QUASI-BOUNDED GEOMETRY OF THE BERGMAN METRIC
AND EQUIVALENCE OF INVARIANT METRICS

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ABSTRACT. We provide the tractable computational method to show the equivalence of invariant metrics on large classes of hyperbolic \(\mathbb{K}\)ähler manifolds with the Bergman metric. Precisely, we consider a class of bounded weakly pseudo-convex domains \(\Omega \subset \mathbb{C}^n\) with Lipschitz boundaries \(\partial \Omega\) having three properties: (i) there exists a compact subset \(K \subset \Omega \cup \partial \Omega\) such that for any point \(x \in \Omega\), there exists an automorphism \(\phi: \Omega \to \Omega\) with \(\phi(x) \in \Omega \cap K\), (ii) the holomorphic sectional curvature of the Bergman metric is negative near \(\partial K \cap \partial \Omega\), and (iii) the Bergman metric admits the quasi-bounded geometry. For this class, we show that the Bergman metric is uniformly equivalent to the complete Kähler–Einstein metric with Ricci curvature equal to \(-1\) and to the Kobayashi–Royden metric. As nontrivial example, we consider the 3-dimensional bounded domains

\[ E_{q,r} := \{(x, y, z) \in \mathbb{C}^3 : |x|^2 + |y|^q < 1, |x|^2 + |z|^r < 1\}, q, r > 0.\]

We proceed with the necessary computations to verify all assumptions (i),(ii), and (iii) and show that for each \(q, r > 0\), three invariant metrics are uniformly equivalent on \(E_{q,r}\). Finally, we introduce the symmetric curvature property of the Bergman metric and show that \(E_{q,r}\) satisfies the symmetric curvature property at the \((0,0,0) \in E_{q,r}\) if and only if \(q = r\).

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1. Introduction

Equivalence of classical invariant metrics including the Bergman metric, the complete Kähler–Einstein metric of negative scalar curvature, the Kobayashi–Royden metric, and the Carathéodory–Reiffen metric on hyperbolic complex manifolds has been studied in complex geometry [17]. Some well-known classes having equivalence of these metrics are complex manifolds with uniform squeezing property, smoothly bounded strictly pseudoconvex domains in \( \mathbb{C}^n \), and weakly pseudoconvex domains of finite type in \( \mathbb{C}^2 \) [4, 20]. In complex dimension 3, the equivalence of these metrics breaks down for some weakly pseudoconvex domains with analytic boundary [10].

One method to show the equivalence of the invariant metrics on a complete Kähler manifold \((M, \omega)\) is to prove that the holomorphic sectional curvature of \(\omega\) has a negative range. Then by a fundamental result of Wu and Yau [18], the metric \(\omega\) is uniformly equivalent to the complete Kähler–Einstein metric with Ricci curvature equal to \(-1\) and to the Kobayashi–Royden metric. Hence if one knows a concrete formula of the Bergman kernel on a bounded weakly pseudoconvex domain with Lipschitz boundary, one could attempt to compute the holomorphic sectional curvature of the Bergman metric to establish the equivalence of the invariant metrics.

Indeed, in complex dimension 2, some examples have recently been studied with concrete formulae for Bergman kernels. For complex ellipsoids
\[
E = \{(z, w) \in \mathbb{C}^1 \times \mathbb{C}^1 : |z|^2 + |w|^{2p} < 1\}, \quad p > 0,
\]
this approach was taken in [7] to investigate relations among the invariant metrics by computing the curvature quantities. In [8], it was shown that the holomorphic sectional curvature of the Bergman metric on symmetrized bidisc
\[
G_2 = \{(z + w, zw) \in \mathbb{C}^2 : |z|, |w| < 1\}
\]
is negatively pinched. As result, the equivalence of invariant metrics and other consequences was obtained.

As for complex dimension \(\geq 3\), explicit formulae are recently obtained for the Bergman kernels on certain weakly pseudoconvex domains (e.g., see [1, 2, 9, 16] and references therein). However, it seems to be a daunting task to compute the holomorphic sectional curvatures for these domains.

In this paper, we establish a new method to study relations among the invariant metrics on weakly pseudoconvex domains in \(\mathbb{C}^n\) which might be quite useful to check concretely for \(n \geq 3\), which does not require complete computation of the holomorphic sectional curvature and the information of the squeezing function. This method is still strong enough to deduce the equivalence of invariant metrics. More precisely, we prove the following theorem.

**Theorem 1.** Let \(\Omega\) be a bounded weakly pseudoconvex domain in \(\mathbb{C}^n\) with Lipschitz boundary. Suppose that there exists a compact subset \(K \subseteq \Omega \cup \partial \Omega\) on which the interior of \(K\) does not meet \(\partial \Omega\). Assume (i) for any point \(x \in \Omega\), there exists an automorphism \(\phi : \Omega \to \Omega\) with \(\phi(x) \in K\), (ii) the holomorphic sectional curvature of the Bergman metric is negative near \(\partial K \cap \partial \Omega\), and (iii) the Bergman metric admits
quasi-bounded geometry. Then the Bergman metric is uniformly equivalent to the complete Kähler–Einstein metric with Ricci curvature $-1$ and to the Kobayashi–Royden metric.

The consequence of Theorem 1 concerning the existence of the Kähler-Einstein metric with Ricci curvature $-1$ forces such a bounded domain must be pseudoconvex [15]. Also, the Bergman metric on a bounded weakly pseudoconvex domain in $\mathbb{C}^n$ with Lipschitz boundary is complete [5]. We believe that Theorem 1 covers a large class of pseudoconvex domains (also see Remark 5 as a counter part).

To provide a non-trivial example of Theorem 1, we should proceed with the computational recipe for showing the bounded geometry. For this purpose, we consider a family of intersection of two complex ellipsoids

$$E_{q,r} = \{(x, y, z) \in \mathbb{C}^3 : |x|^2 + |y|^q < 1, |x|^2 + |z|^r < 1\}, q, r > 0,$$

which is interesting in its own right. The result with $E_{q,r}$ is the following:

**Theorem 2.** For each $q, r > 0$, there exists a compact subset $K \subseteq E_{q,r} \cup \partial E_{q,r}$ such that $K$ satisfies assumptions (i),(ii),(iii) stated in Theorem 1. Consequently, the Bergman metric is uniformly equivalent to the complete Kähler–Einstein metric with Ricci curvature $-1$ and to the Kobayashi–Royden metric on $E_{q,r}$.

