PUZZLES, POSITROID VARIETIES, AND EQUIVARIANT $K$-THEORY OF GRASSMANNIANS

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ABSTRACT. Vakil studied the intersection theory of Schubert varieties in the Grassmannian in a very direct way [Va06]: he degenerated the intersection of a Schubert variety $X_\mu$ and opposite Schubert variety $X_\nu$ to a union $\{X_\lambda\}$, with repetition. This degeneration proceeds in stages, and along the way he met a collection of more complicated subvarieties, which he identified as the closures of certain locally closed sets.

We show that Vakil’s varieties are posistroid varieties, which in particular shows they are normal, Cohen-Macaulay, have rational singularities, and are defined by the vanishing of Plücker coordinates [KLS]. We determine the equations of the Vakil variety associated to a partially filled “puzzle” (building on the appendix to [Va06]), and extend Vakil’s proof to give a geometric proof of the puzzle rule from [KaTao03] for equivariant Schubert calculus.

The recent paper [AGriMil] establishes (abstractly; without a formula) three positivity results in equivariant $K$-theory of flag manifolds $G/P$. We demonstrate one of these concretely, giving a corresponding puzzle rule.

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1. Introduction, and statement of results

1.1. Schubert varieties and Vakil’s geometric shifts. Fix a Grassmannian \( \text{Gr}_k(\mathbb{A}^n) \) of \( k \)-planes in affine \( n \)-space over a field. One way to study it is as a quotient of the Stiefel manifold \( \text{St}_{k,n} \subseteq M_{k,n} \) of \( k \times n \) matrices of full rank \( k \); the map \( \text{St}_{k,n} \to \text{Gr}_k(\mathbb{A}^n) \) taking a matrix to its row span is surjective, and exactly mods out the left action of \( GL(k) \), which is by row operations.

We will use Greek letters \( \lambda, \mu, \nu, \ldots \) to mean words of length \( n \) with \( n - k \) 0s and \( k \) 1s. To each one, we associate the varieties of matrices into components, and continue to degenerate them separately.

For each \( \lambda \), we define the varieties of matrices

\[
\mathbb{X}_\lambda : = \{ M \in M_{k,n} : \forall j = 1, \ldots, n, \ \text{rank}(M_{i,j}) \leq \# \text{1s in } \lambda \text{ at or before place } j \text{ in } \lambda \}
\]

where \( M_{i,j} \) indicates the \( k \times (j - i + 1) \) submatrix using columns \( i, i+1, \ldots, j \) of the \( k \times n \) matrix \( M \). Then

\[
X_\lambda := \text{GL}(k) \setminus (\mathbb{X}_\lambda \cap \text{St}_{k,n}), \quad X^\mu := \text{GL}(k) \setminus (\mathbb{X}^\mu \cap \text{St}_{k,n})
\]

are Schubert and opposite Schubert varieties in \( \text{Gr}_k(\mathbb{A}^n) \), and have

\[
\text{codim } X_\lambda = \text{dim } X^\lambda = |\lambda| := \# \{(i,j) : i < j, \lambda_i > \lambda_j \}.
\]

These are well-known to be reduced and irreducible, and the set of Schubert varieties gives a \( \mathbb{Z} \)-basis of the cohomology ring. Moreover, a cohomology class is effective if and only if it is a nonnegative combination of Schubert classes.

The coefficients \( c_{\lambda \mu}^\nu \) in the multiplication \( [X_\lambda][X_\mu] = \sum c_{\lambda \mu}^\nu [X_\nu] \) arise in many contexts \([Fu99]\), and rules for computing them are generically referred to as “Littlewood-Richardson rules”, though we will only apply this term to the Young-tableau-based such rules. In “A geometric Littlewood-Richardson rule” \([Va06]\), Vakil studies this intersection problem in a very direct way; he degenerates by stages the Richardson variety \( X^\mu_\lambda := X_\lambda \cap X^\mu \) to a union of opposite Schubert varieties, in which \( X_\mu \) occurs \( c_{\lambda \mu}^\nu \) times. (To avoid multiplicities cropping up in his degenerate schemes, after each partial degeneration he must break into components, and continue to degenerate them separately.)

Specifically, define the geometric shift \( \text{III}_{i-j}X \) of a subscheme \( X \subseteq \text{Gr}_k(\mathbb{A}^n) \) as the flat limit\(^1\)

\[
\lim_{t \to -\infty} \exp(te_{ij}) \cdot X,
\]

where \( e_{ij} \) is a matrix whose only nonzero entry is at \( (i,j) \). (These are related to the combinatorial shifts pioneered in \([EK61]\), as we intend to explain in a separate paper.) We define also a related operation on subvarieties \( X \subseteq \text{Gr}_k(\mathbb{A}^n) \), the geometric sweep \( \Psi_{i-j}X \), as the closure of \( \bigcup_{t \in \mathbb{A}^1} \exp(te_{ij}) \cdot X \). So \( \Psi_{i-j}X \) is again irreducible, and either \( X = \text{III}_{i-j}X = \Psi_{i-j}X \), or \( \Psi_{i-j}X \) contains \( X \) and \( \text{III}_{i-j}X \) as rationally equivalent divisors.

We can describe already the principal geometric (rather than cohomological or combinatorial) results of this paper:

Theorem 1.1. Recall Vakil’s “degeneration order”, the following list of \( \binom{n}{2} \) pairs:

\[
(n-1 \to n),
\]

\[
(n-2 \to n), (n-2 \to n-1),
\]

\(^1\)Consider pairs \( \{(t, \exp(te_{ij}) \cdot x) : t \in \mathbb{A}^1, x \in X \} \subseteq \mathbb{A}^1 \times \text{Gr}_k(\mathbb{A}^n) \), and let \( F \) be the closure in \( \mathbb{P}^1 \times \text{Gr}_k(\mathbb{A}^n) \). The flat limit is then the scheme-theoretic intersection \( F \cap (\{\infty\} \times \text{Gr}_k(\mathbb{A}^n)) \). The image of \( F \) projected to \( \text{Gr}_k(\mathbb{A}^n) \) is the “geometric sweep” defined in that same paragraph.
Let $\Pi_{#i}, \Psi_{#i}$ denote the shift and sweep operations for the $i$th pair in this list.

Let $X_0$ be a Richardson variety in $\text{Gr}_k(\mathbb{A}^n)$. As $i$ runs from 1 to $\binom{n}{2}$, apply $\Pi_{#i}$ to $X_{i-1}$, then let $X_i$ be an irreducible component of $\Pi_{#i}X_{i-1}$. Vakil proves [Va06, Theorem 5.10, Proposition 5.15] that regardless of these choices, each $\Pi_{#i}X_{i-1}$ is generically reduced, and has at most two components. Also, $X_{\binom{n}{2}}$ is an opposite Schubert variety.

We generalize his process: as $i$ runs from 1 to $\binom{n}{2}$, apply either $\Pi_{#i}$ or $\Psi_{#i}$ to $X_{i-1}$, then let $X_i$ be an irreducible component or (if $\Pi_{#i}$ was used, and the result was reducible) the intersection of the two components. Then:

1. Each $\Pi_{#i}X_{i-1}$ is reduced (not just generically reduced).
2. It is again true that each $\Pi_{#i}X_{i-1}$ has at most two components. If there are two, then their intersection is reduced and irreducible. Again, $X_{\binom{n}{2}}$ is an opposite Schubert variety.
3. Each $X_i$ is a “positroid variety”, which implies [KLS] that it is normal, Cohen-Macaulay, has rational singularities, is defined by the vanishing of Plücker coordinates, and has other admirable qualities described in [KLS].

Despite the fact that Vakil did not study the sweep operations (relevant for equivariant cohomology) or the intersections (relevant for K-theory), we will call any variety $\{X_i\}$ constructed in the theorem above a Vakil variety. So we have, in increasing order of generality, $\{\text{Schubert varieties}\} \subseteq \{\text{Richardson varieties}\} \subseteq \{\text{Vakil varieties}\} \subseteq \{\text{positroid varieties}\}$. In §1.4 we will define yet another class in between the last two.

Our cohomological application of this theorem is to extend Vakil’s rule for the cohomology product to one in equivariant K-theory.

1.2. Puzzles. A puzzle triangle is just an equilateral triangle of side-length $n$, oriented like $\Delta$ (not $\nabla$). It has $\binom{n+2}{2}$ puzzle vertices, connected by $3\binom{n+1}{2}$ puzzle edges parallel to the sides, whose directions we will call by approximate compass directions E/W, NE/SW, NW/SE. In particular, we may refer to the $n$ rows of a puzzle counted from the top down, and its NW/SE columns and NE/SW columns, each counted from left to right. Consider unlabeled puzzle paths $\gamma$ that (as in figure 1) traverse puzzle edges

- starting at the top vertex of the puzzle, then
- head Southeast some distance along the Northeast side of the puzzle,
- head Southwest some distance through the puzzle,
- jog one optional step Southeast along an edge called the kink,
- continue Southwest until they hit the bottom edge,
- and go West until they hit the Southwest corner.

Since we think of $\gamma$ as a directed path, we will talk about one edge of $\gamma$ being “after” another edge, as traversed in the order above. There are two puzzle paths that stay entirely on the boundary: the initial path which follows the NE edge then bottom edge, and the final path which follows the NW edge. While we described the kink as optional, except for final paths $\gamma$ we can always take the last SE step to be the kink.

Label the edges along $\gamma$, each with one of four possible labels:
• 0
• 1
• R (for Rhombus) – this may not occur on the outer boundary of the puzzle
• K (for K-theory) – this may only occur on the kink, and not on the outer boundary of the puzzle.

Only certain labelings are allowed; we detail the conditions in §3. A **puzzle path** will refer to one with an allowed labeling. Some may be seen in figure 4 on p[17] To refer to labeled edges on $\gamma$, we will talk about the $\Phi$-edges, the $R$-edges, etc.

To each puzzle triangle with a (labeled) puzzle path $\gamma$, we will explain in §3 how to select certain horizontal edges in the puzzle triangle, with which to define an upper triangular partial permutation matrix and, eventually, a Vakil subvariety of the Grassmannian.

Each step of Vakil’s geometric algorithm will then correspond to a small change in $\gamma$, with the whole process going from the initial path to the final path. We will record this process by placing “puzzle pieces” in a separate copy of the triangle. The proof that Vakil’s degenerative geometry is captured by the combinatorics of the puzzle pieces will be theorem 6.3

Puzzle pieces come in three types:

- triangles, which may be rotated:

```
0 0
0
```

```
1 1
1
```

```
1 0
R
```

- the **equivariant rhombus**

```
0 1
```

```
1
```

- the **top, middle, and bottom K-rhombi**

```
1 1
0 K
K 0
```

```
1 0
K 0
0 1
```

We will often want a **puzzle rhombus** to refer not only to the equivariant and K-rhombi but also to a $\Delta$ piece atop a $\nabla$ piece, with the labels on the horizontal edges matching. In this paper, “puzzle rhombi” will always have this vertical orientation.

Define, almost, a **puzzle** to be a tiling by these pieces of a large triangle such that edge labels match up, and with only 0,1 labels on external edges. We say “almost” because there are two non-local conditions concerning the placement of K-edges, each of which

![Figure 1. The initial path, a more general puzzle path with a K indicating the kink, and the final path.](image-url)
appears on the kink of a (unique) puzzle path $\gamma$: (1) if $\K$ is due NE of a $\mathcal{R}$, there must be a $\mathcal{R}$ along $\gamma$ somewhere between them, and (2) if indeed $\K$ is NE of an $\mathcal{R}$, there must be a $\mathcal{I}$ along $\gamma$ somewhere between them.

If we disallow $K$-rhombi, then we can glue the triangles with $R$-edges together in pairs to make the rhombi in the $[KnTaoWood04, KnTao03]$ formulations of puzzles. If we instead glue the $K$-rhombi together along their $K$s, in each aggregate the “top” $K$-rhombus will be on top, the “middle” $K$-rhombus occurring several times in the middle (possibly zero), and the “bottom” $K$-rhombus on bottom.

In figure 5 on p21 we give all the puzzles with 0101 and 1010 on the NE and S sides.

For each horizontal edge $e$ in the puzzle, let $i(e)$ denote its NE/SW column and $j(e)$ its NW/SE column. So if we drop lines Southwest and Southeast from the edge, they point to the $i(e)$th and $j(e)$th edges on the bottom; we may refer to the edge or the vertical rhombus it bisects as being in position $(i(e), j(e))$. If we consider the horizontal edges one NE/SW column at a time, rightmost column to leftmost, then down each column (but skipping the bottom edges), their $i(e), j(e)$ correspond to the shifts $(i(e) \to j(e))$ in Vakil’s degeneration order. See figure 2.

![Figure 2](image_url)

**Figure 2.** Vakil’s degeneration order of shifts, thought of as a filling order on the rhombi in the puzzle. The boundary between the rhombi filled so far, and those yet to filled, is an unlabeled puzzle path. In the picture above, $2 \to 4$ is the next to be filled.

