Two-parameter Quantum Affine Algebra $U_{r,s}(\hat{\mathfrak{sl}}_n)$, Drinfel’d Realization and Quantum Affine Lyndon Basis

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Abstract: We further define two-parameter quantum affine algebra $U_{r,s}(\hat{\mathfrak{sl}}_n)$ ($n > 2$) after the work on the finite cases (see [BW1,BGH1,HS,BH]), which turns out to be a Drinfel’d double. Of importance for the quantum affine cases is that we can work out the compatible two-parameter version of the Drinfel’d realization as a quantum affinization of $U_{r,s}(\mathfrak{sl}_n)$ and establish the Drinfel’d Isomorphism Theorem in the two-parameter setting, via developing a new combinatorial approach (quantum calculation) to the quantum affine Lyndon basis we present (with an explicit valid algorithm based on the use of Drinfel’d generators).

1. Introduction

1.1. In 2001, Benkart-Witherspoon investigated the structures of two-parameter quantum groups $U_{r,s}(\mathfrak{g})$ for $\mathfrak{g} = \mathfrak{gl}_n$, or $\mathfrak{sl}_n$ in [BW1] originally obtained by Takeuchi [T], and the finite-dimensional weight representation theory in [BW2], and further obtained some new finite-dimensional pointed Hopf algebras in [BW3] when $rs^{-1}$ is a root of unity, which possess new ribbon elements under some conditions (and will yield new invariants of knots and links). These show that two-parameter quantum groups are well worth further study.

1.2. In 2004, Bergeron-Gao-Hu [BGH1] gave the structures of two-parameter quantum groups $U_{r,s}(\mathfrak{g})$ for $\mathfrak{g} = \mathfrak{so}_{2n+1}$, $\mathfrak{sp}_{2n}$, $\mathfrak{so}_{2n}$, and developed in [BGH2] the highest weight representation theory when $rs^{-1}$ is not a root of unity. Especially, [BGH1] explored the environment condition upon which Lusztig’s symmetries exist for the classical simple Lie algebras $\mathfrak{g}$, namely, they exist as $\mathbb{Q}$-isomorphisms between $U_{r,s}(\mathfrak{g})$ and the
associated object $U_{q^{-1},s^{-1}}(g)$ only when rank $(g) = 2$, and in the case when rank $(g) > 2$, the sufficient and necessary condition for the existence of Lusztig’s symmetries between $U_{r,s}(g)$ and its associated object forces $U_{r,s}(g)$ to take the “one-parameter” form $U_{q^r,q^s}(g)$, where $r = s^{-1} = q$. In other words, when rank $(g) > 2$, the Lusztig’s symmetries exist only for the one-parameter quantum groups $U_{q^r,q^s}(g)$ as $\mathbb{Q}(q)$-automorphisms (rather merely as $\mathbb{Q}$-isomorphisms). In this case, these symmetries give rise to, with respect to modulo some identification of group-like elements, the usual Lusztig symmetries on quantum groups $U_q(g)$ of Drinfel’d-Jimbo type. The Lusztig symmetry property indicates that there do exist remarkable differences between the two-parameter quantum groups in question and the one-parameter quantum groups of Drinfel’d-Jimbo type. Afterwards, Hu-Shi [HS] and Bai-Hu [BH] studied the two-parameter quantum groups for type $G_2$ and $E$ cases. Through these work, we found that the treatments in two-parameter cases are frequently more subtle to follow combinatorial approaches only, for instance, the description of the convex PBW-type basis (cf. [BH]) has to appeal to the use of Lyndon words (see [R2] and references therein) because there is no braid group available in question.

Thereby so far, it seems desirable to extend these kind of two-parameter quantum groups in Benkart-Witherspoon’s sense in finite cases to the affine cases. The present paper is aimed at this purpose for the affine type $A^{(1)}_n$ ($n > 1$) case. To this end, we first give the defining structure of $U_{r,s}(\widehat{sl}_n)$ ($n > 2$) whereas $U_{r,s}(\widehat{sl}_2)$ is essentially isomorphic to $U_{q^r,q^s}(\widehat{sl}_2)$ if set $rs^{-1} = q^2$, which is not considered in the paper).

1.3. As is well-known, the importance of the Drinfel’d generators (in the Drinfel’d realization) for quantum affine algebras is just like that of the loop generators (in the loop realization) for affine Kac-Moody algebras (see [Ga,K]). Early in 1987, Drinfel’d [Dr2] put forward his famous new (conjectural) realization of quantum affine algebras $U_q(\hat{g})$ with $g$ semisimple, because he recognized that the study of finite dimensional representations of $U_q(\hat{g})$ is made easier by the use of this realization on the set of Drinfel’d generators, which is called the Drinfel’d realization of $U_q(\hat{g})$ or the Drinfel’d quantum affinization of $U_q(g)$. Besides this, the Drinfel’d realization also finds its main contribution to the construction of vertex representations for quantum affine algebras $U_q(\hat{g})$ (see [FJ,J1,D2], etc.), as does the loop realization in the vertex representation theory of affine Kac-Moody algebras (see [K]). In 1993, Khoroshkin-Tolstoy [KT] constructed the Drinfel’d realization for the untwisted types using a Cartan-Weyl generators system with no proof. The first perfect proof of the Drinfel’d isomorphism only for the untwisted types was given by Beck [B2] in 1994, making use of his extended Weyl generators, based on the work of Damiani [Da], Levendorskii-Soibelman-Stukopin [LSS] for the case $U_q(\widehat{sl}_2)$. In 1998, Jing [J2] basically adopted the inverse map suggested by Beck for the untwisted types (see the final remark in [B2, Sect. 4]) and gave a combinatorial proof for the Drinfel’d isomorphism for the untwisted types.

1.4. In order to further explore and enrich the structure and representation theory of the two-parameter quantum affine algebras later on, another main result of this paper is to give the Drinfel’d realization of $U_{r,s}(\widehat{sl}_n)$ ($n > 2$). Its definition depends on the self-compatible defining system (Definition 3.1), which in the two-parameter setting, varies dramatically in comparison with the one-parameter cases (see [Dr2], or [B2, Theorem 4.7]) and is nontrivial to match up here and there the whole relations together. Indeed, to invent the two-parameter version of Drinfel’d realization needs some insights, e.g., from the antisymmetric point of view via the $\mathbb{Q}$-algebra antiautomorphism $\tau$, based on
some information from the combinatorial description of the convex PBW-type basis via the Lyndon words (see [R2,BH], etc.), and also, the proof of the Drinfel’d isomorphism in our case depends completely on the combinatorial approach with specific techniques to design those defining relations in order to fit the compatibilities in the whole system. If the readers follow the details, they will find how our quantum calculations (somehow a bit tedious) work well and necessarily for exactly verifying the compatibilities of the defining system. The reason is that the method we expanded, to some extent, essentially follows an approach to a kind of description of the quantum “affine” Lyndon basis. Actually, we can construct explicitly all quantum real and imaginary root vectors using this method (see Lemmas 4.7 & 4.8, together with Definition 3.9).

1.5. The paper is organized as follows. We first give the structure of two-parameter quantum affine algebra \( U_{r,s}(\mathfrak{sl}_n) \) \((n > 2)\) as a Hopf algebra in Sect. 2. We prove that the two-parameter quantum affine algebra \( U_{r,s}(\mathfrak{sl}_n) \) is characterized as a Drinfel’d double \( D(\hat{B}, \hat{B}^\prime) \) of Hopf subalgebras \( \hat{B}, \hat{B}^\prime \) with respect to a skew-dual pairing. In Sect. 3, we explicitly describe the two-parameter Drinfel’d quantum affinization of \( U_{r,s}(\mathfrak{sl}_n) \) \((n > 2)\), that is, the Drinfel’d realization in the two-parameter case which is antisymmetric with respect to the \( \mathbb{Q} \)-algebra antiautomorphism \( \tau \). In the case when \( rs = 1 \), i.e., \( r = s^{-1} = q \), our result modulo some identification yields the usual Drinfel’d realization of a quantum affine algebra \( U_q(\mathfrak{sl}_n) \) of Drinfel’d-Jimbo type (see [Dr1, B2, DI1, J2], etc.). Since Beck’s extended braid group actions approach is invalid for our case, we combine the Lyndon words description ([R2]) with the quantum Lie bracket operation ([J2]) to develop a combinatorial trick in the quantum affine case (we call it quantum calculations), which can be utilized in the construction of all the quantum root vectors (including real and imaginary ones), so that we can formulate and prove the quantum “affine” Lyndon basis for the first time (in a more explicit form than that of [B1]) for \( U_{r,s}(\hat{\mathfrak{sl}}_n) \) based on the Drinfel’d realization in Sect. 3, and further prove the Drinfel’d isomorphism using our combinatorial algorithm in Sect. 4. In fact, our proof also provides a concrete process of how to construct the Drinfel’d generators using the Chevalley-Kac-Lusztig generators.

2. Quantum Affine Algebra \( U_{r,s}(\hat{\mathfrak{sl}}_n) \) and Drinfel’d Double

2.1. Let \( \mathbb{K} = \mathbb{Q}(r, s) \) denote a field of rational functions with two-parameters \( r, s \) \((r \neq \pm s)\). Assume \( \Phi \) is a finite root system of type \( A_{n-1} \) with \( \Pi \) a base of simple roots. Regard \( \Phi \) as a subset of a Euclidean space \( E = \mathbb{R}^n \) with an inner product \((\cdot, \cdot)\). Set \( I = \{1, \ldots, n-1\}, I_0 = \{0\} \cup I \). Let \( \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \) denote an orthonormal basis of \( E \), then we can take \( \Pi = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid i \in I\} \) and \( \Phi = \{\varepsilon_i - \varepsilon_j \mid i \neq j \in I\} \). Let \( \delta \) denote the primitive imaginary root of \( \mathfrak{sl}_n \). Take \( \alpha_0 \equiv \delta - (\varepsilon_1 - \varepsilon_n) \), then \( \Pi' = \{\alpha_i \mid i \in I_0\} \) is a base of simple roots of affine Lie algebra \( \mathfrak{sl}_n \).

Let \( A = (a_{ij}) \) \((i, j \in I_0)\) be a generalized Cartan matrix associated to affine Lie algebra \( \mathfrak{sl}_n \). Let \( \mathfrak{h} \) be a vector space over \( \mathbb{K} \) with a basis \( \{h_0, h_1, \ldots, h_{n-1}, d\} \) and define the linear action of \( \alpha_i \) \((i \in I_0)\) on \( \mathfrak{h} \) by

\[
\alpha_i(h_j) = a_{ij}, \quad \alpha_i(d) = \delta_{i,0}, \quad \text{for } j \in I_0.
\]

Let \( Q = \mathbb{Z}\alpha_0 + \cdots + \mathbb{Z}\alpha_{n-1} \) denote the root lattice of \( \hat{\mathfrak{sl}}_n \). The standard nondegenerate symmetric bilinear form \((\cdot, \cdot)\) on \( \mathfrak{h}^* \) satisfies

\[
(a_i, a_j) = a_{ij}, \quad (\delta, a_i) = (\delta, \delta) = 0, \quad \forall i, j \in I_0.
\]
Definition 2.1. Let \( U = U_{r,s}(\mathfrak{sl}_n) \) (\( n > 2 \)) be the unital associative algebra over \( \mathbb{K} \) generated by the elements \( e_j, f_j, \omega_j^\pm, \omega_j' \) (\( j \in I_0 \)), \( \gamma^\pm, \gamma'^\pm, D^\pm, D'^\pm \) (called the Chevalley-Kac-Lusztig generators), satisfying the following relations:

(A1) \( \gamma^\pm, \gamma'^\pm \) are central with \( \gamma = \omega_0, \gamma' = \omega_0' \), \( \gamma \gamma' = rs \), such that \( \omega_i \omega_i^{-1} = \omega_i' \omega_i'^{-1} = 1 = DD^{-1} = D'D'^{-1} \), and

\[
\begin{align*}
[\omega_i^\pm, \omega_j^\pm] &= [\omega_i^\pm, D^\pm] = [\omega_i'^\pm, D'^\pm] = 0 \\
&= [\omega_i^\pm, \omega_j'^\pm] = [D'^\pm, D^\pm] = [\omega_i'^\pm, \omega_j^\pm].
\end{align*}
\]

(A2) For \( i \in I_0 \) and \( j \in I \),

\[
\begin{align*}
D e_i D^{-1} &= r^{\delta_{0i}} e_i, \\
\omega_j e_i \omega_j^{-1} &= r^{(\varepsilon_j, \alpha_i)} s^{(\varepsilon_{j+1}, \alpha_i)} e_i, \\
\omega_0 e_i \omega_0^{-1} &= r^{-(\varepsilon_{i+1}, \alpha_0)} s^{(\varepsilon_i, \alpha_i)} e_i, \\
D f_i D^{-1} &= r^{-\delta_{0i}} f_i, \\
\omega_j f_i \omega_j^{-1} &= r^{-(\varepsilon_j, \alpha_i)} s^{-(\varepsilon_{j+1}, \alpha_i)} f_i, \\
\omega_0 f_i \omega_0^{-1} &= r^{(\varepsilon_{i+1}, \alpha_0)} s^{-(\varepsilon_i, \alpha_i)} f_i.
\end{align*}
\]

(A3) For \( i \in I_0 \) and \( j \in I \),

\[
\begin{align*}
D' e_i D'^{-1} &= s^{\delta_{0i}} e_i, \\
\omega_j' e_i \omega_j'^{-1} &= s^{(\varepsilon_j, \alpha_i)} r^{(\varepsilon_{j+1}, \alpha_i)} e_i, \\
\omega_0' e_i \omega_0'^{-1} &= s^{-(\varepsilon_{i+1}, \alpha_0)} r^{(\varepsilon_i, \alpha_i)} e_i, \\
D' f_i D'^{-1} &= s^{-\delta_{0i}} f_i, \\
\omega_j' f_i \omega_j'^{-1} &= s^{-(\varepsilon_j, \alpha_i)} r^{-(\varepsilon_{j+1}, \alpha_i)} f_i, \\
\omega_0' f_i \omega_0'^{-1} &= s^{(\varepsilon_{i+1}, \alpha_0)} r^{-(\varepsilon_i, \alpha_i)} f_i.
\end{align*}
\]

(A4) For \( i, j \in I_0 \), we have

\[
[e_i, f_j] = \frac{\delta_{ij}}{r - s} (\omega_i - \omega_i').
\]

(A5) For \( i, j \in I_0 \), but \( (i, j) \notin \{ (0, n - 1), (n - 1, 0) \} \) with \( a_{ij} = 0 \), we have

\[
[e_i, e_j] = 0 = [f_i, f_j].
\]

(A6) For \( i \in I_0 \), we have the \((r, s)\)-Serre relations:

\[
\begin{align*}
& e_i^2 e_{i+1} - (r + s) e_i e_{i+1} e_i + (rs) e_{i+1} e_i^2 = 0, \\
& e_i e_{i+1}^2 - (r + s) e_{i+1} e_i e_{i+1} + (rs) e_i^2 e_{i+1} = 0, \\
& e_{n-1}^2 e_0 - (r + s) e_{n-1} e_0 e_{n-1} + (rs) e_0^2 e_{n-1} = 0, \\
& e_{n-1} e_0^2 - (r + s) e_0 e_{n-1} e_0 + (rs) e_0^2 e_{n-1} = 0.
\end{align*}
\]

(A7) For \( i \in I_0 \), we have the \((r, s)\)-Serre relations:

\[
\begin{align*}
& f_i^2 f_{i+1} - (r^{-1} + s^{-1}) f_i f_{i+1} f_i + (r^{-1} s^{-1}) f_{i+1} f_i^2 = 0, \\
& f_i f_{i+1}^2 - (r^{-1} + s^{-1}) f_{i+1} f_i f_{i+1} + (r^{-1} s^{-1}) f_i^2 f_{i+1} = 0, \\
& f_{n-1}^2 f_0 - (r^{-1} + s^{-1}) f_{n-1} f_0 f_{n-1} + (r^{-1} s^{-1}) f_0 f_{n-1}^2 = 0, \\
& f_{n-1} f_0^2 - (r^{-1} + s^{-1}) f_0 f_{n-1} f_0 + (r^{-1} s^{-1}) f_0^2 f_{n-1} = 0.
\end{align*}
\]
$U_{r,s}(\widehat{sl}_n)$ is a Hopf algebra with the coproduct $\Delta$, the counit $\varepsilon$ and the antipode $S$ defined below: for $i \in I_0$, we have

$$
\Delta(y^{\pm\frac{1}{2}}) = y^{\pm\frac{1}{2}} \otimes y^{\pm\frac{1}{2}}, \quad \Delta(y'^{\pm\frac{1}{2}}) = y'^{\pm\frac{1}{2}} \otimes y'^{\pm\frac{1}{2}},
$$

$$
\Delta(D^{\pm1}) = D^{\pm1} \otimes D^{\pm1}, \quad \Delta(D'^{\pm1}) = D'^{\pm1} \otimes D'^{\pm1},
$$

$$
\Delta(w_i) = w_i \otimes w_i, \quad \Delta(w'_i) = w'_i \otimes w'_i,
$$

$$
\Delta(e_i) = e_i \otimes 1 + w_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes w_i + 1 \otimes f_i,
$$

$$
\varepsilon(e_i) = \varepsilon(f_i) = 0, \quad \varepsilon(y^{\pm\frac{1}{2}}) = \varepsilon(y'^{\pm\frac{1}{2}}) = \varepsilon(D^{\pm1}) = \varepsilon(D'^{\pm1}) = \varepsilon(w_i) = \varepsilon(w'_i) = 1,
$$

$$
S(y^{\pm\frac{1}{2}}) = y^{\mp\frac{1}{2}}, \quad S(y'^{\pm\frac{1}{2}}) = y'^{\mp\frac{1}{2}}, \quad S(D^{\pm1}) = D^{\mp1}, \quad S(D'^{\pm1}) = D'^{\mp1},
$$

$$
S(e_i) = -w_i^{-1}e_i, \quad S(f_i) = -f_i w_i^{-1}, \quad S(w_i) = w_i^{-1}, \quad S(w'_i) = w'_i^{-1}.
$$

2.2. In what follows, we give the skew-pairing and the Drinfel’d double structure.

**Definition 2.2.** A bilinear form $\langle \cdot, \cdot \rangle : \mathfrak{B} \times \mathfrak{A} \to \mathbb{K}$ is called a skew-dual pairing of two Hopf algebras $\mathfrak{A}$ and $\mathfrak{B}$ (see [KS, 8.2.1]), if it satisfies

$$
\langle b, 1_{\mathfrak{A}} \rangle = \varepsilon_{\mathfrak{B}}(b), \quad \langle 1_{\mathfrak{B}}, a \rangle = \varepsilon_{\mathfrak{A}}(a),
$$

$$
\langle b, a_1a_2 \rangle = \langle \Delta^{op}_{\mathfrak{B}}(b), a_1 \otimes a_2 \rangle, \quad \langle b_1b_2, a \rangle = \langle b_1 \otimes b_2, \Delta_{\mathfrak{A}}(a) \rangle,
$$

for all $a, a_1, a_2 \in \mathfrak{A}$ and $b, b_1, b_2 \in \mathfrak{B}$, where $\varepsilon_{\mathfrak{A}}, \varepsilon_{\mathfrak{B}}$ denote the counits of $\mathfrak{A}, \mathfrak{B}$, respectively, and $\Delta_{\mathfrak{A}}, \Delta_{\mathfrak{B}}$ are the respective coproducts.

