Abstract

It is shown that every outerplanar graph $G$ could be linearly embedded in the plane so that the number of distinct distances between pairs of adjacent vertices is at most thirteen and there is no intersection between the image of a vertex and that of an edge not containing it. This settles a problem of Carmi, Dujmović, Morin and Wood.

Keywords: Outerplanar graphs, planar graphs, distance number of a graph, drawing of a graph.

1 Introduction

A linear embedding of a graph $G$ is a mapping of the vertices of $G$ to points in the plane. The image of every edge $uv$ of the graph is the open interval between the image of $u$ and the image of $v$. The length of that interval is called the edge-length of $uv$ in the embedding. A degenerate drawing of a graph $G$ is a linear embedding in which the images of all vertices are distinct. A drawing of $G$ is a degenerate drawing in which the image of every edge is disjoint from the image of every vertex. Observe that in both cases edges are allowed to cross each other so that the embedding is not planar.

The distance-number of a graph is the minimum number of distinct edge-lengths in a drawing of $G$.

An outerplanar graph is a graph that can be embedded in the plane without crossings in such a way that all the vertices lie in the boundary of the unbounded face of the embedding. In [3], Carmi, Dujmović, Morin and Wood ask whether the distance-number of outerplanar graphs is uniformly bounded. We answer this question positively, by showing that the distance number of outerplanar graphs is at most 13. This result is derived by explicitly constructing a degenerate drawing for every such graph.

Theorem 1. Every outerplanar graph has distance-number at most 13.

For matters of convenience, throughout the paper we consider all linear embeddings as mapping vertices to the complex plane.
2 Background and Motivation

The notions of distance-number and degenerate distance-number of a graph were introduced by Carmi, Dujmović, Morin and Wood in [3] in order to generalize several well studied problems. Indeed, Erdős suggested in [4] the problem of determining or estimating the minimum possible number of distinct distances between \( n \) points in the plane. In 2015, Guth and Katz, in a ground-breaking paper [5], established a lower-bound of \( cn/\log n \) on this number, which almost matches the \( O(n/\sqrt{\log n}) \) upper-bound due to Erdős.

Another problem, considered by Szemerédi (See Theorem 13.7 in [7]), is that of finding the minimum possible number of distances between \( n \) non-collinear points in the plane. This problem can be rephrased as finding the distance-number of \( K_n \). One interesting consequence of the known results on these questions is that the distance-number and the degenerate distance-number of \( K_n \) are not the same, thus justifying the two separate notions. For a short survey of the history of both problems, including some classical bounds, the reader is referred to the background section of [3].

Another notion which is generalized by the degenerate distance-number is that of a unit-distance graph, i.e., a graph that can be embedded in the plane so that two vertices are at distance one if and only if they are connected by an edge. Observe that all unit-distance graphs have degenerate distance-number 1 while the converse is not true. This is because in a degenerate drawing of a graph which uses only length 1 edges, there could be points at distance 1 which are not connected by an edge. This distinction has recently been investigated also in higher dimension by Alon and Kupavskii [2]. They show that every graph whose chromatic number is at most \( d \) could be embedded in \( \mathbb{R}^d \) so that the distance between the two sides of every edge is one. They also provide several Ramsey-type results concerning with such graphs and with unit distance graphs in \( \mathbb{R}^d \).

Constructing 'dense' unit-distance graphs is a classical problem. The best construction, due to Erdős [4], gives an \( n \)-vertex unit-distance graph with \( n^{1+c/\log \log n} \) edges, while the best known upper-bound, due to Spencer, Szemerédi and Trotter [8], is \( cn^{4/3} \) (A simpler proof for this bound was found by Székely, see [9]). Note that this implies that the \( k \) most frequent interpoint distances between \( n \) points occur in total no more than \( ckn^{4/3} \) times, and thus that a graph with degenerate distance-number \( k \) cannot have more than \( ckn^{4/3} \) edges. Katz and Tardos gave in [6] another bound on the frequency of interpoint distances between \( n \) points in the plane, which yields that a graph with distance-number \( k \) cannot have more than \( c n^{1.4610^{0.63}} \) edges.

After introducing the notions of distance-number and degenerate distance-number, Carmi, Dujmović, Morin and Wood studied in [3] the behavior of bounded degree graphs with respect to these notions. They show that graphs with bounded degree greater or equal to five can have degenerate distance-number arbitrarily large, giving a polynomial lower-bound for graphs with bounded degree greater or equal to seven. They also give a \( c \log(n) \) upper-bound to the distance-number of bounded degree graphs with bounded treewidth. In the same paper, the authors ask whether this bound can be improved for outerplanar graphs, and in particular whether such graphs have a uniformly bounded
(degenerate) distance-number. Alon and the second author \cite{ref1} gave a partial answer by proving that outerplanar graphs have degenerate distance number at most 3. Here we provide a complete answer to the question by showing that such graphs have distance number at most 13.

3 \hspace{1em} Proof outline

Our proof of Theorem \cite{ref1} is obtained in three steps.

\textbf{Reduction to drawing the rhombus tree.} Firstly, we remind the readers that outerplanar graphs are all sub-graphs of a universal cover called the triangle tree, generated by starting from a single base triangle with a marked edge and iteratively gluing additional triangles along any unglued side of a triangle except for the base edge, ad infinitum. We observe that this graph could also be represented as a tree of rhombi denoted by $H^*$, as depicted in Figure 2, so that the theorem reduces to showing that $H^*$ has a degenerate distance number 13.

\textbf{Encoding the rhombi.} To describe each of the rhombus of $H^*$, we encode it by the sequence of descending steps taken to reach it from the root rhombus. These correspond to right steps, where we move to a rhombus glued along the edge to the right of the base edge, left steps, where we move to a rhombus glued along the edge to the left of the base edge and forward steps, where we move to a rhombus glued along the edge parallel to the base edge. These steps are depicted in Figure 2. As every vertex in the tree appears in infinitely many rhombi, we later define a canonical one-to-one mapping between vertices and rhombi (see Observation 4.2).

\textbf{Polynomial embedding.} We define a polynomial mapping $\psi$ which maps every vertex of $H^*$ to the polynomial $C[\{X_i\}_{i \in I}]$ where $|I| = 12$. For every choice of parameters $\{x_i\}_{i \in I}$ in the unit circle, our mapping $\psi_x$ maps every rhombus in $H^*$ to a rhombus in $C$ with side length 1 and diagonal $|x_i - 1|$. We call $i$ the type of the rhombus. It is our purpose to show that for almost every $\{x_i\}_{i \in I}$ in the unit circle, the mapping $\psi_x$ is a drawing, i.e., the images of vertices of $H^*$ through $\psi_x$ do not coincide nor do they intersect with the image of any edges.

\textbf{Avoiding intersections.} In order to show that image of distinct vertices do not coincide for almost every $x$, it suffices to show that the corresponding polynomials do not coincide. This is due to the fact that any two polynomials in $x$ that do not coincide take distinct values for almost every $x$ on the unit circle. Showing that a vertex and an edge cannot coincide is more challenging, and is the main innovation of the paper. We show that if a vertex $v$ lies on the line containing the image of an edge $(u, w)$ for more than a zero measure set of assignments – then $\frac{\psi_x(v)}{\psi_x(w) - \psi_x(v)}$ is a Laurent sum with symmetric coefficient around the zero degree. Our choice for the type of each rhombus is made to guarantee that this cannot occur, unless the vertex is obtained by taking forward steps from that edge or vice versa (in which case, the vertex and the edge clearly cannot coincide).
4 Preliminaries

In this section we repeat many of the notations, definitions and observations used in [1]. These are included in order to make the paper self-contained.

