UNIVERSALLY SYMMETRICALLY NORMING OPERATORS ARE COMPACT

SATISH K. PANDEY

Abstract. We introduce and study the concepts of “universally symmetrically norming operators” and “universally absolutely symmetrically norming operators” on a separable Hilbert space. These refer to the operators that are, respectively, norming and absolutely norming, with respect to every symmetric norm on $B(H)$. We establish a characterization theorem for such operators and prove that these classes are identical and that they coincide with the class of compact operators. In particular, we provide an alternative characterization of compact operators on a separable Hilbert space.

1. Introduction

A bounded linear operator $T : \mathcal{H} \to \mathcal{K}$ between two Hilbert spaces is said to be norming or norm attaining if there is an element $x \in \mathcal{H}$ with $\|x\| = 1$ such that $\|T\| = \|Tx\|$, where $\|T\| = \sup\{\|Tx\|_K : x \in \mathcal{H}, \|x\|_H \leq 1\}$. We say that $T \in B(\mathcal{H}, \mathcal{K})$ is absolutely norming if for every nontrivial closed subspace $M$ of $\mathcal{H}$, $T|_M$ is norming. We let $\mathcal{N}(\mathcal{H}, \mathcal{K})$ (or $\mathcal{N}$) and $\mathcal{AN}(\mathcal{H}, \mathcal{K})$ (or $\mathcal{AN}$) respectively denote the set of norming and absolutely norming operators in $B(\mathcal{H}, \mathcal{K})$.

The class of norming operators on complex Hilbert spaces have been extensively studied and there is a plethora of information on these operators; see, for instance, [4, 3, 2, 1, 5, 10, 11, 12, 13] and references therein. The class of “absolutely norming” operators, however was introduced recently in [6] and a spectral characterization theorem for these operators was established in [9], see [9, Theorem 5.1]. Henceforth, $\mathcal{H}$ will denote a separable Hilbert space and we write $B(\mathcal{H})$ for the set of all bounded linear operators on $\mathcal{H}$.

In [8], we used the theory of symmetrically normed ideals to extend the concept of “norming” and “absolutely norming” from the operator norm to arbitrary symmetric norms that are equivalent to the operator norm, and established a few spectral characterization theorems for operators in $B(\mathcal{H})$ that are absolutely norming with respect to various symmetric norms. It was also shown that for a large family of symmetric norms the absolutely norming operators have the same spectral characterization as proven earlier.

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for the class of operators that are absolutely norming with respect to the usual operator norm.

The subsequent discussion in [8] involves positive operators of the form of a nonnegative scalar multiple of identity plus a positive compact plus a self-adjoint finite-rank. It is not clear, \textit{a priori}, if the operators of this form are absolutely norming with respect to every symmetric norm on $\mathcal{B}(\mathcal{H})$. To our surprise, it turned out that there exists a symmetric norm on $\mathcal{B}(\ell^2(\mathbb{N}))$ such that the identity operator $I$ does not attain its norm; see Theorem 3.4.

In the present paper, we introduce and study the concept of “universally symmetrically norming operators” (see Definition 4.1) and “universally absolutely symmetrically norming operators” (see Definition 4.2) on $\mathcal{B}(\mathcal{H})$. These refer to the operators that are, respectively, norming and absolutely norming, with respect to every symmetric norm. The goal of this paper is to characterize such operators.

Our main result is Theorem 4.6 which states that an operator in $\mathcal{B}(\mathcal{H})$ is universally symmetrically norming if and only if it is universally absolutely symmetrically norming, which holds if and only if it is compact. We hence establish a characterization theorem for such operators on $\mathcal{B}(\mathcal{H})$. This result provides an alternative characterization theorem for compact operators on a separable Hilbert space and may prove useful in future endeavours.

This paper is organized as follows. The section that follows is dedicated to reviewing the background required for this paper. In Section 3 we introduce a certain family of symmetric norms on $\mathcal{B}(\mathcal{H})$ and establish a characterization theorem for operators in $\mathcal{B}(\mathcal{H})$ that are symmetrically norming with respect to every symmetric norm in this family. The operators in $\mathcal{B}(\mathcal{H})$ that are absolutely symmetrically norming with respect to every symmetric norm in this family are also studied in this section, and, in particular, it is shown that an operator is symmetrically norming with respect to every symmetric norm in the family if and only if it is absolutely symmetrically norming with respect to every symmetric norm in the family. The notions of “universally symmetrically norming operators” and “universally absolutely symmetrically norming operators” on a separable Hilbert space are introduced in the final section of this paper and a characterization theorem — our main result — for such operators is presented.

2. Preliminaries

In this section we recall some notions and results concerning the ideal structure of the algebra of all bounded linear operators acting on a separable Hilbert space.

\textbf{Notation 2.1.} Consider the algebra $\mathcal{B}(\mathcal{H})$ of operators on a separable Hilbert space $\mathcal{H}$. We let $\mathcal{B}_{00}(\mathcal{H})$, respectively, $\mathcal{B}_0(\mathcal{H})$ denote the set of all finite-rank operators on $\mathcal{H}$, respectively, the set of compacts. We use $\mathcal{B}_1(\mathcal{H})$ to denote the trace class operators, with the trace norm $\| \cdot \|_1$. By $c_0$ we denote the space of all convergent sequences of real numbers with limit 0 and
we let $c_0 \subseteq c$ be the linear subspace of $c_0$ consisting of all sequences with a finite number of nonzero terms. The positive cone of $c_0$ is denoted by $c_0^+$ and we use $c_0^0 \subseteq c_0^+$ to denote the cone of all nonincreasing nonnegative sequences from $c_0^0$.

We now define the notion of a symmetric norm on a two-sided ideal of $B(H)$. An ideal of $B(H)$ always means a two-sided ideal.

**Definition 2.2 (Symmetric Norm).** Let $\mathcal{I}$ be an ideal of the algebra $B(H)$ of operators on a complex Hilbert space. A symmetric norm on $\mathcal{I}$ is a function $\| \cdot \|_s : \mathcal{I} \rightarrow [0, \infty)$ which satisfies the following conditions:

1. $\| \cdot \|_s$ is a norm.
2. $\|AXB\|_s \leq \|A\|\|X\|\|B\|$ for every $A, B \in B(H)$ and $X \in \mathcal{I}$.
3. $\|X\|_s = \|X\| = s_1(X)$ for every rank-one operator $X \in \mathcal{I}$.

