Detecting Patterns Can Be Hard: Circuit Lower Bounds for the String Matching Problem

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Abstract

Detecting patterns in strings and images is a fundamental and well studied problem. We study the circuit complexity of the string matching problem under two popular choices of gates: De Morgan and threshold gates. For strings of length \( n \) and patterns of length \( \log n \ll k \leq n - o(n) \), we prove super polynomial lower bounds for De Morgan circuits of depth 2, and nearly linear lower bounds for depth 2 threshold circuits. For unbounded depth and \( k \geq 2 \), we prove a linear lower bound for (unbounded fan-in) De Morgan circuits. For certain values of \( k \), we prove a \( \Omega(\sqrt{n}/\log n) \) lower bound for general (no depth restriction) threshold circuits. Our proof for threshold circuits builds on a curious connection between detecting patterns and evaluating Boolean functions when the truth table of the function is given explicitly. Finally, we provide upper bounds on the size of circuits that solve the string matching problem.

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1 Introduction

One of the most basic and frequently encountered problems by minds and machines is that of detecting patterns in perceptual inputs. A classical example of such a pattern recognition problem is that of string matching: deciding whether a given pattern is contained in a given string. The one dimensional setting readily extends to the 2D case where one is looking for a 2D template in a 2D image (e.g., [ABF94]). String-matching problems arise in a host of applications such as speech recognition, text processing, web search and computational biology [NR02, CHL14], and hence it is of interest to devise efficient computational devices for such tasks.

We focus exclusively on the one-dimensional binary case, where strings and patterns are over the binary alphabet. We formalize it in the following definition.

**Definition 1.1.** In the string matching problem, the input is a binary string $I$ of $n$ bits and a binary pattern $P$ of length $k$. The goal is to compute the Boolean string matching function $SM_{n,k}$ which is defined as $SM_{n,k}(x_1\ldots x_n; y_1\ldots y_k) = 1$ if and only if the (binary) pattern $y_1\ldots y_k$ occurs as a substring in the binary string $x_1\ldots x_n$, i.e., for some $1 \leq i \leq n-k+1$ it holds that $y_j = x_{i+j-1}$ for all $j = 1,\ldots,k$.

We study the circuit complexity of the string matching problem. Recall that a linear threshold function (LTF) parameterized by $a_1,\ldots,a_m; \theta \in \mathbb{R}$ is a Boolean function $L_{\bar{a},\theta} = (a_1,\ldots,a_m,\theta)$ over $m$ Boolean variables which on input $(z_1,\ldots,z_m) \in \{0,1\}^m$ outputs 1 if and only if $\sum_{i=1}^m a_i z_i \geq \theta$. A threshold circuit is a Boolean circuit whose gates compute LTF’s. The study of LTF’s and threshold circuits was pioneered by [MP88], who among other results, proved limitations on the expressive power of LTF’s. The similarity between LTF’s and artificial neurons has resulted in a large body of research aimed at applying findings and methods from circuit complexity to understanding neural networks [HMP+93, PS88, Par94, MCPZ13, Mur71]. In particular, the empirical success of deep learning architectures [KSH12, LBH15] has fueled interest in the benefit of depth for neural networks [ES16, RPK+16, SMG13].

We consider two widely studied circuit families: Threshold circuits and De Morgan circuits (i.e., circuits with gates computing AND, OR and NOT). Our main interest is in proving lower bounds, although upper bounds are presented as well. It has been observed that there are relatively few circuit lower bounds results for pattern recognition tasks [LM01, LM02]. Therefore, studying the circuit complexity of Boolean functions that are related to pattern recognition is of potential interest as it can help bridge this gap.

The string matching function admits a nearly linear implementation at low depth. Hence our focus is on fine-grained complexity and we establish mainly linear or sublinear (in terms of $n$) lower bounds for $SM_{n,k}$. While such bounds may appear weak, it is worth noting that for the circuits families that we consider (threshold circuits, De Morgan circuits of unbounded fan-in) there are relatively few explicit examples of functions that are known to have non constant lower bounds and proving such lower bounds is considered to be challenging [ROS94]. Indeed, it is mentioned in [Juk12] with respect to threshold circuits that “even proving non constant lower bounds ... is a nontrivial task”.

1.1 Our results

For ease of exposition, we present our results for different circuit models in separate sections. For certain values of $k$ (for example, for $k = n$ or $k = 1$), it is easy to establish tight asymptotic bounds
on the circuit complexity of $\text{SM}_{n,k}$. For example, for $k = 1$, the $\text{SM}_{n,k}$ function can be computed by a circuit of constant size and depth (obviously, such a circuit has unbounded fan-in) as we only need to check whether the input text $(x_1, \ldots, x_n)$ contains a one or a zero.

$$\text{SM}_{n,k}(x_1, \ldots, x_n; y_1) = \left( \bigvee_{i=1}^n x_i \land y_1 \right) \lor \left( \bigvee_{i=1}^n \bar{x_i} \land \bar{y_1} \right).$$

Here we will mostly assume that $1 < k < n$.

In order to obtain more general results, we prove lower and upper bounds for (almost) all regimes of $k$. Throughout the paper, log denotes the logarithm of base 2.

Depth-1 De Morgan and threshold circuits compute functions monotone in each of their inputs. Since $\text{SM}_{n,k}$ is not monotone in its inputs, it cannot be computed by a depth-1 circuit.

Below we state our results for different circuit families of depth greater than 1. We start with De Morgan circuits.

**Theorem 1.2** (De Morgan circuits of bounded depth). Let $k \leq n$ be parameters for the $\text{SM}_{n,k}$ problem.

- **Depth 2 upper bound:** There exists a De Morgan circuit of depth 2 and size $O(n \cdot 2^k)$ computing $\text{SM}_{n,k}$.
- **Depth 3 upper bound:** There exists a De Morgan circuit of depth 3 and size $O(nk)$ computing $\text{SM}_{n,k}$.
- **Depth 2 lower bound:** Any De Morgan circuit of depth 2 computing $\text{SM}_{n,k}$ must be of size at least

  $$\begin{align*}
  \Omega \left( 2^{2k} \right) & \quad \text{if } k \leq \log n + 1; \\
  \Omega \left( n \cdot 2^k \right) & \quad \text{if } \log n + 1 \leq k \leq \sqrt{n}; \\
  \Omega \left( 2^{2\sqrt{n-k+1}} \right) & \quad \text{if } k \geq \sqrt{n};
  \end{align*}$$

Note that for the regime of $\log n \leq k \leq \sqrt{n}$ our results for depth 2 are tight up to a multiplicative constant. We also note that this theorem gives super-polynomial lower bounds on the size of De Morgan circuits of depth 2 for $\omega(\log n) \leq k \leq n - \omega(\log^2 n)$.

