Randomness extraction involves the processing of purely classical information and is therefore usually studied in the framework of classical probability theory. However, such a classical treatment is generally too restrictive for applications, where side information about the values taken by classical random variables may be represented by the state of a quantum system. This is particularly relevant in the context of cryptography, where an adversary may make use of quantum devices. Here, we show that the well known construction paradigm for extractors proposed by Trevisan is sound in the presence of quantum side information.

We exploit the modularity of this paradigm to give several concrete extractor constructions, which, e.g., extract all the conditional (smooth) min-entropy of the source using a seed of length poly-logarithmic in the input, or only require the seed to be weakly random.

1 Introduction

Randomness extraction is the art of generating (almost) uniform randomness from any weakly random source $X$. More precisely, a randomness extractor (or, simply extractor) is a function $\text{Ext}$ that takes as input $X$ together with a uniformly distributed (and usually short) string $Y$, called the seed, and outputs a string $Z$. One then requires $Z$ to be almost uniformly distributed whenever the min-entropy of $X$ is larger than some threshold $k$, i.e.,

$$H_{\min}(X) \geq k \implies Z := \text{Ext}(X, Y) \text{ statistically close to uniform.}$$

(1)
The min-entropy of a random variable $X$ is directly related to the probability of correctly guessing the value of $X$ using an optimal strategy: $2^{-H_{\text{min}}(X)} = \max_x P_X(x)$. Hence Criterion (1) can be interpreted operationally: if the maximum probability of successfully guessing the input of the extractor, $X$, is sufficiently low then its output is statistically close to uniform.

In most applications, such as privacy amplification \cite{BBR88, BBCM95}, or simply when applying two extractors in succession\footnote{When applying two extractors in succession to the same input, with the goal that the two outputs are jointly uniform, the output of the first extractor needs to be considered as side information when analyzing the second extractor.} to the same input $X$, there is a notion of side information, which describes the information about the input which is contained in the environment, or accessible to an adversary. Notions of randomness such as the guessing probability, min-entropy or the uniformity of a random variable naturally always depend on the side information relative to which they are defined, and in particular one would like the output of the extractor to be uniform \textit{with respect to the side information}. Hence we may make this requirement explicit in our formulation of Criterion (1) by denoting by $E$ all side information with respect to which the extractor’s output should be uniform:

$$H_{\text{min}}(X|E) \geq k \implies Z := \text{Ext}(X, Y) \text{ statistically close to uniform } \quad (2)$$

conditioned on $E$,

where $H_{\text{min}}(X|E)$ is the conditional min-entropy, formally defined in Section 2.2. This conditioning naturally extends the operational interpretation of the min-entropy to scenarios with side information, i.e., $2^{-H_{\text{min}}(X|E)}$ is the maximum probability of correctly guessing $X$, given access to side information $E$ \cite{KRS09}.

Interestingly, the relationship between the two Criteria (1) and (2) depends on the physical nature of the side information $E$, i.e., whether $E$ is represented by the state of a classical or a quantum system. In the case of purely classical side information, $E$ may be modeled as a random variable and it is known that the two criteria are essentially equivalent (see Lemma 3.3 for a precise statement). But in the general case where $E$ is a quantum system, Criterion (2) is strictly stronger than (1); it was shown in \cite{GKK+07} that there exist extractors that fulfill (1) but for which (2) fails (see also \cite{KR07} for a discussion).

Since our world is inherently non-classical, it is of particular importance that (2) rather than the weaker Criterion (1) be taken as the relevant criterion for the definition of extractors. For example, in the context of cryptography, one typically uses extractors to generate secret keys, i.e., randomness that is uniform from an adversary’s point of view. Even if the extractor itself is classical, nothing can prevent an adversary from storing information $E$ in a quantum system, so Criterion (1) does not imply security. Randomness recycling is another simple example involving quantum side information. If we run a (simulation of) a quantum system $E$ using randomness $X$, approximately $H_{\text{min}}(X|E)$ bits of $X$ can be reused. Applying a function $\text{Ext}$ which has been shown to fulfill (1) but not (2) could result in an output $Z$ which is still correlated to the system $E$.

Moreover, since it is known that the smooth conditional min-entropy precisely characterizes the optimal amount of uniform randomness that can be extracted from $X$ while being independent from $E$ \cite{Ren05}, one may argue that Criterion (2) is indeed the correct definition for randomness extraction.
In particular, we would like to point out that the popular bounded storage model — in which the entropy of the source \( H_{\text{min}}(X|E) \) is lower-bounded by \( H_{\text{min}}(X) - H_0(E) \) and \( H_0(E) \) denotes the number of qubits needed to store \( E \) — is strictly weaker: there are sources \( X \) and nontrivial side information \( E \) such that \( H_{\text{min}}(X) - H_0(E) \ll H_{\text{min}}(X|E) \) and extractors which are sound for any input with \( H_{\text{min}}(X) - H_0(E) \geq k \), but cannot be applied to all sources with \( H_{\text{min}}(X|E) \geq k \). An extractor which has only been proven sound in the bounded storage model can thus only extract \( H_{\text{min}}(X) - H_0(E) \) bits of uniform randomness instead of the optimal \( H_{\text{min}}(X|E) \) bits. For the same reason in the purely classical case, no recent work defines classical extractors for randomness sources with side information stored in bounded classical memories.

Furthermore, in applications where extractors are used, the increased generality of the conditional min-entropy over the bounded storage model is often what is needed. For example in quantum key distribution, where extractors are used for privacy amplification \cite{Ren05}, it is generally impossible to bound the adversary’s memory size.

\subsection*{Related results.}
In the standard literature on randomness extraction, constructions of extractors are usually shown to fulfill Criterion (1) for certain values of the threshold \( k \) (see \cite{Zuc90} as well as \cite{Sha02} for an overview). However, only a few constructions have been shown to fulfill Criterion (2) with arbitrary quantum side information \( E \). Among them is two-universal hashing \cite{Ren05,TSSR10} as well as constructions based on the sample-and-hash approach \cite{KR07}.

Recently, Ta-Shma \cite{TaShma09} studied Trevisan’s extractor construction \cite{Tre01} in the bounded quantum storage model. Although his proof requires the output length to be much smaller than the min-entropy of the original data, the result was a breakthrough because it, for the first time, implied the existence of “quantum-proof” extractors requiring only short seeds (logarithmic in the input length). More recently, two of the present authors \cite{DV10} were able to improve the output length that Trevisan’s extractor could provably extract in the presence of a quantum bounded-storage adversary, bringing it close to what is known for the case of classical adversaries. However, both these results are proved in the bounded quantum storage model, which, as discussed previously, only allows the extractor to output at most \( H_{\text{min}}(X) - H_0(E) \) bits. This expression can in general be arbitrarily smaller than \( H_{\text{min}}(X|E) \), and in some cases may even become 0 (or negative) for \( n \)-bit sources for which it is possible to extract \( \Omega(n) \) bits of randomness.

Subsequent to this work, Ben-Aroya and Ta-Shma \cite{BATS10} showed how two versions of Trevisan’s extractor, shown quantum-proof in this paper, can be combined to extract a constant fraction of the min-entropy of an \( n \)-bit source with a seed of length \( O(\log n) \), when \( H_{\text{min}}(X|E) > n/2 \). This is better than the

\footnote{This can easily be seen by considering the following example. Let \( X \) be uniformly distributed on \( \{0,1\}^n \) and \( E \) be \( X \) with each bit flipped with constant probability \( \epsilon < 1/2 \). Then \( H_{\text{min}}(X|E) = \Theta(n) \), but \( H_{\text{min}}(X) - H_0(E) = 0 \).}

\footnote{Restricting the class of randomness sources further than by bounding their min-entropy can have advantages, e.g., if we consider only bit-fixing sources, or sources generated by a random walk on a Markov chain, then the extractor can be deterministic. (See \cite{Sha02} for a brief overview of restricted families of sources studied in the literature.) There is however no known advantage (e.g., in terms of seed length) in considering only input sources with side information stored in memory of bounded size, whether it is classical or quantum memory.}
straightforward application of Trevisan’s extractor analyzed here, which requires $O(\log^2 n)$ bits of seed for the same output size (but works for any $H_{\min}(X|E)$).

**Our results.** In this work, we show that the performance of Trevisan’s extractor does not suffer in the presence of quantum side information. More precisely, we show that the output length of the extractor can be close to the optimal conditional min-entropy $H_{\min}(X|E)$ (see Corollary 5.4 for the exact parameters). This is the first proof of security of an extractor with poly-logarithmic seed meeting Criterion (2) in the presence of arbitrary quantum side information.

More generally, we prove security of a whole class of extractors. It has been observed, by, e.g., Lu and Vadhan [Lu04, Vad04], that Trevisan’s extractor [Tre01] (and variations of it, such as [RRV02]) is a concatenation of the outputs of a one-bit extractor with different pseudo-random seeds. Since the proof of the extractor property is independent of the type of the underlying one-bit extractor (and to some extent the construction of the pseudo-random seeds), our result is valid for a generic scheme (defined in Section 4.1, Definition 4.2). We find that the performance of this generic scheme in the context of quantum side information is roughly equivalent to the (known) case of purely classical side information (Section 4.2, Theorem 4.6).

Our argument follows in spirit the work of De and Vidick [DV10]. Technically, the proof is essentially a concatenation of the two following ideas.

- In the first part of the original proof of Trevisan [Tre01], it is shown that the ability to distinguish the extractor’s output from uniform implies the ability to distinguish the output of the underlying one-bit extractor from uniform (a list-decodable code in Trevisan’s original scheme). Ta-Shma has argued that this claim is still true in the context of quantum side information [TS09], by treating the adversary as an oracle and measuring its memory size by counting the queries to the oracle. We extend this result to the case of arbitrary quantum side information, where the entropy of the source is measured with the conditional min-entropy, and show that it still holds even if the seed of the underlying one-bit extractor is not fully uniform.

- This reduces the problem to proving that the one-bit extractor used in the construction is quantum-proof. However, because for one-bit extractors, the more general Criterion (2) is essentially equivalent to the usual Criterion (1), as shown by König and Terhal [KT08], the claim follows from known classical results on one-bit extractors with only a small loss in the error parameter.