**Remark 2.3.** In the above definition, if we consider the ideal $\mathcal{I}$ to be $B(H)$, then it is said to be a symmetric norm on $B(H)$. Moreover, the following observations are obvious:

1. the usual operator norm on any ideal $\mathcal{I}$ of $B(H)$ is a symmetric norm; and
2. every symmetric norm on $B(H)$ is topologically equivalent to the ordinary operator norm.

We next use the concept of symmetric norm to introduce the notion of a symmetrically normed ideal of $B(H)$.

**Definition 2.4 (Symmetrically-Normed Ideals).** A symmetrically-normed ideal (or an s.n. ideal) is an ideal $\mathcal{S}$ of the algebra $B(H)$ with a symmetric norm $\| \cdot \|_S$ such that $\mathcal{S}$ is complete in the metric given by this norm.

**Definition 2.5.** We say that two ideals $\mathcal{S}_I$ and $\mathcal{S}_{II}$ coincide elementwise if $\mathcal{S}_I$ and $\mathcal{S}_{II}$ consist of the same elements.

**Definition 2.6 (Symmetric norming function).** [7, Chapter 3, Page 71] A function $\Phi : c_0^0 \rightarrow [0, \infty)$ is said to be a symmetric norming function (or, s.n. function) if it satisfies the following properties.

1. $\Phi$ is a real norm.
2. $\Phi(1,0,0,...) = 1$.
3. $\Phi(\xi_1,\xi_2,...,\xi_n,0,0,...) = \Phi(|\xi_{j_1}|,|\xi_{j_2}|,...,|\xi_{j_n}|,0,0,...)$ for every $\xi \in c_0^0$ and $n \in \mathbb{N}$, where $j_1,j_2,...,j_n$ is any permutation of the integers $1,2,...,n$.

**Remark 2.7.** A moment’s thought will convince the readers that an s.n. function can be uniquely defined by its values on the cone $c_0^0$. Here the minimal s.n. function $\Phi_\infty : c_0^+ \rightarrow [0, \infty)$ is defined by $\Phi_\infty(\xi) = \xi_1$ for every $\xi = (\xi_j)_j \in c_0^0$ and the maximal s.n. function $\Phi_1 : c_0^0 \rightarrow [0, \infty)$ is defined by $\Phi_1(\xi) = \sum_j \xi_j$ for every $\xi = (\xi_j)_j \in c_0^0$. For any s.n. function $\Phi$, we have $\Phi_\infty \leq \Phi \leq \Phi_1$ (see [7, Chapter 3, Section 3, Relation 3.12, Page 76]).
**Definition 2.8** (Equivalence of s.n. functions). [7, Chapter 3, Page 76] Two s.n. functions $\Phi$ and $\Psi$ are said to be **equivalent** if
\[
\sup_{\xi \in c_{00}} \Phi(\xi) < \infty \quad \text{and} \quad \sup_{\xi \in c_{00}} \Psi(\xi) < \infty.
\]

**Definition 2.9.** Let $\Phi$ and $\Psi$ be two s.n. functions. We say that $\Phi \leq \Psi$ if for every $\xi \in c_{00}$, we have $\Phi(\xi) \leq \Psi(\xi)$.

For a given s.n. function, we next recall the notion of its adjoint.

**Definition 2.10.** The **adjoint** $\Phi^*$ of the s.n. function $\Phi$ is given by
\[
\Phi^*(\eta) = \max \left\{ \sum_j \eta_j \xi_j : \xi \in c_{00}^*, \Phi(\xi) = 1 \right\}, \quad \text{for every } \eta \in c_{00}^*.
\]

**Remark 2.11.** That $\Phi^*$ is an s.n. function is a trivial observation. Also, the adjoint of $\Phi^*$ is $\Phi$. In particular, the minimal and maximal s.n. functions are the adjoint of each other, that is, $\Phi^*_1 = \Phi_{\infty}$ and $\Phi_{\infty}^* = \Phi_1$. Therefore, when an s.n. function is equivalent to the maximal(minimal) one, its adjoint is equivalent to the minimal(maximal) one.

**Remark 2.12.** It is evident that every s.n. ideal gives rise to an s.n. function. Conversely, to every s.n. function $\Phi$ we associate an s.n. ideal $\mathcal{S}_\Phi$, which is referred to as the s.n. ideal generated by the s.n. function $\Phi$. For a detailed exposition of the construction of the s.n. ideal from an s.n. function we refer the reader to Gohberg and Krein’s text [7, Chapter 3] which elaborately discusses the theory of s.n. ideals. An abridged outline of this construction has also been discussed in [8, Section 6, Notation 6.4]. Since we will be dealing with this theory extensively, we have attempted to duplicate their notation wherever possible.

If $\Phi, \Psi$ are s.n. functions and $\mathcal{S}_\Phi, \mathcal{S}_\Psi$ are the s.n. ideals generated by these s.n. functions respectively, then $\mathcal{S}_\Phi$ and $\mathcal{S}_\Psi$ coincide elementwise if and only if $\Phi$ and $\Psi$ are equivalent. In particular, if $\Phi$ is an s.n. function equivalent to $\Phi_1$, then $\mathcal{S}_\Phi$ and $\mathcal{B}_1(\mathcal{H})$ coincide elementwise and when $\Phi$ is equivalent to $\Phi_{\infty}$, $\mathcal{S}_\Phi$ and $\mathcal{B}_0(\mathcal{H})$ coincide elementwise.

Here is an often needed elementary piece of folklore from [7].

**Proposition 2.13.** [7, Chapter 3, Theorems 12.2 and 12.4] Let $\Phi$ be an arbitrary s.n. function.

(1) If $\Phi$ is not equivalent to the maximal s.n. function, then the general form of a continuous linear functional $f$ on the separable space $\mathcal{S}^{(0)}_{\Phi}$ is given by
\[
f(X) = \text{Tr}(AX), \quad \text{for some } A \in \mathcal{S}_{\Phi^*}.
\]

Thus, the space adjoint to the space $\mathcal{S}^{(0)}_{\Phi}$ is isometrically isomorphic to $\mathcal{S}_{\Phi^*}$, that is, $\mathcal{S}^{(0)*}_{\Phi} \cong \mathcal{S}_{\Phi^*}$. In particular, if both functions $\Phi$ and
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\(\Phi^*\) are mononormalizing, the space \(S_\Phi\) is reflexive.

