ON MULTIPLICITIES OF MAXIMAL WEIGHTS OF $\widehat{\mathfrak{sl}}(n)$-MODULES

REBECCA L. JAYNE AND KAILASH C. MISRA

Abstract. We determine explicitly the maximal dominant weights for the integrable highest weight $\widehat{\mathfrak{sl}}(n)$-modules $V((k-1)\Lambda_0 + \Lambda_s)$, $0 \leq s \leq n-1$, $k \geq 2$. We give a conjecture for the number of maximal dominant weights of $V(k\Lambda_0)$ and prove it in some low rank cases. We give an explicit formula in terms of lattice paths for the multiplicities of a family of maximal dominant weights of $V(k\Lambda_0)$. We conjecture that these multiplicities are equal to the number of certain pattern avoiding permutations. We prove that this conjecture holds for $k = 2$.

1. Introduction

We consider the affine Kac-Moody algebra $\widehat{\mathfrak{sl}}(n)$. Let $P^+$ denote the set of dominant integral weights and for $\Lambda \in P^+$, let $V(\Lambda)$ denote the integrable highest weight $\widehat{\mathfrak{sl}}(n)$-module. Let $P(\Lambda)$ denote the set of weights of $V(\Lambda)$ and $\delta$ denote the null root. A weight $\mu \in P(\Lambda)$ is maximal if $\mu + \delta \not\in P(\Lambda)$. Let $\text{max}(\Lambda)$ denote the set of maximal weights in $P(\Lambda)$. It is known (see [5]) that the weights in $P(\Lambda)$ are $\delta$ shifts of maximal weights. Furthermore, any weight in $P(\Lambda)$ is Weyl group conjugate to a dominant weight in $P(\Lambda) \cap P^+$. Hence to determine the set of weights $P(\Lambda)$ it is sufficient to obtain explicitly the set of maximal dominant weights $\text{max}(\Lambda) \cap P^+$. It is known that this is a finite set (see [5]). However, neither the explicit descriptions nor the multiplicities of these weights are known in general.

In [1] a non-recursive criterion is given to decide whether a weight is in $P(\Lambda)$. Also, a combinatorial algorithm is given to obtain these weights. However, obtaining the set of weights $\text{max}(\Lambda) \cap P^+$ explicitly for arbitrary rank and arbitrary level using the algorithm given in [1] is difficult. In [7], Tsuchioka determined explicitly the maximal dominant weights of the $\widehat{\mathfrak{sl}}(p)$-modules $V(\Lambda_0 + \Lambda_s)$ for any prime $p$. One of the goals in this paper is to give explicit descriptions of the maximal dominant weights of the $\widehat{\mathfrak{sl}}(n)$-modules $V((k-1)\Lambda_0 + \Lambda_s)$, $0 \leq s \leq n-1$, $k \geq 2$ (Theorem 3.5). Our approach is rather simple and different from [1]. We also conjecture a closed form formula for the number of maximal dominant weights of $V(k\Lambda_0)$ and prove this conjecture for $k \leq 3$.

Determining the multiplicities of the weights of $V(\Lambda)$ is an important problem. In [7], Tsuchioka showed that the multiplicities of the maximal dominant weights of $V(2\Lambda_0)$ are given by the Catalan numbers. The second goal of this

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paper is to study the multiplicities of the maximal dominant weights of \( V(k\Lambda_0) \) using the extended Young diagram realizations of the crystal bases for \( V(k\Lambda_0) \), given in [4]. In particular, we give an explicit formula in terms of lattice paths to determine the multiplicities of a large family of maximal dominant weights of \( V(k\Lambda_0) \) (Theorem 4.6). We conjecture that these multiplicities can be given by certain pattern avoiding permutations. Using the bijection given in [2], we show that this conjecture holds for \( k = 2 \), recovering the result in [7] from a different viewpoint. We also give multiplicity tables as evidence for the validity of our conjecture when \( k > 2 \).

## 2. Preliminary

Let \( \mathfrak{g} = \widehat{sl}(n) \) be the affine Kac-Moody Lie algebra with Cartan datum \( \{ A, \Pi, \Pi', P, P' \} \) and index set \( I = \{0, 1, \ldots, n-1\} \). Here \( A = (a_{ij})_{i,j=0}^{n-1} \) is the generalized Cartan matrix where \( a_{ii} = 2, a_{ij} = -1 \) for \( |i - j| = 1, a_{0,n-1} = a_{n-1,0} = -1, \) and \( a_{ij} = 0 \) otherwise. The sets \( \Pi = \{\alpha_0, \alpha_1, \ldots, \alpha_{n-1}\} \) and \( \Pi' = \{h_0, h_1, \ldots, h_{n-1}\} \) are the simple roots and simple coroots, respectively. Note that \( \alpha_j(h_i) = a_{ij} \) and that \( Q = \mathbb{Z}\alpha_0 \oplus \mathbb{Z}\alpha_1 \oplus \ldots \mathbb{Z}\alpha_{n-1} \) is the root lattice. The weight lattice and coweight lattice are \( P = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \ldots \mathbb{Z}\Lambda_{n-1} \oplus \mathbb{Z}\delta \) and \( P' = \mathbb{Z}h_0 \oplus \mathbb{Z}h_1 \oplus \ldots \mathbb{Z}h_{n-1} \oplus \mathbb{Z}d \), respectively, where \( \Lambda_i, \ i \in I, \) defined by \( \Lambda_i(h_j) = \delta_{ij}, \Lambda_i(d) = 0 \) for all \( j \in I, \) are the fundamental weights, \( \delta = \alpha_0 + \alpha_1 + \ldots + \alpha_{n-1} \) is the null root, and \( d \) is a degree derivation. The Cartan subalgebra of \( \mathfrak{g} \) is \( \mathfrak{h} = \text{span}_\mathbb{C}\{h_0, h_1, \ldots, h_{n-1}, d\} \). Note that \( P \subset \mathfrak{h}^* \) and \( P' \subset \mathfrak{h} \). Let \( (\mid ) : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C} \) denote the nondegenerate symmetric bilinear form (see [5]) on \( \mathfrak{g} \). We denote the induced form on \( \mathfrak{h}^* \) by the same notation \( (\mid ) \).

It is known that \( \mathfrak{g} \cong \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus Cd, \) where \( \mathfrak{g} = \widehat{sl}(n) \) is the simple Lie algebra of \( n \times n \) trace zero matrices, \( c = h_0 + h_1 + \ldots + h_{n-1} \) is the canonical central element, and \( d = 1 \otimes \frac{dt}{t} \) is the degree derivation. The Cartan subalgebra of \( \mathfrak{g} \) is \( \mathfrak{h} = \text{span}_\mathbb{C}\{h_1, h_2, \ldots, h_{n-1}\} \). The submatrix \( \tilde{A} = (a_{ij})_{i,j=1}^{n-1} \) is the Cartan matrix for \( \hat{\mathfrak{g}} \). We define \( \mathfrak{h}_\mathbb{R} = \text{span}_\mathbb{R}\{h_1, h_2, \ldots, h_{n-1}\} \subset \mathfrak{h} \) and hence \( \mathfrak{h}_\mathbb{R}^* \subset \mathfrak{h}^* \).

A weight \( \Lambda \in P \) is of level \( k \) if \( \Lambda(c) = k \). The set \( P^+ = \{ \lambda \in P \mid \lambda(h_i) \geq 0 \text{ for all } i \in I \} \) is the set of dominant integral weights. For any \( \Lambda \in P^+ \), we denote by \( V(\Lambda) \) the integrable highest weight \( \mathfrak{g} \)-module of level \( k = \Lambda(c) \).

For \( \mu \in \mathfrak{h}^* \) to be a weight of \( V(\Lambda) \), we have \( V(\Lambda)_\mu = \{ v \in V(\Lambda) \mid h(v) = \mu(h)v, \text{ for all } h \in \mathfrak{h} \} \neq 0 \). Any weight \( \mu \) of \( V(\Lambda) \) is of the form \( \mu = \Lambda - \sum_{i=0}^{n-1} m_i \alpha_i \), where \( m_i \) is a nonnegative integer for all \( i \in I \). The dimension of the \( \mu \)-weight space \( V(\Lambda)_\mu \) is called the multiplicity of \( \mu \) in \( V(\Lambda) \), denoted by \( \text{mult}_\Lambda(\mu) \). A weight \( \mu \) of \( V(\Lambda) \) is a maximal weight if \( \mu + \delta \) is not a weight of \( V(\Lambda) \). We denote the set of all maximal weights of \( V(\Lambda) \) by \( \text{max}(\Lambda) \). Hence, \( \text{max}(\Lambda) \cap P^+ \) is the set of all maximal dominant weights of \( V(\Lambda) \). We define the orthogonal projection \( \pi : \mathfrak{h}^* \to \mathfrak{h}_\mathbb{R}^* \) by \( \lambda \mapsto \pi = \lambda - \lambda(c)\Lambda_0 - (\lambda|\Lambda_0)\delta \) ([6], Equation 6.2.7) and denote \( \overline{\mathfrak{g}} \) to be the orthogonal projection of \( Q \) on \( \mathfrak{h}_\mathbb{R}^* \). Note that \( \theta = \alpha_1 + \alpha_2 + \ldots + \alpha_{n-1} \) is the highest root of \( \hat{\mathfrak{g}} \). We define \( kC_{\alpha} = \{ \lambda \in \mathfrak{h}_\mathbb{R}^* \mid \lambda(h_i) \geq 0, (\lambda|\theta) \leq k \} \). Then we have the following proposition.
Proposition 2.1. ([5], Proposition 12.6) The map \( \lambda \mapsto \overline{\lambda} \) is a bijection from \( \max(\Lambda) \cap \mathbb{P}^+ \) to \( kC_{af} \cap (\overline{\Lambda} + \overline{Q}) \), where \( k \) is the level of \( \Lambda \). In particular, the set \( \max(\Lambda) \cap \mathbb{P}^+ \) is finite.

