Will jams get worse when slow cars move over?

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Abstract. – Motivated by an analogy with traffic, we simulate two species of particles (‘vehicles’), moving stochastically in opposite directions on a two-lane ring road. Each species prefers one lane over the other, controlled by a parameter $0 \leq b \leq 1$ such that $b = 0$ corresponds to random lane choice and $b = 1$ to perfect ‘laning’. We find that the system displays one large cluster (‘jam’) whose size increases with $b$, contrary to intuition. Even more remarkably, the lane ‘charge’ (a measure for the number of particles in their preferred lane) exhibits a region of negative response: even though vehicles experience a stronger preference for the ‘right’ lane, more of them find themselves in the ‘wrong’ one! For $b$ very close to 1, a sharp transition restores a homogeneous state. Various characteristics of the system are computed analytically, in good agreement with simulation data.

Driven diffusive systems [1,2] have been widely studied since they are amongst the simplest models which settle into nontrivial nonequilibrium steady states. These are of fundamental interest to physicists in their quest to formulate a theoretical framework for nonequilibrium behavior, on a par with Gibbs ensemble theory. Moreover, such models serve to describe numerous scientific or engineering situations involving net currents of mass and/or energy. Examples include colloids sedimenting under gravity [3], molecular motors moving along a microtubule [4], or traffic flowing on a highway [5].

The behavior of such systems becomes especially interesting when two (or more) different components (‘species’) move preferentially in different directions, such as, e.g., positive and negative charges in a uniform electric field. In the traffic analogy, the two species of particles can be interpreted as ‘fast cars’ and ‘slow trucks’, viewed from a co-moving frame. In the simplest model, cars and trucks may pass each other with a small rate and change lanes randomly. For ring roads with two or more lanes and no exits, two distinct phases are observed: on one side of the phase boundary, typical configurations are disordered and particles move freely; on the other side, configurations are spatially inhomogeneous and jammed up [6,7]. In particular, Monte Carlo simulations for two-lane ($L \times 2$) roads show one large macroscopic jam, of size $O(L)$, containing almost all vehicles. Only a few particles (‘travellers’) are found outside the jam, having just escaped by repeated passing.

We should note that recent analytic [8] and simulational [9] studies suggest that this jam is ‘merely’ a finite-size effect: As $L \to \infty$, the system consists of a distribution of jam-sizes...
controlled by a finite length \( \ell_0 \). However, \( \ell_0 \) may be as large as \( 10^{70} \) [8], rendering the study of systems with physically accessible sizes of great interest, as is our focus here.

It is natural to ask whether the jams will get shorter when drivers develop a lane preference. For simplicity, we explore this question for a two-lane periodic system, modeling cars/trucks preferring the ‘fast/slow’ lane. We introduce a parameter, \( b \), for the bias towards the preferred lane. Thus, \( b = 0 \) denotes no preference (random lane changes) while \( b = 1 \) corresponds to perfect lane segregation (‘laning’) of cars and trucks. With no vehicles in the ‘wrong’ lane, there will be no jams for \( b = 1 \). As \( b \) increases from zero, we may expect the number of vehicles in the ‘wrong’ lane to decrease, creating fewer ‘obstacles’ for those in the ‘right’ lane, and leading to smaller jams. In fact, we observe just the opposite: As \( b \) increases, the cluster grows in size until a sharp first order transition, very close to \( b = 1 \), restores the system to homogeneity. Even more remarkably, the number of vehicles in their preferred lane depends non-monotonically on \( b \), exhibiting a region of negative response. Clearly, the system is more complex than one might have anticipated.

This letter is organized as follows. We first describe our model and a few technical details of the simulations. We then present our data, followed by supporting analytic arguments. We conclude with a summary and open questions. More details will be reported elsewhere [10].