**Outerplanarity, \(\Delta\)-trees and \(T^*\).** An outerplanar graph is a graph that can be embedded in the plane without crossings so that all its vertices lie in the boundary of the unbounded face of the embedding. The edges which border this unbounded face are uniquely defined, and are called the external edges of the graph; the rest of the edges are called internal. A Graph is outerplanar if and only if it has a planar linear embedding mapping its vertices to any distinct set of roots of unity. Such a graph is said to be complete if all of the bounded faces of its embedding are triangular.

Let \(\Delta\) be the triangle graph, that is, a graph on three vertices \(v_0, v_1, \) and \(v_2\), whose edges are \(v_0v_1, v_0v_2\) and \(v_2v_1\). A graph is said to be a \(\Delta\)-tree if it can be generated from \(\Delta\) by iterations of adding a new vertex and connecting it to both ends of some external edge other than \(v_0v_1\). The adjacency graph of the bounded faces of such a graph forms a binary tree, that is – a rooted tree of maximal degree 3. By repeating the above glowing procedure ad infinitum, leaving no external edges, one obtains an infinite complete outerplanar graph \(T^*\) which contains all \(\Delta\)-trees as subgraphs. The adjacency graph of the faces of this graph form a complete infinite binary tree. The root of \(T^*\) is denoted by \(T^*_\text{root}\).

It is a classical fact, which can be proved using induction, that the triangulation of every outerplanar graph is a \(\Delta\)-tree. All outerplanar graphs are therefore subgraphs of \(T^*\). Thus, Theorem [1] reduces to the following proposition.

**Proposition 4.1.** For almost every set \(\{a_i \in (0,1)\}_{i\in[13]}\), the graph \(T^*\) has a drawing using only edge-lengths from the set \(\{a_i\}_{i\in[13]}\).

**The rhombus graph \(H\), Covering \(T^*\) by rhombi.** In order to prove the above proposition, we construct an explicit embedding of \(T^*\) in \(\mathbb{C}\). To do so we introduce a covering of \(T^*\) by copies of a particular directed graph \(H\) which we call a rhombus. We then embed \(T^*\) into \(\mathbb{C}\), one copy of \(H\) at a time.

![Figure 1: The rhombus graph \(H\).](image)

The rhombus directed graph \(H\), is defined to be the graph satisfying \(V_H = \{v_0, v_1, v_2, v_3\}\) and \(E_H = \{v_0v_1, v_0v_2, v_2v_3, v_1v_3, v_2v_1\}\). We call \(v_0\) and \((v_0, v_1)\) the base vertex and base edge of \(H\), respectively.
Define $H^*$ to be the infinite directed trinary tree whose nodes are copies of $H$, labeling the three arcs emanating from every node by $v_0v_2$, $v_2v_3$ and $v_1v_3$. We write $L(a)$ for the label of an arc $a$. Let $N$ be a node of $H^*$, and let $v_i v_j \in E_H$; we call a pair $(N, v_i)$ a vertex of $H^*$, and a pair $(N, v_i v_j)$, an edge of $H^*$. Notice the distinction between arcs of $H^*$ and edges of $H^*$, and the distinction between nodes and vertices. The root of $H^*$ is denoted by $H^*_{root}$. A portion of $H^*$ is depicted in Figure 2.

There exists a natural map $\pi$ from the vertices of $H^*$ to the vertices of $T^*$ which maps each node of $H^*$ to a pair of adjacent triangles of $T^*$. $\pi$ is defined in such a way that $H^*_{root}$ is mapped to $T^*_{root}$ and to one of its neighboring triangles, and every directed arc $MN$ of $H^*$, satisfies $\pi((M, L(MN))) = \pi((N, v_0v_1))$ (in the sense of mapping origin to origin and destination to destination). In the rest of the paper we extend $\pi$ naturally to edges and subgraphs, and abridge $\pi((N, v))$ to $\pi(N, v)$. A portion of $T^*$ and its covering by $H^*$ through $\pi$ are depicted in Figure 2.

![Figure 2](image-url)

Figure 2: A portion of $T^*$ and the corresponding covering by $H^*$. The light gray, dark gray and black arrows represent the $v_0v_2 \rightarrow v_0v_1$, $v_2v_3 \rightarrow v_0v_1$ and $v_1v_3 \rightarrow v_0v_1$ edges of the nodes $H \in H^*$ respectively. The light gray vertex and the rhombus with light gray sides are encoded by the labels $S = (v_0v_2, v_0v_1, v_0v_2, v_0v_1)$. These are also QR-encoded by $((1,1,0),(0,0),2)$. Observe that this encoding is the unique encoding of the corresponding rhombus, but that the base vertex of this rhombus could be encoded in many ways (e.g. as $((1,1),(0,1))$).

**Encoding the rhombi.** In order to embed $T^*$ into $\mathbb{C}$, rhombus-by-rhombus, a way to refer to every node $N \in H^*$ is called for. We first encode $N$ by the sequence of labels on the path from $H^*_{root}$ to $N$. This trinary sequence is denoted by $S_N$. Observe that the map $N \rightarrow S_N$ is a bijection. Indeed, one may think of each label in $S_N$ as a direction, "left", "right" or "forward", in which one must descend $H^*$ until finally arriving at $N$. 

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To simplify our proofs, we further encode $S_N$ using a so called \textit{QR-encoding} which encodes the path to each node via "how many forward steps to take between each turn left or right" and "whether the $i$-th turn left or right". Formally, the QR-coding of $S_N = \{s_1, \ldots, s_{m(N)}\}$ consists of a triple $\{(q_i(N))_{i=1}^{m(N)+1}, (\rho_i(N))_{i=1}^{m(N)}, m(N)\}$. We set $q_i(N)$ to be the number of $v_2v_3$-labeled steps between the $(i-1)$-th non-$v_2v_3$ label in $S_N$ and the $i$-th one (with $q_1$ being the number of $v_2v_3$-labeled steps before the first non-$v_2v_3$ label in $S_N$ and $q_{m(N)+1}$ being their number after the last non-$v_2v_3$ label in $S_N$). We set $\rho_i(N)$ to be 0 if the $i$-th non-$v_2v_3$ element is $v_0v_2$ and 1 if it is $v_1v_3$. We call the triple $\{(q_i(N))_{i=1}^{m(N)+1}, (\rho_i(N))_{i=1}^{m(N)}, m(N)\}$ the \textit{QR-encoding} of $N$ denoting it by $QR(N)$. In particular, in the case where $N = H^*_{\text{root}}$ we use the encoding $\{(0), \{\}, 0\}$. When $q_m = 0$, we omit it in the QR-encoding, writing $QR(N) = \{(q_i)_{i\in[m]}, (\rho_i)_{i\in[m]}, m\}$ for $m = m(N)$.

In accordance with our informal introduction, a QR-encoding $\{(q_i)_{i\in[m+1]}, (\rho_i)_{i\in[m]}, m\}$ should be interpreted as taking $q_1$ steps forward, then turning left or right according to $\rho_1$ being 0 or 1 respectively, then taking another $q_2$ steps forward in the new direction and so on and so forth. Observe that the QR-encoding of each node is unique.

**Encoding the vertices of $T^*$.** The encoding of the nodes of $H^*$ naturally extends to an encoding of the vertices of $T^*$ by defining $QR(u) = \{QR(N) : \pi(N, v_0) = u\}$ for $u \in T^*$. This is indeed an encoding of all the vertices of $T^*$, as for every vertex $u \in T^*$ there exists at least one node $N$ such that $\pi(N, v_0) = u$. However, the encoding of every vertex is non-unique as an infinite number of nodes encode each vertex as a base vertex. As a unique encoding of every vertex is desirable for our purpose, we make the following observation.

