Super-PP-wave Algebra
from
Super-AdS$\times S$ Algebras in Eleven-dimensions

Machiko Hatsuda, Kiyoshi Kamimura† and Makoto Sakaguchi

Theory Division, High Energy Accelerator Research Organization (KEK),
Tsukuba, Ibaraki, 305-0801, Japan
† Department of Physics, Toho University, Funabashi, 274-8510, Japan

E-mails: mhatsuda@post.kek.jp, kamimura@ph.sci.toho-u.ac.jp, Makoto.Sakaguchi@kek.jp

Abstract

Maximally supersymmetric spacetime algebras in eleven-dimensions, which are
the isometry superalgebras of Minkowski space, AdS$_7 \times S^4$, AdS$_4 \times S^7$ and pp-wave
background, are related by Inönü-Wigner contractions. The super-AdS$_{4(7)} \times S^{7(4)}$
algebras allow to introduce two contraction parameters, the one for the flat limit to
the super-Poincaré algebra and the other for a Penrose limit to the super-pp-wave
algebra. Under these contractions supersymmetries are maintained because the
Jacobi identity of three supercharges holds for any values of contraction parameters.

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1 Introduction

Recently, maximally supersymmetric pp-wave (plane-fronted gravitational wave with parallel rays) backgrounds of supergravity theories in eleven- and ten-dimensions have attracted great interests. In [1], maximally supersymmetric backgrounds in eleven-dimensions were classified into four types: flat Minkowski space (and their toroidal compactifications), AdS$_4 \times S^7$, AdS$_7 \times S^4$ and the super-pp-wave background. This super-pp-wave background in eleven-dimensions was reconsidered recently in [2]. For the type-IIB supergravity, a super-pp-wave background was discovered in [3]. The type-IIB superstrings in this background were shown to be exactly solvable [6] in spite of the presence of the RR five-form field strength. This model is expected to provide some hints for the study of superstrings on more general backgrounds [4] [5] [6].

The Penrose limit plays a central role in these recent studies of the pp-wave backgrounds. It was known that any solution of Einstein gravity admits plane-wave backgrounds in the Penrose limit [10]. This was extended to the solutions of supergravities in [11] and was studied in the context of the Wess-Zumino-Witten models in [12]. It was shown that the super-pp-wave background can be derived by the Penrose limit from the super-AdS$\times S$ backgrounds in [13] [14] and this was investigated further in [15]. In addition, the full supersymmetry algebra of the super-pp-wave background of the type-IIB theory was shown to be derived by an Inönü-Wigner (IW) contraction [16] from the super-AdS$_5 \times S^5$ algebra of SU(2,2|4) in [17]. The Penrose limit was recognized to be important to explore the AdS/CFT correspondence [18] beyond massless string modes in [19]. This line was pursued further in [20].

In this paper, we consider the Penrose limit from an algebraic point of view, which is coordinate-independent and manifestly supersymmetric. We show that the isometry superalgebra of the super-pp-wave background in eleven-dimensions is obtained from the super-AdS$_{4(7)} \times S^{7(4)}$ algebras by IW contractions. The supersymmetries are maintained during contractions because the Jacobi identity of three supercharges holds for any values of contraction parameters. In section 2, the pp-wave algebra is derived from the AdS$_{4(7)} \times S^{7(4)}$ algebras. This is extended to the maximally supersymmetric case in section 3. The last section is devoted to a summary and discussions.

2 PP-wave Algebra from AdS$_{4(7)} \times S^{7(4)}$ Algebras

The AdS$_{p+2} \times S^{9-p}$ algebra, which is the isometry algebra of the AdS$_{p+2} \times S^{9-p}$ space, is given, in terms of dimensionless momenta $P$’s and rotations $J$’s, as

\[
\begin{align*}
[P_a, P_b] &= 4\epsilon^2 J_{ab}, & [P_a, P_b'] &= -\epsilon^2 J_{a'b'}, \\
[J_{ab}, P_c] &= 2\eta_{ac} P_b, & [J_{a'b'}, P_{c'}] &= 2\eta_{b'c'} P_{a'}, \\
[J_{ab}, J_{cd}] &= 4\eta_{ad} J_{bc}, & [J_{a'b'}, J_{c'd'}] &= 4\eta_{a'd'} J_{b'c'},
\end{align*}
\]

