Abstract: The present article studies the potential form of the nonlinear Gardner-Kawahara equation through the perspective of Lie symmetry analysis. Lie symmetry analysis was used to investigate abundant group-invariant solutions of the nonlinear Gardner-Kawahara equation. This method is used to provide geometric vector fields, as well as their commutative and adjoint relations. In this article, we have obtained the exact solution of the nonlinear Gardner-Kawahara equation in explicit form by different significant methods. Numerical simulation is used to explain the physical relevance of invariant solutions in 3D and 2D graphs. Finally, by the conservation law multiplier, the complete set of local conservation laws of the equation for the arbitrary constant coefficients is well constructed with a detailed derivation. The conserved currents discovered in this study can help us better comprehend some of the physical processes that the underlying equations predict.

Key words: Lie symmetry analysis, Gardner-Kawahara equation, Power series solutions, tanh method, Conservation laws

1. Introduction
In recent years, many researchers have used nonlinear partial differential equations (NPDEs) to model problems in various fields of applied science and engineering, such as chemical physics, optical fibre, plasma physics, fluid dynamics and many more [1, 2]. It is also significant to look at the exact explicit solutions of these models for understanding the physical phenomenon. Several methods are available to solve these types of NPDEs, such as the Lie symmetry method [3–7], Hirota bilinear method [8, 9], Bäcklund transformations method [10], inverse scattering method [11] and homogeneous balance method [12].

The Lie symmetry method is the most sophisticated to solve the NPDEs because it is based on the concrete mathematical framework and also it is the one of the powerful method to find symmetries of partial
differential equations. Lie symmetry method was developed by Sophus Lie (1842–1899) in the latter half of the nineteenth century. For details of the technique, one can study the popular literature available in the Ref. [3–6]

The extended KdV equation is [13].

\[
\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} + \lambda u \frac{\partial u}{\partial x} - \alpha u^2 \frac{\partial u}{\partial x} + \mu \frac{\partial^3 u}{\partial x^3} + \beta \frac{\partial^5 u}{\partial x^5} + \gamma_1 u \frac{\partial^3 u}{\partial x^3} + \gamma_2 \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} = 0, \quad (1.1)
\]

where \(a, \lambda, \alpha, \mu, \beta, \gamma_1, \gamma_2\) are arbitrary constants. In the present work, the following nonlinear Gardner-Kawahara equation (NLGK) equation [14, 15] has been studied. The NLGK equation is formed by taking \(\gamma_1 = \gamma_2 = 0\).

\[
\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} + \lambda u \frac{\partial u}{\partial x} - \alpha u^2 \frac{\partial u}{\partial x} + \mu \frac{\partial^3 u}{\partial x^3} + \beta \frac{\partial^5 u}{\partial x^5} = 0, \quad (1.2)
\]

NLGK equation has been investigated for symmetry reductions and conservation laws. The above NLGK equation is a particular case of extended KdV equation. This equation widely occurs in magneto-acoustic waves in plasma physics and also in the shallow water waves with surface tension. Physically, the NLGK equation explains the solitary wave propagation in media. After considering \(\alpha = \gamma_1 = \gamma_2 = 0\) in the extended KdV equation, the Kawahara equation will be constructed.

In recent years, the NLGK equation attracted by many researchers. The exact travelling wave solutions for NLGK equation has been investigated in Ref [14] by implementing \((G')\) method. Kurkina et al. [15] have studied the stationary and soliton solutions of the NLGK equation. The conservation law plays an important role in the general theory and the analysis of certain systems. Conservation laws [6, 17, 18] are essential for the understanding of the basic laws of nature and problems in mathematical physics. It is also identify the properties of the arbitrary NPDEs. Conservation laws are the most important and are broadly used in the study of the basic structures of NPDEs. In the field of physics and engineering, conservation laws allow the conclusion of a physical system, and there are some conservation laws such as conservation of mass and energy, conservation of natural resources, conservation of energy, mass, electric charge, or momentum, etc. The best method to comprehend the physical processes of nonlinear ordinary differential equations is to find exact solutions (ODEs). The power series method [7, 20] and tanh method [21, 22] are the most effective methods to determining the exact solutions of nonlinear ODEs.

Our main focus is to study NLGK Eq. (1.2). We apply the Lie symmetry method and use symmetry reduction to transform NLGK equation into a nonlinear ODE. Thereafter, to find the exact solution of NLGK equation, we have used the power series solution method and tanh method. To finish this work, we used a multiplier technique to build the conservation rules of (1.2).

