Light-cone Hamiltonian flow for positronium.
The numerical solutions

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15 January, 1999

Abstract

The effective Hamiltonian, as obtained from applying the Hamiltonian flow equations to front form QED, are solved numerically for positronium. Both the exchange and the annihilation channels are included. The impact of different similarity functions is explicitly studied. Perfect numerical agreement with other methods is found.

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1 Introduction

Light-front frame is believed to be a useful tool for solving bound state problems in QCD. Generally, bound state calculations in the field theory have two sources of complexity - they are relativistic and of many-body type. The method of flow equations copes with both of them, at least to the definite order in perturbation theory. One transforms the Hamiltonian to eliminate interactions changing particle number, reducing thus the bound state problem to a few-body problem. Simultaneously ultraviolet divergencies occur, originating from the high-energy region. To complete renormalization one uses either coupling coherence or fixes counterterms to provide finite values for physical observables and to retain symmetries violated by the procedure [5]. Both type of flow equations of renormalization type and for the new (particle number conserving) interactions appear together. This program can be fulfilled in perturbation theory expansion. As a result, the bound state problem is approximated by a set of renormalized, effective interactions that do not change particle number.

The sensitive tool in the light-front frame to check how accurately one describes bound states by these effective interactions is to measure the violation of rotational symmetry. This symmetry is linked to a dynamical operator on the light-front, since rotations are dynamical, i.e. depend on the interaction. The symmetry may be spoiled in two steps: first, regularization and renormalization; and second, reduction to the effective few-body interactions with particle number conservation. The nonperturbative renormalization flow is of crucial importance for QCD [2, 4, 8], but one can disregard this point in QED bound state calculations, that disentangles the two problems mentioned above. To the leading order the results for QED are obtained in [4], where positronium system is described approximately by the effective electron-positron interaction. In the nonrelativistic limit the results for positronium spectrum agree with the results of covariant calculations.

There are at least two other alternative approaches to solve for bound states in the light-front dynamics, the scheme of similarity renormalization of Glazek and Wilson [6] and the method of iterated resolvents of Pauli [8]. In both schemes calculations of the effective electron-positron interaction are performed and the question of rotational invariance for positronium spectrum is investigated.

Calculations done so far in the similarity renormalization scheme use the nonrelativistic limit to find corresponding eigenvalues in the bound state perturbation theory [4]. Analytical calculations are performed there for ground state: ground triplet levels are degenerate, indicating that rotational symmetry is restored [4]. Performing similarity renormalization one eliminates high-energy modes and absorbs relativistic effects into an effective band-diagonal Hamiltonian, which describes bound state creation at nonrelativistic energy scales. It is a well working scheme for such systems as positronium [4]; therefore nonrelativistic approximations done in this approach to extract eigenvalues from effective Hamiltonian are quite natural there. In general, it is not always the case. In fact rotational symmetry becomes kinematic one, like light-front boost in the nonrelativistic limit, i.e. total momentum and its projection can be considered approximately as quantum numbers, that makes simpler to trace rotational invariance in these calculations.

In the method of iterated resolvents an effective electron-positron potential is obtained and exact numerical solution of positronium bound state equation with the given poten-
tial is done \[10, 11\]. Degenerate multiplets for ground as well as for exited states are obtained \[10\]. It is convenient to perform relativistic calculations in the light-front frame, which effectively has nonrelativistic kinematics.

In the present work we perform relativistic few-body calculations for positronium spectrum numerically in the spirit of the work Trittmann et.al. \[10\] (and using numerical code \[11\]), based on the effective electron-positron Hamiltonian obtained by the flow equations \[15\]. Effective interaction was derived there for different cutoff functions. The requirement of block-diagonalization of the Hamiltonian determines the generator only up to a unitary transformation of the blocks; this explains why the effective interaction may depend on the cutoff function. The question we investigate is to what extent rotational invariance is violated on the level of positronium spectrum and how does it depend on the choice of the cutoff function. We are not able to trace rotational invariance during the calculations, since it is dynamical operator. This is an excellent test for the method of flow equations itself and the control of approximations done during the calculations.

2 Formulation of the problem

We address to solve a light-front Hamiltonian bound state equation

\[ H|\psi\rangle = E|\psi\rangle \]  

(1)

for positronium. Using flow equations we transform the QED Hamiltonian \( H \) to a block-diagonal effective Hamiltonian, which reduces positronium problem to a bound state problem in the electron-positron sector. The effective Hamiltonian for an electron and a positron is

\[ H_{\text{eff}} = H_0 + U_{\text{eff}} \]  

(2)

where \( H_0 \) is the kinetic energy, and \( U_{\text{eff}} \) includes effective interactions generated by the flow equations in the second order in coupling constant. The integral bound state equation is written

\[ E\langle p_1, p_2; \lambda_1, \lambda_2 | \psi \rangle = (E_{p_1} + E_{p_2})\langle p_1, p_2; \lambda_1, \lambda_2 | \psi \rangle + \sum_{\lambda'_1, \lambda'_2} \int d^3p'_1 d^3p'_2 \langle p_1, p_2; \lambda_1, \lambda_2 | U_{\text{eff}} | p'_1, p'_2; \lambda'_1, \lambda'_2 \rangle \langle p'_1, p'_2; \lambda'_1, \lambda'_2 | \psi \rangle \]  

(3)

where the effective Hamiltonian picks out from the positronium wave function \( |\psi\rangle \) the lowest \( e\bar{e} \)-component \( \langle p_1, p_2; \lambda_1, \lambda_2 | \psi \rangle \) with \( p_i, \lambda_i \) being the light-front three-momenta and helicities, respectively, carried by an electron \((i = 1)\), and a positron \((i = 2)\). The primed quantities refer to the initial state, the unprimed ones to the final state. The effective interaction \( \langle p_1, p_2; \lambda_1, \lambda_2 | U_{\text{eff}} | p'_1, p'_2; \lambda'_1, \lambda'_2 \rangle = U_{\text{eff}} \delta(p_1 + p_2 - p'_1 - p'_2) \) will be specified below. In order to deduce a Lorentz invariant energy we consider the bound state equation written for operator \( P^- P^+ \), corresponding to the invariant mass-squared \( M^2 \) on the light-front, rather than for the light-front Hamiltonian operator \( H = P^- \). The light-front

\[ \text{Numerical solution of positronium bound state problem in the light-front frame can be found also in \[13, 14\].} \]
integral equation,
\[
M^2 \langle x, \vec{k}_\perp; \lambda_1, \lambda_2 | \psi \rangle = \frac{m^2 + \vec{k}_\perp^2}{x(1-x)} \langle x, \vec{k}_\perp; \lambda_1, \lambda_2 | \psi \rangle + \sum_{x', \lambda'_1, \lambda'_2} \int dx' d^2 \vec{k}_\perp' \langle x, \vec{k}_\perp; \lambda_1, \lambda_2 | V_{\text{eff}} | x', \vec{k}_\perp'; \lambda'_1, \lambda'_2 \rangle \langle x', \vec{k}_\perp'; \lambda'_1, \lambda'_2 | \psi \rangle
\]
(4)

is independent of the total momentum \(P^+\) and \(\vec{P}_\perp\). We introduced \(V_{\text{eff}} = P^{\prime+} U_{\text{eff}}\). In that equation only intrinsic transversal momenta \(\vec{k}_\perp\) and longitudinal momentum fractions \(x = p^+_1/P^+\) appear (\(p^\prime_1 = (xP^+, x\vec{P}_\perp + \vec{k}_\perp, p^-_1)\)). Its spectrum is thus manifestly independent of the kinematical state of the bound system, particularly of \(P^+\) and \(\vec{P}_\perp\), which reflects on the boost invariance peculiar to the light-front form [1]. The integration domain \(D\) is restricted by the covariant cutoff condition of Brodsky and Lepage [3],
\[
\frac{m^2 + \vec{k}_\perp^2}{x(1-x)} \leq \Lambda^2 + 4m^2,
\]
which allows for states having a kinetic energy below the bare cutoff \(\Lambda\). The effective interaction between electron and positron, being a kernel in the integral equation (4), is generated by the flow equations [15]
\[
V_{\text{eff}} = -\frac{\alpha}{4\pi^2} \langle \gamma^\mu \gamma^\nu \rangle_{\text{ex}} g_{\mu\nu} \left[ \frac{\Theta_{ee}^2}{Q_e^2} + \frac{\Theta_{\bar{e}e}^2}{Q_{\bar{e}}^2} \right] + \eta_{\mu\nu} \frac{\delta Q^2}{q^2 + Q_e^2} \left[ -\frac{\Theta_{ee}}{Q_e^2} - \frac{\Theta_{\bar{e}e}}{Q_{\bar{e}}^2} \right]
-\frac{\alpha}{4\pi^2} \langle \gamma^\mu \gamma^\nu \rangle_{\text{an}} g_{\mu\nu} \left[ \frac{\Theta_{ab}}{M_a^2} + \frac{\Theta_{ba}}{M_b^2} \right] - \eta_{\mu\nu} \frac{\delta M^2}{p^+ + M_a^2 - M_b^2}
\]
(6)

with the generator of unitary transformation
\[
\eta (l) = -\frac{1}{D} \left( \frac{d \ln f (D; l)}{dl} \right) g(l).
\]
(7)

where \(g(l)\) is the coupling constant as a function of flow parameter \(l\), and \(f (D; l)\) is the cutoff function specified below. In Eq. (6) subscript \(\text{ex}\) refers to the exchange part, and \(\text{an}\) to the annihilation part. The null vector \(\eta^\mu\) has components \((\eta^+, \eta_\perp, \eta^-) = (0, \vec{0}, 2)\) and is specific to the light-front calculations. The light-front metric tensor is denoted by \(g_{\mu\nu}\). The current-current tensors in the two channels are
\[
\langle \gamma^\mu \gamma^\nu \rangle_{\text{ex}} = \frac{(\bar{u}(p_1, \lambda_1) \gamma^\mu u(p'_1, \lambda'_1)) (\bar{u}(p'_2, \lambda'_2) \gamma^\nu v(p_2, \lambda_2))}{\sqrt{xx'(1-x)(1-x')}}
\]
\[
\langle \gamma^\mu \gamma^\nu \rangle_{\text{an}} = \frac{(\bar{u}(p_1, \lambda_1) \gamma^\mu v(p_2, \lambda_2)) (\bar{u}(p'_2, \lambda'_2) \gamma^\nu u(p'_1, \lambda'_1))}{\sqrt{xx'(1-x)(1-x')}}
\]
(8)

where the fermion momenta were defined after Eq. (3). The remaining definitions are as follows. The energy differences along the electron and the positron line,
\[
D_e = p'_1^- - p_1^- - (p'_1 - p_1)^-,
D_{\bar{e}} = p_2^- - p'_2^- - (p_2 - p'_2)^-.
\]
(9)
respectively, have a simple relation to the (Feynman-) 4-momentum transfers along the two lines

\[
Q_e^2 = -(p_1' - p_1)^2 = -q^+ D_e,
Q_e^2 = -(p_2' - p_2)^2 = -q^+ D_e. \tag{10}
\]

