Categorical Physics

Louis Crane
Department of Mathematics
Kansas State University
Manhattan, KS 66506

I. Introduction

The purpose of this letter is to outline a rather novel suggestion as to what form the quantum theory of gravity should take. The suggestion has a certain mathematical elegance, but that was not the motivation which brought it into being. Rather, it was motivated by the juxtaposition of two ideas:

(A) the work of Rovelli and Smolin on the loop representation (1) which showed that a state for the quantum theory of gravity could be described by an invariant of links in a 3-manifold, given by integrating a measure on the space of connections on the manifold against the traced holonomy around the link.

(B) the work of E. Witten (2) which recovered a generalized Jones polynomial as precisely such an integral; using the Chern-Simons invariant to create a measure.

The “measures”, in A and B above are, of course, formal. It was in order to make mathematical sense of Witten’s Chern-Simons invariant that I showed in (3) that any modular tensor category gave rise to a 3−D Topological Quantum Field Theory, i.e. allows us to reproduce all the results of Witten’s path integral by rigorous methods.

The juxtaposition of ideas (A) and (B) tells us that the Chern Simons measure gives us a state for quantum gravity on a closed 3-manifold, which can be reproduced entirely from an algebraic structure called a modular tensor category, first described in (4). The construction gives us more than a state (i.e. link invariant) on a closed 3-manifold; it also gives us vector valued invariants of links in 3-manifolds with boundary, and a nice factorization when we join along boundary components. This is what we mean when we say that we can construct a topological quantum field theory.

What I want to propose is that the universe as a whole is in a Chern-Simons state (presumably for a larger group than \(SU(2)\), in order to include matter - more on this latter).

The connection of the Chern Simons state to a TQFT is what allows us to experience a world of changing phenomena in a universe which is not changing as a whole. Manifolds with boundary correspond to observers, and observations take place in the Hilbert spaces which our TQFT assigns to boundaries. Thus, in order to quantize gravity (or gravity plus matter) we have to abandon the
idea of measurement at a distance, and create a relative form of the probability interpretation of quantum mechanics; with a Hilbert space for each observer, and consistency relations which are best expressed in categorical language. We can say that a state for quantum gravity is given by a functor from the category of observers to the category of vector spaces; this coincides with the notion of a TQFT. Time evolution must be given by a natural transformation between functors.

This is obviously an ambitious program. As yet, I am not able to compute the result of any experiment in this framework, but I can see a path which could lead to that goal. In the rest of this letter, I summarize what I can understand of various aspects of this program, and suggest various directions for further research.

II. The Category of Observers

In this section, I make a series of fundamental definitions. On the one hand, they are motivated by the mathematical juxtaposition mentioned above; on the other, they have, I believe, a certain physical plausibility.

Definition: An Observer is an oriented 3-manifold with boundary containing an embedded labeled framed graph which intersects the boundary in isolated labeled points.

(The labeling sets for the edges and vertices of the graph are finite, and need to be chosen for all of what ensues.)

Definition: A skin of observation is a closed oriented surface with labeled punctures.

Definition: If A and B are skins of observation an inspection, $\alpha$, of B by A is an observer whose boundary is identified with $\overline{A} \cup B$ (i.e. reverse the orientation on A) such that the labelings of the components of the graph which reach the boundary of $\alpha$ match the labelings in $\overline{A} \cup B$.

This definition requires that the set of labelings possess an involution corresponding to reversal of orientation on a surface.

To every inspection $\alpha$ of B by A there corresponds a dual inspection of A by B, given by reversing orientation and dualizing on $\alpha$.

Definition: The category of observation is the category whose objects are skins of observation and whose morphisms are inspections.

Definition: If $M^3$ is a closed oriented 3-manifold the category of observation in $M^3$ is the relative (i.e. embedded) version of the above.

Of course, we can speak of observers, etc. in $M^3$.

Definition: A state for quantum gravity (in $M^3$) is a functor from the category of observation (in $M^3$) to the category of vector spaces.

Let us reiterate that all the above definitions should be extended by specifying the labeling set with involution.

Observation: Any modular tensor category gives rise to a state for quantum gravity.

