Cooperative effects in one-dimensional random atomic gases: Absence of single-atom limit

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Abstract – We study superradiance in a one-dimensional geometry, where \( N \gg 1 \) atoms are randomly distributed along a line. We present an analytic calculation of the photon escape rates based on the diagonalization of the \( N \times N \) coupling matrix \( U_{ij} = \cos x_{ij} \), where \( x_{ij} \) is the dimensionless random distance between any two atoms. We show that unlike a three-dimensional geometry, for a one-dimensional atomic gas the single-atom limit is never reached and the photon is always localized within the atomic ensemble. This localization originates from long-range cooperative effects and not from disorder as expected on the basis of the theory of Anderson localization.

Introduction. – Cooperative effects such as superradiance and subradiance [1,2] originate from indirect interactions between atoms through the radiation field. These effects show up as a multi-atomic coherent emission which is qualitatively different from that of a single atom [3,4]. Cooperative effects have been studied both theoretically and experimentally in various systems, such as quantum dots [5], Bose-Einstein condensate [6,7], cold atoms [8,9] and Rydberg gases [10].

In the context of cold atoms, photon localization, which occurs as a decrease of photon escape rates from disordered media, has been investigated [11]. It has been shown that for a three-dimensional atomic system, photon localization is primarily determined by cooperative effects rather than by disorder. Moreover, localization shows up as a crossover between delocalized and localized photons and not as a disorder-driven phase transition as for Anderson localization.

In this letter we study photon escape rates from a disordered one-dimensional atomic gas and compare them to those obtained in a three-dimensional geometry. We will show that unlike a three-dimensional geometry, for a one-dimensional atomic gas the single-atom limit is never reached and the photons are always localized. This localization stems only from long-range cooperative effects and not from disorder as expected from the theory of Anderson localization.

Model. – We are interested in the dipolar interaction of \( N \gg 1 \) identical atoms with a scalar radiation field. Here, atoms are taken as non-degenerate, two-level systems. The energy separation between the excited state \( |e\rangle \) and the ground state \( |g\rangle \), including radiative shift, is \( \hbar \omega_0 \) and the inverse lifetime of the excited level is \( \Gamma_0 \). Indeed, this two-level atom model neglects the energy structure of a real atom, but as selection rules restrict the allowed transitions between states, this approximation is more than a mathematical convenience.

We consider a one-dimensional geometry where the atoms are randomly distributed along a line. Moreover, only modes of field that belong to an elongated pencil-shaped radiation pattern parallel to the inter-atomic axis are taken into account. This radiation pattern, obtained in a pencil-shaped cavity, corresponds to the directional emission along the cavity axis [12,13].

We neglect recoil effects and the Doppler shift by assuming that the typical speed of the atoms is large compared to \( \hbar k/\mu \) but small compared to \( \Gamma_0/k \), where \( k \) is the radiation wave number and \( \mu \) is the mass of the atom. Additionally, we neglect retardation effects so each atom is allowed to influence the others instantaneously.

Dicke states and cooperative effects. – The absorption of a photon by a pair of atoms, each in its ground state and, respectively, located at \( \mathbf{r}_1 \) and \( \mathbf{r}_2 \), leads
to a configuration where one atom is excited while the other is de-excited. The possible configurations can be represented by the Dicke states [1]. The singlet Dicke state is

\[ |\rangle = |00\rangle = \frac{1}{\sqrt{2}}[|e_1e_2\rangle - |g_1e_2\rangle] \] (1)

and the triplet Dicke states are

\[ |01\rangle = |e_1e_2\rangle, \quad |10\rangle = \frac{1}{\sqrt{2}}[|e_1e_2\rangle + |g_1e_2\rangle], \quad |11\rangle = |g_1e_2\rangle. \] (2)

These states are characterized by an effective interaction potential and a modified lifetime as compared to independent atoms. For the one-dimensional geometry considered here, the cooperative spontaneous emission rate or the inverse lifetime of the states |±⟩ is [13]

\[ \frac{\Gamma^\pm}{\Gamma_0} = 1 \pm \cos k_0r, \] (3)

where \( k_0 = \omega_0/c \) and \( r = |r_1 - r_2| \). The corresponding cooperative radiative level shift or the interaction potential is given by \( \Delta E^\pm = \pm \frac{G_0}{2} \sin k_0r \).

