Convergence of Mesh Adaptive Algorithms for Elliptic Singularly Perturbed Boundary Value Problems with Exponential Boundary Layer

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Abstract. In this paper we are considering the Galerkin finite element method as applied to self-adjoint singularly perturbed boundary value problems on Bakhvalov and Shishkin meshes. The method of mesh posterior adaptation in the case of an unknown boundary layer border is analyzed. We have formulated the theorem on the convergence of approximate solutions and the mesh convergence theorem as applied to the extremum partitions and conventional ε-uniform estimates of approximate solution errors on extremum partitions. We cite a scheme for proving these theorems and the results of numerical experiments.

1. Introduction

In the projection-grid methods (PGM) for solving problems with singularities (particularly singularly perturbed boundary value problems SPBVP) the adaptive moving mesh method is extensively used [1]. However, despite the widest choice of literature on this issue (see for example [2] and bibliography therein), the issues of theoretical justification of moving mesh convergence to extremum partition are much less studied. Despite quite a number of papers on the posterior estimates of errors and adaptation algorithms connected with them, it is recognized in [3] that “the only one strict published result of this type is the mesh adaptive algorithm in [4], and the corresponding problem is qualified as unsolved”. Please note that a difference scheme for SPBVP of a very special form is analyzed in [4], and on the finite partitioning the estimates of the first-order errors are obtained, and the number of adaptation algorithm steps depends on a small parameter. Issues close to finite difference methods were also analyzed in [5]-[6], see also the bibliography of [7]. It was shown in [7]-[10] that the Galerkin projection method can be the basis of adaptation algorithm convergence proof on the Bakhvalov and Shishkin meshes. In this paper we apply this method to the elliptic self-adjoint SPBVP in the case of Bakhvalov and Shishkin meshes being used. The mesh convergence theorem is announced as applied to the extreme partitioning at the number of steps independent of a small parameter, and the conventionally ε-uniform second-order error estimates in the C-norm were obtained. The results of numerical experiments are cited.
2. Setting of the Problem and the PGM

Let us analyze the boundary value problem

\[ L_u = -\varepsilon^2 \Delta u + p(x,y)u = f(x,y), \quad (x,y) \in \Omega, \ u_\Gamma = 0 \]  

Here \( \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \Omega \) is a bounded simply connected domain in \( \mathbb{R}^2, \Gamma = \partial \Omega \) - a rather differentiable closed curve. It is supposed that \( p(x,y) \) is a rather differentiable function on the strength of all the arguments with \( p(x,y) > 0 \) at \( (x,y) \in \Omega \). Let us assume that \( \min_{(x,y)\in \Gamma} p(x,y) = p_0 > 0 \). With \( C_0(\Omega) \) we denote the set of functions \( u \in C(\Omega) : u_\Gamma = 0 \).

Let us assume that \( z = (x,y), \rho(z,\Gamma) \) is the distance from \( z \) to \( \Gamma \), \( \theta \) - curve \( \Gamma \) coordinate, i.e. arc \( \Gamma \) length counted off from a fixed point \( \Gamma \).

From the results of [11] it follows that at the made assumptions problem (1) has at small \( \varepsilon > 0 \) the only solution \( u_\varepsilon \), given that the following estimates are valid:

\[ \left| \frac{\partial^i u_\varepsilon}{\partial n^i}(z) \right| \leq C \left( 1 + \varepsilon^{-1} \exp \left( \frac{-p_0}{\varepsilon \rho(z,\Gamma)} \right) \right), \quad 0 \leq i \leq 2, \]  

\[ \left| \frac{\partial^i u_\varepsilon}{\partial \theta^i}(z) \right| \leq C, \quad \rho(z,\Gamma) \leq \frac{2}{p_0} \varepsilon \ln \varepsilon, \quad 0 \leq i \leq 2, \]  

\[ \left| \frac{\partial^\alpha u_\varepsilon}{\partial z^\alpha}(z) \right| \leq C, \quad \alpha = (\alpha_1,\alpha_2), |\alpha| = \alpha_1 + \alpha_2, \rho(z,\Gamma) > \frac{2}{p_0} \varepsilon \ln \varepsilon, \quad 0 \leq \alpha \leq 2, \]  

where \( n \) is the normal line to \( \Gamma \) passing through point \( z \).

