MASS AND EXTREMALS ASSOCIATED WITH THE HARDY–SCHRÖDINGER OPERATOR ON HYPERBOLIC SPACE

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Abstract. We consider the Hardy–Schrödinger operator

\[ L_\gamma = -\Delta - \gamma V(x) \]

on the Poincaré ball model of the Hyperbolic space \( B^n \) \((n \geq 3)\). Here \( V(x) \) is a well chosen radially symmetric potential, which behaves like the Hardy potential around its singularity at 0, i.e., \( V(0) \sim \frac{1}{r^2} \). Just like in the Euclidean setting, the operator \( L_\gamma \) is positive definite whenever \( \gamma < \frac{(n-2)^2}{4} \), in which case we exhibit explicit solutions for the Sobolev critical equation

\[ L_\gamma u = V(0) u^{2^*(s)-1} \]

in \( B^n \), where \( 0 \leq s < 2 \), \( 2^*(s) = \frac{2(n-s)}{n-2} \), and \( V(0) \) is a weight that behaves like \( \frac{1}{r^s} \) around 0. In dimensions \( n \geq 5 \), the above equation in a domain \( \Omega \) of \( B^n \) containing 0 and away from the boundary, has a ground state solution, whenever \( 0 < \gamma \leq \frac{(n-4)}{4} \), and provided \( L_\gamma \) is replaced by a linear perturbation \( L_\gamma - \lambda u \), where \( \lambda > \frac{n-2}{n-4} \left( \frac{n(n-4)}{4} - \gamma \right) \). On the other hand, in dimensions 3 and 4, the existence of solutions depends on whether the domain has a positive “hyperbolic mass,” a notion that we introduce and analyze therein.

1. Introduction

Hardy–Schrödinger operators on manifolds are of the form \( \Delta - V \), where \( \Delta \) is the Laplace–Beltrami operator and \( V \) is a potential that has a quadratic singularity at some point of the manifold. For hyperbolic spaces, Carron [8] showed that, just like in the Euclidean case and with the same best constant, the following inequality holds on any Cartan–Hadamard manifold \( M \),

\[ \frac{(n-2)^2}{4} \int_M \frac{u^2}{d_g(o,x)^2} \, dv_g \leq \int_M |\nabla_g u|^2 \, dv_g \]

for all \( u \in C^\infty_c(M) \),

where \( d_g(o,x) \) denotes the geodesic distance to a fixed point \( o \in M \). There are many other works identifying suitable Hardy potentials, their relationship with the elliptic operator on hand, as well as corresponding energy inequalities [2, 3, 9, 13, 15, 16, 20]. In the Euclidean case, the Hardy potential \( V(x) = \frac{1}{|x|^2} \) is distinguished by the fact that \( \frac{u^2}{|x|^2} \) has the same homogeneity as \( |\nabla u|^2 \), but also \( \frac{u^{2^*(s)}}{|x|^s} \), where \( 2^*(s) = \frac{2(n-s)}{n-2} \)

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and \(0 \leq s < 2\). In other words, the integrals \(\int_{\mathbb{R}^n} \frac{u^2}{|x|^2} \, dx\), \(\int_{\mathbb{R}^n} |\nabla u|^2 \, dx\) and \(\int_{\mathbb{R}^n} \frac{u^2(s)}{|x|^s} \, dx\) are invariant under the scaling \(u(x) \rightarrow \lambda^{\frac{n-2}{2}} u(\lambda x)\), \(\lambda > 0\), which makes corresponding minimization problem non-compact, hence giving rise to interesting concentration phenomena. In [1], Adimurthi and Sekar use the fundamental solution of a general second order elliptic operator to generate natural candidates and derive Hardy-type inequalities. They also extended their arguments to Riemannian manifolds using the fundamental solution of the \(p\)-Laplacian. In [9], Devyver, Fraas and Pinchover study the case of a general linear second order differential operator \(P\) on non-compact manifolds. They find a relation between positive super-solutions of the equation \(Pu = 0\), Hardy-type inequalities involving \(P\) and a weight \(W\), as well as some properties of the spectrum of a corresponding weighted operator.

In this paper, we shall focus on the Poincaré ball model of the hyperbolic space \(\mathbb{B}^n\), \(n \geq 3\), that is the Euclidean unit ball \(B_1(0) := \{x \in \mathbb{R}^n : |x| < 1\}\) endowed with the metric \(g_{\mathbb{B}^n} = \left(\frac{2}{1 + \rho^2}\right)^2 g_{\text{Eucl}}\). This framework has the added feature of radial symmetry, which plays an important role and contributes to the richness of the structure. In this direction, Sandeep and Tintarev [18] recently came up with several integral inequalities involving weights on \(\mathbb{B}^n\) that are invariant under scaling, once restricted to the class of radial functions (see also Li and Wang [15]). As described below, this scaling is given in terms of the fundamental solution of the hyperbolic Laplacian \(\Delta_{\mathbb{B}^n}\). Indeed, let

\[
f(r) := \frac{(1 - r^2)^{n-2}}{r^{n-1}} \quad \text{and} \quad G(r) := \int_r^1 f(t) \, dt,
\]

where \(r = \sqrt{\sum_{i=1}^n x_i^2}\) denotes the Euclidean distance of a point \(x \in B_1(0)\) to the origin. It is known that \(\frac{1}{n^{n-1}} G(r)\) is a fundamental solution of the hyperbolic Laplacian \(\Delta_{\mathbb{B}^n}\). As usual, the Sobolev space \(H^1(\mathbb{B}^n)\) is defined as the completion of \(C^\infty_c(\mathbb{B}^n)\) with respect to the norm \(\|u\|_{H^1(\mathbb{B}^n)}^2 = \int_{\mathbb{B}^n} |\nabla_{\mathbb{B}^n} u|^2 \, dv_{g_{\mathbb{B}^n}}\). We denote by \(H^1_+(\mathbb{B}^n)\) the subspace of radially symmetric functions. For functions \(u \in H^1_+(\mathbb{B}^n)\), we consider the scaling

\[
u_\lambda(r) = \lambda^{-\frac{1}{2}} u \left(G^{-1}(\lambda G(r))\right), \quad \lambda > 0.
\]

In [18], Sandeep–Tintarev have noted that for any \(u \in H^1_+(\mathbb{B}^n)\) and \(p \geq 1\), one has the following invariance property:

\[
\int_{\mathbb{B}^n} |\nabla_{\mathbb{B}^n} u_\lambda|^2 \, dv_{g_{\mathbb{B}^n}} = \int_{\mathbb{B}^n} |\nabla_{\mathbb{B}^n} u|^2 \, dv_{g_{\mathbb{B}^n}} \quad \text{and} \quad \int_{\mathbb{B}^n} V_p |u_\lambda|^p \, dv_{g_{\mathbb{B}^n}} = \int_{\mathbb{B}^n} V_p |u|^p \, dv_{g_{\mathbb{B}^n}},
\]

where

\[
V_p(r) := \frac{f(r)^2(1 - r^2)^2}{4(n - 2)^2 G(r)^{\frac{p}{p-2}}}.
\]

In other words, the hyperbolic scaling \(r \rightarrow G^{-1}(\lambda G(r))\) is quite analogous to the Euclidean scaling. Indeed, in that case, by taking \(G(\rho) = \rho^{2-n}\), we see that \(G^{-1}(\lambda G(\rho)) = \tilde{\lambda} = \lambda^{\frac{1}{2-n}}\) for \(\rho = |x|\) in \(\mathbb{R}^n\). Also, note that \(\tilde{G}\) is – up to a constant – the fundamental solution of the Euclidean Laplacian \(\Delta\) in \(\mathbb{R}^n\). The weights \(V_p\) have the following asymptotic behaviors, for \(n \geq 3\) and \(p > 1\).

\[
V_p(r) = \begin{cases} 
\frac{c_0(n, p)}{r^{n(1-p/2)}} (1 + o(1)) & \text{as } r \to 0, \\
\frac{c_1(n, p)}{(1 - r)^{(n-1)(p-2)/2}} (1 + o(1)) & \text{as } r \to 1.
\end{cases}
\]

In particular for \(n \geq 3\), the weight \(V_2(r) = \frac{1}{4(n-2)^2} \left(\frac{f(r)(1-r^2)}{G(r)}\right)^2 \sim_{r \to 0} \frac{1}{1 + r^2}\), and at \(r = 1\) has a finite positive value. In other words, the weight \(V_2\) is qualitatively similar to the Euclidean Hardy weight, and Sandeep–Tintarev have indeed established the following Hardy inequality on the hyperbolic space \(\mathbb{B}^n\) (Theorem 3.4 of...
Also, see [9] where they deal with similar Hardy weights.
\[
\frac{(n-2)^2}{4} \int_{\mathbb{B}^n} V_2 |u|^2 dv_{g_{\mathbb{B}^n}} \leq \int_{\mathbb{B}^n} |\nabla_{\mathbb{B}^n} u|^2 dv_{g_{\mathbb{B}^n}} \quad \text{for any } u \in H^1(\mathbb{B}^n).
\]

They also show in the same paper the following Sobolev inequality, i.e., for some constant \( C > 0 \),
\[
\left( \int_{\mathbb{B}^n} V_2 |u|^{2^*} dv_{g_{\mathbb{B}^n}} \right)^{2/2^*} \leq C \int_{\mathbb{B}^n} |\nabla_{\mathbb{B}^n} u|^2 dv_{g_{\mathbb{B}^n}} \quad \text{for any } u \in H^1(\mathbb{B}^n),
\]
where \( 2^* = \frac{2n}{(n-2)} \). By interpolating between these two inequalities taking \( 0 \leq s \leq 2 \), one easily obtain the following Hardy–Sobolev inequality.

**Lemma 1.1.** If \( \gamma < \frac{(n-2)^2}{4} \), then there exists a constant \( C > 0 \) such that, for any \( u \in H^1(\mathbb{B}^n) \),
\[
C \left( \int_{\mathbb{B}^n} V_{2^*} |u|^{2^*(s)} dv_{g_{\mathbb{B}^n}} \right)^{2/2^*(s)} \leq \int_{\mathbb{B}^n} |\nabla_{\mathbb{B}^n} u|^2 dv_{g_{\mathbb{B}^n}} - \gamma \int_{\mathbb{B}^n} V_2 |u|^2 dv_{g_{\mathbb{B}^n}},
\]
where \( 2^*(s) := \frac{2(n-s)}{(n-2)} \).