In the next section, we will give explicit descriptions for the maximal dominant weights of the integrable highest weight \( g \)-modules \( V((k - 1)\Lambda_0 + \Lambda_s) \), where \( k \geq 2 \) and \( 0 \leq s \leq n - 1 \).

3. Maximal dominant weights of \( V((k - 1)\Lambda_0 + \Lambda_s) \)

In order to explicitly determine the maximal dominant weights of \( V((k - 1)\Lambda_0 + \Lambda_s) \), where \( k \geq 2 \) and \( 0 \leq s \leq n - 1 \), we need to introduce the following notations.

For fixed positive integers \( p, q, i_{\text{min}}, i_{\text{max}} \), \( 1 \leq p, q \leq n - 1 \), \( i_{\text{min}} \leq i_{\text{max}} \), we define \( I(p, i_{\text{max}} : q, i_{\text{min}}) \) to be the set of all \( (q - p + 1) \)-tuples \((x_p, \ldots, x_q)\) satisfying:

- \( i_{\text{max}} \geq x_i - x_{i-1} \geq i_{\text{min}} \) for \( p < i < q \), and
- \( x_i - x_{i-1} \geq x_{i+1} - x_i \) for \( p < i < q \).

Define the set \( I^*(p, i_{\text{min}} : q, i_{\text{max}}) \) to be those \( (q - p + 1) \)-tuples \((x_p, x_{p+1}, \ldots, x_q)\) such that \((x_q, x_{q-1}, \ldots, x_p) \in I(q, i_{\text{max}} : p, i_{\text{min}}) \). Notice that elements of \( I^*(p, i_{\text{min}} : q, i_{\text{max}}) \) are strictly decreasing sequences of nonnegative integers.

For given \( \Lambda = (k - 1)\Lambda_0 + \Lambda_s \), \( k \geq 2 \), \( 0 \leq s \leq n - 1 \), we choose a pair of integers \((x_1, x_{n-1})\) such that \( x_1, x_{n-1} \geq \delta_s,0 \) and \( x_1 + x_{n-1} \leq k - 1 + \delta_s,0 \).

We define sets of \((n - 1)\)-tuples of nonnegative integers \( M_1, M_2, M_3, M_4, \) and \( M_5 \) as follows. First, we define \( M_1 = M_1(s, n : x_1, x_{n-1}) \) to be the set with elements of the form

\[
(x_1, x_2, \ldots, x_p = x_{p+1} = \cdots = x_q = \ell_1, x_{q+1}, \ldots, x_s, x_{s+1} = x_s - t_1, \ldots, x_{n-1})
\]

such that \((x_1, x_2, \ldots, x_p) \in I(1, x_1 : p, 1), q \neq s, (x_q, x_{q+1}, \ldots, x_s) \in I^*(q, 1 : s, t_1 + 1), (x_s, x_{s+1}, \ldots, x_{n-1}) \in I^*(s, t_1 : n - 1, x_{n-1}), \) for all \( x_s, t_1, \ell_1 \) satisfying \( x_{n-1} + n - s - 1 \leq x_s \leq \min\{x_1(s - 1), x_{n-1}(n - s)\} \), \( \max\{1, x_s - x_{n-1}(n - s - 1)\} \leq \ell_1 \leq \left\lfloor \frac{x_s - x_{n-1}}{n - s} \right\rfloor \), and \( \max\{x_1, x_s + 1\} \leq \ell_1 \leq \max\{a, a + m_1 - (t_1 + 1)\} \), where \( m_1 \) is such that \((s(t_1 + 1) + x_s) \equiv m_1 \) (mod \( t_1 + x_s + 1 \)), and \( a = \frac{x_s((t_1 + 1)s + x_s - m_1)}{x_1 t_1 + 1} \). Note that when \( s = n - 1 \), \( t_1 = x_{n-1} = x_s \).

Similarly, we define \( M_2 = M_2(s, n : x_1, x_{n-1}) \) to be the set of \((n - 1)\)-tuples of nonnegative integers of the form

\[
(x_1, x_2, \ldots, x_p = x_{p+1} = \cdots = x_q = \ell_2, x_{q+1}, \ldots, x_s = x_{s+1} = \cdots = x_r, x_{r+1}, \ldots, x_{n-1}),
\]

where \( s \neq r, q \neq s, (x_1, x_2, \ldots, x_p) \in I(1, x_1 : p, 1), (x_q, x_{q+1}, \ldots, x_s) \in I^*(q, 1 : s, 1), \) and \((x_r, x_{r+1}, \ldots, x_{n-1}) \in I^*(r, 1 : n - 1, x_{n-1}), \) for all \( x_r, x_{r+1} \) such that \( \max\{x_{n-1}, x_1 - s + 1\} \leq x_s \leq \min\{x_{n-1}(n - s - 1), x_1(s - 1) - 1\} \) and \( \max\{x_1, x_s + 1\} \leq \ell_2 \leq \left\lfloor \frac{x_1}{x_1 + 1}(s + x_s) \right\rfloor . \)
We define $M_3 = M_3(s, n : x_1, x_{n-1}) = M_1(n - s, n : x_{n-1}, x_1)$ and $M_4 = M_4(s, n : x_1, x_{n-1}) = M_2(n - s, n : x_{n-1}, x_1)$.

Now, we define $M_5 = M_5(s, n : x_1, x_{n-1})$ to be the set of $(n - 1)$-tuples of nonnegative integers of the form

$$(x_1, x_2, \ldots, x_q = x_{q+1} = \cdots = x_r \in \ell_5, x_{r+1}, \ldots, x_{n-1})$$

such that $(x_1, x_2, \ldots, x_q) \in I(1, x_1 : q, 1)$, $(x_r, x_{r+1}, \ldots, x_{n-1}) \in I^*(r, 1 : n - 1, x_{n-1})$, where $\ell_5$ satisfy

$$\max \{x_1, x_{n-1}\} \leq \ell_5 \leq \min \{sx_1, (n - s)x_{n-1}\}$$

and $q \leq r$ if $s > 0$ and $\ell_5$ satisfy $\max \{x_1, x_{n-1}\} \leq \ell_5 \leq \max \left\{\frac{x_1(x_1 + n - m_0)}{x_1 + x_{n-1}}, \frac{x_1(x_1 + n - m_0)}{x_1 + x_{n-1}} + m_5 - x_{n-1}\right\}$ with $x_1n \equiv m_5 \pmod{x_1 + x_{n-1}}$ if $s = 0$.

For $s > 0$ we observe that $x_s \neq \ell_j$, $\min \{i \mid x_i = \ell_j\} < s$ in $M_1, M_2$ and $\min \{i \mid x_i = \ell_j\} > s$ in $M_3, M_4$. By definition, $M_1 \cap M_2 = \emptyset$ and hence $M_3 \cap M_4 = \emptyset$. Thus, by the above observation, $M_1, M_2, M_3, M_4$, and $M_5$ are disjoint when $s > 0$. Observe that when $s = 0$, $M_1 = M_2 = M_3 = M_4 = \emptyset$ and we only have the set of $(n - 1)$-tuples $M_5$ nonempty.

Before proving the main theorem of this section, we need the following lemmas.

**Lemma 3.1.** Let $c, d, e, f$ be nonnegative integers ($c, d > 0$). The largest nonnegative integer value of $\ell$ satisfying

$$\left\lfloor \frac{\ell}{c} \right\rfloor + \left\lceil \frac{\ell - e}{d} \right\rceil \leq f$$

is $\ell = \max \left\{\frac{c(df + e - m)}{c + d}, \frac{c(df + e - m)}{c + d} + \frac{m - d}{d}\right\}$, where $df + e \equiv m \pmod{c + d}$.

