REMARKS ON THE CRITICAL COUPLING STRENGTH FOR THE CUCKER-SMALE MODEL WITH UNIT SPEED

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Abstract. We present a non-trivial lower bound for the critical coupling strength to the Cucker-Smale model with unit speed constraint and short-range communication weight from the viewpoint of a mono-cluster (global) flocking. For a long-range communication weight, the critical coupling strength is zero in the sense that the mono-cluster flocking emerges from any initial configurations for any positive coupling strengths, whereas for a short-range communication weight, a mono-cluster flocking can emerge from an initial configuration only for a sufficiently large coupling strength. Our main interest lies on the condition of non-flocking. We provide a positive lower bound for the critical coupling strength. We also present numerical simulations for the upper and lower bounds for the critical coupling strength depending on initial configurations and compare them with analytical results.

1. Introduction. Collective self-driven synchronized motions such as the aggregation of bacteria, flocking of birds and swarming of fish are often observed in biological complex system [19, 20, 21, 37, 44, 48, 49, 50, 51]. They have been extensively studied in an engineering domain, because of their potential applications to unmanned aerial vehicles and client network equipments, etc. [37, 45, 44]. In a half century ago, Winfree and Kuramoto [34, 51] proposed several agent-based models which are studied extensively both numerically and analytically. Recently, several Vicsek type particle models with unit speed were proposed in literatures...

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for the study of velocity alignment. Motivated by a quantum synchronization model \[38, 39\], one of the authors \[14\] discussed the mono-cluster flocking for the Cucker-Smale model with unit speed constraint. As far as the authors know, constant speed constraint matters a lot in flocking modeling due to the historical development of flocking in physics literature. The modeling of flocking phenomena was first introduced by Vicsek’s group \[50\] in the physics community and the unit speed constraint was employed in relation with the phase models for synchronization, e.g., the Kuramoto model \[35, 36\]. This Vicsek’s work advanced further research \[23, 32, 33, 37, 40, 43, 44, 46\] on the collective dynamics of interacting multi-agent systems in engineering and physics.

Let $x_i$ and $v_i$ be the position and velocity of the $i$-th particle, respectively. Then, the dynamics of $(x_i, v_i)$ is governed by the following system of ODEs:

$$
\dot{x}_i = v_i, \quad t > 0, \quad i = 1, 2, \ldots, N,
$$

$$
\dot{v}_i = \frac{\kappa}{N} \sum_{k=1}^{N} \psi(||x_k - x_i||) \left( \frac{v_k - (v_k \cdot v_i)}{v_i \cdot v_i} v_i \right),
$$

(1.1)

where $\kappa$ is a positive coupling strength, and $\psi$ is the short-range communication weight satisfying the positivity, continuity, monotonicity, and integrability conditions:

$$
0 < \psi(r) \leq \psi(0) := 1, \quad r \geq 0, \quad (\psi(r_2) - \psi(r_1))(r_2 - r_1) \leq 0, \quad r_1, r_2 \geq 0,
$$

$$
||\psi||_{L^{ip}} = \sup_{r_1 \neq r_2} \frac{|\psi(r_2) - \psi(r_1)|}{|r_2 - r_1|} < \infty, \quad ||\psi||_{L^1} := \int_0^\infty \psi(s)ds < \infty.
$$

(1.2)

The Cucker-Smale model and its variants have been extensively studied in previous literature \[1, 2, 4, 6, 8, 9, 10, 12, 13, 15, 16, 17, 22, 24, 26, 29, 30, 31, 41, 42, 47\] from the viewpoint of mono-cluster flocking in terms of initial configurations and communication weights. For example, for a long-range communication weight with an infinite $L^1$-norm, mono-cluster(global) flocking occurs for any initial configurations. In contrast, for a short-range communication weight, numerical simulations in \[12, 13\] illustrate that mono-cluster(global) flocking does not occur for some initial configurations. There are also interesting works on optimal control of Cucker-Smale model \[3, 5, 7\]. Recently, the authors \[28\] present a lower bound of the critical coupling strength for the Cucker-Smale model. Our main purpose in this paper is to look for the existence of a critical coupling strength of the Cucker-Smale model with unit speed constraint.

In this paper, we provide the existence of the critical coupling strength $\kappa_c$ for short-range communication weights. More precisely, we provide a positive lower bound $\kappa_0$ for $\kappa_c$ depending on the initial configurations. For a given initial configuration, our strategy for estimating a critical coupling strength can be summarized. We first classify an ensemble of particles into several sub-ensembles according to the initial velocities, i.e., two particles are in the same sub-ensemble if and only if they have the same initial velocities, and we then prove that if $\kappa < \kappa_0$, then any two particles in different groups will not flock forever. An interesting question is whether the particles in the same sub-ensemble will flock or not. This motivates us to study the phenomena of multi-cluster flocking. We allow different particles to have different initial velocities and we assume that the differences among particles in same groups is small comparing to the differences between particles in different…
groups, then multi-cluster flocking will appear if each group departs from the others (see Theorem 4.2).

The rest of the paper is organized as follows. In Section 2, we review some basic properties of the Cucker-Smale model with a short-range communication weight \( \psi \) which will be useful in the following sections. In Section 3, we show that mono-cluster flocking will not emerge in a small coupling strength regime. In Section 4, we prove the emergence of multi-cluster flocking for suitable small coupling strength. In Section 5, we provide several numerical simulations and compare them to our analytic results in previous sections. Finally, Section 6 is devoted to the summary of our main results.

**Notation.** Throughout this paper, we use the superscript to denote the component of vector and subscript to denote the ordering of particles. For vectors \( x = (x^1, \cdots, x^d), v = (v^1, \cdots, v^d) \in \mathbb{R}^d, \) \( l^2 \)-norm, and inner product are defined as follows:

\[
\|x\| := \sqrt{\sum_{j=1}^{d} |x^j|^2}, \quad \langle x, v \rangle := \sum_{j=1}^{d} x^j v^j.
\]

2. **Preliminaries.** In this section, we first recall the concepts of multi-cluster flocking, critical coupling strength, basic a priori estimates for (1.1), and then summarize previous results on the flocking estimate.

2.1. **The Cucker-Smale model with unit speed.** In this subsection, we briefly discuss the basic a priori estimates for (1.1). We set

\[
A(v) := \min_{i \neq j} \langle v_i(t), v_j(t) \rangle.
\]

**Lemma 2.1.** [14] Let \( (x_i(t), v_i(t)) \) be a global solution to system (1.1) with initial data with unit speed constraint:

\[
\|v^0_i\| = 1, \quad 1 \leq i \leq N, \quad A(v^0) > 0.
\]

Then, we have

(i) \( \|v_i(t)\| = 1 \), for all \( t \geq 0 \), \( i = 1, \cdots, N \),

(ii) \( A(v(t)) \geq A(v^0), \quad t \geq 0 \).

**Proof.** (i) We multiply equation (1.1) by \( 2v_i(t) \) and sum the results together to have

\[
\frac{d}{dt} \|v_i(t)\|^2 = 2\kappa \frac{N}{N} \sum_{k=1}^{N} \psi(||x_k(t) - x_i(t)||) \left( \langle v_k(t), v_i(t) \rangle - \frac{\langle v_k(t), v_i(t) \rangle}{\langle v_i(t), v_i(t) \rangle} \langle v_i(t), v_i(t) \rangle \right) = 0.
\]

Hence we conclude that \( \|v_i(t)\| = \|v^0_i\| \), for all \( t \geq 0 \) and \( i = 1, \cdots, N \).

(ii) For the proof of non-increasing property of \( A(v) \), we refer to Lemma 2.2 in [14]. \( \square \)

Next, we briefly discuss the relationship between our C-S model with unit speed constraint and the flocking model introduced in [27] for two-dimensional case \( d = 2 \): We set

\[
v_i = (\cos \theta_i, \sin \theta_i).
\]
Then, we substitute the ansatz (2.1) into the equation (1.1) to obtain
\[ (-\sin \theta_i, \cos \theta_i) \dot{\theta}_i = \frac{\kappa}{N} \sum_{k=1}^{N} \psi_{ik} \left[ \cos \theta_k, \sin \theta_k \right] - \left[ \cos \theta_i \cos \theta_k + \sin \theta_i \sin \theta_k \right] (\cos \theta_k, \sin \theta_k). \]

We take an inner product with \((-\sin \theta_i, \cos \theta_i)\), then we obtain
\[ \frac{d\theta_i}{dt} = \frac{\kappa}{N} \sum_{k=1}^{N} \psi_{ik} \sin(\theta_k - \theta_i). \]

Thus, our proposed model (1.1) becomes the model in [27]:
\[ \frac{dx_i}{dt} = e^{\sqrt{-\kappa} \theta_i}, \quad \frac{d\theta_i}{dt} = \frac{\kappa}{N} \sum_{k=1}^{N} \psi_{ik} \sin(\theta_k - \theta_i). \]

2.2. Review on the previous results. In this subsection, we recall definitions of mono-cluster flocking and multi-cluster flocking, and summarize the result in [14] for the mono-cluster flocking.

**Definition 2.2.** [12, 13] Let \( G := \{(x_i, v_i)\}_{i=1}^{N} \) be an ensemble of the Cucker-Smale flocking group.

1. The configuration \( G \) tends to a mono-cluster (global) flocking configuration asymptotically, if the following two conditions hold:
   \[ \sup_{0 \leq t < \infty} \|x_i(t) - x_j(t)\| < \infty, \quad \lim_{t \to \infty} \|v_i(t) - v_j(t)\| = 0. \]

2. The configuration \( G \) tends to a multi-cluster (local) flocking configuration asymptotically, if there exist subclasses \( G_\alpha = \{(x_{\alpha i}(t), v_{\alpha i}(t))\}_{i=1}^{N_\alpha}, \quad \alpha = 1, 2, \ldots, n \) such that
   (i) \( |G_\alpha| \geq 1, \quad \sum_{\alpha=1}^{n} |G_\alpha| = N, \quad G = \bigcup G_\alpha, \)
   \[ \sup_{0 \leq t < \infty} \|x_{\alpha i}(t) - x_{\alpha j}(t)\| < \infty, \quad \lim_{t \to \infty} \|v_{\alpha i}(t) - v_{\alpha j}(t)\| = 0, \]
   for any \( \alpha \in \{1, 2, \ldots, n\} \), and \( 1 \leq i \neq j \leq N_\alpha \).
   (ii) \( \sup_{0 \leq t < \infty} \|x_{\alpha i}(t) - x_{\beta j}(t)\| = \infty, \quad \liminf_{t \to \infty} \|v_{\alpha i}(t) - v_{\beta j}(t)\| > 0, \)
   for any \( \alpha \neq \beta, \quad 1 \leq i \leq N_\alpha, \) and \( 1 \leq j \leq N_\beta \).

