A new concept for the geometrisation of electromagnetic interaction is proposed. Instead of the concept "extended field–point sources", interacting Maxwell’s and Dirac’s fields are considered as a unified closed noneuclidean and nonriemannean space–time 4-manifold. This manifold can be considered as geometrical realisation of the ”dressed electron” idea. Within this approach, the Dirac equation proves to be a relation that accounts for topological and metric characteristics of this manifold. Dirac’s spinors serve as basis vectors of its fundamental group representation, while the electromagnetic field components prove to be components of a curvature tensor of the manifold covering space. Energy, momentum components, mass, charge, spin and particle–antiparticle states appear to be geometrical characteristics of the above manifold.

Introduction

First attempts of the electromagnetic field geometrisation were undertaken just after the appearance of the theory of general relativity. The goal was to unify gravitation and electromagnetism within one geometrical approach (Weyl, Kaluza, Einstein, Fok, Wheeler and others). It was expected that gravitation and electromagnetism can be considered as a manifestation of noneuclidean properties of the physical space–time as it is for gravitational field alone (see, for example, [1-3]). Later there were also attempts of the gauge fields geometrisation where these fields were interpreted as connections in a space of the local gauge symmetry group [4,5].

We showed early that the equation for free Dirac’s field can be interpreted as a relation that accounts for the topological and metric properties of the nonorientable space–time 4–manifold which fundamental group is generated by four glide reflections [6–8]. (Two dimentional analog of such manifold is a Klein bottle [9]). We also noticed there that Maxwell’s equations for a free electromagnetic field can be interpreted as a group–theoretic relations describing the orientable 4–manifold which fundamental group is generated
by four parallel translations (two dimensional analog of such manifold is a torus [9]).

We shall show now that the system of equations for interacting Dirac and Maxwell fields can be also considered as a topological encoding of some unified closed connected nonorientable space–time 4–manifold.

**Free electromagnetic field as an orientable space–time 4–manifold**

Before the interacting fields consideration we shall firstly show that Maxwell’s equations for free electromagnetic field can be interpreted as relations describing topological properties of some orientable 4–manifold. For more visualization let us consider the two-dimensional orientable manifold that is homeomorphic (topologically equivalent) to a torus. Torus can be represented as a product of two circles $S_1 \times S_2$ [8]. Let $L_1$ and $L_2$ be the circles lengths. Suppose that $L_1 = L_2$. Let us find out relations expressing topological (orientable) and metric ($L_1 = L_2$) invariants of the manifold and let us show that such relations are formally analogous to Maxwell’s equations.

One of the manifold topological invariants is its fundamental group. This group elements are classes of pathes starting and finishing at the same point [8]. There are two classes for our two dimensional torus and corresponding pathes are homeomorphic to the circles $S_1$ and $S_2$. This group is isomorphic (assume one–to–one correspondence) to the group of two parallel translations $L_1$ and $L_2$ along the Cartesian coordinates $0X$ and $0Y$ on euclidean plane (this plane is said to be a covering surface for our torus [8]). As the above group representation we take operators for the $L_1$– and $L_2$–translations along $0X$ and $0Y$

$$T_x = -\frac{iL_1}{2\pi} \frac{\partial}{\partial x}, \quad T_y = -\frac{iL_2}{2\pi} \frac{\partial}{\partial y}. $$

It is easy to verify that the basic vectors for this representation have the form

$$\varphi = \exp[2\pi i(\frac{x}{L_1} + \frac{y}{L_2})].$$

Therefore, the conditions imposed by the manifold fundamental group (parallel–translations group) and by the metric restriction ($L_1 = L_2$) can be formulated with the help of the one relation

$$\frac{\partial \varphi}{\partial x} = \frac{\partial \varphi}{\partial y}. \quad (1)$$
Eq.(1) does not however represent the fact that our geometrical object is a orientable manifold and does not therefore allow to fix the orientation. The reason is that we choosed scalars as basic vectors for the manifold fundamental group representation. It is known that orientable manifolds need more complex tensors for its representation, namely antisymmetric second rank tensors $F_{ik}$ (bivectors)[9,11]. The bivector components are defined by two vectors $a_i$ and $b_k$ as

$$F_{ik} = a_i b_k - a_k b_i,$$

where for two-dimentional space $i = x, y; k = x, y$.