(2) If \(\Phi\) is equivalent to the maximal s.n. function, then the general form of a continuous linear functional \(f\) on the separable space \(S_\Phi\) is given by

\[
f(X) = \text{Tr}(AX)\]

for some \(A \in \mathcal{B}(H)\) and

\[
\|f\| := \sup \{|\text{Tr}(AX)| : X \in S_\Phi, \|X\|_\Phi \leq 1 \} = \|A\|_{\|\cdot\|_{\Phi^*}}.
\]

Thus, the dual space \(S_\Phi^*\) is isometrically isomorphic to \((\mathcal{B}(H), \|\cdot\|_{\Phi^*})\), that is, \(S_\Phi^* \cong (\mathcal{B}(H), \|\cdot\|_{\Phi^*})\).

Given an arbitrary s.n. function \(\Phi\) that is equivalent to the maximal s.n. function, we now recall the definition of operators in \(\mathcal{B}(H)\) that attain their \(\Phi^*\)-norm.

**Definition 2.14.** [8, Definitions 6.11, 6.14] Let \(\Phi\) be an s.n. function equivalent to the maximal s.n. function. An operator \(T \in (\mathcal{B}(H), \|\cdot\|_{\Phi^*})\) is said to be \(\Phi^*\)-norming or symmetrically norming with respect to the symmetric norm \(\|\cdot\|_\Phi\) if there exists an operator \(K \in S_\Phi = \mathcal{B}_1(H)\) with \(\|K\|_\Phi = 1\) such that

\[
|\text{Tr}(TK)| = \|T\|_{\Phi^*}.
\]

We say that \(T \in (\mathcal{B}(H), \|\cdot\|_{\Phi^*})\) is absolutely \(\Phi^*\)-norming or absolutely symmetrically norming with respect to the symmetric norm \(\|\cdot\|_\Phi\) if for every nontrivial closed subspace \(M\) of \(H\), \(TP_M \in \mathcal{B}(H)\) is \(\Phi^*\)-norming (here \(P_M\) is the orthogonal projection onto \(M\)).

We let \(\mathcal{N}_{\Phi^*}(H)\) and \(\mathcal{AN}_{\Phi^*}(H)\) respectively denote the set of \(\Phi^*\)-norming and absolutely \(\Phi^*\)-norming operators in \(\mathcal{B}(H)\). Needless to mention, every absolutely \(\Phi^*\)-norming operator is \(\Phi^*\)-norming, that is, \(\mathcal{AN}_{\Phi^*}(H) \subseteq \mathcal{N}_{\Phi^*}(H)\).

3. Characterization of Symmetrically Norming Operators Affiliated to Strictly Decreasing Weights

In this section we introduce a certain family of symmetric norms on \(\mathcal{B}(H)\) and establish a characterization theorem for operators in \(\mathcal{B}(H)\) that are symmetrically norming with respect to every symmetric norm in this family. This section also studies the operators in \(\mathcal{B}(H)\) that are absolutely symmetrically norming with respect to every symmetric norm in the family and presents a characterization theorem for those as well. It turns out that an operator is symmetrically norming with respect to every symmetric norm in the family if and only if it is absolutely symmetrically norming with respect to every symmetric norm in the family. This “characterization theorem” is the main theorem of this section.

We begin by recalling the following result.

**Theorem 3.1.** [8, Theorem 6.17] Let \(\Phi\) be an arbitrary s.n. function equivalent to the maximal s.n. function. If \(T \in (\mathcal{B}(H), \|\cdot\|_{\Phi^*})\) is a compact operator, then \(T \in \mathcal{AN}_{\Phi^*}\), that is, \(\mathcal{B}_0(H) \subseteq \mathcal{AN}_{\Phi^*}(H)\).
Following well established precedent, we use \( s_j(T) \) to denote the \( j \)th singular value of \( T \in \mathcal{B}(\mathcal{H}) \). The following proposition allows us to concentrate on the positive operators that are symmetrically norming.

**Proposition 3.2.** Let \( \Phi \) be an s.n. function equivalent to the maximal s.n. function. Then \( T \in \mathcal{N}_{\Phi^*}(\mathcal{H}) \) if and only if \(|T| \in \mathcal{N}_{\Phi^*}(\mathcal{H})\).

Proof. We first assume that \( T \in \mathcal{N}_{\Phi^*}(\mathcal{H}) \) and observe that \(|T|\Phi^* = \|T\|\Phi^* \) since for each \( j \), \( s_j(T) = s_j(|T|) \). Then there exists \( K \in \mathcal{B}_1(\mathcal{H}) \) with \( \|K\|\Phi = 1 \) such that \(|T|\Phi^* = |\text{Tr}(TK)|\). If \( T = U|T| \) is the polar decomposition of \( T \), then \(|\text{Tr}(T)\Phi^* = |\text{Tr}(TK)| = |\text{Tr}(U|T|K)| = |\text{Tr}(|T|KU)|\) where \( KU \in \mathcal{B}_1(\mathcal{H}) \) with \( \|KU\|\Phi = \|IKU\|\Phi \leq \|T\|\|K\|\Phi \|U\| = \|K\|\Phi = 1 \). In fact, \( \|KU\|\Phi = 1 \); for if not, then the operator \( S := KU/\|KU\|\Phi \in \mathcal{B}_1(\mathcal{H}) \) satisfies \( \|S\|\Phi = 1 \) and yields

\[
|\text{Tr}(|T|S)| = \left|\text{Tr} \left( \frac{|T|KU}{\|KU\|\Phi} \right) \right|
\]

\[
= \frac{1}{\|KU\|\Phi} |\text{Tr}(|T|KU)|
\]

\[
> |\text{Tr}(|T|KU)| = \|T\|\Phi^*,
\]

which contradicts the fact that the supremum of the set \( \{|\text{Tr}(|T|X)| : X \in \mathcal{B}_1(\mathcal{H}), \|X\|\Phi \leq 1\} \) was attained at \( KU \). This shows that \(|T| \in \mathcal{N}_{\Phi^*}(\mathcal{H})\).

Conversely, if \(|T| \in \mathcal{N}_{\Phi^*}(\mathcal{H})\), then by replacing \( T \) by \(|T| \) in the above argument using \(|T| = U^*T\), we can prove the existence of \( \tilde{K} \in \mathcal{B}_1(\mathcal{H}) \) with \( \|\tilde{K}\|\Phi = 1 \) such that \(|T|\Phi^* = |\text{Tr}(TKU^*)|\) where \( KU^* \in \mathcal{B}_1(\mathcal{H}) \) with \( \|KU^*\|\Phi \leq 1 \). It can then be shown that \( \|\tilde{K}U^*\|\Phi = 1 \) and the result follows. \( \Box \)

In the remainder of this section we want to introduce a certain family of symmetric norms on \( \mathcal{B}(\mathcal{H}) \) and study the operators that are symmetrically norming with respect to each of these norms. Before we can proceed we need one more result concerning the computation of the symmetric norm of an operator.

