Formalism Locality in Quantum Theory and Quantum Gravity

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Abstract

We expect a theory of Quantum Gravity to be both probabilistic and have indefinite causal structure. Indefinite causal structure poses particular problems for theory formulation since many of the core ideas used in the usual approaches to theory construction depend on having definite causal structure. For example, the notion of a state across space evolving in time requires that we have some definite causal structure so we can define a state on a space-like hypersurface. We will see that many of these problems are mitigated if we are able to formulate the theory in a formalism local (or F-local) fashion. A formulation of a physical theory is said to be F-local if, in making predictions for any given arbitrary space-time region, we need only refer to mathematical objects pertaining to that region. This is a desirable property both on the grounds of efficiency and since, if we have indefinite causal structure, it is not clear how to select some other space-time region on which our calculations may depend. The usual ways of formulating physical theories (the time evolving state picture, the histories approach, and the local equations approach) are not F-local.

We set up a framework for probabilistic theories with indefinite causal structure. This, the causaloid framework, is F-local. We describe how Quantum Theory can be formulated in the causaloid framework (in an F-local fashion). This provides yet another formulation of Quantum Theory. This formulation, however, may be particularly relevant to the problem of finding a theory of Quantum Gravity.

1 Introduction

The problem of Quantum Gravity is to find a theory which reduces in appropriate limits to General Relativity and Quantum Theory (including, at least, those situations where those two theories have been experimentally confirmed). To be significant, the theory must also make correct predictions for new experiments.
in the future. The problem of combining two less fundamental theories into a more fundamental one is not something for which a simple algorithm can exist and thus we need a motivating idea to get started. Here we note that General Relativity and Quantum Theory are each conservative and radical in complementary ways. General Relativity is conservative in that it is deterministic but radical in that it has non-fixed causal structure (whether a particular interval is time-like is not fixed in advance but can only be decided after we have solved for the metric). Quantum Theory is conservative in that it has fixed causal structure built in, but radical in that it is inherently probabilistic (standard Quantum Theory cannot be formulated without reference to probabilities). It seems likely that a theory of Quantum Gravity must inherit the radical features of the two component theories. Hence, we are looking for a theoretical structure that

1. is probabilistic
2. has non-fixed causal structure

In fact, we expect the situation to be even more radical. In General Relativity the causal structure is not fixed in advance, but, once determined, there is a definite answer to the question of whether an interval is time-like or not. However, in Quantum Theory any quantity that is subject to variation is also subject to quantum uncertainty. This means that, in a theory of Quantum Gravity, there may be no matter of fact as to whether a particular interval is time-like or not. It is likely that the causal structure is not only non-fixed, but also indefinite. The fact that we expect the conservative features of each component theory to be replaced by the radical features in the other suggests that a theory of Quantum Gravity cannot be entirely formulated within General Relativity or Quantum Theory. In this, our program differs from String Theory [1] and Loop Quantum Gravity [2] where the attempt is to formulate Quantum Gravity within Quantum Theory (though there are other approaches which, to varying extents, do not assume Quantum Theory will remain intact [3][4][5][6]).

One signature of the fixed causal structure in Quantum Theory is the fact that we have a fixed background time $t$ used to evolve the state $|\psi(t)\rangle = U(t)|\psi(0)\rangle$. A deeper signature of fixed causal structure in Quantum Theory can be seen by considering the different ways in which operators can be put together. The operators corresponding to two space-like separated regions are combined with the tensor product $A \otimes B$. If a system passes through two immediately sequential time-like separated regions then the appropriate way to combine the corresponding operators is with the direct product $CB$. And if a system passes through two time-like separated regions which have a gap in between (i.e. they are not immediately sequential) then the appropriate way to combine the operators is with what we will call the question mark product $[DB]$. This linear operator is defined by $[DB]C \equiv DCB$. For each situation, we must combine the associated operators in a way that depends on the causal relationship between the two regions. It would be good to have a mathematical framework which treats each type of situation on an equal footing since then
the fixed causal background need not be ingrained into the very structure of the theory.

The task becomes one of finding a theoretical framework for probabilistic theories with indefinite causal structure that correlate recorded data. The causaloid framework set up in [7] (see also [8]) does this. In this framework the causaloid product is defined. This unifies the three products mentioned above from Quantum Theory (in the context of a more general mathematical framework).

In this paper I will discuss the challenges posed by having indefinite causal structure and show how the causaloid formalism deals with them. I will indicate how the Quantum Theory of pairwise interacting qubits can be dealt with in this formalism (this is an important example since we can use it to do universal quantum computation). Finally we will look at the road to formulating Quantum Gravity in this framework.

2 Dealing with indefinite causal structure

Indefinite causal structure is much more radical than merely having non-fixed causal structure as in General Relativity (GR). In GR the causal structure, whilst not given in advance, is part of the solution. After solving Einstein’s field equations we know the metric and therefore the causal structure.

