THETA SERIES ASSOCIATED WITH THE SCHRÖDINGER-WEIL REPRESENTATION

JAE-HYUN YANG

ABSTRACT. In this paper, we define the Schrödinger-Weil representation for the Jacobi group and construct covariant maps for the Schrödinger-Weil representation. Using these covariant maps, we construct Jacobi forms with respect to an arithmetic subgroup of the Jacobi group.

1. Introduction

For a given fixed positive integer $n$, we let

$$H_n = \{ \Omega \in \mathbb{C}^{(n,n)} | \Omega = t\Omega, \quad \text{Im } \Omega > 0 \}$$

be the Siegel upper half plane of degree $n$ and let

$$Sp(n, \mathbb{R}) = \{ g \in \mathbb{R}^{(2n,2n)} | t^g J_n g = J_n \}$$

be the symplectic group of degree $n$, where $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring $F$, $t^M$ denotes the transposed matrix of a matrix $M$, $\text{Im } \Omega$ denotes the imaginary part of $\Omega$ and

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$ 

We see that $Sp(n, \mathbb{R})$ acts on $H_n$ transitively by

$$g \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1},$$

where $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$ and $\Omega \in H_n$.

For two positive integers $n$ and $m$, we consider the Heisenberg group

$$H^{(n,m)}_\mathbb{R} = \{ (\lambda, \mu; \kappa) | \lambda, \mu \in \mathbb{R}^{(m,n)}, \kappa \in \mathbb{R}^{(m,m)}, \kappa + \mu^t \lambda \text{ symmetric} \}$$

endowed with the following multiplication law

$$(\lambda, \mu; \kappa) \circ (\lambda', \mu'; \kappa') = (\lambda + \lambda', \mu + \mu'; \kappa + \kappa' + \lambda^t \mu' - \mu^t \lambda').$$

We let

$$G^J = Sp(n, \mathbb{R}) \ltimes H^{(n,m)}_\mathbb{R}$$

be the Jacobi group endowed with the following multiplication law

$$\left( g, (\lambda, \mu; \kappa) \right) \cdot \left( g', (\lambda', \mu'; \kappa') \right) = \left( gg', (\bar{\lambda} + \lambda', \bar{\mu} + \mu'; \kappa + \kappa' + \bar{\lambda}^t \mu' - \bar{\mu}^t \lambda') \right).$$

Subject Classification: Primary 11F27, 11F50

Keywords and phrases: the Schrödinger-Weil Representation, covariant maps, the Schrödinger representation, the Weil representation, Jacobi forms, Poisson summation formula.
with \( g, g' \in \text{Sp}(n, \mathbb{R}), (\lambda, \mu; \kappa), (\lambda', \mu'; \kappa') \in H^{(n,m)}_{\mathbb{R}} \) and \( (\tilde{\lambda}, \tilde{\mu}) = (\lambda, \mu)g' \). We let \( \Gamma_n = \text{Sp}(n, \mathbb{Z}) \) be the Siegel modular group of degree \( n \). We let
\[
\Gamma^J = \Gamma_n \ltimes H^{(n,m)}_Z
\]
be the Jacobi modular group. Then we have the natural action of \( G^J \) on the Siegel-Jacobi space \( \mathbb{H}_{n,m} := \mathbb{H}_n \times \mathbb{C}^{(m,n)} \) defined by
\[
(g, (\lambda, \mu; \kappa)) \cdot (\Omega, Z) = \left( g \cdot \Omega, (Z + \lambda \Omega + \mu)(C \Omega + D)^{-1} \right),
\]
where \( g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(n, \mathbb{R}), (\lambda, \mu; \kappa) \in H^{(n,m)}_{\mathbb{R}} \) and \( (\Omega, Z) \in \mathbb{H}_{n,m} \). We refer to [19]-[25] for more details on materials related to the Siegel-Jacobi space.

The Weil representation for the symplectic group was first introduced by A. Weil in [13] to reformulate Siegel’s analytic theory of quadratic forms (cf. [12]) in terms of the group theoretical theory. It is well known that the Weil representation plays a central role in the study of the transformation behaviors of the theta series. In this paper, we define the Schrödinger-Weil representation for the Jacobi group \( G^J \). The aim of this paper is to construct the covariant maps for the Schrödinger-Weil representation, and to construct Jacobi forms with respect to an arithmetic subgroup of \( \Gamma^J \) using these covariant maps.

This paper is organized as follows. In Section 2, we discuss the Schrödinger representation of the Heisenberg group \( H^{(n,m)}_{\mathbb{R}} \) associated with a symmetric nonzero real matrix of degree \( m \). In Section 3, we review the concept of a Jacobi form briefly. In Section 4, we define the Schrödinger-Weil representation \( \omega_M \) of the Jacobi group \( G^J \) associated with a symmetric positive definite matrix \( M \) and provide some of the actions of \( \omega_M \) on the representation space \( L^2(\mathbb{R}^{(m,n)}) \) explicitly. In Section 5, we construct the covariant maps for the Schrödinger-Weil representation \( \omega_M \). In the final section we construct Jacobi forms with respect to an arithmetic subgroup of \( \Gamma^J \) using the covariant maps obtained in Section 5.

**Notations:** We denote by \( \mathbb{Z} \) and \( \mathbb{C} \) the ring of integers, and the field of complex numbers respectively. \( \mathbb{C}^\times \) denotes the multiplicative group of nonzero complex numbers. \( T \) denotes the multiplicative group of complex numbers of modulus one. The symbol “:=” means that the expression on the right is the definition of that on the left. For two positive integers \( k \) and \( l \), \( F^{(k,l)} \) denotes the set of all \( k \times l \) matrices with entries in a commutative ring \( F \). For a square matrix \( A \in F^{(k,k)} \) of degree \( k \), \( \sigma(A) \) denotes the trace of \( A \). For any \( M \in F^{(k,l)} \), \( tM \) denotes the transposed matrix of \( M \). \( I_n \) denotes the identity matrix of degree \( n \). We put \( i = \sqrt{-1} \). For \( z \in \mathbb{C} \), we define \( z^{1/2} = \sqrt{z} \) so that \(-\pi/2 < \arg(z^{1/2}) \leq \pi/2 \). Further we put \( z^{\kappa/2} = (z^{1/2})^\kappa \) for every \( \kappa \in \mathbb{Z} \).

2. The Schrödinger Representation of \( H^{(n,m)}_{\mathbb{R}} \)

First of all, we observe that \( H^{(n,m)}_{\mathbb{R}} \) is a 2-step nilpotent Lie group. The inverse of an element \((\lambda, \mu; \kappa) \in H^{(n,m)}_{\mathbb{R}}\) is given by
\[
(\lambda, \mu; \kappa)^{-1} = (-\lambda, -\mu; -\kappa + \lambda^t \mu - \mu^t \lambda).
\]
Now we set
\[ [\lambda, \mu; \kappa] = (0, \mu; \kappa) \circ (\lambda, 0; 0) = (\lambda, \mu; \kappa - \mu^\prime \lambda). \]
Then \( H^{(m,m)} \) may be regarded as a group equipped with the following multiplication
\[ [\lambda, \mu; \kappa] \circ [\lambda_0, \mu_0; \kappa_0] = [\lambda + \lambda_0, \mu + \mu_0; \kappa + \kappa_0 + \lambda^\prime \mu_0 + \mu_0^\prime \lambda]. \]
The inverse of \([\lambda, \mu; \kappa] \in H^{(n,m)} \) is given by
\[ [\lambda, \mu; \kappa]^{-1} = [-\lambda, -\mu; \kappa + \lambda^\prime \mu + \mu^\prime \lambda]. \]
We set
\[ L = \left\{ [0, \mu; \kappa] \in H^{(n,m)} \mid \mu \in \mathbb{R}^{(m,n)}, \kappa = \kappa R \in \mathbb{R}^{(m,m)} \right\}. \]
Then \( L \) is a commutative normal subgroup of \( H^{(n,m)} \). Let \( \hat{L} \) be the Pontrajagin dual of \( L \), i.e., the commutative group consisting of all unitary characters of \( L \). Then \( \hat{L} \) is isomorphic to the additive group \( \mathbb{R}^{(m,n)} \times \text{Symm}(m, \mathbb{R}) \) via
\[ \langle a, \hat{a} \rangle = e^{2\pi i \sigma(\mu, \mu + \kappa^\prime)}, \quad a = [0, \mu; \kappa] \in L, \quad \hat{a} = (\hat{\mu}, \hat{\kappa}) \in \hat{L}, \]
where \( \text{Symm}(m, \mathbb{R}) \) denotes the space of all symmetric \( m \times m \) real matrices.
We put
\[ S = \left\{ [\lambda, 0; 0] \in H^{(n,m)} \mid \lambda \in \mathbb{R}^{(m,n)} \right\} \cong \mathbb{R}^{(m,n)}. \]
Then \( S \) acts on \( L \) as follows:
\[ \alpha_{\lambda}([0, \mu; \kappa]) = [0, \mu; \kappa + \lambda^\prime \mu + \mu^\prime \lambda], \quad [\lambda, 0, 0] \in S. \]
We see that the Heisenberg group \( \left( H^{(n,m)}, \circ \right) \) is isomorphic to the semi-direct product \( S \rtimes L \) of \( S \) and \( L \) whose multiplication is given by
\[ (\lambda, a) \cdot (\lambda_0, a_0) = (\lambda + \lambda_0, a + \alpha_{\lambda}(a_0)), \quad \lambda, \lambda_0 \in S, \quad a, a_0 \in L. \]
On the other hand, \( S \) acts on \( \hat{L} \) by
\[ \alpha_{\lambda}^\ast(\hat{a}) = (\hat{\mu} + 2\kappa \lambda, \hat{\kappa}), \quad [\lambda, 0; 0] \in S, \quad a = (\hat{\mu}, \hat{\kappa}) \in \hat{L}. \]
Then, we have the relation \( \langle \alpha_{\lambda}(a), \hat{a} \rangle = \langle a, \alpha_{\lambda}^\ast(\hat{a}) \rangle \) for all \( a \in L \) and \( \hat{a} \in \hat{L} \).