**Proof.** We show that the given value of $\ell$ satisfies the inequality (3.3) when $m > d$. The case for $m \leq d$ is similar.

$$\left\lfloor \frac{\ell}{c} \right\rfloor + \left\lceil \frac{\ell - e}{d} \right\rceil = \left\lfloor \frac{c(df + e - m)}{c + d} + \frac{m - d}{d} \right\rceil = \left\lfloor \frac{c(df + e - m)}{c + d} + \frac{m - d - e}{d} \right\rceil - 1$$

$$= \frac{df + e - m}{c + d} + \left\lfloor \frac{m - d}{c} \right\rceil + \left\lceil \frac{c(df + e - m) + (c + d)(m - e)}{d(c + d)} \right\rceil - 1 = \frac{df + e - m}{c + d} + \left\lfloor \frac{(c + d)f - df + m - e}{c + d} \right\rceil - 1 = \frac{df + e - m}{c + d} + \left\lfloor \frac{e}{c + d} \right\rceil$$

$$= \frac{df + e - m}{c + d} + f = \frac{-df + m - e}{c + d} = f$$

□

**Lemma 3.2.** Let $x = (x_1, x_2, \ldots, x_{n-1})^T \in \mathbb{Z}_{\geq 0}^{n-1}$ and let $(\hat{A}x)_i$ denote the $i^{th}$ row entry in $\hat{A}x$. For convenience, we assume $x_0 = x_n = 0$. The following statements are true.

1. Suppose $(\hat{A}x)_i \geq 0$ for $1 \leq i \leq n - 1$. Then $x_{j+1} - x_j \leq x_j - x_{j-1}$ for all $1 \leq j \leq n - 1$. 

\[ \]
(2) Suppose for some $1 \leq r \leq n-1$, $(\hat{\mathbf{A}}\mathbf{x})_i \geq 0$, $1 \leq i \neq r \leq n-1$ and $(\hat{\mathbf{A}}\mathbf{x})_r \geq -1$. Then for all $1 \leq j \leq n-1$,
\[
\begin{cases}
x_{j+1} - x_j \leq x_j - x_{j-1}, & \text{if } j \neq r \\
x_{j+1} - x_j \leq 1 + x_j - x_{j-1}, & \text{if } j = r .
\end{cases}
\]

Proof. We will prove the second statement. Suppose for some $1 \leq r \leq n-1$, $(\hat{\mathbf{A}}\mathbf{x})_i \geq 0$, $1 \leq i \neq r \leq n-1$ and $(\hat{\mathbf{A}}\mathbf{x})_r \geq -1$. For $j \neq r$, $0 \leq -x_{j-1} + 2x_j - x_{j+1}$, which implies $x_{j+1} - x_j \leq x_j - x_{j-1}$. For $j = r$, $0 \leq 1 - x_{r-1} + 2r - x_{r+1}$ and so $x_{r+1} - x_r \leq 1 + x_r - x_{r-1}$. The proof of the first statement is similar.

Lemma 3.3. The $(q - p + 1)$-tuples in $\mathcal{I}(p, i_{\max} : q, i_{\min})$, $\mathcal{I}^*(p, i_{\min} : q, i_{\max})$, as well as the $(q - p + 1)$-tuple $(a, a, \ldots, a)$, satisfy the system of inequalities $-x_j + 2x_{j+1} - x_{j+2} \geq 0$, for $p \leq j \leq q - 2$.

Proof. Let $j$ be such that $p \leq j \leq q - 2$. Consider $(x_p, x_{p+1}, \ldots, x_q) \in \mathcal{I}(p, i_{\max} : q, i_{\min})$. Then $x_{j+1} = x_j + \alpha$, for some $i_{\min} \leq \alpha \leq i_{\max}$ and $x_{j+2} = x_j + \alpha + (\alpha - \beta)$ for some $0 \leq \beta \leq \alpha - i_{\min}$. Then $-x_j + 2x_{j+1} - x_{j+2} = \beta \geq 0$. Since the tuples in $\mathcal{I}^*(p, i_{\min} : q, i_{\max})$ can be obtained by reversing the order of the tuples in $\mathcal{I}(q, i_{\max} : p, i_{\min})$, the rest of the lemma follows.

Lemma 3.4. For $n \geq 2$, $0 \leq s \leq n - 1$, $x_1, x_{n-1} \in \mathbb{Z}$ such that $x_1, x_{n-1} \geq \delta_{s, 0}$ and $x_1 + x_{n-1} \leq k - 1 + \delta_{s, 0}$, let $S = \{ \mathbf{x} = (x_1, x_2, \ldots, x_n) : (\hat{\mathbf{A}}\mathbf{x})_i \geq 0 \text{ for } i \neq s, (\hat{\mathbf{A}}\mathbf{x})_s \geq -1 \}$. Then $S = M_1 \cup M_2 \cup M_3 \cup M_4 \cup M_5$, where $M_j, 1 \leq j \leq 5$ are the tuples given above.

Proof. First, let us show that $M_1 \cup M_2 \cup M_3 \cup M_4 \cup M_5 \subseteq S$. Let $\mathbf{x} = (x_1, x_2, \ldots, x_{n-1}) \in M_1 \cup M_2 \cup M_3 \cup M_4 \cup M_5$. Since the $M_j$’s are disjoint, $\mathbf{x} \in M_j$ for some $j$. Suppose $\mathbf{x} \in M_1$. Then $(x_1, x_2, \ldots, x_p) \in \mathcal{I}(1, x_1 : p, 1)$, $(x_p, x_{p+1}, \ldots, x_q) = (\ell_1, \ell_1, \ldots, \ell_1)$, $(x_q, x_{q+1}, \ldots, x_s) \in \mathcal{I}^*(q, 1 : s, t_1 + 1)$, and $(x_s, x_{s+1}, \ldots, x_{n-1}) \in \mathcal{I}^*(s, t_1 : n - 1, x_{n-1})$. It follows from Lemma 3.3 that $(\hat{\mathbf{A}}\mathbf{x})_i \geq 0$ for all $i \neq 1, p, q, s, n-1$, so we must check these values of $i$. Now, $(\hat{\mathbf{A}}\mathbf{x})_1 = 2x_1 - x_2 \geq 0$ since $(x_1, x_2, \ldots, x_p) \in \mathcal{I}(1, x_1 : p, 1)$; similarly $(\hat{\mathbf{A}}\mathbf{x})_{n-1} = -x_{n-2} + 2x_{n-1} \geq 0$. Additionally, $(\hat{\mathbf{A}}\mathbf{x})_p = -x_{p-1} + 2x_p - x_{p+1} = (x_p - x_{p-1}) + (x_p - x_{p+1}) \geq 0$ since $(x_1, x_2, \ldots, x_p) \in \mathcal{I}(1, x_1 : p, 1)$ and either $x_p = x_{p+1}$ or $(x_p, x_{p+1}, \ldots, x_s) \in \mathcal{I}^*(p, 1 : s, t_1 + 1)$. Similarly, $(\hat{\mathbf{A}}\mathbf{x})_q \geq 0$. Finally, $(\hat{\mathbf{A}}\mathbf{x})_s = -x_{s-1} + 2x_s - x_{s+1} = -(x_{s-1} - x_s) \geq -(t_1 + 1) + t_1 = -1$. Thus $\mathbf{x} \in S$. Similarly, if $\mathbf{x} \in M_j, j = 2, 3, 4, 5$, it can be shown that $\mathbf{x} \in S$.

Now, we show that $S \subseteq M_1 \cup M_2 \cup M_3 \cup M_4 \cup M_5$. Let $\mathbf{x} = (x_1, x_2, \ldots, x_{n-1}) \in S$. We wish to show that $\mathbf{x} \in M_j$ for some $j = 1, 2, 3, 4, 5$. Denote $\max\{x_i \mid i = 1, 2, \ldots, n-1\}$ by $\ell$. Note that $\ell \geq \max\{x_1, x_{n-1}\}$.

Suppose $s = 0$ and suppose further that $r$ is the smallest integer such that $x_r > x_{r+1}$. Then by Lemma 3.2, $(x_r, x_{r+1}, \ldots, x_{n-1}) \in \mathcal{I}^*(r, 1 : n - 1, x_{n-1})$. Then $x_{r-1} \leq x_r$. Suppose that $q$ is the largest integer such that $x_q < x_q$. Then $x_q = x_{q+1} = \cdots = x_r = \ell$. By Lemma 3.2, $(x_1, x_2, \ldots, x_q) \in \mathcal{I}(1, x_1 : q, 1)$. Hence, $\mathbf{x}$ has the
structure of an element in \( M_5 \). Observe that the largest value of \( \ell \) occurs when we increase by \( x_1 \) and decrease by \( x_{n-1} \) as many times as possible. Therefore, \( \ell \) satisfies the inequality \( \left\lceil \frac{\ell - x_\ell}{x_1} \right\rceil + 1 + \left\lceil \frac{\ell - x_{n-1}}{x_{n-1}} \right\rceil \leq n - 1 \), which is equivalent to \( \left\lceil \frac{\ell}{x_1} \right\rceil + \left\lceil \frac{\ell}{x_{n-1}} \right\rceil \leq n \). So, by Lemma 3.1 we have \( \ell \leq \max \left\{ \frac{x_\ell (x_\ell - n - m)}{x_1 + x_{n-1}}, \frac{x_{n-1} (x_{n-1} - n - m)}{x_1 + x_{n-1}} + m - x_{n-1} \right\} \), where \( x_\ell n \equiv m \pmod{x_1 + x_{n-1}} \). A similar argument can be made in the case in which \( s > 0 \) and \( x_s = \ell \).

