On a continuum limit of discrete Schrödinger operators on square lattice

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Abstract

The norm resolvent convergence of discrete Schrödinger operators to a continuum Schrödinger operator in the continuum limit is proved under relatively weak assumptions. This result implies, in particular, the convergence of the spectrum with respect to the Hausdorff distance.

1 Introduction

We consider a Schrödinger operator

\[ H = H_0 + V(x), \quad H_0 = -\Delta, \quad x \in \mathbb{R}^d, \]

on \( \mathcal{H} = L^2(\mathbb{R}^d) \), where \( d \geq 1 \), and corresponding discrete Schrödinger operators: We set \( h > 0 \) be the mesh size, and we write

\[ \mathcal{H}_h = \ell^2(h\mathbb{Z}^d), \quad h\mathbb{Z}^d = \{(hz_1, \ldots, hz_d) \mid z \in \mathbb{Z}^d\}, \]

with the norm \( \|v\|_h^2 = h^d \sum |v(hz)|^2 \) for \( v \in \mathcal{H}_h \). We denote the standard basis of \( \mathbb{R}^d \) by \( e_j = (\delta_{ik})_{k=1}^d \in \mathbb{R}^d, j = 1, \ldots, d \). Our discrete Schrödinger operator is

\[ H_h = H_{0,h} + V(z), \quad z \in h\mathbb{Z}^d, \]

where

\[ H_{0,h}v(z) = h^{-2} \sum_{j=1}^d (2v(z) - v(z + he_j) - v(z - he_j)), \quad v \in \mathcal{H}_h. \]

We suppose

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**Assumption A.** $V$ is a real-valued continuous function on $\mathbb{R}^d$, and bounded from below. $(V(x) + M)^{-1}$ is uniformly continuous with some $M > 0$, and there is $c_1 > 0$ such that

$$c_1^{-1}(V(x) + M) \leq V(y) + M \leq c_1(V(x) + M), \quad \text{if } |x - y| \leq 1.$$  

The above assumption implies $V$ is slowly varying in some sense, and uniformly continuous relative to the size of $V(x)$. Under the assumption, $H$ is essentially self-adjoint, and $H_h$ is self-adjoint. The assumption is satisfied if $V$ is bounded and uniformly continuous. $V(x) = a \langle x \rangle \mu$ with $a, \mu > 0$, also satisfies the assumption.

For $\varphi \in S(\mathbb{R}^d)$, $h > 0$ and $z \in h\mathbb{Z}^d$, we set

$$\varphi_{h,z}(x) = \varphi(h^{-1}(x - z)), \quad x \in \mathbb{R}^d,$$

and we define $P_h = P_h \varphi : \mathcal{H} \to \mathcal{H}_h$ by

$$P_h u(z) := h^{-d} \int_{\mathbb{R}^d} \overline{\varphi_{h,z}(x)} u(x) dx, \quad h > 0, \ z \in h\mathbb{Z}^d.$$  

The adjoint operator is given by

$$P_h^* v(x) = \sum_{z \in h\mathbb{Z}^d} \varphi_{h,z}(x) v(z), \quad h > 0, \ v \in \mathcal{H}_h.$$  

It is easy to observe that $P_h^*$ is an isometry and hence $P_h$ is an orthogonal projection if and only if $\{ \varphi_{1,z} \ | \ z \in \mathbb{Z}^d \}$ is an orthonormal system. This condition is also equivalent to the condition:

$$\sum_{n \in \mathbb{Z}^d} |\hat{\varphi}(\xi + n)|^2 = 1 \quad \text{for } \xi \in \mathbb{R}^d,$$  

(1.1)

where $\hat{\varphi}$ is the Fourier transform:

$$\hat{\varphi}(\xi) = \mathcal{F} \varphi(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} \varphi(x) dx, \quad \xi \in \mathbb{R}^d.$$  

