New $p+1$ Dimensional Non-relativistic Theories from Euclidean Stable and Unstable Dp-Branes

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Abstract: In this paper we continue the study of non-relativistic $p+1$ dimensional theories that we started at arXiv:0904.1343. We extend the analysis presented there to the case of stable and unstable Dp-branes.

Keywords: D-branes
1. Introduction

In past few months P. Hořava formulated new interesting theories with anisotropic scaling between time and spatial dimensions in series of papers [1, 2, 3, 4]. This phenomenon is well known from the study of condensed matter systems at quantum criticality [35]. The similar systems have been investigated from the point of view of non-relativistic form of AdS/CFT correspondence [36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 60, 61, 62, 63, 64, 65, 66, 67].

The construction of the theories with anisotropic scaling is based on the following question: Is it possible to find a $p + 1$ dimensional quantum theory such that its ground state wave functional reproduces the partition function of $p$ dimensional theory?

One particular example of such $p$ dimensional theories was studied in our previous paper [34] where we constructed new non-relativistic $p + 1$ dimensional theories from the Nambu-Gotto action for p-branes in general background. The goal of this paper is to implement similar ideas to the case when $p$ dimensional system is either stable or unstable D(p-1)-brane action. We also presume that this action is embedded in general background with non-trivial dilaton, gravity and NSNS two-form field. As in case of p-brane theory we should stress that this $p$ dimensional action is highly non-linear with all well known consequences for the renormazibility and quantum analysis of given action. Further, even if we consider Dp-brane actions that are well known from superstring theories we do not address the question how these new theories can be realized in superstring ones. In particular,
we do not worry about relation between dimensions of Dp-branes and their realizations in either IIA or type IIB theories.

Even if we leave the question of explicit realization of these theories in the framework of string theory open we mean that these new theories are interesting in own sense. These $p + 1$ dimensional theories are invariant under spatial diffeomorphism and spatial gauge transformations as well under global time translation. Then following [3] we show that these symmetries can be extended to the space-time diffeomorphism that respect the preferred codimension-one foliation of $p + 1$ dimensional space-time by the slices at fixed time known as \textit{foliation-preserving diffeomorphism}. Note that these transformations consist a space-time dependent spatial diffeomorphism together with time-dependent reparameterization of time. Further, we also extend the spatial dependent gauge transformations to the space-time dependent ones. Then in order to achieve of the invariance of $p + 1$ dimensional action under these symmetries we introduce gauge fields $N^i$ and $N$ to maintain its invariance under foliation preserving diffeomorphism and $A_t$ to maintain gauge invariance. As in case of non-critical $p$-brane theory [34] these new gauge fields will be crucial for the correct Hamiltonian formulation of the theory as the theory of constraint systems and we show that the Hamiltonian can be expressed as a linear combination of constraints.

We also address the question of the tachyon condensation on the world-volume of $p + 1$ dimensional theory. We show that this theory has a natural spatial dependent solution known as a tachyon kink and we argue that the resulting codimension one defect corresponds to stable D$(p-1)$-brane at criticality.

The organization of this paper is as follows. In the next section (2) we formulate the non-critical $p + 1$ dimensional theories that obey detailed balance condition in the sense that their potential term is proportional to the variation of D$(p-1)$-brane action. In section (3) we generalize the gauge symmetries of this $p + 1$ dimensional theory when we extend rigid time translation and spatial diffeomorphism to the foliation-preserving diffeomorphism and spatial dependent gauge symmetry to the space-time dependent one. In section (4) we develop the Hamiltonian formalism for given theory and we calculate the algebra of constraints. In section (5) we study the tachyon condensation on the world-volume of $p + 1$ dimensional Dp-brane that contains tachyon and we argue that it leads to the emergence of stable $p$ dimensional D$(p-1)$-brane theory at criticality. Finally in section (??) we outline our results and suggest possible extension of this work.

2. Stable and Unstable D-branes at Criticality

In this section we formulate $p+1$ dimensional Dp-brane theories that obey detailed balance conditions. We closely follow [34].

Let us consider Euclidean DBI action for unstable D$(p-1)$-brane that is embedded in general 10-dimensional background

\[
W = \tau_{p-1} \int d^p x e^{-\Phi} V(T) \sqrt{\det A} ,
\]

\[
A_{ij} = \partial_i Y^M \partial_j Y^N g_{MN} + 2\pi\alpha' \mathcal{F}_{ij} + 2\pi\alpha' \partial_i T \partial_j T ,
\]
\[ F_{ij} = \partial_i A_j - \partial_j A_i - (2\pi\alpha')^{-1} b_{MN} \partial_i Y^M \partial_j Y^N, \]

(2.1)

where \( \tau_{p-1} = \frac{\sqrt{2}}{(2\pi)^{p-1}\alpha'\pi} \) is an unstable D(p-1)-brane tension, \( x = (x^1, \ldots, x^p) \) are the world-volume coordinates, \( Y^M, M, N = 1, \ldots, 10 \) are world-volume scalars that parameterize the embedding of the D(p-1)-brane in target space-time. Further, \( T \) is world-volume tachyon mode with the potential \( V(T) \) that is even function of \( T \) with the property

\[ V(T \to \pm \infty) = 0, \quad V(0) = 1. \]

(2.2)

Finally, \( A_i \) is a world-volume gauge field and \( g_{MN}, b_{MN} \) and \( \Phi \) are background metric, NS-NS two-form and dilaton respectively. Note that the stable D(p-1)-brane can be derived from (2.1) be striping the tachyon contribution \( (V(T) = 1, T = 0) \) and replacing \( \tau_{p-1} \) with \( \tau_{p-1} = \frac{1}{\sqrt{2}} \tau_{p-1} \). For simplicity we consider the background with zero Ramond-Ramond fields even if the analysis can be easily generalized to this case as well.

Let us now discuss symmetries of the action (2.1). By construction this action is invariant under world-volume diffeomorphism:

\[ x'^i = x'^i(x) \]

(2.3)

under which the world-volume fields transform as

\[ Y'^M(x') = Y^M(x), \quad T'(x') = T(x), \quad A_i(x') = A'_i(x)(D^{-1})^i_j, \]

(2.4)

where we introduced \( p \times p \) matrix \( D^i_j = \frac{\partial x'^i}{\partial x^j} \) that is generally function of \( x \). Then it is easy to show that \( G_{ij} \) and \( B_{ij} \) defined as

\[ G_{ij}(x) = g_{MN}(Y(x)) \partial_i Y^M(x) \partial_j Y^N(x), \quad B_{ij}(x) = b_{MN}(Y(x)) \partial_i Y^M(x) \partial_j Y^N(x) \]

(2.5)

together with \( F_{ij} = \partial_i A_j - \partial_j A_i \) transform under (2.3) as

\[ G'_{ij}(x') = G_{kl}(x)(D^{-1})^k_i(D^{-1})^l_j, \]
\[ B'_{ij}(x') = B_{kl}(x)(D^{-1})^k_i(D^{-1})^l_j, \]
\[ F'_{ij}(x') = F_{kl}(x)(D^{-1})^k_i(D^{-1})^l_j \]