We have three types of \( S \)-orbits in \( \hat{L} \).

**TYPE I.** Let \( \kappa \in \text{Symm}(m, \mathbb{R}) \) be nondegenerate. The \( S \)-orbit of \( \hat{a}(\kappa) = (0, \kappa) \in \hat{L} \) is given by
\[ \hat{O}_{\kappa} = \left\{ (2\kappa \lambda, \hat{\kappa}) \in \hat{L} \mid \lambda \in \mathbb{R}^{(m,n)} \right\} \cong \mathbb{R}^{(m,n)}. \]

**TYPE II.** Let \( (\hat{\mu}, \hat{\kappa}) \in \mathbb{R}^{(m,n)} \times \text{Symm}(m, \mathbb{R}) \) with degenerate \( \hat{\kappa} \neq 0 \). Then
\[ \hat{O}_{(\hat{\mu}, \hat{\kappa})} = \left\{ (\hat{\mu} + 2\hat{\kappa} \lambda, \hat{\kappa}) \mid \lambda \in \mathbb{R}^{(m,n)} \right\} \subset \mathbb{R}^{(m,n)} \times \{ \hat{\kappa} \}. \]

**TYPE III.** Let \( \hat{y} \in \mathbb{R}^{(m,n)} \). The \( S \)-orbit \( \hat{O}_{\hat{y}} \) of \( \hat{a}(\hat{y}) = (\hat{y}, 0) \) is given by
\[ \hat{O}_{\hat{y}} = \{ (\hat{y}, 0) \} = \hat{a}(\hat{y}). \]
We have
\[
\hat{L} = \left( \bigcup_{\hat{\kappa} \in \text{Symm}(m,\mathbb{R})} \mathcal{O}_{\hat{\kappa}} \right) \cup \left( \bigcup_{\hat{\gamma} \in \mathbb{R}^{(m,n)}} \mathcal{O}_{\hat{\gamma}} \right) \cup \left( \bigcup_{(\hat{\mu},\hat{\kappa}) \in \mathbb{R}^{(m,n)} \times \text{Symm}(m,\mathbb{R})} \mathcal{O}(\hat{\mu},\hat{\kappa}) \right)
\]
as a set. The stabilizer $S_\kappa$ of $S$ at $\hat{a}(\hat{\kappa}) = (0,\hat{\kappa})$ is given by
\[
S_\kappa = \{0\}.
\]
And the stabilizer $S_{\hat{\gamma}}$ of $S$ at $\hat{a}(\hat{\gamma}) = (\hat{\gamma},0)$ is given by
\[
S_{\hat{\gamma}} = \left\{ \left(\lambda,0;0\right) \mid \lambda \in \mathbb{R}^{(m,n)} \right\} = S \cong \mathbb{R}^{(m,n)}.
\]

In this section, for the present being we set $H = H^H_{\mathbb{R}}(n,m)$ for brevity. We see that $L$ is a closed, commutative normal subgroup of $H$. Since $(\lambda,\mu;\kappa) = (0,\mu;\kappa + \mu^t\lambda) \circ (\lambda,0;0)$ for $(\lambda,\mu;\kappa) \in H$, the homogeneous space $X = L \setminus H$ can be identified with $\mathbb{R}^{(m,n)}$ via
\[
Lh = L \circ (\lambda,0;0) \longrightarrow \lambda, \quad h = (\lambda,\mu;\kappa) \in H.
\]
We observe that $H$ acts on $X$ by
\[
(Lh) \cdot h_0 = L(\lambda + \lambda_0,0;0) = \lambda + \lambda_0,
\]
where $h = (\lambda,\mu;\kappa) \in H$ and $h_0 = (\lambda_0,\mu_0;\kappa_0) \in H$.

If $h = (\lambda,\mu;\kappa) \in H$, we have
\[
l_h = (0,\mu;\kappa + \mu^t\lambda), \quad s_h = (\lambda,0;0)
\]
in the Mackey decomposition of $h = l_h \circ s_h$ (cf. [8]). Thus if $h_0 = (\lambda_0,\mu_0;\kappa_0) \in H$, then we have
\[
s_h \circ h_0 = (\lambda,0;0) \circ (\lambda_0,\mu_0;\kappa_0) = (\lambda + \lambda_0,\mu_0;\kappa_0 + \lambda^t\mu_0)
\]
and so
\[
(2.1) \quad l_{s_h h_0} = (0,\mu_0;\kappa_0 + \mu_0^t\lambda_0 + \lambda^t\mu_0 + \mu_0^t\lambda).
\]

For a real symmetric matrix $c = \,^t\!c \in \text{Symm}(m,\mathbb{R})$ with $c \neq 0$, we consider the unitary character $\chi_c$ of $L$ defined by
\[
\chi_c ((0,\mu;\kappa)) = e^{\pi i \sigma(ck)} I, \quad (0,\mu;\kappa) \in L,
\]
where $I$ denotes the identity mapping. Then the representation $\mathcal{W}_c = \text{Ind}_L^H \chi_c$ of $H$ induced from $\chi_c$ is realized on the Hilbert space $H(\chi_c) = L^2(X,d\hat{h},\mathcal{C}) \cong L^2(\mathbb{R}^{(m,n)},dx)$ as follows. If $h_0 = (\lambda_0,\mu_0;\kappa_0) \in H$ and $x = Lh \in X$ with $h = (\lambda,\mu;\kappa) \in H$, we have
\[
(2.3) \quad (\mathcal{W}_c(h_0)f)(x) = \chi_c(l_{s_h h_0})(f(xh_0)), \quad f \in H(\chi_c).
\]
It follows from (2.1) that
\[
(2.4) \quad (\mathcal{W}_c(h_0)f)(\lambda) = e^{\pi i \sigma \{\sigma(\kappa_0 + \mu_0^t\lambda_0 + 2\lambda^t\mu_0)\}} f(\lambda + \lambda_0),
\]
where $h_0 = (\lambda_0,\mu_0;\kappa_0) \in H$ and $\lambda \in \mathbb{R}^{(m,n)}$. Here we identified $x = Lh$ (resp. $xh_0 = Lhh_0$) with $\lambda$ (resp. $\lambda + \lambda_0$). The induced representation $\mathcal{W}_c$ is called the Schrödinger representation of $H$ associated with $\chi_c$. Thus $\mathcal{W}_c$ is a monomial representation.
Theorem 2.1. Let $c$ be a positive definite symmetric real matrix of degree $m$. Then the Schrödinger representation $\mathcal{W}_c$ of $H$ is irreducible.

Proof. The proof can be found in [14], Theorem 3. □

Remark. We refer to [14]-[18] for more representations of the Heisenberg group $H_{(n,m)}$ and their related topics.

3. Jacobi Forms

Let $\rho$ be a rational representation of $GL(n, \mathbb{C})$ on a finite dimensional complex vector space $V_\rho$. Let $\mathcal{M} \in \mathbb{R}^{(m,m)}$ be a symmetric half-integral semi-positive definite matrix of degree $m$. Let $C^\infty(\mathbb{H}_{n,m}, V_\rho)$ be the algebra of all $C^\infty$ functions on $\mathbb{H}_{n,m}$ with values in $V_\rho$. For $f \in C^\infty(\mathbb{H}_{n,m}, V_\rho)$, we define

$$
(f|_{\rho,\mathcal{M}}[(g, (\lambda, \mu; \kappa))])(\Omega, Z)
$$

(3.1)

$$
e^{-2\pi i \sigma(\mathcal{M}(\lambda^T \Omega + \mu))(\Omega + D)^{-1}C^\dagger(Z + \lambda \Omega + \mu)} \times e^{2\pi i \sigma(\mathcal{M}(\lambda^T \Omega + \mu))}
$$

$$
\times \rho(C\Omega + D)^{-1}f(g, \Omega, (Z + \lambda \Omega + \mu)(C\Omega + D)^{-1}),
$$

where $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$, $(\lambda, \mu; \kappa) \in H_{R}^{(n,m)}$ and $(\Omega, Z) \in \mathbb{H}_{n,m}$.

