Nonfragile observer-based guaranteed cost finite-time control of discrete-time positive impulsive switched systems

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Abstract: This paper considers the nonfragile observer-based guaranteed cost finite-time control of discrete-time positive impulsive switched systems (DPISS). Firstly, the positive observer and nonfragile positive observer are designed to estimate the actual state of the underlying systems, respectively. Secondly, by using the average dwell time (ADT) approach and multiple linear co-positive Lyapunov function (MLCLF), two guaranteed cost finite-time controller are designed and sufficient conditions are obtained to guarantee the corresponding closed-loop systems are guaranteed cost finite-time stability (GCFTS). Such conditions can be solved by linear programming. Finally, a numerical example is provided to show the effectiveness of the proposed method.

Keywords: Guaranteed cost finite-time control, Nonfragile positive observer design, Discrete-time positive impulsive switched systems, Average dwell time

MSC: 15-xx, 65-xx, 93-xx

1 Introduction

The switched system is a type of hybrid systems. It comprises a set of a differential or difference equations and a switched controller, which designate the switching between subsystems at a specific interval of time. It has been studied very well, see [1-4]. As a special kind of switched systems, the positive switched systems whose output and state are non-negative have been paid much attention and adopted to many practical applications, such as communication networks [5], viral mutation [6], formation flying [7], and so on. There have been many available results about discrete-time positive switched systems [8-10]. Because of sudden changes in the state of the system at certain instants of switching, many practical switched systems exhibit impulsive dynamical behavior, these systems are usually called impulsive switched systems [11]. For discrete-time positive impulsive switched system, a few results have been obtained, see [12,13]. In [12], the exponential stability for a class of discrete-time positive impulsive switched linear systems was studied. In [13], the finite-time stability for a class of discrete-time positive impulsive switched time-delay systems under asynchronous switching was discussed. However, [12] and [13] are based on the assumption of the known state.

Moreover, it is necessary to design a state observer, because the states of the systems are not all measurable in practice [14]. Recently, the exponential stability property of the proposed switching observer
was discussed, and an LMI-based algorithm was given for discrete-time impulsive switched nonlinear systems with time-varying delays in [15]. The problem of state observation for continuous-time and discrete-time impulsive switched systems was investigated in [16]. Furthermore, when the state observer gain variations could not be avoided, a kind of nonfragile state observers for discrete-time switched nonlinear systems was proposed in [17]. However, [15-17] were involved in non-positive systems and the design of finite time controller was not considered.

On the other hand, in most practical applications, the researchers are more interested in designing the control system, which is not only finite-time stable but also guarantees an adequate level of performance. One method to this problem is so-called guaranteed cost finite time control. Some remarkable results have been presented, see [18-23]. These results mainly focus on non-positive systems. Very recently, In [24], guaranteed cost finite-time control was extended to fractional-order positive switched systems and a cost function for fractional-order positive switched systems (or fractional-order positive switched systems) was proposed. In [25], the problem of guaranteed cost finite-time control for positive switched linear systems with time-varying delays was considered and a cost function of positive systems (or positive switched systems) was also presented. It is worth noting that [24] and [25] are involved in continuous-time positive switched systems with known states. To the best of our knowledge, the problem of observer-based guaranteed cost finite-time control of discrete-time impulsive switched systems has not been fully investigated, especially for DPISS, which motivates us for this study.

In this paper, the problem of nonfragile observer-based guaranteed cost finite-time control of DPISS is considered. The co-positive type Lyapunov function with average dwell time (ADT) technique is constructed. The main contributions lie in two aspects: 1) For DPISS, the design methods of positive observer and nonfragile positive observer are firstly proposed, respectively. 2) Two types of guaranteed cost finite-time controller are designed to guarantee the corresponding closed-loop systems are GCFTS, the obtained conditions can be easily solved by linear programming. The rest of the paper is organised as follows: Section 2 gives some necessary preliminaries and problem statements. In Section 3, the main results are given. In Section 4, a numerical example is provided. Section 5 concludes the paper.

