Maximal nonassociativity via fields

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Abstract
We say that \((x, y, z) \in Q^3\) is an associative triple in a quasigroup \((Q, \ast)\) if \((x \ast y) \ast z = x \ast (y \ast z)\). Let \(a(Q)\) denote the number of associative triples in \(Q\). It is easy to show that \(a(Q) \geq |Q|\), and we call the quasigroup maximally nonassociative if \(a(Q) = |Q|\). It was conjectured that maximally nonassociative quasigroups do not exist when \(|Q| > 1\). Drápal and Lisoněk recently refuted this conjecture by proving the existence of maximally nonassociative quasigroups for a certain infinite set of orders \(|Q|\). In this paper we prove the existence of maximally nonassociative quasigroups for a much larger set of orders \(|Q|\). Our main tools are finite fields and the Weil bound on quadratic character sums. Unlike in the previous work, our results are to a large extent constructive.

Keywords Nonassociativity · Latin square · Finite field · Weil bound

1 Maximally nonassociative quasigroups

A quasigroup \((Q, \ast)\) is a set \(Q\) with a binary operation \(\ast\) such that for all \(a, b \in Q\) there exist unique \(x, y \in Q\) such that \(a \ast x = b\) and \(y \ast a = b\). Hence a binary operation on a finite set yields a quasigroup if and only if its multiplication table is a Latin square. In this paper we only deal with finite quasigroups.

We call a triple \((x, y, z) \in Q^3\) associative if \((x \ast y) \ast z = x \ast (y \ast z)\), and we denote the number of associative triples in \(Q\) by \(a(Q)\). For each \(c \in Q\) there exist \(x, y\) such that \(c \ast x = c\) and \(y \ast c = c\). Then \((y, c, x)\) is an associative triple and it follows that \(a(Q) \geq |Q|\). We call \(Q\) maximally nonassociative when \(a(Q) = |Q|\). The existence of such quasigroups has been investigated for at least four decades \([10,11]\). It was shown \([9]\) that quasigroups with few associative triples can be used in the design of hash functions in cryptography. Grošek and Horáš conjectured that maximally nonassociative quasigroups do not exist when \(|Q| > 1\).
[9, Conjecture 1.2]. An example of a maximally nonassociative quasigroup of order 9 was found recently [4]. No example of order greater than 1 and less than 9 exists, and no example of order 10 exists [3,4]. It is known [9, Theorem 1.1] that any maximally nonassociative quasigroup must be idempotent, that is, \( x \ast x = x \) for all \( x \in Q \).

Very recently Drápal and Lisoněk [2] used Dickson’s quadratic nearfields to prove existence of an infinite set of maximally nonassociative quasigroups. Specifically they proved:

**Theorem 1.1** [2, Corollary 5.8] Let \( m = 2^kr \) where \( k \geq 0 \) is an integer and \( r \) is odd. There exists a maximally nonassociative quasigroup of order \( m^2 \).

In this paper we greatly extend the set of orders for which the existence of maximally nonassociative quasigroups can be proved (see Theorem 4.1 below). As well, our approach is more constructive in comparison to the methods of [2].

### 1.1 Prescribed automorphism groups

The method of prescribed automorphism group has been widely used to construct combinatorial objects with distinguished properties. One starts with some permutation group \( G \), and with the assumption that \( G \) is a group of symmetries of the desired object (but perhaps not its full automorphism group). It is then sufficient to specify the behaviour of the object on the orbits of \( G \), thus cutting down the complexity of computer search or theoretical proofs. Often there is some natural choice for \( G \), which may or may not work. If \( G \) itself does not work, then one uses proper subgroups of \( G \). This process will proceed by increasing the index of the subgroup, from smaller index to larger index. The reason for proceeding in this particular way is that the number of orbits (and thus the complexity of computer search or theoretical proofs) will generally increase with the index of the subgroup.

Let \( L \) be a binary operation on the set \( Q \), i.e., a mapping \( L \) from \( Q \times Q \) to \( Q \). In the sequel, we will use only \( L \) instead of “\( Q \) with the operation \( L \)” An automorphism of \( L \) is a bijection \( f \) of \( Q \) onto \( Q \) such that \( L(f(x), f(y)) = f(L(x, y)) \) for all \( x, y \in Q \). We note that \((x, y, z)\) is associative if and only if \((f(x), f(y), f(z))\) is associative.

We wish to apply the idea of prescribed automorphism group \( G \) to constructing maximally nonassociative quasigroup \( L \). Since \( L \) is known to be idempotent, we need to specify only the values \( L(x, y) \) for \( x \neq y \). The most desirable choice for \( G \) would be a group that acts primitively transitively on the ordered pairs \((x, y)\) with \( x \neq y \), known as sharply 2-transitive (S2T) group. It is known that finite S2T groups are in one-to-one correspondence with nearfields, and this underlies the approach taken in [2], where maximally nonassociative quasigroups are obtained from proper nearfields, that is, those which are not fields.

