Approximate Solutions of Some Linear Boundary Value Problems Using the Homotopy Analysis Method Combined with Sumudu Transform

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Abstract. This paper considers a combined form of the Sumudu transform with the modified homotopy analysis method (MHAM) to find approximate and analytical solutions for linear two-point boundary value problems. This method is called the modified Sumudu transform homotopy analysis method (MSTHAM). The suggested technique relies on the freedom of the homotopy analysis method (HAM) by introducing an appropriate initial approximation and furthermore, the residual error will be canceled in some points of the interval (RECP). Only a first order approximation of MSTHAM will be required, as compared to STHAM, which needs more iterations for the same cases of study. The proposed method has been applied to solve two examples and the results have been compared with those obtained by the standard STHAM and the exact solutions, where it is found that the proposed solutions are of high accuracy and, therefore, MSTHAM is extremely efficient, simple and can be applied to other types of problems.

1. Introduction
A kind of analytic technique, namely the homotopy analysis method was proposed by Liao in 1992 [1], Where he used the basic concept of the homotopy in topology to create a general analytical method for linear and nonlinear problems [2–4]. Researchers who have used the HAM have frequently reported the following advantages for this method. HAM enjoys great freedom in choosing initial approximations and auxiliary linear operators. The HAM can guarantee the convergence of the series solutions by auxiliary parameters especially the so-called convergence-controller parameter $h$ [5]. Also, the HAM avoids discretization and provides an efficient approximate solution with high accuracy and minimal calculations [6]. Unlike perturbation methods, the validity of the HAM is independent of whether or not there exist small parameters in the considered equations [7]. Although the HAM contains the auxiliary parameter $h$ to adjust and control the convergence region, this auxiliary parameter is not sufficient to ensure the convergence of the solution for some problems in a wide region.

In recent years, many authors pay the attention to study the solutions of linear and nonlinear differential equations by using a combination of Sumudu transform with various methods, such as Sumudu transform Homotopy perturbation method (STHPM) [8, 9], Sumudu transform variational iteration method (STVIM) [10–12] and Sumudu transform Homotopy analysis method (STHAM) [13, 14]. Amongst all these methods, the STHAM has been considered as one of the most popular one, due to its simplicity and its wide range of applications. The
STHAM was suggested by Rathore in [13] to find the solution of nonlinear partial differential equations with the initial conditions, and it has been presented by many authors to be a powerful mathematical tool for solving a wide range of nonlinear operator equations [15–18]. The main advantage of combining Sumudu transform with standard HAM is that it avoids integration at each step of finding the solution which makes the calculation easier and reduces the amount of the computational work. Despite its advantages, STHAM still suffers instability in the choice of the initial approximation. In addition, it needs an infinite number of iterations to obtain the desired approximate solution. However, to overcome these disadvantages, a modified of STHAM is introduced in this paper to improve the analytical solution of linear second-order two-point boundary value problems. A suitable initial approximation will be introduced; in addition, the residual error in several points of the interest interval will be canceled. Also, the size of computational work has been reduced to the first order approximate solution.

In this paper the definition of Homotopy, the Sumudu transform and the Homotopy analysis method are presented in section 1, 2 and 3. Section 4 and 5 are discussed the Sumudu transform Homotopy analysis Method and the modified Sumudu transform Homotopy analysis method. The numerical application of the method is illustrated by two test examples to demonstrate the efficiency of the method in section 6. Conclusion is given in section 7.

2. Definition of Homotopy
A homotopy between two continuous functions $f(t)$ and $g(t)$ from a topological space $T$ to a topological space $Y$ is formally defined to be a continuous function $H : T \times [0, 1] \rightarrow Y$ from the product of the space $T$ with the unit interval $[0,1]$ to $Y$ such that, if $t \in T$ then

$$H(t, 0) = f(t) \quad \text{and} \quad H(t, 1) = g(t).$$

3. Sumudu Transform (ST)
Watugala in 1993 introduced a new integral transform, named the Sumudu transform (ST) and further applied it to the solution of ordinary differential equation in control engineering problems. The Sumudu transform is defined by the following formula [19]

$$F(\eta) = S(f(t)) = \int_0^\infty \frac{1}{\eta^n} f(t) dt. \quad (1)$$

For any function $f(t)$, and $-\tau_1 \leq \eta \leq \tau_2$. In this work we used the following properties of ST:

