FREEDOM WITH AMALGAMATION, LIMIT THEOREMS AND S-TRANSFORM IN NON-COMMUTATIVE PROBABILITY SPACES OF TYPE B

MIHAI POPA

Abstract. The present material addresses several problems left open in the Trans. AMS paper ”Non-crossing cumulants of type B” of P. Biane, F. Goodman and A. Nica. The main result is that a type B non-commutative probability space can be studied in the framework of freeness with amalgamation. This view allows easy ways of constructing a version of the S-transform as well as proving analogue results to Central Limit Theorem and Poisson Limit Theorem.

1. Introduction

The present material addresses several problems left open in the paper ”Non-crossing cumulants of type B” of P. Biane, F. Goodman and A. Nica (reference [4]).

The type $A, B, C$ and $D$ root systems determine correspondent lattices of non-crossing partitions (see [8], [2]). The type $A_{n+1}$ corresponds to the lattice of non-crossing partitions on the ordered set $[n] = 1 < \cdots < n$; the types $B_n$ and $C_n$ determine the same lattice of non-crossing partitions on $[\overline{n}] = 1 < \cdots < n < -1 < \cdots < -n$, namely the partitions with the property that if $V$ is a block, then $-V$ (the set containing the opposites of the elements from $V$) is also a block; the type $D$ corresponds to a lattice of the symmetric non-crossing partitions with the property that if there exists a symmetric block, then it has more than 2 elements and contains $-n$ and $n$. (see again [8], [2], [3]).

The lattices of type $A$ and type $B$ non-crossing partitions are self-dual with respect to the Kreweras complementary. In the type $A$ case, the lattice structure was known to be connected the combinatorics of Free Probability Theory (see [7]). For the type $B$ case, the properties of the lattice allow also a construction, described in [4], of some associated non-commutative probability spaces, with a similar apparatus as in the type $A$ case (such as $F$-transform and boxed convolution). The paper [4] leaves open some questions on these objects: possible connections to other types of independence, limit theorems, $S$-transform. The main observation of the present material is that a type $B$ non-commutative probability space can be studied in the framework of freeness with amalgamation, that gives fast answers to the rest of the problems.

The material is organized as follows: second section reviews some results from [4]; third section presents the connection with freeness with amalgamation; forth section is briefing the construction of the $S$-transform for the type B non-commutative probability spaces, utilizing the commutativity of the matrix algebra $C$; fifth and,
respectively, sixth section are presenting limit results: analogues of central limit theorem, respectively Poisson limit theorem.

2. PRELIMINARY RESULTS

**Definition 2.1.** A non-commutative probability space of type B is a system 
\((\mathcal{A}, \varphi, \mathcal{X}, f, \Phi)\), where:

(i) \((\mathcal{A}, \varphi)\) is a non-commutative probability space (of type A), i.e. \(\mathcal{A}\) is a complex unital algebra and \(\varphi : \mathcal{A} \longrightarrow \mathbb{C}\) is a linear functional such that \(\varphi(1) = 1\).

(ii) \(\mathcal{X}\) is a complex vector space and \(f : \mathcal{X} \longrightarrow \mathbb{C}\) is a linear functional.

(iii) \(\Phi : \mathcal{A} \times \mathcal{X} \times \mathcal{A} \longrightarrow \mathcal{A}\) is a two-sided action of \(\mathcal{A}\) on \(\nu\) (when there is no confusion, it will be written \(\nu \cdot a \xi b\) instead of \(\Phi(\nu \cdot a \xi b)\), for \(a, b \in \mathcal{A}\) and \(\xi, \eta \in \nu\)).

On the vector space \(\mathcal{A} \times \mathcal{X}\) it was defined a structure of unital algebra considering the multiplication:

\[(a, \xi) \cdot (b, \eta) = (ab, a\eta + \xi b), \quad a, b \in \mathcal{A}, \xi, \eta \in \mathcal{X}\]

The above algebra structure can be obtained when \((a, \xi) \in \mathcal{A} \times \mathcal{X}\) is identifies with a \(2 \times 2\) matrix,

\[(a, \xi) \leftrightarrow \begin{bmatrix} a & \xi \\ 0 & a \end{bmatrix} \in M_2(\mathbb{C})\]

