Displacement Profile of Charge Density Waves and Domain Walls at Critical Depinning

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The influence of a strong surface potential on the critical depinning of an elastic system driven in a random medium is considered. If the surface potential prevents depinning completely the elastic system shows a parabolic displacement profile. Its curvature \( C \) exhibits at zero temperature a pronounced rhombic hysteresis curve of width \( 2f_c \) with the bulk depinning threshold \( f_c \). The hysteresis disappears at non-zero temperatures if the driving force is changed adiabatically. If the surface depins by the applied force or thermal creep, \( C \) is reduced with increasing velocity. The results apply, e.g., to driven magnetic domain walls, flux-line lattices and charge-density waves.

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The driven viscous motion of an interface in a medium with random pinning forces is one of the paradigms of condensed matter physics \[1, 2\]. This problem arises, e.g., in the domain wall motion of magnetically or structurally ordered systems with impurities \[3\] or when an interface between two immiscible fluids is pushed through a porous medium \[4\]. Closely related problems are the motion of other elastic systems like a vortex line in an impure superconductor \[5\], of a dislocation line in a solid \[6\] or driven charge density waves (CDWs) \[7\]. For a constant external driving force this problem has been considered close to the zero temperature critical depinning threshold \[8, 9, 10, 11\] and in the creep region \[7, 12\].

It was a tacit assumption of these investigations that the motion of the elastic system is not hindered by effects from surfaces or internal grain boundaries. Surface barriers are however known to be relevant in all cases mentioned above. In superconductors they prevent the penetration of new flux lines into the probe \[13\]. In CDWs normal electrons have to be converted into those con-

The random force \( \rho \) is assumed to be Gaussian distributed with \( \langle g \rangle_d = 0 \) and \( \langle g(x, \varphi)g(x', \varphi') \rangle_d = \hat{g}(x - x')\Delta_0(\varphi - \varphi') \) where \( \langle \ldots \rangle_d \) denotes the random average. For domain walls \( \Delta_0(\varphi) = \Delta_0(-\varphi) \) is an analytical monotonically decreasing function of \( \varphi \) which decays to zero over a finite distance \( l \). For CDWs \( g \propto \sin(\varphi - \alpha(x)) \) with a random phase \( \alpha(x) \in [0, 2\pi] \) and therefore \( \Delta_0(\varphi) \) is periodic with \( \Delta_0(\varphi) = \Delta_0(\varphi + 2\pi \mathbb{Z}) \).

The surface potential \( V_s(\varphi) \) is assumed to act only in the vicinity (\( a \ll L \)) of \( x_1 = 0 \) and \( x_1 = L \), e.g., \( \rho(x) = e^{-x_1/a} + e^{-x_1-L)/a} \), and favors the values of \( \varphi(0, x_1) \) and \( \varphi(L, x_1) \) at \( 2\pi \mathbb{Z} \). The details of the interaction between the elastic system and the surface depend on the specific system under consideration. We will restrict ourselves here to a periodic surface potential which has applications in type-II superconductors and may also serve as a first step for the treatment of conversion phenomena in CDWs.

The case \( V_s \equiv 0 \) was considered previously in \[8, 9, 10, 11\]. It was shown that at zero temperature the system undergoes a depinning transition at a critical value \( f_c \). For \( f > f_c \) the velocity \( v = \langle \dot{\varphi} \rangle \) increases as \( v \sim (f - f_c) \beta \).
with the critical exponent $\beta$ calculated in an expansion in $D = 4 - \epsilon$ dimensions. The average displacement profile is macroscopically flat. At non-zero temperatures the depinning transition is smeared out and goes over into a creep motion for $f \ll f_c$.

In this Letter we will consider the opposite case where a strong surface potential $V_s$ slows down or prevents completely the motion of the elastic object. Then the displacement profile becomes parabolic with a history-dependent curvature $C(f, t)$ . The steady state solution for the average phase is given by

$$
\varphi_0 = \langle \varphi \rangle = vt + \frac{c_s(f)}{2}(L - x_1) x_1 ,
$$

(3)

where $c_s(f) = C_s(f, t \to \infty)$ is the saturation value of the curvature.