The mathematical motivation for the above definitions should be clear; it is just what we know how to produce.
In our language, a State for quantum gravity is the same thing as a $3-D$ TQFT. We should note some facts about the development of the Ashtekar variables for general relativity, which make this physically plausible. Although a state is given by any invariant of links, since the Hamiltonian constraint only gives a condition at the vertices of a graph, the states which are supported only on links seem to have unsatisfactory properties. In searching for solutions of the Hamiltonian constraint which were nontrivial on graphs Gambini et al (5) found themselves reproducing the perturbative terms of the CSW invariant. Thus, it could well be that CSW gives the only physically interesting states. Should the connection with TQFT then be taken as pure coincidence?

Let us now consider the physical ideas which underlie these definitions. The first departure is attaching vector (“Hilbert”) spaces to skins of observation, rather than observers. This means that if we divide the universe into “system” and “observer” the observer no longer measures the state of the system, but only of that part of it which impinges instantaneously on the observer. This is commonly neglected in ordinary quantum mechanics, but it seems reasonable that in quantizing general relativity we would need to consider it. I refer to this as abandoning “measurements at a distance”.

We are left with a picture in which states for a whole universe are fixed things constructed out of special algebraic structures, while the relational states which we measure live in spaces constructed by a different recipe from the same structures.

Here we see a second physical departure. The state of the universe as a whole becomes part of the framework of laws by which the parts interact.

It has been noted for some time that when considering the entire universe, the relationship between laws and initial conditions could be different from what physicists commonly expect. Hawking has suggested determining initial conditions from laws. I am proposing, rather that the “initial” state of the universe (which does not change in time) acts as a law.

It is not hard to create a quantum mechanical system in which the initial state of a combined system acts as a law of motion for the two parts [10], so perhaps this notion is not so implausible.

Before discussing how this theory might meet some of the titanic challenges which await it, let me allow myself a few philosophic observations.

Category theory was invented largely in order to rewrite mathematics in the most coordinate invariant way possible. The notion that it was desirable to write things invariantly is something which 20th century mathematics inherited from relativity. (The first abstract definition of a vector space is in Weyl’s exposition of relativity Zeit, Raum, Materie.)

Thus, if the fully adequate formulation of relativity uses categories, we would be coming full circle.

Beside the functorial nature of the result, i.e. of a state of quantum gravity; there is also a categorical structure, namely a modular tensor category, at the heart of the construction [3] of a $3-D$ TQFT, which we are reinterpreting here.

I want to argue that the entire CSW invariant is more interesting than its perturbative terms. This is because modular tensor categories are only slight
modifications of Tannakian categories. [6] Tannakian categories are simply the categories of representations of groups. Thus, tensor product categories are expressions of symmetries. Modular tensor categories are “purely quantum” versions of the same thing. Thus, we can think of them as “quantum symmetries”. The fact that we can construct our invariant states from MTC’s can be expressed (somewhat lavishly) by saying that in the quantum domain invariant states are determined by symmetry.

Enough philosophy. How can this theoretical model be compared with experiment? I believe that there are three main problems which must be confronted: (A) making some physical interpretation of the states which we attach to skins of observation (B) the problem of introducing time and (C) how to include matter. In the next 3 paragraphs, I shall say what I know about these problems.

III. States as Quantum Geometries. Deformed Spin Networks

The idea I discuss here was already proposed in [7]. I shall outline it here for completeness, and to set it in a theoretical context.

The invariants which CSW theory assigns to embedded knotted labeled graphs are deformed generalizations of the number called the “evaluation”, which Penrose [8] assigned to a labeled graph called a spin network.

Regge and Ponzano [9], were able to interpret the evaluation as a sort of discrete path integral for $3-d$ euclidean quantum gravity. The formula which Regge and Ponzano found for the evaluation of a graph

$$\# = \sum_{\text{labelings}} \prod_{\text{int. edges}} \text{qdim} J \prod_{\text{Tetrachedra}} \{6J\}, 1$$

follows from some elementary properties of representation theory. In (1), we have placed our trivalent labeled graph on the boundary of a 3-manifold with boundary, then cut the interior up into tetrahedra, labeling the edges of all the internal lines with arbitrary spins. (The Clebsch-Gordon Relations imply that only finitely many terms in (1) are non-zero.)

The evaluation of a tetrahedron for a spin network is a $6J$ symbol for the group $SU(2)$.

The derivation of formula (1) uses only basic properties of the category of representations of $SU(2)$. Thus, we can repeat the derivation of (1) for CSW invariants of trivalent graphs, provided that we are careful not to embed the graph in such a way that we have to unbraid it in order to put it on a boundary. This is not an issue for spin networks, since over and under crossings are equivalent (i.e. the tensor category is symmetric).

