Essential dimension of the spin groups in characteristic 2

Burt Totaro

The essential dimension of an algebraic group $G$ is a measure of the number of parameters needed to describe all $G$-torsors over all fields. A major achievement of the subject was the calculation of the essential dimension of the spin groups over a field of characteristic not 2, started by Brosnan, Reichstein, and Vistoli, and completed by Chernousov, Merkurjev, Garibaldi, and Guralnick [3, 4, 7, 17, Theorem 9.1].

In this paper, we determine the essential dimension of the spin group $\text{Spin}(n)$ for $n \geq 15$ over an arbitrary field (Theorem 2.1). We find that the answer is the same in all characteristics. In contrast, for the groups $O(n)$ and $SO(n)$, the essential dimension is smaller in characteristic 2, by Babic and Chernousov [1].

In characteristic not 2, the computation of essential dimension can be phrased to use a natural finite subgroup of $\text{Spin}(2r + 1)$, namely an extraspecial 2-group, a central extension of $(\mathbb{Z}/2)^{2r}$ by $\mathbb{Z}/2$. A distinctive feature of the argument in characteristic 2 is that the analogous subgroup is a finite group scheme, a central extension of $(\mathbb{Z}/2)^r \times (\mu_2)^r$ by $\mu_2$, where $\mu_2$ is the group scheme of square roots of unity.

In characteristic not 2, Rost and Garibaldi computed the essential dimension of $\text{Spin}(n)$ for $n \leq 14$ [6, Table 23B], where case-by-case arguments seem to be needed. We show in Theorem 3.1 that for $n \leq 10$, the essential dimension of $\text{Spin}(n)$ is the same in characteristic 2 as in characteristic not 2. It would be interesting to compute the essential dimension of $\text{Spin}(n)$ in the remaining cases, $11 \leq n \leq 14$ in characteristic 2.

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1 Essential dimension

Let $G$ be an affine group scheme of finite type over a field $k$. Write $H^1(k, G)$ for the set of isomorphism classes of $G$-torsors over $k$ in the fppf topology. For $G$ smooth over $k$, this is also the set of isomorphism classes of $G$-torsors over $k$ in the etale topology.

Following Reichstein, the essential dimension $\text{ed}(G)$ is the smallest natural number $r$ such that for every $G$-torsor $\xi$ over an extension field $E$ of $k$, there is a subfield $k \subset F \subset E$ such that $\xi$ is isomorphic to some $G$-torsor over $F$ extended to $E$, and $F$ has transcendence degree at most $r$ over $k$. (It is essential that $E$ is allowed to be any extension field of $k$, not just an algebraic extension field.) There are several survey articles on essential dimension, including [18, 16].
For example, let \( q_0 \) be a quadratic form of dimension \( n \) over a field \( k \) of characteristic not 2. Then \( O(q_0) \)-torsors can be identified with quadratic forms of dimension \( n \), up to isomorphism. (For convenience, we sometimes write \( O(n) \) for \( O(q_0) \).) Thus the essential dimension of \( O(n) \) measures the number of parameters needed to describe all quadratic forms of dimension \( n \). Indeed, every quadratic form of dimension \( n \) over a field of characteristic not 2 is isomorphic to a diagonal form \( \langle a_1, \ldots, a_n \rangle \). It follows that the orthogonal group \( O(n) \) in characteristic not 2 has essential dimension at most \( n \); in fact, \( O(n) \) has essential dimension equal to \( n \), by one of the first computations of essential dimension [18 Example 2.5]. Reichstein also showed that the connected group \( SO(n) \) in characteristic not 2 has essential dimension \( n - 1 \) for \( n \geq 3 \) [18 Corollary 3.6].

For another example, for a prime number \( p \) and any field \( k \) of characteristic \( n \)th roots of unity is smooth over \( k \) if and only if \( n \) is invertible in \( k \). Independent of that, \( H^1(k, \mu_n) \) is always isomorphic to \( k^*/(k^*)^n \). From that description, it is immediate that \( \mu_n \) has essential dimension at most 1 over \( k \). It is not hard to check that the essential dimension is in fact equal to 1.

One simple bound is that for any generically free representation \( V \) of a group scheme \( G \) over \( k \) (meaning that \( G \) acts freely on a nonempty open subset of \( V \)), the essential dimension of \( G \) is at most \( \dim(V) - \dim(G) \) [17 Proposition 5.1]. It follows, for example, that the essential dimension of any affine group scheme of finite type over \( k \) is finite.

For a prime number \( p \), the \( p \)-essential dimension \( ed_p(G) \) is a simplified invariant, defined by “ignoring field extensions of degree prime to \( p \)”. In more detail, for a \( G \)-torsor \( \xi \) over an extension field \( E \) of \( k \), define the \( p \)-essential dimension \( ed_p(\xi) \) to be the smallest number \( r \) such that there is a finite extension \( E'/E \) of degree prime to \( p \) such that \( \xi \) over \( E' \) comes from a \( G \)-torsor over a subfield \( k \subset F \subset E' \) of transcendence degree at most \( r \) over \( k \). Then the p-essential dimension \( ed_p(G) \) is defined to be the supremum of the \( p \)-essential dimensions of all \( G \)-torsors over all extension fields of \( k \).

The spin group \( Spin(n) \) is the simply connected double cover of \( SO(n) \). It was a surprise when Brosnan, Reichstein, and Vistoli showed that the essential dimension of \( Spin(n) \) over a field \( k \) of characteristic not 2 is exponentially large, asymptotic to \( 2^{n/2} \) as \( n \) goes to infinity [3]. As an application, they showed that the number of “parameters” needed to describe all quadratic forms of dimension \( 2r \) in \( I^3 \) over all fields is asymptotic to \( 2^r \).

We now turn to quadratic forms over a field which may have characteristic 2. Define a quadratic form \( (q, V) \) over a field \( k \) to be nondegenerate if the radical \( V^\perp \) of the associated bilinear form is 0, and nonsingular if \( V^\perp \) has dimension at most 1 and \( q \) is nonzero on any nonzero element of \( V^\perp \). (In characteristic not 2, nonsingular and nondegenerate are the same.) The orthogonal group is defined as the automorphism group scheme of a nonsingular quadratic form [12 section VI.23]. For example, over a field \( k \) of characteristic 2, the quadratic form

\[
x_1x_2 + x_3x_4 + \cdots + x_{2r-1}x_{2r}
\]

is nonsingular of even dimension \( 2r \), while the form

\[
x_1x_2 + x_3x_4 + \cdots + x_{2r-1}x_{2r} + x_{2r+1}^2
\]
is nonsingular of odd dimension $2r+1$, with $V^\perp$ of dimension 1. The split orthogonal group over $k$ is the automorphism group of one of these particular quadratic forms.

