ON THE SEMIGROUP OF INJECTIVE ENDOMORPHISMS OF THE SEMIGROUP $B^F_{\omega n}$ WHICH IS GENERATED BY THE FAMILY $F_n$ OF INITIAL FINITE INTERVALS OF $\omega$

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ABSTRACT. In the paper we describe injective endomorphisms of the inverse semigroup $B^{\omega}_{\mathbb{Z}}$, which is introduced in the paper [O. Gutik and M. Mykhalenych, On some generalization of the bicyclic monoid, Visnyk Lviv. Univ. Ser. Mech.-Mat. 90 (2020), 5–19 (in Ukrainian)], in the case when the family $F_n$ is generated by the set $\{0, 1, \ldots, n\}$. In particular we show that the semigroup of injective endomorphisms of the semigroup $B^{\omega}_{\mathbb{Z}}$ is isomorphic to $(\omega, +)$. Also we describe the structure of the semigroup $\text{End}(B_{\lambda})$ of all endomorphisms of the semigroup of $\lambda \times \lambda$-matrix units $B_{\lambda}$.

1. Introduction, motivation and main definitions

We shall follow the terminology of [3, 4, 15, 18]. By $\omega$ we denote the set of all non-negative integers.

Let $\mathcal{P}(\omega)$ be the family of all subsets of $\omega$. For any $F \in \mathcal{P}(\omega)$ and $n, m \in \omega$ we put $n - m + F = \{n - m + k : k \in F\}$ if $F \neq \emptyset$ and $n - m + \emptyset = \emptyset$. A subfamily $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is called $\omega$-closed if $F_1 \cap (-n + F_2) \in \mathcal{F}$ for all $n \in \omega$ and $F_1, F_2 \in \mathcal{F}$.

We denote $[0; 0] = \{0\}$ and $[0; k] = \{0, \ldots, k\}$ for any positive integer $k$. The set $[0; k]$, $k \in \omega$, is called an initial interval of $\omega$.

A partially ordered set (or shortly a poset) $(X, \leq)$ is the set $X$ with the reflexive, antisymmetric and transitive relation $\leq$. In this case the relation $\leq$ is called a partial order on $X$. A partially ordered set $(X, \leq)$ is linearly ordered or is a chain if $x \leq y$ or $y \leq x$ for any $x, y \in X$. A map $f$ from a poset $(X, \leq)$ onto a poset $(Y, \lesssim)$ is said to be an order isomorphism if $f$ is bijective and $x \leq y$ if and only if $f(x) \lesssim f(y)$. A partial order isomorphism $f$ from a poset $(X, \leq)$ into a poset $(Y, \lesssim)$ is an order isomorphism from a subset $A$ of a poset $(X, \leq)$ onto a subset $B$ of a poset $(Y, \lesssim)$. For any elements $x$ of a poset $(X, \leq)$ we denote

$$\uparrow_{\leq} x = \{ y \in X : x \leq y \}.$$

A nonempty set $S$ with a binary associative operation is called a semigroup. By $(\omega, +)$ we denote the set $\omega$ with the usual addition $(x, y) \mapsto x + y$.

A semigroup $S$ is called inverse if for any element $x \in S$ there exists a unique $x^{-1} \in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. The element $x^{-1}$ is called the inverse of $x \in S$. If $S$ is an inverse semigroup, then the mapping $\text{inv}: S \rightarrow S$ which assigns to every element $x$ of $S$ its inverse element $x^{-1}$ is called the inversion.

If $S$ is a semigroup, then we shall denote the subset of all idempotents in $S$ by $E(S)$. If $S$ is an inverse semigroup, then $E(S)$ is closed under multiplication and we shall refer to $E(S)$ as a band (or the band of $S$). Then the semigroup operation on $S$ determines the following partial order $\preceq$ on $E(S)$: $e \preceq f$ if and only if $ef = fe = e$. This order is called the natural partial order on $E(S)$. A semilattice is a commutative semigroup of idempotents. By $(\omega, \min)$ we denote the set $\omega$ with the semilattice operation $x \cdot y = \min\{x, y\}$.

For semigroups $S$ and $T$, a map $\mathfrak{h} : S \rightarrow T$ is called:

- a homomorphism if $\mathfrak{h}(s_1 \cdot s_2) = \mathfrak{h}(s_1) \cdot \mathfrak{h}(s_2)$ for all $s_1, s_2 \in S$;
- an annihilating homomorphism if $\mathfrak{h}$ is a homomorphism and $\mathfrak{h}(s_1) = \mathfrak{h}(s_2)$ for all $s_1, s_2 \in S$;
For a semigroup \( S \) a homomorphism (an isomorphism) \( h: S \to T \) is a bijective homomorphism.

A congruence on a semigroup \( S \) is an equivalence relation \( \mathcal{E} \) on \( S \) such that \( (s, t) \in \mathcal{E} \) implies that
\[
(as, at), (sb, tb) \in \mathcal{E}
\]
for all \( a, b \in S \). Every congruence \( \mathcal{E} \) on a semigroup \( S \) generates the associated natural homomorphism \( S \to S/\mathcal{E} \) which assigns to each element \( s \) of \( S \) its congruence class \([s]_\mathcal{E}\) in the quotient semigroup \( S/\mathcal{E} \). Also every homomorphism \( h: S \to T \) of semigroups \( S \) and \( T \) generates the congruence \( \mathcal{E}_h \) on \( S \) as follows: \((s_1, s_2) \in \mathcal{E}_h\) if and only if \((s_1)h = (s_2)h\).

A nonempty subset \( I \) of a semigroup \( S \) is called an ideal of \( S \) if \( SIS = \{asb: s \in I, a, b \in S\} \subseteq I \). Every ideal \( I \) of a semigroup \( S \) generates the congruence \( \mathcal{E}_I = (I \times I) \cup \Delta_S \) on \( S \), which is called the Rees congruence on \( S \).