We next present a concept of the critical coupling strength for the emergence of mono-cluster flocking as follows.

**Definition 2.3.** For a given initial configuration \((x^0, v^0)\), a nonnegative constant \( \kappa_c = \kappa_c(x^0, v^0) \) is a critical coupling strength for mono-cluster flocking if and only if the following two criterions hold.

1. If \( \kappa > \kappa_c \), then initial configuration \((x^0, v^0)\) tends to a mono-cluster flocking asymptotically.
2. If \( \kappa < \kappa_c \), then initial configuration \((x^0, v^0)\) does not tend to a mono-cluster flocking asymptotically.

Before the end of this subsection, we recall the flocking estimates on the mono-cluster formation for (1.1). We set
\[ D(v) := \max_{i,j} \|v_i - v_j\|. \]
Theorem 2.4. [14] Suppose that the coupling strength and initial configuration \((x^0, v^0)\) satisfy
\[
\kappa > 0, \quad \|v^0_i\| = 1, \quad \min_{i \neq j} \langle v^0_i, v^0_j \rangle > 0, \quad 0 < D(v^0) < \frac{\kappa A(v^0)}{2} \int_{D(x^0)} \psi(s) \, ds,
\]
Then, for any solution \((x(t), v(t))\) to system (1.1), there exists a positive constant \(d_x^\infty\) such that
\[
\sup_{0 \leq t < \infty} D(x(t)) \leq d_x^\infty, \quad t \geq 0, \quad D(v(t)) \leq D(v^0) e^{-\kappa C_0 \psi(d_x^\infty t)}.
\]

Remark 2.5. Note that Theorem 2.4 yields a sufficient condition for a mono-cluster flocking. For a small coupling strength \(\kappa \ll 1\), bi-cluster and multi-cluster flockings can emerge from some initial configurations. It has been shown that local flocking, in particular bi-cluster flocking, can emerge from some well-prepared configurations close to bi-cluster configurations [12, 13].

3. A necessary condition for a mono-cluster flocking. In this section, we provide a framework for the non-existence of mono-cluster flocking and state a necessary condition for the emergence of a mono-cluster flocking.

3.1. A framework and main result. In this subsection, we will introduce a framework for the non-existence of mono-cluster flocking. Let \(G := \{(x^0_i, v^0_i)\}_{i=1}^N\) be an initial non-flocking configuration of the ensemble of C-S particles. Then, we set sub-ensembles \(G_1, \ldots, G_n\) of the total ensemble \(G\) according to initial velocity: for \(\alpha = 1, \ldots, n,\)
\[
(x_{\alpha i}, v_{\alpha i}), (x_{\alpha j}, v_{\alpha j}) \in G_\alpha \iff v^0_{\alpha i} = v^0_{\alpha j}, \quad \text{for all } i, j \leq |G_\alpha| =: N_\alpha.
\]
Since we assume the initial configuration is not in the mono-cluster flocking state, we have \(n \geq 2\), and the original system (1.1) can be rewritten as:
\[
\begin{align*}
\dot{x}_{\alpha i} &= v_{\alpha i}, \quad t > 0, \quad i = 1, 2, \ldots, N_\alpha, \\
\dot{v}_{\alpha i} &= \frac{\kappa}{N} \sum_{k=1}^{N_\alpha} \psi(||x_{\alpha k} - x_{\alpha i}||) \left( v_{\alpha k} - \frac{\langle v_{\alpha k}, \alpha i \rangle}{\langle v_{\alpha i}, v_{\alpha i} \rangle} v_{\alpha i} \right) \\
&\quad + \frac{\kappa}{N} \sum_{\beta \neq \alpha} \sum_{k=1}^{N_\beta} \psi(||x_{\beta k} - x_{\alpha i}||) \left( v_{\beta k}(t) - \frac{\langle v_{\beta k}, \alpha i \rangle}{\langle v_{\alpha i}, v_{\alpha i} \rangle} v_{\alpha i} \right), \\
(x_{\alpha i}(0), v_{\alpha i}(0)) &= (x^0_{\alpha i}, v^0_{\alpha i}), \quad \|v^0_{\alpha i}\| = 1.
\end{align*}
\]
Here we assume the short-range communication weight such that
\[
\psi(s) = \frac{1}{(1 + s^2)^{\frac{\beta}{2}}}, \quad \beta \geq 1.
\]
For conveniences, we introduce local averages:
\[
x^c_\alpha := \frac{1}{N_\alpha} \sum_{i=1}^{N_\alpha} x_{\alpha i}, \quad v^c_\alpha := \frac{1}{N_\alpha} \sum_{i=1}^{N_\alpha} v_{\alpha i}.
\]
Now, we describe the geometry of initial separation between sub-ensembles. For a given initial configuration \( (x^0, v^0) \), we set
\[
\theta_0(x^0, v^0) := \min_{\beta \neq \alpha} \arccos (v^0_0(0) \cdot v^0_\alpha(0)), \quad D(x^0) := \max_{\beta \neq \alpha,i,k} \| x^0_\alpha - x^0_{\beta k} \|, \quad T_0(x^0, v^0) := \max_{\beta \neq \alpha,i,k} \left\{ 0, -\frac{(x^0_\alpha - x^0_{\beta k}) \cdot v^0_\alpha(0)}{\lambda_0} \right\},
\]
where \( \lambda_0 := \cos \frac{\theta_0}{4} - \cos \frac{3\theta_0}{8} \).

For notational simplicity, we suppress \( (x^0, v^0) \) dependence in \( T_0, \kappa_0 \) in the following:
\[
\theta_0 := \theta_0(x^0, v^0), \quad T_0 := T_0(x^0, v^0).
\]

**Remark 3.1.** We can easily see that \( \theta_0 \in (0, \pi] \).

We next introduce a coupling strength \( \kappa_0(x^0, v^0) \) depending on the geometry of the initial configuration \( (x^0, v^0) \).

- If initial configuration satisfies
  \[
  \min_{\beta \neq \alpha,i,k} (x^0_\alpha - x^0_{\beta k}) \cdot v^0_\alpha(0) < 0,
  \]
  then, we set
  \[
  \kappa_0(x^0, v^0) := \min \left\{ 1 - \cos \frac{\theta_0}{8}, \frac{\cos \theta_0 - \cos \frac{\theta_0}{4}}{D(x^0) + 2T_0}, \frac{\lambda_0(\cos \frac{\theta_0}{8} - \cos \frac{\theta_0}{4})}{(1 - \gamma_N) \int_0^\infty \psi(s) ds} \right\},
  \]
  \[
  \gamma_N := \min \frac{N_\beta}{N},
  \]
- If initial configuration satisfies
  \[
  \min_{\beta \neq \alpha,i,k} (x^0_\alpha - x^0_{\beta k}) \cdot v^0_\alpha(0) \geq 0,
  \]
  then, we set
  \[
  \kappa_0(x^0, v^0) := \frac{\tilde{\lambda}_0(1 - \cos \frac{\theta_0}{8})}{(1 - \gamma_N) \int_0^\infty \psi(s) ds}, \quad \tilde{\lambda}_0 = \cos \frac{\theta_0}{8} - \cos \frac{7\theta_0}{8}.
  \]

Now we are ready to state our main result as follows.

**Theorem 3.2.** Let \( (x, v) \) be a global solution to (1.1) with initial data satisfying
\[
\max_{i \neq j} \| u_i^0 - u_j^0 \| > 0.
\]
If \( \kappa < \kappa_0(x^0, v^0) \), then we have
\[
\min_{\alpha \neq \beta,1,k} \sup_{0 \leq t < \infty} \| x_{\alpha i}(t) - x_{\beta k}(t) \| = \infty, \quad \min \lim \inf_{\alpha \neq \beta,1,k} t \to \infty \| v_{\alpha i}(t) - v_{\beta k}(t) \| > 0.
\]
i.e., mono-cluster flocking does not occur asymptotically. Moreover, each groups are separating.

### 3.2. Dynamics of local averages and fluctuations.

In this subsection, we provide estimates on the local averages and fluctuations. In particular, we introduce a useful function which is crucial for the study of the Cucker-Smale model with unit speed:
\[
v_{\alpha i}(t) := \min_{1 \leq t \leq N_\alpha} v_{\alpha i}(t) \cdot e, \quad t \geq 0,
\]
where \( e \) represents a unit constant vector, which will be replaced later by a fixed vector \( e_\alpha(T_0) \) depending on initial data. Then, we have the following proposition with respect to function \( v_{\alpha i}^m(t) \) defined in (3.4) as follows.
Proposition 3.3. Let \((x_\alpha, v_\alpha)\), \(\alpha = 1, \cdots, n\) be a solution to system (3.1). Then, for any \(\alpha\) and \(e\), we have
\[
\dot{v}_\alpha^m(t) \geq -\kappa(1 - \gamma_N)\psi_M(t), \quad t \in [0, T), \quad \psi_M(t) := \max_{\beta \neq \alpha, 1, k} \psi(||x_\beta^k(t) - x_\alpha(t)||),
\]
where \(T\) is the time satisfying \(\langle v_\alpha(t), e \rangle \geq 0\) for all \(t \in [0, T)\).

Proof. Each \(v_\alpha(t)\) (respectively, each \(v_\alpha(t) \cdot e\)) is a real analytic function with values in \(\mathbb{R}^d\) (respectively, in \(\mathbb{R}\)) on \(t \in [0, +\infty)\). Thus, there exist time steps \(0 = t_{\alpha 0} < t_{\alpha 1} < t_{\alpha 2} < \cdots\) such that for each \(k \in \{0, 1, 2, \cdots\}\), there exists \(i_k \in \{1, \cdots, N_\alpha\}\) satisfying \(t_\alpha^m(t) = v_\alpha^{i_k}(t) \cdot e\) for all \(t \in [t_{\alpha k}, t_{\alpha (k+1)}]\). For shorthand, we assume that
\[
v_\alpha(t) \cdot e = v_\alpha^m(t), \quad t \in [t_{\alpha k}, t_{\alpha (k+1)}].
\]

Note that \(v_\beta - \langle v_\alpha, v_\beta^k \rangle v_\alpha\) is the component of \(v_\beta\) which is orthogonal to \(v_\alpha\). Then,
\[
||v_\beta - \langle v_\alpha, v_\beta^k \rangle v_\alpha|| \leq 1,
\]
and
\[
\langle v_{\alpha k}, e \rangle - \langle v_{\alpha k}, v_\alpha \rangle \langle v_\alpha, e \rangle \geq \langle v_{\alpha k}, e \rangle - \langle v_\alpha, e \rangle \geq 0.
\]

