COMPLEX MONGE-AMPÈRE EQUATIONS ON QUASI-PROJECTIVE VARIETIES

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ABSTRACT. We introduce generalized Monge-Ampère capacities and use these to study complex Monge-Ampère equations whose right-hand side is smooth outside a divisor. We prove, in many cases, that there exists a unique normalized solution which is smooth outside the divisor.

1. INTRODUCTION

Let \((X, \omega)\) be a compact Kähler manifold of complex dimension \(n\) and let \(D\) be a divisor on \(X\). Let \(f\) be a non-negative function such that \(\int_X f \omega^n = \int_X \omega^n\). Consider the following complex Monge-Ampère equation

\[ (\omega + dd^c \varphi)^n = f \omega^n. \]

When \(f\) is smooth and positive on \(X\), it follows from the seminal work of Yau [37] that (1.1) admits a unique normalized smooth solution \(\varphi\) such that \(\omega + dd^c \varphi\) is a Kähler form. Recall that this result solves in particular the Calabi conjecture and allows to construct Ricci flat metrics on \(X\) whenever \(c_1(X) = 0\).

It is very natural to look for a similar result when \(f\) is merely smooth and positive on the complement of \(D\), e.g. when studying Calabi’s conjecture on quasi-projective manifolds (see e.g. [33, 34, 35] and [25] for recent developments). The study of conical Kähler-Einstein metrics (Kähler-Einstein metrics in the complement of a divisor with a precise behavior near \(D\)) has played a major role in the resolution of the Yau-Tian-Donaldson conjecture for Fano manifolds (see [20], [21], [14], [15], [16], [32]).

However no systematic study of the regularity of solutions to such complex Monge-Ampère equations has ever been done, this is the main goal of this article. It follows from [24] that (1.1) has a unique (up to an additive constant) solution in the finite energy class \(\mathcal{E}(X, \omega)\). We say that the solution is normalized if \(\sup_X \varphi = 0\). The problem thus boils down to showing that such a normalized solution is smooth in \(X \setminus D\) and understanding its asymptotic behavior along \(D\).

As in the classical case of Yau [37] the main difficulty is in establishing a priori \(C^0\) bounds. Since, in general the solution \(\varphi\) is unbounded, the idea is to bound \(\varphi\) from below by some (singular) \(\omega\)-psh function.

Our first main result shows that the solution \(\varphi\) is smooth in \(X \setminus D\) when \(f\) satisfies the mild condition \(\mathcal{H}_f\):

\[ f = e^{\psi^+ - \psi^-}, \quad \psi^\pm \text{ are quasi plurisubharmonic on } X, \quad \psi^- \in L^\infty_{\text{loc}}(X \setminus D). \]

Let us stress that \(D\) is here an arbitrary divisor.
Theorem 1. Assume that \( 0 < f \in C^\infty(X \setminus D) \) satisfies Condition \( \mathcal{H}_f \). Then the solution \( \varphi \) is also smooth on \( X \setminus D \).

In Theorem 1, the density \( f \) is only in \( L^1(X) \) and there is no regularity assumption on \( D \). Hence we do not have any information about the behavior of \( \varphi \) near \( D \).

If we assume more regularity on \( f \) and \( D \), we will get more precise \( C^0 \)-bounds.

Assume that \( D = \sum_{j=1}^N D_j \) is a simple normal crossing divisor (snc for short). For each \( j = 1, \ldots, N \), let \( L_j \) be the holomorphic line bundle defined by \( D_j \). Let \( s_j \) be a holomorphic section of \( L_j \) such that \( D_j = \{ s_j = 0 \} \). Fix a hermitian metric \( h_j \) on \( L_j \) such that \( |s_j| := |s_j|_{h_j} \leq 1/e \).

We say that \( f \) satisfies Condition \( S(B, \alpha) \) for some constants \( B > 0 \), \( \alpha > 0 \) if

\[
f \leq B \prod_{j=1}^N |s_j|^2 (-\log |s_j|)^{1+\alpha}.
\]

Theorem 2. Assume that \( f \leq e^{-\phi} \) for some quasi-plurisubharmonic function \( \phi \). Then for each \( a > 0 \) there exists \( A > 0 \) depending on \( \int_X e^{-2\phi/a} \omega^n \) such that

\[
\varphi \geq a \phi - A.
\]

More precisely, if \( f \) satisfies Condition \( S(B, \alpha) \) for some \( \alpha > 0 \), \( B > 0 \), then the following holds:

(a) if \( \alpha > 1 \) then \( \varphi \) is continuous on \( X \), \( \varphi \geq -C \), with \( C = C(B, \alpha) \).

(b) if \( \alpha = 1 \) then there exists \( A_1, A_2 > 0 \) depending on \( B \) such that

\[
\varphi \geq \sum_{j=1}^N -A_1 [\log(\log |s_j|) + A_2],
\]

(c) if \( \alpha \in (0,1) \) then for each \( \beta \in (1-\alpha, 1) \) and \( a > 0 \) there exists \( A > 0 \) depending on \( a, \alpha, \beta, B \) such that

\[
\varphi \geq \sum_{j=1}^N -a(\log |s_j|)^\beta - A.
\]

Remark. It follows from Skoda’s theorem \[30\] that \( \int_X e^{-2\phi/a} \omega^n \) is finite for all \( a > 0 \), since \( \varphi \in \mathcal{E}(X, \omega) \) has zero Lelong number at all points \[24\].

When the behavior of \( f \) near the divisor \( D \) looks exactly like

\[
\frac{1}{\prod_{j=1}^N |s_j|^2 \log |s_j|^{1+\alpha}}, \quad \alpha \in (0, 1]
\]

we show in Proposition \[42\] and Proposition \[43\] that \( \varphi(x) \) converges to \(-\infty \) as \( x \) approaches \( D \) with precise rates. In particular there is no bounded solution to \( (1.1) \).

When \( f \in L^p(\omega^n) \) for some \( p > 1 \), it follows from the work of Kołodziej \[27\] that the solution of \( (1.1) \) is actually uniformly bounded (and even Hölder continuous) on the whole of \( X \).

In our result, the density \( f \) is merely in \( L^1 \). The first part of Theorem \[2\] says that when \( \alpha > 1 \) the solution is continuous on \( X \). Kołodziej’s result \[27\] Theorem 2.5.2] also applies when \( f \) satisfies \( S(B, \alpha) \) for \( \alpha > n \) but can not be applied to a density \( f \) as above if \( \alpha \leq n \).
Observe furthermore that $\alpha = 1$ is a critical exponent as is easily seen when $n = 1$. In any dimension, when $f$ has singularities of Poincaré type,

$$\frac{1/C}{\prod_{j=1}^{N} |s_j|^2 |\log |s_j||^2} \leq f(z) \leq \frac{C}{\prod_{j=1}^{N} |s_j|^2 |\log |s_j||^2}$$

along $D$ we show in Section 4.3 that the solution is locally uniformly bounded on compact subsets of $X \setminus D$ and goes to $-\infty$ along $D$ with a certain rate. If moreover $f$ has a "very precise" behavior near $D$ it follows from the recent work of Auvray (see [1]) that $\phi$ goes to $-\infty$ along $D$ like $\sum_{j=1}^{N} -\log(-\log |s_j|)$. We stress that this condition is very restrictive while in our result we only need a very weak condition on the density. Recall also that in [33] the authors constructed "almost complete" Kähler Einstein metrics of negative Ricci curvature on $X \setminus D$. In this case the $C^0$ estimate follows easily from the maximum principle.

In order to prove the $C^0$-estimate we follow and generalize Kołodziej’s approach. We introduce and study the $\psi$-Capacity of a Borel subset $E \subset X$,

$$\text{Cap}_\psi(E) := \sup \left\{ \int_E (\omega + dd^c u)^n \mid u \in \text{PSH}(X, \omega), \psi - 1 \leq u \leq \psi \right\}$$

where $\psi \in \text{PSH}(X, \omega)$ and here $(\omega + dd^c u)^n$ is the nonpluripolar Monge-Ampère measure of $u$ (see Section 2 for the definition). When $\psi$ is constant, $\psi \equiv C$, we recover the Monge-Ampère capacity,

$$\text{Cap}_\omega = \text{Cap}_C.$$

A similar notion has been studied in [13] in a local context. These generalized capacities are interesting for themselves. In this paper we only need some of their properties and refer the reader to [18] for a more systematic study.

One of the advantages of the Kołodziej’s approach for the $C^0$ estimates is that it also works in the case of semipositive and big classes as shown in [4], [22] and [8]. Thus it is not surprising that our method is still valid in this situation.

Let $\theta$ be a smooth semipositive form on $X$ such that $\int_X \theta^n > 0$. Let $f$ be a nonnegative function such that $\int_X f \omega^n > 0$. Consider the following degenerate complex Monge-Ampère equation

$$(\theta + dd^c \varphi)^n = f \omega^n.$$  

(1.2)

It follows from [6] that (1.2) admits a unique normalized solution $\varphi \in \mathcal{E}(X, \theta)$. As in the Kähler case, it is interesting to investigate the regularity properties of $\varphi$ if we know that the density $f$ is smooth, strictly positive outside a divisor $D$ and verifies Condition $\mathcal{H}_f$. We can not expect $\varphi$ to be smooth on $X \setminus D$ since $\theta$ may be zero somewhere there. Our result below shows that the solution is smooth on $X \setminus (D \cup E)$, where $E$ is an effective simple normal crossing divisor on $X$ such that $\{\theta\} - c_1(E)$ is ample.

