S \)-graded ring \( R \) is said to be graded von Neumann regular if \( x \in xRx \) for every homogeneous element \( x \in R \). Under the assumption that \( S \) is cancellative, we characterize \( S \)-graded rings which are graded von Neumann regular. If a ring is \( S \)-graded von Neumann regular, and if \( S \) is cancellative, then \( S \) is such that for every \( s \in S \), there exist \( s^{-1} \in S \) and idempotent elements \( e, f \in S \) for which \( es = sf = s, fs^{-1} = s^{-1}e = s^{-1}, ss^{-1} = e \) and \( s^{-1}s = f \), which is a property enjoyed by Brandt groupoids. We observe a Leavitt path algebra of an arbitrary non-null directed graph over a unital ring as a ring graded by a Brandt groupoid over the additive group of integers \( \mathbb{Z} \), and we prove that it is graded von Neumann regular if and only if its coefficient ring is von Neumann regular, thus generalizing the recently obtained result for the canonical \( \mathbb{Z} \)-grading of Leavitt path algebras.

1 Introduction

Throughout the paper, all rings are assumed to be associative, and, unless otherwise stated, without a unity (non-unital). A ring with unity is said to be unital.

Assigning various algebraic structures to directed graphs is widely present in the literature (see \([36]\)\). Given a field \( K \) and a directed graph \( E \), a specific \( K \)-algebra associated to \( E \) can be constructed, called the Leavitt path algebra, denoted by \( L_K(E) \). Leavitt path algebras are introduced independently in \([2, 10]\) as algebraic analogues of graph \( C^* \)-algebras. One obtains Leavitt algebras of type \((1, n)\) \([47]\) as a particular case, see for instance \([7]\). Leavitt path algebras over coefficient rings other than fields have been considered as well. Commutative unital rings are considered in \([53]\), the ring of integers in \([30, 31]\), and Leavitt path algebras over arbitrary unital rings in \([22]\).

Although introduced relatively recently, Leavitt path algebras have received a lot of attention, see for instance \([14, 17]\) and references therein. In particular, it is of interest to relate various combinatorial properties of the graph \( E \) with the algebraic properties of the Leavitt path algebra of \( E \) (as for instance in \([26]\)). It is also of interest to investigate various algebraic properties of the Leavitt path algebra \( L_R(E) \) with respect to the correspondind properties of the coefficient ring \( R \) (as for instance in \([23]\)).

Recall that a ring \( R \) is said to be von Neumann regular if \( x \in xRx \) for every \( x \in R \). It is proved in \([9]\) that \( L_K(E) \) is von Neumann regular if and only if \( E \) is acyclic. If \( R \) is a \( G \)-graded ring, where \( G \) is a group with identity \( e \), we recall that \( R \) is said to be graded von Neumann regular \([48]\) if every homogeneous element \( x \in R \) belongs to \( xRx \). (Note that the graded von Neumann regularity can be defined the same way for any grading set.) As it is well-known, since \( G \) is a group, this is equivalent to the existence of a homogeneous element \( y \in R \) such that \( x = xyx \). Now, for any ring \( R \) and a directed graph \( E \), the Leavitt path algebra \( L_R(E) \) is equipped with a natural \( \mathbb{Z} \)-grading, induced by the lengths of the paths of \( E \), where \( \mathbb{Z} \) denotes the additive group of integers, see for instance \([16]\). If \( K \) is a field, it was reasonable to ask what could be said of the \( \mathbb{Z} \)-graded von Neumann regularity of \( L_K(E) \), that is, whether it is true that for every homogeneous element \( x \in L_K(E) \) there exists a homogeneous element \( y \in L_K(E) \) such that \( x = xyx \). This question was raised and answered in the affirmative in \([23]\).

Theorem 1.1 (Theorem 10 in \([23]\)). For a field \( K \) and a directed graph \( E \), the \( \mathbb{Z} \)-graded Leavitt path algebra \( L_K(E) \) is graded von Neumann regular.

Let us mention that, recently, in \([54, 55]\), a \( \mathbb{Z} \)-graded unit regular Leavitt path algebra \( L_K(E) \) is characterized in terms of the properties of \( E \), as well as the other graded cancellation properties of \( L_K(E) \), including the graded cleanness property from \([35]\) (see also \([30, 31]\)).

Let us return to graded von Neumann regularity. If \( G \) is a group with identity \( e \) and \( R \) is a strongly

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G-graded ring with unity, then, it is well-known by Corollary C.I.1.5.3 in [18] that $R$ is graded von Neumann regular if and only if $R_e$ is von Neumann regular. This result was established by using Dade’s theorem, and later on, Theorem 3 in [56] provided us with a direct, element-wise proof of this fact. Non-unital graded von Neumann regular rings are studied in [23] in order to get Theorem 1.14 which is recently extended in [46] to the Leavitt path algebras over arbitrary unital rings with the help of a generalization of Theorem 3 in [50]. Namely, the following characterization holds.

**Theorem 1.2** (Theorem 1.4 in [46]). Let $R$ be a unital ring and $E$ a directed graph, which is not null. Then, the $\mathbb{Z}$-graded Leavitt path algebra $L_R(E)$ is graded von Neumann regular if and only if $R$ is von Neumann regular.

Now, let $K$ be a field and let $A_n$ be the oriented $n$-line graph with $n$ vertices and $n-1$ edges:

$$v_1 \xrightarrow{\alpha_1} v_2 \xrightarrow{\alpha_2} v_3 \xrightarrow{\alpha_3} \cdots \xrightarrow{\alpha_{n-2}} v_{n-1} \xrightarrow{\alpha_{n-1}} v_n.$$  

It is well-known, and easy to check, that $L_K(A_n)$ is isomorphic to the full matrix ring $M_n(K)$ (see Proposition 1.3.5 in [7]). The isomorphism is given by mapping each vertex $v_i$ to the standard matrix unit $e_{i,i}$, and by mapping the edges $\alpha_i$ and the corresponding ghost edges $\alpha^*_i$ to $e_{i,i+1}$ and $e_{i+1,i}$, respectively.

It is also well-known that $M_n(K)$ can be observed as a $\mathbb{Z}$-graded ring $\bigoplus_{k \in \mathbb{Z}} (M_n(K))_k$, where $(M_n(K))_k$ consists of matrices $(a_{ij})$ for which $a_{ij} = 0$ whenever $i, j$ are such that $i \neq j$, for $-n \leq k \leq n-1$, and $(M_n(K))_0$ is the zero matrix for $|k| \geq n$. Moreover, the above isomorphism gives rise to a $\mathbb{Z}$-graded isomorphism between $L_K(A_n)$ and $M_n(K)$. However, $M_n(K)$ can be observed as a ring graded by a partial groupoid. In this article, by a groupoid we mean a set with a binary operation.

Let $(M_n(K))_{(i,j)-i,j}$ be a subset of $M_n(K)$ which consists of matrices with an entry from $K$ in the $(i,j)$ position and zeroes elsewhere, where $i,j = 1, \ldots, n$. Then, for the additive group of $M_n(K)$ we have that $M_n(K) = \bigoplus_{(i,j),i} (M_n(K))_{(i,j)-i,j}$, and, moreover, $(M_n(K))_{(i,j)-i,j}(M_n(K))_{(k,l)-k,l} = \delta_{j,k}(M_n(K))_{(i,i)-i,i}$, where $\delta_{j,k}$ is the Kronecker delta. This induces a partial operation on the set $S = \{(i,j),i \mid i,j = 1, \ldots, n\}$.

In fact, $S$ is a Brandt groupoid. For a vertex $v_i$ of $A_n$, let us put $w(v_i) = (i,0,i) \in S$, for an edge $\alpha_i$, put $w(\alpha_i) = (i,1,i+1)$, and for the corresponding ghost edge $\alpha^*_i$, put $w(\alpha^*_i) = (i+1,-1,i)$. Then, it is easy to see that $L_K(A_n)$ is also an $S$-graded ring $\bigoplus_{(i,j),i \in S} (L_K(A_n))_{(i,j)-i,j}$, where $(L_K(A_n))_{(i,j)-i,j}$ is the $K$-linear span of monomials $\mu\eta^*$, where $\mu, \eta$ are paths in $A_n$ such that $w(\mu\eta^*) = (i,j,i)$. With thus defined grading, the above isomorphism tells us that $L_K(A_n)$ and $M_n(K)$ are also graded isomorphic as $S$-graded rings.

As we know, since $K$ is von Neumann regular, $M_n(K)$ is von Neumann regular. In particular, for every homogeneous element $A$ of an $S$-graded ring $M_n(K)$, there exists an element $A' \in M_n(K)$, which is homogeneous, such that $A = AA'A$. Hence, $L_K(A_n)$ is graded von Neumann regular as an $S$-graded ring. Note that $S$ is cancellative and that it is such that for every $s \in S$, there exist $s^{-1} \in S$ and idempotent elements $e, f \in S$ for which $es = sf = s, fs^{-1} = s^{-1}e = s^{-1}, ss^{-1} = e$ and $s^{-1}s = f$, in which case we say in this article that $S$ satisfies the (LRI) condition (existence of left and right inverses). A cancellative groupoid $S$ needs to be such in order for an $S$-graded ring to be graded von Neumann regular (Proposition 1.14).

Let $B$ be an arbitrary Brandt groupoid over $\mathbb{Z}$. The given example triggers out the following natural questions.

**Given an arbitrary directed graph $E$, which is not null, and a unital von Neumann regular ring $R$, is it possible to observe $L_R(E)$ as a ring graded by $B$ so that $L_R(E)$ is $B$-graded von Neumann regular?**

**Conversely, if $L_R(E)$ is $B$-graded von Neumann regular, is $R$ von Neumann regular?**

Theorem 3.10 answers these questions in the affirmative, thus generalizing Theorem 1.2. The Brandt groupoid gradings on Leavitt path algebras of the desired kind are explained in Section 3. The approach for proving Theorem 3.10 inspired by that of [23], is similar to the approach taken in the proof of Theorem 1.2. Namely, the result is first established for finite graphs, and then, for arbitrary graphs by observing the Leavitt path algebra as a direct limit of Leavitt path algebras over finite graphs.
graphs.

In category theoretic terminology, Brandt groupoids are connected small categories all of whose morphisms are invertible, which are also known simply as groupoids. However, in this article, as we have already pointed out, by a groupoid we mean a set with a binary operation, which is also known in the literature as magma. Let \( K \) be a field and \( E \) a directed graph. The Leavitt path algebra \( L_K(E) \) has already been observed as a ring graded by a small category all of whose morphisms are invertible via partial actions. Namely, in [18], the Leavitt path algebra \( L_K(E) \) is realized as a partial skew groupoid ring \( D(X) \star_\lambda G(E) \), where \( D(X) \) is a certain \( K \)-algebra and \( \lambda \) is a partial action of the free path groupoid \( G(E) \) on \( D(X) \). Under some restrictions, in [12], the \( G(E) \)-graded von Neumann regularity of \( L_K(E) \) is characterized in terms of \( D(X) \). Namely, let \( E \) be a connected graph and let the set of vertices \( E^0 \) be finite. Then, \( L_K(E) \) is \( G(E) \)-graded von Neumann regular if and only if \( D(X) \) is von Neumann regular, provided that the characteristic of the field \( K \) does not divide \( |E^0|1_K \).

One of the key notions used in obtaining Theorem 1.2 is the notion of a nearly epsilon-strongly group graded ring, which is introduced in [51] as a generalization of an epsilon-strongly group graded ring [51], class of which, in turn, contains all strongly group graded unital rings. Let \( G \) be a group with identity \( e \). According to Theorem 1.2 in [46], we have that \( R = \bigoplus_{g \in G} R_g \) is graded von Neumann regular if and only if \( R \) is nearly epsilon-strongly graded and \( R_e \) is von Neumann regular. As a corollary, one also obtains that an epsilon-strongly graded ring \( R = \bigoplus_{g \in G} R_g \) is graded von Neumann regular if and only if \( R_e \) is von Neumann regular (Corollary 3.11 in [46]). For results concerning rings graded by finite small categories all of whose morphisms are invertible, the reader is referred to [12,52].