This claim is well-known, but we give its proof in Appendix for the completeness (Lemma A.1). By this observation, we learn that there is a large class of $\varphi$'s satisfying the above condition. In this paper, we use $P_h$ to identify $\mathcal{H}_h$ with a subspace of $\mathcal{H}$. We suppose:

**Assumption B.** $\varphi$ satisfies the condition (1.1), and $\text{supp} [\hat{\varphi}] \subset (-1, 1)^d$.

**Theorem 1.1.** Suppose Assumptions A and B. Then, for any $\mu \in \mathbb{C} \setminus \mathbb{R}$,

$$\|P_h^*(H_h - \mu)^{-1} P_h - (H - \mu)^{-1}\|_{\mathcal{B}(\mathcal{H})} \to 0 \quad \text{as } h \to 0.$$  

Furthermore, if $(V(x) + M)^{-1}$ is uniformly Hölder continuous of order $\alpha \in (0, 1]$ (with some $M > 0$), then for any $0 < \beta < \alpha$,

$$\|P_h^*(H_h - \mu)^{-1} P_h - (H - \mu)^{-1}\|_{\mathcal{B}(\mathcal{H})} \leq C_\mu h^\beta \quad \text{as } h \to 0.$$  

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Here $\mathcal{B}(X)$ denotes the Banach space of the operators on a Banach space $X$. Combining this with the argument of Theorem VIII.23 (b) in [10], we obtain the following corollary. We denote the spectrum of a self-adjoint operator $A$ by $\sigma(A)$, and the spectral projection by $E_A(\Omega)$ for $\Omega \subset \mathbb{R}$.

**Corollary 1.2.** Suppose Assumptions A and B. Let $a, b \in \mathbb{R}$, $a < b$, be not in $\sigma(H)$. Then $a, b \notin \sigma(H)$ for sufficiently small $h$ and

$$\|P_h^* E_{H_h}((a, b)) P_h - E_H((a, b))\|_{\mathcal{B}(\mathcal{H})} \to 0 \quad \text{as } h \to 0.$$ 

We denote the Hausdorff distance of sets $X, Y \subset \mathbb{C}$ by

$$d_H(X, Y) = \max \left\{ \sup_{x \in X} d(x, Y), \sup_{y \in Y} d(y, X) \right\},$$

where $d(\cdot, \cdot)$ denotes the standard distance in $\mathbb{C}$. It is not difficult to show $d_H(\sigma(A), \sigma(B)) \leq \|A - B\|$ for normal operators $A$ and $B$ (see Lemma A.2 in Appendix). Thus we also have the following result.

**Corollary 1.3.** Suppose Assumptions A and B. Then for $M \gg 0$,

$$d_H\left(\sigma((H_h + M)^{-1}), \sigma((H + M)^{-1})\right) \to 0 \quad \text{as } h \to 0.$$ 

There are studies concerning continuum limits of NLS equations, in many cases, mainly with applications to numerical analysis. We refer Bambusi and Penati [2], Hong and Yang [4] and references therein. For linear discrete Schrödinger operators, Rabinovich [9] has studied the relation between the essential and discrete spectra of the discrete and continuum Schrödinger operators, provided $V$ is bounded and uniformly continuous.

In Section 2, we give the proof of our main theorem, and proofs of several technical lemmas are given in Appendix.

## 2 Proof

We denote the discrete Fourier transform $F_h : \mathcal{H}_h \to \hat{\mathcal{H}}_h = L^2(h^{-1}\mathbb{T}^d)$, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, by

$$F_h v(\zeta) = h^d \sum_{z \in h\mathbb{Z}^d} e^{-2\pi iz \cdot \zeta} v(z), \quad \zeta \in h^{-1}\mathbb{T}^d, \ v \in \mathcal{H}_h.$$ 

$F_h$ is unitary, and its adjoint is given by

$$F_h^* g(z) = \int_{h^{-1}\mathbb{T}^d} e^{2\pi iz \cdot \zeta} g(\zeta) d\zeta, \quad z \in h\mathbb{Z}^d, \ g \in \hat{\mathcal{H}}_h.$$
2.1 Convergence of the free Hamiltonian

