Structure of T - and S - Matrices in Unphysical Sheets and Resonances in Three – Body Systems

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Abstract. Algorithm, based on explicit representations for the analytic continuation of Faddeev components of the three-body T-matrix in unphysical energy sheets, is employed to study mechanism of disappearance and formation of the Efimov levels of the helium \textsuperscript{4}He trimer.

1 Introduction

Explicit representations for the Faddeev components of the three-body T-matrix continued analytically into unphysical sheets of the energy Riemann surface have been formulated and proved recently in Ref. \cite{1}. According to the representations, the T-matrix in unphysical sheets is explicitly expressed in terms of its components only taken in the physical sheet. Analogous explicit representations were also found for the analytic continuation of the three-body scattering matrices. These representations disclose the structure of kernels of the T- and S-matrices after continuation and give new capacities for analytical and numerical studies of the three-body resonances. In particular the representations imply that the resonance poles of the S-matrix as well as T-matrix in an unphysical sheet correspond merely to the zeros of the suitably truncated three-body scattering matrix taken in the physical sheet. Therefore, one can search for resonances in a certain unphysical sheet staying always, nevertheless, in the physical sheet and only calculating the position of zeros of the appropriate truncation of the total three-body scattering matrix. This statement holds true not only for the case of the conventional smooth quickly decreasing interactions but also for the case of the singular interactions described by different variants of the Boundary Condition Model, in particular for the inter–particle interactions of a hard-core nature like in most molecular systems.

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As a concrete application of the method, we present here the results of our numerical study of the simplest truncation of the scattering matrix in the $^4\text{He}$ three-atomic system, namely of the $(2 + 1 \to 2 + 1)$ S-matrix component corresponding to the scattering of a $^4\text{He}$ atom off a $^4\text{He}$ dimer. The point is that there is already a series of works [2]–[4] (also see Refs. [5]–[7]) showing that the excited state of the $^4\text{He}$ trimer is initiated by the Efimov effect [8]. In these works, various versions of the $^4\text{He}–^4\text{He}$ potential were employed. However, the basic result of Refs. [2]–[4] on the excited state of the helium trimer is the same: this state disappears after the interatomic potential is multiplied by the increasing factor $\lambda$ when it approaches the value about 1.2. It is just such a nonstandard behavior of the excited-state energy as the coupling between helium atoms becomes more and more strengthening, points to the Efimov nature of the trimer excited state. The present work is aimed at elucidating the fate of the trimer excited state upon its disappearance in the physical sheet when $\lambda > 1$ and at studying the mechanism of arising of new excited states when $\lambda < 1$. As the interatomic $\text{He}–\text{He}$ potential, we use the potential HFD-B [4]. We have established that for such $\text{He}–\text{He}$-interactions the trimer excited-state energy merges with the two-body threshold $\epsilon_d$ at $\lambda \approx 1.18$ and with further decreasing $\lambda$ it transforms into a virtual level of the first order (a simple real pole of the analytic continuation of the $(2 + 1 \to 2 + 1)$ scattering matrix component) lying in the unphysical energy sheet adjoining the physical sheet along the spectral interval between $\epsilon_d$ and the three-body threshold. We trace the position of this level for $\lambda$ increasing up to 1.5. Besides, we have found that the excited (Efimov) levels for $\lambda < 1$ also originate from virtual levels of the first order that are formed in pairs. Before a pair of virtual levels appears, there occurs a fusion of a pair of conjugate resonances of the first order (simple complex poles of the analytic continuation of the scattering matrix in the unphysical sheet) resulting in the virtual level of the second order.

