Multi-variable reductions of the dispersionless DKP hierarchy

V Akhmedova¹, T Takebe¹ and A Zabrodin¹,²

¹ National Research University Higher School of Economics, 20 Myasnitskaya Ulitsa, Moscow 101000, Russia
² ITEP, 25 B.Cheremushkinskaya, Moscow 117218, Russia

E-mail: valeria-58@yandex.ru, ttakebe@hse.ru and zabrodin@itep.ru

Received 26 July 2017, revised 15 October 2017
Accepted for publication 18 October 2017
Published 9 November 2017

Abstract
We consider multi-variable reductions of the dispersionless DKP hierarchy (the dispersionless limit of the Pfaff lattice) in the elliptic parametrization. The reduction is given by a system of elliptic Löwner equations supplemented by a system of partial differential equations of hydrodynamic type. The compatibility conditions for the elliptic Löwner equations are derived. They are elliptic analogues of the Gibbons–Tsarev equations. We prove solvability of the hydrodynamic type system by means of the generalized hodograph method. The associated diagonal metric is proved to be of the Egorov type.

Keywords: dispersionless DKP hierarchy, multi-variable reduction, elliptic Löwner equation, Gibbons–Tsarev equation, Egorov type metric

1. Introduction

The DKP hierarchy is one of the integrable hierarchies with $D_\infty$ symmetries introduced by M Jimbo and T Miwa in 1983 [1]. It was subsequently rediscovered and came to be also known as the coupled KP hierarchy [2] and the Pfaff lattice [3, 4], see also [5–7]. The solutions and the algebraic structure were studied in [8–10], the relation to matrix integrals was elaborated in [3–5, 11, 12].

The dispersionless version of the DKP hierarchy (the dDKP hierarchy) was suggested in [13, 14]. It is an infinite system of differential equations for a real-valued function $F = F(t)$ of the infinite number of (real) ‘times’ $t = \{t_0, t_1, t_2, \ldots\}$. The differential equations are obtained by expanding equations

$$e^{D(z)D(\zeta)F} \left(1 - \frac{1}{z^2\zeta^2} e^{2\partial_z(2\partial_{\zeta} + D(z) + D(\zeta))F}\right) = 1 - \frac{\partial_t D(z)F - \partial_z D(\zeta)F}{z - \zeta},$$

(1)
\[
e^{-D(z)D(\zeta)\frac{2e^{-2\partial_{\zeta}D(\zeta)F} - \zeta^2 e^{-2\partial_{\zeta}D(\zeta)F}}{z - \zeta}} = z + \zeta - \partial_{\zeta} (2\partial_{\zeta} + D(z) + D(\zeta)) F, \tag{2}
\]

where

\[
D(z) = \sum_{k\geq 1} \frac{z^{-k}}{k} \partial_{\zeta} \tag{3}
\]

in powers of \(z, \zeta\). The function \(F\) corresponds to the logarithm of the tau function in the case of the dispersionless KP hierarchy (see for example, [15, 16]).

In [17, 18] it was shown that equations (1) and (2), when rewritten in an elliptic parametrization in terms of Jacobi’s theta-functions \(\theta_a(u, \tau)\), assume a nice and suggestive form which looks like a natural elliptic extension of the dispersionless KP hierarchy:

\[
(z^{-1} - \zeta^{-1}) e^{(\partial_{\zeta} + D(z)\partial_{\zeta} + D(z))F} = \frac{\theta_1(u(z) - u(\zeta), \tau)}{\theta_4(u(z) - u(\zeta), \tau)} \tag{4}
\]

Here the function \(u(z)\) is defined by

\[
e^{\theta_4(\partial_{\zeta} + D(z))F} = z \frac{\theta_1(u(z), \tau)}{\theta_4(u(z), \tau)}. \tag{5}
\]

The modular parameter \(\tau\) is a dynamical variable: \(\tau = \tau(t)\). This feature suggests some similarities with the genus 1 Whitham equations [19] and the integrable structures behind boundary value problems in plane doubly-connected domains [20]. We assume that \(\tau\) is purely imaginary.

One may look for solutions of the hierarchy such that \(u(z, t)\) and \(\tau(t)\) depend on the times through a single variable \(\lambda = \lambda(t)\): \(u(z, t) = u(z, \lambda(t))\), \(\tau(t) = \tau(\lambda(t))\). In [17] it was shown that such one-variable reductions are classified by solutions of a differential equation which is an elliptic analogue of the famous Löwner equation (see, e.g. [21, chapter 6]). In complex analysis, this ‘elliptic Löwner equation’ is also known as the Goluzin–Komatu equation [22, 23], see also [24–27]:

\[
4\pi i \partial_{\lambda} u(z, \lambda) = \left[-\zeta_i \left(u(z, \lambda) + \xi(\lambda), \frac{\tau}{2}\right) + \zeta_1 \left(\xi(\lambda), \frac{\tau}{2}\right)\right] \frac{\partial \tau}{\partial \lambda}, \tag{6}
\]

where \(\zeta_i(u, \tau) := \partial_{\zeta} \log \theta_1(u, \tau)\) and \(\xi(\lambda)\) is an arbitrary (continuous) function of \(\lambda\) (the ‘driving function’). This equation is the basic element of the theory of parametric conformal maps from doubly connected slit domains to annuli. A similar relation between the chordal Löwner equation and one-variable reductions of the dKP hierarchy was known since the seminal papers by Gibbons and Tsarev [28, 29]. Further developments are discussed in [30–34].

In this paper we study diagonal \(N\)-variable reductions of the dDKP hierarchy when \(u\) depends on the times through \(N\) real variables \(\lambda_j\). The starting point is the system of \(N\) elliptic Löwner equations which characterize the dependence of \(u(z)\) on the variables \(\lambda_j\):

\[
4\pi i \partial_{\lambda_j} u(z, \{\lambda_j\}) = \left[-\zeta_i \left(u + \xi_j, \frac{\tau}{2}\right) + \zeta_1 \left(\xi_j, \frac{\tau}{2}\right)\right] \frac{\partial \tau}{\partial \lambda_j}. \tag{7}
\]

Their compatibility condition is expressed as the elliptic Gibbons–Tsarev system (see (42) and (43) below). The time dependence of the variables \(\lambda_j\) is fixed by a system of quasi-linear partial differential equations of the form

\[
\frac{\partial \lambda_j}{\partial t_k} = \phi_{jk}(\{\lambda_j\}) \frac{\partial \lambda_j}{\partial t_0}, \tag{8}
\]
with $\phi_{j,k}(\{\lambda_i\})$ defined with the help of ‘elliptic Faber functions’. We show that the system (8) is compatible and the associated diagonal metric is of Egorov type. The system (8) can be solved by the generalized hodograph method developed by Tsarev in [35]. For the general theory of equations of hydrodynamic type see also [36–38].

The paper is organized as follows. In section 2 we review the algebraic and elliptic formulations of the dDKP hierarchy. In section 3 we define the $N$-variable reductions with the help of a system of elliptic Löwner equations. Their compatibility condition (the elliptic analogue of the Gibbons–Tsarev system) is derived in section 4. Section 5 is devoted to the generalized hodograph method. In section 6 we prove that the associated diagonal metric is of Egorov type and find its potential function. Finally, in section 7 we discuss conserved quantities. Some long calculations with elliptic and theta functions are contained in the appendices.

2. The dispersionless DKP hierarchy

We begin with the algebraic form of the dDKP hierarchy. In what follows we use the differential operator
\[
\nabla(z) = \partial_0 + D(z)
\]
which in the dDKP case is more convenient than $D(z)$. Introducing the functions
\[
p(z) = z - \partial_t \nabla(z) F, \quad w(z) = z^2 e^{-2\partial_t \nabla(z) F},
\]
we can rewrite equations (1) and (2) in a more compact form
\[
e^{D(z)D(\zeta)F} \left( 1 - \frac{1}{w(z)w(\zeta)} \right) = \frac{p(z) - p(\zeta)}{z - \zeta},
\]
\[
e^{-D(z)D(\zeta)F + 2\partial_t^2 F} \frac{w(z) - w(\zeta)}{z - \zeta} = p(z) + p(\zeta).
\]
Multiplying the two equations, we get the relation
\[
p^2(z) - e^{2F_{00}} \left( w(z) + w^{-1}(z) \right) = p^2(\zeta) - e^{2F_{00}} \left( w(\zeta) + w^{-1}(\zeta) \right)
\]
from which it follows that $p^2(z) - e^{2F_{00}} \left( w(z) + w^{-1}(z) \right)$ does not depend on $z$ (here and below we use the short-hand notation $F_{mn} = \frac{\partial^2 F}{\partial t_m \partial t_n}$). Tending $z$ to infinity, we find that this expression is equal to $F_{02} - 2F_{11} - F_{01}^2$. Therefore, we conclude that $p(z), w(z)$ satisfy the algebraic equation [14]
\[
p^2(z) = R^2 \left( w(z) + w^{-1}(z) \right) + V,
\]
where
\[
R = e^{F_{00}}, \quad V = F_{02} - 2F_{11} - F_{01}^2,
\]
are real numbers depending on the times ($R$ is positive). This equation defines an elliptic curve, with $w, p$ being algebraic functions on this curve.

A natural further step is to uniformize the curve through elliptic functions. This provides the elliptic formulation of the dDKP hierarchy which was suggested in [17], see also [18]. To this end, we use the standard Jacobi theta functions $\theta_a(u) = \theta_a(u, \tau)$ ($a = 1, 2, 3, 4$). (Their definition is given in appendix A.) The elliptic parametrization of (13) is as follows:
where \( u(z) = u(z, t) \) is some function of \( z \), \( \gamma \) is a \( z \)-independent factor, and

\[
R = \gamma \theta_2(0) \theta_3(0), \quad V = -\gamma^2 \left( \theta_2^4(0) + \theta_3^4(0) \right).
\]  

(16)

Here \( \gamma \) is an arbitrary real parameter but we will see that it can not be put equal to a fixed number like 1 because it is a dynamical variable, as well as the modular parameter \( \tau: \tau = \gamma(t), \tau = \tau(t) \). The reality of the coefficients \( R^2, V \) implies certain restrictions on possible values of \( \tau \). The sufficient condition is that \( \tau \) is purely imaginary, which we assume in what follows. It is convenient to normalize \( u(z) \) by the condition \( u(\infty) = 0 \), with the expansion around \( \infty \) being

\[
u(z, t) = \frac{c_1(t)}{z} + \frac{c_2(t)}{z^2} + \ldots
\]  

(17)

with real coefficients \( c_i \).