So if we are going to change in (1) scalar for bivector we have to introduce into the theory two vectors on the $X,Y$–plane defining the manifold orientation (up–down). One of the vectors is the vector of parallel translations $(\partial/\partial x, \partial/\partial y)$. Another vector has to be introduced as additional topological propriety of the manifold. Denote this vector by $A$. Then we have for $F_{ik}$

$$F_{ik} = \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x}. \quad (3)$$

Let us change in (1) scalar $\varphi$ for bivector $F_{ik}$ and extend our two-dimentional consideration to the analogous four-dimentional manifold with the pseudoeuclidean covering space. In other words we rewrite Eqs.(1) and (3) as

$$\frac{\partial F_{ik}}{\partial x_i} = 0, \quad F_{ik} = \frac{\partial A_k}{\partial x_i} - \frac{\partial A_i}{\partial x_k}, \quad (4)$$

where $i, k = 0, 1, 2, 3; \ x_0 = ct, \ c$ is a light velocity. Here (and later on) the summation is supposed to be going over repeating indices.

We see that Eqs.(4) coincide exactly with Maxwell’s equations for free field if we consider $F_{ik}$ as the electric and magnetic fields tensor and $A_i$ as 4–potentials [10]. This coincidence means that we can interpret the free electromagnetic field as the orientable closed connected space—time manifold which fundamental group is generated by four parallel translations. The field energy and momentum appear here as the manifold topological invariants and the energy conservation law appears as an additional metric restriction.

We showed earlier that the equation for free Dirac’s field can be considered as a relation describing topological and metric characteristics of the another type manifold (nonorientable one) [6–8]. It is usefull now to repeat shortly
the argumentation of the above interpretation. This equation has the form [12]:

\[ \gamma^l p_l \psi = m \psi, \]  

(5)

where \( \gamma^l p_l = p_0 \gamma^0 - p_1 \gamma^1 - p_2 \gamma^2 - p_3 \gamma^3. \)

Here \( m \) is a mass and \( \psi \) is the four–component first rank spin–tensor. It can be represented by the matrix with four rows and one column

\[ \psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \]  

(6)

where \( \xi, \eta \) are two–component spinors (dotted and undotted ones). Here \( p_l = i \partial / \partial x^l \) are the 4–momentum operators, \( x^0 = t, \ x^1 = x, \ x^2 = y, \ x^3 = z \), and \( \gamma^l \ (l = 0,1,2,3) \) are the Dirac four–row matrices. If we choose bispinors in the form (6) then the matrices \( \gamma_l \) can be written as

\[ \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \gamma^\alpha = \begin{pmatrix} 0 & -\sigma^\alpha \\ \sigma^\alpha & 0 \end{pmatrix}, \]  

(7)

where \( \alpha = 1,2,3 \) and \( \sigma^\alpha \) are two-row Pauli matrices. We write here four–row matrices as two–row ones: each symbol in (7) corresponds to a two–row matrix. Here and later on \( \hbar = c = 1, \ \hbar \) is the Planck constant.