Since the Feynman-momentum transfer \(Q\) is more physical quantity than the energy difference, we will use the former as far as possible. In fact, in our formulae we make use of the mean-square momentum transfer and the mean-square difference,

\[
Q^2 = \frac{1}{2}(Q_e^2 + Q_e^2) = -\frac{q^+}{2}(D_e + D_e),
\]

\[
\delta Q^2 = \frac{1}{2}(Q_e^2 - Q_e^2) = -\frac{q^+}{2}(D_e - D_e), \tag{11}
\]

respectively. The dependence of the effective interaction Eq. (6) on the cutoff function \(f(D; l)\) is carried by the factor

\[
\Theta(D_e, D_e) = -\int_0^\infty dl' \frac{df(D_e; l')}{dl'} f(D_e; l') \equiv \Theta_{e\bar{e}}, \tag{12}
\]

which is asymmetric in the arguments but which satisfies

\[
\Theta(D_e, D_e) + \Theta(D_e, D_e) = \Theta_{e\bar{e}} + \Theta_{\bar{e}e} = 1. \tag{13}
\]

The latter combination obeys

\[
\frac{\Theta_{e\bar{e}}}{Q_e^2} + \frac{\Theta_{\bar{e}e}}{Q_e^2} = \frac{Q^2}{Q_e^2 Q_e^2} \left( 1 - \frac{\delta Q^2}{Q^2} (\Theta_{e\bar{e}} - \Theta_{\bar{e}e}) \right),
\]

\[
\frac{\Theta_{e\bar{e}}}{Q_e^2} - \frac{\Theta_{\bar{e}e}}{Q_e^2} = -\frac{\delta Q^2}{Q_e^2 Q_e^2} \left( 1 - \frac{Q^2}{\delta Q^2} (\Theta_{e\bar{e}} - \Theta_{\bar{e}e}) \right). \tag{14}
\]

that we use further.

For the annihilation term we define the energy differences as

\[
D_a = p_1'^- + p_2'^- - (p_1' + p_2')^-,
D_b = p_1^- + p_2^- - (p_1 + p_2)^-. \tag{15}
\]

They are related to the 4-momentum \(p^\mu\) of the photon and to the free invariant mass-squares of the initial and final states

\[
M_a^2 = (p_1' + p_2')^2 = p^+ D_a,
M_b^2 = (p_1 + p_2)^2 = p^+ D_b, \tag{16}
\]

as well as to their mean and difference

\[
M^2 = \frac{1}{2}(M_a^2 + M_b^2) = \frac{p^+}{2}(D_a + D_b),
\]

\[
\delta M^2 = \frac{1}{2}(M_a^2 - M_b^2) = \frac{p^+}{2}(D_a - D_b), \tag{17}
\]
Effective interaction \( g_{\mu\nu} \) part insures the Bohr spectrum and is responsible for the spin splittings; \( \eta_{\mu}\eta_{\nu} \) term is diagonal in spin space and vanishes for real processes, i.e. on mass shell with \( \delta Q^2 = 0 \), making the effective interaction to coincide with the Tamm-Dancoff approximation [15]. The explicit \( x \)-dependence in the denominator of Eq. (8) looks like the only remnant of the light-front formulation; all other quantities are Lorentz scalars. One can absorb this dependence by redefining the wave function in the integral equation Eq. (4). We introduce instead of Jacobi momentum \( (x, \vec{\kappa}_\perp) \) the three momentum in the center of mass frame \( \vec{p} = (p_z, \vec{\kappa}_\perp) \) as follows

\[
x = \frac{1}{2} \left( 1 + \frac{p_z}{\sqrt{\vec{p}^2 + m^2}} \right),
\]

where the Jacobian of this transformation \( dx/dp_z \) is

\[
J = \frac{1}{2} \frac{\vec{\kappa}_\perp^2 + m^2}{(\vec{p}^2 + m^2)^{3/2}} = \frac{x(1-x)}{E},
\]

and it holds in this frame

\[
x(1-x) = \frac{1}{4} \frac{\vec{\kappa}_\perp^2 + m^2}{\vec{p}^2 + m^2},
E = \sqrt{\vec{p}^2 + m^2}.
\]

The connection between ‘old’ and ‘new’ wave functions and the interaction matrix elements are

\[
\langle x, \vec{\kappa}_\perp | \psi \rangle = \frac{\langle \vec{p} | \psi' \rangle}{\sqrt{x(1-x)}},
\]

\[
\langle x, \vec{\kappa}_\perp | V_{\text{eff}} | x', \vec{\kappa}_\perp' \rangle = \frac{\langle \vec{p} | V_{\text{eff}}' | \vec{p}' \rangle}{\sqrt{x(1-x)x'(1-x')}}.
\]

Integral equation (14),

\[
M^2 \langle \vec{p}; \lambda_1, \lambda_2 | \psi' \rangle = 4(\vec{p}^2 + m^2) \langle \vec{p}; \lambda_1, \lambda_2 | \psi' \rangle + \sum_{\lambda'_1, \lambda'_2} \int \frac{d^3 \vec{p}'}{2E} \langle \vec{p}; \lambda_1, \lambda_2 | V_{\text{eff}}' | \vec{p}'; \lambda'_1, \lambda'_2 \rangle \langle \vec{p}'; \lambda'_1, \lambda'_2 | \psi' \rangle
\]

is written in a rotationally covariant form.

### 3 Rotational invariance

Integral equation (14) has rotationally covariant but still not rotationally invariant form because of the interaction kernel \( \hat{V}_{\text{eff}} \), written in the light-front frame. Let us extract the part of interaction, which has manifestly rotational symmetry.
Quite generally Θ factor Eq. (12) is a function of the ratio of its two arguments. Therefore, making use of Eq. (14), the effective interaction Eq. (6) is given as

\[ V_{\text{eff}} = -\frac{\alpha}{4\pi^2} \langle \gamma^\mu \gamma^\nu \rangle_{\text{ex}} B_{\mu\nu} - \frac{\alpha}{4\pi^2} \langle \gamma^\mu \gamma^\nu \rangle_{\text{an}} C_{\mu\nu}, \]  

(23)

where the exchange part is defined

\[ B_{\mu\nu} = \frac{g_{\mu\nu}}{Q^2} + \left( \frac{g_{\mu\nu} - \eta_\mu\eta_\nu}{p^{+2}} \right) \frac{\xi^2 - \xi \vartheta(\xi)}{1 - \xi^2} \]

\[ \vartheta(\xi) = \Theta_{ee} - \Theta_{\bar{e}e}, \quad \xi = \frac{\delta Q^2}{Q^2}, \]  

(24)

and the annihilation part

\[ C_{\mu\nu} = \frac{g_{\mu\nu}}{M^2} + \left( \frac{g_{\mu\nu} - \eta_\mu\eta_\nu}{p^{+2}} \right) \frac{\beta^2 - \beta \chi(\beta)}{1 - \beta^2} \]

\[ \chi(\beta) = \Theta_{ab} - \Theta_{ba}, \quad \beta = \frac{\delta M^2}{M^2}. \]  

(25)

The terms in the effective interaction proportional to \( \vartheta(\xi), \chi(\beta) \) depend explicitly on the choice of cut-off function and arise from \( l \)-ordering of the generator in the operator of unitary transform [15]. Explicit form of the effective interaction with different cut-off functions is given in Appendix A.

Define energy denominators in equation (25). Due to the three-momentum conservation on the light-front, \( p_1 + p_2 = p_1' + p_2' \) for longitudinal and transversal components, one has \( D_e - D_{\bar{e}} = D_a - D_b \), where the energy denominators \( D_k \) in both channels are given in Eq. (9) and Eq. (15). Therefore

\[ \delta Q^2 = \left( -\frac{q^+}{p^+} \right) \delta M^2. \]  

(26)

where \( \delta M^2 \) – the total energy difference between initial and final states shows the 'off-shellness' of process.

Using the parametrization Eq. (18) one has for the energy denominators

\[ Q^2 = \tilde{q}^2 - p_z p_z' \frac{(M_a - M_b)^2}{M_a M_b} \]

\[ \delta Q^2 = -\left( \frac{p_z'}{M_a} - \frac{p_z}{M_b} \right) \delta M^2 \]

\[ M_a^2 = 4(\vec{p}^2 + m^2) \]

\[ M_b^2 = 4(\vec{p}^2 + m^2), \]  

(27)

where \( q = p' - p = (q_z, q_\perp) \) is the three-momentum transfer of the photon, and the relations between mean-squared and difference momenta and corresponding energy differences are given in Eq. (11) and Eq. (12) for exchange and annihilation channels, respectively.

The second term in Eq. (23) is obviously not rotational invariant. For the real processes, \( \delta M^2 = 0 \), the second term vanishes for both channels, and the effective interaction
is independent on the cutoff function and coincide with the result of Tamm-Dancoff approximation

\[ V_{\text{eff}} = -\frac{\alpha}{4\pi^2} \frac{\langle \gamma^\mu \gamma^\mu \rangle}{q^2}, \] (28)

and similarly in annihilation channel. The same holds to the leading order of nonrelativistic expansion \( \bar{p}^2/m^2 \ll 1 \) [15].

