HOMOGENIZATION THEORY OF ELLIPTIC SYSTEM WITH LOWER ORDER TERMS FOR DIMENSION TWO

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ABSTRACT. In this paper, we consider the homogenization problem for generalized elliptic systems
\[ L_\varepsilon = -\text{div}(A(x/\varepsilon)\nabla + V(x/\varepsilon)) + B(x/\varepsilon)\nabla + c(x/\varepsilon) + \lambda \]
with dimension two. Precisely, we will establish the \( W^{1,p} \) estimates, Hölder estimates, Lipschitz estimates and \( L^p \) convergence results for \( L_\varepsilon \) with dimension two. The operator \( L_\varepsilon \) has been studied by Qiang Xu with dimension \( d \geq 3 \) in [22, 23] and the case \( d = 2 \) is remained unsolved. As a byproduct, we will construct the Green functions for \( L_\varepsilon \) with \( d = 2 \) and their convergence rates.

Keywords: Homogenization, elliptic systems, lower order terms, two dimension.

1. Introduction and Main results

The homogenization theory, which has been gradually established since the 1970s, has high research value both in mathematics and mechanics. Its direct background is the equivalent research scheme of non-uniform material (static loading) or wave or medium (wave or oscillation) forces, elastic waves, and so on. In recent years, considerable advances have been made in the theory of homogenization for second-order linear elliptic systems in the divergence form with rapidly oscillating periodic coefficients,
\[ L_\varepsilon = -\text{div}(A(x/\varepsilon)\nabla ) - \frac{\partial}{\partial x_i} \left\{ a^{\alpha \beta}_{ij}(\frac{x}{\varepsilon}) \frac{\partial}{\partial x_j} \right\} , \varepsilon > 0, \]
in a bounded domain \( \Omega \) in \( \mathbb{R}^d \), where \( 1 \leq i, j \leq d \) and \( 1 \leq \alpha, \beta \leq m \).

This paper concerns uniform regularity estimates for second order elliptic systems with lower order terms in a bounded domain in \( \mathbb{R}^2 \). More precisely, we consider
\[ L_\varepsilon = -\text{div}(A(x/\varepsilon)\nabla + V(x/\varepsilon)) + B(x/\varepsilon)\nabla + c(x/\varepsilon) + \lambda, \]
for the case that \( x \in \Omega \subset \mathbb{R}^d \), where \( \Omega \) is a bounded domain in \( \mathbb{R}^d \) and \( \lambda > 0 \) is a sufficiently large constant.

Let \( 1 \leq i, j \leq d \) and \( 1 \leq \alpha, \beta \leq m \), where \( m \geq 1 \) denotes the number of equations in the system. Assume that the measurable functions \( A = (a^{\alpha \beta}_{ij}(x)) : \mathbb{R}^d \to \mathbb{R}^{m \times d} \), \( V = (V^{\alpha i}_i(x)) : \mathbb{R}^d \to \mathbb{R}^{m \times d} \), \( B = (B^\alpha_i(x)) : \mathbb{R}^d \to \mathbb{R}^{m \times d} \) and \( c = (c^\alpha(x)) : \mathbb{R}^d \to \mathbb{R}^m \) are coefficients of the elliptic operator \( L_\varepsilon \).

Furthermore, we can assume that the coefficients satisfy the following conditions:

1. The uniform elliptic condition
\[ \mu |\xi|^2 \leq a^{\alpha \beta}_{ij}(y)\xi_i \xi_j \leq \mu^{-1} |\xi|^2 \]
for all \( y \in \mathbb{R}^d \) and \( \xi = (\xi^\alpha) \in \mathbb{R}^{m \times d} \), where \( \mu > 0 \) is a positive constant;

2. The periodicity condition
\[ A(y + z) = A(y), V(y + z) = V(y), B(y + z) = B(y), c(y + z) = c(y) \]
for all \( y \in \mathbb{R}^d \) and \( z \in \mathbb{Z}^d \);

3. The boundedness condition
\[ \max \left\{ \|V\|_{L^\infty(\mathbb{R}^d)}, \|B\|_{L^\infty(\mathbb{R}^d)}, \|c\|_{L^\infty(\mathbb{R}^d)} \right\} \leq \kappa_1, \]

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where $\kappa_1 > 0$ is a positive constant;

(4) the Hölder regularity

$$\max \{ \| A \|_{C^{\alpha, \tau}(\mathbb{R}^d)}, \| V \|_{C^{\alpha, \tau}(\mathbb{R}^d)}, \| B \|_{C^{\alpha, \tau}(\mathbb{R}^d)} \} \leq \kappa_2,$$

where $\tau \in (0,1)$ and $\kappa_2 > 0$ are positive constants.

Let $\kappa = \max \{ \kappa_1, \kappa_2 \}$ and we call $A \in \Lambda(\mu, \tau, \kappa)$ if $A = A(y)$ satisfies (1.2), (1.3) and (1.5).

The uniform regularity estimates of the operator (1.1) have been studied by Qiang Xu in [22, 23]. In [22], the author concentrated on the Dirichlet problems and derived uniform $W^{1,p}$ Hölder and Lipschitz estimates for the case $d \geq 3$. The reason for the assumption $d \geq 3$ is that the Sobolev embedding theorem $W^{1,2}(\mathbb{R}^d) \subset L^{\frac{2d}{d-2}}(\mathbb{R}^d)$, $(d \geq 3)$ is not valid for the case $d = 2$. The author also pointed out in [22] that for $d = 1, 2$, results in the paper are still true. For $d = 1$, we see that it is actually an ODE problem and it should not be hard to solve. Therefore, in this paper, we mainly consider the case that $d = 2$. The following are the main results of the paper.

**Theorem 1.1 (W$^{1,p}$ estimates).** Suppose that $A \in \text{VMO}(\mathbb{R}^2)$ satisfies (1.2), (1.3), other coefficients of $\mathcal{L}_e$, $V, B, c$ satisfy (1.4) and $\Omega$ is a bounded $C^{1,\eta}$ ($0 < \eta < 1$) domain in $\mathbb{R}^2$. Let $1 < p < \infty$, $f \in L^p(\Omega; \mathbb{R}^{m \times 2})$, $F \in L^q(\Omega; \mathbb{R}^m)$ and $g \in B^{1,p-\frac{2}{p}}(\partial \Omega; \mathbb{R}^m)$, where $q = \frac{2p}{p+2}$ if $p > 2$, $1 < q < \infty$ if $p = 2$ and $q = 1$ if $1 < p < 2$. Then the Dirichlet problem

$$\begin{cases}
\mathcal{L}_e(u_e) = \text{div}(f) + F & \text{in } \Omega, \\
u_e = g & \text{on } \partial \Omega,
\end{cases}$$

has a unique weak solution $u_e \in W^{1,p}(\Omega; \mathbb{R}^m)$, whenever $\lambda \geq \lambda_0$ and $\lambda_0 = \lambda_0(\mu, \kappa, m)$ is sufficiently large. Furthermore, the solution satisfies the uniform estimate

$$\| \nabla u_e \|_{L^p(\Omega)} \leq C \left\{ \| f \|_{L^p(\Omega)} + \| F \|_{L^q(\Omega)} + \| g \|_{B^{1,p-\frac{2}{p}}(\partial \Omega)} \right\},$$

where $C$ depends only on $\mu, \omega(t), \kappa, \lambda, p, q, m$ and $\Omega$.

Note that $A \in \text{VMO}(\mathbb{R}^d)$ if $A$ satisfy

$$\sup_{x \in \mathbb{R}^d, \rho < t} \int_{B(x, \rho)} \left| A(y) - \int_{B(x, \rho)} A \, dy \right| \omega(t), \text{ and lim } t \to 0 \omega(t) = 0,$$

where $\omega(t)$ is a continuous nondecreasing function. $B^{s,p}(\partial \Omega; \mathbb{R}^m)$ denotes the $L^p$ Besov space with order $\alpha$ (see [1]). For the $W^{1,p}$ estimates of the elliptic operator $\mathcal{L}_e$ without lower order terms, one can refer to [15, 16, 17, 18]. We mention that the proof of $W^{1,p}$ estimates for the case $d = 2$ is almost the same as that for $d = 3$ given in [22]. The only difference is that the indices $p$ and $q$ are changed due to different Sobolev embedding theorems.

**Theorem 1.2.** Suppose that $A \in \text{VMO}(\mathbb{R}^2)$ satisfies (1.2), (1.3), and other coefficients of $\mathcal{L}_e$, $V, B, c$ satisfy (1.4) and $\Omega$ is a bounded $C^{1,\eta}$ ($0 < \eta < 1$) domain in $\mathbb{R}^2$. Let $f \in L^p(\Omega; \mathbb{R}^{m \times 2})$, $F \in L^q(\Omega; \mathbb{R}^m)$ and $g \in C^{0,\sigma}(\partial \Omega; \mathbb{R}^m)$, where $2 < p < \infty$, $q = \frac{2p}{p+2}$ and $\sigma = 1 - \frac{2}{p}$. Then the weak solution to (1.6) satisfies the uniform estimate

$$\| u_e \|_{C^{0,\sigma}(\Omega)} \leq C \left\{ \| F \|_{L^q(\Omega)} + \| f \|_{L^p(\Omega)} + \| g \|_{C^{0,\sigma}(\partial \Omega)} \right\},$$

where $C$ depends only on $\mu, \omega(t), \kappa, \lambda, m, p, q, \sigma$ and $\Omega$.

Note that the estimate in Theorem 1.2 is sharp. If we change the $C^{0,\sigma}$ norm of $g$ to $C^{0,1}$ norm, (1.9) is easy to verify by using Sobolev embedding theorem, since there exists an extension function $G \in C^{0,1}(\Omega; \mathbb{R}^m)$ such that $G = g$ on $\partial \Omega$ and $\| G \|_{C^{0,1}(\Omega)} \leq C \| g \|_{C^{0,\sigma}(\partial \Omega)}$. To get the sharp estimate, the authors of [22] constructed the corresponding Green functions. The main difference between the cases $d \geq 3$ and $d = 2$ is that the Green functions are different from each other. In this paper, we turn to study the Green functions for $\mathcal{L}_e$ with $d = 2$. More precisely we can derive the following result.

**Theorem 1.3 (Green functions of $\mathcal{L}_e$ with $d = 2$).** Suppose that $A$ satisfies (1.2), (1.3), VMO condition (1.8), other coefficients of $\mathcal{L}_e$, $V, B, c$ satisfy (1.4) and $\Omega$ is a bounded $C^{1,\eta}$ ($0 < \eta < 1$) domain in $\mathbb{R}^2$. If $\lambda \geq \lambda_0$, then there exists a unique Green function $G_e : \Omega \times \Omega \to \mathbb{R}^{m^2} \cup \{ \infty \}$ such that for all $u_e$ being the
weak solution for the Dirichlet problem \( L_\varepsilon(u_\varepsilon) = F \) in \( \Omega \) and \( u_\varepsilon = 0 \) on \( \partial \Omega \), where \( F \in L^p(\Omega; \mathbb{R}^m) \), \( p > 1 \), we have

\[
\tag{1.10}
 u_\varepsilon(x) = \int_\Omega G_\varepsilon(x,y) F(y) dy.
\]

Furthermore, for the Green function \( G_\varepsilon^*(x,y) \) corresponding to the dual operator of \( L_\varepsilon \) denoted as \( L_\varepsilon^* \), we have \( G_\varepsilon^*(x,y) = G_\varepsilon(y,x) \). For all \( \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma \in (0,1) \), we have

\[
\tag{1.11}
 G_\varepsilon(\cdot, y) \in \text{BMO}(\Omega) \text{ that is, } \|G_\varepsilon(\cdot, y)\|_\ast \leq C, \text{ uniformly for } y \in \Omega,
\]

\[
\tag{1.12}
 |G_\varepsilon(x, y)| \leq \frac{C}{|x - y|^\sigma}, \text{ for any } x, y \in \Omega,
\]

\[
\tag{1.13}
 |G_\varepsilon(x, y)| \leq \frac{C|\delta(x)|^{\sigma_1}}{|x - y|^{\sigma_1}}, \text{ if } \delta(x) < \frac{1}{4}|x - y|,
\]

\[
\tag{1.14}
 |G_\varepsilon(x, y)| \leq \frac{C|\delta(y)|^{\sigma_2}}{|x - y|^{\sigma_2}}, \text{ if } \delta(y) < \frac{1}{4}|x - y|,
\]

\[
\tag{1.15}
 |G_\varepsilon(x, y)| \leq \frac{C|\delta(x)|^{\sigma_1}|\delta(y)|^{\sigma_2}}{|x - y|^{\sigma_1 + \sigma_2}}, \text{ if } \delta(x) < \frac{1}{4}|x - y| \text{ or } \delta(y) < \frac{1}{4}|x - y|,
\]

\[
\tag{1.16}
 |G_\varepsilon(x, y)| \leq C \left(1 + \ln \left(\frac{\text{diam}(\Omega)}{|x - y|}\right)\right), \text{ if } \delta(x) \geq \frac{1}{4}|x - y| \text{ and } \delta(y) \geq \frac{1}{4}|x - y|,
\]

where \( \delta(x) = \text{dist}(x, \partial \Omega) \) denote the distance from \( x \) to the boundary of \( \Omega \), \( \text{diam}(\Omega) \) denotes the diameter of \( \Omega \) and \( C \) is a constant depending only on \( \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma, \mu, \omega(t), \kappa, \lambda, m \) and \( \Omega \).

Here, the space \( \text{BMO}(\Omega) \) (functions of bounded mean oscillation), is defined in Definition 4.1. To construct the Green functions with \( d = 2 \), we will use the methods in [6, 7, 21], which are mainly some arguments related to \( \text{BMO} \) space. In [22], the methods for the proof of Lipschitz estimates can be applied to the case \( d = 2 \) without any difficulty. In this paper, we do not repeat the proof for the sake of simplicity. The theorem is stated as follows.

**Theorem 1.4** (Lipschitz estimates). Suppose that \( A \in \Lambda(\mu, \tau, \kappa), V \) satisfies (1.3), (1.5), \( B, c \) satisfy (1.4), \( \lambda \geq \lambda_0 \) and \( \Omega \) is a bounded \( C^{1,\eta} \) \((0 < \eta < 1)\) domain in \( \mathbb{R}^2 \). Let \( p > 2 \) and \( 0 < \sigma \leq \eta \). Then, for any \( f \in C^{0,\sigma}(\Omega; \mathbb{R}^m), F \in L^p(\Omega; \mathbb{R}^m) \), and \( g \in C^{1,\sigma}(\partial \Omega; \mathbb{R}^m) \), the weak solution to (1.6) satisfies the uniform estimate

\[
\tag{1.19}
 \|\nabla u_\varepsilon\|_{L^\infty(\Omega)} \leq C \left\{ \|f\|_{C^{0,\sigma}(\Omega)} + \|F\|_{L^p(\Omega)} + \|g\|_{C^{1,\sigma}(\partial \Omega)} \right\},
\]

where \( C \) depends only on \( \mu, \tau, \kappa, \lambda, p, m, \sigma, \eta \) and \( \Omega \).

The uniform regularity estimates for the operators \( L_\varepsilon \) with lower order terms are first studied by Fanghua Lin and Marco Avellaneda in [2]. For the operator with the non-divergence form, see [3]. Because of Theorem 1.4, we can establish the Lipschitz estimates for the Green functions with \( d = 2 \) constructed in Theorem 1.3. To this point, we can obtain the following results.

**Theorem 1.5** (Lipschitz estimates for Green functions). Suppose that \( A \in \Lambda(\mu, \tau, \kappa), V \) satisfies (1.3), (1.5), \( B, c \) satisfy (1.4), \( \lambda \geq \lambda_0 \) and \( \Omega \) is a bounded \( C^{1,\eta} \) \((0 < \eta < 1)\) domain in \( \mathbb{R}^2 \). Then the Green functions
$G_\varepsilon(x, y)$ constructed in Theorem 1.3 satisfy the uniform estimates

\begin{align}
(1.20) & \quad |G_\varepsilon(x, y)| \leq \frac{C|\delta(x)|}{|x-y|}, \text{ if } \delta(x) < \frac{1}{4}|x-y|, \\
(1.21) & \quad |G_\varepsilon(x, y)| \leq \frac{C|\delta(y)|}{|x-y|}, \text{ if } \delta(y) < \frac{1}{4}|x-y|, \\
(1.22) & \quad |G_\varepsilon(x, y)| \leq \frac{C|\delta(x)||\delta(y)|}{|x-y|^2}, \text{ if } \delta(x) < \frac{1}{4}|x-y| \text{ or } \delta(y) < \frac{1}{4}|x-y|.
\end{align}

Furthermore, the Green functions satisfy the following uniformly Lipschitz estimates,

\begin{align}
(1.23) & \quad |\nabla_x G_\varepsilon(x, y)| \leq \frac{C}{|x-y|} \min \left \{ 1, \frac{\delta(y)}{|x-y|} \right \}, \\
(1.24) & \quad |\nabla_y G_\varepsilon(x, y)| \leq \frac{C}{|x-y|} \min \left \{ 1, \frac{\delta(x)}{|x-y|} \right \}, \\
(1.25) & \quad |\nabla_x \nabla_y G_\varepsilon(x, y)| \leq \frac{C}{|x-y|^2},
\end{align}

where $C$ depends on $\mu, \tau, \kappa, \lambda, m, \sigma, \eta$ and $\Omega$.

**Theorem 1.6** (Nontangential maximal function estimates). Suppose that $A \in \Lambda(\mu, \tau, \kappa), V, B$ satisfy (1.3) and (1.5), $c$ satisfies (1.4), $\lambda \geq \lambda_0$ and $\Omega$ is a bounded $C^{1,\eta}$ ($0 < \eta < 1$) domain in $\mathbb{R}^d$ with $d \geq 2$. Let $1 < p < \infty$, and $u_\varepsilon$ be the solution of the $L^p$ Dirichlet problem $L_\varepsilon(u_\varepsilon) = 0$ in $\Omega$ and $u_\varepsilon = g$ on $\partial \Omega$ with $(u_\varepsilon)^* \in L^p(\partial \Omega)$, where $g \in L^p(\partial \Omega; \mathbb{R}^m)$ and $(u_\varepsilon)^*$ is the nontangential maximal function. Then

\begin{equation}
(1.26) \quad \| (u_\varepsilon)^* \|_{L^p(\partial \Omega)} \leq C_p \| g \|_{L^p(\partial \Omega)},
\end{equation}

where $C_p$ depends on $\mu, \tau, \kappa, \lambda, d, m, p, \eta$ and $\Omega$. Furthermore, if $g \in L^\infty(\partial \Omega; \mathbb{R}^m)$, we have

\begin{equation}
(1.27) \quad \| u_\varepsilon \|_{L^\infty(\Omega)} \leq C \| g \|_{L^\infty(\partial \Omega)},
\end{equation}

where $C$ depends on $\mu, \tau, \kappa, \lambda, d, m, \eta$ and $\Omega$.

The estimate (1.27) is known as the Agmon-Miranda maximum principle, and $(u_\varepsilon)^*$ is the nontangential maximal function defined as follows.

\begin{equation}
(1.28) \quad (u_\varepsilon)^*(x) = \sup \{ |u_\varepsilon(y)| : y \in \Omega \text{ and } |y-x| \leq C_0 \text{ dist}(y, \partial \Omega) \},
\end{equation}

for any $x \in \partial \Omega$, where $C_0$ is a sufficiently large constant number. Because of Theorem 1.5, (1.26) follows directly by constructing Poisson kernel $P_\varepsilon(x, y)$ associated with $L_\varepsilon$. The Poisson kernel is defined as follows.

\begin{equation}
(1.29) \quad P_\varepsilon^{T, \beta}(x, y) = -n_j(y)a_{ij}^{\beta}(y/\varepsilon)\partial_{y_j}\{G_\varepsilon^{2\gamma}(x, y)\} - n_j(y)B_j^{\beta}(y/\varepsilon)G_\varepsilon^{2\gamma}(x, y),
\end{equation}

with $n(x) = (n_i(x))_{i=1}^d$ being the outward unit normal to $\partial \Omega$. We will omit the proof for Theorem 1.6 since the proof for the case $d = 2$ is almost the same as the case $d \geq 3$, which is already given in [22]. The proof is related to the Lipschitz estimates for the Green functions, for which we assume that $V$ is divergence-free.

