Random quantum correlations and density operator distributions

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Abstract
Randomly correlated ensembles of two quantum systems are investigated, including average entanglement entropies and probability distributions of Schmidt-decomposition coefficients. Maximal correlation is guaranteed in the limit as one system becomes infinite-dimensional. The reduced density operator distributions are compared with distributions induced via the Bures and Hilbert-Schmidt metrics.

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1 Introduction
This Letter is primarily motivated by the following question: what statistical ensemble corresponds to minimal prior knowledge about a quantum system? Such an ensemble may be identified as the most random ensemble of possible states of the system. It would provide, for example, a natural benchmark for assessing how "random" a given evolution process is; a
worst-case scenario for general schemes for extracting information about the system [2]; and a natural unbiased measure over the set of possible states of the system (which would allow one to calculate, e.g., the average effectiveness of a general scheme for distinguishing between quantum states [3]).

For the case where the system is known to in fact be in a pure state, there is an obvious answer to the question. In particular, identifying minimal knowledge with maximal symmetry, it is natural to require that the ensemble be invariant under the full group of unitary transformations (thus there is no preferred measurement basis for extracting information). This requirement yields a unique probability distribution over the set of pure states of the system [2, 4], which has found applications in quantum inference [2], quantum chaos [1], and quantum information [5].

However, as pointed out by Wootters [1], there does not appear to be a natural generalisation of the above ensemble when the restriction of pure states is removed. Indeed, if general states described by density operators are allowed, the requirement of unitary invariance only implies that the probability measure over the set of possible states is a function of the density operator eigenvalue spectrum alone. Hence a unique probability measure can be specified only via some further principle or restriction, to be motivated on physical or conceptual grounds.

In this Letter two possible approaches to the question are examined. The first is motivated by recent work of Braunstein [3], and is considered in sections 2 and 3 below. It corresponds to assuming that the quantum system is randomly correlated with a second system, where the composite system is in a pure state. The reduced ensemble of the system is characterised by the distribution of Schmidt-decomposition coefficients of the composite system, and is explicitly calculated for the 2-dimensional case. This further allows calculation of the average “entanglement entropy” [6] of the systems. In the limit as the dimension of the auxiliary system becomes infinite, the systems become maximally correlated with probability unity.

The second approach, studied in section 4 below, is more formal in nature. It relies on choosing a metric on the space of density operators of the system; the random ensemble then corresponds to the (normalised) volume element on this metric space. There are strong information-theoretic

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1 If the average density operator of the ensemble is also known, then the above-mentioned pure-state ensemble may be modified to give a corresponding maximally-random or “Scrooge” ensemble, with the property of being that ensemble of pure states on which measurement yields the least possible information consistent with the prior knowledge [4].
grounds for motivating the choice of the Bures “distinguishability” metric \[^8\]. Moreover, for a 2-dimensional system, this metric corresponds to the conceptually satisfying case of a maximally symmetric space, with no preferred locations or directions in the space of density operators. The ensemble induced by the Hilbert-Schmidt metric is also considered.

Finally, in Section 5 comparisons are made between the above two approaches for the two-dimensional case. It is argued that it is the second approach, based on the Bures metric, which yields the desired “minimal knowledge” ensemble in this case.

2 Randomly correlated ensembles

Let quantum system \( S \), with Hilbert space \( H_S \), be correlated with an auxiliary system \( A \), with Hilbert space \( H_A \) (for example, a similar system, a measuring apparatus, or the environment). A general pure state of the composite system then has the form

\[
| \psi \rangle = \sum_{i=1}^{M} \sum_{j=1}^{N} c_{ij} | u_i \rangle \otimes | v_j \rangle,
\]

where \( \{ | u_i \rangle \} \) and \( \{ | v_j \rangle \} \) denote orthonormal bases for \( H_S \) and \( H_A \) respectively, and \( M \) and \( N \) are the respective dimensions of \( H_S \) and \( H_A \).