**Observation 4.2.** Let $u \in T^*$, there exists a unique node $N$ such that

$$QR(N) = \{(q_i)_{i\in[m]}, (\rho_i)_{i\in[m]}, m\} \in QR(u)$$

and either $q_m > 0$ or $m = 1$. We call this encoding the proper encoding of $N$.

**Proof.** Firstly, observe that the only proper encodings of the vertices $\pi(H^*_{\text{root}}, v_0)$ and $\pi(H^*_{\text{root}}, v_1)$ are $((0,0), (0,1))$ and $((0,0), (1,1))$ respectively.

Next, from our construction of $H^*$, we see that for every vertex $u \in T^*$, except for the base edges of the root rhombus, $\pi(H^*_{\text{root}}, v_0)$ and $\pi(H^*_{\text{root}}, v_1)$, there exists a unique node $N_u \in H^*$ satisfying that $\pi(N_u, v_i) = u$ for some $i \in \{2, 3\}$. Let $\sim$ denote the concatenation operation between sequences. Using this notation we have that either $S(N_u) \sim v_2v_3 \sim v_0v_2$ or $S(N_u) \sim v_2v_3 \sim v_1v_3$ encode a node whose base vertex is mapped by $\pi$ to $u$. One may verify from the definition of QR-encodings that $S_N$ ending with either $v_2v_3 \sim v_0v_2$ or with $v_2v_3 \sim v_1v_3$ is equivalent to $q_{m+1} = 0$ and $q_m > 0$.  

**Polynomial embedding.** Denote by $\mathbb{T}^d$ the $d$-dimensional unit torus in $\mathbb{C}^d$, i.e.

$$\mathbb{T}^d = \{(x_1, \ldots, x_d) \in \mathbb{C}^d : \forall i \in \{1, \ldots, d\}, |x_i| = 1\}.$$  

We begin by defining a degenerate polynomial embedding, as an auxiliary in defining a full polynomial embedding.
**Definition 4.3.** A \( d \)-degenerate polynomial embedding of a graph \( G \) using \( d + 1 \) edge-lengths is a one-to-one mapping \( \psi : V_G \to \mathbb{C}[x_1, \ldots, x_d] \) with integer coefficients, such that for every fixed \( x \in \mathbb{T}^d \) the map \( v \mapsto \psi(v)(x) = \psi_x(v) \) is a linear embedding using at most \( d + 1 \) non-zero edge-lengths.

The importance of \( d \)-degenerate polynomial embeddings to our purpose stems from the following proposition, given in [1] Proposition 1:

**Proposition 4.4.** If \( \psi \) is a \( d \)-degenerate polynomial embedding of a graph \( G \), then for almost every \( x = (x_1, \ldots, x_d) \in \mathbb{T}^d \), \( \psi_x \) is a degenerate drawing of \( G \).

**Proof.** For any \( v, w \in V_G \), the polynomials \( \psi(v)(x) \) and \( \psi(w)(x) \) may coincide only on a set of measure 0 in \( \mathbb{T}^d \). Taking union over all the pairs \( v_1, v_2 \), we get that outside an exceptional set of measure zero in \( \mathbb{T}^d \), the map \( \psi_x \) is one-to-one. \( \square \)

**Definition 4.5.** A \( d \)-polynomial embedding of a graph \( G \) using \( d + 1 \) edge-lengths is a \( d \)-degenerate polynomial embedding that satisfies that, for every vertex \( v \) and edge \((u, w)\) in \( T^* \), for almost every \( x \in \mathbb{T}^d \) we have

\[
\psi_x(v) \not\in \{\alpha \psi_x(w) + (1 - \alpha) \psi_x(u) : \alpha \in (0, 1)\}.
\]  

(1)

This definition guarantees the following

**Observation 4.6.** If \( \psi \) is a \( d \)-polynomial embedding of a graph \( G \), then \( \psi_x \) is a drawing of \( G \) for almost every \( x = (x_1, \ldots, x_d) \in \mathbb{T}^d \).

Denote \( f(x) = \frac{\psi_x(v) - \psi_x(w)}{\psi_x(u) - \psi_x(w)} \), and observe that if \( f(x) \) is not a real function we obtain that \( \psi_x(v) \in \{\alpha \psi_x(w) + (1 - \alpha) \psi_x(u) : \alpha \in (0, 1)\} \) only on a set of measure 0 in \( \mathbb{T}^d \). Hence we make the following observation:

**Observation 4.7.** To show that If \( \psi \) is a \( d \)-polynomial embedding of a graph \( G \) is would suffice to show for every \((u, w) \in E(G)\) and \( v \in E(G)\)

1. either that \( \psi_x(v) \not\in \{\alpha \psi_x(w) + (1 - \alpha) \psi_x(u) : \alpha \in (0, 1)\} \),

2. or that the rational function \( f(x) = \frac{\psi_x(v) - \psi_x(w)}{\psi_x(u) - \psi_x(w)} \) is not a real function.

**Real rational functions and central palindromicity.** In this section we establish criteria for a rational function to be real. We later apply this together with Observation 4.7 to establish the fact that our construction consists of a polynomial embedding. Let \( P \) be a multivariate polynomial with leading monomial \( N \). The ray \( P + \alpha N \), for \( \alpha \in \mathbb{R}_+ \), will be called the main ray of \( P \).

Denote by \( M^d(x_1, \ldots, x_n) \) the set of monomials in \( x_1, \ldots, x_n \) of total degree at most \( d \) and write \( M = M^{\infty} \). A rational function \( g(x_1, \ldots, x_n) \in \mathbb{F}(x_1, \ldots, x_n) \) with coefficients in a field \( \mathbb{F} \) is called central palindromic of order-\( d \), if it could be written as \( g = \sum_{M \in M^d(x_1, \ldots, x_n)} a_M (ML + (ML)^{-1}) \)
for some real coefficients $a_M$, where $L^2 \in \mathcal{M}^4$ (here the role of $L$ is to allow central palindromic polynomial functions containing monomials of fractional degree of basis 2).

We say that a monomial $L_0$ is the symmetric monomial of $L_1$ relative to $M$ if $L_0L_1 \equiv M^2$. When $M$ is understandable from the context, $L_0$ will simply be called the symmetric monomial of $L_1$. Observe that symmetric monomials in $P$ relative to $M$ have the same coefficient if and only if $P/M$ is central palindromic.

**Proposition 4.8.** Let $P(x)$ and $M(x)$ be a polynomial and a monomial with coefficients in $\mathbb{R}$, respectively. Denote $g(x) = \frac{P}{M}(x)$. Then $g$ is real over the unit torus $\mathbb{T}^d$ if and only if $g$ is central palindromic, i.e. the coefficients of symmetric monomials relative to $M$ are equal.

**Proof.** The fact that a central palindromic function $g$ is real over $\mathbb{T}^d$ follows directly from the fact that for all $x \in \mathbb{T}^d$ we have $g(x) = g(x^{-1})$ which implies that on the unit torus $g(x) = g(\bar{x}) = \bar{g(x)}$.

To see the other implication, let $g(x) = \frac{P(x)}{M(x)}$ be a rational function taking real values over $\mathbb{T}^d$ and let $d$ be the maximum total degree of $P$. Denote $P(x) = \sum_{L \in \mathcal{M}^d(x)} a_L L$. Since $g(x)$ is analytic, we have $g(x) = \bar{(g)(x)}$ so that $g(x) = g(x^{-1})$ on $\mathbb{T}^d$. Hence,

$$\frac{\sum_{L \in \mathcal{M}^d(x)} a_L L}{M} = \frac{\sum_{L \in \mathcal{M}^d(x)} a_L L^{-1}}{M^{-1}}.$$ 

Hence,

$$\sum_{L \in \mathcal{M}^d(x)} a_L LM^{-1} = \sum_{L \in \mathcal{M}^d(x)} a_L L^{-1} M. \quad (2)$$

The proposition then follows by comparing the coefficients in (2). \qed

We further observe that central palindromy is preserved under substitution of monomials.