$$(i) \quad S(t^n) = n! \eta^n \quad (2)$$

$$(ii) \quad S(f^{(n)}(t)) = \frac{1}{\eta^n} F(\eta) - \frac{1}{\eta^n} \sum_{k=0}^{n-1} \eta^k f^{(k)}(0) \quad (3)$$

where $f^{(0)}(0) = f(0)$, $f^{(k)}(0)$, $k = 1, 2, 3, \ldots, n - 1$ are the $k^{th}$ derivatives of the function $f(t)$, and $S(f^{(n)}(t)) = F(\eta)$. If $F(\eta)$ is the Sumudu transform of $f(t)$, then $f(t)$ is called the inverse Sumudu transform of $F(\eta)$ and is expressed by $f(t) = S^{-1}(F(\eta))$, where the inverse Sumudu transform operator is $S^{-1}$.

4. Homotopy Analysis Method (HAM)
The homotopy analysis method (HAM) is a general analytic approach to obtain series solutions of various types of differential equations. The HAM is based on the concept of the homotopy, a fundamental concept in topology and differential geometry. Further improvements of this
method were also proposed by [20] to adjust and control the convergence of series solution by introducing the nonzero auxiliary parameter $h$ into the traditional way of constructing a homotopy.

To clarify the basic principles of the HAM, consider the following nonlinear differential equation [14]:

$$\mathcal{N}[u(t)] = 0,$$  \hspace{1cm} (4)

where $\mathcal{N}$ is a nonlinear operator and $u(t)$ is an unknown function of the independent variable $t$. In the frame of HAM, first, the zeroth-order deformation equation is constructed as follows:

$$(1-q)L[\phi(t, q) - u_0(t)] = qhH(t)\mathcal{N}[\phi(t, q)],$$  \hspace{1cm} (5)

where $q \in [0,1]$ is an embedding parameter, $\sim \neq 0$ is an auxiliary parameter, $L$ is an auxiliary linear operator, $H(t) \neq 0$ is an auxiliary function, $\phi(t, q)$ is an unknown function and $u_0(t)$ is the initial guess of the solution $u(t)$.

Clearly, when $q = 0$ and $q = 1$, the following equations hold

$$\phi(t, 0) = u_0(t), \quad \phi(t, 1) = u(t),$$  \hspace{1cm} (6)

respectively. Expanding $\phi(t, q)$ using the Taylor series with respect to $q$, yields:

$$\phi(t, q) = u_0(t) + \sum_{m=1}^{\infty} u_m(t)q^m,$$  \hspace{1cm} (7)

where

$$u_m(t) = \frac{1}{m!} \frac{\partial^m \phi(t, q)}{\partial q^m}\bigg|_{q=0}.$$  \hspace{1cm} (8)

Assuming that the linear operator $L$, the initial approximation $u_0(t)$, the auxiliary parameter $h$ and the auxiliary function $H(t)$ are properly chosen, then the series (7) converge at $q = 1$, and we have:

$$\phi(t, q) = u_0(t) + \sum_{m=1}^{\infty} u_m(t).$$  \hspace{1cm} (9)

According to the definition (8), the governing equation can be deduced from the zeroth-order deformation equation (5). Define the vector

$$\vec{u}_n = \{u_0(t), u_1(t), u_2(t), ..., u_n(t)\}.$$  \hspace{1cm} (10)

Differentiating Eq.(5) $m$ times with respect to $q$, then dividing by $m!$ and finally substituting $q = 0$, we have the so-called $m^{th}$-order deformation equation

$$L[u_m(t) - \chi_m u_{m-1}(t)] = hH(t)R(\vec{u}_{m-1}), \quad m = 1, 2, 3, ..., n,$$  \hspace{1cm} (11)

where

$$R(\vec{u}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^m \phi(t, q)}{\partial q^m}\bigg|_{q=0},$$  \hspace{1cm} (12)

and

$$\chi_m = \begin{cases} 0 & \text{if } m \leq 1, \\ 1 & \text{if } m > 1. \end{cases}$$  \hspace{1cm} (13)

Finally, the $m^{th}$-order approximate solution $u(t)$ is given by

$$u(t) = \sum_{k=0}^{m} u_k(t).$$  \hspace{1cm} (14)
5. Sumudu Transform Homotopy Analysis Method (STHAM)

In this section, to figure out how STHAM works, consider a general nonlinear differential equation in the form:

\[ L(u) + N(u) = f(t), \]  \hspace{1cm} (15)

with the following boundary conditions

\[ \beta(u, \partial u/\partial t), \; t \in \Gamma, \]  \hspace{1cm} (16)

with solution \( u(t) \), where \( L \) and \( N \) is a linear and nonlinear operators respectively, \( f(t) \) is a known analytical function, \( \beta \) is a boundary operator and \( \Gamma \) is the domain boundary for \( \Omega \).

