Precontinuity and applications

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Introduction

In this note, a map \( f \) acting between metric (or topological) spaces is referred to be pre-continuous at a point \( x \) if, for some sequence \( (x_n) \) of points \( x_n \) different from \( x \) and converging to \( x \), the sequence \( (f(x_n)) \) converges to \( f(x) \) (section 2, Definition 1). We observe that this rather weak property enjoys every function with a dense graph, and a function is not pre-continuous at a point if, and only if, the respective point of its graph is isolated. In particular every additive, exponential, logarithmic, and multiplicative function is pre-continuous at every point. As a matter of fact, these functions have a stronger property, namely, they are uniformly pre-continuous (section 3, Definition 2, and Definition 3).

Another definition of continuity, using the notion of a preopen set, was introduced by Mashhour, Abd El-Monsof, and El-Deep [1] (Remark 1).

In section 4 we show that pre-continuity can be useful in solving some functional equations. Applying the property of uniform (and one-sided) pre-continuity, we determine the translative beta type functions considered in [2], the homogeneous multiplicative Cauchy quotients, and a topic leading to the Pexider equation.

Recently the family of beta-type means considered in [3,4] was applied in [5] (Remark 6).

Pre-continuous functions

We introduce the following

Definition 1

Let \( (X,d_X),(Y,d_Y) \) be metric spaces. A function \( f : X \rightarrow Y \) is called pre-continuous at the point \( x \in X \), if there exists a sequence \( (x_n) \), \( x_n \in X \setminus \{x\} \) for \( n \in \mathbb{N} \) such that \( \lim_{n \to \infty} x_n = x \) and \( \lim_{n \to \infty} f(x_n) = f(x) \). The function \( f \) is called pre-continuous if it is pre-continuous at every point of \( X \).

It is easy to construct examples of functions that are not pre-continuous. For instance, every function \( f : \mathbb{R} \rightarrow \mathbb{R} \) that is increasing at \( 0 \), i.e. such that \( \limsup_{x \to 0^-} f(x) < f(0) < \liminf_{x \to 0^+} f(x) \), is not pre-continuous at \( 0 \). On the other hand, even extremely discontinuous functions are pre-continuous. Namely, we have the following

Theorem 1

Let \( (X,d_X),(Y,d_Y) \) be metric spaces and let \( f : X \rightarrow Y \) be an arbitrary function.

(i) If the graph \( f \) is dense in the product metric space \( X \times Y \), then \( f \) is pre-continuous.
(ii) The function $f$ is pre-continuous at a point $x \in X$ if and only if $(x, f(x))$ is not an isolated point of the graph $f$.

(iii) If $f$ is continuous at an accumulation point $x$, then $f$ is pre-continuous at the point $x$.

Proof. (i) Take an arbitrary point $x \in X$. The density of the graph of $f$ there exists a sequence $(x_n, f(x_n)) \in X \times Y$ such that $\lim_{n \to \infty} (x_n, f(x_n)) = (x, f(x))$ in the product metric, that is such that $\lim_{n \to \infty} d_X(x_n, x) = 0$ and $\lim_{n \to \infty} d_Y(f(x_n), f(x)) = 0$, which completes the proof.

(ii) If $f$ is pre-continuous at a point $x \in X$ then, by Definition 1, there is a sequence $(x_n)$, $x_n, x \in X \setminus \{x\}$ for $n \in \mathbb{N}$ such that $\lim_{n \to \infty} x_n = x$ and $\lim_{n \to \infty} f(x_n) = f(x)$. Without any loss of generality, we can assume that $(x_n)$ is one-to-one. Then $(x_n, f(x_n)) \in X \times Y$, $n \in \mathbb{N}$, is a sequence of different points of the graph $f$ converging $(x, f(x))$ in the product topology. The converse implication follows from the definition of the product topology.

We omit an easy argument for (iii).