Let \( \mathcal{I}_\lambda \) denote the set of all partial one-to-one transformations of \( \lambda \) together with the following semigroup operation:
\[
x(\alpha\beta) = (xa)\beta \quad \text{if} \quad x \in \text{dom}(\alpha\beta) = \{y \in \text{dom} \alpha: y\alpha \in \text{dom} \beta\}, \quad \text{for} \quad \alpha, \beta \in \mathcal{I}_\lambda.
\]
The semigroup \( \mathcal{I}_\lambda \) is called the symmetric inverse semigroup over the cardinal \( \lambda \) (see [3]). For any \( \alpha \in \mathcal{I}_\lambda \) the cardinality of \( \text{dom} \alpha \) is called the rank of \( \alpha \) and it is denoted by \( \text{rank} \alpha \). The symmetric inverse semigroup was introduced by V. V. Wagner [25] and it plays a major role in the theory of semigroups.

Put \( \mathcal{I}_\lambda^n = \{\alpha \in \mathcal{I}_\lambda: \text{rank} \alpha \leq n\} \), for \( n = 1, 2, 3, \ldots \). Obviously, \( \mathcal{I}_\lambda^n \) \((n = 1, 2, 3, \ldots)\) are inverse semigroups, \( \mathcal{I}_\lambda^n \) is an ideal of \( \mathcal{I}_\lambda \), for each \( n = 1, 2, 3, \ldots \). The semigroup \( \mathcal{I}_\lambda^n \) is called the symmetric inverse semigroup of finite transformations of the rank \( \leq n \) [10]. By
\[
\begin{pmatrix}
x_1 & x_2 & \cdots & x_n \\
y_1 & y_2 & \cdots & y_n
\end{pmatrix}
\]
we denote a partial one-to-one transformation which maps \( x_1 \) onto \( y_1 \), \( x_2 \) onto \( y_2 \), \ldots, and \( x_n \) onto \( y_n \). Obviously, in such case we have \( x_i \neq x_j \) and \( y_i \neq y_j \) for \( i \neq j \) \((i, j = 1, 2, 3, \ldots, n)\). The empty partial map \( \emptyset: \lambda \to \lambda \) is denoted by \( 0 \). It is obvious that \( 0 \) is zero of the semigroup \( \mathcal{I}_\lambda^n \).

For a partially ordered set \((P, \leq)\), a subset \( X \) of \( P \) is called order-convex, if \( x \leq z \leq y \) and \( x, y, z \in X \) implies that \( z \in X \), for all \( x, y, z \in P \) [11]. It is obvious that the set of all partial order isomorphisms between convex subsets of \((\omega, \leq)\) under the composition of partial self-maps forms an inverse subsemigroup of the symmetric inverse semigroup \( \mathcal{I}_\omega \) over the set \( \omega \). We denote this semigroup by \( \mathcal{I}_\omega(\text{conv}^\omega) \). We put \( \mathcal{I}_\omega^n(\text{conv}^\omega) = \mathcal{I}_\omega(\text{conv}^\omega) \cap \mathcal{I}_\omega^n \) and it is obvious that \( \mathcal{I}_\omega^n(\text{conv}^\omega) \) is closed under the semigroup operation of \( \mathcal{I}_\omega^n \) and the semigroup \( \mathcal{I}_\omega^n(\text{conv}^\omega) \) is called the inverse semigroup of convex order isomorphisms of \((\omega, \leq)\) of the rank \( \leq n \). Obviously that every non-zero element of the semigroup \( \mathcal{I}_\omega^n(\text{conv}^\omega) \) of the rank \( k \leq n \) has a form
\[
\begin{pmatrix}
i & i+1 & \cdots & i+k-1 \\
j & j+1 & \cdots & j+k-1
\end{pmatrix}
\]
for some \( i, j \in \omega \).

The bicyclic monoid \( C(p, q) \) is the semigroup with the identity 1 generated by two elements \( p \) and \( q \) subjected only to the condition \( pq = 1 \). The semigroup operation on \( C(p, q) \) is determined as follows:
\[
q^kp^l \cdot q^mp^n = q^{k+m-\min\{l,m\}}p^{l+n-\min\{l,m\}}.
\]
It is well known that the bicyclic monoid \( C(p, q) \) is a bisimple (and hence simple) combinatorial \( E \)-unitary inverse semigroup and every non-trivial congruence on \( C(p, q) \) is a group congruence [3].

On the set \( B_\omega = \omega \times \omega \) we define the semigroup operation "\( \cdot \)" in the following way
\[
(i_1, j_1) \cdot (i_2, j_2) = \begin{cases} (i_1 - j_1 + i_2, j_2), & \text{if } j_1 \leq i_2; \\ (i_1, j_1 - i_2 + j_2), & \text{if } j_1 \geq i_2. \end{cases}
\]
It is well known that the semigroup \( B_\omega \) is isomorphic to the bicyclic monoid by the mapping \( h: C(p, q) \to B_\omega, q^kp^l \mapsto (k, l) \) (see: [3, Section 1.12] or [18, Exercise IV.1.11(ii)]).

Next we shall describe the construction which is introduced in [6].
Let $B_\omega$ be the bicyclic monoid and $\mathcal{F}$ be an $\omega$-closed subfamily of $\mathcal{P}(\omega)$. On the set $B_\omega \times \mathcal{F}$ we define the semigroup operation "." in the following way

$$(i_1, j_1, F_1) \cdot (i_2, j_2, F_2) = \begin{cases} 
(i_1 - j_1 + i_2, j_2, (j_1 - i_2 + F_1) \cap F_2), & \text{if } j_1 \leq i_2; \\
(i_1 - j_1 + i_2 + j_2, F_1 \cap (i_2 - j_1 + F_2)), & \text{if } j_1 \geq i_2. 
\end{cases}$$