It is always possible to choose orthonormal bases \( \{ | u_i^* \rangle \}, \{ | v_j^* \rangle \} \) for \( H_S \) and \( H_A \) in which \( | \psi \rangle \) has the Schmidt-decomposition form \[^8\]

\[
| \psi \rangle = \sum_{k=1}^{K} \sqrt{\lambda_k} | u_k^* \rangle \otimes | v_k^* \rangle,
\]

where \( K \leq \min(M,N) \), and the Schmidt coefficients \( \{ \lambda_k \} \) are non-zero and unique up to permutations. These coefficients are just the (non-zero) eigenvalues of the reduced density operators \( \rho_S = tr_A[| \psi \rangle \langle \psi |] \) and \( \rho_A = tr_S[| \psi \rangle \langle \psi |] \) of \( S \) and \( A \) respectively, as may be verified directly from Eq.(3).

The quantity

\[
E_\psi = - \sum_k \lambda_k \log_2 \lambda_k
\]

is called the “entanglement entropy” of the two systems \[^{[1]}\], and is a useful measure of the degree of correlation between \( S \) and \( A \) \[^{[3, 6, 9]}\].

It will now be assumed that \( S \) and \( A \) are randomly correlated, by which it is meant that the composite system is a member of the maximally random
pure-state ensemble discussed in the Introduction, described by a uniform distribution over the pure states of \( H_S \otimes H_A \). Such correlations may arise if the composite system is “chaotic” in the sense of Schack and Caves \(^{[10]}\) (i.e., if its state is randomised over the Hilbert space by stochastic fluctuations). They are also relevant if two observers observe the respective \( S \) and \( A \) components of an ensemble of pure composite systems (possibly for cryptographic key generation \(^{[6]}\)), in the case of minimal knowledge about the ensemble. Further, for \( M = 2 \), Braunstein has used such randomly correlated ensembles to numerically generate reduced density operators of \( S \), to test the average effectiveness of a general measurement scheme for distinguishing between two states of \( S \) \(^{[4]}\).

Now, for a quantum system of dimension \( D \), the maximally random pure-state ensemble over states \(| \sigma \rangle \) of the system is described by the probability measure \(^{[2, 4]}\)

\[
d\Omega_\sigma = K_D \delta(\langle \sigma | \sigma \rangle - 1) \prod_{d=1}^D d\text{Re}\{\sigma_d\}d\text{Im}\{\sigma_d\},
\]

where the \( \{\sigma_d\} \) are the coefficients of \(| \sigma \rangle \) with respect to some (arbitrary) orthonormal basis, and the normalisation factor \( K_D \) is given by

\[
K_D = (D - 1)!/\pi^D.
\]

Hence the randomly correlated ensemble is described by the corresponding probability measure \( d\Omega_\psi \) over the pure states \(| \psi \rangle \) of \( H_S \otimes H_A \):

\[
d\Omega_\psi = 2^{-MN}K_{MN}\delta(\langle \psi | \psi \rangle - 1) \prod_{ij} dp_{ij}d\phi_{ij},
\]

where the coefficients \( c_{ij} \) in Eq. \(^{[4]}\) have the polar form \((p_{ij})^{1/2}\exp(i\phi_{ij})\).

It proves useful to rewrite Eq. \(^{[6]}\) via the definitions

\[
\begin{align*}
x_i &= \sum_j p_{ij}, \quad (7) \\
P_{ij} &= \frac{p_{ij}}{x_i}, \quad (8) \\
| \alpha_i \rangle &= \sum_j \sqrt{P_{ij}} \exp(i\phi_{ij}) | v_j \rangle, \quad (9) \\
d\Omega_{\alpha_i} &= 2^{-N}K_N \delta(\langle \alpha_i | \alpha_i \rangle - 1) \prod_j dP_{ij}d\phi_{ij}, \quad (10)
\end{align*}
\]
where \( d\Omega_{\alpha_i} \) is the uniform measure over pure states \( \{|\alpha_i\rangle\} \) of \( H_A \). In particular, if for each \( i \) the variables \( p_{ij} \) in Eq. (6) are transformed to the variables \( P_1, \ldots, P_{N-1} \) and \( x_i \), one obtains

\[
\prod_{j=1}^{N} dp_{ij} = (x_i)^{N-1} dx_i \prod_{j=1}^{N-1} dP_{ij},
\]