If we set \( H_0(\xi) = |2\pi \xi|^2 \), it is well-known that \( H_0 = \mathcal{F}^*H_0(\cdot)\mathcal{F} \) on \( \mathcal{H} \). Similarly, if we set
\[
H_{0,h}(\zeta) = 2h^{-2} \sum_{j=1}^{d} (1 - \cos(2\pi h\zeta_j)), \quad \zeta \in h^{-1}\mathbb{T}^d,
\]
then \( H_{0,h} = F_h^*H_{0,h}(\cdot)F_h \). We denote
\[
Q_h := F_hP_hF^* : \hat{\mathcal{H}} \rightarrow \hat{\mathcal{H}}.
\]
The following formula is convenient in the following argument. It is well-known in signal analysis (see, e.g., [7]), but we give a proof in Appendix for the completeness.

**Lemma 2.1.** For \( f \in S(\mathbb{R}^d) \),
\[
Q_h f(\zeta) = \sum_{n \in \mathbb{Z}^d} \hat{\varphi}(h\zeta + n)f(\zeta + h^{-1}n), \quad \zeta \in h^{-1}\mathbb{T}. \tag{2.1}
\]
For \( g \in \hat{\mathcal{H}}_h \),
\[
Q_h^* g(\xi) = \hat{\varphi}(h\xi)\tilde{g}(\xi), \quad \xi \in \mathbb{R}^d, \tag{2.2}
\]
where \( \tilde{g} \) is the periodic extension of \( g \) on \( \mathbb{R}^d \).

**Lemma 2.2.** For \( \mu \in \mathbb{C} \setminus \mathbb{R}_+ \) there is \( C > 0 \) such that
\[
\|(1 - P_h^*P_h)(H_0 - \mu)^{-1}\|_{\mathcal{B}(\mathcal{H})} \leq Ch^2, \quad h > 0.
\]

**Proof.** We first note
\[
\|(1 - P_h^*P_h)(H_0 - \mu)^{-1}\|_{\mathcal{B}(\mathcal{H})} = \|(1 - Q_h^*Q_h)(|2\pi \xi|^2 - \mu)^{-1}\|_{\mathcal{B}(\hat{\mathcal{H}})}
\]
where \( \hat{\mathcal{H}} = \mathcal{F}[\hat{\mathcal{H}}] = L^2(\mathbb{R}^d) \). Let \( f \in \hat{\mathcal{H}} \) and \( g = (|2\pi \xi|^2 - \mu)^{-1}f \). Then we have, by using the above lemma,
\[
(1 - Q_h^*Q_h)g(\xi) = (1 - |\hat{\varphi}(h\xi)|^2)g(\xi) - \hat{\varphi}(h\xi)\sum_{n \neq 0} \hat{\varphi}(h\xi + n)g(\xi + h^{-1}n).
\]
For the first term in the right hand side, we observe by Assumption B that \( |\hat{\varphi}(h\xi)| = 1 \) if \( |\xi| \leq h^{-1}\delta \) with some \( \delta > 0 \). Then we learn
\[
\|(1 - |\hat{\varphi}(h\xi)|^2)g(\xi)\|_{\hat{\mathcal{H}}} \leq \sup_{|\xi| > h^{-1}\delta} \|2\pi \xi|^2 - \mu|^{-1} \|f\|_{\hat{\mathcal{H}}} \leq Ch^2\|f\|_{\hat{\mathcal{H}}}. \]
For the second term, we note that the terms in the summation vanish except for \( n \in \{0, \pm 1\}^d \setminus 0 \). Using the support condition of \( \hat{\varphi} \) again, we learn that \( \hat{\varphi}(h\xi)\hat{\varphi}(h\xi + n) = 0 \) if \( |\xi + h^{-1}n| \leq h^{-1}\delta \) with some \( \delta > 0 \). Thus we can use the same argument to show that the second term is bounded by \( Ch^2 \). \( \square \)
Lemma 2.3. For $\mu \in \mathbb{C} \setminus \mathbb{R}_+$ there is $C > 0$ such that
$$\|(H_{0,h} - \mu)^{-1}P_h - P_h(H_0 - \mu)^{-1}\|_{\mathcal{B}(\mathfrak{H}^d)} \leq Ch^2, \quad h > 0.$$  