Definition 3.1. Let $\rho$ and $\mathcal{M}$ be as above. Let

$$H_{\mathcal{M}}^{(n,m)} = \{(\lambda, \mu; \kappa) \in H_{R}^{(n,m)} | \lambda, \mu \in \mathbb{Z}^{(m,m)}, \kappa \in \mathbb{Z}^{(m,m)} \}.$$

A Jacobi form of index $\mathcal{M}$ with respect to $\rho$ on a subgroup $\Gamma$ of $\Gamma_n$ of finite index is a holomorphic function $f \in C^\infty(\mathbb{H}_{n,m}, V_\rho)$ satisfying the following conditions (A) and (B):

(A) \quad $f|_{\rho,\mathcal{M}}[\tilde{\gamma}] = f$ for all $\tilde{\gamma} \in \Gamma \times H_{\mathcal{M}}^{(n,m)}$.

(B) \quad For each $M \in \Gamma_n$, $f|_{\rho,\mathcal{M}}[M]$ has a Fourier expansion of the following form :

$$
(f|_{\rho,\mathcal{M}}[M])(\Omega, Z) = \sum_{R \in \mathbb{Z}^{(n,m)}} \sum_{T = t, T \geq 0 \text{ half-integral}} \sum_{R} c(T, R) \cdot e^{\frac{2\pi i}{\lambda_1} \sigma(T \Omega)} \cdot e^{2\pi i \sigma(RZ)}
$$

with a suitable $\lambda_1 \in \mathbb{Z}$ and $c(T, R) \neq 0$ only if \(\begin{pmatrix} \frac{1}{2} T & \frac{1}{2} R \\ \frac{1}{2} T^T & \mathcal{M} \end{pmatrix} \geq 0\).

If $n \geq 2$, the condition (B) is superfluous by Koecher principle (cf. [26] Lemma 1.6). We denote by $J_{\rho,\mathcal{M}}(\Gamma)$ the vector space of all Jacobi forms of index $\mathcal{M}$ with respect to $\rho$ on $\Gamma$. Ziegler (cf. [26] Theorem 1.8 or [2] Theorem 1.1) proves that the vector space $J_{\rho,\mathcal{M}}(\Gamma)$ is finite dimensional. In the special case $\rho(A) = (\det(A))^k$ with $A \in GL(n, \mathbb{C})$ and a fixed $k \in \mathbb{Z}$, we write $J_{k,\mathcal{M}}(\Gamma)$ instead of $J_{\rho,\mathcal{M}}(\Gamma)$ and call $k$ the weight of the corresponding Jacobi forms. For more results on Jacobi forms with $n > 1$ and $m > 1$, we refer to [19]-[22] and [26].

Definition 3.2. A Jacobi form $f \in J_{\rho,\mathcal{M}}(\Gamma)$ is said to be a cusp (or cuspidal) form if

$$
\begin{pmatrix} \frac{1}{2} T & \frac{1}{2} R \\ \frac{1}{2} T^T & \mathcal{M} \end{pmatrix} > 0 \quad \text{for any } T, R \text{ with } c(T, R) \neq 0.
$$

A Jacobi form $f \in J_{\rho,\mathcal{M}}(\Gamma)$ is said to
According to Schur's lemma, we have a map $c$.

Therefore $R$ is a fixed element $g H$. Here $\text{Id}$

We allow a weight $k$ to be half-integral.

**Definition 3.3.** Let $\Gamma \subset \Gamma_n$ be a subgroup of finite index. A holomorphic function $f : \mathbb{H}_{n,m} \rightarrow \mathbb{C}$ is said to be a Jacobi form of a weight $k \in \frac{1}{2}\mathbb{Z}$ with level $\Gamma$ and index $M$ if it satisfies the following transformation formula

$$f(\tilde{\gamma} : (\Omega, Z)) = \chi(\tilde{\gamma}) J_{k,M}(\tilde{\gamma}, (\Omega, Z))f(\Omega, Z) \quad \text{for all } \tilde{\gamma} \in \tilde{\Gamma} = \Gamma \times H^{(n,m)}_\mathbb{Z},$$

where $\chi$ is a character of $\tilde{\Gamma}$ and $J_{k,M} : \tilde{\Gamma} \times \mathbb{H}_{n,m} \rightarrow \mathbb{C}^\times$ is an automorphic factor defined by

$$J_{k,M}(\tilde{\gamma}, (\Omega, Z)) = e^{2\pi i \sigma(M(\delta + \lambda \Omega + \mu)(\lambda \Omega + D))^{-1}C^{-1}(Z + \lambda \Omega + \mu)}$$

$$\times e^{-2\pi i \sigma(M(\lambda \Omega + 2\lambda Z + \kappa + \mu \lambda))} \det(C \Omega + D)^{k}$$

with $\tilde{\gamma} = (\gamma, (\lambda, \mu; \kappa)) \in \tilde{\Gamma}$ with $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$, $(\lambda, \mu; \kappa) \in H^{(n,m)}_\mathbb{Z}$ and $(\Omega, Z) \in \mathbb{H}_{n,m}$.

4. The Schrödinger-Weil Representation

Throughout this section we assume that $M$ is a symmetric integral positive definite $m \times m$ matrix. We consider the Schrödinger representation $\mathcal{W}_M$ of the Heisenberg group $H^{(n,m)}_\mathbb{R}$ with the central character $\mathcal{W}_M((0, 0; \kappa)) = \chi_M((0, 0; \kappa)) = e^{\pi i \sigma(Mn)}, \kappa \in \text{Symm}(m, \mathbb{R})$ (cf. (2.2)).

We note that the symplectic group $Sp(n, \mathbb{R})$ acts on $H^{(n,m)}_\mathbb{R}$ by conjugation inside $G^J$. For a fixed element $g \in Sp(n, \mathbb{R})$, the irreducible unitary representation $\mathcal{W}_M^g$ of $H^{(n,m)}_\mathbb{R}$ defined by

$$\mathcal{W}_M^g(h) = \mathcal{W}_M(ghg^{-1}), \quad h \in H^{(n,m)}_\mathbb{R}$$

has the property that

$$\mathcal{W}_M^g((0, 0; \kappa)) = \mathcal{W}_M((0, 0; \kappa)) = e^{\pi i \sigma(Mn)} \text{Id}_{H(\chi_M)}, \quad \kappa \in \text{Symm}(m, \mathbb{R}).$$

Here $\text{Id}_{H(\chi_M)}$ denotes the identity operator on the Hilbert space $H(\chi_M)$. According to Stone-von Neumann theorem, there exists a unitary operator $R_M(g)$ on $H(\chi_M)$ such that $R_M(g)\mathcal{W}_M(h) = \mathcal{W}_M^g(h)R_M(g)$ for all $h \in H^{(n,m)}_\mathbb{R}$. We observe that $R_M(g)$ is determined uniquely up to a scalar of modulus one. From now on, for brevity, we put $G = Sp(n, \mathbb{R})$. According to Schur's lemma, we have a map $c_M : G \times G \rightarrow T$ satisfying the relation

$$R_M(g_1g_2) = c_M(g_1, g_2)R_M(g_1)R_M(g_2) \quad \text{for all } g_1, g_2 \in G.$$ 

Therefore $R_M$ is a projective representation of $G$ on $H(\chi_M)$ and $c_M$ defines the cocycle class in $H^2(G, T)$. The cocycle $c_M$ yields the central extension $G_M$ of $G$ by $T$. The group $G_M$ is a set $G \times T$ equipped with the following multiplication

$$(g_1, t_1) \cdot (g_2, t_2) = (g_1g_2, t_1t_2 c_M(g_1, g_2)^{-1}), \quad g_1, g_2 \in G, \quad t_1, t_2 \in T.$$
We see immediately that the map \( \tilde{R}_M : G_M \rightarrow GL(H(\chi_M)) \) defined by
\[
(4.2) \quad \tilde{R}_M(g, t) = t R_M(g) \quad \text{for all } (g, t) \in G_M
\]
is a true representation of \( G_M \). As in Section 1.7 in [7], we can define the map \( s_M : G \rightarrow T \) satisfying the relation
\[
c_M(g_1, g_2)^2 = s_M(g_1)^{-1} s_M(g_2)^{-1} s_M(g_1 g_2) \quad \text{for all } g_1, g_2 \in G.
\]
Thus we see that
\[
G_{2, M} = \{ (g, t) \in G_M \mid t^2 = s_M(g)^{-1} \}
\]
is the metaplectic group associated with \( M \) that is a two-fold covering group of \( G \). The restriction \( R_{2, M} \) of \( \tilde{R}_M \) to \( G_{2, M} \) is the Weil representation of \( G \) associated with \( M \). Now we define the projective representation \( \pi_M \) of the Jacobi group \( G^J \) by
\[
(4.3) \quad \pi_M(hg) = \mathcal{W}_M(h) R_M(g), \quad h \in H^{(n, m)}_R, \ g \in G.
\]
The projective representation \( \pi_M \) of \( G^J \) is naturally extended to the true representation \( \omega_M \) of the group \( G^J_{2, M} = G_{2, M} \ltimes H^{(n, m)}_R \). The representation \( \omega_M \) is called the Schrödinger-Weil representation of \( G^J \). Indeed we have
\[
(4.4) \quad \omega_M(h \cdot (g, t)) = t \mathcal{W}_M(h) R_M(g), \quad h \in H^{(n, m)}_R, \ (g, t) \in G_{2, M}.
\]

