GLOBAL AND LOCAL SCALING LIMITS FOR THE $\beta = 2$ STIELTJES–WIGERT
RANDOM MATRIX ENSEMBLE

PETER J. FORRESTER

Abstract. The eigenvalue probability density function (PDF) for the Gaussian unitary ensemble has a well known analogy with the Boltzmann factor for a classical log-gas with pair potential $-\log |x - y|$, confined by a one-body harmonic potential. A generalisation is to replace the pair potential by $-\log |\sinh(\pi(x - y)/L)|$. The resulting PDF first appeared in the statistical physics literature in relation to non-intersecting Brownian walkers, equally spaced at time $t = 0$, and subsequently in the study of quantum many body systems of the Calogero-Sutherland type, and also in Chern-Simons field theory. It is an example of a determinantal point process with correlation kernel based on the Stieltjes–Wigert polynomials. We take up the problem of determining the moments of this ensemble, and find an exact expression in terms of a particular little $q$-Jacobi polynomial. From their large $N$ form, the global density can be computed. Previous work has evaluated the edge scaling limit of the correlation kernel in terms of the Ramanujan ($q$-Airy) function. We show how in a particular $L \to \infty$ scaling limit, this reduces to the Airy kernel.

1. Introduction

Highly recognisable in mathematical physics is the probability density function (PDF)

$$p_N^{(G)}(x_1, \ldots, x_N) = \frac{1}{C_N^{(G)}} \prod_{l=1}^N e^{-x_l^2} \prod_{1 \leq j < k \leq N} (x_k - x_j)^2, \quad x_l \in \mathbb{R} (l = 1, \ldots, N)$$

with normalisation

$$C_N^{(G)} = 2^{-N(N-1)/2} \prod_{j=1}^N j!.$$ (1.2)

In a somewhat disguised form (1.1) first (to this author’s knowledge) arose in the 1940 work of Husimi [35], who was studying properties of the ground state wave function $\psi_0(x_1, \ldots, x_N)$ for $N$ spin polarised fermions in one-dimension with a harmonic confining potential. For this problem, using dimensionless units, Husimi gave $\psi_0$ in the Slater determinant form

$$\psi_0(x_1, \ldots, x_N) = \frac{1}{\sqrt{N!}} \det \left[ \phi_{j-1}(x_k) \right]_{j,k=1}^N.$$ (1.3)
where, with $H_l(x)$ denoting the Hermite polynomials,

$$
\phi_l(x) = \frac{1}{\sqrt{2^{l!} \pi^{1/2}}} H_l(x) e^{-x^2/2}.
$$

(1.4)

To understand this formula, note that the Schrödinger equation for $N$ particles on a line with a confining harmonic potential factorises as the sum of $N$ single particle Schrödinger equations

$$
-\frac{1}{2} \left( \frac{d^2}{dx^2} - x^2 \right) \phi_l(x) = \epsilon_l \phi_l(x), \quad \epsilon_l = l + 1/2 \ (l = 0, 1, \ldots).
$$

(1.5)

For (1.5) $\{\phi_l(x)\}_{l=0}^{N-1}$ are the normalised wave functions corresponding to the lowest allowed energy levels in order; forming the determinant ensures that the many body wave function is anti-symmetric as required for fermions.

Factoring the normalisation in (1.4) from each row of (1.3), and the Gaussian from each column shows

$$
\psi_0(x_1, \ldots, x_N) = \prod_{l=1}^N \frac{e^{-x_l^2/2}}{\sqrt{2^{l!} \pi^{1/2}}} \det \left[ H_{j-1}(x_k) \right]_{j,k=1}^N.
$$

(1.6)

Introducing the monic Hermite polynomials $p_l^{(G)}(x)$ we have $H_l(x) = 2lp_l^{(G)}(x)$ and thus

$$
\det \left[ H_{j-1}(x_k) \right]_{j,k=1}^N = 2^{N(N-1)/2} \det \left[ p_{j-1}^{(G)}(x_k) \right]_{j,k=1}^N.
$$

(1.7)

But for general monic polynomials $\{p_l(x)\}_{l=0,1,\ldots}$

$$
\det \left[ p_{j-1}(x_k) \right]_{j,k=1}^N = \det \left[ x_{j-1}^{-1} \right]_{j,k=1}^N = \prod_{1 \leq j < k \leq N} (x_k - x_j),
$$

(1.8)

where the first equality follows by successive elementary row operations to eliminate all but the leading monomial from each row, and the second equality is the Vandermonde determinant identity (see e.g. [21] Ex. 1.9 q.1). Substituting (1.8) in (1.7), and the result in (1.6), we see that

$$
|\psi_0(x_1, \ldots, x_N)|^2 = p_N^{(G)}(x_1, \ldots, x_N),
$$

(1.9)

or in words, the square of the ground state wave function for spin polarised fermions in one-dimension with a harmonic confining potential is given by the PDF (1.4).

There are other interpretations of (1.1) in theoretical/mathematical physics. Let $X$ be an $N \times N$ standard complex Gaussian matrix, and define the random Hermitian matrix $H$ by $H = \frac{1}{2} (X + X^\dagger)$. The set of such matrices is said to form the Gaussian unitary ensemble
we see that with (GUE). It is a well known result that the eigenvalue PDF for the GUE is given by (1.1); see e.g. [21] Prop. 1.3.4. Exponentiating the product over pairs in (1.1) shows

\[ p_N^{(G)}(x_1, \ldots, x_N) \propto e^{-\beta U(x_1, \ldots, x_N)}, \quad U := \frac{1}{2} \sum_{j=1}^{N} x_j^2 - \sum_{1 \leq j < k \leq N} \log |x_k - x_j|, \quad \beta = 2, \quad (1.10) \]

thus revealing an analogy with the Boltzmann factor for a classical gas, in equilibrium at inverse temperature \( \beta = 2 \), and interacting via a repulsive logarithmic pair potential, and an attractive one-body harmonic potential towards the origin. This analogy was used extensively in random matrix theory by Dyson [16].

As a further interpretation, consider \( N \) Brownian walkers in one-dimension, and thus individually with a PDF \( u_t(x^0, x) \) obeying the diffusion equation

\[ \frac{1}{D} \frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2}, \quad (1.11) \]

starting from the points \( x^0 = (x_1^{(0)}, \ldots, x_N^{(0)}) \), \( (x_1^{(0)} < \cdots < x_N^{(0)}) \). The Karlin–MacGregor formula [38] tells us that the PDF for the event the walkers arrive at \( x = (x_1, \ldots, x_N) \) without intersecting is

\[ G_t(x^0; x) = \det \left[ u_t(x_j^{(0)}; x_k) \right]_{j,k=1}^{N}. \quad (1.12) \]

Using this, it can be shown [39] (see also [21] Prop. 10.1.12) that the PDF for the event that \( N \) non-intersecting Brownian walkers all starting at the origin, arrive at position \( x \) after time \( t \), with the non-intersecting condition required for all times \( T \) \( (T \to \infty) \), is equal to (1.1).

