Visualization of internal forces inside the proton in a classical relativistic model

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Abstract: A classical model of a stable particle of finite size is studied. The model parameters can be chosen such that the described particle has the mass and radius of a proton. Using the energy-momentum tensor (EMT), we show how the presence of long-range forces alters some notions taken for granted in short-range systems. We focus our attention on the 2D and 3D distributions of energy, angular components of the EMT, as defined by the stress tensor, i.e. the total pressure, \( p(r) \) and shear force \( s(r) \), are defined through components of the stress tensor, i.e. the \( T_{ij} \) components of the EMT, as

\[
T_{ij} = \left( e_i^p e_j^p - \frac{1}{3} \delta_{ij} \right) s(r) + p(r) \delta_{ij},
\]

where \( e_i^p \) is the unit vector in the radial direction. The total pressure, \( p(r) = p_{\text{scal}}(r) + p_{\text{vec}}(r) + p_{\text{Coul}}(r) \), receives contributions from fields which are given by

\[
p_{\text{scal}}(r) = -\frac{1}{6} \phi'(r)^2 - \frac{1}{2} m_S^2 \phi(r)^2,
\]

\[
p_{\text{vec}}(r) = \frac{1}{6} V_0'(r)^2 + \frac{1}{2} m_V^2 V_0(r)^2,
\]

\[
p_{\text{Coul}}(r) = \frac{1}{6} A_0'(r)^2.
\]

As can be seen in Eqs. (2–4), the scalar meson contribution is always negative, which corresponds to attractive forces directed towards the inside. On the other hand, the contributions of the vector mesons and the Coulomb field are always positive, which corresponds to repulsive forces directed towards the outside. When we integrate \( \int_0^\infty dr r^2 p_i(r) \) we get

\[
T_{ij} = \left( e_i^p e_j^p - \frac{1}{3} \delta_{ij} \right) s(r) + p(r) \delta_{ij},
\]
−10.916 MeV from the scalar fields, 10.891 MeV from the vector field, and a miniscule 0.025 MeV from the Coulomb field. This reflects that the proton is a bound state of strong forces and the electromagnetic contribution plays a minor role. But no contribution, no matter how small, can be neglected as these numbers must add up exactly to zero and fulfill von Laue condition,
\[ \int_0^\infty dr r^2 p(r) = 0, \]
which shows that the internal forces balance each other and is a necessary condition for mechanical stability \([21]\). The von Laue condition is exactly satisfied in the BB-model \([1, 2]\).

The shear force is \( s(r) = \phi'(r)^2 - V_0'(r)^2 - A_0'(r)^2 \). Notice that the dust particles do not contribute to \( s(r) \) and \( p(r) \). The pressure and shear force are not independent but connected by \( \frac{3}{2} s(r) + \frac{\alpha}{2} p(r) = 0 \) due to EMT conservation. The model results are shown in Fig. 1b. The pressure inside the proton obtained from this model is an order of magnitude smaller than in the chiral quark soliton model \([21]\) or that inferred from experiment \([18]\). This is because the BB-model is based on “residual nuclear forces” which are about an order of magnitude weaker than the strong forces among quarks inside the proton.

The results for \( s(r) \) and \( p(r) \) in Fig. 1b are qualitatively very similar to what was found in other theoretical studies \([21, 42]\). In order to see the impact of long-range forces, it is necessary to look more closely at the region of large \( r \) which we shall do in the next section.

3. Effects of long-range forces on the EMT

In previous studies of strongly interacting systems governed by short-range forces, three common features were observed. The first feature is that the shear force is always positive. The second feature is that the pressure has one node at some point \( r_0 \) with \( p(r_0) > 0 \) when \( r < r_0 \) and the pressure is less than zero for \( r > r_0 \). This property arises from the fact that the pressure must have at least one node to satisfy the von Laue condition, and the ground state exhibits a single node. Finally, the combination of \( \frac{3}{2} s(r) + p(r) \), which is normal force per unit area, is always positive.

The BB-model is different from other studies, as it includes long-range Coulomb forces. From the model expressions for \( T_{00}(r) \), \( s(r) \) and \( p(r) \), we obtain the long-distance behavior which holds numerically for \( r > 2 \) fm,

\[ T_{00}(r) = \frac{1}{2} \frac{\alpha}{4\pi} \frac{\hbar c}{r^4} + \ldots, \]
\[ s(r) = -\frac{\alpha}{4\pi} \frac{\hbar c}{r^4} + \ldots, \]
\[ p(r) = \frac{1}{6} \frac{\alpha}{4\pi} \frac{\hbar c}{r^4} + \ldots, \]

where the dots indicate contributions from the strong fields which are exponentially suppressed, and \( \alpha \) is the fine-structure constant. We observe that \( T_{00}(r) \) is always greater than zero which is in agreement with all prior studies. Because of the \( \frac{1}{r^4} \) decay of \( T_{00}(r) \), the total energy converges but the mean square radius of the energy density diverges.

