Application of two-sublattice bilinearly coupled Heisenberg model to the description of certain ferrimagnetic materials

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Abstract

We study phenomenologically on the basis of two bilinearly coupled Heisenberg models the phase diagram of some ferrimagnetic substances. Calculations are performed with the help of Landau energy obtained through applying the Hubbard-Stratonovich transformation to the initial microscopic Heisenberg Hamiltonian. The phase transitions within the model are of second order with the emergence of a compensation point at lower temperatures for some values of parameters of the system. The main phase is a two-sublattice collinear ferrimagnet but also a metastable non-collinear phase is present within the exchange approximation presented here. The numerical results give a detailed description of temperature dependence of magnetization on the strength of intersublattice interaction and the difference between the effective exchanges of two ferromagnetically ordered sublattices.

1 Introduction

Ferrimagnets are substances made of various components having different magnetic properties. The differences in magnetic moments lead to a geometric frustration that may arise because either different elements occupy the lattice sites or the same element occupies nonequivalent crystallographic sites surrounded by a different number or type of non-magnetic ions, which effectively results in different magnetic properties. For complex alloys a combination of both may take place (For an extensive review see Ref. 1 and references therein). Within mean-field approach it is generally accepted that ferrimagnets can be modeled with the help of several interpenetrating sublattices each ordered ferromagnetically with effective antiferromagnetic coupling between them. In the pioneering works of Néel on ferrimagnetism within
the molecular field approach (see e.g. [2]), a two sublattice model is used to compute
the thermal magnetization behaviour of ferrimagnets, and six possible magnetization
curves were derived. Special attention there is paid to iron garnets, where the
spontaneous magnetization in comparison with experiment can be interpreted by
applying a three-sublattice model. In view of experimental study of magnetocaloric
effect of rare-earth based ferrimagnets [3] which has great potential for technologi-
cal applications in environmentally-friendly refrigeration, the theoretical mean-field
description of such alloys with three sublattices, is further elaborated [4]. Such stud-
ies are based on considering microscopic classical Heisenberg models with different
exchange and spin-orbit interactions depending on the crystal structure, chemical
composition of the particular alloy under study.

There is another theoretical mean field approach based on considering mixed spin
Ising model for description of ferrimagnets, see for example [5, 6, 7], where a very de-
tailed review of the literature on this approach is presented. In the present paper we
will consider ferrimagnets which can be described by different magnetic ions sitting
on two interpenetrating sublattices in a body centred cubic structure. The interaction
of ions on each sublattice is supposed ferromagnetic, while ions on the different
sublattices are coupled antiferromagnetically. The magnetic properties will be inves-
tigated on the basis of bilinearly coupled Heisenberg classical model in a mean-field
approximation which is treated using the Hubbard-Stratonovich transformation for
obtaining the respective Landau free energy and its analysis.

The rest of the paper is organized as follows: in Section 2 we describe in detail how
we calculate the Landau free energy from classical Heisenberg model with competing
interactions on the basis of previously applied approach [8] for derivation of mean
field approximation. In Section 3 the solutions of equations of state obtained by
the minimization of Landau energy derived in Section 1 are discussed. Section 4
summarizes the results both in strong and weak-coupling limit for ferrimagnetic
substances under consideration. Section 5 generalizes the conclusions and possible
further development of our study.

2 The model and derivation of Landau free energy

The microscopic Heisenberg Hamiltonian which describes two coupled subsystems
consisting of classical spins with different magnetic properties which antiferromag-
netically between them through a bilinear term can be written in the following form:

\[ H = -\frac{1}{2} \sum_{ij}^{2N} \left[ J_{ij}^{(1)} S_i^{(1)} \cdot S_j^{(1)} + J_{ij}^{(2)} S_i^{(2)} \cdot S_j^{(2)} + 2K_{ij} S_i^{(1)} \cdot S_j^{(2)} \right]. \]  \hspace{1cm} (1)

Here \( S_j^{(1,2)} \) are \( n \)-component classical Heisenberg spins whose magnitude is normalized on the unit sphere in spin space through the condition \( |S_j^{(1,2)}| = 1 \). The exchange parameters \( J_{ij}^{(1,2)}, K_{ij} \) in the general case are \( N \times N \) symmetric matrices with \( N \) - the number of lattice sites considered equal for both subsystems. This condition simplifies the consideration as makes the system symmetric with respect to the interchange of the subsystems. The exchange matrices \( J_{ij}^{(1,2)} \) denote the interaction between magnetic atoms of the same sort and \( K_{ij} \) - between magnetic atoms of different sorts.