**Proposition 3.3.** Let \( \Phi \) be an s.n. function equivalent to the maximal s.n. function and let \( T \in \mathcal{B}(\mathcal{H}) \). Then

\[
\|T\|\Phi^* = \sup \left\{ \sum_j s_j(T)s_j(K) : K \in \mathcal{B}_1(\mathcal{H}), K = \text{diag}\{s_j(K)\}_j, \|K\|\Phi = 1 \right\}
\]

Proof. Since \( \Phi \) is equivalent to the maximal s.n. function, we know that \( \mathcal{S}_\Phi^* \cong (\mathcal{B}(\mathcal{H}), \|\cdot\|\Phi^*) \), and by Definition 2.13 the \( \|\cdot\|\Phi^* \) norm for any operator \( T \in \mathcal{B}(\mathcal{H}) \) is given by \( \|T\|\Phi^* = \sup\{|\text{Tr}(TK)| : K \in \mathcal{S}_\Phi, \|K\|\Phi = 1\} \). But the ideal \( \mathcal{B}_1(\mathcal{H}) \) and \( \mathcal{S}_\Phi \) coincide elementwise and hence \( \|T\|\Phi^* = \sup\{|\text{Tr}(TK)| : K \in \mathcal{B}_1(\mathcal{H}), \|K\|\Phi = 1\} \).

First we claim that \( \alpha := \sup\{|\text{Tr}(TK)| : K \in \mathcal{B}_1(\mathcal{H}), \|K\|\Phi = 1\} = \sum_j s_j(T)s_j(K) : K \in \mathcal{B}_1(\mathcal{H}), \|K\|\Phi = 1 \} =: \beta \). That \( \alpha \leq \beta \) is a trivial
It follows then that $\beta \leq \alpha$, let us choose an operator $K \in B_1(\mathcal{H})$ with $\|K\|_\Phi = 1$. An easy computation yields

$$
\sum_j s_j(T)s_j(K) = \left\langle \begin{bmatrix} s_1(T) \\ \vdots \\ s_j(T) \end{bmatrix}, \begin{bmatrix} s_1(K) \\ \vdots \\ s_j(K) \end{bmatrix} \right\rangle \\
\leq \Phi^* \begin{bmatrix} s_1(T) \\ \vdots \\ s_j(T) \end{bmatrix} \Phi \begin{bmatrix} s_1(K) \\ \vdots \\ s_j(K) \end{bmatrix} \\
= \Phi^* \begin{bmatrix} s_1(T) \\ \vdots \\ s_j(T) \end{bmatrix} = \|T\|_{\Phi^*} \\
= \sup\{|\text{Tr}(TK)| : K \in B_1(\mathcal{H}), \|K\|_\Phi = 1\} = \alpha.
$$

It follows then that $\beta \leq \alpha$ and this proves our first claim.

We next let $\gamma := \sup\left\{\sum_j s_j(T)s_j(K) : K \in B_1(\mathcal{H}), K = \text{diag}\{s_j(K)\}, \|K\|_\Phi = 1\right\}$ and prove that $\gamma = \beta$. That $\gamma \leq \beta$ is obvious. To prove $\beta \leq \gamma$, we choose an operator $K \in B_1(\mathcal{H})$ with $\|K\|_\Phi = 1$ and define

$$
\tilde{K} := \begin{pmatrix} s_1(K) & 0 \\ s_2(K) & \ddots \\ 0 & \ddots & s_j(K) \end{pmatrix}.
$$

Notice that for every $j$, we have $s_j(\tilde{K}) = s_j(K)$ which implies that $\|\tilde{K}\|_\Phi = \|K\|_\Phi = 1$. Even more, $\tilde{K} \in B_1(\mathcal{H})$ and hence $\sum_j s_j(T)s_j(K) = \sum_j s_j(T)s_j(\tilde{K}) \leq \sup\left\{\sum_j s_j(T)s_j(K) : K \in B_1(\mathcal{H}), K = \text{diag}\{s_j(K)\}, \|K\|_\Phi = 1\right\} = \gamma$. It then immediately follows that $\beta \leq \gamma$ and consequently establishes our second claim.

From the above two observations we conclude that $\alpha = \gamma$ which proves the assertion.

As we mentioned in the introduction to this paper, one of the most non-intuitive and important results that motivated this work is that there exists a symmetric norm on $B(\mathcal{H})$ with respect to which even the identity operator $I$ does not attain its norm. We recall the result that illustrates this fact.
Theorem 3.4. [8] Proposition 1.3] There exists a symmetric norm $\| \cdot \|_{\Phi_\pi}$ on $B(\ell^2(\mathbb{N}))$ such that $I \notin \mathcal{N}_{\Phi_\pi}(\ell^2(\mathbb{N}))$.

Remark 3.5. The proof of this theorem is constructive and illustrates an elegant technique of producing symmetric norms on $B(\mathcal{H})$ with respect to which the identity operator $I$ does not attain its norm. In particular, the proof demonstrates a family of s.n. functions — s.n. functions affiliated to strictly decreasing weights — which naturally generate such symmetric norms. This family of s.n. functions lies at the very foundation of the results we prove in the remainder of this section.

Let $\hat{\Pi}$ denote the set of all strictly decreasing convergent sequence of positive numbers with their first term equal to 1 and positive limit, that is, $\hat{\Pi} = \{ \pi := (\pi_n)_{n \in \mathbb{N}} : \pi_1 = 1, \lim_{n \to \infty} \pi_n > 0, \text{ and } \pi_k > \pi_{k+1} \text{ for each } k \in \mathbb{N} \}$. (We have used $\Pi$, in [8], to denote the set of all nonincreasing sequences of positive numbers with their first term equal to 1, and hence, in accordance with that notation, we have $\hat{\Pi} \subseteq \Pi$.) For each $\pi \in \hat{\Pi}$, let $\Phi_\pi$ denote the symmetrically norming function defined by $\Phi_\pi(\xi_1, \xi_2, \ldots) = \sum_{j=1}^{\infty} \pi_j \xi_j$ and observe that $\Phi_\pi$ is equivalent to the maximal s.n. function $\Phi_1$.