In the following theorem, we consider pairs $\gamma, \gamma'$ of puzzle paths whose symmetric difference $p$ is either a $\Delta$ piece or two triangles stacked in a vertical rhombus, as in figure 3. In this situation, say that $p$ added to $\gamma$ (on the right of $p$) gives $\gamma'$ (on the left of $p$), where $p$ is the one or two puzzle pieces. It is easy to see from the allowed shapes of $\gamma, \gamma'$ that there is a unique location one might add some $p$ to $\gamma$, either filling in the triangle at the bottom of a NE/SW column or moving the kink SW one rhombus.

**Theorem 1.2.** To each puzzle path $\gamma$, there is a way given in §3 to associate an “interval rank variety” $\Pi_r \subseteq M_{k \times n}$, defined by rank conditions on intervals of columns, whose associated “interval positroid variety” $\Pi_r := GL(k) \setminus (\Pi_r \cap St_{k,n})$ in the Grassmannian will turn out to be a Vakil variety. If $\gamma$ is initial, $\Pi_r$ is a Richardson variety. If $\gamma$ is terminal, $\Pi_r$ is an opposite Schubert variety.
Figure 3. Each picture contains the superposition of two puzzle paths $\gamma$ and $\gamma'$ agreeing away from a puzzle triangle or rhombus, $p$, which added to $\gamma$ (on the right of $p$) gives $\gamma'$ (on the left of $p$).

If $\gamma$ is not terminal, take its last SE edge to be the kink. If the next step $\sigma$ is due West, there exists a unique triangular puzzle piece to add to $\gamma$, obtaining a new puzzle path $\gamma'$. This $\gamma'$ has the same associated interval rank variety.

If the next step $\sigma$ is SW, and the kink and $\sigma$ are not labeled 0 and 1 respectively, there exists a unique puzzle rhombus to add to $\gamma$, obtaining a new puzzle path $\gamma'$. This $\gamma'$ has the same associated interval rank variety. This situation occurs iff $\Pi_{i(e)} \rightarrow j(e) \Pi_r = \Pi_r$, where $e$ is the horizontal edge crossing the rhombus.

If the next step $\sigma$ is SW, and the kink and $\sigma$ are labeled 0 and 1 respectively, there exist multiple puzzle rhombi to add to $\gamma$, obtaining new puzzle paths $\gamma'$. Each such $\gamma'$ has a different associated interval rank variety (and all are different from that of $\gamma$). This situation occurs iff $\Pi_{i(e)} \rightarrow j(e) \Pi_r \neq \Pi_r$, where $e$ is the horizontal edge crossing the rhombus. Indeed $\Psi_{i(e)} \rightarrow j(e) \Pi_r$ is the Vakil variety constructed from adding the equivariant piece to $\gamma$, whereas $\Pi_{i(e)} \rightarrow j(e) \Pi_r$ is the union of the Vakil varieties associated to the other possible additions.

Readers wishing to see a detailed example may jump directly to §4.

1.3. Positivity and puzzle statements in various cohomology theories. Let $P_{\lambda\mu}^\nu$ be the set of puzzles with labels $\lambda$ on the NW side, $\mu$ on the NE side, $\nu$ on the S side, all read left to right. For each of the cohomology theories $E^*$ discussed below, and each rhombus puzzle piece, we associate an element $\Phi(E^*, \rho)$ of $E^*(pt)$ (possibly 0), so that the formula

$$\sum_{\rho \in P_{\lambda\mu}^\nu} \prod_{\rho \in P} \Phi(E^*, \rho)$$

will turn out to compute a coefficient of interest.

1.3.1. Ordinary cohomology. As already mentioned, the Schubert cycles $\{X_\lambda\}$ define a $\mathbb{Z}$-basis of the cohomology ring of the Grassmannian. In this theory, the Littlewood-Richardson coefficients $\{c_{\lambda\mu}^\nu \in \mathbb{N}\}$ show up in two expansions:

$$[X_\lambda][X_\mu] = \sum_{\nu} c_{\lambda\mu}^\nu [X_\nu], \quad [X_\mu] = \sum_{\lambda} c_{\lambda\mu}^\nu [X_\lambda].$$

In essence, it is the latter expansion that Vakil studies, largely because the intersection $X_\mu \cap X_\lambda$ is transverse and the intersection $X_\lambda \cap X_\mu$ is not.

Both expansions are consequences of the alternate definition

$$c_{\lambda\mu}^\nu = \int_{\text{Gr}_k(\mathbb{A}^n)} [X_\lambda][X_\mu][X_\nu]$$

and the dual-basis relation $\int_{\text{Gr}_k(\mathbb{A}^n)} [X_\lambda][X_\mu] = \delta_{\lambda\nu}$. 

Theorem 1.3. [KnTaoWood04] Let the factors $\Phi(H^*, \rho)$ be 0 for equivariant and K-rhombi. Then $c_{\lambda\mu}^y = \sum_{p \in P_{\lambda\mu}} \prod_{p \in P} \Phi(H^*, \rho) = \text{the number of puzzles } P \in P_{\lambda\mu} \text{ using only triangles}.$

1.3.2. Equivariant cohomology. Since the Schubert and Richardson varieties are invariant under the action of the torus $T \leq \text{GL}(n)$ of diagonal matrices, they define also a basis of the equivariant cohomology ring $H^*_T(\text{Gr}_k(\mathbb{A}^n))$, considered as a module over $H^*_T(\text{pt}) \cong \mathbb{Z}[y] := \mathbb{Z}[y_1, \ldots, y_n]$. We do not need to introduce new notation; the “equivariant numbers” $c_{\lambda\mu}^y(y) \in \mathbb{Z}[y]$ specialize to the ordinary numbers $c_{\lambda\mu}^T \in \mathbb{Z}$ by specializing each $y_i \mapsto 0$. In particular, $c_{\lambda\mu}^y(y) \neq 0$ implies $|\lambda|+|\mu| \geq |\nu|$, and if $|\lambda|+|\mu| = |\nu|$ then $c_{\lambda\mu}^y(y) = c_{\lambda\mu}^y$.

The Schubert and opposite Schubert varieties are related

$$X^\lambda = w_0 \cdot X^\lambda_{\text{reversed}}$$

by the long element $w_0 = (1 \leftrightarrow n)(2 \leftrightarrow n-1) \cdots$ of $S_n$, and hence define the same element of the cohomology ring. For that reason, one may wonder why we used both $[X^\lambda]$s and $[X^\nu]$s in the equations in 1.3.2 rather than stating everything in one basis. This is because the Schubert and opposite Schubert varieties do not define the same elements in equivariant cohomology, and only when written in the form above do the relations extend to equivariant cohomology.

It was proven abstractly for generalized flag manifolds $G/P$ [Gr00], and combinatorially for $\text{Gr}_k(\mathbb{A}^n)$ [KnTao03], that the equivariant numbers $\{c_{\lambda\mu}^y(y)\}$ can be written as $\mathbb{N}$-combinations of products of distinct positive roots $y_i - y_j$, $i > j$. (The reference [Gr00] makes the weaker claim that $c_{\lambda\mu}^y(y)$ is an $\mathbb{N}$-combination of products of simple roots, but the proof there gives this more precise result.)

Theorem 1.4. [KnTao03] Let the factors $\Phi(H^*_T, \rho)$ be 0 for the K-pieces. For an equivariant rhombus, the factor is $y_{ji} - y_{ij}$. Then $c_{\lambda\mu}^y(y) = \sum_{p \in P_{\lambda\mu}} \prod_{p \in P} \Phi(H^*_T, \rho)$.

Equivalently, one can work with only triangles and their rotations, plus the equivariant piece (no rotations), which is very nearly the viewpoint of [KnTao03].

1.3.3. (Nonequivariant) K-theory. Let $[O^\lambda], [O^\nu]$ denote the classes in $K(\text{Gr}_k(\mathbb{A}^n))$ of the structure sheaves $O^\lambda, O^\nu$ of the Schubert and opposite Schubert varieties. These are not dual bases:

$$\left\{ [O^\lambda][O^\nu] = \begin{cases} 1 & \text{if } X^\lambda \cap X^\nu \neq \emptyset, \quad \text{i.e. } \lambda \leq \nu \text{ in Bruhat order} \\ 0 & \text{otherwise.} \end{cases} \right.$$ 

Here $\tilde{K} : K(\text{Gr}_k(\mathbb{A}^n)) \to K(\text{pt}) \cong \mathbb{Z}$ denotes the pushforward to a point in K-theory, giving the “holomorphic Euler characteristic” of a sheaf.

Consequently, there is another basis of $K(\text{Gr}_k(\mathbb{A}^n))$ to consider; the dual basis $\{[\xi^\nu]\}$ satisfying $\tilde{K}[O^\lambda][\xi^\nu] = \delta_{\lambda\nu}$. (Right now “[\xi^\nu]” is just a K-class; in a moment we will define an actual sheaf $\xi^\nu$. Using the known Möbius function of the Bruhat order, one can show that

$$[O^\lambda] = \sum_{\nu \geq \lambda} [\xi^\lambda], \quad [\xi^\lambda] = \sum_{\nu \geq \lambda} (-1)^{|\nu|-|\lambda|} [O^\lambda].$$

It is a pleasant fact [GrKu08, Proposition 2.1] that this K-class $[\xi^\nu]$ is actually the K-class of a sheaf $\xi^\nu$, the subsheaf of $O^\nu$ consisting of functions vanishing on $\partial X^\nu := \bigcup_{\lambda < \nu} X^\lambda$. 
If we define the coefficients $g, e$ by

$$[O_\lambda][O_\mu] = \sum_{\nu} g_{\lambda\mu}^\nu [O_\nu], \quad [O_\mu][O^\nu] = \sum_{\lambda} e_{\mu\lambda}^\nu [O^\lambda]$$

then

$$g_{\lambda\mu}^\nu = \oint [O_\lambda][O_\mu][\xi^\nu], \quad e_{\mu\lambda}^\nu = \oint [O_\mu][O^\nu][\xi_\lambda] \quad \text{so } e_{\mu\lambda}^\nu = g_{\lambda\mu}^{\nu \text{ reversed}}.$$ 

We will extend Vakil’s techniques to study the $e$ coefficients, and thereby obtain the $g$ coefficients as well.

The coefficients $g_{\lambda\mu}^\nu \in \mathbb{Z}$ turn out to be nonnegative once multiplied by $(-1)^{|\nu|-|\lambda|-|\mu|}$, as was first shown combinatorially in the Grassmannian case in [Buc02], and then geometrically for arbitrary G/P in [Bri02]. The condition $g_{\lambda\mu}^\nu \neq 0$ implies that $|\lambda| + |\mu| \leq |\nu|$ – the opposite inequality we had for equivariant cohomology – and if $|\lambda| + |\mu| = |\nu|$ then $g_{\lambda\mu}^\nu = e_{\lambda\mu}^\nu = c_{\lambda\mu}^\nu$. 

Now that we have another basis $\{[\xi_\lambda]\} \subseteq K(\text{Gr}_k(\mathbb{A}^n))$, we can consider also its structure constants

$$[\xi_\lambda][\xi_\mu] = \sum_{\nu} p_{\lambda\mu}^\nu [\xi_\nu]$$

(though it is perhaps a bit weird to do so, as $1$ is not an element of this basis). Once again, these are nonnegative once multiplied by $(-1)^{|\nu|-|\lambda|-|\mu|}$ [GrKu08, Remark 3.7].

So far everything in this discussion of $K$-theory holds for Schubert classes on arbitrary flag manifolds G/P. We now make use of a special property characterizing minuscule G/P: the two bases have a further relation $[\xi_\lambda] = [O_\lambda](1 - \square)$, where $\square$ denotes the $K$-class of the (unique) Schubert divisor. (On Grassmannians, this fact can be found in [Buc02 §8], where it is used to show a 3-fold symmetry of the $g$ coefficients.) Then

$$\oint [O_\lambda][O_\mu][\xi^\nu] = \oint [O_\lambda][O_\mu][O^\nu](1 - \square) = \oint [O_\lambda][O^\nu][\xi_\lambda]$$

so $g_{\lambda\mu}^\nu = e_{\lambda\mu}^\nu$. We also obtain the relation

$$p_{\lambda\mu}^\nu = \oint [\xi_\lambda][\xi_\mu][O^\nu] = \oint [O_\lambda](1 - \square)[O_\mu](1 - \square)[O^\nu] = \oint [O_\lambda](1 - \square)[O_\mu][\xi^\nu] = g_{\lambda\mu}^\nu - g_{\lambda\mu}^{\nu \text{ reversed}}.$$ 

and both of the latter terms (the second one, a structure constant for a triple product) have the right sign for $p_{\lambda\mu}^\nu$. Hence, in the case G/P minuscule, the positivity property of the $p$ coefficients follows from that of the $g$.

**Theorem 1.5.*** (An analogue of [Va06, Theorem 3.6].) Let the factors $\Phi(K, \rho)$ be 0 for the equivariant piece, $-1$ for the top $K$-piece, and 1 for the others.