Next, we prove that the circuit complexity of $\text{SM}_{n,k}$ for De Morgan circuits with no restrictions either on the depth or fan-in must be at least linear in $n$ (a nearly matching upper bound was given in [Gal85]).

**Theorem 1.3** (De Morgan circuits with no depth restrictions). Let $2 \leq k \leq n$ be parameters for the $\text{SM}_{n,k}$ problem.

- **Upper bound:** There exists a De Morgan circuit of size $O\left( \min(n \cdot \text{polylog}(n), nk) \right)$ computing $\text{SM}_{n,k}$.
- **Lower bound:** Any De Morgan circuit computing $\text{SM}_{n,k}$ must be of size at least $n/2$.\(^1\)

\(^1\)In fact, a similar argument proves a slightly stronger lower bound of $n$ for $\log n \leq k \leq n - \log n$. Since in this paper we are focused on asymptotic results for all regimes of $k$, we omit this proof.
Turning to threshold circuits we prove the following theorem.

Theorem 1.4 (Threshold circuits computing SM$_{n,k}$). Let $k \leq n$ be parameters for the SM$_{n,k}$ problem.

- **Upper bound:** There exists a threshold circuit of depth 2 and size $O(n - k)$ computing SM$_{n,k}$.

- **Lower bound for unbounded depth:** Let $k$ be even and let $k \leq n \leq \frac{k}{2} \cdot 2^{k/2}$. Then, any threshold circuit computing SM$_{n,k}$ must be of size at least $\Omega\left(\sqrt{n/k} \cdot \log(n/k)\right)$.

- **Lower bound for depth 2:** Let $k$ be even and let $k \leq n \leq \frac{k}{2} \cdot 2^{k/2}$. Then any threshold circuit of depth 2 computing SM$_{n,k}$ must be of size at least $\Omega\left nosotros(n/k)\right)$.

In particular, for certain values of $k$ any threshold circuit computing SM$_{n,k}$ has size $\Omega(\sqrt{n/k} \cdot \log(n/k))$.

If we restrict our attention to depth–2 threshold circuits, then for certain $k$’s we have a nearly linear (e.g., $\Omega(n/\log^2 n)$) lower bound for depth–2 threshold circuits computing SM$_{n,k}$. We stress that there are no restrictions on the weights of the threshold circuits in these lower bounds. Note that our proof of lower bounds works only for the parameters $k \leq n \leq \frac{k}{2} \cdot 2^{k/2}$, that is, roughly for the regime of $k$ between $2\log(n)$ and $n$.

For the proof techniques, we start our study of the complexity of SM$_{n,k}$ for De Morgan circuits of depth 2. We first prove a lower bound on the size of minterms and maxterms of SM$_{n,k}$ (see Section 3 for precise definitions), then prove an estimate on the number of zeros and ones of SM$_{n,k}$, and then use them to derive lower bounds on the size of De Morgan circuits. In Section 5, we prove lower bounds for threshold circuits by reducing the problem of computing a “hard” function to computing SM$_{n,k}$. Perhaps surprisingly, we show that the string matching problem can encode a truth table of an arbitrary Boolean function. We also observe that our lower bounds apply to 2-dimensional pattern matching problems without much difficulty (we omit the details).

All of our upper bounds for circuits computing SM$_{n,k}$ are straightforward and may have been discovered before. We include them here as other than [Gal85], we have not been able to find a source indicating circuit upper bounds for SM$_{n,k}$.

## 2 Related work

Devising lower bounds for threshold circuits has proven to be a challenge. Recall that the inner product function (a.k.a IP2) over the field with two elements $Z_2$ receives $x, y \in Z_2^n$ and computes their inner product $\sum_{i=1}^n x_i y_i \mod 2$. In [HMP+93], an exponential (in $n$) lower bound was given for the inner product function over the field with two elements (IP2) for depth 2 threshold circuits with polynomial (in $n$) weights. These results have been strengthened by [FKL+01] where it was shown that the exponential lower bounds for IP2 hold for depth 2 threshold circuits even if the weights of the middle (hidden) layer are unbounded. Superlinear lower bounds on the number of gates of arbitrary depth 2 threshold circuits as well as depth 3 threshold circuits (with polynomial

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2That is, if $2 \log(n) \leq k \leq 2.1 \log(n)$, then we have a lower bound $\sqrt{n}/k$, and for $k > 2.1 \log(n)$ we have a lower bound of $\sqrt{n/k}$.

3That is, if $2 \log(n) \leq k \leq 2.1 \log(n)$, then we have a lower bound $\frac{n}{k}$, and for $k > 2.1 \log(n)$ we have a lower bound of $\frac{n}{k^2}$.
weights on the top layer) were proven recently in [KW16]. On the positive side, several arithmetic functions such as division and multiplication were shown to admit efficient implementations using threshold circuits of constant depth [SBKH93]. For De Morgan circuits, the celebrated Håstad’s switching lemma [Hås87] established exponential lower bounds for bounded depth circuits computing explicit functions (e.g., majority, parity). We note that in contrast to the parity function, the string matching function admits a polynomial size circuit of depth 3. We are not aware of previous works regarding circuit lower bounds for string matching (neither for De Morgan nor for threshold circuits).

Using gate elimination, it was proven in [GT93] that any threshold circuit (with no depth restrictions) computing IP2 requires size $\Omega(n)$. A communication complexity approach was used in [ROS94] to prove nearly linear lower bounds for Boolean functions such as equality (deciding whether two strings of length $n$ are identical). These authors considered threshold functions with polynomial weights (for unrestricted weights the equality function has a threshold circuit of constant size). Nisan [Nis93] proved that a Boolean function $f : \{0, 1\}^n \to \{0, 1\}$ with $\Omega(n)$ two-way communication complexity (for randomized protocols) cannot be computed by unrestricted threshold circuits of size $O(n/\log n)$. It follows that several well studied functions such as Disjointness and Generalized Inner Product have $\Omega(n/\log n)$ lower bounds on the size of threshold circuits computing them. However, both the gate elimination method and the communication complexity approach do not seem to give non constant bounds for threshold circuits for the string matching problem. In particular, the gate elimination methods build on discrepancy bounds (e.g., Lindsey Lemma [Juk12]) utterly fail for the string matching problem (SM$_{n,k}$ may have very large monochromatic rectangles).

**Upper bounds** on the circuit complexity of 2D image matching problem under projective transformations was studied in [Ros16]. In this problem, which is considerably more complicated than the pattern matching problems we study, the goal is to find a projective transformation $f$ such that $f(A)$ “resembles” $B$ for two images $A, B$. Here, images are 2D square arrays of dimension $n$ containing discrete values (colors). In particular, it is proven that this image matching problem is an $TC0$ (it admits a threshold circuit of polynomial size and logarithmic depth in $n$). It is also shown how to implement majority gates using gates that can solve the aforementioned image matching problem with a polynomial overhead.

