A note on the uniqueness of minimal length carrier graphs

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Abstract

We show that minimal length carrier graphs are not unique, but if $M$ is in a large class of hyperbolic 3-manifolds, including the geometrically finite ones, then $M$ has only finitely many minimal length carrier graphs and no two of them are homotopic. As a corollary, we obtain a new proof that the isometry group of a geometrically finite 3-manifold is finite.

Let $M$ be a hyperbolic 3-manifold. A carrier graph for $M$ is a finite graph $X$ along with a map $f: X \to M$ such that $f_*: \pi_1(X) \to \pi_1(M)$ is surjective. We will assume that $\pi_1(M)$ is finitely generated, and it is then clear that a carrier graph for $M$ exists. We will only consider carrier graphs with $\text{rank}(\pi_1(X)) = \text{rank}(\pi_1(M))$. The length of an edge $e$ of $X$, $\text{len}_f(e)$, is the length of the path $f|_e$ (we assume $f$ takes edges to rectifiable paths in $M$), and $\text{len}_f(X)$ is the sum of the lengths of the edges of $X$. A minimal length carrier graph is a carrier graph with length less than or equal to the length of any other (minimal rank) carrier graph for $M$. In [8], White showed that if $M$ is closed, then it has a minimal length carrier graph, and in [5], it is shown how to extend White’s argument to a much larger class of hyperbolic 3-manifolds. Minimal length carrier graphs have nice geometric properties and were first used by White [8] to prove that if $M$ is closed, then it has a nontrivial loop whose length is bounded above in terms of nothing more than the rank of $\pi_1(M)$. They have subsequently been used, for example, to show that rank equals Heegaard genus for large classes of hyperbolic 3-manifolds in [6], [4], and [3].

Following Souto [6], we say that two carrier graphs $f: X \to M$ and $g: Y \to M$ are equivalent if there exists a homotopy equivalence $\eta: X \to Y$ so that $f$ and $g\eta$ are freely homotopic. We will say that $f$ and $g$ are strongly
equivalent if there is a homeomorphism $\eta: X \to Y$ such that $f = g\eta$. In this note, we consider the following two uniqueness questions:

1. Must any two minimal length carrier graphs for $M$ be strongly equivalent?

2. Must any two carrier graphs which both have minimal length within the same equivalence class be strongly equivalent?

The answer to both questions is no, according to the examples in Section 1. However, we will prove two weaker uniqueness results in Section 2. In order to state the results, we need one more (very strong) notion of equivalence. Two carrier graphs $f, g : X \to M$ are essentially equivalent if $f = g\eta$, where $\eta : X \to X$ is a homeomorphism that fixes vertices and leaves edges and their orientations invariant. In other words, $f$ and $g$ are the same except for reparameterizing the edges. Carrier graphs are essentially distinct if they are not essentially equivalent.

**Theorem 1.** Let $M$ be a hyperbolic 3-manifold and let $f : X \to M$ and $g : X \to M$ be carrier graphs, which either each have minimal length within their equivalence classes or each have minimal length globally. If $f$ and $g$ are homotopic, then they are essentially equivalent.

And although there may be more than one carrier graph of minimal length globally or within an equivalence class, we show

**Theorem 2.** Let $M$ be a hyperbolic 3-manifold that does not have a simply-degenerate, $\pi_1$-surjective NP-end. Then $M$ has only finitely many essentially distinct minimal length carrier graphs, and each equivalence class of carrier graphs can have only finitely many essentially distinct minimal length representatives.

The fact that a manifold satisfying the hypotheses of this theorem has a minimal length carrier is proved in [5]. In Section 3, we will look at the action of the isometry group $\text{Isom}(M)$ of $M$ on the set of minimal length carrier graphs and derive a new proof of the following fact:

**Corollary 1.** If $M$ does not have a simply-degenerate, $\pi_1$-surjective NP-end, then $\text{Isom}(M)$ is finite.

Note that this corollary applies, for example, to all geometrically finite 3-manifolds.

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1 Non-uniqueness example

Proposition 1. Let $M$ be a hyperbolic 3-manifold with $\text{rank}(\pi_1(M)) = 2$. Suppose that $M$ has a minimal length carrier graph $f: X \to M$ and that $M$ has a fixed-point free isometry $h$ of finite order not divisible by 3. Then $hf$ is a minimal length carrier graph not strongly equivalent to $f$.

