DYNAMIC ANALYSIS OF AN SEIR EPIDEMIC MODEL WITH A TIME LAG IN AWARENESS ALLOCATED FUNDS

RIANE HAJJAMI, MUSTAPHA EL JARROUDI, AADIL LAHROUZ AND ADEL SETTATI
LMA, Department of Mathematics, FST, B.P. 416-Tanger Principale
Tanger, Morocco

MOHAMED EL FATINI
Ibn Tofail University, Laboratory of PDE, Algebra and Spectral Geometry
Department of Mathematics, Faculty of Sciences, Kénitra, BP 133, Morocco

KAI WANG
Department of Applied Mathematics, Anhui University of Finance and Economics
Bengbu 233030, China

(Communicated by Xiaoying Han)

Abstract. The spread of infectious diseases is often accompanied by a rise in the awareness programs to educate the general public about the infection risk and suggest necessary preventive practices. In the present paper we propose to study the impact of awareness on the dynamics of the classical SEIR by considering the budget allocation to warn people as a new dynamic variable. In the model formulation, it is assumed that the susceptible individuals contract the infection via a direct contact with infected individuals, and that the transmission rate is presented by a general decreasing function of the availability of funds. We further introduced a time delay in the growth rate of the budget allocation related to the number of reported infected cases. The existence and the stability criteria of the equilibrium states are obtained in terms of the basic reproduction number $R_a$. It is shown that $R_a \leq 1$ is a necessary and sufficient condition for the global stability of the disease-free equilibrium, and by application of the geometric approach based on the third additive compound matrix we derived sufficient conditions for the global stability of the positive equilibrium state in the absence of delay. Our analysis reveals that awareness programs have the ability to reduce the infection prevalence. However, delay in providing funds destabilizes the system and give rise to periodic oscillations through Hopf-bifurcation. The direction and the stability of the bifurcating periodic solutions are investigated by using the normal form theory and central manifold theorem. Numerical simulations and sensitive analysis are provided to illustrate the theoretical findings.

1. Introduction. Infectious diseases continue to be one of the most important health problems worldwide. On the emergence of any infectious disease, countries struggle to raise the funds required for the health services their populations need. Yet, the health-care expenditure of pharmaceutical interventions keep increasing

2020 Mathematics Subject Classification. 92B05, 39A30, 37G15, 37G05.
Key words and phrases. Epidemic model, awareness, Stability, Delay, Hopf-bifurcation, geometric approach.

* Corresponding author: Mohamed El Fatini.
through years. In order to reduce this economic burden, the governments look actually for non-pharmaceutical interventions and awareness can have a considerable capacity to halt the spread of many communicable diseases. Indeed, creation of disease awareness programs, among others, can be an effective approach to educate the general public on the infection risk and outbreak severity, and suggest the necessary preventive practices such as social distancing, wearing protective masks, practice of better hygiene, voluntary quarantine, and other precautionary measures which reduce the exposure to the causative agent and the possibility of infection. Such programs can be implemented either physically (with health professionals on site), or through news media including fliers, posters, newspapers, television, social media, with the common goals of communicating accurate and timely information about the disease to the public and directing people toward appropriate prevention and intervention during a disease outbreak. As the awareness disseminates, people respond toward it and eventually will change their behavior to alter their susceptibility (which was been perceived during the outbreak of many diseases such as SARS [35], influenza A(H1N1)[11], AIDS [28], and recently COVID-19 ).

Mathematical models have provided a useful tool to gain insights into the transmission characteristics of diseases and potentially evaluate the effectiveness and implications of various preventive and control strategies. In the recent past, mathematical studies associated with awareness programs in epidemic outbreaks have been discussed extensively (see for instance [10, 34, 30, 31, 24, 20] and the references therein). In fact, several researchers focused their attention on the contact rate and considered that awareness will certainly decrease it. Based on this assumption, Cui et al. [6] have considered the contact rate as a decaying exponential function of the number of infective individuals of the form \( \beta e^{-mI} \), with the parameter \( m > 0 \) reflects how strongly media coverage can affect contact infection. With this rate, the analysis showed that the proposed model may exhibit periodic oscillations for weak media effects, while it may have three endemic equilibria for strong media effects. Similarly, Liu et al. [21] divided the total population into exposed (E), infectious (I) and hospitalized (H), and considered \( \beta e^{-\alpha_1 E - \alpha_2 I - \alpha_3 H} \) as the transmission rate. In their paper, they explored the potential for multiple outbreaks and sustained oscillations of emerging infectious diseases due to the psychological impact from reported numbers of infectious and hospitalized individuals. The form \( \beta - \beta_1 \frac{I}{m+I} \) was also used as the transmission rate, with the deduction \( \beta_1 \frac{I}{m+I} \) due to media use [8, 22, 33], where \( \beta \) is the transmission rate of the disease under consideration in the absence of awareness, \( \beta_1 \) represents the reduced maximum value of the transmission rate when the number of infectives \( I \) becomes large, and \( m \) reflects the reactive velocity of media coverage and individuals to the epidemic disease. A general transmission rate of the form \( \beta - \beta_1 f(I) \) was adopted by Berrhazi et al. [4] to reflect the effect of awareness in their stochastic epidemiological model [3, 5, 9].

In the previous cited works, the contact rate was considered as a decaying function of infective individuals due to media coverage. Another suggested approach to incorporate the effect of awareness is by adding a new compartment to the epidemiological model under consideration, to describe the awareness density and to assume that the transmission rate decreases in terms of the density of awareness programs. In particular, in [27] Pawelek et al. have assumed that the direct contact between susceptible and infected individuals \( \beta \), decreases by a factor \( e^{-\alpha T} \) due to the behavior change of public after reading the tweets about influenza, where \( \alpha \) determines how effective is the disease related information in influencing the transmission rate,
and \( T \) is the number of the tweets that contain influenza. Their findings suggest that social media programs such as twitter may have substantial influence on influenza virus infection during an epidemic season. Authors in [15] have adopted the same transmission rate, that is \( \beta e^{-\alpha M} \) to study the impact of media on the dynamics of an \( SEIS \) model, where \( M \) is the number of disease-related messages provided by media. They showed that a rich dynamical behavior such as backward-bifurcation and Hopf-bifurcation may arise from their model, and that media reduces the disease prevalence and can serve as a good indicator in controlling its emergence. It may be noted that generally the number of infective cases known to the policy makers are some time old and thus the number of awareness programs. Therefore, it is more plausible to consider a time delay in the growth rate of awareness programs related to the number of reported infected cases. In this context, some mathematical models proposed to assess the effect of time delay in execution of awareness programs (see for instance [32, 1, 25, 26, 29, 12, 2]). It was found that incorporation of time delay in the modeling process destabilizes the system and periodic solutions may arise through Hopf-bifurcation. To warn people requires funds, which is made available by the government. This aspect was lately considered by [26] where they proposed an SIS model with budget allocations (considered as a dynamic variable) and studied the effect of time delay in budget allocations to warn people. They showed that the increase in funds to make people aware reduces the number of infected individuals however delay in providing the funds destabilizes the interior equilibrium and may cause stability switches.

Motivated by the above works, we consider in the present paper a nonlinear \( SEIR \) model (Susceptible, Exposed, Infectious, Removed) which is more appropriate for diseases causing latent period such as measles, influenza, tuberculosis etc, with an eventual delay in providing the funds required to warn people about the emergence of an infectious disease. Furthermore, we approach the transmission rate by a general decreasing function of the availability of funds.

2. Mathematical model formulation. We begin by considering the basic \( SEIR \) model [18]:

\[
\begin{align*}
\frac{dS(t)}{dt} &= \mu - \mu S(t) - \beta S(t)I(t), \\
\frac{dE(t)}{dt} &= \beta S(t)I(t) - (\mu + \alpha)E(t), \\
\frac{dI(t)}{dt} &= \alpha E(t) - (\mu + \gamma)I(t), \\
\frac{dR(t)}{dt} &= \gamma I(t) - \mu R(t),
\end{align*}
\]

where, \( S(t), \ E(t), \ I(t), \) and \( R(t) \) are the densities of the susceptible, exposed( in the latent period), infective and removed individuals in a population respectively, in the region under consideration at any time \( t \). The total population is assumed to be constant so that the parameter \( \mu > 0 \) represents the recruitment rate and the death rate of the population at the same time. It is also assumed that the disease spreads due to direct contact between the susceptible and infective individuals with only a positive transmission rate denoted \( \beta \). The positive parameter \( \alpha \) is the rate of the exposed to the disease becoming infectious, and \( \gamma > 0 \) is the rate of infected individuals being removed. To control the spread of a disease, governments allocate some funds to make public aware by awareness campaigns through mass media,
print media, social media, etc. Let $M(t)$ be the budget allocated at time $t$ to aware people in the considered region, and we suppose that in the presence of awareness the transmission rate $\beta$ decreases as a function of the availability of funds $M(t)$. Specifically, we adopt the following assumption:

$$\beta(M) \in C^1([0, M_{\max}], \mathbb{R}^+), \text{ and } \beta'(M) \leq 0.$$ 

The growth rate of the budget allocation is assumed to be proportional to the number of infective individuals in the region. We further assume that a certain level of funds allocation is always assigned to maintain some awareness programs, since there are diseases such as Tuberculosis (TB), influenza, measles, dengue, etc., that are common endemic in many parts of the world, and so require continuous awareness. These assumptions lead to the following model:

Specifically, we adopt the following assumption:

$$\gamma I(t) - \mu R(t),$$

$$\frac{dM(t)}{dt} = a_1 I(t) + a_2(M_0 - M(t)),$$

where, $a_1$ represents the per-capita growth rate in the budget allocation to warn people due to increase in infected individuals, $a_2$ is the rate with which the budget allocation is diminishing and $M_0$ is the baseline budget always available. In reality the reported cases of infected individuals are not instantaneous, so the increment by government in the budget allocation depends on old data. To incorporate this fact in the model (2), we assume that the growth rate of $M(t)$ due to the increase in infected individuals is proportional to $I(t - \tau)$, where $\tau > 0$ is some time delay. Thus, model (2) becomes as follows

$$\frac{dS(t)}{dt} = \mu - \mu S(t) - \beta(M)S(t)I(t),$$

$$\frac{dE(t)}{dt} = \beta(M)S(t)I(t) - (\mu + \alpha)E(t),$$

$$\frac{dI(t)}{dt} = \alpha E(t) - (\mu + \gamma)I(t),$$

$$\frac{dR(t)}{dt} = \gamma I(t) - \mu R(t),$$

$$\frac{dM(t)}{dt} = a_1 I(t) + a_2(M_0 - M(t)),$$

where $\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)^T \in BC$, and $BC$ denotes the Banach space $BC((-\infty, 0], \mathbb{R}_+^4)$ of bounded continuous functions mapping the interval $(-\infty, 0]$ into the non-negative cone $\mathbb{R}_+^4$ equipped with the sup-norm:

$$\|\varphi\| = \sup_{-\tau \leq \theta \leq 0} \{ |\varphi_1(\theta)|, |\varphi_2(\theta)|, |\varphi_3(\theta)|, |\varphi_4(\theta)| \}.$$
The equation of $R(t)$ is omitted because of its independence from the other equations. In what follows we consider $a_0 = a_2M_0$.

3. Equilibrium states and stability analysis. Before analyzing the system model (3), it is necessary to validate it mathematically and biologically. That is, we need to show that the model (3) under initial conditions (4) has a unique solution defined for all $t \geq 0$ which remains non-negative and bounded.

Lemma 3.1. The system (3) with initial condition (4) has a unique solution $(S(t), E(t), I(t), M(t))$ defined for all $t \geq 0$, and the set
\[
\Omega = \left\{ (S, E, I, M) \in [0, +\infty]^4 : S + E + I \leq 1, \frac{a_0}{a_2} \leq M \leq \frac{a_0 + a_1}{a_2} \right\},
\]
is positively invariant.