**Definition 2.3.** For any two Hopf algebras $\mathfrak{A}$ and $\mathfrak{B}$ skew-paired by $\langle \cdot, \cdot \rangle$, there exists a Drinfel’d quantum double $\mathcal{D}(\mathfrak{A}, \mathfrak{B})$ which is a Hopf algebra whose underlying coalgebra is $\mathfrak{A} \otimes \mathfrak{B}$ with the tensor product coalgebra structure, whose algebra structure is defined by

$$
(a \otimes b)(a' \otimes b') = \sum \langle S_{\mathfrak{B}}(b_{(1)}), a'_{(1)} \rangle \langle b_{(2)}, a'_{(3)} \rangle a a'_{(2)} \otimes b_{(2)} b',
$$

for $a, a' \in \mathfrak{A}$ and $b, b' \in \mathfrak{B}$, and whose antipode $S$ is given by

$$
S(a \otimes b) = (1 \otimes S_{\mathfrak{B}}(b))(S_{\mathfrak{A}}(a) \otimes 1).
$$

Let $\widehat{B}$ (resp. $\widehat{B}'$) denote the Hopf (Borel-type) subalgebra of $U_{r,s}(\widehat{sl}_n)$ generated by $e_j, \omega_j^{\pm1}, y^{\pm\frac{1}{2}}, D^{\pm1}$ (resp. $f_j, \omega_j'^{\pm1}, y'^{\pm\frac{1}{2}}, D'^{\pm1}$) with $j \in I_0$.

**Proposition 2.4.** There exists a unique skew-dual pairing $\langle \cdot, \cdot \rangle : \widehat{B}' \times \widehat{B} \to \mathbb{K}$ of the Hopf subalgebras $\widehat{B}$ and $\widehat{B}'$ such that:

1. 

$$
\langle f_i, e_j \rangle = \delta_{ij} \frac{1}{S - r}, \quad (i, j \in I_0),
$$

2. 

$$
\langle \omega'_i, \omega_j \rangle = \begin{cases} 
    r^{(e_j, \alpha_i)} s^{(e_j + 1, \alpha_i)}, & (i \in I_0, j \in I) \\
    r^{-(e_j + 1, \alpha_0)} s^{(e_1, \alpha_i)}, & (i \in I_0, j = 0), 
\end{cases}
$$
that the relations in among the

\[ B \]

The uniqueness assertion is clear, as any skew-dual pairing of bialgebras is determined by the values on the generators. We proceed to prove the existence of the pairing.

The pairing defined on generators as (1)—(8) may be extended to a bilinear form on \( \hat{B} \times \hat{B} \) in a way such that the defining properties in Definition 2.2 hold. We will verify that the relations in \( \hat{B} \) and \( \hat{B}' \) are preserved, ensuring that the form is well-defined and is a skew-dual pairing of \( \hat{B} \) and \( \hat{B}' \).

First, it is straightforward to check that the bilinear form preserves all the relations among the \( \omega_i^{\pm 1} \), \( \gamma_i^{\pm \frac{1}{2}} \), \( D_i^{\pm 1} \) in \( \hat{B} \) and the \( \omega_i^{\pm 1} \), \( \gamma_i^{\pm \frac{1}{2}} \), \( D_i^{\pm 1} \) in \( \hat{B}' \). Next, we observe that the identities hold: for \( i, j \in I \),

\[
(e_j, \alpha_i) = -(e_{i+1}, \alpha_j), \quad (e_j, \alpha_0) = -(e_1, \alpha_j),
\]

which ensure the compatibility of the form defined above with the relations of (A2) and (A3) in \( \hat{B} \) or \( \hat{B}' \) respectively. This fact is easily checked by definition (see (1)—(8)). So we are left to verify that the form preserves the \( (r, s) \)-Serre relations in \( \hat{B} \) and \( \hat{B}' \).

For \( 1 \leq i < n \), \( (r, s) \)-Serre relations in \( \hat{B} \) and \( \hat{B}' \) have been checked in [BW1]. Here we need only to verify the relations involving index \( i = 0 \) in \( \hat{B} \) and \( \hat{B}' \). It suffices to consider the following case (the remaining case is similar)

\[
\langle X, e_0^2e_{n-1} - (r^{-1} + s^{-1})e_0e_{n-1}e_0 + (rs)^{-1}e_{n-1}e_0^2 \rangle,
\]

where \( X \) is any word in the generators of \( \hat{B}' \). By definition, this equals

\[
\langle \Delta^{(2)}(X), e_0 \otimes e_0 \otimes e_{n-1} - (r^{-1} + s^{-1})e_0 \otimes e_{n-1} \otimes e_0 + (rs)^{-1}e_{n-1} \otimes e_0 \otimes e_0 \rangle,
\]

Proof. The uniqueness assertion is clear, as any skew-dual pairing of bialgebras is determined by the values on the generators. We proceed to prove the existence of the pairing.

The pairing defined on generators as (1)—(8) may be extended to a bilinear form on \( \hat{B} \times \hat{B} \) in a way such that the defining properties in Definition 2.2 hold. We will verify that the relations in \( \hat{B} \) and \( \hat{B}' \) are preserved, ensuring that the form is well-defined and is a skew-dual pairing of \( \hat{B} \) and \( \hat{B}' \).

First, it is straightforward to check that the bilinear form preserves all the relations among the \( \omega_i^{\pm 1} \), \( \gamma_i^{\pm \frac{1}{2}} \), \( D_i^{\pm 1} \) in \( \hat{B} \) and the \( \omega_i^{\pm 1} \), \( \gamma_i^{\pm \frac{1}{2}} \), \( D_i^{\pm 1} \) in \( \hat{B}' \). Next, we observe that the identities hold: for \( i, j \in I \),

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For \( 1 \leq i < n \), \( (r, s) \)-Serre relations in \( \hat{B} \) and \( \hat{B}' \) have been checked in [BW1]. Here we need only to verify the relations involving index \( i = 0 \) in \( \hat{B} \) and \( \hat{B}' \). It suffices to consider the following case (the remaining case is similar)

\[
\langle X, e_0^2e_{n-1} - (r^{-1} + s^{-1})e_0e_{n-1}e_0 + (rs)^{-1}e_{n-1}e_0^2 \rangle,
\]

where \( X \) is any word in the generators of \( \hat{B}' \). By definition, this equals

\[
\langle \Delta^{(2)}(X), e_0 \otimes e_0 \otimes e_{n-1} - (r^{-1} + s^{-1})e_0 \otimes e_{n-1} \otimes e_0 + (rs)^{-1}e_{n-1} \otimes e_0 \otimes e_0 \rangle,
\]
where $\Delta$ stands for $\Delta^{op}_{D^r}$. In order for any one of these terms to be nonzero, $X$ must involve exactly two $f_0$ factors, one $f_{n-1}$ factor, and arbitrarily many $\omega_j^{\pm 1}$ ($j \in I_0$), $y^{\prime \pm \frac{1}{2}}$, or $D^{r \pm 1}$ factors. For simplicity, we first consider three key cases:

(i) If $X = f_0^2 f_{n-1}$, then $\Delta^{(2)}(X)$ is equal to

$$
(\omega'_0 \otimes \omega'_0 \otimes f_0 + \omega'_0 \otimes f_0 \otimes 1 + f_0 \otimes 1 \otimes 1)^2 (\omega'_{n-1} \otimes \omega'_{n-1} \otimes f_{n-1}) + \omega'_{n-1} \otimes f_{n-1} \otimes 1 + f_{n-1} \otimes 1 \otimes 1).
$$

The relevant terms of $\Delta^{(2)}(X)$ are

$$
\begin{align*}
&f_0 \omega'_0 \omega'_{n-1} \otimes f_0 \omega'_{n-1} \otimes f_{n-1} + \omega'_0 f_0 \omega'_{n-1} \otimes f_0 \omega'_{n-1} \otimes f_{n-1} \\
&+ f_0 \omega'_0 \omega'_{n-1} \otimes f_0 \omega'_{n-1} \otimes f_0 + \omega'_0 f_0 \omega'_{n-1} \otimes f_0 \omega'_{n-1} \otimes f_0 \\
&+ \omega'_0 f_{n-1} \otimes f_0 \omega'_{n-1} \otimes f_0 + \omega'_0 f_{n-1} \otimes f_0 \omega'_{n-1} \otimes f_0.
\end{align*}
$$

Therefore, (2.2) becomes

$$
\begin{align*}
\langle f_0 \omega'_0 \omega'_{n-1}, e_0 \rangle \langle f_0 \omega'_{n-1}, e_0 \rangle \langle f_{n-1}, e_{n-1} \rangle \\
+ \langle f_0 \omega'_0 \omega'_{n-1}, e_0 \rangle \langle f_0 \omega'_{n-1}, e_0 \rangle \langle f_{n-1}, e_{n-1} \rangle \\
= \frac{1}{s-r} \left\{ 1 + \langle \omega'_0, \omega_0 \rangle - (r^{-1} s^{-1}) \langle \omega'_0, \omega_{n-1} \rangle \right\} \\
+ \langle \omega'_0, \omega_{n-1} \rangle \langle \omega'_0, \omega_{n-1} \rangle \\
= \frac{1}{s-r} \left\{ 1 + r s^{-1} - (r^{-1} s^{-1}) \right\} = 0.
\end{align*}
$$

(ii) When $X = f_0 f_{n-1} f_0$, it is easy to get the relevant terms of $\Delta^{(2)}(X)$:

$$
\begin{align*}
\omega'_0 \omega'_{n-1} f_0 \otimes f_0 \omega'_{n-1} \otimes f_{n-1} + f_0 \omega'_{n-1} \omega'_0 \otimes \omega'_{n-1} f_0 \otimes f_{n-1} \\
+ \omega'_0 \omega'_{n-1} f_0 \otimes f_0 \omega'_{n-1} \otimes f_0 + f_0 \omega'_{n-1} \omega'_0 \otimes f_0 \omega'_{n-1} \otimes f_0 \\
+ \omega'_0 f_{n-1} \omega'_0 \otimes f_0 \omega'_0 \otimes f_0 + \omega'_0 f_{n-1} \omega'_0 \otimes f_0 \omega'_0 \otimes f_0.
\end{align*}
$$

Thus, (2.2) becomes

$$
\begin{align*}
\frac{1}{(s-r)^3} \left\{ \langle \omega'_0, \omega_0 \rangle \langle \omega'_0, \omega_{n-1} \rangle + \langle \omega'_0, \omega_0 \rangle \\
- (r^{-1} s^{-1}) \langle \omega'_0, \omega_0 \rangle \langle \omega'_0, \omega_{n-1} \rangle + 1 \right\} \\
+ (r s)^{-1} \left\{ \langle \omega'_0, \omega_{n-1} \rangle \langle \omega'_0, \omega_0 \rangle + \langle \omega'_0, \omega_{n-1} \rangle \right\} \\
= \frac{1}{(s-r)^3} \left\{ (r s^{-1} - (r^{-1} s^{-1}) (r s^{-1} - (r^{-1} s^{-1} s) + (r s)^{-1} (s^2 + s r s^{-1}) \right\} \\
= 0.
\end{align*}
$$
(iii) If \( X = f_{n-1}f_0^2 \), one can similarly get that (2.2) vanishes.

Finally, if \( X \) is any word involving exactly two \( f_0 \) factors, one \( f_{n-1} \) factor, and arbitrarily many factors \( \omega_j^{\pm 1} (j \in I_0), \gamma_j^{\pm \frac{1}{2}} \) and \( D_j^{\pm 1} \), then (2.2) will just be a scalar multiple of one of the quantities we have already calculated, and then will be 0.

Analogous calculations show that the relations in \( \hat{B}' \) are preserved. \( \square \)

**Theorem 2.5.** \( \mathcal{D}(\hat{B}, \hat{B}') \) is isomorphic to \( U_{r,s}(\hat{\mathfrak{sl}}_n) \) as Hopf algebras.

**Proof.** We denote the image \( e_i \otimes 1 \) of \( e_i \) in \( \mathcal{D}(\hat{B}, \hat{B}') \) by \( \hat{e}_i \) and similarly for \( \omega_i^{\pm 1}, \gamma_i^{\pm \frac{1}{2}}, D_i^{\pm 1} \), denote the image \( 1 \otimes f_i \) of \( f_i \) in \( \mathcal{D}(\hat{B}, \hat{B}') \) by \( \hat{f}_i \) and similarly for \( \omega_i^{\prime \pm 1}, \gamma_i^{\prime \pm \frac{1}{2}}, D_i^{\prime \pm 1} \).

Define a map \( \varphi : \mathcal{D}(\hat{B}, \hat{B}') \rightarrow U_{r,s}(\hat{\mathfrak{sl}}_n) \) by

\[
\varphi(\hat{e}_i) = e_i, \quad \varphi(\hat{f}_i) = f_i, \quad \varphi(\omega_i^{\pm 1}) = \omega_i^{\pm 1}, \quad \varphi(\gamma_i^{\pm \frac{1}{2}}) = \gamma_i^{\pm \frac{1}{2}}, \quad \varphi(D_i^{\pm 1}) = D_i^{\pm 1}.
\]

The remaining argument is analogous to that of [BGH1, Theorem 2.5]. \( \square \)

**Remark 2.6.** (1) Up to now, we have completely solved the compatibility problem on the defining relations of our two-parameter quantum affine algebra \( U_{r,s}(\hat{\mathfrak{sl}}_n), n > 2 \).

This is done in two steps: the proof of Theorem 2.5 indicates that the cross relations between \( \hat{B} \) and \( \hat{B}' \) are half of the relations (A1)—(A4), and the proof of Proposition 2.4 shows the remaining relations, including the remaining half of relations (A1)—(A4) and the (r, s)-Serre relations (A5)—(A7).

(2) When \( r = s^{-1} = q \), the Hopf algebra \( U_{q,q^{-1}}(\hat{\mathfrak{sl}}_n) \) modulo the Hopf ideal generated by the set \{ \( \omega_i' - \omega_i^{\pm 1} (i \in I_0), \gamma_i^{\prime \pm \frac{1}{2}} - \gamma_i^{\pm \frac{1}{2}}, D_i^{\prime \pm 1} \) \} is the usual quantum affine algebra \( U_q(\mathfrak{sl}_n) \) of Drinfeld-Jimbo type.

Let \( U^0 = K[\omega_0^{\pm 1}, \ldots, \omega_n^{\pm 1}, r_0^{\pm 1}, \ldots, r_n^{\pm 1}] \), \( U_0 = K[\omega_0^{\pm 1}, \ldots, \omega_n^{\pm 1}] \), and \( U_0' = K[\omega_0^{\pm 1}, \ldots, \omega_n^{\pm 1}] \) denote the Laurent polynomial subalgebras of \( U_{r,s}(\hat{\mathfrak{sl}}_n), \hat{B} \), and \( \hat{B}' \) respectively. Clearly, \( U^0 = U_0 U_0' = U_0' U_0 \). Furthermore, let us denote by \( U_{r,s}(\hat{\mathfrak{n}}^-) \) (resp. \( U_{r,s}(\hat{\mathfrak{n}}^-) \)) the subalgebra of \( \hat{B} \) (resp. \( \hat{B}' \)) generated by \( e_i \) (resp. \( f_i \)) for all \( i \in I_0 \). Thus, by definition, we have \( \hat{B} = U_{r,s}(\hat{\mathfrak{n}}^-) \times U_0 \) and \( \hat{B}' = U_0' \times U_{r,s}(\hat{\mathfrak{n}}^-) \), so that the double \( \mathcal{D}(\hat{B}, \hat{B}') \cong U_{r,s}(\hat{\mathfrak{n}}^-) \otimes U^0 \otimes U_{r,s}(\hat{\mathfrak{n}}^-) \) as vector spaces. On the other hand, if we consider \( (\cdot \cdot)^- : \hat{B}' \otimes \hat{B} \rightarrow K \) by \( (b', b)^- : = (S(b'), b) \), the convolution inverse of the skew-dual pairing \( (\cdot \cdot) \) in Proposition 2.4, the composition with the flip mapping \( \sigma \) then gives rise to a new skew-dual pairing \( (\cdot \cdot) := (\cdot \cdot)^{-1} \circ \sigma : \hat{B} \times \hat{B} \rightarrow K \), given by \( (b | b') = (S(b'), b) \). As a byproduct of Theorem 2.5, similar to [BGH1, Cor. 2.6], we get the standard triangular decomposition of \( U_{r,s}(\hat{\mathfrak{sl}}_n) \).