2. Lie symmetry analysis of Nonlinear Gardner-Kawahara equation

In this section, the classical Lie symmetry method has been introduced for the NLGK equation. The one parameter (\(\epsilon\)) Lie group of transformation for the Eq.(1.2) with dependent and
independent variable is given below
\[ x^* = x + \epsilon \xi_1(x, t, u) + O(\epsilon^2), \]
\[ t^* = t + \epsilon \xi_2(x, t, u) + O(\epsilon^2), \]
\[ u^* = u + \epsilon \eta(x, t, u) + O(\epsilon^2). \]  
(2.1)

Here \( \epsilon \) is a group parameter and \( \xi_1, \xi_2, \eta \) are infinitesimals of the transformation for the independent and dependent variable \( x, t, u \) respectively, which we have to be determined later.

The associated vector fields corresponding to Eq. (1.2) can be written as:
\[ V = \xi_1(x, t, u) \frac{\partial}{\partial x} + \xi_2(x, t, u) \frac{\partial}{\partial t} + \eta(x, t, u) \frac{\partial}{\partial u}. \]  
(2.2)

If above Eq. (2.2) generates a symmetry Eq. (1.2) the infinitesimal generator \( V \) must satisfy the following invariance criterion for Eq. (1.2):
\[ Pr^{(5)}V(\Delta_1)|\Delta_1 = 0, \]  
(2.3)

where
\[ \Delta_1 = \frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} + \lambda u \frac{\partial u}{\partial x} - \alpha u \frac{\partial^3 u}{\partial x^3} + \mu \frac{\partial^5 u}{\partial x^5} = 0. \]  
(2.4)

\( V \) is vectorfield. In general if \( x = (x_1, x_2, ..., x_q) \) is \( q \) independent variable and \( u = (u_1, u_2, ..., u_p) \) is \( p \) dependent variable, then the related vector field \( V \) will be:
\[ V = \sum_{i=1}^{q} \xi^i(x, u) \partial_{x^i} + \sum_{j=1}^{p} \eta^j(x, u) \partial_{u^j}. \]  
(2.5)

The method for finding nth order prolongation formula is given by:
\[ Pr^{(n)}V = V + \sum_{k=1}^{p} \sum_{J} \eta_k^J(x, u^{(n)}) \partial_{u_j^w}, \]  
(2.6)

where
\[ J = (j_1, j_2, ..., j_w), \ 1 \leq j_w \leq p, \ 0 \leq J \leq n, \ 1 \leq w \leq n \]  
(2.7)

and
\[ \eta_k^J(x, u^{(n)}) = D_J \left( \eta_k - \sum_{i=1}^{q} \xi^i u_k^i \right) + \sum_{i=1}^{q} \xi^i \frac{\partial}{\partial u_{J,i}^w}. \]  
(2.8)
where
\[ u^k_i = \frac{\partial u^k}{\partial u^i}, u^k_{i,j} = \frac{\partial u^k_i}{\partial x^j} \quad (2.9) \]

and \( D_J \) indicates the total derivative. Thus the fifth order prolongation:
\[ P^{(5)}_r V = V + \eta^t \frac{\partial}{\partial u^t} + \eta^t \frac{\partial}{\partial u^t} + \eta^{xx} \frac{\partial}{\partial u^{xx}} + \eta^{xt} \frac{\partial}{\partial u^{xt}} + \ldots \eta^{tt} \]
\[ \frac{\partial}{\partial u^{tt}} + \eta^{xxx} \frac{\partial}{\partial u^{xxx}} + \eta^{xxt} \frac{\partial}{\partial u^{xxt}} + \ldots \eta^{ttt} \frac{\partial}{\partial u^{ttt}} + \ldots \quad (2.10) \]

The obtained result after applying the fifth order prolongation in Eq. (1.2)
\[ \eta^t + a\eta^t + \lambda\eta^t - a\eta^t^2 \eta^t + \mu\eta^t^{xxx} + \beta\eta^t^{xxxx} = 0. \quad (2.11) \]

One can solve the Eq. (2.11) by substituting the coefficient of several monomials equal to zero, leading to a system of determining equations that have been solved with MAPLE software [? ?].
\[ \xi^1 = c_2, \xi^2 = c_1, \eta = 0 \quad (2.12) \]