(2.6)

and consequently

\[ \sqrt{\text{det } A'(x')} = \frac{1}{|\text{det } D|} \sqrt{\text{det } A(x)}. \]

(2.7)

Then using the fact that the element \( d^p x \) transforms as \( d^p x' = |\text{det } D| d^p x \) under (2.3) we find that

\[ d^p x' \sqrt{\text{det } A'(x')} = d^p x \sqrt{\text{det } A(x)}. \]

(2.8)
As the next step in the construction of the wave functional Ψ[Y(x), T(x), A(x)] that has the form

\[ \Psi[Y(x), T(x), A(x)] = \exp(-W) , \]

(2.13) where \( W \) is defined in (2.1). The second one is the definition of the operators \( \hat{Q}_M(x), \hat{Q}_T(x), \hat{Q}_i(x) \)

\[ \hat{Q}_M(x) = -i\hat{\Pi}_M(x) + \frac{1}{2} \frac{\delta \hat{W}}{\delta \hat{Y}^M(x)} , \]

(2.14)
\[ \hat{Q}_T(x) = -i \hat{\Pi}_T(x) + \frac{1}{2} \frac{\delta \hat{W}}{\delta T(x)}, \]
\[ \hat{Q}^i(x) = -i \hat{\Pi}^i(x) + \frac{1}{2} \frac{\delta \hat{W}}{\delta A_i(x)}. \]

(2.14)

Then we finally presume that the quantum mechanical Hamiltonian density of \( p + 1 \)-dimensional theory takes the form

\[ \hat{\mathcal{H}} = \kappa^2 \left( \hat{Q}_M \hat{G}^{MN} \hat{Q}_N + \hat{Q}_T \hat{G}^{TT} \hat{Q}_T + \hat{Q}^i \hat{G}^{ij} \hat{Q}^j \right), \]

(2.15)

where \( \kappa \) is the coupling constant, and where \( \hat{G}^{MN}, \hat{G}^{TT} \) and \( \hat{G}^{ij} \) that generally depend on of \( \hat{Y}^M, \hat{T}, \hat{A}_i \) will be defined below when we will discuss the classical Lagrangian form of the theory. Note that in the Schrödinger representation the operators defined in (2.14) take the form

\[ \hat{Q}_M(x) = \frac{\delta}{\delta Y^M(x)} + \frac{1}{2} \frac{\delta W}{\delta Y^M(x)}, \]
\[ \hat{Q}_T(x) = \frac{\delta}{\delta T(x)} + \frac{1}{2} \frac{\delta W}{\delta T(x)}, \]
\[ \hat{Q}^i(x) = \frac{\delta}{\delta A^i(x)} + \frac{1}{2} \frac{\delta W}{\delta A_i(x)}. \]

(2.16)

and it is clear that these operators annihilate the wave functional \( \Psi[Y(x), A(x), T(x)] = \exp(-W) \) and consequently this state is eigenstate of the Hamiltonian with zero energy. However it is crucial that this is only formal construction. For example, we do not discuss the issue whether this state is normalizable. We should rather consider this construction as the motivation for the particular form of the quantum Hamiltonian density (2.15) that implies corresponding form of the classical Hamiltonian density. In fact, in what follows we restrict to the classical description when we replace operators with their classical analogues.

At the classical level we also specify the form of the matrices \( G \) that appear in (2.13) and we suggest that they have the form

\[ G^{MN} = \frac{2\pi \alpha' g^{MN}}{\tau_{p-1} e^{-\Phi V(T) \sqrt{\det A}}}, \]
\[ G^{TT} = \frac{1}{\tau_{p-1} e^{-\Phi V(T) \sqrt{\det A}}}, \]
\[ G^{ij} = \frac{A^{ij}_S}{2\pi \alpha' \tau_{p-1} e^{-\Phi V(T) \sqrt{\det A}}}, \]

(2.17)

where we defined symmetric and anti-symmetric part of the matrix \( A_{ij} \) as

\[ A^{S}_{ij} = \frac{1}{2} (A_{ij} + A_{ji}), \quad A^{A}_{ij} = \frac{1}{2} (A_{ij} - A_{ji}). \]

(2.18)
Let us say few comments that explain our choice of the matrices (2.17). First of all it was shown in \[80\] that the natural open string metric on the world-volume of Dp-brane is given by combination $G_{ij} + F_{ij}$. Further, it was also argued in many papers \[81, 82, 83, 84, 85, 86\] that the tachyon kinetic term should appear in the linear combination with $G_{ij}$ and $F_{ij}$. These arguments explain our choice of $A_{ij}^S$. Note also that presence of the factor $\tau_{p-1}V(T)$ in (2.17) is dictated by requirement that the Lagrangian of non-relativistic $p+1$ dimensional theory contains an overall factor $\tau_{p-1}V(T)$ that is crucial for the correct interpretation of the tachyon kink on the world-volume of $p+1$ dimensional theory. Finally, since we want the Lagrangian density with the factor $e^{-\phi}$ we had to include the factor $e^\phi$ into the definition of the Hamiltonian density. In summary, we argue that the classical Hamiltonian density takes the form

$$\mathcal{H} = \frac{\kappa^2}{2} \left( i\Pi_M(x) + \frac{1}{2} \frac{\delta W}{\delta Y^M(x)} \right) \frac{2\pi\alpha' g^{MN}(x)}{\tau_{p-1}e^{-\phi(x)}V(T)\sqrt{\det A(x)}} \left( -i\Pi_N(x) + \frac{1}{2} \frac{\delta W}{\delta Y^N(x)} \right) +$$

$$\frac{\kappa^2}{2} \left( i\Pi_T(x) + \frac{1}{2} \frac{\delta W}{\delta T(x)} \right) \frac{1}{\tau_{p-1}e^{-\phi(x)}V(T)\sqrt{\det A(x)}} \left( -i\Pi_T(x) + \frac{1}{2} \frac{\delta W}{\delta T(x)} \right) +$$

$$\frac{\kappa^2}{2} \left( i\Pi^i(x) + \frac{1}{2} \frac{\delta W}{\delta A_i(x)} \right) \frac{(A^S)_{ij}(x)}{2\pi\alpha'\tau_{p-1}e^{-\phi(x)}V(T)\sqrt{\det A(x)}} \left( -i\Pi^j(x) + \frac{1}{2} \frac{\delta W}{\delta A_j(x)} \right).$$

(2.19)

