Time dependent mean-field games with logarithmic nonlinearities

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Abstract

In the present paper, we prove the existence of classical solutions for time dependent mean-field games with a logarithmic nonlinearity and subquadratic Hamiltonians. Because the logarithm is unbounded below, this nonlinearity poses substantial mathematical challenges that have not been addressed in the literature. Our result is proven by recurring to a delicate argument, which combines Lipschitz regularity for the Hamilton-Jacobi equation with estimates for the nonlinearity in suitable Lebesgue spaces. Lipschitz estimates follow from an application of the nonlinear adjoint method. These are then combined with a-priori bounds for solutions of the Fokker-Planck equation and a concavity argument for the nonlinearity.

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1 Introduction

In this paper, we discuss time dependent mean-field games with logarithmic nonlinearities. More precisely, we study the following mean-field game (MFG, for short) problem:

$$\begin{align*}
-u_t + H(x, Du) &= \Delta u + g[m] \\
m_t - \text{div}(D_pH(x, Du)m) &= \Delta m,
\end{align*}$$

(1)
where \( g[m] = \ln m \). The previous system is endowed with initial-terminal boundary datum

\[
\begin{align*}
 u(x, T) &= u_T(x) \\
m(x, 0) &= m_0(x),
\end{align*}
\]

(2)

where the terminal time \( T > 0 \) is fixed. For simplicity, we work in the periodic setting, i.e., the variable \( x \) takes values in the \( d \)-dimensional flat torus \( \mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d \). We consider Hamiltonians \( H \) which satisfy subquadratic growth conditions, as well as a number of additional hypotheses discussed in Section 2.1. A model Hamiltonian that satisfies these is

\[
H(x, p) = a(x)(1 + |p|^2)^{\gamma/2} + V(x),
\]

with \( a, V : \mathbb{T}^d \to \mathbb{R}, a, V \in C^\infty \), with \( a > 0 \), and \( 1 < \gamma < \frac{5}{2} \). As usual, we assume that \( u_T \) and \( m_0 \) are smooth functions, with \( m_0 > 0 \).

Because the logarithm is unbounded by below, existence of solutions was an open question until now. Indeed, for time dependent problems, earlier results in the literature rely on the fact that the nonlinearity has uniform lower bounds. The question of existence of solutions for mean-field games systems has been investigated by several authors since the seminal works of J-M. Lasry and P-L. Lions \( \text{[LL06a, LL06b, LL07]} \) and M. Huang, P. Caines and R. Malhamé \( \text{[HMC06, HCM07]} \). See, for an account of the recent developments in this direction, \( \text{[LLG10, Car11, Ach13, or GSL14]} \), the lectures by P-L. Lions \( \text{[Lio11, Lio12]} \) and the monograph by A. Bensoussan, J. Frehse and P. Yam \( \text{[BFY13]} \).

In previous works \( \text{[1, 2]} \) has been considered under the hypothesis that the nonlinearity \( g \) is bounded by below. A typical choice is the power nonlinearity \( g(z) = z^\alpha \), for some \( \alpha > 0 \). See \( \text{[LL06b, LL07, Por13, Por14, GPSM14, GPSM13, GP14a, and GP14b, CGP14, GP]} \). Firstly, lower bounds for \( g \) imply that the solutions of the Hamilton-Jacobi equation are bounded by below. Then, because of the optimal control nature of the problem, estimates for solutions of the Hamilton-Jacobi equation in \( L^\infty(\mathbb{T}^d \times [0, T]) \) can be proved under various conditions. However, if \( g \) fails to be bounded by below, those estimates are no longer valid. Moreover, the logarithmic structure of the nonlinearity produces further difficulties. The polynomial estimates for the solutions of the Fokker-Planck equation no longer lead to bounds for the nonlinearity \( g \) in the appropriate Lebesgue spaces. In the stationary case, the logarithmic nonlinearity was investigated in \( \text{[GSM14, GPSM12, and GPV14]} \). The stationary obstacle problem with power or logarithmic nonlinearity dependence was studied in \( \text{[GP13]} \). The congestion problem was studied in \( \text{[GM]} \).

To overcome the difficulties caused by the unboundedness of the logarithm, we recur to the nonlinear adjoint method \( \text{[Eva10]} \). This yields estimates for the solutions \( u \) of the Hamilton-Jacobi equation in \( L^\infty(\mathbb{T}^d \times [0, T]) \) in terms of both norms of \( g \) in Lebesgue spaces and the adjoint variable. A further study of the regularity of the adjoint variable improves these estimates. Then, to obtain a bound for \( g \) in \( L^\infty(0, T; L^p(\mathbb{T}^d)) \), we explore the concavity properties of the logarithm combined with a-priori bounds for \( \frac{1}{m} \), using the Fokker-Planck
equation. A further application of the adjoint method yields Lipschitz regularity for the solutions of the Hamilton-Jacobi equation in terms of norms of the nonlinearity $g$ in Lebesgue spaces.

