The mechanism of enhancing the noise nonlinear effect by symmetric parabolic potential function

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Abstract. The bistable potential function is divided into inner and outer sides. Inside and outside functions are both turned to the form of parabola, then symmetric parabolic potential function is constructed. Based on the Kramers escape theory, the Kramers escape ratio and SNR of different potential functions are analyzed and compared. The theoretical analysis and numerical simulation manifest that symmetric parabolic potential function is much better than other potential functions. The mechanism is explained which the symmetric parabolic potential function can enhance the noise effects from the perspective of dynamics. The strength of the noise effect is related to the functional form of the inside and outside potential function. And the inside of symmetric parabolic potential function plays a greater role.

1. Introduction

Nowadays, the research of nonlinear science shows noise does not always play a destructive role, sometimes it helps system establish a new order. For example, stochastic resonance can improve the signal-to-noise ratio of the output signal with noise, so as to realize the detection of weak signal [1].

Many theoretical and experimental studies have been carried out around the noise nonlinear effect of the classical bistable potential function [2-4]. The potential energy and displacement of classical bistable system are the function relation of biquadratic. There is a saturation characteristic when displacement is large, small increments of the displacement will lead to a sharp increase of potential energy, the range of the particle motion will be limited, and the output SNR decreases rapidly.

In order to overcome the saturation property, the form of potential function can be changed. Zhao Wenli and so on [5] proposed a piecewise mixed potential function, which changed the classical bistable potential to a linear one, avoiding the saturation characteristic, but the SNR was not improved obviously. Zhang Jinyan [6] proposed the quadratic segmented bistable potential function, which improved the output SNR to some extent, but the level of improvement is little. It shows that the change of potential function can overcome the saturation characteristics and enhance the nonlinear effect of noise. The key ideas of this research are summarized below:

a. The impact of medial function on the enhancement of the nonlinear noise effect.
b. The impact of inside and outside functions when they change simultaneously.

2. Symmetric parabolic potential function

Under the combined action of periodic signal and noise, the dynamic equation of particle under potential is as follows:
\[ \frac{dx}{dt} = -dU(x)/dx + A\cos(2\pi f_0 t) + N(t) \]  

In the equation, \( U(x) \) is a potential function. \( -dU(x)/dx \) is a potential field force. \( A\cos(2\pi f_0 t) \) is the cyclical driving force provided by the outside world. \( A \) is the amplitude of the input signal. \( f_0 \) is the frequency of the input signal. \( N(t) \) is a random force acting on noise. The expression of the classical bistable potential function is as follows [6]:

\[ U(x) = -\frac{1}{2}ax^2 + \frac{1}{4}bx^4 \]

Where \( a \) and \( b \) are system parameters, and \( a > 0, b > 0 \). We enumerate the bistable potential functions in figure 1. According to the location of dynamic equilibrium point, the potential function is divided into inside and outside.

![Figure 1. Bistable potential function curve.](image)

There exists a functional relation between potential energy and displacement in the classical bistable potential function. The input and output of the system are restricted by the saturation property of the four variables. When the amplitude of input signal reaches a certain value, the magnitude of input signal has little effect on the output of the system. It is difficult to enhance the random resonance effect in the system with noise. In order to change this phenomenon, we choose to change the form of potential function. The results show that the potential function is better than the two power form. In this paper, the quadratic potential function of the outer side is constructed. The equation is

\[
U_o(x) = \begin{cases} 
\frac{a}{4}(x + \sqrt{a/b})^2 - \frac{a^2}{4b} & x < -\sqrt{a/b} \\
\frac{1}{2}ax^2 + \frac{1}{4}bx^4 & -\sqrt{a/b} \leq x \leq \sqrt{a/b} \\
\frac{a}{4}(x - \sqrt{a/b})^2 - \frac{a^2}{4b} & x > \sqrt{a/b} 
\end{cases}
\]

Further, the inside function is changed into the two power form, and the symmetric parabolic potential function is constructed. The equation is

\[
U(x) = \begin{cases} 
\frac{a}{4}(x + \sqrt{a/b})^2 - \frac{a^2}{4b} & x \leq 0 \\
\frac{a}{4}(x - \sqrt{a/b})^2 - \frac{a^2}{4b} & x > 0 
\end{cases}
\]

Where \( a > 0, b > 0 \). Let \( dU(x)/dx = 0 \), then \( x_i = 0 \) is a dynamic unbalance point of the potential field. And the potential function has a maximum value at \( x_i \). Besides, \( x_2 = -\sqrt{a/b} \), \( x_3 = \sqrt{a/b} \) are dynamic equilibrium points. And there is a minimum in the place. When \( a = 1, b = 1 \), the curve of the symmetric parabolic potential function is shown in figure 2.
3. Mechanism analysis of enhancing the nonlinear effect of noise

3.1. Kramers escape rate

\( R^\pm \) and \( R^\mp \) are the escape rates of Brown particles from the bottom of the potential well respectively. The Kramers escape rate equation from bistable system can be obtained [6]:

\[
R^{-1} = \frac{1}{D} \int_{a}^{\lambda} e^{-U(x)/D} dx \int_{-\infty}^{\lambda} e^{-U(x)/D} dx
\]

(5)

\( D \) is the noise intensity. The escape rate of Kramers indicates that the particle flows into the unstable region at a certain probability. The larger the escape rate of Kramers is, the faster the particle transitions between the two potential wells. According to equation (4), the escape rate of Kramers is related to the specific form of potential function. The form of the potential function is different, the Kramers escape rate is different, and the response rate of the system to the weak periodic signal is different, which affects the strong degree of the nonlinear effect of the noise. Therefore, we need to compare the degree of Kramers escape rate of different nonlinear potential functions in order to estimate the intensity of noise effects.

