Superintegrable and shape invariant systems with position dependent mass

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Abstract
Second order integrals of motion for 3d quantum mechanical systems with position dependent masses (PDM) are classified. Namely, all PDM systems are specified which, in addition to their rotation invariance, admit at least 1 second order integral of motion. All such systems appear to be also shape invariant and exactly solvable. Moreover, some of them possess the property of double shape invariance and can be solved using two different superpotentials. Among them there are systems with double shape invariance which present nice bridges between the Coulomb and isotropic oscillator systems. A simple algorithm for calculating the discrete spectrum and the corresponding state vectors for the considered PDM systems is presented and applied to solve five of the found systems.

Keywords: superintegrability, shape invariance, exact solutions, position dependent mass

1. Introduction

There are three differently defined global properties which are possessed by some quantum mechanical systems: superintegrability, supersymmetry and exact solvability.

A system with $d$ degrees of freedom is called superintegrable if it admits more than $d$ integrals of motion (including Hamiltonian). Moreover, exactly $d$ of them should commute with each other. The maximal admissible number of integrals of motion is equal to $2d − 1$, and the related systems are called maximally superintegrable.

The system is treated as supersymmetric in two cases: when its integrals of motion form a superalgebra, or (and) the Hamiltonian has a specific symmetry with respect to the Darboux transform, called shape invariance.
Maximally superintegrable or shape invariant systems as a rule are exactly solvable. It means that all their energy levels can be calculated algebraically, and the corresponding wave functions can be found in explicit form.

On the other hand, a tight coupling exists between the maximal superintegrability and shape invariance [1]. The list of systems which are both supersymmetric and shape invariant includes the hydrogen atom, isotropic harmonic oscillator, 2d and 3d superintegrable systems with spin [2–4], arbitrary dimensional systems with spin 1/2 [5], and others. New examples of such coupling are presented in the current paper.

There are various reasons to search for superintegrable and shape invariant systems. First, many such systems are very important from a physical viewpoint. Secondly, as a rule they can be solved analytically in a way free of uncertainties generated by various approximate approaches. In addition, these systems present a nice field for application of symmetries in physics.

The systematic search for integrable and superintegrable systems in quantum mechanics started with fundamental papers [6] and [7] where the second order integrals of motion for planar quantum mechanical systems have been classified. We will not discuss the very inspiring history of this research which is still continuing now, see survey [8]. Let us only mention that there is a great number of papers devoted to this subject. In particular, the contemporary field of superintegrable models includes the systems with spin [3, 4, 9–12]. For the classification of shape invariant systems with spin see [13] and [14].

In the present paper we continue the search for superintegrable Schrödinger equations with position dependent mass (PDM), started in paper [15]. In contrast with the systems with a constant mass, superintegrability aspects of such equations were not studied systematically, although special classes of maximally superintegrable systems are well known, see, e.g. [16]–[19] and references cited therein. Let us remind ourselves that just the PDM systems are intensively used in modern physics. They are applied for modeling of condensed–matter systems, namely, semiconductors [20, 21], quantum liquids [22] and metal clusters [23], quantum wells, wires and dots [24, 25] and many, many others. In addition, thanks to the additional randomness connected with a non–fixed mass, the PDM systems present a much more rich field for application of symmetry methods than the systems with constant masses.

The present paper includes the complete classification of 3d PDM systems which are invariant with respect to the rotation group and admit second order integrals of motion. Up to equivalence, 20 such systems are specified. Moreover, 16 of them are defined up to an arbitrary parameter, and 4 of them include pairs of arbitrary parameters.

We will show that all obtained systems are both maximally superintegrable, shape invariant and exactly solvable. In other words, we prove that if a PDM system is rotationally invariant and admits second order integrals of motion additional to polynomials in angular momentum and dilatation, it possesses all global properties discussed at the beginning of this paper. Using these properties, we calculate the energy spectra for some of the obtained systems and construct their solutions explicitly. We also present a simple algorithm for the construction of discrete spectrum solutions for any of the presented systems.

Shape invariance is a fine symmetry which presents very convenient tools for constructing exact solutions of systems which possess this property. We will see that the PDM systems include ones which have doubled hidden supersymmetry. In other words, the corresponding radial equations are shape invariant with respect to two different Darboux transforms.

It is necessary to note that an important and in some sense completed class of rotationally invariant superintegrable systems was presented in papers [16]–[19]. These systems are quantized versions of classical ones, obtained starting with the Coulomb and oscillator
systems in curved spaces. Being superintegrable and rotationally invariant, they naturally appear in the results of our research. In particular, we specify those which admit second order integrals of motion and are shape invariant. We also show that such rotationally invariant and superintegrable systems exist which do not belong to the class introduced in [16]–[19]. A more detailed discussion of these points is presented in section 7.

2. Rotationally invariant PDM Schrödinger equations

We will study stationary Schrödinger equations with position dependent mass, which can be represented in the following form:

\[ \hat{H} \psi = E \psi, \]  

(1)

where

\[ \hat{H} = p_a f(x)p_a + V(x). \]  

(2)

Here \( x = (x^1, x^2, x^3) \), \( p_a = -i \partial_a \), \( V(x) \) and \( f(x) = \frac{1}{2m(x)} \) are arbitrary functions associated with the effective potential and inverse effective PDM, and summation from 1 to 3 is imposed over the repeating index \( a \).

In paper [15] all equations (1) admitting at least one first order integral of motion have been classified. In particular all rotationally invariant systems with different symmetries were presented there. In general such systems are characterized by the following \( x \)-dependence of \( f \) and \( V \):

\[ f = f(x), \quad V = V(x), \quad x = \sqrt{x_1^2 + x_2^2 + x_3^2}. \]  

(3)

Hamiltonians (2), (3) by construction are invariant with respect to group SO(3) whose generators are components of the angular momentum vector:

\[ J_a = \epsilon_{abc} x_b p_c \]  

(4)

where \( \epsilon_{abc} \) is the Levi-Civita tensor.

In accordance with [15] there are exactly four such Hamiltonians which have a more extended symmetry. They are specified by the following inverse masses and potentials:

\[ f = x^2, \quad V = 0, \]  

(5)

\[ f = (1 + x^2)^2, \quad V = -6x^2, \]  

(6)

\[ f = (1 - x^2)^2, \quad V = -6x^2, \]  

(7)

\[ f = x^4, \quad V = -6x^2. \]  

(8)

Hamiltonians (2) whose arbitrary elements are fixed by equations (5), (6), (7) and (8) admit additional integrals of motion, i.e.

\[ D = x \cdot p - \frac{3i}{2}, \]  

(9)

\[ N_a = \frac{1}{2} (K_a - p_a). \]  

(10)
In the case (5) the rotation symmetry is extended by the scaling transformations while for cases (6) and (7) the corresponding integrals of motion (10), (4) and (11), (4) form bases of algebras so(4) and so(1,3) correspondingly [15]. Thanks to their high symmetry equations (1) with potentials (5)–(7) are exactly solvable, for their explicit solutions see [15].

3. Determining equations

Let us search for second order integrals of motion for equation (1), i.e. for commuting with $H$ differential operators of second order of the following generic form:

$$Q = \mu_{ab} \partial_a \partial_b + \xi^a \partial_a + \eta$$

where $\mu^{ab} = \mu^{ba}$, $\xi^a$ and $\eta$ are functions of $x$ and summation from 1 to 3 is imposed over all repeating indices.