Thus, we have
\[
\dot{v}_\alpha(t) \cdot e = \frac{K}{N} \sum_{k=1}^{N_\alpha} \psi(||x_\alpha(t) - x_\alpha(t)||)(v_\alpha(t) - v_\alpha(t))v_\alpha(t) \cdot e
\]
\[
+ \frac{K}{N} \sum_{\beta \neq \alpha} \sum_{k=1}^{N_\beta} \psi(||x_\beta^k(t) - x_\alpha(t)||)(v_\beta^k(t) - v_\alpha(t))v_\alpha(t) \cdot e
\]
\[
\geq -\frac{K}{N} (N - N_\alpha) \psi_M(t)||v_\beta - \langle v_\alpha, v_\beta^k \rangle v_\alpha||
\]
\[
\geq -\frac{K(N - N_\alpha)}{N} \psi_M(t)
\]
\[
\geq -\kappa(1 - \gamma_N)\psi_M(t), \quad t \in [0, T).
\]

\[
\square
\]

3.3. Non-existence of mono-cluster flocking. In this subsection, we will provide the proof of Theorem 3.2. We first briefly outline our strategy as follows. The proof of our main results can be split into three stages. For a given initial configuration \((x^0, v^0)\),

- **Initial stage** (from mixed configuration to segregated configuration): there exists a \(T_0 \geq 0\) such that, for any \(i, k\), and \(\beta \neq \alpha\),

  \[
  (x_\alpha(T_0) - x_\beta^k(T_0)) \cdot v^*_\alpha(T_0) \geq 0.
  \]

- **Intermediate stage** (maintaining segregated configuration): there exists \(T_0^* > T_0\) which has the desired properties for all \(t \in [T_0, T_0^*)\),

  \[
  \min_{\alpha \neq \beta, i, k} \left\{ (v_\alpha(t) - v_\beta^k(t)) \cdot e_\alpha(T_0) \right\} > \lambda_0, \quad ||x_\beta^k(t) - x_\alpha(t)|| > \lambda_0(t - T_0),
  \]

  where \(e_\alpha(T_0)\) is the unit vector in the direction of \(v^*_\alpha(T_0)\).
• Final stage (emergence of non-mono cluster configuration): finally we show that
\[ T_0^* = \infty \]
and obtain the non-existence of mono-cluster flocking.

3.3.1. Emergence of segregated configurations. In this subsection, we will show that the configuration at time \( T_0 \) is well segregated:
\[
\triangle_{\alpha_i,\beta_k}(T_0) := \left( x_{\alpha_i}(T_0) - x_{\beta_k}(T_0) \right) \cdot v^c_{\alpha}(T_0) \geq 0.
\] (3.5)
Recall that \( T_0 := \max_{\beta \neq \alpha, i, k} \left\{ -\triangle_{\alpha_i,\beta_k}(0) \lambda_0, 0 \right\} \).

In the sequel, we assume without loss of generality that \( \triangle_{\alpha_i,\beta_k}(0) < 0 \) so that \( T_0 > 0 \).
Otherwise, \( T_0 = 0 \) and the desired estimate (3.5) holds trivially, and all the lemmas from Lemma 3.4 to Lemma 3.7 can be proved with better estimates. We stated this argument in the proof of Theorem 3.2, at the end of this section. As in the definition of \( e_{\alpha}(T_0) \), we set
\[
e_{\alpha}(t) = \frac{v^c_{\alpha}(t)}{\|v^c_{\alpha}(t)\|}.
\]

Lemma 3.4. Let \((x_{\alpha_i}, v_{\alpha_i})\) be a global solution to (3.1) with non-flocking initial data \((x_{\alpha_i}^0, v_{\alpha_i}^0)\). If the coupling strength \( \kappa \) satisfies
\[
0 < \kappa < \frac{1 - \cos \frac{\theta_0}{8}}{2T_0},
\]
then the following estimates hold: for \( t \in [0, T_0] \) and \( \beta \neq \alpha \),
\[
(i) \quad v_{\alpha_i}(t) \cdot v^c_{\alpha}(t) > \cos \frac{\theta_0}{8}, \quad v_{\beta_k}(t) \cdot v^c_{\alpha}(t) < \cos \frac{7\theta_0}{8},
\]
\[
(ii) \quad v_{\alpha_i}(t) \cdot v_{\beta_k}(t) < \cos \frac{7\theta_0}{8}, \quad e_{\alpha}(t) \cdot e_{\beta}(t) < \cos \frac{5\theta_0}{8}.
\]

Proof. (i) Note that \( v_{\alpha_k}(t) - \langle v_{\alpha_i}(t), v_{\alpha_k}(t) \rangle v_{\alpha_i}(t) \) is the component of \( v_{\alpha_k} \) that is orthogonal to \( v_{\alpha_i} \), thus
\[
\|v_{\alpha_k}(t) - \langle v_{\alpha_i}(t), v_{\alpha_k}(t) \rangle v_{\alpha_i}(t)\| \leq 1.
\]
Now, we use system (3.1), the upper bound of \( \psi \) in (1.2), and the above relation to get for any \( \alpha \in \{1, \ldots, n\} \),
\[
\|v_{\alpha}(t)\| \leq \frac{\kappa N_\alpha}{N} + \frac{\kappa (N - N_\alpha)}{N} = \kappa.
\] (3.6)
Similarly, by direct calculation, we have
\[
\|v^c_{\alpha}(t)\| \leq \kappa.
\] (3.7)
Thus, we combine estimates (3.6) and (3.7) to obtain
\[
\left| \frac{d(v_{\alpha_i}(t) \cdot v^c_{\alpha}(t))}{dt} \right| \leq 2\kappa.
\] (3.8)
Then, we use estimate (3.8) and the assumption of \( \kappa \) to get
\[
v_{\alpha_i}(t) \cdot v^c_{\alpha}(t) \geq v_{\alpha_i}(0) \cdot v^c_{\alpha}(0) - 2\kappa T_0 = 1 - 2\kappa T_0 > \cos \frac{\theta_0}{8}, \quad t \in [0, T_0].
\]
For the second estimate, we use the estimate of $v_{\alpha}^c(t)$ in (i) for all $\alpha \in \{1, \cdots, n\}$ to obtain that for any $\beta \neq \alpha$
\[
\left| \frac{d(v_{\beta k}(t) \cdot v_{\alpha}^c(t))}{dt} \right| \leq |v_{\beta k}(t) \cdot v_{\alpha}^c(t)| + |v_{\beta k}(t) \cdot v_{\alpha}^c(t)| \leq 2\kappa.
\]
Thus, we obtain the following from the assumption of $\kappa$.
\[
v_{\beta k}(t) \cdot v_{\alpha}^c(t) \leq v_{\beta k}(0) \cdot v_{\alpha}^c(0) + 2\kappa T_0 \leq \cos \theta_0 + 2\kappa T_0 \leq \cos \theta_0 + 1 - \cos \frac{\theta_0}{8} \leq \cos \frac{7\theta_0}{8}, \quad t \in [0, T_0].
\]
Here the last inequality is from properties of cosine functions,
\[
\cos \frac{\theta_0}{8} + \cos \frac{7\theta_0}{8} = 2\cos(\frac{\theta_0}{2})\cos(\frac{3\theta_0}{8}) \geq 2\cos(\frac{\theta_0}{2})\cos(\frac{\theta_0}{2}) = 1 + \cos \theta_0
\]
for $\theta_0 \in (0, \pi]$.

(ii) By using a similar analysis as in the second estimate of (i), we can derive the first estimate in (ii). For the last inequality, we use (i) and the definition of $e_{\alpha}(t)$ to see that for any $\alpha \in \{1, \cdots, n\}$,
\[
v_{\alpha i}(t) \cdot e_{\alpha}(t) \geq v_{\alpha i}(t) \cdot v_{\alpha}^c(t) > \cos \frac{\theta_0}{8}, \quad t \in [0, T_0] \tag{3.9}
\]
Now we combine relation (3.9) and the previous estimates to get
\[
\arccos(e_{\alpha}(t) \cdot e_{\beta}(t)) \geq \arccos(v_{\alpha i}(t) \cdot v_{\beta k}(t)) - \arccos(v_{\alpha i}(t) \cdot e_{\alpha}(t)) - \arccos(v_{\beta k}(t) \cdot e_{\beta}(t))
\]
\[
> \frac{7\theta_0}{8} - \frac{\theta_0}{8} - \frac{\theta_0}{8} = \frac{5\theta_0}{8}, \quad t \in [0, T_0].
\]
Hence, we obtain
\[
e_{\alpha}(t) \cdot e_{\beta}(t) < \cos \frac{5\theta_0}{8}, \quad t \in [0, T_0].
\]

\[\square\]

**Lemma 3.5.** Let $(x_{\alpha i}, v_{\alpha i})$ be a global solution to (3.1) with non-flocking initial data $(x_{\alpha i}^0, v_{\alpha i}^0)$. If the coupling strength $\kappa$ satisfies
\[
0 < \kappa < \min \left\{ \frac{1 - \cos \frac{\theta_0}{8}}{2T_0}, \frac{\cos \frac{\theta_0}{8} - \cos \frac{\theta_0}{4}}{D(x_0) + 2T_0} \right\},
\]
then, we have
\[
\min_{\beta \neq \alpha, i, k} \Delta_{\alpha i, \beta k}(T_0) > 0.
\]

**Proof.** For the desired estimate, we claim:
\[
\min_{\beta \neq \alpha, i, k} \frac{d}{dt} \Delta_{\alpha i, \beta k}(t) > \lambda_0, \quad t \in [0, T_0]. \tag{3.10}
\]

**Proof of claim (3.10).** For all $t \in [0, T_0]$, $\alpha \neq \beta$ and $i, k$,
\[
\|x_{\alpha i}(t) - x_{\beta k}(t)\| = \|x_{\alpha i}^0 - x_{\beta k}^0\| + \int_0^t (v_{\alpha i}(s) - v_{\beta k}(s))ds
\leq \|x_{\alpha i}^0 - x_{\alpha i}^0\| + 2T_0
\]
By Lemma 3.4 and the assumption of $\kappa$, we obtain

$$
\frac{d}{dt} \Delta_{\alpha,i,\beta,k}(t) \\
= (v_{\alpha}(t) - v_{\beta}(t)) \cdot \nu_{\alpha}(t) + (x_{\alpha}(t) - x_{\beta}(t)) \cdot \nu_{\alpha}(t) \\
= v_{\alpha}(t) \cdot \nu_{\alpha}(t) - v_{\beta}(t) \cdot \nu_{\beta}(t) + (x_{\alpha}(t) - x_{\beta}(t)) \cdot \nu_{\alpha}(t) \\
> \cos \theta_0 - \cos \frac{7\theta_0}{8} - (D(x_0) + 2T_0)\kappa \\
> \cos \frac{\theta_0}{4} - \cos \frac{7\theta_0}{8} \\
> \lambda_0, \quad t \in [0, T_0].
$$

We now integrate relation (3.10) to obtain

$$
\Delta_{\alpha,i,\beta,k}(t) > \Delta_{\alpha,i,\beta,k}(0) + \lambda_0 t, \quad t \in [0, T_0].
$$