**Theorem 3.** Let $(X, \omega)$ be a compact Kähler manifold of complex dimension $n$ and $D$ be an arbitrary divisor on $X$. Let $E$ be an effective snc divisor on $X$, and $\theta$ be a smooth semipositive form on $X$ such that $\int_X \theta^n > 0$ and $\{\theta\} - c_1(E)$ is ample. Assume that $0 < f \in C^\infty(X \setminus D)$ satisfies Condition $\mathcal{H}_f$. Let $\varphi$ be the unique normalized solution to equation (1.2). Then $\varphi$ is smooth on $X \setminus (D \cup E)$. 
**Remark.** The condition we impose on \( \{ \theta \} \) is natural in studying Kähler Einstein metrics on singular varieties (see [9]).

Let us say some words about the organization of the paper. In Section 2 we introduce the generalized \( \psi \)-Capacity, and establish their basic properties. The proof of Theorem 1 will be given in Section 3. We provide some volume-capacity estimates in Section 4.1. We then use these to prove Theorem 2 and discuss about the asymptotic behavior of solutions near the divisor in Section 4.2. Finally we consider the case of semipositive and big classes in Section 5.

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2. Preliminaries

Let \( (X, \omega) \) be a compact Kähler manifold. We first recall basic facts about finite energy classes of \( \omega \)-psh functions on \( X \). The reader can find more details about these in [24].

2.1. Finite energy classes.

**Definition 2.1.** We let \( \text{PSH}(X, \omega) \) denote the class of \( \omega \)-plurisubharmonic functions (\( \omega \)-psh for short) on \( X \), i.e. the class of functions \( \varphi \) such that locally \( \varphi = \rho + u \), where \( \rho \) is a local potential of \( \omega \) and \( u \) is a plurisubharmonic function.

Let \( \varphi \) be some (unbounded) \( \omega \)-psh function on \( X \) and consider \( \varphi_j := \max(\varphi, -j) \) the canonical approximation by bounded \( \omega \)-psh functions. It follows from [24] that

\[
1_{\{\varphi_j > -j\}}(\omega + dd^c \varphi_j)^n
\]

is a non-decreasing sequence of Borel measures. We denote by \( (\omega + dd^c \varphi)^n \) (or \( \text{MA}(\varphi) \) for short if \( \omega \) is fixed and no confusion can occur) this limit:

\[
\text{MA}(\varphi) = (\omega + dd^c \varphi)^n = \lim_{j \to +\infty} 1_{\{\varphi > -j\}}(\omega + dd^c \varphi_j)^n.
\]

It was shown in [24] that the Monge-Ampère measure \( \text{MA}(\varphi) \) puts no mass on pluripolar sets. This is the non-pluripolar part of the Monge-Ampère of \( \varphi \). Note that its total mass \( \text{MA}(\varphi)(X) \) can take value in \( [0, \int_X \omega^n] \).

**Definition 2.2.** We let \( \mathcal{E}(X, \omega) \) denote the class of \( \omega \)-psh function having full Monge-Ampère mass:

\[
\mathcal{E}(X, \omega) := \left\{ \varphi \in \text{PSH}(X, \omega) \mid \int_X \text{MA}(\varphi) = \int_X \omega^n \right\}.
\]

Let us stress that \( \omega \)-psh functions with full Monge-Ampère mass have mild singularities. Indeed, it was shown in [24, Corollary 1.8] that

\[
\nu(\varphi, x) = 0, \forall \varphi \in \mathcal{E}(X, \omega), \: x \in X.
\]
We also recall that, for every \( \varphi \in E(X, \omega) \) and \( \psi \in \text{PSH}(X, \omega) \), the generalized comparison principle holds (see [8, Corollary 2.3]), namely

\[
\int_{\{\varphi < \psi\}} (\omega + dd^c \psi)^n \leq \int_{\{\varphi < \psi\}} (\omega + dd^c \varphi)^n.
\]

Let \( \chi : \mathbb{R}^- \to \mathbb{R}^- \) be an increasing function such that \( \chi(0) = 0 \) and \( \chi(-\infty) = -\infty \).

**Definition 2.3.** Let \( E_\chi(X, \omega) \) denote the set of \( \omega \)-psh functions with finite \( \chi \)-energy,

\[
E_\chi(X, \omega) := \{ \varphi \in E(X, \omega) \mid \chi(-|\varphi|) \in L^1(\text{MA}(\varphi)) \}.
\]

For \( p > 0 \), we use the notation \( E^p(X, \omega) := E_\chi(X, \omega) \), when \( \chi(t) = -(t)^p \).

2.2. The \( \psi \)-Capacity.

**Definition 2.4.** Let \( \psi \in \text{PSH}(X, \omega) \). We define the \( \psi \)-Capacity of a Borel subset \( E \subset X \) by

\[
\text{Cap}_\psi(E) := \sup \left\{ \int_E \text{MA}(u) \mid u \in \text{PSH}(X, \omega), \psi - 1 \leq u \leq \psi \right\}.
\]

Then the Monge-Ampère capacity corresponds to \( \psi \equiv \text{constant} \) (see [3], [28], [23]). We list below some basic properties of the \( \psi \)-Capacity.

**Proposition 2.5.**

(i) If \( E_1 \subset E_2 \subset X \) then \( \text{Cap}_\psi(E_1) \leq \text{Cap}_\psi(E_2) \).

(ii) If \( E_1, E_2, ... \) are Borel subsets of \( X \) then

\[
\text{Cap}_\psi \left( \bigcup_{j=1}^{\infty} E_j \right) \leq \sum_{j=1}^{+\infty} \text{Cap}_\psi(E_j).
\]

(iii) If \( E_1 \subset E_2 \subset ... \) are Borel subsets of \( X \) then

\[
\text{Cap}_\psi \left( \bigcup_{j=1}^{\infty} E_j \right) = \lim_{j \to +\infty} \text{Cap}_\psi(E_j).
\]

The following results are elementary and important for the sequel. We stress that these results still hold in the case when \( \omega \) is merely semipositive and big rather than Kähler.

**Lemma 2.6.** Let \( \psi \in \text{PSH}(X, \omega) \) and \( \varphi \in \mathcal{E}(X, \omega) \). Then the function

\[
H(t) := \text{Cap}_\psi(\{\varphi < \psi - t\}), \ t \in \mathbb{R},
\]

is right-continuous and \( H(t) \to 0 \) as \( t \to +\infty \).

**Proof.** The right-continuity of \( H \) follows from (iii) of Proposition 2.5. Let us prove the second statement. We can assume that \( \psi \leq 0 \) on \( X \). Fix \( v \in \text{PSH}(X, \omega) \) such that \( \psi - 1 \leq v \leq \psi \). We apply the comparison principle to obtain

\[
\int_{\{\varphi < \psi - t\}} \text{MA}(v) \leq \int_{\{\varphi < \psi - t+1\}} \text{MA}(v) \leq \int_{\{\varphi < -t+1\}} \text{MA}(\varphi).
\]

The last term goes to zero as \( t \) goes to \( +\infty \) since \( \varphi \in \mathcal{E}(X, \omega) \). \( \square \)
Lemma 2.7. Let \((X,\omega)\) be a compact Kähler manifold and \(\psi \in \text{PSH}(X,\omega/2)\). Then we have
\[
\text{Cap}_{\omega/2}(E) \leq \text{Cap}_{\psi}(E),
\]
where \(\text{Cap}_{\omega/2}\) is the Monge-Ampère Capacity with respect to the Kähler metric \(\omega/2\) introduced in [28] and studied in [23], and \(\text{Cap}_{\psi}\) is the generalized \(\psi\)-Capacity with respect to the Kähler metric \(\omega\).

We stress that the above result insures \(\text{Cap}_{\psi}(E) > 0\) for any Borel subset \(E\) which is not pluripolar.

Proof. Let \(u \in \text{PSH}(X,\omega/2)\) be such that \(-1 \leq u \leq 0\). Then \(\varphi := \psi + u\) is a candidate defining \(\text{Cap}_{\psi}\). Using the definition of the Monge-Ampère measure it is not difficult to see that
\[
\int_{\mathbb{C}^n} (\omega/2 + dd^c u)^n \leq \int_{\mathbb{C}^n} (\omega + dd^c \varphi)^n \leq \text{Cap}_{\psi}(E),
\]
and taking the supremum over all \(u\) we get the result. \(\square\)

The following result generalizes Lemma 2.3 in [22].