Theorem 3.4 essentially proves that $I \notin \mathcal{N}_{\Phi_\pi}(\mathcal{H})$ for every $\Phi_\pi$ from the family $\{ \Phi_\pi : \pi \in \hat{\Pi} \}$ of s.n. functions. We know that, in general, $\mathcal{N}_{\Phi_\pi}(\mathcal{H}) \subseteq B_0(\mathcal{H})$ for an arbitrary s.n. function $\Phi$ equivalent to the maximal s.n. function $\Phi_1$. However, it is of interest to know whether $\mathcal{N}_{\Phi_\pi}(\mathcal{H}) \subseteq B_0(\mathcal{H})$ if $\Phi_\pi$ belongs to the family $\{ \Phi_\pi : \pi \in \hat{\Pi} \}$ of s.n. functions; for if the answer to this question is affirmative, then Theorem 3.4 would yield $\mathcal{N}_{\Phi_\pi}(\mathcal{H}) = B_0(\mathcal{H})$ and would thus characterize the $\Phi_\pi^*$-norming operators in $B(\mathcal{H})$. By Proposition 3.2 it suffices to know whether $\mathcal{N}_{\Phi_\pi}(\mathcal{H}) \cap B(\mathcal{H})_+ \subseteq B_0(\mathcal{H})$ where $B(\mathcal{H})_+ = \{ T \in B(\mathcal{H}) : T \geq 0 \}$. The following lemma and example prove the existence of $\pi \in \hat{\Pi}$ such that $\mathcal{N}_{\Phi_\pi}(\mathcal{H}) \nsubseteq B_0(\mathcal{H})$.

Lemma 3.6. [7] Chapter 3, Lemma 15.1, Page 147; see also Page 148-149, the paragraph preceding the Theorem 15.2] If $\Phi_\pi \in \{ \Phi_\pi : \pi \in \hat{\Pi} \}$, then its adjoint $\Phi_\pi^*$ is given by

$$
\Phi_\pi^*(\xi) = \sup_n \left\{ \frac{\sum_{j=1}^{n} \xi_j}{\sum_{j=1}^{n} \pi_j} \right\} \text{ for every } \xi = (\xi_i)_{i \in \mathbb{N}} \in c_0^*.
$$

Moreover, the s.n. function $\Phi_\pi^*$ is equivalent to the minimal s.n. function.
Example 3.7. Consider the positive diagonal operator

\[
P = \begin{bmatrix}
2 & 1 + \frac{1}{2} & 0 \\
1 + \frac{1}{3} & 1 + \frac{1}{4} & \ddots \\
0 & \ddots & \ddots \\
\end{bmatrix} \in B(\ell^2),
\]

with respect to an orthonormal basis \( B = \{v_i : i \in \mathbb{N} \} \). Let \( \pi = (\pi_n)_{n \in \mathbb{N}} \) be a sequence of real numbers defined by \( \pi_n := \frac{1}{2} + \frac{1}{n+2} - \frac{1}{2n} \). That \( \pi \in \hat{\Pi} \) is obvious and hence \( \Phi_\pi \in \{ \Phi_\pi : \pi \in \hat{\Pi} \} \). Consequently, \( \Phi_\pi \) is equivalent to the maximal s.n. function \( \Phi_1 \) and the dual \( S_{\Phi_\pi} \) of the s.n. ideal \( \mathcal{S}_{\Phi_\pi} \) is isometrically isomorphic to \( (B(\ell^2), \| \cdot \|_{\Phi_\pi}^* ) \), that is, \( \mathcal{S}_{\Phi_\pi} \cong (B(\ell^2), \| \cdot \|_{\Phi_\pi}^* ) \).

An easy computation yields

\[
\|P\|_{\Phi_\pi} = \sup_n \left\{ \frac{\sum_{j=1}^n s_j(P)}{\sum_{j=1}^n \pi_j} \right\} = \sup_n \left\{ \frac{n + (1 + 1/2 + \ldots + 1/n)}{2(n + (1 + 1/2 + \ldots + 1/n))} \right\} = 2.
\]

If we define \( K \) to be the diagonal operator given by

\[
K = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
\end{bmatrix} \in B_1(\ell^2) = \mathcal{S}_{\Phi_\pi},
\]

then we have \( \|K\|_{\Phi_\pi} = \sum_j \pi_j s_j(K) = 1 \) and \( |\text{Tr}(PK)| = |\text{Tr}(\text{diag}\{2, 0, 0, \ldots\})| = 2 = \|P\|_{\Phi_\pi} \) which implies that \( P \in \mathcal{N}_{\Phi_\pi}(\mathcal{H}) \). However, \( P \notin B_0(\ell^2) \). This proves the existence of an s.n. function \( \Phi_\pi \in \{ \Phi_\pi : \pi \in \hat{\Pi} \} \) equivalent to the maximal s.n. function such that \( \mathcal{N}_{\Phi_\pi}(\mathcal{H}) \nsubseteq B_0(\mathcal{H}) \).

The above example establishes the fact that even for the family of s.n. functions given by \( \{ \Phi_\pi : \pi \in \hat{\Pi} \} \), it is too much to ask for the set \( \mathcal{N}_{\Phi_\pi}(\mathcal{H}) \) to be contained in the compacts for a given \( \Phi_\pi \) from the family. So let us be more modest and ask whether \( P \in \mathcal{B}(\mathcal{H}) \) is compact whenever \( P \in \mathcal{N}_{\Phi_\pi}(\mathcal{H}) \cap \mathcal{B}(\mathcal{H})_+ \) for every \( \Phi_\pi \in \{ \Phi_\pi : \pi \in \hat{\Pi} \} \). The answer to this question is a resounding yes as is stated in the Theorem 3.11. However, before we prove this theorem rigorously, let us pause to find an s.n. function \( \Phi_\pi \) from the family \( \{ \Phi_\pi : \pi \in \hat{\Pi} \} \) of s.n. functions such that the positive noncompact...
operator $P$ of Example 3.7 does not belong to $\mathcal{N}_{\Phi^+}(\mathcal{H})$. The example which follows illustrates this and hence agrees with the Theorem 3.11.