Our interest in the present work is in the generalisation of (1.4) (set \( c = 1 \) and take \( L \to \infty \))

\[ p_N^{(SW)}(x_1, \ldots, x_N) = \frac{1}{C_{N,c}^{(SW)}} \prod_{l=1}^{N} e^{-c x_l^2} \prod_{1 \leq j < k \leq N} \left( \sinh(\pi(x_k - x_j)/L) \right)^2, \quad (1.13) \]

where

\[ C_{N,c}^{(SW)}(q) = N! 2^{-N(N-1)} \left( \sqrt{\frac{\pi}{c}} \right)^N q^{N^2 - 1/2} q^{-\frac{1}{6} N(N-1)(2N+1)} \prod_{j=1}^{N-1} \left( 1 - q^j \right)^{N-j}, \quad (1.14) \]

with

\[ q = e^{-1/(2k^2)}, \quad k^2 = \frac{cL^2}{2\pi}. \quad (1.15) \]

Changing variables

\[ u_j = e^{\frac{2q}{\pi}}(q + \frac{q}{\pi}) \quad \Rightarrow \quad q = N e^{2\pi x_j/L}, \quad (1.16) \]

we see that

\[ p_N^{(SW)}(x_1, \ldots, x_N) dx_1 \cdots dx_N = p_N^{(SW)}(u_1, \ldots, u_N) du_1 \cdots du_N, \quad (1.17) \]
where
\[ p_N^{(SW)}(u_1, \ldots, u_N) = \frac{1}{C_N^{(SW)}(q)} \prod_{i=1}^{N} w^{(SW)}(u_i; q) \prod_{1 \leq j < k \leq N} (u_k - u_j)^2, \quad u_l \in \mathbb{R}^+ \ (l = 1, \ldots, N). \] 

(1.18)

In (1.18)
\[ w^{(SW)}(u; q) = \frac{k}{\sqrt{\pi}} e^{-k^2 (\log u)^2} \]

(1.19)

and, with \( q \) again given by (1.15),
\[ C_N^{(SW)}(q) = N! q^{-\frac{N}{2}(2N-1)(2N+1)} \prod_{j=1}^{N-1} (1 - q^j)^{N-j}. \]

(1.20)

It is well known [56] that the polynomials on the half line \( u > 0 \), orthonormal with respect to the weight function \( w^{(SW)}(u; q) \), are the Stieltjes–Wigert polynomials
\[ S_l(u; q) := \frac{(-1)^l q^{l/2+1/4}}{\{(1-q)/(1-q^2)\cdots(1-q^l)\}^{1/2}} \sum_{v=0}^{l} \begin{bmatrix} l \\ v \end{bmatrix}_q q^{v^2} (-q^{1/2} u)^v, \]

(1.21)

where, with
\[ [n]_q! := \frac{(q; q)_n}{(1-q)_n}, \quad (u; q)_n := (1-u)(1-qu)\cdots(1-q^{n-1}u), \]

(1.22)

the quantities
\[ \begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{[n]_q!}{[n-m]_q! [m]_q!}, \]

(1.23)

are the \( q \)-binomial coefficients. This is the reason for the label \( (SW) \) in (1.18); the label \( (SW_e) \) used in (1.13) is to indicate the underlying Stieltjes–Wigert polynomials in exponential variables.

In relation to the PDF (1.1) we have indicated four distinct interpretations in the context of theoretical/ mathematical physics. All have analogues for the PDF (1.13). These will be reviewed in Section 2. It is also true that (1.13) has a further interpretation relative to (1.1) — this is in relation to a partition function which occurs in Chern–Simons field theory [46, 58]. The various applications have resulted in a number of works studying properties of (1.13); see for example [18, 20, 47, 34, 15, 50, 5, 59, 29, 60, 53, 7, 48, 57, 61, 13].

Notwithstanding this previous literature, in light of known properties of (1.1), there are still some fundamental properties of (1.13) which remain to be investigated. Here we consider two of these. The first relates to the moments (in exponential variables) of the density for (1.13), or equivalently the usual moments defined as power sum averages of
We will show in Section \ref{sec:scaling}, and their consequence in relation to the computation of the global density in the latter. We are motivated by the fact that the moments for \eqref{eq:momentSG},

\[ m_{2i}^{(G)} = \left\langle \sum_{j=1}^{N} x_j^{2i} \right\rangle_{(G)} \quad (l \in \mathbb{Z}_{\geq 0}), \]  

have the closed form hypergeometric evaluation \cite[Th. 8]{64}

\[ 2^{l} m_{2i}^{(G)} = N \frac{(2l)!}{2^l l!} 2F_1(-l, 1-N; 2; 2). \]  

(With \( x_j \) replaced by \(|x_j|\) in \eqref{eq:momentSG}, this evaluation in fact extends to complex \( l \) \cite{12}.) The second relates to the edge scaling limit of the correlation kernel for \eqref{eq:momentSG}, and its relation to the well known Airy kernel specifying the edge scaling limit of the correlation kernel \( K_{(G)} \) for \eqref{eq:momentSG} \cite{19}

\[ K_{\text{edge}}^{(G)}(X, Y) := \lim_{N \to \infty} \frac{1}{\sqrt{2N^{1/6}}} K_{(G)}^{(X, Y)} \bigg|_{x=\sqrt{2N^{1/6}}y+\sqrt{2N^{1/6}}} = \frac{\text{Ai}(X)\text{Ai}'(Y) - \text{Ai}(Y)\text{Ai}'(X)}{X-Y}. \]  

The functional form of the edge scaling limit of the correlation kernel for \eqref{eq:momentSG}, \( K_{\text{edge}}^{(SW_c)}(X, Y) \), is known from \cite{20}, and is given in terms of the special function (well defined for \(|q| < 1\))

\[ A_q(z) := \sum_{v=0}^{\infty} \frac{q^v z^v}{(q; q)_v}. \]  

The specific problem to be addressed is to identify scaling variables, with \( X, Y \) dependent on \( L \), so that in the limit \( L \to \infty \) the kernel \( K_{\text{edge}}^{(SW_c)}(X, Y) \) reduces to \eqref{eq:edgeSG}.

The moments of interest for the models \( (SW_c) \) and \( (SW) \) are specified by

\[ m_l^{(SW_c)} = \left\langle \sum_{j=1}^{N} e^{\frac{2\pi}{N}(x_j + \pi l)} \right\rangle_{(SW_c)} = \left\langle \sum_{j=1}^{N} u_j^{l} \right\rangle_{(SW)} \quad (l \in \mathbb{Z}). \]  

We will show in Section \ref{sec:scale} that \eqref{eq:momentsSW}, like its GUE counterpart \eqref{eq:momentSG}, admits a hypergeometric evaluation. This involves the \( q \)-generalisation of the Gauss \( 2F_1 \) function

\[ 2F_1\left(\begin{array}{c} a_1, a_2 \\ b_1 \end{array}\right| q; z) = \sum_{n=0}^{\infty} \frac{(a_1; q)_n (a_2; q)_n z^n}{(q; q)_n (b_1; q)_n}, \]  

or equivalently the little \( q \)-Jacobi polynomial

\[ p_n^{(q)}(x; a, b|q) = 2F_1\left(\begin{array}{c} -n, abq^n+1 \\ aq \end{array}\right| q; qx). \]
Proposition 1.1. Let \( q \) be given by (1.15) and let \( l \in \mathbb{Z}^+ \). We have

\[
\frac{1}{N} q^{Nl} m_i^{(SW)} = -\frac{1}{N} \left( \frac{1}{1 - q^{-1/2}} \right) 2\phi_1 \left( q^{-1}; q^{-l} \right) - \frac{1}{N} \left( \frac{1}{1 - q^{-1/2}} \right) p_i^{(lq-1)}(q^{-N}; 1, q^{-1}).
\] (1.31)

Simple manipulation of the series definition of \( 2\phi_1 \) on the RHS of (1.31) shows that it is in fact a function of \( (q^{1/2} - q^{-1/2}) \), and of \( q^{-N} \). This is in keeping with the well known fact that the resolvent corresponding to (1.13), and thus the moments, permit an expansion in \( 1/N^2 \) upon setting \( 2k^2 = N/\lambda \) in (1.15) so that

\[
q = e^{-\lambda/N}.
\] (1.32)

With this choice of \( q \), the large \( N \) scaling limit of (1.31), and thus the leading term in the \( 1/N^2 \) expansion, is almost immediate.

Corollary 1.2. For \( l \in \mathbb{Z}^+ \) we have

\[
\mu_{l,0}^{(SW)} := \lim_{N \to \infty} \frac{q^{Nl}}{N} m_i^{(SW)} \bigg|_{q=e^{-\lambda/N}} = \frac{(-1)^l}{\lambda l} 2F_1(-l, l; 1; e^\lambda).
\] (1.33)

In Section 3.2 the result (1.33) will be used to give a new derivation of the corresponding limiting scaled spectral density of the ensemble specified by (1.18).

The correlation kernel for (1.13) is specified in Section 4.1 and its bulk and edge scaling limits, already known from \([19,20]\), are revised in Section 4.2. In Section 4.3, a suitable asymptotic expansion known from \([32,33]\) is used to deduce the sought \( L \to \infty \) scaling limit of \( K_{edge}^{(SW)} \), reclaiming (1.26).

