Shape of Eigenvectors for the Decaying Potential Model

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Abstract. We consider the 1d Schrödinger operator with decaying random potential and study the joint scaling limit of the eigenvalues and the measures associated with the corresponding eigenfunctions, which is based on the formulation by Rifkind and Virág (Geom Funct Anal 28:1394–1419, 2018). As a result, we have completely different behavior depending on the decaying rate $\alpha > 0$ of the potential: The limiting measure is equal to (1) Lebesgue measure for the supercritical case ($\alpha > 1/2$), (2) a measure of which the density has power-law decay with Brownian fluctuation for critical case ($\alpha = 1/2$), and (3) the delta measure with its atom being uniformly distributed for the subcritical case ($\alpha < 1/2$). This result is consistent with the previous study on spectral and statistical properties.

1. Introduction

In this paper, we consider the one-dimensional Schrödinger operator with random decaying potential:

$$H := -\frac{d^2}{dt^2} + a(t)F(X_t)$$

where $a \in C^\infty(\mathbb{R})$, $a(-t) = a(t)$, $a(t)$ is monotone decreasing for $t > 0$ and

$$a(t) = t^{-\alpha}(1 + o(1)), \quad t \to \infty$$

for some $\alpha > 0$. For $\alpha < 1/2$, we also need $a'(t) = O(t^{-\alpha - 1})$ to use the results in [6]. $F \in C^\infty(M)$ is a smooth function on a torus $M$ such that

$$\langle F \rangle := \int_M F(x)dx = 0$$

and $\{X_t\}_{t \in \mathbb{R}}$ is the Brownian motion on $M$. Since $a(t)F(X_t)$ is a compact perturbation with respect to $(-\triangle)$, the spectrum $\sigma(H) \cap (-\infty, 0)$ on the negative
real axis is discrete. The spectrum $\sigma(H) \cap [0, \infty)$ on the positive real axis is [7]:

$$\sigma(H) \cap [0, \infty) = \begin{cases} \text{a.c.} & (\alpha > 1/2) \\
\text{p.p. on } [0, E_c] \text{ and s.c. on } [E_c, \infty) & (\alpha = 1/2) \\
\text{p.p.} & (\alpha < 1/2) \end{cases}$$

where $E_c$ is a deterministic constant. In fact, it is shown in [7] that the generalized eigenfunctions of $H$ are bounded for supercritical case ($\alpha > 1/2$), have power-law decay for critical case ($\alpha = 1/2$) and are sub-exponentially localized for subcritical case ($\alpha < 1/2$). For the level statistics problem, we consider the point process $\xi_{n,E_0}$ composed of the rescaling eigenvalues $\{n^{\beta}E_j(n) - \sqrt{E_0}\}_j$ of the finite box Dirichlet Hamiltonian $H_n := H_{[0,n]}$ around the reference energy $E_0 > 0$, whose behavior as $n \to \infty$ is given by [5,6,9]

$$\xi_{n,E_0} \overset{d}{\to} \begin{cases} \text{Clock}(\theta(E_0)) & (\alpha > 1/2) \\
\text{Sine}(\beta(E_0)) & (\alpha = 1/2) \\
\text{Poisson}(d\lambda/\pi) & (\alpha < 1/2) \end{cases}$$

where $\text{Clock}(\theta) := \sum_{n \in \mathbb{Z}} \delta_{n\pi + \theta}$, is the clock process for a random variable $\theta$ on $[0, \pi)$, $\text{Sine}(\beta)$ is the Sine-$\beta$ process which is the bulk scaling limit of the Gaussian beta ensemble [13], and Poisson $(\mu)$ is a Poisson process on $\mathbb{R}$ with intensity measure $\mu$. $\theta(E_0)$ has a form of the projection onto the torus $[0, \pi)$ of a time change of a Brownian motion [5]. $\beta(E_0) = \tau(E_0)^{-1}$ is equal to the reciprocal number of $\tau(E_0)$ explicit form of which is given in (4). $\tau(E_0)$ is the “Lyapunov exponent” such that the solution to the Schrödinger equation $H\varphi = E\varphi$ has the power-law decay: $\varphi(x) \simeq |x|^{-\tau(E)}$, $|x| \to \infty$. Since $\lim_{E_0 \to 0} \beta(E_0) = 0$ and $\lim_{E_0 \to \infty} \beta(E_0) = \infty$, repulsion of eigenvalues near $E_0$ is small (resp. large) if $E_0$ is small (resp. large), which is consistent with the following fact [1,10]:

$$\text{Sine}(\beta) \overset{d}{\to} \begin{cases} \text{Poisson}(d\lambda/\pi) & (\beta \downarrow 0) \\
\text{Clock}(\text{unif}[0, \pi)) & (\beta \uparrow \infty) \end{cases} \quad (1)$$

In this paper, we consider the scaling limit of the measure corresponding to the eigenfunction of $H_L$ along the formulation studied by Rifkind-Virag [12]. Let $\{E_j(n)\}_{j \geq 1}$ be the positive eigenvalues of $H_n$, and let $\{\psi_{E_j(n)}^{(n)}\}$ be the corresponding eigenfunctions. We consider the associated random probability measure $\mu_{E_j(n)}^{(n)}$ on $(0,1)$.

$$\mu_{E_j(n)}^{(n)}(dt) := Cn \left( |\psi_{E_j(n)}^{(n)}(nt)|^2 + \frac{1}{E_j(n)} \left| \frac{d}{dt} \psi_{E_j(n)}^{(n)}(nt) \right|^2 \right) dt \quad (2)$$

where $C$ is the normalizing constant. In (2), we consider the derivative of the $\psi_{E_j(n)}^{(n)}$ as well as $\psi_{E_j(n)}^{(n)}$ so that the analysis is reduced to that of the radial part of the Prüfer variable to be introduced in (6). However, since $\psi$ and $\psi'$ have same global behavior, and since we are interested in the global shape of $\psi$, we believe that (2) is a reasonable definition to study the shape of eigenvectors. Let $J := [a,b] \subset (0, \infty)$ be an interval, $\mathcal{E}_J^{(n)} := \{E_j(n)\}_j \cap J$ be the set of
eigenvalues of $H_n$ in $J$, and $E_{J}^{(n)}$ be the random variable uniformly distributed on $E_{J}^{(n)}$. Our aim is to consider the large $n$ limit of the joint distribution of the eigenvalue–eigenvector pairs:

$$Q : \left( E_{J}^{(n)} , \mu_{E_{J}^{(n)}}^{(n)} \right) \overset{d}{\to} ?$$

For $d$-dimensional discrete random Schrödinger operator, if $J$ is in the localized region, we have [8]

$$\left( E_{J}^{(n)} , \mu_{E_{J}^{(n)}}^{(n)} \right) \overset{d}{\to} ( E_{J} , \delta_{\text{uni} f[0,1]^d} ) \quad (3)$$

where $E_J$ is the random variable whose distribution is equal to $N(J)^{-1} 1_J(E) dN(E)$, where $dN$ is the density of states measure. Rifkind-Virag [12] studied the 1-d discrete Schrödinger operator with critical ($\alpha = 1/2$) decaying coupling constant and obtained that the limit of $\mu_{E_{J}^{(n)}}^{(n)}$ is given by an exponential Brownian motion with negative drift which corresponds to the exponential decay of the eigenfunctions.

To state our result, we need notations further. Let $N(E) := \pi^{-1} \sqrt{E}$ be the integrated density of states of $H$, $N(J) := N(b) - N(a)$, and let

$$\tau(E) := \frac{1}{8E} \int_M |\nabla (L + 2i\sqrt{E})^{-1} F|^2 dx \quad (4)$$

where $L$ is the generator of $(X_t)$. Moreover, let $E_J$ be the random variable whose distribution is equal to $N(J)^{-1} 1_J(E) dN(E)$, let $U$ be the uniform distribution on $(0,1)$, and let $Z$ be the two-sided Brownian motion, where $E_J$, $U$, and $Z$ are independent.

**Theorem 1.1**

$$\left( E_{J}^{(n)} , \mu_{E_{J}^{(n)}}^{(n)} \right) \overset{d}{\to} \begin{cases} 
(E_J, 1_{[0,1]}(t)dt) & (\alpha > 1/2) \\
E_J, \exp\left( \frac{2 \tau(E_J) \log \frac{t}{s} - 2\tau(E_J) |\log \frac{t}{s}|}{\frac{\tau(E_J)}{2}} \right) dt & (\alpha = 1/2) \\
(E_J, \delta_U(dt)) & (\alpha < 1/2) 
\end{cases}$$

When $\alpha < 1/2$, this result is the same as (3) and reflects the fact that, in the global scaling limit, eigenfunctions are localized around the localization centers being uniformly distributed, which is typical in Anderson localization [8]. For $\alpha > 1/2$, this result corresponds to the fact that the generalized eigenfunctions are spread over the entire space and is consistent with the extended nature of the system. For $\alpha = 1/2$, since

$$\exp \left[ Z_{\tau(E)} \log \frac{s}{t} - \tau(E) \right] = \exp \left[ Z_{\tau(E)} \log \frac{s}{t} \right] \begin{cases} 
\left( \frac{t}{s} \right)^{\tau(E)} & (t < s) \\
\left( \frac{s}{t} \right)^{\tau(E)} & (s < t) 
\end{cases}$$
this result implies that the center $U$ of the generalized eigenfunction $\psi$ is uniformly distributed and $\psi$ has the power-law decay around $U$ with Brownian fluctuation. Since $\lim_{E \downarrow 0} \tau(E) = \infty$ and $\lim_{E \uparrow \infty} \tau(E) = 0$, $\psi$ is localized (resp. delocalized) as $E \downarrow 0$ (resp. $E \uparrow \infty$) which is consistent with the previous discussion (1).

Remark (1) The results in this paper are announced in [11] without proof.

(2) In “Appendix,” we discuss the continuum one-dimensional operator with decaying coupling constant and have similar results with Theorem 1.1 except that, “$\log \frac{t}{U}$” for $\alpha = \frac{1}{2}$ in Theorem 1.1 is replaced by $t - U$. The conclusion for $\alpha = \frac{1}{2}$ for this model is essentially the same as that in Rifkind-Virag [12].

For the outline of proof, we mostly follow the strategy in [12], that is, (i) first consider the local version of the problem and then (ii) average over the reference energy, which we show more explicitly below.

