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Class fields, Dirichlet characters, and extended genus fields of global function fields

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Abstract
We study the extended genus field of an abelian extension of a rational function field. We follow the definition of Anglès and Jaulent, which uses the class field theory. First, we show that the natural definition of extended genus field of a cyclotomic function field obtained by means of Dirichlet characters is the same as the one given by Anglès and Jaulent. Next, we study the extended genus field of a finite abelian extension of a rational function field along the lines of the study of genus fields of abelian extensions of rational function fields. In the absolute abelian case, we compare this approach with the one given by Anglès and Jaulent.

KEYWORDS
class fields, Dirichlet characters, extended genus fields, genus fields, global fields

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1 | INTRODUCTION

Gauss [8] was the first to study extended genus fields introducing the genus concept in the context of quadratic forms. It seems that Hilbert [12] and then Hecke [11] were the first in translating Gauss’ language of genus theory to the number field setting. Later on, Hasse [9] considered the extended genus field of a quadratic number field by means of the kernel of characters in the context of the class field theory. Starting from here, Leopoldt [14] generalized Hasse’s results defining the extended genus field of a finite abelian extension of the field of rational numbers. In his study, Leopoldt studied the arithmetic of abelian number fields using Dirichlet characters. Fröhlich defined the concept of genus field of an arbitrary number field [5, 6]. Fröhlich’s definition of the genus field of a finite number field \( K \) is \( K_{ge} := KF \) where \( F \) is the maximal abelian extension of the field of rational numbers \( \mathbb{Q} \) contained in the Hilbert Class Field (HCF) \( H_K \) of \( K \). Similarly, the extended genus field of a number field \( K \) is defined as \( K_{gex} := KF^+ \), where \( F^+ \) is the maximal abelian extension of \( \mathbb{Q} \) contained in the extended or narrow HCF \( H^+_K \). Since then, numerous authors have studied both, genus and extended genus fields of number fields.

In the number field setting, the concepts of HCF \( H_K \) and extended HCF \( H^+_K \) of \( K \) are defined canonically as the maximal unramified extension and as the maximal unramified extension at the finite primes of \( K \), respectively. In particular, the concepts of genus field and of extended genus field are canonically defined. We have \( K \subseteq K_{ge} \subseteq H_K \) and the Galois group \( \text{Gal}(H_K/K) \) is isomorphic to the class group \( Cl_K \) of \( K \). The genus field \( K_{ge} \) corresponds to a subgroup \( G_K \) of \( Cl_K \) and we have \( \text{Gal}(K_{ge}/K) \cong Cl_K/G_K \). The degree \( [K_{ge} : K] \) is called the genus number of \( K \) and \( \text{Gal}(K_{ge}/K) \) is called the genus group of \( K \). Similarly, \( K \subseteq K_{gex} \subseteq H^+_K \) and \( K_{gex} \) corresponds to a subgroup \( G^+_K \) of \( \text{Gal}(H^+_K/K) \cong Cl^+_K \).
The function field case is quite different since there are several possible good definitions of HCF and consequently of extended HCF. The definition given depends heavily on which aspect one wants to study. The first possible definition of HCF of a global function field $K$ would be as the maximal abelian extension of $K$. However, this extension is of infinite degree over $K$ because it contains all the extensions of constants. If one wants a finite extension as the HCF, it is necessary to impose a splitting condition on some primes since every prime is eventually inert in the extensions of constants.

The first person who considered genus fields of global function fields was Clement [4] who defined the genus field of a cyclic Kummer extension $K$ of a rational function field $k$ of prime degree. She followed closely the definition given by Hasse in [9]. The next step was given by Bae and Koo in [3] where they developed the concept of genus fields along the lines of Fröhlich’s work. In fact, they defined the extended genus field for an arbitrary global function field $K$ using a generalization of cyclotomic function field extensions given by the Carlitz module. The genus field of a general abelian extension of a global function field over a rational function field using Dirichlet characters can be found in [2, 15, 16].

The next step was given by Bae and Koo in [3] where they developed the concept of genus fields along the acyclic Kummer extension $K$ of a global function field $K$. The genus field of the modulus $1+\cdot$ is the maximal unramified abelian extension of $K$ such that every element of $S$ decomposes fully in $H_{K,S}$. We have that $H_{K,S}/K$ is a finite extension and $\text{Gal}(H_{K,S}/K) \cong \text{Cl}_S$, the class group $\text{Cl}_S$ of the Dedekind domain $\mathcal{O}_S$ of the elements of $K$ with poles contained in $S$.

There is no simple definition of what the extended HCF of a function field $K$ should be. In the number field case, we have that the extended HCF $H^+_K$ of $K$ is the maximal abelian extension of $K$ unramified at the finite primes and it is the ray class field of the modulus $1_+ := \prod_{p \text{ is real}} p$. We have that a place $p$ decomposes fully in $H^+_K/k$ if and only if $p$ is principal generated by a totally positive element. Following this idea, Anglès and Jaulent [1] defined the extended HCF of a global function field $K$ in an analogous way. In fact, the definition of Anglès and Jaulent works for any global field, either numeric or function.

In this paper, we use the above definition as the extended HCF $H^+_K$ of a global function field $K$. Using this definition, we define the extended genus field $K_{gex}$ of $K$ over $k$, where $k = F_q(T)$ and $K/k$ is a finite geometric extension, as $K_{gex} = KF$, where $F$ is the maximal abelian extension of $k$ contained in $H^+_K$. The main purpose of this paper is to show that if $K$ is contained in a cyclotomic function field, then $K_{gex}$ is the maximal abelian extension of $K$ contained in a cyclotomic function field unramified at the finite primes. This is Theorem 4.5, where we obtain that $E_{gex}$ is the field associated with a group of Dirichlet characters. In particular, this definition is the same as the one given in [18] in the case of a field contained in a cyclotomic function field but it might be different in the general case, even for abelian extensions.

The general expression of $K_{gex}$ for a finite abelian extension $K$ of $k$ is given in Theorem 5.1.

2 | PRELIMINARIES AND NOTATIONS

We define by $k = F_q(T)$ the global rational function field with field of constants the finite field of $q$ elements $F_q$. Let $R_T = F_q[T]$ be the ring of polynomials, that is, the ring of integers of $k$ with respect to the pole of $T$, the infinite prime $p_\infty$. Let $R_T^+ := \{P \in R_T \mid P \text{ is monic and irreducible}\}$. The elements of $R_T^+$ are the finite primes of $k$ and $p_\infty$ is the infinite prime of $k$. For $N \in R_T, \Lambda_N$ denotes the $N$th torsion of the Carlitz module. A finite extension $F/k$ will be called cyclotomic if there exists $N \in R_T$ such that $k \subseteq F \subseteq k(\Lambda_N)$.

Given a cyclotomic function field $E$, the group of Dirichlet characters $X$ corresponding to $E$ is the group $X$ such that $X \subseteq (R_T/\langle N \rangle)^\times \cong \text{Gal}(k(\Lambda_N)/k) = \text{Hom}(\langle R_T/\langle N \rangle \rangle^\times, \mathbb{C}^\times)$ and $E = k(\Lambda_N)^H$, where $H = \bigcap_{X \in X} \ker X$. For the basic results on Dirichlet characters, we refer to [21, Ch. 3] or [20, Section 12.6].

For a group of Dirichlet characters $X$, let $Y = \prod_{P \in R_T} X_P$ where $X_P = \{X_P \mid X \in X\}$ and $X_P$ is the $P$th component of $X$: $X = \prod_{P \in R_T^+} X_P$. If $E$ is the field corresponding to $X$, we define $E_{gex}$ as the field corresponding to $Y$. We have that $E_{gex}$ is the maximal unramified extension at the finite primes of $E$ contained in a cyclotomic function field $k(\Lambda_N)$. The infinite prime $p_\infty$ might be ramified in $E_{gex}/k$ (see [15]).