Indefinite causal structure would mark a radical departure from previous physics. Many of our basic concepts and modes of thought rely on having definite causal structure. For example, we often think of quantities being conserved (in time) or increasing (in time), and we think of information flowing (in time). We think of entanglement (across space). And, most crucially, we often think of a state (across space) evolving (in time). However, if we have indefinite causal structure there would, in general, be no matter of fact as to whether a particular interval was space-like or time-like and so all of these concepts and modes of thought would be placed under some tension. Nevertheless, one can make a very strong case that Quantum Gravity (QG) will have indefinite causal structure and so we need to think sufficiently radically to be in a position to deal with this. Most approaches to QG do imagine some form of indefinite causal structure. However, there has been comparatively little thought as to how to really deal with this properly. Generally the conceptual and mathematical tools handed down to us from previous physics, encumbered with ingrained notions of definite causal structure, are used. For example, one might argue that we can model indefinite causal structure by taking a sum over histories each having its own definite causal structure. But why require each history to have definite causal structure rather than giving up this notion all together at the fundamental level. We need to be prepared to think radically about this issue. The causaloid formalism offers a way forward here.

A common attitude is that the equations of physics must tell us how to calculate the evolution of physical systems in time. If there is indefinite causal structure then we cannot think in this way. Instead, we adopt the assertion that a physical theory must correlate recorded data. This does not commit us to a
picture of anything evolving in time. Thus, we might ask what

$$\text{prob}(\text{data}_2|\text{data}_1)$$  \hspace{1cm} (1)

is equal to. If we can deal with all such probabilities for any data then we can say that we have formulated a physical theory (at least that aspect of the theory which can be empirically verified). By thinking about how data might be correlated, we are adopting an operational methodology here. However, this is just a methodology aimed at helping along theory construction. In adopting this approach we do not commit ourselves to operationalism as a fundamental philosophical outlook on the world.

We will now discuss two issues which arise when we think in this way (particularly when there is indefinite causal structure).

2.1 Issue 1: The need for a two-step approach

The first issue is the question of when we have sufficient information to be able to make a prediction. Though this is often not appreciated, physical theories only attempt to answer a very small fraction of the possible questions about the world one might put to them. To see this, consider a spin-half particle subjected to three sequential spin measurements. The probability that spin up is seen at the second position given that spin up was seen at the first position, and given that the angles chosen were $\theta_1$ and $\theta_2$ (in the first and second positions) can be written

$$\text{prob}(+|+, \theta_1, \theta_2).$$  \hspace{1cm} (2)

This probability can be calculated using QT (and is equal to $\cos^2(\frac{\theta_2-\theta_1}{2})$). We can say that this probability is well defined. This is an example of a question which the theory does answer. But now consider the probability that spin up is seen at the third position given that spin up was seen at the first position, and given that the angles chosen were $\theta_1$ and $\theta_3$ (in the first and third positions). We can write this probability as

$$\text{prob}(+|+, \theta_1, \theta_3)$$  \hspace{1cm} (3)

Note that we are not given any information about the second spin measurement. This is not part of the conditioning. Under these circumstances we cannot use quantum theory to calculate this probability. This probability is not well defined. This is a question which QT does not answer. And neither should it. Indeed, generically QT does not answer most questions. This is true of physical theories in general (even deterministic ones). In General Relativity, for example, we can only make predictions about data that may be recorded in some region $R_2$ given data in region $R_1$ if $R_2$ is a domain of dependence of $R_1$.

The key difference between the two situations in the spin example is to do with the causal structure. In the first case one measurement immediately proceeds the other, whereas in the second case there is a gap in time for which we have no information. In order to know whether the probability is well defined
or not we need to know what causal situation pertains. If we have definite causal structure then we can refer to it and know whether we are in the rather special type of situation where we can actually make a prediction. However, if we have indefinite causal structure then we do not know how to proceed.

No doubt there will still be certain conditional probabilities which are well defined even if we go beyond quantum theory and have indefinite causal structure. One way we might deal with this is to mathematize the question. Thus we want a formalism involving two steps

**Two step approach**

**Step 1** We have a mathematical condition that is satisfied if and only if a probability is well defined.

**Step 2** In the case where the condition in step 1 is satisfied, we have a formula for calculating the probability.

The standard picture with definite causal structure is actually an example of this form. Thus, we have the theory of domains of dependence which tell us whether we can make predictions about some region $R_2$ based on data in region $R_1$ by looking at the causal structure. However, we can imagine more general ways in which we might implement this two step approach that do not explicitly refer to causal structure (at least as the latter is usually conceived).