The methodology consists of applying the Sumudu transform first on both sides of Eq.(15) in the following operator form:

\[ S(L(u)) + S(N(u)) = S(f(t)). \]  \hspace{1cm} (17)

Using the differential property of the Sumudu transform, we have

\[ \frac{1}{\eta^n} S(u(t)) - \frac{1}{\eta^n} \sum_{k=0}^{n-1} \eta^k f^{(k)}(0) + S(N(u)) = S(f(t)). \]  \hspace{1cm} (18)

We define the nonlinear operator

\[ N(\phi(\eta; q)) = \phi(\eta; q) - \sum_{k=0}^{n-1} \eta^k \phi^{(k)}(0) + \eta^n (S(N(\phi(t; q))) - S(f(t))), \]  \hspace{1cm} (19)

where \( q \in [0,1] \) is an embedding parameter and \( \phi(\eta; q) \) is the Sumudu transform of a real function of \( \phi(t; q) \). For simplicity, we will consider the auxiliary parameter \( H(t) = 1 \).

The so-called zeroth-order deformation equation of the Eq.(15) has the form:

\[ (1 - q)(\phi(\eta; q) - U_0(\eta)) = q h N(\phi(\eta; q)), \]  \hspace{1cm} (20)

where \( h \) is a nonzero auxiliary parameter and \( U_0(\eta) \) is the Sumudu transform of the initial approximation \( u_0(t) \). Obviously, when \( q = 0 \) and \( q = 1 \), the following hold

\[ \phi(\eta; 0) = U_0(\eta), \; \phi(\eta; 1) = U(\eta), \]  \hspace{1cm} (21)

respectively. Therefore, as \( q \) increases from 0 to 1, the solution varies from initial approximation \( U_0(\eta) \) to the solution \( U(\eta) \). Now, expanding \( \phi(\eta; q) \) in Taylor’s series with respect to \( q \), we have:

\[ \phi(\eta; q) = U_0(\eta) + \sum_{m=1}^{\infty} U_m(\eta)q^m, \]  \hspace{1cm} (22)

where

\[ U_m(\eta) = \frac{1}{m!} \frac{\partial^m \phi(\eta; q)}{\partial q^m} \bigg|_{q=0}. \]  \hspace{1cm} (23)

If the auxiliary linear operator, the initial guess \( U_0(\eta) \), and the auxiliary parameter \( h \) are properly chosen, the series Eq.(22) converges at \( q = 1 \), then we have:

\[ U(\eta) = U_0(\eta) + \sum_{m=1}^{\infty} U_m(\eta), \]  \hspace{1cm} (24)

\[ U(\eta) = U_0(\eta) + \sum_{m=1}^{\infty} U_m(\eta), \]  \hspace{1cm} (24)
which must be one of the solutions of the original nonlinear equations. According to the definition Eq.(23), the governing equation can be deduced from the zeroth-order deformation Eq.(20). Define the vectors
\[ \vec{U}_m = \{ U_0(\eta), U_1(\eta), ..., U_m(\eta) \}. \] (25)
Differentiating the zeroth-order deformation equation Eq.(20) \( m \) times with respect to \( q \) and then dividing them by \( m! \) and finally setting \( q = 0 \), we get the following \( m^{th} \) order deformation equation:
\[ U_m(\eta) = \chi_m U_{m-1}(\eta) + \hbar \mathcal{R}_m(\vec{U}_{m-1}(\eta)), \] (26)
where
\[ \mathcal{R}_m(\vec{U}_{m-1}(\eta)) = \frac{1}{(m-1)!} \frac{\partial^{m-1} \mathcal{N}(\phi(\eta; q))}{\partial q^{m-1}}|_{q=0}, \] (27)
and
\[ \chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \] (28)
Applying the inverse Sumudu transform, then we have the series solution of Eq.(15) as follows:
\[ u(t) = \sum_{k=0}^{\infty} u_k(t). \] (29)

6. Modified Sumudu Transform Homotopy Analysis Method (MSTHAM)
The objective of this section is to show how MSTHAM can be employed to find analytical approximate solutions of ordinary differential equations (ODEs) as Eq.(15). For this purpose MSTHAM follows the same steps of standard STHAM until Eq.(24), but unlike STHAM, the proposed method employs the freedom of HAM, by substituting \( u_0(t) \) for an suitable function \( Z(t) \). An adequate guide for the kind of problems proposed in this work would be to choose a polynomial trial function provided with some unknown parameters \( c_0, c_1, c_2, ... \) to be determined.