Proposition 2.8. Let \(\varphi \in \mathcal{E}(X,\omega), \psi \in \text{PSH}(X,\omega)\). Then for all \(t > 0\) and \(0 \leq s \leq 1\) we have
\[
s^n \text{Cap}_{\psi}(\{\varphi < \psi - t - s\}) \leq \int_{\{\varphi < \psi - t\}} \text{MA}(\varphi).
\]

Proof. Let \(u \in \text{PSH}(X,\omega)\) such that \(-1 \leq u \leq \psi\). Observe the following trivial inclusion
\[
\{\varphi < \psi - t - s\} \subset \{\varphi < su + (1-s)\psi - t\} \subset \{\varphi < \psi - t\}.
\]
It thus follows from the generalized comparison principle (see [3] Corollary 2.3)) that
\[
s^n \int_{\{\varphi < \psi - t - s\}} \text{MA}(u) \leq \int_{\{\varphi < \psi - t - s\}} \text{MA}(su + (1-s)\psi) \\
\leq \int_{\{\varphi < su + (1-s)\psi - t\}} \text{MA}(su + (1-s)\psi) \\
\leq \int_{\{\varphi < \psi - t\}} \text{MA}(\varphi).
\]
By taking the supremum over all candidates \(u\) we get the result. \(\square\)

3. Smooth solution in a general case

In this section we prove Theorem 1. The most difficult part is the \(C^0\) estimate which is followed by a simple observation: if \(\varphi \in \mathcal{E}(X,\omega)\), \(\sup_X \varphi = 0\) is such that \(\text{MA}(\varphi) \leq e^{-\phi} \omega^n\), for some quasi-psh function \(\phi\), then \(\varphi\) is bounded from below by \(a\phi - A\), for some positive constants \(a, A\).
3.1. Uniform estimate. In this subsection we assume that $0 \leq f \in L^1(X)$ is such that $\int_X f \omega^n = \int_X \omega^n$. Let $\varphi \in \mathcal{E}(X, \omega)$ be the unique normalized solution to
\begin{equation}
(\omega + dd^c \varphi)^n = f \omega^n.
\end{equation}
Here we normalize $\varphi$ such that $\sup_X \varphi = 0$. We prove the following $C^0$ estimate:

**Theorem 3.1.** Assume that $f \leq e^{-\phi}$ for some quasi-plurisubharmonic function $\phi$. Let $\varphi \in \mathcal{E}(X, \omega)$ be the unique normalized solution to (3.1). Then for any $a > 0$, there exists $A > 0$ depending on $\int_X e^{-2\varphi/a} \omega^n$ such that
$$\varphi \geq a\phi - A.$$ 

Moreover, if $\phi$ is bounded in a compact subset $K \subset X$ then $\varphi$ is continuous on $K$.

**Proof.** We can assume that $\phi \leq 0$. Observe that it is enough to prove Theorem 3.1 for $a > 0$ small enough. Fix $a > 0$ such that $\psi := a\phi$ belongs to $\text{PSH}(X, \omega/2)$. It follows from Lemma 2.4 that $\text{Cap}_\omega \leq 2^n \text{Cap}_{\omega/2} \leq 2^n \text{Cap}_\psi$. Fix $s \in [0, 1]$, $t > 0$ and apply Proposition 2.8 to get
\begin{equation}
s^n \text{Cap}_\psi(\varphi < \psi - t - s) \leq \int_{\{\varphi < \psi - t\}} \text{MA}(\varphi).
\end{equation}
By assumption on $f$ we have
$$\int_{\{\varphi < \psi - t\}} \text{MA}(\varphi) \leq \int_{\{\varphi < \psi - t\}} e^{-\varphi/a} e^{\psi/a} \text{MA}(\varphi) \leq \int_{\{\varphi < \psi - t\}} e^{-\varphi/a} \omega^n.$$ 
It follows from Lemma 2.3 that
$$\text{Vol}_\omega \leq \exp \left( \frac{-C_1}{\text{Cap}_\omega^{1/n}} \right).$$ 
Thus using Hölder inequality we get from (3.2) that
$$s^n \text{Cap}_\psi(\varphi < \psi - t - s) \leq C_2 (\text{Cap}_\omega(\varphi < \psi - t))^2 \leq C_3 (\text{Cap}_\psi(\varphi < \psi - t))^2,$$
where $C_3$ depends only on $\int_X e^{-2\varphi/a} \omega^n$. Now, consider the following function
$$H(t) = \left[ \text{Cap}_\psi(\{|\varphi < \psi - t\}|) \right]^{1/n}, \ t > 0.$$ 
By the arguments above we get
$$sH(t + s) \leq C_4 H(t)^2, \ \forall t > 0, \forall s \in [0, 1],$$
where $C_4 > 0$ depends only on $\int_X e^{-2\varphi/a} \omega^n$. It follows from Lemma 2.6 that $H$ is right-continuous and $H(+\infty) = 0$. Thus by Lemma 2.4 we get $\varphi \geq \psi - C_5$, where $C_5$ only depends on $\int_X e^{-2\varphi/a} \omega^n$.

Now, assume that $\phi$ is bounded on a compact subset $K \subset X$. Set $\psi := a\phi$ as above. Let us prove that $\varphi$ is continuous on $K$. For convenience, we normalize $\varphi$ so that $\sup_X \varphi = -1$. Let $0 \geq \varphi_j$ be a sequence of continuous $\omega$-psh functions on $X$ decreasing to $\varphi$. Fix $\lambda \in (0, 1)$. For each $j \in \mathbb{N}$ set
$$\psi_j := \lambda \varphi_j + (1 - \lambda)\psi - (1 - \lambda)A - 2.$$ 
Then $\psi_j$ belongs to $\text{PSH}(X, \frac{1+\lambda}{2} \omega)$ and $\psi_j \leq \varphi_j - 2$. Set
$$H_j(t) := \left[ \text{Cap}_{\psi_j}(\{|\varphi < \psi_j - t\}|) \right]^{1/n}, \ t > 0.$$ 
We can argue as above and use Proposition 2.8 to get
$$sH_j(t + s) \leq C_1 H_j(t)^2, \ \forall t > 0, \forall s \in [0, 1],$$
where $C_1 > 0$ depends on $\int_X e^{-2\varphi/(1-\lambda)n}$. Let $\chi : \mathbb{R}^- \to \mathbb{R}^-$ be an increasing convex weight such that $\chi(0) = 0$, $\chi(-\infty) = -\infty$ and $\varphi \in \mathcal{E}_\chi(X,\omega)$. By the comparison principle we also get

$$\text{Cap}_{\varphi_j}(\varphi < \psi_j) \leq \int_{\{\varphi < \psi_j+1\}} \text{MA} (\varphi) \leq \int_{\{\varphi < \psi_j-1\}} \text{MA} (\varphi) \leq \frac{1}{\chi(-1)} \int_X (-\chi \circ (\varphi - \varphi_j)) f\omega^n.$$ 

The latter converges to 0 as $j \to +\infty$, since $\varphi_j$ decreases to $\varphi$. Thus for $j$ big enough we have $H_j(0) \leq 1/(2C_1)$. It then follows from [22] Remark 2.5 that $H_j(t) = 0$ if $t \geq t_\infty$ where $t_\infty \leq C_2 H_j(0)$ and $C_2$ depends on $C_1$. We then get

$$\varphi \geq \lambda \varphi_j + (1-\lambda)\psi - C_2 H_j(0).$$

Now, letting $j \to +\infty$, we get

$$\lim_{j \to +\infty} \inf_K (\varphi - \varphi_j) \geq (\lambda - 1) \sup_K |\psi|. \tag{3.4}$$

Finally, letting $\lambda \to 1$ we get the continuity of $\varphi$ on $K$. \hfill \Box

### 3.2. Laplacian estimate. The following a priori estimate generalizes [29].

**Theorem 3.2.** Let $\mu$ be a positive measure on $X$ of the form $\mu = e^{\psi^+ - \psi^-}\omega^n$ where $\psi^+, \psi^-$ are smooth on $X$. Let $\varphi \in C^\infty(X)$ be such that $\sup_X \varphi = 0$ and

$$(\omega + dd^c \varphi)^n = e^{\psi^+ - \psi^-}\omega^n.$$ 

Assume given a constant $C > 0$ such that

$$dd^c \psi \pm \geq -C\omega, \sup_X \psi^\pm \leq C.$$ 

Assume also that the holomorphic bisectional curvature of $\omega$ is bounded from below by $-C$. Then there exists $A > 0$ depending on $C$ and $\int_X e^{-2(4C+1)}\omega^n$ such that

$$0 \leq n + \Delta_\omega \varphi \leq Ae^{-2\psi^-}.\tag{3.5}$$

We follow the lines in Appendix B of [7]. We recall the following result:

**Lemma 3.3.** Let $\alpha, \beta$ be positive $(1,1)$-forms. Then

$$n \left( \frac{\alpha^n}{\beta^n} \right)^{\frac{1}{n}} \leq \text{tr}_\alpha (\beta) \leq n \left( \frac{\alpha^n}{\beta^n} \right)^{\frac{n}{n-1}}.$$ 

**Proof of Theorem 3.2.** Set $\omega_\varphi := \omega + dd^c \varphi$. Since the holomorphic bisectional curvature of $\omega$ is bounded from below by $-C$, it follows from Lemma 2.2 in [10] that

$$\Delta_{\omega_\varphi} \log \text{tr}_{\omega_\varphi}(\omega_\varphi) \geq \frac{\text{tr}_{\omega_\varphi}(dd^c \psi^+ - dd^c \psi^-)}{\text{tr}_{\omega_\varphi}(\omega_\varphi)} - C\text{tr}_{\omega_\varphi}(\omega). \tag{3.3}$$