Then, the defining relation of $T_0$ in (3.3) implies

$$
\Delta_{\alpha,i,\beta,k}(T_0) > \Delta_{\alpha,i,\beta,k}(0) + \lambda_0 T_0 \geq 0.
$$

We now take a minimum over $\alpha, \beta, i$ and $k$ to obtain the desired result. \qed

### 3.3.2. Proof of Theorem 3.2

In this subsection, we provide the proof of Theorem 3.2. Recall that we defined a normal vector in the direction of $v_{\alpha}(T_0)$:

$$
e_{\alpha}(T_0) := \frac{v_{\alpha}(T_0)}{\|v_{\alpha}(T_0)\|}.
$$

Note that it is a well-defined since $v_{\alpha}(T_0)$ cannot be zero from previous lemmas. We define

$$
T_0^* := \sup \left\{ T \in (T_0, \infty) \left| \min_{\alpha, i} (v_{\alpha}(t) \cdot e_{\alpha}(T_0)) > \cos \frac{\theta_0}{4}, \text{ for all } t \in [T_0, T] \right. \right\}.
$$

**Lemma 3.6.** Let $(x, v)$ be a global solution to (3.1) with non-flocking initial data $(x^0, v^0)$. If the coupling strength $\kappa$ satisfies

$$
0 < \kappa < \frac{1 - \cos \frac{\theta_0}{8}}{2T_0}.
$$

Then we have, for $t \in [T_0, T_0^*)$,

1. \(\max_{\beta,k} (v_{\beta,k}(t) \cdot e_{\alpha}(T_0)) < \cos \frac{3\theta_0}{8},\)

2. \(\min_{\alpha, \beta, i, k} \left\{ (v_{\alpha}(t) - v_{\beta,k}(t)) \cdot e_{\alpha}(T_0) \right\} > \lambda_0,\)

where $\lambda_0 := \cos \frac{\theta_0}{4} - \cos \frac{3\theta_0}{8}$.

**Proof.** (i) We use Lemma 3.4 to get that for $t \in [T_0, T_0^*)$

\[
\arccos (v_{\beta,k}(t) \cdot e_{\alpha}(T_0)) \\
\geq \arccos (e_{\alpha}(T_0) \cdot e_{\beta}(T_0)) - \arccos (v_{\beta,k}(t) \cdot e_{\beta}(T_0)) > \frac{5\theta_0}{8} - \frac{\theta_0}{4} = \frac{3\theta_0}{8}.
\]
Hence, we obtain
\[ v_{\beta k}(t) \cdot e_\alpha(T_0) < \cos \frac{3\theta_0}{8}, \quad t \in [T_0, T_0^*). \]

(ii) By the definition of \( T_0^* \) and estimate (i), assertion (ii) holds trivially. \( \square \)

**Lemma 3.7.** Let \((x, v)\) be a global solution to (1.1) with non-flocking initial data \((x^0, v^0)\). If the coupling strength \( \kappa \) satisfies
\[ 0 < \kappa < \min \left\{ \frac{1 - \cos \frac{\theta_0}{8}}{2T_0}, \frac{\cos \frac{\theta_0}{8} - \cos \frac{\theta_0}{4}}{D(x_0) + 2T_0} \right\}. \]
Then, we have
\[ T_0^* > T_0 \quad \text{and} \quad \psi_M(t) < \psi \left( \lambda_0(t - T_0) \right) \quad \text{for} \quad t \in (T_0, T_0^*). \]

**Proof.** (i) It follows from Lemma 3.4 that we have
\[ v_{\alpha i}(T_0) \cdot e_\alpha(T_0) > \cos \frac{\theta_0}{4}. \]
Thus, there exist \( \delta > 0 \) such that
\[ v_{\alpha i}(t) \cdot e_\alpha(T_0) > \cos \frac{\theta_0}{4} \quad \text{for all} \quad t \in [T_0, T_0 + \delta]. \]
Hence, we have
\[ T_0^* > T_0. \]

(ii) We use Lemma 3.5 and Lemma 3.6 to obtain
\[
\| x_{\alpha i}(t) - x_{\beta k}(t) \|
\geq (x_{\alpha i}(T_0) - x_{\beta k}(T_0)) \cdot e_\alpha(T_0)
= (x_{\alpha i}(T_0) - x_{\beta k}(T_0)) \cdot e_\alpha(T_0) + \int_{T_0}^{t} (v_{\alpha i}(s) - v_{\beta k}(s)) \cdot e_\alpha(T_0) ds
> \lambda_0(t - T_0), \quad t \in (T_0, T_0^*).
\]
Thus, by the non-increasing property of \( \psi(t) \), we have
\[ \psi_M(t) < \psi \left( \lambda_0(t - T_0) \right), \quad \text{for all} \quad t \in (T_0, T_0^*). \]
\( \square \)

We are now ready to provide the proof of Theorem 3.2 as follows.

**The proof of Theorem 3.2.** Let \((x, v)\) be a global solution to (1.1) with non-flocking initial data \((x^0, v^0)\). If the coupling strength \( \kappa \) satisfies
\[ \kappa < \kappa_0. \]
Then, we claim: for \( t \in (T_0, \infty) \),
\[ \min_{\beta \neq \alpha, i, k} \left( v_{\alpha i}(t) - v_{\beta k}(t) \right) \cdot e_\alpha(T_0) > \lambda_0, \quad \| x_{\alpha i}(t) - x_{\beta k}(t) \| > \lambda_0(t - T_0). \]
For the proof of the above claim, we consider two cases:

Either \( T_0(x^0, v^0) > 0 \), or \( T_0(x^0, v^0) = 0 \).

- **Case A.** Suppose that we have
\[ T_0(x^0, v^0) > 0. \]
Then, it follows from the arguments in Lemma 3.7 that
\[ T_0^* > T_0. \]

Suppose that
\[ T_0^* < \infty. \]
Then, by definition in (3.11), there exist \( \alpha, i \) such that
\[ v_{\alpha i}(T_0^*) \cdot e_{\alpha}(T_0) = \cos \frac{\theta_0}{4}. \quad (3.12) \]
On the other hand, we use Proposition 3.3, Lemma 3.4, Lemma 3.7 and the assumption of \( \kappa \) to obtain
\[ v_{\alpha i}(t) \cdot e_{\alpha}(T_0) \geq v_{m \alpha}(t) - \kappa \frac{(N - N_\alpha)}{N} \int_0^t \psi_M(s)ds \]
\[ \geq \cos \frac{\theta_0}{8} - \kappa \frac{(1 - \gamma N)}{\lambda_0} \int_0^\infty \psi(s)ds \]
\[ > \cos \frac{\theta_0}{4}, \quad t \in [T_0, T_0^*]. \]
In particular, we have
\[ v_{\alpha i}(T_0^*) \cdot e_{\alpha}(T_0) > \cos \frac{\theta_0}{4} \]
This contradicts inequality (3.12). Thus, we have \( T_0^* = \infty \). Therefore, the conclusion (ii) of Lemma 3.6 implies the conclusion of Theorem 3.2.

*Case B.* Suppose that we have
\[ T_0(x^0, v^0) = 0. \]
In this case, we do not need Lemma 3.4, Lemma 3.5 again, and \( \cos \frac{\theta_0}{2} \) is replaced by \( \cos \frac{\theta_0}{8} \), \( e_{\alpha}(T_0) \) is replaced by \( e_{\alpha}(0) \) in our definition of \( T_0^* \) in (3.11). Then Lemma 3.6 and Lemma 3.7 hold with \( T_0 = 0, \bar{\lambda}_0 = \cos \frac{\theta_0}{8} = \cos \frac{7\theta_0}{8} \) and without smallness of \( \kappa \). Recall that
\[ \kappa_0 = \frac{1 - \cos \frac{\theta_0}{2}}{1 - \gamma N} \int_0^\infty \psi(s)ds, \quad v_{m \alpha}(0) = 1. \]
Then, for \( \kappa < \kappa_0 \), we use the similar arguments in Case A to obtain \( T_0^* = \infty \). Thus, we have
\[ \|v_{\alpha i}(t) - v_{\beta k}(t)\| > \bar{\lambda}_0, \quad \|x_{\alpha i}(t) - x_{\beta k}(t)\| > \bar{\lambda}_0 t, \quad \text{for all} \quad t > 0. \]
Finally, it follows from Case A and Case B that we complete the proof of Theorem 3.2.

4. **Emergence of multi-cluster flockings.** In this section, we present an emergence of multi-cluster flocking to the Cucker-Smale model (1.1). In Section 3, we divided the particles into \( n \) sub-ensembles \( G_1, \ldots, G_n \) according to their initial velocities, and showed that for a small coupling strength \( \kappa < \kappa_0 \), any two different particles in different groups do not flock. Thus, it is natural to ask whether two different particles in the same group will flock or not in a small coupling regime. In the sequel, we will concentrate this question by allowing the initial velocities of different particles in the same group to be slightly different.
Remark 4.1. Now, we introduce the local fluctuations and total fluctuations of each group:

4.1. A framework and main result.

Consider the Cucker-Smale flocking system with \( n \) sub-ensembles \( \mathcal{G}_\alpha \), \( \alpha = 1, 2, \ldots, n \):

\[
\dot{x}_{\alpha i} = v_{\alpha i}, \quad t \geq 0, \quad \alpha = 1, 2, \ldots, n, \quad i = 1, \ldots, N_\alpha,
\]

\[
v_{\alpha i} = \frac{\kappa}{N} \sum_{k=1}^{N_\alpha} \psi(||x_{\alpha k} - x_{\alpha i}||) \left( v_{\alpha k} - \frac{\langle v_{\alpha i}, v_{\alpha k} \rangle}{\langle v_{\alpha i}, v_{\alpha i} \rangle} v_{\alpha i} \right)
\]

\[
+ \frac{\kappa}{N} \sum_{\beta \neq \alpha} \sum_{k=1}^{N_\alpha} \psi(||x_{\beta k} - x_{\alpha i}||) \left( v_{\beta k} - \frac{\langle v_{\alpha i}, v_{\beta k} \rangle}{\langle v_{\alpha i}, v_{\alpha i} \rangle} v_{\alpha i} \right).
\]

(4.1)

4.1. A framework and main result. As in Section 3, we define some parameters \( \theta_0, \delta_0 \) and \( r_0 \) related to the separations of each sub-ensemble:

\[
\theta_0(x^0, v^0) := \min_{\beta \neq \alpha} \arccos \left( \frac{\langle v_\beta^0(0), v_{\alpha i}^0(0) \rangle}{\|v_\beta^0(0)\| \|v_{\alpha i}^0(0)\|} \right),
\]

\[
\delta_0(x^0, v^0) := \max_{\alpha, i} \arccos \left( \frac{\langle v_\alpha^0, v_{\alpha i}^0(0) \rangle}{\|v_{\alpha i}^0(0)\|} \right),
\]

\[
r_0(x^0, v^0) := \min_{\alpha \neq \beta, i, k} \|x_{\alpha i}^0 - x_{\beta k}^0\| / \|v_{\alpha i}^0(0)\|,
\]

\[
\Lambda_0 := \cos \left( \frac{\theta_0}{3} + \frac{\delta_0}{3} \right) - \cos \left( \frac{2\theta_0}{3} - \frac{\delta_0}{3} \right).
\]

Now, we introduce the local fluctuations and \( l_2 \)-type functionals that measure the total fluctuations of each group:

\[
\dot{x}_{\alpha i} = x_{\alpha i} - x_c, \quad \dot{v}_{\alpha i} = v_{\alpha i} - v_c, \quad x_\alpha := \left( \sum_{i=1}^{N_\alpha} x_{\alpha i}^2 \right)^{1/2}, \quad v_\alpha := \left( \sum_{i=1}^{N_\alpha} v_{\alpha i}^2 \right)^{1/2}.
\]

We next state our framework \((\mathcal{F})\) for a multi-cluster flocking as follows.

