Landau damping of electron-acoustic waves due to multi-plasmon resonances

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The linear and nonlinear theories of electron-acoustic waves (EAWs) are studied in a partially degenerate quantum plasma with two-temperature electrons and stationary ions. The initial equilibrium of electrons is assumed to be given by the Fermi-Dirac distribution at finite temperature. By employing the multi-scale asymptotic expansion technique to the one-dimensional Wigner-Moyal and Poisson equations, it is shown that the effects of multi-plasmon resonances lead to a modified complex Korteweg-de Vries (KdV) equation with a new nonlocal nonlinearity. Besides giving rise to a nonlocal nonlinear term, the wave-particle resonance also modifies the local nonlinear coupling coefficient of the KdV equation. The latter is shown to conserve the number of particles, however, the wave energy decays with time. A careful analysis shows that the two-plasmon resonance is the dominant mechanism for nonlinear Landau damping of EAWs. An approximate soliton solution of the KdV equation is also obtained, and it is shown that the nonlinear Landau damping causes the wave amplitude to decay slowly with time compared to the classical theory.

I. INTRODUCTION

In many equilibrium plasmas, electrons can be divided into two groups with different temperatures. Such a division may be possible as certain electrons are preferentially heated by some external sources like beams or waves until a final equilibrium is reached. The existence of two-temperature electrons was also experimentally observed in a sputtering magnetron plasma. However, in 1977, it was investigated by Watanabe and Tanii that if electrons in a plasma with background ions have two different temperatures, there can exist an electron-acoustic wave (EAW). They also reported that if the hot electron number density is much higher than that of cold ones, the phase velocity of such waves lies in between the thermal velocities of cold and hot species, and the EAWs do not experience strong Landau damping. Later, in 1991, Holloway and Dorning noted that a small-amplitude undamped EAW can exist in plasmas with the dispersion relation of the form $\omega \approx 1.31 k v_{th}$, where $\omega$ is the wave frequency (number) and $v_{th}$ is the electron thermal speed. Recently, based on the observations from the Van Allen Probes, Vasko et al. reported the existence of EAWs in the inner magnetosphere with the similar properties as above but a different linear dispersion law (as per their notations): $\omega = k v_0 (1 - k^2 d^2/2)$, where $v_0$ and $d$ are, respectively, the electron-acoustic velocity and dispersive scale. They also elucidated the observed asymmetric solitary waves in terms of electron-acoustic solitons and shocks. In an another work by Dilard et al., it has been shown by means of theoretical and numerical analysis of hydrodynamic equations that the asymmetric solitary waves can appear due to the steepening of initially quasi-monochromatic EAWs. The beam-generated EAWs due to trapping of electrons by whistler waves has also been reported with the phase velocity slightly large than that of the whistler waves. Although, such EAWs have similar properties with those stated before, their phase velocity is predicted a bit higher, i.e., $1.8 \lesssim \omega/kv_{th} \lesssim 2.2$.

The EAWs are typically electrostatic plasma oscillations which propagate with a frequency below that of the Langmuir wave. The characteristics of EAWs have been extensively studied theoretically by means of Vlasov-Poisson simulation and PIC simulations, as well as experimentally. However, all these investigations were limited to the classical regimes. Recently, a liner theory of EAWs in degenerate plasmas with two groups of electrons has been studied by Fahad et al. They considered a fully degenerate plasma with two different Fermi temperatures of electrons. However, such a model may not be physically sound in the context of quantum mechanics as there should not be more than one species of electrons in a fully degenerate plasma or a zero-temperature Fermi gas.

In plasmas with two-temperature electrons, the nonlinear theory of EAWs has been investigated by many authors, however, using the fluid theory approach, i.e., without the Landau damping effect. On the other hand, after the work of Ott and Sudan on nonlinear theory of ion-acoustic waves with linear Landau damping, many authors have studied the Landau damping of low-frequency electrostatic waves as well as high-frequency wave envelopes, however, no nonlinear theory of EAWs with Landau damping has been developed in the context of quantum plasmas. While the linear plasmon resonance is known in the literature, the multi-plasmon resonance in the nonlinear regime has not yet been fully explored, especially in the context of Korteweg-de Vries (KdV) equation.
In this work, we will generalize the previous works on EAWs to include short-scale quantum effects. This is of interest for dense plasmas where the characteristic de Broglie wavelength of electrons is larger than the nearest neighboring distance. Such dense plasmas include those that appear in various astrophysical environments and certain laboratory systems, e.g., cases where high power lasers compress solid density targets. We consider weakly coupled systems only such that a background distribution, deviating from the thermodynamic equilibrium, can be sustained. More specifically, we study the linear and nonlinear theories of EAWs in a partially degenerate plasma consisting of two groups of electrons with their thermodynamic temperatures \( T_j \) exceeding the Fermi temperature \( T_{Fj} \) somewhat (i.e., \( T_j > T_{Fj} \)) where \( j = l, h \) stand for low and high temperature electrons) and stationary positive ions. The purpose of this study is to find the quantum generalization of known classical results of EAWs including wave-particle interaction. As a consequence, we need to consider the Wigner theory rather than the classical Vlasov or semi-classical Vlasov theories to take into account the effects of quantum dispersion and the wave-particle resonance. A specific and novel feature introduced by the Wigner theory, is a nonlinear wave-particle interaction referred to as multi-plasmon interaction (see e.g. Refs. 22, 23) where we can have simultaneous absorption (or emission) of multiple plasmon quanta. Including this mechanism into the calculation scheme, an additional wave damping due to the two-plasmon resonance is seen to occur in the evolution of EAWs along with the resonance of linear theory. For these types of effect to be significant, we have to focus on the regime of short-wavelengths, i.e., the thermal de Broglie wavelength can not be much smaller than that of EAWs \( \hbar k/m \lesssim v_t \) which still accounts for some quantum effects to enter the picture. Here, \( \hbar = h/2\pi \) is the reduced Planck’s constant, \( k \) is the wave number, \( m \) is the electron mass, and \( v_t \) is the thermal velocity of low-temperature electrons.

An important aspect for the details of EAWs is the properties of the electron background distribution function. For dense plasmas, as can be found, e.g., in the interiors of giant stars like white dwarfs and neutron stars22,23, as well as in gas giants (e.g., Jupiter)24, typically the density is high enough for the electron distribution function to be degenerate or partially degenerate. In our study, the case of partial degeneracy is in the focus. Moreover, we are concerned with the background distributions deviating from the thermodynamic equilibrium. The high density (partially degenerate) plasmas where the electron background deviates from the thermodynamic equilibrium can appear in the context of plasmas produced by lasers or ion beams25,26. Importantly, the system energy flows mostly into the electrons, and, as a result, there may appear partially degenerate electrons with high temperature tails, allowing the electron background to be modeled by two different temperatures. In such systems, EAWs are likely to be excited and play a role in the nonlinear dynamics.

The paper is organized as follows: In Sec. II we present the basic physical model and the underlying assumptions. The linear theory of EAWs comprising the wave dispersion and Landau damping are studied in Sec. III. Section IV presents the derivation of the modified KdV equation and the occurrence of multi-plasmon resonances. We have verified the conservation laws for the KdV equation as demonstrated in Sec. V. In Sec. VI an approximate solution of the KdV equation is obtained and the effect of Landau resonance is discussed. Finally, Sec. VII is left to summarize and conclude the results.

II. THE MODEL

We consider the nonlinear wave-particle interaction and evolution of small-amplitude EAWs in a partially degenerate plasma with two-temperature electrons and stationary ions. Our basic assumptions for the study are the following:

- Under certain conditions electrons have a relatively high-temperature tail such that electrons can be grouped into two distinct components with different thermodynamic temperatures \( T_l \) and \( T_h \) [Hereafter we call low- \((\alpha = l)\) and high-temperature \((\alpha = h)\) electrons with \( T_h > T_l \) and \( T_h \gtrsim T_{F\alpha} \), where \( T_{F\alpha} \) denotes the Fermi temperature of \( \alpha \)-species electrons].

- Although, the theory is independent of the background distribution, however, we consider the initial equilibrium of electrons as given by the Fermi-Dirac distribution at finite temperature \((T_\alpha \neq 0)\). The case of full degeneracy of electrons at zero temperature restricts only one species of electrons and thereby inadmissible to the present study.

