Design of Pressure Actuated Cellular Structures

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Abstract

A novel concept for pressure actuated cellular structures was published by Pagitz et al. 2012 Bioinspir. Biomim. 7. The corresponding mathematical foundation for the simulation and optimization of compliant cellular structures with central cell corner hinges can be found in Pagitz and Hühne 2014 arXiv:1403.2197. The aim of this article is fourfold. First, analytical expressions for optimal material properties of compliant cellular structure are presented. Second, it is shown that, for given target shapes and stiffness requirements, a cellular structure can be either made from a large number of small and highly pressurized cells that consist of a stiff material or from a small number of large and lowly pressurized cells that consist of a soft material. Third, extensions to the previously published numerical model are presented and their application ranges are determined. Fourth, end cap designs for prismatic cells are developed that can withstand substantial differential pressures while being flexible enough to allow large cross sectional shape changes.

Notation

| Symbol | Description |
|--------|-------------|
| $A_c, A_t$ | cross sectional area of cell, end cap tendon |
| $E$ | Young’s modulus |
| $F$ | axial cell side force |
| $L$ | cell side length |
| $M, M_0$ | bending moment of cell side, at cell corner |
| $M_m, M_p$ | cell side bending moment due to end moments, pressure |
| $T$ | transverse forces at cell side ends |
| $V, V_0$ | volume, initial volume of a single cell, end cap |
| $W, W_0$ | section modulus of cell side, at cell corner |
1 Introduction

The multifunctionality and relative simplicity of plant cells is a marvel. Unlike humans and animals, plants do not possess a centralized skeleton and complex control system. Yet they can create their own food through photosynthesis, reproduce, carry considerable external loads and in some cases are even capable of rapid movements. This article is concerned with the nastic movement of plants as known, for example, from Mimosa pudica and Dionaea muscipula. Hence the focus of this article is on the cell geometries and material properties of cell walls.

Figure 1: (a) Neighbouring plant cells are connected to each other via a middle lamella. (b) Cell walls are fiber reinforced composite structures.

A plant cell is surrounded by a cell wall that contains turgor pressures of up to
5 MPa [10]. Prestressing cell walls increases their compression strength which has a positive impact on the plants overall stiffness. Hence it could be argued that plants possess a decentralized skeleton that is formed by a large number of connected pressure vessels. In order to withstand the relatively high tensile stresses, cell walls are made from a composite material as shown in Figure 1. Most of the tensile stresses are carried by the primary cell wall. It is made from a network of microfibrils that are connected to each other by hemicellulose tethers. This network is embedded in a pectin matrix, a hydrated gel that pushes the microfibrils apart, thus easing their sideways motion [2]. Hence, plants can continuously optimize the material properties of their cell walls and thus maximize their stiffness and minimize stress peaks for given loads [5]. Furthermore, the Young’s modulus of cell walls increases with the applied stresses and volume changes of cells due to elastic deformations are negligible [11].

Figure 2: Pressure actuated cellular structures that are made from two separately pressurized cell levels for (a) one- and (b) two-dimensional target shapes.

The nastic movement of plants is driven by pressure changes that require a water flow between neighbouring cells [3]. Skotheim and Mahadevan [9] found that the speed of plant movements increases for decreasing cell sizes and pumping distances [4]. Therefore it is best if water fluxes occur mainly between neighbouring cell layers. Based on these observations, Pagitz et al. developed a novel concept for pressure actuated cellular structures that can alter their shape between any given set of one- [6], [8] and two-dimensional [7] functions. An example of a cellular structure for two one- and two-dimensional target shapes is shown in Figure 2. It can be seen that both structures consist of two layers of equally pressurized cells. Changing the pressure ratio between cell layers alters the shape of the structure. Computing the length and thickness of each cell side allows the creation of structures that can take up desired target shapes for given cell pressures and material properties. Pressure actuated cellular structures possess a large shape changing capability and a high strength to self-weight ratio, energy efficiency. Hence their potential application ranges from passenger seats, hospital beds to leading, trailing edges of aircraft.

Nastic movements are only a small aspect of a plants capabilities. Hence the material and construction principles of a cell are influenced by numerous alternative objectives. In contrast, an exclusive focus on the nastic movement reduces the num-

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1 Motor cells of Mimosa pudica have a diameter of about 20 µm.
ber of objectives and thus increases the range of potential cell-sizes, -materials and -pressures. The remainder of this article is organized as follows: Section 2 investigates the bending of cell sides at cell corners as well as the thickness distribution of cell sides due to differential pressures. Section 3 derives analytical expressions for optimal material properties of cells. Section 4 studies the relation between cell-size, -material and -pressure. Section 5 presents advanced numerical models for the simulation and optimization of compliant pressure actuated cellular structures and investigates their applicability and accuracy. Section 6 introduces a design for end caps of prismatic cells that can withstand large differential pressure while being flexible enough to allow large cross sectional shape changes. Section 7 concludes the paper.

2 Compliant Cellular Structures

2.1 Numerical Models

Cell sides of pressure actuated cellular structures experience large axial tensile stresses. In addition, sides between cell layers or at the boundary of a cellular structure are subjected to differential pressures so that they require an increased central bending stiffness. Both, the axial tensile stresses and the increased central cell side thicknesses promote regions around cell corners where the bending energy is concentrated. This localization allows the use of numerical models that are based on bars and hinges as shown in Figure 3.

![Figure 3](a) Compliant cellular structure. (b) Bending strains are concentrated in regions around cell corners. (c) Numerical model is based on bars and hinges.