**Infinite surface barriers.—** We begin with the case $V_s \to \infty$, where the depinning transition is depressed. To determine $C(f, t)$ we first apply perturbation theory. Using the decomposition $\varphi(x) = \varphi_0(x) + \varphi_1(x)$ with $\langle \varphi_1(x) \rangle = 0$ in the equation of motion and expanding $g(x, \varphi_0(x) + \varphi_1)$ to linear order in $\varphi_1$ we get after averaging over the disorder

$$
\frac{1}{\gamma} \dot{\varphi}_0 = -GC(t) + f + \langle g_{\varphi}(x, \varphi_0(x), t) \varphi_1(x, t) \rangle ,
$$

(4)

where $g_{\varphi}(x, \varphi) = \frac{\partial}{\partial \varphi} g(x, \varphi)$. Calculating $\varphi_1$ also to first order of $g$ we get from (3) and (4) $C_s = C_0 = f/\Gamma$ since $\Delta_0(0) = 0$, i.e., there seems to be no influence of the disorder. Here, the situation is completely analogous to that at the conventional depinning transition $\Sigma$. However, as we know from critical depinning, this is the situation below the Larkin scale $L_p$.

Next we discuss renormalized perturbation theory starting from a situation where $C_s = 0$. As long as $f \leq f_c$, the elastic object is pinned and boundary pinning does not matter, hence $C(f, t) = 0$. At $f = f_c$ the elastic object is in the same critical state as at the depinning transition. Therefore we can use the results of the previous renormalization group calculation in this case. As a result $\gamma$ and $\Delta_0(z)$ are replaced there by the renormalized, momentum dependent quantities

$$
\gamma(p) \simeq \gamma(pL_p)^{-2+\zeta} ,
$$

(5a)

$$
\Delta_p(\varphi) \approx K_D^{-1}(\Gamma/L_p)^{\zeta} p^{4-D-2\zeta} \Delta^*(\varphi(pL_p)^{\zeta}/l) .
$$

(5b)

$\zeta$ denotes the roughness exponent which was calculated for domain walls to order $\epsilon = 4 - D$ in $\Sigma$ and recently to $O(\epsilon^2)$ $\Sigma$. For CDWs $\zeta = 0$ $\Sigma$. The most important feature of $\Delta_p(\varphi)$ is that $\Delta^*(\varphi)$ has a cusp-like singularity at the origin. The renormalized equation for $C(f, t \to \infty)$ is given by

$$
\Gamma C(f, t) = f + f_p \omega_p \int_0^\infty dt' \int_0^\infty d\tilde{p} \times
$$

$$
x \tilde{p}^{1+\zeta - \omega_p \tilde{p}' \Delta^* \left[ (-C(t) + C(t - t')) \frac{x_1}{2} (x_1 - L) \right] ,
$$

where $\tilde{p} = pL_p$, $\omega_p = \gamma f_p/l$ and $f_p = \Gamma L_p^{-2}$. After having increased $f$ adiabatically to a fixed value slightly larger than $f_c$, $C(f, t)$ saturates for $t \to \infty$ and hence the difference $C(t) - C(t - t')$ vanishes. As a result the argument of $\Delta^*$ also vanishes and the right hand side of (5) becomes independent of $x_1$. Since $C(t) > C(t - t')$ the argument of $\Delta^*$ approaches zero from positive values. Thus we get for the saturation value $C_s(f)$

$$
C_s(f) = \frac{f - f_c}{\Gamma} = \frac{1}{L_p^2} \frac{f - f_c}{f_p} , \quad f_c = \frac{f_p}{2 - \zeta} \Delta^*(0^+) .
$$

(7)

One can understand this result in the following way: Using the decomposition $\varphi = \varphi_0 + \varphi_1$ in the asymptotic region, where $C(t)$ saturates, the equation of motion can be written as

$$
\frac{1}{\gamma} \dot{\varphi}_1 = \Gamma \nabla^2 \varphi_1 + f - \Gamma C + g_1(x, \varphi_1) ,
$$