Babic and Chernousov computed the essential dimension of $O(n)$ and the smooth connected subgroup $O^+(n)$ over an infinite field $k$ of characteristic 2 [1]. (We also write $SO(n)$ for $O^+(n)$ by analogy with the case of characteristic not 2, even though the whole group $O(2r)$ is contained in $SL(2r)$ in characteristic 2.) The answer is smaller than in characteristic not 2. Namely, $O(2r)$ has essential dimension $r + 1$ (not $2r$) over $k$. Also, $O^+(2r)$ has essential dimension $r + 1$ for $r$ even, and either $r$ or $r + 1$ for $r$ odd, not $2r - 1$. Finally, the group scheme $O(2r + 1)$ has essential dimension $r + 2$ over $k$, and $O^+(2r + 1)$ has essential dimension $r + 1$. The lower bounds here are difficult, while the upper bounds are straightforward. For example, to show that $O(2r)$ has essential dimension at most $r + 1$ in characteristic 2, write any quadratic form of dimension $2r$ as a direct sum of 2-dimensional forms, thus reducing the structure group to $(\mathbb{Z}/2)^r \times (\mu_2)^r$, and then use that the group $(\mathbb{Z}/2)^r$ has essential dimension only 1 over an infinite field of characteristic 2 [1] proof of Proposition 13.1.

In this paper, we determine the essential dimension of $\text{Spin}(n)$ in characteristic 2 for $n \leq 10$ or $n \geq 15$. Surprisingly, in view of what happens for $O(n)$ and $O^+(n)$, the results for spin groups are the same in characteristic 2 as in characteristic not 2. For $n \leq 10$, the lower bound for the essential dimension is proved by constructing suitable cohomological invariants. It is not known whether a similar approach is possible for $n \geq 15$, either in characteristic 2 or in characteristic not 2.

2 Main result

**Theorem 2.1.** Let $k$ be a field. For every integer $n$ at least 15, the essential dimension of the split group $\text{Spin}(n)$ over $k$ is given by:

$$\text{ed}_2(\text{Spin}(n)) = \text{ed}(\text{Spin}(n)) = \begin{cases} 2^{n-1} - n(n-1)/2 & \text{if } n \text{ is odd;} \\ 2^{(n-2)/2} - n(n-1)/2 & \text{if } n \equiv 2 \pmod{4}; \\ 2^{(n-2)/2} + 2^m - n(n-1)/2 & \text{if } n \equiv 0 \pmod{4}, \end{cases}$$

where $2^m$ is the largest power of 2 dividing $n$.

**Proof.** For $k$ of characteristic 0, this was proved by Chernousov and Merkurjev, sharpening the results of Brosnan, Reichstein, and Vistoli [4 Theorem 2.2]. Their argument works in any characteristic not 2, using the results of Garibaldi and Guralnick for the upper bounds [7]. Namely, Garibaldi and Guralnick showed that for any field $k$ and any $n$ at least 15, $\text{Spin}(n)$ acts generically freely on the spin representation for $n$ odd, on each of the two half-spin representations if $n \equiv 2 \pmod{4}$, and on the direct sum of a half-spin representation and the standard representation if $n \equiv 0 \pmod{4}$. Moreover, for $n$ at least 20 with $n \equiv 0 \pmod{4}$, $H\text{Spin}(n) = \text{Spin}(n)/\mu_2$ (the quotient different from $O^+(n)$) acts generically freely on a half-spin representation [7 Theorem 1.1].

It remains to consider a field $k$ of characteristic 2. Garibaldi and Guralnick’s result gives the desired upper bound in most cases. Namely, for $n$ odd and at least 15, the spin representation has dimension $2^{(n-1)/2}$, and so $\text{ed}(\text{Spin}(n)) \leq 2^{(n-1)/2} - \dim(\text{Spin}(n)) = 2^{(n-1)/2} - n(n-1)/2$. For $n \equiv 2 \pmod{4}$, the half-spin
representations have dimension $2^{(n-2)/2}$, and so $\text{ed}(\text{Spin}(n)) \leq 2^{(n-2)/2} - n(n-1)/2$. For $n = 16$, since the spin group acts generically freely on the direct sum of a half-spin representation and the standard representation, $\text{ed}(\text{Spin}(n)) \leq 2^{(n-2)/2} + n - n(n-1)/2 (= 24)$.

For $n$ at least 20 and divisible by 4, the optimal upper bound requires more effort. The following argument is modeled on Chernousov and Merkurjev’s characteristic zero argument [4, Theorem 2.2]. Namely, consider the map of exact sequences of $k$-group schemes:

$$
\begin{array}{cccccc}
1 & \longrightarrow & \mu_2 & \longrightarrow & \text{Spin}(n) & \longrightarrow & \text{HSpin}(n) & \longrightarrow & 1 \\
\downarrow & & & & & & \downarrow & & \\
1 & \longrightarrow & \mu_2 & \longrightarrow & O^+(n) & \longrightarrow & \text{PGO}^+(n) & \longrightarrow & 1.
\end{array}
$$

Since HSpin($n$) acts generically freely on a half-spin representation, which has dimension $2^{(n-2)/2}$, we have $\text{ed}(\text{HSpin}(n)) \leq 2^{(n-2)/2} - n(n-1)/2$.

By Chernousov-Merkurjev or independently L"otscher, for any normal subgroup scheme $C$ of an affine group scheme $G$ over a field $k$,

$$\text{ed}(G) \leq \text{ed}(G/C) + \max \text{ed}[E/G],$$

where the maximum runs over all field extensions $F$ of $k$ and all $G/C$-torsors $E$ over $F$ [4, Proposition 2.1], [14, Example 3.4]. Thus $[E/G]$ is a gerbe over $F$ banded by $C$.

Identifying $H^2(K, \mu_p)$ with the $p$-torsion in the Brauer group of $K$, we can talk about the index of an element of $H^2(K, \mu_p)$, meaning the degree of the corresponding division algebra over $K$. For a prime number $p$ and a nonzero element $E$ of $H^2(K, \mu_p)$ over a field $K$, the essential dimension (or also the $p$-essential dimension) of the corresponding $\mu_p$-gerbe over $K$ is equal to the index of $E$, by Karpenko and Merkurjev [11, Theorems 2.1 and 3.1].

By the diagram above, for any field $F$ over $k$, the image of the connecting map

$$H^1(F, \text{HSpin}(n)) \rightarrow H^2(F, \mu_2) \subset \text{Br}(F)$$

is contained in the image of the other connecting map

$$H^1(F, \text{PGO}^+(n)) \rightarrow H^2(F, \mu_2) \subset \text{Br}(F).$$

In the terminology of the Book of Involutions, the image of the latter map consists of the classes $[A]$ of all central simple $F$-algebras $A$ of degree $n$ with a quadratic pair $(\sigma, f)$ of trivial discriminant [12, section 29.F]. Any torsor for $\text{PGO}^+(n)$ is split by a field extension of degree a power of 2, by reducing to the corresponding fact about quadratic forms. So ind($A$) must be a power of 2, but it also divides $n$, and so ind($A$) $\leq 2^m$, where $2^m$ is the largest power of 2 dividing $n$. We conclude that

$$\text{ed}(\text{Spin}(n)) \leq \text{ed}(\text{HSpin}(n)) + 2^m \leq 2^{(n-2)/2} - n(n-1)/2 + 2^m.$$ 

This completes the proof of the upper bound in Theorem 2.1.