Depending on the geometry, pressures and material of a cellular structure it is possible to use numerical models of varying complexity as illustrated in Figure 4. Rigid cell sides and central, frictionless hinges at cell corners can be used if the ratio between pressure and elastic energy of a cellular structure is large. A decreasing ratio between pressure and bending energy requires the consideration of rotational springs. This can be done in a first step by assuming rotational springs around central cell corner hinges and, in a second step by additionally taking into account hinge eccentricities. Furthermore, a decreasing ratio between pressure and axial strain energy requires the consideration of axial strains. A detailed study of the various numerical models and their impact on the equilibrium shapes of pressure actuated cellular structure can be found in Section 5.
2.2 Interface Between Compliant Structure and Numerical Model

The geometry of a compliant cellular structure is relatively complex. As a consequence, it is necessary to use a detailed finite element model for each cell corner to obtain accurate values for the hinge eccentricities and rotational springs in order to set up the numerical model. The problem is that the geometry of a compliant cellular structure varies during form finding. Hence it would be necessary to update the geometry and to recompute the parameters of the numerical model after each iteration. It is obvious that such an approach is not practical. This problem can be solved by using an interface model that consists of stiff and elastic rectangular regions as shown in Figure 5. This model is used throughout the simulation, optimization of a cellular structure. The geometry of the compliant structure is determined in a postprocessing step such that it matches the stiffness, eccentricity and maximum stresses of the interface model at all cell corners. Hence it is possible to decouple the complex geometry of a compliant cellular structure from the interface, numerical model without sacrificing accuracy.

Figure 4: Numerical models for varying ratios between pressure and elastic energy. Ratio between pressure and bending energy increases from left to right. Ratio between pressure and axial strain energy increases from top to bottom. A comprehensive theory for (a) and (b) can be found in Pagitz & Hühne [8].
Figure 5: A model that consists of stiff and elastic rectangular regions is used as an interface between compliant structure and numerical model.

### 2.3 Cell Corners

Figure 5 shows that the hinge eccentricities $r$ at cell corners as well as the aspect ratios $\mu$ of the elastic zones are assumed to be constant throughout the structure. Since elastic regions are not allowed to overlap it is possible to determine a smallest possible eccentricity that is proportional to

$$r \propto t_0.$$  \hfill (1)

By using the Euler-Bernoulli beam theory, the rotational spring stiffness $k_b$ of the numerical model can be written as

$$k_b = \frac{Et_0^3}{12\mu}$$ \hfill (2)

where $E$ is the Young’s modulus and $t_0$ the cell side thickness at a cell corner, Figure 5. Hence the corresponding bending moment $M_0$ for a bending angle $\theta_b$ is

$$M_0 = k_b\theta_b = \frac{Et_0^3}{12\mu} \theta_b$$ \hfill (3)

so that the absolute axial $\sigma_{a_0}$ and maximum bending $\sigma_{b_0}$ stresses of an elastic region are

$$\sigma_{a_0} = \frac{|F|}{t_0} \quad \text{and} \quad \sigma_{b_0} = \frac{|M_0|}{W_0} = \frac{E}{2\mu} |\theta_b| \quad \text{where} \quad W_0 = \frac{t_0^3}{6}$$ \hfill (4)

Therefore, the required cell side thickness at a cell corner is
\[ t_0 = \frac{2\mu|F|}{2\mu\sigma_y - E|\theta_b|} \]  

where \( \sigma_y = \sigma_{a0} + \sigma_{b0} \) is the yield strength of the considered material. The bending moment and energy of an elastic region can be written as

\[ M_0 = \frac{\mu E F^2 \theta_b}{3(2\mu\sigma_y - E|\theta_b|)^2} \quad \text{and} \quad \Pi_0 = \frac{M_0 \theta_b}{2} = \frac{\mu E F^2 \theta_b^2}{6(2\mu\sigma_y - E|\theta_b|)^2}. \]  

### 2.4 Cell Sides

Based on the model shown in Figure 7, the thickness distribution of a single cell side is subsequently derived. Equilibrium requires that

\[ T_1 = (L - 2r) \frac{p}{2} \left( \frac{M_{0.1} + M_{0.2}}{L - 2r} \right) \quad \text{and} \quad T_2 = (L - 2r) \frac{p}{2} \left( \frac{M_{0.1} + M_{0.2}}{L - 2r} \right). \]  

The bending moment \( M(\xi) \) along an intermediate cell side due to end moments and a differential pressure \( p \) is

\[ M(\xi) = M_p(\xi) + M_m(\xi) = \frac{p}{2} (L - 2r)^2 (1 - \xi) \xi + M_{0.1}(\xi - 1) + M_{0.2} \xi \]
where $\xi \in [0, 1]$. The corresponding absolute axial and maximum bending stresses are

$$\sigma_a(\xi) = \frac{|F|}{t(\xi)} \quad \text{and} \quad \sigma_b(\xi) = \frac{|M(\xi)|}{W(\xi)} = \frac{|M(\xi)|}{t(\xi)^2}$$

where $W(\xi) = \frac{t(\xi)^2}{6}$. \hspace{1cm} (9)

Hence the required cell side thickness is

$$t(\xi) = \frac{|F| + \sqrt{F^2 + 24|M(\xi)|\sigma_y}}{2\sigma_y}.$$ \hspace{1cm} (10)