In the proof, we use a concrete formula for the Bergman kernels of $(E_{r,q})_{r,q>0}$, which was obtained in [3] to show that the Bergman metrics satisfy all assumptions in Theorem 1. The computational processes that make sure this example satisfies the conditions are really interesting. In particular, since this example is a three-dimensional weakly pseudoconvex domain, the holomorphic sectional curvature generally not known about boundary behavior. Nevertheless, the quasi-bounded geometry and the negative range of holomorphic sectional curvature near the boundary of the Bergman metric can be confirmed from the sophisticated calculation. Finally, not only investigating the boundary behavior of curvature quantities, but we also provide new curvature behavior at the origin of $E_{q,r}$ which we call by the symmetric curvature property that happens only when $q = r$ in the last section of the paper.

In general, this observation is very interesting in that, apart from what people have studied on the boundary tendency of the holomorphic curvature, the interior of the complex manifolds, and it is determined about the space from a single point.

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2. Quasi-bounded geometry

The notion of quasi-bounded geometry is introduced by S. T. Yau and S. Y. Cheng ([6]). We adopt the following formulation. Let $(M, \omega)$ be an $n$-dimensional complete Kähler manifold. For a point $p \in M$, let $B_\omega(p; \rho)$ be the open geodesic ball centered at $p$ in $M$ of radius $\rho$; we omit the subscript $\omega$ if there is no peril of
confusion. Denote by $B_{\mathbb{C}^n}(r)$ the open ball centered at the origin in $\mathbb{C}^n$ of radius $r$ with respect to the standard metric $\omega_{\mathbb{C}^n}$.

**Definition 3.** An $n$-dimensional Kähler manifold $(M, \omega)$ is said to have quasi-bounded geometry if there exist two constants $r_2 > r_1 > 0$ such that for each point $p \in M$, there is a domain $U \subset \mathbb{C}^n$ and a nonsingular holomorphic map $\psi : U \to M$ satisfying the following properties:

1. $B_{\mathbb{C}^n}(r_1) \subset U \subset B_{\mathbb{C}^n}(r_2)$ and $\psi(0) = p$;
2. there exists a constant $C > 0$ depending only on $r_1, r_2, n$ such that $C^{-1}\omega_{\mathbb{C}^n} \leq \psi^*(\omega) \leq C\omega_{\mathbb{C}^n}$ on $U$;
3. for each integer $l \geq 0$, there exists a constant $A_l$ depending only on $l, n, r_1, r_2$ such that
   \[
   \sup_{x \in U} \left| \frac{\partial^{\mu+\nu} g_{\mathcal{J}}}{\partial v^\mu \partial v^\nu} \right| \leq A_l, \text{ for all } |\mu| + |\nu| \leq l,
   \]
   where $g_{\mathcal{J}}$ are the components of $\psi^*\omega$ on $U$ in terms of the natural coordinates $(v^1, \ldots, v^n)$, and $\mu, \nu$ are multiple indices with $|\mu| = \mu_1 + \cdots + \mu_n$. We call $r_1$ a radius of quasi-bounded geometry.

We will apply the following theorem in order to prove that a complete Kähler metric admits quasi-bounded geometry.

**Theorem 4** ([18], Theorem 9). Let $(M, \omega)$ be a complete Kähler manifold. Then the manifold $(M, \omega)$ has quasi-bounded geometry if and only if for each integer $q \geq 0$, there exists a constant $C_q > 0$ such that

\[
\sup_{p \in M} |\nabla^q R_m| \leq C_q, \tag{2.1}
\]

where $R_m = \{R_{\mathcal{J}kl}\}$ denotes the curvature tensor of $\omega$. In this case, the radius of quasi-bounded geometry depends only on $C_0$ and the dimension of $M$.

Note that the left-hand side in (2.1) is the coordinate-free expression. Consequently, one can check (2.1) with any choice of holomorphic orthonormal frames. We will use this observation to prove Theorem 2, especially we apply the Gram–Schmidt process to determine an orthonormal frame $X, Y, Z$ in Section 4.

### 3. Proof of the Main Theorem

**Proof of Theorem 1.** Consider a Kähler metric of the form

\[
g_\Omega := mg_P + g_B, \quad m > 0,
\]

where $g_B$ is the Bergman metric on $\Omega$ and $g_P$ is the Poincaré metric of a ball $D$ in $\mathbb{C}^n$ with $\Omega \subset D$. Since $\partial \Omega$ is Lipshitz, the Bergman metric $g_B$ is complete. Thus $g_\Omega$ is complete on $\Omega$ for any $m > 0$. 


Recall from [19] that a holomorphic sectional curvature is the Gaussian curvature of a holomorphically embedded disk. We denote the Gaussian curvature of any hermitian metric $g$ by $H_g$. First we show that $H_{g_{\Omega}}$ has a negative upper bound for $m$ sufficiently large. For $\epsilon > 0$, let $C_\epsilon := \{ z \in \Omega : H_{g_{\Omega}} > -\epsilon \}$. Then it follows from the formula of Grauert and Reckziegel [14] that, on the set $C_\epsilon$, one has

$$H_{g_{\Omega}} \leq \left( \frac{1}{1 + mu} \right)^2 \| H_{g_{B}} \|_K + \left( \frac{u}{1 + u} \right) \frac{1}{m} H_{g_P}, \quad (3.1)$$

where $u = \min g_P(t,t)$ for all $t \in \mathbb{C}^n$ with $g_B(t,t) = 1$ (see p.280 in [13]). Since the sum of two Kähler metrics with negative upper bounds for holomorphic sectional curvatures has a negative upper bound for the holomorphic sectional curvature (see Lemma 2 in [19]), it is enough to control the quantity of $(3.1)$ on $C_\epsilon$.