- \((\mathcal{F}1)\) (Initial configuration): Initial configuration is well-separated and initial fluctuations are sufficiently small in the sense that

\[
r_0 \geq 0, \quad \delta_0 \in \left[ 0, \frac{1}{2} \theta_0 \right), \quad \Lambda_0^0 \leq \frac{1}{4} \psi(\sqrt{2} x_\alpha^0).
\]

- \((\mathcal{F}2)\) (Coupling strength): The coupling strength takes an intermediate value and the initial distance is large such that

\[
(i) \quad \psi(\sqrt{2}(x_\alpha^0 + A)) \geq \frac{3}{4} \psi(\sqrt{2} x_\alpha^0),
\]

\[
(ii) \quad \kappa < \kappa_1 := \min \left\{ \frac{\Lambda_0 \left( \cos \delta_0 - \cos \left( \frac{\theta_0}{3} + \frac{\delta_0}{3} \right) \right)}{(1 - \gamma N) \int_0^\infty \psi(s)ds}, \frac{\Lambda_0 \psi(\sqrt{2} x_\alpha^0)}{4\sqrt{N}(1 - \gamma N) \int_0^\infty \psi(s)ds} \right\},
\]

where \( A \) and \( \beta_\alpha \) are positive constants defined by the following relations

\[
A := \frac{4\Lambda_0^0}{\beta_\alpha} + 4\kappa\sqrt{N(1 - \gamma N) / \beta_\alpha \Lambda_0} \int_{r_0}^{\infty} \psi(s)ds, \quad \beta_\alpha = \kappa r N \psi(\sqrt{2} x_\alpha^0).
\]

(4.2)

Remark 4.1. (i) By the assumption \((\mathcal{F}1)\), we know that \( \Lambda_0 > 0 \).

(ii) For fixed \( x_\alpha^0, \theta_0, \delta_0, \alpha \), if \( \Lambda_0^0 \ll 1 \) and \( r_0 \gg 1 \), we can always choose such \( \kappa \) satisfying \((\mathcal{F}2)\).

Theorem 4.2. Suppose that the framework \((\mathcal{F})\) holds, and let \( (x_{\alpha i}, v_{\alpha i}) \) be a solution to system \((4.1)\) with initial configuration \( (x_{\alpha i}^0, v_{\alpha i}^0) \). Then, we have the following estimates:

\[
(i) \quad \min_{\beta \neq \alpha, i, k} \|x_{\beta k}(t) - x_{\alpha i}(t)\| > r_0 + \Lambda_0 t, \quad t \in (0, \infty),
\]
\[ \mathcal{X}_\alpha(t) < \lambda_\alpha^0 + A, \]
\[ \mathcal{V}_\alpha(t) \leq C_\alpha \max \left\{ e^{-\frac{\nu(t_0) x_0}{r_0 + \frac{\lambda_0}{2}}}, \psi\left( r_0 + \frac{\lambda_0}{2} \right) \right\}, \text{ for some } C_\alpha > 0, \]
i.e., the multi-cluster flocking emerges.

**Remark 4.3.** In Section 3, it follows from the classification that we assume \( v_0^{\alpha_i} = v^{(0)}_0 \) for any \( \alpha \in \{1, \cdots, n\} \). Thus the initial assumption of \( \delta_0 \) in the above theorem is satisfied naturally.

Theorem 4.2 exhibits the stability of multi-cluster flocking in the following sense.

**Corollary 4.4.** Let \( \bar{G} = \bigcup_{\alpha} \tilde{G}_\alpha \) with \( \tilde{G}_\alpha := (\bar{x}_\alpha(t), \bar{v}_\alpha(t)) \) \( \text{(not necessary to be solutions of system (4.1))} \) be a configuration such that

- \( \bar{v}_\alpha \) is continuous on \( t \in \mathbb{R} \), and \( \bar{x}_\alpha = \bar{v}_\alpha \);
- \( \bar{G} \) tends to a multi-cluster flocking in the sense of Definition 2.2 (2).

Then there exists \( T > 0, \epsilon > 0, \kappa^* > \kappa_\alpha > 0 \) such that if \( \kappa_* < \kappa < \kappa^* \) and \( \|G(0) - \bar{G}(0)\| < \epsilon \),

then solution \( G(t) \) of system (4.1) with initial data \( G(0) \) tends to multi-cluster flocking in the sense of Definition 2.2 (2).

**Proof.** Define \( \bar{\theta}(t), \bar{\delta}(t), \bar{\tau}(t), \bar{\Lambda}(t), \bar{x}_\alpha(t) \) and \( \bar{v}_\alpha(t) \) in the analogue notations with above \( \theta(t), \delta(t), \tau(t), \Lambda(t), x_\alpha(t) \) and \( v_\alpha(t) \) respectively. Then there exists \( \bar{x}_\infty > 0 \), \( \bar{\theta}_\infty > 0 \) such that

\[ \lim_{t \to \infty} \bar{\theta}(t) = \bar{\theta}_\infty > 0, \quad \lim_{t \to \infty} \bar{\delta}(t) = 0, \quad \lim_{t \to \infty} \bar{\tau}(t) = + \infty, \]

\[ \max_{\alpha} \limsup_{t \to \infty} \bar{x}_\alpha(t) < \bar{x}_\infty, \quad \max_{\alpha} \limsup_{t \to \infty} \bar{v}_\alpha(t) = 0, \]

\[ \lim_{t \to \infty} \bar{\Lambda}(t) = \bar{\Lambda}_\infty := \cos \frac{\bar{\theta}_\infty}{2} - \cos \frac{2\bar{\theta}_\infty}{2} > 0. \]

Then by the choice of \( G(0) \) satisfying \( \|G(0) - \bar{G}(0)\| < \epsilon \), we can show that framework \((\mathcal{F})\) holds by direct calculation. Thus, the solution \( G(t) \) of system (4.1) with initial data \( G(0) \) tends to multi-cluster flocking in the sense of Definition 2.2 (2).

### 4.2. Dynamics of local averages and fluctuations

In this subsection, we study the time-evolution of local averages and fluctuations. We use the same definition as in \((3.2)\).

**Lemma 4.5.** Let \( (x_{\alpha i}, v_{\alpha i}) \) be a solution to system (4.1). Then, local averages and fluctuations satisfy

\[
\begin{dcases}
\dot{x}_\alpha^0 = v_\alpha^0, & t \geq 0, \quad \alpha = 1, 2, \cdots, n, \\
\dot{v}_\alpha = \frac{\kappa}{2NN_\alpha} \sum_{k=1}^{N_\alpha} \sum_{i=1}^{N_\alpha} \psi(||x_{\alpha k} - x_{\alpha i}||) |v_{\alpha i} - v_{\alpha k}| v_{\alpha i} \\
+ \frac{\kappa}{NN_\alpha} \sum_{\beta \neq \alpha} \sum_{k=1}^{N_\alpha} \sum_{i=1}^{N_\alpha} \psi(||x_{\beta k} - x_{\alpha i}||) (v_{\beta k} - (v_{\alpha i}, v_{\beta k}) v_{\alpha i}).
\end{dcases}
\]
We sum (3.1) over \( i = 1, \ldots, N_\alpha \) to yield

\[
\dot{x}_\alpha(t) = \dot{v}_\alpha(t), \quad t \geq 0, \quad \alpha = 1, \ldots, n, \quad i = 1, 2, \ldots, N_\alpha,
\]

\[
\begin{align*}
\dot{v}_\alpha(t) &= -\ddot{\psi} + \frac{K}{N} \sum_{k=1}^{N_\alpha} \psi(|x_{ak} - x_{ai}|)(v_{ak} - v_{ai}) \\
&\quad + \frac{K}{N} \sum_{k=1}^{N_\alpha} \psi(|x_{ak} - x_{ai}|)(v_{ai} - v_{ak} - v_{ai}) \\
&\quad + \frac{K}{N} \sum_{\beta \neq \alpha} \sum_{k=1}^{N_\alpha} \psi(|x_{\beta k} - x_{ai}|)(v_{\beta k} - v_{ai}).
\end{align*}
\] (4.4)

**Proof.** (i) (Derivation of (4.3)): It follows from the definition of \( x^c_\alpha \) that we have

\[
\dot{x}_\alpha^c = \frac{1}{N_\alpha} \sum_{i=1}^{N_\alpha} \dot{x}_\alpha = \frac{1}{N_\alpha} \sum_{i=1}^{N_\alpha} v_{ai} = \dot{v}_\alpha^c.
\]

We sum (3.1) over \( i = 1, \ldots, N_\alpha \) to yield

\[
\begin{align*}
\ddot{v}_\alpha &= \frac{1}{N} \sum_{i=1}^{N_\alpha} \ddot{v}_\alpha \big|_{x_i} \\
&= \frac{K}{N_\alpha} \sum_{k=1}^{N_\alpha} \sum_{i=1}^{N_\alpha} \psi(|x_{ak} - x_{ai}|)(v_{ak} - \langle v_{ai}, v_{ak} \rangle v_{ai}) \\
&\quad + \frac{K}{N_\alpha} \sum_{\beta \neq \alpha} \sum_{k=1}^{N_\alpha} \sum_{i=1}^{N_\alpha} \psi(|x_{\beta k} - x_{ai}|)(v_{\beta k} - \langle v_{ai}, v_{\beta k} \rangle v_{ai}) \\
&\quad + \frac{K}{N_\alpha} \sum_{\beta \neq \alpha} \sum_{k=1}^{N_\alpha} \sum_{i=1}^{N_\alpha} \psi(|x_{\beta k} - x_{ai}|)(v_{\beta k} - \langle v_{ai}, v_{\beta k} \rangle v_{ai}).
\end{align*}
\]