All cell sides that are subjected to a non-zero differential pressure possess a varying thickness. However, an average thickness $t^*$ can be computed that possesses the same axial stiffness

$$t^* = \frac{\int_{\xi=0}^{1} \sigma_a(\xi) \, d\xi}{\int_{\xi=0}^{1} \frac{dF}{t(\xi)}}.$$ \hspace{1cm} (11)

Therefore, the elastic energy $\Pi_a$ due to axial strains is

$$\Pi_a = \frac{(L - 2r) F^2}{2Et^*}. \hspace{1cm} (12)$$

3 Cell Material

3.1 Candidate Materials

Potential materials for pressure actuated cellular structures range from plastics over fiber reinforced plastics to metals. The yield strength $\sigma_y$ and Young’s modulus $E$ of various materials is shown in Figure 8. The material parameters that are subsequently considered are dyed in black and summarized in Table 1. Note that these materials possess outstandingly large $\sigma_y/E$ ratios.

![Figure 8: Yield strength $\sigma_y$ versus Young’s modulus $E$ of various materials.](image)

| Material       | $E$ [MPa] | $\sigma_y$ [MPa] |
|----------------|-----------|-----------------|
| TPU            | 1,000     | 50              |
| PPSU           | 2,340     | 70              |
| PES            | 2,800     | 90              |
| PEI            | 3,200     | 105             |
| PA6 GF25       | 6,500     | 130             |
| PA66 GF25      | 7,250     | 145             |
| PEI GF30       | 9,500     | 165             |
| PEEK CF30      | 13,000    | 224             |
| Ti10V 2Fe3Al   | 106,000   | 1,105           |
| X2NiCoMo1895   | 193,000   | 1,815           |

Table 1: Considered materials with large $\sigma_y/E$ ratios.

The yield stresses of the considered materials\footnote{It should be noted that the material parameters were not corrected for the plane strain condition that is dominant in prismatic cellular structure. See Appendix A for further details.} can be interpolated with a power law so that

\begin{align*}
\text{Elastic energy due to axial strains:} \quad \Pi_a &= \frac{(L - 2r) F^2}{2Et^*}. \hspace{1cm} (12) \end{align*}
where $\chi \in [0, 1]$ is a stress safety factor. Based on a least square interpolation the constants $a, b$ result in

$$a = 0.3388$$
$$b = 0.6946.$$

(14)

### 3.2 Optimal Material Properties at Cell Corners

Based on the previous interpolation it is possible to obtain expressions that directly relate the geometry and loading of a cell side to optimal material parameters. The bending energy $\Pi_b$ of an elastic region is

$$\Pi_b = \frac{\mu F^2 \theta_b^2}{6E (2\mu \chi a E_b^{b-1} - |\theta_b|)^2}$$

(15)

which is minimal if

$$\frac{\partial \Pi_b}{\partial E} = \frac{\mu F^2 \theta_b^2 (|\theta_b| - 2 \mu \chi a (2b-1) E_b^{b-1})}{6E^2 (2\mu \chi a E_b^{b-1} - |\theta_b|)^3} = 0$$

(16)

so that

$$E = \left(\frac{|\theta_b|}{2 \mu \chi a (2b-1)}\right)^{\frac{1}{b}} \propto |\theta_b|^{-3.27}$$

(17)

and

$$\Pi_b = \frac{\mu F^2}{24 (1-b)^2} \left( \frac{|\theta_b|}{2 \mu \chi a (2b-1)^{2b-1}} \right)^{\frac{1}{1-b}} \propto |\theta_b|^{3.27}.$$  

(18)

Hence the bending moment that acts on an elastic region is

$$M_0 = \text{sgn} (\theta_b) \frac{\mu F^2}{12 (1-b)^2} \left( \frac{|\theta_b|^b}{2 \mu \chi a (2b-1)^{2b-1}} \right)^{\frac{1}{1-b}} \propto |\theta_b|^{2.27}$$

(19)

and the required cell side thickness at a cell corner for an axial force $F$ and bending angle $\theta_b$ results in

$$t_0 = \frac{F}{1-b} \left( \frac{|\theta_b|^b}{2 \mu \chi a (2b-1)^b} \right)^{\frac{1}{1-b}} \propto |\theta_b|^{2.27}.$$  

(20)

Previous three equations are illustrated in Figure 9. It can be seen that the optimal Young’s modulus decreases exponentially for increasing bending angles. Furthermore,
the bending energy and the required cell side thickness decrease exponentially for decreasing bending angles. Given a cellular structure, each elastic zone usually possesses a different bending angle and axial force. Hence it would be optimal to build each cell side from two different materials. If only one material is used throughout the structure it is best to choose the optimal material for the largest bending angle that occurs in the structure. This procedure is driven by the fact that elastic regions with large bending angles are dominant since $\Pi_0 \propto |\theta_b|^3$.

$$\chi = \frac{1}{2}$$

$X_2NiCoMo1895$

$Ti10V2Fe3Al$

$PEEK CF30$

$PEI$

$TPU$

$$\lim_{n_b \to \infty} \frac{n_s (2n_b)}{n_s (n_b)} = 2$$

where

$$n_s = n_b (1 + 3n_r) + \frac{n_r}{2} (5 - 3n_r).$$

In general, not all cell side lengths and bending angles are exactly halved. However, it is an acceptable assumption in order to get a good overview of the underlying physics.