\textsuperscript{1} For other dimensions see [4] and [5].
where $\epsilon^2 = 1$ for $p = 2$ and $\epsilon^2 = -1$ for $p = 5$. It is understood that the obvious anti-symmetries of indices in the left hand side of equalities in (2.1) are to be implemented on the right hand side with unit weight. For $p = 2$, this algebra is the $\text{AdS}_4 \times S^7$ algebra with the vector index of $\text{AdS}_4$, $a = 0, 1, 2, 3$, and that of $S^7$, $a' = 4, ..., 9, \sharp$. On the other hand, for $p = 5$, this algebra is the $\text{AdS}_7 \times S^4$ algebra with the vector index of $S^4$, $a = \sharp, 1, 2, 3$, and that of $\text{AdS}_7$, $a' = 4, ..., 9, 0$. We use the metric $\eta_{\mu\nu}$ with $\eta_{00} = -1$ otherwise $+1$.

The symmetry group is $\text{SO}(p + 1, 2) \times \text{SO}(10 - p)$ which has a flat limit to $\text{ISO}(p + 1, 1) \times \text{ISO}(9 - p)$ by the following IW contraction

1. Rescale the translation generators $P$’s as
   \[ P_a \rightarrow R P_a, \quad P_{a'} \rightarrow R' P_{a'}. \]  
   where $R/2$ and $R'$ are the radii of the $\text{AdS}_4(S^4)$ and that of $S^7(\text{AdS}_7)$, respectively.

2. Then take $R \rightarrow \infty$ and $R' \rightarrow \infty$.

In this limit, $P$’s become the linear momenta and $J$’s are the Lorentz generators. For the maximal spacetime supersymmetry, $R$ and $R'$ must satisfy

\[ R = R', \quad R^2 = 2 \rho R_S^2 = 2 \frac{\rho}{\rho} R_{\text{AdS}}^2, \quad \rho = \begin{cases} 1/2 & \text{for } p = 2 \\ 2 & \text{for } p = 5 \end{cases} \]  

where $R_S$ and $R_{\text{AdS}}$ are the radii of $S^{9-p}$ and that of $\text{AdS}_{p+2}$, respectively.

Besides the flat limit, as any other metrics, the $\text{AdS} \times S$ metric allows the Penrose limit \[10\] giving a plane wave metric. The Penrose limit can be understood as an IW contraction of the $\text{AdS}_{p+2} \times S^{9-p}$ algebra into the pp-wave algebra.

1. Define the light cone components of the momenta $P$’s and boost generators $P^*$’s as
   \[ P_\pm \equiv \frac{1}{\sqrt{2}} (P_0 \pm P_0), \quad P_m \equiv (P_i, P_{i'}), \]
   \[ P_m^* = \begin{cases} (P_i^* = J_{00}, P_i^* = J_{i'0}) & \text{for } p = 2 \\ (P_i^* = J_{03}, P_i^* = J_{i'0}) & \text{for } p = 5 \end{cases} \]  

   where $i = 1, 2, 3, \ i' = 4, ..., 9$, and $m = (i, i')$.

2. Suppose the plane-wave propagates along $x_+$ time direction. The transverse translation and boost generators are rescaled with a dimensionless parameter $\Omega$ as
   \[ P_+ \rightarrow \frac{1}{\Omega^2} P_+, \quad P_m \rightarrow \frac{1}{\Omega} P_m, \quad P_m^* \rightarrow \frac{1}{\Omega} P_m^*. \]