(11)

where the Jacobian factor \((x_i)^{N-1}\) is most easily evaluated by adding the first \( N-1 \) rows of the Jacobian determinant to the last row. Substituting Eqs. (6)-(10) into Eq. (6), and multiplying by dummy terms of the form \( \delta(\langle \alpha_i | \alpha_i \rangle - 1) dP_{iN} \), yields the final symmetric expression

\[
d\Omega_\psi = K_{MN}/(K_N)^M \delta\left(\sum_{i} x_i - 1\right) \prod_{i=1}^{M} (x_i)^{N-1} dx_i d\Omega_{\alpha_i}
\]

(12)

for the randomly correlated ensemble.

### 3 Statistical properties

Since the measure \( d\Omega_\psi \) is invariant under unitary transformations \([2, 4]\), the reduced ensemble of density operators \( \rho_S = tr_A[|\psi\rangle\langle\psi|] \), corresponding to system \( S \), is similarly invariant under such transformations. It follows that the distribution of density operators \( \rho_S \) is basis-independent, and hence that the reduced ensemble is characterised by a probability distribution over the eigenvalue spectrum of \( \rho_S \). As noted following Eq. (2), this spectrum is determined by the Schmidt-decomposition coefficients of \(|\psi\rangle\), and hence the corresponding probability distribution will be denoted by \( p_{M,N}(\lambda_1, \lambda_2, \ldots) \). Note that the symmetry between systems \( S \) and \( A \) implies that

\[
p_{M,N} \equiv p_{N,M}
\]

(13)

To calculate the distribution \( p_{M,N} \), and hence such quantities such as the average entanglement entropy, note from Eqs. (1) and (6)-(10) that \( \rho_S \) has the general form of an \( M \times M \) matrix, with coefficients

\[
\langle u_i | \rho_S | u_j \rangle = \sqrt{x_i x_j} \langle \alpha_i | \alpha_j \rangle
\]

(14)

with respect to the \( \{|u_i\rangle\} \) basis. Hence, if a general expression for the eigenvalue spectrum of this matrix can be given, \( p_{M,N}(\lambda_1, \lambda_2, \ldots) \) can be
calculated from \( d\Omega_\psi \) in Eq. (12). This approach is successfully followed below for \( M = 2 \), while a less direct approach allows calculation in the limit as \( N \to \infty \).

Suppose then that \( M = 2 \). The eigenvalues of \( \rho_S \) follow from Eq. (14) as

\[
\lambda_1 = \frac{1}{2}(1 + r), \quad \lambda_2 = \frac{1}{2}(1 - r),
\]

(15)

where

\[
r = [1 - 4x_1x_2(1 - |\langle \alpha_1 | \alpha_2 \rangle|^2)]^{1/2}.
\]

(16)

As shown in the Appendix, one finds

\[
p_{2,N}(\lambda_1, \lambda_2) = \frac{(2N - 1)!\delta(\lambda_1 + \lambda_2 - 1)}{2(N - 2)!(N - 1)!} (\lambda_1 - \lambda_2)^2 (\lambda_1 \lambda_2)^{N-2},
\]

(17)

describing a two-dimensional system \( S \) randomly correlated with an \( N \)-dimensional system \( A \).

The average entanglement entropy of \( S \) and \( A \) can be calculated from Eqs. (3) and (17) using standard integrals, with the final result

\[
\langle E_\psi \rangle = \frac{\log_2 e}{4^{N-1}} \frac{(2N - 1)!}{(N - 2)!(N - 1)!} \sum_{s=0}^{N-2} \binom{N - 2}{s} \frac{(-1)^s}{(s + 2)(2s + 3)} \sum_{t=0}^{s+1} \frac{1}{2t + 1}.
\]

(18)

For the case of two randomly correlated qubits (\( N = 2 \)), this yields a value of \((\log_2 e)/3\approx 0.481\) bits, which is about half of the maximum possible value of 1 bit. In the limit as \( N \to \infty \) the average entanglement entropy monotonically approaches this maximum (e.g., for \( N = 100 \) the average entanglement entropy is 0.99 bits). Thus maximal correlation between \( S \) and \( A \) is guaranteed in this limit.

