I. INTRODUCTION

The generic phase diagram of the unconventional superconductors has a dome-structure superconducting phase. It is proximate to an antiferromagnetic insulator or bad-metal state on one underdoped side and to a normal Fermi-liquid state on the other overdoped side. The mother normal states from which the superconductivity emerges can be such as in the cuprate superconductors the mysterious pseudogap state, the strange metallic state or the normal Fermi-liquid state. Although the mother normal states are diverse with some in non-Fermi liquid states, it is remarkable that the emergent superconductivity is normal, where most of the superfluid responses such as the penetration depth, the Andreev reflection, the Josephson effects, and the superfluid responses such as the penetration depth, are followed in element superconductors, phonon-mediated superconductors, etc. It should be noted that Anderson’s resonating valence bond (RVB) theory and its extended concept are partially confirmed by a universal scaling of the condensation energy in different-family superconductors [1],

\[ \frac{U}{\gamma} \frac{U}{\Delta C / T_c} \propto T_c^2, \tag{1} \]

where \( U \) is the condensation energy for superconductivity, \( \gamma \) is the specific heat constant and \( \Delta C \) is the specific heat jump at \( T_c \). This universal scaling behaviour is followed in element superconductors, phonon-mediated metal superconductors, Fe-based superconductors, heavy fermion superconductors, Sr\(_2\)RuO\(_4\), Li\(_{0.1}\)ZrNCl, organic superconductors and optimal-doped cuprate superconductors. This scaling behaviour shows that the superconductivity whatever the mother normal states are comes from the condensation of the Cooper pairs.

How to reconcile the diverse mother normal states and the normal superconductivity is one big issue in the condensed matter filed. One interesting idea for this issue comes from Shen and Sawatzky, who propose that the superconducting phase transition in cuprate superconductors is not only the opening of the superconducting gap but also the emergence of the well-defined coherent quasiparticles. This is one idea in large debate, especially for underdoped cuprate superconductors which are mostly regarded in doped Mott-insulator states. However, experiments seem to support this idea. The coherent weight of the physical electrons near antinodal points \( k = (\pi, 0)/n \pi \) accumulates largely when temperature decreases across \( T_c \). The well-defined Fermi arc in the pseudogap normal state shows the emergence of the nodal quasiparticles, the coherent weight of which is largely suppressed with doping dependence much smaller than \( 2p/(p+1) \). The discrimination of the superconducting gap and the pseudogap as two distinct energy scales strongly shows that the superconducting phase transition is relevant to the opening of the superconducting gap, not as coherence of the preformed Cooper pairs. Uchida’s group find that the superconducting transition \( T_c \) is smaller than both the coherent Fermi temperature \( T_F \) and the pseudogap temperature \( T^* \) and conclude that the superconductivity emerges out of the coherent fermionic quasiparticles.

In this article, we revisit this proposal that the emergence of the low-energy coherent parts of the physical electrons, which survive from the correlations of the electrons, is an essential prerequisite for superconductivity, and the superconductivity is driven by the condensation of the paired coherent parts of the physical electrons. We show that \( T_c \) is strongly influenced by the renormalized coherent weight of the physical electrons and propose that this is one dominant mechanism for the doping variation of \( T_c \) in underdoped unconventional superconductors. Our proposal is made for the unconventional superconductors, such as the cuprate superconductors, the Fe-based superconductors and the heavy fermion superconductors, etc. It should be noted that Anderson’s resonating valence bond (RVB) theory and its extended
gauge theories are one elegant formalism for cuprate superconductors. However, the novel spin-charge separation of the physical electrons in these theories is not confirmed by experiments, which instead favour the integrity of the physical electron as the component of the Cooper pair. There are no large debate on our proposal for the Fe-based superconductors where the interaction correlations are not strong. For the heavy fermion superconductors where antiferromagnetic quantum criticality may be relevant for superconductivity, the normal superconducting responses and the universal scaling behaviour in Eq. \( I \) show that our proposal is one favour mechanism for superconductivity in the heavy fermion superconductors.

This article is arranged as following. In Sec. I we show a novel property in unconventional superconductors that the normal superconductivity can emerge from the diverse mother normal states, some of which are non-Fermi liquid states. In Sec. II we revisit the proposal in detailed that the normal superconductivity in unconventional superconductors is driven by the condensed coherent parts of the physical electrons in pairs. In Subsection II A, we present a schematic formulation to redefine the mother normal states and the superconducting state. In Subsection II B we propose a single-particle spectrum function with the separated coherent and incoherent weights of the physical electrons in the mother normal states. In Subsection II C we study the Thouless’s instability for superconductivity and show how the coherent parts of the physical electrons drive the emergence of the superconductivity. The influence of the incoherent parts of the physical electrons on the superconductivity is also shown to enhance \( T_c \) although they are not a driving factor. In Sec. III we present a reduced model for the renormalized coherent parts of the physical electrons, where we predict their responses in experiments. Sec. IV shows some remarks on the dominant factors on \( T_c \). Some further theoretical problems are also discussed. In A some remarks are given on the theoretical formalisms, the weak-coupling BCS theory, the strong-coupling Eliashberg theory and the macroscopic Ginzburg-Landau functional theory.

