Linear instability of Vlasov-Maxwell systems revisited-A Hamiltonian approach

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Abstract

We consider linear stability of steady states of 1D and 3D Vlasov-Maxwell systems for collisionless plasmas. The linearized systems can be written as separable Hamiltonian systems with constraints. By using a general theory for separable Hamiltonian systems, we recover the sharp linear stability criteria obtained previously by different approaches. Moreover, we obtain the exponential trichotomy estimates for the linearized Vlasov-Maxwell systems in both relativistic and non-relativistic cases.

1 Introduction

Consider a plasma at high temperature, of low density such that collisions can be ignored compared with the electromagnetic forces. Such a collisionless plasma is modeled by the relativistic Vlasov-Maxwell system. In applications, the classical Vlasov-Maxwell system is also considered when the effect of special relativity is negligible. One of the central problems in the theory of plasmas is to understand plasma stability and instability. The stability problem of Vlasov plasmas is complicated partly because of the instability is usually due to the collective behavior of all the particles. This makes the instability problem highly nonlocal and difficult to study analytically. It is also challenging numerically since the distribution is defined in the phase space with a dimension doubling the space dimension. In a series of works
(10, 12, 11), a sharp stability criterion was obtained for certain equilibria of 1 1 2D Vlasov-Maxwell system and 3D relativistic Vlasov-Maxwell system with cylindrical symmetry. More specifically, when the steady distribution function has a monotonic dependence on the particle energy, the number of unstable modes of linearized RVM systems is shown to be equal to $n^-(L^0)$, the number of negative eigenvalues of a self-adjoint operator $L^0$ (see (3.21) and (4.19)) acting on functions depending only on space variables. In these works, the existence of unstable eigenfunctions was shown by introducing a family of non-local self-adjoint operators $A^\lambda$ for electromagnetic potentials, where the positive parameter $\lambda$ is the possible unstable eigenvalue. Then an instability criteria was obtained by using a continuity argument to exploit the gap of numbers of negative eigenvalues of $A^\lambda$ when $\lambda \to \infty$ and $\lambda \to 0^+$. The proof was particularly involved for the 3D Vlasov-Maxwell systems (12) since the self-adjoint formulation of $A^\lambda$ relied on a careful choice of the gauge condition of the electromagnetic potentials. Moreover, the operator $A^\lambda$ in 3D has an infinite number of negative eigenvalues and a truncation of $A^\lambda$ has to be introduced in order to use the continuity argument. The linear stability criterion $L^0 \geq 0$ was proved by studying the invariant functionals of the linearized Vlasov-Maxwell systems.

In this paper, we study the linearized Vlasov-Maxwell systems by using a framework of separable Hamiltonian systems, which was recently developed in [9] when studying the stability of nonrotating stars. Consider a linear Hamiltonian PDEs of the separable form

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & B \\ -B' & 0 \end{pmatrix} \begin{pmatrix} L & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = JL \begin{pmatrix} u \\ v \end{pmatrix},$$

(1.1)

where $u \in X$, $v \in Y$ and $X, Y$ are real Hilbert spaces. The triple $(L, A, B)$ is assumed to satisfy assumptions (G1)-(G3) in Section 2, which roughly speaking require that $B : Y^* \supset D(B) \to X$ is a densely defined closed operator, $L : X \to X^*$ is bounded and self-dual with finitely many nonpositive modes, and $A : Y \to Y^*$ is bounded, self-dual and positive. Then the number of unstable modes of (1.1) is shown to be equal to $n^-(L|_{R(B)})$, which is the number of negative directions of the quadratic form $(L, \cdot, \cdot)$ restricted to the subspace $R(B) \subset X$. Moreover, exponential trichotomy estimates are obtained for the solution group $e^{tJL}$. See Theorem 2.1 for the detailed statements. By using a parity splitting of the distribution function, we are able to rewrite the linearized 1 1 2D and 3D Vlasov-Maxwell systems in the
separable Hamiltonian forms (1.1) with the constraint of the Poisson equation for the electric fields ((3.10) for 1\(\frac{1}{2}\)D and (4.3) for 3D). The assumption (G1-3) can be verified in an energy space \(X\) and the number \(n^-(L|_{\mathbb{R}(B)})\) is shown to be exactly equal to \(n^-(\mathcal{L}^0)\). Then by Theorem 2.1, we recover the stability criterion obtained in [10] and [12]. Moreover, we also obtain the exponential trichotomy estimates for the linearized Vlasov-Maxwell systems. These estimates will be useful for proving nonlinear instability or constructing invariant (stable, unstable and center) manifolds near an unstable steady state. The exponential trichotomy for the linearized relativistic 1\(\frac{1}{2}\)D Vlasov-Maxwell system can be shown ([12] [5]) by using the compact perturbation (\(A\)-smoothing) theory of semigroups, where the separation of characteristics of the relativistic Vlasov equation and Maxwell system played a crucial role in the proof. Such a separation is possible since the particle velocity in the relativistic case is always less than the speed of light which is the propagation speed of the Maxwell systems. However, for the nonrelativistic Vlasov-Maxwell system such a separation of characteristics is no longer true since the particle might travel faster than the speed of light, and as a consequence the same arguments fail. By using the separable Hamiltonian structures, the exponential trichotomy is obtained for both relativistic and nonrelativistic Vlasov-Maxwell system. Moreover, we get more precise growth estimates (i.e. at most quadratic growth) on the center space. In particular, there is Liapunov stability on the center space when \(\mathcal{L}^0\) has no kernel.

We make some comments to compare the Hamiltonian approach and the previous approach. In [10] [12], the instability and stability criteria were obtained in very different ways. In the Hamiltonian approach, both stability and instability information are obtained from the computation of \(n^-(L|_{\mathbb{R}(B)})\). Another difference lies in the treatment of the Poisson constraint. In the Hamiltonian approach, the Poisson constraint is only imposed on the initial data and it does not appear in the Hamiltonian formulation (1.1). Moreover, since the constraint is automatically satisfied on the eigenspaces of nonzero eigenvalues, it does not affect the counting of unstable modes. Thus, we can leave out the Poisson constraint until stating the exponential trichotomy estimates for data satisfying this constraint. We refer to Remark 3.1 for more details. In [10] [12], the Poisson equation is needed to formulate a family of self-adjoint operators \(A^\lambda\) on electromagnetic potentials for the eigenvalue problem. But it requires some careful choice of the gauge condition to make the Poisson equation to be compatible with the current
equation ((3.7) in 1\frac{1}{2}D and (4.1) in 3D). The approach of ([10] [12]) had been extended to Vlasov-Maxwell systems in a bounded domain ([13] [14] [15]). It might still be possible to use the Hamiltonian formulation for models with boundary conditions. The current Hamiltonian approach requires the monotone dependence of steady distribution function on the particle energy. On the other hand, the approach of ([10] [12]) can be used to obtain sufficient instability conditions for non-monotonic steady distribution function. See [7] [6] for the Vlasov-Poisson models, and [12, Section 9] [4] [1] [2] for the Vlasov-Maxwell models. It would be very interesting to explore the Hamiltonian formulations for the non-monotonic cases.

This paper is organized as follows. In Section 2, we state the results of separable Hamiltonian systems to be used in later sections. In Sections 3, we study the 1\frac{1}{2}D Vlasov-Maxwell system. In Sections 4, we study the 3D relativistic Vlasov-Maxwell system with cylindrical symmetry.