In Fig. 1b we saw that \( s(r) \) is positive, which agrees with prior studies. But this is true only up to about 2.1 fm at which point \( s(r) \) changes sign as shown in Fig. 1c. Similarly, the picture of the pressure in the BB-model in Fig. 1b agrees with observations in other studies with \( p(r) \) turning from positive to negative around 0.8 fm. However, looking more closely in the region of larger \( r \) we see that \( p(r) \) exhibits a second node around 2.4 fm, and then remains positive. For completeness, we remark that the normal force, \( \frac{3}{2} s(r) + p(r) \), exhibits an unusual feature and turns negative in the large \( r \) region \([1]\).

In view of what has been learned from other studies based on short-range forces, these three features are counter-intuitive. It is an important observation that the presence of long-range interactions introduces new features which have not been observed in prior studies of EMT densities. One important practical implication is the divergence of \( D \)-term which we shall review in the next section.

**Figure 1.** EMT densities in the BB-model \([1]\). (a) \( T_{00}(r) \) (total) vs. \( r \). (b) \( p(r) \) and \( s(r) \) (total) vs. \( r \) in the region of smaller \( r \) (\( r \lesssim 2 \) fm). (c) \( r^4 p(r) \) and \( r^4 s(r) \) at very large \( r \) (\( r \gtrsim 2 \) fm), where we see the new features (the power \( r^4 \) is included to enhance the features).
4. Divergence of the $D$-term

The $D$-term, “the least known global property [13],” is given in terms of two equivalent definitions (arising from EMT conservation) in terms of shear force and pressure,

$$D = -\frac{4}{15} M \int d^3 r \, r^2 s(r) = M \int d^3 r \, r^2 p(r) . \quad (9)$$

The Coulomb contributions to $s(r)$ and $p(r)$ are minuscule in the region $r < 2 \text{ fm}$, see Fig. 1b, giving the impression that the electromagnetic interaction plays a very small role for the description of the structure of a charged hadron. However, small, the Coulomb contribution cannot be ignored, as it tells is that there is an electric charge. Especially at large $r$, the long-range $\frac{1}{r}$ behavior of the Coulomb contribution takes over which has an important impact on the $D$-term. Because of the asymptotic behavior of $s(r)$ and $p(r)$ at large $r$ in Eqs. (7,8), both expressions for the $D$-term in (9) diverge.

The fact that the $D$-term diverges due to long-range forces is a new result, which has not been seen in prior studies.

In order to obtain a finite (“regularized”) value for the $D$-term, one can introduce a regularization prescription. A unique regularization method can be derived by observing that, if the integrals were finite, then any linear combination of the two equivalent expressions in Eq. (9) would give the same expression for $D$. However, the divergence can be removed by considering one and only one linear combination which leads to finite regularized result for $D$, namely

$$D_{\text{reg}} = M \int d^3 r \, r^2 \left[ \frac{4}{9} s(r) + \frac{8}{3} p(r) \right] . \quad (10)$$

Numerically, we find $D_{\text{reg}} = -0.317$, i.e. this regularization method preserves the negative sign of the $D$-term that has been observed in all prior studies. The numerical value is about an order of magnitude smaller than e.g. in the quark soliton model [21], which is expected as the BB-model is based on “residual nuclear forces” that are weaker than the strong interactions among quarks. It would be interesting to see if other methods exist to regularize these divergences.

The form factor $D(t)$ in the BB-model is negative in a wide range of $t$. Only when $(-t) \lesssim 2.8 \times 10^{-4} \text{ GeV}^2$ does it become positive, and diverges like $D(t) \sim 1/\sqrt{-t}$ for still smaller $t$ [1]. Such small momentum transfers are currently beyond experimental reach. Noteworthy, the regularized value $D_{\text{reg}}$ together with a quadrupole fit, provide a very good approximation to the exact numerical model results for $D(t)$ which confirms the practical usefulness of the regularization method [1].

5. Model independent conclusions

Our results for the EMT densities are model dependent in the region $r < 2-3 \text{ fm}$, where the strong forces dominate. However, at $r \gg 3 \text{ fm}$, exact QED calculations yield the same EMT density results as us, since QED has to reproduce Maxwell’s classical theory at long distances. In particular, the results in Eqs. (6, 7, 8) are model independent and were obtained in QED calculations [44,45]. The divergence of $D(t)$ at small $t$ due to QED effects was also found in chiral perturbation theory calculations for charged pions [46]. When comparing our results for $D(t)$ to those found using effective field theory techniques, we find that in the region $(-t) < 10^{-6} \text{ GeV}^2$, the model exactly reproduces QED [44,45].