The Kramers escape rate of the classical bistable system [7]:

\[
R_1 = a\sqrt{2\pi} e^{\frac{\sigma^2}{4D}} \approx 0.225ae^{\frac{\sigma^2}{4D}}
\]

(6)

The Kramers escape rate of piecewise mixed system [8]:

\[
R_2 = \frac{a^2}{4b\sqrt{a/b(c-a/b)}} e^{\frac{\sigma^2}{4D}} = 0.25ae^{\frac{\sigma^2}{4D}} > R_1
\]

(7)

Where \( c = 2\sqrt{a/b} \).

The Kramers escape rate of the Quadratic segmented bistable system [7]:

\[
R_3 = \frac{1}{\sqrt{a/b(c-a/b)}} e^{\frac{\sigma^2}{4D(c-a)}} \approx \frac{b}{a}e^{\frac{\sigma^2}{4D}} > R_2
\]

(8)

Where \( c = 2\sqrt{a/b} \).

Replace equation (3) into equation (4) form:

\[
R^{-1} = \frac{1}{D} \left[ \int_{-\infty}^{-\Delta U} e^{-U(x)/D} dx \right] \left[ \int_{-\Delta U}^{\lambda} e^{-U(x)/D} dx \right] \times \left[ \int_{\lambda}^{0} e^{-U(x)/D} dx \right] \left[ \int_{-\Delta U}^{0} e^{-U(x)/D} dx \right]
\]

(9)

So, from the equation (8), the Kramers escape rate of the symmetric parabolic system is as follows:

\[
R^{-1} \approx \frac{a}{b} e^{\frac{\sigma^2}{4D}}
\]

(10)
When $a = 1$, $b = 1$, and $c = 2\sqrt{a/b}$, the change rate of Kramers escape rate with noise intensity for different nonlinear systems is shown in figure 3.

\[
R = R_s = \frac{b}{a} e^{-\frac{\omega^2}{48D}} > R_i
\]  

(11)

It can be seen from figure 3 that, under the same noise intensity, the Kramers escape rate of the symmetric parabolic system is the largest, the quadratic segmented bistable system is the second, and the classical bistable system is the smallest. The Kramers escape rate of different nonlinear systems is consistent with the above conclusions. Symmetric parabolic systems greatly enhance the nonlinear effect of noise.

3.2. Signal-to-noise ratio (SNR)

The SNR equation is as follows:

\[
SNR = \lim_{\Delta \omega \rightarrow 0} \int_{\omega - \Delta \omega}^{\omega + \Delta \omega} S(w)dw \approx \pi (Ax_0 / D)^2 R
\]  

(12)

Where $x_0 = \sqrt{a/b}$. The Kramers escape rate in equation (10) is replaced by equation (11), and the output $SNR$ of the symmetric parabolic system is as follows:

\[
SNR = \frac{\pi A^2}{2D^2} e^{-\frac{\omega^2}{48D}}
\]  

(13)

When $A = 0.3$, $a = 1$, $b = 1$, the changes of $SNR$ with $D$ is shown in figure 4. From figure 4, we can see that the output $SNR$ of symmetric parabolic potential function first increases and then decreases with the change of $D$, showing a significant unimodal characteristic. And the peak value of $SNR$ ratio is much larger than the classical bistable system and the quadratic segmented bistable system.

3.3. Constitutive elements of stochastic resonance

From equation (1), we can see that stochastic resonance includes 3 elements: weak periodic signal, noise and nonlinear system. Next, from the perspective of stochastic resonance elements, the effects of periodic signal alone, noise alone and the interaction of periodic signals and noise on the output of the system are analyzed in turn, in order to explain the mechanism of enhancing the noise nonlinear effect by symmetric parabolic potential function.

3.3.1. Periodic signal alone. Let $A = 0.56$, $A_\omega = 0.57$, and $D = 0$, the output time domain diagram of different nonlinear systems is shown in figure 5.
\begin{align*}
(a) & \quad A_1 = 0.56 \\
(b) & \quad A_2 = 0.57
\end{align*}

Figure 5. Time domain diagram of output response.