Proposition 1.3. Let \( K_{edge}^{(SW)}(X, Y) \) be specified by (4.12) below, and thus be determined by the function \( A_q(z) \) as defined in (4.27). Let

\[
\epsilon = \frac{2\pi^2}{cL^2}, \quad X(x, L) = \frac{L}{2\pi} \log \frac{1}{4} - \frac{L}{2\pi} e^{2/3} x, \quad Y(y, L) = \frac{L}{2\pi} \log \frac{1}{4} - \frac{L}{2\pi} e^{2/3} y.
\] (1.34)

We have

\[
-\lim_{L \to \infty} \frac{L}{2\pi} e^{2/3} K_{edge}^{(SW)}(X(x, L), Y(y, L)) = K^{(G)}_{edge}(x, y).
\] (1.35)

2. Interpretations of the PDF (1.13)

2.1. Boltzmann factor of a classical gas. It is immediate that the PDF (1.13) can be written in a Boltzmann factor form analogous to (1.1)

\[
p_N^{(SW)} \propto e^{-\beta U_L(x_1, \ldots, x_n)}, \quad U_L := \frac{c}{2} \sum_{j=1}^N x_j^2 - \sum_{1 \leq j < k \leq N} \log \left| \sinh \left( \frac{\pi(x_k - x_j)}{L} \right) \right|, \quad \beta = 2.
\] (2.1)
What remains is to interpret the pair potential in this expression; the one-body term is simply an harmonic attraction towards the origin as in (1.1), with a scale factor \( c \). For this consider the pair potential \( \Phi(r, r') \) due to a two-dimensional unit charge in the plane at point \( r = (x, y) \), and another at point \( r' = (x', y') \). With \( r' \) regarded as fixed, this pair potential must satisfy the two-dimensional Poisson equation \( \nabla^2 \Phi(r, r') = -2\pi \delta(r' - r) \). Require too that the charges are restricted (at first) to the strip \( 0 \leq y < L \) in the plane, and subject to semi-periodic boundary conditions \( \Phi((x, y + L), (x', y')) = \Phi((x, y), (x', y')) \). Equivalently \( 2\pi y/L \) can be regarded as the angular position on a cylinder, and with \( x \) corresponding to the height. The explicit form of \( \Phi \) is (see e.g. [21] §2.7)

\[
\Phi((x, y), (x', y')) = -\log \left( \frac{1}{|x - x' - i(y - y')/L|} \right), \tag{2.2}
\]

Requiring that all charges, confined at first to the strip \( 0 \leq y < L \), be further confined to the \( x \)-axis (or \( \theta = 0 \) in the cylinder picture, varying only in their height) we have that \( y = y' \) in (2.2) which is then recognised as the pair potential in (2.1). Note that in this circumstance (2.2) exhibits the large separation asymptotic behaviour

\[
\Phi((x, 0), (x', 0)) \bigg|_{|x - x'| \to \infty} \sim -\frac{\pi |x - x'|}{L}, \tag{2.3}
\]

which is in fact proportional to the Coulomb potential in one-dimension.

### 2.2. Ground state wave function

Notwithstanding the determinantal structure associated with \( p^{(SW)}_N \), as evidenced by (1.18) and (1.8), there is no free Fermi system with a ground state wave function equal to either (1.13) or (1.17). The essential point here is that the class of single-particle Schrödinger operators which permit wave functions of the form \( \sqrt{w(x)} p_l(x) \) for some weight function \( w(x) \) and orthogonal polynomials \( \{p_l(x)\} \) is extremely limited — this is quantified by Bocher’s theorem [3].

On the other hand, it turns out [20] that not only \( p^{(SW)}_N \), but also its \( \beta \)-generalisation in the sense of (2.1), has an interpretation of a squared ground state wave function for the particular many body Schrödinger operator of Calogero–Sutherland type (see [41] for an extended account of this class of quantum many body systems)

\[
\mathcal{H} = -\sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} + a^2 \sum_{j=1}^N x_j^2 - \frac{ma}{L} \sum_{1 \leq i < j \leq N} \left( x_i - x_j \right) \coth \left( \frac{x_i - x_j}{2L} \right)
+ \frac{m(m - 1)}{2L} \sum_{1 \leq i < k \leq N} \frac{1}{\sin^2((x_i - x_k)/2L)}, \quad m := \beta/2. \tag{2.4}
\]

In particular when \( \beta = 2 \) as in (1.13), the second pair potential term in (2.4) vanishes but the first such term remains. Denoting this term \( \sum_{1 \leq i < j \leq N} V(x_i - x_j) \), we see that analogous to
2.3. Eigenvalue PDF on the space of complex positive definite matrices. From the viewpoint of the set of positive definite matrices \( \mathcal{P}_N \) as an example of a Riemannian manifold, the squared geodesic distance between \( A, B \in \mathcal{P}_N \), \((d(A, B))^2 \) say, is given by (see e.g. [3])

\[
(d(A, B))^2 = \sum_{j=1}^{N} \left( \log \lambda_j(A^{-1}B) \right)^2,
\]

where \( \{ \lambda_j(M) \}_{j=1}^{N} \) denotes the eigenvalues of \( M \). If the eigenvalue of \( A \) are parametrised as \( \{ e^{x_j} \}_{j=1}^{N} \), and \( B \) is the identity, (2.6) simplifies to read

\[
(d(A, I))^2 = \sum_{j=1}^{N} x_j^2.
\]

In [52], a density for the invariant Riemannian volume element, \( d\mu(A) \) say, on \( \mathcal{P}_N \) proportional to

\[
e^{-c(d(A, I))^2}
\]

is proposed. A decaying density function is in fact necessary: like the Lebesgue measure on \( \mathbb{R} \), \( d\mu(A) \) itself is not normalisable. For complex positive definite matrices the volume element has the explicit form

\[
d\mu(A) = \frac{1}{(\det A)^N} (dA),
\]

where \( (dA) \) denotes the Lebesgue measure for the independent entries of \( A \) (both real and imaginary part for the off diagonal entries). The required invariance \( d\mu(A) = d\mu(M^{1/2}AM^{1/2}) \) can be checked using [21] Exercise 1.3 q.2]; contrast this to the case of \( A \in \text{GL}(\mathbb{C}) \), as discussed in e.g. [27] §2.1], for which the exponent on the RHS of (2.10) is \( 2N \).

Changing variables to the eigenvalues and eigenvectors, it is well known that the eigenvector contribution factorises (see e.g. [21] Prop. 1.3.4) and so can be integrated out. Finally, parametrising the eigenvalues according to (2.7) gives the PDF

\[
\frac{1}{Z_{N, c}} e^{-c \sum_{j=1}^{N} x_j^2} \prod_{1 \leq j < k \leq N} \left( \sinh((x_k - x_j)/2) \right)^2,
\]

and is thus identical to (1.13) with \( L = 2\pi \).
Remark 2.1. Of interest in [52] is \( \langle \sum_{j=1}^{N} x_j^2 \rangle_{(SW_e)} |_{L=2\pi} \). We see from the definitions that

\[
\langle \sum_{j=1}^{N} x_j^2 \rangle_{(SW_e)} |_{L=2\pi} = -\frac{d}{dc} \log Z_{N,c}.
\]

(2.12)

Since \( Z_{N,c} = C_{N,c}^{(SW_e)} (q) |_{L=2\pi} \), it follows from (1.14) that this can be computed exactly.

2. For recent further developments of the theme of this subsection, see [55].

2.4. Non-intersecting Brownian walkers with equal spacing initial condition. Consider the Brownian walker problem of the paragraph containing (1.11). Inserting the explicit form of \( u_t \) in (1.12) shows

\[
G_t(x^{(0)}; x) = \left( \frac{1}{2\pi D t} \right)^{N/2} e^{-\sum_{j=1}^{N} (x_j^2 + (x_j^{(0)})^2)/2D t} \det \left[ e^{x_j^{(0)} x_k}/D t \right]_{j,k=1}^{N}.
\]

(2.13)

In the case of the equal spacing initial condition \( x_j^{(0)} = (j-1)a \) \( j = 1, \ldots, N \), use of the Vandermonde formula (1.7) shows (2.13) simplifies to

\[
G_t(x^{(0)}; x)|_{x_j^{(0)}=(j-1)a} = \left( \frac{1}{2\pi D t} \right)^{N/2} e^{-\sum_{j=1}^{N} (x_j^2 + (N-1)ax_j + (j-1)^2a^2)/2D t} \prod_{1 \leq j < k \leq N} 2 \sinh \frac{a(x_k - x_j)}{2D t}.
\]

(2.14)

Notice that after the simple change of variables

\[
x_j \mapsto x_j - \frac{(N-1)a}{2},
\]

and with

\[
c = 1/D t,
\]

this is proportional to the Boltzmann factor (2.1) with \( \beta = 1 \).