It is well-known that the graph of every discontinuous additive function $\alpha : \mathbb{R} \to \mathbb{R}$ is dense in $\mathbb{R}^2$ (as well as the graphs of discontinuous multiplicative, exponential, and logarithmic functions are dense in the suitable natural subsets $\mathbb{R}^2$).

**Theorem 2**

Let $I \subset \mathbb{R}$ be an interval. For an arbitrary function, $f : I \to \mathbb{R}$ the set of all points $x \in I$ that $f$ is not pre-continuous $x$ is at most countable.

Proof. Let $Z \subset X$ be the set of all $x \in I$ such that $f$ is not pre-continuous at $x$. By Theorem 1(ii), the set $\{(x, f(x)) : x \in Z\}$ is the set of all isolated points of the graph $f$, that are contained $I \times \mathbb{R}$. But, clearly, the set of isolated points of any subset $I \times \mathbb{R}$ is at most countable.

**Remark 1**

Let $(X, T)$, $(Y, S)$ be topological spaces. In [1] a function $f : X \to Y$ is said to be pre-continuous at a point $x \in X$, if for every open set $V \in S$ containing $f(x)$ there is a set $U \subset X$ such that $x \in U$, $U \subset \text{Int}(\text{Cl}(U))$ (preopeness) and $f(U) \subset V$ (see also (6)).

For obvious reasons, the notion of precontinuity proposed in Definition 1 could be called a Heine-type. We omit to discuss the mutual relations between these two concepts.

**Uniform pre-continuous functions**

**Definition 2**

Let $X$ be a subset of a metric group $G$ with an addition “$+$” and neutral element $0$ and $Y$ be a metric space. A function $f : X \to Y$ is called uniformly pre-continuous, if there exists a sequence $z_n \in G \setminus \{0\}$ for all $n \in \mathbb{N}$, with $\lim_{n \to \infty} z_n = 0$ such that $x + z_n \in X$, for all $x \in X$, $n \in \mathbb{N}$, and $\lim_{n \to \infty} f(x + z_n) = f(x)$.

**Remark 2**

Let $X$ be a metric group with an addition “$+$” and neutral element $0$, $Y$ be a metric space, and $f : X \to Y$ be an additive function, i.e.

$$f(x + y) = f(x) + f(y), \quad x, y \in X.$$  

The following two conditions are equivalent

(i) $f$ is pre-continuous at a point;

(ii) $f$ is uniformly pre-continuous.

Proof. To prove (i) $\Rightarrow$ (ii) assume that for some $x_0 \in X$ there is $x_n \in X \setminus \{x_0\}$ for all $n \in \mathbb{N}$ such that $\lim_{n \to \infty} x_n = x_0$

$\lim_{n \to \infty} f(x_n) = f(x_0)$. Putting $z_n := x_n - x_0$ we have $z_n \in X \setminus \{0\}$ for every $n \in \mathbb{N}$ and $\lim_{n \to \infty} z_n = 0$. Hence, for arbitrary $x \in X$, making use of the additivity of $f$ and its oddness, we

$$\lim_{n \to \infty} f(x + z_n) = \lim_{n \to \infty} \left[ f(x) + f(z_n) \right] = f(x) + \lim_{n \to \infty} f(z_n) = f(x) + \lim_{n \to \infty} f(z_n) = f(x) + \lim_{n \to \infty} f(x + z_n - x_0) = f(x) + \lim_{n \to \infty} \left[ f(x_0) - f(x_n) \right] = f(x) + \left[ f(x_0) - f(x) \right] = f(x),$$

Which proves (ii). The converse implication is trivial.

**Remark 3**

In the case of functions of real variable we define a uniformly right-pre-continuous (left-pre-continuous) function, postulating...
that the respective zero sequences \((z_n)\) are positive (resp. negative). In this case, the oddness of the additive function implies that the above result remains true if (i) is replaced by " \(f\) is left- or right-pre-continuous at a point".

**Corollary 1**

Let \(X\) be a metric group with an addition "\(+\)" and a neutral element \(0\). If a function \(f : X \to \mathbb{R}\) is additive and discontinuous at a point, then its graph is dense in the product metric space \(X \times \mathbb{R}\).

