Prequantization of logsymplectic structures.

J. DONGHO

January 12, 2010

Abstract

In this paper, we study quantization condition of logsymplectic structure using integrality of such structure on the complement of associated divisor D.

1 Introduction

The notion of logsymplectic structure taking their origin from particular meromorphic forms having at most simple poles along certain divisor D of a given complex manifold X. Such forms are amply studies in [1]. Using the notion Lie-Rinehart algebra, we give the algebraic generalization of such notion. Giving an K algebra A, and I a non trivial ideal of A, we consider the A-submodule Der_k(log I) of Der_k(A) constituted by δ ∈ Der_k(A) such that δ(I) ⊂ I. It is usually called the A-module of derivation logarithmic along I. An element δ is called principal if for all u ∈ I, δ(u) ∈ uA. We denoted ˆDer_k(log I) the subset of principal elements of Der_k(log I). It is a submodule of Der_k(log I) closed under A−module and k−Lie structure of Der_k(log I).

1.1 The dual of ˆDer_k(log I) is the A−module Ω_k(log I) generated by Ω and the set \( \{ du/u; u ∈ I \setminus \{0\} \} \). The inclusion of ˆDer_k(log I) in Der_k(log I) induce an algebra Lie-Rinehart structure on ˆDer_k(log I). We can then talk about the notion of Lie-Rinehart Poisson structure on Der_k(log I); it is a 2-form  \( \eta: Der_k(log I) \otimes Der_k(log I) \to A \) such that  \( d_\rho \eta = 0 \); where  \( d \) is the Chevaley lie-Rinehart differential of Der_k(log I). A lie-Rinehart Poisson structure  \( \eta \) is called logsymplectic if it is non degenerated; in that case, A is called a logsymplectic algebra.

When I designed the ideal of a giving divisor of some complex manifold X, a Lie-Rinehart-symplectic structure on Der_C(log I) correspond to the notion of logsymplectic structure using by R. Goto in [2]. In this particular case Der_C(log I) is equal to Der_X(log D).

Recalled that giving a symplectic Lie-Rinehart algebra (L, ρ, η) and a ∈ A, there is a unique element  \( \delta_a \) ∈ L such that  \( i_{\delta_a} \eta = d_\rho a \).

Let  \( L_\delta = [i_{\delta}, d_\rho] \); where  \([,] \) denote the commutator and  \( \delta \in L, L_\delta \eta = 0 \). From unicity of  \( \delta_a \), the following bracket is well defined, \( \{a, b\} = \eta(\delta_a, \delta_b) \), which is Poisson structure induced by Lie-Rinehart symplectic structure  \( \eta \). When the Lie-Rinehart symplectic algebra is \( (Der_k(log I), \eta) \), the associated Poisson structure is called logsymplectic Poisson structure.
Logsymplectic Poisson structure represented a particular case of degenerated Poisson structure since there are symplectic on the complement of the associated divisor.

The goal of this paper is to study integral condition of such structure. According to Izu Vaisman in a Poisson manifold \((X, P)\) is prequantization if and only if there exist vector field \(X\) and 2-form \(\omega\) who represented an integrable class such that \(P + [X, P]_{SC} = P^\sharp(\omega)\); where \([-,-]_{SC}\) is well known Schouten bracket. We can remark that Vaisman condition presented 2 main difficulties: solve a partial differential equation in 2 variables \(P + [X, P]_{SC} + Y = 0\) and compute the Poisson cohomology class of inverse of \(Y\) for a giving solution \((X, Y)\). Follow B. Kostant and Soureau, Vaisman condition is equivalent to integral condition of \(\eta\) in the symplectic case. The integral condition only involve the De Rahm cohomology class of \(\eta\). It will be very useful to know how we can change the Vaisman condition by someone filling integral condition; when we have singular Poisson structure.

In this paper, we study the case of singular Poisson structure; a logsymplectic Poisson structure recalled above. Of it, we change the De Rham cohomology by logarithmic De Rham cohomology. Due to the fact that logsymplectic structure are symplectic on the complement of the divisor \(D\), we apply integral condition on the complement of \(D\) and extend the corresponding prequantum line bundle to \(X\).

2 Preliminaries

We begin this section by introducing the notion of Lie-Rinehart-Poisson-Logsymplectic algebra which allow us to give the algebraic definition of logsymplectic structure. This notion breathe in [12]. It is particular case of Lie-Rinehart-Poisson-symplectic algebra fully study in algebra and Poisson geometry.

2.1 On Lie-Rinehart Poisson-logsymplectic algebra.

In what follow, \(A\) is associative, commutative and unitary \(k\)-algebra on a field \(k\) such that \(\text{car}(k) = 0\). Let \(I\) be a nonzero ideal of \(A\). \(\text{Der}_k(A)\) is the \(A\)-module of derivations on \(A\) and \(\text{Der}_k(\log I)\) is its submodule constitute by logarithmic principal derivations along \(I\). We recall that a Lie-Rinehart algebra is a pair \((L, \rho)\); where \(L\) is Lie algebra and \(\rho : L \to \text{Der}_k(A)\) is Lie algebras homomorphism satisfy the following equality.

\[
[l_1, a l_2] = \rho(l_1)(a).l_2 + a[l_1, l_2]
\]

(1)

It follow from this definition that \(\text{Der}_k(A)\) endowed with identity is Lie-Rinehart algebra. In other hand, we can easily prove that the inclusion map of \(\text{Der}_k(\log I)\) endowed it to the structure of Lie-Rinehart algebra. We can then consider on \(\mathcal{L}_{alt}(\text{Der}_k(\log I), A) = \bigoplus \mathcal{L}_{alt}^p(\text{Der}_k(\log I), A)\) the following differential:

\[
(d^\log f)(\delta_0, ..., \delta_p) = \sum_{i=1}^p (-1)^{i+1} \delta_i f(\delta_1, ..., \hat{\delta}_i, ..., \delta_p) + \sum_{i<j} (-1)^{i+j} f([\delta_i, \delta_j], \delta_1, ..., \hat{\delta}_i, ..., \hat{\delta}_j, ..., \delta_p)
\]

(2)
As in general case, we have \( d^{\log} = 0 \); then corresponding cohomology is called logarithmic De Rham cohomology. This cohomology is fully study in the framework of algebraic and complex geometry. For example, if \( I \) denote the definition ideal of reduced divisor \( Y \) of a complex manifold \( X \), \( \text{Der}_C(\log I) \) correspond at each point \( x \) to the \( \mathcal{O}_{X,x} \)-module of logarithmic vector field introduce in \(^{[1]}\) which is denoted \( \text{Der}_{X,x}(\log Y) \) and \( \mathcal{L}_{\text{alt}}(\text{Der}_k(\log I), \mathcal{A}) = \Omega^p_{X}(\log Y) \).

Since \( \text{Der}_k(\log I) \) is Lie-Rinehart algebra, we can define on the notion of Lie-Rinehart-Poisson structure. It is 2-cocycle of \( d^{\log} \). Now, we can give the definition of Lie-Rinehart-Poisson-logsymplectic structure.

**Definition 2.1.** A lie-Rinehart-Poisson-logsymplectic structure is a pair \((\text{Der}_k(\log I), \mu)\) where \( \mu \) is a non degenerated 2-cocycle of \( d^{\log} \).

As in smooth case, Lie-Rinehart-Poisson-logsymplectic structure induce a Poisson structure on \( \mathcal{A} \). Indeed, since the 2-cocycle \( \mu \) is non degenerated, its contraction by logarithmic derivation induce an isomorphism \( i \) of \( \mathcal{A} \)-modules from \( \text{Der}_k(\log I) \) to its algebraic dual \( \text{Der}_k(\log I)^* \). So, according to Hochschild-Konstant-Rosenberg \(^{[22]}\), since \( \text{Der}_k(\mathcal{A}) \) is submodule of \( \text{Der}_k(\log I) \), for all \( a \in \mathcal{A} \), \( d^{\log}(a) \in \Omega_k \), and \( \Omega_k = \text{Der}_k(\mathcal{A}) \) if and only if \( \mathcal{A} \) is regular affine algebra on perfect field \( k \). In this case, we are sure that \( d^{\log}(a) \in \text{Der}_k(\log I)^* \) for all \( a \in \mathcal{A} \). Then there is an unique \( \delta_a \in \text{Der}_k(\log I) \) such that:

\[
i_{(\delta_a)} \mu = d^{\log}(a). \tag{3}\]

\( a \) is called Hamiltonian of Hamiltonian derivation \( \delta_a \).

For all \( a, b \in \mathcal{A} \), consider:

\[
\{a, b\} = -\mu(\delta_a, \delta_b) \tag{4}
\]

We have have the following proposition.