This proof structure results in a very modular extractor construction paradigm, which allows arbitrary one-bit extractors and pseudo-random seeds to be “plugged in,” producing different final constructions, optimized for different needs, e.g., maximizing the output length, minimizing the seed, or even using a non-uniform seed if the underlying one-bit extractor also uses a non-uniform seed. In Table 1 we give a brief overview of the final constructions proposed.

**Organization of the paper.** We first define the necessary technical tools in Section 2 in particular the conditional min-entropy. In Section 3 we give
formal definitions of extractors and discuss briefly how much randomness can be extracted from a given source. Section 4 contains the description of Trevisan’s extractor construction paradigm and a proof that it is still sound in the presence of quantum side information. Then in Section 5 we plug in various one-bit extractors and pseudo-random seed constructions, resulting in, amongst others, a construction which is nearly optimal in the amount of randomness extracted in Section 5.1 (which is identical to the best known bound in the classical case [RRV02] for Trevisan’s extractor), and a construction which is still sound if there is a small linear entropy loss in the seed in Section 5.4. Finally, in Section 6, we mention a few classical results which modify and improve Trevisan’s extractor, but for which the correctness in the presence of quantum side information does not seem to follow immediately from this work.

2 Technical preliminaries

2.1 Notation

We write \([N]\) for the set of integers \(\{1, \ldots, N\}\). If \(x \in \{0, 1\}^n\) is a string of length \(n\), \(i \in [n]\) an integer, and \(S \subseteq [n]\) a set of integers, we write \(x_i\) for the \(i^{\text{th}}\) bit of \(x\), and \(x_S\) for the string formed by the bits of \(x\) at the positions given by the elements of \(S\).

\(\mathcal{H}\) always denotes a finite-dimensional Hilbert space. We denote by \(\mathcal{P}(\mathcal{H})\) the set of positive semi-definite operators on \(\mathcal{H}\). We define the set of normalized quantum states \(\mathcal{S}(\mathcal{H}) := \{\rho \in \mathcal{P}(\mathcal{H}) : \text{tr } \rho = 1\}\) and the set of sub-normalized quantum states \(\mathcal{S}_\leq(\mathcal{H}) := \{\rho \in \mathcal{P}(\mathcal{H}) : \text{tr } \rho \leq 1\}\).

We write \(\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B\) for a bipartite quantum system and \(\rho_{AB} \in \mathcal{P}(\mathcal{H}_{AB})\) for a bipartite quantum state. \(\rho_A = \text{tr}_B(\rho_{AB})\) and \(\rho_B = \text{tr}_A(\rho_{AB})\) denote the corresponding reduced density operators.

If a classical random variable \(X\) takes the value \(x \in \mathcal{X}\) with probability \(p_x\), it can be represented by the state \(\rho_X = \sum_{x \in \mathcal{X}} p_x |x\rangle \langle x|\), where \(|\{x\}_{x \in \mathcal{X}}\rangle\) is an orthonormal basis of a Hilbert space \(\mathcal{H}_X\). If the classical system \(X\) is part of a composite system \(XB\), any state of that composite system can be written as \(\rho_{XB} = \sum_{x \in \mathcal{X}} p_x |x\rangle \langle x| \otimes \rho_B\).
\[ \| \cdot \|_{\text{tr}} \] denotes the trace norm and is defined by \[ \| A \|_{\text{tr}} := \text{tr} \sqrt{A^\dagger A}. \]

### 2.2 Min-entropy

To measure how much randomness a source contains and can be extracted, we need to use the *smooth conditional min-entropy*. This entropy measure was first defined by Renner [Ren05], and represents the optimal measure for randomness extraction in the sense that it is always possible to extract that amount of almost-uniform randomness from a source, but never more.

**Definition 2.1** (conditional min-entropy). Let \( \rho_{AB} \in \mathcal{S}_\leq (\mathcal{H}_{AB}) \). The min-entropy of \( A \) conditioned on \( B \) is defined as

\[
H_{\text{min}}(A|B)_{\rho} := \max_{\sigma_B \in \mathcal{S}(\mathcal{H}_B)} \left\{ \lambda \in \mathbb{R} : 2^{-\lambda} A \otimes \sigma_B \geq \rho_{AB} \right\}.
\]

We will often drop the subscript \( \rho \) when there is no doubt about what underlying state is meant.

This definition has a simple operational interpretation when the first system is classical, which is the case we consider. König et al. [KRS09] showed that for a state \( \rho_{XB} = \sum_{x \in X} p_x |x\rangle \langle x| \otimes \rho_x^B \) classical on \( X \),

\[
H_{\text{min}}(X|B)_{\rho} = -\log p_{\text{guess}}(X|B)_{\rho},
\]

where \( p_{\text{guess}}(X|B)_{\rho} \) is the maximum probability of guessing \( X \) given \( B \), namely

\[
p_{\text{guess}}(X|B)_{\rho} := \max_{\{E_x^B\}_{x \in X}} \left( \sum_{x \in X} p_x \text{tr} (E_x^B \rho_x^B) \right),
\]

where the maximum is taken over all POVMs \( \{E_x^B\}_{x \in X} \) on \( B \). If the system \( B \) is empty, then the min-entropy of \( X \) reduces to the standard definition, \( H_{\text{min}}(X) = -\log \max_{x \in X} p_x \) (sometimes written \( H_{\infty}(X) \)). In this case the connection to the guessing probability is particularly obvious: when no side information is available, the best guess we can make is simply the value \( x \in X \) with highest probability.

As hinted at the beginning of this section, the min-entropy is not quite optimal, in the sense that it is sometimes possible to extract more randomness. However, the *smooth* min-entropy is optimal. This information measure consists in maximizing the min-entropy over all sub-normalized states \( \varepsilon \)-close to the actual state \( \rho_{XB} \) of the system considered. Thus by introducing an extra error \( \varepsilon \), we have a state with potentially much more entropy. (See Section 3.2 for more details.)

**Definition 2.2** (smooth min-entropy). Let \( \varepsilon \geq 0 \) and \( \rho_{AB} \in \mathcal{S}_\leq (\mathcal{H}_{AB}) \), then the \( \varepsilon \)-smooth min-entropy of \( A \) conditioned on \( B \) is defined as

\[
H_{\text{min}}^\varepsilon(A|B)_{\rho} := \max_{\rho_{AB} \in \mathcal{B}^\varepsilon(\rho_{AB})} H_{\text{min}}(A|B)_{\rho},
\]

where \( \mathcal{B}^\varepsilon(\rho_{AB}) \subseteq \mathcal{S}_\leq (\mathcal{H}_{AB}) \) is a ball of sub-normalized states of radius \( \varepsilon \) around \( \rho_{AB} \).

Theoretically any distance measure could be used to define an \( \varepsilon \)-ball. We use the purified distance, \( P(\rho, \sigma) := \sqrt{1 - F(\rho, \sigma)}^2 \), where \( F(\cdot, \cdot) \) is the fidelity, since this measure has some advantages over other metrics such as the trace distance. The only propriety of the purified distance we need in this work is that it is larger than the trace distance, i.e., \( P(\rho, \sigma) \geq \frac{\| \rho - \sigma \|_\text{tr}}{2} \). We refer to [TCR10] for a formal definition of the purified distance (and fidelity) on sub-normalized states and a discussion of its advantages.
3 Extractors

3.1 Extractors, side information, and privacy amplification

An extractor $\text{Ext} : \{0,1\}^n \times \{0,1\}^d \rightarrow \{0,1\}^m$ is a function which takes a weak source of randomness $X$ and a uniformly random, short seed $Y$, and produces some output $\text{Ext}(X,Y)$, which is almost uniform. The extractor is said to be strong, if the output is approximately independent from the seed.

**Definition 3.1 (strong extractor).** A function $\text{Ext} : \{0,1\}^n \times \{0,1\}^d \rightarrow \{0,1\}^m$ is a $(k, \varepsilon)$-strong extractor with uniform seed, if for all distributions $X$ with $H_{\min}(X) \geq k$ and a uniform seed $Y$, we have

$$\frac{1}{2} \| \rho_{\text{Ext}(X,Y)Y} - \rho_U \otimes \rho_Y \|_{\text{tr}} \leq \varepsilon,$$

where $\rho_U$ is the fully mixed state on a system of dimension $2^m$.

Using the connection between min-entropy and guessing probability (see Eq. (3)), a $(k, \varepsilon)$-strong extractor can be seen as a function which guarantees that if the guessing probability of $X$ is not too high ($\rho_{\text{guess}}(X) \leq 2^{-k}$), then it produces a random variable which is approximately uniform and independent from the seed $Y$.

As discussed in the introduction, we consider here a more general situation involving side information, denoted by $E$, which may be represented by the state of a quantum system. We then want to find some function $\text{Ext}$ such that, if the probability of guessing $X$ given $E$ is not too high, $\text{Ext}$ can produce a random variable $\text{Ext}(X,Y)$ which is approximately uniform and independent from the seed $Y$ and the side information $E$. Equivalently, one may think of a privacy amplification scenario [BBR88, BBCM95], where $E$ is the information available to an adversary and where the goal is to turn weakly secret data $X$ into a secret key $\text{Ext}(X,Y)$, where the seed $Y$ is assumed to be public. (In typical key agreement protocols, the seed is chosen by the legitimate parties and exchanged over public channels.)

The following definition covers the general situation where the side information $E$ may be represented quantum-mechanically. The case of purely classical side information is then formulated as a restriction on the nature of $E$.

**Definition 3.2 (quantum-proof strong extractor).** A function $\text{Ext} : \{0,1\}^n \times \{0,1\}^d \rightarrow \{0,1\}^m$ is a quantum-proof $(k, \varepsilon)$-strong extractor with uniform seed, if for all states $\rho_{XE}$ classical on $X$ with $H_{\min}(X|E) \rho \geq k$, and for a uniform seed $Y$, we have

$$\frac{1}{2} \| \rho_{\text{Ext}(X,Y)YE} - \rho_U \otimes \rho_Y \otimes \rho_E \|_{\text{tr}} \leq \varepsilon,$$

where $\rho_U$ is the fully mixed state on a system of dimension $2^m$.

The function $\text{Ext}$ is a classical-proof $(k, \varepsilon)$-strong extractor with uniform seed if the same holds with the system $E$ restricted to classical states.

---

5A more standard classical notation would be $\frac{1}{2} \| \text{Ext}(X,Y) \circ Y - U_m \circ Y \| \leq \varepsilon$, where the distance metric is the variational distance. However, since classical random variables can be represented by quantum states diagonal in the computational basis, and the trace distance reduces to the variational distance, we use the quantum notation for compatibility with the rest of this work.
It turns out that if the system $E$ is restricted to classical information about $X$, then this definition is essentially equivalent to the conventional Definition 3.1.