II. SUPERCONDUCTIVITY FROM MOTHER NORMAL STATES

A. Schematic formulation for the mother normal states and the superconducting state

As there are no unified theory for the diverse mother normal states in unconventional superconductors, we introduce a schematic representation \( |\Psi_m\rangle \) for them,

\[
|\Psi_m\rangle = \mathbb{P}_I |\Psi_0\rangle .
\]  

(2)

Here \( |\Psi_0\rangle \) is the interaction-free ground state and \( \mathbb{P}_I \) is an operation to project the freely interacting ground state into a special mother normal state. \( \mathbb{P}_I \) is interaction relevant.

We should remark that the interactions in a model study are treated in different levels. The most relevant interactions in condensed matters are the electron-electron and electron-ion Coulomb interactions. While the static electron-ion Coulomb interaction has been included priorly in Bloch electronic band structure, the dynamical electron-phonon interactions should be involved exclusively. These interactions are the basic ones and in general have high energy scales. There are some other effective low-energy interactions such as super-exchange antiferromagnetic interactions which come from virtual processes. The relevant interactions in \( \mathbb{P}_I \) are the above two categories, which dominate the physics of the mother normal states. The pairing interactions for superconductivity is separated from the above two categories in our formulation, although in concept they come from the projection of the above two-category interactions in particle-particle channels.

The mother normal states thus defined can be normal Fermi-liquid state, antiferromagnetic state with long-range order, metallic state with strong antiferromagnetic fluctuations, non-Fermi liquid state such as the strangle metallic state in the cuprate superconductors, the heavily renormalized fermionic state in the heavy fermion superconductors, etc. To establish well-defined theories or formalisms for the diverse novel mother normal states is still one challenge.

The superconducting state from the mother normal states can be defined as

\[
|\Psi_{sc}\rangle = \mathbb{P}_{sc} |\Psi_m\rangle ,
\]  

(3)

where \( \mathbb{P}_{sc} \) is the projection of the pairing interactions upon the mother normal state \( |\Psi_m\rangle \) to produce the superconducting state \( |\Psi_{sc}\rangle \). The role of \( \mathbb{P}_{sc} \) is to bind the low-energy coherent parts of the physical electrons into pairs. In general case, the superconducting state \( |\Psi_{sc}\rangle \) thus defined involves both the pairing physics of the low-energy coherent parts of the physical electrons and the normal physics of their high-energy parts. Superconducting phase transition is a pairing physics, which is our assumption and starting point to study the superconductivity in the unconventional superconductors.

In many cases, the pairing interactions in \( \mathbb{P}_{sc} \) are much smaller than the interactions in \( \mathbb{P}_I \). For example, the pairing attractive interaction in normal metal superconductors comes from the effective electron-phonon interactions. It has energy scale of the Debye frequency and is much smaller than the Fermi energy which comes from direct Coulomb interactions. The pairing interaction in heavy fermion superconductors is still in debate. The superconducting temperature \( T_c \sim 1K \), which is much smaller than the antiferromagnetic temperature \( T_N \sim 100 - 1000K \). The separation of the interaction energy scales in \( \mathbb{P}_{sc} \) for superconductivity and in \( \mathbb{P}_I \) for mother normal states in these cases is the basic principle to preserve the reliability of our assumption in Eq. (3). It should be noted that in our assumption \( \mathbb{P}_{sc}\mathbb{P}_I \neq \mathbb{P}_I \mathbb{P}_{sc} \). Thus in our formulation the Anderson’s RVB state de-
defined by $|\Psi_{RVB}⟩ = P_F P_{sc} |Ψ_0⟩$ with $P_F$ the Hubbard-interaction driven Gutzwiller projection is not the physical relevant superconducting state in high-$T_c$ cuprate superconductors. Similarly, the scenario of the performed Cooper pairs in unconventional superconductors is also not relevant in our proposed formulation. Some remarks on the weak-coupling BCS theory and its strong-coupling version, the Eliashberg theory, are given in [X]

B. Single-particle spectrum function of the mother normal states

As we have pointed out in the above sections, the superconductivity in unconventional superconductors is normal, i.e., the superconducting phase transition is a pairing physics and the superconducting responses in both superfluid and quasiparticle relevant channels show BCS-like behaviours. Moreover, the resistivity above $T_c$ involves the scattering effects of the interactions, which should not diminish absolutely when superconductors enter into the superconducting state across $T_c$. It implies that the superconducting state involves both the low-energy dissipation-less superconducting condensate and the high-energy interaction scattering physics, most of the latter which are continuously transited from the mother normal states.

Based upon the above two facts, the normal superconductivity and the interaction scattering physics in superconducting state, we assume that in the mother normal states the single-particle spectrum function of the Green’s function $G_γ(k,τ) = -⟨T_\tau c_kγ(τ)c^\dagger_kγ(0)⟩$ has two parts,

$$A(k,ω) = A_{coh}(k,ω) + A_{inc}(k,ω),$$  \hspace{1cm} (4)

where $A_{coh}(k,ω)$ is the low-energy coherent part defined by

$$A_{coh}(k,ω) = Z_kδ(ω - ε^{\star}_k),$$  \hspace{1cm} (5)

and $A_{inc}(k,ω)$ is the high-energy incoherent part. Here we have assumed that the normal state has no long-range order and the spectrum is spin independent. $Z_k ≡ Z_k(ω)|_{ω=ε_k}$ is the coherent weight of the single physical electrons with renormalized energy $ε_k^{\star}$. $δ_γ(x) = 1/\pi x^{-1}$ is an extended $δ$-function with scattering rate $Γ_k$. Physically, the renormalized energy $ε_k^{\star}$, the scattering rate $Γ_k$ and the coherent weight $Z_k$ of a physical electron are determined by the real scattering processes where the energy conservation law is preserved as described by the Fermi’s Golden rule. The incoherent part $A_{inc}(k,ω)$ comes from the virtual scattering processes where the energy conservation law is broken. In the latter case, the intermediate states for $c_k\dagger$ during scattering may be a composite one such as defined by $\sum_q c_k^{\dagger}_{-q,\sigma} b_q^{\dagger}$ with $b_q^{\dagger} = \sum_{σσ'} c_{k+q,σ'}^{\dagger} c_{k,σ} + \cdots$.