The latter result holds more generally. In fact, for arbitrary \( M \) it can be shown that the reduced ensemble contains only one density operator, \( M^{-1} \hat{1} \), in the limit \( N \to \infty \). The corresponding eigenvalues (and hence the Schmidt-decomposition coefficients) each equal \( M^{-1} \), and hence from Eq. (18) the average entanglement entropy attains its maximum value of \( \log_2 M \), i.e., the systems are maximally-correlated.

To show \( \rho_S \to M^{-1} \hat{1} \), note that integrating over the vectors \( \{| \alpha_i \rangle \} \) in Eq. (12) yields the marginal probability distribution

\[
p(x_1, \ldots, x_M) = K_{MN}(K_N)^{-M} \delta(x_1 + \ldots + x_M - 1)(x_1 \ldots x_M)^{N-1} \]

(19)

for the diagonal elements of \( \rho_S \) in the \( \{| u_i \rangle \} \) basis. Using Stirling’s approximation for \( n! \) in Eq. (18), it follows that this distribution vanishes everywhere
in the limit $N \to \infty$, except for the case $x_i \equiv 1/M$ for all $i$. Since the reduced ensemble is invariant under unitary transformations, the diagonal elements of $\rho_S$ in this limit are therefore equal to $1/M$ relative to any basis. Choosing a basis in which $\rho_S$ is diagonal gives $\rho_S \equiv M^{-1} \mathbb{1}$ as claimed.

The above results imply that the limit $N \to \infty$ does not yield a particularly “random” reduced ensemble of density operators for the system – indeed, it gives an ensemble with only one member. Thus, for example, the numerical evaluation of averages over the reduced ensemble in Section 7 of [3] for $M = 2$, to test the average effectiveness of a particular measurement scheme, is of most value for the maximally random case $N = 2$.

The fact that “randomness” is in fact decreased as the dimension of the auxiliary system increases suggests that a potential candidate for the minimal-knowledge ensemble discussed in the Introduction is the reduced ensemble corresponding to $N=M$, with corresponding eigenvalue distribution $p_{M,M}$ (choosing $N$ less than $M$ would unduly restrict the ensemble to density operators with $M-N$ zero eigenvalues). However, other potential candidates may be generated by a second approach, as shown in the next Section.

4 Metric-induced ensembles

An ensemble of general states of a quantum system is in general described by a probability measure over the density operators of the system. Given that probability measures transform in the same way as volume elements under co-ordinate transformations, and that volume elements are in general properties of metric spaces, this suggests that the distribution of density operators corresponding to a “minimal-knowledge” ensemble may be obtained from the normalised volume element induced by some natural metric on the space of density operators.

A metric of particular interest is the Bures metric [7], where the infinitesimal distance element between two states $\rho$ and $\rho + \delta \rho$ is given by [11]

$$ (ds_B)^2 = 2 \sum_{j,k} (\lambda_j + \lambda_k)^{-1} | \langle j | \delta \rho | k \rangle |^2, $$

(20)

where $\rho$ is diagonal in the orthonormal basis $\{|j\rangle\}$ with eigenvalues $\{\lambda_j\}$. This metric provides a unitarily-invariant measure for distinguishing between two quantum states, and has been strongly motivated as physically relevant both on measurement [12] and statistical [3, 13] grounds.
To calculate the volume element corresponding to the Bures metric, it is useful to decompose $\rho + \delta \rho$ as an infinitesimal shift in the eigenvalues of $\rho$ followed by an infinitesimal unitary transformation:

$$\rho + \delta \rho = (\hat{1} + \delta U)(\rho + \delta \Lambda)(\hat{1} + \delta U)^\dagger = \rho + \delta \Lambda + [\delta U, \rho],$$

(21)

where $\langle j | \delta \Lambda | k \rangle = \delta_{jk} d\lambda_j$, and $(\delta U)^\dagger = -\delta U$ follows from unitarity. Note moreover that the infinitesimal generator $\delta U$ can generally be decomposed as

$$\delta U = \sum_{j \leq k} \left[ (dx_{jk} + idy_{jk}) | j \rangle \langle k | - \text{h.c.} \right]$$

(22)

where $dx_{jk}$ and $dy_{jk}$ are real, and h.c. denotes the Hermitian conjugate of the expression preceding it.