Let us pass to the description of PGM. First let us analyze two auxiliary meshes on the intervals defining the width of the boundary layer. Let us locate natural \( n \). We choose the first mesh according to the known Bakhvalov technique [12]. Let us assume that

\[ a_\varepsilon = \frac{2}{p_0} \varepsilon \ln \varepsilon, \quad b_\varepsilon = a_\varepsilon + \frac{2(1 - \varepsilon)}{p_0}. \]

Let us define function \( \chi(y) \) with formula

\[ \chi(y) = -\frac{2}{p_0} \varepsilon \ln \left( \frac{p_0}{2} \left( y - a_\varepsilon + \frac{2\varepsilon}{p_0} \right) \right), \quad y \in [a_\varepsilon, b_\varepsilon]. \]

It is obvious that \( \chi(y) \in C^1[a_\varepsilon, b_\varepsilon] \) and one-to-one transforms \( [a_\varepsilon, b_\varepsilon] \) to \( [0, a_\varepsilon] \). The required partitioning of interval \( [0, a_\varepsilon] \) we define as \( G = \chi(G_\varepsilon) \), where \( G_\varepsilon \) is the auxiliary partitioning of interval \( [a_\varepsilon, b_\varepsilon] \). Let us define it. Let us assume that \( a_\varepsilon = \tau_n \). Let us represent

\[ \tau_{i-1} = \tau_i + (b_\varepsilon - a_\varepsilon)/n, \quad i = n - 1, n - 2, \ldots, 0. \]

The junctures of required partitioning have the form of

\[ t_i = \chi(\tau_i), \quad i = 0, 1, \ldots, n. \]  

For \( G_\varepsilon \) we take the Shishkin piecewise-uniform mesh [5]. Let us assume that \( \bar{a}_\varepsilon = \frac{2}{p_0} \varepsilon \ln n, \)
\[ t_i = \tilde{a}_e \frac{i}{n}, \quad i = 0, 1, \Lambda, n \]

(5)

As we analyze only a part of the Shishkin mesh on the interval defining the width of boundary layer, so mesh (5) is a uniform partitioning of interval \([0, a_e]\) with the step of \(\tilde{a}_e / n\). Further on the definitions of the spaces of testing functions and setting of the problems for both partitionings are made in the same way and differ only in the defining of junctures \(t_i\) according to formulae (4) and (5) respectively.

Let us define the partitioning of domain \(\Omega\) to the final elements. Let us assume that \(\Omega_\epsilon = \{ z \in \Omega : \rho(z, \Gamma) < a \}\), where \(a = \tilde{a}_e\) and \(a = a_e\) in the case of Bakhvalov and Shishkin meshes respectively. \(\Omega_\Lambda = \Omega \setminus \Omega_\epsilon\). Let us call domain \(\Omega_\Lambda\) the central zone and \(\Omega_\epsilon\) - the boundary layer zone. Let us suppose that parameter \(\epsilon\) is so small, that the intervals of inner normal to \(\Gamma\) do not cross in \(\Omega_\epsilon\). From every point of line \(\Gamma\) we draw a half line to inner normal \(\Gamma\). On every half line we set an interval as long as \((2/p_0)\epsilon \log n = a_e\) in the case of the Bakhvalov mesh, and \((2/p_0)\epsilon \log n = \tilde{a}_e\) in the case of the Shishkin mesh, and divide it by junctures \(t_i\) (4) or (5) respectively on \(n\). The lines that are the geometric loci of the junctures equidistant from \(\Gamma\) we denote by \(\gamma_k\), \(k = 0, 1, \Lambda, n\).

Let us partition contour line \(\Gamma\) into the arcs as long as \(O(1/n)\) and from the dividing points set off half lines \(l_j\) directed to inner normal \(\Gamma\). As a result domain \(\Omega_\epsilon\) will be divided by lines \(\gamma_k\) and \(l_j\) into finite elements \(\omega_i\), each of which is a curvilinear quadrilateral. Let us partition the remaining part of \(\Omega - \Omega_\epsilon\) some way into finite elements \(\omega_i\), being the triangles, the sides of which will be the intervals of right lines, or the arcs of curve \(\gamma_n\). Let us suppose that partitioning \(\Omega_\Lambda\) is quasi-uniform with parameter \(h = 1/n\), i.e. \(\text{mes}\omega_i = O(h^2)\).

Let us denote the formed partitioning as \(\Delta_{n,p_0}\) and call it the Bakhvalov or Shishkin partitioning depending on which out of two auxiliary meshes was applied for its forming.

**Figure 1.** The Bakhvalov mesh for domain with curvilinear boundary.

**Figure 2.** The Shishkin mesh for rectangular domain.
Let us pass to defining the Galerkin test spaces. We define test space functions \( E = E(\varepsilon, h) \) as functions from \( C^0(\Omega) \), linear on each of the interval of normal to \( \Gamma \), included between lines \( \gamma_k \) and \( \gamma_{k+1} \), and linear on each of the sections of line \( \gamma_k \), included between half lines \( l_j \) and \( l_{j+1} \), with respect to variable \( \kappa = \psi_{ij}(z) \) where \( \psi_{ij} \) is a transformation transmitting each point of this interval to its projection along the normal going from this point to bisecant \( \chi_j \), being a segment of broken line inscribed in \( \Gamma \).