We now prove the corresponding lower bound for the 2-essential dimension of the spin group over a field $k$ of characteristic 2. Since $\text{ed}_2(\text{Spin}(n)) \leq \text{ed}(\text{Spin}(n))$,
this will imply that the 2-essential dimension and the essential dimension are both equal to the number given in Theorem 2.1.

Write $O(2r)$ for the orthogonal group of the quadratic form $x_1x_2 + x_3x_4 + \cdots + x_{2r-1}x_{2r}$ over $k$, and $O(2r + 1)$ for the orthogonal group of $x_1x_2 + x_3x_4 + \cdots + x_{2r-1}x_{2r} + x_{2r}^2$. Then we have an inclusion $O(2r) \subset O(2r + 1)$. Note that $O(2r)$ is smooth over $k$, with $O(2r)/O^+(2r) \cong \mathbb{Z}/2$. The group scheme $O(2r + 1)$ is not smooth over $k$, but it contains a smooth connected subgroup $O^+(2r + 1)$ with $O(2r + 1) \cong O^+(2r + 1) \times \mu_2$. It follows that $O(2r)$ is contained in $O^+(2r + 1)$. Using the subgroup $\mathbb{Z}/2 \times \mu_2$ of $O(2)$, we have a $k$-subgroup scheme $K := (\mathbb{Z}/2 \times \mu_2)^r \subset O(2r) \subset O^+(2r + 1)$. Let $G$ be the inverse image of $K$ in the double cover $\text{Spin}(2r + 1)$ of $O^+(2r + 1)$. Thus $G$ is a central extension

$$1 \to \mu_2 \to G \to (\mathbb{Z}/2)^r \times (\mu_2)^r \to 1.$$  

(essentially the same “finite Heisenberg group scheme” appeared in the work of Mumford and Sekiguchi on abelian varieties [19, Appendix A].)

To describe the structure of $G$ in more detail, think of $K = (\mu_2)^r$ as the $2$-torsion subgroup scheme of a fixed maximal torus $T_{SO} \cong (G_m)^r$ in $O^+(2r + 1)$. The character group of $T_{SO}$ is the free abelian group $\mathbb{Z}\{x_1, \ldots, x_r\}$, and the Weyl group $W = N(T_{SO})/T_{SO}$ of $O^+(2r + 1)$ is the semidirect product $S_r \rtimes (\mathbb{Z}/2)^r$. Here $S_r$ permutes the characters $x_1, \ldots, x_r$ of $T_{SO}$, and the subgroup $E_r = (\mathbb{Z}/2)^r$ of $W$, with generators $\epsilon_1, \ldots, \epsilon_r$, acts by: $\epsilon_i$ changes the sign of $x_i$ and fixes $x_j$ for $j \neq i$. The character group of $K = T_{SO}[2]$ is $\mathbb{Z}/2\{x_1, \ldots, x_r\}$. The group $E_r$ centralizes $K$, and the group $(\mathbb{Z}/2)^r \times (\mu_2)^r \subset O^+(2r + 1)$ above is $E_r \times K$.

Let $L$ be the inverse image of $K$ in $\text{Spin}(2r + 1)$, which is contained in a maximal torus $T$ of $\text{Spin}(2r + 1)$, the inverse image of $T_{SO}$. The character group $X^*(T)$ is

$$\mathbb{Z}\{x_1, \ldots, x_r, A\}/(2A = x_1 + \cdots + x_r).$$

Therefore, the character group $X^*(L)$ is

$$\mathbb{Z}\{x_1, \ldots, x_r, A\}/(2x_i = 0, \ 2A = x_1 + \cdots + x_r).$$

(Thus $X^*(L)$ is isomorphic to $(\mathbb{Z}/4) \times (\mathbb{Z}/2)^{r-1}$, and so $L$ is isomorphic to $\mu_4 \times (\mu_2)^{r-1}$.) The Weyl group $W$ of $\text{Spin}(2r + 1)$ is the same as that of $O^+(2r + 1)$, namely $S_r \rtimes E_r$. In particular, the element $\epsilon_i$ of $E_r$ acts on $X^*(T)$ by changing the sign of $x_i$ and fixing $x_j$ for $j \neq i$, and hence it sends $A$ to $A - x_i$.

The subset $S$ of $X^*(L)$ corresponding to characters of $L$ which are faithful on the center $\mu_2$ of $L$ is the complement of the subgroup $X^*(K) = \mathbb{Z}/2\{x_1, \ldots, x_r\}$. Therefore, $S$ has order $2^r$. The group $E_r = (\mathbb{Z}/2)^r$ acts freely and transitively on $S$, since

$$\left(\prod_{i \in I} \epsilon_i\right)(A) = A - \sum_{i \in I} x_i$$

for any subset $I$ of $\{1, \ldots, r\}$.

The group $G = E_r \cdot L$ is the central extension considered above. Now, let $V$ be a representation of $G$ over $k$ on which the center $\mu_2 \subset L$ acts faithfully by scalars. Then the restriction of $V$ to $L$ is fixed (up to isomorphism) by the action of $E_r$ on $X^*(L)$. By the previous paragraph, the $2^r$ 1-dimensional representations of $L$ that are nontrivial on the center $\mu_2$ all occur with the same multiplicity in $V$.  

5
Therefore, $V$ has dimension a multiple of $2^r$. This bound is optimal, since the spin representation $W$ of Spin$(2r + 1)$ has dimension $2^r$ over $k$, and the center $\mu_2$ acts faithfully by scalars on $W$.

We use the following result of Merkurjev’s [15, Theorem 5.2], [11, Remark 4.5]. (The first reference covers the case of the group scheme $\mu_p$ in characteristic $p$, as needed here.)

**Theorem 2.2.** Let $k$ be a field and $p$ be a prime number. Let $1 \to \mu_p \to G \to Q \to 1$ be a central extension of affine group schemes over $k$. For a field extension $K$ of $k$, let $\partial_K: H^1(K, Q) \to H^2(K, \mu_p)$ be the boundary homomorphism in fppf cohomology. Then the maximal value of the index of $\partial_K(E)$, as $K$ ranges over all field extensions of $k$ and $E$ ranges over all $Q$-torsors over $K$, is equal to the greatest common divisor of the dimensions of all representations of $G$ on which $\mu_p$ acts by its standard representation.

As mentioned above, for a prime number $p$ and a nonzero element $E$ of $H^2(K, \mu_p)$ over a field $K$, the essential dimension (or also the $p$-essential dimension) of the corresponding $\mu_p$-gerbe over $K$ is equal to the index of $E$.