It still remains to explore the case when the nearfield is not proper, that is, it is the finite field of order \( q \), denoted \( \mathbb{F}_q \). In this case the corresponding S2T group is the Frobenius group of invertible affine mappings \( x \mapsto \alpha x + \beta \) on \( \mathbb{F}_q \). Let \( G_q \) denote this group. It is shown in [2] that the approach used there does not produce maximally nonassociative quasigroups when applied to the group \( G_q \). Therefore, following the strategy outlined above, one proceeds to proper subgroups of \( G_q \) of low index. Luckily, when \( q \) is odd, there is always a subgroup of index 2, which is the group of invertible affine mappings \( x \mapsto \alpha x + \beta \) where \( \alpha \) is a non-zero square in \( \mathbb{F}_q \). Amazingly, this group works for constructing a very large set of maximally nonassociative quasigroups, and this is the approach that we take in this paper.
2 Constructions

By the parity of \( a \in \mathbb{F}_q \) we mean the squareness of \( a \), hence the parity is either “square” or “non-square.”

**Definition 2.1** Let \( q \) be an odd prime power. For fixed \( a,b \in \mathbb{F}_q \) define the mapping \( L_{a,b} : \mathbb{F}_q \times \mathbb{F}_q \to \mathbb{F}_q \) by

\[
L_{a,b}(x,y) = \begin{cases} 
 x + a(y-x) & \text{if } y-x \text{ is square} \\
 x + b(y-x) & \text{if } y-x \text{ is non-square}.
\end{cases}
\]

Note that \( L_{a,b}(x,x) = x \) for all \( x \in \mathbb{F}_q \). In the special case \( a = b \neq 0,1 \) we get a classical construction of idempotent quasigroups due to Bose et al. [12, p. 288].

**Lemma 2.2** Let \( q \) be an odd prime power and let \( L_{a,b} : \mathbb{F}_q \times \mathbb{F}_q \to \mathbb{F}_q \) be as above. Let \( G^{(2)}_q \) denote the group of mappings \( u \mapsto \alpha u + \beta \) where \( \alpha, \beta \in \mathbb{F}_q \) and \( \alpha \) is a non-zero square. Then \( G^{(2)}_q \) is a group of automorphisms of \( L_{a,b} \).

**Proof** Let \( f \in G^{(2)}_q \), \( f(u) = \alpha u + \beta \). Let \( (x,y) \in \mathbb{F}_q^2 \) and assume \( y-x \) is square. Then \( f(y) - f(x) = \alpha(y-x) \) is also a square, and

\[
f(L_{a,b}(x,y)) = f(x + a(y-x)) = \alpha(x + a(y-x)) + \beta
= \alpha x + \beta + a(\alpha(y-x)) = f(x) + a(f(y) - f(x))
= L_{a,b}(f(x), f(y)).
\]

The case when \( y-x \) is non-square is analogous. \( \square \)

Letting \( G^{(2)}_q \) act naturally on \( \mathbb{F}_q^2 \) we observe that there are three orbits. One orbit contains the diagonal pairs \( (x,x) \), and \( G^{(2)}_q \) acts primitively on the remaining two orbits, one of which contains pairs \( (x,y) \) where \( y-x \) is non-zero square and the other orbit contains pairs where \( y-x \) is non-square.

Evans determined the conditions under which \( L_{a,b} \) is a quasigroup [7].

**Lemma 2.3** [7, Lemma 2(a)] Let \( q \) be an odd prime power. Then \( L_{a,b} \) is a quasigroup if and only if \( ab \) and \( (a-1)(b-1) \) are both non-zero squares.

It turns out that if we restrict attention to the case \( b = a^2 \), then a large family of maximally nonassociative quasigroups can be constructed, and the necessary conditions and proofs simplify considerably. In the rest of the paper we focus on the case \( b = a^2 \). We will first state Evans’ condition for our special case.

**Corollary 2.4** Let \( q \) be an odd prime power, and let \( a \in \mathbb{F}_q \) be such that \( a \neq \{-1,0,1\} \) and both \( a \) and \( a+1 \) are squares. Then \( L_{a,a^2} \) is a quasigroup.

**Proof** By Lemma 2.3 \( L_{a,a^2} \) is a quasigroup if and only if both \( aa^2 = a^3 \) and \( (a-1)(a-1) = (a-1)^2(a+1) \) are non-zero squares. Since \( a^3 \) has the same parity as \( a \), the result follows. \( \square \)

**Theorem 2.5** Let \( q \equiv 1 \pmod{4} \) be a prime power. Let \( a \in \mathbb{F}_q \) be such that \( a \neq -1,0,1 \), and the elements \( a, a+1, a^3-a-1 \) are squares, and the elements \( a-1, a^2+1, a^2-a-1, a^2+a+1, a^2+a-1 \) are non-squares. Then \( L_{a,a^2} \) is a maximally nonassociative quasigroup.

