HOMOGENIZATION ON ARBITRARY MANIFOLDS
PROOF WITH TEST FUNCTIONS

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Abstract. We proof the homogenization of the Hamilton-Jacobi equation on arbitrary compact manifolds using Evans perturbed test function method.

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1. Introduction.

In this paper we proof the homogenization of the Hamilton-Jacobi equation in arbitrary manifolds using Evans perturbed test function method. A setting for generalizations of periodic homogenization results for PDE's in $\mathbb{R}^n$ to arbitrary manifolds was presented in Contreras, Iturriaga, Siconolfi [6] together with a proof of the homogenization of the Hamilton-Jacobi equation on arbitrary manifolds using the convergence of the formula of the solution. See also Contreras [3].

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We review the setting of the homogenization of the Hamilton-Jacobi equation on manifolds. Let \( M \) be a compact boundaryless manifold and \( T^*M \) its cotangent bundle. A Tonelli Hamiltonian on \( M \) is a \( C^2 \) function \( H : T^*M \to \mathbb{R} \) satisfying 

(a) Convexity: \( \frac{\partial H}{\partial p}(x, p) \) is positive definite \( \forall (x, p) \in T^*M \).

(b) Superlinearity: \( \lim_{|p| \to \infty} \frac{H(x, p)}{|p|} = \infty \) uniformly on \( x \in M \).

The maximal abelian cover \( \hat{M} \) is the covering map \( \hat{M} \to M \) with group of Deck transformations \( H_1(M, \mathbb{Z}) \). We see \( \hat{M} \) as an (infinite) repetition of a fundamental domain with a “periodicity” given by the group \( H_1(M, \mathbb{Z}) \). On the covering \( \hat{M} \) we consider the Hamilton-Jacobi equation for \( u \):

\[
\begin{align*}
\partial_t u + \hat{H}(x, \partial_x u) &= 0, \\
 u(x, 0) &= f(x);
\end{align*}
\]

where \( \hat{H} : T^*\hat{M} \to \mathbb{R} \) is the lift of \( H \). In the homogenization problem we want to see \( \hat{M} \) “from far away” and prove that the solution approximates a solution of a simpler equation as we see it from far away.

1.1. Convergence of spaces.

To “see \( \hat{M} \) from far away” is interpreted as multiplying the metric of \( \hat{M} \) by \( \varepsilon \) and letting \( \varepsilon \) tend to zero.

The following definition is inspired in Gromov’s Hausdorff convergence but it is made ad hoc for our homogenization problem. Let \( M_\varepsilon \) be the metric space \((\hat{M}, d_\varepsilon)\) where \( d_\varepsilon := \varepsilon \hat{d} \) and \( \hat{d} \) is the metric on \( \hat{M} \) obtained by lifting the riemannian metric on \( M \). Since the passage of \( \varepsilon \to 0 \) will only see the large scale properties of the space \( M \), our definition will be invariant under quasi-isometries. In fact \( \hat{M} \) is quasi-isometric to the group of Deck transformations \( H_1(M, \mathbb{Z}) \), and \( M_\varepsilon \) is quasi-isometric to \( \varepsilon H_1(M, \mathbb{Z}) \). We want to make a formal definition for the intuitive fact that

\[
\varepsilon H_1(M, \mathbb{Z}) = \varepsilon \mathbb{Z}_{a_1} \oplus \cdots \oplus \varepsilon \mathbb{Z}_{a_k} \oplus \varepsilon \mathbb{Z}^n \xrightarrow{\varepsilon \to 0} \mathbb{R}^n = H_1(M, \mathbb{R}).
\]

Let \((M, d), (M_n, d_n)_{n \in \mathbb{N}}\) be complete metric spaces and \( F_n : (M_n, d_n) \to (M, d) \) continuous functions. We say that \( \lim_n (M_n, d_n, F_n) = (M, d) \) if

1. There are \( B, A_n > 0 \) with \( \lim_n A_n = 0 \) such that

\[
\forall x, y \in M_n : \quad B^{-1} d_n(x, y) - A_n \leq d(F_n(x), F_n(y)) \leq B d_n(x, y) + A_n.
\]

2. For all \( y \in M \) and \( n \in \mathbb{N} \) there are \( x_n \in M_n \) such that \( \lim_n F_n(x_n) = y \).

Observe that (2) is a kind of surjectivity condition. And (1) implies that

\[
\text{diam } F_n^{-1}(y) \leq B A_n \xrightarrow{n \to 0} 0,
\]

a kind of injectivity condition.
If \( \lim_n (M_n, d_n, F_n) = (M, d) \) and \( f_n : (M_n, d_n) \to \mathbb{R}, f : (M, d) \to \mathbb{R} \) are continuous, we say that \( \lim_n f_n = f \) \emph{uniformly on compact sets} if for every compact set \( K \subset M \)
\[
\lim_n \sup_{x \in F_n^{-1}(K)} |f_n(x) - f(F_n(x))| = 0.
\]
We say that the family \( \{f_n\} \) is \emph{equi-Lipschitz} if there is \( K > 0 \) such that
\[
\forall n \quad \forall x, y \in M_n \quad |f_n(x) - f_n(y)| < K d_n(x, y).
\]

Fix \( x_0 \in \hat{M} \). Fix a basis \( c_1, \ldots, c_k \) for \( H^1(M, \mathbb{R}) \). Fix closed 1-forms \( \omega_i \) on \( M \) such that \( c_i = [\omega_i] \). Define \( G : \hat{M} \to H_1(M, \mathbb{R}) = H^1(M, \mathbb{R})^* \) by
\[
G(x) \cdot c_i = \oint_{x_0}^{x} \hat{\omega}_i,
\]
where \( \hat{\omega}_i \) is the pullback of \( \omega_i \) on \( \hat{M} \). Let \( F_\varepsilon : (M_\varepsilon, d_\varepsilon) \to H_1(M, \mathbb{R}) \) be
\[
F_\varepsilon(x) := \varepsilon G(x).
\]

1.1. **Proposition** ([6, p. 241]). \( \lim_{\varepsilon \to 0} (\hat{M}, \varepsilon d, F_\varepsilon) = H_1(M, \mathbb{R}) \).

Here all the torsion of the fundamental group is killed in the limit so it is the same limit for the maximal abelian cover as the limit for the maximal free abelian cover.

In our homogenization result we will have a family of solutions \( v^\varepsilon \) of properly scaled family of Hamilton-Jacobi equations on \( M_\varepsilon \). Then we prove that \( v^\varepsilon \) has a limit \( u \) defined on the limit space \( \lim_{\varepsilon \to 0} M_\varepsilon = H_1(M, \mathbb{R}) \), which is a solution of a simpler Hamilton-Jacobi equation.

1.2. **Invariance under quasi-isometries**.

Given two metric spaces \( (M_1, d_1) \) and \( (M_2, d_2) \) a function \( q : M_1 \to M_2 \) is a \emph{quasi-isometry} if there are constants \( A > 1, B, C > 0 \) such that
\[
\forall x, y \in M_1 \quad A^{-1} d_1(x, y) - B \leq d_2(q(x), q(y)) \leq A d_1(x, y) + B,
\]
\[
\forall z \in M_2 \quad \exists x \in M_1 \quad d_2(z, q(x)) \leq C.
\]
We say that the metric spaces \( (M_1, d_1) \), \( (M_2, d_2) \) are \emph{quasi-isometric} if there exists a quasi-isometry \( q : (M_1, d_1) \to (M_2, d_2) \).

If \( q_1 : (M_1, d_1) \to (M_2, d_2) \) is a quasi-isometry, then there exists a quasi-isometry \( q_2 : (M_2, d_2) \to (M_1, d_1) \). Indeed, given \( z \in M_2 \), by (7) there is \( x \in M_1 \) such that \( d(q_1(x), z) \leq C \). It is enough to take \( q_2(z) = x \). In fact the quasi-isometry is an equivalence relation.

1.2. **Proposition**.

If \( (M_1, d_1) \), \( (M_2, d_2) \) are quasi-isometric and \( \lim_\varepsilon (M_2, \varepsilon d_2) \) exists then
\[
\lim_\varepsilon (M_1, \varepsilon d_1) = \lim_\varepsilon (M_2, \varepsilon d_2).
\]
Proof:
Suppose that \((K,d) = \lim_\varepsilon(M_2, \varepsilon d_2, F_\varepsilon)\) and \(B, A_\varepsilon > 0\) satisfy (1). Then
\[
\forall z, w \in M_2 \quad B^{-1} \varepsilon d_2(z, w) - A_\varepsilon \leq d(F_\varepsilon(z), F_\varepsilon(w)) \leq B \varepsilon d_2(z, w) + A_\varepsilon.
\]
Let \(q : M_1 \to M_2\) be a quasi-isometry and let \(A_1 > 1, B_1, C_1 > 0\) satisfy (6) and (7). Applying inequalities (8) to \(z = q(x), w = q(y)\) we have that for all \(x, y \in M_1\)
\[
B^{-1} \varepsilon d_2(q(x), q(y)) - A_\varepsilon \leq d(F_\varepsilon q(x), F_\varepsilon q(y)) \leq B \varepsilon d_2(q(x), q(y)) + A_\varepsilon,
\]
\[
(BA_1)^{-1} \varepsilon d_2(q(x), q(y)) - (A_\varepsilon + \varepsilon B_1) \leq d(F_\varepsilon q(x), F_\varepsilon q(y)) \leq (BA_1) \varepsilon d_2(q(x), q(y)) + (A_\varepsilon + \varepsilon B_1).
\]
Thus \(F_\varepsilon \circ q : (M_1, \varepsilon d_1) \to (K, d)\) satisfies (1).

Let \(w \in K\), by (2) for every \(\varepsilon\) there is \(z_\varepsilon \in M_2\) such that
\[
\lim_\varepsilon d(F_\varepsilon(z_\varepsilon), w) = 0.
\]
By (7) there is \(x_\varepsilon \in M_1\) such that \(d_2(q(x_\varepsilon), z_\varepsilon) \leq C\). Then
\[
d(F_\varepsilon q(x_\varepsilon), w) \leq d(F_\varepsilon q(x_\varepsilon), F_\varepsilon(z_\varepsilon)) + d(F_\varepsilon(z_\varepsilon), w)
\leq B \varepsilon d_2(q(x_\varepsilon), z_\varepsilon) + A_\varepsilon + d(F_\varepsilon(z_\varepsilon), w)
\leq B \varepsilon C + A_\varepsilon + d(F_\varepsilon(z_\varepsilon), w) \xrightarrow{\varepsilon \to 0} 0.
\]
Therefore \(\lim_\varepsilon(M_1, \varepsilon d_1, F_\varepsilon \circ q) = (K, d)\).

\[\square\]

1.3. The Scaling.
In the usual homogenization of the Hamilton-Jacobi equation in \(\mathbb{R}^n\) the proper scaling to obtain convergence is just to “replace the coefficients \(x\) of the PDE by \(\frac{x}{\varepsilon}\),” this is
\[
\partial_t u^\varepsilon(x, t) + H\left(\frac{x}{\varepsilon}, \partial_x u(x, t)\right) = 0,
\]
\[
u^\varepsilon(x, 0) = f(x).
\]
Our way to interpret this in a manifold is to consider the function
\[
v^\varepsilon(\frac{x}{\varepsilon}, t) := u^\varepsilon(x, t).
\]
The equation (10) becomes
\[
\partial_t v^\varepsilon + H\left(y, \frac{1}{\varepsilon} \partial_y v^\varepsilon\right) = 0,
\]
\[
v^\varepsilon(y, 0) = f(\varepsilon y).
\]
Now equation (11) makes sense in the manifold \(M_\varepsilon\), with \(v^\varepsilon : M_\varepsilon \times [0, +\infty) \to \mathbb{R}\). And equation (12) becomes \(v^\varepsilon(y, 0) = f(F_\varepsilon(y)), y \in M_\varepsilon\), when we interpret “\(\varepsilon y\) := F_\varepsilon(y)”. 

1.4. Viscosity solutions.

Hamilton-Jacobi equations usually do not have smooth solutions due to the intersection or “shocks” of their characteristic lines. Local (smooth) solutions can be found using the characteristic method. Patching local solutions can give a plethora of almost everywhere solutions. A way to recover existence and uniqueness results for the Hamilton-Jacobi equation

\[ \partial_t u + H(x, \partial_x u) = 0, \quad u : \Omega \to \mathbb{R}, \quad \Omega = M \times [0, \infty[. \]

is to consider viscosity solutions.

Write \( \Omega := M \times [0, \infty[. \) A continuous function \( u \in C^0(\Omega, \mathbb{R}) \) is a viscosity subsolution of the Hamilton-Jacobi equation (13) if

\[ \forall \phi \in C^1(\Omega, \mathbb{R}), \text{ if } (x_0, t_0) \in \Omega \text{ is a local maximum of } u - \phi, \]

then

\[ \partial_t \phi(x_0, t_0) + H(x_0, \partial_x \phi(x_0, t_0)) \leq 0. \]

A continuous function \( u \in C^0(\Omega, \mathbb{R}) \) is a viscosity supersolution of the Hamilton-Jacobi equation (13) if

\[ \forall \phi \in C^1(\Omega, \mathbb{R}), \text{ if } (x_0, t_0) \in \Omega \text{ is a local minimum of } u - \phi, \]

then

\[ \partial_t \phi(x_0, t_0) + H(x_0, \partial_x \phi(x_0, t_0)) \geq 0. \]

A function \( u \in C^0(\Omega, \mathbb{R}) \) is a viscosity solution of the Hamilton-Jacobi equation (13) if it is both a viscosity subsolution and a viscosity supersolution of (13).

