Averaged projections, angles between groups and strengthening of Banach property (T)

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Abstract. Recently, Lafforgue introduced a new strengthening of Banach property (T), which he called strong Banach property (T) and showed that this property has implications regarding fixed point properties and Banach expanders. In this paper, we introduce a new strengthening of Banach property (T), called “robust Banach property (T)”, which is weaker than strong Banach property (T), but is still strong enough to ensure similar applications. Using the method of averaged projections in Banach spaces and introducing a new notion of angles between projections, we establish a criterion for robust Banach property (T) and show several examples of groups in which this criterion is fulfilled. We also derive several applications regarding fixed point properties and Banach expanders and give examples of these applications.

1 Introduction

In [12] and [13] V. Lafforgue introduced a very strong variant of property (T), which he named strong Banach property (T) and proved that $SL_3(F)$, where $F$ is a non-archimedean local field, has strong Banach property (T). After Lafforgue’s seminal work, his techniques were developed and generalized in [14], [5] and [7]. We shall start by reviewing the definition of Lafforgue and then introduce a weaker version of this definition which we call robust Banach property (T).

Let $G$ be a locally compact group. Let $\mathcal{F}$ be a family of linear representations on Banach spaces, $\pi : G \to B(X)$, that are continuous with respect to the strong operator topology. Define the norm $\|\|_{\mathcal{F}}$ on $C_0(G)$ as $\|f\|_{\mathcal{F}} = \sup_{\pi \in \mathcal{F}} \|\pi(f)\|$. Define $C_{\mathcal{F}}(G)$ to be the completion of $C_0(G)$ with respect to this norm. If $\mathcal{F}$ is closed under complex conjugation (i.e., $\pi \in \mathcal{F} \Rightarrow \pi^* \in \mathcal{F}$) and under duality (i.e., $\pi \in \mathcal{F} \Rightarrow \pi^* \in \mathcal{F}$), then $C_{\mathcal{F}}(G)$ is a Banach algebra with an involution

$$f^*(g) = f(g^{-1}), \forall g \in G.$$ 

Next, we shall define the notion of strong Banach property (T) introduced by Lafforgue in [12], [13].
For a locally compact group $G$, a length over $G$ is a continuous function $l : G \to \mathbb{R}_+$ such that $l(g) = l(g^{-1})$ and $l(g_1 g_2) \leq l(g_1) + l(g_2)$ for every $g, g_1, g_2 \in G$. Notice that if $l$ is a length $l$ over $G$, then for any $s \geq 0$, $c \geq 0$, $sl + c$ is also a length over $G$. For a family $\mathcal{E}$ of Banach spaces and a length function $l$, denote $F(\mathcal{E}, l)$ to be the family of representations $\pi$ on some $X \in \mathcal{E}$ such that $\|\pi(g)\| \leq e^{l(g)}$ for every $g \in G$. Recall that for any representation $\pi$ on $X$, $X^\pi$ denotes the subspaces of invariant vectors under the action of $\pi$, i.e.,

$$X^\pi = \{ v \in X : \forall g \in G, \pi(g)v = v \}.$$ 

**Definition 1.1.** A group $G$ has strong Banach property (T) if for any class of Banach spaces $\mathcal{E}$ of type $> 1$ that is stable under duality and under complex conjugation and for any length $l$ over $G$, there exists $s_0 > 0$ such that for every $c \geq 0$, there $p \in C_{F(\mathcal{E}, s_0 + c)}$ that is a real, self-adjoint idempotent such that for every $\pi \in F(\mathcal{E}, s_0 + c)$, $\pi(p)$ is a projection on $X^\pi$.

Our results does not achieve strong Banach property (T), but a slightly weaker notion that we shall call robust property (T). We define it as follows:

**Definition 1.2.** A group $G$ has robust Banach property (T) with respect to a class of Banach spaces $\mathcal{E}$, if for any length $l$ over $G$, there exists $s_0 > 0$ and a sequence of real functions $f_n \in C_c(G)$ with the following properties:

1. For every $n$, $f_n$ is symmetric, i.e., $f_n(g) = f_n(g^{-1})$.
2. For every $n$, $\int f_n = 1$.

And such that the sequence $(f_n)$ converges in $C_{F(\mathcal{E}, s_0)}$ to $p$ and $\forall \pi \in F(\mathcal{E}, s_0)$, $\pi(p)$ is a projection on $X^\pi$.

If we assume that $G$ is compactly generated (e.g., if $G$ has property (T)), we can give an equivalent definition that is more convenient:

**Definition 1.3.** Let $G$ be a compactly generated group and let $K$ be some symmetric compact set that generates $G$. For a class of Banach spaces $\mathcal{E}$ and a constant $s_0 \geq 0$, denote $F(\mathcal{E}, K, s_0)$ to be the class of all the representations $\pi$ of $G$ on some $X \in \mathcal{E}$ such that $\sup_{g \in K} \|\pi(g)\| \leq e^{s_0}$.

$G$ has robust Banach property (T) with respect to a class of Banach spaces $\mathcal{E}$, if there exists $s_0 > 0$ and a sequence of real functions $f_n \in C_c(G)$ with the following properties:

1. For every $n$, $f_n$ is symmetric, i.e., $f_n(g) = f_n(g^{-1})$.
2. For every $n$, $\int f_n = 1$.

And such that the sequence $(f_n)$ converges in $C_{F(\mathcal{E}, K, s_0)}$ to $p$ and $\forall \pi \in F(\mathcal{E}, K, s_0)$, $\pi(p)$ is a projection on $X^\pi$.

We’ll leave the proof of equivalence between the two definitions to the reader. Note that in the above definition, $s_0$ depends on the choice of the generating set $K$, but the fact that $G$ has robust Banach property (T) with respect to $\mathcal{E}$ does not depend on this choice.
Remark 1.4. The criteria we’ll give below for robust Banach property (T) assumes compact generation and therefore definition 1.3 is more convenient. We shall use the more general definition 1.2 only when proving the general implications of robust Banach property (T) in the appendix.

Remark 1.5. The reader should note that $E$ in the definition above is not assumed to be closed under duality or complex conjugation. Instead, we added the conditions that the functions $f_n$ are all symmetric and real. This definition was inspired by the definition given by de la Salle in [5] for strong Banach property (T) with respect to a class of Banach spaces.

Remark 1.6. If $G$ has robust Banach property (T) with respect to a class $E$ that is closed under duality and complex conjugation, then the projection $p$ is a central idempotent and therefore a Kazhdan projection (see [8] and reference therein for details on Kazhdan projections).

Remark 1.7. Property (T) and the equivalent property (FH) are usually considered rigid, i.e., they are preserved under small changes. Following this line of thought, de la Salle has recently proven [5] that any group with property (T) will have robust Banach property (T) with respect to a class of Banach spaces that are all small enough deformations of Hilbert spaces. However, the reader should note that different groups with property (T) will allow different extents of deformation. Therefore, the question of what is $E$ in the definition above for a given group remains interesting.

Our main achievement is establishing criteria for robust Banach property (T) for a class of Banach spaces for groups $G$ with the following structure: $G$ is generated by compact subgroups $K_1, ..., K_N$ such that each pair $K_i, K_j$ generate a compact group in $G$. This setup is quite general and apply to a several families of groups such as groups acting on Buildings (under certain assumptions on the action) and Kac-Moody-Steinberg groups defined in [10]. Our criteria relays on bounding the "angle" between $K_i$ and $K_j$ in every unitary representation (on Hilbert spaces) of $\langle K_i, K_j \rangle$ (see exact formulation in theorems 4.9, 4.15 below). This approach is heavily influenced by the work of Ershov and Jaikin-Zapirain in [10] and the work of Kassabov in [11] regarding property (T) via the notion of angle between groups. As far as we can tell, the approach taken in this paper is different from the one taken by Lafforgue and his successors (although it seems to share some common features). We also owe a great debt to the work of de la Salle [5], which provided the necessary machinery that allows us to use information regarding the unitary representations on Hilbert spaces to deduce information regarding representations on Banach spaces.

In order to prove our main results, we also establish a criterion for the convergence of the averaged projections method in Banach spaces that may be of independent interest.

Robust Banach property (T) has two nice applications (which are the same applications derived in [13] for strong property (T)).

1.1 Fixed point property

We recall the following definitions given in [3]:

\[ \text{def} \]
Definition 1.8. Let $G$ be a topological group and let $X$ be a Banach space. $G$ is said to have property $F_X$ if every continuous affine isometric action of $G$ on $X$ has a fixed point.

For $p \in [1, \infty]$, $G$ is said to have property $F_{L^p}$ if every continuous affine isometric action of $G$ on any $L^p$ space has a fixed point.

We note that it was proven in [3] and [2] that if $G$ has property (T), then $G$ has property $F_{L^p}$ for every $p \in [1, 2]$.

Robust Banach property (T) can be used to prove property $F_X$ using the following proposition:

Proposition 1.9. Let $X$ be a Banach space and $G$ be a locally compact group. If $G$ has robust Banach property (T) with respect to $C \oplus X$ with the $l_2$ norm, then $G$ has property $F_X$.

The proof of this proposition needs an (easy) adaptation of the proof given in [13]. For completeness, the proof is provided in the appendix.

Using this application we are able to prove new fixed point theorems. For instance we prove a generalization of the following results:

Theorem 1.10. Let $\Sigma$ be a pure $n$-dimensional simplicial complex that is galley connected. Let $G$ be a group acting simplicially on a $\Sigma$ such that the action is cocompact and the fundamental domain $\Sigma / G$ is a single $n$-dimensional simplex and that the stabilizer of each $(n - 2)$-dimensional simplex is a compact subgroup of $G$. Assume that every 1-dimensional link of $\Sigma$ is a finite connected graph. Assume further that there is a constant $\eta > 1 - \frac{1}{n(n+1)-1}$ such that for every 1-dimensional link the smallest positive eigenvalue of the Laplacian on the link is $\geq \eta$. For every $p_0 > 2$ there is a constant $1 > K(p_0)$ such that if $\eta \geq K(p_0)$, then $G$ has property $F_{L^p}$ for every $p \in [1, p_0]$.

Theorem 1.11. For any Banach space $X$ of type $p_1$ and cotype $p_2$ such that $\frac{1}{p_1} - \frac{1}{p_2} < \frac{1}{4}$ and any $n \geq 3, m \geq 1$, the Steinberg group $St_n(\emptyset[t_1, \ldots, t_m])$ has property $F_X$ provided that $q$ is sufficiently large.

Theorem 1.12. For any $p_0 > 2, n \geq 3, m \geq 1$ there is a constant $q(p, n, m)$ such that for any prime $q \geq q(p, n, m)$, Steinberg group $St_n(\emptyset[t_1, \ldots, t_m])$ has property $F_{L^p}$ for every $p \in [1, p_0]$.

More examples (and a more general statement of the above examples) are given in the last section of this paper.

1.2 Expander graphs

Definition 1.13. Let $X$ be a Banach space and $\{ (V_i, E_i) \}_{i \in \mathbb{N}}$ be a sequence of finite graphs with uniformly bounded degree, such that $\lim_i |V_i| = \infty$. We say that $\{ (V_i, E_i) \}_{i \in \mathbb{N}}$ has a uniform coarse embedding in $X$ if there are functions $\phi_i : V_i \to X$ and functions $\rho_- , \rho_+ : \mathbb{N} \to \mathbb{R}$ such that $\lim_n \rho_-(n) = \infty$ and

$$\forall i \in \mathbb{N}, \forall x, y \in V_i, \rho_-(d_i(x, y)) \leq \| \phi_i(x) - \phi_i(y) \| \leq \rho_+(d_i(x, y)),$$

where $d_i(x, y)$ is the graph distance in $(V_i, E_i)$ between $x$ and $y$.

If $\{ (V_i, E_i) \}_{i \in \mathbb{N}}$ has no uniform coarse embedding in $X$, we shall say that $\{ (V_i, E_i) \}_{i \in \mathbb{N}}$ is a family of $X$-expanders.
Proposition 1.14. Let $G$ be a finitely generated discrete group and let $\{N_i\}_{i\in \mathbb{N}}$ be a sequence of finite index normal subgroups of $G$ such that $\bigcap_i N_i = \{1\}$. Let $\mathcal{E}$ be a class of Banach spaces that is closed under $l_2$ sums. Fix $S$ to be some symmetric generating set of $G$. If $G$ has robust property (T) with respect to $\mathcal{E}$, then the family of Cayley graphs $\{(G/N_i,S)\}_{i\in \mathbb{N}}$ is a family of $X$-expanders for any $X \in \mathcal{E}$.

The proof of this proposition is similar to the proof given in [13]. For completeness, the proof is provided in the appendix.

Using this application we are able construct new examples of families of graph that are expanders with respect to large classes of (non superreflexive) Banach spaces. For instance we can show the following:

Theorem 1.15. For any Banach space $X$ of type $p_1$ and cotype $p_2$ such that $\frac{1}{p_1} - \frac{1}{p_2} < \frac{1}{4}$, we can construct an $X$-expander by taking Cayley graphs of quotients of the group $EL_n(\mathbb{F}_q[t_1,\ldots,t_m])$ provided that $q$ is sufficiently large.

A more general statement of this example is given in the last section of this paper.

Structure of this paper. Section 2 includes all the needed background material. Section 3 is devoted to proving a criterion for quick convergence of the averaged projections method, relaying on the concept of an angle between projections. In section 4, we formulate and prove several criteria for robust Banach property (T). In section 5, we give examples of groups with robust Banach property (T) and construct Banach expanders.

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2 Background

2.1 Projections in a Banach space

Let $X$ be a Banach space. Recall that a projection $P$ is a bounded operator $P \in B(X)$ such $P^2 = P$. Note that $\|P\| \geq 1$ if $P \neq 0$. For subspaces $M,N$ of $X$, we’ll say that $P$ is a projection on $M$ along $N$ if $P$ is a projection such that $\operatorname{Im}(P) = M$, $\operatorname{ker}(P) = N$.

2.2 The Banach-Mazur distance

The Banach-Mazur distance measures a "distance" between finite dimensional Banach spaces:

Definition 2.1. Let $Y_1,Y_2$ be two isomorphic Banach spaces. The (multiplicative) Banach-Mazur distance between $Y_1$ and $Y_2$ is defined as

$$d_{BM}(Y_1,Y_2) = \inf\{\|T\|\|T^{-1}\| : T : Y_1 \to Y_2 \text{ is a linear isomorphism}\}.$$
This distance has a multiplicative triangle inequality:

**Proposition 2.2.** Let $Y_1, Y_2, Y_3$ be isomorphic Banach spaces. Then

$$d_{BM}(Y_1, Y_3) \leq d_{BM}(Y_1, Y_2)d_{BM}(Y_2, Y_3).$$

We leave the proof of the above proposition as an exercise to the reader.

### 2.3 Type and cotype

Let $X$ be a Banach space. For $1 < p_1 \leq 2$, $X$ is said to have (Gaussian) type $p_1$, if there is a constant $T_{p_1}$, such that for $g_1, ..., g_n$ independent standard Gaussian random variables on a probability space $(\Omega, P)$, we have that for every $x_1, ..., x_n \in X$ the following holds:

$$\left( \int_0^1 \left\| \sum_{i=1}^n g_i(\omega)x_i \right\|^2 dP \right)^{\frac{1}{2}} \leq T_{p_1} \left( \sum_{i=1}^n \|x_i\|^{p_1} \right)^{\frac{1}{p_1}}.$$

The minimal constant $T_{p_1}$ such that this inequality is fulfilled is denoted $T_{p_1}(X)$. For $2 \leq p_2 < \infty$, $X$ is said to have (Gaussian) cotype $p_2$, if there is a constant $C_{p_2}$, such that for $g_1, ..., g_n$ independent standard Gaussian random variables on a probability space $(\Omega, P)$, we have that for every $x_1, ..., x_n \in X$ the following holds:

$$C_{p_2} \left( \sum_{i=1}^n \|x_i\|^{p_2} \right)^{\frac{1}{p_2}} \leq \left( \int_0^1 \left\| \sum_{i=1}^n g_i(\omega)x_i \right\|^2 dP \right)^{\frac{1}{2}}.$$

The minimal constant $C_{p_2}$ such that this inequality is fulfilled is denoted $C_{p_2}(X)$. We shall say that a class of Banach spaces, $E$, is of type $> 1$, if there is $p_1 > 1, K > 0$ such that every $X \in E$ is of type $p_1$ with $K^{(q_1)}(X) \leq K$.

**Remark 2.3.** We remark that the Gaussian type and cotype defined above are equivalent to the usual (Rademacher) type and cotype (see [17][pages 311-312] and reference therein).

**Remark 2.4.** Recall that a Banach space is called superreflexive if it is isomorphic to a uniformly convex Banach space. In [19], Pisier and Xu showed that for any $p_2 > 2$ one can construct a non superreflexive Banach space $X$ with type 2 and cotype $p_2$.