In [6] is proved that if the family $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is $\omega$-closed then $(B_\omega \times \mathcal{F}, \cdot)$ is a semigroup. Moreover, if an $\omega$-closed family $\mathcal{F} \subseteq \mathcal{P}(\omega)$ contains the empty set $\emptyset$ then the set $I = \{(i, j, \emptyset) : i, j \in \omega\}$ is an ideal of the semigroup $(B_\omega \times \mathcal{F}, \cdot)$. For any $\omega$-closed family $\mathcal{F} \subseteq \mathcal{P}(\omega)$ the following semigroup

$$B_\omega^\mathcal{F} = \begin{cases} 
(B_\omega \times \mathcal{F}, \cdot) / I, & \text{if } \emptyset \in \mathcal{F}; \\
(B_\omega \times \mathcal{F}, \cdot), & \text{if } \emptyset \notin \mathcal{F}
\end{cases}$$

is defined in [6]. The semigroup $B_\omega^\mathcal{F}$ generalizes the bicyclic monoid and the countable semigroup of matrix units. It is proved in [6] that $B_\omega^\mathcal{F}$ is a combinatorial inverse semigroup and Green's relations, the natural partial order on $B_\omega^\mathcal{F}$ and its set of idempotents are described. The criteria of simplicity, 0-simplicity, bisimplicity, 0-bisimplicity of the semigroup $B_\omega^\mathcal{F}$ and when $B_\omega^\mathcal{F}$ has the identity, is isomorphic to the bicyclic semigroup or the countable semigroup of matrix units are given. In particular in [6] is proved that the semigroup $B_\omega^\mathcal{F}$ is isomorphic to the semigroup of $\omega \times \omega$-matrix units if and only if $\mathcal{F}$ consists of a singleton set and the empty set.

The semigroup $B_\omega^\mathcal{F}$ in the case when the family $\mathcal{F}$ consists of the empty set and some singleton subsets of $\omega$ is studied in [5]. It is proved that the semigroup $B_\omega^\mathcal{F}$ is isomorphic to the subsemigroup $B_\omega^{\mathcal{F}_{\text{min}}}$ of the Brandt $\omega$-extension of the subsemilattice $(\mathcal{F}, \text{min})$ of $(\omega, \text{min})$, where $\mathcal{F} = \bigcup \mathcal{F}$. Also topologizations of the semigroup $B_\omega^\mathcal{F}$ and its closure in semitopological semigroups are studied.

For any $n \in \omega$ we put $\mathcal{F}_n = \{[0; k] : k = 0, \ldots, n\}$. It is obvious that $\mathcal{F}_n$ is an $\omega$-closed family of $\omega$.

In the paper [7] we study the semigroup $B_\omega^{\mathcal{F}_n}$. It is shown that the Green relations $\mathcal{D}$ and $\mathcal{J}$ coincide in $B_\omega^{\mathcal{F}_n}$, the semigroup $B_\omega^{\mathcal{F}_n}$ is isomorphic to the semigroup $\mathcal{J}_n+1(\text{conv} \omega)$, and $B_\omega^{\mathcal{F}_n}$ admits only Rees congruences. Also in [7], we study shift-continuous topologies of the semigroup $B_\omega^{\mathcal{F}_n}$ in particular we prove that for any shift-continuous $T_1$-topology $\tau$ on the semigroup $B_\omega^{\mathcal{F}_n}$ every non-zero element of $B_\omega^{\mathcal{F}_n}$ is an isolated point of $(B_\omega^{\mathcal{F}_n}, \tau)$, $B_\omega^{\mathcal{F}_n}$ admits the unique compact shift-continuous $T_1$-topology, and every $\omega_0$-compact shift-continuous $T_1$-topology is compact. We describe the closure of the semigroup $B_\omega^{\mathcal{F}_n}$ in a Hausdorff semitopological semigroup and prove the criterion when a topological inverse semigroup $B_\omega^{\mathcal{F}_n}$ is $H$-closed in the class of Hausdorff topological semigroups.

Surprisingly, not so many articles are devoted to endomorphisms and automorphisms of semigroups. In particular, in [1] the authors propose a general recipe for calculating the automorphism groups of semigroups consisting of partial endomorphisms of relational structures over a finite set with a single $m$-ary relation for any positive integer $m$, which determine the automorphism groups of the following semigroups: the full transformation semigroup, the partial transformation semigroup, and the symmetric inverse semigroup, the wreath product of two full transformation semigroups, the partial endomorphisms of any partially ordered set, the full spectrum of semigroups of partial mappings preserving or reversing a linear or circular order. In the paper [12] the authors characterize the endomorphisms of the semigroup of all order-preserving partial transformations and of the semigroup of all order-preserving partial permutations of a finite chain. Also the semigroups of a finite chain are described in [16, 21]. Endomorphisms and automorphisms of other types of semigroups are studied in [2, 8, 9, 14, 17, 19, 20, 22–24, 26] and other papers.

This paper is a continuation of the investigation which are presented in [7]. Here we describe injective endomorphisms of the semigroup $\mathcal{J}_n(\text{conv} \omega)$ for a positive integer $n \geq 2$. In particular we show that for $n \geq 2$ the semigroup of injective endomorphisms of the semigroup $B_\omega^{\mathcal{F}_n}$ is isomorphic to $(\omega, +)$. Also we describes the structure of the semigroup $\text{End}(\mathcal{B}_\lambda)$ of all endomorphisms of the semigroup of $\lambda \times \lambda$-matrix units $\mathcal{B}_\lambda$. 

ON THE SEMIGROUP OF INJECTIVE ENDOMORPHISMS OF THE SEMIGROUP $B_\omega^{\mathcal{F}_n}$ 

3
Proposition 1. For any non-negative integer \( n \) and arbitrary \( p \in \omega \) the map \( \epsilon_p : B^{\omega n}_\omega \rightarrow B^{\omega n}_\omega \) defined by the formulae (0)\( \epsilon_p = 0 \) and
\[
(i, j, [0; k]) \epsilon_p = (p + i, p + j, [0; k]),
\]
is an endomorphism of the semigroup \( B^{\omega n}_\omega \).

