Electron-positron bremsstrahlung and pair creation in very high magnetic fields

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ABSTRACT

Cross-sections for Rutherford scattering, Coulomb bremsstrahlung and pair creation, have been calculated at very high magnetic fields in order to investigate the photon-production of protons at the polar caps of pulsars whose spin is antiparallel with the polar magnetic flux density. The Landau-Pomeranchuk-Migdal effect at very high magnetic fields is included in a simple electron Green function.

Key words: pulsars: general - stars: neutron - stars: magnetic field

1 INTRODUCTION

Work on quantum electrodynamic processes at magnetic flux densities of the order of the critical field $B_c = m^2 c^3 / e \hbar = 4.41 \times 10^{13}$ G has been reviewed recently, and very completely, by Harding & Lai (2006). The processes concerned are primarily those relevant to collisions within plasma at low altitudes above the surface of a neutron star and therefore do not include relativistic Coulomb bremsstrahlung and pair creation in the interaction of electrons and photons with ions. There appear to be no published cross-sections for these processes at $B \sim B_c$ apart from a number of papers on the bremsstrahlung energy-loss of non-relativistic electrons. But of these, only Lieu (1981) and Lauer et al (1983) have treated bremsstrahlung as a process of second order in the electron-photon coupling and the work of the latter authors is also limited to collisions in which both initial and final electrons are in the lowest Landau state. By contrast, only Lieu (1981) and Lauer et al (1983) have treated bremsstrahlung as a process of second order in the electron-photon coupling and the work of the latter authors is also limited to collisions in which both initial and final electrons are in the lowest Landau state. Bus-sard (1980) and Langer (1981) have considered photon production in non-relativistic electron-ion plasma collisions but have treated it as a sequence of first-order processes, that is, Coulomb excitation of a higher Landau state followed by cyclotron emission.

However, the reverse flux of electrons or positrons produced by pair creation in the open magnetic flux-line region above neutron-star polar caps must produce electromagnetic showers in the condensed matter at the surface whose properties are determined by the relativistic second-order processes. In neutron stars with spin $\Omega$ such that $\Omega \cdot B > 0$, the outward particle flux consists of electrons so that nuclear reactions within the positron-initiated showers have no obvious observable effect. But in the opposite case, $\Omega \cdot B < 0$, the outward particle flux is positively charged and, with positrons, also includes protons that are produced in the shower and diffuse to the surface. It is likely that the temporal behaviour of these different components does lead to instabilities that are the basis of observable phenomena (Jones 2010). In general, there appears to be no reason a priori why both $\Omega \cdot B$ cases should not be present in the neutron-star population and it is of interest to understand how each might be observed in the electromagnetic spectrum. Estimates of shower development in previous work relied on the correspondence principle, that for large values of the Landau quantum number the zero-field cross-sections are valid, but this is unlikely to be satisfactory at $B \sim B_c$ where the number of contributing Landau states can be small. The present paper attempts to give cross-sections that are rather better than order-of-magnitude estimates but are not of high accuracy owing to the discrete nature of the Landau spectrum and to truncations that are made in performing the calculations. Its purpose is to see whether and in what way the cross-sections differ qualitatively from those at zero field. Nucleon production in electromagnetic showers occurs almost entirely through formation and decay of the giant dipole resonance. Thus the paper is concerned, particularly, with factors that influence the low-energy photon track length and its depth distribution.

With neglect of radiative corrections and of the anomalous magnetic moment, the energy states of an electron in a uniform magnetic field are

$$E = \pm \sqrt{p^2 + 1} + 2nB$$

in which $p$ is the longitudinal momentum component, parallel with $B$, and $n = 0, 1, 2, \ldots$ is the Landau quantum number. Apart from the Appendix and where explicitly stated otherwise, momentum and energy are here expressed in units of $mc$ and $mc^2$; the magnetic flux density $B$ is in units of $B_c$. In electron-photon interactions, the conserved quantities are energy and the longitudinal momentum component. Thus the kinematic behaviour of an electron is that of a par-
particle confined to one dimension and of mass \( m \sqrt{1 + p_\perp^2} \) in which we have defined, purely for ease of notation, a notional transverse momentum component \( p_\perp = \sqrt{2mB} \).

We have adopted the solutions of the Dirac equation found by Johnson & Lippmann (1949) and give a summary of calculational details in the Appendix. These are unremarkable except that owing to the one-dimensional kinematics of electrons and the small number of Landau states that contribute significantly at \( B \sim B_c \), we have found it convenient, given that numerical computation must be involved, not to proceed with the usual approach of covariant formalism and Green functions but instead to write down the transition matrix to second order directly in terms of time-ordered matrix elements calculated using an explicit representation of the Dirac matrices. It is also appropriate to mention here that the states given by equation (1) are two-fold degenerate for \( n > 0 \) only if radiative level shifts, natural widths, and the anomalous magnetic moment of the electron are neglected. But for the high-energy spin-averaged or summed processes considered here, the fine structure is of no consequence and we use only the completeness of the Johnson-Lippmann eigenfunctions. References and some further details are given in the Appendix.

