Non-hermitian tricriticality in the Blume-Capel model with imaginary field

G. von Gehlen

Laboratoire de Physique Théorique, Ecole Normale Supérieure, 46, allée d’Italie, 69364 Lyon Cedex 07, France, and
Physikalisches Institut der Universität Bonn
Nussallee 12, D - 53115 Bonn, Germany

Abstract: Using finite-size-scaling methods, we study the quantum chain version of the spin-1-Blume-Capel model coupled to an imaginary field. The aim is to realize higher order non-unitary conformal field theories in a simple Ising-type spin model. We find that the first ground-state level crossing in the high-temperature phase leads to a second-order phase transition of the Yang-Lee universality class (central charge $c = -22/5$). The Yang-Lee transition region ends at a line of a new type of tricriticality, where the three lowest energy levels become degenerate. The analysis of the spectrum at two points on this line gives good evidence that this line belongs to the universality class of the $\mathcal{M}_{2,7}$-conformal theory with $c = -68/7$. 
1 Introduction

The study of statistical models coupled to imaginary fields can give useful insight on the singularity structure of the partition functions and the corresponding phase transition mechanisms. In 1952 Yang and Lee [1] initiated these investigations considering the Ising model in a complex magnetic field $B$. In the high-temperature regime above the critical temperature $T_c$, the authors of [1] found that the partition function has zeros for purely imaginary fields $B = ih$ only. In the thermodynamical limit these zeros become dense and produce a singularity at $h = \pm h_c(T)$, which for $T \to T_c$ gives rise to the Ising phase transition. Fisher [2] pointed out that the singularity in the $B$-plane discovered by Yang and Lee has many properties of a standard second order phase transition. With the advent of the conformal field theory (CFT)-interpretation of two-dimensional phase transitions [3], Cardy [4] showed that the Yang-Lee singularity is described by the CFT with the simplest possible operator content (just one primary field with conformal dimension $d = -\frac{2}{5}$ apart from the unit operator with $d = 0$) which has the Virasoro central charge $c = -\frac{22}{5}$. This CFT is non-unitary and corresponds to the case $p = 2$, $p' = 5$ of the minimal series $\mathcal{M}_{p,p'}$, with central charge

$$c = 1 - 6(p - p')^2/(pp').$$  (1)

Generally, a minimal theory $\mathcal{M}_{p,p'}$ ($p$ coprime $p'$) is non-unitary if $|p - p'| > 1$, since according to the Kac-formula for the scaling dimensions of the primary fields $h_{r,s}$

$$h_{r,s} = h_{p' - r,p - s} = \frac{(rp - sp')^2 - (p - p')^2}{4pp'}; \quad 1 \leq r \leq p' - 1; \quad 1 \leq s \leq p - 1, \quad r, s \in \mathbb{N}$$  (2)

it contains at least one primary field with negative dimension $h_{r,s}$. If one wants to describe non-unitary minimal models by a coset construction, Kac-Moody algebras at fractional level have to be used [5, 6]. For non-unitary minimal theories the modular invariant partition functions and their relation to the fusion rules have been studied in [7, 8, 9]. Recently, such non-unitary models appeared, coupled to 2-dimensional gravity, in the study of multicritical matrix models [10]. For applications of non-unitary evolution operators in other fields of physics, see e.g. [11].

For $p = 2$ the coprime values for $p'$ are $p' = 2n + 3$, $n = 1, 2, \ldots$ (the first member of this series is the Yang-Lee-singularity), while for $p = 3$ there are two series $p' = 3n + 1$ and $p' = 3n + 2$, etc. So, the simplest non-unitary minimal theories beyond the $\mathcal{M}_{2,5}$ are the $\mathcal{M}_{2,7}$ which has $c = -\frac{68}{7}$ and $\mathcal{M}_{3,5}$ with $c = -\frac{3}{2}$. Lässig [12] has considered the renormalization flow between non-unitary theories caused by the $\phi_{(1,3)}$-perturbation. He found that the flow leading to the $\mathcal{M}_{2,5}$-theory is $\ldots \to \mathcal{M}_{8,11} \to \mathcal{M}_{5,8} \to \mathcal{M}_{2,5}$. So in connection with the flow to the $\mathcal{M}_{2,5}$-theory it may be useful to check the possibility of a $\mathcal{M}_{5,8}$ showing up.