The coherent parts of the physical electrons have dominant roles in the normal pairing superconductivity. The renormalized electronic structure $ε_k^{\star}$ has a well-defined Fermi surface, whose topology combined with the pairing interaction plays important role in the symmetry of the pairing gap potential [11]. The finite coherent weight $Z_k$ has a dominant role in determination of $T_c$ as shown below.

The incoherent part $A_{inc}(k,ω)$ describes the high-energy relevant physics in the single-particle channels and has negligible modification in superconducting phase transition from the mother normal states to the superconducting state. It should be noted that the physical electrons in the mother normal states are strongly entangled even in the normal Fermi-liquid state. The entanglements of the physical electrons which contribute to the incoherent part $A_{inc}(k,ω)$ is an exact N-body problem unsolved.

C. Thouless’s criterion for superconductivity

In this article, we propose that the normal superconductivity is driven by the pairing of the coherent parts of the physical electrons and the finite coherent weight $Z_k$ near Fermi energy is essential prerequisite for superconductivity in unconventional superconductors.

We will use the Thouless’s criterion to study the superconductivity instability of the mother normal states [12, 13]. Consider a simple case where the attractive pairing interaction is dominantly in channel with symmetry function $ϕ_k$.

$$H_p = -\frac{1}{N} \sum_{k_1k_2} gϕ_{k_1}ϕ_{k_2}c_{k_1\uparrow}^{\dagger}c_{-k_2\downarrow}c_{k_2\downarrow}c_{k_1\uparrow}. \hspace{1cm} (6)$$

Introduce the pairing operator

$$Δ = \frac{1}{\sqrt{N}} \sum_k ϕ(k)c_{-k\downarrow}c_k^{\uparrow},$$

we define the pairing susceptibility as

$$χ(τ) = ⟨T_\tau Δ(τ)Δ^\dagger(0)⟩. \hspace{1cm} (7)$$

In random-phase approximation (RPA), the pairing susceptibility is given by

$$χ(ν_n) = \frac{χ_0(ν_n)}{1 - gχ_0(ν_n)}, \hspace{1cm} (8)$$

where $χ_0(ν_n)$ is the bare zero-th order pairing susceptibility. The Thouless’s criterion for superconductivity is defined as [12, 13]

$$1 - gχ_0(0) = 0. \hspace{1cm} (9)$$

Suppose the spectrum function of the mother normal states $A(k,ω) = A_{coh}(k,ω) + A_{inc}(k,ω)$ has following behaviours,

$$A_{coh}(k,ω) = Z_kδ(ω - ε_k^{\star}), A_{inc}(k,ω) = P_kθ_{A}(ω), \hspace{1cm} (10)$$
where \( \theta_A(\omega) = \frac{1}{\Lambda} [\theta(\omega + \Lambda) - \theta(\omega - \Lambda)] \) with \( \theta(x) \) the step function and \( \Lambda \) the effective bandwidth, and \( P_k = 1 - Z_k \). It can be shown that

\[
\chi_0(0) = \chi_0^{(1)} + \chi_0^{(2)} + \chi_0^{(3)},
\]  
(11)

with

\[
\chi_0^{(1)} = \frac{1}{N} \sum_k (\phi_k Z_k)^2 \frac{f_\beta(\varepsilon_k^* - \varepsilon_k)}{\varepsilon_k^*},
\]

\[
\chi_0^{(2)} = \frac{2}{N} \sum_k \phi_k^2 Z_k P_k \int_{-\infty}^{+\infty} d\omega \theta_A(\omega) \frac{f_\beta(\varepsilon_k)}{\varepsilon_k^* + \omega},
\]

\[
\chi_0^{(3)} = \frac{1}{N} \sum_k (\phi_k P_k)^2 \int_{-\infty}^{+\infty} d\omega_1 d\omega_2 \theta_A(\omega_1) \theta_A(\omega_2) \frac{f_\beta(\omega_1) + f_\beta(\omega_2)}{\omega_1 + \omega_2}.
\]

Here \( f_\beta(\omega) \equiv \frac{\tanh(\beta\omega/2)}{2} \). Physically, \( \chi_0^{(1)} \) comes from the pure coherent parts of the physical electrons, \( \chi_0^{(2)} \) is the mixed contribution of the coherent and incoherent parts and \( \chi_0^{(3)} \) is the contribution from the pure incoherent parts. One additional interesting result is that the renormalized factors \( Z_k \) and \( P_k \) appear combined with the pairing symmetry function \( \phi_k \). It implies that the pairing symmetry function, which is dominated by the pairing interaction, can be renormalized further by the renormalized factors. This is an interesting result analogue to the strong-coupling Eliashberg theory, as remarked in Sec. [11]. It should be noted that in the above Thouless’s approximation for the superconducting instability, we only consider the most relevant ladder-diagram RPA contribution in the pairing susceptibility. Moreover, we have defined approximately the single-particle spectrum function following Eq. (10), which involves the universal feature of one finite coherent peak and one broad incoherent background as observed in ARPES. All other novel physics of the normal states are assumed irrelevant to the superconducting instability and are ignored in our study.