The most simple problem here is that of finding the replacement for Rutherford’s scattering formula. Small-angle Coulomb scattering in the zero-field case is replaced at \( B \sim B_c \) by the Coulomb excitation of Landau states. This has been treated by Bussard (1980) and Langer (1981) but only in the non-relativistic limit. The relativistic cross-sections that we require appear not to have been published previously and are given in Section 2. There have been many calculations of transition rates for cyclotron emission, and a summary is contained in the review of Harding & Lai (2006). New transition rates are given in Section 3 to allow comparison with previous work and, more specifically, to obtain the distribution of photon transverse momentum \( k_\perp \) from the decay of \( n > 1 \) Landau states which is of crucial importance in the development of electromagnetic showers at \( B \sim B_c \). Cross-sections for Coulomb pair creation and bremsstrahlung are obtained in Sections 4 and 5. In the case of pair creation, there appears to be no previously published work. For bremsstrahlung, the work of Lieu (1981) and Lauer et al (1983) is in the non-relativistic limit so that comparison is not possible. There is also the further problem that calculations relying on truncation of the number of Landau states lack external tests of their correctness because the zero-field limit is not available. But while minor tests of correctness exist, we find that they are satisfied. Finally, with primary electron energies of the order of \( 10^3 \) GeV and the high condensed matter densities predicted at the neutron star surface, it is essential to consider the Landau-Pomeranchuk-Migdal effect and its influence on Coulomb pair creation and bremsstrahlung cross-sections. Its existence and broad properties at zero field have been verified through experiments at particle accelerators. We refer, in particular, to the recent work of Hansen et al (2004) and to the review of Klein (1999). We find that the effect exists at \( B \sim B_c \) and that the one-dimensional electron kinematics makes possible a particularly simple treatment which is given in Section 6. The cross-section calculations of Sections 4 & 5 have been repeated with allowance for the LPM effect which is shown to significantly change early shower development. The final section gives a qualitative summary of the cross-sections obtained and of the effect of magnetic fields of the order of the critical field on shower development.

2 RUTHERFORD SCATTERING

Small-angle Coulomb scattering at zero field, in which the transverse momentum component is a continuous variable, is replaced by a Landau transition \( n \to n' \) with longitudinal momentum transfer \( q \). Free atoms in high magnetic fields have an axially symmetric but complex electron density distribution. But the departures from uniform electron density in condensed matter are not large except for a small number of inner orbitals and we therefore adopt the potential \( Z e/V \) of a point charge within a spherical Wigner-Seitz unit cell of uniform electron density. Its radius is

\[
r_{WS} = 2.15 \times 10^{-10} Z^{0.23} B^{-0.40} \, \text{cm},
\]

obtained from the ion number density calculated by Medin & Lai (2006) for atomic number \( Z \). Calculational details are summarized in the Appendix. The symmetry of \( \tilde{V} \) indicates that Landau functions in cylindrical polar coordinates are the appropriate choice. The quantum number \( l \) denotes the spatial degeneracy of these states and the procedure is to average the transition rate over all \( l \) in the range \( 0 \leq l \leq l_m \).

Each state has a guiding centre radius \( \sqrt{2l + 1} B R \), where \( RB \) is the cyclotron radius. The incident flux is then \( c \) divided by \( (2l_m + 1) \pi r_B^2 \), and the cross-section at incident momentum \( p > 1 \), with no spin-flip, is the limit \( l_m \to \infty \) of

\[
\sigma_{nL} = \frac{\pi r_B^2 (Z e^2 / \hbar c)^2 (2l_m + 1)}{(l_m + 1)^2} \sum_{l=0}^{l_m} |\langle p + q, n', l' | \tilde{V} | p, n, l \rangle|^2.
\]

This expression is greatly simplified by the selection rule \( \delta(n-l) = 0 \) arising from equation (A3) and from the axial symmetry of \( \tilde{V} \). Spin-flip cross-sections are smaller by a factor of the order of \((g_{L} - p_\perp)^2 / 4E^2\) relative to unity and as our interest is primarily in electrons with \( p \gg mc \) they have not been calculated. The range of initial-state quantum numbers \( l \) should be so large that the cross-section is independent of \( l_m \) in the case of an isolated atom. We have confirmed that, for \( B = B_c \), this condition is satisfied adequately for \( l_m \approx 30 \) at which value guiding centres are at radii approaching the Wigner-Seitz cell radius. Numerical cross-sections \( \sigma_{nL} \) are given in Table 1, for \( n, n' \leq 4 \). They are in units of \( 1 \text{bn} = 10^{20} \text{cm}^{-2} \) and are for \( Z = 26 \) and longitudinal momentum \( p = 20 \), but are almost exactly independent of \( p \). Calculation to the lowest order in \( \tilde{V} \) is satisfactory here, as for the zero-field small-angle Rutherford formula, owing to the pole in the amplitude at zero momentum transfer. Thus the cross-sections are linear in \( Z^2 \).