The idea to lower bound the circuit complexity of Boolean functions that arise in feature detection was studied in [LM01, LM02]. In [LM01, LM02] it is assumed that there are two types of features $a$ and $b$ and detectors corresponding to the two types of features are situated on a 1D or 2D grid. The binary output of these features are represented by an array of $n$ positions: $a_1, \ldots, a_n$ (where $a_i = 1$ if the feature $a$ is detected in position $i$, and $a_i = 0$ otherwise) and an array $b_1, \ldots, b_n$ which is analogously defined with respect to $b$. The Boolean function $P_{LR}$ outputs 1 if there exist $i, j$ with $i < j$ such that $a_i = b_j = 1$, and 0 otherwise. This function is advocated in [LM02] as a simple example of a detection problem in vision that requires to identify spatial relationship among features. It is shown that this problem can be solved by $O(\log n)$ threshold gates. A 2-dimensional analogue where the indices $i = (i_1, i_2)$ and $j = (j_1, j_2)$ represent two-dimensional coordinates and one is interested whether there exist indices $i$ and $j$ such that $a_i = b_j = 1$ and $j$ is above and to the right of the location $i$ is studied in [LM02]. Recently, the two-dimensional version was studied in [UYZ15] where a $O(\sqrt{n})$-gate threshold implementation was given along with a lower bound of $\Omega(\sqrt{n}/\log n)$ for the size of any threshold circuit for this problem. We remark that the problem studied in [LM01, LM02, UYZ15] is different from ours, and different proof ideas are needed for

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4We refer to [Ros16] for the precise definition of distance used there.
establishing lower bounds in our setting.

There are several algorithms that solve the string matching problem in \(O(n)\) time [BM77, KMP77]. There has also been significant interest in obtaining parallel algorithms for string matching. In [Gal85], it is shown that their parallel algorithm gives an \(O(n \log^2 n)\) De Morgan circuit of depth \(O(\log^2 n)\). It is also noted [Gal85] that the classical Boyer-Moore and KMP algorithms do not seem to parallelize, thus implementing these algorithms by small depth circuit of size \(O(n)\) may be difficult or impossible.

3 Bounded depth circuits

In this section we prove Theorem 1.2.

3.1 Constructing bounded depth De Morgan circuits for \(\text{SM}_{n,k}\)

In this section we prove the first two items of Theorem 1.2.

**Theorem 3.1.** For any \(k \leq n\) there exists a De Morgan circuit of depth 2 and size \((n-k+1) \cdot 2^k + 1\) computing \(\text{SM}_{n,k}\).

**Proof.** First we note that equality of two \(k\)-bit strings can be implemented using a DNF of width \(2k\) and size (number of clauses) \(2^k\). Indeed, denoting the two inputs by \(z = (z_1, \ldots, z_k)\) and \(w = w_1, \ldots, w_k\), let

\[
\text{EQ}(z_1, \ldots, z_k; w_1, \ldots, w_k) = \bigvee_{a=(a_1, \ldots, a_k) \in \{0,1\}^k} \left( \land_{i=1}^k (z_i = a_i) \land_{i=1}^k (w_i = a_i) \right),
\]

where \((w_i = a_i)\) is equal to \(w_i\) if \(a_i = 1\), and \(\neg w_i\) otherwise.

For each \(i = 1, \ldots, n-k+1\) let \(\text{EQ}_i\) be the DNF that outputs 1 if and only if \(y = (x_i, \ldots, x_{i+k-1})\). Taking \(\bigvee_{i=1}^{n-k+1} \text{EQ}_i\), we obtain a circuit of depth 3 that computes the \(\text{SM}_{n,k}\) function. In order to turn it into a depth-2 circuit, note that the second and the third layers consist of \(\lor\) gates, and hence can be collapsed to one layer. This way we get a depth-2 circuit of size \((n-k+1) \cdot 2^k + 1\). \(\square\)

The next Theorem is likely to have been discovered multiple times. We attribute it to folklore.

**Theorem 3.2.** There exists a De Morgan circuit of depth 3 and size \(O(nk)\) computing \(\text{SM}_{n,k}\).

**Proof.** First we note that the equality function of two \(k\)-bit strings can be implemented using a CNF of width 2 and size (number of clauses) \(2k\). Indeed, we can check equality of two bits \(z\) and \(w\) using the circuit \((z \lor \neg w) \land (\neg z \lor w)\). Therefore, we can implement equality of two \(k\)-bits strings using the CNF formula

\[
\text{EQ}(z_1, \ldots, z_k; w_1, \ldots, w_k) = \bigwedge_{i=1}^k ((z_i \lor \neg w_i) \land (\neg z_i \lor w_i)) .
\]

From here on we can proceed as in the previous proof, namely, for each \(i = 1, \ldots, n-k+1\) let \(\text{EQ}_i\) be the CNF that outputs 1 is and only if \(y = (x_i, \ldots, x_{i+k-1})\). Taking \(\bigvee_{i=1}^{n-k+1} \text{EQ}_i\), we obtain a circuit of depth 3 that computes the \(\text{SM}_{n,k}\) function. The output gate has fanin \(n-k+1\), the gates in the second layer have fan-in \(2k\), and the gates in the first layer have fan-in 2. Therefore, the total size of the circuit is \(O(nk)\). \(\square\)
3.2 Lower bounds for depth 2 De Morgan circuits

For the lower bound we need the following definition.

**Definition 3.3.** A minterm (maxterm) of a Boolean function \( f \) is a set of variables of \( f \), such that some assignment to those variables makes \( f \) output 1(0) irrespective of the assignment to the other variables. The width of a minterm (maxterm) is the number of variables in it.

First we find the minimal width of minterms and maxterms of \( SM_{n,k} \).

**Lemma 3.4.** For any values of \( n \) and \( k \):

1. Every minterm of \( SM_{n,k} \) has width at least \( 2k \).
2. Every maxterm of \( SM_{n,k} \) has width at least \( 2\sqrt{n - k + 1} \).
3. If \( k \leq \sqrt{n - k + 1} \), then every maxterm of \( SM_{n,k} \) has width at least \( k + \frac{n - k + 1}{k} \).

**Proof.** For a string \( x \) of length \( n \), and integers \( 1 \leq i \leq j \leq n \), by \( x[i,j] \) we denote the substring of \( x \) starting at the position \( i \) and ending at the position \( j \).

1. For every \( n \geq 1 \) and every \( 1 \leq k \leq n \), we show that any assignment \( \rho \) to \( 2k - 1 \) variables can be extended to an assignment \( \pi \) which forces \( SM_{n,k} \) to output 0. We prove this statement by induction on \( n + k \). The base cases \( n + k = 2 \) and \( n + k = 3 \) hold trivially. Indeed, in both cases \( k = 1 \), so \( \rho \) fixes only one variable which clearly is not enough to make \( SM_{n,k} \) constant.