**Definition 2.2.** Let \((\mathcal{A}, \varphi, \mathcal{X}, f, \Phi)\) be a non-commutative probability space of type B. The non-crossing cumulant functionals of type B are the families of multilinear functionals \((\kappa_n : (\mathcal{A} \times \mathcal{X})^n \longrightarrow \mathcal{C})_{n=1}^{\infty}\) defined by the following equations: for every \(n \geq 1\) and every \(a_1, \ldots, a_n \in \mathcal{A}, \xi_1, \ldots, \xi_n \in \mathcal{X}\), we have that:

\[(1) \sum_{\gamma \in NC(\mathcal{A})^{(n)}} \prod_{B \in \gamma} \kappa_{\text{card}(B)} ((a_1, \xi_1) \cdots (a_n, \xi_n)|B) = E ((a_1, \xi_1) \cdots (a_n, \xi_n))\]

where the product on the left-hand side is considered with respect to the multiplication on \(\mathcal{C}\) and the product \((a_1, \xi_1) \cdots (a_n, \xi_n)\) on the right-hand side is considered with respect to the multiplication on \(\mathcal{A} \times \mathcal{X}\) defined above.

Note that the first component of \(\kappa_m ((a_1, \xi_1) \cdots (a_n, \xi_n))\) equals the non-crossing cumulant \(k_m(a_1, \ldots, a_m)\).

We will also use the notation \(\kappa_n(a, \xi)\) for \(\kappa_n ((a, \xi) \cdots (a, \xi))\) and \(M_n\) for \(E ((a, \xi)^n)\).

**Definition 2.3.** Let \(\mathcal{A}_1, \ldots, \mathcal{A}_k\) be unital subalgebras of \(\mathcal{A}\) and let \(\mathcal{X}_1, \ldots, \mathcal{X}_k\) be linear subspaces of \(\mathcal{X}\) such that each \(\mathcal{X}_j\) is invariant under the action of \(\mathcal{A}_j\). We say that \((\mathcal{A}_1, \mathcal{X}_1), \ldots, (\mathcal{A}_k, \mathcal{X}_k)\) are free independent if

\[\kappa_n ((a_1, \xi_1), \ldots, (a_n, \xi_n)) = 0\]

whenever \(a_l \in \mathcal{A}_{i_l}, \xi_l \in \mathcal{X}_{i_l}\) \((l = 1, \ldots, n)\) are such that there exist \(1 \leq s < t \leq n\) with \(i_s \neq i_t\).
For \((a, \xi) \in \mathcal{A} \times \mathcal{X}\) we consider the moment and cumulant or \(R\)-transform, series:

\[
M(a, \xi) = \sum_{n=1}^{\infty} (E((a, \xi)^n)) z^n
\]

\[
R(a, \xi) = \sum_{n=1}^{\infty} \kappa_n(a, \xi) z^n
\]

**Definition 2.4.** Let \(\Theta^{(B)}\) be the set of power series of the form:

\[
f(z) = \sum_{n=1}^{\infty} (\alpha'_n, \alpha''_n) z^n,
\]

where \(\alpha'_n, \alpha''_n\) are complex numbers. For \(p \in NC^{(A)}(n)\) and \(f \in \Theta^{(B)}\), consider

\[
Cf_p(f) = \prod_{B \in p} (\alpha'_{|B|}, \alpha''_{|B|})
\]

(the right-hand side product is in \(\mathbb{C}\).) On \(\Theta^{(B)}\) we define the binary operation \(\ast\) by:

\[
f \ast g = \sum_{n=1}^{\infty} (\gamma'_n, \gamma''_n) z^n\text{ where }
\]

\[
(\gamma'_n, \gamma''_n) = \sum_{p \in NC^{(A)}(n)} Cf_p(f)Cf_{Kr(p)}(g)
\]

**Theorem 2.5.** The moment series \(M\) and \(R\)-transform \(R\) of \((a, \xi)\) are related by the formula

\[
M = R \ast \zeta'
\]

where \(\zeta' \in \Theta^{(B)}\) is the series \(\sum_{n=1}^{\infty} (1, 0) z^n\).