To obtain (2.1) with \( \beta = 2 \), require that after arriving at positions \( x \) in time \( t \), the walkers return to the same equal spacing configuration of their initial condition in further time \( t \). The corresponding PDF is

\[
\left. \frac{G_t(x_0, x) G_t(x, x_0)}{G_2(x_0, x_0)} \right|_{x_j^{(0)}=(j-1)a}.
\]

After the change of variables (2.15), and with \( c \) given by (2.16), this is seen to reduce to (1.13), reproducing too the explicit value of the normalisation (1.14) apart from a factor of \( N! \) which is accounted for by the ordering \( x_1 < \cdots < x_N \) assumed in (2.13).
2.5. Chern–Simons partition function. Consider the Chern–Simons action with gauge group $U(N)$ of the 3-sphere, and with coupling strength $k/4\pi$ (see e.g. [47]). It was shown in [65] that the corresponding partition function, $Z_{S^3}$ say, of significance from the fact that it is a topological invariant, has the evaluation

$$Z_{S^3} = \frac{1}{(k+N)^{N/2}} \sum_{\rho \in P_N} \varepsilon(w) \exp \left( -\frac{2\pi i}{k+N} \rho \cdot w(\rho) \right),$$  

(2.17)

where $P_N$ denotes the set of permutations of $\{1, 2, \ldots, N\}$, $\varepsilon(w)$ is the signature of the permutation $w$ and $\rho = \frac{1}{2}(N-1, N-3, \ldots, -N+1)$ is the Weyl vector of $SU(N)$. The observation of [46] is that the sum in (2.17) can be recognised as the determinant in (3.9) below. In this formula the partition $\kappa$ is to have all parts equal to zero, implying that $Z_{S^3}$ can be expressed in terms of the integral over the Boltzmann factor (2.1). One reads off too that the coupling constants are related by

$$\frac{1}{c} = \frac{4\pi i}{k+N}.$$  

3. Proof of Proposition 1.1 and its consequences

3.1. A Schur function average. Our approach to establishing (1.31) requires knowledge of the evaluation of the average value of the Schur polynomial

$$s_\kappa(u_1, \ldots, u_N) := \frac{\det[u_{k+j-1}^{i}]}{\det[u_{k-1}^{i}]}_k,$$  

(3.1)

where $\kappa$ denotes the partition

$$\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_N), \quad \kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_N,$$  

(3.2)

each $\kappa_i$ being a non-negative integer, with respect to the PDF (1.18). This can be found in the work of Dolivet and Tierz [15]. We will provide a different derivation.

**Proposition 3.1.** Let $\kappa$ be a partition as in (3.2), and denote $|\kappa| = \sum_{j=1}^{N} \kappa_j$. We have

$$q^{N|\kappa|} \langle s_\kappa(u_1, \ldots, u_N) \rangle_{(SW)} = q^{-\frac{1}{2} \sum_{j=1}^{N} \kappa_j^2} \prod_{1 \leq j < k \leq N} \frac{1 - q^{-(\kappa_j - \kappa_k + k)}}{1 - q^{-(k-j)}}.$$  

(3.3)

**Proof.** The change of variables $u_i \mapsto q^{-N} u_i$ shows that

$$\langle s_\kappa(u_1, \ldots, u_N) \rangle_{(SW)} = q^{-N|\kappa|} \langle s_\kappa(u_1, \ldots, u_N) \rangle_{(SW^*)},$$  

(3.4)

where $SW^*$ is specified by the PDF

$$\frac{1}{C_N^{(SW^*)}(q)} \prod_{l=1}^{N} u_l^{-N} w^{(SW)}(u_l; q) \prod_{1 \leq j < k \leq N} (u_j - u_k)^2.$$  

(3.5)
for suitable normalisation \( C_N^{SW}(q) \). The distinguishing feature of (3.5) is that it is unchanged by the mappings \( u_i \mapsto 1/u_i \). We will see later (Remark 3.2) that working with (3.5) gives rise to formulas which can be identified, up to replacing \( q \) by \( q^{-2} \), with those appearing in the computation of the average of a Schur polynomial with respect to a PDF generalising the eigenvalue PDF for Haar distributed random unitary matrices on the unit circle, associated with the Rogers–Szegö orthogonal polynomials.

Evaluating the denominator in (3.1), which is the Vandermonde determinant by the second equality in (1.8), then using the same equality in the reverse direction, we see from (1.18) and (3.1) that

\[
\langle s_k(u_1, \ldots, u_N) \rangle_{SW} = \frac{1}{C_N^{SW}(q)} \times \int_0^\infty \cdots \int_0^\infty \prod_{l=1}^N u_l^{-N} e^{-k^2(\log u_l)^2} \det[u_k^{-1}]_{j,k=1} \det[u_k^{N-N+j-1}]_{j,k=1}. \quad (3.6)
\]

According to the Andréief identity (see e.g. [22]) this multiple integral simplifies to the determinant of single integrals

\[
\langle s_k(u_1, \ldots, u_N) \rangle_{SW} = \frac{N!}{C_N^{SW}(q)} \det \left[ \int_0^\infty w^{SW}(u; q) u^{j-k+\kappa-1} du \right]_{j,k=1}. \quad (3.7)
\]

Each integral in (3.7) is a moment of the log-normal weight (1.20), with the well known evaluation

\[
\int_0^\infty x^nw^{SW}(x; q) \, dx = q^{-(n+1)^2/2}, \quad n \in \mathbb{C}. \quad (3.8)
\]

Thus

\[
\langle s_k(u_1, \ldots, u_N) \rangle_{SW} = \frac{N!}{C_N^{SW}(q)} \det \left[ q^{-\frac{1}{2}(j-k+\kappa)^2} \right]_{j,k=1}. \quad (3.9)
\]

Since

\[
q^{-\frac{1}{2}(j-k+\kappa)^2} = q^{-\frac{1}{2}(j-1)^2-\frac{1}{2}(\kappa_k+1)^2} v_k^{j-1}, \quad v_k = q^{-(\kappa_k+1)}
\]

the determinant in (3.9) can be reduced to the Vandermonde determinant. Evaluating the latter according to the second equality in (1.8) gives

\[
\langle s_k(u_1, \ldots, u_N) \rangle_{SW} = \frac{N!}{C_N^{SW}(q)} \prod_{l=1}^N q^{-\frac{1}{2}(l-1)^2-\frac{1}{2}(\kappa_l-l+1)^2} \prod_{1 \leq j < k \leq N} (q^{-(\kappa_k-k+1)} - q^{-(\kappa_j-j+1)}).
\]

Manipulating the product over \( j < k \), and introducing the normalisation by the requirement that the RHS equals unity for \( \kappa_l = 0 \) (\( l = 1, \ldots, N \)) gives the stated result. \( \square \)
Remark 3.2. In a particular notation for the Jacobi theta functions define

$$\theta_3(z; q) = \sum_{n=0}^{\infty} q^{n^2} z^n. \quad (3.10)$$

For a suitable normalisation $C_{N}^{(RS)}(q)$, where the superscript $(RS)$ stands for Rogers-Szegö in keeping with the name of the underlying orthogonal polynomials, introduce the PDF on $\theta_l \in [0, 2\pi]$ ($l = 1, \ldots, N$)

$$\frac{1}{C_{N}^{(RS)}(q)} \prod_{l=1}^{N} \theta_3(e^{i\theta_l}; q) \prod_{1 \leq j < k \leq N} |e^{i\theta_l} - e^{i\theta_j}|^2. \quad (3.11)$$

It is shown in [51, 11] that

$$\langle s_{\kappa}(e^{i\theta_1}, \ldots, e^{i\theta_N}) \rangle_{(RS)} = q^{\sum_{l=1}^{N} \kappa_l^2} \prod_{1 \leq j < k \leq N} \frac{1 - q^{2(k-j)}}{1 - q^{2(j-k)}}.$$  