We recall that the following matrices
\[
t_0(b) = \begin{pmatrix} I_n & b \\ 0 & I_n \end{pmatrix} \quad \text{with any } b = t b \in \mathbb{R}^{(n, n)},
\]
\[
g_0(\alpha) = \begin{pmatrix} t \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \quad \text{with any } \alpha \in GL(n, \mathbb{R}),
\]
\[
\sigma_{n, 0} = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}
\]
generate the symplectic group \( G = Sp(n, \mathbb{R}) \) (cf. [3] p. 326, [10] p. 210]). Therefore the following elements \( h_t(\lambda, \mu; \kappa), t_M(b), g_M(\alpha) \) and \( \sigma_{n, M} \) of \( G_M \ltimes H^{(n, m)}_R \) defined by
\[
h_t(\lambda, \mu; \kappa) = ((I_{2n}, t), (\lambda, \mu; \kappa)) \quad \text{with } t \in T, \ \lambda, \mu \in \mathbb{R}^{(n, n)} \text{ and } \kappa \in \mathbb{R}^{(m, m)},
\]
\[
t_M(b) = ((t_0(b), 1), (0, 0; 0)) \quad \text{with any } b = t b \in \mathbb{R}^{(n, n)},
\]
\[
g_M(\alpha) = ((g_0(\alpha), 1), (0, 0; 0)) \quad \text{with any } \alpha \in GL(n, \mathbb{R}),
\]
\[
\sigma_{n, M} = ((\sigma_{n, 0}, 1), (0, 0; 0)),
\]
generate the group \( G_M \ltimes H^{(n, m)}_R \). We can show that the representation \( \tilde{R}_M \) is realized on the representation \( H(\chi_M) = L^2(\mathbb{R}^{(m, n)}) \) as follows: for each \( f \in L^2(\mathbb{R}^{(m, n)}) \) and \( x \in \mathbb{R}^{(m, n)} \), the actions of \( \tilde{R}_M \) on the generators are given by
\[
(4.5) \quad \left( \tilde{R}_M(h_t(\lambda, \mu; \kappa)) f \right)(x) = t e^{\pi i \sigma(M(\kappa + \mu; \lambda + 2x t^t \mu))} f(x + \lambda),
\]
We note that the restriction of $\omega$ of $G$ of Remark. In the case $h$ all more details about the Weil representation $\tilde{\omega}_n, Z$ The map $\tilde{\omega}_n, \omega$ is given by Formula (5.2). In other words, $\tilde{\omega}_n, \omega$ are realized on $L^2(JAE-HYUN YANG)$ for the Schrödinger-Weil representation $\tilde{\omega}_n, \omega$ is dealt in [1] and [9]. We refer to [5] and [6] for more details about the Weil representation $R_{2,M}$.

$$R_{2,M} = R_{2,M}^+ \oplus R_{2,M}^-,$$

where $R_{2,M}^+$ and $R_{2,M}^-$ are the even Weil representation and the odd Weil representation of $G$ that are realized on $L^2(JAE-HYUN YANG)$ and $L^2(JAE-HYUN YANG)$ respectively. Obviously the center $\mathcal{Z}_{2,M}$ of $G_{2,M}$ is given by

$$\mathcal{Z}_{2,M} = \{(I_{2n}, 1), (0, 0; \kappa) \in G_{2,M} \} \triangleq \text{Symm}(m, \mathbb{R}).$$

We note that the restriction of $\omega_M$ to $G_{2,M}$ coincides with $R_{2,M}$ and $\omega_M(h) = \mathcal{H}_M(h)$ for all $h \in H_{\mathbb{R}}^{(m,m)}$.

**Remark.** In the case $n = m = 1$, $\omega_M$ is dealt in [1] and [9]. We refer to [5] and [6] for more details about the Weil representation $R_{2,M}$.

### 5. Covariant Maps for the Schrödinger-Weil representation

As before we let $\mathcal{M}$ be a symmetric positive definite $m \times m$ real matrix. We define the mapping $\mathcal{F}(\mathcal{M}) : \mathbb{H}_{n,m} \rightarrow L^2(JAE-HYUN YANG)$ by

$$(5.1) \quad \mathcal{F}(\mathcal{M})(\Omega, Z)(x) = e^{\pi i \sigma(\mathcal{M} x \Omega^i x + 2 x^i Z)}, \quad (\Omega, Z) \in \mathbb{H}_{n,m}, \ x \in \mathbb{R}^{(m,m)}.$$  

For brevity we put $\mathcal{F}_{\Omega, Z}^{(\mathcal{M})} := \mathcal{F}(\mathcal{M})(\Omega, Z)$ for $(\Omega, Z) \in \mathbb{H}_{n,m}$.

We define the automorphic factor $J_M : G^J \times \mathbb{H}_{n,m} \rightarrow \mathbb{C}^\times$ for $G^J$ on $\mathbb{H}_{n,m}$ by

$$(5.2) \quad J_M(\tilde{g}, (\Omega, Z)) = e^{\pi i \sigma(\mathcal{M}(Z + \lambda \Omega^i + \mu)(C \Omega + D)^{-1} C^i (Z + \lambda \Omega^i + \mu))} e^{-\pi i \sigma(\mathcal{M}(\lambda \Omega^i + 2 \lambda^i Z + \kappa + \mu^i \lambda))} \det(C \Omega + D)^\frac{m}{2},$$

where $\tilde{g} = (g, (\lambda, \mu; \kappa)) \in G^J$ with $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(n, \mathbb{R})$, $(\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(m,m)}$ and $(\Omega, Z) \in \mathbb{H}_{n,m}$.

**Theorem 5.1.** The map $\mathcal{F}(\mathcal{M}) : \mathbb{H}_{n,m} \rightarrow L^2(JAE-HYUN YANG)$ defined by (5.1) is a covariant map for the Schrödinger-Weil representation $\omega_M$ of $G^J$ and the automorphic factor $J_M$ for $G^J$ on $\mathbb{H}_{n,m}$ defined by Formula (5.2). In other words, $J_M$ satisfies the following covariance relation.
\begin{equation}
\omega_M(\bar{g}) \mathcal{F}_{\Omega,Z}^{(M)} = J_M(\bar{g}, (\Omega, Z))^{-1} \mathcal{F}_{\bar{g}(\Omega, Z)}^{(M)}
\end{equation}
for all \( \bar{g} \in G^I \) and \((\Omega, Z) \in \mathbb{H}_{n,m} \).

**Proof.** For an element \( \tilde{g} = (g, (\lambda, \mu; \kappa)) \in G^I \) with \( g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R}) \), we put \((\Omega_s, Z_s) = \tilde{g} \cdot (\Omega, Z) \) for \((\Omega, Z) \in \mathbb{H}_{n,m} \). Then we have

\[
\begin{align*}
\Omega_s &= g \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1}, \\
Z_s &= (Z + \lambda \Omega + \mu)(C\Omega + D)^{-1}.
\end{align*}
\]

In this section we use the notations \( t_0(b) \), \( g_0(\alpha) \) and \( \sigma_{n,0} \) in Section 4. Since the following elements \( h(\lambda, \mu; \kappa) \), \( t(b) \), \( g(\alpha) \) and \( \sigma_n \) of \( G^I \) defined by

\[
\begin{align*}
h(\lambda, \mu; \kappa) &= (I_{2m}, (\lambda, \mu; \kappa)) \quad \text{with } \lambda, \mu \in \mathbb{R}^{(m,n)}, \kappa \in \mathbb{R}^{(m,m)}, \\
t(b) &= (t_0(b), (0,0;0)) \quad \text{with } b = t_b \in \mathbb{R}^{(m,m)}, \\
g(\alpha) &= (g_0(\alpha), (0,0;0)) \quad \text{with } \alpha \in GL(n, \mathbb{R}), \\
\sigma_n &= (\sigma_{n,0}, (0,0;0))
\end{align*}
\]
generate the Jacobi group, it suffices to prove the covariance relation (5.3) for the above generators.

**Case I.** \( \bar{g} = h(\lambda, \mu; \kappa) \) with \( \lambda, \mu \in \mathbb{R}^{(m,n)}, \kappa \in \mathbb{R}^{(m,m)} \).