**Lemma 3.3 ([KT08, Proposition 1])**. Any $(k, \varepsilon)$-strong extractor is a classical-proof $(k + \log 1/\varepsilon, 2\varepsilon)$-strong extractor.

However, if the system $E$ is quantum, this does not necessarily hold. Gavinsky et al. [GKK+07] give an example of a $(k, \varepsilon)$-strong extractor, which breaks down in the presence of quantum side information, even when $H_{\min}(X|E)$ is significantly larger than $k$.

**Remark 3.4**. In this section we defined extractors to use a uniform seed, as this is the most common way of defining them. Instead one could use a seed which is only weakly random, but require it to have a min-entropy larger than a given threshold, $H_{\min}(Y) \geq s$. The seed must still be independent from the input and the side information. We redefine extractors formally this way in Appendix A.1. All the considerations of this section, in particular Lemma 3.3 and the gap between classical and quantum side-information, also apply if the seed is only weakly random. In the following, when we simply talk about a strong extractor, without specifying the nature of the seed, we are referring to both uniform seeded and weakly random seeded extractors.

### 3.2 Extracting more randomness

Radhakrishnan and Ta-Shma [RTS00] have shown that a $(k, \varepsilon)$-strong extractor $\text{Ext} : \{0,1\}^n \times \{0,1\}^d \rightarrow \{0,1\}^m$ will necessarily have

$$m \leq k - 2 \log 1/\varepsilon + O(1).$$

(4)

However, in some situations we can extract much more randomness than the min-entropy. For example, let $X$ be distributed on $\{0,1\}^n$ with $\Pr[X = x_0] = 1/n$ and for all $x \neq x_0$, $\Pr[X = x] = \frac{1}{n^2}$. We have $H_{\min}(X) = \log n$, so using a $(\log n, 1/n)$-strong extractor we could obtain at most $\log n$ bits of randomness. But $X$ is already $1/n$-close to uniform, since $\frac{1}{n}\|\rho_X - \rho_U\|_{tr} \leq \frac{1}{n}$. So we already have $n$ bits of nearly uniform randomness, exponentially more than by using a $(\log n, 1/n)$-strong extractor.

In the case of quantum extractors, similar examples can be found, e.g., in [TCR10, Remark 22]. However, an upper bound on the extractable randomness can be obtained by replacing the min-entropy by the smooth min-entropy [Definition 2.2]. More precisely, the total number of $\varepsilon$-uniform bits that can be extracted in the presence of side information $E$ can never exceed $H_{\min}(X|E)$ [Ren05, Section 5.6].

Conversely, the next lemma implies that an extractor which is known to extract $m$ bits from any source such that $H_{\min}(X|E) \geq k$ can in fact extract the same number of bits, albeit with a slightly larger error, from sources which only satisfy $H_{\min}(X|E) \geq k$, a much weaker requirement in some cases.

**Lemma 3.5.** If $\text{Ext} : \{0,1\}^n \times \{0,1\}^d \rightarrow \{0,1\}^m$ is a quantum-proof $(k, \varepsilon)$-strong extractor, then for any state $\rho_{XE}$ and any $\varepsilon' > 0$ with $H_{\min}^{\varepsilon'}(X|E) \rho \geq k$,

$$\frac{1}{2} \|\rho_{\text{Ext}(X,Y)|YE} - \rho_{U_m} \otimes \rho_Y \otimes \rho_E\|_{tr} \leq \varepsilon + 2\varepsilon'.$$
Proof. Let \( \tilde{\rho}_{XE} \) be the state \( \varepsilon' \)-close to \( \rho_{XE} \) for which \( H_{\min}(X|E) \) reaches its maximum. Then

\[
\frac{1}{2} \left| \frac{1}{2} \left| \rho_{\text{Ext}(X,Y)Y} - \rho_{U_m} \otimes \rho_Y \otimes \rho_E \right|_{\text{tr}} \right|
\leq \frac{1}{2} \left| \frac{1}{2} \left| \rho_{\text{Ext}(X,Y)Y} - \tilde{\rho}_{\text{Ext}(X,Y)Y} \right|_{\text{tr}} + \frac{1}{2} \left| \tilde{\rho}_{\text{Ext}(X,Y)Y} - \rho_{U_m} \otimes \rho_Y \otimes \rho_E \right|_{\text{tr}} \right|
\leq \frac{1}{2} \left| \tilde{\rho}_{\text{Ext}(X,Y)Y} - \rho_{U_m} \otimes \rho_Y \otimes \rho_E \right|_{\text{tr}} + \left| \rho_{XE} - \tilde{\rho}_{XE} \right|_{\text{tr}} \leq \varepsilon + 2\varepsilon'.
\]

In the second inequality above we used (twice) the fact that a trace-preserving quantum operation can only decrease the trace distance. And in the last line we used the fact that the purified distance — used to measure the distance between two states (see Definition 2.2) — is larger than the trace distance.

Remark 3.6. Since a \((k, \varepsilon)\)-strong extractor can be applied to any source with smooth min-entropy \( H_{\min}(X|E) \geq k \), we can measure the entropy loss of the extractor — namely how much entropy was not extracted — with

\[
\Delta := k - m,
\]

where \( m \) is the size of the output. From Eq. (4) we have that an extractor has optimal entropy loss if \( \Delta = 2 \log 1/\varepsilon + O(1) \).

4 Constructing \( m \)-bit extractors from one-bit extractors and weak designs

In this section we show how to construct a quantum \( m \)-bit extractor from any (classical) 1-bit strong extractor.

This can be seen as a derandomization of a result by König and Terhal \cite{KT08}, who also extract \( m \) bits in the presence of quantum side information by concatenating \( m \) times a 1-bit extractor. They however choose a different seed for each bit, thus having a seed of total length \( d = mt \), where \( t \) is the length of the seed of the 1-bit extractor. In the case of classical side information, this derandomization was done by Trevisan \cite{Tre01}, who shows how to concatenate \( m \) times a 1-bit extractor using only \( d = \text{poly}(t, \log m) \) bits of seed\footnote{Trevisan’s original paper does not explicitly define his extractor as a pseudo-random concatenation of a 1-bit extractor. It has however been noted in, e.g., \cite{Lu04, Vad04}, that this is basically what Trevisan’s extractor does.}. We combine the weak designs from Raz et al. \cite{RRV02}, which they use to improve Trevisan’s extractor, and a previous observation by two of the authors \cite{DV10}, that since 1-bit extractors were shown to be quantum-proof in \cite{KT08}, Trevisan’s extractor is also quantum-proof.

This results in a generic scheme, which can be based on any weak design and 1-bit strong extractor. We define it in Section 4.1 then prove bounds on the min-entropy and error in Section 4.2.
4.1 Description of Trevisan’s construction

In order to shorten the seed while still outputting $m$ bits, in Trevisan’s extractor construction paradigm the seed is treated as a string of length $d < mt$, which is then split in $m$ overlapping blocks of $t$ bits, each of which is used as a (different) seed for the 1-bit extractor. Let $y \in \{0, 1\}^d$ be the total seed. To specify the seeds for each application of the 1-bit extractor we need $m$ sets $S_1, \ldots, S_m \subset [d]$ of size $|S_i| = t$ for all $i$. The seeds for the different runs of the 1-bit extractor are then given by $y_{S_i}$, namely the bits of $y$ at the positions specified by the elements of $S_i$.

The seeds for the different outputs of the 1-bit extractor must however be nearly independent. To achieve this, Nisan and Wigderson [NW94] proposed to minimize the overlap $|S_i \cap S_j|$ between the sets, and Trevisan used this idea in his original work [Tre01]. Raz et al. [RRV02] improved this, showing that it is sufficient for these sets to meet the conditions of a weak design.

**Definition 4.1** (weak design, [RRV02, Definition 5]). Sets $S_1, \ldots, S_m \subset [d]$ are said to form a weak $(t, r)$-design if

1. For all $i$, $|S_i| = t$.
2. For all $i$, $\sum_{j=1}^{i-1} 2^{|S_j \cap S_i|} \leq rm$.

We can now describe Trevisan’s generic extractor construction.

**Definition 4.2** (Trevisan’s extractor). For a one-bit extractor $C : \{0, 1\}^n \times \{0, 1\}^t \rightarrow \{0, 1\}$, which uses a (not necessarily uniform) seed of length $t$, and for a weak $(t, r)$-design $S_1, \ldots, S_m \subset [d]$, we define the $m$-bit extractor $\text{Ext}_C : \{0, 1\}^n \times \{0, 1\}^d \rightarrow \{0, 1\}^m$ as

$$\text{Ext}_C(x, y) := C(x, y_{S_1}) \cdots C(x, y_{S_m}).$$

**Remark 4.3.** The length of the seed of the extractor $\text{Ext}_C$ is $d$, one of the parameters of the weak design, which in turn depends on $t$, the size of the seed of the 1-bit extractor $C$. In Section 5 we will give concrete instantiations of weak designs and 1-bit extractors, achieving various entropy losses and seed sizes. The size of the seed will always be $d = \text{poly}(\log n)$, if the error is $\epsilon = \text{poly}(1/n)$. For example, to achieve a near optimal entropy loss (Section 5.1), we need $d = O(t^3 \log m)$ and $t = O(\log n)$, hence $d = O(\log^3 n)$.

4.2 Analysis

We now prove that the extractor defined in the previous section is a quantum-proof strong extractor. The first step follows the structure of the classical proof [Tre01, RRV02]. We show that an adversary holding the side information and who can distinguish the output of the extractor $\text{Ext}_C$ from uniform can — given a little extra information — distinguish the output of the underlying 1-bit extractor $C$ from uniform. This is summed up in the following proposition:

7The second condition of the weak design was originally defined as $\sum_{j=1}^{i-1} 2^{|S_j \cap S_i|} \leq r(m-1)$. We prefer to use the version of [HR03], since it simplifies the notation without changing the design constructions.
Proposition 4.4. Let $X$ be a classical random variable correlated to some quantum system $E$, let $Y$ be a (not necessarily uniform) seed, independent from $XE$, and let

$$\|\rho_{\text{Ext}_C(X,Y)E} - \rho_{m} \otimes \rho_{Y \otimes \rho_{E}}\|_{\text{tr}} > \varepsilon,$$

where $\text{Ext}_C$ is the extractor from Definition 4.2. Then there exists a partition of the seed $Y$ in two substrings $V$ and $W$, and a classical random variable $G$, such that $G$ has size $H_0(G) \leq rm$, where $r$ is one of the parameters of the weak design (Definition 4.1), $V \leftrightarrow W \leftrightarrow G$ form a Markov chain and

$$\|\rho_{C(X,V) \otimes WGE} - \rho_{U} \otimes \rho_{V \otimes G \otimes E}\|_{\text{tr}} > \frac{\varepsilon}{m}.$$

We provide a proof of Proposition 4.4 in Appendix B.2, where it is restated as Proposition B.5.