Comparison with (3.3) and (3.4) shows

$$\langle s_{\kappa}(e^{i\theta_1}, \ldots, e^{i\theta_N}) \rangle_{(RS)} = \langle s_{\kappa}(u_1, \ldots, u_N) \rangle_{(SW')} \bigg|_{q \rightarrow q^{-2}}. \quad (3.12)$$

To understand (3.12), first note that with $z_j = e^{i\theta_j}$

$$\prod_{1 \leq j < k \leq N} |e^{i\theta_k} - e^{i\theta_j}|^2 d\theta_1 \cdots d\theta_N = i^{-N} \prod_{l=1}^{N} z_l^{-N} \prod_{1 \leq j < k \leq N} (z_k - z_j)^2 dz_1 \cdots dz_N,$$

which formally is identical to the portion of the PDF in (3.5) excluding $\prod_{l=1}^{N} \omega_l^{(SW)}(u_l; q)$. This is to be combined with the moment formula

$$\int_{|z|=1} \theta_3(z; q) z^n \frac{dz}{iz} = q^{(n+1)^2}, \quad n \in \mathbb{Z}. \quad (3.13)$$

Since (3.13) is identical in value to (3.8), but with $q \rightarrow q^{-2}$ in the latter, (3.12) follows.

The utility of Proposition 3.1 for the purposes of calculating the power sum average in (1.28) stems from the equality (see e.g. [45])

$$\sum_{j=1}^{N} u_j^l = \sum_{r=0}^{\min(l-N-1)} (-1)^r s_{(l-r, 1^r)}(u_1, \ldots, u_N), \quad (3.14)$$

where $(l-r, 1^r)$ denotes the partition with largest part $\kappa_1 = l - r$, and $r$ parts ($r \leq N - 1$) equal to 1. Thus the average of each of the Schur polynomials on the RHS of (3.14) can be read off from (3.3), and moreover identified in terms of particular $q$-binomial coefficients (1.23).
Corollary 3.3. For \( l - r \geq 1, r \leq N - 1 \), we have
\[
q^{Nl} \langle s_{(l-r,r)}(u_1, \ldots, u_N) \rangle_{\text{(SW)}} = q^{-(l-r)^2/2 - r/2} \left[ \frac{N + l - r - 1}{l} \right]_{q^{-1}} \left[ \frac{l - 1}{r} \right]_{q^{-1}}.
\] (3.15)

Summing (3.15) over \( r \) according to (1.22) gives a \( q \)-series which can, after some manipulation be recognised as a particular terminating \( 2\phi_1 \) sum, so establishing Proposition 1.1.

Remark 3.4. 1. In keeping with (3.12), replacing \( q \) by \( q^{-2} \) on the RHS of (3.15) gives the value of \( \langle s_{(l-r,r)}(e^{i\theta_1}, \ldots, e^{i\theta_N}) \rangle_{\text{(RS)}} \) [11 Eq. (4.5)], and this same replacement on the RHS of (1.31) gives \( \langle \sum_{j=1}^{N} e^{i\theta_j} \rangle_{\text{(RS)}} \) [11 Eq. (4.6)].

2. It has been remarked that the PDF specifying \( (SW^*) \) is unchanged by the mappings \( u_j \mapsto 1/u_j \). Hence
\[
q^{-Nl} m^{-1}_{-l}(SW) = q^{Nl} m_l^{(SW)},
\] (3.16)
so extending (1.31) to \( l \) a negative integer.

3. The little \( q \)-Jacobi polynomials in (1.30), being orthogonal polynomials, satisfy a 3-term recurrence [40]
\[
-q^{-N} p_l = A_l p_{l+1} - (A_l + A_{l-1}) p_l + A_{l-1} p_{l-1},
\] (3.17)
valid for \( l \geq 1 \), with initial conditions \( p_0 = 1, p_1 = 1 - q^{-N} \), and where
\[
A_l = \frac{(q^{(l+1)/2} - q^{-(l+1)/2})(q^{l/2} - q^{-l/2})}{(q^{l+1/2} - q^{l-1/2})(q^{l} - q^{-l})}.
\]

It follows that the moments similarly satisfy a 3-term recurrence. For the GUE (1.1), there is a well known 3-term recurrence for the moments due to Harer and Zagier [33]. For a derivation of the latter in the context of the Fermi gas interpretation of (1.1), and an extension, see [23].

4. Orthogonal polynomials from the Askey scheme have occurred in a number of recent works in random matrix theory; see [12, 24, 25, 2, 28].

3.2. Scaled large \( N \) limit. It has already been noted in Corollary 1.2 that with \( q \) given by (1.32) the moments as given by (1.31) admit a well defined large \( N \) limit. In keeping with Remark 3.2 upon replacing \( \lambda \mapsto -\lambda/2 \) this same expression gives the value of the limiting scaled moments for the Rogers-Szegö PDF (3.11) [51].

\[
\mu_{l,0}^{(RS)} := \lim_{N \to \infty} \frac{1}{N} \left\langle \sum_{j=1}^{N} e^{i\theta_j} \right\rangle_{(RS)} = \mu_{l,0}^{(SW^*)} \bigg|_{\lambda \to -\lambda/2}.
\] (3.18)

Defining the scaled densities
\[
\rho_{(1),0}^{(SW^*)}(x) = \lim_{N \to \infty} \frac{1}{N} \rho_{(1),0}^{(SW^*)}(x) \bigg|_{q = e^{-\lambda/N}}, \quad \rho_{(1),0}^{(RS)}(\theta) = \lim_{N \to \infty} \frac{1}{N} \rho_{(1),0}^{(RS)}(\theta) \bigg|_{q = e^{-\lambda/N}}.
\] (3.19)
we have the relations to the scaled moments

\[ \mu_{l,0}^{(SW^*)} = \int_0^{\infty} x^l \rho_{(1),0}^{(SW^*)}(x) \, dx, \quad \mu_{l,0}^{(RS)} = \int_0^{2\pi} e^{i\theta} \rho_{(1),0}^{(RS)}(\theta) \, d\theta. \] (3.20)

The second of these can be immediately inverted to give

\[ \rho_{(1),0}^{(RS)}(\theta) = 1 + \frac{1}{\pi} \sum_{l=1}^{\infty} \mu_{l,0}^{(RS)} \cos \theta l. \] (3.21)

Moreover, the explicit form of \( \{\mu_{l}^{(RS)}\}_{l=1}^{\infty} \) allows for the sum to be computed, with the result

\[ \rho_{(1),0}^{(RS)}(\theta) = \frac{1}{\pi \lambda} \log \left( \frac{1 - \cos \theta_e + 2 \cos \theta + 2 \cos(\theta/2) \sqrt{2 \cos \theta - 2 \cos \theta_e}}{1 + \cos \theta_e} \right) \chi_{\cos \theta \geq \cos \theta_e} \] (3.22)

where \( \cos \theta_e = 2e^{-A/2} - 1 \), and \( \chi_A = 1 \) for \( A \) true, \( \chi_A = 0 \) otherwise.

There is no literal analogue of (3.21) for the inversion of the first relation in (3.20). Instead, the standard strategy is to introduce the generating function

\[ G^{(SW^*)}(y) = \frac{1}{y} + \frac{1}{y} \sum_{l=1}^{\infty} \mu_{l,0}^{(SW^*)} y^{-l}. \] (3.23)

and note from a geometric series expansion that

\[ G^{(SW^*)}(y) = \int_{I} \frac{\rho_{(1),0}^{(SW^*)}(x)}{y - x} \, dx \] (3.24)

(here \( I \) denotes the interval of support of \( \rho_{(1),0}^{(SW^*)} \)). The relation can be inverted by the Sokhotski-Plemelj formula to give

\[ \rho_{(1),0}^{(SW^*)}(x) = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \left( G^{(SW^*)}(x - i\varepsilon) - G^{(SW^*)}(x + i\varepsilon) \right). \] (3.25)

For this to be of practical use we require the closed form of \( G^{(SW^*)} \).

