CANONICAL NONLINEAR CONNECTIONS
ON JET BUNDLES OF FIRST ORDER

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Abstract

The aim of this paper is to open the problem of finding of a nonlinear connection \( \Gamma = (M^{(i)}{}_{\alpha \beta}, N^{(i)}{}_{\alpha j}) \) on 1-jet bundle \( J^1(T, M) \), which to be canonically produced from a given Kronecker \( h \)-regular fundamental vertical metrical d-tensor \( G^{\alpha \beta}{}_{ij} \), possibly provided by multi-time dependent quadratic Lagrangians coming from various branches of theoretical physics: bosonic string theory \( \mathbb{1} \), magneto-hydrodynamics \( \mathbb{2} \), electrodynamics \( \mathbb{3} \) or elasticity \( \mathbb{4} \). From geometrical point of view, the importance of this problem comes from contravariant Riemann-Lagrange geometry of 1-jet spaces \( \mathbb{5} \), and consists in the possibility of construction of distinguished 1-forms \( \delta x^{\alpha} = dx^{\alpha} + M^{(i)}{}_{\alpha \beta} dt^{\alpha} + N^{(i)}{}_{\alpha j} dx^{j} \), necessary for simple local descriptions of geometrical or physical objects studied.

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1 Geometrical and physical aspects

From a physical point of view, we underline that the jet fibre bundle of order one \( J^1(T, M) \) appears as a natural house for geometrical studies of important physical domains (continuum mechanics \( \mathbb{5} \), quantum field theories \( \mathbb{2} \), generalized multi-time relativity and electromagnetism \( \mathbb{3} \) or dynamical relativistic multi-time optics \( \mathbb{8} \)) including natural processes characterized by dependence on position, multi-momentum or partial velocities. At the same time, geometrical studies of first order quadratic Lagrangians from several important branches of theoretical physics were required a profound analysis of the differential geometry of 1-jet spaces \( \mathbb{5} \), and consists in the possibility of construction of distinguished 1-forms \( \delta x^{\alpha} = dx^{\alpha} + M^{(i)}{}_{\alpha \beta} dt^{\alpha} + N^{(i)}{}_{\alpha j} dx^{j} \), necessary for simple local descriptions of geometrical or physical objects studied.

(1.1) \[ G^{\alpha \beta}{}_{ij}(t^{\gamma}, x^{k}, x^{k}_{\gamma}) = h^{\alpha \beta}(t^{\gamma})g_{ij}(t^{\gamma}, x^{k}, x^{k}_{\gamma}), \]

This geometry is naturally produced by a given fundamental vertical metrical d-tensor having the Kronecker \( h \)-regular product form
Remarks 1.1 i) The preceding Kronecker $h$-regular vertical metrical d-tensor may be provided by particular multi-time dependent quadratic Lagrangian functions $L$ on 1-jet bundle $J^1(T, M)$, via the formula $G^{(i)\beta}_{(j)\alpha} = (1/2)(\partial^2 L/\partial x^i_\alpha \partial x^j_\beta)$.

ii) In our opinion, the metrical d-tensor (1.1) may be viewed as an unified gravitational field on $J^1(T, M)$, realized by a temporal gravitational field $h_{\alpha\beta}(t)$ and a spatial gravitational field $g_{ij}(t^\gamma, x^k_\alpha)$ depending on multi-moment, position and partial velocities $x^\alpha_i$. Moreover, the $x^\alpha_i$-dependence may be combined with the abstract concept of partial directions anisotropy.

Concerning the novelty brought in theoretical-physics, we consider that the main and unpublished feature of Riemann-Lagrange geometry on $J^1(T, M)$ is the construction of a large geometrical background for a generalized multi-time field theory, in the sense of generalized Maxwell and Einstein equations, that allows the including of famous equations of mathematical-physics (classical Maxwell or Einstein equations) as particular cases. We recall that the construction of a new field theory, described in multi-time terms on $J^1(T, M)$, was required by geometrical studies of certain famous relativistic invariant equations involving many time variables (chiral fields, sine-Gordon), and of KP-hierarchy of integrable systems in which the arbitrary variables $t^\alpha$ and $t^\beta$ are quite equal in rights and there is no reason to prefer one to another by choosing it as time [1].

From geometrical point of view, a central role in Riemann-Lagrange differential geometry is played by nonlinear connections $\Gamma = (M^{(i)}_{(\alpha)\beta}, N^{(i)}_{(\alpha)j})$ on $J^1(T, M)$, that allow the construction of adapted bases of vector or covector fields [5].