Proof. Since $P_h^*$ is isometric, it suffices to estimate
$$\|(H_{0,h} - \mu)^{-1}P_h - P_h(H_0 - \mu)^{-1}\| = \|P_h^*(H_{0,h} - \mu)^{-1}P_h - P_h^*P_h(H_0 - \mu)^{-1}\| = \|Q_h^*(H_{0,h} - \mu)^{-1}Q_h - Q_h^*Q_h(H_0 - \mu)^{-1}\|.$$  

Then we compute, for $f \in \mathcal{S}(\mathbb{R}^d)$,
$$\left(Q_h^*(H_{0,h} - \mu)^{-1}Q_h - Q_h^*Q_h(H_0 - \mu)^{-1}\right) f(\xi) = \sum_{n \in \mathbb{Z}^d} \hat{\varphi}(h\xi)\hat{\varphi}(h\xi + n)B_h(\xi + h^{-1}n)f(\xi + h^{-1}n),$$  

where $B_h(\xi) := (H_{0,h}(\xi) - \mu)^{-1} - (H_0(\xi) - \mu)^{-1}$. We note, as well as in the proof of Lemma 2.2, $\hat{\varphi}(h\xi)\hat{\varphi}(h\xi - n)$ vanishes except for $n \in \{0, \pm 1\}^d$.

By the Taylor expansion, we have
$$|H_{0,h}(\xi) - H_0(\xi)| \leq Ch^{-2}(h|\xi|)^4 = Ch^2|\xi|^4, \quad h > 0, \quad \xi \in \mathbb{R}^d.$$  

On the other hand, if $h\xi \in \text{supp}[\hat{\varphi}]$, we have $H_{0,h}(\xi) \geq c_0|\xi|^2$ with some $c_0 > 0$. These imply
$$|\hat{\varphi}(h\xi)|^2|B_h(\xi)| \leq Ch^2|\hat{\varphi}(h\xi)|^2, \quad h > 0, \quad \xi \in \mathbb{R}^d,$$

with some $C > 0$. On the support of $\hat{\varphi}(h\xi)\hat{\varphi}(h\xi + n)$, $n \neq 0$, we have $H_{0,h}(\xi + h^{-1}n) \geq c_1h^{-2}$, $H_0(\xi + h^{-1}n) \geq c_2h^{-2}$ with some $c_1 > 0$, and hence $|B_h(\xi)| = O(h^2)$ as $h \to 0$. Combining these, we learn
$$\left|(Q_h^*(H_{0,h} - \mu)^{-1}Q_h - Q_h^*Q_h(H_0 - \mu)^{-1}) f(\xi)\right| \leq Ch^2 \sum_{n \in \{0, \pm 1\}^d} |f(\xi + h^{-1}n)|, \quad \xi \in \mathbb{R}^d,$$

and the assertion follows.  

2.2 Relative boundedness

In this section, we suppose $V \geq 1$ without loss of generality. In particular, $V(x)^{-1}$ is uniformly bounded, and
$$c_1V(x) \leq V(y) \leq c_1V(x) \quad \text{for } x, y \in \mathbb{R}^d, \quad |x - y| \leq 1. \quad (2.3)$$