To prove Theorem 3.10 and also, as one of the aims of this paper, we study the graded von Neumann regularity of rings which are not necessarily graded by a group or by a small category all of whose morphisms are invertible, which is the content of Section 4. More precisely, we are interested in rings graded in the sense of the following definition, which, in particular, includes all the other gradings.

**Definition 1.3** ([39,41]). Let \( R \) be a ring, and \( S \) a partial groupoid, that is, a set with a partial binary operation. Also, let \( \{R_s\}_{s \in S} \) be a family of additive subgroups of \( R \), called **components**. We say that \( R = \bigoplus_{s \in S} R_s \) is \( S \)-graded and \( R \) induces \( S \) (or \( R \) is an \( S \)-graded ring inducing \( S \)) if the following two conditions hold:

i) \( R_sR_t \subseteq R_{st} \) whenever \( st \) is defined;

ii) \( R_sR_t \neq 0 \) implies that the product \( st \) is defined.

The set \( H_R = \bigcup_{s \in S} R_s \) is called the **homogeneous part** of \( R \), and elements of \( H_R \) are called **homogeneous elements** of \( R \).

This definition applies to both associative and nonassociative rings. For the results on nonassociative rings graded by a set, we refer the reader to [14][17][43] and references therein. Note that the associativity of an \( S \)-graded ring inducing \( S \) does not imply the associativity in \( S \). As examples of \( S \)-graded rings inducing \( S \), let us mention a semidirect extension of a ring (in particular, the Dorroh extension), generalized matrix rings (in particular, the already given example of \( \mathbb{M}_n(K) \)), every group or semigroup graded ring, see [39], etc.

We finish the article by applying Theorem 3.10 in order to show that the Leavitt path algebra over a unital von Neumann regular ring enjoys some properties which represent groupoid graded counterparts of those satisfied by von Neumann regular rings. In particular, if \( R \) is von Neumann regular, then one type of the graded Jacobson radical of \( L_R(E) \) is zero, and \( L_R(E) \) is graded semiprime as a Brandt groupoid graded ring (Theorem 5.17).

## 2 Preliminaries

### 2.1 Graded rings

Let \( R = \bigoplus_{s \in S} R_s \) be an \( S \)-graded ring inducing \( S \). The degree \( \deg(a) \) of a nonzero homogeneous element \( a \) of \( R \) is defined to be a unique \( s \in S \) such that \( a \in R_s \). We define \( 0 = \deg(0) \), and we may
without loss of generality assume that $0 \in S$ since the zero element of $R$ can be viewed as a component of $R$. We may also assume that $S \setminus\{0\} = \{s \in S \mid R_s \neq 0\}$. Hence $R_0 = 0$, the zero subring of $R$. Throughout the article, we make $S$ a groupoid by putting $st = 0$ for those pairs $(s,t) \in S \times S$ for which the product $st$ is not originally defined (in which case $R_s R_t = 0$). Also, $s0 = 0s = 0$ for every $s \in S$. This is done for every $S$-graded ring inducing $S$ without further notice. We set $S^\times = S \setminus\{0\}$.

It is clear that $R = \bigoplus_{s \in S} R_s = \bigoplus_{s \in S^\times} R_s$.

**Remark 2.1.** Note that for $s, t, u \in S^\times$, if $R_s R_t R_u \neq 0$, then $(st)u = s(tu)$. In that case, as usual, we write this element as $stu$.

Throughout the article, a groupoid $S$ with zero $0$ is said to be *cancellative* if for $s, t, u \in S$, each of the equalities $0 \neq su = tu \in S$ or $0 \neq us = ut \in S$ implies $s = t$. Also, the set of all idempotent elements of $S$ is denoted by $I(S)$. By $I(S)^\times$ we denote the set $I(S) \setminus\{0\}$.

If $R = \bigoplus_{s \in S} R_s$ is an $S$-graded ring inducing $S$, then $R$ is said to be strongly graded if $R_s R_t = R_{st}$ for every $s, t \in S$.

We also note that the notions of an $S$-graded ring inducing $S$ and of a graded ring studied in [20, 21, 44] are equivalent.

Let $R = \bigoplus_{s \in S} R_s$ be an $S$-graded ring inducing $S$. A right (left, two-sided) ideal $I$ of $R$ is said to be homogeneous if $I = \bigoplus_{s \in S} R_s \cap I$. Also recall that if $I$ is a homogeneous ideal (two-sided) of $R$ and $I_s = R_s \cap I$, then $R/I = \bigoplus_{s \in S} R_s/I_s$ is an $S$-graded ring inducing $S$ [21, 37, 39, 44].

If $R$ is an $S$-graded ring inducing $S$ and $R'$ an $S'$-graded ring inducing $S'$, then a ring homomorphism $f : R \to R'$ is said to be homogeneous [20, 21, 44] if $f(H_R) \subseteq H_{R'}$ and if for $x, y \in H_R$ such that $f(x), f(y) \neq 0$, we have that $\deg(f(x)) = \deg(f(y))$ implies that $\deg(x) = \deg(y)$. It is easy to verify that the $S$-graded rings inducing $S$, together with the homogeneous homomorphisms form a category. In particular, if $S$ is fixed, we denote such category by $S$-RING.

The category $S$-RING has arbitrary direct limits. Namely, let $\{R_i \mid i \in I\}$ be a direct system of $S$-graded rings, where $R_i = \bigoplus_{s \in S} (R_i)_s$. Then it can be easily verified, like in the group graded case, that $A = \varinjlim R_i$ is an $S$-graded ring with the components $A_s = \varinjlim (R_i)_s$.

The following lemma will be used in the proof of Theorem 3.10. Its proof is similar to the group graded case (see Proposition 5.2.14 in [13]) and therefore, omitted.

**Lemma 2.2.** Let $\{R_i \mid i \in I\}$ be a direct system of rings which are objects in $S$-RING. If $R_i$ is graded von Neumann regular as an object in $S$-RING for every $i \in I$, then the direct limit of $\{R_i \mid i \in I\}$ is graded von Neumann regular as an object in $S$-RING too.

### 2.2 The graded Jacobson radical

Throughout the article, the classical Jacobson radical of a ring $A$ is denoted as usual by $J(A)$.

Let $R$ be an $S$-graded ring inducing $S$ and let us assume that $S$ is cancellative. A homogeneous right ideal $I$ of $R$ is said to be a graded modular right ideal [20, 21] if there exists a homogeneous element $u \in R$ such that $ux - x \in I$ for every homogeneous element $x \in R$. The cancellativity of $S$ gives that $\deg(u)$ is an idempotent element of $S$, and that all such elements $u$ are of the same degree, which is referred to as the degree of $I$.

The graded Jacobson radical $J^R(S) = \bigoplus_{s \in S} I_s$, where $I_s = \{x \in R_s \mid \langle v \in I(S) \rangle x H_R \cap I_e \subseteq J(R_e)\}$. It is then easy to verify the following statements. These represent one of the key ingredients we use in a characterization of specific kinds of graded von Neumann regular S-graded rings inducing $S$ (Proposition 1.9 and Theorem 1.17).

**Theorem 2.3.** [20, 21]. Let $R = \bigoplus_{s \in S} R_s$ be an $S$-graded ring inducing $S$, where $S$ is cancellative. Then:
a) \( J^0(R) \cap R_e = J(R_e) \) for all \( e \in I(S) \);

b) \( J^0(R) = 0 \), that is, \( R \) is graded semisimple, if and only if the following two conditions are satisfied:

i) \( J(R_e) = 0 \) for every \( e \in I(S) \), that is, each ring component of \( R \) is semisimple;

ii) For every nonzero homogeneous element \( x \in R \) there exists a homogeneous element \( y \in R \) such that \( xy \) is a nonzero homogeneous element of an idempotent degree;

c) Let \( s \in S \). If \( R_s \) is not contained in the graded Jacobson radical \( J^0(R) \), then there exist elements \( e, f \in I(S) \) and an element \( s^{-1} \in S \) such that \( es = sf = s, fs^{-1} = s^{-1}e = s^{-1}, ss^{-1} = e \), and \( s^{-1}s = f \).

Remark 2.4. Note that there are graded rings \( R \) in which \( J(R) = 0 \) but \( J^0(R) \neq 0 \). For instance, if \( K \) is a field, the \( \mathbb{N}_0 \)-graded polynomial ring \( K[x] = \bigoplus_{n \in \mathbb{N}_0} Kx^n \) is such a ring, where \( \mathbb{N}_0 \) is the additive monoid of nonnegative integers.

2.3 Von Neumann regular rings

Let us recall that a ring \( R \) is said to be von Neumann regular if for every \( x \in R \) we have that \( x \in xRx \). A ring \( R \) is said to be \( s \)-unital if, for every \( x \in R \), there exist \( e, e' \in R \) such that \( ex = xe = xe' \).

It is clear that every von Neumann regular ring is \( s \)-unital (Proposition 2.1 in [46]). The following proposition is established in [46], and represents a generalization of a well-known characterization of von Neumann regularity for unital rings (Theorem 1.1 in [19]) to \( s \)-unital rings.

Proposition 2.5 (Proposition 2.2 in [46]). Let \( R \) be an \( s \)-unital ring. Then the following statements are equivalent:

a) \( R \) is von Neumann regular;

b) Every principal right (left) ideal of \( R \) is generated by an idempotent element;

c) Every finitely generated right (left) ideal of \( R \) is generated by an idempotent element.

Since the Jacobson radical of a ring does not contain nonzero idempotent elements, the following corollary is immediate.

Corollary 2.6 (cf. Corollary 1.2 in [19]). Let \( R \) be an \( s \)-unital ring. If \( R \) is von Neumann regular, then the Jacobson radical \( J(R) \) is zero.

3 A Brandt groupoid graded von Neumann regularity of Leavitt path algebras

Let us recall first the notion of the Leavitt path algebra of a directed graph over a unital ring.

A directed graph \( E = (E^0, E^1, r, s) \) consists of two sets \( E^0, E^1 \) and mappings \( r, s : E^1 \rightarrow E^0 \). When there are more directed graphs observed at the same time, the mappings \( r \) and \( s \) of the graph \( E \) are denoted by \( r_E \) and \( s_E \), respectively. Elements of \( E^0 \) are called vertices and elements of \( E^1 \) are called edges. We say that \( E \) is finite if the cardinal number of \( E^0 \) is finite.

A vertex \( v \) for which \( s^{-1}(v) \) is empty is called a sink, while a vertex \( v' \) for which \( r^{-1}(v') \) is empty is called a source. A vertex \( v \) for which the cardinality of \( s^{-1}(v) \) is infinite, is called an infinite emitter. A vertex \( v \), that is neither a sink nor an infinite emitter, is called a regular vertex.

The set of all sinks in \( E \) is denoted by \( \text{Sink}(E) \), while the set of all regular vertices in \( E \) is denoted by \( \text{Reg}(E) \).

A path \( \mu \) in a graph \( E \) is a sequence of edges \( \mu = \alpha_1 \ldots \alpha_k \) such that \( r(\alpha_i) = s(\alpha_{i+1}), i = 1, \ldots, k-1 \), where \( k \in \mathbb{N} \). The length of \( \mu \) is equal to \( k \). We also put \( s(\mu) := s(\alpha_1) \), and call it the source of \( \mu \) and \( r(\mu) := r(\alpha_k) \) is called the range of \( \mu \). Any vertex \( v \in E^0 \) can be observed as a trivial path of length zero with \( s(v) = r(v) = v \). By \( E^k \) we denote the set of all paths of length \( k \), where \( k \in \mathbb{N}_0 \), and we set
Let $E$ and $F$ be directed graphs. Recall also that by a graph homomorphism $\psi : E \to F$ we mean a pair of mappings $(\psi_0 : E^0 \to F^0, \psi^1 : E^1 \to F^1)$ for which we have that $s(\psi^1(\alpha)) = \psi^0(s(\alpha))$ and $r(\psi^1(\alpha)) = \psi^0(r(\alpha))$ for every $\alpha \in E^1$.