**Theorem 1.7** (Convergence rates). Suppose that $A \in \Lambda(\mu, \tau, \kappa), V, B$ satisfy (1.3) and (1.5), $c$ satisfies (1.4), $\lambda \geq \lambda_0$ and $\Omega$ is a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ with $d \geq 2$. Let $u_\varepsilon$ be the weak solution to $L_\varepsilon(u_\varepsilon) = F$ in $\Omega$ and $u_\varepsilon = 0$ on $\partial \Omega$, where $F \in L^2(\Omega; \mathbb{R}^m)$. Then we have

\begin{equation}
(1.30) \quad \| u_\varepsilon - \Phi_{\varepsilon, \lambda}(0) - (\Phi_{\varepsilon, \kappa} - P_\varepsilon^{T, \beta}) \partial_k u^\beta \|_{H_0^1(\Omega)} \leq C\varepsilon \| F \|_{L^2(\Omega)}
\end{equation}

where $u_0$ satisfies $L_\varepsilon(u_0) = F_\varepsilon$ in $\Omega$ and $u_0 = 0$ on $\partial \Omega$. Moreover, assume that the coefficients of $L_\varepsilon$ satisfy (1.2)-(1.5), then

\begin{equation}
(1.31) \quad \| u_\varepsilon - u_0 \|_{L^q(\Omega)} \leq C\varepsilon \| F \|_{L^p(\Omega)}
\end{equation}

holds for any $F \in L^p(\Omega; \mathbb{R}^m)$, where $q = \frac{pd}{d-p}$ if $1 < p < d$, $q = \infty$ if $p > d$, and $C$ depends on $\mu, \tau, \kappa, \lambda, m, d, p$ and $\Omega$. 
Here, $\Phi_{\varepsilon,k} = (\Phi_{\varepsilon,k}^{\alpha\beta}), 0 \leq k \leq d$ is the Dirichlet corrector, associated with $L_{\varepsilon}$ as follows:

$$L_{\varepsilon}(\Phi_{\varepsilon,0}) = \text{div}(V_{\varepsilon}) \quad \text{in} \quad \Omega, \quad \Phi_{\varepsilon,k} = I \quad \text{on} \partial \Omega,$$

and

$$L_{\varepsilon}(\Phi_{\varepsilon,k}^{\beta}) = 0 \quad \text{in} \quad \Omega, \quad \Phi_{\varepsilon,k} = P_{k}^{\beta} \quad \text{on} \partial \Omega,$$

for $1 \leq k \leq d$, where $V_{\varepsilon}(x) = V(x/\varepsilon), \Phi_{\varepsilon,k}^{\beta} = (\Phi_{\varepsilon,k}^{\alpha\beta}, \ldots, \Phi_{\varepsilon,k}^{m\beta}) \in W^{1,2}(\Omega; \mathbb{R}^{m})$, and $P_{k}^{\beta} = x_{k}e_{\beta}$. Here, $e_{\beta} = (0, \ldots, 1, \ldots, 0)$ with 1 in the $\beta$th position and 0 otherwise.

We remark that (1.30) is a generalization for Theorem 2.4 in [13], which is established of the operator $L_{\varepsilon}$. For $d \geq 3$, it is proved in [22]. For the case that $d = 2$, the proof is almost the same and we will omit it for the sake of simplicity. For (1.31), it is a generalization of Theorem 3.4 in [12], which is about the elliptic operator $L_{\varepsilon}$. In [22], the author gave the simplified proof for $d \geq 3$ without using the convergence rates of the Green functions. In this paper, we will complete the proof of the theorem about convergence rates for the Green functions for $L_{\varepsilon}$, which is mentioned in [22] without proof. The method is from [12].

### 2. Preliminaries

Before the proofs of the main theorems of the paper, we will first introduce some results of the homogenization problems for the operator $L_{\varepsilon}$ with $d \geq 2$. The homogenized operator of $L_{\varepsilon}$ is defined as follows.

$$L_{0} = -\text{div}(\tilde{A}\nabla + \tilde{V}) + \tilde{B}\nabla + \tilde{c} + \lambda I.$$

The coefficients of the homogenized operator $L_{0}, \tilde{A} = (\tilde{a}_{ij}^{\alpha\beta}), \tilde{V} = (\tilde{v}_{i}^{\alpha\beta}), \tilde{B} = (\tilde{b}_{i}^{\alpha\beta})$ and $\tilde{c} = (\tilde{c}^{\alpha\beta})$ are given by

$$\tilde{a}_{ij}^{\alpha\beta} = \int_{Y} [\tilde{a}_{ij}^{\alpha\beta}(y) + a_{ik}^{\alpha\gamma}(y)\partial_{k}\chi_{j}^{\gamma\beta}(y)]dy, \quad \tilde{v}_{i}^{\alpha\beta} = \int_{Y} [\tilde{v}_{i}^{\alpha\beta}(y) + a_{ij}^{\alpha\gamma}(y)\partial_{j}\chi_{i}^{\gamma\beta}(y)]dy,$$

where $\chi_{k}$ are the corresponding correctors for homogenization problems, defined as $\chi_{k} = (\chi_{k}^{\alpha\beta})$ with $0 \leq k \leq d$, satisfying

$$L_{1}(\chi_{0}) = \text{div}(V) \quad \text{in} \quad \mathbb{R}^{d}, \quad \chi_{0} \in W^{1,2}_{\text{per}}(Y; \mathbb{R}^{m^{2}}) \quad \text{and} \quad \int_{Y} \chi_{0}dy = 0,$$

and

$$L_{1}(\chi_{k}^{\beta} + P_{k}^{\beta}) = 0 \quad \text{in} \quad \mathbb{R}^{d}, \quad \chi_{k} \in W^{1,2}_{\text{per}}(Y; \mathbb{R}^{m}) \quad \text{and} \quad \int_{Y} \chi_{k}^{\beta}dy = 0, \quad 1 \leq k \leq d.$$

Here, $Y = [0, 1)^{d} = \mathbb{R}^{d}/\mathbb{Z}^{d}, P_{k}^{\beta} = x_{k}e_{\beta}$, and $W^{1,2}_{\text{per}}(Y; \mathbb{R}^{m})$ denotes the closure of $C_{\text{per}}^{\infty}(\Omega; \mathbb{R}^{m})$ in $W^{1,2}(\Omega; \mathbb{R}^{m})$. Note that $C_{\text{per}}^{\infty}(\Omega; \mathbb{R}^{m})$ is the subset of $C^{\infty}(Y; \mathbb{R}^{m})$, which collects all $Y$-periodic vector-valued functions.

Next, we will give some simple conclusions. These conclusions are basic and important for subsequent proofs.

**Lemma 2.1 ([22], Lemma 2.4).** Let $\Omega$ be a Lipschitz domain in $\mathbb{R}^{d}, d \geq 2$. Suppose that $A$ satisfies the ellipticity condition (1.2), and other coefficients of $L_{\varepsilon}, V, B, c$ satisfy (1.4). Then we have the following properties. For any $u, v \in W^{1,2}_{0}(\Omega; \mathbb{R}^{m})$,

$$||L_{\varepsilon}(u), v|| \leq C ||u||_{W^{1,2}_{0}(\Omega)} ||v||_{W^{1,2}_{0}(\Omega)}, \quad c_{0} ||u||^{2}_{W^{1,2}_{0}(\Omega)} \leq \langle L_{\varepsilon}(u), u \rangle,$$

whenever $\lambda \geq \lambda_{0}$, where $\lambda_{0} = \lambda_{0}(\mu, \kappa, m, d)$ is sufficiently large. Note that $C$ depends only on $\mu, \kappa, \lambda, m, d, \Omega$, and $c_{0}$ depends only on $\mu, \kappa, m, d, \lambda, \Omega$.

Based on the lemma above and by using the Lax-Milgram theorem, we can obtain the following theorem.
Theorem 2.2 ([22], Lemma 2.5). Under the conditions of Lemma 2.1, suppose that \( F \in H^{-1}(\Omega; \mathbb{R}^m) \) and \( g \in W^{1/2}(\Omega; \mathbb{R}^m) \). Then the Dirichlet boundary value problem \( L_\varepsilon(u_\varepsilon) = F \) in \( \Omega \) and \( u_\varepsilon = g \) on \( \partial \Omega \), has a unique weak solution \( u_\varepsilon \in W^{1,2}(\Omega; \mathbb{R}^m) \), whenever \( \lambda \geq \lambda_0 \), and the solution satisfies the uniform estimate

\[
\|u_\varepsilon\|_{W^{1,2}(\Omega)} \leq C \left\{ \|F\|_{H^{-1}(\Omega)} + \|g\|_{W^{1/2}(\partial \Omega)} \right\},
\]

where \( C \) depends only on \( \mu, \kappa, m, d \) and \( \Omega \). Moreover, with one more the periodicity condition (1.3) on the coefficients of \( L_\varepsilon \), we then have \( u_\varepsilon \to u \) weakly in \( W^{1,2}(\Omega; \mathbb{R}^m) \) and strongly in \( L^2(\Omega; \mathbb{R}^m) \) as \( \varepsilon \to 0 \). where \( u \) satisfies \( L_\varepsilon(u_0) = F \) in \( \Omega \), and \( u_0 = g \) on \( \partial \Omega \).

To simplify the notations, we define that for \( x \in \Omega \) and \( 0 < r < \text{diam}(\Omega) \),

\[
\Omega(x, r) = \Omega \cap B(x, r) \quad \text{and} \quad \Delta(x, r) = \partial \Omega \cap B(x, r).
\]

Also, for \( u \in L^p(E) \) with \( E \) being measurable and \( 1 \leq p < \infty \),

\[
\|u\|_{L^p(E)} = \left( \frac{1}{|E|} \int_E |u(x)|^p dx \right)^{\frac{1}{p}}.
\]

Lemma 2.3 (Caccioppoli’s inequality). Assume that \( A \) satisfies (1.2), other coefficients of \( L_\varepsilon \), \( V, B, c \) satisfy (1.4), \( \Omega \) is a \( C^1 \) bounded domain in \( \mathbb{R}^d \) \((d \geq 2)\), \( x_0 \in \Omega \) and \( 0 < r < \text{diam}(\Omega) \). If \( \partial \Omega \cap B(x_0, 2r) \neq \emptyset \), assume that \( u_\varepsilon \in W^{1,2}(\Omega(x_0, 2r); \mathbb{R}^m) \) is a weak solution to

\[
\begin{cases}
L_\varepsilon(u_\varepsilon) = \text{div}(f) + F & \text{in } \Omega(x_0, 2r), \\
u_\varepsilon = 0 & \text{on } \Delta(x_0, 2r),
\end{cases}
\]

with \( f \in L^2(\Omega(x_0, 2r); \mathbb{R}^{m \times d}) \), \( F \in L^q(\Omega(x_0, 2r); \mathbb{R}^m) \), \( q = \frac{2d}{d+2} \) if \( d \geq 3 \) and \( q > 1 \) if \( d = 2 \). If \( \partial \Omega \cap B(x_0, 2r) = \emptyset \), assume that \( u_\varepsilon \in W^{1,2}(B(x_0, 2r); \mathbb{R}^m) \) is a weak solution to

\[
L_\varepsilon(u_\varepsilon) = \text{div}(f) + F \quad \text{in } B(x_0, 2r),
\]

with the same data \( f, F \) and \( g \). Then there exists \( \lambda_0 = \lambda_0(\mu, d, m, \kappa) \), such that for \( \lambda \geq \lambda_0 \), we have the uniform estimate

\[
\left( \int_{\Omega_r} \varphi u_\varepsilon^2 dx \right)^{\frac{1}{2}} \leq C \left\{ \left( \int_{\Omega_{2r}} \varphi^2 dx \right)^{\frac{1}{2}} + C \left( \int_{\Omega_{2r}} |f|^q dx \right)^{\frac{1}{q}} + Cr \left( \int_{\Omega_{2r}} |F|^q dx \right)^{\frac{1}{q}} \right\},
\]

where \( \Omega_r = \Omega(x_0, r) \) and \( C \) depends only on \( \mu, \kappa, \lambda, \mu, m, d, \Omega \).

Proof. For the case \( d \geq 3 \), one can find the proof in [22]. For \( d = 2 \), the proof is almost the same. We only need to adjust the proof for \( d = 3 \) by changing the Sobolev embedding theorem \( W^{1,2}(\mathbb{R}^d) \subset L^{2d/3}(\mathbb{R}^d) \). when \( d \geq 3 \) to \( W^{1,2}(\mathbb{R}^2) \subset L^q(\mathbb{R}^2) \) with \( q \geq 2 \) when \( d = 2 \). For the sake of completeness, we give the proof for \( d = 2 \) as follows. By translation and rescaling, we may assume that \( r = 1 \) and \( x_0 = 0 \). Let \( \varphi \in C_0^\infty(B(0, 2)) \) be a cut-off function satisfying \( \varphi \equiv 1 \) in \( B(0, 1) \), \( \varphi \equiv 0 \) in \( (B(0, \frac{3}{2}))^c \), and \( |\nabla \varphi| \leq C \). Then, by choosing the test function as \( \phi = \varphi^2 u_\varepsilon \), we can obtain

\[
\int_{\Omega_{(0,2)}} [A(x/\varepsilon)\nabla u_\varepsilon + V(x/\varepsilon) u_\varepsilon] \varphi^2 u_\varepsilon dx + 2 \int_{\Omega_{(0,2)}} [A(x/\varepsilon)\nabla u_\varepsilon + V(x/\varepsilon) u_\varepsilon] \nabla \varphi u_\varepsilon \varphi dx
\]

\[
+ \int_{\Omega_{(0,2)}} B(x/\varepsilon) \nabla u_\varepsilon \varphi^2 dx + \int_{\Omega_{(0,2)}} c(x/\varepsilon) u_\varepsilon \varphi^2 + \lambda u_\varepsilon^2 \varphi^2 dx
\]

\[
= \int_{\Omega_{(0,2)}} Fe \varphi^2 dx - \int_{\Omega_{(0,2)}} f \nabla u_\varepsilon \varphi^2 dx - 2 \int_{\Omega_{(0,2)}} f \nabla \varphi u_\varepsilon \varphi dx.
\]

From the ellipticity condition (1.2) and Young’s inequality, we have

\[
\frac{\mu}{4} \int_{\Omega_{(0,2)}} \varphi^2 \nabla u_\varepsilon^2 dx \leq (C' - \lambda) \int_{\Omega_{(0,2)}} \varphi^2 |u_\varepsilon|^2 dx
\]

\[
+ C \int_{\Omega_{(0,2)}} |\nabla \varphi|^2 |u_\varepsilon|^2 dx + C \int_{\Omega_{(0,2)}} \varphi^2 |f|^2 dx + \int_{\Omega_{(0,2)}} \varphi^2 |F||u_\varepsilon| dx,
\]

(2.8)
where $C' = C'(\mu, \kappa, m)$ is a constant. We can derive $\lambda_1$, such that for any $\lambda \geq \lambda_0$, $C' - \lambda < 0$. By the Sobolev embedding theorem, we can choose $1 < q \leq 2$ (if we prove that the results are true for $1 < q \leq 2$, the case for $q > 2$ is trivial)

$$
\int_{\Omega(0,2)} \varphi^2 |F||u_\varepsilon| dx \leq \left( \int_{\Omega(0,2)} (\varphi|u_\varepsilon|^q)^{\frac{1}{q}} dx \right)^\frac{q}{q-1} \left( \int_{\Omega(0,2)} (\varphi|F|^q)^{\frac{1}{q}} dx \right)^{\frac{1}{q}}
$$

$$
\leq C \left( \int_{\Omega(0,2)} |\nabla (\varphi u_\varepsilon)|^2 dx \right)^{\frac{1}{2}} \left( \int_{\Omega(0,2)} |(\varphi|F|)^q| dx \right)^{\frac{1}{2}}
$$

$$
\leq \frac{\mu}{8} \int_{\Omega(0,2)} |\nabla u_\varepsilon|^2 \varphi^2 dx + \frac{\mu}{8} \int_{\Omega(0,2)} |u_\varepsilon|^2 |\nabla \varphi|^2 dx + C \left( \int_{\Omega(0,2)} |\varphi F|^q dx \right)^{\frac{2}{q}}.
$$

Here we use the Sobolev embedding theorem $W^{1,2}_0(\Omega(0,2); \mathbb{R}^m) \subset L^{\frac{2}{q}}(\Omega(0,2); \mathbb{R}^m)$ with $1 < q \leq 2$. By the definition of $\varphi$, (2.8) is true.

**Remark 2.4.** More precisely, we can obtain the inequality from $\Omega(x_0, tr)$ to $\Omega(x_0, sr)$, where $0 < t < s < 1$. In the proof of this, we can choose the cut-off function $\varphi$ satisfying $\varphi = 1$ in $B(x_0, tr)$, $\varphi = 0$ in $(B(x_0, sr))^c$ and $|\nabla \varphi| \leq \frac{C}{(s-t)r}$. Using almost the same arguments, we can obtain

$$
\left( \int_{\Omega_r} |\nabla u_\varepsilon|^2 dx \right)^{\frac{1}{2}} \leq \frac{C}{(s-t)r} \left( \int_{\Omega_r} |u_\varepsilon|^2 dx \right)^{\frac{1}{2}} + C \left( \int_{\Omega_{2r}} |f|^2 dx \right)^{\frac{1}{2}} + C r \left( \int_{\Omega_{2r}} |F|^q dx \right)^{\frac{1}{q}},
$$

where $\Omega_{tr} = \Omega(x_0, tr)$ and $C$ depends only on $\mu, \kappa, \lambda, m, d, \Omega$.

**Remark 2.5.** Under the same conditions of Lemma 2.3, assume that $J \in \mathbb{R}^m$ is an arbitrary vector and $v_\varepsilon = u_\varepsilon - J$, the equation corresponding to $v_\varepsilon$ is stated as follows

$$
\mathcal{L}_\varepsilon(v_\varepsilon) = \text{div}(f + V(x/\varepsilon)J) + F - c(x/\varepsilon)J - \lambda J.
$$

Then, if $\partial \Omega \cap B(x, 2r) = \emptyset$, for $v_\varepsilon$, we can use the Caccioppoli’s inequality (2.8) to obtain

$$
\left( \int_{B_r} |\nabla u_\varepsilon|^2 dx \right)^{\frac{1}{2}} \leq \frac{C}{r} \left( \int_{B_2r} |u_\varepsilon - J|^2 dx \right)^{\frac{1}{2}} + C \left( \int_{B_2r} |f|^2 dx \right)^{\frac{1}{2}} + C r \left( \int_{B_2r} |F|^q dx \right)^{\frac{1}{q}} + C(|J| + |J|r),
$$

where $B_r = B(x_0, r)$ and $C$ depends only on $\mu, \kappa, \lambda, m, p, q, d$.

**Lemma 2.6** ([22], Remark 3.4). Let $\Omega$ be a $C^1$ bounded domain in $\mathbb{R}^d (d \geq 2)$, and $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ with $2 \leq p < \infty$. Then, for all $\delta > 0$, there exists $C_\delta > 0$ depending only on $\delta, m, p, d, \Omega$ such that

$$
\|u\|_{L^p(\Omega)} \leq \delta \|\nabla u\|_{L^p(\Omega)} + C_\delta \|u\|_{L^2(\Omega)}.
$$

Moreover, when $1 \leq p < \infty$ and $q = \frac{pd}{p+d}$, we have

$$
\|u\|_{L^p(\Omega)} \leq C \left\{ \|\nabla u\|_{L^q(\Omega)} + \|u\|_{L^2(\Omega)} \right\},
$$

where $C$ depends only on $m, p, d, \Omega$.

**Remark 2.7.** Obviously, the second constant $C$ in the above Lemma 2.6 is related to the diameter of $\Omega$. Here, for the convenience of later calculation, we need to quantify this. For simplicity, let us assume $\Omega = B(0, 1)$. Then we can get

$$
\|u\|_{L^p(B(0, 1))} \leq C_1 \|\nabla u\|_{L^p(B(0, 1))} + C_2 \|u\|_{L^2(B(0, 1))}.
$$

By choosing $v(x) = u(rx)$ and using the above results, we have

$$
\|u(rx)\|_{L^p(B(0, 1))} \leq C_1 \|r\nabla u(rx)\|_{L^p(B(0, 1))} + C_2 \|u(rx)\|_{L^2(B(0, 1))}.
$$

After changing the variables, it follows that

$$
r^{-\frac{d}{m}} \|u(x)\|_{L^p(B(0,r))} \leq C_1 r^{1-\frac{d}{m}} \|\nabla u(x)\|_{L^p(B(0,r))} + C_2 r^{-\frac{d}{m}} \|u(x)\|_{L^2(B(0,r))}.
$$
This implies that $C_1(r) = C_1$ and $C_2(r) = C_2 r^{\frac{d}{p} - \frac{d}{q}}$. Rewrite the inequality above, we have

$$
\|u\|_{L^p(B(0,r))} \leq C_1 \|\nabla u\|_{L^q(B(0,r))} + C_2 r^{\frac{d}{p} - \frac{d}{q}} \|u\|_{L^2(B(0,r))},
$$

where $C_1, C_2$ depends only on $m, d, p$.

3. $W^{1,p}$ estimates for $\mathcal{L}_\varepsilon$ with $d = 2$

**Theorem 3.1.** Assume that $A \in \text{VMO}(\mathbb{R}^2)$ satisfies (1.2), (1.3), $\Omega$ is a $C^{1,\gamma}$ $(0 < \gamma < 1)$ bounded domain in $\mathbb{R}^2$, and $1 < p < \infty$. Let $f \in L^p(\Omega; \mathbb{R}^{m\times 2})$, $F \in L^q(\Omega; \mathbb{R}^m)$, where $q = \frac{2p}{p+2}$ if $p > 2$, $1 < q < \infty$ if $p = 2$ and $q = 1$ if $1 < p < 2$. Then there exists a unique weak solution $u_\varepsilon \in W^{1,p}(\Omega; \mathbb{R}^m)$ of the Dirichlet problem

$$
\begin{cases}
\mathcal{L}_\varepsilon(u_\varepsilon) = \text{div}(f) + F & \text{in } \Omega, \\
u_\varepsilon = 0 & \text{on } \partial\Omega,
\end{cases}
$$

satisfying the uniform estimate

$$
\|\nabla u_\varepsilon\|_{L^p(\Omega)} \leq C \{\|F\|_{L^q(\Omega)} + \|f\|_{L^p(\Omega)}\},
$$

where $C$ depends only on $\mu, \omega(t), m, d, p, q, \Omega$.

**Proof.** See Theorem 5.3.1 in [19].

Using Theorem 3.1 and some iteration arguments, we can prove the $W^{1,p}$ estimates for the operator $\mathcal{L}_\varepsilon$.