2 Representations for three-body T– and S–matrices in unphysical energy sheets

The method used for calculation of resonances in the present work, is based on the explicit representations [4] for analytic continuation of the T- and scattering matrices in unphysical sheets which hold true at least for a part of the three-body Riemann surface. To describe this part we introduce the auxiliary vector-function $f(z) = (f_0(z), f_{1,1}(z), ..., f_{1,n_1}(z), f_{2,1}(z), ..., f_{2,n_2}(z), f_{3,1}(z), ..., f_{3,n_3}(z))$ with $f_0(z) = \ln z$ and $f_{\alpha,j}(z) = (z - \lambda_{\alpha,j})^{1/2}$. Here, by $z$ we understand the total three-body energy in the c.m. system and by $\lambda_{\alpha,j}$, the respective binding energies of the two-body subsystems $\alpha$, $\alpha = 1, 2, 3, j = 1, 2, ..., n_\alpha, n_\alpha < \infty$. The sheets $\Pi_l$ of the Riemann surface of the vector-function $f(z)$ are numerated by the multi-index $l = (l_0, l_{1,1}, ..., l_{1,n_1}, l_{2,1}, ..., l_{2,n_2}, l_{3,1}, ..., l_{3,n_3})$, where $l_{\alpha,j} = 0$ if the sheet $\Pi_l$ corresponds to the main (arithmetic) branch of the square root $(z - \lambda_{\alpha})^{1/2}$. Otherwise, $l_{\alpha,j} = 1$ is assumed. Value of $l_0$ coincides with the number of the branch of the function $\ln z$, $\ln z = \ln |z| + i2\pi l_0 + i\phi$ where $\phi = \arg z$. For the physical sheet identified by $l_0 = l_{\alpha,j} = 0$, $\alpha = 1, 2, 3$, $j = 1, 2, ..., n_\alpha$, we use the notation $\Pi_0$.

Surely, the structure of the total three-body Riemann surface is essentially
more complicated than that of the auxiliary function \( \mathfrak{f} \). For instance, the sheets \( \Pi_l \) with \( l_0 = \pm 1 \) have additional branching points corresponding to resonances of the two-body subsystems. The part of the total three-body Riemann surface where the representations of Ref. [10] are valid, consists of the sheets \( \Pi_l \) of the Riemann surface of the function \( \mathfrak{f} \) identified by \( l_0 = 0 \) (such unphysical sheets are called two-body sheets) and two three-body sheets identified by \( l_0 = \pm 1 \) and \( l_{\alpha,j} = 1, \alpha = 1, 2, 3, \ j = 1, 2, ..., n_{\alpha} \).

In what follows by \( k_\alpha, p_\alpha \ (k_\alpha, p_\alpha \in \mathbb{R}^3 \) or \( k_\alpha, p_\alpha \in \mathbb{C}^3 \) we understand the standard reduced relative momenta of the three-body system while \( P = \{k_\alpha, p_\alpha\} \ (P \in \mathbb{R}^6 \) or \( P \in \mathbb{C}^6 \) stands for the total relative momentum.

The representations \( \tilde{\Pi} \) for the analytic continuation of the matrix \( \tilde{M}(z) = \{M_{\alpha\beta}(z)\}, \alpha, \beta = 1, 2, 3, \) of the Faddeev components \( M_{\alpha\beta}(z) \) (see \footnote{1}) of the three-body \( T \)-operator, into the sheet \( \Pi_l \) is, as follows:

\[
\tilde{M}(z)\bigg|_{\Pi_l} = M(z) = B^\dagger(z)A(z)L_{\alpha,j}^{-1}(z)\tilde{L}B(z).
\]

Here, the factor \( A(z) \) is the diagonal matrix, \( A(z) = \text{diag}\{A_0(z), A_{1,1}(z), ..., A_{i,n_i}(z)\} \) with \( A_0(z) = -\pi z^2 \) and \( A_{\alpha,j} = -\pi i \sqrt{\lambda_{\alpha,j}} \), \( j = 1, 2, ..., n_{\alpha} \).

Notations \( L \) and \( \tilde{L} \) stand for the diagonal number matrices combined of the indices of the sheet \( \Pi_l \): \( L = \text{diag}\{l_0, l_{1,1}, ..., l_{3,n_3}\} \) and \( \tilde{L} = \text{diag}\{l_0, l_{1,1}, ..., l_{3,n_3}\} \).

By \( S_l(z) \) we understand a truncation of the three-body scattering matrix \( S(z) \)

defined in \( \tilde{\mathcal{G}} = L_2(S^5) \oplus \bigoplus_{\alpha=1}^{n_{\alpha}} L_2(S^2) \) by the equation

\[
S_l(z) = \hat{I} + \tilde{L}[S(z) - \hat{I}]L
\]

where \( \hat{I} \) is the identity operator in \( \tilde{\mathcal{G}} \). Also, we use the notations

\[
B(z) = \left( \begin{array}{c} J_0\Omega \mathbf{M} \\
J_1 \Psi^* [\gamma \mathbf{M} + \nu] \end{array} \right) \quad \text{and} \quad B^\dagger(z) = \left( \begin{array}{c} (M(z)\Omega J_0^\dagger) \nu + (\mathbf{v} + M\gamma\Psi J_1^\dagger) \end{array} \right).
\]