**Observation 4.9.** For any central palindromic function $g(x)$ and any vector $M = (M_1, \ldots, M_n)$ of monomials of the variables $Y = (Y_1, \ldots, Y_m)$, the function $g(M(Y))$ is a central palindromic function.

We also required the following counterpart of proposition 4.8 for quotient of a polynomial and a certain class of binomials.

**Proposition 4.10.** Let $P(x)$ be a complex polynomial and $M(x)$ be a monomial. Denote $g(x) = \frac{P(x)}{M(x)(1-x)}$. Then $g$ is real over $\mathbb{T}^n$ if and only if it is central palindromic, i.e. the coefficients of monomials in $P$ which are symmetric relative to $\frac{M}{\sqrt{x}}$ are additive inverses.

**Proof.** As the proof of Proposition 4.8 we observe that a central palindromic function $g$ is real on $\mathbb{T}^d$ if and only if $g(x) = g(x^{-1})$.

To see the other implication, let $g(x) = \frac{P(x)}{M(x)(1-x)}$ be a rational complex function taking real values over $\mathbb{T}^n$ and let $d$ be the maximum total degree of $P$. Denote $P(x) = \sum_{L \in \mathcal{M}^d(x)} a_L L$. As
before, we have \( g(x) = g(x^{-1}) \) on \( \mathbb{T}^n \) so that
\[
\sum_{L \in M^d(x)} a_{LM} \frac{L}{M(1 - x_1)} = \sum_{L \in M^d(x)} a_{LM} \frac{L^{-1}}{M^{-1}(1 - x_1^{-1})}.
\]

Hence, multiplying by \((1 - x_1)\) and using the fact that \( \frac{1 - x}{1 - x^{-1}} = -x \), we obtain
\[
\sum_{L \in M^d(x)} a_{LM} M^{-1} = \sum_{L \in M^d(x)} -a_{LM} x_1 M^{-1} L^{-1}.
\tag{3}
\]

The proposition then follows by comparing the coefficients in (3). \qed

Our construction will map vertices of \( H^* \) to polynomials satisfying several properties. These are summarized in the following definition.

**Definition 4.11.** A polynomial \( P \in \mathbb{R}[x_0, x_1, x_2] \) with non-negative coefficients is called \((x_0, x_1, x_2)\)-ascending if it can be represented as
\[
P = \sum_{j=1}^{m} \prod_{i=1}^{j-1} x_{i \mod 3} \left( b_{j,0} x_{j \mod 3}^{-1} + b_{j,1} x_{j+1 \mod 3} \right)
\]
such that for every \( j \leq m \), the following holds:

1. \( a_j \in \mathbb{N}, b_j \in \mathbb{N}_0 \).
2. \( b_{m,1} > 0 \).
3. \( a_j = 1 \) if and only if \( b_{j-1,1} > 0 \) and \( b_{j,0} = 0 \).
4. If \( a_j > 1 \) for \( j < m \) then \( b_{j,0} > 0 \).

**Observation 4.12.** If \( P \) is \((x_0, x_1, x_2)\)-ascending then

1. Monomials in \( P \) are ordered by their degree so that the degrees of consecutive monomials in each variables are weakly increasing. Denote this order by \( \prec \). We say that monomials \( N, N' \) are consecutive in \( P \) if they have non-zero coefficients, \( N \prec N' \) and there is no \( N'' \) in \( P \) with non-zero coefficient such that \( N \prec N'' \prec N' \).
2. Let \( N \prec N' \) be two consecutive monomials in \( P \). Then \( \frac{N'}{N} \) is a bivariate monomial of the form
\[
\frac{N'}{N} = x_{j \mod 3} x_{j+1 \mod 3}^k,
\tag{4}
\]
for some \( k \in \mathbb{N}_0 \).
3. Let $N < N' < N''$ be three consecutive monomials in $P$. Then $\frac{N'}{N}$ and $\frac{N''}{N'}$ are bivariate monomials of the form

$$\frac{N'}{N} = x_j \mod 3^{a_{j+1} \mod 3},$$

$$\frac{N''}{N'} = x_{j+1} \mod 3^{a_{j+2} \mod 3},$$

for some $k_1, k_2 \in \mathbb{N}_0$.

**Proof.** The first item is straightforward from the definition. To see the second item, let $j \in M$ and $\ell \in \{0, 1\}$ be such that

$$N = \left( \prod_{i=1}^{j-1} x_i^{a_i \mod 3} \right) x_j^{a_j - 1 + \ell} \mod 3$$

We consider two cases. If $a_{j+1} > 1$, then, by Item 3 of Definition 4.11 we have $b_{j+1,0} > 0$ so that

$$N' \prec \left( \prod_{i=1}^{j} x_i^{a_i \mod 3} \right) x_{j+1}^{a_{j+1} - 1} \mod 3$$

and the item follows. Otherwise $a_{j+1} = 1$ so that, by Item 3 of Definition 4.11 applied to $j + 1$, we obtain $b_{j,1} > 0$, so that if $\ell = 0$ the proposition follows. We are left with the case $\ell = 1$. In this case, if $b_{j+1,1} = 1$, we obtain

$$N' = \left( \prod_{i=1}^{j} x_i^{a_i \mod 3} \right) x_{j+1} \mod 3$$

and the second item follows. On the other hand, if $b_{j+1,1} = 0$, then, by Item 3 of Definition 4.11 applied to $j + 2$, we have $a_{j+2} > 1$ so that, by items 2 and 4 either $b_{j+1,0} > 0$ or $b_{j+1,1} > 0$. In either case we obtain

$$N' = \left( \prod_{i=1}^{j} x_i^{a_i \mod 3} \right) x_{j+1} \mod 3^{a_{j+2} \mod 3}$$

for some $k \in \mathbb{N}_0$. The second item follows. The third item follows from similar arguments, applied to three consecutive terms. \qed

**Proposition 4.13.** Let $P,Q \in \mathbb{R}[x_0, x_1, x_2]$ be two $(x_0, x_1, x_2)$-ascending polynomials with coefficients over $\mathbb{N}_0$. Assume that $\deg(Q) \leq \deg(P)$ and that $\frac{P-Q}{M}(x_0, x_1, x_2)$ is central palindromic, and let $M$ be the leading monomial of $Q$. Then $P - Q = kM$ for a $k \in \mathbb{Z}$.

**Proof.** Throughout the proof we omit the modulo 3 in the indices of $x$. Denote $d = \deg(M)$, by $\mathcal{P}$ the collection of all monomials with positive coefficients in $P$ which are of degree greater or equal than $d$, and by $\bar{\mathcal{P}}$ the collection monomials which are symmetric to monomials in $\mathcal{P}$ with respect to $M$. Observe that the coefficient of every $N \in \mathcal{P} \setminus \{M\}$ in $P - Q$ is the same as its coefficient in $P$. By Proposition 4.8 symmetric monomials of $P - Q$ have the identical coefficients so that all monomials
in \( \mathcal{P} \) have a positive coefficient in \( P - Q \). Since the only positive contribution to monomials in \( P - Q \) comes from their coefficient in \( P \), we deduce that \( \mathcal{P} \) are monomials of positive coefficients in \( P \).