The main result of this paper is the following:

**Theorem 1.1.** Suppose that Assumptions A1-5 from Section 2.1 are satisfied. Then, there exists a classical solution $(u, m)$ for \( (1) \) under the initial-terminal boundary datum \( (2) \).

The Assumptions of the previous Theorem are presented in the next Section. There, we outline its proof and the structure of this paper.

## 2 Main assumptions and outline of the proof

In the sequel, we detail the main hypotheses used throughout the paper.

### 2.1 Main assumptions

We begin by introducing standard assumptions on the Hamiltonian $H$ as well as on the boundary data in \( (2) \):

**A 1.** The Hamiltonian $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$, $d > 2$ is smooth and

1. for fixed $x$, the map $p \mapsto H(x, p)$ is strictly convex;
2. satisfies the coercivity condition

$$
\lim_{|p| \to \infty} \frac{H(x, p)}{|p|} = +\infty,
$$

and, without loss of generality, we suppose additionally that $H \geq 0$.

3. $H$ satisfies the growth condition:

$$
0 \leq H(x, p) \leq C + C|p|^\gamma,
$$

for some $\gamma > 1$, and $C > 0$.

The nonlinearity $g$ in \( (1) \) is

$$
g[m](x, t) \doteq \ln [m](x, t).
$$

We suppose additionally that $u_T, m_0 \in C^\infty(\mathbb{T}^d)$ with $m_0 > 0$.

Note that further conditions will be placed on the parameter $\gamma$ in Assumption A5.

In this paper, we consider $d > 2$. Nevertheless, minor modifications of our arguments, yield similar results for the case $d \leq 2$.

Assumptions A2-4 that follow, impose natural growth conditions on the first derivatives of the Hamiltonian $H$. 
A 2. There exists a constant $C > 0$ such that:

$$D_p H(x, p) p - H(x, p) \geq CH(x, p) - C.$$  

A 3. There exists a constant $C > 0$ such that:

$$|D_p H|^2 \leq C + C|Du|^{2(\gamma - 1)}.$$  

A 4. There exists a constant $C > 0$ such that:

$$|D_x H(x, p)| \leq CH(x, p) + C.$$  

Lastly, we impose a condition on the exponent $\gamma$:

A 5. The exponent $\gamma$ satisfies

$$1 < \gamma < \frac{5}{4}.$$

2.2 Outline of the proof

We consider an approximate problem whose limit solves (1). This is done by replacing the operator $g[m] = \ln[m]$ with

$$g_\epsilon[m](x, t) = \ln[\epsilon + m](x, t).$$

As a consequence, we study the following regularized problem:

$$\begin{align*}
-u_\epsilon^t + H(x, Du_\epsilon) &= \Delta u_\epsilon + g_\epsilon[m_\epsilon] \\
m_\epsilon^t - \text{div}(D_p H(x, Du)m_\epsilon) &= \Delta m_\epsilon.
\end{align*}$$

(3)

For fixed $\epsilon > 0$, because $g_\epsilon$ is bounded by below and with logarithmic growth, the existence of classical solutions for (3)-(2) follows from standard arguments along the same lines of those in [GPSM14].

To prove Theorem 1.1, Lipschitz regularity for $u_\epsilon$ plays a critical role. Indeed, we begin by considering estimates for $g_\epsilon$ in terms of $Du_\epsilon$, as stated in the following Proposition:

Proposition 2.1. Let $(u_\epsilon, m_\epsilon)$ be a solution of (3)-(2) and assume that A1-4 hold. Then, for every $p > 1$,

$$\|g_\epsilon\|_{L^\infty(0,T;L^p(\mathbb{T}^d))} \leq C + C\|Du_\epsilon\|_{L^\infty(\mathbb{T}^d \times [0,T])}^{2(\gamma - 1)}.$$

The proof of Proposition 2.1 is presented in Section 5. Then the nonlinear adjoint method (see [Eva10], as well as [Tra11]) yields the following estimate:

Proposition 2.2. Let $(u_\epsilon, m_\epsilon)$ be a solution of (3)-(2) and assume that A1-5 are satisfied. Then,

$$\|Du_\epsilon\|_{L^\infty(\mathbb{T}^d \times [0,T])} \leq C + C\|g_\epsilon\|_{L^\infty(0,T;L^p(\mathbb{T}^d))} + C\|g_\epsilon\|_{L^\infty(0,T;L^p(\mathbb{T}^d))}\|Du_\epsilon\|_{L^\infty(\mathbb{T}^d \times [0,T])}^{2(\gamma - 1)}.$$

for some $p > 1$. 

