Numerators in parametric representations of Feynman diagrams.

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Abstract

The parametric representation has been used since a long time for the evaluation of Feynman diagrams. As a dimension independent intermediate representation, it allows a clear description of singularities. Recently, it has become a choice tool for the investigation of the type of transcendent numbers appearing in the evaluation of Feynman diagrams. The inclusion of numerators has however stagnated since the ground work of Nakanishi. I here show how to greatly simplify the formulas through the use of Dodgson identities.

Keywords: Feynman integrals, parametric representation, Dodgson identities.

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1 Introduction

In quantum field theories others than the scalar ones, propagators and vertices introduce numerators depending on the various momenta, internal or external, in the integrals described by Feynman diagrams. Many textbooks dismiss such numerators as inessential complications, but practical computations in realistic theories soon encounter the formidable task it is to really compute those numerators. Even for lowest order computations, with only tree diagrams, amplitudes can be tricky to evaluate.

Parametric representations have been an important tool for perturbative computations in Quantum Field Theory since their introduction by Feynman and Schwinger. They are particularly interesting to prove general properties of the amplitudes and have been important in many studies of renormalizability. More recently, they have been instrumental in the proof that a whole category of diagrams can be evaluated in terms of multiple zeta values (MZV) [1, 2]. A specialized program has been developed, which allows for the complete evaluation of the corresponding integrals [3].

The difficulty in having a simple representation of the numerators has hindered the extension of such results to field theories with non-scalar fields. Nakanishi [4] worked out the expression of correlations in terms of graph polynomials, but the straightforward application of these formulas results in far from optimal expressions, with really high powers of the first Symanzik polynomial of the graph in the denominator.

Most present evaluations of Feynman diagrams therefore avoid this kind of computations. Additional edges are added to the diagram, but without introducing new loop momenta, such that all possible scalar products can be expressed as combinations of the squared momenta in all edges of these extended diagrams. The diagram with its numerator can be expressed as a combination of diagrams with positive or negative powers of
the propagators corresponding to all these edges. The whole family of such integrals is
linked by integration by part relations (IBP methods), in order to express all necessary
diagrams with their numerators in terms of master integrals. A recent addition has
been to include higher dimension scalar integrals, allowing for a set of integrals which
are completely finite and therefore easier to evaluate [5]. This program has been quite
fruitful and is at the basis of such computer codes as Mincer [6,7] or more recently
Forcer [8], which allow for the evaluation of any three or four loop massless diagram.

Another direction has been the realization that some amplitudes in gauge theories
are much simpler than their individual components, starting from the maximally helicity
violating (MHV) amplitudes. A whole machinery has been set up which uses twistor
representations of the momenta, recursion relations to circumvent the evaluation of
Feynman diagrams. The extension to loop amplitudes, even if it is mainly limited to
maximally supersymmetric theories, has similarly produced numerous important works.
It would be difficult to do justice to these numerous developments.

Finally, steps have been taken to express the sum of numerators appearing in Yang–
Mills theory [9], but falling short to an explicit expression only in terms of the Schwinger
parameters.

I will show that a parametric representation is possible which is nicer than the one
given by the straightforward application of Nakanishi’s rules. The principal tool is a set
of identities among graph polynomials stemming from the Dodgson identities1. I will
use ideas and notations from Francis Brown’s work [2,11]. One interesting aspect of this
rather direct approach is that it avoids the “black box” aspect of IBP computations,
which makes it difficult to have estimates on the integrals. We also are able, in the
case of the Wess–Zumino model, to obtain direct cancellation of divergences at the level
of the integrand, so that we only deal with finite integrals, without having to rely on
compensations between different divergent terms. This is important for the application I
envision, where Schwinger–Dyson equations are written directly in terms of renormalized
amplitudes, without any regulator.

For scalar massless theories, the integrals only depend on a vacuum graph obtained
by closing the two point functions by an additional edge or joining all exterior legs to
an additional vertex. I will first show how this dependence on only the completed graph
can be obtained in the case of a graph with a single scalar product. The general case
would be difficult to solve using the same method, but a workaround is possible using
scale invariance. In the case of Fermion loops, expanding the traces in terms of scalar
products would produce far too many terms. It is better to start from correlations of
momenta and a nice reduction of the number and complexity of the terms will be shown
in the cases of loops of length four and six.

As a showcase for the power of this approach, I will deal with the three and four loops
primitive contributions to the two point functions in the supersymmetric Wess–Zumino
model, obtaining parametric expressions with the full symmetry of the completed graph
and no spurious divergences.

We limit ourselves here to the case of corrections to propagators for scalar particles,
for which the reduction to vacuum diagrams is straightforward. In the case of propa-

1Some persons object on this name, since Dodgson only used the case of the dimension 2 determinant
in his condensation method for the computation of determinants [10]. Jacobi is at the origin of this
formula and is dutifully referred to by Dodgson. However, anything named after Jacobi requires further
precisions, in view of the range of his mathematics interests, so that I will stick to this name, used by a
number of recent authors. Furthermore, I cannot resist this occasion to refer to the author of Alice in
Wonderland.
gators for non-scalar particles, due attention has to be taken to the different possible tensor structures while vertex type diagrams have their own difficulties. In all cases, reduction to completed diagrams should be possible and allow for similar simplifications in the parametric representations. Marcel Golz has made parallel progress on the explicit parametric expression of a class of diagrams in QED, using some of the same identities \[12\]. Using a description of the Dirac trace problem with chord diagrams, he is able to evaluate this part of the problem for a large number of propagators, even if it is in four dimension for a single Fermionic loop. However he does not look for a reduction to a complete diagram. The terms proportional to external momenta remain in their original form with a maximal power of the denominator $n + 2$ for $n$ Fermionic propagators instead of the $n/2 + 2$ obtained in this work. This higher degree in the denominator would make the integration steps much more difficult.

2 Presentation of the main results.

The notations for graph polynomials will be the ones in the works of Brown and collaborators \[2, 11\]. To each connected graph $G$ a matrix $\tilde{M}_G$ is associated. The matrix $\tilde{M}_G$ is a block matrix with a diagonal block with the Schwinger parameters as diagonal values, two blocks made of the incidence matrix and minus its transpose and a zero block to complete it. This matrix is singular since the sum over the rows indexed by the vertices of the incidence matrix is zero, but the determinant of any matrix $M_G$ obtained by removing the row and column corresponding to any vertex gives the first Symanzik polynomial of the graph $U_G$, independently of the chosen vertex. This polynomial is of degree $L$, with $L$ the dimension of the first homology group of the graph, the loop number in the language of physicists.

The objects which will be used to express our results are the polynomials $U_{I,J,K}^{I,J}$ defined for $I$, $J$ and $K$ subsets of the edges of the graph such that $I$ and $J$ have the same number of elements $k$. The polynomial $U_{K}^{I,J}$ is the determinant of the matrix deduced from the graph matrix by removing the rows associated to $I$, the columns associated to $J$ and setting the variables associated to $K$ to zero and has degree $L - k$. From the symmetry properties of $M_G$, one deduces that $U_{K}^{I,J} = U_{K}^{J,I}$. The graph polynomial does not depend on the order given to edges and vertices and the orientation of the edges, but the sign of $U_{K}^{I,J}$ can change. It will require special attention. In fact, since we will need a definite sign for these expressions, it will be preferable to define $U_{I,J}^{I,J}$ as the determinant of the matrix where the rows indexed by $I$ are replaced with rows with only one non-zero value $1$ in a position corresponding to a row in $J$. Expansion along the columns indexed by $J$ or the rows indexed by $I$ gives back the previous definition up to a sign, but now the result only depends on the orientation of the edges indexed by $I$ and $J$ and the bijection between these subsets defined by the positions of the $1$ in the modified matrix. It appears that Schnetz in \[13\] proposed the same choice of signs, but expressed it as an explicit sign in front the determinant of the reduced matrix rather than the implicit definition given here. Both implicit and explicit definitions have their value.

These polynomials, in the special case where $I$ and $J$ have only one element, give the adjoint elements of the matrix $M_G$, so that the correlation between momenta in rows $i$ and $j$ is given by $U_{\{i\},\{j\}}^{\{i\},\{j\}}/U = U_{i,j}^{I,J}/U$ (we do not keep the braces in the case of singletons to lighten the notations). The Dodgson identities relate determinants of matrices whose elements are $U_{i,j}^{I,J}$ for $i \in I$ and $j \in J$ to the polynomials $U_{I,J}^{I,J}$. In appendix $A$ we precise these identities and their proofs, paying due attention to the question of signs.
The gist of our results is most easily described for a vacuum diagram. We consider a numerator expressed as the product for \( s = (s_0, s_1) \) in a set \( S \) of \( n \) scalar products \( p_{s_0} \cdot p_{s_1} \). A priori, there are no restrictions to the number of times the momentum associated to a given edge can appear in this product of scalar products. It should be even possible to have higher powers of some of the scalar products, but it seems to enter the domain of useless generality.

**Definition 1.** For a non-empty subset \( S_0 \) of \( S \), define \( U^{S_0} \) as the mean of the determinants defined using functions \( \varepsilon \) from \( S_0 \) to the set of edges of the graph such that \( \varepsilon(s) \in \{s_0, s_1\} \), together with the complementary function \( \tilde{\varepsilon} \) such that \( \{\varepsilon(s), \tilde{\varepsilon}(s)\} = \{s_0, s_1\} \):

\[
U^{S_0} = \frac{1}{2\#(S_0)} \sum_{\varepsilon} U^{\varepsilon(S_0), \tilde{\varepsilon}(S_0)}.
\]  

(1)

Since the symmetry of the matrix means changing \( \varepsilon \) to \( \tilde{\varepsilon} \) does not change the term, \( U^{S_0} \) has at most \( 2\#(S_0) - 1 \) different terms. With this objects, we can easily state our main theorem

**Theorem 1.** The integrand of a massless vacuum diagram in space-time dimension \( d \) with numerator expressed as a product of scalar products can be written as a sum over the partitions \( P_k \) of \( S \) in \( k \) parts:

\[
I = \sum_{k=1}^{\#(S)} \frac{(-1)^{\#(S)-k}\Gamma\left(\frac{d}{2} + k\right)}{U^{\frac{d}{2} + k}} \sum_{\{S_1, \ldots, S_k\} \in P_k} \prod_{j=1}^k U^{S_j}.
\]  

(2)

We can observe that this rational function has homogeneity \(-LD/2 - n\). The value of the residue of the diagram is than obtained by integrating \( I \) with the measure \( \prod \frac{1}{\Gamma(\beta)} x_i^{\beta_i - 1} dx_i \), where \( x_i \) is the variable associated to the edge \( i \) in the graph polynomials and \( \beta_i \) is the exponent for the propagator.