2 Separable Linear Hamiltonian PDEs

We briefly describe the results in [9] about general separable Hamiltonian PDEs (1.1). The triple $(L, A, B)$ is assumed to satisfy assumptions:

(G1) The operator $B : Y^* \supset D(B) \to X$ and its dual operator $B' : X^* \supset D(B') \to Y$ are densely defined and closed (and thus $B'' = B$).

(G2) The operator $A : Y \to Y^*$ is bounded and self-dual (i.e. $A' = A$ and thus $\langle Au, v \rangle$ is a bounded symmetric bilinear form on $Y$). Moreover, there exist $\delta > 0$ such that

$$\langle Au, u \rangle \geq \delta \|u\|^2_Y, \forall u \in Y.$$  

(G3) The operator $L : X \to X^*$ is bounded and self-dual (i.e. $L' = L$ etc.) and there exists a decomposition of $X$ into the direct sum of three closed subspaces

$$X = X_- \oplus \ker L \oplus X_+, \dim \ker L < \infty, \ n^-(L) \triangleq \dim X_- < \infty \ (2.1)$$

satisfying

(G3.a) $\langle Lu, u \rangle < 0$ for all $u \in X_- \setminus \{0\}$;
(G3.b) there exists $\delta > 0$ such that

$$\langle Lu, u \rangle \geq \delta \|u\|^2, \text{ for any } u \in X_+.$$

We note that the assumptions $\dim \ker L < \infty$ and $A > 0$ can be relaxed (see [9]). But these simplified assumptions are enough for the applications to Vlasov-Maxwell systems studied in this paper.

**Theorem 2.1** [9] Assume (G1-3) for (1.1). The operator $JL$ generates a $C^0$ group $e^{tJL}$ of bounded linear operators on $X = X \times Y$ and there exists a decomposition

$$X = E^u \oplus E^c \oplus E^s,$$

of closed subspaces $E^{u,s,c}$ with the following properties:

i) $E^c, E^u, E^s$ are invariant under $e^{tJL}$.

ii) $E^u (E^s)$ only consists of eigenvectors corresponding to negative (positive) eigenvalues of $JL$ and

$$\dim E^u = \dim E^s = n^- \left( L_{|R(B)} \right), \quad (2.2)$$

where $n^- \left( L_{|R(B)} \right)$ denotes the number of negative modes of $\langle L_{\cdot \cdot} \rangle_{R(B)}$. If $n^- \left( L_{|R(B)} \right) > 0$, then there exists $M > 0$ such that

$$|e^{tJL}|_{E^u} \leq Me^{-\lambda_u t}, \; t \geq 0; \quad |e^{tJL}|_{E^s} \leq Me^{\lambda_u t}, \; t \leq 0, \quad (2.3)$$

where $\lambda_u = \min \{ \lambda \mid \lambda \in \sigma(JL|_{E^u}) \} > 0$.

iii) The quadratic form $\langle L_{\cdot \cdot} \rangle$ vanishes on $E^{u,s}$, i.e. $\langle Lu, u \rangle = 0$ for all $u \in E^{u,s}$, but is non-degenerate on $E^u \oplus E^s$, and

$$E^c = \{ u \in X \mid \langle Lu, v \rangle = 0, \; \forall \; v \in E^s \oplus E^u \}. \quad (2.4)$$

There exists $M > 0$ such that

$$|e^{tJL}|_{E^c} \leq M(1 + t^2), \; \text{for all } t \in R. \quad (2.5)$$

iv) Suppose $\langle L_{\cdot \cdot} \rangle$ is non-degenerate on $R(B)$, then $|e^{tJL}|_{E^c} \leq M$ for some $M > 0$. Namely, there is Lyapunov stability on the center space $E^c$. 

\[ \]
Remark 2.1 Above theorem shows that the solutions of (1.1) are spectrally stable (i.e. nonexistence of exponentially growing solution) if and only if $L|_{R(B)} \geq 0$. Moreover, $n^-(L|_{R(B)})$ gives the number of unstable modes when $L|_{R(B)}$ has a negative direction. The exponential trichotomy estimates (2.3)-(2.5) are important in the study of nonlinear dynamics near an unstable steady state, such as the proof of nonlinear instability or the construction of invariant (stable, unstable and center) manifolds. If the spaces $E^{u,s}$ have higher regularity, then the exponential trichotomy can be lifted to more regular spaces. We refer to Theorem 2.2 in [8] for more precise statements.

3 1.5 D Vlasov-Maxwell systems

In this section, we consider the stability of a class of equilibria of $1 \frac{1}{2}$D Vlasov-Maxwell systems by using the framework of separable Hamiltonian systems. We largely follow the notations in ([10]). Here, we consider the classical (i.e. nonrelativistic) Vlasov-Maxwell system, while in ([10]) the relativistic Vlasov-Maxwell system was studied. The stability criteria obtained in both cases are very similar.

The $1 \frac{1}{2}$D Vlasov Maxwell system for electrons with a constant ion background $n_0$ is

$$\partial_t f + v_1 \partial_x f - (E_1 + v_2 B) \partial_{v_1} f - (E_2 - v_1 B) \partial_{v_2} f = 0$$

$$\partial_t E_1 = -j_1 = \int v_1 f \, dv, \quad \partial_t B = -\partial_x E_2$$

$$\partial_t E_2 + \partial_x B = -j_2 = \int v_2 f \, dv$$

with the constraint

$$\partial_x E_1 = n_0 - \int f \, dv.$$
with the electron energy and the “angular momentum” defined by

\[ e = \frac{1}{2} |v|^2 - \phi^0(x), \quad p = v_2 - \psi^0(x). \] (3.2)

We assume

\[ \mu \geq 0, \quad \mu \in C^1, \quad \mu_e \equiv \frac{\partial \mu}{\partial e} < 0 \] (3.3)

and, in order for \( \int (|\mu_e| + |\mu_p|) \ dv \) to be finite,

\[ (|\mu_e| + |\mu_p|) (e, p) \leq c(1 + |e|)^{-\alpha} \] for some \( \alpha > 2. \) (3.4)

The linearized Vlasov equation is

\[ (\partial_t + D) f = \mu_e v_1 E_1 - \mu_p v_1 B + (\mu_e v_2 + \mu_p) E_2, \] (3.5)

where \( D \) is the transport operator associated with the steady fields, that is,

\[ D = v_1 \partial_x - (E_1^0 + v_2 B_0^0) \partial_{v_1} + v_1 B^0 \partial_{v_2} \]

\[ = v_1 \partial_x + \partial_x \phi^0 \partial_{v_1} + \partial_x \psi^0 (v_1 \partial_{v_2} - v_2 \partial_{v_1}). \] (3.6)

The linearized Maxwell equations become

\[ \partial_t E_1 = \int v_1 f \ dv, \] (3.7)

\[ \partial_t E_2 + \partial_x B = \int v_2 f \ dv, \] (3.8)

\[ \partial_t B + \partial_x E_2 = 0. \] (3.9)

with the constraint

\[ \partial_x E_1 = - \int f \ dv. \] (3.10)