For readers familiar with Trevisan’s scheme [Tre01, RRV02], we briefly sketch the correspondence between the variables of Proposition 4.4 and quantities analyzed in Trevisan’s construction. Trevisan’s security proof proceeds by assuming by contradiction that there exists an adversary, holding $E$, who can distinguish between the output of the extractor and the uniform distribution (Eq. (5)). Part of the seed is then fixed (this corresponds to $W$ in the above statement) and some classical advice is taken (this corresponds to $G$ in the above statement) to construct another adversary who can distinguish a specific bit of the output from uniform. But since a specific bit of Trevisan’s extractor is just the underlying 1-bit extractor applied to a substring of the seed ($V$ in the above statement), this new adversary (who holds $WGE$) can distinguish the output of the 1-bit extractor from uniform (Eq. (6)).

In the classical case, Proposition 4.4 would be sufficient to prove the correctness of Trevisan’s scheme, since it shows that if an adversary can distinguish $\text{Ext}_C$ from uniform, then he can distinguish $C$ from uniform given a few extra advice bits, which contradicts the assumption that $C$ is an extractor. But since our assumption is that the underlying 1-bit extractor is only classical-proof, we still need to show that the quantum adversary who can distinguish $C(X,V)$ from uniform is no more powerful than a classical adversary, and so if he can distinguish the output of $C$ form uniform, so can a classical adversary. This has already been done by König and Terhal [KT08], who show that 1-bit extractors are quantum-proof.

Theorem 4.5 ([KT08 Theorem III.1]). Let $C : \{0,1\}^n \times \{0,1\}^t \rightarrow \{0,1\}$ be a $(k, \varepsilon)$-strong extractor. Then $C$ is a quantum-proof $(k + \log 1/\varepsilon, 3\sqrt{\varepsilon})$-strong extractor.

We now need to put Proposition 4.4 and Theorem 4.5 together to prove that Trevisan’s extractor is quantum-proof. The cases of uniform and weak random

---

8Three random variables are said to form a Markov chain $X \leftrightarrow Y \leftrightarrow Z$ if for all $x, y, z$ we have $P_{Z|X,Y}(z|y, x) = P_{Z|Y}(z|y)$, or equivalently $P_{Z|X,Y}(z, x|y) = P_{Z|Y}(z|y)P_{X|Y}(x|y)$.

9Note that Ta-Shma [TS09] has already implicitly proved that this proposition must hold in the presence of quantum side information, by arguing that the adversary can be viewed as an oracle. The present statement is a strict generalization of that reasoning, which allows conditional min-entropy as well as non-uniform seeds to be used.

10In the classical case, [Tre01, RRV02] still show that an adversary who can distinguish $C(X,V)$ from uniform can reconstruct $X$ with high probability. But this is nothing else than proving that $C$ is an extractor.

11This result holds whether the seed is uniform or not.
seeds differ somewhat in the details. We therefore give two separate proofs for these two cases in Section 4.2.1 and Section 4.2.2.

4.2.1 Uniform seed

We show that Trevisan’s extractor is a quantum-proof strong extractor with uniform seed with the following parameters.

**Theorem 4.6.** Let $C : \{0, 1\}^n \times \{0, 1\}^t \rightarrow \{0, 1\}$ be a $(k, \varepsilon)$-strong extractor with uniform seed and $S_1, \ldots, S_m \subset \{d\}$ a weak $(t, r)$-design. Then the extractor given in Definition 4.2, $\text{Ext}_C : \{0, 1\}^n \times \{0, 1\}^d \rightarrow \{0, 1\}^m$, is a quantum-proof $(k + rm + \log(1/\varepsilon), 3m\sqrt{\varepsilon})$-strong extractor.

**Proof.** In Proposition 4.4, if the seed $Y$ is uniform, then $V$ is independent from $W$ and hence by the Markov chain property from $G$ as well, so Eq. (6) can be rewritten as

$$\|\rho_{C(X,V)VWGE} - \rho_{V} \otimes \rho_{W} \otimes \rho_{WGE}\|_{tr} > \frac{\varepsilon}{m},$$

which corresponds to the exact security criterion of the definition of an extractor.

Let $C$ be a $(k, \varepsilon)$-strong extractor with uniform seed, and assume that an adversary holds a system $E$ such that

$$\|\rho_{C(X,V)VWGE} - \rho_{V} \otimes \rho_{W} \otimes \rho_{WGE}\|_{tr} > 3m\sqrt{\varepsilon}. \quad (7)$$

Then by Proposition 4.4 and because $Y$ is uniform, we know that there exists a classical system $G$ with $H_0(G) \leq rm$, and a partition of $Y$ in $V$ and $W$, such that,

$$\|\rho_{C(X,V)VWGE} - \rho_{V} \otimes \rho_{W} \otimes \rho_{WGE}\|_{tr} > 3m\sqrt{\varepsilon}.$$

Since $C$ is a $(k, \varepsilon)$-strong extractor, we know from Theorem 4.5 that we must have $H_{\min}(X|WGE) < k + \log(1/\varepsilon)$ for Eq. (7) to hold. Hence by Lemma B.3, $H_{\min}(X|E) = H_{\min}(X|W) \leq H_{\min}(X|WGE) + H_0(G) < k + rm + \log(1/\varepsilon)$. \qed

4.2.2 Weak random seed

We also show that Trevisan’s extractor is a quantum-proof strong extractor with weak random seed, with the following parameters.

**Theorem 4.7.** Let $C : \{0, 1\}^n \times \{0, 1\}^t \rightarrow \{0, 1\}$ be a $(k, \varepsilon)$-strong extractor with an $s$-bit seed — i.e., the seed needs at least $s$ bits of min-entropy — and $S_1, \ldots, S_m \subset \{d\}$ a weak $(t, r)$-design. Then the extractor given in Definition 4.2, $\text{Ext}_C : \{0, 1\}^n \times \{0, 1\}^d \rightarrow \{0, 1\}^m$, is a quantum-proof $(k + rm + \log(1/\varepsilon), 6m\sqrt{\varepsilon})$-strong extractor for any seed with min-entropy $d = (t - s - \log(1/\varepsilon))$.

The main difference between this proof and that of Theorem 4.6 is that since the seed $Y$ is not uniform in Proposition 4.4, the substring $W$ of the seed not used by the 1-bit extractor $C$ is correlated to the seed $V$ of $C$, and acts as classical side information about the seed. To handle this, we show in Lemma A.3 that with probability $1 - \varepsilon$ over the values of $W$, $V$ still contains a lot of min-entropy, roughly $s' - d'$, where $d'$ is the length of $W$ and $s'$ the min-entropy of $Y$. And hence an adversary holding $WGE$ can distinguish the output of $C$ from uniform, even though the seed has enough min-entropy.
Proof of Theorem 4.7. Let $C$ be a $(k, \varepsilon)$-strong extractor with $s$ bits of min-entropy in the seed, and assume that an adversary holds a system $E$ such that

$$\|ho_{\text{Ext}_C(X,Y)E} - \rho_{U_m} \otimes \rho_Y \otimes \rho_E\|_{tr} > 6m\sqrt{\varepsilon}.$$

Then by Proposition 4.4 we have

$$\|ho_{C(X,V)WGE} - \rho_{U_1} \otimes \rho_{VWGE}\|_{tr} > 6\sqrt{\varepsilon}. \quad (8)$$

Since the adversary has classical side-information $W$ about the seed $V$, we need an extra step to handle it. Lemma A.3 tells us that from Eq. (8) and because by Theorem 4.5 $C$ is a quantum $(k + \log 1/\varepsilon, 3\sqrt{\varepsilon})$-strong extractor, we must have either for some $w$,

$$H_{\text{min}}(X|GEW = w) < k + \log 1/\varepsilon$$

and hence

$$H_{\text{min}}(X|E) = H_{\text{min}}(X|GEW = w) \leq H_{\text{min}}(X|GEW = w) + H_0(G) < k + rm + \log 1/\varepsilon,$$

or

$$H_{\text{min}}(V|W) < s + \log \frac{1}{3\sqrt{\varepsilon}},$$

from which we obtain using Lemma B.1

$$H_{\text{min}}(Y) \leq H_{\text{min}}(V|W) + H_0(W) < s + \log \frac{1}{3\sqrt{\varepsilon}} + d - t. \quad \square$$

5 Concrete constructions

Depending on what goal has been set — e.g., maximize the output, minimize the seed length — different 1-bit extractors and weak designs will be needed. In this section we give a few examples of what can be done, by taking various classical extractors and designs, and plugging them into Theorem 4.6 (or Theorem 4.7), to obtain bounds on the seed size and entropy loss in the presence of quantum side information.

The results are usually given using the $O$-notation. This is always meant with respect to all the free variables, e.g., $O(1)$ is a constant independent of the input length $n$, the output length $m$, and the error $\varepsilon$. Likewise, $o(1)$ goes to 0 for both $n$ and $m$ large.

We first consider the problem of extracting all the min-entropy of the source in Section 5.1. This was achieved in the classical case by Raz et al. [RRV02], so we use the same 1-bit extractor and weak design as them.

In Section 5.2 we give a scheme which uses a seed of length $d = O(\log n)$, but can only extract part of the entropy. This is also based on Raz et al. [RRV02] in the classical case.

In Section 5.3 we combine an extractor and design which are locally computable (from Vadhan [Var04] and Hartman and Raz [HR03] respectively), to produce a quantum $m$-bit extractor, such that each bit of the output depends on only $O(\log(m/\varepsilon))$ bits of the input.

And finally in Section 5.4 we use a 1-bit extractor from Raz [Raz05], which only requires a weakly random seed, resulting in a quantum $m$-bit extractor, which also works with a weakly random seed.

