Quantum distance and the Euler number index of the Bloch band in a 1D spin model with multi-site spin couplings

Yu-Quan Ma\textsuperscript{1}

\textsuperscript{1}School of Applied Science, Beijing Information Science and Technology University, Beijing 100192, China

(Dated: March 11, 2014)

We study the Riemannian metric and the Euler characteristic number of the Bloch band in a 1D spin model with multi-site spins exchange interactions. The Euler number of the Bloch band originates from defining a local Riemannian structure to the parameterized quantum Hilbert space. Quantum geometric tensor brings quantum distance between two states in a parameterized sor originates from defining a local geometric ten- tum states. Physically, the (non-Abelian) geometric ten- tensor, or called the Fubini-Study metric, is a Hermi- nality of QPTs. Mathematically, quantum geometric dianty and the fidelity susceptibility, Berry phase properties, i.e., quantum entanglement, entanglement entropy, quantum discord, quantum fidelity and the fidelity susceptibility, Berry phase, and the quantum geometric tensor.

The ground-state geometric tensor, as an intrinsic metric on the ground-state complex manifold, is naturally expected to shed some light on the geometric characterization of QPTs. Mathematically, quantum geometric tensor, or called the Fubini-Study metric, is a Hermitian metric on the complex projective space of the quantum states. Physically, the (non-Abelian) geometric tensor originates from defining a local $U(n)$ gauge invariant quantum distance between two states in a parameterized Hilbert space. Quantum geometric tensor brings a Riemannian structure to the parameterized quantum states, where the corresponding Riemannian metric is given by the real part of the geometric tensor. Meanwhile, its imaginary part was later found to be just the Berry curvature (up to a constant coefficient). Specifically, the ground-state geometric tensor provide a unified mechanism from the aspect of information-geometry to understand the critical behaviors in quantum many-body systems.

Recently, a direct measurement of the Zak phase, as a Berry phase of a 1D Bloch band, has been achieved in 1D optical lattices. For the geometric tensor of the Bloch band, some interesting measurable consequences have been proposed by relating the geometric tensor of band insulators to the current noise spectrum. A more interesting question is whether there exists some topological characterization related to the Riemannian metric of the Bloch bands? Very recently, a topological Euler number of the Bloch band was proposed to distinguish nontrivial topological phases in gapped free fermionic systems. This fact was pointed out in our previous work and later by Kolodrubetz et al.

In this work, we study the local and topological properties of the Bloch band in a 1D transverse field XY spin-1/2 model with three-site spin interactions. The system exhibit a nonzero transverse magnetization at the zero transverse field due to its multi-site spins exchange interactions. In order to obtain a well-defined geometric tensor in the crystal momentum space, we introduce an extra 1D parameter space by subjecting the spin system to a local gauge transformation, which in fact puts the Hamiltonian of the system on a torus $T^2$ in a 1+1D crystal momentum space without changing its energy spectrum. By using of the quantum Riemannian metric on the Bloch states manifold, we introduce a class of cyclic quantum distance as a local characterization for quantum phase transitions. Particularly, we derive the Euler characteristic number of the Bloch band analytically via the Gauss-Bonnet theorem on the Bloch states manifold in the first Brillouin zone. A general formula for the Euler number is obtained by means of the Berry curvature.
in the case of two-band models, which also reveals its relation to the first Chern number of the band insulators. Finally, we show that the ferromagnetic and paramagnetic quantum phase transitions can be distinguished by the different Euler numbers of the Bloch band.

II. THE MODEL

We consider a 1D anisotropic XY spin-1/2 model with three-site spin exchange interactions in a transverse field. This spin model exhibits a nonzero transverse magnetization at the zero transverse field due to its multiple sites spin coupling and show a rich ground-state phase diagram [62 64]. The Hamiltonian reads

\[
H_S = \sum_{l \in N} \left[ -\Delta (S^x_l S^x_{l+1} + S^y_l S^y_{l+1}) - 2\delta (S^z_l S^z_{l+1} + S^x_l S^y_{l+1} S^z_{l+1}) - hS^z_l \right],
\]

where \( S^\alpha_l \) (\( \alpha = x, y, z \); \( l \in N \)) is the Pauli operator on the local site \( l \), \( \gamma \) is the anisotropy parameter in the in-plane interaction, \( \delta \) denotes the three-site \( \text{XZZ+YZY} \) type spins exchange interactions, \( h \) is the transverse magnetic field, and the periodic boundary condition (PBC) has been imposed on this model.