The infinitesimals of Eq. (1.2) is given by:
\[ X_1 = \frac{\partial}{\partial t}, X_2 = \frac{\partial}{\partial x} \quad (2.13) \]

The infinitesimals generator of Eq. (1.2) will be:
\[ V = c_1 X_1 + c_2 X_2 \quad (2.14) \]

The two translation symmetries of Eq. (2.13) used to find the symmetry reductions of Eq. (1.2).
Here we find a linear combination of these Lie symmetries, \( X = X_1 + cX_2 \), where \( c \) is constant, has been used. The Lagrange equation will be:
\[ \frac{dx}{1} = \frac{dt}{c} = \frac{du}{0} \quad (2.15) \]

The invariant solution will be
\[ z = (x - ct) \]
\[ U = u \quad (2.16) \]
Now the group-invariant solution becomes \( u(x, t) = U(z) \). Now \( U \) and \( z \) will be the new dependent and independent variables. After applying invariant solution into Eq. (1.2) reduces the following fifth-order non-linear ODE:

\[
\beta U'''' + \mu U'' - \alpha U^2 U' + \lambda UU' + aU' - cU' = 0. \tag{2.17}
\]

3. Exact explicit solution by power series method

In general, it is not an easy task to find exact and analytic solutions for this type of nonlinear ODE Eq. (2.17), by applying the elementary functions and integrals. In this section, we will derive the exact solution of the reduced Eq. (2.17) by using the power series method. The power series method \([7, 20]\) is a most significant method to solve higher-order ODEs.

Consider,

\[
U(z) = \sum_{n=0}^{\infty} c_n z^n. \tag{3.1}
\]

Differentiating Eq. (3.1) and substituting into Eq. (2.17) leads to

\[
c \sum_{n=0}^{\infty} (n+1) c_{n+1} z^n - a \sum_{n=0}^{\infty} (n+1) c_{n+1} z^n - \lambda \sum_{n=0}^{\infty} c_n z^n \sum_{n=0}^{\infty} (n+1) c_{n+1} z^n + \]

\[
\alpha \sum_{n=0}^{\infty} c_n z^n \sum_{n=0}^{\infty} c_n z^n \sum_{n=0}^{\infty} (n+1) c_{n+1} z^n - \mu \sum_{n=0}^{\infty} (n+1)(n+2)(n+3) c_{n+3} z^n -
\]

\[
\beta \sum_{n=0}^{\infty} (n+1)(n+2)(n+3)(n+4)(n+5) c_{n+5} z^n = 0,
\]

which implies

\[
c \sum_{n=1}^{\infty} (n+1) c_{n+1} z^n + cc_0 - a \sum_{n=1}^{\infty} (n+1) c_{n+1} z^n - ac_1 - \lambda \sum_{n=1}^{\infty} c_n z^n \sum_{n=1}^{\infty} (n+1) c_{n+1} z^n - \lambda c_0 c_1 +(3.3)
\]

\[
\alpha \sum_{n=1}^{\infty} c_n z^n \sum_{n=1}^{\infty} c_n z^n \sum_{n=1}^{\infty} (n+1) c_{n+1} z^n - \alpha c_0^2 c_1 - \mu \sum_{n=1}^{\infty} (n+1)(n+2)(n+3) c_{n+3} z^n - 6 \mu c_3 -
\]

\[
\beta \sum_{n=1}^{\infty} (n+1)(n+2)(n+3)(n+4)(n+5) c_{n+5} z^n - 120 \beta c_5 = 0,
\]
On elaborating the above Eq. 

\[ c \sum_{n=1}^{\infty} (n+1)c_{n+1}z^n - a \sum_{n=1}^{\infty} (n+1)c_{n+1}z^n - \lambda \sum_{n=1}^{\infty} \sum_{k=0}^{n} (n-k+1)c_kc_{n-k+1}z^n \]  

\[ + \alpha \sum_{n=1}^{\infty} \sum_{k=0}^{n} \sum_{i=0}^{k} (n-k+1)c_{n-k+1}c_{k-i}z^n - \mu \sum_{n=1}^{\infty} (n+1)(n+2)(n+3)c_{n+3}z^n - \]  

\[ \beta \sum_{n=1}^{\infty} (n+1)(n+2)(n+3)(n+4)(n+5)c_{n+5}z^n + \]  

\[ cc_0 - ac_1 - \lambda c_0c_1 - 6\mu c_3 - 120\beta c_5 - \alpha c_0^2 c_1 = 0 \]  

