Magneto hydrodynamic and dissipated nanofluid flow over an unsteady turning disk

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Abstract
A modern development in the field of fluid dynamics emphasizes on nanofluids which maintain remarkable thermal conductivity properties and intensify the transport features of heat in fluids. The present communication provides an innovative idea of MHD (magneto-hydrodynamics) unsteady, incompressible nanoliquid flow due to the stretching rotating disk with the effect of Joule heating and dissipation. The engine oil is used as a carrier fluid for an immersed rotating disk. A special type of nanoliquid, which consists of cylindrical shape nano materials CNTs (Carbon Nanotubes), is being taken into an account. The CNTs are assembled of both single and multi-walled carbon nanotubes. Some other factors such as the effect of joule heating and viscous dispassion are also used in this investigation. The foremost set of PDEs (Partial Differential Equations) of our model is reformed to the dimensionless form via invoking suitable variables. The resultant set of equations is sketched out through RK-4 method. Furthermore, the velocity profile and energy distribution versus dimensionless flow factors have been sketched and discussed. Also, the outcomes of important engineering curiosity like Nusselt number and skin friction are depicted and interpreted by taking various model factors. Radial and transverse velocity fields, declines via $M$ (magnetic factor), while the temperature field enhances. Furthermore, the larger estimation of $f$ and $v$ leads to enhance the velocity field while higher estimation of $S$ reducing the velocity and temperature fields. The current work has an extensive verity use such as nano-mechanics and electro-magnetic micro pumps.

Keywords
Unsteady rotating disk, CNTs nanofluid, engine oil, magnetic field and viscous dissipation, RK-4 method