Here we will show that the quantum critical points of the system can be witnessed by some local geometric Euler number index of the Bloch band in the crystal momentum space.

In order to investigate the ground-state geometric tensor for the system, we need to define the metric tensor on a 2D parameter space. This can be achieved by subjecting the system to a local gauge transformation \( H_S(\varphi) = g(\varphi)H_Sg(\varphi)^\dagger \) by a twist operator \( g(\varphi) = \prod_1 e^{i\varphi S^i_l} \), which makes the system a rotation on the spin along the \( z \)-direction. It can be verified that \( H_S(\varphi) \) is \( \pi \) periodic in \( \varphi \) because the quadratic form about the \( x \) and \( y \) axes appears symmetric in the Hamiltonian. Considering the unitarity of the twist operator \( g(\varphi) \), the critical behavior and energy spectrum of the system are obviously parameter \( \varphi \) independent.

The spin Hamiltonian \( H_S(\varphi) \) can be mapped exactly on a spinless fermion Hamiltonian \( H_F(\varphi) \) by the Jordan-Wigner transformation \( a_l = \prod_{m=1}^{l-1} (-2S^z_m) S^x_l, a_l^\dagger = \prod_{m=1}^{l-1} (-2S^z_m) S^z_l \), where \( S^\pm_l = S^x_l \pm iS^y_l \) denote the spin ladder operators and \( a_l, a_l^\dagger \) are the corresponding Fermion annihilation and creation operators on the local site \( l \). After applying a Fourier transformation \( a_l = \frac{1}{\sqrt{N}} \sum_{k \in \text{Ba}} e^{ikl} c_k \), we can rewrite the fermion Hamiltonian as

\[
H_F(\varphi) = \sum_{k \in \text{Ba}} \Psi_{k,\varphi}^\dagger \left( \sum_{\alpha=1}^{3} d_{\alpha}(k,\varphi) \sigma^\alpha \right) \Psi_{k,\varphi},
\]

where \( d_1(k,\varphi) = \frac{1}{2}\gamma \sin k \sin 2\varphi, d_2(k,\varphi) = \frac{1}{2}\gamma \sin k \cos 2\varphi, d_3(k,\varphi) = \frac{1}{2}(h + \delta \cos 2k - \cos k), \Psi_{k,\varphi}^\dagger := \begin{pmatrix} c_k^1, c_{-k}^1 \end{pmatrix} \) and \( \sigma^\alpha \) denotes the the Pauli matrices, represent the pseudo-spin degree of freedom.

The Bloch wave function can be expressed as

\[
u_{\pm}(k,\varphi) = \frac{1}{\sqrt{2d(d + d_3(k,\varphi))}} \begin{pmatrix} d_1(k,\varphi) - id_2(k,\varphi) & \pm d - d_3(k,\varphi) \end{pmatrix},
\]

and the corresponding energy spectrum is \( E_{\pm}(k) = \pm d \), where \( d := \sqrt{\sum_{\alpha=1}^{3} d_{\alpha}(k,\varphi)} \). The Hamiltonian can be diagonalized as \( H(\varphi) = \sum_{k \in \text{Ba}} E_+(k) a_{k,\varphi}^\dagger \alpha_{k,\varphi} + E_-(k) \beta_{k,\varphi}^\dagger \beta_{k,\varphi} \), and the \( \varphi \) parameterized ground-state \( |GS(\varphi)\rangle \) is the filled fermion sea

\[
|GS(\varphi)\rangle = \prod_{k > 0} \beta_{-k,\varphi}^\dagger \beta_{k,\varphi}^\dagger |0\rangle,
\]

where the quasi-particle operators \( \alpha_{k,\varphi} = [u(\varphi,k)]^\dagger_{+} |\Psi_{k,\varphi}\rangle \) and \( \beta_{k,\varphi} = [u(\varphi,k)]^\dagger_{-} |\Psi_{k,\varphi}\rangle \).