\( \square \) Springer
then there exists an automorphism \( f \) hence showing that this case can not occur. Once all eight cases are shown to be impossible, nonexistence of associative triples of the form \((z, \eta, \zeta)\) is established. The details of all eight cases are summarized in Table 1, where S denotes square and N denotes non-square.

| \( z - a \) | \( z - 1 \) | \( L(1, z) \) | \( z^* \) | Contradiction |
|------------|------------|--------------|--------|-------------|
| S          | S          | S            | 0      | \( L(1, z^*) = 1 - a \) is N |
| S          | S          | N            | \( \frac{a-1}{a+1} \) | \( z^* - a = -\frac{a^2+1}{a+1} \) is N |
| S          | N          | S            | \( \frac{a}{a+1} \) | \( z^* - 1 = -\frac{1}{a+1} \) is S |
| S          | N          | N            | \( \frac{a^2+a-1}{a^2+a+1} \) | \( L(1, z^*) = -\frac{a^2-a-1}{a^2+a+1} \) is S |
| N          | S          | S            | None   | \( L(1, z^*) = -a \) is S |
| N          | N          | S            | 0      | \( z^* - a = -a \) is S |
| N          | N          | N            | \( \frac{a-1}{a} \) | \( z^* - 1 = -\frac{1}{a} \) is S |

**Proof** Since the assumptions of Corollary 2.4 are a subset of the assumptions of Theorem 2.5, it follows that \( L_{a,a^2} \) as a quasigroup. For simplicity let us abbreviate \( L_{a,a^2} \) as \( L \).

Since \( q \equiv 1 \) (mod 4), we note that \(-1\) is a square, which will be used often throughout this proof. Let \( \eta \in \mathbb{F}_q \) be a fixed non-square.

First we show that there are no associative triples of the form \((t, t, v)\), where \( t \neq v \). For any such triple there exists an automorphism \( f \) of \( L_{a,a^2} \) such that \( (f(t), f(t), f(v)) = (0, 0, z) \) for some \( z \in \mathbb{F}_q \), \( z \neq 0 \). Towards a contradiction assume that \( L(L(0, 0), z) = L(0, L(0, z)) \). This simplifies to \( L(0, z) = L(0, L(0, z)) \). Let \( \ell = L(0, z) \). From Definition 2.1 we get \( \ell = a\ell \) or \( \ell = a^2\ell \). Since \( a \neq \pm 1 \), we get \( L(0, z) = 0 \). Since \( L(0, 0) = 0 \) and \( L \) is a quasigroup we get \( z = 0 \), a contradiction.

Now let \((t, u, v) \in \mathbb{F}_q^3 \) be an associative triple such that \( t \neq u \). If \( u - t \) is square, then there exists an automorphism \( f \) of \( L_{a,a^2} \) such that \( (f(t), f(u), f(v)) = (0, 1, z) \) for some \( z \in \mathbb{F}_q \). If \( u - t \) is non-square, then there exists an automorphism \( f \) of \( L_{a,a^2} \) such that \( (f(t), f(u), f(v)) = (0, \eta, z) \) for some \( z \in \mathbb{F}_q \). Hence it is sufficient to prove the nonexistence of associative triples of the form \((0, 1, z)\) and \((0, \eta, z)\).

Suppose that \((0, 1, z)\) is an associative triple, that is, \( L(L(0, 1), z) = L(0, L(1, z)) \). Since 1 is square, we get \( L(0, 1) = 0 + a(1 - 0) = a \) and the associative triple condition simplifies to

\[
L(a, z) = L(0, L(1, z)).
\]  

The three values of the function \( L \) seen in Eq. (1) depend on the parities of the three elements

\[
c_1 = z - a, \quad c_2 = z - 1, \quad c_3 = L(1, z).
\]  

Thus there are eight cases to consider. In each case the parities of \( c_1, c_2 \) and \( c_3 \) are fixed, and Eq. (1) can be written down explicitly in terms of \( a \) and \( z \) and no other variables or functions. This equation may be free of \( z \), in which case it turns out to be never satisfied due to the assumptions on \( a \), or the equation is linear in \( z \) and it has a unique root in \( \mathbb{F}_q \) which we denote \( z^* \). The contradiction is then obtained by showing that at least one of the elements \( z^* - a, z^* - 1, L(1, z^*) \) has the opposite parity in comparison to what was originally assumed, hence showing that this case can not occur. Once all eight cases are shown to be impossible, it follows that there is no \( z \in \mathbb{F}_q \) such that \((0, 1, z)\) is an associative triple.