These constructions are summarized in Table 1 on page 5.
5.1 Near optimal entropy loss

To achieve a near optimal entropy loss we need to combine a 1-bit extractor with near optimal entropy loss and a weak \((t, 1)\)-design. We use the same extractor and design as Raz et al. \cite{RRV02} to do so.

**Lemma 5.1** \cite{RRV02} Lemma 17\textsuperscript{12}. For every \( t, m \in \mathbb{N} \) there exists a weak \((t, 1)\)-design \( S_1, \ldots, S_m \subset [d] \) such that \( d = t \left\lceil \frac{1}{m^2} \right\rceil \log 4m = O(t^2 \log m) \). Moreover, such a design can be found in time \( \text{poly}(m, d) \) and space \( \text{poly}(m) \).

As 1-bit extractor, Raz et al. \cite{RRV02} (and Trevisan \cite{Tre01} too) used the bits of a list-decodable code. We give the parameters here as Proposition 5.2 and refer to Appendix C for details on the construction and proof.

**Proposition 5.2.** For any \( \varepsilon > 0 \) and \( n \in \mathbb{N} \) there exists a \((k, \varepsilon)\)-strong extractor with uniform seed \( \text{Ext}_{n, \varepsilon} : \{0, 1\}^n \times \{0, 1\}^d \rightarrow \{0, 1\} \) with \( d = O(\log(n/\varepsilon)) \) and \( k = 3 \log 1/\varepsilon \).

Plugging this into Theorem 4.6 we get a quantum extractor with parameters similar to Raz et al. \cite{RRV02}.

**Corollary 5.3.** Let \( C_{n, \delta} : \{0, 1\}^n \times \{0, 1\}^t \rightarrow \{0, 1\} \) be the extractor from Proposition 5.2 with \( \delta = \frac{\varepsilon^2}{8m^2} \) and let \( S_1, \ldots, S_m \subset [d] \) be the weak \((t, 1)\)-design from Lemma 5.1. Then

\[
\text{Ext} : \{0, 1\}^n \times \{0, 1\}^d \rightarrow \{0, 1\}^m
\]

\[(x, y) \mapsto C(x, ys_1) \cdots C(x, ys_m)\]

is a quantum-proof \((m + 8 \log m + 8 \log 1/\varepsilon + O(1), \varepsilon)\)-strong extractor with uniform seed, with \( d = O(\log^4(n/\varepsilon) \log m) \).

For \( \varepsilon = \text{poly}(1/n) \) the seed has length \( d = O(\log^3 n) \). The entropy loss is \( \Delta = 8 \log m + 8 \log 1/\varepsilon + O(1) \), which means that the input still has this much randomness left in it (conditioned on the output). We can extract a bit more by now applying a second extractor to the input. For this we will use the extractor by Tomamichel et al. \cite{TSSR10}, which is a quantum \((k', \varepsilon')\)-strong extractor with seed length \( d' = O(m' + \log n + \log 1/\varepsilon') \) and entropy loss \( \Delta' = 4 \log 1/\varepsilon' + O(1) \), where \( n' \) and \( m' \) are the input and output string lengths. Since we will use it for \( m' = 8 \log m + 4 \log 1/\varepsilon' + O(1) \), we immediately get the following corollary from Lemma A.3.

**Corollary 5.4.** By applying the extractors from Corollary 5.3 and \cite{TSSR10} Theorem 10\textsuperscript{12} in succession, we get a new function \( \text{Ext} : \{0, 1\}^n \times \{0, 1\}^d \rightarrow \{0, 1\}^m \), which is a quantum-proof \((m + 4 \log 1/\varepsilon + O(1), \varepsilon)\)-strong extractor with uniform seed, with \( d = O(\log^5(n/\varepsilon) \log m) \).

For \( \varepsilon = \text{poly}(1/n) \) the seed has length \( d = O(\log^3 n) \).

The entropy loss is \( \Delta = 4 \log 1/\varepsilon + O(1) \), which is only a factor 2 times larger than the optimal entropy loss. By Lemma 3.5 this extractor can produce \( m = H_{\min}(X|E) - 4 \log 1/\varepsilon - O(1) \) bits of randomness with an error \( 3\varepsilon \).

\textsuperscript{12}Hartman and Raz \cite{HR03} give a more efficient construction of this lemma, namely in time \( \text{poly}(\log m, t) \) and space \( \text{poly}(\log m + \log t) \), with the extra minor restriction that \( m > t^{2048} \).
5.2 Seed of logarithmic size

The weak design used in Section 5.1 requires a seed of length \( d = \Theta(t^2 \log m) \), where \( t \) is the size of the seed of the 1-bit extractor. Since \( t \) cannot be less than \( \log n \), a scheme using this design will always have \( d = \Omega(\log^2 n \log m) \). If we want to use a seed of size \( d = O(\log n) \) we need a different weak design.

**Lemma 5.5 (RRV02 Lemma 15).** For every \( t, m \in \mathbb{N} \) and \( r > 1 \), there exists a weak \((t, r)\)-design \( S_1, \ldots, S_m \subset [d] \) such that \( d = t \left\lceil\frac{m}{\log r}\right\rceil = O\left(\frac{s^2}{\log r}\right) \). Moreover, such a design can be found in time \( \text{poly}(m, d) \) and space \( \text{poly}(m) \).

For the 1-bit extractor we can use the same as in the previous section, Proposition 5.2.

Plugging this into Theorem 4.6 with \( \log r = \Theta(t) \), we get a quantum extractor with logarithmic seed length.

**Corollary 5.6.** If for any constant \( 0 < \alpha \leq 1 \), the source has min-entropy \( H_{\min}(X|E) = n^\alpha \), and the desired error is \( \varepsilon = \text{poly}(1/n) \), then using the extractor \( C_{n, \delta} : \{0,1\}^n \times \{0,1\}^t \rightarrow \{0,1\} \) from Proposition 5.2 with \( \delta = \frac{r^2}{nm^2} \) and the weak \((t, r)\)-design \( S_1, \ldots, S_m \subset [d] \) from Lemma 5.5 with \( r = n^\gamma \) for any \( 0 < \gamma < \alpha \), we have that

\[
\text{Ext} : \{0,1\}^n \times \{0,1\}^d \rightarrow \{0,1\}^m
\]

\[
(x, y) \mapsto C(x, yS_1) \cdots C(x, yS_m)
\]

is a quantum-proof \((n^\gamma m + 8 \log m + 8 \log 1/\varepsilon + O(1), \varepsilon)\)-strong extractor with uniform seed, with \( d = O\left(\frac{1}{\varepsilon} \log n\right) \).

Choosing \( \gamma \) to be a constant results in a seed of length \( d = O(\log n) \). The output length is \( m = n^{\alpha - \gamma} - o(1) = H_{\min}(X|E)^{1-\frac{\gamma}{\alpha}} - o(1) \). By Lemma 3.3 this can be increased to \( m = H_{\min}^\varepsilon(X|E)^{1-\frac{\gamma}{\alpha}} - o(1) \) with an error of \( 3\varepsilon \).

5.3 Locally computable extractor

Another interesting feature of extractors is to be local, that is, the \( m \)-bit output depends only a small subset of the \( n \) input bits. This is useful in, e.g., the bounded storage model (see [Mau92, Lu04, Vad04] for the case of a classical adversary and [KR07] for a general quantum treatment), where we assume a huge source of random bits, say \( n \), are available, and the adversary’s storage is bounded by \( \nu n \) for some constant \( \nu < 1 \). Legitimate parties are also assumed to have bounded workspace for computation. In particular, for the model to be meaningful, the bound is stricter than that on the adversary. So to extract a secret key from the large source of randomness, they need an extractor which only reads \( \ell \ll n \) bits.

**Definition 5.7 (\( \ell \)-local extractor).** An extractor \( \text{Ext} : \{0,1\}^n \times \{0,1\}^d \rightarrow \{0,1\}^m \) is \( \ell \)-locally computable (or \( \ell \)-local), if for every \( r \in \{0,1\}^d \), the function \( x \mapsto \text{Ext}(x, r) \) depends only on \( \ell \) bits of its input, where the bit locations are determined by \( r \).

Lu [Lu04] modified Trevisan’s scheme [Tre01, RRV02] to use a local list-decodable code as 1-bit extractor. Vadhan [Vad04] proposes another construction for local extractors, which is optimal up to constant factors. Both these
constructions have similar parameters in the case of 1-bit extractors. We state the parameters of Vadhan’s construction here and Lu’s constructions in Appendix C.

**Lemma 5.8** ([Vad04, Theorem 8.5]). For any \( \varepsilon > \exp \left( -n/2^{O(\log n)} \right) \), \( n \in \mathbb{N} \) and constant \( 0 < \gamma < 1 \), there exists an explicit \( \ell \)-local \((k, \varepsilon)\)-strong extractor with uniform seed \( \text{Ext}_{\ell, \varepsilon, \gamma} : \{0,1\}^n \times \{0,1\}^d \to \{0,1\} \) with \( d = O(\log(n/\varepsilon)) \), \( k = \gamma n \) and \( \ell = O(\log 1/\varepsilon) \).

Since we assume that the available memory is limited, we also want the construction of the weak design to be particularly efficient. For this we can use a construction by Hartman and Raz [HR03].

**Lemma 5.9** ([HR03, Theorem 3]). For every \( m, t \in \mathbb{N} \), such that \( m = \Omega(t^{\log t}) \) and constant \( r > 1 \), there exists an explicit weak \((t, r)\)-design \( S_1, \ldots, S_m \subseteq [d] \), where \( d = O(t^2) \). Such a design can be found in time \( \text{poly}(m, t) \) and space \( \text{poly}(\log m + \log t) \).

**Remark 5.10.** For the extractor from Lemma 5.8 and an error \( \varepsilon = \text{poly}(1/n) \), this design requires \( m = \Omega((\log n)^{\log \log n}) \). If we are interested in a smaller \( m \), say \( m = \text{poly}(\log n) \), then we can use the weak design from Lemma 5.8 with \( r = n^\gamma \). This construction would require time and space \( \text{poly}(\log n) = \text{poly}(\log 1/\varepsilon) \).

The resulting seed would have length only \( O(\log n) \) instead of \( O(\log^2 n) \).

Plugging this into Theorem 4.6 we get a quantum local extractor.