We consider the initial data satisfying the constraint \( \int B (0, x) \ dx = 0. \) Then by (3.9), \( \int B (t, x) \ dx = 0 \) for all \( t \in \mathbb{R}. \) Let \( \psi (t, x) \) be the magnetic potential function satisfying

\[ \psi_x = B, \quad \int_0^P \psi (t, x) \ dx = - \int_0^t \int_0^P E_2 (s, x) \ dx \ dt. \] (3.11)
Then by (3.9), $\psi_t = -E_2$. Below, we write the linearized equations (3.5) and (3.7)-(3.9) as a separable Hamiltonian system (1.1). We split $f$ into its even and odd parts in the variable $v_1$:

$$f = f_{ev} + f_{od}, \quad \text{where } f_{ev}(x, v_1, v_2) = \frac{1}{2}\{f(x, v_1, v_2) + f(x, -v_1, v_2)\}.$$ 

and define $g_{ev} = f_{ev} + \mu_p\psi$. The operator $D$ takes even functions into odd ones, and vice versa. So from (3.5), we have

$$\partial_t f_{od} = -Df_{ev} + (E_1 + v_2 B)\partial_{v_1} f^0 - v_1 B\partial_{v_2} f^0 = -Df_{ev} + \mu_e v_1 E_1 - \mu_p v_1 \partial_x \psi = -Dg_{ev} + \mu_e v_1 E_1,$$

and

$$\partial_t f_{ev} + Df_{od} = E_2 \partial_{v_2} f^0 = \mu_e v_2 E_2 - \mu_p \partial_t \psi,$$

which yields

$$\partial_t g_{ev} = -Df_{od} + \mu_e v_2 E_2. \quad (3.13)$$

The Maxwell equations (3.7)-(3.9) become

$$\partial_t E_1 = \int v_1 f_{od} dv, \quad (3.14)$$

$$\partial_t E_2 = -\partial_{xx} \psi - \int \mu_p v_2 \psi + \int v_2 g_{ev} dv, \quad (3.15)$$

$$\partial_t \psi = -E_2. \quad (3.16)$$

Define

$$X_{od} = \left\{ f \in L^2_{[p\in]} \ | \ f(x, -v_1, v_2) = -f(x, v_1, v_2) \right\}$$

and

$$X_{ev} = \left\{ f \in L^2_{[p\in]} \ | \ f(x, -v_1, v_2) = f(x, v_1, v_2) \right\}.$$ 

Let $L^2_P, H^1_P$ be the $x-$periodic functions in $L^2$ and $H^1$, and define $X = X_{ev} \times L^2_P \times H^1_P$. Define the operators $L : X \rightarrow X^*$ by

$$L \begin{pmatrix} g_{ev} \\ E_1 \\ \psi \end{pmatrix} = \begin{pmatrix} -\frac{1}{\mu_e} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & L_0 \end{pmatrix} \begin{pmatrix} g_{ev} \\ E_1 \\ \psi \end{pmatrix}, \quad (3.17)$$
where \( L_0 = -\frac{d^2}{dx^2} - \int \mu_p v_2 dv \). Let \( Y = X_{od} \times L_P^2 \) and define the operator \( A : Y \to Y^* \) by

\[
A = \left( \begin{array}{cc} -\frac{1}{\mu_e} & 0 \\ 0 & I \end{array} \right). \tag{3.18}
\]

Note that \( A : Y \to Y^* \) is an isometry. Define \( B : Y^* \supset D(B) \to X \) by

\[
B = \left( \begin{array}{cc} \mu_e^D & \mu_e v_2 \\ -\int \mu_e v_1 \cdot dv & 0 \\ 0 & -I \end{array} \right), \tag{3.19}
\]

and the corresponding dual operator \( B' : X^* \supset D(B') \to Y \) is

\[
B' = \left( \begin{array}{cc} -\mu_e^D & -\mu_e v_1 \\ \int \mu_e v_2 \cdot dv & 0 \\ 0 & -I \end{array} \right).
\]

Let

\[
u = \left( \begin{array}{c} g_{ev} \\ E_1 \\ \psi \end{array} \right) \in X, \quad v = \left( \begin{array}{c} f_{od} \\ E_2 \end{array} \right) \in Y.
\]

Then the linearized 11/2D Vlasov-Maxwell system (3.12)-(3.16) can be written as a separable Hamiltonian form (1.1) with \( \langle L, A, B \rangle \) defined in (3.17)-(3.19). Now we check that the triple \( \langle L, A, B \rangle \) satisfies assumptions (G1-3) in Section 2. Assumptions (G1-2) are obvious. To verify (G3), we note that for any \((g_{ev}, E_1, \psi) \in X\),

\[
\langle L \left( \begin{array}{c} g_{ev} \\ E_1 \\ \psi \end{array} \right), \left( \begin{array}{c} g_{ev} \\ E_1 \\ \psi \end{array} \right) \rangle \tag{3.20}
\]

\[
= \iint \frac{1}{\mu_e} |g_{ev}|^2 dv \, dx + \int |E_1|^2 \, dx + \int |\psi'|^2 \, dx - \int \int \mu_p v_2 |\psi|^2 \, dx \, dv.
\]

Then assumption (G3) follows since the operator \( L_0 = -\frac{d^2}{dx^2} - \int \mu_p v_2 dv \) has finite-dimensional negative and zero eigenspaces. To apply Theorem 2.1 to study the solutions of (3.12)-(3.16), we need to compute \( n^- (L|_{R(B)}) \). First, we introduce some notations as in ([10]). Define the following operators, \( A_1^0, A_2^0, L^0 \) act from \( H_P^2 \) to \( L_P^2 \) and \( B^0, (B^0)^* \) act from \( L_P^2 \) to \( L_P^2 \)

\[
A_1^0 h = -\varphi_x h - \left( \int \mu_e dv \right) h + \int \mu_e \mathcal{P} h \, dv,
\]

\[n^- (L|_{R(B)}) = \cdots\]
\[A^0_2 h = -\partial^2_x h - \left( \int v_2 \mu_2 dv \right) h - \int \mu_2 v_2 \mathcal{P}(\dot{v}_2 h) \, dv,\]

\[B^0 h = \left( \int \mu_2 dv \right) h + \int \mu_2 \mathcal{P}(v_2 h) \, dv - \int \mu_2 (I - \mathcal{P})(v_2 h) \, dv,\]

\[(B^0)^* h = \left( \int \mu_2 dv \right) h + \int v_2 \mu_2 \mathcal{P}(h) \, dv\]

and

\[L^0 = (B^0)^* (A^0_1)^{-1} B^0 + A^0_2,\]

where \(\mathcal{P}\) is the projection operator of \(L^2_{|\mu_\epsilon|}\) onto \(\text{ker} \, D\). Then we have

**Lemma 3.1**

\[n^{-} \left( L|_{R(B)} \right) = n^{-} \left( L^0 \right), \quad \dim \, \text{ker} \, L|_{R(B)} = \dim \, \text{ker} \, L^0.\]

**Proof.** First, for any \(0 \neq u = (g_{ev}, E_1, \psi) \in X\) with \(\langle Lu, u \rangle \leq 0\), it is easy to see from (3.20) that \(\psi \neq 0\). For any \(u = (g_{ev}, E_1, \psi) \in R(B) = R(BA)\), let \(u = BA v\) where \(v = (f_{od}, E_2) \in Y\). Then

\[g_{ev} = -D f_{od} + \mu_2 v_2 E_2, \quad E_1 = \int v_1 f_{od} dv, \quad \psi = -E_2.\]