Now consider the case in which \( s > 0 \) and \( x_s \neq \ell \). Note that either \( \min\{i \mid x_i = \ell\} < s \) or \( \min\{i \mid x_i = \ell\} > s \) and either the value of \( x_s \) consecutively repeats or does not consecutively repeat.

First, consider the case in which \( \min\{i \mid x_i = \ell\} < s \) and the value of \( x_s \) does not consecutively repeat. Suppose \( p \) is the smallest positive integer such that \( x_p = \ell \). Then by Lemma 3.2 \((x_1, x_2, \ldots, x_p) \in \mathcal{I}(1, x_1 : p, 1) \). Now let \( q \) be the smallest positive integer such that \( x_q > x_{q+1} \). Then \( x_p = x_{p+1} = \cdots = x_q = \ell \). By Lemma 3.2 \((x_q, x_{q+1}, \ldots, x_s) \in \mathcal{I}^*(q, 1 : s, t + 1) \) and \((x_s, x_{s+1}, \ldots, x_{n-1}) \in \mathcal{I}^*(s, t : n - 1, x_{n-1}) \), where \( t = x_s - x_{s+1} \). Hence, \( \mathbf{x} \) has the structure of an element of \( M_1 \). Since \((x_s, x_{s+1}, \ldots, x_{n-1}) \in \mathcal{I}^*(s, t : n - 1, x_{n-1}) \) and since we could have \( x_{s-1} = \ell \) and in this case \((x_1, x_2, \ldots, x_{s-1}) \in \mathcal{I}(1, x_1 : s - 1, 1) \), we obtain \( x_{n-1} + n - s - 1 \leq x_s \leq \min\{x_1(s - 1) - 1, x_{n-1}(n - s)\} \).

Now, we consider \( t = x_s - x_{s+1} \). Because \((x_s, x_{s+1}, \ldots, x_{n-1}) \in \mathcal{I}^*(s, t : n - 1, x_{n-1}) \) and \( x_s - t \leq x_{n-1}(n - s - 1) \), \( \max\{x_s - x_{n-1}(n - s - 1)\} \leq t \leq \left\lceil \frac{x_s - x_{n-1}}{n - s - 1} \right\rceil \). Notice that \( \ell \) must satisfy \( \left\lceil \frac{\ell - x_\ell}{x_1} \right\rceil + 1 + \left\lceil \frac{\ell - x_{n-1}}{x_{n-1}} \right\rceil \leq s - 1 \), which expresses increasing by \( x_1 \), the largest possible increase and decreasing by \( t + 1 \), the largest possible decrease, as many times as possible, obtaining \( x_s \). The equation simplifies to \( \left\lceil \frac{\ell}{x_1} \right\rceil + \left\lceil \frac{\ell}{x_{n-1}} \right\rceil \leq s \) and by Lemma 3.1 we obtain \( \max\{x_1, x_{s+1}\} \leq \ell \leq \max\left\{ \frac{x_{s+1}(t+1) + x_{s+1}}{x_{s+1} + (t+1)}, \frac{x_s((t+1) + x_s)}{x_{s+1} + (t+1)} + m - (t + 1) \right\} \), where \( m \) is such that \((s(t+1)+x_s) \equiv m \pmod{t} \).

Now, consider the case in which \( \min\{i \mid x_i = \ell\} > s \), the value of \( x_s \) does consecutively repeat, and \( x_s \neq \ell \). By a similar argument as above, it follows that \((x_1, x_2, \ldots, x_p) \in \mathcal{I}(1, x_1 : p, 1) \), \( x_p = x_{p+1} = \cdots = x_q = \ell \), and \((x_q, x_{q+1}, \ldots, x_s) \in \mathcal{I}^*(q, 1 : s, 1) \). Suppose \( r \) is the smallest integer greater than \( s \) such that \( x_r > x_{r+1} \). Then by Lemma 3.2 \( x_s = x_{s+1} = \cdots = x_r \) and \((x_r, x_{r+1}, \ldots, x_{n-1}) \in \mathcal{I}^*(r, 1 : n - 1, x_{n-1}) \). Therefore, \( \mathbf{x} \) has the form of an element of \( M_2 \). Since \((x_1, x_2, \ldots, x_p) \in \mathcal{I}(1, x_1 : p, 1) \), \( \min\{i \mid x_i = \ell\} < s \), and \((x_r, x_{r+1}, \ldots, x_{n-1}) \in \mathcal{I}^*(r, 1 : n - 1, x_{n-1}) \), \( \max\{x_{n-1}, x_1 - 1\} \leq x_s \leq \min\{x_{n-1}(n - s - 1), x_1(s - 1) - 1\} \). By similar reasoning as above, \( \ell \) must satisfy \( \left\lceil \frac{\ell - x_1}{x_1} \right\rceil + 1 + \ell - x_s \leq s \), giving the condition \( \max\{x_1, x_s + 1\} \leq \ell \leq \left\lceil \frac{x_1 + x_s}{s + x_1} \right\rceil \).

By similar reasoning, if we have an \( \mathbf{x} \) such that \( \min\{i \mid x_i = \ell\} > s \), \( x_s \neq \ell \) and the value of \( x_s \) does not consecutively repeat, we find that \( \mathbf{x} \in M_3 \). If instead, \( \mathbf{x} \) is such that \( \min\{i \mid x_i = \ell\} > s \), \( x_s \neq \ell \) and the value of \( x_s \) does consecutively repeat, \( \mathbf{x} \in M_4 \).

Now, for \( 1 \leq j \leq 5 \), we define the sets of weights \( W_j = \{ \Lambda - \ell_j \omega_0 - \sum_{i=1}^{n-1} (\ell_j - x_i) \omega_i \} \), where \((x_1, x_2, \ldots, x_{n-1}) \in \bigcup_{x_1, x_{n-1}} M_j(s, n : x_1, x_{n-1}) \). Note that if \( s = 0 \), then \( W_1 = W_2 = W_3 = W_4 = \emptyset \).
Theorem 3.5. Let $n \geq 2$, $\Lambda = (k - 1)\Lambda_0 + \Lambda_s$, $k \geq 2$, $0 \leq s \leq n - 1$. Then $\max(\Lambda) \cap P^+ = \{\Lambda\} \cup W_1 \cup W_2 \cup W_3 \cup W_4 \cup W_5$.

Proof. By Proposition 2.1, the map

$$
\max(\Lambda) \cap P^+ \rightarrow kC_{a\ell} \cap (\Lambda + Q)
$$

$$
\lambda \mapsto \overline{\lambda}
$$

is a bijection. We will first find all elements in $kC_{a\ell} \cap (\Lambda + Q)$ and then use the bijection to describe all elements of $\max(\Lambda) \cap P^+$. Since $\overline{\Lambda}_0 = 0$, by definition we have

$$
kC_{a\ell} \cap (\overline{\Lambda}_s + Q) = \left\{ \overline{\lambda} = \overline{\Lambda}_s + \sum_{j=1}^{n-1} x_j \alpha_j \mid \lambda(h_j) \geq 0, 1 \leq j < n, (\lambda|\theta) \leq k \right\}.
$$

For $\overline{\lambda} \in kC_{a\ell} \cap (\overline{\Lambda}_s + Q)$, we denote $x_\overline{\lambda} = (x_1, x_2, \ldots, x_{n-1})$. Then $x_\overline{\lambda}$ satisfies $(\lambda|\theta) = \min\{s, 1\} + x_1 + x_{n-1} \leq k$ and $\lambda(h_j) = \delta_{aj} - x_{j-1} + 2x_j - x_{j+1} \geq 0$, for $1 \leq j \leq n - 2$, where we take $x_0 = x_n = 0$.

These conditions are equivalent to

$$
(\lambda|\theta) \geq 1 + s \leq n - 1,
$$

(3.2)

$$
(\lambda|\theta) \leq 1 = x_1 + x_{n-1} \leq k.
$$

Note that $(\lambda|\theta) \geq -1$ is vacuous when $s = 0$. Since $\lambda$ is a Cartan matrix of finite type and $x_\overline{\lambda}$ satisfies (3.2), we have $x_i \in \mathbb{Z}_{\geq 0}$, $1 \leq i \leq n - 1$. (See proof of Theorem 1.4 in [7].) Consider $x_1$ and $x_{n-1}$. Observe that if $x_1 = x_{n-1} = 0$, then $x_\overline{\lambda} = (0, 0, \ldots, 0)$. In this case, $\overline{\lambda} = \overline{\Lambda}_s$. Suppose $s = 0$. If $x_1 = 0$ or $x_{n-1} = 0$, then $x_1 = 0 = x_{n-1}$; assume $x_1 \geq 1, x_{n-1} \geq 1$. Since $x_\overline{\lambda}$ satisfies the last inequality of (3.2), we also have $x_1 + x_{n-1} \leq k$. When $s > 0$ and either $x_1$ or $x_{n-1}$ is nonzero, by the last inequality of (3.2), we have $1 \leq x_1 + x_{n-1} \leq k - 1$. Hence, by Lemma 3.4, $x_\overline{\lambda} \in M_1 \cup M_2 \cup M_3 \cup M_4 \cup M_5$. Therefore, $kC_{a\ell} \cap (\overline{\Lambda}_s + Q) = \{\overline{\lambda}_s, \overline{\alpha} + \sum_{j=1}^{n-1} x_j \alpha_j \mid (x_1, x_2, \ldots, x_{n-1}) \in M_1 \cup M_2 \cup M_3 \cup M_4 \cup M_5\}.