Proof. From the fundamental theory of functional differential equations [13], we infer that the system (3) with the initial condition (4) admits a unique solution $(S(t), E(t), I(t), M(t))$ defined on a maximal interval $[0, T_e]$. To show the global existence and the non-negativity of the solution we define
\[
T_+ = \inf \left\{ t \in (0, T_e), S(t) > 0 \text{ or } E(t) < 0 \text{ or } I(t) < 0 \text{ or } M(t) < 0 \right\}
\]
Suppose that $T_+ < \infty$. Hence, we have for all $t \in (0, T_+]
\[
S(t), E(t), I(t), M(t) \geq 0,
\]
\[
S(T_+)E(T_+)I(T_+)M(T_+) = 0,
\]
\[
\frac{dS(T_+)}{dt} < 0 \text{ or } \frac{dE(T_+)}{dt} < 0 \text{ or } \frac{dI(T_+)}{dt} < 0 \text{ or } \frac{dM(T_+)}{dt} < 0.
\]
However,
\[
\left\{ \begin{array}{l}
\frac{dS}{dt} \big|_{S=0} = \mu > 0, \\
\frac{dE}{dt} \big|_{E=0} = \beta(M)SI \geq 0, \\
\frac{dI}{dt} \big|_{I=0} = \alpha E \geq 0, \\
\frac{dM}{dt} \big|_{M=0} = a_0 + a_1I(t-\tau) \geq 0.
\end{array} \right.
\]
Therefore, $T_+ = +\infty$ and $T_e = T_+ = +\infty$. That is, the solution $(S(t), E(t), I(t), M(t))$ is non-negative and defined for all $t \geq 0$. Beside, let $(S(0), E(0), I(0), M(0)) \in \Omega$, we have
\[
\frac{d}{dt} (S(t) + E(t) + I(t)) = \mu - \mu (S(t) + E(t) + I(t)) - \gamma I(t) \leq \mu - \mu (S(t) + E(t) + I(t)).
\]
So
\[
S(t) + E(t) + I(t) \leq 1 + (S(0) + E(0) + I(0) - 1) e^{-\mu t} \leq 1.
\]
On the other hand, using the fact that $I \in [0, 1]$ we get
So, if \( M \leq \frac{a_0}{a_2} \), by integration, we obtain
\[
\frac{a_0}{a_2} \leq \frac{a_0}{a_2} + \left( M(0) - \frac{a_0}{a_2} \right) e^{-a_2t} \leq M(t) \leq \frac{a_0 + a_1}{a_2} + \left( M(0) - \frac{a_0}{a_2} \right) e^{-a_2t} \leq \frac{a_0 + a_1}{a_2}.
\]

\[\square\]

**Lemma 3.2.** The model system (3) has exactly two equilibrium states, namely,

i) the disease-free equilibrium state \( E^0 \left( 1, 0, 0, \frac{a_0}{a_2} \right) \).

ii) A unique interior equilibrium state \( E^* \left( S^*, E^*, I^*, M^* \right) \) provided that

\[
\mathcal{R}_a = \frac{\alpha \beta \left( \frac{a_0}{a_2} \right)}{(\mu + \alpha) (\mu + \gamma)} > 1.
\]

**Proof.** Obviously, \( E^0 \) is an equilibrium for system (3). To find an interior equilibrium system (3) we consider the system of equations:

\[
\begin{align*}
I &= \frac{1}{\alpha} (-a_0 + a_2 M), \\
E &= \frac{1}{\alpha} \frac{\mu + \gamma}{\mu} I, \\
S &= -1 - \frac{1}{\mu} E + \frac{1}{\mu} I, \\
0 &= -\frac{(\mu + \alpha)(\mu + \gamma)}{\alpha} + \left[ 1 + \frac{\mu + \alpha}{\mu a_1} \right] \frac{1}{\mu a_1} a_0 + \frac{\mu + \alpha}{\mu a_1} \frac{(\mu + \gamma) a_0}{\alpha}.
\end{align*}
\]

\[\text{(5)}\]

Which is equivalent to the system

\[
\begin{align*}
I &= \frac{1}{\alpha} (-a_0 + a_2 M), \\
E &= \frac{1}{\alpha} \frac{\mu + \gamma}{\mu} I, \\
S &= -1 - \frac{1}{\mu} E + \frac{1}{\mu} I, \\
0 &= -\frac{(\mu + \alpha)(\mu + \gamma)}{\alpha} + \left[ 1 + \frac{\mu + \alpha}{\mu a_1} \right] \frac{1}{\mu a_1} a_0 + \frac{\mu + \alpha}{\mu a_1} \frac{(\mu + \gamma) a_0}{\alpha}.
\end{align*}
\]

Put
\[
F(M) = -\frac{(\mu + \alpha)(\mu + \gamma)}{\alpha} + \left( \omega_0 - \omega_1 M \right) \beta(M),
\]

where
\[
\omega_0 = 1 + \frac{(\mu + \alpha)(\mu + \gamma)}{\mu a_1}, \quad \omega_1 = \frac{(\mu + \alpha)(\mu + \gamma) a_0}{\mu a_1}.
\]

So, if \( M \geq \frac{\omega_0}{\omega_1} \), we have \( F(M) < 0 \). Therefore, there is no interior equilibrium for the system (3) such that \( M \geq \frac{\omega_0}{\omega_1} \). On the other hand, the function \( F(M) \) is decreasing on \( \left[ \frac{a_0}{a_2}, \frac{a_0 + a_1}{a_2} \right] \subset \left[ \frac{a_0}{a_2}, \frac{a_0 + a_1}{a_2} \right] \), with
\[
\begin{align*}
F \left( \frac{\omega_0}{\omega_1} \right) &= -\frac{(\mu + \alpha)}{\alpha} (\mu + \gamma) < 0, \\
F \left( \frac{a_0}{a_2} \right) &= -\frac{(\mu + \alpha)}{\alpha} (\mu + \gamma) + \beta \left( \frac{a_0}{a_2} \right) = \frac{(\mu + \alpha)}{\alpha} (\mu + \gamma) (\mathcal{R}_a - 1).
\end{align*}
\]

Hence, if \( \mathcal{R}_a \leq 1 \) then, \( F(M) < 0 \) for all \( M \in \left( \frac{a_0}{a_2}, \frac{a_0 + a_1}{a_2} \right) \). If not, we have \( F \left( \frac{a_0}{a_2} \right) > 0 > F \left( \frac{\omega_0}{\omega_1} \right) \). Thus, there exists a unique \( M^* \in \left( \frac{a_0}{a_2}, \frac{a_0 + a_1}{a_2} \right) \) such that
$F(M^*) = 0$, and in view of (5), we conclude that the system (3) admit a unique equilibrium $E^*$.

In the rest of this section we shall study of equilibria the stability of the system (3). We begin by the local and global stability of the disease-free equilibrium $E^0$ in term of the basic reproduction number $R_a$.

3.1. Stability analysis of $E^0$. Using the Jacobian $J(E^0)$ at the steady state $E^0$, we obtain the following theorem that gives the local asymptotic stability and the instability of $E^0$ depending on the value of $R_a$.

**Theorem 3.3.** The disease-free steady state $E^0$ is locally asymptotically stable for all $\tau \geq 0$ whenever $R_a < 1$ and is unstable for $R_a > 1$.

**Proof.** The characteristic equation of the system (3) linearized around $E^0$ is given by:

$$
\begin{vmatrix}
\lambda + \mu & 0 & \beta\left(\frac{a_0}{a_2}\right) & 0 \\
0 & \lambda + \mu + \alpha & -\beta\left(\frac{a_0}{a_2}\right) & 0 \\
0 & -\alpha & \lambda + \mu + \gamma & 0 \\
0 & 0 & -a_1e^{-\lambda\tau} & \lambda + a_2
\end{vmatrix} = 0,
$$

which leads to

$$(\lambda + \mu)(\lambda + a_2)\det(A - \lambda I_2) = 0, \quad (6)$$

where

$$A = \begin{pmatrix}
-(\mu + \alpha) & \beta\left(\frac{a_0}{a_2}\right) \\
\alpha & -(\mu + \gamma)
\end{pmatrix}.$$  

We have

$$\text{tr}(A) = -(2\mu + \alpha + \gamma) < 0, \quad \det A = (\mu + \alpha)(\mu + \gamma)(R_a - 1).$$

Hence, $A$ is asymptotically stable if $R_a < 1$ and unstable if $R_a > 1$. Therefore, according to (6), we conclude that the disease-free equilibrium $E^0$ is locally asymptotically stable if $R_a < 1$ and unstable when $R_a > 1$.

**Theorem 3.4.** The disease-free equilibrium $E^0$ of the system (3) is globally asymptotically stable on $\Omega$ if and only if $R_a \leq 1$.

**Proof.** We know from theorem (3.3) that $E^0$ is unstable if $R_a > 1$. Let $R_a \leq 1$, we use the Lyapunov functional method to establish the global asymptotic stability of $E^0$. We have

$$\frac{dE}{dt} = -\left((\mu + \alpha)E + \beta(M)SI\right) \leq -\left((\mu + \alpha)E + \beta\left(\frac{a_0}{a_2}\right)I - \beta\left(\frac{a_0}{a_2}\right)I^2\right), \quad (7)$$

where, $S \leq 1 - I$, $M \geq \frac{a_0}{a_2}$ and the monotony of $\beta$ is used to obtain the last estimate of $\frac{dE}{dt}$.
Combining (7) and the I-equation one can get
\[
\frac{d}{dt} \left( \frac{\mu + \alpha}{\alpha} I + E \right) \leq \frac{\left( \mu + \gamma \right) (\mu + \alpha)}{\alpha} (\mathcal{R}_a - 1) I - \beta \left( \frac{a_0}{a_2} \right) I^2.
\] (8)

On the other hand,
\[
\frac{d}{dt} (1 - S)^2 = -2\mu (1 - S)^2 + 2\beta (M) S (1 - S) I
\[
\leq -2\mu (1 - S)^2 + 2 \beta \left( \frac{a_0}{a_2} \right) S (1 - S) I
\]
\[
\leq -2 \left[ \mu - \varepsilon \beta \left( \frac{a_0}{a_2} \right) \right] (1 - S)^2 + \frac{1}{2} \mu - \varepsilon \beta \left( \frac{a_0}{a_2} \right) I^2,
\] (9)

where the elementary inequality
\[
ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2,
\] (10)

for a small \( \varepsilon > 0 \) such that \( \mu - \varepsilon \beta \left( \frac{a_0}{a_2} \right) > 0 \), is used in the last inequality of (10). Combining (8) and (9), yields that
\[
\frac{d}{dt} \left[ \frac{\mu + \alpha}{\alpha} I + E + \varepsilon (1 - S)^2 \right] \leq -2\varepsilon \left[ \mu - \varepsilon \beta \left( \frac{a_0}{a_2} \right) \right] (1 - S)^2 + \frac{1}{2} \mu - \varepsilon \beta \left( \frac{a_0}{a_2} \right) I^2.
\] (11)

In addition, from the \( M \)-equation we have
\[
\frac{d}{dt} \left( M - \frac{a_0}{a_2} \right)^2 = -2a_2 \left( M - \frac{a_0}{a_2} \right) + 2a_1 \left( M - \frac{a_0}{a_2} \right) I(t - \tau).
\]

Using again the inequality (10), we get
\[
\frac{d}{dt} \left[ \frac{2\varepsilon}{a_1} \left( M - \frac{a_0}{a_2} \right)^2 \right] \leq -\frac{4\varepsilon}{a_1} (a_2 - a_1 \varepsilon) \left( M - \frac{a_0}{a_2} \right)^2 + I^2(t - \tau).
\] (12)