**Proposition 3.5.** We have

\[ G^{(SW^*)}(y) = -\frac{1}{\lambda y} \log \left( \frac{1 + y + \sqrt{(1 + y)^2 - 4ye^y}}{2ye^y} \right). \] (3.26)

**Proof.** It follows from (3.23) and (1.33) that

\[ \frac{d}{dy} \left( yG^{(SW^*)}(y) \right) = -\frac{1}{\lambda y} \sum_{l=1}^{\infty} (-1)^l 2F_1(-l, l; e^{1/y} y^{-l}). \] (3.27)

Recognising that

\[ 2F_1(-l, l; e^{1/y}) = P_l^{(0,-1)}(1 - 2e^1), \]
where $p_{n}^{(a,b)}$ denotes the Jacobi polynomial in usual notation, the sum in (3.27) can be computed according to the standard generating function for the latter. This gives

$$
\frac{d}{dy}(yG_{SW}^{(SW)}(y)) = -\frac{1}{2\lambda y} \left( -1 + \frac{y - 1}{(1 + 2y(1 - 2e^{\lambda}) + y^2)^{1/2}} \right)
= -\frac{1}{\lambda} \frac{d}{dy} \log \left( 1 + y + \sqrt{\frac{(1 + y)^2 - 4ye^{\lambda}}{2ye^{\lambda}}} \right),
$$

where the second equality can be verified by a direct calculation. The formula (3.26) follows.

\[ \square \]

We can apply (3.25) to now deduce the scaled density.

**Corollary 3.6.** With $z = 1 - 2e^{\lambda}$ set $z_{\pm} = -z \pm (z^2 - 1)^{1/2}$. We have

$$
\rho_{SW}^{(SW)}(x) = \frac{1}{\pi \lambda x} \arctan \left( \frac{4e^{\lambda}x - (1 + x)^2}{1 + x} \right) \chi_{z_{-} < x < z_{+}}.
$$

**Proof.** After substituting (3.26) in (3.25), the result follows from the identity

$$
\arctan z = \frac{1}{2i} \log \frac{x - i}{x + i}.
$$

\[ \square \]

**Remark 3.7.** 1. Upon the change of variables $x \mapsto e^{-\lambda}x$, the functional form (3.29) has been derived using a loop equation formalism in [46]. Earlier, this same functional form was known from the computation of the density of the scaled zeros of the Stieltjes-Wigert polynomials [8,42].

2. Let $\hat{S}_{N}(u; q)$ denote the Stieltjes-Wigert polynomials (1.21) scaled so that the coefficient of $u^{l}$. The determinant structure implied by (1.18) gives [21] special case of Prop. 5.1.4]

$$
\left\langle \prod_{l=1}^{N} (x - x_{l}) \right\rangle_{SW}^{(SW)} = \hat{S}_{N}(x; q).
$$

With $q$ given by (1.32), for large $N$ the LHS can to leading order be expressed in terms of the scaled density $\rho_{(1,0)}^{(SW)}$ (see e.g. the discussion in the paragraph below (3.3) of [26]) to give

$$
\hat{S}_{N}(x; e^{-\lambda/N}) \sim \exp \left( N \int_{I} \log(x - y)\rho_{(1,0)}^{(SW)}(y) \, dy \right).
$$

(3.30)

Let us now combine (3.30) with knowledge of the fact that $\hat{S}_{N}$ obeys the second order $q$-difference equation (see e.g. [9])

$$
f(xq) - \frac{1}{x}f(x) + \frac{1}{x}f(x/q) = q^{N}f(x).
$$

(3.31)
Dividing both sides of (3.31) by \( f(x) \), substituting (3.30) and expanding \( q = (1 - \lambda/N + O(1/N^2)) \) shows

\[
e^{-u} + e^{u-y} = e^{-\lambda} + e^{-y}, \quad e^y := x, \quad e^u := \exp \left( \lambda x \int_{t}^{(SW)} (t) \frac{dt}{x-t} \right). \tag{3.32}
\]

With a different derivation (and slightly varying notation), this functional equation has been derived previously \cite{29,46}.

3. As commented below Proposition \cite{11}, the LHS of (1.34) can be expanded in a series in \( 1/N \). Thus, extending the notation for the leading term in (1.33), we have that for large \( N \)

\[
\frac{q^{Nl}}{N} m_l^{(SW)} \bigg|_{q=e^{-\lambda/N}} = \mu_{l,0}^{(SW)} + \frac{1}{N^2} \mu_{l,2}^{(SW)} + \frac{1}{N^4} \mu_{l,4}^{(SW)} + \cdots. \tag{3.33}
\]

For the Rogers-Szegö moments as specified by the average in (3.18), the analogous expansion up to this order has been calculated in \cite{51}. The theory of Remark \cite{32} tells us that replacing \( \lambda \) by \( -2\lambda \) in the latter gives the terms in (3.33). In particular, we read off from \cite{51} that

\[
\mu_{l,2}^{(SW)} = (-1)^l \frac{\lambda^2}{24} \sum_{p=1}^{l} \frac{e^{lp} - 1}{\lambda p} \left( \begin{array}{c} l+p-1 \\ l \end{array} \right) (2p-1)^2 - 2p^2. \tag{3.34}
\]

4. Special functions and the limiting correlation kernel

4.1. Stieltjes–Wigert correlation kernel. The PDF (1.18) specifying the ensemble (SW) is of a standard form familiar in the study of unitary invariant random matrices; see \cite{21} Ch. 5. In particular, the \( k \)-point correlation function \( \rho_{(k)} \), defined as \( N!/(N-k)! \) times the integral over all but the first \( k \) co-ordinates is therefore given as a determinant

\[
\rho_{(k)}^{(SW)}(u_1, \ldots, u_k) = \det \left[ K_{N}^{(SW)}(u_j, u_l) \right]_{j,l=1}^{k}, \tag{4.1}
\]

where \( K_{N}^{(SW)}(x, y) \) is referred to as the correlation kernel. The latter is given in terms of the orthonormal polynomials associated with the weight function \( w^{(SW)}(u) \), i.e. the Stieltjes–Wigert polynomials (1.21), by the sum

\[
K_{N}^{(SW)}(u, v) = \left( w^{(SW)}(u) w^{(SW)}(v) \right)^{1/2} \sum_{j=0}^{N-1} S_j(u; q) S_j(v; q)
\]

\[
= \left( w^{(SW)}(u) w^{(SW)}(v) \right)^{1/2} \frac{C_N}{C_{N-1}} \frac{S_N(u; q) S_{N-1}(v; q) - S_{N-1}(u; q) S_N(v; q)}{u-v}, \tag{4.2}
\]

where the second line follows by the Christoffel–Darboux summation formula (see e.g. \cite{21} Prop. 5.1.3). In this formula \( C_N \) is the coefficient of \( u^N \) in \( S_N(u; q) \), which we read off from
to be given by
\[ C_N = \frac{q^{N^2+N+1/4}}{((1-q) \cdots (1-q^N))^{1/2}}. \] (4.3)

The change of variables (1.16) shows that the correlation kernel for the ensemble \((SW_e)\) is related to (4.2) by
\[ K_N^{(SW_e)}(x, y) = \frac{2\pi}{L} (uvw(u; q)w(v; q))^{1/2} K_N^{(SW)}(u, v) \bigg|_{u=q^{-N,2\pi x/L}, v=q^{-N,2\pi y/L}}. \] (4.4)

### 4.2. Scaling limits.
With \(L\) fixed and \(N\) large, the interpretation of (1.13) in terms of the Boltzmann factor (2.1) gives the prediction \([20]\) that to leading order the density will be supported on the interval \([-\pi N/Lc, \pi N/Lc]\) and is on average uniform. Hence there are two distinct scaling limits. One is when the particles are located in the interior of the support and away from the edges. To accomplish this we choose to locate the particles in the neighbourhood of the origin. The other is when the particles are located a finite distance from one of the edges. We consider each separately.