The aim of this paper is to find simple SU$(2)$-spin quantum chain realisations of non-unitary minimal theories. Except for $\mathcal{M}_{2,5}$, which is well-known to be represented by the Yang-Lee singularity of the Ising chain in an imaginary field, no such quantum chain hamiltonians realizing non-unitary $\mathcal{M}_{p,p}$ theories have been written down in the literature. There is a general, though not simple, way for obtaining a quantum chain model for a $\mathcal{M}_{p,p'}$-theory which is even integrable: The Forrester-Baxter [13] 2-dimensional RSOS-models have critical points realizing $\mathcal{M}_{p,p'}$-theories [14]. Using e.g. the recipes of DeVega [15] one may deduce a quantum chain from the RSOS-transfer matrix. However, this has not been done and certainly will lead to very complicated hamiltonians. In order to get a simple realization of a higher-order non-unitary phase transition we shall instead try to generalize the non-integrable Ising-Yang-Lee model, looking into its spin-1 analog, and there to possible tripl-level crossings, as will be explained below.

Let us briefly review the main features of the Ising representation of the $\mathcal{M}_{2,5}$-model. Already the work of [1] and [2] suggested that the $c = -\frac{22}{5}$ CFT should be approximated on the lattice
by the thermodynamic limit of an Ising-type quantum chain with the Hamiltonian

\[ \mathcal{H}_{LY} = -\frac{1}{2} \sum_{i=1}^{N} (\sigma_i^x + \lambda \sigma_i^x \sigma_{i+1}^z + i h \sigma_i^z). \]  

(3)

at a certain curve \( \lambda(h) \). Here \( \sigma_i^x \) and \( \sigma_i^z \) are standard Pauli matrices acting at site \( i \). \( \lambda \sim T^{-1} \) is the inverse temperature and \( h \) is the imaginary part of the external field. For \( h = 0 \) the model (3) has the \( Z_2 \)-symmetry \( \sigma_i^x \leftrightarrow -\sigma_i^x \), \( \sigma_i^z \leftrightarrow \sigma_i^z \), which for \( h \neq 0 \) survives as an antiunitary symmetry [16, 17]. Due to this, the spectrum of (3) remains unchanged under the reflection \( h \leftrightarrow -h \). The operator content and central charge at a particular critical point of (3) were first checked by Itzykson et al. [7] and were found to agree with the spectrum of the \( c \)-phase transition described by the minimal CFT with \( c = \frac{5}{2} \). For \( h = 0 \) and \( \lambda < 1 \) there is a finite gap. By continuity, in this regime, the finite-temperature phase transition and this property was used in [18] to obtain the FFS determination of the phase transition curve shown in Fig. 1. For \( h = 0 \) and \( \lambda = 1 \) we have the standard Ising phase transition described by the minimal CFT with \( c = \frac{1}{2} \). For \( h = 0 \) and \( \lambda < 1 \) we are in the massive \( Z_2 \)-symmetric high-temperature phase.

Within a mean-field theory or Landau approach the appearance of a phase transition for \( \lambda < 1 \) and \( h \neq 0 \) can be explained as follows [4]: Consider a model for which the free energy admits a Landau expansion in terms of an order parameter \( \varphi \) which couples to a magnetic field \( B \):

\[ \mathcal{F} = \mathcal{F}_0 + B \varphi + \varphi^2 \mathcal{F}_2 + \varphi^4 \mathcal{F}_4 + \ldots \]  

(4)

\( \mathcal{F}_0, \mathcal{F}_2 \) and \( \mathcal{F}_4 \) are functions of the inverse temperature \( \lambda \). For \( B = 0 \) a second order phase transition will take place if there is a \( \lambda = \lambda_c \) with \( \mathcal{F}_2(\lambda_c) = 0 \) and \( \mathcal{F}_4(\lambda_c) > 0 \). Now, for any \( \lambda = \lambda_{LY} \) with \( \mathcal{F}_2(\lambda_{LY}) > 0 \) and \( \mathcal{F}_4(\lambda_{LY}) > 0 \) (this is in the high-temperature phase) criticality of the system can be restored by applying a correctly tuned imaginary magnetic field: shifting the order parameter by the constant \( i \varphi_0 \), i.e. \( \varphi \Rightarrow \bar{\varphi} + i \varphi_0 \), (4) becomes

\[ \mathcal{F} = B(\bar{\varphi} + i \varphi_0) + (\bar{\varphi} + i \varphi_0)^2 \mathcal{F}_2 + (\bar{\varphi} + i \varphi_0)^4 \mathcal{F}_4 + \ldots 
\]

\[ = \ldots + \bar{\varphi} [B + \frac{2i}{\mathcal{F}_2} (\mathcal{F}_2 - 2 \varphi_0 \mathcal{F}_4)] + \bar{\varphi}^3 (\mathcal{F}_2 - 6 \varphi_0^2 \mathcal{F}_4) + 4i \varphi_0^3 \mathcal{F}_4 + \ldots, \]  