Our model is defined on a \( L \times 2 \) lattice with periodic boundary conditions in the \( x \)-direction. With \( x = 0, 1, \ldots, L - 1 \) and \( y = 0, 1 \), each site \( (x, y) \) can be empty or occupied by a positive (+) or a negative (−) particle. Configurations are specified in terms of an occupation variable \( q(x, y) \), taking the values 0 or \( \pm 1 \), depending on the ‘charge’ at the site. For later convenience, we define the local ‘mass’ by \( m(x, y) \equiv |q(x, y)| \), the positive particle density by \( p(x, y) \equiv \frac{1}{2} [m(x, y) + q(x, y)] \) and similarly, the negative case by \( n \equiv \frac{1}{2} [m - q] \). The system is neutral and half-filled; \( N \equiv L/2 \) is the number of positive particles in the system. Our rates are motivated by the traffic analogy. Positive (negative) particles never move left (right). The allowed moves \(+0 \rightarrow 0+\) and \(0- \rightarrow -0\) occur with rate 1, and ‘passing’ \(+- \rightarrow --\) takes place with rate \( \gamma < 1 \). Turning to lane changes, a positive (negative) particle in lane 0 (1) hops into a hole in lane 1 (0) with rate 1, while the opposite move occurs with rate \( 1 - b \); finally, particles can exchange positions across lanes with rate \( \gamma \) or \( \gamma (1 - b) \), depending on whether the move takes them into their preferred lanes or not.

The updating rule is random sequential. We choose a nearest-neighbor pair (bond) at random and attempt to exchange its contents. One Monte Carlo step (MCS) consists of selecting \( L \) bonds. Simulations lasted from \( 5 \times 10^5 \) up to \( 10^7 \) MCS. Typically, \( 2 \times 10^5 \) MCS
were discarded to ensure that the steady state was reached, and data were then taken every 100 MCS. Unless otherwise stated, all measured quantities are time averages in the steady state, denoted by $\langle \cdot \rangle$. Here, we focus on data for $\gamma = 0.1$ and $L = 10^3$; other $L$'s, ranging from 100 to $10^4$ have also been simulated. All show the same qualitative behaviors. Quantitatively, we find only very weak finite-size effects (below 5%) for sizes $250 \leq L \leq 10^4$, giving us confidence in our findings over a significant range of $L$.

For $b = 0$, ordered states display one large cluster. To distinguish such configurations from disordered ones, we define an order parameter, $\Omega(b) \equiv \langle 1/2 \sin(\pi/L) \sum_{x,y} e^{2\pi ix/L} m(x, y) \rangle$ which is $O(L^{-1/2})$ if particles are distributed randomly, and takes its maximum value, 1, for a single jam containing all particles. The measured value $\Omega(0) = 0.940$ indicates that a few particles (the ‘travellers’) remain outside the jam. Remarkably, $\Omega(b)$ increases with $b$, demonstrating that the cluster actually grows in size. This pattern persists up to $b = 0.998 \pm 0.001$ where $\Omega$ drops abruptly (from 0.972 to 0.045) as the system returns to disorder. This sharp transition appears to be first order, since it displays hysteresis [10].

As soon as $b > 0$, particles begin to favor one lane over the other, leading to a ‘charge’ imbalance. Clearly, it suffices to monitor the total charge in, e.g., lane 1, $Q(b) \equiv (2/L) \sum_{x} \langle q(x, 1) \rangle$, normalized such that $0 \leq Q(b) \leq 1$. Shown in Fig. 1, $Q(b)$ first increases monotonically, as one would expect. At $b = 0.95$, however, $Q(b)$ displays a maximum and enters a region of negative response, i.e., $dQ/db < 0$: More vehicles are found in the ‘wrong’ lanes, even though their preference for the ‘right’ lane has increased! Since the order parameter remains perfectly monotonic here, this behavior must be associated with a restructuring of the jam interior. Near $b \approx 0.99$, $Q(b)$ goes through a minimum (inset) and becomes ‘normal’ again. Finally, it jumps abruptly to the disordered value ($\approx 1$), correlated with the drop in $\Omega(b)$.

A detailed view of the jammed state is provided by measuring steady-state mass and charge profiles of each lane, $\langle m(x, y) \rangle$ and $\langle q(x, y) \rangle$. Since the jams form randomly and diffuse slowly during a run, we shift their centers of mass to $x = 0$ before averaging, by tracking the phase of the Fourier transformed mass density.