Our purpose is to show that \( \mathcal{P} \subseteq \{ M \} \), as this would imply that \( P - Q \) has no monomials of positive coefficients of degree greater than \( d \), which would in turn imply, by Proposition 4.8, that \( P - Q = cM \) for \( c \in \mathbb{Z} \) as required. To this end assume towards obtaining a contradiction that \( \mathcal{P} \setminus \{ M \} \neq \emptyset \). Recall Item 1 of Observation 4.12 and observe that every monomial in \( \mathcal{P} \) is lexicographically greater than \( M \), since otherwise, its symmetric monomial will not be lexicographically ordered with respect to it. Let \( N \) be the minimal monomial greater than \( M \) in \( \mathcal{P} \) and denote by \( N' \) its symmetric monomial with respect to \( M \). Then either \( N' \) and \( N \) are consecutive in \( \mathcal{P} \), or \( N' \) is followed by \( M \) which is then followed by \( N \). The former case is impossible by Item 2 of Observation 4.12, since it would imply that \( \frac{N}{N'} \) is of the form \( x_j \mod 3 x_{j+1} \mod 3 \) in which case \( N' \) and \( N \) cannot be symmetric. The latter case could also be ruled out by observing that Item 3 of Observation 4.12 does not allow \( N \) and \( N' \) to be symmetric.

\[ \square \]

5  **Polynomial embedding of degree 12 of all outerplanar graphs**

In this section we prove Proposition 4.1 and thus Theorem 1. To do so, we introduce in Section 5.1 a 12-polynomial embedding \( \psi = \psi(x) : T^* \rightarrow \mathbb{C}[x] \), where

\[ x = \{ x_I \}_{I \in I}, \quad I = \{ 0, 1 \}^2 \times \{ 0, 1, 2 \}. \]  

(5)

In Section 5.2 we then write an explicit formula for the image of every vertex \( v \) under \( \psi \). This we do using the QR-encoding introduced in the preliminaries section. In Section 5.4 we prove that \( \psi \) is a degenerate polynomial embedding. In Section 5.5 we further show that \( \psi \) is a polynomial embedding. Finally, in Section 5.6 we conclude the proof of Proposition 4.1.

5.1  **The definition of \( \psi \)**

In this section we define \( \psi \). The construction is a somewhat more elaborate variant of the construction used in [1]. We start by presenting \( \psi_H(x) \), a 1-polynomial embedding of \( H \) which embeds the rhombus graph onto a rhombus of side length 1 with angle \( x \) (identifying the complex number \( x \) with its angle on the unit circle). We then define a type function \( Ty \) on each node \( v \in T^* \) and a map \( \psi \), which maps the rhombus encoded by \( QR(v) \) to a rotate copy of \( \psi_H(x_{Ty}) \) in the only way which respects \( \pi \), as defined in Section 4. The image of several subsets of \( T^* \) through \( \psi \) is depicted in Figure 4.

**Polynomial embedding of a single rhombus.** We set \( \psi_H(x)(v_0) = 0, \psi_H(x)(v_1) = 1, \psi_H(x)(v_2) = x \) and \( \psi_H(x)(v_3) = x + 1 \). This is indeed a polynomial embedding, mapping the rhombus graph to a rhombus of edge length 1, whose \( v_1v_0v_2 \) angle is \( x \). Figure 3 illustrates the image of \( H \) under \( \psi_H \).
Figure 3: The image of $H$ under $\psi_H$. Observe how $x$ determines the $v_1v_0v_2$ angle of the rhombus.

**Defining $Ty(H)$, the type function.** Let $H_{m+1}$ be a vertex of $H^*$ with the QR encoding

$$QR(H_{m+1}) = (\{q_i\}_{i \in [m+2]}, \{\rho_i\}_{i \in [m+1]}, m+1).$$

We set

$$Ty(H_{m+1}) = (q_m \pmod{2}, \rho_m \pmod{2}, s_m \pmod{3}),$$

(6)

where

$$s_m = \sum_{i=1}^{m} 1\{ q_{i-1} + \rho_{i} > 0 \},$$

(7)

and set $Ty(H_{\text{root}}^*) = (0, 0, 0)$.

Set $\psi(\pi(H_{\text{root}}^*)) = \psi_H(x_{0,0,0})(H)$. Let $M, N \in H^*$ be a pair of nodes such that $MN$ is an arc of $H^*$, and assume that $\psi$ is already defined on the vertices of $\pi(M)$. By the definition of $\pi$, this implies that $\psi(\pi(N, v_0))$ and $\psi(\pi(N, v_1))$ are already defined. We then define $\psi(\pi(N, v_2)), \psi(\pi(N, v_3))$ so that $\psi(\pi(N, v_0)), \psi(\pi(N, v_1)), \psi(\pi(N, v_2)), \psi(\pi(N, v_3))$ form a translated and rotated copy of $H(x_{Ty(N)})$.

As the image of every edge in $T^*$ is isometric to some edge of $H(x_t)$ when $t \in I$, we get

**Observation 5.1.** Every edge of $T^*$ is mapped through $\psi$ to an interval of length in

$$\{1\} \cup \{|x_t - 1| : t \in I\}.$$

While this definition of $\psi(\{x_t\})$ is complete, an explicit formula for every vertex in $T^*$ under $\psi(\{x_t\})$ is required for proving that $\psi$ is indeed a polynomial embedding. We devote the next section to develop this formula.

### 5.2 The image of $\psi$

In this section we state a formula for $\psi \circ \pi$ of every base vertex. An illustration of the image of several such vertices through $\psi \circ \pi$ is given in Figure 4. Let $u \in T^*$ and let $N \in H^*$, such that $QR(N) = (\{q_k\}_{k \in [m]}, \{\rho_k\}_{k \in [m]}, m)$ is the proper encoding of $u$ (as per Observation 4.2). Let $x = \{x_t\}_{t \in I}$, and write $N_i$ for the node in $H^*$ which is encoded by $(\{q_k\}_{k \leq i}, \{\rho_k\}_{k \leq i}, t)$ (where $N_0 = H_{\text{root}}^*$ corresponds to the null sequence). Naturally, $N_m = N$. From (6) we get

$$Ty(N_i) = (q_{i-1}(\pmod{2}), \rho_{i-1}(\pmod{2}), s_{i-1}(\pmod{3})).$$

(8)
Observe that in the embedding of every $H^*$ node through $\psi$, the edges $v_0v_2$, $v_1v_3$ are parallel, as are the edges $v_0v_1$, $v_2v_3$. Towards developing a formula for $\psi_x(u)$, denote $u_i := \pi(N_i, v_0)$ and define

$$P_i(x) = P_i^n(x) := \psi(\pi(N_i, v_1)) - \psi(\pi(N_i, v_0)) = \psi(\pi(N_i-1, v_2)) - \psi(\pi(N_i-1, v_0)).$$

(9)

Notice that, by definition, $P_0(x) = 1$. Thus, $P_i^n$ is a unit vector in the direction of the edges $(v_0, v_1), (v_2, v_3)$ in $\psi(\pi(N_i))$ which, for $i > 0$, is the same as the direction of $(v_0, v_2), (v_1, v_3)$ in $\psi(\pi(N_{i-1}))$.