4 Scaling of Cellular Structures

4.1 Geometry and Pressure

Choosing a certain number of base pentagons for a given set of target shapes and stiffness requirements is an important aspect in designing pressure actuated cellular structures. The effect of a varying number of base pentagons on the geometry, potential energies and weight of a cellular structure is subsequently investigated. It is assumed that all considered structures

- are capable of adopting the same target shapes
- possess the same overall stiffness
- are made from an optimal material that minimizes bending energy.

Two cellular structures with seven and fourteen base pentagons are shown in Figure 10 for the same two target shapes. It can be seen that doubling the number of base pentagons approximately halves the cell side lengths and the bending angles. Furthermore, the total number of cell sides is doubled since

$$\lim_{n_b \to \infty} \frac{n_s (2n_b)}{n_s (n_b)} = 2$$

where

$$n_s = n_b (1 + 3n_r) + \frac{n_r}{2} (5 - 3n_r).$$
Note that \( n_r, n_s, n_b \) are the number of cell rows, cell sides, base pentagons. Therefore it can be concluded that scaling does not affect the sum of all cell side lengths.

\[ L^s = \eta L, \quad \theta^b_s = \eta \theta^b, \quad \text{and} \quad p^s = \eta^{-2} p. \]  

Thus the scaled cell side forces are

\[ F^s = \eta^{-1} F \quad \text{since} \quad F \propto pL \]

and the scaled pressure energy of a single cell \( \Pi^s_p \) and the whole structure \( \Pi^{g,s}_p \) are

\[ \Pi^s_p = \Pi_p \quad \text{and therefore} \quad \Pi^{g,s}_p = \eta^{-1} \Pi^s_p. \]

\section*{4.2 Optimal Properties of Scaled Structures}

Inserting the scaled expressions for bending angles and cell side forces into the equations derived in Section 3 leads to

\[ E^s = \eta^{-4} E \propto \eta^{-3.27} \quad \text{and} \quad t^s_0 = \eta^{\frac{1}{4}} t_0 \propto \eta^{1.27} \]

\[ M^s_0 = \eta^{\frac{3}{4} - 2} M_0 \propto \eta^{0.27} \quad \Pi^s_0 = \eta^{\frac{3}{4} - 1} \Pi_0 \propto \eta^{1.27} \]

and thus

\[ \sigma^s_y = \eta^{\frac{1}{4} - 2} \sigma_y \propto \eta^{-2.27} \quad \text{and} \quad r^s = \eta^{\frac{3}{4} - 1} r \propto \eta^{1.27}. \]  

It can be seen that the decrease of side thicknesses at cell corners is superlinear for an increasing number of base pentagons. Hence, from a geometrical point of view, it
is possible to arbitrarily reduce cell sizes without compromising the overall structural stiffness. The global bending energy $\Pi^{0,s}_b$ of a scaled structure results in

$$\Pi^{0,s}_b = \eta^{-1} \Pi^s_b = \eta^{-1/6} \Pi_b \propto \eta^{0.27}$$  \hspace{1cm} (27)

since the number of elastic regions is inverse proportional to the cell sizes.

The scaled moment $M^s(\xi)$ that acts along a cell side is

$$M^s(\xi) = M_p(\xi) + \eta^{b-2} M_m(\xi)$$  \hspace{1cm} (28)

so that its bounds are

$$\eta^{0.27} \leq \frac{M^s(\xi)}{M^s(\xi)} \leq 1.$$  \hspace{1cm} (29)

The scaled thickness distribution along a cell side for the upper bound of the bending moment is

$$t^s(\xi) = \left|\frac{|F| + \sqrt{F^2 + 24|M(\xi)| \eta^{b-2} \sigma_y}}{2\eta^{2/3} \sigma_y}\right|$$  \hspace{1cm} (30)

which is bounded by

$$t^s(\xi) = \begin{cases} \eta^{b-2} t(\xi) \propto \eta^{1.27} & \text{if } M(\xi) = 0 \\ \eta^{b-2} t(\xi) \propto \eta^{1.14} & \text{if } F = 0. \end{cases}$$  \hspace{1cm} (31)

Hence it can be concluded that

$$t^s(\xi) = \eta^c t(\xi) \quad \text{where } 1.14 \leq c \leq 1.27$$  \hspace{1cm} (32)

and thus

$$\Pi^s_a = \eta^c \Pi_a \quad \text{where } 1 \leq c \leq 1.13$$  \hspace{1cm} (33)

so that the global axial energy $\Pi^{0,s}_a$ is

$$\Pi^{0,s}_a = \eta^{-1} \Pi^s_a = \eta^{c-1} \Pi_a.$$  \hspace{1cm} (34)

However, it should be noted that axial cell side strains decrease for a decreasing scaling factor due to increasing cell side forces. In summary it can be said that an increasing number of base pentagons reduces the volume of the structure and increases the ratio between pressure and elastic energy. This might be an explanation for the small cell sizes and relatively large pressures that are found in nastic plants. Figure\[12\] illustrates the previous results.
Altering the number of base pentagons of a cellular structure for a given set of target shapes and structural stiffness requires changes in geometry, material selection and cell pressures. These changes and their overall impact are subsequently summarized. Doubling the number of base pentagons approximately...

(a) ...quarters the cross sectional area of each cell and halves the total cross sectional area of the cellular structure. Cell side lengths are halved but the total number of cell sides is doubled. Furthermore, bending angles are halved.

(b) ...requires a fourfold increase of cell pressures in order to maintain the pressure potential of single cells and thus to preserve the overall structural stiffness.