Now we determine corresponding Lagrangian density. Using the equation of motion for $Y^M, T, A_i$ we find

$$\partial_t Y^M(x) = \{Y^M(x), H\} = \kappa^2 \frac{2\pi\alpha' g^{MN}(x)}{\tau_{p-1}e^{-\phi(x)}V(T)\sqrt{\det A(x)}} \Pi_N(x),$$

$$\partial_t T(x) = \{T(x), H\} = \kappa^2 \frac{1}{\tau_{p-1}e^{-\phi(x)}V(T)\sqrt{\det A(x)}} \Pi_T(x),$$

$$\partial_t A_i(x) = \{A_i(x), H\} = \kappa^2 \frac{(A^S)_{ij}(x)}{2\pi\alpha'\tau_{p-1}e^{-\phi(x)}V(T)\sqrt{\det A(x)}} \Pi^j(x)$$

(2.20)

and hence the Lagrangian density takes the form

$$\mathcal{L} = \partial_t Y^M \Pi_M + \partial_t T \Pi_T + \partial_t A_i \Pi^i - \mathcal{H} = \mathcal{L}_K + \mathcal{L}_P,$$

$$\mathcal{L}_K = \frac{\tau_{p-1}}{2\kappa^2}e^{-\phi}V(T)\sqrt{\det A} \left[ \frac{1}{2\pi\alpha'} \partial_t Y^M g_{MN} \partial_t Y^N \right. +$$

$$\left. \frac{1}{2\kappa^2}e^{-\phi}V(T)\sqrt{\det A} \partial_t T \partial_t T + \frac{2\pi\alpha' \tau_{p-1}}{2\kappa^2}e^{-\phi}V(T)\sqrt{\det A} \partial_t A_i (A^{-1})_{ij}^S \partial_j \right.$$

$$\left. A_j \right],$$

$$\mathcal{L}_P = -\frac{\kappa^2}{8} \frac{\tau_{p-1}}{2\kappa^2}e^{-\phi}V(T)\sqrt{\det A} \left[ 2\pi\alpha' \mathcal{J}_M g^{MN} \mathcal{J}_N + \mathcal{J}_T \mathcal{J}_T + (2\pi\alpha')^{-1} \mathcal{J}^i A^S_{ij} \mathcal{J}^j \right],$$

(2.21)

where we defined $\mathcal{J}_M, \mathcal{J}_T$ and $\mathcal{J}^i$ as

$$\frac{\delta W}{\delta T(x)} = \tau_{p-1}e^{-\phi}V\sqrt{\det A} \left[ V' - \frac{2\pi\alpha'}{V} \frac{\delta}{\delta T(x)} \partial_t [e^{-\phi}V \partial_j T (A^{-1})_{ij}^S \sqrt{\det A}] \right].$$

\(^3\)For review and extensive list of references, see \[81\].
where the world-volume modes transform as

\[ \mathcal{J}_M'(x') = \mathcal{J}_M(x) , \quad \mathcal{J}_T'(x') = \mathcal{J}_T(x) , \quad \mathcal{J}^{ij}(x') = \mathcal{J}^{ij}(x) D_j^i(x) . \]  

Using these results we consequently find

\[ \mathcal{L}_K(x', t') = \frac{1}{\det D(x)} \mathcal{L}_K(x, t) , \quad \mathcal{L}_P(x', t') = \frac{1}{\det D(x)} \mathcal{L}_P(x, t) \]  

that implies that the action \( S = \int dt dx \mathcal{L} \) is invariant under spatial diffeomorphism \( \mathcal{J}_T \). Further, it can be also easily shown that the action is invariant under global time translation

\[ t' = t + \delta t , \quad \delta t = \text{const} , \]  

where the world-volume modes transform as

\[ Y^M(t', x) = Y^M(t, x) , \quad T'(t', x) = T(t, x) , \quad A_i^j(t', x) = A_i(t, x) . \]  

Finally we determine scaling dimension of world-volume modes and coupling constants \(^4\). To do this let us presume following scaling

\[ x'^i = \lambda^{-1} x^i , \quad t' = \lambda^{-2} t , \quad Y'^M(x') = \lambda^{-1} Y(x) , \quad T'(x') = T(x) , \quad A_i^j(x') = \lambda A_i(x) , \quad (\alpha')^i = \lambda^{-2} \alpha^i , \quad \tau'_p - 1 = \lambda^p \tau_{p-1} , \quad \kappa = \lambda^{[i]} \kappa , \]  

where \( \lambda \) is constant scaling parameter. Let us say few words about this scaling. First of all it is natural to presume that \( x \) have standard scaling dimension \( [x] = -1 \). Then the

\(^4\)Since we consider the classical theory only these dimensions are classical "engineering" dimensions.
anisotropic nature of the theory is reflected by the scaling dimension of $t$ that generally is different from $x$'s. Further, since $Y^M$'s describe the embedding of the D(p-1)-brane into the target space-time it is again natural to presume that it scales as in (2.27). In other words $Y^M$'s have the same dimensions as in D-brane theory. Then we can also demand $T$ and $A_i$ have the standard scaling dimensions as in D-brane theory (2.27). Finally the scaling dimension of $\alpha'$ and $\tau_{p-1}$ is obvious as it follows from their string theory definition.

Let us now discuss the consequence of the scaling (2.27). First of all it implies that

$$A'_{ij}(x') = A_{ij}(x), \quad \tau'_{p-1}d^p x' = \tau_{p-1}d^p x$$

(2.28)

and hence we find that $W$ is invariant under scaling while the currents $J$ scale as

$$J'_M(x') = \lambda J_M(x), \quad J'_T(x') = J_T(x), \quad J'^i(x) = \lambda^{-1} J^i(x).$$

(2.29)

As a consequence we find that the Lagrangian densities (2.21) scale as

$$L'_P(x') = \lambda^{p+2[k]} L_P(x),$$

$$L'_K(x') = \lambda^{p-2[k] + 2z} L_P(x).$$

(2.30)

Then the requirement of the invariance of the action implies the relation between dimension of $\kappa$ and $z$

$$[\kappa] = \frac{z}{2}.$$  

(2.31)

However it is clear from the form of the Lagrangian density (2.21) that the effective coupling constant is

$$\frac{1}{G^2} = \frac{\tau_{p-1}}{\kappa^2}$$

(2.32)

that is dimensionless for

$$2[k] - p = 0$$

(2.33)

that using (2.31) implies the famous relation between scaling dimension of $z$ and dimension of the theory

$$z - p = 0.$$  

(2.34)

However we should stress that the theory still contain dimensionfull coupling $\tau_p$. On the other hand the scaling dimensions of coupling constants are important at the quantum analysis of the theory while our treatment is pure classical.

