Two-parameter twisted quantum affine algebras

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ABSTRACT. We establish the Drinfeld realization for the two-parameter twisted quantum affine algebras using a new method. The Hopf algebra structure for Drinfeld generators is given for both untwisted and twisted two-parameter quantum affine algebras, which include the quantum affine algebras as special cases.

1. Introduction

Drinfeld realization \([\text{Dr}]\) is a quantum loop algebra realization of the quantum affine algebra. It was introduced in studying finite dimensional representations of quantum affine algebras, and has since played an important role in representation theory such as in vertex representations \([\text{FJ}, \text{J1}]\) and in finite dimensional representations of quantum affine algebras for quiver varieties \([\text{GKV, N}]\).

Drinfeld realization was first proved by Beck \([\text{B}]\) using Lusztig’s braid group actions \([\text{L}]\). One of us \([\text{J2}]\) also gave an elementary proof of the untwisted quantum affine algebras using \([q]\)-commutators. Subsequently both of these methods are generalized to twisted quantum affine algebras in \([\text{ZJ, JZ3, JZ4}]\) using the Hopf algebraic structures and braid groups.

Two-parameter quantum enveloping algebras are introduced as a generalization of the one-parameter quantum enveloping algebras \([\text{T, BW, BGH1, BGH2}]\). It was known that the theory has analogous properties with the one-parameter counterpart such as a similar Schur-Weyl duality and the Drinfeld double \([\text{BW}]\). Recent advances on geometric representations \([\text{FL}]\) have realized the two-parameter quantum groups naturally, where the second parameter is closely associated with the Tate twist.

Two-parameter quantum enveloping algebras are known to have a generalized root space structure where the action of Lusztig’s braid groups is not closed, but a Weyl groupoid action sends \([U_{r,s}(g)]\) to \([U'_{r',s'}(g)]\) \([\text{H}]\). Therefore the Drinfeld realizations of two-parameter quantum enveloping algebras can not be studied by the braid group action.

The goal of this paper is twofold. First we extend the \([q]\)-commutator approach \([\text{J2}]\) to derive Drinfeld realizations for all twisted two-parameter quantum affine algebras using a new method, and we establish the isomorphism between the Drinfeld realization and the Drinfeld-Jimbo form of the two-parameter quantum enveloping algebras. We use a simple set of generators to replace the original full set of the generators in the quantized algebra,

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and this enables us to simplify many computations involved with Drinfeld generators and Drinfeld-Jimbo generators. Thus the current work in two parameter cases contains a brand new proof of the Drinfeld realization for quantum affine algebras as a special case.

Secondly, our new method gives explicit formulae for the Hopf algebra structure of $U_{r,s}(\hat{g})$ in terms of the Drinfeld generators. It was recently announced by Guay and Nakajima that the Yangian algebra has a simple Hopf algebra structure in terms of the Drinfeld generators. The special case of our result for the quantum affine algebra will also induce a Hopf algebra structure for the Yangian algebra as well in view of $[GT]$.

The paper is organized as follows. After a quick introduction of preliminaries in Section 2, we give the definition of two-parameter twisted quantum affine algebras in Drinfeld-Jimbo form in Section 3. The loop algebra formulation of the two-parameter twisted quantum affine algebras was given in Section 4. We obtain in Section 5 that the Drinfeld realization is isomorphic to Drinfeld-Jimbo form as associative algebras. This isomorphism is proved using a new method in Sections 6 and 7, which is based on a set of simple generators. In section 8, we define the actions of a commultiplication on the simple generators of Drinfeld realization, thus we can obtain the Hopf algebra structure of Drinfeld realization. Furthermore, we announce that there exists a Hopf algebra isomorphism between the above two realizations.

2. Definitions and Preliminaries

2.1. Finite order automorphisms of $g$. We begin with a brief review of basic terminologies and notations of twisted affine Lie algebras following $[K]$, in particular, on finite order automorphisms of the finite dimensional simple Lie algebra.

Let $g$ be a simple finite-dimensional Lie algebra with a Cartan matrix $A = (A_{ij}), (i, j \in \{1, 2, \ldots, N\}$, of simply laced type $A_N (N \geq 2), D_N (N > 4), E_6$ or $D_4$. Let $\sigma$ be an Dynkin diagram automorphism of $g(A)$ of order $k$. We denote by $I = \{1, 2, \ldots, n\}$ the set of $\sigma$-orbits on $\{1, 2, \ldots, N\}$. The action of $\sigma$ on the Dynkin diagram is listed as follows:

\[
\begin{align*}
A_N & : \sigma(i) = N + 1 - i, \\
D_N & : \sigma(i) = i, 1 \leq i \leq N - 2; \sigma(N - 1) = N, \\
D_4 & : \sigma(1, 2, 3, 4) = (3, 2, 4, 1), \\
E_6 & : \sigma(i) = 6 - i, 1 \leq i \leq 5; \sigma(6) = 6,
\end{align*}
\]

Let $\omega = exp^{2\pi i/k}$ be the primitive $k$th root of unity. Then we have the $\mathbb{Z}/r\mathbb{Z}$-graded decomposition

\[g(A) = \bigoplus_{j \in \mathbb{Z}/k\mathbb{Z}} g_j,\]

where $g_j$ is the eigenspace relative to the eigenvalue $\omega^j$. Subsequently $g_{0}$ is a Lie subalgebra of $g(A)$. Obviously, the nodes of the Dynkin diagram of $g_{0}$ are indexed by $I$.

2.2. Twisted affine Lie algebras. For a non-trivial automorphism $\sigma$ of the Dynkin diagram, the twisted affine Lie algebra $\hat{g}^\sigma$ is realized as follows.

\[\hat{g}^\sigma = \left( \bigoplus_{j \in \mathbb{Z}} g_{(j)} \otimes \mathbb{C}t^j \right) \oplus \mathbb{C}c \oplus \mathbb{C}d,\]
where \( c \) is the central element and \( \text{ad}(d) = t \frac{d}{dt} \). Here we use \( [j] \) to denote \( j(\text{mod} \, k) \). Thus the twisted affine Lie algebra \( \hat{g}^\sigma \) is the universal central extension of the twisted loop algebra. We denote by \( \hat{I} = I \cup \{0\} \) the nodes set of Dynkin diagram of \( \hat{g}^\sigma \).

Furthermore, \( A^\sigma = (a_{ij}^\sigma) \) (\( i, j \in \hat{I} \)) will denote the Cartan matrix of the twisted affine Lie algebra \( \hat{g}^\sigma \) of type \( X_r^{(r)} \). The Cartan matrix \( A^\sigma \) is symmetrizable, that is there exists a diagonal matrix \( D = \text{diag}(d_i) \) (\( i \in \hat{I} \)) such that \( DA^\sigma D^{-1} \) is symmetric. Let \( a_i (i \in \hat{I}) \) be the simple roots of \( g_0 \) and \( \hat{g}^\sigma \) respectively. Then \( \Delta = \{ \alpha_1, \cdots, \alpha_n \} \) and \( \hat{\Delta} = \{ \alpha_0 \} \cup \Delta \), where \( \alpha_0 = \delta - \theta, \delta \) is the simple imaginary root and \( \theta \) is the maximal root of \( g_0 \). We let \( \langle , \rangle \) be the canonical bilinear form of the Cartan subalgebra such that \( \langle 2(\alpha_i, \alpha_j) \rangle = a_{ij}^\sigma, \ i, j \in \hat{I} \). Let \( d_i = \frac{\langle \alpha_i, \alpha_i \rangle}{2}, \ i \in \hat{I} \).

3. Two-parameter twisted quantum affine algebras \( U_{r,s}(\hat{g}^\sigma) \)

We assume that the ground field \( \mathbb{K} \) is \( \mathbb{Q}(r, s) \), the field of rational functions with two indeterminates \( r, s \) (\( r \neq s \)).

3.1. Drinfeld-Jimbo definition of \( U_{r,s}(\hat{g}^\sigma) \). Let \( r_i = r^{d_i}, \ s_i = s^{d_i}, \ i \in \hat{I} \). Introduce the \( (r, s) \)-integer by:

\[
[n]_i = \frac{r^n i - s^n i}{r_i - s_i}, \ [n]_i! = \prod_{k=1}^{n} [k]_i.
\]

**DEFINITION 3.1.** The two-parameter twisted quantum affine algebra \( U_{r,s}(\hat{g}^\sigma) \) is the unital associative algebra over \( \mathbb{K} \) generated by the elements \( e_i, f_i, \omega_i^{\pm 1}, \omega_i'\omega_i'^{-1} (j \in \hat{I}), \gamma_{\pm}^{\pm}, \) \( \gamma'_{\pm} \), satisfying the following relations:

**(X1)** \( \gamma_{\pm}^{\pm}, \gamma'_{\pm} \) are central with \( \gamma = \omega_{\delta}, \gamma' = \omega'_{\delta}, \gamma \gamma' = (rs)^c, \) such that \( \omega_i \omega_i^{-1} = \omega_i', \omega_i'^{-1} = 1, \) and

\[
[\omega_i^{\pm 1}, \omega_i'^{\pm 1}] = [\omega_i^{\pm 1}, \omega_i'^{\pm 1}] = [\omega_i'^{\pm 1}, \omega_i'^{\pm 1}] = 0.
\]

**(X2)** For \( i, j \in \hat{I} \), then we have:

\[
\omega_i e_i \omega_j^{-1} = \langle i, j \rangle e_i, \quad \omega_j f_i \omega_j^{-1} = \langle i, j \rangle^{-1} f_i.
\]

**(X3)** For \( i, j \in \hat{I} \), then we have:

\[
\omega_i' e_i \omega_i'^{-1} = \langle j, i \rangle^{-1} e_i, \quad \omega_j' f_i \omega_j'^{-1} = \langle j, i \rangle f_i.
\]

**(X4)** For \( i, j \in \hat{I} \), we have

\[
[e_i, f_j] = \frac{\delta_{ij}}{r - s} (\omega_i - \omega_j').
\]

**(X5)** For any \( i \neq j \in \hat{I} \), we have the \( (r, s) \)-Serre relations involving \( e_i \):

\[
(\text{ad}_l e_i)^{1 - a_{ij}} (e_j) = 0,
\]

**(X6)** For any \( i \neq j \in \hat{I} \), we get the \( (r, s) \)-Serre relations involving \( f_i \):

\[
(\text{ad}_r f_i)^{1 - a_{ij}} (f_j) = 0,
\]

where we have used the notations of the left-adjoint action \( \text{ad}_l e_i \) and the right-adjoint action \( \text{ad}_r f_i \), which are given in Proposition 3.3 below.

Here the structural constants \( \langle i, j \rangle \) in the relations (X2) and (X3) are given in the two-parameter quantum Cartan matrix \( J \) defined below. In fact, the matrix after removing
the zero row and zero column of $J$ is exactly the two-parameter quantum Cartan matrix of finite dimensional type. While the data of the zero row and the zero column are compatible with the results of the section 6. The two-parameter quantum Cartan matrix $J$ can be realized as the two-parameter quantization of Cartan matrix of a Lie algebra, we list the two-parameter quantum Cartan matrices for our twisted cases for further reference.

For type $A_{2n-1}^{(2)}$,

$$J = \begin{pmatrix}
rs^{-1} & (rs)^{-1} & r^{-1} & \cdots & 1 & (rs)^2 \\
rs & rs^{-1} & r^{-1} & \cdots & 1 & 1 \\
s & s & rs^{-1} & \cdots & 1 & 1 \\
& \cdots & \cdots & \cdots & \cdots & \cdots \\
1 & 1 & 1 & \cdots & rs^{-1} & r^{-2} \\
(rs)^{-2} & 1 & 1 & \cdots & s^2 & r^2s^{-2}
\end{pmatrix}. $$

For type $A_{2n}^{(2)}$,

$$J = \begin{pmatrix}
rs^2 s^{-2} & r^{-2} & 1 & \cdots & 1 & rs \\
s^2 & rs^{-1} & r^{-1} & \cdots & 1 & 1 \\
1 & s & rs^{-1} & \cdots & 1 & 1 \\
& \cdots & \cdots & \cdots & \cdots & \cdots \\
1 & 1 & 1 & \cdots & rs^{-1} & r^{-1} \\
(rs)^{-1} & 1 & 1 & \cdots & s & r^2s^{-2}
\end{pmatrix}. $$

For type $D_{n+1}^{(2)}$,

$$J = \begin{pmatrix}
rs^{-1} & r^{-2} & 1 & \cdots & 1 & rs \\
s^2 & r^2s^{-2} & r^{-2} & \cdots & 1 & 1 \\
1 & s^2 & r^2s^{-2} & \cdots & 1 & 1 \\
& \cdots & \cdots & \cdots & \cdots & \cdots \\
1 & 1 & 1 & \cdots & r^2s^{-2} & r^{-2} \\
(rs)^{-1} & 1 & 1 & \cdots & s^2 & rs^{-1}
\end{pmatrix}. $$

For type $D_4^{(3)}$,

$$J = \begin{pmatrix}
rs^{-1} & r^{-2}s^{-1} & (rs)^{-1} & (rs)^3 \\
r^2s^{-2} & rs^{-1} & r^{-3} & 1 \\
(rs)^{-3} & s^3 & r^3s^{-3}
\end{pmatrix}. $$

For type $E_6^{(2)}$

$$J = \begin{pmatrix}
rs^{-1} & r^{-2}s^{-1} & (rs)^{-1} & (rs)^2 & (rs)^2 \\
r^2s^{-2} & rs^{-1} & r^{-1} & 1 & 1 \\
r^2s^{-2} & s & rs^{-1} & r^{-2} & 1 \\
(rs)^{-2} & 1 & s^2 & r^2s^{-2} & r^{-2} \\
(rs)^{-2} & 1 & 1 & s^2 & r^2s^{-2}
\end{pmatrix}. $$

REMARK 3.2. For later reference, we list relation (X5) case by case explicitly.
**Case (I):** In the case of $A_{2n-1}^{(2)}$, for $i = 1, 2 \cdots , n - 1$ and $j, k \in \hat{I}$ such that $a_{jk}^r = 0$, we have the $(r,s)$-Serre relations:

\[
\begin{align*}
    e_je_k - (k,j)e_je_j &= 0, \\
    e_j^2e_{i+1} - (r_i + s_i)e_ie_{i+1}e_i + (r_is_i)e_{i+1}^2 &= 0, \\
    e_i^2e_{i+1} - (r_{i+1} + s_{i+1})e_{i+1}e_{i+1} + (r_{i+1}s_{i+1})e_{i+1}^2 &= 0, \\
    e_0^2e_2 - (r + s)e_0e_2e_0 + (rs)e_2^2 & = 0 \\
    e_0e_2^2 - (r + s)e_2e_0e_2 + (rs)e_2^3e_0 &= 0.
\end{align*}
\]

**Case (II):** In the case of $A_{2n}^{(2)}$, for $i = 1, 2 \cdots , n - 1$ and $j, k \in \hat{I}$ such that $a_{jk}^r = 0$, we have the $(r,s)$-Serre relations:

\[
\begin{align*}
    e_je_k - (k,j)e_je_j &= 0, \\
    e_j^2e_{i+1} - (r_i + s_i)e_ie_{i+1}e_i + (r_is_i)e_{i+1}^2 &= 0, \\
    e_i^2e_{i+1} - (r_{i+1} + s_{i+1})e_{i+1}e_{i+1} + (r_{i+1}s_{i+1})e_{i+1}^2 &= 0, \\
    e_0^2e_1 - (r + s)^2e_0e_1e_0 + (rs)^2e_1^2 &= 0, \\
    e_0e_1^3 - (rs)[3]e_1e_0e_1^2 + [3]e_1^2e_0e_1 - (rs)^3e_1e_0 &= 0, \\
    e_n^{-1}e_n - (r + s)e_ne_n^{-1}e_n + (rs)e_ne_n^{-2} &= 0, \\
    e_n^{-1}e_n^3 - (rs)[3]e_ne_n^{-1}e_n^2 + [3]e_n^2e_ne_n^{-1} - (rs)^3e_n^3 &= 0.
\end{align*}
\]

**Case (III):** In the case of $D_{n+1}^{(2)}$, for $i = 0, 1 \cdots , n - 1$ and $j, k \in \hat{I}$ such that $a_{jk}^r = 0$, we have the $(r,s)$-Serre relations:

\[
\begin{align*}
    e_je_k - (k,j)e_je_j &= 0, \\
    e_j^2e_{i+1} - (r_i + s_i)e_ie_{i+1}e_i + (r_is_i)e_{i+1}^2 &= 0, \\
    e_i^2e_{i+1} - (r_{i+1} + s_{i+1})e_{i+1}e_{i+1} + (r_{i+1}s_{i+1})e_{i+1}^2 &= 0, \\
    e_n^{-1}e_n - (r^2 + s^2)e_ne_n^{-1}e_n + (rs)^2e_n^2 &= 0, \\
    e_n^{-1}e_n^3 - (rs)[3]e_ne_n^{-1}e_n^2 + [3]e_n^2e_ne_n^{-1} - (rs)^3e_n^3 &= 0.
\end{align*}
\]

**Case (IV):** In the case of $D_{4}^{(3)}$, for $i, j \in \hat{I}$ such that $a_{ij}^r = 0$, we have the $(r,s)$-Serre relations:

\[
\begin{align*}
    e_je_k - (k,j)e_je_j &= 0, \\
    e_0^2e_1 - (rs)[2]e_0e_1e_0 + (rs)^2e_0^2 &= 0, \\
    e_0e_1^2 - (rs)[2]e_1e_0e_1 + (rs)^3e_1^2 &= 0, \\
    e_1e_2^2 - (r^3 + s^3)e_2e_1e_2 + (rs)^3e_2^3e_1 &= 0, \\
    e_1^4e_2 - [4]e_1^3e_2e_1 + (rs)[4][3]e_1^2e_2^2 - (rs)^3e_1^3e_2e_1^3 + (rs)^6e_2e_1^4 &= 0.
\end{align*}
\]
For the case of $D_4^{(3)}$, for $i, j \in \hat{I}$ such that $a_{ij}^0 = 0$, we have the $(r, s)$-Serre relations:

\[ e_i e_j - \langle j, i \rangle e_j e_i = 0, \]

\[ e_0^2 e_1 - rs(r + s)e_0 e_1 + (rs)^2 e_1^2 = 0, \]

\[ e_0 e_1^2 - rs(r + s)e_0 e_1 + (rs)^2 e_1^2 = 0, \]

\[ e_1^2 e_2 - (r + s)e_1 e_2 + (rs)e_2^2 = 0, \]

\[ e_1 e_2^2 - (r + s)e_1 e_2 + (rs)e_2^2 = 0, \]

\[ e_2^2 e_3 - (r^2 + s^2)e_3 + (rs)^2 e_3^2 = 0, \]

\[ e_3^2 e_4 - (r^2 + s^2)e_4 + (rs)^2 e_4^2 = 0, \]

\[ e_4^2 e_5 - (r^2 + s^2)e_5 + (rs)^2 e_5^2 = 0. \]

3.2. Hopf algebra structure. One can check the following fact directly.

**Proposition 3.3.** The two-parameter twisted quantum affine algebra $U_{r, s}(\mathfrak{g}^\sigma)$ is a Hopf algebra with the comultiplication $\Delta$, the counit $\varepsilon$ and the antipode $S$ defined below: for $i \in \hat{I}$, we have

\[ \Delta(\gamma^{i \pm}) = \gamma^{i \pm} \otimes \gamma^{i \pm}, \]

\[ \Delta(\omega_i) = \omega_i \otimes \omega_i, \]

\[ \Delta(e_i) = e_i \otimes 1 + 1 \otimes e_i, \]

\[ \varepsilon(e_i) = 0, \quad \varepsilon(\gamma^{i \pm}) = 1, \]

\[ S(\gamma^{i \pm}) = \gamma^{-i \pm}, \]

\[ S(e_i) = -\omega_i^{-1} e_i, \quad S(f_i) = -f_i \omega_i^{-1}, \quad S(\omega_i) = \omega_i^{-1}, \quad S(\omega_i') = \omega_i'^{-1}. \]

**Remark 3.4.** (1) When $r = q = s^{-1}$, the quotient Hopf algebra of $U = U_{r, s}(\mathfrak{g}^\sigma)$ modulo the Hopf ideal generated by the elements $\omega_i' - \omega_i^{-1}$ ($i \in \hat{I}$) and $\gamma^{i \pm} - \gamma^{-i \pm}$ is the classical twisted quantum affine algebra $U_{r, s}(\mathfrak{g}^\sigma)$ of Drinfeld-Jimbo type.