Let $L_n = k(\Lambda_{1/(T+1)})^{\mathbb{F}_q^n}, n \in \mathbb{N} \cup \{0\}$ where $\mathbb{F}_q^n \subseteq (R_{1/T}/\langle 1/T^{n+1} \rangle)^\times$, is isomorphic to the inertia group of the prime corresponding to $T$ in $K(\Lambda_{1/(T+1)})/k$. The prime $p_\infty$ is the only ramified prime in $L_n/k$ and it is totally and wildly ramified. For $n \in \mathbb{N} \cup \{0\}$ and any finite extension $F/k$, we denote $n F := L_n F$. For $m \in \mathbb{N}$, and for any finite extension $F/k$, $F_m$ denotes the extension of constants: $F_m = F^{\mathbb{F}_q^m}$. In particular, $k_m = F_q(T)$.

Given a finite abelian extension $K/k$, there exist $n \in \mathbb{N} \cup \{0\}, m \in \mathbb{N}$ and $N \in R_T$ such that $K \subseteq L_m k(\Lambda_N)k_m = k(\Lambda_N)_m$ (see [10] or [20, Theorem 12.8.31]). We define $M := L_n k_m$. In $M/k$, no finite prime of $k$ is ramified.
For any extension $E/F$ of global fields and for any place $\mathfrak{p}$ of $E$ and $\mathfrak{p} = \mathfrak{P} \cap F$, the ramification index is denoted by $e_{E/F}(\mathfrak{P}|\mathfrak{p}) = e(\mathfrak{P}|\mathfrak{p})$ and the inertia degree is denoted by $f_{E/F}(\mathfrak{P}|\mathfrak{p}) = f(\mathfrak{P}|\mathfrak{p})$. When the extension is Galois we denote $e_{E/F}(\mathfrak{P}|\mathfrak{p}) = e_{\mathfrak{P}}(E|\mathfrak{p})$ and $f_{E/F}(\mathfrak{P}|\mathfrak{p}) = f_{\mathfrak{P}}(E|\mathfrak{p})$. In particular, for any abelian extension $E/k$, $e_{\mathfrak{P}}(E|\mathfrak{p})$ and $f_{\mathfrak{P}}(E|\mathfrak{p})$ denote the ramification index and the inertia degree of $P \subseteq \mathcal{R}^+$ in $E/k$, respectively, and we denote by $e_{\infty}(E|\mathfrak{p})$ and $f_{\infty}(E|\mathfrak{p})$ the ramification index and the inertia degree of $\mathfrak{p}_{\infty}$ in $E/k$.

For any finite separable extension $K/k$, the finite primes of $K$ are the primes over the primes $P$ in $\mathcal{R}_{F}^+$ and the infinite primes of $K$ are the primes over $\mathfrak{p}_{\infty}$. The HCF $H_K$ of $K$ is the maximal abelian extension of $K$ unramified at every finite prime of $K$ and where all the infinite primes of $K$ are fully decomposed. The genus field $K_{gs}$ of $K/k$ is the maximal extension of $K$ contained in $H_K$ and such that it is the composite $K_{gs} = KF$ where $F/k$ is abelian. We choose $F$ the maximal possible extension. In other words, $F$ is the maximal abelian extension of $k$ contained in $H_K$.

Let $K/k$ be a finite abelian extension. We know that $K_{ge} = K_{E \cdot H_{ge}}$ is the genus field of $K$, where $H$ is the decomposition group of the infinite primes in $KE/K$ and $E := K \cap k(\Lambda_N)$ (see [2]). We also know that $KE_{ge}/K_{gs}$ and $KE/K$ are extensions of constants.

For a local field $F$ with prime $\mathfrak{p}$, we denote by $F(\mathfrak{p}) \cong F_q$ the residue field of $F$ and by $U_n(\mathfrak{p}) = 1 + \mathfrak{p}^n$ the group of $n$th units of $F$, $n \in \mathbb{N} \cup \{0\}$. Let $\pi = \pi_F = \pi_{\mathfrak{p}}$ be a uniformizer for $\mathfrak{p}$, that is, $v_\mathfrak{p}(\pi) = 1$. Then, the multiplicative group of $F$ satisfies $F^* \cong \langle \pi \rangle \times U(1)$ as groups.

For a global field $F$, we denote by $J_F$ the idèle group of $F$ and by $C_F = J_F/F^*$ the idèle class group of $F$.

### 3 Extended Genus Field of a Global Function Field

First, we establish the definition of extended genus fields according to Anglès and Jaulent [1]. We begin recalling two results on the class field theory.

**Proposition 3.1.** Let $F$ be a global (resp. local) field and let $R/F$ be the field corresponding to the subgroup $H < C_F = J_F/F^*$ (resp. $H < F^*$), that is, $H$ is the open subgroup of $C_F$ (resp. open subgroup of $F^*$) such that $H = N_{R/F} C_R$ (resp. $H = N_{R/F} R^*$) and $Gal(R/F) \cong C_F/H$ (resp. $Gal(R/F) \cong R^*/H$). Let $E/F$ be a finite separable extension. Then, the extension $ER/E$ corresponds to the group $N_{E/F}(H)$ of $C_E$ (resp. $E^*$).

**Proof.** See [13, Ch. X, Theorem 6] or [18, Theorem 3.8].

**Proposition 3.2.** Let $L/K$ be a finite separable extension of global fields. Let $H$ be an open subgroup of finite index in $C_L$ and let $L_H$ be its class field. Let $K_0$ be the maximal abelian extension of $K$ contained in $L_H$. Then, the norm group of $K_0$ is $N_{L/K}(H)$.

**Proof.** See [7, Lemma 1] or [19, Proposición 17.6.48].
Definition 3.3. Let $K$ be a global function field and let $S$ be a finite nonempty set of prime divisors of $K$. The HCF of $K$ relative to $S$, $H_{K,S}$, is the maximal abelian unramified extension of $K$ such that every element of $S$ decomposes fully in $H_{K,S}/K$.

In the case of a number field $K$, the extended HCF $H^+_K$ is, by definition, the maximal abelian extension of $K$ such that every finite prime of $K$ is unramified in $H^+_K$. We have that $H^+_K$ corresponds to the idèle group $J^+_K$, that is, a place $\mathfrak{p}$ of $K$ decomposes fully in $H^+_K/K$ if and only if $\mathfrak{p}$ is principal generated by a totally positive element. So, a concept of “totally positive” should be developed in the function field case.

Let $k = \mathbb{F}_q(T)$ be a fixed global rational function field over the field of $q$ elements $\mathbb{F}_q$. Let $\mathfrak{p}_\infty$ be the infinite prime of $k$, that is, the pole of $T$ and let $k_\infty \cong \mathbb{F}_q((1/T))$ be the completion of $k$ at $\mathfrak{p}_\infty$. Let $x \in k_\infty^*$. Then, $x$ is written uniquely as $x = \left(\frac{1}{T}\right)^{n_x} \lambda_x \varepsilon_x$ with $n_x \in \mathbb{Z}$, $\lambda_x \in \mathbb{F}_q^*$ and $\varepsilon_x \in U_\infty^{(1)}$, where $U_\infty^{(1)} = U^{(1)}_{\mathfrak{p}_\infty}$. We write $\pi_\infty := 1/T$, which is a uniformizer at $\mathfrak{p}_\infty$.

Definition 3.4. The sign function is defined as $\phi_\infty : k_\infty^* \rightarrow \mathbb{F}_q^*$ given by $\phi_\infty(x) = \lambda_x$ for $x \in k_\infty^*$. The value $\phi_\infty(x)$ is called the “sign” of $x$. We also write $\text{sgn}(x) = \phi_\infty(x)$.