Since $dd^c \psi^+ \geq -C\omega$, using the trivial inequality $n \leq \text{tr}_{\omega_\varphi}(\omega_\varphi)\text{tr}_{\omega_\varphi}(\omega_\varphi)$ we thus get from (3.3) that

$$\Delta_{\omega_\varphi} \log \text{tr}_{\omega_\varphi}(\omega_\varphi) \geq \frac{\text{tr}_{\omega_\varphi}(C\omega + dd^c \psi^-)}{\text{tr}_{\omega_\varphi}(\omega_\varphi)} - C\text{tr}_{\omega_\varphi}(\omega) \tag{3.4}$$

$$\geq -2C\text{tr}_{\omega_\varphi}(\omega) - \frac{\Delta \psi^-}{\text{tr}_{\omega_\varphi}(\omega_\varphi)}.$$
By assumption we have $0 \leq C\omega + dd^c\psi^- \leq \text{tr}_{\omega_\varphi} (C\omega + dd^c\psi^-)\omega_\varphi$. Applying $\text{tr}_{\omega}$ to the previous inequality yields

$$Cn + \Delta \psi^- \leq (C\text{tr}_{\omega_\varphi}(\omega) + \Delta_{\omega_\varphi}\psi^-)\text{tr}_{\omega}(\omega_\varphi),$$

and hence

$$-\Delta \psi^- \geq -(C\text{tr}_{\omega_\varphi}(\omega) + \Delta_{\omega_\varphi}\psi^-)\text{tr}_{\omega}(\omega_\varphi).$$

Thus, plugging this into (3.4) we obtain

$$(3.5) \quad \Delta_{\omega_\varphi} \log \text{tr}_{\omega}(\omega_\varphi) \geq -3C\text{tr}_{\omega_\varphi}(\omega) - \Delta_{\omega_\varphi}\psi^-.$$ 

We want now to apply the maximum principle to the function

$$H := \log \text{tr}_{\omega}(\omega_\varphi) + 2\psi^-(1 + 4C)\varphi,$$

Let $x_0 \in X$ be such that $H$ achieves its maximum on $X$ at $x_0$. Then at $x_0$ we get

$$0 \geq \Delta_{\omega_\varphi}H \geq \text{tr}_{\omega_\varphi}(\omega) - n(1 + 4C).$$

Furthermore, by Lemma 3.3 we get

$$\text{tr}_{\omega}(\omega_\varphi)(x_0) \leq ne^{\psi^+ - \psi^-}(x_0) \left(\text{tr}_{\omega_\varphi}(\omega)\right)^{n-1}(x_0) \leq A_1 e^{\psi^+ - \psi^-}(x_0),$$

and hence, since $\sup_X \psi^+ \leq C$,

$$\log \text{tr}_{\omega}(\omega_\varphi)(x_0) \leq \log A_1 + \psi^+(x_0) - \psi^-(x_0) \leq A_2 - \psi^- (x_0).$$

It follows that

$$H(x) \leq H(x_0) \leq A_3 + \psi^-(x_0) - (1 + 4C)\varphi(x_0).$$

By assumption and the $C^0$ estimate in Theorem 1.1 we have $\varphi \geq a\psi^- - A_4$, where $a = 1/(4C + 1)$ and $A_4$ depends on $C$ and $\int_X e^{-2\varphi/a} \omega^n$. Thus

$$\log \text{tr}_{\omega}(\omega_\varphi) \leq A_5 - 2\psi^-.$$ 

We finally infer as desired

$$\text{tr}_{\omega}(\omega_\varphi) \leq A_6 e^{-2\psi^-}. \quad \square$$

We are now ready to prove Theorem 1.1.

3.3. **Proof of Theorem 1.1.** Let $\varphi \in \mathcal{E}(X, \omega)$ be the unique normalized solution to

$$(\omega + dd^c\varphi)^n = f \omega^n.$$

By assumption we can write $\log f = \psi^+ - \psi^-$, where $\psi^\pm$ are quasi psh functions on $X$, $\psi^-$ is locally bounded on $X \setminus D$, and there is a uniform constant $C > 0$ such that

$$dd^c\psi^\pm \geq -C\omega, \quad \sup_X \psi^+ \leq C.$$

We now approximate $\psi^\pm$ by using Demailly’s regularization operator $\rho_\varepsilon$. We recall the construction: if $u$ is a quasi-psh function on $X$ and $\varepsilon > 0$ we set

$$\rho_\varepsilon(u)(z) := \frac{1}{\varepsilon^{2n}} \int_{\xi \in TX} u(\text{exph}_z(\xi)) \chi \left( |\xi|^2 / \varepsilon^2 \right) d\lambda(\xi).$$

Here $\chi \in C^\infty(\mathbb{R})$ is a cut-off function supported in $[-1, 1]$, $\int_{\mathbb{R}} \chi(t) dt = 1$, and $\text{exph} : TX \to X$, $\xi \mapsto \text{exph}_z(\xi)$ is the formal holomorph part of the Taylor expansion of the exponential map defined by the metric $\omega$. For more details, see [17]. Observe that by Jensen’s
inequality, \( \rho_\varepsilon(e^u) \geq e^{\rho_\varepsilon(u)} \). Applying this smoothing regularization to \( \psi^\pm \) we get, for \( \varepsilon > 0 \) small enough,
\[
\frac{ddc}{\varepsilon}\rho_\varepsilon(\psi^\pm) \geq -C_1\omega, \quad e^{\rho_\varepsilon(\psi^+ - \psi^-)} \leq e^{-\rho_\varepsilon(\psi^-) + C_1},
\]
where \( C_1 \) depends on \( C \) and the Leelong numbers of the currents \( C\omega + \frac{dd^c\psi^\pm}{\varepsilon} \). Now, for each \( \varepsilon > 0 \), let \( \varphi_\varepsilon \in C^{\infty}(X) \) be the unique normalized solution to
\[
(\omega + \frac{dd^c\varphi_\varepsilon}{\varepsilon})^n = e^{\rho_\varepsilon(\psi^+ - \rho_\varepsilon(\psi^-))}\omega^n = f_\varepsilon\omega^n,
\]
where \( c_\varepsilon > 0 \) is a normalization constant. Since \( e^{\rho_\varepsilon(\log f)} \) converges point-wise to \( f \) on \( X \) and since \( e^{\rho_\varepsilon(\log f)} \leq \rho_\varepsilon(e^f) \), by the General Lebesgue Dominated Convergence Theorem we see that \( e^{\rho_\varepsilon(\log f)} \) converges to \( f \) in \( L^1(X) \) as \( \varepsilon \to 0 \). This implies that \( c_\varepsilon \) converges to \( 1 \) as \( \varepsilon \to 0 \). Then we can assume that \( c_\varepsilon \leq 2 \). Thus we get the following uniform control
\[
f_\varepsilon \leq e^{-\rho_\varepsilon(\psi^-) + C_2}.
\]
By Lemma 3.4 below we know that \( \varphi_\varepsilon \) converges to \( \varphi \) in \( L^1(X) \). Thus the set
\[
\mathcal{U} := \{ \varphi_\varepsilon \mid \varepsilon > 0 \} \cup \{ \varphi \}
\]
is compact in \( L^1(X) \). Then it follows from the uniform Skoda integrability theorem (Lemma 3.5 below) that for any \( A > 0 \) we have
\[
\sup_{\varepsilon > 0} \int_X e^{-A\varphi_\varepsilon}\omega^n < +\infty.
\]
Thus, we can apply Theorem 3.2 to find \( C_3 > 0 \) under control such that
\[
\Delta_\omega\varphi_\varepsilon \leq C_3 e^{-2\psi^-}.
\]

Fix a compact \( K \subset X \setminus D \), \( k \geq 2 \) and \( \beta \in (0, 1) \). Now since \( 0 < f \in C^\infty(X \setminus D) \) we have uniform controls on the derivatives of all orders of \( \log f_\varepsilon \) on \( K \). Using the standard Evans-Krylov method and Schauder estimates we then obtain
\[
\|\varphi_\varepsilon\|_{C^{k,\beta}(K)} \leq C_{K,k,\beta}.
\]
This explains the smoothness of \( \varphi \) on \( X \setminus D \).

**Lemma 3.4.** Let \((X, \omega)\) be a compact Kähler manifold of dimension \( n \). Let \((f_j)\) be a sequence of non-negative functions on \( X \) such that \( \int_X f_j\omega^n = \int_X \omega^n \). Assume that \( f_j \) converges in \( L^1(X) \) and point-wise to \( f \). For each \( j \), let \( \varphi_j \in \mathcal{E}(X, \omega) \) be the unique normalized solution to \( MA(\varphi_j) = f_j\omega^n \). Then \( \varphi_j \) converges in \( L^1(X) \) to \( \varphi \in \mathcal{E}(X, \omega) \) the unique normalized solution to \( MA(\varphi) = f\omega^n \).