### 2.4 Vector valued $L^2$ spaces

Given a measure space $(\Omega, \mu)$ and Banach space $X$, a function $s : \Omega \to X$ is called simple if it is of the form:

$$s(\omega) = \sum_{i=1}^n \chi_{E_i}(\omega)v_i,$$

where $\{E_1, ..., E_n\}$ is a partition of $\Omega$ where each $E_i$ is a measurable set, $\chi_{E_i}$ is the indicator function on $E_i$ and $v_i \in X$.

A function $f : \Omega \to X$ is called Bochner measurable if it is almost everywhere 

the limit of simple functions. Denote \( L^2(\Omega; X) \) to be the space of Bochner measurable functions such that

$$\forall f \in L^2(\Omega; X), \|f\|_{L^2(\Omega; X)} = \left( \int_{\Omega} \|f(\omega)\|_X^2 d\mu(\omega) \right)^{\frac{1}{2}} < \infty.$$  

Given an operator \( T \in B(L^2(\Omega, \mu)) \), we can define \( T \otimes id_X \in B(L^2(\Omega; X)) \) by defining it first on simple functions. We shall need to following facts:

**Lemma 2.5.** [Lemma 3.1] Let \((\Omega, \mu)\) be a measure space, \( C \geq 0 \) and \( T \) a bounded operator on \( L^2(\Omega, \mu) \). The class of Banach spaces \( \theta \), for which \( \|T \otimes id_X\| \leq C \), is stable under quotients, subspaces, \( l_2 \)-sums and ultraproducts.

**Remark 2.6.** The fact that the above class is closed under \( l_2 \) sums, did not appear in [5][Lemma 3.1] and it is left as an exercise to the reader.

**Lemma 2.7.** Let \((\Omega, \mu)\) be a measure space and \( T \) a bounded operator on \( L^2(\Omega, \mu) \). Given two isomorphic Banach spaces \( X, X' \), we have that

$$\|T \otimes id_{X'}\| \leq d_{BM}(X, X') \|T \otimes id_X\|.$$  

**Proof.** Let \( S : X \to X' \) be an isomorphism, then

$$T \otimes id_{X'} = (id_{L^2(\Omega, \mu)} \otimes S) \circ (T \otimes id_X) \circ (id_{L^2(\Omega, \mu)} \otimes S^{-1})$$

and therefore

$$\|T \otimes id_{X'}\| \leq \|id_{L^2(\Omega, \mu)} \otimes S\| \|T \otimes id_X\| \|id_{L^2(\Omega, \mu)} \otimes S^{-1}\| = \|S\| \|S^{-1}\| \|T \otimes id_X\|,$$

and the conclusion of the lemma follows. \( \square \)

### 2.5 Interpolation

Two Banach spaces \( X_0, X_1 \) form a compatible pair \((X_0, X_1)\) if there is a continuous linear embedding of both \( X_0 \) and \( X_1 \) in the same topological vector space. The idea of complex interpolation is that given a compatible pair \((X_0, X_1)\) and a constant \( 0 < \theta < 1 \), there is a method to produce a new Banach space \([X_0, X_1]_\theta\) as a “combination” of \( X_0 \) and \( X_1 \). We will not review this method here, and the interested reader can find more information on interpolation in [5][Section 2.4] and reference therein. The only fact we shall use regarding complex interpolation is the following:

**Lemma 2.8.** [Lemma 3.1] Given a compatible pair \((X_0, X_1)\), a measure space \((\Omega, \mu)\) and an operator \( T \in B(L^2(\Omega, \mu)) \), we have for every \( 0 \leq \theta \leq 1 \) that

$$\|T \otimes id_{[X_0, X_1]_\theta}\|_{B(L^2(\Omega, [X_0, X_1]_\theta))} \leq \|T \otimes id_{X_0}\|_{B(L^2(\Omega, X_0))}^{1-\theta} \|T \otimes id_{X_1}\|_{B(L^2(\Omega, X_1))}^\theta.$$  

### 2.6 \( \theta \)-Hilbertian Banach spaces

The following definition is due to Pisier in [13]: a Banach space \( X \) is called strictly \( \theta \)-Hilbertian for \( 0 < \theta < 1 \), if there is a compatible pair \((X_0, X_1)\) such that \( X_1 \) is a Hilbert space such that \( X = [X_0, X_1]_\theta \). Examples of are \( L^p \) space and non-commutative \( L^p \) spaces when (in these cases \( \theta = \frac{1}{p} \) if \( 2 \leq p < \infty \) and \( \theta = 2 - \frac{2}{p} \) if \( 1 < p \leq 2 \)). In [13], Pisier showed that any superreflexive Banach lattice is strictly \( \theta \)-Hilbertian and conjectured that any superreflexive Banach space is a subspace of a quotient of a strictly \( \theta \)-Hilbertian Banach space for some \( \theta > 0 \).
2.7 Group representations in a Banach space

Let \( G \) be a locally compact group and \( X \) a Banach space. Let \( \pi \) be a representation \( \pi : G \to B(X) \). Throughout this paper we shall always assume \( \pi \) is continuous with respect to the strong operator topology without explicitly mentioning it.

Denote by \( C_c(G) \) the groups algebra of compactly supported continuous functions on \( G \) with convolution. For any \( f \in C_c(G) \) we can define \( \pi(f) \in B(X) \) as
\[
\forall v \in X, \pi(f).v = \int_G f(g)\pi(g).v d\mu(g),
\]
where the above integral is the Bochner integral with respect to the (left) Haar measure \( \mu \) of \( G \).

Recall that given \( \pi \) one can define the following representations:

1. The complex conjugation of \( \pi \), denoted \( \overline{\pi} : G \to B(X) \) is defined as
\[
\overline{\pi}(g).v = \overline{\pi(g).v}, \forall g \in G, v \in X.
\]

2. The dual representation \( \pi^* : G \to B(X^*) \) is defined as
\[
\langle v, \pi^*(g)u \rangle = \langle \pi(g^{-1}).v, u \rangle, \forall g \in G, v \in X, u \in X^*.
\]

Next, we’ll restrict ourselves to the case of compact groups. Let \( K \) be a compact group with a Haar measure \( \mu \) and let \( C_c(K) = C(K) \) defined as above. Let \( X \) be Banach space and let \( \pi \) be a representation of \( K \) on \( X \) that is continuous with respect to the strong operator topology. We shall show that for every \( f \in C_c(K) \) we can bound the norm of \( \pi(f) \) using the norm of \( \lambda \otimes \text{id}_X \in B(L^2(K;X)) \) (the definition of \( L^2(K;X) \) is given in subsection 2.4 above). We shall start with the following result of de la Salle that deals with the case in which \( \pi \) is an isometric representation:

**Proposition 2.9.** Let \( \pi \) be an isometric representation of a compact group \( K \) on a Banach space \( X \). Then for any real function \( f \in C_c(K) \) we have that
\[
\|\pi(f)\|_{B(X)} \leq \|\lambda \otimes \text{id}_X(f)\|_{B(L^2(K;X))},
\]
where \( \lambda \) is the left regular representation of \( K \).

**Remark 2.10.** The above proposition in [5] is phrased in the language of signed Borel measures on \( K \) and not \( f \in C_c(K) \), but this is equivalent, since \( f \) can be thought of as the density function of a signed Borel measure.

We can use the above proposition to bound the norm of \( \pi(f) \) in the more general case in which \( \sup_{g \in K} \|\pi(g)\| < \infty \):

**Corollary 2.11.** Let \( \pi \) be a representation of a compact group \( K \) on a Banach space \( X \). Assume that \( \sup_{g \in K} \|\pi(g)\| < \infty \), then for any real function \( f \in C_c(G) \) we have that
\[
\|\pi(f)\|_{B(X)} \leq \left( \sup_{g \in K} \|\pi(g)\| \right)^2 \|\lambda \otimes \text{id}_X(f)\|_{B(L^2(K;X))},
\]
where \( \lambda \) is the left regular representation of \( K \).
Proof. Define the following norm \( \| \cdot \|' \) on \( X \):
\[
\forall v \in X, \| v \|' = \sup_{g \in K} \| \pi(g) \cdot v \|_X.
\]
Denote \( X' \) to be the Banach space \( X' = (X, \| \cdot \|') \), then \( \pi \) is an isometric representation on \( X' \) and \( d_{BM}(X, X') \leq \sup_{g \in K} \| \pi(g) \| \). By proposition \ref{prop2.9} we have that
\[
\| \pi(f) \|_{B(X')} \leq \| (\lambda \otimes id_{X'}) (f) \|_{B(L^2(K;X'))}.
\]
By lemma \ref{lem2.7} we have that
\[
\| (\lambda \otimes id_{X'}) (f) \|_{B(L^2(K;X'))} \leq \left( \sup_{g \in K} \| \pi(g) \| \right) \| (\lambda \otimes id_{X'}) (f) \|_{B(L^2(K;X'))}.
\]
One can also check that by the definition of \( \| \cdot \|' \), we have that
\[
\| \pi(f) \|_{B(X)} \leq \left( \sup_{g \in K} \| \pi(g) \| \right) \| \pi(f) \|_{B(X')}.
\]
and the conclusion of the corollary follows.

2.8 Spectra of bipartite graphs

Let \( G = (V, E) \) be a graph without loops, multiple edges or isolated vertices. Recall the following definitions:

Definition 2.12. The normalized Laplacian on \( (V, E) \) as the operator \( \Delta \) acting on \( L^2(V) \) as
\[
\Delta \psi(v) = \psi(v) - \frac{1}{d(v)} \sum_{u \in V, \{u,v\} \in E} \psi(u),
\]
where \( d(v) \) is the valency of \( v \). \( \Delta \) is positive operator with respect to the following inner product on \( L^2(V) \):
\[
\langle \psi_1, \psi_2 \rangle = \sum_{v \in V} d(v) \psi_1(v) \psi_2(v).
\]
If \( G \) is connected, then \( 0 \) is an eigenvalue of multiplicity 1 of \( \Delta \) and the smallest positive eigenvalue of \( \Delta \) is called the spectral gap of the Laplacian.

Definition 2.13. The graph \( G \) is called bipartite if there are non empty disjoint sets \( V_1, V_2 \subset V \) such that \( V = V_1 \cup V_2 \) and \( E \subseteq \{ \{v_1, v_2\} : v_1 \in V_1, v_2 \in V_2 \} \), i.e., there are no edges between two vertices of \( V_1 \) or between two vertices of \( V_2 \). A bipartite graph is called semi-regular if for every \( i = 1, 2 \) and every \( v, v' \in V_i \) we have \( d(v) = d(v') \).

The following proposition is well known and the proof is a simple computation which is left for the reader:

Proposition 2.14. Let \( G = (V, E) \) be a bipartite graph with \( V_1, V_2 \subset V \) as in the definition above. If \( \psi \in L^2(V) \) is an eigenvector of Laplacian \( \Delta \) with eigenvalue \( \eta \), then
\[
\psi'(v) = \begin{cases} 
\psi(v) & v \in V_1 \\
-\psi(v) & v \in V_2 
\end{cases}
\]
is an eigenvector of \( \Delta \) with eigenvalue \( 2 - \eta \).
Proposition 2.15. Let $G = (V, E)$ be a bipartite graph with $V_1, V_2 \subset V$ as in the definition above. If $|V_2| < |V_1|$, then the space of eigenfunctions of $\Delta$ with the eigenvalue 1 has a subspace of functions supported on $V_1$ of dimension $\geq |V_1| - |V_2|$.

Proof. Define

$$W = \{ \psi \in L^2(V) : \forall v \in V_2, \psi(v) = 0 \text{ and } \sum_{u \in V_1, \{v,u\} \in E} \psi(u) = 0 \}.$$

We conclude by noticing that for every $\psi \in W$, $\Delta \psi = 0$ and that $W$ is a subspace of $L^2(V)$ defined by $2|V_2|$ linear equations on $|V_1| + |V_2|$ variables and therefore the dimension of $W$ is at least $|V_1| - |V_2|$.

Proposition 2.16. Let $G = (V, E)$ be a semi-regular bipartite graph and let $\psi \in L^2(V)$ be an eigenfunction of $\Delta$ with an eigenvalue $0 < \eta < 2$, then for $i = 1, 2$ we have

$$\sum_{v \in V_i} \psi(v) = 0.$$

Proof. Let $\chi_{V_i}, \chi_{V_1}, \chi_{V_2}$ be the indicator functions of $V, V_1, V_2$. By definition of $\Delta$ we have that $\Delta \chi_V = 0$. Therefore by proposition 2.14 we get that $\Delta (\chi_{V_i} - \chi_{V_2}) = 2(\chi_{V_i} - \chi_{V_2})$. From the fact that $\Delta$ is self-adjoint we deduce that $\langle \psi, \chi_V \rangle = \langle \psi, \chi_{V_i} - \chi_{V_2} \rangle = 0$. Denote by $d_{V_1}, d_{V_2}$ the valency of all the vertices in $V_1$ and $V_2$ respectively. Then

$$0 = \langle \psi, \chi_V \rangle = d_{V_1} \sum_{v \in V_1} \psi(v) + d_{V_2} \sum_{v \in V_2} \psi(v),$$

$$0 = \langle \psi, \chi_{V_i} - \chi_{V_2} \rangle = d_{V_1} \sum_{v \in V_1} \psi(v) - d_{V_2} \sum_{v \in V_2} \psi(v),$$

and the proposition follows.

3 Averaged projections in a Banach space

Given a family of projections $P_1, ..., P_N$ on $M_1, ..., M_N$ in $X$, there is a well known algorithm of finding a projection on $\cap_{k=1}^{N} M_k$, which is known as the method of alternating projections. This algorithm can be stated (in full generality) as follows: let $S(P_1, ..., P_N)$ be the convex hull of the semigroup consisting of all products with factors from $\{P_1, ..., P_N\}$. Let $T \in S(P_1, ..., P_N)$, such that for every $k = 1, ..., n$, $P_k$ appears in some product in the decomposition of $T$ in $S(P_1, ..., P_N)$. Then under some assumptions on $X$ (for instance, if $X$ is uniformly convex and $\|P_k\| = 1$ for every $k$ as in [1]), we have that the sequence $T^n$ converges in the strong operator topology to an operator $T^\infty$ that is a projection on $\cap_{k=1}^{N} M_k$. Below we shall restrict ourselves to a special case of the general alternating projections method described above in which $T = \frac{P_1 + ... + P_N}{N}$. This case is called the averaged projections method.

The rate of converges of this method is either "very slow" or "very fast". To make this statement precise we recall the following definitions and results from [1].
Definition 3.1 (Rates of convergence). Let $X$ be a Banach space and let $(T_n)$ be a sequence of bounded operators on $X$. Assume that $(T_n)$ converges in the strong operator topology to $T^\infty \in \mathcal{B}(X)$. Then we say that:

1. Quick uniform convergence condition holds (abbreviated: (QUC) holds) if there are constants $C \geq 0$, $0 \leq r < 1$ such that $\|T^\infty - T_n\| \leq Cr^n$.

2. Arbitrarily slow convergence condition holds (abbreviated: (ASC) holds) if for every sequence of positive numbers $(a_n)$ such that $\lim_{n \to \infty} (a_n) = 0$ and for every $\varepsilon > 0$, there exists $v \in X$ such that $\|v\| < \sup_n a_n + \varepsilon$ and for every $n$, $\|T^\infty v - T_n v\| \geq a_n$.

Remark 3.2. Note that in the above definition if (ASC) holds, then in particular $(T_n)$ does not converge to $T^\infty$ in the uniform operator topology. Indeed, for any arbitrary $n_0$, one can choose a positive sequence $(a_n)$ such that $\lim_{n \to \infty} (a_n) = 0$, $\sup_n a_n < 1$ and $a_{n_0} = \frac{1}{2}$. Therefore, by (ASC), there is $v \in X$, such that $\|v\| \leq 1$ and $\|T^\infty v - T_{n_0} v\| \geq \frac{1}{2}$. This is true for any $n_0$ and therefore we get that for any $n$, $\|T^\infty - T_n\| \geq \frac{1}{2}$.

The conditions (QUC) and (ASC) represent the two extreme cases of convergence rates of sequences of bounded operators (assuming convergence in the strong operator topology). The theorem below states that in the case that $T_n$ is defined as $T_n = T^n$ for some fixed $T \in \mathcal{B}(X)$, these are the only possibilities of convergence:

Theorem 3.3 (Dichotomy theorem). \cite{1}[Theorem 2.1] Let $X$ be a Banach space and let $T \in \mathcal{B}(X)$. If $T^n$ converges in the strong operator topology to $T^\infty \in \mathcal{B}(X)$ then either (QUC) holds or (ASC) holds.