The equations (11) and (12) are then represented as a single equation:

\[
\left(z_1^{-1} - z_2^{-1}\right) e^{\nabla(z_1)\nabla(z_2)F} = \frac{\theta_1(u(z_1) - u(z_2))}{\theta_4(u(z_1) - u(z_2))}.
\]  

(18)

The limit \( z_2 \to \infty \) in (18) gives the definition of the function \( u(z) \):

\[
e^{\partial_u \nabla(z)F} = \frac{\theta_1(u(z))}{\theta_4(u(z))}
\]  

(19)

(equivalent to the first formula in (15)). The \( z \to \infty \) limit of equation (19) yields:

\[
e^{F_{\nu_0}} = R = \pi c_1 \theta_2(0) \theta_3(0),
\]  

(20)

hence \( c_1(t) = \gamma(t)/\pi \).

Another useful form of equation (18) can be obtained by passing to logarithms and applying \( \nabla(z_3) \) to the both sides. It is convenient to introduce the function

\[
S(u, \tau) := \log \frac{\theta_1(u, \tau)}{\theta_4(u, \tau)},
\]  

(21)

which has the following quasiperiodicity properties: \( S(u + 1, \tau) = S(u, \tau) + i\pi, S(u + \tau, \tau) = S(u, \tau) \). In what follows we write simply \( S(u) = S(u, \tau) \). In terms of this function, the equation (18) means that \( \nabla(z_3)S(u(z_1) - u(z_2)) = \nabla(z_3)\nabla(z_2)\nabla(z_1)F \) is symmetric under permutations of \( z_1, z_2, z_3 \):

\[
\nabla(z_1)S(u(z_2) - u(z_3)) = \nabla(z_2)S(u(z_1) - u(z_3)) = \nabla(z_3)S(u(z_1) - u(z_2)).
\]  

(22)

In particular, as \( z_3 \to \infty \) we get

\[
\nabla(z_1)S(u(z_2)) = \partial_u S(u(z_1) - u(z_2)).
\]  

(23)
Equation (22) are equivalent to the dDKP hierarchy (18). Thus we may say that the pair \( u(z, t), \tau(t) \) satisfying (22) is a solution to the dDKP hierarchy.

In order to connect this with the algebraic formulation, we note that

\[
\log w(z) = -2S(u(z)), \quad p(z) = c_1 S'(u(z)) \tag{24}
\]

where \( S'(u) \equiv \partial_u S(u) \). See [17] for details.

3. From the elliptic Löwner equation to the dDKP hierarchy

In this section, we prove that a solution of a system of elliptic Löwner equations gives a solution to the dDKP hierarchy (the \( N \)-variable diagonal reduction).

Let \( u = u(z, \{ \lambda_j \}) \) be a function of \( z \) and real variables \( \{ \lambda_j \} = \{ \lambda_1, \ldots, \lambda_N \} \). Consider the system of elliptic Löwner (Goluzin–Komatsu) equations

\[
\frac{\partial u}{\partial \lambda_j} = \frac{1}{4\pi i} \left( -\zeta_1(u + \xi_j) - \zeta_4(u + \xi_j) + \zeta_4(\xi_j) + \zeta_4(\xi_j) \right) \frac{\partial \tau}{\partial \lambda_j}, \tag{25}
\]

where \( \zeta_\alpha(x) = \zeta_\alpha(x, \tau) = \partial_\lambda \log \theta_\alpha(x, \tau) \) are analogues of the Weierstrass’ zeta function, \( \xi_j \) and \( \tau \) are functions of \( \{ \lambda_j \} \): \( \xi_j = \xi_j(\{ \lambda_j \}), \tau = \tau(\{ \lambda_j \}) \). We assume that \( \xi_j \) are real-valued functions.

Let us find \( \partial_u \lambda S(u(z_1) - u(z_2)) \):

\[
\partial_u \lambda S(u_1 - u_2) = S'(u_1 - u_2) \left( \frac{\partial u_1}{\partial \lambda_j} - \frac{\partial u_2}{\partial \lambda_j} \right) + S(u_1 - u_2) \frac{\partial \tau}{\partial \lambda_j},
\]

where we abbreviate \( u_1 \equiv u(z_1) \) and \( S(u) = \partial_u S(u, \tau) \). Plugging here the elliptic Löwner equation (25) and the formula

\[
2\pi i S(u) = S'(u) \zeta_2(u, \tau) + \frac{\pi^2}{2} \theta_4^2(0, \tau) \tag{26}
\]

(see [17] for the proof and [39, 40] for the proofs of similar formulae), we have:

\[
\partial_u \lambda S(u_1 - u_2) = \frac{1}{4\pi i} S'(u_1 - u_2) \left[ -\zeta_1(u_1 + \xi_j) - \zeta_4(u_1 + \xi_j) + \zeta_4(\xi_j) + \zeta_4(\xi_j) \right.
\]

\[
+ 2\zeta_2(u_1 - u_2) + \frac{\pi^2}{2} \theta_4^2(0, \tau) \frac{\partial \tau}{\partial \lambda_j}
\]

\[
= \frac{1}{4\pi i} S'(u_1 + \xi_j) S'(u_2 + \xi_j) \frac{\partial \tau}{\partial \lambda_j}, \tag{27}
\]

where we have used the identity (A15) from [17]. In particular, tending \( z_2 \to \infty \), we get

A sketch of proof is as follows. It follows from (23) that \( \nabla(z_1) S_u(z_2)) = \nabla(z_2) S_u(z_1) \). Therefore, there exists a function \( f = f(t) \) such that \( S_u(z(t)) = \nabla(z)(f) \). Substituting this into (23) and integrating with respect to \( \lambda_0 \), we get

\[
\nabla(z_1) S_u(z_2) = \log(z_1^{-1} - z_2^{-1}) - S_u(z_1) - u(z_2)) = c(t_1, z_1, z_2),
\]

where \( \partial_u F_1 = f, c(t, z_1, z_2) = \sum_{n \geq 0} c_n(t) z_1^{-n} z_2^{-m} \) is the integration constant and \( t' = \{ t_1, t_2, \ldots \} \). Applying \( \nabla(z_1) \) to both sides of this equation and using the symmetry of \( \nabla(z_1) S_u(z_1) - u(z_2)) \) under the permutations of \( z_1, z_2, \text{c} \) (equations (22)), we conclude that \( \nabla(z_1) [c(t', z_1, \text{c})] \) is also symmetric under these permutations. Therefore, there exists a function \( g \) such that \( c(t', z_1, \text{c}) = \nabla(z_1) [\nabla(z_2)] g \). Then \( F = F_1 - g \) satisfies (18).
\[ \partial_\tau S(u(z)) = \frac{1}{4\pi i} S'(\zeta) S'(u(z) + \xi) \frac{\partial \tau}{\partial \zeta}. \] (28)

Using the above functions \( u(z, \{ \lambda_i \} ) \), \( \tau(\{ \lambda_i \} ) \) let us construct a solution \( u(z) \), \( \tau \) to the dDKP hierarchy which depends on times through the \( \lambda_j \)'s: \( u(z, t) = u(z, \{ \lambda_i(t) \} ) \), \( \tau(t) = \tau(\{ \lambda_i(t) \} ) \). This is called \( N \)-variable reduction of the hierarchy. In the case of the reduction equation (23) reads

\[ \sum_{j=1}^{N} \nabla(z_1) \lambda_j \cdot \partial_\lambda_j S(u(z_2)) = \sum_{j=1}^{N} \partial_\tau \lambda_j \cdot \partial_\lambda_j S(u(z_1) - u(z_2)). \]

Plugging here (27) and (28), we have:

\[ \sum_{j=1}^{N} \nabla(z_1) \lambda_j \cdot S'(\zeta) S'(u(z_2) + \xi) \frac{\partial \tau}{\partial \zeta} = \sum_{j=1}^{N} \partial_\tau \lambda_j \cdot S'(u(z_1) + \xi) S'(u(z_2) + \xi) \frac{\partial \tau}{\partial \zeta}. \] (29)

Now we see that if we introduce the dependence of the \( \lambda_j \)'s on \( t \) by means of the equation

\[ \nabla(z) \lambda_j = \frac{S'(u(z) + \xi)}{S'(\zeta)} \frac{\partial \lambda_j}{\partial t_0}. \] (30)

Equation (29) is satisfied identically. It is easy to see that equation (22) are also satisfied. Indeed, in the case of the reduction equation (22) reads

\[ \sum_{j=1}^{N} \nabla(z_3) \lambda_j \cdot \partial_\lambda_j S(u(z_1) - u(z_2)) = \text{(symmetric under permutations of } z_1, z_2, z_3). \]

Plugging here (27), we have:

\[ \sum_{j=1}^{N} \nabla(z_3) \lambda_j \cdot S'(u(z_1) + \xi) S'(u(z_2) + \xi) = \text{(symmetric under permutations of } z_1, z_2, z_3), \] (31)

which is true if \( \nabla(z) \lambda_j \) is given by (30). This means that the functions \( u(z, t) = u(z, \{ \lambda_i(t) \} ) \), \( \tau(t) = \tau(\{ \lambda_i(t) \} ) \) obey the dDKP hierarchy.

Equation (30) contains an infinite system of partial differential equations of hydrodynamic type. To write them out explicitly, we introduce elliptic Faber functions \( \Phi_k(u) \) via the expansion \( S(u(z) + w) = \frac{\Phi_k(w)}{\xi^k} \) or

\[ S'(u(z) + w) = S'(w) + \sum_{k=1}^{\infty} \frac{\gamma-k}{k} \Phi_k'(w) \] (32)

(here \( \Phi_k'(w) = \partial_\xi \Phi_k(w) \)). Then the system (30) reads

\[ \frac{\partial \lambda_j}{\partial t_k} = \phi_{jk}(\{ \lambda_i \}) \frac{\partial \lambda_j}{\partial t_0}, \quad \phi_{jk} = \frac{\Phi_k'(\xi)}{S'(\zeta)}, \] (33)

which is an infinite diagonal system of partial differential equations of hydrodynamic type. The \( \lambda_j \)'s play the role of the Riemann invariants. Note that \( \Phi_k'(w) \) depends on the \( \lambda_j \)'s through \( \tau \) and \( u(z) \). The generating function of \( \phi_{jk}(\{ \lambda_i \} ) \) is obtained from (32):
\[ Q(u(z, \{\lambda_i\}, \xi(\{\lambda_i\}), \tau(\{\lambda_i\})) = 1 + \sum_{k \geq 1} \phi_{jk}(\{\lambda_i\}) \frac{z^{k-1}}{k}, \quad Q(u, \xi, \tau) = \frac{S(u + \xi, \tau)}{S(\xi, \tau)}. \]

It is convenient to put \( \phi_{i,0} = 1 \).