Estimate current-current term in exchange channel, Eq. (8), which defines nominator of the interaction Eq. (28). We work in the Lepage-Brodsky convention for the spinors [9]

\[ u(p, \lambda) = \frac{2}{p^+} (p^+ + \beta m + \vec{\alpha} \cdot \vec{p}) \Lambda_+ \chi_\lambda \]

\[ = \frac{2}{\sqrt{N}} (E + \beta m + \vec{\alpha} \cdot \vec{p}) \Lambda_+ \chi_\lambda, \] (29)

and similarly for \( v(p, \lambda) \) with the change \( m \rightarrow -m \) and \( \chi_\lambda \rightarrow \chi_\lambda \) in the above formula. The second expression for spinor holds quite in general for the solution of Dirac equation, where \( \beta = \gamma^0 \), \( \vec{\alpha} = \gamma^0 \vec{\gamma}, \Lambda_+ = 1/2(1 + \alpha^3) \) is the projection operator [1], and the spinor

\[ \chi_\lambda = \left( \begin{array}{c} \xi_\lambda \\ 0 \end{array} \right), \]

is defined through the usual two-component spinors

\[ \xi_1 = \left( \begin{array}{c} 1 \\ 0 \end{array} \right); \quad \xi_2 = \left( \begin{array}{c} 0 \\ 1 \end{array} \right), \]

Using the explicit representation for \( \gamma \) matrices and projection operators \( \Lambda_+, \Lambda_- \) and relations between them [1], one has

\[ \bar{u}(p, \lambda) \gamma^0 u(p', \lambda') = \frac{1}{\sqrt{N N'}} \xi^+_{\lambda'} \left( (E + p_z)(E' + p_z') + (\vec{p} \cdot \vec{p}') + i[\vec{p} \times \vec{p}'] \right) \sigma 
+ i[\vec{p} \times \vec{\sigma}]_z (p_z + m) - i[\vec{p} \times \vec{\sigma}]_z (p_z + m) + m^2 - p_z p'_z) \xi_{\lambda'} 
\]

\[ \bar{u}(p, \lambda) \gamma^i u(p', \lambda') = \frac{1}{\sqrt{N N'}} \xi^+_{\lambda'} \left( (E + p_z)(p^i + i[\vec{p} \times \vec{\sigma}]^i) + (E' + p_z)(p^i - i[\vec{p} \times \vec{\sigma}]^i) 
+ \delta^{ij} (E E' - m^2 - (\vec{p} \cdot \vec{p}')) - i[\vec{p} \times \vec{p}'] \sigma 
+ i[\vec{p} \times \vec{\sigma}]^i (E' - m) - i[\vec{p} \times \vec{\sigma}]^i (E - m) 
+ i\varepsilon^{ij} \sigma^j ((E + p_z)(m + p'_z) - (E' + p'_z)(m + p_z)) \xi_{\lambda'}, \right) \] (30)

where \( i = 1, 2, 3 \); \( p = (p_z, \vec{k}) \) is the three momentum and \( \varepsilon^{ij} = \delta^{ij}, \varepsilon^{12} = -\varepsilon^{21} = 1 \). Introducing three-momentum transfer and its mean

\[ \vec{q} = \vec{p}' - \vec{p} \]
\[ \vec{k} = \frac{1}{2}(\vec{p}' + \vec{p}) \] (31)
where \(4(\tilde{q} \tilde{k}) = \delta M^2 = 2(E^2 - E'^2)\) with \(\delta M^2\) defined above, Eq. (30) is written

\[
\bar{u}(p, \lambda) \gamma^0 u(p', \lambda') &= \frac{1}{\sqrt{NN'}} \xi_\lambda^+ \left( (E + k_z)(E' + k_z) + \tilde{k}^2 - \tilde{q}^2/4 - i[\tilde{q} \times \tilde{k}]\vec{\sigma} \right) \\
&- i[\tilde{q} \times \vec{\sigma}]_z(k_z + m) + q_z((E - E')/2 + i[\tilde{k} \times \vec{\sigma}]_z) + m^2 - k_z^2)\xi_{\lambda'}
\]

\[
\bar{u}(p, \lambda) \gamma^i u(p', \lambda') &= \frac{1}{\sqrt{NN'}} \xi_\lambda^+ \left( ((E + E')/2 + k_z)(2k^i + i[\tilde{q} \times \vec{\sigma}]^i) \right) \\
&+ \left( (E - E') - q_z)(q^i/2 + i[\tilde{k} \times \vec{\sigma}]^i) \right) \\
&+ \delta^{ij}(EE' - m^2 - (\tilde{k}^2 - \tilde{q}^2/4) + i[\tilde{q} \times \tilde{k}]\vec{\sigma} \\
&- i[\tilde{q} \times \vec{\sigma}]^i((E + E')/2 - m) - i[\tilde{k} \times \vec{\sigma}]^i(E - E')) \\
&+ i\varepsilon^{ij}\sigma^i(\tilde{k}_z + m(E - E') + q_z((E + E')/2 - m))\xi_{\lambda'}, \tag{32}
\]

where no approximations are done so far. Excluding the overall normalization factor the first lines in Eq. (32) for scalar and vector current terms contain rotationally invariant parts (except terms proportional to \(k_z\)), which coincide with the corresponding expressions when making use of Bjorken-Drell convention for spinors [12].

Merkel et.al. [12] showed, that as far as the energy is conserved, this part gives rise to familiar spin dependent forces. The rest terms in Eq. (32) are obviously not rotationally invariant, particularly when the spacial rotations are performed perpendicular to the \(z\)-axis. Expanding expression Eq. (32) to the second order in \(|\vec{p}|/m \ll 1\) and performing the unitary transformation in spin space, Brisudova et.al. [3] obtained Breit-Fermi spin-spin and tensor interactions. It seems to be impossible to reproduce full set of Breit-Fermi terms from the second order effective interaction in the light-front gauge. Also it is complicated to cover rotational symmetry on the level of light-front effective Hamiltonians without additional approximations are done. We use directly the effective electron-positron interaction Eq. (23) for numerical calculations of positronium spectrum. We aim to get fine structure and to investigate rotational symmetry on the level of spectrum. The impact of different cutoff functions is also considered. The results of these calculations are presented in the next section.

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2 Spinors as used by Bjorken-Drell are

\[
u(p, \lambda) = \frac{1}{\sqrt{N}}(E + \beta m + \alpha \vec{p})\chi_\lambda,
\]

where \(N = E + m\) and \(\chi_\lambda\) is defined in the main text. The corresponding expressions for current terms are

\[
\bar{u}(p, \lambda) \gamma^0 u(p', \lambda') &= \frac{1}{\sqrt{NN'}} \xi_\lambda^+ (\delta M^2/2)(E^2 - E'^2) + k_z^2 - q_z^2/4 - i[\tilde{q} \times \tilde{k}]\vec{\sigma} \xi_{\lambda'} \\
&+ (E - E')(q^i/2 + i[\tilde{k} \times \vec{\sigma}]^i)\xi_{\lambda'}, \tag{33}
\]

For the energy conserving process this expression was obtained in [12].
### 3 ROTATIONAL INVARIANCE

#### Table 1: Binding coefficients, $B_n = 4(2 - M_n)/\alpha^2$ ($\alpha = 0.3$), for the lowest modes of the positronium spectrum at $J_z = 0$ for the equal time perturbation theory up to order $\alpha^4$ ($B_{ETPT}$) compared to our calculations with exponential ($B_E$), Gaussian ($B_G$) and sharp ($B_S$) cutoffs. $B_G$ is obtained using only $g_{\mu\nu}$ part of interaction; for $B_{G}^{\prime \eta\mu\eta\nu}$ term is included. Exchange channel is considered.

| $n$ | Term   | $B_{ETPT}$ | $B_E$     | $B_G$     | $B_S$     |
|-----|--------|------------|-----------|-----------|-----------|
| 1   | $1^1S_0$ | 1.118125   | 1.049550  | 1.101027  | 1.026170  | 0.920921  |
| 2   | $1^3S_1$ | 0.998125   | 1.001010  | 1.049700  | 0.981969  | 0.885347  |
| 3   | $2^1S_0$ | 0.268633   | 0.260237  | 0.266490  | 0.260642  | 0.242607  |
| 4   | $2^3S_1$ | 0.253633   | 0.253804  | 0.259506  | 0.254765  | 0.234312  |
| 5   | $2^1P_1$ | 0.253633   | 0.257969  | 0.263056  | 0.257664  | 0.237611  |
| 6   | $2^3P_0$ | 0.261133   | 0.260237  | 0.266490  | 0.260642  | 0.242607  |
| 7   | $2^3P_1$ | 0.255508   | 0.259667  | 0.265412  | 0.260127  | 0.238135  |
| 8   | $2^3P_2$ | 0.251008   | 0.255258  | 0.260642  | 0.254765  | 0.234312  |

#### Table 2: Binding coefficients, $B_n = 4(2 - M_n)/\alpha^2$ ($\alpha = 0.3$), for the lowest modes of the positronium spectrum at $J_z = 0$ for our calculations with exponential ($B_E$), Gaussian ($B_G$) and sharp ($B_S$) cutoffs. $B_G$ includes $\eta\mu\eta\nu$ term in exchange channel; $B_G$ does not. Exchange and annihilation channels are considered.

| $n$ | Term   | $B_E$     | $B_G$     | $B_S$     |
|-----|--------|-----------|-----------|-----------|
| 1   | $1^1S_0$ | 1.049550  | 1.026170  | 0.920921  |
| 2   | $1^3S_1$ | 1.001010  | 0.981969  | 0.885347  |
| 3   | $2^1S_0$ | 0.260237  | 0.266490  | 0.260642  |
| 4   | $2^3S_1$ | 0.253804  | 0.259506  | 0.254765  |
| 5   | $2^1P_1$ | 0.257969  | 0.263056  | 0.257664  |
| 6   | $2^3P_0$ | 0.260237  | 0.266490  | 0.260642  |
| 7   | $2^3P_1$ | 0.259667  | 0.265412  | 0.260127  |
| 8   | $2^3P_2$ | 0.255258  | 0.260642  | 0.254765  |