Proof. It is clear that $hf$ is a minimal length carrier graph. Suppose it is strongly equivalent to $f$. Then there exists a homeomorphism $\eta: X \to X$ such that $hf = f\eta$. Note that $\eta$ cannot fix any point $x \in X$, for then $h$ would fix $f(x)$. According to [8], minimal length carrier graphs must be trivalent. There are only two trivalent graphs of rank 2: one that looks like a $\theta$ and one that looks like eye-glasses. The eye-glasses graph does not admit a fixed-point free homeomorphism; so $X$ is the $\theta$ graph. Up to homotopy, $\eta$ must be the homeomorphism that swaps vertices and cyclically permutes the edges.

Let $m$ be the order of $h$. Then $h^m f = f \eta^m$, which is equivalent to $f = f \eta^m$. Since $m$ is not divisible by 3, $\eta^m$ cyclically permutes the edges of $X$. Hence, $f$ must map each edge to the same image, which contradicts $f$ being a carrier graph because $f_*(\pi_1(X))$ would be trivial. 

We can get concrete examples from this proposition. For example, let $M$ be the figure 8 knot complement. Then $M$ is a two-fold cover of the Gieseking manifold, hence it has a fixed-point free isometry $h$ of order 2, and the rank of $\pi_1(M)$ is easily found to be two (from, say, the Wirtinger presentation); so $M$ has non-unique minimal length carrier graphs.

We can also get closed examples. Reid [4] shows how to produce, for any $p > 1$, a closed hyperbolic 3-manifold $M$ with a regular, cyclic cover $N$ of degree $p$ such that $\text{rank}(\pi_1(N)) = 2$. If $p$ is not divisible by 3, then $N$ with its order $p$ deck transformation satisfies the hypotheses of Proposition 1 and thus has non-unique minimal length carrier graphs.

We can take these examples a bit further to get examples of carrier graphs which are minimal in the same equivalence class but are not strongly equivalent. Reid’s manifolds are formed as follows. Let $\varphi$ be a pseudo-Anosov homeomorphism of a punctured torus $T$ and let $M_\varphi$ be the mapping torus of $\varphi$. Let $a, b$ be generators of $\pi_1(T)$. Reid forms a manifold, which we are calling $N$, by taking the obvious $p$-fold cyclic cover of $M_\varphi$ and performing a certain Dehn filling on it. It is shown that $N$ is a $p$-fold cyclic cover of a manifold obtained from Dehn filling $M_\varphi$ and the preimage of the filling torus for $M_\varphi$ is the filling torus of $N$ (in particular, the deck transformations of $N$ leave the filling torus invariant). By abuse of notation, we will use $a$ and $b$ to...
refer to the generators of the fiber subgroup of $M_\phi$ and its cover and to their images in the filled manifold $N$. Reid shows that $a$ and $b$ generate $\pi_1(N)$. Let $H$ be the filling torus of $N$. Then $N \setminus H$ is fiber bundle over $S^1$ with fiber a compact surface of genus 1 and with 1 boundary component. Choose representatives $\alpha$ and $\beta$ of $a$ and $b$, respectively, that lie in a particular fiber $\Sigma$ of $N \setminus H$. If $h$ is an order $p$ deck-transformation of $N$, then $h \circ \alpha$ and $h \circ \beta$ are loops in the fiber $h(\Sigma)$ which also generate $\pi_1(N)$. Notice that there is a submanifold homeomorphic to $\Sigma \times [0, 1] \subset N$ containing $\alpha$, $\beta$, $h \circ \alpha$ and $h \circ \beta$. The manifold $\Sigma \times [0, 1]$ is a genus 2 handlebody and the pairs $\{\alpha, \beta\}$ and $\{h \circ \alpha, h \circ \beta\}$ each generate its fundamental group. It is a well-known fact that any two minimal cardinality generating sets for the fundamental group of a handlebody are Nielsen equivalent; hence, $h$ preserves the Nielsen equivalence class of the generating pair $\{a, b\}$.

Let $f : S^1 \vee S^1 \to N$ be the carrier graph given by mapping one of the $S^1$s to $\alpha$ and the other to $\beta$, and let $f'$ be a carrier graph of minimal length in the equivalence class of $f$. Then $h \circ f'$ has minimal length in the equivalence class of the graph coming from $h \circ \alpha$ and $h \circ \beta$. In [6], Souto shows how to associate an equivalence class of carrier graphs to a Nielsen equivalence class of generators for $\pi_1$ and vice versa. His discussion of this correspondence implies that since $h$ preserves the Nielsen equivalence class of $\{a, b\}$, $f'$ and $h \circ f'$ are equivalent. However, Proposition 1 implies that these carrier graphs are not strongly equivalent. Hence, minimal length carrier graphs are not unique even within an equivalence class.