**Lemma 3.2.** Assume that $A \in \text{VMO}(\mathbb{R}^2)$ satisfies (1.2), (1.3), and other coefficients of $\mathcal{L}_\varepsilon$, $V, B, c$ satisfy (1.4), $\Omega$ is a $C^{1,\gamma}$ $(0 < \gamma < 1)$ bounded domain in $\mathbb{R}^2$, and $1 < p < \infty$. Let $f \in L^p(\Omega; \mathbb{R}^{m\times 2})$. Then, for $\lambda \geq \lambda_0$, there exists a unique weak solution $u_\varepsilon \in W^{1,p}(\Omega; \mathbb{R}^m)$ of the Dirichlet problem

$$
\begin{cases}
\mathcal{L}_\varepsilon(u_\varepsilon) = \text{div}(f) & \text{in } \Omega, \\
u_\varepsilon = 0 & \text{on } \partial\Omega,
\end{cases}
$$

satisfying the uniform estimate

$$
\|\nabla u_\varepsilon\|_{L^p(\Omega)} \leq C \|f\|_{L^p(\Omega)},
$$

where $C$ depends only on $\mu, \omega(t), \kappa, \lambda, m, p$ and $\Omega$.

**Proof.** If $p = 2$, we can derive the estimate (3.2) by the energy inequality (2.5). It is easy to obtain a unique weak solution $u_\varepsilon \in W^{1,2}_0(\Omega; \mathbb{R}^m)$ such that $\|\nabla u_\varepsilon\|_{L^2(\Omega)} \leq C \|f\|_{L^2(\Omega)}$. For $p > 2$, the uniqueness and the existence are trivial. Then, we only need to show the uniform estimate (3.2). Firstly, $u_\varepsilon$ is the weak solution of the Dirichlet problem

$$
\begin{cases}
\mathcal{L}_\varepsilon(u_\varepsilon) = \text{div}(f + V \varepsilon u_\varepsilon) - B \varepsilon \nabla u_\varepsilon + (c_\varepsilon + \lambda)u_\varepsilon & \text{in } \Omega, \\
u_\varepsilon = 0 & \text{on } \partial\Omega.
\end{cases}
$$

From (3.1), we have

$$
\|\nabla u_\varepsilon\|_{L^p(\Omega)} \leq C \left\{\|f\|_{L^p(\Omega)} + \|u_\varepsilon\|_{L^p(\Omega)} + \|u_\varepsilon\|_{L^{\frac{2p}{p+2}}(\Omega)} + \|\nabla u_\varepsilon\|_{L^{\frac{2p}{p+2}}(\Omega)}\right\}.
$$

Then, in view of (2.5), Hölder’s inequality and (2.11), it follows that

$$
\|\nabla u_\varepsilon\|_{L^p(\Omega)} \leq C \left\{\|u_\varepsilon\|_{L^2(\Omega)} + \|\nabla u_\varepsilon\|_{L^2(\Omega)} + \|f\|_{L^p(\Omega)}\right\} \leq C \|f\|_{L^p(\Omega)}.
$$

If $1 < p < 2$, we can derive (3.2) by the duality arguments. \qed

**Lemma 3.3.** Assume that $A \in \text{VMO}(\mathbb{R}^2)$ satisfies (1.2), (1.3), and other coefficients of $\mathcal{L}_\varepsilon$, $V, B, c$ satisfy (1.4), $\Omega$ is a $C^{1,\gamma}$ $(0 < \gamma < 1)$ bounded domain in $\mathbb{R}^2$, and $1 < p < \infty$. Let $F \in L^q(\Omega; \mathbb{R}^m)$, where $q = \frac{2p}{p+2}$ if $p > 2$, $1 < q < \infty$ if $p = 2$, $q = 1$ if $1 < p < 2$. Then, for $\lambda \geq \lambda_0$, there exists a unique weak solution $u_\varepsilon \in W^{1,p}(\Omega; \mathbb{R}^m)$ of the Dirichlet problem

$$
\begin{cases}
\mathcal{L}_\varepsilon(u_\varepsilon) = F & \text{in } \Omega, \\
u_\varepsilon = 0 & \text{on } \partial\Omega,
\end{cases}
$$

satisfying the uniform estimate

$$
\|\nabla u_\varepsilon\|_{L^p(\Omega)} \leq C \|F\|_{L^q(\Omega)},
$$

where $C$ depends only on $\mu, \omega(t), \kappa, \lambda, m, p, q, \Omega$.

**Proof.** See Theorem 5.3.1 in [19].

Using Theorem 3.1 and some iteration arguments, we can prove the $W^{1,p}$ estimates for the operator $\mathcal{L}_\varepsilon$. \qed
where $C$ depends only on $\mu, \omega(t), \kappa, \lambda, m, p$ and $\Omega$.

**Proof.** After getting the estimate (3.4), the uniqueness of (3.3) is trivial to establish. Therefore, we only need to show the existence of the solution. In fact, given $1 < p < \infty$, and $q$ defined above, we can first assume that $F \in L^q(\Omega; \mathbb{R}^m) \cap L^2(\Omega; \mathbb{R}^m)$ and use Theorem 2.2 to establish the existence of (3.3). Then, by using the standard density arguments, we can prove that the result is valid for all $F \in L^q(\Omega; \mathbb{R}^m)$ with the help of the a priori estimate. Now, we will prove the estimate (3.4), where we use the duality arguments and Lemma 3.2. For any $f \in C^1_0(\Omega; \mathbb{R}^{m \times 2})$, there exists a unique $v_\varepsilon \in W^{1,2}_0(\Omega; \mathbb{R}^m)$ such that $L^*_\varepsilon(v_\varepsilon) = \text{div}(f)$ in $\Omega$, and $v_\varepsilon = 0$ on $\partial \Omega$. According to (3.2), we have $\|\nabla v_\varepsilon\|_{L^p(\Omega)} \leq C \|f\|_{L^{p'}(\Omega)}$, where $p' = \frac{p}{p-1}$ denoted as the conjugate number of $p$. Then, by using the definition of $u_\varepsilon$ and $v_\varepsilon$, we can obtain

$$\int_\Omega \nabla u_\varepsilon f dx = - \int_\Omega L^*_\varepsilon(u_\varepsilon)v_\varepsilon dx = - \int_\Omega F v_\varepsilon dx.$$

(1) If $2 < p < \infty$, we choose $q = \frac{2p}{p+2}$. Using Hölder’s inequality, Poincaré-Sobolev inequality and Sobolev embedding theorem, we obtain

$$\left| \int_\Omega \nabla u_\varepsilon f dx \right| \leq \|F\|_{L^{\frac{2p}{p+2}}(\Omega)} \|v_\varepsilon\|_{L^{\frac{2p}{p+2}}(\Omega)} \leq C \|F\|_{L^{\frac{2p}{p+2}}(\Omega)} \|\nabla v_\varepsilon\|_{L^{\frac{p}{p-1}}(\Omega)} \leq C \|F\|_{L^{\frac{2p}{p+2}}(\Omega)} \|f\|_{L^{\frac{p}{p-1}}(\Omega)}.$$ 

Then, using the duality methods, we can obtain that $\|\nabla u_\varepsilon\|_{L^p(\Omega)} \leq C \|F\|_{L^{\frac{2p}{p+2}}(\Omega)}$.

(2) If $p = 2$, choose $1 < q < \infty$, then we have

$$\left| \int_\Omega \nabla u_\varepsilon f dx \right| \leq \|F\|_{L^q(\Omega)} \|v_\varepsilon\|_{L^{\frac{q}{p+2}}(\Omega)} \leq C \|F\|_{L^q(\Omega)} \|\nabla v_\varepsilon\|_{L^\infty(\Omega)} \leq C \|F\|_{L^q(\Omega)} \|f\|_{L^2(\Omega)}.$$ 

Then, we can obtain that $\|\nabla u_\varepsilon\|_{L^2(\Omega)} \leq C \|F\|_{L^q(\Omega)}$.

(3) If $1 < p < 2$, choose $q = 1$, then we have

$$\left| \int_\Omega \nabla u_\varepsilon f dx \right| \leq \|F\|_{L^1(\Omega)} \|v_\varepsilon\|_{L^\infty(\Omega)} \leq C \|F\|_{L^1(\Omega)} \|\nabla v_\varepsilon\|_{L^{\frac{p}{p-1}}(\Omega)} \leq C \|F\|_{L^1(\Omega)} \|f\|_{L^{\frac{p}{p-1}}(\Omega)}.$$ 

Because of cases (1), (2) and (3), we can complete the proof. \hfill \square

**Remark 3.4.** Furthermore, when $p > 2$ and $F \in \dot{W}^{-1,p'}(\Omega)$ (where $\dot{W}^{-1,p'}(\Omega)$ denotes the dual space for the homogeneous Sobolev space $W_0^{1,p}(\Omega)$), we have

$$\|u_\varepsilon\|_{L^p(\Omega)} \leq C \|F\|_{\dot{W}^{-1,p'}(\Omega)}.$$ 

The conclusion is proved by using the dual method. For all $f \in C^1_0(\Omega; \mathbb{R}^m)$ we can choose $v_\varepsilon$ such that $L^*_\varepsilon(v_\varepsilon) = f$ in $\Omega$, and $v_\varepsilon = 0$ on $\partial \Omega$, and

$$\int_\Omega u_\varepsilon f dx = \int_\Omega u_\varepsilon L^*_\varepsilon(v_\varepsilon) = \int_\Omega L^*_\varepsilon(u_\varepsilon)v_\varepsilon dx = \langle F, v_\varepsilon \rangle.$$ 

This implies that

$$\left| \int_\Omega u_\varepsilon f dx \right| = |\langle F, v_\varepsilon \rangle| \leq \|F\|_{\dot{W}^{-1,p'}(\Omega)} \|v_\varepsilon\|_{W_0^{1,p}(\Omega)} \leq \|F\|_{\dot{W}^{-1,p'}(\Omega)} \|f\|_{L^p(\Omega)}.$$

According to the duality property, we have $\|u_\varepsilon\|_{L^p(\Omega)} \leq C \|F\|_{\dot{W}^{-1,p'}(\Omega)}$.

**Proof of Theorem 1.1.** In the case of $g = 0$, we write $v_\varepsilon = u_{\varepsilon,1} + u_{\varepsilon,2}$, where $u_{\varepsilon,1}$ and $u_{\varepsilon,2}$ are the solution in Lemma 3.2 and 3.3, respectively. Then we have

$$\|\nabla v_\varepsilon\|_{L^p(\Omega)} \leq \|\nabla u_{\varepsilon,1}\|_{L^p(\Omega)} + \|\nabla u_{\varepsilon,2}\|_{L^p(\Omega)} \leq C \left\{ \|f\|_{L^p(\Omega)} + \|F\|_{L^q(\Omega)} \right\}.$$ 

For $g \neq 0$, consider the homogeneous Dirichlet problem $L^*_\varepsilon(w_\varepsilon) = 0$ in $\Omega$ and $w_\varepsilon = g$ on $\partial \Omega$, where $g \in B^{1-\frac{2}{p}}(\partial \Omega; \mathbb{R}^m)$. By the properties of boundary Besov space, there exists $G \in W^{1,p}(\Omega; \mathbb{R}^m)$ such that $G = g$ on $\partial \Omega$ and $\|G\|_{W^{1,p}(\Omega)} \leq C \|g\|_{B^{1-\frac{2}{p}}(\partial \Omega)}$. For $h_\varepsilon = w_\varepsilon - G$, we have

\[
\begin{cases}
  L^*_\varepsilon(h_\varepsilon) = \text{div}(A(x/\varepsilon)\nabla G + V(x/\varepsilon)G) - B(x/\varepsilon)\nabla G - (c(x/\varepsilon) + \lambda)G & \text{in } \Omega, \\
  h_\varepsilon = 0 & \text{on } \partial \Omega.
\end{cases}
\]
Recall the case of $g = 0$, in which there exists the unique weak solution $h_\varepsilon \in W_0^{1,p}(\Omega; \mathbb{R}^m)$, satisfying the uniform estimate
\[
\|\nabla h_\varepsilon\|_{L^p(\Omega)} \leq C \|G\|_{W^{1,p}(\Omega)} + C \|\nabla G\|_{L^p(\Omega)} + C \|G\|_{L^p(\Omega)}.
\]

For $p > 2$, choose $q = \frac{2p}{p+2} < p$, for $p = 2$, choose $q < 2 = p$ and for $p < 2$, choose $q = 1 < p$, then
\[
\|\nabla h_\varepsilon\|_{L^p(\Omega)} \leq C \|G\|_{W^{1,p}(\Omega)} + C \|\nabla G\|_{L^p(\Omega)} + C \|G\|_{L^p(\Omega)} \leq C \|G\|_{W^{1,p}(\Omega)} \leq C \|g\|_{B^{1+\frac{1}{p}}(\partial\Omega)},
\]
for any $1 < p < \infty$. This implies
\[
(3.7) \quad \|\nabla v_\varepsilon\|_{L^p(\Omega)} \leq \|\nabla h_\varepsilon\|_{L^p(\Omega)} + C \|\nabla G\|_{L^p(\Omega)} \leq C \|g\|_{B^{1+\frac{1}{p}}(\partial\Omega)}.
\]

Finally, let $u_\varepsilon = v_\varepsilon + w_\varepsilon$. Combining (3.6) and (3.7), we can complete the proof. \(\square\)

**Theorem 3.5** (The Localization of $W^{1,p}$ estimates for $L_\varepsilon$ with $d = 2$). Let $2 \leq p < \infty$. Assume that $A \in \text{VMO}(\mathbb{R}^2)$ satisfies (1.2), (1.3), other coefficients of $L_\varepsilon$, $V, B, c$ satisfy (1.4), $\lambda \geq \lambda_0$, $\Omega$ is a $C^{1,\eta}$ ($0 < \eta < 1$) bounded domain in $\mathbb{R}^2$, $x_0 \in \Omega$ and $0 < r < \text{diam}(\Omega)$. If $\partial\Omega \cap B(x_0, 2r) \neq \emptyset$, assume that $u_\varepsilon \in W^{1,2}(\Omega(x_0, 2r); \mathbb{R}^m)$ is the weak solution to
\[
\begin{aligned}
\mathcal{L}_\varepsilon(u_\varepsilon) &= \text{div}(f) + F \quad \text{in} \quad \Omega(x_0, 2r), \\
\varepsilon^2 u_\varepsilon &= 0 \quad \text{on} \quad \Delta(x_0, 2r),
\end{aligned}
\]
with $f \in L^p(\Omega(x_0, 2r); \mathbb{R}^{m \times 2})$, $F \in L^q(\Omega(x_0, 2r); \mathbb{R}^m)$, $q = \frac{2p}{p+2}$ if $2 < p < \infty$ and $q > 1$ if $p = 2$. If $\partial\Omega \cap B(x_0, 2r) = \emptyset$, assume that $u_\varepsilon \in W^{1,2}(B(x_0, 2r); \mathbb{R}^m)$ is a weak solution to
\[
\mathcal{L}_\varepsilon(u_\varepsilon) = \text{div}(f) + F \quad \text{in} \quad B(x_0, 2r),
\]
with the same data $f, F, p$ and $q$. Then $\nabla u_\varepsilon \in L^p(\Omega(x_0, r); \mathbb{R}^m)$ and
\[
(3.8) \quad \|\nabla u_\varepsilon\|_{L^{\frac{2pq}{p+2}}(\Omega(x_0, r))} \leq \frac{C}{(s-t)^2} \left\{ \|f\|_{L^{\frac{2pq}{p+2}}(\Omega(x_0, r))} + r \|F\|_{L^{\frac{2pq}{p+2}}(\Omega(x_0, r))} \right\} + \frac{C}{(s-t)^\frac{p}{2}} \left\{ \|f\|_{L^{\frac{2pq}{p+2}}(\Omega(x_0, r))} + r \|F\|_{L^{\frac{2pq}{p+2}}(\Omega(x_0, r))} \right\},
\]
where $0 < t < s < 1$, $\Omega_{tr} = \Omega(x_0, tr)$ and $C$ depends only on $\mu, \omega(t), \kappa, \lambda, m, p, q, \Omega$.

**Proof.** If $p = 2$, we can derive this inequality by Caccioppoli’s inequality (2.8). If $p > 2$, by using rescaling and translation, we can assume that $x_0 = 0$ and $r = 1$. We can also assume that $\partial\Omega \cap B(x_0, 2r) \neq \emptyset$, since the other case is almost the same. For $0 < t < s < 1$, we see that $0 < t < \frac{t}{s-t} < s < 1$. We can choose $\varphi \in C^\infty_0(B(0, \frac{t}{s-t}))$ as a cut-off function such that $\varphi \equiv 1$ in $B(0, t)$, $\varphi \equiv 0$ in $B(0, \frac{t}{s-t})$ and $|\nabla \varphi| \leq \frac{C}{(s-t)^{\frac{p}{2}}}$. Then by letting $w_\varepsilon = \varphi u_\varepsilon$, we have
\[
\begin{aligned}
\left\{ \begin{array}{ll}
-\text{div}(A_\varepsilon \nabla w_\varepsilon) &= \text{div}(f \varphi) - \text{div} F \varphi + F \varphi + \tilde{f} \quad \text{in} \quad \Omega(0, 2), \\
\varepsilon^2 w_\varepsilon &= 0 \quad \text{on} \quad \partial \Omega(0, 2),
\end{array} \right.
\end{aligned}
\]
where $A_\varepsilon = A(x/\varepsilon)$, $V_\varepsilon = V(x/\varepsilon)$, $B_\varepsilon = B(x/\varepsilon)$, $\varepsilon_c = c(x/\varepsilon)$, and
\[
\tilde{f} = f \varphi + V_\varepsilon \varphi \delta^\beta \nabla \varphi u_\varepsilon^\beta - V_\varepsilon \delta^\beta \nabla \varphi u_\varepsilon^\beta - V_\varepsilon \delta^\beta \nabla \varphi u_\varepsilon^\beta - V_\varepsilon \delta^\beta \nabla \varphi u_\varepsilon^\beta - V_\varepsilon \delta^\beta \nabla \varphi u_\varepsilon^\beta - \lambda \varepsilon^2 u_\varepsilon^\alpha.
\]

Let $\tilde{f} = f \varphi + V_\varepsilon \varphi \delta^\beta \nabla \varphi u_\varepsilon^\beta$ and
\[
G = -f \nabla \varphi + F \varphi - A_\varepsilon \delta^\beta \nabla \varphi u_\varepsilon^\beta - V_\varepsilon \delta^\beta \nabla \varphi u_\varepsilon^\beta - B_\varepsilon \delta^\beta \nabla u_\varepsilon^\beta - c_\varepsilon \delta^\beta u_\varepsilon^\beta - \lambda \varepsilon^2 u_\varepsilon^\alpha.
\]