**Corollary 5.11.** If for any constant \( 0 < \alpha \leq 1 \), the source has min-entropy \( \text{H}_{\min}(X|E) = \alpha n \), then using the weak \((t, r)\)-design \( S_1, \ldots, S_m \subseteq [d] \) from Lemma 5.8 and the extractor \( C_{\alpha, \delta, \gamma} : \{0,1\}^n \times \{0,1\}^t \to \{0,1\} \) from Lemma 5.8 with \( \delta = \frac{\varepsilon^2}{9m^2} \) and any constant \( \gamma < \alpha \), we have that

\[
\text{Ext} : \{0,1\}^n \times \{0,1\}^d \to \{0,1\}^m
\]

\[
(x, y) \mapsto C(x, y_{S_1}) \cdots C(x, y_{S_m})
\]

is a quantum-proof \( \ell \)-local \((\gamma n + rm + 2\log m + 2\log 1/\varepsilon + O(1))\)-strong extractor with uniform seed, with \( d = O(\log^2 (n/\varepsilon)) \) and \( \ell = O(m \log(m/\varepsilon)) \). Furthermore, each bit of the output depends on only \( O(\log(m/\varepsilon)) \) bits of the input.

With these parameters the extractor can produce up to \( m = (\alpha - \gamma)n/r - O(\log 1/\varepsilon) = (H_{\min}(X|E) - \gamma n)/r - O(\log 1/\varepsilon) \) bits of randomness, with \( \varepsilon = \text{poly}(1/n) \). By Lemma 5.8 this can be increased to \( m = (H_{\min}(X|E) - \gamma n)/r - O(\log 1/\varepsilon) \) with an error of \( 3\varepsilon \).

### 5.4 Weak random seed

Extractors with weak random seeds typically require the seed to have a min-entropy linear in its length. [Theorem 4.7] says that the difference between the length and the min-entropy of the seed needed in Trevisan’s extractor is roughly the same as the difference between the length and min-entropy of the seed of the underlying 1-bit extractor. So we will describe in detail how to modify the

\footnote{If the extractor is used to extract \( m \)-bits, then Vadhan’s scheme reads less input bits and uses a shorter seed than Lu’s.}
construction from Section 5.2 to use a weakly random seed. As that extractor uses a seed of length $O(\log n)$, this new construction allows us to preserve the linear loss in the min-entropy of the seed. Any other version of Trevisan’s extractor can be modified in the same way to use a weakly random seed, albeit with weaker parameters.

We will use a result by Raz [Raz05], which allows any extractor which needs a uniform seed to be transformed into one which can work with a weakly random seed.

**Lemma 5.12** ([Raz05 Theorem 4]). For any $(k, \varepsilon)$-strong extractor $\text{Ext} : \{0, 1\}^n \times \{0, 1\}^t \rightarrow \{0, 1\}^m$ with uniform seed, there exists a $(k, 2\varepsilon)$-strong extractor $\text{Ext} : \{0, 1\}^n \times \{0, 1\}^{t'} \rightarrow \{0, 1\}^m$ requiring only a seed with min-entropy $H_{\min}(Y) \geq (\frac{1}{2} + \beta) t'$, where $t' = 8t/\beta$.

By applying this lemma to the 1-bit extractor given in Proposition 5.2 we obtain the following 1-bit extractor.

**Corollary 5.13.** For any $\varepsilon > 0$ and $n \in \mathbb{N}$ there exists a $(k, \varepsilon)$-strong extractor $\text{Ext}_{n, \varepsilon} : \{0, 1\}^n \times \{0, 1\}^d \rightarrow \{0, 1\}$ requiring a seed with min-entropy $\left(\frac{1}{2} + \beta\right) d$, where $d = O(\frac{1}{\beta} \log(n/\varepsilon))$ and $k = 3\log 1/\varepsilon + 3$.

Plugging this and the weak design from Lemma 5.5 in Theorem 4.7 we get the following extractor with weak random seed.

**Corollary 5.14.** Let $\alpha > 0$ be a constant such that the source has min-entropy $H_{\min}(X|E) = n^\alpha$, and the desired error is $\varepsilon = \text{poly}(1/n)$. Using the extractor $C_{n, \varepsilon} : \{0, 1\}^n \times \{0, 1\}^t \rightarrow \{0, 1\}$ from Corollary 5.13 with $\delta = \frac{\varepsilon}{\text{min}}$ and the weak $(t, r)$-design $S_1, \ldots, S_m \subset [d]$ from Lemma 5.5 with $r = n^{\gamma}$ for any $0 < \gamma < \alpha$, we have that

$$\text{Ext} : \{0, 1\}^n \times \{0, 1\}^d \rightarrow \{0, 1\}^m$$

$$(x, y) \mapsto C(x, y_{S_1}) \cdots C(x, y_{S_m})$$

is a quantum-proof $(n^{\gamma} m + 8 \log m + 8 \log 1/\varepsilon + O(1), \varepsilon)$-strong extractor with an $s$-bit weak random seed, where the seed has length $d = O\left(\frac{1}{\gamma} \log n\right)$ and min-entropy $s = \left(1 - \frac{\gamma - \beta}{\varepsilon}\right) d$, for some constant $\varepsilon$.

Choosing $\beta$ and $\gamma$ to be constants results in a seed of length $d = O(\log n)$ with a possible entropy-loss linear in $d$. The output length is the same as in Section 5.2 $m = n^{\alpha - \gamma} - o(1) = H_{\min}(X|E)^{1-\frac{\delta}{\varepsilon}} - o(1)$.

If we are interested in extracting all the min-entropy of the source, we can combine Lemma 5.12 with the extractor from Section 5.1. The results in a new extractor with seed length $d = O(\log^3 n)$ and seed min-entropy $s = d - O(\sqrt{d})$.

## 6 Other variations of Trevisan’s scheme

There exist many results modifying and improving Trevisan’s extractor. Some of them still follow the “design and 1-bit extractor” pattern — hence our work

\[\text{If we work out the exact constant, we find that } c \approx d/t \approx \frac{8(1+4a)}{3\sqrt{\ln a}}, \text{ for } \varepsilon = n^{-a}.\]
implies that these are immediately quantum-proof with roughly the same parameters — e.g., the work of Raz et al. [RRV02] and Lu [Lu04], which were mentioned in Section 5 and correspond to modifications of the design and 1-bit extractor respectively. Other results such as [RRV02, TSZS06, SU05] replace the binary list-decoding codes with multivariate codes over a field $F$. The connection to 1-bit extractors is not clear anymore, and the security in the presence of quantum side information not guaranteed.

Raz et al. extract a little more randomness than we do in Section 5.1. They achieve this by composing (in the sense described in Appendix A.2) the scheme of Corollary 5.3 with an extractor by Srinivasan and Zuckerman [SZ99], which has an optimal entropy loss of $\Delta = 2 \log 1/\varepsilon + O(1)$. In the presence of quantum side information this extractor has only been proven to have an entropy loss of $\Delta = 4 \log 1/\varepsilon + O(1)$ in [TSSR10], hence our slightly weaker result in Corollary 5.4. This still leaves room for a small improvement.

In the case of a logarithmic seed length, Impagliazzo et al. [ISW00] and Ta-Shma et al. [TSUZ01] modify Trevisan’s extractor to work for a sub-polynomial entropy source, still using a seed of size $d = O(\log n)$. While it is unclear whether these modifications preserve the “design and 1-bit extractor” structure, it is an interesting open problem to analyze them in the context of quantum side information.

### Appendices

#### A More on extractors

##### A.1 Weak random seed

In Section 3.1 we defined extractors as functions which take a uniformly random seed. This is the most common way of defining them, but not a necessary condition. Instead we can consider extractors which use a seed which is only weakly random, but with a bounded min-entropy. We extend Definition 3.1 this way.

**Definition A.1** (strong extractor with weak random seed). A function $\text{Ext} : \{0, 1\}^n \times \{0, 1\}^d \rightarrow \{0, 1\}^m$ is a $(k, \varepsilon)$-strong extractor with an $s$-bit seed, if for all distributions $X$ with $H_{\text{min}}(X) \geq k$ and any seed $Y$ independent from $X$ with $H_{\text{min}}(Y) \geq s$, we have

$$\frac{1}{2} \left\| \rho_{\text{Ext}(X,Y)} - \rho_{U_m} \otimes \rho_Y \right\|_{\text{tr}} \leq \varepsilon,$$

where $\rho_{U_m}$ is the fully mixed state on a system of dimension $2^m$.

If quantum side information about the input is present in a system $E$, then as before, we require the seed and the output to be independent from that side-information.

**Definition A.2** (quantum-proof strong extractor with weak random seed). A function $\text{Ext} : \{0, 1\}^n \times \{0, 1\}^d \rightarrow \{0, 1\}^m$ is a quantum-proof $(k, \varepsilon)$-strong extractor with an $s$-bit seed, if for all states $\rho_{XE}$ classical on $X$ with $H_{\text{min}}(X|E) \geq k$, $H_{\text{min}}(Y|E) \geq s$, we have

$$\frac{1}{2} \left\| \rho_{\text{Ext}(X,Y)} - \rho_{U_m} \otimes \rho_Y \right\|_{\text{tr}} \leq \varepsilon,$$

where $\rho_{U_m}$ is the fully mixed state on a system of dimension $2^m$. 


and for any seed $Y$ independent from $XE$ with $H_{\min}(Y) \geq s$, we have
\[
\frac{1}{2} \left\| \rho_{\text{Ext}(X,Y)Y} - \rho_{U_m} \otimes \rho_Y \otimes \rho_E \right\|_{\text{tr}} \leq \epsilon,
\]
where $\rho_{U_m}$ is the fully mixed state on a system of dimension $2^m$.