We have that $\phi_\infty$ is an epimorphism and $\ker \phi_\infty = \langle \pi_\infty \rangle \times U^{(1)}_{\mathfrak{p}_\infty}$. For example, if $M \in R_T = \mathbb{F}_q[T]$, $M \neq 0$, say $M$ is of degree $d$ and its leader coefficient is $a_d \in \mathbb{F}_q^*$, then $\text{sgn}(M) = a_d$.

Definition 3.5. Let $L$ be a finite separable extension of $k_\infty$. We define the sign of $L^*$ by the morphism $\phi_L := \phi_\infty \circ N_{L/k_\infty} : L^* \rightarrow \mathbb{F}_q^*$. 

Proposition 3.6. Let $k_\infty \subseteq E \subseteq L$. Then, we have $\phi_L = \phi_E \circ N_{L/E}$ and $N_{L/E}(\ker \phi_L) \subseteq \ker \phi_E$.

Proof. [1, Remarque 1.1.1].

Definition 3.7. Let $L/k$ be a finite separable extension and let $P_\infty$ be the set of infinite places of $L$, that is, $P_\infty$ is the set of primes $\mathfrak{p}$ of $L$ such that $\mathfrak{p}|\mathfrak{p}_\infty$. Let $L_\mathfrak{v}$ the completion of $L$ at $\mathfrak{v} \in P_\infty$. We have that $\phi_{L_\mathfrak{v}}(x)$ is well defined for $x \in L_\mathfrak{v}^*$. The element $x \in L^*$ is called totally positive if $\phi_{L_\mathfrak{v}}(x) = 1$ for all $\mathfrak{v} \in P_\infty$. We define $L^+ = \{x \in L^* | x$ is totally positive\}.

Definition 3.8. The group of signs of a global function field is defined by $\text{Sig}_L := L^*/L^+$. For a subgroup $R$ of $L^*$, we define $\text{Sig}_L(R) := RL^+/L^+$, so $\text{Sig}_L(L^*) = \text{Sig}_L$.

Proposition 3.9. We have $\text{Sig}_L(L^*) \cong \prod_{\mathfrak{v} \in P_\infty} \phi_{L_\mathfrak{v}}(L_\mathfrak{v}^*)$.

Proof. [1, Lemme 1.2.1].

Let $\mathcal{O}_L = \{x \in L | v_\mathfrak{p}(x) \geq 0 \text{ for all } \mathfrak{p} \notin P_\infty\}$. Let $P_L^+ = \{x\mathcal{O}_L | x \in L^+\}$ be the principal ideals generated by a totally positive element and we define the extended ideal class group by

$$CL^\text{ext}_L = CL^+_L = I_L/P_L^+,$$

where $I_L$ is the group of fractional ideals of $\mathcal{O}_L$. We have the ideal class group $CL_L = I_L/P_L$, where $P_L = \{(x) = x\mathcal{O}_L | x \in L^*\}$ and $|CL_L| < \infty$.

Definition 3.10. We define the following subgroups of the group of idèles $J_L$ as:

$$U_L := \prod_{\mathfrak{v} \in P_\infty} L_\mathfrak{v}^+ \times \prod_{\mathfrak{v} \notin P_\infty} U_{L_\mathfrak{v}}.$$
The groups $U_L L^*$ and $U_L^+ L^*$ are open subgroups of $J_L$, the idèle group of $L$. The groups $U_L L^*$ and $U_L^+ L^*$ correspond to $J_L^1$ and $J_L^{1+}$, the idèle groups congruent to the modulus 1 and 1+, respectively, in the number field case.

From the canonical isomorphism $I_L \cong J_L / U_L$, we obtain

$$\text{Cl}_L \cong J_L / U_L L^*$$

and since we may find principal idèles with arbitrary signs at the infinite places, it follows that

$$\text{Cl}_L^+ \cong J_L / U_L^+ L^*.$$  

For $S = P_\infty$, we denote $H_L = H_{L,S}$ the HCF of the global function field $L$. By the class field theory, we have

$$\text{Gal}(H_L / L) \cong \text{Cl}_L \cong J_L / U_L L^*.$$  

**Definition 3.11.** Let $H_L^+ = H_L^{\text{ext}}$ be the abelian extension of the global function field $L$ corresponding to the idèle subgroup $U_L^+ L^*$ of $J_L$. The field $H_L^+$ is called the extended HCF of $L$ corresponding to $\mathcal{O}_L$.

We have that $H_L^+ / L$ is an unramified extension at the finite prime divisors of $L$, $H_L \subseteq H_L^+$ and

$$\text{Gal}(H_L^+ / L) \cong J_L / U_L^+ L^* \cong I_L / P_L^+ = \text{Cl}_L^+ = \text{Cl}_L^{\text{ext}}.$$  

Now, we have

$$\frac{U_L}{U_L^+} \cong \prod_{v \in \mathcal{P}_\infty} \frac{L_v^*}{\ker \phi_{L_v}} \cong \prod_{v \in \mathcal{P}_\infty} \phi_{L_v}(L_v^*) \cong \frac{L^*}{L^+}.$$  

**Remark 3.12.** We have $[H_L^+ : H_L] || \text{Sig}(L^*) = \prod_{v \in \mathcal{P}_\infty} |\phi_{L_v}(L_v^*)| |(q - 1)^{|S|}.$

It follows that $H_L^+ / H_L$ is unramified at the finite primes and tamely ramified at the infinite primes. Furthermore,

$$\text{Gal}(H_L^+ / H_L) \cong \ker \left( \text{Gal}(H_L^+ / L) \xrightarrow{\text{rest}} \text{Gal}(H_L / L) \right) \cong \frac{U_L L^*}{U_L^+ L^*} \cong \text{Sig}_E.$$  

**Proposition 3.13.** Let $L/E$ be a finite separable extension of global function fields. Then, $H_L^+ / E$. If $L/E$ is a Galois extension, then $H_L^+ / E$ is also Galois.

**Proof.** We have

Then, $H_L^+ / H_L^+ \cap L$ is an abelian extension unramified at the finite primes, so that $LH_E^+ / L$ is also unramified at the finite primes.
We consider the norm \( N_{L/E} : C_L = J_L/L^* \rightarrow C_E = J_E/E^* \) of idèle's groups and since \( N_{L_0/E_0}(\ker \phi_{L_0}) \subseteq \ker \phi_{E_0} \), \( N_{L/E}(U^+_L) \subseteq U^+_E \) and \( N_{L/E}(L^+) \subseteq E^* \), it follows \( N_{L/E}(U^+_E) \subseteq U^+_E \). Now, the group \( U^+_E \) corresponds to the field \( H^+_E \).

From Proposition 3.1, we have that the norm group of \( \frac{H^+_E}{L} \) is \( \frac{N_{L/K}(U^+_E)}{L} \). Hence, \( LH^+_L \subseteq H^+_L \) and thus \( H^+_L \subseteq H^+_E \).

Now, if \( L/E \) is a Galois extension, we consider \( \sigma : H^+_L \rightarrow H^+_L \) a monomorphism such that \( \sigma|_E = \text{Id}_E \). Here, \( H^+_L \) denotes a fixed algebraic closure of \( H^+_L \). Since \( L/E \) is Galois, we have \( \sigma|_L : L \rightarrow L \) and since \( \sigma|_E = \text{Id}_E \), it follows that \( \sigma(L^+_L) = L^+_L \).

From Proposition 3.14, we have that the norm group of \( \frac{H^+_L}{L} \) is \( \frac{N_{L/K}(L^+_L)}{L} \). Hence, \( LH^+_L \subseteq H^+_L \) and \( H^+_L \subseteq H^+_E \).

A general property of the HCF (resp. of the extended HCF) is given in the following result.

**Remark 3.16.** When \( L/K \) is abelian, \( L_{gex,K} \) is the maximal abelian extension of \( K \) contained in \( H^+_L \).

We also have \( H^+_L/K = \frac{H^+_L}{L_{gex,K}} \).