Then, by using the estimate (3.1), we can obtain
\[
\|\nabla w_\varepsilon\|_{L^p(\Omega(0, t))} \leq C \left\{ \|\tilde{f}\|_{L^p(\Omega(0, t))} + \|G\|_{L^{\frac{2pq}{p+2}}(\Omega(0, t))} \right\},
\]
where we define $\tilde{f} = \frac{1+t}{2}$. This implies that
\[
\|\nabla u_\varepsilon\|_{L^p(\Omega(0, t))} \leq \|\varphi u_\varepsilon\|_{L^p(\Omega(0, t))} + \|\varphi v_\varepsilon\|_{L^p(\Omega(0, t))} + C \|\nabla \varphi u_\varepsilon\|_{L^{\frac{2pq}{p+2}}(\Omega(0, t))} + C \|\varphi u_\varepsilon\|_{L^{\frac{2pq}{p+2}}(\Omega(0, t))} + C \|\varphi v_\varepsilon\|_{L^{\frac{2pq}{p+2}}(\Omega(0, t))} + C \|\tilde{f}\|_{L^{\frac{2pq}{p+2}}(\Omega(0, t))} + C \|G\|_{L^{\frac{2pq}{p+2}}(\Omega(0, t))} + C \|F\|_{L^{\frac{2pq}{p+2}}(\Omega(0, t))}.
\]
Using the definition of the cut-off function \( \varphi \), we have
\[
\| \nabla u_\varepsilon \|_{L^p(\Omega(0,t))} \leq C \| u_\varepsilon \|_{L^p(\Omega(0,T))} + \frac{C}{(s-t)} \| u_\varepsilon \|_{L^p(\Omega(0,T))} + \frac{C}{(s-t)} \| \nabla u_\varepsilon \|_{L^\frac{2p}{p-2}(\Omega(0,T))} \\
+ \frac{C}{(s-t)} \| u_\varepsilon \|_{L^\frac{2p}{p-2}(\Omega(0,T))} + \frac{C}{(s-t)} \| \nabla u_\varepsilon \|_{L^\frac{2p}{p-2}(\Omega(0,T))} + \frac{C}{(s-t)} \| u_\varepsilon \|_{L^\frac{2p}{p-2}(\Omega(0,T))} \\
+ \frac{C}{(s-t)} \| \nabla u_\varepsilon \|_{L^\frac{2p}{p-2}(\Omega(0,T))} + \frac{C}{(s-t)} \| u_\varepsilon \|_{L^\frac{2p}{p-2}(\Omega(0,T))} + \frac{C}{(s-t)} \| \nabla u_\varepsilon \|_{L^\frac{2p}{p-2}(\Omega(0,T))}.
\]
This, together with (2.13) for \( p > 2, q = \frac{2p}{p+2} \) and \( d = 2 \), i.e.
\[
\| u_\varepsilon \|_{L^p(\Omega(0,t))} \leq C \| \nabla u_\varepsilon \|_{L^\frac{2p}{p+2}(\Omega(0,T))} + C \| u_\varepsilon \|_{L^2(\Omega(0,T))},
\]
gives that
\[
\| \nabla u_\varepsilon \|_{L^p(\Omega(0,t))} \leq C \| u_\varepsilon \|_{L^p(\Omega(0,T))} + \frac{C}{(s-t)} \| u_\varepsilon \|_{L^2(\Omega(0,T))} \\
+ \frac{C}{(s-t)} \| \nabla u_\varepsilon \|_{L^2(\Omega(0,T))} + \frac{C}{(s-t)} \| u_\varepsilon \|_{L^2(\Omega(0,T))} + \frac{C}{(s-t)} \| \nabla u_\varepsilon \|_{L^2(\Omega(0,T))}.
\]
(3.9)
For the second inequality of (3.9), we have used the Hölder’s inequality. Finally we can use (2.9) to derive
\[
\| \nabla u_\varepsilon \|_{L^\frac{2p}{p+2}(\Omega(0,t))} \leq \frac{C}{(s-t)} \| u_\varepsilon \|_{L^\frac{2p}{p+2}(\Omega(0,s))} + C \| f \|_{L^2(\Omega(0,s))} + C \| F \|_{L^\frac{2p}{p+2}(\Omega(0,s))},
\]
i.e.
\[
\| \nabla u_\varepsilon \|_{L^\frac{2p}{p+2}(\Omega(0,t))} \leq \frac{C}{(s-t)} \| u_\varepsilon \|_{L^2(\Omega(0,s))} + C \| f \|_{L^2(\Omega(0,s))} + \frac{C}{s^{\frac{2p}{p+2}}} \| F \|_{L^\frac{2p}{p+2}(\Omega(0,s))}.
\]
(3.10)
Combining inequalities (3.9), (3.10) and the fact that \( \frac{7}{8} \leq 1 \), we can obtain
\[
\| \nabla u_\varepsilon \|_{L^p(\Omega(0,\tau))} \leq \frac{C}{(s-t)^{2(1-\frac{1}{p})}} \| u_\varepsilon \|_{L^2(\Omega(0,s))} + \frac{C}{(s-t)} \| f \|_{L^p(\Omega(0,s))} + \frac{C}{(s-t)} \| F \|_{L^\frac{2p}{p+2}(\Omega(0,s))},
\]
where \( C \) depends only on \( \mu, \omega(t), \kappa, \lambda, m, p, q \). Taking the average and using \( t, s < 1 \), we have
\[
\| \nabla u_\varepsilon \|_{L^\frac{2p}{p+2}(\Omega(0,t))} \leq \frac{C}{(s-t)^2} \| u_\varepsilon \|_{L^2(\Omega(0,s))} + \frac{C}{(s-t)^2} \frac{2p}{p+2} \left\{ \| f \|_{L^\frac{2p}{p+2}(\Omega(0,s))} + \| F \|_{L^\frac{2p}{p+2}(\Omega(0,s))} \right\}.
\]
This completes the proof. \( \square \)

**Corollary 3.6.** Let \( 2 < p < \infty \). Assume that \( A \in \text{VMO}(\mathbb{R}^2) \) satisfies (1.2), (1.3), other coefficients of \( L_\varepsilon \) satisfy (1.4), \( \lambda \geq \lambda_0 \), \( \Omega \) is a \( C^{1,\alpha} \) \( (0 < \eta < 1) \) bounded domain in \( \mathbb{R}^2 \), \( x_0 \in \Omega \) and \( 0 < r < \text{diam}(\Omega) \). If \( \partial \Omega \cap B(x_0,2r) \neq \emptyset \), assume that \( u_\varepsilon \in W^{1,2}(\Omega(x_0,2r); \mathbb{R}^m) \) is the weak solution to
\[
\begin{aligned}
L_\varepsilon(u_\varepsilon) &= \text{div}(f) + F \quad \text{in} \quad \Omega(x_0,2r), \\
u_\varepsilon &= 0 \quad \text{on} \quad \Delta(x_0,2r),
\end{aligned}
\]
with \( f \in L^p(\Omega(x_0,2r); \mathbb{R}^{m\times 2}), \quad F \in L^q(\Omega(x_0,2r); \mathbb{R}^m) \) and \( q = \frac{2p}{p+2} \). If \( \partial \Omega \cap B(x_0,2r) = \emptyset \), assume that \( u_\varepsilon \in W^{1,2}(B(x_0,2r); \mathbb{R}^m) \) is a weak solution to
\[
L_\varepsilon(u_\varepsilon) = \text{div}(f) + F \quad \text{in} \quad \Omega(x_0,2r),
\]
with the same data \( f, F \) and \( p \). Then, for \( \sigma = 1 - \frac{2}{p} \),

\[
(3.11) \quad [u_{\varepsilon}]_{C^{0,\sigma}(\Omega_{\varepsilon})} \leq C r^{-\sigma} \left\{ \left( \int_{\Omega_{\varepsilon}} |u_{\varepsilon}|^2 dx \right)^{\frac{1}{2}} + r \left( \int_{\Omega_{\varepsilon}} |f|^p dx \right)^{\frac{1}{p}} + r^2 \left( \int_{\Omega_{\varepsilon}} |F|^q dx \right)^{\frac{1}{q}} \right\}.
\]

In particular, for all \( \overline{\sigma} > 0 \), we have

\[
(3.12) \quad \|u_{\varepsilon}\|_{L^\infty(\Omega_{\varepsilon})} \leq C \left\{ \left( \int_{\Omega_{\varepsilon}} |u_{\varepsilon}|^p dx \right)^{\frac{1}{p}} + r \left( \int_{\Omega_{\varepsilon}} |f|^p dx \right)^{\frac{1}{p}} + r^2 \left( \int_{\Omega_{\varepsilon}} |F|^q dx \right)^{\frac{1}{q}} \right\},
\]

where \( \Omega_{\varepsilon} = \Omega(x_0, r) \) and \( C \) depends only on \( \mu, \omega(t), \kappa, \lambda, m, p, q, \Omega \).

**Proof.** To obtain more precise scale estimates, we need to make a clear exploration of the relationship between the constant of Morrey’s inequality and the domain’s radius. It can be seen that on a ball with a radius of 1, for \( u \in W^{1,p}(B(0,1)) \) and \( p > d \), where \( d \geq 2 \) is the dimension, then according to Morrey’s theorem, we can obtain that

\[
[u]_{C^{0,1-\frac{d}{p}}(B(0,1))} \leq C \|\nabla u\|_{L^p(B(0,1))} + C \|u\|_{L^p(B(0,1))}.
\]

Then, Lemma 2.6 implies that

\[
[u]_{C^{0,1-\frac{d}{p}}(B(0,1))} \leq C \|\nabla u\|_{L^p(B(0,1))} + C \|u\|_{L^2(B(0,1))}.
\]

Taking \( v(x) = u(rx) \) and using the inequality on \( v \), we have

\[
r^{1-\frac{d}{p}} [u]_{C^{0,1-\frac{d}{p}}(B(0,r))} \leq C r^{1-\frac{d}{p}} \|\nabla u\|_{L^p(B(0,1))} + C r^{-\frac{d}{p}} \|u\|_{L^2(B(0,1))},
\]

(3.13)

\[
[u]_{C^{0,1-\frac{d}{p}}(B(0,r))} \leq C \|\nabla u\|_{L^p(B(0,r))} + C r^{-\frac{d}{p}-1} \|u\|_{L^2(B(0,r))}.
\]

Note that if we change \( B(0, r) \) to \( \Omega(0, r) \), (3.13) is still true. Therefore, for \( \Omega(x_0, r) \) such that \( x_0 \in \Omega \), \( 0 < r < \text{diam}(\Omega) \), \( d = 2 \), \( \sigma = 1 - \frac{2}{p} \) and \( 0 < t < s \leq 1 \), we have

\[
[u_{\varepsilon}]_{C^{0,\sigma}(\Omega(x_0, tr))} \leq C \|\nabla u_{\varepsilon}\|_{L^p(\Omega(x_0, tr))} + C (tr)^{-1-\sigma} \|u_{\varepsilon}\|_{L^2(\Omega(x_0, tr))}
\]

\[
\leq C (tr)^{\frac{1}{p}} \|\nabla u_{\varepsilon}\|_{L^p_{\text{avg}}(\Omega(x_0, tr))} + C (tr)^{-\sigma} \|u_{\varepsilon}\|_{L^2_{\text{avg}}(\Omega(x_0, tr))}
\]

\[
\leq C (tr)^{-\sigma} \|u_{\varepsilon}\|_{L^2_{\text{avg}}(\Omega(x_0, tr))}
\]

\[
+ C (tr)^{\frac{1}{p}} \left\{ \frac{C}{(s-t)^2} \|u_{\varepsilon}\|_{L^2_{\text{avg}}(\Omega(0, sr))} + \frac{C}{(s-t)^2} \left\{ \|f\|_{L^p_{\text{avg}}(\Omega(x_0, sr))} + r \|F\|_{L^q_{\text{avg}}(\Omega(x_0, sr))} \right\} \right\}
\]

(3.14)

\[
\leq C r^{-\sigma} \|u_{\varepsilon}\|_{L^2_{\text{avg}}(\Omega(x_0, sr))} + \frac{C (s-t)^{-\sigma}}{(s-t)^2} \left\{ \|f\|_{L^p_{\text{avg}}(\Omega(x_0, sr))} + r \|F\|_{L^q_{\text{avg}}(\Omega(x_0, sr))} \right\},
\]

where, for the second inequality, we use the \( W^{1,p} \) estimate (3.8) and \( C \) depends only on \( \mu, \omega(t), \kappa, \lambda, m, p \).

By choosing special \( t, s \), we can get the first inequality (3.11). For any \( x \in \Omega(x_0, tr) \), by using (3.14), we have

\[
|u_{\varepsilon}(x)| \leq \left| u_{\varepsilon}(x) - \int_{\Omega(x, \frac{x-x_0}{t})} u_{\varepsilon} \right| + \left| \int_{\Omega(x, \frac{x-x_0}{t})} u_{\varepsilon} \right|
\]

\[
\leq C [u_{\varepsilon}]_{C^{0,\sigma}(\Omega(x, \frac{x-x_0}{t}))} r^{\sigma} + \|u_{\varepsilon}\|_{L^2_{\text{avg}}(\Omega(x, \frac{x-x_0}{t}))}
\]

(3.15)

\[
\leq \frac{C}{(s-t)^2} \|u_{\varepsilon}\|_{L^2_{\text{avg}}(\Omega(x_0, sr))} + \frac{C (s-t)^{-\sigma}}{(s-t)^2} \left\{ \|f\|_{L^p_{\text{avg}}(\Omega(x_0, sr))} + r \|F\|_{L^q_{\text{avg}}(\Omega(x_0, sr))} \right\}.
\]

Taking special \( t, s \), we can prove that the second inequality holds when \( \overline{\sigma} = 2 \). Next, we will discuss the case of \( 0 < \overline{\sigma} < 2 \). In fact, this is a standard convexity improvement. What is different from [22] is only the difference of a series of inequality indices when \( d = 2 \). They are not the essential differences. For the sake of completeness, we will prove this conclusion here. We adopt the methods in [14]. Since the index \( \frac{4d}{d-2} \) appears
By (3.16), (3.17) and (3.18), we obtain
\[ \|v_\varepsilon\|_{L^\infty(\Omega(0,1))} \leq C \|v_\varepsilon\|_{L^2_{avg}(\Omega(0,1))}, \]
and
\[ \|w_\varepsilon\|_{L^\infty(\Omega(0,1))} \leq C \left\{ \|w_\varepsilon\|_{L^2_{avg}(\Omega(0,2))} + \|f\|_{L^p_{avg}(\Omega(0,2))} + \|F\|_{L^{2p}_{avg}(\Omega(0,2))} \right\}. \]
Moreover, since \( w_\varepsilon = 0 \) on \( \partial(\Omega(0,2)) \), using Theorem 1.1, we get
\[ \|w_\varepsilon\|_{L^2_{avg}(\Omega(0,2))} \leq C \|v_\varepsilon\|_{L^p_{avg}(\Omega(0,2))} \leq (C \|f\|_{L^p_{avg}(\Omega(0,2))} + C \|F\|_{L^{2p}_{avg}(\Omega(0,2))}). \]
By (3.16), (3.17) and (3.18), we obtain
\[ \|u_\varepsilon\|_{L^\infty(\Omega(0,1))} \leq \|w_\varepsilon\|_{L^\infty(\Omega(0,1))} + \|v_\varepsilon\|_{L^\infty(\Omega(0,1))} \]
\[ \leq C \|v_\varepsilon\|_{L^2_{avg}(\Omega(0,2))} + C \|f\|_{L^p_{avg}(\Omega(0,2))} + C \|F\|_{L^{2p}_{avg}(\Omega(0,2))}. \]
Since \( u_\varepsilon = 0 \) on \( \Delta(0,2) \) and \( v_\varepsilon = u_\varepsilon \) on \( \partial(\Omega(0,2)) \), we have \( L_\varepsilon(v_\varepsilon) = 0 \) in \( \Omega(0,2) \) and \( v_\varepsilon = 0 \) on \( \Delta(0,2) \). We claim that,
\[ \|v_\varepsilon\|_{L^2_{avg}(\Omega(0,2))} \leq C \|v_\varepsilon\|_{L^p_{avg}(\Omega(0,2))}. \]
If we prove the claim, we can get the conclusions. This is because by using (3.19) and (3.20),
\[ \|u_\varepsilon\|_{L^\infty(\Omega(0,1))} \leq C \|v_\varepsilon\|_{L^p_{avg}(\Omega(0,2))} + C \|f\|_{L^p_{avg}(\Omega(0,2))} + C \|F\|_{L^{2p}_{avg}(\Omega(0,2))} \]
\[ \leq C \|u_\varepsilon\|_{L^p_{avg}(\Omega(0,2))} + C \|f\|_{L^p_{avg}(\Omega(0,2))} + C \|F\|_{L^{2p}_{avg}(\Omega(0,2))} + C \|w_\varepsilon\|_{L^2_{avg}(\Omega(0,2))} \]
\[ \leq C \|u_\varepsilon\|_{L^p_{avg}(\Omega(0,2))} + C \|f\|_{L^p_{avg}(\Omega(0,2))} + C \|F\|_{L^{2p}_{avg}(\Omega(0,2))} \]
In this way, the proof of (3.12) comes down to the proof of (3.20). Next, we will prove the claim. We can assume that \( \|v_\varepsilon\|_{L^p_{avg}(\Omega(0,2))} = 1 \), for otherwise, setting \( \bar{v}_\varepsilon = v_\varepsilon/\|v_\varepsilon\|_{L^p_{avg}(\Omega(0,2))} \), we can obtain that \( \|\bar{v}_\varepsilon\|_{L^p_{avg}(\Omega(0,2))} = 1 \) and \( L_\varepsilon(\bar{v}_\varepsilon) = 0 \) in \( \Omega(0,2) \) and \( \bar{v}_\varepsilon = 0 \) on \( \Delta(0,2) \).
By using the results for the case that \( \|v_\varepsilon\|_{L^p_{avg}(\Omega(0,2))} = 1 \), we can prove the general case. Choosing \( p_1 > 2 \), and using (3.15) with \( p = 2 \), we have, for \( 0 < t < s \leq 1 \)
\[ \|v_\varepsilon\|_{L^p_{avg}(\Omega(0,t))} \leq \|v_\varepsilon\|_{L^2_{avg}(\Omega(0,t))} \leq C \]
Take the logarithm on both sides, multiply them by $\frac{1}{s}$, choose $t = s^b$ with $b = \frac{1+\eta}{2}\alpha$ and integrate $s$ from $\frac{1}{b}$ to 1. Then we have
\[
\frac{1}{t} \ln(I(t)) \frac{ds}{s} \leq C + \alpha \int_{\frac{1}{b}}^{1} \ln(I(s)) \frac{ds}{s} + C \ln \|v_\varepsilon\|_{L^p(\Omega(0,1))}.
\]
If there exists $t_0 \in [\left(\frac{1}{b}\right)^b, \frac{1}{b}]$ such that $I(t_0) \leq 1$, then it is easy to get that
\[
(3.21)\quad \left(\int_{\Omega(0,\left(\frac{1}{b}\right)^b)} |v_\varepsilon|^2 \right)^{\frac{1}{2}} \leq CI(t_0) \leq C \leq C \left(\int_{\Omega(0,2)} |v_\varepsilon|^p \right)^{\frac{1}{p}},
\]
where $C$ depends only on $\mu, \omega(t), \kappa, \lambda, m, p, \bar{p}, q, \Omega$. If for any $t \in [\left(\frac{1}{b}\right)^b, 1], I(t) \geq 1$, then
\[
\frac{1}{b} \int_{\frac{1}{b}}^{1} \ln(I(s^b)) \frac{ds}{s} = \frac{1}{b} \int_{\frac{1}{b}}^{1} \ln(I(s)) \frac{ds}{s} \geq \frac{1}{b} \int_{\frac{1}{b}}^{1} \ln(I(s)) ds.
\]
Hence $\frac{1}{b} - \alpha = \frac{\alpha + 1}{2\alpha} - \alpha > 0$, and
\[
(3.22)\quad \left(\frac{1}{b} - \alpha\right) \int_{\frac{1}{b}}^{1} \ln(I(s)) \frac{ds}{s} \leq C + C \ln \|v_\varepsilon\|_{L^p(\Omega(0,1))}.
\]
Moreover, for any $t \in [\frac{1}{b}, 1]$, there exists $C$ depending only on $\mu, \omega(t), \kappa, \lambda, m, p, \bar{p}, q, \Omega$, such that
\[
(3.23)\quad I(t) \geq C \left(\int_{\Omega(0,\left(\frac{1}{b}\right)^b)} |v_\varepsilon|^2 \right)^{\frac{1}{2}}.
\]
In view of (3.22) and (3.23), it can be got that
\[
\ln \left(\left(\int_{\Omega(0,\left(\frac{1}{b}\right)^b)} |v_\varepsilon|^2 \right)^{\frac{1}{2}}\right) \leq C + C \ln \|v_\varepsilon\|_{L^p(\Omega(0,1))},
\]
and then
\[
\left(\int_{\Omega(0,\left(\frac{1}{b}\right)^b)} |v_\varepsilon|^2 \right)^{\frac{1}{2}} \leq C \left(\int_{\Omega(0,1)} |v_\varepsilon|^p \right)^{\frac{1}{p}} \leq C \left(\int_{\Omega(0,2)} |v_\varepsilon|^p \right)^{\frac{1}{p}}.
\]
This, together with (3.21) and almost the same arguments, implies that there is a constant $r_0 = \min\{\frac{1}{10}, (\frac{1}{b})^b\}$ such that for any $x \in \Omega(0, \frac{2}{b})$,
\[
\left(\int_{\Omega(x,r_0)} |v_\varepsilon|^2 \right)^{\frac{1}{2}} \leq C \left(\int_{\Omega(0,2)} |v_\varepsilon|^p \right)^{\frac{1}{p}}.
\]
By covering $\Omega(0, \frac{2}{b})$ with $\Omega(x, r_0)$ defined above, we can complete the proof of claim (3.20).

\[\square\]

**Corollary 3.7.** Suppose that $A \in \text{VMO}(\mathbb{R}^2)$ satisfies (1.2), (1.3), and other coefficients of $L_\varepsilon V, B, c$ satisfy (1.4). $\Omega$ is a $C^{1,\eta}$ $(0 < \eta < 1)$ bounded domain in $\mathbb{R}^2$, $f \in L^p(\Omega; \mathbb{R}^{m \times 2})$, $F \in L^q(\Omega; \mathbb{R}^m)$ and $g \in C^{0,\beta}(\partial \Omega; \mathbb{R}^m)$, where $2 < p < \infty$, $q = \frac{2p}{p+2}$ and $\sigma = 1 - \frac{2}{p}$. Then the weak solution to (1.6) satisfies the uniform estimate
\[
(3.24)\quad \|u_\varepsilon\|_{C^{0,\beta}(\Omega)} \leq C \left\{ \|F\|_{L^q(\Omega)} + \|f\|_{L^p(\Omega)} + \|g\|_{C^{0,\beta}(\partial \Omega)} \right\},
\]
where $C$ depends only on $\mu, \omega(t), \kappa, \lambda, m, p, \sigma$ and $\Omega$.

**Proof.** One can use Sobolev embedding theorem and Theorem 1.1 to finish the proof. For details, one can see Corollary 3.8 in [22].

If we do not assume that $A \in \text{VMO}(\mathbb{R}^2)$, we have the following theorem given by Theorem 4.3.3 in [19],
**Theorem 3.8.** Suppose that $A$ satisfies (1.2), and other coefficients of $L_\varepsilon V, B, \mu$ satisfy (1.4). Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^d$ with $d \geq 2$. Then there exists $\delta \in (0, \frac{1}{2})$, depending only on $\mu$ and $\Omega$, such that for any $F \in W^{-1, \frac{1}{\rho}}(\Omega; \mathbb{R}^m)$ and $g \in B^{1 - \frac{1}{q}}(\partial \Omega; \mathbb{R}^m)$ with $|\frac{1}{p} - \frac{1}{q}| < \delta$, there exists a unique solution in $W^{1-p}(\Omega; \mathbb{R}^m)$ to the Dirichlet problem: $L_\varepsilon u = F$ in $\Omega$ and $u = g$ on $\partial \Omega$. Moreover, the solution satisfies the estimate

$$
\|u\|_{W^{1,p}(\Omega)} \leq C \left\{ \|F\|_{W^{-1,p}(\Omega)} + \|g\|_{B^{1 - \frac{1}{q}}(\partial \Omega)} \right\},
$$

with constant $C$ depending only on $p, \mu$ and $\Omega$.