Combining (12) with
\[
\frac{d}{dt} \int_{t-\tau}^t I^2(x) dx = I^2(t) - I^2(t-\tau),
\]
leads to
\[
\frac{d}{dt} \left[ \frac{2\varepsilon}{a_1} \left( M - \frac{a_0}{a_2} \right)^2 + \int_{t-\tau}^t I^2(x) dx \right] \leq -\frac{4\varepsilon}{a_1} (a_2 - a_1 \varepsilon) \left( M - \frac{a_0}{a_2} \right)^2 + I^2.
\] (13)

We also choose \( \varepsilon \) such that \( a_2 - a_1 \varepsilon > 0 \). Finally, consider the Lyapunov functional
\[
V(t) = \frac{\mu + \alpha}{\alpha} I + E + \varepsilon (1 - S)^2 + \frac{1}{4} \beta \left( \frac{a_0}{a_2} \right) \left[ \frac{2\varepsilon}{a_1} \left( M - \frac{a_0}{a_2} \right)^2 + \int_{t-\tau}^t I^2(x) dx \right].
\]

Using (11) and (13), we obtain
\[
\frac{dV}{dt} = -2\varepsilon \left[ \mu - \varepsilon \beta \left( \frac{a_0}{a_2} \right) \right] (1 - S)^2 - \frac{1}{4} \beta \left( \frac{a_0}{a_2} \right) I^2 - \varepsilon \beta \left( \frac{a_0}{a_2} \right) \left( M - \frac{a_0}{a_2} \right)^2.
\]

Thus, \( \frac{dV}{dt} \leq 0 \). Furthermore, \( \frac{dV}{dt} = 0 \) holds only if \( S = 1, I = 0 \) and \( M = \frac{a_0}{a_2} \). Substituting \( I = 0 \) in the I-equation gives \( E = 0 \). Therefore, the singleton \( \{ E^o \} \) is the
only invariant set of system (3) contained entirely in \( \{ (S, E, I, M) \mid \frac{dV}{dt} = 0 \} \), and consequently by the LaSalle’s invariance principle, \( \{ E^0 \} \) is globally asymptotically stable.

3.2. Stability of the interior steady state for \( \tau = 0 \). We begin by analyzing the local stability of the positive equilibrium of the system (3) in the absence of delay. The characteristic equation of the Jacobian matrix of the system (3) evaluated at \( E^* \) is given by

\[
\lambda^4 + A_3 \lambda^3 + A_2 \lambda^2 + A_1 \lambda + A_0 = 0,
\]

where

\[
A_3 = 3\mu + \gamma + \alpha + a_2 + \beta(M^*)I^*,
A_2 = (\mu + \beta(M^*)I^*) (2\mu + \gamma + \alpha + a_2) + a_2 (2\mu + \gamma + \alpha),
A_1 = a_2 (\mu + \beta(M^*)I^*) (2\mu + \gamma + \alpha) + (\mu + \gamma) (\mu + \alpha) \beta(M^*)I^* - \alpha a_1 \beta(M^*)S^* I^*,
A_0 = a_2 (\mu + \gamma) (\mu + \alpha) \beta(M^*)I^* - \mu \alpha a_1 \beta(M^*)S^* I^*.
\]

Using the fact that \( \beta(M) \) is a positive decreasing function, we can see clearly that all the coefficients \( A_1, A_2, A_3 \) and \( A_4 \) are positive. In addition, by the criterion of Routh-Hurwitz all the roots of the characteristic equation (14) will lie in the left-half of the complex plane provided that:

\[
A_3(A_1A_2 - A_3) - A_1^2A_4 > 0.
\]

Hence, we claim the following theorem:

**Theorem 3.5.** Let \( R_0 > 1 \). The interior equilibrium \( E^* \) is locally asymptotically stable provided that

\[
A_3(A_1A_2 - A_3) - A_1^2A_4 > 0.
\]

To study the global stability of the endemic equilibrium \( E^* \), we shall use the geometrical approach developed by M. Li and J. Muldowney [16, 19]. In what follows, we outline the geometric approach based on the third additive compound matrix. The third additive compound matrix for a \( 4 \times 4 \) matrix \( A = [a_{ij}] \) is defined as

\[
A^{[3]} = \begin{pmatrix}
  a_{11} + a_{22} + a_{33} & a_{34} & -a_{24} & -a_{14} \\
  a_{43} & a_{11} + a_{22} + a_{44} & a_{23} & -a_{13} \\
  -a_{42} & a_{32} & a_{11} + a_{33} + a_{44} & a_{12} \\
  a_{41} & -a_{31} & a_{21} & a_{22} + a_{33} + a_{44}
\end{pmatrix}
\]

Now, we consider the dynamical system

\[
\frac{dX}{dt} = F(X),
\]

where, \( F : \Omega \rightarrow \mathbb{R}^n \) is \( C^1 \) function and \( \Omega \subset \mathbb{R}^n \) is a simply connected open set. Let \( X(t, x_0) \) denotes the solution of system (15) with the initial condition \( X(0) = x_0 \). We assume that

\( (H_1) \) There exists a compact absorbing set \( K \subset \Omega \),
\( (H_2) \) The system (15) has a unique equilibrium point \( X^* \in \Omega \).

For a solution \( X(t, x_0) \) of any initial value problem (15), the linearized system is

\[
Y' = J(X(t, x_0))Y,
\]
and the associated third compound system is

\[ Z' = J^{[3]}(X(t, x_0))Z, \]  

(16)

Where \( J^{[3]} \) is the third compound matrix of the jacobian matrix of the system (15).

**Theorem 3.6.** Assume that (H\(_1\)) and (H\(_2\)) hold and there exists a Lyapunov function \( V(X, Z) \), a function \( K(t) \), and positive constants \( e, k, c \) such that

(i) \( e|z| \leq V(X, Z) \leq c|z| \),
(ii) \( D_+V \leq (K'(t) - k)V \),

where \( D_+ \) is the right hand derivative of \( V \) taken along the direction of system (16). Then, the interior equilibrium \( E^* \) of (15) is globally asymptotically stable.

**Theorem 3.7.** Let \( \tau = 0 \) and \( R_\alpha > 1 \). If the following conditions hold

\[ a_1 \leq \mu, \quad \alpha \leq \gamma + a_2, \quad -\beta(M) \leq \frac{\mu}{\mu + \alpha} - \beta(M) \forall M \in \Omega \quad (A \tau), \]

then, the interior equilibrium \( E^* \) is globally asymptotically stable in \( \bar{\Omega} \).

**Proof.** The system (3), under the assumptions \( \tau = 0 \) and \( R_\alpha > 1 \), satisfies conditions (H\(_1\)) and (H\(_2\)). In fact, from lemma 3.2, \( E^* \) is the only steady state in the interior \( \bar{\Omega} \) of \( \Omega \). Hence, (H\(_2\)) is verified. Moreover, when \( R_\alpha > 1 \), \( E^0 \) is unstable. Therefore, the instability of \( E^0 \) together with \( E^0 \in \partial \Omega \), implies the uniform persistence of solutions of the system (3), that is, there exists a constant \( \epsilon > 0 \) such that

\[ \liminf_{t \to \infty} S(t), \liminf_{t \to \infty} E(t), \liminf_{t \to \infty} I(t), \liminf_{t \to \infty} M(t) > \epsilon. \]

(17)

The uniform persistence, because of the boundedness of \( \Omega \) is equivalent to the existence of a compact set in \( \Omega \) which is absorbing for system (3). Thus, (H\(_1\)) is verified. Now, the Jacobian for system (3) is given by:

\[
J = \begin{pmatrix}
-\mu + \beta(M)I & 0 & -\beta(M)S & -\beta'(M)SI \\
\beta(M)I & - ( \mu + \alpha ) & \beta(M)S & \beta'(M)SI \\
0 & \alpha & - ( \mu + \gamma ) & 0 \\
0 & 0 & a_1 & -a_2
\end{pmatrix}.
\]

The third additive compound matrix of \( J \) is

\[
J^{[3]} = \begin{pmatrix}
-3\mu + \alpha + \beta(M)I & 0 & -\beta'(M)SI & -\beta'(M)SI \\
a_1 & -[\kappa + \beta(M)I] & \beta(M)S & \beta'(M)S \\
0 & \alpha & -[\kappa + \beta(M)I] & 0 \\
0 & 0 & \beta(M)I & -(\kappa + \gamma)
\end{pmatrix},
\]

where, \( \kappa = 2\mu + \alpha + a_2 \), and the associated system is

\[
\begin{align*}
\frac{dZ_1}{dt} &= -[3\mu + \alpha + \beta(M)I]|Z_1| - \beta'(M)SI(Z_3 + Z_4), \\
\frac{dZ_2}{dt} &= a_2Z_1 - [2\mu + \alpha + a_2 + \beta(M)I]Z_2 + \beta(M)S(Z_3 + Z_4), \\
\frac{dZ_3}{dt} &= \alpha Z_2 - [2\mu + \alpha + a_2 + \beta(M)I]Z_3, \\
\frac{dZ_4}{dt} &= \beta(M)IZ_3 - (2\mu + \gamma + \alpha + a_2)Z_4.
\end{align*}
\]

(18)

As in [17], we need to show the uniform global stability of the linear compound system (18). To do this, we choose an associated Lyapunov function

\[ V(t, Z_1, Z_2, Z_3, Z_4) = \max(V_1, V_2), \]
and the coefficient $\theta$ will be determined in the rest of the proof. Based on (17), one can find positive constants $c_1$ and $c_2$ such that

$$c_1 |z| \leq V \leq c_2 |z|,$$

where,

$$|z| = \sum_{i=1}^{4} |z_i|.$$

Next, we calculate the right derivative of $V$ along the trajectory of the compound system (18). We have

$$D_+|Z_1| \leq -[3\mu + \alpha + \beta(M)I]|Z_1| - \beta'(M)S(I)|Z_3| + |Z_4|),$$

$$D_+|Z_2| \leq a_2 |Z_1| - [2\mu + \alpha + a_2 + \beta(M)I]|Z_2| + \beta(M)S(|Z_3| + |Z_4|),$$

$$D_+|Z_3| \leq \alpha|Z_2| - [2\mu + \gamma + a_2 + \beta(M)I]|Z_3|,$$

$$D_+|Z_4| \leq \beta(M)I|Z_3| - (2\mu + \gamma + a_2)|Z_4|.$$

Then

$$D_+(|Z_1| + |Z_2|) \leq -[2\mu + \alpha + \beta(M)I](|Z_1| + |Z_2|) + (a_1 - \mu)|Z_1| - a_2 |Z_2| + |\beta(M) - \beta'(M)I|S(|Z_3| + |Z_4|).$$

Using the assumption ($Ar$), we get

$$D_+(|Z_1| + |Z_2|) \leq -[2\mu + \alpha + \beta(M)I](|Z_1| + |Z_2|) - a_2 |Z_2| + \frac{2\mu + \alpha}{\mu + \alpha} \beta(M)S(|Z_3| + |Z_4|).$$

In addition, we have

$$D_+(|Z_3| + |Z_4|) \leq -(2\mu + \gamma + a_2)(|Z_3| + |Z_4|) + \alpha |Z_4|.$$

Choosing $\theta = \frac{a_2}{\alpha}$ and combining (19) and (20) yields that

$$D_+V_1 \leq -[2\mu + \alpha + \beta(M)I](|Z_1| + |Z_2|) - [2\mu + \gamma + a_1] |Z_3| + |Z_4|) + 2\mu + \alpha \beta(M)S(|Z_3| + |Z_4|).$$

Let $\varepsilon > 0$ the constant of the persistence supposed sufficiently small. Since $\gamma + a_2 > \alpha$, then $\gamma + a_2 > \alpha + \varepsilon$. Furthermore, taking $\varepsilon$ small enough such that $\beta(M)I > \varepsilon$, from (21) we infer that

$$D_+V_1 \leq -(2\mu + \gamma + \varepsilon) V_1 + \frac{2\mu + \alpha}{\mu + \alpha} \frac{\beta(M)SI}{E} V_2.$$