Note that $H_{g_P}$ is constant for any point in $\Omega$. With the invariance of the Bergman metric and the assumption of automorphisms, it suffices to control the upper bound of the following inequality:

$$H_{g_{\Omega}} \leq \left( \frac{1}{1 + mu} \right)^2 \| H_{g_{B}} \|_K + \left( \frac{u}{1 + u} \right) \frac{1}{m} H_{g_P}, \quad (3.2)$$

on $C_\epsilon \cap K$. Notice that the quasi-bounded geometry of $g_B$ forces that $\| H_{g_B} \|_K$ is bounded by some constant. Also, since the interior of $K$ does not meet $\partial \Omega$, the number $u$ can be zero on $C_\epsilon \cap K$ only when $u$ takes (limit) values on $\partial K \cap \partial \Omega$. However, by the assumption (ii), we have $H_{g_B} \leq -\epsilon$ on $\partial K \cap \partial \Omega$ for a sufficiently small $\epsilon > 0$, and $C_\epsilon \cap K$ is away from $\partial K \cap \partial \Omega$. Consequently the minimum and the maximum of $u$ on $C_\epsilon \cap K$ should be both positive by the compactness of $K$. Now we can find sufficiently large $m \gg 0$ so that the right-hand side of $(3.2)$ becomes uniformly negative, and $H_{g_{\Omega}}$ has a negative upper bound for $m$ sufficiently large.

Next, we show that $g_{\Omega}$ admits quasi-bounded geometry by checking conditions in Definition 3. From the quasi-bounded geometry of $g_B$, the first requirement in Definition 3 is clearly satisfied. Since the ball $D$ contains $\Omega$, the Poincaré metric $g_P$ on $\Omega$ is merely a weighted hermitian inner product in $\mathbb{C}^n$, and thus the second requirement is satisfied trivially. For the last requirement, write

$$g_{\Omega} = \sqrt{-1} \partial \bar{\partial} \log K_B K_P^m,$$

where $K_B$ and $K_P$ are the Bergman kernels on $\Omega$ and $D$, respectively. Then $K_B$ satisfies the last requirement, and $K_P^m$ never blows up with any $k$th-order derivative on $\Omega$. This proves that $g_{\Omega}$ admits quasi-bounded geometry.

Also we see that the holomorphic sectional curvature $H_{g_{\Omega}}$ of $g_{\Omega}$ is bounded below. Indeed, it is sufficient to check quasi-bounded geometry of order 2 in (2.1), i.e., the $k$th derivatives of metric $g_{\Omega}$ with respect to the local quasi-coordinates are bounded for $k \leq 2$. Since we already know that $g_{\Omega}$ admits quasi-bounded geometry, it is satisfied.

Now, since the holomorphic sectional curvature of $g_{\Omega}$ is negatively pinched, the metric $g_{\Omega}$ is uniformly equivalent to the Kobayashi–Royden metric and to the complete Kähler–Einstein metric of Ricci curvature $-1$ by Theorems 2 and 3 in [18].
Moreover, by quasi-bounded geometry of $g_B$ and Lemma 20 in [18], we have
\[ C^{-1} \chi_\Omega \leq \sqrt{g_B} \]
where $\chi_\Omega$ denotes the Kobayashi–Royden metric and $C$ is a universal constant. On the other hand, by the negative upper bound of $g_\Omega$ and Lemma 19 in [18], we also have
\[ \sqrt{g_\Omega} \leq C\chi_\Omega. \]
Since $g_B \leq g_\Omega$ from the construction of $g_\Omega$, we obtain in all
\[ C^{-1} \chi_\Omega \leq \sqrt{g_B} \leq \sqrt{g_\Omega} \leq C\chi_\Omega. \]
Hence the Bergman metric is also uniformly equivalent to the Kobayashi–Royden metric. \hfill \Box

**Remark 5.** There are weakly pseudoconvex domains $\Omega$ in $\mathbb{C}^n$ which do not satisfy all assumptions in Theorem 1 ([10], [11]). For example, consider $\Omega := \{(z_1, z_2, z_3) \in \mathbb{C}^3; \Re z_1 + |z_2|^{12} + |z_3|^{12} + |z_2|^4|z_3|^2 + |z_2|^2|z_3|^6 < 0\}$. It was shown that the Bergman metric is not equivalent to the Kobayashi–Royden metric (Theorem 3 in [10]). This implies that the Bergman metric on $\Omega$ does not admit the negative holomorphic sectional curvature and some assumptions in Theorem 1 must be failed.

4. The Intersection of Complex Ellipsoids $E_{q,r}$

We will investigate the intersection of two complex ellipsoids $E_{q,r} = \{(x, y, z) \in \mathbb{C}^3; |x|^2 + |y|^q < 1, |x|^2 + |z|^r < 1\}, q, r > 0$.

Any point $(x, y, z) \in E_{q,r}$ can be mapped onto the form $(0, \tilde{y}, \tilde{z})$ by the following automorphism
\[(x, y, z) \mapsto \frac{x-a}{1-\overline{a}x}, \frac{(1-|a|^2)^{1/q}}{(1-\overline{a}x)^{2/q}}y, \frac{(1-|a|^2)^{1/r}}{(1-\overline{a}x)^{2/r}}z \in \mathbb{C}^3.\]

Furthermore, one can take the rotations as automorphisms to make $y, z$ to be the real valued. Since Bergman metric is preserved by any automorphism, it is natural to take the compact set $K_1 = \{(0, y, z) \in E_{q,r}, y, z \in [0, 1)\}$ with the closure of the usual topology in $\mathbb{C}^3$.