- There exists a low-frequency mode with the frequency \((\omega)\) proportional to the wave number \((k)\) such that the linear wave damping is small and the nonlinear evolution of EAWs can be described by the Korteweg de Vries (KdV) equation.

- Similar to classical plasmas, the equilibrium number density of high-temperature electrons is higher than that of low-temperature species, i.e., \( n_{h0} > n_{l0} \).

- The EAWs are weakly dispersive such that in the regime \( \hbar k/m \lesssim v_t \), the Wigner-Moyal equation is still valid and some quantum effects come into play in the process of wave-particle interactions.
We start with the Wigner-Moyal equation, given by,
\[
\frac{\partial f_\alpha}{\partial t} + v \cdot \nabla f_\alpha + \frac{ie m^3}{(2\pi \hbar)^3} \int d^3r' d^3v' e^{i m (v-v') \cdot r'/\hbar} \times \left[ \phi \left( r + \frac{r'}{2}, t \right) - \phi \left( r - \frac{r'}{2}, t \right) \right] f_\alpha(r, v', t) = 0,
\]
and the Poisson equation
\[
\nabla^2 \phi = 4\pi e \sum_{\alpha = l, h} \left( \int f_\alpha d^3v - n_0 \right),
\]
where \( f_\alpha \) is the Wigner distribution function for \( \alpha \)-species electrons; \( e, m, v \), respectively, denote the charge, mass, and velocity of electrons; \( \phi \) is the self-consistent electrostatic potential, and \( n_0 \) is the number density of background ions. Moreover, we consider the propagation of EAWs along the \( x \)-axis and the background distribution function for \( \alpha \)-species electrons in one-dimension as (writing \( v_x \) as \( v \))
\[
f_\alpha^{(0)}(v) = \int \int f_{3D}(v) dv_y dv_z = \int \int \left[ 1 + \exp \left( \frac{E - \mu_\alpha}{k_B T_\alpha} \right) \right]^{-1} dv_y dv_z = \frac{3 n_{\alpha \alpha} T_\alpha}{4 v_{F\alpha} T_{F\alpha}} \log \left[ 1 + \exp \left( \frac{\frac{\lambda}{2} m v^2 - \mu_\alpha}{k_B T_\alpha} \right) \right],
\]
where \( E = (m/2)(v_x^2 + v_y^2 + v_z^2) \) is the kinetic energy and \( \mu_\alpha \) the chemical potential which satisfies the following charge neutrality condition
\[
n_0 = \sum_\alpha \int f_\alpha^{(0)}(v) dv.
\]
In the nondegenerate limit (\( T_\alpha \gg T_{F\alpha} \)), the parameter \( \xi_\alpha = \mu_\alpha/k_B T_\alpha \) is large and negative, while it is large and positive in the fully degenerate limit (\( T_\alpha \ll T_{F\alpha} \)). We, however, consider the case of \( T_\alpha \gg T_{F\alpha} \) such that \( \xi_\alpha \) (\( < 0 \)) is of moderate value.

We stress here that there are certain parameter restrictions imposed by the Pauli principle, since we cannot have a phase space density of the background Wigner-function exceeding \( 2(m/2\pi \hbar)^3 \). As a result, the parameters of the high- and low-temperature distributions cannot be chosen independently. The strictest criterion appears for \( E = 0 \) and reads
\[
\left[ 1 + \exp \left( -\frac{\mu_\alpha}{k_B T_\alpha} \right) \right]^{-1} \leq 1.
\]
For a partially degenerate low-temperature distribution (i.e., \( k_B T_\alpha \sim E_F, \mu_\alpha \sim 2E_F \)), this condition is typically fulfilled if the high-temperature distribution is not too far from the classical Maxwell-Boltzmann regime such that \( -\mu_\alpha/k_B T_\alpha \gtrsim 3 \).

### III. LINEAR THEORY: WAVE DISPERSION AND ONE-PLOSMON RESONANCE

Before we proceed to the nonlinear theory of wave-particle interactions, it is imperative to study the linear theory of EAWs and the associated wave damping. To this end, we linearize Eqs. (1) and (2) by splitting up the functions \( f \) and \( \phi \) into their equilibrium and perturbation parts, i.e., \( f_\alpha(x, v, t) = f_\alpha^{(0)} + f_\alpha^{(1)} \) and \( \phi(x, t) = \phi^{(1)} \), and assume the perturbations to be of the form \( \exp(i k x - i \omega t) \), i.e., plane waves with frequency \( \omega \) and wave number \( k \). Thus, we obtain the following dispersion relation.
\[
D(\omega, k) \equiv 1 - \sum_{\alpha = l, h} \frac{\omega_\alpha^2}{n_{\alpha \alpha} k^2} \int_{-\infty}^{\infty} \frac{f_\alpha^{(0)}(v)}{(v - \omega/k)^2 - v_q^2} dv = 0,
\]
where \( \omega_\alpha = \sqrt{4\pi n_{\alpha \alpha} e^2/m} \) is the plasma frequency for \( \alpha \)-species electrons and \( v_q = \hbar k/2m \) is the velocity associated with the plasmon quanta.

For weak wave damping of EAWs, the time asymptotic solution for the complex frequency (\( \omega = \omega_0 + i \gamma_L \)) can be obtained by finding the roots of the dispersion equation \( D(\omega, k) \equiv D_r(\omega_r, k) + iD_i(\omega_r, k) + i\gamma_L \partial D_r(\omega_r, k)/\partial \omega_r = 0 \). Thus, separating the real and imaginary parts we obtain
\[
D_r(\omega_r, k) \equiv 1 - \sum_{\alpha = l, h} \frac{\omega_\alpha^2}{n_{\alpha \alpha} k^2} \int_{-\infty}^{\infty} \frac{f_\alpha^{(0)}(v)}{(v - \omega_r/k)^2 - v_q^2} dv = 0,
\]
and the Landau damping rate, given by,
\[
\gamma_L = \frac{D_i(\omega_r, k)}{D_r(\omega_r, k)},
\]
where
\[
D_i = -2\pi v_q \sum_{\alpha = l, h} \frac{\omega_\alpha^2}{n_{\alpha \alpha} k^2} \left[ f_\alpha^{(0)}(v_{+1}^{\text{res}}) - f_\alpha^{(0)}(v_{-1}^{\text{res}}) \right],
\]
in which \( v_{\pm}^{\text{res}} = \omega_r/k \pm v_q \) denotes the plasmon resonance velocity.

Next, substituting the distribution function (3) into Eq. (7) and evaluating the integrals in two different regimes \( |v| < \omega_r/k \pm v_q \) and \( |v| > \omega_r/k \pm v_q \), i.e.,
\[
T \int_{-\infty}^{\infty} = \lim_{\epsilon \to 0^+} \left[ \int_{-\infty}^{-(\lambda \pm v_q) + \epsilon} + \int_{-(\lambda \pm v_q) - \epsilon}^{-(\lambda \pm v_q)} + \int_{-(\lambda \pm v_q)}^{\infty} \right] + \int_{-(\lambda \pm v_q) - \epsilon}^{-(\lambda \pm v_q)} + \int_{-(\lambda \pm v_q)}^{\infty},
\]
and noting that the exponential function in the distribution function (Eq. 3) is small in the partially degenerate
plasma regime, we obtain

\[ D_r(\omega_r, k) \equiv 1 - \frac{3}{4} \sum_{\alpha=l,h} \omega_{p\alpha}^2 e^{\xi_\alpha} \left[ 2\sqrt{2} \frac{\nu_{ta}/v_{F\alpha}}{\omega_r^2 - k^2 v_{th}^2} + \frac{2\pi v_{ta} - \omega_r/k}{k^2 v_{ta}^2 v_{F\alpha}} \right] = 0. \tag{11} \]

Equation (11) describes both the high-frequency and relatively low-frequency wave eigenmodes of EAWs. In the limit of \( \omega_r/k \pm v_{th} \gg v_{th} \), the high-frequency Langmuir wave (LW) mode can be obtained from Eq. (11) by considering the first and the second terms as

\[ \omega_r^2 = k^2 v_{th}^2 + \frac{3}{\sqrt{2}} \sum_{\alpha=l,h} \omega_{p\alpha}^2 v_{ta}^2 e^{\xi_\alpha}, \tag{12} \]

where \( v_{ta} = \sqrt{2k_BT_\alpha/m} \) is the thermal velocity and \( v_{F\alpha} = \sqrt{2k_BT_{F\alpha}/m} \) the Fermi velocity of \( \alpha \)-species electrons.

