On two rationality conjectures for cubic fourfolds

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Motivated by the question of rationality of cubic fourfolds, we show that a cubic $X$ has an associated K3 surface in the sense of Hassett if and only if the variety $F$ of lines on $X$ is birational to a moduli space of sheaves on a K3 surface, but that having $F$ birational to $\text{Hilb}^2(\text{K3})$ is more restrictive. We compare the loci in the moduli space of cubics where each condition is satisfied.

It is widely expected that a smooth complex cubic fourfold $X$ is rational if and only if it has an associated K3 surface in the sense of Hassett [8] or Kuznetsov [11]. New work of Galkin and Shinder [7] suggests instead that if $X$ is rational then the variety $F$ of lines on $X$ is birational to the Hilbert scheme of two points on a K3 surface. The purpose of this note is to clarify the relationship between these two conditions. The latter is somewhat stronger.

First let us recall Hassett’s Noether–Lefschetz divisors $C_d$ in the moduli space $\mathcal{C}$ of cubic fourfolds [8, §3.2]. For a very general cubic $X$, the algebraic lattice $H^{2,2}(X, \mathbb{Z}) := H^{2,2}(X) \cap H^4(X, \mathbb{Z})$ is generated by $h^2$, the square of the hyperplane class. A special cubic of discriminant $d$ is one for which there is a primitive sublattice $K \subset H^{2,2}(X, \mathbb{Z})$ of rank 2 and discriminant $d$ that contains $h^2$. Such cubics form an irreducible divisor $C_d \subset \mathcal{C}$, non-empty if and only if

\[
(*) \quad d > 6 \text{ and } d \equiv 0 \text{ or } 2 \pmod{6}.
\]

Moreover there exists a polarized K3 surface $S$ such that $K^\perp \subset H^4(X, \mathbb{Z})$ is Hodge-isometric to $H^2_{\text{prim}}(S, \mathbb{Z})(-1)$ if and only $d$ satisfies the further condition

\[
(**) \quad d \text{ is not divisible by } 4, 9, \text{ or any odd prime } p \equiv 2 \pmod{3}.
\]
Using the Eisenstein integers one can show that (**) is equivalent to saying that \( d \) is the norm of a primitive vector in the lattice \( A_2 = \left( \begin{array}{cc} 2 & -1 \\ -1 & 2 \end{array} \right) \), or that \( d \) divides \( 2n^2 + 2n + 2 \) for some integer \( n \).

**Theorem 1.** The following are equivalent:

(a) \( X \in C_d \) for some \( d \) satisfying (**).

(b) The transcendental lattice \( T_X \subset H^4(X, \mathbb{Z}) \) is Hodge-isometric to \( T_S(-1) \) for some K3 surface \( S \).

(c) \( F \) is birational to a moduli space of stable sheaves on \( S \).

By a recent result of Bayer and Macrì [5, Thm. 1.2(c)], this last condition is equivalent to saying that \( F \) is isomorphic to a moduli space of Bridgeland-stable objects in the derived category of \( S \). Thus Theorem 1 answers [13, Question 1.2] in the untwisted case.

Hassett [8, Prop. 6.1.3] showed that if the generic \( X \in C_d \) has \( F \) isomorphic to \( \text{Hilb}^2(S) \) for some K3 surface \( S \) then

\[
(***) \quad d \text{ is of the form } \frac{2n^2 + 2n + 2}{a^2} \text{ for some } n, a \in \mathbb{Z},
\]

and proved a partial converse [8, Thm. 6.1.4]. Thanks to the global Torelli theorem for hyperkähler manifolds [10, 15, 19] we can now prove a more complete result:

**Theorem 2.** The following are equivalent:

(a) \( X \in C_d \) for some \( d \) satisfying (**).

(b) \( F \) is birational to \( \text{Hilb}^2(S) \) for some K3 surface \( S \).