Date received: 19 February 2021; accepted: 28 June 2021

Handling Editor: James Baldwin

Introduction
In recent years, researchers have drawn considerable attention to improving heat transfer in engineering and industrial uses. This is because, for instance, the performance of most equipment’s in this field depends significantly on the heat transport rate of electronic devices and heat exchangers. Heat transfer fluids like oil, ethylene glycol, and water limits the heat transfer rate since the thermal conductivity of these fluids are very low. As we know that the thermal performance of fluid may improve in a case of using solid particles. The merger of some solid (metallic, nonmetallic, carbon nanotubes,
carbon nanorods) nanoparticles to upsurge thermal conductivity is considering a new and novel procedure. This merged fluid is called as nanofluid and the small solid or metal particles are called nanoparticles. Nanofluid can be the best possible exploited to be used in heat transmission accounting, industrial cooling applications, bio-medicine and paper production, chemical reactor even in nuclear sectors, fuel cells, domestic refrigerators, automobile thermal management, and many other fields.\textsuperscript{1} It was Choi and Eastman,\textsuperscript{2} who for the first time presented the idea of nanofluid, being used for the enhancement of thermal conductivities then other common liquids. The analytical results for transfer of heat through MHD flow in nanofluid was addressed by Turkyilmazoglu.\textsuperscript{3} Khan et al.\textsuperscript{4} reported the expression of nanofluid flow through swinging sheet. In this direction, several scientists\textsuperscript{5–8} have studied nanofluid flow from several aspects. But in the last few years, the heat transport in carbon-nanofluids has grown an extensive attention among the researchers in different sectors of technologies. Carbon Nanotubes are the simple chemical structure along with the composition of carbon atoms, rolled in cylindrical form. Carbon Nanotubes have superior prominent thermophysical, chemical, and mechanical features, can be utilized easily as a nanoparticle in base fluid. They have unique advantages such as large surface area, tube shape, configuration, chemical stability, hardness, and their smallest dimension over other nanoparticles. Below is the list of analysis of Carbon Nanotubes nanofluid. Haq et al.\textsuperscript{9} examined the numerical outcomes for conducting fluid and heat exchange due to carbon nanotubes merged in various based fluids over an extending sheet using numerical scheme. The influence of carbon nanotube water based nanoliquid (micropolar) on heat transport and MHD flow between two rotating disks using HAM technique was employed by Rehman et al.\textsuperscript{10} Rehman et al.\textsuperscript{10} reported the influence of magnetic field on non-Newtonian fluid containing CNTs. Mahanthesh et al.\textsuperscript{11} studied the nanofluid flow under the impact of heat source, cross diffusion, and exponential space. The entropy generation between two rotating disks of CNTs nanofluid flow in the presence of thermal radiation and MHD was examined by Hosseinzadeh et al.\textsuperscript{12} Chamka and Djezzar,\textsuperscript{13} Reddy et al.,\textsuperscript{14} and Almeshaal et al.\textsuperscript{15} have studied the CNTs Water nanofluid using various mathematical models for the enhancement of heat transfer analysis. Several other valuable scholars’ articles have been available in literature that can be mentioned to references\textsuperscript{16–18} for additional study. The effect of the moving objects in a rotating system produces a force acting vertical to the direction of motion and the axis of rotation called the Coriolis force. The Coriolis force has a vital role in the fluid motion over a rotating disk. Gaspard-Gustave de Coriolis is the pioneer to investigate the consequences of the Coriolis force on the water wheels. The idea of rotation and its significance are mainly used in the automobile industry, rotating machinery, petroleum industry and medical equipment, and so on. Persson\textsuperscript{19} presented the appropriate equation for the Coriolis acceleration which is described by Leonhard Euler in 1749. Durrán\textsuperscript{20} reported that the existence of inertial oscillation is not allied to the inertial forces but actually the Geo-potential surfaces are parallel to gravity force. To confiscate the effect of Coriolis force Lan et al.\textsuperscript{21} used triplet glass. Von Karman\textsuperscript{22} presented the appropriate similarity transformation for magnetic flow over rotating frame. Von Karman transformation is very useful to diminish differential equation into the dimensionless differential equation. The exact solution of laminar flow through revolving stretchable disk is computed by Fang.\textsuperscript{23} Turkyilmazoglu\textsuperscript{24} studied the expression of heat transport and MHD flow over a radially rotating disk. Hatami et al.\textsuperscript{25} discussed the incompressible laminar flow of viscous nanoliquid owing to spinning and expanding disks. Mustafa et al.\textsuperscript{26} described the fluid flow throughout the existence of nanoparticles by stretching the disk. They found that standardized disk stretching has a supreme role in reducing the thickness of the boundary layer. Due to an inclined spinning disk, Sheikholeslami et al.\textsuperscript{27} introduced the nanofluid flow. Magneto-hydrodynamic nanofluid flow has currently been described by Hayat et al.\textsuperscript{28} owing to a spinning disk with slip influences. The latest studies heat transport phenomena by virtue of stretchable rotating disk have been augmented more rapidly in Mahanthes et al.,\textsuperscript{29} Gul et al.,\textsuperscript{30} and Acharya et al.\textsuperscript{31} For the last few decades, the theory of MHD is extremely appreciated for the various engineering and scientific purposes. It is actually the combination of fluid velocity with magnetic field. Such well-organized fact was first applied for different problems related to geophysics and astrophysics. In recent times, the MHD flow and heat exchange have achieved vital roles in agronomic engineering, industry of petroleum, and medical field. In this context, Davidson,\textsuperscript{32} Steg and Sutton\textsuperscript{33} presented several applications appearing in the various fields of medical and engineering as well due to which MHD has grown the consideration of scientist. Hayat et al.,\textsuperscript{34} Pal and Mondal\textsuperscript{35} examined the impact of MHD and heat (source/sink) on 3D water based nanofluid flow. Interesting results of nanofluid flow through stretching surface in the presence of MHD and thermal radiation were communicated by Ramzan et al.\textsuperscript{36} Furthermore, additional interesting and recent studies on MHD and RK-4 method can be seen in Arqub and Rashaidel,\textsuperscript{37} Arqub,\textsuperscript{38} and Momani et al.\textsuperscript{39} Having inspired from the above mention perceptions, in this work we have represented the consequence of MHD on unsteady flow of nanofluids over a
rotating frame. An advanced type of nanoliquid, which consists of cylindrical shape nano materials CNTs is being taken into an account. The CNTs are assembled of both single and multi-walled carbon nanotubes. Some other factors such as the effect of joule heating and viscous dispassion are also used in energy expression. In this study, our core purpose was to expose how the different flow features (velocity, temperature, and heat transport) are affected owing the inserting factors for CNTs nanofluid. Furthermore, the influences of different parameters in the nanofluid flow regimes presented statistically and graphically.