In this case, we have

\[
\begin{align*}
\Omega_s &= \Omega, \\
Z_s &= Z + \lambda \Omega + \mu
\end{align*}
\]
and

\[
J_M(\bar{g}, (\Omega, Z)) = e^{-\pi i \sigma(\mathcal{M}(\lambda \Omega + \mu)^t \lambda \Omega + \mu)}.
\]

According to Formula (4.5), for \( x \in \mathbb{R}^{(m,n)} \),

\[
\begin{align*}
\left( \omega_M(h(\lambda, \mu; \kappa)) \mathcal{F}_{\Omega,Z}^{(M)} \right)(x) &= e^{\pi i \sigma(\mathcal{M}(\lambda \Omega + \mu)^t \lambda \Omega + \mu)} \mathcal{F}_{\Omega,Z}^{(M)}(x + \lambda) \\
&= e^{\pi i \sigma(\mathcal{M}(\lambda \Omega + \mu)^t \lambda \Omega + \mu)} e^{\pi i \sigma(\mathcal{M}(x + \lambda)^t (x + \lambda) + 2(x + \lambda)^t Z)}.
\end{align*}
\]

On the other hand, according to Formula (5.2), for \( x \in \mathbb{R}^{(m,n)} \),

\[
\begin{align*}
J_M(h(\lambda, \mu; \kappa), (\Omega, Z))^{-1} \mathcal{F}_{\bar{g}(\Omega, Z)}^{(M)}(x) &= J_M(h(\lambda, \mu; \kappa), (\Omega, Z))^{-1} \mathcal{F}_{\bar{g}(\Omega, Z)}^{(M)}(x) \\
&= e^{\pi i \sigma(\mathcal{M}((x+\lambda)^t (x+\lambda) + 2(x+\lambda)^t Z + \lambda \Omega + \mu))} e^{\pi i \sigma(\mathcal{M}(x + \lambda)^t (x + \lambda) + 2(x + \lambda)^t Z)}.
\end{align*}
\]

Therefore we prove the covariance relation (5.3) in the case \( \bar{g} = h(\lambda, \mu; \kappa) \) with \( \lambda, \mu, \kappa \) real.

**Case II.** \( \bar{g} = t(b) \) with \( b = t_b \in \mathbb{R}^{(n,n)} \).

\[
\begin{align*}
\end{align*}
\]
In this case, we have
\[ \Omega_* = \Omega + b, \quad Z_* = Z \quad \text{and} \quad J_M(\bar{g}, (\Omega, Z)) = 1. \]
According to Formula (4.6), we obtain
\[
\left( \omega_M(\bar{g}) \mathcal{F}_{\Omega,Z}^{(M)} \right)(x) = e^{\pi i \sigma(M x b \cdot x)} \mathcal{F}_{\Omega,Z}^{(M)}(x), \quad x \in \mathbb{R}^{(m,n)}.
\]
On the other hand, according to Formula (5.2), for \( x \in \mathbb{R}^{(m,n)} \), we obtain
\[
J_M(\bar{g}, (\Omega, Z))^{-1} \mathcal{F}_{\Omega,Z}^{(M)}(x) = (\det \alpha)^{-m/2}.
\]
Therefore we prove the covariance relation (5.3) in the case \( \bar{g} = t(b) \) with \( b = t b \in \mathbb{R}^{(n,n)} \).

**Case III.** \( \bar{g} = g(\alpha) \) with \( \alpha \in GL(n, \mathbb{R}) \).
In this case, we have
\[ \Omega_* = t \alpha \Omega \alpha, \quad Z_* = Z \alpha \]
and
\[ J_M(\bar{g}, (\Omega, Z)) = (\det \alpha)^{-m/2}. \]
According to Formula (4.7), for \( x \in \mathbb{R}^{(m,n)} \),
\[
\left( \omega_M(\bar{g}) \mathcal{F}_{\Omega,Z}^{(M)} \right)(x) = \frac{1}{2} \mathcal{F}_{\Omega,Z}^{(M)}(x^t \alpha) = (\det \alpha)^{-m} \cdot e^{\pi i \sigma(M x \cdot \alpha (x^t \alpha + 2 x^t \alpha t) \cdot Z)}.
\]
On the other hand, according to Formula (5.2), for \( x \in \mathbb{R}^{(m,n)} \),
\[
J_M(\bar{g}, (\Omega, Z))^{-1} \mathcal{F}_{\Omega,Z}^{(M)}(x) = (\det \alpha)^{-m/2} \cdot e^{\pi i \sigma(M x \cdot \alpha (x^t \alpha + 2 x^t \alpha t) \cdot Z)}.
\]
Therefore we prove the covariance relation (5.3) in the case \( \bar{g} = g(\alpha) \) with \( \alpha \in GL(n, \mathbb{R}) \).

**Case IV.** \( \bar{g} = \left( \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}, (0,0;0) \right) \).
In this case, we have
\[ \Omega_* = -\Omega^{-1}, \quad Z_* = Z \Omega^{-1} \]
and
\[ J_M(\bar{g}, (\Omega, Z)) = e^{\pi i \sigma(M Z \Omega^{-1} t) Z} \left( \det \Omega \right)^{-m/2}. \]
In order to prove the covariance relation (5.3), we need the following useful lemma.
Lemma 5.1. For a fixed element $\Omega \in \mathbb{H}_n$ and a fixed element $Z \in \mathbb{C}^{(m,n)}$, we obtain the following property

\begin{equation}
\int_{\mathbb{R}^{(m,n)}} e^{\pi i \sigma(x \Omega^t x + 2x^t Z)} dx_{11} \cdots dx_{mn} = \left(\det \frac{\Omega}{i}\right)^{-\frac{m}{2}} e^{-\pi i \sigma(Z \Omega^{-1} t Z)},
\end{equation}

where $x = (x_{ij}) \in \mathbb{R}^{(m,n)}$.

Proof of Lemma 5.1. By a simple computation, we see that

\[e^{\pi i \sigma(x \Omega^t x + 2x^t Z)} = e^{-\pi i \sigma(Z \Omega^{-1} t Z)} \cdot e^{\pi i \sigma\{(x+Z \Omega^{-1}) \Omega^t (x+Z \Omega^{-1})\}}.\]

Since the real Jacobi group $Sp(n, \mathbb{R}) \ltimes H^{(m,n)}_\mathbb{R}$ acts on $\mathbb{H}_{n,m}$ holomorphically, we may put

$$\Omega = iA^t A, \quad Z = iV, \quad A \in \mathbb{R}^{(n,n)}, \quad V = (v_{ij}) \in \mathbb{R}^{(m,n)}.$$

Then we obtain

\begin{align*}
\int_{\mathbb{R}^{(m,n)}} e^{\pi i \sigma(x \Omega^t x + 2x^t Z)} dx_{11} \cdots dx_{mn} &= e^{-\pi i \sigma(Z \Omega^{-1} t Z)} \int_{\mathbb{R}^{(m,n)}} e^{\pi i \sigma\{x+iV(A^t A)^{-1}\} (iA^t A)^t (x+iV(A^t A)^{-1})} dx_{11} \cdots dx_{mn} \\
&= e^{-\pi i \sigma(Z \Omega^{-1} t Z)} \int_{\mathbb{R}^{(m,n)}} e^{\pi i \sigma\{x+V(A^t A)^{-1}\} A^t A^t (x+V(A^t A)^{-1})} dx_{11} \cdots dx_{mn} \\
&= e^{-\pi i \sigma(Z \Omega^{-1} t Z)} \int_{\mathbb{R}^{(m,n)}} e^{-\pi \sigma\{(uA)^t (uA)\}} du_{11} \cdots du_{mn} \quad \text{(Put } u = x + V(A^t A)^{-1} = (u_{ij})\} \\
&= e^{-\pi i \sigma(Z \Omega^{-1} t Z)} \int_{\mathbb{R}^{(m,n)}} e^{-\pi \sigma(w^t w)} (\det A)^{-m} dw_{11} \cdots dw_{mn} \quad \text{(Put } w = uA = (w_{ij})\} \\
&= e^{-\pi i \sigma(Z \Omega^{-1} t Z)} (\det A)^{-m} \cdot \left(\prod_{i=1}^{m} \prod_{j=1}^{m} \int_{\mathbb{R}} e^{-\pi w_{ij}^2} dw_{ij}\right) \\
&= e^{-\pi i \sigma(Z \Omega^{-1} t Z)} (\det A)^{-m} \quad \text{(because } \int_{\mathbb{R}} e^{-\pi w_{ij}^2} dw_{ij} = 1 \text{ for all } i, j\}
\end{align*}

This completes the proof of Lemma 5.1. \hfill \Box

According to Formula (4.8), for $x \in \mathbb{R}^{(m,n)}$, we obtain

\begin{align*}
\left(\omega_{\mathcal{M}}(\bar{g}) \mathcal{F}_{\Omega, Z}^{(M)}\right)(x) &= \left(\frac{1}{i}\right)^\frac{mn}{2} \left(\det \mathcal{M}\right)^\frac{n}{2} \int_{\mathbb{R}^{(m,n)}} \mathcal{F}_{\Omega, Z}^{(M)}(y) e^{-2\pi i \sigma(M y^t x)} dy \\
&= \left(\frac{1}{i}\right)^\frac{mn}{2} \left(\det \mathcal{M}\right)^\frac{n}{2} \int_{\mathbb{R}^{(m,n)}} e^{\pi i \sigma(M(y \Omega^t y + 2y^t Z))} e^{-2\pi i \sigma(M y^t x)} dy \\
&= \left(\frac{1}{i}\right)^\frac{mn}{2} \left(\det \mathcal{M}\right)^\frac{n}{2} \int_{\mathbb{R}^{(m,n)}} e^{\pi i \sigma(M(y \Omega^t y + 2y^t (Z-x)))} dy.
\end{align*}
If we substitute \( u = M^{1/2} y \), then \( du = (\det M)^{\frac{1}{2}} dy \). Therefore according to Lemma 5.1, we obtain

\[
\left( \omega_M(\tilde{g}) \mathcal{F}_\Omega^M(x) \right) = \left( \frac{1}{i} \right)^{\frac{mn}{2}} \left( \det M \right)^{\frac{1}{4}} \int_{\mathbb{R}^{(m,n)}} e^{\pi i \sigma(u \Omega u' + 2M^{1/2} u'(Z-x))} (\det M)^{-\frac{n}{2}} du
\]

\[
= \left( \frac{1}{i} \right)^{\frac{mn}{2}} \int_{\mathbb{R}^{(m,n)}} e^{\pi i \sigma(u \Omega u + 2u'(M^{1/2}(Z-x)))} du
\]

\[
= \left( \frac{1}{i} \right)^{\frac{mn}{2}} \left( \det \Omega \right)^{-\frac{m}{2}} e^{-\pi i \sigma(M(\tilde{Z}-x)\Omega^{-1}t(\tilde{Z}-x))} (by \ Lemma \ 5.1)
\]

\[
= (\det \Omega)^{-\frac{m}{2}} e^{-\pi i \sigma(M(\tilde{Z}-x)\Omega^{-1}t(\tilde{Z}-x))}
\]