(5)

where the dots stand for terms of order \( \varphi_0 \), and higher than \( \varphi_0^3 \). The coefficient of \( \varphi_0^2 \) will vanish if we choose

\[ \varphi_0 = \pm \sqrt{\mathcal{F}_2(\lambda_{LY})/6 \mathcal{F}_4(\lambda_{LY})}, \]  

(6)

giving rise to the Yang-Lee phase transition. This way, necessarily a term proportional \( \bar{\varphi}^3 \) is introduced. Neglecting the residual \( \bar{\varphi}^3 \)-term, at \( \lambda = \lambda_{LY} \) we have

\[ \mathcal{F} = (B - B_{LY}) \bar{\varphi} + \frac{1}{3} i g \bar{\varphi}^3 + \ldots \]  

(7)

with

\[ B_{LY} = -\frac{4i}{3} \sqrt{\frac{\mathcal{F}_2^3}{6 \mathcal{F}_4}} \quad \text{and} \quad g = \pm \sqrt{24 \mathcal{F}_2 \mathcal{F}_4}. \]  

(8)
We now address the question, whether and how can we get non-unitary multicritical phase transitions in the sense that not only the coefficient of $\varphi^2$ but also higher terms of a Landau expansion come to vanish. Since, in order to achieve this, we will need more degrees of freedom than are available in a spin-$\frac{1}{2}$ chain, we shall study a spin-1 quantum chain placed in an imaginary field which breaks the original $Z_2$-symmetry. Is there a new type of phase transitions which may be called a non-unitary tricritical phase transition and which is produced by the crossing of the three lowest energy levels? In general we should have three parameters in the hamiltonian in order to find such a triple cross-over. Looking into the Fisher’s Landau expansion (4) this has a chance to be realized in a model in which the parameters can be choosen to get vanishing coefficients of $\varphi^2$, $\varphi^4$ and also of $\varphi^3$, so that the first nonvanishing terms is proportional $\varphi^5$ (with a purely imaginary coefficient). In a mean field treatment of the non-hermitian spin-1 model defined in (9) below, this tricritical curve has been determined by K. and M.Becker [16, 17]. So it seems possible to generalize the realizations of the $(A_{m-1}, A_m)$ unitary minimal CFTs (which have $|p - p'| = 1$, $p' \equiv m \geq 3$), by spin-$s$-quantum chains with $s = m/2 - 1$ [16, 17, 19] to the non-unitary series.

Attempts to establish a connection between the mean-field non-hermitian multicritical curve and a particular minimal CFT via a generalization of Zamolodchikov’s composite field representation of the CFT fusion rules, have met various difficulties [16, 17]. So, as a more direct approach, in this paper we treat a suitable non-hermitian quantum chain hamiltonian by FSS methods and determine its non-hermitian tricritical manifold. From the chain size-dependence of the low-lying levels of the spectrum we then determine the CFT’s which describe the spectrum in the different regions of the critical manifold.

The main part of this paper is organized as follows: In Sec. 2 we define the particular spin-1 quantum chain we want to study and review its known zero magnetic field properties. In Sec. 3 we determine the phase transition in the imaginary field from the ground-state level crossings. Sec. 4 is devoted to the calculation of the finite-size spectrum on the two-level crossing phase transition curve and the determination of the universality class involved. Sec. 5 contains the main result of the paper: the determination of the FSS of the spectrum at the tripel level crossings which gives the evidence that there the $M_{2,7}$-minimal theory is realized. Sec. 6 presents our Conclusions.