Next, we want to establish a criterion that guarantees that the convergence is quick uniform in the case of averaged projections where $T = P_1 + \ldots + P_N$ (this case is considered because it will suit our needs later and because it is simple). In the case where $X = H$ is a Hilbert space, criteria for quick uniform are given in \cite{1} and \cite{20} (see also \cite{11} for related results) in terms of angles between subspaces. The basic idea is that if the angles between the subspaces are large enough, then the averaged projections method has quick uniform convergence. The concept of angle considered in the articles mentioned above is the Friedrichs angle defined as follows:

Definition 3.4 (Friedrichs angle). Let $M_1, M_2$ be subspaces of a Hilbert space $H$. Denote $M = M_1 \cap M_2$. If $M_1 = M$ or $M_2 = M$, Friedrichs angle between $M_1$ and $M_2$ is defined to be $\frac{\pi}{2}$. Otherwise, the Friedrichs angle between $M_1$ and $M_2$ is defined as:

$$\angle(M_1, M_2) = \arccos \left( \sup \left\{ \frac{|u, v|}{\|u\|\|v\|} : 0 \neq u \in M_1 \cap M^\perp, 0 \neq v \in M_2 \cap M^\perp \right\} \right).$$

This definition is equivalent to

$$\angle(M_1, M_2) = \arccos \left( \sup \{ \|P_{M_1}(v)\| : v \in M_2 \cap M^\perp, \|v\| = 1 \} \right),$$

where $P_{M_1}$ is the orthogonal projection on $M_1$. We’ll leave this equivalence as an exercise to the reader.
Although several authors gave definitions for angle between subspaces in Banach spaces (see for instance [16]), we could not find such a definition that suits our purpose and allows giving a criterion for quick uniform convergence of the averaged projections method in Banach spaces. There seems to be two major problems with such definitions: first, the lack of the concept orthogonality in Banach spaces (without passing to the dual space) and second, the fact that the projections are not uniquely determined by their image (unlike the case of orthogonal projections in a Hilbert space). In order to circumvent both problems, we shall define an angle between projections and not between subspaces.

**Definition 3.5** (Friedrichs angle between projections). Let $X$ be a Banach space and let $P_1, P_2$ be projections on $M_1, M_2$ respectively. Assume that there is a projection $P_{1,2}$ on $M_1 \cap M_2$ such that $P_{1,2}P_1 = P_{1,2}$ and $P_{1,2}P_2 = P_{1,2}$ and define

$$\cos(\angle(P_1, P_2)) = \max \{ \|P_1(P_2 - P_{1,2})\|, \|P_2(P_1 - P_{1,2})\| \}.$$ 

**Remark 3.6.** In the above definition, we are actually defining the “cosine” of the angle. This is a little misleading, because in some cases $\cos(\angle(P_1, P_2)) > 1$.

**Remark 3.7.** Note that the assumptions on $P_{1,2}$ above imply that $P_1 - P_{1,2}$ and $P_2 - P_{1,2}$ are projections, i.e., $(P_1 - P_{1,2})^2 = P_1 - P_{1,2}$ and $(P_2 - P_{1,2})^2 = P_2 - P_{1,2}$.

**Remark 3.8.** In the case where $X$ is a Hilbert space and $P_1, P_2$ are orthogonal projections on $M_1, M_2$, the orthogonal projection $P_{1,2}$ on $M_1 \cap M_2$ will always fulfill $P_{1,2}P_1 = P_{1,2}$ and $P_{1,2}P_2 = P_{1,2}$. Further more, note that in this case, we have that

$$v \in M_2 \cap (M_1 \cap M_2)^\perp \Leftrightarrow (P_2 - P_{1,2})v = v.$$ 

Therefore

$$\cos(\angle(M_1, M_2)) = \sup \{ \|P_1(v)\| : v \in M_2 \cap M_1^\perp, \|v\| = 1 \}$$

$$= \sup \{ \|P_2(v)\| : (P_2 - P_{1,2})v = v, \|v\| = 1 \}$$

$$= \sup \{ \|P_2(P_2 - P_{1,2})v\| : (P_2 - P_{1,2})v = v, \|v\| = 1 \}$$

$$= \|P_2(P_2 - P_{1,2})\| = \cos(\angle(P_1, P_2)).$$ 

Therefore the above definition is a complete analogue to the definition of Friedrichs angle in Hilbert spaces.

Next, we shall also need the following useful constant $a(P_1, P_2)$:

**Definition 3.9.** Let $X$ be a Banach space and let $P_1, P_2$ be projections on $M_1, M_2$ respectively. Define $a(P_1, P_2)$ as follows:

$$a(P_1, P_2) = \inf \{ \gamma \in [0, \infty) : \forall v \in X, \|(P_1P_2 - P_1)v\| \leq \gamma \|(P_1 - P_2)v\| \}.$$ 

Let $P_1, ..., P_N$ be projections in a Banach space $X$. For $T = \frac{1}{N} \sum_{j=1}^N P_j$, we will show that if for every couple $P_i, P_j$, $\cos(\angle(P_i, P_j))$ is small enough, then the sequence $\{T^n\}$ converges uniformly quickly to $\bigcap_{i=1}^N \text{Im}(P_i)$. We shall prove this in two steps: first, we will show that if for every $P_i, P_j$, $a(P_i, P_j)$ is small enough then the sequence $\{T^n\}$ converges uniformly quickly to $\bigcap_{i=1}^N \text{Im}(P_i)$. 

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Second, we will show $a(P_i, P_j)$ can be bounded from above by (a function of) $\cos(\angle(P_i, P_j))$.

We start the first step by introducing the following function $E : X \to \mathbb{R}$:

$$E(v) = \sum_{1 \leq i < j \leq N} \| (P_i - P_j)v \|.$$  

**Lemma 3.10.** Let $X$ be a Banach space and let $P_1, ..., P_N$ be projections in $X$ (where $N \geq 2$). Denote $T = \frac{P_1 + \ldots + P_N}{N}$, then for every $v \in X$ and every $n \in \mathbb{N}$, we have that

$$E(T^n v) \leq \left( \frac{1 + (N-2)\beta + (2N-3)\alpha}{N} \right)^n E(v).$$

and

$$E(T^n v) \leq 2\beta \left( \frac{N}{2} \right) \left( \frac{1 + (N-2)\beta + (2N-3)\alpha}{N} \right)^n \|v\|.$$

**Proof.** We start by observing that for every $1 \leq i < j \leq N$ we have that

$$(P_i - P_j)(P_i + P_j) = P_i - P_j + P_jP_i - P_jP_j.$$

Therefore, for every $v \in X$, we have that

\begin{equation}
\| (P_i - P_j)(P_i + P_j)v \| \leq \| (P_i - P_j)v \| + \| (P_iP_j - P_jP_j)v \| \leq \| (P_i - P_j)v \| + a(P_i, P_j)\|((P_i - P_j)v)\| \leq (1 + \alpha)\| (P_i - P_j)v \| \tag{1}
\end{equation}

Let $i, j$ as before and let $1 \leq k \leq N$ such that $k \neq i, j$. Observe that

$$(P_i - P_j)P_k = P_iP_k - P_jP_k = P_iP_k - P_kP_i + P_kP_j - P_jP_k = (P_iP_k - P_kP_i) + (P_kP_j - P_jP_k) + P_k(P_i - P_j).$$

Therefore, for every $v \in X$, we have that

\begin{equation}
\| (P_i - P_j)P_kv \| \leq \| (P_iP_k - P_kP_i)v \| + \| (P_iP_j - P_jP_k)v \| + \| P_k(P_i - P_j)v \| \leq a(P_i, P_k)\| ((P_i - P_k)v) \| + a(P_j, P_k)\| (P_j - P_k)v \| + \| P_k \| \| (P_i - P_j)v \| \leq \alpha\| (P_i - P_k)v \| + \| (P_i - P_k)v \| + \beta\| (P_i - P_j)v \|. \tag{2}
\end{equation}

Using (1), (2), we get that

\begin{align*}
\| (P_i - P_j)Tv \| & = \left\| (P_i - P_j)\left( \frac{P_1 + \ldots + P_N}{N} \right)v \right\| \\
& \leq 1 + \frac{(N-2)\beta + \alpha}{N} \| (P_i - P_j)v \| + \frac{\alpha}{N} \sum_{1 \leq k \neq i, j \leq N} \| (P_i - P_k)v \| + \| (P_j - P_k)v \|. 
\end{align*}
By summing this inequalities along all $1 \leq i < j \leq N$, we get that

$$E(Tv) \leq \left(\frac{1 + (N - 2)\beta + (2N - 3)\alpha}{N}\right) E(v).$$

Therefore, by induction we have for every $n \in \mathbb{N}$ that

$$E(T^n v) \leq \left(\frac{1 + (N - 2)\beta + (2N - 3)\alpha}{N}\right)^n E(v).$$

To conclude, notice that for every $v$ we have that

$$\|E(v)\| \leq \left(\frac{N}{2}\right) \frac{2\beta}{N^2} \|v\|.$$

**Lemma 3.11.** Let $X$ be a Banach space and let $P_1, ..., P_N$ be projections in $X$ (where $N \geq 2$). Denote $T = \frac{P_1 + ... + P_N}{N}$. If there are $\alpha < \frac{1}{2N - 3}$ and $\beta < \frac{N - 1 - (2N - 3)\alpha}{N - 2}$, such that

$$\max\{\alpha(P_i, P_j) : 1 \leq i < j \leq N\} \leq \alpha,$$

$$\max\{\|P_1\|, ..., \|P_N\|\} \leq \beta,$$

then there are constants $r' = r'(\alpha, \beta), C' = C'(\alpha, \beta)$ and an operator $T^\infty$, such that

$$0 \leq r' < 1, C' \geq 0$$

and

$$\|T^\infty - T^n\| \leq C'(r')^n. \quad \text{Moreover, } T^\infty \text{ is a projection on } \bigcap_{n=1}^\infty \text{Im}(P_i).$$

**Proof.** Note that for every $v \in X$, we have that

$$\|(T - T^2)v\| = \left\|\sum_{i=1}^N \frac{P_i}{N} (P_i - T)v\right\| =$$

$$= \left\|\sum_{i=1}^N \frac{P_i}{N} \sum_{j=1}^N \frac{P_i - P_j}{N} v\right\| \leq \frac{\beta}{N^2} \sum_{1 \leq i < j \leq N} \|(P_i - P_j)v\| =$$

$$\frac{2\beta}{N^2} \sum_{1 \leq i < j \leq N} \|(P_i - P_j)v\| = \frac{2\beta}{N^2} E(v).$$

Therefore, for any $n \in \mathbb{N}$ and any $v \in X$ we have that

$$\|(T^{n+1} - T^n)v\| = \|(T^2 - T)(T^{n-1}v)\| \leq \frac{2\beta}{N^2} E(T^{n-1}v) \leq$$

$$\frac{(2\beta)^2}{N^2} \binom{N}{2} \left(\frac{1 + (N - 2)\beta + (2N - 3)\alpha}{N}\right)^{n-1} \|v\|,$$

where the last inequality is due to lemma 3.10. Denote

$$r' = \frac{1 + (N - 2)\beta + (2N - 3)\alpha}{N}.$$
Notice that the conditions $\alpha < \frac{1}{2N-3}$ and $\beta < \frac{N-1-(2N-3)\alpha}{N-2}$ stated in the theorem, insure that $r' < 1$. After simplifying (3), we get that
\[
\|T^{n+1} - T^n\| \leq \frac{(N-1)2\beta^2}{N} (r')^{n-1}.
\]
Therefore, for every two integers $m > n$, we have that
\[
\|T^m - T^n\| \leq \|T^m - T^{m-1}\| + \|T^{m-1} - T^{m-2}\| + \ldots + \|T^{n+1} - T^n\| \leq \frac{(N-1)2\beta^2}{N} \frac{1}{(1-r')}^{n-1} (r')^n.
\]
Denote $C' = \frac{(N-1)2\beta^2}{N} \frac{1}{(1-r')}^{n-1}$. We showed that $(T^n)$ is a Cauchy sequence with respect to the operator norm and therefore converges to an operator $T^\infty$, and
\[
\|T^\infty - T^n\| \leq C'(r')^n.
\]
One can easily verify that $(T^\infty)^2 = T^\infty$ and therefore $T^\infty$ is a projection. To see that $\text{Im}(T^\infty) = \bigcap_{i=1}^N \text{Im}(P_i)$, we first note that for every $v \in \bigcap_{i=1}^N \text{Im}(P_i)$, we have that $Tv = v$ and therefore $\bigcap_{i=1}^N \text{Im}(P_i) \subseteq \text{Im}(T^\infty)$. To finish the proof, we need to show that $\text{Im}(T^\infty) \subseteq \bigcap_{i=1}^N \text{Im}(P_i)$. Note that $TT^\infty = T^\infty$ and therefore for every $v \in \text{Im}(T^\infty)$, we have that $Tv = v$. From lemma 3.10 we have that for every $v \in \text{Im}(T^\infty)$, $E(v) = E(Tv) \leq r'E(v)$ and therefore $E(v) = 0$ (recall that $0 \leq r' < 1$). $E(v)$ is defined as
\[
E(v) = \sum_{1 \leq i < j \leq N} \| (P_i - P_j)v \|.
\]
Therefore $E(v) = 0$ implies that $P_1v = P_2v = \ldots = P_Nv$. To finish, we’ll again use the fact that $Tv = v$ and get that for every $1 \leq i \leq N$,
\[
\left( v = Tv = \frac{P_1v + \ldots + P_nv}{N} = \frac{P_1v + \ldots + P_2v}{N} = P_1v \right) \implies v \in \text{Im}(P_i).
\]
Since this is true for all the $i$’s, we got that $\text{Im}(T^\infty) \subseteq \bigcap_{i=1}^N \text{Im}(P_i)$. $\square$

Next, we turn to the second step, bounding $a(P_1, P_2)$ as a function of $\cos(\angle(P_1, P_2))$:

**Lemma 3.12.** Let $X$ be a Banach space and let $P_1, P_2$ be two projections in $X$, such that there is a projection $P_{1,2}$ on $\text{Im}(P_1) \cap \text{Im}(P_2)$ such that $P_{1,2}P_1 = P_{1,2}$ and $P_{1,2}P_2 = P_{1,2}$. Denote $\beta = \max\{\|P_1\|, \|P_2\|\}$. Then
\[
a(P_1, P_2) \leq \frac{2(1 + \beta) \cos(\angle(P_1, P_2))}{1 - \cos(\angle(P_1, P_2))}.
\]

**Proof.** Let $v \in X$. We start by noting that
\[
(P_1P_2 - P_2P_1)v = (P_1P_2 - P_{1,2})v - (P_{1,2}P_1 - P_{1,2})v = P_1(P_2 - P_{1,2})(P_2 - P_{1,2})v - P_2(P_1 - P_{1,2})(P_1 - P_{1,2})v.
\]
This yields that
\[
\| (P_1P_2 - P_2P_1)v \| \leq \cos(\angle(P_1, P_2)) \| (P_1 - P_{1,2})v \| + \| (P_2 - P_{1,2})v \| \leq 2 \cos(\angle(P_1, P_2)) \max\{ \| (P_1 - P_{1,2})v \|, \| (P_2 - P_{1,2})v \| \}.
\]

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Therefore in order to prove the inequality stated in the lemma it is enough to show that 
\[
\max\{\|(P_1 - P_{1,2})v\|, \|(P_2 - P_{1,2})v\|\} \leq \frac{1 + \beta}{1 - \cos(\angle(P_1, P_2))} \|(P_1 - P_2)v\|.
\]
Assume without loss of generality that \(\|(P_1 - P_{1,2})v\| \geq \|(P_2 - P_{1,2})v\|\). Then we have that 
\[
\begin{align*}
\|(P_1 - P_2)v\| &= \|(I - P_2)(P_1 - P_2)v + P_2(P_1 - P_2)v\| \\
\geq\|(I - P_2)(P_1 - P_2)v\| - \|P_2(P_1 - P_2)v\| \\
\geq\|(I - P_2)(P_1 - P_{1,2}, P_2 - P_2)v - \|\beta\|((P_1 - P_2)v\| \\
\geq\|(I - P_2)(P_1 - P_{1,2})v\| - \|\beta\|((P_1 - P_2)v\| \\
\geq\|(P_1 - P_{1,2})v\| - \|P_2(P_1 - P_{1,2})v\| - \|\beta\|((P_1 - P_2)v\| \\
\geq\|(P_1 - P_{1,2})v\| - \|P_2(P_1 - P_{1,2})(P_1 - P_{1,2})v\| - \|\beta\|((P_1 - P_2)v\| \\
\geq(1 - \cos(\angle(P_1, P_2)))\|(P_1 - P_{1,2})v\| - \|\beta\|((P_1 - P_2)v\|
\end{align*}
\]
which yields the necessary inequality to finish the proof. 