Note that, up to a positive constant, we have \( V_{2^*} \sim_r \frac{1}{r^2} \), adding to the analogy with the Euclidean case, where we have for any \( u \in H^1(\mathbb{R}^n) \),
\[
C \left( \int_{\mathbb{R}^n} \frac{|u|^{2^*(s)}(s)}{|x|^s} dx \right)^{2/2^*(s)} \leq \int_{\mathbb{R}^n} |\nabla u|^2 dx - \gamma \int_{\mathbb{R}^n} \frac{|u|^2}{|x|^2} dx.
\]

Motivated by the recent progress on the Euclidean Hardy–Schrödinger equation (See for example Ghoussoub–Robert [12, 11], and the references therein), we shall consider the problem of existence of extremals for the corresponding best constant, that is
\[
\mu_{\gamma, \lambda}(\Omega) := \inf_{u \in H^1_0(\Omega) \setminus \{0\}} \left\{ \int_{\Omega} |\nabla_{\Omega} u|^2 dv_{g_{\Omega}} - \gamma \int_{\Omega} V_2 |u|^2 dv_{g_{\Omega}} - \lambda \int_{\Omega} |u|^2 dv_{g_{\Omega}} \right\}^{2/2^*(s)},
\]
where \( H^1_0(\Omega) \) is the completion of \( C^\infty_c(\Omega) \) with respect to the norm \( \|u\| = \sqrt{\int_{\Omega} |\nabla u|^2 dv_{g_{\Omega}}} \). Similarly to the Euclidean case, and once restricted to radial functions, the general Hardy–Sobolev inequality for the hyperbolic Hardy–Schrödinger operator is invariant under hyperbolic scaling described in (1.2). This invariance makes the corresponding variational problem non-compact and the problem of existence of minimizers quite interesting.

In Proposition 3.1, we start by showing that the extremals for the minimization problem (1.4) in the class of radial functions \( H^1_0(\mathbb{B}^n) \) can be written explicitly as:
\[
U(r) = c \left( G(r)^{-\frac{2-\alpha_-(\gamma)}{4}} + G(r)^{-\frac{2-\alpha_+(\gamma)}{4}} \right)^{-\frac{n-2}{4}},
\]
where \( c \) is a positive constant and \( \alpha_{\pm}(\gamma) \) satisfy
\[
\alpha_{\pm}(\gamma) = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{\gamma}{(n-2)^2}}.
\]
In other words, we show that

$$
\mu_{\gamma,0}^{\text{rad}}(B^n) := \inf_{u \in H^1_r(B^n) \setminus \{0\}} \frac{\int_{B^n} |\nabla B^n u|^2 \, dV_{B^n} - \gamma \int_{B^n} V_2 |u|^2 \, dV_{B^n}}{\left( \int_{B^n} V^{2^*(s)}_2 |u|^{2^*(s)} \, dV_{B^n} \right)^{2/2^*(s)}}
$$

(1.5)

is attained by \( U \).

Note that the radial function \( G^\alpha(r) \) is a solution of \(-\Delta B^n u - \gamma V^2 u = 0\) on \( B^n \setminus \{0\} \) if and only if \( \alpha = \alpha_{\pm}(\gamma) \).

These solutions have the following asymptotic behavior

$$
G(r)_{\alpha_{\pm}(\gamma)} \sim c(n, \gamma) r^{-\beta_{\pm}(\gamma)} \text{ as } r \to 0,
$$

where

$$
\beta_{\pm}(\gamma) = \frac{n - 2}{2} \pm \sqrt{\frac{(n - 2)^2}{4} - \gamma}.
$$

These then yield positive solutions to the equation

$$
-\Delta B^n u - \gamma V^2 u = V^{2^*(s)}_2 u^{2^*(s)-1} \text{ in } B^n.
$$

We point out the paper [14] (also see [5, 6, 10]), where the authors considered the counterpart of the Brezis–Nirenberg problem on \( B^n (n \geq 3) \), and discuss issues of existence and non-existence for the equation

$$
-\Delta u - \lambda u = u^{2^*-1} \text{ in } B^n,
$$

in the absence of a Hardy potential.

Next, we consider the attainability of \( \mu_{\gamma,\lambda}(\Omega) \) in subdomains of \( B^n \) without necessarily any symmetry. In other words, we will search for positive solutions for the equation

$$
\begin{cases}
-\Delta B^n u - \gamma V_2 u - \lambda u = V_2^{2^*(s)} u^{2^*(s)-1} & \text{in } \Omega \\
u \geq 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
$$

(1.6)

where \( \Omega \) is a compact smooth subdomain of \( B^n \) such that \( 0 \in \Omega \), but \( \overline{\Omega} \) does not touch the boundary of \( B^n \) and \( \lambda \in \mathbb{R} \). Note that the metric is then smooth on such \( \Omega \), and the only singularity we will be dealing with will be coming from the Hardy-type potential \( V_2 \) and the Hardy–Sobolev weight \( V_2^{2^*(s)} \), which behaves like \( \frac{1}{r^2} \) (resp., \( \frac{1}{r^s} \)) at the origin. This is analogous to the Euclidean problem on bounded domains considered by Ghoussoub–Robert [12, 11]. We shall therefore rely heavily on their work, at least in dimensions \( n \geq 5 \). Actually, once we perform a conformal transformation, the equation above reduces to the study of the following type of problems on bounded domains in \( \mathbb{R}^n \):

$$
\begin{cases}
-\Delta v - \left( \frac{s}{|x|^s} + h(x) \right) v = b(x) \frac{2^*(s)-1}{|x|^{s}} & \text{in } \Omega \\
v \geq 0 & \text{in } \Omega \\
v = 0 & \text{on } \partial \Omega,
\end{cases}
$$

where \( b \) is a positive function in \( C^1(\overline{\Omega}) \) with

$$
b(0) = \frac{(n-2)^{\frac{n-s}{2}}}{2^{\frac{n-s}{2}}} \text{ and } \nabla b(0) = 0,
$$

(1.7)

and

$$
h_{\gamma,\lambda}(x) = \gamma a(x) + \frac{4\lambda - n(n-2)}{(1 - |x|^2)^2},
$$
Theorem 3. Let $\Omega \subseteq \mathbb{R}^n$ be a smooth domain with $0 \in \Omega$, $0 \leq \gamma < \frac{(n-2)^2}{4}$ and let $\lambda \in \mathbb{R}$ be such that the operator $-\Delta_{\mathbb{R}^n} - \gamma V_2 - \lambda$ is coercive. Then, the best constant $\mu_{\gamma,\lambda}(\Omega)$ is attained under the following conditions:

1. $n > 4$, $\gamma \leq \frac{n(n-4)}{4}$ and $\lambda > \frac{n-2}{n-4} \left( \frac{n(n-4)}{4} - \gamma \right)$.

2. $\max \left\{ \frac{n(n-4)}{4}, 0 \right\} < \gamma < \frac{(n-2)^2}{4}$ and $m^H_{\gamma,\lambda}(\Omega) > 0$.

As mentioned above, the above theorem will be proved by using a conformal transformation that reduces the problem to the Euclidean case, already considered by Ghoussoub–Robert [12]. Actually, this leads to the following variation of the problem they considered, where the perturbation can be singular but not as much as the Hardy potential.
\begin{theorem}
Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^n$, $n \geq 3$, with $0 \in \Omega$ and $0 < \gamma < \frac{(n-2)^2}{4}$. Let $h \in C^1(\Omega \setminus \{0\})$ be such that
\begin{equation}
 h(x) = -C_1|x|^{-\theta} \log |x| + \tilde{h}(x) \quad \text{where} \quad \lim_{x \to 0} |x|^\theta \tilde{h}(x) = C_2 \quad \text{for some} \quad 0 < \theta < 2 \quad \text{and} \quad C_1, C_2 \in \mathbb{R},
\end{equation}
and the operator $-\Delta - \left( \frac{\gamma}{|x|^2} + h(x) \right)$ is coercive. Also, assume that $b(x)$ is a non-negative function in $C^1(\Omega)$ with $b(0) > 0$. Then the best constant
\begin{equation}
 \mu_{\gamma,h}(\Omega) := \inf_{u \in H^1_0(\Omega) \backslash \{0\}} \frac{\int_{\Omega} \left( \nabla u^2 - \left( \frac{\gamma}{|x|^2} + h(x) \right) u^2 \right) dx}{\left( \int_{\Omega} b(x) \frac{|u|^{2^*(s)}}{|x|^s} dx \right)^{2/2^*(s)}}
\end{equation}
is attained if one of the following two conditions is satisfied:
1. $0 < \frac{(n-2)^2}{4} - \frac{(2-\theta)^2}{4}$ and, either $C_1 > 0$ or $\{C_1 = 0, C_2 > 0\}$;
2. $0 < \frac{(n-2)^2}{4} - \frac{(2-\theta)^2}{4} < \gamma < \frac{(n-2)^2}{4}$ and $m_{\gamma,h}(\Omega) > 0$, where $m_{\gamma,h}(\Omega)$ is the mass of the domain $\Omega$ associated to the operator $-\Delta - \left( \frac{\gamma}{|x|^2} + h(x) \right)$.
\end{theorem}

\section{Hardy–Sobolev type inequalities in hyperbolic space}

The starting point of the study of existence of weak solutions of the above problems are the following inequalities which will guarantee that the above functionals are well defined and bounded below on the right function spaces. The Sobolev inequality for hyperbolic space \cite{18} asserts that for $n \geq 3$, there exists a constant $C > 0$ such that
\begin{equation}
 \left( \int_{\mathbb{B}_n} V_2 |u|^{2^*} \, dv_{g_{\mathbb{B}_n}} \right)^{2/2^*} \leq C \int_{\mathbb{B}_n} |\nabla_{\mathbb{B}_n} u|^2 \, dv_{g_{\mathbb{B}_n}} \quad \text{for all} \quad u \in H^1(\mathbb{B}_n),
\end{equation}
where $2^* = \frac{2n}{n-2}$ and $V_2$ is defined in (1.3). The Hardy inequality on $\mathbb{B}_n$ \cite{18} states:
\begin{equation}
 \frac{(n-2)^2}{4} \int_{\mathbb{B}_n} V_2 |u|^2 \, dv_{g_{\mathbb{B}_n}} \leq \int_{\mathbb{B}_n} |\nabla_{\mathbb{B}_n} u|^2 \, dv_{g_{\mathbb{B}_n}} \quad \text{for all} \quad u \in H^1(\mathbb{B}_n).
\end{equation}
Moreover, just like the Euclidean case, $\frac{(n-2)^2}{4}$ is the best Hardy constant in the above inequality on $\mathbb{B}_n$, i.e.,
\begin{equation}
 \gamma_H := \frac{(n-2)^2}{4} = \inf_{u \in H^1(\mathbb{B}_n) \backslash \{0\}} \frac{\int_{\mathbb{B}_n} |\nabla_{\mathbb{B}_n} u|^2 \, dv_{g_{\mathbb{B}_n}}}{\int_{\mathbb{B}_n} V_2 |u|^2 \, dv_{g_{\mathbb{B}_n}}},
\end{equation}
By interpolating these inequalities via Hölder’s inequality, one gets the following Hardy–Sobolev inequalities in hyperbolic space.