### 2.2 Issue 2: The need for F-locality

The second issue is very much related to the first. Imagine we want to calculate probabilities pertaining to some arbitrary space-time region $R$. This space-time region may be of any shape and may be disconnected (in so much as we have a notion of connection in the absence of definite causal structure). For example we may want to know what the probability of seeing a certain outcome in $R$ is given that we performed certain measurements in $R$ and saw certain other outcomes in $R$. In the standard formulation of QT we have a state across space evolving in time. Imagine that $R$ consists of two disconnected parts that are time-like separated. To make a prediction (to say whether the probability is well defined and, if so, what it is equal to) we need to evolve the quantum state through intermediate times. Therefore we necessarily need to refer to mathematical objects and (implicit) data which does not pertain to $R$ in the evolving state picture. As we will see, this is also the case in other types of formulation of physical theories (such as histories approaches). If we have some well defined causal structure then we can use that to tell us what other region, besides $R$, we need to be considering to implement the mathematical machinery of the physical theory. However, this option is not open to us if we have indefinite causal structure. In that case the only clean approach is to insist that, in making predictions for $R$, we only refer to mathematical objects pertaining to $R$ (for, if not, what do we consider). This seems like a useful idea and deserves a name - we will call it formalism locality (or F-locality).
**F-locality:** A formulation of a physical theory is F-local if, in using it to make statements (using the two step approach) about an arbitrary spacetime region $R$, we need only refer to mathematical objects pertaining to $R$.

It is possible that a given physical theory can be formulated in different ways. F-locality is a property of the way the physical theory is formulated rather than of the theory itself. It is possible that any theory admits a formulation which is F-local. In the case where there is a definite causal structure we may be able to provide both F-local and not F-local formulations of a theory. However, if there is indefinite causal structure then it seems likely that any fundamental formulation of the theory will necessarily be F-local.

There are two motivations for attempting to formulate theories in an F-local fashion:

1. We do not need to refer to some definite causal structure to decide what other region (besides the region under consideration) to consider.
2. It is more efficient to consider only mathematical objects pertaining to the given region.

Both these reasons are worth bearing in mind when evaluating formulations which are not F-local.

### 3 How standard formulations of physical theories are not F-local

There are, perhaps, three ways in which physical theories have been formulated to date.

1. The state evolving in time picture.
2. Histories formulations.
3. Local equations approach.

None of these are F-local (the third case is a little more debatable) as we will now see.

In the state evolving in time picture the state is specified at some initial time and it then evolves according to some equations. Imagine we want to make a statement about a space-time region, $R$, consisting of two disconnected parts that are separated in both space and time. To do this we take a state defined across enough of space to encompass both spatial regions and evolve it through enough time to encompass the two regions. Hence, we need to refer to mathematical objects pertaining to a region of space-time $R'$ which includes both spatial and temporal regions that are not part of $R$.

In histories formulations we consider the entire history from some initial to some final time. The physical theory makes statements about such entire
histories (the path integral formulation of Quantum Theory is one example).
If we are only interested in some particular region \( R \) then, clearly, in a history formulation, we need to make reference to mathematical objects which do not pertain to \( R \) and so the formulation is not F-local. One might claim that, since we can take the history across all of space-time, we do not need to refer to any definite causal structure to decide what region to consider - we simply take everything. Even if this does work, it is still more efficient to aim at an F-local formulation. And, in practice, we always take our histories over some limited time interval. Indeed, in the absence of a solution, we may not know the nature of “all of space-time” and so it is difficult to know how to give a histories formulation of the theory.

An example of the local equations approach is Maxwell’s equations. Such equations constitute a set of local statements about the infinitesimal regions making up our region \( R \). To actually make a prediction for region \( R \) we need to use these local statements appropriately. Typically this involves solving the equations with boundary conditions on a boundary that is in the causal past of all of \( R \). Hence, we need to consider a region bigger than \( R \). There may be other ways to utilize local equations to make predictions about arbitrary regions that do not require consideration of a larger region. It is clear, in any case, that a local equations formulation is not explicitly F-local as defined above because it does not come equipped with an F-local technique for making predictions for arbitrary regions. There is one sense in which local equations clearly go against the spirit of F-locality. A local equation relates quantities in regions that are infinitesimally displaced from one another. The property of being infinitesimally displaced relates to causal structure. If the causal structure is indefinite then it is not clear that we can retain this notion (this is one reason that we may expect whatever plays the role of space-time in a theory of Quantum Gravity to be discrete rather than continuous).

4 An outline of the causaloid framework

It is not clear that physical theories can be formulated in an F-local fashion. In [7, 8] a framework for probabilistic theories with indefinite causal structure was given. This framework provides a way of explicitly formulating theories in an F-local fashion. Here we will give a bare-bones outline of this framework. In the next section we will indicate how the QT of interacting qubits can be formulated in the framework.

4.1 Data and regions

Imagine that all the data collected during an experiment is recorded on cards as triples \((x, F_x, Y_x)\). Here \( x \) is some recorded data taken as representing space-time location, \( F_x \) is some choice of experiment (knob setting) at \( x \), and \( Y_x \) is the outcome of some experiment at \( x \). For example, \( x \) might be recorded from a GPS system, \( F_x \) could be the angle at which a Stern-Gerlach apparatus is
set, and \( Y_x \) could be the outcome of the spin measurement. During a typical experiment, data will be recorded at many space-time locations. At the end of one run of the experiment we will collect a stack of cards. Since we are interested in probabilities we can imagine running the experiment many times so we can obtain relative frequencies.

Since \( x \) constitutes recorded data, it will be discrete. Therefore, we can imagine running the experiment many times so we can obtain relative frequencies. We do not assume any a priori causal structure on the \( x \). An arbitrary region \( R_1 \) consists of some set of elementary regions \( R_x \).