Here, \( \mathbf{v} = \text{diag}\{v_1, v_2, v_3\} \) with \( v_\alpha \), the pair potentials, \( \alpha = 1, 2, 3 \). At the same time, \( \Omega = (1, 1, 1), \ \gamma = \{\gamma_{\alpha\beta}\} \) with \( \gamma_{\alpha\beta} = 1 - \delta_{\alpha\beta}, \ \alpha, \beta = 1, 2, 3, \) and \( \Psi = \text{diag}\{\Psi_1, \Psi_2, \Psi_3\} \) where \( \Psi_\alpha, \ \alpha = 1, 2, 3, \) are operators acting on \( f = (f_1, f_2, ..., f_{n_\alpha}) \in \bigoplus_{j=1}^{n_\alpha} L_2(\mathbb{R}^3) \) as \( (\Psi_\alpha f)(P) = \sum_{j=1}^{n_\alpha} \psi_{\alpha,j}(k_\alpha) f_j(k_\alpha) \) where, in turn, \( \psi_{\alpha,j} \) is the bound-state wave function of the pair subsystem \( \alpha \) corresponding to the binding energy \( \lambda_{\alpha,j} \). By \( \Psi^* \) we denote operator adjoint to \( \Psi \). Notation \( J_0(z) \) is used for the operator restricting a function on the energy-shell \( |P|^2 = z \).

The diagonal matrix-valued function \( J_1(z) = (J_{1,1}(z), ..., J_{3,n_3}(z)) \) consists of the operators \( J_{\alpha,j}(z) \) of restriction on the energy surfaces \( |p_\alpha|^2 = z - \lambda_{\alpha,j} \).

The operators \( \Omega^\dagger, J_0^\dagger(z) \) and \( J_1^\dagger(z) \) represent the “transposed” matrices \( \Omega, J_0(z) \) and \( J_1(z) \), respectively. Operators \( J_0(z) \) and \( J_1(z) \) act in the expression for \( B(z) \) (as \( \mathfrak{f} \)) to the left.

\footnote{One assumes that all the pair interactions fall off in the coordinate space not slower than exponentially and, thus, their Fourier transforms \( v_\alpha(k), k \in \mathbb{C}^3, \) are holomorphic functions of the relative momenta \( k \) in a stripe \( |\text{Im} k| < b \) for some \( b > 0 \).}
With some stipulations (see [1]) the representations for the scattering matrix read

\[ S(z)\mid_{\Pi_i} = \mathcal{E}(l) \left\{ \hat{I} + S^{-1}_l(z)[S(z) - \hat{I}]e(l) \right\} \mathcal{E}(l). \]  

(2)

Here, \( \mathcal{E} = \text{diag}\{\mathcal{E}_0, \mathcal{E}_{1,1}, \ldots, \mathcal{E}_{3,n_3}\} \) where \( \mathcal{E}_0 \) is the identity operator in \( L_2(S^5) \) if \( l_0 = 0 \) and inversion, \( (\mathcal{E}_0f)(\hat{P}) = f(-\hat{P}) \), if \( l_0 = \pm 1 \). Analogously, \( \mathcal{E}_{\alpha,j} \) is the identity operator in \( L_2(S^2) \) for \( l_{\alpha,j} = 0 \) and inversion for \( l_{\alpha,j} = 1 \). Notation \( e(l) \) is used for the diagonal number matrix \( e(l) = \text{diag}\{e_0, e_{1,1}, \ldots, e_{3,n_3}\} \) with nontrivial elements \( e_{\alpha,j} = 1 \) if \( l_{\alpha,j} = 0 \) and \( e_{\alpha,j} = -1 \) if \( l_{\alpha,j} = 1 \); for all the cases \( e_0 = 1 \).

It follows from the representations (1) and (2) that the resonances (the nontrivial poles of \( M(z)\mid_{\Pi_i} \) and \( S(z)\mid_{\Pi_i} \)) situated in the unphysical sheet \( \Pi_i \) are those points \( z = z_{\text{res}} \) in the physical sheet where the matrix \( S_l(z) \) has zero as eigenvalue. Therefore, calculation of resonances in the unphysical sheet \( \Pi_i \) is reduced to a search for zeros of the respective truncation \( S_l(z) \) of the scattering matrix \( S(z) \) in the physical sheet.