The same result applies for a propagator style graph, with the graph replaced by the related completed graph, obtained by linking the two exterior legs in a simple edge. In this case, the exponent associated to the added edge must be chosen to make the whole vacuum diagram scale invariant.

In the simple case where all the scalar products are squares of momenta, that is to say, \( s_0 = s_1 \), all the terms in \( U^{S_0} \) are equal and the mean is simply \( U^{S_0} \), which itself is the derivative of \( U \) with respect to the variables associated to the edges in \( S_0 \). Since a numerator of the form \( p_i \cdot p_i \) can also be obtained by lowering by one unit the exponent of the propagator associated to edge \( i \), this allows for a simple proof of theorem [1] in this case, using integration by parts in each of the Schwinger variables associated to the edges in \( S \). In fact, it is this simple case which suggested that a symmetrical and rather simple result could be obtained in the general case.

The interesting feature of Theorem [1] is that, through the identity between the power of the graph polynomial in the denominator and the argument of the \( \Gamma \)-function, it appears as a sum of the evaluation of the graph in the dimensions \( d + 2, d + 4, \) up to \( d + 2m \), if \( m \) is the total number of scalar products in the numerator. The polynomial numerators mean that the propagators can become doubled, tripled, or more, but since each factor is a kind of graph polynomial, of maximal degree 1 in each of its variables, the maximal power of each propagator is \( k + 1 \) for the term in dimension \( d + 2k \), meaning that we remain short of having an infrared divergence on a single propagator. The precise
interplay between this formulation and the structure of sub-divergences in this setting go beyond the purpose of this letter. This structure is similar to the one appearing in the study of the graphs with sub-divergences of \[5\]. As in this study, integration by part identities should allow to relate terms stemming from different monomials in the numerator. On the other end, the explicit evaluation of terms with non trivial numerators in terms of higher dimensional graphs could decrease the total number of terms necessary to obtain a reduction to master integrals.

3 Reducing to the completed diagram.

3.1 A single scalar product.

In the case of a propagator graph \(G\) in a scalar theory, it is known that the residue can be expressed as an integral which only depends on the completed graph \(\tilde{G}\) and its graph polynomial \(\tilde{U}\), with an exponent for the additional edge giving scale invariance to the completed graph, which is then a vacuum graph (see, e.g., \[14\]).

\[
\mathcal{I}_G[\beta_i] = \frac{\Gamma(d/2)}{\prod \Gamma(\beta_i)} \int \prod (dx_i x_i^{\beta_i - 1}) \tilde{U}^{-d/2} \delta(\sum' x_i - 1). \tag{3}
\]

The derivation of formula (3) uses the relation between the graph polynomial \(\tilde{U}\) of the completed graph and the two Symanzik polynomials of the propagator graph: the first Symanzik polynomial is what we have called the graph polynomial up to now and will be noted \(U\), while the second one, in this single scale setting, is \(p^2 V\). The deletion contraction relation then gives that \(\tilde{U} = x_0 U + V\). It is traditional to give the index 0 to the edge completing the graph.

In the case of a non trivial numerator, such an expression depending only on a completed graph seems harder to get by. Let us first study the case where the numerator is simply the scalar product \(p_i \cdot p_j\) and redo the same steps as in the scalar case. The first step is to integrate on the loop momenta. For fixed Schwinger parameters, it is a Gaussian integration with two contributions for the scalar product, one coming from the correlation between \(p_i\) and \(p_j\) which gets a factor of the dimension \(d\), and the other stemming from the part proportional to the exterior momentum in both objects. This latter part can be determined by the electric circuit analogy, where the \(x_i\) parameters play the roles of the resistance of the edges. The fraction of the intensity going through a given edge is given by the ratio of two polynomials of degree \(L\), one dependent on the edge \(V_i\) and the universal \(U\). By this analogy, for given values of the parameters \(x_i\), the mean of the momentum \(p_i\) is given by \(p V_i / U\).

The evaluation of the diagram is then given by:

\[
\mathcal{I}_G(p) = \frac{1}{\prod \Gamma(\beta_i)} \int \prod (dx_i x_i^{\beta_i - 1}) \left( \frac{d}{2} U^{i,j} + \frac{V^{i,j} p^2}{U^{d/2+1}} \right) \exp\left(-p^2 V / U\right). \tag{4}
\]

The two terms do not have the same homogeneity, since the \(V^k\) have the same degree as \(U\), while \(U^{i,j}\) has one degree less. Let us define \(\beta_0\) such that \(\beta_0 + \sum \beta_i = (L + 1)d/2 + 1\), so that the first term is homogeneous of degree \(d/2 - \beta_0\) and the second is homogeneous of degree \(d/2 - \beta_0 + 1\) and integrate on the global scale of the Schwinger parameters. The \(p^2\) already present in the second term makes the two terms proportional to \((p^2)^{\beta_0 - d/2}\) so that the integrand without the \(\Gamma\) prefactors and powers of the \(x_j\) reads:

\[
\frac{d}{2} \frac{\Gamma(d/2 - \beta_0) U^{i,j}}{U^{\beta_0+1} V^{d/2-\beta_0}} + \frac{\Gamma(d/2 - \beta_0 + 1) V^{i,j}}{U^{\beta_0+1} V^{d/2-\beta_0+1}}. \tag{5}
\]
We write \( d/2 \) in the first term as \( \beta_0 + (d/2 - \beta_0) \) and use the functional identity for the \( \Gamma \)-function to obtain \( \beta_0 \Gamma(d/2 - \beta_0) + \Gamma(d/2 - \beta_0 + 1) \). The term with the \( \beta_0 \) factor can be written as an integral on \( x_0 \) as in the scalar case, giving

\[
\int_{x_0=0}^{\infty} dx_0 x_0^{\beta_0-1} \frac{\Gamma(d/2 + 1)}{\Gamma(\beta_0)} \frac{U^{i,j}}{U^{d/2+1}},
\]

The \( \beta_0 \) factor of the numerator can be used to change the argument of the \( \Gamma \)-function in the denominator to \( \beta_0 \).

The other term has a denominator with a global degree in \( U \) and \( V \) of \( d/2 + 2 \) and a simple application of the same transformation would give a higher power of \( \tilde{U} \) in the denominator. We obtain indeed

\[
\Gamma(d/2 - \beta_0 + 1) \frac{V U^{i,j} + V^{i}V^{j}}{U^{d/2+1}},
\]

by putting everything on the same denominator. However, the Dodgson identities allow to simplify the numerator. In the completed graph, we have

\[
\tilde{U} U^{0,0,j} = \tilde{U}^{0,0} \tilde{U}^{i,j} - \tilde{U}^{0,j} \tilde{U}^{i,0}
\]

and consider the part proportional to \( x_0 \) and the one which does not depend on it. The part proportional to \( x_0 \) is trivially true but the other part is quite interesting, since using that \( \tilde{U}^{0,0} = U, \tilde{U}^{0} = V \) and \( \tilde{U}^{0,k} = \tilde{U}^{k,0} = V^{k} \), it can be written:

\[
V U^{i,j} = U \tilde{U}^{i,j} - V^{i}V^{j}
\]

The numerator in Eq. (7) turns out to be proportional to \( U \), reducing the total power of \( U \) and \( V \) in the denominator. The trick of writing this term as an integral over \( x_0 \) will give a contribution similar to the one in Eq. (6), so that both combine to give

\[
\int_{x_0=0}^{\infty} dx_0 x_0^{\beta_0-1} \frac{\Gamma(d/2 + 1)}{\Gamma(\beta_0)} \frac{x_0 U^{i,j} + \tilde{U}^{0,j}}{U^{d/2+1}},
\]

where it is easy to recognize the full \( \tilde{U}^{i,j} \) in the numerator.

### 3.2 The general case.

In the general case, such a step by step reduction of the terms coming from the parts proportional to the exterior momentum would become rapidly very complex, without a clear generalization for an arbitrary numerator. Indeed, when a numerator is a product, one must generate independently all possible powers of \( x_0 \), splitting some terms in numerous parts. It is therefore impossible to separate a phase where one reduces to the vacuum graph and one where the correlations are expressed in terms of the polynomials \( U^{S} \).

However, it is possible to circumvent such a painful computation. We just start from the remark that in a massless theory, the dependence on the exterior momentum is fixed by homogeneity reason to be a power of \( p^2 \) which will be noted \( \beta_0 - d/2 \). If we integrate \( \exp(-p^2)I_G(p) \) on the whole momentum space, one gets \( \Gamma(\beta_0)p^{d/2}/\Gamma(d/2) \) times the residue of the graph, with the \( d \) dependent factor simply the volume of the \( d-1 \)-sphere. On the other hand, we could do the integration over this exterior \( p \) before all parametric integrations, making with the integrations on the momenta on the internal edges the
Gaussian integral over the completed graph $\tilde{G}$. The parameter $x_0$ associated to the additional edge is simply fixed to 1. This constraint on $x_0$ can be transformed in a special case of the reduction of a projective integral if we complete the integrand by a factor that makes it scale invariant. The relation between the original diagram $G$ and the completed one $\tilde{G}$ with one more loop shows that the required factor is $x_0^{\beta_0-1}$. The $\Gamma$ factors in the relation between the residue of $G$ and the integral on $\tilde{G}$ insure that all edges of the completed diagram are associated with integrals with the same factor $1/\Gamma(\beta_i)$. The case of the propagator diagrams is therefore reduced to the one of vacuum diagrams.