**Definition 3.1** (Definition 1.6.2 in [7]). The category $\mathcal{G}$ is defined as the category with pairs $(E, X)$, where $E$ is a directed graph and $X \subseteq \text{Reg}(E)$, as objects, and if $(F, Y)$ and $(E, X)$ are objects of $\mathcal{G}$, then $\psi = (\psi_0, \psi^1)$ is a morphism in $\mathcal{G}$ if the following conditions are satisfied:

a) $\psi : F \to E$ is a graph homomorphism such that $\psi_0$ and $\psi^1$ are injective;

b) $\psi^0(Y) \subseteq X$;

c) For every $v \in Y$, the restriction $\psi^1 : s_{F^1}^{-1}(v) \to s_{E^1}^{-1}(\psi^0(v))$ is a bijection.

A morphism $\psi$ is said to be complete if, for every $v \in F^0$, if $\psi^0(v) \in X$ and $s_{F^1}^{-1}(v) \neq \emptyset$, then $v \in Y$.

**Definition 3.2** (Definition 1.6.8 in [7]). Let $E = (E^0, E^1, r, s)$ be a directed graph and $X \subseteq \text{Reg}(E)$.

A subgraph $F$ of $E$ is said to be $X$-complete if the inclusion mapping $(F, \text{Reg}(F) \cap X) \to (E, X)$ is a complete morphism in the category $\mathcal{G}$.

**Definition 3.3.** For a directed graph $E = (E^0, E^1, r, s)$, a unital ring $R$, and $X \subseteq \text{Reg}(E)$, the Cohn path algebra of $E$ with respect to $X$ [9], denoted by $C_R^X(E)$, is the free algebra generated by the sets $\{v \mid v \in E^0\}$, $\{\alpha \mid \alpha \in E^1\}$ and $\{\alpha^* \mid \alpha \in E^1\}$ with the coefficients in $R$, subject to the relations:

1) $v_iv_j = \delta_{i,j}v_i$ for every $v_i, v_j \in E^0$;

2) $s(\alpha)\alpha = \alpha r(\alpha) = \alpha$ and $r(\alpha)\alpha^* = \alpha^* s(\alpha) = \alpha^*$ for all $\alpha \in E^1$;

3) $\alpha^*\alpha' = \delta_{\alpha,\alpha'}r(\alpha)$ for all $\alpha, \alpha' \in E^1$;

4) $\sum_{\alpha \in E^1, s(\alpha) = v} \alpha\alpha^* = v$ for every $v \in X$.

We let $R$ commute with the set of generators of $C_R^X(E)$. If $X = \text{Reg}(E)$, then $C_R^X(E)$ is called the Leavitt path algebra [2][10][23] of $E$ over $R$, and is denoted by $L_R(E)$.

The elements $\alpha^*$, where $\alpha \in E^1$, are called the ghost edges. The set of all ghost edges is denoted by $(E^1)^*$.

Clearly, $L_R(E)$ is with unity $1$ if and only if $E^0$ is finite, and in that case, $1 = \sum_{v \in E^0} v$ (see for instance Lemma 1.2.12 in [7]).

If $\mu = \alpha_1 \ldots \alpha_k$, where $\alpha_i \in E^1$, is a path in $E$, that is, if $\mu \in E^*$, then, without further notice, we observe $\mu$ as an element of $C_R^X(E)$. Also, if $\mu = \alpha_1 \ldots \alpha_k \in E^*$, then by $\mu^*$ we denote the element $\alpha_k^* \ldots \alpha_1^* \in C_R^X(E)$. We moreover put $u^* = v$ for all $v \in E^0$. According to the condition 3) of Definition [3][33] any word in the generators $\{v, \alpha, \alpha^* \mid v \in E^0, \alpha \in E^1\}$ in $C_R^X(E)$ can be written as $\mu\eta^*$, where $\mu$ and $\eta$ are paths in $E$. Elements of the form $\mu\eta^*$, where $\mu, \eta \in E^*$, are called monomials.

### 3.1 Brandt groupoid gradings on Leavitt path algebras

Let $E = (E^0, E^1, r, s)$ be a directed graph, $R$ a unital ring, $X \subseteq \text{Reg}(E)$, and $C_R^X(E)$ the Cohn path algebra of $E$ with respect to $X$ with coefficients in $R$.

Let $B$ be a Brandt groupoid with the set of all idempotent elements denoted by $I(B)$. Let us recall (see for instance [10]), a Brandt groupoid is a partial groupoid $B$ which satisfies the following axioms:

(B1) If $st = u$ ($s, t, u \in B$), then each of the three elements $s, t, u$ is uniquely determined by the other two;

(B2) Let $s, t, u \in B$.

(i) If $st$ and $tu$ are defined, so are $(st)u$ and $s(tu)$, and $(st)u = s(tu)$;

(ii) If $st$ and $(st)u$ are defined, so are $tu$ and $s(tu)$, and $s(tu) = (st)u$;
(iii) If $tu$ and $s(tu)$ are defined, so are $st$ and $(st)u$, and $(st)u = s(tu)$;

(B3) For every element $s \in B$, there exist unique elements $e$, $f$, $s' \in B$ such that $es = sf = s$ and $s's = f$;

(B4) If $e$, $f \in I(B)$, then there exists $s \in B$ such that $es = sf = s$.

Note that $B$ satisfies the following condition:

(LRI) For every $s \in B$ there exist $s^{-1} \in B$ and $e, f \in I(B)$ such that $es = sf = s$, $fs^{-1} = s^{-1}e = s^{-1}$, $ss^{-1} = e$ and $s^{-1}s = f$.

Let us also recall that $B$ is isomorphic to a partial groupoid $(M(G, I), \circ)$, for some group $G$, and some set $I$. Here, $M(G, I)$ consists of triples $(i, g, j)$ ($g \in G, i, j \in I$), and $\circ$ is defined by $(i, g, j) \circ (k, h, l) = (i, gh, l)$ if $j = k$. Without loss of generality, we may assume that $|I| = |E^0|$.

Throughout this section, let $S = B \cup \{0\}$ and let us make $S$ a groupoid by setting $st = 0$ for all $s, t \in B$ for which $st$ is not defined in $B$, and $0s = s0 = 0$ for all $s \in B$. As it is well-known, $S$ is then a semigroup, known as a Brandt semigroup (see [16]). We also write $S^*$ instead of $B$. Let $I(S) = I(B) \cup \{0\}$. Of course, the semigroup $S$ also satisfies the (LRI) condition.

Since $S$ is cancellative, note that idempotents $e$ and $f$, and element $s^{-1}$ from the condition (LRI) are unique.

We define a weight mapping $w : E^* \cup \{\mu^* \mid \mu \in E^*\} \rightarrow B$ such that

\[
\begin{align*}
w|_{E^0} : E^0 &\rightarrow I(B), \\
w|_{E^1} : E^1 &\rightarrow B, \\
w|_{(E^1)^*} : (E^1)^* &\rightarrow B,
\end{align*}
\]

subject to the following rules:

1) If $\alpha \in E^1$, then $w(\alpha)$ is such that $w(s(\alpha))w(\alpha) = w(\alpha)w(r(\alpha))$.

2) $w(\alpha^*) = (w(\alpha))^{-1}$ for every $\alpha \in E^1$.

Note that the weight mapping from the example given in the Introduction, is in lines with the listed rules.

Now, let $w$ be a weight mapping defined with respect to the rules 1) and 2). For $s \in S^*$ let

\[ (C_R^X(E))^s = \{ \sum_i r_i \mu_i \eta_i^* \mid r_i \in R, \mu_i, \eta_i \in E^*, r(\mu_i) = r(\eta_i), w(\mu_i \eta_i^*) = s \}, \]

where the indicated sums are finite. Moreover, for the zero element 0 of $S$, let $(C_R^X(E))^0 = \{0\}$.

**Definition 3.4.** Let $R = \bigoplus_{s \in T} R_s$ be a $T$-graded ring inducing $T$, where $T$ is cancellative and with the property (LRI). Then an involution mapping $^* : R \rightarrow R$, that is, a mapping for which $(x^*)^* = x$ for every $x \in R$, is said to be anti-graded if $(R_s)^* = R_{s^{-1}}$ for all $s \in T$.

**Proposition 3.5.** The Cohn path algebra $C_R^X(E)$ with respect to $X$ is an $S$-graded ring $C_R^X(E) = \bigoplus_{s \in S} (C_R^X(E))^s$ with an anti-graded involution with respect to the given weight mapping $w$. In particular, the Leavitt path algebra $L_R(E) = C_R^{Res}(E)$ is an $S$-graded ring with an anti-graded involution.

**Proof.** Let $R(Y)$ be a free algebra over $R$ generated by the set $Y = E^0 \cup E^1 \cup (E^1)^*$. Then $R(Y)$ is clearly an $S$-graded ring with respect to $w$. By definition, $C_R^X(E)$ is the factor algebra $(R(Y))/I$, where $I$ is an ideal generated by the elements of the form:

1. $v_iv_j - \delta_{i,j} v_i$ for every $v_i, v_j \in E^0$;
2. $s(\alpha)\alpha - \alpha$, $r(\alpha)\alpha - \alpha$, $r(\alpha)\alpha^* - \alpha^*$, $\alpha^*s(\alpha) - \alpha^*$ for all $\alpha \in E^1$;
3. $\alpha^*\alpha' - \delta_{\alpha,\alpha'} r(\alpha)$ for all $\alpha, \alpha' \in E^1$.
4. $v - \sum_{\alpha \in E^1, s(\alpha) = v} \alpha \alpha^*$ for every $v \in X$.

Now, all of these elements are homogeneous with respect to the given weight mapping $w$. Therefore, $I$ is a homogeneous ideal of $R(Y)$. It follows that $C^X_R(E)$ is $S$-graded. Also, every element of $C^X_R(E)$ is a finite $R$-linear combination of monomials of the form $\mu \eta^*$, where $\mu, \eta \in E^*$, and $r(\mu) = r(\eta)$.

Hence, $C^X_R(E) = \bigoplus_{\alpha \in S} (C^X_R(E))_s$. For $i, j, k \in B$, let us define $(\sum_i r_i \mu_i \eta_i^*)^* = \sum_i r_i \mu_i \eta_i^*$. Then $*: C^X_R(E) \to C^X_R(E)$ is an involution mapping (see also \\[53\]). Let $s \in S$ and $0 \neq x \in (C^X_R(E))_s$. Also, let $e \in I(S)$ be such that $ss^{-1} = e$. Now, $x$ is a finite sum $\sum_i r_i \mu_i \eta_i^*$, where $r_i \in R$, and $\mu_i, \eta_i \in E^*$ are such that $r(\mu_i) = r(\eta_i)$ and $w(\mu_i \eta_i^*) = s$ for every $i$. Then $x^* = \sum_i r_i \mu_i \eta_i^*$. Since $S$ is cancellative, and $(\mu_i) \mu_i \eta_i^* = \mu_i \eta_i^*$, it follows that $w(s(\mu_i)) = e$. Also, since $(\mu_i \eta_i^*) (\eta_i \mu_i^*) = \mu_i \mu_i^*$, by the definition of $w$, we get that $w(\mu_i \eta_i^*) w(\mu_i^*) = w(\mu_i \mu_i^*) = w(s(\mu_i)) = e$. Hence, $w(\eta_i \mu_i^*) = s^{-1}$.

Thus, $(C^X_R(E))^{*}_s = (C^X_R(E))_{s^{-1}}$ for every $s \in S$, that is, $C^X_R(E)$ is equipped with an anti-graded involution. \qed

Remark 3.6. Note that since $C^X_R(E) = \bigoplus_{\alpha \in B} (C^X_R(E))_s = \bigoplus_{\alpha \in S} (C^X_R(E))_s$, we may observe $C^X_R(E)$ as a ring graded by a Brandt groupoid $B = S^{\times}$.

Remark 3.7. In a particular case when $S^{\times} = \mathbb{Z}$ is the additive group of integers, one obtains the most explored canonical $\mathbb{Z}$-grading of $C^X_R(E)$ (see \([10]\)) by defining $w(v) = 0$ for every $v \in E^0$, and $w(\alpha) = 1$ ($w(\alpha^*) = -1$) for every $\alpha \in E^1$.

Let us assume now that $w$, along with $w(1)$ and $w(2)$, moreover satisfies the following rules:

w3) If $e$ is an idempotent element of $B$, then every edge, which starts from the vertex of weight $e$, is of the same weight.

w4) If $e$ is an idempotent element of $B$, then every edge, which ends in the vertex of weight $e$, is of the same weight.