2 Weaker forms of uniqueness

For the proof of Theorem 1 we will need a lemma.

**Lemma 1.** Let $x$, $y$ and $z$ be distinct points in $\mathbb{H}^n$. Let $x'$ (resp. $y'$) be the midpoint of the geodesic between $x$ (resp. $y$) and $z$. Then $d(x', y') \leq \frac{1}{2} d(x, y)$, and equality is achieved exactly when the angle $\angle xzy$ is $0$ or $\pi$.

**Proof.** Let $a = d(x', z)$, $b = d(y', z)$, $c = d(x', y')$ and $\gamma = \angle xzy$. We wish to show that $2c \leq d(x, y)$. This is equivalent to $\cosh(2c) \leq \cosh(d(x, y))$. By the hyperbolic law of cosines,

$$
\cosh(c) = \cosh(a) \cosh(b) - \sinh(a) \sinh(b) \cos(\gamma)
$$

$$
\cosh d(x, y) = \cosh(2a) \cosh(2b) - \sinh(2a) \sinh(2b) \cos(\gamma).
$$

Now we just follow our noses: $\cosh(2c) = 2 \cosh^2(c) - 1$, so we need to
show

$$2 \cosh^2(c) - 1 \leq \cosh(2a) \cosh(2b) - \sinh(2a) \sinh(2b) \cos(\gamma).$$

The left side is equal to

$$2 \cosh^2(a) \cosh^2(b) - 4 \cosh(a) \cosh(b) \sinh(a) \sinh(b) \cos(\gamma) +$$

$$2 \sinh^2(a) \sinh^2(b) \cos^2(\gamma) - 1.$$ 

Notice that

$$\sinh(2a) \sinh(2b) \cos(\gamma) = 4 \cosh(a) \cosh(b) \sinh(a) \sinh(b) \cos(\gamma).$$

So our goal becomes

$$2 \cosh^2(a) \cosh^2(b) + 2 \sinh^2(a) \sinh^2(b) \cos^2(\gamma) - 1 \leq \cosh(2a) \cosh(2b).$$

Using the identity $\cosh(2x) = 2 \cosh^2(x) - 1$ and some algebra, one can see that this is equivalent to

$$\sinh^2(a) \sinh^2(b) \cos^2(\gamma) + \cosh^2(a) + \cosh^2(b) \leq \cosh^2(a) \cosh^2(b) + 1.$$ 

It suffices to prove this inequality with the assumption that $\cos^2(\gamma) = 1$, or equivalently, $\gamma = 0, \pi$. In this case, it is an equality, which follows from the identity $\cosh^2(b) = \sinh^2(b) + 1$. \hfill $\square$

**Proof of Theorem 1.** Suppose $f$ and $g$ are homotopic and essentially distinct. Let $H : X \times [0, 1] \to M$ be a homotopy from $f$ to $g$. The space $X \times [0, 1]$ can be triangulated as follows. Let $e \subset X$ be an edge. Suppose $e$ has distinct endpoints. Then a homeomorphism from $e$ to $[0, 1]$ can be extended to a homeomorphism from $e \times [0, 1]$ to $[0, 1] \times [0, 1]$ (sending $e$ to $[0, 1] \times \{0\}$). The latter space has a triangulation with two triangles obtained by splitting the square along one of its diagonals. This triangulation can be pulled back to a triangulation for $e \times [0, 1]$. If $e$’s endpoints are the same (i.e. $e$ is a loop), then it can be triangulated in essentially the same way, but with $e \times \{0\}$ and $e \times \{1\}$ identified. These triangulations can be glued together in an obvious way to yield a triangulation of $X \times [0, 1]$. The map $H$ can be made simplicially hyperbolic with respect to this triangulation, i.e. it can be made to send edges to geodesic segments and 2-simplices to geodesic triangles. We will assume this has been done. Note that this does not change the ends of the homotopy $(f$ and $g$), since they already have geodesic edges by virtue of having minimal length.
We now construct a new carrier graph \( h : X \to M \). Let \( v \) be a vertex of \( X \). Then \( H \) maps \( \{v\} \times [0,1] \) to a geodesic arc in \( M \). Let \( h(v) \) be the midpoint of that geodesic. Having defined \( h \) on the vertices on \( X \), we can define it on the edges. Let \( e \) be an edge of \( X \) with distinct endpoints \( v \) and \( w \). Then \( e \times [0,1] \) consists of two triangles, which share a common edge. Let \( m \) be the midpoint of the (geodesic) image of that edge under \( H \). There are geodesic arcs \( e_1 \) and \( e_2 \) connecting \( h(v) \) to \( m \) and \( m \) to \( h(w) \) and lying within \( H(e \times [0,1]) \). Let \( h \) map \( e \) homeomorphically to the path formed by concatenating \( e_1 \) and \( e_2 \). See Figure 1. If \( e \) has only one endpoint, then \( h(e) \) is formed similarly; the picture is the same as Figure 1, except that the left and right edges are identified. Thus, the map \( h \) is essentially the midpoint of the homotopy \( H \). It is clear that \( h \) is homotopic to \( f \) and \( g \), which implies that it is a carrier graph in the same equivalence class as \( f \) and \( g \).