**Remark 2.6.** We denote by \(k'_{n,p}\) or, for simplicity, by \(k'_n\), the multilinear functional from \(\mathcal{A}^{p-1} \times \mathcal{X} \times \mathcal{A}^{n-p}\) to \(\mathbb{C}\) which is defined by the same formula as for the (type A) free cumulants \(k^n : \mathcal{A}^n \rightarrow \mathbb{C}\), but where the \(p\)th argument is a vector from \(\mathcal{X}\) and \(\varphi\) is replaced by \(f\) in all the appropriate places. The connexion between the type B cumulants \(\kappa_n\) and the functionals \(k'_n, k''_n\) is given by:

\[
(2)\ \kappa_n((a_1, \xi_1), \ldots, (a_n, \xi_n)) = \left(k_n(a_1, \ldots, a_n), \sum_{p=1}^{n} k'_{n}(a_1, \ldots, a_{p-1}, \xi_p, a_{p+1}, \ldots, a_n)\right)
\]

**Theorem 2.7.** If \((\mathcal{A}_1, \mathcal{X}_1), (\mathcal{A}_2, \mathcal{X}_2)\) are free independent, \((a_1, \xi_1) \in (\mathcal{A}_1, \mathcal{X}_1), (a_2, \xi_2) \in (\mathcal{A}_2, \mathcal{X}_2)\), and \(R_1, \text{ respectively } R_2\) denote the \(R\)-transforms of \((a_1, \xi_1)\) and \((a_2, \xi_2)\), then:

(i) the \(R\)-transform of \((a_1, \xi_1) + (a_2, \xi_2)\) is \(R_1 + R_2\).

(ii) the \(R\)-transform of \((a_1, \xi_1) \cdot (a_2, \xi_2)\) is \(R_1 \ast R_2\).

3. connexion to "freeness with amalgamation"

As shown in [4], Section 6.3, Remark 3, the definitions of the type B cumulants are close to those from the framework of the "operator-valued cumulats", yet some details are different - mainly the map \(E\) is not a conditional expectation and \(\mathcal{A} \times \mathcal{X}\) is not a bimodule over \(\mathbb{C}\). Following a suggestion of Dimitri Shlyakhtenko,
the construction of the type B probability spaces can still be modified in order to overcome these points.

Let $\mathcal{C} = \mathcal{X} \oplus \mathcal{A}$. On $\mathcal{A} \times \mathcal{C}$ we have a $\mathcal{C}$-bimodule structure given by:

$$(x, t)(a, \xi + b) = (a, \xi + b)(x, t) = (ax, at + (\xi + b)x)$$

for any $x, t \in \mathbb{C}, a, b \in \mathcal{A}, \xi \in \mathcal{X}$. Since $\mathcal{A}$ is unital, $\mathcal{C}$ is a subspace of $\mathcal{C}$.

The map $E$ extends to $\mathcal{C}$ via:

$$\tilde{E}(a, \xi + b) = (\varphi(a), f(\xi) + \varphi(b))$$

The extension becomes a conditional expectation, since:

$$\tilde{E}((x, t)(a, \xi + b)) = \tilde{E}(ax, at + (\xi + b)x)$$

$$= (\varphi(ax), \varphi(ta) + f(\xi x) + \varphi(bx))$$

$$= (x\varphi(a), t\varphi(a) + xf(\xi) + x\varphi(b))$$

$$= (x, t)(\varphi(a), f(\xi) + \varphi(b))$$

$$= (x, t)\tilde{E}(a, \xi + b)$$

The equation (1) can naturally be extended in the framework of $\mathcal{C}$ and $\tilde{E}$, framework that reduces the construction to freeness with amalgamation, namely defining the cumulants $\bar{\kappa}$ by the equation:

$$\sum_{\gamma \in NC(A)(n)} \prod_{B \in \gamma} \bar{\kappa}_{\text{card}(B)}((a_1, \xi_1) \cdots (a_n, \xi_n)|B) = \tilde{E}((a_1, \xi_1) \cdots (a_n, \xi_n))$$

If $m : \mathcal{A} \times \mathcal{A} \ni (a, b) \mapsto m(a, b) = ab \in \mathcal{A}$ is the multiplication in $\mathcal{A}$, note that $(\mathcal{A}, \varphi, \mathcal{X} \oplus \mathcal{A}, f \oplus \varphi, \Phi \oplus m)$ is also a type B noncommutative probability space, therefore Remark 2.6 (i.e. Theorem 6.4 from [4]) gives the components of $\bar{\kappa}$:

$$\bar{\k}\kappa_n((a_1, \xi_1 + b_1), \ldots, (a_n, \xi_n + b_n)) =$$

$$\left(k_n(a_1, \ldots, a_n), \sum_{p=1}^{n} k'_n(a_1, \ldots, a_{p-1}, \xi_p + b_p, a_{p+1}, \ldots, a_n)\right)$$

4. THE $S$-TRANSFORM

Utilizing the commutativity of the algebra $\mathcal{C}$, the construction of the $S$-transform is essentially a verbatim reproduction of the type A situation.