In [3], the formula of the Bergman kernel $K_{E_{q,r}}$ on $E_{q,r}$ is explicitly known:
\[ B_{E_{q,r}}((x, y, z), (x, y, z)) = \frac{qr(1-\mu_2)(1-\mu_3) + 2q(1-\mu_2)(1+\mu_3) + 2r(1+\mu_2)(1-\mu_3)}{\pi^3 qr(1-x\overline{x})^{2+\frac{2}{q}+\frac{2}{r}}(1-\mu_2)^3(1-\mu_3)^3}, \]
where $\mu_2, \mu_3$ are given by
\[ \mu_2 = \frac{y\overline{y}}{(1-x\overline{x})^q}, \mu_3 = \frac{z\overline{z}}{(1-x\overline{x})^r}. \]

From the formula of the Bergman kernel in (4.1), one can realize that the derivative with respect to $y, \overline{y}$ or $z, \overline{z}$ variables at any $(0, y, z) \in K_1$ are the same as the derivatives of $B_{E_{q,r}}((0, y, z), (0, y, z))$. 
With
\[ g_\mathcal{J} = \frac{\partial^2 \log B_{E_{q,r}}}{\partial z_i \partial \overline{z}_j}, \]
elementary computations yield the following proposition.

**Proposition 6.** Each components of the Bergman metric \( g_\mathcal{J} \) at \((0, y, z), 0 \leq y, z < 1\) in \( E_{q,r} \) are given as follows:

\[
\begin{align*}
\hat{g}_1 &= \frac{a_1}{q r (y^2 - 1) (z^2 - 1) (q (y^2 - 1) (r (z^2 - 1) - 2 (z^2 + 1)) - 2r (y^2 + 1) (z^2 - 1))} \\
ge_2 &= \frac{a_2}{(y^2 - 1)^2 (q (y^2 - 1) (r (z^2 - 1) - 2 (z^2 + 1)) - 2r (y^2 + 1) (z^2 - 1))^2}, \\
ge_3 &= \frac{-q r y z}{16q r y z} \\
ge_4 &= \frac{a_3}{(z^2 - 1)^2 (q (y^2 - 1) (r (z^2 - 1) - 2 (z^2 + 1)) - 2r (y^2 + 1) (z^2 - 1))^2}, \\
\hat{g}_5 &= 0 \text{ otherwise.}
\end{align*}
\]

Here, \( a_1, a_2, a_3 \) are written as

\[
\begin{align*}
a_1 &= 2q^2 (y^2 - 1)^2 \left(r^2 (z^2 - 1)^2 - 3r (z^4 - 1) + 2 (z^4 + 4z^2 + 1)\right) \\
&- 2qr (y^4 - 1) (z^2 - 1) (3r (z^2 - 1) - 4 (z^2 + 1)) + 4r^2 (y^4 + 4y^2 + 1) (z^2 - 1)^2, \\
a_2 &= 2(q^2 (y^2 - 1)^2 \left(r (z^2 - 1) - 2 (z^2 + 1)\right)^2 - 6qr (y^4 - 1) (z^2 - 1) (r (z^2 - 1) - 2 (z^2 + 1)) \\
&+ 8r^2 (y^4 + y^2 + 1) (z^2 - 1)^2), \\
a_3 &= 2((q - qy^2)^2 \left(r^2 (z^2 - 1)^2 - 6r (z^4 - 1) + 8 (z^4 + z^2 + 1)\right) \\
&- 4qr (y^4 - 1) (z^2 - 1) (r (z^2 - 1) - 3 (z^2 + 1)) + 4r^2 (y^2 + 1)^2 (z^2 - 1)^2).
\end{align*}
\]

One can notice that the matrix above becomes the block matrix having one \( 2 \times 2 \) block which indeed reduces the amounts of necessary computations. We will also write those metric components slightly different ways as follows:

**Proposition 7.** For each fixed \( z \) with \((0, y, z) \in E_{q,r}, 0 \leq y, z < 1, y \text{ around } 1\), we have

\[
\begin{align*}
\hat{g}_1 &= \frac{3}{q (1 - y)} + \text{higher degree terms}, \\
ge_2 &= \frac{3}{4 (1 - y)^2} + \text{higher degree terms}, \\
ge_3 &= \frac{-q z}{r (1 - z)^2} + \text{higher degree terms}, \\
ge_4 &= \frac{2}{(1 - z^2)^2} + \text{higher degree terms}.
\end{align*}
\]

Above, all degrees are taken with respect to \((1 - y)\)-term.
Proposition 8. For each fixed $y$ with $(0, y, z) \in E_{q, r}, 0 \leq y, z < 1, z$ around 1, we have

\begin{align*}
g_{1\bar{1}} &= \frac{3}{r(1 - z)} + \text{higher degree terms}, \\
g_{2\bar{2}} &= \frac{2}{(1 - y)^2} + \text{higher degree terms}, \\
g_{2\bar{3}} - g_{3\bar{2}} &= \frac{-ry}{q(1 - y^2)^2} + \text{higher degree terms}, \\
g_{3\bar{3}} &= \frac{3}{4(1 - z)^2} + \text{higher degree terms}.
\end{align*}

Above, all degrees are taken with respect to $(1 - z)$-term.

Note that by the Hopf-Rinow theorem, the Bergman metric is complete if for each $p \in E_{q, r}$ and $\xi_0 \in \partial E_{q, r}$,

$$\lim_{E_{q, r} \ni \xi \to \xi_0} \text{dist}_{E_{q, r}}(p, \xi) = \infty,$$

(4.2)

where $\text{dist}_{E_{q, r}}(p, \xi)$ is the distance of the Bergman metric between $p$ and $\xi$ and the limit is with respect to the Euclidean topology. Then from the invariance of the Bergman metric by automorphisms, it is enough to check for each $p \in K_1$ and $\xi_0 \in \partial K_1 \cap \partial E_{q, r}$,

$$\lim_{K_1 \ni \xi \to \xi_0} \text{dist}_{E_{q, r}}(p, \xi) = \infty.$$  

(4.3)

Then the completeness of the Bergman metric follows from above two propositions and thereby $E_{q, r}$ is a weakly pseudoconvex domain for each $q, r > 0$ (Theorem 15.1.1 in [12]).