From Theorem 3.8 and the proof of Theorem 1.1, we can obtain the $W^{1,p}$ estimates for the operator $L_\varepsilon$ without the assumption that $A \in \text{VMO}(\mathbb{R}^d)$.

**Theorem 3.9.** Suppose that $A$ satisfies (1.2), other coefficients of $L_\varepsilon V, B, \mu$ satisfy (1.4) and $\Omega$ is a $C^{1, \eta}$ $(0 < \eta < 1)$ bounded domain in $\mathbb{R}^2$. Then, there exists $\delta \in (0, \frac{1}{2})$, depending only on $\mu$ and $\Omega$, such that for any $\frac{1}{p} - \frac{1}{q} < \delta$, the Dirichlet problem $L_\varepsilon(u) = \text{div}(f) + F$ in $\Omega$, $u = g$ on $\partial \Omega$ with $f \in L^p(\Omega; \mathbb{R}^{m \times 2})$, $F \in L^q(\Omega; \mathbb{R}^m)$ and $g \in B^{1 - \frac{1}{q}}(\partial \Omega; \mathbb{R}^m)$ has a unique weak solution $u \in W^{1,p}(\Omega; \mathbb{R}^m)$, whenever $\lambda \geq \lambda_0$. Furthermore, the solution satisfies the uniform estimate

$$
\|\nabla u\|_{L^p(\Omega)} \leq C \left\{ \|f\|_{L^p(\Omega)} + \|F\|_{L^q(\Omega)} + \|g\|_{B^{1 - \frac{1}{q}}(\partial \Omega)} \right\},
$$

where $q = \frac{2p}{p+2}$ if $p > 2$, $1 < q < \frac{1}{1+2\eta}$ if $p = 2$, $q = 1$ if $\frac{1}{1+2\eta} < p < 2$ and $C$ depends only on $\mu, \omega(t), \kappa, \lambda, \rho, \eta, \eta^\ast, m$ and $\Omega$.

By using Sobolev embedding theorem for the case that $d = 2$, we can deduce that $u_\varepsilon$ in Theorem 3.9 is actually Hölder continuous.

### 4. Green functions for the operator $L_\varepsilon$

The following definitions about BMO and atom functions are from [21] and is essential for the construction of the Green functions for the elliptic operator $L_\varepsilon$.

**Definition 4.1** (BMO space and atom functions). For $x_0 \in \mathbb{R}^d, r > 0$ and $\Omega$ a domain in $\mathbb{R}^d$. We denote

$$
\Omega(x_0, r) := \Omega \cap B(x_0, r),
$$

as before. We define that the $\text{BMO}(\Omega)$ is a space containing functions such that

$$
\|u\|_{\ast} = \sup \left\{ \int_{\Omega(x_0, r)} |u - \pi_{x_0, r}| : x_0 \in \Omega, r > 0 \right\}
$$

is finite, where we define

$$
\pi_{x_0, r} := \begin{cases} 0 & \text{if } r \geq \text{dist}(x_0, \partial \Omega), \\
\int_{\Omega(x_0, r)} u & \text{if } r < \text{dist}(x_0, \partial \Omega).
\end{cases}
$$

We call the bounded measurable function a atom function in $\Omega$ if $\text{supp}(a) \subset \Omega(x_0, r)$ with $x_0 \in \Omega$ for $r > 0$ and

$$
\|a\|_{\infty} \leq \frac{1}{|\Omega(x_0, r)|}, \quad \pi_{x_0, r} = 0.
$$

**Lemma 4.2** (Generalized Morrey’s inequality). Let $\Omega$ be a bounded $C^{1, \eta}$ domain in $\mathbb{R}^d (d \geq 2)$ with $0 < \eta < 1$. For $d < p < \infty, \rho > 0$, $u \in W_0^{1,p}(\Omega; \mathbb{R}^m)$, and $x_0 \in \Omega$, we have

$$
\|u - \pi_{x_0, \rho}\|_{L^\infty(\Omega(x_0, \rho))} \leq C \rho^{-\eta} \left( \int_{\Omega(x_0, 2\rho)} |u|^p dy \right)^{\frac{1}{p}},
$$

where $C$ depends only on $m, d, p, \Omega$. 
Proof. If \( \rho \geq \text{dist}(x_0, \partial \Omega) \), we have \( \mathfrak{u}_{x, \rho} = 0 \). Then, we can choose \( \mathfrak{v} \in \partial \Omega \) such that \( \text{dist}(x_0, \partial \Omega) = |x - \mathfrak{v}| \). For \( x \in \Omega(x_0, \rho) \), we have
\[
|u(x)| \leq |x - \mathfrak{v}|^{1 - \frac{d}{p}} |u|_{C^{0,1 - \frac{d}{p}}(\Omega(x_0, 2\rho))} + |u(\mathfrak{v})| \leq C \rho^{1 - \frac{d}{p}} \left( \int_{\Omega(x_0, 2\rho)} |\nabla u|^p \, dy \right)^{\frac{1}{p}}.
\]
If \( \rho < \text{dist}(x_0, \partial \Omega) \) and \( x \in \Omega(x_0, \rho) \), we have
\[
|u(x) - \mathfrak{u}_{x, \rho}| \leq C \rho^{1 - \frac{d}{p}} |u|_{C^{0,1 - \frac{d}{p}}(\Omega(x_0, 2\rho))} \leq C \rho^{1 - \frac{d}{p}} \left( \int_{\Omega(x_0, 2\rho)} |\nabla u|^p \, dy \right)^{\frac{1}{p}}.
\]
This proves the conclusion. \( \square \)

**Lemma 4.3.** Assume that \( A \in \text{VMO}(\mathbb{R}^2) \) satisfies (1.2) and (1.3), other coefficients of \( \mathcal{L}_x \) satisfy (1.4) and \( \Omega \) is a \( C^{1, \eta} \) (0 < \( \eta < 1 \)) bounded domain in \( \mathbb{R}^2 \). Let \( a \) be an atom function in \( \Omega \). If \( u_\varepsilon \) is the unique weak solution for the Dirichlet problem \( \mathcal{L}_x(u_\varepsilon) = a \) in \( \Omega \) and \( u_\varepsilon = 0 \) on \( \partial \Omega \), then there exists a constant \( C \) depending only on \( \mu, \omega(t)\kappa, m, \lambda, p, q, \) and \(\Omega\) such that
\[
\|u_\varepsilon\|_\infty \leq C.
\]

**Proof.** For atom function \( a \), we can assume that
\[
\text{supp}(a) \subset \Omega(x_0, \rho) \quad \text{and} \quad \|a\|_\infty \leq \frac{1}{|\Omega(x_0, \rho)|}
\]
with \( x_0 \in \Omega \) and \( 0 < \rho < \text{diam}(\Omega) \). Fix \( z \in \Omega \), we can choose \( p > 2 \). Using the Sobolev embedding theorem, we have
\[
|u(z)| \leq |u(z) - \mathfrak{u}_{z, \rho}| + |\mathfrak{u}_{z, \rho}| \leq C \left\{ \rho^{1 - \frac{d}{p}} \|\nabla u_\varepsilon\|_{L^p(\Omega)} + \rho^{\frac{d}{p} - 1} \|u_\varepsilon\|_{L^{2p}(\Omega)} \right\}.
\]
From the \( W^{1,p} \) estimates and (3.5), we have
\[
\|\nabla u_\varepsilon\|_{L^p(\Omega)} \leq C \|a\|_{L^{2p}(\Omega)} \leq C \rho^{\frac{d}{p} - 1},
\]
\[
\|u_\varepsilon\|_{L^{2p}(\Omega)} \leq C \|a\|_{W^{-1,p'}(\Omega)}.
\]
Then, we only need to prove that \( \|a\|_{W^{-1,p'}(\Omega)} \leq \rho^{1 - \frac{d}{p}} \). It is because for all \( v \in W_0^{1,p}(\Omega; \mathbb{R}^m) \), we have
\[
\left| \int_{\Omega} a^\alpha v^\alpha \, dy \right| \leq \left| \int_{\Omega} a^\alpha (v^\alpha - \mathfrak{u}_{x, \rho}^\alpha) \, dy \right| \leq C \rho^{1 - \frac{d}{p}} \|\nabla v\|_{L^p(\Omega)},
\]
where we have used (4.2). This completes the results. \( \square \)

**Definition 4.4** (Hardy space). Let \( \Omega \) be a \( C^1 \) domain in \( \mathbb{R}^d \). A function \( f \) is an element in the Hardy space \( \mathcal{H}_1 \), if there exist a sequence of atoms \( \{a_i\}_{i=1}^{\infty} \) and a sequence of real numbers \( \{\lambda_i\} \in l^1 \) such that \( f = \sum_{i=1}^{\infty} \lambda_i a_i \). We define the norm in this space as
\[
\|f\|_{\mathcal{H}_1(\Omega)} = \inf \left\{ \sum_{i=1}^{\infty} |\lambda_i| : f = \sum_{i=1}^{\infty} \lambda_i a_i \right\}.
\]
We notice the expression
\[
\sup \left\{ \int_{\Omega} a(y) u(y) \, dy : a \text{ is an atom in } \Omega \right\},
\]
gives the equivalent norm of \( \text{BMO}(\Omega) \). This is because that \( \text{BMO}(\Omega) \) space can be regarded as the dual space of \( \mathcal{H}_1 \).

Next, we will give a vital property for BMO space. For the complete proof of the theorem, we refer to Corollary 6.22 of [10].
**Theorem 4.5.** Let \( u \in \text{BMO}(\Omega) \) with \( \Omega \) being a bounded domain in \( \mathbb{R}^d \), then \( u \in L^p(\Omega) \) for all \( 1 \leq p < \infty \) and there is a \( C \) depending only on \( p \) and \( d \), such that

\[
(4.4) \quad \left( \int_{\Omega(x_0, r)} |u - \Pi_{x_0, r}|^p \, dx \right)^{\frac{1}{p}} \leq C \|u\|_s ,
\]

for any \( x_0 \in \Omega \), \( 0 < r < \text{diam}(\Omega) \).

**Proof of Theorem 1.3.** Here we follow the proof in [6], [7] and [21] to construct the Green function and get its pointwise estimates. For \( y \in \Omega \), there exists a matrix valued function \( G_{j\rho,\gamma}(\cdot, y) \in W^{1,\infty}(\Omega; \mathbb{R}^m) \), such that for any \( 1 \leq \gamma \leq m \), \( u_\rho = (u_\rho) = (G_{\rho,\gamma}(\cdot, y)) \in W^{1,\infty}(\Omega; \mathbb{R}^m) \) satisfies

\[
(4.5) \quad \langle L_\varepsilon(u_\rho), \varphi \rangle = \int_{\Omega(y, \rho)} \varphi \, dx , \quad \forall \varphi \in W^{1,\infty}(\Omega; \mathbb{R}^m).
\]

Here, we call \( G_{\rho,\gamma}(\cdot, y) \) the average Green matrix for the operator \( L_\varepsilon \). Then for all atom functions in \( \Omega \) denoted as \( a \) such that

\[
\text{supp}(a) \subset \Omega(y, \rho) \quad \text{and} \quad \|a\|_\infty \leq \frac{1}{|\Omega(y, \rho)|},
\]

we can obtain \( v_\varepsilon \in W^{1,\infty}(\Omega; \mathbb{R}^m) \) such that \( L_\varepsilon^*(v_\varepsilon) = a \). Because of the properties of dual operators, we have

\[
\int_{\Omega(y, \rho)} v_\varepsilon dx = \langle L_\varepsilon(u_\rho) \rangle = \langle u_\rho, L_\varepsilon^*(v_\varepsilon) \rangle = \int_{\Omega} G_{\rho,\gamma}(\cdot, y)a(\cdot).
\]

From (4.3), we have \( \|v_\varepsilon\|_\infty \leq C \), where \( C \) depends only on \( \mu, \omega(t), \kappa, m, \lambda \) and \( \Omega \). Then we have

\[
\left| \int_{\Omega} G_{\rho,\varepsilon}(x, y)a(x)dx \right| \leq C.
\]

According to the fact that \( H^1 \) is the dual space of BMO space, we can derive that \( G_{\rho,\varepsilon}(\cdot, y) \) has a uniform boundedness \( C \) in BMO space, where \( C \) depends only on \( \mu, \omega(t), \kappa, m, \lambda \) and \( \Omega \). From Banach-Alaoglu theorem, we have, for all \( y \in \Omega \), there exists a sequence \( \rho_j \) such that \( \rho_j \to 0 \) and functions \( G_{\rho_j,\varepsilon}(\cdot, y) \in \text{BMO}(\Omega) \) such that \( G_{\rho_j,\varepsilon}(\cdot, y) \) converge \( G_{\varepsilon}(\cdot, y) \) in \( \text{BMO}(\Omega) \) space in the sense of the weak-* topology. For \( F \in L^q(\Omega; \mathbb{R}^m) \) where \( q > 1 \), we can choose \( 1 < q_1 < q \), \( p = \frac{2q}{2-q_1} > 2 \) and \( u_\varepsilon \in W^{1,p}(\Omega; \mathbb{R}^m) \), such that \( L_\varepsilon^*(u_\varepsilon) = F \). Then, we have

\[
(4.6) \quad u_\varepsilon = \int_{\Omega(y, \rho)} G_{\rho,\varepsilon}(x, y)F(x)dx.
\]

Of course, this is still a certain distance from the representation theorem. Next, we will prove the representation theorem and the uniqueness of the Green function. Set \( G_{\rho,\varepsilon}^{\alpha\beta}(\cdot, x) \) is the average Green matrix for \( L_\varepsilon \). Then, by the definition of the average Green matrix for \( L_\varepsilon \), we have

\[
\int_{\Omega(y, \rho)} G_{\rho,\varepsilon}^{\alpha\beta}(x, z)dz = \langle L_\varepsilon(G_{\rho,\varepsilon}^{\alpha\beta}(\cdot, y)), G_{\rho,\varepsilon}^{\beta\varepsilon}(\cdot, x) \rangle = \langle G_{\rho,\varepsilon}^{\alpha\varepsilon}(\cdot, y), L_\varepsilon(G_{\rho,\varepsilon}^{\alpha\beta}(\cdot, x)) \rangle = \int_{\Omega(x, \rho)} G_{\rho,\varepsilon}^{\alpha\beta}(z, y)dz.
\]

Choose \( \rho_n \) and \( \tilde{\rho}_n \) are two sequences such that the two average Green matrices converge. Letting \( n \to \infty \) at the same time we have

\[
(4.7) \quad G_{\varepsilon}^{\alpha\beta}(y, x) = G_{\varepsilon}^{\alpha\beta} (x, y)
\]

and the representation theorem can be obtained by this and (4.6). It is easy to verify the uniqueness of the Green function. If we assume that \( \tilde{G}_\varepsilon(x, y) \) is another Green function that satisfies the properties above, we can take \( F \in C^\infty_0(\Omega) \) and the Dirichlet problem \( L_\varepsilon(u_\varepsilon) = F \) in \( \Omega \) with \( u_\varepsilon = 0 \) on \( \partial \Omega \) has the unique solution
(taking by the representation theorem) \( u_e(x) = \int_\Omega G_e(x, y)F(y)dy \) and \( u_e(x) = \int_\Omega \tilde{G}_e(x, y)F(y)dy \). Then we have
\[
\int_\Omega (G_e(x, y) - \tilde{G}_e(x, y))F(y)dy = 0.
\]
According to the arbitrariness of \( F \), the uniqueness of the Green function can be obtained.

Finally, we will prove the pointwise estimates for the Green functions. Namely, we will prove (1.12)-(1.18).

Letting \( x_0, y_0 \in \Omega \) and assuming that \( \delta(x_0) < \frac{1}{4}|x_0 - y_0| = \frac{1}{4}r \), we have \( \Omega(x_0, \frac{r}{2}) \subset \Omega \setminus \{y_0\} \). According to the definition of \( G_e^r(\cdot, y_0) \), we have \( \mathcal{L}_e(G_e^r(\cdot, y_0)) = 0 \) in \( \Omega(x_0, \frac{r}{2}) \) and \( G_e^r(\cdot, y_0) = 0 \) on \( \Omega \setminus B(x_0, \frac{r}{2}) \).

Then, by using (3.12), we have
\[
|G(x_0, y_0)| \leq C \|G(\cdot, y_0)\|_{L^\infty(\Omega(x_0, \frac{r}{2}))} \leq C \int_{\Omega(x_0, \frac{r}{2})} |G(z, y_0)|dz.
\]
According to the definition of \( \text{BMO}(\Omega) \) and noticing that \( G(z, y_0) \neq 0 \) by (4.1), we have
\[
|G(x_0, y_0)| \leq C \int_{\Omega(x_0, \frac{r}{2})} |G(z, y_0) - G(z, y_0)|dz \leq C \|G(\cdot, y_0)\| \leq C,
\]
where \( C \) depends only on \( \mu, \omega(t), \kappa, m, \lambda \) and \( \Omega \). Then for \( \delta(x) < \frac{1}{4}|x - y| \), we can obtain
\[
|G_e(x, y)| \leq C,
\]
where \( C \) depends only on \( \mu, \kappa, m, \lambda, \omega(t) \) and \( \Omega \). Assume that \( \delta(x_0) < \frac{1}{4}|x_0 - y_0| = \frac{1}{4}r \), and \( z_0 \) is chosen such that \( |x_0 - z_0| = \delta(x_0) \), we have \( \mathcal{L}_e(G_e^r(\cdot, y_0)) = 0 \) in \( \Omega(x_0, \frac{r}{2}) \) and \( G_e^r(\cdot, y_0) = 0 \) on \( \Omega \setminus B(z_0, \frac{r}{2}) \).

According to the localized boundary H"older estimates, (3.11), we have, for all \( \sigma_1 \in (0, 1) \) by letting \( v_e(x) = G_e^r(x, y_0) \), there is
\[
|v_e(x_0)| = |v_e(x_0) - v_e(z_0)| + |v_e(z_0)| \leq |x_0 - z_0|^{\sigma_1}\|v_e\|_{C^{0,\sigma_1}(\Omega(z_0, \frac{r}{2}))}
\]
\[
\leq C \left( \frac{\delta(x_0)}{r} \right)^{\sigma_1} \int_{\Omega(z_0, \frac{r}{2})} |v_e|^2dx.
\]
Here, we notice that for all \( x \in \Omega(z_0, \frac{r}{2}) \), \( \delta(x) < \frac{1}{4}r \). Then by using (4.8), we have
\[
|v_e(x_0)| \leq C \left( \frac{\delta(x_0)}{r} \right)^{\sigma_1} \int_{\Omega(z_0, \frac{r}{2})} |v_e|^2dx
\]
\[
\leq C \left( \frac{\delta(x_0)}{r} \right)^{\sigma_1} \left( \frac{\|v_e\|_{C^{0,\sigma_1}(\Omega(z_0, \frac{r}{2}))}}{\|\delta(x_0)\|_{L^\infty(\Omega(z_0, \frac{r}{2}))}} \right).
\]

For all \( y \in B(z_0, \frac{7r}{16}) \cap \Omega \), \( |y - y_0| \geq |x_0 - y_0| - |x_0 - y| \geq \frac{9}{16}r \) and \( \delta(y_0) \leq \frac{1}{4}|x_0 - y_0| = \frac{1}{4}r \leq \frac{7}{16}r \). So according to the above description, we can also obtain that
\[
|v_e(y)| \leq C \left( \frac{\delta(y_0)}{|y - y_0|^{\sigma_2}} \right)^{\sigma_1} \text{ for any } y \in \Omega(z_0, \frac{7r}{16}).
\]
Then in view of (4.9),
\[
|v_e(x_0)| \leq C \left( \frac{\delta(x_0)}{r} \right)^{\sigma_1} \left( \frac{\delta(y_0)^{2\sigma_2}}{|y - y_0|^{2\sigma_2}dx} \right)^{\frac{1}{2}} \leq C \left( \frac{\delta(x_0)}{|x_0 - y_0|^{\sigma_1+\sigma_2}} \right).
\]

At last, when \( \delta(x_0) \geq \frac{1}{4}|x_0 - y_0| \) and \( \delta(y_0) \geq \frac{1}{4}|x_0 - y_0| \), we can choose \( F \in C_0^\infty(\Omega(x_0, \frac{r}{2}); R^m) \). Obviously, here, \( \Omega(x_0, \frac{r}{2}) = B(x_0, \frac{r}{2}) \). Let \( w_e \) satisfies \( \mathcal{L}_e^r(w_e) = F \) in \( \Omega \) and \( w_e = 0 \) on \( \partial \Omega \). From the representation theorem, we can easily obtain that \( w_e(y) = \int_\Omega G_e(z, y)F(z)dz \). Since \( F \equiv 0 \) in \( \Omega \setminus \Omega(x_0, \frac{r}{2}) \), we have,
For all \( p > 2 \) and \( 1 < q < 2 \), by using the \( W^{1,p} \) estimates (3.8) and Sobolev embedding theorem, we have
\[
\left| w_\varepsilon(y_0) \right| \leq C \left( \frac{1}{\varepsilon} \int_{\Omega_{y_0, r_0}} |w_\varepsilon(z)|^2 dz \right)^{\frac{1}{p}} \leq C \left( \frac{1}{\varepsilon} \int_{\Omega_{y_0, r_0}} |w_\varepsilon(z)|^p dz \right)^{\frac{1}{p}} = C \varepsilon^{-\frac{2}{p}} \left( \int_{\Omega} |\nabla w_\varepsilon(z)|^2 dz \right)^{\frac{1}{2}} \leq C \varepsilon^{-\frac{2}{p} + \frac{2}{q} - 1} \left( \int_{\Omega_{y_0, r_0}} |F|^2 dz \right)^{\frac{1}{2}}.
\]
This implies that
\[
\int_{\Omega(x_0, \varepsilon)} G_\varepsilon(z, y_0) F(z) dz \leq C \varepsilon^{-\frac{2}{p} + \frac{2}{q} - 1} \left( \int_{\Omega_{y_0, r_0}} |F|^2 dz \right)^{\frac{1}{2}}.
\]
From (4.10), we have
\[
\left( \int_{\Omega_{x_0, \varepsilon}} |G_\varepsilon(z, y)|^2 dz \right)^{\frac{1}{2}} \leq C \varepsilon^{-\frac{2}{p} + \frac{2}{q} - 1},
\]
by using the duality arguments. For all \( \sigma \in (0, 1) \), we can choose special \( p, q \) such that \( -\frac{2}{p} + \frac{2}{q} - 2 = -\sigma \).
(4.10), together with (3.12) gives the proof of (1.12). Of course, there is still a certain distance between this and (1.16), so we need to make more precise estimates.

