The Critical State as a Steady-State solution of Granular Solid Hydrodynamics

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The approach to the critical state – the transition from partially elastic to perfectly plastic behavior – is considered the most characteristic of granular phenomena in soil mechanics. By identifying the critical state as the steady-state solution of the elastic strain, and presenting the main results as transparent, analytic expressions, the physics of this important phenomenon is clarified.

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Applying a constant shear rate to an elastic body, the shear stress will monotonically increase until the point of breakage. Granular media are different, as they can maintain a uniform state, continually deforming yet of constant stress – which has, for given density, a unique value independent of the shear rate. This perfectly plastic state is hailed as a hallmark of granular behavior in its behavior – including the critical state that is identified as a steady-state solution of granular solid hydrodynamics.

More realistically, one needs to include the density dependence. Assuming cylindrical symmetry, the stress has two independent elements, \( \sigma_1 = \sigma_2 \) and \( \sigma_3 \), in the system in which it is diagonal. The approach to the critical state is then more varied. The experiments are usually performed keeping one stress element constant, say \( \sigma_1 \), while applying a slowly increasing strain, say \( \varepsilon_3 \). Employing a tri-

![Fig. 1](image-url) Seemingly a textbook illustration of the critical state, this is the result of a calculation employing GSH, specifically the solution of Eqs. (3), plotting the stress \( q \equiv \sigma_3 - \sigma_1 \) and the void ratio \( e \) against the strain \( \varepsilon_3 \) in triaxial tests (cylinder axis along 3), at given \( \sigma_1 \) and strain rate \( \varepsilon_3/t \), for an initially dense and loose sample, respectively. Insets are variations of the critical stress \( q_c \) and density \( \rho_c \) with \( \sigma_1 \). See the text for details, and the caption of Fig. 2 for the parameter values.
axial apparatus, one measures the second independent stress element and the density \( \rho \). With \( q \equiv \sigma_3 - \sigma_1 \) and the void ratio \( c \equiv \rho_g/\rho - 1 \) (where \( \rho_g \) is the density of the grains), the results are frequently presented as \( q(\varepsilon_3), \varepsilon(\varepsilon_3) \). Provided no shear zones are formed, the findings are as rendered in Fig. 11 (1) \( q \) increases monotonically in loose systems, but displays a maximum in dense ones. Both approach asymptotically the stationary value \( q_c \). (2) Dense systems dilate, loose systems contract, until a universal \( \varepsilon_c \) (or \( q_c \)) is reached. (3) All stress-strain curves are rate-independent, retaining their form for whatever strain rate. (4) The friction angle \( \varphi_c \), defined as \( \sqrt{3} \tan \varphi_c = q_c/P \), is essentially constant, independent of \( \sigma_1 \) or \( P \), implying \( q_c \sim \sigma_1 \sim P \). The steady state, characterized by \( q_c \) and \( \varepsilon_c \) for given \( \sigma_1 \) or \( P \), is referred to as the critical state, see \([1, 2]\) for more details, and \([3]\) for a presentation catered to physicists.

There are many engineering theories capable of a realistic account of these observations. A mathematically elegant and comparatively simple one is the hypoplastic theory \([2, 4]\) that starts from a postulated relation between the stress \( \sigma_{ij} \) and strain rate \( \dot{v}_{ij} \):

\[
\dot{\sigma}_{ij} = H_{ijkl} \dot{v}_{kl} - \Lambda_{ij} \dot{v}_s,
\]

where \( v_s \equiv \sqrt{\dot{v}_{ij} \dot{v}_{ij}} > 0 \). (The compressional rate, \( \dot{v}_{ij} \approx 0 \), small in the present context, is neglected; the dot in \( \dot{\sigma}_{ij} \) implies an appropriate objective time derivative.) The tensors \( H_{ijkl}, \Lambda_{ij} \) are taken as functions of \( \sigma_{ij}, \rho \), specified using experimental data. The critical state is given by \( \dot{\sigma}_{ij} = 0 \). Nevertheless, being descriptive rather than explaining, the hypoplastic model – same as various elasto-plastic ones – is not a complete theory: As the important features, especially rate-independence and the so-called incremental nonlinearity, are put in via Eq (1), there is no way to understand them. The hypoplastic model is also narrow: None of the rate-dependent phenomena, say dense flow \([6]\), or the transition to elastic behavior are accounted for. (The latter is needed for describing sound propagation \([5]\) and static stress distributions.) More fundamentally, hypoplasticity does not specify a complete set of state variables, and lacks energetic and entropic considerations.