Notations. The representation $A > 0$ ($\geq 0$, $< 0$, $\leq 0$) means that $a_{ij} > 0$ ($\geq 0$, $< 0$, $\leq 0$), which is also applying to a vector. $A \succ B$ ($A \succeq B$) means that $A - B > 0$ ($A - B \geq 0$). $R^n_+$ is the $n$-dimensional non-negative (positive) vector space. $R^{m\times n}$ denotes the space of $m \times n$ matrices with real entries. $A^T$ denotes the transpose of matrix $A$. $N$ and $N^+$ are the sets of non-negative and positive integers. $Z^+$ denotes the set of positive integers. Matrices are assumed to have compatible dimensions for calculating if their dimensions are not explicitly stated.

2 Preliminaries and problem statements

Consider the following DPISS:

\[
\begin{align*}
    x(k + 1) &= A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k), \quad k \neq k_m - 1, \quad m \in Z^+ \\
    x(k + 1) &= E_{\sigma(k)}x(k), \quad k = k_m - 1, \quad m \in Z^+ \\
    y(k) &= C_{\sigma(k)}x(k) \\
    x(k_0) &= x_0
\end{align*}
\]

(1)

where $k \in N$, $x(k) \in R^n$ is the system state, $u(k) \in R^m$ represents the control input. $\sigma(k)$ represents switching signal of system and takes values in a finite set $I = 1, 2, \ldots, S$, $S \in N^+$. In general, $A_i$, $B_i$, $C_i$, $E_i$ are the $ith$ subsystem if $\sigma(k) = i \in I, k_0 = 0$ is the initial time. $k_m(m \in Z^+)$ denotes the $bth$ impulsive switching instant. Moreover, $\sigma(k) = i \in I$ means that the $ith$ subsystem is active. $\sigma(k - 1) = j$ and $\sigma(k) = i(i \neq j)$ indicate that $k$ is a switching instant at which the system is switched from the $jth$ subsystem to the $ith$ subsystem. At switching instants, there exist impulsive jumps described by (1). $A_p$, $B_p$, $C_p$, $E_p$ are constant matrices with suitable dimensions.

Next, we will give some definitions and lemmas for the system (1).
**Definition 1 [5].** System (1) with \( u(k) \geq 0 \) is positive if \( x(k_0) \geq 0 \) and any switching signals \( \sigma(k) \), the corresponding trajectories \( x(k) \geq 0, y(k) \geq 0 \) hold for all \( k \geq k_0 \).

**Lemma 1.** System (1) is positive if and only if \( A_i \geq 0, B_i \geq 0, C_i \geq 0, E_i \geq 0 \) where \( i \in I \).

### 2.1 Positive observer design

We construct the following DPISS (1):

\[
\begin{align*}
\dot{x}(k+1) &= (A_{o(k)} - L_{o(k)}C_{o(k)})\dot{x}(k) + B_{o(k)}u(k) + L_{o(k)}C_{o(k)}x(k), \quad k \neq k_m - 1, \ m \in \mathbb{Z}^+ \\
\dot{\hat{x}}(k+1) &= H_{o(k)}\hat{x}(k), \quad k = k_m - 1, \ m \in \mathbb{Z}^+ \\
u(k) &= K_{o(k)}\hat{x}(k) \\
x(k_0) &= x_0
\end{align*}
\]  

where \( \hat{x}(k) \in \mathbb{R}^n \) is the estimated state vector of \( x(k) \), \( \hat{y}(k) \in \mathbb{R}^p \) is the observer output vector, \( L_i \in \mathbb{R}^{nxp} \) is the observer gain, \( H_i \in \mathbb{R}^{m \times n}, \forall i \in I \) is the matrix to be determined.

**Remark 1.** For DPISS (1), it not only requires that the state of the designed observer converges to that of the considered system, but also guarantees the positivity of the estimated state \( \hat{x}(k) \) of system (2) with \( u(k) \geq 0 \). To this end, it is naturally required, according to Lemma 1, \( A_i - L_iC_i \succeq 0, B_i \geq 0, L_iC_i \succeq 0, H_i \succeq 0, K_i \succeq 0 \), where \( i \in I \).