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Proposition 2.2 is proved in Section 6. By combining Proposition 2.2 with Proposition 2.1 obtain the next Theorem which gives the Lipschitz regularity \( u^\epsilon \).

**Theorem 2.1.** Assume that A1-5 from Section 2.1 hold. There exists a constant \( C > 0 \) such that any solution \((u^\epsilon, m^\epsilon)\) to (3) satisfies \( \|Du^\epsilon\|_{L^\infty} < C \).

**Proof.** By combining Propositions 2.1 and 2.2, we get

\[
\|Du\|_{L^\infty(T^d \times [0,T])} \leq C + C\|Du\|_{L^\infty(T^d \times [0,T])}^{4(\gamma-1)}.
\]

Assumption A5 ensures that \( 4(\gamma-1) < 1 \). Hence, a weighted Young’s inequality yields the result.

Once this is done, to prove Theorem 2.1, we need to establish additional regularity for \((u^\epsilon, m^\epsilon)\). This allows us to pass to the limit \( \epsilon \to 0 \) and to conclude that \((u, m) = \lim \epsilon \to 0 (u^\epsilon, m^\epsilon)\) solves the system (1)-(2) in some appropriate sense as well. Because \((u, m)\) has the same regularity of \((u^\epsilon, m^\epsilon)\), the existence of classical solutions is proven. This argument is set forth in Section 6.

3 **Estimates in** \( L^\infty(T^d \times [0, T]) \)

Fix \( x_0 \in T^d \) and \( 0 \leq \tau < T \). Based upon the ideas in [Eva10] (see also [Tra11]), we begin by introducing the linearized adjoint equation

\[
\begin{aligned}
\rho_t - \text{div}(D_pH(x, Du^\epsilon)\rho) &= \Delta \rho \\
\rho(x, \tau) &= \delta_{x_0},
\end{aligned}
\]

where \( \delta_{x_0} \) is the Dirac delta centered at \( x_0 \).

If \( g_\epsilon \) were bounded by below, the optimal control formulation for the Hamilton-Jacobi equation in (3) would yield immediately an estimate from below for the value function \( u^\epsilon \) in \( L^\infty(T^d \times [0, T]) \). This is not the case in the presence of a logarithmic nonlinearity. The next Proposition investigates upper for the solutions of the Hamilton-Jacobi equation.

**Proposition 3.1.** Let \((u^\epsilon, m^\epsilon)\) be a solution of (3) and assume that A1 is satisfied. Suppose further that \( \rho \) solves (4). Then, for any \( p, q \geq 1 \) such that

\[
\frac{1}{p} + \frac{1}{q} = 1,
\]

and

\[
p > \frac{d}{2},
\]

we have

\[
\|u^\epsilon\|_{L^\infty(T^d \times [0,T])} \leq C + C\|g_\epsilon\|_{L^\infty(0,T;L^p(T^d))} (1 + \|\rho\|_{L^1(0,T;L^q(T^d))}) \cdot
\]

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Proof. For ease of presentation, we drop the $\epsilon$ in this proof. Let $0 \leq \tau \leq T$. If $u_0$ is satisfied, it follows
\[
u(x_0, \tau) \leq C + C\|g\|_{L^\infty(0,T;L^p(T^d))},
\]
see [GPSM14]. Besides, by multiplying the first equation of (3) by $\rho$ and (4) by $u$, adding them and integrating by parts, we obtain that (using $0 \leq \tau \leq T$)
\[
u(x_0, \tau) \geq -C - C\int_0^T \int_{T^d} |g\rho| \ dx \ dt \geq -C - C\|g\|_{L^\infty(0,T;L^p(T^d))}\|\rho\|_{L^1(0,T;L^q(T^d))},
\]
where the second estimate follows from H"older’s inequality. Thus, we have proven (7).

The next Corollary establishes a key estimate.

Corollary 3.1. Let $(u^\epsilon, m^\epsilon)$ be a solution of (3) and assume that $A_{2,3}$ hold. Assume further that $\rho$ solves (4), fix $p, q \geq 1$ such that (5)-(6) are satisfied. Then,
\[
\int_0^T \int_{T^d} H\rho \ dx \ dt \leq C + C\|g\|_{L^\infty(0,T;L^p(T^d))} (1 + \|\rho\|_{L^1(0,T;L^q(T^d))}).
\]

Proof. As before, we omit the $\epsilon$ in this proof. Multiply the first equation of (3) by $\rho$ and (4) by $u$, add them and integrate by parts to obtain
\[
\int_0^T \int_{T^d} H\rho \ dx \ dt \leq C + u^\epsilon(x_0, 0) + C\|g\|_{L^\infty(0,T;L^p(T^d))}\|\rho\|_{L^1(0,T;L^q(T^d))},
\]
where we have used $A_{2,3}$ Proposition 3.1 yields the result.