3. Then take $\Omega \rightarrow 0$ limit.
To see them explicitly the AdS$_{p+2} \times S^{9-p}$ algebra (2.1) is rescaled, following to (2.2) and (2.3), as

\[
[P_i, P_+] = 2\sqrt{2} \frac{e^2 \Omega^2}{R^2} P_i^*, \quad [P_i', P_+] = -\frac{e^2 \Omega^2}{\sqrt{2} R^2} P_i^*,
\]

\[
[P_i^*, P_+] = -\frac{1}{\sqrt{2}} e^2 \Omega^2 P_i, \quad [P_i', P_+] = \frac{1}{\sqrt{2}} e^2 \Omega^2 P_i',
\]

\[
[P_+, P_-] = -2\sqrt{2} \frac{1}{R^2} P_i^*, \quad [P_+, P_-] = -\frac{1}{\sqrt{2} R^2} P_i^*,
\]

\[
[P_i^*, P_-] = \frac{1}{\sqrt{2}} P_i, \quad [P_i^*, P_-] = \frac{1}{\sqrt{2}} P_i',
\]

\[
[P_i^*, P_j] = \frac{1}{\sqrt{2}} \eta_{ij} (e^2 \Omega^2 P_- - P_+), \quad [P_i', P_j'] = -\frac{1}{\sqrt{2}} \eta_{i'j'} (e^2 \Omega^2 P_- + P_+),
\]

\[
[P_i, P_j] = 4 e^2 \Omega^2 \frac{J_{ij}}{R^2}, \quad [P_i', P_j'] = -4 e^2 \Omega^2 \frac{J_{i'j'}}{R^2},
\]

\[
[P_i^*, P_j^*] = e^2 \Omega^2 J_{ij}, \quad [P_i^*, P_j^*] = -e^2 \Omega^2 J_{i'j'},
\]

\[
[J_{ij}, P_k] = 2\eta_{jk} P_i, \quad [J_{i'j'}, P_{k'}] = 2\eta_{j'k'} P_{i'},
\]

\[
[J_{ij}, J_{kl}] = 4\eta_{ij} J_{jk}, \quad [J_{i'j'}, J_{k'l'}] = 4\eta_{i'j'} J_{k'l'},
\]

(2.6)

where we used relation \( R = R' \) of (2.3) for simplicity. By taking \( \Omega \to 0 \) limit, the \( \epsilon \) dependence, which distinguishes the \( p = 2 \) case from the \( p = 5 \) case, disappears and (2.6) becomes the unique plane wave algebra in eleven-dimensions,

\[
[P_i, P_-] = -2\sqrt{2} \frac{1}{R^2} P_i^*, \quad [P_i', P_-] = -\frac{1}{\sqrt{2} R^2} P_i^*,
\]

\[
[P_m^*, P_-] = \frac{1}{\sqrt{2}} P_m, \quad [P_m^*, P_n] = -\frac{1}{\sqrt{2}} \eta_{mn} P_+,
\]

\[
[J_{mn}, P_p] = 2\eta_{np} P_m, \quad [J_{mn}, P_p^*] = 2\eta_{np} P_m^*,
\]

\[
[J_{mn}, J_{pq}] = 4\eta_{mq} J_{np}.
\]

(2.7)

This is the symmetry algebra of the pp-wave metric \( [21] \). The Poincaré algebra of ISO(10,1) group is obtained by taking flat limit \( R \to 0 \) in (2.7) and by making manifest restored symmetry generators such as \( J_{ij'}, J_{k'i} \), etc.

### 3 Super-PP-wave algebra from Super-AdS$_{4(7)} \times S^{7(4)}$ Algebras

We extend the previous analysis to the supersymmetric case and show the maximally supersymmetric eleven-dimensional pp-wave algebra \( [2] \) is obtained by IW contractions of the super-AdS$_{p+2} \times S^{9-p}$ algebras. In this case the Jacobi identities of the superalgebra...
give a restriction on the radii of AdS and $S$, $R_{AdS}$ and $R_S$, as $R_{AdS} = \frac{1}{2} R_S$ for $p = 2$ and $R_{AdS} = 2 R_S$ for $p = 5$, as in (2.3). In eleven-dimensions, the supersymmetry generators $Q_\alpha$ are 32 Majorana spinors. The gamma matrices $\Gamma^\mu$, ($\mu = 0, 1, \ldots, 9, z$), in eleven-dimensions are composed in terms of those of AdS$_4$, $\gamma^a$, ($a = 0, 1, 2, 3$), and of $S^7$, $\gamma^a'$, ($a' = 4, 5, \ldots, 9, z$), or in terms of those of $S^4$, $\gamma^a$, ($a = \xi, 1, 2, 3$), and of AdS$_7$, $\gamma^a'$, ($a' = 4, 5, \ldots, 9, 0$), as