Let \( e(k) = x(k) - \hat{x}(k) \) be the estimated error, then we can obtain the following error system:

When \( k \neq k_m - 1 \), we have

\[
\begin{bmatrix}
e(k+1) \\
\hat{x}(k+1)
\end{bmatrix} =
\begin{bmatrix}
A_i - L_iC_i & 0 \\
L_iC_i & A_i + B_iK_i
\end{bmatrix}
\begin{bmatrix}
e(k) \\
\hat{x}(k)
\end{bmatrix}
\]  

When \( k = k_m - 1 \), we get

\[
\begin{bmatrix}
e(k+1) \\
\hat{x}(k+1)
\end{bmatrix} =
\begin{bmatrix}
E_i & E_i - H_i \\
0 & H_i
\end{bmatrix}
\begin{bmatrix}
e(k) \\
\hat{x}(k)
\end{bmatrix}
\]  

For the sake of convenience, we define \( \hat{x}(k+1) = [e^T(k+1) \hat{x}^T(k+1)]^T \). Thus, equation (3) and (4) can be rewritten as

\[
\begin{align*}
\dot{\hat{x}}(k+1) &= \tilde{A}_i\hat{x}(k), \quad k \neq k_m - 1, \ m \in \mathbb{Z}^+ \\
\hat{x}(k+1) &= \tilde{B}_i\hat{x}(k), \quad k = k_m - 1, \ m \in \mathbb{Z}^+ \\
\hat{x}(k_0) &= \hat{x}_0
\end{align*}
\]  

**Remark 2.** According to Lemma 1, if error dynamic system (5) is positive, then it should guarantee that \( \tilde{A}_i \succeq 0, \tilde{B}_i \succeq 0, \forall i \in I \). It means \( A_i - L_iC_i \succeq 0, L_iC_i \succeq 0, A_i + B_iK_i \succeq 0, E_i \succeq 0, E_i - H_i \succeq 0, H_i \succeq 0 \).

**Definition 2.** For any switching signal \( \sigma(k) \) and any \( k_2 \geq k_1 \geq 0 \), let \( N_o(k_1, k_2) \) denote the switching numbers over the interval \( [k_1, k_2] \). For given \( \tau_a > 0 \) and \( N_0 > 0 \), if the inequality

\[
N_o(k_1, k_2) \leq N_0 + \frac{k_2 - k_1}{\tau_a}
\]  

holds, then \( \tau_a \) is called an average dwell time, and \( N_0 \) is called a chattering bound. Generally, we choose \( N_0 = 0 \).

**Definition 3.** (Finite-Time Stability(FTS)). For a given time \( T_f \) and two vectors \( \alpha \succ \beta \succ 0 \), discrete-time positive impulsive switched systems (5) with is said to be FTS with respect to \( (\alpha, \beta, T_f, \sigma(k)) \), if

\[
\hat{x}^T(0)\beta \leq 1 \Rightarrow \hat{x}^T(k)\alpha < 1, \forall k \in [0, T_f]
\]  

**Remark 3.** In (5), the state \( \hat{x}^T(k) \) includes the \( e(k) \) and \( \hat{x}(k) \). From Definition 3, our goal is that the weighted system \( \hat{x}^T(k)\alpha \) does not exceed threshold 1 in a given time interval \( T_f \), then the estimation error \( e(k) \) might...
not converge to zero in a given time interval $T_f$. If a smaller threshold is chosen, then the estimation error will become very small.

Now we give some new definitions for our further study.

**Definition 4 [24].** Define the cost function of DPISS (5) as follows:

$$J = \sum_{s=0}^{T_f-1} (x^T(s)R_1 + u^T(s)R_2)$$

(8)

where $R_1 \succ 0$ and $R_2 \succ 0$ are two given vectors.

**Definition 5 [24].** (GCFTS) For a given time constant $T_f$ and two vectors $\alpha \succ \beta > 0$, consider DPISS (5) and cost function (8), if there exist a control law $u(k)$ and a positive scalar $J^*$ such that the closed-loop system is FTS with respect to $(\alpha, \beta, T_f, \sigma(k))$ and the cost function satisfies $J \leq J^*$, then the closed-loop system is called GCFTS, where $J^*$ is a guaranteed cost value and $u(k)$ is a guaranteed cost finite-time controller.

**Remark 4.** [14] noted that one could not stabilize any unstable positive system by using extended Luenberger type positive observers. But finite-time stability means that the system state does not exceed the specified boundary within a given time interval and it is different from the asymptotic stability. So the problem of positive observer-based guaranteed cost finite-time control of DPISS is feasible.