By definition, operators $Q$ should commute with $\hat{H}$:

$$[\hat{H}, Q] \equiv \hat{H}Q - Q\hat{H} = 0.$$  

Calculating the commutator and equating the coefficients for different differential operators we come to the following system of determining equations:

$$5\left(\mu_{ab}^{\alpha\beta} + \mu_{a}^{\beta\alpha} + \mu_{b}^{\alpha\beta}\right) \equiv \delta_{ab} \left(\mu_{\alpha\eta}^{\alpha\beta} + 2\mu_{\alpha\beta}^{\alpha\beta}\right) + \delta_{bc} \left(\mu_{\alpha\eta}^{\alpha\beta} + 2\mu_{\alpha\beta}^{\alpha\beta}\right) + \delta_{ac} \left(\mu_{b\beta}^{\alpha\beta} + 2\mu_{b\beta}^{\alpha\beta}\right),$$

$$F^{\alpha} \equiv \left(\mu_{a}^{\alpha\beta} + 2\mu_{b}^{\alpha\beta}\right)f - 5\mu_{\alpha\beta}^{\alpha\beta}x_\alpha f' = 0,$$

$$F^{ab} \equiv \left(\mu_{a}^{\alpha\beta} + \xi^a \partial_a + \xi^b \partial_b\right)f + \left(\mu_{a}^{\alpha\beta}x^\beta - 2\mu_{\alpha\beta} - \delta_{a\beta} \left(\mu_{a\beta}^{\alpha\beta} + \xi^a x^\alpha\right)\right)f'$$

$$+ \left(\mu_{a\beta}x^\alpha + \mu_{\alpha\beta}x^\alpha + \delta_{a\beta} \mu_{\alpha\beta} x^\alpha\right)x^\beta x_\alpha \frac{f'}{f} = 0,$$

$$\hat{F}^{\alpha} \equiv \left(2\mu_{a}^{\alpha\beta} + \xi^a \partial_a\right)f + \left(\xi^a x^\beta - \xi^\beta x^\alpha\right)f' - \frac{1}{x^2} \left(x^\alpha \xi^a x^\beta + 2\mu_{\alpha\beta} x^\alpha + x^\alpha x^\beta\right)(x^2 f' - x f)$$

$$- \frac{1}{x^2} \mu_{a\beta} x^\alpha x^\beta x^\alpha \left(3x^2 f' - 3x^2 f' + 3x f'\right) + 2\mu_{a\beta} x^\beta V',$$

$$x^2 f_{\alpha\beta} + \eta_a x^\alpha x^\beta f' + \left(\xi^a x^\beta + \mu_{\alpha\beta}\right)x V' + \frac{1}{x^2} \mu^{\alpha\beta} x^\alpha x^\beta (x^2 V' - x V') = 0$$

where $f' = \frac{df}{dx}$, $\xi^a = \frac{d\xi}{dx}$, etc.

Thus to classify Hamiltonians (2) admitting second order integrals of motion (13) it is necessary to find all inequivalent solutions of rather complicated systems (15)–(19). Moreover, we will see that equation (19) can be deduced from the remaining ones. The presented system is overdetermined and includes 19 equations for 12 unknown functions $\mu^{ab}$, $\xi^a$, $\eta$, $f$ and $V$. 

$$N_a^+ = \frac{1}{2} \left(\mu^a + p_a\right)$$

and

$$K^a = x^2 p^a - 2x^a D$$

correspondingly.
4. Discussion of the determining equations

The autonomous subsystem (15) defines a conformal Killing tensor. Its general solution is a linear combination of the following tensors (see, e.g. [26])

\[ \mu_{1}^{ab} = \delta^{ab} \varphi_{1}(x) + k \left( x^{a}x^{b} - \delta^{ab}x^{2} \right), \]
\[ \mu_{2}^{ab} = \lambda^{a}x^{b} + \lambda^{b}x^{a} + \delta^{ab}\lambda^{c}x^{c} \varphi_{2}(x), \]
\[ \mu_{3}^{ab} = \left( x^{a}x^{bc} + x^{b}x^{ac} \right)x^{d}, \]
\[ \mu_{4}^{ab} = \left( x^{a}\lambda^{b} + x^{b}\lambda^{a} \right)x^{2} - 4x^{a}x^{b}\lambda^{c}x^{c} + \delta^{ab}\lambda^{c}x^{c} \varphi_{3}(x), \]
\[ \mu_{5}^{ab} = \delta^{ab}\lambda^{cd}x^{c}x^{d} \varphi_{4}(x), \]
\[ \mu_{6}^{ab} = \left( \epsilon^{acd}x^{b} + \epsilon^{bcd}x^{a} \right)x^{d}, \]
\[ \mu_{7}^{ab} = \lambda^{ab}x^{2} - \left( x^{2}\lambda^{bc} + x^{b}\lambda^{ac} \right)x^{c} + \delta^{ab}\lambda^{cd}x^{c}x^{d} \varphi_{5}(x), \]
\[ \mu_{8}^{ab} = 2 \left( x^{a}x^{bc} + x^{b}x^{ac} \right)\lambda^{ab}x^{d} - \left( \epsilon^{acd}x^{b} + \epsilon^{bcd}x^{a} \right)x^{2}, \]
\[ \mu_{9}^{ab} = \lambda^{ab}x^{4} - 2 \left( x^{a}\lambda^{bc} + x^{b}\lambda^{ac} \right)x^{c}x^{d} + 4x^{a}x^{b} + \lambda^{cd}x^{c}x^{d} + \delta^{ab}\lambda^{cd}x^{c}x^{d} \varphi_{6}(x) \]  
(20)

where \( \lambda^{ab} = \lambda^{ba}, \lambda^{a} \) are arbitrary parameters, and \( \varphi_{1}, \ldots, \varphi_{6} \) are arbitrary functions of \( x \).

The next step is to solve the remaining equations (17)–(19) with \( \mu^{ab} \) being linear combinations of tensors (20). Fortunately, this huge problem can be reduced to a series of relatively simple subproblems corresponding to particular linear combinations of these tensors.

Let us specify such linear combinations of tensors (20) which should be considered separately. They should include the terms with the same transformation properties w.r.t. the rotation group. The tensors \( \mu_{1}^{ab}, \mu_{2}^{ab}, \ldots, \mu_{6}^{ab} \) and \( \langle \mu_{5}^{ab}, \mu_{6}^{ab} \rangle \) generate scalar, vector and tensor integrals of motion correspondingly. Separating scalars, vectors and tensors with the same parities, we can specify the following non-equivalent versions of \( \mu^{ab} \):

\[ \mu^{ab} = \mu_{1}^{ab} \]  
(21)
for scalar integrals of motion,

\[ \mu^{ab} = \mu_{3}^{ab} \]  
(22)
for pseudovector integrals of motion,

\[ \mu^{ab} = \nu\mu_{2}^{ab} + \lambda\mu_{4}^{ab} \]  
(23)
for vector integrals of motion,

\[ \mu^{ab} = \nu\mu_{5}^{ab} + \omega\mu_{2}^{ab} + \lambda\mu_{6}^{ab} \]  
(24)
for pseudotensor integrals of motion, and

\[ \mu^{ab} = \nu\mu_{5}^{ab} + \lambda\mu_{8}^{ab} \]  
(25)
for tensor integrals of motion, where \( \nu, \lambda \) and \( \omega \) are arbitrary parameters.

The linear combinations of arbitrary functions appearing in (23) and (24) should be treated as new arbitrary functions.
The next subsystem of the determining equation, i.e. (16), is compatible iff:

\[
\left( \mu_{b}^{an} + 2 \mu_{n}^{an} \right) \mu_{b}^{bn} x_{n} = \left( \mu_{b}^{an} + 2 \mu_{n}^{an} \right) \mu_{b}^{bn} x_{n}.
\]  

(26)

Then, comparing \( \partial_{a} F^{a} \) with \( x_{a} x_{b} F^{ab} \) we find the following differential consequence of (16) and (17):

\[
2 \left( z_{n}^{a} - \mu_{kn}^{an} \right) f' = 3 \left( z_{n}^{a} x_{a} - \mu_{k}^{kn} x_{k} \right) f'.
\]

which is compatible with (16) and (17) in two cases: either

\[
z_{n}^{a} = \mu_{k}^{kn}
\]

(27)
or the vector \( \tilde{z}_{n}^{a} = z_{n}^{a} - \mu_{k}^{kn} \) satisfies the following condition

\[
\left( \tilde{z}_{n}^{a} + \tilde{z}_{b}^{b} \right) f' = \delta_{ab} \tilde{z}_{n}^{a} f'.
\]

which is the necessary condition for coefficients of the first order integrals of motion [15]. Since such integrals of motion had been already classified in [15], we will set \( \tilde{z}_{n}^{a} = 0 \), i.e. impose the condition (27) on coefficients \( z_{n}^{a} \).