Combining the two lemmas above gives raise to the following convergence criterion:

**Theorem 3.13.** Let \(X\) be a Banach space and let \(P_1, ..., P_N\) be projections in \(X\) (where \(N \geq 2\)). Denote \(T = P_1 + ... + P_N\), 
\[
\cos_{\text{max}} = \max\{\cos(\angle(P_i, P_j)) : 1 \leq i < j \leq N\}.
\]
Assume there are constants 
\[
\gamma < \frac{1}{8N - 11} \quad \text{and} \quad \beta < 1 + \frac{1 - (8N - 11)\gamma}{N - 2 + (3N - 4)\gamma}
\]
such that \(\max\{\|P_1\|, ..., \|P_N\|\} \leq \beta\) and \(\cos_{\text{max}} \leq \gamma\). Then there are constants 
\(r = r(\gamma, \beta), C = C(\gamma, \beta)\) and an operator \(T^\infty\), such that \(0 \leq r < 1, C \geq 0\) and 
\(\|T^\infty - T^n\| \leq C r^n\). Moreover, \(T^\infty\) is a projection on \(\bigcap_{i=1}^N \text{Im}(P_i)\).

**Proof.** By lemma 3.12 we get that 
\[
\max\{a(P_i, P_j) : 1 \leq i < j \leq N\} \leq \frac{2(1 + \beta)\cos_{\text{max}}}{1 - \cos_{\text{max}}} \leq \frac{2(1 + \beta)\gamma}{1 - \gamma}.
\]
Denote \(\alpha = \frac{2(1 + \beta)\gamma}{1 - \gamma}\). By lemma 3.11 it is enough to verify that 
\[
\alpha < \frac{1}{2N - 3} \quad \text{and} \quad \beta < \frac{N - 1 - (2N - 3)\alpha}{N - 2},
\]
and then take \(C = C'(\frac{2(1 + \beta)\gamma}{1 - \gamma}, \beta), r = r'(\frac{2(1 + \beta)\gamma}{1 - \gamma}, \beta)\), where \(C'\) and \(r'\) are the constants given in lemma 3.11.

By our notations \(\alpha < \frac{1}{2N - 3}\), is equivalent to 
\[
\frac{2(1 + \beta)\gamma}{1 - \gamma} < \frac{1}{2N - 3}.
\]
Standard algebraic manipulations yields that the above inequality is equivalent to
\[ \beta < 1 + \frac{1 - (8N - 11)\gamma}{(4N - 6)\gamma}. \]
First note that \( \beta \geq 1 \) and therefore without the assumption \( \gamma < \frac{1}{8N-11} \) this inequality cannot hold. Second note that \( \gamma < \frac{1}{8N-11} < 1 \) and therefore the assumption that \( \beta < 1 + \frac{1 - (8N - 11)\gamma}{N - 2 + (3N - 4)\gamma} \) implies the needed inequality.

Next, we need to check that \( \beta < \frac{N - 1 - (2N - 3)\alpha}{N - 2} \), i.e., we need to check that
\[ \beta < 1 + \frac{1 - (8N - 11)\gamma}{N - 2 + (3N - 4)\gamma}, \]
Standard algebraic manipulations yields that this is equivalent to
\[ \beta < 1 + \frac{1 - (8N - 11)\gamma}{N - 2 + (3N - 4)\gamma}, \]
as needed.

4 Robust Banach property (T)

Throughout this section we will work under the following assumptions (and notations): \( G \) is a locally compact group with a Haar measure \( \mu \). Assume that \( G \) generated by compact subgroups \( K_1, \ldots, K_N \), such that \( \mu(K_i) > 0 \) for every \( i = 1, \ldots, N \) and such that for each \( i \neq j \), \( K_{i,j} = \langle K_i, K_j \rangle \) is compact. Denote \( K = \bigcup_{1 \leq i < j \leq N} K_{i,j} \). Define functions \( k_i \in C_c(G), \forall 1 \leq i \leq N \) and \( k_{i,j} \in C_c(G), \forall 1 \leq i < j \leq N \) as
\[
k_i(g) = \begin{cases} \frac{1}{\mu(K_i)} & g \in K_i \\ 0 & g \notin K_i \end{cases},
\]
\[
k_{i,j}(g) = \begin{cases} \frac{1}{\mu(K_{i,j})} & g \in K_{i,j} \\ 0 & g \notin K_{i,j} \end{cases}.
\]

**Proposition 4.1.** Let \( G \) be as above and let \( X \) be a Banach space. For any \( 1 \leq i \leq N \) and any representation \( \pi \) of \( G \) on \( X \), \( \pi(k_i) \) is a projection on \( X^{\pi(K_i)} = \{ v \in X : \forall g \in K_i, \pi(g)v = v \} \). Further more, for any \( 1 \leq i < j \leq N \) and any representation \( \pi \) of \( G \) on \( X \), \( \pi(k_{i,j}) \) is a projection on \( X^{\pi(K_{i,j})} = X^{\pi(K_i)} \cap X^{\pi(K_j)} \) and \( \pi(k_{i,j}) \pi(k_i) = \pi(k_{i,j}), \pi(k_{i,j}) \pi(k_j) = \pi(k_{i,j}) \).

**Proof.** Fix some \( i \). Note that for every \( g \in K_i \), \( g.k_i = k_i \). This implies two things: first, \( (k_i)^2 = k_i \) and therefore \( \pi(k_i)^2 = \pi(k_i) \), i.e., \( \pi(k_i) \) is a projection. Second, for every \( g \in K_i \) and every \( v \in X \), \( \pi(g)\pi(k_i)v = \pi(k_i)v \) and therefore \( \text{Im}(\pi(k_i)) \subseteq X^{\pi(K_i)} \). To see that \( X^{\pi(K_i)} \subseteq \text{Im}(\pi(k_i)) \), notice that for every \( v \in X^{\pi(K_i)} \) we have that
\[
\pi(k_i)v = \int_{g \in K_i} \frac{\pi(g)v}{\mu(K_i)} d\mu(g) = \frac{1}{\mu(K_i)} \int_{g \in K_i} vd\mu(g) = v.
\]

The proof that $\pi(k_{i,j})$ is a projection on $X^{\pi(K_{i,j})}$ is similar and therefore is left for the reader. To see that $\pi(k_{i,j})\pi(k_i) = \pi(k_{i,j})$, we note that in the algebra $C_0(G)$, we have that for any $g \in K_{i,j}$, $k_{i,j}k_i = k_{i,j}, k_{i,j}k_j = k_{i,j}$ and therefore this equality passes to any representation $\pi$.

The above proposition shows that for any representation $\pi$ of $G$, we can define $\cos(\angle(\pi(k_i), \pi(k_j)))$ as in the previous section, i.e.,

$$\cos(\angle(\pi(k_i), \pi(k_j))) = \max\{||\pi(k_i)\pi(k_j) - \pi(k_{i,j})||, ||\pi(k_j)\pi(k_i) - \pi(k_{i,j})||\}.$$ 

Note that this yields that

$$\cos(\angle(\pi(k_i), \pi(k_j))) = \max\{||\pi(k_j - k_{i,j})||, ||\pi(k_j - k_{i,j})||\}.$$ 

In particular, this is true when $G = K_{i,j}$ and therefore $\cos(\angle(\pi(k_i), \pi(k_j)))$ is defined for any representation of $K_{i,j}$. For any $1 \leq i < j \leq N$ and any Banach space $X$, denote by $\lambda_{i,j}$ the left regular representation on $L^2(K_{i,j}; \mu)$ and recall that $\lambda_{i,j} \otimes id_X$ is an isometric representation on $L^2(K_{i,j}; X)$. Denote

$$\cos^X(\angle(k_i, k_j)) = \cos(\angle((\lambda_{i,j} \otimes id_X)(k_i), (\lambda_{i,j} \otimes id_X)(k_j))),$$

$$\cos^X_{\max} = \max_{1 \leq i < j \leq N} \cos^X(\angle(k_i, k_j)).$$

**Remark 4.2.** As noted above

$$\cos^X(\angle(k_i, k_j)) = \max\{||\lambda_{i,j} \otimes id_X(k_i - k_j)||, ||\lambda_{i,j} \otimes id_X(k_i - k_j)||\}.$$ 

Therefore $\cos^X(\angle(k_i, k_j))$ is given as a maximum of two operator of the general form $T \otimes id_X$, where $T$ is an operator on $T \in B(L^2(K_{i,j}; \mu)$ and therefore we can apply the results for vector value $L^2$ spaces to bound $\cos^X(\angle(k_i, k_j))$.

For a class $\mathcal{E}$ of Banach spaces denote

$$\cos_\max^\mathcal{E} = \sup_{X \in \mathcal{E}} \cos^X_{\max}.$$ 

With this notation, we can use the criterion stated in theorem 3.13 to get a criterion for robust Banach property (T):

**Theorem 4.3.** Let $G, K$ be as above and let $\mathcal{E}$ be a class of Banach spaces. Assume that there is $\varepsilon > 0$ such that

$$\cos_\max^\mathcal{E} \leq \frac{1 - \varepsilon}{8N - 11}.$$ 

Then there is $s_0 > 0$ such that the sequence $\left(\frac{h_1 + \ldots + h_N}{N}\right)$ converges to $p$ in $C_0(\mathcal{E}, K, s_0)$ as $n \to \infty$ and $\forall \pi \in \mathcal{F}(\mathcal{E}, K, s_0)$, $\pi(p)$ is a projection on $X^\pi$.

**Proof.** Assume without loss of generality that $\varepsilon < 1$.

**Step 1:** We’ll show that there is $s_1 > 0$ such that for every $\pi \in \mathcal{F}(\mathcal{E}, K, s_1)$, we have that

$$\cos_\max^\mathcal{E} = \max_{1 \leq i < j \leq N} \cos(\angle(\pi(k_i), \pi(k_j))) \leq \frac{1 - \varepsilon}{8N - 11} \tag{4}$$

By definition for every $s_1 > 0$ and every $\pi \in \mathcal{F}(\mathcal{E}, K, s_1)$ we have that

$$\forall 1 \leq i < j \leq N, \sup_{g \in K_{i,j}} ||\pi(g)|| \leq e^{s_1}.$$
Combining the above inequality with corollary 2.11 yields that for every \( f \in C_c(K_{i,j}) \) we have that
\[
\|\pi(f)\|_{B(X)} \leq e^{2s_1}\|(\lambda_{i,j} \otimes id_X)(f)\|_{B(L^2(K_{i,j}; X))}.
\]
By the definition of \( \cos(\mathcal{L}(\pi(k_i), \pi(k_j))) \) and \( \cos^X(\mathcal{L}(k_i, k_j)) \), this yields that for every \( 1 \leq i < j \leq N \) we have that
\[
\cos(\mathcal{L}(\pi(k_i), \pi(k_j))) \leq e^{2s_1} \cos^X(\mathcal{L}(k_i, k_j)) \leq e^{2s_1}\left(\frac{1-\varepsilon}{8N-11}\right).
\]
Therefore choosing
\[
s_1 = \ln\left(1 + \frac{\varepsilon(1-\varepsilon)}{2}\right),
\]
 yields inequality (4) as needed.

**Step 2:** We’ll show that there is \( s_2 > 0 \) such that for every \( \pi \in \mathcal{F}(E, K, s_2) \), we have that
\[
\max_{1 \leq i \leq N}\|\pi(k_i)\| < 1 + \frac{\varepsilon}{8N}.
\] (5)
By definition for every \( s_2 > 0 \) and every \( \pi \in \mathcal{F}(E, K, s_2) \), we have that
\[
\forall 1 \leq i \leq N, \sup_{g \in K_i}\|\pi(g)\| \leq e^{s_2}.
\]
By the definition of the functions \( k_i \), this yields that
\[
\forall 1 \leq i \leq N, \|\pi(k_i)\| \leq e^{s_2}.
\]
Therefore choosing
\[
s_2 = \ln\left(1 + \frac{1}{8N}\right),
\]
 yields inequality (5) as needed.

**Step 3:** To finish, choose \( s_0 = \min\{s_1, s_2\} > 0 \). Denote
\[
\gamma = \frac{1-\varepsilon}{8N-11}, \quad \beta = 1 + \frac{\varepsilon}{8N}.
\]
For every \( \pi \in \mathcal{F}(E, K, s_0) \), we have that by previous steps
\[
\cos^\pi_{\max} \leq \gamma,
\]
\[
\max_{1 \leq i \leq N}\|\pi(k_i)\| \leq \beta.
\]
Also note that
\[
\beta \leq 1 + \frac{\varepsilon}{8N} < 1 + \frac{\varepsilon}{8N - 12} \leq 1 + \frac{\varepsilon}{4N - 6} \leq 1 + \frac{1-(8N-11)\gamma}{N-2+(3N-4)\gamma}.
\]
This implies that the conditions of theorem 3.13 are fulfilled for the projections \( \pi(k_1), \ldots, \pi(k_N) \) and therefore there are \( 0 \leq r = r(\gamma, \beta) < 1, C = C(\gamma, \beta) \geq 0 \) and an operator which we denote as \( \pi(p) \) such that
\[
\left\|\pi(p) - \left(\frac{\pi(k_1) + \ldots + \pi(k_N)}{N}\right)^n\right\| \leq Cr^n,
\]
and such that \( \pi(p) \) is a projection on \( \bigcap_{1 \leq i \leq N} \text{Im}(\pi(k_i)) = \bigcap_{1 \leq i \leq N} X^{\pi(k_i)} = X^{\pi} \) (the last equality is due to the fact that \( K_1, ..., K_N \) generate \( G \)). Note that the constants that bound the rate of convergence \( r, C \) are independent of \( \pi \) and therefore \( \left( \frac{\pi(k_1) + ... + \pi(k_N)}{N} \right)^n \) converges in \( C_{\mathcal{F}(E, K, s_0)} \).

Next we shall address the following question: let \( \mathcal{E} \) be a class of Banach spaces such that \( \cos_{\max}^\mathcal{E} \leq c \), how can we expand \( \mathcal{E} \) to a larger class \( \mathcal{E}' \) such that \( \cos_{\max}^{\mathcal{E}'} \leq c' \) when \( c' \) is a function of \( c \).

First, we note that after finding a class \( \mathcal{E} \) with a bound on \( \cos_{\max}^\mathcal{E} \), we can assume it is stable under certain operations. To be specific, given a class of Banach spaces \( \mathcal{E} \), denote by \( \overline{\mathcal{E}} \) to be the smallest class of Banach spaces that contains \( \mathcal{E} \) and is stable under quotients, subspaces, \( l_2 \)-sums, ultraproducts and complex interpolation for any \( 0 < \theta < 1 \) of any compatible pair \( (X_0, X_1) \) such that \( X_0, X_1 \in \overline{\mathcal{E}} \). The next proposition states that a bound on \( \cos_{\max}^\mathcal{E} \) implies a bound on \( \cos_{\max}^{\mathcal{E}'} \):

**Proposition 4.4.** Let \( \mathcal{E} \) be a class of Banach spaces and let \( c \geq 0 \) be some constant. If \( \cos_{\max}^{\mathcal{E}} \leq c \), then \( \cos_{\max}^{\overline{\mathcal{E}}} \leq c \).

**Proof.** Combine the definition of \( \cos_{\max}^{\mathcal{E}} \) with lemma 2.3 and lemma 2.8.

Second, we observe that considering a neighbourhood of \( \mathcal{E} \) with respect to the Banach-Mazur distance changes \( \cos_{\max}^{\mathcal{E}'} \) by the radius of this neighbourhood. To be precise:

**Proposition 4.5.** Let \( \mathcal{E} \) be a class of Banach spaces and let \( c \geq 0, \delta \geq 0 \) be some constants. Let \( B_{BM}(\mathcal{E}, \delta) \) be the class of Banach spaces defined as:

\[
B_{BM}(\mathcal{E}, \delta) = \{ X : \exists Y \in \mathcal{E}, d_{BM}(X, Y) \leq 1 + \delta \}.
\]

If \( \cos_{\max}^{\mathcal{E}} \leq c \), then \( \cos_{\max}^{B_{BM}(\mathcal{E}, \delta)} \leq c(1 + \delta) \).

**Proof.** Combine the definition of \( \cos_{\max}^{\mathcal{E}} \) with lemma 2.7.

Third, we observe that taking \( \theta \)-interpolation of some \( X \in \mathcal{E} \) changes \( \cos_{\max}^{\mathcal{E}} \) as a function of \( \theta \):

**Proposition 4.6.** Let \( \mathcal{E} \) be a class of Banach spaces and let \( 2 > c \geq 0, 0 < \theta \leq 1 \) be some constants. Let \( Int(\mathcal{E}, \geq \theta) \) be the class of Banach spaces defined as:

\[
Int(\mathcal{E}, \geq \theta) = \{ X : \exists X_1 \in \mathcal{E} \text{ and } X_0 \text{ such that } X = [X_0, X_1]_{\theta'} \text{ for some } \theta' \geq \theta \}.
\]

If \( \cos_{\max}^{\mathcal{E}} \leq c \), then \( \cos_{\max}^{Int(\mathcal{E}, \geq \theta)} \leq 2 \left( \frac{c}{\theta} \right)^{\theta} \).