When \( K = k = \mathbb{F}_q(T) \) and \( L/k \) is a finite separable extension we have \( L_{gex} = L_{gex,k} \).

**Remark 3.17.** We have that if \( L/E \) is a finite separable extension of global function fields, then \( E_{gex} \subseteq L_{gex} \), \( E_{gex} \subseteq L_{gex} \), \( H_E \subseteq H_L \) and \( H^+_E \subseteq H^+_L \). We also have \( K_{gex} = \left( \frac{K}{gex} \right)_{gex} \).

**Remark 3.18.** In [18] we defined the extended genus field \( K^{ext} \) for a finite abelian \( K/k \) as \( K^{ext} : = \frac{E_{gex}K}{E^{ext}K} = \frac{E_{gex}}{E^{ext}}K \), where \( E = K \cap k(\Lambda_N) \) is the corresponding cyclotomic function field and \( E_{gex} = E^{ext} \) is defined as the maximal cyclotomic extension containing \( E \) such that the finite primes are unramified (we will see that this agrees with Definition 3.15).
4 | EXTENDED GENUS FIELDS IN THE CYCLOTOMIC CASE

Let \( E \subseteq k(\Lambda_N) \) be a cyclotomic function field. We have \( \text{Gal}(k(\Lambda_N)/k) \cong (k^T/\langle N \rangle)^* \). Let \( X \) be the group of Dirichlet characters associated with \( E \), that is,

\[
X = \{ \chi : (RT/\langle N \rangle)^* \rightarrow \mathbb{C}^* | \chi \text{ is a group homomorphism such that } \text{Gal}(k(\Lambda_N)/E) \subseteq \ker \chi \}
\]

and \( E = k(\Lambda_N)^H \) is the fixed field under \( H \), where \( H = \bigcap_{\chi \in X} \ker \chi \). We have \( X \cong \hat{\text{Gal}}(E/k) = \text{Hom}(\text{Gal}(E/k), \mathbb{C}^*) \)

The aim of this section is to prove that \( E^{\text{ext}} = E^{\text{gex}} \). In fact, in [18] the extended genus field of \( E/k \) was defined as \( E^{\text{ext}} \) and it was denoted as \( E^{\text{gex}} \) in that paper.

To begin with, we first find the idèle class group associated with a cyclotomic function field \( k(\Lambda_N) \). Let \( N = P_1^{\alpha_1} \cdots P_r^{\alpha_r} \) be the decomposition of \( N \) as a product of irreducible monic polynomials. Set \( R'_T = RT \setminus \{ P_1, \ldots, P_r \} \), \( \pi = 1/T = \pi_{\infty} \) and \( U_{\infty} = U_{\mathfrak{p}_{\infty}} \).

We define

\[
\mathcal{X}_N = \prod_{i=1}^r U_{P_i}^{(\alpha_i)} \times \prod_{P \in R'_T} U_P \times (\langle \pi \rangle \times U_{\infty}^{(1)}) \tag{4.1}
\]

Note that \( \mathcal{X}_N k^*/k^* \cong \mathcal{X}_N \).

Let \( U_T := \{ \tilde{\alpha} \in J_k | \alpha_{\mathfrak{p}_{\infty}} = 1 \text{ and } \alpha_P \in U_P \text{ for all } P \in R'^+_T \} \). Hayes [10] proved that \( U_T \cong G_T = \text{Gal}(k_T/k) \) where \( k_T := \bigcup_{N \in RT} k(\Lambda_N) \).

**Proposition 4.1.** Let \( U' := \prod_{P \in RT} U_P \times (\langle \pi \rangle \times U_{\infty}^{(1)}) \subseteq J_k \). Then, there exists an epimorphism \( \psi_N : U' \twoheadrightarrow \text{Gal}(k(\Lambda_N)/k) =: G_N \) with kernel \( \ker \psi_N = \mathcal{X}_N \) so that, \( U'/\mathcal{X}_N \cong G_N \).

**Proof.** Let \( \tilde{\xi} \in U' \). Then, \( \tilde{\xi}_{P_i} \in U_{P_i} = \{ \sum_{j=0}^\infty a_j P_i^j | a_j \in RT/(P_i), a_0 \neq 0 \}, 1 \leq i \leq r \). Since \( K \preceq k_{P_i} \) is dense, there exists \( Q_i \in RT \) with \( Q_i \equiv \tilde{\xi}_{P_i} \mod P_i^{\alpha_i} \). From the Chinese residue theorem, there exists \( C \in RT \) such that \( C \equiv Q_i \mod P_i^{\alpha_i}, 1 \leq i \leq r \) and therefore \( C \equiv \xi_{P_i} \mod P_i^{\alpha_i}, 1 \leq i \leq r \).

Now, if \( C_1 \in RT \) satisfies \( C_1 \equiv \xi_{P_i} \mod P_i^{\alpha_i}, 1 \leq i \leq r \), then \( P_i^{\alpha_i} | C - C_1 \) for \( 1 \leq i \leq r \). It follows that \( N | C - C_1 \) so that \( C \in RT \) is unique modulo \( N \). On the other hand, \( v_{P_i}(\xi_{P_i}) = 0 \), so that \( P_i \nmid \xi_{P_i} \). It follows that \( \text{gcd}(C, N) = 1 \). In this way, we obtain that \( C \mod N \) defines an element of \( G_N \).

Given \( \sigma \in G_N \), there exists \( C \in RT \) such that \( \sigma \Lambda_N = \Lambda_N \), where \( \Lambda_N \) is a generator of \( \Lambda_N \). Let \( \tilde{\xi} \in U' \) with \( \xi_{P_i} = C, 1 \leq i \leq r \) and \( \xi_{\mathfrak{p}_{\infty}} = 1 = \xi_{\infty} \). Hence, \( \tilde{\xi} \mapsto C \mod N \) and \( \psi_N \) is surjective. Finally, \( \ker \psi_N = \{ \tilde{\xi} \in U' | \xi_{P_i} \equiv 1 \mod P_i^{\alpha_i}, 1 \leq i \leq r \} = \mathcal{X}_N \). The result follows.

Next, we will show that \( U'/\mathcal{X}_N \cong J_k/\mathcal{X}_N k^* \). We have the natural composition

\[
U'^C \xrightarrow{\underline{u}} J_k \xrightarrow{\mu} J_k/\mathcal{X}_N k^*.
\]

with \( \text{im } \mu = U'/\mathcal{X}_N k^*/\mathcal{X}_N k^* \) and \( \ker \mu = U' \cap \mathcal{X}_N k^* \).

Now, \( \mathcal{X}_N \subseteq U' \) which implies \( \mathcal{X}_N \subseteq U' \cap \mathcal{X}_N k^* \). Conversely, if \( \tilde{\xi} \in U' \cap \mathcal{X}_N k^* \), then the components of \( \tilde{\xi} \) are given by

\[
\xi_P = a \cdot \beta_P, \quad P \in RT, \quad \tilde{\beta} \in \mathcal{X}_N, \quad a \in k^*,
\]

\[
\xi_{\mathfrak{p}_{\infty}} = a \cdot \beta_{\mathfrak{p}_{\infty}}, \quad \beta_{\mathfrak{p}_{\infty}} \in \langle \pi \rangle \times U_{\infty}^{(1)}.
\]

Since \( \xi_P, \beta_P \in U_P \) we have that \( v_P(\xi_P) = v_P(\beta_P) = 0 \) for all \( P \in RT \). It follows that \( v_P(a) = 0 \) for all \( P \in RT \). Furthermore, since \( \deg a = 0 \), we have \( v_{\infty}(a) = 0 \) so that \( a \in F_q^* \).
Next, since $\xi, \beta_{\infty} \in \langle \pi \rangle \times U_{\infty}^{(1)} = \ker \phi_{\infty}$, it follows that $1 = \phi_{\infty}(\xi) = \phi_{\infty}(\alpha)\phi_{\infty}(\beta_{\infty}) = \phi_{\infty}(\alpha)$. Thus, $\alpha = 1$. It follows that $\xi \in \mathcal{X}_N$. Therefore, $\ker \mu = \mathcal{X}_N$ and we obtain a monomorphism $U'/\mathcal{X}_N \rightarrow J_k/\mathcal{X}_Nk^*$. It remains to verify that $\bar{\delta}$ is onto. We must prove that $J_k = \mathcal{X}_Nk^* = U'k^*$. We have that $U'$ corresponds to an abelian extension of $k$ unramified at every finite prime. Let $L/k$ be this extension. Now, $U_{\infty}^{(1)}$ corresponds to the first ramification group and therefore corresponds to the wild ramification of $p_{\infty}$. It follows that in $L/k$ there is at most a unique ramified prime and this prime is tamely ramified and of degree $1$ ($p_{\infty}$). It follows that $L/k$ is an extension of constants (see [19, Proposición 10.4.11]).