In contrast, starting from a few ideas about the physics of sand and derived from general principles, including energy conservation and thermodynamics, GSH is a full-fledged theory that aims to model granular behavior in all its facets \([6]\). It was first developed to calculate static stress distribution for various geometries, including sand piles, silos, and point load, achieving results in agreement with observation \([5]\). GSH was then generalized to dynamic situations \([6]\), producing response envelopes strikingly similar to those from hypoplasticity. At the same time, it clarified when rate-independence holds, and why incremental nonlinearity applies. Two recent papers are on elastic waves \([10]\), showing the quantitative agreement to experiment \([6]\), and on GSH presented from the point of view of soil mechanics \([11]\). Preprints on compaction, dense flow, fluidization and jamming have been submitted, that on shear band is being prepared. In this letter, we consider the critical state.

GSH consists of five conservation laws: for the energy \( w \), mass \( \rho \), and momentum \( \rho u \), an evolution equation for the elastic strain \( u_{ij} \), and balance equations for two entropies, or equivalently, two temperatures. Both the true and the granular temperature, \( T \) and \( T_g \), are necessary, because granular media sustain a two-stage irreversibility. Macroscopic energy, kinetic and elastic, dissipates into mesoscopic, inter-granular degrees of freedom, which in turn degrades into microscopic, inner-granular degrees of freedom. The first are granular jiggling, quantified by \( T_g \); the second mostly phonons, quantified by \( T \).

The elastic strain \( u_{ij} \) is defined as the portion of the strain \( \varepsilon_{ij} \) that deforms the grains and leads to reversible storage of elastic energy. The plastic rest, \( \varepsilon_{ij} - u_{ij} \), accounts for rolling and sliding. Because the energy depends on \( u_{ij} \) alone, we take it as a state variable, while excluding the total strain and the plastic one. This is a crucial step that enables us to retain many useful features of elasticity, especially an explicit expression for the stress.

From the equations of GSH as derived in \([9]\), we need three: for \( T_g \), \( u_{ij} \), and the Cauchy stress \( \sigma_{ij}(\rho, u_{kl}) \). Accounting for the first-stage irreversibility, the balance equation for \( T_g \) has a similar structure as that for the true temperature, with the production of granular entropy given as \( R_g = \eta_g v_s^2 - \gamma T_g^2 \). The first term, preceded by the viscosity \( \eta_g \), is positive. It describes how the shear rate \( v_s \) jiggles grains, converting macroscopic kinetic energy into \( T_g \). The second term, negative, accounts for inelastic collisions, as a result of which \( T_g \) diminishes with the rate \( \gamma \). In steady-state, quickly arrived at (say after \( 10^{-4} \) s), we have \( R_g = 0 \), or

\[
T_g = v_s \sqrt{\eta_g/\gamma}. \tag{2}
\]

Dividing \( u_{ij} \) into \( \Delta \equiv -u_{\ell\ell} \) and \( u_{ij}^0 \), \( u_{ij}^0 = u_{ij} + \frac{1}{2} u_{\ell\ell} \delta_{ij} \), the evolution equation for \( u_{ij} \) takes the form \([1, 2]\)

\[
\partial_t \Delta + (1 - \alpha) v_{\ell \ell} - \alpha_2 u_{ij}^0 v_{ij} = -3\Delta/\tau_1 = -3\lambda_1 T_g \Delta,
\]

\[
\partial_t u_{ij}^0 - (1 - \alpha) v_{ij} = -u_{ij}^0/\tau = -\lambda T_g u_{ij}, \tag{3}
\]