On $E_{q, r}$, we will use the following orthonormal basis $X, Y, Z$ that will be used to prove the quasi-bounded geometry and necessary curvature computations. Let $g$ be the Bergman metric, and we proceed the Gram-Schmidts as follows: take the first unit vector field

$$X = \frac{\partial_1}{\sqrt{g_{1\bar{1}}}} = k_1 \partial_1.$$

(4.4)

Then a vector field $\tilde{Y}$ which is orthogonal to $X$ is given by

$$\tilde{Y} = \frac{\partial_2}{\sqrt{g_{2\bar{2}}}} - g\left(\frac{\partial_2}{\sqrt{g_{2\bar{2}}}} X\right) X = a_1 \partial_1 + a_2 \partial_2,$$

where we put

$$a_1 := -\frac{g_{2\bar{2}}}{g_{1\bar{1}} \sqrt{g_{2\bar{2}}}} \quad \text{and} \quad a_2 := \frac{1}{\sqrt{g_{2\bar{2}}}}.$$

Since $g(\tilde{Y}, \tilde{Y}) = a_1 \bar{a}_1 g_{1\bar{1}} + a_1 \bar{a}_2 g_{1\bar{2}} + a_2 \bar{a}_1 g_{2\bar{1}} + a_2 \bar{a}_2 g_{2\bar{2}}$, we take

$$Y = \frac{\tilde{Y}}{\sqrt{g(\tilde{Y})}} = \frac{a_1 \partial_1 + a_2 \partial_2}{\sqrt{a_1 \bar{a}_1 g_{1\bar{1}} + a_1 \bar{a}_2 g_{1\bar{2}} + a_2 \bar{a}_1 g_{2\bar{1}} + a_2 \bar{a}_2 g_{2\bar{2}}}} = t_1 \partial_1 + t_2 \partial_2,$$

(4.5)
where we put
\[ t_i := \frac{a_i}{\sqrt{a_1 a_1 g_{1\bar{1}} + a_1 a_2 g_{1\bar{2}} + a_2 a_1 g_{2\bar{1}} + a_2 a_2 g_{2\bar{2}}}} \quad i = 1, 2. \] (4.6)

Similarly, consider
\[ \tilde{Z} = p_1 \partial_1 + p_2 \partial_2 + p_3 \partial_3, \]
where
\[ p_1 := -\frac{g_{3\bar{3}}}{g_{1\bar{1}} g_{3\bar{3}}} - \frac{t_1}{\sqrt{g_{3\bar{3}}}} (t_1 g_{3\bar{1}} + t_2 g_{3\bar{2}}), \]
\[ p_2 := -\frac{t_2}{\sqrt{g_{3\bar{3}}}} (t_1 g_{3\bar{1}} + t_2 g_{3\bar{2}}), \]
\[ p_3 := \frac{1}{\sqrt{g_{3\bar{3}}}}, \]
and normalizing of \( \tilde{Z} \) yields
\[ Z = s_1 \partial_1 + s_2 \partial_2 + s_3 \partial_3, \] (4.7)
where
\[ s_i := \frac{p_i}{\sqrt{\sum_{k,l=1}^{3} p_k p_l g_{k\bar{l}}}}, \quad i = 1, 2, 3. \]

From elementary computations, we obtain the coefficient functions of orthonormal basis in (4.4),(4.5),(4.7).

**Proposition 9.** For each fixed \( z \) with \( (0, y, z) \in E_{q,r}, 0 \leq y, z < 1, y \) around 1, we have
\[ k_1 = \frac{2}{3} (1 - y)^{1/2} + \text{higher degree terms}, \]
\[ t_2 = \frac{2\sqrt{3}}{3} (1 - y) + \text{higher degree terms}, \]
\[ s_2 = \frac{2\sqrt{2} q}{3r} \frac{z}{1 - z^2} (1 - y^2) + \text{higher degree terms}, \]
\[ s_3 = \frac{1}{\sqrt{2}} (1 - z^2) + \text{higher degree terms}, \]
\[ t_3 = s_1 = 0. \]

Above, all degrees are taken with respect to \((1 - y)\)-term.

Similarly, we also have
Proposition 10. For each fixed $y$ with $(0, y, z) \in E_{q,r}, 0 \leq y, z < 1, z$ around 1, we have

$$k_1 = \frac{r}{3}(1 - z)^{1/2} + \text{higher degree terms},$$

$$t_2 = \frac{1}{\sqrt{2}}(1 - y^2) + \text{higher degree terms},$$

$$s_2 = \frac{ry}{\sqrt{3q}}(1 - z) + \text{higher degree terms},$$

$$s_3 = \frac{2}{\sqrt{3}}(1 - z) + \text{higher degree terms},$$

$$t_3 = s_1 = 0.$$

Above, all degrees are taken with respect to $(1 - z)$-term.

Recall that the Christoffel symbols $\Gamma^k_{ij}$ of a Kähler metric $g = (g_{ij})$ is written in local coordinates by

$$\Gamma^k_{ij} = g^{kl} \partial_i g_{lj}. \quad (4.8)$$

Then elementary computations yield the following two propositions:

Proposition 11. For each fixed $z$ with $(0, y, z) \in E_{q,r}, 0 \leq y, z < 1, y$ around 1, each Christoffel symbols $\Gamma^k_{ij}$ of the Bergman metric is given as follows:

$$\Gamma^1_{12} = \Gamma^1_{21} = \frac{1}{2}(1 - y)^{-1} + \text{higher degree terms},$$

$$\Gamma^2_{22} = (1 - y)^{-1},$$

$$\Gamma^1_{13} = \Gamma^1_{31} = \frac{2qz}{r(1 - z^2)^2}(1 - y) + \text{higher degree terms},$$

$$\Gamma^2_{23} = \Gamma^2_{32} = \frac{2q^2z(r(1 - z^2) - 6(1 + z^2)(1 - y)^2}{3r^2(1 - z^2)^3} + \text{higher degree terms},$$

$$\Gamma^3_{23} = \Gamma^3_{32} = \frac{q(-1 + z^4)}{2r(-1 + z^2)^2} + \text{higher degree terms},$$

$$\Gamma^2_{33} = \frac{8q^2z^2(1 - y)^3}{r^2(1 - z^2)^4} + \text{higher degree terms},$$