We will denote

$$\mathcal{G} = \{\sum_{n=1}^{\infty} \alpha_n z^n, \alpha_n \in \mathbb{C}\}$$

the set of formal series without constant term with coefficients in $\mathbb{C}$, and

$$\mathcal{G}'^{-1} = \{\sum_{n=1}^{\infty} \alpha_n z^n, \alpha_n \in \mathbb{C}, \alpha_1 = \text{invertible}\}$$

the set of all invertible series (with respect to substitutional composition) with coefficients in $\mathbb{C}$ (see [1]).
**Definition 4.1.** Let \((a, \xi) \in A \times X\) such that \(\varphi(a) \neq 0\), that is \((\varphi(a), f(\xi))\) is invertible in \(C\). If \(R(\varphi(\xi))\) is the \(R\)-transform series of \((a, \xi)\), then the \(S\)-transform of \((a, \xi)\) is the series defined by

\[
S(a, \xi)(z) = \frac{1}{z} R'(a, \xi)(z)
\]

**Theorem 4.2.** If \(A_1, X_1, (A_2, X_2) \subset (A, X)\) are free independent and \((x_j, \xi_j) \in (A_j, X_j), j = 1, 2\) are such that \(\varphi(x_j) \neq 0\), then:

\[
S(a_1, \xi_1)(x_2, \xi_2)(z) = S(a_1, \xi_1)(z)S(a_2, \xi_2)(z)
\]

**Proof.** The proof presented in [7], for the type \(A\) case, works also for the freeness with amalgamation over a commutative algebra. Yet, for the convenience of the reader, we will outline the main steps.

Since, for \((a_1, \xi_1), (a_2, \xi_2)\) free, \(R(a_1, \xi_1) \ast R(a_2, \xi_2) = R(a_1, \xi_1) \ast R(a_2, \xi_2)\), it suffices to prove that the mapping

\[
F : G^{-1} \ni f \mapsto \frac{1}{z} f(z) \in G
\]

has the property

\[
(4) \quad F(f \ast g) = F(f)F(g).
\]

Indeed, (4) is equivalent to

\[
(5) \quad z(f \ast g) = f^{(-1)}(f \ast g) \ast g^{(-1)}(g \ast f)
\]

For \(\sigma \in NC(n)\) and \(h = \sum_{n \geq 1} h_n z^n\), we define

\[
Cf_\sigma(h) = \prod_{B \in \sigma} h_{\text{card}(B)} \in C.
\]

Also, for \(f, g \in G\), we denote

\[
(f \ast g)(z) = \sum_{n \geq 1} \lambda_n z^n
\]

where \((K(\sigma)\) is the Kreweras complementary of \(\sigma)\)

\[
\lambda_n = \sum_{\sigma \in NC(n)} Cf_\sigma(f) \cdot Cf_{K(\sigma)}(g)
\]

For \(f = \sum_{n \geq 1} \alpha_n z^n \in G^{-1}\) we have that:

\[
f^{(-1)}(f \ast g) = \alpha_1^{-1}(f \ast g)
\]

since, with the above notations, the coefficient of \(z^m\) in the right hand side is

\[
\sum_{n \geq 1} \alpha_n \alpha_1^{-n} \lambda_{i_1} \cdots \lambda_{i_n}
\]

while the coefficient of \(z^m\) in the left-hand side is

\[
\sum_{n \geq 1} \sum_{1=b_1 \leq \cdots \leq b_n \leq m} \sum_{\pi \in NC(m)} Cf_\pi(f) \cdot Cf_{K(\sigma)}(g)
\]

where \(\pi = (b_1, \ldots, b_n) \in \pi\)
and the equality follows setting $\pi_k = \pi|\{b_k, \ldots, b_{k+1} - 1\}$ (notationally $b_{n+1} = m$) and remarking that $K(\pi)$ is the juxtaposition of $K(\pi_1), \ldots, K(\pi_n)$.