Finally, consider a central extension $1 \to \mu_p \to G \to Q \to 1$ of finite group schemes over a field $k$. Generalizing an argument of Brosnan-Reichstein-Vistoli, Karpenko and Merkurjev showed that the $p$-essential dimension of $G$ (and hence the essential dimension of $G$) is at least the $p$-essential dimension of the $\mu_p$-gerbe over $K$ associated to any $Q$-torsor over any field $K/k$ [11, Theorem 4.2]. By the analysis above of representations of the finite subgroup scheme $G$ of Spin$(2r + 1)$ over a field $k$ of characteristic 2, we find that $ed_2(G) \geq 2^r$. For a closed subgroup scheme $G$ of a group scheme $L$ over a field $k$ and any prime number $p$, we have $ed_p(L) + \dim(L) \geq ed_p(G) + \dim(G)$ [16, Corollary 4.3] (which covers the case of fppf torsors for nonsmooth group schemes, as needed here). Applying this to the subgroup scheme $G$ of Spin$(2r)$, we conclude that $ed_2(\text{Spin}(2r+1)) \geq 2^r - \dim(\text{Spin}(2r+1)) = 2^r - r(2r+1)$. Combining this with the upper bound discussed above, we have

$$ed(\text{Spin}(2r+1)) = ed_2(\text{Spin}(2r+1)) = 2^r - r(2r+1)$$

for $r \geq 7$.

The proof of the lower bound for $ed_2(\text{Spin}(2r))$ when $r$ is odd is similar. The intersection of the subgroup $K = (\mu_2 \times \mathbb{Z}/2)^r \subset O(2r)$ with $O^+(2r)$ is $K_1 \cong (\mu_2)^r \times (\mathbb{Z}/2)^{r-1}$, where $(\mathbb{Z}/2)^{r-1}$ denotes the kernel of the sum $(\mathbb{Z}/2)^r \to \mathbb{Z}/2$. As a result, the double cover Spin$(2r)$ contains a subgroup $G_1$ which is a central extension

$$1 \to \mu_2 \to G_1 \to (\mathbb{Z}/2)^{r-1} \times (\mu_2)^r \to 1.$$ 

In this case, an argument analogous to the one for $G$ shows that every representation of $G_1$ on which the center $\mu_2$ acts by its standard representation has dimension a multiple of $2^{r-1}$ (rather than $2^r$). The argument is otherwise identical to the argument for Spin$(2r + 1)$, and we find that $ed_2(\text{Spin}(2r)) \geq 2^{r-1} - r(2r - 1)$. For $r$ odd at least 9, this agrees with the lower bound found earlier, which proves the theorem on Spin$(n)$ for $n \equiv 0 \pmod{4}$.

It remains to show that for $n$ a multiple of 4, with $2^m$ the largest power of 2 dividing $n$, we have

$$ed_2(\text{Spin}(n)) \geq 2^{(n-2)/2} + 2^m - n(n-1)/2.$$
The argument follows that of Merkurjev in characteristic not 2 \cite[Theorem 4.9]{16}.

Namely, for \( n \) a multiple of 4, the center \( C \) of \( G := \text{Spin}(n) \) is isomorphic to \( \mu_2 \times \mu_2 \), and \( H := G/C \) is the group \( \text{PGO}^+(n) \). An \( H \)-torsor over a field \( L \) over \( k \) is equivalent to a central simple algebra \( A \) of degree \( n \) over \( L \) with a quadratic pair \( (\sigma, f) \) and with trivialized discriminant, meaning an isomorphism from the center of the Clifford algebra \( C(A, \sigma, f) \) to \( L \times L \) \cite[section 29.F]{12}. The image of the homomorphism from \( C^* \cong (\mathbb{Z}/2)^2 \) to the Brauer group of \( L \) is equal to \( \{0, [A], [C^+], [C^-] \} \), where \( C^+ \) and \( C^- \) are the simple components of the Clifford algebra; each is a central simple algebra of degree \( 2^{(n-2)/2} \) over \( L \). By Merkurjev, there is a field \( L \) over \( k \) and an \( H \)-torsor \( E \) over \( L \) such that \( \text{ind}(C^+) = \text{ind}(C^-) = 2^{(n-2)/2} \) and \( \text{ind}(A) = 2^m \) \cite[section 4.4 and Theorem 5.2]{15}. We use the following result \cite[Example 3.7]{16}:

**Lemma 2.3.** Let \( L \) be a field, \( p \) a prime number, and \( r \) a natural number. Let \( C \) be the group scheme \((\mu_p)^r\), and let \( Y \) be a \( C \)-gerbe over \( L \). Then the \( p \)-essential dimension of \( Y \), and also the essential dimension of \( Y \), is the minimum, over all bases \( u_1, \ldots, u_r \) for \( C^* \), of \( \sum_{i=1}^{r} \text{ind}(u_i(Y)) \).

It follows that the 2-essential dimension of the \((\mu_2)^2\)-gerbe \( E/G \) over \( L \) associated to the \( H \)-torsor \( E \) above is

\[
ed_2(E/G) = \text{ind}(A) + \text{ind}(C^+) = 2^{(n-2)/2} + 2^m.
\]

It follows that

\[
ed(\text{Spin}(n)) \geq \ned_2(\text{Spin}(n)) \\
\geq \ned_2(E/G) - \dim(G/C) \\
= 2^{(n-2)/2} + 2^m - n(n-1)/2.
\]

\[\square\]

### 3 Low-dimensional spin groups

Rost and Garibaldi determined the essential dimension of the spin groups \( \text{Spin}(n) \) with \( n \leq 14 \) in characteristic not 2 \cite[Table 23B]{6}. It should be possible to compute the essential dimension of low-dimensional spin groups in characteristic 2 as well. The following section carries this out for \( \text{Spin}(n) \) with \( n \leq 10 \). We find that in this range (as for \( n \geq 15 \)), the essential dimension of the spin group is the same in characteristic 2 as in characteristic not 2, unlike what happens for \( \text{O}(n) \) and \( \text{SO}(n) \).

For \( n \leq 10 \), we give group-theoretic proofs which work almost the same way in any characteristic, despite the distinctive features of quadratic forms in characteristic 2.

**Theorem 3.1.** For \( n \leq 10 \), the essential dimension, as well as the 2-essential dimension, of the split group \( \text{Spin}(n) \) over a field \( k \) of any characteristic is given
by:

\[
\begin{array}{ccc}
n & \text{ed}(\text{Spin}(n)) \\
\leq 6 & 0 \\
7 & 4 \\
8 & 5 \\
9 & 5 \\
10 & 4 \\
\end{array}
\]

**Proof.** As discussed above, it suffices to consider the case of a field \( k \) of characteristic 2. For \( n \leq 6 \), every \( \text{Spin}(n) \)-torsor over a field is trivial, for example by the exceptional isomorphisms \( \text{Spin}(3) \cong SL(2) \), \( \text{Spin}(4) \cong SL(2) \times SL(2) \), \( \text{Spin}(5) \cong Sp(4) \), and \( \text{Spin}(6) \cong SL(4) \). It follows that \( \text{ed}(\text{Spin}(n)) = 0 \) for \( n \leq 6 \).