Then $g_{\lambda\mu}^\nu(y) = \sum_{p \in P^\nu_{\lambda\mu}} \prod_{\rho \in P} \Phi(K, \rho) = (-1)^{|\nu|-|\lambda|-|\mu|} \# \{\text{puzzles } P \text{ using only these pieces}\}$.

### 1.3.4. Equivariant $K$-theory

Our reference for this subject is [AGriMil].

The base ring $K_T(pt)$ for T-equivariant $K$-theory is the representation ring of T, and isomorphic to a Laurent polynomial ring. Since we use $y_1, \ldots, y_n$ to denote an additive basis of the weight lattice of T, we will instead use $\exp(y_1), \ldots, \exp(y_n)$ to denote the corresponding elements of $K_T(pt) \cong \mathbb{Z}[e^y] := \mathbb{Z}[\exp(\pm y_1), \ldots, \exp(\pm y_n)]$. The sheaves $O_\lambda, \xi_\lambda, O^\nu, \xi^\nu$ are T-equivariant, and so define classes in $K_T(\text{Gr}_k(\mathbb{A}^n))$, for which we use the same notation $[O_\lambda], [\xi_\lambda], [O^\nu], [\xi^\nu]$ as before.
The structure constants $e^y_{\lambda\mu}$, $p^y_{\lambda\mu}$, $g^y_{\lambda\mu}$, $h^y_{\lambda\mu}$ generalize to $e^y_{\lambda\mu}(e^y)$, $p^y_{\lambda\mu}(e^y)$, $g^y_{\lambda\mu}(e^y)$, and again the latter specialize to the former under $y \mapsto 0$, $e^y \mapsto 1$. Each family has the same positivity statement:
\[
e^y_{\lambda\mu}(e^y), p^y_{\lambda\mu}(e^y), g^y_{\lambda\mu}(e^y) \in (-1)^{|\lambda|+|\mu|-|\nu|} N[[\exp(y_1 - y_j) - 1]]_{j>0}
\]
but these appear to require three different proofs [AGriMil, Corollaries 5.1-5.3]. I do not know (even conjecturally) the proper analogue of the “products of distinct positive roots” property mentioned at the end of §1.3.2.

Vakil’s geometric techniques generalize most easily to studying the $e^y_{\lambda\mu}$ structure constants, and we confine ourselves to that problem in this paper.

**Theorem 1.6.** Let $\beta(\rho) = y_{i(e)} - y_{j(e)}$ for $\rho$ a vertical rhombus, and $e$ the edge bisecting it. Let the factors $\Phi(K_T, \rho)$ be as follows:

\[
\Phi(K_T, \rho) = \begin{cases} 
1 - e^{\beta(\rho)} & \text{if } \rho \text{ is an equivariant piece} \\
e^{\beta(\rho)} & \text{if } \rho \text{ is another rhombus with } 1, 0 \text{ on its right side} \\
-1 & \text{if } \rho \text{ is the top } K\text{-piece} \\
1 & \text{otherwise.}
\end{cases}
\]

Then $e^y_{\lambda\mu}(e^y) = \sum_{\rho \in \mathcal{P}_y} \prod_{\rho \in \rho} \Phi(K_T, \rho)$.

To be sure that these formulæ have the desired positivity properties, we give a lemma, which can be proved (though we won’t do so) by the techniques from [KnTao03 §4].

**Lemma 1.7.** Let $\Delta$ be a puzzle, with $\lambda, \mu, \nu$ the strings of labels on the NW, NE, S sides respectively, all read left-to-right. Then

\[|\nu| + \#\{\text{equivariant rhombi in } P\} = |\lambda| + |\mu| + \#\{\text{top } K\text{-rhombi in } P\}.
\]

1.4. **Interval rank varieties.** Given a matrix $M \in M_{k, n}$, associate an upper triangular interval rank matrix $r(M)$ by

\[
r(M)_{ij} := \begin{cases} 
\text{rank}(M_{[i,j]}) & \text{if } i \leq j \\
0 & \text{if } i > j
\end{cases}
\]

**Theorem 1.8.**

- For any $M \in M_{k, n}$, there exists a unique upper triangular $n \times n$ partial permutation matrix $\bar{J}(r)$ such that

\[
r(M)_{ij} = |[i, j]| - \#\{1s \text{ in } \bar{J}(r) \text{ southwest of } (i, j)\}.
\]

- Every upper triangular $n \times n$ partial permutation matrix with at least $n - k$ 1s arises this way.

- If we fix an interval rank matrix $r$ that actually arises for some $M$, the interval rank variety

\[\prod_r := \{N : r(N) \leq r \text{ entrywise}\}
\]

is isomorphic to a certain “Kazhdan-Lusztig variety” in a flag manifold. Hence it is reduced, irreducible, normal, Cohen-Macaulay, and has rational singularities, and there is a good formula for its $T$-equivariant Hilbert series.

\[\text{meaning, at most one } 1 \text{ in each row and column}\]
The quotient
\[ \Pi_r := \text{GL}(k) \setminus (\Pi_r \cap \text{St}_{k,n}) \]
is a special case of a “positroid” subvariety of the Grassmannian. Positroid varieties are defined by rank conditions on all cyclic intervals of columns, i.e. including \( i, i+1, \ldots, n, n-1, n, 1, 2, \ldots, j \). We studied these in [KLS], where we showed they are reduced, irreducible, normal, and Cohen-Macaulay with rational singularities. Unfortunately, we don’t know this upstairs in \( M_{k,n} \) (just in \( \text{St}_{k,n} \)), when cyclic conditions are used, so we make use of the connection to Kazhdan-Lusztig varieties in the flag manifold to give an independent proof.

We will call these \( \{\Pi_r\} \) interval positroid varieties. It will turn out that each Vakil variety is of the form \( \Pi_r \) for some \( r \).

In [HoSu04] they determine the components of the subscheme of \( M_{k \times n} \) defined by asking that each connected \( k \times k \) minor vanish. Via the connection to positroid varieties, one can show that each of these components is an interval rank variety.

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## 2. Interval Rank Varieties

Let \( B_- \), respectively \( B_+ \), denote the groups of lower, respectively upper, triangular matrices in \( \text{GL}(N) \). A **Schubert variety** in the flag manifold \( B_- \setminus \text{GL}_N \) is the closure
\[ X_{\pi} := \overline{B_- \pi B_+ / B_+} \]
where \( \pi \) is a permutation matrix. These are well-known to be Cohen-Macaulay with rational singularities (see e.g. [Bri05]). An **opposite Schubert cell** is an orbit of \( B_+ \),
\[ X_{\rho}^- := B_+ \rho B_+ / B_+, \]
and a **Kazhdan-Lusztig variety** (terminology from [WooY08]) is the intersection
\[ X_{\rho \pi}^\circ := X_{\pi} \cap X_{\rho}^- . \]
It is of dimension \( \ell(\rho) - \ell(\pi) \), where
\[ \ell(\pi) := \#\{ (i, j) : i < j, \pi(i) > \pi(j) \}. \]
This variety \( X_{\rho \pi}^\circ \) is used to study the singularities of \( X_{\pi} \) near the point \( \rho B_+ / B_+ \). These affine varieties have the same good properties as the Schubert varieties, and have nice degenerations to unions of coordinate spaces [Kn, §7.3].

It will be convenient to study \( X_{\pi} \) via its **matrix Schubert variety** [Fu92, KnMil05]:
\[ \overline{X}_{\pi} := \overline{B_- \pi B_+ \subseteq M_{N \times N}} \]
Fulton [Fu92] determined the equations defining \( \overline{X}_{\pi} \):
\[ \overline{X}_{\pi} := \{ M \in M_{N \times N} : \text{rank}(M_{\leq i, \leq j}) \leq \text{rank}(\pi_{\leq i, \leq j}), i, j \leq n \} \]
where \( M_{\leq i, \leq j} \) denotes the upper left \( i \times j \) submatrix. It is enough to take \( (i, j) \) in Fulton’s **essential set**, the Southeast corners of \( \pi \)'s Rothe diagram. Fulton also proves (after [Fu92]}
Lemma 6.1] that $X_n$ is itself a Kazhdan-Lusztig variety for $GL(2N)$, without using that language.

**Proof of theorem**

An interval rank matrix is easily seen to satisfy the following properties:

1. The diagonal entries are 0 or 1.
2. Each entry is either 0 or 1 more than the entries West and South of it.
3. If $r(M)_{ij} = r(M)_{i-1,j} = r(M)_{i,j+1}$, then $r(M)_{ij} = r(M)_{i-1,j+1}$.

Let $J = J(r)$ be the upper triangular matrix with 1 at $(i,j)$ iff $r(M)_{ij} = r(M)_{i,j-1} = r(M)_{i+1,j} = r(M)_{i+1,j-1} + 1$, and 0 otherwise. Then $J$ is a partial permutation matrix, and $r(M)_{ij} = |i,j| - \#\{1s in \pi \ \text{southwest of} \ (i,j)\}$. We refer to [KLS corollaries 3.10-3.12] for the proof of a similar but more general statement.

We will show that $\Pi_r$ is isomorphic to a Kazhdan-Lusztig variety $X_{\rho}^\rho_n$ in $GL(k + n)/B$. The upper index $\rho$ will not depend on $r$: in one-line notation it is $n+1 \ n+2 \ldots \ n+k \ 1 \ldots \ n$, of length $kn$. There is a handy subset $C \subseteq GL(k + n)$ that projects isomorphically to $X_{\rho}^\rho_n = B_+\rho B_+/B_+$:

$$C := \left\{ \begin{bmatrix} N & \Id_k \\ \Id_n & 0 \end{bmatrix} : N \in M_{k \times n} \right\} \text{ where the } \Id_k, \Id_n \text{ are identity matrices.}$$

We still need to define $\pi$ from $J$, which we recall has at least $n - k$ 1s, thus at most $k$ empty rows and $k$ empty columns. Loosely speaking, put $J$ in the lower left of $\pi$ and extend it to a permutation matrix in the unique way with fewest inversions:

$$\pi = \begin{bmatrix} A_1 & 0 & 0 \\ J & \Id_k & 0 \\ A_2 \end{bmatrix}, \quad s = \text{rank}(J) - (n - k).$$

In more detail, $A_1$ is the $k \times n$ partial permutation matrix whose $j$th row ($j \leq k - s$) has a 1 in the $j$th empty column of $J$, and $A_2$ is the $n \times k$ partial permutation matrix whose $j$th column has a 1 in the $j$th empty row of $J$. It is easy to see that $\pi$’s Rothe diagram lies in the first $n - k$ columns, has no essential boxes above the $k$th row, and has no boxes in the lower triangle of the $J$ square. In particular, Fulton’s description of $X_\pi$ implies

$$M \in X_\pi \iff \forall 0 \leq i < j \leq n, \text{rank}(M_{\leq k, i \leq j}) \leq \#\{j' \leq j : \text{column } j' \text{ of } J \text{ is zero} \} + \text{rank}(J_{i \leq i, j})$$

$$= j - \text{rank}(J_{i \leq i, j})$$

$$= i + |i + 1, j| - \text{rank}(J_{i \leq i, j})$$

$$= i + r_{i+1,j}.$$

Rather than computing $X_{\rho}^\rho_n \cap X_\pi$ down in $GL(k + n)/B$, we will compute up in $GL(k + n)$ and project. So we intersect $C$ (which maps isomorphically to its projection in $GL(n + k)/B$) and $X_\pi$ (which inside $GL(k + n)$, is a union of fibers of the projection):

$$C \cap X_\pi = \left\{ M = \begin{bmatrix} N \in M_{k \times n} & \Id_k \\ \Id_n & 0 \end{bmatrix} : \forall 0 \leq i < j \leq n, \text{rank}(M_{\leq k, i \leq j}) \leq i + r_{i+1,j} \right\}$$

$$= \left\{ M = \begin{bmatrix} N \in M_{k \times n} & \Id_k \\ \Id_n & 0 \end{bmatrix} : \forall 0 \leq i < j \leq n, \text{rank}(N_{i \leq i, j}) \leq r_{i+1,j} \right\}$$

$$\cong \{ N \in M_{k \times n} : \forall 1 \leq i' \leq j \leq n, \text{rank}(N_{i' \leq i, j}) \leq r_{i' \leq i, j} \} \quad (i' = i + 1)$$

$$= \Pi_r.$$
Since $C$ is projecting isomorphically to $X^\rho_\pi$, this intersection is projecting isomorphically to $X^\rho_\pi$.

Our running example will be the following $r$ on the left, giving the $\pi$ on the right,

$$
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}
$$

**Corollary 2.1** (of the proof). Partially order the set of interval rank matrices by $r \leq r'$ if $\overline{\Pi}_r \supseteq \overline{\Pi}_{r'}$ (the reversal is to match Bruhat order). Then $r \leq r'$ is a covering relation iff one of the following possibilities holds:

1. $J(r)$ and $J(r')$ agree, except on a rectangle in which $J(r)$ has 1s only in the NW and SE corners, whereas $J(r')$ has 1s only in the SW and NE corners.
2. $J(r)$ and $J(r')$ agree, except that a 1 in $J(r)$ has moved one column to the left (into a column that was previously zero), or one row down (into a row that was previously zero).