**Example 3.8.** Let $\pi = (\pi_n)_{n \in \mathbb{N}}$ be a sequence defined by

$$\pi_n := \frac{1}{3} + \frac{1 - 1/3}{n} = \frac{n + 2}{3n}. $$

Then $\pi \in \hat{\Pi}$, $\Phi_\pi \in \{\Phi_\pi : \pi \in \hat{\Pi}\}$ and $\mathfrak{S}_{\Phi_\pi} \cong (B(\ell^2), \|\cdot\|_{\Phi_\pi^+}).$ We consider the operator $P$ of Example 3.7 and prove that $P \notin \mathcal{N}_{\Phi^+}(\ell^2)$. To show this, we assume that $P \in \mathcal{N}_{\Phi^+}(\ell^2)$, that is, the supremum,

$$\sup \left\{ \sum_j s_j(P)s_j(K) : K \in B_1(\ell^2), K = \text{diag}\{s_1(K), s_2(K), \ldots\}, \|K\|_{\Phi_\pi} = 1 \right\},$$

is attained, and we deduce a contradiction from this assumption. So there exists $K = \text{diag}\{s_1(K), s_2(K), \ldots\} \in B_1(\ell^2)$ with $\|K\|_{\Phi_\pi} = \sum_j \pi_j s_j(K) = 1$ such that $\|P\|_{\Phi_\pi} = |\text{Tr}(PK)| = \sum_j s_j(P)s_j(K)$. Since $K \in B_1(\mathcal{H}) \subseteq B_0(\mathcal{H})$, we have $\lim_{j \to \infty} s_j(K) = 0$. This forces the existence of a natural number $M$ such that $s_M(K) > s_{M+1}(K)$. All that remains is to show the existence of an operator $\tilde{K} \in B_1(\mathcal{H})$, $\|\tilde{K}\|_{\Phi_\pi} = 1$ of the form $\tilde{K} = \text{diag}\{s_1(\tilde{K}), s_2(\tilde{K}), \ldots\}$ such that $\sum_j s_j(P)s_j(\tilde{K}) > \sum_j s_j(P)s_j(K)$.

If we define

$$t := \frac{\sum_{j=M}^{M+1} \pi_j s_j(K)}{\sum_{j=M}^{M+1} \pi_j} = \frac{\pi_M s_M(K) + \pi_{M+1} s_{M+1}(K)}{\pi_M + \pi_{M+1}},$$

and let $\tilde{K}$ be the diagonal operator defined by

$$\tilde{K} := \begin{pmatrix} s_1(K) & \cdots & \cdots \\ \cdots & s_{M-1}(K) & t \\ \cdots & t & s_{M+1}(K) \\ \cdots & \cdots & \cdots \end{pmatrix},$$

then for every $j$, $s_j(\tilde{K}) = s_j(K)$ which implies that $\|\tilde{K}\|_{\Phi_\pi} = \|K\|_{\Phi_\pi} = 1$ so that $\tilde{K} \in B_1(\ell^2)$ and is of the form $\tilde{K} = \text{diag}\{s_1(\tilde{K}), s_2(\tilde{K}), \ldots\}$. We now prove that $\tilde{K}$ is the required candidate. It is not too hard to see that

$$\frac{\pi_M}{\pi_{M+1}} > \frac{s_M(P)}{s_{M+1}(P)},$$

which yields,

$$\pi_M s_{M+1}(P)(s_M(K) - s_{M+1}(K)) > \pi_{M+1} s_M(P)(s_M(K) - s_{M+1}(K)).$$
Simplification and rearrangement of terms in the above inequality gives

\[ (s_M(P) + s_{M+1}(P)) \left[ \frac{\pi_M s_M(K) + \pi_{M+1} s_{M+1}(K)}{\pi_M + \pi_{M+1}} \right] > s_M(P)s_M(K) + s_{M+1}(P)s_{M+1}(K). \]

But the left hand side of the above inequation is actually

\[ s_M(P)s_M(K) + s_{M+1}(P)s_{M+1}(K), \]

which implies that

\[ s_M(P)s_M(K) + s_{M+1}(P)s_{M+1}(K) > s_M(P)s_M(K) + s_{M+1}(P)s_{M+1}(K). \]

It then immediately follows that \( \sum_j s_j(P)s_j(K) > \sum_j s_j(P)s_j(K) \) which contradicts the assumption that \( \sum_j s_j(P)s_j(K) \) is the supremum of the set \( \{ \sum_j s_j(P)s_j(K) : K \in B_1(\ell^2), K = \text{diag}\{s_1(K), s_2(K), \ldots\}, \|K\|_{\Phi^*} = 1 \} \) and this is precisely the assertion of our claim.

The working rule of the above example is illuminating. The sequence \( \pi = (\pi_n)_{n \in \mathbb{N}} \in \tilde{\Pi} \) has been cleverly chosen to construct the example. The significance of choosing this sequence lies in the fact that it guarantees the existence of a natural number \( M \) so that \( s_M(K) > s_{M+1}(K) \) as well as \( \frac{s_M(P)}{s_{M+1}(P)} \). We use this example as a tool to prove the following proposition.

**Proposition 3.9.** Let \( P \in B(\mathcal{H}) \) be a positive operator. If \( \pi \in \tilde{\Pi} \) such that

\[ \frac{s_n(P)}{s_{n+1}(P)} > s_n(P) \]

for every \( n \in \mathbb{N} \), then \( P \notin \mathcal{N}_{\Phi^*(\mathcal{H})} \).

**Proof.** To show that \( P \notin \mathcal{N}_{\Phi^*(\mathcal{H})} \), we assume that \( P \in \mathcal{N}_{\Phi^*(\mathcal{H})} \), and we deduce a contradiction from this assumption. If \( P \in \mathcal{N}_{\Phi^*(\mathcal{H})} \), then there exists \( K = \text{diag}(s_1(K), s_2(K), \ldots) \in B_1(\mathcal{H}) \) with \( \|K\|_{\Phi^*} = \sum_j s_j(K) = 1 \) such that \( \|P\|_{\Phi^*} = |\text{Tr}(PK)| = \sum_j s_j(P)s_j(K) \). Since \( K \in B_1(\mathcal{H}) \subseteq B_0(\mathcal{H}) \), we have \( \lim_{j \to \infty} s_j(K) = 0 \). This forces the existence of a natural number \( M \) such that \( s_M(K) > s_{M+1}(K) \). We complete the proof by establishing the existence of an operator \( \tilde{K} \in B_1(\mathcal{H}) \), \( \|\tilde{K}\|_{\Phi^*} = 1 \) of the form \( \tilde{K} = \text{diag}(s_1(\tilde{K}), s_2(\tilde{K}), \ldots) \) such that \( \sum_j s_j(P)s_j(K) > \sum_j s_j(P)s_j(K) \).