**Proposition 2.2.** Let \( \mathcal{A} \) be a regular affine algebra on perfect field \( k \) endowed with a lie-Rinehart-Poisson-logsymplectic structure \( \mu \). \( \{a, b\} = -\mu(\delta_a, \delta_b) \) is a well defined logarithmic Poisson structure on \( \mathcal{A} \).

The hypotheses of the above proposition is satisfy when \( \mathcal{A} \) is the sheaf of holomorphic functions on logarithmic manifold \( X \). Indeed, It is proven it \(^{[1]}\) that the pair \( \{\text{Der}_X(\log Y), \Omega_X(\log Y)\} \) is reflexive. Then for all \( f \in \mathcal{O}_X \), \( d^{\log}(f) \in \Omega_X(\log Y) = \text{Der}_X(\log Y)^* \). An then each \( f \in \mathcal{O}_X \) is associated to an unique \( \delta_f \in \text{Der}_X(\log Y) \). Therefore, the above bracket is well defined.

More generally, if the pair \( \{\text{Der}_k(\log I), \Omega_k(\log I)\} \) is reflexive, the above proposition is true and besides, for all \( u \in I - 0 \), \( \frac{d^{\log} u}{u} \in \Omega_k(\log I) = \text{Der}_k(\log I)^* \).

Therefore, there exist an unique \( \delta_u \in \text{Der}_k(\log I) \) such that \( i_{\delta_u} \mu = \frac{d^{\log} u}{u} \). In this case, since \( I \) is subset of \( \mathcal{A} \), for all \( u \in I \) there exist \( \delta_u \) such that \( i_{\delta_u} \mu = \frac{d^{\log} u}{u} \). It is easy to prove that \( \delta_u = u \delta_u \). We can then consider the following bracket:

\[
\{a, b\}^{\text{sing}} := \begin{cases} 
\frac{1}{uv} \{u, v\} & \text{if} \quad a = u, b = v \in I - 0 \\
\frac{1}{u} \{u, b\} & \text{if} \quad a = u \in I - 0, b \in \mathcal{A} - I \\
\{a, b\} & \text{if} \quad a, b \in \mathcal{A} - I
\end{cases} \tag{5}
\]

We have the following proposition.
Proposition 2.3. If the pair \( \{\text{Der}_k(\log I), \Omega_k(\log I)\} \) is reflexive then each Lie-Rinehart-Poisson-Logsymplectic structure on \( \mathcal{A} \) induce two Lie structure \( \{-,-\} \) and \( \{-,-\}_{\text{sing}} \) such that for all \( u, v \in I = 0 \),

i) \( i(\delta_{(u,v)}-uv\delta_{(u,v)_{\text{sing}}})\mu = \{u,v\}\left(\frac{d\log u}{u} + \frac{d\log v}{v}\right)\)

ii) \( \{uv, a\}_{\text{sing}} = \{u + v, a\}_{\text{sing}}; \forall a \in \mathcal{A} \lor I \).

iii) \( \{a, b\} = \delta_a(b) \)

iv) \( \left[\delta_a, \delta_b\right] = \delta_{\{a, b\}} \)

v) \( \delta_{(u,v)} = uv[\delta_u, \delta_v] + \{u, v\}(\delta_v + \delta_u) \)

Proof. It straightforward

Let \( \mathcal{H}^{\log}(\mathcal{A}, I) := \{\delta \in \text{Der}_k(\log I); \exists a \in \mathcal{A}; \delta = \delta_a\} \). According to above proposition, \( \mathcal{H}^{\log}(\mathcal{A}, I) \) is sub-Lie-algebra of \( \text{Der}_k(\log I) \) and we have the following exact sequence of Lie algebras.

\[
\begin{array}{cccccc}
O & \longrightarrow & k & \longrightarrow & (\mathcal{A}, \{-,-\}) & \longrightarrow & \mathcal{H}^{\log}(\mathcal{A}, I) & \longrightarrow & 0
\end{array}
\]  

(6)

According to Dirac principe of quantization, \((\mathcal{A}, \{-,-\})\) is quantization if there exist an representation \((\mathcal{H}, \varphi)\) where \( \mathcal{H} \) is Lie-Rinehart extension of \( \mathcal{A} \) along \( \text{Der}_k(\log I) \) such that the following diagram commute.

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathcal{A} & \xrightarrow{f} & \mathcal{H} & \xrightarrow{g} & \text{Der}_k(\log I) & \longrightarrow & 0
\end{array}
\]

(7)

The morphism \( \varphi \) is called quantization formula and it satisfy:

\[
\varphi(as) = \nabla_{v(a)}s + 2i\pi as
\]  

(8)

\( a \in \mathcal{A}, \nabla \) is a section of \( g \); and \( v(a) = \{a, -\} \).

In general, \( \nabla \) is only an \( \mathcal{A} \)-module homomorphism. Obstruction to become Lie-morphism is measured by cohomology class of an 2-cocycle \( K_\nabla \); it is usually called curvature of \( \nabla \) on \( \mathcal{H} \).

When \((\mathcal{H}, \varphi)\) exist and \( \varphi \) satisfy \( \Box \) the triplet \((\mathcal{H}, \nabla, K_\nabla)\) is called prequantum representation of \( \mathcal{A} \). The following paragraph is devoted to the construction of \( \mathcal{H} \) when \( \mu := \omega \) is a logsymplectic structure on a even dimensional complex manifold \( X \) with reduced divisor \( D \).

2.2 Logarithmic differential operator.

In this paragraph, \((X, \omega, D)\) is logsymplectic manifold. Our stumbles is to construct a solution \((\mathcal{H}, \varphi)\) to the above problem when \( \varphi \) is given by \( \Box \).

Our construction of \( \mathcal{H} \) is motivated by the notion of logarithmic connection which is sufficiently studied in Complex and Algebraic geometry. We will also denote \( \mathcal{E} \) a locally free \( \mathcal{O}_X \)-module of rank 1 and \( D = \{h = 0\} \) a divisor of \( X \).
2.2.1 Logarithmic connection.

The notion of logarithmic connection is original in the work of P. Deligne when he formulated and proved the theorem establishing a Riemann-Hilbert correspondence between monodromy groups and Fuchsian systems of integrable partial equations or flat connections on complex manifolds. He also gave a treatment of the theorem of Griffiths which states that the Gauss-Manin or Picard-Fuchs systems of differential equations are regular singular.

**Definition 2.4.** Let $\mathcal{M}$ be $\mathcal{O}_X$-module. A connection on $\mathcal{M}$ with logarithmic poles along $D$ is a $\mathbb{C}$-linear homomorphism $\nabla : \mathcal{M} \to \Omega^1_X(\log D) \otimes \mathcal{M}$ that satisfies Leibniz’s identity:

$$\nabla(fm) = df \cdot m + f \nabla(m)$$

where $d$ is the exterior derivative over $\mathcal{O}_X$.

This is equivalent to a linear map $\Delta : \operatorname{Der}_X(\log D) \to \operatorname{End}(\mathcal{E})$ satisfy the following

$$\Delta_X(fs) = f \Delta_X s + X(f)s$$

If $\nabla$ is logarithmic connection $K_\nabla$ will denoted its curvature and the pair $(\mathcal{M}, \nabla)$ will refer to logarithmic connection on a locally free $\mathcal{O}_X$-module of rank 1 $\mathcal{M}$.

**Lemma 2.5.** If $(\mathcal{M}, \nabla)$ is rank one connection on $X$, Then for all closed 1-form $\tau \in H^0(X, \Omega^1_X(\log D))$ $(\mathcal{M}, \nabla + \tau \otimes \text{id})$ is a connection with curvature form $K = K_\nabla$

**Proof.** Suppose that $\nabla$ is define by

$$\nabla(s) = \sigma \otimes s$$

for a giving nowhere vanish section $s$

$$(\nabla + \tau \otimes \text{id})(s) = \nabla(s) + \tau \otimes s = \sigma \otimes s + \tau \otimes s = (\sigma + \tau) \otimes s$$

And

$$(\nabla + \tau \otimes \text{id})(fs) = \nabla(fs) + \tau \otimes \text{id}(fs) = df \otimes s + f \sigma \otimes s + f\tau \otimes s = df \otimes s + f(\nabla + \tau \otimes \text{id})s$$

Let $s_0 \in H^0(U_i, \mathcal{M})$ such that $0 \notin s_0(U_i)$.

There exist $\sigma \in H^0(U_i, \Omega^1_X(\log D))$ such that $\nabla s_0 = \sigma \otimes s_0$. Then $K_\nabla = ds_0$.