Remark 3.8. Note that in the case of a connected graph which contains loops, we have that every vertex is of the same weight and that every edge is of the same weight. Therefore, in that case, if we take $B = M(\mathbb{Z}, I)$ to be a Brandt groupoid over $\mathbb{Z}$, then we obtain the canonical $\mathbb{Z}$-grading of $L_R(E)$ by putting $w(v) = (i, 0, i)$ for every $v \in E^0$ and $w(\alpha) = (i, 1, i)$ ($w(\alpha^*) = (i, -1, i)$) for every $\alpha \in E^1$, for some $i \in I$.

Of course, there are many ways to define $w$ which satisfies the rules w1)-w4). For instance, let $v \in E^0$ be a vertex for which $s^{-1}(v) \neq 0$. Take $e \in I(S)^{\times}$ and put $w(v) = e$.

Case 1. $v$ belongs to a component which contains loops. Then, by Remark 3.8, the component which contains vertex $v$ is such that every vertex is of the same weight $e$, and every edge is of the same weight $s$ such that $es = se = s$.

Case 2. $v$ belongs to a component which contains no loops. Let $e \neq f \in I(S)^{\times}$, and let $s \in S$ be such that $es = sf = s$. Then, for every edge $\alpha \in s^{-1}(v)$, we put $w(\alpha) = s$ and $w(r(\alpha)) = f$. Further, for each $\alpha \in s^{-1}(v)$, if $s^{-1}(r(\alpha)) \neq 0$, we put $w(\gamma) = t$ for every $\gamma \in s^{-1}(r(\alpha))$, where $t \in S$ is such that $ft = t$, and so on. Note that $st \neq 0$.

Of course, if we start with $f = e$, it leads us to the weight mapping from the Case 1.

If $v'$ is a vertex for which $r^{-1}(v') \neq 0$, then we proceed analogously in the opposite direction with the given rules in mind. For instance, let $w(v') = e'$. Again, if $e'$ belongs to a component with loops, then each vertex of that component is of weight $e'$ and each edge of that component is of weight $s'$, where $s'$ is such that $s' e' = e' s' = s'$. Assume that $v'$ belongs to a component without loops. Take $e' \neq f' \in I(B)$. Then, for every $\beta \in r^{-1}(v')$, we put $w(\beta) = s$, where $s$ is such that $s e' = f' s = s$, etc.

Finally, we put $w(\alpha^*) = (w(\alpha))^{-1}$ for every $\alpha \in E^1$.

In view to Remark 3.8, we introduce the following notion.

Definition 3.9. Let $S = B \cup \{0\}$ be a Brandt semigroup over the additive group of integers $\mathbb{Z}$, that is, $B = \{0\}$, for some set $I$. Let $w: E^0 \cup \{\mu\mid \mu \in E^1\} \to B$ be the weight mapping which satisfies the rules w1)-w4), and it is such that the edges have the weights of the form $(i, 1, j) \in B ((j, -1, i)$ for the corresponding ghost edges) and vertices have the weights of the form $(i, 0, i)$. Then an $S$-graded algebra $C^X_R(E)$, with respect to $w$, in the sense of Proposition 3.8, is said to be canonically $S$-graded.
We now state the result which characterizes canonically \(S\)-graded von Neumann regularity of the Leavitt path algebras of arbitrary non-null directed graphs in terms of von Neumann regularity of the coefficient unital rings.

**Theorem 3.10.** Let \(R\) be a unital ring and let \(E\) be a directed graph distinct from the null graph. If \(S\) is a Brandt semigroup over the additive group of integers, then the Leavitt path algebra \(L_R(E)\) is graded von Neumann regular as a canonically \(S\)-graded ring if and only if \(R\) is von Neumann regular.

In particular, since a field \(K\) is von Neumann regular, then \(L_K(E)\) is graded von Neumann regular as a canonically \(S\)-graded ring. Therefore, by Remark 3.7 as a corollary to Theorem 3.10 one obtains Theorem 1.1 as well as Theorem 1.2.

Also, following the setting from Remark 3.8 in the case \(E\) is a connected graph with loops, Theorem 3.10 coincides with Theorem 1.2.

**Remark 3.11.** As it is noted in Remark 4.7 in [46], in case \(E\) is the null graph, the Leavitt path algebra is the zero ring over any ring \(R\). Hence, \(L_R(E)\) is then graded von Neumann regular as an \(S\)-graded ring for any \(R\), which means that Theorem 3.10 does not hold for the null graphs.

**Remark 3.12.** Theorem 3.10 cannot be extended to arbitrary gradings by Brandt semigroups. Namely, let \(S = M(Z, I)\) be a Brandt semigroup over \(Z\), and let \(E = R_1\) be the graph with one vertex \(v\) and one loop \(e\). Also, let \(K\) be a field, and let us put \(w(v) = w(e) = e = (i, 0, i)\), for some \(i \in I\). Note that \(w\) satisfies \(w(1)\) and \(w(2)\). Then, \(L_K(R_1)\) is trivially \(S\)-graded in the sense of Proposition 3.5 \((L_K(R_1))_e = L_K(R_1)\), and \((L_K(R_1))_s = 0\) for all \(e \neq s \in S\). However, as it is well-known, \(L_K(R_1) \cong K[x, x^{-1}]\), and \(K[x, x^{-1}]\) is not a von Neumann regular ring.

## 4 Groupoid graded von Neumann regular rings

In this section, rings are graded in the sense of Definition 1.3

**Definition 4.1.** Let \(R\) be an \(S\)-graded ring inducing \(S\). We say that \(R\) is graded von Neumann regular if \(x \in xRx\) for every \(x \in H_R\).

The following characterization, as in the case of group graded rings, is clear, but we include its proof for the sake of completeness.

**Proposition 4.2.** Let \(R = \bigoplus_{s \in S} R_s\) be an \(S\)-graded ring inducing \(S\), and let \(S\) be cancellative. Then \(R\) is graded von Neumann regular if and only if for every \(x \in H_R\) there exists \(y \in H_R\) such that \(x = yx y\).

**Proof.** The ‘if’ part is obvious by the definition of a graded von Neumann regular ring. Now, let us assume that \(R\) is graded von Neumann regular, and let \(x \in H_R\). Without loss of generality, we may assume that \(x \neq 0\). Since \(R\) is graded von Neumann regular, there exists \(y \in R\) such that \(x = yx y\). Let \(y = \sum_{s \in S} y_s\) be a unique homogeneous decomposition of \(y\). Then \(x = yx y = \sum_{s \in S} xy s y x\). By the hypothesis, \(S\) is cancellative. Therefore, if \(s \) and \(t\) are distinct elements of \(S\) such that \(R_{\text{deg}(x)} R_s R_{\text{deg}(x)}\), \(R_{\text{deg}(x)} R_t R_{\text{deg}(x)} \neq 0\), it follows by Remark 2.1 that \(\text{deg}(x)s \text{deg}(x) \neq \text{deg}(x)t \text{deg}(x)\). Hence, from \(x = \sum_{s \in S} xy s y x\) we get that there exists \(s \in S\) such that \(x = yx y x\), which concludes the proof.

**Lemma 4.3.** Let \(R = \bigoplus_{e \in S} R_e\) be an \(S\)-graded ring inducing \(S\), and let us suppose that \(S\) is cancellative. If \(R\) is graded von Neumann regular, then \(R_e\) is von Neumann regular for every \(e \in I(S)\).

**Proof.** Let \(e\) be an arbitrary idempotent element from \(S^x\), and \(0 \neq x \in R_e\). Since \(R\) is graded von Neumann regular, Proposition 4.2 implies that there exists \(y \in H_R\) such that \(x = yx y\). Now, since \(x \neq 0\), we get by Remark 2.1 that \(e = e \text{deg}(y) e\). By the cancellativity of \(S\), it follows that \(\text{deg}(y) = e\). Hence, \(R_e\) is a von Neumann regular ring.

It is known from the group graded case that the reverse statement of the previous lemma does not hold (see for instance Example 3.1 in [46]).

The following notion generalizes the notion of a nearly epsilon-strongly group graded ring from [50] to the case of \(S\)-graded rings inducing \(S\).
Definition 4.4. Let \( R = \bigoplus_{s \in S} R_s \) be an \( S \)-graded ring inducing \( S \), where \( S \) is cancellative. We say that \( R \) is nearly epsilon-strongly graded if the following conditions are satisfied:

i) \( S \) satisfies (LRI);

ii) For every \( s \in S \) and \( x \in R_s \) there exist \( \epsilon(x) \in R_s R_{s-1} \) and \( \epsilon'(x) \in R_{s-1} R_s \) such that \( \epsilon(x)x = x = x\epsilon'(x) \).

Remark 4.5. If \( R = \bigoplus_{s \in S} R_s \) is nearly epsilon-strongly graded, we note that \( R_s R_f = 0 \) for all distinct \( e, f \in I(S) \). Namely, let \( e, f \in I(S)^\times \) and \( e \neq f \). If \( R_s R_f \neq 0 \), then there exist \( x \in R_s \) and \( y \in R_f \) such that \( xy \neq 0 \), and \( \deg(xy) = ef \). Since \( R \) is nearly epsilon-strongly graded, there exists \( \epsilon(x) \in R_s \) such that \( \epsilon(x)x = x \). Hence, \( \epsilon(x)xy = xy \neq 0 \). So, we get that \( ef = f \). Since \( S \) is cancellative, it follows that \( ef = f \). Hence, \( e = f \), a contradiction. Therefore, \( R_s R_f = 0 \) for all distinct \( e, f \in I(S) \).

Example 4.6. Let \( R \) be an \( s \)-unital ring, and let \( \mathbb{M}_2(R) = \left( \begin{array}{cc} R & R \\ R & R \end{array} \right) \) be the ring of \( 2 \times 2 \) matrices with the usual matrix addition and multiplication over \( R \). Also, let \( (\mathbb{M}_2(R))_{(i,j)} \) be the subset of \( \mathbb{M}_2(R) \) which consists of matrices with entries from \( R \) in the \((i, j)\) position and zeroes elsewhere, where \( i, j = 1, 2 \). Then, like in the example from Section 1 of \[23\], \( \mathbb{M}_2(R) = \bigoplus_{i,j=1,2} (\mathbb{M}_2(R))_{(i,j)} \) is a strongly \( S \)-graded ring inducing \( S \), where \( S^\times = \{ (i, j) \mid i, j \in \{1, 2\} \} \), with respect to \((i, j)(k, l) = \delta_{j,k}(i, l)\), for all \( i, j, k, l \) and \( i, j, k, l = 1, 2 \) (see \[39\] and \[41\]). Namely, since \( R \) is \( s \)-unital, we have

\[
(\mathbb{M}_2(R))_{(i,j)}(\mathbb{M}_2(R))_{(k,l)} = \begin{cases} 
(\mathbb{M}_2(R))_{(i,l)} & \text{if } j = k; \\
O & \text{otherwise},
\end{cases}
\]

for all \( i, j, k, l = 1, 2 \), where \( O \) denotes the zero matrix. Also, \( S \) is cancellative, satisfies (LRI), and, of course, \( S^\times \) is not a group. Let \((i, j) \in S^\times \), and let \( X \in (\mathbb{M}_2(R))_{(i,j)} \) be a nonzero matrix with a nonzero entry \( x_{(i,j)} \in R \). Since \( R \) is \( s \)-unital, there exist \( e_{(i,j)}, f_{(i,j)} \in R \) such that \( e_{(i,j)}x_{(i,j)} = x_{(i,j)} = x_{(i,j)}f_{(i,j)} \).

Now, let \( \epsilon(X) \in (\mathbb{M}_2(R))_{(i,j)} \) be a nonzero matrix with a nonzero entry \( e_{(i,j)} \) and let \( \epsilon'(X) \in (\mathbb{M}_2(R))_{(i,j)} \) be a nonzero matrix with a nonzero entry \( f_{(i,j)} \). Then \( \epsilon(X)X = X = X\epsilon'(X) \). Hence, \( \epsilon_2(R) \) is a nearly epsilon strongly \( S \)-graded ring inducing \( S \).

The following result justifies the usage of the (LRI) condition in the first place.

Proposition 4.7. Let \( R = \bigoplus_{s \in S} R_s \) be an \( S \)-graded ring inducing \( S \), and let us assume that \( S \) is cancellative. If \( R \) is graded von Neumann regular, then \( R \) is nearly epsilon-strongly graded.