For simplicity, we will consider the auxiliary parameter \( H(t) = 1 \). The so-called zero-order deformation equation of the Eq.(15) has the form:
\[ (1 - q)(\phi(\eta; q) - \mathcal{Z}(\eta)) = qh \mathcal{N}(\phi(\eta; q)), \] (30)
where \( \hbar \) is a nonzero auxiliary parameter and \( \mathcal{Z}(\eta) \) is the Sumudu transform of trial function \( \mathcal{Z}(t) \). Obviously, when \( q = 0 \) and \( q = 1 \) it holds:
\[ \phi(\eta; 0) = \mathcal{Z}(\eta), \quad \phi(\eta; 1) = U(\eta), \] (31)
respectively. Thus, as \( q \) increases from 0 to 1, the solution varies from initial guess \( \mathcal{Z}(\eta) \) to the solution \( U(\eta) \). Now, expanding \( \phi(\eta; q) \) on Taylor’s series with respect to \( q \), we have:
\[ \phi(\eta; q) = \mathcal{Z}(\eta) + \sum_{m=1}^{\infty} U_m(\eta)q^m, \] (32)
where
\[ U_m(\eta) = \frac{1}{m!} \frac{\partial^m \phi(\eta; q)}{\partial q^m}|_{q=0}. \] (33)
If the auxiliary linear operator and the auxiliary parameter \( \hbar \) are properly chosen, the series Eq.(32) converges at \( q = 1 \), then we have
\[ U(\eta) = \mathcal{Z}(\eta) + \sum_{m=1}^{\infty} U_m(\eta), \] (34)
which must be one of the solutions of the original nonlinear equations. According to the definition Eq. (34), the governing equation can be deduced from the zero-order deformation Eq. (30). Define the vectors
\[ \vec{U}_m = \{ \phi(\eta), U_1(\eta), ..., U_m(\eta) \} \]  
(35)
Differentiating the zeroth-order deformation equation Eq. (30) \( m \) times with respect to \( q \) and then dividing them by \( m! \) and finally setting \( q = 0 \), we get the following \( m \)th order deformation equation:
\[ U_m(\eta) = \chi_m U_{m-1}(\eta) + h \mathcal{R}_m(\vec{U}_{m-1}(\eta)) , \]  
(36)
where
\[ \mathcal{R}_m(\vec{U}_{m-1}(\eta)) = \frac{1}{(m-1)!} \frac{\partial^{m-1} \mathcal{N}(\phi(\eta); q)}{\partial q^{m-1}} |_{q=0} . \]  
(37)
and
\[ \chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \]  
(38)
Applying the inverse Sumudu transform, then we have the series solution of Eq. (30) as the following
\[ u(t) = \sum_{k=0}^{\infty} u_k(t). \]  
(39)
The values of \( c_0, c_1, c_2, \ldots \) are adequately calculated by solving the algebraic system, Equation (39) should satisfy the boundary conditions at the end points of the interval. Also, In order to determine the values of all the parameters, we need to solve more algebraic equations.

7. CASE STUDIES
In this section, to illustrate the efficiency of MSTHAM, we solve two numerical examples involving linear second-order two-point BVPs. In these examples we applied the STHAM discussed in section 5, and applied the proposed method using linear and quadratic trial functions \( \phi \).
We make a comparison of the corresponding absolute error for the standard STHAM and the MSTHAM.

Example 1 Consider the following linear boundary value problem in the form [21]:
\[ u''(t) - u(t) - 1 = 0, \quad 0 \leq t \leq 1, \]  
(40)
with boundary conditions
\[ u(0) = 0, \quad u(1) = e - 1. \]
The exact solution of this problem is given by \( u(t) = e^t - 1 \).
We define the operators as:
\[ L(u) = \frac{d^2 u(t)}{dt^2}, \]  
(41)
\[ \mathcal{N}(\phi(t; q)) = \frac{d^2 \phi(t; q)}{dt^2} - \phi(t; q) - 1. \]  
(42)
The STHAM method
Applying the Sumudu transform (ST) to Eq. (40), we obtain:
\[ S(u''(t) - u(t) - 1) = 0. \]  
(43)
Next, employing the differential property of ST for $n = 2$, we have:

$$
\frac{1}{\eta^2} U(\eta) - \frac{1}{\eta^2} u(0) - \frac{1}{\eta} u'(0) - S (u(t) + 1)) = 0,
$$

that gives

$$
U(\eta) = \alpha \eta - \eta^2 S (u(t) + 1) = 0,
$$

obtained upon solving for $U(\eta)$, where we define $\alpha = u'(0)$, and using $u(0) = 0$