**Corollary 2.7.** \( U_{r,s}(\hat{\mathfrak{sl}}_n) \cong U_{r,s}(\hat{\mathfrak{n}}^-) \otimes U^0 \otimes U_{r,s}(\hat{\mathfrak{n}}^-) \), as vector spaces. \( \square \)

**Corollary 2.8.** For any \( \zeta = \sum_{i=0}^n \zeta_i \omega_i \in Q \) (the root lattice of \( \hat{\mathfrak{sl}}_n \)), the defining relations (A2) and (A3) in \( U_{r,s}(\hat{\mathfrak{sl}}_n) \) take the form:

\[
\omega_\zeta \ e_i \omega_\zeta^{-1} = \langle \omega_i', \omega_\zeta \rangle e_i, \quad \omega_\zeta \ f_i \omega_\zeta^{-1} = \langle \omega_i', \omega_\zeta \rangle^{-1} f_i,
\]
\[
\omega_i' \ e_i \omega_i'^{-1} = \langle \omega_i', \omega_i \rangle^{-1} e_i, \quad \omega_i' \ f_i \omega_i'^{-1} = \langle \omega_i', \omega_i \rangle f_i.
\]
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$U_{r,s}(\widehat{n}^\pm) = \bigoplus_{\eta \in Q^+} U^\eta_{r,s}(\widehat{n}^\pm)$ is then $Q^\pm$-graded with

$U^\eta_{r,s}(\widehat{n}^\pm) = \left\{ a \in U_{r,s}(\widehat{n}^\pm) \mid \omega_\xi a \omega_\xi^{-1} = (\omega'_{\eta'}, \omega_\xi) a, \omega'_\xi a \omega'_\xi^{-1} = (\omega'_\xi, \omega_\eta)^{-1} a \right\}$.

for $\eta \in Q^+ \cup Q^-$. Furthermore, $U = \bigoplus_{\eta \in Q} U^\eta_{r,s}(\widehat{n})$ is $Q$-graded with

$U^\eta_{r,s}(\widehat{n}) = \left\{ \sum F_\alpha \omega'_\mu \omega_\nu E_\beta \in U \mid \omega_\xi (F_\alpha \omega'_\mu \omega_\nu E_\beta) \omega_\xi^{-1} = (\omega'_{\beta-\alpha}, \omega_\xi) F_\alpha \omega'_\mu \omega_\nu E_\beta, \right.$

$
\omega'_\xi (F_\alpha \omega'_\mu \omega_\nu E_\beta) \omega'_\xi^{-1} = (\omega'_\xi, \omega_\beta-\alpha)^{-1} F_\alpha \omega'_\mu \omega_\nu E_\beta, \text{ with } \beta - \alpha = \eta \left\}$,

where $F_\alpha$ (resp. $E_\beta$) runs over monomials $f_1 \cdots f_i$ (resp. $e_j \cdots e_{j_m}$) such that $\alpha_1 + \cdots + \alpha_i = \alpha$ (resp. $\alpha_j + \cdots + \alpha_{j_m} = \beta$).

Definition 2.9. Let $\tau$ be the $Q$-algebra anti-automorphism of $U_{r,s}(\widehat{n})$ such that $\tau(r) = s, \tau(s) = r, (\langle \omega'_j, \omega_i \rangle \pm 1) = \langle \omega'_j, \omega_i \rangle \mp 1$, and

$\tau(e_i) = f_i, \tau(f_i) = e_i, \tau(\omega_i) = \omega'_j, \tau(\omega'_j) = \omega_i, \tau(\gamma) = \gamma', \tau(D) = D', \tau(D') = D$.

Then $\widehat{B}' = \tau(\widehat{B})$ with those induced defining relations from $\widehat{B}$, and those cross relations in (A2)—(A4) are antisymmetric with respect to $\tau$.

3. Drinfel’d Realization of $U_{r,s}(\widehat{n})$ and Quantum Affine Lyndon Basis

3.1. For the two-parameter quantum affine algebra $U_{r,s}(\widehat{n})$ ($n > 2$) we defined in Sect. 2, we give the following definition of its Drinfel’d realization. In the two-parameter case, the defining relations (D2), (D6), (D7) and (D8) below appear to vary dramatically in comparison with the one-parameter cases (see (d2), (d6), (d7) and (d8) in Remark 3.3) where the compatibilities for the whole system are based on some intrinsic considerations as indicated in the sequel.

We briefly write $\langle i, j \rangle := \langle \omega'_j, \omega_i \rangle$.

Definition 3.1. Let $U_{r,s}(\widehat{n})$ ($n > 2$) be the unital associative algebra over $\mathbb{K}$ generated by the elements $x_i^\pm(k), a_i(\ell), \omega_i^{\pm 1}, \omega_i^{\pm 1}, \gamma^{\pm 1}, \gamma^{\pm 1}, D^{\pm 1}, D'^{\pm 1}$ ($i \in I, k, k' \in \mathbb{Z}, \ell, \ell' \in \mathbb{Z}\backslash\{0\}$), subject to the following defining relations:

(D1) $\gamma^{\pm 1}, \gamma^{\pm 1}$ are central with $\gamma \gamma' = rs, \omega_i \omega^{-1} = \omega'_j \omega'^{-1} = 1 = DD^{-1} = D'D'^{-1}(i, j \in I)$, and

$[ \omega_i^{\pm 1}, \omega_j^{\pm 1}] = [ \omega_i^{\pm 1}, D^{\pm 1}] = [ \omega_j^{\pm 1}, D^{\pm 1}] = [ \omega_i^{\pm 1}, D'^{\pm 1}] = 0$

$[ \omega_i^{\pm 1}, \omega_j^{\pm 1}] = [ \omega_j^{\pm 1}, D'^{\pm 1}] = [D'^{\pm 1}, D^{\pm 1}] = [ \omega_i^{\pm 1}, \omega_j^{\pm 1}]$.

(D2)

$[ a_i(\ell), a_j(\ell') ] = \delta_{\ell+\ell',0} \frac{(rs)^{\frac{|i|}{2}}(\langle i, i \rangle \frac{\ell_{ij}}{2} - \langle i, i \rangle \frac{\ell_{ij}}{2})}{|\ell|(r-s)} \cdot \frac{\gamma^{\ell} - \gamma'^{\ell}}{r-s}$. 
where $\omega$ with $\omega^\pm_0 = \{ (j, i)^\pm_0 x_j^\pm(k), \omega_i x_j^\pm(k) \omega_i^{-1} = (j, i)^\mp_1 x_j^\pm(k) \}$.

\[ [a_i(\ell), x_j^\pm_0] = [a_i(\ell), \omega^\pm_0] = 0. \]

\[ D x_i^\pm(k) D^{-1} = r^k x_i^\pm(k), \quad D' x_i^\pm(k) D'^{-1} = s^k x_i^\pm(k), \]
\[ D a_i(\ell) D^{-1} = r^\ell a_i(\ell), \quad D' a_i(\ell) D'^{-1} = s^\ell a_i(\ell). \]

\[ \omega_i x_j^\pm(k) \omega_i^{-1} = (j, i)^\pm_0 x_j^\pm(k), \quad \omega_i' x_j^\pm(k) \omega_i'^{-1} = (j, i)^\mp_1 x_j^\pm(k). \]

\[ [a_i(\ell), x_j^\pm(\ell)] = \left( \frac{r s}{\ell} \right)^{\frac{\epsilon a_{ij}}{r s}} (i, i) \frac{\epsilon a_{ij}}{r s} - (j, j) \frac{\epsilon a_{ij} - \epsilon a_{ij}}{r s} \gamma^\pm_\ell x_j^\pm(\ell + \ell_k), \quad \text{for } \ell < 0. \]

\[ [a_i(\ell), x_j^\pm(\ell)] = \left( \frac{r s}{\ell} \right)^{\frac{\epsilon a_{ij}}{r s}} (i, i) \frac{\epsilon a_{ij}}{r s} - (j, j) \frac{\epsilon a_{ij} - \epsilon a_{ij}}{r s} \gamma^\pm_\ell x_j^\pm(\ell + \ell_k), \quad \text{for } \ell > 0. \]

\[ x_i^\pm(k + 1) x_j^\pm(k') - (j, i)^\pm_0 x_j^\pm(k') x_i^\pm(k + 1) \]
\[ = - (j, i) (i, j) \frac{1}{(r - s)} \left( x_j^\pm(k' + 1) x_i^\pm(k) - (i, j)^\pm_0 x_i^\pm(k) x_j^\pm(k' + 1) \right). \]

\[ [x_i^\pm(k), x_j^\pm(k')] = \frac{\delta_{ij}}{r - s} \left( \gamma^{' - k} \gamma^{k k'} \omega_i(k + k') - \gamma^{' k'} \gamma^{' - k k'} \omega_i'(k + k') \right), \]

where $\omega_i(m), \omega'_i(-m)$ ($m \in \mathbb{Z}_{\geq 0}$) with $\omega_i(0) = \omega_i$ and $\omega'_i(0) = \omega'_i$ are defined by:

\[ \sum_{m=0}^\infty \omega_i(m) z^{-m} = \omega_i \exp \left( (r - s) \sum_{\ell=1}^\infty a_i(\ell) z^{-\ell} \right); \]
\[ \sum_{m=0}^\infty \omega'_i(-m) z^{m} = \omega'_i \exp \left( -(r - s) \sum_{\ell=1}^\infty a_i(-\ell) z^{\ell} \right), \]

with $\omega_i(-m) = 0$ and $\omega'_i(m) = 0$, $\forall m > 0$.

\[ x_i^\pm(m) x_j^\pm(k) = x_j^\pm(k) x_i^\pm(m), \quad \text{for } a_{ij} = 0, \]

\[ Sym_{m_1, m_2} \left( x_i^\pm(m_1) x_j^\pm(m_2) x_i^\pm(k) - (r^\pm_1 + s^\pm_1) x_i^\pm(m_1) x_j^\pm(k) x_i^\pm(m_2) \right) + (r s)^\pm_1 x_i^\pm(k) x_i^\pm(m_1) x_i^\pm(m_2) = 0, \quad \text{for } a_{ij} = -1, 1 \leq i < j < n, \]
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(D93)

\[ \text{Sym}_{m_1, m_2} \left( x_i^\pm(m_1)x_i^\pm(m_2)x_j^\pm(k) - (r^{\pm 1} + s^{\mp 1}) x_i^\pm(m_1)x_j^\pm(k)x_i^\pm(m_2) \right) + (rs)^{\mp 1} x_j^\pm(k)x_i^\pm(m_1)x_i^\pm(m_2) = 0, \text{ for } a_{ij} = -1, 1 \leq j < i < n, \]

Sym denotes symmetrization with respect to the indices \((m_1, m_2)\).

As one of crucial observations of the compatibilities of the defining system above, we have

**Proposition 3.2.** There exists the \(\mathbb{Q}\)-algebra antiautomorphism \(\tau\) of \(U_{r,s}(\widehat{sl}_n)\) \((n > 2)\) such that \(\tau(r) = s, \tau(s) = r, \tau((\omega_i^+, \omega_j)_{\pm 1}) = (\omega_j^+, \omega_i)_{\mp 1}\) and

\[
\begin{align*}
\tau(\omega_i) &= \omega_i^+,& \tau(\omega_i^-) &= \omega_i^-,& \\
\tau(\gamma) &= \gamma^+,& \tau(\gamma') &= \gamma^-,& \\
\tau(D) &= D',& \tau(D') &= D, & \\
\tau(a_i(\ell)) &= a_i(-\ell),& \\
\tau(x_i^+(m)) &= x_i^-(m),& \\
\tau(\omega_i(m)) &= \omega_i^-(m),& \tau(\omega_i^-(m)) &= \omega_i(m),
\end{align*}
\]

and \(\tau\) preserves each defining relation \((Dn)\) in Definition 3.1 for \(n = 1, \ldots, 9\). \(\square\)

**Remark 3.3.** (1) Note that the defining relations \((D1)\)---\((D5), (D7), (D8), \text{and } (D9)\) are self-compatible each under the \(\mathbb{Q}\)-algebra antiautomorphism \(\tau\), while the couple of the defining relations \(((D61),(D62))\) is compatible with each other with respect to \(\tau\). Using such a \(\tau\), it is sufficient to consider the compatibility for half of the relations, e.g., those relations involving in +-parts for \(x_i^\pm(m)\), or in positive \(\ell\)'s for \(a_i(\ell)\) (for instance, see \((D62))\).

(2) The constraint condition \(\gamma^+\gamma' = rs\) in \((D1)\) is required intrinsically by the compatibilities among \((D1), (D3), (D5), (D6), (D7) \& (D8)\). For instance, by \((D7)\), we have \([x_i^-(0), x_j^-(1)]_{(i, j)} = ((j, i)(i, j))^\frac{1}{2} [x_i^-(1), x_j^-(0)]_{(j, i)^{-1}}\). Thus, using the property \((3.5)\) in Definition 3.4 below and \((D8) \& (D5)\), we get

\[
\begin{align*}
[x_j^+(0), [x_i^-(0), x_j^-(1)]_{(i, j)}] &= ((j, i)(i, j))^{\frac{1}{2}} [x_i^-(1), [x_j^+(0), x_j^-(0)]_{(j, i)^{-1}} \\
&= ((j, i)(i, j))^{\frac{1}{2}} x_i^-(1), r - s, \omega_j - \omega_i^{\prime - 1} \\
&= ((j, i)(i, j))^{\frac{1}{2}} - ((j, i)(i, j))^{-\frac{1}{2}} x_i^-(1)\omega_j/r - s.
\end{align*}
\]

However, using \((3.5), (D8), (D3), (D5) \& (D62)\), we can follow another way to expand \([x_j^+(0), [x_i^-(0), x_j^-(1)]_{(i, j)}]\) directly as

\[
\begin{align*}
[x_j^+(0), [x_i^-(0), x_j^-(1)]_{(i, j)}] &= [x_i^-(0), [x_j^+(0), x_j^-(1)]_{(i, j)}] \\
&= \gamma^{\frac{1}{2}} [x_i^-(0), a_j(1)] \omega_j \\
&= (rs)^{\frac{1}{2}} (\gamma^+\gamma')^{\frac{1}{2}} (j, j)^{a_{ji}}\omega_j^+ - (j, j)^{a_{ji}}\omega_j^{\prime - 1} x_i^-(1)\omega_j/r - s.
\end{align*}
\]
Therefore, we obtain that \( \gamma' \gamma = rs \) and \( (i, j) (j, i) = (i, i)^{a_{ij}} \), for any \( i, j \in I \).

(3) As a glimpse of the compatibility of (D2) with (D61), (D62) and (D8), we have the following: By (D8), we get \( a_i(1) = a_i^{-1} \gamma \frac{1}{2} \{ x_i^+(0), x_i^-(1) \} \) and \( a_i(-1) = a_i^{-1} \gamma \frac{1}{2} - D \gamma \gamma^+ \{ x_i^+(0), x_i^-(1) \} \). Then using one of these expressions of \( a_i(\pm 1) \) and using (D61) (or (D62)) and (D8) again, we may expand the Lie bracket \([a_i(1), a_j(-1)]\) in two manners to get to the same formula as (D2). One is to expand \( a_i(1) \) first, and then to use (D61) & (D8) as follows:

\[
[a_i(1), a_j(-1)] = a_i^{-1} \gamma \frac{1}{2} \{ [x_i^+(0), x_i^-(1)], a_j(-1) \} = a_i^{-1} \gamma \frac{1}{2} \{ [x_i^+(0), a_j(-1)], x_i^-(1) + [x_i^-(1), a_j(-1)] \} = a_i^{-1} \gamma \frac{1}{2} \{ -a_{ij}(i, i) \omega_i^{-1} \{ \gamma \omega_i - \gamma \omega_i \frac{1}{r - s} - \gamma \omega_i - \omega_i \frac{1}{r - s} \} = \{ -a_{ij}(i, i) \frac{\gamma' - \gamma}{r - s} = \{ a_{ij}(i, i) \frac{\gamma - \gamma'}{r - s} \}
\]

where \( \{ (a_{ij})_{(i, i)} = (r s) \frac{1}{r - s} \{ a_{ij}(i, j), (t; a_{ij} \{ i, i \} = \{ a_{ij} \{ i, i \}
\]

Expanding \( a_j(-1) \) instead and using (D62) & (D8), we get the same result. More compatibilities will be clearer in the proof of the Drinfel’d isomorphism theorem.

(4) Another observation is the following: When \( r = s^{-1} = 1 \), the algebra \( \mathcal{U}_q(q^{-1}(\mathfrak{sl}_n)) \) modulo the ideal generated by \( \{ \alpha_{ij}(i, i) \} \) \( \{ \omega_i^1 - \gamma q^{-1}, \gamma' \frac{1}{r - s} - \gamma^{-1} \frac{1}{r - s}, D' - D^{-1} \} \) is exactly the usual Drinfel’d realization \( \mathcal{U}_q(\mathfrak{sl}_n) \) defined below (cf. [B2]).