$$\Gamma^3_{33} = \frac{2z}{1 - z^2} + \text{higher degree terms}$$

$$\Gamma^k_{ij} = 0, \text{otherwise.}$$

Above, all degrees are taken with respect to $(1 - y)$-term.
Above, all degrees are taken with respect to $(1 - z)\text{-term}$. Hence we can deduce that the 0-th order bounded geometry can be shown to involve the abusing of notation, let us consider only one term in $V_f$ differentiation, each term has four Christoffel symbols, that is, with regarding the Christoffel symbol is one of the second term and the third term in the right-hand side of \((4.9)\) are similar as above and one can observe that each term possesses the specific pattern, that is, with regarding the Christoffel symbol is one differentiation, each term has four $f_i$'s, two differentiations, and one metric component. Hence we can deduce that the 0-th order bounded geometry can be shown to verify each term consists of 4 number of $f_i$s, 2 number of differentiation, and one
metric component is bounded. Furthermore, by extending the same reasoning, we can deduce that for any \( n \)-th order bounded geometry can be shown to verify each term consists of \( n + 4 \) number of \( f_i \)'s, \( n + 2 \) number of differentiation, and one metric component is bounded.

Fix any non-negative integer \( n \). By the invariance of the Bergman metric, it is enough to prove the bounded geometry on \( K_1 = \{(0, y, z) \in E_{q,r}, y, z \in [0, 1]\} \). Since any order derivative of Bergman kernel is clearly well-defined on the interior of \( K_1 \), it suffices to check the bounded geometry on \( \partial K_1 \cap \partial E_{q,r} \). Thus there are only two cases for the boundary points of \( K_1 \) to check, either with fixed \( z \) with \((0, y, z) \in E_{q,r}, 0 \leq y, z < 1, \) and \( y \) goes to 1 or with fixed \( y \) with \((0, y, z) \in E_{q,r}, 0 \leq y, z < 1, \) and \( z \) goes to 1.

Let’s consider the former case, i.e., with fixed \( z \) with \((0, y, z) \in E_{q,r}, 0 \leq y, z < 1, \) and \( y \) goes to 1. If \( n = 0 \), one can see that from (4.10), it is not hard to see that any terms consist of 4 coefficient functions, 2 differentiations, and one metric are all bounded by the following reasoning: there are metric components (here, \( g_{1\mathcal{T}}, g_{2\mathcal{T}} \)) contributing to \(-1\) degree of \((1 - y)\) and some Christoffel symbols that yield \(-1\) degree of \((1 - y)\) with lower indices \( i, j = 1, 2 \) for \( \Gamma^k_{ij} \), 4 coefficient functions with 2 differentiation (regarding one Christoffel symbol as one differentiation) must compensate the degree of \((1 - y)\) between 2 to 3. Then one can see that from 6 terms in the right-hand side in (4.10), indices \( i, j, k, l, m, q \) in those terms show that it is impossible to have any term with four coefficient functions consist of \( k_1 \)'s only. Hence with Proposition 7, Proposition 9, and Proposition 11, there is no term in the right-hand side of (4.10) that can be blown up as \( y \to 1 \).

Furthermore, from the formula of the Bergman kernel in (4.1), the derivative with respect to \( y, \bar{y} \) or \( z, \bar{z} \) variables at any \((0, y, z) \in K_1 \) are the same as the derivative of \( B_{E_{q,r}}((0, y, z), (0, y, z)) \). Hence we can use exactly the same propositions to track the degree of \((1 - y)\) of necessary components. Now notice that any derivatives with respect to \( x, \bar{x} \), or \( z, \bar{z} \) does not change the degrees of \((1 - y)\) in any terms listed in Proposition 7, Proposition 9, and Proposition 11. Thus from the same propositions, one can see that the negative degree of \( 1 - y \) could happen only when derivatives of \( k_1 \) with respect to \( y, \bar{y} \) appears. However, taking one derivative of \( k_1 \) with respect to \( y, \bar{y} \) should be coming from either \( \nabla_y(\cdot) \) or \( \nabla_{\bar{y}}(\cdot) \), hence a derivative of \( k_1 \) should be multiplied by either \( t_2, \) or \( s_2, \) and thus, each differentiation with respect to any variables does not yield a negative degree of \( 1 - y \). Since this argument can be repeated with any further covariant derivatives with \( X, Y, Z, \) consequently the boundedness of any term in the \( n \)th order covariant derivative of the curvature tensor follows when \( y \to 1 \).

The latter case, i.e., with fixed \( y \) with \((0, y, z) \in E_{q,r}, 0 \leq y, z < 1, \) and \( z \) goes to 1 case can be argued in the similar way with Proposition 8, Proposition 10, and Proposition 12 and the bounded geometry of infinite order follows. \( \square \)
The following two propositions are obtained by computations with Mathematica 10.4.

**Proposition 14.** On \((0, y, z) \in \partial K_1\) with \(y \to 1\),

\[
H(X) := R(X, \bar{X}, X, \bar{X}) = -2/3 \\
H(Y) := R(Y, \bar{Y}, Y, \bar{Y}) = -2/3 \\
H(Z) := R(Z, \bar{Z}, Z, \bar{Z}) = -1 \\
B(X, Y) := R(X, \bar{X}, Y, \bar{Y}) = -1/3 \\
B(X, Z) := R(X, \bar{X}, Z, \bar{Z}) = 0, \\
B(Y, Z) := R(Y, \bar{Y}, Z, \bar{Z}) = 0,
\]

\[
R(X, \bar{X}, X, \bar{Y}) = R(Y, \bar{Y}, Y, \bar{X}) = R(Y, \bar{X}, X, \bar{Y}) = R(X, \bar{X}, X, \bar{Z}) \\
= R(Z, \bar{Z}, Z, \bar{X}) = R(Z, \bar{X}, Z, \bar{X}) = R(Y, \bar{Y}, Y, \bar{Z}) = R(Z, \bar{Z}, Z, \bar{Y}) \\
= R(Z, \bar{Y}, Z, \bar{Y}) = R(X, \bar{X}, Y, \bar{Z}) = R(Y, \bar{Y}, X, \bar{Z}) = R(Z, \bar{Z}, X, \bar{Y}) = 0.
\]