To this end we define

\[ t := \frac{\sum_{j=M}^{M+1} \pi_j s_j(K)}{\sum_{j=M}^{M+1} \pi_j}, \]
and let
\[
\tilde{K} := \begin{pmatrix} s_1(K) & \cdots & \cdots & t \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ t & \cdots & \cdots & s_{M+1}(K) \\ \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix},
\]
then for every \( j \), \( s_j(\tilde{K}) = s_j(K) \) which implies that \( \|\tilde{K}\|_{\Phi_n} = \|K\|_{\Phi_n} = 1 \) so that \( \tilde{K} \in B_1(\ell^2) \) and is of the form \( \tilde{K} = \text{diag}\{s_j(\tilde{K})\} \). However, since
\[
\frac{\pi_n}{\pi_{n+1}} > \frac{s_n(P)}{s_{n+1}(P)} \quad \text{for every} \quad n \in \mathbb{N},
\]
it follows that
\[
\frac{\pi_M}{\pi_{M+1}} > \frac{s_M(P)}{s_{M+1}(P)},
\]
and thus we have,
\[
s_M(P)s_M(\tilde{K}) + s_{M+1}(P)s_{M+1}(\tilde{K}) > s_M(P)s_M(K) + s_{M+1}(P)s_{M+1}(K),
\]
which yields
\[
\sum_j s_j(P)s_j(\tilde{K}) > \sum_j s_j(P)s_j(K) = \|P\|_{\Phi_n},
\]
which contradicts the assumption that \( \sum_j s_j(P)s_j(K) \) is the supremum of the set
\[
\left\{ \sum_j s_j(P)s_j(K) : K \in B_1(\ell^2), K = \text{diag}\{s_1(K), s_2(K), \ldots\}, \|K\|_{\Phi_n} = 1 \right\}.
\]
This proves our assertion.

\begin{flushright}
\Box
\end{flushright}

**Theorem 3.10.** Let \( P \in B(\mathcal{H}) \) be a positive operator and \( \lim_{j \to \infty} s_j(P) \neq 0 \), that is, \( P \) is not compact. Then there exists \( \pi \in \hat{\Pi} \) such that
\[
\frac{\pi_n}{\pi_{n+1}} > \frac{s_n(P)}{s_{n+1}(P)} \quad \text{for every} \quad n \in \mathbb{N}.
\]

Alternatively, if \( P \in B(\mathcal{H}) \) is positive noncompact operator then there exists \( \pi \in \hat{\Pi} \) such that \( P \notin \mathcal{N}_{\Phi_n}(\mathcal{H}) \).

**Proof.** Since \( P \geq 0 \) and \( \lim_{j \to \infty} s_j(P) \neq 0 \), there exists \( s > 0 \) such that \( \lim_{j \to \infty} s_j(P) = s \). If we take \( \alpha_n := \frac{1}{e^{1/n^2}} \) for all \( n \in \mathbb{N} \) and define a sequence \( \pi = (\pi_n)_{n \in \mathbb{N}} \) recursively by
\[
\pi_1 = 1 \quad \text{and} \quad \frac{\pi_{n+1}}{\pi_n} := \alpha_n \frac{s_{n+1}(P)}{s_n(P)} \quad \text{for all} \quad n \in \mathbb{N},
\]
we have $\alpha_n < 1$ for all $n \in \mathbb{N}$. Then the fact that $s_n(P)$ is a nonincreasing sequence implies that $\frac{s_{n+1}(P)}{s_n(P)} < \frac{s_{n+1}(P)}{s_n(P)}$ for all $n \in \mathbb{N}$. Therefore, $\frac{s_{n+1}(P)}{s_n(P)} > \frac{s_{n+1}(P)}{s_n(P)}$ for every $n \in \mathbb{N}$. All that remains is to show that $\pi \in \hat{\Pi}$. That $\pi_1 = 1$ and $(\pi_n)_{n \in \mathbb{N}}$ is a strictly decreasing sequence of positive real numbers are trivial observations. We complete the proof by showing that $\lim_{n \to \infty} \pi_n > 0$.

An easy calculation shows that

$$\pi_{n+1} = \left( \prod_{m=1}^{n} \alpha_m \right) \frac{s_{n+1}(P)}{s_1(P)}$$

for each $n \in \mathbb{N}$.

Let $x_n = \left( \prod_{m=1}^{n} \alpha_m \right)$ for every $n \in \mathbb{N}$ and observe that

$$\pi_{n+1} = \left( x_n \right) \left( \frac{s_{n+1}(P)}{s_1(P)} \right),$$

which yields

$$\lim_{n \to \infty} (\pi_{n+1}) = \frac{1}{s_1(P)} \lim_{n \to \infty} x_n \lim_{n \to \infty} s_{n+1}(P).$$

This observation, together with the facts that $s_1(P) > 0$ and $\lim_{n \to \infty} s_{n+1}(P) = \alpha$ allows us to infer that $\lim_{n \to \infty} (\pi_{n+1}) > 0$ if and only if $\lim_{n \to \infty} x_n > 0$.

But

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} \frac{1}{\pi_1^{\sum_{m=1}^{\infty} 1/m^2}} = \frac{1}{\pi/6} > 0,$$

and we conclude that $\lim_{n \to \infty} (\pi_n) > 0$. This completes the proof.

We are now in a position to prove a key result — a characterization theorem for positive operators in $\{ N_{\Phi_n}^\pi(\mathcal{H}) : \pi \in \hat{\Pi} \}$ — which answers the question we asked in the paragraph preceding the Example 3.8. Moreover, this result is a special case of a more general result that will be presented in the next section (see Theorem 4.4).

**Theorem 3.11.** Let $P$ be a positive operator on $\mathcal{H}$. Then the following statements are equivalent.

1. $P \in B_0(\mathcal{H})$.
2. $P \in AN_{\Phi_n}^\pi(\mathcal{H})$ for every $\pi \in \hat{\Pi}$.
3. $P \in N_{\Phi_n}^\pi(\mathcal{H})$ for every $\pi \in \hat{\Pi}$.

**Proof.** (1) implies (2) follows from Theorem 3.1. (2) implies (3) is obvious.
(3) implies (1) is a direct consequence of the Theorem 3.10. □

We conclude this section by proving the following result that extends the above theorem to bounded operators in $\mathcal{B}(\mathcal{H})$, the above theorem required the operator to be positive. This is the main theorem of this section.