(2) In the Hopf algebra $U_{r, s}(\mathfrak{g}^\sigma)$, there exist left-adjoint and right-adjoint actions defined by the Hopf algebra structure:

\[ \text{ad}_a a(b) = \sum_{(a)} a_{(1)} b a_{(2)}, \quad \text{ad}_a a(b) = \sum_{(a)} S(a_{(1)}) b a_{(2)}, \]

where $\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}$ for any $a, b \in U_{r, s}(\mathfrak{g}^\sigma)$.

3.3. Triangular decomposition of $U_{r, s}(\mathfrak{g}^\sigma)$. Two-parameter twisted quantum affine algebra $U_{r, s}(\mathfrak{g}^\sigma)$ is endowed with the Drinfeld double structure (see Proposition 3.5), which is similar to the non-twisted cases. The following statement can be proved by standard arguments as in the untwisted types (see [HRZ]).

**Proposition 3.5.** $U_{r, s}(\mathfrak{g}^\sigma)$ is isomorphic to its Drinfeld double as a Hopf algebra.

Let $U^0 = \mathbb{K}[\omega_0^{\pm 1}, \cdots, \omega_n^{\pm 1}, \omega'_0^{\pm 1}, \cdots, \omega'_n^{\pm 1}, \gamma, \gamma']$ denote the Cartan subalgebra of $U_{r, s}(\mathfrak{g}^\sigma)$. 
Furthermore, let us denote by \( U_{r,s}(\hat{g}^\sigma) \) (resp. \( U_{r,s}(\hat{g}^{-\sigma}) \)) the subalgebra of \( U_{r,s}(\hat{g}^\sigma) \) generated by \( e_i \) (resp. \( f_i \)) for all \( i \in I \). Then, we get the standard triangular decomposition of \( U_{r,s}(\hat{g}^\sigma) \).

**Corollary 3.6.** \( U_{r,s}(\hat{g}^\sigma) \cong U_{r,s}(\hat{g}^{-\sigma}) \otimes U^0 \otimes U_{r,s}(\hat{g}) \) as vector spaces. \( \square \)

**Definition 3.7.** (cf. [HRZ]) Let \( \tau \) be the \( \mathbb{Q} \)-algebra anti-automorphism of \( U_{r,s}(\hat{g}^\sigma) \) such that \( \tau(r) = s, \tau(s) = r, \tau(\omega^\prime, \omega^\prime) = (\omega^\prime, \omega^\prime)^\pm 1, \) and

\[
\tau(e_i) = f_i, \quad \tau(f_i) = e_i, \quad \tau(\omega_i) = \omega_i^\prime, \quad \tau(\omega_i^\prime) = \omega_i, \quad \tau(\gamma) = \gamma^\prime, \quad \tau(\gamma^\prime) = \gamma.
\]

In fact \( \tau \) is an analog of the Chevalley anti-involution on \( U_{r,s}(\hat{g}^\sigma) \).

### 4. Drinfeld realizations of two-parameter twisted quantum affine algebras

In this section, we would like to generalize the result for untwisted cases [HZ] to the twisted cases and state the Drinfeld realization for the two-parameter twisted quantum affine algebras.

For convenience, if \( \alpha = \alpha_{i_1} + \cdots + \alpha_{i_m}, \beta = \alpha_{j_1} + \cdots + \alpha_{j_n} \) are the decompositions into simple roots, we denote \( (\alpha, \beta) = \prod_{k=1}^m \prod_{l=1}^n \langle i_k, j_l \rangle \)

**4.1. Generating functions for two-parameter cases.** In order to state Drinfeld realization of two-parameter quantum affine algebra \( U_{r,s}(\hat{g}^\sigma) \), we need to define some functions \( g^\pm_{ij}(z) \).

For \( i, j = 1, \cdots, n \), let

\[
F_{ij}^\pm(z, w) := \prod_{l \in \mathbb{Z}/k\mathbb{Z}} (z - \omega^l((i, \sigma^l(j)))(\sigma^l(j), i))^\pm \frac{1}{2} w)
\]

\[
G_{ij}^\pm(z, w) := \prod_{l \in \mathbb{Z}/k\mathbb{Z}} ((\sigma^l(j), i)^\pm 1 z - ((\sigma^l(j), i)(\sigma^l(j))^{-1})^\pm \frac{1}{2} w).
\]

For two simple roots \( \alpha_i, \alpha_j \in \Delta \), we set \( g_{ij}^\pm(z) = \sum_{n \in \mathbb{Z}_+} c_{ij}^{\pm} \alpha_i, \alpha_j, z^n := \sum_{n \in \mathbb{Z}_+} c_{ij}^{\pm} z^n \), a formal power series in \( z \), where the coefficients \( c_{ij}^{\pm} \) are determined from the Taylor series expansion in the variable \( \xi \) at \( 0 \in \mathbb{C} \) of the function

\[
\sum_{n \in \mathbb{Z}_+} c_{ij}^{\pm} z^n = g_{ij}^\pm(\xi) = \frac{G_{ij}^\pm(\xi, 1)}{F_{ij}^\pm(\xi, 1)}
\]

To write the relations in compact form, we use the following generating functions:

\[
x_{ij}^\pm(z) = \sum_{k \in \mathbb{Z}} x_{ij}^\pm(k)z^{-k}, \quad \phi_i(z) = \sum_{m \in \mathbb{Z}_+} \phi_i(m)z^{-m}, \quad \varphi_i(z) = \sum_{n \in \mathbb{Z}_+} \varphi_i(n)z^{-n}, \quad \delta(z) = \sum_{n \in \mathbb{Z}} z^n.
\]

**Definition 4.1.** The two-parameter twisted quantum affine algebra \( \hat{U}_{r,s}(\hat{g}^\sigma) \) is the associative algebra with unit 1, and the generators

\[
\{ x_{ij}^\pm(w), \phi_i(z), \varphi_i(z), \gamma^\frac{1}{2}, \gamma^\frac{1}{2} | i = 1, \cdots, n \}
\]
satisfying the defining relations:

(4.1) \[ \gamma' \pm \frac{1}{\gamma}, \gamma' \pm \frac{1}{\gamma} \text{ are central and invertible such that } \gamma' \gamma = (rs)^r, \]

(4.2) \[ x_{\sigma(i)}^\pm (k) = \omega^k x_i^\pm (k), \: \varphi_{\sigma(i)}^\pm (m) = \omega^m \varphi_i^\pm (m), \: \phi_{\sigma(i)}^\pm (n) = \omega^n \phi_i^\pm (n), \]

(4.3) \[ \varphi_i^0 (\phi_j^0) = \phi_j^0 (\phi_i^0), \]

(4.4) \[ [\varphi_i(z), \varphi_j(w)] = [\phi_i(z), \phi_j(w)] = 0, \]

(4.5) \[ g_{ij}(zw^{-1} (\gamma' (\gamma)^r) \varphi_i^0 (w) \phi_j^0 (w) = g_{ij}(zw^{-1} (\gamma' (\gamma)^r) \phi_i^0 (w) \varphi_j^0 (z), \]

(4.6) \[ \varphi_i(z) x_i^\pm(w) \varphi_i(z)^{-1} = g_{ij} \left( \frac{z}{w} (\gamma (\gamma')^r) \right)^{\pm s} x_i^\pm(w), \]

(4.7) \[ \phi_i(z) x_i^\pm(w) \phi_i(z)^{-1} = g_{ij} \left( \frac{w}{z} (\gamma (\gamma')^r) \right)^{\pm s} x_i^\pm(w), \]

(4.8) \[ [x_i^\pm(z), x_j^\pm(w)] = \frac{\delta_{ij}}{r_i - s_i} \left( \delta(zw^{-1} (\gamma) \phi_i^0 (w) \gamma) - \delta(zw^{-1} (\gamma) \varphi_i^0 (z) \gamma' \gamma) \right), \]

(4.9) \[ F_{ij}^\pm(z, w) x_i^\pm(z) x_j^\pm(w) = G_{ij}^\pm(z, w) x_j^\pm(w) x_i^\pm(z), \]

(4.10) \[ x_i^\pm(z) x_j^\pm(w) = (j, i)^{\pm 1} x_i^\pm(w) x_j^\pm(z), \quad \text{for } a_{ij}^0 = 0, \]

(4.11) \[ \text{Sym}_{z_1, z_2, z_3} \{ (r_s^2)^{\pm t} z_1 - (r_s^2 + s^2) z_2 + (r_s^2 - s^2) z_3 \} x_i^\pm(z_1) x_j^\pm(z_2) x_k^\pm(z_3) = 0, \]

for \( A_{i, \sigma(i)} = -1, \]

(4.12) \[ \text{Sym}_{z_1, z_2} \left\{ P_{ij}^\pm(z_1, z_2) \sum_{t=0}^{t=2} (-1)^{(r_i s_i)^{\pm t}(t+1)} \left[ \frac{2}{t+1} \right] x_i^\pm(z_1) \cdots x_i^\pm(z_t) \times x_j^\pm(w) x_j^\pm(z_{t+1}) \cdots x_j^\pm(z_2) \right\} = 0, \]

for \( A_{i, j} = -1, \quad 1 \leq j < i \leq N \) such that \( \sigma(i) \neq j, \]

(4.13) \[ \text{Sym}_{z_1, z_2} \left\{ P_{ij}^\pm(z_1, z_2) \sum_{t=0}^{t=2} (-1)^{(r_i s_i)^{\pm t}(t+1)} \left[ \frac{2}{t+1} \right] x_i^\pm(z_1) \cdots x_i^\pm(z_t) \times x_j^\pm(w) x_j^\pm(z_{t+1}) \cdots x_j^\pm(z_2) \right\} = 0, \]

for \( A_{i, j} = -1, \quad 1 \leq i < j \leq N \) such that \( \sigma(i) \neq j, \]

where \([l]_{\pm t} = \frac{r_i^{\pm t} - s_i^{\pm t}}{r_i^{\pm t} - s_i^{\pm t}}, \quad [l]_{\mp t} = \frac{r_i^{\pm t} - s_i^{\pm t}}{r_i^{\pm t} - s_i^{\pm t}}, \quad t = 0, \quad d_{ij} = k; \]

If \( \sigma(i) = i, \) then \( P_{ij}^\pm(z, w) = 1, \) \( d_{ij} = k; \]

If \( A_{i, \sigma(i)} = 0, \) \( \sigma(j) = j, \) then \( P_{ij}^\pm(z, w) = \frac{z^r (rs^{-1})^{\pm k} - w^r}{z(rs^{-1})^{\pm 1} - w}, \) \( d_{ij} = k; \]

If \( A_{i, \sigma(i)} = 0, \) \( \sigma(j) \neq j, \) then \( P_{ij}^\pm(z, w) = 1, \) \( d_{ij} = 1/2; \]

If \( A_{i, \sigma(i)} = -1, \) then \( P_{ij}^\pm(z, w) = z(rs^{-1})^{\pm k/4} + w, \) \( d_{ij} = k/2. \]

**Remark 4.2.** Note that in relation (4.1), \( \gamma \) and \( \gamma' \) are related by the central element \( c. \) In the classical cases, the central element \( c \) is absent in the relation, since \( \gamma \) and \( \gamma' \) are inverse to each other.

We now give the two-parameter Drinfeld realization which will be proved to be equivalent to the earlier definition.
DEFINITION 4.3. The unital associative algebra $\mathcal{U}_{r,s}(\hat{g}^\gamma)$ over $\mathbb{K}$ is generated by the elements $x_i^\pm(k), a_i(\ell), \omega_i^\pm, \omega_i, \gamma_i^\pm, \gamma_i, (i \in I, k, k' \in \mathbb{Z}, \ell, \ell' \in \mathbb{Z}\setminus\{0\})$, subject to the following defining relations:

(D1) $\gamma_i^\pm, \gamma_i$ are central such that $\gamma_i \gamma_j = (rs)^c, \omega_i \omega_i^{-1} = \omega_i' \omega_i'^{-1} = 1 (i \in I)$, and for $i, j \in I$, one has

$$\left[ \omega_i^\pm \pm, \omega_j^\pm \pm \right] = \left[ \omega_i^\pm \pm, \omega_j^\pm \pm \right] = \left[ \omega_i^\pm \pm, \omega_j^\pm \pm \right] = 0.$$  

(D2)

$$x_{\sigma(i)}(l) = \omega^i x_i^l(l), a_{\sigma(i)}(m) = \omega^m a_i(m)$$

(D3)

$$[a_i(\ell), a_j(\ell')] = \delta_{\ell + \ell', 0} \sum_{t=0}^{k-1} \frac{(\gamma_i^t)}{2} \frac{A_{\ell, \sigma(i)}(t)}{[\ell]} \frac{A_{-\ell, \sigma(i)}(t)}{[\ell]} \gamma_i^{\ell - \ell'} \left[ \gamma_i \right] \frac{r - s}{r - s}.$$  

(D4)

$$\left[ a_i(\ell), \omega_j^\pm \pm \right] = \left[ a_i(\ell), \omega_j^\pm \pm \right] = 0.$$  

(D5)

$$\omega_i x_j^\pm(k) \omega_i^{-1} = \sum_{t=0}^{k-1} \langle \gamma_i^t \rangle \langle j \rangle \pm x_j^\pm(k),$$  

$$\omega_i' x_j^\pm(k) \omega_i'^{-1} = \sum_{t=0}^{k-1} \langle i \rangle \langle \gamma_i^t \rangle \pm x_j^\pm(k).$$  

(D61)

$$\left[ a_i(\ell), x_j^\pm(k) \right] = \pm \sum_{t=0}^{k-1} \frac{(\ell A_{\ell, \sigma(i)}(t))}{2} \frac{A_{-\ell, \sigma(i)}(t)}{[\ell]} (\gamma_i^t) \frac{2}{2} \gamma_i^{\ell - \ell'} x_j^\pm(\ell+k),$$  

for $\ell > 0$,

(D62)

$$\left[ a_i(\ell), x_j^\pm(k) \right] = \pm \sum_{t=0}^{k-1} \frac{(\ell A_{\ell, \sigma(i)}(t))}{2} \frac{A_{-\ell, \sigma(i)}(t)}{[\ell]} (\gamma_i^t) \frac{2}{2} \gamma_i^{\ell - \ell'} x_j^\pm(\ell+k),$$  

for $\ell < 0$,

(D7)

$$F_{ij}^\pm(z, w) x_i^\pm(z) x_j^\pm(w) = G_{ij}^\pm(z, w) x_j^\pm(w) x_i^\pm(z),$$

(D8)

$$[x_i^\pm(k), x_j^\pm(k')] = \frac{\delta_{ij}}{r_i - s_i} \left( \gamma_i^{-k} \gamma_i^{-k'} \phi_i(k + k') - \gamma_i^k \gamma_i^{-k'} \varphi_i(k + k') \right),$$

where $\phi_i(m), \varphi_i(-m) (m \in \mathbb{Z}_{>0})$ such that $\phi_i(0) = \omega_i$ and $\varphi_i(0) = \omega_i'$ are defined as below:

$$\sum_{m=0}^{\infty} \phi_i(m) z^{-m} = \omega_i \exp \left( \sum_{\ell=1}^{\infty} a_i(\ell) z^{-\ell} \right), \quad (\omega_i(-m) = 0, \forall m > 0);$$

$$\sum_{m=0}^{\infty} \varphi_i(-m) z^m = \omega_i' \exp \left( -\sum_{\ell=1}^{\infty} a_i(-\ell) z^\ell \right), \quad (\omega_i'(m) = 0, \forall m > 0).$$

(D91)

$$x_i^\pm(m) x_j^\pm(k) = \langle j, i \rangle \pm x_j^\pm(k) x_i^\pm(m),$$

for $a_{ij}^\sigma = 0$,

(D92)

$$\text{Sym}_{z_1, z_2, z_3} \left\{ (rs^{-2})\tilde{z} \right\} = 0,$$

for $A_{i, \sigma(i)} = -1$. 


(D9.3) \[ \text{Sym}_{z_1, z_2} \left\{ P_{ij}^\pm (z_1, z_2) \sum_{t=0}^2 (-1)^t (r_i s_i)^{t+1} 2 \choose t \frac{2}{ \pm i } (z_1) \cdots x_i^\pm (z_2) \right\} = 0, \]
for \( A_{i,j} = -1, \) and \( 1 \leq j < i \leq N \) such that \( \sigma(i) \neq j, \)

(D9.4) \[ \text{Sym}_{z_1, z_2} \left\{ P_{ij}^\pm (z_1, z_2) \sum_{t=0}^2 (-1)^t (r_i s_i)^{t+1} 2 \choose t \frac{2}{ \pm i } (z_1) \cdots x_i^\pm (z_2) \right\} = 0, \]
for \( A_{i,j} = -1, \) and \( 1 \leq i < j \leq N \) such that \( \sigma(i) \neq j, \)

where \( [l]_{\pm i} = \frac{r^{\pm i} - s^{\pm i}}{r_i^{\pm i} - s_i^{\pm i}}, \) \( [l]_{\pm i} = \frac{r^{\pm i} - s^{\pm i}}{r_i^{\pm i} - s_i^{\pm i}}, \) and \( \)

If \( \sigma(i) = i, \) then \( P_{ij}^\pm (z, w) = 1, \) \( d_{ij} = k, \)

If \( A_{i,\sigma(i)} = 0, \) \( \sigma(j) = j, \) then \( P_{ij}^\pm (z, w) = \frac{z^r (r s^{-1})^{\pm k} - w^r}{z (r s^{-1})^{\pm 1} - w}, \) \( d_{ij} = k, \)

If \( A_{i,\sigma(i)} = 0, \) \( \sigma(j) \neq j, \) then \( P_{ij}^\pm (z, w) = 1, \) \( d_{ij} = 1/2, \)

If \( A_{i,\sigma(i)} = -1, \) then \( P_{ij}^\pm (z, w) = z (r s^{-1})^{\pm k/4} + w, \) \( d_{ij} = k/2. \)

4.2. The anti-involution \( \tau. \) The following analog of Chevalley anti-homomorphism can be checked directly.

**Proposition 4.4.** There exists the \( \mathbb{Q} \)-algebra anti-automorphism \( \tau \) of \( \mathcal{U}_{r,s}(\mathfrak{g}^r) \) such that \( \tau(r) = s, \tau(s) = r, \tau([\omega_i^j, \omega_j^l]) = [\omega_j^l, \omega_i^j] = \omega_i^j \pm 1 \) and

\[
\tau(\omega_i) = \omega_i^l, \quad \tau(\omega_i^l) = \omega_i^l, \\
\tau(\gamma) = \gamma^l, \quad \tau(\gamma^l) = \gamma, \\
\tau(a_i(l)) = a_i(-l), \\
\tau(x_i^\pm(m)) = x_i^\mp(-m), \\
\tau(\phi_i(m)) = \varphi_i(-m), \quad \tau(\varphi_i(m)) = \phi_i(m),
\]

and \( \tau \) preserves each defining relation (Dn) in Definition 4.3 for \( n = 1, \cdots, 9. \) \( \square \)

4.3. Triangular decomposition of \( \mathcal{U}_{r,s}(\mathfrak{g}^r). \) Let \( \mathcal{U}_{r,s}(\mathfrak{g}^r) \) denote the subalgebra of \( \mathcal{U}_{r,s}(\mathfrak{g}^r) \) generated by \( x_i^\pm(0), \omega_i, \omega_i^l, (i \in I). \) Clearly \( \mathcal{U}_{r,s}(\mathfrak{g}^r) \cong U_{r,s}(\mathfrak{g}^r), \) the subalgebra of \( U_{r,s}(\mathfrak{g}^r) \) generated by \( e_i, f_i, \omega_i, \omega_i^l (i \in I). \)

Using the defining relations (D1)—(D9), one can easily shows that \( \mathcal{U}_{r,s}(\mathfrak{g}^r) \) has a triangular decomposition:

\[
\mathcal{U}_{r,s}(\mathfrak{g}^r) = \mathcal{U}_{r,s}(\mathfrak{n}^-) \otimes \mathcal{U}_{r,s}(\mathfrak{g}) \otimes \mathcal{U}_{r,s}(\mathfrak{n}^+),
\]

where \( \mathcal{U}_{r,s}(\mathfrak{n}^-) = \bigoplus_{a \in \mathbb{Q}^+} \mathcal{U}_{r,s}(\mathfrak{n}^a) \) is generated respectively by \( x_i^\pm(k) (i \in I) \), \( \) and \( \mathcal{U}_{r,s}(\mathfrak{g}) \) is the subalgebra generated by \( \omega_i^\pm, \omega_i^{\pm 1}, \gamma^\pm, \gamma^\pm \), and \( a_i(\pm \ell) \) for \( i \in I, \ell \in \mathbb{N}. \)

Namely, \( \mathcal{U}_{r,s}(\mathfrak{g}) \) is generated by the subalgebra \( \mathcal{U}_{r,s}(\mathfrak{g}) \) and the quantum Heisenberg subalgebra \( \mathcal{H}_{r,s}(\mathfrak{g}) \), which is generated by the quantum imaginary root vectors \( a_i(\pm \ell) \) \( (i \in I, \ell \in \mathbb{N}). \)
5. Two-parameter Drinfeld isomorphism theorem

5.1. Quantum Lie bracket. We recall the quantum Lie bracket from \[J2\].

**Definition 5.1.** For \( q_i \in \mathbb{K}^* = \mathbb{K}\setminus\{0\} \) and \( i = 1, 2, \ldots, s - 1 \), the Lie q-brackets \( [a_1, a_2, \ldots, a_s]_{(q_1, q_2, \ldots, q_{s-1})} \) and \( [a_1, a_2, \ldots, a_s]_{(q_1, q_2, \ldots, q_{s-1})} \) are defined inductively by

\[
[a_1, a_2]_{q_1} = a_1a_2 - q_1a_2a_1, \\
[a_1, a_2, \ldots, a_s]_{(q_1, q_2, \ldots, q_{s-1})} = [a_1, [a_2, \ldots, a_s]_{(q_2, \ldots, q_{s-1})}]_{q_1},
\]

It follows from the definition that the quantum brackets satisfy the following identities.

\[
[a, bc]_v = [a, b]_q c + q b [a, c]_q, \\
[ab, c]_v = a [b, c]_q + q [a, c]_q b,
\]

\[
[a, [b, c]]_w = [a, b]_q c + q [a, c]_q b, \\
[[a, b], c]_w = [a, [b, c]]_q + q [a, [b, c]]_q b.
\]

In particular, we have that

\[
[a, [b_1, \ldots, b_s]_{(v_1, \ldots, v_{s-1})}] = \sum_i [b_1, \ldots, [a, b_i], \ldots, b_s]_{(v_1, \ldots, v_{s-1})},
\]

\[
[a, a, b]_{(u, v)} = a^2b - (u+v)aba + (uv)ba^2 = (uv)[b, a, a]_{(u^{-1}, v^{-1})},
\]

\[
[a, a, a, b]_{(u^2, uv, v^2)} = a^3b - [3]_{u,v} a^2ba + (uv)[3]_{u,v} a^2ba^2 - (uv)^3ba^3,
\]

\[
[a, a, a, b, c]_{(u^3, u^2v, uv^2, v^3)} = a^4b - [4]_{u,v} a^3ba + uv \left\lfloor \frac{4}{2} \right\rfloor_{u,v} a^2ba^2 - (uv)^3ba^3
\]

\[ - (uv)^3[4]_{u,v} a^2ba^3 + (uv)^3ba^4.
\]

where \([n]_{u,v} = \frac{u^n - v^n}{u - v}, [n]_{u,v} := [n]_{u,v} [m]_{u,v} := \frac{[n]_{u,v}}{[n-m]_{u,v}}.
\]

5.2. Quantum root vectors. In this paragraph, we define the quantum root vectors using the q-bracket. For our purpose, we need to fix a reduced expression of the longest element in the finite Weyl group.