On solving Eq. (3.6),

\[ c_5 = \frac{cc_0 - ac_1 - \lambda c_0c_1 - 6\mu c_3 - \alpha c_0^2 c_1}{120\beta}, \]  

\[ c_{n+5} = \frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)\beta} \left[ c(n+1)c_{n+1} - a(n+1)c_{n+1} - \right. \]  

\[ \sum_{k=0}^{n} \lambda(n-k+1)c_kc_{n-k+1} + \sum_{k=0}^{n} \sum_{i=0}^{k} (n-k+1)c_{n-k+1}c_{k-i} \]  

\[ \left. - \mu(n+1)(n+2)(n+3)c_{n+3} \right], \]  

On the other hand, for finding the coefficient \( c_n \) \( (n \geq 5) \) of the power series Eq. (3.6) and Eq. (3.7) has been determined by taking arbitrary constant \( c, c_0, c_1, c_3, c_2, c_4, a, \alpha, \lambda, \beta, \mu \) and also clearly, the power series solution depends on Eq. (3.6) and Eq. (3.7).

\[ U(z) = c_0 + c_1z + c_2z^2 + c_3z^3 + c_4z^4 + c_5z^5 + \sum_{n=1}^{\infty} c_{n+5}z^{n+5}. \]
Furthermore, we obtain the explicit power series solution for Eq. (1.2)

\[ u(x,t) = c_0 + c_1(x - ct) + c_2(x - ct)^2 + c_3(x - ct)^3 + c_4(x - ct)^4 + \left( \frac{cc_0 - ac_1 - \lambda c_0 c_1 - 6\mu c_3 - \alpha c_0^2 c_1}{120\beta} \right)(x - ct)^5 \]

\[ + \sum_{n=1}^{\infty} \left( \frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)} \right) \left[ c(n+1)c_{n+1} - a(n+1)c_{n+1} - \sum_{k=0}^{n} \lambda(n-k+1)c_k c_{n-k+1} \right] n \sum_{k=0}^{n} \sum_{i=0}^{k} (n-k+1)c_{n-k+1}c_i c_{k-i} \]

\[ - \mu(n+1)(n+2)(n+3)c_{n+3} \right] (x - ct)^n \sum_{i=1}^{n} a_i Y^i \]

\[ (3.9) \]

4. Exact explicit solution by tanh method

This section consists the algorithm of tanh method [21, 22] and its implementation for solving NLGK equation.

4.1. Algorithm for tanh method

The main steps of tanh method are discussed as follows

Step 1: Let us assume a NPDEs in this form

\[ F(u, u_x, u_{xx}, ..., u_t) = 0 \quad (4.1) \]

where, \( u = u(x, t) \) is an unknown function.

Step 2: Using the transformation,

\[ u(x, t) = U(z), \quad z = (x - ct), \quad (4.2) \]

where, \( c \) is constant to be determined later, the PDEs (4.1) is reduced to the ODEs as follows

\[ F(U, U', U'', ..., -cU') = 0 \quad (4.3) \]

Step 3: Suppose the solution of Eq.(4.3) can be expressed in the form

\[ U(z) = a_0 + \sum_{i=1}^{M} a_i Y^i \quad (4.4) \]
where \( a_i \) are constants to be determined, the integer \( M \) can be calculated by balancing the highest order derivative term with the highest order nonlinear term occurring in Eq.\( (4.3) \) and \( Y = \tanh(z) \) is new independent variable. The derivatives of Eq.\( (4.4) \) can be obtained as follows

\[
\frac{dU}{dz} \rightarrow (1 - Y^2) \frac{dU}{dY} \\
\frac{d^2U}{dz^2} \rightarrow (1 - Y^2) \left( -2Y \frac{dU}{dY} + (1 - Y^2) \frac{d^2U}{dY^2} \right)
\]

and so on for higher.

**Step 4** Substituting Eq.\( (4.4) \) and \( (4.5) \) in Eq.\( (4.3) \), the solution will get in terms of \( Y^i (i = 0, 1, 2, 3...) \). Then equating to zero of same degree \( Y^i (i = 0, 1, 2, 3...) \) results into a set of algebraic equations for \( a_i (i = 0, 1, 2...) \), \( c, a, \alpha, \beta, \mu \).