**Proof.** The covering relations in $S_n$ Bruhat order, when expressed in terms of permutation matrices, are exactly as described in (1). When we embed $J(r), J(r')$ into $(n+k) \times (k+n)$ permutation matrices as in the proof of theorem 1.8, we acquire more 1s in the permutation matrix, hidden in the rectangles $A_1$ and $A_2$ (and $A_1', A_2'$). The covering relations of type (2) are the ones that involve moving these hidden 1s. (The covering relations involving them are rather limited by the fact that the 1s in those rectangles are arranged NW/SE.)

We denote covering relations by $r \lessdot r'$.

**Lemma 2.2.** The intersection of two interval rank varieties is a reduced union of other interval rank varieties. The same follows for their positroid varieties inside $Gr_k(\mathbb{A}^n)$.

**Proof.** The intersection $X_w \cap X_v$ of two matrix Schubert varieties is a reduced union of other matrix Schubert varieties (by [Fu92, proof of lemma 3.11 after lemma 6.1] and [Ra85, theorem 3]), namely those $X_u$ where $u$ is a least upper bound in $S_n$ of $w, v$. Let $\pi_1, \pi_2$ be the $(k+n) \times (n+k)$ matrices associated in the proof of theorem 1.8 to two interval rank varieties.

If neither $w$ nor $v$ have a descent between positions $i, i+1$, then each $u$ won’t either. (Proof: the descent condition says that the corresponding Schubert varieties $X_u, X_v \subseteq \text{Gr}_n/B$ are unions of fibers of the map $\text{GL}_n/B \to \text{Gr}_n/P_i$, hence their intersection is too, hence any component of it is too.) Using transpose, the same holds for $w^{-1}, v^{-1}$. So for each component $X^\rho_\pi$ of $X^\pi_{\pi_1} \cap X^\pi_{\pi_2}$, the permutation $\rho$ has no descents in positions $1 \ldots k$, and $\rho^{-1}$ has none in positions $n+1 \ldots n+k$. Thus it is necessarily of the same form as the $\pi$ from the proof of theorem 1.8 and hence $C \cap X^\rho_\pi$ is again an interval rank variety.
To compute the corresponding intersection inside $\text{Gr}_k(\mathbb{A}^n)$, we intersect with the copy of the Stiefel manifold inside $C$. (This drops those components with $s > 0$, for the $s$ from the proof of theorem 1.8.)

We now define the “essential set” for an interval rank matrix, an analogue of Fulton’s essential set for a Northwest rank matrix (used to define matrix Schubert varieties). First draw lines strictly to the South, and strictly to the West, of each 1 in $J(r)$, which we think of as crossing out boxes. Then also cross out any empty row or column (with no 1). Call the remaining matrix entries (which includes all the 1s) the strict S/W diagram of $J(r)$. Define the essential set to be the Northeast corners of the strict S/W diagram. (Fulton’s essential set, on his different sort of rank matrix, is the SE corners of the weak S/E diagram.)

In the running example above, the essential rank conditions are $r_{33} \leq 0$, $r_{15} \leq 3$.

**Proposition 2.3.** The interval rank variety $\Pi _r = \{ N \in M_{k \times n} : \tau(N) \leq r \text{ entrywise} \}$ is defined as a scheme already by the rank conditions $\tau(N)_{ij} \leq \tau_{ij}$ for essential boxes $(i, j)$ in $J(r)$’s strict S/W diagram.

*Proof.* First, let $(i, j)$ be a matrix entry not lying in the strict S/W diagram at all. Thus $(i, j)$ is crossed out, say from the North, either by a 1 to the North at $(i' < i, j)$ or because it is in an empty column. (The cases of being crossed out from the East will work the same way.) Therefore there is no 1 lying weakly South of $(i, j)$. Hence $\tau_{ij} = r_{i, j-1} + 1$, so the $(i, j)$ rank condition is implied by the $(i, j - 1)$ rank condition.

Now assume $(i, j)$ is in the strict S/W diagram, but is not a NE corner. Then there is another diagram box at $(i - 1, j)$ or $(i, j + 1)$; we treat the first case. Since $(i - 1, j)$ is not crossed out, it has a 1 weakly to its West, at some $(i - 1, j' < j)$. Hence $\tau_{ij} = r_{i-1,j}$, so the $(i, j)$ rank condition is implied by the $(i - 1, j)$ rank condition.

This lets us trace each rank condition $(i, j)$ outside the “essential set” to another rank condition with the same rank bound to the North or East, or, to one with a lower rank bound to the South or West. Clearly this process must terminate, at an essential box. So the rank conditions from the essential boxes imply all the others. \qed

The term “essential”, taken from [Fu92], is misleading; if $r_{i,k} = r_{ij} + r_{j+1,k}$, and the latter two define essential rank conditions, the $r_{i,k}$ condition is certainly implied but may also be “essential”, as occurs in the example in [3]. (This phenomenon does not occur in Fulton’s context [Fu92] lemma 3.14.)

Given a subvariety $Y \subseteq \text{Gr}_k(\mathbb{A}^n)$, let $\lambda(Y)$ denote the maximum $\lambda$ such that $Y \subseteq X_\lambda$. (Proof of existence: if $Y \subseteq X_\lambda$ and $Y \subseteq X_\mu$, then since $Y$ is irreducible it is contained in one of the components $X_\nu$ of $X_\lambda \cap X_\mu$, and $\nu \geq \lambda, \mu$.) Similarly, let $\mu(Y)$ denote the minimum $\mu$ such that $Y \subseteq X_\mu^\dagger$. If $Y$ is T-invariant, then $\lambda(Y), \mu(Y)$ are the minimum and maximum of $Y^\dagger$ in Bruhat order. Call $X^{\mu(Y)}_{\lambda(Y)}$ the Richardson envelope of $Y$; it is the unique smallest Richardson variety containing $Y$.

**Proposition 2.4.** Let $r$ be an interval rank matrix of size $n$ with $r_{1n} = k$. Then $\lambda(\Pi_r)$ has its 1s in the empty rows of $J(r)$, and $\mu(\Pi_r)$ has its 1s in the empty columns of $J(r)$. The codimension of $\Pi_r$ inside its Richardson envelope $X^{\mu(\Pi_r)}_{\lambda(\Pi_r)}$ is the number of pairs of 1s in $J(r)$ arranged NE/SW.

In our running example, this number of pairs is 1. The Richardson envelope $X_{10101}^{11010}$ is defined by $r_{13} \leq 2$, $r_{35} \leq 2$, $r_{15} \leq 3$. In that larger variety (or really its Stiefel cone in matrix
space), the middle column is a vector contained in the 2-plane spanned by the left three columns, intersect the 2-plane spanned by the right three columns, inside the ambient 3-space, making that column unique up to scale. Hence imposing $r_{33} = 0$, that that column be the zero vector, only drops the dimension by 1, which is the computed codimension in the Richardson envelope.

\textbf{Proof.} The minimum $\lambda(\Pi_r)$ is determined by the rank conditions $\overline{X}_r$ satisfies on its initial intervals $\{[i, j]\}$, i.e. the first row $(r_{ij})$ of $r$. That, in turn, is determined by the empty columns of $J(r)$. The same analysis connects $\mu(\Pi_r)$ to the last column $(r_{jn})$ of $r$ to the empty rows of $J(r)$.

Then, using theorem \[1.8\]

$$\dim \Pi_r = \dim \overline{X}_r - \dim \text{GL}(k) = \dim X^o_{\rho o} - k^2 = \ell(\rho) - \ell(\pi) - k^2 = k(n-k) - \ell(\pi)$$

where (as in its proof)

$$\pi = \begin{bmatrix} A_1 & 0 & \vdots \\ J & A_2 \end{bmatrix}, \quad \rho = n+1 \ldots n+k 1 2 \ldots n.$$

To determine $\ell(\pi)$, we count inversions, i.e. pairs of 1s in $\pi$ aligned NE/SW rather than NW/SE. These pairs come in three types: one in the $A_1$ block and one in the $J$ block, one in the $A_2$ block and one in the $J$ block, or both in the $J$ block.

Each such pair with one 1 in the $A_1$ block and one 1 in the $J$ block corresponds to a 1 in $\mu(\Pi_r)$ occurring before a 0, so the number of them is $\operatorname{codim}(X^{\mu(\Pi_r)} \subseteq \text{Gr}_k(\mathbb{A}^n))$. Similarly, the number of such pairs with one 1 in the $A_2$ block and one 1 in the $J$ block is $\operatorname{codim}(X^{\mu(\Pi_r)} \subseteq \text{Gr}_k(\mathbb{A}^n))$. Hence $\ell(\pi) = \operatorname{codim}(X^{\mu(\Pi_r)} \subseteq \text{Gr}_k(\mathbb{A}^n)) + c$, where $c$ is the number of NE/SW pairs in $J(r)$. Finally,

$$\operatorname{codim}(\Pi_r \subseteq X^{\mu(\Pi_r)}_{\lambda(\Pi_r)}) = \dim X^{\mu(\Pi_r)}_{\lambda(\Pi_r)} - \dim \Pi_r$$

$$= \dim \text{Gr}_k(\mathbb{A}^n) - \operatorname{codim}(X^{\mu(\Pi_r)}_{\lambda(\Pi_r)} \subseteq \text{Gr}_k(\mathbb{A}^n)) - k(n-k) + \ell(\pi)$$

$$= c. \qed$$

For later use, we will want some handle on the $T$-fixed points $(\Pi_r)^T \subseteq \text{Gr}_k(\mathbb{A}^n)^T = \{k\text{-dimensional coordinate subspaces}\} \cong \{\text{words } \lambda \text{ of length } n \text{ with } n-k \text{ 0s and } k \text{ 1s}\}$.

\textbf{Lemma 2.5.} Let $r$ be an interval rank matrix, let $\lambda$ be a word of length $n$ with $n-k$ 0s and $k$ 1s, and $V_\lambda$ the corresponding $k$-dimensional coordinate subspace.

1. If for some $(i, j)$, $r_{ij} < \sum_{k \in [i,j]} \lambda_k$, then $V_\lambda \not\subseteq \Pi_r$.

2. Let $\supp(J(r))$ denote the locations of the 1s in the partial permutation matrix corresponding to $r$, and $m : \supp(J(r)) \to \{1, \ldots, n\}$ an injection such that $m((i, j)) \in \{i, \ldots, j\}$. Let $\lambda = 0$ on the image of $m$ (for \textbf{matching}), and 1 on the complement. Then $V_\lambda \subseteq \Pi_r$.

3. (Hall’s marriage theorem for interval rank varieties) If $V_\lambda \in \Pi_r$, then there exists a matching $m$ as described above.

4. Let $r' \geq r$ be a covering relation as in corollary \[2.7\] and $d$ be the position of the unique 1 (if type (2)) or the Southwestern of the two 1s (if type (1)) in $\supp(J(r')) \setminus \supp(J(r))$. Let $m$ be a matching of $r$, and think of it as a map from $\supp(J(r))$ to the diagonal of $r$, whose image has complement $\lambda$. 

If any 1 in a row above \(d\) is matched to an entry in a row above \(d\), and any 1 in a column to the right of \(d\) is matched to an entry in a column to the right of \(d\), then \(V_\lambda \in \Pi_r \setminus \Pi_r'.

Proof. Let \(M_\lambda\) be the \(k \times n\) matrix with the \(k \times k\) identity matrix in the \(k\) columns \(\{i : \lambda_i = 1\}\), and other columns 0. Then \(V_\lambda\) is the row-span of \(M_\lambda\), so \(V_\lambda \in \Pi_r\) iff \(M_\lambda \in \Pi_r\). Then since 
\[
\tau(M_\lambda)_{ij} = \sum_{k \in [i,j]} \lambda_k,
\]
we have \(M_\lambda \in \Pi_r\) iff \(\tau_{ij} \geq \sum_{k \in [i,j]} \lambda_k\) for all \(i \leq j\).

(1) If some \(\tau_{ij} < \tau(M_\lambda)_{ij}\), then \(M_\lambda \not\in \Pi_r\), hence \(V_\lambda \not\in \Pi_r\).

(2) For each \(i \leq j\), 
\[
\tau(M)_{ij} = \sum_{k \in [i,j]} \lambda_k = |[i,j] \setminus m(\text{supp}(\tau(r)))| 
\leq |[i,j] \setminus m(\text{supp}(\tau(r)))\text{ southwest of } (i,j) | 
= |[i,j]| - |\text{supp}(\tau(r))\text{ southwest of } (i,j)| = \tau_{ij}.
\]

Part (3) is trivial if \(\tau(r)\) is diagonal, and \(m\) is the identity map, if we think of its target set as the diagonal entries \(\{(i,i)\}\). We use this as the base case for an induction.

The induction step is to prove that if \(r' \triangleright r\) is a covering relation of type (1) or (2) in the sense of corollary 2.1, then the set of coordinate subspaces in \(\Pi_r\) with matchings is contained in the set of coordinate subspaces in \(\Pi_r\) with matchings.