We can ask what \( \text{prob}(Y_2|Y_1,F_1,F_2) \) is equal to. That is what is the probability of seeing outcomes \( Y_2 \) in region \( R_2 \) given that chose \( F_1 \) in region \( R_1 \), chose \( F_2 \) in \( R_2 \), and saw outcomes \( Y_1 \) in region \( R_1 \) (this is a more sophisticated version of (1))? Here we are trying to make statements about region \( R_1 \cup R_2 \). We will now outline how we go about doing this in an F-local way for \( R_1 \cup R_2 \).

### 4.2 p-type vectors and r-type vectors

Let \( V \) be the union of all elementary regions. We will assume that the probabilities

\[
\text{prob}(Y_V|F_V)
\]

are well defined (we are glossing over subtle points that are covered in [7]). We can write

\[
\text{prob}(Y_V|F_V) = \text{prob}(Y_{R_1}, Y_{V-R_1}|F_{R_1}, F_{V-R_1})
\]

We will now label each possible \( (Y_{R_1}, F_{R_1}) \) combination in region \( R_1 \) with \( \alpha_1 \in \mathcal{Y}_1 \). This label runs over all possible (outcome, choice) combinations in region \( R_1 \). We write

\[
p_{\alpha_1} = \text{prob}(Y^{\alpha_1}_{R_1}, Y_{V-R_1}|F^{\alpha_1}_{R_1}, F_{V-R_1})
\]

We can regard \( (Y_{V-R_1}, F_{V-R_1}) \) in \( V-R_1 \) as a kind of generalized preparation for region \( R_1 \) (it is generalized since it pertains to both the future and the past in so far as those concepts have meaning). Associated with each generalized preparation is a state. We define the state for region \( R_1 \) to be that thing which is represented by any mathematical object which can be used to calculate an
arbitrary probability $p_{\alpha_1}$. Clearly the object
\[
\begin{pmatrix}
\vdots \\
p_{\alpha_1} \\
\vdots
\end{pmatrix}
\quad \alpha_1 \in \Upsilon_1
\]  
(9)
suffices (since it simply lists all the probabilities). However, in general, we
expect that this is much more information than necessary. In general, in physical
theories, all quantities can be calculated from a subset of quantities. We call this
physical compression. In our particular case we expect that a general probability
$p_{\alpha_1}$ can be calculated from a subset of these probabilities. We will restrict
ourselves to linear physical compression (where the probabilities are related by
linear equations). We set
\[
\mathbf{p} = \begin{pmatrix}
\vdots \\
p_{k_1} \\
\vdots
\end{pmatrix}
\quad k_1 \in \Omega_1 \subseteq \Upsilon_1
\]  
(10)
such that a general probability $p_{\alpha_1}$ can be calculated from the $p_{k_1}$ (with $k_1 \in \Omega_1$)
by a linear equation
\[
p_{\alpha_1} = r_{\alpha_1} \cdot \mathbf{p}
\]  
(11)
We choose the fiducial set $\Omega_1$ of labels such that there is no other choice with
smaller $|\Omega_1|$ (this means that every probability in $\mathbf{p}$ is necessary in the speci-
fication of the state). There may be many possible choices for the fiducial set $\Omega_1$. We simply choose one (for each region) and stick with it. We do not lose
generality by imposing linearity here. In the worst case $\Omega_1 = \Upsilon_1$. It is possible
that nonlinear compression is more efficient. However, for probabilities this is
not the case as long as one can form arbitrary mixtures. In particular, in Quan-
tum theory (and Classical Probability Theory) linear compression is optimal (so
long as we allow mixed states rather than restrict ourselves to pure states).

Associated with each region $R_1$ is a real vector space of dimension $|\Omega_1|$. Fur-
ther, associated with each $(Y_1, F_1)$ combination there is a $r$-type vector (which
lives in a dual space to the $p$-type vectors representing the states) which we can
write as
\[
r_{(Y_1, F_1)}(R_1)
\]  
(12)
or $r_{\alpha_1}$ for short. It is these $r$-type vectors that we use in real calculations.
The state vector, $\mathbf{p}$, is akin to scaffolding - it can be dispensed with once the
structure of the $r$-type vectors is in place as we will see shortly.

4.3 The causaloid product
If we have two disjoint regions $R_1$ and $R_2$, then we can consider the region
$R_{12} \equiv R_1 \cup R_2$ as a region in its own right. We can denote the outcome and
knob settings for $R_{12}$ as $Y_1 \cup Y_2$ and $F_1 \cup F_2$ (perhaps we are slightly abusing the
∪ notation here). Since \( R_1 \cup R_2 \) is a region in its own right we will have \( r \)-type vectors associated with each (outcome, choice) combination in this region also.

We can label the (outcome, choice) pairs \((Y_1 \cup Y_2, F_1 \cup F_2)\) with \( \alpha_1 \alpha_2 \in \Upsilon_1 \times \Upsilon_2 \) (where \( \times \) denotes the cartesian product of ordered pairs taken from the two sets). We also have the fiducial set \( \Omega_{12} \) for this region. There is an important theorem - namely that it is always possible to choose \( \Omega_{12} \subseteq \Omega_1 \times \Omega_2 \).