(8)

where $g_1(x, \varphi_1) = g(x, \varphi_0(x) + \varphi_1(x))$. $g_1(x, \varphi)$ and $g(x, \varphi)$ have the same statistical properties. According to $\Sigma$ the force acting on the field $\varphi_1$ is now reduced by the curvature force $-\Gamma \nabla^2 \varphi_1$. The depinning of the $\varphi_1$-field seems hence to occur at $f \gtrsim f_c = f_c + \Gamma C_s$. However, since the boundary conditions fix $\varphi_1(0) = \varphi_1(L) = 0$ and hence $\langle \dot{\varphi}_1 \rangle = 0$ for all values of $f$, the system is always at its depinning transition, which implies $\Sigma$. Starting from some $f < f_c$ and $C_s = 0$, $C_s$ will stay at this value until $f$ reaches $f_c$. For $f > f_c$, $C_s$ obeys $\Sigma$. The same argument can be used for negative forces $f < 0$. Then we find for $f < -f_c$: $\Gamma C_s(f) = f + f_c = -(|f| - f_c) \Delta^*(0^+) = -\Delta^*(0^+)$. A scaling argument supports the validity of eq. (8) to all orders in $g$: Close to the $V_s = 0$ depinning transition the correlation length $\xi$ diverges as $\xi \approx L_p ((f - f_c)/f_c)^{-\nu}$. For $L < \xi$ the roughness – the mean square displacement of a piece of linear size $L'$ of the elastic object – scales as $w^2(L') \approx L'^2 (L'/L_p)^{2\zeta} \Sigma$ $\Sigma$ $\Sigma$. If we choose the system size $L \approx \xi$ we expect that the roughness scales as the height of the parabolic $\varphi$-profile on the same scale, $w(\xi) \approx C^2 \Sigma$, which is indeed fulfilled if we use the scaling law $\nu = 1/(2 - \zeta)$

$$
C = \frac{w(\xi)}{\xi^2} = \frac{1}{L_p^2} \left( f - f_c \right)^{\nu - 2} \approx \frac{f - f_c}{\Gamma} .
$$

(9)

**Hysteresis.—** Next we consider the case that we increase $f$ adiabatically from $f \leq f_c$ to a value $f_{max}$, where $C(f, t)$ reaches $C_{max}$ and then decrease $f$ again. In this case $C(f, t) < C(f, t - t')$ and hence the argument of $\Delta^*$ becomes negative. Instead of $\Sigma$ we get from (8)

$$
\Gamma C_{max} \equiv \max \left( f - f_c = f + f_c \right) .
$$

(10)

The effective force acting on the elastic object is now given by $f - \Gamma C_{max}$. Further decreasing $f$, there is
no change of $C(f, t)$ until the effective force reaches the threshold $-f_c = f - \Gamma c_{\text{max}} = f - (f_{\text{max}} - f_c)$. According to the last relation this happens at $f = f_{\text{max}} = f_{\text{max}} - 2f_c$. Analogous arguments can be used for reversing the fields from $f < 0$ to $f > 0$. Thus $C_s$ undergoes a hysteresis which consists of the two parallel segments given by $C_s = (f + f_c)$ and two horizontal segments determined by $C_{\text{max}} = (f_{\text{max}} - f_c)/\Gamma$ and $C_{\text{min}} = (f_{\text{min}} + f_c)/\Gamma$, respectively. Indeed, similar hysteresis effects of the strain have been observed in CDWs [16].

These findings are fully supported by numerical simulations, as shown in fig. 1. For integrating the equation of motion, the $x$-coordinate is discretized with a lattice constant $\alpha$ and the simulation time is measured in units of a time $\tau_0$ (the dimensionless lattice laplacian for $D = 1$ is given by $\nabla^2 \tilde{\varphi}_i = \tilde{\varphi}_{i+1} + \tilde{\varphi}_{i-1} - 2\tilde{\varphi}_i$, with lattice sites $i = 0, \ldots, L$). $\alpha$ and $\tau_0$ are chosen such that $\frac{\alpha \tau_0}{D} = 1$ and the dimensionless stochastic forces $\tau_0 \gamma g(x, \varphi) \in [-1/2, 1/2]$ (the dimensionless driving force is $\tau_0 f$).