The above Hamiltonian may be applied to the description of magnetic systems which consist of two different magnetic materials and no matter what is the microscopic origin of this difference, it is effectively described by different exchange interactions within the two subsystems. There may be other situations when the substance is made only of one type of magnetic ions but they occupy two different crystallographic positions in the Bravais lattice and are separated by a number of non-magnetic atoms. Such substance may also be considered as built of two magnetic subsystems with different exchange interactions within them.

In order to analyse the behaviour of magnetization and the phase transitions in systems that can be described by the above microscopic Hamiltonian we have to find the mean-field energy for the Hamiltonian (1) by calculating the partition function which in this case is represented by functional integral in \( n \)-dimensional spin space, where \( n \) - is the number of spin components. To do this we apply the Hubbard-Stratonovich transformation; see for example [10] and the papers cited therein. We have used this approach in [8] for ferromagnetic coupling between the two magnetic subsystems where a detailed description of procedure is given. Here we will just outline the important steps in the derivation of Landau free energy, especially in relation of antiferromagnetic coupling between the subsystems.

We present the two interacting different magnetic subsystems on a body-centered crystal lattice, for which the corners of elementary cube are occupied by one sort of magnetic atoms, and at the center of the cube atoms of different magnetic sort are located. So the nearest neighbours belong to different magnetic subsystems and the next-nearest neighbour to subsystems 1 and 2, respectively. Thus the system may be described as two interpenetrating sublattices, consisting of different magnetic atoms and we assume that the interaction within the sublattices \( J_{ij}^{1,2} \) is ferromagnetic and...
between them, \( K_{ij} \), it is antiferromagnetic. The Hubbard-Stratonovich transformation renders the initial microscopic Hamiltonian in new \( n \)-component variables \( \Psi^{(1)}_i, \Psi^{(2)}_i \) defined in real space, directly connected with the initial spins (see [8]), namely:

\[
\mathcal{H} = \frac{1}{2} \sum_{ij} ^{2N} \left( J^{(1)}_{ij} \Psi^{(1)}_i \cdot \Psi^{(1)}_j + J^{(2)}_{ij} \Psi^{(2)}_i \cdot \Psi^{(2)}_j + 2K_{ij} \Psi^{(1)}_i \cdot \Psi^{(2)}_j \right) - \ln \left[ \sum_{i} I_{n/2-1}(x_i^{(1)})(\frac{x_i^{(1)}}{2})^{-n/2} \Gamma(\frac{n}{2}) \right] - \ln \left[ \sum_{i} I_{n/2-1}(x_i^{(2)})(\frac{x_i^{(2)}}{2})^{-n/2} \Gamma(\frac{n}{2}) \right] .
\]

Here \( I_{n/2-1}(x_i^{(1,2)}) \) is the modified Bessel function, and \( \Gamma(\frac{n}{2}) \) is the Gamma function. In the above expression the exchange parameters \( J^{(1,2)}_{ij} \) and \( K_{ij} \) are connected to those in the initial Hamiltonian (1) by the relations:

\[
J_{ij}^{(1,2)} = \frac{J_{ij}^{(1,2)}}{T}, \quad K_{ij} = \frac{K_{ij}}{T}
\]

with \( T \) - the temperature.