By the bijection, $\lambda = \Lambda + \sum_{j=0}^{n-1} q_j \alpha_j \in \max(\Lambda) \cap P^+$ (with $q_j \in \mathbb{Z}_{\leq 0}, 1 \leq j \leq n - 1$) maps to $\overline{\lambda} = \overline{\Lambda}_s + \sum_{j=1}^{n-1} x_j \alpha_j \in kC_{a\ell} \cap (\overline{\Lambda}_s + Q)$, where $x_j = q_j - q_0$, $1 \leq j \leq n - 1$ (see [7]). Hence $(q_1, q_2, \ldots, q_{n-1}) = (x_1 + q_0, x_2 + q_0, \ldots, x_{n-2} + q_0, x_{n-1} + q_0)$. Let $\ell = \max\{x_i \mid 1 \leq i \leq n - 1\}$. Suppose $x_\ell = \ell$. Then $q_0 = -\ell - r$, where $r = -q_\ell \geq 0$. Suppose $r > 0$. Then $\lambda + \delta = \Lambda + \sum_{j=0}^{n-1} (q_j + 1) \alpha_j = \Lambda + (\ell(\ell - (r - 1)) \alpha_0 + \sum_{j=1}^{n-1} (x_j - \ell - (r - 1)) \alpha_j \leq \Lambda$, since $x_j \leq \ell, 1 \leq j \leq n - 1$. Notice that $\lambda + \delta \in P^+$. Hence, by (3.5, Proposition 12.5), $\lambda + \delta$ is a weight of $V(\Lambda)$ which is a contradiction since $\lambda \in \max(\Lambda)$. Therefore $r = 0$ and $\lambda = \Lambda - \ell \alpha_0 - (\ell - x_1) \alpha_1 - (\ell - x_2) \alpha_2 - \ldots - (\ell - x_{n-1}) \alpha_{n-1}$. Thus, $\lambda \in \{\Lambda\} \cup W_1 \cup W_2 \cup W_3 \cup W_4 \cup W_5$. □
Remark 3.6. Note that by the symmetry of the Dynkin diagram, we also have a description of \( \max(\Lambda) \cap P^+ \) for all \( \Lambda = (k - 1)\Lambda_i + \Lambda_{s+1}, 0 \leq i \leq n - 1, 0 \leq s \leq n - 1 \).

Consider the case \( k = 2 \). When \( s = 0 \), we have \( x_1 = 1, x_{n-1} = 1 \) and

\[
M_5(0, n : 1, 1) = \{(1, 2, \ldots, \ell_5 - 1, \ell_5, \ell_5, \ldots, \ell_5, \ell_5 - 1, \ldots, 1) \mid 1 \leq \ell_5 \leq \left\lfloor \frac{n}{2} \right\rfloor \}.
\]

When \( s > 0 \), we have the cases \( x_1 = 0, x_{n-1} = 1 \) and \( x_1 = 1, x_{n-1} = 0 \). If \( x_1 = 0 \) and \( x_{n-1} = 1 \), \( x_1 = x_2 = \ldots = x_s = 0 \). Thus, the maximum \( x_i \) must occur to the right of position \( s \) and the value \( x_s \) is repeated. Thus \( M_3(s, n : 0, 1) = M_5(s, n : 0, 1) = \emptyset \) and

\[
M_4(s, n : 0, 1) = \{(0, 0, \ldots, 0, 1, 2, \ldots, s - 2\ell_4 + 1 \ell_4 - 1, \ell_4, \ell_4, \ldots, \ell_4, \ell_4 - 1, \ldots, 1) \mid 1 \leq \ell_4 \leq \left\lfloor \frac{n - s}{2} \right\rfloor \}.
\]

Similarly, when \( x_1 = 1, x_{n-1} = 0 \), \( M_1(s, n : 1, 0) = M_5(s, n : 0, 1) = \emptyset \) and

\[
M_2(s, n : 1, 0) = \{(1, 2, \ldots, \ell_2, \ell_2, \ldots, \ell_2, s - 2\ell_2 + 1 \ell_2 - 1, \ell_2 - 2, \ldots, 2, 1, 0, 0 \ldots 0) \mid 1 \leq \ell_2 \leq \left\lfloor \frac{s}{2} \right\rfloor \}.
\]

Hence, in this case, we have \( W_1 = \emptyset, W_3 = \emptyset \), and

\[
W_2 = \{2\Lambda_0 - \ell_2 \alpha_0 - ((\ell_2 - 1)\alpha_1 + (\ell_2 - 2)\alpha_2 + \cdots + \alpha_{\ell_2 - 1}) + \alpha_{\ell_2 - \ell_2 + 1} + 2\alpha_{\ell_2 - \ell_2 + 2} + \cdots + (\ell_2 - 2)\alpha_{s - 2} + (\ell_2 - 1)\alpha_{s - 1} + \ell_2 \alpha_s + \cdots + \ell_2 \alpha_{n - 1}) \mid 1 \leq \ell_2 \leq \left\lfloor \frac{s}{2} \right\rfloor \},
\]

\[
W_4 = \{2\Lambda_0 - \ell_4 \alpha_0 - (\ell_4 \alpha_1 + \cdots + \ell_4 \alpha_s + (\ell_4 - 1)\alpha_{s + 1} + (\ell_4 - 2)\alpha_{s + 2} + \cdots + \alpha_{\ell_4 + s - 1} + \alpha_{n - \ell_4 + 1} + \cdots + (\ell_4 - 2)\alpha_{n - 2} + (\ell_4 - 1)\alpha_{n - 1}) \mid 1 \leq \ell_4 \leq \left\lfloor \frac{n - s}{2} \right\rfloor \},
\]

and

\[
W_5 = \{2\Lambda_0 - \ell_5 \alpha_0 - ((\ell_5 - 1)\alpha_1 + (\ell_5 - 2)\alpha_2 + \cdots + \alpha_{\ell_5 - 1} + \alpha_{\ell_5 - \ell_5 + 1} + \cdots + (\ell_5 - 2)\alpha_{n - 2} + (\ell_5 - 1)\alpha_{n - 1}) \mid 1 \leq \ell_5 \leq \left\lfloor \frac{n}{2} \right\rfloor \}.
\]

Therefore, we have the following Corollary which agrees with the result in [7] when \( n \) is prime.
Corollary 3.7. Let \( n \geq 2, 0 \leq s \leq n - 1 \), \( \Lambda = \Lambda_0 + \Lambda_s \). Then

\[
\max(\Lambda) \cap P^+ = \{\Lambda\} \cup \begin{cases} 
W_2 \cup W_4, & \text{if } s > 0, \\
W_5, & \text{if } s = 0.
\end{cases}
\]

We have the following conjecture for the number of the maximal dominant weights of the \( \widehat{\mathfrak{sl}}(n) \)-module \( V(k\Lambda_0) \) for \( k \geq 1, n \geq 2 \).

Conjecture 3.8. For fixed \( n \geq 2 \), the number of maximal dominant weights of the \( \widehat{\mathfrak{sl}}(n) \)-module \( V(k\Lambda_0) \) is

\[
\frac{1}{n+k} \sum_{d|\gcd(n,k)} \phi(d) \left( \frac{n+k}{d} \right),
\]

where \( \phi \) is the Euler phi function.

Clearly the conjecture holds for \( k = 1 \). We consider the \( k = 2 \) case. The maximal dominant weights of the \( \widehat{\mathfrak{sl}}(n) \)-module \( V(2\Lambda_0) \) are described in Corollary 3.7. There is one maximal dominant weight for each value of \( \ell_5 \), \( 1 \leq \ell_5 \leq \left\lfloor \frac{n}{2} \right\rfloor \). Thus, counting \( k\Lambda_0 \), there are \( \left\lfloor \frac{n}{2} \right\rfloor + 1 \) maximal dominant weights of \( V(2\Lambda_0) \), which agrees with the conjectured formula.

Now we consider the case \( k = 3 \). The set of maximal dominant weights of the \( \widehat{\mathfrak{sl}}(n) \)-module \( V(3\Lambda_0) \) is \( W_5 \), which is in bijection with the set of \((n-1)\)-tuples in \( U_n = M_5(n : 0,0) \cup M_5(n : 1,1) \cup M_5(n : 1,2) \cup M_5(n : 2,1) \), where \( M_5(n : x_1,x_{n-1}) = M_5(0,n : x_1,x_{n-1}) \). Let \( u_n \) denote the number of maximal dominant weights of \( V(3\Lambda_0) \). Then \( u_n = |U_n| \). Since \( |M_5(n : 0,0)| = 1 \) and \( |M_5(n : 1,2)| = |M_5(n : 2,1)| \), we will focus on counting the tuples in \( M_5(n : 1,1) \) and \( M_5(n : 1,2) \). Any tuple in these sets is of the form \((1,2,3,\ldots,\ell,\ldots,\ell,\ldots,\ell_{n-2},x_{n-1})\), where \( x_{n-1} = 1 \) or \( 2 \) and \( x_{n-1} \leq \ell \leq \left\lfloor \frac{x_{n-1}n}{x_1+x_{n-1}} \right\rfloor \). The decrease from \( \ell \) to \( x_{n-1} \) in the last part of the tuple can be first by steps of one, possibly followed by steps of \( x_{n-1} \).