(c) ...affects the optimal material for a cellular structure. This is caused by the twofold increase of cell side forces, a reduction of cell side thicknesses by 55-59% and halved bending angles. Optimal materials for a structure with a larger number of base pentagons have a higher Young’s modulus and yield strength. Hence, for given stiffness requirements and target shapes, a cellular structure can either be made from a few rubber cells or a large number of steel cells.

(d) ...does not considerably affect the total weight of a cellular structure. This is due to the fact that a material with a larger Young’s modulus usually possesses a larger density.

(e) ...doubles the pressure potential while the bending energy reduces by about 17% and the axial strain energy by about 0-9%. Hence the ratio between pressure and elastic energy potential increases by more than a factor of two. Consequently, the bending energy of a cellular structure can be neglected if it is made from a large number of base pentagons.
4.3 Pressure Medium

Reducing cell sizes while preserving the overall structural stiffness requires increasing cell pressures which has an impact on the optimal pressure medium. Gasses are the best choice for large cell volumes and small pressures whereas incompressible liquids are better for small volumes and large pressures. For example, the pressure of a given mass of gas for a varying volume $V$ is

$$p = p_0 \frac{V_0}{V}$$

(35)

where $p_0$ and $V_0$ are the reference pressure and volume. Hence the change of pressure potential due to a volume change from $V_0$ to $V_1$ is

$$\Delta \Pi_p = \int_{V_0}^{V_1} pdV = p_0 V_0 \int_{V_0}^{V_1} \frac{1}{V} dV = p_0 V_0 \ln \left( \frac{V_1}{V_0} \right).$$

(36)

In contrast, the pressure potential of an incompressible hydraulic fluid is always zero. Furthermore, the weight of air at 30 MPa exceeds 300 kg/m$^3$ which is about one third of the weight of common hydraulic fluids. Therefore, the compressibility of gasses limits the usable upper pressure for most applications to about 1 MPa due to safety and energy efficiency concerns. In contrast, hydraulic systems are often operated at pressures of up to 70 MPa.

5 Application Range of Numerical Models

5.1 Example Structure

The influences of axial cell side strains, compliant hinges and hinge eccentricities on the equilibrium configurations of a cellular structure are subsequently studied. This is done by using an example structure that consists of two cell rows and 60 base pentagons as shown in Figure 12. It can be seen that all pentagonal and hexagonal cells are identical. For given cell pressures, rigid cell sides and frictionless central cell corner hinges the cell side lengths are chosen such that the structure deforms approximately into a half, full circle. The reference configuration is chosen in between both equilibrium configurations in order to minimize variations of cell corner angles. Axial cell side strains, compliant hinges or hinge eccentricities affect the subtended angles of both equilibrium configurations. Hence, the deviations $\Delta \varphi_\pi, \Delta \varphi_2\pi$ from the original subtended angles $\varphi_\pi = \pi, \varphi_2\pi = 2\pi$ can be used to quantify their influence.

5.2 Axial Stiffness

Axial cell side strains alter the cell side lengths and thus change the subtended angles of both equilibrium configurations. The numerical model used to simulate their influence is shown in Figure 13. It is based on elastic bars and central frictionless cell corner hinges. An average thickness $t^*$ is determined for cell sides with a varying thickness. It can be seen that the deviations from the full circle $|\Delta \varphi_2\pi|$ are larger than from the half circle $|\Delta \varphi_\pi|$. Furthermore, both deviations decrease for an increasing Young’s modulus and a decreasing safety factor $\chi$.

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5 An ideal gas law and constant temperature is assumed.
Figure 12: Example structure consists of two cell rows with identical pentagonal, hexagonal cells. Structure deforms approximately into half, full circle for given cell pressures, frictionless central cell corner hinges and rigid cell sides.

Figure 13: (a) Example structure with frictionless central cell corner hinges and elastic cell sides. (b) Deviations from original equilibrium configurations for varying Young’s modulus.
The sensitivity to axial strains can be reduced by individually foreshortening all cell sides. This is illustrated in Figure 14 for a single cell side that is subjected to two different axial forces. The foreshortening is performed such that it compensates the average strain. However, it should be noted that this procedure alters the cell corner angles and thus the bending energy of a cellular structure. Consequently, if axial and bending energies of a cellular structure need to be considered it is best to use a numerical model where cell corner angles and side lengths are fully coupled.

\[ F_1, F_2, L_0, \Delta L_1, \Delta L_2 \]

Figure 14: (a) Elongation of an elastic cell side due to different axial forces. Foreshortening of cell side (b) minimizes deviations from reference length and thus, (c) on a global scale, reduces deviations from original equilibrium configurations.

5.3 Compliant Cell Corner Hinges

The influence of compliant cell corner hinges on the subtended angles of both equilibrium configurations is studied next. The used numerical model is shown in Figure 15. It consists of rigid bars, frictionless central cell corner hinges and rotational springs between neighbouring cell sides. The thickness of each cell side is optimized such that the material is fully utilized at either equilibrium configuration. As a consequence, the bending energy of a cellular structure is minimized. Details of the underlying optimization approach can be found in Appendix B. It should be noted that the assumed aspect ratio of the elastic regions is \( \mu = 2 \). As predicted by the analytical investigation of Section 3, the deviations from the target shapes and thus the bending energy is minimal for a certain Young’s modulus. Note that the optimal Young’s modulus decreases and target shape deviations increase for a decreasing safety factor \( \chi \). Furthermore, the required cell side thicknesses decrease for an increasing Young’s modulus.