**Lemma A.3.** Let $\text{Ext} : \{0,1\}^n \times \{0,1\}^d \to \{0,1\}^m$ be a quantum-proof $(k, \varepsilon)$-strong extractor with an $s$-bit seed. Then for any classical $X$, $Y$, and $Z$, and quantum $E$, such that $XE$ and $Y$ are independent, $Y \leftrightarrow Z \leftrightarrow E$ form a Markov chain if $H_{\min}(Y|Z) \geq s + \log 1/\varepsilon$, and for all $z \in Z$, $H_{\min}(X|EZ = z) \geq k$, we have
\[
\frac{1}{2} \left\| \rho_{\text{Ext}(X,Y)YEZ} - \rho_U \otimes \rho_{YZE} \right\|_{\text{tr}} \leq 2\varepsilon.
\]

**Proof.** For any two classical systems $Y$ and $Z$, we have
\[
2^{-H_{\min}(Y|Z)} = \sum_{z \in Z} 2^{-H_{\min}(Y|Z = z)},
\]
so by Markov’s inequality,
\[
\Pr_{z \in Z}[H_{\min}(Y|Z) \leq H_{\min}(Y|Z) - \log 1/\varepsilon] \leq \varepsilon.
\]
And since $Y \leftrightarrow Z \leftrightarrow E$ form a Markov chain, we have for all $z \in Z$,
\[
\rho_{YZE|Z = z} = \rho_{Y|Z = z} \otimes \rho_{E|Z = z}.
\]
Hence
\[
\frac{1}{2} \left\| \rho_{\text{Ext}(X,Y)Y} - \rho_U \otimes \rho_{Y} \right\|_{\text{tr}} = \frac{1}{2} \sum_{z \in Z} P_Z(z) \left\| \rho_{\text{Ext}(X,Y)Y|Z = z} - \rho_U \otimes \rho_{Y|Z = z} \right\|_{\text{tr}} \leq 2\varepsilon. \quad \Box
\]

The case of quantum side information correlated to both the input and the seed is out of the scope of this work.

### A.2 Composing extractors

If an extractor does not have optimal entropy loss, a useful approach to extract more entropy is to apply a second extractor to the original input, trying to extract the randomness that remains when the output of the first extractor is known. This was first proposed in the classical case by Wigderson and Zuckerman [WZ99], and improved by Raz et al. [RRV02]. König and Terhal [KT08] gave the first quantum version for composing $m$ times quantum 1-bit extractors. We slightly generalize the result of König and Terhal [KT08] to the composition of arbitrary quantum extractors.

---

11 A ccq state $\rho_{XYE}$ forms a Markov chain $X \leftrightarrow Y \leftrightarrow E$ if it can be expressed as $\rho_{XYE} = \sum_{x,y} \rho_{X|Y = y} \langle x | y \rangle \otimes \rho_E^x$. 

---
Lemma A.4. Let \( \text{Ext}_1 : \{0,1\}^n \times \{0,1\}^{d_1} \to \{0,1\}^{m_1} \) and \( \text{Ext}_2 : \{0,1\}^n \times \{0,1\}^{d_2} \to \{0,1\}^{m_2} \) be quantum-proof \((k, \varepsilon_1)\)- and \((k-m_1, \varepsilon_2)\)-strong extractors. Then the composition of the two, namely

\[
\text{Ext}_3 : \{0,1\}^n \times \{0,1\}^{d_1} \times \{0,1\}^{d_2} \to \{0,1\}^{m_1} \times \{0,1\}^{m_2}
\]

\((x, y_1, y_2) \mapsto (\text{Ext}_1(x, y_1), \text{Ext}_2(x, y_2))\),

is a quantum-proof \((k, \varepsilon_1 + \varepsilon_2)\)-strong extractor.

Proof. We need to show that for any state \( \rho_{XE} \) with \( H_{\min}(X|E) \geq k \),

\[
\frac{1}{2} \left\| \rho_{\text{Ext}_1(X,Y_1)\text{Ext}_2(X,Y_2)Y_1E} - \rho_{U_1} \otimes \rho_{U_2} \otimes \rho_{Y_1} \otimes \rho_{Y_2} \otimes \rho_E \right\|_{\text{tr}} \leq \varepsilon_1 + \varepsilon_2. \tag{9}
\]

The left-hand side of Eq. (9) can be upper-bounded by

\[
\frac{1}{2} \left\| \rho_{\text{Ext}_1(X,Y_1)Y_1E} - \rho_{U_1} \otimes \rho_{Y_1} \otimes \rho_E \right\|_{\text{tr}} + \frac{1}{2} \left\| \rho_{\text{Ext}_2(X,Y_2)Y_2E} - \rho_{U_2} \otimes \rho_{Y_2} \otimes \rho_{\text{Ext}_1(X,Y_1)Y_1E} \right\|_{\text{tr}}. \tag{10}
\]

By the definition of \( \text{Ext}_1 \) the first term in Eq. (10) is upper-bounded by \( \varepsilon_1 \). For the second term we use Lemma B.3 and get

\[
H_{\min}(X|\text{Ext}_1(X,Y_1)Y_1E) \geq H_{\min}(X|Y_1E) - H_0(\text{Ext}_1(X,Y_1)) \geq H_{\min}(X|E) - H_0(\text{Ext}_1(X,Y_1)) \geq k - m_1.
\]

By the definition of \( \text{Ext}_2 \) the second term in Eq. (10) can then be upper-bounded by \( \varepsilon_2 \).

B Technical lemmas

B.1 Min-entropy chain rules

We use the following “chain-rule type” statement about the min-entropy. The proofs for the two first can be found in [Ren05].

Lemma B.1 ([Ren05, Lemma 3.1.10]). For any state \( \rho_{ABC} \),

\[
H_{\min}(A|BC) \geq H_{\min}(AC|B) - H_0(C),
\]

where \( H_0(C) = \log \text{rank}(\rho_C) \).

Lemma B.2 ([Ren05, Lemma 3.1.9]). For any state \( \rho_{ABZ} \) classical on \( Z \),

\[
H_{\min}(AZ|B) \geq H_{\min}(A|B).
\]

Lemma B.3. For any state \( \rho_{ABZ} \) classical on \( Z \),

\[
H_{\min}(ABZ) \geq H_{\min}(AB) - H_0(Z),
\]

where \( H_0(Z) = \log \text{rank}(\rho_Z) \).

Proof. Immediate by combining Lemma B.1 and Lemma B.2 \( \square \)
B.2 Security reduction

To show that an adversary who can distinguish the output of Ext\_C (defined in Definition 4.2 on page 10) from uniform can also guess the output of the extractor C, we first show that such an adversary can guess one of the bits of the output of Ext\_C given some extra classical information. This is a quantum version of a result by Yao [Yao82].

Lemma B.4. Let a player hold a quantum state ρ\_B correlated with a classical random variable Z on m-bit strings, such that he can distinguish Z from uniform with probability greater than ε. Then there exists a bit i ∈ [m] such that when given the previous i − 1 bits of Z, he can distinguish the i\textsuperscript{th} bit of Z from uniform with probability greater than \(\frac{ε}{m}\). In other words, if \(\|ρ\_ZB − ρ\_U\_m \otimes ρ\_B\|_{\text{tr}} > ε\), then there exists an \(i \in [m]\) such that

\[
\left\| \sum_{z \in Z, z_i=0} \frac{p_z}{2^m} \left| z[i] \rangle \langle z[i-1] \right| \otimes \rho\_B^z - \sum_{z \in Z, z_i=1} \frac{p_z}{2^m} \left| z[i] \rangle \langle z[i-1] \right| \otimes \rho\_B^z \right\|_{\text{tr}} > \frac{ε}{m}. \tag{11}
\]

Proof. The proof uses a hybrid argument. Let

\[
σ_i = \sum_{z \in Z, r \in \{0,1\}^m} \frac{p_z}{2^m} \left| z[i]; r[(i+1),...,m] \right\rangle \langle z[i]; r[(i+1),...,m] \right\| \otimes \rho\_B^z.
\]

Then

\[
ε < \|ρ\_ZB - ρ\_U\_m \otimes ρ\_B\|_{\text{tr}} = \|σ_m - σ_0\|_{\text{tr}} \leq \sum_{i=1}^m \|σ_i - σ_{i-1}\|_{\text{tr}} \leq m \max_i \|σ_i - σ_{i-1}\|_{\text{tr}}.
\]

By rearranging \(\|σ_i - σ_{i-1}\|_{\text{tr}}\) we get the lhs of Eq. (11).

We now need to bound the size of this extra information, the “previous i − 1 bits”, and show that when averaging over all the seeds of Ext\_C, we average over all the seeds of C, which means that guessing a bit of the output of Ext\_C corresponds to distinguishing the output of C from uniform. For the reader’s convenience we now restate Proposition 4.4 and give its proof.

Proposition B.5. [Proposition 4.4] Let X be a classical random variable correlated to some quantum system E, let Y be a (not necessarily uniform) seed, independent fromXE, and let

\[
\|ρ\_\text{Ext\_C}(X,Y)E - ρ\_U\_m \otimes ρ\_Y \otimes ρ\_E\|_{\text{tr}} > ε, \tag{12}
\]

previous expression. For the reader’s convenience we now restate Proposition 4.4 and give its proof.

\[
\|ρ\_XQ - ρ\_U\_1 \otimes ρ\_Q\|_{\text{tr}} = \|ρ\_0ρ\_Q^0 - ρ\_1ρ\_Q^1\|_{\text{tr}}.
\]

[16]To simplify the notation, the statement of this lemma uses the fact that for any binary random variable X and quantum system Q, the following equality holds:

\[
\|ρ\_XQ - ρ\_U\_1 \otimes ρ\_Q\|_{\text{tr}} = \|ρ\_0ρ\_Q^0 - ρ\_1ρ\_Q^1\|_{\text{tr}}.
\]

21
where \( \text{Ext}_{C} \) is the extractor from [Definition 4.3]. Then there exists a partition of the seed \( Y \) in two substrings \( V \) and \( W \), and a classical random variable \( G \), such that \( G \) has size \( H_{0}(G) \leq rm \), where \( r \) is one of the parameters of the weak design [Definition 4.1]. \( V \leftrightarrow W \leftrightarrow G \) form a Markov chain, and

\[
\| \rho_{C}(X,V) \|_{tr} > \epsilon m. \tag{13}
\]

Proof. We apply Lemma B.4 to Eq. (12) and get that there exists an \( i \in [m] \) such that

\[
\sum_{x,y} p_{x}q_{y}|C(x,y_{S_{i}})\cdots C(x,y_{S_{i-1}}),y\rangle\langle C(x,y_{S_{i}})\cdots C(x,y_{S_{i-1}}),y| \otimes \rho^{x} - \sum_{x,v,w} p_{x}q_{v}|C(x,v)\cdots C(x,v_{S_{i-1}}),v\rangle\langle C(x,v)\cdots C(x,v_{S_{i-1}}),v| \otimes \rho^{x} \|_{tr} > \frac{\epsilon}{m}, \tag{14}
\]

where \( \{p_{x}\}_{x \in X} \) and \( \{q_{y}\}_{y \in Y} \) are the probability distributions of \( X \) and \( Y \) respectively.