Remark 1.2 Concerning the physical aspects of nonlinear connections on 1-jet spaces, we believe that these prescribe possible intrinsic interactions between the temporal and spatial gravitational fields.

In our opinion, the importance of adapted bases attached to nonlinear connections on $J^1(T, M)$ may be justified in various ways.

Firstly, the simple tensorial local transformations of their elements [10] imply simple adapted local descriptions of geometrical objects with physical meaning studied in Riemann-Lagrange geometry. At the same time, these allow beautiful and natural adapted local descriptions of the fundamental equations (generalized Maxwell or Einstein equations [5]) that dominate the abstract multi-time field theory created by Riemann-Lagrange geometrical instruments.

Secondly, the use of adapted bases produced by canonical nonlinear connections on 1-jet spaces (i. e., which are built from the given multi-time dependent Lagrangian function $L$ or the fundamental vertical metrical d-tensor (1.1)) offers a metrical character to Riemann-Lagrange theory of multi-time physical fields, according to field theories classification from [2]. In this direction, we point out that the construction of canonical nonlinear connections on $ML^n_p$ spaces [5] is a solved problem, while the construction of canonical nonlinear connections on $GML^n_p$ spaces [5] is still an open problem. In what follows, we try to present a deep exposition of main geometrical results already obtained, together with possible future directions of research, concerning the construction of canonical nonlinear connections on $ML^n_p$ or $GML^n_p$ spaces.

In this way, let us consider the first jet fibre bundle $J^1(T, M) \rightarrow T \times M$, whose local coordinates $(t^\alpha, x^i, x^\alpha_i)$, where $\alpha = \overline{1,p}$, $i = \overline{1,n}$, transform by the rules.
Definition 1.1 A pair $\Gamma = (M(t), N)$ consisting of local functions on 1-jet bundle $E = J^1(T, M)$, whose transformation rules are given by

$$
\tilde{M}^{(j)}_{(\beta)\mu} \frac{\partial \tilde{t}^\mu}{\partial t^\alpha} = M^{(k)}_{(\gamma)\alpha} \frac{\partial \tilde{t}^\gamma}{\partial t^\beta} - \frac{\partial \tilde{x}^i}{\partial t^\beta} = N^{(k)}_{(\gamma)\alpha} \frac{\partial \tilde{x}^i}{\partial t^\beta} - \frac{\partial \tilde{x}^\gamma}{\partial x^\beta},
$$

is called a nonlinear connection on $E$. The components $M^{(i)}_{(\alpha)\beta}$ (resp. $N^{(i)}_{(\alpha)\beta}$) are called the temporal (resp. spatial) components of the nonlinear connection $\Gamma$.

Example 1.1 If $h_{\alpha\beta}(t^\mu)$ (resp. $\varphi_{ij}(x^m)$) is a semi-Riemannian metric on the temporal (resp. spatial) manifold $T$ (resp. $M$), and $H^\gamma_{\alpha\beta}(t^\mu)$ (resp. $\gamma^k_{ij}(x^m)$) are their Christoffel symbols, then the pair of local functions $\Gamma_0 = (M^{(j)}_{(\beta)\alpha}, N^{(j)}_{(\beta)\alpha})$, where

$$
(1.2) \quad 0^{(j)}_{(\mu)\alpha} = -H^\gamma_{\alpha\beta} x^\gamma_j, \quad 0^{(j)}_{(\beta)\alpha} = \gamma^j_{ik} x^k_{\beta},
$$

represents a nonlinear connection on $E$, which is called the canonical nonlinear connection of the semi-Riemannian metrics $h_{\alpha\beta}$ and $\varphi_{ij}$.

In this context, an important geometrical concept used in our studies, whose physical meaning is intimately connected by the concept of energy, is introduced by

Definition 1.2 A smooth map $f \in C^\infty(T, M)$, whose local components verify the PDEs system of order two

$$
h^{\alpha\beta}(x^i, \gamma x^i, g_{ij}(\gamma, x^k)x^i_{\alpha}x^j_{\beta} + U_{(i)}^\alpha(\gamma, x^k)x^i_{\alpha} + F(\gamma, x^k), \quad p \geq 2,
$$

where $U_{(i)}^\alpha(\gamma, x^k)$ is a d-tensor on $J^1(T, M)$ and $F$ is a smooth function on $T \times M$.