The cross-sections of Table 1 are valid in the relativistic limit of \( p \gg 1 \) so that comparison with the near-threshold cross-sections found by Bussard (1980) and Langer (1981) is not possible. But the standard Rutherford cross-section at small angles can be expressed in terms of a continuous variable \( n \) through the correspondence principle relation \( p_\perp = \sqrt{2mB} \). It is \( d\sigma / dn = 0.50 Z^2 / n^2 B \) bn. The numerical
### 3 CYCLOTRON EMISSION

Transition rates for relativistic cyclotron emission are obviously relevant to the low-altitude plasma above polar caps at $B \sim B_c$, and therefore have been calculated by a number of authors (Herold, Ruder & Wunner 1982, Latal 1986, Baring, Gonthier & Harding 2005; see also the review of Harding & Lai 2006). The principal reason for our calculation here, which is limited to unpolarized electrons, is to investigate the sequence of partial transition rates by which an electron in Landau state $n$ decays to $n = 0$. These determine the distribution of the transverse photon momentum $k_\perp = k \sin \theta$, where $\theta$ is the photon angle with respect to $B$, and hence electromagnetic shower development at $B \sim B_c$.

In the rest frame of the initial electron, the relation between momentum $k$ and $\theta$ is,

$$k \sin^2 \theta = E_0 - (E_0^2 - \sin^2 \theta(p_\perp^2 - p_\parallel^2))^{1/2},$$

in which $E_0 = \sqrt{1 + p_\perp^2}$. The partial transition rate for unpolarized electrons is

$$\Gamma_{n'n} = \frac{1}{\hbar} \int_{-1}^{1} d(\cos \theta) \frac{k^2}{2\pi \hbar^3 c} E' \left| \frac{1}{2(l_m + 1)} \sum_{s,s'} \sum_{l,l'} \langle l', n', s', k, c|A|n, l, s \rangle \right|^2,$$

summed over photon polarization states $\epsilon_\parallel$ and $\epsilon_\perp$, respectively parallel with and perpendicular to $k \times B$, and over final electron spins. The radiation field vector potential is $A\parallel$ given by equation (A6) and the final-state electron energy is $E'$. The transition rate is expressed here as an average over $l_m + 1$ spatially degenerate initial states and we have verified that the computed rates are $l_m$-independent, as they should be, and have used the values $l_m = 0$ and $l_m' = 12$. The computed values, in units of $10^{-3}$, are given in Table 2 for $n \leq 4$. The maximum photon transverse momentum occurs at $\theta = \pi/2$ and is,

$$k_{\perp,\text{max}} = \sqrt{1 + p_\perp^2} - \sqrt{1 + p_\parallel^2}.$$
Coulomb pair creation in a uniform magnetic field requires a longitudinal momentum transfer $q$ from the nucleus, 

\[ q = k(1 - \cos \theta) - \frac{1}{2p_\|} \left(1 + \frac{p_{\perp+}^2}{p_\|^2}\right) - \frac{1}{2p_\perp} \left(1 + \frac{p_{\perp-}^2}{p_\perp^2}\right), \quad (7) \]

an expression valid for final-state electron and positron momenta $p_{\perp \pm} > 1$, and therefore differs from the zero-field case in being a function of both $k$ and $k_{\perp}$. The momentum transfer is a minimum for $p_\| = p_\perp$ and $l_{\perp+} = p_{\perp+} = 0$ for which conditions it falls to zero at $k_{\perp} = 2$, the kinematic threshold for magnetic pair production. The transition matrix element to second order for an isolated atom is, 

\[ M_{\|\perp}^P = \langle p_{\perp-} n_- l_{\perp-} s_- | - Z e^2 \hat{V} | p_\| - q, n_- l_{\perp-} s_- \rangle \]

\[ - (E_\| (p_\|) - E_{\perp} (p_{\perp-} - q + i\eta))^{-1} \]

\[ \langle p_\| - q, n_- l_{\perp+} s_+ | \mathbf{\mathbf{e}_z} \cdot \mathbf{\hat{A}} | p_{\perp+}, n_+, l_{\perp+}, s_+ \rangle \]

\[ \sum_{l_{\perp-}} \sum_{n_-=0}^{l_{\perp-}} \sum_{l_{\perp+}}^{l_{\perp+}} |M_{\|\perp}^P|^2 \quad (9) \]

for pair creation by a photon of momentum $k$, transverse momentum $k_{\perp}$, and of polarization $e_{\|}$ or $e_{\perp}$. In this expression, the charges have been factored out from the matrix element (8). There are a number of approximations here that merit some consideration. Firstly, truncation of the Landau quantum numbers to $n_{\perp \pm} \leq 4$ has been tested by examination of the computed cross-section as a function of $n_m$ for $0 \leq n_m \leq 4$. The basis for this truncation is that, for fixed $k$, the amplitude given by equation (8) is approximately $\propto p_{\perp+}^{-1}$, also that matrix elements tend rapidly to zero as they are constructed between states with increasingly different quantum numbers and hence nodal surfaces.

There is evidence of reasonably satisfactory convergence, but cross-sections obtained here should be seen as lower limits. Truncation of the sum over the spatial degeneracy quantum numbers to $l_{\perp} \leq 12$ again produces fair convergence. But in this case, there is a natural upper limit determined by the radius of the Wigner-Seitz cell. Finally, the matrix elements of the Coulomb potential $Z e^2 \hat{V}$ have been limited to those with $\delta n = 0$. This is satisfactory, for the purposes of the present work, because these are typically much larger than off-diagonal matrix elements, as shown by the cross-sections of Table 1. But errors arising from this latter truncation are not positive-definite.