3 Method for search of resonances in a three–body system on the basis of the Faddeev differential equations

In this work we discuss the example of the three-atomic \( ^4 \)He system at the total angular momentum \( L = 0 \). We consider the case where the interatomic interactions include a hard core component and, outside the hard core domain, are described by conventional smooth potentials. In this case, the angular partial analysis reduces the initial Faddeev equation for three identical bosons to a system of coupled two-dimensional integro-differential equations (see Ref. [3] and references therein)

\[ \left[ \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + l(l + 1) \left( \frac{1}{x^2} + \frac{1}{y^2} \right) - E \right] F_l(x, y) \]

\[ = \left\{ \begin{array}{ll}
-V(x)\Psi_l(x, y), & x > c \\
0, & x < c.
\end{array} \right. \]

(3)

Here, \( x, y \) stand for the standard Jacobi variables and \( c \), for the core range. At \( L = 0 \) the partial angular momentum \( l \) corresponds both to the dimer subsystem and a complementary atom. For the \( S \)-state three-boson system \( l \) is even, \( l = 0, 2, 4, \ldots \). In our work, the energy \( z \) can get both real and complex values. The He–He potential \( V(x) \) acting outside the core domain is assumed to be central. The partial wave function \( \Psi_l(x, y) \) is related to the Faddeev components \( F_l(x, y) \) by \( \Psi_l(x, y) = F_l(x, y) + \sum_{l'} \int_{-1}^{+1} d\eta h_{l'l}(x, y, \eta) F_{l'}(x', y') \)

where \( x' = \left( \frac{1}{4} x^2 + \frac{3}{4} y^2 - \frac{\sqrt{3}}{2} xyz \right)^{1/2} \), \( y' = \left( \frac{3}{4} x^2 + \frac{1}{4} y^2 + \frac{\sqrt{3}}{2} xyz \right)^{1/2} \), and \( 1 \leq \eta \leq 1 \). The explicit form of the function \( h_{l'l} \) can be found in Refs. [10, 11]. The functions \( F_l(x, y) \) satisfy the boundary conditions

\[ F_l(x, y) \mid_{x=0} = F_l(x, y) \mid_{y=0} = 0, \quad \Psi_l(c, y) = 0 \]

(4)
Note that the last of these conditions is a specific condition corresponding to the hard-core model (see Ref. [4]).

Here we only deal with a finite number of equations (3), assuming that \( l \leq l_{\text{max}} \) where \( l_{\text{max}} \) is a certain fixed even number. The condition \( 0 \leq l \leq l_{\text{max}} \) is equivalent to the supposition that the potential \( V(x) \) only acts in the two-body states with \( l = 0, 2, \ldots, l_{\text{max}} \). The spectrum of the Schrödinger operator for a system of three identical bosons with such a potential is denoted by \( \sigma_{3B} \).

We assume that the potential \( V(x) \) falls off exponentially and, thus, \( |V(x)| \leq C \exp(-\mu x) \) with some positive \( C \) and \( \mu \). For the sake of simplicity we even assume sometimes that \( V(x) \) is finite, i.e., \( V(x) = 0 \) for \( x > r_0, r_0 > 0 \). Looking ahead, we note that, in fact, in our numerical computations of the \( ^4\text{He}_3 \) system at complex energies we make a “cutoff” of the interatomic He–He–potential at a sufficiently large \( r_0 \). The asymptotic conditions as \( \rho \to \infty \) and/or \( y \to \infty \) for the partial Faddeev components of the \( (2+1 \to 2+1; 1+1+1) \) scattering wave functions for \( z = E + i0, E > 0 \), read (see, e.g., Ref. [10])

\[
F_l(x, y; z) = \delta_{l0} \psi_d(x) \left\{ \sin(\sqrt{\rho - \epsilon_d} y) + \exp(i\sqrt{\rho - \epsilon_d} y) [a_0(z) + o(1)] \right\} + \frac{\exp(i\sqrt{\rho})}{\sqrt{\rho}} [A_l(z, \theta) + o(1)].
\] (5)