### 2.2 Nonfragile positive observer design

If the state observer gain variations could not be avoided, a kind of nonfragile state observer will be designed as follows

$$\dot{x}(k+1) = (A_{\sigma(k)} - (L_{\sigma(k)} + \triangle L_{\sigma(k)}))\dot{x}(k) + B_{\sigma(k)}u(k) + (L_{\sigma(k)} + \triangle L_{\sigma(k)})C_{\sigma(k)}x(k)$$

$$k \neq k_m - 1, \ m \in Z^+$$

$$\dot{x}(k+1) = H_{\sigma(k)}\dot{x}(k), \ k = k_m - 1, \ m \in Z^+$$

$$u(k) = K_{\sigma(k)}\dot{x}(k)$$

$$x(k_0) = x_0$$

(9)

where $\triangle L_i \in R^{n \times p}$ are uncertain real-valued matrices which satisfy $\triangle L_i \in [L'_1, L'_2], L'_1 \in R^{n \times p}, L'_2 \in R^{n \times p}$.

We can obtain the following error system

$$\begin{cases}
\dot{\hat{x}}(k+1) = \hat{A}_i\hat{x}(k), \ k \neq k_m - 1, \ m \in Z^+ \\
\dot{\hat{x}}(k+1) = \hat{B}_i\hat{x}(k), \ k = k_m - 1, \ m \in Z^+ \\
\hat{x}(k_0) = \hat{x}_0
\end{cases}$$

(10)

where

$$\hat{A}_i = \begin{bmatrix} A_i - (L_i + \triangle L_i)C_i & 0 \\ (L_i + \triangle L_i)C_i & A_i + B_iK_i \end{bmatrix}, \hat{B}_i = \begin{bmatrix} E_i & E_i - H_i \\ 0 & H_i \end{bmatrix}$$

**Remark 5.** According to Lemma 1, if observer system (9) and error dynamic system (10) are positive, then it should guarantee that $\hat{A}_i \geq 0, \hat{B}_i \geq 0, \forall i \in I$ (it means $A_i - (L_i + \triangle L_i)C_i \geq 0, (L_i + \triangle L_i)C_i \geq 0, A_i + B_iK_i \geq 0, B_i \geq 0, E_i \geq 0, E_i - H_i \geq 0, K_i \geq 0, H_i \geq 0$).

The aim of this paper is to design the positive observer and nonfragile positive observer based on state feedback controller, and find a class of switching signals $\sigma(k)$ for systems (2) and (6) such that the corresponding closed-loop systems are GCFTS, respectively.

### 3 Main results

#### 3.1 Observer-based guaranteed cost finite-time stability analysis

In this subsection, we will focus on the problem of GCFTS for DPISS (5). The following theorem gives sufficient conditions of GCFTS for system (5).

...
From (11) and (12), we have

\[ (A_i^T - C_i^T L_i^T) v_i + R_1 + C_i^T L_i^T u_i - \xi v_i < 0 \]  
\[ (A_i^T + K_i^T B_i^T) u_i + R_1 + K_i^T R_2 - \xi u_i < 0 \]
\[ E_i^T v_j - \mu v_i < 0 \]
\[ (E_i^T - H_i^T) v_j + H_i^T u_j - \mu v_i < 0 \]
\[ \phi_1 \alpha < \psi_i < \phi_2 \beta \]
\[ A_i - L_i C_i > 0 \]
\[ A_i + B_i K_i > 0 \]
\[ E_i - H_i > 0 \]
\[ \phi_1 > \phi_2 \xi T_f \]

\[ \psi_i = [\psi_{i1}, \psi_{i2}, \ldots, \psi_{im}]^T \]

\[ \psi_i \] represents the ith elements of the vectors \( \psi_i \), respectively, then under the following ADT scheme

\[ T_a > T_a^* = \frac{T_r \ln \mu}{\ln \phi_1 - \ln \phi_2 - T_r \ln \xi} \]

the system (5) is GCFTS with respect to \( \alpha, \beta, T_f, \sigma(k) \) and the guaranteed cost value of system (5) is given by

\[ J = \sum_{s=0}^{T_r-1} (x^T(s)R_1 + u^T(s)R_2) < \xi T_r \mu^\frac{T_r}{T_r - 1} \phi_2 \]

**Proof.** Construct the following multiple linear co-positive Lyapunov function for the systems (5) as follows:

\[ V_i \tilde{x}(k) = \tilde{x}^T (k) \psi_i \]

where \( \psi_i = [v_i^T, u_i^T]^T \), \( i \in I \).