### 3. Extension of the symmetries

In this section we extend the symmetries of the non-relativistic action introduced above. Recall that this action is invariant under global time translations (2.25) and under local spatial diffeomorphism (2.3). Further, the action is also invariant under spatial dependent gauge transformation with parameter $\epsilon(x)$. Following the logic given in [3] we would like
to extend this spatial diffeomorphism and rigid time translation to the foliation preserving
diffeomorphism defined as

$$\delta t = t' - t = f(t) , \quad \delta x^i = x'^i - x^i = \zeta^i(x, t) . \quad (3.1)$$

Note that under these transformations the modes $Y^M, T$ transform as

$$\partial_\nu Y^M(x', t') = \partial_\nu Y^M(x, t) - \partial_\nu Y^M(x, t) f(t) - \partial_\nu Y^M(x, t) \partial_\nu \zeta^i(x, t) ,$$

$$\partial_\nu T(x', t') = \partial_\nu T(x, t) - \partial_\nu T(x, t) f - \partial_\nu T(x, t) \partial_\nu \zeta^i(x, t) ,$$

$$\partial_{\gamma'} Y^M(x', t') = \partial_{\gamma'} Y^M(x, t) - \partial_{\gamma'} Y^M(x, t) \partial_{\gamma'} \zeta^j(x, t) ,$$

$$\partial_{\gamma'} T(x', t') = \partial_{\gamma'} T(x, t) - \partial_{\gamma'} T(x, t) \partial_{\gamma'} \zeta^j(x, t) , \quad (3.2)$$

where $\dot{f} = \frac{df}{dt}$. Before we proceed to the determination of the transformation rules for
$A_i$ we extend the spatial dependent gauge symmetry with parameter $\epsilon(x)$ to the gauge
symmetry that is time dependent as well so that

$$A'_i(x, t) = A_i(x, t) + \partial_\nu \epsilon(x, t) . \quad (3.3)$$

However then the requirement of the invariance of the action under time dependent gauge
transformation implies that we have to introduce the gauge field $A_i(t, x)$ that under gauge
transformation transforms as

$$A'_i(x, t) = A_i(x, t) + \partial_\nu \epsilon(x, t) . \quad (3.4)$$

Using this gauge field we replace $\partial_\nu A_i$ with the object $E_i$ defined as

$$E_i(x, t) = \partial_\nu A_i(x, t) - \partial_\nu A_i(x, t) \quad (3.5)$$

that is invariant under the time dependent gauge transformations (3.3) and (3.4). As the
next step we determine how the vectors $A_i(x, t), A_i(x, t)$ transform under (3.1). Obviously
it is natural to demand that they transform as vectors

$$A'_i(x', t) = A_i(x, t) - A_j(x, t) \partial_\nu \zeta^j(x, t) ,$$

$$A'_i(x', t') = A_i(x, t) - A_i(x, t) f(t) - A_j(x, t) \partial_\nu \zeta^j(x, t) \quad (3.6)$$

and consequently

$$F'_{ij}(x', t') = F_{ij}(x, t) - F_{ik}(x, t) \partial_\nu \zeta^k(x, t) - \partial_\nu \zeta^k(x, t) F_{kj}(x, t) . \quad (3.7)$$

Then using (3.2) and (3.7) we easily find that

$$d^p x' \sqrt{\det A'(x', t')} = d^p x \sqrt{\det A(x, t)} . \quad (3.8)$$
Further, using (3.6) we determine the transformation property of $E_i$

$$E_i'(x', t') = E_i(x, t) - E_i(x, t)\dot{f}(t) - E_j(x, t)\partial_i\zeta^j(x, t) - F_{ji}(x, t)\dot{\zeta}^j(x, t) .$$  

(3.9)

Alternatively, we can determine the transformation rules of $A_i, A_t$ from the relativistic transformation law for $p + 1$ dimensional vector $A_i, A_0$, following the analysis performed in [3]. Explicitly, let us consider covariant $p + 1$ dimensional vector $A_\mu = (A_i, A_0)$ with corresponding field strength $F_{\mu\nu}$

$$F_{ij} = \partial_i A_j - \partial_j A_i , \quad F_{0i} = \partial_0 A_i - \partial_i A_0 ,$$  

(3.10)

where $x^0 = ct$. Since it is a covariant object it transforms under general diffeomorphism as

$$F_{\mu\nu}'(x', ct') = F_{\kappa\rho}(x, ct)\frac{\partial x^\kappa}{\partial x'^\mu} \frac{\partial x^\rho}{\partial x'^\nu} .$$  

(3.11)

It is a simple task to show that (3.11) imply the transformation rules for $F_{ij}$ given (3.7).

In order to determine the transformation of $E_i$ note that we can write $F_{0i}$ as

$$F_{0i} = \partial_0 A_i - \partial_i A_0 = \frac{1}{c}\partial_t A_i - \frac{1}{c}\partial_i A_t = \frac{1}{c}F_{ti} ,$$  

(3.12)

where due to the fact that $x^0 = ct$ we replaced $A_0$ with $A_0 = \frac{1}{c}A_t$ holding $A_t$ fixed in the limit $c \to \infty$. Then it is easy to see that when we take $\mu = 0, \nu = i$ in (3.11) and consider the limit $c \to \infty$ we find

$$F_{0i}'(x', t') = \frac{1}{c}F_{ti}'(x', t') =$$

$$= \frac{1}{c}F_{ti}(x, t) - \frac{1}{c}F_{ij}(x, t)\partial_i\zeta^j(x, t) - \frac{1}{c}F_{ti}(x, t)\dot{f}(t) - \frac{1}{c}F_{ji}(x, t)\dot{\zeta}^j(x, t)$$  

(3.13)

that reproduces (3.3).

As we argued above the invariant spatial volume element is $d^p x \sqrt{\det A}$. It is also easy to see that under foliation preserving diffeomorphism the matrix $(A^{-1})$ transforms as

$$(A^{-1})^{ij}(x', t') = A^{ij}(x, t) + \partial_i\zeta^j(x, t) (A^{-1})^{ij}(x, t) + (A^{-1})^{il}(x, t)\partial_j\zeta^l(x, t)$$  

(3.14)

and hence its symmetric form transforms in the same way as spatial components of $p + 1$ dimensional metric. On the other hand in order to have kinetic part of the Lagrangian invariant under foliation preserving diffeomorphism we have to introduce two gauge fields $N(x, t)$ and $N^i(t)$ that transform under (3.1) as

$$N^i(x', t') = N^i(x, t) + N^j(x, t)\partial_j\zeta^i(x, t) - N^i(x, t)\dot{f}(x, t) - \dot{\zeta}^i(x, t) ,$$

$$N^t(x', t') = N(t) - N(t)\dot{f}$$  

(3.15)
and replace $\partial_t Y^M, \partial_t T$ and $E_i$ with

\[
\begin{align*}
\partial_t T & \to \frac{1}{N} (\partial_t T - N^i \partial_i T) \equiv \mathcal{D}_t T , \\
\partial_t Y^M & \to \frac{1}{N} (\partial_t Y^M - N^i \partial_i Y^M) \equiv \mathcal{D}_t Y^M , \\
E_i & \to \frac{1}{N} (E_i - N^k F_{ki}) \equiv \mathcal{D}_t A_i .
\end{align*}
\]

Then using (3.2), (3.7), (3.9) and (3.15) we find that

\[
\begin{align*}
\mathcal{D}_t Y^M(x', t') & = \mathcal{D}_t Y^M(x, t) - \mathcal{D}_t Y^M(x, t) \partial_i \zeta^i(x, t) , \\
\mathcal{D}_t T(x', t') & = \mathcal{D}_t T(x, t) - \mathcal{D}_t T(x, t) \partial_i \zeta^i(x, t) , \\
\mathcal{D}_t A_i(x', t') & = \mathcal{D}_t A_i(x, t) - \mathcal{D}_t A_j(x, t) \partial_i \zeta^j(x, t) .
\end{align*}
\]