The unital associative algebra \( \mathcal{U}_q(\mathfrak{sl}_n) \) over \( \mathbb{Q}(q) \) is generated by the elements \( x_i^{\pm}(k) \), \( a_i(\ell), \omega_i^{\pm 1}, \gamma \pm \frac{1}{2}, D^{\pm 1}, (i \in I, k \in \mathbb{Z}, \ell \in \mathbb{Z} \backslash \{0\}) \) subject to the following defining relations:

\[
\text{(d1)} \quad \gamma \pm \frac{1}{2} \text{ are central}, \quad \omega_i^{-1} \gamma^{-1} = 1 = DD^{-1} \quad (i \in I), \quad \text{and for } i, j \in I, \quad \text{one has}
\[
[a_i^{\pm 1}, \omega_j^{\pm 1}] = [a_i^{\pm 1}, D^{\pm 1}] = 0.
\]

\[
\text{(d2)} \quad [a_i(\ell), a_j(\ell')] = \delta_{\ell + \ell', 0} \frac{1}{\ell} \frac{[a_{ij}]}{q^{1/2}}, \quad \gamma^\ell = \gamma^{-\ell}, \quad \frac{1}{q^{n}} = \frac{q^n}{q - q^{-1}}.
\]

\[
\text{(d3)} \quad [a_i(\ell), \omega_j^{\pm 1}] = 0.
\]

\[
\text{(d4)} \quad D x_i^{\pm}(k) D^{-1} = q^k x_i^{\pm}(k), \quad D a_i(\ell) D^{-1} = q^\ell a_i(\ell).
\]

\[
\text{(d5)} \quad \omega_i x_j^{\pm}(k) \omega_i^{-1} = q^{\pm a_{ij}} x_j^{\pm}(k).
\]
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(d6)

\[ [ a_i(\ell), x_j^\pm(k) ] = \pm \frac{[ \ell a_{ij} ]}{\ell} \gamma^\frac{\ell}{\ell} x_j^\pm(\ell+k). \]

(d7)

\[
x_i^\pm(k+1)x_j^\pm(k') - q^\pm a_{ij} x_j^\pm(k')x_i^\pm(k+1)
= q^\pm a_{ij} x_i^\pm(k)x_j^\pm(k'+1) - x_j^\pm(k'+1)x_i^\pm(k).
\]

(d8)

\[ [ x_i^+(k), x_j^-(k') ] = \frac{\delta_{ij}}{q - q^{-1}} \left( \gamma^{k-k'} \omega_i(k+k') - \gamma^{k'-k} \omega_i^{-1}(k+k') \right), \]

where \( \omega_i(m) \) and \( \omega_i^{-1}(-m) \) (\( m \in \mathbb{Z}_{\geq 0} \)) with \( \omega_i(0) = \omega_i \) and \( \omega_i^{-1}(0) = \omega_i^{-1} \) are defined by:

\[
\sum_{m=0}^{\infty} \omega_i(m) z^{-m} = \omega_i \exp \left( (q-q^{-1}) \sum_{\ell=1}^{\infty} a_i(\ell) z^{-\ell} \right), \quad (\omega_i(-m) = 0, \ \forall \ m > 0); \]

\[
\sum_{m=0}^{\infty} \omega_i^{-1}(-m) z^{-m} = \omega_i^{-1} \exp \left( -(q-q^{-1}) \sum_{\ell=1}^{\infty} a_i(-\ell) z^{-\ell} \right), \quad (\omega_i^{-1}(m) = 0, \ \forall \ m > 0).
\]

(d91)

\[ x_i^+(m)x_j^+(k) = x_j^+(k)x_i^+(m), \quad \text{for} \ a_{ij} = 0, \]

(d92)

\[ \text{Sym}_{m_1, m_2} \left( x_i^+(m_1)x_i^+(m_2)x_j^+(k) - (q^{\pm 1} + q^{\mp 1}) x_i^+(m_1)x_j^+(k)x_i^+(m_2) + x_j^+(k)x_i^+(m_1)x_i^+(m_2) \right) = 0, \quad \text{for} \ a_{ij} = -1, \ 1 \leq i < j < n, \]

(d93)

\[ \text{Sym}_{m_1, m_2} \left( x_i^+(m_1)x_i^+(m_2)x_j^+(k) - (q^{\mp 1} + q^{\pm 1}) x_i^+(m_1)x_j^+(k)x_i^+(m_2) + x_j^+(k)x_i^+(m_1)x_i^+(m_2) \right) = 0, \quad \text{for} \ a_{ij} = -1, \ 1 \leq j < i < n. \]

3.2. Before putting forward the Drinfel’d isomorphism theorem, that is, showing that the \( \overline{Q}(r, s) \)-algebra \( \mathcal{U}_{r,s}(\widehat{sl}_n) \) in Definition 3.1 is exactly the Drinfel’d realization of the two-parameter quantum affine algebra \( \mathcal{U}_{r,s}(\widehat{sl}_n) \) defined in Definition 2.1, we need to make some preliminaries on Lyndon words, and to adapt a definition of quantum Lie bracket borrowed from [J2] to give our definition about “affine” quantum Lie bracket (see Definition 3.6) which enables us to derive an interesting description on the quantum affine Lyndon basis in the quantum affine cases for the first time.

Note that the (affine) quantum Lie bracket possesses some advantages in calculations such as less related to degrees of elements (see the properties (3.3) & (3.4) below). This generalized quantum Lie bracket, like the one used in the usual construction of the quantum Lyndon basis (for definition, see [R2]), is consistent with the process when adding the bracketing on those corresponding Lyndon words. This is crucial to the quantum calculations we develop later on.
**Definition 3.4 ([12])**. The quantum Lie bracket \([ a_1, a_2, \cdots, a_s ]_{(q_1, q_2, \cdots, q_{s-1})}\) is defined inductively by
\[
[ a_1, a_2 ]_q = a_1 a_2 - q a_2 a_1, \quad \text{for } q \in \mathbb{K}\setminus\{0\},
\]
\[
[ a_1, a_2, \cdots, a_s ]_{(q_1, q_2, \cdots, q_{s-1})} = [ a_1, [ a_2, \cdots, a_s ]_{(q_1, q_2, \cdots, q_{s-2})} ]_{q_{s-1}}, \quad \text{for } q_i \in \mathbb{K}\setminus\{0\}.
\]
The following identities follow from the definition:
\[
[a, bc]_v = [a, b]_x c + x [a, c]_x, \quad x \neq 0, \tag{3.1}
\]
\[
[ab, c]_v = a [b, c]_x + x [a, c]_x b, \quad x \neq 0, \tag{3.2}
\]
\[
[a, [b, c]_u]_v = [[a, b]_x, c]_{uv} + x [a, [b, c]_v]_x, \quad x \neq 0, \tag{3.3}
\]
\[
[[a, b]_u, c]_v = [a, [b, c]_x]_{uv} + x [[a, c]_x, b]_v, \quad x \neq 0, \tag{3.4}
\]
\[
[ a, b_1, \cdots, b_s ]_{(v_1, \cdots, v_{s-1})} = \sum_i [b_1, \cdots, [a, b_i], \cdots, b_s ]_{(v_1, \cdots, v_{s-1})}, \tag{3.5}
\]
\[
[a, a, b]_{(u, v)} = [a, a, b]_{(v, u)} = a^2 b - (u + v) aba + (uv) ba^2. \tag{3.6}
\]

**Definition 3.5.** For the generators system of the algebra \(U_{r,s}(\widehat{\mathfrak{sl}}_n)\), we define the \(\hat{Q}\)-gradation (where \(\hat{Q}\) is the root lattice of \(\mathfrak{sl}_n\)) as follows:
\[
deg(\omega_i^{\pm 1}) = \deg(\omega_j^{\pm 1}) = \deg(\gamma^{\pm 1}) = \deg(D^{\pm 1}) = \deg(D^{\pm 1}) = 0, \tag{3.7}
\]
\[
deg(a_i(\pm \ell)) = 0, \quad \deg(x_i^{\pm}(k)) = \pm a_i. \tag{3.8}
\]
Hence, the defining relations (D1)–(D9) ensure that \(U_{r,s}(\widehat{\mathfrak{sl}}_n)\) has a triangular decomposition:
\[
U_{r,s}(\widehat{\mathfrak{sl}}_n) = U_{r,s}(\widehat{\mathfrak{sl}}_n^{-}) \otimes U^0_{r,s}(\widehat{\mathfrak{sl}}_n) \otimes U_{r,s}(\widehat{\mathfrak{sl}}_n),
\]
where \(U_{r,s}(\widehat{\mathfrak{sl}}_n^{-}) = \bigoplus_{\alpha \in \hat{Q}^+} U_{r,s}(\widehat{\mathfrak{sl}}_n^{-})_\alpha\) is generated respectively by \(x_i^{\pm}(k)\) \((i \in I)\), and \(U^0_{r,s}(\widehat{\mathfrak{sl}}_n)\) is the subalgebra generated by \(\omega_i^{\pm 1}, \omega_j^{\pm 1}, \gamma^{\pm 1}, \gamma^{\pm 1}, D^{\pm 1}, D^{\pm 1}\), and \(a_i(\pm \ell)\) for \(i \in I, \ell \in \mathbb{N}\). Namely, \(U^0_{r,s}(\widehat{\mathfrak{sl}}_n)\) is generated by the toral subalgebra \(U_{r,s}(\widehat{\mathfrak{sl}}_n)\) and the quantum Heisenberg subalgebra \(H_{r,s}(\widehat{\mathfrak{sl}}_n)\) generated by those quantum imaginary root vectors \(a_i(\pm \ell)\) \((i \in I, \ell \in \mathbb{N})\).

**Definition 3.6.** For \(\alpha, \beta \in \hat{Q}^+\) (a positive root lattice of \(\mathfrak{sl}_n\)), \(x_\alpha^{\pm}(k)\), \(x_\beta^{\pm}(k') \in U_{r,s}(\widehat{\mathfrak{sl}}_n)\), we define their “affine” quantum Lie bracket as follows:
\[
[x_\alpha^{\pm}(k), x_\beta^{\pm}(k')]_{(\omega_\alpha', \omega_\beta')_{\mp 1}} := x_\alpha^{\pm}(k) x_\beta^{\pm}(k') - \langle \omega_\alpha', \omega_\beta' \rangle_{\mp 1} x_\beta^{\pm}(k') x_\alpha^{\pm}(k). \tag{3.9}
\]
By Definition 3.6, the formula (D7) will take the convenient form as
\[
[x_i^{\pm}(k), x_j^{\pm}(k+1)]_{(j, i)_{\mp 1}} = -\left(\left(\langle j, i \rangle \langle i, j \rangle^{-1}\right)^{\mp 1/2}\right) x_j^{\pm}(k') x_i^{\pm}(k+1)_{(j, i)_{\mp 1}}. \tag{3.10}
\]
By (3.6), the \((r, s)\)-Serre relations (D92) & (D93) for \(m_1 = m_2\) in the case of \(a_{ij} = -1\) can be reformulated as:
\[
[x_i^{\pm}(m), x_j^{\pm}(m)]_{(r^{\pm 1}, s^{\pm 1})} = 0, \quad \text{for } 1 \leq i < j < n, \tag{3.11}
\]
\[
[x_i^{\pm}(m), x_j^{\pm}(m), x_k^{\pm}(k)]_{(s^{\mp 1}, r^{\mp 1})} = 0, \quad \text{for } 1 \leq j < i < n. \tag{3.12}
\]
Remark 3.7. (1) For any nonsimple root \( \alpha (\neq \alpha_i) \) (\( i \in I \)), the meaning of notation \( x^{\pm}_\alpha(k) \) (resp. \( x^{-}_\alpha(k) \)) in Definition 3.6 has a bit of ambiguity, as is well-known even for quantum “classical” root vectors \( x^{\pm}_\alpha(0) \) which have different linearly-independent choices. However, the combinatorial approach to Lyndon words, together with the “affine” quantum Lie bracket, will give us a valid and specific choice for \( x^{\pm}_\alpha(k) \) which leads to a construction of quantum “affine” Lyndon basis for \( U_{r,s}(\bar{\alpha}) \), on which acting \( \tau \) will yield a corresponding construction of quantum “affine” Lyndon basis for \( U_{r,s}(\bar{\alpha}^-) \) (see Proposition 3.10 & Theorem 3.11 below).

(2) In fact, (3.8) describes a kind of consistent constraints of quantum affine root vectors defined by some Lyndon words of different levels (if say, \( x^{\pm}_\alpha(k) \) have level \( k \)) which obey the defining rule of Lyndon basis (see below) via Lyndon words as in the classical types, since from (3.8), we get the level-shifting formula

\[
\left[ x^{\pm}_i(k), x^{\pm}_j(k'+1) \right]_{(i,j)\equiv 1} = (i,i)^{\pm ij} \left[ x^{\pm}_i(k+1), x^{\pm}_j(k') \right]_{(i,j)\equiv 1} + \left( (i,j)\equiv 1 - (j,i)^{\pm 1} \right) x^{\pm}_i(k') x^{\pm}_j(k+1)).
\]

(3) Let \( U_{r,s}(n) \) denote the subalgebra of \( U_{r,s}(\bar{n}) \), generated by \( x^{\pm}_i(0) \) (\( i \in I \)). By definition, it is clear that \( U_{r,s}(n) \cong U_{r,s}(n) \), the subalgebra of \( U_{r,s}(\mathfrak{sl}_n) \) generated by \( e_i \) (\( i \in I \)) (see [BGH1, Remarks (2), p. 391]). Now let us recall the construction of a Lyndon basis. The natural ordering \( \prec \) in \( I \) gives a total ordering of the alphabet \( A = \{ x^{\pm}_1(0), \ldots, x^{\pm}_n(0) \} \). Let \( A^* \) be the set of all words in the alphabet \( A \) (including the vacuum 1) and let \( u < v \) denote that word \( u \) is lexicographically smaller than word \( v \). Recall that a word \( \ell \in A^* \) is a Lyndon word if it is lexicographically smaller than all its proper right factors (cf. [LR,R2,BH]). Let \( \mathbb{K}[A^*] \) be the associative algebra of \( \mathbb{K} \)-linear combinations of words in \( A^* \) whose product is juxtaposition, namely, a free \( \mathbb{K} \)-algebra. Let \( J \) be the \((r,s)\)-Serre ideal of \( \mathbb{K}[A^*] \) generated by elements \( \{ (ad_i x^{\pm}_j(0))^{1-\alpha_{ij}} x^{\pm}_i(0) \} \) \( 1 \leq i \neq j \leq n-1 \). Clearly, \( U_{r,s}(n) = \mathbb{K}[A^*]/J \) now given another ordering \( \preceq \) in \( A^* \), introducing a usual length function \( | \cdot | \) for each word \( u \in A^* \). We say \( u \preceq w \) if \( |u| < |w| \) or \( |u| = |w| \) and \( u \geq w \). Then we call a (Lyndon) word to be good with respect to the \((r,s)\)-Serre ideal \( J \) if it cannot be written as a sum of strictly smaller words modulo \( J \) with respect to the ordering \( \preceq \). From [R2], the set of quantum Lie brackets (or say, \( q \)-bracketings) of all good Lyndon words consists of a system of quantum root vectors of \( U_{r,s}(n) \). More precisely, we have a construction for any quantum root vector \( x^{\pm}_\alpha(0) \) with \( \alpha \in \hat{\Delta}^+ \) (a positive root system of \( \mathfrak{sl}_n \)) in the following. \( \square \)

Take a corresponding ordering (compatible with the natural ordering \( \prec \) on \( I \)) of \( \hat{\Delta}^+ = \{ \alpha_{ij} := \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1} = \varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq n \} \) with \( \alpha_{i,i+1} = \alpha_i \) as follows (see [H, p. 533]):

\[
\alpha_{12}, \alpha_{13}, \alpha_{14}, \cdots, \alpha_{1n}, \alpha_{23}, \alpha_{24}, \cdots, \alpha_{2n}, \cdots, \alpha_{n-1,n},
\]

(3.11)

which is a convex ordering on \( \hat{\Delta}^+ \) (for definition, see [R2, Sect. 6]). Hence, for each \( \alpha = \alpha_{ij} \in \hat{\Delta}^+ \), by [R2], we can construct the quantum root vector \( x^{\pm}_\alpha(0) \) as a
(r, s)-bracketing of a good Lyndon word in the inductive fashion:

\[
x_{\alpha_{ij}}^+ (0) := \left[ x_{\alpha_{ij-1}}^+ (0), x_{ij-1}^+ (0) \right]_{(\omega_{\alpha_{ij-1}}, \omega_{j-1})^{-1}} = \left[ \cdots \left[ x_i^+ (0), x_{i+1}^+ (0) \right]_{(i, i+1)^{-1}}, \cdots, x_{j-1}^+ (0) \right]_{(\omega_{\alpha_{ij-1}}, \omega_{j-1})^{-1}} = \left[ \cdots \left[ x_i^+ (0), x_{i+1}^+ (0) \right]_r, \cdots, x_{j-1}^+ (0) \right]_r.
\]

Applying \( \tau \) to (3.12), we can obtain the definition of quantum root vector \( x_{\alpha_{ij}}^- (0) \) as below:

\[
x_{\alpha_{ij}}^- (0) = \tau \left( x_{\alpha_{ij}}^+ (0) \right) = \left[ x_{j-1}^- (0), \cdots, \left[ x_{i+1}^- (0), x_i^- (0) \right]_s \cdots \right]_s.
\]

**Theorem 3.8.** (i) The set

\[
\left\{ x_{\alpha_{n-1,n}}^+ (0) \ell_{n-1,n} \cdots x_{\alpha_{r+1}}^+ (0) \ell_{r+1} \cdots x_{\alpha_{12}}^+ (0) \ell_{12} \cdots x_{\alpha_{1n}}^+ (0) \ell_{1n} \left| \ell_{ij} \geq 0 \right. \right\}
\]

is a Lyndon basis of \( \mathcal{U}_{r,s}(n) \).