We assume that the \( ^4\text{He} \) dimer has an only bound state with an energy \( \epsilon_d \), \( \epsilon_d < 0 \), and wave function \( \psi_d(x) \), \( \psi_d(x) = 0 \) for \( 0 \leq x \leq c \). The notations \( \rho, \rho = \sqrt{x^2 + y^2} \), and \( \theta, \theta = \arctan(y/x) \), are used for the hyperradius and hyperangle. The coefficient \( a_0(z), z = E + i0 \), for \( E > \epsilon_d \) is the elastic scattering amplitude. The functions \( A_l(E + i0, \theta) \) provide us, at \( E > 0 \), the corresponding partial Faddeev breakup amplitudes. For real \( z = E + i0, E > \epsilon_d \), the \( (2+1 \to 2+1) \) component of the \( s \)-wave partial scattering matrix for a system of three helium atoms is given by the expression

\[
S_0(z) = 1 + 2ia_0(z).
\]

Our goal is to study the analytic continuation of the function \( S_0(z) \) into the physical sheet. As it follows from the results of Refs. [4], the function \( S_0(z) \) is just that truncation of the total scattering matrix whose roots in the physical sheet of the energy \( z \) plane correspond to the location of resonances situated in the unphysical sheet adjoining the physical one along the spectral interval \( (\epsilon_d, 0) \).

There are the following three important domains in the physical sheet.

1°. The domain \( \Pi^{(\Phi)} \) where the Faddeev components \( F_l(x, y; z) \) (and, hence, the wave functions \( \Phi_l(x, y; z) \) can be analytically continued in \( z \) so that the differences \( \Phi_l(x, y; z) = F_l(x, y; z) - \delta_{l0} \psi_d(x) \sin(\sqrt{\rho - \epsilon_d} y) \) at \( z \in \Pi^{(\Phi)} \setminus \sigma_{3B} \) are square integrable. This domain is described by the inequality

\[
\text{Im} \sqrt{\rho - \epsilon_d} < \min \left\{ \frac{3}{2} \mu, \sqrt{3} \sqrt{|\epsilon_d|} \right\}.
\]

2°. The domain \( \Pi^{(A)} \) where both the elastic scattering amplitude \( a_0(z) \) and the Faddeev breakup amplitudes \( A_l(z, \theta) \) can be analytically continued in \( z \),
z \not\in \sigma_{3B}$, and where the continued functions $F_l(x, y; z)$ still obey the asymptotic formulas \((3)\). This domain is described by the inequalities
\[ \text{Im } \sqrt{z} + \frac{1}{2} \text{Im } \sqrt{z - \epsilon_d} < \frac{\sqrt{3}}{2} |\epsilon_d|, \quad \text{Im } \sqrt{z} + \text{Im } \sqrt{z - \epsilon_d} < \frac{\sqrt{3}}{2} \mu. \]

3°. And finally, we distinguish the domain $\Pi^{(S)}$, most interesting for us, where the analytic continuation in $z$, $z \not\in \sigma_{3B}$, can be only done for the amplitude $a_0(z)$ (and consequently, for the scattering matrix $S_0(z)$); the analytic continuability of the amplitudes $A_l(z, \theta)$ in the whole domain $\Pi^{(S)}$ is not required. The set $\Pi^{(S)}$ is a geometric locus of points obeying the inequality
\[ \text{Im } \sqrt{z - \epsilon_d} < \min \left\{ \frac{1}{\sqrt{3}} \sqrt{|\epsilon_d|}, \frac{\sqrt{3}}{2} \mu \right\}. \]

Since the spherical wave $\exp(i \sqrt{z \rho})/\sqrt{\rho}$ in Eq. (3) is a function rapidly decreasing in all the directions, the use of the asymptotic condition (3) is justified even if $z \in \Pi^{(S)} \setminus \Pi^{(A)}$. Outside of the domain $\Pi^{(S)}$ the numerical construction of $S_0(z)$ by solving the Faddeev differential equations is, in general, impossible.

4 Numerical results
In the present work we search for the resonances of the $^4$He trimer including the virtual levels as roots of $S_0(z)$ and for the bound-state energies as positions of poles of $S_0(z)$. All the results presented below are obtained for the case $l_{\text{max}} = 0$. In all our calculations, $\hbar^2/m = 12.12$ KÅ$^2$. As the interatomic He–He–He interaction we employed the HFD-B potential constructed by R. A. Aziz and co-workers [2].