On the other hand, for \( \omega_r/k \pm v_{th} \ll v_{th} \), the first and the third terms of Eq. (11) can be combined to yield the following dispersion relation for the EAW mode.

\[ \omega_r = k \left( \frac{2\pi}{\sqrt{2}} \sum_{\alpha=l,h} \omega_{p\alpha}^2 e^{\xi_\alpha} \right)^{1/2} / \sum_{\alpha=l,h} \omega_{p\alpha}^2 v_{ta} v_{F\alpha}. \tag{13} \]

From Eq. (13) we note that, similar to the ion-acoustic waves in an electron-ion plasma, the EAW in a two-temperature-electron plasma has also the cubic order of dispersion and for small wave numbers \( \omega_r/k \) approaches a constant value, i.e., the EAW becomes dispersionless. These are the prerequisites for the description of small-amplitude EAWs by a KdV-like equation. Further inspecting on Eq. (13) we find that for typical plasma parameters \( n_{t0} = 2 \times 10^{24} \text{ cm}^{-3}, n_{00} = 10 - 12n_{t0}, T_I = 10^6 \) K and \( T_h = 2.5 - 3T_I \) and with \( k_{th}/\omega_{ph} \equiv k/\lambda_0 \ll 1 \), the ratio of the two terms (without the factor \( k \) and the term involving \( k^2 \)) on the right-hand side of Eq. (13) roughly varies in between 2.2 to 2.5 times the thermal velocity of high-temperature species \( (v_{th}) \). In this situation, Eq. (13) reduces to \( \omega_r \approx 2.5kv_{th} \). Thus, it follows that the phase velocity of EAWs in a plasma with finite temperature degeneracy is a bit higher than predicted in the classical theory \( \omega_r \approx 1.31kv_{th} \). Figure 1 displays the dispersion curves for high-frequency LWs and low-frequency EAWs [panels (a) and (c)], as well as the Landau damping rates for EAWs [panels (b) and (d)]. It is seen that with lower values of the temperature ratio \( T = T_I/T_h \) or with higher values of the density ratio \( N = n_{00}/n_{t0} \), critical value \( K_c \) of \( K = k/\lambda_0 \) exists, beyond which the EAW frequency can turn over, going to zero, and then assume negative values. Such a nature of the wave frequency, which is unusual in classical plasmas where the high- and low-frequency branches join to form a thumb-like curve, may be due to finite temperature degeneracy of background electrons. As a result, the linear damping rate \( |\gamma_L/\omega_{ph}| \), which was initially low \((\sim 0.01)\), starts increasing with certain values of \( K < K_c \) and then decreases at relatively higher values of \( K > K_c \). This enables us to consider the regime \( 0 \lesssim K \lesssim 0.5 \) or \( \hbar k/mv_{th} \leq 0.2 \) where the linear Landau damping rate of EAWs remains weak.

It is to be noted that the approximate dispersion Eqs. (12) and (13) are obtained by assuming the smallness of the exponential function in the distribution function \( f_\alpha \) as applicable in the partially degenerate regime. However, the Fermi-Dirac distribution approaches the Maxwell-Boltzmann distribution in the limit of low density or high temperature, i.e., when the integrand in Eq. (13) is much smaller than the unity. So, the classical results of LWs and EAWs cannot be recovered directly from Eqs. (12) and (13).

In what follows, we calculate the equilibrium chemical potentials from the following relation.

\[ \sum_{\alpha=l,h} \int_{-\infty}^{\infty} f_\alpha^{(0)}(v)dv = n_0, \tag{14} \]

which gives

\[ \frac{3}{4\sqrt{\pi}} \sum_{\alpha=l,h} \left( \frac{T_\alpha}{T_{F\alpha}} \right)^{3/2} n_{a\alpha} e^{\xi_\alpha} = n_0. \tag{15} \]

IV. NONLINEAR THEORY: THE KDV EQUATION WITH MULTI-PLASMON RESONANCE

On the basis of the results of the linear theory of EAWs (Sec. III), i.e., EAW has a cubic order dispersion and the Landau damping rate is small, we proceed to the nonlinear theory of small-amplitude EAWs. Here, we derive an evolution equation for the weakly nonlinear EAWs in a degenerate plasma using the multiple-scale perturbation technique. Some special attention must be devoted to the higher order (in the amplitude) resonances that occur in the Wigner theory. In particular, due to the nonlinearities, we will have Landau resonances with resonant velocities that are shifted an amount \( \pm n_0v_0 \) in momentum space (See e.g., Ref. 22) compared to the resonant velocity of classical theory (i.e., the phase velocity). Here, \( n = 1 \) gives the velocity shift already appeared in the linear theory as described in the previous section III. Close to the resonant velocities, the Wigner equation must be analyzed in more detail. As a starting point, we will divide the velocity space into resonant and non-resonant regions. While a rigorous derivation of the nonlinear higher order resonances, based on the Wigner equation, has been given in Ref. 22, we will here give a heuristic derivation based on elementary quantum mechanics, following the steps of Ref. 22. Our starting point is the quantum mechanical modifications of the classical linear resonance, where the resonant velocity is adjusted according to

\[ v_{res}/k \rightarrow v_{res}/k = \frac{\omega \pm \hbar k}{2m}. \tag{16} \]
Let us study the physical meaning of this modification. When a particle absorbs or emits a wave quantum it can increase or decrease the momentum according to

\[ \hbar k_1 \pm \hbar k = \hbar k_2, \]  

and at the same time the energy changes according to

\[ \hbar \omega_1 \pm \hbar \omega = \hbar \omega_2. \]  

Next we identify \( \hbar k_1 / m \) (or equally well \( \hbar k_2 / m \)) with the resonant velocity \( v_{\text{res}} \) and note that for small amplitude waves, the particle frequencies and wavenumbers (\( \omega_{1,2}, k_{1,2} \)) obey the free particle dispersion relation \( \omega_{1,2} = \hbar k_{1,2}^2 / 2m \). Using these relations, we immediately see that the energy momentum relations Eqs. (17) and (18) imply the modification of the resonant velocity seen in Eq. (10). An interesting possibility, which was studied in Ref. 21, is the simultaneous absorption (or emission) of multiple wave quanta, rather than a single wave quantum at a time. In that case Eqs. (17) and (18) are replaced by

\[ \hbar k_1 \pm n \hbar k = \hbar k_2, \]  

and

\[ \hbar \omega_1 \pm n \hbar \omega = \hbar \omega_2, \]

where \( n = 1, 2, 3, \ldots \) is an integer. Accordingly, performing the same algebra as for the linear case, the resonant velocities now becomes

\[ v_{\text{res}}^{\pm n} = \frac{\omega}{k} \pm n \frac{\hbar k}{2m}. \]  

When we pick the minus sign in Eq. (21), the resonant velocity for absorbing multiple wave quanta can be considerably lower, provided the wavelengths are short. As a consequence, in the case of Langmuir waves, the damping rate due to absorption of multiple wave quanta can be much larger than the standard linear damping rate. This is due to the larger number of resonant particles in the former case.