Let $p$ be a point of $D$ and $(z^1_\lambda)$ logarithmic coordinate system along $D$ at $p$.

$$\sigma = \sum_{i=1}^{r} a_i \frac{dz^1_\lambda}{\lambda} + \sum_{i=r+1}^{2n} a_i dz^i_\lambda$$

where $a_i \in H^0(X, \mathcal{O}_X)$. It follow that
Lemma 2.6. Let $D$ be a normal crossing divisor and $\alpha \in H^0(X, \Omega^1_X(\log D))$. If $d\alpha = 0$ then the residue of $\alpha$ is constant on any component of singular locus of $D$. Any such form with at least one nonzero residue admits representation

$$\alpha = \sum_{j=1}^r \frac{d\alpha_j}{f_j}, \quad \alpha_1, ..., \alpha_r \in \mathbb{C}$$  \hspace{1cm} (12)

Proof. Let $p$ be a point of $D$ and $U_\lambda$ an open coordinate neighborhood of $p$. We have:

$$\alpha = \text{Res}(\alpha) dh_p + \alpha_{\text{reg}}$$

Where $\text{Res}(\alpha)$ is the residue of $\alpha$. $d\alpha = 0$ imply $d(\text{Res}\alpha) = 0$, (since $\text{Res}$ commute with $d$). However, From Theorem 2.9 in [1], $\text{Res}(\Omega^1_X(\log D)) = \mathcal{O}_X$, $d\text{Res}(\alpha) = 0$ imply $\text{Res}\alpha \in \mathbb{C}$. \hfill \square

Proposition 2.7. $K_\nabla = 0$ if and only if $\sigma = \sum_{i=1}^r a_i \frac{dz_i}{z_i^\lambda}$ with $a_i \in \mathbb{C}$

Proof. $K_\nabla = d\sigma$; where $\sigma$ is the connection one form of $\nabla$. The result is then a consequence of Lemma 2.6 \hfill \square

Definition 2.8. Let $(\mathcal{M}, \nabla)$ and $(\mathcal{N}, \delta)$ be two connections. An homomorphism from $(\mathcal{M}, \nabla)$ to $(\mathcal{N}, \delta)$ is a sheaf homomorphism $\varphi : \mathcal{M} \to \mathcal{N}$ such that the following diagram commute.

\[
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\varphi} & \mathcal{N} \\
\nabla \downarrow & & \downarrow \delta \\
\Omega^1_X(\log D) \otimes \mathcal{O}_X & \longrightarrow & \Omega^1_X(\log D) \otimes \mathcal{O}_X
\end{array}
\]

Definition 2.9. Let $(\mathcal{M}, \nabla)$ be a connection on $X^*$. A meromorphic prolongation of $(\mathcal{M}, \nabla)$ is a meromorphic connection $(\bar{\mathcal{M}}, \bar{\nabla})$ on $X$ such that the restriction is an isomorphism.

2.2.2 Module of logarithmic differential operator.

Let $\mathcal{A}$ be a commutative ring. For a pair of $\mathcal{A}$-modules $M, N$ we define module $\text{Diff}^k_{\mathcal{A}}(M, N)$ inductively by putting

1. $\text{Diff}^0_{\mathcal{A}}(M, N) = \text{Hom}_{\mathcal{A}}(M, N)$

2. $\text{Diff}^k_{\mathcal{A}}(M, N) = \{ \text{additive maps } u : M \to N \text{ s.t. } \forall a \in \mathcal{A} au - ua \in \text{Diff}^{k-1}_{\mathcal{A}}(M, N) \}$

Elements of $\text{Diff}^k_{\mathcal{A}}(M, N)$ are called $k$-order differential operator from $M$ to $N$. We note $\text{Diff}^k_{\mathcal{A}}(M)$ for $\text{Diff}^k_{\mathcal{A}}(M, M)$.

Replacing $M$ by $\mathcal{E}$, the above definition became;

Definition 2.10. A $r$-order differential operator on $\mathcal{E}$ is a $\mathbb{C}$-linear map $\varphi : \mathcal{E} \to \mathcal{E}$ such that $s \mapsto \varphi(f s) - f \varphi(s)$ is an $(r-1)$-order differential operators on $\mathcal{E}$; for all $f \in \mathcal{O}_X$.
In the previous paragraph, we see that each logarithmic connection induce a morphism \( \Delta \) from \( \text{Der}_X(\log D) \) such that for all \( f \in \mathcal{O}_X, X \in \text{Der}_X(\log D) \), \( \Delta_X(f s) - f \Delta_X(s) = X(f)s \). It follow that \( \Delta_X \in \text{End}(\mathcal{A}) \) and the map \( \Delta^f_X : s \mapsto \Delta_X(f s) - f \Delta_X(s) \) is zero order differential operator on \( \mathcal{E} \) and \( \Delta^f_X(h) \in h\mathcal{O} \).

Therefore, for each an unique \( f_h \in \mathcal{O} \) such that \( h^{-1}[\Delta^f_X(h)]s = f_h s \). In other words, \( h^{-1}[\Delta^f_X(h)] \) is zero order operator. This motivate the following definition.

**Definition 2.11.** A \((r)\)-order differential operator \( \varphi \) is logarithmic along \( D \) if \( s \mapsto [\varphi(hs) - h\varphi(s)]h^{-1} \) is an \((r)\)-order differential operators on \( \mathcal{E} \).

**Notation 2.12.** We denote \( \text{Diff}^r(\mathcal{E}) \) the set of \( r \)-order differential operators and \( \text{Diff}^{1}_{\text{log}}(\mathcal{E}) \) is the subset of \( r \)-order differential operators logarithmic along \( D \).

According to what precedes, \( \Delta_X \in \text{Diff}^{1}_{\text{log}}(\mathcal{E}) \); For all \( X \in \text{Der}_X(\log D) \).

**Lemma 2.13.** Let \( \varphi \) be a first order differential operator logarithmic along \( D \) for all sections \( f \) of \( \mathcal{O}_X \), There exists unique \( \tilde{f} \in \mathcal{O}_X \) such that \( [\varphi(f s) - f \varphi(s)] = m_{\tilde{f}} s \).

**Proof.** \( [s \mapsto \varphi(f s) - f \varphi(s)] \in \text{Diff}^0_{\text{log}}(\mathcal{E}) \) then there exist \( \tilde{f} \in \mathcal{O}_X \) such that \( [\varphi(f s) - f \varphi(s)] = m_{\tilde{f}} s \). If \( g \) is another section of \( \mathcal{O}_X \) such that \( [\varphi(f s) - f \varphi(s)] = m_g s \). Then \( f s = g s \) for all \( s \in \mathcal{E} \); i.e; \( \tilde{f} = g \).

**Corollary 2.14.** If \( \varphi \) is a first order operator logarithmic along \( D \) then \( \tilde{h} \in h\mathcal{O}_X \)

**Proof.** For all \( s \in \mathcal{E}, \varphi(hs) - h\varphi(s) = \tilde{h} s \) and there exist \( g \in \mathcal{O}_X \) such that \( \varphi(hs) - h\varphi(s) = hgs \). Therefore, \( (\tilde{h} - hg)s = 0 \) for all \( s \).

It follows that any first order differential operator logarithmic along \( D \), \( \varphi \) gives rise to a map \( \sigma_{\varphi} : \mathcal{O}_X \to \mathcal{O}_X \) defined by \( \sigma_{\varphi}(f) = \tilde{f} \) such that \( [\varphi(f s) - f \varphi(s)] = \tilde{f} s \) for all \( s \in \mathcal{E} \).

**Lemma 2.15.** For all \( \varphi \in \text{Diff}^1_{\text{log}}(\mathcal{E}), \sigma_{\varphi} \in H^0(\text{Der}_X^1(\log D)) \)

**Proof.**

\[
\sigma_{\varphi}(f,g)s = \varphi(f(gs)) - fg\varphi(s) \\
= \sigma_{\varphi}(f)(gs) + f\varphi(gs) - fg\varphi(s) \\
= \sigma_{\varphi}(f)(gs) + f(\varphi(gs) - g\varphi(s)) \\
= (\sigma_{\varphi}(f)g + f\sigma_{\varphi}(g))s
\]

in other hand, we have;

\[
\sigma_{\varphi}(h)s = \varphi(hs) - h\varphi(s) \\
= h\tilde{m}_h(s)
\]

Then \( (\sigma_{\varphi}(h) - h\tilde{m}_h)s = 0 \) for all \( s \)

Therefore, \( \sigma_{\varphi}(h) \in h\mathcal{O}_X \); i.e., \( \sigma_{\varphi} \in H^0(\text{Der}_X^1(\log D)) \).