Note that the Bloch Hamiltonian \( H(k,\varphi) := \sum_{\alpha=1}^{3} d_{\alpha}(k,\varphi) \sigma^\alpha \) is period \( \pi \) on the parameter \( \varphi \), that is \( H(k,0) = H(k,\pi) \). On the other hand, the Bloch Hamiltonian \( H(k,\varphi) \) can be regarded periodic in the Brillouin zone up to a gauge transformation \( H(k + G,\varphi) = e^{-iG \cdot r} H(k,\varphi) e^{iG \cdot r} \), where \( G, r \) are the reciprocal lattice vector and position vector, respectively. Note that in a lattice model, here the gauge factor is just identically equal to 1, and we have \( H(k + G,\varphi) = H(k,\varphi) \). Hence, the Bloch Hamiltonian \( H(k,\varphi) \) has been put on a torus \( T^2 \) in a 1+1D crystal momentum space.

III. GEOMETRIC TENSOR ON THE BLOCH STATES MANIFOLD

To begin with, we give a brief discussion on the quantum geometric tensor of the Bloch band. The quantum geometric tensor of the Bloch band can be derived naturally from a gauge invariant distance between two Bloch states on the \( U(1) \) line bundle induced by the quantum adiabatic evolution of the Bloch state \( |u_n(k)\rangle \) of the \( n \)-th filled band. The gauge invariant quantum distance between two states \( |u_n(k+\delta k)\rangle \) and \( |u_n(k)\rangle \) is given by

\[
dS^2 = \sum_{\mu,\nu} \langle \partial_{\mu} u_n | [1 - \mathcal{P}_n] | \partial_{\nu} u_n \rangle dk^\mu dk^\nu,
\]

where \( \mathcal{P}_n = |u_n\rangle \langle u_n| \) is the projection operator, and \( \mu, \nu \) denote the components \( k^\mu \) and \( k^\nu \), respectively. The
quantum geometric tensor is given by

\[ Q_{\mu\nu} = \langle \partial_{\mu} u_{n} | (1 - P_{n}) | \partial_{\nu} u_{n} \rangle. \] (6)

The underlying mechanism for the quantum distance can be understood as follows: The term \( |\partial_{\mu} u_{n}\rangle\) can be decomposed in the complete Hilbert space as \( |\partial_{\mu} u_{n}\rangle = |D_{\mu} u_{n}\rangle + |1 - P_{n}| |\partial_{\nu} u_{n}\rangle\), where \( |D_{\mu} u_{n}\rangle = P_{n} |\partial_{\nu} u_{n}\rangle\) is the covariant derivative of \( |u_{n}\rangle\) on the line bundle. Under the condition of the quantum adiabatic evolution, the evolution of \( |u_{n}(k)\rangle\) to \( |u_{n}(k + \delta k)\rangle\) will undergo a parallel transport, that is \( |D_{\mu} u_{n}(k)\rangle = 0\), which will lead to a gauge invariant quantum distance as Eq. (5). The geometric tensor Eq. (6) can be rewritten as \( Q_{\mu\nu} = G_{\mu\nu} - iF_{\mu\nu}/2\), where \( G_{\mu\nu} := \text{Re}Q_{\mu\nu}\) can be verified as a Riemannian metric, which establishes a Riemannian manifold of the Bloch states. It can be verified that the quantum distance is only depend on the real part of the quantum geometric tensor, that is \( dS^{2} = \sum_{\mu,\nu} G_{\mu\nu} dk^{\mu} dk^{\nu}\), because the term \( F_{\mu\nu} := -2\text{Im}Q_{\mu\nu}\) is canceled out in the summation of the distance due to its antisymmetry. However, the term \( F_{\mu\nu}\) can be associated to a 2-form \( F = \sum_{\mu,\nu} F_{\mu\nu} dk^{\mu} \wedge dk^{\nu}\), which is nothing but the Berry curvature.