Consider a simple s-wave case with \( k \)-independent renormalized factor,

\[
\phi_k = 1, \ Z_k = Z,
\]
(12)

and in a weak-coupling limit where the pairing interaction \( g \) is mainly within an energy range \( \omega_g \ll \Lambda \). In this case, \( \chi_0^{(1)} \) shows singularity at low temperature \( T \to 0^+ \),

\[
\chi_0^{(1)} = Z^2 N_F \ln \left( \frac{\omega_g}{0^+} \right).
\]
(13)

However, in the same limit, \( \chi_0^{(2)} \) and \( \chi_0^{(3)} \) shows non singularity,

\[
\chi_0^{(2)} = 4(\ln 2)ZPN_F \frac{\omega_g}{\Lambda^2},
\]
(14)

\[
\chi_0^{(3)} = (\ln 2)P^2 \frac{\omega_g}{\Lambda^2}.
\]
(15)

Here \( N_F \) is the density of state (DOS) at Fermi energy of the mother normal state with renormalized dispersion \( \varepsilon_k^* \) and \( P = 1 - Z \). The logarithm singularity of \( \chi_0^{(1)} \) implies that at some finite \( T_c \), the Thouless’s condition of the superconductivity Eq. (9) can be satisfied. \( T_c \) is the critical temperature for superconducting phase transition. Since there are non singularity in \( \chi_0^{(2)} \) and \( \chi_0^{(3)} \), the superconductivity is dominantly driven by the coherent parts of the physical electrons. As \( \chi_0^{(2)} \) and \( \chi_0^{(3)} \) are shown to be positive, the Thouless’s criterion Eq. (9) can be modified into the form:

\[
(1)T - g\chi_0^{(1)} = 0,
\]
(16)

where (1)\( _T \) is a renormalized value by the incoherent parts of the physical electrons defined by \( (1)T \equiv 1 - g(\chi_0^{(2)} + \chi_0^{(3)}) \). The superconducting transition \( T_c \) follows

\[
T_c = \alpha \omega_g e^{-\frac{(1)T_c}{\beta \omega_g N_F}},
\]
(17)

where the constant \( \alpha \approx 1.13 \) for s-wave attractive interaction.

There are two dominant factors for \( T_c \), one comes from the pairing interaction, such as the interaction range \( \omega_g \), the interaction constant \( g \) and the pairing symmetry \( \phi_k \), and the other comes from the quasiparticle properties of the mother normal state, such as the renormalized DOS and the coherent weight \( Z_k \). Larger pairing interaction range and stronger interaction constant favour higher \( T_c \). More normal DOS and larger coherent weight lead to higher \( T_c \). One additional interesting factor comes from the incoherent parts of the physical electrons in (1)\( _T \). Although it is not a driving factor for superconductivity, the renormalization effects of the incoherent parts of the physical electrons can enhance \( T_c \).

In the above weak-coupling limit, the renormalized DOS can be approximated constant near Fermi energy. In the reverse strong-coupling limit with \( \omega_g \gg \Lambda \), if the DOS is weakly energy-dependent in such as quasi-two-dimensional system, the pairing susceptibilities \( \chi_0^{(1)} \) have similar singularity behaviours with just a substitution of \( \omega_g \) by \( \Lambda \). In this case, the coherent parts of the physical electrons can also lead to the dominant singularity of the superconducting pair instability. A general strong-coupling limit with strong frequency-dependent DOS should be studied extensively as there may be a crossover from wide-band BCS to narrow-band BEC.

The mother normal states of the underdoped unconventional superconductors have more abnormal physics beyond the Fermi-liquid state. The interaction correlations strongly suppress the coherent parts of the physical electrons and thus reduce the coherent weight \( Z_k \). The renormalized reduced \( Z_k \) largely suppresses the superconducting temperature \( T_c \). With doping increases, the electron correlations become weaker and \( Z_k \) increases. This leads to an enhanced \( T_c \) with increased doping. Therefore, the doping variation of \( T_c \) in underdoped unconv-
III. A REDUCED MODEL FOR THE
RENORMALIZED NORMAL
SUPERCONDUCTIVITY

In this section, we consider a simple case where the roles of $P_I$ can be approximated by a coherent projection. Consider a superconductor with Hamiltonian $H = H_I + H_p$, where the kinetic part $H_I$ is given by $H_I = \sum_{k\sigma} \varepsilon_k c_{k\sigma}^\dagger c_{k\sigma}$ and the pairing interaction $H_p$ is defined by Eq. (6). Here the interactions in $P_I$ of Eq. (2) are not explicitly given out. The renormalization effects of the projection $P_I$ on the Hamiltonian $H$ can be described by $\tilde{H} = P_I^\dagger H P_I$. Consider the following projection of the operator,

$$P_I^\dagger c_{k\sigma} P_I = Z_k^{1/2} c_{k\sigma} + \cdots , \quad (18)$$

the coherent part of the renormalized Hamiltonian $\tilde{H}$ can be simplified as

$$\tilde{H}_{coh} = \tilde{H}_I + \tilde{H}_p,$$  \quad (19)

where $\tilde{H}_I$ and $\tilde{H}_p$ are given by

$$\tilde{H}_I = \sum_{k\sigma} \varepsilon_k c_{k\sigma}^\dagger c_{k\sigma},$$  \quad (20)

$$\tilde{H}_p = -\frac{1}{N} \sum_{k_1k_2} \beta \tilde{\phi}_{k_1}^\dagger \tilde{\phi}_{k_2} c_{k_1\uparrow}^\dagger c_{-k_1\downarrow} c_{-k_2\downarrow} c_{k_2\uparrow}^\dagger.$$  \quad (21)

