A TOTAL CUNTZ SEMIGROUP FOR $C^*$-ALGEBRAS OF STABLE RANK ONE

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Abstract. In this paper, we show that for unital, separable $C^*$-algebras of stable rank one and real rank zero, the unitary Cuntz semigroup functor and the functor $K_0$ are naturally equivalent. Then we introduce a refinement of the unitary Cuntz semigroup, say the total Cuntz semigroup, which is a new invariant for separable $C^*$-algebras of stable rank one, is a well-defined continuous functor from the category of $C^*$-algebras of stable rank one to the category $Cu$. We prove that this new functor and the functor $K_0$ are naturally equivalent for unital, separable, $K$-pure $C^*$-algebras. Therefore, the total Cuntz semigroup is a complete invariant for a large class of $C^*$-algebras of real rank zero.

INTRODUCTION

The Cuntz semigroup is an invariant for $C^*$-algebras that is intimately related to Elliott’s classification program for simple, separable, nuclear $C^*$-algebras. Its original construction $W(A)$ resembles the semigroup $V(A)$ of Murray-von Neumann equivalence classes of projections, is a positively ordered, abelian semigroup whose elements are equivalent classes of positive elements in matrix algebras over $A$ [9]. This was remedied in [8] by constructing an ordered semigroup, termed $Cu(A)$, in terms of countably generated Hilbert modules. The semigroup $Cu(A)$ is order isomorphic to $W(A \otimes K)$, where $K$ is the $C^*$-algebra of compact operators on a separable Hilbert space, it can also be regarded as a completion of $W(A)$ [3, 8]. Moreover, a Cuntz category was described to which the Cuntz semigroup belongs and as a functor into which it preserves inductive limits. There are also many interesting consequences for the Cuntz semigroup of $C^*$-algebras of stable rank one ([2, 25]).

The Cuntz semigroup contains a great deal of the structure of $C^*$-algebras, but the main limitation is that it fails to capture the $K_1$ information. Recently, Cantier introduced the unitary Cuntz semigroup, denoted by $Cu_1$, as an invariant which captures the $K_1$ information [6], and aimed to classify certain non-simple $C^*$-algebras. A unitary Cuntz category was also defined, as an analogy to the Cuntz category.

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In this paper, we first investigate the unitary Cuntz semigroup and show the close links between $\text{Cu}_{1,u}$ and $K_*$ for the real rank zero case, this extends Theorem 5.20 of [6].

**Theorem 0.1.** Upon restriction the class of unital, separable $C^*$-algebras of stable rank one and real rank zero, the functors $\text{Cu}_{1,u}$ and $K_*$ are naturally equivalent.

Based on the above result, $\text{Cu}_1$ is not a complete invariant for AD algebras of real rank zero, then we introduce a refinement of unitary Cuntz semigroup, denoted by $\text{Cu}$, and establish the functorial properties. We show that $\text{Cu}$ is a continuous functor from the category of unital, separable $C^*$-algebras of stable rank one, denoted by $C^*_{sr}$, to a subcategory of $\text{Cu}^\sim$. We have

**Theorem 0.2.** Let $A = \lim\limits_{\rightarrow}(A_i, \phi_{ij})$ be an inductive system in the category $C^*_{sr}$. Then

$$\text{Cu} - \lim\limits_{\rightarrow} \text{Cu}(A_i) \cong \text{Cu}(A).$$

Then we show that the total K-theory can be recovered funtorially from $\text{Cu}$. Furthermore, we have

**Theorem 0.3.** Upon restriction the class of unital, separable $K$-pure (or $AHD$) algebras with stable rank one and real rank zero, the functors $\text{Cu}_u$ and $K$ are naturally equivalent. Therefore, for these algebras, $K$ is a classifying functor, if, and only if, so is $\text{Cu}_u$.

Note that the total K-theory is a complete invariant for $AHD$ algebras (including AD algebras and AH algebras with slow dimension growth, see [19]) of real rank zero, then $\text{Cu}_u$ is also a complete invariant for this class. The question whether all the unital, separable $C^*$-algebras of stable rank one and real rank zero can be classified by the total K-theory is still open.

This paper is organized as follows: In the first two sections, we present definitions and basic tools, then from a functorial point of view, we prove that the unitary Cuntz semigroup and graded K-theory determine each other for the real rank zero case.

In Section 3, we construct a total Cuntz semigroup and prove that it satisfies the axioms (O1)–(O4). Then, we find a suitable category, called the total Cuntz category. In Section 4, we use the eventually-increasing sequence to obtain the continuity of the functor $\text{Cu}$ from the category of unital, separable $C^*$-algebras of stable rank one to the total Cuntz category.

Finally, we show that this new invariant performs well in classification of certain non-simple $C^*$-algebras. We can recover the total Cuntz semigroup from the total K-theory for a large class of $C^*$-algebras, but the whole relation between these invariants remains unknown.

1. **Prelimaries**

1.1. Let $A$ be a unital $C^*$-algebra. $A$ is said to have stable rank one, written $sr(A) = 1$, if the set of invertible elements of $A$ is dense. $A$ is said to have
real rank zero, written \( rr(A) = 0 \), if the set of invertible self-adjoint elements is dense in the set \( A_{sa} \) of self-adjoint elements of \( A \). If \( A \) is not unital, let us denote the minimal unitization of \( A \) by \( A^\sim \). A non-unital C*-algebra is said to have stable rank one (or real rank zero) if its unitization has stable rank one (or real rank zero). Denote by \( C_{sr}^+ \) the category of unital, separable C*-algebras with stable rank one.

1.2 ([14]). An approximatively dimension-drop algebra, written AD algebra, is an inductive limit of finite direct sums of the form \( M_n(I_p) \) and \( M_n(C(X)) \), where

\[
I_p = \{ f \in C([0, 1], M_p(\mathbb{C})) \mid f(0) = 0 \text{ and } f(1) \in \mathbb{C} \cdot 1_p \}
\]

is the Elliott-Thomsen dimension-drop interval algebra and \( X \) is one of the following finite connected CW complexes: \( \{pt\}, \top, [0, 1] \). (In [6, 5.4], AD algebra is called AH\(_d\)-algebra.)

1.3. (The Cuntz semigroup of a C*-algebra) Denote the cone of positive elements in \( A \) by \( A_+ \). Let \( a, b \in A_+ \). One says that \( a \) is \textit{Cuntz subequivalent} to \( b \), denoted by \( a \lesssim_{Cu} b \), if there exists a sequence \( (r_n) \) in \( A \) such that \( r_n^* b r_n \to a \). One says that \( a \) is \textit{Cuntz equivalent} to \( b \), denoted by \( a \sim_{Cu} b \), if \( a \lesssim_{Cu} b \) and \( b \lesssim_{Cu} a \). The \textit{Cuntz semigroup} of \( A \) is defined as \( Cu(A) = (A \otimes \mathcal{K})_+ / \sim_{Cu} \). We will denote the class of \( a \in (A \otimes \mathcal{K})_+ \) in \( Cu(A) \) by \( [a] \). \( Cu(A) \) is a positively ordered abelian semigroup when equipped with the addition: \( [a] + [b] = [a \oplus b] \), and the relation:

\[
[a] \leq [b] \iff a \lesssim_{Cu} b, \quad a, b \in (A \otimes \mathcal{K})_+.
\]

1.4. (The category \( Cu \)) Let \( (S, \leq) \) be a positively ordered semigroup such that the suprema of increasing sequences always exists in \( S \). For \( x \) and \( y \) in \( S \), let us say that \( x \) is compactly contained in \( y \) (or \( x \) is way-below \( y \)), and denote it by \( x \ll y \), if for every increasing sequence \( (y_n) \) in \( S \) such that \( y \leq \sup_{n \in \mathbb{N}} y_n \), then there exists \( k \) such that \( x \leq y_k \). This is an auxiliary relation on \( S \), called the compact containment relation. If \( x \in S \) satisfies \( x \ll x \), we say that \( x \) is compact.

We say that \( S \) is a \( Cu \)-semigroup of the Cuntz category \( Cu \), if it has a 0 element and satisfies the following order-theoretic axioms:

(01): Every increasing sequence of elements in \( S \) has a supremum.

(02): For any \( x \in S \), there exists a \( \ll \)-increasing sequence \( (x_n)_{n \in \mathbb{N}} \) in \( S \) such that \( \sup_{n \in \mathbb{N}} x_n = x \).

(03): Addition and the compact containment relation are compatible.

(04): Addition and suprema of increasing sequences are compatible.

A \( Cu \)-morphism between two \( Cu \)-semigroups is a positively ordered monoid morphism that preserves the compact containment relation and suprema of increasing sequences.

Let \( S \) be a \( Cu \)-semigroup, we say that \( M \) is an \( Cu \)-ideal of \( S \), if \( M \) is a sub-\( Cu \)-semigroup of \( S \) and order-hereditary (for any \( a, b \in S, a \leq b \) and \( b \in M \) imply \( a \in M \)).
1.5. Let $A$ be a $C^*$-algebra and let $\text{Lat}(A)$ denote the collection of ideals in $A$, equipped with the partial order given by inclusion of ideals. For any ideal $I$ in $A$, then $\text{Cu}(I)$ is an ideal of $\text{Cu}(A)$. (See [3, 5.1] and [7, 3.1] for more details.) For any $x \in \text{Cu}(A)$, write $I_x = \{ y \in \text{Cu}(A) \mid y \leq \infty x \}$ as the ideal of $\text{Cu}(A)$ generated by $x$, where

$$\infty x = \bigoplus_{n=1}^{\infty} \frac{1}{2^n} a \in \text{Cu}(A),$$

for any $a \in (A \otimes K)_+$ with $[a] = x$.

For any $C^*$-algebra $A$, $I \rightarrow \text{Cu}(I)$ defines a lattice isomorphism between the lattice $\text{Lat}(A)$ of closed two-sided ideals of $A$ and the lattice $\text{Lat}(\text{Cu}(A))$ of ideals of $\text{Cu}(A)$, denote this isomorphism by $\Gamma$, i.e.,

$$\Gamma : \text{Lat}(A) \rightarrow \text{Lat}(\text{Cu}(A)), \quad \Gamma(I) = \text{Cu}(I).$$

There will be an abuse of notations from now on, for any $a \in A_+$, we write $I_a, I_{[a]}$ for the respective ideals in $A$ and $\text{Cu}(A)$, but we will indistinguishably use $I_a$ or $I_{[a]}$, referring to one or the other.

1.6. Let $A = \lim(A_n, \phi_{nm})$ be an inductive limit of $C^*$-algebras. It is shown in [8, Theorem 2] that the Cuntz semigroup functor preserves inductive limits of sequences (this was generalized to arbitrary inductive limits in [3, Corollary 3.2.9], see also [26, Lemma 3.8]) and each element $s \in \text{Cu}(A)$ can be represented by an increasing sequence $(s_1, s_2, \cdots)$, with $s_1 \in \text{Cu}(A_1), s_2 \in \text{Cu}(A_2), \cdots$ increasing in the sense that for each $i$, the image of $s_i \in \text{Cu}(A_i)$ in $\text{Cu}(A_{i+1})$ is less than or equal to $s_{i+1}$. Recall that two sequences $(s_i) \leq (t_i)$ if for any $i$ and any $s \in S_i$ with $s \ll s_i$, eventually $s \ll t_j$ (in $S_j$). We say $(s_i) \sim (t_i)$, if $(s_i) \leq (t_i)$ and $(t_i) \leq (s_i)$. We denote the class of $(s_i)$ in $\text{Cu}(A)$ by $[(s_i)]$, then

$$\text{Cu}(A) = \text{Cu} - \lim \text{Cu}(A_i).$$

Let $(s_1, s_2, \cdots)$ be an increasing sequence with $[(s_i)] = s$, where $s_i \in \text{Cu}(A_i)$ for any $i$. Then $I_s \cong \text{Cu} - \lim I_{s_i}$, where $I_{s_i}$ is the ideal generated by $s_i$ in $\text{Cu}(A_i)$. For the corresponding ideals in $C^*$-algebras, we simply write $I_s = C^* - \lim I_{s_i}$.

The following theorem is [4, Theorem 2.8].

**Theorem 1.7.** If $B$ is a full hereditary $C^*$-subalgebra of $A$ and if each of $A$ and $B$ has a strictly positive element, then $B$ is stably isomorphic to $A$.

For any $a \in A_+$, the hereditary algebra $\text{her}(a)$ generated by $a$ and the ideal $I_a$ generated by $a$ are stably isomorphic, and hence, have the same $K$-theory. We will use this fact frequently.
1.8. (Unitary Cuntz semigroup) Recall the definition of unitary Cuntz semigroup introduced by Cantier in [6, 3.3]. Let $A$ be a $C^*$-algebra of stable rank one and let

$$H(A) := \{(a, u) \mid a \in (A \otimes K)_+, u \in U(\text{her}(a)^\sim)\}.$$ 

Write $(a, u) \lesssim_1 (b, v)$ if $a \lesssim_{Cu} b$ and $[\theta_{ab, \alpha}(u)] = [v]$ in $K_1(\text{her}(b))$, where $\theta_{ab, \alpha}$ is an explicit injection from her($a$) to her($b$) (see [6, 2.2]), we say $(a, u) \sim_1 (b, v)$ if $(a, u) \lesssim_1 (b, v)$ and $(b, v) \lesssim_1 (a, u)$.

Define the unitary Cuntz semigroup of $A$ by

$$\text{Cu}_1(A) := H(A)/ \sim_1.$$ 

The equivalent class of an element $(a, u)$ in $H(A)$ is denoted by $[(a, u)]$.

For any $[(a, u)], [(b, v)] \in \text{Cu}_1(A)$, we write $[(a, u)] \leq [(b, v)]$, if $[(a, u)] \lesssim_1 [(b, v)]$, and we set $[(a, u)] + [(b, v)] = [(a \oplus b, u \oplus v)]$.

Then $(\text{Cu}_1(A), +, \leq)$ is a partially ordered monoid.

Let $A$ be a $C^*$-algebra of stable rank one. Denote Lat$_f(A)$ the subset of Lat($A$) consisting of all the ideals of $A$ that contains a full positive element. Hence, any ideal in Lat$_f(A)$ is singly-generated. Define $\text{Cu}_f(I) := \{[a] \in \text{Cu}(A) \mid I_a = I\}$. In other words, $\text{Cu}_f(I)$ consists of the elements of $\text{Cu}(A)$ that are full in $\text{Cu}(I)$.

The following theorem illustrates the main structure of $\text{Cu}_1(A)$.

**Theorem 1.9.** ([6, Theorem 4.8]) Let $A$ be a separable $C^*$-algebra of stable rank one, then there is a $\text{Cu}^\sim$-isomorphism:

$$\text{Cu}_1(A) \cong \coprod_{I \in \text{Lat}_f(A)} \text{Cu}_f(I) \times K_1(I).$$

The addition and order on $\coprod_{I \in \text{Lat}_f(A)} \text{Cu}_f(I) \times K_1(I)$ are defined as follows:

For any $(x, k) \in \text{Cu}_f(I_x) \times K_1(I_x)$ and $(y, l) \in \text{Cu}_f(I_y) \times K_1(I_y)$,

$$\begin{cases} (x, k) \leq (y, l), & \text{if } x \leq y \text{ and } \delta_{I_x I_y}(k) = l, \\ (x, k) + (y, l) = (x + y, \delta_{I_x I_y}(k) + \delta_{I_y I_x + y}(l)) & \end{cases},$$

where $\delta_{IJ} = K_1(I \overset{\sim}{\rightarrow} J)$, for any $I, J \in \text{Lat}(A)$ such that $I \subset J$.

If there is an $I \in \text{Lat}(A) \setminus \text{Lat}_f(A)$, we have $\text{Cu}_f(I) = \emptyset$. We mention that whenever $A$ is separable, then Lat($A$) = Lat$_f(A)$.

1.10. [3, 5.5] Denote Mon$_\leq$ the category of ordered monoids and POM the category consisting of positively ordered monoids. Given $M \in \text{POM}$, denote its Cu-completion by Cu($M$) (the set of any increasing sequence in $M$ modulo equivalent relation) (see more details in [3, 3.1.6]). Thus, we obtain a functor

$$\text{Cu} : \text{POM} \rightarrow \text{Cu}, \quad M \mapsto \text{Cu}(M), \quad \text{for all } M \in \text{POM},$$

mapping the POM-morphism $f : M \rightarrow N$ into the induced Cu-morphism $\text{Cu}(f) : \text{Cu}(M) \rightarrow \text{Cu}(N)$. 
Conversely, given a Cu-semigroup $S$, we denote by $S_c$ the set of compact elements in $S$. It is easy to see that $S_c$ is a submonoid of $S$ and we equip it with the order induced by $S$. Then we get another functor

$$\mathcal{C}: \text{Cu} \rightarrow \text{POM}, \quad S \mapsto S_c,$$

for all $S \in \text{Cu}$.

It is obvious that the two functors Cu and $\mathcal{C}$ establish an equivalence of the category POM and the full subcategory of Cu consisting of algebraic Cu-semigroups (every element is the supremum of an increasing sequence of compact elements).

1.11. The unitary Cuntz category, written $\text{Cu}^\sim$, is the subcategory of $\text{Mon}_{\leq}$ whose objects are ordered monoids satisfying the axioms (O1)- (O4) and such that $0 \ll 0$ and morphisms are $\text{Mon}_{\leq}$-morphisms that respect suprema of increasing sequences and the compact containment relation.

Let $S, T \in \text{Cu}^\sim$ and $f : S \rightarrow T$ be a $\text{Cu}^\sim$-morphism, then $S_c, T_c$ are ordered submonoids of $S, T$, respectively. Denote the restriction map of $f$ by $f_c : S_c \rightarrow T_c$, and denote the induced Grothendieck map of $f_c$ by $\text{Gr}(f_c) : \text{Gr}(S_c) \rightarrow \text{Gr}(T_c)$.

**Definition 1.12.** Give $S \in \text{Cu}^\sim$, we say that $S$ is an algebraic $\text{Cu}^\sim$-semigroup if every element in $S$ is the supremum of an increasing sequence of compact elements, that is, an increasing sequence in $S_c$.

**Corollary 1.13.** ([6, Corollary 3.25]) Let $A \in C^*_{sr1}$. Then $A$ has real rank zero if and only if $\text{Cu}_1(A)$ is algebraic if and only if $\text{Cu}(A)$ is algebraic.

**Definition 1.14.** Let $C, D$ be arbitrary categories, and let $I : C^* \rightarrow C$ and $J : C^* \rightarrow D$ be covariant functors. Let $H : C \rightarrow D$ be a functor such that there exists a natural isomorphism $\eta : H \circ J \simeq I$. Then we say we can recover $I$ from $J$ through $H$.

2. Cu$_1$ and K$_*$

Cantier has shown that for any unital, separable $C^*$-algebras of stable rank one, the invariant K$_*$ can be recovered from Cu$_1$, in this section, under the real rank zero setting, we prove the converse.

**Definition 2.1.** [16, Definition 1.2.1] Let $A$ be a (unital) $C^*$-algebra and set $K_*(A) = K_0(A) \oplus K_1(A)$. Define $K^+_k(A) = \{(p), [u \oplus (1_k - p)]\}$, where $p \in M_k(A)$ is a projection, $u \in pM_k(A)p$ is a unitary and $1_k \in M_k(A)$ is the unit. Note that we also use $[u]$ or $[u]_{K_1(A)}$ to denote the K$_1$ class $[u + (1_k - p)]_{K_1(A)}$.