Finally, since $d = \min\{n \in \mathbb{N} | \deg \bar{\alpha} = n, \bar{\alpha} \in U'\} = 1$, the field of constants of $L$ is $\mathbb{F}_q$ and therefore $L = k$. It follows that $N_L/k = C_L \cong \mathbb{F}_q^*$ and consequently, the idèle class subgroup of $C_k$ corresponding to $k(\Lambda_N)$ is $\mathcal{X}_Nk^*/k^* \cong \mathcal{X}_N$.

**Corollary 4.3.** The idèle subgroup corresponding to the real cyclotomic function field $k(\Lambda_N)^+$ is $\mathcal{X}_N^+k^*$, where

$$\mathcal{X}_N^+ = \prod_{i=1}^{r} U_{\pi_i}^{(\alpha_i)} \times \prod_{\mathfrak{p} \in R'_{T}} U_{\mathfrak{p}} \times k_{\infty}^*.$$ 

**Proof.** If $L_1$ is the field associated with $\mathcal{X}_N^+k^*$, then following the previous argument we obtain that the field of constants of $L_1$ is $\mathbb{F}_q$ and we have

$$\frac{\mathcal{X}_N^+}{\mathcal{X}_N} \cong \frac{k_{\infty}^*}{(\pi) \times U_{\infty}^{(1)}} \cong \mathbb{F}_q^*,$$ 

which implies that $[L : L_1] = q - 1$. Furthermore, $p_{\infty}$ decomposes totally in $L_1$ and it has ramification index $q - 1$ in $L$. It follows that $[L : L_1] = q - 1$ and $L_1 = k(\Lambda_N)^+$. In fact, $\mathcal{X}_N^+ \cap k^* = \mathbb{F}_q^*$.

We return to our study of $E_{gex}$ for $E \subseteq k(\Lambda_N)$. First, we give a result needed for the main result.

**Proposition 4.4.** Let $E$ be a cyclotomic function field, $E \subseteq k(\Lambda_N)$. If $\mathfrak{p}$ is an infinite prime in $E$ then $1/T$ is a norm from $E_{\mathfrak{p}}$, that is, there exists $x \in E_{\mathfrak{p}}$ such that $N_{E_{\mathfrak{p}}/k_{\infty}} x = 1/T$.

**Proof.** Since $E$ is cyclotomic, we have that $E \subseteq k_{\infty}(\sqrt[4]{-1/T})$ ([3, Section 3] or [19, Proposición 9.3.20]). Hence, $E_{\mathfrak{p}} = k_{\infty}(\sqrt[4]{-1/T})$ with $e \mid q - 1$. The result follows. See also [1, Théorème 1.1.2 (ii)].

Let $\Gamma \subseteq J_E$ be the idèle subgroup of $E$ that corresponds to $H_E^+$. This group satisfies $N_{H_E^+/E} C_{H_E^+} = E^* \Gamma/E^*$. Then, by definition, we may take $\Gamma = U_E^+$. Observe also that since $E/k$ is abelian, $H_E^+ = E_{gex}$ over $k$ (see [1, Proposition 2.1.3] or [19, Proposición 17.6.48]). It follows that the idèle class subgroup of $C_k$ that corresponds to $H_E^+/k = E_{gex}$ over $k$ is precisely $N_E/k (U_E^+)k^*/k^* \cong N_{E/k} U_E^+$. We have $U_E^+ = \prod_{\mathfrak{p}_{\infty}} \ker \phi_{\mathfrak{p}} \times \prod_{\mathfrak{p} \neq \infty} U_{\mathfrak{p}}$, where we denote $\infty = \mathfrak{p}_{\infty}$.
If \( \mathfrak{p} \) is unramified, that is, \( \mathfrak{p} \notdivides \infty \) and \( \mathfrak{p} \nmid P_i, 1 \leq i \leq r \), where \( N = P_1^{\alpha_1} \cdots P_r^{\alpha_r} \), then we have \( N_{E_P/k}\mathfrak{p}(U_{\mathfrak{p}}) = U_{\mathfrak{p}} \), where \( \mathfrak{p} \cap k = P \in R_T^+ \) ([17, II, Corollary 4.4] or [19, Teorema 17.2.17]). If \( \mathfrak{p}\nmid P_i \) for some \( 1 \leq i \leq r \), we have \( [U_{\mathfrak{p}} : N_{E_{\mathfrak{p}}/k\mathfrak{p}}]_{U_{\mathfrak{p}}} = e_{E_{\mathfrak{p}}/k\mathfrak{p}}(p|P_i) = \Phi(P_i^{\alpha_i}) = q^{(\alpha_i - 1)d_i}(q - 1) \), where \( d_i = \deg P_i \). Now, because \( E \subseteq k(\Lambda_N) \), we obtain that \( E_{\mathfrak{g}E} \subseteq k(\Lambda_N) \) and the subgroup of \( J_E \) that corresponds to \( E_{\mathfrak{h}E} \) is \( U_{E}E^* \) and therefore the idèle class subgroup of the idèle class group \( C_k \) corresponding to \( E_{\mathfrak{g}E} \) is \( N_{E/k}(U_{E})k^*/k^* \). We have

\[
N_{E/k} U_E = \prod_{\mathfrak{p}|\infty} N_{E_{\mathfrak{p}}/k_\infty} \cdot E^* \times \prod_{P \in R_T} \prod_{\mathfrak{p}|P} N_{E_{\mathfrak{p}}/k_P} U_{\mathfrak{p}}.
\]

We also have

\[
N_{E/k} U_E^+ = \prod_{\mathfrak{p}|\infty} N_{E_{\mathfrak{p}}/k_\infty}(\ker \phi_{E_{\mathfrak{p}}}) \times \prod_{P \in R_T} \prod_{\mathfrak{p}|P} N_{E_{\mathfrak{p}}/k_P} U_{\mathfrak{p}}.
\]

If we write \( R'_T := R_T \setminus \{P_1, \ldots, P_r\} \), then

\[
\prod_{P \in R'_T} U_P \times \prod_{i=1}^r U_{P_i}^{(\alpha_i)} \subseteq \left( \prod_{P \in R_T} \prod_{\mathfrak{p}|P} N_{E_{\mathfrak{p}}/k_P} U_{\mathfrak{p}} \right)^{k^*}. 
\]

(4.2)

Let \( \mathfrak{p} \) be an infinite prime in \( E \). Note that since \( E \) is cyclotomic, we have \( E_{\mathfrak{p}} \subseteq k(\sqrt[\alpha]{-1/T}) \) and it follows that \( 1/T \) is a norm from \( E_{\mathfrak{p}} \) (Proposition 4.4. See also [1, Théorème 1.1.2]). Because \( E_{\mathfrak{p}}/k_\infty \) is totally ramified, we have \( N_{E_{\mathfrak{p}}/k_\infty} \geq (\pi_\infty) \times U_{\infty}^{(n_0)} \) for some \( n_0 \in \mathbb{N} \cup \{0\} \) ([17, II, Theorem 7.17] or [19, Teorema 17.5.47]).