Step 1: Local version

We first consider the local version $\Xi^{(n)}$ of our problem: Take $E_0 > 0$ as the reference energy and let

$$\Xi^{(n)}_{E_0} := \sum_j \delta \left( n \left( \sqrt{E_j(n)} - \sqrt{E_0} \right) + \theta_j, \mu_j^{(n)} \right)$$

which is a point process on $\mathbb{R} \times \mathcal{P}(0,1)$ where $\mathcal{P}(0,1)$ is the space of probability measures on $(0,1)$ with the vague topology. $\theta$ is a random variable with $\theta \sim \text{unif}[0,\pi)$ for $\alpha > 1/2$ which is independent from $(X_t)$, and $\theta = 0$ otherwise. The motivation to consider $\Xi^{(n)}_{E_0}$ is to study the behavior of eigenvalues lying in the $O(n^{-1})$-neighborhood of $E_0$ and the measures associated with them. Random variable $\theta$ (for $\alpha > 1/2$) has the role of making the $n \to \infty$ limit being independent of the choice of subsequence. We then have the following result which is of independent interest:

Theorem 1.2 $\Xi^{(n)} \xrightarrow{d} \Xi$, where

$$\Xi = \left\{ \sum_{j \in \mathbb{Z}} \delta_{j \pi + \theta} \otimes \delta_{1[0,1]}(t) dt \left( \begin{array}{c} \exp(2\tilde{r}_t(\lambda)) dt \\ \lambda \end{array} \right) \right\} \begin{cases} \sum_{\lambda \in \text{Sine}_\beta} \delta_{\lambda} \otimes \delta_{ \left( \begin{array}{c} \exp(2\tilde{r}_t(\lambda)) dt \\ \lambda \end{array} \right) } & (\alpha = 1/2) \\ \sum_{j \in \mathbb{Z}} \delta_{P_j} \otimes \delta_{\tilde{P}_j} & (\alpha < 1/2) \end{cases}$$

where $\theta \sim \text{unif}[0,\pi)$, $\tilde{r}_t(\lambda)$ is characterized by the following equation:

$$d\tilde{r}_t(\lambda) = \frac{\tau(E_0)}{t} dt + \sqrt{\frac{\tau(E_0)}{t}} dB^\lambda_t, \quad t > 0, \quad \lambda \in \mathbb{R}, \quad \tau(E_0)$$
and \( \{B_t^\lambda\}_\lambda \) is a family of Brownian motion. Moreover, \( \{P_j\} \sim \text{Poisson}(d\lambda/\pi) \) and \( \{\tilde{P}_j\} \sim \text{Poisson}(1_{[0,1]}(t)dt) \). The intensity measure of \( \Xi \) is given by

\[
\mathbb{E} \left[ \int G(\lambda, \mu) d\Xi(\lambda, \mu) \right] = \frac{1}{\pi} \begin{cases} \int d\lambda \mathbb{E} \left[ G(\lambda, 1_{[0,1]}(t)dt) \right] & (\alpha > 1/2) \\ \int d\lambda \mathbb{E} \left[ G \left( \frac{\exp \left( 2Z_{\tau(E_0)} \log \frac{s}{U} \right) - 2\tau(E_0) \log |s| }{f_0 \exp \left( 2Z_{\tau(E_0)} \log \frac{s}{U} \right) - 2\tau(E_0) \log |s| } \right) dt \right] & (\alpha = 1/2) \\ \int d\lambda \mathbb{E} \left[ G(\lambda, \delta_U) \right] & (\alpha < 1/2) \end{cases}
\]

for \( G \in C_b(\mathbb{R} \times \mathcal{P}(0,1)) \), where \( U := \text{unif}[0,1] \).

We note that Eq. (5) determines \( \tilde{r}_t(\lambda) \) up to constant and hence by normalization it determines \( \exp[2\tilde{r}_t(\lambda)]dt/\int_0^1 \exp[2\tilde{r}_s(\lambda)]ds \) uniquely. We also note that the intensity measure of \( \Xi_E \) gives the limit of the measure part \( \mu_E^{(n)} \) in Theorem 1.1. The main technical problems we have for the proof are: (1) For the critical case, the radial component of the generalized eigenfunction of \( H \) are divergent so that the argument in [12], which works for decaying coupling constant model (DC model, in short), is not directly applicable. Thus, we need to renormalize it by canceling out with the term coming from the normalization part. (2) For the subcritical case, we obtained that the joint limit of the pair of rescaling eigenvalues and corresponding localization centers converge to a Poisson process on \( \mathbb{R} \times [0,1] \), which has been proved for a class of random Schrödinger operators [2,4,8]. However, in [2,4,8] one used the stationarity of the random potential and Minami’s estimate which is known to hold mostly for discrete models only. Here, we have a general argument such that if (i) the reference energy lies in the localized regime and if (ii) the point process of rescaled eigenvalues of any subsystems, of which the volume is comparable with that of the original one, converge to a Poisson process on \( \mathbb{R} \), then the joint limit of eigenvalues and localization centers also converge to a Poisson process on \( \mathbb{R} \times [0,1] \).

Step 2: Average over the reference energy

Next, as is done in Rifkind-Virag [12], we take the “average” the result of Theorem 1.2 over the reference energy \( E_0 \) w.r.t. the density of states measure \( dN \), leading to the conclusion of Theorem 1.1. This is a model-independent, general argument and if (i) one has the limit of the local version \( \Xi_E^{(n)} \) and its intensity measure, and if (ii) \( G_n(E) \) (defined in Sect. 3) is uniformly integrable w.r.t. \( dN \times \mathcal{P} \), then the distribution of the limit of the measure part \( \mu_E^{(n)} \) is given by the intensity measure.

The rest of this paper is organized as follows. In Sects. 2 and 3, we prove Theorems 1.1, 1.2 respectively. In “Appendix,” we state the results for DC model and prove uniform integrability mentioned above. We sometimes use momentum variable \( \kappa = \sqrt{E} \) instead of energy variable \( E \).
2. Local Version

In this section, we prove Theorem 1.2 separately for $\alpha > 1/2$, $\alpha = 1/2$, and $\alpha < 1/2$.

2.1. Preliminary

We adopt the following version of Prüfer coordinate for the solution $\psi_E$ to the Schrödinger equation $H\psi_{\kappa^2}(t) = \kappa^2\psi_{\kappa^2}(t)$, $\kappa > 0$:

$$
\begin{pmatrix}
\psi_{\kappa^2}(t) \\
\psi'_{\kappa^2}(t)/\kappa
\end{pmatrix}
= R_t(\kappa)
\begin{pmatrix}
sin \theta_t(\kappa) \\
cos \theta_t(\kappa)
\end{pmatrix}.
$$

(6)

Introducing $\tilde{\theta}_t(\kappa)$, $r_t(\kappa)$ by

$$
\theta_t(\kappa) =: kt + \tilde{\theta}_t(\kappa), \quad R_t(\kappa) =: \exp[r_t(\kappa)]
$$

we have [7]

$$
r_t(\kappa) = \frac{1}{2\kappa} \text{Im} \int_0^t a(s) e^{2i\theta_s(\kappa)} F(X_s) ds,
$$

(7)

$$
\tilde{\theta}_t(\kappa) =: \frac{1}{2\kappa} \text{Re} \int_0^t \left(e^{2i\theta_s(\kappa)} - 1\right) a(s) F(X_s) ds.
$$

(8)

2.2. Supercritical Case

Proof of Theorem 1.2 for supercritical case

In general, for a sequence $\xi_n$ of the point processes on the metric space $X$ and a point process $\xi$, $\xi_n \xrightarrow{d} \xi$ is equivalent to $\int hd\xi_n \xrightarrow{d} \int hd\xi$ for any $h \in C_c(X)$ (the space $P(X)$ of probability measures on $X$ is endowed with the vague topology). It thus suffices to show

$$
\int hd\Xi_n E_0 \xrightarrow{d} \int hd\Xi E_0, \quad \Xi E_0 := \sum_{j \in \mathbb{Z}} \delta_{j\pi + \theta} \otimes \delta_{1[0,1]}(t) dt, \quad \theta \sim \text{unif}[0, \pi)
$$

for any $h \in C_c(\mathbb{R} \times P[0,1])$. Also, it is sufficient to assume that $h(\lambda, \mu) = h_1(\lambda) \cdot h_2(\mu)$ with $h_1 \in C_c(\mathbb{R})$, $h_2 \in C(P(0,1))$. We first work under a subsequence $\{n_k\}_k$ which satisfies the following condition: $\lim_{k \to \infty} n_k = \infty$ and there exist $\{m_k\}_k \subset \mathbb{N}$ and $\beta \in [0, \pi)$ such that

$$
n_k\sqrt{E_0} = m_k\pi + \beta + o(1), \quad k \to \infty.
$$

(9)

Here, we make use of the facts that, for a.s., $\tilde{\theta}_t(\kappa) \xrightarrow{t \to \infty} \tilde{\theta}_\infty(\kappa)$ for locally uniformly w.r.t. $\kappa$ [7], and $E_j(n) \to E_0$ for all $j$’s such that $n \left(\sqrt{E_j(n) - \sqrt{E_0}}\right) + \theta \in \text{supp} h_1$ ([5], Lemma 4.1). By Sturm’s oscillation theory, we have