For \( x_0, y_0 \in \Omega \), let \( r_1 = \frac{1}{2} |x_0 - y_0| \), we see that if \( \delta(x_0) < \frac{1}{4} |x_0 - y_0| \), according to (4.8), we can obtain \( |G_\varepsilon(x, y_0)| \leq C \). If \( \delta(x_0) \geq \frac{1}{2} |x_0 - y_0| \), we can consider the sequence of subsets of \( \Omega \) denoted as \( \Omega_j = \Omega(x_0, 2^j r_1) \) with \( j = 0, 1, ..., N \) such that \( 2^N r_1 \geq \text{diam}(\Omega) \). Note that \( N \leq C \left( 1 + \ln \left( \frac{\text{diam}(\Omega)}{|x_0 - y_0|} \right) \right) \).
According to the fact that \( G_\varepsilon(\cdot, y_0) \in \text{BMO}(\Omega) \), we obtain
\[
\left| \int_{\Omega_j} G_\varepsilon(x, y_0) - \int_{\Omega_{j+1}} G_\varepsilon(x, y_0) dx \right| \leq C \| G_\varepsilon(\cdot, y_0) \|_s \leq C.
\]
With the choice of \( N \), we get
\[
\int_{\Omega_N} |G_\varepsilon(x, y_0)| dx = \int_{\Omega} |G_\varepsilon(x, y_0)| dx \leq C \| G_\varepsilon(\cdot, y_0) \|_s \leq C.
\]
Setting \( J = \int_{\Omega_0} G_\varepsilon(x, y_0) dx \) and \( \theta_\varepsilon = G_\varepsilon(x, y_0) - J \), we have
\[
L_\varepsilon(\theta_\varepsilon) = \text{div}(V(x/\varepsilon) J + (-c(x/\varepsilon) J - \lambda J) \in B(x_0, r_1).
\]
In the view of (3.12), we can choose \( p > 2, q = \frac{2p}{p+2}, f = V(x/\varepsilon) J \) and \( F = -c(x/\varepsilon) J - \lambda J \). Then, we can obtain
\[
|\theta_\varepsilon(x_0)| \leq C \left\{ \int_{B(x_0, r_1)} |\theta_\varepsilon| dx + \int_{B(x_0, r_1)} |f|^p dx \right\}^{\frac{1}{p}} + r_1^2 \left( \int_{B(x_0, r_1)} |F|^q dx \right)^{\frac{1}{q}} \leq C \left\{ \int_{B(x_0, r_1)} \left( G_\varepsilon(x, y_0) - \int_{\Omega_0} G_\varepsilon(z, y_0) dz \right) dx + |J| r_1 + |J| r_1^2 \right\},
\]
(4.14)

The third inequality in (4.14) is derived from (4.12) and
\[
\int_{B(x_0, r_1)} \left| G_\varepsilon(x, y_0) - \int_{\Omega_0} G_\varepsilon(z, y_0) dz \right| dx \leq C \| G_\varepsilon(\cdot, y_0) \|_s \leq C.
\]
Then we can see that
\[
\left| G_\varepsilon(x_0, y_0) - \int_{\Omega_0} G_\varepsilon(z, y_0) dz \right| \leq C(1 + |J| r_1 + |J| r_1^2).
\]
According to (4.12) and (4.13), we have
\[
|G_\varepsilon(x_0, y_0)| \leq C \sum_{j=1}^{N} \left| \int_{\Omega_j} G_\varepsilon(x, y_0)dx - \int_{\Omega_{j+1}} G_\varepsilon(x, y_0)dx \right| + \int_{\Omega_N} |G_\varepsilon(x, y_0)|dx
\]
\[
\leq C \left( 1 + \ln \left( \frac{\text{diam}(\Omega)}{|x_0 - y_0|} \right) \right) + C \left| G_\varepsilon(x_0, y_0) - \int_{\Omega_0} G_\varepsilon(x, y_0)dx \right|,
\]
where we denote \( f_{\Omega_0} G_\varepsilon(x, y_0) = G_\varepsilon(x_0, y_0) \), and \( C \) depends only on \( \mu, \omega(t), \kappa, \lambda, m, p \). Set \( \sigma \in (0, 1) \), by (1.12), we get
\[
|J| = \left| \int_{\Omega_0} G_\varepsilon(x, y_0)dx \right| \leq \int_{\Omega_0} |G_\varepsilon(x, y_0)|dx \leq C \int_{B(x_0, r_1)} |x - y|^{-\sigma}dx \leq C r_1^{-\sigma}.
\]
From (4.14), we can obtain \(|\theta_\varepsilon(x_0)| \leq C(1 + r_1^{-\sigma} + r_1^{2-\sigma}) \leq C \) since \( r_1 \leq \text{diam}(\Omega) \). Then
\[
|G_\varepsilon(x_0, y_0) - \int_{\Omega_0} G_\varepsilon(x, y_0)dx| \leq C.
\]
Therefore, we can obtain the proof of (1.16), that is \(|G_\varepsilon(x, y)| \leq C \left( 1 + \ln \left( \frac{\text{diam}(\Omega)}{|x - y|} \right) \right) \), where \( C \) depends only on \( \mu, \omega(t), \kappa, \lambda, m, p, \Omega \). Finally, we need to prove (1.17)-(1.18). We only need to prove (1.17) and (1.18) follows directly from (4.7). For \( 0 < \sigma_3 < 1 \), \( x_0, y_0, z_0 \in \Omega \) such that \(|x_0 - z_0| < \frac{1}{2}|x_0 - y_0| \), we can define \( r_2 = |x_0 - y_0| \) and have
\[
|G_\varepsilon(x_0, y_0) - G_\varepsilon(z_0, y_0)| \leq |G_\varepsilon(x_0, y_0)|_{C^0, \sigma_3(\Omega(x_0, \frac{2}{3}r_2))}|x_0 - z_0|^{\sigma_3}.
\]
If \( \delta(x_0) < \frac{2}{3}r_2 \), noticing that \( L_\varepsilon(G_\varepsilon^\gamma(x, y_0)) = 0 \) in \( \Omega(x_0, \frac{2}{3}r_2) \) and \( G_\varepsilon^\gamma(\cdot, y_0) = 0 \) on \( \partial \Omega \cap B(x_0, \frac{2}{3}r_2) \), we get
\[
|G_\varepsilon(x_0, y_0) - G_\varepsilon(z_0, y_0)| \leq |G_\varepsilon(x_0, y_0)|_{C^0, \sigma_3(\Omega(x_0, \frac{2}{3}r_2))}|x_0 - z_0|^{\sigma_3}
\]
\[
\leq C|x_0 - z_0|^{\sigma_3}r_2^{1-\sigma_3} \left( \int_{\Omega(x_0, \frac{2}{3}r_2)} |G_\varepsilon(x, y_0)|^2dx \right)^{\frac{1}{2}}
\]
\[
\leq C|x_0 - z_0|^{\sigma_3}r_2^{1-\sigma_3} \left( \int_{\Omega(x_0, \frac{2}{3}r_2)} |G_\varepsilon(x, y_0) - G_\varepsilon(x_0, y_0)|_{\Omega(x, \frac{4}{3}r_2)}^2dx \right)^{\frac{1}{2}}
\]
\[
\leq C|x_0 - z_0|^{\sigma_3},
\]
where we have used (3.11), (4.4) and (4.1). If \( \delta(x_0) > \frac{2}{3}r_2 \), we can choose \( J_3 = G_\varepsilon^\gamma(x_0, y_0)_{x_0, \frac{4}{3}r_2} \) and have
\[
L_\varepsilon(G_\varepsilon^\gamma(x, y_0) - J_1) = \text{div}(V(x/\varepsilon)J_1) - c(x/\varepsilon)J_1 - \lambda J_1.
\]
From (1.12) and the fact that \( r_2 \leq \text{diam}(\Omega) \), we can obtain
\[
(4.16) \quad r_2|J_1| + r_2^2|J_1| \leq C.
\]
From (3.11), we have
\[
|G_\varepsilon(x_0, y_0)|_{C^0, \sigma_3(\Omega(x_0, \frac{2}{3}r_2))} \leq C r_1^{-\sigma_3} \left( \int_{\Omega(x_0, \frac{2}{3}r_2)} |G_\varepsilon(x, y_0) - J_1|^2dx \right)^{\frac{1}{2}} + r_2|J_1| + r_2^2|J_1|.
\]
This, together with (4.4) and (4.16), we can obtain
\[
(4.17) \quad |G_\varepsilon(x, y_0)|_{C^0, \sigma_3(\Omega(x_0, \frac{4}{3}r_2))} \leq C|x_0 - y_0|^{-\sigma_3}.
\]
Combining (4.16) and (4.17), we can obtain (1.17) and finish the proof.

**Remark 4.6.** From (1.13)-(1.16), we can obtain that for any \( 0 < \sigma, \sigma_1, \sigma_2 < 1 \),
\[
(4.18) \quad |G_\varepsilon(x, y)| \leq \frac{C}{|x - y|^\sigma} \min \left\{ \frac{|\delta(x)|^{\sigma_1}}{|x - y|^{\sigma_1}}, \frac{|\delta(y)|^{\sigma_2}}{|x - y|^{\sigma_2}}, \frac{|\delta(x)|^{\sigma_1} |\delta(y)|^{\sigma_2}}{|x - y|^{\sigma_1 + \sigma_2}} \right\},
\]
where \( C \) depends only on \( \sigma_1, \sigma_2, \sigma, \mu, \omega(t), \kappa, \lambda, m, \) and \( \Omega \).
Remark 4.7. For $R \leq \delta(y)$, $B(y, R)$ being a ball with center $y$, radius $R$, and $F \in C_0^\infty(\Omega; \mathbb{R}^m)$, we can consider $L_\varepsilon^*(u_\varepsilon) = F$ in $\Omega$ and $u_\varepsilon = 0$ on $\partial \Omega$. Then, there exists a unique $u_\varepsilon \in H_0^1(\Omega; \mathbb{R}^m)$ such that

$$u_\varepsilon(y) = \int_\Omega G_\varepsilon(x, y) F(x) \, dx.$$ 

If $F \in C_0^\infty(\Omega; \mathbb{R}^m)$ and supp($F$) $\subseteq B(y, R) \subseteq \Omega$, then, by (3.12), we have

$$\left| \int_\Omega G_\varepsilon(x, y) F(x) \, dx \right| \leq \|u_\varepsilon\|_{L^\infty(B(y, R))} \leq C \left( \int_{B(y, \frac{1}{2}R)} |u_\varepsilon|^2 \, dx \right)^{\frac{1}{2}} + CR^2 \left( \int_{B(y, \frac{1}{4}R)} |F|^2 \, dx \right)^{\frac{1}{2}}.$$ 

Note that $u_\varepsilon = 0$ on $\partial \Omega$, we can use Poincaré’s inequality and obtain

$$\left( \int_{B(y, \frac{1}{2}R)} |u_\varepsilon|^2 \, dx \right)^{\frac{1}{2}} \leq CR^{-1} \left( \int_{B(y, \frac{1}{2}R)} |u_\varepsilon^\ast|^2 \, dx \right)^{\frac{1}{2}} \leq C \delta(y) R^{-1} \left( \int_\Omega |\nabla u_\varepsilon|^2 \, dx \right)^{\frac{1}{2}} \leq C \delta(y) R^{-1} \left( \int_\Omega |\nabla u|^2 \, dx \right)^{\frac{1}{2}}.$$ 

From Theorem 2.2, we have

$$\left( \int_{B(y, \frac{1}{2}R)} |u_\varepsilon|^2 \, dx \right)^{\frac{1}{2}} \leq C \delta(y) R^{-1} \left( \int_\Omega |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \leq C \delta(y) R^{-1} \|F\|_{H^{-1}(\Omega)}.$$ 

On the other hand, since $F \in L^2(\Omega; \mathbb{R}^m)$, we note that actually, $(F, \varphi)_{H^{-1}(\Omega) \times H^1_0(\Omega)} = \int_\Omega F(x) \varphi(x) \, dx$. Then, by using Poincaré’s inequality,

$$\|F\|_{H^{-1}(\Omega)} = \sup_{\|\varphi\|_{H^1_0(\Omega)} = 1} \left| \int_\Omega F(x) \varphi(x) \, dx \right| \leq \|F\|_{L^2(B(y, R))} \|\varphi\|_{L^2(B(y, R))} \leq CR \|\varphi\|_{L^2(B(y, R))} \|\nabla \varphi\|_{L^2(B(y, R))} \leq CR \|\varphi\|_{L^2(B(y, R))}.$$ 

This, together with (4.19) implies that

$$\left( \int_{B(y, \frac{1}{2}R)} |u_\varepsilon|^2 \, dx \right)^{\frac{1}{2}} \leq C R \delta(y) \left( \int_{B(y, R)} |F|^2 \, dx \right)^{\frac{1}{2}}.$$ 

Then, $\left| \int_\Omega G_\varepsilon(x, y) F(x) \, dx \right| \leq C R \delta(y) \left( \int_{B(y, R)} |F|^2 \, dx \right)^{\frac{1}{2}}$, where $C$ depends only on $\mu, \kappa, \lambda, m, \omega(t)$ and $\Omega$.

Using the duality methods, we have

$$\left( \int_{B(y, \frac{1}{2}R)} |G_\varepsilon(x, y)|^2 \, dx \right)^{\frac{1}{2}} \leq C \frac{\delta(y)}{R},$$

where $C$ depends only on $\mu, \omega(t), \kappa, \lambda, m$, and $\Omega$. If $\delta(y) \leq CR$, we have

$$\left( \int_{B(y, R)} |G_\varepsilon(x, y)|^2 \, dx \right)^{\frac{1}{2}} \leq C.$$ 

The following lemma is essential for the proof of Theorem 1.2. This lemma is from [19]. The author gave the proof for the elliptic operator without lower order terms. What is remarkable is that, in [19], for the elliptic operator $L_\varepsilon$ without lower order terms, the author does not give the proof for the case $d = 2$.

**Lemma 4.8.** Suppose that $A$ satisfies (1.2), (1.3) and VMO condition (1.8). Other coefficients of $L_\varepsilon$, $V, B, c$ satisfy (1.4). $\Omega$ is a $C^{1, \eta}$ $(0 < \eta < 1)$ bounded domain in $\mathbb{R}^2$. Then, for all $\sigma \in (0, 1)$, we have

$$\int_\Omega |\nabla y G_\varepsilon(x, y)| |\delta(y)|^{\sigma-1} \, dy \leq C[\delta(x)]^\sigma,$$ 

where $C$ depends only on $\mu, \omega(t), \kappa, \lambda, m, \sigma$ and $\Omega$. 

Proof. Set $x_0 \in \Omega$ and $r = \frac{1}{2}\sigma(x_0)$. For all $R < \frac{1}{4}\delta(x_0)$, we consider the annulus $B(x_0, 2R)\setminus B(x_0, R)$. We can use small two dimensional balls with radius of $\frac{1}{8}R$ whose center are on the circle $B(x_0, \frac{5}{8}R)$ to cover the annulus $B(x_0, 2R)\setminus B(x_0, R)$. We denote these small balls as $\{B(x_i, \frac{5}{8}R)\}_{i=1}^N$. Obviously, we have $N \leq C \frac{(4R^2-R^2)}{\pi(\frac{5}{8}R)^2} \leq C$, where $C$ is a constant, independent of $R$. The specific positional relationship is shown in Figure 1, where $r_1 = \frac{2}{3}R$, $r_2 = \frac{5}{8}R$ and $r_3 = 2R$.

Figure 1. Balls $B(x_i, \frac{5}{8}R)$ construct a covering of $B(x_0, 2R)\setminus B(x_0, R)$.

Then, for $u_\varepsilon(y) = G_\varepsilon(x_0, y)$, we have $\mathcal{L}^*_\varepsilon(u_\varepsilon) = 0$ in $B(x_0, 3R)\setminus B(x_0, \frac{1}{3}R)$. Using Hölder’s inequality, we have

$$\int_{B(x_0, 2R)\setminus B(x_0, R)} |\nabla_y G_\varepsilon(x_0, y)| dy \leq C \left( \int_{B(x_0, 2R)\setminus B(x_0, R)} |\nabla_y G_\varepsilon(x_0, y)|^2 dy \right)^{\frac{1}{2}} R \leq C \sum_{i=1}^N \left( \int_{B(x_i, \frac{5}{8}R)} |\nabla_y G_\varepsilon(x_0, y)|^2 dy \right)^{\frac{1}{2}} R.$$

Using the previous annotation of Caccioppoli’s inequality (2.10), we have

$$\left( \int_{B(x_i, \frac{5}{8}R)} |\nabla_y G_\varepsilon(x_0, y)|^2 dy \right)^{\frac{1}{2}} \leq \frac{C}{R} \left( \int_{B(x_i, \frac{5}{8}R)} \left| G_\varepsilon(x_0, y) - \int_{B(x_i, \frac{5}{8}R)} G_\varepsilon(x_0, z) dz \right|^2 dy \right)^{\frac{1}{2}} + C \left| \int_{B(x_i, \frac{5}{8}R)} G_\varepsilon(x_0, z) dz \right| R + C \left| \int_{B(x_i, \frac{5}{8}R)} G_\varepsilon(x_0, z) dz \right| R^2.$$

For any $\sigma_1 \in (0, 1)$, by using (1.12), we have

$$\text{(4.22)} \quad \left| \int_{B(x_i, \frac{5}{8}R)} G_\varepsilon(x_0, z) dz \right| \leq \int_{B(x_i, \frac{5}{8}R)} |G_\varepsilon(x_0, z)| dz \leq CR^{-\sigma_1}.$$

Since $G_\varepsilon(x_0, y) \in \text{BMO}(\Omega)$, in view of (4.4), we have

$$\text{(4.23)} \quad \left( \int_{B(x_i, \frac{5}{8}R)} \left| G_\varepsilon(x_0, y) - \int_{B(x_i, \frac{5}{8}R)} G_\varepsilon(x_0, z) dz \right|^2 dy \right)^{\frac{1}{2}} \leq C \|G_\varepsilon(x_0, y)\|_\ast \leq C.$$
Combining it with (4.22), (4.23) and the fact that $R \leq \text{diam}(\Omega)$, we can obtain

$$\left( \int_{B(x, \frac{3}{2}R)} |\nabla_y G_\varepsilon(x_0, y)|^2 dy \right)^{\frac{1}{2}} \leq C,$$

where $C$ depends only on $\mu, \omega(t), \kappa, \lambda, m, \sigma_1$ and $\Omega$. Then

$$\int_{B(x_0, 2R) \setminus B(x_0, R)} |\nabla_y G_\varepsilon(x_0, y)| dy \leq CR.$$

Therefore, we have

$$\int_{B(x_0, r)} |\nabla_y G_\varepsilon(x_0, y)| dy \leq \sum_{j=0}^{\infty} \int_{B(x_0, 2^{-j}r) \setminus B(x_0, 2^{-j-1}r)} |\nabla_y G_\varepsilon(x_0, y)| dy$$

$$\leq C \sum_{j=1}^{\infty} 2^{-j} r + \int_{B(x_0, r) \setminus B(x_0, 4r)} |\nabla_y G_\varepsilon(x_0, y)| dy$$

$$\leq Cr + \left( \int_{B(x_0, r) \setminus B(x_0, 4r)} |\nabla_y G_\varepsilon(x_0, y)|^2 dy \right)^{\frac{1}{2}} r$$

$$\leq Cr + \left( \int_{B(x_0, 4r) \setminus B(x_0, 4r)} |G_\varepsilon(x_0, y)|^2 dy \right)^{\frac{1}{2}} r \leq Cr.$$

For the last inequality, we have used (4.20). Then, we have

$$(4.24) \int_{B(x_0, r)} |\nabla_y G_\varepsilon(x_0, y)| |\delta(y)|^{\sigma-1} dy \leq Cr^\sigma.$$