### 4. The Gibbons–Tsarev system

The Gibbons–Tsarev system is the compatibility condition for the system of elliptic Löwner equations (25):

\[ \frac{\partial u}{\partial \xi} = \frac{1}{4\pi} \left( -\zeta_1(u + \xi, \tau) - \zeta_2(u + \xi, \tau) + \zeta_3(u + \xi, \tau) + \zeta_4(u, \xi, \tau) \right) \frac{\partial \tau}{\partial \xi}, \]

\[ (35) \]

Here and below we abbreviate \( \tau' = \frac{\tau}{\xi} \). The compatibility condition is

\[ F_{jk}(u) := \frac{\partial}{\partial \lambda_j} \frac{\partial u}{\partial \lambda_k} - \frac{\partial}{\partial \lambda_k} \frac{\partial u}{\partial \lambda_j} = 0. \]

The left hand side is of the form

\[ F_{jk}(u) = F_{jk}^{(1)} \frac{\partial \xi_j}{\partial \lambda_k} \frac{\partial \tau}{\partial \lambda_k} - F_{jk}^{(1)} \frac{\partial \xi_j}{\partial \lambda_k} \frac{\partial \tau}{\partial \lambda_j} + F_{jk}^{(2)} \frac{\partial^2 \tau}{\partial \lambda_j \partial \lambda_k} + G_{jk} \frac{\partial \tau}{\partial \lambda_j} \frac{\partial \tau}{\partial \lambda_k}. \]

\[ (36) \]

The coefficients are:

\[ F_{jk}^{(1)} = \frac{1}{4\pi i} \left( \varphi_1(u + \xi, \tau') - \varphi_1(\xi, \tau') \right), \]

\[ F_{jk}^{(2)} = \frac{1}{4\pi i} \left( -\zeta_1(u + \xi, \tau') + \zeta_1(\xi, \tau') + \zeta_1(u + \xi, \tau') - \zeta_1(\xi, \tau') \right), \]

\[ G_{jk} = \frac{1}{2(4\pi i)^2} \left( \varphi_1(u + \xi, \tau') - \varphi_1'(u + \xi, \tau') - \varphi_1'(\xi, \tau') + \varphi_1'(\xi, \tau') \right) \]

\[ + \frac{1}{(4\pi i)^2} \left( \zeta_1(u + \xi, \tau') - \zeta_1(u + \xi, \tau') + \zeta_1(\xi, \tau') \right) \varphi_1(u + \xi, \tau') \]

\[ - \frac{1}{(4\pi i)^2} \left( \zeta_1(u + \xi, \tau') - \zeta_1(u + \xi, \tau') + \zeta_1(\xi, \tau') \right) \varphi_1(u + \xi, \tau') \]

\[ + \frac{1}{(4\pi i)^2} \left( -\zeta_1(\xi, \tau') \varphi_1(\xi, \tau') + \zeta_1(\xi, \tau') \varphi_1(\xi, \tau') \right). \]

where \( \varphi_2(x, \tau) = -\partial_\tau \zeta_3(x, \tau), \varphi_3'(x, \tau) = \partial_\tau \varphi_2(x, \tau) \). Here we have used the equation

\[ 4\pi i \frac{\partial \zeta_1(u, \tau')}{\partial \tau} = -\zeta_1(u, \tau') \varphi_1(u, \tau') - \frac{1}{2} \varphi_1(u, \tau') \]

\[ (37) \]

(equivalent to equation (A11) in [17]). Note that:
• $\varphi_1(u, \tau')$ is an elliptic function with periods 1 and $\tau'$. Hence $\varphi'_{1}(u, \tau')$ is also an elliptic function with the same periods.

• $\zeta_1(u + 1, \tau') = \zeta_1(u, \tau')$ and $\zeta_1(u + \tau', \tau') = \zeta_1(u, \tau') - 2\pi i$, which implies that $\zeta_1(u + \xi_k, \tau') = \zeta_1(u + \xi, \tau')$ is an elliptic function of $u$ with periods 1 and $\tau'$.

Therefore, $F_{jk}^{(1)}, F_{jk}^{(2)}, G_{jk}$ and consequently $F_{jk}(u)$ are all elliptic functions of $u$ with periods 1 and $\tau'$. Possible poles in the parallelogram spanned by 1 and $\tau'$ are at $u = -\xi_k$ and $u = -\xi_j$.

The formulae of expansions around $u = 0$ are:

$$\zeta_1(u, \tau') = \frac{1}{u} + O(u), \quad \varphi_1(u, \tau') = \frac{1}{u^2} + O(1), \quad \varphi_1'(u, \tau') = -\frac{2}{u^3} + O(u).$$

(38)

Using them, we have the expansions near $u = -\xi_k$:

$$F_{jk}^{(1)} = \frac{1}{4\pi i} \left( \frac{1}{(u + \xi_k)^2} + O(1) \right),$$

(39)

$$F_{jk}^{(2)} = \frac{1}{4\pi i} \left( -\frac{1}{u + \xi_k} + O(1) \right),$$

(40)

$$G_{jk} = \frac{1}{(4\pi i)^2} \left( \frac{\zeta_1(\xi_k, \tau') - \zeta_1(-\xi_k + \xi_j, \tau')}{(u + \xi_k)^2} + \frac{2\varphi_1(\xi_j - \xi_k, \tau')}{u + \xi_k} + O(1) \right).$$

(41)

Substituting them into (36), we can expand $F_{jk}(u)$ around $u = -\xi_k$ as

$$F_{jk}(u) = \frac{f_2}{(u + \xi_k)^2} + \frac{f_1}{u + \xi_k} + O(1),$$

where

$$f_2 = \frac{1}{4\pi i} \frac{\partial \tau}{\partial \xi_k} \left( \frac{\partial \xi_k}{\partial \lambda_j} + \frac{1}{4\pi i} \left( \zeta_1(\xi_j, \tau') - \zeta_1(-\xi_k + \xi_j, \tau') \right) \frac{\partial \tau}{\partial \lambda_j} \right),$$

$$f_1 = \frac{1}{4\pi i} \left( \frac{2}{4\pi i} \frac{\partial \tau}{\partial \lambda_j} \frac{\partial \tau}{\partial \lambda_k} \varphi_1(\xi_j - \xi_k, \tau') - \frac{\partial^2 \tau}{\partial \lambda_k \partial \lambda_j} \right).$$

Therefore, if

$$\frac{\partial \xi_k}{\partial \lambda_j} = \frac{1}{4\pi i} \left( \zeta_1(-\xi_k + \xi_j, \tau') - \zeta_1(\xi_j, \tau') \right) \frac{\partial \tau}{\partial \lambda_j},$$

(42)

$$\frac{\partial^2 \tau}{\partial \lambda_k \partial \lambda_j} = \frac{1}{2\pi i} \varphi_1(\xi_k - \xi_j, \tau') \frac{\partial \tau}{\partial \lambda_k} \frac{\partial \tau}{\partial \lambda_j}$$

(43)

for all $j = 1, \ldots, N$, $j \neq k$, then $F_{jk}(u)$ is regular at $u = -\xi_k$. Similarly, if these equations with $j$ and $k$ exchanged are satisfied for all $k = 1, \ldots, N$, $k \neq j$, then $F_{jk}(u)$ is regular at $u = -\xi_j$.

Assume that equations (42) and (43) hold for all $j, k = 1, \ldots, N$, $j \neq k$. Then $F_{jk}(u)$ is a regular elliptic function, which is nothing but a constant. It is easy to see that $F_{jk}(0) = 0$. So under the conditions (42) and (43), $F_{jk}(u) = 0$ which means that the elliptic Löwner system (25) is compatible.
The system of equations (42) and (43) is the elliptic analogue of the famous Gibbons–Tsarev system [28, 29]. They already appeared in the literature [41–43].

5. Generalized hodograph method

In the previous sections we have reduced the dDKP hierarchy to the system of elliptic Löwner equations and the auxiliary equations

\[
\frac{\partial \lambda_i(t)}{\partial n} = \phi_{i,n}(\{\lambda_j(t)\}) \frac{\partial \lambda_i(t)}{\partial t_0},
\]

where \(\phi_{i,n}\) are as in (33). We are going to show that this system of the first order partial differential equations is consistent and can be solved by Tsarev’s generalized hodograph method [35].

As is easy to see, the compatibility condition of the system (44) is

\[
\frac{\partial \lambda_j}{\partial \phi_{i,n}} \phi_{i,n} - \phi_{i,n} = \frac{\partial \lambda_j}{\partial \phi_{i,n'}} \phi_{i,n'} - \phi_{i,n'}
\]

for all \(i \neq j, n, n'\).

In other words, we should show that

\[
\Gamma_{ij} := \frac{\partial \lambda_j}{\partial \phi_{i,n}} \phi_{i,n} - \phi_{i,n}
\]

does not depend on \(n\), i.e. (45) holds for all \(n\) simultaneously. This is equivalent to the statement that the ratio

\[
\frac{\partial \lambda_j Q(u(z), \xi_i, \tau)}{Q(u(z), \xi_j, \tau) - Q(u(z), \xi_i, \tau)},
\]

where \(Q\) is the generating function (34), is independent of \(z\). The independence of (46) of \(z\) is proved in appendix B by a direct calculation, and the coefficients \(\Gamma_{ij}\) are found. The result is

\[
\Gamma_{ij} = -\frac{1}{4\pi i} \frac{S'(\xi)}{S(\xi)} \frac{\tau}{S'(\xi)} S''(\xi_i - \xi_j) \frac{\partial \tau}{\partial \lambda_j}.
\]

Proposition 5.1. Consider the following system for \(R_i = R_i(\{\lambda_j\}), i = 1, \ldots, N:\)

\[
\frac{\partial R_i}{\partial \lambda_j} = \Gamma_{ij}(R_j - R_i), \quad i, j = 1, \ldots, N, \quad i \neq j,
\]

where \(\Gamma_{ij}\) is defined as in (47) (when \(N = 1\), the condition (48) is void). Then the following holds.