On the other hand, the right derivative of $V_2$ is

$$D_+V_2 = \left( \frac{1}{E} \frac{dE}{dt} - \frac{1}{T} \frac{dI}{dt} \right) V_2 + \frac{E}{T} D_+(|Z_3| + |Z_4|),$$

$$\leq \left( \frac{1}{E} \frac{dE}{dt} - \frac{1}{T} \frac{dI}{dt} \right) V_2 - (2\mu + \gamma + a_2) V_2 + \frac{aE}{T} |Z_2|. $$
Now, to estimate $D_+ V$ we will discuss two cases:

**case 1** If $V_2 \leq V_1$, then $V = V_1$ and we have from (22) that

$$D_+ V \leq - (2\mu + \alpha + \varepsilon) V + \frac{2\mu + \alpha}{\mu + \alpha} \frac{\beta(M)S}{E} V.$$ 

Beside, from the $E$-equation, one can write

$$\frac{\beta(M)S}{E} = \frac{1}{E} \frac{dE}{dt} + (\mu + \alpha).$$

Hence

$$D_+ V \leq (K'(t) - \varepsilon)V,$$

where

$$K(t) = \frac{2\mu + \alpha}{\mu + \alpha} \log(E(t)).$$

**case 2** If $V_1 \leq V_2$, then $V = V_2$ and we have from (23) that

$$D_+ V \leq \left( \frac{1}{E} \frac{dE}{dt} - \frac{1}{I} \frac{dI}{dt} \right) V - (2\mu + \gamma + a_1) V + \frac{\alpha E}{I} V.$$ 

Using the $I$-equation, one can write

$$\frac{\alpha E}{I} = \frac{1}{I} \frac{dI}{dt} + (\mu + \gamma).$$

Thereby

$$D_+ V \leq \left( \frac{1}{E} \frac{dE}{dt} - (\mu + a_1) \right) V.$$ 

In both cases we have

$$D_+ V \leq (K'(t) - \varepsilon)V.$$ 

Thus, according to 3.6, we conclude that the unique interior equilibrium $E^*$ of the system (3) is globally asymptotically stable in $\Omega$. □

### 3.3. Local stability and Hopf-bifurcation at $E^*$

Having established the stability of the endemic state $E^*$ for $\tau > 0$, the next step in our analysis is to investigate whether this steady state can lose its stability for $\tau > 0$. By introducing $U_1 = S - S^*, U_2 = E - E^*, U_3 = I - I^*$ and $U_4 = M - M^*$, the system (3) becomes

\[
\begin{align*}
\frac{dU_1(t)}{dt} &= -(\mu + I^* \beta(M^*))U_1(t) - S^* \beta(M^*)U_3(t) - S^* I^* \beta'(M^*)U_4(t) - \beta(M^*)U_1(t)U_3(t) \\& - \beta'(M^*) (I^*U_1(t) + S^*U_3(t)) U_4(t) - \frac{S^* I^* \beta^2(M^*)}{2} U_1^2(t) - \beta'(M^*) U_1(t)U_3(t)U_4(t) \\& - \frac{\beta^2(M^*)}{2} (S^*U_3(t) + I^*U_1(t)) U_4^2(t) - \frac{\beta^2(M^*)}{2} U_1(t)U_3(t)U_4^2(t) + h.o.t., \\
\frac{dU_2(t)}{dt} &= I^* \beta(M^*)U_1(t) - (\mu + a_2)U_2(t) + S^* \beta(M^*)U_3(t) + S^* I^* \beta'(M^*)U_4(t) + \beta(M^*)U_1(t)U_3(t) \\& + \beta'(M^*) (I^*U_1(t) + S^*U_3(t)) U_4(t) + \frac{S^* I^* \beta^2(M^*)}{2} U_1^2(t) + \beta'(M^*) U_1(t)U_3(t)U_4(t) \\& + \frac{\beta^2(M^*)}{2} (S^*U_3(t) + I^*U_1(t)) U_4^2(t) + \frac{\beta^2(M^*)}{2} U_1(t)U_3(t)U_4^2(t) + h.o.t., \\
\frac{dU_3(t)}{dt} &= \alpha U_2(t) - (\mu + \gamma)U_3(t), \\
\frac{dU_4(t)}{dt} &= a_1 U_3(t) - \tau - a_2 U_4(t).
\end{align*}
\]

The characteristic equation of the system (24) at $(0, 0, 0, 0)$ is given by
Adding up the squares of the corresponding sides of the above equations leads to
\[
\begin{vmatrix}
\lambda + \mu + \beta(M^*)I* & 0 & \beta(M^*)S^* & \beta'(M^*)S^*I^*\\
-\beta(M^*)I^* & \lambda + \mu + \alpha & -\beta(M^*)S^* & -\beta'(M^*)S^*I^* \\
0 & -\alpha & \lambda + \mu + \gamma & 0 \\
0 & 0 & -a_1e^{-\lambda \tau} & \lambda + a_2
\end{vmatrix} = 0,
\]
which has the explicit form
\[
\lambda^4 + B_3\lambda^3 + B_2\lambda^2 + B_1\lambda + B_0 + (C_1\lambda + C_0)e^{-\lambda \tau} = 0. \tag{25}
\]

In order for the steady state \(E^*\) to lose its stability when \(\tau\) exceeds 0, some of the eigenvalues determined by the equation (25) must cross the imaginary axis. Obviously, \(i\omega\) (\(\omega > 0\)) is a root of (25) if and only if \(\omega\) satisfies the following equation
\[
\omega^4 + B_3i\omega^3 - B_2\omega^2 + B_1i\omega + B_0 + (C_1i\omega + C_0)(\cos(\omega \tau) - i \sin(\omega \tau)) = 0. \tag{26}
\]
Separating the real and imaginary parts of gives
\[
\begin{cases}
\omega^4 - B_2\omega^2 + B_0 = -C_1 \omega \sin(\omega \tau) - C_0 \cos(\omega \tau), \\
B_3\omega^3 - B_1 \omega = C_1 \omega \cos(\omega \tau) - C_0 \sin(\omega \tau).
\end{cases} \tag{27}
\]
Adding up the squares of the corresponding sides of the above equations leads to
\[
(\omega^4 - B_2\omega^2 + B_0)^2 + (B_3\omega^3 - B_1 \omega)^2 = C_1^2\omega^2 + C_0^2,
\]
which implies
\[
\omega^8 + a\omega^6 + b\omega^4 + c\omega^2 + d = 0, \tag{28}
\]
where
\[
a = B_3^2 - 2B_2, \\
b = 2B_0 + B_2^2 - 2B_1B_1, \\
c = B_1^2 - 2B_0B_2 - C_0^2, \\
d = B_0^2 - C_0^2.
\]
Let \(\Upsilon = \omega^2\), then the equation (28) is equivalent to the following fourth degree polynomial equation
\[
\Upsilon^4 + a\Upsilon^3 + b\Upsilon^2 + c\Upsilon + d = 0. \tag{29}
\]
It is easy to show that if \(d < 0\), then the equation (29) has at least one positive real root. Using the fact that \(\beta(M)\) is a positive decreasing function, we have
\[
B_0 + C_0 = a_2(\mu + \gamma)(\mu + \alpha)\beta(M^*)I^* - a_1\alpha\mu\beta'(M^*)S^*I^* > 0.
\]
Therefore, \(d = (B_0 - C_0)(B_0 + C_0)\) is strictly negative if and only if \(B_0 - C_0 < 0\), which is insured if \(\beta^2(M^*) < -\frac{m_{\alpha}}{\alpha^2}\beta'(M^*)\). Now, we investigate the existence of positive roots of (29) when \(d \geq 0\). Denote \(h(\Upsilon) = \Upsilon^4 + a\Upsilon^3 + b\Upsilon^2 + c\Upsilon + d\).

Then \(h'(\Upsilon) = 4\Upsilon^3 + 3a\Upsilon^2 + 2b\Upsilon + c\).

Let \(y = \Upsilon + \frac{a}{4}\), then the equation \(h'(\Upsilon) = 0\) becomes
\[
y^3 + py + q = 0,
\]
where \(p = \frac{b}{2} - \frac{3}{16}a^2\), \(q = \frac{a^3}{32} - \frac{ab}{8} + c/4\).

Define \(D = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 = \frac{-\Delta}{27}\), where \(\Delta\) is the discriminant of the equation (30). Using the Cardano formula for the three-degree algebraic equation (30), the solutions are given by
\[
y_k = j^k \sqrt[3]{-\frac{q}{2} + \sqrt[3]{\frac{-q^2}{4} - \frac{-p}{2}}} + j^{-k} \sqrt[3]{-\frac{q}{2} - \sqrt[3]{\frac{-q^2}{4} - \frac{-p}{2}}}, \quad k = 0, 1, 2
\]
where \(j = e^{\frac{2\pi}{3}} = \frac{-1}{2} + i\frac{\sqrt{3}}{2}\). Define
\[
\Upsilon_k = -\frac{a}{4} + j^k \sqrt[3]{\frac{q}{2} + \sqrt[3]{\frac{-q^2}{4} - \frac{-p}{2}}} + j^{-k} \sqrt[3]{\frac{q}{2} - \sqrt[3]{\frac{-q^2}{4} - \frac{-p}{2}}}, \quad k = 0, 1, 2.
\]
We have three cases to discuss:

1) Assume \(D < 0\), the equation \(h'(\Upsilon) = 0\) has three distinct real roots:
\[
\Upsilon_k = -\frac{a}{4} + 2 \text{Re} \left( j^k \sqrt[3]{\frac{-q}{2} + i\sqrt[3]{D}} \right),
\]
which can be rewritten after taking \(j^k u = \sqrt[3]{\frac{-q}{2} + i\sqrt[3]{D}}\) in the trigonometric form as follows:
\[
\Upsilon_k = -\frac{a}{4} + 2 \sqrt[3]{\frac{-p}{3}} \cos \left( \frac{1}{3} \arccos \left( \frac{-q}{2} \sqrt[3]{\frac{-27}{p^3}} + \frac{2k\pi}{3} \right) \right), \quad k = 0, 1, 2.
\]
\(\Upsilon_0, \Upsilon_1, \Upsilon_2\) are the extreme points of \(h(\Upsilon)\). Put
\[
\Upsilon^* = \max \{\Upsilon_0, \Upsilon_1, \Upsilon_2\}, \quad \Upsilon_* = \min \{\Upsilon_0, \Upsilon_1, \Upsilon_2\}.
\]
Note that \(h(\Upsilon)\) is strictly monotonously increasing when \(\Upsilon > \Upsilon^*\), and we have the following sub-cases:

- If \(\Upsilon^* \leq 0\), then \(h(\Upsilon) > h(0) = d \geq 0\), which means that \(h(\Upsilon)\) doesn’t admit any positive real root.
- If \(\Upsilon_* < 0 < \Upsilon^*\). By studying the sign of \(h'\), we deduce that \(h\) attains its minimum at 0 or \(\Upsilon^*\) according to the sign of \(h'(0) = c\). Since \(h(0) = d \geq 0\) and \(h(\infty) = \infty\), the polynomial \(h\) has at least one positive root if an only if \(h(\Upsilon^*) \leq 0\).
- If \(\Upsilon_* > 0\). In this case, the global minimum of \(h\) is \(\min \{h(\Upsilon_*), h(\Upsilon^*)\}\). Thus, \(h\) has at least one positive root if an only if \(\min \{h(\Upsilon_*), h(\Upsilon^*)\} \leq 0\).
2) Suppose $D = 0$, then the equation $h'(Υ) = 0$ has two real roots one simple and the other one is double given as follows.

$$Υ_0 = -\frac{a}{4} + 2\sqrt{-\frac{q}{2}}, \quad Υ_2 = Υ_1 = -\frac{a}{4} - \frac{3\sqrt{-q}}{2}.$$ 

Similarly to the first case, we can show that $h(Υ)$ has positive real roots if and only if $Υ_1 ≤ Υ_0$, $h(Υ_0) ≤ 0$ or $Υ_1 > 0$, and $\min(h(Υ_0), h(Υ_1)) ≤ 0$.