Numerical values of the isolated-atom total cross-sections are given in Table 3 for ranges of values of $B$, $k$ and $k_{\perp}$ and for both photon polarizations. They are for $Z = 26$ but being of second order in electron-photon coupling, are linear in $Z^2$. The differential cross-sections are unremarkable symmetric functions of the final-state energies $E_\perp$, not strongly peaked within the kinematic limits present in the integral of equation (9). They are not given here primarily because in the relevant case of condensed matter they may be unpredictably distorted. The explanation for this is a consequence of the partial order which may be present in neutron-star surface condensed matter, and is deferred until Section 7.

A problem in this Section is that there are no external tests of the validity of expressions derived from equations (8) and (9) or of their numerical evaluation. But we note that the differential cross-sections are symmetric in $E_\perp$ and that under the restriction $n_m = 0$, the cross-section is $\sigma_{\|0}^P = 0$ for photons polarized parallel with $k \times \mathbf{B}$. This is consistent with the fact that for this polarization, the matrix element (A7) should be exactly zero for $n_\perp = n_\| = 0$ (see Semionova & Leahy 2001).

The cross-sections in Table 3 are quite slowly varying functions of $k$ and can be compared with the asymptotic $k \gg 1$ Bethe-Heitler value of $\sigma^P = 5.0$ bn for unpolarized photons incident on a screened $Z = 26$ nucleus in zero magnetic field. The correspondence principle indicates that the Bethe-Heitler expression with modified screening should be a fair approximation at moderate fields, of the order of $10^{12}$ G, for which the Landau quantum number associated with $p_{\perp \pm} \sim 1$ is large. It is difficult to obtain cross-sections for fields between $10^{12}$ G and those of the Table because the truncations we have used are unlikely to be satisfactory approximations. But as a function of decreasing $B$, the
cross-sections in the Table are not inconsistent with smooth convergence at intermediate fields toward the Bethe-Heitler value. The most noticeable features of the Table are: (i) the rapid increase of cross-section as $k_\perp$ approaches the $n\perp$-dependent thresholds for magnetic pair production at which $q$ and hence the energy denominators in equation (8) become zero; and (ii), the rapid decrease of cross-sections at $B > B_c$.

5 BREMSSTRAHLUNG

The emission of a photon of momentum $k$ at an angle $\theta$ with $B$, and a change of Landau quantum number $n \rightarrow n'$, requires a longitudinal momentum transfer from the atom of

$$q = -k(1-\cos\theta) + \frac{1+p^2l}{2p} - \frac{1+p'^2l}{2p'},$$

where the initial electron longitudinal momentum is $p \gg mc$ and the final momentum is $p' = p + q - k \cos\theta$.

The transition matrix element, to second order for an isolated atom, is

$$M_{\parallel\perp}^{B} = \langle p', k, n', l', s' | \alpha \cdot \hat{A} | p, q, n, l, s \rangle$$

$$= (E(p) - E(p + q) + i\gamma)^{-1}$$

$$\langle p + q, n, l, s | - Z e^2 \hat{V} | p, n, l, s \rangle +$$

$$\langle p', n', l', s' | - Z e^2 \hat{V} | p' - q, n', l', s' \rangle$$

$$= (E(p) - E(p' - q) + i\gamma)^{-1}$$

$$\langle p' - q, k, n', l', s' | \alpha \cdot \hat{A} | p, n, l, s \rangle,$$

in which the subscripts $\parallel$ and $\perp$ again refer to the photon polarization with respect to $k \times B$. As in the previous Section, and for the same reason, we have neglected two time-ordered terms in the amplitude that involve virtual pair creation in the Coulomb field. The existence of single-photon magnetic pair creation thresholds implies that the bremsstrahlung cross-section in a high magnetic field is best expressed in terms of the independent variables $k$ and $k_\perp$.

The cross-section defined by the energy-loss rate (equivalent to the radiation length) is then given by an integration over photon energy,

$$\sigma_{Rad} = \int_0^{k_m} dk \frac{k \, d\sigma_B}{p} dk,$$

in which the differential bremsstrahlung cross section for an electron in Landau state $n$, summed over both photon polarization states, is,

$$\frac{d\sigma_B}{dk} = (2l_m + 1)\frac{\alpha}{\pi} Z^2 \frac{\gamma}{\hbar c} \left( \frac{1 + \delta_{n0}}{4} \right) \int \frac{k_\perp}{k} dk_\perp$$

$$\sum_{l,l'=0}^{l_m} \sum_{s,s'=0}^{l} M^B_{l,l'}^{s,s'}|^2,$$

The summations have been truncated to $n_m = 4$ and $l_m = 12$, as in the case of pair creation, but this approximation is much less satisfactory for bremsstrahlung and has lead us to omit values for $B < B_c$. Bearing in mind the importance of the variable $k_\perp$, the cross-section $\sigma_{Rad}$ given by equations (12) and (13) has been divided into two components: $\sigma^B_{Rad}$ is for the interval $0 < k_\perp < 2.5$ and $\sigma^S_{Rad}$ for $2.5 < k_\perp < 5.0$ mc. The computed values in units of $10^{-24}$ cm$^{-2}$, for electrons in an initial $n = 0$ Landau state incident on isolated atoms are given in the upper sector of Table 4. They are summed over both photon polarizations, but we note that the $\gamma_{\perp}$ cross-section is typically an order of magnitude larger than that for $\gamma_{\parallel}$. The computed values are almost completely independent of $p$ and there is little cross-section outside the intervals of $k_\perp$ considered. The total cross-section is simply defined here as the sum of $\sigma^B_{Rad}$. Cross-sections for initial Landau states $n > 0$ are of similar magnitude but are not given here. These initial states are, of course, naturally unstable against photon decay, which appears as a zero in the denominators of equation (11) and is discussed further in Section 7.