Proof. We prove that the conditions i) and ii) of Definition 4.4 are satisfied. Let \( s \in S^\times \) and \( 0 \neq x \in R_s \). Since \( R \) is graded von Neumann regular, by Proposition 4.2 there exist \( t \in S \) and \( y \in R_t \) such that \( x = xyx \). Then \( s = sts \) by Remark 2.1. It follows that \( e := st \in I(S)^\times \) and that \( f := ts \in I(S)^\times \). Hence, \( es = s \) and \( sf = s \). Since \( S \) is cancellative, we get that \( t = tst \). Hence, \( t = tst = te = ft \).

Therefore, i) holds true for \( s^{-1} := t \). Now, \( \epsilon(x) := xy \in R_s R_t \) is such that \( x = xyx = \epsilon(x)x \), and \( \epsilon'(x) := yx \in R_t R_s \) is such that \( x = x(yx) = x\epsilon'(x) \). Therefore, ii) holds true as well. Thus, \( R \) is indeed nearly epsilon-strongly graded.

We also note that the following characterization of graded von Neumann regular rings which are nearly epsilon-strongly \( S \)-graded rings inducing \( S \) holds true (cf. Theorem 1.1 in [19], Proposition 1 in [23] and Proposition 2.5).

Proposition 4.8. Let \( R = \bigoplus_{s \in S} R_s \) be an \( S \)-graded ring inducing \( S \) which is nearly epsilon-strongly graded. Then the following statements are equivalent:

i) \( R \) is graded von Neumann regular;

ii) Every principal right (left) homogeneous ideal of \( R \) is generated by a homogeneous idempotent element;

iii) Let \( I \) be a right (left) ideal of \( R \) which is generated by finitely many homogeneous elements, say \( \{x_1, \ldots, x_n\} \), such that for all \( i \in \{1, \ldots, n\} \) we have that \( \deg(x_i)(\deg(x_i))^{-1} = e \), for some \( e \in I(S)^\times \). Then \( I \) is generated by a homogeneous idempotent element.
Proof. Implications $i) \implies ii)$ and $iii) \implies i)$ are clear like in the non-graded case. The proof of implication $ii) \implies iii)$ is also similar to the non-graded case but there are certain details that should be addressed. So, let $x$ and $y$ be nonzero homogeneous elements of $R$, say $\deg(x) = s$, $\deg(y) = t$, and such that $ss^{-1} = tt^{-1} = e \in I(S)$. Then, $es = s$ and $et = t$. Since there exist $e'(x) \in R_{s^{-1}}R_s$ and $e'(y) \in R_{t^{-1}}R_t$ such that $x = xe'(x)$ and $y = ye'(y)$, we have that $xR + yR$ is a homogeneous right ideal of $R$ generated by the set $\{x, y\}$. Now, by $ii)$, there exists a homogeneous idempotent element $a \in R$ such that $xR = aR$. Hence, $x = ax$. It follows that $s = \deg(x) = \deg(a)\deg(x)$. Since $es = s$, the cancellativity of $S$ implies that $\deg(a) = e$. Further, note that then $y - ay \in xR + yR$ is a homogeneous element of $R$, since $et = t$. Now, reasoning similarly, there exists a homogeneous idempotent element $b \in R$, orthogonal to $a$, and of degree $e$, such that $(y - ay)R = bR$. It follows that $c = b - ba$ is a homogeneous idempotent element of degree $e$, which is also orthogonal to $a$, and $cR = bR = (y - ay)R$. Therefore, like in the non-graded case, $xR + yR = (a + c)R$. This concludes the proof, since $\deg(a) = \deg(c) = e$ and $a + c$ is an idempotent element of $R$.

Let us recall, if $R$ is an $S$-graded ring inducing $S$, then $R$ is said to be graded semiprime [27] if for any homogeneous ideal $I$ of $R$ we have that $I^n \subseteq R$ implies that $I \subseteq R$, where $n$ is a positive integer.

**Proposition 4.9.** Let $R = \bigoplus_{s \in S} R_s$ be an $S$-graded ring inducing $S$. If $R$ is graded von Neumann regular, then the following assertions hold:

i) Every homogeneous right (left) ideal $I$ of $R$ is idempotent, that is, $I^2 = I$;

ii) Every homogeneous two-sided ideal of $R$ is graded semiprime.

If, moreover, $R$ is nearly epsilon-strongly graded, then:

iii) The graded Jacobson radical $J^0(R)$ of $R$ is zero.

Proof. $i)$ and $ii)$ follow as in the non-graded case, see Corollary 1.2 in [19]. As for $iii)$, let us assume that there exists a nonzero homogeneous element $0 \neq x \in J^0(R)$ of degree, say $s$. Since $R$ is by the hypothesis nearly epsilon-strongly graded, there exists an element $e'(x) \in R_{s^{-1}}R_s$ such that $x = xe'(x)$. Then $xR$ is a homogeneous right ideal of $R$ generated by $x$. By Proposition 4.8 there exists a homogeneous idempotent element $0 \neq y \in R$ such that $xR = yR$. Since $y$ is a nonzero idempotent element of $R$, its degree is a nonzero idempotent of $S$. Let $\deg(y) = e$. Again, since $R$ is nearly epsilon-strongly graded, and since $e^{-1} = e$, by the cancellativity of $S$, there exists $e'(y) \in R_{e^{-1}}R_e \subseteq R_e$ such that $y = ye'(y)$. Now, $x \in J^0(R)$, and so, we get that $xR = yR \subseteq J^0(R)$. Hence, $y \in J^0(R)$. It follows that $y \in J^0(R) \cap R_e = J(R_e)$, according to Theorem 2.3. By Lemma 1.3 we have that $R_e$ is von Neumann regular. It follows by Corollary 2.0 that $J(R_e) = 0$. Hence, $y = 0$, a contradiction. Therefore, indeed $J^0(R) = 0$, as claimed.

We now state and prove the main result of this section which characterizes graded von Neumann regular $S$-graded rings inducing $S$ in terms of nearly epsilon-strongly graded rings.

**Theorem 4.10.** Let $R = \bigoplus_{s \in S} R_s$ be an $S$-graded ring inducing $S$ with a cancellative $S$. Then $R$ is graded von Neumann regular if and only if $R$ is nearly epsilon-strongly graded and each ring component of $R$ is von Neumann regular.

Proof. ($\Rightarrow$) Let $R$ be graded von Neumann regular. Then, by Proposition 4.7 we have that $R$ is nearly epsilon-strongly graded. Also, Lemma 4.3 implies that $R_e$ is von Neumann regular for every $e \in I(S)$.

($\Leftarrow$) Let us assume that $R$ is nearly epsilon-strongly graded and that $R_e$ is a von Neumann regular ring for every idempotent element $e \in S$. We may use the approach of the proof of Theorem 3 in [56] (see also the proofs of Lemma 3.9 and Proposition 3.10 in [16]) in order to prove that $R$ is graded von Neumann regular. Let $x \in R$ be a nonzero homogeneous element of $R$, say $x \in R_s$. Since $R$ is nearly epsilon-strongly graded, there exist unique $s^{-1} \in S$ and unique idempotent elements $e$ and $f$ from $S$ such that $ss^{-1} = e$, $s^{-1}s = f$, $fs^{-1} = s^{-1}e = s^{-1}$, and $es = sf = s$. Then, $R_{s^{-1}}x$ is a left ideal of $R_f$. Indeed, since $s^{-1}s = f$, we get that $R_{s^{-1}}x \subseteq R_{s^{-1}}R_s \subseteq R_f$. It is clear that $R_{s^{-1}}x$ is an additive subgroup of $R_f$. Now, take $r_{s^{-1}}x \in R_{s^{-1}}x$ and $r_f \in R_f$ arbitrarily. Then
r_{f/r_s^{-1}}x = (r_{f/r_s^{-1}})x \in (R_f R_{s^{-1}})x \subseteq R_{s^{-1}}x. However, r_{s^{-1}} = s^{-1}. Therefore, r_{f/r_s^{-1}}x \in R_{s^{-1}}x. So, R_{s^{-1}}x is indeed a left ideal of R_f. We claim that R_{s^{-1}}x is generated by an idempotent element in R_f.

By the hypotheses, R is nearly epsilon-strongly graded. Hence, there exists \( e' \) such that \( xe' = x \). It follows that \( e' = \sum x_i y_i \), where \( x, y \) are some positive integers. We get \( x = xe' = \sum x_i y_i \). Note that for each i there exists \( y_i \in R_{s^{-1}} \). Therefore, for every \( t \in S^{-x} \), we have that \( R_t \) is finitely generated as a left \( R_{g^{-1}} \)-module, where \( g \in I(S) \) is such that \( gt = t \). In particular, \( R_{s^{-1}} \) is finitely generated by a left \( R_{g^{-1}} \)-module, say \( \{v_1, \ldots, v_n\} \subseteq R_{s^{-1}} \) for some positive integer n. Then \( R_{s^{-1}} = \sum R f v_i x \). However, \( s^{-1}x = f \), and so, \( v_i x \in R_f, i = 1, \ldots, n \). Therefore, \( R_{s^{-1}} \) is a finitely generated left ideal of \( R_f \). Since \( R_f \) is von Neumann regular, by Proposition 2.3, we get that \( R_{s^{-1}} \) is indeed generated by an idempotent element of \( R_f \). Hence, \( R_{s^{-1}} = R_{g^{-1}} \), for some \( a^2 = a \in R_f \), since \( a = a^2 \in R_{g^{-1}} \). Therefore, \( a = yx \), for some \( y \in R_{s^{-1}} \). Moreover, we have that \( R_{s^{-1}} = R_{g^{-1}} \), since \( sf = s \). Since \( x = \epsilon(x)x \in R_{s^{-1}}x \), it follows that there exists \( z \in R_s \) such that \( x = za \). Therefore, \( xa = (za)a = za = x \), and so, \( x = xa = xyx \).

We finish this section by characterizing graded von Neumann regular \( S \)-graded rings inducing \( S \), which are epsilon-strongly graded, thus generalizing Corollary 3.11 in [46] to \( S \)-graded rings inducing \( S \).

**Definition 4.11.** Let \( R = \bigoplus_{s \in S} R_s \) be an \( S \)-graded ring inducing \( S \), where \( S \) is cancellative. Then \( R \) is said to be epsilon-strongly graded if the following conditions are satisfied:

i) \( S \) satisfies (LRI);

ii) For every \( s \in S \), there exists \( \epsilon(s) \in R_s R_{s^{-1}} \) such that for every \( x \in R_s \) we have that \( \epsilon(s)x = x = xe(s^{-1}) \).

**Corollary 4.12.** Let \( R = \bigoplus_{s \in S} R_s \) be an epsilon-strongly \( S \)-graded ring inducing \( S \). Then \( R \) is graded von Neumann regular if and only if each ring component of \( R \) is von Neumann regular.

**Proof.** It is obvious that every epsilon-strongly graded ring is nearly epsilon-strongly graded. Therefore, the statement is immediate by Theorem 4.10.

### 4.1 Pseudo-unitary \( S \)-graded rings inducing \( S \)

It is already mentioned that a unital strongly group graded ring is graded von Neumann regular if and only if the ring component is von Neumann regular (Corollary C.I.1.5.3 in [46]). The aim of this subsection is to characterize the graded von Neumann regularity of strongly graded \( S \)-graded rings inducing \( S \), which are pseudo-unitary.

**Definition 4.13 (21).** Let \( R = \bigoplus_{s \in S} R_s \) be an \( S \)-graded ring inducing \( S \) with a cancellative \( S \). We say that \( R \) is pseudo-unitary or pseudo-unital if the following conditions are satisfied:

i) For every \( e \in I(S) \), the ring \( R_e \) is a ring with unity \( 1_e \);

ii) For every \( x \in H_R \) there exist \( e, f \in I(S) \) such that \( 1_e x = x = 1_f \).