In contrast to (**), it is hard to tell at a glance whether a number \( d \) satisfies (**). On the one hand (***) implies (**), but it is strictly stronger: Hassett remarks in [8, §6.1] that 74 satisfies (***) but not (**). To address the question systematically, observe that \( d \) satisfies (***), if and only if there is an integral solution to the Pell-type equation \( m^2 - 2da^2 = -3 \); just substitute \( m = 2n + 1 \). If such an equation has any solution then it has one with \( a \) below an explicit bound [2, Thm. 4.2.7]. It is then straightforward to have a computer search for solutions up to this bound. Table 1 lists all \( d \) up to 200 that satisfy (*), indicating whether they satisfy (**) and (***) I do not know any nice characterization of (***) in terms of the \( A_2 \) lattice.
Table 1: Comparison of numerical conditions.

| $d$ | (**) | (****) | $d$ | (**) | (****) | $d$ | (**) | (****) |
|-----|------|--------|-----|------|--------|-----|------|--------|
| 8   | 74   | x      | 12  | 78   | x      | 14  | 80   | x      |
| 14  | x    | x      | 18  | x    | x      | 20  | 86   | x      |
| 24  | x    | 90     | 26  | x    | x      | 30  | 96   | x      |
| 32  | x    | 102    | 36  | 104  | x      | 38  | 108  | x      |
| 44  | x    | 110    | 48  | x    | x      | 50  | 114  | x      |
| 54  | x    | 116    | 56  | x    | x      | 60  | 120  | x      |
| 62  | x    | 122    | 62  | x    | x      | 66  | 126  | x      |
| 68  | 134  | x      | 72  | 138  | x      |

Outline

In §1 we review Markman’s Mukai lattice for a variety $Y$ of $K3^{[n]}$-type, which governs the global Torelli theorem for such varieties. We give criteria in terms of this lattice for $Y$ to be birational to a moduli space of sheaves or Hilbert scheme of $n$ points on a K3 surface.

In §2 we review Kuznetsov’s K3 category $\mathcal{A}$ associated to $X$, the special classes $\lambda_1, \lambda_2 \in K_{num}(\mathcal{A})$, and the Mukai lattice $K_{top}(\mathcal{A})$ introduced in [1]. We prove that

$H^2(F, \mathbb{Z})(1) \cong \lambda_1^\perp \subset K_{top}(\mathcal{A})$. 
This extends Beauville and Donagi’s result [6, Prop. 6] that $H^2_{\text{prim}}(F, \mathbb{Z})(1) \cong H^1_{\text{prim}}(X, \mathbb{Z})(2)$, since the latter is Hodge-isometric to $\langle \lambda_1, \lambda_2 \rangle \perp \subset K_{\text{top}}(A)$. From (1) we deduce that $K_{\text{top}}(A)(-1)$ is the Markman–Mukai lattice of $F$. All this is consistent with Kuznetsov and Markushevich’s result [12, §5] that $F$ is a moduli space of objects in the numerical class $\lambda_1 \in K_{\text{num}}(A)$.

With this lattice theory in hand, we prove Theorems 1 and 2 in §3.

**Convention**

Since we are speaking about transcendental lattices and moduli spaces of sheaves, we will take all K3 surfaces to be projective unless otherwise stated.

1. **The Markman–Mukai lattice of a variety of K3$^n$-type**

A *variety of K3$^n$-type* is a smooth projective variety $Y$ deformation-equivalent to the Hilbert scheme of $n$ points of a K3 surface, $n \geq 2$. The second cohomology group $H^2(Y, \mathbb{Z})$ carries a quadratic form $q$, the *Beauville–Bogomolov–Fujiki form*, under which it is a lattice of discriminant $-2n + 2$ and signature $(3, 20)$. Markman [15, §9] has described an extension of lattices and weight-2 Hodge structures $H^2(Y, \mathbb{Z}) \subset \tilde{\Lambda}$ with the following properties:

**Theorem 3 (Markman$^1$).**

(a) As a lattice, $\tilde{\Lambda}$ is isomorphic to $U^4 \oplus (-E_8)^2$.

(b) The orthogonal $H^2(Y, \mathbb{Z}) \perp \subset \tilde{\Lambda}$ is generated by a primitive vector of square $2n - 2$.

