A quark confinement potential is derived from QCD Lagrangian under the two assumptions in the energy region relevant to quark confinement. The possibility of perturbation in QCD is assumed, and the momentum of the gluon transferred by the quark is larger than the mass of the internal quark, so that the expansion of $m^2/q^2$ is assumed to be reliable. This potential comes from the one loop diagrams in the expansion of $m^2/q^2$, especially here, the vacuum polarization diagram is examined. This explains the quark confinement mechanism on the basis of QCD and the Regge trajectories which is governed by a linear potential.

PACS numbers: 12.39.Pn, 12.38.Aw, 12.38.Bx

Quarks are known as the constituents of the nucleon and the meson, and their interaction is believed to be explained by quantum chromodynamics (QCD). However there is no direct experimental evidence of detecting an isolated single quark. So it is said that quarks are always confined in the hadron. QCD tells us that quarks behave as weakly interacting particles in high energy states, which is referred to as “asymptotic freedom”. In this energy states, perturbative calculations are possible so that perturbative QCD has been actively researched. In order to motivate this letter, it is useful to cite “If indeed the quarks behave approximately like free particles and their masses are rather small, the critical question, then, is why don’t we see free quarks in the final state? This is the well-known problem of quark confinement.”. In low energy physics, however, perturbation is questionable because of the behavior of the larger coupling constant of QCD. So quark degree of freedom doesn’t appear manifest. It is possible to describe nuclear phenomena with the nucleon and the meson, i.e., meson theories instead of quarks. In order to describe low energy nuclear phenomena with quarks, several potential models are used such as bag models, harmonic oscillator potential, linear confinement potential model and so on. Another kind of model is the string model from which a linear confinement potential is derived naturally, but in which the concept of a particle should be changed into a string. All those potential models have their own quark confinement conditions.

If QCD is a true theory for the interaction of quarks and gluons, why isn’t the quark confinement phenomenon explained well with it? The hint to this question is the Lamb shift. With this hint, it is shown that QCD has a certain confinement potential which can be derived from QCD Lagrangian only. Consequently, this potential gives the reason why quarks are always confined in the hadron. Quark confinement is explained by the one loop effect of QCD, as if the Lamb shift is done by that of quantum electrodynamics (QED). The essential difference between QCD and QED in the two phenomena is the magnitude of the coupling constants rather than the difference between the Abelian and non-Abelian gauge theories. In QED, a hydrogen atom is the well-known bound state of a proton and an electron. The coupling constant of QED is so small compared to that of QCD that the size of the hydrogen atom should be much larger than that of the nucleon which is a bound state of quarks. The larger coupling constant will cause the larger momentum of the gluon which is transferred by the quark in order to localize quarks in the size of a nucleon.

In the calculation of the Lamb shift, the momentum of the photon transferred by the electron is so small that the one loop diagrams are calculated in the expansion of $q^2/m^2$, and the result is the famous splitting of the 2p and 2s states of the energy levels of a hydrogen atom. This splitting is generated by the following effective potential:

$$V(r) = -\frac{\alpha}{r} - \frac{4\alpha^2}{15}\delta(r)$$  \hfill (1)

in configuration space. In momentum space, this potential corresponds to the propagator of the photon which is modified by the one loop effect, i.e., the vacuum polarization:

$$D_{\mu\nu} = \frac{g_{\mu\nu} - q^\mu q^\nu}{q^2}(1 - \frac{e^2}{60\pi^2m^2}q^2).$$  \hfill (2)
However what is calculated in such an expansion in QCD may be inappropriate to explain the hadronic phenomena, because quarks are interacting so strongly with each other. The quarks confined in the hadron may not be in low energy-momentum states because of the large coupling constant which will produce a large momentum transfer. So the basic assumptions of this letter are that the quarks inside a hadron are so relativistic that perturbation is meaningful and the expansion of the inverse momentum of the gluon, that is, \( m^2/q^2 \) is reasonable rather than \( q^2/m^2 \).