2 Canonical nonlinear connections on $ML^p_n$ spaces

Concerning the construction of a canonical nonlinear connection $\Gamma_L$ from a given multi-time dependent Lagrangian $\mathcal{L} = L\sqrt{|h|}$, let us consider the $ML^p_n$ space (i. e., the metrical multi-time Lagrange space) $ML^p_n = (J^1(T, M), L)$, where $\dim T = p$, $\dim M = n$, whose multi-time dependent Lagrangian function $L$ is of the form [1]:

$$
L(t^\gamma, x^k, x^k_{\gamma}) = \begin{cases} L(t, x^i, y^i), & p = 1 \\ h^{\alpha\beta}(t^\gamma)x^i_{\alpha}x^j_{\beta} + U_{(i)}^\alpha(\gamma, x^k)x^i_{\alpha} + F(\gamma, x^k), & p \geq 2, \end{cases}
$$

where $U_{(i)}^\alpha(\gamma, x^k)$ is a d-tensor on $J^1(T, M)$ and $F$ is a smooth function on $T \times M$. 


From our point of view, it is very interesting that the construction of $\Gamma_L$ is strongly connected of the critical points of the energy action functional of $L$, via its attached generalized harmonic maps. To be more clearly, assume that the semi-Riemannian temporal manifold $(T, h)$ is compact and orientable. In this context, we define the \textit{multi-time relativistic h-energy functional} of the Lagrangian function $L$:

$$\mathcal{E}_L : C^\infty(T, M) \to \mathbb{R}, \quad \mathcal{E}_L(f) = \int_T L(t^\alpha, x^i, x^i_\alpha) \sqrt{|h|} \, dt^1 \wedge dt^2 \wedge \ldots \wedge dt^p,$$

where the smooth map $f$ is locally expressed by $(t^\alpha) \to (x^i(t^\alpha))$. Now, using an important result proved in [11], we immediately find

**Theorem 2.1** The extremals of the multi-time relativistic h-energy functional $\mathcal{E}_L$ of the metrical multi-time Lagrange space $ML^n_p$ are equivalent with the h-generalized harmonic maps of the nonlinear connection $\Gamma_L$ defined by the components:

$$M_{(\alpha)\beta}^{(i)} = -H^\gamma_{\alpha\beta} x^i_\gamma, \quad p \geq 1,$$

$$N_{(\alpha)j}^{(i)} = \begin{cases} \frac{g^{ik}}{4} \left[ \frac{\partial^2 L}{\partial x^j \partial y^k} y^j - \frac{\partial L}{\partial x^j} + \frac{\partial^2 L}{\partial t \partial y^k} H_{11}^1 + 2h^{11} H_{11}^1 g_{kl} y^l \right], & p = 1 \\ \Gamma_{kj}^i x^k + \frac{g^{ik}}{2} \frac{\partial g_{jk}}{\partial t^\alpha} + \frac{g^{ik}}{4} h_{\alpha\gamma} U_{(k)j}^{(\gamma)}, & p \geq 2, \end{cases}$$

where $\Gamma_{ij}^k = (g^{li}/2)(\partial g_{lj}/\partial x^k + \partial g_{ik}/\partial x^j - \partial g_{jk}/\partial x^i), U_{(i)j}^{(\alpha)} = \partial U_{(i)}^{(\alpha)}/\partial x^j - \partial U_{(j)}^{(\alpha)}/\partial x^i$.

**Definition 2.1** The nonlinear connection $\Gamma_L$ from Theorem 2.1 is called the \textit{canonical nonlinear connection} of the metrical multi-time Lagrange space $ML^n_p$.

**Remarks 2.1** i) In the particular case of usual time axis $(T, h) = (\mathbb{R}, \delta)$, the canonical nonlinear connection $\Gamma_L = (0, N_{(i)j}^{(i)})$, produced by the time dependent Lagrangian $L = L \sqrt{|h|}$ of a relativistic rheonomic Lagrange space $\mathcal{L}$

$$RL^n = (J^1(\mathbb{R}, M) \equiv \mathbb{R} \times TM, L),$$

generalizes, in relativistic dynamical terms, canonical nonlinear connections used in theory of classical rheonomic Lagrange spaces [6], Ch. XIII.

ii) Note that, in the case $p \geq 2$, the construction of $\Gamma_L$ on a $ML^n_p$ space essentially relies on Kronecker $h$-regularity of the fundamental vertical metrical d-tensor produced by $L$ (i.e., $G_{(i)(j)}^{(\alpha)(\beta)} = (1/2)(\partial^2 L/\partial x^\alpha \partial x^\beta) = h^{\alpha\beta}(t^\gamma)g_{ij}(t^\gamma, x^k))$.