3) Suppose $D > 0$, then the equation has only one positive real root:

$$Υ_0 = -\frac{a}{4} + \frac{\sqrt{-q}}{2} + \sqrt{D} + \frac{\sqrt{-q}}{2} - \sqrt{D},$$

in this case one can show in a similar way as the approach above that $h(Υ)$ has a positive real root if and only if $Υ_0 > 0$ and $h(Υ_0) ≤ 0$.

We summarize the above results in the following lemma:

**Lemma 3.8.** The equation (29) has at least one positive real root if and only one of the following conditions holds:

i) $d < 0$.

ii) $d ≥ 0$, $D > 0$, $Υ_0 > 0$, and $h(Υ_0) ≤ 0$.

iii) $d ≥ 0$ and $D ≤ 0$, $Υ_0 ≤ 0 < Υ^*$, and $h(Υ^*) ≤ 0$.

iv) $d ≥ 0$ and $D ≤ 0$, $Υ_0 > 0$, and $\min(h(Υ^*, h(Υ_0))) ≤ 0$, where $Υ^*$ and $Υ_0$ are defined by (31).

In what follows we set the conditions

(H3) $β^2(M^*) < \frac{-μ_{a2}}{μ_{a2}} β^2(M^*)$,

(H4) $β^2(M^*) ≥ \frac{-μ_{a2}}{μ_{a2}} β^2(M^*)$, $D > 0$, $Υ_0 > 0$, and $h(Υ_0) ≤ 0$,

(H5) $β^2(M^*) ≥ \frac{-μ_{a2}}{μ_{a2}} β^2(M^*)$, $D ≤ 0$, $Υ_0 ≤ 0 < Υ^*$, and $h(Υ^*) ≤ 0$.

(H6) $β^2(M^*) ≥ \frac{-μ_{a2}}{μ_{a2}} β^2(M^*)$, $D ≤ 0$, $Υ_0 > 0$, and $\min(h(Υ^*, h(Υ_0))) ≤ 0$.

We know that if one of the above conditions holds, the equation (28) has at least one positive real root such that the equation (25) has a pair of purely imaginary roots.

Without loss of generality one can assume that the equation (28) has 8 different positive real roots $ω_i$, $i = 1, 2, ... , 8$, for each of those roots we can solve the system of equations (27) to find the corresponding critical value of the the delay

$$τ_{n,i} = \frac{1}{ω_i} \left[ \cos^{-1} \left( \frac{(A_3C_1 - C_0)ω_i + (B_3C_0 - B_1C_1)ω_i^2 - B_0C_0}{C_0^2 + C_1^2ω_i^2} \right) + 2nπ \right], \quad n ∈ \mathbb{N}, i = 1, 2, ..., 8.$$ 

Define

$$τ_0 = τ_{n_0,i_0} = \min \{ τ_{n,i} \}, \quad ω_0 = ω(τ_0).$$

By Butler’s lemma [36] the endemic equilibrium remains stable for $τ < τ_0$. In order to establish whether the Hopf bifurcation actually occurs at $τ = τ_0$, one has to determine the sign of $\frac{dRe(λ(τ_0))}{dτ}$. Differentiating the characteristic equation with respect to $τ$, we get

$$[4λ^3 + 3B_3λ^2 + 2B_2λ + B_1] \frac{dλ}{dτ} + C_1e^{-λτ} \frac{dλ}{dτ} - \left( λ + τ \frac{dλ}{dτ} \right) [C_1λ + C_0] e^{-λτ} = 0.$$
This gives
\[ \left[ \frac{d\lambda}{d\tau} \right]^{-1} = \frac{C_1 e^{-\lambda \tau} - (4\lambda^3 + 3B_3\lambda^2 + 2B_2\lambda + B_1)}{(C_1\lambda^2 + C_0\lambda)e^{-\lambda \tau}} - \frac{\tau}{\lambda} \]
\[ = \frac{C_1}{C_1\lambda^2 + C_0\lambda} - \frac{4\lambda^3 + 3B_3\lambda^2 + 2B_2\lambda + B_1}{(C_1\lambda^2 + C_0\lambda)e^{-\lambda \tau}} - \frac{\tau}{\lambda} \]
\[ = \frac{C_0}{\lambda^2(C_1\lambda + C_0)} - \frac{3\lambda^4 + 2B_3\lambda^3 + B_2\lambda^2 - B_0}{\lambda^2(\lambda^4 + B_3\lambda^3 + B_2\lambda^2 + B_1\lambda + B_0)} - \frac{\tau}{\lambda}. \]

Therefore
\[
\text{sign} \left\{ \frac{d(\text{Re}\lambda)}{d\tau} \right\}_{\tau=\tau_0} = \text{sign} \left\{ \text{Re} \left[ \frac{d\lambda}{d\tau} \right]^{-1} \right\}_{\lambda_0=\omega_0, \tau=\tau_0}
\]
\[ = -\text{sign} \left\{ \text{Re} \left[ \frac{C_0}{\lambda^2(C_1\lambda + C_0)} + \frac{3\lambda^4 + 2B_3\lambda^3 + B_2\lambda^2 - B_0}{\lambda^2(\lambda^4 + B_3\lambda^3 + B_2\lambda^2 + B_1\lambda + B_0)} \right] \right\}_{\lambda_0=\omega_0}
\]
\[ = \text{sign} \left\{ \frac{3\omega^4 + 2a\omega^3 + 2b\omega^2 + d}{\omega^2_0(B_1\omega_0^2 + C_0)} \right\}
\]
\[ = \text{sign} \left\{ \frac{4\omega^4 + 2a\omega^3 + 2b\omega^2 + c\omega_0^2}{\omega^2_0(B_1\omega_0^2 + C_0)} \right\}
\]
\[ = \text{sign} \left\{ \frac{h'(\omega_0)}{B_1\omega_0^2 + C_0} \right\}
\]
\[ = \text{sign} \left\{ h'(\omega_0) \right\}. \]

Hence, if \( h'(\omega_0) > 0 \) by continuity, the real part of \( \lambda(\tau) \) becomes positive when \( \tau > \tau_0 \) and the steady state becomes unstable. Moreover, a Hopf bifurcation occurs when \( \tau \) passes through the critical value \( \tau_0 \). Now we can state the following theorem:

**Theorem 3.9.** Suppose that \( R_a > 1 \) and the condition \( (A\tau) \) of the theorem 3.6 holds, then the following results are true

**i)** if none of the conditions \( (H_3), (H_4), (H_5), (H_6) \) is satisfied, then the endemic steady state \( E^* \) of the model (3) is locally asymptotically stable for all \( \tau \geq 0 \).

**ii)** if one of the conditions \( (H_3), (H_4), (H_5) \) or \( (H_6) \) holds then the positive equilibrium of the model (3) is stable when \( \tau \in [0, \tau_0) \), and unstable when \( \tau > \tau_0 \). If in addition \( h'(\omega_0) > 0 \), the system (3) undergoes a Hopf bifurcation at \( E^* \) when \( \tau = \tau_0 \), and a family of periodic solutions bifurcates from the positive equilibrium near \( \tau = \tau_0 \).

### 3.4. Stability and direction of the Hopf-bifurcation

In the previous section we have obtained conditions for Hopf-bifurcation to occur at the endemic equilibrium \( E^* \) when \( \tau \) takes certain critical values. In this section, using the center manifold argument and the normal form theory presented by Hassard et al. [14], we will establish an explicit formula in determining the direction and stability of periodic solution bifurcating from the positive equilibrium at a Hopf bifurcation value, say \( \tau = \tau_0 \). For convenience, we rescale the time by \( t \rightarrow \frac{t}{\tau} \) to normalize the delay, and let \( \tau = \tau_0 + \nu, \nu \in \mathbb{R} \), so that \( \nu = 0 \) is the Hopf bifurcation value of the system (3). Then system (24) can be transformed in the phase space \( C = C([1−, 0], \mathbb{R}^4_+) \) (with the norm \( ||\varphi||_C = \sup_{-1<\theta<0}\varphi(\theta) \)) into an FDE as

\[ U'(t) = L_\nu(U_\nu) + F(\nu, U_\nu), \]

(32)
where, $U(t) = (U_1(t), U_2(t), U_3(t), U_4(t))^T \in \mathbb{R}_+^4$, and $L_\nu : C \to \mathbb{R}_+^4$, $F : \mathbb{R} \times C \to \mathbb{R}_+^4$ are given respectively by

$$L_\nu(\varphi) = (\tau_0 + \nu) \left[ \begin{array}{cccc} -\mu - \beta(M^*)I^* & 0 & -\beta(M^*)S^* & -\beta'(M^*)S^*I^* \\ \beta(M^*)I^* & -\alpha - \mu & \beta(M^*)S^* & -\beta'(M^*)S^*I^* \\ 0 & \alpha & -\gamma - \mu & 0 \\ 0 & 0 & 0 & -a_2 \end{array} \right] \delta(\theta)$$

$$+ (\tau_0 + \nu) \left[ \begin{array}{c} \varphi_1(-1) \\ \varphi_2(-1) \\ \varphi_3(-1) \\ \varphi_4(-1) \end{array} \right]$$

and

$$F(\nu, \varphi) = (\tau_0 + \nu) \left[ \begin{array}{c} F_1(\varphi) \\ 0 \\ 0 \end{array} \right],$$

where

$$F_1(\varphi) = \beta(M^*)\varphi_1(0)\varphi_3(0) + \beta'(M^*) (I^*\varphi_1(0) + S^*\varphi_3(0))\varphi_4(0) + \frac{S^*I^*\beta(2)(M^*)}{2} \varphi_3(0)$$

$$+ \beta'(M^*)\varphi_1(0)\varphi_3(0)\varphi_4(0) + \frac{\beta(2)(M^*)}{2} (S^*\varphi_3(0) + I^*\varphi_1(0))\varphi_4(0)$$

$$+ \frac{\beta(2)(M^*)}{2} \varphi_1(0)\varphi_3(0)\varphi_4(0) + h.o.t,$$

$$F_2(\varphi) = -F_1(\varphi).$$

Clearly $L_\nu$ is a linear continuous operator from $C$ to $\mathbb{R}_+^4$. By the Riesz representation theorem, there exists a matrix whose components are bounded variation functions $\eta(\nu, \theta)$ for $\theta \in [-1, 0]$, such that

$$L_\nu(\varphi) = \int_{-1}^0 d\eta(\theta, \nu) d\varphi(\theta), \quad \text{for } \varphi \in C. \quad (35)$$

In fact, we can choose

$$\eta(\theta, \nu) = (\tau_0 + \nu) \left[ \begin{array}{cccc} -\mu - \beta(M^*)I^* & 0 & -\beta(M^*)S^* & -\beta'(M^*)S^*I^* \\ \beta(M^*)I^* & -\alpha - \mu & \beta(M^*)S^* & -\beta'(M^*)S^*I^* \\ 0 & \alpha & -\gamma - \mu & 0 \\ 0 & 0 & 0 & -a_2 \end{array} \right] \delta(\theta)$$

$$+ (\tau_0 + \nu) \left[ \begin{array}{c} 0 \\ 0 \\ 0 \\ a_1 \end{array} \right] \delta(\theta + 1),$$

where $\delta$ denotes the Dirac delta function. The infinitesimal generator $A$ of the continuous semigroup generated by the linear operator in (32) assumes the form:

$$A(\nu)\varphi = \left\{ \begin{array}{ll} \frac{d\varphi(\theta)}{d\theta}, & \theta \in [-1, 0), \\
\int_{-1}^0 d\eta(s, \nu)\varphi(s), & \theta = 0. \end{array} \right.$$

We further define the generator $R(\nu)$ as:

$$R(\nu)\varphi = \left\{ \begin{array}{ll} 0, & \theta \in [-1, 0) \\
F(\nu, \varphi), & \theta = 0. \end{array} \right.$$
Then, we can transform (32) into an operator equation as follows:

$$U'(t) = A(\nu)U_t + R(\nu)U_t,$$

(37)

where, $U_t(\theta) = U(t + \theta)$, for $\theta \in [-1, 0]$. In the following we set $\nu = 0$.