#### 4.2.1. Bulk scaling.
Define
\[ \ell(y; q) = e^{\xi \pi y / 2 L} q^{-1/16} \sum_{v=-\infty}^{\infty} (-1)^v q^{(v+1/4 - c Ly/2\pi)^2} \]
\[ = e^{(\xi+1)\pi y / 2 L} e^{-c y^2 / 2} \theta_3(-q^{1/2} e^{\pi Ly / L}; q), \] (4.5)
\[ \hat{\ell}(y; \hat{q}) = e^{-\xi \pi y / 2 L} \theta_1 \left( \pi \left( \frac{\pi}{4} + \frac{c Ly}{2\pi} \right) |\hat{q}^2 \right), \] (4.6)

where in (4.6) we have adopted the particular notation for the Jacobi theta function
\[ \theta_1(x|q) := -i \sum_{n=-\infty}^{\infty} (-1)^n q^{(n+1/2)^2} e^{2i(n+1/2)x} \] (4.7)

(note that this convention differs from that used in the second line of (4.5) as specified by (3.10)). The bulk scaling limit of the kernel (4.4) has been established in terms of these functions in [20].

**Proposition 4.1.** Let \(\xi = 1\) for \(N\) even, and \(\xi = -1\) for \(N\) odd. Also define
\[ \hat{q} = e^{-c L^2 / 2}. \] (4.8)
We have

\[
K^{(\text{SW})}_{\text{bulk}}(x, y) := \lim_{N \to \infty} K^{(\text{SW})}(x, y) = \left( \frac{c}{\pi} \right)^{1/2} \frac{1}{(q ; q)_\infty^3} \frac{\ell(x ; q) \ell(-y ; q) - \ell(y ; q) \ell(-x ; q)}{2 \sinh(\pi(x - y) / L)} = \frac{1}{L} \frac{1}{\theta'_1(0|q^2)} \frac{\hat{\ell}(x ; q) \hat{\ell}(-y ; q) - \hat{\ell}(y ; q) \hat{\ell}(-x ; q)}{2 \sinh(\pi(x - y) / L)}.
\] (4.9)

Remark 4.2. 1. The first of the equalities in (4.12), in the case \( N \) odd, has also been obtained in [57]. Moreover, asymptotic estimates from this latter reference show that rate of convergence to the limit has a correction term proportional to \( q^N \), and is thus exponentially small.

2. The bulk density is obtained by taking the limit \( y \to x \) in (4.12). The resulting expression is a non-trivial periodic function of period \( 2 \pi / cL \). Indeed a crystalline phase is a characteristic of the many body state resulting from the \( |x| \) potential in one-dimension [6].

3. It follows from the final equality in (4.12) and (4.6) and (4.7) that

\[
\lim_{L \to \infty} \frac{2 \pi}{cL} K^{(\text{SW})}_{\text{bulk}}(\frac{2 \pi}{cL} x, \frac{2 \pi}{cL} y) = \frac{\sin \pi(x - y)}{\pi(x - y)},
\] (4.10)

(this result is also deduced in [57] by a somewhat complicated calculation using working based on the first of the evaluations in (4.12)). The kernel in (4.10) is well known as specifying the bulk scaling limit of Hermitian random matrices; see e.g. [17].

4.2.2. Edge scaling. Writing \( x = -\pi N / Lc + X \) as appropriate for the analysis of the (left) edge scaling (recall the discussion at the beginning of §4.2), we see from (4.4) and (4.2) that relevant is the large \( N \) form of \( S_N(z ; q) \) with \( z, q \) fixed. In relation to this, it is easy to check from the definition (1.21) that

\[
S_N(z ; q) = (q ; q)_N^{1/2} \frac{(-1)^N q^{N/2 + 1/4} \left( A_q(q^{1/2}z) - \frac{q^{1+N}}{1-q} A_q(q^{-1/2}z) + O(q^{2N}) \right)}{(q ; q)_\infty},
\] (4.11)

where \( A_q(y) \) is specified by (1.27). In fact the limit implied by the leading term is already in Wigert’s original paper [62].

From the expansion (4.11) the scaling limit of (4.4) is immediate (this limit formula was presented in [20] but with (4.11) only implicit).
Proposition 4.3. We have

$$K_{\text{edge}}^{(SW)}(X, Y) := \lim_{N \to \infty} K^{(SW)}(x, y) \Big|_{x = -\pi N/2 + X, y = -\pi N/2 + Y}$$

$$= \left( \frac{c}{\pi} \right)^{1/2} \frac{1}{(q; q)_{\infty}} e^{-c(X^2 + Y^2)/2} \sinh(\pi(X - Y)/L) \times \left( A_q(q^{1/2}e^{2\pi X/L}; q) A_q(q^{-1/2}e^{2\pi Y/L}; q) - A_q(q^{1/2}e^{2\pi X/L}; q) A_q(q^{-1/2}e^{2\pi Y/L}; q) \right).$$

(4.12)

Remark 4.4. 1. The order of the remainder in (4.11) implies that the convergence to the limit happens at a rate proportional to $q^N$ and is thus exponentially fast.

2. It is noted in [26] that the first evaluation in Proposition 4.1 can be reclaimed for $N$ even (odd) from (4.12) by making the replacements $X \mapsto M + x$ ($M - 1/2 + x$), $Y \mapsto M + y$ ($M - 1/2 + y$) and taking the limit $M \to \infty$.

3. The edge scaling limit of the Christoffel-Darboux kernel for both the $q$-Hermite and $q$-Laguerre orthogonal polynomial systems has been shown in [36] to have explicit forms also involving the function $A_q(z)$, but which are distinct from each other, and distinct from (4.12).

4. According to (4.4) and (4.2), a direct analysis of the right edge scaling limit requires the large $N$ form of $S_N(q^{-2N}; z; q)$ with $z, q$ fixed. This expansion can be found in [36], which is obtained from the series (1.21) by first replacing $\nu$ by $l - \nu$. As commented in [66], the latter replacement implies the symmetry

$$S_n(z; q) = (-zq^n)^n S_n \left( \frac{1}{zq^{2n}}; q \right).$$

This symmetry used in (4.4) and (4.2) maps the right edge to the left edge, showing both are equivalent, as can be anticipated from (4.13).

Associated with the (left) edge scaled kernel $K_{\text{edge}}^{(SW)}$ is the gap probability

$$E_{\text{edge}}^{(SW)}(0; (-\infty, s)) = \det \left( I - K_{\text{edge}}^{(SW)} \right|_{(-\infty, s)}.$$

(4.13)

Here $K_{\text{edge}}^{(SW)} \left|_{(-\infty, s)}$ denotes the integral operator on $(-\infty, s)$ with kernel $K_{\text{edge}}^{(SW)}(X, Y)$. The gap probability in turn determines the probability density function of the scaled position of the leftmost particle, $p_{\text{left}}^{(SW)}(s)$ say, by a simple differentiation

$$p_{\text{left}}^{(SW)}(s) = -\frac{d}{ds} E_{\text{edge}}^{(SW)}(0; (-\infty, s)).$$

(4.14)
The gap probability admits an expansion in terms of the edge correlations, obtained by substituting \(K_{\text{edge}}^{(SW)}\) in (4.1) (see e.g. [21] Eq. (9.4))

\[
K_{\text{edge}}^{(SW)}(0; (-\infty, s)) = 1 - \int_{-\infty}^{s} K_{\text{edge}}^{(SW)}(x, x) \, dx + \frac{1}{2} \int_{-\infty}^{s} dx_1 \int_{-\infty}^{s} dx_2 \det \begin{bmatrix} K_{\text{edge}}^{(SW)}(x_1, x_1) & K_{\text{edge}}^{(SW)}(x_1, x_2) \\ K_{\text{edge}}^{(SW)}(x_2, x_1) & K_{\text{edge}}^{(SW)}(x_2, x_2) \end{bmatrix} - \cdots
\]

(4.15)

It follows from (4.14) and (4.15) that

\[
p_{\text{left}}^{(SW)}(s) \sim K_{\text{edge}}^{(SW)}(s, s),
\]

(4.16)

and so recalling (4.12) and (1.27) we have that \(p_{\text{left}}^{(SW)}(s)\) exhibits a leading order Gaussian decay \(e^{-c^2}\) in its left tail. This is in agreement with the prediction from [14] Eq. (8)] for the classical one-dimensional Coulomb gas in a confining harmonic potential of strength 1/4 (therefore, upon comparing with (3.1), we must set \(c = 1/2\) for the results of [14]). This same reference also predicts the leading behaviour in the right tail

\[
p_{\text{left}}^{(SW)}(s) \sim \exp \left( -s^3/24\alpha + O(s^2) \right), \quad \alpha = \pi/L.
\]

(4.17)

While we know of no direct way to establish this result from (4.13), by modifying the statistical mechanics model (3.1) to its natural two-dimensional extension (recall the discussion of §2.1), an analytic derivation of (4.17) is possible; see the Appendix.