**Proposition 2.16.** \( \text{Diff}^1_{\text{log}}(\mathcal{E}) \) is closed under commutator.
Proof. Let \( \varphi_1, \varphi_2 \) be two sections of \( \text{Diff}^1_{\log}(\mathcal{E}) \); we have:

\[
\varphi_1 \varphi_2(f) = \varphi_1 \left( f \varphi_2(s) + \bar{f}^2 s \right) = f \varphi_1(f \varphi_2(s) + \varphi_1(\bar{f}^2 s)) = f \varphi_1(\varphi_2(s)) + f^1 \varphi_2(s) + f^2 \varphi_1(s) + \bar{f}^1 s
\]

In the same way, we obtain:

\[
\varphi_2 \varphi_1(f) = f \varphi_2(\varphi_1(s)) + \bar{f}^2 \varphi_1(s) + f^1 \varphi_2(s) + \bar{f}^2 s
\]

therefore

\[
\varphi_1 \varphi_2(s) - \varphi_2 \varphi_1(s) = f (\varphi_1 \varphi_2 - \varphi_2 \varphi_1)(s) = (\bar{f}^1 - \bar{f}^2)s
\]

In other hand, since \( \varphi_1, \varphi_2 \) are sections of \( \text{Diff}^1_{\log}(\mathcal{E}) \), there exist \( h_1, h_2 \in \mathcal{O}_X \) such that:

\[
[\varphi_2(hs) - h \varphi_2(s)]_{h^2} = h_2 s \quad \text{and} \quad [\varphi_1(hs) - h \varphi_1(s)]_{h^2} = h_1 s
\]

i.e. \( \bar{h}^2 = hh_2 \) and \( \bar{h}^1 = hh_1 \). In the same way, there exist \( h_{21}, h_{12} \in \mathcal{O}_X \) such that \( \bar{h}^1 = hh_{21} \) and \( \bar{h}^2 = hh_{12} \).

Therefore,

\[
\varphi_1 \varphi_2(hs) - \varphi_2 \varphi_1(hs) - h(\varphi_1 \varphi_2 - \varphi_2 \varphi_1)(s) = (\bar{h}^1 - \bar{h}^2)s = h[h_{21} - h_{12}]s
\]

Proposition 2.17. \( \text{Diff}^1_{\log}(\mathcal{E}) \) is a logarithmic Lie-Rinehart algebra.

Proof. According to above results, we have the following map.

\[
\text{Diff}^1_{\log}(\mathcal{E}) \rightarrow \text{Der}_X(\log D)
\]

\[
\varphi \mapsto \sigma_\varphi
\]

For all \( f \in \mathcal{O}_X, s \in \mathcal{E} \), we have:

\[
\sigma_{\varphi_1, \varphi_2}(f)s = [\varphi_1, \varphi_2](f)s - f[\varphi_1, \varphi_2](s)
\]

\[
= \varphi_1 \varphi_2(f)s - \varphi_2 \varphi_1(f)s - f \varphi_1 \varphi_2(s) + \varphi_2 \varphi_1(s)
\]

\[
= \varphi_1(\sigma_{\varphi_2}(f)s + f \varphi_2(s)) - \varphi_2(\sigma_{\varphi_1}(f)s + f \varphi_1(s)) - f[\varphi_1, \varphi_2]s
\]

\[
= \varphi_1(\sigma_{\varphi_1}(f)s + f \varphi_1(s)) - \varphi_2(\sigma_{\varphi_2}(f)s + f \varphi_2(s)) - f \varphi_1 \varphi_2(s) - f \varphi_2 \varphi_1(s)
\]

\[
= \varphi_1(\sigma_{\varphi_1}(f)s - \sigma_{\varphi_1}(f) \varphi_2(s)) - \varphi_2(\sigma_{\varphi_2}(f)s - \sigma_{\varphi_2}(f) \varphi_1(s)) - f[\varphi_1, \varphi_2]s
\]

\[
= [\sigma_{\varphi_1, \varphi_2}(f)]s
\]

In other hand, for all \( \varphi_1, \varphi_2 \in \text{Diff}^1_{\log}(\mathcal{E}), f \in \mathcal{O} \) and \( s \in \mathcal{E} \) we have:

\[
[\varphi_1, f \varphi_2] = \varphi_1(f \varphi_2(s)) - (f \varphi_2)(\varphi_1(s))
\]

\[
= f(\varphi_1(\varphi_2(s)) + \sigma_{\varphi_1}(f) \varphi_2(s)) - f \varphi_2(\varphi_1(s))
\]

\[
= \sigma_{\varphi_1}(f)(\varphi_2(s)) + f[\varphi_1, \varphi_2]
\]

From above results, we deduce the following exact sequence of Lie-Rinehart algebras

\[
0 \longrightarrow \mathcal{O}_X \overset{m}{\longrightarrow} \text{Diff}^1_{\log}(\mathcal{E}) \overset{\sigma}{\longrightarrow} \text{Der}_X(\log D) \longrightarrow 0
\]

Therefore, if we replace in \( \mathcal{A} \) and \( k \), respectively by \( \mathcal{O}_X \) and \( \mathcal{C} \) then Dirac Proceedings of prequantization comes back to determine a locally free rank 1 \( \mathcal{O}_X \)-module \( \mathcal{E} \) endowed with connection \( \nabla \) such that \( \text{Diff}^1_{\log}(\mathcal{E}) \) is faithful representation of \( (\mathcal{O}_X, \omega) \).

In the following paragraph, we study the obstruction to existence of solution to this problem.
3 Prequantization.

Suppose that the logsymplectic manifold \((X, \omega, D)\) admit a prequantum representation \((\text{Diff}^1_{\log}(\mathcal{E}), \nabla, K_\nabla)\).

For all \(f, g \in H^0(X, \mathcal{O}_X)\) and \(s \in \mathcal{E}\),

\[
\varphi(f)\varphi(g)s = \varphi(f)(\varphi(g)s)
= \varphi(f)[\nabla_{\varphi(g)s} + 2\pi igs]
= \nabla_{\varphi(f)}(\nabla_{\varphi(g)s} + 2\pi igs) + 2\pi i(f\nabla_{\varphi(g)s} + 2\pi igs)
= \nabla_{\varphi(f)}\nabla_{\varphi(g)s} + 2\pi i\nabla_{\varphi(f)}(gs) + 2\pi i\nabla_{\varphi(g)s} - 4\pi^2 fs
= \nabla_{\varphi(f)}\nabla_{\varphi(g)s} + 2\pi i(H(df), g)s + 2\pi ig\nabla_{\varphi(f)s} + 2\pi i(f\nabla_{\varphi(g)s} - 4\pi^2 fg
\]

Changing the role of \(f\) and \(g\), we obtain:

\[
\varphi(g)\varphi(f)s = \nabla_{\varphi(g)}\nabla_{\varphi(f)s} + 2\pi i(H(dg), f)s + 2\pi ig\nabla_{\varphi(g)s} + 2\pi ig\nabla_{\varphi(f)s} - 4\pi^2 gfs
\]

Therefore

\[
[\varphi(f), \varphi(g)]s = [\nabla_{\varphi(f)}, \nabla_{\varphi(g)}]s + 4\pi i\omega(v(f), v(g))s
\]

In other hand,

\[
\varphi\{f, g\} = \nabla_{\varphi\{f, g\}}s + 2\pi i\{f, g\}s
= \nabla_{\nabla_{\varphi\{f, g\}}s} + 2\pi i\{f, g\}s
= [\nabla_{\varphi(f)}, \nabla_{\varphi(g)}] - K_\nabla(v(f), v(g))s + 2\pi i\{f, g\}s
= [\varphi(f), \varphi(g)]s + 2\pi i\{f, g\}s - K_\nabla(v(f), v(g))s
\]

It follow that \(\varphi\) is prequantum map of \((X, \omega, D)\) if and only if

\[
K_\nabla = 2\pi i\omega \quad (13)
\]

Since \(\omega \in H^0(X, \Omega^2_X(\log D))\), relation \((13)\) imply that \(K_\nabla\) and then \(\nabla\) are logarithmic forms.

**Definition 3.1.** We refer to prequantum sheaf on \((X, \omega, D)\) a rank 1 connection \((\mathcal{M}, \nabla)\) satisfy \((13)\).

### 3.1 Extension of prequantum sheaf.

Our main objective being to determine existence condition of prequantum sheaf \((\mathcal{M}, \nabla)\) on \((X, D, \omega)\) satisfy \((13)\), we intend in a first times to determine in which case integral condition of \(\omega\) on \(X - D\) could be extended to entire \(X\). Of cause we shall know in how and when it is possible to prolong connection on \(X - D\) to logarithmic connection on \(D\). First about, we recall the following of S. Litaka proved in [17].