We denote by \( x_i^{(1)} \) and \( x_i^{(2)} \) in (2) the following expressions:

\[
x_i^{(1)} = \left| J_{ij}^{(1)} \Psi^{(1)}_j + K_{ij} \Psi^{(2)}_j \right|; \quad x_i^{(2)} = \left| J_{ij}^{(2)} \Psi^{(2)}_j + K_{ij} \Psi^{(1)}_j \right|
\]

The terms containing Bessel functions in (2) will be further used only in the form of expansion with respect to \( x_i^{(1,2)} \) up to forth order by using the relation:

\[
\Gamma(\frac{n}{2})(\frac{2}{x})^{(n/2-1)} I_{n/2-1}(x) = 1 + \sum_{k=1}^{\infty} \frac{(x^2/4)^k}{k!(k+n/2+1)(k+n/2-2)...n/2}.
\]

The next step is to perform Fourier transformation to \( k \)-space, and pass to continuum limit in \( k \) as the finite size effects will not be considered at this stage. The quadratic part of the obtained Hamiltonian again contains a bilinear term with respect to \( \Psi^{(1,2)}(k) \) and we have to diagonalise it. This is done with the help of unitary matrix \( S \):

\[
\hat{S} = \begin{pmatrix} S_0(k) & -S_1(k) \\ S_1^*(k) & S_0(k) \end{pmatrix}.
\]

The eigenvalues of the matrix \( \hat{S} \) read:

\[
\lambda_{1,2}(k) = \frac{1}{2} \left[ J_1(k) + J_2(k) \pm \sqrt{(J_1(k) - J_2(k))^2 + 4K^2(k)^2} \right],
\]

where \( J_1(k) = \frac{1}{2} \sum_{ij} ^{2N} x_i^{(1)}(J^{(1)}_{ij} + K_{ij}) \) and \( J_2(k) = \frac{1}{2} \sum_{ij} ^{2N} x_i^{(2)}(J^{(2)}_{ij} + K_{ij}) \).
where $J_{1,2}(k)$ and $K(k)$ are the Fourier transforms of $J_{ij}^{(1,2)}$ and $K_{ij}$, respectively. In order to compute the integral we use the steepest-descent method, i.e. the integration contour is taken around the maxima of the eigenfunctions [5]. The calculation for bcc structure show that if we take the nearest neighbour interaction between atoms of the same sort and the nearest neighbour interaction between the atoms of different sort, $\lambda_{1,2}(k)$ has a maximum in the centre of the Brillouin zone that gives ferromagnetic ordering for the sublattices with antiferromagnetic $K < 0$ interaction between them which is the focus of our consideration below. We should note that $\lambda_{1,2}(k)$ has a maximum also at the border of the Brillouin zone $k = \pi/a$ ($a$ is the lattice constant) which supposes antiferromagnetically ordered sublattices. There may be also some local maximum inside the Brillouin zone, which may give some incommensurate ordering within the sublattices, but this case is beyond the scope of the present study.

After performing the reverse Fourier transform to real space we obtain the dimensionless Landau energy in the following form:

$$
\frac{F}{T} = \frac{t_1}{2} \psi_1^2 + \frac{t_2}{2} \psi_2^2 + \frac{g}{4} \left[ (\psi_1^2)^2 + (\psi_2^2)^2 \right] + \frac{b}{2} \psi_1^2 \psi_2^2 + b(\psi_1 \cdot \psi_2)^2 + w(\psi_2^2 - \psi_1^2)(\psi_1 \cdot \psi_2).
$$

The coefficients of the Landau energy are expressed by the components of the matrix (4) and its eigenvalues (5) for $k = 0$:

$$
S_0 = \frac{1}{D} \left( J_1 - J_2 + \sqrt{(J_1 - J_2)^2 + 4K^2} \right),
$$

$$
S_1 = \frac{2K}{D}
$$

where $D$ is introduced to satisfy the condition $\|\hat{S}\| = 1$, namely $S_0^2 + S_1^2 = 1$:

$$
D = \sqrt{2} \left[ (J_1 - J_2)^2 + 4K^2 \right]^{1/4} \left[ J_1 - J_2 + \sqrt{(J_1 - J_2)^2 + 4K^2} \right]^{1/2}.
$$