Lemma 2.4. Suppose Assumption A. Then $V$ is $H$-bounded, and hence $H_0$ is also $H$-bounded.
Proof. By the quadratic inequality, it is easy to observe $V^{1/2}$ and $(H_0 + 1)^{1/2}$ are $H^{1/2}$-bounded. Let $\eta \in C_0^\infty(\mathbb{R}^d)$ be a smooth cut-off function such that $\eta(x) \geq 0$, $\text{supp}[\eta] \subset \{|x| \leq 1\}$. Then we set $\tilde{V} = \eta \ast V$, and we use $\tilde{V} \geq 1$ as a smooth weight function comparable to $V$. By (2.3), we have

$$c_1^{-1}V(x) \leq \tilde{V}(x) \leq c_1V(x), \quad x \in \mathbb{R}^d.$$ 

By elementary computation, we also have

$$|\partial^\alpha_x \tilde{V}(x)| \leq C_\alpha \tilde{V}(x), \quad x \in \mathbb{R}^d$$

with some $C_\alpha > 0$, where $\alpha \in \mathbb{Z}_+^d$. It suffices to show $\tilde{V}$ is $H$-bounded.

We write $W(x) = \tilde{V}(x)^{1/2} \geq 1$, and compute

$$\tilde{V} H^{-1} = W H^{-1} W + W [W, H^{-1}]$$

$$= (WH^{-1/2})(WH^{-1/2})^* + WH^{-1}[H, W] H^{-1}.$$ 

The first term in the right hand side is bounded since $W$ is $H^{1/2}$-bounded. We note

$$[H, W] = -\partial_x \cdot \partial_x W(x) - \partial_x W(x) \cdot \partial_x,$$

and $\partial_x$ is $H^{1/2}$-bounded. We also note

$$|\partial_x W(x)| = \frac{1}{2} \tilde{V}^{-1/2}(x)|\partial_x \tilde{V}(x)| \leq CW(x)$$

with some $C > 0$, and hence $\partial_x W$ is $H^{1/2}$-bounded. Thus we learn

$$WH^{-1}[H, W] H^{-1} = (WH^{-1/2})(\partial_x H^{-1/2})^* (\partial_x W) H^{-1/2}$$

$$- (WH^{-1/2})(\partial_x W) H^{-1/2})^* (\partial_x H^{-1/2}) H^{-1/2}$$

is bounded, and hence $\tilde{V}$ is $H$-bounded. \qed

Lemma 2.5. Suppose Assumption A. Then $V$ is $H_h$-bounded uniformly in $h > 0$, and hence $H_{0, h}$ is also $H_h$-bounded uniformly in $h > 0$.

Proof. The proof is analogous to that of Lemma 2.4. We note $W = \tilde{V}^{1/2}$ and $H_0^{1/2}$ are uniformly $H^{1/2}$-bounded. We similarly have

$$\tilde{V} H_h^{-1} = (WH_h^{-1/2})(WH_h^{-1/2})^* + WH_h^{-1}[H_h, W] H_h^{-1},$$

and the first term in the right hand side is uniformly bounded.

For the second term, we recall that $H_{0, h} = \sum_{j=1}^d \nabla^* \nabla_j$, where

$$\nabla_j v(z) := \frac{1}{h} (v(z + h e_j) - v(z)), \quad v \in H_h.$$ 

Then we learn

$$[W, H_h] = \sum_{j=1}^d (\nabla_j, W]^* \nabla_j - \nabla_j^* [\nabla_j, W].$$ 

By elementary computations, we can show $[\nabla_j, W] W^{-1}$ is bounded uniformly in $h$, and hence $WH_h^{-1}[H_h, W] H_h^{-1}$ is bounded uniformly in $h$. \qed
2.3 Proof of Theorem 1.1

**Lemma 2.6.** If $G$ is a uniformly continuous function, then

$$\|GP_h - P_h G\|_{\mathcal{B}(\mathcal{H}, \mathcal{H})} \to 0, \quad h \to 0.$$  

If, in addition, $G$ is uniformly Hölder continuous of order $\alpha \in (0, 1]$, then

$$\|GP_h - P_h G\|_{\mathcal{B}(\mathcal{H}, \mathcal{H})} \leq C\varepsilon h^{\alpha - \varepsilon}, \quad h > 0,$$

with any $\varepsilon > 0$.

