LINEARITY DEFECT OF THE RESIDUE FIELD OF SHORT LOCAL RINGS

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Abstract. Let \((R, m, k)\) be a Noetherian local ring with maximal ideal \(m\) and residue field \(k\). The linearity defect of a finitely generated \(R\)-module \(M\), which is denoted \(\text{ld}_R(M)\), is a numerical measure of how far \(M\) is from having linear resolution. We study the linearity defect of the residue field. We give a positive answer to the question raised by Herzog and Iyengar of whether \(\text{ld}_R(k) < \infty\) implies \(\text{ld}_R(k) = 0\), in the case when \(m^4 = 0\).

1. Introduction and notation

This paper is concerned with the notion of the linearity defect of the residue field of a commutative Noetherian local ring. This invariant was introduced by Herzog and Iyengar \([3]\) and has been further studied by Iyengar and Römer \([4]\), Şega \([6]\) and Nguyen \([5]\). Let us recall the definition of the linearity defect. Throughout this paper \((R, m, k)\) will denote a commutative Noetherian local ring with maximal ideal \(m\) and residue field \(k\). Let 

\[ F : \cdots \to F_n \xrightarrow{\partial_n} F_{n-1} \xrightarrow{\partial_{n-1}} \cdots \to F_1 \xrightarrow{\partial_1} F_0 \to 0 \]

be a minimal complex (i.e. \(\partial_i(F_i) \subseteq mF_{i-1}\) for all \(i \geq 0\)) of finitely generated free \(R\)-modules. Then the complex has a filtration \(\{\delta^p F\}_{p \geq 0}\) with \((\delta^p F)_i = m^{p-i}F_i\) for all \(p\) and \(i\) where, by convention, \(m^j = R\) for all \(j \leq 0\). The associated graded complex with respect to this filtration is called the linear part of \(F\) and denoted by \(\text{lin}^R(F)\). Let \(N\) be an \(R\)-module. The notation \(R^g\) will stand for the associated graded ring \(\bigoplus_{i \geq 0} m^i/m^{i+1}\) and \(N^g\) for the associated graded \(R^g\)-module \(\bigoplus_{i \geq 0} m^iN/m^{i+1}N\). By construction, \(\text{lin}^R(F)\) is a graded complex of graded free \(R^g\)-modules and has the property that \(\text{lin}^R(F)_i = F^g_i(-n)\), for all \(n\). For more information about this complex, we again refer to \([3]\) and \([4]\). Let \(M\) be a finitely generated \(R\)-module. The linearity defect of \(M\) is defined to be the number

\[ \text{ld}_R(M) := \sup \{i \in \mathbb{Z} | \text{H}_i(\text{lin}^R(F)) \neq 0\}, \]

where \(F\) is a minimal free resolution of \(M\). By definition, \(\text{ld}_R(M)\) can be infinite and \(\text{ld}_R(M) \leq d\) is finite if and only if \((\text{Syz}_d(M))^g\) has a linear resolution over the standard graded algebra \(R^g\), where \(\text{Syz}_d(M)\) is the \(d\)th syzygy module of \(M\). In particular, \(\text{ld}_R(M) = 0\) if and only if \(M^g\) has a linear resolution over \(R^g\) and then \(\text{lin}^R(F)\) is a minimal graded free resolution of \(M^g\). The notion of the linearity defect can be defined, in the same manner, for graded modules over a standard graded

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algebra $A$ over a field $k$. In [3], the authors proved that if $\text{ld}_A(k) < \infty$, then $\text{ld}_A(k) = 0$. Motivated by this known result in the graded case, the following natural question raised in [3].

**Question 1.** If $\text{ld}_R(k) < \infty$, does it follow that $\text{ld}_R(k) = 0$?

If $R^k$ is Cohen-Macaulay, Şega [6] showed that the question has positive answer in the case that $R$ is a complete intersection. Also, she gave an affirmative answer when $m^3 = 0$. Another positive answer to the question is given by the author and Rossi [1] when $R$ is of homogeneous type, that is $\dim_k \text{Tor}_i^R(M, k) = \dim_k \text{Tor}_i^{R^k}(k, k)$ for all $i$.

In this paper we show that this problem has an affirmative answer when $m^4 = 0$. The proof relies on the existence of a DG algebra structure of a minimal free resolution of residue field $k$.

### 2. Preliminaries and the main result

Şega provided an interpretation of linearity defect in terms of vanishing of special maps. For each $n \geq 0$ and $i \geq 0$ we consider the map

$$v_i^n(M) : \text{Tor}_i^R(M, R/m^{n+1}) \to \text{Tor}_i^R(M, R/m^n)$$

induced by the natural surjection $R/m^{n+1} \to R/m^n$. For simplicity, we set $v_i^n := v_i^n(k)$ when $M = k$.

**Theorem 2.1.** [6, Theorem 2.2] Let $M$ be a finitely generated $R$-module and $d$ be an integer. Then the following conditions are equivalent.