For the non-resonance region of velocity space we have \( |v - v_{\text{res}}^{\pm n}| \gg o(\epsilon) \) with \( \epsilon > 0 \) denoting the scaling parameter and \( v_{\text{res}}^{\pm n} = \omega/k \pm n \hbar k / 2m \), \( n = 0, 1, 2, 3, \ldots \) the resonant velocity. In the non-resonance regime, we consider the following expansions for \( f_\alpha \) and \( \phi \).

\[ f_\alpha(x, v, t) = f_\alpha^{(0)} + \epsilon^{1/2} f_\alpha^{(1)} + \epsilon^2 f_\alpha^{(2)} + \cdots \]

\[ \phi(x, t) = \epsilon \phi^{(1)} + \epsilon^{3/2} \phi^{(2)} + \cdots \]

However, in the resonance region where \( |v - v_{\text{res}}^{\pm n}| \sim o(\epsilon) \), a slightly different ordering for both \( f_\alpha \) and \( \phi \) needs to be considered. This will be verified shortly. The idea of considering different orderings for the distribution function and the electrostatic potential in the resonance and non-resonance regions was proposed and rigorously discussed by Vandam and Tanuity on the consideration that the Landau damping of KdV solitons is a far-field approximation of the Vlasov equation. Although, the present
nonlinear theory considers a similar approach, the mathematical treatment is a bit different from the above due to the Wigner function. A particular ordering for each of $f_\alpha$ and $\phi$ cannot give rise the resonance and nonresonance contributions properly rather with different orderings, the terms involving the resonance and nonresonance parts can be combined with the closure equation, i.e., the Poisson equation which involves both the resonance and nonresonance contributions of the distribution function. Also, the Gardner-Morikawa transformation, $\xi = e^{1/2}(x - \lambda t)$, $\tau = e^{3/2}t$, which is applicable for the fluid theory of a KdV equation, may not be directly applied to the Vlasov-Poisson or Wigner-Poisson systems because of the singularities in the integrals due to resonant particles. Thus, we modify the Gardner-Morikawa transformation as \[ \xi = e^{1/2}x, \quad \sigma = e^{1/2}t, \quad \tau = e^{3/2}t \] (23) and assume the following multi-scale Fourier-Laplace transforms for the perturbations. \[ f^{(n)}(v, \xi, \sigma, \tau) = \frac{i}{(2\pi)^2} \int_W d\omega \int_{-\infty}^{\infty} dk \tilde{f}^{(n)}(v, k, \omega, \tau) \times (\omega - \lambda k)^{-1} \exp[i(k\xi - \omega\sigma)] \] \[ \phi^{(n)}(\xi, \sigma, \tau) = \frac{i}{(2\pi)^2} \int_W d\omega \int_{-\infty}^{\infty} dk \tilde{\phi}^{(n)}(v, k, \omega, \tau) \times (\omega - \lambda k)^{-1} \exp[i(k\xi - \omega\sigma)], \] (24)

where $W$ is the usual Laplace transform contour which excludes the poles of the integrals, and $f^{(n)}_\alpha$ and $\tilde{\phi}^{(n)}$ are analytic for all $\omega$. Equation (24) represents a plane wave propagating along the $\xi$-direction with the phase speed $\lambda$.

Substituting the stretched coordinates (23) and the expansions (22) into Eqs. (1) and (2) we obtain in one-dimensional geometry

\[
\left( \epsilon^{3/2} \frac{\partial}{\partial \sigma} + \epsilon^{1/2} \frac{\partial}{\partial \tau} \right) \left( f^{(0)}_\alpha + \epsilon^{1/2} f^{(1)}_\alpha + \epsilon^2 f^{(2)}_\alpha + \ldots \right) + \epsilon^{1/2} \nu \frac{\partial}{\partial \xi} \left( f^{(0)}_\alpha + \epsilon^{1/2} f^{(1)}_\alpha + \epsilon^2 f^{(2)}_\alpha + \ldots \right) \\
+ \frac{e_m \omega}{2\pi \hbar^2} \int d\nu' d\nu'' \epsilon^{im(v-v')} \epsilon^{i-1}/h \left[ \left\{ \epsilon \phi^{(1)}(\xi + \nu', \sigma, \tau) + \epsilon^{3/2} \phi^{(2)}(\xi + \nu', \sigma, \tau) + \ldots \right\} \\
- \left\{ \epsilon \phi^{(1)}(\xi - \nu', \sigma, \tau) + \epsilon^{3/2} \phi^{(2)}(\xi - \nu', \sigma, \tau) + \ldots \right\} \right] \left( f^{(0)}_\alpha + \epsilon^{1/2} f^{(1)}_\alpha + \epsilon^2 f^{(2)}_\alpha + \ldots \right) = 0, \tag{25}
\]

and

\[- \epsilon \frac{\partial^2}{\partial \xi^2} \left( \phi^{(1)} + \epsilon^{3/2} \phi^{(2)} + \ldots \right) = 4\pi e \sum_{\alpha = \ell, h} \int f^{(0)}_\alpha + \epsilon^{1/2} f^{(1)}_\alpha + \epsilon^2 f^{(2)}_\alpha + \ldots \right) \right) dv, \tag{26}\]

where the symbol $\equiv$ is used to denote the equality in the weak sense. In the following subsections [IV.1] [IV.2] we obtain different expressions for the first and second order perturbations to obtain the modified KiV equation, as well as the conditions for multi-plasmon resonances.

1. **First order perturbations**

Equating the coefficients of $\epsilon$, we get from Eq. (25)

\[
\frac{\partial f^{(1)}_{\alpha}}{\partial \sigma} + \frac{\partial f^{(1)}_{\alpha}}{\partial \xi} + \frac{e_m \omega}{2\pi \hbar^2} \int d\nu' d\nu'' \epsilon^{im(v-v')} \epsilon^{i-1}/h \\
\times \left[ \phi^{(1)}(\xi + \lambda/2) - \phi^{(1)}(\xi - \lambda/2) \right] f^{(0)}_\alpha = 0, \tag{27}\]

By applying to Eq. (27) the Fourier-Laplace transform, given by Eq. (24), we obtain

\[ f^{(1)}_{\alpha} = -\frac{\epsilon}{\hbar} \frac{f^{(0)}_\alpha (v + \frac{hk}{2m}) - f^{(0)}_\alpha (v - \frac{hk}{2m})}{\omega - k\nu} \phi^{(1)}(\xi). \tag{28}\]

In the lowest order of $\epsilon$, i.e., $\epsilon^{1/2}$, the Poisson equation (26) gives

\[ 4\pi e \sum_{\alpha = \ell, h} \int f^{(1)}_\alpha dv = 0. \tag{29}\]

From Eqs. (28) and (29) eliminating $f^{(1)}_{\alpha}$, we obtain the following expression for the wave speed $\lambda$.

\[ \sum_{\alpha = \ell, h} \frac{\omega^2_p}{n_{\alpha} k^2} \int_{\text{n.r.}} \frac{f^{(0)}_\alpha(v)}{\lambda - v^2 - \hbar^2 k^2/4m^2} dv = 0, \tag{30}\]

where ‘n.r.’ stands for the non-resonance region. Equation (30) resembles the quasineutrality condition which agrees with the linear dispersion equation (7) except the term ‘1’ which appears as the effect of dispersion due to the charge separation of electrons. Thus, from Eq. (30) it follows that such a dispersion is a higher-order effect [at least $O(\epsilon^2)$]. Note that in Eq. (30) we only integrate over the non-resonant region, and thus, in order to
2. Second order perturbations

Equating the coefficients of $e^{5/2}$, we obtain from Eq. (28)

$$
\frac{\partial f^{(2)}_\alpha}{\partial \tau} + \frac{e}{\hbar} \frac{\partial f^{(2)}_\alpha}{\partial \xi} + \frac{e m}{2 \pi \hbar^2} \int dr' dr'' e^{i m (v-v') r'/\hbar} \times \left[ \phi^{(2)}_\alpha \left( \xi + \frac{\lambda}{2} \right) - \phi^{(2)}_\alpha \left( \xi - \frac{\lambda}{2} \right) \right] f^{(0)}_\alpha = 0.
$$

Transforming Eq. (31) in view of Eq. (24) and eliminating $f^{(1)}$ by Eq. (28), we obtain

$$
\hat{f}^{(2)}_\alpha = -\frac{(e/\hbar)}{\omega - kv} \left[ f^{(0)}_\alpha \left( v + \frac{\hbar k}{2m} \right) - f^{(0)}_\alpha \left( v - \frac{\hbar k}{2m} \right) \right] \hat{\phi}^{(2)}.
$$

Next, from Eq. (26) equating the coefficients of $e^2$, we get

$$
\frac{\partial^2 \phi^{(1)}}{\partial \xi^2} = -4 \pi e \sum_{\alpha=\pm1,\pm2} \int_{-\infty}^{\infty} f^{(2)}_\alpha dv.
$$