With the same notation above, one can also get

**Proposition 15.** On \((0, y, z) \in \partial K_1\) with \(z \to 1\),

\[
H(X) = R(X, \bar{X}, X, \bar{X}) = -2/3 \\
H(Y) = R(Y, \bar{Y}, Y, \bar{Y}) = -1 \\
H(Z) = R(Z, \bar{Z}, Z, \bar{Z}) = -2/3 \\
B(X, Z) = R(X, \bar{X}, Z, \bar{Z}) = -1/3 \\
B(X, Y) = R(X, \bar{X}, Y, \bar{Y}) = 0, \\
B(Y, Z) = R(Y, \bar{Y}, Z, \bar{Z}) = 0,
\]

\[
R(X, \bar{X}, X, \bar{Y}) = R(Y, \bar{Y}, Y, \bar{X}) = R(Y, \bar{X}, X, \bar{Y}) = R(X, \bar{X}, X, \bar{Z}) \\
= R(Z, \bar{Z}, Z, \bar{X}) = R(Z, \bar{X}, Z, \bar{X}) = R(Y, \bar{Y}, Y, \bar{Z}) = R(Z, \bar{Z}, Z, \bar{Y}) \\
= R(Z, \bar{Y}, Z, \bar{Y}) = R(X, \bar{X}, Y, \bar{Z}) = R(Y, \bar{Y}, X, \bar{Z}) = R(Z, \bar{Z}, X, \bar{Y}) = 0.
\]

Finally, let us check (iii) stated in Theorem 1.

**Proposition 16.** The holomorphic sectional curvature near \(\partial K_1\) is negatively pinched.

**Proof.** Take any unit vector field \(V = aX + bY + cZ\) with respect to the Bergman metric with \(|a|^2 + |b|^2 + |c|^2 = 1\). Then at any point, by using symmetrics of the Riemann curvature tensor of a Kähler metric, one can write

\[
R(V, \bar{V}, V, \bar{V}) = |a|^4 H(X) + |b|^4 H(Y) + |c|^4 H(Z) \\
+ 4|a|^2 |b|^2 B(X, Y) + 4\text{Re}(\bar{a}b) (|a|^2 R(X, \bar{X}, X, \bar{Y}) + |b|^2 R(Y, \bar{Y}, Y, \bar{X})) + 2\text{Re}(\bar{a}b^2) R(Y, \bar{X}, X, \bar{X}) \\
+ 4|a|^2 |c|^2 B(X, Z) + 4\text{Re}(\bar{a}c) (|a|^2 R(X, \bar{X}, X, \bar{Z}) + |c|^2 R(Z, \bar{Z}, Z, \bar{X})) + 2\text{Re}(\bar{a}c^2) R(Z, \bar{X}, X, \bar{X}) \\
+ 4|b|^2 |c|^2 B(Y, Z) + 4\text{Re}(\bar{b}c) (|b|^2 R(Y, \bar{Y}, Y, \bar{Z}) + |c|^2 R(Z, \bar{Z}, Z, \bar{Y})) + 2\text{Re}(\bar{b}c^2) R(Z, \bar{Y}, Z, \bar{Y}) \\
+ 4|a|^2 \text{Re}(\bar{b}c) R(X, \bar{X}, Y, \bar{Z}) + 4|b|^2 \text{Re}(\bar{a}c) R(Y, \bar{Y}, X, \bar{Z}) + 4|c|^2 \text{Re}(\bar{a}b) R(Z, \bar{Z}, X, \bar{Y}).
\]
Note that there are only two types of the boundary points of $K_1$, that is, either with fixed $z$ with $(0, y, z) \in E_{q,r}, 0 \leq y, z < 1$, and $y$ goes to 1 or with fixed $y$ with $(0, y, z) \in E_{q,r}, 0 \leq y, z < 1$, and $z$ goes to 1, by Proposition 14 and Proposition 15,

$$R(V, \bar{V}, V, \bar{V}) = |a|^4 H(X) + |b|^4 H(Y) + |c|^4 H(Z) + 4|a|^2 |b|^2 B(X, Y) + 4|a|^2 |c|^2 B(X, Z) + 4|b|^2 |c|^2 B(Y, Z).$$

With same propositions, one can easily conclude that

$$H_{gB} \leq -\epsilon$$

for some $\epsilon > 0$ on $\partial K_1$. 

5. **Symmetric curvature property of the Bergman metric on $E_{q,r}$**

In this section, let’s address some interesting consequences on $E_{q,r}$ from the computations of holomorphic sectional curvature in the previous section. To do so, let us introduce the notion of symmetric curvature property on complex manifolds.

**Definition 17.** Let $(M^n, \omega_B)$ be a complex manifold with the Bergman metric $\omega_B$, $n \geq 2$. Let $p \in M$. We say that $M$ has the symmetric curvature property at $p$ (briefly s.c.p at $p$) if there are two orthonormal vector fields $X, Y$ around $p$ such that the holomorphic sectional curvatures $h(X)$ and $h(Y)$ with respect to $\omega_B$ coincide at $p$.