The renormalized variables are defined as following:

$$\varepsilon_k = Z_k P_I^\dagger \varepsilon_k P_I, \quad \tilde{g} = P_I^\dagger g P_I, \quad \tilde{\phi}_k = Z_k P_I^\dagger \phi_k P_I.$$  \quad (22)

It should be noted that there are two renormalization factors in the band dispersion $\varepsilon_k$, one comes from the projection coherent weight $Z_k$ and the other, denoted as $P_I^\dagger \varepsilon_k P_I$, stems from other interaction effects of $P_I$. These two factors are also relevant to the renormalization of DOS.

The above coherent projection approximation comes from the following assumption for the ground state:

$$\langle \Psi_{sc}| H |\Psi_{sc}\rangle \simeq \langle \Psi_0 | \tilde{H}_{coh} | \Psi_0 \rangle \simeq \langle \Psi_{sc}^{(0)} | \tilde{H}_{coh} | \Psi_{sc}^{(0)} \rangle.$$  \quad (22)

Here $| \Psi_{sc}^{(0)} \rangle$ is the weak-coupling BCS mean-filed ground state corresponding to the renormalized coherent Hamiltonian $\tilde{H}_{coh}$. In the above derivation, we use $P_I^\dagger H P_{sc} = H$ since only the pairing interaction $H_p$ involved in Hamiltonian $H$.

Now let us extend the above approximation at ground state into the finite-temperature case. For any operator $O$, the thermal average $\langle O \rangle$ is defined and approximated as following:

$$\langle O \rangle \equiv \frac{\sum_{r} e^{-\beta (H+H_I) O}}{\sum_{r} e^{-\beta (H+H_I)}} \quad = \frac{\sum_{r} e^{-\beta H O P_I}}{\sum_{r} e^{-\beta H P_I}} \quad \simeq \langle \tilde{O} \rangle_{coh},$$  \quad (23)

where $\tilde{O} = P_I^\dagger O P_I$ and $\langle \tilde{O} \rangle_{coh} \equiv \sum_{r} \tilde{\rho}_{coh} \tilde{O}$ with $\tilde{\rho}_{coh} = e^{-\beta \tilde{H}_{coh}}$. In the above approximation, the interactions in $P_I$ are only approximated by the projection operation $P_I$ and the high-energy incoherent physics is neglected. In our following discussion, the coherent approximation in Eq. (22) and Eq. (23) is assumed.

A. Pairing order parameter

Introduce a pairing order parameter $\Delta_c$ as

$$\Delta_c = -\frac{1}{N} \sum_{k} \beta \tilde{\phi}_k \langle c_{-k\downarrow} c_{k\uparrow} \rangle,$$  \quad (24)

we can decouple the renormalized pairing interaction $\tilde{H}_p$ into a mean-field approximation

$$\tilde{H}_p = \sum_{k} \Delta_k \left( c_{k\uparrow}^\dagger c_{-k\downarrow} + c_{-k\downarrow} c_{k\uparrow}^\dagger \right),$$  \quad (25)

where $\Delta_k = \Delta_c \tilde{\phi}_k$. We can then obtain a self-consistent equation for the pairing order parameter,

$$1 = \frac{1}{N} \sum_{k} \beta \tilde{\phi}_k^2 \tanh \left( \frac{\beta E_k}{2} \right),$$  \quad (26)

where $E_k = \sqrt{\varepsilon_k^2 + |\Delta_k|^2}$ is the energy of the coherent parts of the Bogoliubov quasiparticles.

Eq. (26) shows that the pairing order parameter $\Delta_c$ has a BCS mean-field solution. Moreover, the pairing symmetry function $\Delta_k$ is renormalized by the coherent weight $Z_k$. This latter fact is important since it shows that the pairing symmetry function $\Delta_k$ observed in ARPES is not only determined by the pairing interaction but also renormalized by the coherent weight. This result is analogue to that in the strong-coupling Eliashberg theory. If the higher-order renormalization effects of the interactions in $P_I$ can be ignored, it predicts from Eq. (26) that

$$\frac{\Delta_m}{T_c} = \text{const.}$$  \quad (27)

where $\Delta_m \equiv \Delta_c |\tilde{\phi}_k|_{max}$ and $T_c$ is given by Eq. (17) with $\langle 1 \rangle_{T_c} \rightarrow 1$. 

ventional superconductors is strongly influenced by the interaction renormalization coherent weight $Z_k$. 

In cuprate superconductors, ARPES shows that the pairing symmetry function has exact d-wave symmetry, \( \Delta \sim (\cos k_x - \cos k_y) \). It is consistent exactly to the theoretical prediction from the hypothesis that the nearest-neighbour antiferromagnetic interaction is the pairing interaction. This result shows that the coherent weight \( Z_k \) in cuprate superconductors is unusual \( k \) independent,

\[
Z_k = Z. \tag{28}
\]

The \( k \)-independent coherent weight \( Z \) implies that the dominant interactions in \( P_l \) are on-site ones, which seems rule out the antiferromagnetic fluctuations with characteristic momentum \( Q = (\pi, \pi) \) as the dominant interactions for the abnormal mother normal states in cuprate superconductors.