Supposing a switching sequence \( 0 = k_0 \leq k_1 \leq k_m \leq k_{m+1} \leq \ldots < T_f \). Without loss of generality, we assume that subsystem \( i \) is activated at the switching instant \( k_{m-1} \) and the subsystem \( j \) is activated at the switching instant \( k_m \).

When \( k \in [k_{m-1}, k_m - 1], m \in N, \sigma(k) = \sigma(k + 1) = i \), along the trajectory of system (5), the difference of the MLCLF is

\[ \Delta V_i(\tilde{x}(k)) + x^T(k)R_1 + x^T(k)K_i^T R_2 \]
\[ = \tilde{x}^T(k + 1) \psi_i - \tilde{x}^T(k) \psi_i + x^T(k)R_1 + x^T(k)K_i^T R_2 \]
\[ = \tilde{x}^T(k) \begin{bmatrix} (A_i^T - C_i^T L_i^T) v_i + R_1 + C_i^T L_i^T u_i \\ (A_i^T + K_i^T B_i^T) u_i + R_1 + K_i^T R_2 \end{bmatrix} - \tilde{x}^T(k) \psi_i \]

From (11) and (12), we have

\[ \Delta V_i(\tilde{x}(k)) + x^T(k)R_1 + x^T(k)K_i^T R_2 \leq (\xi - 1) \tilde{x}^T(k) \psi_i \]
it implies
\[ V_j(\tilde{x}(k+1)) \leq \xi \tilde{x}^T(k) \psi_i \]  
(25)

When \( k = k_m - 1 \), \( \sigma(k+1) = \sigma(k_m) = j \), \( \sigma(k) = \sigma(k_m - 1) = i \), \( i \neq j \). Along the trajectory of system (5), the difference of MLCLF is
\[ V_j(\tilde{x}(k+1)) - \mu V_j(\tilde{x}(k)) = \tilde{x}^T(k+1)\psi_j - \mu \tilde{x}^T(k)\psi_i = \tilde{x}^T(k+1)\psi_j - \tilde{x}(k)\psi_i \leq \tilde{x}^T(k) \begin{bmatrix} E_i^T v_j - \mu v_j \\ (E_j^T - H_j^T)v_j + H_j^T v_j - \mu v_j \end{bmatrix} \]  
(26)

From (13) and (14), we have
\[ V_j(\tilde{x}(k+1)) \leq \mu V_j(\tilde{x}(k)), \ i \neq j. \]  
(27)

So, when \( k \in [k_m, k_{m+1}) \), from (27), we get
\[ V_{\sigma(k)}(\tilde{x}(k)) < \xi^{k-k_0} V_{\sigma(k_0)}(\tilde{x}(k_m)) < \mu \xi^{k-k_0} V_{\sigma(k_{m-1})}(\tilde{x}(k_{m-1})) \]  
(28)

Repeating the procedure of (28) and noting \( \xi > 1 \), we obtain
\[ V_{\sigma(k)}(\tilde{x}(k)) < \xi^{k-k_0} V_{\sigma(k_0)}(\tilde{x}(k_m)) < \mu \xi^{k-k_0} \xi^{k_{m-1}-k_0} V_{\sigma(k_{m-1})}(\tilde{x}(k_{m-1})) \]  
(29)

By iterative operation, we get
\[ V_{\sigma(k)}(\tilde{x}(k)) < \mu^2 \xi^{k-k_0} V_{\sigma(k_0)}(\tilde{x}(k_m)) \leq \ldots \leq \mu^{N_{\sigma}(k_0)} \xi^{k-k_0} V_{\sigma(k_0)}(\tilde{x}(k_0)) = \mu^{N_{\sigma}(k_0)} \xi^{k-k_0} V_{\sigma(k_0)}(\tilde{x}(k_0)) \]  
(30)