Because of the skew-symmetric property of \( \psi(|x_{ak} - x_{ai}|) \), the first term on the above equation becomes zero. And note that

\[
\langle v_{ai}, v_{ai} - v_{ak} \rangle = \langle v_{ak}, v_{ak} \rangle - \langle v_{ak}, v_{ai} \rangle.
\]

Thus, it holds that

\[
\langle v_{ai}, v_{ai} - v_{ak} \rangle = \frac{1}{2} \|v_{ai} - v_{ak}\|^2 = \frac{1}{2} \|\ddot{v}_{ai} - \ddot{v}_{ak}\|^2.
\] (4.5)

Hence, we have

\[
\begin{align*}
\dot{v}_\alpha^c &= \frac{K}{2N_\alpha} \sum_{k=1}^{N_\alpha} \sum_{i=1}^{N_\alpha} \psi(|x_{ak} - x_{ai}|) \|v_{ai} - v_{ak}\|^2 v_{ai} \\
&\quad + \frac{K}{N_\alpha} \sum_{\beta \neq \alpha} \sum_{k=1}^{N_\alpha} \sum_{i=1}^{N_\alpha} \psi(|x_{\beta k} - x_{ai}|)(v_{\beta k} - \langle v_{ai}, v_{\beta k} \rangle v_{ai}).
\end{align*}
\]
(ii) (Derivation of (4.4)): The first equation easily follows from the definitions of fluctuations. It follows from equality (3.1)

\[
\dot{v}_{\alpha i} = \dot{v}_{\alpha i} - \dot{v}_{\alpha i}^c = -\dot{v}_{\alpha i}^c + \frac{K}{N} \sum_{k=1}^{N_\alpha} \psi(||x_{\alpha k} - x_{\alpha i}||)(v_{\alpha k} - v_{\alpha i})
\]

\[
+ \frac{K}{N} \sum_{k=1}^{N_\alpha} \psi(||x_{\alpha k} - x_{\alpha i}||)(v_{\alpha i}, v_{\alpha i} - v_{\alpha k})v_{\alpha i}
\]

\[
+ \frac{K}{N} \sum_{\beta \neq \alpha, k=1}^{N_\beta} \psi(||x_{\beta k} - x_{\alpha i}||)(v_{\beta k} - (v_{\alpha i}, v_{\beta i})v_{\alpha i}).
\]

Then, we complete the proof of this Lemma.

In the following proposition, we derive estimates on the time-derivatives of \(X_\alpha\) and \(V_\alpha\).

**Proposition 4.6.** Let \((x_{\alpha i}, v_{\alpha i}), \alpha = 1, \cdots, n\) be a solution to system (3.1). Then we have, for any \(\alpha\),

\[(i) \quad \frac{dX_\alpha}{dt} \leq V_\alpha, \quad \text{a.e. } t \in [0, \infty),\]

\[(ii) \quad \frac{dV_\alpha}{dt} \leq -\frac{\kappa N_\alpha}{N}(\psi(\sqrt{2}X_\alpha) - V_\alpha)V_\alpha + \kappa \sqrt{N}(1 - \gamma_N)^2 \psi_M,\]

where \(\psi_M := \max_{\beta \neq \alpha, i, k} \psi(||x_{\beta k} - x_{\alpha i}||)\).

**Proof.** (i) We multiply equation (4.4) by \(2x_{\alpha i}(t)\), and add the results together over \(i = 1, \cdots, N_\alpha\) yields

\[
\left| \frac{dX_\alpha^2}{dt} \right| = \sum_{i=1}^{N_\alpha} \frac{d||x_{\alpha i}||^2}{dt} = 2 \sum_{i=1}^{N_\alpha} \langle x_{\alpha i}, \dot{x}_{\alpha i} \rangle
\]

\[
\leq 2 \left( \sqrt{\sum_{i=1}^{N_\alpha} x_{\alpha i}^2} \right) \left( \sqrt{\sum_{i=1}^{N_\alpha} \dot{x}_{\alpha i}^2} \right) = 2X_\alpha V_\alpha. \tag{4.6}
\]

Then, we divide (4.6) by \(2|X_\alpha(t)|\) to obtain

\[
\left| \frac{dX_\alpha}{dt} \right| \leq V_\alpha.
\]

(ii) We multiply equation (4.4) by \(2\dot{v}_{\alpha i}(t)\) and sum the resulting relation over \(i = 1, \cdots, N_\alpha\) to obtain

\[
\frac{dV_\alpha^2}{dt} = -2 \sum_{i=1}^{N_\alpha} \langle \dot{v}_{\alpha i}, \dot{v}_{\alpha i}^c \rangle
\]

\[
+ \frac{2K}{N} \sum_{k=1}^{N} \sum_{i=1}^{N_\alpha} \psi(||x_{\alpha k} - x_{\alpha i}||)(\hat{v}_{\alpha i}, \dot{v}_{\alpha k} - \dot{v}_{\alpha i})
\]
In the last inequality, we used \( \sum \) to get that
\[
\psi
\]
Thus, we use the upper bound of
\[
\| I \| (Estimate on \| I \|)
\]
Hence, we obtain
\[
\kappa \psi - N - 2 M + 2 N\alpha\frac{\kappa}{N} \psi = 0
\]
\[
= I_{11} + I_{12} + I_{13} + I_{14}.
\]
\bullet (Estimate on \( I_{11} \)): It is easy to see that
\[
I_{11} = 0.
\]
\bullet (Estimate on \( I_{12} \)): We exchange \( i \leftrightarrow k \) to get
\[
I_{12} = - \kappa N \sum_{k=1}^{N} \sum_{i=1}^{N} \psi(\| x_{ak} - x_{ai} \|) \| \tilde{v}_{ai} - \tilde{v}_{ak} \| \leq - \frac{2 N\alpha \kappa}{N} \psi(\sqrt{2} \kappa) \psi^2.
\]
In the last inequality, we used \( \sum \tilde{v}_{ai} = 0 \) and get
\[
\sum_{i,k=1}^{N} \| \tilde{v}_{ai} - \tilde{v}_{ak} \| = \sum_{i,k=1}^{N} (\| \tilde{v}_{ai} \|^2 + \| \tilde{v}_{ak} \|^2) = 2 N\alpha \psi^2.
\]
\bullet (Estimate on \( I_{13} \)): We use relation \( \langle \tilde{v}_{ai}, v_{ai} - v_{ak} \rangle = \frac{1}{2} \| \tilde{v}_{ai} - \tilde{v}_{ak} \|^2 \) derived in (4.5) to get that
\[
I_{13} = - \kappa N \sum_{k=1}^{N} \sum_{i=1}^{N} \psi(\| x_{ak} - x_{ai} \|) \| \tilde{v}_{ai} - \tilde{v}_{ak} \|^2 \langle \tilde{v}_{ai}, v_{ai} \rangle.
\]
Thus, we use the upper bound of \( \psi \) and \( v_{ai} \) to obtain
\[
| I_{13} | \leq \kappa N \sum_{k=1}^{N} \sum_{i=1}^{N} \| \tilde{v}_{ai} - \tilde{v}_{ak} \|^2 \| \tilde{v}_{ai} \|
\]
\[
= \kappa N \sum_{k=1}^{N} \sum_{i=1}^{N} (\| \tilde{v}_{ai} \|^2 + \| \tilde{v}_{ak} \|^2) \| \tilde{v}_{ai} \|
\]
\[
\leq \kappa N (\sqrt{N\alpha} + \sqrt{N\alpha}) \leq \frac{2 N\alpha \kappa}{N} \psi^2,
\]
where in the second equality we have used that
\[
-2 \sum_{k=1}^{N} \sum_{i=1}^{N} \langle \tilde{v}_{ai}, \tilde{v}_{ak} \rangle \| \tilde{v}_{ai} \| = -2 \left( \sum_{k=1}^{N} \sum_{i=1}^{N} \langle \tilde{v}_{ak}, \tilde{v}_{ai} \rangle \| \tilde{v}_{ai} \| \right) = 0.
\]
\bullet (Estimate on \( I_{14} \)): Note that \( v_{\beta k} - \langle v_{ai}, v_{\beta k} \rangle v_{ai} \) is the component of \( v_{\beta k} \) that is orthogonal to \( v_{ai} \), thus we have
\[
\| v_{\beta k} - \langle v_{ai}, v_{\beta k} \rangle v_{ai} \| \leq 1.
\]
Hence, we obtain
\[
| I_{14} | \leq \frac{2 \kappa \psi M}{N} \sum_{\beta \neq \alpha} \sum_{k=1}^{N} \sum_{i=1}^{N} \| \tilde{v}_{ai} \| \| v_{\beta k} - \langle v_{ai}, v_{\beta k} \rangle v_{ai} \| \leq \frac{2 \sqrt{N\alpha (N - N\alpha)} \kappa}{N} \psi M \psi M.
\]
In (4.7), we combine all estimates of $I_{1i}$, $i = 1, \cdots, 4$ to obtain
\[
\frac{dV^2}{dt} \leq -\frac{2N_\alpha \kappa}{N} \psi(\sqrt{2}X_\alpha) + \frac{2N_\alpha \kappa}{N} V^2 + \frac{2\sqrt{N_\alpha(N - N_\alpha) \kappa}}{N} \psi V_{\alpha} \\
\leq -\frac{2\kappa N_\alpha}{N} (\psi(\sqrt{2}X_\alpha) - V_\alpha) V^2 + 2\kappa \sqrt{N}(1 - \beta) \psi V_{\alpha}.
\]

We now divide the above relation by $2V^2_\alpha$ to obtain the desired estimate. \hfill \square

4.3. **Proof of Theorem 4.2.** In this subsection, we prove the emergence of multi-cluster flocking configurations for the Cucker-Smale dynamics.

**Definition 4.7.** Define
\[
T^*_1 := \sup \left\{ T \in (0, \infty) \left| \min_{\alpha, i} (v_{\alpha i}(t) \cdot e_\alpha(0)) > \cos(\frac{\theta_0}{3} + \frac{\delta_0}{3}), \quad t \in [0, T] \right. \right\}, \tag{4.8}
\]
where $e_\alpha(0)$ is the unit vector in the direction of $v^\alpha_\alpha(0)$ as before.