\begin{lemma}
Let $2^*(s) = \frac{2(n-s)}{n-2}$ where $0 \leq s \leq 2$. Then, there exist a positive constant $C$ such that
\begin{equation}
 C \left( \int_{\mathbb{B}_n} V_{2^*(s)} |u|^{2^*(s)} \, dv_{g_{\mathbb{B}_n}} \right)^{2/2^*(s)} \leq \int_{\mathbb{B}_n} |\nabla_{\mathbb{B}_n} u|^2 \, dv_{g_{\mathbb{B}_n}} \quad \text{for all} \quad u \in H^1(\mathbb{B}_n).
\end{equation}
If $\gamma < \gamma_H := \frac{(n-2)^2}{4}$, then there exists $C_\gamma > 0$ such that
\begin{equation}
 C_\gamma \left( \int_{\mathbb{B}_n} V_{2^*(s)} |u|^{2^*(s)} \, dv_{g_{\mathbb{B}_n}} \right)^{2/2^*(s)} \leq \int_{\mathbb{B}_n} |\nabla_{\mathbb{B}_n} u|^2 \, dv_{g_{\mathbb{B}_n}} - \gamma \int_{\mathbb{B}_n} V_2 |u|^2 \, dv_{g_{\mathbb{B}_n}} \quad \text{for all} \quad u \in H^1(\mathbb{B}_n).
\end{equation}
\end{lemma}
Proof. Note that for $s = 0$ (resp., $s = 2$) the first inequality is just the Sobolev (resp., the Hardy) inequality in hyperbolic space. We therefore have to only consider the case where $0 < s < 2$ where $2^*(s) > 2$. Note that $2^*(s) = \left(\frac{s}{2}\right)^2 + \left(\frac{2-s}{2}\right)^2$, and so

$$V_{2^*(s)} = \frac{f(r)^2(1-r)^2}{4(n-2)^2G(r)} \left(\frac{1}{\sqrt{G(r)}}\right)^{2^*(s)}$$

$$= \left(\frac{f(r)^2(1-r)^2}{4(n-2)^2G(r)}\right)^{s/2 + \frac{2-s}{2}} \left(\frac{1}{\sqrt{G(r)}}\right)^{s/2} \left(\frac{f(r)^2(1-r)^2}{4(n-2)^2G(r)}\right)^{1/2} \left(\frac{1}{\sqrt{G(r)}}\right)^{1/2}$$

$$= V_{2^*(s)}^s V_{2^*(s)}^{2-s}.$$

Applying H"older’s inequality with conjugate exponents $\frac{s}{2}$ and $\frac{2-s}{2}$, we obtain

$$\int_{\mathbb{B}^n} V_{2^*(s)}^{|u|^{2^*(s)}} \, dv_{g_{hn}} = \int_{\mathbb{B}^n} \left|u\right|^2 \leq \left(\int_{\mathbb{B}^n} V_{2^*(s)}^{|u|^{2^*(s)}} \, dv_{g_{hn}}\right)^{\frac{s}{2}} \left(\int_{\mathbb{B}^n} \left|u\right|^{2^*(s)} \, dv_{g_{hn}}\right)^{\frac{2-s}{2}}$$

$$\leq C^{-1} \left(\int_{\mathbb{B}^n} \left|\nabla_{\mathbb{B}^n} u\right|^2 \, dv_{g_{hn}}\right)^{\frac{s}{2}} \left(\int_{\mathbb{B}^n} \left|u\right|^{2^*(s)} \, dv_{g_{hn}}\right)^{\frac{2-s}{2}}$$

It follows that for all $u \in H^1(\mathbb{B}^n)$,

$$\int_{\mathbb{B}^n} \left|\nabla_{\mathbb{B}^n} u\right|^2 \, dv_{g_{hn}} - \gamma \int_{\mathbb{B}^n} V_2 u^2 \, dv_{g_{hn}} \leq \left(1 - \frac{\gamma}{\gamma_H}\right) \int_{\mathbb{B}^n} \left|\nabla_{\mathbb{B}^n} u\right|^2 \, dv_{g_{hn}} \left(\int_{\mathbb{B}^n} V_{2^*(s)}^{|u|^{2^*(s)}} \, dv_{g_{hn}}\right)^{2/2^*(s)}.$$

Hence, (2.1) implies (2.2) whenever $\gamma < \gamma_H := \frac{(n-2)^2}{4}$. \hfill \Box

The best constant $\mu_\gamma(\mathbb{B}^n)$ in inequality (2.2) can therefore be written as:

$$\mu_\gamma(\mathbb{B}^n) = \inf_{u \in H^1(\mathbb{B}^n) \setminus \{0\}} \frac{\int_{\mathbb{B}^n} \left|\nabla_{\mathbb{B}^n} u\right|^2 \, dv_{g_{hn}} - \gamma \int_{\mathbb{B}^n} V_2 u^2 \, dv_{g_{hn}}}{\left(\int_{\mathbb{B}^n} V_{2^*(s)}^{|u|^{2^*(s)}} \, dv_{g_{hn}}\right)^{2/2^*(s)}}.$$

Thus, any minimizer of $\mu_\gamma(\mathbb{B}^n)$ satisfies –up to a Lagrange multiplier– the following Euler–Lagrange equation

$$-\Delta_{\mathbb{B}^n} u - \gamma V_2 u = V_{2^*(s)}^{|u|^{2^*(s)-2}} u,$$

where $0 \leq s < 2$ and $2^*(s) = \frac{2(n-s)}{n-2}$.
3. The explicit solutions for Hardy–Sobolev equations on $\mathbb{B}^n$

We first find the fundamental solutions associated to the Hardy–Schrödinger operator on $\mathbb{B}^n$, that is the solutions for the equation $-\Delta_{\mathbb{B}^n} u - \gamma V_2 u = 0$.

**Lemma 3.1.** Assume $\gamma < \gamma_H := \frac{(n-2)^2}{4}$. The fundamental solutions of $-\Delta_{\mathbb{B}^n} u - \gamma V_2 u = 0$ are given by

$$u_\pm(r) = G(r)^{\alpha_\pm(\gamma)} \sim \begin{cases} 
\left(\frac{1}{n-2}\right)^{\frac{2-n}{n-2}} r^{2-n} & \text{as } r \to 0,
\left(\frac{2}{n-1}\right)^{\frac{n-2}{n-1}} (1-r)^{n-1} & \text{as } r \to 1,
\end{cases}$$

where

$$\alpha_\pm(\gamma) = \frac{\beta_\pm(\gamma)}{n-2} \quad \text{and} \quad \beta_\pm(\gamma) = \frac{n-2}{2} \pm \sqrt{\frac{(n-2)^2}{4} - \gamma}.$$ (3.1)

**Proof.** We look for solutions of the form $u(r) = G(r)^{-\alpha}$. To this end we perform a change of variable $\sigma = G(r)$, $v(\sigma) = u(r)$ to arrive at the Euler-type equation

$$(n-2)^2 v''(\sigma) + \gamma \sigma^{-2} v(\sigma) = 0 \quad \text{in } (0, \infty).$$

It is easy to see that the two solutions are given by $v(\sigma) = \sigma^\pm$, or $u(r) = c(n, \gamma) r^{-\beta_\pm}$ where $\alpha_\pm$ and $\beta_\pm$ are as in (3.1).

**Remark 3.1.** We point out that $u_\pm(r) \sim c(n, \gamma) r^{-\beta_\pm(\gamma)}$ as $r \to 0$.

**Proposition 3.1.** Let $-\infty < \gamma < \frac{(n-2)^2}{4}$. The equation

$$-\Delta_{\mathbb{B}^n} u - \gamma V_2 u = V_2 \ast (u^{2^*(s)-1}) \quad \text{in } \mathbb{B}^n,$$ (3.2)

has a family of positive radial solutions which are given by

$$U(G(r)) = c \left( G(r)^{-\frac{2-n}{n-2} \alpha_-(\gamma)} + G(r)^{-\frac{2-n}{n-2} \alpha_+(\gamma)} \right)^{-\frac{n-2}{2}},$$

where $c$ is a positive constant and $\alpha_\pm(\gamma)$ and $\beta_\pm(\gamma)$ satisfy (3.1).

**Proof.** With the same change of variable $\sigma = G(r)$ and $v(\sigma) = u(r)$ we have

$$(n-2)^2 v''(\sigma) + \gamma \sigma^{-2} v(\sigma) + \sigma^{-\frac{2^*(s)+2}{2}} v^{2^*(s)-1}(\sigma) = 0 \quad \text{in } (0, \infty).$$

Now, set $\sigma = r^{2-n}$ and $w(\tau) = v(\sigma)$

$$r^{1-n} (r^{n-1} w'(\tau))' + \gamma \tau^{-2} w(\tau) + w(\tau)^{2^*(s)-1} = 0 \quad \text{on } (0, \infty).$$

The latter has an explicit solution

$$w(\tau) = c \left( r^{\frac{2-n}{2} \beta_-(\gamma)} + \frac{1}{r^{\frac{2-n}{2} \beta_+(\gamma)}} \right)^{-\frac{n-2}{2}},$$

where $c$ is a positive constant. This translates to the explicit formula

$$u(r) = c \left( G(r)^{-\frac{2-n}{2} \alpha_-(\gamma)} + G(r)^{-\frac{2-n}{2} \alpha_+(\gamma)} \right)^{-\frac{n-2}{2}}$$

$$= c \left( G(r)^{-\frac{2-n}{(n-2)^2} \beta_-(\gamma)} + G(r)^{-\frac{2-n}{(n-2)^2} \beta_+(\gamma)} \right)^{-\frac{n-2}{2}}.$$
Remark 3.2. We remark that, in the special case $\gamma = 0$ and $s = 0$, Sandeep–Tintarev [18] proved that the following minimization problem

$$\mu_0(\mathbb{B}^n) = \inf_{u \in H^1(\mathbb{B}^n) \setminus \{0\}} \frac{\int_{\mathbb{B}^n} |\nabla u|^2 \, dv_{g_{\mathbb{B}^n}}}{\int_{\mathbb{B}^n} V_2 |u|^{2^*(s)-1} \, dv_{g_{\mathbb{B}^n}}}$$

is attained.