Proof. It is obvious that \((0) \epsilon_p \cdot (0) \epsilon_p = 0 \cdot 0 = 0 = (0) \epsilon_p = (0 \cdot 0) \epsilon_p \) and
\[
(0) \epsilon_p \cdot (i, j, [0; k]) \epsilon_p = 0 \cdot (p + i, p + j, [0; k]) = 0 = (0) \epsilon_p = (0 \cdot (i, j, [0; k])) \epsilon_p,
\]
for any non-zero element \((i, j, [0; k])\) of the semigroup \( B^{\omega n}_\omega \). Also, for any non-zero elements \((i_1, j_1, [0; k_1])\) and \((i_2, j_2, [0; k_2])\) of the semigroup \( B^{\omega n}_\omega \) we have that
\[
(i_1, j_1, [0; k_1]) \epsilon_p \cdot (i_2, j_2, [0; k_2]) \epsilon_p = (p + i_1, p + j_1, [0; k_1]) \cdot (p + i_2, p + j_2, [0; k_2]) =
\]
\[
= \begin{cases} 
(p + i_1 - (p + j_1) + p + i_2, p + j_2, (p + j_1 - (p + i_2) + [0; k_1]) \cap [0; k_2]), & \text{if } p + j_1 < p + i_2; \\
(p + i_1, p + j_2, [0; k_1] \cap [0; k_2]), & \text{if } p + j_1 = p + i_2; \\
(p + i_1, p + j_1 - (p + i_2) + p + j_2, (p + i_1 + j_1 - (p + i_2) + [0; k_2]) \cap [0; k_2]), & \text{if } p + j_1 > p + i_2
\end{cases}
\]
and
\[
((i_1, j_1, [0; k_1]) \cdot (i_2, j_2, [0; k_2]) \epsilon_p = \begin{cases} 
(i_1 - j_1 + i_2, j_2, (i_1 - j_2 + [0; k_1]) \cap [0; k_2]) \epsilon_p, & \text{if } j_1 < i_2; \\
(i_1, j_2, [0; k_1] \cap [0; k_2]) \epsilon_p, & \text{if } j_1 = i_2; \\
(i_1, j_1 - i_2 + j_2, [0; k_2] \cap (i_2 - j_1 + [0; k_2]) \epsilon_p, & \text{if } j_1 > i_2.
\end{cases}
\]
and hence the map \( \epsilon_p \) is an endomorphism of the semigroup \( B^{\omega n}_\omega \). \( \square \)

By Theorem 1 of \([7]\) for any \( n \in \omega \) the semigroup \( B^{\omega n}_\omega \) is isomorphic to the semigroup \( \mathcal{I}^{n+1}(\mathcal{C}_\omega \text{conv}) \) by the mapping \( \mathcal{I} : B^{\omega n}_\omega \rightarrow \mathcal{I}^{n+1}(\mathcal{C}_\omega \text{conv}) \), defined by the formulae (0)\( \mathcal{I} = 0 \) and
\[
(i, j, [0; k]) \mathcal{I} = (i_{i+1} j_{i+1} \cdots i_{j+k}).
\]
This and Proposition 1 imply the following corollary.

Corollary 1. For any positive integer \( n \) and arbitrary \( p \in \omega \) the map \( \epsilon_p : \mathcal{I} \rightarrow \mathcal{I} \) defined by the formulae (0)\( \epsilon_p = 0 \) and
\[
(i, j, [0; k]) \epsilon_p = (i_{i+1} j_{i+1} \cdots i_{j+k}),
\]
is an endomorphism of the semigroup \( \mathcal{I} \).

Later we shall study endomorphisms of the semigroup \( \mathcal{I}(\mathcal{C}_\omega \text{conv}) \).

Lemma 1. Let \( n \) be any positive integer and \( a \) be an arbitrary non-annihilating endomorphism of the semigroup \( \mathcal{I}^{n}(\mathcal{C}_\omega \text{conv}) \). Then \( (0) a = 0 \).

Proof. Since \( 0 \) is an idempotent of \( \mathcal{I}^{n}(\mathcal{C}_\omega \text{conv}) \), so is the image \( (0) a \). Suppose to the contrary that \( (0) a = e \neq 0 \). By Theorem 3 from [7] the image of \( \mathcal{I}^{n}(\mathcal{C}_\omega \text{conv}) \) under the endomorphism \( a \) is isomorphic to the semigroup \( \mathcal{I}^{m}(\mathcal{C}_\omega \text{conv}) \) for some positive integer \( m \leq n \). Hence the subsemigroup \( \mathcal{I}^{n}(\mathcal{C}_\omega \text{conv}) a \) of \( \mathcal{I}^{n}(\mathcal{C}_\omega \text{conv}) \) has infinitely many idempotents. But by Theorem 1 and Lemma 1 from [7] the set \( \uparrow_{\mathcal{I}} e \) is finite, a contradiction. The obtained contradiction implies the equality \( (0) a = 0 \). \( \square \)
Lemma 1 implies the following corollary.

**Corollary 2.** Let $n$ be any positive integer and $a$ be an endomorphism of the semigroup $\mathcal{I}_n^{\omega}(\text{conv})$. If $(0)a \neq 0$ then $a$ is annihilating.

**Lemma 2.** Let $n$ be any positive integer $\geq 2$ and $a$ be an arbitrary non-annihilating endomorphism of the semigroup $\mathcal{I}_n^{\omega}(\text{conv})$. If $(0)a = (0)$ then $a$ is the identity automorphism of $\mathcal{I}_n^{\omega}(\text{conv})$.

**Proof.** First we shall show that the restriction of the endomorphism $a$ onto the band $E(\mathcal{I}_n^{\omega}(\text{conv}))$ is the identity map of $E(\mathcal{I}_n^{\omega}(\text{conv}))$.

The definition of the natural partial order $\preccurlyeq$ on $E(\mathcal{I}_n^{\omega}(\text{conv}))$ implies that

$$\uparrow \preccurlyeq (0) = \{(0), (0 \ 1), \ldots, (0 \ 1 \cdots n-1)\}.$$  

By Proposition 1.14.21(6) of [15] every homomorphism of inverse semigroups preserves the natural partial order, and hence $(0)a \preccurlyeq (0 \ 1)a$, because $(0) \preccurlyeq (0 \ 1)$. Also, by Proposition 4 of [7] every congruence on the semigroup $\mathcal{I}_n^{\omega}(\text{conv})$ is Rees, which implies that $(0)a \neq (0 \ 1)a$. Hence we obtain that $(0 \ 1)a = (0 \ 1)$. Similarly by induction we get that $(0 \ 1 \cdots k)a = (0 \ 1 \cdots k)$ for any $k = 2, \ldots, n-1$.

