A REMARK ON $C^{1,\alpha}$-REGULARITY FOR DIFFERENTIAL INEQUALITIES IN VISCOSITY SENSE

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Abstract. We prove interior $C^{1,\alpha}$-regularity for solutions

$-\Lambda \leq F(D^2 u) \leq \Lambda$

where $\Lambda$ is a constant and $F$ is fully nonlinear, 1-homogeneous, uniformly elliptic.

The proof is based on a reduction to the homogeneous equation $F(D^2 u) = 0$ by a blow-up argument – i.e. just like what is done in the case of viscosity solutions $F(D^2 u) = f$ for $f \in L^\infty$.

However it was not clear to us that the above inequality implies $F(D^2 u) = f$ for some bounded $f$ (as would be the case for linear equations in distributional sense by approximation). Nor were we able to find the literature on $C^{1,\alpha}$-regularity for viscosity inequalities. So we thought this result might be worth recording.

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1. Introduction

It is a classical result in the regularity theory of viscosity solutions that viscosity solutions $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ to a large class of fully nonlinear elliptic equation

(1.1) \[ F(D^2 u) = f \quad \text{in } \Omega \]

actually have H"older continuous gradient, see e.g. [3, Theorem 8.3]. See Section 2 for the precise definition of $F$ we consider here.
Let us recall that a viscosity solution to (1.1) is a map $u \in C^0(\Omega)$ such that

$$F(D^2u) \leq f, \quad \text{and} \quad F(D^2u) \geq f$$

both hold in viscosity sense. And $F(D^2u) \leq f$ holds in viscosity sense if for any $\varphi \in C^2(\mathbb{R}^n)$ such that $\varphi - u$ attains its maximum in some $x_0 \in \Omega$ we have

$$F(D^2\varphi(x_0)) \leq f(x_0).$$

Similarly, $F(D^2u) \geq f$ holds in viscosity sense if for any $\varphi \in C^2(\mathbb{R}^n)$ such that $\varphi - u$ attains its minimum in some $x_0 \in \Omega$ we have

$$F(D^2\varphi(x_0)) \geq f(x_0).$$

For an introduction to the theory of viscosity solutions we refer e.g. to [3, 8, 9].

In this small note we want to record that the $C^{1,\alpha}$-regularity theory for equations $F(D^2u) = f$ also holds for differential inequalities. More precisely we have

**Theorem 1.1.** Assume that $u \in C^0(\Omega)$ for some $\beta > 0$ solves in viscosity sense

(1.2) \quad $- \Lambda \leq F(D^2u) \leq \Lambda$ \quad in $\Omega$,

where $F$ is a uniformly elliptic operator and 1-homogeneous (see Section 2), and $\Lambda < \infty$ is a constant. Then $u \in C^{1,\alpha}(\Omega)$ for some $\alpha < 1$.

Let us remark that Theorem 1.1 does not seem to follow (even in the linear case $F(D^2u) = \Delta u$ and even with right-hand side in $f \in L^\infty$) only from considering incremental quotients and using Harnack inequality (as in [3, §5.3] where the right-hand side is zero). The incremental quotient of $f$ is not uniformly bounded and blows up as $h \to 0$.

The problem that lead us to searching in the literature for Theorem 1.1 is the following: in [7] Khomrutai and the author study a geometric obstacle problem. In this geometric problem one is lead to consider obstacle problems for obstacles $\psi \in C^2$ where the energies is of the form

$$\int |\nabla u|^2 + u^2g \quad \text{where} \quad u \geq \psi.$$ 

For $g \geq 0$ and $g \in L^1$ one can show boundedness of $u$. If one has $g$ bounded one obtains Hölder continuity of $u$. In particular, in the latter case one obtains in viscosity sense the following three inequalities.

$$\Delta u \leq ug \quad \text{in} \quad \Omega$$

$$\Delta u = ug \quad \text{in} \quad \{u > \psi\}$$

$$\Delta u \geq \Delta \psi \quad \text{in} \quad \{u = \psi\}.$$ 

That is, one can find $\Lambda$ such that

$$\Delta u \leq \Lambda,$$

and

$$\Delta u \geq \Lambda,$$

where $F$ is a uniformly elliptic operator and 1-homogeneous (see Section 2), and $\Lambda < \infty$ is a constant. Then $u \in C^{1,\alpha}(\Omega)$ for some $\alpha < 1$. 

both hold in viscosity sense, but it is not obvious how to find a priori a function \( f \) such that \( \Delta u = f \in L^\infty \). If these inequalities were to hold for distributional solutions one easily gets \( C^{1,\alpha} \)-regularity, cf. Theorem 1.2. For this linear problem one might hope to use an argument as in [6] for the \( p \)-Laplacian to show that the inequality is actually true also in a weak sense.