**Proposition 3.2.** [17]

Let \(F\) be a closed subset of a nonsingular variety \(X\) with \(F \neq X\) if \(\omega_1\) and \(\omega_2\) are rational \(q\)-forms such that \(\omega_1|_{X - F} = \omega_2|_{X - F}\), then \(\omega_1 = \omega_2\)

**Proposition 3.3.** If \((\nabla, \nabla)\) is extension of a prequantum sheaf \((\mathcal{N}, \nabla)\) on \(X\) and \(D\) a closed reduced divisor of \(X\), then \((\nabla, \nabla)\) is a prequantum sheaf of \(X\).
Proof. Since $D$ is a simple normal crossing divisor and $(\tilde{N}, \nabla)$ is an extension of $(N, \nabla)$ then $K_{\tilde{N}} |_{X^*} = K_{\tilde{N}} = 2\pi i\omega$. The result follows from the proposition 3.2.

Corollary 3.4. If $(\tilde{N}, \nabla)$ is extension of prequantum sheaf of $X^*$ $(N, \nabla)$ and $D$ is simple normal crossing, then there exists strictly close logarithmic form $\tau$ such that $\tilde{\sigma} = \sigma + \tau$

Proof. From Proposition 3.3 we have: $d(\sigma - \sigma^0) = 0$. The existence of $\tau$ follow from lemma 2.6.

Lemma 3.5. Let $(N, \nabla^0, K_{\nabla^0})$ be a sheaf of locally free $O_{X^*}$-module of rank 1. If $(M, \nabla, K_{\nabla})$ be a sheaf of locally free $O_{X}$-module of rank 1 such that $\nabla = \nabla^0 + \tau \otimes id$, with $\tau$ a close logarithmic 1-form, then $(M, \nabla, K_{\nabla})$ is prequantum sheaf of $(X, \omega, D)$ if and only if

$$K_{\nabla} = 2\pi i\omega$$

Proof. The prequantum map 13 become

$$\tilde{\nu} : O_X \to End_C(\mathcal{E})$$

$$f \mapsto v(f).\tau \otimes s + \nabla^0 s + 2\pi ifs$$

(14)

And we have by simple calculation

$$\tilde{f}(\tilde{g}s) = v(g)v(f)\tau s + v(f)\nabla v_0(f) s + v(g)v(f)\tau s + 2\pi iv(f)\tau s + v(g)v(f)\nabla v_0(f) s + v(f)\nabla v_0(f) s + 2i\pi g v(f)s + 2i\pi f v(g)s + 2i\pi f g s - 4\pi^2 g f s$$

and

$$\tilde{g}(\tilde{f}s) = v(f)v(g)\tau s + v(g)v(f)\nabla v_0(f) s + v(f)v(g)\tau s + 2\pi iv(g)v(f)\tau s + v(f)v(g)\nabla v_0(f) s + v(g)v(f)\nabla v_0(f) s + 2i\pi f v(g)s + 2i\pi f v(f)s + 2i\pi f g s - 4\pi^2 f g s$$

then

$$[\tilde{f}, \tilde{g}] s = [\nabla v_0(f), \nabla v_0(g)] s + [v(f), v(g)] s + 4i\pi \omega(v(f), v(g))$$

in other hand,

$$\{\tilde{f}, \tilde{g}\} = [\nabla v_0(f), \nabla v_0(g)] s - K_{\nabla}(v(f), v(g)) s + [v(f), v(g)] s + 2i\pi \omega(v(f), v(g))$$

Now, we can give the main theorem of this section.

Theorem 3.6. Let $(X, D, \omega)$ be a log symplectic manifold such that:

1. $D$ is closed reduced divisor of $X$
2. $X - D$ is even dimensional complex sub manifold of $X$
3. integral of $\omega$ in all closed connected surface of $X - D$ is integers multiple of $2\pi i$. Then the symplectic manifold $(X - D, \omega)$ is prequantizable and if its prequantum connection spreads on $X$, the $(X, D, \omega)$ is prequantizable.
Proof. Since \((X - D, \omega)\) is symplectic manifold, condition 3. of Theorem is B. Kostant quantization condition. If the prequantique sheaf of \((X - D, \omega)\) spreads on \(X\), their associated curvature coincided on \(X - D\). An then it follow from (3.2) that their are equal.

It follow from above result that prequantization of \(X - D\) informe us on the on of \(X\). Since \((X - D, \omega)\) is symplectic, it is prequantizable if and only if the cohomology class \([\omega] \in H^2(X - D, \mathbb{C})\) live in \(i_*(H^2(X - D, \mathbb{Z}))\). We shall be careful on the fat that Obstruction coming from logarithmic De Rham cohomology and not from De Rham cohomology of \(X - D\). In general the two cohomology are not equal. We need the Logarithmic Comparison Theorem before use only cohomology of \(X - D\). Nowadays, it is prove that if the divisor \(D\) is locally quasi-homogeny and free, the cohomology of logarithmic De Rham complex is equal to the De Rham cohomology of the complement of divisor. If we suppose that \(D\) is closed and locally quasi-homogeny, then the \(X - D\) is prequantization if the cohomology class of \(\omega\) in \(X - D\) is integer. We can deduce the following proposition.

**Corollary 3.7.** If \(D\) is close, locally quasi-homogeny and free, if the de Rham cohomology class of \(\omega\) on \(X - D\) is integral, then there exist a prequantum sheaf on \(X - D\). Besides, if prequantum connection on \(X - D\) extending on \(X\), then \((X, D, \omega)\) have prequantum sheaf.

We remark that the problem of extending connection is fundamental in our approach. In the following paragraph, we will study the case where \(D\) is a Normal Crossing Divisor.

### 3.2 The normal Crossing Divisor Case.

Throughout this section \(X\) denotes connected complex analytic compact manifold of dimension \(2n\) and \(D = \sum_{i=1}^{s} v_i D_i\) an effective normal crossing divisor on \(X\), i.e. an effective divisor locally with nonsingular components meeting transversally; \(\omega\) is a logsymplectic structure on \(X\). Using main result of P.Deligne and B.Malgrange about extension of connection on \(X - D\) we prove the sufficient condition of prequantization of \((X, D, \omega)\).

First about, we recall the notion of extension of connection.

**Definition 3.8.** If \((\mathcal{M}, \nabla)\) is connection on \(X - D\), we called meromorphic extension of \(\mathcal{M}\) and \(\mathcal{O}_X[D]\)-coherent module \(\tilde{\mathcal{M}}\) provided with isomorphism \(\tilde{\mathcal{M}}|_{X - D} = \mathcal{M}\).

Thanks to Hilbert Nullstellensatz Theorem, B. Malgrange prove in [23] the following Lemma.

**Lemma 3.9.** [23] A coherent \(\mathcal{O}_X[D]\)-module \(M\) whose support is contained in \(D\) is trivial.

It follow that extension of connection \((\mathcal{M}, \nabla)\) when it exist it is unique. Therefore, If \((X - D, \omega)\) is symplectic manifold, then the prequantum connection of \((X, D, \omega)\). The following P.Deligne Theorem assure the existence of extension of each connection on \(X - D\) when \(D\) is normal crossing divisor on \(X\).
Theorem 3.10. [23] Let $D$ be a divisor with normal crossing of $X$, and $(\mathcal{M}, \nabla)$ a connection on $X - D$. There exists a free extension $(G, \triangle)$ of $(\mathcal{M}, \nabla)$ on $X$, unique up unique isomorphism such that

1. $\nabla$ has logarithmic pole with respect to $G$
2. The eigenvalues of the residues of $\nabla$ with respect to $G$ belong to the image of $\tau$ the section of $\mathbb{C} \to \mathbb{C}/\mathbb{Z}$

We can then state the main theorem of this section.

Theorem 3.11. Let $(X, D, \omega)$ be a log symplectic manifold such that:

1. $D$ is normal crossing divisor of $X$
2. $X - D$ is even dimensional complex sub manifold of $X$
3. integral of $\omega$ in all closed connected surface of $X - D$ is integers multiple of $2\pi i$

when it exist is unique. According to above results, if extension of each connection on $X - D$ exist, then existence of prequantum connection on $X - D$ imply that of $X$. Since the unique extension of prequantum connection $(\mathcal{M}, \nabla)$ on $X - D$ will agree on $X - D$ with $(\mathcal{M}, \nabla)$.

Example 3.12. If $\omega$ is exact, then $(X, \omega, D)$ is prequantizable.

Proof. If $\omega$ is exact, then $[\omega] \in H^2(X - D, \mathbb{C})$ is integer. Therefore the Kostant theorem imply that there exist integrable connection $(\mathcal{N}, \nabla)$ on $X^*$ and from Deligne theorem, $(\mathcal{N}, \nabla)$ extend to $(\tilde{\mathcal{N}}, \tilde{\nabla})$ where the $\tilde{\nabla}$ is integrable logarithmic connection on $X$.]