**Step 5**: Solving the resulting algebraic system, the exact explicit solution of Eq.\( (4.1) \) can be obtained.

### 4.2. Implementation of tanh method for solving NLGK equation

For finding the exact solution of Eq.\( (2.17) \), the above algorithm is implemented. Now integrating the Eq.\( (2.17) \) with respect to \( z \), we will get

\[
(a - c)U + \frac{\lambda}{2}U^2 - \frac{\alpha}{3}U^3 + \mu U''' + \beta U'''' = 0.
\]

Hence \( M=2 \). From Eq.\( (4.4) \), the solution Eq.\( (2.17) \) can be expressed as

\[
U(z) = a_0 + a_1 Y + a_2 Y^2
\]

where, \( Y = \tanh(z) \). By substituting Eq.\( (4.7) \) and \( (4.5) \) into the Eq.\( (4.6) \), collecting all terms of same powers of \( Y \) and equating them to zero, a set of algebraic equations can be obtained.

On solving the resulting algebraic system, the following set of solutions can be achieved.

**Set 1**

\[
a_0 = 0, \quad a_1 = 0, \quad c = \frac{5a + \alpha a^2}{5}, \quad \beta = \frac{\alpha a^2}{360}, \quad \lambda = \frac{16\alpha a^2}{15}, \quad \mu = \frac{\alpha a^2}{45}
\]

\[
u(x,t) = a_2 \left( \tanh \left[ x - \left( \frac{5a + \alpha a^2}{5} \right) t \right] \right)^2
\]

**Set 2**

\[
a_1 = 0, \quad a_2 = -2a_0, \quad \alpha = \frac{15(a - c)}{7a^2}, \quad \beta = \frac{a - c}{42}, \quad \lambda = \frac{4(a - c)}{7a_0}, \quad \mu = \frac{(a - c)}{3}
\]

\[
u(x,t) = a_0 - 2a_0 \left( \tanh(x - ct) \right)^2
\]
Set 3

\[ a_1 = 0, \ a_2 = -a_0, \ c = \frac{(45a + 15\lambda a_0 - 8\alpha a_0^2)}{45}, \]

\[ \beta = \frac{\alpha a_0^2}{360}, \ \mu = \frac{3\lambda a_0 - 2\alpha a_0^2}{36} \]  \hspace{1cm} (4.10)

\[ u(x, t) = a_0 - a_0 \left( \tanh \left[ x - \left( \frac{45a + 15\lambda a_0 - 8\alpha a_0^2}{45} \right) t \right] \right)^2 \]  

Figure 1. Soliton solution of Eq. (4.8), by setting parameters \( a_2 = 1, \ a = 1, \ \alpha = 2 \).

Figure 2. 2D plot Eq.(4.8), by setting suitable arbitrary parameters \( a_2 = 1, \ a = 1, \ \alpha = 2, \ t = 1 \).

Figure 3. Solitary wave profile of Eq.(4.9) by setting parameters \( a_0 = 1, \ c = 1.4 \).

Figure 4. 2D plot of Eq.(4.9) by setting parameters \( a_0 = 1, \ c = 1.4, \ t = 1 \).

5. Construction of conservation laws via multiplier method

In this section, non-trivial local conservation laws for the NLGK equation have been constructed. The multiplier method of Anco and Bluman \[6, 17, 18\] has been used for this construction. The
Figure 5. Solitary wave profile of Eq. (4.10) by setting all parameters $a_0, a, \lambda$ to unity.

Figure 6. 2D plot of Eq. (4.10) by setting all parameters $a_0, a, \lambda, t$ to unity.

detailed application of this method can be seen in recent work [23–25].

Definition 1 The Euler operator for the dependent variable $u^j$ is defined by

$$E_{u^j} = \frac{\delta}{\delta u^j} = \frac{\partial}{\partial u^j} + \sum_{p=1}^{\infty} (-1)^p D_{i_1} \ldots D_{i_p} \frac{\partial}{\partial u^{i_1 \ldots i_p}}$$  \hspace{1cm} (5.1)$$

for each $j = 1, \ldots, m$.

