Some Algebraic Geometry Aspects of Gravitational Theories with Covariant and Contravariant Connections and Metrics (GTCCCM) and Possible Applications to Theories with Extra Dimensions

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Abstract

On the base of the distinction between covariant and contravariant metric tensor components, an approach from algebraic geometry will be proposed, aimed at finding new solutions of the Einstein’s equations both in GTCCCM and in standard gravity theory, if these equations are treated as algebraic equations.

As a partial case, some physical applications of the approach have been considered in reference to theories with extra dimensions. The s.c. "length function" $l(x)$ has been introduced and has been found as a solution of quasilinear differential equations in partial derivatives for two different cases, corresponding to "compactification + rescaling" and "rescaling + compactification" of the type I low-energy string theory action. New (although complicated) relations between the parameters in the action have been found, valid also for the standard approach in theories with extra dimensions.

1 Introduction and Statement of the Problem

Inhomogeneous cosmological models have been intensively studied in the past in reference to colliding gravitational waves [1] or singularity structure and generalizations of the Bondi - Tolman and Eardley-Liang-Sachs metrics [2, 3]. In these models the inhomogeneous metric is assumed to be of the form [2]

$$ds^2 = dt^2 - e^{2\alpha(t,r,y,z)}dr^2 - e^{2\beta(t,r,y,z)}(dy^2 + dz^2)$$ (1.1)
(or with $r \rightarrow z$ and $z \rightarrow x$) and it is called the Szafron-Szekeres metric [4-7]. In [7], after an integration of one of the components - $G_0^1$ of the Einstein’s equations, a solution in terms of an elliptic function is obtained.

This is an important point since valuable cosmological characteristics for observational cosmology such as the Hubble’s constant $H(t) = \frac{\dot{R}(t)}{R(t)}$ and the deceleration parameter $q = -\frac{\ddot{R}(t)R(t)}{R^2(t)}$ may be expressed in terms of the Jacobi’s theta function and of the Weierstrass elliptic function respectively [8]. In the same paper, the expression for the metric of the anisotropic cosmological model in the Szafron-Szekeres approach has been obtained in terms of the Weierstrass elliptic function after reducing the component $G_0^1$ of the Einstein’s equations [7, 8] to the nonlinear differential equation

$$\left(\frac{\partial \Phi}{\partial t}\right)^2 = -K(z) + 2M(z)\Phi^{-1} + \frac{1}{3}\Lambda\Phi^2.$$ (1.2)

Then if one sets up

$$g_2 = \frac{K^2(z)}{12}; \quad g_3 = \frac{1}{216}K^3(z) - \frac{1}{12}\Lambda M^2(z),$$ (1.3)

the resulting two-dimensional cubic algebraic equation

$$y^2 = 4x^3 - g_2x - g_3$$ (1.4)

according to the standard algebraic geometry prescription (see [9] for a contemporary introduction into algebraic geometry) can be parametrized as

$$x = \rho(z), \quad y = \rho'(z),$$ (1.5)

where $\rho(z)$ is the well-known Weierstrass elliptic function

$$\rho(z) = \frac{1}{z^2} + \sum_\omega \left[ \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right]$$ (1.6)

and the summation is over the poles in the complex plane. However, it is important to remind the standard definition of the elliptic curve (1.5), according to which the points $z$ on the complex plane, factorized by the lattice $\Lambda = \{ m\omega_1 + n\omega_2 \mid m, n \in \mathbb{Z}; \omega_1, \omega_2 \in \mathbb{C}, \text{Im} \omega_1 \omega_2 > 0 \}$, are mapped via the mapping $f : C/\Lambda \rightarrow \mathbb{C}P^2$ onto the points $(x, y) = (\rho(z), \rho'(z))$ of the two dimensional complex projective space $\mathbb{C}P^2$, which satisfy the curve (1.5). This definition in fact imposes an important restriction on the functions $g_2$ and $g_3$, namely they should be equal to the s.c. "Eisenstein series"

$$g_2 = 60 \sum_{\omega \in \Gamma} \frac{1}{\omega}, \quad g_3 = 140 \sum_{\omega \in \Gamma} \frac{1}{\omega^3}.$$ (1.7)