**Proof.** We note

$$(GP_h - P_h G)u(z) = \int_{\mathbb{R}^d} K(x, z; h)u(x)\,dx,$$

where

$$K(x, z; h) := h^{-d}(G(z) - G(x))\varphi(h^{-1}(x - z)).$$

By Schur’s lemma, we have

$$\|GP_h - P_h G\| \leq \sqrt{K_1K_2},$$

where

$$K_1 = \sup_{z \in h\mathbb{Z}^d} \int_{\mathbb{R}^d} |K(x, z)|\,dx, \quad K_2 = \text{ess sup}_{x \in \mathbb{R}^d} h^d \sum_{z \in h\mathbb{Z}^d} |K(x, z)|.$$

We set

$$R(\delta) := \sup_{x, y \in \mathbb{R}^d, |x - y| < \delta} |G(x) - G(y)|$$

and we choose $n > d$. Then we have

$$\int_{\mathbb{R}^d} |K(x, z)|\,dx = \int_{|x - z| < \delta} |K(x, z)|\,dx + \int_{|x - z| \geq \delta} |K(x, z)|\,dx \leq CR(\delta) \int_{|y| < \delta} (hy)^{-n}h^{-d}\,dy + C \int_{|y| \geq \delta} (hy)^{-n}h^{-d}\,dy \leq C'R(\delta) + C'h^{-1}\delta^{-(n-d)}.$$

By the same computation, we also have

$$h^d \sum_{z \in h\mathbb{Z}^d} |K(x, z)| \leq CR(\delta) + C'h^{-1}\delta^{-(n-d)}.$$

Combining these and setting $\delta = h^\gamma$ with $\gamma \in (0, 1)$, we obtain

$$\|GP_h - P_h G\| \leq CR(\delta') + Ch^{(1-\gamma)(n-d)}.$$

By the assumption, $R(\delta) \to 0$ as $\delta \to 0$, and we conclude the first assertion.

If $G$ is uniformly Hölder continuous of order $\alpha$, then $R(\delta) \leq C\delta^\alpha$, and hence the right hand side of the above estimate is $O(h^{\alpha\gamma}) + O(h^{(1-\gamma)(n-d)})$. We can choose $\gamma$ very close to 1, and $n$ very large so that $\alpha\gamma \geq \alpha - \varepsilon$ and $(1 - \gamma)(n - d) \geq \alpha - \varepsilon$, and we have the second assertion. $\square$
Proof of Theorem 1.1. We compute
\[
P^*_h (H_h - \mu)^{-1} P_h - (H - \mu)^{-1}
= P^*_h (H_h - \mu)^{-1} P_h - (H_h - \mu)^{-1} - (1 - P^*_h P_h)(H - \mu)^{-1}
= P^*_h (H_h - \mu)^{-1} (P_h H - H_h P_h)(H - \mu)^{-1} - (1 - P^*_h P_h)(H - \mu)^{-1}.
\]
By Lemmas 2.2 and 2.4, we learn
\[
\| (1 - P^*_h P_h)(H - \mu)^{-1} \| \leq C h^2.
\]
The other term is estimated as follows:
\[
\| (H_h - \mu)^{-1} (P_h H - H_h P_h)(H - \mu)^{-1} \|
\leq \| (H_h - \mu)^{-1} (P_h H_0 - H_0 P_h)(H - \mu)^{-1} \|
+ \| (H_h - \mu)^{-1} (P_h V - V_h P_h)(H - \mu)^{-1} \|
\leq C \| (H_0 - \mu)^{-1} (P_h H_0 - H_0 P_h)(H_0 - \mu)^{-1} \|
+ C \| (V - \mu)^{-1} (P_h V - V P_h)(V - \mu)^{-1} \|
= C \| (H_0 - \mu)^{-1} P_h - P_h (H_0 - \mu)^{-1} \|
+ C \| (V - \mu)^{-1} P_h - P_h (V - \mu)^{-1} \|,
\]
where we have used Lemmas 2.4 and 2.5 for the second inequality. The two terms in the right hand side are estimated using Lemmas 2.3 and 2.6, respectively, to complete the proof. \qed

A Appendix

Here we give the proofs of several technical lemmas.