Another approach to prove Theorem 1.1 might be to appeal to the relation between Viscosity solutions and pointwise strong solutions as in [4], and show this to hold for inequalities.

Our choice of proof for Theorem 1.1 is very similar to the usual arguments used for equations \( F(D^2u) = f \in L^\infty \), namely one uses a blow-up procedure to reduce the regularity theory to the homogeneous solutions. We saw similar arguments appear e.g. in [1, 11, 10, 2].

However, while Hölder continuity for solutions of viscosity inequalities are well-established and easily citable, e.g. in [3], we were not able to find in the literature a statement regarding Hölder continuity for the gradient of solutions to such inequalities. The author would have appreciated such a statement recorded somewhere, and thought it might be useful also for others.

Let us also remark that in the weak sense a theorem similar to Theorem 1.1 holds true – simply by approximation.

**Theorem 1.2.** Let \( A \in \mathbb{R}^{n \times n} \) be a symmetric positive definite matrix, and let \( u \in W^{1,2}(\Omega) \), \( \Omega \subset \mathbb{R}^n \) open, solve

\[
f_1 \leq \text{div}(A\nabla u) \leq f_2 \quad \text{in } \Omega
\]

that is we have for any \( \varphi \in C^\infty_c(\Omega), \varphi \geq 0, \)

\[
-\int \langle A\nabla u, \nabla \varphi \rangle \leq \int f_2 \varphi,
\]

and

\[
-\int \langle A\nabla u, \nabla \varphi \rangle \geq \int f_1 \varphi.
\]

Then for every Ball \( B(2r) \subset \Omega, \)

\[
\|\nabla^2 u\|_{L^p(B(2r))} \lesssim \|f_1\|_{L^p(B(2r))} + \|f_2\|_{L^p(B(2r))} + \|u\|_{L^2(B(2r))}.
\]

In particular, by Sobolev embedding, if \( p > n \) we obtain \( C^{1,\alpha} \)-regularity estimates for \( u \).

**Proof.** Let \( \eta \in C^\infty_c(B(0,1)), \eta \equiv 1 \) on \( B(0,1/2) \), and \( 0 \leq \eta \leq 1 \) on \( B(0,1) \) be the usual mollifying kernel and set \( \eta_\varepsilon := \varepsilon^{-n}\eta(\cdot/\varepsilon) \). Denote the convolutions with \( \eta_\varepsilon \) by \( u_\varepsilon := \eta_\varepsilon * u \) and \( \varphi_\varepsilon := \eta_\varepsilon * \varphi \). Moreover we define

\[
(1.3) \quad g_\varepsilon := \text{div}(A\nabla u_\varepsilon) \in C^\infty(\Omega_{-\varepsilon}).
\]

Here

\[
\Omega_{-\varepsilon} := \{ x \in \Omega, \text{dist} (x, \partial \Omega) > \varepsilon \}.
\]
We have for any $\varphi \in C^\infty_c(\Omega)$, $\varphi \geq 0$,
$$\int g_\varepsilon \varphi = -\int \langle A \nabla u_\varepsilon, \nabla \varphi \rangle = -\int \langle A \nabla u, \nabla \varphi_\varepsilon \rangle \leq \int f_2 * \eta_\varepsilon \varphi.$$  
and likewise
$$\int g_\varepsilon \varphi \geq \int f_1 * \eta_\varepsilon \varphi.$$  
With the same argument that one uses to prove the fundamental theorem of calculus, namely letting $\varphi$ approximate the dirac-function, we obtain
$$f_1 * \eta_\varepsilon \leq g_\varepsilon \leq f_2 * \eta_\varepsilon \quad \text{pointwise everywhere in } \Omega - \varepsilon.$$  
In particular, for $\varepsilon < r$ and $B(2r) \subset \Omega$ we readily obtain for any $p \in (1, \infty)$
$$\|g_\varepsilon\|_{L^p(B(r))} \lesssim \|f_1\|_{L^p(B(2r))} + \|f_2\|_{L^p(B(2r))}.$$  
Thus, from standard Calderon-Zygmund elliptic theory for the (constant coefficient-) equation (1.3) we find
$$\|\nabla^2 u_\varepsilon\|_{L^p(B(r))} \lesssim \|f_1\|_{L^p(B(2r))} + \|f_2\|_{L^p(B(2r))} + \|u\|_{L^2(B(2r))}$$
with constants independent of $\varepsilon$. Since $u_\varepsilon \xrightarrow[\varepsilon \to 0]{} u$ in $W^{1,2}_\text{loc}(\Omega)$ we obtain from the boundedness of the $W^{2,p}$-norm of $u_\varepsilon$ that the weak limit $u \in W^{2,p}_\text{loc}(\Omega)$. Moreover, from weak convergence we have the estimate
$$\|\nabla^2 u\|_{L^p(B(r))} \lesssim \liminf_{\varepsilon \to 0} \|\nabla^2 u_\varepsilon\|_{L^p(B(r))} \leq \|f_1\|_{L^p(B(2r))} + \|f_2\|_{L^p(B(2r))} + \|u\|_{L^2(B(2r))}$$
□