If \(r' \triangleright r\) is a type (2) covering relation, one \(1\) in \(\tau(r')\) moves one step North or East to give \(\tau(r)\), giving a simple correspondence \(c\) between their sets of 1s. If \(m'\) is a matching for \(r'\), then \(m := m' \circ c\) is a matching for \(r\); the \(1\) in \(\tau(r)\) first maps South or West one step via \(c\), then maps SW via \(m'\).

If \(r' \triangleright r\) is a type (1) covering relation, then two 1s in \(\tau(r')\) at positions p-NE-of-q move to give those in \(\tau(r)\), at positions s-NW-of-t. To define a correspondence \(c\) as in the type (2) case, and take \(m := m' \circ c\), we need to choose \(p \mapsto s, q \mapsto t\) or \(p \mapsto t, q \mapsto s\). There are three possibilities:

- \(m'(p)\) is SW of q, hence SW of both s and t. Then either possibility for c produces a suitable m.
- \(m'(p)\) is SW of s, but not SW of t. Then c should take \(p \mapsto s, q \mapsto t\).
- \(m'(p)\) is SW of t, but not SW of s. Then c should take \(p \mapsto t, q \mapsto s\).

For example, if \(r'\) is the \(5 \times 5\) matrix in our running example and \(m'\) maps each both \(p\) and \(q\) due West (so \(\lambda = 01011\)), then we face the second possibility. Whereas if \(m'\) maps each due West (so \(\lambda = 11010\)), we face the third. The first possibility cannot occur for this \(r'\) since \(m'(q)\) is forced and \(m'\) is injective.

For part (4), let \(i, j\) be the row and column of \(d\). By the condition on \(m\), \(\tau(M_{ij})_{ij} = \tau_{ij}\). But by the condition on \(d\), \(\tau_{ij} = \tau_{ij} - 1\). So \(\tau(M_{ij})_{ij} > \tau_{ij}'\), hence \(V_\lambda \not\in \Pi_r'\).

As we explained in the introduction, the varieties \(\Pi_r\) are a special class of “positroid varieties” [KLS], which in their full generality allow for rank conditions on cyclic intervals. There is another connection between the \(\Pi_r\) and positroid varieties (much like the double connection between matrix Schubert varieties and Schubert varieties), as follows. Embed \(\Pi_r \times M_{k \times k}\) into \(M_{k \times (n+k)}\) by imposing no rank conditions on the last \(k\) columns; this is \(\Pi_{r'}\) for a suitable \(r'\). Then \(\Pi_r\) is isomorphic to the affine open set on the positroid variety \(GL(k) \setminus (\Pi_{r'} \cap St_{k,n+k})\) where one asks that the last \(k\) columns are linearly independent. In
her thesis [Sn]. Snider shows more generally that the natural affine patches on arbitrary positroid varieties are isomorphic to certain Kazhdan-Lusztig varieties in the affine flag manifold.

3. The Vakil Variety of a Puzzle Path

Let $\gamma$ be a (labeled) puzzle path, as defined in §1.2. In this section we will use $\gamma$ to single out $n - k$ horizontal edges in the puzzle triangle (possibly along the bottom), and use them to define interval rank conditions.

There are a number of conditions that we require the labels on $\gamma$ to satisfy, all of which are implied by “there should be a way to successively add puzzle pieces to $\gamma$, culminating in a final path”. These conditions are:

- On the boundary of the puzzle, there are only 0s and 1s, no R or Ks. (In particular, on initial or final paths there are only 0s and 1s, no Rs.)
- The only place a K may appear is on the kink, so $\_K$.
- Say the first $i$ steps of $\gamma$ are SE. The number of 0s on those edges should be at least the number of −0− in the last $i$ steps (necessarily all West).
- The number of 0s should equal the number of $/_R$ plus the number of $−0−$. (In particular, on final paths as on initial paths, there are only 0s and 1s, no Rs or Ks.)
- If the kink is $\_R$ or $\_K$, after it there must be a $/1$ or $−1$− before there is any $/_R$ or $−0−$.
- If the kink is $\_0$ or $\_K$, after it there must be an $/_R$ or $−0−$ before there is any $−1$− (as implied by the previous two conditions).

3.1. Pink rays and pink dots. We first draw $n - k$ pink rays aligned NE/SW, and $n - k$ more aligned NW/SE. At certain crossing points of these rays, we will place $n - k$ pink dots. These define a partial permutation as in theorem 1.8 giving rank conditions

$$\text{rank}(M_{[i,j]} \leq |[i,j]| - \#\{\text{pink dots on edges } e \text{ with } i \leq i(e) \leq j(e) \leq j\}$$

where $(i(e), j(e))$ were defined in §1.2.

Each pink ray emanates from the midpoint of an edge, so does not follow puzzle edges (rather, it is only parallel to them). Cut the puzzle triangle into a left half and right half along $\gamma$. On the left side of $\gamma$:

- The rays are aligned NW & SW.
- Each $\_0$ (which may include the kink) has a SW-pointing ray.
- Each $\_0$- and $\_R$ has a NW-pointing ray.
- If the kink is $\_R$ or $\_K$, it gets a SW-pointing ray, and causes the next $/1$ South of it to get a NW-pointing ray.

On the right of $\gamma$:

- The rays are aligned NE & SE.
- Each $\_0$ has a SE-pointing ray.
- If the kink is $\_1$, and there is a $\_0$ somewhere above it, the $\_1$ gets a NE-pointing ray.
- Immediately to the right of the rightmost edge of $\gamma$ on the bottom of the triangle, enough other NE-pointing rays are placed to match the number of SE-pointing rays. These are the only rays that come out of puzzle edges not in $\gamma$. 
We will not extend these rays forever, but only to certain crossings, which we will label with “pink dots”.

**Theorem 3.1.** Let $\gamma$ be a puzzle path, with pink rays attached as described above. Assume first that the kink is not labeled $\mathbb{K}$.

Then there is a unique way to pair up the pink rays, such that each pair of paired rays are extended to intersect at a pink dot (on a horizontal edge), and there are no other intersections of pink rays.

If the kink is labeled $\mathbb{K}$, then almost the same is true: there is one crossing (with no pink dot), of the pink ray coming SW out of the $\mathbb{K}$ and the one coming NW out of the next $\mathbb{I}$.

No pink dot is in the same column (NW/SE or NE/SW) as another.

**Proof.** Throughout this proof we use the conditions on the labeling of $\gamma$, generally without comment.

We worry first about matching the ray out of the kink (if any).

- If the kink is a $\mathbb{Q}$ or $\mathbb{R}$, it has a pink ray going SW. Further South along $\gamma$, there must be a positive number of pink rays going NW (either from the first $\mathbb{I}$ after the $\mathbb{R}$, or from an $\mathbb{R}$ or $\mathbb{Q}$). The first such NW ray below the kink must meet the SW ray out of the kink – there are no other rays beforehand for the SW ray to collide with. So
declare those two rays matched up, place a pink dot where they cross, and extend them no further.

- If the kink is a $\kappa$, then further South along $\gamma$, there must be at least two pink rays going NW (by the last condition on puzzle paths). The SW ray from the $\kappa$ is required to cross through the first NW ray, and as above must be matched up with the second NW ray.
- If the kink is a $\mid$, and there is no $\emptyset$ above it, then there is no pink ray out of the $\mid$ to consider.
- If the kink is a $\mid$, and there is a $\emptyset$ above it, then there is a pink ray NE out of the $\mid$ which must be matched up with the SE ray out of the $\emptyset$ most closely above it.

Now we match up the remaining pink rays on the left half of the puzzle triangle (as cleft by $\gamma$). The SW rays come from $\emptyset$s on the NE side of the puzzle triangle, and the NW ones come from $\kappa$s along $\gamma$ and $\emptyset$s on the South side of the puzzle triangle. By assumption on $\gamma$, there are the same number of these rays (which involves a small case check over the possibilities for the kink). To avoid creating crossings, the SW rays must be matched with the NW rays in order, giving the uniqueness. To ensure that the $k$th SW ray crosses the $k$th NW ray at all (each extended infinitely), we use the first condition on $\gamma$.

Wholly independently, we match up the remaining pink rays on the right half of the puzzle triangle. Here the number of NE rays from the bottom edge of the puzzle triangle (not on $\gamma$) was chosen to match the number of SE rays from $\emptyset$s along $\gamma$. Again, to avoid creating crossings, the SW rays must be matched with the NW rays in order, giving the uniqueness.

We must check that no two pink dots are in the same NE/SW or NW/SE column. Group the dots into Left, Kink, and Right according to their NE/SW column. Obviously two pink dots in different groups cannot be in the same NE/SW column, and it is easy to see also that two Left pink dots cannot be in the same NE/SW or NW/SE column, nor can two Right pink dots. There is at most one pink dot in the Kink group.

It remains to show that no two pink dots in different groups can be in the same NW/SE column. Drop the Kink dot (if any) into the Left group or Right group depending on which side of $\gamma$ it lies on. If a Left dot were NW of a Right dot (SE being obviously impossible), the NW ray pointing to the Left dot would emanate from the same $\gamma$-edge as the SE pointing to the Right dot, a contradiction.

### 3.2. The Vakil variety

Let $\Pi_\gamma \subseteq M_{k \times n}$ denote the interval rank variety and $\Pi_\gamma \subseteq Gr_k(\mathbb{A}^n)$ its associated Vakil subvariety of the Grassmannian, using the rank conditions from the $n-k$ pink dots placed according to theorem 3.1. It is easy to carry over the definition of “essential set” from proposition 2.3 to puzzle paths and their pink dots: cross out all NW/SE columns and NE/SW columns with no pink dots, and strictly SW,SE of each pink dot. Then the essential rank conditions correspond to the locally Northernmost horizontal edges remaining, which we will call **essential edges**.

**Proposition 3.2.** If $\gamma$ has no kink, it is easy to compute the codimension of $X_\gamma$ inside its Richardson envelope: it is the number of pairs “$\kappa$ above $\emptyset$” occurring along $\gamma$.

If $\gamma$ has a kink, we must add correction terms:
• If the kink is \(\gamma\), add the number of \(\mathcal{R}\)s above the last \(\mathcal{P}\) above the kink, and if there is a \(\mathcal{P}\) above the \(\gamma\), add also the number of \(\mathcal{P}\)s below the \(\gamma\).

• If the kink is \(\mathcal{P}\),
  add the number of \(\mathcal{R}\)s above the kink, plus the number of \(\mathcal{P}\)s below the first \(\mathcal{R}\) below the \(\mathcal{P}\).

• If the kink is \(\mathcal{K}\),
  add the number of \(\mathcal{R}\)s above the kink, plus the number of \(\mathcal{P}\)s below the first \(\mathcal{J}\) below the \(\mathcal{K}\).

Proof. First consider the case of no kink. The noncrossing condition on the pink rays implies that two pink dots both on the left, or both on the right, of \(\gamma\) will not contribute to the codimension (as computed in proposition 2.4). A pink dot on the left is NW of a \(\mathcal{R}\), and a pink dot on the right is SE of a \(\mathcal{P}\); such a pair only contributes if the \(\mathcal{R}\) occurs above the \(\mathcal{P}\).

If there is a kink, split the pink dots into three classes:

1. those in the NE/SW columns to the left of \(\gamma\),
2. the at most one pink dot in the NE/SW column of the kink, and
3. those in the NE/SW columns to the right of \(\gamma\).

Again, there can be no contribution from pairs of pink dots in the same class. Case-by-case analysis comparing groups 1 and 2, 1 and 3, 2 and 3 gives the rest. \(\square\)

Proposition 3.3. Let \(\gamma\) be the initial path, with a labeling, \(\mu\) on the NE side of the puzzle triangle and \(\nu\) along the bottom edge (both read left to right). Then \(X_\gamma\) is the Richardson variety \(X_\nu^\mu\).

Proof. Proposition 3.2 easily gives that \(X_\gamma\) is codimension 0 in its Richardson envelope, so we merely have to determine that envelope.

Each pink dot is NW of a \(\mathcal{P}\) and SW of a \(\mathcal{Q}\). With this, we can determine \((r_{1j})\) and \((r_{in})\), hence the Richardson envelope \(X_\nu^\mu\). \(\square\)

Proposition 3.4. Let \(\gamma\) be the final path, with a labeling \(\lambda\) on the NW side of the puzzle triangle (read left to right). Then \(X_\gamma\) is the opposite Schubert variety \(X_\lambda^\gamma\).

Proof. Again, proposition 3.2 gives that \(X_\gamma\) is codimension 0 in its Richardson envelope. Let \(\gamma'\) be the initial path with labels \(0^{n-k}1^k\) on the NE side and \(\lambda\) on the S side. It is easy to check that the pink dots for \(\gamma\) are in the same locations as for \(\gamma'\). Now apply proposition 3.3

Alternately, apply proposition 2.3 to see that the only essential rank conditions are from \((r_{1j})\), and check that those define \(X_\lambda^\gamma\). \(\square\)

The next proposition is crucial to the puzzle combinatorics: it will say that a Vakil variety associated to a puzzle path is III-invariant unless there are multiple ways to fill in the next puzzle piece.