6 THE LANDAU-POMERANCHUK-MIGDAL EFFECT

In both this and previous work, we have adopted expressions for the zero-pressure ion number density,

$$N = 2.5 \times 10^{28} Z^{-0.7} B^{1/2} \text{ cm}^{-3},$$

and for the interatomic separation in a linear chain, parallel with $B$,

$$a_s \approx 1.5 \times 10^{-10} Z^{1/2} B^{-1/2} \text{ cm},$$

conveniently summarizing the calculated values of Medin & Lai (2006). We anticipate that the bremsstrahlung and pair creation cross-sections obtained in previous Sections for isolated atoms will be considerably modified in the condensed
We refer to Section 6 for further details.

| p  | B  | $\sigma_{1}^{Rad}$ | $\sigma_{2}^{Rad}$ | $\sigma^{Rad}$ |
|----|----|-------------------|-------------------|---------------|
| 200| 1.0| 4.83              | 1.85              | 6.68          |
|    | 3.0| 0.77              | 1.44              | 2.21          |
|    | 10.0| 0.08             | 0.25              | 0.33          |
| 2000| 1.0| 4.82              | 1.85              | 6.67          |
|    | 3.0| 0.77              | 1.44              | 2.21          |
|    | 10.0| 0.08             | 0.25              | 0.33          |
| 20000| 1.0| 4.82              | 1.85              | 6.67          |
|     | 3.0| 0.77              | 1.44              | 2.21          |
|     | 10.0| 0.08             | 0.25              | 0.33          |

Table 4. Numerical values of the cross-section equivalent to the radiation length, defined by equation (12), are given in the upper sector of the Table for electrons in the $n = 0$ Landau state incident on isolated atoms of $Z = 20$. The values of electron momentum $p$ are in units of $mc$, and $B$ is in units of the critical field $B_{c} = 4.41 \times 10^{13}$ G. Columns 3 and 4 are partial cross-sections defined for the photon transverse momentum intervals $0 < k_{\perp} < 2.5$ and $2.5 < k_{\perp} < 5.0$ mc, respectively. The total cross-section in column 5 is defined here as the sum of these partial cross-sections. The 15 rows in the lower sector of the table give the above cross-sections re-calculated in condensed matter assuming a static structure function $S(q) = 1$ but with inclusion of the Landau-Pomeranchuk-Migdal effect. We refer to Section 6 for further details.

The essence of the effect is that, for example in bremsstrahlung and at high energies, the isolated-atom longitudinal coherence length (of the order of $q^{-1}$) can be so long that the coherence that would be present in the case of an isolated atom is partially removed by multiple scattering of the initial or final state electron within the medium. We refer to the review by Klein (1999) for further details, and for a discussion of other processes in condensed matter, such as the dielectric effect, that are not relevant here. In the present Section, we examine the LPM effect at high magnetic fields and find that the one-dimensional nature of kinematics at $B \sim B_{c}$ allows a particularly simple description.

We begin by considering the effect of scattering of an electron with initial longitudinal momentum $p \gg mc$ by a (dimensionless) potential $U^1 = -Z e^2 V / hc$ whose origin is at $z = z_{i}$. We consider transitions with no change of Landau quantum number so that the final state is given by the Green function,

$$G = -i\theta(t-t') \sum_{p' > 0} \psi_n(r_{\perp}) \psi_{n'}(r'_{\perp}) e^{ip'(z-z')-iEt'(t-t')},$$

constructed from the Landau functions given by equation (A3). The truncation to $\delta n = 0$ is possible because, as we found in Section 2, the $\delta n \neq 0$ transition rates are at least an order of magnitude smaller. It also excludes spin-flip and back-scattering ($p' < 0$) which are both negligible in the present context. The scattered state is

$$\int dq dq' dt t' G(r, t, r', t') U^1 \psi_{n'}(r'_{\perp}) e^{ip'z'-iEt'} =$$

$$\int dq \frac{U^1_{r}}{2\pi - q + i\eta} e^{i(q(z-z_{i})} \left( \psi_{n}(r_{\perp}) e^{ipz-iEt} \right),$$

in which $U^1_{r}$ is the matrix element of $U^1$ between states $p$ and $p' = p + q$, and $\eta > 0$ is infinitesimal. It is reached by introducing the integral representation of the Heaviside function $\theta$, followed by integration over $r'$ and $t'$. Equation (17) shows that the electron has components of momentum $p + q$ and that these are significant at distances $z-z_{i} < q^{-1}$. Either by performing the integral, or taking the limit $z \to \infty$, we can see that the prefactor outside the brackets evolves to an amplitude $U^1_{0}$, as expected. Apart from the forward-scattered component of the wave-function given by equation (17), flux conservation shows that there is also an undisturbed component of amplitude $(1 - |U^1_{0}|^{2})^{1/2}$. Equation (17), giving the forward-scattered state, is no more than a concrete expression of the uncertainty principle, but it shows that forward scattering by a sequence of atoms, though unobservable in the motion of the electron itself, has the effect of broadening the distribution of $q$ and so reducing the small-$q$ amplitudes that mediate most of the bremsstrahlung cross-section at high energies.