The QCD Lagrangian is

\[
\mathcal{L} = \langle \bar{\psi}(\partial_\mu - igT^a A_\mu^a)\gamma^\mu \psi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu},
\]

\[
F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c,
\]

where the index \( a \) runs 1,2,...,8. The effective potential to one loop order can be calculated from the Feynman diagrams of Fig. 1. Those diagrams contribute to the following transition amplitude:

\[
\mathcal{M} = g^2 \bar{u}(p_2)\{\gamma_\mu F_1(q^2) + \frac{\kappa}{2m} F_2(q^2\sigma_{\mu\nu}g^a)\} T^a u(p_1) D^{\mu\nu}_{ab} J_\nu^b,
\]

where \( J_\nu^b \) means the quark current which consists of the bispinor of an antiparticle for the meson or the composite of the bispinors of two quarks and gluons for the nucleon. This amplitude is not a direct observable in the case of a bound state but an analogy of the amplitude of a scattering process. As the purpose of this letter is to show a qualitative confinement potential form QCD, the vacuum polarization diagram is considered only. It is calculated as

\[
\Pi^{ab}_{\mu\nu}(q^2) = \frac{ig^2}{16\pi^2} (q^2 g_{\mu\nu} - q_\mu q_\nu) \int \frac{d^4k}{(2\pi)^4} \frac{1}{q^2 - m^2} \frac{1}{k^2 - m^2} \int_0^1 dx (1-x) \log(q^2/(1-x)q^2 - m^2) + \cdots
\]

\[
= \frac{g^2 n_f \delta_{ab}}{4\pi^2} (q^2 g_{\mu\nu} - q_\mu q_\nu) \int_0^1 dx (1-x) \log(q^2/(1-x)q^2 - m^2) + \cdots
\]

\[
= \frac{g^2 n_f \delta_{ab}}{4\pi^2} (q^2 g_{\mu\nu} - q_\mu q_\nu) \int_0^1 dx (1-x) \log(q^2/(1-x)q^2 - m^2) + \cdots
\]

\[
= \frac{g^2 n_f \delta_{ab}}{4\pi^2} (q^2 g_{\mu\nu} - q_\mu q_\nu) \int_0^1 dx (1-x) \log(q^2/(1-x)q^2 - m^2) + \cdots
\]

If it is expanded in \(-4m^2/q^2\) which is called the expansion of \( m^2/q^2 \) throughout this letter, it is easy to calculate the effective potentials. This vacuum polarization modifies the gluon propagator as

\[
D^{\mu\nu}_{ab}(q^2) = \frac{\delta^{ab}}{q^2} + \frac{\delta^{ac}(g^{\mu\rho} - q_\rho q_\mu)}{q^2} \Pi_{\rho\sigma} \frac{\delta^{bd}(g^{\sigma\nu} - q_\sigma q_\nu)}{q^2} + \cdots
\]

\[
= \frac{\delta^{ab}}{q^2} + \frac{\delta^{ac}(g^{\mu\rho} - q_\rho q_\mu)}{q^2} \left[ 1 + \frac{g^2 n_f}{4\pi^2} \left\{ \frac{5}{18} - \frac{1}{6} \log(\frac{m^2}{q^2}) + \frac{m^2}{q^2} - \frac{1}{2} \right\} \right]
\]

\[
= \frac{\delta^{ab}}{q^2} + \frac{\delta^{ac}(g^{\mu\rho} - q_\rho q_\mu)}{q^2} \left[ 1 + \frac{g^2 n_f}{4\pi^2} \left\{ \frac{5}{18} - \frac{1}{6} \log(\frac{m^2}{q^2}) + \frac{m^2}{q^2} - \frac{1}{2} \right\} \right]
\]

where the coupling constant \( g \) is renormalized one from now on, since the infinite part of the vacuum polarization is canceled by the suitable counter term which comes from the redefinition of the physical parameters, i.e., wavefunctions, \( m, g \) and so on. In the center of mass frame of a quark and the remaining part, the four momentum of the gluon transferred by the quark is reduced to a three momentum, because the energy of the quark before the interaction with the gluon is exactly equal to that after the interaction. Now the Fourier transforms of each term are

\[
\int \frac{d^3q}{(2\pi)^3} \frac{1}{q^2} e^{iq.r} = \frac{1}{4\pi r},
\]

\[
\int \frac{d^3q}{(2\pi)^3} \frac{1}{q^2} \log(q^2) e^{iq.r} = -\frac{1}{2\pi r} \log(\gamma r),
\]
\[ \int \frac{d^3q}{(2\pi)^3} \frac{1}{q^2} e^{iq \cdot r} = \frac{1}{8\pi r}, \]
\[ \int \frac{d^3q}{(2\pi)^3} \frac{1}{q^6} e^{iq \cdot r} = \frac{1}{4!4\pi r^3}, \]
\[ \int \frac{d^3q}{(2\pi)^3} \frac{1}{q^2} \log(q^2) e^{iq \cdot r} = \frac{1}{4!2\pi} \left( \frac{25}{12} - \log(\gamma r) \right) r^3, \]