With this in mind, it is possible to describe the change between $P_{i-1}$ and $P_i$, namely

$$P_i(x) = P_{i-1}(x) \cdot x_{Ty(N_{i-1})}.$$

(10)
For $0 \leq i \leq m$ write

$$Q_i(x) = Q_i^*(x) := \psi_x(u_i) = \psi_x(\pi(N_i, v_0)),$$

and observe that $Q_0 = \psi(\pi(H^*_\text{root}, v_0)) = 0$. Next, we describe how to get $Q_i$ from $Q_{i-1}$ using $((\{q_k\}, \{\rho_k\}, m)$. By definition,

$$Q_i(x) - Q_{i-1}(x) = \psi(\pi(N_i, v_0)) - \psi(\pi(N_{i-1}, v_0)).$$

Thus, $Q_i(x) - Q_{i-1}(x)$ could be computed from the labels of the edges along the path connecting $(N_{i-1}, v_0)$ and $(N_i, v_0)$. Each edge labeled $v_1v_3$ contributes to this difference $P_i$, and thus in total such edges contribute $q_i \cdot P_i$. An edge with label $v_1v_3$ contributes $P_i/x_{Ti(N_{i-1})} = P_{i-1}$, while an edge labeled $v_0v_2$ does not change the base vertex at all.

Applying this to the encoding, we get that

$$Q_i - Q_{i-1} = q_i \cdot P_i + \rho_i \cdot P_i - 1.$$

Summing this over $1 \leq i \leq m$, we get:

$$\psi_x(u) = Q_m = \sum_{i=1}^{m} (q_i P_i + \rho_i P_{i-1})$$

$$= \rho_1 + \sum_{i=1}^{m-1} (q_i + \rho_{i+1}) P_i + q_m P_m$$

Equivalently, setting $\rho_{m+1} = 0$ and $q_0 = 0$, we have

$$\psi_x(u) = \sum_{i=0}^{m} (q_i + \rho_{i+1}) P_i(x) = \sum_{i=0}^{m} c_i P_i(x), \quad (11)$$

where

$$c_i = q_i + \rho_{i+1}. \quad (12)$$

Observe that for every $u \in T^*$, $\psi_x(u)$ is a polynomial in $\{x_i\}_{i \in I}$ (as $P_i$ are monomials). Also observe that the total degree of $P_i$, which we denote by $\deg P_i$, obeys $\deg P_i = \deg P_{i-1} + 1$. Therefore $c_i$ is the coefficients of the only $i$-th degree monomial in the polynomial $\psi_x(u)$.

Note that, in particular, using the above notation, Observation 1.2 and the fact that the encoding $((\{q_k\}, \{\rho_k\}, m)$ is proper, yield that for $m > 1$,

$$c_m = q_m > 0. \quad (13)$$

### 5.3 Relating $\psi$ to ascending polynomials

In this section we show that, under the substitution $x_{i,j,k} = y_k$, the image of each vertex of $T^*$ under $\psi$ forms a $(y_0, y_1, y_2)$-ascending polynomial. To this end let $y_0, y_1, y_2 \in \mathbb{T}^3$ and write $y = (y_0, y_1, y_2)$. Denote $x(y) \in \mathbb{T}^{12}$ for the vector satisfying

$$x_{i,j,k}(y) = y_k,$$
and write
\[ \psi^*(y) = \psi(x(y)). \]

**Proposition 5.2.** For every vertex \( u \) the polynomial \( \psi^*_u(y) \) is \((y_0, y_1, y_2)\)-ascending.

**Proof.** Our purpose is to verify that \( \psi^*_u(y) \) satisfies the conditions of Definition 4.11. Denote \( P^*_i(y) \) for the monomial of total degree \( i \) in \( \psi^*_u(y) \) as per (9). Recall (6) and (10), and use that fact that \( s_m = \sum_{i=1}^{m} (q_i - 1 + \rho_i) > 0 \) and \( s_m \in \{s_{m-1}, s_{m-1} + 1\} \) to deduce that there exist \( j \geq 0 \) and a sequence \( \{a_i\}_{i \leq j} \) of natural numbers such that

\[ P^*_\ell(y) = \prod_{i=1}^{j} y_i^{a_i} \mod 3. \]

By (11), we deduce that \( \psi^*_u(y_0, y_1, y_2) \) can be written

\[ \psi^*_u(y_0, y_1, y_2) = \sum_{j=1}^{m-1} \prod_{i=1}^{j} y_i^{a_i} \mod 3 \left( b_{j,0} y_{j,0}^{a_j-1} + b_{j,1} y_{j,1}^{a_j} \right), \]

where, by (13), \( b_{m,a_m} > 0 \), and \( b_{j,0} = 0 \) if \( a_j = 1 \). Thus Items 1 and 2 of the definition are satisfied.

Next, observe that

\[ b_{j,k} = 0 \text{ if and only if } q_\ell + \rho_{\ell+1} = 0 \text{ for } \ell = \sum_{i=1}^{j-1} a_i + k. \] (14)

Further observe that, by (7) and (10), we have \( \frac{P^*_\ell+1}{P^*_\ell} = \frac{P^*_\ell+1}{P^*_{\ell-1}} \) if and only if \( s_\ell = s_{\ell-1} \), so that

\[ s_\ell = s_{\ell-1} \mod 3 \text{ if and only if there exists } j \leq m \text{ such that } \ell = \sum_{i=1}^{j} a_i. \] (15)

Next, let \( j \leq m \) and write \( \ell = \sum_{i=1}^{j-1} a_i \). To obtain Item 3 in definition 4.11 observe that, by putting together (14), (15) and the definition of \( s_\ell \), we have \( a_j = 1 \) if and only if \( b_{j,1} = q_\ell + \rho_{\ell+1} > 0 \).

To obtain Item 4 assume that \( a_j > 1 \) and denote \( \ell = \sum_{i=1}^{j-1} a_i + k \). Observe that, by (15), \( s_\ell = s_{\ell-1} + 1 \mod 3 \). By the definition of \( s_\ell \) this implies that \( q_{\ell-1} + \rho_\ell \neq 0 \) so that, by (14), \( b_{j,0} > 0 \).

### 5.4 \( \psi \) is a degenerate polynomial embedding

In this section we show that the image of the vertices of \( T^* \) under \( \psi \) are all distinct. Relation (11) and Observation 5.1 imply that if this is the case, then \( \psi \) is a 12-degenerate polynomial embedding of \( T^* \).

The proof is a significantly simplified version of the proof used in [1] (n.b. that the result there is not directly applicable to our modified construction). The main proposition of this section is the following:
Proposition 5.3. Let $u, w \in T^*$ be two distinct vertices. Then $\psi_x(u)$ and $\psi_x(w)$ are distinct polynomials in $x = \{x_i\}_{i \in I}$.

Proof. Assume that $\psi_x(u) \equiv \psi_x(w)$ and let the proper QR-encoding of the two vertices $u, w$ be

$$QR(u) = (\{q_i^u\}_{i \in [m_u]}, \{\rho_i^u\}_{i \in [m_u]}, m_u),$$
$$QR(w) = (\{q_i^w\}_{i \in [m_w]}, \{\rho_i^w\}_{i \in [m_w]}, m_w).$$

Further assume that $m_u, m_w > 1$, as otherwise $\psi_x$ is a constant function and the proposition is straightforward. As per (11), we write

$$\psi_x(u) = \sum_{i=0}^{m_u} c_i^u i P_i^u,$$
$$\psi_x(w) = \sum_{i=0}^{m_w} c_i^w i P_i^w.$$  \hspace{1cm} (16)