Remark 3.3. The change of variable $\sigma = G(r)$ offers a nice way of viewing the radial aspect of hyperbolic space $\mathbb{B}^n$ in parallel to the one in $\mathbb{R}^n$ in the following sense.

- The scaling $r \mapsto G^{-1}(\lambda G(r))$ for $r = |x|$ in $\mathbb{B}^n$ corresponds to $\sigma \mapsto \lambda \sigma$ in $(0, \infty)$, which in turn corresponds to $\rho \mapsto \lambda \rho = G^{-1}(\lambda G(\rho))$ for $\rho = |x|$ in $\mathbb{R}^n$, once we set $G(\rho) = \rho^{2-n}$ and $\lambda = \lambda^{2-n}$.
- One has a similar correspondence with the scaling-invariant equations: if $u$ solves

$$-\Delta_{\mathbb{B}^n} u - \gamma V_2 u = V_2^*(s) u^{2^*(s)-1} \quad \text{in } \mathbb{B}^n,$$

then

1. as an ODE, and once we set $v(\sigma) = u(\rho)$, $\sigma = G(\rho)$, it is equivalent to

$$- (n-2) \sigma''(\sigma) - \gamma \sigma^{-2} v(\sigma) = \sigma^{-2^*(s)+2} v(\sigma) 2^*(s)-1 \quad \text{on } (0, \infty);$$

2. as a PDE on $\mathbb{R}^n$, and by setting $v(\sigma) = u(\rho)$, $\sigma = G(\rho)$, it is in turn equivalent to

$$-\Delta v - \frac{\gamma}{|x|^2} v = \frac{1}{|x|} v^{2^*(s)-1} \quad \text{in } \mathbb{R}^n.$$

This also confirm that the potentials $V_2^*(s)$ are the “correct” ones associated to the power $|x|^{-s}$.

- The explicit solution $u$ on $\mathbb{B}^n$ is related to the explicit solution $w$ on $\mathbb{R}^n$ in the following way:

$$u(r) = w \left( G(r) - \frac{1}{|x|} \right).$$

- Under the above setting, it is also easy to see the following integral identities:

$$\int_{\mathbb{B}^n} |\nabla u|^2 \, dv_{g_{\mathbb{B}^n}} = \int_0^\infty v'(\sigma)^2 \, d\sigma$$

$$\int_{\mathbb{B}^n} V_2 u^2 \, dv_{g_{\mathbb{B}^n}} = \frac{1}{(n-2)^2} \int_0^\infty \frac{v^2(\sigma)}{\sigma^2} \, d\sigma$$

$$\int_{\mathbb{B}^n} V_{\mathbb{R}^n} u^p \, dv_{g_{\mathbb{B}^n}} = \frac{1}{(n-2)^2} \int_0^\infty \frac{v^p(\sigma)}{\sigma^{2n-2p}} \, d\sigma,$$

which, in the same way as above, equal to the corresponding Euclidean integrals.

4. The Corresponding Perturbed Hardy–Schrödinger Operator on Euclidean Space

We shall see in the next section that after a conformal transformation, the equation (1.6) is transformed into the Euclidean equation

$$\begin{cases}
-\Delta u - \left( \frac{\gamma}{|x|^2} + h(x) \right) u = b(x) \frac{2^*(s)-1}{|x|^2} \quad &\text{in } \Omega, \\
u > 0 \quad &\text{in } \Omega, \\
u = 0 \quad &\text{on } \partial \Omega,
\end{cases}$$

(4.1)
where $\Omega$ is a bounded domain in $\mathbb{R}^n$, $n \geq 3$, $h \in C^1(\overline{\Omega} \setminus \{0\})$ with $\lim_{|x| \to 0} |x|^2 h(x) = 0$ is such that the operator $-\Delta - \left(\frac{\gamma}{|x|^2} + h(x)\right)$ is coercive and $b(x) \in C^1(\Omega)$ is non-negative with $b(0) > 0$. The equation (4.1) is the Euler–Lagrange equation for following energy functional on $D^{1,2}(\Omega)$,

$$ J_{\gamma,h}^\Omega(u) := \int_\Omega \left( |\nabla u|^2 - \left( \frac{\gamma}{|x|^2} + h(x) \right) u^2 \right) dx - \left( \int_\Omega b(x) \frac{|u|^2}{|x|^s} dx \right)^{2/(2s)} . $$

Here $D^{1,2}(\Omega)$ or $H^1_0(\Omega)$ if the domain is bounded – is the completion of $C_0^\infty(\Omega)$ with respect to the norm given by $||u|| = \int_\Omega |\nabla u|^2 dx$. We let

$$ \mu_{\gamma,h}(\Omega) := \inf_{u \in D^{1,2}(\Omega) \setminus \{0\}} J_{\gamma,h}^\Omega(u) $$

A standard approach to find minimizers is to compare $\mu_{\gamma,h}(\Omega)$ with $\mu_{\gamma,0}(\mathbb{R}^n)$. It is know that $\mu_{\gamma,0}(\mathbb{R}^n)$ is attained when $\gamma \geq 0$, are explicit and take the form

$$ U_\varepsilon(x) := c_{\gamma,s}(n) \cdot \varepsilon^{-\frac{2\gamma}{n-2}} U \left( \frac{x}{\varepsilon} \right) = c_{\gamma,s}(n) \cdot \left( \varepsilon^{\frac{2\gamma}{n-2}} \frac{\beta_+(\gamma) - \beta_-(\gamma)}{|x|^{n-2} (\beta_+(\gamma) - \beta_-(\gamma)) \left( \frac{2\gamma}{n-2} \right) + |x|^{\frac{2\gamma}{n-2}}} \right)^{\frac{2\gamma}{n-2}} $$

for $x \in \mathbb{R}^n \setminus \{0\}$, where $\varepsilon > 0$, $c_{\gamma,s}(n) > 0$, and $\beta_\pm(\gamma)$ are defined in (3.1). In particular, there exists $\chi > 0$ such that

$$ -\Delta U_\varepsilon - \frac{\gamma}{|x|^2} U_\varepsilon = \chi U_\varepsilon^{2^*(s)-1} \quad \text{in } \mathbb{R}^n \setminus \{0\}. \quad (4.2) $$

We shall start by analyzing the singular solutions and then define the mass of a domain associated to the operator $-\Delta - \left(\frac{\gamma}{|x|^2} + h(x)\right)$.

**Proposition 4.1.** Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^n$ such that $0 \in \Omega$ and $\gamma < \frac{(n-2)^2}{4}$. Let $h \in C^1(\overline{\Omega} \setminus \{0\})$ be such that $\lim_{|x| \to 0} |x|^\tau h(x)$ exists and is finite, for some $0 \leq \tau < 2$, and that the operator $-\Delta - \left(\frac{\gamma}{|x|^2} + h(x)\right)$ is coercive. Then

1. There exists a solution $K \in C^\infty(\overline{\Omega} \setminus \{0\})$ for the linear problem

$$ \begin{cases} 
-\Delta K - \left(\frac{\gamma}{|x|^2} + h(x)\right) K = 0 & \text{in } \Omega \setminus \{0\} \\
K > 0 & \text{in } \Omega \setminus \{0\} \\
K = 0 & \text{on } \partial \Omega,
\end{cases} \quad (4.3) $$

such that for some $c > 0$,

$$ K(x) \asymp_{x \to 0} \frac{c}{|x|^\beta_+(\gamma)}. \quad (4.4) $$

Moreover, if $K' \in C^\infty(\overline{\Omega} \setminus \{0\})$ is another solution for the above equation, then there exists $\lambda > 0$ such that $K' = \lambda K$.

2. Let $\theta = \inf \{\theta \in [0,2] : \lim_{|x| \to 0} |x|^{\theta} h(x) \text{ exists and is finite}\}$. If $\gamma > \frac{(n-2)^2}{4} - \frac{(2-\theta)^2}{4}$, then there exists $c_1, c_2 \in \mathbb{R}$ with $c_1 > 0$ such that

$$ K(x) = \frac{c_1}{|x|^\beta_+(\gamma)} + \frac{c_2}{|x|^\beta_-(\gamma)} + o \left( \frac{1}{|x|^\beta_-(\gamma)} \right) \quad \text{as } x \to 0. \quad (4.5) $$
The ratio \( \frac{\eta}{\Theta} \) is independent of the choice of \( K \). We can therefore define the mass of \( \Omega \) with respect to the operator \(-\Delta - \left( \frac{\gamma}{|x|^2} + h(x) \right)\) as \( m_{\gamma,h}(\Omega) := \frac{c_2}{c_1} \).

(3) The mass \( m_{\gamma,h}(\Omega) \) satisfies the following properties:
- \( m_{\gamma,h}(\Omega) \leq 0 \),
- If \( h \leq h' \) and \( h \neq h' \), then \( m_{\gamma,h}(\Omega) < m_{\gamma,h'}(\Omega) \),
- If \( \Omega' \subset \Omega \), then \( m_{\gamma,h}(\Omega') < m_{\gamma,h}(\Omega) \).

Proof. The proof of (1) and (3) is similar to Proposition 2 and 4 in [12] with only a minor change that accounts for the singularity of \( h \). To illustrate the role of this extra singularity we prove (2). For that, we let \( \eta \in C^\infty_c(\Omega) \) be such that \( \eta(x) \equiv 1 \) around 0. Our first objective is to write \( K(x) := \frac{\eta(x)}{|x|^{\alpha+\gamma}} + f(x) \) for some \( f \in H^1_0(\Omega) \).

Consider the function \( g(x) = -\left( -\Delta - \left( \frac{\gamma}{|x|^2} + h(x) \right) \right)(\eta|x|^{-\beta_+(\gamma)}) \) in \( \Omega \setminus \{0\} \).