The novelty of the present work is pointed out by making extension of the published work of Turkyilmazoglu\(^24\) as:

1. The recent work is based on the unsteadiness while the published work\(^24\) is the steady case.
2. The magnetic field is added in the present work.
3. Turkyilmazoglu\(^24\) is extended with the viscous dissipation.
4. In the present study the CNTs (Carbon Nanotubes)-Engine Oil nanofluid has been focused for the heat transfer enhancement applications.
5. Finally, to the best of authors, information, not any investigation as deliberated in this study have far been communicated and analysis on engine oil based CNTs nanofluids problems is very rare.
6. The main objective of the present work is to fill the gap in the literature, so we apply the numerical scheme. The transformed system of equations is tackled with the RK-4 method.\(^{40,41}\)

### Mathematical formulation

#### Statement of the problem

Let us presume the unsteady flow of Single Walled Carbon Nanotubes (SWCNTs-engine oil) and Multi Walled Carbon Nanotubes (SWCNTs-engine oil) nanofluid over an axisymmetric disk as shown in geometry. To formulate our work, we have considered some assumptions as

1. We have assumed that the disk rotates about its own axis through an angular velocity \(\Omega\), where \(\Omega\) is the disk rotating rate and \(\beta\) is constant.
2. Here \(c\) is the stretching rate and \(u = \frac{\varepsilon}{1 - \beta t}\) is the stretching velocity.

iii. The \((B(t) = B_0/\sqrt{(1 - \beta t)})\) is employed perpendicular to the disk as in Khan and Nadeem.\(^{40}\)
iv. The pressure term is assumed to constant, so that its gradient vanishes.

v. Free stream and the disk surface temperature are specified through \((T_s)\) and \((T_w)\).
vi. We consider the Joule heating and dissipation in energy expression.

vii. No chemical reaction and no slip occur among nanoparticles and base fluid.

Under these assumptions, the law of conservation of mass, momentum, and energy takes the following form\(^{24,42–44}\):

\[
\begin{align*}
\frac{\partial u}{\partial r} + \frac{u}{r} \frac{\partial w}{\partial z} + \frac{\partial w}{\partial z} & = 0, \\
n_{nf} \left[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2} \right] & = \frac{\partial p}{\partial r} + \mu_{nf} \left[ \frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial z^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2} \right] - \sigma_{nf} B^2(t) u, \\
n_{nf} \left[ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial r} + w \frac{\partial v}{\partial z} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{v}{r^2} \right] & = \frac{\partial p}{\partial r} + \mu_{nf} \left[ \frac{\partial^2 v}{\partial r^2} + \frac{\partial^2 v}{\partial z^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{v}{r^2} \right] - \sigma_{nf} B^2(t) v, \\
n_{nf} \left[ \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{w}{r^2} \right] & = \frac{\partial p}{\partial z} + \mu_{nf} \left[ \frac{\partial^2 w}{\partial r^2} + \frac{\partial^2 w}{\partial z^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{w}{r^2} \right], \\
(pcp)_{nf} \left[ \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial r} + w \frac{\partial T}{\partial z} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{T}{r^2} \right] & = k_{nf} \frac{\partial^2 T}{\partial z^2} + \sigma_{nf} B^2(t) (u^2 + v^2) + \mu_{nf} \left[ \frac{\partial u}{\partial z} \right]^2 + \left[ \frac{\partial v}{\partial z} \right]^2,
\end{align*}
\]

Here \((p, T)\) are the pressure and temperature of fluid. Furthermore, the density, dynamic viscosity, specific heat, thermal conductivity, and electrical conductivity of nanofluid are \(\rho_{nf}, \mu_{nf}, (Cp)_{nf}, k_{nf}, \sigma_{nf}\) respectively.