In other words, if the parametrization (1.5) is possible also for the nonlinear differential equation (1.2), then the Eisenstein series imply additional restrictions.
on the functions in the right-hand side of (1.3), which has not been taken into account in the paper [8]. However, there is another reasoning if one admits that the Weierstrass function and its derivatives can parametrize a cubic algebraic equation of the type (1.3) with non-constant coefficient functions (depending on the complex variable $z$) - to a certain extent this problem has been analysed in [10].

2 Fundamentals of the New Algebraic Geometry Approach in Gravity Theory

The main goal of the present paper and of the preceding ones [10, 11] is to propose a new algebraic geometry approach for finding new solutions of the Einstein’s equations by representing these equations in an algebraic form. The approach also is based on the gravitational theory with covariant and contravariant metrics and connections (GTCCMC) [12]. This theory makes a clear distinction between covariant $g_{ij}$ and contravariant metric tensor components $\tilde{g}^{is}$, which means that $\tilde{g}^{is}$ should not be considered to be the inverse ones to the covariant components $g_{ij}$, consequently $\tilde{g}^{is}g_{im} \equiv f_{m}^{s}(x)$. It is elementary to prove [11,12] that if the components of $f_{m}^{s}(x)$ are considered to be functions, then two connections should be introduced - one connection $\Gamma^{s}_{\alpha\beta}$ for the case of the parallel transport of covariant basic vectors $\nabla_{e_{\beta}} e_{\alpha} = \Gamma_{\alpha\beta}^{\gamma} e_{\gamma}$ and a separate connection $P_{\alpha\beta}^{s}$ for the contravariant basic vector $e^{\alpha}$, the defining equation for which is $\nabla_{e_{\beta}} e^{\alpha} = P_{\alpha\beta}^{s} e^{\gamma}$. However, if $f_{m}^{s}(x)$ are considered to be tensor components, then by simple covariant differentiation of $\tilde{g}^{is}g_{im} \equiv f_{m}^{s}(x)$ it can be proved that it is not obligatory to introduce a second connection $P_{\alpha\beta}^{s}$, i.e. it can be assumed that $P_{\alpha\beta}^{s} = -\Gamma_{\alpha\beta}^{s}$, as it is the case in the standard Einsteinian theory of gravity. This shall further be exploited, when assuming that the contravariant metric tensor can be represented in the form of the factored product $\tilde{g}^{ij} = dX^{i}dX^{j}$, where the differentials $dX^{i}$ remain in the tangent space $T_{X}$ of the generalized coordinates $X^{i} = X^{i}(x_{1}, x_{2}, ..., x_{n})$, defined on the initially given manifold. Also, the existence of different from $g^{ij}$ contravariant metric tensor components $\tilde{g}^{ij}$ means that another connection

$$\tilde{\Gamma}^{s}_{kl} \equiv \tilde{g}^{is}\Gamma_{i;kl} = \tilde{g}^{is}g_{im}\Gamma_{kl}^{m} = \frac{1}{2}\tilde{g}^{is}(g_{ik,l} + g_{il,k} - g_{kl,i}) \quad (2.1)$$

not consistent with the initial metric $g_{ij}$, can be introduced. By substituting $\tilde{\Gamma}^{s}_{kl}$ in the expression for the ”tilda” Ricci tensor $\tilde{R}_{ij}$ and requiring the equality of the ”tilda” scalar curvature $\tilde{R}$ with the usual one $R$, i.e. $\tilde{R} = R$, one can obtain the s.c. ”cubic algebraic equation for reparametrization invariance of the gravitational Lagrangian” [10]

$$dX^{i}dX^{l} \left( P_{ijkl} d^{2}X^{k} - \Gamma_{ik}^{r}g_{lr}d^{2}X^{k} - \Gamma_{ik}^{r}(g_{lr}d^{2}X^{k}) - dX^{i}dX^{l}R_{il} = 0 \quad (2.2)$$