$$
j\pi = \theta_{n_k} \left(\sqrt{E_j(n_k)}\right) = \sqrt{E_j(n_k)} n_k + \tilde{\theta}_{n_k} \left(\sqrt{E_j(n_k)}\right)
= \sqrt{E_j(n_k)} n_k + \tilde{\theta}_\infty \left(\sqrt{E_0}\right) + o(1), \quad k \to \infty
$$
and thus together with (9),
\[ n_k \left( \sqrt{E_j(n_k)} - \sqrt{E_0} \right) + \theta \Rightarrow_{a.s.} (j - m_k)\pi - \tilde{\theta}_\infty \left( \sqrt{E_0} \right) - \beta + \theta + o(1). \]
which yields
\[ \sum_j h_1 \left( n_k \left( \sqrt{E_j(n_k)} - \sqrt{E_0} + \theta \right) \right) \Rightarrow_{a.s.} \sum_j h_1 \left( (j - m_k)\pi - \tilde{\theta}_\infty \left( \sqrt{E_0} \right) - \beta + \theta \right). \]
\[ (10) \]
To study the measure part, we introduce measures \( \nu^{(n)}_E, \mu^{(n)}_E \) on \((0, 1)\):
\[ dr^{(n)}_E = n \cdot R_{nt}(\sqrt{E})^2 dt, \quad \mu^{(n)}_E := \frac{\nu^{(n)}_E}{\nu^{(n)}_E(0, 1)} \]
so that \( \mu^{(n)}_E \) is equal to the measure part \( \mu^{(n)}_E \) of \( \Xi^{(n)}_E \) if \( E \) is equal to an eigenvalue \( E_j(n) \) of \( H_n \). Then, by Lemma 2.1, for a.s. and for locally uniformly w.r.t. \( E \), we have
\[ h_2(\mu^{(n)}_E) \Rightarrow h_2(1_{[0, 1]}(t) dt). \]
\[ (11) \]
Therefore, by (10) and (11),
\[ \sum_j h_1 \left( n_k \left( \sqrt{E_j} - \sqrt{E_0} \right) + \theta \right) h_2(\mu^{(n)}_{E_j(n_k)}) \Rightarrow_{a.s.} \sum_{j \in \mathbb{Z}} h_1 \left( (j\pi - \tilde{\theta}_\infty \left( \sqrt{E_0} \right) - \beta + \theta \right) h_2(1_{[0, 1]}(t) dt). \]
Here, we note that \( (\beta - \tilde{\theta}_\infty(\kappa) + \theta)\pi \mathbf{Z} \overset{d}{=} \theta \), where \( (x)\pi \mathbf{Z} := x - \max\{k \in \mathbb{Z} \mid k \leq x\} \in [0, \pi) \) is the “fractional part” of \( x \) modulo \( \pi \mathbf{Z} \), which yields
\[ \int h \, d\Xi^{(n)}_{E_0} \overset{d}{=} \int h \, d\Xi_{E_0}, \quad h \in C_c(\mathbb{R} \times \mathcal{P}(0, 1)). \]
Since the limit in distribution is independent of the subsequence, this convergence holds for the whole limit so that we have \( \Xi^{(n)}_{E_0} \overset{d}{=} \Xi_{E_0} \). For the intensity measure, we note \( \theta \sim \text{unif}[0, \pi) \) and compute:
\[ \mathbb{E} \left[ \int G(\lambda, \mu) \, d\Xi_{E_0} \right] = \int_\mathbf{Z} \int_0^\pi \frac{d\theta}{\pi} G(j\pi + \theta, 1_{[0, 1]}(t) dt) = \frac{1}{\pi} \int d\lambda G(\lambda, 1_{[0, 1]}(t) dt). \]
\[ \square \]
**Lemma 2.1.** For a.s., we have
\[ d\mu^{(n)}_E \overset{u}{\Rightarrow} 1_{[0, 1]}(t) dt. \]
locally uniformly w.r.t. \( E \).

**Proof.** Let \( \kappa := \sqrt{E} \). Since \( r_t(\kappa) \overset{t \to \infty}{\Rightarrow} r_\infty(\kappa) \) locally uniformly w.r.t. \( \kappa \) for almost surely [7], for any \( \epsilon > 0 \) there exists \( T_\epsilon > 0 \) s.t. \( t > T_\epsilon \) implies \( |r_t(\kappa) - \]
\[ \square \]
Proof. Letting $r_{\infty}(\kappa) < \epsilon$. By definition, $\nu_{E}^{(n)}(a, b) = \int_{a}^{b} e^{2\nu_{\kappa}(\kappa)} n du = \int_{na}^{nb} e^{2\nu_{\kappa}(\kappa)} ds$ so that for $na > T_{\epsilon}$,

$$e^{2r_{\infty}(\kappa) - 2\epsilon} n(a - b) \leq \int_{na}^{nb} e^{2\nu_{\kappa}(\kappa)} ds \leq e^{2r_{\infty}(\kappa) + 2\epsilon} n(a - b). \quad (12)$$

To estimate $\nu_{E}^{(n)}(0, 1) = \int_{0}^{1} e^{2\nu_{\kappa}(\kappa)} n du = \int_{0}^{n} e^{2\nu_{\kappa}(\kappa)} ds$, we note that $\sup_{n, t} |r_{nt}(\kappa)| =: M < \infty$, and we divide the domain of integral as $(0, n) = (0, T_{\epsilon}] \cup [T_{\epsilon}, n)$. We then have

$$e^{2r_{\infty}(\kappa) - 2\epsilon} (n - T_{\epsilon}) + T_{\epsilon} e^{-2M} \leq \int_{0}^{n} e^{2\nu_{\kappa}(\kappa)} ds \leq e^{2r_{\infty}(\kappa) + 2\epsilon} (n - T_{\epsilon}) + T_{\epsilon} e^{2M}. \quad (13)$$

By (12), (13)

$$e^{-4\epsilon} (a - b) \leq \liminf_{n \to \infty} \frac{\nu_{E}^{(n)}(a, b)}{\nu_{E}^{(n)}(0, 1)} \leq \limsup_{n \to \infty} \frac{\nu_{E}^{(n)}(a, b)}{\nu_{E}^{(n)}(0, 1)} \leq e^{4\epsilon} (a - b).$$

Since $\epsilon > 0$ is arbitrary, we have

$$\lim_{n \to \infty} \frac{\nu_{E}^{(n)}(a, b)}{\nu_{E}^{(n)}(0, 1)} = (a - b), \; a.s.$$  

2.3. Critical Case

In Sect. 2.2, we renormalize the radial component of $\psi_{\kappa^{2}}$ in Sect 2.2.1, and prove Theorem 1.2 for $\alpha = 1/2$ in Sect 2.2.2.

2.3.1. Renormalize the Radial Component. Let $r_{t}(\kappa), \tilde{\theta}_{t}(\kappa)$ defined in (7), (8).

And let $\tilde{r}_{t}^{(n)}(\kappa) := r_{nt}(\kappa) - \langle F_{g_{\kappa}} \rangle \int_{0}^{n} a(s)^{2} ds$ be the “renormalized” radial part of $\psi_{\kappa^{2}}(t)$, where $g_{\kappa} := (L + 2\kappa) F_{g_{\kappa}}^{-1} F$, $t \in (0, 1)$. To study the local version of our problem, we work under the following notation: $\kappa_{c} := \kappa_{0} + \frac{\pi}{\kappa_{0}}$, $\kappa_{0} := \sqrt{E_{0}}$, $c \in \mathbb{R}$. We then have

**Lemma 2.2** If $\alpha = 1/2$, then there exists subsequence $\{n_{k}\}_{k \geq 1}$ and continuous function-valued process $\tilde{r}_{t}(c)$ such that

$$\tilde{r}_{t}^{(n_{k})}(\kappa_{c}) \xrightarrow{d} \tilde{r}_{t}(c), \quad \text{locally uniformly in } t \in (0, 1), c \in \mathbb{R}$$

$$d\tilde{r}_{t}(c) = \frac{\tau(\kappa_{c})}{t} dt + \sqrt{\frac{\tau(\kappa_{c})^{2}}{t}} dB_{t}^{e}, \quad t > 0$$

where $\{B_{t}^{e}\}$ is a family of Brownian motion.

**Proof.** Letting

$$J_{t}(\kappa) := \int_{0}^{t} a(s)e^{2i\tilde{\theta}_{s}(\kappa)} F(X_{s}) ds, \quad J_{t}(0) := \int_{0}^{t} a(s) F(X_{s}) ds$$

we have

$$r_{t}(\kappa) = \frac{1}{2\kappa} Im J_{t}(\kappa), \quad \tilde{\theta}_{t}(\kappa) = \frac{1}{2\kappa} Re (J_{t}(\kappa) - J_{t}(0)).$$

Here, we use the following lemma in [5].
Lemma 6.2 in [5]

(1) \[ \int_0^t a(s) e^{2i\theta_s(\kappa)} F'(X_s) ds = -\frac{i}{2\kappa} \int_0^t a(s)^2 Fg_{\kappa}(X_s) ds + Y_t(\kappa) + \delta_t(\kappa) \]

where \[ Y_t(\kappa) := \int_0^t a(s) e^{2i\theta_s(\kappa)} \nabla g_{\kappa}(X_s) dX_s \]

\[ \delta_t(\kappa) := [a(s) e^{2i\theta_s(\kappa)} g_{\kappa}(X_s)]^t_0 - \int_0^t a'(s) e^{2i\theta_s(\kappa)} g_{\kappa}(X_s) ds \]

\[ -\frac{i}{\kappa} \int_0^t a(s)^2 \left( \frac{e^{2i\theta_s(\kappa)}}{2} - 1 \right) e^{2i\theta_s(\kappa)} Fg_{\kappa}(X_s) ds, \]

(2) \[ \lim_{t \to \infty} \delta_t(\kappa) = \delta_\infty(\kappa), \text{ a.s. for some } \delta_\infty(\kappa), \]

(3) \[ \lim_{n \to \infty} E \left[ \max_{0 \leq t \leq T} |\delta_{nt}(\kappa) - \delta_{nt}(\kappa_0)|^2 \right] = 0. \]

By Ito’s formula,

\[ Fg_{\kappa}(X_s) ds = (Fg_{\kappa}) ds + dL^{-1}(Fg_{\kappa} - \langle Fg_{\kappa} \rangle) - \nabla L^{-1}(Fg_{\kappa} - \langle Fg_{\kappa} \rangle) dX_s \quad (14) \]

by which we further integrate by parts.

\[ \int_0^t a(s)^2 Fg_{\kappa}(X_s) ds = \langle Fg_{\kappa} \rangle \int_0^t a(s)^2 ds + \tilde{\delta}_t(\kappa) \]

where \[ \tilde{\delta}_t(\kappa) := [a(s)^2 L^{-1}(Fg_{\kappa} - \langle Fg_{\kappa} \rangle)]^t_0 \]

\[ -\int_0^t (a(s)^2)' L^{-1}(Fg_{\kappa} - \langle Fg_{\kappa} \rangle) ds \]

\[ -\int_0^t a(s)^2 \nabla L^{-1}(Fg_{\kappa} - \langle Fg_{\kappa} \rangle) dX_s. \]

\[ \tilde{\delta}_t(\kappa) \] is a sum of convergent terms and one with finite quadratic variation so that it converges almost surely:

\[ \lim_{t \to \infty} \tilde{\delta}_t(\kappa) = \tilde{\delta}_\infty(\kappa), \text{ a.s.} \]