**Lemma 4.8.** Let $(x_{\alpha i}(t), v_{\alpha i}(t))$, $\alpha = 1, \cdots, n$ be the solution to system (4.1) with initial data satisfying $(F1)$. Then, we have for $t \in [0, T^*_1)$,
\[
(i) \quad \max_{\beta, k} (v_{\beta k}(t) \cdot e_\alpha(T_0)) < \cos(\frac{2\theta_0}{3} - \frac{\delta_0}{3}), \\
(ii) \quad \min_{\beta \neq \alpha, i, k} (v_{\alpha i}(t) - v_{\beta k}(t)) \cdot e_\alpha(0) > \Lambda_0,
\]
where $\Lambda_0$ is a constant depending on $\theta_0$ and $\delta_0$:
\[
\Lambda_0 := \cos\left(\frac{\theta_0}{3} + \frac{\delta_0}{3}\right) - \cos\left(\frac{2\theta_0}{3} - \frac{\delta_0}{3}\right).
\]

**Proof.** (i) For any $\beta \neq \alpha$ and $1 \leq k \leq N_\beta$, we can get that
\[
\arccos\left(v_{\beta k}(t) \cdot e_\alpha(0)\right) \geq \arccos\left(e_\alpha(0) \cdot e_\beta(0)\right) - \arccos\left(v_{\beta k}(t) \cdot e_\beta(0)\right)
\]
\[
\geq \theta_0 - \frac{1}{3}(\theta_0 + \delta_0) = \frac{1}{3}(2\theta_0 - \delta_0), \quad t \in [0, T^*_1).
\]
Thus, we have
\[
\max_{\beta, k} (v_{\beta k}(t) \cdot e_\alpha(T_0)) < \cos\left(\frac{2\theta_0}{3} - \frac{\delta_0}{3}\right), \quad t \in [0, T^*_1).
\]

(ii) By the definition of $T^*_1$ and estimate (i), assertion (ii) holds trivially. \hfill \square

**Lemma 4.9.** Let $(x_{\alpha i}(t), v_{\alpha i}(t))$, $\alpha = 1, \cdots, n$ be the solution to system (4.1) with initial data satisfying $(F1)$. Then, we have
\[
T^*_1 > 0 \quad \text{and} \quad \psi_M(t) < \psi(r_0 + \Lambda_0 t) \quad \text{for} \quad t \in (0, T^*_1).
\]

**Proof.** (i) By the assumptions $(F1)$ on initial data, we have
\[
\arccos\left(v_{\alpha i}(0) \cdot e_\alpha(0)\right) \leq \delta_0 < \frac{1}{3}(\theta_0 + \delta_0).
\]
Thus, by the continuity, we can conclude $T^*_1 > 0$. 

\[
\text{(4.7)} \quad \frac{dV^2}{dt} \leq -\frac{2N_\alpha \kappa}{N} \psi(\sqrt{2}X_\alpha) + \frac{2N_\alpha \kappa}{N} V^2 + \frac{2\sqrt{N_\alpha(N - N_\alpha) \kappa}}{N} \psi V_{\alpha} \\
\leq -\frac{2\kappa N_\alpha}{N} (\psi(\sqrt{2}X_\alpha) - V_\alpha) V^2 + 2\kappa \sqrt{N}(1 - \beta) \psi V_{\alpha}.
\]
By the initial assumptions and Lemma 4.8, for any $\beta \neq \alpha$, $1 \leq i \leq N_\alpha$ and $1 \leq k \leq N_\beta$,
\[
\|x_{\alpha i}(t) - x_{\beta k}(t)\| \geq (x_{\alpha i}(t) - x_{\beta k}(t)) \cdot e_\alpha(0) = (x_{\alpha i}(0) - x_{\beta k}(0)) \cdot e_\alpha(0) + \int_0^t (v_{\alpha i}(s) - v_{\beta k}(s)) \cdot e_\alpha(0)
\geq r_0 + \Lambda_0 t, \quad t \in (0, T_1^*)
\]
Thus, by the non-increasing property of $\psi(t)$, we have
\[
\psi_M(t) < \psi(r_0 + \Lambda_0 t), \quad t \in (0, T_1^*)
\]

We are now ready to prove Theorem 4.2.

**The proof of Theorem 4.2.** Suppose that the framework $(F)$ holds, and let $(x_{\alpha i}(t), v_{\alpha i}(t))$ be a solution to system (4.1) with initial configuration $(x_{\alpha i}^0, v_{\alpha i}^0)$. Then, we claim $T_1^* = +\infty$ and

(i) $\min_{\beta \neq \alpha, i, k} \|x_{\alpha i}(t) - x_{\beta j}(t)\| > \Lambda_0 t + r_0$,

(ii) $X_\alpha(t) < X_\alpha^0 + A$, where $A = 4V_0 \frac{\Lambda_0}{\beta_\alpha} + \frac{4\kappa \sqrt{N}(1 - \gamma_N)}{N_\beta \Lambda_0} \int_{r_0}^{+\infty} \psi(s)ds$,

with $\beta_\alpha = \kappa N_\alpha \psi(\sqrt{2}X_\alpha^0)$,

(iii) $V_\alpha(t) \leq C_\alpha \max \left\{ e^{-\frac{\kappa(t-r_0)}{2}}, \psi(r_0 + \frac{\Lambda_0 t}{2}) \right\}$, for some $C_\alpha > 0$.

• (Estimate of estimate (i)): It follows from Lemma 4.9 that we have

$T_1^* > 0$.

Now suppose that $T_1^* < +\infty$.

Then by definition in (4.8), there exists $\alpha \in \{1, \cdots, n\}$ and $1 \leq i_0 \leq N_\alpha$ such that

\[
v_{\alpha i_0}(T_1^*) \cdot e_\alpha(0) = \cos(\frac{\theta_0}{3} + \frac{\delta_0}{3}).
\]

On the other hand, we use Proposition 3.3 and Lemma 4.9 to have that for any $\alpha \in \{1, \cdots, n\}$

\[
v^\alpha(t) \geq -\kappa(1 - \gamma_N)\psi(r_0 + \Lambda_0 t), \quad t \in [0, T_1^*].
\]

Thus, we use relation (4.10) and the assumption of $\kappa$ to obtain

\[
v^\alpha_{\alpha i}(T_1^*) \geq v^\alpha_{\alpha i}(0) - \kappa(1 - \gamma_N) \int_0^{T_1^*} \psi(r_0 + \lambda_0 t)dt
\geq \cos \delta_0 - \kappa(1 - \gamma_N) \int_0^{+\infty} \psi(s)ds
\geq \cos(\frac{\theta_0}{3} + \frac{\delta_0}{3}).
\]

Then, we have

\[
v_{\alpha i}(T_1^*) \cdot e_\alpha(0) \geq v^\alpha_{\alpha i}(T_1^*) > \cos(\frac{\theta_0}{3} + \frac{\delta_0}{3}).
\]

This contradicts to relation (4.9). Thus we have $T_1^* = +\infty$. 


Then, we apply the same arguments in Lemma 4.9 to derive the estimate:
\[ \| x_{\beta k}(t) - x_{\alpha_0}(t) \| > r_0 + \Lambda_0 t, \quad t > 0. \]

- (Estimate of estimate (ii)): Firstly, we claim that
\[
V_\alpha(t) < V_\alpha^0 + \frac{\kappa N(1 - \gamma N)}{\Lambda_0} \int_0^\infty \psi(s)ds, \quad V_\alpha(t) < V_\alpha^0 + A, \tag{4.11}
\]
where \( A \) and \( \beta_0 \) are defined in (4.2).

To prove Claim (4.11), we set
\[
\hat{T}^* := \sup\{ T \geq 0 \mid \text{Claim (4.11) holds, for all } t \in [0, T] \}.
\]
Thus, we only need to prove \( \hat{T}^* = +\infty \).

\text{Step 1. (} \hat{T}^* > 0): By the continuity of \( V(t) \) and \( \psi(t) \), there exists \( T_1 > 0 \) such that, for all \( t \in [0, T_1] \),
\[
V_\alpha(t) < V_\alpha^0 + \frac{\kappa N(1 - \gamma N)}{\Lambda_0} \int_0^\infty \psi(s)ds, \quad V_\alpha(t) < V_\alpha^0 + A,
\]
Thus, we get that \( \hat{T}^* > 0 \).

\text{Step 2. (} \hat{T}^* = +\infty): First, we assume that \( \hat{T}^* < +\infty \). Then by definition of \( \hat{T}^* \), we have
\[
V_\alpha(\hat{T}^*) = V_\alpha^0 + \frac{\kappa N(1 - \gamma N)}{\Lambda_0} \int_0^\infty \psi(s)ds \quad \text{or} \quad V_\alpha(\hat{T}^*) = V_\alpha^0 + A. \tag{4.12}
\]
On the other hand, we use assumption (F1) and (F2) to get that, for \( t \in [0, \hat{T}^*] \),
\[
\psi(\sqrt{2} \Lambda_\alpha) - V_\alpha \geq \psi(\sqrt{2}(\Lambda_\alpha^0 + A)) - V_\alpha^0 + \frac{\kappa N(1 - \gamma N)}{\Lambda_0} \int_0^\infty \psi(s)ds \\
> \frac{3}{4} \psi(\sqrt{2} \Lambda_\alpha^0) - \frac{1}{4} \psi(\sqrt{2} \Lambda_\alpha^0) - \frac{1}{4} \psi(\sqrt{2} \Lambda_\alpha^0) \\
= \frac{1}{4} \psi(\sqrt{2} \Lambda_\alpha^0).
\]
Thus, by Proposition 4.6 and Lemma 4.9, we have
\[
\frac{dV_\alpha(t)}{dt} \leq -\frac{\kappa N_\alpha}{N} (\psi(\sqrt{2} \Lambda_\alpha) - V_\alpha) V_\alpha(t) + \kappa N(1 - \gamma N)\psi(\Lambda_0 t + r_0) \\
< -\frac{\beta_0}{4} V_\alpha(t) + \kappa N(1 - \gamma N)\psi(\Lambda_0 t + r_0), \quad t \in [0, \hat{T}^*],
\tag{4.13}
\]
where \( \beta_0 = \kappa N_\alpha \psi(\sqrt{2} \Lambda_\alpha^0) \). We Integrate (4.13) directly and apply Gronwall's inequality to obtain
\[
V_\alpha(t) < V_\alpha^0 e^{-\frac{\beta_0}{4} t} + \kappa N(1 - \gamma N) \int_0^t \psi(\Lambda_0 s + r_0)e^{-\frac{\beta_0}{4} (t-s)}ds, \quad t \in (0, \hat{T}^*].
\tag{4.14}
\]
In particular, we obtain
\[
V_\alpha(\hat{T}^*) < V_\alpha^0 + \frac{\kappa N(1 - \gamma N)}{\Lambda_0} \int_0^\infty \psi(s)ds.
\tag{4.15}
\]
And it follows from the inequality (4.14) that we have
\[ |X_\alpha(t) - X_\alpha^0| \leq \int_0^t \frac{d}{ds} X_\alpha(s) \, ds \leq \int_0^t \mathcal{V}_\alpha(s) \, ds \]

\[ \leq \int_0^t e^{-\frac{2\beta\alpha}{\kappa N}(s-t)} \, ds \]

\[ = \frac{4\sqrt{N}(1-\gamma N)}{\beta_\alpha} \int_{r_0}^{+\infty} \psi(s) \, ds, \quad t \in (0, \hat{T}^*]. \]