**Proof.** Assume that \(f : X \to \mathbb{R}\) is additive and discontinuous at a point. Of course, \(f\) is discontinuous \(0\) (see, for instance [7]). Since \(f(0) = 0\), there is a sequence \((z_n)\) with \(\lim_{n \to \infty} z_n = 0\) such that either \(\lim_{n \to \infty} f(z_n)\) is a finite nonzero real number or \(\lim_{n \to \infty} f(z_n) = \infty\).

In the first case, we can assume that \(\lim_{n \to \infty} f(z_n) = 1\)

If the second case holds, choosing a sequence of real rational numbers \((r_n)\), \(r_n \neq 0\) for all \(n \in \mathbb{N}\), such that

\[
\lim_{n \to \infty} f(z_n) = 1
\]

and putting

\[w_n := \frac{z_n}{r_n}, \quad n \in \mathbb{N},\]

we have

\[\lim_{n \to \infty} w_n = 0,\]

and, making use of the rational homogeneity of \(f\) (Aczél [8] p. 32, Kuczma [7] p. 121, Theorem 1),

\[\lim_{n \to \infty} f(w_n) = \lim_{n \to \infty} \frac{f(z_n)}{r_n} = \lim_{n \to \infty} \frac{f(z_n)}{r_n} = 1.\]

Thus, in both possible cases, there exists a sequence \((z_n)\) such that

\[\lim_{n \to \infty} f(z_n) = 0\]

Hence, by the rational homogeneity of \(f\), every rational number \(r \in \mathbb{Q}\), we have

\[\lim_{n \to \infty} f(rz_n) = r \lim_{n \to \infty} f(z_n) = r,\]

which implies that every point \(\{0\} \times \mathbb{R}\{0, t : 0 \in X \land t \in \mathbb{R}\}\) is an accumulation point of the graph \(f\). Through the additivity \(f\) we have

\[f(x + rz_n) = f(x) + f(rz_n), \quad x \in X, r \in \mathbb{Q}\]

so, for every point \(x \in X\), the set \(\{x\} \times \mathbb{R}\) is contained in the closure of the graph \(f\). This completes the proof.

**Definition 3**

Let \(X\) be a subset of a metric group \(G\) with a multiplication "\(*)" and neutral element \(1\) and \(Y\) be a metric space. A function \(f : X \to Y\) is called uniformly pre-continuous, if there exists a sequence \((r_n)\) for all \(n \in \mathbb{N}\), with \(\lim_{n \to \infty} r_n = 1\) such that for all \(x \in X\), \(n \in \mathbb{N}\), we have \(x \cdot r_n \in X\), and \(\lim_{n \to \infty} f(x \cdot r_n) = f(x)\).

**Remark 4**

Here, in the case of functions of real variable we define uniformly right-pre-continuous (left-pre-continuous) functions, postulating that the respective zero sequences \((z_n)\) are such that \(z_n > 0\) (resp. \(z_n < 0\)) for all \(n \in \mathbb{N}\).

**Theorem 3**

Every additive function \(\alpha : \mathbb{R} \to \mathbb{R}\) is uniformly pre-continuous.

Every additive function \(\alpha : (0, \infty) \to \mathbb{R}\) is right-uniformly pre-continuous.

Indeed, assume that \(\alpha : \mathbb{R} \to \mathbb{R}\) is additive and take

\[\frac{1}{n} = \alpha(1) \quad \text{for} \quad n \in \mathbb{N}\]

As every additive function is rationally homogeneous, we have for every \(x \in \mathbb{R}\) and \(n \in \mathbb{N}\),

\[\alpha \left( x + \frac{1}{n} \right) = \alpha(x) + \alpha \left( \frac{1}{n} \right) = \alpha(x) + \frac{1}{n} \alpha(1),\]

Whence \(\lim_{n \to \infty} \alpha \left( x + \frac{1}{n} \right) = \alpha(x)\). Since \(\mathbb{R}\) the addition is a group of the neutral element \(0\), and the euclidean topology, satisfies the conditions of Definition 2, the function \(\alpha\) is uniform and pre-continuous \(\mathbb{R}\).