We show all details of the proof for two of the eight cases; the other cases are similar. The details of all eight cases are summarized in Table 1, where S denotes square and N denotes non-square.
First assume that \(c_1 = z - a\) is non-square and \(c_2 = z - 1\), \(c_3 = L(1, z)\) are both square. Applying Definition 2.1 we get \(L(1, z) = 1 + a(z - 1)\) and then applying Definition 2.1 to Eq. (1) we get

\[
L(a, z) = L(0, L(1, z)) \\
a + a^2(z - a) = 0 + a(1 + a(z - 1) - 0) \\
a^2(a - 1) = 0
\]

which is impossible since \(a \neq 0, 1\). This is a case where no \(z^*\) exists.

Next assume that \(c_1 = z - a\) is non-square, \(c_2 = z - 1\) is square, and \(c_3 = L(1, z)\) is non-square. Applying Definition 2.1 we get \(L(1, z) = 1 + a(z - 1)\) and then applying Definition 2.1 to Eq. (1) we get

\[
L(a, z) = L(0, L(1, z)) \\
a + a^2(z - a) = 0 + a^2(1 + a(z - 1) - 0) \\
a(a - 1)(az + 1) = 0.
\]

Since \(a \neq 0, 1\), the last equation has the unique solution \(z^* = -1/a\). By assumption, any \(z\) occurring in this subcase satisfies that \(z - 1\) is square, hence we can evaluate

\[
L(1, z^*) = 1 + a(z^* - 1) = 1 + a(-1/a - 1) = 1 - 1 - a = -a.
\]

Since \(-1\) is square when \(q \equiv 1 \pmod{4}\) and \(a\) is assumed to be square, it follows that \(-a\), and hence also \(L(1, z^*)\), is square. This contradicts the assumption that \(L(1, z)\) is non-square.

The remaining six cases are similar. The details of all eight cases are recorded in Table 1.

In the second half of the proof suppose that \((0, \eta, z)\) is an associative triple, that is, \(L(L(0, \eta), z) = L(0, L(\eta, z))\). Since \(\eta\) is non-square, we get \(L(0, \eta) = 0 + a^2(\eta - 0) = a^2\eta\) and the associative triple condition simplifies to

\[
L(a^2\eta, z) = L(0, L(\eta, z)).
\]

The three values of the function \(L\) seen in Eq. (3) now depend on the parities of the three elements

\[
d_1 = z - a^2\eta, \quad d_2 = z - \eta, \quad d_3 = L(\eta, z).
\]

Again we decompose this half of the proof into eight cases that cover all possible combinations of parities of \(d_1, d_2, d_3\) and we drive each case to a contradiction. The details are recorded in Table 2.

Note that all field elements recorded in Tables 1 and 2 exist, that is, there is no division by zero. The denominators occurring in the tables are \(a, a + 1\) and \(a^2 + a + 1\). The first two can not be zero because \(a \neq 0, -1\), and the last one can not be zero because \(a^2 + a + 1\) is assumed to be non-square. This completes the proof of the theorem.

\[\square\]

**Theorem 2.6** Let \(q \equiv 3 \pmod{4}\) be a prime power. Let \(a \in \mathbb{F}_q\) be such that \(a \neq -1, 0, 1\), and the elements \(a, a + 1\), \(a - 1\), \(a^2 + 1\), \(a^3 + a^2 + 1\) are squares, and the elements \(a^2 - a + 1\), \(a^2 + a + 1\), \(a^2 + a - 1\) are non-squares. Then \(L_{a, a^2}\) is a maximally nonassociative quasigroup.

**Proof** The structure of the proof is very similar to the case \(q \equiv 1 \pmod{4}\). The reason why we need a separate proof for \(q \equiv 3 \pmod{4}\) is, of course, that \(-1\) is now non-square.

As in the previous proof we first note that \(L_{a, a^2}\) is a quasigroup, and again we will abbreviate \(L_{a, a^2}\) as \(L\). Again let \(\eta \in \mathbb{F}_q\) denote a fixed non-square. We could have achieved
some simplifications by taking $\eta = -1$, however we chose to keep the symbol $\eta$ to illustrate the full analogy with the case $q \equiv 1 \pmod{4}$.

The proof of non-existence of associative triples of the form $(t, t, v)$, where $t \neq v$, which was given for $q \equiv 1 \pmod{4}$ above, does not require any changes. The rest of the proof is organized exactly as in Theorem 2.5.

Of course, the values of $z^*$ in all 16 cases are the same as those found in the proof of Theorem 2.5. The details are recorded in Tables 3 and 4.

There is no division by zero in the tables for the same reasons as in the case $q \equiv 1 \pmod{4}$, namely it is assumed that $a \neq 0, -1$ and $a^2 + a + 1$ is non-square. □

### 3 Applying the Weil bound

We state the well known Weil bound on multiplicative character sums.