The definition of the natural partial order $\preccurlyeq$ on $E(\mathcal{I}_n^{\omega}(\text{conv}))$ implies that $0 \preccurlyeq (1) \preccurlyeq (0 \ 1)$. The above part of the proof, Lemma 1 and Proposition 4 of [7] imply that

$$0 = (0)a \preccurlyeq (1)a \preccurlyeq (0 \ 1)a = (0 \ 1).$$

Again, by the definition of the natural partial order $\preccurlyeq$ on $E(\mathcal{I}_n^{\omega}(\text{conv}))$ we have that the inequalities $0 \preccurlyeq x \preccurlyeq (0 \ 1)$ have two solutions either $x = (0)$ or $x = (1)$. Then Proposition 4 of [7] implies that $(1)a = (1)$. Similar arguments and the following conditions

$$(1) = (1)a \preccurlyeq (1 \ 2)a \preccurlyeq (0 \ 1 \ 2)a = (0 \ 1 \ 2)$$

imply that $(1 \ 2)a = (1 \ 2)$. Next by induction we get that $(1 \ 2 \cdots k+1)a = (1 \ 2 \cdots k+1)$ for any $k = 2, \ldots, n-1$.

We observe that the proof of the step of induction: the equalities

$$(p \ p) = (p \ p), \quad (p \ p+1) = (p \ p+1), \quad \ldots, \quad (p \ p+1 \ldots p+n-1) = (p \ p+1 \ldots p+n-1)$$

hold for $p \leq m$, imply that these equalities hold for $p = m + 1$, is similar to the above part of the proof.

Fix an arbitrary $x \in \mathcal{I}_n^{\omega}(\text{conv}) \setminus E(\mathcal{I}_n^{\omega}(\text{conv}))$ with rank $x = k$, $k = 1, \ldots, n$. Since $x$ is a partial convex order isomorphism of $(\omega, \preccurlyeq)$, there exist $s, t \in \omega$ such that $x = (s \ s+1 \cdots s+k-1)$.

Since $xx^{-1}, x^{-1}x \in E(\mathcal{I}_n^{\omega}(\text{conv}))$, by Proposition 1.14.21(1) of [15] we have that

$$(x)a \cdot ((x)a)^{-1} = (x)a \cdot (x^{-1})a =$$

$$= (xx^{-1})a =$$

$$= xx^{-1} =$$

$$= (s \ s+1 \cdots s+k-1) \cdot (s \ s+1 \cdots s+k-1)^{-1} =$$

$$= (s \ s+1 \cdots s+k-1) \cdot (s \ s+1 \cdots s+k-1) =$$

$$= (s \ s+1 \cdots s+k-1) \cdot (s \ s+1 \cdots s+k-1) =$$

and

$$(x)^{-1} \cdot (x)a = (x^{-1})a \cdot (x)a =$$

$$= (x^{-1}x)a =$$

$$= x^{-1}x =$$

$$= (s \ s+1 \cdots s+k-1)^{-1} \cdot (s \ s+1 \cdots s+k-1) =$$

$$= (t \ t+1 \cdots t+k-1) \cdot (s \ s+1 \cdots s+k-1) =$$

$$= (t \ t+1 \cdots t+k-1) \cdot (s \ s+1 \cdots s+k-1) =$$

$$= (t \ t+1 \cdots t+k-1).$$
The above equalities imply that
\[ \text{dom}((x)a) = \text{dom}((x)a \cdot (x)a^{-1}) = \{s, \ldots, s + k - 1\} \]
and
\[ \text{ran}((x)a) = \text{dom}((x)a)^{-1} \cdot (x)a = \{t, \ldots, t + k - 1\}. \]
Since \((x)a\) is a partial convex order isomorphism of \((\omega, \leq)\), we get that \((x)a = (s \ s+1 \ \ldots \ s+k-1)\), which completes the proof of the lemma. \qed

For visually simplify of the proof of Theorem 1, we schematically present the natural partial order on the semilattice \(E(I^\omega(\text{conv}))\) on Figure 1.

![Figure 1](image-url)

For any \(i_0 \in \omega\) we define the endomorphism \(e_{i_0} : I^\omega(\text{conv}) \to I^\omega(\text{conv})\) in the following way
\[
(0)c_{i_0} = 0, \quad (i \ j) c_{i_0} = \begin{cases} (i+i_0) & \text{if} \ i_0 = 0, \\ (i+1+j_0) & \text{if} \ i_0 = 1, \\ \vdots & \text{if} \ i_0 = k-1. \end{cases}
\]

**Theorem 1.** Let \(n\) be any positive integer \(\geq 2\). For every injective endomorphism \(a : I^\omega(\text{conv}) \to I^\omega(\text{conv})\) there exists \(i_0 \in \omega\) such that \(a = e_{i_0}\).

**Proof.** By Lemma 1 we get that \((0)a = 0\).

It is obvious that
\[ \mathcal{M} = \{(i \ i+1 \ \ldots \ i+n-1) : i \in \omega\} \]
is the set of all maximal idempotents of \(E(I^\omega(\text{conv}))\), and moreover every maximal chain in the semilattice \(E(I^\omega(\text{conv}))\) contains \(n + 1\) idempotents. Hence
\[ L_0 = \{(0,0), (0,1), \ldots, (0,1 \ldots n-1)\} \]
are maximal chains in $E(\mathcal{F}_n^0(\text{conv}))$. Since $\mathbf{a}$ is an injective endomorphism of the semigroup $\mathcal{F}_n^0(\text{conv})$, Proposition 1.14.21(6) of [15] implies that the images $(L_0)\mathbf{a}$ and $(L_1)\mathbf{a}$ are maximal chains in $E(\mathcal{F}_n^0(\text{conv}))$.

Put $(\frac{1}{i\cdot n-1}) \mathbf{a} = (i_0 \ i_0 \ i_0 + 1 \ i_0 + 2 \ i_0 + n - 1) \in \mathcal{M}$. Since rank $(\frac{1}{i\cdot n-1}) = n - 1$, $(\frac{1}{i\cdot n-1}) \preceq (\frac{1}{2\cdot n-1})$ and $(\frac{1}{\cdot n-1}) \preceq (\frac{1}{2\cdot n-1})$, the definition of the natural partial order on the semilattice $E(\mathcal{F}_n^0(\text{conv}))$ and Proposition 1.14.21(6) of [15] imply that either $(\frac{1}{i\cdot n-1}) \mathbf{a} = (i_0 \ i_0 \ i_0 + 1 \ i_0 + n - 2)$ or $(\frac{1}{i\cdot n-1}) \mathbf{a} = (i_0 + 1 \ i_0 + 2 \ i_0 + n - 1) \mathbf{a}$.