#### Table 3: Difference in the corresponding energy levels between $J_z = 0$ and $J_z = 1$ states for exponential ($\delta B_E$), Gaussian ($\delta B_G$) and sharp ($\delta B_S$) cutoffs. Exchange channel is considered.

| $n$ | Term   | $\delta B_E$   | $\delta B_G$   | $\delta B_S$   |
|-----|--------|---------------|---------------|---------------|
| 2   | $1^3S_1$ | $6.30 \times 10^{-4}$ | $1.76 \times 10^{-3}$ | $1.18 \times 10^{-3}$ |
| 4   | $2^3S_1$ | $8.40 \times 10^{-5}$ | $1.77 \times 10^{-4}$ | $9.0 \times 10^{-5}$ |
| 5   | $2^1P_1$ | $-1.30 \times 10^{-5}$ | $-7.47 \times 10^{-4}$ | $-9.1 \times 10^{-5}$ |
| 7   | $2^3P_1$ | $-4.08 \times 10^{-4}$ | $-4.08 \times 10^{-4}$ | $1.4 \times 10^{-4}$ |
| 8   | $2^3P_2$ | $5 \times 10^{-6}$ | $7.7 \times 10^{-5}$ | $4.15 \times 10^{-4}$ |
| $n$ | Term  | $\delta B_E$   | $\delta B_G$   | $\delta B_S$   |
|-----|-------|----------------|----------------|----------------|
| 2   | $1^3S_1$ | $-1.411 \times 10^{-3}$ | $-7.86 \times 10^{-4}$ | $-1.65 \times 10^{-3}$ |
| 4   | $2^3S_1$ | $-4.1 \times 10^{-5}$  | $-4.0 \times 10^{-5}$  | $-1.15 \times 10^{-4}$ |
| 5   | $2^1P_1$ | $-6.4 \times 10^{-5}$  | $-6.52 \times 10^{-4}$ | $-4.60 \times 10^{-4}$ |
| 7   | $2^3P_3$ | $-4.69 \times 10^{-4}$ | $-4.74 \times 10^{-4}$ | $-1.40 \times 10^{-4}$ |
| 8   | $2^3P_2$ | $-1.96 \times 10^{-4}$ | $-1.36 \times 10^{-4}$ | $-2.44 \times 10^{-4}$ |

Table 4: Difference in the corresponding energy levels between $J_z=0$ and $J_z=1$ states for exponential ($\delta B_E$), Gaussian ($\delta B_G$) and sharp ($\delta B_S$) cutoffs. Exchange and annihilation channels are considered.

## 4 Mass spectrum of positronium

We solve the integral equation (5), with interaction kernel given in Eq. (4), for positronium spectrum numerically. Effective interaction with different choice of cutoffs is summarized in Appendix A.

In polar coordinates the light-front variables are ($\vec{\kappa}_\perp; x$) = ($\kappa_\perp, \varphi; x$); therefore the matrix elements of the effective interaction Eq. (12) depend on the angles $\varphi$ and $\varphi'$, i.e. $\langle x, \kappa_\perp, \varphi; \lambda_1, \lambda_2 | V_{\text{eff}} | x', \kappa'_\perp, \varphi'; \lambda'_1, \lambda'_2 \rangle$. In order to introduce the spectroscopic notation for positronium mass spectrum we integrate out the angular degree of freedom, $\varphi$, introducing a discrete quantum number $J_z = n$, $n \in \mathbb{Z}$ (actually for the annihilation channel only $|J_z| \leq 1$ is possible),

$$
\langle x, \kappa_\perp; J_z, \lambda_1, \lambda_2 | V_{\text{eff}} | x', \kappa'_\perp, J'_z, \lambda'_1, \lambda'_2 \rangle
= \frac{1}{2\pi} \int_{0}^{2\pi} d\varphi e^{-iL_ze\varphi} \int_{0}^{2\pi} d\varphi' e^{iL'_ze\varphi'} \langle x, \kappa_\perp, \varphi; \lambda_1, \lambda_2 | V_{\text{eff}} | x', \kappa'_\perp, \varphi'; \lambda'_1, \lambda'_2 \rangle
$$

(35)

where $L_z = J_z - S_z$; $S_z = \frac{1}{2} + \frac{1}{2}$ and the states can be classified (strictly speaking only for rotationally invariant systems) according to their quantum numbers of total angular momentum $J$, orbit angular momentum $L$, and total spin $S$. Definition of angular momentum operators in light-front dynamics is problematic because they include interactions.

The matrix elements of the effective interaction before integrating over the angles, $\langle x, \kappa_\perp, \varphi; \lambda_1, \lambda_2 | V_{\text{eff}} | x', \kappa'_\perp, \varphi'; \lambda'_1, \lambda'_2 \rangle$, and after the integration introducing the total momentum, $J_z$, $\langle x, \kappa_\perp; J_z, \lambda_1, \lambda_2 | V_{\text{eff}} | x', \kappa'_\perp; J'_z, \lambda'_1, \lambda'_2 \rangle$ for different cutoff functions are given in the exchange and annihilation channels in Appendices B and C, respectively.

Now we proceed to solve for the positronium spectrum in all sectors of $J_z$. For this purpose we formulate the light-front integral equation Eq. (4) in the form where the integral kernel is given by the effective interaction for the total momentum $J_z$, Eq. (5). After the change of variables Eq. (14) we parametrize $\vec{p} = (\vec{k}_\perp, p_z) = (\mu \sin \theta \cos \varphi, \mu \sin \theta \sin \varphi, \mu \cos \theta)$. The Jacobian of this transformation Eq. (15) is given

$$
J = \frac{1}{2} \frac{m^2 + \mu^2 \sin^2 \theta}{(m^2 + \mu^2)^{3/2}}.
$$

(36)
One obtains then the integral equation

\[
(M_n^2 - 4(m^2 + \mu^2))\tilde{\psi}_n(\mu, \cos \theta; J_z, \lambda_1, \lambda_2) + \sum_{J'_z, \lambda'_1, \lambda'_2} \int_D d\mu' \int_{-1}^{+1} d\cos \theta' \frac{m^2 + \mu'^2(1 - \cos^2 \theta')}{(m^2 + \mu'^2)^{3/2}} \\
\times \langle \mu, \cos \theta; J_z, \lambda_1, \lambda_2 | \tilde{V}_{\text{eff}} | \mu', \cos \theta'; J'_z, \lambda'_1, \lambda'_2 \rangle \tilde{\psi}_n(\mu', \cos \theta'; J'_z, \lambda'_1, \lambda'_2) = 0. \tag{37}
\]

The integration domain \( D \), defined in Eq. (5), is given now by \( \mu \in [0; \Lambda^2] \). Neither \( L_z \) nor \( S_z \) are good quantum numbers; therefore we set \( L_z = J_z - S_z \).

The integral equation Eq. (37) is used to calculate positronium mass spectrum numerically. Note, that if one succeeds to integrate out the angular degrees of freedom for the effective interaction Eq. (35) analytically, one has 2-dimensional integration in Eq. (37) instead of 3-dimensional one in the original integral equation (4) to perform numerically.

We use the numerical code [11], worked out by Uwe Trittmann for the similar problem [10]. This code includes for the numerical integration the Gauss-Legendre algorithm (Gaussian quadratures). To improve the numerical convergence the technique of Coulomb counterterms is included. The problem has been solved for all components of the total angular momentum, \( J_z \).

Positronium spectrum is mainly defined by the Coulomb singularity

\[
\tilde{q} \rightarrow 0, \tag{38}
\]

which is an integrable one analytically and also, by use of technique of Coulomb counterterms, numerically. In this region \( \delta Q^2 \rightarrow 0 \) and the energy denominator \( Q^2 \rightarrow \tilde{q}^2 \) Eq. (27), giving rise to the leading order Coulomb behavior for the effective interaction Eq. (28), independent on the cutoff function. We use therefore standard Coulomb counterterms, introduced for the Coulomb problem Eq. (28) [10, 11], in the case of all cutoffs. Basing also on the argument Eq. (38), we expect the same pattern of levels for different cutoffs, that we prove numerically to be true.

Another important limiting case to study effective interaction Eq. (23), namely its exchange part, is the collinear limit

\[
q^+ \rightarrow 0, \tag{39}
\]

that is special for light-front calculations. Because of Eq. (23) the variable \( \xi^2 \sim q^+ q^- \), resulting for the \( '\eta_{\mu} \eta_{\nu}' \) part of effective interaction to be

\[
\frac{\eta_{\mu} \eta_{\nu}}{q^+ q^-} \xi^2 (1 - \vartheta'(0)), \tag{40}
\]

which is finite in this limit. This is true for the regular cutoff functions, as in the case of exponential and gaussian cutoffs, where the derivative \( d\vartheta(0)/d\xi \) is well defined. For sharp cutoff this condition is not fulfilled, and the effective interaction contains the \( 1/q^+ \) type of singularity in this case (see Appendix A). We do not associate any physics with this singularity, considering it as a consequence of artificial choice of cutoff, which corresponds to singular generator of unitary transformation Eq. (7). We omit the \( '\eta_{\mu} \eta_{\nu}' \) term in exchange channel for sharp cutoff in numerical calculations.
We argued that the region of Coulomb singularity, and hence $g'_{\mu\nu}$ part of effective interaction, determines mainly the positronium spectrum. However, including $\eta_{\mu}\eta'_{\nu}$ part for gaussian cutoff shifts all levels as a whole of about $5-7\%$, since this part is diagonal in spin space (Appendix B), and improves the data to be near the result obtained in covariant equal time calculations (Table 1). Presumably, it is necessary to take into account $\eta_{\mu}\eta'_{\nu}$ term in exchange channel also for sharp cutoff after the proper regularization of infrared longitudinal divergences is done.