There is a natural generalization of spin networks to larger groups, called fabrics by Moussouris [11]. A formula like (1) can be similarly deduced, except that the “labelings” include assigning intertwining operators to internal faces. (These are unique for $SU(2)$.)
Thus, a formula like (1) but with more general labelings, including labels on faces, can be derived for CSW invariants of graphs.

Formula (1) has the same form as the invariant of 3-manifolds due to Viro and Turaev, but the geometric setting is different, and we are obtaining the CSW invariant, instead of its absolute square.

Regge and Ponzano reinterpreted (1) as a discrete path integral for $3-d$ quantum gravity.

The essential step for them was to reinterpret the spin label on each edge as a length. Assigning lengths to the edges of a triangulation is an approximation to selecting a metric; summing over all labelings then approximates a path integral over metrics.

Thus a labeling of the edges of a graph gives a sort of discretized probability density of metrics or “quantum geometry”, to the space it surrounds. For ordinary spin networks, these sums peak around flat metrics, so the classical “equations of motion” match those for $3-d$ euclidean general relativity.

Applying the same approach to states on the vector space attached to a boundary surface yields a “quantum geometry” on part of the interior of a 3-manifold with boundary. A small improvement of the form of (1) would suffice to give a quantum geometry on the entire interior. Studying the recursion relations for quantum $6J$ symbols leads me to believe the improvement exists, but I have not constructed it, and do no pursue it here. At any rate, being able to interpret a state on the boundary as probabilistic information about the geometry on the interior brings us much closer to relating our formalism to experiment. If we could relate states on a boundary for CSW theory with a larger group to a more complex geometry in the interior, mixing intrinsic with extrinsic geometry (i.e. coupling gravity to Yang-Mills) it would be even a larger step.

This brings us to the question of introducing matter into our picture.

IV. Matter and Larger Groups

The $2+1-D$ TQFT constructed by means of a modular tensor category (or quantum group, etc.), can equally well be defined for some larger semisimple compact lie group as for $SU(2)$. [Noncompact forms are much thornier.] It is tempting (to anyone who finds this paper tempting), to try to interpret them as “states of gravity plus matter.” This raises a key issue.

Issue: Are TQFT's associated to Larger Groups States for some Quantum Theory?

If the answer to this question is yes, then it is not necessary to worry too much about finding the Yang-Mills Lagrangian in the theory, since the action of the renormalization group will bring us to it if our theory has the right symmetries.

As it turns out, it is possible to approach this problem, although not in a mathematically rigorous way. Peldan [12] has written down a Hamiltonian constraint for an arbitrary lie group, which reduces to the Ashtekar form for $SU(2)$, and produces no second class constraints. We can then translate our question into the simpler question of whether the Chern-Simons State $\ell^2 \text{tr}e^{ikcx(A)}$
solves Peldan’s generalized Ashtekar Hamiltonian constraint (with cosmological constant). This reduces to some algebraic relations on the group, currently under study. This would not, of course, provide a rigorous argument that the TQFT associated to some special group is a quantum state for gravity-plus-matter, but it would be very suggestive.

V. The Problem of Time

The description of the world which the picture we are discussing provides would not, (even given the most optimistic outcome of the unsettled questions) at this point, resemble the world we see. This is because the relative states which live on surfaces have no time evolution. Past, present and future are all one in them.

This is not a new problem in attempts to quantize gravity. It is a consequence of the vanishing of the Hamiltonian on physical states, which is ultimately an expression of $4 - D$ diffeomorphism invariance.

There is a standard resolution of this question among relativists. This is to treat one part of the variables as a clock. The relative states of the rest of the variables then turn out to solve a Schrödinger equation [13].

Thus, if we are to interpret the states in our model as giving an evolution in $4 - D$ space time, we have to solve the problem of representing clocks within the states of a TQFT.

There is another fundamental question which must be faced here: Can one have a clock in a purely gravitational universe, or are clocks necessarily material? As Einstein once phrased a similar question: If matter disappeared from the universe, would time as well?

Moncrief [14] has taken rather heroic measures, to construct a clock from a gravitational universe. However, it is hard to imagine doing general relativity with only one clock, and measuring the volume of the universe has little to do with what an observer timing some event normally does. I therefore believe that the most likely route to an interpretation of the picture we are developing must go through interpreting a larger group as gravity plus matter, and constructing clocks from matter.

Although it is hard to construct a state involving time explicitly, it is easy to see what mathematical form such a thing should take. We formulate such a picture, then make some speculations as to how it might be constructed.

