In 2005 Serre in [8] introduced the notion of complete reducibility in spherical buildings. He went on to point out the following conjecture [8, Conjecture 2.8] which he attributes to Tits from the 1950’s.

**Conjecture 1 (Tits’ Centre Conjecture).** Suppose that $\Delta$ is a spherical building and $\Omega$ is a convex subcomplex of $\Delta$. Then (at least) one of the following holds:

(a) for each simplex $A$ in $\Omega$, there is a simplex $B$ in $\Omega$ which is opposite to $A$ in $\Delta$; or

(b) there exists a nontrivial simplex $A'$ in $\Omega$ fixed by any automorphism of $\Delta$ stabilizing $\Omega$.

If possibility (a) in the conjecture arises we say that $\Omega$ is **completely reducible** and if (b) is the case, then the simplex $A'$ is called a **centre** of $\Omega$. If alternative (a) holds then $\Omega$ is a possibly thin subbuilding of $\Delta$ (see [7]).

If $G$ is an algebraic group with associated building $\Delta$, then a subgroup $H$ of $G$ is called **completely reducible** provided that whenever it is a subgroup of a parabolic subgroup of $G$ it is contained in a Levi complement of that parabolic subgroup. In this case, the convex subcomplex of $\Delta$ fixed by $H$ is completely reducible. Conversely if the subcomplex of $\Delta$ fixed by a subgroup $H$ of a parabolic subgroup of $G$ is completely reducible, then so is $H$. This relationship between complete reducibility of subcomplexes of the building and completely reducible subgroups of parabolic subgroups has lead to a source of fruitful research of which we particularly mention [1, Theorem 3.1] in which they prove the conjecture in the case that $\Omega$ is the fixed point set of some subgroup $H$.

In the more general setting, for the classical buildings and buildings of rank 2 the conjecture was proved by Mühlherr and Tits [4] in 2006. For buildings of exceptional type $E_6$, $E_7$ and $E_8$ the conjecture has been proved by Leeb and Ramos Cuevas [3, 6] using, in part, some of the observations presented in this paper. They also include the proof of the conjecture for buildings of type $F_4$, which was first presented by the authors at a meeting in Oberwolfach in January 2007 [4]. All of the investigations of the Centre Conjecture have used the lemma of Serre’s [8] which states that $\Omega$ is completely reducible if every vertex of $\Omega$ has an opposite. For chamber complexes, we can prove the following stronger
assertion and thereby obtain a very short proof of the Centre Conjecture for convex chamber subcomplexes of classical buildings.

**Theorem 2.** Let $\Delta$ be an irreducible spherical building of type $(W, I)$. Let $\Omega$ be a convex chamber subcomplex of $\Delta$. If for some $k \in I$ every vertex of type $k$ in $\Omega$ has an opposite in $\Omega$, then $\Omega$ is completely reducible.

Notice that the hypothesis that $\Delta$ is irreducible in Theorem 2 may not be dropped as is easily seen by taking a product of two buildings and choosing a convex subcomplex which is completely reducible in one factor and has a centre in the second factor. Our notation follows [9]. So given a simplex $R$ of type $J \subseteq I$, the collection of all simplices containing $R$ form a building $\text{St}R$ of type $(W_{(I \setminus J)}, I \setminus J)$. Of particular importance to us are the projection maps: given simplices $R$ and $S$, $\text{proj}_R(S)$ is the unique simplex of $\text{St}R$ which is contained in every shortest gallery from $S$ to $R$ (see [9, Proposition 2.29]) and is called the projection of $S$ to $R$. Note that if $\Omega$ is a convex subcomplex of $\Delta$ then, for all simplices $R$ and $S$ in $\Omega$, we have $\text{proj}_RS \in \Omega$ and this is the crucial property of convexity that we use in the proof of Theorem 2. We refer the reader to [9, 2.30 and 2.31] for many properties of projection maps. Two chambers in $\Delta$ are opposite in $\Delta$ provided their convex hull is an apartment of $\Delta$. Two simplices $R$ and $R'$ of $\Delta$ are opposite if every chamber of $\text{St}R$ has an opposite in $\text{St}R'$.

**Lemma 3.** Suppose that $x$ and $y$ are opposite chambers in $\Delta$. Let $\Sigma$ be the convex hull of $x$ and $y$ in $\Delta$ and $R$ be a simplex in $\Sigma$. Then $\text{proj}_R(x)$ and $\text{proj}_R(y)$ are opposite in $\text{St}R$.

*Proof.* Set $x_1 = \text{proj}_R(x)$ and $y_1 = \text{proj}_R(y)$. Then $x_1$ and $y_1$ are chambers by [9, Proposition 2.29]. Let $z$ be opposite $x_1$ in $\text{St}R$. Then we have $\text{dist}(x, z) = \text{dist}(x, x_1) + \text{dist}(x_1, z)$ and $\text{dist}(y, z) = \text{dist}(y, y_1) + \text{dist}(y_1, z)$ by [9, 2.30.6]. Therefore $\text{dist}(x, y) = \text{dist}(x, x_1) + \text{dist}(x_1, z) + \text{dist}(y_1, z)$ as every chamber of $\Sigma$ is on a shortest gallery between $x$ and $y$ by [9, 2.35 (iv)]. On the other hand, as $x_1$ and $z$ are opposite in $\text{St}R$, $\text{dist}(x_1, y_1) \leq \text{dist}(x_1, z)$ and so
\[
\text{dist}(x, y) = \text{dist}(x, x_1) + \text{dist}(x_1, y_1) + \text{dist}(y_1, y) \\
\leq \text{dist}(x, x_1) + \text{dist}(x_1, z) + \text{dist}(y, y_1).
\]
It follows that $\text{dist}(y_1, z) = 0$ and hence $z = y_1$ as claimed. \qed

The following observation is especially important to us.