The argument for the second result is analogous.
Corollary 2

Every exponential function \( f : \mathbb{R} \to (0, \infty) \), i.e. such that \( f(x + y) = f(x)f(y) \), \( x, y \in \mathbb{R} \), is uniformly pre-continuous (in the additive group \( \mathbb{R} \)).

Proof. It \( f : \mathbb{R} \to (0, \infty) \) is exponential then \( \alpha := \log \circ \circ \exp \) is additive in \( \mathbb{R} \), and \( f = \exp \circ \alpha \). Thus, for all \( x \in \mathbb{R} \) and \( n \in \mathbb{N} \), by the additivity of \( \alpha \), similarly as above, we have
\[
\lim_{n \to \infty} f\left( x + \frac{1}{n} \right) = f(x).
\]

Corollary 3

Every logarithmic function \( f : (0, \infty) \to \mathbb{R} \), i.e. such that \( f(x + y) = g(x)g(y) \), \( x, y \in (0, \infty) \), is uniformly pre-continuous (in the multiplicative group \( \mathbb{R}^{+} \)).

Proof. The interval \((0, \infty)\) with the multiplication, neutral element 1, and the euclidean topology, satisfies the conditions of Definition 2. It \( f : (0, \infty) \to \mathbb{R} \) is logarithmic then \( \alpha : \mathbb{R} \to \mathbb{R} \) defined by \( \alpha := \log \circ \circ \exp \) is additive, and \( f = \exp \circ \alpha \circ \log \). Taking \( z_n := \exp \left( \frac{1}{n} \right) \) we have \( \lim_{n \to \infty} z_n = 1 \) and for all \( x > 0 \) and \( n \in \mathbb{N} \),
\[
f(x \cdot z_n) = f(x) + f(z_n) = f(x) + \alpha(\log z_n) = f(x) + \alpha \left( \log e^1 \right)
\]
\[
= f(x) + \alpha \left( \frac{1}{n} \right) = f(x) + \frac{\alpha(1)}{n}
\]
whence \( \lim_{n \to \infty} f(x \cdot z_n) = f(x) \), which, in view of Definition 3, shows that the function \( f \) is uniformly pre-continuous.

Corollary 4

Every multiplicative function \( f : (0, \infty) \to (0, \infty) \), i.e. such that
\[
f(x \cdot y) = f(x)f(y), \quad x, y \in (0, \infty),
\]
is uniformly pre-continuous (in the multiplicative group \( (0, \infty) \)).

Proof. The function \( \alpha : \mathbb{R} \to \mathbb{R} \) defined by \( \alpha := \log \circ \circ \exp \) is additive, and \( f = \exp \circ \alpha \circ \log \) we can argue similarly as in the proof of Corollary 3.

Examples of applications

To illustrate the possible advantages of the introduced notions we begin with the following

Proposition 1

The functions \( f, g : (0, \infty) \to \mathbb{R} \) satisfy the equation
\[
f(x + y) + g(z) = f(z + y) + g(x), \quad x, y, z \in (0, \infty),
\]
and \( f \) is uniformly right-pre-continuous, if and only if
\[
f = \alpha + b, \quad g = \alpha + c
\]
for some additive function \( \alpha : (0, \infty) \to \mathbb{R} \) and \( b, c \in \mathbb{R} \).