On the other hand, according to Formula (5.2), for \( x \in \mathbb{R}^{(m,n)} \),

\[
J_M(\tilde{g},(\Omega,Z))^{-1} \mathcal{F}_\bar{\Omega}^M(x)
= e^{-\pi i \sigma(M Z \Omega^{-1}t'Z)} (\det \Omega)^{-\frac{m}{2}} \mathcal{F}_\bar{\Omega}^M(\tilde{g},(\Omega,Z)-1)\Omega^{-1}Z\Omega^{-1}(x)
\]

\[
= (\det \Omega)^{-\frac{m}{2}} e^{-\pi i \sigma(M Z \Omega^{-1}t'Z + x\Omega^{-1}t'x - 2Z\Omega^{-1}t'x)}
\]

Therefore we prove the covariance relation (5.3) in the case \( \tilde{g} = \sigma_n \). Since \( J_M \) is an automorphic factor for \( G^J \) on \( \mathbb{H}_{n,m} \), we see that if the covariance relation (5.3) holds for for two elements \( \tilde{g}_1, \tilde{g}_2 \) in \( G^J \), then it holds for \( \tilde{g}_1\tilde{g}_2 \). Finally we complete the proof. \( \square \)

6. Construction of Jacobi Forms

Let \( (\pi, V_\pi) \) be a unitary representation of \( G^J \) on the representation space \( V_\pi \). We assume that \( (\pi, V_\pi) \) satisfies the following conditions (A) and (B):

(A) There exists a vector valued map

\[
\mathcal{F}: \mathbb{H}_{n,m} \rightarrow V_\pi, \quad (\Omega, Z) \mapsto \mathcal{F}_\Omega Z := \mathcal{F}(\Omega, Z)
\]

satisfying the following covariance relation

\[
(\pi(\tilde{g})\mathcal{F}_\Omega Z = \psi(\tilde{g}) J(\tilde{g},(\Omega,Z))^{-1} \mathcal{F}_{\tilde{g}\bar{\Omega}} Z) \quad \text{for all} \quad \tilde{g} \in G^J, \quad (\Omega, Z) \in \mathbb{H}_{n,m},
\]

where \( \psi \) is a character of \( G^J \) and \( J: G^J \times \mathbb{H}_{n,m} \rightarrow GL(1, \mathbb{C}) \) is a certain automorphic factor for \( G^J \) on \( \mathbb{H}_{n,m} \).

(B) Let \( \tilde{\Gamma} \) be an arithmetic subgroup of \( \Gamma^J \). There exists a linear functional \( \theta: V_\pi \rightarrow \mathbb{C} \) which is semi-invariant under the action of \( \tilde{\Gamma} \), in other words, for all \( \tilde{g} \in \tilde{\Gamma} \) and \( (\Omega, Z) \in \mathbb{H}_{n,m} \),

\[
\langle \pi(\tilde{g})\theta, \mathcal{F}_\Omega Z \rangle = \langle \theta, \pi(\tilde{g})^{-1} \mathcal{F}_{\tilde{g}\bar{\Omega}} Z \rangle = \chi(\tilde{g}) \langle \theta, \mathcal{F}_{\Omega Z} \rangle,
\]
where $\pi^*$ is the contragredient of $\pi$ and $\chi : \Gamma \to T$ is a unitary character of $\Gamma$.

Under the assumptions (A) and (B) on a unitary representation $(\pi, V_\pi)$, we define the function $\Theta$ on $\mathbb{H}_{n,m}$ by

$$
(6.3) \quad \Theta(\Omega, Z) := \langle \theta, \mathcal{F}_{\Omega,Z} \rangle = \theta(\mathcal{F}_{\Omega,Z}), \quad (\Omega, Z) \in \mathbb{H}_{n,m}.
$$

We now shall see that $\Theta$ is an automorphic form on $\mathbb{H}_{n,m}$ with respect to $\Gamma$ for the automorphic factor $J$.

**Lemma 6.1.** Let $(\pi, V_\pi)$ be a unitary representation of $G^J$ satisfying the above assumptions (A) and (B). Then the function $\Theta$ on $\mathbb{H}_{n,m}$ defined by (6.3) satisfies the following modular transformation behavior

$$
(6.4) \quad \Theta(\gamma \cdot (\Omega, Z)) = \psi(\gamma)^{-1} \chi(\gamma)^{-1} J(\gamma, (\Omega, Z)) \Theta(\Omega, Z)
$$

for all $\gamma \in \Gamma$ and $(\Omega, Z) \in \mathbb{H}_{n,m}$. 

**Proof.** For any $\gamma \in \Gamma$ and $(\Omega, Z) \in \mathbb{H}_{n,m}$, according to the assumptions (6.1) and (6.2), we obtain

$$
\begin{align*}
\Theta(\gamma \cdot (\Omega, Z)) &= \langle \theta, \mathcal{F}_{\gamma \cdot (\Omega, Z)} \rangle \\
&= \langle \theta, \psi(\gamma)^{-1} J(\gamma, (\Omega, Z)) \pi(\gamma) \mathcal{F}_{\Omega,Z} \rangle \\
&= \psi(\gamma)^{-1} J(\gamma, (\Omega, Z)) \langle \theta, \mathcal{F}_{\Omega,Z} \rangle \\
&= \psi(\gamma)^{-1} \chi(\gamma)^{-1} J(\gamma, (\Omega, Z)) \langle \theta, \mathcal{F}_{\Omega,Z} \rangle \\
&= \psi(\gamma)^{-1} \chi(\gamma)^{-1} J(\gamma, (\Omega, Z)) \Theta(\Omega, Z).
\end{align*}
$$

Now for a positive definite integral symmetric matrix $M$ of degree $m$, we define the holomorphic function $\Theta_M : \mathbb{H}_{n,m} \to \mathbb{C}$ by

$$
(6.5) \quad \Theta_M(\Omega, Z) := \sum_{\xi \in \mathbb{Z}^{(n,m)}} e^{\pi i \sigma(M(\xi^t \xi + 2 \xi^t Z))}, \quad (\Omega, Z) \in \mathbb{H}_{n,m}.
$$

**Theorem 6.1.** Let $M$ be a symmetric positive definite, unimodular even integral matrix of degree $m$. Then for any $\gamma = (\gamma, (\lambda, \mu; \kappa)) \in \Gamma^J$ with $\gamma \in \Gamma_n$ and $(\lambda, \mu; \kappa) \in H_{Z}^{(n,m)}$, the function $\Theta_M$ satisfies the functional equation

$$
(6.6) \quad \Theta_M(\gamma \cdot (\Omega, Z)) = \rho_{M}(\gamma) J_M(\gamma, (\Omega, Z)) \Theta_M(\Omega, Z), \quad (\Omega, Z) \in \mathbb{H}_{n,m},
$$

where $\rho_M(\gamma)$ is a uniquely determined character of $\Gamma^J$ with $|\rho_M(\gamma)|^2 = 1$ and $J_M : G^J \times \mathbb{H}_{n,m} \to \mathbb{C}^\times$ is the automorphic factor for $G^J$ on $\mathbb{H}_{n,m}$ defined by the formula (5.2).

**Proof.** For an element $\gamma = (\gamma, (\lambda, \mu; \kappa)) \in \Gamma^J$ with $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$ and $(\lambda, \mu; \kappa) \in H_{Z}^{(n,m)}$, we put $(\Omega_*, Z_*) = \gamma \cdot (\Omega, Z)$ for $(\Omega, Z) \in \mathbb{H}_{n,m}$. Then we have

$$
\begin{align*}
\Omega_* &= \gamma \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1}, \\
Z_* &= (Z + \lambda \Omega + \mu)(C\Omega + D)^{-1}.
\end{align*}
$$
We define the linear functional $\vartheta$ on $L^2(\mathbb{R}^{(m,n)})$ by

$$
\vartheta(f) = \langle \vartheta, f \rangle := \sum_{\xi \in \mathbb{Z}^{(m,n)}} f(\xi), \quad f \in L^2(\mathbb{R}^{(m,n)}).
$$

We note that $\Theta_M(\Omega, Z) = \vartheta(\mathcal{F}_{\Omega, Z}^{(M)})$. Since $\mathcal{F}^{(M)}$ is a covariant map for the Schrödinger-Weil representation $\omega_M$ by Theorem 5.1, according to Lemma 6.1, it suffices to prove that $\vartheta$ is semi-invariant for $\omega_M$ under the action of $\Gamma^J$, in other words, $\vartheta$ satisfies the following semi-invariance relation

$$
(6.7) \quad \left\langle \vartheta, \omega_M(\tilde{\gamma}) \mathcal{F}_{\Omega, Z}^{(M)} \right\rangle = \rho_M(\tilde{\gamma})^{-1} \left\langle \vartheta, \mathcal{F}_{0, Z}^{(M)} \right\rangle
$$

for all $\tilde{\gamma} \in \Gamma^J$ and $(\Omega, Z) \in H_{n,m}$.