The former estimates depend both on $g_\epsilon$ as well as on the adjoint variable $\rho$. In the next Section, we investigate the regularity of $\rho$ in an attempt to remove this dependence.

4 Regularity for the adjoint variable

The following Proposition is critical in investigating the regularity of the solutions $u^\epsilon$ to the regularized Hamilton-Jacobi equation in (3).

Proposition 4.1. Let $(u^\epsilon, m^\epsilon)$ be a solution of (3) and assume that $A_{2,3}$ holds. Let $0 \leq \tau \leq T$ and $\rho$ be a solution of (4). Then,
\[
\int_\tau^T \int_{T^d} \left| D \left( \rho^2 \right) \right|^2 \ dx \ dt \leq C + C\|Du^\epsilon\|_{L^\infty(T^d \times [0,T])}^{2(\gamma-1)}.
\]
Proof. As before, we omit the $\epsilon$ throughout the proof. For $0 < \nu < 1$, multiply \((4)\) by $\nu \rho^{\nu-1}$ and integrate by parts to obtain, after some elementary estimates, and using Assumption 3
\[
\int_T^T \int_{T^d} |D(\rho^{\frac{\nu}{2}})|^2 \, dx \, dt \leq C + C \int_T^T \int_{T^d} |D_p H|^2 \rho'' \, dx \, dt \leq C \|Du\|_{L^\infty(T^d \times [0,T])}^{2(\gamma-1)},
\]
because $\rho$ is a probability measure for each fixed time and $0 < \nu < 1$.

We state now a technical lemma:

Lemma 4.1. Suppose $q, b, \lambda \in \mathbb{R}$ satisfy
\[
q, b \geq 1, \quad 0 < \lambda < 1, \quad q < \frac{db}{bd - 2b}. \tag{9}
\]
Denote by $2^*$ the Sobolev conjugated exponent $2^* = \frac{2d}{d-2}$. Then there exists $a, M \geq 1, Q \geq q, 0 < \kappa < 1$ and $0 < \tilde{\nu} < 1$ such that
\[
\frac{1}{M} = \frac{\lambda}{b}, \tag{10}
\]
\[
\frac{1}{Q} = 1 - \lambda + \frac{\lambda}{a}, \tag{11}
\]
\[
\frac{1}{a} = 1 - \kappa + \frac{2\kappa}{2^* \tilde{\nu}}, \tag{12}
\]
and
\[
\frac{\kappa b}{\tilde{\nu}} \leq 1. \tag{13}
\]

Proof. The Lemma is established by elementary computations and can be checked easily by recurring to the software Mathematica.

The bounds in Section 3 depend on norms of $\rho$ in $L^1(0,T;L^q(T^d))$, which are estimated in the next Lemma.

Lemma 4.2. Let $(u^\epsilon, m^\epsilon)$ be a solution of \((3)\) and assume that $A3$ holds. Assume further that $\rho$ solves \((4)\). Suppose $q, b, \lambda \in \mathbb{R}$ satisfy \((1)\). Then
\[
\|\rho\|_{L^1(0,T;L^q(T^d))} \leq C + C \|Du^\epsilon\|_{L^\infty(T^d \times [0,T])}^{2(\gamma-1)}.
\]

Proof. Since \((3)\) holds, fix $M, Q, a, \kappa,$ and $\tilde{\nu}$ as in the statement of Lemma 4.1. Hence, by Hölder’s inequality combined with \((10)\) and \((11)\), we have
\[
\|\rho\|_{L^1(0,T;L^q(T^d))} \leq \|\rho\|_{L^M(0,T;L^q(T^d))} \leq \|\rho\|_{L^\infty(0,T;L^1(T^d))}^{\gamma-1} \|\rho\|_{L^{M}(0,T;L^q(T^d))}^\gamma.
\]
Using Hölder’s inequality once more together with \((12)\) yields
\[
\left(\int_{T^d} \rho^\alpha \, dx\right)^{\frac{1}{\alpha}} \leq \left(\int_{T^d} \rho \, dx\right)^{1-\kappa} \left(\int_{T^d} \rho^2 \tilde{\nu} \, dx\right)^{\frac{2\kappa}{2^*\tilde{\nu}}}.\]
We have, by Sobolev’s Theorem, that
\[
\left( \int_{\mathbb{T}^d} \rho^{2+\frac{2}{d}} \right)^{\frac{d}{2+\frac{2}{d}}} \leq C + C \left( \int_{\mathbb{T}^d} |D\left(\rho^\frac{2}{d}\right)|^2 \right)^{\frac{d}{2}} ,
\]
and, therefore,
\[
\int_0^T \left( \int_{\mathbb{T}^d} \rho^2 \right)^{\frac{d}{2}} \leq C + C \int_0^T \left( \int_{\mathbb{T}^d} |D\left(\rho^\frac{2}{d}\right)|^2 \right)^{\frac{d}{2}}.
\]