If we also take into account the fact that $N'(t')dt' = N dt$ we obtain that the action for non-relativistic $p + 1$-brane theory that is invariant under foliation preserving diffeomorphism takes the form

\[
S = \int dt d^p x N \mathcal{L} , \quad (3.18)
\]

where

\[
\begin{align*}
\mathcal{L} & = \mathcal{L}_K + \mathcal{L}_P , \\
\mathcal{L}_K & = \frac{\tau_{p-1}}{2\kappa^2} e^{-\phi} V(T) \sqrt{\det A} \left[ 1 \frac{1}{2\pi \alpha'} \mathcal{D}_t Y^M g_{MN} \mathcal{D}_t Y^N + \mathcal{D}_t T \mathcal{D}_t T + 2\pi \alpha' \mathcal{D}_t A_i (A^{-1})^{ij} \mathcal{D}_t A_j \right] , \\
\mathcal{L}_P & = \frac{-\kappa^2 \tau_{p-1}}{8} e^{-\phi} V(T) \sqrt{\det A} \{ 2\pi \alpha' J_M g^{MN} J_N + J_T J_T + (2\pi \alpha')^{-1} J^i A_{ij} S_j J^j \} .
\end{align*}
\]

In the next section we determine the Hamiltonian formalism that follows from the Lagrangian density (3.19).

4. Hamiltonian analysis

To begin with we introduce the momenta $\Pi_M, \Pi_T, \Pi^i, \pi_0, \pi_N$ that are conjugate to $Y^M, A_i, T, A_t$ and $N^i, N$ and that have following non-zero Poisson brackets

\[
\begin{align*}
\{ Y^M(x), \Pi_N(y) \} & = \delta^M_N \delta(x-y) , \quad \{ T(x), \Pi_T(y) \} = \delta(x-y) , \\
\{ A_i(x), \Pi^j(y) \} & = \delta_i^j \delta(x-y) , \quad \{ A_0(x), \Pi^0(y) \} = \delta(x-y) , \\
\{ N^i(x), \pi_j(y) \} & = \delta_i^j \delta(x-y) , \quad \{ N, \pi_N \} = 1 .
\end{align*}
\]

First of all since the Lagrangian density (3.19) does not contain time derivative of $N^i, N$ and $A_0$ the conjugate momenta form a primary constraints of the theory

\[
\pi_i(x) \approx 0 , \quad \pi_N \approx 0 , \quad \Pi^0(x) \approx 0 . \quad (4.2)
\]
Further, the momenta conjugate to $Y_M, T$ and $A_i$ takes the form

$$
\begin{align*}
P_M(t, x) &= \frac{\tau_p^{-1}}{2\pi\alpha'\kappa^2} e^{-\phi V(T)} \sqrt{\det A} g_{MN} D_Y Y^N , \\
P_T(t, x) &= \frac{\tau_p^{-1}}{2\pi\alpha'\kappa^2} e^{-\phi V(T)} \sqrt{\det A} D_T T , \\
\Pi_i(t, x) &= \frac{2\pi\alpha'\tau_p^{-1}}{\kappa^2} e^{-\phi V(T)} \sqrt{\det A} (A^{-1})^{ij}_S D_i A_j .
\end{align*}
$$

(4.3)

Then it is simple task to find corresponding Hamiltonian density from (3.19)

$$
H = \partial_t Y^M \Pi_M + \partial_t T \Pi_T + \partial_t A_i \Pi^i - \mathcal{L} =
\begin{align*}
&= N \left[ \frac{\kappa^2}{2\tau_p^{-1} e^{-\phi V(T)} \sqrt{\det A}} \left( 2\pi\alpha' \Pi_M g^{MN} \Pi_N + 2\pi\alpha' \Pi_T \Pi_T + \frac{1}{2\pi\alpha'} \Pi^j (A^S)^{ij}_S \Pi_j \right) + \\
&+ \frac{\kappa^2 \tau_p^{-1}}{8} e^{-\phi V(T)} \sqrt{\det A} [2\pi\alpha' J_M g^{MN} J_N + J_T J_T + (2\pi\alpha')^{-1} J^i (A_S)^{ij}_S] + \\
&+ N^i (\partial_i Y^M \Pi_M + \partial_i T \Pi_T + F_{ij} \Pi^j) + \partial_t A_0 \Pi^i = \\
&= NT_0 + N^i T_i + \partial_t A_0 \Pi^i .
\end{align*}
$$

(4.4)

The primary constraints (4.2) have to be preserved during the time evolution of the system. In other words we require

$$
\partial_t \pi_i(x) = \{\pi(x), H\} \approx 0 , \quad \partial_t \pi_N = \{\pi, H\} \approx 0 , \quad \partial_t \pi^0(x) = \{\pi^0(x), H\} \approx 0 .
$$

(4.5)

Then using the canonical Poisson brackets and the form of the Hamiltonian density (4.4) we obtain that the theory should be supplemented with secondary constraints in the form

$$
T = \int d^p x T_0(x) \approx 0 , \quad T_i(x) \approx 0 .
$$

(4.6)

and finally the consistency of the constraint $\pi^0 \approx 0$ with its time evolution implies the constraint

$$
G(x) \equiv \partial_t \pi^i(x) = 0 .
$$

(4.7)

On the other hand it turns out [3] that instead to impose the constraint $T \approx 0$ it is much more convenient to introduce equivalent set of constraints $Q$ using the fact that $T_0$ can be written as

$$
T_0 = \frac{\kappa^2}{2} \left[ \frac{2\pi\alpha' g^{MN}}{\tau_p^{-1} e^{-\phi V(T)} \sqrt{\det A}} Q_M + \\
+ \frac{2\pi\alpha'}{\tau_p^{-1} e^{-\phi V(T)} \sqrt{\det A}} Q_T + Q_i \frac{(A^{-1})^{ij}_S}{2\pi\alpha' \tau_p^{-1} e^{-\phi V(T)} \sqrt{\det A}} Q_j \right] ,
$$

(4.8)
where

\[
\begin{align*}
Q_M &= -i\Pi_M + \frac{\tau_{p-1}}{2} e^{-\phi} V(T) \sqrt{\det A} J_M, \\
Q_T &= -i\Pi_T + \frac{\tau_{p-1}}{2} e^{-\phi} V(T) \sqrt{\det A} J_T, \\
Q^i &= -i\Pi^i + \frac{\tau_{p-1}}{2} e^{-\phi} V(T) \sqrt{\det A} J^i.
\end{align*}
\]

(4.9)