The disordered state is spatially uniform. In contrast, profiles in the jammed phase are highly nontrivial and depend strongly on $b$. Due to an underlying symmetry, it suffices to focus our attention on just one lane, say, lane 1 (preferred by positive charges). Fig. 2 shows four charge profiles $\langle q(x, 1) \rangle$, at different values of $b$. Focusing first on common features, each profile displays two distinct regions. Outside the large cluster, there are few travellers. Denoted by $m^*$ and $q^*$, the mass and charge densities are uniform. Inside the cluster, positive/negative

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Fig. 2 – Charge profiles $\langle q(x, 1) \rangle$ vs $x$ for different $b$; $L = 10^3$ and $\gamma = 0.1$. For comparison, the solid (dashed) lines are the mass profiles $\langle m(x, y) \rangle$ of lane 1 (0), for $b = 0.9989$. 

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charges tend to accumulate at its left/right edge, so that \( \langle q(x, 1) \rangle \) decreases from left to right. The mass profiles (plotted only for \( b = 0.9989 \)) show essentially no holes in the interior for any \( b \). At the jam edges, \( \langle q(x, 1) \rangle \) varies rapidly, reminiscent of a shock. Beyond these global features, remarkable differences emerge. For \( b = 0 \), the traveller region is neutral in each lane, and \( \langle q(x, 1) \rangle \) is odd with respect to the center of mass, \( x = 0 \). As \( b \) increases, the traveller region becomes charged, reaching a maximum of \( q^+(b) \approx 0.032 \) at \( b = 0.81 \). The remainder of the profile looks similar to its \( b = 0 \) counterpart, but shifted upwards by a constant as positive particles begin to favor this lane. As \( b \) approaches 0.95, i.e., the maximum of \( Q(b) \) (cf. Fig. 1), the profile changes shape dramatically inside the cluster, while the local charge outside is now lower: \( q^+(0.95) \approx 0.029 \). The region of negative response in \( Q \) is associated with this ‘reshaping’ of the interior profile. As \( b \) increases further, beyond \( b \approx 0.99 \), each lane develops an extended block which is (almost) entirely occupied by its preferred species. As we will see below, these solid domains act as blockages through which no currents can flow. The remainder of the cluster is still mixed, containing both positive and negative particles with a nontrivial charge gradient. At the jam edges, another new feature emerges here, namely, the shock in lane 0 is distinctly offset from its counterpart in lane 1. As illustrated in Fig. 2 for \( b = 0.9989 \), the left edge of the cluster in lane 1 is located 47 ± 1 lattice sites further left than its counterpart in lane 0! The traveller region is now almost perfectly charge-segregated, with \( m^+(0.9989) \approx q^+(0.9989) \approx 0.026 \). For profiles of this type, \( Q_1(b) \) is, once again, monotonically increasing. Eventually, at \( b = 0.9990 \), the large jam disintegrates and the system reverts to disorder.

A remarkable feature of this system is the presence of highly nonuniform current densities, both in the longitudinal (in-lane) as well as the transverse (cross-lane) directions. Focusing on just the positive particles, say, we let \( J^+(x-1, x; y) \) denote the (average) in-lane current from site \( x-1 \) to \( x \) in lane \( y \), and \( j^+(x) \) denote the cross-lane current from lane 0 into lane 1, at \( x \). Thus, \( j^+ > 0 \) indicates a net current into the preferred lane. In the steady state, particle conservation ensures that \( J^+(x-1, x; y) - J^+(x, x+1; y) = (-1)^y j^+(x) \) for all \( (x,y) \). In the extreme cases with \( b \approx 1 \) (e.g., 0.9989), particles in preferred lanes form solid domains making up half of the jam. Since bonds occupied by two identical particles carry no current, the in-lane current is essentially zero through these solid domains. Yet, the traveller region clearly carries a nonzero current. As a result, particles outside the jam flow mainly in their preferred lane, and then jump – via a suppressed transition! – into the ‘wrong lane’ so as to circumvent the solid blockage. Such highly suppressed moves are facilitated by the offset of the cluster boundaries in the two lanes. In the ‘wrong lane,’ particles move through the mixed region of the cluster, and return to the preferred lane near the center of the cluster. This behavior is understandable, since a particle in the ‘wrong lane’ faces a solid block of the opposite species. By crossing back into the ‘right lane,’ they make their way through a mixed region and exit the jam. This phenomenon is well illustrated in Fig. 3, in which the solid circles show \( j^+ < 0 \) in a sizable region around \( x = -250 \), followed by a positive peak near \( x = 0 \). As a contrast, for \( b \lesssim 0.8 \) cases, where such solid blockages have not formed yet, the currents are quite different (open squares): Positive charges still jump into the unfavored lane near the left edge of the cluster, but the associated peak in \( j^+ \) is much narrower. Since they face no solid domains in either lane, there is no need to return to the favored lane until the end of the jam. The analogue in circuits for this two-lane jam would be two resistors in parallel, with the incoming current split appropriately between them at one end, only to rejoin at the other end. In all other parts of the system, the cross-lane currents are statistically indistinguishable from zero, so that the \( J^+ \)’s are essentially uniform (within such regions).