**Proof.** We can assume that \( \varphi_j \) converges in \( L^1(X) \) to \( \psi \in \text{PSH}(X, \omega) \). It follows from the Hartogs lemma that \( \sup_X \psi = 0 \). For each \( j \in \mathbb{N} \) set
\[
\psi_j := \left( \sup_{k \geq j} \varphi_j \right)^* \quad \text{and} \quad u_j := \max(\psi_j, \varphi - 1).
\]
Then we see that \( \psi_j \downarrow \psi \) and \( u_j \downarrow u := \max(\psi, \varphi - 1) \in \mathcal{E}(X, \omega) \). We also have that \( \sup_X u = 0 \). It follows from the comparison principle that
\[
MA(u_j) \geq \min \left( f, \inf_{k \geq j} f_k \right) \omega^n = g_j\omega^n.
\]
By the continuity of the Monge-Ampère operator along decreasing sequences in $\mathcal{E}(X, \omega)$ we get
\[
\text{MA}(u) = \lim_{j \to +\infty} \text{MA}(u_j) \geq \lim_{j \to +\infty} g_j \omega^n = f \omega^n.
\]
Then the equality holds since they have the same total mass. Finally, by the uniqueness result in the class $\mathcal{E}(X, \omega)$ (see [19]) we deduce that $u = \varphi$, which implies that $\psi = \varphi$. The proof is thus complete.

By [24], functions in $\mathcal{E}(X, \omega)$ have zero Lelong number at every point on $X$.

Thus the following lemma is a direct consequence of the uniform Skoda integrability theorem due to Zeriahi [38]:

**Lemma 3.5.** Let $U$ be a compact family of functions in $\mathcal{E}(X, \omega)$. Then for each $C_1 > 0$ there exists $C_2$ depending on $C_1$ and $U$ such that
\[
\int_X e^{-C_1 \phi} \omega^n \leq C_2, \forall \phi \in U.
\]

### 4. Asymptotic behavior near the divisor

In Theorem 3.1 we have given a very general $C^0$ estimate. We only assumed that the density $f$ is bounded by $e^{-\phi}$ for some quasi plurisubharmonic function $\phi$, and there is no regularity assumption on $D$. It is therefore natural to investigate the asymptotic behavior of the solution near $D$ when we have more information about $D$ and about the behavior of $f$ near $D$.

Let $X$ be a compact Kähler manifold of dimension $n$ and let $\omega$ be a Kähler form on $X$. Let $D = \sum_{j=1}^N D_j$ be a simple normal crossing divisor on $X$. Here "simple normal crossing" means that around each intersection point of $k$ components $D_{j_1}, ..., D_{j_k}$ ($k \leq N$), we can find complex coordinates $z_1, ..., z_n$ such that for each $l = 1, ..., k$ the hypersurface $D_{j_l}$ is locally given by $z_l = 0$. For each $j$, let $L_j$ be the holomorphic line bundle defined by $D_j$. Let $s_j$ be a holomorphic section of $L_j$ defining $D_j$, i.e. $D_j = \{ s_j = 0 \}$. We fix a hermitian metric $h_j$ on $L_j$ such that $|s_j|_{h_j} \leq 1/e$.

We say that $f$ satisfies Condition $S(B, \alpha)$ for some $B > 0, \alpha > 0$ if
\[
(4.1) \quad f \leq \frac{B}{\prod_{j=1}^N |s_j|^2(-\log |s_j|)^{1+\alpha}}.
\]

#### 4.1. Volume-capacity domination

**Lemma 4.1.** Assume that $f$ satisfies (4.1) for some $B > 0, \alpha > 0$. Then for each $0 < \gamma < \alpha$ we can find $A > 0$ which only depends on $B, \alpha, \gamma, \omega$ such that
\[
\text{Vol}_f(E) := \int_E f \omega^n \leq A \text{Cap}_\omega(E)^\gamma, \forall E \subset X,
\]
where $\text{Cap}_\omega$ is the Monge-Ampère capacity introduced in [28], [23].

Before giving the proof of the lemma, let us recall the definition and basic facts about Cegrell’s classes. We refer the reader to [11, 12] for more details.

Let $\Omega$ be a bounded hyperconvex domain in $\mathbb{C}^n$. The class $\mathcal{E}_0(\Omega)$ consists of bounded psh functions which vanish on the boundary and have finite total mass.
We say that \( u \in \mathcal{E}^p(\Omega) \) if there exists a sequence \((u_j) \subset \mathcal{E}_0(\Omega)\) decreasing to \( u \) such that

\[
\sup_j \int_{\Omega} (-u_j)^p (dd^c u_j)^n < +\infty.
\]

A function \( u \) belongs to \( \mathcal{F}(\Omega) \) if there exists a sequence \((u_j) \subset \mathcal{E}_0(\Omega)\) decreasing to \( u \) such that

\[
\sup_j \int_{\Omega} (dd^c u_j)^n < +\infty.
\]

We recall the local Monge-Ampère capacity introduced in \[3\]: for any Borel subset \( E \subset \Omega \), we define

\[
\text{Cap}_{BT}(E, \Omega) := \sup \left\{ \int_E (dd^c u)^n \mid u \in \text{PSH}(\Omega), -1 \leq u \leq 0 \right\}.
\]

The relative extremal function of \( E \) with respect to \( \Omega \) is

\[
u_{E, \Omega} := \sup \left\{ u \in \text{PSH}(\Omega) \mid u \leq 0 \text{ on } \Omega, u \leq -1 \text{ on } E \right\}.
\]

**Proof of Lemma 4.1.** It follows from \(\ref{28}\) that \(\text{Cap}_{\omega} \) is comparable to the local capacity \(\text{Cap}_{BT}(\cdot, \Omega)\), where \( \Omega \) is an open subset contained in a local chart. By considering \( E \) a small subset contained in a local chart we reduce the problem to showing that

\[
\text{Vol}_g(E) \leq A_1 \text{Cap}_{BT}(E, D^n)^{\alpha}, \forall E \Subset D^n \Subset \mathbb{D}^n,
\]

where \( \mathbb{D}^n \) is the unit polydisk in \( \mathbb{C}^n \), \( \delta > 0 \) small enough and fixed, and

\[
g(z) = g(z_1, \ldots, z_n) := \frac{1}{\prod_{j=1}^k |z_j|^2(1 - \log |z_j|)^{1+\alpha}}, k \leq n.
\]

We prove \((\ref{4.2})\) by induction using the ideas in \(\ref{2}\). We start with the case \( n = 1 \).

Set \( E_r := E \cap \partial \mathbb{D}_r \), for any \( r \in [0, t] \). Define now \( E := \{ r \in [0, t] \mid E_r \neq \emptyset \} \) and denote by \( l(E) \) the length of \( E \). Since the function \( r \mapsto \frac{1}{r(1 - \log r)^{1+\alpha}} \) is non-increasing when \( r \) is small, we obtain

\[
\int_E g(z) dV(z) = \int_0^{2\pi} \int_{E_r} \frac{drd\theta}{r(1 - \log r)^{1+\alpha}} \leq 2\pi \int_0^{l(E)} \frac{dr}{r(1 - \log r)^{1+\alpha}} \leq \frac{C_1}{(\log l(E))^{\alpha}} \leq C_2 \text{Cap}_{BT}(E, D)^{\alpha},
\]

where the last inequality follows from \(\ref{29}\) p.1336.

Assume that the result holds for \( n - 1 \). Let us prove it for \( n \). Without loss of generality we can assume that \( E \) is compact in \( \mathbb{D}^n \). We can also assume that \( k = n \) (if \( k < n \) the situation is much easier). Set \( h = h_{E, D^n} \) the relative extremal function of \( E \). Consider

\[
g_n(w) := \frac{1}{|w|^2(1 - \log |w|)^{1+\alpha}} \quad g_{n-1}(z) := \frac{1}{\prod_{j=1}^{n-1} |z_j|^2(1 - \log |z_j|)^{1+\alpha}}.
\]

For each \( w \in \mathbb{D} \) set

\[
E_w = \{ z \in \mathbb{D}^{n-1} \mid h(z, w) \leq -1 \} \quad \text{and} \quad h_w = h(\cdot, w).
\]
By induction hypothesis we get
\[
\text{Vol}_g(E) = \int_{\mathbb{D}} \text{Vol}_{g_{n-1}}(E_w)g_n(w)dV_2(w) \leq A_1 \int_{\mathbb{D}} \left[\text{Cap}_{BT}(E_w, \mathbb{D}^{n-1})\right]^{-1} g_n(w)dV(w).
\]

Fix now \(w \in \mathbb{D}\) and denote by \(u = h^*_{E_w, w}\) the relative extremal function of \(E_w\). Since \(h \in \mathcal{F}(\mathbb{D}^n)\) it follows from [2, Theorem 3.1] that \(h_w \in \mathcal{E}(\mathbb{D}^{n-1})\). We also have \(h_w \leq u\) and \(h_w = -1\) on \(E_w\). Using integration by parts we get
\[
\text{Cap}_{BT}(E_w, \mathbb{D}^{n-1}) \leq \int_{\mathbb{D}^{n-1}} (-h_w)(dd^c u)^{n-1} \leq \int_{\mathbb{D}^{n-1}} (-h_w)(dd^c h_w)^{n-1} =: -\varphi(w).
\]

By [2, Theorem 3.1] we know that \(\varphi \in \mathcal{F}(\mathbb{D})\). Moreover, we also have \(\varphi \geq -A_0\) for some universal constant \(A_0\) (here \(A_0\) depends on \(\delta\)). Indeed, let \(v\) be the relative extremal function of \(\mathbb{D}^n\) with respect to \(\mathbb{D}^n\). Since \(h \geq v\), it is easy to see that for each \(w \in \mathbb{D}, h_w \geq v_w\). From this we get a uniform lower bound for \(\varphi\). Since \(E\) is compact in \(\mathbb{D}^n\) we also get
\[
\mu = \int_{\mathbb{D}} dd^c \varphi = \int_{\mathbb{D}^n} (dd^c h)^n = \text{Cap}_{BT}(E, \mathbb{D}^n).
\]

Thus, using the previous part (when \(n = 1\)) we obtain
\[
\text{Vol}_g(E) \leq A_1 \int_{\mathbb{D}} (-\varphi(w))^{-1} g_n(w)dV_2(w)
= A_2 \int_0^{A_0} t^{\gamma - 1} \text{Vol}_{g_n}(\varphi < -t)dt
\leq A_3 \int_0^{A_0} t^{\gamma - \beta_1 - 1} \mu^{\beta_1} dt
= A_4 \left[\text{Cap}_{BT}(E, \mathbb{D}^n)\right]^{\beta_1}.
\]

Here, we choose \(\beta_1 < \gamma\) so that the integrals converge. In the above we have used the fact that
\[
\text{Cap}_{BT}(v < -t) \leq \frac{1}{t} \int_{\mathbb{D}} dd^c v, \ \forall v \in \mathcal{F}(\mathbb{D}), \ \forall t > 0.
\]

Since \(\beta_1\) can be chosen arbitrarily near \(\gamma\) (and the constant \(A_4\) will increase), the result follows.