**Lemma 5.2.** Let \( i \in I \), there exists a sequence of indices \( i = i_1, i_2, \ldots, i_{h-1} \) such that \( \theta = \alpha_{i_1} + \cdots + \alpha_{i_{h-1}} \), and \( (\alpha_{i_1} + \cdots + \alpha_{i_{h-1}}, \alpha_i) = \epsilon_k \neq 0, (2 \leq k \leq h-1) \), where \( \theta \) is the maximal root of Lie algebra \( g_0 \) and \( h \) is the Coxeter number.

**Remark 5.3.** (1) For the case of \( \mathbb{A}_{2n-1}^{(2)} \), we fix a root chain as follows

\[
\alpha_1 \to \alpha_2 \to \cdots \to \alpha_{n-1} \to \alpha_n \to \alpha_{n-1} \to \cdots \to \alpha_2
\]

(2) For the case of \( \mathbb{A}_{2n}^{(2)} \), we fix a root chain as follows

\[
\alpha_1 \to \alpha_2 \to \cdots \to \alpha_n \to \alpha_n \to \alpha_{n-1} \to \cdots \to \alpha_1
\]

(3) For the case of \( \mathbb{D}_{n+1}^{(2)} \), we fix a root chain as follows

\[
\alpha_n \to \alpha_{n-1} \to \cdots \to \alpha_1
\]

(4) For the case of \( \mathbb{E}_6^{(2)} \), we fix a root chain as follows

\[
\alpha_1 \to \alpha_2 \to \alpha_3 \to \alpha_4 \to \alpha_2 \to \alpha_3 \to \alpha_2 \to \alpha_1
\]
In particular, the quantum vectors associated to the decomposition of the maximal root are quantum root vectors associated to the roots $\theta$ and $\theta - \beta + \delta$. We choose a decomposition of the quantum root vector case by case for later reference.

**Case(I)** For $A_{2n-1}^{(2)}$, if $\alpha_{11} = \alpha_1 + \cdots + \alpha_t (2 \leq t \leq n)$ and $\alpha_{12} = \alpha_1$, we define the quantum root vectors associated to the roots $\delta - \alpha_{12}$ and $-\delta + \alpha_{1t}$ inductively as follows:

$$x_{\alpha_{1t}}^{-1}(1) = x_{\alpha_{1t}}^{-1}(1) = [x^*_t(0), x^*_t(1)](t,t-1)\cdots(t,1),$$

$$x_{\alpha_{1t}}^{+1}(-1) = x_{\alpha_{1t}}^{+1}(-1) = [x^+_t(-1), x^+_t(0)](t-1,t-1)\cdots(1,1).$$

If $\beta_{1t} = \alpha_1 + \cdots + \alpha_n + \alpha_{n-1} + \cdots + \alpha_t (2 \leq t \leq n-1)$, so $\beta_{1(n-1)} = \alpha_{1n} + \alpha_{n-1}$, and $\theta = \beta_{12}$. We define the quantum root vectors associated to the roots $\delta - \beta_{1t}$ and $-\delta + \beta_{1t}$ inductively as follows:

$$y_{\beta_{1t}}^{-1}(1) = y_{\beta_{1t}}^{-1}(1) = [x^*_t(0), y_{\beta_{1t}(-1)}](t,t+1)\cdots(t,n)(t,n-1)\cdots(1,1),$$

$$y_{\beta_{1t}}^{+1}(-1) = y_{\beta_{1t}}^{+1}(-1) = [y^*_t(-1), x^+_t(0)](t+1,t)\cdots(n,t)\cdots(n-1,t)\cdots(1,1).$$

In particular

$$x^{-1}(1) = y_{12}^{-1}(1)$$

$$= [x^*_2(0), \cdots, x^*_n(0), x^*_n(0), \cdots, x^*_1(1)](s, s, s^2, r^{-1}, \cdots, r^{-1}),$$

$$x^+(1) = y_{12}^{+1}(1)$$

$$= [x^+_2(-1), \cdots, x^+_n(0), x^+_n(0), \cdots, x^+_1(0)](r, r, r^2, s^{-1}, \cdots, s^{-1}).$$

**Case(II)** For $A_{2n}^{(2)}$, if $\alpha_{1t} = \alpha_1 + \cdots + \alpha_t (2 \leq t \leq n)$ and $\alpha_{11} = \alpha_1$, we define the quantum root vectors associated to the roots $\delta - \alpha_{12}$ and $-\delta + \alpha_{1t}$ inductively as follows:

$$x_{\alpha_{1t}}^{-1}(1) = x_{\alpha_{1t}}^{-1}(1) = [x^*_t(0), x^*_t(1)](t,t-1)\cdots(t,1),$$

$$x_{\alpha_{1t}}^{+1}(-1) = x_{\alpha_{1t}}^{+1}(-1) = [x^+_t(-1), x^+_t(0)](t-1,t-1)\cdots(1,1).$$

If $\beta_{1t} = \alpha_1 + \cdots + \alpha_n + \alpha_{n-1} + \cdots + \alpha_t (1 \leq t \leq n)$, so $\beta_{1n} = \alpha_{1n} + \alpha_n$, and $\theta = \beta_{11}$. We define the quantum root vectors associated to the roots $\delta - \beta_{1t}$ and $-\delta + \beta_{1t}$ inductively as follows:

$$y_{\beta_{1t}}^{-1}(1) = y_{\beta_{1t}}^{-1}(1) = [x^*_t(0), y_{\beta_{1t}(-1)}](t,t+1)\cdots(t,n)(t,n-1)\cdots(1,1),$$

it holds that $x^{-1}(1) = y_{11}^{-1}(1)$.

$$y_{\beta_{1t}}^{+1}(-1) = y_{\beta_{1t}}^{+1}(-1) = [y^*_t(-1), x^+_t(0)](t+1,t)\cdots(n,t)\cdots(n-1,t)\cdots(1,1),$$

it follows $x^+(-1) = y_{11}^+(-1)$. 

(5) For the case of $D_4^{(3)}$, we fix a root chain as follows

$$\alpha_1 \to \alpha_2 \to \alpha_1$$
In particular
\[ x_\theta^-(1) = y_1^+(1) = [ x_1^-(0), \cdots, x_n^-(0), x_n^+(0), \cdots, x_1^+(1) ] (s, \cdots, s, \theta) \]
\[ x_\theta^+(1) = y_1^+(1) = [ x_1^+(0), \cdots, x_n^+(0), x_n^+(0), \cdots, x_1^+(1) ] (s, \cdots, s, \theta) \]
\[ x_\theta^-(1) = y_1^+(1) = [ x_1^-(0), \cdots, x_n^-(0), x_n^+(0), \cdots, x_1^+(1) ] (s, \cdots, s, \theta) \]
\[ x_\theta^+(1) = y_1^+(1) = [ x_1^+(0), \cdots, x_n^+(0), x_n^+(0), \cdots, x_1^+(1) ] (s, \cdots, s, \theta) \]

Case(III) For \( D_{n+1}^{(2)} \), if \( \alpha_{nt} \equiv \alpha_n + \alpha_{n-1} + \cdots + \alpha_t \) for \( 1 \leq t \leq n \) and \( \alpha_{nn} \equiv \alpha_n \), so \( \theta = \alpha_{n1} \). We define the quantum root vectors associated to the roots \( \delta - \alpha_{nt} \) and \( -\delta + \alpha_{nt} \) inductively as follows:
\[ x_{nt}^-(1) = x_{n1}^+(1) = [ x_{t1}^-(0), x_{n(t-1)}^-(1) ] (t, t+1, \cdots, n) \]
\[ x_{nt}^+(1) = x_{n1}^-(1) = [ x_{t1}^+(0), x_{n(t-1)}^+(1) ] (t+1, t-1, \cdots, n) \]

Case(IV) For \( E_6^{(2)} \), if \( \eta_j \equiv \alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_j} \) where \( \{ i_k \} \in \{ 1, 2, 3, 4, 5, 6 \} \), \( \eta_j \equiv \eta_8 \). We define the quantum root vectors associated to the roots \( \delta - \eta_j \) and \( -\delta + \eta_j \) inductively as follows:
\[ z_{j}^-(1) = x_{nj}^+(1) = [ x_{nj}^-(0), x_{n(j-1)}^+(1) ] (j, j-1, \cdots, n) \]
\[ z_{j}^+(1) = x_{nj}^-(1) = [ x_{nj}^+(0), x_{n(j-1)}^-(1) ] (j, j-1, \cdots, n) \]

Case(V) For \( D_{4}^{(2)} \), here we only consider the quantum root vectors associated to the roots \( \delta - \alpha_1 + \alpha_2, -\delta + \alpha_1 + \alpha_2 \), where \( \theta = 2 \alpha_1 + \alpha_2 \) is the maximal root of \( G_2 \).
\[ x_{12}^-(1) = [ x_{2}^-(0), x_{1}^-(1) ] s^1 \]
and
\[ x_{12}^+(1) = [ x_{2}^+(-1), x_{1}^+(0) ] s^3 \]
On the other hand,
\[ x_{12}^-(1) = [ x_{2}^-(0), x_{1}^-(1) ] (s^3, r^2, s^3) ] \]
and
\[ x_{12}^+(1) = [ x_{2}^+(0), x_{1}^+(0) ] (s^3, r^2, s^3) ] \]
5.3. Two-parameter Drinfeld isomorphism theorem. In this subsection, we establish the isomorphism between the two-parameter quantum affine algebra \( U_{r,s}(\widehat{\mathfrak{g}}^\sigma) \) and the \((r,s)\)-analogue of Drinfeld quantum affinization of \( U_{r,s}(\widehat{\mathfrak{g}}^\sigma) \). The identification of these two forms has been proved for the case of \( \mathfrak{sl}_n \) in [HRZ]. We will give a new proof for the most general case in the next two sections.

We keep the same notations and assumptions as above. In particular, \( \ell_1, \ldots, \ell_{h-1} \) is the fixed sequence associated with the maximum root given in Lemma 5.2 (or the fixed reduced expression for the longest element in the Weyl group).

For simplicity, we denote

\[
\langle i_j, i_{j-1} \cdots i_2 i_1 \rangle \doteq \langle i_j, i_{j-1} \rangle \cdots \langle i_j, i_1 \rangle
\]

and

\[
\langle i_1 i_2 \cdots i_{j-1}, i_j \rangle^{-1} = \langle i_1, i_j \rangle^{-1} \cdots \langle i_{j-1}, i_j \rangle^{-1}.
\]

For \( j = 2, \ldots, h - 1 \), let \( t_{i_j} = \frac{q_{i_j} - p_{i_j}}{r_{i_j} - s_{i_j}} \).

\[
p_{i_j} = \langle i_j, i_{j-1} \cdots i_2 i_1 \rangle,
\]

and

\[
q_{i_j} = \langle i_1 i_2 \cdots i_{j-1}, i_j \rangle^{-1}.
\]

Now we arrive at our main theorem as follows.

**Theorem 5.4.** Let \( \theta = \alpha_{i_1} + \cdots + \alpha_{i_{h-1}} \) be the maximal positive root of a simple Lie algebra \( \mathfrak{g} \) associated with a fixed reduced expression of \( w_0 \). Then there exists an algebra isomorphism \( \Psi : U_{r,s}(\widehat{\mathfrak{g}}^\sigma) \to U_{r,s}(\widehat{\mathfrak{g}}^\sigma) \) given as follows. For each \( i \in I \),

\[
\begin{align*}
\omega_i & \mapsto \omega_i, \\
\omega'_i & \mapsto \omega'_i, \\
\omega_0 & \mapsto \gamma' \omega_0^{-1}, \\
\omega'_0 & \mapsto \gamma \omega_0^{-1}, \\
\gamma^\pm \frac{1}{2} & \mapsto \gamma^\pm \frac{1}{2}, \\
\gamma'^\pm \frac{1}{2} & \mapsto \gamma' \pm \frac{1}{2}, \\
e_i & \mapsto x_i^+(0), \\
f_i & \mapsto \frac{1}{p_i} x_i^-(0), \\
e_0 & \mapsto a x_0^{-1} (1) \cdot (\gamma' \omega_0^{-1}), \\
f_0 & \mapsto (\gamma \omega_0^{-1}) \cdot x_0^+ (-1)
\end{align*}
\]

where \( \omega_0 = \omega_{i_1} \cdots \omega_{i_{h-1}}, \omega'_0 = \omega'_{i_1} \cdots \omega'_{i_{h-1}} \) and \( a = t_{i_2} \cdots t_{i_{h-1}} \).

\[
p_i = \begin{cases} r, & \text{if } \sigma(i) = i; \\ 1, & \text{otherwise} \end{cases}
\]
Here we list the constant $a \in \mathbb{K}$ case by case:

$$a = \begin{cases} 
(r s)^{n-2}, & \text{for } A_{2n-1}^{(2)}; \\
(r s)^{n-2}[2]_{n}^{-2}, & \text{for } A_{2n}^{(2)}; \\
(r s)^{2n-1}, & \text{for } D_{n+1}^{(2)}; \\
(rs)^2, & \text{for } D_{11}^{(3)}; \\
(rs)^{5}[2]_{3}^{-1}, & \text{for } E_6^{(2)}. 
\end{cases}$$

We divide the proof into three steps given by the following three theorems: (Theorems $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$), which will be proved in the following two sections.

6. $\Psi$ is an algebra homomorphism

In this section, we show that $\Psi$ is an algebra homomorphism (Theorem $\mathcal{A}$). The proof will be divided into the following five cases.

Theorem $\mathcal{A}$. The map $\Psi$ defined above is an algebra homomorphism from $U_{r,s}(\hat{\mathfrak{g}}^\sigma)$ to $U_{r,s}(\hat{\mathfrak{g}}^\tau)$.

Let $E_i, F_i, \omega_i, \omega'_i$ denote the images of $e_i, f_i, \omega_i, \omega'_i$ ($i \in \hat{I}$) in the algebra $U_{r,s}(\hat{\mathfrak{g}}^\sigma)$ under the map $\Psi$ respectively. We shall check that the elements $E_i, F_i, \omega_i, \omega'_i$ ($i \in \hat{I}$), $\gamma^\pm \frac{1}{2}, \gamma'^\pm \frac{1}{2}$ satisfy the defining relations (X1)–(X6), where (X = $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}$) are given in Definition 3.1. First of all, the defining relations (X1)–(X3) can be verified directly as in the untwisted case [HRZ], so we are left to check the relations (X4)–(X6) involved with $i = 0$ case by case.

6.1. Proof of Theorem $\mathcal{A}$ for the case of $U_{r,s}(A_{2n-1}^{(2)})$. For relation (A4), when $i \neq 0$, one gets by definition that

$$[E_0, F_i] = a[\frac{1}{p_i}x_i^{-1}(0) \cdot (\gamma'^{-1}\omega'^{-1}), \frac{1}{p_i}x_i^{-1}(0)]$$

$$= -a\frac{1}{p_i} [x_i^{-1}(0), x_i^{-1}(1)\gamma'^{-1}\omega'^{-1}).$$

To prove this, we need the following technical lemmas, which can be easily proved as in the cases of untwisted types (see [HZ]).

Lemma 6.1. The following identities are true.

(6.1) $[x_i^{-1}(0), y_{i-1+i+1}^{-1}(1)]_{s-1} = 0, \quad 1 < i < n,
(6.2) [x_i^{-1}(0), y_{i-1+i}^{-1}(1)]_{(rs)-1} = 0, \quad 1 \leq i \leq n - 1,
(6.3) [x_i^{2}(0), x_i^{13}(1)] = 0,
(6.4) [x_i^{2}(0), y_{i-1+i+2}^{1}(1)] = 0, \quad 1 \leq i \leq n - 2
(6.5) [x_i^{2}(0), y_{i-1+i+2}^{1}(1)] = 0, \quad 2 \leq i \leq n - 2.

The following three lemmas are needed for later reference.

Lemma 6.2. It is clear that $[x_i^{0}(0), x_i^{(r-1)=-1}(0), x_i^{0}(0), x_i^{0}(-1), x_i^{0}(0)]_{(rs-1,-1,1)} = 0.$
PROOF. Using (5.3) and the Serre relations, one gets
\[
\begin{align*}
[x_i^-(0), [x_{i-1}^-, x_i^-(0), x_{i+1}^-(0)]_{(r^{-1}, s^{-1})}]^r_s &= 0 \text{ (using (5.3))} \\
+ r^{-1}[x_i^-(0), x_i^-(0), [x_{i-1}^-, x_{i+1}^-(0)]_{(1, s^{-1})}] &= 0 \text{ (by the Serre relation)}
\end{align*}
\]

which implies that \((1 + r^{-1}s)[x_i^-(0), [x_{i-1}^-, x_i^-(0), x_{i+1}^-(0)]_{(r^{-1}, s^{-1})}] = 0\). Thus if \(r \neq -s\), it follows that
\[
[x_i^-(0), [x_{i-1}^-, x_i^-(0), x_{i+1}^-(0)]_{(r^{-1}, s^{-1})}] = 0.
\]

\[\square\]

LEMMA 6.3. Under the previous notations, we have \([x_{n-1}^-, x_{1\ n-1}^-]_r = 0\).

PROOF. If we combine (5.4) with the Serre relations, then we get
\[
\begin{align*}
[x_{n-1}^-, x_{1\ n-1}^-]_r &= [x_{n-1}^-, [x_{n-1}^-, x_{n-2}^-, x_{1\ n-3}^-]_{(s, s)}]_r \text{ (by definition)} \\
&= [x_{n-1}^-, [x_{n-1}^-, x_{n-2}^-, x_{1\ n-3}^-]_{(s, s)}]_r \text{ (using (5.4))} \\
&= s[x_{n-1}^-, [x_{n-1}^-, x_{n-2}^-]_{s}, x_{1\ n-3}^-]_{(s, r)} \text{ (using (5.4))} \\
&= s[x_{n-1}^-, [x_{n-1}^-, x_{n-2}^-]_{s}, x_{1\ n-3}^-]_{(s, r)} \text{ (by the Serre relation)} \\
&= [x_{n-1}^-, [x_{n-1}^-, x_{n-2}^-]_{s}, x_{1\ n-3}^-]_{(1, r)} \text{ (by the Serre relation)} \\
&= 0 \text{ (by the Serre relation)}
\end{align*}
\]

\[\square\]

LEMMA 6.4. We have that \([x^-_n(0), x^-_{1\ n}(1)]_{r^2} = 0\).