(ii) The set

\[
\left\{ x_{\alpha_{12}}^- (0) \ell_{12} x_{\alpha_{13}}^- (0) \ell_{13} \cdots x_{\alpha_{1n}}^- (0) \ell_{1n} x_{\alpha_{23}}^- (0) \ell_{23} \cdots x_{\alpha_{n-1,n}}^- (0) \ell_{n-1,n} \left| \ell_{ij} \geq 0 \right. \right\}
\]

is a Lyndon basis of \( \mathcal{U}_{r,s}(n^-) \). \( \square \)

**Definition 3.9.** For \( \alpha_{ij} \in \hat{\Delta}^+ \), we define the quantum affine root vectors \( x_{\alpha_{ij}}^\pm (k) \) of nontrivial level \( k \) by

\[
x_{\alpha_{ij}}^+ (k) := \left[ \cdots \left[ x_i^+ (k), x_{i+1}^+ (0) \right]_r, \cdots, x_{j-1}^- (0) \right]_r, \quad x_{\alpha_{ij}}^- (k) := \left[ x_{j-1}^- (0), \cdots, \left[ x_{i+1}^- (0), x_i^- (k) \right]_s \cdots \right]_s,
\]

where \( \tau \left( x_{\alpha_{ij}}^\pm (\mp k) \right) = x_{\alpha_{ij}}^\mp (\mp k) \).

For each fixed \( \alpha \in \hat{Q}^+ \), let us denote by \( \mathcal{U}_{r,s}(\tilde{n})_\alpha \) the subspace of \( \mathcal{U}_{r,s}(\tilde{n})_\alpha \), consisting of elements of level \( k \). Hence, \( \mathcal{U}_{r,s}(\tilde{n})_\alpha = \bigoplus_{k \in \mathbb{Z}} \mathcal{U}_{r,s}^{(k)}(\tilde{n})_\alpha \). When \( \alpha = \alpha_i \in \hat{\Delta}^+ \) is a simple root, by definition, \( \dim \mathcal{U}_{r,s}^{(k)}(\tilde{n})_{\alpha_i} = 1 \) for any level \( k \). However, for any nonsimple root \( \alpha \neq \alpha_i (i \in I) \), \( \dim \mathcal{U}_{r,s}^{(k)}(\tilde{n})_\alpha = \infty \) for any level \( k \). In this case, given a positive root \( \alpha = \alpha_{ij} \in \hat{\Delta}^+ \), we call a tuple \( (\beta_1, \cdots, \beta_v) (v \geq 1) \) a partition of root \( \alpha_{ij} \) if \( \beta_j < \cdots < \beta_j \) in the ordering given in (3.11) such that \( \beta_1 + \cdots + \beta_v = \alpha_{ij} \). If \( v > 1 \), we say this partition is proper. Denote by \( \mathcal{Q}^\circ (\alpha) \) the set of all proper partitions of root \( \alpha \). Obviously, we have \( \mathcal{U}_{r,s}^{(k_1)}(\tilde{n})_{\beta_{j_1}} \cdots \mathcal{U}_{r,s}^{(k_v)}(\tilde{n})_{\beta_{j_v}} \subseteq \mathcal{U}_{r,s}(\tilde{n})_\alpha \) if \( k_1 + \cdots + k_v = k \). Now we write

\[
\Omega^{(k)}_\alpha (\tilde{n}) := \sum_{(\beta_{j_1}, \cdots, \beta_{j_v}) \in \mathcal{Q}^\circ (\alpha)} \mathcal{U}_{r,s}^{(k_1)}(\tilde{n})_{\beta_{j_1}} \cdots \mathcal{U}_{r,s}^{(k_v)}(\tilde{n})_{\beta_{j_v}} \subseteq \mathcal{U}_{r,s}(\tilde{n})_\alpha
\]
for the subspace of \( \mathcal{U}_{r,s}^{(k)}(\bar{n})_\alpha \) spanned by basis elements’ products of level \( k \) from those proper partitions pertaining to \( \alpha \). Using the \( \mathbb{Q} \)-antiautomorphism \( \tau \) on \( \Omega_{\alpha}^{(s)}(\bar{n}) \), we get

\[
\Omega_{\alpha}^{(k)}(\bar{n}^-) := \tau \left( \Omega_{\alpha}^{(s)}(\bar{n}) \right).
\]

Then we have the following description on \( \mathcal{U}_{r,s}^{(k)}(\bar{n}^\pm)_\alpha \) for \( \alpha \in \hat{\Delta}^+ \), whose proof shows that Definition 3.9 makes sense.

**Proposition 3.10.** For \( 1 \leq i < j \leq n \) and \( \alpha_{ij} \in \hat{\Delta}^+ \) (a positive root system of \( \mathfrak{sl}_n \)), we have

(i) \( \mathcal{U}_{r,s}^{(k)}(\bar{n})_{\alpha_{ij}} = \mathbb{K} x_{\alpha_{ij}}^{+}(k) \bigoplus \Omega_{\alpha_{ij}}^{(k)}(\bar{n}) \),

(ii) \( \mathcal{U}_{r,s}^{(k)}(\bar{n}^-)_{\alpha_{ij}} = \mathbb{K} x_{\alpha_{ij}}^{-}(k) \bigoplus \Omega_{\alpha_{ij}}^{(k)}(\bar{n}^-) \).

**Proof.** (i) We will use an induction on rank \( n \), where \( n \geq 2 \). Assume that \( i < j \) and \( k' > 0 \), then by (3.10), we have

\[
\begin{align*}
\left[ x_i^+(k), x_j^+(k') \right]_{(i,j)^{-1}} &= (i,i)^{-a_{ij}} \left[ x_i^+(k+1), x_j^+(k'-1) \right]_{(i,j)^{-1}} \\
&+ (i,j)^{-1} - (j,i) \ x_j^+(k'-1) x_i^+(k+1), \quad (3.16)
\end{align*}
\]

\[
\begin{align*}
\left[ x_i^+(k), x_j^+(-k') \right]_{(i,j)^{-1}} &= (i,i)^{-a_{ij}} \left[ x_i^+(k-1), x_j^+(-k'+1) \right]_{(i,j)^{-1}} \\
&+ (j,i) - (i,j)^{-1} \ x_j^+(k') x_i^+(k), \quad (3.17)
\end{align*}
\]

When \( n = 2 \), for any \( k' \in \mathbb{N} \), repeatedly using (3.16) \& (3.17), we get

\[
\begin{align*}
\left[ x_1^+(k), x_2^+(k') \right]_r &= \langle 1, 1 \rangle \frac{k'}{2} x_{\alpha_{13}}^+(k+k') \\
&+ \sum_{t=1}^{k'} \langle 1, 1 \rangle \frac{k'-t+1}{2} (r-s) x_2^+(t-1) x_1^+(k+k'-t+1) \\
&\equiv \langle 1, 1 \rangle \frac{k'}{2} x_{\alpha_{13}}^+(k+k') \mod \Omega_{\alpha_{13}}^{(k+k')}(\bar{n}), \quad (3.18)
\end{align*}
\]

\[
\begin{align*}
\left[ x_1^+(k), x_2^+(k'-1) \right]_r &= \langle 1, 1 \rangle \frac{k'}{2} x_{\alpha_{13}}^+(k-k') \\
&+ \sum_{t=1}^{k'} \langle 1, 1 \rangle \frac{k'-t}{2} (s-r) x_2^+(-t) x_1^+(k-k'+t) \\
&\equiv \langle 1, 1 \rangle \frac{k'}{2} x_{\alpha_{13}}^+(k-k') \mod \Omega_{\alpha_{13}}^{(k-k')}(\bar{n}),
\end{align*}
\]

which means that in both cases, we have

\[
\left[ x_1^+(k), x_2^+(k') \right]_r = \langle 1, 1 \rangle \frac{k'}{2} x_{\alpha_{13}}^+(k+k') \mod \Omega_{\alpha_{13}}^{(k+k')}(\bar{n}), \quad \text{for any } k' \in \mathbb{Z}.
\]

Therefore, in rank 2 case, any elements (except for \( x_2^+(k') x_1^+(k) \)) of degree \( \alpha_{13} \) generated by \( x_1^+(k) \) and \( x_2^+(k') \) are of the form: \( \left[ x_1^+(k), x_2^+(k') \right]_a \) for any \( a \in \mathbb{K} \); however,

\[
\begin{align*}
\left[ x_1^+(k), x_2^+(k') \right]_a &= \left[ x_1^+(k), x_2^+(k') \right]_r + (r-a) x_2^+(k') x_1^+(k) \\
&\equiv (rs^{-1}) \frac{k'}{2} x_{\alpha_{13}}^+(k+k') \mod \Omega_{\alpha_{13}}^{(k+k')}(\bar{n}).
\end{align*}
\]
This fact shows that

\[ U_{r,s}^{(k)}(\overline{n})_{\alpha_{13}} = \mathbb{K}x_{\alpha_{13}}^+(k) \bigoplus \Omega_{\alpha_{13}}^{(k)}(\overline{n}) \]

as vector spaces. Dually, we also have \( U_{r,s}^{(k)}(\overline{n}^-)_{\alpha_{13}} = \mathbb{K}x_{\alpha_{13}}^-(k) \bigoplus \Omega_{\alpha_{13}}^{(k)}(\overline{n}^-) \) as vector spaces.

Now we assume that we have proved the results for rank \( \leq n \), that is, for those \( \alpha_{ij} \) with \( 1 \leq i < j < n \). For the rank \( n \) case, owing to the ordering given in (3.11), we are left to prove the remaining cases: \( U_{r,s}^{(k)}(\overline{n})_{\alpha_{ij}} \) with \( 1 \leq i < j = n \).

In view of the same observation as (3.18), we need only consider the following elements of degree \( \alpha_{ij} \) and level \( k + k' \) generated by \( x_{\alpha_{ij},n-1}^+(k) \) and \( x_{n-1}^+(k') \) for \( 1 \leq i < n \):

\[
\begin{bmatrix}
  x_{\alpha_{ij},n-1}^+(k), x_{n-1}^+(k') \\
  x_{\alpha_{ij},n-2}^+(k), x_{n-2}^+(0), x_{n-1}^+(k')
\end{bmatrix}_{(\omega_{ij,n-1}, \omega_{n-1})} = \begin{bmatrix}
  x_{\alpha_{ij},n-1}^+(k), x_{n-1}^+(k')
\end{bmatrix}_r.
\]

By definition (see (3.14)) and using (3.4), (3.5) & (3.1), we have

\[
\begin{align*}
&= \begin{bmatrix}
  x_{\alpha_{ij},n-2}^+(k), x_{n-2}^+(0), x_{n-1}^+(k')
\end{bmatrix}_r \\
&\quad + r \begin{bmatrix}
  x_{\alpha_{ij},n-2}^+(k), x_{n-1}^+(k')
\end{bmatrix}_r
\end{align*}
\]

(2nd term = 0 by (3.5) & (D91))

\[
= \begin{bmatrix}
  x_{\alpha_{ij},n-2}^+(k), x_{n-2}^+(0), x_{n-1}^+(k')
\end{bmatrix}_r
\]

(3.18): rank 2 case

\[
= (rs^{-1})^{\frac{k'}{2}} \begin{bmatrix}
  x_{\alpha_{ij},n-2}^+(k), x_{n-2}^+(k')
\end{bmatrix}_r, x_{n-1}^+(0)
\]

(using the inductive hypothesis)

\[
+ \sum_t \ast_t (rs^{-1}) \begin{bmatrix}
  x_{\alpha_{ij},n-2}^+(k), x_{n-1}^+(t), x_{n-2}^+(k'-t)
\end{bmatrix}_r
\]

(3.1)

\[
= (rs^{-1})^{\frac{k'+(n-1)}{2}} \begin{bmatrix}
  x_{\alpha_{ij},n-1}^+(k+k'), x_{n-1}^+(0)
\end{bmatrix}_r \mod \Omega_{\alpha_{ij},n-1}^{(k+k')}(\overline{n}), x_{n-1}^+(0)
\]

(by definition)

\[
+ \sum_t \ast_t (rs^{-1}) \begin{bmatrix}
  x_{\alpha_{ij},n-1}^+(k), x_{n-2}^+(k'-t)
\end{bmatrix}_r
\]

(using the inductive hypothesis)

\[
= (rs^{-1})^{\frac{k'+(n-1)}{2}} x_{\alpha_{ij}}^+(k+k') \mod \Omega_{\alpha_{ij}}^{(k+k')}(\overline{n})
\]

\[
+ \sum_t \ast'_t (rs^{-1}) x_{\alpha_{ij},n-1}^+(k+k'-t) \mod x_{n-1}^+(t) \mod \Omega_{\alpha_{ij},n-1}^{(k+k'-t)}(\overline{n})
\]

\[
= (rs^{-1})^{\frac{k'+(n-1)}{2}} x_{\alpha_{ij}}^+(k+k') \mod \Omega_{\alpha_{ij}}^{(k+k')}(\overline{n})
\]
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where in the 1st ‘≡’, we used the following fact:

\[
\begin{bmatrix}
    x^+_{\alpha_{i,n-2}}(k), x^+_{n-1}(t) x^+_{n-2}(k'-t) \\
    x^+_{\alpha_{i,n-2}}(k) x^+_{n-2}(k'-t)
\end{bmatrix}_r = x^+_{n-1}(t) \begin{bmatrix}
    x^+_{\alpha_{i,n-2}}(k), x^+_{n-2}(k'-t) \\
    x^+_{\alpha_{i,n-2}}(k) x^+_{n-2}(k'-t)
\end{bmatrix}_r \\
+ \begin{bmatrix}
    x^+_{\alpha_{i,n-2}}(k), x^+_{n-1}(t) x^+_{n-2}(k'-t) \\
    x^+_{\alpha_{i,n-2}}(k) x^+_{n-2}(k'-t)
\end{bmatrix}_r
\]

(2nd term = 0 by (3.5) & (D91))

while in the 2nd ‘≡’, we used the facts:

\[
\begin{bmatrix}
    \Omega^{(k+k')}_n(\tilde{n}), x^+_{n-1}(0) \\
    x^+_{n-1}(t) \Omega^{(k+k'-t)}_n(\tilde{n})
\end{bmatrix}_r \subseteq \Omega^{(k+k')}_n(\tilde{n}),
\]

\[
x^+_{n-1}(t) \Omega^{(k+k'-t)}_n(\tilde{n}) \subseteq \Omega^{(k+k')}_n(\tilde{n}).
\]

The latter is clear, due to the definition of \(\Omega^{(k+k')}_n(\tilde{n})\). As for the first inclusion, we have the following argument provided that we notice the basis elements’ constituents of \(\Omega^{(k+k')}_n(\tilde{n})\). Indeed, for any basis element

\[
x^+_{\alpha_{i,v-1},\ell_v}(k_v) x^+_{\alpha_{\ell_{v-1},\ell_1}}(k_{v-1}) \cdots x^+_{\alpha_{i,\ell_1}}(k_1) \in \Omega^{(k+k')}_n(\tilde{n})
\]
of level \(k + k'\) pertaining to a partition of \(\alpha_{i,n-1}\), using (3.2), we have

\[
\begin{bmatrix}
    x^+_{\alpha_{\ell_v-1},\ell_v}(k_v) x^+_{\alpha_{\ell_{v-1},\ell_1}}(k_{v-1}) \cdots x^+_{\alpha_{i,\ell_1}}(k_1), x^+_{n-1}(0) \\
    x^+_{\alpha_{\ell_v-1},\ell_v}(k_v) x^+_{\alpha_{\ell_{v-1},\ell_1}}(k_{v-1}) \cdots x^+_{\alpha_{i,\ell_1}}(k_1)
\end{bmatrix}_r
\]

(by definition)

\[
\begin{bmatrix}
    x^+_{\alpha_{\ell_v-1},\ell_v}(k_v) x^+_{\alpha_{\ell_{v-1},\ell_1}}(k_{v-1}) \cdots x^+_{\alpha_{i,\ell_1}}(k_1), x^+_{n-1}(0)
\end{bmatrix}_r
\]

(2nd term = 0 by (3.5) & (D91) since \(\ell_1 < \cdots < \ell_v < n-1\))

\[
= x^+_{\alpha_{\ell_v-1},\ell_v}(k_v) x^+_{\alpha_{\ell_{v-1},\ell_1}}(k_{v-1}) \cdots x^+_{\alpha_{i,\ell_1}}(k_1) \\
\in \Omega^{(k+k')}_n(\tilde{n}),
\]

here \(k_1 + \cdots + k_v = k + k'\).

Up to now, we have finished the proof of (i). Using \(\tau\) to (i), we can get the second statement (ii). □

The argument above (in fact used the so-called quantum calculations) implies the important conclusions about the quantum affine Lyndon basis we present below.