Using (13) in the previous computation combined with Proposition 4.1, we obtain
\[
\|\rho\|_{L^2(0,T;L^b(\mathbb{T}^d))} \leq C + C \left\|Du^\epsilon\right\|_{L^\infty(0,T;L^p(\mathbb{T}^d))}^{2(\gamma-1)} \left(1 + \left\|Du^\epsilon\right\|_{L^\infty(0,T;L^p(\mathbb{T}^d))}^{2(\gamma-1)} \left(1 + \left\|Du^\epsilon\right\|_{L^\infty(0,T;L^p(\mathbb{T}^d))} \right) \right),
\]
which concludes the proof.

**Corollary 4.1.** Let \((u^\epsilon, m^\epsilon)\) be a solution of (6) and assume that A1-2 hold. Assume further that \(\rho\) solves (4) and that \(\lambda, p\) and \(b\) satisfy
\[
0 < \lambda < 1, \quad p > \frac{d}{2\lambda}, \quad 1 < b < \frac{2\lambda p}{d}.
\]

Then,
\[
\int_0^T \int_{\mathbb{T}^d} H \rho dx dt \leq C + C \left\|u^\epsilon\right\|_{L^\infty(0,T;L^p(\mathbb{T}^d))} \left(1 + \left\|Du^\epsilon\right\|_{L^\infty(0,T;L^p(\mathbb{T}^d))}^{2(\gamma-1)} \left(1 + \left\|Du^\epsilon\right\|_{L^\infty(0,T;L^p(\mathbb{T}^d))} \right) \right).
\]

**Proof.** The result follows by combining Corollary 3.1 with Lemma 4.2 by observing that (9), (5), and (6) are equivalent to (14) and (5).

## 5 Estimates for the Fokker-Planck equation

Next, we establish several estimates concerning the integrability of solutions of the Fokker-Planck equation. The proof of Proposition 2.1 closes this section.

**Lemma 5.1.** Let \((u^\epsilon, m^\epsilon)\) be a solution of (3). Then,
\[
\frac{d}{dt} \left[ \ln \left( \int_{\mathbb{T}^d} \frac{1}{m + \epsilon} dx \right) \right] \leq C \left\|D_pH^2\right\|_{L^\infty(\mathbb{T}^d)} + C.
\]

**Proof.** Notice that
\[
\frac{d}{dt} \int_{\mathbb{T}^d} \frac{1}{m + \epsilon} dx = - \int_{\mathbb{T}^d} \frac{\text{div}(D_pHm)}{(m + \epsilon)^2} dx - \int_{\mathbb{T}^d} \frac{\Delta m}{(m + \epsilon)^2} dx.
\]

Integration by parts yields
\[
- \int_{\mathbb{T}^d} \frac{\text{div}(D_pHm)}{(m + \epsilon)^2} dx = -2 \int_{\mathbb{T}^d} \frac{D_pHmDm}{(m + \epsilon)^{\frac{2}{2}}(m + \epsilon)^{\frac{2}{2}}}.
\]
and
\[-\int_{\mathbb{T}^d} \frac{\Delta m}{(m + \epsilon)^2} = -2 \int_{\mathbb{T}^d} \frac{|Dm|^2}{(m + \epsilon)^3} dx.\]

Hence, for some \( C, c > 0, \)
\[
\frac{d}{dt} \int_{\mathbb{T}^d} \frac{1}{m + \epsilon} dt \leq C \int_{\mathbb{T}^d} \frac{|D_pH|^2 m^2}{(m + \epsilon)^3} dx - c \int_{\mathbb{T}^d} \frac{|Dm|^2}{(m + \epsilon)^3} dx
\]
\[
\leq C \left\| |D_pH|^2 \right\|_{L^\infty(\mathbb{T}^d)} \int_{\mathbb{T}^d} \frac{1}{m + \epsilon} dx.
\]

Consequently,
\[
\frac{d}{dt} \left[ \ln \left( \int_{\mathbb{T}^d} \frac{1}{m + \epsilon} dx \right) \right] \leq C \left\| |D_pH|^2 \right\|_{L^\infty(\mathbb{T}^d)}.
\]

\[\Box\]