**Proof.** Note that for any Banach space \( X \), we have for every \( 1 \leq i < j \leq N \) that

\[
\| (\lambda_{i,j} \otimes id_X)(k_i) \| = \left\| \int_{g \in K_i} \frac{(\lambda_{i,j} \otimes id_X)(g)}{\mu(K_i)} v \right\| \mu(g) \right\| \leq \int_{g \in K_i} \left\| (\lambda_{i,j} \otimes id_X)(g) \right\| v \right\| d\mu(g) = \int_{g \in K_i} \frac{\| v \|}{\mu(K_i)} d\mu(g) \right\| \| v \| = \| v \|.
\]

Therefore

\[
\| (\lambda_{i,j} \otimes id_X)(k_i) \| \leq 1
\]
and similarly
\[
\| (\lambda_{i,j} \otimes id_X)(k_j) \| \leq 1, \| (\lambda_{i,j} \otimes id_X)(k_{i,j}) \| \leq 1.
\]
This yields that for every \( 1 \leq i < j \leq N \) we have that
\[
\| \lambda_{i,j}(k_{i,j} - k_{i,j}) \otimes id_X \| \leq 2, \| \lambda_{i,j}(k_{i} - k_{i,j}) \otimes id_X \| \leq 2.
\]
Let \( X_1 \in \mathcal{E} \) and \( X_0 \) be a Banach space such that \((X_0, X_1)\) are a compatible pair. By lemma 2.8 we have that for every \( \theta \leq \theta' \leq 1 \) and every \( 1 \leq i < j \leq N \) that
\[
\| \lambda_{i,j}(k_{i,j} - k_{i,j}) \otimes id_{[X_0, X_1]_{\theta'}} \| \leq 2^{1-\theta'} \theta' \| \lambda_{i,j}(k_{i} - k_{i,j}) \otimes id_{X_0} \| \| \leq 2^{1-\theta'} \theta' = 2 \left( \frac{c}{2} \right)^{\theta'}.
\]
Similarly,
\[
\| \lambda_{i,j}(k_{j}k_{i} - k_{i,j}) \otimes id_{[X_0, X_1]_{\theta'}} \| \leq 2 \left( \frac{c}{2} \right)^{\theta},
\]
and we are done by the definition of \( \cos_{\text{max}}^{[X_0, X_1]_{\theta'}} \).

Combining all the above propositions yields the following:

**Corollary 4.7.** Let \( G \) be as above and let \( \mathcal{E} \) be a class of Banach spaces. Assume that there is a constant \( c \geq 0 \) such that
\[
\cos_{\text{max}}^{\mathcal{E}} \leq c < \frac{1}{8N-11}.
\]
Let \( c' \) be a constant such that \( c \leq c' < \frac{1}{8N-11} \). Denote \( \delta = \frac{c'}{c} - 1, \theta = \frac{\ln(2)-\ln(c')}{\ln(2)-\ln(c)} \) and
\[
\mathcal{E}' = \mathcal{B}_{BM}(\mathcal{E}, \delta) \cup \text{Int}(\mathcal{E}, \geq \theta).
\]
Then
\[
\cos_{\text{max}}^{\mathcal{E}'} \leq c' < \frac{1}{8N-11},
\]
and there is \( s_0 > 0 \) such that the sequence \( (\frac{k_1+...+k_N}{N})^n \) converges to \( p \) in \( C_F(\mathcal{E}', K, s_0) \) as \( n \to \infty \) and \( \forall \pi \in F(\mathcal{E}', K, s_0), \pi(p) \) is a projection on \( X^\pi \).

**Proof.** Combing propositions 4.4, 4.5, 4.6 and theorem 4.3.

The above corollary gives us a way to get a class of Banach spaces \( \mathcal{E}' \) for which \( G \) has robust Banach property (T) providing that we have a class of of Banach spaces \( \mathcal{E} \) such that \( \cos_{\text{max}}^{\mathcal{E}} \) is small. We are left with the question of how to produce such a class \( \mathcal{E} \). Below we shall describe two methods to do so based on our knowledge of unitary representations of \( K_{i,j} \)'s in Hilbert spaces. These methods can be summarized as follows:

- **Method 1:** take \( \mathcal{E} = \mathcal{H} \) as the class of all Hilbert spaces. In this case \( \cos_{\text{max}}^{\mathcal{H}} \) can be bounded via analysing the classical Friedrichs angles between fixed subspaces in Hilbert spaces. This in turn can be done via analysing angles in irreducible representations of \( K_{i,j} \).
Theorem 4.9. Let $G$ be as above. Assume that there is some constant $c < \frac{1}{8N^{-11}}$ such that for any $1 \leq i < j \leq N$ and any unitary representation of $K_{i,j}$ on a Hilbert space $H$ we have that $\cos(\angle(H^{\pi(K_i)}, H^{\pi(K_j)})) \leq c$. Let $c'$ be a constant such that $c \leq c' < \frac{1}{8N^{-11}}$. Denote $\delta = \frac{c'}{c} - 1$, $\theta = \frac{\ln(2) - \ln(c')}{\ln(2) - \ln(c)}$ and 

$E = B_{BM}(\mathcal{H}, \delta) \cup \text{Int}(\mathcal{H}, \geq \theta),

where $\text{Int}(\mathcal{H}, \geq \theta)$ is the class of all the $\theta'$-Hilbertian Banach spaces with $\theta' \geq \theta$ and $B_{BM}(\mathcal{H}, \delta)$ is the class of all the spaces $X$ isomorphic to some Hilbert space $H = H(X)$ such that $d_{BM}(X, H) \leq 1 + \delta$. Then there is $s_0 > 0$ such that the sequence $\left(\frac{k_1 + \ldots + k_n}{N}\right)^n$ converges to $p$ in $C_F(\mathcal{E}, K, s_0)$ as $n \to \infty$ and $\forall \pi \in F(\mathcal{E}, K, s_0), \pi(p)$ is a projection on $X^\perp$.

This theorem has a nice corollary regarding fixed point properties:

Corollary 4.10. Let $G$ be as above. Assume that there is some constant $c < \frac{1}{8N^{-11}}$ such that for any $1 \leq i < j \leq N$ and any unitary representation of $K_{i,j}$ on a Hilbert space $H$ we have that $\cos(\angle(H^{\pi(K_i)}, H^{\pi(K_j)})) \leq c$. Denote $\delta = \frac{1}{c(8N^{-11})} - 1$, $\theta = \frac{\ln(2) + \ln(8N^{-11})}{\ln(2) - \ln(c)}$. If $X$ is a Banach space of one of the following types:

1. $X$ is isomorphic to a Hilbert space $H$ with $d_{BM}(X, H) < 1 + \delta$.
2. $X$ is $\theta'$-Hilbertian with $\theta' > \theta$.

Then $G$ has property $F_X$, i.e., every continuous affine isometric action of $G$ on $X$ has a fixed point.

Next, we shall give a detailed account on each method.

4.1 Robust Banach property (T) via bounding $\cos^H_{\max}$

Let $\mathcal{H}$ be the class of all Hilbert spaces. Bounding $\cos^H_{\max}$ is achieved by classical Friedrichs angles in Hilbert spaces, by the following observation:

Observation 4.8. Notice that for any $H \in \mathcal{H}$, we have that for any $1 \leq i < j \leq N$, $L^2(K_{i,j}; H)$ is a Hilbert space and $\lambda_{i,j} \otimes \text{id}_H$ is a unitary representation on this space. Also note that since $\lambda_{i,j}(k_i) \otimes \text{id}_H$, $\lambda_{i,j}(k_i) \otimes \text{id}_H$, $\lambda_{i,j}(k_{i,j}) \otimes \text{id}_H$ are all projections of norm 1 and therefore they are orthogonal projections.

Therefore for any $1 \leq i < j \leq N$, bounding

$$
\cos^H(\angle(k_i, k_j)) = \sup_{H \in \mathcal{H}} \cos^H(\angle(k_i, k_j))
$$

boils down to bounding the (classical) Friedrichs angle $\cos(\angle(H^{\pi(K_i)}, H^{\pi(K_j)}))$ for any unitary representation $\pi$ on some Hilbert space $H$.

Combining the above observation with corollary 4.10 gives the following theorem:

Theorem 4.9. Let $G$ be as above. Assume that there is some constant $c < \frac{1}{8N^{-11}}$ such that for any $1 \leq i < j \leq N$ and any unitary representation of $K_{i,j}$ on a Hilbert space $H$ we have that $\cos(\angle(H^{\pi(K_i)}, H^{\pi(K_j)})) \leq c$. Let $c'$ be a constant such that $c \leq c' < \frac{1}{8N^{-11}}$. Denote $\delta = \frac{c'}{c} - 1$, $\theta = \frac{\ln(2) - \ln(c')}{\ln(2) - \ln(c)}$ and 

$$
E = B_{BM}(\mathcal{H}, \delta) \cup \text{Int}(\mathcal{H}, \geq \theta),
$$

where $\text{Int}(\mathcal{H}, \geq \theta)$ is the class of all the $\theta'$-Hilbertian Banach spaces with $\theta' \geq \theta$ and $B_{BM}(\mathcal{H}, \delta)$ is the class of all the spaces $X$ isomorphic to some Hilbert space $H = H(X)$ such that $d_{BM}(X, H) \leq 1 + \delta$. Then there is $s_0 > 0$ such that the sequence $\left(\frac{k_1 + \ldots + k_n}{N}\right)^n$ converges to $p$ in $C_F(\mathcal{E}, K, s_0)$ as $n \to \infty$ and $\forall \pi \in F(\mathcal{E}, K, s_0), \pi(p)$ is a projection on $X^\perp$.

This theorem has a nice corollary regarding fixed point properties:

Corollary 4.10. Let $G$ be as above. Assume that there is some constant $c < \frac{1}{8N^{-11}}$ such that for any $1 \leq i < j \leq N$ and any unitary representation of $K_{i,j}$ on a Hilbert space $H$ we have that $\cos(\angle(H^{\pi(K_i)}, H^{\pi(K_j)})) \leq c$. Denote $\delta = \frac{1}{c(8N^{-11})} - 1$, $\theta = \frac{\ln(2) + \ln(8N^{-11})}{\ln(2) - \ln(c)}$. If $X$ is a Banach space of one of the following types:

1. $X$ is isomorphic to a Hilbert space $H$ with $d_{BM}(X, H) < 1 + \delta$.
2. $X$ is $\theta'$-Hilbertian with $\theta' > \theta$.

Then $G$ has property $F_X$, i.e., every continuous affine isometric action of $G$ on $X$ has a fixed point.
Proof. Note that in the above theorem \( \mathcal{E} \) contain every Hilbert space and in particular \( \mathbb{C} \in \mathcal{E} \). Also note that \( \mathcal{E} \) is closed under \( l_2 \) sums. Therefore we get the corollary by combining the above theorem with proposition 1.9.

Last, we’ll make two remark regarding bounding \( \cos(\mathcal{L}(H^{\pi(K_i)}, H^{\pi(K_j)})) \) for some fixed \( 1 \leq i < j \leq N \).

Remark 4.11. Observe that due to Peter-Weyl theorem, if for any irreducible unitary representation \( \pi \) we have that
\[
\cos(\mathcal{L}(H^{\pi(K_i)}, H^{\pi(K_j)})) \leq c,
\]
then for any unitary representation \( \pi \) we have
\[
\cos(\mathcal{L}(H^{\pi(K_i)}, H^{\pi(K_j)})) \leq c.
\]
Therefore, it is enough to bound the angle for irreducible representations.

4.2 Robust Banach property (T) via Schatten norms

We start by recalling the following definitions: for a Hilbert space \( H \) and a bounded operator \( T \in B(H) \) and a constant \( r \in [1, \infty] \), the \( r \)-th Schatten norm is defined as
\[
\|T\|_{S^r} = \left( \sum_{i=1}^{\infty} (s_i(T))^r \right)^{\frac{1}{r}},
\]
where \( s_1(T) \geq s_2(T) \geq ... \) are the eigenvalues of \( \sqrt{T^*T} \). An operator \( T \) is said to be of Schatten class \( r \) if \( \|T\|_{S^r} < \infty \).

In [5] the following proposition is proved:

Proposition 4.12. [5][Proposition 3.3] Let \( 1 < p_1 < 2 < p_2 < \infty \) and let \( r \in [2, \infty) \) such that \( \frac{1}{p_1} - \frac{1}{p_2} < \frac{1}{r} \). There is a constant \( M = M(p_1, p_2, r) \geq 0 \) such that the following holds. If \( X \) is a Banach space of type \( p_1 \) and cotype \( p_2 \), \((\Omega, \mu)\) is a measure space and \( T \in B(L^2(\Omega, \mu)) \) of Schatten class \( r \), then
\[
\|T \otimes id_X\|_{B(L^2(\Omega, X))} \leq MT_{p_1}(X)C_{p_2}(X)\|T\|_{S^r}.
\]

Remark 4.13. The constant \( M \) in the above proposition can be computed explicitly. To be precise
\[
M = \sum_{i=1}^{\infty} 2^{\frac{i}{p_1}} \left( \frac{1}{p_1} - \frac{1}{p_2} - \frac{1}{r} \right)^i.
\]

Using the above proposition gives us a way to bound \( \cos^{\mathcal{E}}_{\text{max}} \) using the Schatten norm of \( \lambda(k_i k_j - k_{i,j}) \), \( \lambda(k_i k_j - k_{i,j}) \) in \( B(L^2(K_{i,j}, \mu)) \) for certain classes of Banach spaces. We shall need the following notation: for \( 1 < p_1 \leq 2 < p_2 < \infty \), \( T_{p_1} \geq 1, C_{p_2} \geq 1 \) constants denote \( \mathcal{T}(p_1, p_2, T_{p_1}, C_{p_2}) \) to be the class of Banach spaces of type \( p_1 \) and cotype \( p_2 \) such that for every \( X \in \mathcal{T}(p_1, p_2, T_{p_1}, C_{p_2}) \) we have that \( T_{p_1}(X) \leq T_{p_1}, C_{p_2}(X) \leq C_{p_2} \).
Proposition 4.14. Let $G$ as above. Let $1 < p_1 \leq 2 \leq p_2 < \infty$, $T_{p_1} \geq 1$, $C_{p_2} \geq 1$ constants and let $r \in [2, \infty)$ such that $\frac{1}{p_1} - \frac{1}{p_2} < \frac{1}{r}$. Assume that for every $1 \leq i < j \leq N$ we have that $\lambda(k_i k_j - k_{i,j}) \in B(L^2(K_{i,j}, \mu))$ are of Schatten class $r$. Denote

$$\cos_{\max}^{r} = \max\{\|\lambda(k_i k_j - k_{i,j})\|_{S^r} : 1 \leq i < j \leq N\}.$$

Then for $M = M(p_1, p_2, r)$ as in the proposition above we have that

$$\cos_{\max}^{T(p_1, p_2, T_{p_1}, C_{p_2})} \leq MT_{p_1} C_{p_2} \cos_{\max}^{r}.$$

Proof. Note that for every $1 \leq i < j \leq N$, $\lambda(k_i k_j - k_{i,j})$ is the adjoint operator of $\lambda(k_i k_j - k_{i,j})$, therefore $\|\lambda(k_i k_j - k_{i,j})\|_{S^r} = \|\lambda(k_i k_j - k_{i,j})\|_{S^r}$. Combine proposition 4.12 with the definition of $\cos_{\max}^{r}$.

Combining the above proposition with theorem 4.13 gives the following result:

Theorem 4.15. Let $G$ as above and let $1 < p_1 \leq 2 \leq p_2 < \infty$ and $r \in [2, \infty)$ such that $\frac{1}{p_1} - \frac{1}{p_2} < \frac{1}{r}$. Denote

$$M = \sum_{i=1}^{\infty} 2^{\frac{1}{r} - \left(\frac{1}{p_1} - \frac{1}{p_2}\right) i}.$$

Assume that there is a constant $c < \frac{1}{8N+1}$ such that $M \cos_{\max}^{r} \leq c$, and let $c'$ be a constant such that $c \leq c' < \frac{1}{8N-1}$. Denote

$$\mathcal{E} = \bigcup_{T_{p_1}, C_{p_2} \in [1, \infty), cT_{p_1} C_{p_2} \leq c'} \text{Int} \left( \left\{ T(p_1, p_2, T_{p_1}, C_{p_2}) \geq \frac{\ln(2) - \ln(c')}{\ln(2) - \ln(c T_{p_1} C_{p_2})} \right\} \right).$$

Then there is $s_0 > 0$ such that the sequence $(\frac{\ln(2) + \ln(k)}{N})^n$ converges to $p$ in $C_{\mathcal{E}, K, s_0}$ as $n \to \infty$ and $\forall \pi \in \mathcal{F}(\mathcal{E}, K, s_0)$, $\pi(p)$ is a projection on $X^\pi$.

Proof. Combine the above proposition 4.14, theorem 4.13 and propositions 4.14 and 4.16.

Combining the above theorem with proposition 4.12 yields the following corollary:

Corollary 4.16. Assume the conditions of the above theorem hold and let $\mathcal{E}$ be as in the above theorem, then for any $X \in \mathcal{E}$, $G$ has property $F_X$, i.e., for every continuous affine isometric action of $G$ on $X$ has a fixed point.