(c) If $Y$ is a moduli space of sheaves on a K3 surface $S$ with Mukai vector $v \in H^*(S, \mathbb{Z})$ then the extension $H^2(Y, \mathbb{Z}) \subset \tilde{\Lambda}$ is naturally identified with $v \perp \subset H^*(S, \mathbb{Z})$.

(d) $Y_1$ and $Y_2$ are birational if and only if there is a Hodge isometry $\tilde{\Lambda}_1 \rightarrow \tilde{\Lambda}_2$ taking $H^2(Y_1, \mathbb{Z})$ isomorphically to $H^2(Y_2, \mathbb{Z})$.

Let $\tilde{\Lambda}_{\text{alg}} \supset H^{1,1}(Y, \mathbb{Z})$ denote the algebraic part of $\tilde{\Lambda}$, that is, the integral classes of type $(1, 1)$.

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$^1$This summary is borrowed from [4, §1].
Proposition 4. Let $Y$ be a variety of $K3^{[n]}$-type, $n \geq 2$. Then the following are equivalent:\(^2\)

(a) $\tilde{\Lambda}_{\text{alg}}$ contains a copy of the hyperbolic plane $U = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)$.

(b) The transcendental lattice $T_Y \subset H^2(Y, \mathbb{Z})$ is Hodge-isometric to $T_S$ for some $K3$ surface $S$.

(c) $Y$ is birational to a moduli space of stable sheaves on $S$.

Proof. (c) $\Rightarrow$ (a): This is immediate from Theorem 3, since the algebraic part of $H^*(S, \mathbb{Z})$ contains a copy of $U$ spanned by $H^0$ and $H^4$.

(a) $\Rightarrow$ (b): Let $L = U^\perp \subset \tilde{\Lambda}$. As a lattice this is isomorphic to $U^3 \oplus (-E_8)^2$, so by the global Torelli theorem it is Hodge-isometric to $H^2(S, \mathbb{Z})$ for some analytic $K3$ surface $S$. In fact $S$ is projective, as follows. By Huybrechts’ projectivity criterion [9, Thm. 3.11] there is a $c \in H^{1,1}(Y, \mathbb{Z})$ with $q(c) > 0$. Let $v$ be a primitive generator of $H^2(Y, \mathbb{Z})^\perp \subset \tilde{\Lambda}$; then $q(v) = 2n - 2 > 0$. Thus $c$ and $v$ span a positive definite sublattice of $\tilde{\Lambda}$. This cannot be contained in $U$, which is indefinite, so $(c, v) \cap L$ contains a class of positive square, so $S$ is projective by Huybrechts’ criterion.

Now $T_S$ is the transcendental part of $L$, which is the transcendental part of $\tilde{\Lambda}$, which is $T_Y$.

(b) $\Rightarrow$ (c): We have a Hodge isometry $\varphi: T_Y \to T_S$, and primitive embeddings $T_Y \subset \tilde{\Lambda} \cong U^4 \oplus (-E_8)^2$ and $T_S \subset H^*(S, \mathbb{Z}) \cong U^4 \oplus (-E_8)^2$. The orthogonal $T_S^\perp$ contains a copy of $U$, so by [18, Prop. 3.8] any two primitive embeddings $T_S \hookrightarrow U^4 \oplus (-E_8)^2$ differ by an automorphism of $U^4 \oplus (-E_8)^2$. Thus the lattice isomorphism $\varphi: T_Y \to T_S$ extends to a lattice isomorphism $\tilde{\varphi}: \tilde{\Lambda} \to H^*(S, \mathbb{Z})$. Since $\varphi$ is a Hodge isometry, it takes $H^{2,0}(Y)$ to $H^{2,0}(S)$, so the extension $\tilde{\varphi}$ does as well, so $\tilde{\varphi}$ is a Hodge isometry.