(i) The system (48) is compatible in the sense of [35].

(ii) Assume that \(R_i\) satisfy the system (48). If \(\lambda_i(t)\) is defined implicitly by the hodograph relation

\[
t_0 + \sum_{n \neq 1} \phi_{i,n}(\{\lambda_j\}) t_n = R_i(\{\lambda_j\}),
\]

then \(\lambda_i(t)\) satisfy (44).
For the proof of statement (i) of the proposition we note that $\Gamma_{ij}$ (47) can be expressed as logarithmic derivative of a function as follows:

$$\Gamma_{ij} = \frac{1}{2} \frac{\partial}{\partial \lambda_j} \log g_i,$$

where

$$g_i = \frac{1}{4\pi} (S'(\xi)) \frac{\partial r}{\partial \lambda_j}.$$

The proof of (50) is given in appendix C. It then follows that

$$\frac{\partial \Gamma_{ij}}{\partial \lambda_k} = \frac{\partial \Gamma_{ik}}{\partial \lambda_j}, \quad i \neq j \neq k,$$

which is the Tsarev compatibility condition. This means that the system (44) is semi-Hamiltonian. The main geometric object associated with a semi-Hamiltonian system is a diagonal metric. The quantities $g_i = g_{ii}$ are components of this metric while $\Gamma_{ij} = \Gamma_{ij}$ are the corresponding Christoffel symbols.

In fact the compatibility conditions of the system (48) are (52) together with

$$\frac{\partial \Gamma_{ij}}{\partial \lambda_k} = \Gamma_{ij} \Gamma_{kj} - \Gamma_{ik} \Gamma_{kj}, \quad i \neq j \neq k.$$

As one can see, (52) follows from (53) because the right hand side of (53) is explicitly symmetric under the permutation of $j$ and $k$. As is shown in [35], (53), in its turn, follows from the definition (45) and the condition (52). In appendix D we give an independent direct proof of (53) starting from the explicit form of the $\Gamma_{ij}$.

The proof of statement (ii) of the proposition is almost the same as that of Theorem 10 of Tsarev’s paper [35]. The difference from [35] is that the number of independent variables is infinite in our case. In spite of this difference, Tsarev’s method does work. For completeness, we give the proof here. (The following argument is the same as in [34].) By differentiating the relation (49) by $t_0$ and $t_k$ we obtain

$$\sum_{j=1}^{N} M_{ij} \frac{\partial \lambda_j}{\partial t_0} = 1, \quad \sum_{j=1}^{N} M_{ij} \frac{\partial \lambda_j}{\partial t_k} = \phi_{ij,k},$$

where

$$M_{ij} := \frac{\partial R_i}{\partial \lambda_j} - \sum_{n \geq 1} \frac{\partial \phi_{i,n}}{\partial \lambda_j} t_n.$$

Because of (45) and (48) and the hodograph relation (49) the above expression becomes

$$M_{ij} = \Gamma_{ij}(R_j - R_i) - \sum_{n \geq 1} \Gamma_{ij}(\phi_{i,n} - \phi_{j,n}) t_n$$

$$= \Gamma_{ij} \left((R_j - \sum_{n \geq 1} \phi_{j,n}) - (R_i - \sum_{n \geq 1} \phi_{i,n}) \right) = 0,$$

if $i \neq j$. (The fact that the coefficients $\Gamma_{ij}$ defined by (45) do not depend on $n$ is essential here.) Therefore, (54) reduces to

$$M_{ii} \frac{\partial \lambda_i}{\partial t_0} = 1, \quad M_{ii} \frac{\partial \lambda_i}{\partial t_k} = \phi_{i,k}.$$

This proves (44).
6. The metric coefficients \( g_i \)

**Proposition 6.1.** The metric \( g_i \) is of Egorov type, i.e. it holds:

\[
\frac{\partial g_i}{\partial \lambda_k} = \frac{\partial g_k}{\partial \lambda_i}. \tag{56}
\]

The proof is very simple:

\[
\frac{\partial g_k}{\partial \lambda_i} = g_k \frac{\partial \log g_k}{\partial \lambda_i} = 2g_k \Gamma_{ki} = -2 \left( \frac{4\pi i}{2} \right)^2 S'(\xi_i) S'(\xi_k) S''(\xi_i - \xi_k) \frac{\partial \tau}{\partial \lambda_i} \frac{\partial \tau}{\partial \lambda_k}
\]

which is explicitly symmetric under the permutation of \( i \) and \( k \). Here we use (47) and (51).

The relations (56) imply that the quantities \( g_i \) have a potential function \( G \) such that

\[
g_i = \frac{\partial G}{\partial \lambda_i}. \tag{57}
\]

Let us show that

\[
G = \log R = \frac{\partial^2 F}{\partial \tau_0^2}, \tag{58}
\]

where \( F \) is the tau-function (free energy) of the dDKP hierarchy (see (20)). The starting point is the system of the elliptic Löwner equations

\[
4\pi i \frac{\partial}{\partial \lambda_j} u(z) = \left( -\zeta_1(u(z) + \xi_j, \tau') + \zeta_1(\xi_j, \tau') \right) \frac{\partial \tau}{\partial \lambda_j}, \tag{59}
\]

where \( u(z) \) has the expansion \( u(z) = \frac{c_1}{z} + \frac{c_2}{z^2} + \ldots \) as \( z \to \infty \). Expanding both sides of (59) as \( z \to \infty \) and equating the coefficients in front of \( z^{-1} \), we have

\[
4\pi i \frac{\partial}{\partial \lambda_j} \log c_1 = \varphi_1(\xi_j, \tau') \frac{\partial \tau}{\partial \lambda_j}. \tag{60}
\]

As is shown in [17] (see also equation (20) in the present paper), \( \log(\pi c_1) = \log R - \log \frac{\theta_2(0, \tau')}{2} \).

Therefore,

\[
\frac{\partial}{\partial \lambda_j} \log c_1 = \frac{\partial}{\partial \lambda_j} \log R - 2 \frac{\partial}{\partial \lambda_j} \log \theta_2(0, \tau')
= \frac{\partial}{\partial \lambda_j} \log R - 2 \frac{\partial}{\partial \tau'} \log \theta_2(0, \tau') \frac{\partial \tau'}{\partial \lambda_j}
= \frac{\partial}{\partial \lambda_j} \log R - \frac{\partial}{\partial \tau'} \log \theta_2(0, \tau') \frac{\partial \tau}{\partial \lambda_j}.
\]

From the heat equation \( 4\pi i \frac{\partial}{\partial \tau} \theta_a(x, \tau) = \theta''_a(x, \tau) \) it follows that

\[
4\pi i \frac{\partial}{\partial \tau} \log \theta_2(0, \tau') = -\varphi_2(0, \tau').
\]

Plugging this into (60), we have

\[
4\pi i \frac{\partial}{\partial \lambda_j} \log R = \left( \varphi_1(\xi_j, \tau') - \varphi_2(0, \tau') \right) \frac{\partial \tau}{\partial \lambda_j} = (S'(\xi_j))^2 \frac{\partial \tau}{\partial \lambda_j} = 4\pi i g_i
\]

(see (A.16)).
7. Conserved quantities

As is shown in [35], densities $P$ of the conserved quantities $I = \int P dt_0$ for any semi-hamiltonian system $\partial_0 \lambda_j = \phi_{j,k} \partial_{\lambda_k} \lambda_j$ satisfy the linear differential equation

$$\frac{\partial^2 P}{\partial \lambda_i \partial \lambda_j} = \Gamma_{ij} \frac{\partial P}{\partial \lambda_i} + \Gamma_{ji} \frac{\partial P}{\partial \lambda_j} \quad (i \neq j) \tag{61}$$

and any solution to this equation gives a conserved quantity. Indeed, substituting (45) for $\Gamma_{ij}$, we have

$$\partial_{\lambda_i} \partial_{\lambda_j} P = \frac{\partial \lambda_j}{\partial \lambda_i} \frac{\partial P}{\partial \lambda_j} + \frac{\partial \lambda_i}{\partial \lambda_j} \frac{\partial P}{\partial \lambda_i}$$

which is equivalent to

$$\partial_{\lambda_i} \left( \frac{\partial \lambda_j}{\partial \lambda_i} P \right) = \partial_{\lambda_i} \left( \frac{\partial \lambda_j}{\partial \lambda_i} P \right). \tag{62}$$

This means that there exists a function $A_n$ such that $\partial_{\lambda_i} P \phi_{i,n} = \partial_{\lambda_i} A_n$. Then

$$\frac{\partial P}{\partial t_n} = \sum_{i=1}^N \partial_{\lambda_i} P \partial_{\lambda_i} \lambda_i = \sum_{i=1}^N \partial_{\lambda_i} P \phi_{i,n} \partial_{\lambda_i} \lambda_i = \sum_{i=1}^N \partial_{\lambda_i} A_n \partial_{\lambda_i} \lambda_i = \frac{\partial A_n}{\partial t_0}$$

which means that $P$ is indeed the density of a conserved quantity.

In appendix E we prove that the function $S(u(z))$ satisfies equation (61):

$$\frac{\partial^2 S(u(z))}{\partial \lambda_i \partial \lambda_j} = \Gamma_{ij} \frac{\partial S(u(z))}{\partial \lambda_i} + \Gamma_{ji} \frac{\partial S(u(z))}{\partial \lambda_j} \tag{63}$$

provided $u(z)$ obeys the elliptic Löwner equations. Therefore, $S(u(z))$ is the generating function for densities of conserved quantities. According to the definition (19) of the function $u(z)$,

$$S(u(z)) = -\log z + F_{00} + \sum_{n \geq 1} \frac{z^{-n}}{n} F_{0n}, \tag{64}$$

so the densities are $F_{0n} \quad n \geq 0$. (The fact that densities of conserved quantities are expressed through second order logarithmic derivatives of tau-function is common for integrable hierarchies, see, e.g. [44, 45].) According to (28) we have

$$\partial_{\lambda_j} S(u(z)) = g_j \left( \frac{S'(u(z)) + \xi_j}{S'(\xi_j)} \right) = g_j \left( 1 + \sum_{n \geq 1} \frac{\phi_{j,n} z^{-n}}{n} \right).$$

Comparing with the $\lambda_j$-derivative of (64), we conclude that

$$g_j \phi_{j,n} = \partial_{\lambda_j} F_{0n}, \quad n \geq 0, \tag{65}$$

which generalizes (57) and (58) (obtained at $n = 0$).