We place the results of calculations for three different cutoffs, performed in exchange and including both exchange and annihilation channels, in Tables 1 and 2, respectively. The corresponding set of figures is presented in Fig.1 and Fig.2. We get the ionization threshold at $M^2 \sim 4m^2$, the Bohr spectrum, and the fine structure. Including annihilation part increases the splittings twice as large for the lowest multiplets.

As one can see from presents figures, certain mass eigenvalues at $J_z = 0$ are degenerate with certain eigenvalues at other $J_z$ to a very high degree of numerical precision. As an example, consider the second lowest eigenvalue for $J_z = 0$. It is degenerate with the lowest eigenvalue for $J_z = \pm 1$, and can thus be classified as a member of the triplet with $J = 1$. Correspondingly, the lowest eigenvalue for $J_z = 0$ having no companion can be classified as the singlet state with $J = 0$. Quite in general one can interpret degenerate multiplets as members of a state with total angular momentum $J = 2J_{z,max} + 1$. One can get the quantum number of total angular momentum $J$ from the number of degenerate states for a fixed eigenvalue $M^2_n$. One can make contact with the conventional classification scheme $^{2S+1}L^J_{J_z}$, as indicated in Tables 1–2.

Such pattern of spectrum is driven by rotational invariance. To trace rotational symmetry we calculate the difference of energy levels between $J_z = 0$ and $J_z = 1$ states for the lowest multiplets. The data are given for exchange and including annihilation channel in Tables 3 and 4, respectively. Annihilation part makes corresponding states practically degenerate (see Tables 4 and Figure 2).

5 Conclusion

The numerical solution of positronium bound state problem, with the effective electron-positron interaction obtained by the flow equations, is presented. No approximations along numerical procedure are done.

Concerning the spin-splittings the best agreement with covariant calculations is obtained for gaussian cutoff, the worst results are for sharp cutoff. Rotational invariance is traced on the level of spectrum by studying the degree of degeneracy of corresponding states with the same total momentum but different projection $J_z$ in the multiplet. Again, better results are obtained for exponential and gaussian cutoff functions than for sharp cutoff. This suggests, that smooth cutoff functions are preferable to perform calculations. Including annihilation channel improves the extend of degeneracy.

For the sharp cutoff the lowest multiplet is placed higher than the one in case of exponential and gaussian cutoffs. The reason is in disregarding the infrared divergent part, which is diagonal in spin space and shifts the spectrum as a whole down. The question how to regularize this part and include it in mass spectrum calculations should
be considered. Generally, the impact of the different choice of cutoff functions on the spectrum is small.

In this work we solve the bound state integral equation for the one fixed integration interval. Integration domain introduces the ultraviolet-cutoff dependence of invariant mass squared $M^2(\Lambda)$, that reflects renormalization group properties of the effective coupling constant. We leave this question for the future study.
Figure 1: The invariant mass-squared spectrum $M_i^2$ for positronium versus the projection of the total spin, $J_z$, excluding annihilation with exponential, Gaussian and sharp cutoffs. The number of integration points is $N_1 = N_2 = 21$.

Figure 2: The invariant mass-squared spectrum $M_i^2$ for positronium versus the projection of the total spin, $J_z$, including annihilation with exponential, Gaussian and sharp cutoffs. The number of integration points is $N_1 = N_2 = 21$. 
A Defining different cut-offs

In this appendix we summarize the results for the effective electron-positron interaction, generated by the flow equations with different similarity functions. In the practical work, three different similarity function will be studied explicitly:

(1) the exponential cut-off, (2) the gaussian cut-off, and (3) the sharp cut-off.

(1) Exponential cut-off

\[
\begin{align*}
  f(D; l) &= \exp (-|D|l) \\
  \Theta(D_e, D_e) &= \frac{D_e}{D_e + D_e} \quad ; \quad \vartheta(\xi) = \xi \\
  V_{\text{eff}} &= \frac{\alpha}{4\pi^2} \langle \gamma^\mu \gamma^\nu \rangle_{\text{ex}} \frac{g_{\mu\nu}}{q^+} \frac{D_e}{D_e} + \frac{\alpha}{4\pi^2} \langle \gamma^\mu \gamma^\nu \rangle_{\text{an}} \frac{g_{\mu\nu}}{p^+} \frac{D_e}{D_e}
\end{align*}
\]

(41)

where \( D = 1/2(D_e + D_e) \) and \( \tilde{D} = 1/2(D_a + D_a) \). The first choice of similarity function gives exactly the result of perturbation theory.

(2) Gaussian cut-off

\[
\begin{align*}
  f(D; l) &= \exp (-D^2 l) \\
  \Theta(D_e, D_e) &= \frac{D_e^2}{D_e^2 + D_e^2} \quad ; \quad \vartheta(\xi) = \frac{2\xi}{1 + \xi^2} \\
  V_{\text{eff}} &= \frac{\alpha}{4\pi^2} \langle \gamma^\mu \gamma^\nu \rangle_{\text{ex}} \left[ g_{\mu\nu} \frac{D_e + D_e}{q^+ D_e + D_e^2} \frac{\eta_\mu \eta_\nu}{2q^+ 2D_e + 2D_e^2} \right] \\
  &= \frac{\alpha}{4\pi^2} \langle \gamma^\mu \gamma^\nu \rangle_{\text{an}} \left[ g_{\mu\nu} \frac{D_e + D_e}{p^+ D_a + D_e^2} \frac{\eta_\mu \eta_\nu}{2p^+ 2D_a + 2D_e^2} \right] \\
  &= -\frac{\alpha}{4\pi^2} \langle \gamma^\mu \gamma^\nu \rangle_{\text{ex}} \left[ g_{\mu\nu} \frac{Q^2}{Q^2 + \frac{\eta_\mu \eta_\nu}{q^+} \delta Q^4} \right] \frac{Q^1}{Q^4 + \delta Q^4} \\
  &= -\frac{\alpha}{4\pi^2} \langle \gamma^\mu \gamma^\nu \rangle_{\text{an}} \left[ g_{\mu\nu} \frac{\eta_\mu \eta_\nu}{p^+} \frac{\delta M^4}{M^4} \right] \frac{M^4}{M^4 + \delta M^4}
\end{align*}
\]

(42)

where we understand under \( Q^4 = (Q^2)^2 \) and \( \delta Q^4 = (\delta Q^2)^2 \) with \( Q^2 \) and \( \delta Q^2 \) defined in Eq. (41).

(3) Sharp cut-off

\[
\begin{align*}
  f(D; l) &= \theta (1 - |D|l) \\
  \Theta(D_e, D_e) &= \theta(|D_e| - |D_e|) \quad ; \quad \vartheta(\xi) = \text{sign}(\xi) \\
  V_{\text{eff}} &= \frac{\alpha}{4\pi^2} \langle \gamma^\mu \gamma^\nu \rangle_{\text{ex}} \left[ g_{\mu\nu} \frac{(\theta(|D_e| - |D_e|)}{D_e} \frac{q^+}{\eta_\mu \eta_\nu} (D_e - D_e) \right] \\
  &= \frac{\alpha}{4\pi^2} \langle \gamma^\mu \gamma^\nu \rangle_{\text{an}} \left[ g_{\mu\nu} \frac{\theta(|D_a| - |D_a|)}{D_a} \frac{(\theta(|D_e| - |D_e|)}{D_e} \right] \\
  &= \frac{\alpha}{4\pi^2} \langle \gamma^\mu \gamma^\nu \rangle_{\text{an}} \left[ g_{\mu\nu} \frac{\theta(|D_a| - |D_a|)}{D_a} \frac{\theta(|D_b| - |D_b|)}{D_b} \right]
\end{align*}
\]
The motivation to choose these cutoff functions is the following. Using exponential cutoff in flow equations one generates the same interaction as obtained also in Tamm-Dancoff approach, where numerical calculations of positronium spectrum are performed \cite{10}, and we use this numerical code here. Note also, that for this cutoff the effective interaction looks very much as in covariant calculations: it contains only $'g_{\mu\nu}'$ part, and $'\eta_{\mu}\eta_{\nu}'$ part is identically zero, so that there is no collinear problem. Gaussian cutoff corresponds to the original choice of generator Eq. (7) by Wegner as commutator of diagonal, particle number conserving, and off-diagonal, particle number changing, parts of Hamiltonian. Sharp cutoff is used often in the alternative similarity scheme to perform calculations \cite{6}. 

\begin{equation}
- \frac{\eta_{\mu}\eta_{\nu}}{2p^{+2}}(D_a - D_b) \left( \frac{\theta(|D_a| - |D_b|)}{D_a} - \frac{\theta(|D_b| - |D_a|)}{D_b} \right) \\
= - \frac{\alpha}{4\pi^{2}} \left( \gamma_{\mu}\gamma_{\nu} \right)_ex \left[ \frac{g_{\mu\nu}}{Q^{2}} + \frac{\eta_{\mu}\eta_{\nu}}{q^{+2}} \frac{|\delta Q^{2}|}{Q^{2}} \right] \frac{Q^{2}}{Q^{2} + |\delta Q^{2}|} \\
- \frac{\alpha}{4\pi^{2}} \left( \gamma_{\mu}\gamma_{\nu} \right)_{an} \left[ \frac{g_{\mu\nu}}{M^{2}} - \frac{\eta_{\mu}\eta_{\nu}}{p^{+2}} \frac{|\delta M^{2}|}{M^{2}} \right] \frac{M^{2}}{M^{2} + |\delta M^{2}|},
\end{equation}
B The matrix elements in the exchange channel

In this Appendix we follow the scheme of the work [10] to calculate the matrix elements of the effective interaction in the exchange channel.\(^3\) Here, we list the general, angle-dependent matrix elements defining the effective interaction in the exchange channel and the corresponding matrix elements of the effective interaction for arbitrary \(J_z\), after integrating out the angles. Exchange part of the effective interaction for three different cut-offs Eqs. (41–43) can be written

\[
V_{\text{eff}} = -\frac{\alpha}{4\pi^2} \langle \gamma^\mu \gamma^\nu \rangle B_{\mu\nu},
\]  

where explicitly one has

(1) **Exponential cut-off**

\[
B_{\mu\nu} = \frac{g_{\mu\nu}}{Q^2},
\]  

(2) **Gaussian cut-off**

\[
B_{\mu\nu} = g_{\mu\nu} \text{Re} \left( \frac{1}{Q^2 + i\delta Q^2} \right) - \frac{\eta_{\mu\nu} \delta Q^2}{q^2} \text{Im} \left( \frac{1}{Q^2 + i\delta Q^2} \right),
\]  

where \(\text{Re}\) and \(\text{Im}\) are real and imaginary parts, respectively, and \(i^2 = -1\).