Considering the differential consequence of (17) and (18) in the forms \( \partial_{a} F^{a} = 0 \) and \( \partial_{a} F^{a} = 0 \) we obtain equation (19). So the latter equation is a consequence of (17) and (18) and can be omitted.

Thus the problem of the classification of the rotationally invariant PDM systems admitting second order integrals of motion is reduced to the search for inequivalent solutions of equations (16), (17), and (18) for unknowns \( f, V, \tilde{z}, \) and \( \eta \) for all versions of functions \( \mu^{ab} \) enumerated in (21)–(25). The corresponding calculations are outlined in the appendix, while the classification results are presented in the following section.

Let us note that whenever condition (27) is satisfied and functions \( f, \tilde{V} \) and \( \varphi_{1}, \varphi_{2}, \ldots, \varphi_{6}, \eta \) in (2) and (13), (20) are real, both Hamiltonians \( \hat{H} \) and second order integrals of motion \( Q \) are formally self-adjoint on the standard \( L^{2} \) space with scalar product

\[
\langle \varphi_{1} | \varphi_{2} \rangle = \int_{M} \varphi_{1}^{*} \varphi_{2} d^{3}x.
\]

(28)

Just using the standard scalar product (28) is one of the main points of our approach.

5. Classification results

5.1. Equivalence transformations

It was indicated in [15] that the equivalence group of equation (1) is nothing but C(3), i.e. the conformal group in 3d Euclidean space. It means that by acting on dependent and independent transformations belonging to C(3), we do not change the generic form of Hamiltonian (2) although functions \( f \) and \( \tilde{V} \) can be changed.

However, since we suppose that equation (1) is rotationally invariant, we should restrict ourselves to such equivalence transformations which keep this invariance, i.e. are either pure rotations or transformations commuting with \( J_{a} \) (4). In other words, the equivalence transformations for the considered class of equations are reduced to products of rotations and scalings of independent variables

\[
x_{a} \rightarrow R_{ab} x_{b}, \quad x_{a} \rightarrow \alpha x_{a},
\]

(29)
and the inverse transformation

\[ x_a \rightarrow \tilde{x}_a = \frac{x_a}{\lambda^2}, \quad \psi(x) \rightarrow \tilde{\psi}(x) \]  

(30)

where \( R_{ab} \) is a rotation matrix and \( \omega \) is an arbitrary real parameter. In addition, we will define Hamiltonians \( H \) up to multiplication by a real parameter \( \omega \) and up to a constant shift of potential \( \tilde{V} \). Thus the equivalence transformations (29) and (30) will be extended by the following changes

\[ H \rightarrow \omega H, \quad \tilde{V} \rightarrow \tilde{V} + C \]  

(31)

where \( \nu \) and \( C \) are real constants.

In the following we present the Hamiltonians and the corresponding second order integrals of motion obtained by solving the system (15)–(19). These solutions are defined up to equivalence transformations (29), (30) and (31).

The presentation (2) for Hamiltonians is compact and convenient for our classification procedure. However there exists another and physically motivated representation [20, 29], which is equivalent to (2):

\[ H = m\tilde{p}_a \tilde{p}^{a-2} + V \]  

(32)

where \( m = \frac{1}{2} \) and \( r \) is the ambiguity parameter of the kinetic energy term [20]. We will present the found Hamiltonians in the form (32) with \( r = -1/2 \), i.e.

\[ H = \tilde{f}_a \tilde{p}^a + V. \]  

(33)

It happens that just representation (33) corresponds to the most compact forms of the mass and potential terms The related equation (1) should be rewritten without hats and tildes:

\[ H\psi = E\psi. \]  

(34)

Notice that potentials \( V \) and \( \tilde{V} \) are connected by the following relation:

\[ V = \tilde{V} + \tilde{f}_a \tilde{p}^a + \frac{\tilde{f}_a \tilde{p}^a}{2} - \frac{\tilde{f}_a \tilde{p}^a}{4f}. \]  

(35)

Nonequivalent Hamiltonians (33) admitting second order integrals of motion are presented in the following subsections.

5.2. Vector integrals of motion

In section 3, three classes of second order integrals of motion had been indicated, i.e. scalar, vector and tensor ones. It is possible to show (see appendix) that the scalar integrals of motion are linear combinations of Hamiltonian and squared orbital momentum and so can be treated as trivial. Thus we start with vector symmetries.

In accordance with the analysis presented in section 3 it is possible to specify two tensors \( \mu^{ab} \) which can generate vector integrals of motion, i.e.

\[ \mu^{ab} = \mu_3^{ab} = (x^a \tilde{x}^b + x^b \tilde{x}^a)e^c x^c_\tilde{a} \]  

(36)

and a linear combination of tensors \( \mu_3^{ab} \) and \( \mu_4^{ab} \):

\[ \mu^{ab} = \nu \mu_3^{ab} + \mu_4^{ab} = \nu (\chi^a \chi^b + \chi^b \chi^a - 2\delta^{ab} \chi^c \chi^c) \]

\[ + \mu (x^a \tilde{x}^b + x^b \tilde{x}^a) x^2 - 4x^a x^b \chi^c x^c + 2\delta^{ab} x^2 \chi^c x^c + \delta^{ab} \chi^c \chi^c \psi(x) \]  

(37)

where we use the arbitrariness of \( \psi \) to obtain a convenient realization for \( \mu^{ab} \).
Versions (36) and (37) should be considered separately, since the related integrals of motion have different parities w.r.t. the space inversion.

It is shown in the appendix that integrals of motion corresponding to (36) are admitted only by the systems specified in (5). Moreover, these integrals of motion are nothing but polynomials in the first order symmetries (4) and (9).

Considering integrals of motion corresponding to (37) we can a priori restrict ourselves to the following values of parameters $\alpha$ and $\mu$:

$$\nu = 1, \quad \mu = 0; \quad \text{(38)}$$
$$\nu = \mu = 1; \quad \text{(39)}$$
$$\nu = -\mu = 1. \quad \text{(40)}$$

Then integrals of motion corresponding to arbitrary $\alpha$ and $\mu$ can be obtained by scaling and inversions of independent variables $x_a$, i.e. by products of transformations (29) and (30).

Substituting (37) into (26) we obtain the following equation for function $\varphi$:

$$x(\nu + \lambda x^2)\varphi' = \varphi^2 + \left(3\lambda x^2 - \nu\right)\varphi,$$

whose solutions are:

$$\varphi = 0, \quad \text{and} \quad \varphi = \frac{(\nu + \mu x^2)^2}{\nu - 2\kappa x - \mu x^2} \quad \text{(41)}$$

where $\kappa$ is an integration constant. Then, substituting (37) and (41) into equations (16), (17) and going over values of parameters $\nu$ and $\mu$ specified in (38)–(40), we find admissible functions $f$. The corresponding potentials can be found solving the remaining determining equation, i.e. (18). The results of these calculations (whose details can be found in the appendix are presented in table 1.

In the table all non-equivalent mass and potentials are presented which give rise to superintegrable systems admitting vector integrals of motion. However, this list can be added by the systems whose masses and potentials are specified in (5)–(8). The latter systems also admit second order integrals of motion, which are products of their first order symmetries. Such symmetries are apparent and will not be discussed here.

All presented systems are shape invariant. More exactly, this property is possessed by the corresponding radial equation. The way to obtain the radial equations with the indicated shape invariant potentials is described algorithmically in section 6.1. For the cases enumerated in items 1–4 we have two-fold shape invariance when two different superpotentials can be used to factorize the radial equation. The types of radial potentials are indicated in the last column.

The list of potentials and mass terms presented in table 1 is completed up to equivalence transformations (29), (30) and (31). Using these transformations, it is possible to propagate the obtained systems to families of equivalent ones. For example, starting with the system specified in item 1 and making the inverse transformation (30), we can construct the following Hamiltonian and the related integrals of motion:

$$H = \frac{3}{x^2} p^2 x + \frac{3}{x} \frac{\alpha}{x},$$
$$Q_a = J_{ab} K_b + K_b J_{ab} + \frac{1}{2} \left\{ H, \frac{x_a}{x} \right\}. \quad \text{(42)}$$
The same trick can be made with the systems specified in items 2, 7 and 8 while the remaining systems are invariant w.r.t. the inversion transformation up to signs of Hamiltonian or parameter $\alpha$.