Proposition 3.5. Let \(\gamma\) be a non-final puzzle path, where the next rhombus to be filled has NE/SW column \(i\), NW/SE column \(j\). The essential edges occurring on the right-hand side of \(\gamma\) only occur in the \(i\)th or \((i+1)\)st NE/SW column.
If the kink is $\ \backslash \ \$ and the next edge is $\ \backslash \ \$, there is an essential edge $e$ at $i(e) = i + 1$, $j(e) \geq j$. Otherwise no essential edges $e$ have $i \not\in [i(e), j(e)] \ni j$.

Proof. If an edge $e$ in the $k$th NE/SW column, $k > i + 1$, is not crossed out (as described at the beginning of §3.2), we claim the horizontal edge $e'$ just NW of $e$ is also not crossed out. Proof: since $e$ is not crossed out from the NW, $e'$ is also not crossed out from the NW. Since $e$ is not crossed out from the NE, there is a pink dot in its NE/SW column, weakly SW of it. By the way we placed pink dots on the right side of $\gamma$, there is also a pink dot strictly SW of $e'$. Hence $e'$ is also not crossed out from the NW, so not at all. Since $e'$ is not crossed out, $e$ is not essential.

Now consider essential edges $e$ with $i < i(e) \leq j \leq j(e)$. By the previous paragraph, $i(e) = i + 1$. Since the horizontal edge $e'$ just NW of $e$ is crossed out, necessarily from its NE, either there is no pink dot in the kink column or the only pink dot is strictly NE of $e'$. Either way, the kink must be $\ \backslash \ \$, and every edge SW of the kink is crossed out.

For $e$ to not be crossed out, there must be a pink dot weakly SE of it, so there must be a $\ \backslash \ \$ NW of it. If $i(e) = j$, that $\ \backslash \ \$ is the next edge below the kink $\ \backslash \ \$, which was the possibility singled out. In this case the horizontal edge just SE of that $\ \backslash \ \$, at $(i + 1, j)$, is not crossed out, but the edge just NW of it, at $(i, j)$, is. Hence some edge weakly NE of $(i + 1, j)$ is essential.

The remaining case is $j > i(e)$. Then we have located some $\ \backslash \ \$ above the $\ \backslash \ \$, so there is a pink ray NW from the $\ \backslash \ \$ meeting the first $\ \backslash \ \$ above it. (In particular, there is a pink dot in the kink column.) But for $e$ to be essential, the horizontal edge just NW of $e$ must be crossed out by a pink dot strictly to its NE, which doesn’t fit with that dot being SE of the first $\ \backslash \ \$ above the $\ \backslash \ \$. So $e$ cannot be essential. □

4. A DETAILED EXAMPLE: $\mu = \text{0101}$, $\nu = \text{1010}$

We follow the Vakil degeneration of the Richardson variety $Y_1 := X^{\text{0101}}_{\text{1010}}$, including the sweeps and the intersections of components as in theorem 1.1. By proposition 3.3, $Y_1$ is associated to an initial puzzle path labeled with $\text{0101}$ on NE, $\text{1010}$ on S. As we will prove in general in §5 each shift/sweep operation will correspond to moving this path leftward by adding a rhombus puzzle piece.

Initially, $Y_1$’s essential rank conditions are $r_{12}, r_{34} \leq 1$. There is one more “essential edge” for the corresponding $J$, but that rank condition $r_{14} \leq 2$ is the direct sum of these two. The first shift, $3 \rightarrow 4$, does nothing to the rank conditions or to the placement of the pink dots:

The second shift, $2 \rightarrow 4$, is nontrivial, but preserves the one-dimension-larger interval rank variety with essential rank conditions $r_{12}, r_{24} \leq 1$. So that variety is the sweep...
\( \Psi_{2 \rightarrow 4} Y_1 \); call it \( Y_{1T} \). The shift \( 2 \rightarrow 4 \) of \( r_{12}, r_{34} \leq 1 \) is \( r_{12}, r_{32} \leq 1 \), which is reducible; one component \( Y_2 \) is defined by \( r_{13} \leq 1 \) and the other, \( Y_3 \), by \( r_{22} \leq 0 \). Finally, we need to consider the intersection \( Y_{23} := Y_2 \cap Y_3 \) defined by \( r_{13} \leq 1 \) and \( r_{22} \leq 0 \). These Vakil varieties are associated to the following puzzle paths. (The blue rhombus is there as reminder that we used a sweep, not a shift.)

The next three shifts, \( 2 \rightarrow 3, 1 \rightarrow 4, 1 \rightarrow 3 \) again do nothing:

The final shift, \( 1 \rightarrow 2 \), is only nontrivial on \( Y_3 \) and \( Y_{23} \), as \( Y_{1T} \) and \( Y_2 \) are already opposite Schubert varieties. The sweep \( Y_{3T} \) of \( Y_3 \) coincides with \( Y_{1T} \), and the shift \( Y_4 := \Psi_{1 \rightarrow 2} Y_3 \) is the opposite Schubert variety defined by \( r_{11} \leq 0 \). The sweep \( Y_{23T} \) of \( Y_{23} \) coincides with \( Y_2 \), and the shift \( Y_5 \) of \( Y_{23} \) is the opposite Schubert variety defined by \( r_{11} \leq 0, r_{13} \leq 1 \).

Applying theorems 1.3–1.6, we have

\[
\begin{align*}
& (H^*) \quad [X_{0101}^{1010}] = [X_{0110}^{0110}] + [X_{1001}^{1001}] \\
& (H_T) \quad [X_{0101}^{1010}] = [X_{0110}^{0110}] + [X_{1001}^{1001}] + (y_4 - y_1)[X_{1010}^{1010}] \\
& (K) \quad [X_{0101}^{1010}] = [X_{0110}^{0110}] + [X_{1001}^{1001}] - [X_{0101}^{0101}] \\
& (K_T) \quad [X_{0101}^{1010}] = \exp(y_2 - y_4)(1 - (1 - \exp(y_1 - y_2)))[X_{1001}^{1010}] \\
& \quad \quad \quad + \exp(y_2 - y_4)[X_{0110}^{0110}] \\
& \quad \quad \quad + ((1 - \exp(y_2 - y_4)) - \exp(y_2 - y_4)(1 - \exp(y_1 - y_2)))[X_{1010}^{1010}] \\
& \quad \quad \quad - \exp(y_2 - y_4) \exp(y_1 - y_2)[X_{0101}^{0101}]
\end{align*}
\]

all terms.
5. ADDING A RHOMBUS TO A PUZZLE PATH

The Fizzbinesque [HaCo68] rules for pink rays presented in §3 will be seen to interface very well with the puzzle pieces. We deal with the easy cases first:

**Lemma 5.1.** Let \( \gamma \) be a non-final puzzle path, and call its last SE step the kink. (So even e.g. initial paths get an honorary kink.)

1. If the kink lies just above the bottom edge (hence the next step is West along the bottom), then there is a unique triangle \( P \) to add to \( \gamma \). The resulting \( \gamma' \) has the same pink dots as \( \gamma \).
2. Otherwise the kink is followed by a step Southwest. If the labels on the kink and this Southwest step are not 1, 0 respectively, then there is a unique way to add a rhombus (possibly consisting of two triangles) to add to \( \gamma \). The resulting \( \gamma' \) has the same pink dots as \( \gamma \).

**Proof.** There is probably no substitute for attempting the case check oneself. Nonetheless, we describe the results.

- If the triangle \( P \) added has labels \( 1, 1, +, - \), then no pink rays move, much less any pink dots.
- If \( P \) has labels \( 0, 0, 0 \), then there is a pink dot on its S edge, for both \( \gamma \) and \( \gamma' \).
- If \( P \) has labels \( 0, 0, 1 \), then it has a pink ray going NW out of \( + \) in \( \gamma \) and out of \( \gamma' \) in \( \gamma' \).
- If \( P \) has labels \( K, 1 \), then \( \gamma' \) satisfies the conditions put forth at the beginning of §3. □

**Lemma 5.2.** Let \( \gamma \) be a puzzle path with a \( 1 \) kink, followed by a SW step \( \emptyset \). Then one can add the equivariant piece and obtain a new puzzle path \( \gamma' \).

There are two vertical rhombi made out of triangles with \( 1, \emptyset \) on the right. At least one of those two rhombi can be added to \( \gamma \) to obtain a new puzzle path. Both of those can be added iff the top K-piece can be added.

**Proof.** It is straightforward to check that the equivariant piece can be added, i.e. that the resulting \( \gamma' \) satisfies the conditions put forth at the beginning of §3.

If the \((1, 1, 1)\)-\(\Delta\) atop the \((0, 1, R)\)-\(\nabla\) cannot be added to \( \gamma \), it is because the first \( K \) or \( \emptyset \) below the kink is not preceded by any \( J \). In this case, one cannot add the left K-piece.
If the \( (0,0,0) \)-\( \nabla \) below the \( (0,1,\mathbb{R}) \)-\( \Delta \) cannot be added, it is because there is no \( \mathbb{R} \) below the kink, and the first horizontal edge is \( + \), not \( \emptyset \). In this case also, one cannot add the left \( K \)-piece.

These conditions cannot hold simultaneously: the first horizontal edge would need to be \( + \), so the first \( \mathbb{R} \) below the kink would need to come before any \( \emptyset \), but also there couldn’t be any \( \mathbb{R} \) below the kink, contradiction.

If neither condition holds, one can indeed add the left \( K \)-piece. \( \square \)

Say that \( \gamma \) covers \( \gamma' \) if \( \tau(\gamma) \) covers \( \tau(\gamma') \) in the sense of corollary 2.1, or equivalently, if \( \overline{\Pi}_\gamma \) is a divisor in \( \overline{\Pi}_{\gamma'} \). Note that the rectangles from corollary 2.1 are now aligned with the puzzle columns; see the red parallelograms in figure 6.

**Figure 6.** A puzzle path (upper left) to which four rhombi may be added (in the shaded area): the equivariant piece (top middle), two possibilities of two triangles (bottom left, bottom right), and the top \( K \)-piece (bottom middle). The red parallelograms indicate the covering relations, as in corollary 2.1.
Lemma 5.3. Let $\gamma$ be a puzzle path with a $\lambda$ kink, followed by a SW step $\emptyset$. Add the equivariant piece to it giving the puzzle path $\Psi\gamma$.

If the $(1, 0, R) - \Delta$ and $(0, 0, 0) - \nabla$ pieces may be added, call the resulting path $\gamma_0$; it covers $\Psi\gamma$. If the $(1, 1, 1) - \Delta$ and $(1, 0, R) - \nabla$ pieces may be added, call the resulting path $\gamma_1$; it covers $\Psi\gamma$.

Assume that one can add the top $K$-piece, producing a puzzle path $\gamma_K$. Then $\gamma_K$ covers $\gamma_0$ and $\gamma_1$ (which both exist, by lemma 5.2), and $\Pi_{\gamma_K} = \Pi_{\gamma_0} \cap \Pi_{\gamma_1}$. Moreover, there exist coordinate subspaces $V_{\lambda_0} \in \Pi_{\gamma_0} \setminus \Pi_{\gamma_1}$, $V_{\lambda_1} \in \Pi_{\gamma_1} \setminus \Pi_{\gamma_0}$ such that $\lambda_0 \not\in i, j$ and $\lambda_1 \supseteq j$.

An example is in figure 6. Since $\Pi_{\gamma_0}, \Pi_{\gamma_1}$ are irreducible of the same dimension (codimension 1 in $\Pi_{\gamma}$), neither one contains the other, and since they are defined by the vanishing of Plücker coordinates [KLS, theorem 7.4], the existence of $V_{\lambda_0}, V_{\lambda_1}$ is clear. Rather, the difficult parts of the last conclusion are the conditions on $\lambda_0, \lambda_1$.

Proof. From $\Psi\gamma$ to $\gamma_1$, we flip the pink ray NE from the $\lambda$ in $\Psi\gamma$ to SW from the $\emptyset$ in $\gamma_0$, and the pink dot in the kink NE/SW column is the only one that moves (it moves due SW, to the first $j$ below the kink). This is a type (2) move from corollary 2.1.

From $\Psi\gamma$ to $\gamma_0$ is a type (1) move. The rhombus just filled is in the East corner of the relevant parallelogram, and the West corner is in the same NE/SW column as the last $\emptyset$ before the kink, and the first NW/SE $\emptyset$ after it.

We leave the reader to check that when $\gamma_K$ exists, it covers both $\gamma_0$ and $\gamma_1$. Consequently $\Pi_{\gamma_K} \subseteq \Pi_{\gamma_0} \cap \Pi_{\gamma_1}$. To show equality, we need the stronger statement that the rank matrices $r(\gamma_K) = \min(r(\gamma_0), r(\gamma_1))$ entrywise, which is also straightforward to check.