### 3 Canonical nonlinear connections on $GML^n_p$ spaces

More general, in order to construct a canonical nonlinear connection $\Gamma_G$ on a generalized metrical multi-time Lagrange space $GML^n_p = (J^1(T, M), G_{(i)(j)}^{(\alpha)(\beta)})$, whose Kronecker $h$-regular vertical fundamental metrical d-tensor (1.13) is not necessarily provided by a multi-time dependent Lagrangian function $L$, we point out that the
temporal semi-Riemannian metric $h_{\alpha\beta}$ naturally produces the *temporal components* of the nonlinear connection $\Gamma_G$, taking

\begin{equation}
M^{(i)}_{(\alpha)\beta} = \frac{\partial M^{(i)}_{(\alpha)\beta}}{\partial x^k} = -H^\gamma_{\alpha\beta}x^j.
\end{equation}

Concerning the construction of the *spatial components* of the canonical nonlinear connection $\Gamma_G$, we emphasize that the Riemann-Lagrange geometrical background for the generalized multi-time field theory from [5] relies on the use of an "a priori" given *without torsion* nonlinear connection $\Gamma$, that is the spatial components of $\Gamma$ verify the equalities

$$\frac{\partial N^{(i)}_{(\alpha)j}}{\partial x^k} = \frac{\partial N^{(i)}_{(\alpha)k}}{\partial x^j}.$$  

Consequently, in order to construct the canonical nonlinear connection $\Gamma_G$ of a $GML^n$ space, we need to produce some without torsion spatial components from the fundamental vertical metrical d-tensor (1.1). In this direction, following geometrical ideas from previous section, we introduce the concept of *multi-time relativistic energy Lagrangian function* of a $GML^n$ space, setting

$$E_G = G^{(\mu)(\nu)}_{(m)(r)}x^m_{\mu}x^r_{\nu} = h^{\mu\nu}(t^\gamma)g_{mn}(t^\gamma, x^k, x^r_{\nu})x^m_{\mu},$$

together with its *multi-time relativistic $\psi$-energy action functional* $E_G : C^\infty(T, M) \to \mathbb{R}$, $E_G(f) = \int_T E_G(t^\gamma, x^k(t^\gamma), x^r_{\nu}(t^\mu))\sqrt{|\psi|}dt$.

where $\psi$ is a semi-Riemannian metric on the temporal manifold $T$.

Firstly, suppose that $E_G$ is a Kronecker $\psi$-regular Lagrangian function of the form

$$E_G = h^{\alpha\beta}(t^\gamma)g_{ij}(t^\gamma, x^k, x^k_{\alpha}x^j_{\beta} = \psi^{\alpha\beta}(t^\gamma)x^i_{\alpha}x^j_{\beta},$$

where $\varepsilon_{ij}(t^\gamma, x^k)$ is a symmetric d-tensor of rank $n$, having a constant signature on $T \times M$. In this context, applying Theorem 2.1 to the particular $ML^n_p$ space $MLGL^n_p = (J^1(T, M), E_G)$ produced by the generalized metrical multi-time Lagrange space $GML^n_p$, we can construct certain without torsion spatial components for $\Gamma_G$, via the $\psi$-energy functional $E_G$. Consequently, we may consider that the spatial components of the canonical nonlinear connection $\Gamma_G$ are given by the formulas:

\begin{equation}
N^{(i)}_{(\alpha)j} = \Psi^{ij}_{jm}x^m_{\alpha} + \frac{\varepsilon^{im}\partial \varepsilon_{jm}}{2}t^\alpha,
\end{equation}

where $\Psi^{ij}_{jm}(t^\mu, x^m)$ are *generalized Christoffel symbols* for $\varepsilon_{ij}(t^\gamma, x^k)$.