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In the same way, assuming the contravariant metric tensor components to be equal to the "tilde" ones, the Einstein’s equations in vacuum were derived. They can be derived also in the general case for arbitrary \( \tilde{g}^{ij} \), when the assumption about the factorized representation \( \tilde{g}^{ij} = dX^i dX^j \) is no longer implemented:

\[
0 = \tilde{R}_{ij} - \frac{1}{2} g_{ij} \tilde{R} =
\]

\[
\tilde{R}_{ij} = \tilde{g}^{jl} \left( \Gamma_{r[ij]}^l - \Gamma_{r[j]}^l \right) + \tilde{g}^{lr} \left( \Gamma_{r[ij]}^l - \Gamma_{r[j]}^l \right) \Gamma_{s,lm} - \Gamma_{s,il} \Gamma_{r,km} \}
\]

Interestingly, this "algebraic" system of the Einstein’s equations can be considered as well as a system of fifth-degree algebraic equations with respect to the covariant variables and also as a system of third-rank equations (with non-constant coefficient functions) with respect to the contravariant variables and their derivatives. The mathematical treatment of fifth-degree equations is known since the time of Felix Klein’s famous monograph [13], published in 1884. A way for resolution of such equations has been developed also in [9, 14, 15].

3 Application to Theories with Extra Dimensions. Tensor Length Scale.

In view of the application of the algebraic geometry approach in theories with extra dimensions, the cubic algebraic equation for reparametrization invariance of the gravitational Lagrangian in the general case (for arbitrary \( \tilde{g}^{ij} \)) (eq. (5.2) in [11])

\[
\tilde{g}^{[ij]} g_{[k]}^l \Gamma_{ik}^l g_{rs} + \tilde{g}^{[ij]} g_{[k]}^l \left( \Gamma_{ik}^l \Gamma_{r}^s \right)_{,l} - \tilde{g}^{[ij]} g_{[k]}^l g_{mr} g_{ps} \left( \Gamma_{ik}^l \Gamma_{lm}^p - \Gamma_{il}^p \Gamma_{km}^q \right) - R = 0 .
\]

(3.1)

can be represented in the form [16]

\[
\varepsilon \frac{\partial l}{\partial y} + \frac{h'}{8k} e^{2ky} \left( \frac{\partial l}{\partial x_1} - \frac{\partial l}{\partial x_2} - \frac{\partial l}{\partial x_3} - \frac{\partial l}{\partial x_4} \right) =
\]

\[
(2k - \varepsilon \frac{h'}{h})(l^3 - l)
\]

(3.2)

for the special case of the metric of a 4D flat Minkowski space, embedded in a five-dimensional space

\[
ds^2 = e^{-2k y} \eta_{\mu \nu} dx^\mu dx^\nu + h(y) dy^2 ,
\]

(3.3)

where \( \eta_{\mu \nu} = (+, -, -, -) \) and \( \varepsilon = \pm 1 \). Note that the metric (3.3) will be with a negative constant (scalar) curvature if additionally one can set up \( h(y) = 1 \). The
obtained equation (3.2) is no longer treated as an algebraic one, but as a quasi-linear differential equation in partial derivatives with respect to the function $l(x)$, which was introduced in [10,16] and has been called "the length function". The key idea for its introduction is that the contraction of the covariant metric tensor $g_{ij}$ with the contravariant one $\tilde{g}^{jk} = dX^j dX^k$ gives exactly (when $i = k$) the length interval

$$ l = ds^2 = g_{ij} dx^i dx^j . \quad (3.4) $$

Then naturally, for $i \neq k$ the contraction will give a tensor function $l^i_k = g_{ij} dX^j dX^k$ (since it is assumed also that the differentials $dX^j$ in the tangent space are vectors - in the general case this might not be true), which can be interpreted as a "tensor" length scale for the different directions. Thus in deriving the quasilinear differential equation (3.2) it had been assumed that $\tilde{g}^{ij} g_{ij} = f_m^i(x) := l^i_i = \delta^i_i$. In principle, it can be accepted that a "hint" for the necessity for introduction of such a "tensor scale" is contained in Witten's paper [17], where it was remarked that "the problem might be ameliorated by considering an anisotropic Calabi - Yau with a scale $\sqrt{\alpha}$ in $d$ directions and $\frac{1}{M_{\text{GUT}}}$ in $(6 - d)$ directions".