Define the nonlinear operator $N$ as

$$
N(\phi(\eta, q)) = \phi(\eta, q) - \alpha \eta - \eta^2 S (\phi(\eta, q) + 1).
$$

By using Eq.(46), we construct the zero-order deformation equation as

$$
(1 - q)(\phi(\eta, q) - U_0(\eta)) = h q N(\phi(\eta, q)),
$$

where $U_0(\eta)$ is the Sumudu transform of the initial guess $u_0(t) = \alpha t$. Consequently, the $m^{th}$ order deformation equation of Eq.(47) is given by

$$
U_m(\eta) = (\chi_m + h) U_{m-1}(\eta) - h (1 - \chi_m) \alpha \eta + \eta^2 S (u_{m-1}) + (1 - \chi_m) \eta^2.
$$

Applying the inverse Sumudu transform for $U_m(\eta)$ where $m \geq 1$, we can compute the first order approximate solution for equation Eq.(48) as follows

$$
u_1(t) = h(-\frac{1}{2} t^2 - \frac{\alpha}{6} t^3).
$$

The general form of $m^{th}$ order approximate solution is

$$
u(t) = \sum_{m=0}^{M} u_m(t).
$$

Substituting the values of $u_0(t)$ and $u_1(t)$ into Eq.(50), we obtain:

$$
\nu(t) = \alpha t + \frac{1}{6} (-3t^2 - \alpha t^3) h.
$$

Setting $h = -1.10$ and applying the boundary condition $\nu(1) = e - 1$ on Eq.(51), to calculate the values of $\alpha$, we have:

$$
\alpha = 0.9872804184.
$$

Substituting Eq.(52) into Eq.(51), we obtain:

$$
u_{ST}(t) = 0.98728 t + 0.55 t^2 + 0.181001 t^3.
$$

The MSTHAM method

The aim of this section is to demonstrate how the MSTHAM can improve the solution by employing the freedom of HAM, by choosing the initial approximation as a trial function $\bar{Z}(t)$. So, for this purpose the proposed method follow the same steps of STHAM until Eq.(47).

- Firstly, the trial function is linear in the form:
\[ Z(t) = c_1 t + c_0. \] (54)

Substituting Eq.(54) into Eq.(47), we have:
\[(1 - \beta)(\phi(\eta, q) - Z(\eta)) = hN(\phi(\eta, q)), \] (55)

where \( Z(\eta) \) is the Sumudu transform of the \( Z(t) \). Consequently, the \( m^{th} \) order deformation equation of Eq.(55) is given by
\[ U_m(\eta) = (\chi_m + h)U_{m-1}(\eta) - h(-1 + \chi_m)\alpha \eta + \eta^2 S(u_{m-1}) + (1 - \chi_m)\eta^2). \] (56)

Applying the inverse Sumudu transform, we compute the first order approximate solution for Eq.(47) as follows:
\[ u_1(t) = c_0 + c_1 t + h(c_0 + c_1 t - \alpha t + \frac{1}{6}(-3t^2 - 3c_0 t^2 - c_1 t^3)). \] (57)

The general form of the \( m^{th} \) order approximate solution is
\[ u(t) = \sum_{m=0}^{M} u_m(t). \] (58)

Substituting equations Eq.(54) and Eq.(57) into Eq.(58), we obtain:
\[ u(t) = \alpha t + \frac{c_0}{2} t^2 + \frac{c_1}{6} t^3 - \frac{h}{120}(-60t^2 + 60c_0 t^2 - 20\alpha t^3 + 20c_1 t^3 - 5c_0 t^4 - c_1 t^5). \] (59)

Setting \( h = -0.95 \) and applying the boundary condition \( u(1) = e - 1 \) to Eq.(59) to calculate the values of \( c_0, c_1 \) and \( \alpha \), and following the MSTM algorithm; substituting Eq.(59) into Eq.(40) and evaluating the resultant expression for the values \( t = 0.30 \) and \( t = 0.69 \), which lies in \([0, 1]\), we have a system of equations for \( c_0, c_1 \) and \( \alpha \), to obtain the values:
\[ \alpha = 1.0004730116 \quad c_0 = 0.9069243494 \quad c_1 = 1.589439535. \] (60)