8. Conclusion

We have found sufficient conditions for $N$-variable diagonal reduction of the dDKP hierarchy (the dispersionless limit of the Pfaff lattice) in the elliptic parametrization. The reduction is given by $N$ elliptic Löwner equation (25) for a function $u(z, \lambda_1, \ldots, \lambda_N)$ supplemented by a
diagonal system of hydrodynamic type (33) for the variables $\lambda_j, j = 1, \ldots, N$. We have derived compatibility conditions for the elliptic Löwner equations which are elliptic analogues of the Gibbons–Tsarev equations and have proved solvability of the hydrodynamic type system by means of the generalized hodograph method. The associated diagonal metric is proved to be of the Egorov type.

Acknowledgments

The work of V A was supported in part by the RFBR grant: 16-01-00562. The work of T T was partly prepared within the framework of the Academic Fund Program at the National Research University Higher School of Economics (HSE) in 2015–2016, grant: 15-01-0102) and supported within the framework of a subsidy granted to the HSE by the Government of the Russian Federation for the implementation of the Global Competitiveness Program. The work of A Z was supported by RSF grant: 16-11-10160.

Appendix A. Necessary functions and identities

The Jacobi’s theta-functions $\theta_a(u) = \theta_a(u, \tau), a = 1, 2, 3, 4$, are defined by the formulas

\[ \theta_1(u) = -\sum_{k \in \mathbb{Z}} \exp \left( \pi i \tau (k + \frac{1}{2})^2 + 2\pi i (u + \frac{1}{2})(k + \frac{1}{2}) \right), \]

\[ \theta_2(u) = \sum_{k \in \mathbb{Z}} \exp \left( \pi i \tau (k + \frac{1}{2})^2 + 2\pi i k(k + \frac{1}{2}) \right), \]

\[ \theta_3(u) = \sum_{k \in \mathbb{Z}} \exp \left( \pi i k^2 + 2\pi i u \right), \]

\[ \theta_4(u) = \sum_{k \in \mathbb{Z}} \exp \left( \pi i k^2 + 2\pi i (u + \frac{1}{2})k \right), \]  

(A.1)

where $\tau$ is a complex parameter (the modular parameter) such that $\text{Im} \tau > 0$. The function $\theta_1(u)$ is odd, the other three functions are even. The infinite product representation for the $\theta_1(u)$ reads:

\[ \theta_1(u) = i \exp \left( \frac{\pi \tau}{4} - i\pi u \right) \prod_{k=1}^{\infty} \left( 1 - e^{2\pi i \tau} \right) \left( 1 - e^{2\pi i (k+1-\tau+u)} \right) \left( 1 - e^{2\pi i (k-\tau-u)} \right). \]

(A.2)

We also mention the identity

\[ \theta_1'(0) = \pi \theta_2(0) \theta_3(0) \theta_4(0). \]  

(A.3)

Many useful identities for the theta functions can be found in [46].

All formulas for derivatives of elliptic functions with respect to the modular parameter follow from the ‘heat equation’ satisfied by the theta-functions:

\[ 4\pi i \partial_\tau \theta_a(u) = \partial_u^2 \theta_a(u). \]  

(A.4)

In the main text we use the functions

\[ \zeta_a(x, \tau) = \frac{\partial}{\partial x} \log \theta_a(x, \tau), \quad \phi_a(x, \tau) = -\frac{\partial}{\partial x} \zeta_a(x, \tau), \quad a = 1, 2, 3, 4. \]
Obviously, \( \zeta_a \) are odd functions. In particular, \( \zeta_4(x, \tau) = \frac{1}{x} + O(x) \) as \( x \to 0 \) and \( \zeta_a(0, \tau) = 0 \) for \( a = 2, 3, 4 \).

Let us introduce the function

\[
S(x) = \log \frac{\theta_2(x, \tau)}{\theta_q(x, \tau)}.
\]

We denote \( \partial_\tau S(x) = S'(x) \), \( \partial_\tau^2 S(x) = S''(x) \), \( \partial_x S(x) = \tilde{S}(x) \). One can prove the following formulae (here and below \( \tau' \equiv \frac{\tau}{x} \)):

\[
S'(x) = \pi \theta_2^2(0, \tau) \frac{\theta_1(x, \tau) \theta_3(x, \tau)}{\theta_2(x, \tau)^2},
\]

\[
S''(x) = -\pi^2 \theta_2^2(0, \tau) \theta_3^2(0, \tau) \theta_4^2(0, \tau) \frac{\theta_1(2x, \tau)}{\theta_2(x, \tau) \theta_2^2(x, \tau)}.
\]

\[
2\pi i \tilde{S}(x) = S'(x) \zeta_2(x, \tau) + \frac{\pi^2}{2} \theta_2^2(0, \tau).
\]

\[
2\pi i \tilde{S}'(x) = S''(x) \zeta_2(x, \tau) - S'(x) \varphi_2(x, \tau).
\]

It is clear from (A.6) and (A.7) that \( S'(x + 1) = S'(x), S'(x + \tau') = -S'(x), S''(x + 1) = S''(x), S''(x + \tau') = -S''(x) \). Note that

\[
S'(x) S'(x + \frac{1}{2}) = -\pi^2 \theta_2^2(0, \tau).
\]

As \( x \to 0 \), we have:

\[
S'(x) = \frac{1}{x} + O(x), \quad S''(x) = -\frac{1}{x^2} + O(1).
\]

We mention the identities

\[
S'(x) S''(x) = \frac{1}{2} \varphi_2(x, \tau'),
\]

\[
S'(x) = 2 \zeta_1(x, \tau) - \zeta_1(x, \tau').
\]

It immediately follows from here that

\[
2 \zeta_2(x, \tau) - \zeta_2(x, \tau') = S'(x + \frac{1}{2}),
\]

\[
2 \varphi_2(x, \tau) - \varphi_2(x, \tau') = -S''(x + \frac{1}{2}).
\]

We also need the standard identity

\[
\varphi_2(x, \tau) - \varphi_2(y, \tau) = \frac{(\theta_1(0, \tau))^2 \theta_1(x - y, \tau) \theta_1(x + y, \tau)}{\theta_2^2(x, \tau) \theta_2^2(y, \tau)}.
\]
A particular case is
\[ \varphi_1(x, \tau') - \varphi_2(0, \tau') = (S'(x))^2. \tag{A.16} \]

**Appendix B. The coefficients \( \Gamma_{ij} \)**

Here we show that
\[ \Gamma_{ij} := \frac{\partial_{\lambda_i} \phi_{i,n}}{\phi_{j,n} - \phi_{i,n}} \tag{B.1} \]
does not depend on \( n \), i.e. holds for all \( n \) simultaneously, and find the coefficients \( \Gamma_{ij} \). Passing to the generating function of \( \phi_{i,n} \),
\[ Q(u(z), \xi_i) = 1 + \sum_{n \geq 1} \phi_{i,n} \frac{z^{-n}}{n} = \frac{S'(u(z) + \xi_i)}{S'(\xi_i)}, \]
we can reformulate the statement as \( z \)-independence of the ratio \( \frac{\partial_{\lambda_i} Q(u(z), \xi_i)}{Q(u(z), \xi_i) - Q(u(z), \xi_j)} \). The coefficients \( \Gamma_{ij} \) are then found as
\[ \frac{\partial_{\lambda_i} Q(u(z), \xi_i)}{Q(u(z), \xi_i) - Q(u(z), \xi_j)} = \Gamma_{ij}. \tag{B.2} \]

**B.1. The denominator \( Q(u(x), \xi_j) - Q(u(x), \xi_i) \)**

The denominator of (B.2) is
\[ Q(u(z), \xi_j) - Q(u(z), \xi_i) = \frac{S'(u + \xi_j)S'(\xi_j) - S'(u + \xi_i)S'(\xi_i)}{S'(\xi_j)S'(\xi_i)}. \tag{B.3} \]
the numerator of which is expressed through theta functions thanks to (A.6) as follows:
\[ S'(u + \xi_j)S'(\xi_j) - S'(u + \xi_i)S'(\xi_i) = \pi^2 \theta^2_n(0, \tau) \theta_2(u + \xi_j) \theta_1(u + \xi_j) \theta_1(\xi_j) - \theta_1(u + \xi_j) \theta_1(\xi_j) \theta_2(u + \xi_i) \theta_1(\xi_i). \]
Here and below in this appendix \( \theta_n(x) = \theta_n(x, \frac{\tau}{\tau}) \). By subtracting (R9) in p.20 of [47] from (R8), we have a formula
\[ \theta_2(x) \theta_2(y) \theta_1(u) \theta_1(v) - \theta_1(x) \theta_1(y) \theta_2(u) \theta_2(v) = \theta_2(x_1) \theta_2(y_1) \theta_1(u_1) \theta_1(v_1) - \theta_1(x_1) \theta_1(y_1) \theta_2(u_1) \theta_2(v_1). \tag{B.4} \]
By setting \( x \mapsto u + \xi_j, \ y \mapsto \xi_i, \ u \mapsto u + \xi_i, \ v \mapsto \xi_j \), the arguments in the right hand side become
\[ x_1 = \frac{1}{2} (x + y + u + v) \mapsto u + \xi_j + \xi_i, \]
\[ y_1 = \frac{1}{2} (x + y - u - v) \mapsto 0, \]
\[ u_1 = \frac{1}{2} (x - y + u - v) \mapsto u, \]
\[ v_1 = \frac{1}{2} (x - y - u + v) \mapsto \xi_j - \xi_i. \]
Substituting (B.4) with this specialization, we have:
\[ S'(u + \xi_1)S'(\xi_1) - S'(u + \xi_2)S'(\xi_2) = \pi^2 \theta_2^4(0, \tau) \frac{\theta_2(u + \xi_1 + \xi_2)\theta_1(0)\theta_1(u)\theta_1(\xi_1 - \xi_2)}{\theta_1(u + \xi_1)\theta_2(\xi_1)\theta_1(u + \xi_2)\theta_1(\xi_2)}. \]