**Theorem 3.1** [8, Theorem 6.2.2]

Let $g \in \mathbb{F}_q[x]$ be a polynomial of degree $d > 0$ and $\chi : \mathbb{F}_q^* \to \mathbb{C}^*$ a non-trivial multiplicative character of order $m$ (extended by zero to $\mathbb{F}_q$). Then, if $g$ is not an $m$-th power

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Table 2 Non-existence of associative triples $(0, \eta, z)$ when $q \equiv 1 \pmod{4}$

| $z - a^2 \eta$ | $z - \eta$ | $L(\eta, z)$ | $z^*$ | Contradiction |
|----------------|------------|---------------|------|--------------|
| S              | S          | S             | $-\eta(a-1)$ | $z^* - \eta = -a\eta$ is N |
| S              | S          | N             | 0     | $z^* - \eta = -\eta$ is N |
| S              | N          | S             | $\frac{\eta}{a+1}$ | $L(\eta, z^*) = -\frac{\eta(a^3-a-1)}{a+1}$ is N |
| S              | N          | N             | $\frac{a^2 \eta}{a^2+a+1}$ | $z^* - \eta = -\frac{\eta(a+1)}{a^2+a+1}$ is S |
| N              | S          | S             | None  | |
| N              | S          | N             | $-a\eta$ | $z^* - \eta = -(a+1)\eta$ is N |
| N              | N          | S             | $\frac{(a^2-1)\eta}{a}$ | $z^* - \eta = -\frac{\eta(a^2+a-1)}{a}$ is S |
| N              | N          | N             | 0     | $L(\eta, z^*) = -\eta(a-1)(a+1)$ is S |

Table 3 Non-existence of associative triples $(0, 1, z)$ when $q \equiv 3 \pmod{4}$

| $z - a$ | $z - 1$ | $L(1, z)$ | $z^*$ | Contradiction |
|---------|---------|-----------|------|--------------|
| S       | S       | S         | 0    | $z^* - 1 = -1$ is N |
| S       | S       | N         | $\frac{a-1}{a+1}$ | $z^* - a = -\frac{a^2+1}{a+1}$ is N |
| S       | N       | S         | $\frac{a}{a+1}$ | $z^* - a = -\frac{a^2}{a+1}$ is N |
| S       | N       | N         | $\frac{a^2+a-1}{a^2+a+1}$ | $z^* - a = -\frac{(a+1)(a^2-a-1)}{a^2+a+1}$ is N |
| N       | S       | S         | None  | |
| N       | S       | N         | $-\frac{1}{a}$ | $z^* - 1 = -\frac{a+1}{a}$ is N |
| N       | N       | S         | 0     | $L(1, z^*) = -(a-1)(a+1)$ is N |
| N       | N       | N         | $\frac{a-1}{a}$ | $z^* - a = -\frac{a^2-a+1}{a}$ is S |
For each odd prime power $q$ such that $q \equiv 3 \pmod{4}$, let $f$ be an odd prime power and let $t$ be the union of the sets of roots of $fi(a)$ is non-zero square for $1 \leq i \leq t$ and $f_i(a)$ is non-square for $t + 1 \leq i \leq n$.

**Lemma 3.2** Let $f_1, \ldots, f_n$ be polynomials in $\mathbb{F}_q[x]$ and let $1 \leq t < n$. Define $\kappa : \mathbb{F}_q \to \mathbb{Q}$ by

$$\kappa(x) = \frac{1}{2^n} \prod_{i=1}^n (1 + \epsilon_i \chi(f_i(x))).$$

The number of $a \in \mathbb{F}_q$ such that $f_i(a)$ is non-zero square for $1 \leq i \leq t$ and $f_i(a)$ is non-square for $t + 1 \leq i \leq n$ equals

$$\sum_{x \in \mathbb{F}_q} \kappa(x) - \sum_{r \in R} \kappa(r)$$

where $R$ is the union of the sets of roots of $f_i$ in $\mathbb{F}_q$ for $1 \leq i \leq n$.

**Proof** The result follows from two simple observations: If $a \in \mathbb{F}_q$ is such that $f_i(a) \neq 0$ for $1 \leq i \leq n$ and all $f_i(a)$ have the desired parities, then $\kappa(a) = 1$. If $a \in \mathbb{F}_q$ is such that $f_i(a) \neq 0$ for $1 \leq i \leq n$ and at least one $f_i(a)$ has the wrong parity, then $\kappa(a) = 0$. □

**Lemma 3.3** (i) For each odd prime power $q$ such that $q \leq 7^3$ and $q \neq 3, 5, 7, 11$ there exist $a, b \in \mathbb{F}_q$ such that $L_{a,b}$ is a maximally nonassociative quasigroup.

(ii) For each odd prime power $q$ such that $7^3 < q < 1537^2$ and $q \neq 3^7, 3^9, 3^{11}, 3^{13}$ there exists $a \in \mathbb{F}_q$ such that $L_{a,a^2}$ is a maximally nonassociative quasigroup.