PROOF. Using (5.3) and the Serre relations, we obtain
\[
\begin{align*}
[x^-_n(0), x^-_{1\ n}(1)]_{r^2} &= [x^-_n(0), [x^-_n(0), [x^-_{n-1}(0), x^-_{1\ n-2}(1)]_{(s, s^2)}]_r \text{ (by definition)} \\
&= [x^-_n(0), [x^-_n(0), [x^-_{n-1}(0), x^-_{1\ n-2}(1)]_{(s, s^2)}]_r \text{ (using (5.3))} \\
&= s^2[x^-_n(0), [x^-_n(0), [x^-_{n-1}(0), x^-_{1\ n-2}(1)]_{(s, r^2)}]_r \text{ (using (5.3))} \\
&= [x^-_n(0), [x^-_n(0), [x^-_{n-1}(0), x^-_{1\ n-2}(1)]_{(s, r^2)}]_r \text{ (by the Serre relation)} \\
&= [x^-_n(0), [x^-_n(0), [x^-_{n-1}(0), x^-_{1\ n-2}(1)]_{(s, r^2)}]_r \text{ (by the Serre relation)} \\
&= 0 \text{ (by Serre relation)}
\end{align*}
\]

\[\square\]
Now we turn to the relation (A4).

**Proposition 6.5.** \([x_i^- (0), x_{\theta}^- (1)]_{(i, 0)} = 0, \text{ for } i \in I.\)

**Proof.** (I) If \(i = 1, (1, 0) = rs.\) By the lemma 6.2, fix \(i = 1,\) we immediately have

\([x_1^- (0), x_{\theta}^- (1)]_{(rs)} = [x_1^- (0), y_{i+1}^- (1)]_{(rs)} = 0.\)

(II) When \(i = 2, (2, 0) = s.\) By the definition of quantum root vectors, we get easily

\[
[x_2^- (0), x_{\theta}^- (1)]_{s^{-1}} = [x_2^- (0), x_3^- (0), y_{1}^- (1)]_{(s^{-1}, s^{-1})} = 0 \quad \text{(using 5.3)}
\]

\[
+ r^{-1}[x_2^- (0), x_3^- (0), y_{1}^- (1)]_{(s^{-1})} = 0 \text{ by Lemma 6.4}
\]

\[
+ s^{-1}[x_2^- (0), x_3^- (0)]_{s^{-1}} = 0 \text{ by Lemma 6.4}
\]

\[
= 0.
\]

(III) When \(2 < i < n, (i, 0) = 1.\) First it follows from 5.3 and the definitions that

\[
x_{\theta}^- (1) = [x_2^- (0), \cdots, x_{i-1}^- (0), x_i^- (0), x_{i+1}^- (0), y_{i+2}^- (1)]_{(s^{-1}, s^{-1})} \quad \text{(using 5.3)}
\]

\[
= [x_2^- (0), \cdots, x_{i-1}^- (0), x_i^- (0), x_{i+1}^- (0), y_{1}^- (1), y_{i+2}^- (1)]_{(s^{-1}, s^{-1})} \quad \text{(using 5.3)}
\]

\[
+ r^{-1}[x_2^- (0), \cdots, x_{i-1}^- (0), x_i^- (0), y_{1}^- (1), y_{i+2}^- (1)]_{(s^{-1}, s^{-1})}
\]

\[
\text{= 0 by Lemma 6.1}
\]

\[
= [x_2^- (0), \cdots, x_{i-1}^- (0), x_i^- (0), y_{1}^- (1), y_{i+2}^- (1)]_{(s^{-1}, s^{-1})}
\]

\[
\text{= 0 by Lemma 6.5}
\]

The above result implies that

\[
[x_i^- (0), x_{\theta}^- (1)]_{(i, 0, 0)} = [x_2^- (0), \cdots, x_{i-1}^- (0), x_i^- (0), x_{i+1}^- (0), y_{i+2}^- (1)]_{(s^{-1}, s^{-1})} \quad \text{(by 5.3 and the Serre relation)}
\]

Therefore it suffices to check

\[
[x_i^- (0), [x_{i-1}^- (0), x_i^- (0), x_{i+1}^- (0)]_{(s^{-1}, s^{-1}), y_{i+2}^- (1)}_{(s^{-1}, s^{-1})} = 0.
\]
Actually, it is obvious that
\[
[x_n^-(0), [x_{n-1}^-(0), x_n^-(0), x_{n+1}^-(0)](r^{-1}, r^{-1})], y_{n+2}^-(1)](r^{-1}, 1) = 0 \quad (\text{using } \text{(5.3)})
\]
\[
= \left[ [x_i^-(0), x_{i-1}^-(0), x_i^-(0), x_{i+1}^-(0)](r^{-1}, r^{-1}), y_{i+2}^-(1) \right]_{r^{-1}} = 0 \quad (\text{by Lemma 6.2})
\]
\[
+ [x_{i-1}^-(0), x_i^-(0), x_{i+1}^-(0)](r^{-1}, r^{-1}), [x_i^-(0), y_{i+2}^-(1)]_{r^{-1}} = 0 \quad (\text{by Lemma 6.4})
\]
\[
= 0.
\]

(IV) When \(i = n, (n, 0) = (rs)^{-2}\). Firstly note that
\[
y_{n-1}^-(1) = [x_{n-1}^-(0), x_n^-(0), x_{n-1}^-(1)](s^2, r^{-1}) = [x_{n-1}^-(0), x_n^-(0)]_{r^{-2}}, x_{n-1}^-(1)]_{rs^2}
\]
\[
+ r^{-2}[x_n^-(0), [x_{n-1}^-(0), x_{n-1}^-(1)]_r]_{(rs)^2} = 0 \quad (\text{by Lemma 6.3})
\]
\[
= [x_{n-1}^-(0), x_n^-(0)]_{r^{-2}}, x_{n-1}^-(1)]_{rs^2},
\]
Applying the above result, it is easy to get
\[
[x_n^-(0), y_{n-1}^-(1)]_{s^4} = [x_n^-(0), [x_{n-1}^-(0), x_n^-(0)]_{r^{-2}}, x_{n-1}^-(1)]_{(rs^2, s^4)} = 0 \quad (\text{using } \text{(5.3)})
\]
\[
= [x_n^-(0), [x_{n-1}^-(0), x_n^-(0)]_{r^{-2}}, x_{n-1}^-(1)]_{r^{-2}} = 0 \quad (\text{by the Serre relation})
\]
\[
+ s^2[x_{n-1}^-(0), x_n^-(0)]_{r^{-2}}, [x_n^-(0), x_{n-1}^-(1)]_{s^2} = 0 \quad (\text{using } \text{(5.4)})
\]
\[
= s^2[x_{n-1}^-(0), x_n^-(0)]_{r^{-2}}, [x_{n-1}^-(0), x_{n-1}^-(1)]_{r^{-3}} = 0 \quad (\text{by Lemma 6.4})
\]
\[
+ r^2 s^2[x_{n-1}^-(0), x_{n-1}^-(1)]_{r^{-4}} = 0 \quad (\text{using } \text{(5.4)})
\]
\[
= r^2 s^2[x_{n-1}^-(0), x_{n-1}^-(1)]_{r^{-4}} = 0.
\]
Expanding two sides of the above relation, we have that
\[
(1 + r^{-2}s^2)[x_n^-(0), y_{n-1}^-(1)]_{(rs)^2} = 0.
\]
So, if \(r \neq -s\), it holds that
\[
[x_n^-(0), y_{n-1}^-(1)]_{(rs)^2} = 0.
\]

Consequently, using \textit{(5.3)} and Serre relation repeatedly, our previous result implies immediately that
\[
[x_n^-(0), x_n^-(1)]_{rs} = 0 \quad (\text{by definition})
\]
\[
= [x_n^-(0), [x_2^-(0), \cdots, x_{n-2}^-(0), y_{n-1}^-(1)]_{(r^{-1}, \cdots, r^{-1})]}_{(rs)^2} = 0 \quad (\text{by Lemma 6.5})
\]
\[
= 0.
\]

Hence Proposition \textit{6.5} is proved. \(\square\)

Next, we turn to the commutation relation in \((A4)\) involved with \(i = j = 0\).

**Proposition 6.6.** \([E_0, F_0] = \frac{\gamma^{-1} \omega^{-1} - \gamma^{-1} \omega^{-1}}{r-s}\).
PROOF. First we consider
\[ [E_0, F_0] = (rs)^{-2} \left[ x^{-1}_0(1) \gamma^{-1} \omega^{-1}, \gamma^{-1} \omega^{-1} x_0^+(1) \right] \]
\[ = (rs)^{-2} \left[ x^{-1}_0(1), x_0^+(1) \right] \cdot \left( \gamma^{-1} \gamma^{-1} \omega^{-1} \omega^{-1} \right). \]

Recall the constructions of \( x^{-1}_0(1) \) and \( x^+_0(1) \) in the present case.
\[ x^{-1}_0(1) = y^{-1}_{12}(1) = [x^{-1}(0), \cdots, x^{-1}(0), x^{-1}_{1n}(1)]_{s_2, r_1, \cdots, r_1}, \]
\[ x^+_0(1) = y^+_1(1) = [[x^+_0(0), x^+_0(0)], x^+_0(0)]_{s_2, r_1, \cdots, r_1}. \]

Thanks for the result of the case \( A^{(1)}_{n-1} \) \([HRZ]\), one has
\[ [x^{-1}_{1n}(1), x^+_{1n}(1)] = \frac{\gamma \omega_{\alpha_{1n-1}} - \gamma' \omega_{\alpha_{1n-1}}}{r-s}. \]
Furthermore, we get from similar calculation that
\[ [x^{-1}_{1n}(1), x^+_{1n}(1)] \quad \text{(by definition)} \]
\[ =[[x^{-1}(0), [x^{-1}_{1n}(1)], s_2, [x^+_0(0)], r_2] \quad \text{(by definition)} \]
\[ =[[x^{-1}(0), [x^{-1}_{1n}(1)], s_2, [x^+_0(0)], r_2] = 0 \quad \text{(by (5.3) and (D9))} \]
\[ + [x^{-1}_{1n}(1), [x^{-1}_{1n}(1)], s_2, [x^+_0(0)], r_2] \quad \text{(using (5.3) and (D9))} \]
\[ + [x^{-1}_{1n}(1), x^{-1}(0), [x^{-1}_{1n}(1)], s_2, [x^+_0(0)], r_2] \quad \text{(using (D9) and (D6))} \]
\[ = \frac{\gamma \omega_{\alpha_{1n-1}} - \gamma' \omega_{\alpha_{1n-1}}}{r-s} \]
\[ = \frac{\gamma \omega_{\alpha_{1n-1}} - \gamma' \omega_{\alpha_{1n-1}}}{r-s}. \]

We can proceed in the same way to obtain
\[ [y^{-1}_{1n}(1), y^+_{1n}(1)] \quad \text{(by definition)} \]
\[ =[[y^{-1}(0), [y^{-1}_{1n}(1)], s_2, [y^+_0(0)], r_2] \quad \text{(using (5.3))} \]
\[ =[[y^{-1}(0), [y^{-1}_{1n}(1)], s_2, [y^+_0(0)], r_2] \quad \text{(using (5.3), (D9) and (5.5))} \]
\[ + [y^{-1}_{1n}(1), [y^{-1}_{1n}(1)], s_2, [y^+_0(0)], r_2] \quad \text{(by the above result and (D6))} \]
\[ + [y^{-1}_{1n}(1), [y^{-1}_{1n}(0), [y^{-1}_{1n}(1)], s_2, [y^+_0(0)], r_2] \quad \text{(by (D9) and (D6))} \]
\[ + [y^{-1}_{1n}(1), [y^{-1}_{1n}(1), y^{-1}_{1n}(0)], r_2] \quad \text{(using (5.3), (D9) and (5.5))} \]
\[ = (rs)^{-1} \frac{\gamma \omega_{\alpha_{1n}}}{r-s} \cdot \frac{\omega_{n-1} - \omega_{n-1}}{r-s} + (rs)^{-1} \frac{\gamma \omega_{\alpha_{1n}}}{r-s} \cdot \frac{\omega_{n-1} - \omega_{n-1}}{r-s} \]
\[ = (rs)^{-1} \frac{\gamma \omega_{\alpha_{1n}}}{r-s} \cdot \frac{\omega_{n-1} - \omega_{n-1}}{r-s}. \]

By the inductive steps it is obvious that
\[ [y^{-1}_{12}(1), y^+_{12}(1)] = (rs)^{2-n} \frac{\gamma \omega_{\alpha_{12}} - \gamma' \omega_{\alpha_{12}}}{r-s}. \]
Hence we arrive at the required relation. \( \square \)

Let us pause to recall the following fact, which will be used in the sequel.
LEMMA 6.7. If $X \in U_q(g_0)^+$, and $[A, F_k] = 0, \forall k \in I$, then $A = 0$. On the other hand, If $X \in U_q(g_0)^-$, and $[A, E_k] = 0, \forall k \in I$, then $A = 0$

We now return to check relations (A5) and (A6). Indeed, relations (A5) or (A6) can be obtained from another one by applying the action of $\tau$. The following three relations are the main statements, others can be verified by similar calculations.

LEMMA 6.8. Using the above notations, we have the following relations:

1. $[E_0, E_n]_{(rs)}^{-2} = 0$,
2. $[E_2, E_2, E_0]_{(s^{-1}, r^{-1})} = 0$,
3. $[F_{n-1}, F_{n-1}, F_{n-1}, F_n]_{(r^{-2}, (rs)^{-1}, s^{-2})} = 0$,

PROOF. (1) Combining the definitions and the Drinfeld relations, we see that

$$[E_0, E_n]_{(rs)}^{-2} = a [x_0^-(1), x_0^+(0)] \cdot \gamma'^{-1} \omega_0^{-1}$$

$$= a [x_0^-(0), \cdots, x_{n-1}^-(0), [x_0^-(0), x_0^+(0)], x_1^{-1}(1)]_{(s^{-1}, r^{-1})} \cdot \gamma'^{-1} \omega_0^{-1}$$

$$= [x_0^-(0), \cdots, x_{n-1}^-(0), x_1^{-1}(1)] \cdot \gamma'^{-1} \omega_0^{-1}$$

$$= 0,$$

where the last step follows from the following calculations

$$[x_{n-1}^-(0), x_1^{-1}(1)] = [x_{n-1}^-(0), [x_{n-2}^-(0), x_{n-3}^{-1}(1)]_{(s, s)}]$$ (using (5.3))

$$= [x_{n-1}^-(0), [x_{n-2}^-(0), x_1^{-1}(1)]_{(s, s)} + s[x_{n-1}^-(0), x_{n-2}^{-1}(0), [x_{n-3}^-(0), x_1^{-1}(1)]_{(1, r)}]$$ (0 by (D10))

$$= [[x_{n-1}^-(0), x_{n-2}^-(0)]_{(s, r)}, x_1^{-1}(1)]_{s}$$ (using (5.3))

$$+ r[[x_{n-1}^-(0), x_{n-2}^-(0)]_{s}, [x_{n-1}^-(0), x_1^{-1}(1)]_{r^{-1}s} = 0.$$

(2) Relation (D9) yields directly that,

$$[x_2^+(0), x_0^- (1)] = [[x_2^+(0), x_0^- (1)], y_{13}^{-1}(1)]_{r^{-1}}$$

$$+ [x_2^+(0), \cdots, x_0^- (0), \cdots, x_3^-(0), [x_2^+(0), x_3^- (0)], x_1^-(1)]_{(s, \cdots, s, s^2, r^{-1}, \cdots, r^{-1})}$$

$$= (rs)^{-1} y_{13}^{-1}(1) \omega_2.$$

The statement (2) follows from the above results and $[x_2^+(0), y_{13}^{-1}(1)] = 0$, which holds by direct calculation.

(3) Denote that $X = [F_{n-1}, F_{n-1}, F_{n-1}, F_n]_{(r^{-2}, (rs)^{-1}, s^{-2})}$. It is obvious that $X \in U_{r,s}(g^0)^-$. Therefore, by Lemma 6.7 in order to prove $X = 0$, it suffices to check $[x_2^+(0), X] = 0$ for $i \in I$. For $i$ not equal to $n - 1$ or $n$, the claim is obvious. So here we only verify the cases of $i = n - 1$ and $i = n$. 


Firstly, we get from relation (D9) in the case of $i = n$,

$$\left[ x_n^+(0), X \right]$$

$$= \frac{1}{2} \left( \left[ x_n^+(0), x_{n-1}^+(0), x_{n-1}^-(0), x_{n-1}^-(0), x_n^-(0) \right]_{(r^{-2}, (rs)^{-1}, s^{-2})} \right)$$

$$= \frac{1}{2} \left( \left[ x_{n-1}^-(0), x_{n-1}^- (0), x_{n-1}^- (0), x_n^-(0) \right]_{(r^{-2}, (rs)^{-1}, s^{-2})} \right)$$

$$= \frac{-r^{-2}s^{-6}}{2} \omega_n' \left[ x_{n-1}^-(0), x_{n-1}^- (0), x_{n-1}^- (0) \right]_{(r^{-1}s, 1)} = 0.$$

One see that the same is true for $i = n - 1$,

$$\left[ x_{n-1}^+(0), X \right]$$

$$= \frac{1}{2} \left( \left[ x_{n-1}^+(0), x_{n-1}^-(0), x_{n-1}^- (0), x_{n-1}^- (0), x_n^- (0) \right]_{(r^{-2}, (rs)^{-1}, s^{-2})} \right)$$

$$= \frac{1}{2} \left( \left[ x_{n-1}^- (0), x_{n-1}^- (0), x_{n-1}^- (0), x_n^-(0) \right]_{(r^{-2}, (rs)^{-1}, s^{-2})} \right)$$

$$+ \left[ x_{n-1}^- (0), x_{n-1}^- (0), x_{n-1}^- (0), x_n^-(0) \right]_{(r^{-2}, (rs)^{-1}, s^{-2})}$$

$$+ \left[ x_{n-1}^- (0), x_{n-1}^- (0), x_{n-1}^- (0), x_n^-(0) \right]_{(r^{-2}, (rs)^{-1}, s^{-2})}$$

$$= \frac{(rs)^{-2}(r+s)}{2} \left[ x_{n-1}^- (0), x_{n-1}^- (0), x_n^-(0) \right]_{(r^{-2}, (rs)^{-1}, s^{-2})}$$

$$= 0.$$

Consequently, theorem $A$ has been proved for the case of $A_{2n-1}^{(2)}$. $\square$

6.2. Proof of theorem $A$ for the case of $U_{r,s}(A_{2n}^{(2)})$. Let us turn to the case of $A_{2n}^{(2)}$. Similarly we only show some key relations (B4)–(B6) involving $i = 0$.

Similarly, when $i \neq 0$, observe that

$$\left[ E_0, F_i \right] = a \left[ x_i^- (1) \cdot (\gamma'^{-1} \omega_i^{-1}), \frac{1}{p_i} x_i^- (0) \right]$$

$$= -a \frac{1}{p_i} \left[ x_i^- (0), x_i^- (1) \right]_{(\gamma'^{-1} \omega_i^{-1})}.$$  

Hence, in order to verify relation (B4), it is enough to check the following Proposition

PROPOSITION 6.9. \( \left[ x_i^- (0), x_i^- (1) \right]_{(i, 0)^{-1}} = 0, \) for $i \in I$.

Before giving the proof of Proposition 6.9, we need the following crucial lemmas which can be proved directly.

LEMMA 6.10. For $1 < i < n$, it holds that $\left[ x_{i-1}^-(0), y_{1+i}^{-1} (1) \right] = 0$.

LEMMA 6.11. For $1 \leq i \leq n - 1$, it follows that $\left[ x_i^- (0), y_i^{-1}(1) \right]_{s^{-1}} = 0$.

LEMMA 6.12. It yields $\left[ x_{n-1}^- (0), x_{1-n}^-(1) \right] = 0$.

LEMMA 6.13. One checks directly $\left[ x_{n-1}^- (0), x_{1-n}^-(1) \right] = 0$. 

In the case of $i = 0$ (the case of $A_{2n}^{(2)}$), we have

$$\left[ x_0^- (0), x_0^- (1) \right]_{(0, 0)^{-1}} = 0.$$
LEMMA 6.14. Using the above notations, one has $[x_i^{-1}(0), y_i^{-1}(1)] = 0$.