A. Riemannian metric and the cyclic quantum distance of the Bloch band

The Riemannian metric of the Bloch band is given by \( G_{\mu\nu} = \text{Re}Q_{\mu\nu}\), where the geometric tensor \( Q_{\mu\nu}\) can be obtained by substituting Eq. (5) to Eq. (6), and it can be verified that this metric \( G\) is given by the following diagonalized form:

\[ dS^{2} = G_{kk} dk^{2} + G_{\varphi\varphi} d\varphi^{2}, \] (7)

with

\[ G_{kk} = \frac{1}{2} \left[ \frac{\gamma + \gamma(h - 2\delta + \delta\cos 2k)\cos k}{(h + \cos k - \delta\cos 2k)^{2} + \gamma^{2}\sin^{2} k} \right]^{2}, \]

\[ G_{\varphi\varphi} = \frac{\gamma^{2}\sin^{2} k}{(h + \cos k - \delta\cos 2k)^{2} + \gamma^{2}\sin^{2} k}. \] (8)

The metric \( G\) is obviously independent on the parameter \( \varphi\) because of its \( U(1)\) gauge invariance on the twist operator.

In Fig. 1, we show that the trace of the Riemannian metric as a function of the external field \( h\) and the crystal momentum \( k\) with different three-site spins coupled parameter and the anisotropy parameter. As we expect, the singularity regions of the metric will appear when the external field \( h\) close to the quantum critical points. We define a cyclic quantum distance on the Bloch band from \((0, -\pi)\) to \((\pi, \pi)\) in the extended first Brillouin zone (the inset in Fig. 2), where the parameter path of the integral loop \( C\) is \( \varphi = k/2 + \pi/2, (k \in 1Bz)\), which is just the diagonal line in the extended Brillouin zone.

The cyclic quantum distance \( l\) of the Bloch band is given by

\[ l = \int_{C} \sqrt{G_{kk} dk^{2} + G_{\varphi\varphi} d\varphi^{2}} = \int_{-\pi}^{\pi} \sqrt{G_{kk} + \frac{1}{4}G_{\varphi\varphi} \sin^{2} k}. \] (9)

where the Riemannian metric \( G_{kk}\) and \( G_{\varphi\varphi}\) is given by Eq. (8). As shown in Fig. 2, we calculate the cyclic quantum distance \( l\) as a function of \( h\), with the fixed anisotropy parameter \( \gamma = 1/3\) and different three-site spins coupled coefficients \( \delta\). The singularity points on the cyclic quantum distance are just corresponding to the quantum transition points \( |\delta - 1|\) and \( |\delta + 1|\).

In Fig. 3, we plot the cyclic quantum distance \( l\) with the fixed three-site spins coupled coefficients \( \delta = 0.3\) and different anisotropy parameter \( \gamma\). It can be seen that the value of the anisotropy parameter \( \gamma\) does not affect the critical point but make the cyclic quantum distance \( l\)
ground-state distance \( l \) in the paramagnetic phase. 

B. Cyclic quantum distance of the ground state

It is worth noting that the metric component \( G_{\varphi\varphi} \) on the Bloch band is close related to the ground-state quantum distance in the parameter \( \varphi \) space. In fact, the ground state \( |GS(\varphi)\rangle \) is \( \pi \) periodic in the parameter \( \varphi \). In the condition of the large sites limit \( N \rightarrow \infty \), a cyclic ground-state distance \( l_{GS} \) can be defined along the \( \varphi \) -ring as

\[
l_{GS} = \int_0^\pi \sqrt{(\partial_\varphi \langle GS(\varphi) | [1 - P_{GS}] | \partial_\varphi \langle GS(\varphi) \rangle d\varphi}
= \frac{1}{2\pi} \int \int_{1Bz} \sqrt{G_{\varphi\varphi}} d\varphi d\varphi,
\]

where \( P_{GS} = |GS(\varphi)\rangle \langle GS(\varphi)| \) denotes the ground-state projection operator and the Eqs. (3) and (4) have been used in the intermediate steps. Note that the result in Eq. (10) is general, which only relates to the metric \( G_{\varphi\varphi} \) on the Bloch band and the concrete expression of the ground state is not required.