5.4 Hinge Eccentricities

Finally, the relation between hinge eccentricities and subtended angles of both equilibrium configurations is investigated. This is a purely geometric problem so that the used numerical model, shown in Figure 16, does not incorporate any elastic parts. It can be seen that deviations from the full, half circle increase nearly linearly for increasing hinge eccentricities.
Figure 15: (a) Example structure with rigid bars, frictionless central cell corner hinges and rotational springs. (b) Cell side thicknesses are optimized to fully utilize material strength and to minimize bending energy. (c) Deviations from original equilibrium configurations for varying Young’s modulus and aspect ratio $\mu = 2$. (d) Maximum cell side thickness at cell corners.
It is subsequently assumed that hinge eccentricities \( r \) are equal to the maximum cell side thickness \( t_0 \) of the cellular structure. Based on the cell side thicknesses from Section 5.3, Figure 17 shows the influence of eccentric cell corner hinges on the deviations from the original equilibrium configurations. It can be seen that the deviations decrease for an increasing Young’s modulus and increase for a decreasing safety factor \( \chi \).

In summary, it can be said that deviations from the original equilibrium configurations are mainly caused by compliant cell corner hinges and hinge eccentricities. Furthermore, a decreasing safety factor \( \chi \) increases the influence of compliant cell corner hinges and hinge eccentricities whereas it decreases the influence of axial cell side strains. Since compliant hinges are exposed to fatigue loads it is likely that safety factors \( \chi < 1/2 \) are required. Hence, it can be concluded that the effect of axial strains is negligible in most applications.
6 End Caps for Prismatic Cells

Pressurized prismatic cells are sealed at both ends via end caps. These end caps have to sustain significant differential pressures while, at the same time, being flexible enough to allow large shape changes. Two different end cap designs are subsequently considered. The geometry of a prismatic cell with a pentagonal cross section as shown in Figure 18 is used as a basis for both designs. It can be seen that a reflection symmetry is assumed so that a single angle $\theta$ is sufficient to define the cross sectional geometry. Furthermore the width of each cell side reduces symmetrically towards the end. The region with a reduced width allows a continuous adaption between the non-uniform thicknesses of a cell side and the constant thickness of an end cap. Its geometry is described by a cubic polynomial that possesses a point reflection symmetry so that

\[
\zeta(\xi) = \Delta z ((\psi - 1) (2\xi - 3) \xi + \psi) \xi \quad \text{for} \quad \xi \in [0, 1] \tag{37}
\]

where $\psi$ is the gradient of the polynomial at $\xi = 0, 1$. An optimal end cap for a prismatic cell with a fixed cross sectional geometry possesses, like a soap film, an isotropic stress state. Deviations from this reference configuration introduce additional, non-isotropic stresses. However, these deviations are usually small so that minimal surfaces are an excellent basis for end cap designs. The shapes of end caps with an isotropic stress state are computed with the updated reference strategy by Bletzinger & Ramm [1]. The subsequently used parameters for end caps are summarized in Table 2.

The initial discretization of the cross sectional reference configuration ($\theta = 0$), the resulting minimal surface and the corresponding stresses after ten iterations with the updated reference strategy are shown in Figure 19. It can be seen that the membrane and tendon stresses are nearly uniform. Furthermore, the tendons carry the forces from the differential pressure directly to the cell side centers where the thickness and thus the axial stiffness is maximal.

The total material volume, deformation energy and the smallest possible cross sectional tendon area $A_t$ are shown in Figure 20 for a varying membrane thickness $t_m$. It

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6Only membrane stresses are considered. Bending stresses in membrane and tendons are neglected.
Geometry
\[ t_m = 1.2.6 \text{ mm} \] membrane thickness

Material
\[ E = 2,000 \text{ MPa} \] Young’s modulus of membrane and tendons
\[ \nu = 0 \] Poisson’s ratio of membrane
\[ \sigma_y = 30 \text{ MPa} \] target stress for form finding

Loading
\[ p = 2 \text{ MPa} \] cell pressure
\[ \theta = \pm 0.75^\circ \] state angle of cell geometry.

Table 2: Parameters for geometry, material and loading of end caps.

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Figure 19: (a) Initial discretization of reference configuration with membrane and bar elements. (b) Minimal surface for \( t_m = 1.5 \text{ mm} \) and \( A_t = 4.4 \text{ mm}^2 \). (c) Membrane and (d) tendon stresses are nearly uniform after ten iterations with the updated reference strategy.

should be noted that, without using tendons, the smallest possible membrane thickness is about \( t_m = 1.6 \text{ mm} \). Further reducing the membrane thickness requires the use of tendons that locally reinforce the membrane. It can be seen that an end cap without tendons and a thickness of \( t_m = 1.8 \text{ mm} \) possesses the smallest weight. However, although the use of tendons increases the weight of an end cap, it reduces the required deformation energy due to the smaller membrane thickness. In general it can be found that it is advantageous to use materials for end caps with large \( \sigma_y/E \) ratios\(^7\). The argument for this is as follows. The membrane thickness of an end cap is proportional to

\[ t_m \propto \frac{1}{\sigma_y} = \frac{1}{\chi a E^b}. \] (38)

Hence, the additional energy density of a membrane due to an unidirectional axial strain \( \varepsilon \) is

\[ d\Pi_a = \frac{1}{2}E t_m \varepsilon^2 \propto E^{1-b}. \] (39)

\(^7\)Note that, for example, TPU possesses a larger \( \sigma_y/E \) ratio than Titan.
Note that $\theta$ and thus $\varepsilon$ are independent of the membrane thickness. Hence, increasing the Young’s modulus increases the energy required to deform an end cap.