Let \( E^* = (\pi_{E_{\mathfrak{p}}}) \times F_{\mathfrak{p}}^{(1)} \times U_{\mathfrak{p}}^{(1)} \) and \( \phi_{\mathfrak{p}} := \phi_{E_{\mathfrak{p}}} \circ N_{E_{\mathfrak{p}}/k_\infty} \). We have \( \ker \phi_{\mathfrak{p}} = N_{E_{\mathfrak{p}}/k_\infty}(\ker \phi_{\infty}) = N_{E_{\mathfrak{p}}/k_\infty}(\pi_{E_{\mathfrak{p}}}) \times U_{\infty}^{(1)} \).

Since \( k \subseteq E \subseteq k(\Lambda_N) \), it follows that \( k_\infty \subseteq E_{\mathfrak{p}} \subseteq k(\Lambda_N)_{\mathfrak{p}} = k(\sqrt[\alpha]{-1/T}) \) where \( \mathfrak{p} \) is a prime in \( k(\Lambda_N) \) such that \( \mathfrak{p}|\infty \) and \( \mathfrak{p} \cap E \). The field associated with \( \ker \phi_{\infty} \) is \( k(\sqrt[\alpha]{-1/T}) \) and the field associated with \( \ker \phi_{\mathfrak{p}} = N_{E_{\mathfrak{p}}/k_\infty}(\ker \phi_{\infty}) \) is \( E_{\mathfrak{p}}(\sqrt[\alpha]{-1/T}) = k(\sqrt[\alpha]{-1/T}) \) (Proposition 3.1 or [1, Corollaire 1.1.3]). It follows that \( N_{E_{\mathfrak{p}}/k_\infty}(\ker \phi_{\mathfrak{p}}) = k(\Lambda_N) \).

Therefore,

\[
N_{E/k} U_E^+ = \prod_{P \in R_T} \prod_{\mathfrak{p}|P} N_{E_{\mathfrak{p}}/k_\infty} U_P \times \prod_{\mathfrak{p}|\infty} N_{E_{\mathfrak{p}}/k_\infty}(\ker \phi_{\mathfrak{p}}) = \prod_{P \in R_T} \prod_{\mathfrak{p}|P} N_{E_{\mathfrak{p}}/k_P} U_{\mathfrak{p}} \times (\ker \phi_{\infty}).
\]

From Equation (4.2), we obtain that

\[
\prod_{P \in R_T} U_P \times \prod_{i=1}^r U_{P_i}^{(\alpha_i)} \times (\pi_\infty) \times U_{\infty}^{(1)} \subseteq \left( \prod_{P \in R_T} \prod_{\mathfrak{p}|P} N_{E_{\mathfrak{p}}/k_P} U_{\mathfrak{p}} \times \prod_{\mathfrak{p}|\infty} N_{E_{\mathfrak{p}}/k_\infty}(\ker \phi_{\mathfrak{p}}) \right)^{k^*}.
\]

Hence, \( \mathfrak{N} k^* \subseteq (N_{E/k} U_E^+)^{k^*} \). Therefore, \( E_{\mathfrak{g}E} \subseteq k(\Lambda_N) \).

Now, \( E_{\mathfrak{g}E} \) is unramified at the finite primes and if \( L \) is the field associated to \( Y = \prod_{P \in R_T} X_P \), \( L \) is the maximal abelian extension of \( E \) contained in \( k(\Lambda_N) \) unramified at the finite primes. Therefore, \( E_{\mathfrak{g}E} \subseteq L \).

To show \( L \subseteq E_{\mathfrak{g}E} \), it suffices to prove that \( L \subseteq H_E^+ \) since \( L/k \) is abelian and \( E_{\mathfrak{g}E} \subseteq H_E^+ \) is the maximal abelian extension of \( k \) contained in \( H_E^+ \).

Now, to show that \( L \subseteq H_E^+ \), we must prove that \( N_{L/E} C_L \supseteq N_{H_E^+/E} C_{H_E^+} = U_E^+ E^*/E^* \), where \( U_E^+ = \prod_{\mathfrak{p}|\infty} \ker \phi_{\mathfrak{p}} \times \prod_{\mathfrak{p}|\infty} U_{\mathfrak{p}} \).

Let \( N_{L/E} C_L = \Lambda E^*/E^* \) where \( \Lambda = N_{L/E} J_L \). It suffices to prove that \( U_E^+ \subseteq \Lambda \). Since \( L/E \) is unramified at the finite primes, if \( \mathfrak{p} \) is a finite prime of \( E \) and \( \mathfrak{p} \) is a prime of \( L \) over \( \mathfrak{p} \), then \( N_{L_\mathfrak{p}/E_{\mathfrak{p}}} U_{\mathfrak{p}} = U_{\mathfrak{p}} \), where we denote \( U_{\mathfrak{p}} = U_{L_\mathfrak{p}} \) and \( U_{\mathfrak{p}} = U_{E_{\mathfrak{p}}} \) ([17, II, Corollary 4.4] or [19, Teorema 17.2.17]). In particular, \( N_{L_\mathfrak{p}/E_{\mathfrak{p}}} U_{\mathfrak{p}} \supseteq U_{\mathfrak{p}} \) and \( N_{L/E} J_L \supseteq \prod_{\mathfrak{p}|\infty} U_{\mathfrak{p}} \).

On the other hand, \( L/E_{\mathfrak{g}E} \) is totally ramified at the infinite primes and the infinite primes of \( E \) decompose fully in \( E_{\mathfrak{g}E} \), so that \( (E_{\mathfrak{g}E})_\mathfrak{p} = E_{\mathfrak{p}} \) for \( \mathfrak{p}|\infty \) and the uniformizer of \( \mathfrak{p} \) in \( E \) is also a uniformizer for \( \mathfrak{p} \) in \( E_{\mathfrak{g}E} \).
From the local class field theory, ([17, II, Theorem 7.17] or [19, Teorema 17.5.47]), \(\pi_{E,\infty} := \pi_{E_p}\) is a norm from \(L\) with \(\mathfrak{p}\). If \((L_{\mathfrak{q}}/E_p)\) represents the Artin local map, then \((U_{E_p}^{(1)}, L_{\mathfrak{q}}/E_p) = G^1(L\mathfrak{q}/E_p)\), the first ramification group of \(L\mathfrak{q}/E_p\). Since the infinite primes are tamely ramified in \(L\mathfrak{q}/E_p\), it follows that \(G^1(L\mathfrak{q}/E_p) = \{1\}\) and that \(U_{E_p}^{(1)} \subseteq N_{L\mathfrak{q}/E_p} L_{\mathfrak{q}}^\ast\).

The field of constants of \(E\) and of \(k\) is \(\mathbb{F}_q\) and every infinite prime has degree 1 in \(E\), so that if \(\mathfrak{p}\) is an infinite prime of \(E\), \(E^\ast_{\mathfrak{p}} = \langle \pi_{E,\infty} \rangle \times \mathbb{F}_q^\ast \times U(1)_{E_p}\). Note that \([E_{\mathfrak{p}} : k_{\infty}] = e f = e = [k_{\infty}^\ast : N_{E_{\mathfrak{p}}/k_{\infty}} (E_{\mathfrak{p}}^\ast)]\) and \(N_{E_{\mathfrak{p}}/k_{\infty}} (E_{\mathfrak{p}}^\ast) = (\mathbb{F}_q^\ast)^e\). It follows that \([k_{\infty}^\ast : N_{E_{\mathfrak{p}}/k_{\infty}} (E_{\mathfrak{p}}^\ast)] = e\) and that \(k_{\infty}^\ast \subseteq N_{L_{\mathfrak{q}}/E_p} L^\ast_{\mathfrak{q}}\).