5.3. Tensor integrals of motion

Consider now determining equations (16)–(19) with tensor $\mu^{ab}$ given by formula (24), i.e.

$$\mu^{ab} = \nu^{ab} + \omega \left( \lambda^{ab} x^2 - \left( x^2 \lambda^{bc} + x^b \lambda^{ca} \right) x^c - 2\delta^{cd} x^c x^d \right)$$

$$+ \mu \left( \lambda^{ab} x^4 - 2 \left( x^a \lambda^{bc} + x^b \lambda^{ca} \right) x^c x^2 + 4 x^a x^b \lambda^{cd} x^c x^d \right) + \delta^{cd} x^c x^d \varphi(\chi).$$

(43)

The term multiplied by $\omega$ is not essential since it corresponds to integrals of motion proportional to $J^a J^b$, which are accepted by any Hamiltonian (33) thanks to its rotational invariance. However, the presence of this term helps to write some of the integrals of motion in a more compact form.

### Table 1

Functions $f$ and $V$ specifying non-equivalent Hamiltonians (33) and the corresponding vector integrals of motion. Where $\alpha$ and $\kappa \neq \pm 1$ are arbitrary constants, the symbol $\{,\}$ denotes anticommutator, $J_{ab} = e_{abc} J_c$, while operators $K_{ab}$, $N^{ab}_c$ and $J_a$ are defined in equations (10)–(12) and (4).

| No | $f$ | $V$ | Integrals of motion | Solution approach | Effective potentials |
|----|-----|-----|---------------------|-------------------|---------------------|
| 1. | $x$ | $\alpha x$ | $Q_a = \left( p_x, J_{ab} \right) + \frac{1}{2} \left[ H, \frac{x_a}{x} \right]$ | direct or two-step | 3d oscillator or Coulomb |
| 2. | $x^4$ | $\alpha x$ | $Q_a = \left( K_{ab}, J_{ab} \right) - \alpha x^a$ | direct or two-step | Coulomb or 3d oscillator |
| 3. | $x(x-1)^2$ | $\frac{\alpha x}{(x+1)^2}$ | $Q_a = \left( J_{ab}, N^a_b \right) + \frac{1}{2} \left[ H, x_a \right]$ | direct or two-step | Eckart or hyperbolic Pöschl–Teller |
| 4. | $x(x+1)^2$ | $\frac{\alpha x}{(x-1)^2}$ | $Q_a = \left( J_{ab}, N^a_b \right) + \frac{1}{2} \left[ H, x_a \right]$ | direct or two-step | Eckart or trigonometric Pöschl–Teller |
| 5. | $(1 + x^2)^2$ | $\frac{\alpha (1-x^2)}{x}$ | $Q_a = \left( J_{ab}, N^a_b \right) - \alpha \frac{x^a}{x}$ | direct | trigonometric Rosen–Morse |
| 6. | $(1 - x^2)^2$ | $\frac{\alpha (1-x^2)}{x}$ | $Q_a = \left( J_{ab}, N^a_b \right) - \alpha \frac{x^a}{x}$ | direct | Eckart |
| 7. | $\frac{x}{x+1}$ | $\alpha x$ | $Q_a = \left( J_{ab}, p_b \right) + \frac{1}{2} \left[ H, \frac{x_a}{x} \right]$ | two-step | Coulomb |
| 8. | $\frac{x}{x-1}$ | $\alpha x$ | $Q_a = \left( J_{ab}, p_b \right) + \frac{1}{2} \left[ H, \frac{x_a}{x} \right]$ | two-step | Coulomb |
| 9. | $\frac{(x^2 - 1)^2 x}{x^2 - 2kx + 1}$ | $\frac{\alpha x}{x^2 - 2kx + 1}$ | $Q_a = \left( J_{ab}, N^a_b \right) + \frac{1}{2} \left[ H, \frac{x_a}{x} \right]$ | two-step | Eckart |
| 10. | $\frac{(x^2 + 1)^2 x}{x^2 - 2kx - 1}$ | $\frac{\alpha x}{x^2 - 2kx - 1}$ | $Q_a = \left( J_{ab}, N^a_b \right) + \frac{1}{2} \left[ H, \frac{x_a}{x} \right]$ | two-step | trigonometric Rosen–Morse |
\begin{table}
\centering
\begin{tabular}{lllll}
\hline

No & $f$ & $V$ & Integrals of motion & Solution approach & Effective radial potential \\
\hline
1. & $\frac{1}{x^2}$ & $\frac{a}{x^2}$ & $Q_{ab} = R_b P_b$ & direct or two-step & Coulomb or 3d oscillator \\
2. & $x^4$ & $-\frac{a}{x^2}$ & $Q_{ab} = K_a K_b - ax^2$ & direct or two-step & 3d oscillator or Coulomb \\
3. & $(x^2 - 1)^2$ & $\frac{ax^2}{(x^2 + 1)^2}$ & $Q_{ab} = \left\{N_a^+, N_b^+\right\} + \frac{2a^2 x^2}{(x^2 + 1)^2}$ & direct or two-step & Eckart or hyperbolic Pöschl–Teller \\
4. & $(x^2 + 1)^2$ & $\frac{ax^2}{(x^2 - 1)^2}$ & $Q_{ab} = \left\{N_a^-, N_b^+\right\} + \frac{2a^2 x^2}{(x^2 - 1)^2}$ & direct or two-step & Eckart or trigonometric Pöschl–Teller \\
5. & $(x^4 - 1)^2$ & $\frac{a(x^4 + 1)}{x^2}$ & $Q_{ab} = K_a K_b + p_a P_b$ & direct & Eckart \\
6. & $(x^4 + 1)^2$ & $\frac{a(x^4 - 1)}{x^2}$ & $Q_{ab} = K_a K_b - p_a P_b$ & direct & trigonometric Rosen–Morse \\
7. & $\frac{1}{x^2 + 1}$ & $\frac{a}{x^2 + 1}$ & $Q_{ab} = p_a P_b$ & two-step & 3d oscillator \\
8. & $\frac{1}{x^2 - 1}$ & $\frac{a}{x^2 - 1}$ & $Q_{ab} = p_a P_b$ & two-step & 3d oscillator \\
9. & $(x^4 - 1)^2$ & $\frac{a x^2}{x^4 - 2a x^2 + 1}$ & $Q_{ab} = K_a K_b + p_a P_b$ & two-step & Eckart \\
\hline
\end{tabular}
\caption{Functions $f$ and $V$ specifying non-equivalent Hamiltonians (33) which admit second order pseudotensor integrals of motion.}
\end{table}
| No. | $f$ | $V$ | Integrals of motion | Solution approach | Effective radial potential |
|-----|-----|-----|---------------------|-------------------|---------------------------|
| 10. | $\frac{(x^4 + 1)^2}{x^4 - 2cx^2 - 1}$ | $\frac{ax^2}{x^4 - 2cx^2 - 1}$ | $Q_{ab} = K_a K_b - p_a p_b$ | two-step | trigonometric Rosen–Morse |

$$Q_{ab} = K_a K_b - p_a p_b - \frac{1}{2} (H + 6\epsilon + \frac{p_a x_b}{x^2} + \frac{p_b (x^4 - 1) p_x}{x^2})$$
As per the previous section it is sufficient to restrict ourselves to the values of parameters $\nu$ and $\mu$ fixed in equations (38), (39) and (40). Solving the corresponding equations (16)–(18) we find all systems admitting pseudotensor integrals of motion, see the appendix for calculation details. The classification results are presented in table 2.

6. Shape invariance and exact solutions

All systems presented in tables 1 and 2 are maximally superintegrable and can be solved exactly. In addition, all of them appear to be shape invariant and so can be solved using tools of SUSY quantum mechanics. Moreover, some of the presented systems are characterized by the multiple shape invariance, i.e. they can be solved using more then one superpotential.