Next, to estimate the integral on $\Omega \setminus B(x_0, r)$, we observe that if $Q$ is a cube in $\mathbb{R}^2$ with the property $3Q \subset \Omega \setminus \{x_0\}$ and its side length $l(Q) \sim \text{dist}(Q, \partial \Omega)$, then

$$\int_{Q} |\nabla_y G_\varepsilon(x_0, y)| |\delta(y)|^{\sigma-1} dy \leq C[l(Q)]^{\sigma-1}|Q| \left( \int_{Q} |\nabla_y G_\varepsilon(x_0, y)|^2 dy \right)^{\frac{1}{2}}$$

$$\leq C[l(Q)]^{\sigma-2}|Q| \left( \int_{2Q} |G_\varepsilon(x_0, y)|^2 dy \right)^{\frac{1}{2}},$$

where we have used Caccioppoli’s inequality (2.8) for the last step. This, together with the pointwise estimates (1.13)-(1.15), gives

$$\int_{Q} |\nabla_y G_\varepsilon(x_0, y)| |\delta(y)|^{\sigma-1} dy \leq Cr^{\sigma_1} [l(Q)]^{\sigma_2+\sigma_2-2}|Q| \left( \int_{2Q} \frac{dy}{|x_0-y|^{\sigma_1+\sigma_2}} \right)^{\frac{1}{2}}$$

$$\leq Cr^{\sigma_1} [l(Q)]^{\sigma_2+\sigma_2-2} \int_{2Q} \frac{dy}{|x_0-y|^{\sigma_1+\sigma_2}}$$

$$(4.25) \leq Cr^{\sigma_1} \int_{\Omega \setminus \{x_0\}} \frac{|\delta(y)|^{\sigma_2+\sigma_2-2}}{|x_0-y|^{\sigma_1+\sigma_2}} dy,$$

where $0 < \sigma_1, \sigma_2 < 1$, and we have used the observation that $\delta(y) \sim l(Q)$ for $y \in 2Q$ and $|x-y| \sim |x-z|$ for any $y, z \in 2Q$. Finally, we perform a Whitney decomposition on $\Omega$ (see [20]). This gives $\Omega = \bigcup_{j} Q_j$ where $\{Q_j\}$ is a sequence of (closed) non-overlapping cubes with the property that $4Q_j \subset \Omega$ and $l(Q_j) \sim \text{dist}(Q_j, \partial \Omega)$. Let

$$\mathcal{O} = \cup_{3Q_j \subset \Omega \setminus \{x_0\}} Q_j.$$

Note that if $y \in \Omega \setminus \mathcal{O}$, then $y \in Q_j$ for some $Q_j$ such that $x_0 \in 3Q_j$. It follows that $|y-x_0| \leq Cl(Q_j) \leq C\delta(x_0)$. Hence

$$\int_{\Omega \setminus \mathcal{O}} |\nabla_y G_\varepsilon(x_0, y)| |\delta(y)|^{\sigma-1} dy \leq Cr^\sigma.$$
by (4.20) and (4.24). By the summation, the estimate (4.25) leads to

\[(4.26) \quad \int_{\Omega \setminus B(x_0, r)} |\nabla_y G_\varepsilon(x_0, y)||\delta(y)||^{\sigma - 1} dy \leq C_{\varepsilon} \int_{\Omega} \frac{[\delta(y)]^{\sigma + \sigma_2 - 2}}{|x_0 - y + r|^{\sigma_1 + \sigma_2}} dy,
\]

where we have used the fact that $|x_0 - y| \geq c_1 \delta(x_0) = c_1 r$ for any $y \in \Omega \setminus B(x_0, r)$ with some positive constant $c_1$. Since $\Omega$ is $C^{1,\alpha}$, the integral in the RHS of (4.26) is bounded by

\[
C \int_0^\infty \frac{t^{\sigma + \sigma_2 - 2}}{(t + r)^{\sigma_1 + \sigma_2}} dt \leq C \int_0^\infty \frac{t^{\sigma + \sigma_2 - 2}}{(t + r)^{\sigma_1 + \sigma_2 - 1}} dt \leq C r^{\sigma - \sigma_1},
\]

where we have chosen $\sigma_1, \sigma_2 \in (0, 1)$ so that $\sigma_1 + \sigma_2 > 1$ and $\sigma < \sigma_1 < 1$. This, together with (4.24) and (4.26), completes the proof.

Using the above lemma, we will prove Theorem 1.2.

The proof of Theorem 1.2. First we assume that $u_{\varepsilon, 1}$ satisfies the equation

\[L_\varepsilon(u_{\varepsilon, 1}) = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad u_{\varepsilon, 1} = g \quad \text{on} \quad \partial \Omega.
\]

Choose $v$ such that $\Delta v^\alpha = 0$ in $\Omega$ and $v^\alpha = g^\alpha$ on $\partial \Omega$. By the classical interior Lipschitz estimates for harmonic functions, Caccioppoli’s inequality (2.8) and Hölder estimate:

\[
|v|_{C^{0, \alpha}(\Omega)} \leq C \|g\|_{C^{0, \alpha}(\partial \Omega)},
\]

we have

\[
|\nabla v(x)| \leq C \left( \int_{B(x, \delta(x))} |\nabla v|^2 dy \right)^{\frac{1}{2}} \leq \frac{C}{\delta(x)} \left( \int_{B(x, \frac{\delta(x)}{2})} |v(y) - v(x)|^2 dy \right)^{\frac{1}{2}} \leq C[\delta(x)]^{\sigma - 1} |v|_{C^{0, \alpha}(\Omega)} \leq C[\delta(x)]^{\sigma - 1} \|g\|_{C^{0, \alpha}(\partial \Omega)},
\]

for all $\sigma \in (0, 1)$. Let $w_\varepsilon = u_{\varepsilon, 1} - v$, we get

\[L_\varepsilon(w_\varepsilon) = -L_\varepsilon(v) \quad \text{in} \quad \Omega \quad \text{and} \quad w_\varepsilon = 0 \quad \text{on} \quad \partial \Omega.
\]

Using the normalization, we can assume that $\|g\|_{C^{0, \alpha}(\Omega)} = 1$. According to the representation theorem (1.10), we obtain

\[
w_\varepsilon(x) = -\int_{\Omega} \nabla_y G_\varepsilon(x, y)[A(y/\varepsilon)\nabla v + V(y/\varepsilon)v] dy - \int_{\Omega} G_\varepsilon(x, y)[B(y/\varepsilon)\nabla v + (c(y/\varepsilon) + \lambda)v] dy,
\]

which implies that

\[
|w_\varepsilon(x)| \leq C \int_{\Omega} |\nabla_y G_\varepsilon(x, y)||\delta(y)||^{\sigma - 1} dy + C \int_{\Omega} |G_\varepsilon(x, y)||\delta(y)||^{\sigma - 1} dy
\]

\[
+ C \int_{\Omega} \int |\nabla_y G_\varepsilon(x, y)| + |G_\varepsilon(x, y)| dy \leq I_1 + I_2 + I_3,
\]

where we use the fact that $v$ is bounded in $\Omega$ as a result of the maximum principle. In view of (4.21), we have $I_1 \leq C[\delta(x)]^\sigma$. Next, we will estimate $I_2$. In fact, by using (4.18), we get

\[
I_2 \leq \left( \int_{\Omega \setminus \{\delta(y) < \frac{1}{4}|x - y|\}} \right) |G_\varepsilon(x, y)||\delta(y)||^{\sigma - 1} dy
\]

\[
\leq \int_{\Omega \setminus \{\delta(y) < \frac{1}{4}|x - y|\}} \frac{C[\delta(x)]^\sigma |\delta(y)||^{\sigma - 1}}{|x - y|^2} dy + \int_{\Omega \setminus \{\delta(y) \geq \frac{1}{4}|x - y|\}} \frac{C[\delta(x)]^\sigma |\delta(y)||^{\sigma - 1}}{|x - y|^2} dy \leq C[\delta(x)]^\sigma.
\]

Here, we choose $\sigma_1 \in (0, 1)$ such that $\sigma_1 + \sigma > 1$ and $\sigma_2 \in (0, 1)$. Then, we have $I_2 \leq C[\delta(x)]^\sigma$. Moreover, as $I_3 \leq C(I_1 + I_2)$, we have $I_3 \leq C[\delta(x)]^\sigma$. Therefore,

\[
(4.27) \quad |w_\varepsilon(x)| \leq C[\delta(x)]^\sigma.
\]
For $u_\varepsilon$, we can choose $u_{\varepsilon,2} = u_\varepsilon - u_{\varepsilon,1}$, then, $\mathcal{L}_\varepsilon(u_{\varepsilon,2}) = \text{div}(f) + F$ in $\Omega$ and $u_{\varepsilon,2} = 0$ on $\partial \Omega$. From the Hölder estimate (3.24), we get

$$
\|u_{\varepsilon,2}\|_{C^{\alpha,\sigma}(\Omega)} \leq C \left\{ \|F\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)} \right\}.
$$

So the problem is reduced down to the proof of $\|u_{\varepsilon,1}\|_{C^{\alpha,\sigma}(\Omega)} \leq C$. By the definition of $v$, we only need to show $\|w_\varepsilon\|_{C^{\alpha,\sigma}(\Omega)} \leq C$. We divide it into the following three cases: (1) $|x - y| \leq \frac{1}{4}\delta(x)$; (2) $|x - y| \leq \frac{1}{4}\delta(y)$; (3) $|x - y| > \frac{1}{4}\delta(x)$ and $|x - y| \leq \frac{1}{4}\delta(y)$. For the first case, choose $r = \delta(x)$ and in view of $\mathcal{L}_\varepsilon(u_\varepsilon) = -\mathcal{L}_\varepsilon(v)$ with (3.11), we have

$$
|w_\varepsilon(x) - w_\varepsilon(y)| \leq |w_\varepsilon|_{C^{\alpha,\sigma}(B(x, \frac{r}{4}))} |x - y|^\sigma
$$

$$
\leq C|x - y|^\sigma \left\{ r^{-\sigma} \left( \int_{B(x, \frac{r}{4})} |w_\varepsilon|^2 \,dz \right)^{\frac{1}{2}} + r^{1-\sigma} \left( \int_{B(x, \frac{r}{4})} (|\nabla v| + |v|^p) \,dz \right)^{\frac{1}{q}} \right\}
$$

$$
+ C|x - y|^\sigma r^{2-\sigma} \left( \int_{B(x, \frac{r}{4})} (|\nabla v| + |v|^q) \,dz \right)^{\frac{1}{q}}
$$

$$
\leq C|x - y|^\sigma
$$

where $q = \frac{2p}{p+d}$ and $\sigma = 1 - \frac{2}{p}$. The case (2) is similar to the case (1) and for the case (3), we obtain

$$
|w_\varepsilon(x) - w_\varepsilon(y)| \leq |w_\varepsilon(x)| + |w_\varepsilon(y)| \leq C|x - y|^\sigma.
$$

This completes the proof of Theorem 1.2. \hfill \Box

Using standard localization arguments in [22], we can obtain the following result.

**Theorem 4.9** (localization of Lipschitz estimates). Suppose that $A \in \Lambda(\mu, \tau, \kappa)$, $V$ satisfies (1.3), (1.5), $B$ and $c$ satisfy (1.3), (1.4), $\lambda > \lambda_0$ and $\Omega$ is a bounded $C^{1,\eta}$ domain in $\mathbb{R}^d$ with $(0 < \eta < 1)$, $d \geq 2$. Let $x_0 \in \Omega$ and $0 < r < \text{diam}(\Omega)$. If $\partial \Omega \cap B(x_0, 2r) \neq \emptyset$, assume that $u_\varepsilon \in W^{1,2}(\Omega(x_0, 2r); \mathbb{R}^m)$ is the weak solution to

$$
\begin{cases}
\mathcal{L}_\varepsilon(u_\varepsilon) = F & \text{in } \Omega(x_0, 2r), \\
u_\varepsilon = 0 & \text{on } \Delta(x_0, 2r),
\end{cases}
$$

with $F \in L^p(\Omega(x_0, 2r); \mathbb{R}^m)$ and $p > d$. If $\partial \Omega \cap B(x_0, 2r) = \emptyset$, assume that $u_\varepsilon \in W^{1,2}(B(x_0, 2r); \mathbb{R}^m)$ is a weak solution to

$$
\mathcal{L}_\varepsilon(u_\varepsilon) = F \quad \text{in} \quad B(x_0, 2r),
$$

with the same data $F$ and $p$, then

$$
\|\nabla u_\varepsilon\|_{L^\infty(\Omega(x_0, r))} \leq C r \left\{ \left( \int_{\Omega(x_0, 2r)} |u_\varepsilon|^2 \,dx \right)^{\frac{1}{2}} + r^2 \left( \int_{\Omega(x_0, 2r)} |F|^p \,dx \right)^{\frac{1}{p}} \right\},
$$

where $C$ depends only on $\mu, \tau, \kappa, \lambda, p, d, m, \sigma, \eta$ and $\Omega$.

**Proof of Theorem 1.5.** For (1.20), (1.21) and (1.22), the proof is trivial since we just need to change localization of the Hölder estimates (3.11) to the localization of (4.28). Also, we note that the constant $\frac{1}{r}$ in (1.20)-(1.22) can be replaced by $\frac{1}{4}$. Then, we only need to show (1.23)-(1.25). For $x_0, y_0 \in \Omega$, we can set $r = |x_0 - y_0|$. If $\delta(x_0) < \frac{1}{2}r$, we have, $\mathcal{L}_\varepsilon(G_{\varepsilon}^{0}(\cdot, y_0)) = 0$ in $\Omega(x_0, \frac{r}{2})$ and $G_{\varepsilon}^{0}(\cdot, y_0) = 0$ on $\partial \Omega \cap B(x_0, \frac{r}{2})$. Then, by using (4.8) and (4.28), we can obtain that

$$
|\nabla_x G_{\varepsilon}(x_0, y_0)| \leq C r \left( \int_{B(x_0, \frac{r}{4})} |G_{\varepsilon}(x, y_0)|^2 \,dx \right)^{\frac{1}{2}} \leq \frac{C}{|x_0 - y_0|},
$$

where $C$ depends only on $\mu, \tau, \kappa, \lambda, p, d, m, \sigma, \eta$ and $\Omega$.

If $\delta(y_0) < \frac{1}{2}r$, for any $x \in B(x_0, \frac{r}{4})$, we have $|x - y_0| \geq |x_0 - y_0| - \frac{1}{2}r = \frac{3}{4}r$ and $\delta(y_0) < \frac{1}{2}|x - y_0| \leq \frac{1}{4}|x - y_0|$. From (1.21) and (4.28), it can be obtained that

$$
|\nabla_x G_{\varepsilon}(x_0, y_0)| \leq \frac{C}{r} \left( \int_{B(x_0, \frac{r}{4})} |G_{\varepsilon}(x, y_0)|^2 \,dx \right)^{\frac{1}{2}} \leq \frac{C\delta(y_0)}{|x_0 - y_0|^2}.
$$
Then according to the fact that \( \frac{1}{4} r \leq \delta(x_0) \) and \( \frac{1}{4} r \leq \delta(y_0) \), we have
\[(4.31) \quad \frac{1}{4} |x_0 - y_0| \leq \frac{3}{8} |x_0 - y_0| = \frac{1}{2} \left( r - \frac{1}{4} r \right) \leq \frac{1}{2} |x - y_0| \quad \text{for any } x \in B \left( x_0, \frac{1}{4} r \right).
\]

At first, there exists a point \( \overline{y} \in \partial \Omega \) (see Figure 2), such that
\[(4.32) \quad (x_0 - \overline{y})/ (x_0 - y_0) \text{ and } (x_0 - \overline{y}) \cdot (x_0 - y_0) > 0.
\]

Then according to the fact that \( \frac{1}{4} |x_0 - y_0| = \frac{1}{4} r \leq |y_0 - \overline{y}| \), there always exists a positive integer \( N \in \mathbb{N} \) and a sequence of points \( \{ y_j \}_{j=1}^N \) such that
\[
y_j = x_0 + \frac{5}{4} (y_{j-1} - x_0), \quad j = 1, \ldots, N,
\]
\[
\frac{1}{4} |x_0 - y_{N-1}| \leq |y_{N-1} - \overline{y}| \quad \text{and} \quad \frac{1}{4} |x_0 - y_N| > |y_N - \overline{y}|
\]

Moreover, we have
\[(4.33) \quad \mathcal{L}_\varepsilon (G_\varepsilon (x, y_0) - G_\varepsilon (x, y_1)) = 0 \quad \text{in } B \left( x_0, \frac{1}{4} r \right).
\]

From (1.18) and (4.31), since \( |y_1 - y_0| = \frac{1}{2} |x_0 - y_0| \leq \frac{1}{2} |x - y_0| \), we can obtain that for \( \sigma \in (0, 1) \),
\[(4.34) \quad |G_\varepsilon (x, y_0) - G_\varepsilon (x, y_1)| \leq \frac{C |y_1 - y_0|^{\sigma}}{|x - y_0|^{\sigma}} \leq C \quad \text{for any } x \in B \left( x_0, \frac{1}{4} r \right).
\]

By applying (4.28) to (4.33), we can obtain
\[(4.35) \quad |\nabla_x (G_\varepsilon (x, y_0) - G_\varepsilon (x, y_1))| \leq \frac{C}{r} \left( \int_{B(x_0, \frac{1}{4} r)} |G_\varepsilon (x, y_0) - G_\varepsilon (x, y_1)|^2 \right)^{\frac{1}{2}} \leq \frac{C}{|x_0 - y_0|}
\]

Similarly, owing to the fact that \( |y_{j+1} - y_j| \leq \frac{1}{4} |x_0 - y_j| \), we have
\[(4.36) \quad |\nabla_x (G_\varepsilon (x_0, y_i) - G_\varepsilon (x_0, y_{i+1}))| \leq \frac{C}{r} \left( \int_{B(x_0, \frac{1}{4} r)} |G_\varepsilon (x_0, y_i) - G_\varepsilon (x_0, y_{i+1})|^2 \right)^{\frac{1}{2}} \leq \frac{C}{|x_0 - y_i|}
\]

where \( i = 1, 2, \ldots, N - 1 \). Finally, because of the simple observation that
\[
\frac{1}{4} |x_0 - y_N| > |y_N - \overline{y}| \geq \delta(y_N),
\]

we can deduce from (4.30) that
\[(4.37) \quad |\nabla_x G_\varepsilon (x_0, y_N)| \leq \frac{C \delta(y_N)}{|x_0 - y_N|^2} \leq \frac{C}{|x_0 - y_N|}.
\]

From (4.35)-(4.37), we get
\[(4.38) \quad |\nabla_x G_\varepsilon (x_0, y_0)| \leq \sum_{i=0}^{N} \frac{C}{|x_0 - y_i|} \leq \frac{C}{r} \sum_{i=0}^{N} \left( \frac{4}{5} \right)^i \leq \frac{C}{r} \leq \frac{C}{|x_0 - y_0|}.
\]

Therefore,
\[
|\nabla_x G_\varepsilon (x_0, y_0)| \leq \frac{C}{r}, \quad \text{when } \frac{1}{4} r \leq \delta(x_0) \text{ and } \frac{1}{4} r \leq \delta(y_0).
\]

This, together with (4.29) and (4.30), gives the proof of (1.23). Then, (1.24) follows directly from (1.23) by considering the adjoint Green function \( G_\varepsilon^*(x, y) \). Finally, by applying (4.28) to \( \nabla_y G_\varepsilon^* (\cdot, y_0) \) and using (4.29), we can prove (1.25). \( \square \)
5. \( L^p \) Convergence Rates

In this section, we will consider the convergence of the Green functions for \( \mathcal{L}_\varepsilon \).

To handle the convergence rates of \( \mathcal{L}_\varepsilon \), we define some auxiliary functions via

(5.1) \[ b_{ik}^{\alpha \gamma} (y) = \bar{a}_{ik}^{\alpha \gamma} (y) - a_{ij}^{\alpha \beta} (y) \partial_j \chi_k^{\beta \gamma} (y), \]

where

and

(5.2) \[ \Delta \Theta_i^{\alpha \gamma} = \bar{B}_i^{\alpha \gamma} (y) - B_i^{\alpha \beta} (y) \partial_j \chi_i^{\beta \gamma} (y) \quad \text{in} \ \mathbb{R}^d, \quad \int_Y \Theta_i^{\alpha \beta} (y) dy = 0, \]

(5.3) \[ \Delta \Theta_0^{\alpha \gamma} = \bar{c}^{\alpha \gamma} (y) - c^{\alpha \beta} (y) \partial_j \chi_0^{\beta \gamma} (y) \quad \text{in} \ \mathbb{R}^d, \quad \int_Y \Theta_0^{\alpha \beta} (y) dy = 0, \]

with \( 1 \leq i \leq d \). We mention that the existence of \( \Theta_k \) is given by Theorem 4.28 in [4] on account of

\[ \int_Y \Theta_k^{\alpha \gamma} (y) dy = 0 \quad \text{for} \quad k = 0, 1, \ldots, d. \]

Furthermore, it is not hard to see that \( \Theta_k^{\alpha \gamma} \) is periodic and belongs to \( W_{1,2}^{1,2} (\mathbb{R}^d) \).