Therefore, \(\phi_p(E_{\mathfrak{p}}^\ast) = \phi_\infty(N_{E_{\mathfrak{p}}/k_{\infty}} E_{\mathfrak{p}}^\ast) = \phi_\infty \left( [\pi_{E,\infty}] \times (\mathbb{F}_q^\ast)^e \times U(1)_{E_p}^{(1)} \right) = (\mathbb{F}_q^\ast)^e\). It follows that \(\ker P_p = [\pi_{E,\infty}] \times R \times U(1)_{E_p}\), where \(R = \{\lambda \in \mathbb{F}_q^\ast : \lambda ^{e} = 1\} = (\mathbb{F}_q^\ast)^{(q-1)/e}\). Here, \(e\) denotes the ramification index of \(\mathfrak{p}\) in \(E/k\).

We note that if \(\mathfrak{p}\) is an infinite prime of \(L, N_{L\mathfrak{q}/E_p} F_{\mathfrak{q}}^\ast = (\mathbb{F}_q^\ast)^r\) where \(e' := e_{\infty}(L/E_{\mathfrak{q}}) = e_{\infty}(L/E)\). Then, \(e'e/q - 1\) and \((q - 1)/e\). Let \(F_{\mathfrak{q}}^\ast = \langle \beta \rangle\) with \(o(\beta) = q - 1\). Let \(\lambda \in R\), \(\lambda ^{e'} = 1\). We have \(\lambda = \beta^s\) for some \(s\). Therefore, \(\lambda ^{e'} = \beta ^{se'} = 1\) and \((q - 1)|es\). Since \(e'|q - 1\) it follows that \(e'|se\) and \(e'\)|\(s\). In this way, we obtain that \(\lambda = \beta^s = (\beta^{e'/e'})^{e'} \subseteq N_{L\mathfrak{q}/E_p} L^\ast_{\mathfrak{q}}\).

Hence, \(\ker P_p \subseteq N_{L\mathfrak{q}/E_p} L^\ast_{\mathfrak{q}}\) and \(U_{E} = \prod_{p|\infty} \ker P_p \times \prod_{p|\infty} U_p \subseteq N_{L/k} J_L\). We then obtain that \(L \subseteq H^+_E\) and therefore \(L = E_{ge}\).

We have proved.

Theorem 4.5. Let \(E \subseteq k(\Lambda_N)\). Then, the extended genus field \(E_{ge}\) relative to \(k\) is the field associated with the group of Dirichlet characters \(Y = \prod_{p \in \text{Ker}^T} X_p\), where \(X\) is the group of Dirichlet characters associated with the field \(E\).

5 \hspace{1cm} FINITE ABELIAN EXTENSIONS

Consider \(K/k\) a finite abelian extension. Let \(n \in \mathbb{N} \cup \{0\}, m \in \mathbb{N}\) and \(N \in \mathbb{R}_T\) such that \(K \subseteq k(\Lambda_N)^m\). Let \(E := KM \cap k(\Lambda_N)^m\). We have that \(EK/K\) is an extension of constants, in particular an unramified extension (see [2]). Let \(H\) be the decomposition group of the infinite primes of \(K\) in \(EK/K\). We have \(H\) is canonically isomorphic with the decomposition group of the infinite primes of \(E_{ge}K/K\). We have that \(|H|\) equals the inertia degree of the infinite primes of \(K\) in \(EK/K\) and in \(E_{ge}K/K\). We also have that \(K_{ge} = E_{ge}^H.K\).

Theorem 5.1. With the above notations, we have that \(K_{ge} = DK\) with \((E_{ge}^H)_{ge} \subseteq D \subseteq E_{ge}\). In particular, when \(H = \{1\}\), we have \(K_{ge} = E_{ge}K\).

Proof. We know that \(E_{ge} = K_{ge}M\) (see [2, p. 2111]). Set \(C := K_{ge}M \cap k(\Lambda_N) \supseteq K_{ge}M \cap k(\Lambda_N) = E_{ge}M \cap k(\Lambda_N) \supseteq E_{ge} \cap k(\Lambda_N) = E_{ge}\).

Note that \(K_{ge}M/K_{ge}M\) is unramified at the finite primes because it is so in \(K_{ge}/K_{ge}\). We also have that \(E_{ge}M/E_{ge}\) is unramified at the finite primes. In particular, \(C/E_{ge}\) is unramified at the finite primes and \(C \subseteq k(\Lambda_N)\). Since \(E_{ge}/k\) is the maximal cyclotomic extension with \(E_{ge}/E\) unramified at the finite primes, it follows that \(C \subseteq E_{ge}\).
Therefore, \(\tilde{E}_{ge} \subseteq C \subseteq \tilde{E}_{gex}\) and \(\| \subseteq \| \subseteq \| \subseteq \). In particular, \(K_{gex}M \subseteq E_{gex}M\).

In the particular case that \(H = \{1\}\), we have that \(E \subseteq K_{ge} \subseteq K_{gex}\) so that \(E_{gex} \subseteq K_{gex}\) and \(K_{gex} = E_{gex}K\).

In the general case, \(E_{gex}M \subseteq D \subseteq E_{gex}\) and \(E_{gex}/E_{ge}M\) is totally ramified at the infinite primes. Since \(E_{gex}M \subseteq K_{gex}\), it follows that \((E_{gex})_{gex} \subseteq K_{gex}\) so that \(E_{gex} = E_{gex}K\).

Example 5.2. Let \(K := k(\sqrt[2]{\gamma D})\) where \(l\) is a prime number such that \(l|q - 1\), \(D \in R_T\), \(P_i \in R^+_T\) and \(1 \leq \alpha_i \leq l^2 - 1\), \(1 \leq i \leq r\). We assume that \(degD = l\delta_1\) with \(l \nmid \delta_1\), that \(l \nmid degP_i\), that \(gcd(\alpha_i, l) = 1\), \(1 \leq i \leq s\), and that \(l|\alpha_j\), \(s + 1 \leq j \leq r\). Therefore, \(e_{P_i}(K/k) = l^2\) for \(1 \leq i \leq s\), \(e_{P_i}(K/k) = l\) for \(s + 1 \leq j \leq r\), and

\[e_{P_i}(K/k) = \begin{cases} l^2 & \text{if } 1 \leq i \leq s, \\ l & \text{if } s + 1 \leq j \leq r. \end{cases}\]

Let \(D := K_{gex} \cap k(\Lambda_N) = \tilde{E}_{ge}K \cap k(\Lambda_N) \supseteq \tilde{E}_{ge}K = \tilde{E}_{ge}K\). Thus, \(\tilde{E}_{ge} \subseteq D\) and \(K_{gex} = DK\).

We have that \(EM/E\) is unramified at the finite primes. Therefore, \(DEM = DKM/EM = KM\) is unramified at the finite primes since \(K_{gex}/K\) is so. It follows that \(DE/E\) is unramified at the finite primes so that \(D \subseteq \tilde{E}_{gex}\). Hence, \(K_{gex} = DK \subseteq E_{gex}K\).

In the particular case that \(H = \{1\}\), we have that \(E \subseteq K_{ge} \subseteq K_{gex}\) so that \(E_{gex} \subseteq K_{gex}\) and \(K_{gex} = E_{gex}K\).

In the general case, \(E_{gex}M \subseteq D \subseteq E_{gex}\) and \(E_{gex}/E_{ge}M\) is totally ramified at the infinite primes. Since \(E_{gex}M \subseteq K_{gex}\), it follows that \((E_{gex})_{gex} \subseteq K_{gex}\). Hence, \((E_{gex})_{gex} \subseteq D\).