Lemma A.1. Let \( \varphi \in \mathcal{S}(\mathbb{R}^d) \). Then, the following are equivalent.

1. \( P^*_h \) is isometric.
2. \( \text{Ran } P_h = \mathcal{H}_h \).
3. \( \int_{\mathbb{R}^d} \varphi(x) \overline{\varphi(x - n)} dx = \delta_{n,0} \) for \( n \in \mathbb{Z}^d \).
4. \( \sum_{n \in \mathbb{Z}^d} |\hat{\varphi}(\xi + n)|^2 = 1 \) for \( \xi \in \mathbb{R}^d \), where \( \hat{\varphi} = \mathcal{F}\varphi \).

Proof. (1) and (2) are equivalent by the standard properties of adjoint operators. Since (2) implies the orthonormality of the basis \( \{ h^{-\frac{d}{2}} \varphi_{h,x} \}_{z \in \mathbb{Z}^d} \), we learn
\[
\int_{\mathbb{R}^d} \varphi(x) \overline{\varphi(x - n)} dx = h^d \int_{\mathbb{R}^d} \varphi_{h,0}(x) \overline{\varphi_{h,hn}(x)} dx = \delta_{0,n},
\]
which implies (3). For the equivalence of (3) and (4), we learn by Parseval’s identity
\[
\int_{\mathbb{R}^d} \varphi(x)\overline{\varphi(x-n)}dx = \int_{\mathbb{R}^d} \hat{\varphi}(\xi)e^{-2\pi in\cdot \xi}\hat{\varphi}(\xi)d\xi
\]
\[
= \int_{\mathbb{R}^d} e^{2\pi in\cdot \xi} |\hat{\varphi}(\xi)|^2 d\xi
\]
\[
= \int_{\mathbb{T}^d} \sum_{m \in \mathbb{Z}^d} e^{2\pi in\cdot (\xi + m)} |\hat{\varphi}(\xi + m)|^2 d\xi
\]
\[
= \int_{\mathbb{T}^d} e^{2\pi in\cdot \xi} \sum_{m \in \mathbb{Z}^d} |\hat{\varphi}(\xi + m)|^2 d\xi,
\]
where $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d \simeq [0,1)^d$. Since \(\{e^{2\pi in\cdot \xi}\}_{n \in \mathbb{Z}^d}\) is a complete orthonormal basis of $L^2(\mathbb{T}^d)$, we conclude that (3) is equivalent to (4). \(\square\)

**Lemma A.2.** For normal operators $A$ and $B$, $d_H(\sigma(A), \sigma(B)) \leq \|A - B\|$.

**Proof.** It suffices to show that $d(\mu, \sigma(B)) > \|A - B\|$ implies $\mu \notin \sigma(A)$. This condition implies $\|(A - B)(B - \mu)^{-1}\| < 1$ and hence the Neumann series
\[
(A - \mu)^{-1} = (B - \mu + A - B)^{-1}
\]
\[
= (B - \mu)^{-1}(1 + (A - B)(B - \mu)^{-1})^{-1}
\]
\[
= (B - \mu)^{-1}\sum_{n=0}^{\infty}(-1)^n((A - B)(B - \mu)^{-1})^n
\]
converges, and thus we learn $\mu \notin \sigma(A)$. \(\square\)