Again let $v \in \tilde{\Lambda}$ be a primitive generator of $H^2(Y, \mathbb{Z})^\perp \subset \tilde{\Lambda}$, and write $\tilde{\varphi}(v) = (r, c, s) \in H^*(S, \mathbb{Z})$. I claim that either $r > 0$, or we can modify $v$ and $\tilde{\varphi}$ to make it so. If $r < 0$, replace $v$ with $-v$. If $r = 0$ and $s \neq 0$, compose $\tilde{\varphi}$ with the Mukai reflection through $(1, 0, 1) \in H^*(S, \mathbb{Z})$, so now $\tilde{\varphi}(v) = (-s, c, 0)$ and we are reduced to the previous case. If $r = s = 0$, note that $c^2 = q(v) = 2n - 2 > 0$, and compose $\tilde{\varphi}$ with multiplication by $\exp(c) = (1, c, n - 1)$, so now $\tilde{\varphi}(v) = (0, c, n - 1)$ and we are reduced to the previous case.

Now $\tilde{\varphi}(v)$ is a Mukai vector of positive rank, so for a generic polarization of $S$ the moduli space $M$ of stable sheaves on $S$ with Mukai vector $\tilde{\varphi}(v)$ is

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\(^2\)Mongardi and Wandel have proved a similar result independently in [16, Prop. 2.3].
smooth and non-empty [17]. By construction, $\tilde{\varphi}$ is a Hodge isometry from $\tilde{\Lambda}$ to $H^*(S, \mathbb{Z})$ taking $H^2(Y, \mathbb{Z})$ isomorphically to $\tilde{\varphi}(v)^\perp$, so $Y$ is birational to $M$ by Theorem 3.

\section{The Markman–Mukai lattice of $F$}

Recall that $X$ is a smooth cubic fourfold and $F$ is the variety of lines on $X$. Kuznetsov has observed that the triangulated category

$$A := \langle \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) \rangle^\perp \subset D^b(\text{Coh}(X))$$

$$:= \{ E \in D^b(\text{Coh}(X)) : \text{Ext}^*(\mathcal{O}_X(i), E) = 0 \text{ for } i = 0, 1, 2 \}$$

is like the derived category of a K3 surface in that it has the same Serre functor and Hochschild homology and cohomology, and has conjectured that $X$ is rational if and only if $A$ is equivalent to the derived category of an actual K3 surface [11]. By [1], this is essentially equivalent to having $X \in \mathcal{C}_d$ for some $d$ satisfying (**) .

Let $K_{\text{num}}(A)$ be the numerical Grothendieck group of $A$, that is, $K(A)$ modulo the kernel of the Euler pairing. Let $\lambda_1, \lambda_2 \in K_{\text{num}}(A)$ be the classes of the projections of $\mathcal{O}_L(1)$ and $\mathcal{O}_L(2)$ into $A$, where $L$ is any line on $X$. The Euler pairing on the sublattice $\langle \lambda_1, \lambda_2 \rangle$ is $-A_2 = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$.
A Mukai lattice for $\mathcal{A}$ was introduced in [1, Def. 2.2]:

$$K_{\text{top}}(\mathcal{A}) := \{ \kappa \in K_{\text{top}}(X) : \chi([\mathcal{O}_X(i)], \kappa) = 0 \text{ for } i = 0, 1, 2 \}.$$  

Here $K_{\text{top}}(X)$ is the Grothendieck group of topological vector bundles and $\chi$ is the Euler pairing, which is integer-valued and extends the Euler pairing on $K_{\text{num}}(X)$. It has a Hodge structure of K3 type pulled back via the Chern character or the Mukai vector $K_{\text{top}}(\mathcal{A}) \otimes \mathbb{C} \hookrightarrow \bigoplus H^{2i}(X, \mathbb{C})(i)$.  

In [1] this was called a weight-two Hodge structure, but it should really be called weight-zero. We will need the following properties:

**Theorem 6** (Addington, Thomas [1, §§2.3–2.4]).

(a) As a lattice, $K_{\text{top}}(\mathcal{A})$ is isomorphic to $U^4 \oplus E_8^2$.

(b) The algebraic part of $K_{\text{top}}(\mathcal{A})$ is isomorphic to $K_{\text{num}}(\mathcal{A})$.

(c) $\langle \lambda_1, \lambda_2 \rangle \perp \subset K_{\text{top}}(\mathcal{A})$ is Hodge-isometric to $H^4_{\text{prim}}(X, \mathbb{Z})(2)$.