Suppose that $(\frac{1}{i\cdot n-1}) \mathbf{a} = (i_0 \ i_0 \ i_0 + 1 \ i_0 + n - 2)$). Since $(\frac{1}{2\cdot n}) \mathbf{a} \in \mathcal{M}$ we have that $(\frac{1}{2\cdot n}) \mathbf{a} \in \mathcal{M}$, and the definition of the natural partial order on the semilattice $E(\mathcal{F}_n^0(\text{conv}))$ and Proposition 1.14.21(6) of [15] imply that $(\frac{1}{2\cdot n}) \mathbf{a} = (i_0 \ i_0 \ i_0 + 1 \ i_0 + n - 2)$). Again, by the definition of the natural partial order on the semilattice $E(\mathcal{F}_n^0(\text{conv}))$ and Proposition 1.14.21(6) of [15] we obtain that $(\frac{1}{2\cdot n}) \mathbf{a} = (i_0 \ i_0 \ i_0 + 1 \ i_0 + n - 2)$ because rank $(\frac{1}{i\cdot n-1}) = n - 1$ and $(\frac{1}{2\cdot n}) \mathbf{a} \preceq (\frac{1}{2\cdot n}) \mathbf{a}$. Since $(\frac{1}{i\cdot n-1}) \mathbf{a} \preceq (\frac{1}{2\cdot n}) \mathbf{a}$ the above arguments imply that $(\frac{1}{3\cdot n+1}) \mathbf{a} = (i_0 \ i_0 \ i_0 \ i_0 + n - 2)$ and $(\frac{1}{3\cdot n+1}) \mathbf{a} = (i_0 \ i_0 \ i_0 \ i_0 + n - 2)$.

Next, we extend the above procedure step-by-step using the definition of the natural partial order on the semilattice $E(\mathcal{F}_n^0(\text{conv}))$ and Proposition 1.14.21(6) of [15] we get that $(\frac{i_0 + 1}{i_0 + 1} \ i_0 + 1 \ i_0 + n - 1) \mathbf{a} = (\frac{1}{i_0 + 1} \ i_0 + 2 \ i_0 + n - 2)$ and $(\frac{i_0 + 1}{i_0 + 1} \ i_0 + 1 \ i_0 + n - 1) \mathbf{a} = (\frac{i_0 + 1}{i_0 + 1} \ i_0 + 1 \ i_0 + n - 2)$.

The inequality $(\frac{1}{i\cdot n-1}) \mathbf{a} \preceq (\frac{1}{i\cdot n-1}) \mathbf{a}$ implies that $(\frac{1}{i\cdot n-1}) \mathbf{a} \preceq (\frac{1}{i\cdot n-1}) \mathbf{a}$, and hence the definition of the natural partial order on $E(\mathcal{F}_n^0(\text{conv}))$, injectivity of $\mathbf{a}$, Proposition 1.14.21(6) of [15] and the equality $(\frac{1}{i\cdot n-1}) \mathbf{a} = (\frac{1}{i_0 + 1} \ i_0 + 2 \ i_0 + n - 1) \mathbf{a}$ imply that $(\frac{1}{i\cdot n-1}) \mathbf{a} = (\frac{1}{i_0 + 1} \ i_0 + 1 \ i_0 + n - 2)$.

Again, since $(\frac{1}{2\cdot n-1}) \mathbf{a} \preceq (\frac{1}{2\cdot n-1}) \mathbf{a}$ and $(\frac{1}{2\cdot n-1}) \mathbf{a} \preceq (\frac{1}{2\cdot n-1}) \mathbf{a}$ we obtain that $(\frac{1}{2\cdot n-1}) \mathbf{a} \preceq (\frac{1}{2\cdot n-1}) \mathbf{a}$ and $(\frac{1}{2\cdot n-1}) \mathbf{a} \preceq (\frac{1}{2\cdot n-1}) \mathbf{a}$.

The above two inequalities and the equalities

Thus, we show that the initial case of induction holds.

Next we shall prove that the induction step holds: if for positive integer the equalities

hold for $p = 0, 1, \ldots, k$, then they hold for $p = k + 1$. 
By the assumption of induction we have that 
\[(k k +1 \ldots k n +n -1) a = (k+i_0 k+i_0+1 \ldots k+i_0+n-1) \quad \text{and}\quad \]
\[(k k +1 \ldots k n +n -2) a = (k+i_0 k+i_0+1 \ldots k+i_0+n-2).
\] Since the endomorphism \(a\) is injective, this, the inequalities \((k k +1 \ldots k n +n -2) k k +1 \ldots k n +n -1) \leq (k k +1 \ldots k n +n -1)\) and \((k k +1 \ldots k n +n -1) k k +1 \ldots k n +n -2) \leq (k k +1 \ldots k n +n -1)\), the definition of the natural partial order on the semilattice \(E(\mathcal{I}_\omega^n(\converse{\omega}))\) and Proposition 1.14.21(6) of [15] imply that 
\[(k k +1 \ldots k n +n -1) a = (k+i_0 k+i_0+1 \ldots k+i_0+n-1).
\] Again, since \((k k +1 \ldots k n +n -1) k k +1 \ldots k n +n -2) a = (k+i_0 k+i_0+1 \ldots k+i_0+n-2)\), the unique idempotent of \(E(\mathcal{I}_\omega^n(\converse{\omega}))\) which is greater than \((k k +1 \ldots k n +n -1)\) and it is distinct from the idempotent \((k k +1 \ldots k n +n -1)\), the definition of the natural partial order on the semilattice \(E(\mathcal{I}_\omega^n(\converse{\omega}))\) and Proposition 1.14.21(6) of [15] imply that 
\[(k k +1 \ldots k n +n -1) a = (k+i_0 k+i_0+1 \ldots k+i_0+n-1)\) and the above presented argument imply that 
\[(k k +1 \ldots k n +n -1) a = (k+i_0 k+i_0+1 \ldots k+i_0+n-2)\), and by the similar way step-by-step we obtain that the following equalities
\[
\begin{align*}
(k k +1 \ldots k n +n -2) a &= (k+i_0 k+i_0+1 \ldots k+i_0+n-2), \\
(k k +1 \ldots k n +n -1) a &= (k+i_0 k+i_0+1 \ldots k+i_0+n-1), \\
\ldots & \ldots \\
(k k +1) a &= (k+i_0+1 k+i_0+2).
\end{align*}
\]
hold, and hence we proved the step of induction.