Definition: A **process** is an oriented 4-manifold with boundary with an embedded labeled branched surface which meets the boundary transversely in a labeled knotted graph.

Definition: An observer with no boundary is a **moment**.

Definition: If $A$ and $B$ are moments and $P$ is a process with boundary $\overline{A} \cup B$, then we say $P$ is an **intervening process** between $A$ and $B$.

If $P$ is an intervening process between $A$ and $B$, then $\overline{P}$ is an intervening process between $B$ and $\overline{A}$ ("time reversal").

Now let us define the structure which seems natural to express time evolution. Suppose given a vector field $\phi$ which is the gradient of a morse function on $\mathcal{P}$ which intersects $\overline{A}$ and $B$ transversely, pointing "inward" on $\overline{A}$ and "outward"
on $B$. Dragging along $\phi$ then gives a functor $F_\phi$ from the relative category of observation $O_A$ to $O_B$.

**Definition:** We say that a state of gravity $S$ is **augmented** if for every two moments $A, B$, process $P$ intervening between them, and vector field $\phi$ as above, we are given a natural transformation $N_{P,\phi} : S_A \to S_B \circ F_\phi$. We require the assignment to be natural, in the sense that the composition of processes and vector fields (obvious definition) is assigned the composition of natural transformations.

Let us translate into more prosaic language. A natural transformation is given by maps between the images of objects, which make certain squares commute. These maps are between the Hilbert spaces of skins of observation and their “time evolutions”, i.e. draggings by $\phi$, and should be thought of as giving time evolution for relative states. The commuting squares give a consistency relation, in which the state $M$ observes in $N$ changes to the same state in $F_\phi(M)$ as the state the later observer $F_\phi(M)$ would observe from the changed state in $F_\phi(N)$, i.e. the following diagram commutes:
Conjecture: The natural world is described by an augmented state of quantum gravity.

We should note that even if we could construct an augmented state, the physical interpretation of it would still involve all the subtleties of clocks, lapses, shifts, etc. with which general relativity is so richly endowed.

VI. Some Observations on Augmented States

The suggestion here is that quantum gravity is to be understood, not as a $3+1-D$ TQFT, but as a $(2+1)-D$ TQFT with some extension into four dimensions. The extension is very “categorical”, a $2+1-D$ TQFT is a set of functors, we now need functors plus natural transformations.

This is reminiscent of the conjecture, [3], since demonstrated by K. Walker [15], that the $3-D$ TQFT’s constructed from modular tensor categories related to WZW models can be interpreted as containing information about a bounding 4-manifold, (thus removing the phase ambiguity).

More recently, I. Frenkel and the present author [16] have been able to associate to a quantum group a tensor 2-category, which is a sort of “recategorification” of a modular tensor category. The relationship between augmented states and 2-categories is very suggestive. For those not versed in category theory, let me remark that both (skins of observation, inspections, processes with corners) and (categories, functors, natural transformations) are 2-categories [17].

VII. Possible Approximations

How could we compute anything in this picture? One suggestion, due to L. Smolin, is to try to model an asymptotically flat situation.

We could imagine picking a particular weave to model the state in the flat region, and couple it to our deformed spin networks in the interior.

In general, it is possible that experiments could be modeled after some hybrid of the loop states [18] in one half of the universe with the TQFT state on the other. This would correspond to breaking gauge invariance in the “apparatus”.

It may be possible to formulate a simplified model of matter by using the representations of an inhomogeneous quantum group instead of a larger simple group.
VIII. Perspective and Apologia

One way of looking at the proposal in this paper is that it is an attempt to fuse the many worlds interpretation of quantum mechanics with general relativity. States become correlated only locally, i.e. only on skins. Somehow, the classical equation of motion must emerge in a situation where many bodies measure one another’s state very often.

The great hope of the subject would be that once we introduce clocks Einstein’s equations in $4 - D$ emerge by a process analogous to the emergence of flat 3-geometries in the work of Regge and Ponzano.

Finally, an apologia. This is a very sweeping proposal, motivated partly by work in quantum gravity, but also largely by mathematical elegance, and the existence of certain elegant algebraic structures. It has no doubt escaped no one’s attention that much of the necessary delicate analysis is not done. Nevertheless, my intuitive feeling is that enough of the ingredients of nature have mathematical analogs here to make the ideas worth pursuing. Mathematical elegance has served physics well enough in the past.

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