On elements \( \omega_j \subset \Omega_{\Delta} \), and also on half lines \( l_j \) it is also calculated by way of linear interpolation. On the intervals of lines \( \gamma_k \), included between \( l_j \) and \( l_{j+1} \) the calculation of linear function given by \( \chi_j \) and projecting along the normal of bisecant point \( \chi_j \) on line \( \gamma_k \).

![Figure 3](image3.png)

**Figure 3.** Graphic illustration to the definition of trial space functions.

Thus, if function \( v \in E \) is defined in the junctures of partitioning, then its calculation at any point \( z \in \Omega \) is reduced to calculating the values of three linear functions.

![Figure 4](image4.png)

**Figure 4.** Supports of trial space basis functions.

Space \( E \) is a fortiori defined. The Galerkin method for problem (1) consists in the search of function \( u_n \in E \) that is for any \( w \in E \)

\[
\varepsilon^2(\nabla u_n, \nabla w) + (p(x, y), w) = (f(x, y), w),
\]

where the scalar product grows in the sense of \( L^2(\Omega) \).
Theorem 1. There can be found such numbers as $\varepsilon_0 > 0$, $h_0 > 0$, $\gamma_0 > 0$, $C > 0$ that for any $\varepsilon \in (0, \varepsilon_0]$, $h \in (0, h_0] : \varepsilon < \gamma_0 h$ there exists the only solution $u_m(x)$ of problem (6), for which in the case of the Shishkin partitioning the following estimates are valid:

$$\|u_m - u_\varepsilon\|_{C(\Omega)} \leq Ch^2 \ln h,$$

and in the case of the Bakhvalov partitioning, the following estimates:

$$\|u_m - u_\varepsilon\|_{C(\Omega)} \leq Ch^2.$$

The proof of this and the following theorems are carried out using the Galerkin projections [8]-[10].

3. Mesh adaptation algorithm

Now let us analyze the mesh adaptation algorithm in the case of an unknown boundary layer border. Let us in symbol

$$f(x, y)|_{k, \theta, n} = \tilde{f}(0, \theta), \quad p(x, y)|_{k, \theta, n} = \tilde{p}(0, \theta),$$

where $\rho$, $\theta$ are the local coordinates from (2). Then the dominant term of the boundary layer component of asymptotic decomposition [11] is as follows

$$-\tilde{f}(0, \theta)e^{-\sqrt{p(0, \theta)/\varepsilon}}.$$

Premise 1. For adaptation algorithms let us suppose that at $(x, y) \in \Gamma$ formula $f(x, y) = \tilde{f}(0, \theta) \neq 0$ is valid.

Definition 1. Let us state that $\phi = \phi(\varepsilon, n)$ is the $n$-edge of the boundary layer, in the case of estimate

$$\max_{\rho \in [0, \phi]} e^{-p_0 \rho^2 / \varepsilon} \leq \frac{1}{n}$$

is valid.

Definition 2. Let us call number $\tilde{\phi} = \tilde{\phi}(\varepsilon, n) = \sup_{\rho \neq 0} \phi(\varepsilon, n)$ the precise $n$-border of the boundary layer. It is obvious that

$$\tilde{\phi} = \frac{2}{p_0} \varepsilon \ln n$$

Let us suppose that the location of boundary layer is known to us (the neighbourhood of border $\Gamma$, but the precise $n$-border (or parameter $p_0$) is unknown. Let us equate the approximate search algorithm of this $n$-border.

Step 1. We define a sufficiently great $p_0 \geq p_0$. We assume that $k = 0$.

Step 2. We define partitioning $\Delta_{n, p_0}$ as the Bakhvalov and Shishkin partitioning, in the forming of which we change parameter $p_0$ to $p_k$.

Step 3. We find solution $u_{n, p_0}(x, y)$ on partitioning $\Delta_{n, p_0}$.

Step 4. We assume that $p_{k+1} = p_k - \tau_k$, where $\tau_k$ is chosen so that $t_{n-1, k+1} - t_{n-1, k} = \varepsilon \ln \ln n$ in the case of the Bakhvalov partitioning and $t_{n, k+1} - t_{n, k} = \varepsilon \ln \ln n$ in the case of the Shishkin partitioning (see (4) or (5)).