The following proposition is well-known, see [21, Proposition 4].

**Proposition 2.2.** Let

$$0 \rightarrow A \rightarrow E \rightarrow B \rightarrow 0$$

be a short exact sequence of $C^*$-algebras. Let $\delta_j : K_j(B) \rightarrow K_{1-j}(A)$ for $j = 0, 1$, be the index maps of the sequence.
(i) Assume that $A$ and $B$ have real rank zero. Then the following three conditions are equivalent:

(a) $\delta_0 \equiv 0$,
(b) $rr(E) = 0$,
(c) all projections in $B$ are images of projections in $E$.

(ii) Assume that $A$ and $B$ have stable rank one. Then the following are equivalent:

(a) $\delta_1 \equiv 0$,
(b) $sr(E) = 1$.

If, in addition, $E$ (and $B$) are unital then (a) and (b) in (ii) are equivalent to
(c) all unitaries in $B$ are images of unitaries in $E$.

Example 2.3. For any $C^*$-algebra $A$ of stable rank one, $\text{Cu}(A)$ has weak cancellation (see [15], [23]), i.e., $x + z \ll y + z$ implies $x \leq y$ for $x, y, z \in \text{Cu}(A)$, but this is not true for $\text{Cu}_1(A)$, we present an example here.

Set

$E = \{ f \in M_2(C[0,1]) \mid f(0) = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ and $f(1) = \begin{pmatrix} \nu & 0 \\ 0 & \mu \end{pmatrix}, \lambda, \mu, \nu \in \mathbb{C} \}.$

We have the following short exact sequence

$$0 \to M_2(C_0(0,1)) \xrightarrow{\iota} E \xrightarrow{\pi} \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \to 0,$$

where $\iota$ is the natural embedding map and $\pi(f) = (\lambda, \nu, \mu)$ for $f \in E$. Then one has the six-term exact sequence

$$0 \to K_0(E) \overset{\pi_*}{\to} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\delta_0} \mathbb{Z} \xrightarrow{\iota_*} K_1(E) \to 0,$$

where $\delta_0 = (1, -1, 0)$.

Such an $E$ is an Elliott-Thomsen algebra, one can see [20] for more details. Here, we have $K_1(E) = 0$ ($\delta_0$ is surjective, and hence, $\iota_* = 0$) and $sr(E) = 1$ ($E$ is an extension of two stable rank one algebras and $\delta_1 \equiv 0 : K_1(C \oplus C \oplus \mathbb{C}) \to K_0(M_2(C_0(0,1)))$, see Proposition 2.2 (ii)).

Let $q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in E$ be a projection, then $\text{her}(q) \cong C(\mathbb{T})$ and $K_1(I_q) = K_1(\text{her}(q)) = K_1(C(\mathbb{T})) = \mathbb{Z}$.

Take

$([q], 0), ([q], 1) \in \text{Cu}_1(E)$.

Since the natural embedding map from $I_q$ to $E \otimes K$ induces $\delta_{I_qE} : K_1(I_q) \to K_1(E) = 0$, then

$([q], 0) + ([1_E], 0) = ([q] + [1_E], \delta_{I_qE}(0) + 0) = ([q] + [1_E], 0),$

$([q], 1) + ([1_E], 0) = ([q] + [1_E], \delta_{I_qE}(1) + 0) = ([q] + [1_E], 0).$

But $([q], 0) \neq ([q], 1)$.

This means $\text{Cu}_1(E)$ doesn’t satisfy the cancellation of compact elements in the sense of [6, Proposition 5.16]. The basic reason is that the natural
map $\delta_{I_0E} = K_1(\iota) : K_1(I_0) \to K_1(E)$ is not injective, though $\iota : I_q \to E$ itself is injective and $K_0(\iota) : K_0(I_0) \to K_0(E)$ is injective.

Moreover, consider the following map

$$\alpha : Cu_1(A)_c \to K^+_1(A)$$

$$([q], [u]) \mapsto ([q], \delta_{I_qA}([u]))$$

In general, $\alpha$ is just a surjective map, not an isomorphism (such as $A = E$). This means the proof of [6, Theorem 5.20] is not entirely correct. We point out that the conclusion is still true, as one can still prove $(Gr(Cu_1(A)_c), \rho(Cu_1(A)_c)) \cong (K_*(A), K^+_*(A))$ without the injectivity of $\alpha$, where $\rho : Cu_1(A)_c \to Gr(Cu_1(A)_c)$ is the natural map ($\rho(x) = [(x, 0)]$). We will write down a proof and also a refinement version.

The following result is partially proved in [6, Theorem 5.20], but there is one deficient part left for the proof, now we present our method.

**Theorem 2.4.** Let $A$ be a unital, separable $C^*$-algebra with stable rank one. Then

$$(Gr(Cu_1(A)_c), \rho(Cu_1(A)_c)) \cong (K_*(A), K^+_*(A)).$$

Moreover, if $A$ is of real rank zero, then

$$(Cu_1(A)_c, ([1_A], 0)) \cong (K^+_1(A), [1_A]).$$

**Proof.** Let $[(a, u)]$ be a compact element in $Cu_1(A)$, by [6, Corollary 3.5], $[a]$ is a compact element of $Cu(A)$. Since $A$ has stable rank one, use [5, Theorem 5.8], there exists a projection $p \in A \otimes K$ such that $[p] = [a]$. Then every compact element in $Cu_1(A)$ can be written as $[(p, u)]$, where $p$ is a projection in $A \otimes K$ and $u$ is a unitary element in $her(p)$. From the view of Theorem 1.9, $[(p, u)]$ can also be written as $([p], [u]_{K_1(I_p)})$. $(K_1(I_p)$ and $K_1(her(p))$ are naturally isomorphic.) Then the map

$$\alpha : Cu_1(A)_c \to K^+_1(A)$$

$$([p], [u]_{K_1(I_p)}) \mapsto ([p], \delta_{I_pA}([u]_{K_1(I_p)}))$$

is a monoid morphism, where $\delta_{I_pA} : K_1(I_p) \to K_1(A)$.

Note that for any $([q], [v]_{K_1(I_q)}) \in K^+_1(A)$, where $q$ is a projection in $A \otimes K$ and $v$ is a unitary in $her(q)$, $([q], [v]_{K_1(I_q)})$ is a compact element in $Cu_1(A)_c$. By the inclusion from $I_q$ to $A \otimes K$, we have $\delta_{I_qA}([v]_{K_1(I_q)}) = [v]_{K_1(A)}$. Then $\alpha$ is surjective.

Suppose that

$$\alpha([p], [u]_{K_1(I_p)}) = \alpha([q], [v]_{K_1(I_q)}).$$

Then for $([1_A], 0) \in Cu_1(A)_c$, we have

$$([p] + [1_A], \delta_{I_pA}([u]_{K_1(I_p)})) = ([q] + [1_A], \delta_{I_qA}([v]_{K_1(I_q)})),$$

which is

$$([p], [u]_{K_1(A)}) + ([1_A], 0) = ([q], [v]_{K_1(A)}) + ([1_A], 0).$$
Hence, the images of \([p], [u]_{K_1(A)}\) and \([q], [v]_{K_1(A)}\) in \(\text{Gr}(\text{Cu}_1(A)_c)\) are the same.

Consider the natural map \(\rho : \text{Cu}_1(A)_c \to \text{Gr}(\text{Cu}_1(A)_c)\), if \(\rho(x) = \rho(y)\), then there exists \(z \in \text{Cu}_1(A)_c\) such that \(x + z = y + z\), hence,

\[\alpha(x) + \alpha(z) = \alpha(y) + \alpha(z).\]

Recall that \((K_\ast, K_1^+(A))\) is an ordered group, then \(\alpha(x) = \alpha(y)\).

Now we have

\[
\text{Gr}(\text{Cu}_1(A)_c) \cong \text{Gr}(\alpha(\text{Cu}_1(A)_c)) = \text{Gr}(K_1^+(A))
\]

and

\[
(\text{Gr}(\text{Cu}_1(A)_c), \rho(\text{Cu}_1(A)_c)) \cong (K_\ast, K_1^+(A)).
\]

Generally speaking, \(\alpha\) is not always injective, but under the assumption of real rank zero, this is true.

Suppose that

\[
\alpha([p], [u]_{K_1(I_p)}) = \alpha([q], [v]_{K_1(I_q)}).
\]

Then \(p \sim_{Cu} q\), and hence, \(I_p = I_q\). By Lemma 2.2, we have \(\delta_{I_p, A}\) and \(\delta_{I_q, A}\) are the same injective map, which means we can identify \([u]_{K_1(I_p)}\), \([u]_{K_1(A)}\), \([v]_{K_1(A)}\) and \([v]_{K_1(I_q)}\) as a same element.

Since \(A\) has stable rank one, Murray von Neumann equivalence and Cuntz equivalence agree on projections. That is, \(([p], [u]_{K_1(I_p)}) = ([q], [v]_{K_1(I_q)})\), and hence, \(\alpha\) is injective. Therefore, \(\alpha\) is an isomorphism.

Note that \(K_1^0(A)\) is a sub-cone of \(K_1^+(A)\), it induces an order on \(K_1^+(A)\), hence, in \(K_1^+(A)\) we say

\[
([p], [u]_{K_1(A)}) \leq ([q], [v]_{K_1(A)}) \iff [p] \leq [q], [u]_{K_1(A)} = [v]_{K_1(A)}.
\]

Then \(\alpha\) becomes an ordered isomorphism from \(\text{Cu}_1(A)_c\) to \(K_1^+(A)\).

Note that \(\text{Cu}_1(A)\) is algebraic and completely determined by \(\text{Cu}_1(A)_c\), meanwhile, \(K_\ast\) is the Grothendieck group of \(K_1^+(A)\). Moreover, \(\alpha\) can be regarded as an ordered isomorphism:

\[
(\text{Cu}_1(A)_c, ([1_A], 0)) \cong (K_1^+(A), [1_A]).
\]

This completes the proof.

\[\square\]

**Proposition 2.5.** Let \(A\) be a unital, separable \(C^*\)-algebra of stable rank one and real rank zero. Then \(\text{Cu}_1(A)\) has weak cancellation, i.e., \(x + z \ll y + z\) implies \(x \leq y\) for \(x, y, z \in \text{Cu}_1(A)\).

**Proof.** For each ideal \(I\) of \(A\), by Proposition 2.2, we will identify \(K_1(I)\) with its natural image in \(K_1(A)\).

Suppose that

\[
([a], [u]_{K_1(I_p)}) + ([c], [w]_{K_1(I_q)}) \ll ([b], [v]_{K_1(I_q)}) + ([c], [w]_{K_1(I_q)}),
\]

then from [6, Proposition 4.3], we have

\[
[a] + [c] \ll [b] + [c] \quad (\text{in } \text{Cu}(A))
\]
Therefore, and

$$\{ u \}_{K_1(I_a)} + \{ w \}_{K_1(I_b)} = \{ v \}_{K_1(I)} + \{ w \}_{K_1(I_b)}.$$ 

By the weak cancellation of \( \text{Cu}(A) \) (see 2.3), we have \( [a] \leq [b] \). Note that we also have

$$\{ u \}_{K_1(I_a)} = \{ v \}_{K_1(I_b)}.$$ 

Therefore,

$$([a], \{ u \}_{K_1(I_a)}) \leq ([b], \{ v \}_{K_1(I_b)}).$$

\(\square\)

**Remark 2.6.** In general, weak cancellation may not hold for algebraic \( \text{Cu}-\)semigroup (\( \text{Cu}\sim-\)semigroup). For each \( k \geq 1 \), the semigroup \( E_k = \{0, 1, \cdots, k, \infty\} \) with the natural order and with \( a + b \) defined as \( \infty \) if usually one would have \( a + b \geq k + 1 \) is an algebraic \( \text{Cu}-\)semigroup (\( \text{Cu}\sim-\)semigroup), and all elements are compact (see [3, 5.1.16]). Then \( E_k \) doesn’t have weak cancellation.

2.7. ([6, 3.9]) Let \( S \in \text{Mon}_\leq \), denote \( \text{Cu}^\sim(S) := \gamma^\sim(S, \leq) \) the \( \text{Cu}\sim\)-completion of \((S, \leq)\), then the assignment \( \text{Cu}^\sim \) from an ordered monoid \( S \) to \( \text{Cu}^\sim(S) \) is a functor. Denote \( \iota_S : S \to \text{Cu}^\sim(S) \) the natural embedding map.

**Proposition 2.8.** ([6, Proposition 3.23])

(i) Let \( M \in \text{Mon}_\leq \). Then \( \text{Cu}^\sim(M) \) is an algebraic \( \text{Cu}^\sim\)-semigroup and, moreover, there is a natural identification between \( M \) and the order monoid of compact elements of \( \text{Cu}^\sim(M) \).

(ii) For any algebraic \( \text{Cu}^\sim\)-semigroup \( S \), we have \( \text{Cu}^\sim(S_c) \cong S \) naturally as \( \text{Cu}^\sim\)-semigroups.

2.9. Let \( S \) be a \( \text{Cu}^\sim\)-semigroup. We say that \( S \) is positively directed, if for any \( x \in S \), there exists \( p_x \in S \) such that \( x + p_x \geq 0 \).

Consider the semigroup \( S = \mathbb{Z} \cup \{ \infty \} \) with the natural order. Then \( S \) is a positively directed algebraic \( \text{Cu}^\sim\)-semigroup with \( S_c = \mathbb{Z} \), but

$$\rho(S_c) \cap \{-\rho(S_c)\} = \mathbb{Z} \neq \{0\},$$

where \( \rho : S_c \to \text{Gr}(S_c) \) is the natural map \( \rho(x) = [(x, 0)] \). Hence, \( (\text{Gr}(S_c), \rho(S_c)) = (\mathbb{Z}, \mathbb{Z}) \) is not an ordered abelian group.

For any separable unital \( C^*\)-algebra \( A \) of stable rank one, on one hand, \( \text{Cu}_1(A) \) is positively directed [6, Lemma 5.2]; on the other hand, from the facts that \( K_1^+(A) \cap \{-K_1^+(A)\} = \{0\} \) ([16, 1.2.2]) and that

$$(\text{Gr}(\text{Cu}_1(A)_c), \rho(\text{Cu}_1(A)_c)) \cong (K_+(A), K_1^+(A)),$$

we have \( \rho(\text{Cu}_1(A)_c) \cap \{-\rho(\text{Cu}_1(A)_c)\} = \{0\} \).

2.10. The category of ordered groups with ordered unit, written \( \text{AbGp}_u \), is the category whose objects are ordered groups with order-unit, and morphisms are ordered group morphisms that preserve the order-unit.

Let \( S \) be a positively directed \( \text{Cu}^\sim\)-semigroup satisfying \( \rho(S_c) \cap \{-\rho(S_c)\} = \{0\} \). Also suppose that \( S_\pm \) admits a compact order-unit. We say that \((S, u)\)
is a \( \text{Cu}^- \)-semigroup with compact order-unit. Now, a \( \text{Cu}^- \)-morphism preserves the order-unit between two \( \text{Cu}^- \)-semigroups with compact order-unit \((S, u), (T, v)\) is a \( \text{Cu}^- \)-morphism \( \alpha : S \to T \) such that \( \alpha(u) \leq v \).

Denote \( \text{Cu}^- \) the category whose objects are \( \text{Cu}^- \)-semigroups with compact order-unit and morphisms are \( \text{Cu}^- \)-morphisms that preserve the order-unit.

Recall the functors defined in [6, 5.4],

\[
\begin{align*}
\text{K}_s &: C_{sr1}^* \to \text{AbGp}_u \\
A &\mapsto (K_s(A), K_s^+(A), [1_A]) \\
\phi &\mapsto K_s(\phi).
\end{align*}
\]

\[
\begin{align*}
\text{Cu}_{1,u} &: C_{sr1}^* \to \text{Cu}_u^- \\
A &\mapsto (\text{Cu}_1(A), ([1_A], 0)) \\
\phi &\mapsto \text{Cu}_1(\phi).
\end{align*}
\]

\[
\begin{align*}
H_s &: \text{Cu}_u^- \to \text{AbGp}_u \\
(S, u) &\mapsto (\text{Gr}(S_c), \rho(S_c), u) \\
\alpha &\mapsto \text{Gr}(\alpha_c).
\end{align*}
\]

Note that if we restrict the domain of \( H_s \) to the full subcategory of unitary Cuntz semigroups of separable, unital \( C^* \)-algebras of stable rank one and real rank zero, then Proposition 2.5 and 2.9 imply that \( H_s \) is a faithful functor. (The proof of [6, Lemma 5.19] holds in this case.)

**Theorem 2.11.** Upon restriction to the class of unital, separable \( C^* \)-algebras of stable rank one and real rank zero, there are natural equivalences of functors:

\[
H_s \circ \text{Cu}_{1,u} \simeq K_s \quad \text{and} \quad G \circ K_s \simeq \text{Cu}_{1,u}.
\]

**Proof.** We restrict the domain of \( H_s \) to the full subcategory whose objects are unitary Cuntz semigroup of separable \( C^* \)-algebras of stable rank one and real rank zero together with compact order-unit, while the codomain of \( H_s \) is the category of the \( K_s \) of the same class. We will prove that \( H_s \) yields an equivalence of functors between \( \text{Cu}_{1,u} \) and \( K_s \) under our assumption. We only need to show that the restriction functor of \( H_s \), which we will still call \( H_s \), is a full, faithful and dense functor.

From the last statement in 2.10, \( H_s \) is faithful. For any unital, separable \( C^* \)-algebra \( A \) of stable rank one and real rank zero, denote the canonical isomorphism we obtained from Lemma 2.4 as

\[
\alpha_A^* : (\text{Gr}(\text{Cu}_1(A)_c), \rho(\text{Cu}_1(A)_c)) \cong (K_s(A), K_s^+(A)),
\]

which forms

\[
\alpha_A^* : H_s(\text{Cu}_1(A)_c, ([1_A], 0)) \cong (K_s(A), K_s^+(A), [1_A]).
\]

This means that \( H_s \) is dense. It remains to show that \( H_s \) is full.
Since $A$ has stable rank one and real rank zero, $K^+_+(A)$ is a subset in $K_+(A)$, then for any order morphism

$$\xi : (K_+(A), K^+_+(A), [1_A]) \to (K_+(B), K^+_+(B), [1_B])$$

with $\xi|_{K^+_+(A)} : K^+_+(A) \to K^+_+(B)$, $\xi([1_A]) \leq [1_B]$ and

$$\xi|_{K^+_+(A)}(K^+_0(A)) \subset K^+_0(B).$$

By the functoriality of $Cu^\sim$ (see 2.7), $\xi|_{K^+_+(A)}$ induces an ordered $Cu^\sim$-morphism

$$\gamma^\sim(\xi|_{K^+_+(A)}) : (Cu^\sim(K^+_+(A), ([1_A], 0)) \to (Cu^\sim(K^+_+(B), ([1_B], 0)),$$

where the order on $K^+_+(A)$ is induced by $K^+_0(A)$.

By Proposition 2.8 and Lemma 2.4, we have

$$Cu_1(A) \cong Cu^\sim(Cu_1(A)_c) \cong Cu^\sim(K^+_+(A))$$

and

$$(Cu_1(A)_c, ([1_A], 0)) \cong (K^+_+(A), [1_A]).$$

Denote $i_A : Cu_1(A) \to Cu^\sim(K^+_+(A))$ the canonical $Cu^\sim$-isomorphism and denote $\alpha_A : (Cu_1(A)_c, ([1_A], 0)) \to (K^+_+(A), [1_A])$ the canonical ordered monoid isomorphism.