#### Thermo-physical properties

The several effective models of (Carbon nanotubes) CNTs nanofluid are described in the Table 1. Also, different thermo-physical information of both (SWCNTs and MWCNTs) small nano-size particles are listed in Table 2. Where ‘\(\phi\)’ describes the volume fraction of...
The temperature at the surface of the disk and the angular velocity approach to diminish our model in non-dimensional transformations. The unsteady initial and boundary conditions are:

\[ u = \frac{cr}{1-\beta t}, \quad v = \frac{\Omega r}{1-\beta t}, \quad w = 0, \quad T = T_w, \quad z \to 0 \]

\[ u \to 0, \quad v \to 0, \quad T = T_z, \quad z \to \infty. \]

The \( c \) is denoted the stretching constant. Where \( T_w \) is the temperature at the surface of the disk and the angular velocity is denoted by \( \Omega \).

### Transformations

Where, we apply the Karman appropriate scaling approach to diminish our model in non-dimensional form

\[ u = \frac{cr}{1-\beta t} F(\eta), \quad v = \frac{\Omega r}{1-\beta t} G(\eta), \]

\[ w = \sqrt{\frac{\Omega h}{1-\beta t}} H(\eta), \quad \eta = z \sqrt{\frac{\Omega}{v_j(1-\beta t)}}, \]

\[ \Theta = \frac{T - T_\infty}{T_w - T_\infty}, \quad p = \frac{v_j(r C_F)}{(1-\beta t) \delta \Omega^2} p^* \]

Now, with the help of these transformations as in equation (7), equations (1)–(5) are altered as:

\[ F'' - (1-\phi)^{2.5} \left[ (1-\phi) + \phi \frac{\rho_{CNT}}{\rho_f} \right] \]

\[ \left[ F^2 - G^2 + HF'' + S \left( F + \frac{\eta}{2} F' \right) \right] - (1-\phi)^{2.5} M F = 0, \]

\[ G'' - (1-\phi)^{2.5} \left[ (1-\phi) + \phi \frac{\rho_{CNT}}{\rho_f} \right] \]

\[ \left[ S \left( G + \frac{\eta}{2} G' \right) + 2FG + HG' \right] = 0, \]

\[ H'' - (1-\phi)^{2.5} \left[ (1-\phi) + \phi \frac{\rho_{CNT}}{\rho_f} \right] \]

\[ \left[ \frac{S}{2} (H + \eta H') + HH' \right] - (\lambda + MH)(1-\phi)^{2.5} = 0, \]

\[ \frac{k_{nf}}{k_f} \Theta'' - Pr \left[ (1-\phi) + \phi \frac{(pc\rho)_{CNT}}{(pc\rho)_f} \right] \]

\[ \left[ \frac{S}{2} \eta \Theta' + H \Theta' \right] + Ec \left( (F^2) + (G')^2 \right) \]

\[ + M Ec (F^2 + G^2) = 0. \]

The physical conditions after transformation are:

\[ F(0) = 1, \quad H(0) = 0, \quad G(0) = 0, \quad \Theta(0) = 1, \]

\[ F(\infty) = 0, \quad G(\infty) = 0, \quad \Theta(\infty) = 0. \]

Where, the dimensionless factors of our investigations are:

\[ \omega = \frac{\Theta}{c}, \quad M = \frac{\alpha_{\theta} B_0^2}{\Omega \rho_f}, \quad S = \frac{B}{\Omega}, \]

\[ Ec = \frac{\mu_j (r \Omega)^2}{k_j(T_w - T_\infty)}, \quad Pr = \frac{v_j (p C_F)}{k_f}, \quad \lambda = \frac{1}{\Omega^2 \delta \rho_f} \]

### Solution methodology

#### Solution by RK-4 method

In the RK4-method the higher order derivatives are reduced in the first order system of the differential equations. The variables are designated as:

\[ x_1 = \eta, x_2 = F, x_3 = F', x_4 = G, x_5 = G', x_6 = H, \]

\[ x_7 = H', x_8 = \Theta, x_9 = \Theta'. \]
The equations (8)–(11) are transformed into the first-order system and using the RK4-technique.