Here, we replace \( t \) by \( nt \), and take \( n \to \infty \) limit with \( t \) being fixed. To cancel out the divergent term, we subtract \( \langle Fg_{\kappa} \rangle \int_0^n a(s)^2 ds \) from both sides and obtain

\[ \int_0^{nt} a(s)^2 Fg_{\kappa}(X_s) ds - \langle Fg_{\kappa} \rangle \int_0^n a(s)^2 ds = \langle Fg_{\kappa} \rangle \log t + \tilde{\delta}_\infty(\kappa) + \epsilon_n(t), \]

\[ \lim_{n \to \infty} \epsilon_n(t) = 0. \]

Therefore,

\[ J_{nt}(\kappa) = -\frac{i}{2\kappa} \langle Fg_{\kappa} \rangle \log t + Y_{nt}(\kappa) + A_n + \epsilon'_n(t), \quad \lim_{n \to \infty} \epsilon'_n(t) = 0. \]

where \[ A_n := -\frac{i}{2\kappa} \left( \langle Fg_{\kappa} \rangle \int_0^n a(s)^2 ds + \tilde{\delta}_\infty(\kappa) \right) + \delta_\infty(\kappa) \]
We remark that $A_n$ is random but does not depend on $t$. Let

$$r_t^{(n)}(\kappa) := r_{nt}(\kappa).$$

Then, we have

$$r_t^{(n)}(\kappa) = \frac{1}{2\kappa} \text{Im} \left[ J_{nt}(\kappa) \right] = \tilde{r}_t^{(n)}(\kappa) + \bar{A}_n + \bar{\varepsilon}_n(t),$$

where $\tilde{r}_t^{(n)}(\kappa) := \frac{1}{2\kappa} \text{Im} \left( -\frac{i}{2\kappa} \langle Fg_\kappa \rangle \log t + Y_{nt}(\kappa) \right)$

and $\bar{A}_n := \frac{1}{2\kappa} \text{Im} \left[ A_n \right]$, $\lim_{n \to \infty} \sup_{\frac{\sqrt{\log n}}{n} \leq t} \bar{\varepsilon}_n(t) = 0$.

The density function of $\mu^{(n)}_E$ is equal to

$$\frac{\exp[2r_t^{(n)}]}{\int_0^1 \exp[2r_s^{(n)}]ds} = \frac{\exp[2r_t^{(n)}]}{\int_0^{\sqrt{\log n}} \exp[2r_s^{(n)}] - 2\bar{A}_n - 2\bar{\varepsilon}_n(t) ds + \int_{\sqrt{\log n}}^1 \exp[2r_s^{(n)}] - 2\bar{\varepsilon}_n(t) + 2\bar{\varepsilon}_n(s) ds}.$$

To estimate the first term in the denominator, we note that

$$|r_s^{(n)}| \leq \frac{1}{2\kappa} \left| \text{Im} \int_0^n e^{2it\theta_u(\kappa)} a(u)F(X_u)du \right| \leq (\text{Const.}) \sqrt{\log n}, \quad 0 \leq s \leq \frac{\sqrt{\log n}}{n}.$$

Moreover, $A_n = O(\log n)$, and denoting by $\sigma_F$ the spectral measure of $L$ w.r.t. $F$, we have

$$\tilde{A}_n = \text{Im} \left[ -\frac{i}{2\kappa} \langle Fg_\kappa \rangle \right] = \frac{1}{2\kappa} \int_{-\infty}^0 \frac{-\lambda}{\lambda^2 + (2\kappa)^2} d\sigma_F(\lambda) > 0,$$

which yields

$$\int_0^{\sqrt{\log n}} \exp[r_s^{(n)} - \tilde{A}_n - \bar{\varepsilon}_n(t)] ds \leq \frac{\sqrt{\log n}}{n} \cdot \exp[-(\text{Const.}) \log n] = O(n^{-\delta}),$$

for some $\delta > 0$. On the other hand,

$$\int_0^{\sqrt{\log n}} \exp[\tilde{r}_s^{(n)}] ds = \int_0^{\sqrt{\log n}} \exp[r_s^{(n)} - \tilde{A}_n - \bar{\varepsilon}_n(s)] ds = O(n^{-\delta'})$$

Thus, we have

$$\frac{e^{2r_t^{(n)}(\kappa)}}{\int_0^1 e^{2r_s^{(n)}(\kappa)} ds} - \frac{e^{2\tilde{r}_t^{(n)}(\kappa)}}{\int_0^1 e^{2\tilde{r}_s^{(n)}(\kappa)} ds} \overset{a.s.}{=} o(1),$$

so that we henceforth consider $\tilde{r}_t^{(n)}(\kappa)$ instead of $r_t^{(n)}(\kappa)$. Let

$$\Theta_t^{(n)}(\kappa) := \theta_{nt}(\kappa_c) - \theta_{nt}(\kappa_0), \quad Y_t^{(n)}(\kappa_c) := Y_{nt}(\kappa_c).$$
Then,
\[ Y_t^{(n)}(\kappa_c) = \int_0^{nt} a(s)e^{2i\theta_s(\kappa_c)} \nabla g_{\kappa_c}(X_s) dX_s \]
\[ = \int_0^{nt} a(s)e^{2i(\theta_s(\kappa_c) - \theta_s(\kappa_0))} e^{2i\theta_s(\kappa_0)} \nabla g_{\kappa_c}(X_s) dX_s \]

and by [5] Proposition 9.1 and Lemma 9.3, we have \( \Theta_n^{(c)}(c) \xrightarrow{d} \Theta_u(c), \theta_{nu}(\kappa_0) \xrightarrow{d} U \). Where \( U \sim \text{unif}[0, \pi) \), \( \Theta_u(c) \) and \( U \) are independent, and these convergence is uniform w.r.t. \( u \in [0, 1] \) and \( c \in \mathbb{R} \). Moreover, \( \Theta_t(c) \) is characterized by the following SDE.
\[ d\Theta_t(c) = cdt + \sqrt{\tau(E_0)Re \left[ \left( e^{2i\Theta_s(c)} - 1 \right) \frac{dZ_t}{\sqrt{t}} \right]}, \quad \Theta_0(c) = 0. \]

Zt is a complex Brownian motion. By Skorohod’s theorem, we can assume that \( \Theta_n^{(c)}(c) \xrightarrow{a.s.} \Theta_u(c), \theta_{nu}(\kappa_0) \xrightarrow{a.s.} U \). Thus, for \( 0 < s < t \), we use (14) and integrate by parts to yield
\[ \left\langle Y^{(n)}(\kappa_{c_1}), Y^{(n)}(\kappa_{c_2}) \right\rangle_t - \left\langle Y^{(n)}(\kappa_{c_1}), Y^{(n)}(\kappa_{c_2}) \right\rangle_s = \int_s^t a(nu)^2 \exp \left[ 2i \left( \Theta_u^{(n)}(c_1) - \Theta_u^{(n)}(c_2) \right) \right] \langle g_{\kappa_{c_1}}, g_{\kappa_{c_2}} \rangle (X_{nu}) ndu. \]

and together with martingale inequality, we obtain
\[ \mathbb{E} \left[ |Y_t^{(n)}(\kappa_c) - Y_s^{(n)}(\kappa_c)|^4 \right] \leq C(t-s)^2, \quad 0 < s < t. \]

Therefore, for any fixed \( \epsilon > 0 \), and for any \( t, s \in [\epsilon, 1] \), we have a tightness condition.

1. \( \lim_{A \to \infty} \sup_{n > 0} \mathbb{P}(|\tilde{r}_{nt}(\kappa)| \geq A) = 0 \)
2. \( \lim_{\delta \to 0} \limsup_{n \to \infty} \left( \sup_{t, s \in [\epsilon, 1], |t-s| < \delta} |\tilde{r}_t^{(n)}(\kappa) - \tilde{r}_s^{(n)}(\kappa)| > \rho \right) = 0, \quad \rho > 0. \)

Let \( \tilde{r}_t(c) \) be a limit point of \( \tilde{r}_t^{(n)}(\kappa_c) \). We then have
\[ d\tilde{r}_t(c) = \frac{e(\kappa_0)}{t} dt + \frac{f(\kappa_0)}{\sqrt{t}} Im \left[ e^{2i\Theta_t(c)} dZ_t \right], \quad t > 0 \]
where
\[ e(\kappa) := -\frac{1}{2\kappa} Im \left[ \frac{2i}{2\kappa} \cdot \frac{1}{2} \langle Fg_{\kappa} \rangle \right] = f(\kappa)^2 = \tau(\kappa^2) \]
\[ f(\kappa) := \frac{1}{2\kappa} \sqrt{\frac{[g_{\kappa}, g_{\kappa}]}{2}} = \sqrt{\tau(\kappa^2)} \]
Letting \( B_i^c := Im[e^{2i\Theta_t(c)} Z_t] \), we complete the proof of Lemma 2.2. \( \square \)
We next consider \( \phi_t(c) := \frac{\partial \Theta_t(c)}{\partial c} \) which satisfies

\[
d\phi_t(c) = dt - \sqrt{\frac{\tau(\kappa_0^2)}{t}} dB_t^c \cdot 2\phi_t(c), \quad \phi_0(c) = 0.
\]

Since the joint distribution of \( \tilde{r}_t(c) \) and \( \phi_t(c) \) does not depend on \( c \), we ignore the \( c \)-dependence of \( B_c^t \). They now satisfy

\[
d\tilde{r}_t(c) = \frac{\tau(\kappa_0^2)}{t} dt + \sqrt{\tau(\kappa_0^2)} dB_c^t, \quad t > 0,
\]

\[
d\phi_t(c) = dt - \sqrt{\frac{\tau(\kappa_0^2)}{t}} dB_c^t \cdot 2\phi_t(c), \quad \phi_0(c) = 0.
\]

By the change of variable,

\[
t = t(v) = \exp \left[ \frac{v}{\tau(\kappa_0^2)} \right], \quad v(t) = \tau(\kappa_0^2) \log t, \quad t \in (0, 1), \quad v \in (-\infty, 0)
\]

we have

\[
d\tilde{r}(c) = dv + dB_v,
\]

\[
d\phi = \frac{t}{\tau(\kappa_0^2)} dv - 2\phi \cdot dB_v
\]

so that

\[
\phi_w = \int_{-\infty}^{w} \frac{t(v)}{\tau(\kappa_0^2)} e^{\tilde{r}_v - \tilde{r}_w} dv
\]

which is the same equation satisfied by \( r_t(c), \phi_t(c) \) in DC model derived in [12], except that we have \( v \in (-\infty, 1) \), while \( t \in [0, 1] \) in DC model. As is done in [12], by Girsanov’s formula we have

**Lemma 2.3.** Let \( \epsilon > 0 \). Let \( R(\omega) \) be the distribution of \( \tilde{r}_v(c) \) as an element of \( C[v(\epsilon), v(1)] \). Then, under \( dQ(\omega) := e^{\omega_v - \omega_v(1)} dR(\omega) \) we have

\[
\tilde{r}_s = f^v(s) + \tilde{B}_s, \quad f^v(s) := \sqrt{v} - |s - v|, \quad s \in [v(\epsilon), v(1)]
\]

where \( \tilde{B}_s \) is a Brownian motion.