Remark 4.17. One should note that due to Peter-Weyl theorem finding the eigenvalues of $\lambda((k_i k_j - k_{i,j})^*(k_i k_j - k_{i,j}))$ in $L^2(K_{i,j}, \mu)$ in order to calculate that Schatten norm boils down to finding those eigenvalues in every irreducible representation of $K_{i,j}$.
4.3 Angle between groups and Schatten norm in Hilbert spaces using combinatorial data

In the two methods described above we deduced robust Banach property (T) using knowledge of the spectrum of \( \lambda((k_ik_j - k_{ij})^* (k_ik_j - k_{ij})) \) for each \( 1 \leq i < j \leq N \). One way to obtain such knowledge is analysing the unitary representations of \( K_{i,j} \). Below we present a more combinatorial way for analysing the spectrum of \( \lambda((k_ik_j - k_{ij})^* (k_ik_j - k_{ij})) \) by analysing the spectrum of the Laplacian of a graph constructed using \( K_{i,j}, K_i, K_j \).

**Definition 4.18.** Let \( K_{1,2} \) be a compact group and let \( K_1, K_2 \) be finite index subgroups of \( K_{1,2} \) such that \( K_{1,2} = \langle K_1, K_2 \rangle \). Define a bipartite graph \( \mathcal{G} = (V, E) \) as follows:

- For \( i = 1, 2 \), \( V_i \) is the set for right cosets:
  \[ V_i = \{ K_ig : g \in K_{1,2} \}, \]
  and \( V = V_1 \cup V_2 \).
- \( K_1g \) and \( K_2g' \) are connected by an edge if \( K_1g \cap K_2g' \neq \emptyset \). In other words, if \( l_i = [K_i : K_1 \cap K_2] \) and \( h_1^i, ..., h_{l_i}^i \) are representatives in \( K_i \) such that \( K_i = \bigcup (K_1 \cap K_2) h_j^i \). Then \( K_1g \) is connected to \( K_2h_1^i g, ..., K_2h_{l_i}^i g \) and \( K_2g \) is connected to \( K_1h_1^i g, ..., K_1h_{l_i}^i g \).

Notice that \( \mathcal{G} \) above is semi-regular, since for every \( i = 1, 2 \) and every \( v \in V_i \), \( d(v) = l_i \).

The next lemma connects the eigenvalues of the Laplacian on \( \mathcal{G} \) to the eigenvalues of \( \lambda((k_ik_2 - k_{1,2})^* (k_1k_2 - k_{1,2})) \). A weaker form of this connection already appeared in [9], where the spectral gap of the Laplacian on \( \mathcal{G} \) was used to bound the norm of \( \lambda((k_ik_2 - k_{1,2})) \).

**Lemma 4.19.** Let \( K_{1,2}, K_1, K_2 \) be as above and let \( \lambda \) be the left regular representation on \( L^2(K_{1,2}, \mu) \), where \( \mu \) is the Haar measure of \( K_{1,2} \). Let \( \Delta \) be the graph Laplacian of \( \mathcal{G} \) defined above. Let \( 0 = \eta_1 < \eta_2 \leq \eta_3 \leq ... \leq \eta_{|V_1| - 1} < \eta_{|V_2|} = 2 \) be the eigenvalues (including multiplicities) of \( \Delta \). Then for \( k_1, k_2, k_{1,2} \in C_\nu(K_{1,2}) \) defined as in the previous section we have that the non-trivial eigenvalues of \( \lambda((k_ik_2 - k_{1,2})^* (k_1k_2 - k_{1,2})) \) are the zero values of

\[
(1 - \eta_1)^2, ..., (1 - \eta_{\min(|V_1|,|V_2|)})^2,
\]
accounting for multiplicities.

**Proof.** Assume without loss of generality that \([K_{1,2} : K_2] \leq [K_{1,2} : K_1] \), i.e., assume that \([V_2] \leq |V_1| \) (if \([K_{1,2} : K_2] > [K_{1,2} : K_1] \) we repeat the argument below for \( k_2k_1 - k_{1,2} \) instead of \( k_1k_2 - k_{1,2} \)).

We start by exploring operators on \( L^2(V) \) that we will later connect to \( \lambda(k_1) \) and \( \lambda(k_2) \). Abusing notation, we define for \( i = 1, 2 \) the operator \( \chi_{V_i} \in L^2(V) \) as multiplying by the indicator function \( \chi_{V_i} \), i.e., \( \chi_{V_i}\psi(v) = \chi_{V_i}(v)\psi(v) \). Define \( M_1, M_2 \) acting on \( L^2(V) \) as \( M_1 = \chi_{V_1}(I - \Delta)\chi_{V_2} \) and \( M_2 = \chi_{V_2}(I - \Delta)\chi_{V_1} \). In other words:

\[
M_1\psi(v) = \begin{cases} 
0 & \text{if } v \in V_1, \\
\frac{1}{d(v)} \sum_{u \in V_2, (v,u) \in E} \psi(u) & \text{if } v \in V_2
\end{cases},
\]

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$$M_2\psi(v) = \begin{cases} \psi(v), & v \in V_1 \\ \frac{1}{d(v)} \sum_{u \in V_1, [v,u] \in E} \psi(u), & v \in V_2. \end{cases}$$

By the definition of $M_1$ and $M_2$, we get that $M_2M_1 = \chi_{V_1}(I-\Delta)\chi_{V_1}(I-\Delta)\chi_{V_2}$. For every function $\psi \in L^2(V)$, if $\psi$ is supported on $V_1$ then $M_2M_1\psi = 0$. Therefore $M_2M_1$ has at most $|V_2|$ non zero eigenvalues (accounting for multiplicities). For an eigenfunction $\psi$ of $\Delta$ with $\lambda\psi = \eta\psi$ we have that $M_2M_1(\chi_{V_2}\psi) = (1-\eta)^2(\chi_{V_2}\psi)$. Let $\psi_1 = \chi_{V_1}\psi_2, \ldots, \psi_{|V_2|}$ be the eigenfunctions of $0 = \eta_1, \ldots, \eta_{|V_2|}$. By propositions 2.11 and 2.13, the space of functions supported on $V_2$ is spanned by $\chi_{V_1}\psi_1, \chi_{V_1}\psi_2, \ldots, \chi_{V_1}\psi_{|V_2|}$.

Next, we connect $M_1$ and $M_2$ to the operators $\lambda(k_1)$ and $\lambda(k_2)$ acting on $L^2(K_{1,2,\mu})$. For $i = 1, 2$ we denote

$$L^2(V)_i = \{ f \in L^2(V) : supp(\psi) \subseteq V_i \},$$

$$L^2(K_{1,2,\mu})_i = \{ f \in L^2(K_{1,2,\mu}) : \forall g \in K_{1,2,\mu}, g' \in K_{1,2,\mu}, f(g') = f(g') \}.$$ By the definition of the vertex sets $V_i$ and $V_2$, there are natural identifications $F_i : L^2(K_{1,2,\mu})_i \rightarrow L^2(V)_i$. We will show that

$$\forall f \in L^2(K_{1,2,\mu})_1, \lambda(k_2)f = F_2^{-1}M_2F_1 f.$$  

First, we note that for every $f \in L^2(K_{1,2,\mu})$, we have that $\lambda(k_2)f \in L^2(K_{1,2,\mu})_2$. Fix $g, g'$ such that $K_2g = K_2g'$. There is $h' \in K_2$ such that $g = h'g'$, therefore

$$\int_{h \in K_2} f(h^{-1}g)d\mu(h) = \int_{h' \in K_2} f(h'^{-1}g')d\mu(h') = \int_{h \in K_2} f(h^{-1}g')d\mu(h) = (\lambda(k_2)f)(g').$$

This yields that for every $f \in L^2(K_{1,2,\mu})$, $\lambda(k_2)f$ is fixed on right cosets of $K_2$ as we claimed.

Next, denote $l_2 = |K_2 : K_1 \cap K_2|$ and fix $h_1^2, \ldots, h_{l_2}^2$, such that $K_2 = \bigcup_{j=1}^{l_2} (K_1 \cap K_2)h_j^2$. Let $f \in L^2(K_{1,2,\mu})_1$, then for every $g \in K_{1,2}$ the following holds:

$$\int_{h \in K_2} f(h^{-1}g)d\mu(h) = \int_{h \in K_2} \frac{1}{\mu(K_2)} f(h^{-1}g)d\mu(h) = \int_{h' \in K_2} \frac{1}{\mu(K_2)} f(h'^{-1}g')d\mu(h') = \sum_{j=1}^{l_2} \int_{h \in K_1 \cap K_2} \frac{1}{\mu(K_2)} f(h^{-1}h_j^2g)d\mu(h).$$

Note that $f \in L^2(K_{1,2,\mu})_1$ and therefore for every $h \in K_{1,2}$ and for every $j$, $f(h^{-1}h_j^2g) = f(h_j^2g)$. Therefore

$$\sum_{j=1}^{l_2} \int_{h \in K_1 \cap K_2} \frac{1}{\mu(K_2)} f(h^{-1}h_j^2g)d\mu(h) = \sum_{j=1}^{l_2} \mu(K_1 \cap K_2) \mu(K_2) f(h_j^2g) = \sum_{j=1}^{l_2} \frac{1}{l_2} f(h_j^2g) = (F_2^{-1}M_2F_1 f)(g),$$
as needed. Similarly

$$\forall f \in L^2(K_{1,2}, \mu)_2, \lambda(k_1)f = F_1^{-1}M_1F_2f.$$ 

Therefore,

$$\forall f \in L^2(K_{1,2}, \mu)_2, \lambda(k_2)\lambda(k_1)f = F_2^{-1}M_2M_1F_2f.$$ 

To finish, we notice that

$$\lambda((k_1k_2 - k_1, 2)^* (k_1k_2 - k_1, 2)) = \lambda(k_2k_1)\lambda(k_2 - k_1, 2).$$

This implies that the non-trivial eigenvalues of $\lambda((k_1k_2 - k_1, 2)^* (k_1k_2 - k_1, 2))$ are in $(\text{Ker}(\lambda(k_2 - k_1, 2)))^\perp = \text{Im}(\lambda(k_2 - k_1, 2)) = \text{Im}(\lambda(k_2)) \cap \text{Im}(I - \lambda(k_2, 1, 2))$. By the definition of $k_1, 2$ and $k_2$ we have that $\text{Im}(\lambda(k_2)) \cap \text{Im}(I - \lambda(k_1, 2))$ is

$$\text{Im}(\lambda(k_2)) \cap \text{Im}(I - \lambda(k_1, 2)) = \{ f \in L^2(K_{1,2}, \mu)_2 : \int_{K_{1,2}} f = 0 \}.$$ 

Therefore $\text{Im}(\lambda(k_2)) \cap \text{Im}(I - \lambda(k_1, 2))$ is identified by $F_2$ with the space

$$\{ \psi \in L^2(V)_2 : \sum_{v \in V_2} \psi(v) = 0 \} = \text{span}\{ \chi_{V_2}\psi_2, ..., \chi_{V_2}\psi_{|V_2|} \},$$

where the last equality is due to proposition 2.13. Therefore the non-trivial spectrum of $\lambda((k_1k_2 - k_1, 2)^* (k_1k_2 - k_1, 2))$ is the same as the non trivial spectrum of $M_2M_1$ on $\text{span}\{ \chi_{V_2}\psi_2, ..., \chi_{V_2}\psi_{|V_2|} \}$ as needed.

\[ \square \]

**Corollary 4.20.** Let $K_{1,2}, K_1, K_2$ be as above and let $\lambda$ be the left regular representation on $L^2(K_{1,2}, \mu)$, where $\mu$ is the Haar measure of $K_{1,2}$. Let $\Delta$ be the graph Laplacian of $\mathcal{G}$ defined above. Let $0 = \eta_1 < \eta_2 \leq \eta_3 \leq ... < \eta_{|V_1|-1} < \eta_{|V_1|} = 2$ be the eigenvalues (including multiplicities) of $\Delta$. Then for $k_1, k_2, k_1, 2 \in C_c(K_{1,2})$ defined as in the previous section we have that

$$\| \lambda(k_1k_2 - k_1, 2) \| \leq 1 - \eta_2,$$

and for every $1 \leq r < \infty$

$$\| \lambda(k_1k_2 - k_1, 2) \|_s \leq ((1 - \eta_2)^r + (1 - \eta_3)^r + ... + (1 - \eta_{\min{|V_{1,1}|,|V_{2,1}|}})^r)^{\frac{1}{r}} \leq (1 - \eta_2) \min\{|V_{1,1}|^{\frac{1}{r}}, |V_{2,1}|^{\frac{1}{r}}\}.$$ 

## 5 Examples and applications

### 5.1 Groups acting on simplicial complexes

We'll start by recalling some basic definitions regarding simplicial complexes. Let $\Sigma$ be a purely $n$-dimensional simplicial complex (i.e., every simplex is a face of an $n$-simplex). $\Sigma$ is called gallery connected if for every two $n$-dimensional simplices $\sigma, \sigma'$, there is a finite sequence of $n$-dimensional simplices $\sigma = \sigma_1, \sigma_2, ..., \sigma_m = \sigma'$ such that for every $i$, $\sigma_i$ and $\sigma_{i+1}$ share an $(n - 1)$-dimensional face.

Recall that for every simplex $\sigma$ in $\Sigma$ one can define a new simplicial complex $\text{link}(\sigma)$ as the sub-complex of $\Sigma$ that contains all the simplices $\sigma'$ that are
disjoint from $\sigma$ such that there is an $n$-dimensional simplex that contains both $\sigma$ and $\sigma'$. Observe that the dimension of $\text{link}(\sigma)$ is always $n - \text{dim}(\sigma) - 1$. In particular, the 1-dimensional links of $\Sigma$ are the links of simplices of dimension $n - 2$.

Assume that $\Sigma$ is a pure $n$-dimensional simplicial complex that is gallery connected such that the 1-dimensional links of $\Sigma$ are finite connected graphs. Assume further that $G$ is a group acting on $\Sigma$ simplicially such that the fundamental domain $\Sigma/G$ is a single $n$-dimensional simplex and such that the stabilizers of all the $(n - 2)$-simplices of $\Sigma$ are compact. Fix $\{v_1, \ldots, v_{n+1}\} \subseteq \Sigma(n)$ and denote $\hat{K}_{i,j}$ to be the stabilizer of $\{v_1, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_{n+1}\}$. The assumption that $\Sigma$ is gallery connected implies that $K_1, \ldots, K_{n+1}$ generate $G$. Also, the assumptions on the 1-dimensional links of $\Sigma$ imply that $K_1, K_2, \ldots, K_n, K_{n+1}$ are compact groups, $K_i, K_j$ are finite index subgroups of $K_{i,j}$ and $K_{i,j} = (K_i, K_j)$. Further more, the 1-dimensional link of $\{v_1, \ldots, \hat{v}_i, \ldots, v_j, \ldots, v_{n+1}\}$ can be identified with the graph $\mathcal{G}$ defined by $K_i, K_j$ and $K_{i,j}$ (see definition [11.8 above]).

Using corollary [12.20 above, we can state the following theorem generalizing theorem [1.10 stated in the introduction:

**Theorem 5.1.** Let $\Sigma$ be a pure $n$-dimensional simplicial complex that is gallery connected. Let $G$ be a group acting simplicially on $\Sigma$ such that the action is cocompact and the fundamental domain $\Sigma/G$ is a single $n$-dimensional simplex $\{v_1, \ldots, v_{n+1}\}$ and such that the stabilizer of every $(n - 2)$-dimensional simplex of $\Sigma$ is a compact subgroup of $G$. Assume that for every $v_i, v_j$, $\{v_1, \ldots, v_i, \ldots, v_j, \ldots, v_{n+1}\}$, the link of $v_{i,j}$ is a finite connected (bipartite) graph $(V_{i,j}, E_{i,j})$, where $V_{i,j}, V_{i,j}$ are the two sides of this graph. Denote

$$L = \max_{1 \leq i < j \leq n+1} \min\{|V_{i,j}|, |V_{i,j}|\}.$$

Assume further that there is a constant $\eta > 1 - \frac{1}{8(n+1)-11}$ such that for every $1 \leq i < j \leq n+1$, the smallest positive eigenvalue of the Laplacian on the link of $v_{i,j}$ is $\geq \eta$. Fix a constant $c'$ such that $1 - \eta \leq c' < \frac{1}{\sqrt{1 + (n+1)-11}}$. Define the following Banach classes:

1. The class $E_1 = \overline{B_{BM}(\mathcal{H}, \delta)} \cup \text{Int}(\mathcal{H}, \geq \theta)$, where $\delta = \frac{c'}{1-\eta} - 1, \theta = \frac{\ln(2) - \ln(c')}{\ln(2) - \ln(1-\eta)}$. Define $\mathcal{H}$ is the class of all Hilbert spaces.