Within topological interpretation the difference between Dirac’s Eq.(5) and Maxwell’s Eqs.(4) is that in (4) we have a bivector \( F_{ik} \) but we have the first–rank spin–tensor \( \psi \) in (5). And we have the parallel translation operator \( p_l \) in (4) instead of the product \( p_l \gamma_l \) in (5). Any first–rank spin–tensor (considered as linear geometrical object) corresponds to the geometrical structure that restores its position after rotation by \( 4\pi \) (not \( 2\pi \)) [9,11]. Such behaviour is a feature of the nonorientable geometrical objects. (The simplest example is the Möbius strip [9,10])

On the other hand the \( \gamma_l \) matrices can be considered within spinor basis as a representation for the product of three symmetries with respect to hyperplanes containing \( 0X \) axes [6-8]. It means that the product \( p_l \gamma_l \) in (5) is a representation for the glide reflection group. Therefore, Dirac’s Eq.(5) can be interpreted as a metric relation for some nonorientable space–time 4–manifold which fundamental group is generated by four glide reflections.
and which covering space is the physical space–time (Minkowski space). The Klein bottle is a two–dimensional analog of this manifold [9,10].

Thus we have shown that equations for free Dirac’s field and free Maxwell’s field can be interpreted as a specific mathematical description of some special closed space–time 4–manifolds. Mass, energy and momentum components appear here as elements of this manifold fundamental group with dimensions of length. Note that the closeness of a manifold in pseudoeuclidean space does not imply any constraints on the manifold extension over the time axis. For example, a circle in pseudoeuclidean plane is mapped into an equilateral hyperbola in the usual plane [11]. In space our manifolds are closed and bounded but they do not have a definite shape (as any nonmetrized manifold): all manifolds obtained from some initial one by the deformation without a damage are equivalent [9]. Nevertheless, it is possible to indicate for these manifolds some characteristic sizes which defined by metric conditions corresponding within geometrical approach to the energy and momentum conservation laws. It is a wave length of the electromagnetic field or the particle wave length $\bar{h}/p$.

Geometrical interpretation of interacting electromagnetic and electron–positron fields

Let us now consider a question of ”switching on” interactions in the geometrical representation of the above considered free fields. In other words let us try to find out the geometrical interpretation of the following known equations for Maxwell’s and Dirac’s interacting fields [12]

$$i\gamma^l(\frac{\partial}{\partial x^l} + ieA_l)\psi = m\psi,$$  \hfill (8)  

$$\frac{\partial F_{ik}}{\partial x^i} = j_k.$$  \hfill (9)  

$$F_{ik} = \frac{\partial A_i}{\partial x^k} - \frac{\partial A_k}{\partial x^i}. $$  \hfill (10)

Here $e$ is an electron charge, $m$ is an electron mass, and the current $j_k$ is defined as

$$j_k = e\psi^+\gamma^k\psi,$$

where $\psi^+ = \psi^*\gamma^0$ is so called Dirac’s conjugate spinor ($\psi^*$ is a complex conjugate spinor). Here and later on we use the system where $\hbar = c = 1$. 

5
We shall show now that Eqs.(8-10) can be considered as relations describing topological properties of one closed 4–manifold that has some features of both above considered ones (corresponding to electromagnetic and electron–positron fields). This conclusion seems inevitable within our topological approach because it is difficult to suggest something else. Up to now the topology for 4–manifolds is developed not so good as for two-dimensional ones. For two-dimensional manifolds a detailed classification is worked out and their main topological invariants are defined [9,10]. Therefore, we shall try to use any possible parallels between our problem and corresponding problem within two-dimensional topology. We have in mind here that a usefulness of low-dimensional considerations is one of the geometrical approach advantages. So let us see what could be the result of a unification in one geometrical object proprieties of two two-dimensional manifolds, orientable and nonorientable ones. What kind of object will be the hybrid of torus and the Klein bottle and how can we reflect mathematically its topological peculiarities?

According to topological classification a two-dimensional torus is a "sphere with one handle" and the Klein bottle is a "sphere with two holes covered by cross–caps or Möbius films" [9,10]. As a hybrid of a torus and the Klein bottle it is natural to consider a sphere with one handle and two cross–caps. The covering space for this nonorientable manifold is a hyperbolic plane and the manifold fundamental group is generated by glide reflections [10,14]. Let us suppose that there is some analogy between above hybrid–manifold and the one which can represent Dirac’s and Maxwell’s interacting fields. Then we can assume that Eq.(8) may be interpreted as a relation describing some nonorientable manifold whose covering space is a four-dimensional analog of a hyperbolic plane. Such analog is a conformal pseudoeuclidean space (the Lobachevskian space is one of the examples [11]).