Thus

\[\langle Lu, u \rangle = \int \int \frac{1}{|\mu_\epsilon|} |D f_{od} - \mu_2 v_2 E_2|^2 dv dx + \int |\partial_x E_2|^2 dx + \int \int v_1 f_{od} dv \bigg|^2 dx - \int \int \mu_2 v_2 |E_2|^2 dv dx := W (f_{od}, E_2).\]

It was shown in ([10, P. 751-752]) that \(W (f_{od}, E_2) \geq (L^0 E_2, E_2)\). Therefore, \(\langle Lu, u \rangle \geq (L^0 \psi, \psi)\) for any \(u \in R(B)\), and also for any \(u \in R(B)^*\) by the density argument. Thus, \(n^{-0} \left( L|_{R(B)} \right) \leq n^{-0} (L^0)\), where \(n^{-0} \left( L|_{R(B)} \right)\) and \(n^{-0} (L^0)\) denote the maximal dimensions of subspaces where the quadratic forms \(\langle L^0, \cdot \rangle|_{R(B)}\) and \(\langle L^0, \cdot \rangle\) are nonpositive.
Next we show that $n^{\leq 0} \left( L_{H(R(B))} \right) \geq n^{\leq 0} (L^0)$, which then implies that $n^{\leq 0} \left( L_{H(R(B))} \right) = n^{\leq 0} (L^0)$. For any $\psi \in H^1_p$, define

$$\phi^\psi = - (A_0^0)^{-1} B^0 \psi, \quad f^\psi = \mu_p \psi - \mu_e \phi^\psi + \mu_e P (v_2 \psi + \phi^\psi). \quad (3.22)$$

Then by the definition of $\phi^\psi$

$$\frac{d^2}{dx^2} \phi^\psi (x) = \int f^\psi dv.$$ 

Let

$$E_1^\psi = \frac{d}{dx} \phi^\psi (x), \quad g_{ev}^\psi = - f^\psi + \mu_p \psi. \quad (3.23)$$

We show that $u^\psi = (g_{ev}^\psi, E_1^\psi, \psi) \in R(B)$. Indeed, since $g_{ev}^\psi \in X_{ev}$ and

$$g_{ev}^\psi + \mu_e v_2 \psi = \mu_e (I - P)(v_2 \psi + \phi^\psi) \in R(D),$$

there exists a sequence $\{ h_{od}^n \} \in X_{od} \cap Dom(D)$ such that

$$\| - D h_{od}^n - (g_{ev}^\psi + \mu_e v_2 \psi) \|_{L^2_{\langle v_1 \rangle}} \to 0, \text{ when } n \to \infty.$$ 

We can choose $h_{od}^n$ such that

$$\int \int v_1 h_{od}^n dxdv = 0. \quad (3.24)$$

To show this, we claim that there exists an odd (in $v_1$) function $\chi \in \ker D$ such that $\int \int v_1 \chi dxdv \neq 0$. Therefore, we can adjust $h_{od}^n$ by $c\chi$ to ensure (3.24). Indeed, a function $\chi \in \ker D$ if and only if it takes constant values on each particle trajectory $(X(t), V_1(t), V_1(t))$ in the steady electromagnetic fields

$$(E_1^0, E_2^0, B^0) = (-\partial_x \phi^0, 0, \partial_x \psi^0),$$

that is,

$$\dot{X}(t) = V_1, \quad \dot{V}_1 = - (E_1^0(X) + V_2 B^0(X)), \quad \dot{V}_2 = V_1 B^0(X).$$

In particular, $\chi$ can take opposite constants on two untrapped particle trajectories with the same particle energy $e$ and momentum $p$ (defined in 3.2) satisfying

$$e > \max \left[ (p + \psi^0)^2 - \phi^0 (x) \right]$$

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but with different sign of \( v_1 \). By choosing \( \chi \in \ker D \) to be zero on the trapped region, take nonnegative values on the untrapped trajectory with positive \( v_1 \) and opposite values on the other untrapped trajectory with negative \( v_1 \), we can ensure that \( \int \int \mu_e v_1 \chi \, dx \, dv < 0 \). We note that this also implies that \( v_1 \notin \overline{R(D)} = (\ker D)^\perp \).

Let

\[
E_1^n = \int v_1 h_{od}^n \, dv, \quad g^n_{ev} = -D h_{od}^n - \mu_e v_2 \psi,
\]

then

\[
u_n = (g^n_{ev}, E^n_1, \psi) = BA \left( \begin{array}{c} h_{od}^n \\ -\psi \end{array} \right) \in R(B).
\]

Moreover, the property (3.24) implies that \( E_1^n = \frac{d}{dx} \phi^n \), where

\[
\frac{d^2}{dx^2} \phi^n = -\int (g^n_{ev} + \mu_e v_2 \psi) \, dv = \frac{d}{dx} \int v_1 h_{od}^n \, dv.
\]

Since

\[
\frac{d^2}{dx^2} \phi^n = \int f^n \psi \, dv = -\int (g^n_{ev} - \mu_p \psi) \, dv = -\int (g^n_{ev} + \mu_e v_2 \psi) \, dv
\]

and \( \|g^n_{ev} - g^n_{ev}\|_{L^2_{(0)}} \to 0 \), thus \( \|E_1^n - E_1^n\|_{L^2} \to 0 \) when \( n \to \infty \). This shows that \( \|u^n - u^n\|_X \to 0 \) and \( u^n \in \overline{R(B)} \). As shown in the proof of Lemma 2.8 in [10], we have

\[
(L^n_0 E_2, E^n_2) = L(u^n, u^n).
\]

Thus \( n^{\leq 0}(L^n_0) \leq n^{\leq 0}(L|_{R(B)}) \) which implies \( n^{\leq 0}(L^n_0) = n^{\leq 0}(L|_{R(B)}) \). To show that \( n^{-}(L|_{R(B)}) = n^{-}(L^n_0) \), it remains to show that

\[
\dim \ker L|_{R(B)} = \dim \ker L^n_0.
\]

We note that \( u \in \ker L|_{R(B)} \) is equivalent to \( u = (g_{ev}, E_1, \psi) \in \overline{R(B)} \cap \ker (B'L) \). So

\[
D g_{ev} - \mu_e v_1 E_1 = 0 \quad (3.26)
\]

and

\[
L_0 \psi + \int v_2 g_{ev} \, dv = 0 \quad (3.27)
\]