The value of the core range $c$ is chosen to be so small that its further decrease does not appreciably influence the dimer binding energy $\epsilon_d$ and the trimer ground-state energy $E_t^{(0)}$. Unlike the paper [3], where $c$ was taken to be equal 0.7 Å, now we take $c = 1.3$ Å. We have found that such a value of $c$ provides at least six reliable figures of $\epsilon_d$ and three figures of $E_t^{(0)}$.

Since the statements of Sect. 3 are valid, generally speaking, only for the potentials decreasing not slower than exponentially, we cut off the potential HFD-B setting $V(x) = 0$ for $x > r_0$. We have established that this cutoff for $r_0 \gtrsim 95$ Å provides the same values of $\epsilon_d$ ($\epsilon_d = -1.68541$ mK), $E_t^{(0)}$ ($E_t^{(0)} = -0.096$ K) and scattering phases which were obtained in earlier calculations [2] performed with the potential HFD-B. Also, we have found that the trimer excited-state energy $E_t^{(1)} = -2.46$ mK. Comparison of these results with results of other researchers can be found in Ref. [3]. In all the calculations of the present work we take $r_0 = 100$ Å. Note that if the formulas from Sect. 3 describing the holomorphy domains $\Pi^{(V)}$, $\Pi^{(A)}$ and $\Pi^{(S)}$ are used for finite potentials, one should set in them $\mu = +\infty$.

A detailed description of the numerical method we use is presented in Ref. [3]. When solving the boundary-value problem (3), we carry out its finite-difference approximation in polar coordinates $\rho$ and $\theta$. In this work, we used
Figure 1. Root locus curves of the real and imaginary parts of the scattering matrix $S_0(z)$. The solid lines correspond to $\text{Re} \, S_0(z) = 0$, while the tiny dashed lines, to $\text{Im} \, S_0(z) = 0$. The numbers 1, 2 and 3 denote the boundaries of the domains $\Pi^{(\theta)}$, $\Pi^{(S)}$ and $\Pi^{(A)}$, respectively. Complex roots of the function $S_0(z)$ are represented by the intersection points of the curves $\text{Re} \, S_0(z) = 0$ and $\text{Im} \, S_0(z) = 0$ and are located at $(-2.34 + i \, 0.96) \, \text{mK}$, $(-0.59 + i \, 2.67) \, \text{mK}$, $(2.51 + i \, 4.34) \, \text{mK}$ and $(6.92 + i \, 6.10) \, \text{mK}$.

Because of the symmetry relationship $S_0(z) = S_0(\overline{z})$ we performed all the calculations for $S_0(z)$ only at $\text{Im} \, z \geq 0$. First, we calculated the root lines of the functions $\text{Re} \, S_0(z)$ and $\text{Im} \, S_0(z)$. For the case of the grid parameters $N_\theta = N_\rho = 600$ and $\rho_{\text{max}} = 600 \, \text{Å}$ these lines are depicted in Fig. 1. Both resonances (roots of $S_0(z)$) and bound-state energies (poles of $S_0(z)$) of the $^4\text{He}$ trimer are associated with the intersection points of the curves $\text{Re} \, S_0(z) = 0$ and $\text{Im} \, S_0(z) = 0$. In Fig. 1, along with the root lines we also plot the boundaries of the domains $\Pi^{(S)}$, $\Pi^{(A)}$ and $\Pi^{(\theta)}$. One can observe that the “good” domain $\Pi^{(S)}$ includes none of the points of intersection of the root lines $\text{Re} \, S_0(z) = 0$ and $\text{Im} \, S_0(z) = 0$. The caption for Fig. 1 points out positions of four “res-
Table 4.1. Dependence of the dimer binding energy $\epsilon_d$ and the differences $\epsilon_d - E_t^{(1)}$, $\epsilon_d - E_t^{(2)}$, $\epsilon_d - E_t^{(2)\ast}$ and $\epsilon_d - E_t^{(2)\ast\ast}$ (all in mK) between this energy and the trimer exited-state energies $E_t^{(1)}$, $E_t^{(2)}$ and the virtual-state energies $E_t^{(2)\ast}$, $E_t^{(2)\ast\ast}$ on the factor $\lambda$.