B. Experimental responses

Firstly, let us consider the superfluid responses in the penetration depth measurement. The diamagnetic current corresponding to the model Hamiltonian \( \tilde{H} \) is given by \( J_{\text{diam}}(\mathbf{q}) = -\Lambda_\alpha \mathbf{A}(\mathbf{q}) \), where \( \mathbf{A}(\mathbf{q}) \) is the vector potential for electromagnetic field, and \( \Lambda_\alpha = \sum_{k\sigma} T_\alpha(k) (c_{k\sigma}^\dagger c_{k\sigma}) \) with \( T_\alpha(k) \) a kinetic energy \( c_k \) relevant parameter. It can be shown that the penetration depth \( \lambda_\alpha \) follows

\[
\lambda_\alpha = \frac{1}{\sqrt{\mu_0 \Lambda_\alpha}}, \tag{29}
\]

where \( \Lambda_\alpha = \sum_{k\sigma} T_\alpha(k) Z_k (c_{k\sigma}^\dagger c_{k\sigma})_{\text{coh}} \) with \( T_\alpha(k) = P_l^l T_\alpha(k) P_l \), and \( \mu_0 \) is the vacuum permeability. For the case with \( Z_k \) nearly \( k \) independent, we have

\[
\lambda_\alpha = \frac{1}{\sqrt{\mu_0 \Lambda_{\alpha,\text{coh}}}}, \tag{30}
\]

where \( \lambda_{\alpha,\text{coh}} = \lambda_\alpha / \sqrt{Z} \) with \( \Lambda_{\alpha,\text{coh}} = \sum_{k\sigma} T_\alpha(k) (c_{k\sigma}^\dagger c_{k\sigma})_{\text{coh}} \).

In cuprate superconductors, since the nodal Fermi velocity is nearly doping independent \([17, 18]\), the renormalized low-energy dispersion and the renormalized coefficient of the diamagnetic current \( Z_k T_\alpha(k) \) should be both nearly doping independent. Therefore, in cuprate superconductors, it seems that the doping dependence of \( \lambda_\alpha^2 \) should only be dominated by the charge-carrier density and the renormalization influence from \( P_l \) could be nearly neglectable. Why the linear doping dependence of the coherent weight \( Z \) can coexist with the weakly linear doping dependence of the nodal Fermi velocity in underdoped cuprates is still one mysterious problem. A formal study on this issue is one future subject.

Secondly, let us consider the low-energy and low-temperature thermodynamic responses. The free-energy corresponding to the coherent Hamiltonian \( \tilde{H}_{\text{coh}} \) is given by \( \tilde{F}_{\text{coh}} = -\frac{1}{\beta} \ln T_r \left[ e^{-\beta \tilde{H}_{\text{coh}}} \right] \). In the superconducting state with finite pairing order parameter, the free-energy is approximated further as

\[
\tilde{F}_{\text{coh}}(0) = -\frac{1}{\beta} \ln T_r \left[ e^{-\beta \tilde{H}_{\text{mf}}} \right], \tag{31}
\]

where \( \tilde{H}_{\text{mf}} \) is the corresponding superconducting mean-field Hamiltonian of \( \tilde{H}_{\text{coh}} \). Since all of the thermodynamic variables such as the entropy \( S \) and the specific heat \( C \) etc. can be obtained by the derivatives of the free energy, the renormalization effects of the interactions in \( P_l \) on the thermodynamic responses are dominantly manifested by the energy spectrum \( E_k \) and there are no additional renormalized factor such as \( \sqrt{Z} \) in penetration depth of Eq. (30).

Thirdly, let us consider the single-particle responses in such as ARPES and STM experiments. For the single-particle Green’s function \( G_\alpha(\mathbf{k}, \tau) = -\langle T_\tau c_{\mathbf{k}\sigma}(\tau) c_{\mathbf{k}\sigma}^\dagger(0) \rangle \), in the coherent approximation defined by Eq. (22) and Eq. (23), we have

\[
G_\sigma(\mathbf{k}, \tau) = Z_k \tilde{G}_{\sigma,\text{coh}}(\mathbf{k}, \tau) \tag{32}
\]

where \( \tilde{G}_{\sigma,\text{coh}}(\mathbf{k}, \tau) = -\langle T_\tau c_{\mathbf{k}\sigma}(\tau) c_{\mathbf{k}\sigma}^\dagger(0) \rangle_{\text{coh}} \). Therefore, the spectrum function in ARPES \( A(\mathbf{k}, \omega) = -\frac{1}{\pi} \text{Im} G_\sigma(\mathbf{k}, \omega) \) is approximated at low-energy in superconducting state as

\[
A(\mathbf{k}, \omega) = Z_k \tilde{A}_{\text{coh}}(\mathbf{k}, \omega), \tag{33}
\]

where \( \tilde{A}_{\text{coh}}(\mathbf{k}, \omega) = u_k^2 \delta(\omega - E_k) + v_k^2 \delta(\omega + E_k) \) and \( u_k^2, v_k^2 = \frac{1}{2} \left( 1 \pm \frac{\omega}{E_k} \right) \).