6.1. Two strategies in the construction of exact solutions

Let us consider equations (34) where $H$ are hamiltonians (33) whose mass and potential terms are specified in the presented tables. We will search for square integrable solutions of these systems vanishing at $x = 0$.

First let us transform (34) to the following equivalent form

$$\hat{H}\Psi = E\Psi,$$

where

$$\hat{H} = \sqrt{f} \left( \frac{1}{\sqrt{f}} \frac{d}{dx} \right) = \frac{d}{dx} + V,$$

$$\Psi = \sqrt{f} \psi.$$

Then, introducing spherical variables and expanding solutions via spherical functions $Y_m^l$

$$\Psi = \frac{1}{x} \sum_{l,m} \phi_{lm}(x) Y_m^l,$$

we obtain the following equation for radial functions:

$$- \frac{d^2\phi_{lm}}{dx^2} + \left( \frac{fl(l+1)}{x^2} + V \right) \phi_{lm} = E\phi_{lm}. \quad (47)$$

We will search for normalizable solutions of equations (44) and (52). In accordance with (28) and (45) the corresponding scalar products look as follows:

$$\langle \Psi_1 | \Psi_2 \rangle = \int_{M} \Psi_1^{*} \Psi_2 f^{-1} d^{3}x. \quad (48)$$

and

$$\langle \phi_{lm}^{(1)} | \phi_{lm}^{(2)} \rangle = \int_{0}^{R} \frac{\phi_{lm}^{(1)}(x) \phi_{lm}^{(2)}(x)}{f(x)} dx \quad (49)$$

respectively, where the integration limit $R$ is equal to 1 or $R \to \infty$ depending on a concrete problem. All solutions presented in the following text are normalizable with respect to the scalar product (49) with $R = \to \infty$.

Let us present two possible ways to solve equation (47). They can be treated as particular cases of Liouville transformation (refer to [28] for definitions) and include commonly known steps. But it is necessary to fix them as concrete algorithms to obtain shape invariant potentials presented in the tables.
The first way (which we call direct) includes consequent changes of independent and dependent variables:

\[ \phi_{lm} \rightarrow \Phi_{lm} = f^1 \phi_{lm}, \quad \frac{\partial}{\partial x} \rightarrow f^1 \frac{\partial}{\partial x} f^{-1} = \frac{\partial}{\partial x} + \frac{f'}{4f} \quad (50) \]

and then

\[ x \rightarrow y(x), \quad (51) \]

where \( y \) solves the equation \( \frac{dy}{dx} = \frac{1}{\sqrt{f}} \). As a result equation (46) will be reduced to a more customary form

\[ -\frac{\partial^2 \Phi_{lm}}{\partial y^2} + V\Phi_{lm} = E\Phi_{lm} \quad (52) \]

where \( \tilde{V} \) is an effective potential

\[ \tilde{V} = V + f \left( \frac{1}{x^2} \left[ l(l + 1) - \left( \frac{f'}{4f} \right) - \left( \frac{f'}{4f} \right)^2 \right] \right), \quad x = x(y). \quad (53) \]

Equations (44), (45) with functions \( f \) and \( V \) specified in items 1–6 of both tables 1 and 2 can be effectively solved using the presented reduction to radial equation (52). All the corresponding potentials (53) appear to be shape invariant, and just these potentials are indicated in the fifth columns of the tables. The related equations (52) are shape invariant too and can be solved using the SUSY routine.

However, if we apply the direct approach to the remaining systems (indicated in items 7–10 of both tables), we come to equations (52) which are not shape invariant and are hardly solvable, if at all. To solve these systems we need a more sophisticated procedure which we call the two-step approach. To apply it we multiply (47) by \( V^{-1} \) and obtain the following equation:

\[ -\tilde{f} \frac{\partial^2 \phi_{lm}}{\partial x^2} + \left( \tilde{f} l(l + 1) + \tilde{V} \right) \phi_{lm} = \mathcal{E}\phi_{lm} \quad (54) \]

where \( \tilde{f} = \frac{f'}{f} \), \( \tilde{V} = -\frac{df}{f} \) and \( \mathcal{E} = -\alpha \). Then treating \( \mathcal{E} \) as an eigenvalue and solving equation (54) we can find \( \alpha \) as a function of \( E \), which defines admissible energy values at least implicitly. To do it is convenient to make changes (50) and (51) where \( f \rightarrow \tilde{f} \).

The presented trick with a formal changing of the roles of constants \( \alpha \) and \( E \) is well known. Our point is that any of the presented superintegrable systems can be effectively solved using either the direct approach presented in equations (45)-(53), or the two-step approach. Moreover, some of the presented systems can be solved using both the direct and two-step approaches, as indicated in the fourth columns of table 1 and 2. In all cases we obtain shape invariant effective potentials and can use tools of SUSY quantum mechanics.

6.2. A system with two-fold shape invariance

Let us apply the presented algorithms to selected superintegrable systems. We start with the following Hamiltonian

\[ H = \frac{1}{x^2} \frac{1}{x} + \frac{\alpha}{x^2} \quad (55) \]
which corresponds to functions $f$ and $V$ specified in the first item of table 2. The corresponding radial equation (47) takes the following form

$$\frac{1}{x^2} \frac{d^2 \phi_{lm}}{dx^2} + \left( \frac{l(l+1)}{x^4} + \frac{\alpha}{x^2} \right) \phi_{lm} = E \phi_{lm} \tag{56}$$

Equation (56) can be effectively solved using the direct method presented in the previous section. Making changes (50), (51) with $y = \frac{x^2}{2}$ we obtain the following version of equation (52):

$$\mathcal{H}_\mu \phi_{lm} \equiv \left( -\frac{\partial^2}{\partial y^2} + \frac{\mu(\mu + 1)}{y^2} + \frac{\alpha}{2y} \right) \phi_{lm} = E \phi_{lm} \tag{57}$$

where

$$\mu = \frac{l}{2} - \frac{1}{4}. \tag{58}$$

Up to the meaning of parameter $\mu$ equation (57) formally coincides with the radial equation for the hydrogen atom, provided $\alpha < 0$. Hamiltonian $\mathcal{H}_\mu$ is shape invariant, so we can construct exact solutions of (57) using the simple and regular procedure presented, e.g. in [30]. As a result we obtain the admissible eigenvalues $E = E_n$ and the corresponding eigenvectors $\psi_n$ in the following form:

$$E_n = -\frac{\alpha^2}{(4n + 2l + 3)^2} \tag{59}$$

and

$$\psi_n^{lm} = C_n^{lm} \frac{2^{l+1}}{4} \exp \left( -\frac{\alpha}{2} \right) F \left( -n, l + \frac{3}{2}, z \right) \tag{60}$$

correspondingly, where $F$ is the confluent hypergeometric function and $z = \frac{-\alpha^2}{(4n + 2l + 3)}$.

Notice that energy levels (59) differ from the levels of the hydrogen atom, thus the similarity of equation (57) with the radial equation for a Coulomb system is indeed only formal. Moreover, in contrast with the standard Coulomb system, the states with odd and even states are logically separated thanks to the specific dependence of parameter $\mu$ on orbital quantum number $l$. More detailed discussion of this point is presented in [27], while the formal separation of odd and even energy levels in SUSY quantum mechanics was discussed in [31].

Alternatively, equation (56) can be solved using the two-step approach discussed in section 6.1. Indeed, multiplying it by $x^2$ we immediately come to the following equation:

$$\mathcal{H}_l \phi_{lm} \equiv \left( -\frac{d^2}{dx^2} + \frac{l(l+1)}{x^2} + \frac{\omega^2}{4x^2} \right) \phi_{lm} = E \phi_{lm} \tag{61}$$

where we denote $-E = \omega^2$ and $-\alpha = E$.

In other words, we come to another shape invariant system which is nothing but a 3d isotropic harmonic oscillator. Using again the standard techniques of SUSY quantum mechanics [30] we can find admissible eigenvalues $E$ and the corresponding state vectors in the following form:

$$E = \omega(2n + l + 3/2) \tag{62}$$
and

$$\phi_{lm}^{n} = C_{lm}^{n} x^{l+1} e^{\frac{-a x^2}{2}} L_n^{l+\frac{1}{2}} \left( \frac{a x^2}{2} \right)$$

(63)

where $L_n^{l+\frac{1}{2}} \left( \frac{a x^2}{2} \right)$ are Laguerre polynomials.