We see that the following elements $h(\lambda, \mu; \kappa)$, $t(b)$, $g(\alpha)$ and $\sigma_n$ of $\Gamma^J$ defined by

$$
\begin{align*}
&h(\lambda, \mu; \kappa) = (I_{2\mu}, (\lambda, \mu; \kappa)) \text{ with } \lambda, \mu \in \mathbb{Z}^{(m,n)} \text{ and } \kappa \in \mathbb{Z}^{(m,m)}, \\
t(b) = (t_0(b), (0, 0; 0)) \text{ with any } b = t b \in \mathbb{Z}^{(n,n)}, \\
g(\alpha) = (g_0(\alpha), (0, 0; 0)) \text{ with any } \alpha \in GL(n, \mathbb{Z}), \\
\sigma_n = (s_n, 0, (0, 0; 0))
\end{align*}
$$

generate the Jacobi modular group $\Gamma^J$. Therefore it suffices to prove the semi-invariance relation (6.7) for the above generators of $\Gamma^J$.

**Case I.** $\tilde{\gamma} = h(\lambda, \mu; \kappa)$ with $\lambda, \mu \in \mathbb{Z}^{(m,n)}$, $\kappa \in \mathbb{Z}^{(m,m)}$.

In this case, we have $\Omega_* = \Omega$, $Z_* = Z + \lambda \Omega + \mu$ and

$$
J_M(\tilde{\gamma}, (\Omega, Z)) = e^{-\pi i \sigma(M(\lambda \Omega^t \lambda + 2 \lambda^t Z + \kappa + \mu^t \kappa))}.
$$

According to the covariance relation (5.3),

$$
\begin{align*}
&\left\langle \vartheta, \omega_M(\tilde{\gamma}) \mathcal{F}_{0, Z}^{(M)} \right\rangle \\
= &\left\langle \vartheta, J_M(\tilde{\gamma}, (\Omega, Z))^{-1} \mathcal{F}_{0, Z}^{(M)} \right\rangle \\
= &J_M(\tilde{\gamma}, (\Omega, Z))^{-1} \left\langle \vartheta, \mathcal{F}_{\Omega, Z+\lambda \Omega+\mu}^{(M)} \right\rangle \\
= &J_M(\tilde{\gamma}, (\Omega, Z))^{-1} \sum_{A \in \mathbb{Z}^{(m,n)}} e^{\pi i \sigma(M(A^t A + 2 A^t (Z + \lambda \Omega + \mu)))} \\
= &J_M(\tilde{\gamma}, (\Omega, Z))^{-1} \cdot e^{-\pi i \sigma(M(\lambda \Omega^t \lambda + 2 \lambda^t Z))} \\
&\times \sum_{A \in \mathbb{Z}^{(m,n)}} e^{2 \pi i \sigma(MA^t \mu)} e^{\pi i \sigma(M((A + \lambda) \Omega(A + \lambda) + 2 (A + \lambda)^t Z))} \\
= &e^{\pi i \sigma(M(\kappa + \mu^t \kappa))} \left\langle \vartheta, \mathcal{F}_{0, Z}^{(M)} \right\rangle.
\end{align*}
$$

Here we used the fact that $\sigma(MA^t \mu)$ is an integer. We put $\rho_M(\tilde{\gamma}) = \rho_M(h(\lambda, \mu; \kappa)) = e^{-\pi i \sigma(M(\kappa + \mu^t \kappa))}$. Therefore $\vartheta$ satisfies the semi-invariance relation (6.7) in the case $\tilde{\gamma} = h(\lambda, \mu; \kappa)$ with $\lambda, \mu \in \mathbb{Z}^{(m,n)}$, $\kappa \in \mathbb{Z}^{(m,m)}$. 
Case II. \( \tilde{\gamma} = t(b) \) with \( b = t'b \in \mathbb{Z}^{(n,n)} \).

In this case, we have

\[
\Omega_* = \Omega + b, \quad Z_* = Z \quad \text{and} \quad J_M(\tilde{\gamma}, (\Omega, Z)) = 1.
\]

According to the covariance relation (5.3), we obtain

\[
\langle \vartheta, \omega_M(\tilde{\gamma}) \mathcal{F}^{(M)}_{\Omega, Z} \rangle = \langle \vartheta, J_M(\tilde{\gamma}, (\Omega, Z))^{-1} \mathcal{F}^{(M)}_{\tilde{\gamma}(\Omega, Z)} \rangle = \langle \vartheta, \mathcal{F}^{(M)}_{\Omega + b, Z} \rangle = \sum_{A \in \mathbb{Z}^{(m,n)}} e^{\pi i \sigma \{ M(\Omega + b, Z) A + 2A^t A \}} = \langle \vartheta, \mathcal{F}^{(M)}_{\Omega, Z} \rangle.
\]

Here we used the fact that \( \sigma(MA b^t A) \) is an even integer. We put \( \rho_M(\tilde{\gamma}) = \rho_M(t(b)) = 1 \).

Therefore \( \vartheta \) satisfies the semi-invariance relation (6.7) in the case \( \tilde{\gamma} = t(b) \) with \( b = t'b \in \mathbb{Z}^{(n,n)} \).

Case III. \( \tilde{\gamma} = g(\alpha) \) with \( \alpha \in GL(n, \mathbb{Z}) \).

In this case, we have

\[
\Omega_* = t\alpha \Omega \alpha, \quad Z_* = Z\alpha
\]

and

\[
J_M(\tilde{\gamma}, (\Omega, Z)) = (\det \alpha)^{-\frac{m}{2}}.
\]

According to the covariance relation (5.3), we obtain

\[
\langle \vartheta, \omega_M(\tilde{\gamma}) \mathcal{F}^{(M)}_{\Omega, Z} \rangle = \langle \vartheta, J_M(\tilde{\gamma}, (\Omega, Z))^{-1} \mathcal{F}^{(M)}_{\tilde{\gamma}(\Omega, Z)} \rangle = \langle \vartheta, \mathcal{F}^{(M)}_{t\alpha \Omega \alpha, Z\alpha} \rangle = \langle \vartheta, \mathcal{F}^{(M)}_{t\alpha \Omega \alpha, Z\alpha} \rangle = \langle \vartheta, \mathcal{F}^{(M)}_{\Omega, Z} \rangle.
\]

Here we put \( \rho_M(\tilde{\gamma}) = \rho_M(g(\alpha)) = (\det \alpha)^{-\frac{m}{2}} \).

Therefore \( \vartheta \) satisfies the semi-invariance relation (6.7) in the case \( \tilde{\gamma} = g(\alpha) \) with \( \alpha \in GL(n, \mathbb{Z}) \).
Case IV. \( \tilde{\gamma} = \left( \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}, (0,0;0) \right) \).

In this case, we have
\[ \Omega_* = -\Omega^{-1}, \quad Z_* = Z \Omega^{-1} \]
and
\[ J_M(\tilde{\gamma}, (\Omega, Z)) = e^{\pi i \sigma(M Z \Omega^{-1} t Z)} (\det \Omega)^{\frac{m}{2}}. \]

In the process of the proof of Theorem 5.1, using Lemma 5.1, we already showed that
\[ (6.8) \int_{\mathbb{R}^{(m,n)}} e^{\pi i \sigma(M(y \Omega t y + 2y^t Z))} dy = (\det M)^{-\frac{m}{2}} (\det \Omega)^{-\frac{m}{2}} e^{-\pi i \sigma(M Z \Omega^{-1} t Z)}. \]
By (6.8), we see that the Fourier transform of \( \mathcal{F}_{\Omega, Z}^{(M)} \) is given by
\[ (6.9) \mathcal{F}_{\Omega, Z}^{(M)}(x) = (\det M)^{-\frac{m}{2}} (\det \Omega)^{-\frac{m}{2}} e^{-\pi i \sigma(M (Z-x) \Omega^{-1} t (Z-x))}. \]