(d) $X \in C_d$ if and only if there is a primitive sublattice $M \subset K_{\text{num}}(\mathcal{A})$ of rank 3 and discriminant $d$ that contains $\lambda_1$ and $\lambda_2$.

**Proposition 7.** Let $P \subset F \times X$ be the universal line and $p: P \to F$ and $q: P \to X$ the two projections. Then the map $\varphi$ from $\lambda_1 \perp \subset K_{\text{top}}(\mathcal{A})$ to $H^2(F, \mathbb{Z})(1)$ defined by $\varphi(\kappa) = c_1(p_*q^*\kappa)$ is a Hodge isometry.

**Proof.** Both $\lambda_1 \perp$ and $H^2(F, \mathbb{Z})(1)$ are lattices of rank 23 and discriminant 2. It is enough to show that $\varphi$ is a Hodge isometry when tensored with $\mathbb{Q}$; a priori this only implies that $\varphi$ embeds $\lambda_1 \perp$ as a finite-index sublattice of $H^2(F, \mathbb{Z})(1)$, but since they have the same discriminant the index must in fact be 1.

By the Riemann–Roch formula [3, §3], $\varphi(\kappa)$ is the degree-2 part of

$$p_*(q^*(\text{ch}(\kappa)) \cup \text{td}(T_p)),$$

where $T_p$ is the relative tangent bundle of the $\mathbb{P}^1$-bundle $p: P \to F$. First we calculate $\text{td}(T_p)$. Let $h \in H^2(X, \mathbb{Z})$ be the hyperplane class. Let $S$ be the restriction to $F$ of the tautological sub-bundle on $\text{Gr}(2, 6)$. Then $g := -c_1(S) \in H^2(F, \mathbb{Z})$ is the hyperplane class in the Plücker embedding. The
universal line $P$ is the projectivization $\mathbb{P} S$, and $O_{\mathbb{P} S}(1) = q^* \mathcal{O}_X(1)$. Since $T_p$ is line bundle, we can take determinants in the Euler sequence

$$0 \to O_{\mathbb{P} S} \to O_{\mathbb{P} S}(1) \otimes p^* S \to T_p \to 0$$

to get $T_p = O_{\mathbb{P} S}(2) \otimes p^* \det S$. Thus

$$(3) \quad \text{td}(T_p) = 1 + \frac{1}{2}(2q^* h - p^* g) + \frac{1}{12}(2q^* h - p^* g)^2 + \cdots.$$

The orthogonal to $\lambda_1$ in $\langle \lambda_1, \lambda_2 \rangle$ is generated by $\lambda_1 + 2\lambda_2$. Since we are tensoring with $\mathbb{Q}$, we have orthogonal direct sums

$$(4) \quad \lambda_1 \perp = \langle \lambda_1 + 2\lambda_2 \rangle \oplus \langle \lambda_1, \lambda_2 \rangle \perp$$

$$(5) \quad H^2(F, \mathbb{Q}) = \langle g \rangle \oplus H^2_{\text{prim}}(F, \mathbb{Q}).$$

By [1, Prop. 2.3], the Chern character$^3$ gives a Hodge isometry from the second summand of (4) to $H^4_{\text{prim}}(X, \mathbb{Q})(2)$. By [6, Prop. 6], $p_* q^*$ gives a Hodge isometry from this to the second summand of (5). Since the degree-0 part of $\text{td}(T_p)$ is 1, we see that for $\alpha \in H^4(X, \mathbb{Q})$, the degree-2 part of $p_*(q^* \alpha \cup \text{td}(T_p))$ is just $p_* q^* \alpha$. Thus $\varphi$ gives a Hodge isometry from the second summand of (4) to the second summand of (5).