   We now prove the induction step. Consider the first character \( x_1 \) of the text.

   If \( \rho \) does not assign a value to \( x_1 \), then we reduce the problem to a problem on strings \( x[2,n] \) and \( y \) with the same number of assigned variables. Indeed, by the induction hypothesis we can find an assignment \( \pi' \) to the variables \( x[2,n], y[1,k] \) such that \( x[2,n] \) does not contain \( y \). Now we set \( x_1 = \neg y_1 \) to ensure that \( x \) does not start with the string \( y \).

   If \( \rho \) does assign a value to \( x_1 \) then we consider two cases. If \( \rho \) assigns a value to \( y_1 \), then we apply the induction hypothesis to the strings \( x[2,n] \) and \( y[2,k] \). Indeed, we have at most \( 2k - 3 \) fixed variables among those variables, thus we have an assignment \( \pi' \) such that \( x[2,n] \) does not contain \( y[2,k] \). This implies that \( x \) does not contain \( y \) as a substring. In the remaining case where \( \rho \) assigns a value to \( x_1 \) but not to \( y_1 \), we set \( y_1 = \neg x_1 \) and use the induction hypothesis for the strings \( x[2,n] \) and \( y \) (these two strings have at most \( 2k - 1 \) fixed variables).

2. Consider a substitution \( \rho \) which fixes \( n_1 \) variables in the text and \( k_1 \) variables in the pattern. In order to force \( SM_{n,k} \) output 0, for every shift \( 1 \leq i \leq n - k + 1 \) there must be an index \( 1 \leq j \leq k \) such that \( \rho \) assigns a value to \( y_j \) and \( x_{i+j} \). Thus, every of \( n_1 \) assigned variables in the text “covers” at most \( k \) shifts. Since the total number of shifts is \( n - k + 1 \), this implies that \( n_1 \cdot k_1 \geq n - k + 1 \). Therefore, by the the inequality of arithmetic and geometric means we get that \( n_1 + k_1 \geq 2\sqrt{n_1 \cdot k_1} \geq 2\sqrt{n - k + 1} \).

3. Since \( n_1 \cdot k_1 \geq n - k + 1 \) we get that \( n_1 + k_1 \geq \frac{n - k + 1}{k_1} + k_1 \). The desired bound follows by noting that the function \( f(k_1) = \frac{n - k + 1}{k_1} + k_1 \) is monotone decreasing for \( k_1 < \sqrt{n - k + 1} \).

Next we prove tight bounds on the number of satisfying and non-satisfying inputs of \( SM_{n,k} \).
Lemma 3.5. For $k \leq n$, let $O$ and $Z$ be the preimages of 1 and 0 of $SM_{n,k}$, respectively. That is,

$$O = \{(x, y) \in \{0, 1\}^{n+k} : SM_{n,k}(x; y) = 1\} \quad Z = \{(x, y) \in \{0, 1\}^{n+k} : SM_{n,k}(x; y) = 0\}.$$  

$$|O| = \Theta\left(2^n \cdot \min(2^k, n - k + 1)\right) \quad \text{for every } k;$$

$$|Z| = \Theta\left(2^{n+k}\right) \quad \text{for } k \geq \log n + 1;$$

$$|Z| \geq \Omega\left(2^n(1 - 2^{-k})^n\right) \quad \text{for every } k.$$

Proof. In order to estimate the number of satisfying inputs of $SM_{n,k}$, we use the following result from [GW07]:

Theorem 3.6 (Corollary 2.2 in [GW07]). Let $x$ be a uniformly distributed random string of length $n$, and let $X_{n,k}$ denote the number of distinct substrings of $x$ of length $k$. Then there exist $0 < \mu < 1$ and $0 < \epsilon < 0.5$, such that for every values of $n$ and $k$:

$$\mathbb{E}[X_{n,k}] = 2^k - 2^k(1 - 2^{-k})^{n-k+1} + O(n^{-\epsilon} \mu^k).$$

The upper bound on $|O|$ follows immediately from the following observation. A string of length $n$ can contain at most $\min(2^k, n - k + 1)$ different substrings of length $k$. For the lower bound on $|O|$ we consider two regimes of $k$.

First, if $(n - k + 1)/2^k \geq 1$, then from Theorem 3.6 we have

$$|O| \geq 2^n \cdot \mathbb{E}[X_{n,k}] \geq 2^{n+k}(1 - (1 - 2^{-k})^{n-k+1})$$

$$\geq 2^{n+k}(1 - \exp(-(n - k + 1)/2^k)) \geq 2^{n+k}(1 - 1/e) .$$

If $(n - k + 1)/2^k \leq 1$, then again from Theorem 3.6 we have

$$|O| \geq 2^n \cdot \mathbb{E}[X_{n,k}] \geq 2^n(2^k - 2^k(1 - 2^{-k})^{n-k+1})$$

$$\geq 2^n \left( 2^k - 2^k \left( 1 - \frac{n - k + 1}{2^k} + \frac{(n - k + 1)^2}{2 \cdot 2^{2k}} \right) \right)$$

$$\geq \Omega(2^n(n - k + 1)) .$$

From the relation $|Z| + |O| = 2^{n+k}$ and a trivial upper bound on $|O| \leq 2^n \cdot n$, we have that $|Z| \geq \Omega(2^{n+k})$ for $k \geq \log n + 1$.

In order to prove the lower bound $|Z| = \Omega\left(2^n(1 - 2^{-k})^n\right)$, we consider the pattern string $y_0 = 0^k$. The number $F_n$ of strings $x$ of length $n$ which do not contain $y_0$ satisfies the generalized Fibonacci recurrence:

$$F_n = \sum_{i=1}^{k} F_{n-i} .$$

From the known bounds on the generalized Fibonacci numbers (see, e.g., Lemma 3.6 in [Wol96]) we have $F_n \geq \Omega(2^n(1 - 2^{-k})^n)$, which implies the lower bound on $|Z|$.

Theorem 3.7. We have the following bounds for depth-2 De Morgan circuits for $SM_{n,k}$.

1. The DNF-size of $SM_{n,k}$ is

$$\text{DNF}(SM_{n,k}) \geq \Omega\left(2^k \cdot \min(2^k, n - k + 1)\right).$$
2. For CNF’s computing $\text{SM}_{n,k}$ we have the following lower bounds on their sizes.

$$\text{CNF}(\text{SM}_{n,k}) \geq \Omega \left( \frac{2^{\sqrt{n}}}{\sqrt{n/k}} \right) \quad \text{if } k \leq \log n + 1;$$

$$\text{CNF}(\text{SM}_{n,k}) \geq \Omega \left( \frac{2^{k+n/k}}{2^k} \right) \quad \text{if } \log n + 1 \leq k \leq \sqrt{n};$$

$$\text{CNF}(\text{SM}_{n,k}) \geq \Omega \left( 2^{\sqrt{n-k+1}} \right) \quad \text{if } k \geq \sqrt{n}.$$

Theorem 3.7 implies Theorem 1.2 since any De Morgan circuit of depth 2 is either a CNF or a DNF.