**Remark 4.14.** Note that if \( R \) is a pseudo-unitary \( S \)-graded ring inducing \( S \), then \( ef = 0 \) for all distinct idempotent elements \( e, f \in S \). As we have already recalled, the Leavitt path algebra of a graph \( E \) is unital if and only if \( E \) is finite. This is the property shared by the pseudo-unitary rings in general, that is, a pseudo-unitary ring is a ring with unity 1 if and only if \( I(S) \) is finite, and in that case, \( 1 = \sum e \in I(S) \), where \( e \in [20][21] \).

Also, note that if \( R \) is epsilon-strongly graded, then it is pseudo-unitary. This can be verified similarly to the fact that every epsilon-strongly group graded ring is unital (Proposition 3.8 in [19]). Moreover, it can be easily seen that every strongly graded pseudo-unitary \( S \)-graded ring inducing \( S \), for which the condition (LRI) holds, is epsilon-strongly graded. The converse does not hold as it is known from the group graded case (see Example 2.8 in [46]).
The notion of pseudo-unitary also served as a motivation to introduce the notion of a pseudo-unitary homogeneous semigroup in [32]. For the study of homogeneous semigroups [29] in case when they are graded by a group or by a small category all of whose morphisms are invertible, we refer the reader to [15, 26]. Cayley graphs of homogeneous semigroups are studied in [32, 34], inspired by the results which can be found in [42].

A similar concept to pseudo-unitary rings has been used in [4, 5, 8]. Namely, let $S$ be an l.i.-semigroup, that is, a semigroup with zero 0 for which there exists a set $E$ of nonzero orthogonal idempotent elements such that for every $s \in S^\times$ there exist $e, f \in E$ such that $es = sf = s$. A ring graded by an l.i.-semigroup is said to be locally unital [4, 5, 8] if $R_0 = 0$ and if for every idempotent element $e \in E$ there exists an element $1_e \in R_e$ such that for every $s \in S^\times$ with $esf = s$, where $e$, $f \in E$, and every $x \in R_e$ we have that $1_x = xy = x$. Therefore, in case $S$ is a cancellative semigroup, a pseudo-unitary ring is locally unital, and if $I(S)$ is moreover finite, it is unital, and vice-versa.

**Example 4.15.** Let $R$ be a ring with unity 1, and let $M_2(R)$ be the ring of $2 \times 2$ matrices with the usual matrix addition and multiplication over $R$, and observe $M_2(R)$ as an $S$-graded ring inducing $S$ as described in Example 4.6. It is strongly graded since $R$ is with unity. The ring components are $(M_2(R))_{(1,1)}$ and $(M_2(R))_{(2,2)}$. They are both unital with unities $1_{(1,1)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $1_{(2,2)} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, respectively. Moreover, it can be easily checked that $1_{(1,1)}(M_2(R))_{(1,2)} = (M_2(R))_{(1,2)} = (M_2(R))_{(1,2)}1_{(2,2)}$ and $1_{(2,2)}(M_2(R))_{(2,1)} = (M_2(R))_{(2,1)} = (M_2(R))_{(2,1)}1_{(1,1)}$. So, $M_2(R)$ is pseudo-unitary since $S$ is cancellative.

This can be easily generalized to the full matrix ring $M_n(R)$ of $n \times n$ matrices over $R$, for a natural number $n$. In particular, if $K$ is a field, then $M_n(K)$ and $L_K(A_n)$, where $A_n$ is the oriented $n$-line graph having $n$ vertices and $n - 1$ edges, are isomorphic as $S$-graded rings inducing $S$, and are graded von Neumann regular (see Section 1). Notice that the ring components of $L_K(A_n)$ are isomorphic to $K$, hence, von Neumann regular, and that for every nonzero homogeneous element $x$ there exists a homogeneous element $y$ such that $xy$ is a nonzero homogeneous element of an idempotent degree. If $E$ is a directed graph, another example of a strongly graded von Neumann regular ring which is pseudo-unitary, with these two properties, is the Cohn path algebra of $E$, that is, the contracted semigroup ring $K_0[S(E)]$ (see for instance [25]) where $S(E)$ is the graph inverse semigroup of $E$ (see [11]). We will soon prove that these two conditions are both necessary and sufficient for a strongly graded pseudo-unitary $S$-graded ring inducing $S$ to be graded von Neumann regular.

**Remark 4.16.** It is well-known that a unital ring $R = \bigoplus_{g \in G} R_g$, graded by a group $G$, is strongly graded if and only if its unity belongs to $R_gR_{g^{-1}}$ for every $g \in G$ (see for instance Proposition 1.1.1 in [39]). In the previous example, $1_{(i,i)} \in (M_2(R))_{(i,j)}(M_2(R))_{(j,i)}$ for all $i, j = 1, 2$, and $S$ satisfies (LRI). Let $R$ be a pseudo-unitary $S$-graded ring inducing $S$, and let us assume that $S$ satisfies (LRI). Then we note that $R$ is strongly graded if and only if $1_e \in R_eR_{e^{-1}}$ for every $e \in I(S)^\times$ and every $s \in S$ such that $ss^{-1} = e$. Namely, let us assume that for every $e \in I(S)^\times$ and every $s \in S$ such that $ss^{-1} = e$, we have that $1_e \in R_eR_{e^{-1}}$. Let $s, t \in S$ be such that $R_sR_t \neq 0$. Also, let $e, f \in I(S)^\times$ be such that $tt^{-1} = e$, $t^{-1}t = f$, and $et = t = tf$, and $ft^{-1} = t^{-1} = t^{-1}e$. Since $0 \neq R_sR_t = R_s(R_t1_f) \subseteq R_sR_tR_f$, it follows by Remark 2.1 that $st = s(tf) = (st)f$. Also, since $R_sR_t \neq 0$, we have that $R_sR_eR_t \neq 0$ since $R_sR_t = R_s(1_eR_t) \subseteq R_sR_tR_e$. Hence, we get that $st = (st)(se)t$. The cancellativity of $S$ implies that $se = s$. Hence, $0 \neq R_s = R_s1_e \subseteq R_sR_tR_{-e}$. Therefore, $s = se = s(tt^{-1}) = (st)t^{-1}$. It follows that $R_st = R_{st}1_f \subseteq R_{st}R_{t^{-1}}R_t \subseteq R_{(st)t^{-1}}R_t \subseteq R_{st}R_t$. Hence, $R_sR_t = R_{st}$. The converse statement is obvious.

We now state and prove a characterization of graded von Neumann regular $S$-graded rings inducing $S$ which are strongly graded and pseudo-unitary, thus generalizing a well-known group grading counterpart, namely Corollary C.I.1.5.3 in [48].

**Theorem 4.17.** Let $R = \bigoplus_{e \in S} R_e$ be a strongly graded pseudo-unitary $S$-graded ring inducing $S$. Then $R$ is graded von Neumann regular if and only if the following conditions are satisfied:

a) For every nonzero homogeneous element $x \in R$ there exists a homogeneous element $y \in R$ such that $xy$ is a nonzero homogeneous element of $R$ of an idempotent degree;
b) Each ring component of $R$ is von Neumann regular.

Proof. ($\Rightarrow$) Let $R$ be graded von Neumann regular. Then we get $b)$ by Theorem 4.10. Moreover, by the same theorem we get that $R$ is nearly epsilon-strongly graded. Therefore, $S$ satisfies (LRI). Let $x \neq 0$ be a homogeneous element of $R$, say $x \in R_s$. Also, let $s^{-1} \in S$ and $e, f \in I(S)$ be such that $ss^{-1} = e, s^{-1}s = f, es = sf = s$ and $fs^{-1} = s^{-1}e = s^{-1}$. Then, since $R$ is pseudo-unitary, $x = x1f$. By the hypothesis, $R$ is strongly graded. Therefore, $1_f \in R_{s-1}R_s$. It follows that $1_f = \sum_{i=1}^{n}x_iy_i$, for some $x_i \in R_{s-1}, y_i \in R_s$ and a positive integer $n$. Now, $x = x1f = \sum_{i=1}^{n}(xx_i)y_i$. Since $x \neq 0$, we get that there exists $i$ such that $xx_i \neq 0$. On the other hand, $xx_i \in R_{s}R_{s-1} = R_s$, that is, $xx_i$ is a nonzero homogeneous element of an idempotent degree. Hence, $a)$ holds true as well.

($\Leftarrow$) Let us assume that $a)$ and $b)$ hold. Since $R$ is by the hypothesis pseudo-unitary, for every $e \in I(S)$ we have that $R_e$ is a ring with unity $1_e$. According to Corollary 2.6, the condition $b)$ implies that $J(R_e) = 0$ for every idempotent element $e \in S$. This, together with $a)$ implies that $J^g(R) = 0$, according to Theorem 2.3). Take $s \in S^\times$. Then $R_s$ is not contained in $J^g(R)$. Hence, by Theorem 2.3), we have that there exist elements $e, f \in I(S)$ and an element $s^{-1} \in S$ such that $es = sf = s, f s^{-1} = s^{-1}e = s^{-1}, ss^{-1} = e, and s^{-1}s = f$. Therefore, $S$ satisfies (LRI). It follows that $1_e x = x = x1_f$ for every $x \in R_s$. However, $R$ is strongly graded. Hence, $1_e \in R_e = R_sR_{s-1}$ and $1_f \in R_f = R_{s-1}R_s$. Therefore, $ii)$ of Definition 1.11 holds true as well, which implies that $R$ is nearly epsilon-strongly graded (as already concluded in Remark 1.11). Thus, $R$ is graded von Neumann regular by Theorem 4.10.

5 Proof of Theorem 3.10

Throughout this section, we keep the notation and agreements set in Section 3. So, $R$ is a unital ring, $E = (E^0, E^1, \mu, r, s)$ a directed graph, $S = M(\mathbb{Z}, I) \cup \{0\}$ a Brandt semigroup over $\mathbb{Z}$ (although some statements in this section hold true for a Brandt semigroup over an arbitrary group), and the Cohn path algebra $C^X_R(E)$ with respect to $X \subseteq \text{Reg}(E)$, in particular, the Leavitt path algebra $L(E)$, is observed as a canonically $S$-graded ring (with respect to the weight mapping $w : E^* \cup \{\mu^* | \mu \in E^*\} \to S^\times$, in the sense of Proposition 3.10 and Definition 3.11). Let us recall that $L(E)$ is equipped with an anti-graded involution mapping $*$ that sends each $x = \sum_{i=1}^{n}r_i \mu_i \eta_i^\delta \in L(E)$ to $x^* = \sum_{i=1}^{n}r_i \eta_i \mu_i^\delta \in L(E)$ (see the proof of Proposition 3.10).

Following the approach of both 23 and 40, we prove Theorem 3.10 by establishing it first for the special case of a finite graph. The case of an arbitrary graph is handled by observing $L(E)$ as a direct limit of Leavitt path algebras over finite graphs (similarly to 23 and 40).

One of the key steps in the proof is to establish that $L(E)$ is nearly epsilon-strongly graded.

5.1 Nearly epsilon-strongly groupoid graded Leavitt path algebras

Theorem 4.2 in 50 asserts that any group graded Leavitt path algebra is nearly epsilon-strongly graded. The approach of preordering the set of all monomials $\mu \eta^\star$, where $\mu, \eta \in E^*$ and $r(\mu) = r(\eta)$, taken in 50, works perfectly fine in proving that the Leavitt path algebra, observed as an $S$-graded ring, is nearly epsilon-strongly graded in the sense of Definition 4.11. We include the key steps for the readers’ convenience.

Definition 5.1 (cf. Definition 4.3 in 50). Let $X$ denote the set of all monomials $\mu \eta^\star$, where $\mu, \eta \in E^*$ and $r(\mu) = r(\eta)$. Take $s \in S$ and put $X_s = \{x \in X | w(x) = s\}$. Suppose that $\mu, \eta, \zeta, \theta \in E^*$ are such that $\mu \eta^\star, \zeta \theta^\star \in X_s$. Then we put $\mu \eta^\star \leq \zeta \theta^\star$ if and only if $\mu$ is the initial subpath of $\zeta$. Also, we define $\mu \eta^\star \sim \zeta \theta^\star$ if and only if $\mu = \zeta$.

Of course, for every $s \in S$, we have that $(L(R(E)))_s$ is the $R$-linear span of $X_s$.

It can be easily verified that the following statements hold (cf. Proposition 4.2 in 50):

a) The relation $\leq$ is a preorder on $X_s$;

b) The relation $\sim$ is an equivalence relation on $X_s$;

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c) The quotient relation $\preceq$ on the factor set $X_s/\sim$, induced by $\preceq$, is a partial order.