Proof. Assume that \( f, g \) satisfy this equation (1) and \( f \) is uniformly right-pre-continuous. Writing this in the form
\[
f(x + y) - g(x) = f(z + y) - g(z), \quad x, y, z \in (0, \infty),
\]
We see that the difference \( f(x + y) - g(x) \) does not depend on \( x \), so the function \( h : (0, \infty) \to \mathbb{R} \) given by
\[
h(y) := f(x + y) - g(x)
\]
is well defined and, consequently, the Pexider functional equation
\[
f(x + y) = g(x) + h(y), \quad x, y \in (0, \infty)
\]
is satisfied. In view of Definition 1 (see also Remark 2), there exists a positive sequence \( \left( z_n \right) \) tending to 0 such that for every \( x > 0 \),
\[
\lim_{n \to \infty} f(x + z_n) = f(x).
\]
Setting \( y = z_n \) in (2) we have, for every \( x > 0 \),
\[ f(x + z_n) = g(x) + h(z_n), \quad n \in \mathbb{N}, \]

and letting \( n \to \infty \), we obtain conclude that

\[ f(x) = g(x) + h_0, \quad x \in (0, \infty), \quad \text{(3)} \]

where

\[ h_0 := \lim_{n \to \infty} h(z_n) \]

exists and does not depend on \( x \).

Similarly, taking \( x := z_n \) in (2), we have

\[ f(z_n + y) = g(z_n) + h(y), \quad n \in \mathbb{N}, \]

and letting \( n \to \infty \), we obtain

\[ f(y) = g_0 + h(y), \quad y > 0, \quad \text{(4)} \]

where

\[ g_0 := \lim_{n \to \infty} g(z_n) \]

is a real constant. From (2), (3), and (4), setting

\[ b := g_0 + h_0, \]

we get

\[ f(x + y) - b = \left[ f(x) - b \right] + \left[ f(y) - b \right], \quad x, y \in (0, \infty), \]

which shows that \( \alpha := f - b \) is an additive function, and

\[ f = \alpha + b. \]

Setting this function into equation (1) gives

\[ g(x) - \alpha(x) = g(z) - \alpha(z), \quad x, z \in (0, \infty), \]

that is \( g - \alpha = c \) for some real \( c \). Thus

\[ g(x) = \alpha(x) + c, \quad x \in (0, \infty). \]

The converse implication follows from the fact that \( \alpha \) is uniformly right-pre-continuous (Theorem 2).

For a function \( f : (1, \infty) \to (0, \infty) \) define the bivariate function \( P_f : (1, \infty)^2 \to (0, \infty) \) by

\[ P_f(x, y) := \frac{f(tx) f(ty)}{f(t^2)}, \quad x, y > 1. \]

**Proposition 2**

Let \( f : (1, \infty) \to (0, \infty) \) be uniformly right-pre-continuous and \( m : (1, \infty) \to (0, \infty) \) be an arbitrary function.

Then the following conditions are equivalent

(i) the function \( P_f \) is \( m \)-homogeneous, i.e.

\[ P_f(tx, ty) = m(t) P_f(x, y), \quad t, x, y > 1; \quad \text{(5)} \]

(ii) the function \( m \equiv 1 \) and there is \( b > 0 \) such that the function \( \frac{f}{b} \) is multiplicative, i.e.

\[ bf(xy) = f(x) f(y), \quad x, y > 1. \]

**Proof.** Assume (i). Then for all \( s, t, x, y > 1 \) we have

\[ m(st) = \frac{P_f(stx, sty)}{P_f(x, y)} \frac{P_f(tx, ty)}{P_f(x, y)} = m(s) m(t), \]

so \( m \) is multiplicative.

The interval \((1, \infty)\) is a subset of the multiplicative group \((0, \infty), \cdot\) with neutral element \( 1 \). Let \( z_n \) be a sequence satisfying the conditions of Definition 3 of uniform right-precontinuity of \( f \) in \((1, \infty)\); in particular \( z_n > 1 \) for \( n \in \mathbb{N} \) and \( \lim_{n \to \infty} z_n = 1 \). Using the definition of \( P_f \) and setting \( y := z_n \) in (5) we have

\[ \frac{f(tx) f(tz_n)}{f(t^2)} = m(t) \frac{f(x) f(z_n)}{f(z_n)}, \quad t, x > 1, n \in \mathbb{N}. \]