A similar approach is also possible in the case of non-scalar propagators. Lorentz invariance and the presence of a single available vector make the finding of a proper Ansatz for the non-trivial tensor structure of propagators simple: in the case of a photon propagator, the diagram should be proportional to $(p^2\eta_{\mu\nu} - p_\mu p_\nu)(p^2)^{\beta_0-d/2}$ and contraction with the photon propagator to get a vacuum diagram would produce a $d-1$ factor, while independence on the gauge parameter in this additional propagator would verify the transversality hypothesis.

4 Proof of the general result

4.1 The terms to find

From the preceding section, we know that we only have to consider correlations in the numerator, since the cases we study can be reduced to vacuum diagrams. Let us consider a given term among the $(2n - 1)!!$ possible matchings induced by the correlations. We can associate to it a permutation $\sigma$ on the set of scalar products as well as a choice of a couple of complementary functions from the set of products to the set of edges $(\varepsilon, \tilde{\varepsilon})$, such that the correlations are given by the products

$$2^{-n} \prod_{s \in S} U^{\varepsilon(s), \tilde{\varepsilon}(\sigma(s))}/U$$

(11)

Starting from any correlation, we can always suppose that one of the terms is given by $\varepsilon(s)$ and the other $\tilde{\varepsilon}(s')$, defining the pair $(\varepsilon, \tilde{\varepsilon})$ on the points $s$ and $s'$ (which may be identical). The action of $\sigma$ on $s$ is thus defined. The element to which $\varepsilon(s')$ is correlated than allow to extend the definition of both $\sigma$ and $\varepsilon$ until the correlations point back to $\tilde{\varepsilon}(s)$. Then, either all the correlations have been covered and we are done, or we start from another scalar product a new chain until every element of $S$ is covered.

For a given term, the permutation $\sigma$ as well as the choice of functions $(\varepsilon, \tilde{\varepsilon})$ are not unique: in the case where $\sigma$ is the identity permutation, the product will not depend on $(\varepsilon, \tilde{\varepsilon})$ and in any case, exchanging the two functions $(\varepsilon, \tilde{\varepsilon})$ and changing $\sigma$ to its inverse leave the term unchanged. More generally, the number of different choices of the function $\varepsilon$ compatible with a given term is equal to $2^c$, where $c$ is the number of cycles of the permutation $\sigma$. If we take the average over all the possible values of the pair $(\varepsilon, \tilde{\varepsilon})$, this gives the factor $2^{-n}$ in Eq. (11), but we must further divide by $2^c$ to account for the multiple appearance of the terms with a permutation of $c$ cycles. This number of cycles also appears in the dependence on the dimension: correlations and scalar products imply that all vectors get the same spacetime index in a cycle, but it is not further constrained, giving a factor $d$ for each cycle. Combining these two factors, the terms with a permutation of $c$ cycles must be multiplied by $(\frac{d}{2})^c$. The signature of
A permutation has also a simple expression in terms of the number of cycles; it is simply the parity of \( n - c \), with \( n \) the number of objects on which the permutation acts.

### 4.2 The terms in \( U^{S_0} \)

We now start from the expression appearing in Theorem 1. Since all terms are subject to the same averaging on all the possible values of the \( \varepsilon \) function, we can just forget about the different maps \( \varepsilon \) and consider a fixed one. If we put every term on the same denominator \( U^{d/2 + \#(S)} \), each \( U^{\varepsilon(S_i), \varepsilon(S_i)} \) can receive a factor \( U^{\#(S_i) - 1} \) and be converted by the Dodgson identities into the determinant of a matrix with correlators \( U^{i,j} \) as elements.

If we now expand each of these determinants in terms of permutations, we obtain terms which are entirely similar to the ones obtained in the preceding section. A given permutation of the whole set \( S \) can be obtained in a number of different ways according to its cycle structure. Indeed, a given cycle structure can only be obtained from the product of determinants associated to a partition of \( S \) in subsets which are unions of (the sets subjacent to) cycles. The identity is, for example, compatible with any partition of \( S \), while a permutation with only one cycle is only compatible with the trivial partition with only one subset.

For a given permutation, the signs coming from the various determinants always combine to the signature of the complete permutation, since the signature is multiplicative. It is given by the parity of the size of \( S \) minus the number of cycles. Combined with the sign in our formulas, we obtain that the unique term with the finest partition compatible with the cycle structure will have a positive sign and the factor \( \Gamma(d/2 + c) \), with \( c \) the number of cycles in the permutation. We therefore will have contributions from partitions of \( S \) with \( c \) down to 1 parts, which comes with alternating signs. All terms with the same number of parts in the partition are identical. Their numbers are given by the Stirling numbers of the second kind \( \{c\}_{p} \), which count the number of ways the \( c \) cycles of the permutation can be collected in \( p \) parts. We therefore obtain the following coefficients for the terms with \( c \) cycles:

\[
\sum_{p=1}^{c} (-1)^{c-p} \binom{c}{p} \Gamma(d/2 + p).
\]  

Now \( \Gamma(d/2 + p) \) can be converted, using the functional equation for \( \Gamma \), in the product of \( \Gamma(d/2) \) and the raising factorial or Pochhammer symbol \( (d/2)^p \). It is a well-known property of the Stirling numbers of the second kind that they allow to convert from falling factorials to plain powers, as it was the reason for their introduction by Stirling in [10], or by a simple change of signs, from raising factorials to plain powers: all we need is the product rule \( x \times x^p = x^{p+1} - px^p \) and the corresponding one for \( x \times x^2 \) and the triangle property of the Stirling number to obtain that the sum in equation [12] will give \( (d/2)^c \Gamma(d/2) \). This completes our proof of Theorem 1.

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2The notations used are the one proposed by Donald Knuth in [13]. They have the advantage of circumventing the notation clash between combinatorists and analysts which use the notation \((x)_n\) for respectively falling and raising factorials.
5 Fermion loops.

Fermion loops could simply be reduced to a sum of terms of the kind just studied, but this would completely defeat the interest of the method, which is to reduce the number of different polynomial factors we have to compute, particularly the ones with the highest degree corresponding to a single correlation $U_{i,j}$. We limit ourselves to the massless case, to be able to still use the method of section 3.2 to only consider vacuum diagrams, with the added benefit that only loops with even number of propagators have to be considered in the Yukawa case. It is however interesting to keep the feature that the dimension of space only appears in the $\Gamma(d/2 + k)$ factor in the term with the power $d/2 + k$ of the Symanzik polynomial in the denominator. We therefore start from a set of correlations between numerators in the Fermion loop and evaluate the dimension depending traces coming from the contractions of gamma–matrices with the metric associated to the correlation. It is then possible to use sums of products of correlations stemming from the Dodgson identities to hopefully obtain a nicer presentation of the result. The difficulty here is that there is not an obviously better presentation. As the number of terms to consider grows rapidly with the number of terms in the loop, we shall present only the case of the loops of four and six propagators, with only Yukawa couplings. Gauge couplings rapidly lead to more cumbersome situations and will not be dealt with here.

5.1 Loop of four Fermionic propagators

In a simple 4 propagator loop, $\text{Tr}(\not p_a \not p_b \not p_c \not p_d)$, we obtain three possible terms corresponding to the possible positions of the correlations, that is $U_{a,b}U_{c,d}$, $U_{b,c}U_{d,a}$ and $U_{a,c}U_{b,d}$, which will get multiplied respectively by $\text{Tr}(\gamma_\mu \gamma_\mu \gamma_\nu \gamma_\nu)$, $\text{Tr}(\gamma_\nu \gamma_\mu \gamma_\mu \gamma_\nu)$ and $\text{Tr}(\gamma_\mu \gamma_\nu \gamma_\mu \gamma_\nu)$, evaluating easily to $d^2 \text{Tr}(1)$ for the first two and $d(2 - d) \text{Tr}(1)$ for the last. The Dodgson identities then allow to write $U_{a,b}U_{c,d} - U_{a,c}U_{b,d}$ as $UU_{ad,bc}$ and $U_{b,c}U_{d,a} - U_{a,c}U_{b,d}$ as $UU_{ba,cd}$. The question then is which product will be taken and which ones will be eliminated with these relations. The product $U_{a,c}U_{b,d}$ would seem to be nicer, since it does not break the symmetries of the loop, however, depending on correlations of non-adjacent momenta, it may involve terms with different signs when the correlations of adjacent momenta are merely obtained by differentiations of the polynomial $U$. Without the overall factor $\text{Tr}(1)$, we can make the following transformations of the numerator associated to this Fermion loop, including the one fourth factor coming in correlators:

$$N = \frac{d^2}{4} U_{a,b}U_{c,d} + \frac{d^2}{4} U_{b,c}U_{d,a} + \frac{d(2 - d)}{4} U_{a,c}U_{b,d}$$

$$= \frac{d^2}{4} U_{a,b}U_{c,d} + \frac{d^2}{4} U U_{ba,cd} + \frac{d}{2} U_{a,c}U_{b,d}$$

$$= \frac{d}{2} \left( \frac{d}{2} + 1 \right) U_{a,b}U_{c,d} + \frac{d^2}{4} U U_{ba,cd} - \frac{d}{2} U U_{ad,bc}$$

$$= \frac{d}{2} \left( \frac{d}{2} + 1 \right) \left( U_{a,b}U_{c,d} + U U_{ba,cd} \right) - \frac{d}{2} U \left( U_{ba,cd} + U U_{ad,bc} \right)$$

In this form, we obtain readily the factors necessary to convert $\Gamma(d/2)$ to $\Gamma(d/2 + 1)$ or $\Gamma(d/2 + 2)$. Separating the term proportional to $U U_{ba,cd}$ in two parts, so that the dependence on $d$ is purely through the $\Gamma$ coefficients, appears to be necessary to obtain nice results. In the next subsection, we will show the nice properties of the numerators coming from this choice. Different forms of the numerator can be obtained if one chooses...
to use different Dodgson identities, but insisting that terms quadratic in \( d \) always come as part of \( d(d + 2) \) makes the remaining terms proportional to \( d \) independent of these choices.