Let $x = \mu\eta^* \in X_s$, and let $e \in I(S)$ be such that $ss^{-1} = e$. According to the proof of Proposition 5.2, we have that $w(xx^*) = e$. Having this in mind, the proofs of the next two results can be carried out as in [50] and are omitted.

**Proposition 5.2** (cf. Proposition 4.3 in [50]). Let $s \in S$ and $e \in I(S)$ be such that $ss^{-1} = e$. Then, the mapping $N_s : X_s/\sim \to (L_R(E))_e$, given by $N_s([x]_\sim) = xx^*$, $([x]_\sim \in X_s/\sim)$, is well defined.

**Proposition 5.3** (cf. Proposition 4.4 in [50]). Let $s \in S$ and $x, y \in X_s$. Then the following statements hold:

a) If $[x]_\sim \preceq [y]_\sim$, then $N_s([x]_\sim)y = y$;

b) If $[x]_\sim \not\preceq [y]_\sim$ and $[y]_\sim \not\preceq [x]_\sim$, then $N_s([x]_\sim)y = 0$.

**Theorem 5.4.** Let $R$ be a unital ring, and let $E$ be a directed graph. Then the Leavitt path algebra $L_R(E)$ is nearly epsilon-strongly graded as an $S$-graded ring.

**Proof.** We may follow the technique presented in the proof of Theorem 4.2 in [50]. Let $s \in S$ and $x \in (L_R(E))_s$. Then $x = \sum_{i=1}^n r_i\mu_i\eta_i^*$ for some natural number $n$, and $r_i \in R$, $\mu_i\eta_i^* \in X_s$. Since $n$ is finite, there exists a subset $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$ such that $\{[\mu_i\eta_i^*]_{j=1}^n\}$ is the set of minimal elements of the set $\{[\mu_i\eta_i^*]_{j=1}^n\}$ with respect to $\preceq$. Let $\epsilon_s(x) := \sum_{j=1}^k N_s([\mu_i\eta_i^*])$. Then $\epsilon_s(x) \in (L_R(E))_s(L_R(E))_{s^{-1}}$. Like in the proof of Theorem 4.2 in [50], with the help of Proposition 5.3, we get that $\epsilon_s(x) = x$. Note that $\epsilon_s(x^*) = \epsilon_s(x)$.

**Remark 5.5.** We note that in the case of a finite graph $E$, the Leavitt path algebra $L_R(E)$ is epsilon-strongly graded as an $S$-graded ring (cf. Theorem 4.1 in [50]).

### 5.2 Case of a finite graph

In this subsection we prove that Theorem 5.10 holds true for finite directed graphs. To this end, we describe the ring components of the $S$-graded Leavitt path algebras of finite graphs and prove they are von Neumann regular if the coefficient ring is von Neumann regular. This is done by showing that these ring components are ultramatricial, that is, direct limits of matricial rings, which generalizes Corollary 2.1.16 in [7], as well as Lemma 4.4 in [46], to canonically $S$-graded Leavitt path algebras. The converse holds for an arbitrary non-null graph, that is, if the ring components are von Neumann regular, then the coefficient ring is von Neumann regular.

Given an idempotent $e \in I(S)^X$, we want to describe nonzero $(L_R(E))_e$. We achieve that by describing nonzero $(C_R^X(E))_e$, where $X$ is a subset of $\text{Reg}(E)$. So, let $X \subseteq \text{Reg}(E)$, and let $e \in I(S)^X$ be such that $(C_R^X(E))_e \neq 0$. We start with the following lemma.

**Lemma 5.6.** If $\sum_i r_i\mu_i\eta_i^*$ is an element of the $e$-component $(C_R^X(E))_e$ of the $S$-graded algebra $C_R^X(E)$, that is, an element of the $R$-linear span of $X_e = \{\mu\eta^* \mid \mu \in E^*, r(\mu) = r(\eta), w(\mu\eta^*) = e\}$, then $w(\mu_i) = w(\eta_i)$ for every $i$.

**Proof.** Let $\mu\eta^* \in X_e$ with $\mu = \alpha_1 \ldots \alpha_m$ and $\eta = \beta_1 \ldots \beta_n$, where $\alpha_i, \beta_j \in E^1$, $i = 1, \ldots, m$, $j = 1, \ldots, n$. Let $s(\beta_1) = v$. Then

$$\mu\eta^* = \alpha_1 \ldots \alpha_m\beta_n^* \ldots \beta_1^* = \alpha_1 \ldots \alpha_m\beta_n^* \ldots \beta_1^* v \neq 0.$$  

Hence, 

$$e = w(\mu\eta^*) = w(\mu\eta^*)w(v) = ew(v).$$
Since $S$ is cancellative, it follows that $w(v) = e$. By the definition of the weight mapping $w$, we have that $ew(\beta_1) = w(\beta_1)$. Now,

$$\mu\eta^*\beta_1 = \alpha_1 \ldots \alpha_m\beta_n^* \ldots \beta_n^*r(\beta_1) \neq 0.$$ 

However, as we know, $r(\beta_1) = s(\beta_2)$. Hence,

$$w(\beta_1) = ew(\beta_1) = w(\alpha_1) \ldots w(\alpha_m)w(\beta_n)^{-1} \ldots w(\beta_2)^{-1}w(s(\beta_2)).$$

Further,

$$\mu\eta^*\beta_1\beta_2 = \alpha_1 \ldots \alpha_m\beta_n^* \ldots \beta_3^*r(\beta_2) \neq 0.$$ 

Since $r(\beta_2) = s(\beta_3)$, we get

$$w(\beta_1)w(\beta_2) = ew(\beta_1)w(\beta_2) = w(\alpha_1) \ldots w(\alpha_m)w(\beta_n)^{-1} \ldots w(\beta_3)^{-1}w(s(\beta_3)).$$

Continuing this way, we eventually obtain that

$$w(\beta_1) \ldots w(\beta_n) = w(\alpha_1) \ldots w(\alpha_m)f,$$

where $f = w(r(\beta_n)) = w(r(\eta)) = w(r(\mu)) = w(r(\alpha_m))$. By the definition of $w$, we get that $w(\alpha_m)f = w(\alpha_m)$. Hence, $w(\beta_1) \ldots w(\beta_n) = w(\alpha_1) \ldots w(\alpha_m)$, that is, $w(\mu) = w(\eta)$. \hfill \qed

Let $S_E$ be the set of elements $s \in S^\times$ such that $w(\mu) = s$ for some path $\mu$ in $E$, that is,

$$S_E := \{s \in S^\times \mid (\exists \mu \in E^*)w(\mu) = s\},$$

and let

$$S_e := \{s \in S_E \mid ss^{-1} = e\}.$$ 

Also, for $s, t \in S_e$ we put $s \leq t$ if and only if every path of weight $t$ contains an initial subpath of weight $s$.

**Lemma 5.7.** $S_e$ is a directed set with respect to $\leq$.

**Proof.** Obviously, $\leq$ is both reflexive and transitive. Now, let $s$ and $t$ be distinct elements of $S_e$. Then $es = s$ and $et = t$. Hence, every path of weight $s$ has a source of weight $e$, and every path of weight $t$ has a source of weight $e$. By the very definition of the weight mapping $w$, it follows that either every path of weight $t$ contains an initial subpath of weight $s$ or every path of weight $s$ contains an initial subpath of weight $t$. Equivalently, either $s \leq t$ or $t \leq s$. Thus, $S_e$ is a directed set with respect to $\leq$. \hfill \qed

Let $e \neq s \in S_e$, and let $P$ be a finite set of paths of $E$ of weight $\leq s$. For $t \leq s$, let $P_t$ be the set of initial paths of weight $t$ of elements of $P$, and let $Q_t$ be the set of edges $\alpha$ for which there exist paths $\mu'$ and $\mu''$ such that $w(\mu'\alpha) = t$ and $\mu = \mu'\alpha\mu'' \in P$. If $\mu \in P$ and $w(\mu) \geq t$, then by $\mu_t$ we denote the initial subpath of $\mu$ of weight $t$. Of course, $P_e$ is the set of the sources of paths from $P$.

**Definition 5.8** (cf. Definition 2.1.11 in [7]). We say that $P$ is an $X$-complete subset of $E^*$ if the following conditions are satisfied:

1. All the paths in $P$ of weight less than $s$ end in a sink;
2. For every $\mu \in P$, every $t < w(\mu)$ such that $r(\mu_t) \in X$, and every $\alpha \in s^{-1}(r(\mu_t))$, we have that $\mu_t\alpha = \eta w(\alpha)$, for some $\eta \in P$;
3. For every $\mu \in P_t \setminus E^0 (t < s)$ and every $\alpha \in Q_{tw(\alpha)}$ such that $r(\mu) = s(\alpha)$, we have that $\mu\alpha \in P_{tw(\alpha)}$.

If $F$ is a finite $X$-complete subgraph of $E$ and $e \neq s \in S_e$, then, by using the way $w$ is defined, it can be shown (cf. Proposition 2.1.12 in [7]) that there exists an $X$-complete subset of $E^*$ of paths of weight $\leq s$ which contains all the paths of weight $s$ of $F$ as well as all the paths of weight $< s$ that end in a sink of $E$. Then, following the technique presented in the proof of Proposition 2.1.14 in [7], one obtains the following result. The proof is similar and we omit it.
Proposition 5.9 (cf. Proposition 2.1.14 in [7]). Let \( E \) be an arbitrary graph, \( R \) a unital ring, and let \( X \subseteq \text{Reg}(E) \). Also, let \( P \) be an \( X \)-complete finite subset of \( E^* \) consisting of paths of weight \( \leq s \in S_e \). Define \( G(P) \) to be the \( R \)-linear span of monomials \( \mu \eta^s \), \( \mu, \eta \in P \), such that \( w(\mu) = w(\eta) \in S_e \). For \( e \neq t \leq s \), define \( F_t(P) \) to be the \( R \)-linear span in \( C_X^R(E) \) of the elements \( \mu \left( v - \sum_{a \in Q_t, s(a) = e} \alpha \right)^{\eta^s} \), where \( \mu, \eta \in P \), \( t'w(\alpha) = t \), \( r(\mu) = r(\eta) = v \notin X \), and \( Q_t \cap s^{-1}(v) \neq \emptyset \). Also, define
\[
F(P) = G(P) + \sum_{e \neq t \leq s} F_t(P).
\]

Then \( F(P) \) is a matricial \( R \)-algebra. Moreover, \( (C_X^R(E))_e \) is the direct limit of the subalgebras \( F(P) \), where \( P \) ranges over all the \( X \)-complete finite subsets of \( E^* \) whose weights belong to \( S_e \).

Recall that if \( A \) is a ring, the full matrix ring over \( A \) of \( n \times n \) matrices is denoted by \( M_n(A) \).

Lemma 5.10. Let \( E \) be a finite directed graph, and \( R \) a unital von Neumann regular ring. Then, for every idempotent element \( e \in S \), the component \( (L_R(E))_e \) of a canonically \( S \)-graded ring \( L_R(E) \) is von Neumann regular.