To construct the required $\lambda_0, \lambda_1$, we construct matchings of their complements as in lemma 2.5; each pink dot must be matched with an edge on the bottom of puzzle in the range $[i(d), j(d)]$. The matchings we will use match each dot due Southwest (to $i(d)$) or due Southeast (to $j(d)$). Define them by

$$m_0(d) = \begin{cases} i(d) & \text{if } j(d) < j \\ j(d) & \text{if } j(d) \geq j, \end{cases} \quad m_1(d) = \begin{cases} i(d) & \text{if } i(d) < i \\ j(d) & \text{if } i(d) \geq i, \end{cases}$$

(where the ds are the pink dots of $\gamma_0, \gamma_1$ respectively). It is trivial to check that these are injective, so the complements $\lambda_0, \lambda_1$ of their images give the coordinates of some subspaces $V_{\lambda_0} \in \Pi_{\gamma_0}$, $V_{\lambda_1} \in \Pi_{\gamma_1}$. To check that $V_{\lambda_0}, V_{\lambda_1} \not\subseteq \Pi_{\gamma_K}$, we use criterion (4) of lemma 2.5. We leave the reader to check that $\lambda_0 \not\in i, j, \lambda_1 \supseteq j$. \hfill $\square$

In particular, the union $\Pi_{\gamma_1} \cup \Pi_{\gamma_0}$ is Cohen-Macaulay, as it is a union of two C-M schemes along a C-M divisor. In the proof of theorem 6.3 in the next section we will show that $\Pi_{i \leftrightarrow j} \Pi_{\gamma} = \Pi_{\gamma_1} \cup \Pi_{\gamma_0}$.

6. PROOF OF THE COHOMOLOGICAL FORMULAE 1.3, 1.6

We need a couple of lemmas about geometric shifts.

Lemma 6.1. Let $S$ be a subset of $\{1, \ldots, n\}$, and $r \in \mathbb{N}$. Let

$$\overline{B}_{S \leq r} := \{ M \in M_{k \times n} : \text{rank}(k \times |S| \text{ submatrix of } M \text{ with columns } S) \leq r \}.$$ 

Then if $i \in S$ or $j \not\in S$, $\Pi_{i \leftrightarrow j} \overline{B}_{S \leq r} = \overline{B}_{S \leq r}$. Otherwise $\Pi_{i \leftrightarrow j} \overline{B}_{S \leq r} = \overline{B}_{S \setminus \{j \cup i\} \leq r}$. 


Proof. Recall that \( III_{i\to j}X := \lim_{t \to \infty} \exp(te_{ij}) \cdot X \), and
\[
\exp(te_{ij}) \cdot X = \{ M + t(\text{column } i \text{ added to column } j) : M \in X \}.
\]
If \( j \not\in S \), then \( \text{rank}(\text{submatrix of } M \text{ with columns } S) \) is unaffected by adding \( t \)-column \( i \) to column \( j \), hence \( \exp(te_{ij}) \cdot X = X \) for all \( t \). The same is true if \( i \in S \).

For the interesting case, we need to look closer at the equations defining \( X \): for each \( C \subseteq S \) and \( R \subseteq \{1, \ldots, k\} \), with \( |C| = |R| = r + 1 \), the \( M \)-minor \( d_{R,C} \) using rows \( R \) and columns \( C \) vanishes. Then the equations defining \( \exp(te_{ij}) \cdot X \) are \( d_{R,C} = 0 \) for \( j \not\in R \), and \( d_{R,C} + td_{R \setminus j \cup i,C} = 0 \) for \( j \in R \). Rescaling the latter by \( t^{-1} \), and taking \( t \to \infty \), we find what are a priori some of the equations on \( III_{i\to j}X \):
\[
III_{i\to j}X \subseteq \{ M \in M_{k \times n} : \text{rank}(k \times |S| \text{ submatrix of } M \text{ with columns } S \setminus j \cup i) \leq r \}.
\]
Since \( X \) and \( III_{i\to j}X \) are conical affine schemes with the same Hilbert series (one being a degeneration of the other), and this upper bound also has that same Hilbert series by \( S_n \)-symmetry, the upper bound must be tight. \( \square \)

(A more general statement is true: if \( III_{i\to j}X = X \), then \( III_{i\to j}X = (i \leftrightarrow j) \cdot X \).) Notice that the shift \( i \to j \) on columns acts backwards on these “basic” rank conditions; when possible, the \( j \in S \) turns into an \( i \).

For calculations in \( H^*(Gr_k(A^n)) \) or \( K(Gr_k(A^n)) \), we have the equation on classes
\[
[X] = [III_{i\to j}X]
\]
but for equivariant calculations we need the following lemma.

Lemma 6.2. Let \( X \subseteq Gr_k(A^n) \) be a \( T \)-invariant subvariety. Consider the space of pairs
\[
F := \{(t, \exp(te_{ij}) \cdot x) : t \in A^1, x \in X \} \subseteq A^1 \times Gr_k(A^n) \subseteq \mathbb{P}^1 \times Gr_k(A^n).
\]
Let \( \pi_1, \pi_2 \) denote the projections of \( F \subseteq \mathbb{P}^1 \times Gr_k(A^n) \) to \( \mathbb{P}^1 \), \( Gr_k(A^n) \). Let \( Y := \Psi_{i\to j}X \) be the image \( \pi_2(F) \).

If the map \( F \to Y \) has degree \( d \) (taken to be 0 if the fibers are \( \mathbb{P}^1 \)s), then we have the following equality between \( H^*_T(Gr_k(A^n)) \)-classes:
\[
[X] = d(y_j - y_i)[Y] + [III_{i\to j}X].
\]

Identify \( Gr_k(A^n)^T \), the set of coordinate \( k \)-planes, with the collection of \( k \)-element subsets of \( \{1, \ldots, n\} \). If \( \lambda \in X^T \) is a point such that \( i \in \lambda, j \not\in \lambda, \) and \( (i \leftrightarrow j) \cdot \lambda \not\in X^T \), then \( \dim Y = \dim X + 1 \) and \( d = 1 \).

Now assume that \( Y := \Psi_{i\to j}X \) has rational singularities, and that \( d \) is indeed 1. Then in \( K_T(Gr_k(A^n)) \) one has
\[
[X] = (1 - \exp(y_i - y_j))[Y] + \exp(y_i - y_j)[III_{i\to j}X].
\]

Proof. If we let \( T \) act on this \( \mathbb{P}^1 \) by \( \text{diag}(t_1, \ldots, t_n) \cdot z := t_it_j^{-1}z \), and hence let \( T \) act on \( \mathbb{P}^1 \times Gr_k(A^n) \) diagonally, then \( F \) is \( T \)-invariant.

The \( H^*_T \) and \( K_T \) calculations are very similar, so we do the harder one, \( K_T \). We must be careful to distinguish between \( K_T \) (cohomology) and \( K^T \) (homology) in the following, because \( F \) is unlikely to be smooth. Take the equation in \( K_T(\mathbb{P}^1) \)
\[
1 - \exp(y_i - y_j) = [0] - \exp(y_i - y_j)[[\infty]],
\]
pull back with $\pi_1^*$ to $K_T(F)$, and cap with the fundamental class to get an equation in $K^T(F)$:

$$(1 - \exp(y_i - y_j))[F] = [(0) \times X] - \exp(y_i - y_j)[[\infty] \times \text{III}_{i \to j}X]$$

Push forward with $(\pi_2)_*$ to get an equation in $K^T(\text{Gr}_k(\mathbb{A}^n))$:

$$(1 - \exp(y_i - y_j))(\pi_2)_*[F] = [X] - \exp(y_i - y_j)[\text{III}_{i \to j}X].$$

If $\text{III}_{i \to j}X = X$, then $F = \mathbb{P}^1 \times X$, and $(\pi_2)_*[F] = [Y] = [X]$, and this is trivial. So assume not. Then $\dim F = \dim Y = 1 + \dim X$.

Since the map $\pi_2 : F \to Y$ is birational (by $d = 1$), and since $Y$ was assumed to have rational singularities, $(\pi_2)_*[F] = [Y]$. Finally we use smoothness of $\text{Gr}_k(\mathbb{A}^n)$ to move the equation from $K^T(\text{Gr}_k(\mathbb{A}^n))$ to $K_T(\text{Gr}_k(\mathbb{A}^n))$.

The derivation for $H^*_T$ proceeds from the $H^*_T(\mathbb{P}^1)$-equation

$$y_j - y_i = [[0]] - [[\infty]]$$

and the calculation $(\pi_2)_*[F] = d[Y]$.

It remains to prove the second claim, that $\dim Y = \dim X + 1$ and $d = 1$. Since $\text{III}_{i \to j}X \supseteq \text{III}_{i \to j}\lambda = (i \leftrightarrow j) \cdot \lambda \notin X$, the (irreducible) image $Y$ of $F$ contains $X$ strictly, so is of larger dimension. But $\dim F = \dim X + 1$, so its image $Y$ can only be one dimension larger.

To show that $d = 1$, we show that the preimage in $F$ of $(i \leftrightarrow j) \cdot \lambda \in Y$ is the single, reduced point $(\infty, \lambda)$. The two extreme cases of $X$ will turn out to be $X_\lambda := \{\lambda\}$ and $X_\lor := \{V : p_{(i \leftrightarrow j)\cdot\lambda}(V) = 0\}$, where $p_\lambda$ is the corresponding Plücker coordinate. By assumption, $X_\lambda \subseteq X$. The open set $p_{(i \leftrightarrow j)\cdot\lambda}(V) = 0$ in $\text{Gr}_k(\mathbb{A}^n)$ is also the open Bialynicki-Birula stratum for a one-parameter subtorus of $T$, with $(i \leftrightarrow j) \cdot \lambda$ the attractive fixed point. By the $T$-invariance of $X$, if $X$ were to intersect this open set, it would contain $(i \leftrightarrow j) \cdot \lambda$. Since $X$ is reduced, it must lie in the divisor complementary to this open set.

Hence we can trap $X$ in $X_\lambda \subseteq X_\lor \subseteq X_\lor$. Let $F_\lambda, F_\lor$ be the corresponding families, so we can similarly trap the fiber $F_{(i \leftrightarrow j)\cdot\lambda}$ over $(i \leftrightarrow j) \cdot \lambda$ in

$$(F_\lambda)_{(i \leftrightarrow j)\cdot\lambda} \subseteq F_{(i \leftrightarrow j)\cdot\lambda} \subseteq (F_\lor)_{(i \leftrightarrow j)\cdot\lambda}.$$ The lower bound contains the point $(\infty, (i \leftrightarrow j) \cdot \lambda)$. The large family $F_\lor$ is defined by the equation

$$F_\lor = \{([a, b], V) : b p_{(i \leftrightarrow j)\cdot\lambda}(V) = 0\}$$

(a particular case of the calculation in the proof of lemma 6.1). Over the point $(i \leftrightarrow j) \cdot \lambda$, this equation is $b = 0$, defining the same point $(\infty, (i \leftrightarrow j) \cdot \lambda)$. \hfill $\square$

**Theorem 6.3.** Let $\gamma$ be a non-final puzzle path, whose kink is followed by a SW edge. Let $p_1, \ldots, p_d$ be the rhombi that can be added to $\gamma$, giving the puzzle paths $\gamma_1', \ldots, \gamma_d'$. Then each $\Pi_{\gamma_i}'$ is $\text{III}_{(p_i) \to (p_d)}$-invariant. Let $\Phi(E^*, \rho) \in E^*(pt)$ be the factors associated to the cohomology theory $E^* \in \{H^*, H^*_T, K, K_T\}$, as defined in 6.1.3.

Then we have the following equality between classes in $E^*(\text{Gr}_k(\mathbb{A}^n))$:

$$[\Pi_\gamma] = \sum_{i=1}^d \Phi(E^*, \rho_i) [\Pi_{\gamma_i}']$$

**Proof.** If the labels on the kink-then-SW-edge are not $\emptyset$, then lemma 5.1 applies: there is a unique $p = p_1$, with $\Phi(K_T, p_1) = 1$, and $\Pi_\gamma = \Pi_{\gamma_1}'$. The equation on $E^*(\text{Gr}_k(\mathbb{A}^n))$ classes
is then trivial. So we assume hereafter that we are in the interesting case, that the labels on the kink-then-SW-edge are indeed \( \emptyset, \emptyset \).

We have already analyzed the interesting case on the puzzle side, in lemma \[5.2\]. To replicate the 2 or 4 fillings that show up there, we will apply lemma \[6.2\] to \( \Pi_\gamma \). To analyze the sweep and shift of \( \Pi_\gamma \), we will need the puzzle-theoretic version of corollary \[2.1\] in which the rectangles in the partial permutation matrix \( J(r) \) are replaced by parallelograms in the puzzle (edges parallel to the rhombi). Type (2) covering relations from the corolla now correspond to pink dots moving SW or SE.

Let \( \Psi_\gamma \) denote the path constructed by adding the equivariant rhombus to \( \gamma \). Let its NE/SW and NW/SE columns be \((i,j)\). Let \( r(\gamma), r(\Psi_\gamma) \) be the interval rank matrices associated to these two puzzle paths as in \[3.2\].