From (15) and (22), we have
\[ V_{\sigma(k)}(\tilde{x}(k)) = \tilde{x}^T(k)\psi_{\sigma(k)} \leq \phi_1 \tilde{x}^T(k) \alpha \]  
(31)
\[ V_{\sigma(k_0)}(\tilde{x}(k_0)) = \tilde{x}^T(0)\psi_{\sigma(k_0)} \leq \phi_2 \tilde{x}^T(0) \beta \]  
(32)

From (30)-(32), we obtain
\[ \tilde{x}^T(k)\alpha \leq \frac{1}{\phi_1} \frac{\theta_1}{\mu} \xi^{k-k_0} \phi_2 \tilde{x}^T(0) \beta \]  
(33)

Substituting (20) into (33), one has
\[ \tilde{x}^T(k)\alpha < 1 \]  
(34)

According to Definition 3, we conclude that the system (5) with \( u(k) = 0 \) is FTS with respect to \( (\alpha, \beta, T_f, \sigma(k)) \).

Next, we will give the guaranteed cost value of system (5).

When \( k \in [k_{m-1}, k_m - 1) \), \( m \in N \), according to (24), we know
\[ V_j(\tilde{x}(k)) \leq \xi^k \tilde{x}^T(k) v_j - x^T(k) R_1 - x^T(k) K_1^T R_2 \]  
(35)

Similar to the proof process of (25)-(30), for any \( k \in [0, T_f) \) and \( \mu > 1 \), we can obtain
\[ V_{\sigma(k)}(\tilde{x}(k)) < \mu^{N_{\sigma}(k_0)} \xi^{k-k_0} V_{\sigma(k_0)}(\tilde{x}(k_0)) - \sum_{s=k_0}^{k-1} x^T(s)(R_1 + K_1^T R_2) \]  
(36)
Noting that \( V_{\varphi}(\tilde{x}(k)) > 0 \), (36) can be rewritten as
\[
0 < \mu^{N_c(k, k_0)} \xi^{k-k_0} V_{\varphi}(k_0)(\tilde{x}(k_0)) - \sum_{s=0}^{k-1} x(s)^T R_1 + K_f^T R_2)
\] (37)

Letting \( k = T_f \), we get
\[
\sum_{s=0}^{T_f-1} x(s)^T R_1 + K_f^T R_2 < \mu^{N_c(k, k_0)} \xi^{k-k_0} (V_{\varphi}(k_0)) - \mu^{N_c(T_f, k_0)} \xi^{T_f-k_0} (V_{\varphi}(k_0))
\] (38)

Then we can obtain
\[
J = \sum_{s=0}^{T_f-1} x^T(s)(R_1 + K_f^T R_2) \leq J^* = \xi^{T_f} \mu^{T_f} \phi_2
\] (39)

Therefore, according to Definition 5, we can conclude that the system (5) is GCFTS. Thus, the proof is completed.

In Theorem 1, (11) and (12) are nonlinear matrix inequalities, there are no effective methods to solve \( K_i \) and \( L_i \). So, an algorithm is presented to obtain the feedback gain matrices \( K_i, L_i, i \in I \).

**Algorithm 1**

**Step 1.** By adjusting the parameters \( \xi, \mu \) and Letting \( f_p = K_p^T B_p^T v_p \), then solving (13)-(19) via linear programming, positive vectors \( v_p, u_p, K_p \) and \( K_p \) can be obtained.

**Step 2.** Substituting \( v_p \) and \( K_p \) into \( f_p = K_p^T B_p^T u_p, f_p \) can be obtained. The gain \( L_p, v_p \) and \( u_p \) are substituted into (12).

**Step 3.** If \( f_p - f_p \leq 0 \) and (12) are all satisfied at the same time, then \( K_p \) and \( L_p \) are admissible. Otherwise, return to Step 1.

### 3.2 Nonfragile observer-based guaranteed cost finite-time stability analysis

Now we consider system (10); the following theorem gives sufficient conditions of GCFTS for system (10).