The simplicity of equation (17) enables us to find the Green function for an electron undergoing a sequence of forward scatters and so provides an easy way to calculate bremsstrahlung cross sections including the LPM effect. We begin by noting that the distribution in the variable $q$ in equation (17) is unchanged in form by multiple forward scattering as is shown by the amplitude for a second and successive scattering at $z_{j}$,

$$\int_{-\infty}^{\infty} \frac{du}{2\pi} \int_{-\infty}^{\infty} \frac{dq}{2\pi} e^{i(u(z-z_{j}))} U^1_{r} e^{iq(z-z_{i})} U^1_{r'} =$$

$$i \int_{-\infty}^{\infty} \frac{dU^1_{r}}{2\pi} \frac{dU^1_{r'}}{2\pi} e^{i(u(z-z_{j}))} \frac{U^1_{r}}{-q + i\eta}.$$
square value of $|U_0^l|^2$, satisfies the condition $\langle |U_0|^2 \rangle \ll 1$, as it does here. Therefore, we can find the Green function that couples the electron to a final state of total longitudinal momentum $K_z$. It is given by,

$$
\sum_j \int_{z_j}^\infty \text{d}z e^{-iK_z z} \int_{-\infty}^{\infty} \frac{\text{d}q}{2\pi} \frac{U_0^j}{q - q + i\eta} e^{i(qz_0 + iq(z - z_j)) - (z - z_j)/2\lambda} = \sum_j \left( \frac{-1}{p - K_\parallel + i/2\lambda} \right) U_0^j e^{i(p - K_\parallel)z_j},
$$

in which $\lambda = \langle a \rangle /\langle |U_0|^2 \rangle$, where $\langle a \rangle$ is the mean interatomic separation $z_j - z_{j-1}$ and we have integrated first over $z$ and then over $q$ by completion of the contour in the semi-infinite upper half of the complex $q$-plane. The right-hand side of equation (19) contains, within brackets, the momentum-space Green function for an electron propagating without back-scattering, spin-flip or space Green function for an electron propagating without

$$
\langle |U_0|^2 \rangle
$$

equation (19) contains, within brackets, the momentum-structure function for a linear chain of $N_a$ homogeneous atoms in which $\langle a \rangle$, the mean interatomic separation $jz - z_{j-1}$ and eventually the $\lambda$ in which this cancellation is not present, so producing the square value of $|U_0|^2$. The revised cross-sections given in the lower sectors of these Tables are for $Z = 26$ and require some comment.

In the case of pair production, the re-calculated cross-sections are given in Table 3 just for $B = B_c$ and show that, owing to the very high matter density ($\sim 2 \times 10^{12}$ g cm$^{-3}$) the LPM effect becomes noticeable at $k = 4000$ and reduces cross-sections by more than an order of magnitude at $k = 40000$. However, its effect on bremsstrahlung is less significant. Its effect can be studied simply by reference to the structure factor given by equation (21) neglecting, for the moment, any inhomogeneity in atomic number. Consider, for example, the structure factor for a finite linear chain of $N_a$ homogeneous atoms with spacing $z_j - z_{j-1} = a_s$,

$$
S(q) = \frac{1}{N_a} \left( \frac{\sin(N_\alpha qa_s/2)}{\sin(qa_s/2)} \right)^2.
$$

This has maxima, $S(q) = N_a$, when $q$ coincides with an integral multiple of the basic reciprocal-lattice wavenumber $g = 2\pi/a_s$, but has zeros at $\sin(qa_s/2) = 0$. Its effect is primarily to significantly distort spectral shapes given by isolated-atom differential cross-sections. Unfortunately, it is not easy to be specific about changes in total cross-sections because they are dependent on the (unknown) structure function and on all other parameters, but with a change of variable in a particular differential cross-section to $q$ we can note that the modified total cross-section is,

$$
\sigma^S = \int S(q) \frac{\text{d}q}{dq} dq,
$$

taken over the complete interval of $q$ for the process, and that this differs from the $S = 1$ cross-section by a factor of order unity, not $N_a$.

7 CONCLUSIONS

The cross-sections obtained here are those that, with Compton scattering, determine electromagnetic shower development. For magnetic flux densities of the order of $10^{12}$ G, Landau quantum numbers $n \sim 10^2$ are associated with transverse momenta $p_\perp \sim mc$. Thus, on the basis of the correspondence principle, zero-field expressions are used with some degree of confidence. But in many neutron stars, fields more than two orders of magnitude larger have been inferred from the observed spin-down rate. For this reason, it is necessary to be aware of any field-dependence of these cross-sections that would qualitatively change the nature of shower development. In this respect, the processes that appear not to have been considered previously are those of Sections 2, 4 and 5. Of these, Rutherford scattering merits little comment except to note that its cross-section decreases with increasing magnetic field in the region $B \sim B_c$. Pair creation is a little more complex because the amplitude (equation 8) becomes singular at the threshold values of $k_\perp$ for magnetic pair creation, beyond which it becomes no more than a Coulomb field-dependent correction to that process. Thus we are concerned with the cross-section for $k_\perp$ below the first two thresholds discussed following equation
(6) in Section 3. This is the explanation for the intervals of $k_\perp$ adopted for columns 3-5 of Table 3.