(8)

where the table of integrals of Ref. \[\Box\] is used and \(\gamma\) is the Euler constant in the second Fourier transform. The first one is the usual Coulomb potential as encountered in QED. The second one is known as the Uehling potential \[\Box\]. The third one is the linear potential which confines quarks in the hadron forever. The last two equations are only for reference, which give higher order confinement potentials. Alternatively, the linear confinement potential can be also estimated from the exact equation \[\Box\] by ignoring the last term which behaves approximately like \(\frac{1}{r^2} \log(q^2)\) at large momentum in the gluon propagator, so one can check that the integration interval in the Fourier transformations and higher power terms in Eq. \[\Box\] do not matter. From the Fourier transform of the propagator of the gluon, the effective potential by which a quark is influenced is calculated as

\[ V(r) = -\frac{\alpha_s}{r} + \frac{5c_F n_f \alpha_s^2}{18\pi} + \frac{c_F n_f \alpha_s^2}{3\pi} \log(\gamma mr) + \frac{c_F n_f}{2\pi} m^2 \alpha_s^2 r + \cdots, \]

(9)

where \(c_F\) is the color factor which is \(\frac{4}{3}\) for the meson and \(\frac{2}{3}\) for the baryon and \(\alpha_s = \frac{g^2}{4\pi}\). The non-Abelian nature of QCD gives only a constant, such as the color factor from the trace in Eq. \[\Box\]. The diagrams Fig. 1(e) and (f) which are quite different from QED do not give any confinement potential in the gluon propagator but give only Uehling potentials because there is no internal quark line. The complete effective potential may be needed to consider further through out the remaining diagrams of Fig. 1 and bremsstrahlung diagrams.

There is the string model that has such a linear potential \[\Box\]. This model explains a linear dependence of the angular momentum of the states of highest angular momentum, on the square of the mass, namely, \(J = \alpha' E^2 + \text{constant}\). This relation holds for the case of a constant energy density \(\rho\) of the string and \(\alpha'\) is the string coupling constant which is produced by the force mediated by pions, vector mesons and so on by coupling to nucleons. The coupling constant of \(g_{NN}^2/(4\pi)\) is about 14.3 in the literature \[\Box\]. The pion mass is the reason why there is no such a potential. The pion propagator corrected by the one loop effect in the expansion of \(m^2/q^2\) is

\[ D(q^2) = \frac{1}{q^2 - m^2} + \frac{1}{q^2 - m^2} \sum \left\{ \frac{1}{q^2 - m^2} + \cdots \right\}, \]

(10)

where \(q^2 = 0\) should not be expanded because of renormalizability. The Fourier transform of the term which has the coefficient \(C_3\) and is expected to give a confinement potential like Eq. \[\Box\] still has an exponential decay factor due to the mass of the pion as followings:

\[ \int \frac{d^3q}{(2\pi)^3} \frac{m^2}{q^2(q^2 + m^2)} e^{iq \cdot r} = \frac{1}{4\pi r} (1 - e^{-mr}). \]

(11)

The higher order terms of \(m^2/q^2\) also do not give a confinement potential, which can be proved by differentiating the above Fourier transform twice with respect to \(r\). Therefore the N-N potential has no confinement potential, though it has a large coupling constant.

However there remains still a question: how large coupling constant produces such a confining potential? Since such a potential can be also derived in QED, if the possibility of the expansion of \(m^2/q^2\) is forced to assume, it is necessary to justify the assumptions of this letter. The running coupling constant of QCD is derived in the perturbative region,
so in the non-perturbative region, it is useful to use that the velocity of the electron $v$ is equal to the coupling constant $\alpha$ in the Bohr model. Using the relativistic energy momentum relation $p_{H}^2 = M_{H}^2 = (p_e + p_p)^2$ in the center of mass frame, the identity is calculated as

$$v_{e}^{non} = \frac{|p_e|}{\mu_e} = \alpha = \frac{(m_e + m_p)\sqrt{[|M_{H}^2 - m_e^2 - m_p^2|^2 - 4m_e^2m_p^2]]}}{2M_Hm_em_p} \approx (137.02)^{-1},$$

where the subscripts $H, e$ and $p$ mean the hydrogen, electron and proton, respectively, and $\mu_e$ is the reduced mass of the electron. Solving this equation for the mass of the hydrogen, and it is expanded with respect to the coupling constant and $m_e/m_p$ as

$$M_H = m_p + m_e - \frac{1}{2} \alpha^2 m_e - \frac{1}{8} \alpha^4 m_e + \frac{3}{8} \alpha^4 m_e \frac{m_e}{m_p} - \frac{1}{16} \alpha^6 m_e + \cdots .$$

where the fine structure term ($\alpha^4$), which consists of the spin orbit coupling and relativistic corrections, can be derived correctly for the ground state. The hyperfine structure term appears less reliable, because the spin-spin interaction is not applied to the equation. The equation is also applied to the positronium as

$$M_{posi} = 2m_e - \frac{1}{4} \alpha^2 m_e - \frac{1}{64} \alpha^4 m_e + \cdots .$$