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Since \( u \in R(B) \), we have
\[
\mathcal{P} (g_{ev} + \mu ev_2 \psi) = 0, \tag{3.28}
\]
\[
\frac{d}{dx} E_1 = - \int (g_{ev} + \mu ev_2 \psi) \, dv. \tag{3.29}
\]
Let \( \phi \) be such that
\[
\phi_{xx} = - \int (g_{ev} + \mu ev_2 \psi) \, dv. \tag{3.30}
\]
Then \( E_1 = \phi_x + k \) where \( k = \frac{1}{h} \int_0^P E_1 \, dx \). By (3.26), we have \( D (g_{ev} - \mu e \phi) = k \mu ev_1 \) which implies that \( k = 0 \) since \( \mu v_1 \notin R(D) \). Thus \( D (g_{ev} - \mu e \phi) = 0 \), that is, \((I - \mathcal{P}) (g_{ev} - \mu e \phi) = 0\). Combining with (3.28), we get
\[
gev = \mu e \phi - \mu e \mathcal{P} (v_2 \psi) - \mu e \mathcal{P} \phi. \tag{3.31}
\]
Plugging above into (3.30), we get \( A^0_1 \phi = B^0 \psi \) and \( \phi = (A^0_1)^{-1} B^0 \psi \). Then by combining with (3.27) and (3.31), it yields \( L^0 \psi = 0 \). On the other hand, if \( L^0 \psi = 0 \), define \( E^\psi_1 \) and \( g^\psi_{ev} \) as in (3.22) and (3.23). Then \( (g^\psi_{ev}, E^\psi_1, \psi) \in R(B) \). By reversing the above computation, it can be checked that (3.26) and (3.27) are satisfied. This shows that \( (g^\psi_{ev}, E^\psi_1, \psi) \in ker \left( L\big|_{R(B)} \right) \). Thus \( ker \left( L\big|_{R(B)} \right) \) and \( ker L^0 \) have the same dimension. This finishes the proof of the lemma. \( \blacksquare \)

**Remark 3.1** We make some comments on the constraint (3.10) which becomes
\[
\partial_x E_1 = - \int (g_{ev} - \mu p \psi). \tag{3.32}
\]
This constraint is preserved by the system (3.12)-(3.16) in the sense that
\[
\partial_t \left( \partial_x E_1 + \int (g_{ev} - \mu p \psi) \right) = 0.
\]
In particular, this implies that for any nonzero eigenvalue \( \lambda \) of (3.12)-(3.16), the constraint (3.32) is satisfied on the corresponding eigenspace. Therefore, the same dimension formula (2.2) is true under the constraint (3.32). The exponential trichotomy estimates (2.3)-(2.5) remain the same by restricting to initial data satisfying the constraint (3.32). The same remark applies to the constraint \( \int_0^T B(x,t) \, dx = 0 \).
We can apply Theorem 2.1 to the linearized system (3.12)-(3.16) with initial data satisfying the constraints $\int_0^P B(x,0)\,dx = 0$ and (3.32). To be more convenient for potential applications to nonlinear problems, we state the results without the even and odd splitting of $f$. Let $\psi(x,t)$ be the magnetic potential defined in (3.11) and define $g = f + \mu_p \psi$. Then $g$ satisfies the equation
\[ g_t = -Dg + \mu_e v_1 E_1 + \mu_e v_2 E_2 \] (3.33) by (3.12) and (3.13). The Maxwell system becomes
\begin{align*}
\partial_t E_1 &= \int v_1 g \, dv, \\
\partial_t E_2 &= -\partial_{xx} \psi - \int \mu_p v_2 \psi + \int v_2 g \, dv, \\
\partial_t \psi &= -E_2,
\end{align*}
with the constraint
\[ \partial_x E_1 = -\int (g - \mu_p \psi). \] (3.34)

**Theorem 3.1** Consider the above equivalent linearized Vlasov-Maxwell systems for $(g, E_1, E_2, \psi)$ in the space
\[ Z = L^2_{|p|} \times L^2_P \times L^2_P \times H^1_P, \]
with initial data satisfying the constraint (3.34). Then
i) The solution mapping is strongly continuous in the space $Z$ and there exists a decomposition
\[ Z = E^u \oplus E^c \oplus E^s, \]
of closed subspaces $E^{u,s,c}$ with the following properties:
ii) $E^c, E^u, E^s$ are invariant under the linearized system.
iii) $E^u (E^s)$ only consists of eigenvectors corresponding to negative (positive) eigenvalues of the linearized system and
\[ \dim E^u = \dim E^s = n^- (L^0), \]
where $L^0$ is defined in (3.21). In particular, $L^0 \geq 0$ implies spectral stability.
iv) The exponential trichotomy is true in the space $Z$ in the sense of (2.3)-(2.5). Moreover, if $\ker L = \{0\}$, then Liapunov stability is true under the norm $\|\|_Z$ on the center space $E^c$. 14
By assuming \( \int \frac{|\mu|^2}{|\mu|} dv < \infty \), above Theorem implies the exponential trichotomy for the linearized VM system (3.5), (3.7)-(3.9) for \((f, E_1, E_2, B)\) in the norm
\[
\|f\|_{L^2} + \|E_1\|_{L^2} + \|E_2\|_{L^2} + \|B\|_{L^2}.
\]

4 3D Vlasov-Maxwell systems

The case of 3D Vlasov-Maxwell is rather similar to the 1.5D case. So we will be more sketchy and only give details when there are significant differences.

As in \([10]\) and \([12]\), we consider the 3D relativistic Vlasov-Maxwell system (RVM) for a non-neutral electron plasma with external fields
\[
\partial_t f + \hat{v} \cdot \nabla_x f - (E + E^\text{ext} + \hat{v} \times (B + B^\text{ext})) \cdot \nabla_v f = 0
\]
\[
\partial_t E - \nabla \times B = \int \hat{v} f \ dv = -j
\]
\[
\partial_t B + \nabla \times E = 0, \quad \nabla \cdot B = 0
\]
\[
\nabla \cdot E = -\int f \ dv = \rho
\]
where \(x \in \mathbb{R}^3, v \in \mathbb{R}^3\). Denote \((r, \theta, z)\) to be the cylindrical coordinates. The equilibrium distribution function with cylindrical symmetry is assumed to have the form \(f^0 = \mu(e, p)\), where
\[
e = \sqrt{1 + \|v\|^2} - \phi^0(r, z) - \phi^\text{ext}(r, z),
\]
\[
p = r \left(v_\theta - A_\theta^0(r, z) - A_\theta^\text{ext}(r, z)\right),
\]
are particle energy and momentum, and \((\phi^0(r, z), A_\theta^0(r, z))\) and \((\phi^\text{ext}(r, z), A_\theta^\text{ext}(r, z))\) are self-generated and external electromagnetic potentials. The steady electromagnetic fields are given by
\[
E^0 = -\partial_v \phi^0 \hat{e}_r - \partial_z \phi^0 \hat{e}_z, \quad B^0 = -\partial_z A_\theta^0 \hat{e}_r + \frac{1}{r} \partial_r \left(r A_\theta^0\right) \hat{e}_z.
\]
The steady potentials \((A_\theta^0, \phi^0)\) satisfy the elliptic system
\[
\Delta \phi^0 = \partial_{zz} \phi^0 + \partial_{rr} \phi^0 + \frac{1}{r} \partial_r \phi^0 = \int \mu dv,
\]
\[
\left( \Delta - \frac{1}{r^2} \right) A^0_\theta = \partial_{zz} A^0_\theta + \partial_{rr} A^0_\theta + \frac{1}{r} \partial_r A^0_\theta - \frac{1}{r^2} A^0_\theta = \int \hat{v}_\theta \mu dv. \tag{4.5}
\]