**Lemma 5.2.** Let \( m : \mathbb{T}^d \rightarrow \mathbb{R}_0^+ \) be integrable with \( \int_{\mathbb{T}^d} m = 1. \) Then,
\[
\int_{\mathbb{T}^d} |\ln(m + \epsilon)|^p dx \leq C + C \int_{m^+ \leq 1} \left( \frac{1}{m + \epsilon} \right)^p dx.
\]

**Proof.** We have
\[
\int_{\mathbb{T}^d} |\ln(m + \epsilon)|^p dx = \int_{m^+ \leq 1} \left( \frac{1}{m + \epsilon} \right)^p dx + \int_{m^+ > 1} |\ln(m + \epsilon)|^p dx.
\]
Because \( \ln(m + \epsilon) \leq C \delta (m + \epsilon) \delta \) for every \( \delta > 0, \) provided \( (m + \epsilon) > 1, \) we conclude
\[
\int_{\mathbb{T}^d} |\ln(m + \epsilon)|^p dx = \int_{m^+ \leq 1} \left( \frac{1}{m + \epsilon} \right)^p dx + C.
\]

\[\Box\]

**Lemma 5.3.** There exists \( 0 < A, \) depending solely on \( p, \) such that \( (\ln z)^p \) is a concave function for \( z > \frac{1}{A}. \)

**Proof.** A straightforward computation implies
\[
[(\ln z)^p]'' = \frac{p(p-2)}{z^2} [p - 1 - \ln z].
\]
For \( z > e^{p-1}, \) we have that
\[
[(\ln z)^p]'' < 0.
\]
Therefore, the result holds for \( A = e^{1-p}. \)

\[\Box\]

**Lemma 5.4.** Let \( (u^*, m^*) \) be a solution of (8). Then,
\[
\int_{m^+ \leq 1} \left( \frac{1}{m(x, \tau) + \epsilon} \right)^p dx \leq C + C \left\| |D_pH|^2 \right\|_{L^\infty(\mathbb{T}^d \times [0, T])}.
\]

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Proof. Clearly,
\[ \int_{m + \epsilon \leq A} \left( \ln \frac{1}{m + \epsilon} \right)^p \, dx = \int_{A \leq m + \epsilon \leq 1} \left( \ln \frac{1}{m + \epsilon} \right)^p \, dx + \int_{m + \epsilon < A} \left( \ln \frac{1}{m + \epsilon} \right)^p \, dx, \]
for every \( 0 < A < 1 \). Select \( A \) as in Lemma 5.3. First note that,
\[ \int_{A \leq m + \epsilon \leq 1} \left( \ln \frac{1}{m + \epsilon} \right)^p \, dx \leq C \max_{A \leq m + \epsilon \leq 1} \left| \ln \frac{1}{m + \epsilon} \right|^p \leq C. \]
Define \( \Psi(z) = (\ln z)^p \), for \( z > \frac{1}{A} \), and extend it continuously and linearly for \( z < \frac{1}{A} \). It is possible to do so in such a way that \( \Psi \) is globally concave and increasing.