We will show that \( \text{len}_h(X) < \frac{1}{2}(\text{len}_f(X) + \text{len}_g(X)) \). Because \( \text{len}_f(X) = \text{len}_g(X) \) (since they both have minimal length), \( h \) will be shorter than both, which will contradict minimality and complete the proof. Still referring to Figure 1, we can lift to \( \mathbb{H}^3 \) and apply Lemma 1 to the left and right triangles to get

\[
\text{len}_h(e) = \text{len}(e_1) + \text{len}(e_2) \leq \frac{1}{2}\text{len}_g(e) + \frac{1}{2}\text{len}_f(e) \quad (1)
\]

with equality if and only if \( \theta, \varphi \in \{0, \pi\} \). Summing over all edges, we get \( \text{len}_h(X) \leq \frac{1}{2}(\text{len}_f(X) + \text{len}_g(X)) \). In order for this to be a strict inequality, we need for there to be at least one edge for which (1) is a strict inequality.

For any vertex \( v \in X \), let \( H_v = H(\{v\} \times [0,1]) \). There must be some vertex \( v_0 \) such that \( H_v \) is not just a point. For otherwise, \( f \) and \( g \) would
agree on the vertices of \( X \) and for each edge \( e \) of \( X \), \( f(e) \) and \( g(e) \) would be geodesic segments homotopic relative to their endpoints. Hence, \( f(e) \) would be the same as \( g(e) \). If \( e \) has distinct endpoints, then it is clear that on \( e \), \( f \) and \( g \) differ by an orientation preserving homeomorphism. If \( e \) is a loop, then perhaps \( f \) and \( g \) map \( e \) to the same geodesic, but with opposite orientation. Since \( f \) and \( g \) are homotopic (via a homotopy that does not move the vertices), that would imply that the loop \( f(e) \) is homotopic to its inverse. Since \( \pi_1(M) \) is torsion-free and simple loops in \( X \) map to non-nullhomotopic loops in \( M \), this cannot happen. Therefore, \( f \) and \( g \) must be essentially equivalent, which is a contradiction. 

Note that if \( H_v \) is not a single point, it is a geodesic path.

Pick an edge \( e \) with (not necessarily distinct) endpoints \( v \) and \( w \), such that \( H_v \) is not a single point. If the inequality (1) for \( e \) is strict, then we are done. If it is an equality, then the angles \( \theta \) and \( \varphi \) must each be either 0 or \( \pi \). This implies that the angle between \( H_v \) and \( f(e) \) is either 0 or \( \pi \). The vertex \( v \) must have some other edge \( e' \neq e \) adjacent to it. In [8], it is shown that the angle between any two edges sharing a vertex in a minimal length carrier graph is \( 2\pi/3 \). In particular, the angle between \( f(e') \) and \( f(e) \) is \( 2\pi/3 \). Thus, the angle between \( f(e') \) and \( H_v \) cannot be 0 or \( \pi \), and so for \( e' \), the inequality (1) must be strict. Hence, we get the desired contradiction

\[
\text{len}_{h}(X) < \frac{1}{2}(\text{len}_f(X) + \text{len}_g(X)) = \text{len}_f(X).
\]

In the examples following Proposition 1, we found that minimal length carrier graphs were not unique because we can compose them with ambient isometries to get new carrier graphs. It is perhaps natural to wonder if any two minimal length carrier graphs are related in this way (up to reparametrizing their edges). If this were true, then the well-known fact that for a large class of hyperbolic 3-manifolds \( M \), Isom(\( M \)) is finite, would imply Theorem 2 that there are only finitely many minimal length carrier graphs. We will prove this theorem directly using Theorem 1 via the following proposition and then prove the finiteness of isometry groups as a corollary in the next section.