The elastic energy required to deform an end cap by $\pm \theta$ is shown in Figure. The corresponding change of pressure energy is subsequently derived in order to get a comparative value. The cross sectional area $A_c$ of a pentagonal cell is

$$A_c = L_2 \cos (\theta) (L_1 - L_2 \sin (\theta)) + \left( \frac{L_1}{2} - L_2 \sin (\theta) \right) \sqrt{L_2^3 - \left( \frac{L_1}{2} - L_2 \sin (\theta) \right)^2}.$$  
(40)

Therefore, the change of cell area for $\theta = \pm 3/4^\circ$ is

$$\Delta A_c (\theta) = A_c (\pm 3/4^\circ) - A_c (0) = -\frac{32.7}{30.1} \text{ mm}^2$$  
(41)

so that the change of pressure energy of a one meter long cell is

$$\Delta \Pi_p = -\frac{65.4}{60.2} \text{ J}$$  
(42)

which is significantly larger than the energy needed to deform the end caps. Hence it can be concluded that the energy required to deform well designed end caps can be neglected for large cell lengths.

7 Conclusions

This article presented analytical expressions for optimal material properties of pressure actuated compliant cellular structures. It was shown that, for given target shapes and stiffness requirements, a cellular structure can be made from a wide range of cell sizes. Furthermore, it was shown that the use of small cells increases the ratio between pressure and elastic energy. This might be an explanation for the relatively small cell sizes found in nastic plants. The required complexity of a numerical model for the accurate simulation of cellular structures depends on the geometry, material and cell pressures. Six different numerical models were presented and their application range was studied by means of an example structure. Finally, two different end cap designs for the termination of prismatic cells were investigated. It was found that it is possible to construct end caps such that the required deformation energy is small compared to changes in pressure energy.

A Corrections of Material Parameters for Plane Strain

Previously summarized material data needs to be modified for the simulation and optimization of compliant cellular structures. Particularly the Young’s moduli need to be recomputed to take into account the stiffening effect of three dimensional stress states. The stress-strain relationship for plane stress, $\sigma_z = 0$ is
Figure 20: (a) Smallest possible cross sectional area of tendons, (b) material volume and (c) deformation energy as a function of membrane thickness $t_m$. (d-g) End caps for different membrane thicknesses.
so that for $\sigma_y = 0$ previous equation reduces to

$$\sigma_x = E \varepsilon_x.$$  \hfill (44)

Note that the subscripts $x$, $y$, $z$ in this appendix refer to the axes of a cartesian coordinate system. In a similar manner, the stress-strain relationship for plane strain, $\sigma_z \neq 0$ is

$$\begin{bmatrix} \sigma_x \\ \sigma_y \end{bmatrix} = \frac{E}{(1 + \nu)(1 - 2\nu)} \begin{bmatrix} 1 - \nu & \nu \\ \nu & 1 - \nu \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \end{bmatrix}$$ \hfill (45)

which reduces to

$$\sigma_z = \frac{E}{1 - \nu^2} \varepsilon_z$$ \hfill (46)

for $\sigma_y = 0$. Hence the Young’s modulus $E^*$ that needs to be used for the computation of the elastic energy of pressure actuated cellular structures is

$$E^* = \frac{E}{1 - \nu^2}. \hfill (47)$$

The stress in thickness direction for plane strain is

$$\sigma_z = \nu (\sigma_x + \sigma_y)$$ \hfill (48)

so that the von Mises yield criterion for $\sigma_y = 0$ is

$$\sigma_v = \sigma_x \sqrt{1 - \nu (1 - \nu)}.$$ \hfill (49)

It can be seen that the equivalent tensile stress $\sigma_v$ decreases. However, this stress reduction can not be justified for all possible stress states of a cellular structure. Hence, for the sake of safety, it should not be taken into account.

### B Optimal Cell Side Thicknesses

Optimized cell side lengths are a function of the considered target shapes, cell pressures, material parameters and the assumed cell side thicknesses $t_0$ at cell corners. Hence it is necessary to iteratively compute the side thicknesses for a given stress constraint such that the bending energy is minimized. It is subsequently assumed that state angles and axial cell side forces from a previous simulation, optimization run are invariant to thickness changes. Although this is not true in general it is a necessary simplification that allows the use of a staggered algorithm. In contrast, a completely coupled approach would require the solution of a system of equations that are not only
a function of the rotational degrees of freedom and cell side lengths but also of the cell side thicknesses. The drawback of a staggered approach is that cell side lengths need to be recomputed after each thickness update. Furthermore, the update of cell side thicknesses needs to be damped in order to limit the stiffness variations of the structure and thus to ensure convergence. However, overall solution time is small so that the ease of a staggered implementation outweighs the increased speed of a completely coupled approach. It is subsequently shown how optimal cell side thicknesses are computed for given state angles, cell side forces and material parameters.

\[
\begin{bmatrix}
\theta_{b,1} \\
\theta_{b,2} \\
\theta_{b,3}
\end{bmatrix} = \frac{1}{k_{b,1} + k_{b,2} + k_{b,3}} \begin{bmatrix}
k_{b,3} & -k_{b,2} & 0 \\
k_{b,3} & k_{b,1} + k_{b,3} & 0 \\
-k_{b,1} & -k_{b,2} & 0
\end{bmatrix} \begin{bmatrix}
\Delta \theta_1 \\
\Delta \theta_2 \\
\Delta \theta_3
\end{bmatrix}
\]  