Note that

$$i_B^{-1} \circ \gamma^\sim(\xi|_{K^+_+(A)}) \circ i_A : (Cu_1(A), ([1_A], 0)) \to (Cu_1(B), ([1_B], 0))$$

is an ordered $Cu^\sim$-morphism. After identifying $Cu_1(A)_c$ and $Cu_1(B)_c$ with $K^+_+(A)$ and $K^+_+(B)$ through $\alpha_A$ and $\alpha_B$, respectively, we have

$$(i_B^{-1} \circ \gamma^\sim(\xi|_{K^+_+(A)}) \circ i_A)_c = \xi|_{K^+_+(A)}.$$

Using the functoriality of $H_*$, then

$$Gr((i_B^{-1} \circ \gamma^\sim(\xi|_{K^+_+(A)}) \circ i_A)_c) : (K_+(A), K^+_+(A), [1_A]) \to (K_+(B), K^+_+(B), [1_B])$$

is an ordered morphism. Now we have

$$H_*(i_B^{-1} \circ \gamma^\sim(\xi|_{K^+_+(A)}) \circ i_A) = Gr((i_B^{-1} \circ \gamma^\sim(\xi|_{K^+_+(A)}) \circ i_A)_c) = Gr(\xi|_{K^+_+(A)}) = \xi.$$

Then $H_*$ is full.

Therefore, by standard category theory, there exists a functor $G$ such that $H_* \circ G$ and $G \circ H_*$ are naturally equivalent to the respective identities. Then we have

$$H_* \circ Cu_{1,u} \simeq K_* \quad \text{and} \quad G \circ K_* \simeq Cu_{1,u}.$$

$\square$

**Corollary 2.12.** Let $A, B$ be unital, separable $C^*$-algebra with stable rank one and real rank zero. Then $(Cu_1(A), ([1_A], 0)) \cong (Cu_1(B), ([1_A], 0))$ if, and only if, $(K_+(A), K^+_+(A), [1_A]) \cong (K_+(B), K^+_+(B), [1_B]).$
Remark 2.13. It was shown in [17, Example 2.19] that $K_*$ is not a complete invariant for AD algebras of real rank zero (see also [13, Theorem 3.3]). Then Corollary 5.21 in [6] the author obtained as the classification theorem for real rank zero AD algebras (which is called AH$_d$ algebras in [6]) in terms of unitary Cuntz semigroup, is not entirely correct. He used Theorem 7.3 of [14] to get the corollary. However it was pointed out by Elliott-Gong-Su [17] that, Theorem 7.3 (and Theorem 7.1) of [14] is only true for simple case. Then $\text{Cu}_1$ is not a complete invariant for AD algebras (AH$_d$ algebras).

Remark 2.14. It was also proved that in [17] the total K-theory is a complete invariant for AD algebras of real rank zero. The total K-theory works quite well in the classification of non-simple C$^*$-algebras (see [18, 12, 1]). Then it is necessary to give a generalized version of Cuntz semigroup for $K$ just like $\text{Cu}_1$ for $K_*$.

3. The total Cuntz semigroup

In this section, we introduce the total Cuntz semigroup, which is a refinement of the unitary Cuntz semigroup. We show that $\text{Cu}$ is a functor from the category of unital, separable C$^*$-algebras of stable rank one to the total Cuntz category.

3.1. (The total K-theory) [12, Section 4]. For $n \geq 2$, the mod-$n$ K-theory groups are defined by

$$K_*(A; \mathbb{Z}_n) = K_*(A \otimes C_0(W_n)),$$

where $W_n$ denotes the Moore space obtained by attaching the unit disk to the circle by a degree $n$-map, such as $f_n : \mathbb{T} \to \mathbb{T}, e^{i\theta} \mapsto e^{i n \theta}$. The C$^*$-algebra $C_0(W_n)$ of continuous functions vanishing at the base point is isomorphic to the mapping cone of the canonical map of degree $n$ from $C(\mathbb{T})$ to itself.

In the setting of [24], the mod-$n$ K-theory groups are defined by

$$K_i(A; \mathbb{Z}_n) = K_i(A \otimes C_0(W_n)), \quad i = 0, 1.$$

Let $K_*(A; \mathbb{Z}_n) = K_0(A; \mathbb{Z}_n) \oplus K_1(A; \mathbb{Z}_n)$. For $n = 0$, we set $K_*(A; \mathbb{Z}_n) = K_*(A)$ and for $n = 1$, $K_*(A; \mathbb{Z}_n) = 0$.

For a C$^*$-algebra $A$, one defines the total K-theory of $A$ by

$$K(A) = \bigoplus_{n=0}^{\infty} K_*(A; \mathbb{Z}_n).$$

It is a $\mathbb{Z}_2 \times \mathbb{Z}^+$ graded group. It was shown in [24] that the coefficient maps

$$\rho : \mathbb{Z} \to \mathbb{Z}_n, \quad \rho(1) = [1],$$

$$\kappa_{mn,m} : \mathbb{Z}_m \to \mathbb{Z}_{mn}, \quad \kappa_{mn,m}(1) = n[1],$$

$$\kappa_{n,mn} : \mathbb{Z}_{mn} \to \mathbb{Z}_n, \quad \kappa_{n,mn}(1) = [1],$$

induce natural transformations

$$\rho^i_n : K_i(A) \to K_i(A; \mathbb{Z}_n)$$

The Bockstein operation
\[ \beta_n^i : K_i(A; \mathbb{Z}_n) \to K_{i+1}(A) \]
appears in the six-term exact sequence
\[ K_i(A) \xrightarrow{\times n} K_i(A) \xrightarrow{\rho_n} K_i(A; \mathbb{Z}/n) \xrightarrow{\beta_n^i} K_{i+1}(A) \xrightarrow{\times n} K_{i+1}(A) \]
induced by the cofibre sequence
\[ A \otimes SC_0(T) \to A \otimes C_0(W_n) \xrightarrow{\beta} A \otimes C_0(T) \xrightarrow{n} A \otimes C_0(T), \]
where \( SC_0(T) \) is the suspension algebra of \( C_0(T) \).

There is a second six-term exact sequence involving the Bockstein operations. This is induced by a cofibre sequence
\[ A \otimes SC_0(W_n) \to A \otimes C_0(W_n) \to A \otimes C_0(W_{mn}) \to A \otimes C_0(W_n) \]
and takes the form:
\[ K_{i+1}(A; \mathbb{Z}_n) \xrightarrow{\beta_{mn,n}^i} K_i(A; \mathbb{Z}_m) \xrightarrow{\kappa_{mn,m}^i} K_i(A; \mathbb{Z}_mn) \xrightarrow{\kappa_{n,n}^i} K_i(A; \mathbb{Z}_n) \]
where \( \beta_{mn,n}^i = \rho_{m+1}^i \circ \beta_n^i \).

The collection of all the transformations \( \rho, \beta, \kappa \) and their compositions is denoted by \( \Lambda \). \( \Lambda \) can be regarded as the set of morphisms in a category whose objects are the elements of \( \mathbb{Z}_2 \times \mathbb{Z}^+ \). Abusing the terminology \( \Lambda \) will be called the category of Bockstein operations. Via the Bockstein operations, \( KK(A) \) becomes a \( \Lambda \)-module. It is natural to consider the group \( \text{Hom}_\Lambda(K(A), K(B)) \) consisting of all \( \mathbb{Z}_2 \times \mathbb{Z}^+ \) graded group morphisms which are \( \Lambda \)-linear, i.e. preserve the action of the category \( \Lambda \).

The Kasparov product induces a map
\[ \gamma_n^i : KK(A, B) \to \text{Hom}(K_i(A; \mathbb{Z}/n), K_i(B; \mathbb{Z}/n)) \]
Then \( \gamma_n = (\gamma_n^0, \gamma_n^1) \) will be a map
\[ \gamma_n : KK(A, B) \to \text{Hom}(K_s(A; \mathbb{Z}_n), K_s(B; \mathbb{Z}_n)). \]
Note that if \( n = 0 \) then \( K_s(A, \mathbb{Z}_n) = K_s(A) \) and the map \( \gamma_0 \) is the same as the map \( \gamma \) from the universal coefficient theorem (UCT) of Rosenberg and Schochet [22]. We assemble the sequence \( (\gamma_n) \) into a map \( \Gamma \). Since the Bockstein operations are induced by multiplication with suitable KK elements and since the Kasparov product is associative, we obtain a map
\[ \Gamma : KK(A, B) \to \text{Hom}_\Lambda(K(A), K(B)). \]
For the sake of simplicity, if \( \alpha \in \text{KK}(A, B) \), then \( \Gamma(\alpha) \) will be often denoted by \( \alpha_s \).

Assume that \( A \) is a separable \( C^* \)-algebra of stable rank one, the following is a positive cone for total K-theory of \( A \) ([12, Definition 4.6]):
\[ K(A)_+ = \{ (\theta, u, \oplus_{p=1}^\infty (s_p, s^p)) : (\theta) \in K_0^+(A), ([\theta], u, \oplus_{p=1}^\infty (s_p, s^p)) \in K_{1e}(A) \}, \]
where $u \in K_1(A)$, $s_p \in K_0(A; Z_p)$ and $s^p \in K_1(A; Z_p)$ are equivalent classes, $L_e$ is the ideal of $A \otimes K$ generated by $e$ and $K_{I_e}(A)$ is the image of $K(I_e)$ in $K(A)$.

**Theorem 3.2.** ([12, Proposition 4.8–4.9]) Suppose that $A$ is of stable rank one and has an approximate unit $(e_n)$ consisting of projections. Then

(i) $K(A) = K(A)_+ - K(A)_+$;

(ii) $K(A)_+ \cap \{-K(A)_+\} = \{0\}$, and hence, $(K(A), K(A)_+)$ is an ordered group;

(iii) For any $x \in K(A)$, there are positive integers $k, n$ such that $k[e_n] + x \in K(A)_+$.

**Proposition 3.3.** ([12, 4.10–4.11]) $K(\cdot)$ and $K(\cdot)_+$ are continuous functors for $C^*$-algebras of stable rank one.

### 3.4. Let $A$ be a separable $C^*$-algebra of stable rank one and let $I$ and $J$ be ideals of $A \otimes K$ such that $I \subset J$. Denote $\theta_{I,J} : I \to J$ the natural embedding map and denote $\delta_{I,J}$ the map $K(\theta_{I,J}) : K(I) \to K(J)$, i.e.,

$$
\delta_{I,J}([e], u, \bigoplus_{p=1}^{\infty} (s_p, s^p)) = (K_0(\theta_{I,J})([e]), K_1(\theta_{I,J})(u), \bigoplus_{p=1}^{\infty} (K_0(\theta_{I,J}; Z_p)(s_p), K_1(\theta_{I,J}; Z_p)(s^p))).
$$

where the maps

$K_0(\theta_{I,J}; Z_p) : K_0(I; Z_p) \to K_0(J; Z_p)$, $K_1(\theta_{I,J}; Z_p) : K_1(I; Z_p) \to K_1(J; Z_p)$

are induced by $\theta_{I,J}$.

In this paper, we will always identify $K_1(I) \times \bigoplus_{n=1}^{\infty} K_*(I; Z_p)$ with its natural image in $K(I) = \bigoplus_{n=0}^{\infty} K_*(I; Z_p)$, i.e., each $(u, \bigoplus_{p=1}^{\infty} (s_p, s^p)) \in K(I)$ then the restriction of $\delta_{I,J}$ also induces a map $K_1(I) \times \bigoplus_{p=1}^{\infty} K_*(I; Z_p) \to K_1(J) \times \bigoplus_{p=1}^{\infty} K_*(J; Z_p)$, which is still denoted by $\delta_{I,J}$.

**Definition 3.5.** Let $A$ be a separable $C^*$-algebra of stable rank one. Define

$$
Cu(A) \triangleq \coprod_{I \in \text{Lat}_f(A)} Cu_f(I) \times K_1(I) \times \bigoplus_{p=1}^{\infty} K_*(I; Z_p).
$$

We equip $Cu(A)$ with addition and order as follows: For any

$$(x, u, \bigoplus_{p=1}^{\infty} (s_p, s^p)) \in Cu_f(I_x) \times K_1(I_x) \times \bigoplus_{p=1}^{\infty} K_*(I_x; Z_p)$$

and

$$(y, v, \bigoplus_{p=1}^{\infty} (t_p, t^p)) \in Cu_f(I_y) \times K_1(I_y) \times \bigoplus_{p=1}^{\infty} K_*(I_y; Z_p),$$

then

$$(x, u, \bigoplus_{p=1}^{\infty} (s_p, s^p)) + (y, v, \bigoplus_{p=1}^{\infty} (t_p, t^p)) = (x + y, \delta_{I_x, I_{x+y}}(u, \bigoplus_{p=1}^{\infty} (s_p, s^p)) + \delta_{I_y, I_{x+y}}(v, \bigoplus_{p=1}^{\infty} (t_p, t^p))).$$
and
\[(x, u, \oplus_{p=1}^{\infty}(s_p, s^p)) \leq (y, v, \oplus_{p=1}^{\infty}(t_p, t^p)),\]
if
\[x \leq y \text{ and } \delta_{I_x I_y}(u, \oplus_{p=1}^{\infty}(s_p, s^p)) = (v, \oplus_{p=1}^{\infty}(t_p, t^p)),\]
where \(\delta_{I_x I_y}\) is the natural map \(K(I_x) \to K(I_y)\) (see 3.4), \(I_x\) and \(I_y\) are ideals in \(A \otimes K\) generated by \(x\) and \(y\), respectively.

It is easy to see that \((Cu(A), +, \leq)\) is a partially ordered monoid, and \(Cu(A)_{+} \triangleq \{x \in Cu(A) : 0 \leq x\} = Cu(A)\). If \(A\) is simple, we will have
\[Cu(A) = (Cu(A) \setminus \{0\}) \times K_1(A) \times \bigoplus_{p=1}^{\infty} K_*(A, \mathbb{Z}_p) \cup \{0, 0, \bigoplus_{p=1}^{\infty}(0, 0)\}.

**Proposition 3.6.** Let \(A\) be a separable \(C^\ast\)-algebra of stable rank one, the addition and order for \(Cu(A)\) are compatible, i.e., for any
\[(x_1, u_1, \oplus_{p=1}^{\infty}(s_1, s^p)) \leq (y_1, v_1, \oplus_{p=1}^{\infty}(t_1, t^p))\]
and
\[(x_2, u_2, \oplus_{p=1}^{\infty}(s_2, s^p)) \leq (y_2, v_2, \oplus_{p=1}^{\infty}(t_2, t^p)),\]
we have
\[(x_1, u_1, \oplus_{p=1}^{\infty}(s_1, s^p)) + (x_2, u_2, \oplus_{p=1}^{\infty}(s_2, s^p)) \leq (y_1, v_1, \oplus_{p=1}^{\infty}(t_1, t^p)) + (y_2, v_2, \oplus_{p=1}^{\infty}(t_2, t^p)).\]

**Proof.** From assumption, we have
\[x_1 \leq y_1, \quad x_2 \leq y_2,\]
\[\delta_{I_{x_1} I_{y_1}}(u_1, \oplus_{p=1}^{\infty}(s_1, s^p)) = (v_1, \oplus_{p=1}^{\infty}(t_1, t^p))\]
and
\[\delta_{I_{x_2} I_{y_2}}(u_2, \oplus_{p=1}^{\infty}(s_2, s^p)) = (v_2, \oplus_{p=1}^{\infty}(t_2, t^p)).\]

Since
\[(x_1, u_1, \oplus_{p=1}^{\infty}(s_1, s^p)) + (x_2, u_2, \oplus_{p=1}^{\infty}(s_2, s^p)) = (x_1 + x_2, \delta_{I_{x_1} I_{y_1}}(u_1, \oplus_{p=1}^{\infty}(s_1, s^p)) + \delta_{I_{x_2} I_{y_2}}(u_2, \oplus_{p=1}^{\infty}(s_2, s^p)))\]
and
\[(y_1, v_1, \oplus_{p=1}^{\infty}(t_1, t^p)) + (y_2, v_2, \oplus_{p=1}^{\infty}(t_2, t^p)) = (y_1 + y_2, \delta_{I_{y_1} I_{y_2}}(v_1, \oplus_{p=1}^{\infty}(t_1, t^p)) + \delta_{I_{y_1} I_{y_2}}(v_2, \oplus_{p=1}^{\infty}(t_2, t^p))).\]

Note that the following diagram is naturally commutative.

\[
\begin{array}{cccc}
K(I_{x_1}) & \rightarrow & K(I_{y_1}) & \\
\downarrow & & \downarrow & \\
K(I_{x_1 + x_2}) & \rightarrow & K(I_{y_1 + y_2}) & \\
\downarrow & & \downarrow & \\
K(I_{x_2}) & \rightarrow & K(I_{y_2}) & \\
\end{array}
\]
Then we have
\[ \delta_{I_{x_1+x_2}I_{y_1+y_2}}(\delta_{I_{x_1}I_{x_1+x_2}}(u_1, \oplus_{p=1}^\infty (s_{1,p}\oplus s_{1,p}^2)) + \delta_{I_{x_2}I_{x_1+x_2}}(u_2, \oplus_{p=1}^\infty (s_{2,p}\oplus s_{2,p}^2))) \]
\[ = \delta_{I_{y_1}I_{y_1+y_2}}(\delta_{I_{y_1}I_{y_1}}(v_1, \oplus_{p=1}^\infty (t_{1,p}\oplus t_{1,p}^2))) + \delta_{I_{y_2}I_{y_1+y_2}}(\delta_{I_{y_2}I_{y_1}}(v_2, \oplus_{p=1}^\infty (t_{2,p}\oplus t_{2,p}^2))) \]
\[ = \delta_{I_{y_1}I_{y_1+y_2}}((v_1, \oplus_{p=1}^\infty (t_{1,p}\oplus t_{1,p}^2))) + \delta_{I_{y_2}I_{y_1+y_2}}((v_2, \oplus_{p=1}^\infty (t_{2,p}\oplus t_{2,p}^2))) \]
\[ = (v_1, \oplus_{p=1}^\infty (t_{1,p}\oplus t_{1,p}^2)) + (v_2, \oplus_{p=1}^\infty (t_{2,p}\oplus t_{2,p}^2)). \]

Combining this with
\[ x_1 + x_2 \leq y_1 + y_2 \in \text{Cu}(A), \]
the conclusion is true.

We now show that \((\text{Cu}(A), +, \leq)\) satisfies the axioms in 1.4. The proofs are similar to what has been done for the unitary Cuntz semigroup in [6].

**Lemma 3.7.** Let \(A\) be a separable \(C^*\)-algebra of stable rank one, \(x_n \in \text{Cu}(A)\) be an increasing sequence with supremum \(x\). Then for any \(u \in K_1(I_x), (s_p, s^p) \in K_s(I_x; \mathbb{Z}_p)\), there exists \(N \in \mathbb{N}\) such that for all \(n > N\), there exist
\[ u_n \in K_1(I_{x_n}), (s_{n,p}, s^{n,p}) \in K_s(I_{x_n}; \mathbb{Z}_p) \]
such that
\[ (x_n, u_n, \oplus_{p=1}^\infty (s_{n,p}\oplus s^{n,p})) \leq (x, u, \oplus_{p=1}^\infty (s_p\oplus s^p)). \]

**Proof.** Since \(x = \sup_n x_n \in \text{Cu}(A)\), we have \(I_x = \bigcap_{n \geq 1} I_{x_n}\). Then by continuity listed in Proposition 3.3, we have
\[ K(I_x) = \lim_{n \to \infty} (K(I_{x_n}), \delta_{I_{x_n}I_{x_n}}), \]
where \(K(I_x)\) is an algebraic limit.