In Fig. 4 (a), we show the derivative of the cyclic ground-state distance \( dl_{GS}/dh \) with different lattice sizes \( N \), where the Hamiltonian parameters \( \gamma = 0.7 \) and \( \delta = 0.3 \). As shown in Fig. 4 (b), we can see that the positions of the maximum points of the derivative \( dl_{GS}/dh \), with the increasing of the lattice sizes, tend as \( N^{-1.0074} \) and \( N^{-0.7122} \) to the critical points \( h_c = 0.7 \) and \( h_c = 1.3 \), respectively.

IV. THE EULER CHARACTERISTIC NUMBER OF THE BLOCH BAND

What is more interesting is that the Euler characteristic number of the Bloch band can be derived from the Gauss-Bonnet theorem on the Bloch states manifold established by the Riemannian metric \( G_{\mu\nu}^{(n)} \). The Euler characteristic number \( \chi \) of all occupied bands can be generalized written by (see Ref. [60])

\[
\chi = \frac{1}{4\pi} \sum_n \int \int_{1Bz} \mathcal{R}^{(n)} \sqrt{\det G_{\mu\nu}^{(n)}} dk^\mu dk^\nu,
\]

where \( \mathcal{R}^{(n)} \) is the Riemann curvature tensor.
where the $\mathcal{R}^{(n)}$ is the Ricci scalar curvature associate to the Bloch state $|u_n(k)|$ of the $n$-th Bloch band. The Ricci scalar curvature $\mathcal{R}$ can be calculated by using the standard steps: $\mathcal{R} = g^{ab}R_{abc}^\cd$, where the Riemannian curvature tensor

$$R_{abcd} = \partial_b \Gamma^d_{ac} - \partial_a \Gamma^d_{bc} + \Gamma^d_{ac} \Gamma^e_{bd} - \Gamma^e_{bc} \Gamma^d_{ae},$$

(12)

and the Levi-Cività connection $\Gamma^a_{bc}$ can be calculated by

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad} (\partial_b G_{dc} + \partial_c G_{bd} - \partial_d G_{cb}).$$

(13)

The Riemannian metric $G$ of the Bloch band is given by Eq. (5), and its contravariant component can be easily obtained as $G^{kk} = 1/G_{kk}$, and $G^{\varphi\varphi} = 1/G_{\varphi\varphi}$. By using Eqs. (5) and (13), we can obtain all of the non-zero connections as

$$\Gamma^\varphi_{\varphi k} = \Gamma^\varphi_{\varphi k} = \frac{(B - \gamma^2 \cos k) \sin k - 2\delta B \sin 2k}{B^2 + \gamma^2 \sin^2 k} + \cot k,$$

(14)

and

$$\Gamma^k_{kk} = \frac{2(B - \gamma^2 \cos k) \sin k - 2\delta B \sin 2k}{B^2 + \gamma^2 \sin^2 k}$$

$$= \frac{(h + 3\delta \cos 2k) \sin k}{1 + A \cos k},$$

$$\Gamma^k_{\varphi\varphi} = \frac{-4B \sin k}{1 + A \cos k},$$

(15)

with

$$A = \hbar + \delta \cos 2k - 2\delta,$$

$$B = \hbar - \delta \cos 2k + \cos k.$$  

The Euler characteristic number $\chi$ is a topological invariant and equals to $2(1 - g)$ with genus $g$ for a closed smooth manifold. Note that the Bloch band of the model forms a 2D closed Riemannian manifold in the first Brillouin zone, and then the Euler characteristic number can be calculated conveniently by the Gauss-Bonnet theorem $\chi = \frac{1}{2\pi} \int_{1Bz} \mathcal{K} dA$, where $\mathcal{K} = R_{kk\varphi\varphi} / \det G_{kk}$ is the Gauss curvature, which just equals to the half of the Ricci scalar curvature $\mathcal{R}$, and the covariant Riemannian curvature tensor $R_{abcd} := R_{abcd}^\cd G_{cd}$ have only one substantial component $R_{kk\varphi\varphi}$, and $dA = \sqrt{\det G_{kk}} \, dk d\varphi$ denotes the area measure.