4.3. Scaling limit of \(A_q(z)\) and the Airy kernel. It is well known that for the PDF (1.1) the Christoffel–Darboux kernel \(K^{(G)}(x, y)\) has a bulk scaling limit equal to the RHS of (4.10); see e.g. [21] Ch. 7]. We know from (4.10) that an appropriate \(L \to \infty\) scaling of \(K_{\text{bulk}}^{(SW)}\) reclaims this functional form. To leading order the edges of the spectrum for (1.1) interpreted as an eigenvalue PDF are at \(\pm\sqrt{2N}\), and \(K^{(G)}\) admits the edge scaling limit (1.26). The question to be addressed is to identify a scaling of \(X, Y\) such that for \(L \to \infty\) the kernel \(K_{\text{edge}}^{(SW)}(X, Y)\) reduces to the Airy kernel (1.26). This is answered in Proposition 1.3. For the proof, appropriate asymptotic properties of the special function \(A_q(z)\) (1.27) are required.

Before introducing these asymptotic properties, which fortunately are available in the literature [44 32 31], some contextual information relating to \(A_q(z)\) is appropriate. Firstly, names associated with \(A_q(z)\) are the Ramanujan function, and the \(q\)-Airy function; see [36] for the underlying reasons. In relation to the latter, it is important to be aware that there are other candidates which qualify for the title of \(q\)-Airy functions, see in particular [39] for special function solutions of the \(q\)-Painlevé II equation which are shown to limit to the Airy
function. Actually these various candidates can be related \cite{37}; see also \cite{49}. We remark too that \(A_q(z)\) satisfies the functional equation (\(q\)-difference equation)

\[ qxu(q^2x) - u(qx) + u(x) = 0, \]

and can also be regarded as a degeneration of the basic hypergeometric function (1.29), being given by

\[ A_q(x) = \phi_1\left( \frac{-}{0} | q; -qx \right), \quad \phi_1\left( \frac{-}{b} | q; z \right) := \sum_{n=0}^\infty \frac{q^{n(n-1)}}{(q;q)_n(b;q)_n} z^n. \]

**4.3.1. Proof of Proposition 1.3** The key to establishing (1.35) is an asymptotic formula contained in \cite{37}. Specifically, with

\[ q = e^{-\epsilon}, \quad \alpha = 1 - 4z, \quad \beta = \frac{\log(z)^2}{4} + \frac{\pi^2}{12}, \]

we have from \cite{37} Theorem 4.7.1, after simplification of (4.55) (see also \cite{32}, and compare the leading term with \cite{44} Th. 2]), that for \(\epsilon \to 0^+\), with \(\alpha / \epsilon^{2/3}\) fixed

\[ A_q(z) = \frac{1}{2} (q; q)_\infty \epsilon^{\beta / \epsilon} \left( \text{Ai} \left( \frac{\alpha}{\epsilon^{2/3}} \right) \epsilon^{1/3} - \text{Ai}' \left( \frac{\alpha}{\epsilon^{2/3}} \right) \epsilon^{2/3} \right) \left( 1 + O(\epsilon) \right). \]

To see the relevance of (4.19) in relation to (4.12), note that with \(X(x, l)\) as in (1.34)

\[ e^{(2\pi/L)X(x, l)} = \frac{1}{4} (1 - e^{2/3} x) + O(e^{4/3}), \]

and so considering \(A_q(e^{(2\pi/L)X(x, l)})\) leads to (4.19) with \(\alpha = 2/3 x\). A (minor) detail is that (4.12) requires not \(A_q(e^{(2\pi/L)X(x, l)})\) but rather \(A_q(q^{-1/2}e^{(2\pi/L)X(x, l)})\). By a first order Taylor expansion, this changes the prefactor of \(\text{Ai}'\) in (4.19) from \(-1\) to \(-\frac{1}{2}\) and \(-\frac{3}{2}\) respectively. Noting this, substituting in (4.12), and recalling too the standard \(\epsilon \to 0^+\) asymptotic formula

\[ \log(q; q)_\infty = -\frac{\pi^2}{6\epsilon} + \log \sqrt{\frac{2\pi}{\epsilon}} + O(\epsilon), \]

which follows from the functional equation for the Dedekind eta function \cite{63}, we find after some minor simplification that (1.35) results.

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Appendix

The statistical mechanical model on a cylinder, as referred to in Section 2.1, was shown to be exactly solvable at \( \beta = 2 \) by Choquard \cite{10}, and further elaborated on in \cite{11}. Define

\[
W_L := \frac{1}{4} \sum_{j=1}^{N} x_j^2 + \sum_{1 \leq j < k \leq N} \Phi((x_j, y_j), (x_j', y_j')) \tag{A.1}
\]

with \( \Phi \) given by (3.7), so that the Boltzmann factor with \( \beta = 2 \) is \( e^{-2W_L} \). For this system, the probability \( E^{(2d)}(0, (-\infty, s) \times (0, L)) \) say that there are no particles in the region \( x \in (-\infty, s), \ y \in [0, L) \), is given by

\[
E^{(2d)}(0, (-\infty, s) \times (0, L)) = Q_N(s)/Q_N(-\infty), \tag{A.2}
\]

where

\[
Q_N(s) := \int_{s}^{\infty} dx_1 \cdots \int_{s}^{\infty} dx_N \int_{0}^{L} dy_1 \cdots \int_{0}^{L} dy_N e^{-2W_L}. \tag{A.3}
\]

The integration technique of \cite{10}, \cite{11}, which uses the fact that with \( z_j := e^{2\pi i (y_j + ix_j)/L} \),

\[
e^{-2W_L} \propto \prod_{l=1}^{N} e^{-\sum_{j=1}^{N} (x_j^2 - 4\pi (N-1)x_j/L)/2} \prod_{1 \leq j < k \leq N} (z_k - z_j)(\bar{z}_k - \bar{z}_j),
\]

then replaces each product with a Vandermonde determinant according to (1.7), allows \( Q_N(s) \) to be computed explicitly as a product of one-dimensional integrals. Substituting in (A.2) shows

\[
E^{(2d)}(0, (-\infty, s) \times (0, L)) = \prod_{l=1}^{N} \sqrt{\frac{2}{\pi}} \int_{s}^{\infty} e^{-(x-2\pi(N-2l+1)/L)^2/2} \ dx. \tag{A.4}
\]

In the notation of (3.1), (A.1) corresponds to a harmonic potential of strength \( c = 1/2 \), so from the discussion of §4.2 the left edge occurs at \( s^* := -2\pi N/L \). It follows from (A.4) that

\[
E^{(2d)}_{\text{edge}}(0, (s, \infty)) := \lim_{N \to \infty} E^{(2d)}(0, (s^* + s, \infty) \times (0, L)) = \prod_{l=0}^{\infty} \sqrt{\frac{2}{\pi}} \int_{s}^{\infty} e^{-(x-2\pi(2l+1)/L)^2/2} \ dx. \tag{A.5}
\]

For \( s \) large in this expression, see that the term \( l \) in the product contributes of order \( e^{-(s-2\pi(2l+1)/L)^2/2} \) for \( l \) up to the value \( sL/2\pi \) (appropriately rounded), and unity after this. Hence to leading order

\[
E^{(2d)}_{\text{edge}}(0, (s, \infty)) \sim \prod_{l=0}^{\left\lfloor sL/2\pi \right\rfloor} e^{-(s-2\pi(2l+1)/L)^2/2} \sim e^{-s^2L/24\pi}, \tag{A.6}
\]

where the final asymptotic expression follows by summing the exponents in the expression before, observing it can be written as a Riemann sum. This gives agreement with (4.17).
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School of Mathematics and Statistics, ARC Centre of Excellence for Mathematical and Statistical Frontiers, University of Melbourne, Victoria 3010, Australia

Email address: pjforr@unimelb.edu.au