Substituting Eq.(60) into Eq.(59), we obtain:
\[ u_{LF}(t) = 1.00047t + 0.497673t^2 + 0.171654t^3 + 0.0358991t^4 + 0.0125831t^5. \] (61)

- Secondly, we choose the trial function as quadratic
\[ Z(t) = c_2 t^2 + c_1 t + c_0. \] (62)

Substituting Eq.(62) into Eq.(55) and applying the inverse Sumudu transform, we compute the approximate solution as follows:
\[ u_1(t) = -\frac{h}{360}(-180t^2 + 180c_0 t^2 - 60\alpha t^3 + 60c_1 t^3 - 15c_0 t^4 + 30c_2 t^4 - 3c_1 t^5 - c_2 t^6). \] (63)

The general form of the \( m^{th} \) order approximate solution
\[ u(t) = \sum_{m=0}^{M} u_m(t). \] (64)
Substituting equation Eq.(62) and Eq.(63) int Eq.(64), we have:

\[
  u(t) = \alpha t + \frac{c_0}{2} t^2 + \frac{c_1}{6} t^3 + \frac{c_2}{12} t^4 - \frac{h}{360} (-180 t^2 + 180 c_0 t^2 - 60 \alpha t^3 + 60 c_1 t^4 - 15 c_0 t^4 + 30 c_2 t^4 - 3 c_1 t^5 - c_2 t^6).
\]  

(65)

Setting \( h = -0.97 \) and Applying the boundary conditions \( u(0) = 0 \) and \( u(1) = e - 1 \) to Eq.(65) to calculate the values of \( c_0, c_1, c_2 \) and \( \alpha \), and following the MSTHAM algorithm, substituting Eq.(65) into Eq.(40) and evaluate the resultant expression for the values \( t = 0.18, t = 0.51 \) and \( t = 0.82 \), which lies in \([0,1]\). Upon following the above procedure, we have a system of equations for \( c_0, c_1, c_2 \) and \( \alpha \), to obtain the values:

\[
  \begin{align*}
  \alpha &= 0.999976357 \\
  c_0 &= 1.0153044651 \\
  c_1 &= 0.8364229701 \\
  c_2 &= 0.8537269969.
  \end{align*}
\]  

(66)

Substituting Eq.(66) into Eq.(65), we obtain:

\[
  u_{QF}(t) = 0.999976 t + 0.50023 t^2 + 0.165845 t^3 + 0.0431695 t^4 + 0.00676109 t^5 + 0.00230032 t^6.
\]  

(67)

### Table 1: Absolute error on \([0,1]\) for Example 1

| \( t \) | \( |u_{\text{exact}} - u_{ST}| \) | \( |u_{\text{exact}} - u_{LF}| \) | \( |u_{\text{exact}} - u_{QF}| \) |
|---|---|---|---|
| 0.0 | 0.000E + 00 | 0.000E + 00 | 0.000E + 00 |
| 0.1 | 7.619E – 04 | 2.848E – 05 | 7.549E – 07 |
| 0.2 | 4.987E – 04 | 3.346E – 05 | 1.624E – 07 |
| 0.3 | 7.124E – 04 | 2.968E – 05 | 3.541E – 07 |
| 0.4 | 2.672E – 03 | 2.590E – 05 | 4.491E – 07 |
| 0.5 | 5.044E – 03 | 2.712E – 05 | 2.421E – 07 |
| 0.6 | 7.346E – 03 | 3.547E – 05 | 7.194E – 09 |
| 0.7 | 8.927E – 03 | 4.960E – 05 | 2.373E – 08 |
| 0.8 | 8.956E – 03 | 6.238E – 05 | 4.302E – 07 |
| 0.9 | 6.399E – 03 | 5.683E – 05 | 9.074E – 07 |
| 1.0 | 0.000E + 00 | 0.000E + 00 | 0.000E + 00 |
Figure 1: (a) Comparison between exact solution of Eq.(40) and approximate solutions $u_{ST}(t)$, $u_{LF}(t)$ and $u_{QF}(t)$ on $[0,1]$, (b) The zoom for exact solution and approximate solutions $u_{ST}(t)$, $u_{LF}(t)$ and $u_{QF}(t)$ for Example 1.