Then Jensen’s inequality leads to
\[ \frac{1}{|\{m + \epsilon \leq A\}|} \int_{m + \epsilon \leq A} \left( \ln \frac{1}{m + \epsilon} \right)^p \, dx \leq \Psi \left( \frac{1}{|\{m + \epsilon \leq A\}|} \int_{\tau \leq m + \epsilon} \frac{1}{m + \epsilon} \, dx \right). \]
Since \( \Psi \) is increasing we conclude that
\[ \frac{1}{|\{m + \epsilon \leq A\}|} \int_{m + \epsilon \leq A} \left( \ln \frac{1}{m + \epsilon} \right)^p \, dx \leq \Psi \left( \frac{1}{|\{m + \epsilon \leq A\}|} \int_{\tau \leq m + \epsilon} \frac{1}{m + \epsilon} \, dx \right). \]
Now there are two cases, either
\[ \frac{1}{|\{m + \epsilon \leq A\}|} \int_{\tau \leq m + \epsilon} \frac{1}{m + \epsilon} < \frac{1}{A}, \]
from what it follows
\[ \int_{m + \epsilon \leq A} \left( \ln \frac{1}{m + \epsilon} \right)^p \, dx \leq \Psi \left( \frac{1}{A} \right), \]
or
\[ \frac{1}{|\{m + \epsilon \leq A\}|} \int_{\tau \leq m + \epsilon} \frac{1}{m + \epsilon} > \frac{1}{A}. \]
In this case,
\[ \Psi \left( \frac{1}{|\{m + \epsilon \leq A\}|} \int_{\tau \leq m + \epsilon} \frac{1}{m + \epsilon} \right) = \left[ \ln \left( \frac{1}{|\{m + \epsilon \leq A\}|} \right) + \ln \left( \int_{\tau \leq m + \epsilon} \frac{1}{m + \epsilon} \right) \right]^p. \]
Therefore, since \( |\{m + \epsilon \leq A\}| \leq 1 \),
\[ \int_{m + \epsilon \leq A} \left( \ln \frac{1}{m + \epsilon} \right)^p \, dx \leq \Psi \left( \frac{1}{A} \right) + C_p \left[ \ln \left( \int_{\tau \leq m + \epsilon} \frac{1}{m + \epsilon} \right) \right]^p + C_p |\{m + \epsilon \leq A\}| \left[ \ln \left( \frac{1}{|\{m + \epsilon \leq A\}|} \right) \right]^p. \]
Because
\[
\frac{1}{\{m + \epsilon \leq A\}} \geq 1,
\]
 it follows that
\[
\ln\left(\frac{1}{\{m + \epsilon \leq A\}}\right) \leq C \left(\frac{1}{\{m + \epsilon \leq A\}}\right)^\delta,
\]
for every \(\delta > 0\). By choosing \(\delta = \frac{1}{p}\) one obtains
\[
\int_{m + \epsilon \leq A} \left(\ln \frac{1}{m + \epsilon}\right)^p \, dx \leq C + C \left(\int_{T \times [0, T]} \frac{1}{m + \epsilon}\right)^p
\]
that is,
\[
\int_{m + \epsilon \leq A} \left(\ln \frac{1}{m + \epsilon}\right)^p \, dx \leq C + C \left(\int_{T \times [0, T]} \frac{1}{m + \epsilon}\right)^p,
\]
which concludes the proof, using Lemma 5.1.

We end this section with the proof of Proposition 2.1.

Proof of Proposition 2.1. By combining Lemmas 5.2 and 5.4 one obtains that
\[
\int_{T \times [0, T]} |\ln(m_\epsilon + \epsilon)|^p \, dx \leq C + C \left\|D_pH\right\|^p_{L^\infty(T^d \times [0, T])}.
\]
The Proposition is then implied by A3.

6 Lipschitz regularity for the Hamilton-Jacobi equation

In the present Section, we obtain estimates for \(Du\) in \(L^\infty(T^d \times [0, T])\), uniformly in \(\epsilon\). We begin with a technical lemma, followed by the proof of Proposition 2.2.

This Section ends with the proof of Theorem 2.1.

Lemma 6.1. For \(d > 2\) there exist real numbers \(\lambda, b, \tilde{q}, \theta, \bar{\nu}\) such that
\[
\frac{1}{p} + \frac{1}{\tilde{q}} = \frac{1}{2}, \quad \tilde{q} \geq 1
\]
\[
\frac{1}{\tilde{q} \left(\frac{1 - \nu}{2}\right)} = 1 - \theta + \frac{2\theta}{\tilde{q} \nu}, \quad 0 < \theta < 1
\]
\[
\theta = \frac{\bar{\nu}}{2 - \bar{\nu}}, \quad 0 < \bar{\nu} < 1
\]
hold simultaneously.
Proof. The Lemma is established by elementary computations and can be checked easily by recurring to the software Mathematica.

Proof of Proposition 2.2. For ease of notation, we omit $\epsilon$ throughout the proof. Choose $\lambda, b, p, q, \theta$ and $\bar{\nu}$ as in Lemma 6.1. Let $\rho$ be a solution of (4). Fix a unit vector $\xi$ and differentiate the first equation in (3) in the $\xi$ direction. Multiply it by $\rho$ and (4) by $u_\xi$. By adding them and integrating by parts, it follows that

$$u_\xi(x_0, \tau) = \int_T \int_{T^d} -D_\xi H \rho + g_\xi \rho \,dx\,dt + \int_{T^d} (u_T)\xi \rho(x, T) \,dx.$$  

Assumption A4 implies

$$\left| \int_T \int_{T^d} -D_\xi H \rho \,dx\,dt \right| \leq C + C \int_0^T \int_{T^d} H \rho \,dx\,dt \leq C + C\|g\|_{L^\infty(0,T;L^p(T^d))} \left( 1 + \|Du_T\|_{L^\infty(T^d \times [0,T])}^{2(\lambda - 1)} \right),$$

by Corollary 4.1. Moreover,

$$\left| \int_{T^d} (u_T)\xi \rho(x, T) \,dx \right| \leq C,$$

as it only depends on the terminal data. It remains after that to address the term

$$\int_0^T \int_{T^d} g_\xi \rho \,dx.$$

Integrating by parts, we have the following estimate

$$\left| \int_0^T \int_{T^d} g_\xi \rho \,dx \right| \leq C\|g\|_{L^\infty(0,T;L^p(T^d))}\|\rho\|_{L^2(0,T;L^4(T^d))} \|D\rho_T\|_{L^2(T^d \times [0,T])},$$

where $\tilde{q}$ is given by (15). From estimate (8) in Proposition 4.1 it follows

$$\|D\rho_T\|_{L^2(T^d \times [0,T])} \leq C + C\|Du\|_{L^\infty(T^d \times [0,T])}^{-1}.$$