### 2.3.2. Local Version for the Critical Case

In this section, we prove Theorem 1.2 for \( \alpha = 1/2 \). Let \( \{n_k\}, \tilde{r}_t(c) \) be those in Lemma 2.2.

**Lemma 2.4.** Let

\[
\Lambda_{n,E_0} := \left\{ \lambda \left( \sqrt{E_j(n)} - \sqrt{E_0} \right) \right\}_{j \geq 1}.
\]

Then, we have

\[
\left\{ \left( \lambda, e^{\tilde{r}_n^j(n_k)}(\kappa_\lambda) \right) \mid \lambda \in \Lambda_{n_k,E_0} \right\} \xrightarrow{d} \left\{ \left( \lambda, e^{\tilde{r}_n^j(\lambda)} \right) \mid \lambda \in \text{Sine}_\beta \right\}
\]

\[
\left\{ \left( \lambda, \mu_{n_k}^{(\kappa_\lambda)} \right) \mid \lambda \in \Lambda_{n_k,E_0} \right\} \xrightarrow{d} \left\{ \left( \lambda, \int_0^1 e^{2\tilde{r}_n^j(\lambda) ds} \right) \mid \lambda \in \text{Sine}_\beta \right\}
\]

in the sense of convergence in distribution on the sequences of point processes on \( \mathbb{R} \times C(0, 1) \) and \( \mathbb{R} \times \mathcal{P}(0, 1) \), respectively.
Any limit point $\tilde{r}_t(c)$ of $\tilde{r}_t^{(n)}(\kappa_c)$ satisfies (5) so that they only differ up to constant. Hence, $e^{2\tilde{r}_t(\lambda)} dt/ \int_0^1 e^{2\tilde{r}_s(\lambda)} ds$ is uniquely determined and we do not need to take subsequence anymore which yields

Corollary 2.5.\]
\[
\left\{ \left( \lambda, \mu_{\kappa_c}^{(n)} \right) \middle| \lambda \in \Lambda_{n,E_0} \right\} \overset{d}{\to} \left\{ \left( \lambda, \frac{e^{2\tilde{r}_t(\lambda)} dt}{\int_0^1 e^{2\tilde{r}_s(\lambda)} ds} \right) \middle| \lambda \in \text{Sine}_\beta \right\}.
\]

Proof of Lemma 2.4.

As in the proof of Theorem 1.2 for supercritical case, we use $\xi_n \overset{d}{\to} \xi \iff \xi_n(f) \overset{d}{\to} \xi(f), f \in C_c(\mathbb{R} \times C(0,1))$, and assume $f(\lambda, \phi) = h_1(\lambda) \cdot h_2(\phi), h_1 \in C_c(\mathbb{R}), h_2 \in C(C[0,1])$. Let $\theta_t^{(n)}(\kappa) := \theta_{nt}(\kappa)$ and we recall

\[
\lambda \in \Lambda_{n,E_0} \iff \Theta_1^{(n)}(\lambda) \in \pi \mathbb{Z} - (\theta_1^{(n)}(\kappa_0))_e \mathbb{Z} \\
\lambda \in \text{Sine}_\beta \overset{d}{\iff} \Theta_1(\lambda) \in \pi \mathbb{Z} + U \\
\theta_t^{(n)}(\kappa_0) \overset{d}{\to} U
\]

where $U \sim \text{unif}[0, \pi]$ and independent from $\Theta_1(\lambda)$. Since $\tilde{r}_t^{(n)}(\kappa_c)$ converges to $\tilde{r}_t(c)$ locally uniformly w.r.t. $t \in (0, 1)$ and $c \in \mathbb{R}$, [5] Lemma 3.1 yields

\[
\sum_{\lambda: \Theta_1^{(n)}(\lambda) \in \pi \mathbb{Z} - (\theta_1^{(n)}(\kappa_0))_e \mathbb{Z}} h_1(\lambda) h_2(\tilde{r}_t^{(n)}(\kappa_\lambda)) \overset{d}{\to} \sum_{\lambda: \Theta_1(\lambda) \in \pi \mathbb{Z} - \text{unif}[0, \pi]} h_1(\lambda) h_2(\tilde{r}_t(\lambda))
\]

which implies

\[
\left\{ \left( \lambda, \tilde{r}_t^{(n)}(\kappa_\lambda) \right) \middle| \lambda \in \Lambda_{n,E_0} \right\} \overset{d}{\to} \left\{ \left( \lambda, \tilde{r}_t(\lambda) \right) \middle| \lambda \in \text{Sine}_\beta \right\}.
\]

We note that $\tilde{r}_t^{(n)} \overset{\text{loc.unif}}{\to} \tilde{r}_t$ implies $e^{\tilde{r}_t^{(n)}} \overset{\text{loc.unif}}{\to} e^{\tilde{r}_t}$, and furthermore $e^{\tilde{r}_t^{(n)}} \overset{\text{loc.unif}}{\to} e^{\tilde{r}_t}$ in turn implies $e^{\tilde{r}_t^{(n)}} dt \overset{v}{\sim} e^{\tilde{r}_t} dt$. By using continuous mapping theorem twice, we have

\[
e^{\tilde{r}_t^{(n)}} dt \overset{v}{\sim} e^{\tilde{r}_t} dt
\]

which proves the first statement. The second one is proved similarly. \hfill \Box

We turn to derive the intensity measure of $\Xi_{E_0}$.

Lemma 2.6. For $G \in C_b(\mathbb{R} \times C(0,1))$, one has

\[
\mathbb{E} \left[ \sum_{\lambda \in \text{Sine}_\beta} G(\lambda, \tilde{r}(\lambda)) \right] = \frac{1}{\pi} \int_{\mathbb{R}} d\lambda \mathbb{E} \left[ G(\lambda, B_v(\cdot) + v(U) - |v(\cdot) - v(U)|) \right].
\]

where

\[
v(t) = \tau(n_0^2) \log t, \quad U \sim \text{unif}[0, 1].
\]

Proof. The proof is almost parallel as that given in [12]. In fact,

\[
\mathbb{E} \left[ \sum_{\lambda \in \text{Sine}_\beta} G(\lambda, \tilde{r}(\lambda)) \right] = \mathbb{E} \left[ \int_0^\pi \frac{du}{\pi} \sum_{\lambda: \Theta_1(\lambda) \in \pi \mathbb{Z} + u} G(\lambda, \tilde{r}(\lambda)) \right]
\]

\[
= \mathbb{E} \left[ \frac{1}{\pi} \int_{\mathbb{R}} du \sum_{\lambda: \Theta_1(\lambda) = u} G(\lambda, \tilde{r}(\lambda)) \right]
\]

\[
= \frac{1}{\pi} \int_{\mathbb{R}} d\lambda \mathbb{E} \left[ G(\lambda, \tilde{r}(\lambda)) \left| \frac{\partial \Theta_1(\lambda)}{\partial \lambda} \right| \right]
\]

(16)
where we set \( u = \frac{\partial \Theta_1(\lambda)}{\partial \lambda} d\lambda \). By (15) and Lemma 2.3

\[
E\left[ G(\lambda, \tilde{r}(\lambda)) \left| \frac{\partial \Theta_1(\lambda)}{\partial \lambda} \right. \right] = \lim_{\epsilon \downarrow 0} \int_{v(e)}^{v(1)} \frac{t(v)}{\tau(E_0)} E\left[ G(\lambda, \tilde{r}(\lambda)) \exp \left( \tilde{r}_v - \tilde{r}_{v(1)} \right) \right] dv
\]

\[
= \lim_{\epsilon \downarrow 0} \int_{v(e)}^{v(1)} \frac{t(v)}{\tau(E_0)} E\left[ G(\lambda, B_+ + v - |v|) \right] dv
\]

\[
= \lim_{\epsilon \downarrow 0} \int_{v(e)}^{v(1)} E\left[ G(\lambda, B_{v(\cdot)} + v(t) - |v(\cdot) - v(t)|) \right] dt
\]

Substituting this equation into (16) yields the conclusion.

To drive the intensity measure of \( \Xi_{E_0} \), let \( G \in C_b(\mathbb{R} \times \mathcal{P}(0, 1)) \). By

\[
\int G(\lambda, \mu) d\Xi_{E_0}(\lambda, \mu) = \sum_{\lambda \in \text{Sine}_\beta} G\left( \lambda, \frac{\exp \left[ 2(B_{v(\cdot)} + v(U) - 2|v| - v(U)) \right]}{\int_0^1 \exp \left[ 2(B_{v(s)} + v(U) - 2|v| - v(U)) \right] ds} \right)
\]

and by Lemma 2.6, we have

\[
E\left[ \int G(\lambda, \mu) d\Xi_{E_0}(\lambda, \mu) \right]
\]

\[
= \frac{1}{\pi} \int_{\mathbb{R}} d\lambda E\left[ G\left( \lambda, \frac{\exp \left[ 2(B_{v(\cdot)} + v(U) - 2|v| - v(U)) \right]}{\int_0^1 \exp \left[ 2(B_{v(s)} + v(U) - 2|v| - v(U)) \right] ds} \right) \right].
\]

Here, we cancel \( v(U) \) out and use \( B_t \overset{d}{=} Z_{t-v(U)} + \text{(random constant)} \) yielding

\[
= \frac{1}{\pi} \int_{\mathbb{R}} d\lambda E\left[ G\left( \lambda, \frac{\exp \left[ 2(Z_{v(\cdot)} - v(U)) - 2|v| - v(U)) \right]}{\int_0^1 \exp \left[ 2(Z_{v(s)} - v(U)) - 2|v| - v(U)) \right] ds} \right) \right]
\]

where \( Z \) is a two-sided Brownian motion. Together with Lemma 2.4, we complete the proof of Theorem 2.1 for \( \alpha = 1/2 \).