$$
\Gamma^a = \gamma^a \otimes 1, \quad \Gamma^a' = \gamma_5 \otimes \gamma^{a'}, \quad \gamma_5 = \left\{ \begin{array}{ll}
i\gamma^{0123} & \text{for } p = 2 \\ \gamma^{123} & \text{for } p = 5 \end{array} \right.. \quad (3.1)
$$

They satisfy $\{\Gamma^\mu, \Gamma^{\nu}\} = 2\eta^{\mu\nu}$. The charge conjugation matrix $C$ in eleven-dimensions is taken as

$$
C = C \otimes C' \quad (3.2)
$$

where $C$ and $C'$ are the charge conjugation matrices for AdS$_4$ ($S^4$) and $S^7$ (AdS$_7$) spinors respectively.

The bosonic part of the super-AdS$_{p+2} \times S^{9-p}$ algebra is (2.1). In addition to it, the odd generators $Q_\alpha$ satisfy

$$
[P_a, Q] = i\epsilon Q \gamma_5 \gamma_a \otimes 1, \quad [P_{a'}, Q] = \frac{i}{2} \epsilon Q 1 \otimes \gamma_{a'}, \quad [J_{ab}, Q] = \frac{1}{2} Q \gamma_{ab} \otimes 1, \quad [J_{a'b'}, Q] = \frac{1}{2} Q 1 \otimes \gamma_{a'b'},
$$

$$
\{Q, Q\} = -2C \gamma^a \otimes C' P_a - 2C \gamma_5 \otimes C' \gamma^{a'} P_{a'} - 2i\epsilon C \gamma_5 \gamma^{ab} \otimes C' J_{ab} + i\epsilon C \gamma_5 \otimes C' \gamma^{a'b'} J_{a'b'}. \quad (3.3)
$$

The Jacobi identities are satisfied as long as $\epsilon^2 = 1$ for $p = 2$ and $\epsilon^2 = -1$ for $p = 5$. Hereafter, we choose $\epsilon = 1$ for $p = 2$ and $\epsilon = -i$ for $p = 5$, for presentation.

For the present purpose, we rewrite this superalgebra in terms of eleven-dimensional covariant gamma matrices, $\Gamma'$'s, instead of $\gamma'$'s. It is convenient to introduce following matrix

$$
I = \Gamma^{123}, \quad \Gamma'^{123} = \Gamma^d I = -i\epsilon \gamma_5 \otimes 1, \quad \Gamma^d = \left\{ \begin{array}{ll}
\Gamma^0 & \text{for } p = 2 \\ -\Gamma^d & \text{for } p = 5 \end{array} \right.. \quad (3.4)
$$

$^2$ For the gamma-matrices in four-dimensions, we use a Majorana representation for AdS$_4$ ($a = 0, 1, 2, 3$) and a pseudo-symplectic representation for $S^4$ ($a = \xi, 1, 2, 3$) which satisfy, in both cases,

$$
\tilde{C} = -C, \quad \tilde{\gamma}^a = -C \gamma^a C^{-1}, \quad \{\gamma^a, \gamma^b\} = 2\eta^{ab}
$$

and $C\gamma^{a_1 \cdots a_N}$ is symmetric iff $N = 1, 2 \mod 4$. For the gamma-matrices in seven-dimensions, we use a pseudo-Majorana representation for $S^7$ ($a' = 4, 5, \ldots, 9, z$) and a symplectic Majorana representation for AdS$_7$ ($a' = 4, 5, \ldots, 9, 0$) which satisfy, in both cases,

$$
\tilde{C}' = +C', \quad \tilde{\gamma}'^{a'} = -C' \gamma'^{a'} C'^{-1}, \quad \{\gamma'^{a'}, \gamma'^{b'}\} = 2\eta'^{a'b'}
$$

and $C'\gamma'^{a_1 \cdots a_N'}$ is symmetric iff $N = 0, 3 \mod 4$. 