In Table 1, comparisons of the absolute error are made for STHAM and MSTHAM using linear and quadratic trial functions $Z(t)$, denoted by $|u_{exact} - u_{ST}(t)|$, $|u_{exact} - u_{LF}(t)|$ and $|u_{exact} - u_{QF}(t)|$, respectively. We can note generally that MSTHAM is more accurate than the STHAM, also, the results show that the MSTHAM is more accurate in the case of raising the degree of the trial function from linear to quadratic. This is attributed to the fact that the introduction of the trial function with unknown parameters contributed to the increase in the order of the polynomial approximation and its number of terms, which in turn, accelerated the convergence and increased the accuracy. It is clear that the advantage of the MSTHAM over the standard STHAM is through the reduction of the amount of computational work to obtain the first order approximate solution which greatly accelerates the convergence of the solution. Figure 1 (a) and (b) show the approximate solutions and the exact solution for Eq.(40) using STHAM and MSTHAM with linear and quadratic trial functions, respectively. Figure 1 (a) visually shows a good compatibility between the approximate solutions and the exact, while Figure 1 (b) which is the zoom for Figure 1 (a) shows that the accuracy of the approximate solution $u_{ST}(t)$ less than the accuracy of approximate solutions $u_{LF}(t)$ and $u_{QF}(t)$ respectively, which demonstrate a remarkable accuracy for such approximate solutions using the trial function.

Example 2 We will now extend our analysis to the following second-order linear BVP [22]:

$$u''(t) - u(t) = \cos(t), \quad 0 \leq t \leq 1,$$
$$u(0) = 1, \quad u(1) = 1.$$  \hspace{1cm} (68)

The exact solution is

$$u(t) = \frac{-3\cosh(1)+3\sinh(1)+\cos(1)+2}{4\sinh(1)}e^t + \frac{3\cosh(1)+3\sinh(1)-\cos(1)-2}{4\sinh(1)}e^{-t} - \frac{\cos(t)}{2}.$$  

The STHAM method

Applying the standard STHAM to Eq.(68), we obtain the approximate solution as the following

$$u(t) = 1 + \alpha t + \left(\frac{t^2}{2} + \frac{\alpha}{6}t^3 - \cos(t)\right)h.$$  \hspace{1cm} (69)
Setting $h = -1.19$ and applying the boundary condition $u(1) = 1$ to Eq.(69), to calculate the values of $\alpha$, we have

$$\alpha = -0.9530238576. \tag{70}$$

Substituting Eq.(70) into Eq.(69), we obtain

$$u_{ST}(t) = 2.00 - 0.719436t + 0.259738t^3 - \cos(t). \tag{71}$$

### The MSTHAM method

Applying the MSTHAM algorithm on Eq.(68) using the linear trial function $\mathcal{F}(t) = c_0 + c_1t$, we obtain the approximate solution

$$u(t) = 1 + \alpha t + c_0 t^2 - \frac{c_1}{6}t^3 + (1 + \frac{t^2}{2} - \frac{c_0}{2}t^2 + \frac{\alpha}{6}t^3 - \frac{c_1}{6}t^3 + \frac{c_0}{24}t^4 + \frac{c_1}{120}t^5 - \cos(t))h. \tag{72}$$

Setting $h = -0.95$ and Applying the boundary condition $u(1) = e - 1$ to Eq.(72), to calculate the values of $c_0$, $c_1$ and $\alpha$. In addition, following MSTHAM algorithm, by substituting Eq.(72) into Eq.(68) and evaluate the resultant expression for the values $t = 0.44$ and $t = 0.91$, which lies in $[0, 1]$. Upon following the above procedure, we have a system of equations for $c_0$, $c_1$ and $\alpha$, to obtain the values

$$\alpha = -0.8883711851 \quad c_0 = 1.9276107412 \quad c_1 = -0.4415003289. \tag{73}$$

Substituting Eq.(73) into Eq.(72), we obtain:

$$u_{LF}(t) = 1.95 - 0.888371t + 0.52319t^2 - 0.144338t^3 + 0.07630134 - 0.00349521t^5 - 0.95\cos(t). \tag{74}$$