Substitution of Eqs. (21) and (22) into Eq. (20) yields

$$\begin{align*}
(ds_B)^2 &= \sum_j \frac{(d\lambda_j)^2}{\lambda_j} + 4 \sum_{j < k} \frac{(\lambda_j - \lambda_k)^2}{\lambda_j + \lambda_k} [(dx_{jk})^2 + (dy_{jk})^2],
\end{align*}$$

(23)

from which one immediately extracts the volume element

$$dV_B = \frac{d\lambda_1 \ldots d\lambda_M}{(\lambda_1 \ldots \lambda_M)^{1/2}} \prod_{j < k} \frac{(\lambda_j - \lambda_k)^2}{\lambda_j + \lambda_k} dx_{jk} dy_{jk}.$$ 

(24)

Normalising $dV_B$ yields the desired probability distribution over the space of density operators.

Since the metric is invariant under unitary transformations, the corresponding ensemble is characterised by the marginal probability distribution $p_B(\lambda_1, \ldots, \lambda_M)$ over the eigenvalue spectrum of the density operators describing the system (see also Section 3). This distribution can be obtained from Eq. (24) by integrating over the (compact) space of unitary transformations (parametrised by $\{x_{jk}, y_{jk}\}$), and normalising, to give

$$p_B(\lambda_1, \ldots, \lambda_M) = C_M \frac{\delta(\lambda_1 + \ldots + \lambda_M - 1)}{(\lambda_1 \ldots \lambda_M)^{1/2}} \prod_{j < k} \frac{(\lambda_j - \lambda_k)^2}{\lambda_j + \lambda_k}.$$ 

(25)

where $C_M$ is a normalisation constant, and the condition $tr[\rho] = 1$ has been made explicit.

Eq. (25) will be compared with Eq. (17) in the following Section for the case $M = 2$. This Section is concluded by noting that in principle there
are many possible choices of metric, each leading to a possible “random” ensemble. One particularly simple choice is the Hilbert-Schmidt metric, with infinitesimal distance element

\[ (ds_{HS})^2 = tr[(\delta \rho)^2]. \]  

(26)

Following essentially the same procedure as above for the Bures metric (where the trace is evaluated in the \{\ket{j}\} basis), one finds the corresponding probability distribution

\[ p_{HS}(\lambda_1, \ldots, \lambda_M) = C'_M \delta(\lambda_1 + \ldots + \lambda_M - 1) \prod_{j<k}(\lambda_j - \lambda_k)^2 \]  

(27)

for the density operator eigenvalue spectrum. This is also considered in the following Section for the case \(M = 2\).

5 Two-dimensional comparisons

The states \(\rho\) of a two-dimensional system may be parametrised in the Bloch representation as

\[ \rho = \frac{1}{2}(1 + \sigma \cdot r), \]  

(28)

where \(\sigma\) is the 3-vector of Pauli matrices and \(r\) is a 3-vector of modulus \(r \leq 1\). The eigenvalues of \(\rho\) are related to \(r\) as per Eq. (15).