By choosing \( \phi^{ext}, A^{ext}_\theta \) and \( \mu \) properly, steady solutions satisfying (4.4)-(4.5) were constructed in [10] with a compact support \( S \) for \( f^0 \) in the \((x,v)\) space and \( f^0, E^0, B^0 \) to be differentiable in the whole space. We assume that \( \mu_e < 0 \) on the support \( \{ \mu > 0 \} \). The linearized VM systems are

\[
\partial_t f + D f - (\mathbf{E} + \hat{v} \times \mathbf{B}) \cdot \nabla_v f^0 = 0, \tag{4.6}
\]

coupled with the Maxwell systems (4.1)-(4.3). Here,

\[
D = \hat{v} \cdot \nabla_x - \left( \mathbf{E}^0 + \mathbf{E}^{ext} + \hat{v} \times \left( \mathbf{B}^0 + \mathbf{B}^{ext} \right) \right) \cdot \nabla_v
\]

is the transport operator with the steady electromagnetic fields. We consider axi-symmetric perturbations and decompose such \( f \) as \( f = f_{od} + f_{ev} \) where \( f_{od} \) (f_{ev}) is odd (even) in \((v_r, v_z)\). Then the linearized Vlasov equation (4.6) can be written as (see [10])

\[
\partial_t f_{od} + D f_{ev} = \mu_e \left( \hat{v}_r E_r + \hat{v}_z E_z \right) - \mu_p r \left( \hat{v}_r B_z - \hat{v}_z B_r \right), \tag{4.7}
\]

and

\[
\partial_t f_{ev} + D f_{od} = \mu_e \hat{v}_\theta E_\theta + \mu_p r E_\theta. \tag{4.8}
\]

Introduce the magnetic potential function \( A_\theta \) such that \( B_r = -\partial_z A_\theta, \ B_z = \frac{1}{r} \partial_r (r A_\theta) \) and

\[
\partial_t A_\theta = -E_\theta. \tag{4.9}
\]

Define \( g_{ev} = f_{ev} + r \mu_p A_\theta \) and note that \( r \left( \hat{v}_r B_z - \hat{v}_z B_r \right) = D (r A_\theta) \), then we can get from (4.7)-(4.8)

\[
\partial_t f_{od} = -D g_{ev} + \mu_e \left( \hat{v}_r E_r + \hat{v}_z E_z \right) \tag{4.10}
\]

\[
\partial_t g_{ev} = -D f_{od} + \mu_e \hat{v}_\theta E_\theta. \tag{4.11}
\]

The Maxwell system (4.1)-(4.3) is reduced to

\[
\partial_t E_r = -\partial_z B_\theta + \int \hat{v}_r f_{od} \ dv, \ \partial_t E_z = \frac{1}{r} \partial_r (r B_\theta) + \int \hat{v}_z f_{od} \ dv, \tag{4.12}
\]

\[
\partial_t B_\theta = -\partial_z E_r + \partial_r E_z, \tag{4.13}
\]

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\[
\partial_t E_\theta = \partial_z B_r - \partial_r B_z + \int \hat{\nu}_e f_{\nu e} dv = L_0 A_\theta + \int \hat{\nu}_g g_{\nu e} dv,
\]

(4.14)

\[
L_0 = -\partial_{zz} - \partial_{rr} - \frac{1}{r} \partial_r + \frac{1}{r^2} - \int \hat{\nu}_\theta \mu_p dv r,
\]

with the constraint

\[
\nabla \cdot E = \frac{1}{r} \partial_r (r E_r) + \partial_z E_z = -\int g_{\nu e} dv + \int r \mu_p dv A_\theta.
\]

(4.15)

Define

\[
X_{od} = \left\{ f \in L^2_{\nu e} (\mathbb{R}^3 \times \mathbb{R}^3) \mid f(r, z, -v_r, v_\theta, -v_z) = -f(r, z, v_r, v_\theta, v_z) \right\}
\]

and

\[
X_{ev} = \left\{ f \in L^2_{\nu e} (\mathbb{R}^3 \times \mathbb{R}^3) \mid f(r, z, -v_r, v_\theta, -v_z) = f(r, z, v_r, v_\theta, v_z) \right\}.
\]

Let \( V^1 \) to be the space of cylindrically symmetric functions \( h(r, z) \) such that

\[
\| h \|_{V^1} = \left( \int \left( \frac{1}{r} \nabla h \cdot (r h) \right)^2 + |\nabla h|^2 \right)^{\frac{1}{2}} dx = \| \nabla (he^{i\theta}) \|_{L^2(\mathbb{R}^3)} < \infty,
\]

and \( L^2_{\nu e} \) be the space of cylindrically symmetric functions in \( L^2(\mathbb{R}^3) \). Let \( X = X_{ev} \times (L^2_{\nu e})^2 \times V^1 \) and \( Y = X_{od} \times (L^2_{\nu e})^2 \). Define the isometry operator \( A : Y \to Y^* \) by

\[
A = \begin{pmatrix}
\frac{1}{\mu_e} & 0 & \frac{1}{\mu_e} \\
0 & I & 0 \\
\frac{1}{\mu_e} & 0 & \frac{1}{\mu_e}
\end{pmatrix}
\]

(4.16)

and \( L : X \to X^* \) by

\[
L = \begin{pmatrix}
-\frac{1}{\mu_e} & I \\
I & L_0
\end{pmatrix}.
\]

(4.17)

Define \( B : Y^* \supset D(B) \to X \) by

\[
B = \begin{pmatrix}
\mu_e D & \mu_e \hat{\nu}_\theta & 0 \\
-\int \hat{\nu}_z \cdot dv & 0 & \hat{\theta}_z \\
-\int \hat{\nu}_r \cdot dv & 0 & \frac{1}{r} \partial_r (r \cdot)
\end{pmatrix}
\]

(4.18)
and the dual operator $B' : X^\ast \supset D(B') \to Y$ is

$$B' = \begin{pmatrix} -\mu_e D & -\mu_e \hat{v}_r & -\mu_e \hat{v}_z & 0 \\ \int \mu_e \hat{v}_\theta \cdot dv & 0 & 0 & -I \\ 0 & \partial_z & -\partial_r & 0 \end{pmatrix}.$$  

Let $u = (g_{ev}, E_r, E_z, A_\theta) \in X$ and $v = (f_{od}, E_\theta, B_\theta) \in Y$, then the linearized 3D relativistic Vlasov-Maxwell system (4.9)-(4.14) can be written as a separable Hamiltonian form (1.1) with $\langle L, A, B \rangle$ defined in (4.16)-(4.18). We check that the triple $\langle L, A, B \rangle$ satisfies assumptions (G1-3) in Section 2. We note that for any $u = (g_{ev}, E_r, E_z, A_\theta) \in X$,

$$\langle Lu, u \rangle = \int \int \frac{1}{|\mu_e|} |g_{ev}|^2 dv dx + \int |E_r|^2 dx + \int |E_z|^2 dx + \langle L_0 A_\theta, A_\theta \rangle,$$

where

$$\langle L_0 A_\theta, A_\theta \rangle = \int \left( |\partial_z A_\theta|^2 + \frac{1}{r} \partial_r (r A_\theta) \right)^2 dx - \int \int r \hat{v}_\theta \mu_p |A_\theta|^2 dx dv.$$

Note that since $f^0 = \mu(e, p)$ has compact support in $x, v$, we have

$$\left| \int r \hat{v}_\theta \mu_p |A_\theta|^2 dx dv \right| \lesssim \| A_\theta \|_{L^6}^2 \lesssim \| \nabla (A_\theta e^{i\theta}) \|_{L^2(R^3)}^2 = \| A_\theta \|_{V^1}^2.$$

Moreover, by Lemma 3.1 of [10] and its proof, $\sigma_{ess}(L_0) = [0, \infty)$ and $L_0$ is a relative compact perturbation of

$$(-\Delta)_{mag} := -\partial_{zz} - \partial_{rr} - \frac{1}{r} \partial_r + \frac{1}{r^2}.$$

Thus there exists a finite co-dimensional subspace $V \subset V^1$ such that

$$\langle L_0 A_\theta, A_\theta \rangle \geq c_0 \| A_\theta \|_{V^1}^2, \quad \forall \ A_\theta \in V,$$

for some constant $c_0 > 0$. This proves assumption (G3) and assumptions (G1)-(G2) are obvious.