For the first summands of (4) and (5), observe that the Euler square of $\lambda_1 + 2\lambda_2$ is $-6$, and by [8, §2.1] we have $q(g) = -6$ as well (the minus sign comes because we have twisted down to weight zero). Thus it is enough to show that

$$(6) \quad \varphi(\lambda_1 + 2\lambda_2) = g.$$

To calculate $\text{ch}(\lambda_1 + 2\lambda_2)$, recall that $\lambda_i$ is the class of the left mutation of $\mathcal{O}_L(i)$ past $\mathcal{O}_X(2)$, $\mathcal{O}_X(1)$, and $\mathcal{O}_X$, where $L$ is any line on $X$, so a straightforward calculation gives

$$\lambda_1 = [\mathcal{O}_L(1)] - [\mathcal{O}_X(1)] + 4[\mathcal{O}_X]$$
$$\lambda_2 = [\mathcal{O}_L(2)] - [\mathcal{O}_X(2)] + 4[\mathcal{O}_X(1)] - 6[\mathcal{O}_X]$$

and thus

$$\text{ch}(\lambda_1 + 2\lambda_2) = -3 + 3h - \frac{1}{2}h^2 + \cdots.$$

$^3$In fact [1, Prop. 2.3] says that the Mukai vector gives such a Hodge isometry, but since $\text{td}(X)$ is a polynomial in $h$, multiplying by $\sqrt{\text{td}(X)}$ does not affect $H^4_{\text{prim}}(X, \mathbb{Q})$. 
By [8, §2.1] we have $p_*q^*h^2 = g$. We also have $p_*q^*h = 1$: to see this, take a smooth hyperplane section $X \cap H$ and take its preimage under $q$; this is the blow-up of $F$ along the surface of lines contained in the cubic threefold $X \cap H$, hence is generically 1-to-1 over $F$. Combining these facts with (2) and (3) we get (6).

Corollary 8. The embedding $H^2(F, \mathbb{Z}) \subset K_{\text{top}}(\mathcal{A})(-1)$ given by the previous proposition can be identified with Markman’s embedding $H^2(F, \mathbb{Z}) \subset \tilde{\Lambda}$ discussed in §1.

Proof. If $n = 2$ or if $n - 1$ is a prime power then for any $Y$ of $K3^{[n]}$-type, any two primitive embeddings of $H^2(Y, \mathbb{Z})$ into $U^4 \oplus (-E_8)^2$ differ by an automorphism of the latter [14, §4.1].

3. Proofs of Theorems 1 and 2

Theorem 1. The following are equivalent:

(a) $X \in C_d$ for some $d$ satisfying (**).

(b) The transcendental lattice $T_X \subset H^4(X, \mathbb{Z})$ is Hodge-isometric to $T_S(-1)$ for some $K3$ surface $S$.

(c) $F$ is birational to a moduli space of stable sheaves on $S$.

Proof. By [1, Thm. 3.1], condition (a) holds if and only if $K_{\text{num}}(\mathcal{A})$ contains a copy of $U \cong -U$. Moreover we have $T_X \cong T_F(-1)$. Thus the theorem follows from Corollary 8 and Proposition 4.

To prove Theorem 2 we will have to work in a basis:

Lemma 9. If $X \in C_d$ then there is a $\tau \in K_{\text{num}}(\mathcal{A})$ such that $\langle \lambda_1, \lambda_2, \tau \rangle$ is a primitive sublattice of discriminant $d$ with Euler pairing

$$\begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 2k \end{pmatrix}$$

when $d = 6k$, or

$$\begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & 2k \end{pmatrix}$$

when $d = 6k + 2$. 

□
\textbf{Proof.} By Theorem 6(d), we can choose a $\tau \in K_{num}(A)$ such that $\langle \lambda_1, \lambda_2, \tau \rangle$ is a primitive sublattice of discriminant $d$. Write the Euler pairing as
\[
\begin{pmatrix}
-2 & 1 & a \\
1 & -2 & * \\
a & * & *
\end{pmatrix}
\]
for some $a \in \mathbb{Z}$. Replace $\tau$ with $\tau - a\lambda_2$; then the Euler pairing becomes
\[
\begin{pmatrix}
-2 & 1 & 0 \\
1 & -2 & 3b + c \\
0 & 3b + c & *
\end{pmatrix}
\]
for some $b$ and some $-1 \leq c \leq 1$. Replace $\tau$ with $\tau + b(\lambda_1 + 2\lambda_2)$; then the Euler pairing becomes
\[
\begin{pmatrix}
-2 & 1 & 0 \\
1 & -2 & c \\
0 & c & 2k
\end{pmatrix}
\]
for some $k$, since $K_{num}(A)$ is an even lattice. If $c = 0$ this has determinant $6k$. If $c = 1$ this has determinant $6k + 2$. If $c = -1$, replace $\tau$ with $-\tau$ to get back to the previous case. \hfill \Box