(3) **Sharp cut-off**

\[
B_{\mu\nu} = g_{\mu\nu} \left( \frac{\theta(-\delta Q^2) + \theta(\delta Q^2)}{Q^2 - \delta Q^2} \right) - \frac{\eta_{\mu\nu} \delta Q^2}{q^2} \left( \frac{\theta(-\delta Q^2) + \theta(\delta Q^2)}{Q^2 - \delta Q^2} \right),
\]  

where \(q = p'_1 - p_1\) is the momentum transfer; and \(\langle \gamma^\mu \gamma^\nu \rangle\) for the exchange channel is given in Eq. (8). We omit index ‘ex’ everywhere.

It is convenient to extract the angular dependence in the functions

\[
Q^2_e = a_1 - b \cos t
\]

\[
Q^2_\bar{e} = a_2 - b \cos t
\]

\[
t = \varphi - \varphi',
\]

where we define

\[
\vec{k}_\perp = k_\perp (\cos \varphi, \sin \varphi)
\]

in polar system; here the terms are given

\[
a_1 = \frac{x'k^2_\perp + xk'^2_\perp + m^2(x - x')^2}{xx'}
\]

\(^3\) Some of these calculations can be found in [16].
B. THE MATRIX ELEMENTS IN THE EXCHANGE CHANNEL

\begin{align*}
  a_2 &= k_\perp^2 + k_\perp'^2 + (x - x') \left( k_\perp^2 \left( \frac{1}{x} \right) - k_\perp'^2 \left( \frac{1}{x'} \right) \right) + m^2 \frac{(x - x')^2}{xx'} \\
  b &= 2k_\perp k_\perp' 
\end{align*}

Then the functions in Eqs. (45–47) are given

\begin{align*}
  Q^2 &= a - b \cos \theta \\
  \delta Q^2 &= \delta a 
\end{align*}

where

\begin{align*}
  a &= \frac{1}{2} (a_1 + a_2) \\
  \delta a &= \frac{1}{2} (a_1 - a_2) 
\end{align*}

It is useful to display the matrix elements of the effective interaction in the form of tables. The matrix elements depend on the one hand on the momenta of the electron and positron, respectively, and on the other hand on their helicities before and after the interaction. The dependence on the helicities occur during the calculation of these functions \( E(x, k_\perp; \lambda_1, \lambda_2 | x', k_\perp'; \lambda'_1, \lambda'_2) \) in part I and \( G(x, k_\perp; \lambda_1, \lambda_2 | x', k_\perp'; \lambda'_1, \lambda'_2) \) in part II as different Kronecker deltas \([9]\). These functions are displayed in the form of helicity tables. We use the following notation for the elements of the tables

\[ F_i(1, 2) \rightarrow E_i(x, k_\perp; x', k_\perp'); G_i(x, k_\perp; x', k_\perp') \]

Also we have used in both cases for the permutation of particle and anti-particle

\[ F_3^*(x, k_\perp; x', k_\perp') = F_3(1 - x, -k_\perp; 1 - x', -k_\perp') \]

one has the corresponding for the elements of arbitrary \( J_z \); in the case when the function additionally depends on the component of the total angular momentum \( J_z = n \) we have introduced

\[ \tilde{F}_i(n) = F_i(-n) \]

B.1. The helicity table

To calculate the matrix elements of the effective interaction in the exchange channel we use the matrix elements of the Dirac spinors listed in Table 1 \([9]\). Also the following holds

\[ \bar{v}_{\lambda'}(p) \gamma^\alpha v_\lambda(q) = \bar{u}_\lambda(q) \gamma^\alpha u_{\lambda'}(p). \]

We introduce for the matrix elements entering in the effective interaction Eqs. (15–17)

\[ 2E^{(1)}(x, k_\perp; \lambda_1, \lambda_2 | x', k_\perp'; \lambda'_1, \lambda'_2) = \langle \gamma^\mu \gamma^\nu \rangle g_{\mu\nu} \]
THE MATRIX ELEMENTS IN THE EXCHANGE CHANNEL

Table 5: Matrix elements of the Dirac spinors.

\[
\langle \gamma_\mu \gamma_\nu \rangle g_{\mu\nu} = \frac{1}{2} \langle \gamma^+ \gamma^- \rangle + \frac{1}{2} \langle \gamma^- \gamma^+ \rangle - \langle \gamma^1 \rangle - \langle \gamma^2 \rangle \quad \text{and} \quad
2E^{(2)}(x, \vec{k}_\perp; \lambda_1, \lambda_2 | x', \vec{k}'_\perp; \lambda'_1, \lambda'_2) = \langle \gamma^\mu \gamma^\nu \rangle \eta_\mu \eta_\nu \frac{1}{q_+}, \tag{57}
\]

with \( \langle \gamma^\mu \gamma^\nu \rangle \eta_\mu \eta_\nu = \langle \gamma^+ \gamma^+ \rangle \); where

\[
\langle \gamma^\mu \gamma^\nu \rangle = \left( \frac{\langle \bar{u}(x, \vec{k}_\perp; \lambda_1) \gamma^\mu u(x', \vec{k}'_\perp; \lambda'_1) \rangle (\bar{v}(1-x', -\vec{k}'_\perp; \lambda'_2) \gamma^\nu v(1-x, -\vec{k}_\perp; \lambda_2))}{\sqrt{x'x}} \right) \left( \frac{1}{\sqrt{(1-x)(1-x')}} \right) \tag{58}
\]

These functions are displayed in Table 6.

| final : initial | \((\lambda'_1, \lambda'_2) = \uparrow\uparrow\) | \((\lambda'_1, \lambda'_2) = \uparrow\downarrow\) | \((\lambda'_1, \lambda'_2) = \downarrow\uparrow\) | \((\lambda'_1, \lambda'_2) = \downarrow\downarrow\) |
|----------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| \((\lambda_1, \lambda_2) = \uparrow\uparrow\) | \(E_1(1, 2)\) | \(E_3(1, 2)\) | \(E_3(1, 2)\) | \(0\) |
| \((\lambda_1, \lambda_2) = \uparrow\downarrow\) | \(E_3^*(2, 1)\) | \(E_2(1, 2)\) | \(E_4(1, 2)\) | \(-E_3(2, 1)\) |
| \((\lambda_1, \lambda_2) = \downarrow\uparrow\) | \(E_3(2, 1)\) | \(E_4(1, 2)\) | \(E_2(1, 2)\) | \(-E_3^*(2, 1)\) |
| \((\lambda_1, \lambda_2) = \downarrow\downarrow\) | \(0\) | \(-E_3(1, 2)\) | \(-E_3^*(1, 2)\) | \(E_1(1, 2)\) |

Table 6: General helicity table defining the effective interaction in the exchange channel.

The matrix elements \(E^{(n)}_i(1, 2) = E^{(n)}_i(x, \vec{k}_\perp; x', \vec{k}'_\perp)\) with \(n = 1\) and \(n = 2\) for \(g_{\mu\nu}\) and \(\eta_\mu \eta_\nu\) terms, respectively, are the following

\[
E^{(1)}_1(x, \vec{k}_\perp; x', \vec{k}'_\perp) = m^2 \left( \frac{1}{xx'} + \frac{1}{(1-x)(1-x')} \right) + \frac{k_\perp k'_\perp}{xx'(1-x)(1-x')} e^{-i(\varphi - \varphi')}.
\]
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\[ E_2^{(1)}(x, \vec{k}_\perp; x', \vec{k}'_\perp) = m^2 \left( \frac{1}{xx'} + \frac{1}{(1-x)(1-x')} \right) + k_{\perp}^2 \frac{1}{x(1-x)} + k_{\perp}^2 \frac{1}{x'(1-x')} \]
\[ + \ k_{\perp} k_{\perp}' \left( \frac{e^{i(\varphi-\varphi')}}{xx'} + \frac{e^{-i(\varphi-\varphi')}}{(1-x)(1-x')} \right) \]
\[ E_3^{(1)}(x, \vec{k}_\perp; x', \vec{k}'_\perp) = -m \frac{m}{xx'} \left( k_{\perp}' e^{i\varphi'} - k_{\perp} \frac{1 - x'}{1 - x} e^{i\varphi} \right) \]
\[ E_4^{(1)}(x, \vec{k}_\perp; x', \vec{k}'_\perp) = -m^2 \frac{(x - x')^2}{xx'(1-x)(1-x')} , \tag{59} \]

and

\[ E_1^{(2)}(x, \vec{k}_\perp; x', \vec{k}'_\perp) = E_2^{(2)}(x, \vec{k}_\perp; x', \vec{k}'_\perp) = \frac{2}{(x - x')^2} \]
\[ E_3^{(2)}(x, \vec{k}_\perp; x', \vec{k}'_\perp) = E_4^{(2)}(x, \vec{k}_\perp; x', \vec{k}'_\perp) = 0 . \tag{60} \]

B.2 The helicity table for arbitrary \( J_z \).