If \( T_g \) is finite, grains jiggles and briefly lose contact with one another, during which their unstressed form will be partially restored. Macroscopically, this shows up as a slow relaxation of \( \Delta \) and \( u_{ij}^0 \), with relaxation rates that grow with \( T_g \), and vanish for \( T_g = 0 \), we therefore take \( 1/\tau = \lambda T_g \) and \( 1/\tau_1 = \lambda_1 T_g \). Eliminating \( T_g \) using Eq (2), we have \( \lambda T_g = \lambda_1 v_s \), with \( \Delta \equiv \lambda \sqrt{\eta_g/\gamma}, \lambda_1 \equiv 3\lambda_1 \sqrt{\eta_g/\gamma} \). Because the relaxation of \( u_{ij} \) should come to a halt at the random closed-packed density \( \rho_{sp} \), where the system turns elastic, we take \( \lambda, \lambda_1 \sim (\rho_{sp} - \rho) \). \( \alpha, \alpha_2 > 0 \) are Onsager coefficients. The first reduces the portion of \( v_{ij} \) that deforms...
the grains and changes the elastic strain $u_{ij}$; the second quantifies dilatancy, the cross effect typical of granular media that a shear flow $v_{ij}^0$ leads to a compressional rate, $\partial_t \Delta$. For the reason discussed below Eq (4), we also take $\alpha_2 \sim (\rho_{cp} - \rho)$. The Cauchy stress is

$$
\sigma_{ij}(u_{ij}, \rho) = (1 - \alpha)\pi_{ij} - \alpha_2 u_{ij}^0 \pi_{tt}/3,
$$

where $\pi_{ij}(u_{ij}, \rho) \equiv -\partial w/\partial u_{ij}$, $w$ being the elastic energy. Viscous terms (large only for the high shear rates typical of dense flows) are neglected. That the same $\alpha, \alpha_2$ appear here is a result of the Onsager reciprocity relation. The term $\sim \alpha_2$ may usually be neglected, as it is an order higher in $u_{ij}$, which is always smaller than $10^{-3}$.

Eqs (3) with $T_g$ eliminated are algebraically simple and physically transparent. It is therefore instructive that they lead directly to Eq (1), the postulated hypoplastic relation, and provide expressions for $H_{ijtt}, \Lambda_{ij}$, see [8]. Although these tensors appear different from the ones used in modern hypoplastic models, the calculated response envelopes (i.e. closed stress or strain curves) are strikingly similar, see [8]. This agreement indicates the appropriateness of our starting points, especially the exclusion of the plastic strain as a state variable.

Next, we evaluate the steady-state solution for the elastic strain, given by taking $\partial_t \Delta, \partial_t u_{ij}^0 = 0$ in Eqs (3). Assuming a stationary shear flow, $v_{ij}^2 \equiv v_{ij}v_{ij}$, $v_{tt} = 0$, and denoting $u_{ij}^* \equiv u_{ij}u_{ij}$, we find

$$
u_e = \frac{1 - \alpha}{\Lambda} \sqrt{\frac{\gamma}{\eta_g}} \equiv \frac{1 - \alpha}{\Lambda} \Delta_c, \quad \Delta_c / u_e = \frac{\alpha_2}{3\Lambda_1} \sqrt{\frac{\gamma}{\eta_g}} \equiv \frac{\alpha_2}{\Lambda_1},
$$

and $u_{ij}^0 / u_s = v_{ij} / v_s$. Reusing the parameter values of [9]: $\Lambda = 10^2, \Lambda_1 = 30$ for $\rho/\rho_{cp} = 0.96, \alpha = 0.8$ (which are the high-$T_g$ limit of $\alpha$ and $\eta_g/\gamma$), we obtain $u_e = 2 \times 10^{-3}$, $\Delta_c / u_e = 0.1 \alpha_2$. As the friction angle $\varphi_c$ is a function of $\Delta_c / \nu_e$ alone, see Eq (9), it determines the value of $\alpha_2$. The approach to the steady state is given by solving Eqs (3) for $u_s(t), \Delta(t)$, at constant $\rho, v_s \equiv \varepsilon_s / t$, with the initial conditions: $\Delta = \Delta_0, u_s = 0$. The solution is

$$
u_s(t) - \nu_e = -e^{-\Delta_c t},$$

$$\Delta(t) - \Delta_c = \frac{\alpha_2 u_e e^{-\Delta_c t}}{\Lambda - \Lambda_1} + \left[\frac{\Delta_0 - \Lambda \Delta_c}{\Lambda - \Lambda_1}\right] e^{-\Lambda_1 \varepsilon_s}. $$

As we shall soon see, these simple expressions, giving $u_{c}, \Delta_c, u_{s}(t), \Delta(t)$ as functions of $\rho$ and $\varepsilon_s \equiv v_s t$, are a complete account of the the critical state and the approach to it. They describe exponential decays of $u_{s}(t), \Delta(t)$ to $u_s, \Delta_c$, and are shear rate-independent, because they depend on $\varepsilon_s$, not $v_s$. Given $\rho, \Delta, u_s$, the stress $\sigma_{ij}$ is known via Eq (4).