When \(\alpha = 1\) we get the following estimate.

Lemma 4.2. Let \(\mu = f\omega^n, f = \prod_{j=1}^n \frac{1}{|s_j|^{\gamma - (-\log |s_j|)^2}}\). Then there exists \(A > 0\) such that for every Borel subset \(E \subset X\) we have
\[
\mu(E) \leq A \cdot [\eta + (-\log \eta)^n \text{Cap}_\omega(E)], \ \forall \eta \in (0, 1/e).
\]

Proof. We only give a sketch of the proof since it is essentially a copy of the proof of Lemma 4.1 with a small change. We also use the same notation as there. Without loss of generality we can assume that \(E \in \mathbb{D}_0^n \subset \mathbb{D}^n\) for some small fixed \(\delta\). The function \(\varphi\) belongs to \(\mathcal{F}(\mathbb{D})\). The same arguments as in Lemma 4.1 show that \(\varphi\) is also bounded from below by \(-A_1\) for some universal constant \(A_1 > 0\). In the final
We get
\[ \text{Vol}_g(E) \leq A_2 \int_D (\eta + \log \eta)^{n-1}(\varphi(w)) g_\eta(w) dV_2(w) \]
\[ = A_3\eta + A_2(-\log \eta)^{n-1} \int_0^{A_1} \text{Vol}_{g_\eta}(\varphi < -t) dt \]
\[ \leq A_3\eta + A_4\eta^2(-\log \eta)^{n-1} + A_5(-\log \eta)^{n-1} \int_{\eta^2}^{A_1} \text{Cap}_\eta(\varphi < -t) dt \]
\[ \leq A_6\eta + A_5(-\log \eta)^{n-1} \int_{\eta^2}^{A_1} \frac{1}{t} \left[ \int_D dd^c \varphi \right] dt \]
\[ \leq A_6\eta + A_7(-\log \eta)^n \int_D dd^c \varphi. \]
\[ \square \]

**Lemma 4.3.** Let \( \varphi \in \mathcal{E}(X, \omega) \) be such that \( \sup_X \varphi = 0 \) and \( \mu = MA(\varphi) \) satisfies (4.3) for some \( A > 0 \). Then there exists \( C, c > 0 \) depending on \( A \) such that
\[ \text{Cap}_\omega(\varphi < -t) \leq Ce^{-ct}, \forall t > 0, \]
In particular, if \( \beta < c \) then
\[ \int_X e^{-\beta \varphi} d\mu \leq C', \text{ with } C' = C(\beta, A) > 0. \]

**Proof.** Fix \( s, t > 1 \). By standard application of the comparison principle we get
\[ (4.4) \quad \text{Cap}_\omega(\varphi < -t-s) \leq \int_{\{\varphi < -t\}} \left( \omega + \frac{1}{s} dd^c \varphi \right)^n \]
\[ \leq \frac{1}{s^n} \int_{\{\varphi < -t\}} \sum_{k=0}^{n} C_n^k (s-1)^k \omega^k \wedge \omega^{n-k} \]
\[ \leq \int_{\{\varphi < -t\}} \omega^n + \frac{2^n}{s} \int_{\{\varphi < -t\}} \text{MA}(\varphi), \]
where the last inequality follows from the partial comparison principle (see [19, Theorem 2.3]). It follows from [23] that
\[ \int_{\{\varphi < -t\}} \omega^n \leq C_1 e^{-at}, a > 0. \]
Choose \( s := 2^n Ae \) and fix \( \varepsilon < \min(1, a, 1/s) \). Set
\[ F(t) := \frac{e^{ct}}{tn} \text{Cap}_\omega(\varphi < -t), t \geq 1. \]
Now, if we choose \( \eta = e^{-t} \) in (4.3) and plug (4.3) into (4.4) we get
\[ F(t^2 + s) \leq C_2 + bF(t), \]
where \( b = 2^n Ae^{\varepsilon/s} / s < 1 \). This yields \( \sup_{t \geq 1} F(t) \leq C_3 \), for some \( C_3 > 0 \) depending on \( A \). We finally get
\[ \text{Cap}_\omega(\varphi < -t) \leq Ce^{-ct}, c < \varepsilon. \]
The last statement easily follows since it follows from [23, Lemma 2.3] that
\[ \int_{\{\varphi < -t\}} \text{MA}(\varphi) \leq t^n \text{Cap}_\omega(\varphi < -t), \forall t \geq 1. \]
\[ \square \]
4.2. **Proof of Theorem 2.** Assume in this section that \( f \) satisfies Condition \( S(B, \alpha) \) for some \( B > 0, \alpha > 0 \). The first part of Theorem 2 was proved in Theorem 3.1. We divide the remaining parts into three cases depending on the value of \( \alpha \).

4.2.1. **The case when \( \alpha > 1 \).** The continuity of \( \varphi \) and the \( C^0 \) estimate follow directly from Lemma 4.1 and Kolodziej’s classical result (see [27]).

4.2.2. **The case when \( 0 < \alpha < 1 \).** Fix \( \beta \in (1-\alpha, 1) \) and set \( \delta = \alpha + \beta - 1 \), and
\[
 u_\beta := \sum_{j=1}^{N} -a(-\log |s_j|)^{\beta},
\]
where \( a > 0 \) is small enough so that \( u_\beta \in \text{PSH}(X, \omega) \). By Theorem 3.1 we have
\[
 \varphi \geq \sum_{j=1}^{N} \log |s_j| - C_0,
\]
for some positive constant \( C_0 \) depending on \( B \). By simple computations we obtain
\[
 \text{MA}(\varphi) \leq \frac{C_1 f_1^{-\beta} \omega^n}{(-\varphi)^\delta},
\]
for some positive constant \( C_1 \) depending on \( C_0 \). Here for each \( r > 0 \), we set \( f_r := \prod_{i=1}^{N} |s_j|^2 (-\log |s_j|)^{1+r} \).

We also get
\[
 \text{MA}(u_\beta - C_2) \geq \frac{C_1 f_1^{-\beta} \omega^n}{(-u_\beta + C_2)^\delta},
\]
where \( C_2 > 0 \) depends on \( C_1, \delta \). The comparison principle yields that \( \varphi \geq u_\beta - C_2 \).

4.2.3. **The case when \( \alpha = 1 \).** Consider the model function
\[
 \psi := -A_1 \sum_{j=1}^{N} \log(-\log |s_j| + A_2),
\]
where \( A_1 > 0 \) is big and \( A_2 \) is chosen so that \( \psi \) is \( \omega/2 \)-psh on \( X \). It follows from Lemma 4.2 and Lemma 4.3 that \( \int_X e^{-c\varphi} f \omega^n < C_1 \) for some small constant \( c > 0 \) depending on \( B \). Here \( C_1 \) depends on \( c \) and \( B \). Thus, for \( t > 0, p > 1 \), by Hölder inequality we get
\[
 \int_{\{\varphi < \psi - t\}} \text{MA}(\varphi) \leq \int_{\{\varphi < \psi - t\}} e^{-c\varphi/p} e^{\psi/p} f \omega^n \leq \left( \int_X e^{-c\varphi} f \omega^n \right)^{1/p} \left( \int_{\{\varphi < \psi - t\}} e^{\psi/(p-1)} f \omega^n \right)^{1-1/p} \leq C_2 \left( \text{Cap}_\psi(\varphi < \psi - t) \right)^\gamma,
\]
where \( \gamma < A_1 c/p + (p-1)/p \) and \( C_2 > 0 \) is a universal constant. The last inequality follows from the volume-capacity domination (Lemma 4.1) and from Lemma 2.7.

Now if \( A_1 c > 1 \) we can choose \( \gamma > 1 \) and the result follows as in Theorem 3.1.
4.3. Regularity near the divisor $D$. In this subsection we will discuss about the behavior of the solution to equation \eqref{eq:1.1} near the divisor $D$. We prove the following result when $\alpha < 1$.