For $\psi \in C([-1, 0], R^*_+)$, Define

$$A^* \psi = \left\{ \begin{array}{ll} -\frac{d\psi(s)}{ds}, & s \in (0, 1], \\
\int_{-1}^{0} d\eta(t, 0)\psi(-t), & s = 0, \end{array} \right.$$  

and the bilinear form

$$<\psi, \varphi> = \overline{\psi(0)\psi(0)} - \int_{-1}^{0} \int_{0}^{\theta} \psi(s)\varphi(s)ds, s \in [0, 1],$$

where $\eta(\theta) = \eta(\theta, 0)$. Then, $A(0)$ and $A^*$ are adjoint operators. By the discussion in Section 3.3, we know that $\pm i\omega_0\tau_0$ are eigenvalues of $A(0)$. Hence, they are also eigenvalues of $A^*$.

To determine the Poincare normal form of operator $A$, we need to calculate the eigenvector $q$ of $A$ associated with the eigenvalue $i\omega_0\tau_0$ and the eigenvector $q^*$ of $A^*$ associated with eigenvalue $-i\omega_0\tau_0$.

Assume that $q(\theta) = (1, q_1, q_2, q_3)^T e^{i\omega_0\tau_0\theta}$ is the eigenvector of $A(0)$ corresponding to $i\omega_0\tau_0$, then $A(0)q(\theta) = i\omega_0\tau_0q(\theta)$, and from the definition of $A(0)$, (33), (35) and (36) we get

$$\tau_0 \begin{pmatrix}
-\mu - \beta(M^*)I^* & 0 & -\beta(M^*)S^* & -\beta'(M^*)S^*I^* \\
\beta(M^*)I^* & -\alpha - \mu & \beta(M^*)S^* & \beta'(M^*)S^*I^* \\
0 & \alpha & -\gamma - \mu & 0 \\
0 & 0 & 0 & -a_2 \\
\end{pmatrix}
\begin{pmatrix}
q(0) \\
q(-1) = i\omega_0\tau_0q(0). \\
\end{pmatrix}$$

(39)

For $q(-1) = e^{-i\omega_0\tau_0}q(0)$, we obtain

$$\begin{align*}
q_1 &= \frac{(\mu + \beta(M^*)I^* + i\omega_0\tau_0) \left(ia_2i\omega_0\tau_0 + e^{i\omega_0\tau_0}(\gamma + \mu)(a_2 + i\omega_0\tau_0)\right)}{\alpha \beta(M^*)S^*(a_2 + i\omega_0\tau_0)e^{i\omega_0\tau_0} + a_1^2 \beta'(M^*)S^*I^*}, \\
q_2 &= \frac{(a_2 + i\omega_0\tau_0)(\mu + \beta(M^*)I^* + i\omega_0\tau_0)e^{i\omega_0\tau_0}}{\beta(M^*)S^*(a_2 + i\omega_0\tau_0)e^{i\omega_0\tau_0} + a_1^2 \beta'(M^*)S^*I^*}, \\
q_3 &= \frac{(\mu + \beta(M^*)I^*)a_1 + i\omega_0\tau_0a_1}{\beta(M^*)S^*(a_2 + i\omega_0\tau_0)e^{i\omega_0\tau_0} + a_1^2 \beta'(M^*)S^*I^*}. \\
\end{align*}$$

(40)

Similarly, we can obtain the eigenvector $q^*(s) = \chi(1, q_1^*, q_2^*, q_3^*)e^{-i\omega_0\tau_0s}$ of $A^*$ corresponding to $-i\omega_0\tau_0$, where

$$\begin{align*}
q_1^* &= \frac{(\mu + \beta(M^*)I^* - i\omega_0\tau_0)}{\beta(M^*)I^*}, \\
q_2^* &= \frac{(\alpha + \mu - i\omega_0\tau_0)(\mu + \beta(M^*)I^* + i\omega_0\tau_0)}{\alpha \beta(M^*)I^*}, \\
q_3^* &= \frac{(\mu - i\omega_0\tau_0)\beta'(M^*)S^*I^*}{(a_2 - i\omega_0\tau_0)\beta(M^*)I^*}, \\
\end{align*}$$

(41)
In order to normalize \( q \) and \( q^* \) we need to determine the value of \( \chi \) which satisfies the condition \( <q, q^*> = 1 \). By (38) we have

\[
<q^*(s), q(\theta) > = \chi(1, q_1, q_2, q_3)^T - \int_{-\infty}^{0} \int_{-\infty}^{0} \chi(1, q_1, q_2, q_3) e^{-i\omega_{\gamma_0}(s-\theta)} d\eta(\theta) (1, q_1, q_2, q_3)^T e^{i\omega_{\gamma_0} \tau_0} d\epsilon \\
= \chi(1 + \overline{q_1}q_1 + \overline{q_2}q_2 + \overline{q_3}q_3) - \int_{-\infty}^{0} \chi(1, q_1, q_2, q_3) e^{-i\omega_{\gamma_0}(s-\theta)} d\eta(\theta) (1, q_1, q_2, q_3)^T \\
= \chi \left( 1 + \overline{q_1}q_1 + \overline{q_2}q_2 + \overline{q_3}q_3 + \tau_0 a_1 \overline{q_3} q_2 e^{-i\omega_{\gamma_0} \tau_0} \right).
\]

(42)

Therefore, we can choose \( \chi \) as

\[
\chi = \frac{1}{1 + \overline{q_1}q_1 + \overline{q_2}q_2 + \overline{q_3}q_3 + \tau_0 a_1 \overline{q_3} q_2 e^{-i\omega_{\gamma_0} \tau_0}}.
\]

Next, we construct the coordinates of the center manifold \( C_0 \) at \( \nu = 0 \) which is a local invariant, attracting, four-dimensional manifold [14]. Define

\[
z(t) = <q^*, U_t >, \quad W(t, \theta) = U_t(\theta) - 2 \text{Re} \{ z(t, q(\theta)) \} \tag{43}
\]

On the center manifold \( C_0 \), \( W(t, \theta) = W(z(t), \overline{z}(t), \theta) \), where

\[
W(z, \overline{z}, \theta) = W_{20}(0) \frac{z^2}{2} + W_{11}(0) z \overline{z} + W_{02}(0) \frac{\overline{z}^2}{2} + W_{30}(0) \frac{z^3}{6} + ... \tag{44}
\]

where \( z \) and \( \overline{z} \) are local coordinates for center manifold \( C_0 \) in \( C \) in the direction \( q^* \) and \( \overline{q}^* \). Note that \( W \) is real if \( U_t \) is real. In fact, we shall consider real solutions only. For \( U_t \in C_0 \) solution of (37), since \( \nu = 0 \), we have

\[
z'(t) = <q^*, AU_t + RU_t > \\quad = i \omega_0 \tau_0 z + \overline{q}^*(0) F(0, W(t, 0) + 2 \text{Re} \{ z(t, q(0)) \}) \\
= i \omega_0 \tau_0 z + \overline{q}^*(0) F_0(z, \overline{z}),
\]

which we can write in an abbreviated form as

\[
z'(t) = i \omega_0 \tau_0 z + g(z, \overline{z}), \tag{45}
\]

where

\[
g(z, \overline{z}) = g_{20} \frac{z^2}{2} + g_{11}(\theta) z \overline{z} + g_{02} \frac{\overline{z}^2}{2} + g_{21} \frac{z^2 \overline{z}}{2} + ... \tag{46}
\]

Then,

\[
g(z, \overline{z}) = \overline{q}^*(0) F(0, U_t) \\quad = \tau_k D(1 - \overline{q}^*_1) F_1(U_t), \tag{47}
\]

Where

\[
F_1(U_t) = \beta(M^*) U_{1,t}(0) U_{3,t}(0) + \beta^*(M^*) (I^* U_{1,t}(0) + S^* U_{3,t}(0)) U_{4,t}(0) \\
+ \frac{S^* I^* \beta(2)(M^*)}{2} U_{4,t}^2(0) \\
+ \beta^*(M^*) U_{1,t}(0) U_{3,t}(0) U_{4,t}(0) + \frac{\beta(2)(M^*)}{2} (S^* U_{3,t}(0) + I^* U_{1,t}(0)) U_{4,t}^2(0) \\
+ \frac{\beta(2)(M^*)}{2} U_{1,t}(0) U_{3,t}(0) U_{4,t}^2(0) + h.o.t \tag{48}
\]

On the other hand, equation (43) indicates that

\[
U_t(\theta) = W(z, \overline{z}, \theta) - z q(\theta) - \overline{z} q(\theta).
\]
In fact, it follows together with (44) that
\[
U_{i,t}(0) = W_{20}^{(i)}(0) z^2 e^{-i\omega_0 T_0} + W_{11}^{(i)}(0) z z^* + W_{02}^{(i)}(0) e^{-z^2/2} + z + z + \cdots,
\]
\[
U_{i,t}(0) = W_{20}^{(i)}(0) z^2 e^{-i\omega_0 T_0} + W_{11}^{(i)}(0) z z^* + W_{02}^{(i)}(0) e^{-z^2/2} + q_i z + q_i z + \cdots \quad (i = 2, 3, 4).
\]
(49)
Substituting equations (49) into equation (47) and comparing the coefficients with equation (46), we get
\[
g_{20} = 2\tau_0 q_0 (q_0^2 - 1) \left[ \beta' (M^*) S^* q_0 q_3 + \beta' (M^*) I^* q_3 + \beta (M^*) q_2 + \frac{\beta^2 (M^*)}{4} S^* I^* q_3^2 \right],
\]
\[
g_{11} = 2\tau_0 q_0 (q_0^2 - 1) \left[ \beta' (M^*) S^* (q_3) + \beta (M^*) I^* (q_3) + \frac{\beta^2 (M^*)}{4} S^* I^* (q_3) \right],
\]
\[
g_{02} = 2\tau_0 q_0 (q_0^2 - 1) \left[ \beta' (M^*) S^* (q_3 q_3) + \beta' (M^*) I^* (q_3 q_3) + \beta (M^*) q_2 + \frac{\beta^2 (M^*)}{4} S^* I^* (q_3 q_3) \right],
\]
\[
g_{21} = 2\tau_0 q_0 (q_0^2 - 1) \left[ \beta' (M^*) S^* (W^{(i)}(0) q_3) + \beta (M^*) I^* (W^{(i)}(0) q_3) + \frac{\beta^2 (M^*)}{4} S^* I^* (W^{(i)}(0) q_3)^2 \right],
\]
\[
\quad + \beta' (M^*) I^* (W^{(i)}(0) q_3) + \beta (M^*) q_2 + \frac{\beta^2 (M^*)}{4} S^* I^* (W^{(i)}(0) q_3)^2.
\]
(50)
Since $W_{20}(0)$ and $W_{11}(0)$ are in $g_{21}$, we still need to compute them. Substituting (43) and (37) into $W' = U'_t - z' q - \nabla^2 \phi$ leads to
\[
W' = \begin{cases} 
AW - 2 Re \{q F_0 q(\theta)\}, & \theta \in [-1, 0), \\
AW - 2 Re \{q F_0 q(0)\} + F_0 & \theta = 0,
\end{cases}
\]
(51)
which we rewrite as
\[
W' = AW + H(z, \nabla, \theta),
\]
(52)
where
\[
H(z, \nabla, \theta) = H_{20}(\theta) \frac{z^2}{2} + H_{11}(\theta) z \nabla + H_{02}(\theta) \frac{\nabla^2}{2} + \cdots.
\]
(53)
Expanding the above series and comparing the coefficients, one obtains
\[
(A - 2i\omega_0 \eta_0 I) W_{20}(\theta) = -H_{20}(\theta), \quad AW_{11}(\theta) = -H_{11}(\theta).
\]
(54)
From (51), we know that for $\theta \in [-1, 0)$,
\[
H(z, \nabla, \theta) = -q F_0 q(\theta) - q^* F_0 q(\theta) = -q(z, \nabla) q(\theta) - \nabla^2 \phi(z, \nabla) q(\theta).
\]
Comparing the coefficients with Equation (53) gives
\[
H_{20}(\theta) = -q_{20} q(\theta) - \nabla q_{20} (\theta), \quad H_{11}(\theta) = -q_{11} q(\theta) - \nabla q_{11} (\theta).
\]
(55)
Equations (54) and (55) yield that
\[
W'_{20}(\theta) = 2i\omega_0 \eta_0 W_{20}(\theta) + q_{20} q(\theta) + \nabla q_{20} (\theta).
\]
(56)
Noting that $q(\theta) = q(0)e^{i\omega_0 T_0}$ and solving (56), we deduce
\[
W_{20}(\theta) = \frac{i q_{20}}{\omega_0 T_0} q(0) e^{i\omega_0 T_0} + \frac{i q_{20}}{3\omega_0 T_0} \nabla q(0) e^{-i\omega_0 T_0} + E_1 e^{2i\omega_0 T_0}.
\]
(57)
Similarly, we obtain