Proof of Proposition 6.9 (I) When $i = 1$, $(1, 0) = s^2$, one gets from the above lemma 6.10

$$[x_i^{-1}(0), x_i^{-1}(1)]_{r^{-1}s} \quad \text{(by definition)}$$

$$= [x_i^{-1}(0), [x_i^{-1}(0), x_i^{-1}(0), y_i^{-1}(1)]_{r^{-1}(rs)^{-1}}]_{s^{-2}} \quad \text{(using} \ (5.3) \text{)}$$

$$= [x_i^{-1}(0), [[x_i^{-1}(0), x_i^{-1}(0)]_{r^{-1}}, y_i^{-1}(1)]_{(rs)^{-1}}]_{s^{-2}} \quad \text{(using} \ (5.3) \text{)}$$

$$+r^{-1}[x_i^{-1}(0), x_i^{-1}(1), y_i^{-1}(1)]_{(1, s^{-2})} = 0 \quad \text{by Lemma} \ 6.10$$

$$= [[x_i^{-1}(0), x_i^{-1}(0), x_i^{-1}(0)]_{(r^{-1})s}, y_i^{-1}(1)]_{r^{-1}s} = 0 \quad \text{(by the Serre relation)}$$

$$+s^{-1}[x_i^{-1}(0), x_i^{-1}(1)]_{r^{-1}s} \quad \text{(by the definition)}$$

$$= s[[x_i^{-1}(0), x_i^{-1}(1)]_{r^{-1}}, y_i^{-1}(1)]_{(rs)^{-1}} = 0 \quad \text{by Lemma} \ 6.10$$

$$= s[x_i^{-1}(0), y_i^{-1}(1)]_{(rs)^{-1}} + [x_i^{-1}(0), y_i^{-1}(1)]_{r^{-1}s} \quad \text{(by the definition)}$$

$$= [y_i^{-1}(1), x_i^{-1}(0)]_{r^{-1}s}.$$ 

As a consequence of above result, it yields that $(1 + r^{-1}s)[x_i^{-1}(0), y_i^{-1}(1)] = 0$. Under the condition $r \neq -s$, it follows

$$[x_i^{-1}(0), y_i^{-1}(1)] = 0.$$

Using the above result, it yields from an immediate calculations that

$$[x_i^{-1}(0), x_i^{-1}(1)] \quad \text{(by the definition)}$$

$$= [x_i^{-1}(0), \ldots, x_i^{-2}(0), [x_i^{-1}(0), y_i^{-1}(1)]_{(r^{-1}, \ldots, r^{-1})}] = 0.$$
When \( i = n, \ (n, 0) = (rs)^{-1} \). Observe that,

\[
y_{-1}^{-1}(1) = [x_{-1}(0), x_{-1}(0), x_{-1}(1)]_{s, (rs)^{-1}, r^{-1}} \quad \text{(using (5.3))}
\]

\[
= [x_{-1}(0), x_{-1}(0), x_{-1}(1)]_{s, r^{-1}, x_{-1}(1)}_{s} \quad \text{(using (5.3))}
\]

\[
+ r^{-1}[x_{-1}(0), [x_{-1}(0), x_{-1}(0), x_{-1}(1)]_{s, (s, 1)}_{s}]_{r^{-1}, x_{-1}(1)}_{s} \quad \text{(= 0 by Lemma 6.12)}
\]

\[
= [x_{-1}(0), x_{-1}(0), x_{-1}(1)]_{s} \quad \text{(= 0 by Lemma 6.12)}
\]

\[
+ (r s)^{-\frac{1}{2}}[x_{-1}(0), [x_{-1}(0), [x_{-1}(0), x_{-1}(1)]_{s, 1}]_{r^{-1}, x_{-1}(1)}_{s}]_{(rs)^{-1}, x_{-1}(1)}_{s} \quad \text{(= 0 by Lemma 6.13)}
\]

Applying the above result, one sees that

\[
[x_{-1}(0), y_{-1}^{-1}(1)]_{s} \quad = \quad [x_{-1}(0), [x_{-1}(0), x_{-1}(0)]_{r^{-1}, x_{-1}(0)}_{s}]_{(rs)^{-1}, x_{-1}(1)}_{s} \quad \text{(using (5.3))}
\]

\[
= [x_{-1}(0), [x_{-1}(0), x_{-1}(0)]_{r^{-1}, x_{-1}(0)}_{s} \quad \text{(using (5.3))}
\]

\[
+ s [x_{-1}(0), x_{-1}(0)]_{r^{-1}, x_{-1}(0)}_{s} \quad \text{(using (5.3))}
\]

\[
= [x_{-1}(0), x_{-1}(0)]_{s} \quad \text{(= 0 by Lemma 6.14)}
\]

\[
+ rs [x_{-1}(0), y_{-1}^{-1}(1)]_{rs} \quad \text{(by the definition)}
\]

\[
+ rs^2 [x_{-1}(0), y_{-1}^{-1}(1)]_{rs} \quad \text{(by the definition)}
\]

\[
= rs [y_{-1}^{-1}(1), x_{-1}(0)]_{rs} + rs^2 [y_{-1}^{-1}(1), x_{-1}(0)]_{rs} \quad \text{(= 0 by Lemma 6.12)}
\]

The above equation means that

\[
(1 + r^{-\frac{1}{2}}s^\frac{1}{2} + r^{-1}s)[x_{-1}(0), y_{-1}^{-1}(1)]_{rs} = 0.
\]

So, if \((r/s)^{3/2} \neq 1\), it follows that

\[
[x_{-1}(0), y_{-1}^{-1}(1)]_{rs} = 0.
\]
By this result it follows from (5.3) and the Serre relation that,

\[
[x_{-n}^-(0), x_{-n}^- (1)]_{rs} \quad \text{(by definition)}
= [x_{-n}^-(0), [x_{-n}^- (0), \cdots, x_{-n}^- (-2), y_{1n}^- (1)]_{(r-1, \ldots, r-1)}]_{rs}
= [x_{-n}^- (0), \cdots, x_{-n}^- (-2), [x_{-n}^- (0), y_{1n}^- (1)]_{(r-1, \ldots, r-1)}]_{rs}
= 0.
\]

Thus Proposition 6.9 has been proved.

Now, we are ready to check the commutation relation (B4), that is,

**Proposition 6.15.** \([E_0, F_0] = \frac{\gamma^{-1} \omega^{-1} - \gamma^{-1} \omega^{-1}}{r-s}.\)

**Proof.** First we observe that

\[
[E_0, F_0] = (r s)^{n-2} \left[ x_{\theta}^-(1), \gamma^{-1} \omega^{-1} \right] = (r s)^{n-2} \left[ x_{\theta}^-(1), x_{\theta}^+(1) \right] \cdot (\gamma^{-1} \omega^{-1} x_{\theta}^-(1)).
\]

Hence we have to compute the bracket \([x_{\theta}^-(1), x_{\theta}^+(1)].\) Recall the construction of \(x_{\theta}^-(1)\) and \(x_{\theta}^+(1).\) By the induction step, one has,

\[
x_{\theta}^-(1) = y_{12}^-(1) = [x_{-2}^-(0), \cdots, x_{-n}^-(0), x_{1n}^- (1)]_{(s^2, r^{-1}, \ldots, r^{-1})},
\]

\[
x_{\theta}^+(1) = \tau(x_{\theta}^- (1)) = y_{12}^+(1).
\]

As a consequence of the case \(A_{n-1}^{(1)} \text{ [HRZ]},\) one has

\[
[x_{1n}^- (1), x_{1n}^+ (1)] = \frac{\gamma \omega_{\alpha_{1n}^-} - \gamma' \omega_{\alpha_{1n}^-}}{r-s}.
\]

Next, we consider

\[
[x_{1n}^-(1), x_{1n}^+ (1)] \quad \text{(by definition)}
= [\left[ x_{n}^- (0), x_{1n}^- (1) \right]_{s}, [x_{1n}^+ (1), x_{n}^+ (0)]_{r}] \quad \text{(using (5.3))}
= [\left[ x_{n}^- (0), x_{1n}^- (1) \right]_{s}, [x_{1n}^+ (1), x_{n}^+ (0)]_{r}] \quad (=0 \text{ by (5.3) and (D9))}
+ [\left[ x_{n}^- (0), x_{1n}^- (1) \right]_{s}, [x_{1n}^+ (1), x_{n}^+ (0)]_{r}] \quad \text{(using (5.3), (D6) and (D9))}
+ [\left[ x_{1n}^- (1), [x_{n}^- (0), x_{1n}^+ (0)], x_{1n}^- (1) \right]_{s}, [x_{1n}^+ (1), x_{n}^+ (0)]_{r}] \quad \text{(using (D9), (D6) and (5.5))}
+ [\left[ x_{1n}^- (1), [x_{n}^- (0), x_{1n}^+ (0)], x_{1n}^- (1) \right]_{s}, [x_{1n}^+ (1), x_{n}^+ (0)]_{r}] \quad (=0 \text{ by (5.3) and (D9))}
= \gamma \omega_{\alpha_{1n}^-} \frac{\omega' - \omega}{r^2 - s^2} + \gamma' \omega_{\alpha_{1n}^-} \frac{\omega' - \omega}{r^2 - s^2}
\]

\[
= \frac{\gamma' \omega_{\alpha_{1n}^-}}{r^2 - s^2}.
\]
Repeating the above steps, one also gets that

\[ [y_{1,n}^{-1}(1), y_{1,n}^+(1)] \] (by definition)

\[ = [[x_n^{-1}(0), x_{1,n}^+(1)]_{(rs)^{-\frac{1}{2}}}, [x_{1,n}^{-1}(1), x_n^+(0)]_{(rs)^{-\frac{1}{2}}} \] (using (5.3))

\[ = [[x_n^{-1}(0), x_{1,n}^+(1)], x_{1,n}^{-1}(1)]_{(rs)^{-\frac{1}{2}}, x_n^+(0)]_{(rs)^{-\frac{1}{2}}} \] (using (5.3), (D9) and (5.5))

\[ + [[x_n^{-1}(0), [x_{1,n}^{-1}(1), x_n^+(1)]_{(rs)^{-\frac{1}{2}}}, x_n^+(0)]_{(rs)^{-\frac{1}{2}}} (=0 by the above result and (D6))

\[ + [x_{1,n}^{-1}(1), [x_n^{-1}(0), x_n^+(0)], x_{1,n}^{-1}(1)]_{(rs)^{-\frac{1}{2}}}, x_n^+(0)]_{(rs)^{-\frac{1}{2}}} (=0 by (D9) and (D6))

\[ + [x_{1,n}^{-1}(1), [x_n^{-1}(0), x_n^+(0)], [x_{1,n}^{-1}(1), x_n^+(0)]_{(rs)^{-\frac{1}{2}}}, x_n^+(0)]_{(rs)^{-\frac{1}{2}}} \] (using (5.3), (D9) and (5.5))

\[ = \frac{r - s}{r^\frac{1}{2} - s^\frac{1}{2}} \gamma \omega_{\alpha_2} - \gamma' \omega_{\alpha_3} - \gamma \omega_{\alpha_1} - \gamma' \omega_{\alpha_3} - \omega_{\alpha_1} \]

\[ = \frac{r - s}{r^\frac{1}{2} - s^\frac{1}{2}} \gamma \omega_{\beta_2} - \gamma' \omega_{\beta_1} - \omega_{\alpha_1} \].

Now we arrive at

\[ [y_{1,n}^{-1}(1), y_{1,n}^+(1)] \] (by definition)

\[ = [[x_n^{-1}(0), y_{1,n}^{-1}(1), [y_{1,n}^+(0), y_{1,n}^{-1}(0)]]_{s^{-1}} \] (using (5.3))

\[ = [[x_n^{-1}(0), y_{1,n}^{-1}(1)], y_{1,n}^+(1), [y_{1,n}^{-1}(0), y_{1,n}^+(0)]_{s^{-1}} (=0 by using (5.3) and (D9))

\[ + [[x_n^{-1}(0), [y_{1,n}^{-1}(1), y_{1,n}^+(0)]_{s^{-1}}}, x_{1,n}^{-1}(0)]_{s^{-1}} (=0 by the above result and (D6))

\[ + [y_{1,n}^+(1), [x_n^{-1}(0), x_{1,n}^{-1}(0)], y_{1,n}^+(1)]_{s^{-1}} (=0 by (D9) and (D6))

\[ + [y_{1,n}^+(1), [x_n^{-1}(0), x_{1,n}^{-1}(0)], [y_{1,n}^+(1), x_{1,n}^{-1}(0)]_{s^{-1}} (=0 by (D9) and (5.5))

\[ = (rs)^{-1}(r - s) \frac{\gamma \omega_{\alpha_2} - \gamma' \omega_{\alpha_3} - \omega_{\alpha_1}}{r - s} + (rs)^{-1}(r - s) \frac{2 \gamma \omega_{\alpha_2} - \gamma' \omega_{\alpha_3} - \omega_{\alpha_1}}{r - s}

\[ = (rs)^{-1}(r - s) \frac{\gamma \omega_{\alpha_2} - \gamma' \omega_{\alpha_3} - \omega_{\alpha_1}}{r - s}.

At last, the following identity holds:

\[ [y_{1,1}(1), y_{1,1}^{-1}(1)] = (rs)^{2-n} \frac{\gamma \omega_{\alpha_2} - \gamma' \omega_{\alpha_3} - \omega_{\alpha_1}}{r - s}.

So we have completed the proof of Proposition 6.15. \qed

We now proceed to check the relation (B5). It suffices to verify two key Serre relations, others are similar.

**Lemma 6.16.** The following relations yield the Serre relations

1. \[ [E_1, E_1, E_1, E_0]_{(r^{-1}s^{-2}), (rs)^{-2}, r^{-2}s^{-1}} = 0, \]
2. \[ [E_n, E_n, E_n, E_{n-1}]_{(r^{-1}, (rs)^{-\frac{1}{2}}, s^{-1})} = 0, \]

**Proof.** (1) First we consider

\[ [x_n^+(0), x_n^-(1)] \]

\[ = [[x_n^+(0), x_n^-(1)], y_{2,1}^{-1}(1)]_{(rs)^{-\frac{1}{2}}} (=0 by (D9) and (D6))

\[ + [x_n^+(0), \cdots, x_n^-(0), x_n^+(0), \cdots, x_n^-(0)], [x_n^+(0), x_n^-(1)] \] (using (5.3), (D9) and (5.5))

\[ = - (\gamma')^{-\frac{1}{2}} y_{2,1}^{-1}(1). \]
where

\[ y_{21}^1(1) = [x_1^1(0), \cdots, x_n^1(0), x_n^- (0), \cdots x_{2}^3(0), x_2^2(1)]_{(s, \cdots, s, (rs)^\frac{1}{2}, r^{-1}, \cdots, r^{-1}, 1, r^{-2})}. \]

Applying the above result, one has,

\[
\begin{align*}
[ x_1^+ (0) , x_1^+ (0) , x_0^- (1) ]_{(1, r^{-1} s)} \\
= - (\gamma \gamma')^{-\frac{1}{2}} [ [ x_1^+ (0) , x_0^- (0) ] , y_{22}^2 (1) ]_{r^{-2} \omega^1} & \quad \text{(using (D9) and (D6))}
= - (\gamma \gamma')^{-\frac{1}{2}} (rs)^{-2} (r + s) y_{22}^2 (1) \omega^1,
\end{align*}
\]

where

\[ y_{22}^2(1) = [x_2^2(0), \cdots, x_n^- (0), x_n^- (0), \cdots x_{3}^3(0), x_3^2(1)]_{(s, \cdots, s, (rs)^\frac{1}{2}, r^{-1}, \cdots, r^{-1}, 1)}. \]

As a consequence of these results, it follows that

\[
\begin{align*}
& [ E_1 , E_1 , E_1 , E_0 ]_{(r^{-1} s^{-2}, (rs)^{-2}, r^{-2} s^{-1})} \\
& = a [ x_1^+ (0) , x_1^+ (0) , x_0^- (1) ]_{(1, r^{-1} s^{-2}, r^{-2} s^{-1})} \gamma^{-1} \omega^{-1} \\
& = - a (\gamma \gamma')^{-\frac{1}{2}} (rs)^{-2} (r + s) [ x_1^+ (0) , y_{22}^2 (1) ] \omega^2 \\
& = 0.
\end{align*}
\]

(2) Denote \( Y = [E_n, E_n, E_n, E_n - 1]_{(r^{-1}, (rs)^{-\frac{1}{2}}, s^{-1})}. \) It is clear that \( Y \in \mathbb{U}_r(s^r)^+ \).

By Lemma 6.7, in order to prove \( Y = 0 \), it suffices to check \([x_i^- (0), Y] = 0\) for \( i \in I \), which is trivial for \( i \) not equal to \( n - 1 \) or \( n \).

In the case of \( i = n - 1 \), one gets that

\[
\begin{align*}
& [ x_{n-1}^+ (0), Y ] \\
= & [ x_n^- (0), x_n^+ (0), x_n^- (0) , [ x_{n-1}^- (0), x_{n-1}^- (0) ] ]_{(r^{-1}, (rs)^{-\frac{1}{2}}, s^{-1})} \quad \text{(using (D9) and (D6))}
= & - r^{-1} s^{-3} \omega_{n-1} [ x_n^- (0), x_n^+ (0), x_n^- (0) ]_{(r^{-1}, (rs)^{-\frac{1}{2}}, 1)} \\
= & 0.
\end{align*}
\]

In the case of \( i = n \), it is easy to see that

\[
\begin{align*}
& [ x_n^- (0), Y ] \\
= & [ x_n^- (0), x_n^+ (0), x_n^- (0) , x_n^- (0) ]_{(r^{-1}, (rs)^{-\frac{1}{2}}, s^{-1})} \\
= & [ x_n^- (0), x_n^+ (0), x_n^- (0) , x_n^- (0) ]_{(r^{-1}, (rs)^{-\frac{1}{2}}, s^{-1})} \quad \text{(using (D9) and (D6))}
+ [ x_n^+ (0), x_n^- (0) , x_n^- (0) , x_n^- (0) ]_{(r^{-1}, (rs)^{-\frac{1}{2}}, s^{-1})} \quad \text{(using (D9) and (D6))}
+ [ x_n^+ (0), x_n^- (0) , x_n^- (0) , x_n^- (0) ]_{(r^{-1}, (rs)^{-\frac{1}{2}}, s^{-1})} \quad \text{(using (D9) and (D6))}
= & - (rs)^{-1} [ 2] a [ x_n^+ (0), x_n^- (0) , x_n^- (0) ]_{(r^{-1}, (rs)^{-\frac{1}{2}}, \omega^1_n)} \\
+ & (rs)^{-1} [ 2] a [ x_n^+ (0), x_n^- (0) , x_n^- (0) ]_{(r^{-1}, (rs)^{-\frac{1}{2}}, \omega^1_n)}
= & 0.
\end{align*}
\]

\( \square \)
6.3. Proof of Theorem A for the case of $U_{r,s}(D^2_{n+1})$. We now proceed to check those relations of (C4)–(C6) involving $i = 0$ in the present case.

Similarly, in order to verify (C4), it is enough to check the following proposition

**Proposition 6.17.** $[x_i^{-1}(0), x_i^0(1)]_{(i,0)-1} = 0$, for $i \in I$.

The proof of this proposition uses the following crucial but technical lemmas.

**Lemma 6.18.** $[x_i^0(0), x_i^{-1}(0), x_i^0(0), x_{i+1}^0(0)]_{(r-2, r-2)} = 0$ for $i = 2, \cdots, n-1$.

**Proof.** Combining (5.5) with the Serre relations, one gets

$$[x_i^{-1}(0), x_i^{-1}(0), x_i^0(0), x_{i+1}^0(0)]_{(r-2, r-2)} = 0 \quad \text{(using (5.5))}$$

which implies that $(1 + r^{-2}s^2)[x_i^{-1}(0), x_i^{-1}(0), x_i^0(0), x_{i+1}^0(0)]_{(r-2, r-2)} = 0$. Thus, when $r \neq -s$, it follows that

$$[x_i^{-1}(0), x_i^{-1}(0), x_i^{-1}(0), x_{i+1}^0(0)]_{(r-2, r-2)} = 0.$$

□

**Proof of Proposition 6.17** (I) When $i = 1$, $\langle 1, 0 \rangle = s^2$, one has,

$$[x_1^{-1}(0), x_1^{-1}(0), x_1^0(0), x_{n+1}^0(1)]_{(r-2, r-2)} = 0 \quad \text{(using (5.5))}$$

which implies that $(1 + r^{-2}s^2)[x_1^{-1}(0), x_1^{-1}(0), x_1^0(0), x_{n+1}^0(1)]_{(r-2, r-2)} = 0$. Thus, when $r \neq -s$, it follows that

$$[x_1^{-1}(0), x_1^{-1}(0), x_1^{-1}(0), x_{n+1}^0(1)]_{(r-2, r-2)} = 0.$$

□

II) When $1 < i < n$, $\langle i, 0 \rangle = 1$. Thanks to Lemma 6.18, it is easy to see that,

$$[x_i^{-1}(0), x_i^0(1)] = 0.$$

Furthermore, we are ready to show the communication relation (C4).