Thus, we get
\[ X_\alpha(\hat{T}^*) < X_\alpha^0 + \frac{4\sqrt{N}(1-\gamma N)}{\beta_\alpha N_{\beta_\alpha}} \int_{r_0}^{+\infty} \psi(s) \, ds. \] (4.16)

The inequalities (4.15) and (4.16) contradict the assertion (4.12). Thus we obtain \( \hat{T}^* = +\infty. \)

(iii) For all \( t \in (0, +\infty), \) we use assertions (4.14) to get that
\[ \mathcal{V}_\alpha(t) < \mathcal{V}_\alpha^0 e^{-\frac{2\beta\alpha}{\kappa N}(1-\gamma N)} \psi(\Lambda_0 t + r_0) \int_0^t \psi(\Lambda_0 s + r_0) e^{-\frac{2\beta\alpha}{\kappa N}(t-s)} \, ds \]

\[ \leq \frac{4\sqrt{N}(1-\gamma N)}{\beta_\alpha} \int_{r_0}^{+\infty} \psi(s) \, ds \left( \psi(r_0) e^{-\frac{2\beta\alpha}{\kappa N} t} + \psi\left(\frac{\Lambda_0}{2} t + r_0\right)\right). \]

Thus we have \( \mathcal{V}_\alpha(t) \to 0, \) as \( t \to +\infty. \)

5. Numerical simulations. In this section, we present several numerical examples and compare them with analytical results in the previous sections, in particular Theorem 3.2 and Theorem 4.2. For numerical integrations, we use the fourth-order Runge-Kutta method and well prepared initial configurations and parameter values in the model (1.1) as follows:

\[ \Delta t = 0.01, \quad d = 2, \quad \psi(s) = \frac{1}{(1 + s^2)}, \quad \text{for} \ t \in [0, 2000] \]

in order to get clear visualizations and computations.

5.1. Non-existence of mono-cluster flocking. Recall that Theorem 3.2 deals with initial conditions leading to the complete separation of each ensemble of particles. In the first simulation, we start with some ensembles of particles whose initial velocities are same in each ensemble. The number of particles are not that important for the framework in Theorem 3.2, thus we choose the following parameters of initial data:

\[ N = 10, \quad N_1 = 3, \quad N_2 = 4, \quad N_3 = 3. \]

Figure 1(a) represents initial spatial configuration. The initial positions are chosen randomly based on the fixed central positions of groups, whereas initial velocities are chosen to collide with other groups. In this situation, the relative parameters employed in the simulation are

\[ \theta_0 = 1.5708, \quad T_0 = 122.18, \quad \kappa_0 = 0.0147, \quad \kappa = 0.9 \times \kappa_0 \]
Figure 1. Position-velocity configurations

and they satisfy the sufficient condition in Theorem 3.2. On the other hand, Figure 1(b) illustrates the conclusion of Lemma 3.7, which means that each group is separating at least after the time $T_0$. Here Figure 1(b) tells us that our $T_0$ is quite bigger (nearly 10 times bigger) than the first separation time. However, this ratio is reasonable value according to the proof of Lemma 3.7. It is also clear that each group is separating at the instant $T_0$.

Figure 2 and 3 denote the temporal evolutions of $D(x(t))$ and $\theta(x(t), v(t))$, which measure the diameters of positions and velocities of the whole ensemble, respectively. For a small time interval, $D(x(t))$ begins to decrease, since initially all the particles are gathering. Because $\kappa$ is not big enough to bond the whole ensemble, they just pass each other and separate to the infinity. These mechanisms are well represented in Figure 2. Along this procedure, the velocities are hard to change its value, since $\kappa$ is small. This is the basic idea of proofs on Lemma 3.4, Lemma 3.5, Lemma 3.6, and Lemma 3.7, since the whole separation is trivial for $\kappa = 0$. This also can be seen in Figure 3, which shows that the minimal difference of velocity angles...
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\[ \theta(x(t), v(t)) \] does not change much. It begins to decrease, since initial effects of global attraction, but large separation of positions makes it increase since each group’s internal attraction is bigger than inter-groups attraction.

5.2. **Total separation of particles.** In Theorem 3.2, the definition of group was quite restricted, two particles are in the same group if they have the same velocity initially. This has enough implication due to two reasons. First, people usually assume general configuration of initial velocity. Second, our objective is to verify the non-existence of flocking. In this subsection, we choose initial data randomly to see the behavior of generic initial configurations. The initial positions and velocities are chosen uniformly in \([-1, 1]^2\) and \(S^1\), respectively.

Figure 4(a) shows one simulation of initial data. In this case, the relative positions and velocities have no good relation, therefore the values of \(T_0\) and \(\kappa_0\) are as in the estimation. This typical example shows that the condition for Theorem 3.2 is as...
Figure 3. Temporal evolution of $\theta(x(t), v(t))$ for time $t \in (0, 20)$

follows.

$$\kappa_0 = 1.1658 \times 10^{-8}, \quad T_0 = 18884.$$ Figure 4(b) shows that total separation condition is satisfied near $t = 30$ for $\kappa = 0.9 \times \kappa_0$. The notable point is, however, that Theorem 3.2 is generally satisfied with respect to initial data if we give small $\kappa$. On the other hand, the result of Theorem 4.2 cannot be applied to general initial data, since the existence of local flocking itself is not guaranteed for every initial configuration.

5.3. Emergence of multi-cluster flockings. In this subsection, we consider Theorem 4.2, whose conclusion is on the local flocking in each groups with global non-flocking between groups. The results are similar to the subsection 5.3, but the difference comes from the fluctuation of initial velocities. Since Theorem 4.2 has restriction on $X^0$ and $V^0$, the $L^2$-norm of fluctuation, its result depend on the number of particles $N$. Hence we set small number of particles as follows, for the
convenience of visualization.

\[ N = 10, \quad N_1 = 3, \quad N_2 = 4, \quad N_3 = 3. \]

Initial configuration is also chosen in a similar way as in subsection 5.1. The reference positions and velocities of each group are fixed and we give small random fluctuation for each particle. In contrast to the previous setting, we set separating initial conditions to see the behavior after separation.

Figure 5(a) shows the initial spatial distribution which is separating. Relative positions are scaled larger to guarantee the conditions of Theorem 4.2. This configuration has the following parameters,

\[ \delta_0 = \frac{\pi}{10000}, \quad \theta_0 = 1.5708, \quad X^0 = 0.1579, \quad Y^0 = 1.6203 \times 10^{-4}, \quad A = 0.2548, \]
where the coupling strengths are chosen as follows.

\[ \kappa_1 = 0.0251, \quad \kappa = 0.9 \times \kappa_1 < \kappa_1. \]

Hence all the restrictions for Theorem 4.2 are satisfied. Here \( \delta_0 \) is chosen to be quite small value in order to satisfy the condition (F2).

In Figure 5(b), we can observe the minimal difference of velocity angle is nearly constant, which implies \( \kappa \) is so small that the interaction between each groups are little. On the other hand, \( \kappa \) is large enough to make local flocking of each group, as we can see Figure 6. Figure 6(a) shows position fluctuation is bounded on each group and Figure 6(b) implies that the velocities are gathering on each group with algebraic decay rate. Therefore, we can see the strong evidence of the multi-cluster flocking in this simulation.
Figure 6. Emergence of local flocking

Figure 7 shows the behavior of solutions when we have smaller $\kappa$, namely, $\kappa = 0.09 \ast \kappa_1$. The graphs are plotted with 10 times larger time axis, in order to investigate the trends of variables in a time scale $t/\kappa$. The system seems to exhibit three clusters although the coupling strength is small enough to violate the condition ($F_2$) of Theorem 4.2. In this case, however, the motion of $\mathcal{X}(t)$ does not match with the conclusion of Theorem 4.2, as we easily can see

$$\mathcal{X}(t) \geq \mathcal{X}(0) + A = 0.1579 + 0.1052$$

for some $t$.

Therefore, the conditions of Theorem 4.2 is not close from optimal to the multi-cluster flocking, but it will destroy the parameters we assumed for the bootstrapping arguments of local flocking phenomena. It suggests that we need different approaches to get a closer guess on the critical value of the coupling strength.

Much smaller value of $\kappa = 0.009 \ast \kappa_1$ seems to get a larger number of clusters as in Figure 8. The finite-time simulation can not prove the non-flocking, but $\mathcal{X}(t)$
looks unbounded and $V(t)$ seems to converge a positive value in the same time scale $t/\kappa$ when we compare it with Figure 6 and 7.

6. **Conclusion.** In this paper, we presented a quantitative estimate on the critical coupling strength for the transition from local flocking state to global flocking state, and we also provided a sufficient framework for the multi-cluster flocking. More precisely, we first provided a possible range of the critical coupling strength for the mono-cluster flocking. For a long-ranged communication weight, any positive coupling strength can push initial data to the corresponding mono-cluster flocking state asymptotically. Thus, in this case, the critical coupling strength can set as zero. However, this scenario is completely different for a short-ranged communication weight. As noticed in [27], even for two-dimensional setting and short-ranged communication weight, a sufficiently large coupling strength is required to guarantee the emergence of mono-cluster flocking. It seems that finding the exact form of the critical coupling strength looks a pretty challenging task, if possible, because it might depend on the delicate geometric information of initial configuration. In this paper, we instead tried to estimate the range of the critical coupling strength for a given initial data. In Theorem 3.2, we have shown that if $\kappa < \kappa_0$, then mono-cluster
flocking cannot happen. Thus, the critical coupling strength should be larger than \( \kappa_0 \) depending on the initial configurations, especially for the distance on positions and velocities of individual particles.

Second, we generalized bi-cluster flocking result in [13], which treated the same unit-speed model in two-dimensional setting. More precisely, we provided a sufficient framework for the emergence of multi-cluster flocking in terms of \( \kappa \). Unlike to the result of the Cucker-Smale model in [28], the bounds for \( \kappa \) both depend on the configurations of positions and velocities. The flow of the proof contains the same idea as in [28], i.e., we employed the Lyapunov functional approach with continuity arguments which describe expected particle formations after sufficiently long time. However, the interactions on unit-speed model is quite different from the Cucker-Smale model, for example, oppositely directed two particles do not have any interacting force. Motivated by this simple example, the condition on multi-cluster flocking is more restricted from the Cucker-Smale model. One of the problems is that the velocity cannot guarantee sufficient distance between groups since all the particles have unit-speed. This is why Lemma 3.7 has more rough estimate on relative velocities. In summary, we have provided the first analytic method on the emergent behaviors with respect to general initial data on the agent-based models with velocity restrictions. Of course, there are still many unresolved issues related to the topics treated in this paper. For example, as aforementioned, the possible range of the critical coupling strength for mono-cluster is not optimal, thus it will be interesting problem to optimize the critical coupling strength using optimization theory. This will be treated in future.

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