**Proof.** First we prove the lower bound for DNFs. From Lemma 3.4, every minterm of $\text{SM}_{n,k}$ has width at least $2k$, and thus in any DNF computing $\text{SM}_{n,k}$ every clause must be of width at least $2k$. Therefore, every clause in such a DNF evaluates to 1 on at most $2^{n+k}/2^k = 2^{n-k}$ inputs of the $\text{SM}_{n,k}$ function. By Lemma 3.5, $|O| \geq \Omega \left(2^n \cdot \min(2^k, n-k+1)\right)$. Thus, every DNF for $\text{SM}_{n,k}$ must contain at least $|O|/2^{n-k} = \Theta \left(2^k \cdot \min(2^k, n-k+1)\right)$ clauses.

The proof idea for CNFs is similar. We say that a clause *covers an input* $w \in \{0,1\}^{n+k}$ if this clause evaluates to 0 on $w$. Note that a clause of width $c$ covers at most $2^{n+k-c}$ elements in $\{0,1\}^{n+k}$. For the parameters $k \leq n$ we claim first that every clause of a CNF computing $\text{SM}_{n,k}$ must be of width at least $c = c(k,n)$ depending on the range of $k$ (as follows from Lemma 3.4). This implies that the number of clauses in any CNF computing $\text{SM}_{n,k}$ is at least $|Z|/2^{n+k-c}$. Below, we use Lemma 3.5 and Lemma 3.4 to estimate $c$ and $|Z|$ for different ranges of $k$.

If $k \leq \log n + 1$, then $|Z| \geq \Omega \left(2^n \left(1 - \frac{n}{2^k}\right)^n\right)$ by Lemma 3.5. By Lemma 3.4, the width of each maxterm is at least $c \geq k + \frac{n-k+1}{k}$. Thus, the number of clauses in any CNF computing $\text{SM}_{n,k}$ must be at least

$$\Omega \left(\frac{|Z|}{2^{n+k-c}}\right) \geq \Omega \left(\frac{|Z|}{2^{n-n/k}}\right) \geq \Omega \left(2^{n/k} \left(1 - 2^{-k}\right)^n\right) \geq \Omega \left(2^{n/10k}\right),$$

where the last bound follows from the inequality $2^{1/k} \cdot \left(1 - 2^{-k}\right) \geq 2^{1/10k}$ which holds for all $k \geq 2$.

For $k \geq \log n + 1$, Lemma 3.5 gives us an $\Omega(2^{n+k})$ lower bound on $|Z|$. Lemma 3.4 provides a lower bound on the width $c$ of maxterms: for $\log n + 1 \leq k \leq \sqrt{n}$, $c \geq k + n/k - 1$, and for $k \geq \sqrt{n}$, $c \geq 2\sqrt{n} - k + 1$. The desired bounds on the number of clauses in any CNF computing $\text{SM}_{n,k}$ now follow immediately. \qed

### 4 Lower bound for unbounded depth De Morgan circuits

In this section we prove the lower bound of Theorem 1.3 stating that any De Morgan circuit computing $\text{SM}_{n,k}$ must be of size at least $n/2$. The $O(nk)$ upper bound follows from the construction for depth 3 circuits given in Theorem 3.2, and the $n \text{polylog}(n)$ bound was given in [Gal85].

**Proof.** Suppose that a circuit $C$ computes the $\text{SM}_{n,k}$ function, and consider an input $(x, y)$ to the circuit. We prove that $C$ has at least $n/2$ gates using the gate elimination method. Specifically, we show that for any fixing of the bits $x_1, x_3, x_5, \ldots, x_{2t-1}$ for $1 \leq t \leq n/2 - 1$, the restricted function depends on the bit $x_{2t+1}$. Since the function depends on $x_{2t+1}$, any circuit computing it must have $x_{2t+1}$ or $\neg x_{2t+1}$ among its inputs. Without loss of generality we assume that $x_{2t+1}$ appears as an input. Now we show that we can fix the input $x_{2t+1}$ so that at least one gate of the circuit is removed.
Indeed, if \( x_{2t+1} \) appears as an input to an AND gate, we can set \( x_{2t+1} = 0 \), hence setting the output of the gate to be 1. This way we can remove the gate from the circuit by setting the output of the AND gate to be 1, and propagate it. (It is possible that we also affect other gates). Similarly, if \( x_{2t+1} \) appears as an input to an OR gate, we can set \( x_{2t+1} = 1 \), hence setting the outputs of the OR gate to be 0, and remove the gate from the circuit. Therefore, we can remove at least \( n/2 - 1 \) gates from the circuit, and hence the size of the original circuit computing \( SM_{n,k} \) was at least \( n/2 \).

Therefore, it is left to prove the following claim

**Claim 4.1.** Let \( k \geq 2 \). For any fixing of the bits \( x_1, x_3, x_5, \ldots, x_{2t-1} \) for \( 1 \leq t \leq n/2 - 1 \), the restricted function depends on the bit \( x_{2t+1} \).

**Proof.** Let \( x^* = (x_1^*, x_3^*, \ldots, x_{2t-1}^*) \) be the values of the \( t \) fixed bits of \( x \). In order to show that the restricted function depends on \( x_{2t+1} \), we show that there exist two inputs: \((x, y)\) and \((x', y)\), such that \( 0 = SM_{n,k}(x, y) \neq SM_{n,k}(x', y) = 1 \) and \((x, y)\) and \((x', y)\) are extensions of \( x^* \) which differ only in the position \( 2t+1 \) : \( x_{2t+1} \neq x'_{2t+1} \).

We set all non-fixed bits of \( x \) to 0, except for \( x_{2t+2} = 1 \). Now we set \( x' \) to be equal to \( x \) everywhere except for the position \( 2t+1 \), where \( x'_{2t+1} = 1 \). Now we see that the string \( x \) does not contain two ones in a row, while \( x' \) does. Since \( k \geq 2 \), we can set \( y \) to be an arbitrary substring of \( x' \) of length \( k \) which contains \( x'_{2t+1} \) and \( x'_{2t+2} \). By the definition of \( y \) we have \( SM_{n,k}(x', y) = 1 \) and \( SM_{n,k}(x, y) = 0 \) because \( x \) does not contain the substring 11.