As expected, formulae (62) and (59) are in perfect agreement: solving (62) for $\omega$ we find that $\omega^2 = E_n$.

We see that Hamiltonian (55) has rather specific properties. Namely, it gives rise to two different shape invariant radial equations, which formally coincide with equations for the Coulomb and 3d oscillator problems. Let us note that the same property, i.e. the existence of more than one shape invariant effective potential, is possessed by all systems specified in items 1–4 of both tables 1 and 2. Moreover, the system presented in item 1 of table 1 also presents a bridge between oscillator and Coulomb systems.

6.3. Systems with two arbitrary parameters

Among the Hamiltonians specified in tables 1 and 2 there are four operators including pairs of arbitrary parameters, namely, $\alpha$ and $\kappa$, see items 9 and 10 of both tables.

Here just these systems including pairs of arbitrary parameters are discussed. All of them are exactly solvable. Moreover, to find their solutions it is reasonable to use the two-step approach outlined in section 6.1.

Let us start with the systems specified in item 10 of table 2. The corresponding Hamiltonian (45) and radial equation (47) have the following form:

$$H = \frac{(x^4 + 1)^2}{x^4 - 2\kappa x^2 - 1} p^2 + \frac{a x^2}{x^4 - 2\kappa x^2 - 1}$$

and

$$\left( -\frac{(x^4 + 1)^2}{x^4 - 2\kappa x^2 - 1} \left( \frac{\partial^2}{\partial x^2} - \frac{l(l + 1)}{x^2} \right) + \frac{a x^2}{x^4 - 2\kappa x^2 - 1} \right) \phi_{lm} = E \phi_{lm},$$

(64)

Multiplying (64) from the left by $\frac{x^4 - 2\kappa x^2 - 1}{x^4}$ we come to the following equation:

$$\left( -\frac{(x^4 + 1)^2}{x^2} \left( \frac{\partial^2}{\partial x^2} = \frac{l(l + 1)}{x^2} \right) + \frac{\tilde{a}(x^4 - 1)}{x^2} \right) \phi_{lm} = \tilde{E} \phi_{lm},$$

(65)

where

$$\tilde{a} = -E \quad \text{and} \quad \tilde{E} = -\alpha - 2\kappa E.$$  

(66)

Notice that equation (65) with $\tilde{a} \rightarrow \alpha$ and $\tilde{E} \rightarrow E$ is needed also to find eigenvectors of the Hamiltonian whose mass and potential terms are specified in item 6 of table 2.

Making transformations (50) and (51) with $f = \frac{1}{x^2} + \frac{1}{y^2}$ and $y = \frac{1}{2} \arctan(x^2)$ we reduce equation (65) to the following form:

$$-\frac{\partial^2 \phi_{lm}}{\partial y^2} + \left( \mu(\mu - 4)\csc^2(4y) + 2\tilde{a}\cot(4y) \right) \phi_{lm} = \tilde{E} \phi_{lm}$$

(67)
Thus we again have an equation with a shape invariant (Rosen–Morse I) potential. It is consistent provided parameters $\tilde{\alpha}$ and $\mu$ are positive. The corresponding eigenvalues and eigenvectors are (see, e.g. [30])

$$E = (\mu + 4n)^2 - \frac{\tilde{\alpha}^2}{(\mu + 4n)^2},$$

and

$$\Phi_{lm} = \frac{1}{(z^2 - 1)^{\frac{1}{4}(\mu + 4n)}} \exp(\lambda y) P_n^{\frac{1}{2}(i\lambda - \mu - 4n), -\frac{1}{2}(i\lambda + \mu + 4n)}(z)$$

where $P_n^{a,-1}(z)$ are Jacobi polynomials, $\lambda = \frac{\tilde{\alpha}}{\mu + 4n}$ and $z = \cot(4y)$.

The initial wave functions $\phi_{lm}^n$ written in terms of the initial variable $x$ can be presented by the following expressions:

$$\phi_{lm}^n = \frac{x^2 - 1}{1 + x^4} \Phi_{lm}^n(z), \quad z = \frac{1 + x^4}{2x^2}.$$  

Thus we find eigenvalues $E = E_n$ and the corresponding state vectors $\phi_{lm}^n$ for radial equation (64). One more effort is needed to find the explicit expression for eigenvalues $E = E_n$, which can be found as solutions of the system of algebraic equations (66), (68) and (69):

$$E_n = (2l + 3 + 4n)^2 \left( \kappa + \sqrt{\kappa^2 + 1 + \frac{\alpha - 4}{(2l + 3 + 4n)^2}} \right)$$

In order for equation (67) to be consistent, both parameters $\tilde{\alpha}$ and $\mu$ should be positive [30]. It means that $E_n$ should be negative, which is guaranteed if parameters $\alpha$ and $\kappa$ satisfy the following condition:

$$\alpha > -5 - \frac{9}{2} (k - |k|)k.$$  

In complete agreement with the above it is possible to solve equations (44) for all cases specified in items 9 and 10 of table 1 and item 9 of table 2. To save room we restrict ourselves to the presentation of our final results.

The energy spectrum of Hamiltonian (45) with mass and potential terms fixed in item 9 of table 2 is given by the following formula:

$$E_n = (2l + 3 + 4n)^2 \left( \kappa + \sqrt{\kappa^2 - 1 + \frac{\alpha + 4}{(2l + 3 + 4n)^2}} \right)$$

while the related radial state vectors are:

$$\phi_{lm}^n = x \left( \frac{x^4 - 1}{x^2} \right)^{-\frac{1}{2}} \left( \frac{x^2 + 1}{x^2 - 1} \right)^\frac{\kappa}{2} P_n^{\frac{1}{2}, -1} \exp(\lambda y) P_n^{\frac{1}{2}(i\lambda - \mu - 4n), -\frac{1}{2}(i\lambda + \mu + 4n)}(z)$$

where $z = \frac{1 + x^4}{2x^2}$ and $N = 2l + 3 + 4n$. Moreover, parameters $\alpha$ and $\kappa$ are restricted by the following conditions:
\[ \alpha > 5 - \frac{9}{2}(\kappa + |\kappa|)\kappa, \quad |\kappa| \geq 1. \]

The eigenvalues and radial state vectors of Hamiltonian (45) specified in item 10 of table 1 are:
\[ E_n = 4(l + 1 + n)^2 \left( \kappa - \frac{\alpha - 1}{4(l + 1 + n)^2} \right), \quad \alpha > -3 - 2(k - |k|)k \quad (75) \]

and
\[ \phi_{llm}^n = x \left( \frac{x^2 + 1}{x} \right)^{-n-l} e^{-\frac{E_n \arctan(x)}{2M}} P_0^{\left(-M-\frac{\kappa}{M}+M+\frac{\kappa}{M}\right)} (i\zeta) \quad (76) \]

where \( \zeta = \frac{1-x^2}{2x} \) and \( M = n+l+1 \).

Finally, for the Hamiltonian (45) whose mass and potential are fixed in item 9 of table 1 we obtain the following energy spectrum:
\[ E_n = 4(l + 1 + n)^2 \left( \kappa + \frac{\kappa^2 - 1 + \frac{\alpha + 1}{4(l + 1 + n)^2}}{\sqrt{\kappa^2 + 1}} \right), \quad \alpha > 3 - 2(k + |k|)k, \quad |\kappa| \geq 1. \quad (77) \]

The corresponding eigenvectors are:
\[ \phi_{llm}^n = x \left( \frac{x^2 - 1}{x} \right)^{-n-l} \left( \frac{x - 1}{x + 1} \right)^{\frac{\kappa}{2M}} P_0^{\left(-M-\frac{\kappa}{M}+M+\frac{\kappa}{M}\right)} (\tilde{\zeta}) \quad (78) \]

where \( \tilde{\zeta} = \frac{x^2 + 1}{2x} \).