**Proof of Lemma 2.1.** We compute
\[
Q_h f(\zeta) = h^d \sum_{z \in h\mathbb{Z}^d} e^{-2\pi iz\cdot \zeta} \left( h^{-d} \int_{\mathbb{R}^d} \overline{\varphi_{h,z}(x)} \int_{\mathbb{R}^d} e^{2\pi i x\cdot \xi} f(\xi)d\xi dx \right)
\]
\[
= \sum_{z \in h\mathbb{Z}^d} e^{-2\pi iz\cdot \zeta} \left( \int_{\mathbb{R}^d} \overline{\varphi_{h,z}(x)} \int_{\mathbb{R}^d} e^{2\pi i x\cdot \xi} f(\xi)d\xi dx \right)
\]
\[
= h^d \sum_{z \in h\mathbb{Z}^d} \int_{\mathbb{R}^d} e^{2\pi i z\cdot (\xi - \zeta)} \overline{\hat{\varphi}(h\xi)} f(\xi)d\xi
\]
\[
= h^d \sum_{z \in h\mathbb{Z}^d} \sum_{n \in \mathbb{Z}^d} \int_{h^{-1}\mathbb{T}^d + n} e^{2\pi i z\cdot (\xi - \zeta)} \overline{\hat{\varphi}(h\xi + n)} f(\xi + h^{-1}n)d\xi
\]
\[
= h^d \sum_{z \in h\mathbb{Z}^d} \sum_{n \in \mathbb{Z}^d} \int_{h^{-1}\mathbb{T}^d} e^{2\pi i z\cdot (\xi - \zeta)} \sum_{n \in \mathbb{Z}^d} \overline{\hat{\varphi}(h\xi + n)} f(\xi + h^{-1}n)d\xi
\]
\[
= \sum_{n \in \mathbb{Z}^d} \hat{\varphi}(h\zeta + n) f(\zeta + h^{-1}n).
\]
We have used the Fourier inversion formula for the last equality. We also have
\[
\langle Q_h^* g, f \rangle = \int_{h^{-1} \mathbb{T}^d} \sum_{n \in \mathbb{Z}^d} g(\zeta) \hat{\phi}(h(\zeta + h^{-1}n)) \bar{f}(\zeta + h^{-1}n) d\zeta \\
= \int_{\mathbb{R}^d} \tilde{g}(\xi) \hat{\phi}(h\xi) \bar{f}(\xi) d\xi,
\]
and this implies (2.2).

References

[1] K. Ando, H. Isozaki, H. Morioka: Spectral properties of Schrödinger operators on perturbed lattices. Ann. Henri Poincaré 17 (2016), 2103–2171.

[2] D. Bambusi, T. Penati: Continuous approximation of breathers in one- and two-dimensional DNLS lattices. Nonlinearity 23 (2010), 143–157.

[3] A. Boutet de Monvel, J. Sahbani: On the spectral properties of discrete Schrödinger operators: (The multi-dimensional case). Rev. Math. Phys. 11 (1999), 1061–1078.

[4] Younghun Hong, Changhun Yang: Strong Convergence for Discrete Nonlinear Schrödinger equations in the Continuum Limit. arXiv:1806.07542.

[5] H. Isozaki, I. Korotyaev: Inverse Problems, Trace Formulae for Discrete Schrödinger Operators. Ann. Henri Poincaré 13 (2012), 751–788.

[6] S. Nakamura: Modified wave operators for discrete Schrödinger operators with long-range perturbations. J. Math. Phys. 55 (2014), 112101 (8 pages).

[7] A. V. Oppenheim, R. W. Schafer, J. R. Buck: Discrete-Time Signal Processing (2nd Edition), Prentice-Hall, 1998.

[8] D. Parra, S. Richard: Spectral and scattering theory for Schrödinger operators on perturbed topological crystals. Rev. Math. Phys. 30 (2018), 1850009-1 – 1850009-39.

[9] V. Rabinovich: Wiener algebra of operators on the lattice $(\mu \mathbb{Z})^n$ depending on the small parameter $\mu > 0$. Complex Variables and Elliptic Equations 58 (2013), No. 6, 751–766.

[10] M. Reed, B. Simon: The Methods of Modern Mathematical Physics, Volume I, Functional Analysis, Academic Press, 1979.