\[
W_{11}(\theta) = -\frac{iq_{11}}{\omega_0 \tau_0} q(0) e^{i \omega_0 \tau_0 \theta} + \frac{iq_{11}}{\omega_0 \tau_0} \bar{q}(0) e^{-i \omega_0 \tau_0 \theta} + E_2, \tag{58}
\]

where, \(E_1 = (E_1^{(1)}, E_1^{(2)}, E_1^{(3)}, E_1^{(4)})^T\) and \(E_2 = (E_2^{(1)}, E_2^{(2)}, E_2^{(3)}, E_2^{(4)})^T\) are constant vectors.

In what follows, we shall seek appropriate \(E_1\) and \(E_2\). From the definition of \(A\) and equation (54) we have

\[
\int_{-1}^0 d\eta(\theta) W_{20}(\theta) = AW_{20}(0) = 2i\omega_0 \tau_0 W_{20}(0) - H_{20}(0), \tag{59}
\]

\[
\int_{-1}^0 d\eta(\theta) W_{11}(\theta) = AW_{11}(0) = -H_{11}(0).
\]

From (51), we know that when \(\theta = 0\),

\[
H(z, \tau, \theta) = -\bar{q}(0) F_0 q(0) - q^*(0) \bar{F}_0 \bar{q}(0) = g(z, \tau) q(0) - \bar{g}(\tau, z) \bar{q}(0) + F_0, \tag{60}
\]

that is

\[
H_{20}(0) = \frac{z^2}{2} + H_{11}(0) z \bar{z} + H_{02}(0) \frac{\bar{z}^2}{2} + \cdots = \left( g_{20} \frac{z^2}{2} + g_{11}(\theta) z \bar{z} + g_{02} \frac{\bar{z}^2}{2} + \cdots \right) q(0) + \left( g_{20} \frac{\bar{z}^2}{2} + g_{11}(\theta) z \bar{z} + g_{02} \frac{\bar{z}^2}{2} + \cdots \right) \bar{q}(0) + F_0, \tag{61}
\]

where

\[
F_0 = \begin{bmatrix}
F_1(U_1) \\
-F_1(U_1) \\
0 \\
0
\end{bmatrix}.
\]

Substituting (49) in the expression of \(F_0\), we obtain

\[
F_0 = \tau_0 \begin{bmatrix}
1 \\
-1 \\
0 \\
0
\end{bmatrix} \begin{bmatrix}
1 \\
1 \\
0 \\
0
\end{bmatrix} z \bar{z} + \cdots, \tag{62}
\]

where

\[
\begin{align*}
\varphi_1 &= -\beta^* (M^*)^* S^* q_3 + \beta^* (M^*)^* T^* q_3 + \beta (M^*) q_2 + \frac{\beta (M^*)^*}{4} S^* T^* q_3^2, \\
\varphi_2 &= -\beta^* (M^*)^* S^* \text{Re}(\bar{q}_3) + \beta^* (M^*)^* T^* \text{Re}(\bar{q}_3) + \beta (M^*) \text{Re}(q_2) + \frac{\beta (M^*)^*}{4} S^* T^* \text{Re}(\bar{q}_3). 
\end{align*}
\]

Hence, by (61) we deduce

\[
H_{20}(0) = -g_{20} q(0) - \bar{g}_{02} \bar{q}(0) + 2\tau_0 \begin{bmatrix}
1 \\
-1 \\
0 \\
0
\end{bmatrix}, \tag{63}
\]

and

\[
H_{11}(0) = -g_{11} q(0) - \bar{g}_{11} \bar{q}(0) + 2\tau_0 \begin{bmatrix}
1 \\
-1 \\
0 \\
0
\end{bmatrix}. \tag{64}
\]

Substituting equations (63) and (57) into (59) and noting that

\[
\left( i\omega_0 \tau_0 I - \int_{-1}^0 e^{i\omega_0 \tau_0 \theta} d\eta(\theta) \right) q(0) = 0, \quad \left( -i\omega_0 \tau_0 I - \int_{-1}^0 e^{-i\omega_0 \tau_0 \theta} d\eta(\theta) \right) \bar{q}(0) = 0,
\]

we obtain
we obtain
\[
\left(2i\omega_0\tau_0 I - \int_{-1}^{0} e^{i\omega_0 \tau_0 \theta} d\eta(\theta) \right) E_1 = 2\tau_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.
\]
That is
\[
\begin{bmatrix}
2i\omega_0 + \mu + \beta(M^*)I^* & 0 & \beta(M^*)S^* \\
-\beta(M^*)I^* & 2i\omega_0 + \alpha + \mu & 2i\omega_0 + \gamma + \mu \\
0 & -\alpha & a_1 e^{-2i\omega_0 \tau_0} + 2i\omega_0 + a_2 \\
\end{bmatrix}
E_1 = 2\tau_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.
\]
It follows that
\[
E_1^{(1)} = \frac{2\tau_1}{M_1}
\begin{bmatrix}
1 & 0 & \beta(M^*)S^* & \beta'(M^*)S^*I^* \\
-1 & 2i\omega_0 + \alpha + \mu & -\beta(M^*)S^* & -\beta'(M^*)S^*I^* \\
0 & -\alpha & 2i\omega_0 + \gamma + \mu & 0 \\
0 & 0 & -a_1 e^{-2i\omega_0 \tau_0} & 2i\omega_0 + a_2 \\
\end{bmatrix},
\]
\[
E_1^{(2)} = \frac{2\tau_1}{M_1}
\begin{bmatrix}
2i\omega_0 + \mu + \beta(M^*)I^* & 1 & \beta(M^*)S^* & \beta'(M^*)S^*I^* \\
-\beta(M^*)I^* & -1 & -\beta(M^*)S^* & -\beta'(M^*)S^*I^* \\
0 & 0 & 2i\omega_0 + \gamma + \mu & 0 \\
0 & 0 & -a_1 e^{-2i\omega_0 \tau_0} & 2i\omega_0 + a_2 \\
\end{bmatrix},
\]
\[
E_1^{(3)} = \frac{2\tau_1}{M_1}
\begin{bmatrix}
2i\omega_0 + \mu + \beta(M^*)I^* & 0 & 1 & \beta'(M^*)S^*I^* \\
-\beta(M^*)I^* & 2i\omega_0 + \alpha + \mu & -1 & -\beta'(M^*)S^*I^* \\
0 & -\alpha & 0 & 0 \\
0 & 0 & 2i\omega_0 + a_2 & 0 \\
\end{bmatrix},
\]
\[
E_1^{(4)} = \frac{2\tau_1}{M_1}
\begin{bmatrix}
2i\omega_0 + \mu + \beta(M^*)I^* & 0 & \beta(M^*)S^* & 1 \\
-\beta(M^*)I^* & 2i\omega_0 + \alpha + \mu & -\beta(M^*)S^* & -1 \\
0 & -\alpha & 0 & 0 \\
0 & 0 & a_1 e^{-2i\omega_0 \tau_0} & 2i\omega_0 + a_2 \\
\end{bmatrix},
\]
where
\[
M_1 =
\begin{bmatrix}
2i\omega_0 + \mu + \beta(M^*)I^* & 0 & \beta(M^*)S^* & \beta'(M^*)S^*I^* \\
-\beta(M^*)I^* & 2i\omega_0 + \alpha + \mu & -\beta(M^*)S^* & -\beta'(M^*)S^*I^* \\
0 & -\alpha & 2i\omega_0 + \gamma + \mu & 0 \\
0 & 0 & a_1 e^{-2i\omega_0 \tau_0} & 2i\omega_0 + a_2 \\
\end{bmatrix}.
\]
Similarly by substituting (58) and (64) into (59) we get
\[
\begin{bmatrix}
\mu + \beta(M^*)I^* & 0 & \beta(M^*)S^* & \beta'(M^*)S^*I^* \\
-\beta(M^*)I^* & \alpha + \mu & -\beta(M^*)S^* & -\beta'(M^*)S^*I^* \\
0 & -\alpha & \gamma + \mu & 0 \\
0 & 0 & -a_1 & a_2 \\
\end{bmatrix}
E_2 = 2\tau_2 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}.
Consequently,

\[ E_2^{(1)} = \frac{2\partial_2}{M_2} \begin{pmatrix}
1 & 0 & \beta(M^*)S^* & \beta'(M^*)S^*I^* \\
-1 & \alpha + \mu & -\beta(M^*)S^* & -\beta'(M^*)S^*I^* \\
0 & -\alpha & \gamma + \mu & 0 \\
0 & 0 & -a_1 & a_2
\end{pmatrix}, \]

\[ E_2^{(2)} = \frac{2\partial_2}{M_2} \begin{pmatrix}
\mu + \beta(M^*)I^* & 0 & \beta(M^*)S^* & \beta'(M^*)S^*I^* \\
-\beta(M^*)I^* & 1 & -\beta(M^*)S^* & -\beta'(M^*)S^*I^* \\
0 & \alpha + \mu & 0 & 0 \\
0 & 0 & -\alpha & a_2
\end{pmatrix}, \]

\[ E_2^{(3)} = \frac{2\partial_2}{M_2} \begin{pmatrix}
\mu + \beta(M^*)I^* & 0 & 1 & \beta'(M^*)S^*I^* \\
-\beta(M^*)I^* & \alpha + \mu & 0 & 0 \\
0 & 0 & -\alpha & 0 \\
0 & 0 & \gamma + \mu & 0 \\
0 & 0 & -a_1 & a_2
\end{pmatrix}, \]

\[ E_2^{(4)} = \frac{2\partial_2}{M_2} \begin{pmatrix}
\mu + \beta(M^*)I^* & 0 & \beta(M^*)S^* & 1 \\
-\beta(M^*)I^* & \alpha + \mu & -\beta(M^*)S^* & -1 \\
0 & 0 & -\alpha & 0 \\
0 & 0 & \gamma + \mu & 0 \\
0 & 0 & -a_1 & 0
\end{pmatrix}, \]

with

\[ M_2 = \begin{pmatrix}
\mu + \beta(M^*)I^* & 0 & \beta(M^*)S^* & \beta'(M^*)S^*I^* \\
-\beta(M^*)I^* & \alpha + \mu & -\beta(M^*)S^* & -\beta'(M^*)S^*I^* \\
0 & 0 & -\alpha & 0 \\
0 & 0 & \gamma + \mu & 0 \\
0 & 0 & -a_1 & a_2
\end{pmatrix}. \]