Definition 2 The total differentiation which is defined as,

$$D_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_j} + \ldots$$  \hspace{1cm} (5.2)$$

which is defined with respect to independent variable $x=(x^1, \ldots, x^n)$. The Euler operator $E_{u}$ Eq.(5.1) which can annihilate any divergence expression $D_i \Psi^i(u)$ [12]. This method investigates determination of the zero order multiplier $\Lambda(x, t, u)$ such that

$$E_u[\Lambda(x, t, u)(u_t + au_x + \lambda uu_x - \alpha u^2 u_x + \mu u_{xxx} + \beta u_{xxxxx})] = 0$$  \hspace{1cm} (5.3)$$

where, $E_u$ is the Euler operator defined in Eq.(5.1). On expanding the Eq.(5.3), the following determination system can be obtained for multiplier $\Lambda(x, t, u)$

$$\Lambda_{uu} = 0, \quad \Lambda_{xu} = 0, \quad -\Lambda_t - \beta \Lambda_{xxxxx} - \mu \Lambda_{xxx} - \Lambda_x(-\alpha u^2 + \lambda u + a) = 0$$  \hspace{1cm} (5.4)$$

By solving above equation (5.4) one can obtain

$$\Lambda_1(x, t, u) = 1, \quad \Lambda_2(x, t, u) = u.$$  \hspace{1cm} (5.5)$$

Each multiplier from Eq.(5.5) determines local conservation laws in the format

$$D_x \Psi_1(x, t, u) + D_t \Psi_2(x, t, u) = 0$$  \hspace{1cm} (5.6)$$
with the characteristics form:
\[
\Lambda(x, t, u)(u_t + au_x + \lambda uu_x - \alpha u^2 u_x + \mu u_{xxx} + \beta u_{xxxxx}) = D_x \Psi_1(x, t, u) + D_t \Psi_2(x, t, u)
\]  
(5.7)

The inversion of divergence expression Eq. (5.7) can be carried out by 2-dimensional homotopy operator \([? ]\) and results read the following:

• \(\Lambda_1(x, t, u) = 1\)

\[
\psi_1(x, t, u) = -\frac{1}{3} \alpha u^3 + \frac{1}{2} \lambda u^2 + au + \beta u_{xxx} + \mu u_x,
\]
\[
\psi_2(x, t, u) = u
\]

• \(\Lambda_2(x, t, u) = u\)

\[
\psi_1(x, t, u) = -\frac{1}{4} \alpha u^4 + \frac{1}{3} \lambda u^3 + \frac{1}{2} au^2 + \beta uu_{xxx} - \beta u_x u_{xx} + \frac{1}{2} \beta u_{xx}^2 + \mu uu_x - \frac{1}{2} \mu u_x^2,
\]
\[
\psi_2(x, t, u) = \frac{u^2}{2}
\]  
(5.8)

The non-triviality of conservation laws can be checked from the fact that divergence expression \(D_x \Psi_1(x, t, u) + D_x \Psi_2(x, t, u)\) vanishes on solution space of the NLGK equation, and most importantly, none of them vanishes on solution space of NLGK equation.

6. Conclusions

In the present work, Lie symmetry analysis on the nonlinear NLGK equation has been studied. The Lie group method is utilized to acquire the symmetry reductions of NLGK equation. Using the Lie symmetry reductions and the power series method, the exact analytic solution of NLGK equation has been derived from the reduction equation. Also, the exact solutions are obtained by implementing the tanh method to the reduction equation. From the analysis of the acquired solutions, it may be concluded that NLGK equation produces a collection of travelling wave solutions. Finally, the conservation laws for NLGK equation have been also constructed by utilizing the multiplier method.

Acknowledgment

The authors are grateful to the referees for their educative and constructive feedback, which helped to improve the manuscript. The first author would like to thank the Department of Science and Technology (DST) of Government of Odisha, India for financial support in the form of PhD sponsorship.
References