**Proposition 6.19.** $[E_0, F_0] = \gamma^{-1} \omega_0^{-1} \gamma^{-1} \omega_0^{-1}$. 

Proof. First, using the relation (D6), it holds that by induction,
\[
\begin{align*}
[E_0, F_0] &= (rs)^{2(n-1)} [x_0^- (1) \gamma'^{-1} \omega_{1}^{-1}, \gamma_{0}'^{-1} \omega_{0}'^{-1} x_0^+ (-1)] \\
&= (rs)^{2(n-1)} [x_0^- (1), x_0^+ (-1)] \cdot (\gamma_{0}'^{-1} \omega_{0}'^{-1} \omega_{0}^{-1}).
\end{align*}
\]

Now we recall the notations mentioned above.
\[
x_0^- (1) = x_{n-1}^- (1) = [x_{n-1}^-(0), \cdots, x_{n-1}^-(0), x_{n}^- (1)]_{(r-2, \cdots, r-2)},
\]
\[
x_0^+ (-1) = \tau(x_0^- (1)) = x_{n-1}^+ (-1)
\]
\[
= [x_{n-1}^+ (1), x_{n-1}^+ (0), \cdots, x_{n-1}^+(0)]_{(s-2, \cdots, s-2)}.
\]

The first induction step is to check,
\[
x_{n-1}^- (1), x_{n-1}^+ (-1)
\]
(by definition)
\[
= [x_{n-1}^- (0), x_{n-1}^+ (0)]_{r-2}, [x_{n-1}^- (-1), x_{n-1}^+ (0)]_{s-2}
\]
(by using (5,3))
\[
= [x_{n-1}^- (0), x_{n-1}^+ (1), x_{n-1}^+ (-1)]_{r-2}, x_{n-1}^+ (0)]_{s-2}
\]
(by using (5,3))
\[
+ [x_{n-1}^+ (-1), [x_{n-1}^- (0), x_{n-1}^+ (0)], x_{n-1}^+ (0)]_{s-2}
\]
(by using (5,3))
\[
= (rs)^{-2} \gamma \omega'_{\alpha_{n}-1} \omega_{n-1} - \gamma' \omega_{\alpha_{n}-1} \omega_{n-1}
\]
\[
= (rs)^{-2} \gamma \omega'_{\alpha_{n}-1} - \gamma' \omega_{\alpha_{n}-1}.
\]

Repeating the above steps, one obtains
\[
x_{n-2}^- (1), x_{n-2}^+ (-1)
\]
(by definition)
\[
= [x_{n-2}^- (0), x_{n-2}^+ (1)]_{r-2}, [x_{n-2}^- (-1), x_{n-2}^+ (0)]_{s-2}
\]
(by using (5,3))
\[
= [x_{n-2}^- (0), x_{n-2}^+ (1), x_{n-2}^+ (-1)]_{r-2}, x_{n-2}^+ (0)]_{s-2}
\]
(by using (5,3))
\[
+ [x_{n-2}^+ (-1), [x_{n-2}^- (0), x_{n-2}^+ (0)], x_{n-2}^+ (0)]_{s-2}
\]
(by using (5,3))
\[
= (rs)^{-2} \gamma \omega'_{\alpha_{n}-2} - \gamma' \omega_{\alpha_{n}-2}
\]
\[
= (rs)^{-4} \gamma \omega'_{\alpha_{n}-1} - \gamma' \omega_{\alpha_{n}-1}.
\]

By induction we arrive at the following
\[
x_{n-1}^- (1), x_{n-1}^+ (-1) = (rs)^{-2(n-1)} \gamma \omega'_{\alpha_{n}} - \gamma' \omega_{\alpha_{n}}
\]
which completes the proof of Proposition 6.19.

The last part of this subsection is to check the Serre relation (C5). Here we only verify the following two crucial Serre relations. Other Serre relations are checked similarly.

Lemma 6.20. We have that (1) \([E_1, E_1, E_0]_{(r-2, s-2)} = 0\),
(2) \([E_n, E_n, E_n, E_{n-1}]_{(r-2, (rs)-1, s-2)} = 0\).

Proof. (1) Before checking the first one, we need the relation
\[
x_{n-1}^+ (0), x_{n-1}^- (1)
\]
\[
= [x_{n-1}^+(0), x_{n-1}^+(0)]_{r-2} (using (D9) and (D6))
\]
\[
= (rs)^{-2} y_{n-2}^- (1) \omega_1.
\]
Consequently, it follows that

\[
\begin{align*}
[ E_1, E_1, E_0 ]_{(r^{-2}, s^{-2})} \\
= & a \left[ x_1^+(0), x_1^+(0), x_0^+ (1) \right]_{(1, r^{-2}s^2) \gamma^{-1} \omega^{-1}} \\
= & a(rs)^{-2} \left[ x_1^+(0), y_{n+2}^- (1) \right] \omega_1 \gamma^{-1} \omega^{-1} (=0 \text{ by } (D9)) \\
= & 0.
\end{align*}
\]

(2) By using Lemma 6.7, one can check the second result. Set

\[
Z = [ E_n, E_n, E_n, E_{n-1} ]_{(r^{-2}, (rs)^{-1}, s^{-2})}.
\]

It is clear that \( Z \in U_{r,s}(\mathfrak{g}^+) \). In order to prove \( Z = 0 \), it suffices to check \([ x_i^- (0), Z ] = 0 \) for \( i \in I \), which is trivial for \( i \) not equal to \( n - 1 \) or \( n \).

As for the case of \( i = n - 1 \),

\[
\begin{align*}
[ x_{n-1}^+ (0), Z ] \\
= & [ x_{n-1}^+ (0), x_n^+ (0), x_n^+ (0), x_{n-1}^+ (0) ]_{(r^{-2}, (rs)^{-1}, s^{-2})} \\
= & - r^{-2} s^{-6} \omega_{n-1} [ x_n^+ (0), x_n^+ (0), x_n^+ (0) ]_{(r^{-2}, s^{-1})} \\
= & 0.
\end{align*}
\]

In the case of \( i = n \), one computes directly that

\[
\begin{align*}
[ x_n^- (0), Z ] \\
= & [ x_n^- (0), x_n^+ (0), x_n^+ (0), x_n^+ (0), x_{n-1}^+ (0) ]_{(r^{-2}, (rs)^{-1}, s^{-2})} \\
= & [ [ x_n^- (0), x_n^+ (0) ] , x_n^+ (0), x_n^+ (0), x_{n-1}^+ (0) ]_{(r^{-2}, (rs)^{-1}, s^{-2})} \quad (\text{using } (D9) \text{ and } (D6)) \\
+ & [ x_n^+ (0), [ x_n^- (0), x_n^+ (0) ] , x_n^+ (0), x_{n-1}^+ (0) ]_{(r^{-2}, (rs)^{-1}, s^{-2})} \quad (=0 \text{ by } (D9) \text{ and } (D6)) \\
+ & [ x_n^+ (0), x_n^+ (0), [ x_n^- (0), x_n^+ (0) ] , x_{n-1}^+ (0) ]_{(r^{-2}, (rs)^{-1}, s^{-2})} \quad (\text{using } (D9) \text{ and } (D6)) \\
= & - (rs)^{-1} (r+s) [ x_n^+ (0), x_n^+ (0) ]_{(r^{-2}, (rs)^{-1})} \omega_n' \\
& + (rs)^{-1} (r+s) [ x_n^+ (0), x_n^+ (0) ]_{(r^{-2}, (rs)^{-1})} \omega_n' \\
& = 0.
\end{align*}
\]

\[ \square \]

6.4. Proof of Theorem \( \mathcal{A} \) for the case of \( U_{r,s}(\mathfrak{d}_4^{(3)}) \). The aim of this subsection is to check the relations \((D4)-(D6)\) involving \( i = 0 \) for the case of \( \mathfrak{d}_4^{(3)} \).

Similar to the above cases, the following proposition 6.21 implies relation \((D4)\). Recall the notation defined in the subsection 5.2.

\[
x_0^- (1) = [ x_1^- (0), x_2^- (0), x_1^- (1) ]_{(s^3, r^{-2}s^{-1})}.
\]

**Proposition 6.21.** \([ x_i^- (0), x_0^- (1) ]_{(i, 0)^{-1}} = 0 \), for \( i \in \{ 1, 2 \} \).
PROOF. (I) When \( i = 1 \), \( (1, 0) = rs^2 \). By repeatedly using (5.3), one has,

\[
\begin{align*}
[x_1^{-1}(0), x_0^{-1}(1)]_{r^{-1}s^{-2}} &= [x_1^{-1}(0), [x_1^{-1}(0), x_2^{-1}(0), x_1^{-1}(1)]_{(s^3, r^{-2}s^{-1})}]_{r^{-1}s^{-2}} \\
&= [x_2^{-1}(0), [x_1^{-1}(0), x_2^{-1}(0)]_{r^{-3}}, x_1^{-1}(1)]_{r^{-2}s^{-2}} \\
&= [x_1^{-1}(0), [x_2^{-1}(0), x_1^{-1}(1)]_{(rs)2}]_{r^{-1}s^{-2}} = 0 \quad (\text{by the Serre relation})
\end{align*}
\]

Thus if \( r \neq s \), it is easy to see that \( [x_1^{-1}(0), x_0^{-1}(1)]_{r^{-1}s^{-2}} = 0 \).

(II) When \( i = 2 \), \( (2, 0) = (rs)^{-3} \). Using (5.3) and the Serre relation, one gets,

\[
\begin{align*}
[x_2^{-1}(0), x_0^{-1}(1)]_{s^6} &\quad (\text{by definition}) \\
&= [x_2^{-1}(0), [x_1^{-1}(0), x_2^{-1}(0), x_1^{-1}(1)]_{(s^3, r^{-2}s^{-1})}]_{s^6} \\
&= [x_2^{-1}(0), [x_1^{-1}(0), x_2^{-1}(0)]_{r^{-3}}, x_1^{-1}(1)]_{s^6} \\
&= [x_2^{-1}(0), [x_1^{-1}(0), x_2^{-1}(0)]_{(rs)3}]_{r^{-1}s^{-2}} = 0 \quad (\text{by (5.3) and the Serre relation})
\end{align*}
\]

The result yields that \((1 + r^{-3}s^3)[x_2^{-1}(0), x_0^{-1}(1)]_{(rs)3} = 0 \), which implies that for \( r \neq -s \)

\[
[x_2^{-1}(0), x_0^{-1}(1)]_{(rs)3} = 0.
\]

\( \square \)

Further, we check the following relations:

**Proposition 6.22.** \( [E_0, F_0] = \frac{\gamma^{-1} - 1}{s^{-2} - 1} \).
PROOF. Similarly, we prove the relation by induction. First, note that
\[
\left[ E_0, F_0 \right] = (rs)^2 \left[ x_0^- (1), \gamma^{-1} \omega_0^{-1}, \gamma^{-1} \omega_0^{-1} x_0^+ (-1) \right] \\
= (rs)^2 \left[ x_0^- (1), x_0^+ (-1) \right] \cdot (\gamma^{-1} \gamma^{-1} \omega_0^{-1} \omega_0^{-1}).
\]

Next, consider the first step,
\[
\left[ x_2^- (0), x_1^+ (1) \right], x_2^+ (0), r \right] \text{ (using (5.3))} \\
\left[ x_2^- (0), x_1^+ (1) \right], x_2^+ (0) \right] \text{ (0 by (5.3) and the Serre relation)} \\
+ \left[ x_1^+ (-1), x_2^- (0), x_2^+ (0) \right], x_1^+ (1) \right] \text{ (using (5.3) and (D6))} \\
= \gamma \omega_1' \omega_2' - \gamma \gamma \omega_1 \omega_2 \\
= \frac{\gamma \omega_1' \omega_2' - \gamma \gamma \omega_1 \omega_2}{r - s}.
\]

Furthermore, one obtains that
\[
\left[ x_0^- (1), x_0^+ (-1) \right] \text{ (by definition)} \\
\left[ x_1^+ (1) \right], x_2^- (1) \right] \text{ (0 by (5.3))} \\
\left[ x_1^+ (1) \right], x_2^- (1) \right] \text{ (0 by (5.3) and (D6))} \\
+ \left[ x_1^+ (1) \right], x_2^- (1) \right] \text{ (0 by the above result)} \\
\left[ x_1^+ (0), x_2^- (1) \right], x_2^+ (0) \right] \text{ (using (5.3), (D9) and (D6))} \\
= (rs)^{-2} \gamma \omega_1' \omega_2' - \gamma \gamma \omega_1 \omega_2 \\
= (rs)^{-2} \gamma (\omega_1')^2 \omega_2' - \gamma (\omega_1)^2 \omega_2 \\
= \frac{(rs)^{-2} \gamma (\omega_1')^2 \omega_2' - \gamma (\omega_1)^2 \omega_2}{r - s}.
\]

So one obtains the required conclusion and the proof of Proposition 6.22 is completed. □

In the remaining part of this section, we check some key Serre relations of (D5).

LEMMA 6.23. One gets by direct calculations,
(1) \( [E_0, E_2]_{(rs)^{-3}} = 0 \),
(2) \( [E_1, E_1, E_0]_{(r^{-2}s^{-1}, r^{-1}s^{-2})} = 0 \),
(3) \( [E_1, E_1, E_1, E_2, E_2]_{(s^3, rs^2, r^2s, r^3)} = 0 \).

PROOF. (1) For the first relation, one immediately gets from the construction of \( E_0 \),
\[
\left[ E_0, E_2 \right]_{(rs)^{-3}} \\
= (rs)^{-2} \left[ x_0^- (1), x_0^+ (0) \right] \cdot \gamma'^{-1} \omega_0^{-1} \\
= \left[ x_1^+ (0), x_2^- (0), x_1^+ (1) \right]_{(s^3, r^{-2}s^{-1})} \cdot \gamma'^{-1} \omega_0^{-1} \\
= \left[ x_1^+ (0), x_2^- (0), x_2^+ (0), x_1^+ (1) \right]_{(s^3, r^{-2}s^{-1})} \cdot \gamma'^{-1} \omega_0^{-1} \\
= \left[ x_1^+ (0), x_1^- (1) \right] \cdot r_{s^{-1}} \omega_2 \gamma'^{-1} \omega_0^{-1} \\
= 0.
\]
(2) To verify the second relation, one first computes that
\[
[x^+_1(0), x^-_0(1)]_1 = ([x^+_1(0), x^-_1(0), x^-_2(0), x^-_1(1)]_{s^3, r^2s^2, r^2s^2}) + [x^+_1(0), x^-_2(0), [x^+_1(0), x^-_1(1)]]_{s^3, r^2s^2, r^2s^2}
\]
\[
= (rs)^{-2} \omega_1 [x^-_2(0), x^-_1(1)]_{s^3} - 3(\gamma')^{-2}[x^+_1(0), x^-_2(1)]_{r^2s^2, r^2s^2}
\]
\[
= (rs)^{-2} \omega_1 [x^-_2(0), x^-_1(1)]_{s^3}.
\]
Then we have
\[
[E_1, E_1, E_0]_{r^2s^2, r^2s^2, r^2s^2} = a [x^+_1(0), x^-_0(1)]_{1, r^2s^2, r^2s^2}
\]
\[
= a [x^+_1(0), \omega_1 [x^-_2(0), x^-_1(1)]_{s^3}]_{r^2s^2, r^2s^2}
\]
\[
= a \omega_1 [x^+_1(0), [x^-_2(0), x^-_1(1)]_{s^3}]_1
\]
\[
= a \omega_1 [x^-_2(0), x^+_1(0), x^-_1(1)]_{s^3}
\]
\[
= -a (\gamma')^{-2}(\omega_1)^2 x^-_2(1) = 0
\]

(3) Denote that \(X' = [E_1, E_1, E_1, E_2]_{(s^3, r^2s^2, r^2s^2, r^2s^2)}\). It is obvious that \(X' \in \mathcal{U}_{r, s}(g^{n^*})^+\), so by Lemma 6.2, it suffices to show \([x^-_i(0), X'] = 0\) for \(i \in I\). In fact, it is enough to check the cases of \(i\) equal to 1 and 2.

For \(i = 2\), one has
\[
[x^-_2(0), X'] = [x^-_2(0), [x^+_1(0), x^+_1(0), x^+_1(0), x^+_1(0), x^-_2(0)]_{(s^3, r^2s^2, r^2s^2, r^2s^2)}]
\]
\[
= [x^-_1(0), [x^+_2(0), x^+_1(0), x^+_1(0), x^+_1(0), x^-_2(0)]_{(s^3, r^2s^2, r^2s^2, r^2s^2)}]
\]
\[
= 3[x^+_1(0), x^+_1(0), x^+_1(0), x^+_1(0), x^-_2(0)]_{(r^2s^2, r^2s^2, r^2s^2, r^2s^2)}
\]
\[
= 0.
\]
As for \(i = 1\),
\[
[x^-_1(0), X'] = [x^-_1(0), [x^+_1(0), x^+_1(0), x^+_1(0), x^+_1(0), x^-_2(0)]_{(s^3, r^2s^2, r^2s^2, r^2s^2)}]
\]
\[
= [x^-_1(0), [x^+_1(0), x^+_1(0), x^+_1(0), x^+_1(0), x^-_2(0)]_{(s^3, r^2s^2, r^2s^2, r^2s^2)}]
\]
\[
+ [x^+_1(0), [x^-_1(0), x^+_1(0), x^+_1(0), x^+_1(0), x^-_2(0)]_{(s^3, r^2s^2, r^2s^2, r^2s^2)}]
\]
\[
+ [x^+_1(0), [x^-_1(0), x^-_1(0), x^+_1(0), x^+_1(0), x^-_2(0)]_{(s^3, r^2s^2, r^2s^2, r^2s^2)}]
\]
\[
+ [x^+_1(0), [x^-_1(0), x^-_1(0), x^-_1(0), x^+_1(0), x^-_2(0)]_{(s^3, r^2s^2, r^2s^2, r^2s^2)}]
\]
\[
= s^3 - - s^3 - s^3 - s^3 - s^3 - s^3
\]
\[
= 0.
\]
Therefore, we have finished the proof of theorem \( \mathcal{A} \) for the case of \( U_{r,s}(D^{(3)}_4) \). \( \square \)

### 6.5. Proof of theorem \( \mathcal{A} \) for the case of \( U_{r,s}(E_6^{(2)}) \)

Now we are left to check the last case of type \( E_6^{(2)} \). As before we only check those nontrivial relations of \((\mathcal{E}4)-(\mathcal{E}6)\)

involving \( i = 0 \).

We begin by listing the following simple lemmas.

**Lemma 6.24.** It is easy to verify that \( [x_0^-(0), z_6^-(1)]_{(rs)^{-1}} = 0 \).

**Lemma 6.25.** An immediate compute that \( [x_0^-(0), z_6^-(1)]_{r^{-1}s^{-2}} = 0 \).

**Lemma 6.26.** It follows that \( [x_0^-(0), z_6^-(1)]_{(rs)^2} = 0 \).

**Lemma 6.27.** It holds that \( [x_0^-(0), z_6^-(1)]_{(rs)} = 0 \).

The following Proposition yields the relation \((\mathcal{E}4)\) for the case of \( i \neq 0 \) and \( j = 0 \).