1.3. Lemma.

In the definition of viscosity subsolution (supersolution) it is equivalent

(1) to take a strict local maximum (strict local minimum).

(2) to take Lipschitz \( C^1 \) test functions.

Proof:

(1). Suppose the definition for subsolution holds for strict local maxima. If \( u - \varphi \) with \( \phi \in C^1 \) has a local maximum at \( (x, t) \), define in local coordinates \( \phi(y, s) := \varphi(y, t) + |x - y|^2 + |s - t|^2 \). Then \( u - \phi \) has a strict local maximum at \( (x, t) \) and inequality (14) holds for \( \phi \). But the derivatives of \( \phi \) and \( \varphi \) at \( (x, t) \) are equal. Then inequality (14) holds for \( \varphi \) at \( (x, t) \). The converse is easier.

(2). Suppose the definition for subsolutions holds for Lipschitz test functions. Given a test function \( \phi \) since \( \phi \in C^1(\Omega, \mathbb{R}) \) we have that \( \phi \) is Lipschitz in a neighborhood of \( U \) of \( (x_0, t_0) \). We can extend \( \phi|_U \) to a Lipschitz \( C^1 \) function \( \varphi \) on \( \Omega \). Then inequality (14) holds for \( \varphi \) and \( D\varphi(x_0, t_0) = D\phi(x_0, t_0) \). Thus inequality (14) holds for \( \phi \).

We shall prove the following Theorem using Evans perturbed test function method.
1.4. **Theorem** (Contreras, Iturriaga, Siconolfi [6]).

Let $M$ be a closed Riemannian manifold. Let $H : T^* M \to \mathbb{R}$ be a Tonelli Hamiltonian and $f_\varepsilon : (M_\varepsilon, d_\varepsilon) \to \mathbb{R}$ an equi-Lipschitz family such that $\lim_\varepsilon f_\varepsilon = f$ uniformly, with $f : H_1(M, \mathbb{R}) \to \mathbb{R}$.

Let $\hat{H}$ be the lift of $H$ to $\hat{M}$ and let $v^\varepsilon$ be the Lipschitz viscosity solution to the problem

$$
\begin{align*}
\frac{\partial v^\varepsilon}{\partial t} + \hat{H}(y, \frac{1}{\varepsilon} \partial_y v^\varepsilon) &= 0, \\
v^\varepsilon(y, 0) &= f_\varepsilon(y).
\end{align*}
$$

Then the family $v^\varepsilon : \hat{M}_\varepsilon \times ]0, +\infty[ \to \mathbb{R}$ is equi-Lipschitz and

$$
\lim_{\varepsilon \to 0} v^\varepsilon = u : H_1(M, \mathbb{R}) \to \mathbb{R}
$$

uniformly on compact sets of $H_1(M, \mathbb{R}) \times ]0, +\infty[$, where $u$ is the solution to

$$
\begin{align*}
\frac{\partial u}{\partial t} + \overline{H}(\partial_x u) &= 0, \\
u(x, 0) &= f(x);
\end{align*}
$$

and $\overline{H} : H^1(M, \mathbb{R}) \to \mathbb{R}$ is $\overline{H} = \alpha$ Mather’s alpha function.

Recall from [4] Corollary 1, that Mather’s alpha function is $\alpha : H^1(M, \mathbb{R}) \to \mathbb{R}$

$$
\alpha(c) = \inf_{[\omega]=c} \sup_{x \in M} H(x, \omega(x)),
$$

where $\omega$ are closed 1-forms of cohomology $c$. Mather’s alpha function is also the convex dual of Mather’s beta function, and it can be written in terms of minimizing invariant measures, see Mather [8], Theorem 1.

The same proof applies to (abelian) subcovers of $\hat{M}$ with the statement given in [6], Theorem 1.4. These may be more intuitive than $\hat{M}$. For example, let $M = T^2 \# T^2$ be the bi-torus, the compact oriented surface of genus 2. Cut $M$ along the “ears” to get a surface $F = (S^1 \times I) \# (S^1 \times I)$ homeomorphic to the boundary of a tubular neighborhood of a cross in $\mathbb{R}^3$, which is a fundamental domain of an abelian cover $\tilde{M}$ of $M$ with group of deck transformations $\mathbb{Z}^2$. The cover $\tilde{M}$ is homeomorphic to the boundary of a small tubular neighborhood of

$$(\mathbb{R} \times \mathbb{Z} \times \{0\}) \cup (\mathbb{Z} \times \mathbb{R} \times \{0\})$$

in $\mathbb{R}^3$. Nevertheless the maximal abelian cover $\hat{M}$ has group of deck transformations $\mathbb{Z}^4 = H_1(M, \mathbb{Z})$. In this example $M_\varepsilon = (\tilde{M}, \varepsilon d_M)$ converges to $\mathbb{R}^2$. We see that the limit process of $M_\varepsilon$ destroys all the differentiable structure of $\tilde{M}$. Nevertheless there is the homogenization limit because the Hamilton-Jacobi equation responds to a large scale variational principle.
1.5. **The cell problem.**

The cell problem in the torus $\mathbb{T}^n$ is written as

$$H(x, P + Dw(x)) = \overline{\Pi}(P), \quad x \in \mathbb{T}^n, \ P \in \mathbb{R}^n, \ w : \mathbb{T}^n \to \mathbb{R}. \quad (18)$$

The real number $\overline{H}(P) = \alpha(P)$ is the unique constant for which equation (18) has a viscosity solution. Thus the cell problem consists in finding a “correcting” exact form $dx_w$ such that the form $P + dx_w$ is a solution of the Hamilton-Jacobi equation in the cohomology class $P$.

In the manifold case we don’t have a standard canonical basis for $H^1(M, \mathbb{R})$. So we use the same basis $c_i = [\omega_i]$ used for the isomorphism $H_1(M, \mathbb{R}) = \mathbb{H}^1(M, \mathbb{R})^*$ and the definition of $G$ in (4). Let $\Omega^1(M)$ be the space of smooth closed 1-forms in $M$. Let $g : H^1(M, \mathbb{R}) \to \Omega^1(M)$ be

$$g\left( \sum_i p_i c_i \right) = \sum_i p_i \omega_i.$$ 

Writing $c_i \in H^1(M, \mathbb{R}) = H_1(M, \mathbb{R})^*$, the definition (4) of $G : \hat{M} \to H_1(M, \mathbb{R})$ is equivalent to

$$c_i \cdot G(x) = \int_{x_0}^{x} \hat{\omega}_i.$$ 

Then the i-th coordinate of the derivative $DG$ applied on the vector $h \in T_x \hat{M}$ is

$$i \cdot DG(x)(h) = \hat{\omega}_i(x) \cdot h.$$ 

This implies that

$$\sup_{x \in \hat{M}} \|DG(x)\| \leq \max_i \max_{x \in M} \|\omega_i(x)\|. \quad (19)$$

For a cohomology class $P = \sum_i p_i c_i \in H^1(M, \mathbb{R})$ we have that

$$P \cdot DG(x) = \sum_i p_i c_i \cdot DG(x) = \sum_i p_i \hat{\omega}_i(x)$$

$$= \sum_i p_i \omega_i(\pi(x)) \circ d\pi = g(P)(\pi x) \circ d\pi = \pi^*(g(P))(x), \quad (20)$$

where $\pi : \hat{M} \to M$ is the covering map.

Given $P \in H^1(M, \mathbb{R})$ our cell problem will be

$$H(x, g(P) + Dw(x)) = \overline{\Pi}(P), \ w : M \to \mathbb{R}. \quad (21)$$

Using (20), a solution of the cell problem (21) lifts to $\hat{M}$ as $\hat{w} = w \circ \pi$ which satisfies

$$\hat{H}(x, P \cdot DG(x) + D\hat{w}(x)) = \overline{\Pi}(P), \quad x \in \hat{M}. \quad (22)$$

because $P \cdot DG = \pi^*(g(P))$ and $D\hat{w} = \pi^*(Dw)$. 


2. The projections $F_\varepsilon$.

The universal cover $\tilde{M}$ of $M$ is the set of homotopy classes with fixed endpoints of the curves in $C^0(([0,1],0),(M,\pi x_0))$. The projection $p : \tilde{M} \to M$ is $p([\gamma]) = \gamma(1)$. We also name $x_0 \in \tilde{M}$ the class of the constant curve $\gamma(t) \equiv \pi x_0$ in $M$.

The maximal abelian cover or universal abelian cover $\hat{M}$ is the subcover given by $\tilde{M}/\sim$, where $\sim$ is the equivalence relation given by $[\gamma] \sim [\eta]$ iff $\gamma(1) = \eta(1)$ and $\gamma * \eta^{-1}$ is null homologous (in the base $M$), or equivalently, its homotopy class is in the center $H = [\pi_1, \pi_1] \subset \pi_1(M,\pi x_0)$ of the fundamental group. This implies that a continuous curve $\Gamma : [0,1] \to \hat{M}, \Gamma(0) = x_0$ is closed iff its projection to the base $M$ has homology zero.

Statement (23) holds at other base points because the subgroup $H \triangleleft \pi_1(M,\pi x_0)$ is normal. Observe that $H_1(\hat{M},\mathbb{Z})$ may be non-trivial, i.e. a closed curve in $\hat{M}$ may not be homologous to zero in $\hat{M}$, but its projection to $M$ must be homologous to zero in $M$. The projection $\pi : \hat{M} \to M$ is also $\pi([\gamma]) = \gamma(1)$. The metric on $\hat{M}$ is obtained by lifting the riemannian metric on $M$ using $\pi$.

Property (23) also implies that if a (closed) form $\hat{\omega} \in \Omega^1(\hat{M})$ is a lift of a closed 1-form $\omega \in \Omega^1(M)$ in the base $M$, i.e. $\hat{\omega} = \pi^*(\omega)$, then the function $f_\omega : \hat{M} \to \mathbb{R}$,

$$f(x) := \int_{x_0}^x \hat{\omega},$$

is well defined in $\hat{M}$, i.e. it does not depend on the curve $\hat{\gamma} : x_0 \to x$ used to compute the integral, because for another curve $\hat{\eta} : x_0 \to x$

$$\int_{\hat{\gamma} - \hat{\eta}} \hat{\omega} = \int_{\pi \circ \hat{\gamma} - \pi \circ \hat{\eta}} \omega = 0,$$

because the chain $\pi \circ \hat{\gamma} - \pi \circ \hat{\eta}$ is homologous to zero.

2.1. Lemma.

(a) The map $G : \tilde{M} \to H_1(M, \mathbb{R})$ in (4), is Lipschitz.

(b) The map $F_\varepsilon : (M_\varepsilon, d_\varepsilon) \to H_1(M, \mathbb{R})$ in (5), is Lipschitz.

Proof:

(a). For the $i$-th coordinate of $G(x)$ we have that

$$|G(y) \cdot c_i - G(x) \cdot c_i| = \left| \int_{x_0}^x \hat{\omega}_i - \int_{x_0}^y \hat{\omega}_i \right| \leq \|\hat{\omega}_i\| d(x, y) \leq \|\omega_i\|_M d(x, y).$$

Thus $G$ has Lipschitz constant $K := \max_i \|\omega_i\|$. 


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(b).

\[ |F_\varepsilon(x) - F_\varepsilon(y)| = |\varepsilon G(x) - \varepsilon G(y)| \leq \varepsilon K d_\hat{M}(x, y) \leq K d_\varepsilon(x, y). \]

\[ \square \]

2.2. Proposition. The map \( F_\varepsilon : M_\varepsilon \to H_1(M, \mathbb{R}) \) is proper.

Proof:

We have to prove that the pre-image of compact sets are compact. Since \( F_\varepsilon \) is continuous it is enough to prove that the pre-image of bounded sets are bounded. Since \( F_\varepsilon = \varepsilon G \) and \( d_\varepsilon = \varepsilon d_\hat{M} \), it is enough to prove it for \( G \).

By the classification of finitely generated abelian groups there is an isomorphism \( f : H_1(M, \mathbb{Z}) \to \mathbb{Z}^k \oplus \mathbb{T} \), where the torsion \( \mathbb{T} = \mathbb{Z}_{a_1} \oplus \cdots \oplus \mathbb{Z}_{a_q} \) is the finite group of the elements in \( H_1(M, \mathbb{Z}) \) with finite order:

\[ \mathbb{T} = \{ h \in H_1(M, \mathbb{Z}) : \exists m \in \mathbb{N} \quad m \cdot h = 0 \}. \]

Let \( \gamma_i \) be a closed curve in \( M \) based at \( \pi x_0 \in M \) representing the homology class \([\gamma_i] = f^{-1}(e_i \oplus 0)\), where \( e_i = (0, \cdots, 1, \cdots, 0) \) are the canonical basis vectors of \( \mathbb{Z}^k \).