Now we compute $n^\perp \left( L |_{\mathcal{R}(B)} \right)$. As in [10], we define the following operators acting on the cylindrically symmetric functions $h = h(r, z) \in L^2(\mathbb{R})$ by

$$A_1^0 h = -\partial_{zz} h - \partial_{rr} h - \frac{1}{r} \partial_r h - \int \mu_e dv h + \int \mu_e \mathcal{P}(h) dv,$$
\[ A^0_2 h = -\partial_{zz} h - \partial_{rr} h - \frac{1}{r} \partial_r h + \frac{1}{r^2} h - \int \hat{v}_\theta \mu_e \partial_r h - \int \hat{v}_\theta \mu_e \mathcal{P}(\hat{v}_\theta h) \, dv, \]
\[ \mathcal{B}^0 h = \int \mu_e \mathcal{P}(\hat{v}_\theta h) \, dv - \int \hat{v}_\theta \mu_e h, \]
and
\[ \mathcal{L}^0 = (\mathcal{B}^0)^* (A^0_1)^{-1} \mathcal{B}^0 + A^0_2, \tag{4.19} \]
where \( \mathcal{P} \) is the projection operator of \( L^2_{|\mu_e|} \) onto \( D \). The properties of these operators are studied in Lemma 3.1 of [10]. As in the \( 1\frac{1}{2}D \) case, we have

**Lemma 4.1** For \( L, B \) defined in (4.17) and (4.18), we have
\[ n^- \left( L|_{R(B)} \right) = n^- \left( \mathcal{L}^0 \right), \quad \dim \ker L|_{R(B)} = \dim \ker \mathcal{L}^0. \]

**Proof.** The proof is similar to that of Lemma 3.1. For any \( u = (g_e, E_r, E_z, A_\theta) \in R(B) = R(BA) \), let \( u = BAv \) where \( v = (f_{od}, E_\theta, B_\theta) \in Y \). Then
\[ g_e = -Df_{od} + \mu_e \hat{v}_\theta E_\theta, \quad A_\theta = -E_\theta, \]
\[ E_r = -\partial_z B_\theta + \int \hat{v}_r f_{od} \, dv, \quad E_z = \frac{1}{r} \partial_r (rB_\theta) + \int \hat{v}_z f_{od} \, dv, \]
and
\[ \langle Lu, u \rangle = \iint \frac{1}{|\mu_e|} \left[ |Df_{od} - \mu_e \hat{v}_\theta E_\theta|^2 \right] dx dv - \int r \hat{v}_\theta \mu_p |E_\theta|^2 \, dv + \int \left( \frac{1}{r} |\partial_r (rE_\theta)|^2 \right) dx \]
\[ + \int \left( |\partial_z B_\theta + \int \hat{v}_r f_{od} \, dv|^2 + \frac{1}{r} |\partial_r (rB_\theta) + \int \hat{v}_z f_{od} \, dv|^2 \right) dx \]
\[ := W(f_{od}, E_\theta, B_\theta). \]
It was shown in [10] that \( W(f_{od}, E_\theta, B_\theta) \geq (\mathcal{L}^0 E_\theta, E_\theta) \). Thus \( \langle Lu, u \rangle \geq (\mathcal{L}^0 A_\theta, A_\theta) \) for any \( u = (g_e, E_r, E_z, A_\theta) \in R(B) \), which yields \( n^\leq (L|_{R(B)} \leq n^\leq (\mathcal{L}^0) \) as in the \( 1\frac{1}{2}D \) case.

Next, we show \( n^\leq (L|_{R(B)} \geq n^\leq (\mathcal{L}^0) \). For any \( A_\theta \in V^1 \), we define
\[ \phi^{A_\theta} = - (A^0_1)^{-1} \mathcal{B}^0 A_\theta, \quad f^{A_\theta} = \tau \mu_p A_\theta - \mu_e \phi^{A_\theta} + \mu_e \mathcal{P}(\hat{v}_\theta A_\theta + \phi^{A_\theta}). \]
By the definition of $\phi^A$, we have $\Delta \phi^A = \int f^A dv$. Define

$$E^A_r = \partial_r \phi^A, \quad E^A_z = \partial_z \phi^A, \quad g^A_{ev} = -f^A + r\mu A \theta.$$  \hspace{1cm} (4.20)

Then $u^A = (g^A_{ev}, E^A_r, E^A_z, A^A) \in \overline{R(B)}$. We skip the proof since it is similar to the $1\frac{1}{2}$D case. We only point out that the following observation is used.

Let $h \in \text{Dom}(D) \cap L^2_{|r|}$, if $\Delta \phi = \int Dhdv$ and $(E_r, 0, E_z) = \nabla \phi$, then there exists $B_\theta \in L^2_s$ such that

$$E_r = -\partial_z B_\theta + \int \hat{v}_r h \ dv, \quad E_z = \frac{1}{r} \partial_r (r B_\theta) + \int \hat{v}_z h \ dv,$$

which is due to

$$\frac{1}{r} \partial_r \left( r \left( E_r - \int \hat{v}_r h \ dv \right) \right) + \partial_z \left( E_z - \int \hat{v}_z h \ dv \right) = \Delta \phi - \int Dhdv = 0.$$  \hspace{1cm} (4.21)

It is easy to check that $(\mathcal{L}^0 A_\theta, A_\theta) = L \left( A^A, u^A \right)$. This shows that $n^{\leq 0} \left( L \big|_{R(B)} \right) \geq n^{\leq 0} (\mathcal{L}^0)$ and consequently $n^{\leq 0} \left( L \big|_{R(B)} \right) = n^{\leq 0} (\mathcal{L}^0)$.