We first recall some general definitions. For a field \( k \) of characteristic \( p > 0 \), let \( H^{i,j}(k) \) be the etale motivic cohomology group \( H^{i}_{\text{et}}(k, \mathbb{Z}/p(j)) \), or equivalently

\[
H^{i}_{\text{et}}(k, \mathbb{Z}/p(j)) \cong H^{i-j}_{\text{et}}(k, \Omega^{j}_{\log}),
\]

where \( \Omega^{j}_{\log} \) is the subgroup of the group \( \Omega^{j} \) of differential forms on the separable closure \( k_{s} \) over \( \mathbb{F}_{p} \), spanned by products \( (da_{1}/a_{1}) \wedge \cdots \wedge (da_{j}/a_{j}) \) with \( a_{1}, \ldots, a_{j} \in k_{s}^{*} \) [9]. The group \( H^{i,j}(k) \) is zero except when \( i \) equals \( j \) or \( j + 1 \), because \( k \) has \( p \)-cohomological dimension at most 1 [20, section II.2.2]. The symbol \( \langle a_{1}, \ldots, a_{n-1}, b \rangle \) denotes the element of \( H^{n,n-1}(k) \) which is the product of the elements \( a_{i} \in k^{*}/(k^{*})^{p} \cong H^{1-1}(k) \) and \( b \in k/\{a^{p} - a : a \in k\} \cong H^{1,0}(k) \).

Also, for a field \( k \) of characteristic 2, let \( W(k) \) denote the Witt ring of symmetric bilinear forms over \( k \), and let \( I_{q}(k) \) be the Witt group of nondegenerate quadratic forms over \( k \). (By the conventions in section [1] \( I_{q}(k) \) consists only of even-dimensional forms.) Then \( I_{q}(k) \) is a module over \( W(k) \) via tensor product [5, Lemma 8.16]. Let \( I \) be the kernel of the homomorphism rank: \( W(k) \to \mathbb{Z}/2 \), and let

\[
I^{m}_{q}(k) = I^{m-1} \cdot I_{q}(k),
\]

following [5, p. 53]. To motivate the notation, observe that the class of an \( m \)-fold quadratic Pfister form \( \langle a_{1}, \ldots, a_{m-1}, b \rangle \) lies in \( I^{m}_{q}(k) \). By definition, for \( a_{1}, \ldots, a_{m-1} \in k^{*} \) and \( b \) in \( k \), \( \langle a_{1}, \ldots, a_{m-1}, b \rangle \) is the quadratic form \( \langle a_{1} \rangle_{b} \otimes \cdots \otimes \langle a_{m-1} \rangle_{b} \otimes \langle b \rangle \) of dimension \( 2^{m} \), where \( \langle a \rangle_{b} \) is the bilinear form \( \langle 1, a \rangle \) and \( \langle b \rangle \) is the quadratic form \( [1, b] = x^{2} + xy + by^{2} \).

In analogy with the Milnor conjecture, Kato proved the isomorphism

\[
I_{q}^{m}(F)/I_{q}^{m+1} \cong H^{m,m-1}(F)
\]

for every field \( F \) of characteristic 2 [5, Fact 16.2]. The isomorphism takes the quadratic Pfister form \( \langle a_{1}, \ldots, a_{m-1}, b \rangle \) to the symbol \( \{a_{1}, \ldots, a_{m-1}, b \} \). (For this paper, it would suffice to have Kato’s homomorphism, without knowing that it is an isomorphism.)

We will use the following standard approach to bounding the essential dimension of a group.

**Lemma 3.2.** Let \( G \) be an affine group scheme of finite type over a field \( k \). Suppose that \( G \) acts on a \( k \)-scheme \( Y \) with a nonempty open orbit \( U \). Suppose that for every \( G \)-torsor \( E \) over an infinite field \( F \) over \( k \), the twisted form \( (E \times Y)/G \) of \( Y \) over
\[ \begin{array}{ccc}
\text{n} & \text{char } k \neq 2 & \text{char } k = 2 \\
6 & SL(3) \cdot (G_a)^3 & \text{same} \\
7 & G_2 & \text{same} \\
8 & \text{Spin}(7) & \text{same} \\
9 & \text{Spin}(7) & \text{same} \\
10 & \text{Spin}(7) \cdot (G_a)^8 & \text{same} \\
11 & SL(5) & \mathbb{Z}/2 \ltimes SL(5) \\
12 & SL(6) & \mathbb{Z}/2 \ltimes SL(6) \\
13 & SL(3) \times SL(3) & \mathbb{Z}/2 \ltimes (SL(3) \times SL(3)) \\
14 & G_2 \times G_2 & \mathbb{Z}/2 \ltimes (G_2 \times G_2) \\
\end{array} \]

Table 1: Generic stabilizer of spin (or half-spin) representation of Spin(n)

\( F \) has a Zariski-dense set of \( F \)-points. Finally, suppose that \( U \) has a \( k \)-point \( x \), and let \( N \) be the stabilizer \( k \)-group scheme of \( x \) in \( G \). Then

\[ H^1(F, N) \to H^1(F, G) \]

is surjective for every infinite field \( F \) over \( k \) (or for every field \( F \) over \( k \), if \( G \) is smooth and connected). As a result, \( \text{ed}_k(G) \leq \text{ed}_k(N) \).

The proof is short, the same as that of [6, Theorem 9.3]. (Note that even if \( k \) is finite, we get the stated upper bound for the essential dimension of \( G \): a \( G \)-torsor over a finite field \( F \) that contains \( k \) causes no problem, because \( F \) has transcendence degree 0 over \( k \).) If \( G \) is smooth and connected, then \( H^1(F, G) \) is in fact trivial for every finite field \( F \) that contains \( k \), by Lang [13]; that implies the statement in the theorem that \( H^1(F, N) \to H^1(F, G) \) is surjective for every field \( F \) over \( k \).

The assumption about a Zariski-dense set of rational points holds, for example, if \( Y \) is a linear representation \( V \) of \( G \), or if \( Y \) is the associated projective space \( P(V) \) to a representation, or (as we use later) a product \( P(V) \times P(W) \).

We use Garibaldi and Guralnick’s calculation of the stabilizer group scheme of a general \( k \)-point in the spin (for \( n \) odd) or a half-spin (for \( n \) even) representation \( W \) of the split group \( \text{Spin}(n) \), listed in Table 1 here [7, Table 1]. Here \( \text{Spin}(n) \) has an open orbit on the projective space \( P(W) \) of lines in \( W \) if \( n \leq 12 \) or \( n = 14 \), and an open orbit on \( W \) if \( n = 10 \). (To be precise, we will use that even if \( k \) is finite, there is a \( k \)-point in the open orbit for which the stabilizer \( k \)-group scheme is the split group listed in the table.)