It is however interesting to further express the polynomials \( U^{ad,bc} \) in terms of the spanning forest polynomials introduced by Brown and Yeats \[11\]. These polynomials have the advantage of having all their coefficients positive. They are no longer defined by sets of edges but by a partition of the subset of vertices of the graph which are endpoints of the edges in \( I \cup J \). Then the forest polynomials, as their names indicate, are defined from forests with the same number of components as the partition used to define them. If we have a partition with \( k \) components, we consider the set of covering \( k \)-trees of \( G \setminus (I \cup J) \) such that all elements of a part of the partition belong to a corresponding tree in the \( k \)-tree. To each such \( k \)-tree we associate a term in the forest polynomial, the product of the variables associated to the edges not in the \( k \)-tree.

A given forest polynomial associated with a partition in \( k \) parts will contribute to a Dodgson polynomial \( U^{I,J} \) if \( I \) and \( J \) have \( k - 1 \) elements and if the elements of \( I \) provide links between the different part of the partition. The same must be true for the elements of \( J \). This means that both \( I \) and \( J \) can complete the \( k \)-trees appearing in the definition of the forest polynomial into covering trees.

In the particular case of \( U^{ad,bc} \), we need to consider the set of vertices \( \{0, 1, 2, 3\} \) such that \( a \) is the edge \((0, 1)\) up to \( d \) being \((3, 0)\). In this case, each of the \( U^{ad,bc} \) polynomials corresponds to a unique spanning forest polynomial, but for the six-propagator case, it will be more natural to express the numerators in terms of these ‘vertex’ polynomials, combining the coefficients of different ‘edge’ polynomials.

Indeed, since there are two elements in either of the sets \( I \) and \( J \), the vertices must be in three distinct sets, meaning that the possible polynomials have just a pairing of two vertices, the other two being independent. We obtain that

\[
\begin{align*}
U^{ad,bc} &= U^{(0,2),1,3} \quad (16) \\
U^{ba,cd} &= U^{(1,3),0,2} \quad (17)
\end{align*}
\]

Our notation for the forest polynomial is that we have as indices the subsets in the partition, but we lighten it by letting down the braces for the singletons. This ends up the general properties of the four propagator Fermion loop.

### 5.2 Dimensional factors for six Fermionic propagators

This case is more complex, since we now have fifteen different terms. The same sets of pairing of edges can than be interpreted either as giving products of scalar products or as sets of correlations. And in the end, we should produce three different numerators for the terms with denominators the Symanzik polynomial to the powers \( d/2 + 1, d/2 + 2 \) and \( d/2 + 3 \), interpretable as scalar integrals in dimensions \( d + 2, d + 4 \) and \( d + 6 \). Things can be made manageable if one considers the symmetries by translation along the loop. This may not be a symmetry for the whole diagram, but it will collect terms with similar factors, obtained from similar functions. The fifteen possible terms are grouped in five collections, with one, two, two times three and six pairing schemes. It is helpful to have a graphical description of the possible pairing schemes. The edges of the Fermion loop are represented by the summits of an hexagon and the correlations are indicated by lines. The five correlation schemes are therefore represented as:

\[
\begin{align*}
a) & \quad \quad \quad \quad \quad b) & \quad \quad \quad \quad c) & \quad \quad \quad \quad d) & \quad \quad \quad \quad e) \quad \quad \quad
divisional_factors_for_six_Fermionic_propagators.png
\end{align*}
\]
It is easy to see that indeed case a) is invariant by rotation, case b) has two versions, cases c) and d) three and that for e), all six possible rotations are distinct. We also need the associated dimensional factors, when taking the trace of the product of six $\gamma$ matrices with the given pattern of contractions. All results are given without the factor of the trace of the identity in spinor space. In cases b) and c), the factor is simply $d^3$, cases d) and e) only use the relation $\gamma_\mu \gamma_\rho \gamma_\nu = (2 - d) \gamma_\rho$ to obtain the respective factors $d(2-d)^2$ and $d^2(2-d)$. The only factor remaining to compute is for a) and requires some work with the anticommutation relations of Dirac matrices to obtain $-d(d^2 - 6d + 4)$. It is remarkable that the same results can be obtained from the requirement that the sum of all coefficients be $d(d + 2)(d + 4)$, similarly to the four propagator case where the sum of the coefficients is $2d^2 + d(2 - d) = d(d + 2)$. This relation ensures that, after using Dogson identities to keep only one product of three correlations, its coefficient will be proportional to $d(d + 2)(d + 4)$, suitable to be interpreted as the $\Gamma$ factor for a $(d + 6)$-dimensional integral. Remembering that each correlator comes with an $1/2$ factor, the different correlators will come with the following dimensionally dependent factors, expressed in terms of the raising factorials of $d/2$:

$$
a) \quad -(\frac{d}{2})^3 + 6(\frac{d}{2})^2 - 5(\frac{d}{2}) \quad (19)$$

$$
b), c) \quad + (\frac{d}{2})^3 - 3(\frac{d}{2})^2 + (\frac{d}{2}) \quad (20)$$

$$
d) \quad + (\frac{d}{2})^3 - 5(\frac{d}{2})^2 + 4(\frac{d}{2}) \quad (21)$$

$$
e) \quad - (\frac{d}{2})^3 + 4(\frac{d}{2})^2 - 2(\frac{d}{2}) \quad (22)$$

These presentations in terms of the raising factorials will guide our search for the expression of the numerators, with the terms proportional to $d/2$ the most constrained, since they must be expressed as sums of simple Dogson polynomials.

5.3 Dodgson polynomials for the six propagator loop.

As in the case of the loops of four propagators, we will take as reference situations the ones with only nearest neighbour correlations, i.e., configurations of type b). Configurations c) and e) have a nearest neighbour correlation, so that they can be obtained from a product of this nearest neighbour correlator and a simple Dogson polynomial involving only four of the propagators of the loop, similar to the ones appearing in the preceding subsection. The new objects we must consider therefore have to produce configurations a) and d). They will involve Dogson polynomials with sets $I$ and $J$ forming a partition of the six edges in the fermionic loop. There are ten different partitions, but three of them have no interest for us: they are the ones with contiguous parts (of the model [abc][def]) which do not allow for three nearest neighbour links between the two parts. The seven others are divided between a highly symmetric one, [ace][bdf] which produces, inter alia, configuration a) and the six rotations of [adf][bce] which can produce the configurations of set d). The lower number of configurations in set d) with respect to the partitions of this subset means that some latitudes remain in our choice of Dogson polynomials generating the required terms.

It will be convenient to have a graphical representation of the different terms coming from a given Dogson polynomial. The edges from one set will be marked by a white blob and we will show the correlators coming from the terms of the determinant associated
to each permutation. We obtain for this two kind of partitions the following set of correlators, with the terms aligned according to the permutation they come from:

\[
\begin{align*}
\begin{array}{c}
\circ \quad \circ \\
\circ \quad \circ \\
\end{array}
\end{align*}
\]

\(=\)

\[
\begin{align*}
\begin{array}{c}
\circ \quad \circ \\
\circ \quad \circ \\
\end{array}
\end{align*}
\]

\(+\)

\[
\begin{align*}
\begin{array}{c}
\circ \quad \circ \\
\circ \quad \circ \\
\end{array}
\end{align*}
\]

\(+\)

\[
\begin{align*}
\begin{array}{c}
\circ \quad \circ \\
\circ \quad \circ \\
\end{array}
\end{align*}
\]

\(−\)

\[
\begin{align*}
\begin{array}{c}
\circ \quad \circ \\
\circ \quad \circ \\
\end{array}
\end{align*}
\]

\(−\)

\[
\begin{align*}
\begin{array}{c}
\circ \quad \circ \\
\circ \quad \circ \\
\end{array}
\end{align*}
\]

\(−\)

\[
\begin{align*}
\begin{array}{c}
\circ \quad \circ \\
\circ \quad \circ \\
\end{array}
\end{align*}
\]

\(−\),

(23)

\[
\begin{align*}
\begin{array}{c}
\circ \quad \circ \\
\circ \quad \circ \\
\end{array}
\end{align*}
\]

\(=\)

\[
\begin{align*}
\begin{array}{c}
\circ \quad \circ \\
\circ \quad \circ \\
\end{array}
\end{align*}
\]

\(+\)

\[
\begin{align*}
\begin{array}{c}
\circ \quad \circ \\
\circ \quad \circ \\
\end{array}
\end{align*}
\]

\(+\)

\[
\begin{align*}
\begin{array}{c}
\circ \quad \circ \\
\circ \quad \circ \\
\end{array}
\end{align*}
\]

\(−\)

\[
\begin{align*}
\begin{array}{c}
\circ \quad \circ \\
\circ \quad \circ \\
\end{array}
\end{align*}
\]

\(−\)

\[
\begin{align*}
\begin{array}{c}
\circ \quad \circ \\
\circ \quad \circ \\
\end{array}
\end{align*}
\]

\(−\),

(24)

In the case of the symmetric partition, equation (23) shows that one gets the correlation of type \(a\), the two ones of type \(b\) as well as the three ones of type \(c\) with a minus factor. The fact that only full sets appear was predictable, since there is only one partition of this kind. In the case of the other kind of partitions, we obtain one of the elements of group \(d\) as well as the element of group \(c\) sharing the same diagonal correlator with a minus sign. We also have an element of group \(b\) as well as three in group \(e\), but two with a minus sign and one with a plus sign.

We are now in a position to express the terms with a factor \(d/2\). For the correlation of type \(a\), which comes with the coefficient \(-5\), we only need to take \(-5\) times the Dodgson polynomial with the symmetric partition. The coefficient 4 for the correlation of type \(d\) will be obtained by adding two times each of the six possible Dodgson polynomials of the other kind. We then obtain 4 times each of the correlations of type \(d\) as well as those of type \(c\), but with a minus sign, since two of the six positions contribute to each element in this case. For type \(c\), combined with the 5 coming from the other type of polynomial, one obtains the required factor 1. For type \(b\), there are three positions which contribute to each one, giving a factor 6 which combines with the \(-5\) to give the required 1. Finally, each object of type \(e\) will see one configuration giving a positive contribution and two a negative one, totalling the expected \(-2\) coefficient. We were therefore able to give all these contributions as linear combinations of Dodgson polynomials. This was predictable from theorem [1] applied to the fifteen different products of scalar products coming from the expansion of the trace, but such a derivation would be much more cumbersome.