This completes the proof of Theorem 1.3. \( \square \)

## 5 Threshold circuits

In this section we prove Theorem 1.4.

### 5.1 Construction of a depth 2 threshold circuits

We start with the construction stated in the first item.

**Lemma 5.1.** There exists a threshold circuit of depth 2 and size \( O(n - k + 1) \) computing \( SM_{n,k} \).

**Proof.** Let \( GEQ \) be the gate that gets \( 2k \) bits \( z_1, \ldots, z_k; w_1, \ldots, w_k \) and evaluates to 1 if and only if the number represented in binary by the bits \( z_1, \ldots, z_k \) is greater than or equal to the number represented by \( w_1, \ldots, w_k \). Note that \( GEQ \) can be implemented by one threshold gate as follows:

\[
GEQ(z_1, \ldots, z_k; w_1, \ldots, w_k) = 1 \text{ if and only if } (z_1 + 2 \cdot z_2 + 4 \cdot z_3 + \ldots + 2^{k-1} \cdot z_k) - (w_1 + 2 \cdot w_2 + 4 \cdot w_3 + \ldots + 2^{k-1} \cdot w_k) \geq 0.
\]

In order to describe the circuit computing \( SM_{n,k}(x_1, \ldots, x_n; y_1, \ldots, y_k) \) let the first layer contain \( 2(n - k + 1) \) \( GEQ \) gates \( g_i \) and \( \ell_i \) for \( i = 1, \ldots, n - k + 1 \), where each \( g_i \) gets as inputs \( x_i, \ldots, x_{i+k-1}; y_1, \ldots, y_k \) (that evaluates to 1 if and only if the number represented by the corresponding bits of \( x \) is at least that represented by \( y \)), and \( \ell_i \) gets has inputs \( y_1, \ldots, y_k; x_1, \ldots, x_{i+k-1} \) (that evaluates to 1 if and only if the number represented by the corresponding bits of \( x \) is at most that represented by \( y \)). The second (output) layer evaluates to 1 if and only if \( \sum_{i=1}^{n-k+1} (g_i + \ell_i) \geq n - k + 2 \). Clearly the circuit contains \( 2n - 2k + 3 \) gates.

In order to prove correctness, we note that for every \( i \), at least one of \( g_i \) and \( \ell_i \) evaluates to 1. Also, both \( g_i \) and \( \ell_i \) are 1 if and only if the corresponding substring of \( x \) equals \( y \). Therefore, \( \sum_{i=1}^{n-k+1} (g_i + \ell_i) > n - k + 1 \) if and only if \( y \) is a substring of \( x \), i.e., \( SM_{n,k}(x, y) = 1 \). \( \square \)
5.2 Lower bounds for threshold circuits: the $n = (k/2) \cdot 2^{k/2}$ case

Below we prove the lower bounds stated in items 2 and 3 of Theorem 1.4 only for the setting of parameters where $n = (k/2) \cdot 2^{k/2}$. Later, we show how to extend this idea to the entire range of $k$ and $n$ stated in the theorem. The lower bound is shown via a reduction from a hard function $f : \{0, 1\}^{k/2 - 1} \to \{0, 1\}$. Specifically, we show that for the setting of even $k$ and $n = (k/2) \cdot 2^{k/2}$ as above if $\text{SM}_{n,k}$ can be solved using a circuit of size $s$, then every function $f : \{0, 1\}^{k/2 - 1} \to \{0, 1\}$ has a circuit of size $s$ computing it. In particular, since we know that there are functions $f : \{0, 1\}^{k/2 - 1} \to \{0, 1\}$ that require large threshold circuits, it follows that a related lower bound holds also for the $\text{SM}_{n,k}$ function. We focus on even $k$ for notational convenience: our ideas easily extend to odd $k$ as is explained in the note in the end of this section.

**The reduction** Below we prove a lower bound on $\text{SM}_{n,k}$ for the parameters $k$ and $n$ where $k$ is even and $n = (k/2) \cdot 2^{k/2}$.

Let $\ell = k/2 - 1$. Given a string $a \in \{0, 1\}^\ell$ define $\text{dup}(a) \in \{0, 1\}^k$ to be the string obtained from $a$ by repeating each bit of $a$ twice, and concatenating it with 01 in the end. (Note that $2\ell + 2 = k$ by the choice of $\ell$). For example $\text{dup}(010) = 00110001$. Also, for an integer $\ell$ define $z(\ell) \in \{0, 1\}^k$ to be the string $0101\ldots01$ of length $2\ell + 2$, i.e., the string where the substring 01 is repeated $\ell + 1$ times.

Let $f : \{0, 1\}^\ell \to \{0, 1\}$ be an arbitrary function. For each $a = (a_1, \ldots, a_\ell) \in \{0, 1\}^\ell$ an input to $f$, define

$$x_a = \begin{cases} \text{dup}(a), & \text{if } f(a) = 1; \\ z(\ell), & \text{if } f(a) = 0. \end{cases}$$

We make the following simple observation.

**Observation 5.2.** Given a function $f : \{0, 1\}^\ell \to \{0, 1\}$, define $x_f \in \{0, 1\}^{2^\ell \cdot (2\ell + 2)}$ to be the concatenation of $x_a$’s for all $a \in \{0, 1\}^\ell$ in some fixed order, e.g., the lexicographic order on $\{0, 1\}^\ell$. Note that $|x_f| = 2^\ell \cdot (2\ell + 2) = 2^{k/2 - 1} \cdot k = n$. Then, for any $y \in \{0, 1\}^\ell$ it holds that $f(y) = 1$ if and only if $\text{SM}_{n,k}(x_f, \text{dup}(y)) = 1$.

Indeed, it is immediate to see that if $f(y) = 1$ then $\text{SM}_{n,k}(x_f, \text{dup}(y)) = 1$. Duplicating every bit in $a$ and adding 01 to the end of the resulting pattern are done to ensure that if $f(y) = 0$ there will not be a copy of $\text{dup}(y)$ in $x_f$.

Given the observation above, it is not difficult to see that any lower bound on the size of the circuit computing $f$ implies a lower bound on $\text{SM}_{n,k}$.

**Proposition 5.3.** For $\ell \in \mathbb{N}$ let $k = 2\ell + 2$, and let $n = 2^\ell \cdot (2\ell + 2)$. Let $f : \{0, 1\}^\ell \to \{0, 1\}$ be a Boolean function. Let $C$ be a threshold circuit computing $\text{SM}_{n,k}$. Then, there exists a threshold circuit $C'$ computing $f$ such that $|C'| \leq |C|$.