\textbf{Theorem 2.} The following are equivalent:

(a) $X \in C_d$ for some $d$ satisfying (***).

(b) $F$ is birational to $\text{Hilb}^2(S)$ for some K3 surface $S$.

\textbf{Proof.} We will show that condition (a) holds if and only if there is a $w \in K_{num}(A)$ such that

\begin{equation}
\chi(\lambda_1, w) = 1 \quad \text{and} \quad \chi(w, w) = 0.
\end{equation}

Then the theorem follows from Corollary 8 and Proposition 5.

If there is such a $w$, let $L = \langle \lambda_1, \lambda_2, w \rangle \subset K_{num}(A)$. By hypothesis, the Euler pairing on $L$ is
\[
\begin{pmatrix}
-2 & 1 & 1 \\
1 & -2 & n \\
1 & n & 0
\end{pmatrix}
\]
for some $n \in \mathbb{Z}$, so $\text{disc}(L) = 2n^2 + 2n + 2$. Let $M$ be the saturation of $L$, let $a$ be the index of $L$ in $M$, and let $d = \text{disc}(M)$. Then $\text{disc}(L) = a^2d$, and $X \in C_d$ by Theorem 6(d).
Conversely, suppose $X \in C_d$ for some $d$ satisfying (**). Choose integers $n$ and $a$ such that

$$da^2 = 2n^2 + 2n + 2.$$  

Recall that $d$ is even. Since $2n^2 + 2n + 2$ satisfies (**) we see that $a$ is a product of primes $p \equiv 1 \pmod{3}$, and in particular $a \equiv 1 \pmod{3}$. We consider three cases.

Case 1: $n \equiv 1 \pmod{3}$. In this case we find that $d \equiv 0 \pmod{6}$. Write $d = 6k$. By Lemma 9 there is a $\tau \in K_{\text{num}}(A)$ such that the Euler pairing on $\langle \lambda_1, \lambda_2, \tau \rangle$ is

$$
\begin{pmatrix}
-2 & 1 & 0 \\
1 & -2 & 0 \\
0 & 0 & 2k
\end{pmatrix}.
$$

Let $m = (n - 1)/3$, which is an integer; then we find that

$$w := m\lambda_1 + (2m + 1)\lambda_2 + a\tau$$

satisfies (7).

Case 2: $n \equiv 2 \pmod{3}$. In this case we find that $d \equiv 2 \pmod{6}$. Write $d = 6k + 2$. By Lemma 9 there is a $\tau \in K_{\text{num}}(A)$ such that the Euler pairing on $\langle \lambda_1, \lambda_2, \tau \rangle$ is

$$
\begin{pmatrix}
-2 & 1 & 0 \\
1 & -2 & 1 \\
0 & 1 & 2k
\end{pmatrix}.
$$

Let $m = (a - n - 2)/3$, which is an integer; then we find that

$$w := m\lambda_1 + (2m + 1)\lambda_2 + a\tau$$

satisfies (7).

Case 3: $n \equiv 0 \pmod{3}$. Again we find that $d \equiv 2 \pmod{6}$. Argue as in the previous case but with $m = (a + n - 1)/3$. □

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References

[1] N. Addington and R. Thomas, Hodge theory and derived categories of cubic fourfolds. Duke Math. J., 163 (2014), no. 10, 1885–1927. Also arXiv:1211.3758.

[2] T. Andreescu, D. Andrica, and I. Cucurezeanu, An introduction to Diophantine equations. Birkhäuser Verlag, New York (2010). A problem-based approach.