**Proposition 6.28.** \( [x_0^-(0), x_0^-(1)]_{(i, 0)} = 0 \) for \( i \in \{1, 2, 3, 4\} \).

**Proof.** (I) When \( i = 1, (1, 0) = rs^2 \). We computes that,

\[
[x_0^-(0), x_0^-(1)]_{r^{-1}s^{-2}}
\]

\[
= [x_0^-(0), [x_0^-(0), x_2^-(0), z_6^-(1)]_{(r^{-2}s^{-1}, r^{-2}s^{-1})}]_{r^{-1}s^{-2}} \quad \text{(using (5.3))}
\]

\[
= [x_0^-(0), [x_0^-(0), x_2^-(0), z_6^-(1)]_{(r^{-3}s^{-2}, r^{-1}s^{-2})}]_{r^{-1}s^{-2}} \quad \text{(using (5.3))}
\]

\[
+ r^{-1} [x_0^-(0), [x_2^-(0), x_0^-(0), z_6^-(1)]_{(rs)^{-1}}]_{r^{-1}s^{-2}} \quad \text{(=0 by Lemma 6.24)}
\]

\[
= [x_0^-(0), x_1^-(0), [x_2^-(0), x_0^-(0)]_{(r^{-1}s^{-1})}, z_6^-(1)]_{r^{-4}s^{-3}} \quad \text{(=0 by Serre relation)}
\]

\[
+ s^{-1} [x_0^-(0), x_2^-(0), [x_0^-(0), x_2^-(0), z_6^-(1)]_{(rs)^{-1}}]_{r^{-4}s^{-3}} \quad \text{(=0 by Lemma 6.24)}
\]

\[
= 0.
\]

(II) When \( i = 2, (2, 0) = rs \). In order to check \([x_0^-(0), x_0^-(1)]_{(rs)^{-1}} = 0\), one has,

\[
[x_0^-(0), x_0^-(1)]_{r^{-2}}
\]

\[
= [x_0^-(0), [x_0^-(0), x_2^-(0), z_6^-(1)]_{(r^{-2}s^{-1}, r^{-2}s^{-1})}]_{r^{-2}} \quad \text{(using (5.3))}
\]

\[
= [x_0^-(0), [x_0^-(0), x_2^-(0), z_6^-(1)]_{(r^{-3}s^{-2}, r^{-2})}]_{r^{-2}} \quad \text{(using (5.3))}
\]

\[
+ r^{-1} [x_0^-(0), [x_2^-(0), x_0^-(0), z_6^-(1)]_{(rs)^{-1}}]_{r^{-2}} \quad \text{(=0 by Lemma 6.24)}
\]

\[
= [x_2^-(0), x_2^-(0), x_1^-(0)]_{(r^{-1}s^{-1})}, z_6^-(1)]_{r^{-5}s^{-3}} \quad \text{(=0 by the Serre relation)}
\]

\[
+ s [x_0^-(0), x_2^-(0), [x_0^-(0), z_6^-(1)]_{r^{-2}s^{-1}}]_{(rs)^{-3}} \quad \text{(By the definition and (5.3))}
\]

\[
= s [x_0^-(0), x_0^-(0), [x_2^-(0), z_6^-(1)]_{r^{-3}s^{-1}}]_{(rs)^{-3}} \quad \text{(=0 by Lemma 6.25)}
\]

\[
= (rs)^{-1} [x_0^-(0), x_2^-(0), x_0^-(0)]_{s^2} \quad \text{(By the definition)}
\]

\[
= (rs)^{-1} [x_0^-(0), x_2^-(0)]_{s^2}
\]

which implies that \((1 + r^{-1}s)[x_0^-(0), x_0^-(1)]_{(rs)^{-1}} = 0\). Consequently, for \( r \neq -s \)

\[
[x_0^-(0), x_0^-(1)]_{(rs)^{-1}} = 0.
\]
(III) When \(i = 3, \langle 3, 0 \rangle = (rs)^{-2} \), observe that,
\[
\left[ x_3^{-}(0), x_\theta^{-}(1) \right]_{(rs)^2} = \left[ 1 \right]_{(rs)^2} = 0 \text{ by the Serre relation}
\]

(IV) When \(i = 4, \langle 4, 0 \rangle = (rs)^{-2} \). Using the above results, one has,
\[
\left[ x_4^{-}(0), x_\theta^{-}(1) \right]_{(rs)^2} = \left[ x_4^{-}(0), [x_1^{-}(0), z_7^{-}(1)]_{r^{-2}s^{-1}} \right]_{(rs)^2} = \left[ x_4^{-}(0), x_\theta^{-}(1) \right]_{(rs)^2} = 0 \text{ by Lemma 6.26}
\]

Relation (E4) for the case of \(i, j = 0 \) follows from proposition 6.29

**Proposition 6.29.** \[ E_0, F_0 \] = \( \frac{\gamma^{-1}\omega_\alpha^{-1} - \gamma^{-1}\omega_\beta^{-1}}{r^{-s}} \).

**Proof.** We argue by induction. First, using relation (D6) it follows that,
\[
\left[ E_0, F_0 \right] = (rs)^{5}[2]_2^{-1} \left[ x_\theta^{-}(1) \gamma^{-1}\omega_\alpha^{-1}, \gamma^{-1}\omega_\beta^{-1}, x_\theta^{+}(-1) \right] = (rs)^{5}[2]_2^{-1} \left[ x_\theta^{-}(1), x_\theta^{+}(-1) \right] \cdot (\gamma^{-1}\omega_\alpha^{-1}\omega_\beta^{-1}).
\]

Now recall the notations mentioned above.
\[
x_\theta^{-}(1) = z_8^{-}(1)
\]
\[
= [x_1^{-}(0), x_2^{-}(0), x_3^{-}(0), x_4^{-}(0), x_5^{-}(1), \cdots, x_1^{-}(1) \cdots, x_8^{-}(1)]_{(s, s^2, r^{-1}, s^2, r^{-2}s^{-1}, r^{-2}s^{-1})},
\]
\[
x_\theta^{+}(-1) = z_8^{+}(-1)
\]
\[
= [x_1^{+}(-1), \cdots, x_4^{+}(0), x_2^{+}(0), x_3^{+}(0), x_4^{+}(0), x_5^{+}(0), x_6^{+}(0), x_7^{+}(0), x_8^{+}(0)]_{(r, 2r, s^{-1}, r^2, r^{-1}s^{-1}, r^{-2}s^{-2})}.
\]
Consider the first step,
\[
\begin{align*}
[z_2^-(1), z_2^+(−1)] \\
= &\left[\left[ z_1^+(0), z_2^-(1) \right]_{r-1}, \left[ z_2^+(−1), z_1^+(0) \right]_{s-1}\right] \quad \text{(using (5.3))} \\
= &\left[\left[ z_1^+(0), z_2^-(1), z_2^+(−1), z_1^+(0) \right]_{s-1}\right] \quad \text{(using (D6) and (D9))} \\
+ &\left[ \left[ z_2^+(−1), \left[ z_1^+(0), z_2^-(1), z_2^+(−1) \right]_{s-1}\right]_{s-1}\right] \quad \text{(using (D6) and (D9))}
\end{align*}
\]
\[
= (rs)^{-1} \gamma \omega_2 \cdot \frac{\omega_1' − \omega_1}{r − s} + (rs)^{-1} \frac{\gamma \omega_2'}{r − s} \omega_1 \\
= (rs)^{-1} \frac{\gamma \omega_2' \omega_2}{r − s},
\]

For simplicity, we list the results for the intermediate steps.
\[
\begin{align*}
[z_3^-(1), z_3^+(−1)] & = (rs)^{-1} \frac{\gamma \omega_{13}'}{r − s}, \\
[z_4^-(1), z_4^+(−1)] & = (rs)^{-1} \frac{\gamma \omega_{24}'}{r − s}, \\
[z_5^-(1), z_5^+(−1)] & = (rs)^{-1} \frac{\gamma \omega_{35}'}{r − s}, \\
[z_6^-(1), z_6^+(−1)] & = (rs)^{-1} \frac{\gamma \omega_{46}'}{r − s}, \\
[z_7^-(1), z_7^+(−1)] & = (rs)^{-3} \frac{\gamma \omega_{57}'}{r − s}.
\end{align*}
\]

One has from the last induction step.
\[
\begin{align*}
\left[ x_9^-(1), x_9^+(−1) \right] & \quad \text{(by definition)} \\
= &\left[\left[ x_1^+(0), z_9^-(1) \right]_{r-2s-1}, \left[ z_9^+(−1), x_9^+(0) \right]_{r-1s-2}\right] \quad \text{(using (5.3))} \\
= &\left[\left[ z_9^+(−1), \left[ z_1^+(0), x_9^+(0) \right]_{r-2s-1}, z_9^+(1) \right]_{r-1s-2}\right]_{r-1s-2} \quad \text{(using (5.3), (D9) and (D6))} \\
+ &\left[\left[ x_1^+(0), z_9^+(−1), z_9^+(1), z_1^+(0) \right]_{r-2s-1}, z_9^+(0) \right]_{r-1s-2} \quad \text{(=0 by (D9) and (D6))} \\
+ &\left[\left[ x_1^+(0), z_9^+(−1), z_9^+(1), z_1^+(0) \right]_{r-1s-1}, x_9^+(0) \right]_{r-1s-2} \quad \text{(=0 by the above result and (D5))} \\
+ &\left[\left[ x_1^+(0), z_9^+(−1), z_9^+(1), z_1^+(0) \right]_{r-2s-1}, x_9^+(0) \right]_{r-1s-2} \quad \text{(using (D6) and (D9))}
\end{align*}
\]
\[
= (rs)^{-5} \frac{2}{} \gamma \omega_{1r} \cdot \frac{\omega_1' − \omega_1}{r − s} + (rs)^{-5} \frac{2}{} \gamma \omega_{1r} \frac{\omega_1'}{r − s} \omega_1 \\
= (rs)^{-5} \frac{2}{} \gamma \omega_{1r} \frac{\omega_1'}{r − s}.
\]

So we have obtained the required conclusion. 

In the last part of this section, we check the remaining Serre relations of \((E5)\).

**LEMA 6.30.** For the case of \(U_{r,s}(E_6^{(2)})\), one has,
\[
\begin{align*}
(1) & \quad \left[ E_1, E_1, E_0 \right]_{(r-2s-1, r-1s-2)} = 0, \\
(2) & \quad \left[ E_2, E_2, E_2, E_3 \right]_{(s^2, rs, r^2)} = 0.
\end{align*}
\]
To prove $= = = 0$

For $i$ (3) Using Lemma 6.7, one can show the third relation. Denote that $Y' = [E_2, E_2, E_3]_{(s^2, rs, r^2)}$. To prove $[x_3^-(0), Y'] = 0$ for $i \in I$, it is enough to check the cases of $i$ being 2 and 3.

For $i = 3$, one immediately has,

\[
[x_3^-(0), Y] = [x_3^-(0), x_2^+(0), x_2^+(0), x_3^+(0)]_{(s^2, rs, r^2)}
\]

\[
= [x_3^+(0), x_2^+(0), x_2^+(0), x_3^+(0)]_{(s^2, rs, r^2)}
\]

\[
= - \omega_3[x_3^+(0), x_2^+(0), x_3^+(0)]_{(r^2, r, 1)}
\]

\[
= 0.
\]

For $i = 2$, one gets from the following direct calculations,

\[
[x_2^-(0), Y]
\]

\[
= [x_2^-(0), x_2^+(0), x_2^+(0), x_3^+(0)]_{(s^2, rs, r^2)}
\]

\[
= [x_2^-(0), x_2^+(0), x_2^+(0), x_2^+(0)]_{(s^2, rs, r^2)}
\]

\[
+ [x_2^+(0), x_2^+(0), x_2^+(0), x_2^+(0)]_{(s^2, rs, r^2)}
\]

\[
+ [x_2^+(0), x_2^+(0), x_2^+(0), x_2^+(0)]_{(s^2, rs, r^2)}
\]

\[
= - (r + s)[x_2^+(0), x_2^+(0), x_3^+(0)]_{(s^2, rs)}
\]

\[
+ (r + s)[x_2^+(0), x_2^+(0), x_3^+(0)]_{(s^2, rs)}
\]

\[
= 0.
\]
So far, we have finished the proof of Theorem A for all twisted cases, that is, there exists an algebra homomorphism $\Psi$ between the two realizations of the two-parameter twisted quantum affine algebras. In the next section, we will give a new proof of the other two theorems.

7. The inverse map $\Phi$ of $\Psi$

The goal of this section is to obtain the inverse map of $\Psi$, more precisely, there exists an algebra isomorphism from Drinfeld realization to Drinfeld-Jimbo realization of two-parameter twisted quantum affine algebras. We remark that the proof works not only for untwisted cases but also for twisted cases. In particular, we give a new proof of Drinfeld isomorphism for the quantum affine algebras of the classical types special cases, which were proved using the braid group [1]. From now on, we denote by $\hat{g}$ for any affine Lie algebra.

Fix $k \in I$, denote by $U_{r,s}^k(\hat{g})$ the subalgebra of $U_{r,s}(\hat{g})$ generated by $x_i^\pm(0)$, $x_k^\pm(1)$, $\omega_i^{\pm1}$, $\omega_i'^{\pm1}$ ($i \in I$), and $\gamma^{\pm1/2}$, $\gamma'^{\pm1}$, satisfying the relations $(D1) - (D9)$, that is,

$$U_{r,s}^k(\hat{g}) := \left\langle x_i^\pm(0), x_k^\pm(1), \omega_i^{\pm1}, \omega_i'^{\pm1}, \gamma^{\pm1/2}, \gamma'^{\pm1}, \Big| i \in I \right\rangle / \sim.$$  

In fact, we have the following result.

**PROPOSITION 7.1.** $U_{r,s}^k(\hat{g}) = U_{r,s}(\hat{g})$

**PROOF.** It suffices to show all other generators of $U_{r,s}(\hat{g})$ are in the algebra $U_{r,s}^k(\hat{g})$. It follows from relation $(D9)$ that

$$a_k(1) = \omega_i^{-1}\gamma^{1/2} \left[ x_i^+(0), x_j^-(1) \right] \in U_{r,s}^k(\hat{g}),$$

$$a_k(-1) = \omega_i'\gamma^{1/2} \left[ x_j^+(1), x_k^-(0) \right] \in U_{r,s}^k(\hat{g}).$$

Then relation $(D7)$ implies that for $j \in I$ such that $a_{jk} \neq 0$,

$$x_j^+(0) = -(rksk)^{-a_{jk}} \left[ a_{jk} \right]^{-1} \gamma^{1/2} \left[ a_k(1), x_j^+(0) \right] \in U_{r,s}^k(\hat{g}),$$

$$x_j^-(1) = -(rksk)^{a_{jk}} \left[ a_{jk} \right]^{-1} \gamma^{-1/2} \left[ a_k(1), x_j^-(0) \right] \in U_{r,s}^k(\hat{g}).$$

Repeating the above two steps, one obtains that for all $i \in I$, $a_i(1), a_i(-1), x_i^-(1)$ and $x_i^+(1)$ are in $U_{r,s}^k(\hat{g})$. At the same time, $x_i^+(1)$ and $x_i^-(1)$ both are in $U_{r,s}^k(\hat{g})$ from $(D7)$. Therefore, all degree one generators are in the subalgebra.

Next for $\ell \in \mathbb{Z}/\{0\}$, suppose that $x_i^\pm(\ell) \in U_{r,s}(\hat{g})$ and $a_i(\ell) \in U_{r,s}^k(\hat{g})$. Using relation $(D9)$, one has,

$$U_{r,s}^k(\hat{g}) \ni \left[ x_i^+, x_i^- \right] \quad = \quad * a_i(\ell + 1) + \sum_{1 \leq t \leq \ell + 1, \sum l_k = \ell + 1} *' a_i(\ell_{l_1} \cdots a_i(\ell_{l_{\ell + 1}}),$$

where scalars $*, *' \in \mathbb{K}/\{0\}$, hence $a_i(\ell + 1) \in U_{r,s}^k(\hat{g})$.

It follows from relation $(D7)$,

$$U_{r,s}^k(\hat{g}) \ni \left[ a_i(\ell), x_i^\pm(1) \right] = * x_i^\pm(\ell + 1),$$

where scalars $* \in \mathbb{K}/\{0\}$, therefore $x_i^\pm(\ell + 1) \in U_{r,s}^k(\hat{g})$. So all generators of $U_{r,s}(\hat{g})$ are in $U_{r,s}^k(\hat{g})$ by induction.  

\[ \square \]
Remark 7.2. As a consequence of the above proposition, there exist \( n \) subalgebras \( \mathcal{U}_{r,s}(\hat{g}) \) which are all isomorphic to \( \mathcal{U}_{r,s}(\hat{g}) \), or rather there are \( n \) sets of generators \( \mathcal{U}_{r,s}(\hat{g}) \) of degree bounded by \( \pm 1 \). From now on, fix \( k \in I \), we use the presentation (subalgebra) \( \mathcal{U}_{r,s}(\hat{g}) \) instead of \( \mathcal{U}_{r,s}(\hat{g}) \).

We keep the previous assumptions and notations, and let \( i_1, \ldots, i_{h-1} \) be a sequence of indices of the fixed reduced expression given in Lemma 5.2. We also need a few more notations for our purpose.

\[
p_i, q_i = (i_j, i_{j-1} \cdots i_2 i_1), \quad p_i' = (i_j, i_{j-1} \cdots i_2 i_1), \quad p_i'' = (i_j, i_{j+1} \cdots i_2 i_1),
\]

\[
q_i = (i_1 i_2 \cdots i_{j-1}, i_j)^{-1}, \quad q_i' = (i_1 i_2 \cdots i_{j-1}, i_j)^{-1}, \quad q_i'' = (i_1 i_2 \cdots i_{j+1}, i_j)^{-1}.
\]

Denote \( t_i = \frac{q_i - p_i'}{q_i - q_i} \).

We now define the inverse map of \( \Psi \), more precisely, we have the following statement.

Theorem 3. Fix \( k \in I \), let \( k = i_1, i_2, \cdots, i_{h-1} \) be the sequence of indices of a fixed reduced expression in Lemma 5.2. Then \( \Phi : \mathcal{U}_{r,s}(\hat{g}) \to \mathcal{U}_{r,s}(\hat{g}) \) is an epimorphism such that for all \( i \in I \)

\[
\Phi(\gamma) = \gamma, \quad \Phi(\gamma') = \gamma', \quad \Phi(x^+_i(0)) = e_i,
\]

\[
\Phi(x^-_i(0)) = f_i, \quad \Phi(\omega_i) = \omega_i, \quad \Phi(\omega'_i) = \omega'_i,
\]

\[
\Phi(x^-_k(1)) = t_i^{-1} \cdots t_{i_2}^{-1} [e_{i_2}, e_{i_3}, \cdots, e_{i_{h-1}}, e_0] (p_i', p_i'') \gamma \omega_k, \quad \Phi(x^+_k(-1)) = \gamma \omega'_i f_0, f_{i_{h-1}}, f_{i_{h-2}}, \cdots, f_{i_2}, (q_i', q_i'').
\]

Proof. We show that \( \Phi \) satisfies the relations (D1) – (D9). From our previous discussion it is enough to check the relations only involving \( x^+_k(-1) \) and \( x^-_k(1) \). In particular, we verify the relation (D8).

For \( 1 < \ell < h \), let us denote

\[
\tilde{e}_i(1) = [e_{i_2}, e_{i_3}, \cdots, e_{i_{h-1}}, e_0] (p_i', p_i''), \quad \tilde{f}_i(1) = [f_{i_{h-1}}, i_{h-2}, \cdots, f_{i_2}, (q_i', q_i'').
\]

Then \( \Phi(x^-_k(1)) = t_i^{-1} \cdots t_{i_2}^{-1} \tilde{e}_i(1) \gamma \omega_k, \text{ and } \Phi(x^+_k(-1)) = t_i^{-1} \cdots t_{i_2}^{-1} \gamma \omega_k \tilde{f}_i(1). \)
We quickly get by direct computation that

\[
[\tilde{f}_{ih-1}(-1), \tilde{e}_{ih-1}(1)] = \left[ [f_0, f_{ih-1}] q'_{ih-1}, [e_{ih-1}, e_0] p'_{ih-1} \right]
\]

\[
= \left[ [f_0, f_{ih-1}] q'_{ih-1}, [e_{ih-1}, e_0] p'_{ih-1} \right]
+ \left[ [f_0, [f_{ih-1}, e_{ih-1}]] q'_{ih-1}, e_0 \right] p'_{ih-1}
+ \left[[f_0, [f_{ih-1}, e_{ih-1}]] q'_{ih-1}, [e_0, f_{ih-1}] q'_{ih-1} \right] p'_{ih-1}
+ \left[[e_{ih-1}, [f_0, f_{ih-1}]] q'_{ih-1} \right] p'_{ih-1}
\]

\[
= -t'_{ih-1} \omega_{ih-1} \frac{\omega_0 - \omega_0}{r - s} + t'_{ih-1} \frac{\omega_{ih-1} - \omega_{ih-1}'}{r - s} \omega_0
\]

where the first summand and the last summand are null due to the communicative relation.