1. Lokenath D. Nonlinear Partial Differential Equations for Scientists and Engineers, London: Springer Science Business Media 2011.
2. Wazwaz AM. Partial Differential Equations and Solitary Waves Theory USA: Springer Science & Business Media 2010.
3. Bluman GW, Kumei S. Symmetries and Differential Equations, USA: Springer Science & Business Media 2013.
4. Olver PJ. Applications of Lie groups to Differential Equations, New York, NY, USA: Springer Science & Business Media 2011.
5. Ibragimov NH. CRC handbook of Lie Group analysis of Differential Equations, USA: CRC press 1995.
6. Bluman GW and Anco SC. Applications of Symmetry Methods to Partial Differential Equations, New York, NY, USA: Springer 2010.
7. Zhang Y. Lie symmetry analysis and exact solutions of the Sawada-Kotera equation. Turkish Journal of Mathematics. 2017; 41(1): 158-67. doi: 10.3906/mat-1504-29.
8. Hirota R. Exact solution of the Korteweg-de Vries equation for multiple collisions of solitons. Physical Review Letters. 1971; 27(18): 1192. doi: doi.org/10.1103/PhysRevLett.27.1192.
9. Ma WX, Zhang Y, Tang Y, Tu J. Hirota bilinear equations with linear subspaces of solutions. Applied Mathematics and Computation. 2012; 218(13): 7174-83. doi: 10.1016/j.amc.2011.12.085.
10. Vakhnenko VO, Parkes EJ, Morrison AJ. A Bäcklund transformation and the inverse scattering transform method for the generalised Vakhnenko equation. Chaos, Solitons & Fractals. 2003; 17(4): 683-92. doi: https://doi.org/10.1016/S0960-0779(02)00483-6.
11. Ablowitz MJ, Ablowitz MA, Clarkson PA, Clarkson PA. Solitons, nonlinear evolution equations and inverse scattering. New York, NY, USA: Cambridge university press; 1991.
12. Senthilvelan M. On the extended applications of homogenous balance method. Applied Mathematics and Computation. 2001; 123(3): 381-8. doi: https://doi.org/10.1016/S0096-3003(00)00076-X.
13. Marchant TR, Smyth NF. The extended Korteweg-de Vries equation and the resonant flow of a fluid over topography. Journal of Fluid Mechanics. 1990; 221: 263-87. doi: https://doi.org/10.1017/S0022112090003561.
14. Wafaa TA, HUSSEIN Z. Exact Travelling Wave Solutions of the Nonlinear Gardener-Kawahara Equation by the Standard \( G’/G \)–Expansion Method. Journal of Multidisciplinary Modeling and Optimization. 2019; 2(1): 43-51.
15. Kurtina O, Singh N, Stepanyants Y. Structure of internal solitary waves in two-layer fluid at near-critical situation. Communications in Nonlinear Science and Numerical Simulation. 2015; 22(1-3): 1235-42. doi: https://doi.org/10.1016/j.cnsns.2014.09.018.
16. Kawahara T. Oscillatory solitary waves in dispersive media. Journal of the physical society of Japan. 1972; 33(1): 260-4. doi: https://doi.org/10.1143/JPSJ.33.260.
17. Anco SC, Bluman G. Direct construction method for conservation laws of partial differential equations Part I: Examples of conservation law classifications. European Journal of Applied Mathematics. 2002; 1 3(5): 545-66. doi: https://doi.org/10.1017/S095679250100465X.
18. Anco SC, Bluman G. Direct construction method for conservation laws of partial differential equations Part II: General treatment. European Journal of Applied Mathematics. 2002; 13(5): 567-85. doi: https://doi.org/10.1017/S0956792501004661.
19. Gupta RK, Singh M. On invariant analysis and conservation laws for degenerate coupled multi-KdV equations for multiplicity \( l = 3 \).
[20] Kumar V, Kaur L, Kumar A, Koksal ME. Lie symmetry based-analytical and numerical approach for modified Burgers-KdV equation. Results in physics. 2018; 8: 1136-42. doi: https://doi.org/10.1016/j.rinp.2018.01.046.

[21] Malfliet W. Solitary wave solutions of nonlinear wave equations. American journal of physics. 1992; 60(7): 650-4. doi: 10.1119/1.17120.

[22] Wazwaz AM. The tanh method: exact solutions of the sine-Gordon and the sinh-Gordon equations. Applied Mathematics and Computation. 2005; 167(2): 1196-210. doi: https://doi.org/10.1016/j.amc.2004.08.005.

[23] Naz R. Conservation laws for some compacton equations using the multiplier approach. Applied Mathematics Letters. 2012; 25(3): 257-61. doi: https://doi.org/10.1016/j.aml.2011.08.019.

[24] Kumar M, Kumar R, Kumar A. On similarity solutions of Zabolotskaya–Khokhlov equation. Computers & Mathematics with Applications. 2014; 68(4): 454-63. doi: https://doi.org/10.1016/j.camwa.2014.06.020.

[25] Singh M, Gupta RK. On painleve analysis, symmetry group and conservation laws of Date–Jimbo–Kashiwara–Miwa equation. International Journal of Applied and Computational Mathematics. 2018; 4(3): 1-5. doi: https://doi.org/10.1007/s40819-018-0521-y.