Fix an arbitrary non-idempotent element \(x = (a a+1 \ldots a+m)\) of the semigroup \(\mathcal{I}_\omega^n(\converse{\omega})\), for some \(a, b \in \omega\) and \(m = 0, 1, \ldots, n - 1\). Then \(x x^{-1} = (a a+1 \ldots a+m)\) and \(x^{-1} x = (b b+1 \ldots b+m)\), and hence by the previous part of the proof we have that 
\[(x x^{-1}) a = (i_0+a i_0+a+1 \ldots i_0+a+m)\quad \text{and}\quad (x^{-1} x) a = (i_0+b i_0+b+1 \ldots i_0+b+m).
\]
Since \(\mathcal{I}_\omega^n(\converse{\omega})\) is an inverse subsemigroup of the symmetric inverse monoid \(\mathcal{I}_\omega\) over \(\omega\), we conclude that 
\[
\text{dom}((x) a) = \text{dom}((x x^{-1}) a) = \{i_0 + a, i_0 + a + 1, \ldots, i_0 + a + m\}
\]
and 
\[
\text{ran}((x) a) = \text{ran}((x^{-1} x) a) = \{i_0 + b, i_0 + b + 1, \ldots, i_0 + b + m\}.
\]
Now, the definition of the semigroup \(\mathcal{I}_\omega^n(\converse{\omega})\) implies that 
\[(x) a = (i_0+a i_0+a+1 \ldots i_0+a+m).
\]
By Corollary 1, \(a = e_{i_0}\) is an endomorphism of the semigroup \(\mathcal{I}_\omega^n(\converse{\omega})\), which completes the proof of the theorem. \(\square\)

Lemma 2 and Theorem 1 imply

**Corollary 3.** For any positive integer \(n \geq 2\) every automorphism of the semigroup \(\mathcal{I}_\omega^n(\converse{\omega})\) is the identity map of \(\mathcal{I}_\omega^n(\converse{\omega})\).

For any positive integer \(n\) and any injective endomorphisms \(e_{i_1}\) and \(e_{i_2}\) of the semigroup \(\mathcal{I}_\omega^n(\converse{\omega})\) simple calculations show that 
\[e_{i_1} \circ e_{i_2} = e_{i_1+i_2} = e_{i_2} \circ e_{i_1}.
\]
This and Theorem 1 imply

**Theorem 2.** For any positive integer \(n \geq 2\) the semigroup of injective endomorphisms of the semigroup \(\mathcal{I}_\omega^n(\converse{\omega})\) is isomorphic to the semigroup \((\omega, +)\). In particular the group of automorphisms of \(\mathcal{I}_\omega^n(\converse{\omega})\) is trivial.

Since by Theorem 3 of [7] for any \(n \in \omega\) the semigroup \(B_{\mathcal{I}_\omega^n(\converse{\omega})}\) is isomorphic to the semigroup \(\mathcal{I}_\omega^{n+1}(\converse{\omega})\), Corollary 3 and Theorem 2 imply the following two corollaries.
Corollary 4. For any positive integer \( n \) every automorphism of the semigroup \( B_{\omega}^{\times n} \) is the identity map of \( B_{\omega}^{\times n} \).

Corollary 5. For any positive integer \( n \) the semigroup of injective endomorphisms of the semigroup \( B_{\omega}^{\times n} \) is isomorphic to the semigroup \((\omega,+)\). In particular the group of automorphisms of \( B_{\omega}^{\times n} \) is trivial.

3. ON ENDOmorphISMS OF THE SEMIGROUP OF \( \lambda \times \lambda \)-matrix units

Let \( \lambda \) be a non-zero cardinal and \( 0 \not\in \lambda \times \lambda \). The set \( \mathcal{B}_\lambda = \lambda \times \lambda \cup \{0\} \) with the following semigroup operation

\[
(a, b) \cdot (c, d) = \begin{cases} 
(a, d), & \text{if } b = c; \\
0, & \text{otherwise}
\end{cases}
\]

and

\[
(a, b) \cdot 0 = 0 \cdot (a, b) = 0 \cdot 0 = 0,
\]

for all \( a, b, c, d \in \lambda \), is called the semigroup of \( \lambda \times \lambda \)-matrix units [3]. It is well known that \( \mathcal{B}_\lambda \) is a combinatorial, congruence-free, primitive, completely \( 0 \)-simple inverse semigroup [15,18], and moreover \( \mathcal{B}_\lambda \) is isomorphic to the semigroup \( \mathcal{I}_\lambda \). By Proposition 4 of [6] the semigroup \( B_{\omega}^{\times n} \) is isomorphic to the semigroup of \( \omega \times \omega \)-matrix units \( \mathcal{B}_\omega \) if and only if \( \mathcal{F} = \{F, \emptyset\} \), where \( F \) is a singleton subset of \( \omega \).

For a non-zero cardinal \( \lambda \) we denote by \( \mathcal{I}_\lambda \) the group of bijective transformations of \( \lambda \) and by \( \mathcal{A}(\mathcal{B}_\lambda) \) the semigroup of injective transformation of \( \lambda \).

Theorem 3. The semigroup \( \mathcal{E}n(\mathcal{B}_\lambda) \) of injective endomorphisms of \( \mathcal{B}_\lambda \) is isomorphic to \( \mathcal{I}_\lambda \), and moreover the group \( \mathcal{A}(\mathcal{B}_\lambda) \) of automorphisms of \( \mathcal{B}_\lambda \) is isomorphic to \( \mathcal{I}_\lambda \).