For comparison, in a three-dimensional system, the cooperative spontaneous emission rate is [14]

\[ \frac{\Gamma^\pm}{\Gamma_0} = 1 \pm \frac{\sin k_0r}{k_0r}, \] (4)

and the corresponding cooperative radiative level shift is given by \( \Delta E^\pm = \frac{G_0}{2} \cos k_0r/k_0r \).

When the atoms are close enough \( (k_0r << 1) \), the Dicke limit is obtained in both geometries, namely, \( \Gamma^\pm = (1 \pm 1)\Gamma_0 \). But, when the atoms are well separated \( (k_0r >> 1) \), the single-atom limit is not recovered in eq. (3) since the one-dimensional inverse lifetime is a periodic function of the inter-atomic distance, while the three-dimensional one falls off with the inter-atomic separation. Similarly, the range of the one-dimensional interaction potential is infinite, while it is finite in the three-dimensional case. This fundamental difference will be the driving effect in the calculation of photon escape rates from an atomic gas in the next section.

**Photon escape rates from atomic gases.** – To go beyond the case of two atoms, we follow [11,12] who studied the equation of motion for the reduced atomic density operator \( \rho \) of a gas of atoms with a single excitation. The time evolution of the ground-state population associated with \( \rho \) is given by

\[ \frac{d(G|\rho|G)}{dt} = \Gamma_0 \sum_{ij} U_{ij} G|S^+_i \rho S^+_j |G\rangle, \] (5)

where \( |G\rangle = |g_1, g_2, \ldots, g_N\rangle \) and \( S^+_i \) is the raising (lowering) operator of atom \( i \). \( U \) is an \( N \times N \) Euclidean random matrix as defined hereafter. For the one-dimensional geometry

\[ U_{ij} = \cos k_0r_{ij}, \] (6)

while for the three-dimensional gas

\[ U_{ij} = \frac{\sin k_0r_{ij}}{k_0r_{ij}}, \] (7)

where \( r_{ij} = |r_i - r_j| \) is the random distance between any two atoms. With the help of the eigenvalue equation of \( U \), namely \( \sum_{n=1}^N U_{ij} u_n^{(n)} = \Gamma_n u_n^{(n)} \), we rewrite eq. (5) as \( d(G|\rho|G)/dt = \Gamma_0 \sum_{n=1}^N \Gamma_n |G|S_n \rho S^+_n |G\rangle \) where \( S_n^\pm = \sum_{n=1}^N u_n^{(n)} S^\pm \) are the collective raising and lowering operators. Thus, we can interpret the eigenvalues \( \Gamma_n \) of the coupling matrix \( U \) as the photon escape rates from the gas and define the dimensionless average density of photon escape rates as

\[ P(\Gamma) = \frac{1}{N} \sum_{n=1}^N \delta(\Gamma - \Gamma_n), \] (8)

where the average, denoted by \( \sum \), is taken over the spatial configurations of the atoms.

The quantity used as a measure of photon localization is the normalized function \( C = 1 - 2 \int_0^\infty d\Gamma P(\Gamma) \). \( C \) thus measures the relative number of states having a vanishing escape rate. In the three-dimensional case, discussed in [11], the function \( C \) exhibits a scaling behavior over a broad range of system size and disorder. The scaling variable is \( N/N_\perp \), where \( N_\perp \) is the number of transverse photon modes in the system. \( C \) allows to compare the different contributions of disorder and cooperative effects to photon localization, although an unambiguous distinction between the two mechanisms cannot be achieved in the three-dimensional geometry [11]. In this letter, we will show that in the one-dimensional case \( C \) exhibits a scaling behavior as well, and the scaling variable is \( N \). We will also show that the expression of \( C \) obtained in the one-dimensional geometry is valid for both ordered and disordered systems, thus it is possible to unambiguously discern between the contributions of disorder and cooperative effects to photon localization.