It follows that, if $\{\alpha_n\}_{n \geq 1}, \{\beta_n\}_{n \geq 1}$ are respectively the coefficients of $f$ and $g$, (3) is equivalent to

$$(f\hat{\star}g)(z) = \alpha_1 \beta_1 \cdot (f\hat{\star}g)(z)$$

The coefficient of $z^{m+1}$ on the left-hand side is

$$\sum_{n=1}^{m} \sum_{\pi \in NC(n)} \sum_{\rho \in NC(m+n-1)} CF_{\pi}(f) \cdot CF_{K(\pi)}(g) \cdot CF_{\rho}(g) \cdot CF_{K(\rho)}(f)$$

while the coefficient of $z^{m+1}$ on the right-hand side is

$$\sum_{\sigma \in NC(m)} \alpha_1 \beta_1 \cdot CF_{\sigma}(f) \cdot CF_{K(\sigma)}(g).$$

As shown in [7], the conclusion follows from the bijection between the index sets of the above sums. More precisely, if $1 \leq n \leq m$, to the pair consisting on $\pi \in NC(n)$ and $\rho \in NC(m+1-n)$ both contain the block $(1)$, we associate the partition from $NC(n+m-1)$ obtained by juxtaposing $\pi \setminus (1)$ and $K(\rho)$. □

5. CENTRAL LIMIT THEOREM

**Theorem 5.1.** Let $\{(A_k, X_k)\}_{k \geq 1} \subset (A, \mathcal{X})$ be type $B$ free independent and $(x_k, \xi_k) \in (A_k, X_k)$ identically distributed such that $\varphi(x_k) = f(\xi_k) = 0$ and $\varphi(x_k^2) = f(\xi_k^2) = 1$. The limit distribution moments of

$$\frac{(a_1, \xi_1) + \cdots + (a_N, \xi_N)}{\sqrt{N}}$$

are $\{m_n, m_n\}_{n}$, where $\{m_n\}_{n}$ are the moments of the semicircular distribution and

$$m_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \binom{2k}{k+1} & \text{if } n = 2k \text{ is even.} \end{cases}$$

**Proof.** Note $S_N = \frac{(a_1, \xi_1) + \cdots + (a_N, \xi_N)}{\sqrt{N}}$ and $R_N = R(S_N)$. Theorem 2.7 implies

$$\lim_{N \to \infty} R_N = (1, 1)z^2$$

The first component of the limit distribution is the Voiculescu’s semicircular distribution. To compute the second component of the moments, we will use the equation (1), which becomes:

$$E((a_1, \xi_1)^n) = \sum_{\gamma \in NC_{A}(n)} \kappa_2 ((a_1, \xi_1))^\gamma$$
It follows that all the odd moments are zero, and, since in $C$, $(a,b)^n = (a^n, na^{n-1}b)$, the even moments are given by:

$$m_{2n} = nC_n, \text{ where } C_n \text{ stands for the } n\text{-th Catalan number}$$

$$= \frac{n}{n+1} \binom{2n}{n}$$

$$= \binom{2n}{n+1}.$$

\[\square\]

**Remark 5.2.** The second components of the above limit moments are not the moments of positive Borel measure on $\mathbb{R}$. Yet, they are connected to the moments of another remarkable distribution appearing in non-commutative probability - the central limit distribution for monotonic independence.

For variables that are monotonically independent (see [5], [6]), the limit moments in the Central Limit Theorem are given by the "arsine law", i.e. the $n$-th moment $\mu_n$ is given by

$$\mu_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (2k) = (k+1)C_k & \text{if } n = 2k \text{ is even}. \end{cases}$$

Hence $\mu_n = m_n + m_n$, which implies the following:

**Corollary 5.3.** On $\mathcal{A} \oplus \mathcal{X}$ consider the algebra structure given by:

$$(a + \xi)(b + \eta) = ab + \xi b + a\eta$$

and $\Psi : \mathcal{A} \oplus \mathcal{X} \ni a + \xi \mapsto \varphi(a) + f(\xi) \in \mathbb{C}$.

Let $(a_j, \xi_j)_j = 1^\infty$ be a family from $\mathcal{A} \oplus \mathcal{X}$ such that $\varphi(a_j) = f(\xi_j) = 0$ and $(a_j, \xi_j)$ are type B free in $(\mathcal{A}, \varphi, \mathcal{X}, f, \Phi)$.

Then the limit in distribution of

$$\frac{a_1 + \xi_1 + \ldots + a_N + \xi_N}{\sqrt{N}}$$

is the "arsine law".

6. **Poisson limit theorem**

We will consider an analogue of the classical Bernoulli distribution in a type B probability space.