Show that Dirac’s equation (8) can indeed be interpreted in this way. Conformal euclidean space is a space that assumes conformal mapping onto euclidean space. This means that for every point $M(x)$ of conformal euclidean space there is a point $M_E$ in euclidean space where arc’s differentials are connected by the relation [11]

$$ds^2_E = f(x^0, x^1, x^2, x^3)ds^2,$$

where $ds^2 = g_{ik}dx^idx^k$ defines the conformal euclidean space metrics, $ds^2_E =$
$g^E_{ik}dx^i dx^k$ is the arc’s differential squared (into our pseudoeuclidean space $g^E_{00} = 1, g^E_{11} = g^E_{22} = g^E_{33} = -1, g^E_{ik} = 0, i \neq k$).

Consider the left side of Eq.(8). As compared with Eq.(5) for free electron–positron field it contains expression $(\partial/\partial x^l + ieA^l)$ instead of usual derivative $\partial/\partial x^l$. It is customary to call this expression ”covariant derivative” because it looks like covariant derivative $\nabla_l$ of covariant vector field $B_m$ [9-11]

$$\nabla_l B_m = \frac{\partial B_m}{\partial x^l} + \Gamma^s_{ml} B_s,$$

(12)

where $\Gamma^s_{ml}$ is a connection.

The connection geometrical meaning is that the covariant derivative plays the role of the parallel translation generator for the conventional tensor field defined on some manifold (the connection for euclidean space is zero and the parallel translation generator is a ”usual” derivative $\partial/\partial x^l$) [9-11]. But there does not exist a connection of this kind into arbitrary space for spintensors (in particular for 4-component Dirac’s spinors). The reason is that spintensors are the euclidean (not affine) tensors. The transformation law for their components is defined by the rotation group representation and it can not be extended to the group of all linear transformations [13]. This means that spintensors can be compared in two different points only if orthogonal frames remain orthogonal after corresponding transfer through the space.

But for particular cases—for conformal euclidean space, for example, we can always map the vicinity of any point $M$ onto vicinity any another point $M'$ in such way that the orthogonal frame at $M$ remains orthogonal in $M'$ [11]. Therefore the parallel translation for spinors in this space can be defined by the same formulas as for any other tensors and then only the connection components $\Gamma^p_{lp}$ will be nonzero [15].

Recall that conformal euclidean space would not be here the physical space–time but the manifold covering space. This space is only a mathematical instrument for the manifold fundamental group description. And only this nonmetrized 4–manifold (that is not a riemann space at all) represents interacting electromagnetic and electron–positron fields. Suppose now that we can consider $ieA^l$ in (8) as a connection $\Gamma^p_{lp}$ in the space like the conformal euclidean one.(Later we shall call this space as ”conformal pseudoeuclidean” though it is not even supposed to be riemann space). Then we can interpret the expression $(\partial/\partial x^l + ieA^l)$ in (8) as a generator of infinitesimal translations in conformal pseudoeuclidean space. The parallel translation generator
multiplied by the reflection operator $\gamma^l$ gives within the spinor basis the local glide reflection operator [6,9,10]. It leads to the main conclusion: Dirac’s equation (8) can be interpreted as a relation describing topological and metric properties of some 4–manifolds. The manifold fundamental group is a local glide reflection and manifold’s covering space is conformal pseudo-euclidean space. This manifold is infinitely connected in contrast to four–connected manifolds describing free electron–positron and electromagnetic fields.