We converted the equations (7)–(11) into the system of first-order equations as:

\[
\begin{pmatrix}
x_1' \\
x_2' \\
x_3' \\
x_4' \\
x_5' \\
x_6' \\
x_7' \\
x_8' \\
x_9'
\end{pmatrix} = \begin{pmatrix}
1 \\
(1 - \phi)^{2.8} \left(1 - \phi + \phi \frac{Pr_{nf}}{Pr}\right) \left(x_1^2 - x_1^4 + x_2x_3 + S(x_2 + \frac{1}{2}x_1)\right) + (1 - \phi)^{2.8}M_{x_2} \\
(1 - \phi)^{2.8} \left(1 - \phi + \phi \frac{Pr_{nf}}{Pr}\right) \left(x_6x_5 + 2x_2x_4 + S(x_4 + \frac{1}{2}x_3)\right) \\
(1 - \phi)^{2.8} \left(1 - \phi + \phi \frac{Pr_{nf}}{Pr}\right) \left(x_6x_5 + 2x_2x_4 + S(x_4 + \frac{1}{2}x_3)\right) \\
(1 - \phi)^{2.8} \left(1 - \phi + \phi \frac{Pr_{nf}}{Pr}\right) \left(x_6x_5 + 2x_2x_4 + S(x_4 + \frac{1}{2}x_3)\right) + (1 - \phi)^{2.8} \left(\lambda + M_{x_6}\right) \\
Pr \left(1 - \phi + \phi \frac{Pr_{nf}}{Pr}\right) \left(x_1x_9 + x_9x_6\right) - \frac{D_c}{\left(k_{nf}/k_f\right)} \left[(x_1')^2 + (x_9')^2\right] + M \left(x_2^2 + x_3^2\right)
\end{pmatrix} \quad , \quad \begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
x_7 \\
x_8 \\
x_9
\end{pmatrix} = \begin{pmatrix}
0 \\
1 \\
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
x_7 \\
x_8 \\
x_9
\end{pmatrix}.
\]

**Engineering curiosity**

The drag force and heat transfer rates are defined as:

\[
C_f = \sqrt{\frac{\tau_{wr}}{\rho_f (\Omega r)^2}}, \quad Nu = \frac{r q_w}{k_f (T_w - T_c)}.
\]

The radial stress is \(\tau_{wr}\), similarly the transverse shear stress is \(\tau_{wp}\) and \(q_w\) is used to show the heat flux.

\[
\tau_{wr} = \mu_{nf} \left(\frac{d u}{d z} + \frac{d w}{d \phi}\right)_{z=0},
\]

\[
\tau_{wp} = \mu_{nf} \left(\frac{d v}{d z} + \frac{1}{r} \frac{d w}{d \phi}\right)_{z=0} \cdot q_w = -\frac{k_{nf}}{k_f} \left(\frac{d T}{d z}\right)_{z=0}.
\]

Physical parameters of interest, including skin friction and heat transfer rates are reflected as:

\[
Re C_f = \frac{\sqrt{(F'(0))^2 + (G'(0))^2}}{(1 - \phi)^{2.8}},
\]

\[
Re^2 Nu = -\frac{k_{nf}}{k_f} \Theta(0), \quad Re = \frac{\Omega r^2}{v_f}.
\]