The quasilinear differential equation (3.2) has the following solution for the square of the length function

$$ l^2 = \frac{1}{1 - \text{const.} \ v^{24} \ k \ v} \quad \varepsilon = \pm 1 \quad , \quad (3.5) $$

From a physical point of view it is interesting to note that the "scale function" will indeed be equal to one (i.e. we have the usual gravitational theory with $\tilde{g}^{ij} = g^{ij}$ and $l = 1$) for $\varepsilon = -1$ and $v \to \infty$ (the s.c. infinite extra dimensions).

4 Compactification in Low Energy Type I Ten-Dimensional String Theory - A Brief Review of the Standard Approach

The standard approach is based on the low-energy action of type I string theory in ten dimensions [18, 19, 20]

$$ S = \int d^{10}x \left( \frac{m_s^8}{(2\pi)^7 \lambda} R + \frac{1}{4} \frac{m_s^6}{(2\pi)^7 \lambda} F^2 + ... \right) = \int d^4x V_6(...) , \quad (4.1) $$

where $\lambda \sim \exp(\Phi)$ is the string coupling, $m_s$ is the string scale, which we can identify with $m_{\text{grav}}$. Compactifying to 4 dimensions on a manifold of volume $V_6$, one can identify the resulting coefficients in front of the $R$ and $\frac{1}{4} F^2$ terms with $M^2_{(4)}$ and $\frac{1}{24}$, from where one obtains [18]

$$ M^2_{(4)} = \frac{(2\pi)^7}{V_6 m_s^3 g_s^2} ; \quad \lambda = \frac{g_s^2 V_6 m_s^6}{(2\pi)^7} \quad . \quad (4.2) $$
Note that although one of the key assumptions in extra-dimensional theories is that the volume $V_6$ is very large (and thus the scale for the four-dimensional gravitational constant $M_{(4)}^2$ is considerably lowered), there is a large indeterminacy, since the exact value of $V_6$ is not known.

5 "Compactification + Rescaling" and "Rescaling + Compactification" - A Proposed New Approach

The essence of the proposed new approach is that the operation of compactification is "supplemented" by the additional operation of "rescaling" of the contravariant metric tensor components in the sense, which was clarified in Section 2. One can discern two cases:

1st case - "compactification + rescaling". One starts from the "unrescaled" ten-dimensional action (4.1), then performs a compactification to a five-dimensional manifold and afterwards a transition to the usual "rescaled" scalar quantities $\tilde{R}$ and $\tilde{F}^2$. Then it is required that the "unrescaled" ten-dimensional effective action (4.1) (i.e. the L.H.S of (2.1)) is equivalent to the five-dimensional effective action after compactification, but in terms of the rescaled quantities $\tilde{R}$ and $\tilde{F}^2$ in the R.H.S of (4.1). This can be expressed as follows

$$S = \int d^{10}x \left( \frac{m_s^8}{(2\pi)^7 \lambda^2} R + \frac{1}{4} \frac{m_s^6}{(2\pi)^7 \lambda} F^2 \right) = \int d^4 x V_6 (\ldots) = \int d^4 x \left( M_{(4)}^2 \tilde{R} + \frac{1}{4} \frac{1}{g_4^2} \tilde{F}^2 \right).$$

(5.1)