Next we have to consider the terms proportional to \((d/2)^2\). We certainly need 6 of the symmetric Dodgson polynomials to get the proper number of objects of type \(a\). However, obtaining the \(-5\) coefficient for type \(d\) is not so clear, since an equal distribution among the 6 Dodgson polynomials would introduce half integers. As in the four fermion propagator case, we will show that reducing the symmetry allows for a reduction of the number of different terms. We therefore split this set between odd and even ones. For definiteness, say that the one represented in equation (24) is an even one, as well as the two ones obtained by rotations by 2 or 4 units. There is a corresponding split of sets \(b\) and \(e\) into even and odd parts, with the even polynomial giving the even element of set \(b\), two even elements of set \(e\) with a minus sign and one odd element of set \(e\) with a plus sign. Suppose now that we take \(-3\) times the even partitions and \(-2\) times the odd ones. For sets \(c\) and \(d\), only the total number counts and we get \(-5\) times each element of \(d\) as well as \(5\) times the elements of \(c\), which combined with the \(-6\) from the symmetric partition gives \(-1\). For the even element of \(b\), we get \(-9\) from the 3 even partitions, which combined with the \(6\) gives the required \(-3\). For the odd one, the odd partitions only give \(-6\) which cancels the other contribution. In the case of the elements of \(e\), we have the same pattern that we have the proper count for the even elements and a missing factor for the odd ones. Indeed, the even elements receive a factor 6 from the even partition and \(-2\) from the odd ones, adding to 4, while for the odd elements, the factors are \(-3\) and 4, adding to 1.
After using the Dodgson polynomials involving the six propagators of the loop, we therefore still have to account for $-3$ times the odd element of $b$, $3$ times every odd elements of $e$) as well as $-2$ times the elements of $c)$. However, for these terms, we can use products of a nearest neighbour correlator and a simpler Dodgson polynomial with the sets $I$ and $J$ having two elements. We can produce in this way the difference between the odd element of $b$) and each of the odd elements of $e)$ or the difference between an element of $c)$ and a corresponding odd element of $e)$. One of the set of odd elements of $e)$ will be paired with the odd element of $b)$, giving terms of type B, i.e., $U^{a,b} U^{c,d,e}$ and two translates. The two other sets will fit with the elements of $c)$, giving two times the terms of type A, $U^{a,b} U^{c,d,e}$. This ends the reduction of the numerator in the $d + 4$ term.

The last point is to consider the terms with a $(d/2)^3$ factor. The $-1$ factor for the type $a$) term is easily accounted for according to our now usual way. The term of type $d$) asks for a subtler treatment. If only even polynomials were used, we would end up with too large an imbalance between even and odd terms for the sets $b$) and $e)$. It is however possible to get all three elements of type $d$) with two polynomials of even type and one of odd type. The even polynomial of type $b$) is obtained two times and with the $-1$ coming with the type $a$) term, we have the proper number. For the odd one, the two contributions from these Dodgson polynomials compensate and the balance is closed by the product of three correlators, giving the only truly 10 dimensional term. For the elements of type $e)$, this choice of Dodgson polynomials does not produce a net contribution for the odd ones and gives the required $-1$ factor for the even ones. As in the preceding case, the odd elements of type $e)$ and the elements of type $c)$ can be combined to give the terms of type A. One can remark that in the end, we only need to explicitly compute the three ‘odd’ correlators appearing in the odd element of type $b)$, since only these ‘odd’ correlators appear in the composite terms of types A and B.

5.4 Forest polynomials

Evaluation of Dodgson polynomials from their definitions as determinants remains challenging. The forest polynomials suggested in [11] can however be evaluated from simple combinatorial rules, especially when their polynomial degree remains low, as in the application we have here. The only delicate point is in the determination of the signs which appear in the decomposition of the Dodgson polynomials, but they can be determined once for all situations. The last point is that in our example, the contributions from different Dodgson polynomials to each forest polynomial add to simple results.

The first step is to enumerate the possible forest structures. The six intermediate vertices in the cycle of fermionic propagators must be grouped in four trees. There are therefore two possibilities, either three isolated vertices and a tree grouping the three other ones or two isolated vertices and two pairs. With the additional constraints that two neighbouring vertices cannot be isolated since no coloring of the three adjacent edges allow them to be connected to the rest of the diagram with the two choices of colours, and that two neighbouring vertices cannot be in the same tree since a loop would necessarily be made, we get the following four types of forests:

![Forest Polynomials Diagram](25)

All these configurations contribute to the symmetric Dodgson polynomial. The configuration 1) is the only possible one with three isolated points and has two versions,
that we will call even (the one represented) and odd. The even one only contributes to
the even polynomials and the odd one to the odd ones. The relative positions in equa-
tions (24) and (25) are chosen so that the vertices and edges, all represented by dots,
alternate on a circle if we overlapped the figures. When the two isolated vertices are at
distance two, the only possibility for the other groups is given by the configuration 2),
with six different possibilities. Here again, we can divide them into even ones and odd
ones. The one represented is an even one, and all even configurations contribute to all
even polynomials but not to the odd ones and vice versa. Finally, the configurations
3) and 4) share diametrically opposed isolated points, have three different versions each
of which contributes to one odd and one even polynomials. The pictured cases are the
one contributing to the configuration appearing in equation (24). The only remaining
point is the sign affecting all these forest polynomials. The situation is simple, since all
contributions come with a plus sign, except for the ones of type 4) which have the minus
sign, independently on the Dodgson polynomial considered. The best way to compute
these signs is to consider the diagram with the six edges of the Fermion loop and the
trees of the forest as vertices. The signs then result from the computation of a product
of three by three determinants.

With these relations between the Dodgson polynomials and the forest polynomials,
it is easy to express the results of the previous subsection in terms of forest polynomials.
For types 3) and 4), even and odd Dodgson polynomials have the same contribution, so
that the resulting coefficient is just the sum of the coefficients for correlators of type a)
and type d), with a minus sign for type 4). We therefore have no contribution in the
term with the factor \((d/2)^3\), a factor +1 for type 3) in the dimension 8 term and for
type 4) in the dimension 6 term and a factor \(-1\) for the remaining cases of 3) in 6 and 4)
in 8. For type 1) and 2), the only relevant information is whether they are even or odd
and the total number of Dodgson polynomials of the same parity if we make the odd
convention that the symmetric Dodgson polynomial is both even and odd. This results
in a factor 1 for all these forest polynomials in the \(d/2\) term, a zero net contribution for
the odd ones in the two other terms, while the even forest polynomials have coefficients 1
in the \((d/2)^3\) term and \(-3\) in the \((d/2)^2\) one. The remarkable point is that, at the level
of these forest polynomials, the coefficients are notably simpler, only 0 and 1 in absolute
value, apart from the single \(-3\).

This will be sufficient for the applications we present in this work, but we must
remember that this applies only to the case of a closed fermion loop. In some diagrams,
we way have to deal with a fermionic trace which is supported by an open path in the
diagram, in which case new forest polynomials could appear, where the two ends of the
path do not belong to the same tree.

Finally, the different coefficients are summed up in table 1. The incomplete subsets
used for the highest dimensional terms are represented by 1/3 and 2/3, since the details
of the used Dodgson polynomials have no influence on the expression in terms of forest
polynomials. Finally, A and B indicate the terms introduced in the previous subsection
which are products of a correlator and a Dodgson polynomial. We do not introduce the
expression of the Dodgson polynomial as sum of forest polynomials. The single term in
dimension ten without a \(U_G\) factor is not written as well as the \(U_G\) factors in the other
terms.
The importance of the supersymmetric Wess–Zumino model stems from the fact that it is the first supersymmetric model in four spacetime dimensions to be found, before the advent of the supersymmetric extensions of gauge field theory or gravity. Its field content is quite simple, with only scalar fields and Fermion fields of the same mass and Yukawa interactions between Fermions and scalars and quartic self-interactions of the scalars. Supersymmetry implies that the quartic scalar coupling is the square of the Yukawa coupling. The supersymmetric transformation of the Fermionic field involves a quadratic polynomial in the scalar fields and the algebra of supersymmetric transformations is only verified up to the classical equations of motion. Both these facts make the study of the quantum case difficult, but can be solved by the introduction of an auxiliary field. As the name implies, such a field has classical equations of motion which do not allow for an independent propagation, since they constraint it to be a quadratic polynomial of the scalars. The quartic coupling of scalars is decomposed in two cubic couplings of an auxiliary field with two scalar field, each of the same strength as the Yukawa coupling.

One further advantage of this formulation with auxiliary fields is that the interaction vertex do not get any corrections: the divergence in the quartic scalar interactions is simply one in the propagator of the auxiliary field, which is therefore no longer constant. To ensure supersymmetry, it is then sufficient that the ratios of the inverse propagators of all components of the supermultiplet to their free counterparts be equal. This ratio is also the only necessary ingredient to compute the renormalisation group functions, as explained in our previous works [17, 18]. All considered graphs are bipartite ternary graphs, but to a single topology, we associate a collection of Feynman graphs with all possible particle assignments for each edge. The numerators we will compute the sum of the contributions for a given topology.

An important property of these numerators is that they only depend on the completed graph. The position of the added edge in the completed graph does not matter. We only consider graphs without propagator insertions, since only primitive graphs are studied, while the Schwinger–Dyson equations take care of the decorations of the propagators.