**Proof.** Suppose there exists a circuit $C$ of size at most $s$ computing $\text{SM}_{n,k}$. We denote the input variables of the pattern $y$ by $y_1, y_2, \ldots, y_{2\ell + 2}$. We show how to convert it into a circuit $C'$ computing $f$ by fixing some of the input variables of $C$. This is done by (1) fixing the “text part” (the variables corresponding to $x$) of the input of $\text{SM}_{n,k}$ to be $x_f$ as defined in Observation 5.2, and (2) replacing every pair of variables $y_{2i-1}$ and $y_{2i}$ for all $i = 1, \ldots, \ell$ by a single variable $\tilde{y}_i$ that is fed to all gates that have inputs $y_{2i-1}$ or $y_{2i}$ (with a proper adjustment of the weight if both $y_{2i-1}$ and $y_{2i}$ are inputs of the gate). Finally, fix $y_{2\ell+1} = 0$ and $y_{2\ell+2} = 1$. It is now easy to see that $C'$ computes $f$. \qed
In order to complete the proof of Theorem 1.4 we need to show that there exists a function \( f : \{0,1\}^\ell \rightarrow \{0,1\} \) that requires large threshold circuit to compute it. These results are known (see, for example, [Juk12, KW16]), but we sketch the proofs here for completeness.

**Proposition 5.4.** Let \( \ell \in \mathbb{N} \) be sufficiently large.

1. There exists a function \( f : \{0,1\}^\ell \rightarrow \{0,1\} \) such that any threshold circuit (with no restrictions on its depth) computing \( f \) must be of size at least \( \Omega(2^{\ell/2}/\sqrt{\ell}) \).

2. There exists a function \( f : \{0,1\}^\ell \rightarrow \{0,1\} \) such that any threshold circuit of depth 2 computing \( f \) must be of size at least \( \Omega(2^{\ell}/\ell^2) \).

**Proof of Proposition 5.4.** We first upper bound the number of functions that can be represented by threshold circuits of size at most \( s \). This can be obtained by the following result due to [RSO94].

**Theorem 5.5.** Let \( f_1 \ldots f_s : \{0,1\}^\ell \rightarrow \{0,1\} \) be a set of \( s \) Boolean functions. Then, the number of Boolean functions which are realized by a threshold gate \( g : \{0,1\}^\delta \rightarrow \{0,1\} \) whose \( s \) inputs are \( f_1 \ldots f_s \) is at most \( 2^{O(\ell s)} \).

It follows from Theorem 5.5 that the number of distinct Boolean functions with \( \ell \) variables computed by a threshold circuit of size \( s \) is \( 2^{O(\ell^2 s)} \), as there are at most \( 2^{O(\ell s)} \) choices for every gate and there are \( s \) gates. Since the total number of Boolean functions \( f : \{0,1\}^\ell \rightarrow \{0,1\} \) is \( 2^{2^\ell} \) it follows that there exists a function that cannot be computed by a threshold circuit of size \( s = \Omega(\sqrt{2^\ell/\ell}) \).

Another implication of Theorem 5.5 (observed in [KW16]) is that there are at most \( 2^{O(\ell^2 s)} \) distinct Boolean functions (over \( \ell \) variables) computed by a depth-2 threshold circuit of size \( s \). Indeed, for each of the \( s \) gates on the first layer (closest to the input) there are at most \( 2^{O(\ell^2)} \) threshold functions, as for each of the gates we can set \( s = \ell \), and apply Theorem 5.5. The bound follows as there are \( s \) gates in the first layer of the circuit and at most \( 2^{O(\ell s)} \) choices for the output gate. Therefore, similarly to the above, since the total number of Boolean functions \( f : \{0,1\}^\ell \rightarrow \{0,1\} \) is \( 2^{2^\ell} \) it follows that there exists a function that cannot be computed by a depth-2 threshold circuit of size \( s = \Omega(2^{\ell/\ell^2}) \).

**Proof of Theorem 1.4 for \( k \approx 2 \log n \).** Recall that in the reduction above we take a function \( f : \{0,1\}^\ell \rightarrow \{0,1\} \), and show that if computing \( f \) requires a circuit of size \( s \), then the same lower bound holds also for the SM\(_{n,k}\) function with \( k = 2\ell + 2 \), and \( n = (k/2) \cdot 2k/2 \).

By Proposition 5.4 for any sufficiently large \( \ell \) there exists a hard function \( f : \{0,1\}^\ell \rightarrow \{0,1\} \) such that any threshold circuit computing \( f \) has size at least \( \sqrt{2^\ell/\ell} \). Therefore, by Proposition 5.3 any circuit computing SM\(_{n,k}\) has size at least \( \Omega(\sqrt{2^\ell/\ell}) \geq \Omega(2k/\ell) = \Omega(\sqrt{n/\log(n)}) \).

The lower bound for depth 2 circuits follows similarly by applying the \( \Omega(2^{\ell/\ell^2}) \geq \Omega(2k/\ell^2) = \Omega(n/\log^3(n)) \) lower bound from Proposition 5.4 for threshold circuits of depth 2.

Note: one can easily adapt our proof to the case of pattern of odd length, by concatenating to the end of \( dup(a) \) the string 010 (as opposed to 01) and modifying \( z(l) \) to end in 010 instead of 01. We omit the details.
5.3 Lower bound for threshold circuits: the general case

Below we prove items 2 and 3 for all ranges of \( k \) stated in the theorem.

Proof of Theorem 1.4. Let \( \ell \in \mathbb{N} \) be a parameter, and let \( F_{\ell,t} = \{ f: \{0,1\}^\ell \to \{0,1\} : |f^{-1}(1)| = t \} \). We prove lower bounds via a reduction from a hard function \( f \in F_{\ell,t} \). Specifically, setting \( k = 2\ell + 2 \) we show that for \( n = t \cdot k \) if \( \text{SM}_{n,k} \) can be solved using a circuit of size \( s \), then every function \( f \in F_{\ell,t} \) has a circuit of size \( s \) computing it. Then, we show that there are functions in \( F_{\ell,t} \) that require large threshold circuits, which implies a related lower bound for the \( \text{SM}_{n,k} \) function.

Let \( \ell, t \in \mathbb{N} \) be parameters such that \( t < 2^{\ell-1} \), and let \( k = 2\ell + 2 \), and \( n = tk \).

Given a string \( a \in \{0,1\}^\ell \) define \( \text{dup}(a) \in \{0,1\}^k \) as before to be the string obtained from \( a \) by repeating each bit of \( a \) twice, and concatenating it with 01 in the end.

Observation 5.6. Given a function \( f: \{0,1\}^\ell \to \{0,1\} \) define \( x_f \in \{0,1\}^{\ell k} \) to be the concatenation of \( \text{dup}(a) \) for all \( a \in f^{-1}(1) \) in some fixed order, e.g., the lexicographic order on \( \{0,1\}^\ell \). Note that \( |x_f| = tk = n \). Then, for any \( y \in \{0,1\}^\ell \) it holds that \( f(y) = 1 \) if and only if \( \text{SM}_{n,k}(x_f, \text{dup}(y)) = 1 \).