4 Lie algebroid formalism

Definition 4.1. [7] A Lie Algebroid on $X$ is an $\mathcal{O}_X$-module $L$ equipped with a Lie algebra bracket $[,]$ and an $\mathcal{O}_X$-linear morphism of Lie algebras $\rho : L \to T_X$ such that for $l_1, l_2 \in \mathcal{O}_X$ one has

$$[l_1, fl_2] = f[l_1, l_2] + \rho(l_1)(f)l_2$$

(15)

$\rho$ is called anchor.

We remark that Lie-Rinehart algebra is Lie-algebroid on affine scheme.

There exists many examples of Lie-algebroid in literature:

Example 4.2. 1. Let $X$ be a smooth scheme. Then sheaf $T_X$ of tangent vector fields is Lie-algebroid with anchor $Id_{T_X}$.

2. Any real Lie-algebra $g$ is Lie-algebroid on $X = \{\bullet\}$ the associated anchor is zero map.

3. Let $(X, D)$ be a logarithmic manifold. The sheaf $T_X(\log D)$ of logarithmic vector fields endowed with the inclusion morphism $T_X(\log D) \hookrightarrow T_X$

The notion of module on Lie-Rinehart algebra is generalize by
Definition 4.3. [7] Let $(L, \rho, [., .])$ be a Lie-algebroid. An $L$-module is an $O_X$-module $M$ equipped with a Lie action; $\omega : L \to \text{End}_{O_X}(M)$ which is a homomorphism of Lie-algebras; such that for all $f \in O_X, l \in L$ and $x \in M$:

1. $(\omega(fl))(x) = f((\omega(l))(x))$
2. $(\omega(l))(fx) = \rho(l)(f)x + (\omega(fl))(x)$

To simplify the notation we will denote $(\omega(l))(x) = l(x)$.

Definition 4.4. Let $(L, \rho, [., .])$ be a Lie-algebroid. Universal object of $L$ is a sheaf of $O_X$-algebras $U(L)$ equipped with a morphism $i_{O_X} : O_X \to U(L)$ of $O_X$-algebras and a morphism $i_L : L \to U(L)$ of Lie-algebras having the following properties:

\[ [i_L(l), i_{O_X}(x)] = i_{O_X}(l(x)) \quad i_{O_X}(f)i_L(l) = i_L(fl) \]  

(16)

and $(U(L), i_{O_X}, i_L)$ is universal among triplet $(V, \alpha, \beta)$ satisfying\[10, 11\].

The notion of universal enveloping algebra of Lie-algebroid is very useful when we define Poisson cohomology in the framework of Algebraic Geometry, we refer the reader to [7] for more explanation. Another notion that we need is extension of Lie-algebroid.

Definition 4.5. Let $L$ be Lie-algebroid on $X$. An extension of $L$ by an $O_X$-module $F$ is an exact sequence of Lie-algebroids:

\[ 0 \to F \to E \xrightarrow{i} L \to 0 \]  

(17)

where $E$ is Abelian Lie-algebroid.

Many notion are related to Lie-algebroid extension. For example:

A transverse of an extension of Lie-algebroid $L$ is a morphism of $O_X$-modules $\chi : L \to E$ such that $\pi \circ \chi$.

A back-transverse of an extension of Lie-algebroid is a morphism of $O_X$-modules $\lambda : E \to L'$ such that $\lambda \circ i = \text{Id}_E$.

A transverse is flat if it homomorphism of Lie-algebroid.

Proposition 4.6. There is one to one correspondence between the transverses $\chi$ and back-transverses $\lambda$ giving by

\[ i \circ \lambda + \chi \circ \pi = \text{Id}_E \]  

such that $\lambda \circ \chi = 0$.

Proof. It is an adaptation of proof in the smooth case.

When $\pi$ is a submersion one to of fiber bundles, transverse all way exist and the choice of the transverse determine an isomorphism $E \cong F \oplus L$. The following proposition is an generalization of Proposition 2.13 (see [12]).

Proposition 4.7. For any Lie-algebroid $L$ and an $O_X$-module locally free $\mathcal{E}$, there is up to congruence extensions at most one extension of $L$ by $\text{End}_{O_X}(\mathcal{E})$. 

\[ \square \]
Proof. Let \((L, [-, -]_\rho, \rho)\) be an Lie algebroid on \(X\) and \(\mathcal{M}\) an \(\mathcal{O}_X\)-module. Define on \(\mathcal{End}_C(\mathcal{M}) \oplus \mathcal{L}\) the following bracket

\[
[(\beta, l), (\beta', l')] = (\beta\beta' - \beta'\beta, [l, l']_\rho)
\]  
(19)

For each open subset \(U\) of \(X\) define

\[
\Gamma(U, \mathcal{A}(\mathcal{M})) = \{(\beta, l) \in \mathcal{End}_C(\mathcal{M}) \oplus \mathcal{L}, \beta(fm) = \rho(l)(f)m + f\beta(m)\}.
\]

\(\mathcal{A}_i\) is a sheaf of \(\mathcal{O}_X\)-module

Indeed, for all \(f \in \mathcal{O}_X\), \((\beta, l) \in \mathcal{A}(\mathcal{M})\). Define \(f(\beta, l) := (f\beta, fl)\).

Since

\[
f\beta(gm) = \beta(fgm) = f(\rho(l)(g)m + g\beta(m)) = f(\rho(l)(g)m + fg\beta(m)) = \rho(fl)(g)m + fg\beta(m) = \rho(fl)(g)m + g(f\beta)(m))
\]

Then \(f(\beta, l) \in \mathcal{A}(\mathcal{M})\).

\(\mathcal{A}_i\) is a Lie algebroid.

Indeed, let \((\beta, l), (\beta', l') \in \mathcal{A}(\mathcal{M})\).

We have:

\[
\beta\beta'(fm) = \beta(\beta(fm)) = \beta(\rho(l')(f)m + \beta\beta'(m)) = \beta(\rho(l')(f)m) + \beta(\beta\beta'(m)) = \beta(\rho(l')(f)m) + \rho(l)(f)(\beta\beta'(m)) + f(\beta\beta'(m)) = \rho(l)(\rho(l')(f)m + \rho(l)(f)(\beta\beta'(m)) + f(\beta\beta'(m)))
\]

In the same manner, we obtain

\[
\beta\beta'(fm) = \rho(l')(\rho(l)(f)m + \rho(l)(f)(\beta\beta'(m)) + f(\beta\beta'(m)))
\]

Then \([\beta, \beta'](fm) = (\rho(l)(\rho(l')(f)m + \rho(l)(f)(\beta\beta'(m)) + f(\beta\beta'(m))))\).

Therefore, \([(\beta, l), (\beta', l')] = (\beta\beta' - \beta'\beta, [l, l']_\rho) \in \mathcal{A}(\mathcal{M})\)

**Leibniz property**

Let \((\beta, l), (\beta', l') \in \mathcal{A}(\mathcal{M}), f \in \mathcal{O}_X\). \([(\beta, l), f(\beta', l')] = (\beta f\beta' - f\beta'\beta, [l, f\beta']_\rho)\).

We have \(\beta(f\beta')(m) - (f\beta')\beta(m) = \rho(l)(f)(\beta\beta'(m)) + f(\beta\beta'(m) - \beta'\beta(m))\).

Then \(\beta(f\beta') - (f\beta')\beta = \rho(l)(f)\beta' + f(\beta, \beta')\).

Then \((\beta f\beta' - f\beta'\beta, [l, f\beta']_\rho) = \rho(l)(f)(\beta\beta') + f(\beta\beta', [l, f\beta']_\rho)\).

Therefore the Lie Leibniz property is satisfy we anchor \(\Phi(\beta, l) = \rho(l)\).

**Proposition 4.8.**

1. For all \((\beta, l) \in \mathcal{A}(\mathcal{M})\), \(\beta \in \text{Diff}^1(\mathcal{M})\)

2. If \(L = \text{Der}(\log D)\) then \(\beta \in \text{Diff}^1_{\log}(\mathcal{M})\)

3. the map \(\varphi((\beta, l)) = \beta \) is Lie-algebroid homomorphism

**Proof.**

1. Let \((\beta, l) \in \mathcal{A}(\mathcal{M}), \beta(fs) = (\rho(l)(f))s + f(\beta(s))\)

Then \(\beta(fs) - f(\beta(s)) = (\rho(l)(f))s\)

Therefore \(\beta \in \text{Diff}^1(\mathcal{M})\)

2. Suppose that \(L = \text{Der}(\text{Log}D)\), then \(\rho(l) \in \text{Der}(\text{Log}D)\) and then \(\rho(l)(h) \in \mathcal{O}_X\) for all \(l \in \text{Der}(\text{Log}D)\).