We will use the notation \( l_1 l_2 \in \Omega_1 \times \Omega_2 \) and \( k_1 k_2 \in \Omega_{12} \).

In the case that \( \Omega_{12} = \Omega_1 \times \Omega_2 \) we have no extra physical compression when two regions are considered together. However, if \( \Omega_{12} \subset \Omega_1 \times \Omega_2 \) then there is an extra physical compression (second level compression) for the composite region over and above the physical compression (first level compression) for each component region considered separately. Physically, this non-trivial case corresponds to causal adjacency such as when a qubit passes through two sequential regions with no gap in between.

Second level compression is quantified by a matrix \( \Lambda_{k_1 k_2}^{l_1 l_2} \) (which depends on the composite region under consideration) such that

\[
\left. r_{\alpha_1 \alpha_2} \right|_{k_1 k_2} = \sum_{l_1, l_2 \in \Omega_1 \times \Omega_2} \left. r_{\alpha_1} \right|_{l_1} \left. r_{\alpha_2} \right|_{l_2} \Lambda_{l_1 l_2}^{k_1 k_2}
\]

where \( \left. r_{\alpha_1} \right|_{l_1} \) denotes the \( l_1 \) component of \( r_{\alpha_1} \). We write

\[
r_{\alpha_1 \alpha_2} = r_{\alpha_1} \otimes^A r_{\alpha_2}
\]

where the components are given in (13). Accordingly, we have defined a new type of product denoted by \( \otimes^A \). We call this the causaloid product. It is the sought after unification of the various products \( A \otimes B \), \( AB \), and \([A?B]\) from quantum theory mentioned in the introduction (though in a more general framework). The form of the product is the same regardless of the causal structure. However, since the \( \Lambda \) matrix can differ for different composite regions, we can still encode the different products in this one product.

If we have a region regarded as being composed of more than two regions then we can generalize the above ideas appropriately (calling on, in general, a matrix of the form \( \Lambda_{l_1 l_2 l_3 \ldots}^{k_1 k_2 k_3 \ldots} \)).

### 4.4 The two-step approach in the causaloid framework

We note that, using Bayes rule,

\[
\text{Prob}(Y_2|Y_1, F_1, F_2) = \frac{\text{Prob}(Y_2, Y_1|F_1, F_2)}{\sum_{X_2 \sim F_2} \text{Prob}(X_2, Y_1|F_1, F_2)}
\]

\[
= \frac{r(Y_2 \cup Y_1, F_1 \cup F_2) \cdot p}{\sum_{X_2 \sim F_2} r(X_2 \cup Y_1, F_1 \cup F_2) \cdot p}
\]

where the notation \( X_2 \sim F_2 \) denotes all outcomes in \( R_2 \) which are consistent with the choice \( F_2 \) in \( R_2 \). In order that the probability \( \text{Prob}(Y_2|Y_1, F_1, F_2) \) be
well defined, it must depend only on the given conditioning in \( R_1 \cup R_2 \) and not depend on what happens outside this region. The state, on the other hand, is associated with some generalized preparation in \( V - R_1 - R_2 \). Hence, this probability is only well defined if there is no dependence on the state \( p \) (strictly we should have included the conditioning in \( V - R_1 - R_2 \) and then shown that it is irrelevant if there is no dependence on \( p \)). We can use this observation to implement a two step approach.

**Two step approach**

1. \( \text{Prob}(Y_2|Y_1, F_1, F_2) \) is well defined if and only if

\[
\mathbf{r}(Y_2 \cup Y_1, F_1 \cup F_2) \text{ is parallel to } \sum_{X_2 \sim F_2} \mathbf{r}(X_2 \cup Y_1, F_1 \cup F_2) \tag{18}
\]

(since this is the necessary and sufficient condition for there being no dependence on \( p \)).

2. If these vectors are parallel, then the probability is given by

\[
\text{Prob}(Y_2|Y_1, F_1, F_2) = \frac{|\mathbf{r}(Y_2 \cup Y_1, F_1 \cup F_2)|}{|\sum_{X_2 \sim F_2} \mathbf{r}(X_2 \cup Y_1, F_1 \cup F_2)|} \tag{19}
\]

We see that this two-step approach does not require us to refer to some given definite causal structure (at least as the latter is usually conceived).

We see that the formulation is F-local since, to make predictions for an arbitrary space-time region (in this case the region \( R_1 \cup R_2 \)), we need only consider mathematical objects pertaining to this region (the \( \mathbf{r} \) vectors). Strictly speaking, we need to be sure we can calculate the \( \mathbf{r} \) vectors for arbitrary regions without referring to mathematical objects pertaining to other regions to assert that the formulation is fully F-local. This will be addressed below.

4.5 **The causaloid**

We can calculate \( \mathbf{r} \) vectors for an arbitrary region by starting from the \( \mathbf{r} \) vectors for the elementary regions comprising that region and using the causaloid product. Hence, we can calculate any \( \mathbf{r} \)-vector if we know

1. All the vectors \( \mathbf{r}_{\alpha_x} \) (which can be regarded as a matrix \( A_{\alpha_x}^{k_x} \)) for all elementary regions, \( R_x \).