**Proposition 2.** Let \( M \) be a finite volume hyperbolic 3-manifold and let \( L > 0 \). There are only finitely many carrier graphs which are minimal length within their equivalence class and have length less than or equal to \( L \).

**Proof.** Suppose \( M \) has an infinite sequence of carrier graphs \( f_i : X \to M \), each of minimal length within its equivalence class and each with length less than or equal to \( L \).
than or equal to $L$. Being minimal length implies the graphs are trivalent. There are only finitely many trivalent graphs of a particular rank; so we may pass to a subsequence and assume that every $X_i$ is homeomorphic to a particular graph $X$. We will continue to call this sequence $f_i$. Since the $f_i$ have bounded length and a carrier graph cannot be contained in a cusp (because this would imply that $\pi_1(M)$ is a quotient of the cusp group $\mathbb{Z}^2$), there is a bound on how deep into a cusp neighborhood the image of any $f_i$ may penetrate. Hence, the $f_i$ all map into one compact subset of $M$. Additionally, the bound on the length of the $f_i$ implies that the sequence is equicontinuous. We can now apply the Arzelà-Ascoli theorem to get that a subsequence of $\{f_i\}$ converges uniformly. Therefore, for some large $i$ and $j$, $f_i$ is sufficiently close to $f_j$ that the two maps must be homotopic. This contradicts Theorem 1. 

**Proof of Theorem 2.** Let $M$ be a finite volume hyperbolic 3-manifold. Let $C$ be the set of minimal length carrier graphs for $M$, and let $L$ be the length of any element of $C$. Elements of $C$ clearly have minimal length within their equivalence classes. Thus, $C$ is contained in the set of carrier graphs which are of minimal length in their equivalence classes and have length less than or equal to $L$. The latter set is finite, by virtue of Proposition 2.

Similarly, the set of carrier graphs of minimal length within a particular equivalence class is seen to be finite by letting $L$ be the length of any minimal length representative and applying Proposition 2 in the same way.

3 An application to isometry groups

We will now give a new proof of the previously known result that a hyperbolic 3-manifold that does not have a simply-degenerate, $\pi_1$-surjective NP-end has finite isometry group. The proof is simple and follows from Theorem 2 and basic facts about minimal length carrier graphs. It requires knowing that such manifolds have minimal length carrier graphs, which was proved in [5]. That proof relies on the proof of the tameness theorem by Agol and Calegari-Gabai. However, the tameness theorem is not needed for the case in which the 3-manifold is geometrically finite. Thus, when the following theorem is restricted to geometrically finite 3-manifolds, its proof is entirely elementary.

**Corollary 2** (Corollary 1). *If $M$ does not have a simply-degenerate, $\pi_1$-surjective NP-end, then $\text{Isom}(M)$ is finite.*
Proof. Let $\mathcal{C}$ be the set of essential equivalence classes of minimal length carrier graphs in $M$. By Theorem 1 of [5], this set is nonempty, and by Theorem 2, this set is finite. It is clear that $\text{Isom}(M)$ acts on $\mathcal{C}$, which gives a map from $\text{Isom}(M)$ to the finite group of permutations of $\mathcal{C}$. Let $K$ be the kernel of this map. It suffices to show that $K$ is finite. Isometries of $M$ that are in $K$ fix a minimal length carrier graph $f:X \to M$ up to essential equivalence. In particular, for some vertex $v \in X$, they fix $f(v)$ and permute the images of the three edges attached to $v$. This gives a map from $K$ to $S_3$, the permutation group on three elements. An element $h$ of the kernel of this map would fix $f(v)$ and the three tangent vectors at $f(v)$ corresponding to the three edges of $X$ attached to $v$. Since the angles between these edges are all $2\pi/3$, these tangent vectors span a plane in the tangent space. Lifting to $\mathbb{H}^3$, $\tilde{h}$ would fix some preimage of $f(v)$ and fix a hyperplane going through $f(v)$ pointwise (since it is an isometry and fixes the tangent plane). Hence, $h$ must be the identity map. Thus, $K$ injects into $S_3$, which means that $K$ is finite. Therefore, $\text{Isom}(M)$ is finite. \hfill \Box

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