2.4. Subcritical Case

Let \( \{E_j(n)\} \) be the positive eigenvalues of \( H_n \), let \( \psi_{E_j(n)} \) be the normalized eigenfunction corresponding to \( E_j(n) \), and let \( x_j(n) \) be a maximal point of \( |\psi_{E_j(n)}(x)|^2 \). Since \( \psi_{E_j(n)} \) satisfies the sub-exponential decay estimate: \( |\psi_{E_j(n)}(x)| \leq C \exp \left[ -D|x - x_j|^\gamma \right] \), \( \gamma := 1 - 2\alpha \), maximal points of \( |\psi_{E_j(n)}| \) have a same limit point when they are divided by \( n \), so that we have no ambiguity in choosing \( x_j(n) \), which we call the localization center of \( E_j(n) \). Let \( \xi_n \) be the point process of pairs of rescaled eigenvalues and corresponding localization centers, and let \( \xi \) be a Poisson process.

\[
\xi_n := \sum_j \delta_{(n(\sqrt{E_j - \mathbb{E}_0}), x_j/n)} \quad \xi := \sum_j \delta_{(\tilde{p}_j, \tilde{p}_j)} \sim \text{Poisson}(d\lambda/\pi \times 1_{[0,1]}(x)dx).
\]

Theorem 1.2 for \( \alpha < 1/2 \) will follow from the following Proposition.

**Proposition 2.7** . For any bounded intervals \( I(\subset \mathbb{R}) \), \( B = [a, b](\subset (0, 1)) \), we have

\[
(1) \quad \lim_n P(\xi_n(I \times B) = 0) = P(\xi(I \times B) = 0)
\]

\[
(2) \quad \lim_n \mathbb{E}[\xi_n(I \times B)] = \mathbb{E}[\xi(I \times B)].
\]

Furthermore, \( \xi_n \overset{d}{\to} \xi \).
This result was expected to hold true in [6]. For proof, we take $0 < \delta < 1$ and let $C$ (resp. $D$) be an interval by eliminating (resp. adding) a small interval of width $n^\delta$ from (resp. to) $nB := n[a,b]$.  

\[ C := [an + n^\delta, bn - n^\delta], \quad D := [an - n^\delta, bn + n^\delta]. \]

Let $H_C := H_n|_C$, $H_D := H_n|_D$ with Dirichlet boundary condition, and let $\xi_n^C$, $\xi_n^D$ be the point processes such that $E_j(n)$ in the definition of $\xi_n$ is replaced by the eigenvalues $E_j^C(n)$, $E_j^D(n)$ of $H_C$, $H_D$ respectively. By a localization argument, for an bounded interval $I(\subset \mathbb{R})$, we can find intervals $I'$, $I''$ such that $I' = I - O(\exp[-(\text{const.})n^\delta])$, $I'' = I + O(\exp[-(\text{const.})n^\delta])$ with 

\[
\xi_n^C(I' \times [0,1]) \leq \xi_n(I \times B) \leq \xi_n^D(I'' \times [0,1]).
\]

In fact, as is discussed in [2,8] for instance, for each eigenvalues of $H_C$ in $I$, by smoothing argument near the boundary, the corresponding eigenfunction becomes an approximate eigenfunction of $H_n$ so that $H_n$ has eigenvalues in $I + O(\exp[-(\text{const.})n^\delta])$ with those localization centers in $B$. Moreover, for each eigenvalues of $H_n$ in $I$ localized in $B$, by cutting off argument we can construct approximate eigenfunctions of $H_D$ with eigenvalues in $I + O(\exp[-(\text{const.})n^\delta])$. On the other hand, for the eigenvalue process of $H_C$, $H_D$, we have the Poisson statistics as for $H_n$ proved in [6]. That is, the point processes $\eta_n^C, \eta_n^D$ whose atoms are composed of the rescaled eigenvalues of $H_C, H_D$ respectively converge to Poisson $(|B|d\lambda/\pi)$. In fact, the key to the proof for $H_n$ is that the jump point of the processes $[\Theta_t(\lambda)^{(n)}(c)/\pi]$ and $[\Theta_t(\lambda)^{(n)}(c') - \Theta_t(\lambda)^{(n)}(c)/\pi]$ converge to Poisson processes and they are asymptotically independent(Proposition 5.7, Remark 5.1 and Lemma 5.11 in [6]). And we can show the same statement for the processes $\Theta_{\delta}^{(n)}(\lambda)$, $\xi = C, D$ where the starting time 0 in $\Theta_t(\lambda)^{(n)}$ is replaced by $an \pm n^\delta$ which satisfy the same SDE as for $\Theta_t^{(n)}(\lambda)$. Then, we can show

**Lemma 2.8**. For $\xi = C, D,$

1. \[\lim_n P(\xi_n^C(I \times [0,1])) = 0 = P(\xi(I \times B) = 0)\]
2. \[\lim_n E[\xi_n^C(I \times [0,1])] = \frac{|I| \cdot |B|}{\pi} = E[\xi(I \times B)].\]

Now, letting $n \to \infty$ in

\[ P(\xi_n^D(I'' \times [0,1])) = 0 \leq P(\xi_n(I \times B) = 0) \leq P(\xi_n^C(I' \times [0,1]) = 0), \]

we have

\[ \lim_n P(\xi_n(I \times B) = 0) = P(\xi(I \times B) = 0) \]

and similarly we have

\[ \lim_n E[\xi_n(I \times B)] = E[\xi(I \times B)] = \frac{|I| \cdot |B|}{\pi} \]

yielding Proposition 2.7(1), (2). Therefore, by [3] Theorem 4.7, $\xi_n$ converges to a Poisson process whose intensity measure is equal to $d\lambda/\pi \times 1_{[0,1]}(x)\, dx$.

**Remark 1** It is sufficient to show $\lim \sup E[\xi_n \times B] \leq E[\xi \times B]$ for the proof of Proposition 2.7 but we will need equality later for the proof of Theorem 1.1.

**Remark 2** The argument of proof of Proposition 2.7 is almost model-independent. If (i) eigenfunctions are exponentially localized, and (ii) if any subsystm of size $O(n)$ we have Poisson statistics for the eigenvalue process, then we have the Poisson convergence for the pairs of eigenvalues and localization centers.

Theorem 1.2 for $\alpha < 1/2$ follows easily from Proposition 2.7.
Proof of Theorem 1.2 for $\alpha < 1/2$

As the other cases, we show $\int F(\lambda, \mu) d\Xi^{(n)}(\lambda, \mu) \xrightarrow{d} \int F(\lambda, \mu) d\Xi(\lambda, \mu)$ for $F \in C_c(\mathbb{R} \times \mathcal{P}(0, 1))$. By Proposition 2.7, we can assume that these atoms $(n(\sqrt{E_j} - \sqrt{E_0}), x_j/n)$ of $\xi_n$ converges in distribution to those $(P_j, \tilde{P}_j)$ of $\xi$. Then, by Lemma 2.9 below, we have

$$\mu^{(n)}_{E_j(n)} \xrightarrow{d} \delta_{\tilde{P}_j}$$

so that for $F \in C_c(\mathbb{R} \times \mathcal{P}(0, 1))$,

$$\sum_j F\left(n\left(\sqrt{E_j(n)} - \sqrt{E_0}\right), \mu^{(n)}_{E_j(n)}\right) \xrightarrow{d} \sum_j F\left(P_j, \delta_{\tilde{P}_j}\right).$$

For the intensity measure of $\Xi$, we use the fact that $\{(P_j, \tilde{P}_j)\} \sim \text{Poisson}(d\lambda/\pi \times 1_{[0,1]}(x)dx)$ and yield

$$E\left[\int F(\lambda, \mu) d\Xi_E\right] = E\left[\sum_j F(P_j, \delta_{\tilde{P}_j})\right] = \int d\lambda E[F(\lambda, \delta_U), \ U \sim \text{unif } [0, 1]].$$

Lemma 2.9 Suppose $(n(\sqrt{E_j} - \sqrt{E_0}), x_j/n) \xrightarrow{d} (P_j, \tilde{P}_j)$. Then, we have

$$\mu^{(n)}_{E_j(n)} \xrightarrow{d} \delta_{\tilde{P}_j}.$$

Proof. Let $\xi = \sum_j \delta_{(P_j, \tilde{P}_j)}$, $(P_j, \tilde{P}_j) \in \mathbb{R} \times [0, 1]$ be a Poisson process in Proposition 2.7. Then, by Skorohod’s theorem, we may assume

$$\left(n\left(\sqrt{E_j(n)} - \sqrt{E_0}\right), \frac{x_j^{(n)}}{n}\right) \xrightarrow{a.s.} \left(P_j, \tilde{P}_j\right).$$

(19)

For simplicity, we write $x_j := x_j^{(n)}$. We represent the normalized eigenfunction $\psi_j^{(n)}$ around its localization center $x_j$ as $\psi_j^{(n)}(x) =: g_j^{(n)}(x - x_j)$. For $f \in C_c(\mathbb{R})$,

$$\int_{\mathbb{R}} f(x) |\psi_j^{(n)}(nx)|^2 dx = \int_{\mathbb{R}} f\left(\frac{x_j}{n} + \frac{y}{n}\right) |g_j^{(n)}(y)|^2 dy \quad (y = nx - x_j).$$

Here, we use the following estimate: For any $\epsilon > 0$ we can find $R_\epsilon > 0$ such that

$$\int_{|y| \geq R_\epsilon} |g_j^{(n)}(y)|^2 dy < \epsilon.$$

In fact, since we only consider eigenvalues $E_j(n)$ in the $O(n^{-1})$-neighborhood of $E_0$, we can bound $|g_j^{(n)}(x)| \leq C(E_0) \exp[-D(E_0)|x|^{1-2\alpha}]$ so that $R_\epsilon$ can be taken uniformly w.r.t. $n, j$. We then have

$$\left|\int_{\mathbb{R}} f(x) |\psi_j^{(n)}(nx)|^2 dx - f(\tilde{P}_j)\right| = \left|\int_{\mathbb{R}} \left\{ f\left(\frac{x_j}{n} + \frac{y}{n}\right) - f(\tilde{P}_j)\right\} |g_j^{(n)}(y)|^2 dy\right|

\leq \sup_{|y| \leq R_\epsilon} \left| f\left(\frac{x_j}{n} + \frac{y}{n}\right) - f(\tilde{P}_j)\right| + 2\|f\|_{L^\infty} \epsilon.$$
Since $f$ is uniformly continuous,
\[
\sup_{|y| \leq R_n} \left| f \left( \frac{x_j}{n} + \frac{y}{n} \right) - f(\tilde{P}_j) \right| \leq \sup_{|y| \leq R_n} \left| f \left( \frac{x_j}{n} + \frac{y}{n} \right) - f \left( \frac{x_j}{n} \right) \right| + \sup_{|y| \leq R_n} \left| f \left( \frac{x_j}{n} \right) - f(\tilde{P}_j) \right|
\]
\[\xrightarrow{n \to \infty} 0.\]