On \( H_1(M, \mathbb{R}) \approx \mathbb{R}^k \) we are using the coordinates \( c_i \cdot h \) and we use the norm

\[ (24) \quad \|h\| = \sum_{i=1}^{k} |c_i \cdot h|. \]

We have that a basis for \( H_1(M, \mathbb{R}) \) is \( ([\gamma_i])_{i=1}^{k} \). If \( h = \sum_{i=1}^{k} r_j [\gamma_j] \) then

\[ c_i \cdot h = \sum_{j=1}^{k} r_j (c_i \cdot [\gamma_j]) = \sum_{j=1}^{k} a_{ij} r_j, \]

where

\[ a_{ij} = c_i \cdot [\gamma_j] = \oint_{\gamma_j} \omega_i. \]

Since this is a change of basis the matrix \( A = [a_{ij}] \) is non-singular. The norm (24) of \( h \) is \( \|h\| = |A\mathbb{T}| \).

The group of deck transformations of \( \hat{M} \) is isomorphic to \( H_1(M, \mathbb{Z}) \). We write \( x \mapsto \zeta \cdot x \) the deck transformation associated to \( \zeta \in H_1(M, \mathbb{Z}) \).

Suppose that \( x \in \hat{M} \) satisfies

\[ \|G(x)\| \leq R. \]

Let \( \alpha : [0, 1] \to M \) be a minimizing geodesic in \( M \) joining \( \pi(x) \) to \( \pi x_0 \). Let \( \hat{\alpha} \) be the lift of \( \alpha \) to \( \hat{M} \) with \( \hat{\alpha}(0) = x \). We have that \( \hat{\alpha}(1) \in \pi^{-1}(\pi x_0) = H_1(M, \mathbb{Z}) \cdot x_0 \) and
\[ d(x, \hat{\alpha}(1)) \leq \ell(\alpha) = d(\pi(x), \pi x_0) \leq \text{diam } M. \] Therefore there are \( \overline{\pi} \in \mathbb{Z}^k \) and \( \tau \in \mathbb{T} \) such that
\[ d_{\hat{M}}(x, (\overline{\pi} + \tau) \cdot x_0) \leq \text{diam } M. \] (25)

Since the torsion group \( \mathbb{T} \) is finite we have that
\[ Q_1 := \text{diam } M + \max\{ d(x_0, \tau \cdot x_0) : \tau \in \mathbb{T} \} < \infty. \] (26)

Since the deck transformations are isometries we have that
\[ d_{\hat{M}}(x, \overline{\pi} \cdot x_0) \leq \text{diam } M + (Q_1 - \text{diam } M) = Q_1. \]

\[ c_i \cdot G(\overline{\pi} \cdot x_0) = c_i \cdot G(x) + \int_{\overline{\pi} x_0}^\tau w_i \]
\[ \leq c_i \cdot G(x) + Q_1 \| \omega_i \|, \]
\[ \| G(\overline{\pi} \cdot x_0) \| \leq \| G(x) \| + k Q_1 \max_{1 \leq i \leq k} \| \omega_i \| \]
\[ \leq R + B, \] (27)

where \( B := k Q_1 \max_{1 \leq i \leq k} \| \omega_i \| \).

Let \( \Lambda : [0, 1] \to \hat{M} \) be a minimizing geodesic in \( \hat{M} \) from \( x_0 \) to \( \overline{\pi} \cdot x_0 \). Its projection \( \lambda = \pi \circ \Lambda \) has homology class \( \overline{\pi} = \sum_i n_i [\gamma_i] \in \mathbb{Z}^k \oplus 0 \subset H_1(M, \mathbb{Z}) \). Let \( \Gamma := \gamma_1^{n_1} \cdots \gamma_k^{n_k} \), a loop in \( (M, \pi x_0) \) with homology \( \overline{\pi} \). Let \( \hat{\Gamma}^{-1} \) be a lift of \( \Gamma^{-1} \) with \( \hat{\Gamma}^{-1}(0) = \overline{\pi} \cdot x_0 \). Since \( \Lambda \ast \hat{\Gamma}^{-1} \) has projection \( \lambda \ast \Gamma^{-1} \) homologous to zero, the curve \( \Lambda \ast \Gamma^{-1} \) is a closed loop in \( (\hat{M}, x_0) \). Then
\[ d_{\hat{M}}(x_0, \overline{\pi} \cdot x_0) = d(x_0, \Lambda(1)) = d(\hat{\Gamma}^{-1}(1), \hat{\Gamma}^{-1}(0)) \]
\[ \leq \text{length}(\Gamma) \leq \sum_{i=1}^k |n_i| \ell(\gamma_i). \] (28)

The coordinates of \( G(\overline{\pi} \cdot x_0) \) are
\[ c_i \cdot G(\overline{\pi} \cdot x_0) = c_i \cdot \sum_{j=1}^k n_j [\gamma_j] = \sum_{j=1}^k a_{ij} n_j, \quad \overline{\pi} = A^{-1} [c_i \cdot G(\overline{\pi} \cdot x_0)] \]
\[ |\overline{\pi}| \leq \| A^{-1} \| \| G(\overline{\pi} \cdot x_0) \| \leq \| A^{-1} \| (R + B), \quad \text{using (27)}. \]
\[ d_{\hat{M}}(x_0, x) \leq d(x_0, \overline{\pi} \cdot x_0) + d(\overline{\pi} \cdot x_0, x) \]
\[ \leq |\overline{\pi}| \max_i \ell(\gamma_i) + Q_1. \]
\[ d_{\hat{M}}(x_0, x) \leq \| A^{-1} \| (R + B) \max_i \ell(\gamma_i) + Q_1 < \infty. \]

Therefore \( G^{-1} \{ h \in H_1(M, \mathbb{R}) : \| h \| \leq R \} \) is bounded. \( \square \)
2.3. Corollary.
If $\varepsilon > 0$, $(y_0, t_0) \in M_\varepsilon \times \mathbb{R}^+$, $0 < r < t_0$ then
$$(F_\varepsilon \times \text{id})^{-1}(B_r(y_0, t_0)) \subset M_\varepsilon \times \mathbb{R}^+$$
is compact.

3. Lipschitz

Let $L : TM \to \mathbb{R}$ be the lagrangian of $H$
$$L(x, v) := \sup \{ p(v) - H(x, p) : p \in T^*_x M \} \quad (x, v) \in TM;$$
and let $L^\varepsilon : T\hat{M} \to \mathbb{R}$ be the lagrangian of the hamiltonian $H^\varepsilon(x, p) = \tilde{H}(x, \frac{1}{\varepsilon}p)$,
$$L^\varepsilon(x, v) := \sup \{ p(v) - H^\varepsilon(x, p) : p \in T^*\hat{M} \}, \quad (x, v) \in T\hat{M}.$$

Then $L$ and $L^\varepsilon$ are also convex and superlinear. Observe that for $\hat{L} = L \circ d\pi$,
$$(29) \quad L^\varepsilon(x, v) = \hat{L}(x, \varepsilon v).$$

Then the energy function $E^\varepsilon(x, v) = v \cdot \partial_v L^\varepsilon(x, v) - L^\varepsilon(x, v)$ is
$$(30) \quad E^\varepsilon(x, v) = \hat{E}(x, \varepsilon v),$$
where $E = v \cdot L_v - L$ is the energy function of $L$ and $\hat{E} = E \circ d\pi$.

The solution to equation (16) is given by the Lax formula
$$(31) \quad u^\varepsilon(x, t) = \min \left\{ f_\varepsilon(y) + \int_0^t L^\varepsilon(\gamma, \dot{\gamma}) : \gamma \in C^1([0, t], M_\varepsilon; (y_0, t_0)); (M_\varepsilon, y_0, t_0)) \right\}.$$

In section §7, Corollary 7.3, we prove that there is a unique Lipshitz viscosity solution to the problem (16) and in Proposition 3.1 we prove that $u^\varepsilon$ is Lipschitz. So we will have that $u^\varepsilon = v^\varepsilon$ is the solution to problem (16) in Theorem 1.4.

Define the Lax-Oleinik operator $\mathcal{L}^\varepsilon_t$ by
$$(32) \quad (\mathcal{L}^\varepsilon_t f)(x) = \min \left\{ f(y) + \int_0^t L^\varepsilon(\gamma, \dot{\gamma}) : \gamma \in C^1([0, t], M_\varepsilon; (y_0, t_0)); (M_\varepsilon, y_0, t_0)) \right\}.$$

The minimum in (32) is always attained by one or more minimizers $\gamma$. Then $\mathcal{L}_t$ is a semigroup, meaning that $\mathcal{L}^\varepsilon_t \circ \mathcal{L}^\varepsilon_s = \mathcal{L}^\varepsilon_{s+t}$. We have that
$$u^\varepsilon(x, t) = (\mathcal{L}^\varepsilon_t f_\varepsilon)(x),$$
and then
$$(33) \quad \forall s \leq t \quad \forall x \in M_\varepsilon \quad u^\varepsilon(x, t) = \mathcal{L}^\varepsilon_{t-s} u^\varepsilon(\cdot, s)(x).$$
3.1. Proposition.

The functions $u^\varepsilon$ in (31) are equi-Lipschitz, i.e.

$$\exists Q > 0 \quad \forall \varepsilon > 0 \quad \forall (x, t), (y, s) \in M_\varepsilon \times [0, +\infty[,$$

$$|u^\varepsilon(x, t) - u^\varepsilon(y, s)| \leq Q|t - s| + d_\varepsilon(x, y).$$

Proof:

Step 1: There are

(34) $k_0 > 0, \quad a_0 > 0$

such that all the minimizers $\gamma$ in (31) satisfy $E^\varepsilon(\gamma, \dot{\gamma}) < k_0$ and $\varepsilon |\dot{\gamma}| < a_0$.

Taking a constant curve $\gamma(s) = x$, $s \in [0, t]$ in (31), we have that

(35) $\forall (x, t) \in M_\varepsilon \times \mathbb{R}^+ \quad u^\varepsilon(x, t) \leq f_\varepsilon(x) + \hat{L}(x, 0) t.$

Let $K$ be a uniform Lipschitz constant for the functions $f_\varepsilon$:

$$\forall x, y \in M_\varepsilon \quad |f_\varepsilon(x) - f_\varepsilon(y)| \leq K d_\varepsilon(x, y) = K \varepsilon d(x, y).$$

Let $A > 0$ be such that

(36) $A > K.$

By the superlinearity of $L$ there is $B > 0$ such that

$$\forall (x, v) \in TM \quad L(x, v) > 2A |v| - B.$$

Therefore

$$\forall (x, v) \in TM_\varepsilon \quad L^\varepsilon(x, v) > 2A \varepsilon |v| - B.$$

Let $a_1 > 0$ be such that

(37) $A a_1 - B > \sup_{x \in M} |L(x, 0)|.$

Let $k_0 > 0$ be such that

$(x, v) \in TM, \quad E(x, v) \geq k_0 \implies |v| > a_1.$

Then

(38) $E^\varepsilon(x, v) = E(x, \varepsilon v) \geq k_0 \implies \varepsilon |v| > a_1.$

Let $a_0 > 0$ be such that

(39) $E(x, v) \leq k_0 \implies |v| < a_0.$

Given $y \in M_\varepsilon$ if a curve $\gamma \in C^1([0, t], 0, t); (M_\varepsilon, y, x)$ has energy $E^\varepsilon(\gamma, \dot{\gamma}) \geq k_0$, by (38) it has speed

$$\forall s \in [0, t] \quad \varepsilon |\dot{\gamma}(s)| > a_1.$$
For \( t > 0 \) we have that
\[
\int_0^t L^\varepsilon(\gamma, \dot{\gamma}) \geq \int_0^t 2A \varepsilon |\dot{\gamma}| - B \\
\geq \int_0^t A \varepsilon |\dot{\gamma}| + \int_0^t A \varepsilon \left( \frac{a_1}{\varepsilon} \right) - B \\
\geq A \varepsilon d(x, y) + (Aa_1 - B)t \\
> A d_\varepsilon(x, y) + \tilde{L}(x, 0)t 
\]
by (37).

\[
f_\varepsilon(y) + \int_0^t L^\varepsilon(\gamma, \dot{\gamma}) > f_\varepsilon(x) - K d_\varepsilon(x, y) + A d_\varepsilon(x, y) + \tilde{L}(x, 0)t \\
= f_\varepsilon(x) + (A - K) d_\varepsilon(x, y) + \tilde{L}(x, 0)t \\
\geq f_\varepsilon(x) + \tilde{L}(x, 0)t \\
\geq u^\varepsilon(x, t) 
\]
by (35).

Therefore \( \dot{\gamma} \) can not be a minimizer in (31). Thus any minimizer \( \gamma \) must have energy \( E^\varepsilon(\gamma, \dot{\gamma}) < k_0 \). By (39) it must satisfy \( \forall s \in [0, t] \ \varepsilon |\dot{\gamma}(s)| < a_0 \).