We will write here the explicit expressions for the coefficients of landau energy as we will need them further in solving the mean field equations and discussion of obtained results:

$$
t_{1,2} = \frac{1}{\lambda_{1,2}} - \frac{1}{n}\quad (9a)
$$

$$
g = \frac{u}{2} (S_0^4 + S_1^4)\quad (9b)
$$

$$
b = u S_1 S_0 S_1^2\quad (9c)
$$

$$
w = \frac{u}{2} S_0 S_1 (S_0^2 - S_1^2);\quad (9d)
here \( u = n^2(n + 2) \) with \( n \) - the number of order parameter components. The real vector fields \( \vec{\psi}_1 \) and \( \vec{\psi}_2 \) in the Landau free energy \( \mathbf{6} \) play the role of two coupled order parameters, and the averaged sublattice magnetizations are related to them by the equations:

\[
\begin{align*}
\vec{m}_1 &= \frac{S_0}{\lambda_1} \vec{\psi}_1 - \frac{S_1}{\lambda_2} \vec{\psi}_2 \\
\vec{m}_2 &= \frac{S_1}{\lambda_1} \vec{\psi}_1 + \frac{S_0}{\lambda_2} \vec{\psi}_2
\end{align*}
\] (10a)

and

\[
\begin{align*}
\vec{m}_3 &= \frac{S_2}{\lambda_1} \vec{\psi}_1 + \frac{S_0}{\lambda_2} \vec{\psi}_2 \\
\vec{m}_4 &= \frac{S_0}{\lambda_1} \vec{\psi}_1 - \frac{S_2}{\lambda_2} \vec{\psi}_2
\end{align*}
\] (10b)

### 3 Solving mean-field equations

The initial microscopic Hamiltonian is symmetric with respect to the rotation of all spins through the same angle. The application of Hubbard-Stratonovich transformation for derivation of landau free energy, given in previous section, preserves the symmetry of initial hamiltonian also with respect to field variables \( \vec{\psi}_{1,2} \) which means that we can find the magnitude and the mutual orientation between order parameters \( \vec{\psi}_{1,2} \) but not their orientation with respect to crystallographic axes. This may be done for particular magnetic substance by including in the initial microscopic Hamiltonian terms accounting for the magnetic anisotropy. For pure exchange interactions we can introduce the following notations \[8\]:

\[
\psi_{1,i} = |\psi_1|\beta_i \quad \text{and} \quad \psi_{2,i} = |\psi_2|\delta_i,
\]

where \( |\psi_1| = \psi_1, \quad |\psi_2| = \psi_2 \) are the magnitudes of the vector fields, and \( \beta_i, \delta_i \) are the respective direction cosines, which fulfil the condition:

\[
\sum_{i=1}^{3} \beta_i^2 = 1 \quad \text{and} \quad \sum_{i=1}^{3} \delta_i^2 = 1.
\] (11)

The equations of state then will be:

\[
\frac{\partial f}{\partial X_i} = 0, \quad \text{where} \quad X_i = \{\psi_1, \psi_2, \beta_i, \delta_i\}
\] (12)

Solving the above equations with respect to direction cosines \( \beta_i, \delta_i \) gives two possible orientations between the vector fields \( \vec{\psi}_1, \vec{\psi}_2 \) for \( K < 0 \):

1. The collinear phase with \( \sum_i \beta_i \delta_i = -1 \), that is, \( \vec{\psi}_1 \) and \( \vec{\psi}_2 \) are antiparallel, and

2. The non-collinear phase with \( \sum_i \beta_i \delta_i = 0 \), that is, \( \vec{\psi}_1 \) and \( \vec{\psi}_2 \) are perpendicular.