In fact, both supersymmetry and independence from the added edge stem from the same fundamental graphical computation. Let us consider the star with vertices \((a, b, c, d)\) and the edges \((a, b), (a, c)\) and \((a, d)\), such that \((a, b)\) is the scalar edge added to complete the graph for the auxiliary field propagator. All edges in this star are therefore of scalar type. Suppose now that we start from vertex \(c\) and there is a auxiliary field edge \((c, f)\), which contribute a factor \(p_{c,f}^2\). Using momentum conservation at the vertex

### Table 1: Recapitulation of factors for the loop of six fermions

| Correlations | Dodgson | Forest |
|--------------|---------|--------|
| \(a\) | \(b,c\) | \(d\) | \(e\) | \(s\) | \(odd\) | \(even\) | \(A\) | \(B\) | \(1,2odd\) | \(1,2even\) | \(3\) | \(4\) |
| \(d + 6\) | -1 | +1 | +1 | -1 | 1/3 | 2/3 | 1 | 0 | 0 | 1 | 0 | 0 |
| \(d + 4\) | 6 | -3 | -5 | 4 | 6 | -2 | -3 | -2 | -1 | 0 | -3 | 1 | -1 |
| \(d + 2\) | -5 | 1 | 4 | -2 | -5 | 2 | 2 | 0 | 0 | 1 | 1 | -1 | 1 |

6 Applications to the Wess–Zumino model.

6.1 Supersymmetry and completion independence.
\[ \hat{f}, p_{c.f} = p_{f,g} + p_{f,h}, \] one can write
\[ \hat{p}^2_{c,f} = \hat{p}_{c,f} \hat{p}_{f,g} + \hat{p}_{c,f} \hat{p}_{f,h}. \] (26)

One can therefore replace the single auxiliary field numerator stemming from the auxiliary field on the edge \((c, f)\) by the sum of two numerators stemming respectively from Fermionic paths \((c, f, g)\) and \((c, f, h)\). If the end of any of these paths contacts an auxiliary field edge, the same procedure can be used to extend the path by two more edges in two different ways. This extension procedure will end in one of two different ways: either one reaches vertex \(b\), which is the only vertex with only scalar incident edges in the completed graph which is of the good color (\(a\) is unreachable), or one encounters a Fermionic line. This Fermionic line can either be a preexisiting Fermion loop or the line extending from \(c\). Any graph with a Fermionic line extending from the vertex \(c\) is obtained once and only once through this construction, since the graph from which it is deduced can be recovered by replacing the line with alternating auxiliary field and bosonic field edges.

In the case of paths which do not extend to vertex \(b\), the same set of Fermionic edges can be obtained in three different ways. Topologically, they form a stem (which can be empty) starting form \(b\), plus a loop: the loop can be either a preexising Fermion loop, or be covered in the two possible directions. The product \(\Pi\) of the propagators in the loop, being an even number of terms proportional to \(\gamma_\mu\), can be expressed on the basis of the even part of the Clifford algebra:
\[ \Pi = A \text{Id} + B^{\mu\nu} \gamma_{\mu\nu} + C \gamma_5. \] (27)

In this equation, \(A\), \(B^{\mu\nu}\) and \(C\) are functions of the momenta associated to the edges in the loop. One can see that the term proportional to \(\gamma_{\mu\nu}\) changes sign when the loop is reversed and disappears in the sum over the two loops. When applying a chiral projector, \(\text{Id}\) and \(\gamma_5\) become proportional, so that the sum of the two lassos is \(2A \pm 2C\) while the trace of the matrix on the loop becomes also \(2A \pm 2C\). With the minus sign associated to each fermion loop, these three contributions exactly cancel, independently on the field assignments on the other edges of the graph.

We therefore obtain the same numerator when considering either all possible graphs with a Fermionic path from \(c\) to \(b\) or the ones without numerators on all neighbors of the edge \((a, b)\). We could obtain in a similar way the sum of the graphs with a Fermionic path from \(c\) to \(b\) from the graphs which single out the edge \((c, a)\). We therefore have that the numerators are the same if the completed graph was obtained by adding the edge \((a, b)\) or the edge \((a, c)\).

Since we can move the singled out edge one step, it can be moved on any position by combining such steps. This implies in particular the absence of vertex subdivergences, since by placing the singled out edge in a vertex subgraph, it becomes ultraviolet convergent by a simple dimensional argument.

Supersymmetry is also a simple consequence. The Fermionic two-point function has a Fermionic path from \(a\) to \(b\). It can either begin by the edge \((a, c)\), which is completed by a path from \(c\) to \(b\), or \((a, d)\), completed by a path form \(d\) to \(b\). Since the two sums, the one including all the paths from \(c\) to \(b\) and the one with all the paths from \(d\) to \(b\) give the same numerator, the one appearing in the case of the auxiliary field propagator, the Fermionic two-point function is the same one multiplied by \(\hat{p}_{a,c} + \hat{p}_{a,d}\), which is \(\hat{p}_{b,a}\), the contraction of the external momentum by Dirac matrices, through momentum conservation.
Similarly, the Bosonic propagator has either an auxiliary field on the edge \((a, c)\), an auxiliary field on the edge \((a, d)\) or a Fermion loop including both edges. In all three cases, the sum on the configurations of the other edges will give the common numerator of the graph, multiplied respectively by \(p_{a,c}^2, p_{a,d}^2\) and \(-2p_{a,a} p_{a,c}\). The sum of these three factors is the exterior momentum squared, ending the general proof of supersymmetry at the integrand level. The situation would be more involved if we were dealing with the massive case. Indeed, mass terms change the chirality of the Fermion, convert scalars to auxiliary fields, so that new topologies become possible.

### 6.2 Result for the three loop primitive divergence.

The first correction to the Schwinger–Dyson equation for the Wess–Zumino model happens at 3 loop order. The completion of the relevant graph is the highly symmetric complete bipartite graph \(K_{3,3}\), as already pointed out in [13]. Its symmetry group is indeed the product of two permutation groups \(S_3\) acting independently on the two subsets of vertices. The numerators we want to compute will share this high level of symmetry, due to the invariance property proved in the preceding subsection. This will give an important check for our computation.

Let us start by fixing some notations. The \(K_{3,3}\) is both highly symmetric and non-planar, so that clear pictorial depiction is not easy. It is better to say that we have vertices of one kind numbered from 1 to 3, the ones of the other kind numbered from 4 to 6. Then the edge between vertices \(i\) of the first kind and \(j\) of the other can be numbered \(3i + j - 6\).

The Symanzik polynomial associated to this diagram becomes, with these notations:

\[
U_K = x_1 x_2 x_3 x_5 + x_1 x_2 x_4 x_0 + x_1 x_2 x_4 x_2 + x_1 x_2 x_4 x_9 + x_1 x_2 x_5 x_6 + x_1 x_2 x_5 x_7 \\
+ x_1 x_2 x_5 x_9 + x_1 x_2 x_6 x_7 + x_1 x_2 x_6 x_8 + x_1 x_2 x_7 x_8 + x_1 x_2 x_7 x_9 + x_1 x_2 x_8 x_9 \\
+ x_1 x_3 x_5 x_1 + x_1 x_3 x_5 x_6 + x_1 x_3 x_5 x_8 + x_1 x_3 x_5 x_9 + x_1 x_3 x_5 x_{10} + x_1 x_3 x_5 x_7 \\
+ x_1 x_3 x_5 x_9 + x_1 x_3 x_6 x_7 + x_1 x_3 x_6 x_8 + x_1 x_3 x_7 x_8 + x_1 x_3 x_7 x_9 + x_1 x_3 x_8 x_9 \\
+ x_1 x_4 x_5 x_8 + x_1 x_4 x_5 x_9 + x_1 x_4 x_6 x_8 + x_1 x_4 x_6 x_9 + x_1 x_5 x_6 x_8 + x_1 x_5 x_6 x_9 \\
+ x_1 x_5 x_7 x_8 + x_1 x_5 x_7 x_9 + x_1 x_5 x_8 x_9 + x_1 x_6 x_7 x_8 + x_1 x_6 x_7 x_9 + x_1 x_6 x_8 x_9 \\
+ x_2 x_3 x_4 x_5 + x_2 x_3 x_4 x_6 + x_2 x_3 x_4 x_8 + x_2 x_3 x_4 x_9 + x_2 x_3 x_5 x_6 + x_2 x_3 x_5 x_7 \\
+ x_2 x_3 x_5 x_9 + x_2 x_3 x_6 x_7 + x_2 x_3 x_6 x_8 + x_2 x_3 x_7 x_8 + x_2 x_3 x_7 x_9 + x_2 x_3 x_8 x_9 \\
+ x_2 x_4 x_5 x_7 + x_2 x_4 x_5 x_9 + x_2 x_4 x_6 x_7 + x_2 x_4 x_6 x_9 + x_2 x_4 x_7 x_8 + x_2 x_4 x_7 x_9 \\
+ x_2 x_4 x_8 x_9 + x_2 x_5 x_6 x_7 + x_2 x_5 x_6 x_8 + x_2 x_6 x_7 x_8 + x_2 x_6 x_7 x_9 + x_2 x_6 x_8 x_9 \\
+ x_3 x_4 x_5 x_7 + x_3 x_4 x_5 x_8 + x_3 x_4 x_6 x_7 + x_3 x_4 x_6 x_8 + x_3 x_4 x_7 x_8 + x_3 x_4 x_7 x_9 \\
+ x_3 x_4 x_8 x_9 + x_3 x_5 x_6 x_7 + x_3 x_5 x_6 x_8 + x_3 x_5 x_7 x_8 + x_3 x_5 x_7 x_9 + x_3 x_5 x_8 x_9 \\
+ x_4 x_5 x_7 x_8 + x_4 x_5 x_7 x_9 + x_4 x_5 x_8 x_9 + x_4 x_6 x_7 x_8 + x_4 x_6 x_7 x_9 \\
+ x_4 x_6 x_8 x_9 + x_5 x_6 x_7 x_8 + x_5 x_6 x_7 x_9 + x_5 x_6 x_8 x_9 \\
(28)
\]

If we single out the edge 9 between vertices 3 and 6, the numerator comes from the loop \((1, 4, 5, 2)\), with either two of the edges with an auxiliary field or the entire loop as a fermionic loop. The numerators coming from the auxiliary fields are easy to obtain as derivatives of the Symanzik polynomial \(U_K\). Using the notation \(\partial_i\) for the derivation with respect to \(x_i\), the numerators corresponding to these cases are simply \(\partial_i U_K \partial_j U_K + \partial_j U_K \partial_i U_K\) for the eight dimensional case, and \(\partial_i \partial_j U_K + \partial_j \partial_i U_K\) for the six dimensional one. Using the results of section 5.1, the fermionic numerator will need
the forest polynomials $U^{1,2,\{4,5\}}$ and $U^{4,5,\{1,2\}}$. The link between 4 and 5 in the first one is necessarily over the vertex 3, so that we have to separate the vertices 1, 2 and 3, all linked to vertex 6 through the edges 7, 8 and 9. This allows us to compute the first forest polynomial as well as the second one which is similar:

\[
U^{1,2,\{4,5\}} = x_7x_8 + x_7x_9 + x_8x_9, \\
U^{4,5,\{1,2\}} = x_3x_6 + x_3x_9 + x_6x_9. 
\] (29)