Notice that potential and inverse mass terms fixed in the last lines of tables 1 and 2 are singular at \( x^2 = \kappa + \sqrt{\kappa^2 + 1} \) and \( x^2 = \kappa + \sqrt{\kappa^2 + 1} \) correspondingly. However, solutions obtained in this section are regular, while the corresponding solutions \( \psi = f^{-1}\Psi \) of the initial equations (34) are equal to zero in these points.

### 7. Discussion

The main goal of the present paper was to make the next step to the complete classification of superintegrable PDM systems admitting second order integrals of motion. Namely, we classify rotationally invariant systems having this property. The complete list of such systems is presented in tables 1 and 2. Thus the first statement we prove is that there are no other systems of the kind specified below which are nonequivalent to the presented ones. The equivalence relations of considered equations (1), (2) are given by relations (29), (30) and (31), or, more generally, by arbitrary transformations belonging to the 3d conformal group \( C(3) \) [15]. Notice that in the latter case the formal rotation invariance can be lost.

Thus we present all nonequivalent rotationally invariant PDM systems admitting second-order integrals of motion. The related integrals of motion are presented explicitly in the fourth columns of the mentioned tables.

In addition to its Hamiltonian, any of the found systems admit four algebraically independent integrals of motion, two of which commute between themselves. The commuting integrals of motion are, say, \( J_3 \) and \( J_1^2 + J_2^2 + J_3^2 \) where \( J_1, J_2 \) and \( J_3 \) are components of angular momentum (4). Two additional independent integrals of motion can be chosen as \( Q_1 \) and \( Q_3 \) or as \( Q_{33} \) and \( Q_{12} \) for systems specified in table 1 or table 2 correspondingly. In other
words, all these systems are maximally superintegrable. Thus it is possible to formulate the second statement: if the PDM system is rotationally invariant and admits at least one second order integral of motion which is not a product of its first order symmetries, it also admits two more such integrals of motion and is maximally superintegrable.

Notice that almost all rotationally invariant PDM systems admitting first order additional integrals of motion are maximally superintegrable too, see equations (6)–(8) and (10)–(12). The only exception is the system specified by equations (5) which is superintegrable but not maximally superintegrable.

The next goal of this paper was to study the relations between the superintegrability and supersymmetry of the PDM systems. As expected these relations appear to be very close. Namely, absolutely all classified systems are also supersymmetric since their effective potentials are shape invariant. The same is true for the first order systems (6)–(8).

Many of the presented systems are characterized by analogous effective potentials. Indeed, in the last columns of tables 1 and 2 we can find eight cases of Eckart potential, six cases of Coulomb and the same number of oscillator potentials, etc. However, the systems with the same named effective potentials are essentially different. In some cases (like ones enumerated in items 1 and 2 of table 2) the same potentials correspond to different, i.e. direct and two-step solution approaches and generate absolutely different energy spectra. In the other cases the effective potentials have the same names but include different parameters. For example, comparing item 2 of table 1 and item 1 of table 2, in both cases we find the Coulomb effective potential appearing in the direct approach. However, in the case indicated in table 1 the potential parameter \( \mu \) is given by equation (58) while in the case presented in table 1 we obtain equation (57) with \( \mu = l \).

Some of the discussed systems have a rather specific property which we call two-fold shape invariance. Namely, they possess extended hidden supersymmetry and can be solved using two different superpotentials. One such system which is well known and is related to the well known Coulomb-oscillator duality, is discussed in section 6.2. The other systems with the two-fold shape invariance are specified in items 1–4 of table 1 and items 2–4 of table 2.

Thanks to their extended symmetries the presented systems are exactly solvable. Moreover, their supersymmetries make it possible to construct solutions in a very easy way by using tools of SUSY quantum mechanics. We find these solutions for the most complicated systems whose Hamiltonians are defined up to two arbitrary parameters, see section 6.3.

We also present a simple algorithm for the construction of exact solutions for any of the considered systems, which reduces this construction to a simple algebraic procedure since the related effective potentials are shape invariant, see section 6.1. For alternative ways to solve 1d PDM Schrödinger equations see [32–34].

The rotationally invariant and superintegrable PDM systems were discussed in numerous interesting papers by Ballesteros, Enciso, Herranz, Ragnisco and Riglioni. Selected papers of this Spanish-Italian team are presented in the reference list, see [16]–[19]. Thanks to efforts of the mentioned authors and their collaborators, such systems became a well studied field. However, we believe that the present paper makes a non-trivial contribution into this field in accordance with the following arguments.

In papers [16]–[19] the main accent is made on classical Hamiltonian systems. Quantum mechanical systems are considered also, but they appear as a result of the quantizing of classical ones.

The number of various second quantization procedures is rather extended. Moreover, starting with a particular classical system, we can obtain a lot of its quantum mechanical counterparts, which in general are not equivalent between themselves. In particular, the direct
application of the so called ‘PDM quantization’ leads to the loss of the superintegrability property of the systems considered in [18]. To keep this property it is necessary to make a specific modification of potentials [18]. Thus it is desirable to have a priori classification of all non-equivalent superintegrable quantum systems.

Just such classification for rotationally invariant PDM systems admitting second order integrals of motion is given in the present paper. Moreover, we write these systems in maximally simple forms, which do not include arbitrary parameters whose values can be fixed using equivalence transformations. The list of superintegrable PDM systems given in the present paper includes two new families of such systems including pairs of arbitrary parameters, see the last items in both tables 1 and 2.

In the past the shape invariance of particular superintegrable PDM systems was sporadically used to construct their exact solutions. We declare the existence of shape invariance for all superintegrable rotationally invariant systems admitting second order integrals of motion and fix the types of the corresponding superpotentials. Moreover, we specify the position dependent masses and potentials which correspond to Hamiltonians with two-fold shape invariance.

In the present paper we restrict ourselves to 3d PDM systems. However, our results admit a direct generalization to systems with arbitrary dimension $d > 3$.

It would be interesting to classify PDM Hamiltonians which lead to other shape invariant potentials, including potentials with spin classified in [13] and [14]. Some elements of such classification for systems with constant masses can be found in [35] and [36].

One more challenge is to extend the classification of the second order integrals of motion to the case of generic PDM systems which are not rotationally invariant. This work is in progress.

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Appendix A. Solutions for determining equations

A.1. Scalar integrals of motion

Integrals of motion (13) are scalars w.r.t. rotations iff $\mu^{ab}$ is reduced to tensor $\mu_1^{ab}$ given by equation (20). Substituting $\mu^{ab} = \mu_1^{ab}$ into equation (16) we obtain the following solution for $f$:

$$f = \varphi(x)$$

where $\varphi(x) = \varphi_1(x)$ is an arbitrary function of $x$.

The corresponding integral of motion (13) takes the following form:

$$Q = -H - k_1\mathbf{J}^2 + ...$$  \hspace{1cm} (A.1)

where the dots denote the term with the first and and zeroth order differentials commuting with $H$. Since the first and second terms in the r.h.s. evidently commute with $H$, the terms denoted by dots also should be (first order) integrals of motion. Such integrals of motion had been classified in [15] and presented above in section 2.
A.2. Vector integrals of motion

Vector integrals of motion are associated with tensors $\mu^{ab}$ presented in (36) and (37). Moreover, versions (36) and (37) corresponding to vector and pseudovector integrals of motion should be considered separately.

Let us start with tensor (36). Substituting it into (16) we come to the condition $f' = 2xf$ and so $f = ax^2$. Using this expression for $f$ and substituting (36) into (17) we obtain the following equations for unknowns $\xi^a$:

$$\xi^a = 3\xi^c x_c,$$

and so $\xi^a = \lambda^a x^2 - 2x^2\lambda^b x^b$, where $\lambda^a$ are arbitrary constants.

The remaining determining equations, i.e., (18) and (19), generate the following conditions: $V = C$ and $\lambda^a = \lambda^a$. In other words, we obtain known solutions (5) while the corresponding second order integrals of motion are reduced to anticommutators of the first order symmetries (4) and (9). This result is trivial.