Now we consider,

\[
[\tilde{f}_i(-1), \tilde{e}_i(1)] = \left[ [\tilde{f}_{i-1}(-1), f_i] q_i, [e_{i-1}, \tilde{e}_{i-1}(1)] p_i \right]
\]

\[
= \left[ [\tilde{f}_{i-1}(-1), f_i] q_i, [e_{i-1}, \tilde{e}_{i-1}(1)] p_i \right]
+ \left[[\tilde{f}_{i-1}(-1), [f_i, e_i]] q_i, \tilde{e}_{i-1}(1)] p_i \right]
+ \left[[e_{i-1}, [\tilde{f}_{i-1}(-1), e_i]] f_i q_i, e_i \right] p_i \right]
+ \left[[e_{i-1}, [\tilde{f}_{i-1}(-1), [f_i, e_i]]] q_i \right] p_i \right]
\]

Inductively we arrive at

\[
[\tilde{f}_{ii}(-1), \tilde{e}_{ii}(1)] = t'_{ih-1} \cdots t'_{ii} \frac{\omega_{i-1} \cdots \omega_{ih-1} \omega_0 - \omega_{i-1}' \cdots \omega_{ih-1}' \omega_0'}{r - s}.
\]

As a consequence of the above results, we get immediately,

\[
\Phi([x_k^-(-1), x_k^+(1)])
\]

\[
= -t'_{ih-1} \cdots t'_{ii} \gamma' \gamma \omega_k \omega_k' \tilde{f}_{ii}(-1), \tilde{e}_{ii}(1)]
\]

\[
= \Phi(\frac{\gamma \omega_k - \gamma \omega_k'}{r - s})
\]

where we have used \(\omega_0 = \gamma^{-1} \omega_0^{-1}\) and \(\omega_0' = \gamma^{-1} \omega_0'^{-1}\).

In fact, the map \(\Psi\) is the inverse of \(\Phi\).

**Theorem C.** \(\Psi = \Phi^{-1}\).

**Proof.** It suffices to check the actions of \(\Psi \Phi\) on the generators are identities. Most of these are trivial except for the generators \(x_k^-(-1)\) and \(x_k^+(-1)\), and we directly compute that \(\Psi \Phi(x_k^-(-1)) = x_k^-(1)\) and \(\Psi \Phi(x_k^+(-1)) = x_k^+(1)\).
In fact, for \(2 \leq j \leq h - 1\), set
\[
\tilde{y}_{j-1}^{-1}(1) = x_{k}^{-1}(1)\gamma'\omega_{k}^{-1},
\]
and
\[
\tilde{y}_{j-1}^{-1}(1) = [x_{j}^{-1}(0), \cdots, x_{2}^{-1}(0), x_{j}^{-1}(1)](p_{j}, \cdots, p_{i})\gamma'\omega_{i}^{-1}\cdots\omega_{j}^{-1}.
\]
Using these new notation, we can write that
\[
\Psi(\tilde{y}_{1,h-1}^{-1}(1)) = \Psi(e_0).
\]
Firstly, note that for \(2 \leq j \leq h - 1\)
\[
[x_{j}^{+}(0), \tilde{y}_{j-1}^{-1}(1)](p_{j}) = [x_{j}^{+}(0), \tilde{y}_{j-1}^{-1}(1)]\gamma'\omega_{i}^{-1}\cdots\omega_{j}^{-1}
\]
\[
= t_{j} \tilde{y}_{j-1}^{-1}(1)
\]
It holds by direct computation,
\[
\Psi\Phi(x_{k}^{-1}(1)) = \Psi\left(t_{h-1}^{-1} \cdots t_{2}^{-1}\left[e_{2}, e_{3}, \cdots, e_{h-1}, e_{0}\right](p_{h-1}, \cdots, p_{2})\gamma'\omega_{k}\right)
\]
\[
= t_{h-2}^{-1} \cdots t_{2}^{-1}\left[x_{2}^{+}(0), x_{3}^{+}(0), \cdots, x_{h-1}^{+}(0), \Psi(e_0)\right](p_{h-1}, \cdots, p_{2})\gamma'\omega_{k}
\]
\[
= t_{h-2}^{-1} \cdots t_{2}^{-1}\left[x_{2}^{+}(0), x_{3}^{+}(0), \cdots, x_{h-2}^{+}(0), \tilde{y}_{1,h-2}^{-1}(1)\right](p_{h-2}, \cdots, p_{2})\gamma'\omega_{k}
\]
\[
= \tilde{y}_{1,1}^{-1}(1)\gamma'\omega_{k}
\]
\[
= x_{k}^{-1}(1)
\]
The action of \(\Psi\Phi\) on \(x_{k}^{+}(-1)\) can be checked similarly. Consequently, \(\Psi\) and \(\Phi\) are inverse.

\[\square\]

So far, we have proved there exists an algebra isomorphism between their two realizations of two-parameter twisted quantum affine algebras. In particular, we also prove the Drinfeld isomorphism for quantum affine algebras as a special case.

8. The Hopf algebra structure of \(\mathcal{U}_{r,s}(\hat{g})\)

In the last section, we will discuss the Hopf structure of the subalgebra \(\mathcal{U}_{r,s}(\hat{g})\) of Drinfeld realization. It is well-known that Drinfeld-Jimbo realization \(U_{r,s}(\hat{g})\) admits a Hopf algebra structure. The aim of the present section is to establish the Hopf algebra structure of Drinfeld realization \(\mathcal{U}_{r,s}(\hat{g})\). Furthermore, we announce that there exists a Hopf isomorphism between these two realizations. It is worthy to point out that these results are totally available for classical quantum affine algebras as a special case.

We freely use the notations and assumptions of the previous section. In particular, fix \(k \in I\), let \(k = i_1, i_2, \cdots, i_{h-1}\) be a sequence of index satisfying Lemma 5.2. For our purpose, let me introduce a few notations.

Define \(x_{i_1, l}(1)\) and \(x_{i_1, l}^{+}(-1)\) for \(2 \leq l \leq h - 1\) inductively by,
\[
x_{i_1, l}^{-1}(1) = [x_{i_2}^{-1}(0), \cdots, x_{i_2}^{-1}(0), x_{i_1}^{-1}(1)](p_{i_2}, \cdots, p_{i_1}),
\]
\[
x_{i_1, l}^{+}(1) = [x_{i_1}^{+}(-1), x_{i_2}^{+}(0), \cdots, x_{i_1}^{+}(0)](q_{i_2}, \cdots, q_{i_1}).
\]
Denote \(x_{i_1, l}^{+}(0)\) and \(x_{i_1, l}^{-}(0)\) for \(2 \leq l \leq h - 1\) as follows,
\[
x_{i_2}^{+}(0) = [x_{i_2}^{+}(0), x_{i_3}^{+}(0), \cdots, x_{i_2}^{+}(0)](u_{i_2-1}, \cdots, u_{i_2}),
\]
\[
x_{i_2}^{-}(0) = [x_{i_2}^{-}(0), x_{i_3}^{-}(0), \cdots, x_{i_2}^{-}(0)](v_{i_2-1}, \cdots, v_{i_2}).
\]
where \( u_{ij} = p'_{ij} q''_{ij} \) and \( v_{ij} = q'_{ij} p''_{ij} \), here we keep the notations from the last section. Now we can define the actions of a comultiplication on the simple generators of the algebra \( \mathcal{U}_r \).

**Definition 8.1.** Fix \( k \in I \), let \( k = i_1, i_2, \ldots, i_{h-1} \) be a sequence of indices satisfying Lemma 5.2. Define the actions of the comultiplication \( \Delta' \) on the generators of the algebra \( \mathcal{U}_r \) as below, for \( i \in I \),

\[
\Delta'(\omega_i) = \omega_i \otimes \omega_i, \quad \Delta'(\omega_i') = \omega_i' \otimes \omega_i',
\]

\[
\Delta'(\gamma^\pm_\frac{1}{2}) = \gamma^\pm_\frac{1}{2} \otimes \gamma^\pm_\frac{1}{2}, \quad \Delta'(\gamma'^\pm_\frac{1}{2}) = \gamma'^\pm_\frac{1}{2} \otimes \gamma'^\pm_\frac{1}{2},
\]

\[
\Delta'(D^{\pm 1}) = D^{\pm 1} \otimes D^{\pm 1}, \quad \Delta'(D'^{\pm 1}) = D'^{\pm 1} \otimes D'^{\pm 1},
\]

\[
\Delta'(x^+_i(0)) = x^+_i(0) \otimes 1 + \omega_i \otimes x^+_i(0), \quad \Delta'(x^-_i(0)) = x^-_i(0) \otimes u_i + 1 \otimes x^-_i(0),
\]

\[
\Delta'(x^-_i(1)) = x^-_i(1) \otimes \gamma \omega_k + 1 \otimes x^-_i(1) + \sum_{l=2}^{h-1} \xi_l x^-_{i,l}(1) \otimes x^+_i(0) \gamma \omega_k
\]

\[
\Delta'(x^+_i(1)) = x^+_i(-1) \otimes 1 + \gamma \omega_k \otimes x^+_i(-1) + \sum_{l=2}^{h-1} \xi_l \gamma \omega_k x^-_{i,l}(0) \otimes x^+_i(-1)
\]

where

\[
\xi_l = (q'_{i_1} - p'_{i_1}) t_{i_2} \cdots t_{i_{l-1}}^{-1} q''_{i_2} \cdots q''_{i_{l-1}},
\]

\[
\xi_l = (p'_{i_1} - q'_{i_1}) t_{i_2} \cdots t_{i_{l-1}}^{-1} p''_{i_2} \cdots p''_{i_{l-1}}.
\]

The definition of the above comultiplication \( \Delta' \) is well-defined, which will be verified by the following proposition.

**Proposition 8.2.** Fix \( k \in I \), let \( k = i_1, i_2, \ldots, i_{h-1} \) be a sequence of indices satisfying Lemma 5.2. The algebra \( \mathcal{U}_r \) is a Hopf algebra with the above comultiplication \( \Delta' \), the counit \( \varepsilon \) and the antipode \( S \) defined below, for \( i \in I \), we have

\[
\varepsilon(x^+_i(0)) = \varepsilon(x^-_i(0)) = \varepsilon(x^+_i(-1)) = \varepsilon(x^-_i(1)) = 0,
\]

\[
\varepsilon(\gamma^\pm_\frac{1}{2}) = \varepsilon(\gamma'^\pm_\frac{1}{2}) = \varepsilon(D^{\pm 1}) = \varepsilon(D'^{\pm 1}) = \varepsilon(\omega_i) = \varepsilon(\omega_i') = 1,
\]

\[
S(\gamma^\pm_\frac{1}{2}) = \gamma^\mp_\frac{1}{2}, \quad S(\gamma'^\pm_\frac{1}{2}) = \gamma'^\mp_\frac{1}{2}, \quad S(D^{\pm 1}) = D^{\mp 1}, \quad S(D'^{\pm 1}) = D'^{\mp 1},
\]

\[
S(\omega_i) = \omega_i^{-1}, \quad S(\omega_i') = \omega_i'^{-1} \quad S(D^{\pm 1}) = D^{\mp 1}, \quad S(D'^{\pm 1}) = D'^{\mp 1},
\]

\[
S(x^+_i(0)) = -\omega_i^{-1} x^+_i(0), \quad S(x^-_i(0)) = -x^-_i(0) \omega_i^{-1},
\]

\[
S(x^+_i(-1)) = -\gamma \omega_k x^+_i(-1) - \sum_{l=2}^{h-2} \xi_l y^-_{i,l}(0) \gamma^{-1} \omega_i'^{-1} \cdots \omega_i'^{-1} x^+_i,(-1),
\]

\[
S(x^-_i(1)) = -x^-_i(1) \gamma^{-1} \omega_i'^{-1} \cdots \omega_i'^{-1} y^-_{i,l}(0),
\]

where

\[
y^-_{i,1}(0) = a \{ x^-_{i,0}(0), x^-_{i,1}(0), \ldots, x^-_{i,0}(0) \}(v'_{i,1}, \ldots, v'_{i,2})
\]

and

\[
y^+_{i,1}(0) = b \{ x^+_{i,0}(0), x^+_{i,1}(0), \ldots, x^+_{i,0}(0) \}(u'_{i,1}, \ldots, u'_{i,2}),
\]
and for $2 \leq j \leq l-1$, $u'_{i_j} = \langle i_j, i_{j+1} \rangle$, $u'_{i_j} = \langle i_{j+1}, i_j \rangle$, here the constant $a = \prod_{j=2}^{l-1} v_i \cdot \langle i_{j+1} \cdots i_1, i_j \rangle$ and $b = \prod_{j=2}^{l-1} u_i \cdot \langle i_j, i_{j+1} \cdots i_1 \rangle$.

**Proof.** (a) We first show that $\Delta'$ defines a morphism of algebra from $U^k_{r,s}(\hat{g})$ into $U^k_{r,s}(\hat{g}) \otimes U^k_{r,s}(\hat{g})$. Note that the actions on all generators are the same as that of Drinfeld-Jimbo generators except $x^-_k(1)$ and $x^+_k(-1)$. So it is enough to show the relations involving $x^-_k(1)$ and $x^+_k(-1)$, that is,

$$[\Delta'(x^+_k(-1)), \Delta'(x^-_k(1))] = \frac{\Delta'(\gamma'\omega_k) - \Delta'(\gamma\omega_k)}{r_k - s_k}$$

By definition, we have immediately,

$$[\Delta'(x^+_k(-1)), \Delta'(x^-_k(1))] = [x^+_k(-1) \otimes 1 + \gamma\omega_k \otimes x^+_k(-1) + \sum_{l'=2}^{h-2} \zeta_{l'} \gamma\omega_k x^-_{i_{l'} \nu}(0) \otimes x^+_{i_{l'} \nu}(-1),$$

$$\langle x^-_k(1) \otimes \gamma'\omega_k + 1 \otimes x^-_k(1) + \sum_{l=2}^{h-1} \xi_l x^-_{i_{l} \nu}(1) \otimes x^+_{i_{l} \nu}(0) \gamma'\omega_k \rangle]$$

By direct calculations, we pull out the common factors, then the above bracket can be divided four summands, that is,

$$[\Delta'(x^+_k(-1)), \Delta'(x^-_k(1))] = [x^+_k(-1), x^-_k(1)] \otimes \gamma\omega_k + \gamma\omega_k \otimes [x^+_k(-1), x^-_k(1)]$$

$$+ \sum_{l'=2}^{h-2} \sum_{l=2}^{h-1} \left\{ \zeta_{l'} \gamma\omega_k \left[x^-_{i_{l'} \nu}(0), x^-_k(1)\right]_{i_{l} \cdots i_{l'}} \otimes x^+_{i_{l'} \nu}(-1) \gamma'\omega_k \right.$$  

$$+ \xi_l \gamma\omega_k x^-_{i_{l} \nu}(1) \otimes \left[x^+_{i_{l} \nu}(0), x^-_k(-1) \right]_{i_{l} \cdots i_{l'}}$$

$$+ \sum_{l'=2}^{h-2} \sum_{l=2}^{h-1} \left\{ \zeta_{l'} \gamma\omega_k x^+_{i_{l'} \nu}(0) \otimes \left[x^+_{i_{l'} \nu}(0), x^-_k(-1) \right]_{i_{l} \cdots i_{l'}}$$

$$+ \xi_l \left[x^+_{i_{l} \nu}(1), x^-_k(1) \right] \otimes x^+_{i_{l} \nu}(0) \gamma'\omega_k \right.$$  

$$+ \sum_{l'=2}^{h-2} \sum_{l=2}^{h-1} \zeta_{l'} \xi_l \left\{ \gamma\omega_k x^+_{i_{l'} \nu}(0) x^-_{i_{l} \nu}(1) \otimes x^+_{i_{l'} \nu}(-1) x^+_{i_{l} \nu}(0) \gamma'\omega_k$$

$$- x^-_{i_{l} \nu}(1) \gamma\omega_k x^+_{i_{l'} \nu}(0) \otimes x^+_{i_{l} \nu}(0) \gamma'\omega_k x^+_{i_{l} \nu}(1) \right\}$$

In fact, the last three summands are 0. Using Drinfeld relation (D9) and (D5), the second summand and the third summand also vanish. The last summand is 0 for Serre relations. For simplicity, we proceed with the example of the case of $D^{(3)}_4$. 


In this case, the last summand becomes:
\[
\zeta_1 \xi_1 \left\{ \gamma \omega_1' x_2^-(0) [x_2^-(0), x_1^- (1)] \otimes \delta \cdot [x_1^+ (1), x_2^+ (0)] \right\}
\]
\[
- [x_2^-(0), x_1^- (1)] \delta \cdot \{ x_2^+ (0) \otimes x_1^- (1), x_2^- (0) \}
\]
\[
= \zeta_1 \xi_1 \left\{ r^3 \gamma \omega_1' [x_2^-(0), x_1^- (1)] \otimes x_2^+ (0) \otimes s^3 x_2^+ (0) [x_1^+ (1), x_2^- (0)] \right\}
\]
\[
- (rs)^3 \gamma \omega_1' [x_2^-(0), x_1^- (1)] \otimes x_2^+ (0) \otimes [x_1^+ (1), x_2^- (0)] \right\}
\]
\[
= 0.
\]
Therefore, we get the required relation,
\[
[ \Delta'(x_k^+ (-1)), \Delta'(x_k^- (1)) ]
\]
\[
= [ x_k^- (-1), x_k^- (1) ] \otimes \gamma \omega_k + [ x_k^+ (1), x_k^- (1) ]
\]
\[
= \frac{\Delta'(\gamma \omega_k) - \Delta'(\gamma \omega_k')}{r_k - s_k}
\]

(b) Next, we need to show that \( \Delta' \) is coassociative. Similarly it suffices to check the actions of \( \Delta' \) on the generators \( x_k^- (1) \) and \( x_k^+ (-1) \).

For the case of \( x_k^- (1) \), we obtain by definition,
\[
(\Delta' \otimes id)\Delta'(x_k^- (1))
\]
\[
= (\Delta' \otimes id)(x_k^- (1) \otimes \gamma \omega_k + 1 \otimes x_k^- (1) + \sum_{i=2}^{h-1} \xi_i x_{i,1}^- (1) \otimes x_{i,2}^+ (0) \gamma \omega_k)
\]
\[
= \Delta'(x_k^- (1)) \otimes \gamma \omega_k + 1 \otimes 1 \otimes x_k^- (1) + \sum_{i=2}^{h-1} \xi_i \Delta'(x_{i,1}^- (1)) \otimes x_{i,2}^+ (0) \gamma \omega_k.
\]

On the other hand, we have,
\[
(id \otimes \Delta')\Delta'(x_k^- (1))
\]
\[
= (id \otimes \Delta')(x_k^- (1) \otimes \gamma \omega_k + 1 \otimes x_k^- (1) + \sum_{i=2}^{h-1} \xi_i x_{i,1}^- (1) \otimes x_{i,2}^+ (0) \gamma \omega_k)
\]
\[
= x_k^- (1) \otimes \gamma \omega_k \otimes \gamma \omega_k + 1 \otimes \Delta'(x_k^- (1)) + \sum_{i=2}^{h-1} \xi_i x_{i,1}^- (1) \otimes \Delta'(x_{i,2}^+ (0) \gamma \omega_k)
\]

By direct calculation of \( \Delta'(x_{i,1}^- (1)) \) and \( \Delta'(x_{i,2}^+ (0)) \), the two expressions are same. Hence we get the required relation
\[
(\Delta' \otimes id)\Delta'(x_k^- (1)) = (id \otimes \Delta')\Delta'(x_k^- (1)).
\]

The proof for \( x_k^+ (-1) \) is analogue.

(c) It is easy to check that \( \varepsilon \) defines a morphism of algebra from \( \mathfrak{u}^k_{r,s} (\mathfrak{g}) \) onto \( \mathbb{K} \) and satisfies the counit axiom.

(d) It remains to verify that \( S \) defines an antipode for \( \mathfrak{u}^k_{r,s} (\mathfrak{g}) \). First we have to show that \( S \) is a morphism of algebra from \( \mathfrak{u}^k_{r,s} (\mathfrak{g}) \) into \( \mathfrak{u}^k_{r,s} (\mathfrak{g}) \), that is
\[
[ S(x_k^- (1)), S(x_k^- (-1)) ] = \frac{S(\gamma \omega_k) - S(\gamma \omega_k')}{r_k - s_k}
\]
The verification is similar to the above and is left to the reader. To conclude that $S$ is an antipode, it suffices to check that the relations
\[ \sum_{(x)} x' S(x'') = \sum_{(x)} S(x') x'' = \varepsilon(x)1 \]
holds when $x$ is any of the generators. Similarly, we have only to check it on $x_k^-(1)$ and $x_k^+(-1)$. But this follows from the construction easily.

Then we arrive at our second main theorem as follows.

**Theorem 8.3.** The morphisms $\Phi$ and $\Psi$ are two coalgebra homomorphisms, that is,
\[ \Delta' \circ \Psi = (\Psi \otimes \Psi) \Delta, \quad \Delta \circ \Phi = (\Phi \otimes \Phi) \Delta'. \]
In particular, The maps $\Phi$ and $\Psi$ between the algebra $U_{r,s}(\hat{g})$ and $U_{r,s}(\hat{g})$ are two Hopf algebra isomorphisms.

**Proof.** It follows from Proposition 8.2 together with the constructions of $\Delta$ and $\Delta'$.

**Remark 8.4.** The result generalizes the result given in [Dr].

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