Letting \( n \to \infty \) we conclude that

\[ b := \lim_{n \to \infty} f(z_n) \]

exists, is positive, finite, and

\[ \frac{f(tx)}{f(t^2)} = bm(t), \quad t, x > 1. \]

Thus \( \frac{f(tx)}{f(t^2)} \) does not depend on \( x \). So, replacing hereby \( x \), and setting
\[ g(t) := \frac{f(x)f(t)}{f(tx)m(t)}, \quad t > 1, \]
we get
\[ f(tx) = g(t)f(x), \quad t, x > 1. \quad (6) \]
Taking here \( x = z_n, n \in \mathbb{N}, \) as above, we have
\[ f(z_n) = g(t)f(z), \quad t > 1, n \in \mathbb{N}. \]
Letting \( n \to \infty, \) the assumed precontinuity of \( f \) gives
\[ f(t) = bg(t), \quad t > 1, \quad (7) \]
Hence, making use of (6), we have
\[ g(tx) = g(t)g(x), \quad t, x > 1, \quad (8) \]
that is \( g \) is multiplicative.

Applying in turn (5), the definition of \( P_f, \) (7) and (8) we get, for all \( t, x, y > 1, \)
\[ m(t) = \frac{P_f(tx,ty)}{P_f(x,y)} = \frac{f(tx)f(ty)f(xy)}{f(t^2xy)f(x)f(y)} \]
\[ = \frac{g(tx)g(ty)g(xy)}{g(t^2xy)g(x)g(y)} \quad \frac{[g(t)]^2 [g(x)]^2 [g(y)]^2}{[g(t)]^2 [g(x)]^2 [g(y)]^2} \]
\[ = 1 \]
which completes the proof of (ii).

The implication (ii) \( \Rightarrow \) (i) is obvious.

**Remark 5**

Of course, the counterpart of the above result for function \( f : (0,1) \to (0,\infty) \) also holds true.

Let \( f : (0,\infty) \to (0,\infty) \) be an arbitrary function. The two-variable functions \( B_f : (0,\infty)^2 \to (0,\infty) \) given by
\[ B_f(x,y) := \frac{f(x)f(y)}{f(x+y)}, \quad x, y > 0, \]
is called a beta-type function, and \( f \) is referred to as its generator (12).

**Remark 6**

Note that Barczy and Burai [5] have derived strong laws of large numbers and central limit theorems, among others, for a new type family of beta-type means considered in [3] and [4].

A function \( F : (0,\infty)^2 \to \mathbb{R} \) is called translative with respect to a function \( \alpha : (0,\infty) \to \mathbb{R}, \) if

**Remark 7**

If \( F \) is translative with respect to \( \alpha \) then \( \alpha \) is an additive function. If moreover \( F \) nonnegative, then there is a \( \alpha \in \mathbb{R}, \alpha \geq 0 \) such that \( \alpha(t) = at \) for all \( t > 0. \)

**Proof.** Indeed, for all \( x, y, t \in (0,\infty) \) we have
\[ F(x+s+t,y+s+t) = F((x+s)+t,(y+s)+t) = F(x+s,y+s)+\alpha(t) = F(x,y)+\alpha(s)+\alpha(t), \]
and
\[ F(x+s+t,y+s+t) = F(x,y)+\alpha(s+t), \]
whence \( \alpha(s+t) = \alpha(s)+\alpha(t), \) so \( \alpha \) is additive in \((0,\infty).\)

From the transitivity of \( F \) and the just proved additivity of \( \alpha \) we have, for all \( x,y,t > 0 \) and \( n \in \mathbb{N}, \)
\[ F(x+nt,y+mt) = F(x,y)+\alpha(mt) = F(x,y)+n\alpha(t). \]
Clearly, this equality and the assumed nonnegativity \( F \) exclude existence \( t > 0, \alpha(t) < 0. \)

**Proposition 3**

Let \( f : (0,\infty) \to (0,\infty) \) be a (right) uniformly precontinuous function and \( \alpha : (0,\infty) \to \mathbb{R} \) be given functions. The following conditions are equivalent:

(i) the beta-type function \( B_f : (0,\infty)^2 \to (0,\infty) \) is translative with respect to the function \( \alpha; \)

(ii) \( \alpha \equiv 0 \) and, for some \( c > 0, \) the function \( \frac{f}{c} \) is an exponential function, i.e.
\[ cf(x+y) = f(x)f(y), \quad x, y > 0. \]

**Proof.** Assume (i). In view of Remark 4, there is a real
number \( a \geq 0 \) such that \( \alpha(t) = at \) for all \( t > 0 \) and from the assumed transitivity \( B \), we have

\[
\frac{f(x+t)f(y+t)}{f(x+y+2t)} = \frac{f(x)f(y)}{f(x+y)} + at, \quad x, y, t > 0.
\]

Hence, for all \( x > y > 0 \), and \( s > 0 \),

\[
\frac{f(x)f(y+s)}{f(x+y+s)} = \frac{f((x-y)+y)f(x+y)}{f((x-y)+y)+(x+y)} = \frac{f(x-y)f(s)}{f(x-y+s)} + ay.
\]

Setting here \( s = z_n \), where \( z_n \) is a sequence such that \( z_n > 0 \) all \( n \in \mathbb{N} \), \( \lim_{n \to \infty} z_n = 0 \), satisfying the condition of the uniform right-precontinuity, we have

\[
\frac{f(x)f(y+z_n)}{f(x+y+z_n)} = \frac{f(x-y)f(z_n)}{f(x-y+z_n)} + ay, \quad n \in \mathbb{N}, 0 < y < x.
\]

Letting \( n \to \infty \), and making use of the right continuity \( f \), we conclude that the limit

\[
b := \lim_{n \to \infty} f(z_n)
\]

exists, is nonnegative, finite and

\[
f(x)f(y) = b + ay, \quad 0 < y < x.
\]

or, equivalently, that

\[
f(x)f(y) = b + a \min(x, y), \quad x, y > 0.
\]

For arbitrary \( x, y > 0 \), choosing positive \( y \) such that \( y < x \) and \( y < z \), we hence get

\[
f(x)f(y) = b + ay = \frac{f(z)f(y)}{f(z+y)},
\]

whence

\[
f(z+y) = \frac{f(x+y)}{f(x)}.
\]

It follows that the function \( g : (0, \infty) \to (0, \infty) \)

\[
g(y) := \frac{f(x+y)}{f(x)}, \quad y > 0,
\]

is well defined. Since \( fg \) are continuous and satisfy the Pexider functional equation

\[
f(x+y) = f(x)g(y), \quad x, y > 0.
\]

By the symmetry of the left-hand-side \( x \) and \( y \) we have

\[
f(x+y) = f(y)g(x), \quad x, y > 0.
\]

Setting here \( y = z_n \), where the sequence \( y = z_n \) is chosen above, we have

\[
f(x+z_n) = f(z_n)g(x), \quad n \in \mathbb{N}, x, y > 0.
\]

Letting here \( n \to \infty \), and using (9), we get

\[
f(x) = bg(x), \quad x > 0,
\]

which implies that \( b \neq 0 \). Hence, using (11), we obtain

\[
g(x+y) = g(x)g(y), \quad x, y > 0,
\]

which means that \( g \) is an exponential function. From (10) we get \( a = 0 \), and using (12) we conclude (ii).

The implication \((ii) \Rightarrow (i)\) is obvious.

**Proposition 4**

If the functions \( f, g, h : (0, \infty) \to \mathbb{R} \) satisfy the equation

\[
f(x+y) = g(x) + h(y), \quad x, y \in (0, \infty),
\]

then

\[
f = \alpha + b + c, \quad g = \alpha + c, \quad h = \alpha + b
\]

for some additive function \( \alpha : (0, \infty) \to \mathbb{R} \) and \( b, c \in \mathbb{R} \).