Thus the expression (B.3) is factorized as

\[ Q(u, \xi_j) - Q(u, \xi_i) = \frac{\theta_2(u + \xi_1 + \xi_2)\theta_1(u)\theta_1(\xi_1 - \xi_2)}{\theta_1(u + \xi_1)\theta_2(\xi_1)\theta_1(u + \xi_2)\theta_2(\xi_2)} \]

because of (A.6). Passing to \( \tau \) instead of \( \xi \) in \( u \)-dependent factors with the help of the formulae

\[ \theta_1(x)\theta_2(0) = 2\theta_1(x, \tau)\theta_3(x, \tau), \quad \theta_2(x)\theta_2(0) = 2\theta_2(x, \tau)\theta_3(x, \tau), \]

we finally obtain

\[ Q(u, \xi_j) - Q(u, \xi_i) = \frac{\theta_2(u + \xi_1 + \xi_2, \tau)\theta_3(u + \xi_1, \tau)\theta_4(u, \tau)}{\theta_1(u + \xi_1, \tau)\theta_3(u + \xi_2, \tau)\theta_4(u + \xi_2, \tau)} \frac{\theta_1(\xi_1 - \xi_2)\theta_2(0)}{\theta_2(\xi_1)\theta_2(\xi_2)}. \]  

**B.2. The numerator \( \partial_\lambda Q(u(z), \xi_i) \)**

We compute the derivative honestly:

\[ \frac{\partial}{\partial \lambda_j} \frac{S'(u + \xi_1)}{S'(\xi_1)} = \frac{1}{(S'(\xi_1))^2} \left( \frac{\partial S'(u + \xi_1)}{\partial \lambda_j} S'(\xi_1) - \frac{\partial S'(\xi_1)}{\partial \lambda_j} S'(u + \xi_1) \right). \]  

(B.7)

Next we have to:

- use the chain rule and rewrite the expression using \( S'' \), \( \dot{S}' \), \( \frac{\partial u}{\partial \lambda}, \frac{\partial \xi_i}{\partial \lambda}, \frac{\partial \tau}{\partial \lambda}; \)
- use the elliptic Löwner equations (25) and the Gibbons–Tsarev system (42) and (43) to represent the \( \lambda_j \)-derivatives of \( u, \xi_i \) and \( \tau \);
- simplify the derivatives of \( S \) in terms of theta functions.

The derivatives in the right hand side of (B.7) are:

\[ \frac{\partial S'(u + \xi_1)}{\partial \lambda_j} = S''(u + \xi_1) \left( \frac{\partial u}{\partial \lambda_j} + \frac{\partial \xi_i}{\partial \lambda_j} \right) + \dot{S}'(u + \xi_1) \frac{\partial \tau}{\partial \lambda_j}, \]

\[ \frac{\partial S'(\xi_1)}{\partial \lambda_j} = S''(\xi_1) \frac{\partial \xi_i}{\partial \lambda_j} + \dot{S}'(\xi_1) \frac{\partial \tau}{\partial \lambda_j}. \]

The elliptic Löwner equation (25) and the Gibbons–Tsarev equation (42) imply

\[ \frac{\partial u}{\partial \lambda_j} + \frac{\partial \xi_i}{\partial \lambda_j} = \frac{1}{4\pi i} \left( -\xi_1(u + \xi_1) + \zeta_1(\xi_1 - \xi_2) \right) \frac{\partial \tau}{\partial \lambda_j}, \]

where \( \zeta_1(x) = \zeta_1(x, \frac{\tau}{2}) \). Substituting (A.9) for \( \dot{S}' \), we have:

\[ 4\pi i (S'(\xi_1))^2 \frac{\partial S'(u + \xi_1)}{\partial \lambda_j} \]

\[ = \left[ S''(u + \xi_1)S'(\xi_1) \left( -\xi_1(u + \xi_1) + \zeta_1(\xi_1 - \xi_2) + 2\zeta_2(u + \xi_1, \tau) \right) - S''(\xi_1)S'(u + \xi_1) \left( -\xi_1(\xi_1 - \xi_2) + 2\zeta_2(\xi_1, \tau) \right) + 2S'(u + \xi_1)S'(\xi_1) \left( -\xi_1(\xi_1 - \xi_2) + 2\zeta_2(\xi_1, \tau) \right) \right] \frac{\partial \tau}{\partial \lambda_j}. \]  

(B.8)
To proceed further, we need the identity
\[ -\zeta_1(x_1) + \zeta_1(x_2) + 2\zeta_2(x_1 - x_2, \tau) = \pi\theta_2(0, \tau)\theta_3(0, \tau)\theta_4^2(0, \tau) \frac{\partial}{\partial \tau} \frac{\partial}{\partial \xi_1} S' + \xi_1, \tau) = \pi\theta_2(0, \tau)\theta_3(0, \tau)\theta_4^2(0, \tau) f(u), \]  

where
\[ f(u) = S''(u + \xi_1)S'(\xi_1) = \frac{\theta_1(u + \xi_1, \tau)\theta_3(u + \xi_1, \tau)\theta_2(u + 2\xi_1 - \xi_2, \tau)}{\theta_1(u + \xi_1, \tau)\theta_4(u + \xi_1, \tau)\theta_1(\xi_2 - \xi_1, \tau)\theta_4(\xi_2 - \xi_1, \tau)\theta_2(u + \xi_1, \tau)} \]

\[ = -S''(u + \xi_1)S'(\xi_1) = \frac{\theta_1(\xi_2 - \xi_1, \tau)\theta_3(\xi_2 - \xi_1, \tau)\theta_2(2\xi_1 - \xi_2, \tau)}{\theta_1(\xi_2 - \xi_1, \tau)\theta_4(\xi_2 - \xi_1, \tau)\theta_2(\xi_2 - \xi_1, \tau)} \]

\[ = -2\pi S''(u + \xi_1)S'(\xi_1) = \frac{\theta_1(u, \tau)\theta_3(u + 2\xi_1, \tau)}{\theta_2(\xi_2, \tau)\theta_2(u + \xi_1, \tau)}. \]

Plugging here \( S', S'' \) from (A.6) and (A.7), we get:
\[ \frac{4\pi i (S'(\xi_1))^2}{\partial \tau / \partial \xi_1} \frac{\partial}{\partial \xi_1} S'(\xi_1) = \pi\theta_2^2(0, \tau)\theta_3^2(0, \tau) g(u), \]  

where the function \( g(u) \) reads
\[ g(u) = -\frac{\theta_4(2u + 2\xi_1, \tau)\theta_2(2u + 2\xi_1 - \xi_2, \tau)\theta_3(u + \xi_1, \tau)\theta_2(u + \xi_1, \tau)\theta_2(u + \xi_1, \tau)\theta_1(u + \xi_1, \tau)}{\theta_1(u + \xi_1, \tau)\theta_4(u + \xi_1, \tau)\theta_1(u + \xi_1, \tau)\theta_4(u + \xi_1, \tau)\theta_1(u + \xi_1, \tau)\theta_4(u + \xi_1, \tau)} \]

\[ + \frac{\theta_2(2u + 2\xi_1, \tau)\theta_3(u + \xi_1, \tau)\theta_4(2\xi_1 - \xi_2, \tau)\theta_2(2\xi_1 - \xi_2, \tau)\theta_1(\xi_2 - \xi_1, \tau)\theta_4(\xi_2 - \xi_1, \tau)\theta_2(2\xi_1 - \xi_2, \tau)}{\theta_1(u + \xi_1, \tau)\theta_4(u + \xi_1, \tau)\theta_1(u + \xi_1, \tau)\theta_4(u + \xi_1, \tau)\theta_1(u + \xi_1, \tau)\theta_4(u + \xi_1, \tau)} \]

\[ - \frac{\theta_1(u, \tau)\theta_1(u + 2\xi_1, \tau)\theta_2(u + \xi_1, \tau)\theta_2(u + \xi_1, \tau)\theta_3(u + \xi_1, \tau)\theta_3(u + \xi_1, \tau)}{\theta_1(u + \xi_1, \tau)\theta_4(u + \xi_1, \tau)\theta_1(u + \xi_1, \tau)\theta_4(u + \xi_1, \tau)\theta_1(u + \xi_1, \tau)\theta_4(u + \xi_1, \tau)}. \]

It is easy to see that it is an elliptic function of \( u \) with periods 1, \( \tau \) and four simple poles in the fundamental parallelogram at the points \( u = -\xi_1, u = -\xi_2 + \frac{\tau}{2}, u = -\xi_1 + \frac{\tau}{2}, u = -\xi_1 + \xi_2 \). A possible pole at \( u = -\xi_1 + \frac{\tau}{2} \) cancels thanks to the identity
\[ \theta_1(2x, \tau)\theta_2(0, \tau)\theta_3(0, \tau)\theta_4(0, \tau) = 2\theta_1(x, \tau)\theta_2(x, \tau)\theta_3(x, \tau)\theta_4(x, \tau). \]  

One can see that \( g(u) \) has zeros at the four points \( u = 0, u = \frac{1}{2} - \xi_1 - \xi_2, u = -\frac{\tau}{4} + \xi_1 - \xi_2 \) and \( u = \frac{1-\tau}{4} - \xi_1 - \xi_2 \). Therefore, \( g(u) \) can be represented in the form
\[
g(u) = \frac{\theta_1(u, \tau)\theta_3(u, \tau)\theta_2(u + \xi_1 + \xi_j, \tau)\theta_3(u + \xi_1, \tau)}{\theta_1(u + \xi_1, \tau)\theta_3(u + \xi_1, \tau)\theta_3(u + \xi_i, \tau)\theta_3(u + \xi_j, \tau)}
\]

with some constant \(C\). The constant can be easily found by tending \(u \to -\xi_j\):
\[
C = \theta_2(0, \tau)\theta_3(0, \tau)\theta_4(0, \tau)\theta_4(2\xi_i - 2\xi_j, \tau).
\]