Based on the analysis above, determining \( W_{20}(0) \) and \( W_{11}(0) \) from (57) and (58) allows to compute \( g_{21} \) by (50). Thus, we can compute the following quantities

\[ C_1(0) = \frac{i}{2\omega_0\tau_0} \left[ g_{20}g_{11} - 2|g_{02}|^2 \right] + \frac{g_{21}}{2}, \]

\[ \nu_2 = -\frac{\text{Re} \{C_1(0)\}}{\text{Re} \{\lambda'(\tau_0)\}}, \]

\[ \beta_2 = 2\text{Re} \{C_1(0)\}, \]

\[ T_2 = -\frac{\text{Im} \{C_1(0)\} + \nu_2\text{Im} \{\lambda'(\tau_0)\}}{\omega_0\tau_0}, \]

which determine the quantities of bifurcating periodic solutions in the center manifold \( C_0 \) at the critical value \( \tau_0 \). Hence, by the result of Hassard et al.[14], we have the following theorem

**Theorem 3.10.** (i) the sign of \( \nu_2 \) determines the directions of the Hopf bifurcation: if \( \nu_2 > 0 (\nu_2 < 0) \), then the Hopf-bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist for \( \tau > \tau_0 (\tau < \tau_0) \);

(ii) The sign of \( \beta_2 \) determines the stability of the bifurcating periodic solutions: the bifurcating periodic solutions on the center manifold \( C_0 \) are stable (unstable) if \( \beta_2 < 0 (\beta_2 > 0) \);
(iii) $T_2$ determines the period of the bifurcating periodic solutions: the period increases (decreases) if $T_2 > 0$ ($T_2 < 0$).

4. Numerical simulations. In this section we perform numerical experiments to further analyze and confirm our theoretical results. The numerical simulations on the epidemic model (3) have been carried out by considering two specific contact rates $\beta(M) = \beta_0 - \frac{\beta_1 M}{p + M}$ [23] and $\beta(M) = \beta_0 e^{-\delta M}$ [15]. In what follows we present the numerical findings for each of the chosen contact rates apart as examples.

Example 1. Let $\beta(M) = \beta_0 - \frac{\beta_1 M}{p + M}$, where $\beta_0$ is the contact rate in the absence of any budget allocated to control the disease through awareness, $\beta_1$ is the efficacy of budget allocation to reduce the contact between the susceptible and infected individuals and the half saturation constant $p$. As mentioned in [23] this saturated type of functional reflects the fact that the amount of the budget used for the implementation of awareness programs has a limited impact on reducing the contact rate which makes its use plausible. With this contact rate, we first analyze the sensitivity of the basic reproductive number to the controlling parameters related to the funds intended for awareness programs. To do so, we compute the derivatives of $R_a$ with respect to these parameters

$$\frac{\partial R_a}{\partial M_0} = - \frac{\alpha \beta_1 p}{(\mu + \alpha)(\mu + \gamma)(p + M_0)^2},$$

$$\frac{\partial R_a}{\partial \beta_1} = - \frac{\alpha M_0}{(\mu + \alpha)(\mu + \gamma)(p + M_0)}.$$

Hence, $R_a$ decreases as the the baseline budget always available for awareness $M_0$ as well as the efficacy of the allocations to reduce the contact $\beta_1$ increase. To visualize this we have plotted the variation of the basic reproductive number with respect to $M_0$ and $\beta_1$ (see figure 1).

![Figure 1. Tendency of the basic reproductive number with $a_0$ and $\beta_1$ for $\beta(M) = \beta_0 - \frac{\beta_1 M}{p + M}$](image-url)
The parameter values in the numerical simulations for the first example are taken as follows:

| Parameters | $\mu$ | $\beta_0$ | $\beta_1$ | $p$ | $\alpha$ | $\gamma$ | $M_0$ | $a_1$ | $a_2$ |
|------------|-------|-----------|-----------|-----|----------|----------|-------|-------|-------|
| Values     | 0.014 | 0.16      | 0.12      | 2.8 | 0.83     | 0.0601   | 1     | 2.38  | 0.012 |

\(\text{Table 1. Table of the parameter values used in the numerical simulations for } \beta(M) = \beta_0 - \frac{\beta_1 M}{p + M}.\)

To manifest the awareness allocated budget, we sketched a comparison plot of the dynamics of the infected individuals in the absence and presence of budget allocation as shown in figure 2, where we can observe how the awareness may be beneficial in reducing the level of infection. In figure 3, we present the variation of \(I(t)\) for different values of $\beta_1$, $a_1$ and $p$, and as it can be seen, increasing the values of $\beta_1$ and $a_1$ allow the endemic level to be controlled to a much lower level, while the number of the infected individuals increases when the value of $p$ increases.

\[\text{Figure 2. Comparison of infected individuals with respect to time in absence and presence of budget allocation, for } \beta(M) = \beta_0 - \frac{\beta_1 M}{p + M} \text{ (for } \tau = 0).\]
Figure 3. variation of I with respect to time for different $a_1$, $p$, $\beta_1$, for $\beta(M) = \beta_0 - \frac{\beta_1 M}{p + M}$.

It is to be noted that all the previous simulations were considered in the absence of delay ($\tau = 0$) and we can check that with the chosen set of parameters, the condition of the stability of the positive equilibrium $E^*$ cited in the theorem 3.5 holds. Now, to investigate numerically the effect of the delay in reporting the infected cases, we treat the roots of the equation (29). It is found that it admits exactly one positive real root $1.9321e^{-04}$, which gives that the characteristic equation (25) has a pair of purely imaginary roots $\pm 0.0139i$, and the critical value of the delay is found to be $\tau_0 = 79.9$. For $\tau = 69 < \tau_0$, the solutions of the system (3) are stable around the endemic equilibrium as shown in figure 4, however, once the delay exceeds the critical value $\tau_0$ the endemic steady state becomes unstable and the system (3) exhibits stable periodic solutions as illustrated in figure 5, which confirms the results of the theorem 3.9.
Figure 4. Variation of $S(t)$, $E(t)$, $I(t)$, $M(t)$ and the phase portrait in $M - I - E$ and $M - I - S$ spaces for $\tau = 69 < \tau_0 = 79.996$ and $\beta(M) = \beta_0 - \frac{\beta_1 M}{p + M}$, which shows that the system (3) is stable around the endemic equilibrium.
Figure 5. Variation of \( S(t) \), \( E(t) \), \( I(t) \), \( M(t) \) and the phase portrait in \( M - E - I \) and \( M - S - I \) spaces for \( \tau = 86 > \tau_0 \) and \( \beta(M) = \beta_0 - \frac{\beta_1 M}{p + M} \), which depicts that the system (3) exhibit a stable periodic behavior around the endemic equilibrium.
Example 2. We consider here the contact rate \( \beta(M) = \beta_0 e^{-\delta M} \), where \( \delta \) determines how effective is the budget allocated for awareness in influencing the transmission rate of the disease under consideration. It is easy to see that the basic reproductive number \( R_a = \frac{\alpha \beta_0 e^{-\delta M_0}}{(\mu + \alpha)(\mu + \gamma)} \) decreases as much as \( \delta \) and \( M_0 \) increase. We adopt the following set of parameters:

| Parameters | \( \mu \) | \( \beta_0 \) | \( \delta \) | \( \alpha \) | \( \gamma \) | \( M_0 \) | \( a_1 \) | \( a_2 = 0.04 \) |
|------------|-----------|--------------|-------------|-------------|-------------|-----------|--------|-------------|
| Values     | 0.15      | 0.29         | 0.3         | 0.12        | 0.0108      | 0.4       | 0.89   | 0.04        |

**Table 2.** Table of the parameter values used in the numerical simulations for \( \beta(M) = \beta_0 e^{-\delta M} \).

As in the previous example, in order to see the influence of the budget allocation devoted to awareness programs, we plot the variation of the infectious population with respect to time in the presence and absence of budget in figure 7 and for different values of parameters \( \delta \), \( M_0 \) and \( a_1 \) in figure 8. From these figures it can be seen that the awareness significantly delayed the epidemic peak and decreases the intensity of the outbreak.
Figure 7. Comparison of infected individuals with respect to time \( t \) in absence and presence of budget allocation, for \( \beta(M) = \beta_0 e^{-\delta M} \) and \( \tau = 0 \).

Figure 8. Variation of \( I \) with respect to time for different \( \delta, M_0, a_1 \) in the absence of delay \( (\tau = 0) \) with \( \beta(M) = \beta_0 e^{-\delta M} \).
To assess the impact of the delay in reporting the infected cases on the dynamics of the model 3, by keeping the same set of parameters we find that the characteristic equation (25) has the pair of purely imaginary roots $\pm 0.0275i$, to which it is associated the critical delay value $\tau_0 = 46.0022$.

The figures 9 and 10 show that increasing the delay $\tau$ over the critical delay value $\tau_0$ destabilize the positive steady state and give rise to the appearance of periodic solutions and a stable limit cycle which support the theoretical findings.

**Figure 9.** Variation of $S(t)$, $E(t)$, $I(t)$, $M(t)$ and the phase portrait in $I - S - M$ and $M - E - I$ spaces for $\tau = 39 < \tau_0 = 79.996$ and $\beta(M) = \beta_0 e^{-\delta M}$, which shows that the system (3) is stable around the endemic equilibrium
Figure 10. Variation of $S(t)$, $E(t)$, $I(t)$, $M(t)$ and the phase portrait in $M - E - I$ and $I - S - M$ spaces for $\tau = 55 > \tau_0$ and $\beta(M) = \beta_0 e^{-\delta M}$, which depicts that the system (3) exhibit a periodic behavior around the endemic equilibrium.
5. Conclusion. This paper has addressed the impact of delay in providing funds allocated to awareness on epidemic outbreaks. Our analysis is based on an SEIR model where the budget allocation to warn people is considered as a new dynamical variable and the transmission rate is presented by a general decreasing function of the availability of funds. The existence and stability of equilibrium states were derived in terms of the basic reproductive number $R_a$. It is shown that the disease-free equilibrium is globally asymptotically stable when $R_a$ is less or equal to 1 for all time delays $\tau \geq 0$, indicating that time delay does not impact on the stability property of this equilibrium. When $R_a > 1$, the disease-free equilibrium becomes unstable and a unique endemic equilibrium exists. By mean of the geometric approach based on the third additive compound matrix, the positive equilibrium is proved to be globally asymptotically stable under the condition (Aτ) in the absence of delay. However, the introduction of time delay changes the dynamic of the system as the delay parameter crosses a critical threshold and leads to the occurrence of Hopf-bifurcation when the condition ii) of the theorem 3.9 is verified.

By applying the normal form theory and the center manifold theorem, we have defined the explicit formulae that determine the stability and direction of the bifurcating periodic solutions.

To better understand and illustrate the theoretical results, numerical simulations were provided by choosing two specific forms of the transmission rate namely, $\beta(M) = \beta_0 - \frac{\beta_1 M}{p + M}$ and $\beta(M) = \beta_0 e^{-\delta M}$. It is shown that providing funds to warn people can be beneficial in reducing the level of prevalence of the disease. Yet, it may remain endemic in the region if the efficiency of the budget allocation to
reduce the contact rate via propagating awareness is not strong enough. On the other hand, the endemic equilibrium loses its stability and an oscillatory behavior is observed when the delay in providing funds exceeds a critical value. Stable limit cycles appear in both examples for the chosen sets of parameters.

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