As shown in figure 5 (a), the periodic force can drive the particle to cross the barrier of the quadratic segmented bistable system, but it is not enough to drive the particle to cross the potential barrier of the symmetric parabolic system. This is because the particle is close to the potential barrier of the system and is constrained by the force of potential field. The amplitude of the signal is increased by 0.01, that is, \( A_2 = 0.57 \). Small variables of signal amplitude can be understood as noise induced minor interference. As shown in figure 5 (b), when the noise is disturbed, the motion of the particle in the quadratic segmented bistable system does not change much, but the motion in the symmetric parabolic bistable system varies greatly and the range of motion is larger. It can be seen that noise interference has greater impact on symmetric parabolic system, that is, symmetric parabolic system can effectively enhance the nonlinear effect of noise.

3.3.2. Noise alone. The system adds noise input power spectrum and output power spectrum separately, as shown in figure 6. The spectral energy of the input noise is uniformly distributed. The spectral energy of the input noise is uniformly distributed. In figure 6 (b), after dealing with different nonlinear systems, the energy of noise is concentrated in the low frequency region. The peak value of the output noise power spectrum of the symmetric parabolic system is high, which shows that the system has a stronger ability to gather noise energy in the low frequency region, and reflects the strong nonlinear effect of the noise.
3.3.3. The interaction of periodic signals and noise. Under the action of amplitude $A = 0.3$, frequency $f_0 = 0.01$, and noise $D = 0.3$, the output response of different nonlinear systems is shown in figure 7. A comparison is made between figure 7 (a) and (b), and the peak value of output power spectrum is 0.2197 after the quadratic segmented bistable system. The same signal is increased to 0.3557 by the symmetric parabolic system. It can be concluded that the nonlinear effect of noise in a symmetric parabolic system can effectively enhance the power spectrum of the output signal and make the weak periodic signal easier to detect.

![Figure 7](image)

(a) Symmetric parabolic bistable system  
(b) quadratic segmented bistable system 

**Figure 7.** Output response of different nonlinear systems.

4. Numerical simulation and analysis 

For the symmetric parabolic system, the fourth order Runge Kutta algorithm is applied for numerical simulation. Inputted weak periodic signal frequency is $f_0 = 0.01Hz$. Sampling frequency is $f_s = 5Hz$. The curve of output SNR with noise intensity is shown in figure 8. The comparison between figure 4 and figure 8 shows that the simulation results agree with the theoretical analysis.

![Figure 8](image)

**Figure 8.** Simulation curve of output SNR changing with $D$.

5. Comparison of inside and outside functions 

According to equation (11), when signal amplitude and noise intensity are determined, the theoretical value of SNR is proportional to the escape rate of Kramers. Therefore, comparing the influence of internal and external changes on the output response of the system can be measured by the escape rate of Kramers. The Kramers escape rate equation for a symmetric parabolic bistable system is as follows:

$$ R^{-1} = \frac{1}{D} \left[ \int_{-\infty}^{\infty} e^{-U(x)/D} dx \right] \times \left[ \int_{-\infty}^{0} e^{-U(x)/D} dx \right] $$

(14)
Where $U_1(x)$ represents the outside function, and $U_2(x)$ represents the inside function. Then:

$$M = \int_{-\infty}^{\infty} e^{-\frac{U_1(x)}{D}} dx$$

(15)

$$N = \int_{-\infty}^{\infty} e^{-\frac{U_2(x)}{D}} dx$$

(16)

So $R = D/MN$. The influence of inner and outer sides on SNR is compared. So the influence of $M$ and $N$ on Kramers escape rate is compared. In $M$, let $x = x_i$. And in $N$, let $x = x_i + \sqrt{\alpha/b}$. Then:

$$M = \int_{-\infty}^{\infty} e^{-\frac{U_1(x)}{D}} dx$$

(17)

$$N = \int_{-\infty}^{\infty} e^{-\frac{U_2(x)}{D}} dx$$

(18)

Where $U_1(x_i)<0$, $U_2(x_i + \sqrt{\alpha/b})<0$. When $D << 1$, $U_2(x_i + \sqrt{\alpha/b})/D << 0 << -U_1(x_i)/D$, so $0 < e^{U_2(x_i + \sqrt{\alpha/b})/D} << e^{-U_1(x_i)/D}$, $e^{-U_1(x_i)/D}$ and $e^{U_2(x_i + \sqrt{\alpha/b})/D}$ are respectively corresponding to the derivatives of $M$ and $N$. The slope of $M$ is far greater than that of $N$, that is, $l/N$ changes faster. Therefore, $N$ has greater impact on Kramers escape rate, that is, the inside function plays a greater role.

6. Conclusions
The nonlinear effect of noise is related to the specific nonlinear form of potential function. In this paper, a symmetric parabolic potential function is proposed. By comparing the Kramers escape rate of different potential functions, it is shown that the symmetric parabolic potential function is superior to other potential functions and greatly enhances the nonlinear effect of noise. The numerical simulation results are in agreement with the theoretical analysis. Then, the mechanism of enhancing the noise nonlinear effect of symmetric parabolic system is analysed from the constituent elements of stochastic resonance. Finally, for the symmetric parabolic potential function, the influence of the inside and outside functions on the output of the system is discussed, and the inside function plays a greater role. This conclusion provides a new theoretical method for enhancing the application of noise nonlinear effects.

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