### B.3. Calculation of \(\Gamma_j\)

Dividing (B.9) by (B.6) we see that the \(u\)-dependent factors cancel (and, therefore, the \(z\)-dependence disappears) and we are left with
\[
\frac{4\pi(S'({\xi}))^2}{\partial \tau/\partial \lambda_j} \Gamma_j = \frac{8\pi^2 S_2^3(0, \tau)\theta_2^2(0, \tau)}{\theta_2^2(0, \frac{\xi}{2})} \frac{\theta_4(2\xi - 2\xi_j, \tau)\theta_2(\frac{\xi}{2})\theta_2(\frac{\xi}{2})\theta_1(\xi_j - \frac{\xi}{2})\theta_1(\xi_j - \frac{\xi}{2})}{\theta_1(\frac{\xi}{2})\theta_1(\xi_j - \frac{\xi}{2})\theta_1(\xi_j - \frac{\xi}{2})}.
\]

Using (A.6) and (A.7) and some identities for theta-constants, we obtain the final result for \(\Gamma_j\):
\[
\Gamma_j = -\frac{1}{4\pi} S'({\xi}) S''(\xi_j - \xi_j) \frac{\partial \tau}{\partial \lambda_j}. \tag{B.11}
\]

### Appendix C. Proof of \(\Gamma_j = \frac{1}{2} \partial_{\lambda_j} \log g_i\)

Here we prove that
\[
\Gamma_j = -\frac{1}{4\pi} S'({\xi}) S''(\xi_j - \xi_j) \frac{\partial \tau}{\partial \lambda_j} = \frac{1}{2} \partial_{\lambda_j} \log g_i, \tag{C.1}
\]
where
\[
g_i = \frac{1}{4\pi} (S'({\xi}))^2 \frac{\partial \tau}{\partial \lambda_j}.
\]

Let us take the derivative:
\[
\frac{\partial}{\partial \lambda_j} \left( \log S'({\xi}) + \frac{1}{2} \log \frac{\partial \tau}{\partial \lambda_j} \right) = \frac{S''(\xi_j)}{S'(\xi_j)} \frac{\partial \xi_j}{\partial \lambda_j} + \frac{S'(\xi_j)}{S'(\xi_j)} \frac{\partial \tau}{\partial \lambda_j} + \frac{1}{2(\partial \tau/\partial \lambda_j)} \frac{\partial^2 \tau}{\partial \lambda_j \partial \lambda_j}.
\]

Using the elliptic Gibbons–Tsarev system
\[
\frac{\partial \xi_j}{\partial \lambda_j} = \frac{1}{4\pi} \left( \zeta_1(-\xi_i + \xi_j, \tau') - \zeta_1(\xi_j, \tau') \right) \frac{\partial \tau}{\partial \lambda_j} \tag{C.2}
\]

\[
\frac{\partial^2 \tau}{\partial \lambda_i \partial \lambda_j} = \frac{1}{2\pi} \mathcal{V}_i(\xi_i - \xi_j, \tau') \frac{\partial \tau}{\partial \lambda_i} \frac{\partial \tau}{\partial \lambda_j} \tag{C.3}
\]

and the formula (A.9) for \(S'\), we get
\[
\frac{1}{2} \frac{\partial}{\partial \lambda_j} \log g_i = \frac{1}{4\pi i S'({\xi})} \frac{\partial \tau}{\partial \lambda_j} \left( S''(\xi_j)(\zeta_1(\xi_j - \xi_j, \tau') - \zeta_1(\xi_j, \tau')) \right)
\]
\[+ S'(\xi_i)(\psi_1(\xi_i - \xi, \tau') - 2\psi_2(\xi_i, \tau)) + 2S''(\xi_i)\zeta_2(\xi_i, \tau)\].

Comparing with (C.1), we see that we should prove the identity
\[S''(\xi_i)\zeta_1(\xi_i - \xi, \tau') - S''(\xi_i)\zeta_1(\xi_i, \tau') + S'(\xi_i)\psi_1(\xi_i - \xi, \tau')
+ 2S''(\xi_i)\zeta_2(\xi_i, \tau) - 2S''(\xi_i)\psi_2(\xi_i, \tau) + S'(\xi_i)S''(\xi_i - \xi) = 0.\]  
(C.4)

The way to prove it is standard. It is easy to see that as a function of \(\xi_i\) the left hand side is an elliptic function with periods 1, \(\tau'\). It may have singularities at the points \(\xi_i = \xi_i\) and \(\xi_i = 0\) only. Setting \(\xi_i = \xi_i + \epsilon\) and \(\xi_i = \epsilon\), one can see that the principal parts as \(\epsilon \to 0\) (double and simple poles) cancel, so the left hand side is a regular function and thus it is constant in \(\xi_i\).

To find the constant, let us evaluate this expression at the regular point \(\xi_i = \xi_i + \frac{1}{2}\). We have, using (A.13) and (A.14):
\[S''(\xi_i)\left(2\zeta_2(\xi_i, \tau) - \zeta_2(\xi_i, \tau')\right) - 2S''(\xi_i)\psi_2(\xi_i, \tau) + S''\left(\frac{1}{2}\right)S''(\xi_i + \frac{1}{2}) + \psi_1(\frac{1}{2}, \tau')S'(\xi_i)
= S''(\xi_i)S'(\xi_i + \frac{1}{2}) + S''\left(\frac{1}{2}\right)S'(\xi_i + \frac{1}{2})
= -S'(\xi_i)\left(\psi_2(\xi_i, \tau') - \psi_2(0, \tau')\right) + S''\left(\frac{1}{2}\right)S'(\xi_i + \frac{1}{2}).\]  
(C.5)

Now, using (A.6), (A.7) and (A.15) we conclude that this is equal to zero.

**Appendix D. Proof of** \(\partial_{\lambda_i} \Gamma_{ij} = \Gamma_{ij, \lambda_k} + \Gamma_{ik} \Gamma_{k,j} - \Gamma_{ik} \Gamma_{ij}\)

Taking the derivative \(\partial_{\lambda_i} \Gamma_{ij}\) (\(\Gamma_{ij}\) is given by (C.1)) with the help of the Gibbons–Tsarev system and equation (A.9), we get
\[\partial_{\lambda_i} \Gamma_{ij} = -\frac{1}{(4\pi i)^2 S'(\xi_i)} \partial_\tau \partial_\tau \partial_{\lambda_i} \partial_{\lambda_j}
\times \left[S'''(\xi_i - \xi, \xi_j - \xi, \tau') - \xi_j \zeta_1(\xi_i - \xi_j, \tau') + 2\zeta_2(\xi_i - \xi_j, \tau')\right]
- 4S''(\xi_i - \xi_j)S'(\xi_i - \xi_j, \tau') - 2S'(\xi_i - \xi_j)S''(\xi_i - \xi_j, \tau')
+ 2S''(\xi_i - \xi_j)S''(\xi_i - \xi_j, \tau')
+ S''(\xi_i - \xi_j)S''(\xi_i - \xi_j, \tau')
- S''(\xi_i - \xi_j)S''(\xi_i - \xi_j, \tau')
+ S''(\xi_i - \xi_j)S''(\xi_i - \xi_j, \tau')
+ 2S''(\xi_i - \xi_j)S''(\xi_i - \xi_j, \tau')\].

19
We recall that $\tau' = \frac{\tau}{2}$. The functions $\zeta_2(x, \tau), \varphi_2(x, \tau)$ and $\varphi_2'(x, \tau)$ can be transformed to the functions $\zeta_2(x, \tau'), \varphi_2(x, \tau')$ and $\varphi_2'(x, \tau')$ with the help of (A.13) and (A.14) (and derivative of (A.14)). The expressions containing products of derivatives of the $S$-function then cancel by virtue of (A.10) (one should take derivatives of this equation). In this way we get

$$-\partial_\lambda \Gamma_y - \Gamma_y \Gamma_y + \Gamma_y \Gamma_y + \Gamma_y \Gamma_y = \frac{S'(\xi)}{(4\pi i)^2 S'(\xi)} \frac{\partial \tau}{\partial \lambda} \frac{\partial \tau}{\partial \lambda} h(\xi, \xi, \xi),$$  

where

$$h = S'''(\xi - \xi) \left( -\zeta_2(\xi) + \zeta_2(\xi) + \zeta_2(\xi - \xi) \right)$$

$$-2S''(\xi - \xi) \varphi_2(\xi - \xi) = S'(\xi - \xi) \varphi_2(\xi - \xi)$$

$$+2S''(\xi - \xi) \varphi_2(\xi - \xi)$$

$$+S''(\xi - \xi) \frac{S''(\xi)}{S'(\xi)} \left( -\zeta_2(\xi) + \zeta_2(\xi - \xi) + \zeta_2(\xi) \right)$$

$$-2S''(\xi - \xi) \varphi_2(\xi - \xi)$$

$$+S''(\xi - \xi) \left( \varphi_2(\xi) - \varphi_2(\xi) \right)$$

$$-S'(\xi) S''(\xi - \xi) S''(\xi - \xi) + S'(\xi) S''(\xi - \xi) S''(\xi - \xi)$$

$$+S''(\xi - \xi) S''(\xi - \xi).$$

Here all $\zeta$- and $\varphi$-functions depend on $\tau'$. As a function of $\xi_k$, $h$ is a double-periodic function with periods 1, $\tau'$ and possible poles at the points $\xi_k = \xi, \xi_k = \xi_k$ and $\xi_k = 0$ in the fundamental parallelogram. One can see that the principal parts of the expansions around these points vanish, so the function is regular and thus it is a constant in $\xi_k$. To find the constant, we evaluate $h$ at the point $\xi_k = \frac{1}{2}$:

$$h(\xi, \xi, \frac{1}{2}) = S''(\xi - \xi) \left( -\zeta_2(\xi) + \zeta_2(\xi) + \zeta_2(\xi - \xi) \right)$$

$$-2S''(\xi - \xi) \varphi_2(\xi - \xi) = S'(\xi - \xi) \varphi_2(\xi - \xi)$$

$$+S''(\xi - \xi) \varphi_2(\xi) + S''(\xi - \xi) \varphi_2(\xi) + S''(\xi - \xi) \left( \varphi_2(\xi - \xi) - \varphi_2(\xi) \right).$$