(50)

where \( \Delta \theta_i = \theta_i - \theta_i^0 \). It should be noted that the bending angles \( \theta_b \) are not only a function of the state angles \( \theta, \theta^0 \) but also of the cell side springs \( k_b \) and therefore of the cell side thicknesses \( t_0 \). Hence, the relation between cell side and state angles can be written as

\[
\begin{bmatrix}
\theta_{b,1} \\
\theta_{b,2} \\
\theta_{b,3}
\end{bmatrix} = \frac{1}{t_{0,1}^2 + t_{0,2}^2 + t_{0,3}^2} \begin{bmatrix}
t_{0,1}^2 & -t_{0,2}^2 & 0 \\
t_{0,2}^2 & t_{0,1}^2 + t_{0,3}^2 & 0 \\
-t_{0,1}^2 & -t_{0,2}^2 & 0
\end{bmatrix} \begin{bmatrix}
\Delta \theta_1 \\
\Delta \theta_2 \\
\Delta \theta_3
\end{bmatrix}
\]  

(51)

so that the change of bending angles \( \theta_b \) with respect to cell side thicknesses \( t_0 \) is

\[
\begin{bmatrix}
\frac{\partial \theta_{b,1}}{\partial t_{0,1}} \\
\frac{\partial \theta_{b,2}}{\partial t_{0,2}} \\
\frac{\partial \theta_{b,3}}{\partial t_{0,3}}
\end{bmatrix} = \frac{2}{(t_{0,1}^2 + t_{0,2}^2 + t_{0,3}^2)^2} \begin{bmatrix}
-t_{0,1}t_{0,3} & t_{0,1}t_{0,2}^2 & 0 \\
-t_{0,2}t_{0,3} & -t_{0,2}t_{0,1}^2 & 0 \\
t_{0,3}t_{0,1}^2 & t_{0,3}t_{0,2} & 0
\end{bmatrix} \begin{bmatrix}
\Delta \theta_1 \\
\Delta \theta_2 \\
\Delta \theta_3
\end{bmatrix}.
\]  

(52)

Note that the first derivatives are identical for all cell side angles \( \theta_b \) so that the corresponding subscript is dropped. Previous equations reduce to

Figure 21: Spring models of a cell corner. (a) Cell corner and (b) cell side springs.
\[
\begin{bmatrix}
\theta_{b,1} \\
\theta_{b,2}
\end{bmatrix}
= \frac{1}{t_{0,1}^2 + t_{0,2}^2} \begin{bmatrix}
t_{0,2}^2 \\
-t_{0,1}^2
\end{bmatrix} \Delta \theta
\quad \text{and} \quad
\begin{bmatrix}
\frac{\partial \theta_b}{\partial t_{0,1}} \\
\frac{\partial \theta_b}{\partial t_{0,2}}
\end{bmatrix}
= \frac{2t_{0,1}t_{0,2}}{(t_{0,1}^2 + t_{0,2}^2)^2} \begin{bmatrix}
t_{0,2} \\
-t_{0,1}
\end{bmatrix} \Delta \theta
\]
if only two cell sides share a common node.

The moment \(M_{0,i}\) that acts on a single cell side at a cell corner is

\[
M_{0,i} = k_{b,i} \theta_{b,i}
\]

so that its first derivative with respect to \(t_{0,j}\) is

\[
\frac{\partial M_{0,i}}{\partial t_{0,j}} = \frac{\partial k_{b,i}}{\partial t_{0,j}} \theta_{b,i} + k_{b,i} \frac{\partial \theta_b}{\partial t_{0,j}}
\quad \text{where} \quad
\frac{\partial k_{b,i}}{\partial t_{0,j}} = \frac{E t_{0,i}}{6 \mu} \delta_{ij}.
\]

Hence the maximum absolute cell side stress \(\sigma_{0,i}\) at a cell corner is

\[
\sigma_{0,i} = \sigma_{0,0,i} + \sigma_{0,1,i} = \frac{|F_i|}{t_{0,i}} + \frac{|M_{0,i}|}{W_{0,i}} = \frac{1}{t_{0,i}} \left( |F_i| + \frac{6}{t_{0,i}} |M_{0,i}| \right)
\]

so that its first derivative with respect to the cell side thicknesses \(t_{0,j}\) results in

\[
\frac{\partial \sigma_{0,i}}{\partial t_{0,j}} = \frac{1}{t_{0,i}^2} \left( 6 \text{sgn} (M_{0,i}) \frac{\partial M_{0,i}}{\partial t_{0,j}} - \left( |F_i| + \frac{12}{t_{0,i}} |M_{0,i}| \right) \delta_{ij} \right).
\]

Previous expression is used to construct a system of equations that relates the change of maximum absolute cell side stresses at cell corners to the change of cell side thicknesses. It should be noted that each equation is based on the pressure set that leads to the largest cell side stress at either end of a cell side.\(^8\) The resulting system of equations is solved with a Newton-Raphson based algorithm for a desired stress \(\sigma_y\). Bounds for maximum and minimum cell side thicknesses are enforced by using Lagrange multipliers.

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\(^8\)The discontinuity of this optimization problem could be slightly alleviated by using independent cell side thicknesses at both ends. However, since the maximum absolute cell side stresses can be caused by different pressure sets it is not possible to eliminate all potential discontinuities.
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