Hence, there exists \(N\) large enough such that for all \(n \geq N\), there exist
\[ u_n \in K_1(I_{x_n}) \quad \text{and} \quad (s_{n,p}, s^{n,p}) \in K_s(I_{x_n}; \mathbb{Z}_p) \]
with
\[ \delta_{I_{x_n}I_x}(u_n, \oplus_{p=1}^\infty (s_{n,p}\oplus s^{n,p})) = (u, \oplus_{p=1}^\infty (s_p\oplus s^p)). \]

Now we get
\[ (x_n, u_n, \oplus_{p=1}^\infty (s_{n,p}\oplus s^{n,p})) \leq (x, u, \oplus_{p=1}^\infty (s_p\oplus s^p)). \]

**Corollary 3.8.** Let \(A\) be a separable \(C^*\)-algebra of stable rank one. Then any increasing sequence \((x_n, u_n, \oplus_{p=1}^\infty (s_{n,p}\oplus s^{n,p}))\) in \(\text{Cu}(A)\) has a supremum \((x, u, \oplus_{p=1}^\infty (s_p\oplus s^p))\). In particular, \(x = \sup_n x_n\), and for any \(n \in \mathbb{N}\),
\[ \delta_{I_{x_n}I_x}(u_n, \oplus_{p=1}^\infty (s_{n,p}\oplus s^{n,p})) = (u, \oplus_{p=1}^\infty (s_p\oplus s^p)). \]
Proof. Since \( \text{Cu}(A) \) satisfies (O1), there exists \( x \in \text{Cu}(A) \) such that \( x = \sup_{n} x_{n} \), and

\[
x = C^{*} - \lim I_{x_{n}}.
\]

Then

\[
K(I_{x}) = \lim \delta_{I_{x_{n}}, I_{x_{m}}}(K(I_{x_{n}}), \delta_{I_{x_{n}}, I_{x_{m}}}).
\]

Note that

\[
\delta_{I_{x_{n}}, I_{x_{m}}}(u_{n}, \oplus_{p=1}^{\infty}(s_{n,p}, s_{n,p}^{p})) = (u_{n+1}, \oplus_{p=1}^{\infty}(s_{n+1,p}, s_{n+1,p}^{p})).
\]

We may choose \((0, u, \oplus_{p=1}^{\infty}(s_{p}, s^{p})) \in K(I_{x})\) be the limit

\[
(0, u, \oplus_{p=1}^{\infty}(s_{p}, s^{p})) := \lim \delta_{I_{x_{n}}, I_{x_{m}}}(0, u_{n}, \oplus_{p=1}^{\infty}(s_{n,p}, s_{n,p}^{p}))
\]

Then for any \( n \in \mathbb{N}, \)

\[
(x_{n}, u_{n}, \oplus_{p=1}^{\infty}(s_{n,p}, s_{n,p}^{p})) \leq (x, u, \oplus_{p=1}^{\infty}(s_{p}, s^{p})).
\]

Now we check that \((x, u, \oplus_{p=1}^{\infty}(s_{p}, s^{p}))\) is in fact the supremum of the sequence, choose any \((y, v, \oplus_{p=1}^{\infty}(t_{p}, t^{p})) \geq (x_{n}, u_{n}, \oplus_{p=1}^{\infty}(s_{n,p}, s_{n,p}^{p}))\) for all \( n \in \mathbb{N}, \) then \( x = \sup_{n \in \mathbb{N}} x_{n} \leq y \) and

\[
\delta_{I_{x_{n}}, I_{y}}(u_{n}, \oplus_{p=1}^{\infty}(s_{n,p}, s_{n,p}^{p})) = (v, \oplus_{p=1}^{\infty}(t_{p}, t^{p})).
\]

It is obvious that

\[
\delta_{I_{x}, I_{y}}(u, \oplus_{p=1}^{\infty}(s_{p}, s^{p})) = (v, \oplus_{p=1}^{\infty}(t_{p}, t^{p})).
\]

Hence,

\[
(x, u, \oplus_{p=1}^{\infty}(s_{p}, s^{p})) \leq (y, v, \oplus_{p=1}^{\infty}(t_{p}, t^{p})).
\]

\[\square\]

Theorem 3.9. Let \( A \) be a separable \( C^{*} \)-algebra of stable rank one and let

\[
(x, u, \oplus_{p=1}^{\infty}(s_{p}, s^{p})), (y, v, \oplus_{p=1}^{\infty}(t_{p}, t^{p})) \in \text{Cu}(A).
\]

Then

\[
(x, u, \oplus_{p=1}^{\infty}(s_{p}, s^{p})) \ll (y, v, \oplus_{p=1}^{\infty}(t_{p}, t^{p})).
\]

if and only if \( x \ll y \) in \( \text{Cu}(A) \) and

\[
\delta_{I_{x}, I_{y}}(u, \oplus_{p=1}^{\infty}(s_{p}, s^{p})) = (v, \oplus_{p=1}^{\infty}(t_{p}, t^{p})).
\]

Proof. Suppose that \((x, u, \oplus_{p=1}^{\infty}(s_{p}, s^{p})) \ll (y, v, \oplus_{p=1}^{\infty}(t_{p}, t^{p})), \) then

\[
\delta_{I_{x}, I_{y}}(u, \oplus_{p=1}^{\infty}(s_{p}, s^{p})) = (v, \oplus_{p=1}^{\infty}(t_{p}, t^{p}))
\]

we only need to prove \( x \ll y. \)

Let \((y_{n})_{n}\) be an increasing sequence in \( \text{Cu}(A) \) with supremum \( y. \) Then

\[
K(I_{y}) = \lim \delta_{I_{y_{n}}, I_{y_{m}}}(K(I_{y_{n}}), \delta_{I_{y_{n}}, I_{y_{m}}}).
\]

There exists \( n_{0} \in \mathbb{N} \) such that for any \( n \geq n_{0}, \)

\[
\delta_{I_{y_{n}}, I_{y}}(u_{n}, \oplus_{p=1}^{\infty}(t_{n,p}, t^{n,p})) = (v, \oplus_{p=1}^{\infty}(t_{p}, t^{p})�)
\]

where \((v_{n}, \oplus_{p=1}^{\infty}(t_{n,p}, t^{n,p}))\) is identified with \((0, v_{n}, \oplus_{p=1}^{\infty}(t_{n,p}, t^{n,p})) \in K(I_{y_{n}}). \)
Then \(((y_n, v_n, \oplus_{p=1}^{\infty} (t_{n,p}, t^{n,p})))\) is an increasing sequence in Cu\((A)\) with supremum \((y, v, \oplus_{p=1}^{\infty} (t_p, t^p))\). From assumption, there exists \(N \in \mathbb{N}\) \((N \geq n_0)\) such that for any \(n \geq N\),

\[
(x, u, \oplus_{p=1}^{\infty} (s_p, s^p)) \leq (y_n, v_n, \oplus_{p=1}^{\infty} (t_{n,p}, t^{n,p})).
\]

Then we have \(x \leq y_n\) for all \(n \geq N\), that is, \(x \ll y\).

Conversely, suppose that \(((y_n, v_n, \oplus_{p=1}^{\infty} (t_{n,p}, t^{n,p})))\) is an increasing sequence in Cu\((A)\) with supremum \((y, v, \oplus_{p=1}^{\infty} (t_p, t^p))\). Then \((y_n)\) is an increasing sequence in Cu\((A)\) with supremum \(y\). Then there exists \(N\) large enough such that for any \(n > N\),

\[
x \leq y_n \leq y, \quad I_x \subset I_{y_n} \subset I_y
\]

and

\[
\delta_{I_{y_n} I_y} (v_n, \oplus_{p=1}^{\infty} (t_{n,p}, t^{n,p})) = (v, \oplus_{p=1}^{\infty} (t_p, t^p)),
\]

where \((v_n, \oplus_{p=1}^{\infty} (t_{n,p}, t^{n,p}))\) is identified as \((0, v_n, \oplus_{p=1}^{\infty} (t_{n,p}, t^{n,p})) \in K(I_{y_n})\).

Now we have

\[
\delta_{I_x I_{y_n}} (u, \oplus_{p=1}^{\infty} (s_p, s^p)) = (v_n, \oplus_{p=1}^{\infty} (t_{n,p}, t^{n,p})),
\]

this means that

\[
(x, u, \oplus_{p=1}^{\infty} (s_p, s^p)) \leq (y_n, v_n, \oplus_{p=1}^{\infty} (t_{n,p}, t^{n,p})) \quad \text{for all } n \geq N.
\]

Hence,

\[
(x, u, \oplus_{p=1}^{\infty} (s_p, s^p)) \ll (y, v, \oplus_{p=1}^{\infty} (t_p, t^p)).
\]

\(\square\)

**Corollary 3.10.** Let \(A\) be a separable C*-algebra of stable rank one, then \((x, u, \oplus_{p=1}^{\infty} (s_p, s^p))\) is compact in Cu\((A)\) if and only if \(x\) is compact in Cu\((A)\).

**Corollary 3.11.** Let \(A\) be a separable C*-algebra of stable rank one, then \(A\) is of real rank zero if and only if Cu\((A)\) is algebraic.

Then we obtain the following theorem.

**Theorem 3.12.** Let \(A\) be a separable C*-algebra of stable rank one, then (Cu\((A)\), +, \(\leq\)) satisfies axioms (O1),(O2),(O3), and (O4).

**Proof.** (O1): It is exactly Corollary 3.8.

(O2): \((x, v, \oplus_{p=1}^{\infty} (s_p, s^p)) \in Cu\((A)\), from the (O2) of Cu\((A)\), we have a \(\ll\)-increasing sequence \((x_n)\) in Cu\((A)\). Then by Lemma 3.7 and Theorem 3.9, we achieve (O2) of (Cu\((A)\), +, \(\leq\)).

(O3): Let

\[
(x_1, u_1, \oplus_{p=1}^{\infty} (s_1, s^1)) \ll (y_1, v_1, \oplus_{p=1}^{\infty} (t_1, t^1))
\]

and

\[
(x_2, u_2, \oplus_{p=1}^{\infty} (s_2, s^2)) \ll (y_2, v_2, \oplus_{p=1}^{\infty} (t_2, t^2)).
\]

We have

\[
(x_1, u_1, \oplus_{p=1}^{\infty} (s_1, s^1)) + (x_2, u_2, \oplus_{p=1}^{\infty} (s_2, s^2)) \leq (y_1, v_1, \oplus_{p=1}^{\infty} (t_1, t^1)) + (y_2, v_2, \oplus_{p=1}^{\infty} (t_2, t^2)).
\]
From (O3) of Cu($A$), we also have $x_1 + x_2 \ll y_1 + y_2$, therefore, by Theorem 3.9, we get (O3) for (Cu($A$), +, ≤).

(O4): Let $((x_n, u_n, \oplus_{p=1}^{\infty} (s_{n,p}, s^{n,p}))_n$ and $((y_n, v_n, \oplus_{p=1}^{\infty} (t_{n,p}, t^{n,p}))_n$ be two increasing sequences in Cu($A$). Let

$$(x, u, \oplus_{p=1}^{\infty} (s_p, s^p)) = \sup_{n \in \mathbb{N}}(x_n, u_n, \oplus_{p=1}^{\infty} (s_{n,p}, s^{n,p}))$$

and

$$(y, v, \oplus_{p=1}^{\infty} (t_p, t^p)) = \sup_{n \in \mathbb{N}}(y_n, v_n, \oplus_{p=1}^{\infty} (t_{n,p}, t^{n,p}))$$

From (O4) of Cu($A$), we have $\sup_{n \in \mathbb{N}}(x_n + y_n) = x + y$. By the compatibility of order ≤ and addition, we know

$$((x_n, u_n, \oplus_{p=1}^{\infty} (s_{n,p}, s^{n,p})) + (y_n, v_n, \oplus_{p=1}^{\infty} (t_{n,p}, t^{n,p})))$$

is also an increasing sequence. Note that

$$\delta_{I_{x_n} I_y}(u_n, \oplus_{p=1}^{\infty} (s_{n,p}, s^{n,p})) = (u, \oplus_{p=1}^{\infty} (s_p, s^p))$$

and

$$\delta_{I_{y_n} I_y}(v_n, \oplus_{p=1}^{\infty} (t_{n,p}, t^{n,p})) = (v, \oplus_{p=1}^{\infty} (t_p, t^p))$$

This implies

$$\delta_{I_{x_n}+y} \delta_{I_{x_n}+y}(u_n, \oplus_{p=1}^{\infty} (s_{n,p}, s^{n,p})) + \delta_{I_{y_n} I_{x_n}+y}(v_n, \oplus_{p=1}^{\infty} (t_{n,p}, t^{n,p})))$$

$$= \delta_{I_{x}+y}(u, \oplus_{p=1}^{\infty} (s_p, s^p)) + \delta_{I_{y} I_{x}+y}(v, \oplus_{p=1}^{\infty} (t_p, t^p))$$

Using Corollary 3.8, we get (O4) for (Cu($A$), +, ≤).

□

**Definition 3.13.** Define the total Cuntz category Cu as follows:

$$\text{Ob(Cu)} = \{ S \in \text{Cu}^\sim | \text{Gr}(S_c) \text{ is a } \Lambda \text{ - module} \};$$

let $X, Y \in \text{Cu}$, we say the map $\phi : X \to Y$ is a Cu-morphism if $\phi$ satisfies the following two conditions:

1. $\phi$ is a Cu-morphism, i.e., $\phi$ is Mon$_{\leq}$-morphism and preserves suprema of increasing sequences and the compact containment relation.

2. The induced Grothendieck map $\text{Gr}(\phi_c) : \text{Gr}(X_c) \to \text{Gr}(Y_c)$ is $\Lambda$-linear, i.e., Gr($\phi_c$) is a morphism of the category $\Lambda$.

**Remark 3.14.** Let $A$ and $B$ be C*-algebras and let $\psi : A \to B$ be a *-homomorphism. We still denote $\psi \otimes \text{id}_K$ by $\psi$. Let $x \in \text{Cu}(A)$ and $y \in \text{Cu}(B)$, suppose that $\psi(I_x) \subseteq I_y$, where $I_x$ and $I_y$ are ideals of $A \otimes K$ and $B \otimes K$ generated by $x$ and $y$, respectively. Denote the restriction map

$$\psi|_{I_x \to I_y} : I_x \to I_y \text{ by } \psi|_{I_x I_y}.$$ 

Then $\psi$ induces the two morphisms:

$$K(\psi) : K(A) \to K(B) \text{ and } K(\psi|_{I_x I_y}) : K(I_x) \to K(I_y).$$

In general, $K(\psi|_{I_x I_y})$ is not the restriction of $K(\psi)$, as $K(I_x)$ is not a subgroup of $K(A)$.
In particular, we define a map \( Cu(\psi) : Cu(A) \to Cu(B) \) by
\[
Cu(\psi)(x, u, \oplus_{p=1}^{\infty}(s_p, s^p)) = (Cu(\psi)(x), K(\psi_{I_x}Cu(\psi)(x))(u, \oplus_{p=1}^{\infty}(s_p, s^p))).
\]
(Note that \((u, \oplus_{p=1}^{\infty}(s_p, s^p))\) is identified with \((0, u, \oplus_{p=1}^{\infty}(s_p, s^p)) \in K(I_x)\).

It is easy to check that
\[
Cu(\psi)(x, u, \oplus_{p=1}^{\infty}(s_p, s^p)) \leq (y, v, \oplus_{p=1}^{\infty}(t_p, t^p))
\]
if and only if \( Cu(\psi)(x) \leq y \in Cu(B) \) and
\[
K(\psi_{I_x}I_y)(0, u, \oplus_{p=1}^{\infty}(s_p, s^p)) = (0, v, \oplus_{p=1}^{\infty}(t_p, t^p)) \in K(I_y).
\]

**Theorem 3.15.** The assignment
\[
\begin{array}{ccc}
Cu : C_{sr1}^* & \to & Cu \\
A & \mapsto & Cu(A) \\
\phi & \mapsto & Cu(\phi)
\end{array}
\]
from the category of unital, separable C*-algebras of stable rank one to the category Cu is a functor.

In particular, we have
\[
(Gr(Cu(A)_c), \rho(Cu(A)_c)) \cong (K(A), K(A)_+),
\]
where \( \rho : S_c \to Gr(S_c) \) is the natural map \( \rho(x) = [(x, 0)] \).

**Proof.** The key point is to prove \( Gr(Cu(A)_c) \cong K(A) \) (canonically), where \( Gr(\cdot) \) is the Grothendick group procedure.

Since \( A \) has stable rank one, any compact element in \( Cu(A) \) can be raised by projections [5, Theorem 5.8], then by Corollary 3.10, every compact element in \( Cu(A) \) has the form \( ([e], u, \oplus_{p=1}^{\infty}(s_p, s^p)) \), where \( e \) is a projection in \( A \otimes K \).

Define
\[
\alpha : Cu(A)_c \to K(A)_+ \\
([e], u, \oplus_{p=1}^{\infty}(s_p, s^p)) \mapsto ([e], \delta_{I_e}A(u, \oplus_{p=1}^{\infty}(s_p, s^p))).
\]

From the definition of \( K(A)_+ \) we listed in 3.1, \( \alpha \) is a surjective monoid morphism. (\( \alpha \) is just surjective, not necessarily injective. The C*-algebra \( E \) in Example 2.3 is such an algebra.)

Note that \( K_0^+(A) \) is a sub-cone of \( K(A)_+ \), it induces an order on \( K(A)_+ \), that is, for \((f, \overline{f}), (g, \overline{g}) \in K(A)_+ \) \((f, g \in K_0^+(A) \) and \( \overline{f}, \overline{g} \in K_1(A) \times \oplus_{p=1}^{\infty}K_*(A; \mathbb{Z}_p) \)\), we say \((f, \overline{f}) \leq (g, \overline{g}) \), if \( f \leq g \) and \( \overline{f} = \overline{g} \).

Then \( \alpha \) is an ordered morphism.

Suppose that
\[
\alpha([e], u, \oplus_{p=1}^{\infty}(s_p, s^p)) = \alpha([q], v, \oplus_{p=1}^{\infty}(t_p, t^p)),
\]
which is
\[
([e], \delta_{I_e}A(u, \oplus_{p=1}^{\infty}(s_p, s^p))) = ([q], \delta_{I_q}A(v, \oplus_{p=1}^{\infty}(t_p, t^p))).
\]
Then for $([1_A], 0, \oplus_{p=1}^{\infty}(0, 0)) \in \Cu(A)_c$, in $\Cu(A)_c$, we have

$$([1_A] + [e], \delta_{I_c}A(u, \oplus_{p=1}^{\infty}(s^p, s^p)))$$

$$= ([1_A] + [q], \delta_{I_c}A(v, \oplus_{p=1}^{\infty}(t_p, t^p))),$$

which is

$$([1_A], 0, \oplus_{p=1}^{\infty}(0, 0)) + ([e], u, \oplus_{p=1}^{\infty}(s^p, s^p))$$

$$= ([1_A], 0, \oplus_{p=1}^{\infty}(0, 0)) + ([q], v, \oplus_{p=1}^{\infty}(t_p, t^p)).$$

This means that for any $x, y \in \Cu(A)_c$, if $\alpha(x) = \alpha(y)$, under the Grothendieck construction, the difference between $x$ and $y$ vanishes.