\section{3. Global Version}

The result for the global version (the statement in Theorem 1.1) follows from that for local version (statement in Theorem 1.2) by a general argument. Following \cite{12}, we introduce
\[
g_1(x) := (1 - |x|)1(|x| \leq 1),
\]
\[
G_n(E) := \sum_{E_j(n) \in J} g_1 \left( n \left( \sqrt{E_j(n)} - \sqrt{E} \right) + \theta \right) \cdot g_2 \left( E_j(n), \mu_{E_j(n)}^{(n)} \right)
\]
where $g_2 \in C_b(\mathbb{R} \times \mathcal{P}(0,1))$. $\theta \sim \text{unif}[0, \pi]$ for $\alpha > \frac{1}{2}$, and $\theta = 0$ otherwise. For the global version, we need to consider $\sum_{E_j \in J} g_2 \left( E_j(n), \mu_{E_j(n)}^{(n)} \right)$. The motivation to consider $G_n(E)$ is to localise this quantity around the reference energy $E$ by multiplying $g_1$. We compute $\int_J dN(E)/N(J) \int d \mathcal{P} G_n(E)$ in two ways by exchanging the order of integrals, and then equate them by the Fubini theorem, which yields the conclusion.

The idea behind this argument is:

1. Integrate w.r.t. $dN(E)/N(J)$ and then take expectation: We first integrate w.r.t. the reference energy around each $E_j$’s of width of order $O(n^{-1})$ which results in to get $n^{-1}$ factor, yielding the quantity we want to compute.
2. Take expectation first and then integrate w.r.t. $dN(E)/N(J)$: We first fix the reference energy, and take expectation first. Since we have $g_1$ factor, we have the intensity measure of the local version. Then, integrating w.r.t. the reference energy gives us the answer.

Therefore, a general principle is that, the answer to our global problem is equal to the integral w.r.t. the reference energy of the intensity measure of the local problem.

Along the idea explained above, we compute $\int_J dN(E)/N(J) \mathbb{E}[G_L(E)]$, $J = [a, b]$ in two ways. We first note that $|n(\sqrt{E_j} - \sqrt{E}) + \theta| \leq 1$ if and only if $|\sqrt{E_j} - \sqrt{E}| \leq (\pi + 1)/n$. Since $\sqrt{E_j} \in (\sqrt{a}, \sqrt{b})$, $a > 0$, we have
\[
\int_J g_1(n \left( \sqrt{E_j} - \sqrt{E} \right) + \theta)dN(E) = \frac{1}{n\pi}.
\]

1. We first integrate w.r.t. $dN(E)$ and then take expectation:
\[
\mathbb{E} \left[ \int_J \frac{dN(E)}{N(J)} G_n(E) \right] = \mathbb{E} \left[ \frac{1}{N(J)} \frac{1}{\pi n} \sum_{E_j(n) \in J} g_2 \left( E_j(n), \mu_{E_j(n)}^{(n)} \right) \right]
\]
\[= \mathbb{E} \left[ \frac{1}{N(H_n, J)} \frac{1}{\pi} \cdot \sum_{E_j(n) \in J} g_2 \left( E_j(n), \mu_{E_j(n)}^{(n)} \right) \right] + o(1). \quad (20)
\]
where we set $N(H_n, J) := \sharp \{\text{eigenvalues of } H_n \text{ in } J \}$. The last equality follows from

$$
\mathbb{E} \left[ \left( \frac{1}{N(H_n, J)} - \frac{1}{N(J)n} \right) \frac{1}{\pi} \sum_{E_j \in J} g_2(E_j, \mu_{E_j}^{(n)}) \right] = o(1). \tag{21}
$$

To show (21), we note that the quantity in the expectation is estimated as

$$
\left| \frac{1}{N(H_n, J)} - \frac{1}{N(J)n} \right| \left| \frac{1}{\pi} \sum_{E_j \in J} \left| g_2(E_j, \mu_{E_j}^{(n)}) \right| \right|
\leq \left| \frac{1}{N(H_n, J)} - \frac{1}{N(J)n} \right| \frac{1}{\pi} \|g_2\|_\infty N(H_n, J)
= \frac{|N(J) - n^{-1}N(H_n, J)|}{N(J)} \frac{1}{\pi} \|g_2\|_\infty
$$

which converges to 0 a.s. by the definition of $N(J)$. On the other hand, since we have

$$
N(H_n, J) \leq \frac{\theta_n(\sqrt{b}) - \theta_n(\sqrt{a})}{\pi}
$$

and since, by examining the integral equation (7) satisfied by $\tilde{\theta}_t(\kappa)$, we have

$$
\theta_n(\sqrt{b}) - \theta_n(\sqrt{a}) \leq Cn
$$

for some deterministic constant $C$, (21) follows from the bounded convergence theorem.

(2) We first expectation and then integrate w.r.t. $dN(E)$:

$$
\frac{1}{N(J)} \int_J dN(E) \mathbb{E}[G_n(E)]
= \frac{1}{N(J)} \int_J dN(E) \mathbb{E} \left[ \sum_{E_j \in J} g_1 \left( n \left( \sqrt{E_j} - \sqrt{E} \right) + \theta \right) g_2(E_j, \mu_{E_j}^{(n)}) \right]
= \frac{1}{N(J)} \int_J dN(E) \mathbb{E} \left[ \sum_{E_j \in J} g_1 \left( n \left( \sqrt{E_j} - \sqrt{E} \right) + \theta \right) \left( g_2(E, \mu_{E_j}^{(n)}) + o(1) \right) \right]
= \frac{1}{N(J)} \int_J dN(E) \mathbb{E} \left[ \sum_{E_j \in J} g_1 \left( n \left( \sqrt{E_j} - \sqrt{E} \right) + \theta \right) \left( g_2(E, \mu_{E_j}^{(n)}) \right) \right] + o(1)
= \frac{1}{N(J)} \int_J dN(E) \mathbb{E} \left[ \int g_1(\lambda + \theta)g_2(E, \mu)d\mathcal{E}_E^{(n)}(\lambda, \mu) \right] + o(1).
$$

where, in the second equality, we used that fact that $E_{j}(n) \to E$ for $j$’s such that $n \left( \sqrt{E_j} - \sqrt{E} \right) + \theta \in \text{supp } g_1$. For the third equality, we used the fact that

$$
\sum_{E_j \in J} g_1 \left( n \left( \sqrt{E_j} - \sqrt{E} \right) + \theta \right) g_2(E_j, \mu_{E_j}^{(n)}) = \sum_{E_j \in J} g_1 \left( n \left( \sqrt{E_j} - \sqrt{E} \right) + \theta \right) g_2(E, \mu_{E_j}^{(n)})
$$

are both uniformly integrable w.r.t. $dN(E) \times \mathbb{P}$ by the argument in “Section 2 in Appendix.”
Since $\Xi^{(n)}_E \xrightarrow{d} \Xi_E$ by Theorem 1.2, and since $\{G_n(E)\}_n$ is uniformly integrable w.r.t. $dN \times P$ to be shown in “Section 2 in Appendix,” we have

$$\int J dN(E) \frac{1}{N(J)} \mathbb{E}[G_n(E)]$$

$$\xrightarrow{n \to \infty} \int J dN(E) \frac{1}{N(J)} \mathbb{E}\left[\int g_1(\lambda + \theta)g_2(E, \mu) d\Xi_E(\lambda, \mu)\right]$$

$$= \int J dN(E) \frac{1}{N(J)} \pi$$

$$\begin{cases}
\int J d\lambda g_1(\lambda) \mathbb{E}\left[g_2(E, 1_{[0,1]}(t) dt)\right] & (\alpha > 1/2) \\
\int J d\lambda g_1(\lambda) \mathbb{E}\left[g_2\left(E, \frac{\exp\left(2Z_\tau(E) \log \frac{\tau}{t}\right)}{\int_0^1 \exp\left(2Z_\tau(E) \log \frac{\tau}{t}\right) dt} \right] \right) & (\alpha = 1/2) \\
\int J d\lambda g_1(\lambda) \mathbb{E}\left[g_2(E, \delta U)\right] & (\alpha > 1/2)
\end{cases}$$

$$= \int J dN(E) \frac{1}{N(J)} \pi$$

$$\begin{cases}
\mathbb{E}\left[g_2\left(E, 1_{[0,1]}(t) dt)\right]\right] & (\alpha > 1/2) \\
\mathbb{E}\left[g_2\left(E, \frac{\exp\left(2Z_\tau(E) \log \frac{\tau}{t}\right)}{\int_0^1 \exp\left(2Z_\tau(E) \log \frac{\tau}{t}\right) dt} \right] \right) & (\alpha = 1/2) \\
\mathbb{E}\left[g_2(E, \delta U)\right] & (\alpha > 1/2)
\end{cases}$$

(22)

(20), (22) yield the statement of Theorem 1.1. \qed

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4. Appendix

4.1. Statement for DC Model

We consider the continuum one-dimensional operator with decaying coupling constant, that is,

$$H_{\alpha, n} = -\frac{d^2}{dt^2} + n^{-\alpha} F(X_t), \quad \text{on } L^2(0, n).$$

with Dirichlet boundary condition. Then, we have the corresponding results for Theorems 1.1, 1.2 which we state here. The conclusion for the critical case is essentially the same as that in Rifkind-Virag [12]. Since the proof is similar for those of Theorems 1.1, 1.2, we omit details.
Theorem 4.1

\[ (E_J^{(n)} , \mu_{E_J^{(n)}}^{(n)}) \]

\[ d \rightarrow \begin{cases} (E_J, 1_{[0,1]}(t)dt) & (\alpha > 1/2) \\ \exp \left( 2Z_{\tau(E_J)(s-U)} - 2|\tau(E_J)(t-U)| \right)_{ds} & (\alpha = 1/2) \\ (E_J, \delta_{unif}[0,1](dt)) & (\alpha < 1/2) \end{cases} \]

For the Local version, we set

\[ \Xi_{E_0}^{(n)} := \sum_j \delta \left( n(\sqrt{E_J^{(n)}} - \sqrt{E_0}) + \theta, \mu_{E_J^{(n)}}^{(n)} \right), \]

where \( \theta \sim \text{unif}[0, \pi) \) for \( \alpha \geq \frac{1}{2} \), \( \theta = 0 \) for \( \alpha < \frac{1}{2} \).