Considering $ieA^l$ as a connection in the manifold covering space we can give a geometrical interpretation for the electric and magnetic fields components (or for components of electric and magnetic fields tensor $F_{ik}$). Let us use for this purpose the relation between the connection $\Gamma_{lm}^k$ and the space curvature tensor $R_{ik,i}^q$ [9,11]

$$R_{ik,i}^q = \left( \frac{\partial \Gamma_{li}^q}{\partial x^k} - \frac{\partial \Gamma_{ki}^q}{\partial x^l} + \Gamma_{lp}^q \Gamma_{ki}^p - \Gamma_{lp}^q \Gamma_{ki}^p \right).$$

(Summation is here going over repeating indices from 0 to 3).

After contraction $R_{ik,i}^q$ over upper and right lower indices one obtains (denote the result as $R_{ik}^0$):

$$R_{ik}^0 = R_{ik,i}^q = \frac{\partial \Gamma_{li}^q}{\partial x^k} - \frac{\partial \Gamma_{ki}^q}{\partial x^l}.\quad (14)$$

Comparing (14) and (10) and taking in mind that $\Gamma_{mq}^q = ieA_m$, we have

$$ieF_{ik} = R_{ik}^0,$$

Comparing Eq. (14) with Eq. (10) and using the fact that $\Gamma_{mq}^q = ieA_m$, one obtains

$$ieF_{ik} = R_{ik}^0,\quad (15)$$

i.e., within the geometrical interpretation, the tensor of electric and magnetic fields coincides, except for the factor $ie$, with certain components of the curvature tensor of a covering surface. Therefore, Maxwell’s Eq. (9) relates the above-mentioned components of curvature tensor to the basis functions of the fundamental group, thereby rendering the system of Eqs. (8)–(10) closed. The curvature tensor for a space with constant curvature $K$ has the form [11]

$$R_{ijkl} = K(g_{ik}g_{jl} - g_{il}g_{jk}).\quad (16)$$
Comparing Eqs. (16) and (15), one arrives at the conclusion that, within the geometrical interpretation, the electric charge $e$ is proportional to the covering space constant curvature $K$.

Finally, Eqs.(8-10) for interacting electromagnetic and electron–positron fields can be written within geometrical approach as

$$i\gamma^i\left(\frac{\partial}{\partial x^i} + \Gamma^p_{ip}\right)\psi = K\psi,$$

(17)

$$R^0_{ik} = \frac{\partial \Gamma^p_{ip}}{\partial x^k} - \frac{\partial \Gamma^p_{kp}}{\partial x^i},$$

(18)

$$\frac{\partial R^0_{ik}}{\partial x^j} = ie^2\psi^+\gamma^j\psi,$$

(19)

**Conclusion**

Finally, we have the following geometrical interpretation of electromagnetic interaction.

1. Electromagnetic field and its sources (electron–positron field) can be considered as a single closed infinitely connected nonorientable nonmetrized 4–manifold.
2. Covering space of this manifold is a conformal pseudoeuclidean space.
3. Potentials $A_k$ is defined by the connection of this space $\Gamma_k$ ($ieA_k = \Gamma_k$).
4. Electric and magnetic field components are defined by the components of the covering space curvature tensor $R^0_{ik}$ ($ieF_{ik} = R^0_{ik}$).
5. Dirac’s and Maxwell’s equations appear as the relations imposing metric restrictions on generators of the manifold fundamental group.
6. Dirac spinors appear as basic vectors for the manifold fundamental group representation.
7. Electron charge appears as a constant covering space curvature.
8. Electron mass appears as a metric parameter of the manifold fundamental group.
9. Particle–antiparticle states and states with different spin projections are the reflection of the manifold nonorientability.
10. In a way above manifold can be considered as ”dressed electron” and it looks like fluctuating shapeless microscopic droplet of space–time points.

One comment in conclusion. Geometrisation of Dirac’s equation introduces new topological interpretation of quantum formalism. But it is important that replacing a ”wave–particle” by a nonmetrized space–time manifold...
does not mean ”more determinism” for the quantum object description and
the topological approach does not introduce any hidden variables and does
not therefore contradict Bell’s and von Neumann’s theorems [16,17].

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