![FIG. 1: Hysteresis of $C(f)$ at $T = 0$ for an one-dimensional interface. The driving force is first increased to $f_{\text{max}} = 0.6$ or $f_{\text{max}} = 0.8$, respectively, and then decreased to $-f_c \approx -0.27$. The arrows show the direction of the hysteresis. The numerical simulation was done for an interface with length $L = 1000$ and averaged over 300 disorder configurations.](image)

Curvature at finite temperature.— Next we want to consider the problem of finite temperatures. Changing $f$ only adiabatically we may use equilibrium statistical mechanics. It is convenient to go over to the field $\tilde{\varphi}(x) = \varphi(x) + \frac{1}{\alpha} x_1 (x_1 - L)$. The Hamiltonian rewritten in $\tilde{\varphi}$ has the same statistical properties as the initial one [11], since $V_{R}(x, \varphi) = - \int \delta \tilde{\varphi} g(x, \varphi')$ is a random function of both arguments. This can most easily seen by using the replica method [12]. The disorder averaged free enthalpy follows from the replica Hamiltonian

$$H_n = \frac{\Gamma}{2} \sum_{a, b = 1}^n \int_x \left\{ (\nabla \tilde{\varphi}_a)^2 \delta_{a, b} - \frac{\Gamma}{T} R(\tilde{\varphi}_a - \tilde{\varphi}_b) \right\}.$$

with $\langle V_R(x, \varphi) V_R(x', \varphi') \rangle_d = \delta^{(D)}(x - x') R(\varphi - \varphi')$ Apparently, the replica Hamiltonian is the same as that following from [11]. It is worth to mention that this is true only if the random potential $V_R(x, \varphi)$ is strictly uncorrelated in $x$-direction. The application of surface barriers implies therefore $C = f/\Gamma$ and $\langle \langle \tilde{\varphi}(x) - \tilde{\varphi}(x') \rangle^2 \rangle_{\text{th}} / \langle \varphi \rangle^2 \approx (L/L_p)^\xi$ where $\xi$ denotes the equilibrium roughness exponent corresponding to Hamiltonian [11]. Thus the displacement profile is the same as in the pure case. For non–adiabatic changes of $f$, traces of the $T = 0$ hysteresis are expected to be seen at non–zero temperatures (cf. fig. 2).

![FIG. 2: Simulation–time resolved coefficient $C(t)$ for a driving force of $f = 0.2$ at various temperatures. The simulation was done for a system of length $L = 1000$ and for one disorder configuration of CDW–type for each temperature (see text).](image)

The numerical solution of equation of motion with thermal noise at finite temperatures and $V_s = \infty$ is in agreement with these analytical considerations. Fig. 2 shows the coefficient $C(t)$ as it approaches its saturation value $C_s = f/(2\Gamma)$ with time. Strictly speaking, we are not in a steady state until $C(t)$ has reached its saturation value and hence the phase profile deviates slightly from the parabolic shape. In fig. 2 $C(t)$ is the least square fit to the profile. Note, that for low temperatures ($T < 5.0$ in the simulation, where $T$ is the dimensionless variance of the thermal noise) this approach is very slow, noticeable by the occurring steps, triggered by avalanches, even at large times. For high $T (T = 5.0)$, one sees that $C(t)$ fluctuates around the saturation value due to thermal noise. Therefore the $T = 0$ hysteresis of $C$ vanishes at finite temperatures.

Critical Depinning.— So far the surface potential was assumed to fix the value of the displacement field $\varphi$ at the surfaces $x_1 = 0$ and $x_1 = L$. We will now assume that the surface potential is reduced (or temperature is raised from zero) such that a macroscopic motion of the elastic object is possible. To determine the mutual interaction between the bulk and the surface we have to consider the effective equation of motion of the surface. Denoting $\varphi(0, x_\perp) = \varphi_s(x_\perp)$ the effective equation of motion of
the surface field can be written as \((\alpha\text{ is assumed to be of the order of the lattice spacing})\)

\[
\frac{1}{\gamma} \varphi_s = \Gamma C \frac{L}{2a} + \Gamma \nabla^2 \varphi_s + f - \frac{\Gamma}{a^2} V'_s(\varphi_s).
\]  

An analogous equation can be written for \(\varphi(L, x)\). In \([12]\) we have replaced the force resulting from the displacement in the bulk by the corresponding average force. In the steady state \(\nabla^2 \varphi_s = 0\) and eq. \([12]\) has a depinning threshold \(f_{s,c} \gg f_c\) determined by

\[
\Gamma C(f_{s,c}) \frac{L}{2a} + f_{s,c} - \frac{\Gamma}{a^2} \max V'_s(\varphi) = 0
\]  