2. Ingredients and definitions

Denote by $\mathcal{S}^n \subset \mathbb{R}^{n \times n}$ the symmetric matrices and let $F: \mathbb{R}^{n \times n} \to \mathbb{R}$ be a uniformly elliptic operator, that is we shall assume there exists ellipticity constants $0 < \lambda_1 < \lambda_2 < \infty$ such that
$$\lambda_1 \text{tr}(N) \leq F(M + N) - F(M) \leq \lambda_2 \text{tr}(N) \quad \forall M, N \in \mathcal{S}^n, \quad N \geq 0.$$  
Moreover, we shall assume that $F$ is 1-homogeneous, i.e. that $F(\sigma N) = \sigma F(N)$.

For solutions $u$ to the homogeneous equations $F(D^2 u) = 0$ we have by e.g. [3, Corollary 5.7.]

**Theorem 2.1** ($C^{1,\alpha}$ for homogeneous equation). Assume that $F$ is as above, $\Omega \subset \mathbb{R}^n$ is open and in viscosity sense $u \in C^0(\Omega)$ solves
$$F(D^2 u) = 0 \quad \text{in } \Omega.$$  
Then $u \in C^{1,\alpha}(\Omega)$ for some $\alpha < 1$.

Theorem 1.1 is thus a consequence of the following
Theorem 2.2. Let $\alpha \in (0, 1]$ and assume that $F$ is a homogeneous, uniformly elliptic operator as above such that every viscosity solution $v \in C^0(\Omega)$ of the homogeneous equation

$$F(D^2v) = 0 \quad \text{in } \Omega$$

satisfies $v \in C^{1,\alpha}$.

Assume that $u \in C^0(\Omega)$ solves in viscosity sense (1.2). Then $u \in C^{1,\beta}(\Omega)$ for any Hölder exponent $\beta \in (0, \alpha)$.

Hölder regularity of solutions $u$ of differential inequalities in viscosity sense are standard, they follow from Harnack’s inequality. See, e.g., [3, Proposition 4.10].

Lemma 2.3 (Uniform Hölder regularity). Let $u$ solve (1.2) for $F$ as above. For some $\gamma \in (0, 1)$ we have $C^\gamma$-regularity, namely for any ball $B(2r) \subset \Omega$ we have

$$[u]_{C^\gamma(B(r))} \leq C(\Lambda, r, \|u\|_{L^\infty(B(2r))})$$

As a last ingredient we need the (standard) result about limits of uniformly converging viscosity (sub/super)-solutions.

Lemma 2.4. Let $\Omega \subset \mathbb{R}^n$ open, $u_k \in C^0(\Omega)$, and $\Lambda_k \in \mathbb{R}$ be a sequence of (viscosity) solutions to

$$F(D^2u_k) \leq \Lambda_k \quad \text{in } \Omega,$$

or

$$F(D^2u_k) \geq \Lambda_k \quad \text{in } \Omega,$$

respectively.