Since \( \eta(x) \equiv 1 \) around 0, we have that
\[
|g(x)| \leq \frac{h(x)}{|x|^{\beta_+(\gamma)}} \leq C|x|^{-(\beta_+(\gamma)+\theta')} \quad \text{as } x \to 0. \quad (4.6)
\]

Therefore \( g \in L^{\frac{2\alpha+\gamma}{\alpha+\gamma}}(\Omega) \) if \( 2\beta_+(\gamma) + 2\theta' < n + 2 \), and this holds since by our assumption \( 2\beta_+ < n - \theta \) and \( 2\theta' < 2 + \theta \). Since \( L^{\frac{2\alpha+\gamma}{\alpha+\gamma}}(\Omega) = L^{\frac{2\alpha+\gamma}{\alpha+\gamma}}(\Omega') \subset H^1_0(\Omega)' \), there exists \( f \in H^1_0(\Omega) \) such that
\[-\Delta f - \left( \frac{\gamma}{|x|^2} + h(x) \right) f = g \quad \text{in } H^1_0(\Omega).
\]

By regularity theory, we have that \( f \in C^2(\overline{\Omega} \setminus \{0\}) \). We now show that \( |x|^{\beta_-(\gamma)}f(x) \) has a finite limit as \( x \to 0 \).

Define \( K(x) = \frac{\eta(x)}{|x|^{\alpha+\gamma}} + f(x) \) for all \( x \in \Omega \setminus \{0\} \), and note that \( K \in C^2(\overline{\Omega} \setminus \{0\}) \) and is a solution to
\[-\Delta K - \left( \frac{\gamma}{|x|^2} + h(x) \right) K = 0.
\]

Write \( g_+(x) := \max\{g(x),0\} \) and \( g_-(x) := \max\{-g(x),0\} \) so that \( g = g_+ - g_- \), and let \( f_1, f_2 \in H^1_0(\Omega) \) be weak solutions to
\[-\Delta f_1 - \left( \frac{\gamma}{|x|^2} + h(x) \right) f_1 = g_+ \quad \text{and} \quad -\Delta f_2 - \left( \frac{\gamma}{|x|^2} + h(x) \right) f_2 = g_- \quad \text{in } H^1_0(\Omega).
\]

In particular, uniqueness, coercivity and the maximum principle yields \( f = f_1 - f_2 \) and \( f_1, f_2 \geq 0 \). Assume that \( f_1 \neq 0 \) so that \( f_1 > 0 \) in \( \Omega \setminus \{0\} \), fix \( \alpha > \beta_+(\gamma) \) and \( \mu > 0 \). Define \( u_-(x) := |x|^{-\beta_-(\gamma)} + \mu|x|^{-\alpha} \) for all \( x \neq 0 \). We then get that there exists a small \( \delta > 0 \) such that
\[
\left( -\Delta - \left( \frac{\gamma}{|x|^2} + h(x) \right) \right) u_-(x) = \mu \left( -\Delta - \frac{\gamma}{|x|^2} \right) |x|^{-\alpha} - \mu h(x)|x|^{-\alpha} - h(x)|x|^{-\beta_-(\gamma)} - \frac{-\mu (\alpha - \beta_+(\gamma))(\alpha - \beta_-(\gamma)) - |x|^2 h(x)(|x|^{\alpha-\beta_-(\gamma)} + \mu)}{|x|^\alpha+2}
\]
\[
< 0 \quad \text{for } x \in B^0(0) \setminus \{0\},
\]
This implies that \( u_- (x) \) is a sub-solution on \( B_1(0) \setminus \{0\} \). Let \( C > 0 \) be such that \( f_1 \geq Cu_- \) on \( \partial B_1(0) \). Since \( f_1 \) and \( Cu_- \in H^1_a(\Omega) \) are respectively super-solutions and sub-solutions to \( \left( -\Delta - \left( \frac{\gamma}{|x|^2} + h(x) \right) \right) u(x) = 0 \), it follows from the comparison principle (via coercivity) that \( f_1 > Cu_- > C|x|^{2-\beta_-(\gamma)} \) on \( B_1(0) \setminus \{0\} \). It then follows from (4.6) that

\[
g_+(x) \leq |g(x)| \leq C|x|^{-(\beta_+(\gamma)+\theta')} \leq C_1|x|^{(2-\theta')-(\beta_+(\gamma)-\beta_-(\gamma))} \frac{f_1}{|x|^2}.
\]

Then rewriting (4.8) as

\[-\Delta f_1 - \left( \frac{\gamma}{|x|^2} + h(x) + \frac{g_+}{f_1} \right) f_1 = 0\]

yields

\[-\Delta f_1 - \left( \frac{\gamma + O (|x|^{(2-\theta')-(\beta_+(\gamma)-\beta_-(\gamma))})}{|x|^2} \right) f_1 = 0.\]

With our choice of \( \theta' \) we can then conclude by the optimal regularity result in [12, Theorem 8] that \( |x|^{\beta_-(\gamma)} f_1 \) has a finite limit as \( x \to 0 \). Similarly one also obtains that \(|x|^{\beta_-(\gamma)} f_2 \) has a finite limit as \( x \to 0 \), and therefore (4.7) is verified.

It follows that there exists \( c_2 \in \mathbb{R} \) such that

\[K(x) = \frac{1}{|x|^{\beta_+(\gamma)}} + \frac{c_2}{|x|^{\beta_-(\gamma)}} + o \left( \frac{1}{|x|^{\beta_-(\gamma)}} \right) \quad \text{as} \quad x \to 0,
\]

which proves the existence of a solution \( K \) to the problem with the relevant asymptotic behavior. The uniqueness result yields the conclusion. \( \square \)

We now proceed with the proof of the existence results, following again [12]. We shall use the following standard sufficient condition for attainability.

**Lemma 4.1.** Under the assumptions of Theorem 4, if

\[
\mu_{\gamma,h}(\Omega) := \inf_{u \in H^1_0(\Omega) \setminus \{0\}} \int_{\Omega} \left( |\nabla u|^2 - \left( \frac{\gamma}{|x|^2} + h(x) \right) u^2 \right) dx < \frac{\mu_{\gamma,0}(\mathbb{R}^n)}{b(0)^{2/2^*(s)}},
\]

then the infimum \( \mu_{\gamma,s}(\Omega) \) is achieved and equation (4.1) has a solution.

**Proof of Theorem 4:** We will construct a minimizing sequence \( u_\varepsilon \) in \( H^1_0(\Omega) \setminus \{0\} \) for the functional \( J^\Omega_{\gamma,h} \) in such a way that \( \mu_{\gamma,h}(\Omega) < b(0)^{-2/2^*(s)} \mu_{\gamma,0}(\mathbb{R}^n) \). As mentioned above, when \( \gamma \geq 0 \) the infimum \( \mu_{\gamma,0}(\mathbb{R}^n) \) is achieved, up to a constant, by the function

\[U(x) := \frac{1}{\left( \frac{(2-s)\beta_+(\gamma)}{n-2} + \frac{(2-s)\beta_-(\gamma)}{n-2} \right)^{\frac{1}{2-s}}} \quad \text{for} \quad x \in \mathbb{R}^n \setminus \{0\}.
\]

In particular, there exists \( \chi > 0 \) such that

\[-\Delta U - \frac{\gamma}{|x|^2} U = \chi \frac{U^{2^*(s)-1}}{|x|^s} \quad \text{in} \quad \mathbb{R}^n \setminus \{0\}. \quad (4.10)\]
Define a scaled version of $U$ by

$$U_\varepsilon(x) := \varepsilon^{-\frac{n-2}{2}} U \left( \frac{x}{\varepsilon} \right) = \left( \frac{\varepsilon^{-\frac{n+2}{2}} (\beta_+(\gamma) - \beta_-(\gamma))}{\varepsilon^{\frac{n-2}{2}} (\beta_+(\gamma) - \beta_-(\gamma)) + |x|^2} \right)^{\frac{n+2}{2}}$$

for $x \in \mathbb{R}^n \setminus \{0\}$. (4.11)

$\beta_\pm(\gamma)$ are defined in (3.1). In the sequel, we write $\beta_+ := \beta_+(\gamma)$ and $\beta_- := \beta_-(\gamma)$. Consider a cut-off function $\eta \in C^\infty_c(\Omega)$ such that $\eta(x) \equiv 1$ in a neighborhood of 0 contained in $\Omega$.

**Case 1:** Test-functions for the case when $\gamma \leq \frac{(n-2)^2}{4} - \frac{(2-\theta)^2}{4}$.

For $\varepsilon > 0$, we consider the test functions $u_\varepsilon \in D^{1,\infty}(\Omega)$ defined by $u_\varepsilon(x) := \eta(x)U_\varepsilon(x)$ for $x \in \mathbb{R}^n \setminus \{0\}$. To estimate $J_{\gamma, h}^\Omega(u_\varepsilon)$, we use the bounds on $U_\varepsilon$ to obtain

$$\int \Omega b(x) \frac{u_\varepsilon^2(x)}{|x|^2} dx = \int_{B_1(0)} b(x) U_\varepsilon^2(x) |x|^2 dx + \int_{\Omega \setminus B_1(0)} b(x) u_\varepsilon^2(x) dx$$

Similarly, one also has

$$\int \Omega (|\nabla u_\varepsilon|^2 - \frac{\gamma}{|x|^2} u_\varepsilon^2) dx = \int_{B_1(0)} (|\nabla U_\varepsilon|^2 - \frac{\gamma}{|x|^2} U_\varepsilon^2) dx + \int_{\Omega \setminus B_1(0)} (|\nabla u_\varepsilon|^2 - \frac{\gamma}{|x|^2} u_\varepsilon^2) dx$$

Estimating the lower order terms as $\varepsilon \to 0$ gives

$$\int \bar{h}(x) u_\varepsilon^2 dx = \begin{cases} 
\varepsilon^{2-\theta} \left[ C_2 \int_{\mathbb{R}^n} \frac{U^2}{|x|^2} dx + o(1) \right] & \text{if } \beta_+ - \beta_- > 2 - \theta, \\
\varepsilon^{2-\theta} \log \left( \frac{1}{\varepsilon} \right) \left[ C_2 o_n - 1 + o(1) \right] & \text{if } \beta_+ - \beta_- = 2 - \theta, \\
O(\varepsilon^{\beta_+ - \beta_-}) & \text{if } \beta_+ - \beta_- < 2 - \theta.
\end{cases}$$