Suppose that \( A \in \Lambda (\mu, \tau, \kappa) \), and \( V \) satisfies (1.3) and (1.5). From the interior Schauder estimate (see [10]), we obtain that

(5.4) \[ \max_{0 \leq k \leq d} \left\{ ||\chi_k||_{L^\infty (Y)}, ||\nabla \chi_k||_{L^\infty (Y)}, ||\nabla \chi_k||_{C^{0, \tau} (Y)} \right\} \leq C (\mu, \tau, \kappa, m, d). \]

By the same argument, it follows from (5.4) that

(5.5) \[ \max_{0 \leq k \leq d} \{ ||\nabla \Theta_k^{\alpha \gamma}||_{L^\infty (Y)} \} \leq C (\mu, \tau, \kappa, m, d). \]

**Theorem 5.1** (Convergence of Green functions for \( \mathcal{L}_\varepsilon \)). Suppose that \( A \in \Lambda (\mu, \tau, \kappa) \), \( V, B \) satisfy (1.3) and

(1.5), \( c \) satisfies (1.3) and (1.4), \( \lambda \geq \lambda_0 \) and \( \Omega \) is a bounded \( C^{1,1} \) domain in \( \mathbb{R}^d \) with \( d \geq 2 \). Then, for \( \mathcal{L}_\varepsilon \) and \( \mathcal{L}_0 \), we have

\[ |G_\varepsilon (x, y) - G_0 (x, y)| \leq \frac{C \varepsilon}{|x - y|^{d-1}}, \]

where \( C \) depends only on \( \mu, \tau, \kappa, \lambda, d, m \) and \( \Omega \).

**Lemma 5.2** ([23], Lemma 2.9). There exist \( E_{ijk}^{\alpha \gamma} \in W_{1,2}^{1,2} (Y) \) with \( k = 0, \ldots, d \), such that

(5.7) \[ b_{ik}^{\alpha \gamma} = \partial_j \left\{ E_{ijk}^{\alpha \gamma} \right\} \quad \text{and} \quad E_{ijk}^{\alpha \gamma} = -E_{ijk}^{\alpha \gamma} \]

where \( 1 \leq i, j \leq d \) and \( 1 \leq \alpha, \gamma \leq m \). Moreover, if \( \chi_k \) is Hölder continuous, then \( E_{ijk}^{\alpha \gamma} \in L^\infty (Y) \).

**Lemma 5.3** ([23], Lemma 5.1). Suppose that \( u_\varepsilon, u_0 \in H^1 (\Omega; \mathbb{R}^m) \) satisfy \( \mathcal{L}_\varepsilon (u_\varepsilon) = \mathcal{L}_0 (u_0) \) in \( \Omega \). Let

\[ u_\varepsilon^\beta = u_\varepsilon^\beta - u_0^\beta - \varepsilon \sum_{k=0}^{d} \chi_k^{\beta \gamma} (x/\varepsilon) \partial_k u_0^\gamma, \]

Then we have

(5.8) \[ \mathcal{L}_\varepsilon (u_\varepsilon)^\alpha (x) = -\partial_\alpha \left\{ K_\alpha^\alpha (x) - \varepsilon (T_\alpha^\alpha (x) + J_\alpha^\alpha (x)) \right\} - \varepsilon (M^\alpha (x) + N^\alpha (x)), \]

where

\[ T_\alpha^\alpha (x) = a_{ij}^{\alpha \beta} (x/\varepsilon) \sum_{k=0}^{d} \chi_k^{\beta \gamma} (x/\varepsilon) \partial_j \partial_k u_0^\gamma + V_i^{\alpha \beta} (x/\varepsilon) \sum_{k=0}^{d} \chi_k^{\beta \gamma} (x/\varepsilon) \partial_k u_0^\gamma, \]

\[ J_\alpha^\alpha (x) = \sum_{k=0}^{d} \partial_\alpha \Theta_k^{\alpha \gamma} (x/\varepsilon) \partial_k u_0^\gamma, \quad K_\alpha^\alpha (x) = \sum_{j=0}^{d} b_{ij}^{\alpha \gamma} (x/\varepsilon) \partial_j u_0^\gamma, \]

\[ M^\alpha (x) = \sum_{k=0}^{d} \left[ \partial_\alpha \Theta_k^{\alpha \gamma} (x/\varepsilon) + B_i^{\alpha \beta} (x/\varepsilon) \chi_k^{\beta \gamma} (x/\varepsilon) \right] \partial_k^2 u_0^\gamma, \]

\[ N^\alpha (x) = \left[ c^{\alpha \beta} (x/\varepsilon) + \lambda \delta^{\alpha \beta} \right] \sum_{k=0}^{d} \chi_k^{\beta \gamma} (x/\varepsilon) \partial_k u_0^\gamma. \]
Remark 5.4. From Lemma 5.3 and (5.7), we have

\begin{equation}
\partial_i (\mathcal{K}_i^\gamma (x)) = \partial_i \left( \sum_{j=0}^{d} b_{ij}^\gamma (x/\varepsilon) \partial_j u_0^\gamma \right) = \varepsilon \partial_j \left( \sum_{k=0}^{d} E_{ijk}^\gamma (x/\varepsilon) \partial_k u_0^\gamma \right).
\end{equation}

Lemma 5.5. Suppose that $A \in \Lambda(\mu, \tau, \kappa), V, B$ satisfy (1.3) and (1.5), $c$ satisfies (1.3) and (1.4), $\lambda \geq \lambda_0$ and $\Omega$ is a bounded $C^{1,\eta}$ $(0 < \eta < 1)$ domain in $\mathbb{R}^d$ with $d \geq 2$. Then, we have

\begin{equation}
\|u_\varepsilon\|_{L^\infty(\Omega(x_0,3r))} \leq C\|f\|_{L^\infty(\Delta(x_0,3r))} + C\int_{\Omega(x_0,3r)} |u_\varepsilon|,
\end{equation}

where $L_\varepsilon(u_\varepsilon) = 0$ in $\Omega(x_0,3r)$, $u_\varepsilon = f$ on $\Delta(x_0,3r)$, $x_0 \in \Omega$, $0 < r < \text{diam}(\Omega)$ and $C$ depends only on $\mu, \tau, \kappa, \lambda, \eta, \Omega$.

Proof. The proof is almost the same as Lemma 6.3.2 in [19]. We only need to change the operator $L_\varepsilon$ in [19] to the operator $L_\varepsilon$ with lower order terms and use (1.27) and (3.12). \qed

Lemma 5.6. Suppose that $A \in \Lambda(\mu, \tau, \kappa), V, B$ satisfy (1.3) and (1.5), $c$ satisfies (1.3) and (1.4), $\lambda \geq \lambda_0$ and $\Omega$ is a bounded $C^{1,\eta}$ $(0 < \eta < 1)$ domain in $\mathbb{R}^d$ with $d \geq 2$. Let $u_\varepsilon \in W^{1,2}(\Omega(x_0,4r); \mathbb{R}^m)$ and $u_0 \in W^{2,p}(\Omega(x_0,4r); \mathbb{R}^m)$ for some $d < p < \infty$. Suppose that

\begin{equation}
L_\varepsilon(u_\varepsilon) = L_0(u_0) \quad \text{in } \Omega(x_0,4r) \quad \text{and} \quad u_\varepsilon = u_0 \quad \text{on } \Delta(x_0,4r).
\end{equation}

Then, we have

\begin{equation}
\|u_\varepsilon - u_0\|_{L^\infty(\Omega(x_0,r))} \leq C\int_{\Omega(x_0,4r)} |u_\varepsilon - u_0| + C\varepsilon \|\nabla u_0\|_{L^\infty(\Omega(x_0,4r))} + C\varepsilon 1 + \frac{1}{d} \|\nabla^2 u_0\|_{L^p(\Omega(x_0,4r))},
\end{equation}

where $x_0 \in \Omega$, $0 < r < \text{diam}(\Omega)$ and $C$ depends only on $\mu, \tau, \kappa, \lambda, d, m, \eta, \Omega$.

Proof. By rescaling and translation, we can assume that $r = 1$. Choose a domain $\tilde{\Omega}$, which is $C^{1,\eta}$, such that $\Omega(0,3) \subset \tilde{\Omega} \subset \Omega(0,4)$. Consider

\begin{equation}
w_\varepsilon = u_\varepsilon - u_0 - \varepsilon \sum_{k=0}^{d} \chi_k(x/\varepsilon) \partial_k u_0^\gamma = w_{\varepsilon,1} + w_{\varepsilon,2} \quad \text{in } \tilde{\Omega},
\end{equation}

where

\begin{equation}
L_\varepsilon(w_{\varepsilon,1}) = L_\varepsilon(w_\varepsilon) \quad \text{in } \tilde{\Omega} \quad w_{\varepsilon,1} \in W^{1,2}(\tilde{\Omega}; \mathbb{R}^m),
\end{equation}

\begin{equation}
L_\varepsilon(w_{\varepsilon,2}) = 0 \quad \text{in } \tilde{\Omega} \quad w_{\varepsilon,2} = w_\varepsilon, \quad \text{on } \partial \tilde{\Omega}.
\end{equation}

Since $w_{\varepsilon,2} = w_\varepsilon = -\varepsilon \sum_{k=0}^{d} \chi_k^\gamma(x/\varepsilon) \partial_k u_0^\gamma$ on $\Delta(0,3)$ and $\|\chi\|_{L^\infty} \leq C$ (by using (3.4)), it follows from (5.11) that

\begin{equation}
\|w_{\varepsilon,2}\|_{L^\infty(\Omega(0,1))} \leq C\varepsilon \|\nabla u_0\|_{L^\infty(\Delta(0,3))} + C\int_{\Omega(0,3)} |w_{\varepsilon,2}| \leq C\varepsilon \|\nabla u_0\|_{L^\infty(\Delta(0,3))} + C\varepsilon \int_{\Omega(0,3)} |w_{\varepsilon,1}| + C\int_{\Omega(0,3)} |w_{\varepsilon,1}| \leq C\varepsilon \int_{\Omega(0,3)} |w_{\varepsilon,1}| + C\int_{\Omega(0,3)} |w_{\varepsilon,1}| + C \int_{\Omega(0,3)} |w_{\varepsilon,1}|.
\end{equation}

This gives

\begin{equation}
\|u_\varepsilon - u_0\|_{L^\infty(\Omega(0,1))} \leq C\int_{\Omega(0,3)} |w_{\varepsilon,1}| + C\varepsilon \|\nabla u_0\|_{L^\infty(\Omega(0,3))} + C \|w_{\varepsilon,1}\|_{L^\infty(\Omega(0,3))}.
\end{equation}

To estimate $w_{\varepsilon,1}$ on $\Omega_3$, we use the Green function representation

\begin{equation}
w_{\varepsilon,1}(x) = \int_{\tilde{\Omega}} G_\varepsilon(x,y) L_\varepsilon(w_\varepsilon) dy,
\end{equation}
where $\tilde{G}_\varepsilon(x,y)$ denotes the matrix of the Green function for $\mathcal{L}_\varepsilon$ in $\tilde{\Omega}$. From (5.9), we obtain

$$
[w_{\varepsilon,1}(x)]^d = \varepsilon \int_{\tilde{\Omega}} \partial_y \tilde{G}_\varepsilon(x,y) \sum_{k=0}^d E_{ijk}^{\alpha} (y/\varepsilon) \partial^2_{jk} u_0(y) dy
$$

$$
- \varepsilon \int_{\tilde{\Omega}} \partial_y \tilde{G}_\varepsilon(x,y) (T_\varepsilon(y) + J_\varepsilon(y)) \varepsilon \int_{\tilde{\Omega}} \tilde{G}_\varepsilon(x,y) (M_\varepsilon(y) + N_\varepsilon(y)) dy.
$$

From Lemma 5.2, we have $\|E\|_{L^\infty(\tilde{\Omega})} \leq C$ and it follows that

$$
|w_{\varepsilon,1}(x)| \leq C \varepsilon \int_{\tilde{\Omega}} |\nabla_y \tilde{G}_\varepsilon(x,y)||\nabla^2 u_0(y)| dy + C \varepsilon \int_{\tilde{\Omega}} |\nabla_y \tilde{G}_\varepsilon(x,y)||\nabla u_0(y)| dy + C \varepsilon \int_{\tilde{\Omega}} |\tilde{G}_\varepsilon(x,y)||\nabla u_0(y)| dy.
$$

Since under the assumption of $\mathcal{L}_\varepsilon$, from (1.12) and (1.24), we have, for $p > d$,

$$
C \varepsilon \int_{\tilde{\Omega}} |\nabla_y \tilde{G}_\varepsilon(x,y)||\nabla^2 u_0(y)| dy + C \varepsilon \int_{\tilde{\Omega}} |\nabla_x \tilde{G}_\varepsilon(x,y)||\nabla^2 u_0(y)| dy
$$

$$
\leq C \varepsilon \|\nabla^2 u_0\|_{L^p(\Omega(0,4))} \left\{ \left( \int_{\tilde{\Omega}} |\nabla_y \tilde{G}_\varepsilon(x,y)|^{p'} dy \right)^{\frac{1}{p'}} + \left( \int_{\tilde{\Omega}} |\tilde{G}_\varepsilon(x,y)|^{p'} dy \right)^{\frac{1}{p'}} \right\}
$$

and

$$
C \varepsilon \int_{\tilde{\Omega}} |\nabla_x \tilde{G}_\varepsilon(x,y)||\nabla u_0(y)| dy + C \varepsilon \int_{\tilde{\Omega}} |\tilde{G}_\varepsilon(x,y)||\nabla u_0(y)| dy
$$

$$
\leq C \varepsilon \|\nabla u_0\|_{L^\infty(\Omega(0,4))} \left( \int_{\tilde{\Omega}} |\nabla_x \tilde{G}_\varepsilon(x,y)| dy + \int_{\tilde{\Omega}} |\tilde{G}_\varepsilon(x,y)| dy \right) \leq C \varepsilon \|\nabla u_0\|_{L^\infty(\Omega(0,4))}.
$$

**Proof of Theorem 5.1.** We first note that under the assumption of $\mathcal{L}_\varepsilon$, $\mathcal{L}_0$ and $\Omega$ in the theorem, the size estimate and $|\nabla \chi_2 G_0(x,y)| \leq C|x - y|^{1-d}$ hold for any $x, y \in \Omega$ and $x \neq y$. We now fix $x_0, y_0 \in \Omega$ and $r = \frac{1}{8}|x_0 - y_0| > 0$. For $F \in C^\infty_0(\Omega(y_0,r); \mathbb{R}^m)$, let

$$
u_\varepsilon(x) = \int_{\tilde{\Omega}} G_\varepsilon(x,y) F(y) dy \quad \text{and} \quad u_0(x) = \int_{\tilde{\Omega}} G_0(x,y) F(y) dy.
$$

Then $\mathcal{L}_\varepsilon(u_\varepsilon) = \mathcal{L}_0(u_0) = F$ in $\Omega$ and $u_\varepsilon = u_0 = 0$ on $\partial \Omega$. Note that since $\Omega$ is $C^{1,1}$,

$$
\|\nabla^2 u_0\|_{L^p(\Omega)} \leq C \|F\|_{L^p(\Omega)} \quad \text{for} \quad 1 < p < \infty,
$$

$$
\|\nabla u_0\|_{L^\infty(\Omega)} \leq CR^{1-d} \|F\|_{L^p(\Omega(y_0,r))} \quad \text{for} \quad p > d.
$$

The inequality (5.16) is the $W^{2,p}$ estimate for the constant second order elliptic operator $\mathcal{L}_0$ in $C^{1,1}$ domains (see [11]) and (5.17) follows from the estimate $|\nabla \chi_2 G_0(x,y)| \leq C|x - y|^{1-d}$ and Hölder’s inequality.

Next, let

$$
w_\varepsilon = u_\varepsilon - u_0 - \varepsilon \sum_{k=0}^d \chi_k^2(x/\varepsilon) \partial_k u_0^2 = v_{\varepsilon,1} + v_{\varepsilon,2},
$$

where $v_{\varepsilon,1} \in W_0^{1,2}(\Omega; \mathbb{R}^m)$ and $\mathcal{L}_\varepsilon(v_{\varepsilon,1}) = \mathcal{L}_\varepsilon(w_\varepsilon)$ in $\Omega$. Observe that by the formula (5.9) for $\mathcal{L}_\varepsilon$,

$$
\|\nabla v_{\varepsilon,1}\|_{L^2(\Omega)} \leq C \varepsilon \|\nabla^2 u_0\|_{L^2(\Omega)} + C \varepsilon \|\nabla u_0\|_{L^2(\Omega)} \leq C \varepsilon \|F\|_{L^2(\Omega(y_0,r))},
$$

where we have used the fact that $\chi_k$ and $E_{kj}$ are bounded, (2.5) and (5.16). By Hölder’s inequality and Sobolev inequalities, this implies that if $d \geq 3$,

$$
\|v_{\varepsilon,1}\|_{L^2(\Omega(y_0,r))} \leq C \varepsilon r^{1+\frac{d}{2}-\frac{d}{2}} \varepsilon \|F\|_{L^p(\Omega(y_0,r))},
$$

for $p > d$. We point out that if $d = 2$, one has

$$
\|v_{\varepsilon,1}\|_{L^2(\Omega(y_0,r))} \leq C \varepsilon r \|F\|_{L^2(\Omega(y_0,r))},
$$
in place of (5.18). To see this we use the $W^{1,p}$ estimates (1.7). Thus there exists some $p < 2$ such that
\begin{align}
\|\nabla v_{x,1}\|_{L^p(\Omega)} \leq C\varepsilon p \|\nabla^2 u_0\|_{L^p(\Omega)} + C\varepsilon p \|\nabla u_0\|_{L^p(\Omega)} \leq C\varepsilon \|F\|_{L^p(\Omega(y_0,r))},
\end{align}
which, by H"older’s inequality and Sobolev inequality, leads to
\begin{align}
\|v_{x,1}\|_{L^2(\Omega(x_0,r))} \leq C r^{-\frac{d}{2}} \|v_{x,1}\|_{L^2(\Omega(y_0,r))} \leq C r^{-1-\frac{d}{2}} \|v_{x,1}\|_{L^2(\Omega)} \leq C r^{-\frac{d}{2}} \|\nabla v_{x,1}\|_{L^p(\Omega)}
\end{align}
(5.20)
where $\frac{1}{q} = \frac{1}{p} - \frac{d}{2}$.

Observe that since $L^p_\varepsilon(v_{x,2}) = 0$ in $\Omega$ and $v_{x,2} = w_\varepsilon$ on $\partial\Omega$, by the maximum principle (1.27), we have
\begin{align}
\|v_{x,2}\|_{L^p(\Omega)} \leq C \|v_{x,2}\|_{L^\infty(\partial\Omega)} \leq C \varepsilon \|\nabla u_0\|_{L^\infty(\partial\Omega)}.
\end{align}
From (5.16)-(5.21), we obtain
\begin{align}
\|u_\varepsilon - u_0\|_{L^2(\Omega(x_0,r))} \leq \|v_{x,1}\|_{L^2(\Omega(x_0,r))} + \|v_{x,2}\|_{L^2(\Omega(x_0,r))} + C \|\nabla u_0\|_{L^2(\Omega(x_0,r))}
\end{align}
\begin{align}
\leq \|v_{x,1}\|_{L^2(\Omega)} + \|v_{x,2}\|_{L^2(\Omega)} + C \varepsilon r^\frac{d}{2} \|\nabla u_0\|_{L^\infty(\Omega)}
\end{align}
\begin{align}
\leq \|v_{x,1}\|_{L^2(\Omega)} + C \varepsilon r^\frac{d}{2} \|\nabla u_0\|_{L^\infty(\Omega)} \leq C \varepsilon r^{1+\frac{d}{2} - \frac{d}{p}} \|F\|_{L^p(\Omega(y_0,r))},
\end{align}
where $p > d$. This, together with Lemma 5.6 and (5.16), gives
\begin{align}
|u_\varepsilon(x_0) - u_0(x_0)| \leq C \varepsilon r^{1-rac{d}{p}} \|F\|_{L^p(\Omega(y_0,r))}.
\end{align}
Then, it follows by the duality arguments that
\begin{align}
\left( \int_{\Omega(y_0,r)} |G_\varepsilon(x_0,y) - G_0(x_0,y)|^p dy \right)^\frac{1}{p} \leq C \varepsilon r^{1-\frac{d}{p}} \text{ for any } p > d.
\end{align}
Finally, since $L^p_\varepsilon(G_\varepsilon(x_0,\cdot)) = L^p_\varepsilon(G_0(x_0,\cdot)) = 0$ in $\Omega(y_0,r)$, we may invoke Lemma 5.6 again to conclude that
\begin{align}
|G_\varepsilon(x_0,y_0) - G_0(x_0,y_0)| \leq \int_{\Omega(y_0,r)} |G_\varepsilon(x_0,y) - G_0(x_0,y)| dy + C \varepsilon \|\nabla G_0(x_0,\cdot)\|_{L^\infty(\Omega(y_0,r))}
\end{align}
\begin{align}
+ C r^{1-rac{d}{p}} \|\nabla G_0(x_0,\cdot)\|_{L^p(\Omega(y_0,r))} \leq C \varepsilon r^{1-d},
\end{align}
where we have used
\begin{align}
\left( \int_{\Omega(y_0,r)} |\nabla G_0(x_0,y)|^p dy \right)^\frac{1}{p} \leq C r^{-2} \|G_0(x_0,\cdot)\|_{L^\infty(\Omega(x_0,2r))} \leq C r^{-d},
\end{align}
obtained by using $W^{1,p}$ estimates on $C^{1,1}$ domains for $L^p_\varepsilon$, which is the localization of the estimates (5.16). $\square$

**Remark 5.7.** (1.31) follows directly from (5.6) by using the standard arguments of Theorem 3.4 in [12].

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