**Proposition 4.4.** Consider $f = \prod_{j=1}^{N} |s_j|^2(-\log|s_j|)^{1+r}$, where $1/B \leq h \leq B$ on $X$ and $\alpha \in (0,1)$. Assume that $f$ is normalized so that $\int_X f \omega^n = \int_X \omega^n$. Let $\varphi \in \mathcal{E}(X, \omega)$ be the unique normalized solution of \eqref{eq:1.1}. Then for each $0 < p < 1 - \alpha$ and each $1 - \alpha < q < 1$, we have

$$-a_1(-\log|s|)^q - A_1 \leq \varphi \leq -a_2(-\log|s|)^p + A_2,$$

where $a_1, A_1 > 0$ depend on $B, \alpha, q$ while $a_2, A_2 > 0$ depend on $B, \alpha, p$. In particular, the solution $\varphi$ goes to $-\infty$ on $D$.

**Proof.** One inequality has been proved in Theorem\footnote{2}. Let us prove the upper bound. We normalize $\varphi$ such that $\sup_X \varphi = -1$. To simplify the notation we denote, for each $r > 0$,

$$f_r := \frac{1}{\prod_{j=1}^{N} |s_j|^2(-\log|s_j|)^{1+r}}.$$

Fix $p \in (0, 1 - \alpha)$ and set $\delta := (1 - \alpha - p)/p > 0$. Consider $u_p := -\sum_{j=1}^{N} a_2(-\log|s_j|)^p$, where $a_2 > 0$ is small so that $u_p$ is $\omega$-psh on $X$. Then we can find $C_3 > 0$ such that

$$\text{MA}(u_p) \leq \frac{C_3 f \omega^n}{(-u_p)^{\delta}},$$

while since $\varphi \leq 0$, for some $A_2 > 0$ big enough (for instance $A_2^6 = C_3$) we have

$$\text{MA}(\varphi - A_2) \geq \frac{C_3 f \omega^n}{(-\varphi + A_2)^{\delta}}.$$

The comparison principle then yields the desired upper bound.

By the same way we obtain a similar upper bound when $\alpha = 1$.

**Proposition 4.5.** Assume that $f$ is normalized so that $\int_X f \omega^n = \int_X \omega^n$ and

$$f \geq \frac{1}{B \prod_{j=1}^{N} |s_j|^2(-\log|s_j|)^2}.$$

Let $\varphi \in \mathcal{E}(X, \omega)$ be the unique normalized solution of \eqref{eq:1.1}. Then for any $p \in (0, 1)$ there exist $a, A > 0$ depending on $B, \alpha$ such that

$$\varphi \leq -a \sum_j |\log(-\log|s_j|)|^p + A.$$

In particular, $\varphi$ is not bounded and goes to $-\infty$ on $D$.

**Proof.** The proof uses the same arguments as in Proposition\footnote{4.4}.

5. The case of semipositive and big classes

In this section we prove Theorem\footnote{3}. For convenience let us recall the setting. We assume that $(X, \omega)$ is a compact Kähler manifold of dimension $n$ and $D$ is an arbitrary divisor on $X$. Let $E = \sum_{j=1}^{M} a_j E_j$ be an effective snc divisor on $X$. Let $\theta$ be a smooth semipositive form on $X$ such that $\int_X \theta^n > 0$ and $\{\theta\} - c_1(E)$ is ample. Consider the following degenerate complex Monge-Ampère equation

\begin{equation}
(\theta + dd^c \varphi)^n = f \omega^n,
\end{equation}
where \(0 \leq f \in L^1(X, \omega^n)\) satisfies the compatibility condition \(\int_X f \omega^n = \int_X \theta^n\).

For each \(j = 1, \ldots, M\) let \(K_j\) be the holomorphic line bundle defined by \(E_j\). Let \(\sigma_j\) be a holomorphic section of \(K_j\) that vanish on \(E_j\). We fix hermitian metric \(h_j\) on \(K_j\) such that \(|\sigma_j| \leq 1/e\). Since \(\{\theta - c_1(E)\} - c_1(E)\) is ample, we can assume that

\[
\theta + dd^c \phi = \omega_0 + |E|,
\]

where \(\omega_0\) is a Kähler form on \(X\) and

\[
\phi := \sum_{j=1}^{M} a_j \log |\sigma_j|.
\]

By rescaling \(\omega\) we can also assume that \(\omega_0 \geq \omega\). Recall that \(f\) satisfies Condition \(H_f\) on \(X\), i.e. there is a constant \(C > 0\) such that

\[
f = e^{\psi^+ - \psi^-}, \quad dd^c \psi^\pm \geq -C\omega, \quad \sup_X \psi^\pm \leq C, \quad \psi^- \in L^\infty_{\text{loc}}(X \setminus D).
\]

### 5.1. Uniform estimate

The following \(C^0\)-lower bound can be proved in the same ways as we have done in Theorem 3.1:

**Theorem 5.1.** Assume that \(D, E\) and \(\theta\) are as above and \(f\) satisfies (5.2). Let \(\varphi\) be the unique normalized solution to equation (5.1). Then \(\varphi\) is uniformly bounded away from \(D \cup E\). More precisely, for any \(a > 0\) there exists \(A > 0\) depending on \(C\) and

\[
\int_X e^{-2\varphi/a} \omega^n
\]

such that

\[
\varphi \geq a\psi^- + \phi - A.
\]

**Proof.** It suffices to prove the result for small \(a > 0\). Fix \(a > 0\) very small so that

\[
\psi := a\psi^- + \frac{1}{2}\phi \in \text{PSH}(X, \theta/2).
\]

It follows from Proposition 3.1 in [22] that

\[
\text{Vol}_\omega \leq C_1 \exp \left( \frac{-C_2}{[\text{Cap}_{\theta/2}]^{1/n}} \right),
\]

for some universal constants \(C_1, C_2 > 0\). Now, the same proof of Lemma 2.7 yields

\[
\text{Cap}_{\theta/2} \leq \text{Cap}_\psi,
\]

where \(\text{Cap}_\psi\) is the generalized capacity defined by the form \(\theta\) and \(\psi\):

\[
\text{Cap}_\psi(E) := \sup \left\{ \int_E (\theta + dd^c u)^n \mid u \in \text{PSH}(X, \theta), \psi - 1 \leq u \leq \psi \right\}.
\]

Then we can repeat the arguments in the proof of Theorem 3.1 to get the result. \(\square\)

### 5.2. Laplacian estimate

We now prove a \(C^2\) a priori estimate in the semipositive and big case. Even when \(f\) is smooth on \(X\), \(\varphi\) is only smooth in the ample locus of \(\theta\). To get rid of this, we replace \(\theta\) by \(\theta + t\omega\), \(t > 0\). In principle, the \(C^2\) estimate will depends heavily on \(t > 0\) and we will have serious problem when \(t \downarrow 0\).

But, fortunately, the so-called Tsuji’s trick (see [36]) allows us to get around this difficulty. In the sequel, we follow essentially the ideas in [8].
Theorem 5.2. Let \( f = e^{\psi^+ - \psi^-} \) where \( \psi^+, \psi^- \) are smooth on \( X \). Fix \( t \in (0,1) \). Let \( \varphi \in C^\infty(X) \) be the unique normalized solution to
\[
(\theta + t\omega + dd^c\varphi)^n = e^{\psi^+ - \psi^-} \omega^n.
\]
Assume given a constant \( C > 0 \) such that
\[
dd^c \psi^\pm \geq -C\omega, \quad \sup_X \psi^+ \leq C.
\]
Assume also that the holomorphic bisectional curvature of \( \omega \) is bounded from below by \(-C\). Then there exists \( A > 0 \) depending on \( C \) and \( \int_X e^{-2(4C+1)} \omega^n \) such that
\[
\Delta_{\omega} \varphi \leq Ae^{-2\psi^- - (4C+1)\varphi}.
\]