2. For $(p_1, p_2, r) \in (1, 2] \times [2, \infty) \times [2, \infty)$ such that $\frac{1}{p_1} - \frac{1}{p_2} < \frac{1}{r}$, the class $\mathcal{E}(p_1, p_2, r)$ defined as follows:

$$\bigcup_{T_{p_1, p_2, r} \in [1, \infty), c(p_1, p_2, r, q)T_{p_1, p_2, r} \leq c'} \text{Int} \left( T(p_1, p_2, T_{p_1, p_2, r}, C_{p_2}) \geq \frac{\ln(2) - \ln(c')}{{\ln(2) - \ln(c(p_1, p_2, r, q)T_{p_1, p_2, r})}} \right),$$

where $c(p_1, p_2, r, q) = L^\frac{1}{r} (1 - \eta) \sum_i 2^i r^{-(1/p_1 + 1/p_2 - 1/q_i)}$. Note that $\mathcal{E}(p_1, p_2, r)$ is non empty only if $c(p_1, p_2, r, q) \leq c'$.

3. The class $A$ defined as follows: denote

$$A(q) = \left\{ (q_1, q_2, r) \in (1, 2] \times [2, \infty) \times [2, \infty) : \frac{1}{q_1} - \frac{1}{q_2} < \frac{1}{r}, c(q_1, q_2, r) \leq c' \right\}.$$

Then $\mathcal{E} = E_1 \cup \bigcup_{(q_1, q_2, r) \in A(q)} \mathcal{E}(q_1, q_2, r)$. 

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Then $G$ has robust Banach property (T) with respect to $E$ and has property $F_X$ for every $X \in E$. In particular, $G$ has property $F_{L^p}$ for any $p < 2 \frac{\ln(2) - \ln(1 - q)}{\ln(2) + \ln(8(n+1) - 11)}$.

Proof. As noted above, by the assumption of the theorem, $G$ is generated by $K_1, \ldots, K_{n+1}$ and for every $1 \leq i < j \leq n + 1$ the link of $\{v_1, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_{n+1}\}$ can be identified with the graph $\mathcal{G}$ generated by $K_1, K_2, K_{i,j}$ as in definition 4.12. Therefore the proof of the theorem follows by applying corollary 4.19 combined with theorems 4.10, 4.15.

Remark 5.2. The reader should note the asymptotic behaviour of the above theorem as $\eta \to 1$. For instance, as $\eta \to 1$, $G$ has property $F_{L^p}$ with $p \to \infty$.

5.2 Groups where the $K_{i,j}$’s are Heisenberg or Abelian

Let $K = K_{1,2}$ be a finite group generated by two subgroups $K_1, K_2$ and let $k_1, k_2, k_{1,2}$ be the functions in $C(K)$ defined in the beginning of the previous section. As we saw in theorems 4.13, 4.15 and corollaries 4.18, 4.19 it is useful to be able to calculate the eigenvalues of $k_1 k_2 - k_{1,2}$ in all the irreducible representations of $K$. There are two cases where calculating those is relatively easy. The first case is $K_{1,2}$ is Abelian (or more generally, when $K_1$ and $K_2$ commute). In this case we have that $k_1 k_2 - k_{1,2} = k_2 k_1 - k_{1,2} = 0$ (all the eigenvalues are 0). The second case is where $K$ is the Heisenberg group modulo $q$, $H_q$, for some prime $q$ and $K_1, K_2$ are the groups generated by the standard generator of $H_q$, i.e.,

$$K_1 = \langle x : x^q = 1 \rangle,$$

$$K_2 = \langle y : y^q = 1 \rangle,$$

$$K = \langle x, y : [x, y], x \rangle = \langle [x, y], y \rangle = x^q = y^q = [x, y]^q = 1 \rangle.$$

In this case, the irreducible representations of $K$ are well known and therefore we can use them to bound $\cos(\angle(H^\pi(K_1), H^\pi(K_2)))$ and calculate the r-Schatten norm for $k_1 k_2 - k_{1,2}, k_2 k_1 - k_{1,2}$ in $L^2(K)$:

1. $K$ has $q^2$ irreducible representations of degree 1. Note that for any representation $\pi$ of degree 1, we always have $\pi(k_1 k_2 - k_{1,2}) = 0$.

2. $K$ has $q - 1$ irreducible representations of degree $q$ described as follows: every non trivial $q$-root of unity $\zeta$, define the representation $\pi_\zeta$ on $\mathbb{C}^q$ as follows: let $e_1, \ldots, e_q$ be the standard basis of $\mathbb{C}^q$, then $\pi_\zeta$ is defined as follows

$$\pi_\zeta(x), e_i = \zeta^i e_i, \pi_\zeta(y), e_i = e_{i+1}.$$

One can calculate that $\pi_\zeta(k_1 k_2 - k_{1,2})$ has $\frac{1}{\sqrt{q}}$ as an eigenvalue of multiplicity 1 and all the other eigenvalues are 0.

3. From the above, we get that for every unitary representation $\pi$ of $K$ on a Hilbert space $H$ we have that

$$\cos(\angle(H^\pi(K_1), H^\pi(K_2))) \leq \frac{1}{\sqrt{q}}.$$
Also, using Peter-Weyl theorem to decompose $L^2(K)$ as matrix coefficients, we can find that

$$
\|\lambda(k_{1}k_{2} - k_{1,2})\|_{S} = \frac{(q^2 - q)^{\frac{1}{2}}}{\sqrt{q}}.
$$

Using the above computation, we can prove the following theorem:

**Theorem 5.3.** Let $G$ be a discrete group generated by finite Abelian subgroups $K_{1}, \ldots, K_{N}$ of order $q$, where $q$ is prime such that $q > (8N - 11)^2$. Assume that for every $1 \leq i < j \leq N$ one of the following holds: either $K_{i}$ and $K_{j}$ commute (and therefore $K_{i,j}$ is $\mathbb{F}_{q} \times \mathbb{F}_{q}$) or $K_{i,j} = H_{q}$ and $K_{i}$, $K_{j}$ are the subgroups generated by the standard generators of $H_{q}$. Fix a constant $c'$ such that $\frac{1}{\sqrt{q}} < c' < \frac{1}{2N - 11}$. Define the following Banach classes:

1. The class $E_{1} = B_{BM}(\mathcal{H}, \delta) \cup \text{Int}(\mathcal{H}, \geq \theta)$, where $\delta = c'\sqrt{q} - 1$, $\theta = \frac{\ln(2) - \ln(c')}{\ln(2) + \ln(\sqrt{q})}$, $\mathcal{H}$ is the class of all Hilbert spaces.

2. For $(p_{1}, p_{2}, r) \in (1, 2] \times [2, \infty) \times [2, \infty)$ such that $\frac{1}{p_{1}} - \frac{1}{p_{2}} < \frac{1}{r}$, the class $E(p_{1}, p_{2}, r)$ defined as follows:

$$
\bigcup_{T_{p_{1}, C_{p_{2}} \in [1, \infty]} \atop c(p_{1}, p_{2}, r, q)T_{p_{1}, C_{p_{2}}}} \text{Int}\left(T(p_{1}, p_{2}, T_{p_{1}, C_{p_{2}}}, T_{p_{2}, C_{p_{2}}}), \frac{\ln(2) - \ln(c)}{\ln(2) - \ln(c)}\right),
$$

where $c(p_{1}, p_{2}, r, q) = \frac{(q^2 - q)^{\frac{1}{2}}}{\sqrt{q}} \sum_{i} \frac{1}{q_{i}}$. Note that $E(p_{1}, p_{2}, r)$ is non empty only if $c(p_{1}, p_{2}, r, q) \leq c'$.

3. The class $E$ defined as follows: denote

$$
A(q) = \left\{ (q_{1}, q_{2}, r) \in (1, 2] \times [2, \infty) \times [2, \infty) : \frac{1}{q_{1}} < \frac{1}{q_{2}} < \frac{1}{r}, c(q_{1}, q_{2}, r) \leq c' \right\}.
$$

Then $E = E_{1} \cup \bigcup_{(q_{1}, q_{2}, r) \in A(q)} E(q_{1}, q_{2}, r)$.

Then $G$ has robust Banach property (T) with respect to $E$ and has property $F_{X}$ for every $X \in E$. In particular, $G$ has property $F_{LP}$ for any $p < 2\frac{\ln(2) + \ln(\sqrt{q})}{\ln(2) + \ln(2N - 11)}$.

**Proof.** Using the above computations for Heisenberg groups $H_{q}$, we have that

- For every $1 \leq i < j \leq N$ and every unitary representation $\pi$ of $K$ on a Hilbert space $H$ we have that

$$
\cos(\zeta(H^\pi(K_{i}), H^\pi(K_{j}))) \leq \frac{1}{\sqrt{q}}.
$$

- For every $r \geq 2$ we have that $\cos_{\text{max}}^{S_{r}} \leq \frac{(q^2 - q)^{\frac{1}{2}}}{\sqrt{q}}$.

Therefore we can apply theorems [19] [15] and proposition [19] to get the above theorem. 

\[\square\]
Therefore for any graph \((V, E)\)

### 5.2.2 The groups

For a ring \(R\) and for \(1 \leq i < j \leq N\), if \(\{i, j\} \notin E\), then \(K_i\) and \(K_j\) commute.

For \(1 \leq i < j \leq N\), if \(\{i, j\} \in E\), then for every \(s_1, s_2 \in R\) we have that \([x_i(s_1), x_j(s_2)] = [x_i(1), x_j(s_1 s_2)]\) and \([K_i, K_j]\) commutes with both \(K_i\) and \(K_j\).

We note that when \(R = \mathbb{F}_q\), then for \(G = G((V, E), \mathbb{F}_q)\) the following holds:

- Every \(K_1, ..., K_N\) are isomorphic to \(\mathbb{F}_q\).
- For \(1 \leq i < j \leq N\), if \(\{i, j\} \notin E\), then \(K_i\) and \(K_j\) commute.
- For \(1 \leq i < j \leq N\), if \(\{i, j\} \in E\), then \(K_{i,j}\) is the \(q\) Heisenberg group \(H_q\) and \(K_i, K_j\) are the subgroups generated by the standard generators of \(H_q\).

Therefore for any graph \((V, E)\), \(G((V, E), \mathbb{F}_q)\) fulfills the conditions of theorem \(5.3\) above.

### 5.2.2 The groups

**St\(_n\)(\(\mathbb{F}_q[t_1, ..., t_m]\)) and \(EL\(_n\)(\(\mathbb{F}_q[t_1, ..., t_m]\))**

For a ring \(R\) and for \(n \geq 3\), the Steinberg group \(St_n(R)\) is a group generated by elements \(x_{i,j}(s)\) where \(1 \leq i \neq j \leq n\) and \(s \in R\) which has the following defining relations:

\[
\forall 1 \leq i \neq j \leq n, \forall s_1, s_2 \in R, x_{i,j}(s_1)x_{i,j}(s_2) = x_{i,j}(s_1 + s_2),
\]
The group of elementary matrices over \( R \), denoted \( EL_n(R) \), is the group of \( n \times n \) matrices with entries in \( R \), generated by the matrices \( e_{i,j}(s) \) for \( 1 \leq i \neq j \leq n \) and \( s \in R \), where \( e_{i,j}(s) \) denotes the elementary matrix with 1 on the diagonal, \( s \) in the \((i,j)\) entry and 0 in all the other entries. One can easily check that the relations specified above for \( St_n(R) \) also hold for \( EL_n(R) \), i.e., that

\[
\forall 1 \leq i \neq j \leq n, \forall q, s_1, s_2 \in R, e_{i,j}(1) e_{i,j}(s_2) = e_{i,j}(s_1 + s_2),
\]

\[
\forall 1 \leq i \neq l \leq n, \forall q, s_1, s_2 \in R, [e_{i,j}(1), e_{l,i}(s_2)] = \begin{cases} 1 & j \neq k \\ e_{i,l}(s_1 s_2) & j = k \end{cases}.
\]

We shall show that for any finitely generated, unital, and associative ring \( R \), the relations specified above for \( St_n(R) \) also hold for \( EL_n(R) \), i.e., that

\[
\forall 1 \leq i \neq j \leq n, \forall q, s_1, s_2 \in R, e_{i,j}(1) e_{i,j}(s_2) = e_{i,j}(s_1 + s_2),
\]

\[
\forall 1 \leq i \neq l \leq n, \forall q, s_1, s_2 \in R, [e_{i,j}(1), e_{l,i}(s_2)] = \begin{cases} 1 & j \neq k \\ e_{i,l}(s_1 s_2) & j = k \end{cases}.
\]

We show that for the ring of polynomials \( \mathbb{F}_q[t_1, \ldots, t_m] \), \( St_n(\mathbb{F}_q[t_1, \ldots, t_m]) \) and \( EL_n(\mathbb{F}_q[t_1, \ldots, t_m]) \) fulfill the conditions of theorem \( 5.3 \). We will show this only for \( St_n(\mathbb{F}_q[t_1, \ldots, t_m]) \) since the proof is identical for both groups. Denote \( t_0 = 1 \in \mathbb{F}_q[t_1, \ldots, t_m] \) and define the following subgroups \( K_1, \ldots, K_{n+m} \) of \( St_n(\mathbb{F}_q[t_1, \ldots, t_m]) \):

\[
\forall 1 \leq i < n, K_i = \{ x_{i,i+1} (a \cdot 1) : a \in \mathbb{F}_p \},
\]

\[
\forall n \leq i \leq n + m, K_i = \{ x_{n,i} (a \cdot t_{i-n}) : a \in \mathbb{F}_p \}.
\]

To see that \( K_1, \ldots, K_{n+m} \) fulfill the conditions of theorem \( 5.3 \), note the following:

- All the \( K_i \)'s are isomorphic to the group \( \mathbb{F}_q \).
- \( K_1, \ldots, K_{n+m} \) generate \( St_n(\mathbb{F}_q[t_1, \ldots, t_m]) \).
- For any \( 1 \leq i < j \leq n + m \), we have the following:

\[
K_{i,j} = \begin{cases} \mathbb{F}_q \times \mathbb{F}_q & 1 \leq i < j < n, j - i > 1 \\ \mathbb{F}_q \times \mathbb{F}_q & 1 \leq i < n - 1, n \leq j \leq n + m \\ \mathbb{F}_q \times \mathbb{F}_q & n \leq i < j \leq n + m \\ H_q & 1 \leq i < n - 1, j = i + 1 \\ H_q & i = 1, n \leq j \leq n + m \\ H_q & i = n - 1, n \leq j \leq n + m \end{cases}.
\]

Applying theorem \( 5.3 \) gives a result which generalizes theorems \( 1.11 \) and \( 1.12 \) stated in the introduction.

**Remark 5.6.** In the case that \( n \geq 4 \), Mimura \( [15] \) using a completely different approach showed that for any finitely generated, unital, and associative ring \( R \), \( EL_n(R) \) and \( St_n(R) \) have fixed point properties for every \( L^p \) space provided that \( p \in [1, \infty) \) and every non commutative \( L^p \) space provided that \( p \in (1, \infty) \) (see \( [15] \) [Corollary 1.4]). Therefore for fixed point properties for \( L^p \) spaces (and non commutative \( L^p \) spaces) Mimura’s results are stronger than ours. However, one should note that our results covers fixed point properties for Banach spaces that are not superreflexive, which are not achieved by Mimura’s work. We also deal with the case \( n = 3 \), which is also not covered by Mimura’s results.

**Remark 5.7.** We chose to phrase our results of the ring \( \mathbb{F}_q[t_1, \ldots, t_m] \), but the same proof the we gave above will also work for the ring \( \mathbb{F}_q[t_1, \ldots, t_m] \).
5.3 Construction of Banach expanders

Here we shall use our results regarding robust Banach property (T) of $EL_n(F_q([t_1, \ldots, t_m]))$ to construct a family of graphs of uniformly bounded valency that are Banach expanders with respect to a large class of Banach spaces generalizing theorems 1.15 given in the introduction. To be specific, for every constants $p_1 \in (1, 2], p_2 \in [2, \infty), T_{p_1} \geq 1, C_{p_2} \geq 2, \theta > 0$, such that $\frac{1}{p_1} - \frac{1}{p_2} < \frac{1}{4}$, we will construct a sequence of expanders that does not uniformly coarsely embed in any $X \in \text{Int}(T(p_1, p_2, T_{p_1}, C_{p_2}), \geq \theta)$.

Let $n \geq 3, m \geq 1$ and let $q$ be some prime. We showed above that $EL_n(F_q([t_1, \ldots, t_m]))$ fulfills the conditions of theorem 5.3. Therefore by remark 5.3 for any constants $p_1 \in (1, 2], p_2 \in [2, \infty), T_{p_1} \geq 1, C_{p_2} \geq 1, \theta > 0$, such that $\frac{1}{p_1} - \frac{1}{p_2} < \frac{1}{4}$, $EL_n(F_q[t_1, \ldots, t_m])$ has robust Banach property (T) with respect to $\text{Int}(\mathcal{T}(p_1, p_2, T_{p_1}, C_{p_2}), \geq \theta)$ provided that $q$ is large enough.