Taking the inverse Fourier-Laplace transform of Eq. (32) and integrating with respect to $v$ over the entire velocity space, we find that the second order density perturbation reduces to

$$
\int_{\text{n.r.}} f^{(2)}_\alpha dv = \frac{e}{\hbar k} \hat{\phi}^{(2)}(\zeta, \tau) \int_{\text{n.r.}} (v - \lambda)^{-1} \left[ f^{(0)}_\alpha \left( v + \frac{\hbar k}{2m} \right) - f^{(0)}_\alpha \left( v - \frac{\hbar k}{2m} \right) \right] dv,
$$

where we have defined a new coordinate $\zeta = \xi - \lambda \sigma$. Some important points are to be noticed. The velocity integral in Eq. (34) can be shown to vanish by the dispersion equation (30). Furthermore, Eq. (32) neither involves the time evolution term nor any other nonlinear terms that involve the first-order perturbations. So, in order to properly include a time evolution term along with the resonance and nonresonance contributions in the nonlinear regime, a slightly different ordering of both $f$ and $\phi$ is necessary. Thus, we consider

$$
\begin{align*}
\hat{f}_\alpha(x, v, t) &= f^{(0)}_\alpha + e f^{(1)}_\alpha + e^2 f^{(2)}_\alpha + \cdots, \\
\hat{\phi}(x, t) &= \epsilon e^{i/2} \phi^{(1)} + e^2 \phi^{(2)} + \cdots.
\end{align*}
$$

Using the expansion (35) and equating the coefficients of $e^{5/2}$ from Eq. (26), we obtain

$$
\begin{align*}
\frac{\partial f^{(2)}_\alpha}{\partial \sigma} + \frac{\partial f^{(2)}_\alpha}{\partial \xi} + \frac{e m}{2 \pi \hbar^2} \int dr' dv' e^{i m (v-v') r'/\hbar} \times \left[ \phi^{(1)} \left( \xi + \frac{r'}{2} \right) f^{(1)}_\alpha - \phi^{(1)} \left( \xi - \frac{r'}{2} \right) f^{(1)}_\alpha \right] = 0.
\end{align*}
$$

By applying the Fourier-Laplace transform to Eq. (36) and eliminating $f^{(1)}$ by Eq. (28), we have

$$
\begin{align*}
\hat{f}^{(2)}_\alpha &= \frac{1}{\omega - kv} \left[ -ie \frac{f^{(0)}_\alpha \left( v + \frac{\hbar k}{2m} \right)}{\hbar (\omega - kv)} \right. \\
&+ \left. - f^{(0)}_\alpha \left( v - \frac{\hbar k}{2m} \right) \right] \frac{\partial \phi^{(1)}}{\partial \tau} + \frac{e^2}{\hbar^2} \hat{\Psi}_\alpha I_\alpha, \\
\hat{\Psi}_\alpha &= i \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\sigma (\omega - \lambda k) \left\{ \phi^{(1)}(\xi, \sigma) \right\}^2 e^{-i(k\xi - \omega \sigma)},
\end{align*}
$$

and

$$
\begin{align*}
I_\alpha(v) &= \left[ f^{(0)}_\alpha \left( v + \frac{\hbar k}{2m} \right) - f^{(0)}_\alpha \left( v - \frac{\hbar k}{2m} \right) \right] \frac{\omega - k \left( v + \frac{\hbar k}{2m} \right)}{\omega - k \left( v - \frac{\hbar k}{2m} \right)} \\
&- f^{(0)}_\alpha \left( v - \frac{\hbar k}{2m} \right) f^{(0)}_\alpha \left( v + \frac{\hbar k}{2m} \right).
\end{align*}
$$

Taking the Fourier-Laplace inverse transform of Eq. (37) and then integrating over the velocity space, the second order density perturbation reduces to

$$
\int_{-\infty}^{\infty} f^{(2)}_\alpha dv = \frac{e}{\hbar} \left[ A_\alpha + \frac{e}{R} B_\alpha \right],
$$

where

$$
\begin{align*}
A_\alpha &= \frac{1}{(2\pi)^2} \int dk d\omega (\omega - \lambda k)^{-1} \frac{\partial \phi^{(1)}}{\partial \tau} \\
&\times J_\alpha(k, \omega) \exp[i(k\xi - \omega \sigma)], \\
J_\alpha(k, \omega) &= \int_{-\infty}^{\infty} (\omega - kv)^{-2} \left[ f^{(0)}_\alpha \left( v + \frac{\hbar k}{2m} \right) \\
&- f^{(0)}_\alpha \left( v - \frac{\hbar k}{2m} \right) \right] dv.
\end{align*}
$$
\[ B_{\alpha} = \frac{i}{2(\pi r)^2} \int \int_{W} dk' d\omega' (\omega' - \pm k')^{-1} \Psi_{\alpha} (\omega', k') \]
\[ \times L_{\alpha}(k, \omega) \exp[i(k' \xi - \omega' \sigma)], \]
\[ L_{\alpha}(k, \omega) = \int_{-\infty}^{\infty} I_{\alpha}(v) (\omega - kv)^{-1} dv. \tag{44} \]

**Multi-plasmon resonance**

Inspecting on the second-order perturbations we find that Eq. (42) can be rewritten as
\[ J_{\alpha}(k, \omega) = \frac{1}{k^2} \int_{-\infty}^{\infty} \left[ \frac{1}{(v - \omega/k - kv)^2} \right] f^{(0)}_{\alpha} dv, \tag{45} \]
Also, from Eqs. (39) and (44) we have
\[ L_{\alpha}(k, \omega) = \frac{1}{k^2v_q} \int_{-\infty}^{\infty} \left( \frac{1}{v - \omega/k - 2v_q} \right) f^{(0)}_{\alpha} dv. \tag{46} \]

Thus, looking at the denominators of the integrands in Eqs. (45) and (46), we note that there are poles at \( v = \omega/k + n v_q, \) where \( n = 0, 1, 2. \) This implies that in the propagation of nonlinear EAWs, the phase velocity resonance, as well as the one- (for \( n = 1 \)) and the two-plasmon (for \( n = 2 \)) resonances can occur in the wave-particle interactions. However, the resonant velocities \( v = \omega/k + n v_q, \) for \( n = 1, 2 \) are not of interest to the present study as the lower resonant velocity \( v = \omega/k - n v_q \) (for \( n = 0, 1, 2 \)) is more likely to cause the necessary wave damping.

In what follows, we calculate the velocity integrals \( J_{\alpha} \) and \( L_{\alpha} \) by considering \( \int_{-\infty}^{\infty} dv = \int_{-\infty}^{0} dv + \int_{0}^{\infty} dv \) and \( \int_{-\infty}^{0} dv \equiv P \int_{-\infty}^{0} dv, \) and choose the Landau contour encircling the pole \( v = \omega/k - n v_q \) below the real axis for \( \Im \omega < 0 . \) Then the integration with respect to \( \omega \) can be performed by closing the \( W \) contour in the lower-half plane encircling the pole at \( \omega = \lambda k. \) Thus, for \( r = 0, 1, 2, \ldots \) and \( n = 1, 2, \) we obtain
\[ \lim_{\omega \rightarrow \lambda k} \int_{-\infty}^{\infty} \frac{f^{(0)}_{\alpha}}{[v - (\omega/k - n v_q)]^{r+1}} dv \]
\[ = P \int_{-\infty}^{\infty} \frac{f^{(0)}_{\alpha}}{[v - (\omega/k - n v_q)]^{r+1}} dv \tag{47} \]
\[ + i \frac{k^2}{|k|^2} \left[ \frac{d}{dv} f^{(0)}_{\alpha}(v) \right]_{v = \lambda k - n v_q}, \]
\[ \lim_{\omega \rightarrow \pm k} J_{\alpha}(k, \omega) \equiv J_{1\alpha}(k, \lambda) + i J_{2\alpha}(k, \lambda), \]
\[ \lim_{\omega \rightarrow \pm k} L_{\alpha}(k, \omega) \equiv L_{1\alpha}(k, \lambda) + i L_{2\alpha}(k, \lambda), \tag{48} \]

where
\[ J_{1\alpha}(k, \lambda) = \frac{1}{k^2} \int C \left[ \frac{1}{(v - \lambda - v_q)^2} - \frac{1}{(v - \lambda + v_q)^2} \right] f^{(0)}_{\alpha} dv, \tag{49} \]
\[ J_{2\alpha}(k, \lambda) = -\frac{\pi}{k^2} \left[ \frac{d}{dv} f^{(0)}_{\alpha}(v) \right]_{v = \lambda - v_q}, \tag{50} \]
\[ L_{1\alpha}(k, \lambda) = \frac{1}{k^2v_q} \int C \left( 1 + \frac{1}{v - \lambda - 2v_q} - \frac{2}{v - \lambda} \right) f^{(0)}_{\alpha}(v) dv, \tag{51} \]
\[ L_{2\alpha}(k, \lambda) = \frac{\pi}{k^2v_q} \left[ f^{(0)}_{\alpha}(v - \lambda + 2v_q) - 2f^{(0)}_{\alpha}(v - \lambda) \right]. \tag{52} \]