**Proof** For each $q$ we found the required maximally nonassociative quasigroup using the computational algebra system Magma [1]. The total computation time was about 40 h on a single CPU. □

| $z - a^2\eta$ | $z - \eta$ | $L(\eta, z)$ | $z^a$ | Contradiction |
|---------------|------------|--------------|-------|--------------|
| S             | S          | S            | $-\eta(a - 1)$ | $z^a - a^2\eta = -\eta(a^2 + a - 1)$ is N |
| S             | S          | N            | 0     | $L(\eta, z^a) = -\eta(a - 1)$ is S |
| S             | N          | S            | $\frac{\eta}{a+1}$ | $z^a - \eta = -\frac{\eta(a+1)}{a+1}$ is S |
| S             | N          | N            | $\frac{a^2\eta}{a^2+a+1}$ | $z^a - a^2\eta = -\frac{a^3\eta(a+1)}{a^2+a+1}$ is N |
| N             | S          | S            | None  | |
| N             | S          | N            | $-a\eta$ | $z^a - a^2\eta = -a\eta(a+1)$ is S |
| N             | N          | S            | $-\frac{(a^2-1)\eta}{a}$ | $z^a - a^2\eta = -\frac{(a^3+a^2-1)}{a}$ is S |
| N             | N          | N            | 0     | $z^a - a^2\eta = -a^2\eta$ is S |

in $\overline{\mathbb{F}}_q[x]$ (where $\overline{\mathbb{F}}_q$ is the algebraic closure of $\mathbb{F}_q$),

$$\left| \sum_{x \in \overline{\mathbb{F}}_q} \chi(g(x)) \right| \leq (d - 1)\sqrt{q}. \quad (5)$$

Let $q$ be an odd prime power and let $\chi$ be the quadratic character on $\mathbb{F}_q^*$, that is, $\chi(u) = 1$ if $u$ is a non-zero square in $\mathbb{F}_q$ and $\chi(u) = -1$ if $u$ is a non-square in $\mathbb{F}_q$. We extend $\chi$ on $\mathbb{F}_q$ by defining $\chi(0) = 0$.

**Lemma 3.2** Let $f_1, \ldots, f_n$ be polynomials in $\mathbb{F}_q[x]$ and let $1 \leq t < n$. Let $\epsilon_i = 1$ for $1 \leq i \leq t$ and $\epsilon_i = -1$ for $t + 1 \leq i \leq n$. Define $\kappa : \mathbb{F}_q \to \mathbb{Q}$ by

$$\kappa(x) = \frac{1}{2^n} \prod_{i=1}^n (1 + \epsilon_i \chi(f_i(x))). \quad (6)$$

The number of $a \in \mathbb{F}_q$ such that $f_i(a)$ is non-zero square for $1 \leq i \leq t$ and $f_i(a)$ is non-square for $t + 1 \leq i \leq n$ equals

$$\sum_{x \in \mathbb{F}_q} \kappa(x) - \sum_{r \in R} \kappa(r) \quad (7)$$

where $R$ is the union of the sets of roots of $f_i$ in $\mathbb{F}_q$ for $1 \leq i \leq n$. □
The following result is well known.

**Lemma 3.4** Suppose that there exist maximally nonassociative quasigroups of order $r$ and $s$, then there exists a maximally nonassociative quasigroup of order $rs$.

**Proof** Let $R$ and $S$ be maximally nonassociative quasigroups of order $r$ and $s$, respectively. Let $R \times S$ be their direct product. The triple $((x_1, x_2), (y_1, y_2), (z_1, z_2))$ is associative in $R \times S$ if and only if $(x_1, y_1, z_1)$ is associative in $R$ and $(x_2, y_2, z_2)$ is associative in $S$. The result follows. \hfill $\square$

Theorems 3.5 and 3.6 are the main results of this section.

**Theorem 3.5** Let $q \equiv 1 \pmod 4$ be a prime power, $q \geq 9$. There exists a maximally nonassociative quasigroup of order $q$.

**Proof** The bulk of the proof is based on the combination of Theorem 2.5, Lemma 3.2 and Theorem 3.1. Some special cases will remain to be handled by other methods, and this will be done at the end of the proof.

To assume the notation of Lemma 3.2, let $f_1(x) = x$, $f_2(x) = x + 1$, $f_3(x) = x^2 - x - 1$, $f_4(x) = x - 1$, $f_5(x) = x^2 + 1$, $f_6(x) = x^2 - x + 1$, $f_7(x) = x + 1$, $f_8(x) = x^2 + x - 1$, and let $t = 3$, that is, $e_1 = e_2 = e_3 = 1$ and $e_4 = e_5 = e_6 = e_7 = e_8 = -1$.