**Corollary 4.** Suppose that $R$, $X$ and $Y$ are simplices in the apartment $\Sigma$ with $X$ opposite $Y$. Then either
\begin{itemize}
  \item[(a)] $\text{proj}_R(X)$ is opposite $\text{proj}_R(Y)$ in $\text{St}R$; or
  \item[(b)] $R = \text{proj}_R(X) = \text{proj}_R(Y)$.
\end{itemize}
Proof. We can pair the chambers containing $X$ and $Y$ into opposite pairs $(x, y)$. Then $\text{proj}_R(x)$ is opposite $\text{proj}_R(y)$ in $\text{St}R$ by Lemma $\Box$. This means every chamber of $\text{proj}_R(X)$ has an opposite in $\text{St}R$ contained in $\text{proj}_R(Y)$. □

We can now prove Theorem $\Box$. So suppose that $\Omega$ is a convex chamber subcomplex of $\Delta$. We recall that $\Omega$ is a subcomplex, means that if a simplex is in $\Omega$ then so are all of its faces and $\Omega$ is a chamber complex means that every simplex is contained in a chamber. We repeatedly use the fact that, as $\Omega$ is convex, projections between simplices of $\Omega$ are contained in $\Omega$.

By hypothesis, we may choose $J \subseteq I$ maximally so that every simplex of type $J$ in $\Omega$ has an opposite in $\Omega$. It suffices to show that $J = I$, as, if a chamber has an opposite, then so does every face of that chamber. So suppose that $J \neq I$. Since $\Delta$ is irreducible there is $i \in I \setminus J$ such that $i$ is a neighbour of some $j \in J$ in the Dynkin diagram of $\Delta$.

Let $z$ be of type $J \cup \{i\}$ in $\Omega$, $x_0$ be the face of $z$ of type $J$, $\ell$ the vertex of $z$ of type $i$ and let $C_0$ be a chamber of $\Omega$ containing $z$. We will construct an opposite for $z$.

Let $p$ be a maximal face of $C_0$ with missing vertex of type $j$ and $x_0'$ be an opposite of $x_0$ in $\Omega$. Then $\ell$ is a vertex of $p$. Put $C_0' = \text{proj}_{x_0'}C_0$ and $C_1 = \text{proj}_pC_0'$. Then, by Corollary $\Box$, $C_0 = \text{proj}_p(x_0) \neq C_1$. Let $x_1$ be the face of $C_1$ of type $J$. So $x_1 \neq x_0$ and setting $y_0 = \text{proj}_{x_1}x_0$ we see that, as the reflections corresponding to $i$ and $j$ do not commute, $y_0$ has $x_1$ as a face and $\ell$ as a vertex. We will first find an opposite of the simplex $y_0$.

Let $y_1 = \text{proj}_{x_1}x_0'$, so $y_1$ and $y_0$ are opposite in $\text{St}x_1$ by Corollary $\Box$. Let $x_1'$ be opposite $x_1$. By [9, Proposition 3.29], we have $y_2 = \text{proj}_{x_1'}(y_1)$ is opposite $y_0$. Since $y_0$ contains the vertex $\ell$, $y_2$ has an opposite of $\ell$ as a vertex and this is contained in $\Omega$.

In order to find an opposite for the simplex $z$, notice that $\text{proj}_\ell x_0 = z$. Let $z_1 = \text{proj}_\ell x_0'$, so $z_1$ and $z$ are opposite in $\text{St}\ell$ by Corollary $\Box$. Using [9, Proposition 3.29] again, the projection of $z_1$ to the opposite of $\ell$ in $\text{St}y_2$ now yields the required opposite of $z$ in $\Omega$. □

Corollary 5. The Centre Conjecture holds for convex chamber subcomplexes of irreducible spherical buildings of classical type.

Proof. For buildings of type $A_n, B_n, C_n$ and $D_n$, we identify the simplices of $\Delta$ with flags of subspaces (singular subspaces, isotropic subspaces) in the appropriate vector spaces. We then consider the vertices of $\Delta$ corresponding to 1-dimensional subspaces (for $A_n$) and 1-dimensional isotropic/singular subspaces in the other cases and call them type 1 vertices.
Since $\Omega$ is a chamber subcomplex, $\Omega$ contains vertices of every type. If every type 1 vertex has an opposite in $\Omega$, then $\Omega$ is completely reducible by Theorem 2. So we suppose that this is not the case and aim to identify a centre.

Suppose that $\Delta$ has type $A_n$ and assume that some type 1 vertex $w$ of $\Omega$ does not have an opposite in $\Omega$. Then $w$ is contained in all the hyperplanes of $\Omega$. Thus the intersection of all hyperplanes of $\Omega$ is the required centre.

Suppose that $\Delta$ has type $B_n, C_n$ or $D_n$. Then a vertex of type 1 in $\Omega$ has no opposite in $\Omega$ if and only if it is collinear with every other vertex of type 1 in $\Omega$. Hence the set of all vertices of type 1 in $\Omega$ having no opposite span a totally isotropic (singular) subspace, and this is the centre. □

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