**Theorem 2.** Consider the system (10), for a given time constant \( T_f \), vectors \( \alpha > \beta > 0, L_1, L_2 \) and \( R_1 > 0, R_2 > 0 \), if there exist a set of positive vectors \( v_1, v_j, u_i, u_j, i \neq j, j \in I \), positive matrices \( K_i, L_i \) and positive constants \( \phi_1, \phi_2, \xi > 1, \mu > 1 \), such that (12)-(15), (17)-(19) and the following inequalities hold:
\[
(A_i^T - C_i^T (L_i + \Delta L_i)^T v_i + R_1 + C_i^T (L_i + \Delta L_i)^T u_i - \xi v_i < 0)
\]
\[
A_i - (L_i + \Delta L_i) C_i > 0
\]
(40)
(41)

then under the ADT scheme (20), the system (10) is GCFTS with respect to \( \alpha, \beta, T_f, \sigma(k) \) and the guaranteed cost value of the system (10) is given by
\[
J = \sum_{s=0}^{T_f-1} x^T(s)R_1 + u^T(s)R_2 \leq J^* = \xi^{T_f} \mu^{T_f} \phi_2
\]
(42)

**Proof.** Replacing \((A_i^T - C_i^T L_i^T)v_i + R_1 + C_i^T L_i^T u_i\) in (23) with \((A_i^T - C_i^T (L_i + \Delta L_i)^T v_i + R_1 + C_i^T (L_i + \Delta L_i)^T u_i\), similar to the proof of Theorem 1, we easily obtain that the resulting closed-loop system (10) is GCFTS.

Thus, the proof is completed.

**Theorem 3.** Consider the system (10), for a given time constant \( T_f \), vectors \( \alpha > \beta > 0, L_1, L_2 \) and \( R_1 > 0, R_2 > 0 \), if there exist a set of positive vectors \( v_i, v_j, u_i, u_j, i \neq j, j \in I \), positive matrices \( K_i, L_i \) and positive constants \( \phi_1, \phi_2, \xi > 1, \mu > 1 \), such that (12)-(15), (17)-(19) and the following inequalities hold:
\[
(A_i^T - C_i^T (L_i + L_1)^T v_i + R_1 + C_i^T (L_i + L_1)^T u_i - \xi v_i < 0)
\]
\[
A_i - (L_i + L_1) C_i > 0
\]
(43)
(44)
then under the ADT scheme (20), the system (10) is GCFTS with respect to \((\alpha, \beta, T_f, \sigma(k))\) and the guaranteed cost value of the system (10) is given by

\[
J = \sum_{s=0}^{T_f-1} \left( x^T(s)R_1 + u^T(s)R_2 \right) \leq J^* = \xi T_f \mu \alpha^T \phi_2
\]

(45)

**Proof.** If \((A_i^T - C_i^T(L_i + L_2)\varepsilon_i + R_1 + C_i^T(L_i + L_2)\varepsilon_i - \xi \varepsilon_i < 0\) and \(A_i - (L_i + L_2)C_i > 0\), we can easily get \((A_i^T - C_i^T(L_i + \triangle L)\varepsilon_i + R_1 + C_i^T(L_i + \triangle L)\varepsilon_i - \xi \varepsilon_i < 0\) and \(A_i - (L_i + \triangle L)C_i > 0\). Then, the conditions of Theorem 2 are satisfied. So the resulting closed-loop system (10) is GCFTS.

To obtain the feedback gain matrices \(K_i\) and \(L_i\) in Theorem 3, an algorithm is presented.

**Algorithm 2**

**Step 1.** By adjusting the parameters \(\xi, \mu\) and Letting \(f_p = K_p^T B_p^T v_p\), then solving (13)-(15), (17) and (19) via linear programming, positive vectors \(v_p, u_p, K_p\) and \(L_p\) can be obtained.

**Step 2.** Substituting \(u_p\) and \(K_p\) into \(f_p = K_p^T B_p^T u_p\), \(f_p\) can be obtained. The gain \(L_p, v_p\) and \(u_p\) are substituted into (43).

**Step 3.** If \(f_p - f_p \leq 0\) and (43) are all satisfied at the same time, then \(K_p\) and \(L_p\) are admissible. Otherwise, return to Step 1.