The behavior rendered in Fig 1 appears more complicated, which stems from two facts of more technical nature: First, $P$ or $\sigma_1$ is usually held constant in stead of the density. As $\Delta, u_s$ change with time, the density compensates to maintain $P(\rho, \Delta, u_s)$. And along with $\rho$, the quantities $\alpha_2, \Lambda, \Lambda_1$, all functions of $\rho$, also change with time. [In addition, there is a change of time scale from the term $\nu_t = -\partial \rho/\partial \rho$ in the first of Eqs (3).]

Second, the stress is measured, not the elastic strain. To calculate $\sigma_{ij}(t)$ employing Eq (1), we need an expression for the elastic energy $w$. The one we have consistently employed [8, 9] is: $w = B\sqrt{\Delta (\varepsilon_2^2 + u_{ij}^2 / \xi)}$ where $B, \xi$ are two elastic coefficients. We fix $\xi = 5/3$, independent of the density, and take $B(\rho) / B_0 = [(\rho - \rho^*)/(\rho_{cp} - \rho)]^{0.15}$, with $9\rho^* \equiv \rho_{cp} + 20(\rho_{cp} - \rho_{cp})$, and $B_0$ around 8 GPa for river sand, 7 GPa for glass beads. The associated pressure $P \equiv \frac{1}{2} \sigma_{tt}$, and shear stress $\sigma_s^2 \equiv \sigma_{ij}^0 \sigma_{ij}^0$ (with $\sigma_{ij}^0 / \sigma_s = u_{ij}^0 / u_s$) are

$$
P = (1 - \alpha)B \Delta^{1.5} \left[1 + u_{ij}^2 / (2\xi \Delta^2)\right],$$

$$\sigma_s = (1 - \alpha)2u_s \sqrt{\Delta B / \xi},$$

$$P / \sigma_s = (\xi/2) \Delta / u_s + (1/4)u_s / \Delta,$$

where the critical pressure and shear stress are

$$P_c = P(\rho, \Delta_c, u_c), \quad \sigma_c = \sigma_s(\rho, \Delta_c, u_c).$$

The expressions Eqs (5, 6, 7, 8, 9, 10), evaluated for constant $\sigma_1$ or $P$, are remarkably similar to textbook illustrations of the critical state, see Fig. 1 and 2. But all the well-taught features of these curves are now easy to understand: We first note that while $u_s(t)$ always increases monotonically, $\Delta(t)$ decreases monotonically only for $f_2 \equiv [\Delta_0 - \Delta_c \Lambda / (\Lambda - \Lambda_1)] > 0$, so $f_1 \equiv \alpha_2 u_s / (\Lambda - \Lambda_1)$ is always positive. We also note that for given $P$, a positive $f_2$ means stronger initial compression, corresponding to a lower density or higher void ratio, while $f_2 < 0$ signifies weaker compression and lower void ratio. At the beginning, the faster relaxation of $f_1$ dominates, so $\Delta, e$ always decrease, irrespective of the void ratio $e$. After $f_1$ has run its course, $\Delta, e$ go on decreasing if $f_2 > 0$, but switch to increasing if $f_2 < 0$, displaying respectively the so-called contractancy and dilatancy, until the critical state, $\Delta = \Delta_c, e = e_c$, is reached.

The shear stress $\sigma_s$ always increases first with $u_s$, until $u_s = u_c$ is reached. The subsequent behavior depends on what $\Delta$ does; cf Eq (6) and the discussion on Coulomb yield below, consider only $u_s / \Delta < \sqrt{2}g$: $\sigma_s$ keeps growing if $\Delta$ decreases [loose case, $f_2 > 0$], but decreases, displaying a peak, if $\Delta$ grows [dense case, $f_2 < 0$].

The friction angle $\tan \varphi_c \equiv \eta_e / \sqrt{3}P = \sigma_s / \sqrt{3}P$ is observed to be essentially independent of the density, or the pressure $P/ \rho$. This is also accounted for by the above results, because the ratio $\sigma_s / P$ depends only on $\Delta_c / u_c = \alpha_2 / \Lambda_1$, see Eqs (5, 9), and we have taken $\alpha_2, \Lambda_1 \sim \rho_{cp} - \rho$. There are two types of density dependence here: the sensitive one via $\rho_{cp} - \rho$, and the weaker one via $\rho$. Neglecting the latter, $\alpha_2 / \Lambda_1$ is a constant.