**Step 2:** The functions \( u^\varepsilon(x, t) \) are equi-Lipschitz in \( x \).

We have to prove that there is a uniform Lipschitz constant for the functions

\[
(M_\varepsilon, d_\varepsilon) \ni x \mapsto u^\varepsilon(x, t), \quad \varepsilon > 0, \quad t \geq 0.
\]

By hypothesis for \( t = 0 \), the functions \( u^\varepsilon(x, 0) := f_\varepsilon(x) \) have uniform Lipschitz constant \( K \) in \( (M_\varepsilon, d_\varepsilon) \). Observe that it is enough to prove that for all \( (x, t) \in M_\varepsilon \times \mathbb{R}^+ \) there is a uniform local Lipschitz constant for \( (M_\varepsilon, d_\varepsilon) \ni x \mapsto u^\varepsilon(x, t) \).

Since the projection \( M \) is compact, by Weiestrass Theorem (cf. Mather [8, p. 175])

\[
(40) \quad \forall A > 0 \quad \exists \tau = \tau(\varepsilon, A) > 0
\]

such that if \( s_0 < t \leq s_0 + \tau \) and \( x, y \in M_\varepsilon \) satisfy \( d_\varepsilon(x, y) < \frac{1}{2} A(t - s_0) \) then there is a unique minimizer \( \zeta \) of the \( L^\varepsilon \)-action in \( C^1([s_0, t], s_0, t); (M_\varepsilon, x, y) \) and moreover \( |\dot{\zeta}|_\varepsilon \leq A \).

Let \( a_0 \) be from (34) in step 1 and take \( \tau = \tau(\varepsilon, 4a_0) \), from (40). Let \( t > 0 \). Shrinking \( \tau \) if necessary we can assume that \( 2\tau < t \). Let \( s_0 = t - \tau \). Let \( x, y \in M_\varepsilon \) be such that

\[
d_\varepsilon(x, y) < a_0 \tau.
\]

Let \( \gamma : [0, t] \to M_\varepsilon \) be the minimizer in (31) such that

\[
u^\varepsilon(x, t) = f_\varepsilon(\gamma(0)) + \int_0^t L^\varepsilon(\gamma, \dot{\gamma}).
\]
By step 1 $|\gamma| < a_0$ and then $d_\varepsilon(\gamma(s_0), \gamma(t)) < a_0 \tau$. Let $\eta \in C^1(([0, 1], 0, 1); (M, x, y))$ a minimizing geodesic from $x$ to $y$. In particular

$$\forall s \in [0, 1] \quad |\partial_s \eta(s)| = d(x, y).$$

Let $h : [0, 1] \times [s_0, t] \to M$ be the variation of $\gamma|_{[s_0, t]}$ such that for every $s \in [0, 1]$, $h(s, \cdot)$ is the minimizer joining $\gamma(s_0)$ to $\eta(s)$. Since $\gamma(t) = x$ we have that

$$d(\gamma(s_0), \eta(s)) \leq d(\gamma(s_0), \gamma(t)) + d(x, y) < 2a_0 \tau.$$ 

Then from (40)

$$\forall (\sigma, \tau) \in [0, 1] \times [s_0, t] \quad \varepsilon |\partial_\tau h(\sigma, \tau)| \leq 4a_0.$$ 

Let

$$b_2 > K \geq \sup_\varepsilon \text{Lip}(f_\varepsilon, d_\varepsilon).$$

be such that

$$\forall (x, w) \in TM \quad |w| \leq 4a_0 \implies |L_\varepsilon(x, w)| < b_2.$$ 

Observe that $b_2$ is independent of $(\varepsilon, t)$. From (42) and (44) we get that

$$\forall (\sigma, \tau) \in [0, 1] \times [s_0, t] \quad |\hat{L}_\varepsilon(h(\sigma, \tau), \varepsilon \partial_\tau h(\sigma, \tau))| < b_2.$$ 

Observe that

$$u_\varepsilon(h(\sigma, t), t) \leq u_\varepsilon(\gamma(s_0), s_0) + \int_{s_0}^{t} L_\varepsilon(h(\sigma, \tau), \partial_\tau h(\sigma, \tau)) \, d\tau,$$

$$u_\varepsilon(x, t) = u_\varepsilon(\gamma(s_0), s_0) + \int_{s_0}^{t} L_\varepsilon(h(0, \tau), \partial_\tau h(0, \tau)) \, d\tau.$$

$$u_\varepsilon(\eta(\sigma), \tau) - u_\varepsilon(x, \tau) \leq \int_{s_0}^{t} \left[ L_\varepsilon(h(\sigma, \tau), \partial_\tau h(\sigma, \tau)) - L_\varepsilon(h(0, \tau), \partial_\tau h(0, \tau)) \right] \, d\tau,$$

$$u_\varepsilon(y, \tau) - u_\varepsilon(x, \tau) \leq \int_{0}^{1} \int_{s_0}^{t} \left( L_\varepsilon \partial_\sigma h + L_\varepsilon \partial_\sigma \partial_\tau h \right) \, d\tau \, d\sigma.$$ 

Since $\partial_\sigma \partial_\tau h = \partial_\tau \partial_\sigma h$, integrating by parts we get

$$\int_{s_0}^{t} \left( L_\varepsilon \partial_\sigma h + L_\varepsilon \partial_\sigma \partial_\tau h \right) \, d\tau \, d\sigma = \int_{s_0}^{t} \left( L_\varepsilon - \frac{d}{d\tau} L_\varepsilon \right) \partial_\sigma h + \frac{d}{d\tau} (L_\varepsilon \partial_\sigma h) \, d\tau.$$ 

Since $\tau \mapsto h(\sigma, \tau)$ is a solution of the Euler-Lagrange equation, the first term is zero. The integral of the second term is

$$\int_{s_0}^{t} L_\varepsilon \partial_\sigma h \, d\tau \, d\sigma = L_\varepsilon \partial_\sigma h \bigg|_{s_0}^{t} = L_\varepsilon(h(\sigma, t), \partial_\tau h(\sigma, t)) \partial_\sigma \eta(\sigma)$$
because \( \partial_s h(\sigma, s_0) \equiv 0 \) and \( \partial_s h(\sigma, t) = \partial_s \eta(\sigma) \). Then from (46) we have that
\[
 u(\sigma, t) - u(\sigma, s) \leq \int_0^1 |L(\eta(\sigma, t), \partial_s h(\sigma, t))| \, d\sigma.
\]
Observe that \( \partial_v L(\sigma, v) = \varepsilon \hat{L}(\sigma, \varepsilon v) \), then using (45) and (41) we get that
\[
d(\sigma, t) < a_0 \tau \implies u(\sigma, t) - u(\sigma, s) \leq \varepsilon b_2 \, d(\sigma, y) = b_2 \, d(\sigma, y).
\]
Since \( a_0 \tau \) does not depend on \((x, y)\), we can interchange the roles of \( x \) and \( y \) and obtain
\[
d(\sigma, t) < a_0 \tau(\varepsilon, a_0, t) \implies |u(\sigma, t) - u(\sigma, y)| \leq b_2 \, d(\sigma, y).
\]
This implies that
\[
\forall x, y \in M \quad |u(\sigma, t) - u(\sigma, y)| \leq b_2 \, d(\sigma, y).
\]
Observe that from (44) the constant \( b_2 \) does not depend on \( t \).

**Step 3:** The functions \( u(\sigma, t) \) are equi-Lipschitz in \( t \).

Comparing with a constant curve in \( x \) the definition of \( u(\sigma, t) \) in (31) implies that
\[
x \in M \quad t \geq s \implies u(\sigma, t) \leq u(\sigma, s) + \hat{L}(x, 0) (t - s),
\]
because \( L(\sigma, 0) = \hat{L}(x, 0) \). Let \( c_2 > 0 \) be such that
\[
\forall (x, v) \in TM \quad L(x, v) > b_2 \, |v| - c_2.
\]
Then
\[
\forall (x, v) \in TM \quad L(x, v) > b_2 \, |v| - c_2.
\]
Let \( \gamma \in C^1([s, t], M) \) be a curve with \( \gamma(t) = x \) where the minimum in (33) is attained, and let \( y := \gamma(s) \). Using (47) and (49),
\[
u(\sigma, t) = u(\sigma, \gamma(s), s) + \int_s^t L(\gamma, \dot{\gamma})
\geq u(\sigma, s) - b_2 \, d(\sigma, y) + \int_s^t b_2 \, |\dot{\gamma}| \, d\tau - c_2 (t - s)
\geq u(\sigma, s) - b_2 \, d(\sigma, y) + b_2 \, d(\sigma, y) - c_2 (t - s)
\geq u(\sigma, s) - c_2 (t - s).
\]
Taking \( c_1 := \max\{c_2, \max_{\sigma \in M} |L(\sigma, 0)|\} \) from (48) and (50), \( t \mapsto u(\sigma, t) \) has the same Lipschitz constant \( c_1 \) for every \( (\varepsilon, x) \).
4. Equicontinuity.

We need the Arzelà-Ascoli Theorem in our context.

Let \((M_n, d_n)_{n \in \mathbb{N}}\) be a sequence of complete metric spaces. Suppose that \(\lim_n (M_n, d_n) = (M, d)\) and that \((M, d)\) is separable. We say that a family of functions \(f_n : (M_n, d_n) \to \mathbb{R}\) is bounded on compact sets if for every compact set \(K \subset M\) the set \(\bigcup_{n \in \mathbb{N}} f_n(F_n^{-1}(K))\) is bounded. Recall that the uniform convergence on compact sets is defined in (3).

We say that the family \(f_n\) is equicontinuous if

\[\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall n \in \mathbb{N} \quad d_n(x, y) < \delta \quad \implies \quad |f_n(x) - f_n(y)| < \varepsilon.\]

4.1. Proposition.

If \(\lim_n (M_n, d_n, F_n) = (M, d)\), with \((M, d)\) separable and \(f_n : (M_n, d_n) \to \mathbb{R}\) is equicontinuous and bounded on compact sets, then there is a subsequence \(n(k)\) such that \(f_{n(k)}\) converges uniformly on compact sets to a continuous function on \(M\).

Proof: Let \(\mathbb{D} = \{x_n\}_{n \in \mathbb{N}}\) be a countable dense subset of \(M\). By (2) for all \(n \in \mathbb{N}\) there are \(x_n^m \in M_m\) such that \(\lim_m F_m(x_n^m) = x_n\). Then for each \(n\), the set \(\bigcup_m \{F_m(x_n^m)\} \cup \{x_n\}\) is compact and hence the sequence \(\{f_m(x_n^m)\}_m\) is bounded. Let \(m(1, k) \in \mathbb{N}\) be an increasing sequence such that

\[\exists \lim_k f_{m(1,k)}(x_1^{m(1,k)}) =: a_1.\]

Let \(m(2, k)\) be an increasing subsequence of \(m(1, k)\) such that

\[\exists \lim_k f_{m(2,k)}(x_2^{m(2,k)}) =: a_2.\]

Inductively, let \(m(\ell, k)\) be an increasing subsequence of \(m(\ell - 1, k)\) such that

\[\exists \lim_k f_{m(\ell,k)}(x_\ell^{m(\ell,k)}) =: a_\ell.\]

Then \(m(k, k) \in \{m(\ell, k)\}_{k \geq \ell}\) and hence \(\{m(k, k)\}_{k \in \mathbb{N}}\) is a subsequence of all the sequences \(m(\ell, \cdot)\). In particular

\[\forall n \in \mathbb{N} \quad \lim_k f_{m(k,k)}(x_n^{m(k,k)}) = a_n.\]

Define \(f : \mathbb{D} \to \mathbb{R}\) by \(f(x_n) := a_n\). We first prove that \(f|_D\) is uniformly continuous. Given \(\varepsilon > 0\) let \(\delta > 0\) be such that

\[\forall n \in \mathbb{N} \quad d_n(x, y) < \delta \quad \implies \quad |f_n(x) - f_n(y)| < \varepsilon.\]

Suppose that \(x_p, x_q \in \mathbb{D}\) and \(d(x_p, x_q) < \delta\). From (1) we have that

\[B^{-1} d_m(x_p^m, x_q^m) - A_m \leq d(F_m(x_p^m), F_m(x_q^m)) \leq B d(x_p^m, x_q^m).\]

\[d_m(x_p^m, x_q^m) \leq A_m B + B d(F_m(x_p^m), F_m(x_q^m))\]

\[\leq 2A_m B + B d(x_p, x_q) \quad \text{for } m \geq m_1 \text{ large enough.}\]
There is $m_2 \geq m_1$ such that
\begin{equation}
\forall m \geq m_2 \quad A_m < \frac{\delta}{8B}.
\end{equation}

If $d(x_p, x_q) < \frac{1}{2B}\delta$ then
\[ 2A_mB + Bd(x_p, x_q) < \frac{1}{4}\delta + \frac{1}{2}\delta < \delta. \]

From (54) we obtain $d(x_p^m, x_q^m) < \delta$ for $m \geq m_2$. Then (53) implies that
\begin{equation}
\forall m \geq m_2 \quad |f_m(x_p^m) - f_m(x_q^m)| < \varepsilon.
\end{equation}

From (52) and (56) we get that
\begin{align*}
|f(x_p) - f(x_q)| &= |a_p - a_q| \\
&= \lim_k |f_{m(k,k)}(x_p^{m(k,k)}) - f_{m(k,k)}(x_q^{m(k,k)})| \\
&\leq \varepsilon.
\end{align*}

Therefore $f|_D$ is uniformly continuous and thus we can extend $f$ to $M$ uniquely by continuity. In particular $f$ is continuous on $M$.