Assume that $\Lambda_k \to \Lambda_\infty \in \mathbb{R}$ and $u_k$ converges locally uniformly to $u_\infty$. Then $u_\infty$ is a solution in viscosity sense of

$$F(D^2u_\infty) \leq \Lambda_\infty \quad \text{in } \Omega,$$

or

$$F(D^2u_\infty) \geq \Lambda_\infty \quad \text{in } \Omega,$$

Proof. This is of course well known, but we repeat the argument for the $\leq$-case.

Let $u_k \in C^0(\Omega)$ converge locally uniformly to $u_\infty \in C^0(\Omega)$, and assume that

$$F(D^2u_k) \leq \Lambda_k$$

in viscosity sense, for some constants $\Lambda_k \xrightarrow{k \to \infty} \Lambda$. We will show that then (also in viscosity sense)

$$F(D^2u) \leq \Lambda.$$

So let $\varphi \in C^2(\Omega)$ be a function testfunction for $u$, i.e. assume that $\varphi \leq u$ and $\varphi(x_0) = u(x_0)$. We need to show that

$$F(D^2\varphi(x_0)) \leq \Lambda.$$
Set 
\[ \tilde{\phi}(x) := \varphi(x) - |x - x_0|^4. \]

Now we observe that for any \( y \) satisfying
\[ (2.4) \quad \tilde{\phi}(y) - u_k(y) \geq \tilde{\phi}(x_0) - u_k(x_0) \]
we also have
\[ \tilde{\phi}(y) - u(y) \geq \tilde{\phi}(x_0) - u(x_0) - 2\|u - u_k\|_{L^\infty}. \]

Since \( u(y) \geq \varphi(y) \) and \( \varphi(x_0) = u(x_0) \) we obtain from the definition of \( \tilde{\phi} \),
\[ -|y - x_0|^4 \geq \varphi(y) - u(y) - |y - x_0|^4 \geq -2\|u - u_k\|_{L^\infty}, \]
that is any \( y \) satisfying (2.4) also satisfies
\[ |y - x_0|^4 \leq 2\|u - u_k\|_{L^\infty} \xrightarrow{k \to \infty} 0. \]

In particular we can find a sequence \( x_k \xrightarrow{k \to \infty} x_0 \) such that
\[ \tilde{\phi}(x_k) - u_k(x_k) = \max_x (\tilde{\phi}(x) - u_k(x)) \geq \tilde{\phi}(x_0) - u_k(x_0) \]
That is, \( \tilde{\phi}(x) \) is a testfunction for \( u_k \) at \( x_k \), and from (2.2) we get
\[ F(D^2\tilde{\phi}(x_k)) \leq \Lambda_k. \]

From the ellipticity condition (2.1) we also obtain (see [3, Lemma 2.2]) for \( M = D^2\tilde{\phi}(x_k) \) and \( N = D^2\varphi(x_0) - D^2\tilde{\phi}(x_k) \)
\[ F(D^2\varphi(x_0)) \leq F(D^2\tilde{\phi}(x_k)) + C(\Lambda) |D^2\tilde{\phi}(x_k) - D^2\varphi(x_0)| \leq \Lambda_k + C(\Lambda) |D^2\tilde{\phi}(x_k) - D^2\varphi(x_0)|. \]

But since \( x_k \xrightarrow{k \to \infty} x_0 \) we have \( D^2\tilde{\phi}(x_k) \xrightarrow{k \to \infty} D^2\tilde{\phi}(x_0) = D^2\varphi(x_0) \). Thus, we obtain (2.3).

3. Proof of the main theorem

The heart of the matter is the following decay estimate for the oscillation, we found this kind of argument in [2, Lemma 3.4].

**Proposition 3.1.** Let \( F \) be as above, and \( \alpha \) as in Theorem 2.2. For any \( \beta < \alpha \) and any \( \lambda_0 \in (0, 1) \) there exists \( \varepsilon > 0 \) and \( \lambda \in (0, \lambda_0) \) such that the following holds.