Proof. Let \( e \in S \setminus \{0\} \) be an idempotent element. If \( (L_R(E))_e = 0 \), there is nothing to prove. So, let \( (L_R(E))_e \neq 0 \). Then, by Lemma 5.6,
\[
(L_R(E))_e = \left\{ \sum_i r_i \mu_i \eta_i^s \mid r_i \in R, \mu_i, \eta_i \in E^*, r(\mu_i) = r(\eta_i), w(\mu_i) = w(\eta_i) \in S_e \right\}.
\]

According to Lemma 5.7, we have that \( S_e \) is a directed set with respect to \( \leq \). Now, for every \( s \in S_e \), define \( D_s \) to be the set
\[
\left\{ \sum_i r_i \mu_i \eta_i^s \mid r_i \in R, \mu_i, \eta_i \in E^*, r(\mu_i) = r(\eta_i), w(\mu_i) = w(\eta_i) = s \right\} \cup \left\{ \sum_i r_i \mu_i \eta_i^s \mid r_i \in R, \mu_i, \eta_i \in E^*, r(\mu_i) = r(\eta_i) \in \text{Sink}(E), w(\mu_i) = w(\eta_i) < s \right\}.
\]

It is easy to verify that \( D_s \) is an \( R \)-subalgebra of \( (L_R(E))_e \). On the other hand, similarly to Corollary 2.1.16 in [7], one concludes that
\[
D_s \cong \bigoplus_{t<s} \bigoplus_{e \in \text{Sink}(E)} M_{|P(t,e)|}(R) \oplus \bigoplus_{e \in E^0} M_{|P(s,e)|}(R),
\]
where \( P(s,e) \) denotes the set of all paths \( \mu \in E^* \) for which \( w(\mu) = s \) and \( r(\mu) = v \). As a corollary to Proposition 5.9, we have that \( (L_R(E))_e \) is the direct limit of the subalgebras \( D_s \), where \( s \) ranges over \( S_e \). In particular,
\[
(L_R(E))_e = \bigcup_{s \in S_e} D_s.
\]
Therefore, by Proposition 5.2.14 in [13], we get that \( (L_R(E))_e \) is von Neumann regular. \( \Box \)

Let \( E \) now be an arbitrary directed graph, distinct from the null graph, and let \( R \) be a unital ring. Also, let us assume that \( (L_R(E))_e \) is a von Neumann regular ring for every idempotent element \( e \in S^* \). Since \( E \) is distinct from the null graph, there exists \( e \in I(S)^* \) such that \( (L_R(E))_e \neq 0 \), that is, there exists a vertex \( v \in E^0 \) such that \( w(v) = e \). Then \( R \) can be embedded in \( (L_R(E))_e \) via the mapping \( x \mapsto xv \) \((x \in R) \). Let \( 0 \neq x \in R \). Since \( (L_R(E))_e \) is von Neumann regular, there exists \( a \in (L_R(E))_e \) such that \( xv = (xv)a(xv) \). We have that \( a \) is a finite sum \( \sum_i r_i \mu_i \eta_i^s \) for some \( r_i \in R \) and \( \mu_i, \eta_i \in E^* \) such that \( w(\mu_i) = w(\eta_i) = s_t, r(\mu_i) = r(\eta_i) \), and \( s_t s_t^{-1} = e \), according to Lemma 5.6. Hence, \( xv = (xv) \left( \sum_i r_i \mu_i \eta_i^s \right) \), and therefore, \( xv = \sum_j x r_j x \mu_j \eta_j^s \), where the sum goes over those \( j \) for which \( s(\mu_j) = s(\eta_j) = v \). We may now proceed exactly as in the proof of Lemma 4.5 in [16] in order to conclude that \( x = x r_k x \) for some \( k \). Therefore, like in the case of the canonical \( Z \)-grading (Lemma 4.5 in [16]), the converse of the previous lemma holds true as well in the case of the canonical \( S \)-grading, regardless of the cardinality of \( E \).
Lemma 5.11. Let $R$ be a unital ring, $E$ a directed graph which is not null, and let us observe $L_R(E)$ as a canonically $S$-graded ring. If $(L_R(E))_e$ is von Neumann regular for every idempotent element $e \in S^*$, then $R$ is von Neumann regular too.

We are now ready to prove Theorem 3.10 in the case of a finite graph.

Theorem 5.12. Let $R$ be a unital ring and let $E$ be a finite directed graph which is not null. Then $L_R(E)$ is graded von Neumann regular as a canonically $S$-graded ring if and only if $R$ is von Neumann regular.

Proof. ($\Rightarrow$) According to the ‘only if’ part of Theorem 4.10 we have that $(L_R(E))_e$ is a von Neumann regular ring for every idempotent element $e \in S$. The ring $R$ is then von Neumann regular by Lemma 5.11.

($\Leftarrow$) Let $R$ be von Neumann regular. Then Lemma 5.11 implies that $(L_R(E))_e$ is von Neumann regular for all $e \in I(S)$. Moreover, according to Theorem 5.11 we have that $L_R(E)$ is nearly epsilon-strongly graded as an $S$-graded ring. Therefore, by the ‘if part’ of Theorem 4.11 we get that $L_R(E)$ is graded von Neumann regular as an $S$-graded ring.

5.3 Case of an arbitrary graph

Let $E$ be a directed graph and $X \subseteq \text{Reg}(E)$. Then, according to Definition 1.5.16 in [46], we define a new graph $E(X)$ in the following way. Let $Y := \text{Reg}(E) \setminus X$, and let us add new vertices $Y' = \{v' \mid v \in Y\}$. The set of vertices $(E(X))^0$ is defined to be the disjoint union of $E^0$ and $Y'$, while the set of edges $(E(X))^1$ is defined to be the disjoint union of $E^1$ and $\{\alpha' \mid \alpha \in E^1, r(\alpha) \in Y\}$, where $\alpha'$ is a new edge starting from $s(\alpha)$ and ending in the new vertex $r(\alpha)' \in Y'$.

If $R$ is a unital ring, then, according to Proposition 4.8 in [46] (cf. Theorem 1.5.8 in [7]), $C^X_R(E)$ and $L_R(E(X))$ are isomorphic as $\mathbb{Z}$-graded rings. This isomorphism $\phi : C^X_R(E) \rightarrow L_R(E(X))$ is given by

$$\phi(v) = \begin{cases} v + v' & \text{if } v \in Y, \\ v & \text{if } v \notin Y, \end{cases} \quad \text{and} \quad \phi(\alpha) = \begin{cases} \alpha + \alpha' & \text{if } r(\alpha) \in Y, \\ \alpha & \text{if } r(\alpha) \notin Y, \end{cases}$$

where $v \in E^0$, $\alpha \in E^1$. Now, let us observe $C^X_R(E)$ as a canonically $S$-graded ring. If $w$ is the weight mapping on $E^* \cup \{\mu^* \mid \mu \in E^*\}$ of this $S$-grading of $C^X_R(E)$, then we put $w(v') = w(v)$ for every $v' \in Y'$ and $w(\alpha') = w(\alpha)$ for every $\alpha' \in \{\alpha' \mid \alpha \in E^1, r(\alpha) \in Y\}$. This extends $w$ to $E(X)^* \cup \{\mu^* \mid \mu \in E(X)^*\}$. Then $L_R(E(X))$ is a canonically $S$-graded ring with respect to $w$, and the mapping $\phi$ is a homogeneous isomorphism with respect to this grading too. Hence, the following proposition holds.

Proposition 5.13. Let $R$ be a unital ring and let $E$ be a directed graph. Then $C^X_R(E)$ and $L_R(E(X))$ are graded isomorphic as $S$-graded rings.

According to Lemma 4.9 in [46], if $\psi = (\psi^0, \psi^1) : (F, Y) \rightarrow (E, X)$ is a morphism in $\mathcal{G}$, and $R$ a unital ring, then there exists an induced $\mathbb{Z}$-graded ring homomorphism $\tilde{\psi} : C^X_R(F) \rightarrow C^X_R(E)$. This homomorphism is given by $\tilde{\psi}(v) = \psi^0(v)$, $\tilde{\psi}(\alpha) = \psi^1(\alpha)$ and $\tilde{\psi}(\alpha^*) = \psi^1(\alpha)^*$ for all $v \in F^0$ and $\alpha \in F^1$. Now, let $\tilde{C}^X_R(F)$ be canonically $S$-graded, with the weight mapping $w_F$. Also, let $C^X_R(E)$ be canonically $S$-graded with respect to the weight mapping $w_E : E^* \cup \{\mu^* \mid \mu \in E^*\} \rightarrow S$ such that $w_E(\psi^0(v)) = w_F(v)$ and $w_E(\psi^1(\alpha)) = w_F(\alpha)$ for all $v \in F^0$ and $\alpha \in F^1$. Then, clearly, $\tilde{\psi}$ is also a morphism in the category $S$-RING. Therefore, the following lemma holds.

Lemma 5.14. If $\psi : (F, Y) \rightarrow (E, X)$ is a morphism in $\mathcal{G}$, and if $R$ is a unital ring, then there exists an induced graded ring homomorphism $\tilde{\psi} : C^X_R(F) \rightarrow C^X_R(E)$ of $S$-graded rings.

Let $R$ be a unital ring. Inspired by Definition 4.10 in [46], we define the $S$-Cohn path algebra functor $C_R : \mathcal{G} \rightarrow S$-RING by $(E, X) \mapsto C^X_R(E)$, and $\tilde{\psi} \mapsto \psi$ for all objects $(E, X)$ of $\mathcal{G}$ and all morphisms $\psi$ of $\mathcal{G}$. Then, with the help of the previous lemma, one may prove that the following result holds by simply observing homogeneous homomorphisms of $S$-graded rings instead of $\mathbb{Z}$-graded ring homomorphisms in the proof of Lemma 4.11 in [46] (see also Proposition 1.6.4 in [7]).
Lemma 5.15. The S-Cohn path algebra functor $C_R$ preserves direct limits.

Proposition 5.16. Let $R$ be a unital ring and let $E$ be a directed graph. If $R$ is von Neumann regular, then $L_R(E)$ is graded von Neumann regular as an $S$-graded ring.

Proof. Of course, if $E$ is the null graph, then $L_R(E)$ is $S$-graded von Neumann regular. So, let $E$ be distinct from the null graph. According to Lemma 1.6.9 in [7], there exists a direct system $\{(F_i, Y_i) \mid i \in I\}$, where each $F_i$ is a finite graph, direct limit of which is $(E, \text{Reg}(E))$. By Lemma 5.15 we get that $L_R(E)$ and $\lim_{i} L_R^{Y_i}(F_i)$ are graded isomorphic as $S$-graded rings. However, by Proposition 5.13 each $C_{R}^{Y_i}(F_i)$ is graded isomorphic as an $S$-graded ring to $L_R(F_i(Y_i))$. Therefore, $L_R(E)$ and $\lim_{i} L_R(F_i(Y_i))$ are graded isomorphic as $S$-graded rings (cf. Proposition 4.12 in [46] and Corollary 1.6.11 in [7]).

On the other hand, since the graphs $F_i$ are finite, by Theorem 5.12 each $L_R(F_i(Y_i))$ is graded von Neumann regular as an $S$-graded ring. Therefore, $L_R(E)$ is indeed graded von Neumann regular as an $S$-graded ring by Lemma 2.2.

We may now prove Theorem 3.10.

Proof of Theorem 3.10. $(\Rightarrow)$ According to the ‘only if’ part of Theorem 4.10 we have that $(L_R(E))_e$ is a von Neumann regular ring for every idempotent element $e \in S$. Hence, $R$ is von Neumann regular by Lemma 5.11.

$(\Leftarrow)$ If $R$ is a von Neumann regular ring, then Proposition 5.16 implies that $L_R(E)$ is graded von Neumann regular as an $S$-graded ring.

Finally, we list some properties of $S$-graded Leavitt path algebras which are consequences of Theorem 3.10 based on the already established results on groupoid graded von Neumann regular rings.

Theorem 5.17. Let $R$ be a unital von Neumann regular ring, $E$ a directed graph, and observe $L_R(E)$ as a canonically $S$-graded ring. Then:

a) Every principal right (left) homogeneous ideal of $L_R(E)$ is generated by a homogeneous idempotent element;

b) Let $I$ be a right (left) ideal of $L_R(E)$ which is generated by finitely many homogeneous elements, say $\{x_1, \ldots, x_n\}$, such that for all $i \in \{1, \ldots, n\}$ we have that $\deg(x_i)(\deg(x_i))^{-1} = e$, for some $e \in I(S)^\times$. Then $I$ is generated by a homogeneous idempotent element;

c) Every homogeneous right (left) ideal of $L_R(E)$ is idempotent;

d) Every two-sided homogeneous ideal of $L_R(E)$ is graded semiprime;

e) $J^q(L_R(E)) = 0$.

Proof. Since $R$ is unital and von Neumann regular, Theorem 3.10 implies that $L_R(E)$ is graded von Neumann regular as an $S$-graded ring. Therefore, c) and d) follow by i) and ii) of Proposition 4.9 respectively. Now, the Leavitt path algebra $L_R(E)$ is nearly epsilon-strongly graded as an $S$-graded ring according to Theorem 5.4. Hence, the assertions a) and b) follow by Proposition 4.8 and e) follows by Proposition 4.9 ii).

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