Bremsstrahlung at $B \sim B_c$ is an even more complex process. It is of second order in the coupling of the electron with the electromagnetic field, as at zero field. But there are two significant differences. The Table 4 cross-sections are for an electron initially in the $n = 0$ Landau ground state, but in the development of a shower in condensed matter, higher-$n$ states are certainly populated owing to the very large Rutherford cross-sections listed in Table 1. Secondly, a pole in the amplitude (equation 11) for such initial states represents the possibility of real intermediate states of $n' \geq 1$. The process we refer to as bremsstrahlung then consists of two separate first-order processes; Rutherford scattering followed by cyclotron emission with the natural widths given in Table 2. The initial electron momentum determines whether or not this is more important than the second-order process. An estimate of the critical value is easily made from the cyclotron emission rates in Table 2 which are all of the order of $10^{-3} \sigma_B$. Although the mean free path $l_{\text{Rad}}$ for Rutherford scattering is always small compared with the radiation length $l_{\text{Rad}}$ which is characteristic of the second-order bremsstrahlung process, time dilation can extend the mean flight path $l_{\text{ce}}$ for natural decay by cyclotron emission to lengths greater than $l_{\text{Rad}}$. Then the first-order processes are the dominant source of photon production provided the initial-state momentum is,

$$\frac{p}{mc} < \frac{l_{\text{Rad}}}{c} \sum_{n'=0}^{n-1} \Gamma_{nn'}.$$  

(23)

From the transition rates given in Table 2 and the values of $\sigma_{\text{Rad}}$ given in Table 4, we can see that its value is not strongly $B$-dependent. Specifically, for $B = B_c$, $Z = 26$, $\sigma_{\text{Rad}} = 22$ bn and $l_{\text{Rad}} = 1.8 \times 10^{-5}$ cm with inclusion of the LPM effect, so that for $n = 4$, the inequality (23) becomes $p < 1100$ mc. At lower momenta, the process of bremsstrahlung becomes simply a sequence of excitation (or de-excitation) of higher-$n$ Landau states by Rutherford scattering and de-excitation by cyclotron emission. At even lower momenta, the balance between excitation and de-excitation changes and the typical values of the Landau quantum number decrease until for $l_{\text{Rad}} \approx l_{\text{ce}}$, that is, for

$$\frac{p}{mc} \approx \frac{l_{\text{Rad}}}{c} \Gamma_{10}.$$  

(24)

they are $n = 1$. With the cross-section $\sigma_{10}$ from Table 1, we find $l_{\text{Rad}} = 4.0 \times 10^{-7}$ cm. The transition rate $\Gamma_{10} = 1.06 \times 10^{-3} \omega_B$ gives $p \approx 11$ mc.

The above summary of the properties of the bremsstrahlung and pair creation processes at high magnetic fields serves to show qualitatively how the characteristics of electromagnetic showers differ from the zero-field case. In the initial stages of an electron-initiated shower of about $10^3$ GeV, the Landau-Pomeranchuk-Migdal (LPM) effect is far more effective in reducing cross-sections owing to the very high density of the condensed matter at the neutron star surface. Therefore, shower development is displaced inward until typical electron or photon energies are reduced to the extent that the effect becomes unimportant. At this stage, electron energy loss becomes rapid, occurring within lengths much shorter than $l_{\text{Rad}}$. Photon mean free paths are dependent on $k_\perp$, but below the lowest threshold for magnetic pair creation at $k_\perp < 2$ they are a little larger than the value of given by the zero-field Bethe-Heitler cross-section.

The principal interest in shower development here is in proton production by decay of the nuclear giant dipole resonance (GDR). Cross-sections for its electro-production are small so that the important shower property is the distribution of total photon track length in the neighbourhood of the resonant momentum, $k \approx 41$ mc. An estimate of this, made under the assumptions that bremsstrahlung occurs through the sequence of first-order processes described above and that the Bethe-Heitler pair creation cross-section is valid, was given by Jones (2010). We have found that shower development at $B \sim B_c$ is more complex than was assumed in that work, the important question being the fraction of photon production in the GDR region that has transverse momentum below the magnetic pair creation thresholds at $k_\perp = 2$ for Landau quantum numbers $n_+ = n_- = 0$ and $k_\perp = 1 + \sqrt{1 + 2B}$ for $n_+ = 0$ with $n_- = 1$.