First claim: \( \Pi_{\Psi_\gamma} \supseteq \Pi_\gamma \). Indeed, transferring the Bruhat order from corollary \[2.1\] on interval rank matrices over to the set of Vakil varieties, we see \( \Pi_\gamma \) covers \( \Pi_{\Psi_\gamma} \); the two pink dots in NE/SW column \( i, i+1 \) exchange their NW/SE columns. So \( \Pi_\gamma \) is codimension 1 in \( \Pi_{\Psi_\gamma} \).

Second claim: there is a T-fixed point \( V_\lambda \in \Pi_\gamma \) such that \( j \notin \lambda, (i \leftrightarrow j) \cdot V_\lambda \notin \Pi_\gamma \). Let \( d \) be the Southern of the two pink dots of \( \gamma \) that move for \( \Psi_\gamma \), and define \( m' : \text{supp}(J(\gamma)) \to \{1, \ldots, n\} \) by

\[
m'(e) = \begin{cases} i(e) & \text{if } j(e) < j(d) \\ j(e) & \text{if } j(e) \geq j(d). \end{cases}
\]

By lemma \[2.5\](2), the complement \( \lambda \) of the image of \( m' \) has \( V_\lambda \in \Pi_\gamma \). Since \( m'(d) = j(d) = j \), we have \( j \notin \lambda \). Now define \( m : \text{supp}(J(\Psi_\gamma)) \to \{1, \ldots, n\} \) by

\[
m(e) = \begin{cases} i(e) & \text{if } j(e) \leq j(d) \\ j(e) & \text{if } j(e) > j(d). \end{cases}
\]

Then \( \text{image}(m) = (i \leftrightarrow j) \cdot \text{image}(m') \), and \( m \) satifies lemma \[2.5\](4), so \((i \leftrightarrow j) \cdot V_\lambda \in \Pi_{\Psi_\gamma} \setminus \Pi_\gamma \).

Third claim: \( \Pi_{\Psi_\gamma} = \Psi_{i \leftrightarrow j} \Pi_\gamma \). By the first claim, \( \Psi_{i \rightarrow j} \Pi_\gamma \subseteq \Psi_{i \leftrightarrow j} \Pi_{\Psi_\gamma} \). By lemma \[5.2\], \( \Psi_\gamma \) has no essential rank conditions on intervals \([k, l]\) with \( i < k \leq j \leq l \), so \( \Pi_{\Psi_\gamma} = \Psi_{i \rightarrow j} \Pi_{\Psi_\gamma} = \Psi_{i \rightarrow j} \Pi_\gamma \). Together, \( \Psi_{i \rightarrow j} \Pi_\gamma \subseteq \Pi_{\Psi_\gamma} \). By the second claim, \( \Pi_\gamma \) is not \( \Pi_{i \rightarrow j} \)-invariant, hence \( \dim \Psi_{i \rightarrow j} \Pi_\gamma = \dim \Pi_\gamma + 1 = \dim \Pi_{\Psi_\gamma} \). Thus the containment of varieties is an equality.

Since \( \Pi_{\Psi_\gamma} \) is a positroid variety, it has rational singularities (\[KLS\] corollary 7.10), or use theorem \[1.8\] and the corresponding fact about Kazhdan-Lusztig varieties). The second claim allows us to apply lemma \[6.2\] to \( \Pi_\gamma \), obtaining

\[
[\Pi_\gamma] = (1 - \exp(y_i - y_j))[\Pi_{\Psi_\gamma}] + \exp(y_i - y_j)[\Pi_{i \rightarrow j} \Pi_\gamma] \in K_T(Gr_k(A^n))
\]

and

\[
[\Pi_\gamma] = (y_j - y_i)[\Pi_{\Psi_\gamma}] + [\Pi_{i \rightarrow j} \Pi_\gamma] \in H^*_T(Gr_k(A^n)).
\]

These are not quite of the form required in the theorem statement, as \( \Pi_{i \rightarrow j} \Pi_\gamma \) is not necessarily irreducible (though, being a flat degeneration of \( \Pi_\gamma \), it is necessarily equidimensional). The remainder of the proof is the analysis of \( \Pi_{i \rightarrow j} \Pi_\gamma \).

Fourth claim: \( r(\gamma)_{i+1,j} < r(\Psi_\gamma)_{i,j-1} \) (see the top two pictures in figure \[6\] for an example). First observe \( r(\gamma)_{i+1,j} = r(\gamma)_{i,j} - 1 \) and \( r(\Psi_\gamma)_{i,j-1} = r(\Psi_\gamma)_{i,j} \), so it is enough to show \( r(\gamma)_{ij} \leq r(\Psi_\gamma)_{ij} \). The pink dots of \( \gamma \) and of \( \Psi_\gamma \) agree except for two arranged roughly east/west
in $\gamma$ that move roughly north/south in $\Psi\gamma$, and give the same count at position $(i, j)$, so $r(\gamma)_{ij} \leq r(\Psi\gamma)_{ij}$.

Now we apply lemma 6.1 since $\Pi_\gamma \subseteq B_{[i+1,j] \leq r(\gamma)_{i+1,j}}$, we know $\Pi_{i\rightarrow j} \Pi_\gamma \subseteq B_{[i,j-1] \leq r(\gamma)_{i+1,j}}$. Hence

$$\Pi_{i\rightarrow j} \Pi_\gamma \subseteq \Pi_{\Psi \gamma} \cap B_{[i,j-1] \leq r(\gamma)_{i+1,j}}$$

and by the fourth claim, the right-hand side is properly contained in $\Pi_{\Psi \gamma}$. Since $\dim \Pi_{i\rightarrow j} \Pi_\gamma = \dim \Pi_\gamma = \dim \Pi_{\Psi \gamma} - 1$ as in the third claim, the equidimensional left side $\Pi_{i\rightarrow j} \Pi_\gamma$ must consist of some of the geometric components of the right-hand side. (The containment will turn out to be an equality.)

Since both $\Pi_{\Psi \gamma}$ and $B_{[i,j-1] \leq r(\gamma)_{i+1,j}}$ are interval positroid varieties, by lemma 2.2 their intersection is a reduced union of interval positroid varieties. Since the dimension count above indicates that the intersection is codimension 1 in $\Pi_{\Psi \gamma}$, we need to look for interval rank matrices covered by $r(\Psi\gamma)$ in the covering relations from corollary 2.1 and they must lower the rank bound on columns $[i, j-1]$. The reader may wish to study figure 6 while following the next argument.

Adding the $(1, 0, R) - \Delta$ and the $(0, 0, 0) - \nabla$. We first consider the covering relations of type (1), coming from a parallelogram in $\Psi\gamma$’s puzzle with pink dots only in the left and right corners. Moreover, the $(i, j-1)$ rhombus should be in the parallelogram but not on its top two edges. That forces the pink dot at position $(i, j)$ in $\Psi\gamma$’s puzzle to be in the rhombus, and moreover to be the pink dot in the right-hand corner.

Which pink dot could be in the left-hand corner? Being left of the $(i, j)$ rhombus, it must be the intersection of a NW- and a SW-pointing ray, along the SW and NW sides of the parallelogram. If we order those dots according to their NW/SE column, the parallelogram with right-hand corner $(i, j)$ and left corner the $p$th dot will contain in its interior the $q$th dot for each $q > p$. Since we only want parallelograms with no pink dots in the interior, we must take the last dot in this order, NW of the first $R$ or $\emptyset$ below the kink. Call the resulting parallelogram the 0-parallelogram for later reference.

If we add the $(1, 0, R) - \Delta$ and the $(0, 0, 0) - \nabla$ triangular pieces to $\gamma$, we get another puzzle path $\gamma_1$ whose pink dots match those of $\Psi\gamma$, except that the pink dots in the left and right of the 0-parallelogram have moved to the top and bottom.

To sum up: there is at least one relevant covering relation of type (1), and it is effected exactly by adding the $(1, 0, R) - \Delta$ and the $(0, 0, 0) - \nabla$ triangular pieces to $\gamma$. (If there is no $R$ or $\emptyset$ below the kink, then adding the $(1, 0, R) - \Delta$ and the $(0, 0, 0) - \nabla$ triangular pieces to $\gamma$ produces an illegal puzzle path.)

Adding the $(1, 1, 1) - \Delta$ and the $(1, 0, R) - \nabla$. Now we consider the covering relations of type (2), which are most easily thought about by adding pink dots just outside the puzzle triangle on the NW and NE sides, in each column (NW/SE or NE/SW) that doesn’t already have a pink dot. We again want a parallelogram in $\Psi\gamma$’s puzzle (now allowed to reach slightly outside) with pink dots only in the left and right corners, such that the $(i, j-1)$ rhombus is in the parallelogram but not on its top two edges. That again forces the right-hand corner to be at position $(i, j)$. The left corner contains the pink dot at position $(0, j')$ with the maximum $j' > j$, i.e. NW of the first $I$ below the kink. Call the resulting parallelogram the 1-parallelogram for later reference.

If we add the $(1, 1, 1) - \Delta$ and the $(1, 0, R) - \nabla$ triangular pieces to $\gamma$, we get another puzzle path $\gamma_0$ whose pink dots match those of $\Psi\gamma$, except that the pink dots in the left
and right of the 1-parallelogram have moved to the top and bottom. Inside the triangle, (only) one pink dot has moved SW from \((i, j)\) to be just NW of the first \(j\) below the kink.

So far we have analyzed the upper bound \(\Pi_{\gamma} \cap B_{[i,-1\leq r(\gamma)_{i+1}, j]}\); it is reduced, and has at most the components \(\Pi_{\gamma_0}, \Pi_{\gamma_1}\) predicted by adding two triangles to \(\gamma\) (in particular, at most two components). Since it contains \(\Pi_{\gamma_1, \gamma}\), whose dimension is \(\dim \Pi_{\gamma_1} - 1\), the upper bound must have at least one of those two possible components.

If \(\Pi_{\gamma} \cap B_{[i,-1\leq r(\gamma)_{i+1}, j]}\) has only one component, say \(\Pi_{\gamma_0}\), then

\[
[\Pi_{\gamma}] = (1 - \exp(y_i - y_j))[\Pi_{\gamma_0}] + \exp(y_i - y_j)[\Pi_{\gamma_1}],
\]

which was to be proved. The \(\Pi_{\gamma_1, \gamma}\) case is exactly the same.

The remaining case is that the upper bound has two components \(\Pi_{\gamma_1} \cup \Pi_{\gamma_0}\) (i.e. both ways of adding two triangles to \(\gamma\) result in valid puzzle paths); we need to show that \(\Pi_{\gamma_1, \gamma}\) contains each entire component. Since \(\Pi_{\gamma_1, \gamma}\) is (set-theoretically) equidimensional, it is enough to show it contains a point in each of \(\Pi_{\gamma_1} \setminus \Pi_{\gamma_0}, \Pi_{\gamma_0} \setminus \Pi_{\gamma_1}\). We did exactly this at the end of lemma \[5,3\] the conditions given there on \(\lambda_0, \lambda_1\) ensure that they lie in \(\Pi_{\gamma_1, \gamma}\). Hence

\[
\Pi_{\gamma_1, \gamma} = \Pi_{\gamma_1} \cup \Pi_{\gamma_0}
\]

and

\[
[\Pi_{\gamma_1, \gamma}] = [\Pi_{\gamma_1}] + [\Pi_{\gamma_0}] - [\Pi_{\gamma_1} \cap \Pi_{\gamma_0}] = [\Pi_{\gamma_1}] + [\Pi_{\gamma_0}] - [\Pi_{\gamma_k}]
\]

the latter by lemma \[5,3\] Consequently

\[
[\Pi_{\gamma}] = (1 - \exp(y_i - y_j))[\Pi_{\gamma_0}] + \exp(y_i - y_j)([\Pi_{\gamma_1}] + [\Pi_{\gamma_0}] - [\Pi_{\gamma_k}])
\]

which was to be proved. □

**Proof of theorems \[1,3,6\]** Let \(\gamma\) be the initial puzzle path having \(\mu\) on the NE side and \(\nu\) on the S side, both read left-to-right. Add rhombi to it in the filling order from figure \[2\] (and add triangles at the end of each NE/SW column). Along the way, there may be choices (exactly when the kink and next edge are \(1, 1\)); a record of the choices made is exactly a puzzle \(P\). At the end, we have a final puzzle path \(\lambda(P)\), whose labels are determined by the NW side of \(P\).

So iterating theorem \[6,3\] \(\binom{n}{2}\) times, we obtain

\[
[\Pi_{\lambda}] = \sum_{\text{puzzles } P \text{ with } \mu \text{ on NE, } \nu \text{ on S}} \left( \prod_{\rho \in P} \Phi(E^*, \rho) \right) [\Pi_{\lambda(P)}]
\]

By propositions \[3,3\] and \[3,4\] \(\Pi_{\lambda} = \chi^\mu_{\lambda}\) and \(\Pi_{\lambda(P)} = \chi^{NW \text{ side of } P}\). So

\[
[\chi^\mu_{\lambda}] = \sum_{\lambda} \sum_{\text{puzzles } P \text{ with } \mu \text{ on NE, } \nu \text{ on S, } \nu \text{ on NW}} \left( \prod_{\rho \in P} \Phi(E^*, \rho) \right) [\chi^\lambda].
\]

□

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