Following the description given in the main text Eq. (35) we integrate out the angles in the effective interaction in the exchange channel. For the matrix elements of the effective interaction for an arbitrary \( J_z = n \) with \( n \in \mathbb{Z} \) we introduce the functions \( G(x, k, \lambda_1, \lambda_2|x', k', \lambda_1', \lambda_2') = \langle x, k; J_z, \lambda_1, \lambda_2|V_{\text{eff}}|x', k', J_z', \lambda_1', \lambda_2' \rangle \) in the exchange channel and obtain the helicity Table 7.

| final : initial | (\( \lambda'_1, \lambda'_2 \)) =↑↑ | (\( \lambda'_1, \lambda'_2 \)) =↑↓ | (\( \lambda'_1, \lambda'_2 \)) =↓↑ | (\( \lambda'_1, \lambda'_2 \)) =↓↓ |
|----------------|-----------------|-----------------|-----------------|-----------------|
| (\( \lambda_1, \lambda_2 \)) =↑↑ | \( G_1(1, 2) \) | \( G_3^*(1, 2) \) | \( G_3(1, 2) \) | 0 |
| (\( \lambda_1, \lambda_2 \)) =↑↓ | \( G_3^*(2, 1) \) | \( G_2(1, 2) \) | \( G_4(1, 2) \) | \( -\tilde{G}_3(2, 1) \) |
| (\( \lambda_1, \lambda_2 \)) =↓↑ | \( G_3(2, 1) \) | \( G_4(1, 2) \) | \( \tilde{G}_2(1, 2) \) | \( -\tilde{G}_3^*(2, 1) \) |
| (\( \lambda_1, \lambda_2 \)) =↓↓ | 0 | \( -\tilde{G}_3(1, 2) \) | \( -\tilde{G}_3^*(1, 2) \) | \( \tilde{G}_1(1, 2) \) |

Table 7: Helicity table of the effective interaction for \( J_z = \pm n, \ x > x' \).

Here, the functions \( G_i(1, 2) = G_i(x, k; x', k') \) are given

\[ G_1(x, k; x', k') = \left( \frac{m^2}{xx'} + \frac{m^2}{(1-x)(1-x')} \right) \text{Int}(|1-n|) \]
\[ + \ \frac{k_{\perp} k_{\perp}'}{xx'(1-x)(1-x')} \text{Int}(|n|) - \frac{2\delta a}{(x - x')^2} \text{Int}(|1-n|) \]
\[ G_2(x, k; x', k') = \left( \frac{m^2}{xx'} + \frac{1}{(1-x)(1-x')} + \frac{k_{\perp}^2}{x(1-x)} + \frac{k_{\perp}^2}{x'(1-x')} \right) \text{Int}(|n|) \]
\[ + \ k_{\perp} k_{\perp}' \left( \frac{1}{xx'} \text{Int}(|1-n|) + \frac{1}{(1-x)(1-x')} \text{Int}(|1+n|) \right) \]
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\[
- \frac{2\delta a}{(x-x')^2} \tilde{I}nt(|n|)
\]

\[
G_3(x, k_\perp; x', k'_\perp) = -\frac{m}{xx'} \left( k'_\perp Int(|1-n|) - k_\perp \frac{1-x'}{1-x} Int(|n|) \right)
\]

\[
G_4(x, k_\perp; x', k'_\perp) = -m^2 \frac{(x-x')^2}{xx'(1-x)(1-x')} Int(|n|)
\]

we define

\[
I(n; a, b) = -\frac{\alpha}{2\pi^2} \int_0^{2\pi} dt \frac{\cos nt}{a - b \cos t},
\]

then in Eq. (61) the following functions are introduced

(1) **Exponential cut-off**

\[
\begin{align*}
\text{Int}(n) &= I(n; a, b) \\
\tilde{\text{Int}}(n) &= 0,
\end{align*}
\]

(2) **Gaussian cut-off**

\[
\begin{align*}
\text{Int}(n) &= \Re I(n; a + i\delta a, b) \\
\tilde{\text{Int}}(n) &= \Im I(n; a + i\delta a, b),
\end{align*}
\]

(3) **Sharp cut-off**

\[
\begin{align*}
\text{Int}(n) &= \theta(-\delta a)I(n; a - \delta a, b) + \theta(\delta a)I(n; a + \delta a, b) \\
\tilde{\text{Int}}(n) &= \theta(-\delta a)I(n; a - \delta a, b) - \theta(\delta a)I(n; a + \delta a, b),
\end{align*}
\]

also \(a + \delta a = a_1\) and \(a - \delta a = a_2\).

Explicitly is used

\[
\begin{align*}
\int_0^{2\pi} dt \frac{\cos nt}{a - b \cos t} &= 2\pi \frac{1}{\sqrt{a^2 - b^2}} \left( \frac{a - \sqrt{a^2 - b^2}}{b} \right)^n \\
\int_0^{2\pi} dt \frac{\sin nt}{a - b \cos t} &= 0,
\end{align*}
\]

where \(a\) can contain imaginary part as in the case of gaussian cutoff.
C The matrix elements in the annihilation channel

We repeat the same calculations for the matrix elements of the effective interaction in the annihilation channel. Annihilation part of the effective interaction can be written

\[ V_{\text{eff}} = -\frac{\alpha}{4\pi^2} (\gamma^\mu \gamma^\nu) C_{\mu\nu}, \]  

(67)

where one has

\[ C_{\mu\nu} = g_{\mu\nu} \left( \frac{\Theta_{ab}}{M_a^2} + \frac{\Theta_{ba}}{M_b^2} \right) - \frac{\eta_\mu \eta_\nu}{p^+}, \]  

(68)

in the frame \( p_\perp = 0 \). Explicitly the annihilation part of the effective interaction for different cut-offs Eq. (41)- Eq. (43) is given

1. **Exponential cut-off**

\[ C_{\mu\nu} = \frac{g_{\mu\nu}}{M^2}, \]  

(69)

2. **Gaussian cut-off**

\[ C_{\mu\nu} = \frac{g_{\mu\nu}}{M^2 + \delta M^2} \frac{M^2}{1 - \beta \chi(\beta)} - \frac{\eta_\mu \eta_\nu}{p^+}, \]  

(70)

3. **Sharp cut-off**

\[ C_{\mu\nu} = \frac{g_{\mu\nu}}{M^2} \left( \frac{\theta(M_a^2 - M_b^2)}{M_a^2} + \frac{\theta(M_b^2 - M_a^2)}{M_b^2} \right) - \frac{\eta_\mu \eta_\nu}{p^+}, \]  

(71)

where \( p^+ = p_1^+ + p_2^+ \) is the total momentum; and \( \langle \gamma^\mu \gamma^\nu \rangle \) for annihilation is defined in Eq. (8). The functions present in Eq. (69)- Eq. (71) are given in the light-front frame

\[ M_a^2 = \frac{k_1^+ m^2}{x(1 - x)}, \]  

\[ M_b^2 = \frac{k_2^+ m^2}{x(1 - x)}, \]  

(72)

Indeed \( \langle \gamma^\mu \gamma^\nu \rangle g_{\mu\nu} = \frac{1}{2} \langle \gamma^+ \gamma^- \rangle + \frac{1}{2} \langle \gamma^- \gamma^+ \rangle + \langle \gamma^\mu \gamma^\nu \rangle g_{\mu\nu} \); therefore it holds

\[ g_{\mu\nu} = g_{\mu\nu}^\perp + \frac{\eta_\mu (p_\nu - p_\nu^\perp) + \eta_\nu (p_\mu - p_\mu^\perp)}{p^+} - \eta_\mu \eta_\nu \frac{p^+ - p^+}{p^+ \eta_\mu \eta_\nu}, \]

The 4-momentum of the photon \( p_\mu \) in the \( t \)-channel can be written \( p_\mu = p_1^\prime \mu + p_2^\prime \mu - \eta_\mu D_a/2 = p_1^\mu + p_2^\mu - \eta_\mu D_b/2 \) with \( D_a, D_b \) defined in Eq. (15). The Dirac equation \( (p_1 + p_2)\bar{u}(p_1)\gamma^\mu v(p_2) = 0 \) allows then to write \( p_\mu \bar{u}(p_1, \lambda_1)\gamma^\mu v(p_2, \lambda_2) = -M^2 (2p^+/\eta_\mu \bar{u}(p_1, \lambda_1)\gamma^\mu v(p_2, \lambda_2)). \) Thus, when \( p_\perp = 0 \), one has

\[ g_{\mu\nu} \rightarrow g_{\mu\nu}^\perp - \frac{\eta_\mu \eta_\nu}{p^+} M^2, \]

where the arrow means that this tensor should be contracted with \( \langle \gamma^\mu \gamma^\nu \rangle \) in the annihilation channel.
we remind also

\[ M^2 = \frac{1}{2}(M_a^2 + M_b^2) \]

\[ \delta M^2 = \frac{1}{2}(M_a^2 - M_b^2) \] (73)

Note that the energy denominators of the effective interaction in the annihilation channel do not depend on the angles \( \varphi, \varphi' \).

Table 8: Matrix elements of the Dirac spinors.

| \( M \) | \( \frac{1}{\sqrt{k+k'}} \bar{v}(k', \lambda') \mathcal{M} u(k, \lambda) \) |
| --- | --- |
| \( \gamma^+ \) | \( 2\delta^\lambda_{-\lambda'} \) |
| \( \gamma^- \) | \( \frac{2}{k+k'} \left[ -\left( m^2 - k_\perp k'_\perp e^{+i\lambda(\varphi-\varphi')}\right) \delta^\lambda_{-\lambda'} - m\lambda \left( k'_\perp e^{-+i\lambda\varphi'} + k_\perp e^{+i\lambda\varphi}\right) \delta^\lambda_{-\lambda'} \right] \) |
| \( \gamma^1_\perp \) | \( \left( \frac{k'_\perp}{k'} e^{-i\lambda\varphi'} + \frac{k_\perp}{k} e^{+i\lambda\varphi} \right) \delta^\lambda_{-\lambda'} - m\lambda \left( \frac{1}{k'} + \frac{1}{k} \right) \delta^\lambda_{-\lambda'} \) |
| \( \gamma^2_\perp \) | \( i\lambda \left( \frac{k'_\perp}{k'} e^{-i\lambda\varphi'} - \frac{k_\perp}{k} e^{+i\lambda\varphi} \right) \delta^\lambda_{-\lambda'} - i m\lambda \left( \frac{1}{k'} + \frac{1}{k} \right) \delta^\lambda_{-\lambda'} \) |

C.1 The helicity table

For the calculation of matrix elements of effective interaction in the annihilation channel we use the matrix elements of the Dirac spinors listed in Table 4 [9]. Also the following holds

\( \langle \bar{v}_\lambda(p) \gamma^\alpha \eta_\lambda(q) \rangle^+ = \bar{u}_\lambda(q) \gamma^\alpha v_\lambda(p) \).