Example 5.3. Let \(l\) be a prime number such that \(l^2|q - 1\). Let \(K = k(\sqrt[2]{\gamma D})\) where \(\gamma \in F_q^*\), \(D = P_1^{\alpha_1} \cdots P_r^{\alpha_r} \in R_T\), \(P_1, \ldots, P_r \in R^+_T\), \(1 \leq \alpha_i \leq l^2 - 1\), \(1 \leq i \leq r\), \(r \geq 2\). We assume that \(degD = ld\) with \(l \nmid d\), that \(l \nmid \alpha_i\), that \(gcd(\alpha_i, l) = 1\), \(1 \leq i \leq s\), \(s \geq 1\) and that \(l|\alpha_j\), \(s + 1 \leq j \leq r\). Then, \(e_{P_i}(K/k) = l^2\) for \(1 \leq i \leq s\), \(e_{P_i}(K/k) = l\) for \(s + 1 \leq j \leq r\), and

\[e_{P_i}(K/k) = \begin{cases} l^2 & \text{if } 1 \leq i \leq s, \\ l & \text{if } s + 1 \leq j \leq r. \end{cases}\]
$e_{\infty}(K/k) = l$. Since $e_P(K/k) = l^2$, the field of constants of $K$ is $F_q$. We also assume $(-1)^{d_D} \not\in (F_q)^l$. Therefore, $F_q(\sqrt[l]{\varepsilon}) = F_q^{l^2}$ where $\varepsilon = (-1)^{d_D}$. 

Then, $M = F_q^{l^2}(T) = k^{l^2}$ and $E = KM \cap k(\Lambda_D)$. Thus, $E = k(\sqrt[l]{(-1)^{d_D} D}) = k(\sqrt[l]{D^r}) \subseteq k(\Lambda_D)$. We have 

$$E K = E(\sqrt[l]{\varepsilon}) = K(\sqrt[l]{\varepsilon}) = E^{l^2} = K^{l^2}.$$ 

We have $f_{\infty}(K/k) = [F_q(\sqrt[l]{\varepsilon}) : F_q] = l$ and therefore $f_{\infty}(E K/k) = \frac{f_{\infty}(E E/k)}{f_{\infty}(K/k)} = \frac{l^2}{l} = l$. Thus, $H \cong C_l$ and $|H| = l$.

We also have $E_{ge} = k\left(\sqrt[l]{P_1^*, \ldots, \sqrt[l]{Q_s^*}, \sqrt[l]{Q_s+1^*}, \ldots, \sqrt[l]{Q_r^*}}\right)$. Since $l \nmid \deg P_1$, it follows that $e_{\infty}\left(k\left(\sqrt[l]{P_1^*}\right) / k\right) = l^2$ and therefore $e_{\infty}(E_{ge}/k) = l^2$. Since $\deg D = ld$ with $l \nmid d$, we have $e_{\infty}(E_{ge}/k) = l$. Now 

$$\deg D = ld = \sum_{i=1}^{r} \alpha_i \deg P_i = \alpha_i \deg P_1 + \sum_{i=2}^{s} \alpha_i \deg P_i + \sum_{j=s+1}^{r} \alpha_j \deg P_j,$$

and since $l \mid \sum_{j=s+1}^{r} \alpha_j \deg P_j$ and $\gcd(\alpha_i, l) = 1$ for $1 \leq i \leq s$, it follows that there exists $2 \leq i \leq s$ with $l \nmid \deg P_i$. Say $l \nmid \deg P_2$.

Let $a, b \in \mathbb{Z}$ with $a \deg P_1 + b l^2 = 1$ (in particular, $\gcd(a, l) = 1$). Therefore, we have for $i \geq 2$ that $\deg P_i - (a \deg P_i) \deg P_1 = b(\deg P_i) l^2$. Set $Q_i = P_i P_2^{z_i}$ with $z_i := -a \deg P_2$ for $2 \leq i \leq r$. Let $L := k\left(\sqrt[l]{P_1^*, \sqrt[l]{Q_2^*}, \ldots, \sqrt[l]{Q_s^*}, \sqrt[l]{Q_s+1^*}, \ldots, \sqrt[l]{Q_r^*}}\right)$. Then, $e_{\infty}(L/k) = l$, $e_{P_1}(L/k) = l^2$ for $2 \leq i \leq s$ and $e_{P_i}(L/k) = l$ for $s + 1 \leq j \leq r$. For $P_1$, we have that since $l \nmid \deg P_2$, $\sqrt[l]{Q_2^*} = \sqrt[l]{P_2^* P_1^{1-a \deg P_2}}$ and $\gcd(l, -a \deg P_2) = 1$ so that $e_{P_1}(k(\sqrt[l]{P_2^*})/k) = l^2$. Hence, $e_{P_1}(L/k) = l^2$. Thus, $E \subseteq L$ and $[E_{ge}: L] = l$, it follows that 

$$L = E_{ge} = k\left(\sqrt[l]{P_1^*, \sqrt[l]{Q_2^*}, \ldots, \sqrt[l]{Q_s^*}, \sqrt[l]{Q_s+1^*}, \ldots, \sqrt[l]{Q_r^*}}\right),$$

and 

$$E_{ge} = k\left(\sqrt[l]{Q_2^*, \ldots, \sqrt[l]{Q_s^*}, \sqrt[l]{Q_s+1^*}, \ldots, \sqrt[l]{Q_r^*}}\right),$$

$$[E_{ge} : E_{ge}] = l = e_{\infty}(E_{ge}/E_{ge}).$$

Now, $E K = K^{l^2}$, so that $H \cong D_{\infty}(EK/K) = \text{Gal}(K^{l^2}/K_l)$, where $D_{\infty}$ denotes the decomposition group of the infinite primes.

$$E K = K_{l^2}$$

We have 

$$E^H = k(\sqrt[l]{D^r}) = k(\sqrt[l]{D}) \quad \text{and} \quad E_{ge}^H = k\left(\sqrt[l]{Q_2^*, \ldots, \sqrt[l]{Q_s^*}, \sqrt[l]{Q_s+1^*}, \ldots, \sqrt[l]{Q_r^*}}\right).$$

We also obtain that $(E_{ge}^H)^{ge} = E_{ge}$ since $[(E_{ge}^H)^{ge} : k] = \prod_{j=1}^{r} e_{P_j}(E_{ge}^H/k) = [E_{ge} : k]$ and $E_{ge}^H \subseteq E_{ge}$. It follows that 

$$K_{ge} = E_{ge} K = k\left(\sqrt[l]{P_1^*, \ldots, \sqrt[l]{Q_s^*}, \sqrt[l]{Q_s+1^*}, \ldots, \sqrt[l]{Q_r^*} / \sqrt[l]{D^r}}\right) = E_{ge}(\sqrt[l]{\varepsilon}).$$
and

$$K_{ge} = E_{ge}^{H}K = k\left(\sqrt[4]{Q_2}, \ldots, \sqrt[4]{Q_s}, \sqrt[4]{\gamma D}\right).$$

Remark 5.4.

(a) When $H = \{1\}$, $K_{gex} = K_{gext} = E_{gex}K$.

(b) In general, we have $f_{\infty}(K_{gex}|K_{ge}) > 1$ (Example 5.3).

(c) The field of constants of $H_K^+$ might be different from the one of $H_K$ (Example 5.3). In that example, $F_{q^4}$ is the field of constants of $H_K$ and $F_{q^2}$ is the field of constants of $H_K^+$.

(d) In general, the extension $K_{gex}/K_{ge}$ might contain ramified extensions. In Example 5.3, $K_{gex} = K_{ge}(\sqrt[4]{P_1})$ and $K_{gex} \subset (K_{ge})^l = K_{ge}(\sqrt[4]{P_1}) \subset K_{gex}$. We have $e_{\infty}(K_{gex}|K_{ge}) = f_{\infty}(K_{gex}|K_{ge}) = l$.

For the description of $K_{gex}$ for a finite separable extension $K/k$, at least in most cases, see [18].

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