Let \( u \in C^0(B(0, 1)) \) with \( \text{osc}_{B(0,1)} u \leq 1 \) and
\[ -\varepsilon \leq F(D^2u) \leq \varepsilon \quad \text{in } B(0, 1) \]

Then there exists \( q \in \mathbb{R}^n \) such that
\[ \text{osc}_{B(\lambda)} (u - \langle q, x \rangle) < \frac{1}{2} \lambda^{1+\beta}. \]
Proof. Assume the claim is false for some fixed $\beta < \alpha$ and $\lambda_0 \in (0, 1)$. Then we find for every $k \in \mathbb{N}$ functions $u_k \in C^0(B(0,1))$ with $\text{osc}_{B(0,1)} u_k \leq 1$ solving
\[-\frac{1}{k} \leq F(D^2 u_k) \leq \frac{1}{k},\]
but for every $\lambda \in (0, \lambda_0)$ we have
\[\inf_{q^* \in \mathbb{R}^n} \text{osc}_{B(\lambda)} (u_k - \langle q^*, x \rangle_{\mathbb{R}^n}) \geq \frac{1}{2} \lambda^{1+\beta}.\]
Without loss of generality we can assume that $u_k(0) = 0$ (since otherwise $u_k - u_k(0)$ satisfies the same assumptions), and since $\text{osc}_{B(0,1)} u_k \leq 1$ we have $\|u_k\|_{\infty} \leq 1$. By Lemma 2.3 the $u_k$ are uniformly bounded in $C^{\alpha}$, for some fixed $\alpha > 0$. By Arzela-Ascoli we thus may assume, up to taking a subsequence, that $u_k \rightarrow u_\infty$ locally uniformly in $B(0,1)$.

In view of Lemma 2.4 we find that $u_\infty$ solves the homogeneous equation
\[F(D^2 u_\infty) = 0 \quad \text{in } B(0,1).\]
From the assumptions of Theorem 2.2 we know that $u_\infty \in C^{1,\alpha}$. From Taylor’s theorem we have thus for any $\lambda \in (0, 1/4)$,
\[\inf_{q^* \in \mathbb{R}^n} \text{osc}_{B(\lambda)} (u_\infty - \langle q^*, x \rangle_{\mathbb{R}^n}) \lesssim [u_\infty]_{C^{1,\alpha}(B(0,1/2))} \lambda^{1+\alpha}.\]
On the other hand, by locally uniform convergence of $u_k$ we have for any $\lambda \in (0, \lambda_0)$.
\[\inf_{q^* \in \mathbb{R}^n} \text{osc}_{B(\lambda)} (u_\infty - \langle q^*, x \rangle_{\mathbb{R}^n}) \geq \frac{1}{2} \lambda^{1+\beta}.\]
That is, we have that for all $\lambda \in (0, 1/4)$, $\lambda < \lambda_0$.
\[\lambda^{\beta - \alpha} \leq [u_\infty]_{C^{1,\alpha}}\]
Since $\beta < \alpha$ this is impossible for very small $\lambda$. \qed

Iterating Proposition 3.1 we obtain

**Corollary 3.2.** Let $F$ be as above, and $\alpha$ as in Theorem 2.2. For any $\beta < \alpha$ and any $\lambda_0 \in (0, 1)$ there exist $\varepsilon > 0$ and $\lambda \in (0, \lambda_0)$ such that the following holds.

Assume $u$ solves
\[
\tag{3.1} -\varepsilon \leq F(D^2 u) \leq \varepsilon \quad \text{in } B(0,1)
\]
and
\[\text{osc}_{B(0,1)} u < 1.\]

Then for any $k \in \mathbb{N} \cup \{0\}$, there exists $q_k \in \mathbb{R}^n$ such that
\[\lambda^{-k(1+\beta)} \text{osc}_{B(\lambda^k)} (u(x) - q_k \cdot x) < 2^{-k}.\]
Proof. Let $\lambda_0$ w.l.o.g. be such that $2\lambda_0^{1-\beta} < 1$ and let $\lambda \in (0, \lambda_0)$ be from Proposition 3.1. For $k \in \mathbb{N} \cup \{0\}$ we set
\[ u_k(x) := 2^k \lambda^{-k(1+\beta)} \left( u(\lambda^k x) - q_k \cdot \lambda^k x \right), \]
where $q_0 = 0$ and $q_k \in \mathbb{R}^n$, $k \geq 1$, remains to be chosen.