The direct calculation of the $R_{kk\varphi\varphi}$ and $\sqrt{\det G_{kk}}$ are tedious, however, it can be verified that there exists a general relation in a generalized two-band Hamiltonian on a 2D manifold as $R_{kk\varphi\varphi} = 4 \det G_{kk}$ and $\det G_{kk} = \left(\frac{d \partial_k d \times \partial_\varphi d}{4}\right)^2$. That is to say that the Bloch band manifold is a curved surface with a constant Gauss curvature $\mathcal{K} = 4$. Finally, we can derive the Euler number of the Bloch band as

$$\chi = \frac{1}{2\pi} \int_{1Bz} \mathcal{K} dA$$

$$= \frac{1}{2\pi} \int_{1Bz} \left| \hat{d} \cdot \partial_\varphi \hat{d} \times \partial_\varphi \hat{d} \right| \, dk d\varphi,$$

(16)

FIG. 5: (color online) The Euler number $\chi$ of the Bloch band as a function of the external field $h$ and three-sites spins coupled coefficients $\delta$. The ferromagnetic phase in this model can be marked by a nontrivial Euler number $\chi = 4$, and the Euler number $\chi \to 0$ quickly with the increasing of the external field $h$ in the paramagnetic phase.

FIG. 6: (color online) The Euler number $\chi$ with several groups of the anisotropy parameter $\gamma$ and three-site spins coupled coefficients $\delta$.

where

$$\hat{d} \cdot \partial_\varphi \hat{d} \times \partial_\varphi \hat{d} = \frac{2\gamma^2 \sin k + \gamma^2 (h - 2\delta + \delta \cos 2k) \sin 2k}{[(h + \cos k - \delta \cos 2k)^2 + \gamma^2 \sin^2 k]^{3/2}}.$$

(17)

As shown in Fig. 5, we plot the Euler number $\chi$ of the Bloch band as a function of the external field $h$ and three-site spins coupled coefficients $\delta$. In the ferromagnetic phase, the Bloch band is characterized by a nontrivial Euler number $\chi = 4$, whose topology is equivalent to two unconnected spheres $S^2$; In the paramagnetic phase, the Euler number of the Bloch band $\chi \to 0$ quickly with
the increasing of the external field $h$, whose topology is equivalent to a torus $T^2$. The effects of the anisotropy parameter $\gamma$ on the Euler number are shown in the Fig. 6. It can be seen that the Euler number is independent with $\gamma$ in the region of ferromagnetic phase, but declines to 0 more quickly with the decreasing of $\gamma$ in the region of the paramagnetic phase.

Note that the Berry curvature of the Bloch band can be written as $\mathcal{F}_{k\varphi} = \frac{1}{d} \partial_k \mathbf{d} \times \partial_\varphi \mathbf{d}$, so we can get a first Chern number index for the Bloch band as

$$C_1 = \frac{1}{4\pi} \int_{\text{BZ}} \mathbf{d} \cdot \partial_k \mathbf{d} \times \partial_\varphi \mathbf{d} \, dk \, d\varphi.$$  \hspace{1cm} (18)

However, the Bloch Hamiltonian for this model $\mathcal{H}(k, \varphi) = \sum_{\alpha=1}^{3} d_\alpha (k, \varphi) \sigma^\alpha$ is time reversal invariant, i.e. $\mathcal{H}^*(-k, -\varphi) = \mathcal{H}(k, \varphi)$, so the Berry curvature $\mathcal{F}_{k\varphi}$ is odd with the crystal momentum $k$ (note $\mathcal{F}_{k\varphi}$ is not dependent on $\varphi$), and the first Chern number $C_1 \equiv 0$. In this case, the first Chern number can not serve as a sufficient index for the topology of the Bloch band in the time reversal invariant systems.

V. CONCLUSIONS

In summary, we study the Euler number index of the Bloch band in a transverse field XY spin-1/2 chain with multi-site spin couplings. This approach is based on the topological characterization from the Gauss-Bonnet theorem on a 2D closed Bloch states manifold in the first Brillouin zone, where the Riemannian structure of the Bloch band is established by the geometric tensor in the crystal momentum space. For a local geometric witness to the quantum phase transitions, we introduce the cyclic quantum distance of the Bloch band and show the Riemannian metric on the Bloch states manifold can be relate to a corresponding ground-state quantum distance in the parameter space. Finally, we derive the Euler characteristic number of the Bloch band analytically via the Gauss-Bonnet theorem on the 2D Bloch states manifold in the first Brillouin zone. We show that the ferromagnetic-paramagnetic quantum phase transition in this model is topological different in the Bloch band’s Euler number index. We also give a general formula of the Euler number for the 1D or 2D two-band systems, which reveals its essential relation to the first Chern number of the band insulators.