Since $\psi_x(u) \equiv \psi_x(w)$ and $c_{m_u}^u, c_{m_w}^w > 0$ (by Observation 4.2), we obtain that $m_u = m_w = m$. By substituting $x_i = y$ for all $i \in I$, and equating coefficients in (16) we get

$$\forall i \in [m] : c_i^u = c_i^w,$$  \hspace{1cm} (17)

where $c_i^u = q_i^u + \rho_{i+1}^u$ and $c_i^w = q_i^w + \rho_{i+1}^w$. Denote $k = \inf\{i \leq m : q_i^u \neq q_i^w \lor \rho_{i+1}^u \neq \rho_{i+1}^w\}$ and $\ell = \inf\{i \geq k : c_i^u \neq 0 \lor c_i^w \neq 0\}$. Assume for the sake of obtaining a contradiction that $k < \infty$ and observe that since $c_{m_u}^u, c_{m_w}^w > 0$, we have $\ell \leq m < \infty$. Observe that, by (10) and the definition of $k$, we have $P_k^u \equiv P_k^w$ and denote this monomial by $P$. Moreover, since

$$q_k^u + \rho_{k+1}^u = c_k^u = c_k^w = q_k^w + \rho_{k+1}^w$$

and either $q_k^u \neq q_k^w$, or $\rho_{k+1}^u \neq \rho_{k+1}^w$, we obtain that both must hold so that $k < \ell$. By the definition of $\ell$, for every $k < i < \ell$ we have

$$q_i^u = \rho_{i+1}^u = q_i^w = \rho_{i+1}^w = 0.$$  

Hence, using (7), (8) and (10),

$$P_u^\ell = P x_q u, s_k x_{0, \rho_{k+1}} \cdot s_{k+1} x_{0, 0, s_{k+1}} x_{0, 0, 0, s_{k+1}} P_x^\ell, P_w^\ell = P x_q w, s_k x_{0, \rho_{k+1}} \cdot s_{k+1} x_{0, 0, s_{k+1}} x_{0, 0, 0, s_{k+1}}.$$  

As follows,

$$\frac{P_u^\ell}{P_w^\ell} = \frac{x_q u, s_k x_{0, \rho_{k+1}} \cdot s_{k+1} x_{0, 0, s_{k+1}}}{x_q w, s_k x_{0, \rho_{k+1}} \cdot s_{k+1} x_{0, 0, s_{k+1}}}.$$  

One may verify that since $q_k^u \neq q_k^w$ and $\rho_{k+1}^u \neq \rho_{k+1}^w$, we must have $P_{\ell}^u \neq P_{\ell}^w$, while the coefficient of at least one of the them is not 0, in contradiction to our assumption that $\psi_x(u) \equiv \psi_x(w)$ for $k < \infty$. The proposition follows. Q.E.D.
\section{\psi is a polynomial embedding}

In this section we study the possible configuration of a vertex under \psi with respect to the line connecting the images of two adjacent vertices in \( T^* \). This is studied separately for edges which form the side of a rhombus in \( H^* \) (Proposition 5.4) and those that form a diagonal (Proposition 5.5). Together with Relation \((11)\) and Observation 5.1, these guarantee that \psi is a 12-polynomial embedding of \( T^* \).

**Proposition 5.4.** Let \( u, w \in T^* \) be two distinct vertices, and denote by \( P(x) \) the leading monomial of \( \psi_x(u) \). If for all \( x \in \mathbb{T}^{12} \) we have

\[
\frac{\psi_x(u) - \psi_x(w)}{P(x)} \in \mathbb{R},
\]

then there exists \( k \in \mathbb{Z} \) such that

\[
\psi_x(u) \equiv \psi_x(w) + kP(x).
\]

**Proof.** Denote \( f := \frac{\psi_x(u) - \psi_x(w)}{P(x)} \), and assume that \( f \) is a real function on \( \mathbb{T}^{12} \), so that, by Proposition 4.8, \f is central palindromic. Further assume that \( \psi_x(u) \neq \psi_x(w) \), as otherwise the proposition is straightforward. Let \( y_0, y_1, y_2 \in \mathbb{T}^3 \) and write \( y = (y_0, y_1, y_2) \). Denote \( x(y) \in \mathbb{T}^{12} \) for the vector satisfying

\[
x_{i,j,k}(y) = y_k,
\]

as in Section 5.3. Denote

\[
\psi^*_u(y) = \psi_{x(y)}(u),
\]

\[
\psi^*_w(y) = \psi_{x(y)}(w).
\]

Observe that the leading monomial of \( \psi^*_u(y) \) is \( P(x(y)) \) and denote the total degree of \( P \) by \( d \). By Proposition 5.2, the polynomials \( \psi^*_u \) and \( \psi^*_w \) are \((y_0, y_1, y_2)\)-ascending. Denote \( g(y) := f(x(y)) \) and observe that, by Observation 4.9, \( g \) is central palindromic. Hence, by Proposition 4.13, \( g(y) = k \), for some \( k \in \mathbb{Z} \). Hence all monomials of total degree other than \( d \) in \( \psi^*_u(y) \) and \( \psi^*_w(y) \) coincide.

Next we wish to lift the above argument to show that all monomials of total degree other than \( d \) in \( \psi(u) \) and \( \psi(w) \) also coincide. Observe that, by \((11)\), \( \psi_u \) and \( \psi_w \) have at most one monomial of each total degree. Let \( d' < d \) and denote by \( P_u^{d'} \) and \( P_w^{d'} \) the unique monomial of total degree \( d' \) in \( \psi_u \) and \( \psi_w \) respectively (which may have coefficient 0). Observe that, for every \( d > d' \) there is at most one monomial of total degree \( d^* \) in the polynomial \( \psi_x(u) - \psi_x(w) \), and thus, by Proposition 4.8, this polynomial contains at most one monomial of each total degree \( d' \). We deduce that \( P_u^{d'} \) and \( P_w^{d'} \) are equal.

Denote \( P' \) for the monomial of total degree \( d \) in \( \psi_u \). It follows that

\[
\psi_x(u) - \psi_x(w) = k_1 P + k_2 P',
\]
for $k_1, k_2 \in \mathbb{Z}$, where $k_1 < 0$ for $P' \neq P$. Since $\frac{\psi_x(u) - \psi_x(w)}{P}$ is central palindromic we deduce that either $k_2 = 0$, or $P = P'$. The proposition follows.

**Proposition 5.5.** Let $u, w \in T^*$ be two distinct vertices, where $w$ is the vertex $v_2$ of a rhombus $H$ in $H^*$, denote $y = x_T y(H)$ and write $P$ for the leading monomial of $\psi_x(w)$. If for all $x \in T_{12}$ we have

$$\frac{\psi_x(u) - \psi_x(w)}{P(1 - y^{-1})} \in \mathbb{R}$$

then there exists $d \in \mathbb{Z}$ such that $\psi_x(u) = \psi_x(w) - dP(1 - y^{-1})$.

The proposition is accompanied by Figure 5.

**Proof.** Denote

$$QR(w) = (\{q_i^w\}_{i \in [m_w]}, \{\rho_i^w\}_{i \in [m_w]}, m_w),$$

$$QR(u) = (\{q_i^u\}_{i \in [m_u]}, \{\rho_i^u\}_{i \in [m_u]}, m_u),$$

$$\psi_x(w) = \sum_{i=0}^{m_w} c_i^w P_i^w,$$

$$\psi_x(u) = \sum_{i=0}^{m_u} c_i^u P_i^u,$$

$$f = \frac{\psi_x(u) - \psi_x(w)}{P(1 - y^{-1})}.$$  

Under the assumptions of the proposition, the function $f$ is real and thus,

$$\frac{\psi_x(u) - \psi_x(w)}{P(1 - y^{-1})} = \frac{\psi_\tau(u) - \psi_\tau(w)}{P(1 - y)}.$$

Observe that the coefficients of the monomials of $\psi_x(u)$ and $\psi_x(w)$ are all positive. The monomials with total degree greater than $d = \deg P$ in the polynomial $\psi_x(u) - \psi_x(w)$ have positive coefficients. Hence, by Proposition 4.10, their symmetric monomials around $\frac{P}{\sqrt{y}}$ have negative coefficients. Moreover, there is at most one monomial of each total degree in the polynomial $\psi_x(u) - \psi_x(w)$. Denote

$$k = \inf\{i \in \{0, \ldots, m_w - 1\} : (q_i^u, \rho_i^u) \neq (q_i^w, \rho_i^w)\},$$

$$\ell = \min\{i > k : c_i^u \neq 0\}.$$  

Observe that $k < \infty$, unless $\psi_x(u) = \psi_x(w)$. Assume towards obtaining a contradiction that $\ell < m_w$. By the definition of $k$ and $\ell$, the monomials $P_k^w$ and $P_k^u$ are equal and we write $P_k = P_k^w = P_k^u$.