[3] M. F. Atiyah and F. Hirzebruch, Vector bundles and homogeneous spaces. In: Proc. Sympos. Pure Math., Vol. III, 7–38, American Mathematical Society, Providence, R.I. (1961).

[4] A. Bayer, B. Hassett, and Y. Tschinkel, Mori cones of holomorphic symplectic varieties of K3 type. Ann. Sci. Éc. Norm. Supér. (4), 48 (2015), no. 4, 941–950. Also arXiv:1307.2291.

[5] A. Bayer and E. Macrì, MMP for moduli of sheaves on K3s via wall-crossing: nef and movable cones, Lagrangian fibrations. Invent. Math., 198 (2014), no. 3, 505–590. Also arXiv:1301.6968.

[6] A. Beauville and R. Donagi, La variété des droites d’une hypersurface cubique de dimension 4. C. R. Acad. Sci. Paris Sér. I Math., 301 (1985), no. 14, 703–706. Also math1.unice.fr/~beauvill/pubs/bd.pdf.

[7] S. Galkin and E. Shinder, The Fano variety of lines and rationality problem for a cubic hypersurface. Preprint, arXiv:1405.5154.

[8] B. Hassett, Special cubic fourfolds. Compositio Math., 120 (2000), no. 1, 1–23. Also math.brown.edu/~bhassett/papers/cubics/cubic.pdf.

[9] D. Huybrechts, Compact hyper-Kähler manifolds: basic results. Invent. Math., 135 (1999), no. 1, 63–113. Also arXiv:alg-geom/9705025.

[10] D. Huybrechts, A global Torelli theorem for hyperkähler manifolds [after M. Verbitsky]. Astérisque, (2012), no. 348, Exp. No. 1040, x, 375–403. Séminaire Bourbaki: Vol. 2010/2011. Exposés 1027–1042. Also arXiv:1106.5573.

[11] A. Kuznetsov, Derived categories of cubic fourfolds. In: Cohomological and geometric approaches to rationality problems, Vol. 282 of Progr. Math., 219–243, Birkhäuser Boston Inc., Boston, MA (2010). Also arXiv:0808.3351.
On two rationality conjectures for cubic fourfolds

[12] A. Kuznetsov and D. Markushevich, *Symplectic structures on moduli spaces of sheaves via the Atiyah class*. J. Geom. Phys., 59 (2009), no. 7, 843–860. Also arXiv:math/0703264.

[13] E. Macrì and P. Stellari, *Fano varieties of cubic fourfolds containing a plane*. Math. Ann., 354 (2012), no. 3, 1147–1176. Also arXiv:0909.2725.

[14] E. Markman, *Integral constraints on the monodromy group of the hyperKähler resolution of a symmetric product of a K3 surface*. Internat. J. Math., 21 (2010), no. 2, 169–223. Also arXiv:math/0601304.

[15] E. Markman, *A survey of Torelli and monodromy results for holomorphic-symplectic varieties*. In: Complex and differential geometry, Vol. 8 of Springer Proc. Math., 257–322, Springer, Heidelberg (2011). Also arXiv:1101.4606.

[16] G. Mongardi and M. Wandel, *Induced automorphisms on irreducible symplectic manifolds*. J. Lond. Math. Soc. (2), 92 (2015), no. 1, 123–143. Also arXiv:1405.5706.

[17] S. Mukai, *On the moduli space of bundles on K3 surfaces. I*. In: Vector bundles on algebraic varieties (Bombay, 1984), Vol. 11 of Tata Inst. Fund. Res. Stud. Math., 341–413, Tata Inst. Fund. Res., Bombay (1987).

[18] D. Orlov, *Equivalences of derived categories and K3 surfaces*. J. Math. Sci. (New York), 84 (1997), no. 5, 1361–1381. Also arXiv:alg-geom/9606006.

[19] M. Verbitsky, *Mapping class group and a global Torelli theorem for hyperkahler manifolds*. Duke Math. J., 162 (2013), no. 15, 2929–2986. Appendix A by Eyal Markman. Also arXiv:0908.4121.

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