2. All the matrices \( A_{\|x'x''\cdots}^{k_x k_{x'} k_{x''} \cdots} \) with \( x, x', x'', \cdots \in \mathcal{O}_1 \), for all \( \mathcal{O}_1 \) with \(|\mathcal{O}_1| \geq 2 \) (since these pertain to composite regions).

This constitutes a tremendous amount of information (the number of matrices is exponential in the number of elementary regions and the size of these matrices grows with the size of the region they pertain to). However, we can apply physical compression by finding relationships between these matrices (we call this third level compression). After applying physical compression, we have a
smaller set of matrices from which all the others can be calculated. We call this smaller set, augmented by a set of rules for implementing decompression, the causaloid and denote it by $\Lambda$. Since we want the framework to be F-local, we require that, in applying decompression to obtain the matrix for a region $R_1$, we only need use matrices pertaining to regions $R_1 \subseteq R_1$.

While we have not shown how to calculate the causaloid in general, it has been shown how to do so for the classical probabilistic theory of pairwise interacting classical bits and the quantum theory of pairwise interacting qubits. We will outline, in the next section, how this works in the quantum case. The classical case is very similar (though we will not outline it in this paper).

5 Formulating Quantum Theory in the causaloid framework

In this section we will content ourselves with simply describing how the Quantum Theory of pairwise interacting qubits can be formulated in the causaloid framework without deriving any of the results. Universal quantum computation can be implemented with pairwise interacting qubits and so arbitrary quantum systems can be simulated to arbitrary accuracy (similar comments apply in the classical case). Hence the case we are studying is more than just an example. It demonstrates (with some appropriate qualifications) that Quantum Theory in general can be formulated in this framework.

Assume we have a large number of qubits moving to the right labelled (from left to right) $u = 1, 2, 3, \ldots$ and a large number of moving to the left labelled (from right to left) $v = 1, 2, 3, \ldots$. If this is plotted against time then we will have a diamond shaped lattice with each vertex corresponding to the interaction of a right moving qubit with a left moving qubit. We can label these vertices with $x \equiv uv$ (the cartesian product of $u$ and $v$ is denoted by $uv$). They correspond to our elementary regions $R_{uv}$.

We imagine that, at each vertex, the two qubits pass through a box which implements a general measurement. The box has a knob which is used to set $F_{uv}$ and a display panel recording the outcome, $Y_{uv}$. As before, we can label all such pairs with $\alpha_{uv}$. For a general quantum measurement, an (outcome, choice) pair is associated with a superoperator $\$ (superoperators are trace non-increasing maps on density operators that take allowed states to allowed states). In this case, we have a superoperator $\$_{\alpha_{uv}}$ associated with $\alpha_{uv}$. In quantum theory we can write a general superoperator on two qubits such as this as a sum of the tensor product of a fiducial set of superoperators acting on each qubit separately:

$$\$_{\alpha_{uv}} = \sum_{k_u, k_v \in \Omega^2} A_{\alpha_{uv}}^{k_u k_v} \$_{k_u} \otimes \$_{k_v} \quad (20)$$

Remarkably, we can only do this if we have complex (rather than real or quaternionic) Hilbert spaces supporting the superoperators. The set $\$_{k_u}$ for $k_u \in \Omega^2$ is a fiducial spanning set of superoperators for the qubit (the superscript denotes
that this is a qubit having Hilbert space dimension 2). We have $|\Omega^N| = N^4$ so for a qubit we have $|\Omega|^2 = 2^4$. This is the number of linearly independent superoperators needed to span the space of superoperators for a qubit. For reasons that will be clear later, we choose $\mathbb{1} = I$ where 1 is the first element of $\Omega^2$ and $I$ is the identity superoperator. We can solve equation (20) to find the $\Lambda$ matrix for each elementary region $R_{uv}$.

Now consider a single right moving qubit as it goes from vertex $(u, v)$ to the next vertex $(u, v+1)$. Assume that it is subject to $l_u \otimes l_v$ at the first vertex and $l'_u \otimes l'_{v+1}$ at the next with $l_u, l'_u, l_v, l'_{v+1} \in \Omega^2$. So far as the right moving qubit, $u$, is concerned, we can ignore the left moving qubits with which it interacts (since the two superoperators just given factorize). The effective superoperator acting on qubit $u$ is $l'_u \circ l_u$. But this is a superoperator belonging to the space of superoperators acting on a single qubit and can, hence, be expanded in terms of the linearly independent fiducial set

$$ l'_u \circ l_u = \sum_{k'_u k_u \in \{1\} \times \Omega^2} \Lambda^{k'_u k_u}_{l'_u l_u} l'_u \circ l_u $$

since we have selected $\mathbb{1}$ to be the identity. We can solve this equation for the matrix $\Lambda^{k'_u k_u}_{l'_u l_u}$.