Proof. Ignoring the dependence on \( t \), we denote \( \omega_{\varphi} := \theta + t\omega + dd^c\varphi \). Consider the following function
\[
H := \log \tr_{\omega}(\omega_{\varphi}) + 2\psi^- - (4C + 1)(\varphi - \phi),
\]
Since \( \phi \) goes to \(-\infty\) on \( E \), we see that \( H \) attains its maximum on \( X \setminus E \) at some point \( x_0 \in X \setminus E \). From now on we carry all computations on \( X \setminus E \). We can argue as in Theorem 5.2 to obtain
\[
(5.3) \quad \Delta_{\omega_{\varphi}} \log \tr_{\omega}(\omega_{\varphi}) \geq -3C \tr_{\omega_{\varphi}}(\omega) - \Delta_{\omega_{\varphi}} \psi^-.
\]
Since \( \omega_0 + t\omega \geq \omega \) we get
\[
(5.4) \quad \Delta_{\omega_{\varphi}} (\varphi - \phi) \leq \tr_{\omega_{\varphi}}(\omega_{\varphi} - \omega_0 - t\omega) \leq n - \tr_{\omega_{\varphi}}(\omega).
\]
Therefore, from (5.3) and (5.4) we deduce that on \( X \setminus E \)
\[
\Delta_{\omega_{\varphi}} H \geq \tr_{\omega_{\varphi}}(\omega) - n(4C + 1).
\]
We now apply the maximum principle to the function \( H \) at \( x_0 \):
\[
0 \geq \Delta_{\omega_{\varphi}} H(x_0) \geq \tr_{\omega_{\varphi}}(\omega)(x_0) - n(4C + 1).
\]
Furthermore, by Lemma 5.3 we get
\[
\tr_{\omega}(\omega_{\varphi})(x_0) \leq ne^{\psi^+ - \psi^-}(x_0)(\tr_{\omega_{\varphi}}(\omega))^{n-1}(x_0) \leq A_1 e^{\psi^+ - \psi^-}(x_0),
\]
and hence, since \( \sup_X \psi^+ \leq C \),
\[
\log \tr_{\omega}(\omega_{\varphi})(x_0) \leq \log A_1 + \psi^+(x_0) - \psi^-(x_0) \leq A_2 - \psi^-(x_0).
\]
It follows that
\[
H(x) \leq H(x_0) \leq A_2 + \psi^-(x_0) - (4C + 1)(\varphi - \phi)(x_0).
\]
By assumption and the \( C^0 \) estimate in Theorem 5.1 we have
\[
\varphi \geq \frac{1}{4C + 1} \psi^- + \phi - A_3,
\]
where \( A_3 \) depends on \( C \) and \( \int_X e^{-2(4C+1)} \omega^n \). Thus
\[
\log \tr_{\omega}(\omega_{\varphi}) \leq A_4 - 2\psi^- + (4C + 1)(\varphi - \phi).
\]
We finally get
\[
\tr_{\omega}(\omega_{\varphi}) \leq A_5 e^{-2\psi^- - (4C+1)\varphi}.
\]
\( \square \)
Proof of Theorem 3. We proceed as in Section 3.3. We also borrow the notations there. Let \( \rho_\varepsilon(\psi^\pm) \) be the Demailly’s smoothing regularization of \( \psi^\pm \). For each \( \varepsilon > 0 \) let \( \varphi_\varepsilon \) be the unique smooth function such that \( \sup_X \varphi_\varepsilon = 0 \) and
\[
(\theta + \varepsilon \omega + dd^c \varphi_\varepsilon)^n = c_\varepsilon e^{\rho_\varepsilon(\psi^+) - \rho_\varepsilon(\psi^-)} \omega^n,
\]
where \( c_\varepsilon \) is a normalization constant. As in Section 3.3 we have a uniform control on the right-hand side:
\[
c_\varepsilon e^{\rho_\varepsilon(\psi^+) - \rho_\varepsilon(\psi^-)} \leq e^{C - \psi^-}.
\]
Now, we can copy the arguments in Section 3.3 since our uniform estimate and laplacian estimate do not depend on \( \varepsilon \). The proof is thus complete. \( \square \)

References

[1] H. Auvray, The space of Poincaré type Kähler metrics on the complement of a divisor, arXiv:1109.3159.
[2] P. Ahag, U. Cegrell, S. Kołodziej, H. H. Pham, A. Zeriahi, Partial pluricomplex energy and integrability exponents of plurisubharmonic functions, Advances Math. 222 (2009), 2036–2058.
[3] E. Bedford, B. A. Taylor, A new capacity for plurisubharmonic functions, Acta Math. 149 (1982), no. 1-2, 1-40.
[4] S. Benelkourchi, V. Guedj, A. Zeriahi, A priori estimates for solutions of Monge-Ampère equations, Annali della Scuola Sup. di Pisa Vol. VII, Issue 1 (2008), 81-96.
[5] S. Benelkourchi, V. Guedj, A. Zeriahi, Plurisubharmonic functions with weak singularities, Acta Univ. Upsaliensis Skr. Uppsala Univ. C Organ. Hist. Vol. 80 (2009), 57-74.
[6] R. J. Berman, S. Boucksom, V. Guedj, A. Zeriahi, A variational approach to complex Monge-Ampère equations, Publ. Math. Inst. Hautes Études Sci. 117 (2013), 179-245.
[7] R. J. Berman, S. Boucksom, P. Eyssidieux, V. Guedj, A. Zeriahi, Kähler-Einstein metrics and the Kähler-Ricci flow on log Fano varieties, arXiv:1111.7158.
[8] S. Boucksom, P. Eyssidieux, V. Guedj, A. Zeriahi, Monge-Ampère equations in big cohomology classes, Acta Math. 205 (2010), no. 2, 199-262.
[9] R. J. Berman, H. Guenancia, Kähler-Einstein metrics on stable varieties and log canonical pairs, arXiv:1304.2087.
[10] F. Campana, H. Guenancia, M. Păun, Metrics with cone singularities along normal crossing divisors and holomorphic tensor fields, arXiv:1104.4879.
[11] U. Cegrell, Pluricomplex energy, Acta Math. 180 (1998), no. 2, 187-217.
[12] U. Cegrell, The general definition of the complex Monge-Ampère operator, Ann. Inst. Fourier (Grenoble) 54 (2004), no. 1, 159-179.
[13] U. Cegrell, S. Kołodziej, A. Zeriahi, Subextension of plurisubharmonic functions with weak singularities, Math. Z. 250 (2005), no. 1, 7-22.
[14] X. X. Chen, S. K. Donaldson, S. Sun, Kähler–Einstein metrics on Fano manifolds, I: approximation of metrics with cone singularities, arXiv:1211.4560.
[15] X. X. Chen, S. K. Donaldson, S. Sun, Kähler–Einstein metrics on Fano manifolds, II: limits with cone angle less than 2\( \pi \), arXiv:1212.3714.
[16] X. X. Chen, S. K. Donaldson, S. Sun, Kähler–Einstein metrics on Fano manifolds, III: limits as cone angle approaches 2\( \pi \) and completion of the main proof, arXiv:1302.0282.
[17] J. P. Demailly, Regularization of closed positive currents and intersection theory, J. Alg. Geom. 1 (1992), no. 3, 361-409.
[18] E. Di Nezza, H. C. Lu, Generalized Monge–Ampère capacities, Preprint (2014).
[19] S. Dinew, Uniqueness in \( C(X, \omega) \), J. Funct. Anal. 256 (2009), no. 7, 2113-2122.
[20] S. K. Donaldson, Kähler metrics with cone singularities along a divisor, Essays in mathematics and its applications, 49-79, Springer, Heidelberg, 2012.
[21] S. K. Donaldson, S. Sun, Gromov-Hausdorff limits of Kähler manifolds and algebraic geometry, arXiv:1206.2669.
[22] P. Eyssidieux, V. Guedj, A. Zeriahi, Singular Kähler Einstein metrics, Journal of the American Mathematical Society, Volume 22, Number 3, (2009), 607-639.
[23] V. Guedj, A. Zeriahi, *Intrinsic capacities on compact Kähler manifolds*, J. Geom. Anal. 15 (2005), no. 4, 607-639.

[24] V. Guedj, A. Zeriahi, *The weighted Monge-Ampère energy of quasiplusharmonic functions*, J. Funct. Anal. 250 (2007), no. 2, 442-482.

[25] H. J. Hein, *Gravitational instantons from rational elliptic surfaces*, J. Amer. Math. Soc. 25 (2012), no. 2, 355-393.

[26] S. Kołodziej, *The Range of the complex Monge-Ampère operator*, Indiana Univ. Math. J. 43 (1994), no. 4, 1321-1338.

[27] S. Kołodziej, *The complex Monge-Ampère equation*, Acta Math. 180 (1998) 69-117.

[28] S. Kołodziej, *The complex Monge-Ampère equation on compact Kähler manifolds*, Indiana Univ. Math. J. 52 (2003), no. 3, 667-686.

[29] M. Păun, *Regularity properties of the degenerate Monge-Ampère equations on compact Kähler manifolds*, Chin. Ann. Math. Ser. B 29 (2008), no. 6, 623-630.

[30] H. Skoda, *Sous-ensembles analytiques d’ordre fini ou infini dans C^n*, Bull. Soc. Math. de France 100 (1972), 353-408.

[31] Y. T. Siu, *Lectures on Hermitian-Einstein metrics for stable bundles and Kähler-Einstein metrics*, Birkhäuser Verlag, 1987.

[32] G. Tian, *K-stability and Kähler-Einstein metrics*, arXiv:1211.4669

[33] G. Tian, S. T. Yau, *Existence of Kähler-Einstein metrics on complete Kähler manifolds and their applications to algebraic geometry*, Mathematical aspects of string theory (San Diego, Calif., 1986), Adv. Ser. Math. Phys., vol. 1, World Sci. Publishing, Singapore, 1987, p. 574-628.

[34] G. Tian, S. T. Yau, *Complete Kähler manifolds with zero Ricci curvature. I.*, J. Amer. Math. Soc. 3 (1990), no. 3, 579-609.

[35] G. Tian, S. T. Yau, *Complete Kähler manifolds with zero Ricci curvature. II*, Invent. Math. 106 (1991), no. 1, 27-60.

[36] H. Tsuji, *Existence and degeneration of Kähler-Einstein metrics on minimal algebraic varieties of general type*, Math. Ann. 281 (1988), 123-133.

[37] S. T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation*, Comm. Pure Appl. Math. 31 (1978), no. 3, 339-411.

[38] A. Zeriahi, *Volume and capacity of sublevel sets of a Lelong class of plurisubharmonic functions*, Indiana Univ. Math. J., 50 (2001), 671-703.

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