| $\lambda$ | $\epsilon_d$ | $\epsilon_d - E_t^{(1)}$ | $\epsilon_d - E_t^{(2)\ast}$ | $\epsilon_d - E_t^{(2)\ast\ast}$ | $\epsilon_d - E_t^{(2)\ast\ast\ast}$ |
|-----------|--------------|---------------------|-----------------|-----------------|---------------------|
| 0.995     | -1.160       | 0.710               | -               | -               | -                   |
| 0.990     | -0.732       | 0.622               | -               | -               | -                   |
| 0.9875    | -0.555       | 0.573               | 0.473           | 0.222           | -                   |
| 0.985     | -0.402       | 0.518               | 0.4925          | 0.097           | -                   |
| 0.980     | -0.170       | 0.39616             | 0.39562         | 0.009435        | -                   |
| 0.975     | -0.036       | 0.2593674545        | 0.2593674502    | -               | 0.00156             |

The other purpose of the present investigation is to determine the mechanism of disappearance of the excited state of the helium trimer when the two-body interactions become stronger owing to the increasing $\lambda > 1$. It turned out that this disappearance proceeds just according to the scheme of the formation of new excited states; only the order of occurring events is inverse. The results of our computations of the energy $E_t^{(1)}$ when $\lambda$ changes from 1.05 to 1.17 are

| $\lambda$ | $\epsilon_d$ | $\epsilon_d - E_t^{(1)}$ | $\epsilon_d - E_t^{(2)\ast}$ | $\epsilon_d - E_t^{(2)\ast\ast}$ | $\epsilon_d - E_t^{(2)\ast\ast\ast}$ |
|-----------|--------------|---------------------|-----------------|-----------------|---------------------|
| 0.995     | -1.160       | 0.710               | -               | -               | -                   |
| 0.990     | -0.732       | 0.622               | -               | -               | -                   |
| 0.9875    | -0.555       | 0.573               | 0.473           | 0.222           | -                   |
| 0.985     | -0.402       | 0.518               | 0.4925          | 0.097           | -                   |
| 0.980     | -0.170       | 0.39616             | 0.39562         | 0.009435        | -                   |
| 0.975     | -0.036       | 0.2593674545        | 0.2593674502    | -               | 0.00156             |

The other purpose of the present investigation is to determine the mechanism of disappearance of the excited state of the helium trimer when the two-body interactions become stronger owing to the increasing $\lambda > 1$. It turned out that this disappearance proceeds just according to the scheme of the formation of new excited states; only the order of occurring events is inverse. The results of our computations of the energy $E_t^{(1)}$ when $\lambda$ changes from 1.05 to 1.17 are...
Figure 2. Graphs of the function $S_0(z)$ at real $z \leq \epsilon_d$ for three values of $\lambda < 1$. The notations used: $E^* = E_t^{(2)*}/|\epsilon_d|$, $E^{**} = E_t^{(2)**}/|\epsilon_d|$.

Table 4.2. Dependence of the dimer energy $\epsilon_d$, the difference $\epsilon_d - E_t^{(1)}$ between $\epsilon_d$ and the trimer exited-state energy $E_t^{(1)}$ and the difference $\epsilon_d - E_t^{(1)*}$ between $\epsilon_d$ and the trimer virtual-state energy $E_t^{(1)*}$ (all in mK) on the factor $\lambda$.

| $\lambda$ | $\epsilon_d$ | $\epsilon_d - E_t^{(1)}$ | $\lambda$ | $\epsilon_d$ | $\epsilon_d - E_t^{(1)*}$ |
|---------|-------------|-------------------|---------|-------------|-------------------|
| 1.05    | -12.244     | 0.873             | 1.18    | -82.927     | 0.001             |
| 1.10    | -32.222     | 0.450             | 1.20    | -99.068     | 0.057             |
| 1.15    | -61.280     | 0.078             | 1.25    | -145.240    | 0.588             |
| 1.16    | -68.150     | 0.028             | 1.35    | -261.393    | 3.602             |
| 1.17    | -75.367     | 0.006             | 1.50    | -490.479    | 12.276            |

given in Table 4.2. In the interval between $\lambda = 1.17$ and $\lambda = 1.18$ there occurs a "jump" of the level $E_t^{(1)}$ on the unphysical sheet and it transforms from the pole of the function $S_0(z)$ into its root, $E_t^{(1)*}$, corresponding to the trimer virtual level. The results of calculation of this virtual level where $\lambda$ changes from 1.18 to 1.5 are also presented in Table 4.2.

More details of our techniques and material presented will be given in an extended article [12].
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