Now we prove that $\lim_k f_{m(k,k)} = f$ uniformly on compact sets. Suppose it is false. Then there is a compact subset $\mathbb{K} \subset M$ and $\theta > 0$ and a subsequence $m_k$ of $\{m(k,k)\}_{k \in \mathbb{N}}$ such that
\begin{equation}
\forall k \in \mathbb{N} \quad \sup_{F_{m_k}(y) \in \mathbb{K}} |f_{m_k}(y) - f(F_{m_k}(y))| > \theta.
\end{equation}

Then there are $y_k \in M_{m_k}$ such that $F_{m_k}(y_k) \in \mathbb{K}$ and
\begin{equation}
\forall k \in \mathbb{N} \quad |f_{m_k}(y_k) - f(F_{m_k}(y_k))| > \theta.
\end{equation}

Since $\mathbb{K}$ is compact by extracting a subsequence of $m_k$ we can assume that $y := \lim_k F_{m_k}(y_k)$ exists. By the continuity of $f$ we have that
\begin{equation}
\lim_k |f(y) - f(F_{m_k}(y_k))| = 0.
\end{equation}

Let $x_n \in D$ be such that $\lim_n x_n = y$. By the continuity of $f$ we have that
\begin{equation}
\lim_n |f(x_n) - f(y)| = 0.
\end{equation}

By the construction above we have that
\begin{equation}
\lim_k f_{m_k}(x_n^{m_k}) = f(x_n) = a_n.
\end{equation}

From (1) we have that
\[ d_{m_k}(y_k, x_n^{m_k}) \leq A_{m_k}B + B d(F_{m_k}(y_k), F_{m_k}(x_n^{m_k})). \]

In the construction above we have that $\lim_k F_{m_k}(x_n^{m_k}) = x_n$, $\lim_k F_{m_k}(y_k) = y$, therefore
\[ \limsup_k d_{m_k}(y_k, x_n^{m_k}) \leq B d(y, x_n) \to 0. \]
By the equicontinuity of \( \{ f_{mk} \}_{k \in \mathbb{N}} \) the last inequality implies that
\[
\lim_{n} \limsup_{k} |f_{mk}(y_k) - f_{mk}(x^m_k)| = 0. 
\]

We have that
\[
|f_{mk}(y_k) - f(F_{mk}(y_k))| \leq |f_{mk}(y_k) - f_{mk}(x^m_k)| + |f_{mk}(x^m_k) - f(x)| + |f(x) - f(y)| + |f(y) - f(F_{mk}(y_k))|. 
\]

Using (60), (58),
\[
\limsup_{k} |f_{mk}(y_k) - f(F_{mk}(y_k))| \leq \limsup_{k} |f_{mk}(y_k) - f_{mk}(x^m_k)| + \limsup_{k} |f(x) - f(y)| + |f(x) - f(y)| + 0. 
\]

Taking \( \lim n \) in the right hand side and using (61) and (59) we obtain
\[
\limsup_{k} |f_{mk}(y_k) - f(F_{mk}(y_k))| = 0, 
\]
which contradicts (57).

\[\Box\]

4.2. Proposition.
Suppose that \( \lim d_\pi(M_n, d, F_n) = (M, d) \), \( (M, d) \) is separable, \( f_n : (M_n, d_n) \rightarrow \mathbb{R} \) is equi-Lipschitz and \( \lim f_n = f \) uniformly compact sets. Then \( f \) is Lipschitz.

**Proof:** By Proposition 4.1, the function \( f \) is continuous. Given \( x, y \in M \), let \( K \subset M \) be a compact set such that \( x, y \in \text{int} K \). Let \( x_n, y_n \in M_n \) be such that \( \lim F_n(x_n) = x \), \( \lim F_n(y_n) = y \) and \( F_n(x_n), F_n(y_n) \in K \). We have that
\[
|f(x) - f(y)| \leq |f(x) - f(F_n(x_n))| + |f(F_n(x_n)) - f(x_n)| + |f(x_n) - f(y_n)| + |f(y_n) - f(F_n(y_n))| + |f(F_n(y_n)) - f(y)|. 
\]

By the continuity of \( f \) and the uniform convergence (3) of \( f_n \) on \( K \), only the third term in the right hand side may not converge to zero. Since the family \( \{ f_n \} \) is equi-Lipschitz, for the third term we have
\[
|f_n(x_n) - f_n(y_n)| \leq Q d_n(x_n, y_n) 
\]
(63)
\[
\leq Q B d(F_n(x_n), F_n(y_n)) + Q B A_n \quad \text{using (1)}. 
\]

Letting \( n \rightarrow \infty \) in (62) and (63) we get
\[
|f(x) - f(y)| \leq Q B d(x, y). 
\]

\[\Box\]
5. Comparison Theorem.

We say that the hamiltonian $H : T^*M \to \mathbb{R}$ is \textit{quadratic at infinity} if there is a riemannian metric on $M$, and $R > 0$ such that if $x \in M$ and $|p|_x > R$ then

$$H(x, p) = \frac{1}{2} |p|_x^2 + p(\xi(x)) + V(x),$$

where $\xi(x)$ is a smooth vector field on $M$ and $V : M \to \mathbb{R}$ is a smooth function.

$5.1. \textbf{Theorem} (Comparison Theorem).

Suppose that $H : T^*M \to \mathbb{R}$ is quadratic at infinity. Let $H_\varepsilon : T^*\hat{M} \to \mathbb{R}$ be

$$H_\varepsilon(x, p) = \hat{H}(x, \frac{1}{\varepsilon}p).$$

Let $\Omega \subset]0, T[\times\hat{M}$ be a compact set. Consider

$$\partial_t u + H_\varepsilon(x, \partial_x u) = 0 \quad x \in \hat{M}. \tag{64}$$

Suppose that $u$ is a Lipschitz viscosity subsolution and $v$ is a Lipschitz viscosity supersolution of (64) such that $u \leq v$ on $\partial \Omega$. Then $u \leq v$ on $\Omega$.

To illustrate the proof we show it in case $u$ and $v$ are differentiable. If there is a point in $\Omega$ where $u > v$ then there is $\lambda > 0$ such that

$$\sup_{\Omega}(u - v - \lambda t) > 0. \tag{65}$$

Let $(x_0, t_0)$ be an (interior) point where the supremum (65) is attained. We have that

$$\partial_t u(x_0, t_0) + H_\varepsilon(x_0, \partial_x u(x_0, t_0)) \leq 0, \tag{66}$$

$$\partial_t v(x_0, t_0) + H_\varepsilon(x_0, \partial_x v(x_0, t_0)) \geq 0.$$ 

$$\partial_x u(x_0, t_0) = \partial_x v(x_0, t_0), \tag{67}$$

$$\partial_t u(x_0, t_0) - \partial_t v(x_0, t_0) = \lambda > 0. \tag{68}$$

Subtracting the equations in (66) we get a contradiction with equations (67), (68).

\textbf{Proof:} Suppose there is a point $(x_0, t_0)$ in $\Omega$ where $u > v$. Then there is $\lambda > 0$ such that

$$\sup_{\Omega}(u - v - 2\lambda t) > 0. \tag{69}$$

For $\delta > 0$ small, define $f_\delta : \Omega \times \Omega \to \mathbb{R}$ by

$$f_\delta(x, t, y, s) := u(x, t) - v(y, s) - \lambda(t + s) - \frac{1}{\delta}(|t - s|^2 + d(x, y)^2).$$

Let $(x_\delta, t_\delta, y_\delta, s_\delta)$ be a maximizing point of $f_\delta$ in $\Omega \times \Omega$.

We have that

$$f_\delta(x_\delta, t_\delta, y_\delta, s_\delta) \geq f_\delta(x_0, t_0, x_0, t_0) > 0. \tag{70}$$
\[ \frac{1}{\delta} |t_\delta - s_\delta|^2 + d(x_\delta, y_\delta)^2 < u(x_\delta, t_\delta) - v(y_\delta, s_\delta) - \lambda (s_\delta + t_\delta) \leq \sup_\Omega |u| + \sup_\Omega |v| + \sup_\Omega 2\lambda |t| =: Q \]

\[ |t_\delta - s_\delta|^2 + d(x_\delta, y_\delta)^2 \leq \delta Q \]

(71)

\[
\max\{|s_\delta - t_\delta|, d(x_\delta, y_\delta)\} \leq Q_1 \sqrt{\delta}, \quad Q_1 := \sqrt{Q}.
\]

Observe that if \((x, t) \in \partial \Omega\) then \(u(x, t) \leq v(x, t)\) and hence, for \(\tau := \inf\{t : (x, t) \in \Omega\}\),

(72)

\[ \forall (x, t) \in \partial \Omega \quad f_\delta(x, t, x, t) \leq -2\lambda t \leq -2\lambda \tau. \]

We show now that both \((x_\delta, t_\delta)\) and \((y_\delta, s_\delta)\) are interior points of \(\Omega\) if \(\delta\) is small enough. Suppose on the contrary that \((x_\delta, t_\delta) \in \partial \Omega\). Let \(K\) be a Lipschitz constant for \(v\). Using (72), we have that

\[
\begin{align*}
    f_\delta(x_\delta, t_\delta, y_\delta, s_\delta) &= f_\delta(x_\delta, t_\delta, y_\delta, s_\delta) - f(x_\delta, t_\delta, x_\delta, t_\delta) + f(x_\delta, t_\delta, x_\delta, t_\delta) \\
    &\leq \left(-v(y_\delta, s_\delta) + v(x_\delta, t_\delta) - \lambda (s_\delta - t_\delta)\right) - 2\lambda \tau \\
    &\leq K \left(d(x_\delta, y_\delta) + |t_\delta - s_\delta|\right) + \lambda |t_\delta - s_\delta| - 2\lambda \tau.
\end{align*}
\]

Then, using (71), there is \(\delta_0 > 0\) such that

\[
0 < \delta < \delta_0 \quad \& \quad (x_\delta, t_\delta) \in \partial \Omega \quad \implies \quad f_\delta(x_\delta, t_\delta, y_\delta, s_\delta) < 0.
\]

This contradicts (70), therefore \((x_\delta, t_\delta) \notin \partial \Omega\) and similarly \((y_\delta, s_\delta) \notin \partial \Omega\) if \(0 < \delta < \delta_0\).

Define \(\phi : [0, T] \times N \to \mathbb{R}\) by

\[
\begin{align*}
    u(x, t) - \phi(x, t) := f_\delta(x, t, y_\delta, s_\delta), \\
    \phi(x, t) &= v(y_\delta, s_\delta) + \lambda(t + s_\delta) + \frac{1}{\delta}(|t - s_\delta|^2 + d(x, y_\delta)^2).
\end{align*}
\]

Observe that \(u - \phi\) has a local maximum at \((x_\delta, t_\delta)\), then

\[
\begin{align*}
    \partial_t \phi(x_\delta, t_\delta) + H_\varepsilon(x_\delta, \partial_x \phi(x_\delta, t_\delta)) &\leq 0, \\
    \lambda + \frac{2}{\delta}(t_\delta - s_\delta) + H_\varepsilon(x_\delta, \frac{2}{\delta} d(x_\delta, y_\delta) \nabla_x d(x_\delta, y_\delta)) &\leq 0.
\end{align*}
\]

(73)

Define \(\psi : [0, T] \times N \to \mathbb{R}\) by

\[
\begin{align*}
    v(y, s) - \psi(y, s) := -f_\delta(x_\delta, t_\delta, y, s), \\
    \psi(x, t) &= u(x, t_\delta) - \lambda(t_\delta + s) - \frac{1}{\delta}(|t_\delta - s|^2 + d(x_\delta, y)^2).
\end{align*}
\]

Then \(v - \psi\) has a local minimum at \((y_\delta, s_\delta)\). Since \(v\) is a supersolution we have that

\[
\begin{align*}
    \partial_s \psi(y_\delta, s_\delta) + H_\varepsilon(y_\delta, \partial_y \psi(y_\delta, s_\delta)) &\geq 0, \\
    -\lambda + \frac{2}{\delta}(s_\delta - t_\delta) + H_\varepsilon(y_\delta, \frac{2}{\delta} d(x_\delta, y_\delta) \nabla_y d(x_\delta, y_\delta)) &\geq 0.
\end{align*}
\]

(74)

Subtracting (73)−(74) we obtain

\[
2\lambda + H_\varepsilon(x_\delta, \frac{2}{\delta} d(x_\delta, y_\delta) \nabla_x d(x_\delta, y_\delta)) - H_\varepsilon(y_\delta, \frac{2}{\delta} d(x_\delta, y_\delta) \nabla_y d(x_\delta, y_\delta)) \leq 0.
\]

(75)
Since $\lambda > 0$, if we can prove that
\[
H_{e}(x_{\delta}, z_{\delta}) \nabla_{x} d(x_{\delta}, y_{\delta}) - H_{e}(y_{\delta}, -z_{\delta}) \nabla_{y} d(x_{\delta}, y_{\delta})
\]
is small for $\delta$ small, we get a contradiction with (75).