We want to show that $h(\xi, \xi, \frac{1}{2}) = 0$, then $h(\xi_k, \xi_k, \xi_k) = 0$ and we are done. Let us denote $H(\xi) = h(\xi, \xi, \frac{1}{2})$. The function $H$ has the following quasiperiodicity properties: $H(\xi + 1) = H(\xi), H(\xi + \tau') = -H(\xi)$. It is enough to show that $H(\xi)$ is regular at the possible singular points $\xi = \xi, \xi = \xi + \frac{1}{2}, \xi = \xi$ and $H(0) = 0$. The regularity is checked by a direct inspection. At $\xi_k = 0$ we have:

$$H(0) = -S'(\xi) \varphi_2(\xi) \varphi_2(0) + S''(\xi) \varphi_2(0) + S''(\xi + \frac{1}{2}) S''(\xi + \frac{1}{2})$$

which is the derivative of (C.5) and thus equal to zero.
Appendix E. Proof of equation (63)

In order to prove (63) we start from (28):

\[ 4\pi i \partial_\lambda S(u) = S'(\xi)S'(u + \xi) \frac{\partial \tau}{\partial \lambda_j}. \]

To take another derivative we need to know \( \partial_\lambda S'(u + \xi) \) and \( \partial_\lambda S'(\xi) \) which are calculated using the elliptic Löwner equation, the Gibbons–Tsarev equations, equations (A.9), (A.13) and (A.14) and the derivative of equation (A.10):

\[ 4\pi i \partial_\lambda S'(u + \xi) = \left[ S''(u + \xi) \left( -\zeta_1(u + \xi) + \zeta_1(\xi - \xi) + \zeta_2(\xi) \right) - S'(u + \xi) \psi_2(u + \xi) \right] \frac{\partial \tau}{\partial \lambda_j}. \]

Here and below all \( \zeta \)- and \( \psi \)-functions have modular parameter \( \frac{\tau}{\xi} \). At \( z = \infty \) (\( u = 0 \)) we get from here

\[ 4\pi i \partial_\lambda S'(\xi_j) = \left[ S''(\xi_j) \left( -\zeta_1(\xi_j) + \zeta_1(\xi_j - \xi_j) + \zeta_2(\xi_j) \right) - S'(\xi_j) \psi_2(\xi_j) \right] \frac{\partial \tau}{\partial \lambda_j}. \]

Combining all necessary equations together, we have:

\[ \partial_\lambda \partial_\lambda S(u) - \Gamma_j \partial_\lambda S(u) - \Gamma_j \partial_\lambda S(u) = \frac{1}{(4\pi i)^2} \frac{\partial \tau}{\partial \lambda_i} \frac{\partial \tau}{\partial \lambda_j} f(\xi_i, \xi_j, u), \]

where

\[ f(\xi_j, \xi_j, u) = 2S'(\xi_j)S'(u + \xi_j) \psi_1(\xi_j - \xi_j) \]
\[ + S'(\xi_j)S''(u + \xi_j) \left( -\zeta_1(u + \xi_j) + \zeta_1(\xi_j - \xi_j) + \zeta_2(u + \xi_j) \right) - S'(\xi_j)S'(u + \xi_j) \psi_2(u + \xi_j) \]
\[ + S''(\xi_j)S'(u + \xi_j) \left( -\zeta_1(\xi_j) + \zeta_1(\xi_j - \xi_j) + \zeta_2(\xi_j) \right) - S'(\xi_j)S'(u + \xi_j) \psi_2(\xi_j) \]
\[ + S'''(\xi_j - \xi_j) \left( S'(\xi_j)S'(u + \xi_j) + S'(\xi_j)S'(u + \xi_j) \right). \]

We want to prove that \( f(\xi_j, \xi_j, u) = 0 \). The argument is standard. As a function of \( \xi_j \) \( f(\xi_j, \xi_j, u) \) is an elliptic function with periods 1 and \( \frac{\tau}{\xi} \) and possible poles at the points \( \xi_j = \xi_j, \xi_j = -u \) and \( \xi_j = 0 \). Expanding around these points, one can see that the principal parts vanish, so the function is constant in \( \xi_j \). To find the constant, we evaluate it at the regular point \( \xi_j = \xi_j + \frac{1}{2} \). We have:

\[ f(\xi_j + \frac{1}{2}, \xi_j, u) = 2S'(\xi_j)S'(u + \xi_j) \psi_2(0) \]
\[ - S'(\xi_j)S'(u + \xi_j) \psi_2(u + \xi_j) - S'(\xi_j)S'(u + \xi_j) \psi_2(\xi_j) \]
\[ + S''(\frac{1}{2}) \left( S'\left(\xi_j + \frac{1}{2}\right)S'(u + \xi_j) + S'(\xi_j)S'(u + \xi_j + \frac{1}{2}) \right) \]
\[ = S'(\xi_j)S'(u + \xi_j) [2\psi_2(0) - \psi_2(u + \xi_j) - \psi_2(\xi_j) \]
\[ + S''(\frac{1}{2}) \left( \frac{S'(\xi_j + \frac{1}{2})}{S'(\xi_j)} + \frac{S'(\xi_j + \frac{1}{2})}{S'(\xi_j + \xi_j)} \right). \]
Using (A.15), (A.6) and (A.7) (from which we find $S''(\frac{1}{2}) = -\pi^2 \theta_1(0, \frac{\tau}{2}) \theta_1(0, \frac{\tau}{2})$), one can see that the expression in the square brackets equals zero, so $f = 0$.

**ORCID iDs**

T Takebe  [https://orcid.org/0000-0002-8338-0285](https://orcid.org/0000-0002-8338-0285)

**References**

[1] Jimbo M and Miwa T 1983 Soliton Equations and Infinite Dimensional Lie Algebras (Kyoto University: Publ. RIMS) vol 19, pp 943–1001
[2] Hirota R and Ohta Y 1991 Hierarchies of coupled soliton equations J. Phys. Soc. Japan 60 798–809
[3] Adler M, Horozov E and van Moerbeke P 1999 The Pfaff lattice and skew-orthogonal polynomials Int. Math. Res. Not. 1999 569–88
[4] Adler M, Shiota T and van Moerbeke P 2002 Pfaff $\tau$-functions Math. Ann. 322 423–76
[5] Takebe T 2009 Orthogonal and symplectic matrix integrals and coupled KP hierarchy J. Phys. Soc. Japan 99 2875–7
[6] van de Leur J 2001 Matrix integrals and the geometry of spinors J. Nonlinear Math. Phys. 8 288–310
[7] Orlov A 2014 Deformed Ginibre ensembles and integrable systems Phys. Lett. A 378 319–28
[8] Kuznetsov V and van Moerbeke P 2002 Rational solutions to the Pfaff lattice and Jack polynomials Ergod. Theor. Dynam. Syst. 22 1365–405
[9] Komatsu Y 1943 Untersuchungen über konforme Abbildung von zweifach zusammenhängenden Gebieten Proc. Phys.-Math. Soc. Japan 25 1–42 (www.jstage.jst.go.jp)
[10] Alexandrov I A 1976 Parametric Continuations in the Theory of Univalent Functions (Moscow: Nauka) (in Russian)
[11] Conteras M D, Diaz-Madrigal S and Gumenyuk P 2013 Löwner theory in annulus I: evolution families and differential equations Trans. Am. Math. Soc. 365 2505–43
Contreras M D, Diaz-Madrigal S and Gumenyuk P 2011 Loewner theory in annulus II: Loewner chains Anal. Math. Phys. 1 351–85
Bracci F, Contreras M D, Diaz-Madrigal S and Vasil’ev A 2013 Classical and Stochastic Löwner–Kufarev Equations (Harmonic and Complex Analysis and Applications) (New York: Birkhäuser-Verlag) pp 39–134
Gibbons J and Tsarev S 1996 Reductions of the Benney equations Phys. Lett. A 211 19–24
Gibbons J and Tsarev S 1999 Conformal maps and reductions of the Benney equations Phys. Lett. A 258 263–71
Mañas M, Martínez-Alonso L and Medina E 2002 Reductions and hodograph solutions of the dispersionless KP hierarchy J. Phys. A: Math. Gen. 35 401–17
Mañas M 2004 S-functions, reductions and hodograph solutions of the rth dispersionless modified KP and Dym hierarchies J. Phys. A: Math. Gen. 37 11191–221
Takasaki K and Takebe T 2006 Radial Löwner equation and dispersionless cmKP hierarchy (arXiv:nlin.SI/0601063)
Takase K, Teo L-P and Zabrodin A 2006 Löwner equation and dispersionless hierarchies J. Phys. A: Math. Gen. 39 11479–501
Takebe T 2014 Dispersionless BKP hierarchy and quadrant Löwner equation SIGMA 10 023
Tsarëv S P 1990 The geometry of Hamiltonian systems of hydrodynamic type. The generalized hodograph method Izv. Akad. Nauk SSSR Ser. Mat. 54 1048–68 (in Russian)
Tsarëv S P 1991 Math. USSR-Izv. 37 397–419 (English translation)
Dubrovin B A and Novikov S P 1989 Hydrodynamics of weakly deformed soliton lattices. Differential geometry and Hamiltonian theory Russ. Math. Surv. 44 35–124
Pavlov M 2007 Algebro-geometric approach in the theory of integrable hydrodynamic-type systems Commun. Math. Phys. 272 469–505
Ferapontov E and Khusnutdinova K 2004 On integrability of (2 + 1)-dimensional quasilinear systems Commun. Math. Phys. 248 187–206
Takasaki K 2001 Painlevé–Calogero correspondence revisited J. Math. Phys. 42 1443–73
Zabrodin A and Zotov A 2012 Quantum Painlevé–Calogero correspondence for Painleve VI J. Math. Phys. 53 073508
Odesskii A and Sokolov V 2009 Systems of Gibbons–Tsarev type and integrable 3-dimensional models (arXiv:nlin/0906.3509)
Odesskii A and Sokolov V 2010 Integrable (2 + 1)-dimensional systems of hydrodynamic type Theor. Math. Phys. 163 549–86
Odesskii A and Sokolov V 2009 Integrable elliptic pseudopotentials Theor. Math. Phys. 161 1340–52
Sato M and Sato Y 1983 Soliton equations as dynamical systems on infinite dimensional Grassmann manifold Lecture Notes in Numerical and Applied Analysis vol 5 pp 259–71
Takebe T 1990 Toda lattice hierarchy and conservation laws Commun. Math. Phys. 129 281–318
Kharchev S and Zabrodin A 2015 Theta vocabulary I J. Geom. Phys. 94 19–31
Mumford D 1983 Tata Lectures on Theta I (Basel: Birkhäuser)