Step 5. We find solution $u_{n, p_{k+1}}(x, y)$ on partitioning $\Delta_{n, p_{k+1}}$.

Step 6. We calculate $\mu_k = \|u_{n, p_{k+1}}(x, y) - u_{n, p_0}(x, y)\|_{C(\Omega)}$, where

$$\Omega_k = \{(x, y) = (\rho, \theta) \in \Omega : t_{n, k} \leq \rho \leq t_{n, k+1}\},$$

and

$$d_k = \|u_{n, p_{k+1}}(x, y) - u_{n, p_0}(x, y)\|_{C(\Omega)}.$$
junctures of the Shishkin mesh in the case of the Shishkin partitioning,
\[ \Omega_k = \{ z = (x, y) = (\rho, \theta) \in \Omega : t_{n_{-1k}} \leq \rho \leq t_{n_{-1k+1}} \} \]

junctures of the Bakhvalov mesh in the case of the Bakhvalov partitioning.

Step 7. Given that \( k = 0 \), then \( k := k + 1 \) and passing to step 2, otherwise to step 8.

Step 8. Given that \( \mu_k > \frac{\ln n}{n^2} \) in the case of the Bakhvalov partitioning and \( \mu_k > \frac{\ln^3 n}{n^2} \), in the case of the Shishkin partitioning, then \( k := k + 1 \) and passing to step 2, otherwise \( \phi \approx t_{n_{-1k}} \) in the case of the Bakhvalov partitioning, \( \phi \approx t_{n_{-1k}} \) in the case of the Shishkin partitioning, and the end-point of the algorithm.

**Theorem 2.** There can be found such numbers as \( \varepsilon_0 > 0 \), \( n_0 \) - natural, \( \gamma_0 > 0 \), \( C_1 > 0 \), \( C_2 > 0 \), \( C_3 > 0 \), that for any \( \varepsilon \in (0, \varepsilon_0) \), \( n \geq n_0 : \varepsilon \| \ln \varepsilon \| \leq \frac{\gamma_0}{n} \) algorithm 1-8 will finish its work at \( k < C_1 \ln n/\ln n \), moreover, estimates
\[
\left\| \phi - t_{n_{-1k}} \right\| \leq C_2 \varepsilon \ln \ln n, \quad \left\| \phi - t_{n_{-1k}} \right\| \leq C_2 \varepsilon \ln \ln n, \quad (10)
\]
\[
\left\| u_{n,\rho'}(x, y) - u_{\varepsilon}(x, y) \right\|_{C[0,1]} \leq C_3 \frac{\ln n}{n^2},
\]
\[
\left\| u_{n,\rho'}(x, y) - u_{\varepsilon}(x, y) \right\|_{C[0,1]} \leq C_3 \frac{\ln^3 n}{n^2}, \quad (11)
\]
will be valid in the case of the Bakhvalov and Shishkin partitionings respectively.

**Remark 1.** Theorems 1 and 2 remain valid in the cases of domain with piecewise-smooth boundary, given that matching conditions [13] are met at the salient points.

The proofs of Theorems 1 and 2 are based on the properties of Galerkin projection connected with problem (6). Let us denote as \( P_n^\varepsilon \) the operator associating the solution of problem (6) with the exact solution of problem (1):
\[ u_n = P_n^\varepsilon u. \]

This operator is called the Galerkin projector.

In [14]-[15] for problems (1) and (6) the existence of Galerkin projector in the case of Bakhvalov mesh was proved, and also evaluation
\[
\left\| P_n^\varepsilon \right\|_{C(\Omega) \rightarrow C(\Omega)} \leq C. \quad (12)
\]

In the case of Shishkin meshes the proof of existence of Galerkin projector and evaluations (12) are cited analogously. Theorem 1 is immediate from evaluations (12).

In [9]-[10] for the ordinary singularly perturbed problems it was shown that evaluations (12) provide the convergence of adaptation algorithms analogous to the one analyzed in this paper. The proof of Theorem 2 is also cited according to mode of operations [9]-[10].