Lemma 3.9. For \( n \geq 6 \), \( u_n = \begin{cases} 
2u_{n-1} + u_{n-2} + 1, & \text{if } n \equiv 0 \text{ or } 2 \pmod{3} \\
2u_{n-1} + u_{n-2} - 1, & \text{if } n \equiv 1 \pmod{3}.
\end{cases} \)

Proof. First, we observe that any tuple in \( U_{n-1} \) corresponds to a tuple in \( U_n \) with the only difference being that the number of \( \ell \)'s exceeds exactly by one. There are also new tuples in \( U_n \) that do not correspond in this way to tuples in \( U_{n-1} \); they arise in two different manners.

One way they arise in \( U_n \) is when the upper bound for \( \ell \) is increased by one. Such a tuple appears in \( M_5(n : 1,1) \) whenever \( n \) is even. Similarly, such a tuple occurs in \( M_5(n : 1,2) \) when \( n \equiv 0 \) or \( 2 \pmod{3} \). This is summarized in the first two rows of Table 1

The other way new tuples arise in \( U_n \) is when there is a tuple in \( M_5(n-1 : 1,2) \) with at least one decrease by a step of two. Here, the tuple in \( M_5(n-1 : 1,2) \) corresponds to the tuple in \( M_5(n : 1,2) \) in which the leftmost decrease
by a step of two is replaced by two decreases of step one. A tuple in \( M_5(n-2:1,2) \) with more than one decrease by two will correspond to a new tuple in \( M_5(n-1:1,2) \) in this manner; this new tuple, in turn, will correspond to a new tuple in \( M_5(n:1,2) \) in the same way. Thus, the number of new tuples in \( U_{n-1} \), \( u_{n-1} - u_{n-2} \), is close to the number of new tuples we obtain in this way, though we must make some adjustments. The value \( u_{n-1} - u_{n-2} \) will count the new tuple in \( M_5(n-1:1,1) \) when \( n \) is odd; thus we must subtract one when \( n \) is odd. Additionally, if a tuple in \( M_5(n-2:1,2) \) has only one decrease by a step of two, we need to account for this. Recall that when \( n \equiv 0 \) or \( 2 \) (mod 3), a new tuple arises in \( M_5(n:1,2) \) because the upper bound for \( \ell \) has increased. This tuple has \( \left\lfloor \frac{\ell}{4} \right\rfloor \) decreases by step two. Because \( n \) and \( \left\lfloor \frac{\ell}{4} \right\rfloor \) have odd/even parity when \( n \equiv 0 \) or \( 2 \) (mod 3), we see that there is a tuple in \( M_5(n-2:1,2) \) with a single decrease by two only when \( n \) is even. Therefore, using the data given in Table 1, we have \( u_n = u_{n-1} + (u_{n-1} - u_{n-2}) + a \), which proves the lemma.

**Table 1.** Recursive Definition of \( u_n \), \( n \geq 6 \)

| \( n \mod 6 \) | 0 | 1 | 2 | 3 | 4 | 5 |
|----------------|---|---|---|---|---|---|
| (1) number of new tuples in \( M_5(n:1,1) \) that arise because the upper bound for \( \ell \) increases | 1 | 0 | 1 | 0 | 1 | 0 |
| (2) twice the number of new tuples in \( M_5(n:1,2) \) that arise because the upper bound for \( \ell \) increases | 2 | 0 | 2 | 0 | 2 | 0 |
| (3) number of new tuples in \( M_5(n-1:1,1) \) | 0 | 1 | 0 | 1 | 0 | 1 |
| (4) twice the number of new tuples in \( M_5(n-2:1,2) \) with exactly one decrease by 2 | 2 | 0 | 2 | 0 | 2 | 0 |
| \( a = (1) + (2) - (3) - (4) \) | 1 | -1 | 1 | 1 | -1 | 1 |

**Lemma 3.10.** For \( n \geq 2 \), \( u_n = \frac{1}{n+3} \sum_{d \mid \gcd(n,3)} \phi(d) \left( \frac{n+3}{d} \right) \).

**Proof.** From Table 2 we see that the statement is true for \( n = 2, 3, 4, 5, 6 \). To prove the statement for \( n \geq 6 \), we will use induction on \( n \). Assume that the statement holds for all \( u_m \) with \( m < n \). We will first prove the case \( n \equiv 0 \) (mod 3). By induction, \( u_n = 2u_{n-1} - u_{n-2} + 1 = \frac{2}{n+2} (\binom{n+2}{3}) - \frac{1}{n+1} (\binom{n+1}{3}) + 1 = \frac{2(n+2)(n+1)n}{6(n+2)} - \frac{(n+1)n(n-1)}{6(n+1)} + 1 = \frac{n^2 + 3n + 6}{6} \).

In this case, \( \frac{1}{n+3} \sum_{d \mid \gcd(n,3)} \phi(d) \left( \frac{n+3}{d} \right) = \frac{1}{n+3} \left( \binom{n+3}{3} + 2 \binom{n+3}{3} \right) = \frac{1}{n+3} \left( \binom{n+3}{3} + 2 \binom{n+3}{3} \right) = \frac{n^2 + 3n + 6}{6} \).

Hence the statement holds for all \( n \geq 6, n \equiv 0 \) (mod 3). The proof for the cases \( n \equiv 1,2 \) (mod 3), \( n \geq 6 \) are similar. \( \square \)
Table 2. $U_n$, $n = 2, 3, 4, 5, 6$

| $n$ | $M_5(n : 0, 0)$ | $M_5(n : 1, 1)$ | $M_5(n : 1, 2)$ | $M_5(n : 2, 1)$ | $u_n$ |
|-----|----------------|----------------|----------------|----------------|------|
| 2   | (0)            | (1)            | -              | -              | 2    |
| 3   | (0,0)          | (1,1)          | (1,2)          | (2,1)          | 4    |
| 4   | (0,0,0)        | (1,1,1)        | (1,2,2)        | (2,2,1)        | 5    |
| 5   | (0,0,0,0)      | (1,1,1,1)      | (1,2,2,2)      | (2,2,2,1)      | 7    |
| 6   | (0,0,0,0,0)    | (1,1,1,1,1)    | (1,2,2,2,1)    | (2,2,2,2,1)    | 10   |

Using MATLAB, we have verified that Conjecture 3.8 holds for $n \leq 20$ and $k \leq 10$. Observe that \( \{k \Lambda_0 - \gamma_\ell \mid 1 \leq \ell \leq \left\lfloor \frac{n}{2} \right\rfloor \} \subseteq \text{max}(k \Lambda_0) \cup P^+$, where \( \gamma_\ell = \ell \alpha_0 + (\ell - 1) \alpha_1 + (\ell - 2) \alpha_2 + \cdots + \alpha_{\ell - 1} + \alpha_{n - \ell + 1} + \cdots + (\ell - 2) \alpha_{n - 2} + (\ell - 1) \alpha_{n - 1} \).

In the next section we study the multiplicities of these maximal dominant weights of $V(k \Lambda_0)$.

4. Multiplicity of weights $k \Lambda_0 - \gamma_\ell$ in $V(k \Lambda_0)$

In this section, we use the explicit realization of the crystal base of $V(k \Lambda_0)$ in terms of extended Young diagrams, given in [4], to study the multiplicity of the maximal dominant weights \( \{k \Lambda_0 - \gamma_\ell \mid 1 \leq \ell \leq \left\lfloor \frac{n}{2} \right\rfloor \} \).

An extended Young diagram $Y = (y_i)_{i \geq 0}$ is a weakly increasing sequence with integer entries such that there exists some fixed $y_\infty$ such that $y_i = y_\infty$ for $i \gg 0$. $y_\infty$ is called the charge of $Y$. Associated with each sequence $Y = (y_i)_{i \geq 0}$ is a unique diagram in the $\mathbb{Z} \times \mathbb{Z}$ right half lattice. For each element $y_i$ of the sequence, we draw a column with depth $y_\infty - y_i$, aligned so the top of the column is on the line $y = y_\infty$. We fill in square boxes for all columns from the depth to the charge and obtain a diagram with a finite number of boxes. We color a box with lower right corner at $(a, b)$ by color $j$, where $(a + b) \equiv j \pmod{n}$. For simplicity, we refer to color $(n - j)$ by $-j$.

The weight of an extended Young diagram of charge $i$ is $\text{wt}(Y) = \Lambda_i - \sum_{j=0}^{n-1} c_j \alpha_j$, where $c_j$ is the number of boxes of color $j$ in the diagram. We denote $Y[n] = (y_i + n)_{i \geq 0}$.