For \(f > f_{s,c} \gg f_c\) the macroscopic velocity is given by the steady state solution \(v = \dot{\varphi}_s\) which follows from integrating \([12]\) with \(\nabla^2 \varphi_s = 0\). The corresponding solution

\[
v(t) = v_p \Phi \left( \frac{\Gamma C \frac{L}{2a} + \frac{f a^2}{V'_{s,max}}}{\Gamma V'_{s,max}}, t \right)
\]  

depends of course on the specific form of the surface potential, \(v_p = \gamma f_p\). Eq. \([14]\) has to be combined with the effective equation for the bulk \((f > f_c)\) \([8]\]

\[
\left( \frac{v(t)}{v_p} \right)^{1/\beta} = \frac{f - f_c}{f_p} - C(t) \frac{L_p^2}{l}
\]  

which follows from \([15]\) and \([17]\). Eqs. \([14]\) and \([15]\) determine both the velocity and the curvature \(C\) as a function of the driving force. If we increase \(f\) from \(f = 0\) with \(C = 0\), \(C\) remains zero until we reach \(f_c\). For \(f_c < f < f_{s,c}\), \(C\) obeys \([17]\). At \(f_{s,c}\) the elastic object is depinned and curvature is reduced with increasing velocity, according to \([15]\). If the surface potential is periodic, also \(v(t)\) will be periodic and the bulk depinning transition is slightly smeared out \([17]\). We will assume that this effect is weak. In principal it can be avoided by adding some randomness to the surface potential.

**Nucleation and Creep.**—At finite but low temperatures the surface field may exhibit a creep motion even if \(f \ll f_{s,c}\). Creep proceeds via the formation of droplets at the surfaces \(x_1 = 0\) and \(x_1 = L\), inside which \(\varphi\) is changed by \(2\pi\), respectively, with respect to the bulk value of \(\varphi\). The droplet consist of a cylindrical piece (the cylinder axis is perpendicular to \(x_1 = 0\)) in the surface layer of height \(a\) and radius \(R\) and an attached semi-sphere with the same radius. The width of the droplet wall confining the cylinder is of the order \(a' = a / \sqrt{V'_{s}}\).

Keeping only the leading order terms we get for the energy of the droplet \(E_{dp}(R) = 2\Gamma R^{D-2} \{ \sqrt{V'_{s}} + \ln \frac{a}{\sqrt{V'_{s}}} - \pi C R L - f R^2 / \Gamma \}\). The critical droplet size \(R_C \ll L\) follows as \(R_c \approx \sqrt{V'_{s}} / (\pi C L)\) or \(R_c = (C L)^{-1}\) in \(D = 3\) or \(D = 2\), respectively. In deriving \(E_{dp}\), we have neglected the contribution from the disorder which is correct as long as \(R_c < L_p\), i.e., \(L \gg L_p\). The nucleation rate of droplets and hence the creep velocity is given by

\[
\frac{v_{D=3}}{v_p} = A \exp \left( -B \frac{V''_{s}}{C L T} \right), \quad \frac{v_{D=2}}{v_p} = A' \left( \frac{C L a'}{\sqrt{V'_{s}}} \right)^{B' T / T}
\]  

which replaces \([13]\) in the case \(f \ll f_{s,c}, T > 0\) (cf. \([14]\)). The present treatment is too crude to give the coefficients \(A', B, B'\). Again, \([16]\) has to be considered together with \([15]\) to determine \(C\) and \(v\). In CDWs, where \(\varphi\) can be multi-valued, nucleation processes also occur deep in the bulk \([14]\). The droplet energy then does not contain a term \(\sim f\), leaving the relations \([16]\) essentially unchanged.

To conclude we have shown that surface pinning of impure elastic systems lead to an onset of curvature \(C\) only above a threshold value \(f_c\) of the random force. In general, \(C\) exhibits a pronounced hysteresis. The curvature is reduced above the surface depinning transition or at finite temperatures when nucleation processes at the surface allow for creep motion.

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