Let us now introduce the smeared form of the constraints defined as

\[
\begin{align*}
T(\zeta) &= \int d^p x \zeta^i(x) T_i(x), \\
Q_Y(\Lambda_Y) &= \int d^p x \Lambda_Y^M(x) Q_M(x), \\
Q_T(\Lambda_T) &= \int d^p x \Lambda_T(x) Q_T(x), \\
G(\epsilon) &= \int d^p x \epsilon(x) G(x), \\
Q_A(\Lambda_A) &= \int d^p x \Lambda_A^i(x) Q^i(x)
\end{align*}
\]

(4.10)

and determine their algebra. First of all we find

\[
\begin{align*}
\{ G(\epsilon), A_i(x) \} &= \partial_i \epsilon(x), \\
\{ G(\epsilon), F_{ij}(x) \} &= 0.
\end{align*}
\]

(4.11)

Then using the fact that \( J \)'s and \( A \) are invariant under gauge transformations we find that

\[
\{ G(\epsilon), Q(\Lambda) \} = 0, \\
\{ G(\epsilon), T(\zeta) \} = 0.
\]

(4.12)

Further, it is straightforward exercise to determine the Poisson brackets of \( T(\zeta) \) with \( T(\xi) \)

\[
\{ T(\zeta), T(\xi) \} = T(\xi^j \partial_j \xi^i - \xi^i \partial_j \xi^j) + G(-2\xi^i F_{ik} \xi^k).
\]

(4.13)

Let us now consider the Poisson brackets that contain \( Q \)'s. It can be easily shown, following [34] that the Poisson brackets of two \( Q \)'s vanish

\[
\begin{align*}
\{ Q_M(x), Q_N(y) \} &= \{ Q_M(x), Q_T(y) \} = \{ Q_M(x), Q^i(y) \} = 0, \\
\{ Q_T(x), Q_T(y) \} &= \{ Q_T(x), Q^i(x) \} = \{ Q^i(x), Q^j(y) \} = 0.
\end{align*}
\]

(4.14)

In order to determine the Poisson bracket between \( T(\zeta) \) and \( Q_M, Q_T \) we again follow [34] and we find

\[
\{ T(\zeta), Q_M(x) \} = -\partial_i Q_M(x) \zeta^i(x) - Q_M(x) \partial_i \zeta^i(x)
\]

(4.15)
\{T(\zeta), Q_T(x)\} = -\partial_i Q_T(x) \zeta^i(x) - Q_T(x) \partial_i \zeta^i(x) .

(4.16)

On the other hand the calculation of the Poisson bracket between \(T(\zeta)\) and \(Q^i(x)\) is more involved however after some effort we find

\{T(\zeta), Q^i(x)\} = -\partial_k \zeta^k(x) Q^i(x) - \zeta^k(x) \partial_k Q^i(x) + \partial_k \zeta^i(x) Q^k(x) + \zeta^i(x) G(x) .

(4.17)

Then finally using these results we find that the smeared form of these Poisson bracket takes the form

\begin{align*}
\{T(\zeta), Q_Y(\Lambda_Y)\} &= Q_Y(\partial_i \Lambda_Y \zeta^i) , \\
\{T(\zeta), Q_T(\Lambda_T)\} &= Q_T(\partial_i \Lambda_T \zeta^i) , \\
\{T(\zeta), Q_A(\Lambda^A)\} &= G(\Lambda^A \zeta^i) + Q_A(\partial_k \Lambda^A_k \zeta^k + \Lambda^A_k \partial_k \zeta^k) .
\end{align*}

(4.18)

These results imply that the requirement that the set of constraints \(T, Q_Y, Q_T, Q_A\) and \(\pi_i, \pi_N, \Pi^0\) is preserved during the time evolution of the system does not generate additional ones. More precisely, note that the total Hamiltonian \(H_T\) can be written as the sum of smeared constraints defined in \([110]\)

\[ H_T = \int d^p x H_T = \]

\[ = \frac{\kappa^2}{2} \left( Q_Y \left( N \frac{2\pi\alpha' Q^\dagger_N g^{NM}}{\tau_{p-1} e^{-\phi V(T) \sqrt{\det A}}} \right) + Q_T \left( N \frac{2\pi\alpha' Q^\dagger_T}{\tau_{p-1} e^{-\phi V(T) \sqrt{\det A}}} \right) + \\
+ Q_A \left( N \frac{Q^\dagger_j (A^{-1})_{ji}^j}{\tau_{p-1} 2\pi\alpha' e^{-\phi V \sqrt{\det A}}} \right) + T(N^i) + G(-A_0) \right) , \]

(4.19)

where the expression in the parenthesis are parameters of corresponding constraints. Then using the form of the algebra of constraints written above we find that their time evolution vanish on constraint surface and consequently these constraints do not generate additional ones. In other words the full set of constraints of the theory is

\[ \pi_i(x), \quad \pi_N, \quad \Pi^0(x), \]
\[ T(\zeta), \quad Q_Y(\Lambda_Y), \quad Q_T(\Lambda_T), \quad Q_A(\Lambda^A), \quad G(\epsilon) . \]

(4.20)

Then in the same way as in \([34]\) we can formally proceed to the quantum formulation of the theory where we replace constraint functions by corresponding operators \(\hat{T}(\zeta), \hat{Q}_Y(\Lambda_Y), \hat{Q}_T(\Lambda_T), \hat{Q}_A(\Lambda^A)\) and \(G(\epsilon)\). Further we can introduce the wave functional

\[ \Psi[Y(x), A(x), T(x)] = \exp(-W) \]

(4.21)
that is clearly annihilated by all constraints functions and at the same time it is a state of the Hamiltonian $H_T$ with zero eigenvalue. However since we are interested in the classical form of the theory we are not going to proceed in this direction further.

5. Tachyon condensation

In this section we will study the spatial dependent tachyon condensation on an unstable non-relativistic Dp-brane with the Lagrangian density given in (3.19). Our goal is to show that the tachyon condensation in the form of kink leads to D(p-1)-brane exactly in the same way as in superstring theory [86, 88]. However due to the complexity of the non-relativistic Lagrangian density (3.19) we restrict ourselves to the case of flat background where $g_{MN} = \delta_{MN}, e^\Phi = e^{\phi_0} = 1, b_{MN} = 0$.