**Proof.** From (13), making use of the commutativity of addition, we have for all \( x, y \in (0, \infty) \)

\[
g(x) + h(y) = f(x + y) = f(y + x) = g(y) + h(x),
\]

whence, for all \( x, y \in (0, \infty) \),

\[
h(x) = g(x) + h(y) - g(y).
\]

Choosing arbitrarily \( y = y_0 > 0 \), we get

\[
h(x) = g(x) + h(y_0) - g(y_0), \quad x > 0.
\]

Setting this into (13) we get

\[
f(x+y) = [g(y_0) + h(y_0)] - [g(x) - g(y_0)] + [g(y) - g(y_0)], \quad x, y > 0,
\]

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whence, setting
\[ f(x) := f(x) - \left[ g\left(y_0\right) + h\left(0\right)\right], \quad \overline{f}(x) := f(x) - g\left(y_0\right), \quad x > 0, \]
we obtain
\[ \overline{f}(x + y) = \overline{f}(x) + \overline{f}(y), \quad x, y > 0. \]
Hence, by induction we get
\[ \overline{f}(x_1 + \ldots + x_n) = \overline{g}(x_1) + \ldots + \overline{g}(x_n), \quad n \in \mathbb{N}, n \geq 2; \quad x_1, \ldots, x_n > 0. \]
Whence
\[ \overline{f}\left(n x\right) = n \underbrace{\overline{g}(x)}_{n}, \quad n \in \mathbb{N}, n \geq 2; \quad x > 0. \]
Replacing hereby \( x \) by \( \frac{x}{n} \), we get
\[ \overline{g}\left(\frac{x}{n}\right) = \frac{\overline{f}(x)}{n}, \quad n \in \mathbb{N}, n \geq 2; \quad x > 0, \]
which implies that
\[ \lim_{n \to \infty} \overline{g}\left(\frac{1}{n}\right) = 0. \]
Now (16) implies that \( \overline{g} \) is uniformly pre-continuous \( z_n = \frac{1}{n}. \) Of course, (16), \( \overline{f} \) is uniformly pre-continuous, and from (14) and (15) it follows that \( f, g, h \) are uniformly pre-continuous with the same sequence \( z_n = \frac{1}{n}. \)

In view of Definition 1 (see also Remark 2), there exists a positive sequence \( (z_n) \) tending to 0 such that for every \( x > 0 \),
\[ \lim_{n \to \infty} f\left(x + z_n\right) = f(x). \]
Setting \( y = z_n \) in (13) we have, for every \( x > 0, \)
\[ f\left(x + z_n\right) = g\left(x\right) + h\left(z_n\right), \quad n \in \mathbb{N}, \]
and letting \( n \to \infty \), we obtain conclude that
\[ f(x) = g(x) + b, \quad x \in (0, \infty), \]
where
\[ b := \lim_{n \to \infty} h\left(z_n\right) \]
exists and does not depend on \( x \).

Similarly, taking \( x := z_n \) in (13), we have
\[ f\left(z_n + y\right) = g\left(z_n\right) + h\left(y\right), \quad n \in \mathbb{N}, \]
and letting \( n \to \infty \), we obtain
\[ f(y) = c + h\left(y\right), \quad y > 0, \]
where
\[ c := \lim_{n \to \infty} g\left(z_n\right) \]
is a real constant. From (13), (17), and (18), setting \( a := b + c \), we get
\[ f\left(x + y\right) - \left(b + c\right) = \left[f\left(x\right) - \left(b + c\right)\right] + \left[f\left(y\right) - \left(b + c\right)\right], \quad x, y \in (0, \infty), \]
which shows that \( \alpha := f - \left(b + c\right) \) is an additive function, and
\[ f = \alpha + \left(b + c\right). \]
Hence, from (17) we get
\[ g\left(x\right) = \alpha\left(x\right) + c, \quad x \in (0, \infty), \]
and from (18),
\[ h\left(x\right) = \alpha\left(x\right) + b, \quad x \in (0, \infty), \]
Which completes the proof.

Final Remark
Following Azad [9] one could try to consider the fuzzy versions of precontinuity.

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