Identifying the expressions in front of the "unrescaled" scalar quantities $F^2$ and $R$, one obtains an algebraic relation and a differential equation in partial derivatives. Combining them, the following differential equation is obtained [16]:

$$\frac{(2\pi)^7 l^3}{m_s^4 V_6 g_4^4} \left( \frac{\partial l}{\partial x^A} g^{AB} \Gamma_C^{AB} - \frac{\partial l}{\partial x^B} g^{AB} \Gamma_C^{AC} \right) + \frac{R P^2}{l^3 (P - N R)^2} \left[ \frac{(2\pi)^7 l^2}{m_s^4 V_6 g_4^4} - M_4^2 \right] +$$

$$+ \frac{(2\pi)^7 l (l - 1)}{m_s^4 V_6 g_4^4} g^{AB} \left( \Gamma_C^{AB} \Gamma_D^{CD} - \Gamma_D^{AC} \Gamma_B^{CD} \right) = 0.$$ (5.2)

2nd case - "rescaling + compactification". In this case, first a rescaling of the contravariant metric components is performed, which means that one starts from the ten-dimensional string theory action with "rescaled" components

$$S = \int d^{10}x \left( \frac{m_s^8}{(2\pi)^7 \lambda^2} \tilde{R} + \frac{1}{4} \frac{m_s^6}{(2\pi)^7 \lambda} \tilde{F}^2 \right) = \int d^4 x V_6 (\ldots) = \int d^5 x \left( M_{(4)}^2 \tilde{R} + \frac{1}{4} \frac{1}{g_4^2} \tilde{F}^2 \right).$$

(5.3)
and after that the compactification is realized, resulting again in the right-hand side of the standard 4D action (5.1), but this time with unrescaled components. In an analogous way, the following algebraic relation can be obtained [16]

\[
\left[ \frac{(2\pi)^7}{V_0 m_4^2 g_4^2} - M_{(4)}^2 \right] R = \frac{(2\pi)^7 (l^2 - 1)}{2 m_4^2 V_4 l^2 g_4^2} g^{AC} g^{BD} (\ldots) ,
\]

where for brevity the brackets denote \((\ldots) := (g_{AD,BC} + g_{BC,AD} - g_{AC,BD} - g_{BD,AC})\). For \(l = 1\), as expected, we obtain the usual relation for \(M_{(5)}^2\) as in (4.2). Now introducing the notation

\[
\beta \equiv \left[ \frac{(2\pi)^7}{V_0 m_4^2 g_4^2} - M_{(4)}^2 \right] m_4^2 V_6 \left( \frac{2}{(2\pi)^7} \right) ,
\]

and assuming a small deviation from the relation \(M_{(4)}^2 = \frac{(2\pi)^7}{V_0 m_2^2 g_4^2}\), i.e. \(\beta \ll 1\), one can express the length scale \(l(x)\) from the algebraic relation (5.4) as

\[
l^2 = \frac{1}{1 - \beta g^{AC} g^{BD} (\ldots)} \approx 1 + \beta \frac{R}{g^{AC} g^{BD} (\ldots)}.
\]

Consequently the deviation from the "standard" length scale \(l = 1\) in the case of a gravitational theory with \(l \neq 1\) in the case of small \(\beta\) shall be proportional to the ratio \(\frac{R}{g^{AC} g^{BD} (\ldots)}\), which for the assumed metric (3.3) with \(h(y) = 1\) is exactly equal to \(\frac{1}{g_{BD,AC}}\) and thus, this constant number will not affect the smallness of the number \(\beta \frac{R}{g^{AC} g^{BD} (\ldots)}\). The above result may have also an important physical meaning - the zero value of the number \(\beta\) (which signifies the fulfillment of the relation \(M_{(4)}^2 = \frac{(2\pi)^7}{V_0 m_2^2 g_4^2}\)) can be related to the usual length scale \(l = 1\) in gravity theory. Conversely, any deviations from the relation (4.2) may be attributed to deviations from the usual scale \(l = 1\). Note also that the assumption \(\beta \ll 1\) poses some restriction on the number \(m_4^2 V_6 \left( \frac{2}{(2\pi)^7} \right)\) in (5.5) - it should not be too large.