Consider now tensor (37). Substituting it into equations (16) and (17) and equating coefficients for linearly independent terms $x^a \lambda^b x_c$, $x^2 \lambda^b x_b$, $(x^a \lambda^b + x^b \lambda^a)$ and $\delta^{ab} \lambda^c x_c$ we come to the following system of algebraic equations for differential variables $f$, $\varphi$ and $\xi^a$:

$$\varphi + 4\mu^2 f - \left(\nu + \mu^2\right)xf' = 0,$$
$$\varphi' - 4\mu f - \left(\varphi - \left(\nu + \mu^2\right)\right)f' = 0,$$
$$2\varphi' x - \varphi' + 2\left(\varphi - 1 - 3\mu^2\right)f' + 2\left(1 - \varphi + \mu^2\right)x f'^2 = 0,$$
$$2(2\mu x + \varphi' x f' + 2\mu x^2 f' - \left(1 + \mu^2\right)xf'^3 = 0. \quad (A.2)$$

The compatibility condition for system (A.2) is given by equation (41). Substituting the expressions (41) for $\varphi$ into one of equations (A.2) and going over all inequivalent versions of parameters $\mu$ and $\nu$ presented in (38) we obtain the following admissible pairs of functions $f$ and $\varphi$:

$$f = x, \quad \varphi = -\frac{1}{4x^2} + \alpha, \quad \mu = 0,$$
$$f = x^3, \quad \varphi = 1, \quad \mu = 0,$$
$$f = x^4, \quad \varphi = \mu = 0,$$
$$f = \frac{x}{x^2 \pm 1}, \quad \varphi = \frac{1}{x^2 \pm 1}, \quad \mu = 0,$$
$$f = \left(1 + \mu^2\right)^2, \quad \varphi = 0, \quad \mu = \pm 1,$$
$$f = \frac{(x^2 - 1)^2 x}{x^2 - 2\alpha x + \mu x^2}, \quad \varphi = \frac{(1 + \mu^2)^2}{\nu - 2\alpha x - \mu x^2}, \quad \mu = \pm 1. \quad (A.3)$$

Thus we specify all functions $f$ for Hamiltonians (2) admitting vector integrals of motion. The generic form of these integrals of motion is given by equations (13) and (37) where
\[ \eta = \lambda^2 x_4 \phi(x) \]  

(A.4)

and \( \phi(x) \) is a yet unknown function.

Substituting (13) and (37) into the remaining determining equation (18) and equating linearly independent terms proportional to \( \lambda^2 \) and \( x_4^2 x_4 \), we come to the following system:

\[
f(2x\phi + 4\phi^r + x\phi_{rr} - 20ux) + f'\left(x\phi^r - \phi - 10ux^2\right) + 2\left(f^r x - x^2 V\right)\left(\mu x^2 - 1\right) = 0,
\]

\[
f\left(2\phi^r x^2 + \phi_{rr} x^2 + 4\phi^r x - 4\phi^r\right) + f'\left(\phi_{rr} x^2 + 14\mu x^2\right) - \left(f^r x - f^r\right)\left(\phi^r x + 6\phi - 4 + 4\mu x^2\right) - \left(f^r x^2 - 3f^r x + 3f^r\right)\phi + 2\left(\phi - 1 + \mu x^2\right)x^2 V' = 0.
\]

(A.5)

Solving this system with all \( f \) and \( \phi \) enumerated in (A.3) we obtain the corresponding potentials \( V \) and functions \( \phi \). Then, making transformation (35) we come to the results presented in table 1.

A.3. Pseudotensor integrals of motion

Consider now determining equations (16)–(18) with tensor \( \mu^{ab} \) given by formula (43). They can be evaluated in complete analogue with the procedure outlined in the previous section, thus we will present the corresponding intervening results without comments.

Equations (16) and (17) result in the following system:

\[
(q' + 8ux)f - \left(2ux^2 + \phi\right)f^r = 0,
\]

(A.6)

\[
\left(2\phi x - 4ux^3\right)f + \left(\mu x^4 - 1\right)f^r = 0,
\]

(A.7)

\[
\left(48ux + \phi_{rr} x + 8\phi^r\right)f - \left(10ux^2 + 5\phi\right)f^r - \left(1 + \mu x^4 + x^2 \phi\right)\left(\frac{f^r}{x}\right)^r = 0,
\]

(A.8)

\[
\left(2\mu x + \phi^r\right)f - 4\mu x^2 f^r - \left(1 - \mu x^4\right)\left(\frac{f^r}{x}\right)^r = 0,
\]

(A.9)

\[
\left(\phi_{rr} x + \phi^r\right)f + 4\mu x^2 f^r - x^2 \left(\phi + 2\mu x^2\right)\left(\frac{f^r}{x}\right)^r = 0.
\]

(A.10)

Equation (A.8) is a linear combination of equations (A.6), (A.9) and (A.10) while (A.9) and (A.10) are differential consequences of (A.7) and (A.6) correspondingly. In other words, we can restrict ourselves to the subsystem (A.6) and (A.7), whose solutions are enumerated in the following formulæ:

\[
\phi = \frac{2\left(\kappa x^4 - 2x^2 + \kappa\right)}{x^4 - 2\kappa x^2 + \mu}, \quad f = C_1 \frac{\left(x^4 - \mu\right)^2}{x^4 - 2\kappa x^2 + \mu}, \quad \mu = \pm 1
\]

\[
\phi = -\frac{\left(x^2 - \mu\right)^2}{x^2}, \quad f = C_2 \frac{\left(x^4 - \mu\right)^2}{x^2}, \quad \mu = \pm 1
\]

\[
\phi = -\frac{1}{x^2 + \kappa}, \quad f = \frac{C_3}{x^2 + \kappa}, \quad \mu = 0.
\]

(A.11)

Here \( C_1, C_2, C_3, \kappa \) and \( k \) are integration constants and conditions (38)–(40) are used.
Thus we fix all possible mass terms for Hamiltonians admitting pseudotensor integrals of motion. The latter ones are given by equations (13) and (43) with

$$\eta = \lambda^{ab}x_ax_b\phi(x). \tag{A.12}$$

Substituting (43), (A.11) and (A.12) into the last remaining determining equation (18) and equating linearly independent terms proportional to $$\lambda^{ab}x_b$$ and $$\lambda^{ac}x_cx_d$$ we obtain the following system of equations for potentials $$V$$ and unknown function $$\phi$$:

$$
\begin{align*}
(4\phi x - 20\mu x + 12\phi' + 2\phi'' \phi) + f(2\phi' - 20\mu x^2) & = 0, \\
+ 2(\mu x^4 - 1) \left( \left( \frac{f}{x} \right) - V' \right) & = 0, \\
f(2\phi' x + 6\left( x^3\phi'' + \phi'' - x\phi' \right)) + (4\mu x + \phi' + \phi') & = 0, \\
(14\mu x^2 + 7\phi + x\phi') + 2(2\phi + 2\mu x^4) & = 0. \\
- \frac{1}{x^3}(1 + \mu x^4 + x^2\phi) & = 0. \tag{A.13}
\end{align*}
$$

Solving this system with all $$f$$ and $$\phi$$ presented in (A.3) we find the corresponding potentials $$V$$ and functions $$\phi$$. Then, making transformation (35) and rescaling independent variables to simplify expressions for $$f$$ and $$V$$ we come to the results presented in table 2.

### A.4. Tensor integrals of motion

The last version of integrals of motion we should consider corresponds to tensor $$\mu^{ab}$$ of type (25), i.e.,

$$
\mu^{ab} = \mu \left( e^{ac} \lambda^{bc}x^c + e^{bcd} \lambda^{ac}x^a \right) + \nu \left( 2 \left( x^a e^{bc} + x^b e^{ac} \right) \lambda^{da} - \left( e^{ack} \lambda^{bk} + e^{bck} \lambda^{ak} \right) x^c x^a \right). \tag{A.14}
$$

Substituting (A.14) into (16) we come to the following equation:

$$4\lambda x_f = (\mu + \nu x^2) f, \quad \text{or} \quad f = C \left( \mu + \nu x^2 \right)^2. \tag{A.15}
$$

Using (A.14), (A.15) and equation (19) we immediately find that $$V = -6\mu x^2$$ In other words, we recover Hamiltonians (6) and (8) while the related symmetry operators (13) are products of the first order integrals of motion (10) and (11) correspondingly.

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