Circumstances are similar if $P$ is the control parameter, because $P \sim B \sim (\rho_{cp} - \rho)^{-0.15}$ for constant $u_{ij}$, so changing $P$ only changes $\rho_{cp} - \rho$. 

The relation of $\varphi_c$ to Coulomb yield is instructive. Given Eq. (9), the ratio $P/\sigma_s$ has the minimal value $\sqrt{\xi/2}$, at $u_c/\Delta = \sqrt{\xi}$. It is larger for $u_c/\Delta < \sqrt{\xi}$, and unstable for $u_c/\Delta > \sqrt{\xi}$ (as the energy $u$ is then concave, see [3]). This may be identified with (the Druck-Prager version of the) Coulomb yield – an energetic instability that ensures that no elastic solution exists for $P/\sigma_s < \sqrt{\xi/2}$, or $u_c/\Delta > \sqrt{\xi}$. Remarkably, this instability is determined by the elasticity coefficient $\xi$, while the friction angle is given by transport coefficients, especially $\alpha_2$. There is no a priori link between both, as they are based on different physics. So the critical state may be stable and observable, or not. For the first case, we have $u_c/\Delta_c = \Lambda_1/\alpha_2 < \sqrt{\xi}$, implying $\tan \varphi_c < 1/\sqrt{\xi}$, or $\varphi_c < 38^\circ$ for $\xi = 5/3$. (Taking as before $\Lambda_1 = 30$, this is equivalent to $\alpha_2 > 3\sqrt{30}$.) Clearly, if critical is meant to imply marginal stability, it is a misnomer – energetic stability is essential for the existence of the critical state.

The yield point discussed in the introduction is also a dubious concept. Lacking an obvious choice in the upper two curves of Fig[1] one frequently takes it as given by $q^*$'s maximum for dense sand – the vague rational being the fact that in an elastic medium, the positivity of the stiffness coefficient, $\partial q/\partial \varepsilon_3|_{\varepsilon_3} = \partial \sigma_3/\partial \varepsilon_3 > 0$, is required by energetic stability, so a negative slope implies instability and is never observed. However, since $u_{ij}$ is the state variable and not $\varepsilon_{ij}$, stability only requires $\partial \sigma_3/\partial u_3$ to be positive, freeing $\sigma_3/\partial \varepsilon_3$ to be negative, as observed.

In spite of the energetic stability of the critical state, the yield surface may of course still be breached at some point. Shear bands (considered in a forthcoming paper) will then appear, destroying the uniformity of the system. This is most likely to happen around $q^*$’s maximum for the densest sand, because $u_c$ meets the smallest $\Delta$ there.

Summary: Sheared granular media, if uniform, will approach the continually deforming critical state that is widely believed to be on the margin of stability. Employing GSH, a broadly applicable granular theory, we find the critical state well accounted for, with all facets transparently explained, and given by the steady-state solution of the elastic strain that is both continuous and stable.

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[12] As $\alpha_2$ was neglected in [7], we revisit the derivation. Denoting $\partial u_{ij} = v_{ij} = X_{ij}$, $\sigma_{ij} = \pi_{ij} - \alpha_1^{ij}$, the entropy production, $R = \sigma_{ij}^{ij}v_{ij} + X_{ij}\pi_{ij} + \cdots$, implies

\[
X_{ij} = \beta_1\pi_{ij}^0 + \beta_1\delta_\varepsilon_{ij}\pi_{kk} + \alpha_{ij\beta\ell}u_{ij\beta\ell} + \eta_{ij}^0 + \alpha_{ij\eta\ell}\pi_{kk},
\]

with the Onsager relation, $\alpha_{ij\ell} = -\alpha_{\varepsilon\ell\varepsilon}$. Taking $\alpha_{ij\varepsilon\ell} = \delta_{i\ell\varepsilon} + \alpha_{ij\varepsilon\ell} + (\alpha_3/3)\delta_{i\varepsilon}\delta_{\varepsilon\ell}$, i.e. including one more element than in [7], stabilizes the steady-state solution.