Consider the natural map $\rho : \Cu(A)_c \to \Gr(\Cu(A)_c)$, if $\rho(x) = \rho(y)$, then there exists $z \in \Cu(A)_c$ such that $x + z = y + z$, hence,

$$\alpha(x) + \alpha(z) = \alpha(y) + \alpha(z),$$

by 3.2, then $\alpha(x) = \alpha(y)$.

Then we have

$$\Gr(\Cu(A)_c) \cong \Gr(\alpha(\Cu(A)_c)) = \Gr(\Cu(A)_c) = K(A)$$

and

$$(\Gr(\Cu(A)_c), \rho(\Cu(A)_c)) \cong (K(A), K(A)_+).$$

Since $K(A)$ is a $\Lambda$-module, then $\Gr(\Cu(A)_c)$ is also a $\Lambda$-module, now we have $\Cu(A) \subset \Cu_c$. Suppose that $\phi : A \to B$ is a $*$-homomorphism, it is obvious that the induced map $\Cu(\phi) : \Cu(A) \to \Cu(B)$ is a $\Cu^*$-morphism.

For any $A \in C_{sr1}$, write the canonical isomorphism induced by $\alpha$ as

$$\alpha^*_{\phi} : (\Gr(\Cu(A)_c), \rho(\Cu(A)_c)) \to (K(A), K(A)_+).$$

We identify $\Gr(\Cu(A)_c)$ and $\Gr(\Cu(B)_c)$ with $K(A)$ and $K(B)$ through $\alpha_A^*$ and $\alpha_B^*$, respectively. Then the Grothendieck map

$$\Gr(\phi) : K(A) \to K(B)$$

is exactly the map $K(\phi)$, and hence, is a $\Lambda$-linear map.

Then $\Cu(\phi)$ is a $\Cu^*$-morphism.

Now we give another picture of the total Cuntz semigroup, which seems a natural construction like $\leq_1$ given in [6, 3.1].

3.16. ([10]) For each $C^*$-algebra $A$, Cuntz defined the $C^*$-algebra $QA = A \ast A$ as the free product of $A$ by itself. $QA$ is generated as a $C^*$-algebra by $\{i(a), i(a) | a \in A\}$ with respect to the largest $C^*$-norm, where $i$ and $i$ denote the inclusions of two copies of $A$ into the free product. He also defined $qA$ as the ideal of $QA$ generated by the differences $\{i(a) - i(a) | a \in A\}$ and showed that for any $C^*$-algebras $A, B$,

$$\KK(A, B) = [qA, B \otimes K],$$

where $[qA, B \otimes K]$ consists of all the homotopy classes of homomorphisms from $qA$ to $B \otimes K$. 
Let \( \psi : A \to B \) be a homomorphism, denote \( SA \) the suspension algebra of \( A \), i.e., \( A \otimes C_0(0, 1) \) and set \( S\psi = \psi \otimes \text{id}_{C_0(0,1)} \). Then we have

\[
K_*(A; \mathbb{Z}_p) = K_0(A; \mathbb{Z}_p) \oplus K_1(A; \mathbb{Z}_p)
= KK(\mathbb{C}, A \otimes C_0(W_p)) \oplus KK(\mathbb{C}, SA \otimes C_0(W_p))
= [q\mathbb{C}, A \otimes C_0(W_p) \otimes \mathcal{K}] + [q\mathbb{C}, SA \otimes C_0(W_p) \otimes \mathcal{K}].
\]

**Definition 3.17.** Let \( A \) be a separable \( C^* \)-algebras of stable rank one. Let \( a, b \in A \otimes \mathcal{K} \), let \( u, v \) be unitary elements of \( I_a^- \) and \( I_b^- \), respectively and let

\[
s_p : q\mathbb{C} \to I_a \otimes C_0(W_p), \quad t_p : q\mathbb{C} \to I_b \otimes C_0(W_p)
\]

be homomorphisms for \( p = 1, 2, \cdots \).

We say \((a, u, \oplus_{\infty=p=1} s_p, t_p) \leq_T (b, v, \oplus_{\infty=p=1} (t_p, t_p)) \), if

\[
a \leq_{C_0} b, \quad [\theta_{I_a I_b}(u)] = [v] \in K_1(I_b),

[(\theta_{I_a I_b} \otimes \text{id}_{C_0(W_p)}) (s_p)] = [t_p] \in K_0(I_b, \mathbb{Z}_p)
\]

and

\[
[(\theta_{I_a I_b} \otimes \text{id}_{C_0(W_p)}) (s_p)] = [t_p] \in K_1(I_b, \mathbb{Z}_p), \quad p = 1, 2, \cdots,
\]

where \( \theta_{I_a I_b} \) (see \( 3.4 \)) is the natural embedding map from \( I_a \) to \( I_b \). \( (\theta_{I_a I_b} \) induces the map \( \delta_{I_a I_b} \), i.e., \( K(\theta_{I_a I_b}) = \delta_{I_a I_b} \).)

It is obvious that the relation \( \leq_T \) is reflexive and transitive.

**Definition 3.18.** Let \( A \) be a separable \( C^* \)-algebra of stable rank one. Denote \( \mathcal{A}(A) \) the set of the all tuples \((a, u, \oplus_{\infty=p=1} (s_p, t_p)) \), where \( a \in (A \otimes \mathcal{K})_+ \), \( u \in \mathcal{U}(I_a^-) \) and \( s_p \in \text{Hom}(q\mathbb{C}, I_a \otimes C_0(W_p)) \), \( s_p \in \text{Hom}(q\mathbb{C}, I_a \otimes C_0(W_p)) \).

By antisymmetrizing the \( \leq_T \) relation, we define an equivalent relation \( \sim_T \) on \( \mathcal{A}(A) \) called the total Cuntz equivalence, and denote \([[(a, u, \oplus_{\infty=p=1} (s_p, t_p))] \) the equivalent class of \((a, u, \oplus_{\infty=p=1} (s_p, t_p)) \). We construct a semigroup of \( A \) as follows:

\[
\mathcal{T}(A) := \{[[a, u, \oplus_{\infty=p=1} (s_p, t_p))] \mid (a, u, \oplus_{\infty=p=1} (s_p, t_p)) \in \mathcal{A}(A)\} / \sim_T.
\]

For any two \([[a, u, \oplus_{\infty=p=1} (s_p, t_p)]] , [[b, v, \oplus_{\infty=p=1} (t_p, t_p)]] \in \mathcal{T}(A) \), we say

\[
[[a, u, \oplus_{\infty=p=1} (s_p, t_p)]] \leq [[b, v, \oplus_{\infty=p=1} (t_p, t_p)]],
\]

if

\[
(a, u, \oplus_{\infty=p=1} (s_p, t_p)) \leq_T (b, v, \oplus_{\infty=p=1} (t_p, t_p)).
\]

The following addition is well-defined and compatible with the relation \( \leq \) on \( \mathcal{T}(A) \):

\[
[[a, u, \oplus_{\infty=p=1} (s_p, t_p)]] + [[b, v, \oplus_{\infty=p=1} (t_p, t_p)]]
= [[a \oplus b, u \oplus v, \oplus_{\infty=p=1} (s_p + t_p, s_p + t_p)]],
\]

where

\[
s_p + t_p = (\theta_{I_a I_b} \otimes \text{id}_{C_0(W_p)}) (s_p) \oplus (\theta_{I_b I_a \oplus b} \otimes \text{id}_{C_0(W_p)}) (t_p)
\]
and
\[
\overline{s^p + t^p} = (S\theta_{I_a, I_{a+b}} \otimes \text{id}_{C_0(W_p)}) \circ s^p \oplus (S\theta_{I_b, I_{a+b}} \otimes \text{id}_{C_0(W_p)}) \circ t^p.
\]

Note that \([0, 1_C, \oplus_{p=1}^{\infty}(0, 0)]\) is the neutral element and we obtain a partially ordered monoid \((\mathcal{T}(A), +, \leq)\).

**Theorem 3.19.** Let \(A\) be a separable \(C^*\)-algebra of stable rank one. We have a \(\text{Cu}^\sim\)-isomorphism
\[
\xi : \mathcal{T}(A) \to \text{Cu}(A).
\]

**Proof.** It has been shown that \(\text{Cu}(A)\) is an object in \(\text{Cu}\), and hence, \(\text{Cu}(A)\) is also an object in \(\text{Cu}^\sim\).

Set
\[
\xi([([a, u, \oplus_{p=1}^{\infty}(s_p, t^p)]]) = ([a], u, \oplus_{p=1}^{\infty}(s_p, t^p)),
\]
where \(a \in (A \otimes K)_+, u \in \mathcal{U}(I_a^\sim), s_p \in \text{Hom}(qC, I_a \otimes C_0(W_p)), t^p \in \text{Hom}(qC, I_a \otimes C_0(W_p)), [a] \in \text{Cu}(A), u = [u] \in K_1(I_a), s_p = [s_p] \in K_0(I_a; Z_p)\) and \(t^p = [t^p] \in K_1(I_a; Z_p)\).

It is immediate that \(\xi\) is a well-defined set map from \(\mathcal{T}(A)\) to \(\text{Cu}(A)\), which is order preserving and injective. The fact that \(\mathcal{K}(\theta_{I_a, I_b}) = \delta_{I_a, I_b}\) for \(a \lesssim_{\text{Cu}} b\) implies that \(\xi\) is additive.

We show the surjectivity. Given any \((x, u, \oplus_{p=1}^{\infty}(s_p, t^p)) \in \text{Cu}(A)\), by Definition 3.5, we have
\[
(x, u, \oplus_{p=1}^{\infty}(s_p, t^p)) \in \text{Cu}_f(I_x) \times K_1(I_x) \times \bigoplus_{p=1}^{\infty} K_\ast(I_x; Z_p).
\]

Then we can lift \(x\) to \(a \in (A \otimes K)_+\) with \(x = [a]\) and \(I_x = I_a\). Next we lift \(u\) to \(u \in \mathcal{U}(I_a^\sim)\) with \([u]_{K_1(I_a)} = [u]\). For any \(p \geq 1\), we have
\[
K_\ast(I_a; Z_p) \cong K_0(I_a; Z_p) \oplus K_1(I_a; Z_p)
\]
and
\[
s_p \in K_0(I_a; Z_p), \quad t^p \in K_1(I_a; Z_p).
\]

Then by 3.16, there exist \(s_p \in \text{Hom}(qC, I_a \otimes C_0(W_p)), t^p \in \text{Hom}(qC, I_a \otimes C_0(W_p))\), satisfying that
\[
[s_p] = s_p \in K_0(I_b; Z_p), \quad [t^p] = t^p \in K_1(I_b; Z_p).
\]

Now we have
\[
\xi([([a, u, \oplus_{p=1}^{\infty}(s_p, t^p)]]) = (x, u, \oplus_{p=1}^{\infty}(s_p, t^p)).
\]

That is, we have \(\xi\) is an ordered monoid isomorphism, by [6, Lemma 4.1], \(\xi\) becomes a \(\text{Cu}^\sim\)-isomorphism and \(\mathcal{T}(A) \in \text{Cu}^\sim\). This implies that
\[
\text{Gr}(\xi_c) : \text{Gr}(\mathcal{T}(A)_c) \to \text{Gr}(\text{Cu}(A)_c)
\]
is a group isomorphic map and in fact, \(\text{Gr}(\xi_c)\) is the identity map of \(\text{K}(A)\), which is of course \(\Lambda\)-linear. This completes the proof. □
Remark 3.20. Thus, using this new picture, the Cu-morphism \( \text{Cu}(\psi) \) induced by a homomorphism \( \psi \) in Remark 3.14 and Theorem 3.15 can also be written as

\[
\text{Cu}(\psi)([[a, u, \oplus_{p=1}^{\infty}(s_p, s^p)])]
= \left[ (\psi(a), \psi^{-1}(u), \oplus_{p=1}^{\infty}(\psi I, I_{\psi(a)} \otimes \text{id}_{C_0(W_p)} \circ s_p, (S \psi I, I_{\psi(a)} \otimes \text{id}_{C_0(W_p)} \circ s^p)) \right].
\]

4. Continuity of the functor \( \text{Cu} \)

It is proved in section 3.8 of [6] that \( \text{Cu}_1 \) is a continuous functor, now we show that \( \text{Cu} \) is also a continuous functor from the category of \( C^* \)-algebras of stable rank one to the category \( \text{Cu} \).

Definition 4.1. Let \( S_1 \xrightarrow{\beta_{12}} S_2 \xrightarrow{\beta_{23}} \cdots \) be a sequence in the category \( \text{Cu}^\sim \). We say \( (s_1, s_2, \cdots) \) is an eventually-increasing sequence if, for \( s_1 \in S_1, s_2 \in S_2, \cdots \) and there exists \( k \in \mathbb{N} \) such that for any \( j > i > k \), \( \beta_{ij}(s_i) \leq s_j \). Denote \( S \) the collection of all the eventually-increasing sequences.

Let \( (s_i), (t_i) \in S \), define the addition operation

\[
(s_i) + (t_i) = (s_i + t_i),
\]

(it is easy to see that \( (s_i + t_i) \) belongs to \( S \)) and the pre-order relation

\[
(s_i) \leq (t_i),
\]

if there exists \( k \in \mathbb{N} \) such that for any \( i \geq k \) and \( x \in S_i \) with \( x \ll s_i \), there is an \( m \geq i \) such that \( \beta_{ij}(x) \ll t_j \) (in \( S_j \)) for any \( j \geq m \). We say \( (s_i) \sim (t_i) \), if \( (s_i) \leq (t_i) \) and \( (t_i) \leq (s_i) \).

For convenience, for any \( i \in \mathbb{N} \) and \( s \in S_i \), we will denote all those \( \beta_{ij}(s) \) by just \( s \) for all \( j \geq i \).

Remark 4.2. Every increasing sequence defined in 1.6 is an eventually-increasing sequence. Every eventually-increasing sequence is equivalent to a sequence of the form

\[
(0, 0, \cdots, 0, s_k, s_{k+1}, \cdots),
\]

where \( s_k \leq s_{k+1} \leq \cdots \). We may denote this sequence by \( (s_i)_{i \geq k} \) (the notation \( i \geq k \) means this sequence is increasing from \( i = k \)).

The eventually-increasing for \( \text{Cu}^\sim \)-category is inspired by [8, Theorem 2]. Note that for particular cases in \( \text{Cu}^\sim \)-category, we may even have \( \phi_{k_0-1} \cap s_{k_0} = \emptyset \), where \( s_{k_0} \triangleq \{ s \in S_{k_0} : s \leq s_{k_0} \} \). This means the notion of eventually-increasing is necessary.

To build such an example, one can pick \( S_1 = \mathbb{N} \cup \{ \infty \} \) and

\[
S_i = (([N \setminus \{0\}) \cup \{ \infty \}) \times \mathbb{Z}^{i-1} \cup \{(0, 0, \cdots, 0)\}, \quad \text{for } i \geq 2.
\]

For \( (m_1, m_2, \cdots, m_i), (n_1, n_2, \cdots, n_i) \in S_i \), we say

\[
(m_1, m_2, \cdots, m_i) \leq (n_1, n_2, \cdots, n_i),
\]
there exist two rapidly increasing sequences \((s_i)\) from sequence. That is, given an eventually-increasing sequence \((l_i)\). Then denote equivalent with an eventually-increasing sequence for the inductive system in \(\text{Cu}^-\)-category. (Denote \(X_i\) to be the one point union of \(i - 1\) circles, one can pick \(S'_i = \text{Cu}_1(C(X_i))\), for \(i \geq 1\), which is also a similar example.)

**Proposition 4.3.** With the assumptions in Definition 4.1, the pre-order relation on \(S\) is reflexive, transitive and compatible with the addition.

**Proof.** The reflexivity and transitivity of the pre-order relation on \(S\) is immediate, we only need to check the compatibility with the addition.

Suppose we have \((s_i) \leq (g_i)\) and \((t_i) \leq (h_i)\) in \(S\) and we will show that \((s_i + t_i) \leq (g_i + h_i)\). Assume that \((s_i)\), \((t_i)\) and \((s_i + t_i)\) are all increasing from \(i = k\). Then given any \(i \geq k\) and \(x \ll s_i + t_i\). Note that \(s_i, t_i \in S_i\), there exist two rapidly increasing sequences \((s^n_i)_n\) and \((t^n_i)_n\) in \(S_i\) with

\[
\sup_{n \in \mathbb{N}} s^n_i = s_i \quad \text{and} \quad \sup_{n \in \mathbb{N}} t^n_i = t_i.
\]

Then denote \(l^n_i = s^n_i + t^n_i\), we have

\[
\sup_{n \in \mathbb{N}} l^n_i = s_i + t_i.
\]

There exists an \(n_0\) such that for all \(n \geq n_0\), we have \(x \leq l^n_i\). In particular,

\[
x \leq s^{n_0} + t^{n_0} \ll s^{n_0+1} + t^{n_0+1}.
\]

Now \(s^{n_0+1} \ll s_i\) and \(t^{n_0+1} \ll t_i\) in \(S_i\). By Definition 4.1, we may find a common \(m \geq i\) such that \(s^{n_0+1} \ll g_m\) and \(t^{n_0+1} \ll h_m\) in \(S_m\). Then

\[
x \ll s^{n_0+1} + t^{n_0+1} \ll g_m + h_m.
\]

Hence,

\[
(s_i + t_i) \leq (g_i + h_i).
\]

This ends the proof. \(\square\)

With a similar proof of the first part of [8, Theorem 2], we have the following:

**Lemma 4.4.** Let \(S_1 \rightarrow S_2 \rightarrow \cdots\) be a sequence in the category \(\text{Cu}^-\). Then

\[
\text{Cu}^- \lim_{\longrightarrow} S_i \cong S/\sim.
\]

**Proof.** We divide the proof into several steps.

**Step 0 :** As a trick, let us show that any eventually-increasing sequence is equivalent with an eventually-\(\ll\)-increasing (eventually and rapidly increasing) sequence. That is, given an eventually-increasing sequence \((s_i)\), there...
is an eventually-increasing sequence \((t_i)\) with \(t_i \ll t_{i+1}\) for all large enough \(i\) satisfying that
\[(s_i) \sim (t_i).\]
To achieve this, we can assume that the given eventually-increasing sequence \((s_i)\) \((s_i \text{ is increasing from } i = k)\) is an increasing sequence \((k = 1)\), otherwise, we just begin constructing \((t_i)\) from the \(k\)th coordinate.

Note that for each \(i \geq 1\), \(s_i \in S_i\). As \(S_i\) belongs to the \(\text{Cu}^\sim\)-category, by \((O2)\), there exists a \(\ll\)-increasing sequence \((s^n_i)_{n \in \mathbb{N}}\) in \(S_i\) such that \(\sup_{n \in \mathbb{N}} s^n_i = s_i\). For \(i = 1\), we keep the \(\ll\)-increasing sequence \((s^n_1)\) in \(S_1\) fixed; for \(i = 2\), from the fact \(s^n_1 \ll s_1 \leq s_2\), we can pick a sub-sequence \((s^n_2)\) of \((s^n_2)\) in \(S_2\) such that
\[s^j_1 \ll s^n_2, \quad \forall j \in \mathbb{N},\]
and we still denote this new sub-sequence \((s^n_2)\) by \((s^n_2)\); inductively, we can always assume we have
\[s^n_i \ll s^n_{i+1}, \quad \forall i, n \in \mathbb{N}.\]

Note that
\[s^j_i \ll s^j_{i+1} \ll s^{j+1}_{i+1}, \quad \forall i \in \mathbb{N},\]
By choosing the diagonal sequence,
\[t_i = s^i_i, \quad \forall i \in \mathbb{N}\]
we get a rapidly increasing sequence \((t_i)\).