According to the covariance relation (5.3), Formula (6.9) and Poisson summation formula, we obtain
\[ \langle \vartheta, \omega_M(\tilde{\gamma}) \mathcal{F}_{\Omega, Z}^{(M)} \rangle \]
\[ = \langle \vartheta, J_M(\tilde{\gamma}, (\Omega, Z))^{-1} \mathcal{F}_{\Omega, Z}^{(M)} \rangle \]
\[ = J_M(\tilde{\gamma}, (\Omega, Z))^{-1} \langle \vartheta, \mathcal{F}_{-\Omega^{-1} t, Z \Omega^{-1} t}^{(M)} \rangle \]
\[ = (\det \Omega)^{-\frac{m}{2}} e^{-\pi i \sigma(M Z \Omega^{-1} t Z)} \sum_{A \in \mathbb{Z}^{(m,n)}} e^{-\pi i \sigma(M (A \Omega^{-1} t A - 2A \Omega^{-1} t Z))} \]
\[ = (\det \Omega)^{-\frac{m}{2}} \sum_{A \in \mathbb{Z}^{(m,n)}} e^{-\pi i \sigma(M (Z \Omega^{-1} t Z + A \Omega^{-1} t A - 2A \Omega^{-1} t Z))} \]
\[ = (\det \Omega)^{-\frac{m}{2}} \sum_{A \in \mathbb{Z}^{(m,n)}} e^{-\pi i \sigma(M (Z-A) \Omega^{-1} t (Z-A))} \]
\[ = (\det \Omega)^{-\frac{m}{2}} (\det M)^{\frac{m}{2}} (\det \Omega)^{-\frac{m}{2}} \sum_{A \in \mathbb{Z}^{(m,n)}} \mathcal{F}_{\Omega, Z}^{(M)}(A) \quad (\text{by Formula (6.9)}) \]
\[ = (\det M)^{\frac{m}{2}} (\det \Omega)^{-\frac{m}{2}} \sum_{A \in \mathbb{Z}^{(m,n)}} \mathcal{F}_{\Omega, Z}^{(M)}(A) \quad (\text{by Poisson summation formula}) \]
\[ = (\det M)^{\frac{m}{2}} (-i)^{\frac{m}{2}} i^{\frac{m}{2}} \langle \vartheta, \mathcal{F}_{\Omega, Z}^{(M)} \rangle \]
\[ = (-i)^{\frac{m}{2}} i^{\frac{m}{2}} \langle \vartheta, \mathcal{F}_{\Omega, Z}^{(M)} \rangle. \]

Here we used the fact that \( \det M = 1 \) because \( M \) is unimodular. We put \( \rho_M(\tilde{\gamma}) = \rho_M(\sigma_n) = (-i)^{-\frac{m}{2}}. \) Therefore \( \vartheta \) satisfies the semi-inversion relation (6.7) in the case \( \tilde{\gamma} = \sigma_n. \) The proof of Case IV is completed. Since \( J_M \) is an automorphic factor for \( G' \) on \( \mathbb{H}_{m,n} \), we see that if the formula (6.6) holds for two elements \( \tilde{\gamma}_1, \tilde{\gamma}_2 \) in \( \Gamma' \), then it holds for \( \tilde{\gamma}_1 \tilde{\gamma}_2. \) Finally we complete the proof of Theorem 6.1. \( \square \)
Remark 6.1. For a symmetric positive definite integral matrix $M$ that is not unimodular even integral, we obtain a similar transformation formula like (6.6). If $m$ is odd, $\Theta_{M}(\Omega, Z)$ is a Jacobi form of a half-integral weight $\frac{m}{2}$ and index $\frac{\det M}{2}$ with respect to a suitable arithmetic subgroup $\Gamma_{\Theta, M}^J$ of $\Gamma^J$ and a character $\rho_{M}$ of $\Gamma_{\Theta, M}^J$.

For instance, we obtain the following:

Theorem 6.2. Let $M$ be a symmetric positive definite integral matrix of degree $m$ such that $\det(M) = 1$. Let $\Gamma_{1,2}$ be an arithmetic subgroup of $\Gamma_n$ generated by all the following elements

$$t(b) = \begin{pmatrix} I_n & b \\ 0 & I_n \end{pmatrix}, \quad g(\alpha) = \begin{pmatrix} t\alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \quad \sigma_n = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

where $b = t^t b \in \mathbb{Z}^{(n,n)}$ with even diagonal and $\alpha \in \mathbb{Z}^{(n,n)}$. We put

$$\Gamma_{1,2}^J := \Gamma_{1,2} \rtimes H_{\mathbb{Z}}^{(n,m)}.$$

Then $\Theta_{M}$ satisfies the transformation formula (6.6) for all $\tilde{\gamma} \in \Gamma_{1,2}^J$. Therefore $\Theta_{M}$ is a Jacobi form of weight $\frac{m}{2}$ with level $\Gamma_{1,2}$ and index $\frac{\det M}{2}$ for the uniquely determined character $\rho_{M}$ of $\Gamma_{1,2}^J$.

Proof. The proof is essentially the same as the proof of Theorem 6.1. We leave the detail to the reader. □

References

[1] R. Berndt and R. Schmidt, Elements of the Representation Theory of the Jacobi Group, Birkhäuser, 1998.
[2] M. Eichler and D. Zagier, The Theory of Jacobi Forms, Progress in Math., 55, Birkhäuser, Boston, Basel and Stuttgart, 1985.
[3] E. Freitag, Siegelsche Modulfunktionen, Grundlehren de mathematischen Wissenschaften 55, Springer-Verlag, Berlin-Heidelberg-New York (1983).
[4] E. Hecke, Herleitung des Euler-Produktes der Zetafunktion und einiger L-Reihen aus ihrer Funktionalgleichung, Math. Ann. 119 (1944), 266-287 (=Werke, 919-940).
[5] S. Gelbart, Weil's Representation and the Spectrum of the Metaplectic Group, Lecture Notes in Math., 530, Springer-Verlag, Berlin and New York, 1976.
[6] M. Kashiwara and M. Vergne, On the Segal-Shale-Weil Representations and Harmonic Polynomials, Invent. Math. 44 (1978), 1-47.
[7] G. Lion and M. Vergne, The Weil representation, Maslov index and Theta series, Progress in Math., 6, Birkhäuser, Boston, Basel and Stuttgart, 1980.
[8] G. W. Mackey, Induced Representations of Locally Compact Groups I, Ann. of Math., 55 (1952), 101-139.
[9] J. Marklof, Pair correlation densities of inhomogeneous quadratic forms, Ann. of Math., 158 (2003), 419-471.
[10] D. Mumford, Tata Lectures on Theta I, Progress in Math. 28, Boston-Basel-Stuttgart (1983).
[11] G. Shimura, On modular forms of half integral weight , Ann. of Math., 97 (1973), 440-481; Collected Papers, 1967-1977, Vol. II, Springer-Verlag (2002), 532-573.
[12] C. L. Siegel, Indefinite quadratische Formen und Funktionentheorie I and II, Math. Ann. 124 (1951), 17-54 and Math. Ann. 124 (1952), 364-387; Gesammelte Abhandlungen, Band III, Springer-Verlag (1966), 105-142 and 154-177.
[13] A. Weil, *Sur certains groupes d'operateurs unitaires*, Acta Math., 111 (1964), 143–211; Collected Papers (1964-1978), Vol. III, Springer-Verlag (1979), 1-69.
[14] J.-H. Yang, *Harmonic Analysis on the Quotient Spaces of Heisenberg Groups*, Nagoya Math. J., 123 (1991), 103–117.
[15] J.-H. Yang, *Harmonic Analysis on the Quotient Spaces of Heisenberg Groups II*, J. Number Theory, 49 (1) (1994), 63–72.
[16] J.-H. Yang, *A decomposition theorem on differential polynomials of theta functions of high level*, Japanese J. of Mathematics, the Mathematical Society of Japan, New Series, 22 (1) (1996), 37–49.
[17] J.-H. Yang, *Fock Representations of the Heisenberg Group $H_{g,h}^R$*, J. Korean Math. Soc., 34, no. 2 (1997), 345–370.
[18] J.-H. Yang, *Lattice Representations of the Heisenberg Group $H_{g,h}^R$*, Math. Annalen, 317 (2000), 309–323.
[19] J.-H. Yang, *The Siegel-Jacobi Operator*, Abh. Math. Sem. Hamburg 63 (1993), 135–146.
[20] J.-H. Yang, *Remarks on Jacobi forms of higher degree*, Proc. of the 1993 Workshop on Automorphic Forms and Related Topics, the Pyungsan Institute for Mathematical Sciences, Seoul (1993), 33–58.
[21] J.-H. Yang, *Singular Jacobi forms*, Trans. of American Math. Soc. 347, No. 6 (1995), 2041-2049.
[22] J.-H. Yang, *Construction of vector valued modular forms from Jacobi forms*, Canadian J. of Math. 47 (6) (1995), 1329-1339.
[23] J.-H. Yang, *A note on a fundamental domain for Siegel-Jacobi space*, Houston Journal of Mathematics, Vol. 32, No. 3 (2006), 701–712.
[24] J.-H. Yang, *Invariant metrics and Laplacians on Siegel-Jacobi space*, Journal of Number Theory, 127 (2007), 83–102 or [arXiv:math.NT/050215](http://arxiv.org/abs/math.NT/050215).
[25] J.-H. Yang, *A partial Cayley transform of Siegel-Jacobi disk*, J. Korean Math. Soc. 45, No. 3 (2008), 781-794.
[26] C. Ziegler, *Jacobi Forms of Higher Degree*, Abh. Math. Sem. Hamburg 59 (1989), 191–224.

Department of Mathematics, Inha University, Incheon 402-751, Korea
E-mail address: jhyang@inha.ac.kr