Theorem 4.2 \( \Xi_{E_0}^{(n)} = d \Xi_{E_0} \), where

\[ \Xi_{E_0} = \begin{cases} \sum_{j \in \mathbb{Z}} \delta_j \pi + \theta \otimes \delta_1 \left( 1_{[0,1]}(t)dt \right) & (\alpha > 1/2) \\ \sum_{\lambda : \text{Sch}} \delta_\lambda \otimes \delta \left( \exp(2\tilde{r}_\lambda(t))_{dt} \right) & (\alpha = 1/2) \\ \sum_{j \in \mathbb{Z}} \delta P_j \otimes \delta P_j & (\alpha < 1/2) \end{cases} \]

where for \( \alpha > 1/2 \), \( \theta \sim \text{unif}[0, \pi) \). For \( \alpha = 1/2 \),

\[ \text{Sch}^* := \{ \lambda \in \mathbb{R} | \Psi_1(\lambda) \in 2j\pi + \text{unif}[0, 2\pi), j \in \mathbb{Z} \}, \]

and \( \{ \Psi_t(\lambda) \} \) is an increasing function-valued process given in Eq. (1.2) in [9]. \( \tilde{r}_t(\lambda) \) is characterized by the solution to the following equation:

\[ d\tilde{r}_t(\lambda) = \tau(E_0)dt + \sqrt{\tau(E_0)}dB_t^\lambda, \quad t > 0 \]

where \( \{ B_t^\lambda \}_\lambda \) is a family of Brownian motion. For \( \alpha < 1/2 \), \( \{ P_j \} : \text{Poisson}(d\lambda/\pi), \{ P_j \} : \text{Poisson}(1_{[0,1]}(t)dt) \) where Poisson(\( \mu \)) is the Poisson process with intensity measure \( \mu \). The intensity measure of \( \Xi_{E_0} \) is given by

\[ \mathbb{E} \left[ \int G(\lambda, \nu)d\Xi_{E_0}(\lambda, \nu) \right] \]

\[ = \frac{1}{\pi} \begin{cases} \int d\lambda \mathbb{E} \left[ G(\lambda, 1_{[0,1]}(t)dt) \right] & (\alpha > 1/2) \\ \int d\lambda \mathbb{E} \left[ G(\lambda, \exp(2Z_{\tau(E_0)(t-U)} - 2|\tau(E_0)(s-U)|_{ds}) \right) & (\alpha = 1/2) \\ \int d\lambda \mathbb{E} \left[ G(\lambda, \delta U) \right] & (\alpha > 1/2) \end{cases} \]

where \( U := \text{unif}[0, 1] \).

4.2. Uniform Integrability

In this subsection, we show the uniform integrability of \( G_n(E) \) w.r.t. \( dN \times P \). Since \( \text{supp } g_1 \subset \{|\lambda| \leq 1\} \), by setting \( N(H_n, J) := \{ \text{eigenvalues of } H_n \text{ in } J \} \), we have for \( c \geq 1 \),

\[ |G_n(E)| \leq \| g_1 \|_\infty \| g_2 \|_\infty N \left( H_n, \sqrt{E_0} + \frac{1}{n}(-c, c) \right) \]

\[ \leq \| g_1 \|_\infty \| g_2 \|_\infty \left( \frac{1}{\pi} \left( \Theta^{(n)}_t(c) - \Theta^{(n)}_t(-c) \right) \right) \tag{23} \]
so that it suffices to show the uniform integrability of \( \{ \Theta^{(n)}_t(c) \} \) w.r.t. \( dN \times P \), which in turn follows from either one of the following two statements:

1. \[
\int_j dN(E)E[\Theta^{(n)}_t(c)] \rightarrow \int dN(E)E[\Theta_t(c)]
\]

2. \[
\sup_n \int_j dN(E)E \left[ \Theta^{(n)}_t(c)^{1+\delta} \right] \text{ for some } \delta > 0
\]

where we note that \( \Theta_t(c) \geq 0 \) for \( c \geq 0 \). We shall show (24) or (25) in Sects. 4.2.1, 4.2.2 for supercritical and critical cases respectively. For subcritical case, we can show the uniform integrability directly to be done in Sect. 4.2.3.

### 4.2.1. Supercritical Case

We show (25) in supercritical case. By definition,

\[
\Theta^{(n)}_t(c) = ct + \tilde{\theta}_{nt}(\kappa_c) - \tilde{\theta}_{nt}(\kappa)
\]

and we write \( \kappa := \kappa_0 \) in this section. By the integral Eq. (7) satisfied by \( \tilde{\theta}_t(\kappa_c) \),

\[
\tilde{\theta}_{nt}(\kappa_c) - \tilde{\theta}_{nt}(\kappa) = \frac{1}{2\kappa} \text{Re} \left( J^{(n)}_t(\kappa_c) - J^{(n)}_t(\kappa) \right) + \frac{-2 \cdot \frac{\kappa}{2\kappa_c \cdot 2\kappa}}{2\kappa_c \cdot 2\kappa} \int_0^{nt} \text{Re} \left( e^{2i\theta_s(\kappa_c)} - 1 \right) a(s)F(X_s)ds.
\]

where \( J^{(n)}_t(\kappa_c) := \int_0^{nt} a(s)e^{2i\theta_s(\kappa_c)}F(X_s)ds \)

The second term goes to 0 as \( n \to \infty \) uniformly w.r.t. \( (\kappa, \omega) \), so that it suffices to show the uniform integrability of \( J^{(n)}_t(\kappa_c) \) for any \( c \geq 0 \). We use “Ito’s formula”

\[
e^{2i\kappa s}F(X_s)ds = d\left( e^{2i\kappa s}g_\kappa(X_s) \right) - e^{2i\kappa s}\nabla g_\kappa(X_s)dX_s
\]

and compute the integral by parts:

\[
J^{(n)}_t(\kappa_c) = \left[ a(s)e^{2i\theta_s(\kappa_c)}g_\kappa(X_s) \right]_0^{nt} - \int_0^{nt} a'(s)e^{2i\theta_s(\kappa_c)}g_\kappa(X_s)ds
\]

\[
- \frac{2i}{2\kappa_c} \int_0^{nt} \text{Re} \left( e^{2i\theta_s(\kappa_c)} - 1 \right) e^{2i\theta_s(\kappa_c)}a(s)^2F(X_s)g_\kappa(X_s)ds
\]

\[
-2i \cdot \frac{c}{n} \int_0^{nt} a(s)e^{2i\theta_s(\kappa_c)}g_\kappa(X_s)ds
\]

\[
- \int_0^{nt} a(s)e^{2i\theta_s(\kappa_c)}\nabla g_\kappa(X_s)dX_s
\]

\[= J_1 + \cdots + J_5.\]

Here, we use the notation \( O(1) \) if the quantity in question is uniformly bounded w.r.t. \( (\kappa, \omega) \in J \times \Omega \). Then, we have

\[J_1 = O(1)\]

\[|J_2| \leq \int_0^{nt} a'(s) \left| e^{2i\theta_s(\kappa_c)}g_\kappa(X_s) \right| ds \leq (\text{Const.}) \int_0^{nt} a'(s)ds = O(1)\]

\[|J_3| \leq (\text{Const.}) \int_0^{nt} a(s)^2 ds = O(1)\]

\[|J_4| \leq (\text{Const.}) \frac{1}{n} \int_0^{nt} a(s)ds = O(n^{-\alpha})\]

\[|J_5|^2 \leq (\text{Const.}) \int_0^{nt} a(s)^2 ds = O(1).\]
Getting together, we have
\[ \sup_n \int_J dN(E) E \left[ |J_t^{(n)}(\kappa_c)|^2 \right] < \infty. \]

4.2.2. Critical Case. We show (24) for the critical case. In fact, [5] Lemma 6.3 says
\[ \Theta_t^{(n)}(c) = 2ct + \text{Re} \epsilon_t^{(n)} + \frac{1}{\kappa} \text{Re} V_t^{(n)}(c) + \frac{1}{\kappa} \text{Re} (\delta_{nt}(\kappa_c) - \delta_{nt}(\kappa)) \]
where \[ |\epsilon_t^{(n)}| \leq \frac{C}{n} \int_0^{nt} a(s) ds \xrightarrow{n \to \infty} 0 \]
\[ V_t^{(n)}(c) : \text{Martingale so that } E[V_t^{(n)}(c)] = 0 \]
\[ E\left[ \max_{0 \leq t \leq T} |\delta_{nt}(\kappa_c) - \delta_{nt}(\kappa)|^2 \right] \xrightarrow{n \to \infty} 0 \]
(26)
which implies \[ E[\Theta_t^{(n)}(c)] \xrightarrow{n \to \infty} E[\Theta_1(c)] = 2ct. \] Moreover, by examining the proof of Lemma 6.3 in [5], the LHS of (26) is locally bounded w.r.t. \( E \), and so is \( E[\Theta_t^{(n)}(c)] \). By the bounded convergence theorem, we now have
\[ \int_J dN(E) E[\Theta_t^{(n)}(c)] \to \int_J dN(E) E[\Theta_1(c)] = 2ct. \]

4.2.3. Subcritical Case. We show the uniform integrability directly for the subcritical case. By (23), it suffices to show the uniform integrability of \( N(H_n, \sqrt{E_0} + \frac{1}{n} (-c, c), E) \) \( \xi_n(((-c, c) \times [0, 1]) \). In Sect. 2, it has been shown that \( \lim_n E[\xi_n(I \times B)] = E[\xi(I \times B)] \). The quantities in LHS are all locally bounded for \( E \). In fact, \( \xi_n(I \times B) \) is governed by the number of jump points of \( t \mapsto \Theta_{nt}(c)/\pi \), and the SDE satisfied by \( \Theta_t(c) \) is determined by \( E \) and \( \langle F g \sqrt{E} \rangle \) only, and \( \langle F g \sqrt{E} \rangle \) is bounded for \( E \in J \). Here, we used the condition that the left end \( a \) of the interval \( J \) is positive. Then, by the bounded convergence theorem, we have
\[ \lim_{n \to \infty} \int_J dN(E) E[\xi_n I \times B] = \int_J dN(E) E[\xi I \times B], \quad J = [a, b]. \]

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