We now begin to compute the essential dimension of the split group \( G = \text{Spin}(7) \) over a field \( k \) of characteristic 2. Let \( W \) be the 8-dimensional spin representation of \( G \). Then \( G \) has an open orbit on the projective space \( P(W) \) of lines in \( W \). By Table 1, there is a \( k \)-point \( x \) in \( W \) whose image in \( P(W) \) is in the open orbit such that the stabilizer of \( x \) in \( G \) is the split exceptional group \( G_2 \). Since \( G \) preserves a quadratic form on \( W \), the stabilizer \( H \) of the corresponding \( k \)-point in \( P(W) \) is at most \( G_2 \times \mu_2 \). In fact, \( H \) is equal to \( G_2 \times \mu_2 \), because the center \( \mu_2 \) of \( G \) acts trivially on \( P(W) \).

By Lemma 3.2, the inclusion \( G_2 \times \mu_2 \hookrightarrow G \) induces a surjection

\[ H^1(F, G_2 \times \mu_2) \to H^1(F, G) \]
for every field $F$ over $k$. Over any field $F$, $G_2$-torsors up to isomorphism can be identified with 3-fold quadratic Pfister forms $\langle \langle a_1, a_2, b \rangle \rangle$ (with $a_1, a_2 \in F^*$ and $b \in F$), and so $G_2$ has essential dimension 3 \cite{10} Théorème 11. Since $\mu_2$ has essential dimension 1, the surjectivity above implies that $G = \text{Spin}(7)$ has essential dimension at most 4.

Next, a $G$-torsor determines two quadratic forms of dimension 8. Besides the obvious homomorphism $\chi_1: G \to \text{Spin}(8) \to \text{SO}(8)$ (which is trivial on the center $\mu_2$ of $G$), we have the spin representation $\chi_2: G \to \text{SO}(8)$, on which $\mu_2$ acts faithfully by scalars. Thus a $G$-torsor $u$ over a field $F$ over $k$ determines two quadratic forms of dimension 8 over $F$, which we call $q_1$ and $q_2$.

To describe these quadratic forms in more detail, use that every $G$-torsor comes from a torsor for $G_2 \times \mu_2$. The two homomorphisms $G_2 \hookrightarrow G \to \text{SO}(8)$ (via $\chi_1$ and $\chi_2$) are both conjugate to the standard inclusion. Also, $\chi_1$ is trivial on the $\mu_2$ factor, while $\chi_2$ acts faithfully by scalars on the $\mu_2$ factor. It follows that $q_1$ is a quadratic Pfister form, $\langle \langle a, b, c \rangle \rangle$ (the form associated to a $G_2$-torsor), while $q_2$ is a scalar multiple of that form, $d\langle \langle a, b, c \rangle \rangle$.

Therefore, a $G$-torsor $u$ canonically determines a 4-fold quadratic Pfister form,

$$q_1 + q_2 = \langle \langle d, a, b, c \rangle \rangle.$$

Define $f_4(u)$ to be the associated element of $H^{4,3}(F)$,

$$f_4(u) = \{d, a, b, c\}.$$

By construction, this is well-defined and an invariant of $u$. This invariant is normalized (zero on the trivial $G$-torsor) and not zero. (By considering the subgroup $G_2 \times \mu_2 \subset \text{Spin}(7)$, where there is a $G_2 \times \mu_2$-torsor associated to any elements $a, b, d$ in $F^*$ and $c$ in $F$, we see that $a, b, c, d$ can be chosen arbitrarily. By taking $F$ to be the rational function field $k(a, b, c, d)$, we see that the element $f_4(u) = \{d, a, b, c\}$ of $H^{4,3}(F)$ can be nonzero. For that, one can use the computation of $H^{n,n-1}$ of a rational function field by Izhboldin \cite{10}.)

Therefore, $G = \text{Spin}(7)$ has essential dimension at least 4. The opposite inequality was proved above, and so $\text{Spin}(7)$ has essential dimension equal to 4. Since the lower bound is proved by constructing a mod 2 cohomological invariant, this argument also shows that $\text{Spin}(7)$ has 2-essential dimension equal to 4. For the same reason, the computations of essential dimension below (for $\text{Spin}(n)$ with $8 \leq n \leq 10$) also give the 2-essential dimension.

Next, we turn to $\text{Spin}(8)$. At first, let $G = \text{Spin}(2r)$ for a positive integer $r$ over a field $k$ of characteristic 2. Let $V$ be the standard $2r$-dimensional representation of $G$. Then $G$ has an open orbit in the projective space $P(V)$ of lines in $V$. The stabilizer $k$-group scheme $H$ of a general $k$-point in $P(V)$ is conjugate to $\text{Spin}(2r-1) \cdot Z$, where $Z$ is the center of $\text{Spin}(2r)$, with $\text{Spin}(2r-1) \cap Z = \mu_2$. (In more detail, a general line in $V$ is spanned by a vector $x$ with $q(x) \neq 0$, where $q$ is the quadratic form on $V$. Then the stabilizer of $x$ in $\text{SO}(V)$ is isomorphic to $\text{SO}(S)$, where $S := x^\perp$ is a hyperplane in $V$ on which $q$ restricts to a nonsingular quadratic form of dimension $2r - 1$, with $S^\perp$ equal to the line $k \cdot x \subset S$.) Here

$$Z \cong \begin{cases} \mu_2 \times \mu_2 & \text{if } r \text{ is even} \\ \mu_4 & \text{if } r \text{ is odd.} \end{cases}$$
In particular, if \( r \) is even, then \( H \cong \text{Spin}(2r-1) \times \mu_2 \). Thus, for \( r \) even, the inclusion \( \text{Spin}(2r-1) \times \mu_2 \to G \) induces a surjection

\[
H^1(F, \text{Spin}(2r-1) \times \mu_2) \to H^1(F, G)
\]

for every field \( F \) over \( k \), by Lemma 3.2.

It follows that, for \( r \) even, the essential dimension of \( \text{Spin}(2r) \) is at most 1 plus the essential dimension of \( \text{Spin}(2r-1) \). Since \( \text{Spin}(7) \) has essential dimension 4, \( G = \text{Spin}(8) \) has essential dimension at most 5.

Before proving that equality holds, let us analyze \( G \)-torsors in more detail. We know that \( H^1(F, \text{Spin}(7) \times \mu_2) \to H^1(F, G) \) is onto, for all fields \( F \) over \( k \). Also, we showed earlier that \( H^1(F, G_2 \times \mu_2) \to H^1(F, \text{Spin}(7)) \) is surjective. Therefore,

\[
H^1(F, G_2 \times \mu_2 \times \mu_2) \to H^1(F, G)
\]

is surjective for all fields \( F \) over \( k \), where \( Z = \mu_2 \times \mu_2 \) is the center of \( G \). As discussed earlier, \( G_2 \)-torsors up to isomorphism can be identified with 3-fold quadratic Pfister forms. It follows that every \( G \)-torsor is associated to some 3-fold quadratic Pfister form \( \langle a, b, c \rangle \) and some elements \( d, e \) in \( F^* \), which yield elements of \( H^1(F, \mu_2) = F^*/(F^*)^2 \).