**Theorem 3.11.** (i) The set

\[
\left\{ \left( \prod_{i \in \mathbb{Z}} x^+_{\alpha_{i+n-1}}(i)^{\ell_{i+n-1}} \right) \cdots \left( \prod_{i \in \mathbb{Z}} x^+_{\alpha_{i+1}}(i)^{\ell_{i+1}} \right) \left( \prod_{i \in \mathbb{Z}} x^+_{\alpha_{i}}(i)^{\ell_{i}} \right) \mid \ell_{i+1} \geq 0 \right\}
\]

is an “affine” Lyndon basis of \(\mathcal{U}_{r,s}(\tilde{n})\), where each index set \(I_{\alpha_{st}} = \{ i \in \mathbb{Z} \mid \ell_{i} \neq 0 \}\) is finite.
(ii) The set

$$\left\{ \left( \prod_{i \in \mathbb{Z}} x_{-1}^{-1} (i) \ell_{12}^{(i)} \right) \cdots \left( \prod_{i \in \mathbb{Z}} x_{-1}^{-1} (i) \ell_{1n}^{(i)} \right) \cdots \left( \prod_{i \in \mathbb{Z}} x_{-1}^{-1} (i) \ell_{m,n}^{(i)} \right) \right\} \ | \ \ell_{m,n}^{(i)} \geq 0$$

is an “affine” Lyndon basis of $U_{r,s} (\widetilde{t}^{-})$, where each index set $I_{\alpha} = \{ i \in \mathbb{Z} \ | \ \ell_{st}^{(i)} \neq 0 \}$ is finite.  □

3.3. The following main theorem establishes the Drinfel’d isomorphism between the two-parameter quantum affine algebra $U_{r,s} (\hat{\mathfrak{sl}}_n)$ (in Definition 2.1) and the $(r, s)$-analogue of Drinfel’d quantum affinization of $U_{r,s} (\hat{\mathfrak{sl}}_n)$ (in Definition 3.1), which affords the two-parameter Drinfel’d realization of $U_{r,s} (\hat{\mathfrak{sl}}_n)$ as required.

**Theorem 3.12** (Drinfel’d Isomorphism). For Lie algebra $\mathfrak{sl}_n$ with $n > 2$, let $\theta = \alpha_{1n}$ be the maximal positive root. Then there exists an algebra isomorphism $\Psi : U_{r,s} (\hat{\mathfrak{sl}}_n) \longrightarrow U_{r,s} (\hat{\mathfrak{sl}}_n)$ defined by: for each $i \in I$,

$$\begin{align*}
\omega_i & \mapsto \omega_i \\
\omega_i' & \mapsto \omega_i' \\
\omega_0 & \mapsto \gamma'^{-1} \omega_0^{-1} \\
\omega_0' & \mapsto \gamma^{-1} \omega_0'^{-1} \\
\gamma^{\pm \frac{1}{2}} & \mapsto \gamma^{\pm \frac{1}{2}} \\
\gamma'^{\pm \frac{1}{2}} & \mapsto \gamma'^{\pm \frac{1}{2}} \\
D^{\pm 1} & \mapsto D^{\pm 1} \\
D'^{\pm 1} & \mapsto D'^{\pm 1} \\
e_i & \mapsto x_i^+ (0) \\
f_i & \mapsto x_i^- (0) \\
e_0 & \mapsto x_{\alpha_{1n}}^- (1) \cdot (\gamma'^{-1} \omega_{\theta}^{-1}) = x_{\theta}^- (1) \cdot (\gamma^{-1} \omega_{\theta}^{-1}) \\
f_0 & \mapsto (\gamma^{-1} \omega_{\theta}^{-1}) \cdot x_{\alpha_{1n}}^+ (-1) = \tau \left( x_{\alpha_{1n}}^- (1) \cdot (\gamma'^{-1} \omega_{\theta}^{-1}) \right),
\end{align*}$$

where $\omega_0 = \omega_1 \cdots \omega_{n-1}$, $\omega_0' = \omega_1' \cdots \omega_{n-1}'$.  □

Since Lusztig’s symmetry of the braid group for the two-parameter cases is no more available when the rank of $g$ is bigger than 2 (see [BGH1, Sect. 3]), this means that Beck’s approach (using the extended braid group actions (see [B2]) to prove the Drinfel’d Isomorphism Theorem) is not yet valid for the two-parameter cases here. Our treatment in the next section in fact develops a valid and interesting algorithm on the quantum calculations, which, as the reader has seen, is also a successful application to the combinatorial approach to the quantum “affine” Lyndon basis (based on the Drinfel’d generators) we introduced above. In some sense, our method also provides another new combinatorial proof via the quantum “affine” Lyndon basis even in the one-parameter setting.
4. Proof of the Drinfeld’s Isomorphism Theorem

4.1. Let $E_i$, $F_i$, $\omega_i$, $\omega'_i$ denote the images of $e_i$, $f_i$, $\omega_i$, $\omega'_i$ ($i \in I_0$) in the algebra $\mathcal{U}_{r,s}(\hat{\mathfrak{sl}_n})$ under the mapping $\Psi$, respectively.

Denote by $\mathcal{U}'_{r,s}(\hat{\mathfrak{sl}_n})$ the subalgebra of $\mathcal{U}_{r,s}(\hat{\mathfrak{sl}_n})$ generated by $E_i$, $F_i$, $\omega_i^\pm 1$, $\omega'_i^\pm 1$ ($i \in I_0$), $\gamma^{\frac{1}{2}}, \gamma'^{\frac{1}{2}}$, $D^\pm 1$ and $D'^\pm 1$, that is,

$$\mathcal{U}'_{r,s}(\hat{\mathfrak{sl}_n}) := \left\{ E_i, F_i, \omega_i^\pm 1, \omega'_i^\pm 1, \gamma^{\frac{1}{2}}, \gamma'^{\frac{1}{2}}, D^\pm 1, D'^\pm 1 \mid i \in I_0 \right\}.$$

Thereby, to prove the Drinfeld’s Isomorphism Theorem (Theorem 3.12) is equivalent to proving the following three theorems:

Theorem 4.1. $\Psi : \mathcal{U}_{r,s}(\hat{\mathfrak{sl}_n}) \rightarrow \mathcal{U}'_{r,s}(\hat{\mathfrak{sl}_n})$ is an epimorphism.

Theorem 4.2. $\mathcal{U}'_{r,s}(\hat{\mathfrak{sl}_n}) = \mathcal{U}_{r,s}(\hat{\mathfrak{sl}_n})$.

Theorem 4.3. $\Psi : \mathcal{U}_{r,s}(\hat{\mathfrak{sl}_n}) \rightarrow \mathcal{U}_{r,s}(\hat{\mathfrak{sl}_n})$ is injective.

4.2. Proof of Theorem 4.1. We shall check that the elements $E_i$, $F_i$, $\omega_i$, $\omega'_i$ ($i \in I_0$), $\gamma^{\frac{1}{2}}, \gamma'^{\frac{1}{2}}$, $D^\pm 1$, $D'^\pm 1$ satisfy the defining relations of (A1)–(A7) of $\mathcal{U}_{r,s}(\hat{\mathfrak{sl}_n})$.

First of all, the defining relations of $\mathcal{U}_{r,s}(\hat{\mathfrak{sl}_n})$ imply that $E_i$, $F_i$, $\omega_i$, $\omega'_i$ ($i \in I$) generate a subalgebra $\mathcal{U}_{r,s}(\hat{\mathfrak{sl}_n})$ of $\mathcal{U}_{r,s}(\hat{\mathfrak{sl}_n})$, which is isomorphic to $\mathcal{U}_{r,s}(\hat{\mathfrak{sl}_n})$. So we are left to check the relations involving the index $i = 0$.

Obviously, the relations of (A1) hold, according to the defining relations of $\mathcal{U}_{r,s}(\hat{\mathfrak{sl}_n})$.

For (A2): we just check the following three relations involving $i = 0$, the remaining relations in (A2) are parallel to check. Using (D4), we get

$$DE_0D^{-1} = D x_0^{-1}(1) D^{-1} \cdot (\gamma'^{-1} \omega^{-1}) = r E_0.$$

For $0 \leq j < n$, noting that $\langle \omega^{-1}_0, \omega_j \rangle = \langle \gamma^{-1} \omega^{-1}_0, \omega_j \rangle = \langle \omega'_0, \omega_j \rangle$ (by Proposition 2.4), we have

$$\omega_j E_0 \omega_j^{-1} = \omega_j x_0^{-1}(1) (\gamma'^{-1} \omega^{-1}_0) \omega_j^{-1} = \langle \omega'_{n-1}, \omega_j \rangle^{-1} \cdots \langle \omega'_1, \omega_j \rangle^{-1} E_0 = \langle \omega'_0, \omega_j \rangle E_0.$$
For (A4): first of all, when \( i \neq 0 \), we see that

\[
[ E_0, F_i ] = \left[ x^-_0(1) \cdot (\gamma'^{-1} \omega^{-1}_0), x^-_i(0) \right] = - \left[ x^-_i(0), x^-_\theta(1) \right]_{(\omega', \omega_0)} (\gamma'^{-1} \omega^{-1}_0).
\]

According to the corresponding cross relations held in \( U_{r,s}(\mathfrak{sl}_n) \), we claim the following crucial lemma, whose proof using the typical quantum calculations is technical.

**Lemma 4.4.** \( [ x^-_i(0), x^-_\theta(1) ]_{(\omega', \omega_0)} = 0 \), for \( i \in I \).

**Proof.** (I) When \( i = 1, \langle \omega'_1, \omega_0 \rangle = \langle \omega'_2, \omega_1 \rangle = s \), and \( \langle \omega'_1, \omega_0 \rangle = s^{-1} \). By (3.8) & (3.9), we have

\[
\left[ x^-_1(0), x^-_{\alpha_13}(1) \right]_{s^{-1}} = \left[ x^-_1(0), \left[ x^-_2(0), x^-_1(1) \right]_{(2,1)} \right]_{s^{-1}} - \left[ x^-_1(0), \left[ x^-_2(0), x^-_1(1) \right]_{(2,1)} \right]_{s^{-1}} = -\left( (1,2)^{-1}(2,1) \right)^{\frac{1}{2}} \left[ x^-_1(0), \left[ x^-_1(0), x^-_2(1) \right]_{(1,2)} \right]_{s^{-1}} = -(rs)^{\frac{1}{2}} \left[ x^-_1(0), x^-_1(0), x^-_2(1) \right]_{(r^{-1},s^{-1})} = 0. \text{ (by (3.9))}
\]

Hence, repeatedly using (3.3), we have

\[
\left[ x^-_1(0), x^-_{\alpha_1n}(1) \right]_{s^{-1}} = \left[ x^-_1(0), x^-_{n-1}(0) \right]_{s^{-1}} \left[ x^-_{\alpha_1,n-1}(1) \right]_{s^{-1}} = 0 \text{ (by (D9))}
\]

\[
\left[ x^-_{n-1}(0), x^-_{n-2}(0), \ldots, \left[ x^-_1(0), x^-_{\alpha_13}(1) \right]_{s^{-1}} \right]_{(s,\ldots,s)} = 0.
\]

(II) When \( i = n-1, \langle \omega'_{n-1}, \omega_0 \rangle = r^{-1} \), that is, \( \langle \omega'_{n-1}, \omega_0 \rangle = r \). By (3.3), (3.9) & (D9), we have

\[
\left[ x^-_{n-1}(0), x^-_{\alpha_1n}(1) \right]_r = \left[ x^-_{n-1}(0), \left[ x^-_{n-1}(0), \left[ x^-_{n-2}(0), x^-_{\alpha_1,n-2}(1) \right]_{s} \right]_s \right]_r \text{ (by definition)}
\]

\[
= \left[ x^-_{n-1}(0), \left[ x^-_{n-1}(0), \left[ x^-_{n-2}(0), x^-_{\alpha_1,n-2}(1) \right]_{s} \right]_s \right]_r \text{ (using (3.3))}
\]

\[
= \left[ x^-_{n-1}(0), \left[ x^-_{n-1}(0), \left[ x^-_{n-2}(0), x^-_{\alpha_1,n-2}(1) \right]_{s} \right]_s \right]_r \text{ (this term using (3.3))}
\]

\[
+ s \left[ x^-_{n-1}(0), \left[ x^-_{n-1}(0), \left[ x^-_{n-2}(0), x^-_{\alpha_1,n-2}(1) \right]_{s} \right]_s \right]_r \text{ (by (3.5), (D9))}
\]
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\[
= \left[ x_{n-1}^{-}(0), x_{n-1}^{-}(0), x_{n-2}^{-}(0) \right]_{(s,r)}, x_{\alpha_{1,n-2}}^{-} (1) \quad \text{(this term= 0 by (3.9))}
\]

\[
+ r \left[ x_{n-1}^{-}(0), x_{n-2}^{-}(0) \right]_{s}, \left[ x_{n-1}^{-}(0), x_{\alpha_{1,n-2}}^{-} (1) \right]_{r^{-1}s} \quad (= 0 by (3.5), (D91))
\]

\[
= 0.
\]

(III) When \(1 < i < n - 1\), \(\langle \omega_i', \omega_0 \rangle = 1\), that is, \(\langle \omega_i', \omega_0 \rangle = 1\). In order to derive the required result, we first need to make two claims below:

Claim (A). \(\left[ x_i^{-}(0), x_{\alpha_{1,i+1}}^{-} (1) \right]_{\langle \omega_i', \omega_{\alpha_{1,i+1}} \rangle} = \left[ x_i^{-}(0), x_{\alpha_{1,i+1}}^{-} (1) \right]_{r} = 0, \text{ for } i \geq 2.\)

In fact, by (3.3), (3.9) & (D91), we have

\[
\left[ x_i^{-}(0), x_{\alpha_{1,i+1}}^{-} (1) \right]_{r} \quad \text{(by definition)}
\]

\[
= \left[ x_i^{-}(0), \left[ x_i^{-}(0), \left[ x_{i-1}^{-}(0), x_{\alpha_{1,i-1}}^{-} (1) \right]_{s} \right] \right]_{r} \quad \text{(using (3.3))}
\]

\[
= \left[ x_i^{-}(0), \left[ x_i^{-}(0), x_{i-1}^{-}(0) \right]_{s}, x_{\alpha_{1,i-1}}^{-} (1) \right]_{r} \quad \text{(this term using (3.3))}
\]

\[
+ s \left[ x_i^{-}(0), \left[ x_{i-1}^{-}(0), \left[ x_i^{-}(0), x_{\alpha_{1,i-1}}^{-} (1) \right] \right] \right]_{r} \quad (= 0 by (3.5), (D91))
\]

\[
= \left[ x_i^{-}(0), x_i^{-}(0), x_{i-1}^{-}(0) \right]_{(s,r)}, x_{\alpha_{1,i-1}}^{-} (1) \quad (= 0 by (3.9))
\]

\[
+ r \left[ x_i^{-}(0), x_{i-1}^{-}(0) \right]_{s}, \left[ x_i^{-}(0), x_{\alpha_{1,i-1}}^{-} (1) \right]_{r^{-1}s} \quad (= 0 by (3.5), (D91))
\]

\[
= 0.
\]

Claim (B). \(\left[ x_i^{-}(0), x_{\alpha_{1,i+2}}^{-} (1) \right]_{\langle \omega_i', \omega_{\alpha_{1,i+2}} \rangle} = \left[ x_i^{-}(0), x_{\alpha_{1,i+2}}^{-} (1) \right]_{r} = 0 (i \geq 2, \text{ if } r \neq -s).\)

By definition, we note that \([ b, a ]_{\alpha} = -u [ a, b ]_{\alpha-1}\). So, we get

\[
\left[ x_i^{-}(0), x_{i+1}^{-}(0), x_{i}^{-}(0) \right]_{(s,r^{-1})} = -s \left[ x_i^{-}(0), x_{i}^{-}(0), x_{i+1}^{-}(0) \right]_{(s^{-1},r^{-1})} \quad \text{(by (3.6))}
\]

\[
= -s \left[ x_i^{-}(0), x_{i+1}^{-}(0), x_{i}^{-}(0) \right]_{(r^{-1},s^{-1})} \quad \text{(by (3.9))}
\]

\[
= 0.
\]
We then consider the following deduction:

$$\begin{align*}
\left[ x_i^- (0), x_{α_1,i+2}^- (1) \right]_{r-1s} &= \left[ x_i^- (0), \left[ x_{i+1}^- (0), \left[ x_i^- (0), x_{α_{li}}^- (1) \right]_s \right]_s \right]_{r-1s} \quad \text{(by (3.3))} \\
&= \left[ x_i^- (0), \left[ x_{i+1}^- (0), x_i^- (0) \right]_s, x_{α_{li}}^- (1) \right]_{r-1s} \quad \text{(using (3.3))} \\
&\quad + s \left[ x_i^- (0), \left[ x_i^- (0), x_{i+1}^- (0), x_{α_{li}}^- (1) \right] \right]_{r-1s} \quad \text{(this term = 0 by (3.5), (D91))} \\
&= \left[ x_i^- (0), x_{i+1}^- (0), x_i^- (0) \right]_{s,r-1}, x_{α_{li}}^- (1) \quad \text{(this term = 0 by the above)} \\
&\quad + r^{-1} \left[ x_{i+1}^- (0), x_i^- (0) \right]_{s, r-1}, x_{α_{li+1}}^- (1) \quad \text{(using (3.4))} \\
&= r^{-1} \left[ x_{i+1}^- (0), x_i^- (0), x_{α_{li+1}}^- (1) \right]_{s^2} + \left[ x_{α_{li+2}}^- (1), x_i^- (0) \right]_{r-1s} \\
&= x_{α_{li+2}}^- (1), x_i^- (0) \quad \text{(1st term = 0 by Claim (A))}.
\end{align*}$$

Expanding both sides of the above equation according to definition, we easily get

$$(1 + r^{-1}s) \left[ x_i^- (0), x_{α_{li+2}}^- (1) \right] = 0.$$

Thus the required result is obtained under the assumption.

Now applying (3.5), we can get

$$\left[ x_i^- (0), x_θ^- (1) \right] = x_{n-1}^- (0), \ldots, x_{i+2}^- (0), \left[ x_i^- (0), x_{α_{li+2}}^- (1) \right]_{(s, \ldots, s)} = 0 \quad \text{(by Claim (B)).}$$

This completes the proof of Lemma 4.4. □

Next, we turn to check the relation below, whose argument (using the quantum calculations) is crucial to our verification on compatibilities of the defining relations system of $\mathcal{U}_{r,s}$ mentioned in Remark 3.3.