Therefore \([\beta(fs) - f(\beta(s))h^{-1}] \in \mathcal{O}_X\)
3. Consider the map
\[ \varphi : \mathcal{A}(\mathcal{M}) \to \text{Diff}^1(\mathcal{M}) \]

\[ (\beta, l) \mapsto \beta \]

\[ \varphi \] is an homomorphism of Lie-algebroids.

Indeed for all \((\beta, l) \in \mathcal{A}(\mathcal{M})\), \(s \in \mathcal{M}\):

\[ \sigma \circ \varphi((\beta, l))(f)(s) = \sigma_{\varphi(\beta, l)}(f)(s) = \sigma_\beta(f)(s) = \beta(fs) - f\beta(s) = (\rho(l)(f))(s) = (\Phi(\beta, l)(f))(s) \]

then \(\sigma \circ \varphi = \Phi\) where \(\Phi(\beta, l) = \rho(l)\)

**Corollary 4.9.** For all \(\mathcal{O}_X\)-module locally free of rank 1, \(\mathcal{M}, \text{Diff}^1(\log_\mathcal{M}), \mathcal{A}_{\log}(\mathcal{M}) \in \text{Ext}^1(\mathcal{O}_X, \text{Der}(-\log D))\)

**Proof.** It follow from the five lemma and above corollary.

It follows from this proposition that giving a complex line bundle \(L\) on a logarithmic manifold \((X, D)\), there exist up to congruence of extensions at most one extension of the Lie algebroid \(T_X(-\log D)\)

**Corollary 4.10.** For all \(\mathcal{O}_X\)-module locally free of rank one \(L\) there exist an exact sequence of Lie algebras

\[ 0 \to \text{End}(\Gamma(E)) \xrightarrow{1} A_{\log}(E) \xrightarrow{\pi} \text{Der}(-\log D) \to 0 \quad (20) \]

Therefore, we have the following definition

**Definition 4.11.** An extension \(A_{\log}(E)\) of \(\text{Der}(-\log D)\) giving by relation \((20)\) is called Atiyah logarithmic algebroid of the invertible sheaf \(E\).

The existence of \(A_{\log}(E)\) allowed us to think about representation of logsymplectic Poisson algebra \((\mathcal{O}_X, \omega)\) by \(A_{\log}(E)\).

In this case prequantum representation shall commuted the following diagram.

\[ 0 \to \mathcal{O}_X \to A_{\log}(E) \to \text{Der}(-\log D) \to 0 \quad (21) \]

**Lemma 4.12.** Let \(\nabla : \text{Der}(\log D) \to \text{End}(\mathcal{M})\) be a logarithmic connection on \(\mathcal{M}\). For all \(\delta \in \text{Der}(-\log D)\)

1. \(\sigma_{\nabla_\delta} = \delta\)
2. For all \(\varphi \in A_{\log}(\mathcal{M}) , \nabla_{\sigma_\varphi} - \varphi \in \ker(\sigma) \subset \mathcal{O}_X\)

**Proof.** Let \(\nabla\) be a logarithmic connection on \(\mathcal{M}\) and \(\delta \in \text{Der}(-\log D)\).
1. \( \nabla_\delta(fs) = f\nabla_\delta s + \delta(f)s \); i.e. \( \delta(f)s = \nabla_\delta(fs) - f\nabla_\delta s \).

Therefore, there exist \( g \in \mathcal{O}_X \) such that \( [\nabla_\delta hs - h\nabla_\delta s] h^{-1} = gs \); i.e. 
\( \nabla_\delta \in A_{\log D}(\mathcal{M}) \) we can compute its image by \( \sigma \). We have:

\[
\begin{align*}
\sigma \nabla_\delta(f)(s) &= \nabla_\delta(fs) - f\nabla ss \\
&= (\delta(f))s \quad \forall f \in \mathcal{O}_X, s \in \mathcal{M}
\end{align*}
\]

Then \( \sigma \nabla_\delta = \delta \)

2. \( \forall \varphi \in A_{\log}(\mathcal{M}), \)

\[
\begin{align*}
\sigma \nabla_{\sigma \varphi}(f)(s) &= \nabla_{\sigma \varphi}(fs) - f\nabla_{\sigma \varphi} s \\
&= f\nabla_{\sigma \varphi}s + (\sigma \varphi(f))s - f\nabla_{\sigma \varphi} s \\
&= (\sigma \varphi(f))s
\end{align*}
\]
i.e., \( \sigma \nabla_{\sigma \varphi} - \sigma \varphi = 0 \)
i.e., \( \nabla_{\sigma \varphi} - \varphi \in \ker(\sigma) \)

It follow that there exist \( m(\varphi) \in \mathcal{O}_X \) such that

\[
\varphi = \nabla_{\sigma \varphi} + m(\varphi) \quad (22)
\]

**Lemma 4.13.** Let \( \omega \) be a logsymplectic form on \( X \), the following morphism \( \text{Der}(\log D) \to \Omega^1_X(\log D) \) by \( \delta \mapsto i_\delta \omega \) is an isomorphism if \( D \) is free.

It follow from this lemma that \( \forall f \in \mathcal{O}_X \) there exist \( \delta_f \in \text{Der}_X(\log D) \) such that \( i_{\delta_f} \omega = df \)

**Lemma 4.14.** There exist a map \( \lambda : (\mathcal{O}_X, \omega) \to A_{\log}(\mathcal{M}) \) who commute the following diagram

\[
\begin{array}{ccc}
(\mathcal{O}_X, \omega) & \xrightarrow{\lambda} & A_{\log}(\mathcal{M}) \\
\gamma \downarrow & & \sigma \downarrow \\
\text{Der}(\log D) & &
\end{array}
\]

**Proof.** For all \( f \in \mathcal{O}_X \) since \( \sigma \) is onto, it admit a section \( \tau \) such that \( \sigma \circ \tau = Id \) we denoted \( \lambda = \tau \circ \gamma \)

For all \( f \in \mathcal{O}_X, \lambda_f \) satisfy equation \( (22) \) i.e. \( \lambda_f = \nabla_{\sigma \lambda_f} + m(f) \) where we have replaced \( m(\lambda_f) \) by \( m(f) \).

**Corollary 4.15.** \( m \) is \( \mathcal{O}_X \)-linear on \( \mathcal{M} \)

**Proof.** We have

\[
\begin{align*}
\lambda_f(gs) &= \nabla_{\sigma \lambda_f}(gs) + m(f)(gs) \\
&= g\nabla_{\sigma \lambda_f} s + \sigma \lambda_f(g)s + m(f)(gs) \\
\lambda_f(gs) - g\lambda_f(s) &= g\nabla_{\sigma \lambda_f} s + \sigma \lambda_f(g)s + m(f)(gs) - g\nabla_{\sigma \lambda_f} s + g\sigma \lambda_f \\
\lambda_f(gs) - g\lambda_f(s) &= \sigma \lambda_f(g)(s) + (m(f))(gs) - g(m(f))(s) \\
\text{i.e.} (m(f))(gs) - g(m(f))(s) &= 0
\end{align*}
\]
Corollary 4.16. 1. The map $Q$ in diagram (21) $Q: (\mathcal{O}_X, \omega) \to A_{\log}(\mathcal{M})$ have the form $Q(f) = \nabla_{\delta_f} + \alpha m(f)$ for some constant $\alpha$

2. $Q(f) = \nabla_{\delta_f} + \alpha m(f)$ is prequantization representation if and only if $m : (\mathcal{O}_X, \omega) \to (\mathcal{O}_X, \omega)$ is $\mathbb{C}$-linear and for any $f, g \in (\mathcal{O}_X, \omega)$ and

$$K_C(\delta_f, \delta_g) = \delta_g m(f) - \delta_f m(g) + m(f, g)$$

(23)

Let us recall that the Poisson structure $\{ -, - \}_\omega$ induced by $\omega$ induce on $\mathcal{O}_X$ a structure of $\mathcal{O}_X$-module defining by $f, g := \delta_f g = \{f, g\}$. So, we can consider the cohomology of the complex $\text{Alt}^*_{\log}(\mathcal{O}_X) = \bigoplus_{k=0}^\infty \text{Alt}^k_{\log}(\mathcal{O}_X)$ where

$\text{Alt}^k_{\log}(\mathcal{O}_X) = \{m : \mathcal{O}_X \times \ldots \times \mathcal{O}_X \to \mathcal{O}_X \text{ alternating multilinear}\}$

the associated differential is: $d_{\log} m(f_0, \ldots, f_r) = \sum_{k=0}^r \text{Alt}^k m(f_0, \ldots, f_k, \ldots, f_r)$ and $m(f_0, \ldots, f_r) := \{f, m(g)\}$ Moreover, any logarithmic $r$-form $\eta \in \Omega^r(\log D)$ define a $r$-cochain $K_\eta \in \text{Alt}^r_{\log}(\mathcal{O}_X)$ via

$K_\eta(f_1, \ldots, f_r) := \eta(\delta_{f_1}, \ldots, \delta_{f_r})$

It follow that $m$ is $1$-cochain and the condition that $Q$ be a representation then becomes

$$d_{\log} m(f, g) = \alpha K_{K_\eta}(f, g)$$

(24)

Proposition 4.17. The map $K : \Omega^*_{\log}(\mathcal{O}_X) \to \text{Alt}^*_{\log}(\mathcal{O}_X)$ $\eta \mapsto K_\eta$ preserves the wedge product and is an injection of cochain complex when $D$ is free

before giving the integrality condition of $\omega$, we shall first recall the following useful notions

Definition 4.18. 1. The polynomial $h(z_1, \ldots, z_n) = \sum a_{i_1 \ldots i_n} z_1^{i_1} \ldots z_n^{i_n} \in \mathcal{O}_{\mathbb{C}^n}$ is weighted homogeneous if there exist positive integer weights $w_1, \ldots, w_n$ such that $h(z_1^{w_1}, \ldots, z_n^{w_n})$ is homogeneous.