### 4 Numerical example

We present a numerical example to show the effectiveness of the proposed approach. Without loss of generality, we consider the case of nonfragile positive observer of DPISS (10) with the parameters as follows:

\[
A_1 = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.3 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.3 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0.2 & 0.3 \\ 0.3 & 0.2 \end{bmatrix}, \quad H_1 = \begin{bmatrix} 0.1 & 0.2 \\ 0.1 & 0.1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0.15 & 0.2 \end{bmatrix},
\]

\[
A_2 = \begin{bmatrix} 0.2 & 0.15 \\ 0.3 & 0.2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.1 & 0.2 \\ 0.2 & 0.1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0.3 & 0.4 \\ 0.2 & 0.5 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 0.2 & 0.2 \\ 0.1 & 0.4 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0.1 & 0.3 \end{bmatrix},
\]

\[
R_1 = \begin{bmatrix} 0.5 \\ 0.6 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}, \quad L_1 = \begin{bmatrix} 0.03 \\ 0.02 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 0.04 \\ 0.03 \end{bmatrix},
\]

\[
a = \begin{bmatrix} 0.2, 0.35, 0.1, 0.4 \end{bmatrix}^T, \quad \beta = \begin{bmatrix} 0.4, 0.1, 0.2, 0.1 \end{bmatrix}^T,
\]

Choosing \(T_f = 5, \xi = 1.1\) and \(\mu = 1.05\). Solving the inequalities in Theorem 3 by linear programming, we have

\[
v_1 = \begin{bmatrix} 0.1522 \\ 0.1594 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0.1522 \\ 0.1593 \end{bmatrix}, \quad u_1 = \begin{bmatrix} 0.0817 \\ 0.0869 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0.0815 \\ 0.0888 \end{bmatrix},
\]

\[
K_1 = \begin{bmatrix} 0.0381 & 0.1498 \\ 0.0052 & 0.1234 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0.0583 & 0.1505 \\ 0.0235 & 0.1244 \end{bmatrix}, \quad L_1 = \begin{bmatrix} 0.3829 \\ 0.4170 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 0.3953 \\ 0.4258 \end{bmatrix},
\]

\[
f_1 = \begin{bmatrix} 0.0194 \\ 0.0825 \end{bmatrix}, \quad f_2 = \begin{bmatrix} 0.0257 \\ 0.0929 \end{bmatrix}.
\]

It is easy to find that \(\tilde{f}_p - f_p \leq 0\) and (10) are satisfied. then \(K_p\) and \(L_p\) are admissible. According to (20), we get \(T_a = 1.7\).

The simulation results are shown in Figs. 1-5, where the initial conditions of the system (10) are \(\hat{x}(0) = \begin{bmatrix} 2, 2, 1, 1 \end{bmatrix}^T\), which meet the condition \(\hat{x}(0)\beta < 1\). The state trajectory of \(x_1\) and state observation trajectory \(\hat{x}_1\) are shown in Fig. 1. The state trajectory of \(x_2\) and state observation trajectory \(\hat{x}_2\) are depicted in Fig. 2. Fig. 3 plots the state of the error dynamic system. The switching signal \(\sigma(k)\) is depicted in Fig. 4. Fig. 5 plots the evolution of \(\hat{x}(t)\alpha\), which implies that the corresponding closed-loop system is GCFTS with respect to \((\alpha, \beta, T_f, \sigma(k))\) to \((\alpha, \beta, T_f, \sigma(k))\), and the cost value \(J^* = 2.33\), which can be obtained by (45).
Fig. 1. The state trajectory of $x_1$ and state observation trajectory $\hat{x}_1$.

Fig. 2. The state trajectory of $x_2$ and state observation trajectory $\hat{x}_2$. 
Fig. 3. State of the error dynamic system.

Fig. 4. Switching signal of system (5) with ADT.
5 Conclusions

In this paper, we have considered the issue of nonfragile observer-based guaranteed cost finite-time control for DPISS. Based on the ADT approach and co-positive type Lyapunov function technique, two types of guaranteed cost finite-time controller based on positive observer and nonfragile positive observer are designed, and sufficient conditions are obtained to guarantee the corresponding closed-loop systems are guaranteed cost finite-time stability (GCFTS), respectively. Such conditions can be solved by linear programming. Finally, a numerical example is given to illustrate the effectiveness of the proposed method.

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