The STM experimental signal is related to the local DOS \( \rho(\omega) = \frac{2}{\pi} \sum_k A(\mathbf{k}, \omega) \). In the coherent approximation, the low-energy local DOS is approximate as

\[
\rho(\omega) = Z \tilde{\rho}_{\text{coh}}(\omega), \tag{34}
\]

where \( \tilde{\rho}_{\text{coh}}(\omega) = \frac{2}{\pi} \sum_k \tilde{A}_{\text{coh}}(\mathbf{k}, \omega) \). In Eq. (34) we have considered a simple case that the coherence weight \( Z_k \) is \( k \) independent.

Lastly, let us focus on the low-energy and low-temperature magnetic responses in such as Knight shift and \( T_1/T \) from NMR and the inelastic spectra from neutron scatterings. These magnetic responses are related to the spin susceptibility defined by \( \chi_{\alpha\beta}(\mathbf{q}, \tau) = \langle T_\tau S_\alpha(q, \tau) S_\beta(-q, 0) \rangle \), where the spin operator \( S_\alpha(\mathbf{q}) = \frac{i}{\sqrt{Z}} \sum_{k\sigma\sigma'} c_{k\sigma}^\dagger (\frac{\mathbf{q}}{2})_{\sigma\sigma'} c_{\mathbf{k}+\mathbf{q}\sigma'} \). It is easily shown that in the coherent approximation and with \( k \)-independent quasiparticle coherent weight, the spin response susceptibility at low-energy and low-temperature is approximate as

\[
\chi_{\alpha\beta}(\mathbf{q}, \tau) = Z^2 \tilde{\chi}_{\alpha\beta,\text{coh}}(\mathbf{q}, \tau), \tag{35}
\]
where $\chi^{(s)}_{a3, coh}(\mathbf{q}, \tau) = \langle T_\tau S_\alpha (\mathbf{q}, \tau) S_\beta (-\mathbf{q}, 0) \rangle_{coh}$. It should be noted that only the contribution from the itinerant coherent parts of the physical electrons has been considered in Eq. (35) to the low-energy and low-temperature magnetic responses, which will be largely suppressed in superconducting state due to the opening of the superconducting gap.

**C. Remarks**

In the above discussion, the coherent parts of the physical electrons are simply described by a reduced coherent Hamiltonian $\tilde{H}_{coh}$ defined in Eq. (19), where the higher-order renormalization effects other than the projection $P_l$ from the incoherent parts of the physical electrons are neglected. For realistic superconductors, when the physical electrons are in strong dynamical entanglement, the roles of the interactions in $P_l$ may not be well described by the simple coherent projection approximation defined in Eq. (22) and Eq. (24). In this case, the entanglement physics requires a theory where both the normal superconductivity and its strongly correlated mother normal states should be involved, such as the strong-coupling Eliashberg theory for the superconductors with a Fermi-liquid mother normal state.

Before the end of this section, some remarks on the renormalization in the RVB formalisms may be useful. In the RVB formalisms, the strong Mott-Hubbard interaction correlations are assumed to the driven factors for both the unconventional superconductivity and the abnormal mother normal states in cuprate superconductors [10, 19, 20]. In a simplified treatment on the t-J model, two renormalized factors are introduced to treat the Mott-Hubbard Gutzwiller projection effects [21]. One is $g_\alpha = \frac{\varepsilon_p}{2(1+\rho)}$ for the kinetic energy and the other is $g_\sigma = \frac{1}{1+\rho}$ for the spin-spin Heisenberg energy ($\rho$ is the hole doping concentration). $g_\alpha$ introduces a new energy scale which renormalizes the superconducting pairing gap function with linear doping dependence in underdoped cuprates. In a more formal slave-boson theory for the RVB, the physical electron is assumed to be decoupled into charge-carrier holon and spin-carrier spinon [20]. Superconductivity can emerge when the spinons are paired and the holons are in coherent motion with Bose–Einstein condensation. Thus there are two different mechanisms for the superconducting phase transition in cuprates. While the superconducting phase transition in overdoped case is driven by the pairing formation of the spinons, it is dominated by the coherent condensation of the holons in the underdoped case. In the latter case, the pairing gap and the superconducting phase transition $T_c$ are two independent energy scales. This is basically different to our ideas in this article, here we assume that in all cases, the superconducting $T_c$ is one energy scale for the emergence of the pairing of the coherent parts of the physical electrons. It should be noted that the brilliant RVB proposal and the various relevant theoretical formalisms are still in doubt, as the predicted spinons and holons have not been found in experiments.

**IV. DISCUSSIONS AND REMARKS**

In this article, we propose that the finite coherent parts of the physical electrons in the mother normal states are prerequisite for superconductivity. By considering the Thouless’s criterion for superconductivity instability, we show that the finite coherent parts of the physical electrons drive the emergence of the superconductivity. The superconducting $T_c$ is determined by two factors, the attractive pairing interaction and the quasiparticle properties of the physical electrons. The former involves the interaction range, the interaction constant and the pairing symmetry function; the latter involves the coherent weight $Z_\mathbf{k}$, the DOS and the Fermi-surface topology of the electronic structure $\varepsilon_\mathbf{k}$. Larger interaction range and larger interaction constant favour higher $T_c$, and more renormalized DOS and larger $Z_\mathbf{k}$ lead to higher $T_c$. Moreover, although the incoherent parts of the physical electrons play no dominant role for superconductivity, they can also enhance $T_c$ by a renormalization effect. We show that the reduced coherent weight $Z_\mathbf{k}$ can be one dominant factor for the superconducting phase transition in underdoped unconventional superconductors.