Moreover,

$$\left( \int_{T^d} \rho^{\tilde{q}} \right)^{\frac{1}{\tilde{q}}} \leq \left( \int_{T^d} \rho \right)^{1-\theta} \left( \int_{T^d} \rho^{\frac{2}{\tilde{q}}} \,dx \right)^{\frac{\tilde{q}}{2}},$$

provided (16) holds. Then Sobolev’s Theorem yields

$$\left( \int_{T^d} \rho^{\tilde{q}} \right)^{\frac{1}{\tilde{q}}} \leq C + C \left( \int_{T^d} |D\rho_T|^2 \right)^{\frac{\tilde{q}}{2}}.$$
Consequently,

\[
\left( \int_{\mathbb{T}^d} \rho^{\hat{\theta}}(2 - \hat{\nu}^2) \right)^{\frac{2}{\hat{\theta}}} \leq C + C \left( \int_{\mathbb{T}^d} |D(\rho^\hat{\theta})|^2 \right)^{\frac{(2 - \hat{\theta})}{\hat{\theta}}}. 
\]

By setting \( \hat{\theta} \) as in (17), and recurring to Proposition 4.1 one obtains that

\[
\|\rho^{1 - \hat{\theta}}\|_{L^2(0,T;L^4(\mathbb{T}^d))} \leq C + C\|D\|^{|1 - \gamma|^{-1}}_{L^{\infty}(\mathbb{T}^d \times [0,T])}. 
\]

By gathering the previous computation, we conclude that

\[
|u_\xi(x,\tau)| \leq C + C\|g\|_{L^{\infty}(0,T;L^p(\mathbb{T}^d))} \left(1 + \|D\|^{2(\gamma-1)}_{L^{\infty}(\mathbb{T}^d \times [0,T])}\right) + C\|g\|_{L^{\infty}(0,T;L^p(\mathbb{T}^d))} \left(1 + \|D\|^{2(\gamma-1)}_{L^{\infty}(\mathbb{T}^d \times [0,T])}\right),
\]

which becomes

\[
|u_\xi(x,\tau)| \leq C + C\|g\|_{L^{\infty}(0,T;L^p(\mathbb{T}^d))} + C\|g\|_{L^{\infty}(0,T;L^p(\mathbb{T}^d))}\|D\|^{2(\gamma-1)}_{L^{\infty}(\mathbb{T}^d \times [0,T])},
\]

once we take into account that

\[
\frac{2\lambda(\gamma - 1)}{b} < 2(\gamma - 1).
\]

\[ \square \]

**Proof of Theorem (1.1).** We first notice that \( \ln(m^\epsilon + \epsilon) \) is in \( L^p \), uniformly in \( \epsilon \). On the other hand, \( Du^\epsilon \) is bounded in \( L^\infty \), uniformly in \( \epsilon \). Therefore, from standard regularity theory for the heat equation, we have \( D^2u^\epsilon \) uniformly bounded in \( L^p \), for every \( 1 < p < \infty \).

On the other hand, consider the Hopf-Cole transformation \( v^\epsilon \doteq \ln(m^\epsilon + \epsilon) \). It follows from elementary computations that

\[ v^\epsilon_t - D_pH Dv^\epsilon - \text{div}(D_pH) = |Dv^\epsilon|^2 + \Delta v, \]

which implies that \( v^\epsilon \) is bounded in \( L^\infty \), uniformly in \( \epsilon \) (see [GPSM14]). Therefore, \( m^\epsilon \) is also bounded by below uniformly in \( \epsilon \). Once this is established, because \( \ln(m + \epsilon) \) is uniformly bounded by below and has sub-polynomial growth, the techniques in [GPSM14] can be applied without any substantial change. Therefore, we conclude that \( u^\epsilon \) is smooth, with norms uniformly bounded in every Sobolev space. From this, it follows that, through some subsequence, \( u^\epsilon \to u \) in any Sobolev space. As a consequence, we obtain that \( m^\epsilon \to m \), also in the strong sense in any Sobolev space. Hence, the limit \( (u, m) \) is a classical solution of (1)-(2), and the proof is complete. \[ \square \]
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