6. Conclusion

In Ref. [1], we used a classical model [2] which includes long-range forces through the Coulomb contribution to calculate the $D$-term. The classical character of the model was not an impediment. It allowed us to investigate properties affected by the presence of long-range forces without worrying about the technical difficulties which arise when studying more complicated quantum systems. We found that the $D$-term of the proton diverges, in direct contrast to the convergent results of previous studies. This feature is due to the infinite range of the electromagnetic interaction and model independent. In fact, the model gives $T_{\text{BB}}(r)$, $s(r)$, $p(r) \sim \frac{1}{r}$ at large $r$ [1] which agrees with QED calculations [44,45]. In the model, we were able to derive a unique regularization prescription to obtain a meaningful, finite, negative value for the $D$-term in agreement with other studies. Without such a regularization, the form factor $D(t)$ changes sign and diverges at very small momentum transfers below $-t < 10^{-4} \text{ GeV}^2$.

While this $t$-region is currently out of reach experimentally, it indicates that it may be necessary to refine the definitions of the EMT properties in the presence of long-range forces. It is currently an open question how to do this in a model-independent way, or whether the divergence of $D(t)$ may be remedied by considering QED radiative corrections.

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References

1. Mira Varma and Peter Schweitzer. Effects of long-range forces on the $D$-term and the energy-momentum structure. Phys. Rev. D, 102(1):014047, 2020.
2. Iwo Bialynicki-Birula. Simple relativistic model of a finite size particle. Phys. Lett. A, 182:346–352, 1993.
3. I. Yu. Kobzarev and L. B. Okun. Gravitational Interaction of Fermions. Zh. Ekspl. Teor. Fiz., 43:1904–1909, 1962.
4. H. Pagels. Energy-Momentum Structure Form Factors of Particles. Phys. Rev., 144:1250–1260, 1966.
5. X.-D. Ji. Off forward parton distributions. J. Phys. G, 24:1181–1205, 1998.
6. A. V. Radyushkin. Generalized parton distributions. 10 2000.
7. K. Gohe, M. V. Polyakov, and M. Vanderhaegen. Hard exclusive reactions and the structure of hadrons. Prog. Part. Nucl. Phys., 47:401–515, 2001.
8. M. Diehl. Generalized parton distributions. Phys. Rept., 388:41–277, 2003.
9. A. V. Belitsky and A. V. Radyushkin. Unraveling hadron structure with generalized parton distributions. Phys. Rept., 418:1–387, 2005.
10. S. Boffi and B. Pasquini. Generalized parton distributions and the structure of the nucleon. Riv. Nuovo Cim., 30(9):387–448, 2007.
11. M. V. Polyakov and C. Weiss. Skewed and double distributions in pion and nucleon. Phys. Rev. D, 60:114017, 1999.
12. M. V. Polyakov. Generalized parton distributions and strong forces inside nucleons and nuclei. Phys. Lett. B, 555:57–62, 2003.
13. M. V. Polyakov and P. Schweitzer. Forces inside hadrons: pressure, surface tension, mechanical radius, and all that. Int. J. Mod. Phys. A, 33(26):1830025, 2018.
14. C. Lorcé, Hervé Moutarde, and A. P. Trawiński. Revisiting the mechanical properties of the nucleon. Eur. Phys. J. C, 79(1):89, 2019.
15. A. Freese and G. A. Miller. Forces within hadrons on the light front. Phys. Rev. D, 103:094023, 2021.
16. J. Yu. Panteleeva and M. V. Polyakov. Forces inside the nucleon on the light front from 3D Breit frame force distributions: Abel tomography case. Phys. Rev. D, 104(1):014008, 2021.
17. S. Kumano, Qin-Tao Song, and O. V. Teryaev. Hadron tomography by generalized distribution amplitudes in pion-pair production process $\gamma^*\gamma \rightarrow \pi^0\pi^0$ and gravitational form factors for pion. Phys. Rev. D, 97(1):014020, 2018.
18. V. D. Burkert, L. Elouadrhiri, and F. X. Girod. The pressure distribution inside the proton. Nature, 557(7705):396–399, 2018.
19. K. Kumerički. Measurability of pressure inside the proton. Nature, 570(7759):E1–E2, 2019.
20. V. D. Burkert, L. Elouadrhiri, and F. X. Girod. Determination of shear forces inside the proton. 4 2021.