The eigenvalues of \( U \) in eq. (7) have been obtained numerically in [11]. Recently, based on the Marchenko-Pastur law [15], the authors of [16] have approximated the spectrum of this matrix in the limit of large systems. Here, we will provide an analytic diagonalization of (6) that holds for an arbitrary system size.

To that purpose, we consider \( N \gg 1 \) atoms confined in a one-dimensional system of length \( L = 2\pi a/k_0 \). The atoms are randomly distributed with a uniform density \( N/L \) and the corresponding coupling matrix is given by eq. (6). The average density of photon escape rates, obtained for many random configurations of the atoms, is presented in fig. 1 for different values of \( W = N/2\pi a \) and \( a \). A remarkable difference between the one and the three-dimensional
case [11] is observed. Unlike the three-dimensional geometry, the single-atom limit is never reached and the photons are always localized in the atomic gas.

Let us distinguish between two regimes, Dicke regime where $a \ll 1$ and the large-sample regime, where $a \gg 1$. In Dicke regime, the coupling matrix is

$$U = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}. \quad (9)$$

Thus, the average density of photon escape rate is given by

$$P(\Gamma) = \frac{1}{N}[(N-1)\delta(\Gamma) + \delta(\Gamma - N)], \quad (10)$$

as presented in fig. 1(d). Equation (10) holds in Dicke regime of the three-dimensional case as well. The spectrum of $U$ given above yields $C = 1 - 2/N$. For the current case where $N \gg 1$, $C = 1$ indicating photon localization.

Away from the Dicke limit, in the large-sample regime shown in fig. 1(a)–(c), $P(\Gamma)$ is calculated as follows. The $N \times N$ matrix $U_{ij} = \cos k_0 r_{ij}$ may be rewritten as $U = \frac{1}{2} A^\dagger A$, where $A$ is the $2 \times N$ matrix defined by $A_{0j} = e^{ik_0 r_{ij}}$ and $A_{1j} = e^{-ik_0 r_{ij}}$. As $U$ is a real symmetric matrix, its non-vanishing eigenvalues can be found from those of $U^\dagger$, given by

$$U^\dagger = \frac{1}{2} \begin{pmatrix} N & M \\ M^* & N \end{pmatrix}. \quad (11)$$

Here $M = \sum_{k=1}^{N} e^{2i k_0 r_k}$ is a random variable where $k_0 r_k$ is uniformly distributed over $[0, 2\pi a]$. Since the two eigenvalues of eq. (11) are

$$\lambda_{\pm} = \frac{N \pm |M|}{2}, \quad (12)$$

the spectrum of $U$ is given by

$$P(\Gamma) = \frac{1}{N} \left[ (N-2)\delta(\Gamma) + \delta(\Gamma - \lambda_+) + \delta(\Gamma - \lambda_-) \right]. \quad (13)$$

We can estimate $|M|$ by writing

$$|M|^2 = N + \sum_{p \neq q} e^{2ik_0(r_p - r_q)}, \quad (14)$$

where the second term involves $N(N-1)$ terms. On average over non-correlated disorder the second term vanishes so that $|M| \sim \sqrt{N}$. For the spectrum of $U$ given in eq. (13) it is evident that $C = 1 - 4/N$. Thus, for large values of $N$ the photons are localized in the gas.