A distribution over \(\rho\) may therefore be written as a distribution over \(r\). Moreover, since unitary transformations of \(\rho\) correspond to rotations of \(r\), it follows that distributions corresponding to unitarily-invariant ensembles depend only on the modulus \(r\), being uniform with respect to direction. Hence the distributions over \(r\) corresponding to eigenvalue distributions Eqs. (17) (with \(N = 2\), (25) and (27) are given respectively by

\[ p_{2,2}(r) = 3/(4\pi), \]  

(29)

\[ p_B(r) = (4/\pi)(1 - r^2)^{-1/2}, \]  

(30)

\[ p_{HS}(r) = 3/(4\pi). \]  

(31)

It is seen that the first and third distributions are uniform over the unit 3-ball, while the distribution corresponding to the Bures metric is sharply peaked at the surface of the ball (corresponding to pure states of the system). This raises the question of which is the more “random”? I shall argue here for the latter, due to its greater symmetry.
In particular, the Bures metric for a two-dimensional system corresponds to the surface of a unit 4-ball \( [1] \), i.e., to the maximally symmetric 3-dimensional space of positive curvature \([4]\) (and may be recognized as the spatial part of the Robertson-Walker metric in general relativity \([14]\)). This space is homogenous and isotropic, and hence the Bures metric does not distinguish a preferred location or direction in the space of density operators. Indeed, as well as rotational symmetry in Bloch co-ordinates (corresponding to unitary invariance), the metric has a further set of symmetries generated by the infinitesimal transformations \([14]\)

\[
r \rightarrow r + \epsilon (1 - r^2)^{1/2} a
\]  

(32)

(where \(a\) is an arbitrary 3-vector).

Taking the viewpoint that maximal randomness corresponds to an ensemble with maximal symmetry, it follows that the distribution of Eq. \((30)\), in corresponding to a maximally symmetric space, is in fact more “random” than the distributions of Eqs. \((29)\) and \((31)\). This strongly suggests, at least for two-dimensional quantum systems, that the minimal-knowledge ensemble discussed in the Introduction is the one induced by the Bures metric.

Finally, note that the existence of various candidates for the minimal-knowledge ensemble discussed in the Introduction begs the question as to whether there exists some natural physical process for generating ensembles of quantum systems, which can be identified with maximal randomness. This would allow experimental determination of the minimal-knowledge ensemble. This is, however, beyond the scope of this Letter.

**Acknowledgement**

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**Appendix**

To derive Eq. \((17)\), note from Eqs. \((15)\) and \((16)\) that the eigenvalue distribution can be calculated if the joint distribution of the variables

\[
X = x_1, Y = | \langle \alpha_1 | \alpha_2 \rangle |^2
\]

is known. From Eq. \((12)\) the statistics of \(X\) and \(Y\) are independent, and hence this joint distribution has the factored form

\[
p(X, Y) = p(X)q(Y).
\]

(34)

From Eqs. \((12)\) and \((33)\) one immediately has

\[
p(X) = K_{2N}(K_N)^{-2}[X(1 - X)]^{N-1}.
\]

(35)
To find $q(Y)$, fix an orthonormal basis $\{ |v_j\rangle \}$ in $H_A$, and for a given unit vector $|\alpha_1\rangle$ let $U$ be a unitary transformation which maps $|\alpha_1\rangle$ to $|v_1\rangle$ and define $|\beta\rangle = U |\alpha_2\rangle$. Thus
\[
Y = | \langle \alpha_1 | U^\dagger U | \alpha_2 \rangle |^2 = | \langle v_1 | \beta \rangle |^2.
\] (36)

Since $d\Omega_{\alpha_2}$ is invariant under unitary transformations, writing $\langle v_j | \beta \rangle = (w_j)^{1/2} \exp(i\theta_j)$ yields
\[
d\Omega_{\alpha_2} = d\Omega_{\beta} = 2^{-N} K_N \delta(\langle \beta \mid \beta \rangle - 1) \prod_j dw_j d\theta_j
\]
\[
= 2^{-N} K_N \delta(\sum_{j=2}^{N} w_j - (1 - Y)) dY d\theta_1 \prod_{j=2}^{N} dw_j d\theta_j
\]
\[
= 2^{-N} K_N \delta(\sum_{j=2}^{N} W_j - 1) (1 - Y)^{N-2} dY d\theta_1 \prod_{j=2}^{N} dW_j d\theta_j
\]
\[
= 2^{-1} (K_N / K_{N-1})(1 - Y)^{N-2} dY d\theta_1 d\Omega_\gamma,
\] (37)

where $W_j = w_j / (1 - Y)$, and $d\Omega_\gamma$ is the uniform measure over pure states of the $(N-1)$-dimensional space spanned by $\{ |v_2\rangle \ldots |v_N\rangle \}$. Multiplying this expression by $d\Omega_{\alpha_1}$ and integrating over all variables except $Y$ then gives
\[
q(Y) = \pi (K_N / K_{N-1})(1 - Y)^{N-2}.
\] (38)