Therefore by proposition 5.8 in order to construct a sequence of expanders with respect to $\text{Int}(\mathcal{T}(p_1, p_2, T_{p_1}, C_{p_2}), \geq \theta)$ it is enough to find a sequence of normal finite index groups $N_i < EL_n(F_q[t_1, \ldots, t_m])$ such that $\bigcap_i N_i = \{1\}$. This can be achieved by considering principal congruence subgroups of $EL_n(F_q[t_1, \ldots, t_m])$: for every $i \in \mathbb{N}$, denote $I_i$ to be the two sided ideal of $F_q[t_1, \ldots, t_m]$ that is generated by all the monomials in $t_1, \ldots, t_m$ of degree $i$. Let $\psi_i$ be the homomorphism:

$$\psi_i : EL_n(F_q[t_1, \ldots, t_m]) \to EL_n(F_q[t_1, \ldots, t_m]/I_i),$$

and let $N_i = \text{ker}(\psi_i)$. From the fact that $|F_q[t_1, \ldots, t_m]/I_i| < \infty$, we get that $N_i$ is always of finite index and one can easily see that $\bigcap_i N_i = \{1\}$. Therefore we can conclude this discussion by stating our result:

**Proposition 5.8.** Let $p_1 \in (1, 2], p_2 \in [2, \infty), T_{p_1} \geq 1, C_{p_2} \geq 1, \theta > 0$ be constants such that $\frac{1}{p_1} - \frac{1}{p_2} < \frac{1}{4}$. For any $n \geq 3$ and any $m \geq 1$, there is a large enough prime $q$ such that for any fixed symmetric generating set $S$ of $EL_n(F_q[t_1, \ldots, t_m])$, we have that the Cayley graphs $\{EL_n(F_q[t_1, \ldots, t_m])/N_i, S\}_{i \in \mathbb{N}}$ is a family of $X$-expanders for any $X \in \text{Int}(\mathcal{T}(p_1, p_2, T_{p_1}, C_{p_2}), \geq \theta)$.

A Applications of robust Banach property (T)

In this appendix, we'll prove the applications of robust Banach property (T) for fixed point properties and for Banach expanders. In both cases the proofs are just minor adaptations of the proofs of Lafforgue in 13.

A.1 Fixed point property application

We shall prove the following:

**Proposition A.1.** Let $X$ be a Banach space and let $G$ be a locally compact group. If $G$ has robust property (T) with respect to $C \oplus X$ with the $l_2$ norm, then any affine isometric action of $G$ on $X$ has a fixed point.

**Proof.** Let $\rho$ be an isometric action of $G$ on $X$. Let $\overline{0} \in X$ be the zero of (the underlying vector space of) $X$ and define a length $l$ over $G$ as

$$l(g) = \max\{\|\rho(g)\overline{0}\|, 1\}.$$

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$G$ has robust property (T) with respect to $\mathbb{C} \oplus X$ and therefore there is some $s_0 > 0$ and a sequence of positive symmetric real functions $f_n \in C_c(G)$ with $\int f_n = 1$ such for every representation $\pi$ of $G$ on $\mathbb{C} \oplus X$ and for any $0 \leq s \leq s_0$, if $\|\pi(g)\| \leq e^{sl(g)}$, then $\pi(f_n)$ converges to $\pi(p)$ that is a projection on $\left((\mathbb{C} \oplus X)^s\right)$.

Fix $D > 1$ to be a constant whose value will be determined later and define a representation $\pi$ as on $\mathbb{C} \oplus X$ as follows:

$$\pi$$ is the unique representation on $\mathbb{C} \oplus X$ such that $\forall g \in G, \forall v \in X, \pi(g)(D, v) = (D, \rho(g)v)$.

In other words, $\pi$ is the representation that keeps $D \oplus X$ invariant and acts on it via $\rho$. Next, we’ll show that

$$\|\pi(g)\| \leq 1 + \sqrt{\frac{3}{D}} l(g), \forall g \in G.$$

Indeed,

$$\frac{\|\pi(g)(D, v)\|^2}{\|(D, v)\|^2} = \frac{D^2 + \|\rho(g)v\|^2}{D^2 + \|v\|^2} \leq \frac{D^2 + (\|v\| + l(g))^2}{D^2 + \|v\|^2} = 1 + \frac{2\|v\|l(g) + l(g)^2}{D^2 + \|v\|^2} \leq 1 + \frac{2\|v\| + 1}{D^2 + \|v\|^2} l(g)^2.$$

(6)

Note that if $\|v\| \geq D$, then

$$\frac{2\|v\| + 1}{D^2 + \|v\|^2} \leq \frac{3\|v\|}{\|v\|^2} \leq \frac{3}{D}.$$

On the other hand, if $\|v\| < D$, then

$$\frac{2\|v\| + 1}{D^2 + \|v\|^2} \leq \frac{3D}{D^2} = \frac{3}{D}.$$

Therefore, we have that for all $v$ that

$$\frac{2\|v\| + 1}{D^2 + \|v\|^2} \leq \frac{3}{D}.$$

Combined with (6), this yields

$$\frac{\|\pi(g)(D, v)\|^2}{\|(D, v)\|^2} \leq 1 + \frac{3}{D} l(g)^2,$$

and therefore

$$\frac{\|\pi(g)(D, v)\|}{\|(D, v)\|} \leq 1 + \sqrt{\frac{3}{D}} l(g).$$

The above inequality implies that

$$\|\pi(g)\| \leq 1 + \sqrt{\frac{3}{D}} l(g), \forall g \in G,$$

as needed. By choosing $D$ large enough, we can therefore insure that we’ll have

$$\|\pi(g)\| \leq 1 + s_0 l(g) \leq e^{s_0 l(g)}, \forall g \in G.$$
Therefore \( \pi \) meets the condition for robust Banach property (T) for \( C \sqcup X \). Let \( \{f_n\} \) be the sequence as in the definition of robust Banach property (T). Note that for every \( n \) and every \( v \in X \), \( \pi(f_n)(D,v) \in D \sqcup X \), since for every \( n \), \( \sum f_n = 1 \). Fix some \( v \in X \) and note that \( \pi(p)(D,v) = \lim_n \pi(f_n).v \in D \sqcup X \) and therefore there is some \( v_0 \in X \) such that \( \pi(p)(D,v) = (D,v_0) \). By the definition of \( \pi \), \( v_0 \) is a fixed point of the action of \( G \) on \( X \) through \( \rho \) and we are done.

\( \square \)

A.2 Banach expanders application

We shall prove the following:

Proposition A.2. Let \( G \) be a finitely generated discrete group and let \( \{N_i\}_{i \in \mathbb{N}} \) be a sequence of finite index normal subgroups of \( G \) such that \( \bigcap_i N_i = \{1\} \). Let \( \mathcal{E} \) be a class of Banach spaces that is closed under \( l_2 \) sums. Fix \( S \) to be some symmetric generating set of \( G \). If \( G \) has robust Banach property (T) with respect to \( \mathcal{E} \), then the family of Cayley graphs \( \{(G/N_i,S)\}_{i \in \mathbb{N}} \) is a family of \( X \)-expanders for any \( X \in \mathcal{E} \).

We shall start by proving the following:

Proposition A.3. Let \( G \) be a discrete finitely generated group with a symmetric generating set \( S \) and let \( N_i \) be a sequence of finite index normal subgroups of \( G \) such that \( \bigcap_i N_i = \{1\} \). Denote by \((V_i,E_i)\) the Cayley graph of \( G/N_i \) with respect to \( S \). Let \( \mathcal{E} \) be a class of Banach spaces such that \( \mathcal{E} \) is closed under \( l_2 \) sums. Assume that \( G \) has robust Banach property (T) with respect to \( \mathcal{E} \), then there is a constant \( C \) such that for every \( X \in \mathcal{E} \), every \( i \) and every map \( \phi : (V_i,E_i) \to X \), we have that there is some \( v(\phi) \in X \) such that

\[
\sum_{x \in V_i} \|\phi(x) - v(\phi)\|_X^2 \leq C \sum_{(x,y) \in E_i} \|\phi(x) - \phi(y)\|_X^2.
\]

Proof. Fix some \( X \in \mathcal{E} \) and some \( i \). Consider \( L^2(G/N_i,X) \) with the representation \( \pi : G \to L^2(G/N_i,X) \), defined as \( \pi(g).\phi(x) = \phi(g.x) \). Then \( L^2(G/N_i,X) \) is the \( l_2 \) sum of \( \{G : N_i\} \) copies of \( X \) and therefore \( L^2(G/N_i,X) \in \mathcal{E} \). Note that \( \pi \) is an isometric representation on \( L^2(G/N_i,X) \) and therefore \( \pi \in \mathcal{F}(\mathcal{E},0) \).

From the fact that \( G \) has robust Banach property (T) on \( \mathcal{E} \), we get that there is \( p \in C_{\mathcal{F}(\mathcal{E},0)} \) such that \( \pi(p) \) is the projection on \( L^2(G/N_i,X) \). Note that the space of invariant vectors under \( \pi \) is exactly the space of constant functions, so for every \( \phi \in L^2(G/N_i,X) \), we can define \( v(\phi) \in X \) as the constant value of \( \pi(p).\phi \). By the definition of robust Banach property (T) there is a real function \( f \in C_{\mathcal{F}(\mathcal{E},0)} \) such that \( \int f = 1 \) and \( \|p - f\|_{\mathcal{F}(\mathcal{E},0)} \leq \frac{1}{2} \).

Note that for every \( \phi \) we have that \( \pi(f).\pi(p).\phi = \pi(p).\phi \). Using this equality we get that \( (\pi(f) - \pi(p)).\phi = (\pi(f) - \pi(p)).(\phi - \pi(p).\phi) \). Therefore

\[
\|\phi - \pi(p).\phi\|_{L^2(G/N_i,X)} \leq \|
\|\phi - \pi(f).\phi\|_{L^2(G/N_i,X)} + \|\pi(f).\phi - \pi(p).\phi\|_{L^2(G/N_i,X)} = \\
\|\phi - \pi(f).\phi\|_{L^2(G/N_i,X)} + \|\pi(f) - \pi(p)\|_{L^2(G/N_i,X)} \leq \\
\|\phi - \pi(f).\phi\|_{L^2(G/N_i,X)} + \frac{1}{2}\|\phi - \pi(p).\phi\|_{L^2(G/N_i,X)}.
\]
This yields that
\[ \| \phi - \pi(p) \phi \|_{L^2(G/N_i, X)} \leq 2 \| \phi - \pi(f) \phi \|_{L^2(G/N_i, X)} \]
or equivalently
\[ \| \phi - \pi(p) \phi \|_{L^2(G/N_i, X)}^2 \leq 4 \| \phi - \pi(f) \phi \|_{L^2(G/N_i, X)}^2. \]

Note that by definition
\[ \| \phi - \pi(p) \phi \|_{L^2(G/N_i, X)}^2 = \sum_{g \in G/N_i} \| \phi(g) - \pi(g) \phi \|^2_X = \sum_{x \in V_i} \| \phi(x) - v(\phi) \|^2_X. \]

Therefore
\[ \sum_{x \in V_i} \| \phi(x) - v(\phi) \|^2_X \leq 4 \| \phi - \pi(f) \phi \|_{L^2(G/N_i, X)}^2. \]  \tag{7}

Recall that \( f \) is compactly supported and therefore there is some \( k \in \mathbb{N} \) such that \( \text{supp}(f) \subseteq S^k \). Note that for every \( g \in S^k \), we have by the triangle inequality that
\[ \| \phi - \pi(g) \phi \|_{L^2(G/N_i, X)} \leq k \sum_{s \in S} \| \phi - \pi(s) \phi \|_{L^2(G/N_i, X)}. \]

Therefore if we denote \( M = \max_{g \in G} |f(g)| \), we get that
\[ \| \phi - \pi(f) \phi \|_{L^2(G/N_i, X)} \leq Mk \sum_{s \in S} \| \phi - \pi(s) \phi \|_{L^2(G/N_i, X)}. \]

This yields that
\[ \| \phi - \pi(f) \phi \|_{L^2(G/N_i, X)}^2 \leq \left( Mk \sum_{s \in S} \| \phi - \pi(s) \phi \|_{L^2(G/N_i, X)} \right)^2 \]
\[ \leq M^2 k^2 |S| \sum_{s \in S} \| \phi - \pi(s) \phi \|_{L^2(G/N_i, X)}^2 = M^2 k^2 |S| \sum_{g \in G/N_i} \| \phi(g) - \pi(g) \phi \|^2_X = M^2 k^2 |S| \sum_{g \in G/N_i} \sum_{s \in S} \| \phi(g) - \phi(g s) \|^2_X \]
\[ = 2M^2 k^2 |S| \sum_{(x, y) \in E_i} \| \phi(x) - \phi(y) \|^2_X. \]

We are done by combining the above with (7). Note that \( M, k, |S| \) are all independent of the choice of \( N_i \) and therefore taking \( C = 8M^2 k^2 |S| \) gives a constant that is uniform for all \( N_i \)'s. \( \square \)

Using this proposition, we can prove proposition A.2 by proving the following lemma:

**Lemma A.4.** Let \( X \) be a Banach space and let \( \{(V_i, E_i)\}_{i \in \mathbb{N}} \) be a family of graphs with uniformly bounded valency such that \( |V_i| \rightarrow \infty \) as \( i \rightarrow \infty \). Assume that there is a constant \( C \) such that for every \( i \in \mathbb{N} \) and every \( \phi : (V_i, E_i) \rightarrow X \) there is some \( v(\phi) \in X \) such that
\[ \sum_{x \in V_i} \| \phi(x) - v(\phi) \|^2_X \leq C \sum_{(x, y) \in E_i} \| \phi(x) - \phi(y) \|^2_X. \]

Then \( \{(V_i, E_i)\}_{i \in \mathbb{N}} \) is a family of \( X \)-expanders.
Proof. Let \( D \geq 2 \) be a uniform bound on the valency of \( \{(V_i, E_i)\}_{i \in \mathbb{N}} \). Assume towards contradiction that there is a sequence of maps \( \phi_i : V_i \to \mathbb{X} \) and functions \( \rho_-, \rho_+ : \mathbb{N} \to \mathbb{R} \) such that \( \lim_k \rho_-(k) = \infty \) and

\[
\forall i \in \mathbb{N}, \forall x, y \in V_i, d_-(d_i(x, y)) \leq \|\phi_i(x) - \phi_i(y)\| \leq \rho_+(d_i(x, y)),
\]

where \( d_i(x, y) \) is the distance in the graph \( (V_i, E_i) \) between \( x \) and \( y \). By replacing \( \phi_i \) by \( \phi_i - v(\phi_i) \), we can assume that for every such \( \phi_i \), we have that

\[
\sum_{x \in V_i} \|\phi_i(x)\| \leq C \sum_{(x, y) \in E_i} \|\phi_i(x) - \phi_i(y)\|.
\]

Note that

\[
\sum_{(x, y) \in E_i} \|\phi_i(x) - \phi_i(y)\| \leq |E_i| \rho_+(1)^2 \leq \frac{D|V_i|}{2} \rho_+(1)^2.
\]

Therefore

\[
\sum_{x \in V_i} \|\phi_i(x)\|^2 \leq \frac{|V_i|^2}{2} (CD \rho_+(1)^2).
\]

Consider the median value of the multiset \( \{\|\phi_i(x)\| : x \in V_i\} \). If this median is strictly greater than \( \sqrt{CD \rho_+(1)} \), we get a contradiction to the above inequality. Therefore, there is a set \( U_i \subseteq V_i \) such that \( \lfloor \frac{|V_i|}{2} \rfloor \) and

\[
\forall x \in U_i, \|\phi_i(x)\| \leq \sqrt{CD \rho_+(1)}.
\]

Therefore by triangle inequality

\[
\forall x, y \in U_i, \|\phi_i(x) - \phi_i(y)\| \leq 2\sqrt{CD \rho_+(1)}.
\]

On the other hand, since the valency in all the graphs is bounded by \( D \), we have that

\[
\forall i \in \mathbb{N}, \forall k \in \mathbb{N}, \forall x \in V_i, |\{y \in V_i : d_i(x, y) < k\}| < D^k.
\]

Denote \( \text{diam}(U_i) \) to be the diameter of \( U_i \) in \( V_i \), then by the above inequality we get that

\[
\text{diam}(U_i) \geq \ln(|U_i|) = \ln(|V_i|) - \ln(D).
\]

Therefore there are \( x, y \in U_i \) such that

\[
\rho_\epsilon \left( \frac{\ln(|V_i|)}{\ln(D)} \right) \leq \|\phi_i(x) - \phi_i(y)\|.
\]

Combining this with \( \square \) yields that for every \( i \) we have that

\[
\rho_\epsilon \left( \frac{\ln(|V_i|)}{\ln(D)} \right) \leq 2\sqrt{CD \rho_+(1)}.
\]

But from the assumption that \( \lim_i |V_i| = \infty \) we get a contradiction to the assumption that \( \lim_k \rho_\epsilon(k) = \infty \). \( \square \)

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