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4
The commutation relations (3.3) for the super-AdS$_{p+2}$ $\times$ $S^{9-p}$ algebra are rewritten as

$$[P_a, Q] = -Q \Gamma^a \Gamma_a, \quad [P_{a'}, Q] = -\frac{1}{2} Q \Gamma^a \Gamma_{a'}, \quad [J_{mn}, Q] = \frac{1}{2} Q \Gamma_{mn}$$

$$\{Q, Q\} = -2C \Gamma^a P_\mu + 2C \Gamma^a \Gamma^{ab} J_{ab} - C \Gamma^a \Gamma^{a'b'} J_{a'b'}. \quad (3.5)$$

Associating with the rescaling of $P$’s by $R$ as in (2.2) with (2.3), the dimensionless supercharges $Q_\alpha$ are rescaled as

$$P_a \to RP_a, \quad P_{a'} \to RP_{a'}, \quad Q_\alpha \to \sqrt{R} Q_\alpha. \quad (3.6)$$

Corresponding to the rescaling of bosonic generators (2.5), the components of the supercharges must be rescaled with proper weights. To do this, we decompose supercharges $Q$ as

$$Q = Q_+ + Q_-, \quad Q_\pm = Q_\pm \mathcal{P}_\pm, \quad (3.7)$$

using the light cone projection operators

$$\mathcal{P}_\pm = \frac{1}{2} \Gamma_\pm \Gamma_\mp, \quad \Gamma_\pm \equiv \frac{1}{\sqrt{2}}(\Gamma_\pm \pm \Gamma_0). \quad (3.8)$$

In order to obtain the well defined limit of the super-AdS$_{p+2}$ $\times$ $S^{9-p}$ algebra, the supercharges turn out to be rescaled as

$$Q_+ \to \frac{1}{\Omega} Q_+, \quad Q_- \to Q_- \quad (3.9)$$

The super-AdS$_{p+2}$ $\times$ $S^{9-p}$ algebra (3.3), after the rescaling (3.6) by $R$ and (2.3) and (3.3) by $\Omega$, becomes

$$[P_+, Q_+] = \frac{\Omega^2 \epsilon^2}{2\sqrt{2} R} Q_+ I, \quad [P_-, Q_+] = -\frac{3}{2\sqrt{2} R} Q_+ I,$$

$$[P_i, Q_-] = \frac{1}{\sqrt{2} R} Q_+ \Gamma^- I \Gamma_i, \quad [P_{i'}, Q_-] = \frac{1}{2\sqrt{2} R} Q_+ \Gamma^- I \Gamma_{i'},$$

$$[P_{m'}, Q_-] = \frac{1}{2\sqrt{2} R} Q_+ \Gamma_m \Gamma^- I, \quad [J_{mn}, Q_\pm] = \frac{1}{2} Q_\pm \Gamma_{mn},$$

$$[P_+, Q_-] = \frac{3\Omega^2 \epsilon^2}{2\sqrt{2} R} Q_- I, \quad [P_-, Q_-] = -\frac{1}{2\sqrt{2} R} Q_- I,$$

$$[P_i, Q_+] = -\frac{\Omega^2 \epsilon^2}{\sqrt{2} R} Q_- \Gamma^+ I \Gamma_i, \quad [P_{i'}, Q_+] = -\frac{\Omega^2 \epsilon^2}{2\sqrt{2} R} Q_- \Gamma^+ I \Gamma_{i'},$$