Since by (71) $d(x_{\delta}, y_{\delta})$ is very small we have that

\[
\nabla_{y} d(x_{\delta}, y_{\delta}) = \gamma_{xy}(d(x_{\delta}, y_{\delta})),
\]

where $\gamma_{xy}(t)$ is the unit speed geodesic going from $x_{\delta}$ to $y_{\delta}$.

Consider the local parametrization of $\hat{M}$ given by the exponential map

\[
\exp_{x_{\delta}} : T_{x_{\delta}} \hat{M} \to \hat{M}.
\]

In these coordinates we have that

\[
\frac{2}{\delta} d(x_{\delta}, y_{\delta}) \nabla_{y} d(x_{\delta}, y_{\delta}) = \frac{2}{\delta} (y_{\delta} - x_{\delta}), \quad x_{\delta} = 0 \in T_{x_{\delta}} \hat{M},
\]

\[
\frac{2}{\delta} d(x_{\delta}, y_{\delta}) \nabla_{y} d(x_{\delta}, y_{\delta}) = -\frac{2}{\delta} (y_{\delta} - x_{\delta}), \quad x_{\delta} = 0 \in T_{x_{\delta}} \hat{M}.
\]

In these coordinates (76) becomes

\[
H_{e}(x_{\delta}, p_{\delta}) - H_{e}(y_{\delta}, p_{\delta}), \quad p_{\delta} := -\frac{2}{\delta} (y_{\delta} - x_{\delta}).
\]

Since $H$ is quadratic at infinity, there is $A > 0$ such that

\[
|H_{e}(x, p) - H_{e}(y, p)| \leq A \left(1 + \frac{1}{\delta} |p|^{2}\right) d(x, y),
\]

and this constant $A$ is uniform for the unit ball $B(0, 1) \subset T_{x} \hat{M}$ in the coordinates (77) for every $x \in \hat{M}$. When $\delta \to 0$ the points $x_{\delta}$ and $y_{\delta}$ move, but inequalities (71) and (79) remain valid with the same constants $Q_{1}$ and $A$.

We need to bound the distance from $(x_{\delta}, t_{\delta})$ to $\partial \Omega$. If $(z, r) \in \partial \Omega$ by (72) we have that

\[
0 \leq f(x_{\delta}, t_{\delta}, y_{\delta}, s_{\delta}) - f(z, r, z, r) + f(z, r, z, r),
\]

\[
0 \leq u(x_{\delta}, t_{\delta}) - v(y_{\delta}, s_{\delta}) - \lambda (t_{\delta} + s_{\delta}) - \frac{1}{\delta} (|t_{\delta} - s_{\delta}|^{2} + d(x_{\delta}, y_{\delta})^{2}) - u(z, r) + v(z, r) + \lambda 2r - 2 \lambda \tau,
\]

\[
0 \leq K(d[(x_{\delta}, t_{\delta}), (z, r)] + d[(y_{\delta}, s_{\delta}), (z, r)]) + |\lambda||t_{\delta} - r| + |s_{\delta} - r| - 2 \lambda \tau.
\]

From (71)

\[
d[(y_{\delta}, s_{\delta}), (z, r)] \leq d[(x_{\delta}, t_{\delta}), (z, r)] + 2Q_{1} \sqrt{\delta},
\]

\[
|s_{\delta} - r| \leq |t_{\delta} - r| + Q_{1} \sqrt{\delta}.
\]

Therefore

\[
2 \lambda \tau - (2K + \lambda) 2Q_{1} \sqrt{\delta} \leq 2Kd[(x_{\delta}, t_{\delta}), (z, r)] + 2 \lambda |t_{\delta} - r|.
\]

Then for $\delta$ small enough

\[
\lambda \tau \leq 2(K + \lambda) d[(x_{\delta}, t_{\delta}), (z, r)].
\]
This implies that there are $a > 0$ and $\delta_1 > 0$ such that

$$\forall \delta \in ]0, \delta_1[ \quad d((x_\delta, t_\delta), \partial \Omega) > a.$$  

This and (71) imply that for $\delta$ small enough $(x_\delta, s_\delta) \in \text{int } \Omega$.

Since $(x_\delta, t_\delta, y_\delta, s_\delta)$ is a maximum on $\Omega \times \Omega$, $v$ is Lipschitz and $(x_\delta, s_\delta) \in \text{int } \Omega$, we have that

$$0 \leq f_\delta(x_\delta, t_\delta, y_\delta, s_\delta) - f_\delta(x_\delta, t_\delta, x_\delta, s_\delta)$$

$$0 \leq (v(x_\delta, s_\delta) - v(y_\delta, s_\delta)) - \frac{1}{\delta} d(x_\delta, y_\delta)^2$$

$$\frac{1}{\delta} d(x_\delta, y_\delta)^2 \leq K d(x_\delta, y_\delta)$$

$$d(x_\delta, y_\delta) \leq K \delta.$$  

(80)

Using (78), (79), (80) we obtain

$$|H_\varepsilon(x_\delta, p_\delta) - H_\varepsilon(y_\delta, p_\delta)| \leq A (1 + \frac{1}{\epsilon} 4K^2)K \delta \xrightarrow{\delta \downarrow 0} 0.$$  

Since (78) is equal to (76), we get that (76) is arbitrarily small for $\delta$ small enough and this contradicts (75).

5.2. Corollary.

If $H$ is quadratic at infinity, $\Omega \subset]0, T[ \times \hat{M}$ is a compact set, $u$ is a Lipschitz viscosity subsolution of (64) and $v$ is a Lipschitz viscosity supersolution of (64) then

$$\max_{\Omega}(u - v) \leq \max_{\partial \Omega}(u - v).$$  

Proof:

Observe that if $a \in \mathbb{R}$ then $v + a$ is also a supersolution of (64). Apply this to $a = \max_{\partial \Omega}(u - v)$. We get that on $\partial \Omega$, $u \leq v + a$, then from Theorem 5.1, $u \leq v + a$ on $\Omega$, this is

$$\max_{\Omega}(u - v) \leq a = \max_{\partial \Omega}(u - v).$$  

□

6. Proof with test functions.

Here we prove Theorem 1.4. By Proposition 3.1 and Proposition 4.1 in order to prove that $\lim_{\varepsilon} v^\varepsilon = u$ it is enough to prove that there is a unique possible limit of subsequences $v^{\varepsilon_n}$. Since the equation (17) has a unique solution, the following proposition finishes the proof of Theorem 1.4.
6.1. Proposition.

If a subsequence \( u^{\varepsilon_n} \) of the family \( u^{\varepsilon} \) in (31) (of solutions to (16)) converges uniformly on compact subsets to a function \( u = \lim_n u^{\varepsilon_n} \), then \( u \) satisfies in the viscosity sense the equation

\[
\partial_t u + \overline{H}(\partial_x u) = 0, \\
u(x,0) = f(x).
\]

(81)

6.2. Lemma.

It is enough to prove Proposition 6.1 for hamiltonians quadratic at infinity.

Proof:

By Proposition 3.1 there is a uniform Lipschitz constant \( Q \) for all the solutions \( u^{\varepsilon} \). By Rademacher Theorem \( \partial_y u^{\varepsilon} \) is defined almost everywhere and it is a weak derivative of \( u^{\varepsilon} \).

Observe that

\[
Q \geq \| \partial_y u^{\varepsilon} \| = \sup_{|v|_\varepsilon = 1} |\partial_y u^{\varepsilon} \cdot v| = \sup_{|v|_\varepsilon = 1} \frac{1}{\varepsilon} \| \partial_y u^{\varepsilon} \|_1.
\]

Thus in equation (16)

\[
\partial_t u^{\varepsilon} + \hat{H}(y, \frac{1}{\varepsilon} \partial_y u^{\varepsilon}) = 0,
\]

we only use the Hamiltonian \( H \) on co-vectors \( (y,p) \in T^*M \) with \( |p|_y \leq Q \).

By Proposition 4.2 the limit function \( u \) has some Lipschitz constant \( QB \), with \( B > 1 \) from (1). Thus the effective hamiltonian \( \overline{H} \) in equation (81) for \( u = \lim u^{\varepsilon_n} \) is only used on co-vectors \( c \in H_1(M, \mathbb{R})^* = H^3(M, \mathbb{R}) \) with norm \( |c| \leq QB \).

On the other hand, by Proposition 4.2, the function \( f \) is Lipschitz. Since Mather’s \( \alpha \) and \( \beta \) functions \( \overline{H} \) and \( \overline{L} \) are convex and superlinear, by Proposition 3.1 there is \( Q_1 > Q > 0 \) such that any solution of the problem (81) has Lipschitz constant \( Q_1 \).

The effective hamiltonian \( \overline{H} \) is Mather’s alpha function which satisfies (see [4] Cor. 1):

\[
\overline{H}(c) = \inf_{|y| = c} \sup_{x \in M} H(x, \eta(x)).
\]

Therefore

\[
H_1 = H \quad \text{on} \quad [H \leq \overline{H}(c)] \implies \overline{H}_1(c) = \overline{H}(c).
\]

Let \( h_0 := \sup_{|c| \leq Q_1 B} \overline{H}(c) \). Then if \( H_1 = H \) on \([H \leq h_0]\), any limit function \( u = \lim_n u^{\varepsilon_n} \) satisfies the problem (81) if and only if it satisfies the problem with \( H_1 \):

\[
\partial_t u + \overline{H}_1(\partial_x u) = 0, \\
u(x,0) = f(x).
\]

(82)

Moreover, the solutions of problems (81) and (82) are equal.

By the superlinearity of \( H \) there is \( R_0 > Q_1 \) such that \([H \leq h_0] \subset [||p||_x \leq R_0] \). Therefore we can replace \( H \) with a hamiltonian \( H_1 \) quadratic at infinity such that \( H_1 = H \) on
$|p|_x \leq R_0$ and we will have that the families $u^\varepsilon$, the limits $u$ and any solution of (81) will be the same for both hamiltonians. For a construction of such hamiltonian quadratic at infinity see Proposition 18 in [5].

\[\]

**Proof of Proposition 6.1:**

By Lemma 6.2 we can assume that $H$ is quadratic at infinity, so we can apply Corollary 5.2. We prove that $u$ is a viscosity subsolution of (81). The proof for supersolution will be similar. By Lemma 1.3, it is enough to use Lipschitz test functions giving strict local maxima.

Let $\phi \in C^1(H_1(M, \mathbb{R}) \times ]0, +\infty[, \mathbb{R})$ be such that $\phi$ is Lipschitz and

\begin{equation}
(83)
\end{equation}

\[u - \phi\] has a strict local maximum at $(y_0, t_0)$.

We want to show that

\begin{equation}
(84)
\end{equation}

\[\partial_t \phi(y_0, t_0) + \overline{H}(D_x \phi(y_0, t_0)) \leq 0.\]

Assume that (84) is not true, i.e. there is $\theta > 0$ such that

\begin{equation}
(85)
\end{equation}

\[\partial_t \phi(y_0, t_0) + \overline{H}(D_x \phi(y_0, t_0)) = \theta > 0.\]

Let $w : M \rightarrow \mathbb{R}$ be a viscosity solution to the cell problem

\begin{equation}
(86)
\end{equation}

\[H(x, g(P) + Dw(x)) = \overline{H}(P), \quad P := D_x \phi(y_0, t_0) \in H^1(M, \mathbb{R}),\]

and let $\hat{w} = w \circ \pi$ be its lift to $\hat{M}$.

Define $\phi^\varepsilon : M_\varepsilon \times \mathbb{R}_+ \rightarrow \mathbb{R}$, $M_\varepsilon = (\hat{M}, d_\varepsilon)$, by

\begin{equation}
(87)
\end{equation}

\[\phi^\varepsilon(x, t) := \phi(F_\varepsilon(x), t) + \varepsilon \hat{w}(x).\]

By (22) the lift $\hat{w} = w \circ \pi$ is a viscosity solution of

\begin{equation}
(88)
\end{equation}

\[\hat{H}(x, P \cdot DG(x) + D\hat{w}(x)) = \overline{H}(P), \quad x \in \hat{M}, \quad P = D_x \phi(y_0, t_0) \in H^1(M, \mathbb{R}).\]

Observe that in (88) $P = D_x \phi(y_0, t_0) : H_1(M, \mathbb{R}) = \mathbb{R}^k \rightarrow \mathbb{R}$ is a linear functional on $H_1(M, \mathbb{R}) = \mathbb{R}^k$ and also on each tangent space $T_z H_1(M, \mathbb{R})$, even when $z \neq y_0$, this is

\[P = D_x \phi(y_0, t_0) \in H^1(M, \mathbb{R}) = H_1(M, \mathbb{R})^*.\]

1. **Claim.** For some $r, \varepsilon_0 > 0$ small and $\hat{H} := H \circ (\pi^*)^{-1}$, the lift of $H$,

\begin{equation}
(89)
\end{equation}

\[\forall \varepsilon < \varepsilon_0 \quad \partial_t \phi^\varepsilon + \hat{H}(x, \frac{1}{\varepsilon} D_x \phi^\varepsilon) \geq \frac{\theta}{2} \quad (\text{supersolution})\]

in the viscosity sense in $(F_\varepsilon \times id)^{-1}(B_r(y_0, t_0))$.\]
**Proof of the Claim:**

Suppose that $\phi^\varepsilon - \psi^\varepsilon$ has a minimum at $(x_1^\varepsilon, t_1) \in M_\varepsilon \times \mathbb{R}_+$ with $F_\varepsilon(x_1^\varepsilon) \rightarrow y_1$ and $(y_1, t_1)$ near $(y_0, t_0)$. We have to prove that

$$
\partial_t \psi^\varepsilon(x_1^\varepsilon, t_1) + \hat{H}(x_1^\varepsilon, \frac{1}{\varepsilon} D_x \psi^\varepsilon(x_1^\varepsilon, t_1)) \geq \frac{\theta}{2}.
$$

We have that

$$
\phi^\varepsilon - \psi^\varepsilon \geq \phi^\varepsilon(x_1^\varepsilon, t_1) - \psi^\varepsilon(x_1^\varepsilon, t_1).
$$

Using (87), this is

$$
\phi(F_\varepsilon(x), t) + \varepsilon \hat{w}(x) - \psi^\varepsilon(x, t) \geq \phi(F_\varepsilon(x_1^\varepsilon), t_1) + \varepsilon \hat{w}(x_1^\varepsilon) - \psi^\varepsilon(x_1^\varepsilon, t_1).
$$

Define $\eta: \hat{M} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ by

$$
\eta(x, t) := \frac{1}{\varepsilon} [\psi^\varepsilon(x, t) - \phi(F_\varepsilon(x), t)].
$$

Then $\hat{w} - \eta$ has a local minimum at $(x_1^\varepsilon, t_1)$.