Proof. Let \( \epsilon \) be an injective endomorphism of \( \mathcal{B}_\lambda \). Then \( (0)\epsilon = 0 \) and the restriction of \( \epsilon \) onto \( E(\mathcal{B}_\lambda) \setminus \{0\} \) is an injection, i.e., there exists an injective transformation \( \iota_\epsilon : \lambda \to \lambda \) such that \( (a, a)\epsilon = ((a)i_\epsilon, (a)i_\epsilon) \) for any \( a \in \lambda \). It is obvious that \( \iota_\epsilon \in \mathcal{I}_\lambda \). Since the composition \( \iota_1 \circ \iota_2 \) of two injective endomorphisms \( \iota_1 \) and \( \iota_2 \) of \( \mathcal{B}_\lambda \) is an injective endomorphism,

\[
(a, a)(\iota_1 \circ \iota_2) = (((a)i_\iota_1), (a)i_\iota_2)\iota_2 = (((a)i_\iota_1)i_\iota_2, ((a)i_\iota_1)i_\iota_2),
\]

and hence \( \iota_1 \circ \iota_2 = \iota_1 \circ \iota_2 \) is an injective map of \( \lambda \). This implies that the such defined map \( \mathcal{J} : \mathcal{E}n(\mathcal{B}_\lambda) \to \mathcal{I}(\lambda) \), \( \epsilon \mapsto \iota_\epsilon \) is a homomorphism. Next we shall show that the homomorphism \( \mathcal{J} \) is surjective. Fix an arbitrary injective map \( \iota : \lambda \to \lambda \). We claim that the mapping \( \iota_\iota : \mathcal{B}_\lambda \to \mathcal{B}_\lambda \) by the formulae

\[
(a, b)\iota = ((a)i_\iota, (b)i_\iota) \quad \text{for all } a, b \in \lambda \quad \text{and} \quad (0)\iota = 0,
\]

is an injective endomorphism of the semigroup \( \mathcal{B}_\lambda \). Indeed, since the mapping \( \iota : \lambda \to \lambda \) is injective,

\[
(a, b)\iota \cdot (c, d)\iota = ((a)i_\iota, (b)i_\iota) \cdot ((c)i_\iota, (d)i_\iota) =
\]

\[
= \begin{cases} 
((a)i_\iota, (d)i_\iota), & \text{if } (b)i_\iota = (c)i_\iota; \\
0, & \text{otherwise}
\end{cases} =
\]

\[
= \begin{cases} 
(a, d)\iota, & \text{if } b = c; \\
0, & \text{otherwise}
\end{cases} =
\]

\[
= ((a, b) \cdot (c, d))\iota,
\]

and

\[
(a, b)\iota \cdot (0)\iota = (a, b)\iota \cdot 0 = 0 = (0)\iota = ((a, b) \cdot 0)\iota;
\]

\[
(0)\iota \cdot (a, b)\iota = 0 \cdot (a, b)\iota = 0 = (0)\iota = (0 \cdot (a, b))\iota;
\]

\[
(0)\iota \cdot (0)\iota = 0 \cdot 0 = 0 = (0)\iota = (0 \cdot 0)\iota,
\]

and hence \( \iota_\iota \) is an endomorphism of \( \mathcal{B}_\lambda \). It is obvious that the injectivity of \( \iota \) implies that the endomorphism \( \iota_\iota \) is injective, too.

Simple verifications show that if \( \epsilon \) be an automorphism of \( \mathcal{B}_\lambda \) then the mapping \( \iota_\iota : \lambda \to \lambda \) is bijective, and the bijectivity of the mapping \( \iota : \lambda \to \lambda \) implies that \( \iota_\iota \) is an automorphism of \( \mathcal{B}_\lambda \). This completes the proof of the last statement. \( \square \)
Recall [3], a semigroup $S$ is said to be left (right) cancellative if for all $a, b, c \in S$, the equality $ab = ac$ ($ba = ca$) implies $b = c$. We remark that simple verification show that the semigroup $\mathcal{I}_T(\lambda)$ (and hence $\mathcal{E}nd^{inj}(\mathcal{B}_\lambda)$) is left cancellative, but $\mathcal{I}_T(\lambda)$ is not right cancellative.

It is well known that the semigroup $\mathcal{B}_\lambda$ of $\lambda \times \lambda$-matrix units is congruence-free, i.e., $\mathcal{B}_\lambda$ has only two congruence: the identity and the universal congruence. This implies that every endomorphism of $\mathcal{B}_\lambda$ is either injective (i.e., is an isomorphism “into”) or annihilating.

By $\mathcal{E}nd^{ann}(\mathcal{B}_\lambda)$ we denote the semigroup of all annihilating endomorphisms of $\mathcal{B}_\lambda$.

It is obvious that for every annihilating endomorphism $a$ of $\mathcal{B}_\lambda$ there exits an idempotent $x \in \mathcal{B}_\lambda$ such that $(y)a = x$ for all $y \in \mathcal{B}_\lambda$, and later such endomorphism we denote by $a_x$. This implies that $\mathcal{E}nd^{ann}(\mathcal{B}_\lambda) = \{a_0\} \cup \{a_{(a,a)} : a \in \lambda\}$.

It is obvious that $\mathcal{E}nd^{ann}(\mathcal{B}_\lambda)$ is a right zero semigroup, $\mathcal{E}nd^{ann}(\mathcal{B}_\lambda)$ is left simple and hence it is simple. For any $c \in \mathcal{E}nd^{inj}(\mathcal{B}_\lambda)$ and $a_x \in \mathcal{E}nd^{ann}(\mathcal{B}_\lambda)$ we have that $c \circ a_x = a_x$ and $a_x \circ c = a_{(x)c}$.

The above arguments we summarize in the following theorem:

**Theorem 4.** The semigroup $\mathcal{E}nd(\mathcal{B}_\lambda)$ of all endomorphisms of the semigroup of $\lambda \times \lambda$-matrix units $\mathcal{B}_\lambda$ is the union of the semigroups $\mathcal{E}nd^{inj}(\mathcal{B}_\lambda)$ and $\mathcal{E}nd^{ann}(\mathcal{B}_\lambda)$. Moreover, $\mathcal{E}nd^{inj}(\mathcal{B}_\lambda)$ a left cancellative semigroup and $\mathcal{E}nd^{ann}(\mathcal{B}_\lambda)$ is the minimal ideal of $\mathcal{E}nd(\mathcal{B}_\lambda)$ which is a right zero semigroup.

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