21. K. Goeke, J. Grabis, J. Ossmann, M. V. Polyakov, P. Schweitzer, A. Silva, and D. Urbano. Nucleon form-factors of the energy momentum tensor in the chiral quark-soliton model. Phys. Rev. D, 75:094021, 2007.
22. K. Goeke, J. Grabis, J. Ossmann, P. Schweitzer, A. Silva, and D. Urbano. The pion mass dependence of the nucleon form-factors of the energy momentum tensor in the chiral quark-soliton model. Phys. Rev. C, 75:055207, 2007.
23. C. Cebulla, K. Goeke, J. Ossmann, and P. Schweitzer. The Nucleon form-factors of the energy momentum tensor in the Skyrme model. Nucl. Phys. A, 794:87–114, 2007.
24. J.-H. Jung, U. Yakhshiev, and H.-C. Kim. Energy-momentum form factors of the nucleon within a $\pi^+\rho^-\omega$ soliton model. J. Phys. G, 41:055107, 2014.
25. H.-C. Kim, P. Schweitzer, and U. Yakhshiev. Energy-momentum tensor form factors of the nucleon in nuclear matter. Phys. Lett. B, 718:625–631, 2012.
26. J.-H. Jung, U. Yakhshiev, H.-C. Kim, and P. Schweitzer. In-medium modified energy-momentum tensor form factors of the nucleon within the framework of a $\pi^+\rho^-\omega$ soliton model. Phys. Rev. D, 89(11):114021, 2014.
27. M. Mai and P. Schweitzer. Energy momentum tensor, stability, and the D-term of Q-balls. Phys. Rev. D, 86:076001, 2012.
28. M. Mai and P. Schweitzer. Radial excitations of Q-balls, and their D-term. Phys. Rev. D, 86:096002, 2012.
29. M. Cantara, M. Mai, and P. Schweitzer. The energy-momentum tensor and D-term of Q-clouds. Nucl. Phys. A, 953:1–20, 2016.
30. I. E. Gulamov, E. Ya. Nugayev, A. G. Panin, and M. N. Smolyakov. Some properties of U(1) gauged Q-balls. Phys. Rev. D, 92(4):045011, 2015.
31. E. Ya. Nugayev and A. V. Shkerin. Review of Nontopological Solitons in Theories with U(1)-Symmetry. J. Exp. Theor. Phys., 130(2):301–320, 2020.
32. J. Hudson and P. Schweitzer. D term and the structure of pointlike and composed spin-0 particles. Phys. Rev. D, 96(11):114013, 2017.
33. J. Hudson and P. Schweitzer. Dynamic origins of fermionic D-terms. Phys. Rev. D, 97(5):056003, 2018.
34. P. E. Shanahan and W. Detmold. Pressure Distribution and Shear Forces inside Proton. Phys. Rev. Lett., 122(7):072003, 2019.
35. P. E. Shanahan and W. Detmold. Gluon gravitational form factors of the nucleon and the pion from lattice QCD. Phys. Rev. D, 99(1):014511, 2019.
36. I. V. Anikin. Gravitational form factors within light-cone sum rules at leading order. Phys. Rev. D, 99(9):094026, 2019.
37. M. J. Neubelt, A. Sampino, J. Hudson, K. Tezgin, and P. Schweitzer. Energy momentum tensor and the D-term in the bag model. Phys. Rev. D, 101(3):034013, 2020.
38. K. Azizi and U. Özdem. Nucleon’s energy-momentum tensor form factors in light-cone QCD. Eur. Phys. J. C, 80(2):104, 2020.
39. J. Yu. Panteleeva and M. V. Polyakov. Quadrupole pressure and shear forces inside baryons in the large $N_c$ limit. Phys. Lett. B, 809:135707, 2020.
40. J.-Y. Kim, H.-C. Kim, M. V. Polyakov, and H.-D. Son. Strong force fields and stabilities of the nucleon and singly heavy baryon $\Sigma_c$. Phys. Rev. D, 103(1):014015, 2021.
41. D. Chakrabarti, C. Mondal, A. Mukherjee, S. Nair, and X. Zhao. Gravitational form factors and mechanical properties of proton in a light-front quark-diquark model. Phys. Rev. D, 102:113011, 2020.
42. S. Owa, A. W. Thomas, and X. G. Wang. Effect of the pion field on the distributions of pressure and shear in the proton. 6 2021.
43. J. More, A. Mukherjee, S. Nair, and S. Saha. Gravitational form factors and mechanical properties of a quark at one loop in light-front Hamiltonian QCD. 12 2021.
44. J. F. Donoghue, B. R. Holstein, B. Garbrecht, and T. Konstandin. Quantum corrections to the Reissner-Nordström and Kerr-Newman metrics. Phys. Lett. B, 529:132–142, 2002. [Erratum: Phys.Lett.B 612, 311–312 (2005)].
45. A. Metz, B. Pasquini, and S. Rodini. The gravitational form factor $D(t)$ of the electron. Phys. Lett. B, 820:136501, 2021.
46. B. Kubis and U.-G. Meißner. Virtual photons in the pion form-factors and energy momentum tensor. Nucl. Phys. A, 671:332–356, 2000. [Erratum: Nucl.Phys.A 692, 647–648 (2001)].