In the remainder of this letter, we present selected analytic results for the steady state, with details to be reported elsewhere [10]. Specifically, we aim to compute \( Q(b) \) over a range
of $b$ within a simple mean-field theory, using some input from the simulations. Omitting the 
$\langle \cdot \rangle$ brackets from now on, we let $p(x, y)$ and $n(x, y)$ denote the configurational averages of the local
densities of positive and negative particles, with $m(x, y)$ and $q(x, y)$ defined similarly.
With the center of mass still at $x = 0$, symmetry dictates $p(x, 0) = n(-x, 1)$ and $p(x, 1) = n(-x, 0)$.
Outside the jam, the densities are uniform, to be denoted by $p^*(y)$ and $n^*(y)$,
with $p^*(0) = n^*(1)$ and $p^*(1) = n^*(0)$. The traveller mass densities in each lane are equal:
$p^*(0) + n^*(0) = p^*(1) + n^*(1) \equiv m^*$, while $p^*(1) - n^*(1) = n^*(0) - p^*(0) \equiv q^*$
denotes the traveller charge. The interior of the cluster contains essentially no holes whence
$m(x, y) \simeq 1$ there. As a result, for each value of $b$, we need to determine only two independent densities,
say, $m^*$ and $q^*$ to characterize the traveller region, and a single function, e.g., $p(x, 0)$, to
characterize the cluster. Of course, the traveller region and the cluster are coupled to one
another, through currents flowing along the lanes and from one lane to the other. In mean-field
theory, the net in-lane current of positive particles, from site $x - 1$ to $x$, can be written
as $J^+(x - 1, x; y) = p(x - 1, y) [1 - m(x, y)] + \gamma p(x - 1, y) n(x, y)$. Here, the first term reflects
the move of a positive particle from $(x - 1, y)$ into an empty site at $(x, y)$, and the second term
reflects the exchange of a positive and a negative particle. Similarly, we can write the net cross-
lane current of positive particles, from lane 0 into lane 1, as $j^+(x) = p(x, 0) [1 - m(x, 1)] - (1 - b) p(x, 1) [1 - m(x, 0)] + \gamma p(x, 0) n(x, 1) - \gamma (1 - b) p(x, 1) n(x, 0)$ with all other currents following by symmetry. In particular, when summed over both lanes, the net mass current vanishes,
and the net charge current is constant, independent of $x$.

Considering the traveller region first, having constant densities there implies $0 = j^+(x)$
which is consistent with the current data (Fig. 3). This allows us to compute all densities in
the traveller region in terms of the single parameter $m^*$. The resulting expressions simplify
for small $\gamma$; we find, e.g., $q^* = bm^*/(2 - b) + O(\gamma)$. Both $q^*$ and $m^*$ are easily measured so
that this relation can be tested: It holds remarkably well over the full range of $b$ [10], giving
us confidence in our mean-field theory.