Observe that since, $c_{k+1}^u = 0, \ldots, c_{\ell}^u = 0$, we have

$$q_{k+1}^u + \rho_{k+2}^u = \cdots = q_{\ell-1}^u + \rho_{\ell}^u = 0. \quad (18)$$

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Thus, by (7), the monomials $P^w_\ell$ and $P^u_\ell$ satisfy:

$$P^u_\ell = P_k \prod_{i=k}^{\ell-1} x_{q_i^w, \rho_i^w, s_i^w} = P_k \cdot x_{q_k^w, \rho_k^w, s_k^w} \cdot x_{0, \rho_{k+1}^u, s_{k+1}^u} \cdot (x_{0, 0, s_{k+1}^w})^{\ell-k-2},$$

$$P^w_\ell = P_k \prod_{i=k}^{\ell-1} x_{q_i^w, \rho_i^w, s_i^w}.$$

From the assumption that $\ell < m_w$ and Proposition 4.10, it follows that $c^u_\ell - c^w_\ell < 0$ and hence, $P^u_\ell = P^w_\ell$. We obtain that the total degree of the variables $x_{s, s_k^w}$ in the monomials $P^u_\ell$ and $P^u_\ell$ are equal. Moreover, the monomial $\frac{P^u_\ell}{P^w_\ell}$ does not contain terms of the form $x_{s, s_k^w}$ so that $\frac{P^u_\ell}{P^w_\ell}$ does not contain such terms as well. The contradiction will now follow from a case study of $P^u_\ell$.

Recall that $s_{k+1} = s_k + 1, 0 \leq k \leq k+1$, so that if $q_k^u > 0$, then $s_{k+1}^u = s_k^u + 1$. Hence, in this case $x_{q_k^w, \rho_k^w, s_k^w}$ must be the only variable of the form $x_{s, s_k^w}$ in the monomials $P^w_\ell$, so that it must be equal to $x_{q_k^w, \rho_k^w, s_k^w}$, in contradiction with the definition of $k$. On the other hand, if $q_k^u = 0$, then $s_{k+1}^u = s_k^u + 1$. Hence, $P^u_\ell = P_k \prod_{i=k}^{\ell-1} x_{q_i^w, \rho_i^w, s_i^w} = P_k x_{0, \rho_k^w, s_k^w} (x_{0, 0, s_k^w})^{\ell-k-1},$ so that by (7) and (18), we have $x_{q_k^w, \rho_k^w, s_k^w} = x_{q_k^w, \rho_k^w, s_k^w}$, in contradiction with our definition of $s$. We conclude that $\ell > m_w$. By the fact that $c^u_{m_w} > 0$ and $c^u_\ell - c^w_\ell < 0$ we obtain that $\ell = m_w$. By the antisymmetry of coefficients we deduce that $c^u_{m_w} > 0$. By the definition of $k$ and $\ell$ we obtain that either $\psi_u(u) = \psi_w(w)$ or $k = m_w - 1$. Therefore,

$$\psi_u - \psi_w = dP - ey^{-1}P$$

where $d, e \in \mathbb{Z}$. From Proposition 4.10 we have $d = e$. The proposition follows.

\[ \square \]

5.6 Thirteen Distances Suffice for Degenerate Drawings

We are now ready to present the proof of Proposition 4.1 and thus conclude the proof of Theorem 1.

\textbf{Proof of Proposition 4.1}. By proposition 5.3 $\psi$ is a one-to-one mapping. By Definition 4.3 and Observation 5.1 $\psi$ is a 12-degenerate polynomial embedding of every finite subgraph $G \subseteq T^*$.

To show that $\psi$ is a polynomial embedding, we will verify that for every vertex $v$ and every edge $(u, w) \in T^*$ we have that

For all $x$ in $\mathbb{T}^d$ we have either $\psi_x(v) \in \{\psi_x(u) + k(\psi_x(w) - \psi_x(u)) \mid k \in \mathbb{Z}\}$,

or $f(x) = \psi_x(v) - \psi_x(u) \over \psi_x(w) - \psi_x(u)$ is not a real function on $\mathbb{T}^d$, (19)

as per definition 4.5
Figure 5: \( u \) and \( w \) are two distinct vertices in \( T^* \), where \( w \) is the vertex \( v_2 \) of the rhombus. \( P \) is the leading monomial of \( \psi_x(w) \) and \( y \) is the angle between it and the base edge.

Every edge \( e = (u, w) \) except the base edge of \( T^* \) plays either the role of a \( v_0v_2 \) edge, or a \( v_2v_3 \) edge, or a \( v_1v_3 \) edge or a \( v_1v_2 \) edge of some rhombus \( H = v_0v_1v_2v_3 \) in \( H^* \). Denote by \( P \) the leading monomial of \( \psi_x(v_2) \) and \( y = x_{TyH} \). Observe that

\[
P = \psi_x(v_2) - \psi_x(v_0) = \psi_x(v_3) - \psi_x(v_1),
\]

\[
P(1 - y^{-1}) = \psi_x(v_2) - \psi_x(v_1).
\]

Proposition 5.4 guarantees that (19) is satisfied in case that \((u, w)\) plays the role of either \(v_0v_2\), \(v_2v_3\) or \(v_1v_3\) in \(H\). While Proposition 5.5 guarantees that (19) is satisfied in case that \((u, w)\) plays the role of \(v_1v_2\). Thus \(\psi\) is a 12-polynomial embedding of every finite subgraph \(G \subseteq T^*\). By Observation 4.6, the set of \(x \in T_{12}\) such that \(\psi(x)\) is a drawing is of full measure, and by Observation 5.1 each of these drawings uses only side lengths \(\{|x_t - 1|\}_{t \in I}\) and 1. To see that this is the case for almost every choice of lengths in \([0, 1]\), let \(a \in (0, 1)\) and observe that the embedding \(a \cdot \psi\), i.e. the composition of a multiplication by \(a\) with \(\psi\), is a degenerate drawing of \(G\) for almost every \(x\), using the side lengths \(a, \{|a|1 - x_t|\}_{t \in I}\). The proposition follows. \(\square\)

6 Open problem

We conclude the paper with an open problem whose solution would naturally extend our work.

In [3] the original paper of Carmi, Dujmović, Morin and Wood the authors have shown the following theorem.

**Theorem.** Let \(G\) be a graph with \(n\) vertices, maximum degree \(\Delta\), and treewidth \(k\). Then the distancenumber of \(G\) is \(O(\Delta k \log n)\).
The authors use this result to provide a similar bound for the distance number of outerplanar graphs, as these have treewidth at most 2. Our work demonstrates that outerplanar graphs have, in fact bounded distance number. It is therefore natural to ask if our result could be generalized to every graph with bounded degree and treewidth.

Conjecture 1. Let $k, \ell \in \mathbb{N}$, The distance number of all graphs with treewidth at most $k$ and degree at most $\ell$ is bounded by $dn(k, \ell)$, where $dn(k, \ell)$ is a universal constant depending only on $k$ and $\ell$.

Naturally, it would also be of interest to find other natural families of graphs with bounded distance number.

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