Here, \( C \) is the Landau contour which excludes the poles at \( v = \lambda \pm n v_q, n = 0, 2. \)

**The KdV equation**

Noting that
\[ \frac{i}{(2\pi i)} \int dk \int d\omega (\omega - \lambda k)^{-1} e^{i(k \xi - \omega \sigma)} \frac{k}{|k|} \Psi_{\alpha}(k, \omega) \]
\[ = -i \frac{P}{\pi} \int d\zeta \left\{ \frac{\phi^{(1)}(\zeta, \lambda)}{\zeta - \zeta'} \right\}^2, \tag{53} \]
we finally obtain from Eq. (43)
\[ B_{\alpha} = \left\{ \phi^{(1)} \right\}^2 L_{1\alpha} + L_{2\alpha} \frac{P}{\pi} \int d\zeta \left\{ \frac{\phi^{(1)}(\zeta, \lambda)}{\zeta - \zeta'} \right\}^2. \tag{54} \]

Next, differentiating \( A_{\alpha} \) and \( B_{\alpha} \) with respect to \( \zeta = \xi - \lambda \sigma \) we get
\[ \frac{\partial A_{\alpha}}{\partial \zeta} = k J_{1\alpha}(k, \lambda) \frac{\partial \phi^{(1)}}{\partial \tau}, \tag{55} \]
\[ \text{and} \]
\[ \frac{\partial B_{\alpha}}{\partial \zeta} = 2 \phi^{(1)} \frac{\partial \phi^{(1)}}{\partial \zeta} L_{1\alpha} \]
\[ + L_{2\alpha} \frac{P}{\pi} \int \frac{1}{\zeta - \zeta'} \frac{\partial}{\partial \zeta} \left\{ \frac{\phi^{(1)}(\zeta, \tau)}{\zeta - \zeta'} \right\}^2 d\zeta'. \tag{56} \]

In the frame \( \zeta = \xi - \lambda \sigma, \) Eq. (53) reduces to
\[ \frac{\partial^2 \phi^{(1)}}{\partial \zeta^2} = -4 \pi e \sum_{\alpha} \int_{-\infty}^{\infty} f^{(2)}_{\alpha}(v) dv. \tag{57} \]

Differentiating Eq. (57) once with respect to \( \zeta \) and using the relations [i.e., Eqs. (35), (40), (55), and (56)], we obtain the following modified KdV equation with nonlinear Landau damping \( \propto \Gamma \)
\[ \frac{\partial \phi}{\partial \tau} + A \frac{\partial^3 \phi}{\partial \zeta^3} + B \frac{\partial \phi}{\partial \zeta} + \Gamma P \int_{-\infty}^{\infty} \frac{\partial^2 \phi (\zeta', \tau)}{\partial \zeta'} \frac{1}{\zeta - \zeta'} d\zeta' = 0, \tag{58} \]
where \( \phi \equiv \phi^{(1)} \), \( \mathcal{P} \) denotes the Cauchy principal value, and the coefficients of the KdV equation are \( A = 1/P \), \( B = Q/P \) and \( \Gamma = R/P \) with

\[
P = \frac{4\pi e^2}{\hbar} \sum_{\alpha = l,h} \int_C \left[ \frac{1}{(v - \lambda - v_q)^2} - \frac{1}{(v - \lambda + v_q)^2} \right] f^{(0)}_\alpha \, dv + i \text{sgn}(k) \frac{4\pi^2 e^2}{\hbar} \sum_{\alpha = l,h} \frac{d}{dv} f^{(0)}_\alpha(v) \bigg|_{v=\lambda-v_q},
\]

(59)

\[
Q = \frac{16\pi me^3}{(\hbar k)^3} \sum_{\alpha = l,h} \int_C \left( \frac{1}{v - \lambda - 2v_q} + \frac{1}{v - \lambda + 2v_q} - \frac{2}{v - \lambda} \right) f^{(0)}_\alpha(v) dv
\]

(60)

and

\[
R = \frac{8\pi me^3}{(\hbar k)^3} \sum_{\alpha = l,h} \left[ f^{(0)}_\alpha(v - \lambda + 2v_q) - 2f^{(0)}_\alpha(v - \lambda) \right].
\]

(61)

From the coefficients of the KdV equation, we note that \( P \) becomes complex due to the one plasmon resonance, and so are the dispersive (\( \propto A \)), local nonlinear (\( \propto B \)), and the nonlocal nonlinear (\( \propto \Gamma \)) terms. It is interesting to note that although the form of the KdV equation looks similar to that first obtained by Ott and Sudan, and later by many authors (See, e.g., Refs. 16,18) in classical plasmas, however, the Landau damping term \( \propto \Gamma \) appears here as nonlinear due to the phase velocity resonance as well as the two-plasmon resonance processes in the wave-particle interactions. The appearance of such a nonlocal nonlinearity not only modifies the propagation of EAWs but also introduces a new wave damping mechanism which has not been studied before in the context of KdV theory of small-amplitude waves.

To get further insight into the KdV equation, we simplify the coefficients \( P, Q \) and \( R \) by assuming the smallness in the argument of the logarithmic function in Eq. (58). This is valid since \( \xi_\alpha \equiv \mu_\alpha/kBT_\alpha \) may be large and negative for \( T_\alpha > T_{fa} \). Also, we evaluate the integrals by means of Eq. (10) and approximate them as

\[
\mathcal{P} = \int_{-\infty}^{\infty} \frac{\exp(-v^2/v_{ta}^2)}{v - \lambda \pm v_q} dv
\]

\[
= \left\{ \begin{array}{ll}
\frac{\sqrt{\pi}}{2} & -2\sqrt{\pi} \delta_{\alpha+} - \frac{14}{9}\delta_{\alpha\pm} + \cdots, \\
-\frac{1}{2v_{ta}} & \delta_{\alpha+} - 3\delta_{\alpha\pm} + \cdots, \\
\delta_{\alpha\pm} & \delta_{\alpha\pm} > 1,
\end{array} \right.
\]

(62)

where \( \delta_{\alpha\pm} = (\lambda \pm v_q)/\sqrt{2}v_{ta} \). In this way, the expressions for \( P, Q \) and \( R \) reduce to

\[
\Re P = -\frac{6\pi e^2}{m} \sum_{\alpha = l,h} \frac{n_{\alpha}T_\alpha e^{\xi}}{v_{fa}T_{fa}} \left[ \frac{5(\lambda^2 - v_q^2) + 2\lambda v_{ta}}{(\lambda^2 - v_q^2)^2} \right],
\]

V. CONSERVATION LAWS

It is imperative to check whether some conservation laws hold for the nonlocal KdV equation (58). Since we look for a solitary wave solution of Eq. (58), it may be assumed that the function \( \phi(\zeta, \tau) \) and its derivatives with respect to \( \zeta \) vanish as \( \zeta \to \pm \infty \). So, integrating Eq. (68) with respect to \( \zeta \), we obtain

\[
\frac{\partial}{\partial \tau} \int_{-\infty}^{\infty} \phi(\zeta, \tau)d\zeta + A \int_{-\infty}^{\infty} \frac{\partial^2 \phi}{\partial \zeta^2}d\zeta + B \int_{-\infty}^{\infty} \frac{\partial \phi}{\partial \zeta}d\zeta + C \int_{-\infty}^{\infty} \frac{1}{\zeta - \zeta'} \mathcal{P} \, d\zeta' = 0.
\]