The s.c.p notion might be quite interesting to investigate on several classes of hyperbolic complex manifolds. For example, $\mathbb{C}^1 \times \mathbb{C}^1 : |z|^2 + |w|^{2p} < 1$, $p > 0$ admits the s.c.p at the origin if and only if $p = 1$ (see proposition 13 in [7]). From here, we can see that only special complex manifold las the Poincaré ball in $\mathbb{C}^2$ admits the s.c.p at the origin. We will see that s.c.p at the origin in $E_{q,r}$ also characterizes $q = r$ with the orthonormal basis $Y, Z$ that we used in the previous section. The computation yields the following proposition.
Proposition 18. With same \(Y, Z\) as before, the difference of the holomorphic sectional curvature \(h(Y) - h(Z)\) at \((0, 0, x)\) is precisely given by

\[
\frac{a_1(x)}{b_1(x)} - \frac{a_2(x)}{b_2(x)},
\]

\[
a_1(x) = (r^6(x^2 - 1)^6 - 18r^5(x^2 + 1)(x^2 - 1)^5 + 24r^4(5x^4 + 8x^2 + 5)(x^2 - 1)^4
- 96r^3(4x^6 + 9x^4 + 9x^2 + 4)(x^2 - 1)^3 + 48r^2(13x^8 + 44x^6 + 54x^4 + 44x^2 + 13)(x^2 - 1)^2
+ 128(x^{12} + 12x^{10} + 15x^8 + 16x^6 + 15x^4 + 12x^2 + 1) - 96r(5x^{12} + 24x^{10} + 9x^8 - 9x^4 - 24x^2 - 5)q^6
+ 12r(x^2 - 1)(r^5(x^2 - 1)^5 - 15r^4(x^2 + 1)(x^2 - 1)^4 + 16r^3(5x^4 + 8x^2 + 5)(x^2 - 1)^3
- 48r^2(4x^6 + 9x^4 + 9x^2 + 4)(x^2 - 1)^2 - 16(5x^{10} + 29x^8 + 38x^6 + 38x^4 + 29x^2 + 5) + 16r(13x^{10}
+ 31x^8 + 10x^6 - 10x^4 - 31x^2 - 13)q^5 + 12r^2(x^2 - 1)^2(5r^4(x^2 - 1)^4 - 60r^3(x^2 + 1)(x^2 - 1)^3
+ 48r^2(5x^4 + 8x^2 + 5)(x^2 - 1)^2 - 96r(4x^8 + 5x^6 - 5x^2 - 4) + 16(13x^8 + 44x^6 + 54x^4 + 44x^2 + 13)q^4
+ 32r^3(x^2 - 1)^3(5r^3(x^2 - 1)^3 - 45r^2(x^2 + 1)(x^2 - 1)^2 - 24(4x^6 + 9x^4 + 9x^2 + 4)
+ 24r(5x^6 + 3x^4 - 3x^2 - 5)q^3 + 48r^2(x^2 - 1)^4(5r^2(x^2 - 1)^2 - 30r(x^4 - 1) + 8(5x^4 + 8x^2 + 5))q^2
+ 192r^5(x^2 - 1)^5(r(x^2 - 1) - 3(x^2 + 1))q + 64r^6(x^2 - 1)^6),
\]

\[
b_1(x) = ((r^2(x^2 - 1)^2 - 6r(x^4 - 1) + 8(x^4 + x^2 + 1))q^2 + 4r(-3x^4 + r(x^2 - 1)^2 + 3)q + 4r^2(x^2 - 1)^2)^3,
\]

\[
a_2(x) = (q(r + 2)(x^2 - 1) - 2r(x^2 + 1))^2(8(4x^2 + 1)r^2 - 24q(r + 2)x^2r + 2q^2(r + 2)^2(2x^2 - 1))(x^2 - 1)^2
- 4(2r - q(r + 2))x^2(x^2 + 1) - q(r + 2)(x^2 - 1)(-12q(r + 2)x^3 + 2q^2(r + 2)(x^2 - 1)x
+ 8r^2(2x^3 + x))(x^2 - 1)^2 + 6q(r + 2) - 2r^2x^2(8(x^4 + x^2 + 1)r^2
- 6q(r + 2)(x^4 - 1)r + q^2(r + 2)^2(x^2 - 1)^2)(x^2 - 1)^2
- 2(2r - q(r + 2))(2r(x^2 + 1) - q(r + 2)(x^2 - 1))(8(x^4 + x^2 + 1)r^2
- 6q(r + 2)(x^4 - 1)r + q^2(r + 2)^2(x^2 - 1)^2)(x^2 - 1)^2
- \frac{c(x)}{8(x^4 + x^2 + 1)r^2 - 6q(r + 2)(x^4 - 1)r + q^2(r + 2)^2(x^2 - 1)^2}
+ 4x(1 - x^2)(q(r + 2)(x^2 - 1) - 2r(x^2 + 1))^2(-12q(r + 2)x^3 + 2q^2(r + 2)^2(x^2 - 1)x + 8r^2(2x^3 + x))
+ 6x^2(q(r + 2)(x^2 - 1) - 2r(x^2 + 1))^2(8(x^4 + x^2 + 1)r^2 - 6q(r + 2)(x^4 - 1)r + q^2(r + 2)^2(x^2 - 1)^2)
+ 2(1 - x^2)(q(r + 2)(x^2 - 1) - 2r(x^2 + 1))^2(8(x^4 + x^2 + 1)r^2 - 6q(r + 2)(x^4 - 1)r
+ q^2(r + 2)^2(x^2 - 1)^2 - 8x(2r - q(r + 2)x)(1 - x^2)(2r(x^2 + 1) - q(r + 2)(x^2 - 1))(8(x^4 + x^2 + 1)r^2
- 6q(r + 2)(x^4 - 1)r + q^2(r + 2)^2(x^2 - 1)^2)
\]

\[
b_2(x) = 2(8(x^4 + x^2 + 1)r^2 - 6q(r + 2)(x^4 - 1)r + q^2(r + 2)^2(x^2 - 1)^2)^2
\]

\[
c(x) = 4x^2(-8(2x^6 + 3x^4 + 6x^2 + 1)r^3 + 4q(r + 2)(5x^6 + 3x^4 - 3x^2 - 5)r^2
- 2q^2(r + 2)^2(x^2 - 1)^2(4x^2 + 5)r + q^2(r + 2)^3(x^2 - 1)^3)^2.
\]
Corollary 19. With same $Y, Z$ as before, the difference of the holomorphic sectional curvature $h(Y) - h(Z)$ at $(0, 0, 0)$ are given by

$$12 \left( q^4(-r - 2)^2 + 8q^3 r(r + 2) - q^2 r^3(r + 8) - 4qr^3(r + 4) - 4r^4 \right) \over (q(r) - 2q - 4r)^2(-q(r + 4) - 2r)^2$$

Consequently, $E_{q,r}$ admits the s.c.p at the origin if and only if $q = r$.

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