Below we will discuss in detail the non-collinear phase 2. The angle between the order parameters \( \vec{\psi}_1 \) and \( \vec{\psi}_2 \), \( i.e. \), is:

\[
\sum_{i}^{3} \beta_1 \delta_i = -\frac{w(\psi_2^2 - \psi_1^2)}{2b\psi_1\psi_2}
\]
and is defined only when \( \psi_1 \neq 0 \) and \( \psi_2 \neq 0 \). For \( K < 0 \), the analysis shows that the non-collinear phase exists only when the order parameters \( \psi_1 \) and \( \psi_2 \) are of equal magnitudes, meaning that the order parameters \( \psi_1 \) and \( \psi_2 \) are mutually perpendicular. The magnitude \( \psi = \psi_1 = \psi_2 \) for the non-collinear phase in analytical form reads:

\[
\psi^2 = -\frac{t_1 + t_2}{u}.
\]

Then the sublattice magnetization magnitudes calculated using the above expressions for the non-collinear phase will be:

\[
|m_1| = \psi \sqrt{\frac{S_0^2}{\lambda_1^2} + \frac{S_1^2}{\lambda_2^2}}, \quad (14a)
\]

\[
|m_2| = \psi \sqrt{\frac{S_1^2}{\lambda_1^2} + \frac{S_0^2}{\lambda_2^2}}. \quad (14b)
\]

Note that the sublattice magnetizations are not perpendicular but form an angle \( \angle (m_1, m_2) = \gamma \) with each other, expressed by

\[
\cos(\gamma) = \frac{S_0 S_1 (\lambda_1^2 - \lambda_2^2)}{\sqrt{(S_0^2 \lambda_2^2 + S_1^2 \lambda_1^2)(S_1^2 \lambda_2^2 + S_0^2 \lambda_1^2)}}.
\]

The calculations show that this non-collinear phase for \( K < 0 \) within the exchange approximation has no domain of stability. We should mention here that the free energy \( (3) \) is very sensitive to the sign of interaction \( K \) between the sublattices. When \( K > 0 \), \( i.e., \) the interaction between the sublattices is ferromagnetic there is small domain in which the respective non-collinear phase is stable \( (9) \).

For antiparallel \( \psi_1 \) and \( \psi_2 \) it is obvious that the sublattice magnetizations \( (10) \) will be also antiparallel. We may write the resulting equations for the magnitudes of the order parameters \( \psi_1 \) and \( \psi_2 \) of the collinear phase and \( K < 0 \) in the following form:

\[
t_1 \psi_1 + g \psi_1^3 + 3b \psi_1 \psi_2^2 - w \psi_2(\psi_2^2 - 3\psi_1^2) = 0, \quad (15a)
\]

\[
t_2 \psi_2 + g \psi_2^3 + 3b \psi_2 \psi_1^2 - w \psi_1(3\psi_2^2 - \psi_1^2) = 0, \quad (15b)
\]

with the stability conditions given by:

\[
t_1 + 3g \psi_1^2 + 3b \psi_2^2 + 3w \psi_1 \psi_2 > 0 \quad (16)
\]

\[
(t_1 + 3g \psi_1^2 + 3b \psi_2^2 + 6w \psi_1 \psi_2)(t_2 + 3g \psi_2^2 + 3b \psi_1^2 - 6w \psi_1 \psi_2) - 9[w(\psi_1^2 - \psi_2^2) + 2b \psi_1 \psi_2]^2 \geq 0 \quad (17)
\]

We will make some remarks on the dependence of solutions of above system on the magnitude of exchange parameters \( J_1, J_2 \) and \( K \). When \( J_1 < |K|, J_2 < |K|, \)
the leading interaction is determined by the antiferromagnetic coupling between the two sublattices. This may be called a strong coupling limit for which the eigenvalue \( \lambda_2(k = 0) \)

\[
\lambda_2 = \frac{1}{2} \left[ J_1 + J_2 - \sqrt{(J_1 - J_2)^2 + 4K^2} \right],
\]

becomes negative. This is equivalent to the inequality \( K^2 - J_1J_2 > 0 \). The coefficient \( t_2 \) in front of \( \psi_2^2 \) becomes positive; see (9), and the field \( \bar{\psi}_2 \) becomes redundant. The Landau free energy (13) will be:

\[
\left( \frac{F}{T} \right)_s = f_s = \frac{t_1}{2} \bar{\psi}_1^2 + \frac{g}{4} (\bar{\psi}_1^2)^2
\]

(19)

The minimization of above equation gives for \( \bar{\psi}_1 \) the solution:

\[
(\bar{\psi}_1)^2 = -\frac{t_1}{g}
\]

which exists and is stable for \( t_1 < 0 \).