**Remark 3.1** A canonical nonlinear connection $\Gamma_L$, derived from a multi-time dependent quadratic Lagrangian function of general form

$$L = G_{(\alpha)(\beta)}^{(i)(j)}(t^\gamma, x^k)x^i_{\alpha}x^j_{\beta} + U^{(\alpha)(i)}(t^\gamma, x^k)x^i_{\alpha} + F(t^\gamma, x^k)$$

and a given temporal semi-Riemannian metric $h_{\alpha\beta}(t^\gamma)$, may be naturally produced, using preceding ideas for the metrical d-tensor $G_{(i)(j)}^{(\alpha)(\beta)} = h^{\alpha\beta}h_{\mu\nu}G_{(i)(j)}^{(\mu)(\nu)}$.  

\[5\]
Secondly, suppose that $E_G$ is not a Kronecker $\psi$-regular Lagrangian function for any semi-Riemannian metric $\psi$. Under these assumptions, we are forced to give "ab initio" the without torsion spatial components of the nonlinear connection $\Gamma_G$.

**Example 3.1** For the study of particular $GML^n_p$ spaces $GRGML^n_p$ (i.e., that represents a geometrical model for multi-time Relativity and Electromagnetism [8]) and $RGOGML^n_p$ (i.e., that represents a geometrical background for dynamical relativistic multi-time optics [8]), it is convenient to use the "a priori" given without torsion spatial components

$N_{(\alpha)j} = N_{(\alpha)j} = \gamma^i_{jm} x^m$  

which are directly provided by the Kronecker $h$-regular fundamental vertical metrical d-tensors of these spaces. The main physical motives, together with their intrinsic difficulties, that determined us to use these spatial components in the geometrical study of the spaces $GML^n_p$ and $RGOGML^n_p$, are deeply exposed in [6], [8].

Taking into account the preceding discussions related to the construction of canonical nonlinear connections on $GML^n_p$ spaces, we consider important to study the conditions that must be imposed to the multi-time relativistic energy Lagrangian function $E_G$ of a generalized metrical multi-time Lagrange space, in order to obtain its Kronecker $\psi$-regularity, where $\psi$ is an arbitrary semi-Riemannian metric on $T$. We recall that, under the Kronecker $\psi$-regularity assumption, $E_G$ produces without torsion spatial components for the canonical nonlinear connection $\Gamma_G$ of the space $GML^n_p$, via the formulas (3.2).

In order to do this study, let us suppose $p = \dim T \geq 2$. In this case, the characterization theorem of $ML^n_p$ spaces from [3] imply that the Kronecker $\psi$-regularity of $E_G$ reduces to the existence of certain d-tensors $\varepsilon_{ij}(t^\gamma, x^k)$, $U^{(\alpha)}_{(i)}(t^\gamma, x^k)$ on $J^1(T, M)$, together with a smooth function $F: T \times M \to \mathbb{R}$, such that

$$E_G = \psi^{\alpha\beta}(t^\gamma) \varepsilon_{ij}(t^\gamma, x^k) x^k_i x^j_\beta + U^{(\alpha)}_{(i)}(t^\gamma, x^k) x^j_\alpha + F(t^\gamma, x^k).$$

In other words, $E_G$ is a Kronecker $\psi$-regular Lagrangian function if the following equality is true for some $\varepsilon_{ij}$, $U^{(\alpha)}_{(i)}$ and $F$:

$$h^{\alpha\beta}(t^\gamma) g_{ij}(t^\gamma, x^k, x^\gamma) x^k_i x^\gamma_j = \psi^{\alpha\beta}(t^\gamma) \varepsilon_{ij}(t^\gamma, x^k) x^k_i x^j_\beta + U^{(\alpha)}_{(i)}(t^\gamma, x^k) x^j_\alpha + F(t^\gamma, x^k).$$

Suppose that the spatial metrical d-tensor $g_{ij}$ of the space $GML^n_p$ does not depend on partial directions $x^i_\alpha$. It is obvious that, taking $\psi^{\alpha\beta} = h^{\alpha\beta}$, $\varepsilon_{ij} = g_{ij}$, $U^{(\alpha)}_{(i)} = 0$ and $F = 0$, we can conclude that $E_G$ is a Kronecker $h$-regular Lagrangian function. Therefore, in this situation, using the formulas (3.2), we can build the spatial components of the canonical nonlinear connection $\Gamma_G$.

**Open problems.** i) In the case $p = \dim T \geq 2$, are there spatial metrical d-tensors $g_{ij}$ on $GML^n_p$ spaces, depending effectively on partial directions $x^i_\alpha$, such that $E_G$ to be a Kronecker $\psi$-regular Lagrangian function?

ii) Is it possible to construct, in a natural way, without torsion spatial components of a nonlinear connection $\Gamma_G$, which to be canonically produced by a Kronecker $h$-regular vertical fundamental metrical d-tensor $G$ on $J^1(T, M)$?


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