Next, observe that a \( G \)-torsor determines several quadratic forms. Besides the obvious double covering \( \chi_1 : G \to \text{SO}(8) \), the two half-spin representations of \( G \) give two other homomorphisms \( \chi_2, \chi_3 : G \to \text{SO}(8) \). (These three homomorphisms can be viewed as the quotients of \( G \) by the three \( k \)-subgroup schemes of order 2 in \( Z \). They are permuted by the group \( S_3 \) of “triality” automorphisms of \( G \).) Thus a \( G \)-torsor \( u \) over a field \( F \) over \( k \) determines three quadratic forms of dimension 8, which we call \( q_1, q_2, q_3 \).

To describe how these three quadratic forms are related, use that every \( G \)-torsor comes from a torsor for \( G_2 \times \mu_2 \times \mu_2 \). The three homomorphisms \( G_2 \to G \to \text{SO}(8) \) (via \( \chi_1, \chi_2, \) and \( \chi_3 \)) are all conjugate to the standard inclusion, whereas the three homomorphisms send \( \mu_2 \times \mu_2 \) to the center \( \mu_2 \subset \text{SO}(8) \) by the three possible surjections. It follows that the three quadratic forms can be written as \( q_1 = d\langle a, b, c \rangle, q_2 = c\langle a, b, c \rangle, \) and \( q_3 = de\langle a, b, c \rangle \).

Note that a scalar multiple of a quadratic Pfister form, \( q = d\langle a_1, \ldots, a_{m-1}, b \rangle \) (as a quadratic form up to isomorphism), uniquely determines the associated quadratic Pfister form \( q_0 = \langle a_1, \ldots, a_{m-1}, b \rangle \) up to isomorphism. (Proof: it suffices to show that if \( q \) and \( r \) are \( m \)-fold quadratic Pfister forms over \( F \) with \( aq \cong r \) for some \( a \) in \( F^* \), then \( q \cong r \). Since \( r \) takes value 1, so does \( aq \), and so \( q \) takes value \( a^{-1} \). But then \( a^{-1}q \cong q \) by the multiplicativity of quadratic Pfister forms [5 Corollary 9.9]. Therefore, \( r \cong aq \cong q \).)

We now define an invariant for \( G = \text{Spin}(8) \) over \( k \) with values in \( H^{5,A} \). Given a \( G \)-torsor \( u \) over a field \( F \) over \( k \), consider the three associated quadratic forms \( q_1, q_2, q_3 \) as above. By the previous paragraph, \( q_1 = d\langle a, b, c \rangle \) determines the quadratic Pfister form \( q_0 = \langle a, b, c \rangle \). So \( u \) determines the 5-fold quadratic Pfister form

\[
q_0 + q_1 + q_2 + q_3 = \langle d, e, a, b, c \rangle.
\]

The associated class

\[
f_5(u) = \{ d, e, a, b, c \} \in H^{5,A}(F)
\]

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is therefore an invariant of $u$.

The invariant $f_5$ is normalized and not 0, as shown by considering the subgroup $G_2 \times Z \subset G = \text{Spin}(8)$, where $Z = \mu_2 \times \mu_2$: there is a $G_2 \times Z$-torsor associated to any elements $a, b, d, e$ in $F^*$ and $c$ in $F$, and $f_5$ of the associated $G$-torsor is $\{d, e, a, b, c\}$ in $H^{5,4}(F)$. Therefore, $G$ has essential dimension at least 5. Since the opposite inequality was proved above, $G = \text{Spin}(8)$ has essential dimension over $k$ equal to 5.

Next, let $G = \text{Spin}(9)$ over a field $k$ of characteristic 2. Let $W$ be the spin representation of $G$, of dimension 16, corresponding to a homomorphism $G \to \text{SO}(16)$. (A reference for the fact that this self-dual representation is orthogonal in characteristic 2, as in other characteristics, is [3, Theorem 9.2.2].) By Table 1, $G$ has an open orbit on the space $P(W)$ of lines in $W$, and the stabilizer in $G$ of a general $k$-point in $W$ is conjugate to $\text{Spin}(7)$. (This is not the standard inclusion of $\text{Spin}(7)$ in $\text{Spin}(9)$, but rather a lift of the spin representation $\chi_2: \text{Spin}(7) \to \text{SO}(8)$ to $\text{Spin}(8)$ followed by the standard inclusion $\text{Spin}(8) \hookrightarrow \text{Spin}(9)$. In particular, the image of $\text{Spin}(7)$ does not contain the center $\mu_2$ of $G = \text{Spin}(9)$.) Since $G$ preserves a quadratic form on $W$, it follows that the stabilizer in $G$ of a general $k$-point in $P(W)$ is conjugate to $\text{Spin}(7) \times \mu_2$, where $\mu_2$ is the center of $\text{Spin}(9)$ (which acts faithfully by scalars on $W$). Therefore, by Lemma [3,2] the inclusion of $\text{Spin}(7) \times \mu_2$ in $G = \text{Spin}(9)$ induces a surjection

$$H^1(F, \text{Spin}(7) \times \mu_2) \to H^1(F, G)$$

for every field $F$ over $k$.

Since $\text{Spin}(7)$ has essential dimension 4 over $k$ as shown above, $G = \text{Spin}(9)$ has essential dimension at most $4 + 1 = 5$.

Next, a $G$-torsor determines several quadratic forms. Besides the obvious homomorphism $R: G \to \text{Spin}(10) \to \text{SO}(10)$, we have the spin representation $S: G \to \text{SO}(8)$. Thus a $G$-torsor over a field $F$ over $k$ determines a quadratic form $r$ of dimension 10 and a quadratic form $s$ of dimension 16.

To describe how these forms are related, use that every $G$-torsor comes from a torsor for the subgroup $\text{Spin}(7) \times \mu_2$ described above. The restriction of $R$ to the given subgroup $\text{Spin}(7)$ is the composition of the spin representation $\chi_2: \text{Spin}(7) \to \text{SO}(8)$ with the obvious inclusion $\text{SO}(8) \hookrightarrow \text{SO}(10)$. The restriction of $S$ to the given subgroup $\text{Spin}(7)$ is the direct sum of the standard representation $\chi_1: \text{Spin}(7) \to \text{SO}(8)$ and the spin representation $\chi_2: \text{Spin}(7) \to \text{SO}(8)$. Finally, $R$ is trivial on the second factor $\mu_2$ (the center of $G$), whereas $S$ acts faithfully by scalars on $S$.