**Graphical results and discussions**

In this section, the physical trend of the engaged parameters on heat transport and nanofluid velocity has been investigated for both CNTs nanoparticles that is SWCNTs and MWCNTs. The outcomes of this analysis are exhibited comprehensively through tables and graphs for two types of CNTs recognized as single and multi-walled CNTs. Several models of CNTs nanofluids are described in Table 1. Also, thermo-physical information of both small nano-size particles are listed in Table 2. Various information of thermal conductivity values at different volume fraction are itemized in Table 3 for single and multi-walled CNTs. The present work has been compared with the existing literature Turkyilmazoglu\textsuperscript{24} in Table 4 by eliminating the extension terms. The numerical results for surface drag force \(\frac{1}{(1 - \phi)^2} f'(0)\) using different factors \(\phi, S, M\) different values of \(\phi, S, M\) subject to SWCNTs and MWCNTs are presented in Table 5. Here we can see that surface drag force is greater for higher estimation of \(\phi, S, M\) and subject to both nanoparticles (SWCNTS and MWCNTs). So, in Table 7 we perceived that \(\frac{1}{(1 - \phi)^2} G'(0)\) (surface drag force) is more for greater approximation of \(\phi, S, M\). Table 6 presents the computational data for \(\frac{1}{(1 - \phi)^2} G'(0)\) different values of \(\phi, S, M\) subject to both nanoparticles (SWCNTS and MWCNTs). It is scrutinized from table that \(-\left(\frac{k_d}{k_f}\right) \Theta(0)\) enhance for various estimations of \(\phi, Pr, Ec\). Figure 1 shows the geometry of the of the nanofluid flow problem.

Figures 2 and 3 exposes that the radial velocity component \((F)\) of nanofluid declines gradually as the value of \(M\) growing and the transverse component of
velocity \( (G) \) reduces respectively for single and multi-walled CNTs-engine oil nanofluids. The basic fact of such reducing situation is the opposing force, which is generated by virtue of Lorentz force. It is observed that

**Table 4.** Comparison of the present work with the existing literature when \( \phi = M = S = 0 \), \( Pr = 6,2 \), \( Ec = 0.2 \), \( \omega = 1 \).

| Turkyilmazoglu \( ^{24} \) | Present |
|--------------------------|---------|
| \( F'(0) \)            | 0.736   | 0.736210 |
| \(-G'(0)\)             | -1.819  | -1.819141 |
| \(-\Theta'(0)\)        | -2.375  | -2.375171 |

**Table 5.** Skin friction \( \frac{1}{(1-\phi)^{1.5}} F'(0) \) versus various values of embedded parameters.

| \( \phi \) | \( S \) | \( M \) | \( \frac{1}{(1-\phi)^{1.5}} F'(0) \) \_SWCNTs | \( \frac{1}{(1-\phi)^{1.5}} F'(0) \) \_MWCNTs |
|-----------|-----|-----|----------------|----------------|
| 0.01      | 0.2 | 0.4 | -0.931056      | -0.888095      |
| 0.02      | 0.2 | 0.4 | -1.124151      | -1.03779       |
| 0.03      | 0.2 | 0.4 | -1.381831      | -1.25175       |
| 0.01      | 0.2 | 0.3 | -0.931056      | -0.888095      |
| 0.01      | 0.4 | 0.4 | -0.996306      | -0.947680      |
| 0.01      | 0.5 | 0.4 | -0.962795      | -0.920066      |

**Table 6.** Skin friction \( \frac{1}{(1-\phi)^{1.5}} G'(0) \) versus various values of embedded parameters.

| \( \phi \) | \( S \) | \( M \) | \( \frac{1}{(1-\phi)^{1.5}} G'(0) \) \_SWCNT | \( \frac{1}{(1-\phi)^{1.5}} G'(0) \) \_MWCNT |
|-----------|-----|-----|-----------------|-----------------|
| 0.01      | 0.2 | 0.4 | -0.566315      | -0.542234      |
| 0.02      | 0.2 | 0.4 | -0.704297      | -0.656168      |
| 0.03      | 0.2 | 0.4 | -0.895518      | -0.823448      |
| 0.01      | 0.2 | 0.3 | -0.566315      | -0.542234      |
| 0.01      | 0.4 | 0.4 | -0.573214      | -0.548600      |
| 0.01      | 0.5 | 0.4 | -0.580080      | -0.554938      |

**Figure 1.** Geometry of the problem.

**Figure 2.** \( M \) versus \( F \). When \( Ec = 10 \), \( Pr = 7 \), \( \phi = 0.02 \), \( \omega = 4 \), \( S = 0.3 \).

**Figure 3.** \( M \) versus \( G \). When \( Ec = 10 \), \( Pr = 7 \), \( \phi = 0.02 \), \( \omega = 4 \), \( S = 0.3 \).