Note that the quadratic character $\chi$ remains multiplicative after extension at 0: we have $\chi(uv) = \chi(u)\chi(v)$ for all $u, v \in \mathbb{F}_q$ (not just on $\mathbb{F}_q^\times$). By expanding the right-hand side of (6) and rearranging the sums we get

$$\kappa(x) - \frac{1}{2^8} = \frac{1}{2^8} \sum_{\emptyset \neq I \subseteq \{1, \ldots, 8\}} \prod_{i \in I} (e_i \chi(f_i(x))) = \frac{1}{2^8} \sum_{\emptyset \neq I \subseteq \{1, \ldots, 8\}} \left( \prod_{i \in I} e_i \right) \chi \left( \prod_{i \in I} f_i(x) \right).$$

After summing over all $x \in \mathbb{F}_q$ and denoting $S = \sum_{x \in \mathbb{F}_q} \kappa(x)$ we get

$$S - \frac{q}{2^8} = \frac{1}{2^8} \sum_{\emptyset \neq I \subseteq \{1, \ldots, 8\}} \left( \prod_{i \in I} e_i \right) \left( \sum_{x \in \mathbb{F}_q} \chi \left( \prod_{i \in I} f_i(x) \right) \right). \quad (8)$$

We would like to apply the Weil bound (Theorem 3.1) on the character sums $\sum_{x \in \mathbb{F}_q} \chi \left( \prod_{i \in I} f_i(x) \right).$ We exclude the cases where $\prod_{i \in I} f_i(x)$ may be a perfect square over some algebraic extension of $\mathbb{F}_q$, for some $I$, as follows. We know that the discriminant of a non-constant polynomial which is a perfect square is equal to 0. Consider a fixed index set $I$. We consider the polynomials $f_i$ over $\mathbb{Z}$ and we compute the discriminant of $\prod_{i \in I} f_i(x)$, let us say the discriminant is the integer $D$. Then the discriminant of $\prod_{i \in I} f_i(x)$ with $f_i$ over $\mathbb{F}_q$ equals $D \pmod p$, where $p$ is the characteristic of $\mathbb{F}_q$. In other words, the prime factors of $D$ are the characteristics in which the Weil bound may not be applicable, and we will deal with those characteristics separately at the end of the proof.

By computing the discriminant of $\prod_{i \in I} f_i(x)$ over $\mathbb{Z}$ for all non-empty subsets $I$ of $\{1, \ldots, 8\}$ such that the degree of $\prod_{i \in I} f_i(x)$ is even (which is a necessary condition for a polynomial to be a perfect square) we find that the only primes that divide any of these discriminants are 2, 3, 5 and 23. Only the last three are odd and relevant to this theorem.

First assume that the characteristic of $\mathbb{F}_q$ is different from 3, 5 and 23. Then the Weil bound applies to each character sum on the right-hand side of (8). Applying Theorem 3.1...
to each sum \( \sum_{x \in \mathbb{F}_q} \chi (\prod_{i \in I} f_i(x)) \) and adding over all non-empty subsets of \( \{1, \ldots, 8\} \) we get

\[
S \geq \frac{1}{2^8}(q - 1537\sqrt{q})
\]

(9)

where \( 1537 = 2^7(1 + 1 + 1 + 2 + 2 + 2 + 2 + 3) - 2^8 + 1 \) is the sum of \( \deg(\prod_{i \in I} f_i(x)) - 1 \) over all \( \emptyset \neq I \subseteq \{1, \ldots, 8\} \).

In preparation for application of Lemma 3.2 we will now evaluate the term \( \sum_{r \in \mathbb{R}} \kappa(r) \) occurring in (7). We could have upper bounded this term as \( \sum_{r \in \mathbb{R}} \kappa(r) \leq \frac{1}{\mathbb{Z}} \sum_{i=1}^{8} \deg f_i = 7 \) but we will do a more precise evaluation, which allows one to slightly reduce the range of values \( q \) for which the claim has to be verified computationally (Lemma 3.3). We show that in fact \( \sum_{r \in \mathbb{R}} \kappa(r) = 0 \), as follows.

For each \( i = 1, \ldots, 8 \), one by one, we will show that the assumption \( f_i(a) = 0 \) and \( a \in \mathbb{F}_q \) leads to the conclusion \( \kappa(a) = 0 \), by showing that for at least one \( j \) we have \( 1 + \varepsilon_j \chi(f_j(a)) = 0 \), that is, \( f_j(a) \neq 0 \) and \( f_j(a) \) has the wrong parity. For this reason, in the rest of this paragraph “square” is an abbreviation for “non-zero square”.