- Secondly, we choose the trial function as quadratic $\mathcal{F}(t) = c_0 + c_1t + c_2t^2$, we have approximate solution as the following

$$u(t) = 1 + \alpha t + \frac{B}{2}t^2 + \frac{C}{6}Ct^3 + \frac{D}{12}Dt^4 + (-1 - \frac{t^2}{2} + \frac{B}{2}t^2 - \frac{\alpha}{6}t^3 + \frac{C}{6}t^3 - \frac{B}{24}t^4 + \frac{D}{12}t^4 - \frac{C}{120}t^5 - \frac{D}{360}t^6 + \cos(t))h. \tag{75}$$

Setting $h = -1.5$ and Applying the boundary condition $u(1) = e - 1$ on Eq.(75), to calculate the values of $c_0$, $c_1$, $c_2$ and $\alpha$. In addition, following MSTHAM algorithm, by substituting Eq.(75) into Eq.(68) and evaluate the resultant expression for the values $t = 0.24$, $t = 0.52$ and $t = 0.84$, which lies in $[0, 1]$. Upon following the above procedure, we have a system of equations for $c_0$, $c_1$, $c_2$ and $\alpha$, to obtain the values:

$$\alpha = -0.8887460324 \quad c_0 = 1.9968698426 \quad c_1 = -0.8557966727 \quad c_2 = 0.3957313004. \tag{76}$$

Substituting Eq.(76) into Eq.(75), we obtain:

$$u_{QF}(t) = 1.94 - 0.888746t + 0.529906t^2 - 0.147795t^3 + 0.0801894t^4 - 0.00670374t^5 + 0.0010333t^6 - 0.94\cos(t). \tag{77}$$
Table 2: Absolute error on [0,1] for Example 2

| t  | |u_{exact} - u_{ST}| | |u_{exact} - u_{LF}| | |u_{exact} - u_{QF}| |
|----|--------|--------|--------|
| 0.0 | 2.220E-16 | 2.220E-16 | 2.220E-16 |
| 0.1 | 4.577E-03 | 2.394E-05 | 5.601E-07 |
| 0.2 | 5.724E-03 | 2.858E-05 | 5.155E-07 |
| 0.3 | 4.006E-03 | 2.393E-05 | 3.831E-07 |
| 0.4 | 1.877E-04 | 1.539E-05 | 2.649E-07 |
| 0.5 | 4.768E-03 | 6.006E-06 | 1.382E-07 |
| 0.6 | 9.710E-03 | 2.061E-06 | 9.117E-09 |
| 0.7 | 1.330E-02 | 6.961E-06 | 4.578E-08 |
| 0.8 | 1.404E-02 | 7.405E-06 | 6.004E-08 |
| 0.9 | 1.022E-02 | 3.757E-06 | 2.534E-07 |
| 1.0 | 0.000E+00 | 2.220E-16 | 2.220E-16 |

Figure 2: (a) Comparison between exact solution of Eq.(68) and approximate solutions $u_{ST}(t)$, $u_{LF}(t)$ and $u_{QF}(t)$ on [0,1], (b) The zoom for exact solution and approximate solutions $u_{ST}(t)$, $u_{LF}(t)$ and $u_{QF}(t)$ for Example 2.

In Table 2 we computed the absolute error for STHAM, MSTHAM using linear and quadratic trial functions $Z(t)$ respectively. We note that the absolute error decrease monotonically with the increase in the degree of the trial function. The advantage of the MSTHAM over the standard STHAM clearly appears reduction in the computational work to obtain the first approximate solution which accelerates the convergence of the solution. Figure 2 shows the approximate solutions $u_{ST}$, $u_{LF}$, $u_{QF}$ and exact solution for Eq.(68) using STHPM and MSTHAM, respectively. In fact, it can be observed that the order of the polynomial approximation and its number of terms were increased in equations (74) and (77) due to of the use of the trial function in the solution algorithm, which in turn led to accelerating the convergence and increased the accuracy, where the error can be made smaller by substituting
the initial approximation by the trial function such as $\mathcal{Z}'(t)$. The result in this figure confirms that the MSTHAM can efficiently provide us a good accuracy of approximate solutions and the accuracy can be improved by increasing the degree of the trial function.

8. Conclusions
In this paper, the Coupling of Sumudu transform (ST) and Homotopy analysis method (HAM) using the trial function as alternative to initial approximation proved very effective to solve the linear second-order two-point BVPs. The proposed algorithm is applied in a direct way without any limitations. The results obtained using the scheme presented here agree well with the exact solutions. It is worth mentioning that the method is capable of reducing the volume of the computational work as compared to the classical methods. The results reveal that the MSTHAM a very powerful and efficient technique in finding analytical solutions for wide classes of differential equations.

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