It is significant that cyclotron decay, which is dominant in the later stages of shower development, always leads to a decrease in Landau quantum number, so bringing about a convergence of the shower toward small $k_\perp$. The only qualification here is the tendency, noted by Harding & Lai (2006) and confirmed in Table 2, for partial transition rates direct to the ground state $n = 0$ to be large at $B \gg B_c$. Complete and quantitative calculations of shower development would require an expansion of Tables 1, 2 and 4 to much larger initial-electron values of the Landau quantum number, and a Monte Carlo calculation using them would be a substantial undertaking which has not been attempted here. Qualitatively, we can conclude that the photon track length distribution assumed in previous work is valid at $B \approx B_c$. For very high fields at $B \gg B_c$, the pair creation cross-sections become small and the photon mean free path at transverse momenta below magnetic pair creation thresholds are then determined by Compton scattering. Therefore, we expect that the effect of the large cyclotron partial transition rates to $n = 0$ will begin to reduce total photon track length in the GDR region of momentum at $B \gg B_c$.

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APPENDIX A: SOLUTIONS OF THE DIRAC EQUATION

The solutions used here are the spinors obtained by Johnson & Lippmann (1949) for a free electron in a uniform magnetic field with the gauge $A = (B \times r)/2$. As these authors noted, they can be easily expressed in terms of any complete set of eigenfunctions of the non-relativistic Schrödinger Hamiltonian, whether in cartesian or polar coordinates. The Dirac equation is,

$$ (\alpha \cdot \pi + \beta m) \Psi = E \Psi, \quad (A1) $$

in which $\pi = p + eA$ with $e > 0$ and $c = 1$. The solutions for $E > 0$ and for Landau quantum number $n \geq 0$ are,

$$ \Psi_{-1,n} = \begin{bmatrix} 0 \\ (m + E)\psi_{n-1} \\ p_\perp \psi_{n-1} \\ -\psi_{n-1} \end{bmatrix}, \quad \Psi_{1,n-1} = \begin{bmatrix} (m + E)\psi_{n-1} \\ 0 \\ \psi_{n-1} \\ p_\perp \psi_{n-1} \end{bmatrix} $$

each multiplied by the common normalization factor $G = 1/\sqrt{2E(E+m)}$. The longitudinal momentum component is $p$. These solutions have eigenvalues given by equation (1) and are two-fold degenerate for $n > 0$, but they are not identical with the true physical states owing to radiative corrections that have been neglected. Natural line-widths, reflecting the shape of the electron orbitals in the magnetic field. (We neglect any small component with a lat-

The negative energy solutions given by the transformation $\Psi \to \Psi \rightarrow -\Psi$, for $n > 0$, are also two-fold degenerate. In addition, the quantum number $l$ represents the infinite spatial degeneracy of all Landau states whose guiding centres can be at radii $\sqrt{2l+1}r_B$ from the z-axis. The polynomials $f_n$ are,

$$ f_0 = 1, \quad f_1 = l - \xi, \quad f_2 = l(l-1) - 2\xi + \xi^2, \quad f_3 = l(l-1)(l-2) - 3l(\xi - 3\xi^2 - \xi^3), $$

and $f_4$ is

$$ + 4l(l-1)^2 - 4l^3 + 4l^4 $$

(A4)

in obvious sequence and they satisfy,

$$ f_n(l, \xi) = (-\xi)^{n-l} f_l(n, \xi), \quad (A5) $$

which is worth noting in relation to states with $l < n$.

We adopt gaussian cgs units so that the fine structure constant is $e^2/\hbar c$. Perturbations of the correct sign are introduced into equation (A1) by modifying the energy-momentum four-vector, $A \to A + \hat{A}$ and $E \to E - Ze^2\tilde{V}$, to include the radiation field $\hat{A}$ and the atomic Coulomb potential for nuclear charge $Z$. Thus the absorption of a photon with wave-vector $k$ and angular frequency $\omega$ is represented by the matrix element of

$$ e \alpha \cdot \hat{A} = \left( \frac{2\pi\hbar c^2}{\omega} \right)^{1/2} e \alpha \cdot e^{ik \cdot r}, \quad (A6) $$

in which $\epsilon$ is the polarization vector either parallel with, or perpendicular to, the vector $k \times \hat{B}$.

The creation of a pair in states specified by $(p_\perp, n_\perp, l_\perp, s_\perp)$, as in lines three and six of equation (8), is given by the matrix element

$$ \int r_\perp dr_\perp d\phi dz\Psi_{-s_\perp,n_\perp}^\dagger(p_\perp, E_-, l-) e\alpha \cdot \hat{A} \Psi_{s_\perp,n_\perp}(-p_\perp, -E_+, l_+), \quad (A7) $$

in which the positron energy is $E_+ > 0$. Evaluation of this proceeds by matrix multiplication and immediately yields, for each set of values of $s_\perp$ and of photon polarization, a linear combination of terms each of which, apart from kinematic factors, contains an integral of the form

$$ I(k_\perp, n_\perp, l_\perp, n_+, l_+) = \int r_\perp dr_\perp d\phi d\omega \psi_{s_\perp,l_\perp}(r_\perp) \psi_{n_+,l_+}(r_\perp). $$

These expressions, and the integrals contained in them, have all been evaluated numerically. Therefore, we are unable to give useful analytical expressions for any of the cross-sections calculated in this work. Evaluation is the more tedious because the transverse dipole approximation is not adequate so that there is no radiative transition selection rule that is effective for physically significant values of $k_\perp$. But
matrix elements of the Coulomb potential satisfy $\delta(n-l) = 0$
strictly for isolated atoms; also in condensed matter to the
extent that components with the symmetry of the crystalline
field are small.

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