Next, we turn to the interior of the cluster. The expressions for the currents simplify
considerably here, due to the absence of holes. For $b \leq 0.9$, the data suggest $j^+(x) \simeq 0$ inside
the cluster. Exploiting this observation for $x = 0$ and invoking symmetries, we can compute
the local charge at the center of the cluster: $q(0, 1) = (1 - \sqrt{1 - b})^2 / b$. Moreover, for $b \leq 0.9$, $q(x, 1) - q(0, 1)$ is, to good approximation, an odd function of $x$ inside the jam. If we assume,
for simplicity, that the travellers can be neglected so that the cluster extends over $L/2$ sites,
we may express the total charge in lane 1 as \( Q(b) = (2/L) \sum_{x=-L/2}^{L/2} q(x,1) \simeq (2/L)q(0,1) \) which is plotted in Fig. 1 as a solid line. Clearly, it follows the data remarkably well, up to \( b \leq 0.9 \).

Next, we focus on profiles just before the onset of disorder. For \( b \gtrsim 0.99 \), the traveller region is essentially charge-segregated, i.e. \( p^*(1) \simeq n^*(0) \simeq m^* \). As a result, the \( \pm \) current there is given by \( J^+(x-1,x,y) \simeq m^*(1-m^*) \). When these particles encounter the solid block of positive charge which forms the left edge of the cluster, they change lanes and enter the mixed region of the cluster in lane 0. Since there is no leakage current into the other lane here, the mixed region can be viewed as a single species asymmetric exclusion process in the maximal current phase, whence \( J^+(x-1,x,y) = \gamma/4 \), up to finite size corrections. Equating these two currents yields \( m^* = (1-\sqrt{1-\gamma})/2 = 0.026 \) for \( \gamma = 0.1 \), in excellent agreement with the data. Next, we consider the cross-lane current which is localized in the offset region. Here, correlations turn out to be essential: If a positive particle has just performed a suppressed jump from lane 1 into 0, it leaves a hole behind. If this bond is selected again, before either particle or hole have moved away, the particle will return to its original location, and no contribution to the current has resulted. If we approximate the number of suppressed jumps over the length \( \Delta \) of the offset by its mean-field form \( \Delta(1-b)(1-m^*) \), the total net cross-lane current is simply the fraction \( \alpha \) of such moves which are not ‘undone’, i.e., \( \alpha \Delta(1-b)(1-m^*) \).

We have not been able to find a good analytic estimate for \( \alpha \), but it is easily measured in simulations: \( \alpha = 1 - N_f/N_q \), where \( N_f \) \((N_q) \) is the number of favored (suppressed) moves between the lanes in the offset region during a run. For example, we measure \( \alpha = 0.503 \pm 0.003 \) for \( b = 0.9989 \). Equating currents again gives \( \Delta = m^*/\alpha(1-b) \). Thus, we predict \( \Delta = 46.3 \), for \( b = 0.9989 \) and \( \gamma = 0.1 \), in excellent agreement with the simulation result \( \Delta = 47 \pm 1 \).

Similar arguments \cite{10} yield the lane charge of such profiles as \( Q(b) = 1/(2-2m^*) + \Delta/L \). Assuming \( \alpha \simeq 1/2 \), this expression compares very well with the data for \( b \geq 0.995 \) (dashed line in the inset of Fig. 1).

To summarize, we consider two species of particles (‘vehicles’), biased to move in opposite directions along a two-lane road. Vehicles can pass each other (with a small rate \( \gamma \)) and change lanes, controlled by a parameter \( b \). We find that (i) for all \( b < 0.9996 \), the system displays one large cluster (‘jam’) whose size increases monotonically with \( b \). This is quite remarkable, given that a larger \( b \) reduces the number of obstacles which a particle encounters as it travels down its preferred lane. For \( b \) very close to unity, a first order transition restores homogeneity. (ii) The lane ‘charge’ \( Q \) exhibits a region of negative response for \( 0.950 \leq b \leq 0.992 \). This is arguably our most surprising result: even though the particles experience a stronger preference for their own lane, more of them find themselves in the ‘wrong’ lane. Preliminary data for systems with different numbers of positive and negative particles show that this behavior persists for a significant range of system ‘charge’. (iii) The system exhibits nontrivial mass current loops: particles change into the unfavored and back into the favored lane at well-defined locations in the system. This observation, in conjunction with particle conservation, allows us to compute the lane charge analytically, in good agreement with the data except in the regime displaying negative response. Here, a very complex restructuring of the cluster interior takes place, requiring further analysis \cite{10}.

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