If \( a = 0 \), then \( a - 1 = -1 \) is square.
If \( a + 1 = 0 \), then \( a^2 + a - 1 = -1 \) is square.
If \( a^3 - a - 1 = 0 \), then \( a^4 = a^2 + a \) and \( a^2 + a \) is square. Since \( 0 = a^3 - a - 1 = (a^2 + a)(a - 1) - 1 \), we get \( (a^2 + a)(a - 1) = 1 \) and \( a - 1 \) is square.
If \( a - 1 = 0 \), then \( a^2 + a - 1 = 1 \) is square.
If \( a^2 + 1 = 0 \), then \( a = a^2 + a + 1 \) and we can not have \( a \) square and \( a^2 + a + 1 \) non-square.
If \( a^2 - a - 1 = 0 \), then \( a = (a + 1)(a - 1) \) and it can not be that \( a \) is square, \( a + 1 \) is square and \( a - 1 \) is non-square.
If \( a^2 + a + 1 = 0 \), then \( a = -(a^2 + 1) \) and it can not be that \( a \) is square and \( a^2 + 1 \) is non-square.
If \( a^2 + a - 1 = 0 \), then \( a - 1 = -a^2 \) is square. This finishes the verification that \( \sum_{r \in \mathbb{R}} \kappa(r) = 0 \).

Now for \( q > 1537^2 \) we have \( \frac{1}{\mathbb{Z}}(q - 1537\sqrt{q}) > 0 \), hence for such \( q \) it follows from Theorem 2.5 and Lemma 3.2 that there exists \( a \in \mathbb{F}_q \) such that \( L_{a,a^2} \) is maximally nonassociative quasigroup. For odd prime powers \( 9 \leq q < 1537^2 \) with \( q \equiv 1 \pmod{4} \) the existence of maximally nonassociative quasigroup of order \( q \) follows from Lemma 3.3, with the exceptions stated there. Since we have to deal with characteristic 3 separately anyway, the exceptions of Lemma 3.3 do not bother us.

We will now deal with the characteristics 3, 5 and 23. By Lemma 3.3, maximally nonassociative quasigroups of orders \( p^2 \) and \( p^3 \) exist for \( p = 3 \) and \( p = 5 \). Then the existence of maximally nonassociative quasigroup of orders \( p^e \) for \( e \geq 2 \) follows from Lemma 3.4.

Again by Lemma 3.3, maximally nonassociative quasigroup of order 23 exists, hence there exist maximally nonassociative quasigroups of order \( 23^e \) for all \( e \geq 1 \), again by Lemma 3.4.

\( \square \)

**Theorem 3.6** Let \( q \equiv 3 \pmod{4} \) be a prime power, \( q \geq 19 \). There exists a maximally nonassociative quasigroup of order \( q \).

**Proof** The structure of the proof is analogous to the proof of Theorem 3.5. Instead of Theorem 2.5 we now use Theorem 2.6. The multiset of degrees of the polynomials \( f_i \) is the same as in the case \( q \equiv 1 \pmod{4} \), hence the Weil bound produces the same results; when applying Lemma 3.2 one can again show that \( \sum_{r \in \mathbb{R}} \kappa(r) = 0 \).

Again we need to consider the special characteristics in which the Weil bound may not apply. By computing the discriminants of products \( \prod_{i \in I} f_i(x) \) of even degree over \( \mathbb{Z} \) we find
that the only primes that divide any of these discriminants are 2, 3, 5, 7 and 23. Only the characteristic 7 is new and needs to be considered. By Lemma 3.3, maximally nonassociative quasigroups of orders $7^2$ and $7^3$ exist hence maximally nonassociative quasigroup of order $7^e$ exists for each $e \geq 2$.

\[\square\]

4 Main result

For a positive integer $n$ and a prime $p$ let $v_p(n)$, the $p$-adic valuation of $n$, be defined as the largest integer $e$ such that $p^e$ divides $n$.

**Theorem 4.1** Let $n$ be a positive integer such that $v_p(n) \neq 1$ for $p = 3, 5, 7, 11$ and $v_2(n)$ is even and different from $2, 4$. There exists a maximally nonassociative quasigroup of order $n$.

**Proof** Examples of order $2^6$ were constructed from Dickson nearfield of that order [2, Lemma 5.7]. Examples of orders $2^8$ and $2^{10}$ can be obtained easily by prescribing an automorphism group consisting of affine mappings $x \mapsto ax + \beta$ where $a$ is a non-zero cube. The rest of the statement follows from Lemma 3.4 and Theorems 3.5 and 3.6.

\[\square\]

Some time after the first preprint version of the current paper was published, Drápal and Wanless published the preprint [5] in which they further extend the spectrum of orders $|Q|$ for which a maximally nonassociative quasigroup $Q$ exists. In another preprint Drápal and Wanless [6] determine that if one draws uniformly at random one pair, say $(a^*, b^*)$, from the set of all pairs $(a, b) \in \mathbb{F}_q^2$ such that $L_{a,b}$ is a quasigroup (see Lemma 2.3), then the probability that $L_{a^*, b^*}$ is a maximally nonassociative quasigroup is about $1/8.596$ if $q \equiv 1$ (mod 4), and it is about $1/19.86$ if $q \equiv 3$ (mod 4).

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