The numerator for the term with $U^3_K$ in the denominator can then be computed, remembering that due the chirality condition, the trace of the identity in spinor space is 2, as:

\[
N^3 = \partial_1\partial_5 U_K \partial_2\partial_4 U_K - 2(U^{1,2,\{4,5\}} + U^{4,5,\{1,2\}})
+ x_1x_5 + x_1x_6 + x_8x_1 + x_9x_1 + x_4x_2 + x_2x_6 + x_7x_2 + x_2x_9 + x_4x_3 \\
+ x_5x_3 + x_7x_3 + x_8x_3 + x_8x_4 + x_9x_4 + x_7x_5 + x_5x_9 + x_7x_6 + x_8x_6.
\] (31)

This expression has the full symmetry of the $K_{3,3}$ graph. Indeed, the 36 different pairs of edges of this graph fall in only two classes under the action of the symmetry group, the pairs with a common vertex and the other ones, each with 18 elements, and $N^3$ is the sum of the products of variables associated to the pairs in the second group. Furthermore, this structure prevents any subdivergence in the corresponding integral. These two properties of independence on the choice of a special edge and absence of subdivergences must both hold for the complete integrand, due to the properties established in the previous subsection, but it is remarkable that our choices make them hold for the two terms independently.

As can be expected, the numerator for the $U^4_K$ is more complex, due to its higher degree. Nevertheless, since we only need one of the forest polynomial $U^{4,5,\{1,2\}}$ and derivatives of the Symanzik polynomial $U_K$, it is not so hard to compute with an algebraic manipulation program.

\[
N^4 = \partial_1U_K\partial_5U_K + \partial_2U_K\partial_4U_K - 2(U_KU^{4,5,\{1,2\}})
+ \frac{1}{2}(\partial_1U_K + \partial_2U_K - \partial_3U_K)\frac{1}{2}(\partial_4U_K + \partial_5U_K - \partial_6U_K). 
\] (32)

The resulting expression, of degree 6, has 729 terms and is too large to print here, but can be found in our supplementary material in file FZ33. Using the symmetry of the expression, a description with words is possible, the interested reader can find it in Appendix B.

### 6.3 Setting for the four loop primitive divergence.

At the following perturbative order, there is also a single contributing diagram with a highly symmetric completion, the cube. In this case, it will be convenient to have a graphical representation of the diagram, even if the plane drawing has not the full symmetry of the diagram. The indexation of the edges is chosen in order to make
explicit a symmetry of our computation.

The generic way of determining the numerator would be to choose one particular edge, say 1, as the special one, with all its neighbours bosonic. The possible decorations of the diagram are therefore a fermionic loop of six edges (3,4,6,8,10,11), two with a fermionic loop of four and one auxiliary field (loop 3,4,6,7 and 10 or loop 7,8,10,11 and 4) and three with three auxiliary fields (4,8,11 and 3,6,10 and 4,7,10). However, the same numerator can be obtained with only one fermionic loop of six and two configuration of three auxiliary fields, involving only the even numbered edges, giving a simpler path to the numerator. This have the added advantage of making a six-fold symmetry explicit.

Let us start with the fermionic loop, \( \text{Tr}(p_2 p_4 p_6 p_8 p_{10} p_{12}) \). Then \( p_2 \) and \( p_{12} \) can be expressed through momentum conservation respectively as \( p_4 - p_3 \) and \( p_{10} - p_{11} \). We therefore obtain four terms, three of which can be simplified from the identity \( q \cdot q = q^2 \).

The trace of the product of two terms just gives the scalar product with a factor 2 due to the chiral trace, \( \text{Tr}(q \cdot r) = 2 q \cdot r \), while the identity \( p_6 = p_7 + p_8 \) can then be used to express the trace of products of 4 terms with closed loops. We therefore have:

\[
\text{Tr}(p_2 p_4 p_6 p_8 p_{10} p_{12}) = 2 p_2^2 (p_6 \cdot p_8) p_{10}^2 \quad \text{Tr}(p_4 p_8 p_{10} p_{11}) - p_2^2 \text{Tr}(p_3 p_4 p_6 p_8 p_{10} p_{11}) + Tr(p_3 p_4 p_6 p_8 p_{10} p_{11})
\]

We recognize in the final form the loop of six propagators as well as the two terms with a loop of 4 propagators in the other formulation of this numerator. The differing sign before one of the loop comes from the reversal of \( p_7 \) in this loop in our conventions. Three terms with a scalar product remain. But we know that we must apply the same identity on \( p_2 \) and \( p_{12} \) in the two terms coming from auxiliary fields, \( p_2^2 p_6^2 p_{10}^2 \) and \( p_2^2 p_8^2 p_{10}^2 \). The expansions of \( p_2^2 \) and \( p_{12}^2 \) give three terms for each: the scalar products give terms which compensate two of these three terms, one of the squared momenta in each case gives one of the auxiliary field terms in the original formulation and we are left with two terms having the factor \( p_2^2 p_{10}^2 \). These two terms and the remaining term with a scalar product in the expansion of the fermionic loop combine to give the last one of the terms with only auxiliary fields \( p_2^2 p_6^2 p_8^2 \).

### 6.4 Evaluation of the numerators.

The Symanzik polynomial \( U_C \) for the cube will not be explicitly written, it will simply be available in the additional files as \( G_{cu} \), since it has 384 terms.
The numerator for the term of dimension 6 is only of degree 2 and is quite easy to compute. The bosonic terms are obtained from third derivatives of the Symanzik polynomial and only two different kinds of forest polynomials contribute to the fermionic loop. Indeed, opposing points in the fermionic loop cannot be joined by paths avoiding this loop, so that only forest polynomials of types 1) and 4) are non null. The three polynomials of type 4) are just the product of the variables associated to two diametrically opposed odd edges, giving \( x_1 x_7, x_3 x_9 \) and \( x_5 x_{11} \). Their sum will be denoted as \( S \). The odd polynomial of type 1) is the symmetric polynomial of degree two on \( (x_1, x_5, x_9) \) denoted \( O \) and the even one is based on the remaining odd numbered variables, denoted \( E \).

\[
\begin{align*}
O &= x_1 x_5 + x_1 x_9 + x_5 x_9, \\
E &= x_3 x_7 + x_3 x_{11} + x_7 x_{11}.
\end{align*}
\]

The numerator for the dimension 6 term can then be written as the sum of the bosonic terms, easily expressed by derivation of the Symanzik polynomial and \(-2\) times the listed forest polynomials.

\[
N^{3c} = \partial_2 \partial_6 \partial_{10} U_C + \partial_4 \partial_8 \partial_{12} U_C - 2 (S + O + E)
\]

\[
= x_1 x_3 + x_1 x_4 + x_1 x_6 + x_1 x_8 + x_1 x_{10} + x_1 x_{11} + x_2 x_5 + x_2 x_6 + x_2 x_7 + x_2 x_9
\]

\[
+ x_2 x_{10} + x_2 x_{11} + x_3 x_5 + x_3 x_6 + x_3 x_8 + x_3 x_{10} + x_3 x_{12} + x_4 x_7 + x_4 x_8
\]

\[
+ x_4 x_9 + x_4 x_{11} + x_4 x_{12} + x_5 x_7 + x_5 x_8 + x_5 x_{10} + x_5 x_{12} + x_6 x_9 + x_6 x_{10}
\]

\[
+ x_6 x_{11} + x_7 x_9 + x_7 x_{10} + x_7 x_{12} + x_8 x_{11} + x_8 x_{12} + x_9 x_{11} + x_9 x_{12}
\]

The result is highly symmetric, with 36 different terms characterized by the fact that the two edges associated to the variables are neither adjacent nor diametrically opposed. It is pitifully the last polynomial that I can explicitly present in this paper, since the two other ones are far too big.

For the next term, we will need additional forest polynomials. Even for the bosonic terms, some additional complexity appears since we have three terms from each of the possible field configurations, corresponding to the choice of the indices in the second derivative. Choosing only one of them, the two other ones are easily deduced by an order three rotation, easily represented by a shift of \( 4 \) in the indices.

For the fermionic part, we also need the terms involving a nearest neighbour correlator. Again, we only write explicitly the one involving the correlator \( U^{10,12} \), knowing there will be two other ones obtained by rotation. This correlator will be multiplied by the two Dodgson polynomials \( A = U^{24,8,6} \) coming with a factor 2 to complete the terms of type \( c \) as well as \( B = U^{28,6,4} \) for the terms of type \( b \). These two terms will be expressed through a new set of forest polynomials and will use the labels of vertices in the figure \[33\] to present them. If we put the two end points \( a \) and \( e \) of the path \( (2 4 6 8) \) in the same set, we would be back to the situation for the four-propagator loop and end up with the forest polynomials \( U^{\{a,e\}} \{b,d\},c \) for \( A \) and \( U^{\{a,c,e\}} \{b,d\} \) for \( B \), both with a plus sign, but other ones are possible. In the case of Dodgson polynomial \( A \), one can also have crossed associations giving the polynomial \( U^{\{a,d\}} \{b,e\},c \) with a minus sign. However for the cube, this polynomial is zero, since there is no way to have paths from \( a \) to \( d \) and \( b \) to \( e \) which do not cross. In the case of \( B \), there are two additional polynomials, \( U^{\{a,c\}} \{b,e\},d = x_7 x_9 x_{12} \) and \( U^{\{a,d\}} \{b,e\},e = x_1 x_3 x_{10} \).