It should be noted that the collective modes of the superconducting condensate may be another important factor for superconductivity. Most of the unconventional superconductors are quasi-two-dimensional in crossover from two- to three-dimensions. Hohenberg theory [22] states that there is no finite-$T_c$ superconductivity in one- and two-dimensional superconductors since the collective modes can easily destroy the long-range superconducting order. The roles of the collective modes of the superconducting condensate have been investigated by Emery and Kivelson as a proposal to account for the reduction of $T_c$ in underdoped cuprate superconductors [23]. Which one, the reduced coherent weight of the physical electrons or the thermal fluctuations of the collective modes, plays the dominant roles in the reduction of $T_c$ in unconventional superconductors is still an issue to be studied.

It is known that the well-established BCS formalisms, the weak-coupling BCS theory and the strong-coupling Eliashberg theory, can not predict reliable $T_c$ for a given superconductor. From our study we can understand why the prediction of $T_c$ is so unreliable. Most of the attractive pairing interactions for the Cooper pairs are residual ones generated from high-energy scattering processes and thus are hardly defined definitely for a realistic superconductor. The single-particle coherent weight $Z_\mathbf{k}$ and the renormalized band structure $\varepsilon^*_\mathbf{k}$ are strongly correlated to the high-energy physics in the mother normal states where a well-defined theory is absent. The feedback effects of the collective modes of the paired condensate involve dimensionality. All these factors in a realistic su-
perconductor are too complex to be definitely defined. In future, we have some further problems to be considered following our proposal, such as to study the experimental responses of the paired coherent parts of the physical electrons which drive the emergence of the superconductivity and to show how these coherent parts survive from the correlated diverse mother normal states.

It should also be noted that in this article we introduce the single-particle spectrum function $A(k, \omega)$ to describe the low-energy coherent physics and the high-energy incoherent entanglement of the physical electrons. One underlying well-established formalism is the strong-coupling Eliashberg theory where the extended self-energy and the full single-particle spectrum function of the Nambu’s spinor for the mixed particles and holes are self-consistently constructed. However in the Eliashberg theory the mother normal state is assumed the Fermi-liquid state. Moreover only the single-particle excitations are involved directly in the Eliashberg theory and the underlying ground state of the superconductors are not known as discussed in Appendix A: Remarks on weak-coupling BCS theory, strong-coupling Eliashberg theory and Ginzburg-Landau functional theory. The effects of the interactions in $P_I$ on the superconductivity have been implicitly included in parameters of the renormalized quasiparticles. The adiabatic principle and the Pauli exclusion principle preserve the reliability of the weak-coupling BCS theory in good metal superconductors.

The strong-coupling Eliashberg theory is an updated version of the weak-coupling BCS theory. In the strong-coupling Eliashberg theory, the time retarded formation of the Cooper pairs and the scattering renormalization of the single-particle excitations are additionally included upon the weak-coupling approximations. While the weak-coupling BCS theory is a mean-field theory with static pairing potential, the strong-coupling Eliashberg theory can be regarded as a dynamical mean-field theory with dynamical pairing potential. In the latter theory, although the mother normal state is also the Fermi-liquid state, the physics of $P_{sc}\cdot P_I$ are treated unifiedly with the dynamical retarded pairing physics and the scattering renormalization of the single-particle excitations included simultaneously. However, unlike in the weak-coupling BCS theory where the ground and excited states can be given out clearly, in the strong-coupling Eliashberg theory the ground and excited states can not be directly shown, while only the physics of the particle and hole excitations are involved in a perturbative self-consistent theory with the dynamical pairing potential in an extended self-energy.

Physically, both the weak-coupling BCS theory and the strong-coupling Eliashberg theory are in BCS formalisms where the pairing of the physical electrons is the essential concept for superconductivity. It should be noted that in the above two theoretical formalisms, only the inner-pair physics of the Cooper pairs is involved. The center-of-mass physics of the Cooper pairs is described by the macroscopic Ginzburg-Landau functional theory, where the detailed inner-pair physics as given in microscopic BCS formalisms is neglected. An updated formalism to unify both the microscopic pairing formation and the macroscopic pair condensate of the superconductivity can be established in a path integral method where both the single-particle and the pairing physics are treated unifiedly. The pairing fluctuations should be carefully included such as done in Reference [13].

V. ACKNOWLEDGEMENTS

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Appendix A: Remarks on weak-coupling BCS theory, strong-coupling Eliashberg theory and Ginzburg-Landau functional theory

There are some well-established theories for superconductivity, the weak-coupling BCS theory and the strong-coupling Eliashberg theory in BCS formalism from microscopic perspective, and the Ginzburg-Landau functional theory from the macroscopic perspective. In the weak-coupling BCS theory, the mother normal state is approximated as Fermi-liquid state. The $P_I$ is now defined as the perturbation evolution $S$-matrix, $P_I = S(0, -\infty)$. The Fermi-liquid ground state is adiabatically evolved from the interaction-free ground state and the low-energy excitations are the so-called quasiparticles, which can be regarded as freely independent excitations with interaction renormalization included. The superconductivity in weak-coupling BCS theory comes from the pairing of the nearly independent quasiparticles near Fermi energy. The effects of the interactions in $P_I$ on the superconductivity have been implicitly included in parameters of the renormalized quasiparticles. The adiabatic principle and the Pauli exclusion principle preserve the reliability of the weak-coupling BCS theory in good metal superconductors.

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