This generalizes to more than two sequential vertices in the obvious way. For three sequential vertices we have

$$ l'_u \circ l'_v \circ l_u = \sum_{k'_u k'_v k_u \in \{1\} \times \{1\} \times \Omega^2} \Lambda^{k'_u k'_v k_u}_{l'_u l'_v l_u} l'_u \circ l'_v \circ l_u $$

and so on. It can be shown that, for three sequential vertices,

$$ \Lambda^{k'_u k'_v k_u}_{l'_u l'_v l_u} = \sum_{n' \in \Omega^2} \Lambda^{k'_u n'_u}_{l'_u l_u} \Lambda^{k'_v n'_v}_{l'_v l_u} \Lambda^{k_u n'_u}_{l_u l_u} $$

For four sequential vertices,

$$ \Lambda^{k''_u k'_u k'_v k_v}_{l''_u l'_u l'_v l_v} = \sum_{n''' \in \Omega^2} \sum_{n'' \in \Omega^2} \Lambda^{k''_u n'''_u}_{l''_u l'u} \Lambda^{k'_u n''_u}_{l'_u l_u} \Lambda^{k'_v n''_v}_{l'_v l_v} \Lambda^{k_v n''''_v}_{l''''_v l_v} $$

and so on. The derivation of these equations relies only on the combinatorics of how the labels combine rather than on any particular details of quantum theory. The same equations are found in the treatment of interacting classical bits.

We may have need of the matrix $\Lambda^{k_u}_{l_u}$ (where $l_u, k_u \in \Omega^2$) for a single vertex for a right moving qubit. Since

$$ l_u = \sum_{k_u \in \Omega^2} \Lambda^{k_u}_{l_u} l_u $$

we have

$$ \Lambda^{k_u}_{l_u} = \delta^{k_u}_{l_u} $$
For left moving qubits we have equations \((21-26)\) but with \(v\) replacing \(u\).

As will be described, the composite region \(\Lambda\) matrices for all situations that do not involve sequential vertices on the same qubit are given by multiplying the \(\Lambda\) matrix components for different clumps of vertices. This means that the causaloid is given by

\[
\Lambda = (\Lambda_{\alpha u v}, \Lambda_{t_{u'} t_{v'}}^{k_{u} k_{v}} \forall \text{RSV}, \Lambda_{t_{l'} t_{v}}^{k_{l} k_{v}} \forall \text{LSV}; \text{clumping method}) \quad (27)
\]

where \((\text{L})\text{RSV}\) stands for pairs of sequential vertices on (left) right moving qubits. The clumping method allows us to calculate the \(\Lambda\) matrix for an arbitrary region \(R_1\) with \(x(=uv) \in O_1\) from this causaloid as follows

1. For each qubit (both left and right moving) circle all complete groups of sequential vertices (call these clumps) in \(O_1\). There must be a gap of at least one vertex between each clump for any given qubit.

2. Calculate the \(\Lambda\) matrix components for each circled clump for each qubit using \((23)\) or one of its generalizations (for a clump of one vertex use \((26)\) and for a clump of two vertices take \(\Lambda_{t_{u'} t_{v}}^{k_{u} k_{v}}\) directly from the specification of the causaloid \((27)\)).

3. Multiply together all \(\Lambda\) matrix components for all circled clumps (note we are multiplying components rather than performing matrix multiplication). This gives the components of the \(\Lambda\) matrix for \(R_1\).

We note that the clumping method respects F-locality since, in calculating the \(\Lambda\) matrix for \(R_1\), we only use \(\Lambda\) matrices pertaining to regions \(\tilde{R}_1 \subseteq R_1\).

It is worth examining the causaloid given in \((27)\) a little more. First we note that we are only required to specify a tiny subset of the exponential number of possible \(\Lambda\) matrices - there is a tremendous amount of third level compression. Second, we note the symmetry property that, according to \((20)\) and \((21)\), the \(\Lambda\) matrices of each type are the same. Hence, we can actually specify the causaloid by

\[
\Lambda = (\Lambda_{\alpha_{11}}, \Lambda_{t_{1} t_{1}}^{k_{1} k_{1}}; \text{symmetry, clumping method}) \quad (28)
\]

where \(\text{symmetry}\) denotes the property just noted and \(\Lambda_{t_{1} t_{1}}^{k_{1} k_{1}}\) is one instance of the \(\Lambda\) matrix for a pair of right (or left) sequential vertices.

Given this causaloid we can employ the standard techniques of the causaloid framework (the causaloid product and the two step approach) to calculate whether an arbitrary probability is well defined and, if so, what it is equal to. In this sense we can say that this causaloid fully specifies the quantum theory of interacting qubits. In particular, note that we separate out the specification of the theory (the causaloid will be different for different physical theories) from the way the causaloid is used to make predictions (it is used in the same way for any physical theory).

The causaloid formulation of Quantum Theory treats arbitrary regions on an equal footing. In this it is similar to the time-symmetric approach of Aharanov
and co-workers (particularly the latest version due to Aharanov, Popescu, Tollaksen, and Vaidman in [9] which allows states and measurements to be defined for arbitrary regions), and the general boundary formulation of quantum theory due to Oeckl [10]. Of related interest is the quantum causal histories approach of Markopoulou [11] and the quantum causal networks of Leifer [12].