**Theorem 3.12.** If $T \in \mathcal{B}(\mathcal{H})$, then the following statements are equivalent.

1. $T \in B_0(\mathcal{H})$.
2. $T \in AN_{\Phi_n}^\pi(\mathcal{H})$ for every $\pi \in \hat{\Pi}$.

We refer to the next section for the proof of this theorem.
(3) \(T \in \mathcal{N}_{\Phi_1}^{\ast}(\mathcal{H})\) for every \(\pi \in \hat{\Pi}\).

**Proof.** (2) implies (3) is obvious, as is (1) implies (2) from the Theorem 3.1. The Proposition 3.2 along with the Theorem 3.10 proves (3) implies (1). □

The above result, although very important, is transitory. We will see a much more general result than this — the characterization theorem for universally symmetrically norming operators (see Theorem 4.6).

### 4. Characterization of Universally Symmetrically Norming Operators

In the preceding section we considered a certain family \(\{\Phi_\pi : \pi \in \hat{\Pi}\}\) of s.n. functions and a family of symmetric norms on \(\mathcal{B}(\mathcal{H})\) generated by the dual of these, and we studied the symmetrically norming operators and absolutely symmetrically norming operators with respect to each of these symmetric norms. The fact that each member of the family \(\{\Phi_\pi : \pi \in \hat{\Pi}\}\) is equivalent to the maximal s.n. function \(\Phi_1\) suggests the possibility of extending the Theorem 3.12 to a larger family of s.n. functions. With this in mind, our attention is drawn to the family of all s.n. functions that are equivalent to the maximal s.n. function, that is, the family \(\{\Phi : \Phi\) is equivalent to \(\Phi_1\}\) of s.n. functions. This larger family of s.n. functions provides us with the leading idea on which we develop the notions of “universally symmetrically norming operators” and “universally absolutely symmetrically norming operators” on a separable Hilbert space. The study of these operators are taken up in this section. Our main result is Theorem 4.6 which states that an operator in \(\mathcal{B}(\mathcal{H})\) is universally symmetrically norming if and only if it is universally absolutely symmetrically norming, which holds if and only if it is compact.

We begin by defining the relevant classes of operators.

**Definition 4.1.** An operator \(T \in \mathcal{B}(\mathcal{H})\) is said to be **universally symmetrically norming** if \(T \in \mathcal{N}_{\Phi_1}^{\ast}(\mathcal{H})\) for every s.n. function \(\Phi\) equivalent to the maximal s.n. function \(\Phi_1\). Alternatively, an operator \(T \in (\mathcal{B}(\mathcal{H}))\) is said to be universally symmetrically norming if \(T \in \mathcal{N}_{\Phi_1}(\mathcal{H})\) for every \(\Phi\) from the family \(\{\Phi : \Phi\) is equivalent to \(\Phi_1\}\) of s.n. functions.

**Definition 4.2.** An operator \(T \in \mathcal{B}(\mathcal{H})\) is said to be **universally absolutely symmetrically norming** if \(T \in \mathcal{A}\mathcal{N}_{\Phi_1}(\mathcal{H})\) for every s.n. function \(\Phi\) equivalent to the maximal s.n. function \(\Phi_1\).

**Remark 4.3.** Since every symmetric norm on \(\mathcal{B}(\mathcal{H})\) is topologically equivalent to the usual operator norm, it follows that \(T \in \mathcal{B}(\mathcal{H})\) is universally symmetrically norming (respectively universally absolutely symmetrically norming) if and only if \(T\) is symmetrically norming (respectively absolutely symmetrically norming) with respect to every symmetric norm on \(\mathcal{B}(\mathcal{H})\). Another important observation worth mentioning here is that every universally absolutely symmetrically norming operator is universally symmetrically norming.
The following theorem gives a useful characterization of positive universally symmetrically norming operators in $\mathcal{B}(\mathcal{H})$.

**Theorem 4.4.** Let $P$ be a positive operator on $\mathcal{H}$ and let $\Phi_1$ denote the maximal s.n. function. Then the following statements are equivalent.

1. $P \in \mathcal{B}_0(\mathcal{H})$.
2. $P$ is universally absolutely symmetrically norming, that is, $P \in \mathcal{A}N_{\Phi^*}(\mathcal{H})$ for every s.n. function $\Phi$ equivalent to $\Phi_1$.
3. $P$ is universally symmetrically norming, that is, $P \in \mathcal{N}_{\Phi^*}(\mathcal{H})$ for every s.n. function $\Phi$ equivalent to $\Phi_1$.

**Proof.** The implication (1) $\implies$ (2) is an immediate consequence of Theorem 3.1 and (2) $\implies$ (3) is straightforward. To prove (3) $\implies$ (1), assume that the positive operator $P$ is universally symmetrically norming on $\mathcal{H}$. Then the statement (3) of Theorem 3.11 holds which implies that $P$ is compact and the proof is complete. $\square$

We next establish the following result which allows us to extend the above theorem to operators that are not necessarily positive.

**Proposition 4.5.** An operator $T \in \mathcal{B}(\mathcal{H})$ is universally symmetrically norming if and only if $|T|$ is universally symmetrically norming.

**Proof.** This follows immediately from the Proposition 3.2. $\square$

We are now prepared to extend the Theorem 4.4 for an arbitrary operator on a separable Hilbert space.

**Theorem 4.6** (Characterization Theorem for Universally Symmetrically Norming Operators). Let $T \in \mathcal{B}(\mathcal{H})$ and let $\Phi_1$ denote the maximal s.n. function. Then the following statements are equivalent.

1. $T \in \mathcal{B}_0(\mathcal{H})$.
2. $T$ is universally absolutely symmetrically norming, that is, $T \in \mathcal{A}N_{\Phi^*}(\mathcal{H})$ for every s.n. function $\Phi$ equivalent to $\Phi_1$.
3. $T$ is universally symmetrically norming, that is, $T \in \mathcal{N}_{\Phi^*}(\mathcal{H})$ for every s.n. function $\Phi$ equivalent to $\Phi_1$.

**Proof.** Theorem 4.4 and the preceding proposition yield this result. $\square$

**Remark 4.7.** The preceding result provides an alternative characterization of compact operators.

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Satish K. Pandey
Department of Pure Mathematics
University of Waterloo
Waterloo, Ontario, N2L 3G1, Canada
E-mail address: satish.pandey@uwaterloo.ca