(67)
The second and third integrals in Eq. (57) can be made zero using the above boundary conditions. Also, the fourth integral, after changing the order of integration, becomes

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial^2 \phi}{\partial \zeta^2} (\zeta', \tau) P \frac{1}{\zeta - \zeta'} d\zeta d\zeta' \]

(68)

and noting that the integral involving \( P \) vanishes identically. Thus, we have

\[ \frac{\partial}{\partial \tau} \int_{-\infty}^{\infty} \phi (\zeta, \tau) d\zeta = 0. \]  

(69)

From the linear theory (Sec. III) after Fourier analyzing the Poisson equation \( (2) \) one obtains \(-k^2 \phi^{(1)} = 4\pi \sum_{a=1,h} f_0^{(1)} ds = 4\pi \pi e \sum_{a=1,h} n_0 n_a = 4\pi e (n - n_0)\), where \( \sum_{a=1,h} n_0 n_a = n \) is the total number density of particles. Since \( n_0 \) is independent on time, the vanishing of the time derivative of \( \phi = \phi^{(1)} \) in Eq. (69) gives that of the total number density \( n \). Thus from Eq. (69), it follows that the KdV equation (58) conserves the total number of particles.

Next, we multiply Eq. (58) by \( \phi^* (\zeta, \tau) \) and the conjugate of Eq. (58) by \( \phi (\zeta, \tau) \) successively, and add the resulting equations. After integration with respect to \( \zeta \) yields

\[ \frac{\partial}{\partial \tau} \int_{-\infty}^{\infty} |\phi (\zeta, \tau)|^2 d\zeta + \int_{-\infty}^{\infty} \left( A \phi^* \frac{\partial^2 \phi}{\partial \zeta^2} + A^* \phi \frac{\partial^2 \phi^*}{\partial \zeta^2} \right) d\zeta \]

\[ + \int_{-\infty}^{\infty} \left( B |\phi|^2 \frac{\partial \phi}{\partial \zeta} + B^* |\phi|^2 \frac{\partial \phi^*}{\partial \zeta} \right) d\zeta \]

\[ + \int_{-\infty}^{\infty} \left( \Gamma \phi^* (\zeta, \tau) \frac{\partial^2 \phi (\zeta', \tau)}{\partial \zeta'} P \frac{1}{\zeta - \zeta'} \right) d\zeta d\zeta' = 0. \]

(70)

The second and third pairs of integrals of Eq. (70) vanish by evaluating the integrals by parts and using the boundary conditions stated above. The fourth pair of integrals of Eq. (70) can be evaluated by using the fact that \( \phi^* (k, \tau) = \hat{\phi}^* (-k, \tau) \), \( \hat{\phi} (k, \tau) = -2ik |\hat{\phi} (k, \tau)|^2 \), \( k \text{sgn}(k) = |k| \), and

\[ \int_{-\infty}^{\infty} \phi^* (\zeta, \tau) P \frac{1}{\zeta - \zeta'} d\zeta = i\pi \int_{-\infty}^{\infty} \text{sgn}(k) \phi^* (k, \tau) e^{ik\zeta'} dk. \]

(71)

So, we obtain

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi^* (\zeta, \tau) \frac{\partial^2 \phi (\zeta', \tau)}{\partial \zeta'} P \frac{1}{\zeta - \zeta'} d\zeta d\zeta' = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi (\zeta, \tau) \frac{\partial^2 \phi^* (\zeta', \tau)}{\partial \zeta'} P \frac{1}{\zeta - \zeta'} d\zeta d\zeta' \]

(72)

\[ = -4\pi \int_{-\infty}^{\infty} k^2 |\hat{\phi} (k, \tau)|^2 |\hat{\phi}^2 (k, \tau)|^2 dk < 0. \]

Since \( \Gamma = R/P \) with \( R > 0 \) and \( \Re P < 0 \), we have \( \Gamma + \Gamma^* < 0 \). Then Eq. (70) reduces to

\[ \frac{\partial}{\partial \tau} \int_{-\infty}^{\infty} |\phi (\zeta, \tau)|^2 d\zeta \]

\[ + \hat{\Gamma} \int_{-\infty}^{\infty} k^2 |\hat{\phi} (k, \tau)|^2 |\hat{\phi}^2 (k, \tau)|^2 dk = 0, \]

(73)

where \( \hat{\Gamma} = -4\pi (\Gamma + \Gamma^*) > 0 \). The first term in Eq. (73) represents the rate of change of wave energy whereas the second one is the rate of change of energy of resonant particles. Thus, the total energy is conserved. However, since the second term is positive definite, by the \( H \)-theorem we have

\[ \frac{\partial}{\partial \tau} \int_{-\infty}^{\infty} |\phi (\zeta, \tau)|^2 d\zeta \leq 0, \]

(74)

where the equality holds at \( \phi = 0 \). This means that the wave energy decays with time and hence a steady-state solution with finite wave energy does not exist.

VI. SOLITARY WAVE SOLUTION

We note that in absence of the damping effect, i.e., \( \Gamma = 0 \), Eq. (58) reduces to the usual KdV equation:

\[ \frac{\partial \phi}{\partial \tau} + A \frac{\partial^3 \phi}{\partial \zeta^3} + B \phi \frac{\partial \phi}{\partial \zeta} = 0. \]

(75)

A traveling solitary wave solution of Eq. (75) is known, i.e.,

\[ \phi = \phi_0 \text{sech}^2 \left( \zeta - U_0 \tau \right), \]

(76)

where \( \phi_0 = 3U_0/B \) is the constant amplitude, \( W = \sqrt{12A/\phi_0 B} = \sqrt{4A/U_0} \) is the constant width, and \( U_0 = \phi_0 B/3 \) is the constant phase speed of electron-acoustic solitary waves.

Our aim is to find an approximate solitary wave solution of the KdV equation (58) with a small effect of the nonlinear Landau damping (\( \propto \Gamma \)) due to the phase velocity and multi-plasmon resonances. We also assume that \( |A|, |B| \gg |\Gamma| \sim \epsilon \). From Eq. (74) it is evident that an initial perturbation of the form of Eq. (76) will decay to zero with time. So, in presence of the nonlinear Landau damping the solitary wave amplitude \( \phi_0 \) is no longer a constant (but decreases with time), i.e., \( \phi_0 \equiv \phi_0 (\tau) \). Thus, we introduce a new space co-ordinate \( z \) in a frame moving with the solitary wave and normalized to its width as

\[ z = \left( \zeta - \frac{1}{3} B \int_{0}^{\tau} \phi_0 (\tau) d\tau \right) / W, \]

(77)

and two time scales, namely \( \eta_0 = \tau \) and \( \eta_1 = \Gamma \tau \). Next, we expand the solution with the scaling parameter \( \Gamma \sim \epsilon \) as

\[ \phi (z, \tau) = \phi^{(0)} + \Gamma \phi^{(1)} + \Gamma^2 \phi^{(2)} + \ldots. \]  