In order to calculate exactly the distribution function of $|M|$, first we assume that $a$ is an integer. In this special case, the distribution function is just the Rayleigh distribution,

$$P(|M|) = \frac{2 |M|}{N} e^{-|M|^2/N}, \quad (15)$$

whose mode is $\sqrt{N}/2$. Figure 2 shows the eigenvalue distribution spectrum of $U$ for $a = 1$ (excluding the degenerate subradiant mode at $\Gamma = 0$) as well as the calculated $P(\Gamma)$ given by eqs. (12) and (15).

In the general case, for an arbitrary value of $a$, we follow [17], as described below. As $N \gg 1$, according to the Central-Limit theorem, $\text{Re}(M)$ is normally distributed with mean $m_1$ and variance $\nu_1$. Similarly, $\text{Im}(M)$ is
distribution function is obtained by an angular integration case, considered earlier, where

\[ P(\Gamma) = \text{const} \]

and the corresponding calculated (excluding the degenerate subradiant mode at \( \Gamma = 0 \)) as

\[ m \sim \text{normal distribution with mean } m \text{ and variance } \nu. \]

This unambiguous distinction between the contributions of disorder and cooperative effects to photon localization cannot be achieved in the three-dimensional geometry, where the role played by each of these two mechanisms cannot be determined separately [11].

The distribution of resonance widths in one-dimensional disordered media has also been studied in [18], where it has been shown that it follows a power law \( P(\Gamma) \sim \Gamma^{-\alpha} \) decay. The spectrum in eq. (13) does not, however, obey this power law. The difference stems from the fact that the authors of ref. [18] have calculated \( P(\Gamma) \) using the real part of the spectrum of the complex-valued Green matrix \( \exp(ik\Omega_{ij}) \), which describes propagation of a wave scattered by a dipole at \( r_i \) to a dipole at \( r_j \). Here, we have taken a different approach [11,12] and studied the time evolution of the ground-state population associated with the reduced atomic density operator of the system. As explained earlier, in our treatment one can interpret the eigenvalues of the real-valued matrix \( \cos(\pi r_{ij}) \) as the photon escape rates from the atomic gas. According to [18], the \( P(\Gamma) \sim \Gamma^{-\alpha} \) behavior can be interpreted as an unambiguous signature of Anderson localization of light in random systems. The fact that our result does not follow this power law supports the claim that cooperative effects and not disorder is the mechanism that leads to photon localization in the case studied here.

It is interesting to compare these results to the two-dimensional case, where \( U_{ij} = J_0(x_{ij}) \) [19] where \( x_{ij} \) is the dimensionless random distance between any two atoms. In this geometry, when the atoms are close enough, the Dicke limit is reached and eq. (10) holds, as in the other geometries. In the opposite limit, \( \bar{U} \) can be approximated as \( U_{ij} \simeq \sqrt{2/\pi x_{ij}} \cos(x_{ij} - \pi/4) \) and since it falls off with the square root of the inter-atomic separation, the single-atom limit can be reached. We conclude that the absence of single-atom limit is specific to the one-dimensional geometry.

Recently, the authors of ref. [20] have studied the interplay of disorder and superradiance in a one-dimensional Anderson model in which all the sites are coupled to a common decay channel with equal coupling strength.
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By diagonalizing the corresponding non-Hermitian Hamiltonian, the participation ratio of the eigenstates have been obtained. It has been shown that while superradiant states become localized as disorder increases, subradiant states remain delocalized. These results differ substantially from ours. The difference stems from the fact that the authors in ref. [20] have considered an Anderson model with on-site disorder and assumed that the sites are coupled to the continuum with equal coupling strength, while in our treatment only the position-dependent continuum coupling is taken into account.

**Summary.** – We have studied cooperative effects in a one-dimensional random atomic system. By an analytic diagonalization of the Euclidean random matrix $U_{ij} = \cos x_{ij}$, where $x_{ij}$ is the dimensionless random distance between any two atoms, we have calculated the photon escape rates from the gas for an arbitrary system size. We have shown that the single-atom limit is never reached and the photon is always localized. This localization stems from long-range cooperative effects and not from disorder as expected on the basis of the theory of Anderson localization.

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