Regardless of the choice of the constant vector $q_k$ we obtain from (3.1), for every $k \in \mathbb{N} \cup \{0\}$,
\[-2^k \lambda^{k(1-\beta)} \varepsilon \leq F(D^2 u_k) \leq 2^k \lambda^{k(1-\beta)} \varepsilon \quad \text{in } B(0, 1).\]
By the choice of $\lambda_0$ and since $\lambda \in (0, \lambda_0)$ we have in particular for every $k \in \mathbb{N} \cup \{0\}$,
\[-\varepsilon \leq F(D^2 u_k) \leq \varepsilon \quad \text{in } B(0, 1).\]

The claim follows, once we show
\[ \text{osc}_{B(0, 1)} u_k < 1 \quad \text{for all } k \in \mathbb{N}. \]

We prove (3.3) by induction, for $k = 0$ this holds already by assumption. Fix $k \in \mathbb{N}$. As induction hypothesis we assume the following holds
\[ \text{osc}_{B(0, 1)} u_{k-1} < 1. \]
In view of (3.2) we can apply Proposition 3.1, and find $\tilde{q}_k \in \mathbb{R}^n$ such that
\[ 2\lambda^{-1-\beta} \text{osc}_{B(\lambda)} (u_{k-1} - \langle \tilde{q}_k, x \rangle_{\mathbb{R}^n}) < 1. \]
That is
\[ 2\lambda^{-1-\beta} \text{osc}_{B(1)} (u_{k-1}(\lambda \cdot) - \langle \lambda \tilde{q}_k, x \rangle_{\mathbb{R}^n}) < 1. \]
By the definition of $u_{k-1}$,
\[ 2^k \lambda^{-k(1+\beta)} \text{osc}_{B(1)} (u(\lambda^k x) - \langle q_{k-1} - 2^{1-k} \lambda^{(k-1)(1+\beta)} \lambda^{1-k} \tilde{q}_k, \lambda^k x \rangle_{\mathbb{R}^n}) < 1. \]
so if we set
\[ q_k := q_{k-1} - 2^{1-k} \lambda^{(k-1)(1+\beta)} \lambda^{1-k} \tilde{q}_k, \]
we have obtained
\[ \text{osc}_{B(1)} (u_k) < 1. \]
That is, by induction, (3.3) holds for any $k \in \mathbb{N} \cup 0$. \qed

Corollary 3.3. Let $F$ be as above, and $\alpha$ as in Theorem 2.2. For any $\beta < \alpha$ let $u$ solve for some ball $B(R) \subset \Omega$
\[-\Lambda \leq F(D^2 u) \leq \Lambda \quad \text{in } B(R)\]
Then
\[ \sup_{r < R} r^{-1-\beta} \inf_{q \in \mathbb{R}^n} \text{osc}_{B(r)} (u - \langle q, x \rangle_{\mathbb{R}^n}) \leq C(\beta, \alpha, \Lambda, R, \text{osc}_{B(R)} u). \]
Proof. By otherwise considering $u_{\kappa,R} := \kappa^{-1} u(Rx)$ for

$$
\kappa := \frac{\Lambda}{\varepsilon} + R^2 + \text{osc}_{B(R)} u + 1,
$$

we can assume that $R = 1$, $\Lambda < \varepsilon$ and $\text{osc}_{B(1)} u < 1$. Here $\varepsilon$ is from Corollary 3.2.

Denoting for the ball $B(r)$

$$
\Phi(B(r)) := r^{-1-\beta} \inf_{q \in \mathbb{R}^n} (u_\infty - \langle q, x \rangle_{\mathbb{R}^n}),
$$

we get from Corollary 3.2 for any $r \in (\lambda^{k-1}, \lambda^k)$

$$
\Phi(B(r)) \leq \lambda^{-1-\beta} \Phi(B(\lambda^k)) \leq C(\lambda) 2^{-k} \Phi(B(1)) \leq C(\lambda) r^{-\log \frac{2}{\log \lambda}} \Phi(B(1)).
$$

This implies for $\sigma := \frac{-\log 2}{-\log \lambda} > 0$

$$
\sup_{r < R} r^{-\sigma} \Phi(B(r)) \leq C(\lambda) \Phi(B(1)).
$$

Dropping the $\sigma$, the claim is now proven. □

Proof of Theorem 2.2. Let $K \subset \Omega$ be a compact set. By a covering argument for any $\beta < \alpha$ we obtain from Corollary 3.3

$$
\sup_{x \in K, r < \text{dist}(x, \partial \Omega)} r^{-1-\beta} \inf_{q \in \mathbb{R}^n} (u - \langle q, x \rangle_{\mathbb{R}^n}) < \infty
$$

This readily implies that $u \in C^{1,\beta}(K)$ for any $\beta < \alpha$, see, e.g. [5]. See also [12, Theorem 4.4]. □

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