1. $\text{ld}_R(M) \leq d$;
2. $v_i^n(M) = 0$ for all $i \geq d + 1$ and all $n \geq 0$.

**Remark 2.2.** Let $i \geq 0$. Assume that $F$ is a minimal free resolution of a finitely generated $R$-module $M$. Then by [6, 2.3 (2′)], the following statements are equivalent.

1. $v_1^1(M) = 0$;
2. if $x \in F_i$ satisfies $\partial_i(x) \in m^2F_{i-1}$, then $x \in mF_i$.

Let $S$ be a unitary commutative ring. Given an $S$-complex $C$, we write $|c| = i$ (the homological degree of $c$) when $c \in C_i$. When we write $c \in C$ we mean $c \in C_i$ for some $i$. A (graded commutative) **DG algebra** over $S$ is a non-negative $S$-complex $(D, \partial)$ with a morphism of complexes called the product

$$\mu^D : D \otimes_S D \to D$$

$$a \otimes b \mapsto ab$$

satisfying the following properties:

1. **unital**: there is an element $1 \in D_0$ such that $1a = a1 = a$ for $a \in D$;
2. **associative**: $a(ba) = (ab)c$ for all $a, b, c \in D$;
3. **graded commutative**: $ab = (-1)^{|a||b|}ba \in D_{|a|+|b|}$ for all $a, b \in D$ and $a^2 = 0$ when $|a|$ is odd.

The fact that $\mu$ is a morphism of complexes is expressed by the **Leibniz rule**:

$$\partial(ab) = \partial(a)b + (-1)^{|a|}a\partial(b)$$

For more information on DG algebras we refer to [2].
Remark 2.3. If $(D, \partial)$ is a DG algebra over $S$. Using Leibniz rule, one can see that the subcomplex of cycles $Z(D)$ is a DG subalgebra of $D$ and the boundaries $B(D)$ is a DG ideal of $Z(D)$. Thus the product on $D$ induces a product on the homology $H(D) = Z(D)/B(D)$. In particular, $\bigoplus_{i \geq 0} H_n(D)$ is a graded module over commutative ring $H_0(D)$.

Tate constructed a DG algebra (free) resolution of $k$. Furthermore, such a resolution can be chosen to be minimal, see [2, Theorem 6.3.5], which we refer a minimal Tate resolution of $k$ over $R$.

The following lemma shows that the linear part of a minimal Tate resolution of $k$ inherits a DG algebra structure from that of the resolution.

Lemma 2.4. Let $(F, \partial)$ be a minimal Tate resolution of $k$. Then $\text{lin}^R(F)$ has a DG algebra structure induced by that of $F$.

Proof. Let $\mu^F : F \otimes_R F \to F$ be a morphism of complexes which defines the product on $F$. Set $S = R^\mathbb{Z}$. Since $F$ is minimal we see that $F \otimes_R F$ is a minimal complex as well. Hence the morphism induces a morphism of graded $S$-complexes $\text{lin}^R(\mu^F) : \text{lin}^R(F \otimes_R F) \to \text{lin}^R(F)$ such that if $i, n \geq 0$ and $x^*$ is the image of an element $x \in m^i(F \otimes_R F)_n$ in $m^i(F \otimes_R F)_n/m^{i+1}(F \otimes_R F)_n$, then $\text{lin}^R(\mu^F)$ maps $x^*$ into the image of $\mu(x)$ in $m^iF_n/m^{i+1}F_n$. There is also a natural isomorphism of graded $S$-complexes $\lambda : \text{lin}^R(F) \otimes_S \text{lin}^R(F) \to \text{lin}^R(F \otimes_R F)$ such that if $i, j, n, m \geq 0$ and $x \in m^iF_n$ and $y \in m^jF_m$, the image of $x^* \otimes y^*$ in $m^iF_n/m^{i+1}F_n$ and $y^*$ in $m^jF_m/m^{j+1}F_m$ respectively, then $\lambda$ maps $x^* \otimes y^*$ into the image of $x \otimes y$ in $m^{i+j}(F \otimes_R F)_{n+m}/m^{i+j+1}(F \otimes_R F)_{n+m}$, see [4, Lemma 2.7]. Now, define (the product) $\mu^{\text{lin}^R(F)} : \text{lin}^R(F) \otimes_S \text{lin}^R(F) \to \text{lin}^R(F)$ as the composition $\text{lin}^R(\mu^F) \circ \lambda$. Since $\mu$ satisfies conditions $(i), (ii), (iii)$ of the definition of DG algebras, one can see that $\mu^{\text{lin}^R(F)}$ satisfies the same properties as well. Therefore the linear part of $F$ is a DG algebra over $S$ augmented to $k$.

Let $m^*$ denote the homogeneous maximal ideal of $R^\mathbb{Z}$. The following is a direct consequence of the above lemma.