And

$$\int_{\Omega} \log \frac{|x|}{|x|^2} u_\varepsilon^2 dx = \begin{cases} 
C_1 \varepsilon^{2-\theta} \log \left( \frac{1}{\varepsilon} \right) \left[ \int \mathbb{R}^n \frac{U^2}{|x|^2} dx + o(1) \right] & \text{if } \beta_+ - \beta_- > 2 - \theta, \\
C_1 \varepsilon^{2-\theta} \left( \log \left( \frac{1}{\varepsilon} \right) \right)^2 \left[ \frac{\omega_{n-1}}{2} + o(1) \right] & \text{if } \beta_+ - \beta_- = 2 - \theta, \\
O(\varepsilon^{\beta_+ - \beta_-}) & \text{if } \beta_+ - \beta_- < 2 - \theta.
\end{cases}$$
Note that $\beta_+ - \beta_- \geq 2 - \theta$ if and only if $\gamma < \frac{(n-2)^2}{4} - \frac{(2-\theta)^2}{4}$. Therefore,

$$
\int_{\Omega} h(x)u_{\varepsilon}^2 \, dx = \begin{cases} \\
\varepsilon^{2-\theta} \int_{\mathbb{R}^n} \frac{U^2}{|x|^\theta} \, dx \left[ C_1 \log \left( \frac{1}{\varepsilon} \right) (1 + o(1)) + C_2 + o(1) \right] & \text{if } \gamma < \frac{(n-2)^2}{4} - \frac{(2-\theta)^2}{4}, \\
\varepsilon^{2-\theta} \log \left( \frac{1}{\varepsilon} \right) \frac{\omega_{n-1}}{2} \left[ C_1 \log \left( \frac{1}{\varepsilon} \right) (1 + o(1)) + 2C_2 + o(1) \right] & \text{if } \gamma = \frac{(n-2)^2}{4} - \frac{(2-\theta)^2}{4}.
\end{cases}
$$

Combining the above estimates, we obtain as $\varepsilon \to 0$,

$$
J_{\gamma,h}^\Omega (u_{\varepsilon}) = \frac{\int_{\Omega} (|\nabla u_{\varepsilon}|^2 - \gamma \frac{u_{\varepsilon}^2}{|x|^\theta} - h(x)u_{\varepsilon}^2) \, dx}{\left( \int_{\Omega} b(x) \left| \frac{u_{\varepsilon}}{|x|^{\beta + 2}} \right|^2 / x^s \right)^{2/2^*}}
= \frac{\mu_{\gamma,h}(\mathbb{R}^n)}{b(0)^{2/2^*}} - \begin{cases} \\
\int_{\mathbb{R}^n} \frac{U^2}{|x|^\theta} \, dx \left[ C_1 \log \left( \frac{1}{\varepsilon} \right) (1 + o(1)) + C_2 + o(1) \right] & \text{if } \gamma < \frac{(n-2)^2}{4} - \frac{(2-\theta)^2}{4}, \\
2 \left( b(0) \int_{\mathbb{R}^n} \frac{U^2}{|x|^\theta} \, dx \right)^{2/2^*} \varepsilon^{2-\theta} \log \left( \frac{1}{\varepsilon} \right) \left[ C_1 \log \left( \frac{1}{\varepsilon} \right) (1 + o(1)) + 2C_2 + o(1) \right] & \text{if } \gamma = \frac{(n-2)^2}{4} - \frac{(2-\theta)^2}{4},
\end{cases}
$$

as long as $\beta_+ - \beta_- \geq 2 - \theta$. Thus, for $\varepsilon$ sufficiently small, the assumption that either $C_1 > 0$ or $C_2 = 0$, $C_2 > 0$ guarantees that

$$
\mu_{\gamma,h}(\Omega) \leq J_{\gamma,h}^\Omega (u_{\varepsilon}) < \frac{\mu_{\gamma,h}(\mathbb{R}^n)}{b(0)^{2/2^*}}.
$$

It then follows from Lemma 4.1 that $\mu_{\gamma,h}(\Omega)$ is attained.

**Case 2:** Test-functions for the case when $\frac{(n-2)^2}{4} - \frac{(2-\theta)^2}{4} < \gamma < \frac{(n-2)^2}{4}$.

Here $h(x)$ and $\theta$ given by (1.11) satisfy the hypothesis of Proposition (4.1). Since $\gamma > \frac{(n-2)^2}{4} - \frac{(2-\theta)^2}{4}$, it follows from (4.5) that there exists $\beta \in D^{1,2}(\Omega)$ such that

$$
\beta(x) \asymp_{x \to 0} \frac{m_{\gamma,h}(\Omega)}{|x|^\beta_-}.
$$

The function $K(x) := \frac{\eta(x)}{|x|^\beta} + \beta(x)$ for $x \in \Omega \setminus \{0\}$ satisfies the equation:

$$
\begin{cases} \\
-\Delta K - \left( \frac{\eta(x)}{|x|^\theta} + h(x) \right) K = 0 & \text{in } \Omega \setminus \{0\} \\
K > 0 & \text{in } \Omega \setminus \{0\} \\
K = 0 & \text{on } \partial \Omega.
\end{cases}
$$

Define the test functions

$$
u_{\varepsilon}(x) := \eta(x)U_{\varepsilon} + \varepsilon^{\theta - \beta_-} \beta(x)
$$

for $x \in \Omega \setminus \{0\}$.

The functions $u_{\varepsilon} \in D^{1,2}(\Omega)$ for all $\varepsilon > 0$. We estimate $J_{\gamma,h}^\Omega (u_{\varepsilon})$. 

Step 1: Estimates for $\int_{\Omega} \left( |\nabla u_{\varepsilon}|^2 - \left( \frac{\gamma}{|x|^2} + h(x) \right) u_{\varepsilon}^2 \right) \, dx$.

Take $\delta > 0$ small enough such that $\eta(x) = 1$ in $B_\delta(0) \subset \Omega$. We decompose the integral as

$$\int_{\Omega} \left( |\nabla u_{\varepsilon}|^2 - \left( \frac{\gamma}{|x|^2} + h(x) \right) u_{\varepsilon}^2 \right) \, dx$$

$$= \int_{B_\delta(0)} \left( |\nabla u_{\varepsilon}|^2 - \left( \frac{\gamma}{|x|^2} + h(x) \right) u_{\varepsilon}^2 \right) \, dx + \int_{\Omega \setminus B_\delta(0)} \left( |\nabla u_{\varepsilon}|^2 - \left( \frac{\gamma}{|x|^2} + h(x) \right) u_{\varepsilon}^2 \right) \, dx.$$ 

By standard elliptic estimates, it follows that $\lim_{\varepsilon \to 0} \frac{\int_{\Omega \setminus B(0)} \left( |\nabla u_{\varepsilon}|^2 - \left( \frac{\gamma}{|x|^2} + h(x) \right) u_{\varepsilon}^2 \right) \, dx}{\frac{\varepsilon^{\beta_+ - \beta_-}}{2}} = K$ in $C^2_{\text{loc}}(\Omega \setminus \{0\})$. Hence

$$\lim_{\varepsilon \to 0} \int_{B(0)} \left( |\nabla u_{\varepsilon}|^2 - \left( \frac{\gamma}{|x|^2} + h(x) \right) u_{\varepsilon}^2 \right) \, dx = \int_{B(0)} \left( |\nabla K|^2 - \left( \frac{\gamma}{|x|^2} + h(x) \right) K^2 \right) \, dx$$

$$= \int_{B(0)} \left( -\Delta K - \left( \frac{\gamma}{|x|^2} + h(x) \right) K \right) K \, dx + \int_{\partial B(0)} K \partial_{\nu} K \, d\sigma$$

$$= \int_{\partial B(0)} K \partial_{\nu} K \, d\sigma = -\int_{\partial B(0)} K \partial_{\nu} K \, d\sigma.$$ 

Since $\beta_+ + \beta_- = n - 2$, using elliptic estimates, and the definition of $K$ gives us

$$K \partial_{\nu} K = -\frac{\beta_+}{|x|^{1+2\beta_+}} - (n - 2) \frac{m_{\gamma,h}(\Omega)}{|x|^{n-1}} + o \left( \frac{1}{|x|^{n-1}} \right) \text{ as } x \to 0.$$

Therefore,

$$\int_{B(0)} \left( |\nabla u_{\varepsilon}|^2 - \left( \frac{\gamma}{|x|^2} + h(x) \right) u_{\varepsilon}^2 \right) \, dx = \varepsilon^{\beta_+ - \beta_-} \omega_{n-1} \left( \frac{\beta_+}{\beta_+ - \beta_-} - (n - 2)m_{\gamma,h}(\Omega) + o_1(1) \right)$$

Now, we estimate the term $\int_{B(0)} \left( |\nabla u_{\varepsilon}|^2 - \left( \frac{\gamma}{|x|^2} + h(x) \right) u_{\varepsilon}^2 \right) \, dx$.

First, $u_{\varepsilon}(x) = U_{\varepsilon}(x) + \varepsilon^{\beta_+-\beta_-} \beta(x)$ for $x \in B(0)$, therefore after integration by parts, we obtain

$$\int_{B(0)} \left( |\nabla u_{\varepsilon}|^2 - \left( \frac{\gamma}{|x|^2} + h(x) \right) u_{\varepsilon}^2 \right) \, dx = \int_{B(0)} \left( |\nabla U_{\varepsilon}|^2 - \left( \frac{\gamma}{|x|^2} + h(x) \right) U_{\varepsilon}^2 \right) \, dx$$

$$+ 2\varepsilon^{\beta_+-\beta_-} \int_{B(0)} \left( \nabla U_{\varepsilon} \cdot \nabla \beta - \left( \frac{\gamma}{|x|^2} + h(x) \right) U_{\varepsilon} \beta \right) \, dx$$

$$+ \varepsilon^{\beta_+ - \beta_-} \int_{B(0)} \left( |\nabla \beta|^2 - \left( \frac{\gamma}{|x|^2} + h(x) \right) \beta^2 \right) \, dx$$

$$= \int_{B(0)} \left( -\Delta U_{\varepsilon} - \frac{\gamma}{|x|^2} U_{\varepsilon} \right) U_{\varepsilon} \, dx + \int_{\partial B(0)} U_{\varepsilon} \partial_{\nu} U_{\varepsilon} \, d\sigma$$

$$- \int_{B(0)} h(x) U_{\varepsilon}^2 \, dx + 2\varepsilon^{\beta_+-\beta_-} \int_{B(0)} \left( -\Delta U_{\varepsilon} - \frac{\gamma}{|x|^2} U_{\varepsilon} \right) \beta \, dx$$

$$- 2\varepsilon^{\beta_+-\beta_-} \int_{B(0)} h(x) U_{\varepsilon} \beta \, dx + 2\varepsilon^{\beta_+-\beta_-} \int_{\partial B(0)} \beta \partial_{\nu} U_{\varepsilon} \, d\sigma$$

$$+ \varepsilon^{\beta_+ - \beta_-} \int_{B(0)} \left( |\nabla \beta|^2 - \left( \frac{\gamma}{|x|^2} + h(x) \right) \beta^2 \right) \, dx.$$
We now estimate each of the above terms. First, using equation (4.2) and the expression for $U_\varepsilon$, we obtain

$$
\int_{B_\varepsilon(0)} \left( -\Delta U_\varepsilon - \frac{\gamma}{|x|^2} U_\varepsilon \right) U_\varepsilon \ dx = \chi \int_{B_\varepsilon(0)} \frac{U_\varepsilon^{2}(s)}{|x|^s} \ dx
$$

$$
= \chi \int_{\mathbb{R}^n} \frac{U_\varepsilon^{2}(s)}{|x|^s} \ dx + O \left( \varepsilon^{\frac{n-1}{2}(\beta_+ - \beta_-)} \right),
$$

and

$$
\int_{\partial B_\varepsilon(0)} U_\varepsilon \partial_n U_\varepsilon \ d\sigma = -\beta_+ \omega_{n-1} \varepsilon^{\beta_+ - \beta_-} + o_\delta \left( \varepsilon^{\beta_+ - \beta_-} \right) \quad \text{as } \varepsilon \to 0.
$$