By the viscosity property (15) on the lift (22) of (86), we have that

$$
\hat{H}(x_1^\varepsilon, P \cdot DG(x_1^\varepsilon) + D_x \eta(x_1^\varepsilon, t_1)) \geq \mathcal{P}(P),
$$

where $P = D_x \phi(y_0, t_0) \in H^1(M, \mathbb{R})$ and $\hat{H} = H \circ (\pi^*)^{-1}$. Then by (85),

$$
\partial_t \phi(y_0, t_0) + \hat{H}(x_1^\varepsilon, P \cdot DG(x_1^\varepsilon) + D_x \eta(x_1^\varepsilon, t_1)) \geq \partial_t \phi(y_0, t_0) + \mathcal{P}(P) = \theta.
$$

Using (5), observe that

$$
D_x \eta(x, t) = -D_x \phi(F_\varepsilon(x), t) DG(x) + \frac{1}{\varepsilon} D_x \psi^\varepsilon(x, t).
$$

Thus

$$
\partial_t \phi(y_0, t_0) + \hat{H}[x_1^\varepsilon, P \cdot DG(x_1^\varepsilon) - D_x \phi(F_\varepsilon(x_1^\varepsilon), t_1) DG(x_1^\varepsilon) + \frac{1}{\varepsilon} D_x \psi^\varepsilon(x_1^\varepsilon, t_1)] \geq \theta.
$$

Writing $y_1^\varepsilon = F_\varepsilon(x_1^\varepsilon)$ and recalling that $P = D_x \phi(x_0, t_0)$, we have that

$$
|\partial_t \phi(y_0, t_0) - \partial_t \phi(y_1^\varepsilon, t_1)| + \partial_t \phi(y_1^\varepsilon, t_1) +
$$

$$
+ \hat{H}[x_1^\varepsilon, (D_x \phi(y_0, t_0) - D_x \phi(y_1^\varepsilon, t_1)) DG(x_1^\varepsilon) + \frac{1}{\varepsilon} D \psi^\varepsilon(x_1^\varepsilon, t_1)] \geq \theta.
$$

In (83) we have that $\phi \in C^1(H_1(M, \mathbb{R}) \times \mathbb{R}_+, \mathbb{R})$ and by (19), $\|DG\|$ is bounded. Thus, if $d[(y_0, t_0), (y_1^\varepsilon, t_1)] < r$ is small enough, then

$$
\partial_t \phi(y_1^\varepsilon, t_1) + \hat{H}[x_1^\varepsilon, \frac{1}{\varepsilon} D_x \psi^\varepsilon(x_1^\varepsilon, t_1)] \geq \frac{\theta}{2}.
$$

Since $\hat{w} - \eta$ has a local minimum at $(x_1^\varepsilon, t_1)$ and $\hat{w}$ is time-independent, we have that

$$
\partial_t \eta(x_1^\varepsilon, t_1) = \frac{1}{\varepsilon} [\partial_t \psi^\varepsilon(x_1^\varepsilon, t_1) - \partial_t \phi(F_\varepsilon(x_1^\varepsilon), t_1)] = 0.
$$

This is $\partial_t \psi^\varepsilon(x_1^\varepsilon, t_1) = \partial_t \phi(y_1^\varepsilon, t_1)$. Therefore we get (90):

$$
\partial_t \psi^\varepsilon(x_1^\varepsilon, t_1) + \hat{H}(x_1^\varepsilon, \frac{1}{\varepsilon} D_x \psi^\varepsilon(x_1^\varepsilon, t_1)) \geq \frac{\theta}{2}.
$$
This proves the claim, i.e. \( \phi^\varepsilon \) satisfies (89) in the viscosity sense in a small neighborhood \((F_\varepsilon \times \text{id})^{-1}(B_r(y_0, t_0))\).

\[ \triangle \]

Let \( x_0^\varepsilon \in M_\varepsilon \) be such that \( \lim_\varepsilon F_\varepsilon(x_0^\varepsilon) = y_0 \). By Claim 1, the function \( \phi^\varepsilon \) is a viscosity supersolution of

\[ \partial u + H(x, \frac{1}{\varepsilon} D_x u) = 0 \]

nearby \((x_0^\varepsilon, t_0)\).

By the choice of \( \phi \) in (83), \( \phi \) is Lipschitz. The solution \( w \) to the cell problem (86), in \( M \) is also Lipschitz. By Lemma 2.1, the map \( F_\varepsilon \) is Lipschitz on \((M_\varepsilon, d_\varepsilon)\). Then by (87) \( \phi^\varepsilon \) is Lipschitz on \((M_\varepsilon, d_\varepsilon)\). By Proposition 3.1 the functions \( u^\varepsilon \) are Lipschitz on \((M_\varepsilon, d_\varepsilon)\).

By Corollary 2.3, if \( 0 < r < t_0 \) the set \((F_\varepsilon \times \text{id})^{-1}(B_r(y_0, t_0))\) is compact. Since \( \phi^\varepsilon \) is a supersolution and \( u^\varepsilon \) is a subsolution of (91) and \( F_\varepsilon(x_0^\varepsilon) \xrightarrow{\varepsilon} y_0 \); by Corollary 5.2 for \( \varepsilon \) small enough we have that

\[ u^\varepsilon(x_0^\varepsilon, t_0) - \phi^\varepsilon(x_0^\varepsilon, t_0) \leq \sup_{\partial(F_\varepsilon \times \text{id})^{-1}(B_r(y_0, t_0))} (u^\varepsilon - \phi^\varepsilon). \]

Let \((z^\varepsilon, s^\varepsilon) \in \partial(F_\varepsilon \times \text{id})^{-1}(B_r(y_0, t_0))\) be such that

\[ u^\varepsilon(x_0^\varepsilon, t_0) - \phi^\varepsilon(x_0^\varepsilon, t_0) \leq u^\varepsilon(z^\varepsilon, s^\varepsilon) - \phi^\varepsilon(z^\varepsilon, s^\varepsilon). \]

Observe that from (87), if \( F_\varepsilon(x^\varepsilon) \to y \) and \( t^\varepsilon \to t \) then \( \phi^\varepsilon(x^\varepsilon, t^\varepsilon) \to \phi(y, t) \). Let \( \varepsilon_i \to 0 \) be a sequence such that the following holds:

(a) \( \lim_i u^\varepsilon_i(x_0^\varepsilon_i, t_0) = u(y_0, t_0) \),

(b) the limit \((z_0, s_0) := \lim_i(F_\varepsilon_i(z^\varepsilon_i), s^\varepsilon_i)\) exists,

(c) \( \lim_i u^\varepsilon_i(z^\varepsilon_i, s^\varepsilon_i) = u(z_0, s_0) \).

Then we have that \( d((y_0, t_0), (z_0, s_0)) = \varepsilon \) and

\[ u(y_0, t_0) - \phi(y_0, t_0) \leq u(z_0, s_0) - \phi(z_0, s_0). \]

This contradicts (83), the strict local maximum property of \((y_0, t_0)\). Thus (85) is impossible, and hence we get (84):

\[ \partial_t \phi(y_0, t_0) + \bar{\Pi}(D_x \phi(y_0, t_0)) \leq 0 \]

whenever \( \phi \in C^1 \), \( \phi \) is Lipschitz, and \( u - \phi \) has a strict local maximum at a point \((y_0, t_0)\). Therefore \( u \) is a viscosity subsolution of (81).

The proof that \( u \) is a viscosity supersolution of (81) is similar. \( \square \)
7. Uniqueness.

We prove here the uniqueness of Lipschitz viscosity solutions to the initial value problem of the evolutive Hamilton-Jacobi equation on the abelian cover \( \hat{M} \).

7.1. Lemma.

Suppose that \( u : \hat{M} \times [0,T] \to \mathbb{R} \) is a continuous function which is a viscosity subsolution of (64) on the open interval \( t \in ]0,T[ \). If \( \phi \) is \( C^1 \) and \( u - \phi \) attains a local maximum at a point \( (x_0,T) \), then

\[
\partial_t \phi(x_0, T) + H(x_0, \partial_x \phi(x_0, T)) \leq 0.
\]

A corresponding statement holds for supersolutions.

Proof:

Replacing \( \phi \) by \( \phi(x,t) + d(x,x_0)^2 + |t-T|^2 \) we can assume that \( u - \phi \) attains a strict local maximum at \( (x_0,T) \). For \( \delta > 0 \) let \( \phi_\delta(x,t) := \phi(x,t) + \frac{\delta}{T-t} \).

Then the function \( u - \phi_\delta \) attains a local maximum at a point \( (x_\delta, t_\delta) \) with

\[
t_\delta < T, \quad (x_\delta, t_\delta) \longrightarrow (x_0, T) \quad \text{as} \quad \delta \to 0^+.
\]

Since \( u \) is a viscosity subsolution on \( t < T \), we have that

\[
\partial_t \phi(x_\delta, t_\delta) + H(x_\delta, \partial_x \phi(x_\delta, t_\delta)) = \partial_t \phi_\delta(x_\delta, t_\delta) + H(x_\delta, \partial_x \phi_\delta(x_\delta, t_\delta)) - \frac{\delta}{(T-t_\delta)^2} \leq 0.
\]

Letting \( \delta \to 0 \) we obtain (93). \( \square \)

7.2. Proposition.

Suppose that \( H : T^* M \to \mathbb{R} \) is quadratic at infinity.

Let \( H_\varepsilon : T^* \hat{M} \to \mathbb{R} \) be \( H_\varepsilon(x,p) = \hat{H}(x, \frac{1}{\varepsilon} p) \). Consider the equation

\[
\partial_t u + H_\varepsilon(x, \partial_x u) = 0. \tag{94}
\]

Suppose that \( u \) is a Lipschitz viscosity subsolution of (94) and \( v \) is a Lipschitz viscosity supersolution of (94). If \( u \leq v \) on \( \hat{M} \times \{0\} \) then \( u \leq v \).

Proof:

Assume that \( u(x,0) \leq v(x,0) \) for all \( x \in \hat{M} \). Suppose by contraposition that there is a point in \( (x_0,t_0) \in \hat{M} \times [0,T] \) where \( u > v \). Then there is \( \lambda > 0 \) such that

\[
(u - v - 2\lambda t)(x_0, t_0) =: \sigma > 0.
\]

The function \( g(x) := d(x,x_0) \) is not differentiable at \( x_0 \) and the points in the cut locus of \( x_0 \). It is Lipschitz and hence weakly differentiable and \( \| \nabla_x d(x,x_0) \| = 1 \) almost
Then we have that
\begin{equation}
\forall x \in \hat{M} \quad \|\nabla_x f\| \leq 2, \\
\end{equation}

\begin{equation}
f(x_0) = 0.
\end{equation}

Let \( K \) be a Lipschitz constant for \( u \) and for \( v \). Then
\begin{equation}
u(x_\delta, t_\delta) - u(y_\delta, s_\delta) = u(x_\delta, t_\delta) - u(x_\delta, 0) + u(x_\delta, 0) - u(y_\delta, 0) \\
+ u(y_\delta, 0) - v(y_\delta, 0) + v(y_\delta, 0) - v(y_\delta, s_\delta)
\end{equation}

\begin{equation}
\leq K|t_\delta| + Kd(x_\delta, y_\delta) + 0 + K|s_\delta|.
\end{equation}

For \( 0 < \delta < 1 \) small, define \( f_\delta : (\hat{M} \times [0, T])^2 \rightarrow \mathbb{R} \) by
\begin{equation}
f_\delta(x, t, y, s) := u(x, t) - v(y, s) - \lambda(t + s) - \frac{1}{\delta^2} (|t - s|^2 + d(x, y)^2) - \delta (f(x)^2 + f(y)^2).
\end{equation}

Since \( u \) and \( v \) have linear growth and by (95), \( f(x)^2 \) has quadratic growth; there is a point \((x_\delta, t_\delta, y_\delta, s_\delta)\) which maximizes \( f_\delta \) in \( \hat{M} \times [0, T] \).