**Proposition 4.5.** $[ E_0, F_0 ] = \frac{ω_0 - ω_0'}{r-s}$.  

**Proof.** Using (D1) & (D5), we have

$$\begin{align*}
\left[ E_0, F_0 \right] &= \left[ x_{α_{in}}^- (1) y'^{-1} ω_0^{-1}, y'^{-1} ω_0^{-1} x_{α_{in}}^+ (1) \right] \\
&= \left[ x_{α_{in}}^- (1), x_{α_{in}}^+ (1) \right] \cdot (y'^{-1} y'^{-1} ω_0^{-1} ω_0'^{-1}). \quad \text{(4.1)}
\end{align*}$$

Note that for $j \geq 1$, we have

$$\begin{align*}
\left[ x_{j+1}^- (0), ω_j \right]_s &= (r - s) ω_j x_{j+1}^- (0), \quad \left[ x_{j+1}^- (0), ω_j \right]_s = 0, \quad \text{(4.2)} \\
\left[ ω_j X, x_{j+1}^+ (k) \right]_r &= ω_j \left[ X, x_{j+1}^+ (k) \right], \quad \text{(4.3)} \\
\left[ x_{j}^+ (k), Y \omega_{j+1} \right]_r &= \left[ x_{j}^+ (k), Y \right] ω_{j+1}. \quad \text{(4.4)}
\end{align*}$$
So (4.2) implies that there hold
\[
\left[ x_{j+1}^-(0), \left[ x_j^-(0), x_j^+(0) \right] \right]_s = \omega_j^\prime x_{j+1}^-(0),
\]
(4.5)
\[
\left[ x_2^-(0), \left[ x_1^-(1), x_1^+(1) \right] \right]_s = \gamma \omega_1^\prime x_2^-(0),
\]
(4.6)
\[
\left[ x_{j+1}^-(0), x_{j+1}^+(0), x_j^-(0) \right]_s = -x_j^-(0) \omega_{j+1}.
\]
(4.7)

Now let us write briefly
\[
\left[ x_i^-(1), x_i^+(0), \cdots, x_i^+_{i-1}(0) \right]_{(r, \cdots, r)} := \left[ \left[ \cdots \left[ x_i^-(1), x_i^+_{i-1}(0) \right]_r, \cdots \right]_r, x_i^+_{i-1}(0) \right]_r.
\]

Thus, by (3.5), we have
\[
\left[ x_{\alpha_{li}}^-(1), x_{\alpha_{li}}^+(1) \right] = \left[ x_{\alpha_{li}}^-(1), \left[ x_i^+(1), x_i^-(0), \cdots, x_i^+_{i-1}(0) \right]_{(r, \cdots, r)} \right]_{(r, \cdots, r)}
\]
\[
= \left[ \left[ x_{\alpha_{li}}^-(1), x_i^+(1) \right], x_2^+(0), \cdots, x_i^+_{i-1}(0) \right]_{(r, \cdots, r)}
\]
\[
+ \sum_{j=2}^{i-1} \left[ x_j^-(1), x_j^+(0), \cdots, \left[ x_{\alpha_{li}}^-(1), x_j^+(0) \right] \right]_{(r, \cdots, r)}
\]
(4.8)

(i) For \( j = 1 \), by (3.5), (D8) & (4.6), we have
\[
\left[ x_{\alpha_{li}}^-(1), x_i^+(1) \right] = \left[ x_{i-1}^+(0), \cdots, \left[ x_2^+(0), x_i^+(1), x_i^+(1) \right] \right]_{(r, \cdots, r)}
\]
(4.9)
\[
= \gamma \omega_1^\prime x_{\alpha_{li}}^-(0), \quad (i > 2),
\]
so that
\[
M(i) := \left[ \left[ x_{\alpha_{li}}^-(1), x_i^+(1) \right], x_2^+(0), \cdots, x_i^+_{i-1}(0) \right]_{(r, \cdots, r)}
\]
\[
= \gamma \left[ \left[ \omega_1^\prime x_{\alpha_{li}}^-(0), x_2^+(0) \right], \cdots, x_i^+_{i-1}(0) \right]_{(r, \cdots, r)}
\]
\[
= \gamma \omega_1^\prime \left[ \left[ \omega_1^\prime x_{\alpha_{li}}^-(0), x_2^+(0) \right], \cdots, x_i^+_{i-1}(0) \right]_{(r, \cdots, r)}
\]
\[
= \gamma \omega_1^\prime \omega_2^\prime \left[ \left[ \omega_2^\prime x_{\alpha_{li}}^-(0), x_3^+(0) \right], \cdots, x_i^+_{i-1}(0) \right]_{(r, \cdots, r)}
\]
\[
= \gamma \omega_1^\prime \omega_2^\prime \left[ \left[ \omega_3^\prime x_{\alpha_{li}}^-(0), x_4^+(0) \right], \cdots, x_i^+_{i-1}(0) \right]_{(r, \cdots, r)}
\]
\[
= \cdots
\]
\[
= \gamma \omega_1^\prime \cdots \omega_{i-2}^\prime \left[ x_{i-1}^-(0), x_i^+_{i-1}(0) \right]
\]
\[
= \gamma \omega_1^\prime \cdots \omega_{i-2}^\prime \frac{\omega_{i-1}^\prime - \omega_{i-1}}{r - s}, \quad (i > 2),
\]
where we used the following identities, respectively:

\[
\begin{bmatrix}
\omega'_{j-1} x_{\alpha j}^-(0), x_j^+(0) \\
x_{\alpha j}^- (0), x_j^+ (0)
\end{bmatrix}_r = \omega'_{j-1} \begin{bmatrix}
x_{\alpha j}^- (0), x_j^+ (0)
\end{bmatrix}, \quad \text{(by (4.3)),}
\]

\[
\begin{bmatrix}
x_{\alpha j}^- (0), x_j^+ (0)
\end{bmatrix} = \omega_j x_{\alpha j+1}^- (0), \quad \text{(by (3.13) & (4.5)).}
\]

(ii) For \( j = i - 1 \), again by (3.5), (3.3) & (4.7), we get

\[
\begin{bmatrix}
x_{\alpha i-1}^-(1), x_{i-1}^+(0)
\end{bmatrix} = \begin{bmatrix}
x_{i-1}^-(0), x_{i-1}^+(0) \\
x_{i-2}^-(0), x_{\alpha i-2}^- (1)
\end{bmatrix}_s \quad \text{(by (3.3))}
\]

\[
= \begin{bmatrix}
x_{i-1}^-(0), x_{i-1}^+(0) \\
x_{i-2}^-(0), x_{\alpha i-2}^- (1)
\end{bmatrix}_s \quad \text{(by (4.7))}
\]

\[
+ s \begin{bmatrix}
x_{i-2}^-(0), \begin{bmatrix}
x_{i-1}^-(0), x_{i-1}^+(0) \\
x_{\alpha i-2}^- (1)
\end{bmatrix}
\end{bmatrix} = 0
\]

\[
= - \begin{bmatrix}
x_{i-2}^- (0) \omega_{i-1}, x_{\alpha i-2}^- (1)
\end{bmatrix}_s
\]

\[
= - \begin{bmatrix}
x_{i-2}^- (0), x_{\alpha i-2}^- (1)
\end{bmatrix} \omega_{i-1}
\]

\[
= - x_{\alpha i-1}^- (1) \omega_{i-1},
\]

where we notice that (*) \[\begin{bmatrix}
x_{i-1}^-(0), x_{i-1}^+(0) \\
x_{\alpha i-2}^- (1)
\end{bmatrix} = 0.\]

Thereby, we further obtain

\[
N(i) := \begin{bmatrix}
x_1^+(1), x_2^+(0), \ldots, x_{\alpha i-1}^- (1), x_{i-1}^+(0)
\end{bmatrix}_{(r, \ldots, r)}
\]

\[
= - \begin{bmatrix}
x_{\alpha i-1}^- (1), x_{\alpha i-1}^- (1) \omega_{i-1}
\end{bmatrix}_r \quad \text{(by (4.4))}
\]

\[
= - \begin{bmatrix}
x_{\alpha i-1}^- (1), x_{\alpha i-1}^- (1)
\end{bmatrix} \omega_{i-1}
\]

\[
= \begin{bmatrix}
x_{\alpha i-1}^- (1), x_{\alpha i-1}^- (1)
\end{bmatrix} \omega_{i-1}.
\]

(iii) For \( 1 < j < i - 1 \), by (3.5), (3.3), (4.7) & (D91), we obtain

\[
\begin{bmatrix}
x_{\alpha i}^- (1), x_j^+(0)
\end{bmatrix} = \begin{bmatrix}
x_{i-1}^-(0), \ldots, \begin{bmatrix}
x_j^-(0), x_j^+ (0) \\
x_{i-1}^- (0), x_{\alpha i-1}^- (1)
\end{bmatrix}_s
\end{bmatrix}_{(s, \ldots, s)} \quad \text{(by (3.3))}
\]

\[
= \begin{bmatrix}
x_{i-1}^-(0), \ldots, \begin{bmatrix}
x_j^-(0), x_j^+ (0) \\
x_{i-1}^- (0), x_{\alpha i-1}^- (1)
\end{bmatrix}_s
\end{bmatrix}_{(s, \ldots, s)} x_{\alpha i-1}^- (1) \quad \text{(by (4.7))}
\]
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\[ + s \left[ x_{j-1}^-(0), \cdots, \left[ x_{j-1}^-(0), \left[ \left[ x_{j}^- (0), x_{j}^+ (0), x_{\alpha_{1,j-1}}^- (1) \right] \right] (s, \ldots, s) \right] \right] \]

\( = 0 \) by (*).

\[ \begin{align*}
&= - \left[ x_{i}^- (0), \cdots, x_{j+1}^- (0), \left[ x_{j}^- (0) \omega_j, x_{\alpha_{1,j-1}}^- (1) \right] \right] (s, \ldots, s) \\
&= - \left[ x_{i}^- (0), \cdots, \left[ x_{j+1}^- (0), x_{\alpha_{1,j}}^- (1) \omega_j \right] \right] (s, \ldots, s) \\
&= - \left[ x_{i}^- (0), \cdots, \left[ x_{j+1}^- (0), x_{\alpha_{1,j}}^- (1) \right] \omega_j \right] (s, \ldots, s) \\
&= 0,
\end{align*} \]

where in the fourth and fifth equality “=” we used the following identities, respectively:

\[ \begin{align*}
&\left[ x_{j-1}^- (0) \omega_j, x_{\alpha_{1,j-1}}^- (1) \right] = \left[ x_{j-1}^- (0), x_{\alpha_{1,j-1}}^- (1) \right] \omega_j = x_{\alpha_{1,j}}^- (1) \omega_j, \\
&\left[ x_{j+1}^- (0), x_{\alpha_{1,j}}^- (1) \omega_j \right] = \left[ x_{j+1}^- (0), x_{\alpha_{1,j}}^- (1) \right] \omega_j.
\end{align*} \]

As a result of (i), (ii) & (iii), (4.8) becomes

\[ \left[ x_{\alpha_{1i}}^- (1), x_{\alpha_{1i}}^+ (-1) \right] = M(i) + N(i) \]

\[ = M(i) + \left[ x_{\alpha_{1,i-1}}^- (1), x_{\alpha_{1,i-1}}^+ (-1) \right] \omega_{i-1} \]

\[ = M(i) + M(i-1) \omega_{i-1} + \left[ x_{\alpha_{1,i-2}}^- (1), x_{\alpha_{1,i-2}}^+ (-1) \right] \omega_{i-2} \omega_{i-1} \]

\[ = \cdots \]

\[ = M(i) + M(i-1) \omega_{i-1} + M(i-2) \omega_{i-2} \omega_{i-1} + \cdots + M(3) \omega_3 \cdots \omega_{i-1} + \left[ x_{\alpha_{12}}^- (1), x_{\alpha_{12}}^+ (-1) \right] \omega_2 \cdots \omega_{i-1} \]

\[ = \frac{\gamma \omega_{\alpha_{1i}} - \gamma' \omega_{\alpha_{1i}}}{r - s}, \quad (i > 1), \quad (4.9) \]

where we used (D8) to get

\[ \left[ x_{\alpha_{12}}^- (1), x_{\alpha_{12}}^+ (-1) \right] = \frac{\gamma \omega_1' - \gamma' \omega_1}{r - s}. \]

Therefore, by (4.9), (4.1) takes the required formula:

\[ [ E_0, F_0 ] = \frac{\gamma'^{-1} \omega_1^{-1} - \gamma^{-1} \omega_1'^{-1}}{r - s}. \]

The proof of Proposition 4.5 is complete. \( \square \)
For (A5): We need only to verify that \([ E_0, E_j ] = 0\) and \([ F_0, F_j ] = 0\) for \(1 < j < n - 1\). Actually, in the proof of Proposition 4.5, the fact that \(\gamma^{-1} \omega_{\theta}^{-1}\) satisfies exactly those \((r, s)\)-Serre relations in \(U_{r, s}(sl_n)\). So, it is enough to check the \((r, s)\)-Serre relations involving the indices with \(i \cdot j = 0\).

**Lemma 4.6.**

1. \(E_0 E_1^2 - (r + s) E_1 E_0 E_1 + (rs) E_1^2 E_0 = 0\),
2. \(E_0^2 E_1 - (r + s) E_0 E_1 E_0 + (rs) E_1 E_0^2 = 0\),
3. \(E_{n-1}^2 E_0 - (r + s) E_{n-1} E_0 E_{n-1} + (rs) E_0 E_{n-1}^2 = 0\),
4. \(E_{n-1} E_0^2 - (r + s) E_0 E_{n-1} E_0 + (rs) E_0^2 E_{n-1} = 0\),
5. \(F_1^2 F_0 - (r + s) F_1 F_0 F_1 + (rs) F_0 F_1^2 = 0\),
6. \(F_1 F_0^2 - (r + s) F_1 F_0 + (rs) F_0^2 F_1 = 0\),
7. \(F_0 F_{n-1}^2 - (r + s) F_0 F_{n-1} F_0 + (rs) F_{n-1}^2 F_0 = 0\),
8. \(F_0^2 F_{n-1} - (r + s) F_0 F_{n-1} F_0 + (rs) F_{n-1}^2 F_0 = 0\).

**Proof.** The proofs for relations of (5)—(8) follow from taking \(\tau\) on the first four relations (1)—(4). We shall demonstrate the first two \((r, s)\)-Serre relations, the third and fourth ones are similar to the first two relations (1) & (2), which are left to the reader.

1. Observing

\[
\begin{bmatrix}
E_1, x_{\theta}^- (1)
\end{bmatrix}
= \begin{bmatrix}
x_{n-1}^-(0), \cdots, x_1^+ (0), \gamma^{-1} \omega_{\theta}^{-1}
\end{bmatrix}
\]

using (D8),

\[
= \gamma^{-\frac{1}{2}} \begin{bmatrix}
x_{n-1}^-(0), \cdots, x_1^+ (0), \omega_1 a_1 (1)
\end{bmatrix}
\]

using (D5),

\[
= \gamma^{-\frac{1}{2}} s \omega_1 \begin{bmatrix}
x_{n-1}^-(0), \cdots, x_1^+ (0), a_1 (1)
\end{bmatrix}
\]

using (D6),

\[
= -(rs)^{-\frac{1}{2}} x_{\theta_{2n}}^- (1) \omega_1
\]

we have

\[
E_0 E_1^2 - (r + s) E_1 E_0 E_1 + (rs) E_1^2 E_0
= (rs) \left( E_1^2 x_{\theta}^- (1) - (1 + r^{-1}s) E_1 x_{\theta}^- (1) E_1 + (r^{-1}s) x_{\theta}^- (1) E_1^2 \right) (\gamma^{-1} \omega_{\theta}^{-1})
= (rs) \begin{bmatrix}
E_1, x_{\theta}^- (1)
\end{bmatrix}
(\gamma^{-1} \omega_{\theta}^{-1})
= -(rs)^{\frac{1}{2}} \begin{bmatrix}
x_1^+ (0), x_{\theta_{2n}}^- (1) \omega_1
\end{bmatrix}
(\gamma^{-1} \omega_{\theta}^{-1})
= -(rs)^{\frac{1}{2}} \omega_1 (\gamma^{-1} \omega_{\theta}^{-1})
= 0
\]

(by (3.5), (D8)).
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(2) Using the formula of \( [E_1, x_{\theta}^{-}(1)] \) derived in (1) above, we have

\[
E_0^2 E_1 - (r + s) E_0 E_1 E_0 + (rs) E_1 E_0^2
\]

\[
= (rs) \left[ x_{\theta}^{-}(1), x_{\theta}^{-}(1), E_1 \right]_{(1, rs^{-1})} (r')^{-2} \omega_{\theta}^{-2}
\]

\[
= (rs)^{\frac{1}{2}} \left[ x_{\theta}^{-}(1), x_{\alpha_2}^{-}(1) \right]_{rs^{-1}} (r')^{-2} \omega_{\theta}^{-2}
\]

\[
= (rs)^{\frac{1}{2}} \left[ x_{\theta}^{-}(1), x_{\alpha_2}^{-}(1) \right]_{s^{-1}} \omega_1 (r')^{-2} \omega_{\theta}^{-2}
\]

\[
= -(rs^{-1})^{\frac{1}{2}} \left[ x_{\alpha_2}^{-}(1), x_{\theta}^{-}(1) \right]_{s, r} \omega_1 (r')^{-2} \omega_{\theta}^{-2}
\]

\[
= 0, \quad \text{(by Claim (C) below)}
\]

where we used the following claim:

**Claim (C).** \( [x_{\alpha_2}^{-}(1), x_{\alpha_1}^{-}(1)]_{s} = 0, \text{ for } n > 2 \text{ and } r \neq -s. \)

The argument for Claim (C) is technical. Indeed, by induction on \( n \), we have: when \( n = 3 \), by (3.8), one gets

\[
\left[ x_2^{-}(1), x_{\alpha_1}^{-}(1) \right]_s = \left[ x_2^{-}(1), \left[ x_2^{-}(0), x_1^{-}(1) \right]_s \right]_s \quad \text{(by (3.8))}
\]

\[
= -(rs)^{\frac{1}{2}} \left[ x_2^{-}(1), \left[ x_1^{-}(0), x_2^{-}(1) \right]_{r^{-1}} \right]_s
\]

\[
= (rs^{-1})^{-\frac{1}{2}} \left[ x_2^{-}(1), x_2^{-}(0), x_1^{-}(0) \right]_{(r, s)}
\]

\[
= (rs^{-1})^{-\frac{1}{2}} \left[ x_2^{-}(1), x_2^{-}(1), x_1^{-}(0) \right]_{(s, r)} \quad \text{(by (3.6))}
\]

\[
= 0, \quad \text{(by (3.9)),}
\]

which is exactly the \((r, s)\)-Serre relation (see (3.9)).