We split \( y \in \{0,1\}^{d} \) in two strings of \( t = |S_{i}| \) and \( d - t \) bits, and write \( v := y_{S_{i}} \) and \( w := y_{[d] \setminus S_{i}} \). To simplify the notation, we set \( g(w,x,j,v) := C(x,y_{S_{j}}) \). Fix \( w, x \) and \( j \), and consider the function \( g(w,x,j,:) : \{0,1\}^{t} \rightarrow \{0,1\} \). This function only depends on \( |S_{j} \cap S_{i}| \) bits of \( v \). So to describe this function we need a string of length at most \( 2^{|S_{j} \cap S_{i}|} \). And to describe \( g^{w,x}(\cdot) := g(w,x,1,:) \cdots g(w,x,i-1,:) \), which is the concatenation of the bits of \( g(w,x,j,:) \) for \( 1 \leq j \leq i-1 \), we need a string of length at most \( \sum_{j=1}^{i-1} |S_{j} \cap S_{i}| \). So a system \( G \) containing a description of \( g^{w,x} \) has size at most \( H_{0}(G) \leq \sum_{j=1}^{i-1} |S_{j} \cap S_{i}| \). We now rewrite Eq. (14) as

\[
\sum_{x,v,w} p_{x}q_{v,w}|g^{w,x}(v),v,w\rangle\langle g^{w,x}(v),v,w| \otimes \rho^{x} - \sum_{x,v,w} p_{x}q_{v,w}|g^{w,x}(v),v,w\rangle\langle g^{w,x}(v),v,w| \otimes \rho^{x} \|_{tr} > \frac{\epsilon}{m},
\]

By providing a complete description of \( g^{w,x} \) instead of its value at the point.
By rearranging this a little more we finally get
\[
\|p_{C(X,V)} \otimes p_{VWGE} - p_U \otimes p_{VWGE}\|_{tr} > \frac{\varepsilon}{m},
\]
where \( G \) is a classical system of size \( H_0(G) \leq \sum_{i,j=1}^{i,m-1} |S_i \cap S_j| \) and \( V \leftrightarrow W \leftrightarrow G \) form a Markov chain. By the definition of weak designs, we have for all \( i \in [m] \),
\[
\sum_{j=1}^{i-1} |S_j \cap S_i| \leq rm \text{ for some } r \geq 1. \text{ So } H_0(G) \leq rm. \]

C List-decodable codes are one-bit extractors

C.1 Construction

A standard error correcting code guarantees that if the error is small, any string can be uniquely decoded. A list-decodable code guarantees that for a larger (but bounded) error, any string can be decoded to a list of possible messages.

**Definition C.1** (list-decodable code). A code \( C : \{0,1\}^n \to \{0,1\}_{\bar{n}} \) is said to be \((\varepsilon, L)\)-list-decodable if every Hamming ball of relative radius \( 1/2 - \varepsilon \) in \( \{0,1\}^n \) contains at most \( L \) codewords.

Neither Trevisan [Tre01] nor Raz et al. [RRV02] state it explicitly, but both papers contain an implicit proof that if \( C : \{0,1\}^n \to \{0,1\}_{\bar{n}} \) is a \((\varepsilon, L)\)-list-decodable code, then
\[
\text{Ext} : \{0,1\}^n \times [\bar{n}] \to \{0,1\}
\]
\((x, y) \mapsto C(x)_y,\)

is a \((\log L + \log 1/2\varepsilon, 2\varepsilon)\)-strong extractor. We have rewritten their proof in Section C.1.1 for completeness.

There exist list-decodable codes with following parameters.

**Lemma C.2.** For every \( n \in \mathbb{N} \) and \( \delta > 0 \) there is a code \( C_{n,\delta} : \{0,1\}^n \to \{0,1\}^n \), which is \((\delta, 1/\delta^2)\)-list-decodable, with \( \bar{n} = \text{poly}(n, 1/\delta) \). Furthermore, \( C_{n,\delta} \) can be evaluated in time \( \text{poly}(n, 1/\delta) \) and \( \bar{n} \) can be assumed to be a power of 2.

For example, Guruswami et al. [GHSZ02] combine a Reed-Solomon code with a Hadamard code, obtaining such a list-decodable code with \( \bar{n} = O(n/\delta^4) \).

Such codes require all bits of the input \( x \) to be read to compute any single bit \( C(x)_i \) of the output. If we are interested in so-called local codes, we can use a construction by Lu [Lu04].

\[\text{A slightly more general proof, that approximate list-decodable codes are 1-bit extractors can be found in [DV09] Claim 3.7.}\]
Lemma C.3 (Lu14 Corollary 1). For every \( n \in \mathbb{N} \), \( 0 < \delta < 1/m \) and constant \( 0 < \gamma < 1 \), there is a code \( C_{n,\delta,\gamma} : \{0,1\}^n \rightarrow \{0,1\}^{\bar{n}} \), which is \((\delta,2^{\gamma n}/\delta^2)\)-list-decodable, with \( \bar{n} = \text{poly}(n,1/\delta) \). Furthermore, for every \( i \in [\bar{n}] \), \( C_{n,\delta,\gamma}(x)_i \) is the parity of \( \Omega(\log(1/m\delta)) \) bits of \( x \).

C.2 Proof

Theorem C.4. Let \( C : \{0,1\}^n \rightarrow \{0,1\}^{\bar{n}} \) be an \((\varepsilon, L)\)-list-decodable code. Then the function

\[
C' : \{0,1\}^n \times [\bar{n}] \rightarrow \{0,1\}
\]

\[ (x,y) \mapsto C(x)_y, \]

is a \((\log L + \log 1/2\varepsilon, 2\varepsilon)\)-strong extractor\(^{18}\).

To prove this theorem we first show that an adversary who can distinguish the bits of \( C(X) \) from uniform can construct a string \( \alpha \) which is close to \( C(X) \) on average (over \( X \)). Then using the error correcting proprieties of the code \( C \), he can reconstruct \( X \). Hence an adversary who can break the extractor must have low min-entropy about \( X \).

Lemma C.5. Let \( X \) and \( Y \) be two independent random variables with alphabets \( \{0,1\}^n \) and \([n]\) respectively. Let \( Y \) be uniformly distributed and \( X \) be distributed such that

\[
\frac{1}{n}|X_Y \circ Y - U_1 \circ Y| > \delta,
\]

where \( U_1 \) is uniformly distributed on \( \{0,1\} \).

Then there exists a string \( \alpha \in \{0,1\}^n \) with

\[
\Pr \left[ d(X, \alpha) \leq \frac{1}{2} - \frac{\delta}{2} \right] > \delta,
\]

where \( d(\cdot, \cdot) \) is the relative Hamming distance.

Proof. Define \( \alpha \in \{0,1\}^n \) to be the concatenation of the most probable bits of \( X \), i.e., \( \alpha_y := \arg \max_b P_{X}(b) \), where \( P_{X}(b) = \sum_{x \in \{0,1\}^n} P_{X}(x) \).

The average relative Hamming distance between \( X \) and \( \alpha \) is

\[
\sum_{x \in \{0,1\}^n} P_X(x) d(x, \alpha) = \frac{1}{n} \sum_{x \in \{0,1\}^n} P_X(x) \sum_{y=1}^{n} |x_y - \alpha_y| = \sum_{x,y : x_y \neq \alpha_y} P_X(x) = 1 - \frac{1}{n} \sum_{y=1}^{n} P_X(\alpha_y).
\]

And since \( \frac{1}{n}|X_Y \circ Y - U_1 \circ Y| > \delta \) is equivalent to \( \frac{1}{n} \sum_{y=1}^{n} \max_{b \in \{0,1\}} P_{X}(b) \) > \( \frac{1}{2} + \delta \), we have

\[
\sum_{x \in \{0,1\}^n} P_X(x) d(x, \alpha) < \frac{1}{2} - \delta. \quad (15)
\]

\(^{18}\)This theorem still holds in the presence of classical side information with exactly the same parameters.
We now wish to lower bound the probability that the average Hamming distance is less than \( \frac{1}{2} - \frac{\delta}{2} \). Let \( B := \{ x : d(x, \alpha) \leq \frac{1}{2} - \frac{\delta}{2} \} \) be the set of values \( x \in \{0, 1\}^n \) meeting this requirement. Then the weight of \( B \), \( w(B) := \sum_{x \in B} P_X(x) \), is the quantity we wish to lower bound. It is at its minimum if all \( x \in B \) have Hamming distance \( d(x, \alpha) = 0 \). In which case the average Hamming distance is

\[
\sum_{x \in \{0, 1\}^n} P_X(x) d(x, \alpha) > (1 - w(B)) \left( \frac{1}{2} - \frac{\delta}{2} \right).
\]  

(16)

Combining Eqs. (15) and (16) we get

\[ w(B) > \frac{\delta}{1 - \delta} \geq \delta. \]

We are now ready to prove Theorem C.4.

**Proof of Theorem C.4.** We will show that if it is possible to distinguish \( C'(X, Y) \) from uniform with probability at least \( 2\varepsilon \), then \( X \) must have min-entropy \( H_{\text{min}}(X) < \log L + \log 1/2\varepsilon \).

If \( \frac{1}{2} |C'(X, Y) \circ Y - U_1 \circ Y \circ E| > 2\varepsilon \), then by Lemma C.5 we know that there exists an \( \alpha \in \{0, 1\}^n \) such that

\[
\Pr \left[ d(C(X), \alpha) \leq \frac{1}{2} - \varepsilon \right] > 2\varepsilon,
\]

where \( d(\cdot, \cdot) \) is the relative Hamming distance.

This means that with probability at least \( 2\varepsilon \), \( X \) take values \( x \) such that \( d(C(x), \alpha) \leq \frac{1}{2} - \varepsilon \). So for these values of \( X \), if we choose one of the codewords in the Hamming ball of relative radius \( \frac{1}{2} - \varepsilon \) around \( \alpha \) uniformly at random as our guess for \( x \), we will have chosen correctly with probability at least \( 1/L \), since the Hamming ball contains at most \( L \) code words. The total probability of guessing \( X \) is then at least \( 2\varepsilon/L \).

Hence by Eq. (3) \( H_{\text{min}}(X) < \log L + \log 1/2\varepsilon \).

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