We have that
\begin{equation}
f_\delta(x_\delta, t_\delta, y_\delta, s_\delta) \geq f_\delta(x_0, t_0, x_0, t_0) = \sigma > 0.
\end{equation}

Using (97), we have that
\begin{equation}
\frac{1}{\delta^2} (|t_\delta - s_\delta|^2 + d(x_\delta, y_\delta)^2) < u(x_\delta, t_\delta) - v(y_\delta, s_\delta) - \lambda(s_\delta + t_\delta)
\end{equation}

\begin{equation}
\leq K|t_\delta| + Kd(x_\delta, y_\delta) + K|s_\delta| + 0
\end{equation}

\begin{equation}
\leq K d(x_\delta, y_\delta) + 2KT.
\end{equation}

Then
\begin{equation}
\frac{1}{\delta^2} d(x_\delta, y_\delta)^2 \leq K d(x_\delta, y_\delta) + 2KT.
\end{equation}

We have that \( z := d(x_\delta, y_\delta) \) satisfies the inequality
\begin{equation}
z^2 - K\delta^2 z - 2KT\delta^2 \leq 0.
\end{equation}

Then \( z \) must be smaller than the larger root of this quadratic polynomial, i.e.
\begin{equation}
d(x_\delta, y_\delta) = z \leq \frac{1}{2} (K\delta^2 + \sqrt{K^2\delta^4 + 8KT\delta^2}) \leq Q_0 \delta,
\end{equation}

where \( Q_0 = Q_0(T) \). From (99) using (100) and \( \delta \leq 1 \) we also obtain
\begin{equation}
|t_\delta - s_\delta| \leq Q_1 \delta,
\end{equation}

with some \( Q_1 = Q_1(T) \geq Q_0(T) \).

Observe that if \( \delta \) is small enough, by (100) the point \( y_\delta \) is not in the cut locus of \( x_\delta \) and vice versa. Therefore the function \( d(x, y)^2 \) is differentiable at \((x_\delta, y_\delta)\) with partial derivatives \( \nabla_x d(x_\delta, y_\delta) = 2 d(x_\delta, y_\delta) \nabla_x d(x_\delta, y_\delta) \), \( \|\nabla_x d(x, y)\| = 1 \).
We show that the maximum of $f_\delta$ is not attained at the initial time $t = 0$. From (98) we have that
\[\sigma \leq f_\delta(x_\delta, t_\delta, y_\delta, s_\delta) \leq u(x_\delta, t_\delta) - v(y_\delta, s_\delta)\]
\[\leq K|t_\delta| + KQ_0 \delta + K|s_\delta|\]
\[\leq 2K|t_\delta| + KQ_1\delta + KQ_0 \delta\]
using (97) and (100),

A similar inequality holds for $|s_\delta|$. For $\delta > 0$ sufficiently small, this implies that there is $\mu > 0$ such that
\[t_\delta > \mu > 0 \quad \text{and} \quad s_\delta > \mu > 0.\]

We could still have $t_\delta = T$ or $s_\delta = T$.

Define $\phi : \hat{M} \times [0, T] \rightarrow \mathbb{R}$ by
\[u(x, t) - \phi(x, t) := f_\delta(x, t, y_\delta, s_\delta)\]
\[= u(x, t) - v(y_\delta, s_\delta) - \lambda(t + s_\delta) - \frac{1}{2\delta}\left(|t - s_\delta|^2 + d(x, y_\delta)^2\right) - \delta \left(f(x)^2 + f(y_\delta)^2\right).\]

Observe that $\phi$ is $C^1$. We have that $u - \phi$ attains a local maximum in $\hat{M} \times [0, T]$ at $(x_\delta, t_\delta)$. If $(x_\delta, t_\delta)$ is an interior point, since $u$ is a viscosity subsolution we have that
\[\partial_t \phi(x_\delta, t_\delta) + H(x_\delta, \partial_x \phi(x_\delta, t_\delta)) \leq 0.\]

By (102), $t_\delta > 0$ and by Lemma 7.1 equation (103) also holds if $t_\delta = T$. From (103) we get that
\[\lambda + \frac{2}{\sigma_\delta^2}(t_\delta - s_\delta) + H(x_\delta, \frac{2}{\sigma_\delta} d(x_\delta, y_\delta) \nabla_x d(x_\delta, y_\delta) + 2\delta f(x_\delta) \nabla f(x_\delta)) \leq 0.\]

Define $\psi : \hat{M} \times [0, T] \rightarrow \mathbb{R}$ by
\[v(y, s) - \psi(y, s) := -f_\delta(x_\delta, t_\delta, y, s)\]
\[= v(y, s) - u(x_\delta, t_\delta) + \lambda(s + t_\delta) + \frac{1}{2\sigma_\delta}(s - s_\delta)^2 + d(x_\delta, y)^2\]
\[+ \delta \left(f(x_\delta)^2 + f(y)^2\right).\]

Then $\psi(y, s)$ has a local minimum at $(y_\delta, s_\delta)$. Since $v$ is a viscosity supersolution we have that
\[\partial_s \psi(y_\delta, s_\delta) + H(y_\delta, \partial_y \psi(y_\delta, s_\delta)) \geq 0,\]
and then
\[-\lambda + \frac{2}{\sigma_\delta}(s_\delta - t_\delta) + H(y_\delta, -\frac{2}{\sigma_\delta} d(x_\delta, y_\delta) \nabla_y d(x_\delta, y_\delta) - 2\delta f(y_\delta) \nabla y f(y_\delta)) \geq 0.\]

Subtracting (104)–(105) we get
\[2\lambda + H(x_\delta, P_\delta + 2\delta f(x_\delta) \nabla f(x_\delta)) - H(y_\delta, Q_\delta - 2\delta f(y_\delta) \nabla y f(y_\delta)) \leq 0,\]

where
\[P_\delta := \frac{2}{\sigma_\delta} d(x_\delta, y_\delta) \nabla_x d(x_\delta, y_\delta), \quad Q_\delta := -\frac{2}{\sigma_\delta} d(x_\delta, y_\delta) \nabla_y d(x_\delta, y_\delta)\]
and $\lambda > 0$. If we show that the last two terms in (106) are arbitrarily small we get the desired contradiction with $\lambda > 0$. 
By (100), \(x_\delta\) is near \(y_\delta\), and then \(|\nabla_x d(x_\delta, y_\delta)| = |\nabla_y d(x_\delta, y_\delta)| = 1\). Therefore

(108) \[ |P_\delta| = |Q_\delta| = \frac{2}{\sigma^2} d(x_\delta, y_\delta). \]

We need a shaper estimate for \(\frac{1}{\sigma^2} d(x_\delta, y_\delta)\). From (98), (97) and (100) we get that

\[ \delta (f(x_\delta)^2 + f(y_\delta)^2) \leq K d(x_\delta, y_\delta) + 2KT \leq 1 + 2KT, \]

(109) \[ \max \{f(x_\delta), f(y_\delta)\} \leq \frac{Q_3(T)}{\sqrt{\delta}}, \]

(110) \[ 2\delta f(x_\delta) \|\nabla f(x_\delta)\| \leq 4Q_3(T) \sqrt{\delta}, \quad \text{using (96).} \]

Since \(\|\nabla f\| \leq 2\), we have that \(f\) has Lipshitz constant 2. Therefore

\[ |f(x_\delta) - f(y_\delta)| \leq 2d(x_\delta, y_\delta) \leq 2Q_0 \delta \quad \text{using (100),} \]

\[ f(x_\delta) + f(y_\delta) \leq \frac{2Q_3}{\sqrt{\delta}} \quad \text{from (109),} \]

(111) \[ |f(x_\delta)^2 - f(y_\delta)^2| \leq 4Q_0 Q_3 \sqrt{\delta}. \]

We need a sharper estimate for \(d(x_\delta, y_\delta)\). We have that

\[ f(x_\delta, t_\delta, y_\delta, s_\delta) \geq f(x_\delta, t_\delta, x_\delta, t_\delta), \]

\[ v(x_\delta, t_\delta) - v(y_\delta, s_\delta) + \lambda(t_\delta - s_\delta) - \frac{1}{\delta^2} (|t_\delta - s_\delta|^2 + d(x_\delta, y_\delta)^2) + \delta (f(x_\delta)^2 - f(y_\delta)^2) \geq 0. \]

\[ \frac{1}{\delta^2} (|t_\delta - s_\delta|^2 + d(x_\delta, y_\delta)^2) \leq v(x_\delta, t_\delta) - v(y_\delta, s_\delta) + \lambda|t_\delta - s_\delta| + \delta (f(x_\delta)^2 - f(y_\delta)^2) \]

(112) \[ \leq K (|t_\delta - s_\delta| + d(x_\delta, y_\delta)) + \lambda|t_\delta - s_\delta| + 4Q_0 Q_3 \delta^3 \]

\[ \leq Q_4 \delta \quad \text{using (100) and (101),} \]

(113) \[ \max \{|t_\delta - s_\delta|, d(x_\delta, y_\delta)\} \leq Q_5 \delta^\frac{3}{2}. \]

We plug inequality (113) in the right hand side of inequality (112) and get

(114) \[ d(x_\delta, y_\delta) \leq Q_6 \delta^\frac{3}{2}. \]

Then from (108) we get

(115) \[ |P_\delta| = |Q_\delta| \leq Q_7 \delta^{-\frac{1}{4}}. \]

For \(x, y \in \hat{M}\) nearby, let \(\tau_{xy} : T_x \hat{M} \to T_y \hat{M}\) be the parallel transport along the minimal geodesic joining \(x\) to \(y\). Since \(H\) is quadratic at infinity, there is \(A > 0\) such that

(116) \[ |H_\varepsilon(x, p) - H_\varepsilon(y, \tau_{xy}(p))| \leq A \left(1 + \frac{1}{\varepsilon^2} |p|_x^2\right) d(x, y), \]

\[ |H_\varepsilon(x, p) - H_\varepsilon(x, q)| \leq A \frac{1}{\varepsilon^2} (1 + |p|_x + |q|_x) |p - q|_x. \]
Using (115) and (110), we have that
\[ |H_\varepsilon(x_\delta, P_\delta + 2\delta f(x_\delta)\nabla f(x_\delta)) - H_\varepsilon(x_\delta, P_\delta)| \leq A_7(2 + 2|P_\delta|) |2\delta f(x_\delta)\nabla f(x_\delta)| \]
\[ \leq Q_8 \delta^{-\frac{1}{4}} \delta^{\frac{1}{2}} = Q_8 \delta^{\frac{1}{4}}, \]  
(117)

\[ |H_\varepsilon(y_\delta, Q_\delta) - H_\varepsilon(y_\delta, Q_\delta - 2f(y_\delta)\nabla f(y_\delta))| \leq Q_8 \delta^{\frac{1}{4}}. \]
(118)

From (107), (100), \( Q_\delta = \tau_{xy}(P_\delta) \). Then using (116), (115) and (114),
\[ |H_\varepsilon(x_\delta, P_\delta) - H_\varepsilon(y_\delta, Q_\delta)| \leq A(1 + \frac{1}{2}(Q_\gamma)^2 \delta^{-\frac{1}{2}}) Q_6 \delta^{\frac{1}{4}} \leq Q_9 \delta^{\frac{5}{4}}. \]
(119)

Adding the inequalities (117), (118), (119) and comparing with (106) we get
\[ 2\lambda - Q_{10} \delta^{\frac{1}{4}} \leq 0, \]
which is false for \( \delta \) small enough. \( \square \)

7.3. Corollary.

If \( H : TM \to \mathbb{R} \) is a convex superlinear hamiltonian, \( \tilde{H} \) is its lift to \( T\tilde{M} \), and \( f : \tilde{M} \to \mathbb{R} \) is Lipschitz, then there is a unique Lipschitz viscosity solution \( u : \tilde{M} \times \mathbb{R} \to \mathbb{R} \) of the initial value problem
\[ \partial_t u + \tilde{H}(x, \partial_x u) = 0, \]
\[ u(x, 0) = f(x). \]
(120)

**Proof:** By Proposition 3.1, the Lax formula,
\[ u(x, t) := \min \left\{ f(\gamma(0)) + \int_\gamma L : \gamma \in C^1([0, T], (M, x)) \right\} \]
gives a Lipschitz viscosity solution of (120). Suppose that \( u \) and \( v \) are Lipschitz viscosity solutions of (120). Let
\[ R_0 > \|\partial_x u\|_{\text{sup}} + \|\partial_x v\|_{\text{sup}}. \]
There is \( H_1 \), quadratic at infinity such that \( H_1(x, p) = H(x, p) \) if \( |p|_x \leq R_0 \). For the lift \( \tilde{H}_1 := H_1 \circ (\pi^*)^{-1} \) we have that \( u \) and \( v \) are solutions of
\[ \partial_t u + \tilde{H}_1(x, \partial_x u) = 0, \]
\[ u(x, 0) = f(x). \]
(121)

Since \( u \) and \( v \) are Lipschitz solutions of (121), they are both sub- and super- solutions of (121). By Proposition 7.2, \( u \leq v \) and \( v \leq u \). \( \square \)
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