It remains to prove $\text{dim ker} \ L \big|_{R(B)} = \text{dim ker} \mathcal{L}^0$. If $u = (g_{ev}, E_r, E_z, A_\theta) \in \ker L \big|_{R(B)}$, then $u \in \overline{R(B)} \cap \ker (B'L)$. Thus

$$Dg_{ev} - \mu_e \hat{v}_r E_r - \mu_e \hat{v}_z E_z = 0,$$  \hspace{1cm} (4.22)

$$L_0 A_\theta + \int \hat{v}_g g_{ev} dv = 0,$$  \hspace{1cm} (4.23)

By (4.23), there exists a potential function $\phi(r, z)$ such that $E_r = \partial_r \phi$ and $E_z = \partial_z \phi$. Since $u \in \overline{R(B)}$, it follows that

$$\Delta \phi = \nabla \cdot (E_r, 0, E_z) = -\int (g_{ev} + \mu_e \hat{v}_g A_\theta) dv$$  \hspace{1cm} (4.24)

By (4.21), $D \left( g_{ev} - \mu_e \phi \right) = 0$ which implies $(I - \mathcal{P}) \left( g_{ev} - \mu_e \phi \right) = 0$. Since $u \in \overline{R(B)}$, $\mathcal{P} \left( g_{ev} + \mu_e \hat{v}_g A_\theta \right) = 0$. Thus

$$g_{ev} = \mu_e \phi - \mu_e \mathcal{P} \left( \hat{v}_g A_\theta \right) + \mu_e \mathcal{P} \phi.$$  \hspace{1cm} (4.25)
Combining (4.22), (4.24) and (4.25), we get $L^0 A_\theta = 0$. On the other hand, if $L^0 A_\theta = 0$, we define $u^{A_\theta} = (g^{A_\theta}_{ev}, E_r^{A_\theta}, E_z^{A_\theta}, A_\theta) \in \hat{R}(B)$ by (4.20). Then reversing the above computation, we have $u^{A_\theta} \in \ker \left( L|_{\hat{R}(B)} \right)$. This shows that $\dim \ker L|_{\hat{R}(B)} = \dim \ker L^0$.

Let $g = f + r \mu_p A_\theta$, which satisfies

$$g_t = -Dg + \mu_e (\hat{v}_r E_r + \hat{v}_z E_z + \hat{v}_\theta E_\theta).$$

Then we can study the equivalent linearized Vlasov-Maxwell system for $(g, A_\theta, B_\theta, E_\theta, E_r, E_z)$, where $(A_\theta, E_r, E_z, E_\theta, B_\theta)$ satisfy

$$\begin{align*}
\partial_t A_\theta &= -E_\theta, \quad \partial_t B_\theta = -\partial_z E_r + \partial_r E_z, \quad \partial_t E_\theta = L_0 A_\theta + \int \hat{v}_\theta g \, dv \\
\partial_t E_r &= -\partial_z B_\theta + \int \hat{v}_r g \, dv, \quad \partial_t E_z = \frac{1}{r} \partial_r (r B_\theta) + \int \hat{v}_z g \, dv,
\end{align*}$$

with the constraint

$$\frac{1}{r} \partial_r (r E_r) + \partial_z E_z = - \int g dv + \int r \mu_p dv A_\theta. \quad (4.26)$$

As in the $1\frac{1}{2}$D case (Remark 3.1), the constraint (4.26) is automatically satisfied on the eigenspaces of nonzero eigenvalues.

**Theorem 4.1** Consider the linearized relativistic Vlasov-Maxwell system for $(g, A_\theta, B_\theta, E_\theta, E_r, E_z)$, with axi-symmetric initial data in the space

$$Z = L^2_{\mu_e}(\mathbb{R}^3 \times \mathbb{R}^3) \times V^1 \times (L^2_s(\mathbb{R}^3))^4$$

satisfying the constraint (4.26). Then

i) The solution mapping is strongly continuous in the space $Z$ and there exists a decomposition

$$Z = E^u \oplus E^c \oplus E^s,$$

of closed subspaces $E^{u,c}$ with the following properties:

i) $E^c, E^u, E^s$ are invariant under the linearized RVM system.

ii) $E^u (E^s)$ only consists of eigenvectors corresponding to negative (positive) eigenvalues of the linearized system and

$$\dim E^u = \dim E^s = n^- (L^0),$$

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where $\mathcal{L}^0$ is defined in (4.19). In particular, $\mathcal{L}^0 \geq 0$ implies spectral stability.

iii) The exponential trichotomy is true in the space $Z$ in the sense of (2.3)-(2.5). Moreover, if $\ker \mathcal{L} = \{0\}$, then Liapunov stability is true under the norm $\|\cdot\|_Z$ on the center space $E^c$.

By assuming $\int \frac{|\mu|^2}{|\mu|} dv < \infty$, above Theorem implies the exponential trichotomy for the original linearized RVM system (3.5), (3.7)-(3.9) for $(f, E, B)$ in the norm $\|f\|_{L^2} + \|E\|_{L^2} + \|B\|_{L^2}$, where $E = (E_r, E_\theta, E_z)$, $B = (B_r, B_\theta, B_z)$.

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References

[1] Ben-Artzi, Jonathan; Holding, Thomas Instabilities of the relativistic Vlasov-Maxwell system on unbounded domains. SIAM J. Math. Anal. 49 (2017), no. 5, 4024–4063.

[2] Ben-Artzi, Jonathan Instabilities in kinetic theory and their relationship to the ergodic theorem. Complex analysis and dynamical systems VI. Part 1, 25–39, Contemp. Math., 653, Israel Math. Conf. Proc., Amer. Math. Soc., Providence, RI, 2015.

[3] Ben-Artzi, Jonathan, Instability of nonmonotone magnetic equilibria of the relativistic Vlasov-Maxwell system. Nonlinearity 24 (2011), no. 12, 3353–3389.

[4] Ben-Artzi, Jonathan Instability of nonsymmetric nonmonotone equilibria of the Vlasov-Maxwell system. J. Math. Phys. 52 (2011), no. 12, 123703, 21 pp.

[5] Guo, Yan and Strauss, Walter, Magnetically created instability in a collisionless plasma, J. Math. Pures. Appl., 79, no. 10, 975-1009 (2000).

[6] Guo, Yan; Lin, Zhiwu, Unstable and stable galaxy models. Comm. Math. Phys. 279 (2008), no. 3, 789–813
[7] Lin, Zhiwu, *Instability of periodic BGK waves*. Math. Res. Lett. 8 (2001), no. 4, 521–534.

[8] Lin, Zhiwu and Zeng, Chongchun, *Instability, index theorem, and exponential trichotomy for Linear Hamiltonian PDEs*, arXiv:1703.04016, accepted by Mem. Amer. Math. Soc.

[9] Lin, Zhiwu and Zeng, Chongchun, *Separable Hamiltonian PDEs and Turning point principle for stability of gaseous stars*, arXiv: 2005.00973, accepted by Comm. Pure. Appl. Math.

[10] Lin, Zhiwu; Strauss, Walter A. *Linear stability and instability of relativistic Vlasov-Maxwell systems*. Comm. Pure Appl. Math. 60, 724–787 (2007).

[11] Lin, Zhiwu; Strauss, Walter A. *Nonlinear stability and instability of relativistic Vlasov-Maxwell systems*, Comm. Pure. Appl. Math. 60, 789–837 (2007).

[12] Lin, Zhiwu and Strauss, Walter A., *A sharp stability criterion for Vlasov-Maxwell systems*, Invent. Math. 173 (2008), no. 3, 497-546.

[13] Nguyen, Toan T.; Strauss, Walter A. *Linear stability analysis of a hot plasma in a solid torus*. Arch. Ration. Mech. Anal. 211 (2014), no. 2, 619–672.

[14] Nguyen, Toan T.; Strauss, Walter A. *Stability analysis of collisionless plasmas with specularly reflecting boundary*. SIAM J. Math. Anal. 45 (2013), no. 2, 777–808.

[15] Zhang, Katherine Zhiyuan, *Linear stability analysis of the relativistic Vlasov-Maxwell system in an axisymmetric domain*. SIAM J. Math. Anal. 51 (2019), no. 6, 4683–4723.