Note that

$$
\beta_+ - \beta_- < 2 - \theta \iff \gamma > \frac{(n-2)^2}{4} - \frac{(2-\theta)^2}{4} \implies 2\beta_+ + \theta < n.
$$

Therefore,

$$
\int_{B_\varepsilon(h(x)U_\varepsilon^2) \ dx = O \left( \varepsilon^{\beta_+ - \beta_-} \int_{B_\varepsilon(0)} \frac{1}{|x|^{2\beta_+ + \theta}} \ dx \right) = o_\delta \left( \varepsilon^{\beta_+ - \beta_-} \right) \quad \text{as } \varepsilon \to 0.
$$

Again from equation (4.2) and the expression for $U$ and $\beta$, we get that

$$
\int_{B_\varepsilon(0)} \left( -\Delta U_\varepsilon - \frac{\gamma}{|x|^2} U_\varepsilon \right) \beta \ dx = \varepsilon^{\beta_+ - \beta_-} \int_{B_{\varepsilon-\varepsilon}(0)} \left( -\Delta U \ dx - \frac{\gamma}{|x|^2} \right) \beta(\varepsilon x) \ dx
$$

$$
= m_{\gamma,h}(\Omega) \varepsilon^{\beta_+ - \beta_-} \int_{B_{\varepsilon-\varepsilon}(0)} \left( -\Delta U \ dx - \frac{\gamma}{|x|^2} \right) |x|^{-\beta_-} \ dx + o_\delta \left( \varepsilon^{\beta_+ - \beta_-} \right)
$$

$$
= m_{\gamma,h}(\Omega) \varepsilon^{\beta_+ - \beta_-} \int_{B_{\varepsilon-\varepsilon}(0)} \left( -\Delta |x|^{-\beta_-} \ dx - \frac{\gamma}{|x|^2} |x|^{-\beta_-} \right) U \ dx
$$

$$
- m_{\gamma,h}(\Omega) \varepsilon^{\beta_+ - \beta_-} \int_{\partial B_{\varepsilon-\varepsilon}(0)} \frac{\partial U}{|x|^2} \ d\sigma + o_\delta \left( \varepsilon^{\beta_+ - \beta_-} \right)
$$

$$
= \beta_+ m_{\gamma,h}(\Omega) \omega_{n-1} \varepsilon^{\beta_+ - \beta_-} + o_\delta \left( \varepsilon^{\beta_+ - \beta_-} \right).
$$

Similarly,

$$
\int_{\partial B_\varepsilon(0)} \beta \partial_n U_\varepsilon \ d\sigma = -\beta_+ \omega_{n-1} \varepsilon^{\beta_+ - \beta_-} + o_\delta \left( \varepsilon^{\beta_+ - \beta_-} \right).
$$

Since $\beta_+ + \beta_- + \theta = n - (2 - \theta) < n$, we have

$$
\int_{B_\varepsilon(h(x)U_\varepsilon \beta) \ dx = O \left( \varepsilon^{\beta_+ - \beta_-} \int_{B_\varepsilon(0)} \frac{1}{|x|^{\beta_+ + \beta_- + \theta}} \ dx \right)
$$

$$
= o_\delta \left( \varepsilon^{\beta_+ - \beta_-} \right).
$$

And, finally

$$
\varepsilon^{\beta_+ - \beta_-} \int_{B_\varepsilon(0)} \left( |\nabla \beta|^2 - \left( \frac{\gamma}{|x|^2} + h(x) \right) \beta^2 \right) \ dx = o_\delta(\varepsilon^{\beta_+ - \beta_-}).
$$

Combining all the estimates, we get
\[ \int_{B_{\delta}(0)} \left( |\nabla u_{\varepsilon}|^2 - \left( \frac{\gamma}{|x|^2} + h(x) \right) u_{\varepsilon}^2 \right) dx = \chi \int_{\mathbb{R}^n} \frac{U^{2^*(s)}}{|x|^s} dx - \beta_+ \omega_{n-1} \varepsilon^{\beta_+ - \beta_-} + o(\varepsilon^{\beta_+ - \beta_-}). \]

So,
\[ \int_{\Omega} \left( |\nabla u_{\varepsilon}|^2 - \left( \frac{\gamma}{|x|^2} + h(x) \right) u_{\varepsilon}^2 \right) dx = \chi \int_{\mathbb{R}^n} \frac{U^{2^*(s)}}{|x|^s} dx + \omega_{n-1}(n-2)m_{\gamma,h}(\Omega)\varepsilon^{\beta_+ - \beta_-} + o(\varepsilon^{\beta_+ - \beta_-}). \]

**Step 2:** Estimating \( \int_{\Omega} b(x) \frac{u_{\varepsilon}^{2^*(s)}}{|x|^s} dx. \)

One has for \( \delta > 0 \) small
\[
\int_{\Omega} \frac{u_{\varepsilon}^{2^*(s)}}{|x|^s} dx = \int_{B_{\delta}(0)} b(x) \frac{u_{\varepsilon}^{2^*(s)}}{|x|^s} dx + \int_{\Omega \setminus B_{\delta}(0)} b(x) \frac{u_{\varepsilon}^{2^*(s)}}{|x|^s} dx
= \int_{B_{\delta}(0)} b(x) \frac{(U_{\varepsilon}(x) + \varepsilon^{\frac{\beta_+ - \beta_-}{2}} \beta(x))^{2^*(s)}}{|x|^s} dx + o(\varepsilon^{\beta_+ - \beta_-})
= \int_{B_{\delta}(0)} b(x) \frac{U_{\varepsilon}^{2^*(s)}}{|x|^s} dx + \varepsilon^{\frac{\beta_+ - \beta_-}{2}} 2^*(s) \beta + o(\varepsilon^{\beta_+ - \beta_-})
= \int_{B_{\delta}(0)} b(x) \frac{U_{\varepsilon}^{2^*(s)}}{|x|^s} dx + \varepsilon^{\frac{\beta_+ - \beta_-}{2}} \frac{2^*(s)}{\chi} \int_{B_{\delta}(0)} b(x) \left( -\Delta U_{\varepsilon} dx - \frac{\gamma}{|x|^2} U_{\varepsilon} \right) \beta dx + o(\varepsilon^{\beta_+ - \beta_-})
= b(0) \int_{\mathbb{R}^n} \frac{U^{2^*(s)}}{|x|^s} dx + \frac{2^*(s)}{\chi} b(0) \beta \omega_{n-1} \varepsilon^{\beta_+ - \beta_-} + o(\varepsilon^{\beta_+ - \beta_-}). \tag{4.14} \]

So, we obtain
\[
J_{\gamma, \lambda, a}^{\Omega}(u_{\varepsilon}) = \int_{\Omega} \left( |\nabla u_{\varepsilon}|^2 - \gamma \frac{u_{\varepsilon}^2}{|x|^2} - h(x) u_{\varepsilon}^2 \right) dx
= \frac{1}{\left( \int_{\Omega} b(x) \frac{|u_{\varepsilon}|^{2^*(s)}}{|x|^s} dx \right)^{2^{2^*(s)}}} \left( \int_{\mathbb{R}^n} \frac{U^{2^*(s)}}{|x|^s} dx \right)^{2^*(s)}
= \frac{\mu_{\gamma,0}(\mathbb{R}^n)}{b(0)^{2^{2^*(s)}}} - m_{\gamma,h}(\Omega) \frac{\omega_{n-1}(\beta_+ - \beta_-)}{b(0) \int_{\mathbb{R}^n} \frac{U^{2^*(s)}}{|x|^s} dx} \varepsilon^{\beta_+ - \beta_-} + o(\varepsilon^{\beta_+ - \beta_-}). \tag{4.15} \]

Therefore, if \( m_{\gamma,h}(\Omega) > 0 \), we get for \( \varepsilon \) sufficiently small
\[
\mu_{\gamma,h}(\Omega) \leq J_{\gamma, \lambda, a}^{\Omega}(u_{\varepsilon}) < \frac{\mu_{\gamma,0}(\mathbb{R}^n)}{b(0)^{2^{2^*(s)}}},
\]

Then, from Lemma 4.1 it follows that \( \mu_{\gamma,h}(\Omega) \) is attained. \( \Box \)

**Remark 4.1.** Assume for simplicity that \( h(x) = \lambda |x|^{-\theta} \) where \( 0 \leq \theta < 2 \). There is a threshold \( \lambda^*(\Omega) \geq 0 \) beyond which the infimum \( \mu_{\gamma, \lambda}(\Omega) \) is achieved, and below which, it is not. In fact,
\[
\lambda^*(\Omega) := \sup \{ \lambda : \mu_{\gamma, \lambda}(\Omega) = \mu_{\gamma,0}(\mathbb{R}^n) \}.
\]
Performing a blow-up analysis like in [12] one can obtain the following sharp results:

- In high dimensions, that is for \( \gamma \leq \left( \frac{n-2}{4} \right)^2 - \frac{(2-\theta)^2}{4} \) one has \( \lambda^*(\Omega) = 0 \) and the infimum \( \mu_{\gamma, \lambda}(\Omega) \) is achieved if and only if \( \lambda > \lambda^*(\Omega) \).