Now we show \((s_i) \sim (t_i)\). On one hand, if we have \(s \ll t_i\) for some \(i\), we get \(s \ll s^i_i \ll s_i \leq s_j\), for all \(j \geq i\). That is, \((t_i) \leq (s_i)\). On the other hand, if we have \(s \ll s_i\) for some \(i\), from the above construction \((\sup_{n \in \mathbb{N}} s^n_i = s_i)\), there is a large \(n_0\) \((n_0 \geq i)\), such that \(s \ll s^{n_0}_i\). Then \(s \ll s^{n_0}_i \ll s^{n_0}_{n_0} \ll s^j = t_j\), for all \(j \geq n_0\). Hence, \((s_i) \leq (t_i)\).

Let us now show that \(S/ \sim\) is an object in \(\text{Cu}^\sim\)-category. This means we must show that \(S/ \sim\) is an ordered monoid (Step 1), that each increasing sequence in \(S/ \sim\) has a supremum (Step 2—\((O1)\)), that each element in \(S/ \sim\) is a supremum of a rapidly increasing sequence (Step 3—\((O2)\)), and finally, the relations \(\leq\) and \(\ll\) and the operation of passing to the supremum of an increasing sequence are compatible with addition (Step 4—not only \((O3), (O4)\).

**Step 1**: We have a reflexive and transitive relation \(\leq\) on \(S\), then the induced relation on \(S/ \sim\), which we still write as \(\leq\) is also reflexive and transitive, i.e., \(\leq\) is an order relation on \(S/ \sim\). From the fact that the preorder relation \(\leq\) and addition + on \(S\) are compatible (Proposition 4.3), we have the induced addition + on \(S/ \sim\) is well-defined. Of course, the class of \((0, 0, 0, \cdots)\) is the unique zero element and this proves that \(S/ \sim\) is an ordered monoid (not necessarily positive).

**Step 2**: Suppose that \(s^1 \leq s^2 \leq s^3 \leq \cdots\) is an increasing sequence in \(S/ \sim\). Then by Step 0, for each \(s^i\), we pick an eventually-\(\ll\)-increasing sequence \((s^n)_{n \geq k_i}\) such that \([s^n)] = s^i\), where \(s^n \in S_n\) for any \(n\). (Here,
we assume that each \((s^n_i)_{n \geq k_i}\) is \(<\ll\)-increasing from \(n = k_i\). For \((s^1_n)\), it is \(<\ll\)-increasing for \(i \geq i_1 = k_1\), then \(s^1_{i_1} \ll s^1_{i_1+1}\). As \(s^1 \leq s^2\), there exists an integer \(i_2 \geq \max\{i_1 + 1, k_2\}\), such that

\[s^1_{i_1} \ll s^2_{i_2}\ (\text{in } S_{i_2}).\]

Next, we begin with the elements \(s^1_{i_2}\) and \(s^2_{i_2}\). Since \(s^1 \leq s^3\) and \(s^2 \leq s^3\), there exist an \(i_3 \geq \max\{i_2 + 1, k_3\}\) such that

\[s^1_{i_2} \ll s^3_{i_3}\ \text{and} \ s^2_{i_2} \ll s^3_{i_3}\ (\text{in } S_{i_3}).\]

Inductively, for any \(m \in \mathbb{N}\), we have \(i_{m+1} \geq \max\{i_m + 1, k_{m+1}\}\) with

\[s^j_{i_m} \ll s^{m+1}_{i_{m+1}}, \quad j = 1, 2, \ldots, m.\]

Set

\[s_j = \begin{cases} 0, & \text{if } 1 \leq j < i_1 \\ s^1_{i_1} \ (\text{in } S_j), & \text{if } i_1 \leq j < i_2 \\ s^2_{i_2} \ (\text{in } S_j), & \text{if } i_2 \leq j < i_3 \\ \vdots & \vdots \end{cases}\]

Then \((s_1, s_2, \ldots)\) is an eventually-increasing sequence (not necessary a rapid one), we denote \(s\) to be the class of this sequence in \(\mathcal{S}/\sim\) and claim that \(s = \sup_{n \in \mathbb{N}} s^n\).

Note first that, for each \(j\) and each \(n (n \geq k_j)\), \(s^j_n \leq s^n_{j+n}\), hence, \(s^j \leq s^n\), and second, that if \(s^1, s^2, \ldots \leq t = (t_1, t_2, \ldots)\), then for each \(i\), if \(r \ll s_i\) in \(S_i\), eventually \(r \ll t_j\), so \(s \leq t\).

**Step 3**: We say \(t \ll s\) in \(\mathcal{S}/\sim\), if for any increasing sequence \((s^n)\) in \(\mathcal{S}/\sim\) with \(\sup_{n \in \mathbb{N}} s^n = s\), there exists an \(n_0 \in \mathbb{N}\), such that \(t \leq s^n\) for all \(n \geq n_0\).

Suppose we have an \(s \in \mathcal{S}/\sim\), then by Step 0, we have an eventually-\(<\ll\>-increasing sequence \((s_i)\) representing \(s\), i.e., \([s_i) = s\). We assume that \((s_i)\) is \(<\ll\>-increasing from \(i = k\) and let

\[s^1 = [(0, 0, \ldots, 0, s_{k+1}, s_{k+1}, \ldots)],\]

\[s^2 = [(0, 0, \ldots, 0, s_{k+1}, s_{k+2}, s_{k+2}, \ldots)],\]

\[\vdots\]

\[s^n = [(0, 0, \ldots, 0, s_{k+1}, s_{k+2}, \ldots, s_{k+n}, s_{k+n}, \ldots)],\]

\[\vdots\]

Now we prove that the sequence \((s^1, s^2, \ldots)\) is a rapidly increasing sequence in \(\mathcal{S}/\sim\) and has supremum \(s\).

To see that it is rapidly increasing, Suppose that we have an increasing sequence \((t^j)\) in \(\mathcal{S}/\sim\) with \(\sup_{j \in \mathbb{N}} t^j = s^{n+1}\). For each \(t^j \in \mathcal{S}/\sim\), we pick
an eventually-$\ll$-increasing sequence $(t_{i_1}^j, t_{i_2}^j, t_{i_3}^j, \ldots)$ from this class, where $t_{i_1}^j \in S_i$. Then set $i_1 = k + 1$, we repeat the construction in Step 2, we have

$$[(0, 0, \ldots, 0, t_{i_1}^1, t_{i_1}^1, \ldots, t_{i_2}^j, t_{i_2}^j, \ldots, t_{i_j}^j, \ldots)] = s^{n+1}.$$  

In particular,

$$(0, 0, \ldots, 0, s_{k+1}, \ldots, s_{k+n+1}, s_{k+n+1}, \ldots) \preceq (0, 0, \ldots, 0, t_{i_1}^1, t_{i_1}^1, \ldots, t_{i_j}^j, \ldots).$$

Since $s_{k+n} \ll s_{k+n+1}$, by Definition 4.1, we have an $i_m$ such that

$$s_{k+n} \ll t_{i_m}^m \quad (\text{in } S_{i_m}).$$

Then for all $j \geq m$, we have

$$s_{k+n} \ll t_{i_m}^m \ll t_{i_j}^j,$$

which implies $s^n \leq t^j$ for all $j \geq m$. Hence

$$s^n \ll s^{n+1}.$$  

Now we show that $\sup_{n \in \mathbb{N}} s^n = s$. Note that $s_i \ll s_{i+1}$ for all $i \geq k$, by the construction in Step 2, $\sup_{n \in \mathbb{N}} s^n$ contains a representing element as

$$(0, 0, \ldots, 0, s_{k+1}, s_{k+2}, s_{k+3}, \ldots),$$

which is equivalent with $(s_i)$. We do have $\sup_{n \in \mathbb{N}} s^n = s$.

Here, we add one more remark. Suppose that $(r_1, r_2, r_3, \ldots)$ is an eventually-$\ll$-increasing sequence in $S$ with $r_i \in S_i$, $[(r_1, r_2, r_3, \ldots)] = r$ and $(r_i)$ is $\ll$-increasing from $i = k$. If we regard $r_i$ as the element $[(0, 0, \ldots, r_i, 0, 0, \ldots)]$

in $S/\sim$, then

$$(r_k, r_{k+1}, r_{k+2}, \ldots)$$

is a $\ll$-increasing sequence in $S/\sim$, and hence by the above construction, it has supremum

$$[(0, 0, \ldots, r_k, r_{k+1}, \ldots)] = r,$$

and we may simply write $r = \sup_{i \geq k} r_i$.

**Step 4**: Firstly, as we mentioned in Step 1 that the addition on $S/\sim$ is well-defined, the compatibility of the relation $\leq$ and addition on $S/\sim$ just comes from the compatibility of the relation $\leq$ and addition on $S$ (Proposition 4.3).

Secondly, we show that $S/\sim$ satisfies (O4). Suppose that we have two increasing sequence $(s^i)$, $(t^i)$ in $S/\sim$ with $\sup_{i \in \mathbb{N}} s^i = s$ and $\sup_{i \in \mathbb{N}} t^i = t$. Note that $(s^i + t^i)$ is also an increasing sequence in $S/\sim$. We do the
construction in Step 2 to \((s^i), (t^i)\) and \((s^i + t^i)\), we may find a common sequence \((i_n)_{n \in \mathbb{N}}\) such that

\[
\begin{align*}
s &= \{0, 0, \ldots, 0, s_{1_{i_1}}, \ldots, s_{1_{i_2}}, \ldots, s_{1_{i_m}}, \ldots, s_{1_{i_{m+1}}}, \ldots\}, \\
t &= \{0, 0, \ldots, 0, t_{1_{i_1}}, \ldots, t_{1_{i_2}}, \ldots, t_{1_{i_m}}, \ldots, t_{1_{i_{m+1}}}, \ldots\}.
\end{align*}
\]

and

\[
\sup_{i \in \mathbb{N}} \{s^i + t^i\} = \{0, 0, \ldots, 0, s_{1_{i_1}} + t_{1_{i_1}}, \ldots, s_{1_{i_m}} + t_{1_{i_m}}, \ldots, s_{1_{i_{m+1}}}, \ldots\}.
\]

At once, from the definition of addition on \(S/ \sim\), we get

\[
\sup_{i \in \mathbb{N}} \{s^i + t^i\} = s + t = \sup_{i \in \mathbb{N}} s^i + \sup_{i \in \mathbb{N}} t^i.
\]

Thirdly, we prove the compatibility of \(\ll\) with addition. Assume \(s \ll g\) and \(t \ll h\) in \(S/ \sim\), we will show that \(s + t \ll g + h\). We do the construction in Step 3 for both \(g\) and \(h\), we have two rapidly increasing sequences

\[
g^n = \{0, 0, \ldots, 0, g_{k+1}, g_{k+2}, \ldots, g_{k+n}, \ldots\}, \quad n \in \mathbb{N}
\]

and

\[
h^n = \{0, 0, \ldots, 0, h_{k+1}, h_{k+2}, \ldots, h_{k+n}, \ldots\}, \quad n \in \mathbb{N}
\]

with \(\sup_{n \in \mathbb{N}} g^n = g\) and \(\sup_{n \in \mathbb{N}} h^n = h\) in \(S/ \sim\). Then there is an \(n_0\) such that

\[
s \leq g^{n_0} \ll g^{n_0 + 1} \leq g
\]

and

\[
t \leq h^{n_0} \ll h^{n_0 + 1} \leq h.
\]

Note that

\[
g^{n_0} + h^{n_0} = \{0, 0, \ldots, 0, f_{k+1}, f_{k+2}, \ldots, f_{k+n_0}, f_{k+n_0}, \ldots\}
\]

and

\[
g^{n_0 + 1} + h^{n_0 + 1} = \{0, 0, \ldots, 0, f_{k+1}, f_{k+2}, \ldots, f_{k+n_0+1}, f_{k+n_0+1}, \ldots\},
\]

where \(f_n = g_n + h_n\) for any \(n \in \mathbb{N}\). Still from proof of Step 3, we have \(g^{n_0} + h^{n_0} \ll g^{n_0 + 1} + h^{n_0 + 1}\). Then from compatibility of the order relation and addition on \(S/ \sim\) (the first part of Step 4), we have

\[
s + t \leq g^{n_0} + h^{n_0} \ll g^{n_0 + 1} + h^{n_0 + 1} \leq g + h.
\]

Now let us complete the proof by proving that the object \(S/ \sim\) in \(\text{Cu}^\sim\)-category is the inductive limit of the given sequence \(S_1 \rightarrow S_2 \rightarrow \cdots\) in this
category. We must show that for every object $T$ in $\text{Cu}^\sim$-category and every compatible sequence of maps $S_1 \to T$, $S_2 \to T$, $\cdots$, there exists a unique compatible map $S/ \sim \to T$.

Of course, for this to make sense we must have maps $S_i \to S/ \sim$ for all $i$ compatible with maps $S_i \to S_{i+1}$. For each $i$, and each $s \in S_i$, note that the sequence $(0, 0, \cdots, 0, s, s, \cdots, s, \cdots)$ is an eventually-increasing sequence and therefore represents an element of $S/ \sim$. For each fixed $i$, note that for any $j \geq i$, the eventually-increasing sequence $(0, 0, \cdots, 0, s, s, \cdots, s, \cdots)$ is equivalent to the one with $j = i$, this follows immediately from Definition 4.1. This shows that the maps $S_1 \to S/ \sim$, $S_2 \to S/ \sim$, $\cdots$ obtained in this way are compatible with the given sequence. It is easy to see that all these maps are $\text{Cu}^\sim$-morphisms.

Note that the maps $S_i \to T$ preserve the order relation and are compatible as set maps. The definition of a (set) map $S/ \sim \to T$ is immediate if one restricts to eventually-$\ll$-increasing representative sequences for elements of $S/ \sim$ (Step 0). (If $(s_1, s_2, \cdots)$ represents $s$ with $s_i \in S_i$ and $s_k \ll s_{k+1} \ll s_{k+2} \ll \cdots$ for some $k$, let us aim to map $s$ into the supremum of increasing sequence in $T$ consisting of the images of $s_k, s_{k+1}, s_{k+2}, \cdots$ by the maps $S_k \to T$, $S_{k+1} \to T$, $S_{k+2} \to T$, $\cdots$; this of course makes sense if the sequence $(s_1, s_2, \cdots)$ is just eventually-increasing. If $(s'_1, s'_2, \cdots)$ is a second eventually-$\ll$-increasing representative sequence of $s$, assume that $s'_k \ll s'_{k+1} \ll s'_{k+2} \ll \cdots$ for the same $k$ as above. Then for each $i \geq k$, as $s_i \ll s_{i+1}$, there is a large enough $m$ ($m \geq k$) we have $s_i \ll s'_m$. This implies for any $i \geq k$, we have $s_i \leq \sup_{j \geq k} s'_j$ (all these are images and elements in $T$), and hence $\sup_{i \geq k} s_i \leq \sup_{j \geq k} s'_j$ in $T$. By symmetry, the suprema of the images of $s_k, s_{k+1}, s_{k+2}, \cdots$ and of $s'_k, s'_{k+1}, s'_{k+2}, \cdots$ in $T$ are equal.

Let us check that the set map $S/ \sim \to T$ thus defined is compatible with the maps $S_i \to T$ (Step 5, i.e., that the diagram is commutative as a diagram of set maps), and that it is a morphism in the category $\text{Cu}^\sim$ (Step 6), and that it is unique with these properties (Step 7).

**Step 5** : To show compatibility of $S_i \to T$ with $S/ \sim \to T$, we must show, for each fixed $i$, that if $s \in S_i$ then the image of $s$ in $T$ by the given map $S_i \to T$ is the same as the image of $s$ in $T$ by the composed map $s_i \to S/ \sim \to T$. By definition, the image of $s$ in $S/ \sim$ is represented by the sequence $(0, 0, \cdots, 0, s, s, \cdots)$; However, in order to compute the image of this...
element of $S/\sim$ in $T$, we must represent it by an eventually-$\ll$-increasing sequence: let us use the sequence $(0, \ldots, 0, r_i, r_{i+1}, \ldots)$, where $r_i \ll r_{i+1} \ll r_{i+2} \ll \cdots$ is a rapidly increasing sequence in $S_i$ with supremum $s$. By definition, the corresponding element in $T$ is the supremum of the (increasing sequence of) images of the elements $r_i \in S_i$, $r_{i+1} \in S_{i+1}, \ldots$ by the maps $S_i \to T$, $S_{i+1} \to T$, $\cdots$, equivalently (by commutativity) of the images of $r_i, r_{i+1} \cdots \in S_i$ by the map $S_i \to T$, and as this map preserves increasing sequential suprema, the supremum in $T$ in question is just the image of $s$, by the map $S_i \to T$, as desired.

**Step 6**: To show that the map $S/\sim \to T$ is a Cu$^\sim$-morphism, we must show that it preserves addition (1), preserves the order relation (2), preserves suprema of increasing sequences (3), and preserves the order-theoretic relation $\ll$ listed in Step 3 (4), in terms of the two notions just mentioned (the order relation $\leq$ and the operation of sequential increasing supremum). Let us address these issues, briefly, in turn.

1. Given two eventually-$\ll$-increasing sequences $(r_i)$ and $(s_i)$ with $r_i, s_i \in S_i$ for each $i$, we may suppose both of these two sequences are $\ll$-increasing from $i = k$. To check that the sum in $S/\sim$ maps into the sum of images in $T$ it is enough to recall what these images are, and that the operation of passing to the supremum of an increasing sequence in $T$ is compatible with addition in $T$.

2. To check that the relation $[(r_1, r_2, \cdots)] \leq [(s_1, s_2, \cdots)]$ in $S/\sim$ (with $(r_i), (s_i)$ are eventually-$\ll$-increasing) leads to the same relation between the images in $T$, we may suppose both of these two sequences are $\ll$-increasing from $i = k$. Then for any $i \geq k$, as $r_i \ll r_{i+1}$, by Definition 4.1, we have a large $m$ ($m \geq k$) with $r_i \ll s_m$ in $S_m$, from the properties of the map $S_m \to T$, the image of $r_i$ in $T$ is compactly contained in the image of $s_m$ in $T$, which means that $r_i \leq \sup_{j \geq k} s_j$ in $T$. Further, we have $\sup_{i \geq k} r_i \leq \sup_{j \geq k} s_j$ in $T$, in other words, $[(r_1, r_2, \cdots)] \leq [(s_1, s_2, \cdots)]$ in $T$.