(78)
where \( \phi^{(i)} \) is a function of \( \tau = \tau_0, \tau_1, \tau_2, \ldots \) with \( \tau_1 = \Gamma^4 \tau \) for \( i = 0, 1, 2, 3 \ldots \). Substituting Eqs. (77) and (78) into Eq. (68), we get
\[
\left[ \frac{\partial \phi^{(0)}}{\partial \eta} + \frac{\partial^3 \phi^{(0)}}{\partial z^3} + 4 \left( 3 \frac{\phi^{(0)}}{\phi_0} - 1 \right) \frac{\partial \phi^{(0)}}{\partial z} \right] + \Gamma \left\{ \frac{\partial \phi^{(1)}}{\partial \eta} + \frac{\partial}{\partial z} \left[ \frac{\phi^{(0)}}{\phi_0} \right] \right\} = 0 \quad (\text{79}),
\]
where
\[
\rho = \sqrt{A} \sqrt{\frac{12}{\Gamma} \phi_0 B}^{-3/2}, \quad (\text{80})
\]
and
\[
M[\phi^{(0)}] = - \left[ \frac{\partial \phi^{(0)}}{\partial \eta} + \frac{\partial \phi^{(0)}}{2 \partial \eta^{(0)}} \right] \frac{\partial \phi^{(0)}}{\partial z} + \sqrt{\frac{\phi^{(0)} B}{12 A}} \rho \int_{-\infty}^{\infty} \frac{dz}{\partial z} \left\{ \phi^{(0)}(z, \tau) \right\}^2 \frac{dz'}{z - z'} \quad (\text{81}).
\]
Applying the initial and boundary conditions, i.e.,
\[
\phi(z, 0) = \phi_0 \sech^2 z, \quad \phi(\pm \infty, \tau) = 0, \quad (\text{82})
\]
where \( \phi_0 = \phi_0(\tau = 0) \) and solving the order of unity equation we get
\[
\phi^{(0)}(z, \eta_0, \eta_1) = \phi_0(\eta_1) \sech^2 z, \quad (\text{83})
\]
Next, to the order of \( \Gamma \), we have
\[
\frac{\partial \phi^{(1)}}{\partial \eta} + L[\phi^{(1)}] = M[\phi^{(0)}], \quad (\text{84})
\]
with \( \phi^{(1)}(\pm \infty, \eta_0) = 0, \phi^{(1)}(\beta, 0) = 0, \) and
\[
L[\phi^{(1)}] = \frac{1}{\rho} \frac{\partial}{\partial z} \left[ \frac{\phi^{(0)}}{\phi_0} \right] \frac{\partial^2 \phi^{(1)}}{\partial z^2} + 4(3 \sech^2 z - 1) \phi^{(1)}. \quad (\text{85})
\]
For the existence of solutions of Eq. (84), it is necessary that \( M[\phi^{(0)}] \) must be orthogonal to all solutions \( g(z) \) of \( L^+(g) \) which satisfy \( g(\pm \infty) = 0 \), where \( L^+ \) is the operator adjoint to \( L \), defined by,
\[
\int_{-\infty}^{\infty} \psi_1(z) L[\psi_2(z)] dz = \int_{-\infty}^{\infty} \psi_1(z) L^+[\psi_2(z)] dz, \quad (\text{86})
\]
with \( \psi_1(\pm \infty) = \psi_2(\pm \infty) = 0 \) and the only solution of \( L^+(g) = 0 \) is \( g(z) = \sech^2 z \). Thus, we have
\[
\int_{-\infty}^{\infty} \sech^2 z M[\phi^{(0)}] dz = 0, \quad (\text{87})
\]
which gives
\[
\frac{\partial \phi_0}{\partial \tau} + \frac{\Gamma}{2} \sqrt{\frac{B}{3 A}} \phi_0^{5/2} + \frac{\partial \phi_0}{\partial \tau} \sqrt{\frac{B}{3 A}} \phi_0 \frac{\partial}{\partial z} \left[ \frac{\sech^2 z}{\sech^4 z} \right] \frac{dzdz'}{z - z'} = 0. \quad (\text{88})
\]
Finally, we obtain for the KdV equation (58)
\[
\phi_0(\tau) = \phi_0 \left( 1 + \frac{\tau}{\tau_0} \right)^{-2/3}, \quad (\text{89})
\]
where
\[
\tau_0^{-1} = \frac{3}{4} \Gamma \sqrt{\frac{B}{3 A}} \phi_0^{5/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sech^2 z \frac{\partial^2 \phi_0}{\partial z^2} \left( \sech^4 z \right) \frac{dzdz'}{z - z'}. \quad (\text{90})
\]
From Eq. (89) it is noted that the nonlinear Landau damping causes the wave amplitude to decay with time \( \sim (\tau + \tau_0)^{-2/3} \). This decay rate is slower than \( \sim (\tau + \tau_0)^{-2} \) due to linear Landau damping that predicted by Ott and Sudan in classical plasmas.

VII. DISCUSSION AND CONCLUSION

Using the Wigner-Moyal equation, we have studied the linear and nonlinear wave particle-interactions and the evolution of EAWs in a partially degenerate quantum plasma with two groups of electrons having temperatures larger than the Fermi temperature. We have considered the background distributions of electrons as given by the Fermi-Dirac distribution at finite temperature. With this choice the linear resonance can not be ruled out as there always remain a few percentage of particles in the tail of the Fermi-Dirac distribution that can also participate in the wave-particle interactions.

In the linear limit, a general (linear) dispersion relation is derived which manifests the coupling of high-frequency Langmuir waves (LWs) and the relatively low-frequency EAWs. While the high-frequency branch is recovered in the limit \( \omega / k \ll v_q \gg v_{th} \), the low-frequency EAW mode is obtained in the opposite limit, i.e., \( \omega / k \ll v_q \ll v_{th} \). Interestingly, this EAW mode has the requisite properties that apply to the KdV theory, and for small wave number it has the form \( \omega_r \approx 2.5 k v_{th} \), i.e., the wave frequency is bit higher than that of the classical mode \( \omega_r \approx 1.3 k v_{th} \). Furthermore, in contrast to the classical mode and depending on the temperature or density ratios of the two species of electrons, the EAWs can turn over, going to zero, and then assume negative values for some values of the wave number \( K \equiv n_k v_{th} \) beyond its critical value \( K_c \). However, the linear damping rate \( \gamma_{tL} / \omega_{ph} \) remains small \((\sim 0.01)\) at some lower and higher values of \( K \) within \( 0 \ll K < 1 \). This enables us to consider the regime \( 0 \leq K \leq 0.5 \) or \( h k / m v_{th} \leq 0.2 \) such that the Wigner-Moyal equation is still applicable to the present theory.

Using the multiple scale expansion technique, the evolution of weakly nonlinear EAWs is shown to be governed by a modified KdV equation with a nonlocal nonlinearity. The latter appears due to the phase velocity as well as the multi-plasmon resonances. Since the KdV equation accounts for up to the second order perturbations, the lower resonance velocity is that of the two-plasmon processes.
We have also verified the conservation laws. It is found that while the total number of particles is conserved, the wave energy decays with time and so a steady-state solution of the KdV equation with finite wave energy does not exist. An approximate analytic solitary wave solution of the KdV equation is also obtained. It is shown that the wave amplitude decays with time \( \sim (\tau + \tau_0)^{-2/3} \) which is slower than \( \sim (\tau + \tau_0)^{-2} \) (due to linear Landau damping) as predicted by Ott and Sudan in classical plasmas.

In conclusion, we have considered a plasma which is not in thermodynamic equilibrium. The theory of EAWs can be studied for plasmas in thermodynamic equilibrium (i.e., for a single species with a temperature), however, for such a background distribution as Eq. 3, the EAWs tend to have a higher damping rate, making the nonlinear analysis less important. The results are independent of the background distribution of electrons. One can consider any other form of the background distribution function that leads EAWs fulfill the right properties for its description by the KdV equation. For example, Fermi-Dirac distribution with arbitrary degeneracy of electrons. However, in the non-degenerate limit \( (T_\alpha \gg T_{F\alpha}) \), the Fermi-Dirac distribution reduces to that of the Maxwell-Boltzmann for which Vlasov equation applies. The motivation for considering the finite temperature \( (T_\alpha > T_{F\alpha}) \) is that the use of the Wigner-Moyal equation (rather than the Vlasov equation) brings about the appearance of multi-plasmon resonances along with the phase velocity resonance. Also, the lower \([\text{than the phase velocity } (\omega/k)]\) resonant velocities \( \omega/k - n v_q \), \( n = 1, 2 \) that are associated with the resonant particles absorbing wave quanta can give the wave damping more easily, and the magnitudes of the resonance terms with \( \omega/k - n v_q \) are rather higher than that are associated with \( \omega/k \). This means that the two-plasmon resonance process is the dominant wave-damping mechanism in the description of EAWs in quantum plasmas. On the other hand, the choice of a fully degenerate \( (T_\alpha \ll T_{F\alpha}) \) background is also inadmissible as electrons with two groups can not be fully degenerate from quantum mechanical points of view. So, a more general background distribution which will include these as special cases may be considered for better understanding the underlying physics of wave-particle interactions in degenerate plasmas. However, this is left for a future work.

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DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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