Finally, substituting Eqs. (1), (33) and (38) into Eq. (34), the marginal distribution of $r$ in Eq. (14) can be calculated as
\[
p(r) = \frac{1}{2 (N-1)!} (2N - 1)! \left( \frac{1 - r^2}{4} \right)^{N-2},
\] (39)

which with Eq. (15) immediately yields Eq. (17).

References

[1] W.K. Wootters, Found. Phys. 20 (1990) 1365.
[2] K.R.W. Jones, Ann. Phys. (N.Y.) 207 (1991) 140.
[3] S.L. Braunstein, Phys. Lett. A 219 (1996) 169.
[4] S. Skyora, J. Stat. Phys. 11 (1974) 17.

[5] R. Josza, D. Robb and W.K. Wootters, Phys. Rev. A 49 (1994) 668.

[6] C.H. Bennett, H.J. Bernstein, S. Popescu and B. Schumacher, Phys. Rev. A 53 (1996) 2046.

[7] D.J.C. Bures, Trans. Am. Math. Soc. 135 (1969) 199.

[8] e.g., A. Ekert and P.L. Knight, Am. J. Phys. 63 (1995) 415.

[9] G. Lindblad, Commun. Math. Phys. 33 (1973) 305; S.M. Barnett and S.J.D. Phoenix, Phys. Rev. A 44 (1991) 535.

[10] R. Schack and C.M. Caves, Phys. Rev. E 50 (1996) 3257; ibid, Hyper-sensitivity to perturbation: an information-theoretical characterization of classical and quantum chaos, in Quantum Communication, Computing, and Measurement, ed. O. Hirota et al. (Plenum, New York, 1997).

[11] M. Hübner, Phys. Lett. A 163 (1992) 239.

[12] S.L. Braunstein and C.M. Caves, Phys. Rev. Lett. 72 (1994) 3439.

[13] R. Josza, J. Mod. Opt. 41 (1994) 2315.

[14] S. Weinberg, Gravitation and Cosmology (Wiley, New York, 1972), Sections 13.3, 14.2.

[15] P.B. Slater, quant-ph/9703012 (see ADDITIONAL NOTES below).
Equation (17) is a special case of Theorem 3, Sec.III of S. Lloyd and H. Pagels, in Ann. Phys. (NY) 188 (1988) 186. Taking $M \leq N$ without loss of generality (see Equation (13)), one has the general formula

$$p_{M,N}(\lambda_1, \ldots, \lambda_M) = C_{M,N} \delta(\sum \lambda_m - 1) [\prod_{m,n} (\lambda_m - \lambda_n)^2] [\prod_k (\lambda_k)^{N-M}], \quad (40)$$

where $C_{M,N}$ is a normalisation constant.

Equation (18) can be simplified and generalised to calculate the average entanglement entropy for all values of $M$ and $N$, using a formula conjectured by Don Page and elegantly proved by S. Sen in Phys. Rev. Lett. 77 (1996) 1-3. Again taking $M \leq N$, one has

$$\langle E_\psi \rangle_{M,N} = \sum_{m=N+1}^{MN} \frac{1}{m} - \frac{M-1}{2N}. \quad (41)$$

As noted in the published version of the present paper (Phys. Lett. A 242 (1998) 123-129), a recent related preprint seeks to determine the “maximally noninformative” ensemble for a two-dimensional quantum system. This is essentially a different concept from the ”maximally random” ensemble sought here (and identified as corresponding to the Bures volume measure in the 2-D case). Indeed, intuitively one would expect an ensemble consisting solely of the density operator proportional to the density operator to be least ”informative”, as (i) no Shannon information can be gained by measurement on such an ensemble; and (ii) no preferred basis can be singled out by measurement.