2. The divisor $D \subset X$ is locally quasi-homogeneous if for all $x \in D$ there are local coordinates on $X$, centered at $x$, with respect to which $D$ has a weighted homogeneous defining equation.

Proposition 4.19. Let $D$ be a strongly quasihomogeneous free divisor in the complex manifold $X$, let $U$ be the complement of $D$ in $X$, and let $j : U \to X$ be inclusion. Then the natural morphism from the complex $\Omega_X^*(\log D)$ of differential forms with logarithmic poles along $D$ to $Rj^* \mathbb{C}$ is quasi-isomorphism.

Example 4.20.

1. Let $X = \mathbb{C}^3$ and $D$ the divisor of $X$ defined by the equation

$xy(x + y)((x - 2)x + y) = 0$,

$D_{\text{c.r.}}(\log D)$ has free basis $\{\delta_1, \delta_2, \delta_3\}$

$\delta_1 = x\partial_x + y\partial_y$

$\delta_2 = ((z - 2)x + y)\partial_z$

$\delta_3 = x^2\partial_x - y^2\partial_y - (z - 2)(x + y)\partial_z$. Each $\delta_i$ is logarithmic vector field and the determinant of their coefficients is reduced equation of $D$. 17
It follow from K.Saito theorem that the system \((\delta_1, \delta_2, \delta_2)\) is a basis of 
\(\text{Der}(-\log D)\). Therefore, \(D\) is free. Since no linear combination of \((\delta_1, \delta_2, \delta_2)\)
has non-singular linear part, \(D\) can not be quasihomogeneous.

2. All normal crossing divisor is locally quasihomogeneous.

It follow from 4.19 that and Grothendieck’s Comparison Theorem that Cohomology of \(X - D\) compute the one of the complex \((\Omega^*_X(\log D), d)\). We denote 
\(H^i_{\text{DR}}(X)\) the cohomology of the complex \((\Omega^*_X(\log D), d)\)
Let \(X\) be a complex analytic space, \(\mathcal{F}\) a coherent sheaf on \(X\). Denote by 
\(\Phi_k(\mathcal{F}) := \{m \in X; \text{prof}_m \mathcal{F} \leq k\}\).
We saying that the sheaf 
\(\mathcal{F}\) satisfy the condition \((s_k)\) if

\[
\dim \Phi_k(\mathcal{F}) \leq k - 2
\]

**Theorem 4.21.** If \(D\) is zero dimensional locally homogeneous free divisor of \(X\) and if the De Rham cohomology class of \(\omega\) on \(X - D\) live in \(i_* (H^2(X - D), \mathbb{Z})\) then 
\((X, \omega)\) have prequantum bundle if the associated prequantum bundle of \(X - D\)
satisfy the condition \((s_2)\)

**Proof.** Since the De Rham cohomology class of \(\omega\) on \(X - D\) live in \(i_* (H^2(X - D), \mathbb{Z})\), it follow from B.Kostant in [10] that there exist a rank one locally free \(O_{X - D}\)-module \(\mathcal{F}\) such that the curvature satisfy the equation with \(\alpha = -2\pi i\). If \(\mathcal{F}\) satisfy the condition \((s_2)\) and \(D\) is zero dimensional analytic divisor of \(X\), then according to Trautmann Theorem that \(\mathcal{F}\) there exist an unique analytic coherent sheaf \(\tilde{\mathcal{F}}\) on \(X\) extending \(\mathcal{F}\). Since the curvature of \(\mathcal{F}\) coincide on \(X - D\) with curvature of \(\tilde{\mathcal{F}}\) it follow from Proposition 3.2 and to the Logarithmic Comparison Theorem that \(\tilde{\mathcal{F}}\) is prequantum sheaf of \(X\).

**References**

[1] K.Saito. *Theory of logarithmic differential forms and logarithmic vector fields*, Sec. IA, J.Fac.Sci. Univ. Tokyo.27(1980) 265-291.
[2] R. Goto *Rozansky-Witten Invariants of Log symplectic Manifolds*, Contemporary Mathematics, volume 309, 2002
[3] Francisco.J. Calderon-Moreno *Logarithmic differential operators and logarithmic de Rham complexes relative to free divisor
[4] V. Dolgushev., *The Van den Bergh duality and the modular symmetry of a Poisson variety*
[5] J. Huebschmann *Poisson Cohomology and quantization*, J.Reine Angew. Math. 408,57 (1990).
[6] Kosmann-Schwarzbach, Y. and Magri, F., *Poisson-Nijenhuis structures*, Ann. Inst. Henri Poincaré Phys. Théor., 53 (1990), no. 1, 35–81.
[7] A. Polishchuk, *Algebraic geometry of poisson brackets*, Journal of Mathematical Sciences. Vol. 84, No. 5, 1997.
[8] I.M. Gel’fand and I. Ya. Dorfman, Hamiltonian Operator And algebraic Structure Related to Them, Institute of Applied Mathematic, Academy of Sciences of the USSR. Institute of Chemical Physics. Journal of, Academy of Sciences of USSR. Vol.13, No.4, pp. 13-30, October-December 1979.

[9] Magri, F. and Morosi, C., A geometrical characterization of integrable Hamiltonian systems through the theory of Poisson-Nijenhuis manifolds, Quaderno, S. 19 (1984), Univ. of Milan.

[10] Andréé, Lichenerowich Les variés de Poisson et leurs algèbres de Lie associées. (French) J. Differential Geometry 12 (1977), no. 2. 253-300

[11] P. Deligne, Equations Différentielles À Points Singuliers Réguliers. Lecture Notes in Mathematics. Berlin, Heidelberg, New York.

[12] J. Huebschmann Lie -Rinehart algebras, Gerstenhaber algebras and Balolin-Volkovisky algebras, Ann.Inst. Fourier, Grenoble 48, 2 (1998) 425-440

[13] F. Pham, Integrales singulières. ISBN CNRS EDITION 2-271-06186-5

[14] S.K. Donaldson, Nahm’s equations and the classification of monopoles Commun. Math. Phys. 96 (1984). 387-407

[15] M. Atiyah, N. Hitchin, The geometry and dynamics of Magnetic Monopoles Princeton University Press, Princeton, NJ, 1988. viii+134. ISBN 0. 691-08480-7

[16] B. Kostant, Quantization and unitary representation. Part I: Prequantization, "Lecture in modern analysis and applications, III", p. 87-207. Berlin, Springer-Verlag, 1970 (Lecture Notes in Mathematics, 170).

[17] S. Iitaka, Algebraic Geometry An introduction to Birational Geometry of Algebraic Varieties, ISBN 0-387-90546-4 Springer-Verlag New York Heidelberg Berlin, 1981.

[18] R. W. Urwin, The prequantization. Representations of the Poisson Lie Algebra. Adv. in Math., 50, 126-154 (1983)

[19] N.M.J. Woodhouse, Geometric quantization. Oxford Mathematical Monographs. Claredon Press. Oxford, 1992 Second edition.

[20] K. Mackenzie, Lie Groupoids and Lie Algebroids in differential Geometry. London Mathematic Society. Cambridge University press.

[21] Francisco J. Castro-Jimenez, Luis .N, David Mond Cohomology of the complement of free divisor. Transactions of the AMS Vol 348, Number 8, August 1996.

[22] G. Hochschild, B. Kostant and A. Rosenberg. Differential forms on regular affine algebras, Trans. Amer. Math. Soc. 102 (1962), 383-408.

[23] B. Malgrange, " Chap iv: regular connections after Deligne”, in algebraic D-modules, Perspectives in math; Vol.2 Academic regular affine algebra press. Boston, 1967, P151-172