The sublattice magnetizations:

\[
\begin{align*}
\vec{m}_1 &= \frac{S_0}{\lambda_1} \bar{\psi}_1 \\
\vec{m}_2 &= \frac{S_1}{\lambda_1} \bar{\psi}_1
\end{align*}
\]

(21)

will be antiparallel as \( S_1 \sim K/D \) and \( K < 0 \). The phase described by the above equations will be presented by two antiparallel sublattices with different magnitudes of sublattice magnetizations.

In the weak coupling limit for antiparallel configuration, \( i.e. \), when \( J_1 > |K|, J_2 > |K| \), or equivalently \( J_1J_2 > K^2 \), the system of equations (15), together with the stability conditions (18), (19) should be solved. This can be done numerically and the results will be presented in the next section.

4 Results and discussion

The analytical result for sublattice magnetizations in the limiting case of strong coupling (21) gives for the magnitude of total magnetization \( |\vec{M}| = |\vec{m}_1 + \vec{m}_2| \) the following expression:

\[
|\vec{M}| = S_0 \left| \frac{\psi_1}{\lambda_1} \right| 1 + \frac{S_1}{S_0}
\]

with \( |\psi_1|^2 \), given by (20). The phase transition is obviously of second order and the total magnetization behaviour with temperature is smooth resembling the one
of Weiss ferromagnet with the exception that no saturation is reached for $T = 0$. According to the Neel’s classification of ferrimagnets, see [2], the change of magnetization with temperature in the strong coupling limit falls within R-type curve. For example, similar curve is obtained theoretically and compared with the respective experiment for $Y_3Fe_5O_{12}$ [4] where two sublattice model with strong antiferromagnetic coupling is considered.

In the limiting case when $J_1 = J_2 = J$ the relation will hold $S_0 = -S_1 = 1/\sqrt{2}$ and an antiferromagnetic structure with $\vec{m}_1 = -\vec{m}_2$ will appear, only if $\psi_2 \equiv 0$ and $|\psi_1|^2 = -2t_1/u$. The transition temperature for antiferromagnetic ordering will be given by: $t^c_a = (J + |K|)/n$.

Further we will present the numerical results for the temperature dependence of sublattice magnetizations and the total magnetization of the system in the weak-coupling limit which we define here in the following way: $J_1 > |K|$ and $J_2 > |K|$. Such a situation is present, for example in some ferrimagnetic compounds like GdCo$_{12}$B$_6$ [11]. It is experimentally found that the exchange constants within sublattices are ferromagnetic and larger than the antiferromagnetic coupling between the sublattices; moreover there the magnetic anisotropy is small.

Experiments for some R-T compounds where R is a rare earth element and T is a transition element, show that the exchange in the transition metal sublattice is leading in magnitude, while the exchange in the rare-earth ion sublattice can be safely ignored and considered as negligible. The intersublattice interaction is also small see, for example, DyFe$_5$Al$_7$ [12], ErFe$_{11}$TiH [13], RCo$_2$ (R = Tb and Gd and R = Er, Ho, and Dy) [14]. In our notations the relation between the exchange integrals in this case will be $J_1 > |K| \gg J_2$, so this does not fall into our assumption of weak coupling and will not be considered here.

In order to solve numerically the equations of state (15) for weak coupling between sublattices we introduce the following dimensionless parameters.

\begin{align}
  t &= \frac{T}{J_1 + J_2}, \quad (22a) \\
  \alpha &= \frac{J_1 - J_2}{J_1 + J_2}, \quad (22b) \\
  \beta &= \frac{K}{J_1 + J_2}, \quad (22c)
\end{align}

with $t$ – the dimensionless temperature. In the above expression we have supposed that $J_1 > J_2$ which in view of symmetry in interchanging the sublattices does not limit the consideration; then $\alpha > 0$ and $\beta < 0$ as $K < 0$. 