$$[P_{m'}, Q_+] = -\frac{\Omega^2 \epsilon^2}{2\sqrt{2} R} Q_- \Gamma_m \Gamma^+ I, \quad [P_{i'}, Q_+] = \frac{\Omega^2 \epsilon^2}{2\sqrt{2} R} Q_- \Gamma_{i'} \Gamma^+ I,$$

$$\{Q_+, Q_+\} = -2C \Gamma^+ P_+ + \frac{\sqrt{2} \Omega^2 \epsilon^2}{R} C \Gamma^+ \Gamma^+ I j_{ij} - \frac{\Omega^2 \epsilon^2}{\sqrt{2} R} C \Gamma^+ \Gamma^{i'j'} j_{i'j'},$$

$$\{Q_-, Q_-\} = -2C \Gamma^- P_- - \frac{\sqrt{2} \Omega^2 \epsilon^2}{R} C \Gamma^- \Gamma^- I j_{ij} + \frac{1}{\sqrt{2} R} C \Gamma^- \Gamma^{i'j'} j_{i'j'},$$

$$\{Q_+, Q_-\} = \left( -2C \Gamma^m P_m - \frac{4}{R} C \Gamma^i P_i^* - \frac{2}{R} C \Gamma^i P_i^* \right) \mathcal{P}_-. \quad (3.10)$$
It is important that negative power terms of $\Omega$ disappear owing to the presence of the light cone projections. Therefore we can take the consistent Penrose limit $\Omega \to 0$ of the algebra to obtain

$$[P_-, Q_+] = -\frac{3}{2\sqrt{2R}} Q_+ I, \quad [P_-, Q_-] = -\frac{1}{2\sqrt{2R}} Q_- I,$$

$$[P_+, Q_-] = \frac{1}{\sqrt{2R}} Q_+ \Gamma^I I \Gamma^I, \quad [P_+, Q_-] = \frac{1}{2\sqrt{2R}} Q_+ \Gamma^I I \Gamma^I,'$$

$$[P_m^*, Q_-] = \frac{1}{2\sqrt{2}} Q_+ \Gamma_m \Gamma^-, \quad [J_{mn}, Q_-] = \frac{1}{2} Q_+ \Gamma_{mn},$$

$$\{Q_+, Q_+\} = -2C \Gamma^+ P_+$$
$$\{Q_-, Q_-\} = -2C \Gamma^- P_- - \frac{\sqrt{2}}{R} \mathcal{C} \Gamma^- I \Gamma^{ij} J_{ij} + \frac{1}{\sqrt{2R}} \mathcal{C} \Gamma^- I \Gamma^{ij'} J_{ij'},$$
$$\{Q_+, Q_-\} = \left(-2C \Gamma^m P_m - \frac{4}{R} \mathcal{C} \Gamma^i P_i^* - \frac{2}{R} \mathcal{C} \Gamma^{ij'} P_i^*\right) P_-.$$  \hspace{1cm} (3.11)

After the Penrose limit, all $\epsilon$ dependence have disappeared. This reflects the fact that the super-AdS$_4 \times S^7$ background and the super-AdS$_7 \times S^4$ background reduce into the unique super-pp-wave background in the Penrose limit. In this way, we have obtained the super-pp-wave algebra from the super-AdS$_4 \times S^7$ algebra and the super-AdS$_7 \times S^4$ algebra.

Furthermore the flat limit to super-Poincaré algebra can be taken by $R \to \infty$ in (2.7) and (3.11), and by making manifest restored (bosonic) symmetry generators as was mentioned in the last section.

We have established relations of the maximally supersymmetric spacetime algebras in eleven-dimensions; super-Poincaré, super-AdS$_4 \times S^7$, super-AdS$_7 \times S^4$ and super-pp-wave, by IW contractions of superalgebras.