Nusselt number \( -\frac{h_{\infty}}{k} \) \( \Theta'(0) \) versus various values of embedded parameters.

| \( \phi \) | \( Pr \) | \( Ec \) | \( -\frac{h_{\infty}}{k} \) \( \Theta'(0) \) \_SWCNTs | \( -\frac{h_{\infty}}{k} \) \( \Theta'(0) \) \_MWCNT |
|-----------|-----|-----|----------------|----------------|
| 0.01      | 0.7 | 0.3 | 0.223284       | 0.222529       |
| 0.02      | 0.7 | 0.3 | 0.417115       | 0.415745       |
| 0.03      | 0.7 | 0.3 | 1.17605        | 1.17354        |
| 0.01      | 0.7 | 0.8 | 0.223284       | 0.222529       |
| 0.01      | 0.7 | 0.9 | 0.228756       | 0.227885       |
| 0.01      | 0.7 | 0.5 | 0.223284       | 0.222529       |
| 0.02      | 0.7 | 0.5 | 0.218123       | 0.217392       |
| 0.02      | 0.7 | 0.5 | 0.212961       | 0.212255       |
radial and transverse component of velocity decline more rapidly of multi walled CNTs as compared to multi walled CNTs. The influence of \( f \) on the radial velocity component \( (F) \) as well as on transverse velocity component \( (G) \) is elucidated in Figures 4 and 5. Here we observed that \( f \) enhances the radial and transverse component of the velocity field \((F, G)\) for both types of engine oils of nanofluids. In fact, the energy, transportation, and cohesive force between the liquid molecules become greater by suspension of these nanoparticles (SWCNTs, MWCNTs) in nanofluid. The variation in the radial \((F)\) and transverse \((G)\) components of velocity fields relative to change in rotation parameter are revealed in Figures 6 and 7. The rotation parameter is considered to be \((\omega = 3, 4, 5, 6)\), which indicates that the rotation is leading. From the figures, we can observe the radial as well as the transverse component of velocities are increased by intensifying the rotation parameter for both nanoparticles. Figures 8 and 9 are portrayed to illustrate the variation of \( S \) (parameter of unsteadiness), on radial and transverse velocity profiles. The effect of \( S \) dominates on the nanofluid flow, and therefore a decreasing trend of radial velocity profile is observed. While the transverse component of velocity profile decay for snowballing value of unsteadiness parameter \( S \) for both nanoparticles. Physically, it is clearly seen that the radial and transverse velocity profile drop with a rise in the unsteadiness factor \( S \) due to the momentum boundary layer thickness is reduced with an increase in the magnitude of \( S \).

The influence of \( M \) (magnetic factor) on temperature \( \Theta(\eta) \) is presented in Figure 10. We see \( \Theta(\eta) \) is swelled through stronger \( M \). In such circumstances more Lorentz force is originated, which delivers larger resistance to the motion of fluid particles. Therefore, extra heat is produced inside the system and thus temperature \( \Theta(\eta) \) is augmented. Figure 11 is dedicated to examining the salient features of \( Ec \) (Eckert number)
on temperature $\Theta(\eta)$ for both nanoparticles. Here it is examined that the fluid temperature significantly boosts versus greater ($Ec = 1, 2, 3, 4$) Eckert number for CNTs. Physically, in situations of greater $Ec$ (Eckert number), the mechanical energy of nanoparticles transformed in the form of thermal energy due to internal fraction of molecules (atoms). A reverse alteration in the temperature field can be observed for the ($Ec = C_0^{-1}, C_0^{-2}, C_0^{-3}, C_0^{-4}$) Eckert number for both nanoparticles. The variation in thermal field $\Theta(\eta)$ relative to change in Prandtl number $Pr$ is exposed in Figure 12. Here the $\Theta(\eta)$ decays for larger magnitude of Prandtl number $Pr$ of both types of nanoparticles. Physically, both the Prandtl number $Pr$ and thermal diffusivity are inversely related to each other. Hence the fluid with a higher Prandtl number diffuses gradually as compared to the fluid of inferior Prandtl number $Pr$. Thus, this difference results in decline of thermal field $\theta(\eta)$. The effect of $S$ on the temperature profile $\theta(\eta)$ has been scrutinized in Figure 13. The larger amount of the unsteadiness decays the thermal profile near the wall surface and enhance the thermal boundary layer in the rest of the domain.
Conclusion