**4. Results of numerical experiments**

Let us analyze the problem
\[
-\varepsilon^2 \Delta u + u = \frac{1}{16}(2 - x - y)(2 + x + y)(2 - x + y)(2 + x - y), u|_{\Gamma} = 0, \quad (13)
\]
where \( \Gamma = \partial \Omega, \quad \Omega = [-1,1] \times [-1,1] \).
For this example, the first-order matching conditions were met at the angular points of the square [13]. The initial mesh parameter \( p_0 > 0 \), at which the adaptation process began we took as 10, and the step of its measurement was chosen like that for the Bakhvalov mesh \( t_{n+1,k} - t_{n+1,k+1} = \varepsilon \ln n \), i.e.

\[
p^{k+1} = \frac{2p^k \ln \left( \frac{1 - \varepsilon}{n} + \varepsilon \right)}{\ln \left( \frac{1 - \varepsilon}{n} + \varepsilon \right) - p^k \ln n}
\]

and for the Shishkin mesh \( t_{n,k} - t_{n,k+1} = \varepsilon \ln n \), i.e.

\[
p^{k+1} = \frac{2p^k \ln n}{\ln n + p^k \ln n}.
\]

The results of calculations are represented in Table 1. The results for the Bakhvalov mesh are given in the second and the third columns, and for the Shishkin mesh they are in the fourth and the fifth columns. Here \( k \) is the index of iteration, at which the fulfilment of algorithm \( \Delta t = |\tilde{\phi} - t_{n+1,k} | \) for the Bakhvalov mesh stops, \( \Delta t = |\tilde{\phi} - t_{n,k} | \) for the Shishkin mesh is the error of approximate value of the precise boundary layer border.

**Table 1. Results of numerical experiments.**

| \( \varepsilon = 10^{-3} \) | \( \varepsilon = 10^{-4} \) | \( \varepsilon = 10^{-3} \) | \( \varepsilon = 10^{-4} \) |
|-----------------|-----------------|-----------------|-----------------|
| \( n = 4 \)    | \( n = 8 \)    | \( n = 16 \)   | \( n = 32 \)   | \( n = 64 \)   |
| \( \Delta t \)  | \( \Delta t \)  | \( \Delta t \)  | \( \Delta t \)  |
| \( k = 7 \)    | \( k = k \)    | \( k = 1 \)    | \( k = 7 \)    |
| \( p^k \)      | \( p^k \)      | \( p^k \)      | \( p^k \)      |
| \( 1.081480 \) | \( 1.081480 \) | \( 4.585820 \) | \( 1.081271 \) |
| \( k = 5 \)    | \( k = 5 \)    | \( k = 5 \)    | \( k = 5 \)    |
| \( \Delta t \)  | \( \Delta t \)  | \( \Delta t \)  | \( \Delta t \)  |
| \( 0.0000209 \)| \( 0.000021 \) | \( 0.002169 \) | \( 0.000021 \) |
| \( n = 8 \)    | \( n = 16 \)   | \( n = 32 \)   | \( n = 64 \)   |
| \( \Delta t \)  | \( \Delta t \)  | \( \Delta t \)  | \( \Delta t \)  |
| \( k = 5 \)    | \( k = 5 \)    | \( k = 5 \)    | \( k = 5 \)    |
| \( \Delta t \)  | \( \Delta t \)  | \( \Delta t \)  | \( \Delta t \)  |
| \( 0.0000108 \)| \( 0.000011 \) | \( 0.000105 \) | \( 0.000011 \) |
| \( n = 16 \)   | \( n = 32 \)   | \( n = 64 \)   |
| \( \Delta t \)  | \( \Delta t \)  | \( \Delta t \)  |
| \( k = 5 \)    | \( k = 5 \)    | \( k = 5 \)    |
| \( \Delta t \)  | \( \Delta t \)  |
| \( 0.00002 \)  | \( 0.00003 \)  |
| \( n = 32 \)   | \( n = 64 \)   |
| \( \Delta t \)  | \( \Delta t \)  |
| \( k = 5 \)    | \( k = 5 \)    |

The data of calculation experiments are in agreement with the theoretical results. The table shows that moving with step \( \varepsilon \ln n \), point \( t_{n+1,k} \) in the case of the Bakhvalov mesh and \( t_{n,k} \) in the case of the Shishkin mesh comes up to the precise edge of boundary layer \( \tilde{\phi} \) with the error meeting estimate (10) for all the estimates, with constant \( C_2 = 1 \), after which the algorithm finishes its work. Herewith the values of parameter \( p^k \), at which the algorithm finishes its work, appear to be either slightly more, or slightly less than one, i.e. the approximate value of the boundary layer border appears to be either slightly more to the left or slightly more to the right of the precise border.
5. Conclusion
In this investigation we have obtained the following results.
1. The new mesh adaptation algorithms for two classes of singularly perturbed boundary value problems in the case of an unknown boundary layer border have been offered.
2. The apriori estimate approximate solution error of the projection-grid method theorem has been announced.
3. The mesh convergence and approximate solutions on the extremum mesh error estimates theorem have been announced.
4. The results of numerical experiments confirming the theoretical conclusions have been cited.

6. References
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