Now, let $(u_1, e)$ be a $\text{Spin}(7) \times \mu_2$-torsor over $k$, where $u_1$ is a $\text{Spin}(7)$-torsor and $e$ is in $H^1(F, \mu_2) = F^*/(F^*)^2$, which we lift to an element $e$ of $F^*$. By the earlier analysis of the quadratic forms associated to a $\text{Spin}(7)$-torsor, the quadratic form associated to $u_1$ via the standard representation $\chi_1: \text{Spin}(7) \to \text{SO}(8)$ is a 3-fold quadratic Pfister form $\langle\langle a, b, c\rangle\rangle$, while the quadratic form associated to $u_1$ via the spin representation $\chi_2: \text{Spin}(7) \to \text{SO}(8)$ is a multiple of the same form, $d\langle\langle a, b, c\rangle\rangle$.

By the analysis of representations two paragraphs back, it follows that the quadratic form associated to $(u_1, e)$ via the representation $R: G \to \text{SO}(10)$ is $r = H + d\langle\langle a, b, c\rangle\rangle$, where $H$ is the hyperbolic plane. Also, the quadratic form associated to $(u_1, e)$ via the representation $S: G \to \text{SO}(16)$ is $s = e\langle\langle a, b, c\rangle\rangle + de\langle\langle a, b, c\rangle\rangle$.

Next, $r$ determines the quadratic form $r_0 = d\langle\langle a, b, c\rangle\rangle$ by Witt cancellation [5, Theorem 8.4], and that in turn determines the quadratic Pfister form $q_0 = \langle\langle a, b, c\rangle\rangle$. 

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as shown above. Therefore, a $G$-torsor $u$ determines the 5-fold quadratic Pfister form

$$q_0 + r_0 + s = \langle d, e, a, b, c \rangle$$

up to isomorphism.

Therefore, defining

$$f_5(u) = \{d, e, a, b, c\}$$

in $H^{5,4}(F)$ yields an invariant of $u$. By our earlier description of Spin(7)-torsors, we can take $a, b, d, e$ to be any elements of $F^*$ and $c$ any element of $F$. Therefore, $f_5$ is a nonzero normalized invariant of $G$ over $k$ with values in $H^{5,4}$. It follows that $G$ has essential dimension at least 5. Since the opposite inequality was proved earlier, $G = \text{Spin}(9)$ over $k$ has essential dimension equal to 5.

Finally, let $G = \text{Spin}(10)$ over a field $k$ of characteristic 2. Let $V$ be the 10-dimensional standard representation of $G$, corresponding to the double covering $G \to \text{SO}(10)$, and let $W$ be one of the 16-dimensional half-spin representations of $G$, corresponding to a homomorphism $G \to \text{SL}(16)$. (The other half-spin representation of $G$ is the dual $W^\ast$.)

As discussed above for any group $\text{Spin}(2r)$, $G = \text{Spin}(10)$ has an open orbit on $P(V)$, with generic stabilizer $\text{Spin}(9) \cdot \mu_4$. (Here $\mu_4$ is the center of $G$, which contains the center $\mu_2$ of Spin(9).) Consider the action of $G$ on $P(V) \times P(W) \cong \mathbf{P}^9 \times \mathbf{P}^{15}$. As discussed above, Spin(9) (and hence Spin(9) $\cdot \mu_4$) has an open orbit on $P(W)$. As a result, $G$ has an open orbit on $P(V) \times P(W)$. Moreover, the generic stabilizer of Spin(9) on $P(W)$ is Spin(7) $\times \mu_2$, where the inclusion Spin(7) $\hookrightarrow$ Spin(9) is the composition of the spin representation Spin(7) $\hookrightarrow$ Spin(8) with the standard inclusion into Spin(9); in particular, the image does not contain the center $\mu_2$ of Spin(9). Therefore, the generic stabilizer of Spin(9) $\cdot \mu_4 \subset \text{Spin}(10)$ on $P(W)$ is Spin(7) $\times \mu_4$. We conclude that $G$ has an open orbit on $P(V) \times P(W)$, with generic stabilizer Spin(7) $\times \mu_4$. It follows that

$$H^1(F, \text{Spin}(7) \times \mu_4) \to H^1(F, G)$$

is surjective for every field $F$ over $k$, by Lemma 3.2.

The image $H_2$ of the subgroup $H = \text{Spin}(7) \times \mu_4 \subset G$ in $\text{SO}(10)$ is Spin(7) $\times \mu_2$, where Spin(7) is contained in $\text{SO}(8)$ (and contains the center $\mu_2$ of $\text{SO}(8)$) and $\mu_2$ is the center of $\text{SO}(10)$. In terms of the subgroup $\text{SO}(8) \times \text{SO}(2)$ of $\text{SO}(10)$, we can also describe $H_2$ as Spin(7) $\times \mu_2$, where Spin(7) is contained in $\text{SO}(8)$ and $\mu_2$ is contained in $\text{SO}(2)$. Thus $H_2$ is contained in Spin(7) $\times \text{SO}(2)$. Therefore, $H$ is contained in Spin(7) $\times G_m \subset G = \text{Spin}(10)$, where the multiplicative group $G_m$ is the inverse image in $G$ of $\text{SO}(2) \subset \text{SO}(10)$. It follows that

$$H^1(F, \text{Spin}(7) \times G_m) \to H^1(F, G)$$

is surjective for every field $F$ over $k$. Since every $G_m$-torsor over a field is trivial,

$$H^1(F, \text{Spin}(7)) \to H^1(F, G)$$

is surjective for every field $F$ over $k$. 13
Here \( \text{Spin}(7) \) maps into \( \text{Spin}(8) \) by the spin representation, and then \( \text{Spin}(8) \hookrightarrow G = \text{Spin}(10) \) by the standard inclusion. By the description above of the 8-dimensional quadratic form associated to a \( \text{Spin}(7) \)-torsor by the spin representation, it follows that the quadratic form associated to a \( G \)-torsor is of the form \( H + d\langle[a, b, c]\rangle \).

Every 10-dimensional quadratic form in \( I_q^3 \) over a field is associated to some \( G \)-torsor. So we have given another proof that every 10-dimensional quadratic form in \( I_q^3 \) is isotropic. This was proved in characteristic not 2 by Pfister, and it was extended to characteristic 2 by Baeza and Tits, independently [2, pp. 129-130], [21, Theorem 4.4.1(ii)].

Since \( \text{Spin}(7) \) has essential dimension 4, the surjectivity above implies that \( G = \text{Spin}(10) \) has essential dimension at most 4. To prove equality, we define a nonzero normalized invariant for \( G \) with values in \( H^{4,3} \) by the same argument used for \( \text{Spin}(7) \). Namely, a \( G \)-torsor \( u \) over a field \( F \) over \( k \) determines a 4-fold quadratic Pfister form

\[
\langle\langle d, a, b, c\rangle\rangle
\]

up to isomorphism, and hence the element

\[
f_4(u) = \{d, a, b, c\}
\]

in \( H^{4,3}(F) \). This completes the proof that \( G = \text{Spin}(10) \) over \( k \) has essential dimension equal to 4. As in the previous cases, since the lower bound is proved using a mod 2 cohomological invariant, \( G \) also has 2-essential dimension equal to 4. 

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UCLA Mathematics Department, Box 951555, Los Angeles, CA 90095-1555
totaro@math.ucla.edu