2 The spin-1 Blume-Capel quantum chain with an imaginary field

We consider a spin-1 quantum chain which has the property that the characteristic polynomial of the hamiltonian matrix has only real coefficients, so that also in this case the eigenvalues are either real or occur in complex conjugate pairs. This property seems necessary if we want to find universality classes described by minimal CFTs, because there the scaling dimensions of the fields, which are related to the energy gaps of the quantum chain hamiltonian, are always real. We insist in having a $Z_2$-symmetry for non-zero imaginary magnetic field, because the simplest modular invariant partition functions, those of the $p = 2$-series, exhibit a $Z_2$-symmetry.

The non-hermitian symmetric spin-1 hamiltonian we shall consider is defined by

$$H = \sum_{i=1}^{N} \left( \alpha(S_i^z)^2 + \beta S_i^x S_{i+1}^x - S_i^z S_{i+1}^z - i\hbar S_i^y \right).$$

Here $S_i^x$ and $S_i^z$ are standard $3 \times 3$ spin-1 matrices acting at site $i$. We shall choose $S_i^y$ to be diagonal. This $H$ has an antiunitary $Z_2$-symmetry: Denoting by $K$ the operator of complex
conjugation, and defining \[16, 17\]

\[ U = \prod_{i=1}^{N} \left( 2(S^x)^2 - 1 \right)_i, \]

(10)

\( \mathcal{H} \) satisfies

\[ [\mathcal{H}, \Theta] = 0 \quad \text{with} \quad \Theta = KU. \]

(11)

There are three parameters: \( \alpha, \beta \) and \( h \). For \( h = 0 \) this is the standard Blume-Capel \[24\] quantum chain, and certain linear combinations of \( \alpha \) and \( \beta \) can be interpreted as a temperature variable and a vacancy density. In principle, one may add more \( Z_2 \)-symmetrical terms to (9), e.g. terms proportional to \((S^x)^2\) and to \((S^z)^2(S^z_{i+1})^2\) etc. (Blume-Emery-Griffiths model \[21\]). However, for simplicity, here we shall consider the form (9) only.

Let us first review the phase diagram of (9) for \( h = 0 \). The critical behaviour of the Blume-Capel quantum chain (with zero field \( h \)) has been studied first in mean field approximation by Gefen et al. \[20\] and then, using finite-size scaling (FSS) by Alcaraz et al. \[22, 23\] and by one of us \[24\], see also \[25\]. The line \( \beta = 0 \) is classical and is easily seen to contain a first order transition at \( \alpha = 1 \). The spectrum of (9) is unchanged by the reflection \( \beta \to -\beta \). So it is sufficient to consider only \( \beta \geq 0 \). Using the FSS-technique, the transition line for \( \beta \neq 0 \) can be located quite precisely as can be seen from Table 1.

| \( \alpha \) | -1.0 | -0.5 | 0.0 | 0.20 | 0.50 | 0.55 | 0.60 | 0.65 | 0.70 | 0.725 |
| \( \beta_c \) | 1.884 | 1.6266 | 1.3259 | 1.187 | 0.9411 | 0.895 | 0.8448 | 0.7932 | 0.737 | 0.707 |
| \( \alpha \) | 0.76 | 0.80 | 0.84 | 0.85 | 0.90 | 0.9102 | 0.93 | 0.95 | 0.98 | 1.00 |
| \( \beta_c \) | 0.6618 | 0.6070 | 0.5458 | 0.531 | 0.4365 | 0.4157 | 0.368 | 0.3118 | 0.199 | 0.0 |

Table 1: Critical values of the Blume-Capel-quantum chain, eq.(9) with \( h = 0 \), as obtained from finite-size scaling \[24\]. The underlined values of \( \alpha, \beta \) are those of the tricritical point.

There is a single critical curve which starts at \( \alpha = 1 \) and \( \beta = 0 \) and then moves with increasing \( \beta \) towards smaller values of \( \alpha \). It first follows the unit circle in the \( \beta-\alpha \)-plane and then takes off to large negative \( \alpha \), ultimately following the curve \( \beta = \sqrt{2/|\alpha|} \). Still close to \( \alpha = 1 \), it passes a tricritical point at \[21\]

\[ \alpha_t = 0.910207(4), \quad \beta_t = 0.415685(6). \]

(12)

For \( \alpha < \alpha_t \) the phase transition is second order, and, on the transition line, the low-lying part of the spectrum is that of the \( c = \frac{1}{2} \) Ising CFT. The high-lying spectrum should also contain a massive component, since the Ising Majorana fermion needs only two of the three degrees of freedom which are available. At the tricritical point the spectrum is described by the \( c = \frac{7}{10} \) CFT ("tricritical Ising model"). For \( \alpha_t < \alpha < 1 \) the transition is first order. The phase below the critical curve at \( \alpha < 1 \) and \( \beta \geq 0 \) is a low-temperature phase with broken \( Z_2 \)-symmetry.