Let $\Lambda = (\alpha_1, \alpha_2) \in \mathbb{R}^2 \subset \mathbb{C}$. We call an element $(a, \xi) \in \mathcal{A} \times \mathcal{X}$ type B Bernoulli with rate $\Lambda$ and jump size $A$ if

$$E((a, \xi)^n) = \Lambda A^n$$

for some $\Lambda = (\lambda_1, \lambda_2) \in \mathbb{C}$

**Theorem 6.1.** Let $\Lambda \in \mathcal{C}$ and $A \in \mathbb{R}^2$. Then the limit distribution for $N \to \infty$ of the sum of $N$ free independent type B Bernoulli variables with rate $\frac{\Lambda}{N}$ and jump size $A$ has cumulants which are given by $\kappa_n = \Lambda A^n$. 

Proof. We will introduce first several new notations in order to simplify the writing. \( \beta_N \) will stand for a type B Bernoulli variable with rate \( \Lambda \), and \( s_N \) for a sum of \( N \) such free independent variables. \( \mu \) will denote the Möbius function of the lattice \( NC(n) \) and, for \( \pi \in NC(n) \) and \( \beta \in A \times X \), we will use the notation
\[
M_{\pi}(\beta) = \prod_{B=\text{block of } \pi} M_{\text{card}(B)}(\beta)
\]
where \( M_n(\beta) = E(\beta^n) \) is the \( n \)-th moment of \( \beta \).

With the above notations, equation (1) gives
\[
\kappa_n(\beta_N) = \sum_{\pi \in NC(n)} M_{\pi}(\beta_N) \mu(\pi, 1_n) = \frac{\Lambda}{N} A^n + \sum_{\pi \in NC(n)} M_{\pi}(\beta_N) \mu(\pi, 1_n) \]
\[
= \frac{\Lambda}{N} A^n + O\left(\frac{1}{N^2}\right)
\]
Therefore
\[
\lim_{N \to \infty} \kappa_n(s_N) = \lim_{N \to \infty} N \kappa_n(\beta_N) = \Lambda A^n.
\]

Like in the type A case, we have the following:

**Consequence 6.2.** The square of a type B random variable \((a, \xi)\) with distribution given by the central limit theorem such that \( E((a, \xi)^2) = \sigma \in C \) is a type B free Poisson element of rate \( \sigma \) and jump size \((1, 0)\).

**Remark 6.3.** The first component of the moments of a type B free Poisson variable coincides to the type A case, therefore are given by a probability measure on \( \mathbb{R} \). In general, the second component of the moments of a type B free Poisson random variable are not the moments of a real measure.

The first part of the assertion is clear. For the second part, we will consider the particular case when \( \lambda_2 = 0 \) and \( \lambda_1 = \lambda \) is close to 0 and \( \alpha_1 = \alpha_2 = \alpha \). It follows that
\[
\kappa_n = \Lambda A^n = ((\lambda, 0)(\alpha^n, n\alpha^n)) .
\]

Since equation (11) implies
\[
M_2 = \kappa_2 + \kappa_1^2 = (\lambda + \lambda^2)A
\]
\[
M_3 = \kappa_3 + 3\kappa_1\kappa_2 + \kappa_1^3
\]
\[
= (\lambda + 3\lambda^2 + \lambda^3)A^3
\]
\[
M_4 = \kappa_4 + 4\kappa_1\kappa_3 + 2\kappa_2^2 + 6\kappa_2\kappa_1^2 + \kappa_1^4
\]
\[
= (\lambda + 6\lambda^2 + 6\lambda^3 + \lambda^4)A^4
\]
we have that the second components are given by:
\[
m_2 = 2(\lambda + \lambda^2)\alpha^2
\]
\[
m_3 = 3(\lambda + 3\lambda^2 + \lambda^3)\alpha^3
\]
\[
m_4 = 4(\lambda + 6\lambda^2 + 6\lambda^3 + \lambda^4)\alpha^4
\]
A necessary condition for \( \{m_k\}_{k \geq 1} \) to be the moments of a measure on \( \mathbb{R} \) (see \[9\], \[7\]) is that 
\[
m_2 m_4 \geq m_3^2
\]
It amounts to 
\[
8(\lambda + \lambda^2)(\lambda + 6\lambda^2 + 6\lambda^3 + \lambda^4)\alpha^6 \geq 9(\lambda + 3\lambda^2 + \lambda^3)^2 \alpha^6
\]
that is 
\[
8(1 + \lambda)(1 + 6\lambda + 6\lambda^2 + \lambda^3) \geq 9(1 + 3\lambda + \lambda^2)^2
\]
\[
8 + O(\lambda) \geq 9 + O(\lambda)
\]
which, for \( \lambda \) small enough, does not hold true.

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