Example 4.1. The extended young diagram $Y = (-4, -4, -3, -2, -2, 0, 0, 0, \ldots)$ is colored as in Fig. [4] and is of weight $\text{wt}(Y) = \Lambda_0 - 3 \alpha_0 - 2 \alpha_1 - 2 \alpha_2 - 2 \alpha_3 - 4 \alpha_4 - \alpha_{n-3} - 2 \alpha_{n-2} - 2 \alpha_{n-1}$ (for any $n \geq 5$).
Figure 1. Extended Young Diagram representation of $Y = (-4, -4, -3, -2, -2, 0, 0, \ldots)$

The weight of a $k$-tuple of extended Young diagrams $Y = (Y_1, Y_2, \ldots, Y_k)$ is $\text{wt}(Y) = \sum_{i=1}^{k} \text{wt}(Y_i)$. Let $\mathcal{Y}(\Lambda_0)$ denote the set of all $k$-tuples of extended Young diagrams of charge zero. We have the following realization of the crystal for $V(\Lambda_0)$.

**Theorem 4.2.** [4] Let $V(\Lambda_0)$ be an $\hat{sl}(n)$-module and let $B(\Lambda_0)$ be its crystal. Then $B(\Lambda_0) = \{Y = (Y_1, \ldots, Y_k) \in \mathcal{Y}(\Lambda_0) \mid Y_1 \supseteq Y_2 \supseteq \cdots \supseteq Y_k \supseteq Y_1[\ell], \text{and for each } i \geq 0, \exists j \geq 1 \text{ s.t. } (Y_{j+1})_i > (Y_j)_{i+1}\}$.

**Remark 4.3.** Let $B(\Lambda_0)_\mu$ denote the set of $Y \in B(\Lambda_0)$ such that $\text{wt}(Y) = \mu$. Then $\text{mult}_{\Lambda_0}(\mu) = |B(\Lambda_0)_\mu|$.

Now we consider $Y = (\ell, -\ell, -\ell, \ldots, -\ell, 0, 0, \ldots)$, an extended Young diagram which we shift up $\ell$ units to form an $\ell \times \ell$ square in the first quadrant (see Fig. 2). In particular, the bottom left corner now has coordinates $(0,0)$. We draw a sequence of $(k-1)$ lattice paths, $p_1, p_2, \ldots, p_{k-1}$, from the lower left to upper right corner of the square,

Figure 2. $Y$, the $\ell \times \ell$ extended Young diagram

moving only up and to the right in such a way that for each color, the number of colored boxes of that same color below $p_i$ is greater than or equal to the number of colored boxes of that same color below $p_{i-1}$. Take $t^i_j, i \geq 2$ to be the number of $j$-colored boxes between $p_{i-1}$ and $p_{i-2}$. Note that $t^1_2$ is the number of boxes of color $j$ below $p_1$. 

**Definition 4.4.** We call such a sequence of lattice paths $p_1, p_2, \ldots, p_{k-1}$ admissible if it satisfies the following conditions:

1. the first path, $p_1$, must be drawn so that it does not cross the diagonal $y = x$, and
2. for $i$ such that $3 \leq i \leq k-1$,
   
   - $(a)$ $t^i_1 \leq \min \left\{ t^i_{i-1}, \ell - |j| - t^i_2 - \sum_{a=2}^{i-1} t^i_a \right\}$,
   
   - $(b)$ for $j > 0$, $t^i_1 \leq t^i_{j-1} \leq t^i_{j-2} \leq \ldots \leq t^i_1 \leq t^0_0$ and for $j < 0$, $t^i_1 \leq t^i_{j+1} \leq t^i_{j+2} \leq \ldots \leq t^i_{j-1} \leq t^0_0$.

Denote by $T^k_\ell$ the set of admissible sequences of $(k-1)$ paths in an $\ell \times \ell$ square.

**Example 4.5.** Fig. 3a is an element of $T^3_4$, where $p_1$ and $p_2$ are shown in Fig. 3b and Fig. 3c, respectively. Notice that between $p_1$ and $p_2$, there is one box of color 0.

\begin{align*}
\begin{array}{c|cc|c}
 & 0 & 1 & 2 & 3 \\
\hline
-1 & 0 & 1 & 2 \\
-2 & -1 & 0 & 1 \\
-3 & -2 & -1 & 0 \\
\end{array} & \begin{array}{c|cc|c}
 & 0 & 1 & 2 & 3 \\
\hline
-1 & 0 & 1 & 2 \\
-2 & -1 & 0 & 1 \\
-3 & -2 & -1 & 0 \\
\end{array} & \begin{array}{c|cc|c}
 & 0 & 1 & 2 & 3 \\
\hline
-1 & 0 & 1 & 2 \\
-2 & -1 & 0 & 1 \\
-3 & -2 & -1 & 0 \\
\end{array}
\end{align*}

**Figure 3.** Admissible Sequence of Paths

**Theorem 4.6.** Consider the maximal dominant weights $k\Lambda_0 - \gamma_\ell \in \max(k\Lambda_0) \cap P^+$, where $1 \leq \ell \leq \lfloor \frac{\ell}{2} \rfloor$. The multiplicity of $k\Lambda_0 - \gamma_\ell$ in $V(k\Lambda_0)$ is equal to $|T^k_\ell|$.

**Proof.** It is enough to show that the elements in $T^k_\ell$ are in one-to-one correspondence with the $k$-tuples of extended Young diagrams in $B(k\Lambda_0)_{k\Lambda_0 - \gamma_\ell}$.

Let $T \in T^k_\ell$ be an admissible sequence of $(k-1)$ paths. Recall that the paths in $T$ are all drawn in an $\ell \times \ell$ square, $Y$. We construct the $k$-tuple $Y = (Y_1, \ldots, Y_k)$ of extended Young diagrams as follows. First, we remove the boxes below $p_1$ and use these boxes to uniquely form an extended Young diagram of charge zero, which we will denote by $Y_2$. Next, we consider the boxes between $p_1$ and $p_2$ in $T$. Since for $j > 0$, $t^1_3 \leq t^i_{j-1} \leq t^i_{j-2} \leq \ldots \leq t^i_3 \leq t^i_1$ and for $j < 0$, $t^1_3 \leq t^i_{j+1} \leq t^i_{j+2} \leq \ldots \leq t^i_3 \leq t^i_1$, we can use these boxes to form a unique extended Young diagram $Y_3$ of charge zero. Now, by Definition 4.3(2a), $t^1_2 \geq t^i_3$ for all colors $j$, which implies $Y_2 \supseteq Y_3$. We continue this process until the boxes between $p_{k-1}$ and $p_k$ have been used to form the extended Young diagram $Y_k$, with $Y_2 \supseteq Y_3 \supseteq \ldots \supseteq Y_k$. The boxes above $p_{k-1}$ form an extended Young diagram, which we denote by $Y_1$. Since for each color $j$, $t^i_1 \leq \ell - |j| - t^i_2 - \sum_{a=2}^{i-1} t^i_a$, $3 \leq i \leq k-1$ by Definition 4.3(2a), we have $Y_1 \supseteq Y_2$. Note that $Y$
Corollary 4.8. When \( p \) is the \( \gamma \ell \)-avoiding permutation given by \( \mu \gamma \ell \) and \( j, j + 1 \) is the \( \gamma \ell \)-avoiding permutation, we need to construct an admissible sequence \( \mathbf{T} \in \mathcal{T}_\ell^k \) of \((k - 1)\) lattice paths. We take \( \mathbf{T} \) to be an empty diagram and fill it with the boxes in \( \mathbf{Y} \) as follows, maintaining color positions as in Fig. 2. We begin by placing \( \mathbf{Y}_1 \) in \( \mathbf{Y} \), aligning the upper left corners. Next, we draw a lattice path tracing the right edge of \( \mathbf{Y}_1 \) from the lower left to upper right corner and take this path to be \( p_{k-1} \). Now, we take the boxes from \( \mathbf{Y}_k \) and place them in \( \mathbf{Y} \), placing each box of color \( j \) in the leftmost available position for that color. Since each \( \mathbf{Y}_i, 1 \leq i \leq k, \) is an extended Young diagram and since we have exactly \( \ell - |j| \) boxes available of each color, we obtain an extended Young diagram. Thus, we are able to draw a lattice path along the right edge of the new diagram. We take this path to be \( p_{k-2} \). Now, we add the boxes of \( \mathbf{Y}_{k-1} \) in the same manner and draw \( p_{k-3} \). We continue this process until we add in the final boxes of \( \mathbf{Y}_2 \) to make a complete square. Let \( \mathbf{T} \) be the sequence of \( k - 1 \) lattice paths in the square. As before, we define \( t'_i \) to be the number of \( j \)-colored boxes between \( p_{i-1} \) and \( p_{i-2} \). Notice that \( t'_i \) is the number of boxes of color \( j \) in \( \mathbf{Y}_i \). Since each \( \mathbf{Y}_i \) is an extended Young diagram, Definition 4.3(2b) is satisfied. Since \( \mathbf{Y}_1 \supset \mathbf{Y}_2 \supset \cdots \supset \mathbf{Y}_k \), Definition 4.3(2a) and Definition 4.3(1) are satisfied. Hence \( \mathbf{T} \) is an admissible sequence of \( k - 1 \) lattice paths, which completes the proof.