The weak coupling between the sublattices, namely $\mathcal{J}_1 > |\mathcal{K}|$ and $\mathcal{J}_1 > |\mathcal{K}|$ may be expressed by the parameters from (22) by the relation:

$$\alpha^2 + \beta^2 < 1.$$ 

The parameter $\alpha$ is a measure for the difference in exchange parameters of the two sublattices and by its definition $0 < \alpha < 1$.

The total magnetization of the system is the sum of sublattice magnetizations (10):

$$\vec{M} = \vec{m}_1 + \vec{m}_2 = \frac{S_0 + S_1}{\lambda_1} \psi_1 + \frac{S_0 - S_1}{\lambda_2} \psi_2.$$ (23)

Hereafter we will use the following notations for magnitudes of sublattice magnetizations and total magnetization both in the text and in figures:

$$m_1 = |\vec{m}_1|; \; m_2 = |\vec{m}_2|; \; \text{and} \; M = |\vec{M}|$$

The calculations show that the phase transition to ordered ferrimagnetic state occurs at temperature:

$$t_c = \frac{1}{6} (1 + \sqrt{\alpha^2 + \beta^2})$$

which grows when either the difference between the exchange interactions in sublattices grows, or when the antiferromagnetic coupling is bigger, or both. The phase transition from disordered to ordered phase is of second order.

We want to note that within the exchange approximation used here for the regime of weak coupling defined above with the decrease of temperature a compensation point appears no matter how small is the difference between the exchange interactions of sublattices. At the compensation temperature $t_{\text{comp}}$, the sublattice magnetizations $\vec{m}_1$ and $\vec{m}_2$ are equal in magnitude and antiparallel, so $M = 0$. The relation between the order parameters magnitudes there is defined by:

$$\psi_1 = \frac{\lambda_1 (S_0 - S_1)}{\lambda_2 (S_0 + S_1)} \psi_2.$$ (24)

As the calculations show $\psi_2 < \psi_1$ for all values of $\alpha$ and $\beta$, but $\psi_1$ grows with the decrease of temperature in a monotonic way, while $\psi_2$ grows more rapidly. The quantity

$$\frac{\lambda_1 (S_0 - S_1)}{\lambda_2 (S_0 + S_1)} = \frac{(6t_c)^2}{1 - \alpha^2 - \beta^2} \left( \frac{\sqrt{\alpha^2 + \beta^2} - \beta}{\alpha} \right)$$

is always $> 1$ as $\beta < 0 \sim K$ and $\alpha > 0$ so at some temperature $t_{\text{comp}} < t_c$, and values of $\psi_1$, $\psi_2$ the condition (24) is fulfilled. In the following figure we show the change of net magnetization magnitude $M(t)$ with the temperature for
Figure 1: The dependence of net magnetization $M$ on reduced temperature $t$ for for fixed $\alpha$ and the different values of antiferromagnetic coupling $\beta$.

$\alpha = 0.1$, i.e., $J_2 = 0.83J_1$ and different values of $\beta$. It is seen from Fig.1 that the increase of antiferromagnetic coupling between the sublattices slightly shifts the compensation temperature to higher values, and $M(t)$ grows more rapidly below the compensation temperature and reaches higher values as $t \to 0$. We suppose that within the exchange approximation and in weak coupling limit the key factor for the compensation point to appear is the weakness of antiferromagnetic exchange between the sublattices compared to the ferromagnetic exchange of sublattices 1 and 2, respectively.

We will discuss in more detail the influence of difference between the magnitudes of exchange interaction in the sublattices, represented by the parameter $\alpha$ on $M(t)$ and sublattice magnetizations $m_1$, and $m_2$. For $\alpha = 0.08$, i.e., $J_2 = 0.85J_1$, $M(t)$, $m_1$, and $m_2$ are shown in Fig. 2. At $t_c$ the transition is of second order and when lowering the temperature a compensation point $t_{comp}$ appears, which is located close to $t_c$. Sublattice magnetizations change with temperature in monotonic way, and in the temperature interval $t_{comp} < t < t_c$, the relation between sublattice magnetizations is $m_1 > m_2$, as expected as in sublattice 1 the exchange interaction $J_1 > J_2$. Below $t_{comp}$ the magnetization of weaker sublattice $m_2$ becomes bigger than $m_1$.