A reformed unsteady flow model for engine oil composed by CNTs (carbon nanotubes) over a stretchable rotating disk with the effects of MHD, Joule heating and dissipation in energy expression is explored. The comparison of the SWCNTs and MWCNTs nanofluid under the impact of the physical constraints has mainly focused. The published work is extended using the concepts of the unsteadiness, CNTs, Engine oil, and viscous dissipation. The main impacts of the nanofluid flow factors on radial, transverse velocity, and temperature field are pointed out as:

- Radial and transverse velocity fields, drops via $M$ (magnetic factor), while the temperature field enhances.
- The larger estimation of $f$ and $v$ leads to enhance the velocity field while higher estimation of $S$ reducing the velocity and temperature fields.
- The greater $Ec$ (Eckert number) transform the mechanical energy of the nanoparticles into the thermal energy and consequently the temperature field has been observed for the decreasing value of $Ec$.
- The recent study concluded that the impact of the physical constraints is comparatively strong using the SWCNTs. In fact, this is happening due to the strong thermos physical properties of the SWCNTs.

Acknowledgements

The authors are like to acknowledge with gratitude, the financial, and moral support provided by the Directorate of Science and Technology (DoST), Government of Khyber Pakhtunkhwa as this would have not been possible without them. T.G. has collected the data and modeled the problem. MU, SN, AS, IK, AK, and MI have participated in the physical discussion, and also contributed in the numerical computations and plotting the graphical results.

Declaration of conflicting interests

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

Funding

The author(s) received no financial support for the research, authorship, and/or publication of this article.

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Data availability statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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**Appendix**

**Notations**

| Symbol | Description |
|--------|-------------|
| \( u \), \( v \) and \( w \) | velocities components (\( ms^{-1} \)) |
| \( B_0 \) | magnetic field strength (\( Nm^{-1} \)) |
| \( F, H, G \) | dimensional velocity profile |
| \( T_w \) | disk surface temperature (\( K \)) |
| \( T_\infty \) | free surface temperature (\( K \)) |
| \( \Omega \) | disk rotating rate |
| \( M \) | magnetic parameter |
| \( r, \varphi, z \) | cylindrical coordinates |
| \( \text{Pr} \) | Prandtl number |
| \( Re \) | Reynolds number |
| \( c \) | stretching rate |
| \( p \) | pressure of fluid |
| \( (C_p)_f \) | specific heat of base fluid (\( J/kgK \)) |
| \( k_{nf} \) | thermal conductivity (\( Wm^{-1}K^{-1} \)) |
| \( S \) | unsteadiness parameter |
| \( Ec \) | Eckert number |
| \( C_f \) | skin friction |
| \( Nu \) | heat transfer rate |

**Greek symbols**

| Symbol | Description |
|--------|-------------|
| \( \mu_{nf} \) | dynamic viscosity of nanofluid (\( mPa \)) |
| \( \mu_f \) | dynamic viscosity of base fluid (\( mPa \)) |
| \( \rho_{nf} \) | nanofluid density (\( Kg m^{-3} \)) |
| \( \rho_f \) | base fluid density (\( Kg m^{-3} \)) |
| \( \eta \) | similarity variable |
| \( \phi \) | nanoparticle volume fraction |
| \( \beta \) | constant |
| \( \sigma_{nf} \) | electrical conductivity |
| \( \omega \) | rotating parameter |
| \( \lambda \) | pressure gradient |

**Subscripts**

| Symbol | Description |
|--------|-------------|
| \( nf \) | nanofluid |
| \( f \) | base fluid |
| CNT | carbon nanotubes |

**Abbreviation**

| Abbreviation | Description |
|--------------|-------------|
| RK | Runge Kutta Method |
| MHD | magneto-hydrodynamics |
| CNT | carbon nanotubes |
| SWCNT | single wall carbon nanotubes |
| MWCNT | multi wall carbon nanotubes |