As before, any lower bound on the size of the circuit computing \( f \) implies a lower bound on \( \text{SM}_{n,k} \). The argument is exactly the same as for the \( n = \frac{k}{2} \cdot 2^{k/2} \) and we give it here for completeness.

Proposition 5.7. For \( \ell, t \in \mathbb{N} \) let \( k = 2\ell + 2 \), and let \( n = tk \). Let \( f: \{0,1\}^\ell \to \{0,1\} \) be a Boolean function such that \( |f^{-1}(1)| = t \). Let \( C \) be a threshold circuit computing \( \text{SM}_{n,k} \). Then, there exists a threshold circuit \( C' \) computing \( f \) such that \( |C'| \leq |C| \).

In order to complete the proof of Theorem 1.4 we show that there exists a function \( f \in F_{\ell,t} \) that requires large threshold circuit to compute it.

Proposition 5.8. Let \( \ell \in \mathbb{N} \) be sufficiently large, and let \( t \in \mathbb{N} \).

1. There exists a function \( f \in F_{\ell,t} \) such that any threshold circuit (with no restrictions on its depth) computing \( f \) must be of size at least \( \Omega(\sqrt{\ell \cdot \log(t)}/t/\ell) \).

2. There exists a function \( f \in F_{\ell,t} \) such that any threshold circuit of depth 2 computing \( f \) must be of size at least \( \Omega((\ell - \log(t))/t/\ell^2) \).

Proof. It follows from the argument in the proof of Proposition 5.4 that the number of distinct Boolean functions with \( \ell \) variables computed by a threshold circuit of size \( s \) is \( 2^{O(s^2)} \). On the other hand, the total number of Boolean functions \( f \in F_{\ell,t} \) is \( \binom{2^\ell}{t} \leq \left(\frac{2^\ell}{t}\right)^t = 2^{(\ell - \log(t))t} \). Therefore, there exists a function \( f \in F_{\ell,t} \) that cannot be computed by a threshold circuit of size \( s \geq \Omega(\sqrt{\ell \cdot \log(t)/t}/\ell) \).

Similarly, there are at most \( 2^{O(s^2)} \) distinct Boolean functions over \( \ell \) variables computed by a depth-2 threshold circuit of size \( s \). Therefore, there exists a function \( f \in F_{\ell,t} \) that cannot be computed by a threshold circuit of size \( s = \Omega((\ell - \log(t))/t/\ell^2) \).

We now derive our lower bounds on the size of threshold circuits computing the string matching function. For general (unbounded depth) threshold circuit it follows that any threshold circuit computing \( \text{SM}_{n,k} \) has size at least \( \Omega(\sqrt{\ell - \log(t)/\ell}) = \Omega(\sqrt{t - \ell \cdot \log(t)/\ell}) \). Plugging in \( k = 2\ell + 2 \) and \( n = tk \), we get a lower bound of \( s \geq \Omega(\sqrt{\frac{2\ell}{t} - \frac{2n}{k} \cdot \log(\frac{n}{\ell})}) \), as required.

For threshold circuits of depth 2 we have a bound of \( \Omega((\ell - \log(t))/t/\ell^2) = \Omega(t/t - \ell \cdot \log(t)/\ell^2) \). Plugging in \( k = 2\ell + 2 \) and \( n = tk \); we get a lower bound of \( s \geq \Omega(\frac{n}{t^2} - \frac{2n}{k^2} \cdot \log(\frac{k}{\ell})) \), as required.
So far we only proved the theorem for all \( n \leq k \cdot 2^{\ell-1} = \frac{k}{2} 2^{k/2} \) that are divisible by \( k \). Observe, however, that in the reduction above we can pad the string \( x_f \) with zeros in the end, and the reduction will still satisfy the property that \( f(y) = 1 \) if and only if \( SM_{n,k}(x_f, \text{dup}(y)) = 1 \) as in Observation 5.6, and the same lower bound holds (except that \( n \) is now slightly larger, which affects only the constant in the \( O(\cdot) \) notation). This completes the proof of Theorem 1.4.

6 Conclusion

We have proven lower and upper bounds for different circuit families that solve the string matching problem. There are several gaps between our upper and lower bounds for \( SM_{n,k} \), leading to several open questions:

- One previously mentioned question is whether there exists a threshold circuit of size \( o(n) \) that solves \( SM_{n,k} \). In particular, while our lower bounds for unrestricted-depth threshold circuits are weaker than those for depth 2 circuits, it is unclear if depth greater than 2 (or even unrestricted depth) allows one to get better upper bounds than those proven in Lemma 5.1.

- It would be interesting to establish superlinear (in \( n \)) lower bounds for De Morgan circuits of depth 3 that solve \( SM_{n,k} \) when \( k = k(n) \) is some function tending to infinity with \( n \). Alternatively, is there a depth 3 De Morgan circuit computing \( SM_{n,k} \) of size \( O(n) \)? Is there a De Morgan circuit (no depth restrictions) computing \( SM_{n,k} \) of size \( O(n) \)?

- Our methods do not seem to imply any non constant lower bounds for threshold circuits when \( k \) is a constant that does not depend on \( n \). Proving nonconstant lower bounds or improving upon the upper bounds in Lemma 5.1 for constant \( k \) is another direction for future study.

In this work we limited our attention to the exact string matching problem. One can also consider the circuit complexity of the approximate case, where given a string \( x \), a pattern \( y \) and a parameter \( b \), we seek to determine if there is an occurrence of a string of length \( |y| \) in \( x \) whose hamming distance from \( y \) is at most \( b \). When \( b = \Omega(|x|) \) it is easy to prove a lower bound of \( 2^{\Omega(n^{1/d})} \) (where \( n = |x| \)) for depth \( d \) De Morgan circuits by using known lower bounds for the majority function [Häs87] and using a (straightforward) reduction from majority to the approximate string matching problem. For threshold circuits, one can obtain a lower bound of \( \Omega(n/\log n) \) for unrestricted circuits as there is a simple reduction between the problem of computing the Hamming distance of two strings to the approximate string matching problem and then one can use the linear lower bound of \( \Omega(n) \) on the communication complexity of computing the Hamming distance [CR12] to deduce the aforementioned lower bound of \( \Omega(n/\log n) \). Future study could look into devising tight upper bounds for circuits tackling the approximate string matching problem, as well as providing lower bounds for the case where \( b \ll n \) (e.g., \( b = O(1) \)).

Finally, as previously noted, there are relatively few works concerned with the circuit complexity of pattern recognition tasks studied in more applied fields such as computer vision, machine learning, and signal processing, and we believe it is an interesting direction for future research to examine the circuit complexity of such problems.

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