To begin with note that Lagrangian density for stable non-relativistic D(p-1)-brane takes the form

$$ S = \int dt d^{p-1}x N L^st, \quad (5.1) $$

where

$$ L^st = L^st_K + L^st_P, $$

$$ L^st_K = \frac{1}{4\pi\alpha' \kappa^2} \sqrt{\det a} \tilde{D}_t \tilde{Y}^M \tilde{D}_t \tilde{Y}_M + \frac{\pi \alpha'T_p}{\kappa^2} \sqrt{\det a} \tilde{D}_t \tilde{A}_\alpha (a^{-1})^{\alpha\beta}_S \tilde{D}_t \tilde{A}_{\beta}, $$

$$ L_P = -\frac{\kappa^2 T_p}{8} \sqrt{\det a} [2\pi \alpha' \tilde{J}^M \tilde{J}_M + (2\pi \alpha')^{-1} \tilde{J}^i \tilde{a}_j^S \tilde{J}^j] \quad (5.2) $$

where $\tilde{Y}^M, M, N = 1, \ldots, 10$ are modes that parameterize transverse positions of D(p-1)-brane, $\tilde{A}_\alpha$ are gauge fields living on its world volume where the world-volume of D(p-1)-brane is parameterized with coordinates $x^\alpha, \alpha, \beta = 1, \ldots, p-1$. Further, $\tilde{D}_t \tilde{Y}^M, \tilde{D}_t \tilde{A}_\alpha, \tilde{J}_Y$ and $\tilde{J}_A$ are defined as

$$ \tilde{D}_t \tilde{Y}^M = \frac{1}{N} \left( \partial_t \tilde{Y}^M - \tilde{N}^\alpha \partial_\alpha \tilde{Y}^M \right), $$

$$ \tilde{D}_t \tilde{A}_\alpha = \frac{1}{N} \left( \tilde{E}_\alpha - \tilde{N}^\beta \tilde{F}_{\beta\alpha} \right), $$

$$ \tilde{J}_M(x) = \frac{1}{\sqrt{\det a}} \partial_\alpha [\partial_\beta Y_M (a^{-1})^{\beta\alpha}_S \sqrt{\det a}], $$

$$ \tilde{J}^\alpha(x) = \frac{2\pi \alpha'}{\sqrt{\det a}} \partial_\beta [(a^{-1})^{\beta\alpha}_A \sqrt{\det a}], \quad (5.3) $$

where

$$ a_{\alpha\beta} = \partial_\alpha \tilde{Y}^M \partial_\beta \tilde{Y}_M + 2\pi \alpha' \tilde{F}_{\alpha\beta}. \quad (5.4) $$

Let us again consider the action for non-relativistic unstable Dp-brane (3.19) and analyze corresponding equation of motion. Again, for simplicity we consider the equation of motion
for the tachyon only and argue the this equation of motion has the solution in the form of the tachyon kink \[86\]

\[ T = f(ax), \quad \frac{df}{du} = f'(u) > 0, \quad \forall u, \quad f(\pm\infty) = \pm\infty, \]

\[ Y^M = A_i = 0, \quad N = \text{const}, \quad N^i = 0, \]

(5.5)

where \( a \) is a parameter that should be taken to infinity in the end. As the first step we determine the equation of motion for \( N \)

\[
\frac{\tau_{p-1}}{2\kappa^2} V(T) \sqrt{\det A} \left[ \frac{1}{2\pi\alpha'} \partial_i Y^M D_i Y_M + \partial_i T D_i T + 2\pi\alpha' D_i A_i \left( A^{-1} \right)_S^{ij} D_i A_j \right] + \\
+ \frac{\kappa^2 \tau_{p-1}}{8} V(T) \sqrt{\det A} \left[ 2\pi\alpha' \mathcal{J}_M \mathcal{J}^M + \mathcal{J}_T \mathcal{J}_T + (2\pi\alpha')^{-1} \mathcal{J}_i S^{ij} \mathcal{J}_j \right] = 0
\]

(5.6)

while the equation of motion for \( N^i \) implies

\[
\frac{\tau_{p-1}}{2\kappa^2} V(T) \sqrt{\det A} \left[ \frac{1}{2\pi\alpha'} \partial_i Y^M D_i Y_M + \partial_i T D_i T + 2\pi\alpha' F_{ik} \left( A^{-1} \right)_S^{kj} D_i A_j \right] = 0.
\]

(5.7)

Now we see that (5.7) solves the equation (5.7) while for \( D_i T = 0 \) the equation (5.6) reduces into

\[
\frac{\kappa^2 \tau_{p-1}^2}{8} V(T) \sqrt{\det A} \mathcal{J}_T \mathcal{J}_T = 0.
\]

(5.8)

We will argue below that the ansatz (5.3) solves the equation (5.8) as well. In fact, let us now analyze the equation of motion for tachyon

\[
\frac{\tau_{p-1}}{2\kappa^2} N V' \sqrt{\det A} \left( \frac{1}{2\pi\alpha'} D_i Y^M D_i Y_M + D_i T D_i T + D_i A_i \left( A^{-1} \right)_S^{ij} D_i A_j \right) - \\
- \frac{\pi\alpha' \tau_{p-1}}{\kappa^2} \partial_i \left[ N V \partial_j T \left( A^{-1} \right)_S^{ij} \sqrt{\det A} \left( \frac{1}{2\pi\alpha'} D_i Y^M D_i Y_M + D_i T D_i T + D_i A_i \left( A^{-1} \right)_S^{ij} D_i A_j \right) \right] + \\
+ \frac{2\pi\alpha' \tau_{p-1}}{\kappa^2} \partial_i \left[ N V(T) \sqrt{\det A} D_i A_i \left( A^{-1} \right)_S^{ik} \partial_j T \left( A^{-1} \right)_S^{ij} D_i A_j \right] - \\
- \frac{\tau_{p-1}}{\kappa^2} N D_i (V(T) \sqrt{\det A} D_i T) - \\
- \frac{\kappa^2 N \tau_{p-1}}{8} V'(T) \sqrt{\det A} \left[ \mathcal{J}_T \mathcal{J}_T + 2\pi\alpha' \mathcal{J}_M \mathcal{J}^M + \frac{1}{2\pi\alpha'} \mathcal{J}_i \mathcal{J}^i \mathcal{J}_j \right] + \\
+ \frac{\pi\alpha' \kappa^2 \tau_{p-1}^2}{4} \partial_i \left[ N V(T) \partial_j T \left( A^{-1} \right)_S^{ij} \sqrt{\det A} \left( \mathcal{J}_T \mathcal{J}_T + 2\pi\alpha' \mathcal{J}_M \mathcal{J}^M + \frac{1}{2\pi\alpha'} \mathcal{J}_i \mathcal{J}^i \mathcal{J}_j \right) \right] - \\
- \frac{\kappa^2 N \tau_{p-1}^2}{4} V(T) \sqrt{\det A} \left( \frac{V''}{V} - \frac{V'^2}{V^2} \right) \mathcal{J}_T - \\
- \frac{\pi\alpha' \kappa^2 N \tau_{p-1}^2}{2} V(T) \sqrt{\det A} \mathcal{J}_T \left( \frac{V'}{V^2} \sqrt{\det A} \partial_i \partial_j T \left( A^{-1} \right)_S^{ij} \sqrt{\det A} \right) + 
\]
very complicated.

\[ \mathcal{J} \]

contains anti-symmetrization of the matrix \( A \quad \ldots \quad \mathcal{J} \)

where "\( \ldots \)" means terms that arise from the variation of the expressions \( \mathcal{J}^M \mathcal{J}_M \) and \( \mathcal{J}^i \mathcal{J}_j \). We see that even in the flat background the equation of motion for tachyon is very complicated.