Example 4.7. We associate the element of \( \mathcal{T}^3 \) in Fig. 3 with an element of \( B(3\Lambda_0) \) of weight \( 3\Lambda_0 - \gamma_4 \) as follows. First, we remove the boxes below and to the right of \( \mathbf{p}_1 \) and obtain \( \mathbf{Y}_1 \) as in Fig. 3b and \( \mathbf{Y}_2 \) as the second element in Fig. 3. Next, we remove the box that remains below \( \mathbf{p}_2 \) to determine \( \mathbf{Y}_3 \) as in Fig. 4.

Corollary 4.8. When \( k = 2 \), the multiplicity of \( 2\Lambda_0 - \gamma_\ell \in \text{max}(2\Lambda_0) \cap P^+ \) is the number of lattice paths in an \( \ell \times \ell \) square that must stay below, but can touch the diagonal \( y = x \).

Denote a permutation of \( \{1, 2, \ldots, n\} \) by a sequence \( w = w_1w_2 \ldots w_n \) indicating that \( 1 \mapsto w_1, 2 \mapsto w_2, \ldots, n \mapsto w_n \). A \((j, j - 1, \ldots, 1)\)-avoiding permutation is a permutation which does not have a decreasing subsequence of length \( j \). For example, \( w = 1342 \) is a 321-avoiding permutation because it has no decreasing subsequence of length three. Now we have the following conjecture.

Conjecture 4.9. The multiplicities of the maximal dominant weights \( (k\Lambda_0 - \gamma_\ell) \) of the \( \widehat{\text{sl}}(n) \)-modules \( V(k\Lambda_0) \) are given by \( \text{mult}_{k\Lambda_0}(k\Lambda_0 - \gamma_\ell) = \lfloor (k + 1)(k) \ldots 21 \text{-avoiding permutations of } \ell \rfloor \).

By definition, \( \mathcal{T}^\ell \) is the set of all lattice paths in an \( \ell \times \ell \) square that do not cross \( y = x \). As shown in (2, page 361), there is a bijection between the lattice paths in \( \mathcal{T}^\ell \) and the set of 321-avoiding permutations of \( \{1, 2, \ldots, \ell\} \)
as follows. Given $p \in T^2_\ell$ we construct a 321-avoiding permutation $w = w_1 w_2 \ldots w_\ell$ of $[\ell] = \{1, 2, \ldots, \ell\}$. As we traverse $p$ from $(0, 0)$ to $(\ell, \ell)$, let $\{(v_1, j_1), (v_2, j_2), \ldots (v_r, j_r)\}$ be the coordinates at the top of each vertical move, not including $(\ell, \ell)$, which is at the top of the final vertical move. We define $w_{j_1} = v_1 + 1$ for $1 \leq i \leq r$. The remaining $w_j$’s are defined by the unique map $\{1, 2, \ldots, \ell\} \setminus \{j_1, j_2, \ldots, j_r\} \mapsto \{1, 2, \ldots, \ell\} \setminus \{w_{j_1}, w_{j_2}, \ldots, w_{j_r}\}$ in increasing order. It follows from the construction that $w$ is a 321-avoiding permutation.

Conversely, let $w = w_1 w_2 \ldots w_\ell$ be a 321-avoiding permutation of $[\ell]$. Define $C_i = \{j \mid j > i, w_j < w_i\}, c_i = |C_i|$, and $J = \{j_1 < j_2 < \cdots < j_r\} = \{j \mid c_j > 0\}$. We define a path, $p \in T^2_\ell$, from $(0, 0)$ to $(\ell, \ell)$ by the moves given in Table 3.

**Table 3.** Rules for Drawing Lattice Path

| Direction   | From                          | To                           |
|-------------|-------------------------------|------------------------------|
| Horizontal  | $(0, 0)$                      | $(c_{j_1} + j_1 - 1, 0)$     |
| Vertical    | $(c_{j_1} + j_1 - 1, 0)$      | $(c_{j_1} + j_1 - 1, j_1)$   |
| Horizontal  | $(c_{j_1} + j_1 - 1, j_1)$    | $(c_{j_2} + j_2 - 1, j_1)$   |
| Vertical    | $(c_{j_2} + j_2 - 1, j_1)$    | $(c_{j_2} + j_2 - 1, j_2)$   |
|            | $\vdots$                      | $\vdots$                     |
| Horizontal  | $(c_{j_r} + j_r - 1, j_r)$    | $(\ell, j_r)$               |
| Vertical    | $(\ell, j_r)$                | $(\ell, \ell)$              |
In the following two examples we illustrate this bijection. Thus Conjecture [1,3] is true for \( k = 2 \). Furthermore, it is known (c.f. [6]) that the number of 321-avoiding permutations of \( \{1, 2, \ldots, \ell\} \) is equal to the \( \ell^{th} \) Catalan number, which coincides with the result for the multiplicity of \( 2\Lambda_0 - \gamma_\ell \) in [2].

**Example 4.10.** Let \( \ell = 4 \) and consider the 321-avoiding permutation \( w = 1342 \). We obtain the values for \( c_i, i = 1, 2, 3, 4 \) shown in Fig. 5a. Subsequently, we have the path coordinates in Fig. 5b, giving the path shown in Fig. 5c.

![Figure 5](image)

| \( i \) | \( C_i \) | \( c_i \) |
|-------|--------|--------|
| 1     | -0     | 0      |
| 2     | 4      | 1      |
| 3     | 4      | 1      |
| 4     | 0      | 0      |

**Figure 5.** Data for the avoiding permutation 1342

**Example 4.11.** Let \( \ell = 4 \) and consider the admissible path in Fig. 6a. We wish to construct a 321-avoiding permutation \( w = w_1w_2w_3w_4 \) to correspond with the admissible path. For each vertical move in the path, we determine values \( w_i \) as in Fig. 6b. We conclude that the path corresponds with the 321-avoiding permutation \( w = 3142 \).

![Figure 6](image)

In Table 4, using Theorem 4.6 we give the multiplicities of maximal dominant weights \( k\Lambda_0 - \gamma_\ell \) for \( k \leq 9 \) and \( \ell \leq 9 \). We observe that these multiplicities coincide with the number of \( (k+1)(k) \ldots 21 \)-avoiding permutations of \( \{1, 2, \ldots, \ell\} \). For some partial results in the \( k \geq 3 \) case, see [3].

Table 4. Multiplicity Table for $k \Lambda_0 - \gamma \ell$

| $k$ | 3  | 4  | 5  | 6  | 7  | 8  | 9  |
|-----|----|----|----|----|----|----|----|
| $\ell = 2$ | 2  | 2  | 2  | 2  | 2  | 2  | 2  |
| $\ell = 3$ | 6  | 6  | 6  | 6  | 6  | 6  | 6  |
| $\ell = 4$ | 23 | 24 | 24 | 24 | 24 | 24 | 24 |
| $\ell = 5$ | 103 | 119 | 120 | 120 | 120 | 120 | 120 |
| $\ell = 6$ | 513 | 694 | 719 | 720 | 720 | 720 | 720 |
| $\ell = 7$ | 2761 | 4582 | 5003 | 5039 | 5040 | 5040 | 5040 |
| $\ell = 8$ | 15767 | 33324 | 39429 | 40270 | 40319 | 40320 | 40320 |
| $\ell = 9$ | 94359 | 261808 | 344837 | 361302 | 362815 | 362879 | 362880 |
| $\ell = 10$ | 586590 | 2190688 | 3291590 | 3587916 | 3626197 | 3628718 | 3628800 |

References

[1] Barshevskey, O., Fayers, M., Schaps, M.: A non-recursive criterion for weights of a highest-weight module for an affine Lie algebra, arXiv:1002.3457v6 [math.RT] (2011).

[2] Billey, S.C., Jockusch, W., Stanley, R.P.: Some Combinatorial Properties of Schubert Polynomials, J. Alg. Combin. 2 (1993) 345-374.

[3] Jayne, R.L.: Maximal dominant weights of some integrable modules for the special linear affine Lie algebras and their multiplicities. NCSU Ph.D. Dissertation. (2011).

[4] Jimbo, M., Misra, K.C., Miwa, T., Okado, M.: Combinatorics of representations of $U_q (\hat{sl}(n))$ at $q = 0$, Commun. in Math. Phys. 136 (1991) 543-566.

[5] Kac, V.G.: Infinite-dimensional Lie algebras. Third edition. Cambridge University Press, New York, 1990.

[6] Stanley, R.P.: Enumerative Combinatorics. Vol. 2. Cambridge University Press, New York, 1999.

[7] Tsuchioka, S.: Catalan numbers and level 2 weight structures of $A_{p-1}^{(1)}$, RIMS Kôkyûroku Bessatsu. B11 (2009) 145-154.

HAMPDEN-SYDNEY COLLEGE, HAMPDEN-SYDNEY, VA 23943

E-mail address: rjayne@hsc.edu

DEPARTMENT OF MATHEMATICS, NORTH CAROLINA STATE UNIVERSITY, RALEIGH, NC 27695-8205

E-mail address: misra@ncsu.edu