The differential equation in partial derivatives for the scale function \(l(x)\) for this case ("rescaling + compactification") is different from the preceeding one (5.2)

\[
\left[ \frac{(2\pi)^7}{m_4^2 V_4 g_4^2 l^2} - M_{(4)}^2 \right] R + \frac{(2\pi)^7}{m_4^2 V_3 g_4^2} \frac{(l - 1)}{l^2} g^{AB} \left( \Gamma^C_{AB} \Gamma^D_{CD} - \Gamma^D_{AC} \Gamma^C_{BD} \right) +
\]

\[
+ \frac{(2\pi)^7}{m_4^2 V_4 g_4^2} \frac{1}{l^3} \left[ \frac{\partial l}{\partial x^C} g^{AB} \Gamma^C_{AB} - \frac{\partial l}{\partial x^B} g^{AB} \Gamma^C_{AC} \right] = 0 .
\]

3rd case - simultaneous fulfillment of "rescaling + compactification" and "compactification + rescaling". In the formal sense, this may mean that it does not matter in what sequence the two operations are performed, i.e. the process of compactification is accompanied by rescaling. From the simultaneous fulfillment
of the two differential equations (5.2) and (5.7) one obtains the following cubic algebraic equation with respect to $l^2$:

$$M_4^2 l^6 - \frac{(2\pi)^7}{m_4^4 V_6 g_4^4} l^4 + \frac{(2\pi)^7}{m_4^4 V_6 g_4^4} \frac{P^2}{(P - NR)^2} l^2 - \frac{M_4^2}{P - NR} \frac{P^2}{(P - NR)^2} = 0. \quad (5.8)$$

The length function $l(x)$ can be an imaginary one for the case of the imaginary Lobachevsky space [12], realized by all the straight lines outside the absolute cone (on which the scalar product is zero, i.e. $[x, x] = 0$- the length may may take imaginary values in the interval $[0, \frac{\pi k}{2}]$, where $k$ is the Lobachevsky constant). Further we shall assume that $l(x)$ is a real function, which will mean that the roots $l^2(x)$ of the cubic equation have to be positive. From this requirement one obtains the following inequalities, relating the parameters in the low energy type I string theory action

$$p^2 = \frac{b^2}{2} + \frac{a^3}{27} - b \sqrt{\frac{b^2}{2} + \frac{a^3}{27}} > \left[ \frac{a_1 + 6a}{18} + \frac{a_1}{18} \sqrt{a_1^2 + 12a} \right]^3 \quad (5.9)$$

or for

$$p^2 = \frac{b^2}{2} + \frac{a^3}{27} - b \sqrt{\frac{b^2}{2} + \frac{a^3}{27}} < \left[ \frac{a_1 + 6a}{18} - \frac{a_1}{18} \sqrt{a_1^2 + 12a} \right]^3, \quad (5.10)$$

where

$$a \equiv a_2 - \frac{a_2^2}{3} ; \quad b \equiv 2 \frac{a_3}{27} - \frac{a_1 a_2}{3} + a_3 , \quad (5.11)$$

$$a_1 \equiv - \frac{(2\pi)^7}{M_4^2 m_4^4 V_6 g_4^4} ; \quad a_2 \equiv \frac{(2\pi)^7}{M_4^2 m_4^4 V_6 g_4^4 Q^2} ; \quad a_3 \equiv - \frac{g^2}{Q^2} , \quad (5.12)$$

$$Q \equiv \frac{g^{AC} g^{BD} (2\pi)^7 g_4^4 (\ldots)}{[g^{AC} g^{BD} (\ldots) (2\pi)^7 g_4^4 - 2 R ((2\pi)^7 - M_4^2 V_6 m_4^4 g_4^4)]} . \quad (5.13)$$

The last two inequalities (5.9) - (5.10) are new (although rather complicated) inequalities between the parameters in the low-energy type I string theory action, which cannot be obtained in the framework of the usual gravity theory. Also, it is important to stress that since the scale function $l(x)$ does not enter in them, they are valid for the standard gravity theory approach in theories with extra dimensions.

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