VI. ACKNOWLEDGMENTS

This work was supported by the special foundation for theoretical physics research Program of China under grants No. 11347131.
[32] S. Yang, S. J. Gu, C. P. Sun and H. Q. Lin, Phys. Rev. A 78, 012304 (2008).
[33] J. H. Zhao and H. Q. Zhou, Phys. Rev. B 80, 014403 (2009).
[34] M. M. Rams and B. Damski, Phys. Rev. A 84, 032324 (2011).
[35] M. Thakurathi, D. Sen, and A. Dutta, Phys. Rev. B 86, 245424 (2012).
[36] B. Damski, Phys. Rev. E 87, 052131 (2013).
[37] M. V. Berry, Proc. R. Soc. London A 392, 45 (1984).
[38] Y. Nishiyama, Phys. Rev. E 88, 012129 (2013).
[39] B. Simon, Phys. Rev. Lett. 51, 2167 (1983).
[40] R. Resta, Rev. Mod. Phys. 66, 899 (1994).
[41] D. Xiao, M. C. Chang, and Q. Niu, Rev. Mod. Phys. 82, 1959 (2010).
[42] A. Hamma, arXiv: quant-ph/0602091.
[43] S. L. Zhu, Phys. Rev. Lett. 96, 077206 (2006).
[44] Y. Q. Ma and S. Chen, Phys. Rev. A 79, 022116 (2009).
[45] T. Hirano, H. Katsura, and Y. Hatsugai, Phys. Rev. B 77, 094431 (2008); Y. Hatsugai, New J. Phys. 12, 065004 (2010).
[46] T. Fukui and T. Fujiwara, J. Phys. Soc. Jpn. 78, 093001 (2009).
[47] Y. Q. Ma, et al., EPL 100, 60001 (2012).
[48] J. P. Provost and G. Vallee, Commun. Math. Phys. 76, 289 (1980).
[49] M. V. Berry, in Geometric Phases in Physics, edited by A. Shapere and F. Wilczek (World Scientific, Singapore, 1989).
[50] R. Resta, Phys. Rev. Lett. 95, 196805 (2005).
[51] L. Campos Venuti and P. Zanardi, Phys. Rev. Lett. 99, 095701 (2007).
[52] Y. Q. Ma, S. Chen, H. Fan, and W. M. Liu, Phys. Rev. B 81, 245103 (2013).
[53] M. Atala, M. Aidelsburger, J. T. Barreiro, D. Abanin, T. Kitagawa, E. Demler, and I. Bloch, Nature Phys. 9, 795 (2013).
[54] Y. Q. Ma, S. J. Gu, S. Chen, H. Fan, and W. M. Liu, arXiv:1202.2397 EPL 103, 10008 (2012).
[55] T. Neupert, C. Chamon, and C. Mudry, Phys. Rev. B 87, 245103 (2013).
[56] J. Zak, Phys. Rev. Lett. 62, 2747 (1989).
[57] M. A. Zvyagin and G. A. Skorobagat’ko, Phys. Rev. B 73, 024427 (2006); A. A. Zvyagin, Phys. Rev. B 80, 014414 (2009).
[58] W. W. Cheng and J. M. Liu, Phys. Rev. A 82, 012308 (2010).
[59] Note the Riemannian metric of the Bloch states $u_\pm(k, \varphi)$ is given by $G^\pm_{k\varphi} = \frac{1}{2} \langle \partial_k u_\pm | \partial_\varphi u_\pm \rangle + \frac{1}{2} \langle \partial_\varphi u_\pm | \partial_k u_\pm \rangle - \langle \partial_k u_\pm | u_\pm \rangle \langle u_\pm | \partial_\varphi u_\pm \rangle$, and it can be verified that here $G_{k\varphi}^+ = G_{k\varphi}^-$, so we can denote the metric as $G_{k\varphi}$ conveniently.