4 Summary and Discussions

We derived the super-pp-wave algebra form the super-AdS$_{4(7)} \times S^{7(4)}$ algebras in eleven-dimensions by IW contractions which correspond to the Penrose limits of the super-AdS$_{4(7)} \times S^{7(4)}$ backgrounds. The differences between the super-AdS$_4 \times S^7$ algebra and the super-AdS$_7 \times S^4$ algebra, which are essentially caused from interchanging 0-component and $\sharp$-component, are shown to disappear after the contraction resulting to the unique super-pp-wave algebra in eleven-dimensions. A solution of plane wave is a function of $x_+ = t + x^\sharp$ and it is invariant under interchanging $t$ and $x^\sharp$. The Penrose limit brings to such a plane wave space. This naturally explains why the maximally supersymmetric pp-wave algebra is unique in spite of the presence of two distinct super-AdS algebras in eleven-dimensions. This property will be deeply related to the T-duality discussed in [1]. Supersymmetries are maintained during contractions because the Jacobi identities hold...
for any values of contraction parameters, $R$ and $\Omega$, and even for their limits $R \to \infty$ and $\Omega \to 0$.

It is noted that one can relate the super-pp-algebra, (2.7) and (3.11), to the one obtained in [2]. To do this, we rewrite generators as

$$P_m \rightarrow 3\sqrt{2}e_m, \quad P_m \rightarrow 3\sqrt{2}e_m,$$

$$P_\pm \rightarrow \frac{9\sqrt{2}}{2\mu^2}e^*_\pm, \quad P^*_\pm \rightarrow \frac{18\sqrt{2}}{\mu^2}e^*_\pm, \quad Q_\pm \rightarrow \sqrt{6\sqrt{2}Q_\pm} \quad (4.12)$$

where $\mu = 1/R$. Under this, the super-pp-wave algebra (2.7) and (3.11) turns out to be

$$[e_m, e_-] = -e^*_m, \quad [e^*_m, e_-] = \frac{\mu^2}{9}e_i, \quad [e^*_\pm, e_-] = \frac{\mu^2}{36}e^*_\pm,$$

$$[e^*_\pm, e_+] = -\frac{\mu^2}{9}\eta_{ij}e_+, \quad [e^*_\pm, e_+] = \frac{\mu^2}{36}\eta_{ij}e_+,$$

$$\{J_{mn}, e_p\} = 2\eta_{np}e_m, \quad [J_{mn}, e^*_p] = 2\eta_{np}e^*_m, \quad [J_{mn}, J_{pq}] = 4\eta_{np}J_{mq},$$

$$[e_-, Q_+] = -\frac{\mu}{4}Q_+I, \quad [e_-, Q_-] = -\frac{\mu}{12}Q_-I, \quad [e_+, Q_-] = \frac{\mu}{6}Q_+\Gamma^I\Gamma_1,$$

$$[e^*_\pm, Q_+] = \frac{\mu}{12}Q_+\Gamma^I\Gamma_1, \quad [e^*_\pm, Q_-] = \frac{\mu}{18}Q_+\Gamma^I\Gamma_1, \quad [e^*_\pm, Q_-] = \frac{\mu}{36}Q_+\Gamma^I\Gamma_1,$$

$$\{Q_+, Q_+\} = -\mathcal{C}\Gamma^+ e_+, \quad \{Q_-, Q_+\} = -\mathcal{C}\Gamma^- e_- - \frac{\mu}{6}\mathcal{C}\Gamma^-\Gamma^{ij}J_{ij} + \frac{\mu}{12}\mathcal{C}\Gamma^-\Gamma^{ij}J_{ij},$$

$$\{Q_+, Q_-\} = -\mathcal{C}\Gamma^m e_m - \frac{3}{\mu}\mathcal{C}\Gamma^m e^*_m - \frac{6}{\mu}\mathcal{C}\Gamma^m e^*_m \quad (4.13)$$

and is the superalgebra in [3]. Our presentation reveals the way to take the flat limit while the flat limit $\mu \to 0$ is not obvious in the form of (4.13).

The relation between the super-pp wave algebra and the AdS$_{4(7)} \times S^7(4)$ algebra is useful for constructing mechanical actions of M-branes in the super-pp wave background. Any form field in the former can be derived from the corresponding form in the latter. This approach will make the (super)symmetry of the pp-wave systems manifest.

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