Corollary 2.5. If $F$ is a minimal free resolution of $k$, then $m^* H_n(\text{lin}^R(F)) = 0$ for all $n$.

Proof. The assertion follows from Remark 2.3 with considering the fact that $H_0(\text{lin}^R(F)) = R^\mathbb{Z}/m^*$.

In what will follow, let $(F, \partial)$ be a minimal free resolution of residue field $k$ with differential map $\partial$. The differential map of $\text{lin}^R(F)$ which is induced by $\partial$ will be denoted by $\partial^*$. We recall that $\text{lin}^R(F)_n = F^R_n(−n)$. For any $i, n \geq 0$ and $x \in m^iF_n$, $\partial^*$ maps $x + m^{i+1}F_n$, the image of $x$ in $m^iF_n/m^{i+1}F_n$, into the image of $\partial(x)$ in $m^{i+1}F_{n-1}/m^{i+2}F_{n-1}$ that is $\partial(x) + m^{i+2}F_{n-1}$.

Proposition 2.6. Let $d$ be an integer. If $\text{ld}_R(k) \leq d$, then the following hold.

(1) $v_d^* = 0$.
(2) $m^* \text{Ker} \partial^* = m^* \text{Im} \partial^*_{d+1}$.
Proof. For the simplicity, we set \( Z = \text{Ker} \partial_d^* \) and \( B = \text{Im} \partial_{d+1}^* \).

(1) If \( v_1^d \neq 0 \), then there exists an element \( e \in F_d \setminus m F_d \) such that \( \partial_d(e) \in m^2 F_{d-1} \), by 2.2. Let \( e^* \) be the image of \( e \) in the quotient module \( F_d/m F_d \). Then \( \partial_d^*(e^*) = 0 \) and so \( e^* \) is a cycle in \( \text{lin} R(F) \). Applying 2.5, we have \( m^* Z \subseteq B \) and therefore \( m^* e^* \subseteq B \). As \( e^* \) is an element of a basis of the free module \( F_d(-d) \) and \( B \subseteq m^* F_d(-d) \), it is straightforward to see that \( m^* e^* \) is a direct summand of \( B \). Therefore \( k \) has a linear resolution over \( R \) and consequently \( \text{lin} R(F) \) is acyclic.

Hence \( v_1^d = 0 \) and this is a contradiction.

For (2), it is enough to show that \( m^* Z \subseteq m^* B \). First we claim that \( m^* Z \) is generated in degree at least \( d + 2 \) and \( m^* Z \subseteq B \). Indeed since \( v_1^d = 0 \), applying Remark 2.2, one has \( Z \subseteq m^* F_d(-d) \) and consequently \( m^* Z \) is generated in degree at least \( d + 2 \). The second part of the claim follows from Corollary 2.5.

On the other hand, \( B \) has a linear resolution, by the hypothesis. Hence \( B \) is generated by elements of degree \( d + 1 \) and then all its elements of degree at least \( d + 2 \) contained in \( m^* B \). Now, putting these two considerations together, we get \( m^* Z \subseteq m^* B \).

Now, we are ready to prove our main result.

**Theorem 2.7.** Assume that \( R \) is Artinian with \( m^4 = 0 \). If \( \text{ld}_R(k) < \infty \), then \( \text{ld}_R(k) = 0 \).

**Proof.** Let \( d \) be a non-negative integer and \( \text{ld}_R(k) \leq d \). We prove by descending induction on \( d \). The case where \( d = 0 \) is clear. Let \( d > 0 \). Applying Proposition 2.6, we have \( v_1^d = 0 \). Since \( m^4 = 0 \), it follows from [6, Theorem 7.1] that \( v_2^d = 0 \). Again since \( m^4 = 0 \) obviously \( v_i^d = 0 \) for all \( i \geq 3 \), by the definition of the map \( v_i^d \). Therefore, from 2.1 we get \( \text{ld}_R(k) \leq d - 1 \). This completes the induction and finishes the proof. \( \square \)

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REFERENCES

[1] R. Ahangari Maleki, M.E. Rossi, Regularity and linearity defect of modules over local rings, J. Commut. Algebra 6 (2014) 485-504.
[2] L.L. Avramov, Infinite free resolutions, in: Six Lectures on Commutative Algebra, Bellaterra, 1996, in: Progr. Math., vol. 166, Birkhäuser, Basel, 1998, pp. 1-118.
[3] J. Herzog, S. Iyengar, Koszul modules, J. Pure Appl. Algebra 201 (2005) 154–188.
[4] S. Iyengar, T. Römer, Linearity defects of modules over commutative rings, J. Algebra 322 (2009) 3212-3237.
[5] H.D. Nguyen, Notes on the linearity defect and applications, arXiv:1411.0261.
[6] L.M. Şega, On the linearity defect of the residue field, J. Algebra 384 (2013) 276–290.

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