Calculating Eq. (12) for the deuteron, the result is 0.097 which is the order of the pseudovector coupling constant $f^2/(4\pi)$ for the usual definition $f/m_e = g_{eNN}/(2M)$ [10]. The ratio of this result to the usual value 0.079 is similar to the axial vector coupling constant $g_A$.

From the above reasonable results, the strong coupling constant can be estimated to a certain extent as

$$\alpha_s = \sqrt{|M_{\pi_0}^2 - 4m_q^2|} = \frac{2|p_q|}{m_q},$$

where $\pi_0$ has a quark and an anti-quark with the same mass. Since the relativistic velocity is less than 1, the lightest quark mass is allowed to be less than 95.4 MeV just as $v_{q}^{rel} = |p_q|/E_q = \sqrt{|M_{\pi_0}^2 - 4m_q^2|}/M_{\pi_0} < 1$. Hence the coupling constant should be estimated from the current quark masses. Now the momentum of the gluon is estimated as

$$q^2 = -4p_q^2 \sin^2(\theta/2) = -m_q^2 \alpha_s^2 \sin^2(\theta/2),$$

in the center of mass frame. Though this equation is defined for scattering processes, if it is applied to QED and compared with the Bohr radius, then the relation $\sin^2 (\theta/2) = \frac{1}{\beta}$ can be analogized for the ground state ($|q| \approx < \psi_{100}|1/r|\psi_{100} > = 1/a_{Bohr} = m_e \alpha$).

For a numerical example, the coupling constant $\alpha_s$ is estimated as 13.35 for the typical current quark mass $m_q = 10$ MeV and then the gluon momentum $q = 66.74$ MeV from Eqs. (13) and (14). The gluon momentum transferred by an excited quark can be estimated simply as 201.72 MeV (66.74+134.98) and 836.74 MeV, respectively, at the threshold energies of $\pi_0$ and $\rho_0$. The running coupling constant $\alpha_s(q^2) = \frac{\alpha_s}{16\pi^2 \log(q^2/\Lambda^2)}$ can be extrapolated by using unknown parameters $\beta$, $\Lambda$ for $\alpha_s(66.74$ MeV) = 13.35 and $\alpha_s(m_{\pi}) = 0.35$ which is the measured coupling constant at the lowest mass scale [11,12]. The coupling constants can be estimated as $\alpha_s(201.72$ MeV) = 0.988 and $\alpha_s(836.74$ MeV) = 0.451, and the expansion parameters of $m^2/q^2$ are 0.0025 and 0.00014, respectively, in the energy region of the $gq$ creation. Therefore perturbation with respect to the $\alpha_s$ and the expansion of $m^2/q^2$ are reliable simultaneously in the energy region relevant to quark confinement. So, quarks are confined in the hadron by the strong coupling in the non-perturbative region and by the linear confinement potential in the perturbative region. If the assumptions of this letter hold for the lightest meson, they also hold for all the other hadrons, because the mass of the internal quark loop starts from the lightest quark. In QED, on the contrary, the inner most electron of the most heavy atom can not transfer a momentum larger than $2m_e$, because the sum of the binding energy ($13.6 \times 120^2$ eV = 0.196 MeV) for the atomic number $Z = 120$ and the momentum of the photon (0.448 MeV) is estimated as 0.644 MeV. Therefore one can not see any confinement potential in QED, because the expansion of $m^2/q^2$ is not allowed.

As a result of the view of this letter, quark masses are not direct observables but parameters to be obtained from the hadronic spectra by solving the Dirac equation. So quark masses are unknown exactly. The current quark masses support the assumptions of this letter and the mass of the light mesons quite well rather than the constituent quark masses. If the quarks are bound by the usual Coulomb potential, then the mass of the light meson is less than the sum of the masses of its constituent two quarks because of their negative kinetic energies. However the quark are bounded by the confinement potential of this letter, the mass of the meson is larger than the sum of the masses of the two quarks because of their positive kinetic energies.
ACKNOWLEDGMENTS

This work was supported by the Korean Ministry of Education (Project no. BSRI 97-2425).

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FIG. 1. The Feynman diagrams at tree and one loop levels to contribute to the effective potential. The diagrams for the counter terms are not shown.