3. Using the same notations in Step 2, let $s^1 \leq s^2 \leq s^3 \leq \cdots$ be an increasing sequence in $S/\sim$ with supremum $s$, and for each $i$, let $(s^i_n)_{n \geq k_i}$ be an eventually-$\ll$-increasing sequence such that $[(s^i_n)] = s^i$, where $s^i_n \in S_n$ for any $n$. We have an eventually-increasing sequence $(s_1, s_2, \cdots)$ (not necessary a rapid one) as follows:

$$(s_n) = (0, 0, \cdots, 0, s^1_{i_1}, \cdots, s^1_{i_{l_1}}, s^2_{i_2}, \cdots, s^2_{i_{l_2}}, \cdots, s^m_{i_m}, \cdots)_{i_{l_1} \cdots i_{l_n}}$$

where $s_n \in S_n$ and $[(s_1, s_2, \cdots)] = s$. We mention that $s^1_{i_1} \ll s^2_{i_2} \ll s^3_{i_3} \ll \cdots$. Suppose we have an eventually-$\ll$-increasing sequence $(t_n)$ (by definition and $[(t_n)] = s$ (Step 0), denote $\hat{s}$ by the image of $s$ in $T$, use the compatibility of the maps $S_i \to T$ and Definition 4.1, we can prove that $\sup_{j \geq 1} \hat{s}^j_{i_j} = \sup_{n \geq k} \hat{t}_n = \hat{s}$ in $T$ and that
sequences with \( \sup_{j \geq 1} s_i^j = \sup_{n \geq j} s_n^j = \sup_{j \geq 1} s_i^j \) in \( T \), i.e., the supremum of sequences of the images of \( s^j \) in \( T \) is indeed the image of \( s \) in \( T \).

(4) Let \( (r_1, r_2, \cdots) \) and \( (s_1, s_2, \cdots) \) be two eventually-\( \ll \)-increasing sequences with \( r_i, s_i \in S_i \) for each \( i \), satisfying \( r \ll s \), where

\[
[(r_1, r_2, \cdots)] = r, \quad [(s_1, s_2, \cdots)] = s.
\]

We may also suppose both of these two sequences are \( \ll \)-increasing from \( i = k \). We must show that the supremum of the images of \( r_k, r_{k+1}, \cdots \) in \( T \) is compactly contained in the supremum of the images of \( s_1, s_2, \cdots \) in \( T \), in other words, if \( \hat{r} \) denotes the image of \( r = \sup_{i \geq k} r_i \) in \( T \), and \( \hat{s} \) denotes the image of \( s = \sup_{i \geq k} s_i \) in \( T \), then \( \hat{r} \ll \hat{s} \).

Since \( r \ll s \), we have \( r \leq s_i \) in \( S/ \sim \) for some \( i \geq k \). Since the map \( S/ \sim \rightarrow T \) preserves the relation \( \leq \) (Step 6(2)), it follows that \( \hat{r} \leq \hat{s}_i \) in \( T \).

Since \( s_i \ll s_{i+1} \), not only in \( S/ \sim \) but also in \( S_{i+1} \), it follows from the given map \( S_{i+1} \rightarrow T \) preserves the relation \( \ll \) that \( \hat{s}_i \ll \hat{s}_{i+1} \) in \( T \). Since \( s_{i+1} \leq s \) in \( S/ \sim \), we have \( \hat{s}_{i+1} \leq \hat{s} \) in \( T \), and hence \( \hat{r} \leq \hat{s}_i \ll \hat{s}_{i+1} \leq \hat{s} \), then \( \hat{r} \ll \hat{s} \) in \( T \).

**Step 7:** We have shown that the map \( S/ \sim \rightarrow T \) preserves \( \leq \) relation (Step 6(2)) and preserves suprema of increasing sequences (Step 6(3)). Note that from Step 0, for any \( s \in S/ \sim \), we can pick an eventually-\( \ll \)-increasing sequences \( (s_1, s_2, \cdots) \) (\( \ll \)-increasing from \( i = k \)) to represent \( s \). From the last remark in Step 3, we have \( \sup_{i \geq k} s_i = s \) in \( S/ \sim \). Note that the choice of \( \hat{s}_i \) in \( T \) is unique by the given map \( S_i \rightarrow T \). From the uniqueness of the \( \sup_{i \geq k} \hat{s}_i \) in \( T \), we obtain the uniqueness of the map \( S/ \sim \rightarrow T \).

\( \square \)

**Remark 4.5.** Let \( (S_i, \beta_{ij}) \) be an inductive system in \( \text{Cu}^\sim \)-category, and \( S = S/ \sim \). Suppose we have an object \( T \in \text{Cu}^\sim \)-category satisfying that for any \( j \geq i \in \mathbb{N} \), the following diagram is commutative.

\[
\begin{array}{ccc}
S_i & \xrightarrow{\psi_i} & T \\
\beta_{ij} \downarrow & & \downarrow \psi_j \\
S_j & \xrightarrow{\psi_j} & T
\end{array}
\]

Then by universal property of inductive limit, there is a unique \( \text{Cu}^\sim \)-map \( \omega \) such that the following is also commutative for any \( i \in \mathbb{N} \).

\[
\begin{array}{ccc}
S_i & \xrightarrow{\psi_i} & T \\
\beta_{i\omega} \downarrow & & \downarrow \exists \omega \\
S & \xrightarrow{\exists \omega} & T
\end{array}
\]

Given \( s = (s_i)_{i \geq k_0} \in S \), for each \( i \geq k_0 \), we have

\[
\beta_{i\omega}(s_i) = [((\beta_{ij}(s_i)))_{j \geq i}],
\]
and hence from the commutativity,

$$\psi_i(s_i) = \omega([\beta_{ij}(s_i)]_{j \geq i}).$$

Then as

$$[s] = \sup_{i \geq k_0} \beta_{i\infty}(s_i)$$

and \(\omega\) preserves the suprema, we have

$$\omega([s]) = \omega(\sup_{i \geq k_0} \beta_{i\infty}(s_i)) = \sup_{i \geq k_0} \omega(\beta_{i\infty}(s_i)) = \sup_{i \geq k_0} \psi_i(s_i).$$

**Proposition 4.6.** Let \( (S_i, \beta_{ij}) \) be an inductive system in \( \text{Cu-category} \), and

Then

$$\text{Cu}^\sim \text{-} \lim S_i = \text{Cu}^- \text{-} \text{lim} S_i.$$  

**Proof.** Denote \( S = \text{Cu}^\sim \text{-} \text{lim} S_i \). From the last remark of Step 3 in Lemma 4.4, we have

$$S_c = \text{lim} (S_i)_c$$

is an algebraic limit, then

$$\text{Gr}(S_c) = \text{lim} \text{Gr}((S_i)_c)$$

is also an algebraic limit, which becomes a \( \Lambda \)-module.

Suppose \( T \) is an object in \( \text{Cu-category} \) with compatible \( \text{Cu} \)-morphisms

$$\psi_i : S_i \to T.$$  

Denote the \( \text{Cu}^- \)-morphism we obtained in Lemma 4.4 by \( \omega \).

$$\begin{array}{ccc}
S_1 & \xrightarrow{S_2} & \cdots \xrightarrow{T} \\
\uparrow & & \uparrow \\
\omega & & \\
S_c & \to & \text{Gr}(S_c)
\end{array}$$

Since each \( \text{Gr}((\psi_i)_c) : \text{Gr}((S_i)_c) \to \text{Gr}(T_c) \) is \( \Lambda \)-linear and compatible with each \( \text{Gr}((\beta_{ij})_c) : \text{Gr}((S_i)_c) \to \text{Gr}((S_j)_c) \), we have \( \text{Gr}(\omega_c) : \text{Gr}(S_c) \to \text{Gr}(T_c) \) is \( \Lambda \)-linear.

\( \square \)

**Theorem 4.7.** Let \( A = \text{lim}(A_i, \phi_{ij}) \) be an inductive system in the category \( C_{sr}^* \). Then

$$\text{Cu}^- \text{-} \text{lim} \text{Cu}(A_i) \cong \text{Cu}(A).$$

**Proof.** Since \( (\text{Cu}(A_i), \text{Cu}(\phi_{ij})) \) is an inductive system in the category \( \text{Cu} \), denote

$$S = \text{Cu}^- \text{-} \text{lim} \text{Cu}(A_i).$$

For \( \text{Cu}(A) \) in \( \text{Cu} \)-category and the compatible sequence of maps

$$\text{Cu}(\phi_{i\infty}) : \text{Cu}(A_i) \to \text{Cu}(A),$$
by Proposition 4.6, there exists a unique compatible Cu-morphism ω : S → Cu(A) such that the following diagram is commutative:

\[
\begin{array}{ccc}
\text{Cu}(A_i) & \xrightarrow{\text{Cu}(\phi_{i,\infty})} & \text{Cu}(A) \\
\text{Cu}(A_i) & \xrightarrow{\Phi_i} & S \\
\end{array}
\]

Now we are going to prove that ω is a Cu-isomorphism.

Firstly, we show that ω is surjective. For any \((x, u, \oplus_{p=1}^{\infty}(s_p, s^p)) \in \text{Cu}(A)\), by 1.6, there exists an increasing sequence \((x_1, x_2, \ldots)\) with \(x_1 \in \text{Cu}(A_1)\), \(x_2 \in \text{Cu}(A_2)\), \(\cdots\) such that

\[x = [(x_1, x_2, \cdots)],\]

Let \(I_{x_i}\) be the ideal of \(A_i\) generated by \(x_i \in \text{Cu}(A_i)\), and let \(I_x\) be the ideal of \(A\) generated by \(x \in \text{Cu}(A)\). Then

\[I_{x_1} \to I_{x_2} \to \cdots \to I_x \subseteq A,\]

with \(I_x = C^* - \lim I_{x_i}\).

Note that \((0, u, \oplus_{p=1}^{\infty}(s_p, s^p)) \in K(I_x)\), by the continuity of \(K\), there exist \(k \in \mathbb{N}\) and \((0, u_k, \oplus_{p=1}^{\infty}(s_{k,p}, s_k^p)) \in K(I_{x_k})\) satisfying

\[K(\phi_{k_i}I_{x_i}(s_{k_i})) = (0, u_k, \oplus_{p=1}^{\infty}(s_{k,p}, s^p)) \in K(I_{x_k}).\]

and for any \(i > k\), pick

\[(0, u_i, \oplus_{p=1}^{\infty}(s_{i,p}, s^i_p)) = K(\phi_{k_i}I_{x_i}(s_{k_i})) = (0, u_k, \oplus_{p=1}^{\infty}(s_{k,p}, s^p)) \in K(I_{x_i}).\]

Then we obtain an eventually-increasing sequence \(((x_j, u_j, \oplus_{p=1}^{\infty}(s_{j,p}, s^j_p)))\).

Set

\[s = (x_j, u_j, \oplus_{p=1}^{\infty}(s_{j,p}, s^j_p))_{j \geq k};\]

By Lemma 4.4, we have \(|s| \in S\).

Then by the commutativity of the diagram

\[
\begin{array}{ccc}
\text{Cu}(A_i) & \xrightarrow{\text{Cu}(\phi_{i,\infty})} & \text{Cu}(A) \\
\text{Cu}(A_i) & \xrightarrow{\Phi_i} & S \\
\end{array}
\]

we get

\[\omega(|s|) = (x, u, \oplus_{p=1}^{\infty}(s_p, s^p)).\]

Secondly, we show ω is injective. Let \(f = (x_i, u_i, \oplus_{p=1}^{\infty}(s_{i,p}, s^i_p))_{i \geq k_1}\) and \(g = (y_j, u_j, \oplus_{p=1}^{\infty}(t_{j,p}, t^j_p))_{j \geq k_2}\) be two eventually-increasing sequences of inductive system \((\text{Cu}(A_i), \text{Cu}(\phi_{ij}))\) in the category Cu~. Suppose \(\omega([f]) = \omega([g]) \in \text{Cu}(A)\), we need only to prove \(f \sim g\).

From Remark 4.5, we have

\[
\sup_{i \geq k_1} \text{Cu}(\phi_{i,\infty})(x_i, u_i, \oplus_{p=1}^{\infty}(s_{i,p}, s^i_p)) = \sup_{j \geq k_2} \text{Cu}(\phi_{j,\infty})(y_j, u_j, \oplus_{p=1}^{\infty}(t_{j,p}, t^j_p)),
\]

...
then
\[
\{(x_i)\} = \sup Cu(\phi_{i \infty})(x_i) = \sup Cu(\phi_{j \infty})(y_j) = \{(y_j)\}, \text{ i.e., } (x_i) \sim (y_j).
\]
(Here we take \(x_i = 0\) for \(i \leq k_1 - 1\) and \(y_j = 0\) for \(j \leq k_2 - 1\).)
Denote \(z = \{(x_i)\} = \{(y_j)\} \in Cu(A)\), then
\[
I_{x_1} \to I_{x_2} \to \cdots \to I_z < A,
\]
\[
I_{y_1} \to I_{y_2} \to \cdots \to I_z < A
\]
and
\[
I_z = C^* - \lim I_{x_i} = C^* - \lim I_{y_i}.
\]
Now we have
\[
\begin{array}{ccc}
K(I_{x_1}) & \xrightarrow{\phi^*_i} & K(I_{x_2}) \\
\downarrow & & \downarrow \\
K(I_{y_1}) & \xrightarrow{\psi^*_i} & K(I_{y_2}) \\
\end{array}
\]
where the maps \(\phi^*_i\) and \(\psi^*_i\) are in fact \(K(\phi_{ij}I_{x_i}I_{x_j})\) and \(K(\phi_{ij}I_{y_i}I_{y_j})\), respectively.
As \((x_i) \leq (y_i)\), by 1.6, there exists \(k \in \mathbb{N}\) such that for any \(i \geq k\) and \(c \in Cu(A_i)\) with \(c \ll x_i\), there exists \(m_c\) such that \(Cu(\phi_{ij})(c) \ll y_j\) (in \(Cu(A_j)\)), for all \(j \geq m_c\).
Suppose we have \((c, w, \oplus_{p=1}^\infty (r_{i,p}, t_{i,p})) \in Cu(A_i)\) with
\[
(c, w, \oplus_{p=1}^\infty (r_{i,p}, t_{i,p})) \ll (x_i, u_i, \oplus_{p=1}^\infty (s_{i,p}, s^{i,p})),
\]
then we have
\[
\delta_{I_{x_i}}(w, \oplus_{p=1}^\infty (r_{i,p}, t_{i,p})) = (u_i, \oplus_{p=1}^\infty (s_{i,p}, s^{i,p})), \forall i \geq k.
\]
Now we have the natural commutative diagram:
\[
\begin{array}{ccc}
I_c & \xrightarrow{\phi_{ij}I_cCu(\phi_{ij})(c)} & I_{x_i} \\
\downarrow & & \downarrow \\
I_{Cu(\phi_{ij})(c)} & \xrightarrow{\delta_{I_{Cu(\phi_{ij})(c)}I_{x_i}}} & I_{y_j}
\end{array}
\]
Then the following diagram commutes at the level of \(K\):
\[
\begin{array}{ccc}
K(I_c) & \xrightarrow{\delta_{I_cI_{x_i}}} & K(I_{x_i}) \\
\downarrow & & \downarrow \\
K(I_{Cu(\phi_{ij})(c)}) & \xrightarrow{\delta_{I_{Cu(\phi_{ij})(c)}I_{y_j}}} & K(I_{y_j})
\end{array}
\]
Note that
\[
K(\phi_{i \infty}I_{x_i}I_z)(0, u_i, \oplus_{p=1}^\infty (s_{i,p}, s^{i,p})) = K(\phi_{j \infty}I_{y_j}I_z)(0, v_j, \oplus_{p=1}^\infty (t_{j,p}, t^{j,p}) \in K(I_z),
\]
then there exists $m \geq m_c$ such that
\[
\delta_{I_{Cu(\phi_{ij}(c))}} I_{y_j} \circ K(\phi_{ij} I_{Cu(\phi_{ij}(c))})(0, w_i, \oplus_{p=1}^{\infty}(r_{i,p}, t^{i,p}))
\]
\[
= (0, v_j, \oplus_{p=1}^{\infty}(t_{j,p}, t^{j,p})) \in K(I_{y_j})
\]
holds for any $j \geq m$.

Recall that $c \ll y_j$, by Theorem 3.9, we have
\[
Cu(\phi_{ij}) (c, w_i, \oplus_{p=1}^{\infty}(r_{i,p}, t^{i,p})) \ll (y_j, v_j, \oplus_{p=1}^{\infty}(t_{j,p}, t^{j,p}))
\]
holds for any $j \geq m$. Then we get $f \leq g$.

Similarly, $(y_i) \leq (x_i)$ will imply $g \leq f$, then $\omega$ is injective.

Now we have $\omega : S \cong Cu(A)$ as objects in the category $Cu^-$. By Theorem 3.3, Theorem 3.15 and Proposition 4.6, we have
\[
Gr(S_c) \cong \lim_{\longrightarrow} K(A_i) = K(A).
\]
Still by Theorem 3.15, we have $Gr(Cu(A)_c)$ is also canonically isomorphic with $K(A)$. Then $Gr(\omega_c)$ works as the identity map on $K(A)$, which is $A$-linear, and hence, we have $S \cong Cu(A)$ as objects in the category $Cu$. This completes the proof.

\[\square\]

5. Recover the total $K$-theory

In this section, we show that $Cu$ contains more information than $K$, hence, $Cu$ is a complete invariant for certain real rank zero algebras.

5.1. We say $(S, u)$ is a $Cu$-semigroup with order-unit if $(S, u)$ is a positively directed $Cu$-semigroup satisfying $\rho(S_c) \cap \{-\rho(S_c)\} = \{0\}$ with a compact order-unit. Now a $Cu$-morphism $f : S \rightarrow T$ with $f(u) \leq v$ will be called a $Cu$-morphism between $(S, u)$ and $(T, v)$. Denote $Cu_u$ as the category whose objects are $Cu$-semigroups with order-unit and morphisms are $Cu$-morphisms.

After adding a unit, we transform Theorem 3.15 into the following:

Lemma 5.2. The assignment
\[
Cu_u : C_{sr1} \rightarrow Cu_u
\]
\[
A \mapsto (Cu(A), ([1_A], 0, \oplus_{p=1}^{\infty}(0, 0)))
\]
\[
\phi \mapsto Cu(\phi)
\]
from the category of unital, separable $C^*$-algebras of stable rank one to the category $Cu^-$ is a covariant functor.

Proof. Note that $[1_A]$ is an order unit of $K(A)_+$. For any $(x, u, \oplus_{p=1}^{\infty}(s_p, s^p)) \in Cu(A)$,
\[
(x, u, \oplus_{p=1}^{\infty}(s_p, s^p)) + (x, -u, \oplus_{p=1}^{\infty}(-s_p, -s^p)) = (2x, 0, \oplus_{p=1}^{\infty}(0, 0)) \geq 0
\]
implies that $Cu(A)$ is positively directed.
From Theorem 3.15, for any unital, separable C*-algebra $A$ with stable rank one, we have

$$(\text{Gr}(\text{Cu}(A)), \rho(\text{Cu}(A)), ([1_A], 0, \oplus_{p=1}^{\infty}(0, 0))) \cong (K(A), K(A)_+, [1_A]),$$

where $\rho : \text{Cu}(A) \to \text{Gr}(\text{Cu}(A))$ is the natural map ($\rho(x) = [(x, 0)]$). From Theorem 3.2, we have $\rho(K(A)) \cap \{-\rho(K(A))\} = \{0\},$ then $\rho(\text{Cu}(A)) \cap \{-\rho(\text{Cu}(A))\} = \{0\}.$

In conclusion, we have $(\text{Cu}(A), ([1_A], 0, \oplus_{p=1}^{\infty}(0, 0)))$ is a $\text{Cu}$-semigroup with order-unit and $\text{Cu}(\phi)$ is a $\text{Cu}_u$-morphism.

\[ \square \]

Now we can recover the total K-theory from the total Cuntz semigroup.