Note that for (5.5) the matrix \( A \) and its determinant are equal to

\[ A_{xx} = 2\pi\alpha' a^2 f'^2, \quad A_{\alpha\beta} = A_{x\alpha} = 0, \quad \det A = \sqrt{2\pi\alpha' a f'}. \quad (5.10) \]

However the situation is much better when we recognize that for (5.5) the currents \( \mathcal{J} \) vanish. In fact, \( \mathcal{J}^M \) vanishes since it contains derivative of \( Y \) and \( \mathcal{J}^i \) vanishes since contains anti-symmetrization of the matrix \( A \) that for the ansatz (5.5) is zero. Finally if we insert the ansatz (5.5) into \( \mathcal{J}_T \) we obtain

\[ \mathcal{J}_T = \frac{V'}{V} - \frac{2\pi\alpha'}{V\sqrt{2\pi\alpha' a f'}} \partial_x \left[ \frac{V a f' \sqrt{2\pi\alpha' a^2 f'^2}}{2\pi\alpha' a^2 f'^2} \right] = 0. \quad (5.11) \]

Further, as we argued above \( D_t T = 0 \) and hence the first fourth lines vanish in the equation of motion for tachyon. Then however using the fact that \( \mathcal{J}_T = 0 \) we find that the variation of the potential term vanishes as well and hence the ansatz (5.3) solves the tachyon equation of motion.

With analogy with familiar case of the tachyon kink in superstring theory \[86\] we interpret this solution as the stable non-relativistic D(p-1)-brane. To support further this claim we should study the dynamics of the fluctuations around this tachyon kink exactly in the same way as in \[86\]. Explicitly, let us consider following ansatz for fluctuations

\[ T = T(a(x - t(x^\alpha))) , \quad Y^M(t, x) = \tilde{Y}^M(t, x^\alpha) , \quad A_x = 0, \quad A_\alpha(t, x) = \tilde{A}_\alpha(t, x^\alpha) , \quad (5.12) \]

where \( x^\alpha \equiv (x^1, \ldots, x^{p-1}) \). Then we should insert (5.12) to the equations of motion that follow from the variation of the action (3.19) and show that the equations of motion are obeyed on condition that \( \tilde{Y}^M, \tilde{A}_\alpha \) obey the equations of motion that follow from the variation of the action (5.1) and hence their dynamics really describes D(p-1)-brane. However due to the complexity of the action (3.19) we do not proceed in this way. Instead we show that when we insert the ansatz (5.12) into (3.19) we reproduce the non-relativistic D(p-1)-brane action.
The structure of the action (3.19) allows further important simplification. By construction the action (3.19) is invariant under foliation-preserving diffeomorphism. On the other hand we suggested the tachyon fluctuation ansatz in the form \( T = f(a(x - t(x^\alpha))) \).

Then the invariance of the action suggest to interpret \( t \) as a parameter of diffeomorphism transformation and hence it is physically redundant and can be gauged away. Then it is natural to presume that the tachyon has the form \( T = f(a(x)) \). However this result simplifies the analysis considerably since then

\[
\mathbf{A}_{\alpha\beta} = \mathbf{a}_{\alpha\beta} , \quad \mathbf{a}_{\alpha\beta} = \partial_{\alpha} \tilde{Y}_M \partial_{\beta} \tilde{Y}_M + 2\pi \alpha' \tilde{F}_{\alpha\beta} ,
\]

\[
\mathbf{A}_{x\beta} = \mathbf{A}_{\alpha x} = 0 , \quad \mathbf{A}_{xx} = a^2 f'^2 .
\]

Inserting this result into the currents \( \mathcal{J}_T, \mathcal{J}_M \) and \( \mathcal{J}^i \) we find

\[
\mathcal{J}_T = \frac{V'}{V} - \frac{(2\pi \alpha')^2}{V \sqrt{\det A_{xx} \det a}} \partial_x \left[ V f \sqrt{\det \mathbf{A}_{xx}} \right] = 0
\]

(5.13)

and

\[
\mathcal{J}_M = \frac{1}{V(T) \sqrt{\mathbf{A}_{xx} \det a}} \partial_{\alpha} [ V(T) \partial_{\beta} \tilde{Y}_M (\mathbf{A}^{-1})^{ji}_S \sqrt{\mathbf{A}_{xx} \det a} ] = \frac{1}{\sqrt{\det a}} \partial_{\alpha} [ \partial_{\beta} \tilde{Y}_M (\mathbf{a}^{-1})^{ji}_S \sqrt{\det a} ] = \tilde{\mathcal{J}}_M
\]

(5.14)

and

\[
\mathcal{J}^i = - \frac{2\pi \alpha'}{V(T) \sqrt{\det \mathbf{A}}} \partial_i [ V (\mathbf{A}^{-1})^{ij} \sqrt{\det \mathbf{A}} ] = - \frac{2\pi \alpha'}{\sqrt{\det a}} \partial_{\beta} [ (\mathbf{a}^{-1})^{\beta\alpha} \sqrt{\det a} ] = \tilde{\mathcal{J}}^i .
\]

(5.15)

Using these results we easily obtain that

\[
\mathcal{L}_K + \mathcal{L}_P = \tau_{p-1} \sqrt{\mathbf{A}_{xx} \mathbf{V}(f(ax))(\tilde{\mathcal{L}}_K + \tilde{\mathcal{L}}_P)} .
\]

(5.16)

Inserting this expression to the action for an unstable Dp-brane we obtain

\[
S = \int dt d^p x N(\mathcal{L}_K + \mathcal{L}_P) = \tau_{p-1} \int dx f'(ax) \mathbf{V}(f(ax)) \int dt d^{p-1} x N(\mathcal{L}_K^{st} + \mathcal{L}_P^{st}) = \tau_{p-1} \int dt d^{p-1} x N(\mathcal{L}_K^{st} + \mathcal{L}_P^{st}) ,
\]

(5.17)

where by presumption

\[
\tau_{p-1} \int dx f'(ax) \mathbf{V}(af(x)) = \tau_{p-1} \int dm \mathbf{V}(m) = T_{p-2}
\]

(5.18)

is the tension of D(p-1)-brane. In other words we found that the action for fluctuation around the tachyon kink is the same as the action for non-relativistic D(p-1)-brane.
6. Conclusion

Let us outline results derived in this paper. We found an action for stable and unstable \( p+1 \) dimensional system with anisotropic scaling that has the property that the potential is proportional to the variation of stable or unstable Euclidean Dp-brane action. We extended the symmetries of these \( p+1 \) dimensional theories in order to be invariant under foliation preserving diffeomorphism and under space-time dependent gauge transformations. Then we developed the Hamiltonian formalism for these theories and we also studied the tachyon kink on the world-volume of \( p+1 \) dimensional unstable theory. We argued that the tachyon kink corresponds to the stable \( p \) dimensional non-relativistic theory.

Even if it is an open question how these theories can be embedded into superstring theories we mean that they provide an interesting new class of non-relativistic theories that can be studied further.

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