We introduce

\[ 2H^{(1)}(x, k_\perp; \lambda_1, \lambda_2 | x', k'_\perp; \lambda'_1, \lambda'_2) = \langle \gamma^\mu \gamma^\nu \rangle g_{\mu\nu}^{\perp} = -\langle \gamma^1_1 \rangle - \langle \gamma^2_2 \rangle \]

\[ 2H^{(2)}(x, k_\perp; \lambda_1, \lambda_2 | x', k'_\perp; \lambda'_1, \lambda'_2) = \langle \gamma^\mu \gamma^\nu \rangle \eta_\mu \eta_\nu \frac{1}{p^2} \] (74)

where

\[ \langle \gamma^\mu \gamma^\nu \rangle = \frac{(\bar{v}(1-x', -k'_\perp; \lambda'_2) \gamma^\mu u(x', k'_\perp; \lambda'_1)) (\bar{u}(x, k_\perp; \lambda_1) \gamma^\nu v(1-x, -k_\perp; \lambda_2))}{\sqrt{x'(1-x')} \sqrt{x(1-x)}} \] (75)
The matrix elements of the effective interaction for \( J \) are displayed in Table 5.

These functions are displayed in the Table 5.

The matrix elements \( H_i^{(n)}(1, 2) = H_i^{(n)}(x, \vec{k}_\perp; x', \vec{k}_\perp') \) are the following

\[
H_1^{(1)}(x, \vec{k}_\perp; x', \vec{k}_\perp') = -m^2 \left( \frac{1}{x} + \frac{1}{1-x} \right) \left( \frac{1}{x'} + \frac{1}{1-x'} \right)
\]

\[
H_2^{(1)}(x, \vec{k}_\perp; x', \vec{k}_\perp') = -k_\perp k'_\perp \left( \frac{e^{i(\phi-\phi')}}{xx'} \right)
\]

\[
H_3^{(1)}(x, \vec{k}_\perp; x', \vec{k}_\perp') = -m\lambda \left( \frac{1}{x} + \frac{1}{1-x} \right) \frac{k'_\perp}{1-x'} e^{i\phi}
\]

\[
H_4^{(1)}(x, \vec{k}_\perp; x', \vec{k}_\perp') = k_\perp k'_\perp \left( \frac{e^{i(\phi-\phi')}}{x'(1-x)} \right)
\]

and

\[
H_1^{(2)}(x, \vec{k}_\perp; x', \vec{k}_\perp') = H_3^{(2)}(x, \vec{k}_\perp; x', \vec{k}_\perp') = 0
\]

\[
H_2^{(2)}(x, \vec{k}_\perp; x', \vec{k}_\perp') = H_4^{(2)}(x, \vec{k}_\perp; x', \vec{k}_\perp') = 2
\]

### C.2 The helicity table for \(|J_z| \leq 1\)

The matrix elements of the effective interaction for \( J_z \geq 0 \) \( F(x, k_\perp; \lambda_1, \lambda_2|x', k'_\perp; \lambda'_1, \lambda'_2) = \langle x, k_\perp; J_z, \lambda_1, \lambda_2|\bar{V}_{eff}^f|x', k'_\perp; J'_z, \lambda'_1, \lambda'_2 \rangle \) in the annihilation channel (the sum of the generated interaction for \( J_z = +1 \) and instantaneous graph for \( J_z = 0 \)) are given in Table 6.

The function \( F_i(1, 2) = F_i(x, k_\perp; x', k'_\perp) \) are the following

\[
F_1(x, k_\perp; x', k'_\perp) = \frac{\alpha}{\pi} \frac{1}{\Omega \cdot xx'(1-x)(1-x')} \frac{m^2}{\delta|J_z|,1}
\]

\[
F_2(x, k_\perp; x', k'_\perp) = \frac{\alpha}{\pi} \left( \frac{1}{\Omega} \frac{k_\perp k'_\perp}{xx'} \delta|J_z|,1 + 2 \delta_{J_z,0} \right)
\]

\[
F_3(x, k_\perp; x', k'_\perp) = \frac{\alpha}{\pi} \frac{1}{\Omega \lambda_1} \frac{m}{x'(1-x')} \frac{k_\perp k'_\perp}{1-x} \delta|J_z|,1
\]
The matrix elements in the annihilation channel

\[
\begin{align*}
\text{final:initial} & \quad |(\lambda'_1, \lambda'_2) = \uparrow \uparrow\rangle \quad |(\lambda'_1, \lambda'_2) = \uparrow \downarrow\rangle \quad |(\lambda'_1, \lambda'_2) = \downarrow \uparrow\rangle \quad |(\lambda'_1, \lambda'_2) = \downarrow \downarrow\rangle \\
(\lambda_1, \lambda_2) = \uparrow \uparrow & \quad F_1(1, 2) \quad F_3(2, 1) \quad F_3^*(2, 1) \quad 0 \\
(\lambda_1, \lambda_2) = \uparrow \downarrow & \quad F_3(1, 2) \quad F_2^*(1, 2) \quad F_4(2, 1) \quad 0 \\
(\lambda_1, \lambda_2) = \downarrow \uparrow & \quad F_3^*(1, 2) \quad F_4(1, 2) \quad F_2(1, 2) \quad 0 \\
(\lambda_1, \lambda_2) = \downarrow \downarrow & \quad 0 \quad 0 \quad 0 \quad 0 
\end{align*}
\]

Table 10: Helicity table of the effective interaction in the annihilation channel for \(J_z \geq 0\).

\[
F_4(x, k_\perp; x', k'_\perp) = \frac{\alpha}{\pi} \left( -\frac{1}{\Omega} \frac{k_\perp k'_\perp}{x(1-x')} \delta_{|J_z|, 1} + 2\delta_{J_z, 0} \right)
\]

(78)

where we have introduced

(1) \textbf{Exponential cut-off}

\[
\frac{1}{\Omega} = \frac{1}{M^2},
\]

(79)

(2) \textbf{Gaussian cut-off}

\[
\frac{1}{\Omega} = \frac{M^2_a + M^2_b}{M^4_a + M^4_b},
\]

(80)

(3) \textbf{Sharp cut-off}

\[
\frac{1}{\Omega} = \frac{\theta(M^2_a - M^2_b)}{M^2_a} + \frac{\theta(M^2_b - M^2_a)}{M^2_b}.
\]

(81)

The table for \(J_z = -1\) is obtained by inverting all helicities, i.e.

\[
F(J_z = +1; \lambda_1, \lambda_2) = -\lambda_1 F(J_z = -1; -\lambda_1, -\lambda_2),
\]

(82)

The matrix elements of the effective interaction in the annihilation channel are nonzero only for \(|J_z| \leq 1\) due to the restriction on the angular momentum of the photon.
References

[1] S.J. Brodsky, H.C. Pauli, and S.S. Pinsky, *Quantum chromodynamics and other field theories on the light cone*, Physics Reports 301, 299 (1998).

[2] M. Brisudova, and R.J. Perry, [hep-ph/9511443](https://arxiv.org/abs/hep-ph/9511443), Phys.Rev. D54, (1996) 1831; M. Brisudova, R.J. Perry, K.G. Wilson, [hep-ph/9607280](https://arxiv.org/abs/hep-ph/9607280), Phys.Rev.Lett.78, (1997) 1227.

[3] M. Brisudova, and R.J. Perry, [hep-ph/9605363](https://arxiv.org/abs/hep-ph/9605363), Phys.Rev. D54, (1996) 6453.

[4] B. D. Jones, R. G. Perry and S. D. Glazek, Phys.Rev. D55, 6561 (1997); [hep-th/9605231](https://arxiv.org/abs/hep-th/9605231); B. D. Jones and R. G. Perry, [hep-th/9703106](https://arxiv.org/abs/hep-th/9703106).

[5] R.J. Perry, in *Proceedings of Hdress 94*, edited by V. Herscovitz and C. Vasconcellos (World Scientific, Singapore, 1995), [hep-th/9407056](https://arxiv.org/abs/hep-th/9407056).

[6] S.D. Glazek and K.G. Wilson, Phys.Rev. D48, 5863 (1993); S.D. Glazek and K.G. Wilson, Phys.Rev. D49, 4214 (1994).

[7] F. Wegner, Ann.Physik 3,77 (1994).

[8] H.C. Pauli, [hep-th/9608035](https://arxiv.org/abs/hep-th/9608035); [hep-th/9707361](https://arxiv.org/abs/hep-th/9707361); [hep-th/9809003](https://arxiv.org/abs/hep-th/9809003), MPIH-V21-1998, June 1998, to appear in Europhys.Journal (1998).

[9] G.P. Lepage, S.J. Brodsky, Phys.Rev. D22, 2157 (1980).

[10] U. Trittmann and H.C. Pauli, [hep-th/9704215](https://arxiv.org/abs/hep-th/9704215), [hep-th/9705021](https://arxiv.org/abs/hep-th/9705021), [hep-th/9705072](https://arxiv.org/abs/hep-th/9705072).

[11] Computer code is available on the World Wide Web (WWW) under [http://pluto.mpi-hd.mpg.de/~trittman/code.html](http://pluto.mpi-hd.mpg.de/~trittman/code.html).

[12] U. Merkel and H.C. Pauli, [hep-th/9608152](https://arxiv.org/abs/hep-th/9608152), Phys.Rev. D55, (1997), 2486.

[13] M. Krautgärtner, H.C. Pauli and F. Wölz, Phys.Rev. D45, (1992) 3755.

[14] M. Kaluža and H.-J. Pirner, Phys.Rev. D47, 1620 (1993).

[15] E.L. Gubankova, H.C. Pauli, F. Wegner [hep-th/9809143](https://arxiv.org/abs/hep-th/9809143).

[16] E.L. Gubankova, [hep-th/9801018](https://arxiv.org/abs/hep-th/9801018).