For intermediate values of $\alpha = 0.4$, or $J_2 = 0.43J_1$, see Fig.3, with decrease of
temperature below the compensation point the total magnetization rapidly grows in non-monotonic way as $t \to 0$, similar to V-curve according to Neel’s classification. The behaviour of sublattice magnetizations with temperature is quite different; $m_1$, which is the sublattice magnetization with stronger exchange interaction $J_1$ grows in smooth way with decrease of temperature, while $m_2$ for weaker sublattice interaction $J_2$, below compensation point grows drastically in non-monotonic way and in the temperature interval $0 < t < t_{\text{comp}}, m_2 \gg m_1$.

Such behaviour is described in detail in [15] for ferrimagnets with compensation point for many experimentally observed substances. There explanation of $M(t)$ behaviour below compensation point is done by introducing the notion of weak sublattice where depending on the particular substance considered different mechanisms for explanation of this effect are pointed out. Within our exchange model the effect of weak sublattice is readily seen and mainly depends on the difference $J_1 - J_2 \sim \alpha$. When $\alpha$ further grows the compensation point is shifted to lower temperatures and the magnetization $m_1$ of the stronger sublattice decreases for $t \to 0$. This is illustrated in Fig.4 for $J_2 = 0.19 J_1$.  

![Figure 2: The dependence of net magnetization $M$ and sublattice magnetizations $m_1$ and $m_2$ on reduced temperature $t$ for small difference between sublattice exchange parameters.](image)
Figure 3: The dependence of net magnetization $M$ and sublattice magnetizations $m_1$ and $m_2$ on reduced temperature $t$ for intermediate difference between sublattice exchange parameters.

A qualitatively similar behaviour of the total magnetization can be seen in the experiments with ErFe$_2$ [16] and GdCo$_{12}$B$_6$ [11] although direct comparison with the experimental curves can hardly be made, as these substances have crystallographic and magnetic structure that differs from the one assumed within our model.

The influence of parameter $\alpha$ is summarized in the next figure, see Fig. 5, where the net magnetization is displayed for small values of $\beta = 0.08$ and different values of parameter $\alpha$. As the difference between the magnitudes of sublattice exchange interactions grow the transition temperature is shifted to higher values as expected and the compensation temperature is lowered. Another effect is the change of net magnetization behaviour below $t_{comp}$ from monotonic to nearly exponential when $\alpha$ increases.

5 Conclusions

Our mean-field analysis of this relatively simple model with competing interactions on a bcc lattice shows that the behaviour of net magnetization of the two-sublattice ferrimagnet depends essentially on the difference in magnetic interactions between...
Figure 4: Illustration of the dependence of net magnetization $M$ and sublattice magnetizations $m_1$ and $m_2$ on reduced temperature $t$ when the exchange in sublattice 2 is very small compared to sublattice 1.

Within our approach the parameters of Landau energy can be directly related to the averaged microscopic exchange interactions as described in section 2, and this relation is quite simple in the case of high symmetry crystal structure as considered here. It will be of interest also to include the influence of an external magnetic field and to perform calculations for ferrimagnets, for which the relation between exchange interactions fulfills the condition $J_1 > |K| \gg J_2$, that is, when one of the sublattices is very weak.

The model may be generalized to include the expansion of the effective Hamiltonian up to sixth-order terms in $x_i$, $i = 1, 2$ in order to consider the possibility of first order phase transition to ferrimagnetic state due to the influence of external parameters as pressure and change of concentration.

The influence of antiferromagnetic coupling is more prominent when its magnitude is of the same order, or larger than the difference between the exchange parameters of the sublattices. The model may be presented here. The influence of antiferromagnetic coupling is more prominent when its magnitude is of the same order, or larger than the difference between the exchange parameters of the sublattices.
Figure 5: The dependence of net magnetization $M$ for fixed antiferromagnetic exchange $\beta$ and growing difference $\alpha$ between the ferromagnetic exchanges of the two sublattices.

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