6 The road to Quantum Gravity

In order to formulate a theory of Quantum Gravity (QG) we need to have a framework that is hospitable to such a theory in the first place. We expect that QG will be a probabilistic theory with indefinite causal structure. The causaloid framework admits such theories (this does not imply that QG certainly fits in the framework - but at least it is not ruled out from the outset). The most satisfying way to obtain QG in this (or any) framework would be to derive it from a set of well motivated principles (such as a suitably generalized equivalence principle). It may be that appropriate principles will carry over from Quantum Theory and General Relativity (indeed the equivalence principle is true in Newtonian Gravity). Therefore, a careful study of Quantum Theory and General Relativity (GR) may be the best way of coming up with such principles.

One particular route that may be taken to finding QG is illustrated in the following diagram

\[
\begin{array}{c}
\text{QT} \\
\uparrow \\
\text{CProbT} \\
\uparrow \\
\text{ProbGR}
\end{array} \\
\longrightarrow \\
\uparrow \\
\text{QG}
\]

CProbT is classical probability theory (of interacting classical bits for example). QT is Quantum Theory (of interacting qubits for example). ProbGR is an appropriately formulated version of General Relativity in the case where we have arbitrary probabilistic ignorance of the values of certain measurable quantities. We will elaborate on this below. The vertical arrows represent a kind of quantization. The horizontal arrows represent what we might call GR-ization. In quantizing from CProbT to QT we need only alter the structure of the $\Lambda_{\alpha \nu}^{k_1 k_2}$ matrices for the elementary regions (this is what might be regarded as the local structure) by replacing that structure that corresponds to a classical probability simplex with a structure that corresponds to the Bloch sphere of the qubit. The structure above this, for composite regions, is basically constructed in the same way in CProbT and QT. ProbGR has not yet been satisfactorily formulated. However, we can expect that in the GR-ization process from CProbT to ProbGR, the local structure associated with the classical probability simplices will survive for the elementary regions but that the structure associated with composite regions will be different (since the causal structure is not fixed). This suggests that we may be able to get a theory of QG by applying quantization essentially at the local level of the elementary regions and GR-ization at the level of the composite regions. If quantization and GR-ization do not interfere with each other too much then the diagram above may not be too misleading.
This approach depends on having a suitable formulation of ProbGR. An obvious way to give a probabilistic formulation of GR is to have some probabilistic distribution over the 3-metric specified on some initial space-like hypersurface and then evolve the distribution employing a canonical formulation of GR. This is unsatisfactory since (a) it is not F-local, (b) we cannot deal with arbitrary probabilistic ignorance about measurable quantities, and (c) the time label for the space-like hypersurface is not an observable and so it is not clear that the numbers we are calling “probabilities” represent ignorance about something measurable. Another approach is to consider a probabilistic distribution over solutions for the metric for all of space-time. This is problematic since (a) it is manifestly not F-local and (b) it is not clear how the differently weighted solutions match up from the internal point of view of somebody who may be making measurements and so, once again, it is unclear whether the “probabilities” represent ignorance about something measurable. Rather than taking either of these approaches, it seems that we need to build up ProbGR from scratch using F-locality and, maybe, the causaloid formalism as guidance. Such a theory contains no new empirical content over standard GR. However, it is possible that the natural mathematical formulation of ProbGR will look very different from standard GR.

7 Conclusions

Many standard notions in Quantum Theory require reference to some definite causal structure. For example the notion of entanglement requires two space-like separated systems, and the notion of information flow requires a sequence of immediately sequential time-like regions. When we embed QT into the causaloid framework these notions become special cases of a much richer structure. Entanglement is supported by the tensor product of QT, but in the causaloid framework, we have the causaloid product which allows us to talk about joint properties of any two regions regardless of their causal relationship. Information flow is supported by the standard product $\hat{A}\hat{B}$ between sequential time-like separated regions. In the causaloid framework we have, again, the causaloid product. In quantum circuit diagrams we draw wires between boxes denoting the path of the qubit. A pair of boxes either do, or do not, have a wire between them. In the causaloid framework we have a $\Lambda$ matrix (by which the causaloid product is defined). Every pair of boxes (or elementary regions) has a $\Lambda$ matrix between it. This richer structure is likely to help in developing a theory of Quantum Gravity since it provides a way round requiring that the causal structure be definite.

The principle that it should be possible to give an F-local formulation may prove to be powerful in theory construction. It is encouraging that Quantum Theory can be formulated in an F-local fashion. Not only does this add to the list of different ways in which QT can be formulated but also it provides encouragement that a theory of QG may share structural similarities with QT.

The next step on the road to QG, if this approach is pursued, is to construct
ProbGR. In tackling the problem of constructing ProbGR we are likely to encounter many of the same difficulties encountered in constructing a fully fledged theory of QG. However, we know that ProbGR is empirically equivalent to GR (just with arbitrary probabilistic ignorance added) and so we fully expect that this theory exists.

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