**Proposition 5.3.** The assignment

\[ H : \text{Cu}_u \to \Lambda - \text{module} \]

\[ (S, u) \mapsto (\text{Gr}(S_c), \rho(S_c), [u]) \]

\[ \phi \mapsto \text{Gr}(\phi_c), \]

is a functor, where $\rho : S_c \to \text{Gr}(S_c)$ is the natural map ($\rho(x) = [(x, 0)]$).

The functor $H$ yields a natural isomorphism $H \circ \text{Cu}_u \cong K,$ which means, for any unital, separable C*-algebras $A, B$ with stable rank one, if

$$(\text{Cu}(A), ([1_A], 0, \oplus_{p=1}^{\infty}(0, 0))) \cong (\text{Cu}(B), ([1_B], 0, \oplus_{p=1}^{\infty}(0, 0))),$$

then

$$(K(A), K(A)_+, [1_A]) \cong (K(B), K(B)_+, [1_B]).$$

**Proof.** Let $(S, u) \in \text{Cu}_u.$ Then $(\text{Gr}(S_c), \rho(S_c), [u])$ is a $\Lambda$-module with order-unit. Now let $\phi : S \to T$ be a $\text{Cu}_u$-morphism between two $\text{Cu}$-semigroups with order-unit $(S, u), (T, v).$ It follows that $\phi_c : S_c \to T_c$ is a Mon$_{\sim}$-morphism, and the induced Grothendieck map $\text{Gr}(\phi_c) : \text{Gr}(S_c) \to \text{Gr}(T_c)$ is $\Lambda$-linear such that $\text{Gr}(\phi_c)(S_c) \subseteq T_c.$ Finally, using that $\phi(u) \leq \phi(v),$ we obtain $\text{Gr}(\phi_c)(u) \leq v.$ We conclude that $H$ is a well-defined functor.

From Theorem 3.2 and Theorem 3.15, for any $A \in C_{sr1},$ we have

$$\alpha_A^* : (\text{Gr}(\text{Cu}(A)), \rho(\text{Cu}(A)), ([1_A], 0, \oplus_{p=1}^{\infty}(0, 0))) \cong (K(A), K(A)_+, [1_A]).$$

This means we do recover $K$ from $\text{Cu}$ as we anticipated.

\[ \square \]

In general, $H$ is not faithful, but it can be faithful if we restrict the domain of $H$ to a suitable subcategory of $C_{sr1}.$ To achieve a total version of Theorem 2.11, we list the following.

**Definition 5.4.** We say that an ideal $I$ in a C*-algebra $A$ is K-pure if both sequences

$$0 \to K_\delta(I) \to K_\delta(A) \to K_\delta(A/I) \to 0, \quad \delta = 0, 1$$

are pure exact. We say a C*-algebra $A$ is K-pure, if all ideals in $A$ are K-pure.
Lemma 5.5. If an ideal $I$ in a $C^*$-algebra $A$ is K-pure, then for any $p \geq 2$, the sequences

$$0 \to K_\delta(A/I; \mathbb{Z}_p) \to K_\delta(A; \mathbb{Z}_p) \to K_\delta(A/I; \mathbb{Z}_p) \to 0$$

are exact.

Proof. For any $p \geq 2$, we have the following well-known commutative diagram induced by the natural embedding $\iota : I \to A$.

$$
\begin{array}{cccc}
K_\delta(I) & \xrightarrow{x_p} & K_\delta(I) & \xrightarrow{\rho_\delta^I} & K_\delta(I; \mathbb{Z}_p) & \xrightarrow{K_\delta(\iota; \mathbb{Z}_p)} & K_{1-\delta}(I) \\
K_\delta(\iota) & & K_\delta(\iota) & & K_\delta(\iota; \mathbb{Z}_p) & & K_{1-\delta}(\iota) \\
K_\delta(A) & \xrightarrow{x_p} & K_\delta(A) & \xrightarrow{\rho_\delta^A} & K_\delta(A; \mathbb{Z}_p) & \xrightarrow{K_{1-\delta}(\iota)} & K_{1-\delta}(A).
\end{array}
$$

By assumption, we have $K_\delta(\iota)$ are injective for $\delta = 0, 1$. Note that the following diagram is exact, hence, what we need to show is $K_\delta(\iota; \mathbb{Z}_p)$ are injective maps for both $\delta = 0, 1$.

$$
\begin{array}{cccc}
K_\delta(I; \mathbb{Z}_p) & & K_\delta(\iota; \mathbb{Z}_p) & & K_\delta(\iota; \mathbb{Z}_p) & & K_\delta(A; \mathbb{Z}_p) & \xrightarrow{K_{1-\delta}(\iota; \mathbb{Z}_p)} & K_{1-\delta}(I; \mathbb{Z}_p) \\
& & K_{1-\delta}(\iota; \mathbb{Z}_p) & & K_{1-\delta}(\iota; \mathbb{Z}_p) & & K_{1-\delta}(I; \mathbb{Z}_p). & &
\end{array}
$$

Given $a_1 \in K_\delta(I; \mathbb{Z}_p)$ with $K_\delta(\iota; \mathbb{Z}_p)(a_1) = 0$, the standard diagram chasing gives $a_2 \in K_\delta(I) \subset K_\delta(A)$ and $a_3 \in K_\delta(A)$ such that

$$\rho_\delta^I(a_2) = a_1 \text{ and } p \times a_3 = a_2.$$

Since

$$0 \to K_\delta(I) \to K_\delta(A) \to K_\delta(A/I) \to 0, \text{ } \delta = 0, 1$$

are pure exact, i.e., for $p \geq 2$ we have

$$p \times K_\delta(I) = K_\delta(I) \cap (p \times K_\delta(A)),$$

there is an $a_4 \in K_\delta(I)$ such that $p \times a_4 = a_2$. By exactness, we have $a_1 = 0$. \qed

5.6. ([19, 2.2-2.3]) An AH$\mathcal{D}$ algebra is an inductive limit of finite direct sums of the form $M_n(\mathbb{I}_p^n)$ and $PM_n(C(X))P$, where $\mathbb{I}_p^n$ is the Elliott-Thomsen dimension-drop interval algebra and $X$ is one of the following finite connected CW complexes: $\{pt\}$, $\mathbb{T}$, $[0, 1]$, $T_{II,k}$. $P \in M_n(C(X))$ is a projection and $T_{II,k}$ is the 2-dimensional connected simplicial complex with $H^1(T_{II,k}) = 0$ and $H^2(T_{II,k}) = \mathbb{Z}/k\mathbb{Z}$. (In [12], this class is called ASH algebras.)

Proposition 5.7. ([11, Proposition 4.4]) Suppose an extension

$$0 \to I \to A \to A/I \to 0$$

is given. If $A$ is an AH$\mathcal{D}$ algebra of real rank zero, we have
(i) $I$ and $A/I$ are AHD algebras of real rank zero.

(ii) The ideal $I$ in $A$ is K-pure.

(iii) For any $p \geq 2$, the sequences

$$0 \to K_*(I; \mathbb{Z}_p) \to K_*(A; \mathbb{Z}_p) \to K_*(A/I; \mathbb{Z}_p) \to 0$$

are pure exact.

That is, all AHD algebras of real rank zero are K-pure.

**Theorem 5.8.** ([12, Proposition 4.12]) Let $A$, $B$ be separable $C^*$-algebras of real rank zero and stable rank one. Let $\phi : K(A) \to K(B)$ be a morphism of $\mathbb{Z}_2 \times \mathbb{Z}^+_{\text{graded}}$ groups. The following are equivalent:

(i) $\phi(K(A)_+) \subset K(B)_+$.

(ii) $\phi(K_0(A)) \subset K_0(B)$ and $\phi(K_1(A)) \subset K_1(B)$ for all ideals $I$ in $A$.

**Theorem 5.9.** Upon restriction to the class of unital, separable K-pure (or AHD) $C^*$-algebras of stable rank one and real rank zero, there are natural equivalences of functors:

$$H \circ \text{Cu}_u \simeq K \quad \text{and} \quad G \circ K \simeq \text{Cu}_u.$$

Therefore, for these algebras, $K$ is a classifying functor if, and only if, so is $\text{Cu}_u$.

**Proof.** The proof is as same as we did for $K_*$ and $\text{Cu}_1$. Note that the domain will be the full subcategory of $\text{Cu}_u$ whose objects are all those $(\text{Cu}(A), ([1_A], 0, \oplus_{p=1}^\infty (0, 0)))$ ($A$ is a unital, separable, K-pure (or AHD) algebra with stable rank one and real rank zero.), while the codomain is category of all the $K$-invariants for the same class of $C^*$-algebras. We only need to show that the corresponding restriction functor of $H$, which we will still call $H$, is a full, faithful and dense functor.

It was shown in Theorem 3.15 that

$$\alpha : \text{Cu}(A)_c \to K(A)_+,$$

$$(\{e\}, u, \oplus_{p=1}^\infty (s_p, s^p)) \mapsto (\{e\}, \delta_{I_eA}(u, \oplus_{p=1}^\infty (s_p, s^p)))$$

is surjective. We will show that $\alpha$ is injective.

Suppose that

$$\alpha(\{e\}, u, \oplus_{p=1}^\infty (s_p, s^p)) = \alpha(\{q\}, v, \oplus_{p=1}^\infty (t_p, t^p)),$$

which is

$$([e], \delta_{I_eA}(u, \oplus_{p=1}^\infty (s_p, s^p))) = ([q], \delta_{I_qA}(v, \oplus_{p=1}^\infty (t_p, t^p))).$$

Since $A$ is K-pure, for any ideal $I$ of $A$, Lemma 5.5 shows that the group $K_*(I; \mathbb{Z}_p)$ is a sub-group of $K_*(A; \mathbb{Z}_p)$, for all $p \geq 2$, then both $\delta_{I_eA}$ and $\delta_{I_qA}$ are the same injective map. Then we have

$$([e], u, \oplus_{p=1}^\infty (s_p, s^p)) = ([q], v, \oplus_{p=1}^\infty (t_p, t^p)).$$
Thus $\alpha$ is an ordered isomorphism. Combining this with Proposition 5.3, we have
\[ \alpha^*: H(Cu(A)_c, ([1_A], 0, \oplus_{p=1}^{\infty}(0, 0))) \cong (K(A), K(A)_+ + [1_A]). \]

This means that $H$ is dense.

To show that $H$ is faithful. Let $\phi, \psi: Cu(A) \rightarrow Cu(B)$ be two $Cu_u$-morphisms such that $H(\phi) = H(\psi)$. Then $Gr(\phi_c) = Gr(\psi_c)$, and hence,
\[ Gr(\phi_c) \mid_{\rho_A(Cu(A)_c)} = Gr(\psi_c) \mid_{\rho_A(Cu(A)_c)}: \rho_A(Cu(A)_c) \rightarrow \rho_B(Cu(B)_c), \]
where $\rho_A$ and $\rho_B$ are the corresponding maps:
\[ \rho_A: Cu(A)_c \rightarrow \rho_A(Cu(A)_c) \quad \text{and} \quad \rho_B: Cu(B)_c \rightarrow \rho_B(Cu(B)_c). \]

Recall that $K(A)_+$ and $K(B)_+$ are subsets of $K(A)$ and $K(B)$, respectively. Combining this with $Cu(A)_c \cong K(A)_+$ and $Cu(B)_c \cong K(B)_+$, we have $\rho_A$ and $\rho_B$ are injective, which implies $\phi_c = \psi_c$. Note that any morphism between algebraic $Cu_\sim$-semigroups is entirely determined by its restriction to compact elements, then we get $\phi = \psi$.

Now we prove that $H$ is full. Suppose we have an ordered morphism
\[ \phi: (K(A), K(A)_+, [1_A]) \rightarrow (K(B), K(B)_+, [1_B]), \]
which is $\Lambda$-linear. Then
\[ \phi|_{K(A)_+}: K(A)_+ \rightarrow K(B)_+, \phi([1_A]) \leq [1_B], \]
and
\[ \phi|_{K(A)_+}(K_0(A)_+) \subseteq K_0(B)_+. \]

By the functoriality of $Cu_\sim$ (see 2.7), $\phi|_{K(A)_+}$ induces a $Cu_u^\sim$-morphism
\[ \gamma^\sim(\phi|_{K(A)_+}): (Cu^\sim(K^+_+(A)), ([1_A], 0, \oplus_{p=1}^{\infty}(0, 0))) \rightarrow (Cu^\sim(K^+_+(B)), ([1_B], 0, \oplus_{p=1}^{\infty}(0, 0))), \]
where the order on $K(A)_+$ is induced by $K_0^+_+(A)$.

By the above results and Proposition 2.8, we have
\[ Cu(A) \cong Cu^\sim(Cu(A)_c) \cong Cu^\sim(K(A)_+). \]

For each $A$ in our assumption, denote by $i_A: Cu(A) \rightarrow Cu^\sim(K(A))$ the canonical $Cu^\sim$-isomorphism and denote by $\alpha_A: Cu(A)_c \rightarrow K(A)_+$ the canonical ordered monoid isomorphism.

Then
\[ i_A^{-1} \circ \gamma^\sim(\phi|_{K(A)_+}) \circ i_A: (Cu(A), ([1_A], 0, \oplus_{p=1}^{\infty}(0, 0))) \rightarrow (Cu(B), ([1_B], 0, \oplus_{p=1}^{\infty}(0, 0))) \]
is an ordered $Cu^\sim$-morphism. After identifying $Cu(A)_c$ and $Cu(B)_c$ with $K(A)_+$ and $K(B)_+$ through $\alpha_A$ and $\alpha_B$, respectively, we have
\[ (i_B^{-1} \circ \gamma^\sim(\phi|_{K(A)_+}) \circ i_A)_c = \phi|_{K(A)_+}. \]

Note that $Cu(A)_c \subseteq Cu$ and $Gr(i_B^{-1} \circ \gamma^\sim(\phi|_{K(A)_+}) \circ i_A)_c = \phi$ is $\Lambda$-linear by assumption. Then $i_B^{-1} \circ \gamma^\sim(\phi|_{K(A)_+}) \circ i_A$ is a $Cu_u^\sim$-morphism.
Using the functoriality of $H$, we obtain

$$H(i_B^{-1} \circ \gamma(\phi|_K(A)_+) \circ i_A) = \text{Gr}(i_B^{-1} \circ \gamma(\phi|_K(A)_+) \circ i_A)_c = \text{Gr}(\phi|_K(A)_+) = \phi.$$ 

Then $H$ is full.

Therefore, by standard category theory, there exists a functor $G$ such that $H \circ G$ and $G \circ H$ are naturally equivalent to the respective identities. Then we have

$$H \circ \text{Cu}_u \simeq K \quad \text{and} \quad G \circ K \simeq \text{Cu}_u.$$ 

□

**Corollary 5.10.** By restricting to the category $\text{Cu}_u$, we can recover $K$ from $\text{Cu}_u$ through $H$. A fortiori, we have $\text{Cu}_u$ is a complete invariant for unital $\text{AHD}$ algebras of real rank zero with slow dimension growth (see Theorem 9.1 in [12]).

**Remark 5.11.** Suppose that $\phi : K(A) \to K(B)$ is a isomorphism of $\mathbb{Z}_2 \times \mathbb{Z}^+$-graded groups satisfying $\phi(K(A))_+ = K(B)_+$. Then by Theorem 5.8, we get $\phi(K_0^+(A)) = K_0^+(B)$ and $\phi(K_I^+(A)) = K_I^+(B)$ for each ideal $I$ in $A$ and the relative $J$ in $B$.

Note that for any ideal $I$ of $A$, Lemma 5.5 shows that the map $K_*(I; \mathbb{Z}_p)$ is a sub-group of $K_*(A; \mathbb{Z}_p)$, for all $p \geq 2$. So do $J$ and $B$. Thus we have the mod-$p$ diagrams commutative for all $p \geq 2$ and $\delta = 0, 1$.

\[
\begin{array}{ccc}
K_\delta(I) & \longrightarrow & K_\delta(I; \mathbb{Z}_p) \\
\downarrow & & \downarrow \\
K_\delta(A) & \longrightarrow & K_\delta(A; \mathbb{Z}_p) \\
\downarrow & & \downarrow \\
K_\delta(B) & \longrightarrow & K_\delta(B; \mathbb{Z}_p) \\
\downarrow & & \downarrow \\
K_\delta(J) & \longrightarrow & K_\delta(J; \mathbb{Z}_p) \\
\downarrow & & \downarrow \\
K_1-\delta(I) & \longrightarrow & K_1-\delta(A) \\
\downarrow & & \downarrow \\
K_1-\delta(A) & \longrightarrow & K_1-\delta(B) \\
\downarrow & & \downarrow \\
K_1-\delta(J) & \longrightarrow & K_1-\delta(J)
\end{array}
\]

**Remark 5.12.** For any unital, separable $C^*$-algebras with stable rank one and real rank zero, the unitary Cuntz semigroup can be recovered functorially from the invariant $K_*$ (Theorem 2.11). But we don’t know whether the total version Theorem 5.9 is still true or not, if we delete the assumption of “$K$-pure (or $\text{AHD}$ algebras)”.

The key point is that the sequences

$$0 \to K_i(I) \to K_i(A) \to K_i(A/I) \to 0$$

are always exact; while for any $p \geq 2$, we are not sure whether the sequences

$$0 \to K_*(I; \mathbb{Z}_p) \to K_*(A; \mathbb{Z}_p) \to K_*(A/I; \mathbb{Z}_p) \to 0$$

are exact or not.
These above exact sequences will lead the commutativity of mod-$p$ diagram for $\delta = 0, 1$, which will generalize Theorem 5.9.

One more observation is that if we replaced “$K$-pure (or $A\mathcal{HD}$ algebras)” by “UCT”, then by [22], we have

$$K_*(I; \mathbb{Z}_p) \cong K_*(J; \mathbb{Z}_p).$$

The problem is that we don’t know whether this isomorphism is compatible with the other maps in the above mod-$p$ diagram.

At last, we have the following question.

**Question 5.13.** Let $A$ be a unital, separable $C^*$-algebra of stable rank one and real rank zero, does $(K(A), K(A)_+)$ determine $\text{Cu}(A)$?

Note that if we replace the definition of $\text{Cu}(A)$ by

$$\text{Cu}^I(A) \triangleq \prod_{I \in \text{Lat}_f(A)} \text{Cu}_f(I) \times K_1^I(A) \times \bigoplus_{p=1}^{\infty} K_1^I(A; \mathbb{Z}_p),$$

where $K_1^I(A)$ and $K_1^I(A; \mathbb{Z}_p)$ are images of $K_1(I)$ and $K_*(I; \mathbb{Z}_p)$ in $K_1(A)$ and $K_*(A, \mathbb{Z}_p)$, respectively.

We point that $\text{Cu}^I(A)$ is also a continuous invariant, it is routine to get that $\text{Cu}^I(A)$ and $(K(A), K(A)_+)$ are equivalent invariants for any unital, separable $C^*$-algebras of stable rank one and real rank zero. And for both simple case and $A\mathcal{HD}$ of real rank zero case, it is obvious that $\text{Cu}^I$ and $\text{Cu}$ coincide with each other.

As supplementary, we raise one more question.

**Question 5.14.** Let $A$ be a unital, separable $C^*$-algebra of stable rank one, when do we have $\text{Cu}^I(A)$ and $\text{Cu}(A)$ determine each other?

Under the real rank zero setting, an affirmative answer to Question 5.14 will give an affirmative answer to Question 5.13. But in general, these questions remain open for us.

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