For \( n > 3 \), we first notice the fact:

\[
\left[ x_{n-1}^{-}(0), x_{\alpha_2}^{-}(1) \right]_{(\omega_{n-1}, -\omega_{\alpha_2})} = \left[ x_{n-1}^{-}(0), x_{\alpha_2}^{-}(1) \right]_r = 0, \quad \text{for } n > 3,
\]

which can be proved using the same method of the proof of (II) in Lemma 4.4.

We thus have

\[
\left[ x_{\alpha_2}^{-}(1), x_{\theta}^{-}(1) \right]_r = \left[ \left[ x_{n-1}^{-}(0), x_{\alpha_2}^{-}(1) \right]_s, x_{\theta}^{-}(1) \right]_r \quad \text{(by (3.4))}
\]

\[
= \left[ x_{n-1}^{-}(0), \left[ x_{\alpha_2}^{-}(1), x_{\theta}^{-}(1) \right]_1 \right]_{rs}
\]

\[
+ \left[ \left[ x_{n-1}^{-}(0), x_{\theta}^{-}(1) \right]_r, x_{\alpha_2}^{-}(1) \right]_s \quad (= 0 \text{ by Claim (A)})
\]

\[
= x_{n-1}^{-}(0), \left[ \left[ x_{\alpha_2}^{-}(1), x_{n-1}^{-}(0) \right]_s, x_{\alpha_1}^{-}(1) \right]_1 \right]_{rs}
\]

\[
= x_{n-1}^{-}(0), \left[ \left[ x_{\alpha_2}^{-}(1), x_{n-1}^{-}(0) \right]_{s^{-1}}, x_{\alpha_1}^{-}(1) \right]_s \right]_{rs}
\]

\[
\text{(by (3.3))}
\]
Lemma 4.7. In Proposition 4.5, we get an important recursive relation:

\[ x_{n-1}^-(0), \left[ x_{n-1}^-(0), x_{\alpha_{1,n-1}}^-(1) \right]_{s^2} \]

(2nd summand = 0 using induction hypothesis)

\[ = -s^{-1} \left[ x_{n-1}^-(0), x_{\alpha_{2n}}^-(1), x_{\alpha_{1,n-1}}^-(1) \right]_{s^2} \]  
(by (3.3))

\[ = -s^{-1} \left[ x_{n-1}^-(0), x_{\alpha_{2n}}^-(1), x_{\alpha_{1,n-1}}^-(1) \right]_{r,s} \]

\[ = -rs^{-1} \left[ x_{\alpha_{2n}}^-(1), x_{n-1}^-(0), x_{\alpha_{1,n-1}}^-(1) \right]_{r^{-1}s^2} \]

(by definition)

By definition, expanding both sides of the above identity gives us

\[ (1 + rs^{-1}) x_{\alpha_{2n}}^-(1) x_{\alpha_{1,n}}^-(1) = (r + s) x_{\alpha_{1,n}}^-(1) x_{\alpha_{2n}}^-(1), \]

which means \( x_{\alpha_{2n}}^-(1), x_{\alpha_{1,n}}^-(1) \) = 0, under the assumption \( r \neq -s \).

For (A7): The verification is analogous to that of (A6). □

4.3. Proof of Theorem 4.2. We shall show that the algebra \( \mathcal{U}_{r,s}(\mathfrak{sl}_n) \) is generated by \( E_i, F_i, \omega_i^+, \omega_i^-, \gamma_i^+, \gamma_i^-, D_i^+, D_i^- (i \in I) \).

To this end, we need to prove the following results.

Lemma 4.7. (1) \( x_1^- (1) = \left[ E_2, E_3, \ldots, E_{n-1}, E_0 \right]_{(r, \ldots, r)} \gamma' \omega_1 \in \mathcal{U}'_{r,s}(\mathfrak{sl}_n) \),

then for any \( i \in I \), \( x_i^- (1) \in \mathcal{U}'_{r,s}(\mathfrak{sl}_n) \).

(2) \( x_i^+ (-1) = \tau \left[ \left( E_2, E_3, \ldots, E_{n-1}, E_0 \right)_{(r, \ldots, r)} \gamma' \omega_1 \right] = \gamma \omega_i [F_0, F_{n-1}, \ldots, F_3, F_2]_{(s, \ldots, s)} \in \mathcal{U}'_{r,s}(\mathfrak{sl}_n), \)

then for any \( i \in I \), \( x_i^+ (-1) \in \mathcal{U}'_{r,s}(\mathfrak{sl}_n) \).

\[ \text{Proof. (1) Set } \tilde{E}(i) = x_{\alpha_{1,i+1}}^-(1) \omega_{i+1} \cdots \omega_{n-1} \gamma' \omega_1 \text{ for } i \geq 1, \text{ where } \tilde{E}(n-1) = E_0. \text{ Observing that } \left[ x_i^+ (0), x_{\alpha_{1,i+1}} (1) \right] = x_{\alpha_{1,i+1}} (1) \omega_i \text{ in the proof (see case (ii)) of Proposition 4.5, we get an important recursive relation:} \]

\[ \left[ E_i, \tilde{E}(i) \right] = \left[ x_i^+ (0), x_{\alpha_{1,i+1}} (1) \omega_{i+1} \cdots \omega_{n-1} \gamma' \omega_1 \right] \]

\[ = \left[ x_i^+ (0), x_{\alpha_{1,i+1}} (1) \omega_{i+1} \cdots \omega_{n-1} \gamma' \omega_1 \right] \]

\[ = \tilde{E}(i-1). \]  

(4.11)

Recursively using the above relations, we obtain

\[ x_1^- (1) = \tilde{E}(1) \gamma' \omega_1 = \left[ E_2, \tilde{E}(2) \right] \gamma' \omega_1 = \cdots \]

\[ = \left[ E_2, \cdots, E_{n-1}, \tilde{E}(n-1) \right]_{(r, \ldots, r)} \gamma' \omega_1 \]

\[ \in \mathcal{U}'_{r,s}(\mathfrak{sl}_n). \]  

(4.12)
Now suppose that we already have obtained \( x_i^-(1) \in \mathcal{U}'_{r,s}(\hat{\mathfrak{sl}_n}) \) for \( i \geq 1 \). Notice that

\[
x_{i+1}^-(1) = (rs) \left[ \left[ x_i^+(0), x_i^-(0) \right], x_{i+1}^-(1) \right]_{r-1} \omega_i^{-1} \quad \text{(by (3.4))},
\]

\[
= (rs) \left[ x_i^+(0), x_i^-(0), x_{i+1}^-(1) \right]_{(r-1,1)} \omega_i^{-1} \quad \text{(by (3.8))},
\]

\[
= -(rs)^{\frac{1}{2}} \left[ x_i^+(0), x_{i+1}^-(0), x_i^-(1) \right]_{(s,1)} \omega_i^{-1}
\]

\[
= (rs)^{\frac{1}{2}} \left[ \left[ F_{i+1}, x_i^-(1) \right]_s, E_i \right] \omega_i^{-1}
\]

\[
\in \mathcal{U}'_{r,s}(\hat{\mathfrak{sl}_n}),
\]

which gives rise to the recursive construction of some basic quantum real root vectors of level 1. Hence, we obtain the required result.

(2) Set \( \widehat{F}(i) = \tau(\widehat{E}(i)) = \gamma^{-1} \omega_i^{-1} \omega_{i-1}^{-1} \cdots \omega_{i+1}^{-1} x_{i+1}^+(1) \) for \( i \geq 1 \), where \( \widehat{F}(n-1) = F_0 \). Applying \( \tau \) to (4.11), we see that \( \left[ \widehat{F}(i), F_i \right]_s = \widehat{F}(i-1) \) and \( \widehat{F}(1) = \gamma^{-1} \omega_1^{-1} x_1^+(1) \), which implies the first claim.

The remaining claim follows from

\[
x_{i+1}^+(1) = \tau(x_{i+1}^-(1)) = (rs)^{\frac{1}{2}} \omega_i^{-1} \left[ F_i, \left[ x_i^+(1), E_{i+1} \right]_r \right] \in \mathcal{U}'_{r,s}(\hat{\mathfrak{sl}_n}).
\]

This completes the proof of Lemma 4.7. \( \square \)

We observe that Lemma 4.7, together with (4.12), (4.13) & (4.14), gives the construction of the Drinfel’d generators of level 1. Furthermore, the first conclusion of the following lemma gives the construction of the quantum imaginary root vectors of any level \((\neq 0)\), while the second gives the construction of some basic quantum real root vectors of any level.

Actually, as a result of Definition 3.9 and Lemma 4.8 below, this approach also gives the construction of all quantum real root vectors of any level.

**Lemma 4.8.**

1. \( a_i(\ell) \in \mathcal{U}'_{r,s}(\hat{\mathfrak{sl}_n}) \), for \( \ell \in \mathbb{Z}\setminus\{0\} \).
2. \( x_i^\pm(k) \in \mathcal{U}'_{r,s}(\hat{\mathfrak{sl}_n}) \), for \( k \in \mathbb{Z} \).

**Proof.**

(1) At first, it follows from (D8) that

\[
a_i(1) = \omega_i^{-1} \gamma^{1/2} \left[ x_i^+(0), x_i^-(1) \right] \in \mathcal{U}'_{r,s}(\hat{\mathfrak{sl}_n}),
\]

\[
a_i(-1) = \omega_i^{-1} \gamma^{1/2} \left[ x_i^+(1), x_i^-(0) \right] = \tau(a_i(1)) \in \mathcal{U}'_{r,s}(\hat{\mathfrak{sl}_n}).
\]

Suppose that we have already obtained \( a_i(\pm \ell') \in \mathcal{U}'_{r,s}(\hat{\mathfrak{sl}_n}) \) for all \( \ell' \leq \ell \).

Now using (D6\(n\)) & (D8), we have the following expansion (in fact, the expansions of both sides are the same which also show the compatibility between (D6\(n\)) and (D8)
for $n = 1, 2$):
\[
U'_{r,s}(\hat{\mathfrak{sl}}_n) \cong \left[ x_i^+(0), \left[ a_i(\ell), x_i^-(1) \right] \right] \\
= \left[ \left[ x_i^+(0), a_i(\ell) \right], x_i^-(1) \right] + \left[ a_i(\ell), \left[ x_i^+(0), x_i^-(1) \right] \right] \\
= *\gamma^{\frac{1}{2}} \left[ x_i^+(\ell), x_i^-(1) \right] \\
+ \left[ a_i(\ell), \gamma^{-\frac{1}{2}} \omega_i a_i(1) \right] \quad \text{(this term= 0 by (D2))}
\]
\[
= *(\gamma \gamma')^{-\frac{1}{2}} \gamma^{-\frac{1}{2}} \omega_i \left[ a_i(\ell+1) + \sum_{1 \leq p \leq q+1} \omega_j^{(r-s)p-1} a_j(\ell_{j_1}) \cdots a_j(\ell_{j_p}) \right], \\
(4.17)
\]

where scalars $*, *' \in \mathbb{K} \setminus \{0\}$. So $a_i(\ell+1) \in U'_{r,s}(\hat{\mathfrak{sl}}_n)$.

Applying $\tau$ to the above formula, we can get $a_i(-(\ell+1)) \in U'_{r,s}(\hat{\mathfrak{sl}}_n)$. Thereby, $a_i(\ell) \in U'_{r,s}(\hat{\mathfrak{sl}}_n)$, for any $\ell \in \mathbb{Z} \setminus \{0\}$.

(2) follows from (D6) (setting $i = j$ and $k = 0$), together with (1). \(\square\)

4.4. Proof of Theorem 4.3. From Sects. 4.2 & 4.3, we actually get an algebra epimorphism $\Psi : U_{r,s}(\hat{\mathfrak{sl}}_n) \longrightarrow U_{r,s}(\hat{\mathfrak{sl}}_n)$, since both algebras have essentially the same generators system enjoying the defining relations from the former.

Notice that both algebras $U_{r,s}(\hat{\mathfrak{sl}}_n)$ and $\hat{\mathfrak{sl}}_n$ have commonly a natural $Q$-gradation structure (see Corollary 2.8), which is by definition preserved evidently under $\Psi$. On the other hand, both toral subalgebras $U_{r,s}(\hat{\mathfrak{sl}}_n)^0$ and $U_{r,s}(\hat{\mathfrak{sl}}_n)^0$ generated by the same generators system of group-like elements

\[
\left\{ \omega_{i}^{\pm 1}, \omega_{i}^{i \pm 1} (i \in I_0), \gamma^{1 \pm 1}, \gamma'^{1 \pm 1}, D^{\pm 1}, D'^{\pm 1} \right\}
\]

are obviously isomorphic with respect to $\Psi^0 := \Psi|_{U_{r,s}(\hat{\mathfrak{sl}}_n)^0}$.

Assigned to the positive or negative nilpotent Lie subalgebra $\hat{\mathfrak{n}}^{\pm}$ of $\hat{\mathfrak{sl}}_n$ are two subalgebras $U_{r,s}(\hat{\mathfrak{n}}^{\pm})$ and $U_{r,s}(\hat{\mathfrak{n}}^{\pm})$. Both are generated by $\hat{\mathfrak{n}}^{\pm}$ in $U_{r,s}(\hat{\mathfrak{sl}}_n)$ and $\hat{\mathfrak{n}}_{s}^{\pm}$ respectively. Denote $\Psi^{\pm} := \Psi|_{U_{r,s}(\hat{\mathfrak{n}}^{\pm})}$. By Corollary 2.7, the double structure of $U_{r,s}(\hat{\mathfrak{sl}}_n)$ in Theorem 2.5 implies its triangular decomposition structure $U_{r,s}(\hat{\mathfrak{n}}^{\pm}) \otimes U_{r,s}(\hat{\mathfrak{n}}^{\pm})$. This fact likewise indicates that $\Psi$ has a corresponding decomposition $\Psi^{-} \otimes \Psi^{0} \otimes \Psi^{+}$. So, we are left to show $\Psi^{\pm}$ are isomorphic. It suffices to consider the epimorphism $\Psi^{\pm} : U_{r,s}(\hat{\mathfrak{n}}^{\pm}) \longrightarrow U_{r,s}(\hat{\mathfrak{n}}^{\pm})$.

Observe that $U_{r,s}(\hat{\mathfrak{n}}^{\pm})$ (resp. $U_{r,s}(\hat{\mathfrak{n}}^{\pm})$) is generated by elements $e_i$ (resp. $E_i$) for $i \in I_0$ and subject to $(r,s)$-Serre relations (A5) & (A6). To check that $\Psi^{\pm}$ is an isomorphism, now we fix $r = q$ and specialize $s$ at $q^{-1}$ as follows.

Note that $U_{r,s}(\hat{\mathfrak{n}}^{\pm})$ can be viewed as defined over the Laurent polynomials ring $\mathbb{Q}[r^{\pm 1}, s^{\pm 1}]$. Let $\mathcal{A} \subset \mathbb{Q}(r,s)$ be the localization of ring $\mathbb{Q}[r^{\pm 1}, s^{\pm 1}]$ at the maximal ideal $(r s - 1)$. Let $U_{\mathcal{A}}^+$ be the $\mathcal{A}$-subalgebra of $U_{r,s}(\hat{\mathfrak{n}}^{\pm})$ generated by $e_i$ ($i \in I_0$). Let $(r s - 1)U_{\mathcal{A}}^+$ be the ideal generated by $(r s - 1)$ in $U_{\mathcal{A}}^+$. Define the algebra $U_q^+$, the specialization of $U_{r,s}(\hat{\mathfrak{n}}^{\pm})$ at $s = q^{-1}$, by $U_q^+ = U_{\mathcal{A}}^+ / (r s - 1)U_{\mathcal{A}}^+$. Obviously, $U_q^+ \cong U_q(\hat{\mathfrak{n}}^{\pm})$, the
usual one-parameter quantum subalgebra of $U_q(\widehat{sl}_n)$. However, in this case, $\Psi^+$ induces the isomorphism $\Psi^+: U_q(\widehat{\mathfrak{g}}^+) \rightarrow U_q(\widehat{\mathfrak{g}}^+)$ given by the Drinfel’d isomorphism in the one-parameter case (see [B2] or [J2]).

Since specialization doesn’t change the root multiplicities, $\Psi^+: U_{r,s}(\widehat{\mathfrak{g}}^+) \rightarrow U_{r,s}(\widehat{\mathfrak{g}}^+)$ is an isomorphism. □

Up to now, from subsections 4.2—4.4, we have finally established the Drinfel’d isomorphism in the two-parameter case.

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