- In low dimensions, that is for \( \left( \frac{n-2}{4} \right)^2 - \frac{(2-\theta)^2}{4} < \gamma \), one has \( \lambda^*(\Omega) > 0 \) and \( \mu_{\gamma, \lambda}(\Omega) \) is not achieved for \( \lambda < \lambda^*(\Omega) \) and \( \mu_{\gamma, \lambda}(\Omega) \) is achieved for \( \lambda > \lambda^*(\Omega) \). Moreover under the assumption \( \mu_{\gamma, \lambda}(\Omega) \) is not achieved, we have that \( m_{\gamma, \lambda}(\Omega) = 0 \), and \( \lambda^*(\Omega) = \sup\{\lambda : m_{\gamma, \lambda}(\Omega) \leq 0\} \).

5. Existence results for compact submanifolds of \( \mathbb{B}^n \)

Back to the following Dirichlet boundary value problem in hyperbolic space. Let \( \Omega \subset \mathbb{B}^n \) (\( n \geq 3 \)) be a bounded smooth domain such that \( 0 \in \Omega \). We consider the Dirichlet boundary value problem:

\[
\begin{aligned}
-\Delta_{\mathbb{B}^n} u - \gamma V_2 u - \lambda u &= V_{2^{r}(s)} u^{2^{r}(s)-1} \quad \text{in } \Omega \\
\quad u &\geq 0 \quad \text{in } \Omega \\
\quad u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

where \( \lambda \in \mathbb{R} \), \( 0 < s < 2 \) and \( \gamma < \gamma_H := \left( \frac{n-2}{4} \right)^2 \).

We shall use the conformal transformation \( g_{\mathbb{B}^n} = \varphi^{4-n} g_{\text{Eucl}} \) where \( \varphi = \left( \frac{2}{1-r^2} \right)^{\frac{n-2}{2}} \) to map the problem into \( \mathbb{R}^n \). We start by considering the general equation:

\[-\Delta_{\mathbb{B}^n} u - \gamma V_2 u - \lambda u = f(x, u) \quad \text{in } \Omega \subset \mathbb{B}^n, \tag{5.2}\]

where \( f(x, u) \) is a Carathéodory function such that

\[|f(x, u)| \leq C|u| \left( 1 + \frac{|u|^{2^{r}(s)-2}}{r^3} \right) \quad \text{for all } x \in \Omega.\]

If \( u \) satisfies (5.2), then \( v := \varphi u \) satisfies the equation:

\[-\Delta v - \gamma \left( \frac{2}{1-r^2} \right)^2 V_2 v - \left( \lambda - \frac{n(n-2)}{4} \right) \left( \frac{2}{1-r^2} \right)^2 v = \varphi^{\frac{4-n}{2}} f \left( x, \frac{v}{\varphi} \right) \quad \text{in } \Omega.\]

On the other hand, we have the following expansion for \( \left( \frac{2}{1-r^2} \right)^2 V_2 \):

\[
\left( \frac{2}{1-r^2} \right)^2 V_2(x) = \frac{1}{(n-2)^2} \left( \frac{f(r)}{G(r)} \right)^2
\]

where \( f(r) \) and \( G(r) \) are given by (1.1). We then obtain that

\[
\left( \frac{2}{1-r^2} \right)^2 V_2(x) = \begin{cases} \frac{1}{r^2} + \frac{4}{n-2} \left( \frac{1-r^2}{1-r^2} + \frac{4}{1-r^2} \right) & \text{when } n = 3, \\
\frac{1}{r^2} + 8 \log \frac{1}{r} - 4 + g_4(r) & \text{when } n = 4, \\
\frac{1}{r^2} + 4(n-2) + r g_n(r) & \text{when } n \geq 5. \tag{5.3}\end{cases}
\]

where for all \( n \geq 4 \), \( g_n(0) = 0 \) and \( g_n \) is \( C^0([0, \delta]) \) for \( \delta < 1 \).

This implies that \( v := \varphi u \) is a solution to

\[-\Delta v - \gamma \frac{r^2}{r^2} v - \left[ \gamma a(x) + \left( \lambda - \frac{n(n-2)}{4} \right) \left( \frac{2}{1-r^2} \right)^2 \right] v = \varphi^{\frac{4-n}{2}} f \left( x, \frac{v}{\varphi} \right),\]

where \( a(x) \) is defined in (1.8). We can therefore state the following lemma:
Lemma 5.1. A non-negative function $u \in H^1_0(\Omega)$ solves (5.1) if and only if $v := \varphi u \in H^1_0(\Omega)$ satisfies
\[
\begin{cases}
-\Delta v - \left(\frac{\gamma}{|x|^2} + h_{\gamma,\lambda}(x)\right) v = b(x)\frac{u^2(x) - 1}{|x|^2} & \text{in } \Omega \\
v \geq 0 & \text{in } \Omega \\
v = 0 & \text{on } \partial\Omega,
\end{cases}
\]
where
\[h_{\gamma,\lambda}(x) = \gamma a(x) + \frac{4\lambda - n(n-2)}{(1-|x|^2)^2},\]
and $a(x)$ is defined in (1.8), and $b(x)$ is a positive function in $C^1(\Omega)$ with $b(0) = \frac{(n-2)^{\frac{n-s}{2}}}{2^{2-s}}$ and $\nabla b(0) = 0$.
Moreover, the hyperbolic operator $L^\mathbb{H}_n := -\Delta_{\mathbb{H}_n} - \gamma V_2 - \lambda$ is coercive if and only if the corresponding Euclidean operator $L^\mathbb{E}_{n,h} := -\Delta - \left(\frac{\gamma}{|x|^2} + h_{\gamma,\lambda}(x)\right)$ is coercive.

Proof. Note that one has in particular
\begin{align*}
h_{\gamma,\lambda}(x) &= h_{\gamma,\lambda}(r) = \\
&= \begin{cases}
\frac{4\gamma + 4\gamma - 3}{(1-r)^2} + \frac{4\gamma}{1-r} & \text{when } n = 3, \\
\frac{8\gamma \log \frac{1}{1-r} - 4\gamma + 4\gamma - 8}{r(1-r)^2} + \gamma g_4(r) + (4\gamma - 8)\frac{r^2(2-r^2)}{(1-r)^2} & \text{when } n = 4, \\
\frac{4(n-2)}{n-4} \left[ \frac{n-4}{n-2} \lambda + \gamma - \frac{n(n-4)}{4} \right] \gamma g_n(r) + (4\gamma - n(n-2))\frac{r^2(2-r^2)}{(1-r)^2} & \text{when } n \geq 5,
\end{cases}
\end{align*}
with $g_n(0) = 0$ and $g_n$ is $C^0([0,\delta])$ for $\delta < 1$, for all $n \geq 4$.

Let $f(x, u) = V^2 u^{2^*(s)-1}$ in (5.2). The above remarks show that $v := \varphi u$ is a solution to (5.4).

For the second part, we first note that the following identities hold:
\[\int_{\Omega} \left( |\nabla_{\mathbb{H}_n} u|^2 - \frac{n(n-2)}{4} u^2 \right) dv_{g_{2n}} = \int_{\Omega} |\nabla v|^2 \, dx \]
and
\[\int_{\Omega} u^2 dv_{g_{2n}} = \int_{\Omega} v^2 \left( \frac{2}{1-\gamma^2} \right)^2 dx.\]
If the operator $L^\mathbb{H}_n$ is coercive, then for any $u \in C^\infty(\Omega)$, we have $\langle L^\mathbb{H}_n u, u \rangle \geq C\|u\|^2_{H^1_0(\Omega)}$, which means
\[\int_{\Omega} \left( |\nabla_{\mathbb{H}_n} u|^2 - \gamma V_2 u^2 \right) dv_{g_{2n}} \geq C \int_{\Omega} \left( |\nabla_{\mathbb{H}_n} u|^2 + u^2 \right) dv_{g_{2n}}.\]
The latter then holds if and only if
\[\langle L^\mathbb{H}_{n,\varphi} u, u \rangle = \int_{\Omega} \left( |\nabla v|^2 - \left( \frac{2}{1-\gamma^2} \right)^2 \left( \gamma V_2 - \frac{n(n-2)}{4} \right) v^2 \right) \, dx \]
\[\geq C \int_{\Omega} \left( |\nabla v|^2 + \left( \frac{2}{1-\gamma^2} \right)^2 \left( \frac{n(n-2)}{4} + 1 \right) v^2 \right) \, dx \]
\[\geq C' \int_{\Omega} \left( |\nabla v|^2 + v^2 \right) \, dx \geq C\|u\|^2_{H^1_0(\Omega)},\]
where $v = \varphi u$ is in $C^\infty(\Omega)$. This completes the proof. \qed
One can then use the results obtained in the last section to prove Theorems 1, 2 and 3 stated in the introduction for the hyperbolic space. Indeed, it suffices to consider equation (5.4), where \( b \) is a positive function in \( C^1(\overline{\Omega}) \) satisfying (1.7) and \( h_{\gamma,\lambda} \) is given by (5.5).

If \( n \geq 5 \), then \( \lim_{|x| \to 0} h_{\gamma,\lambda}(x) = \frac{4(n-2)}{n-4} \left[ \frac{n-2}{n-4} \lambda + \gamma - \frac{n(n-4)}{4} \right] \), which is positive provided
\[
\lambda > \frac{n-2}{n-4} \left( \frac{n(n-4)}{4} - \gamma \right).
\]
Moreover, since in this case \( \theta = 0 \), Theorem 4 holds when
\[
\gamma \leq \frac{(n-2)^2}{4} - 1 = \frac{n(n-4)}{4}.
\]

If \( n = 3 \), then \( \lim_{|x| \to 0} |x|h_{\gamma,\lambda}(x) = 4\gamma \), hence \( \gamma \) needs to be positive. On the other hand, since we use \( \theta = 1 \), the first option in Theorem 4 cannot occur and to have positive solutions one needs that the mass \( m^H_{\gamma,\lambda}(\Omega) \) to be positive. We note that the mass \( m^H_{\gamma,\lambda}(\Omega) \) associated to the operator \( L_{\gamma,\lambda}^B \) is a positive multiple of mass of the corresponding Euclidean operator. In other words, they both have the same sign.

Similarly, for \( n = 4 \), we have that \( \lim_{|x| \to 0} \frac{|x|h_{\gamma,\lambda}(x)}{\log \frac{1}{|x|}} = 8\gamma \). Hence \( \gamma \) needs to be positive. On the other hand, \( \gamma \) needs to be less than \( \frac{(4-2)^2}{4} - 1 = 0 \), which is not possible. Hence the first option in Theorem 4 cannot occur, and again one needs that the mass \( m^H_{\gamma,\lambda}(\Omega) \) be positive.

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