The numerator for the dimension 8 term will be expressed as the sum of the three images by rotation of the polynomial \( N^{4p} \) minus a contribution written as the product
of the Symanzik polynomial $U_C$ by $6O + 2S$. We have

$$N4p = \partial_{10}U_C \partial_2 \partial_3 U_C + \partial_{12} U_C \partial_3 \partial_4 U_C - 2U^{10,12}(2A + B)$$

$$= \partial_{10}U_C \partial_2 \partial_3 U_C + \partial_{12} U_C \partial_3 \partial_4 U_C - (\partial_{10}U_C + \partial_{12}U_C - \partial_{11}U_C) \cdot$$

$$\left(2(x_5x_{11}(x_1 + x_9 + x_{10} + x_{12}) + x_5x_{10}x_{12} + x_1x_9x_{11}) + (x_3x_7 + x_3x_{11} + x_7x_{11})(x_1 + x_9 + x_{10} + x_{12}) + x_{10}x_{12}(x_3 + x_7)ight)$$

$$+ U^{(a,c)\{b,e\},d} + U^{(a,d)\{b,c\}} \right)$$

(37)

Adding its two images by rotation and removing the terms proportional to $U_C$, we get the numerator $N4c$. With 6516 terms with coefficients ranging from 2 to 24, it even defies the kind of description we gave for $N4$ in appendix [B]. Indeed, if we could give this kind of description for the 168 terms with coefficient 2, which have the type $x^2y^2z^2t$, the 2232 terms with coefficient 3 or the 2976 ones with coefficient 6 are best left to the supplementary files.

Finally, all elements are available for the description of the degree 12 numerator for the term with $U^5_C$ in the denominator. It is paradoxically easier to describe than the previous one, since the $B$-type Dodgson polynomials do not contribute.

$$N5c = \partial_3 U_C \partial_3 U_C \partial_{10} U_C + \partial_4 U_C \partial_5 U_C \partial_{12} U_C - 2U^{2,4}U^{6,8}U^{10,12}$$

$$- 2U^5_C \partial_2 U_C \partial_4 U + 2U^{10,12}A + \text{rotated}.$$  

(38)

The resulting polynomial has again the full cubic symmetry, but with its 207 359 terms and coefficients ranging from 1 to 2064, it is beyond simple description. The coefficient 2064 pertains to the single term with the product of all edge variables, but there are 136 distinct numerical coefficients, 12 of which are common to more than 5232 terms. All these polynomials are available in the supplementary file $F_{cube}$.

7 Conclusion and perspectives

We have presented a way to obtain simpler parametric representations for graphs with numerator. An important aspect is the use of the completion of the graph in the case of propagator style graphs, which allows for a higher symmetry of the result and also limit the highest power of the graph polynomial appearing in the denominator. Knowing the large size of the obtained numerators, this claimed simplicity is not so obvious, but we must not forget that, without the reduction to the completed graph, there would be terms with up to the eighth power of the Symanzik polynomial in the denominator with the concomitant high degree of the numerator. The simplicity is also in the small number of low degree forest polynomials used as building blocks of the numerators apart from the derivatives of the Symanzik polynomial.

Our explicit results for the massless Wess–Zumino model should be essential stepping stones for the explicit study of the corrections to the Schwinger–Dyson equation of this model. In [17], only the lowest term in the equation was included, but already in [18] the effect of these three and four loop corrections to the Schwinger–Dyson equations on the asymptotic properties of the perturbative series was studied. However, it was only a negative result that the leading asymptotic behaviour should not be changed. With the full parametric representation of these diagrams, explicit corrections to this asymptotic behaviour could be computed as well as the subleading contributions to the $\beta$-function itself. One may wonder if the high number of terms in the obtained numerators would
preclude the practical use of such expressions. Combined with the fifth power of the
Symanzik polynomial in the denominator, this could be stretching the capacity of the
presently available programs for the analytic integration of such expressions. However,
the high symmetry and the avoidance of any spurious subdivergence should make this
difficulty less serious. Moreover, some of the important use cases of these expressions as
the determination of the residues of the Mellin transform of the corresponding graphs
will only need the terms with some given power of a variable, selecting a reduced number
of terms. The combination of exact supersymmetry, possible since we remain in four
dimensions, and explicit cancellation of all subdivergences gives tremendous advantages
over the use of dimensional regularisation.

The results obtained for the fermionic loops can also be used for five or six loops
primitive diagrams, but only for their lowest order contribution: we hope to show in a
future publication how to compute the full five or even six loops terms in the $\beta$-function
of the Wess–Zumino model. Having a full description of the five loop primitives would
however be more demanding: in this case, we would need either the case of two independ-
ent loops with four fermionic propagators or the one of the eight fermionic propagator
loops. Both cases introduce additional hurdles for their evaluation, in particular for the
case of two independent loops for which the chirality constraint is more than just halving
the dimension of the spinor space. And there is the additional question of dealing with
the possibly hundreds of millions of terms in the obtained numerators.

Applications of this formalism to other quantum field theories would also be inter-
esting. In the case of Yang–Mills theories, a path towards parametric representations
of arbitrary diagrams (or rather, of the collection of diagrams sharing a common color
structure) has been made through the introduction of corolla polynomials [19, 9], but
additional variables remain. An explicit evaluation of the effect of the differential op-
erators present in this formalism to obtain expressions involving only the parameters
associated to edges would be interesting, especially if the complete result can be shown
to have a high degree of symmetry. Fermion loops in gauge theories would also add to
the complexity, due to the gamma matrix appearing in the vertex. Finally, we limited
ourselves to the massless case, since it allows for the reduction to the completed graph
with the simple proof of subsection 3.2. Nevertheless, similar identities could be used:
terms with the highest possible number of $V^i$ terms are still the ones with the highest
power of the first Symanzik polynomial in the denominator and any way of reducing
this power would be welcomed.

In the Wess–Zumino model our examples are based on, combining the numerators
for the differing assignations of fields to the edges of the graph made perfect sense, since
it allowed to avoid all spurious divergences and the computations are rather straight-
forward with respect to the reductions based on IBP identities. Nevertheless, reducing
the total number of different integrals would be a bonus. It would be interesting to
find ways of combining the explicit description of numerators in the parametric repre-
sentation presented in this work with reduction methods to try to get the best of both
worlds.

\section{Dodgson identities}

In order to obtain exact results, we need the Dogdson identities in a form which allows
to track signs. They are indeed essential to obtain exact results. In other contexts, the
signs can be irrelevant since the Dogdson polynomials will appear squared or as part of
expressions the sign of which is not important, but it is not the case here. Nevertheless, we want to keep things simple and with signs which depend on a minimum number of choices in the presentation of the diagram. In particular, the usual presentation as the determinant of a submatrix with rows and columns removed keeps a dependence on the order of the rows. The convention we proposed in section 2 for $U_{I,J}$ is the determinant of the matrix $M_G$ with the columns indexed by the set $I$ replaced with ones with only a 1 in the position indexed by the corresponding element of $J$. Independence on the order of the edges or the vertices is clear in this case and expansion according to the columns indexed by $I$ trivially gives back the usual definition up to a sign, while multilinearity can be used to show that the same result can be obtained by changing instead the rows indexed by $J$. Finally, in the case where $I$ and $J$ have only one element, $U_{i,j}$ only depends on the orientations of $i$ and $j$.

Our aim is to express some determinants involving correlators $U_{i,j}$. Examining the proof of Dodgson identities, such as the one found in [2], shows that our definition of $U_{I,J}$ naturally appears in them. Indeed, $U_{i,j}$ is the $j, i$ component of the adjugate matrix of $M_G$, the matrix which, multiplied by $M_G$ gives $\det(M_G)$ times the identity. We want to express the determinant of the matrix $M$ with elements $U_{i,j}$ such that $i$ is in $I$ and $j$ is in $J$. We form a matrix of the dimension of $M_G$ by replacing in the identity matrix the columns with index in $J$ by the ones indexed by $I$ of the adjugate matrix. An expansion along the columns coming from the identity matrix shows that this new matrix has the same determinant as $M$. Multiplying by the left by $M_G$, one obtains a matrix which has the columns of $M_G$ except for the columns indexed by $J$ which have $\det(M_G)$ in the position indexed by $I$. We recognize the matrix used to define $U_{J,I}$, apart from the factors $\det(M_G)$, so that we obtain the required Dodgson identities without any sign ambiguity by taking the determinants.

$$\det(M_G) \det\left((U_{i,j})_{i\in I, j\in J}\right) = \det(M_G)^{\#J} U_{I,J}$$

(39)

B Description of $N4$

We use here the high level of symmetry of $N4$ to give it a not so large verbal description. This description also shows that none of the terms can give a contribution with subdivergences. The possible terms are constrained by the facts that the degree in each variable is at most 2 and the combined powers of the variables linked to the three edges meeting at a given vertex cannot be greater than 3. This in particular precludes that the edges associated to two squared variables may meet in a common vertex.

We first have 6 terms of the type $x^2 y^2 z^2$, with coefficient 1: the three corresponding edges do not have any vertex in common. There are $18 \times 15$ terms of type $x^2 y^2 z t$, still with coefficient 1: there are 18 choices of the edges associated to $x$ and $y$ which have no common vertex, as in the polynomial $N3$. Among the 21 pairs of edges possible among the 7 remaining ones, 6 are excluded: the four pairs which have one of the ends of $x$ or $y$ as a common vertex and the two pairs where $x$ and $z$ have a common vertex, $y$ and $t$ also, but there are no other common vertices between the four edges.

There are then $9 \times 41$ terms with exactly one square. They have coefficient 2. The 9 factor comes from the 9 ways of choosing the edge which corresponds to the squared variable. It remains to choose 4 edges among the remaining eight ones. They are divided between the four edges having a vertex in common with the one associated with the squared variables and the remaining four. The basic rule enunciated in the previous
paragraph then limits to two the number of edges in the first group. There is a first possibility with the four edges in the second group, 4 times 4 possibilities where we choose one edge in the first group and exclude one in the second and finally 4 times 6 where we choose one edge from each of the end of the squared edge and the last two among the four of the second group for the announced total of 41 possibilities. Finally, the 84 possible terms with six different variables are all present, but they come with coefficients 8, 6 or 4. It is easier to describe these terms by the variables which are not included. The 6 terms where the excluded variables correspond to edges without common vertices have coefficient 8, the 36 ones where the excluded edges form an unbranched path have coefficient 4 and the remaining 42 ones have coefficient 6.

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