3 Determination of the phase transition curves from lowest level crossings

In analogy to the situation for the Ising chain \[3\], we expect that also in the high-temperature regime of (9) there should be a range in \( h \) where the spectrum remains real despite the hamiltonian
being non-hermitian. The spectrum should become complex where the first lowest level crossings appear.

We have calculated the four lowest energy levels of the translationally invariant states of the hamiltonian \( \mathcal{H} \) with \( N = 2, \ldots, 11 \) sites, for various values of the parameters \( \alpha, \beta \) and \( h \), using a Lanczos technique. For \( N \leq 8 \) we diagonalize the hamiltonian also by exact methods without the restriction to momentum-zero states. In this paper we always take periodic boundary conditions.

As a typical example of the lowest level crossing behaviour at our small values of \( N \), in the last line of Table 2 we show results for a fixed value of \( \alpha \), i.e. \( \alpha = 0.80 \), and the value \( \beta = 0.725 \), which for \( h = 0 \) would be in the disordered phase, slightly above the critical curve (for \( h = 0 \) from Table 1 we have \( \beta_c = 0.6070 \)).

**Two-level cross-over**

Now, switching on \( h \), we find that as long as \( h < \sim 0.24 \), the four lowest energy levels remain real as they were for \( h = 0 \). However, e.g. for \( N = 10 \) sites at \( h_c = 0.025828 \) the two lowest levels meet and form a square-root branch point. For \( h < h_c \) the two eigenvalues of the spectrum with the lowest real parts form a complex conjugate pair. The branch point positions are \( N \)-dependent.

| \( \beta \)  | \( N = 5 \)    | 6   | 7   | 8   | 9   | 10  | 11  | \( \infty \) |
|-------------|---------------|-----|-----|-----|-----|-----|-----|----------|
| 0.675       | 0.023796      | 0.019666 | 0.016984 | 0.015119 | 0.013758 | 0.012725 | 0.011920 | 0.007(1) |
| 0.700       | 0.029975      | 0.025677 | 0.022893 | 0.020961 | 0.019553 | 0.018486 | 0.017655 | 0.012(1) |
| 0.725       | 0.037034      | 0.032733 | 0.030000 | 0.028142 | 0.026815 | 0.025828 | 0.025072 | 0.021(1) |
| \( \alpha = 0.80 \) |               |     |     |     |     |     |     |          |
| \( \alpha = 0.20 \) |               |     |     |     |     |     |     |          |

Table 2: Values \( h_c(N) \) of the first crossing of the two lowest levels of the hamiltonian \( \mathcal{H} \) for \( \alpha = 0.80 \) and three values of \( \beta \), and for \( \alpha = 0.20 \) and two values of \( \beta \). For \( h \) above these values the two lowest eigenvalues form a complex conjugate pair. \( N \) denotes the chain length, the last column \( N = \infty \) gives an estimate of branch point position in the thermodynamic limit.

We use standard extrapolation techniques (both rational polynomials and Van-den-Broeck-Schwarz approximants [26]) to obtain the branch point positions in the limit \( N \to \infty \). In analogy to the situation in the Yang-Lee-model [3] [18], we interpret these as second-order phase transition points.

Table 3 collects our numerical results for these branch point positions \( h_c(\alpha, \beta) \) obtained by extrapolation \( h_c = \lim_{N \to \infty} h_c(N) \) from \( N = 3, \ldots, 11 \) sites. In order to save space, in Table 2 we give some abbreviated data, but for the actual extrapolation we used also \( N = 3 \) and \( N = 4 \) sites and 8-digit precision for the finite-\( N \)-branch point positions.

In Fig. 2 we plot the resulting critical surface in the space of \( \alpha, \beta \) and \( h \). This surface has the shape of wings starting from the \( h = 0 \)-critical curve of Table 1. The wings converge at a vertex which is the standard tricritical Ising point [12] with \( c = \frac{7}{10} \). This shape of the wings is just the same as it is familiar for applied staggered real fields, see e.g. Fig. 44 in Lawrie and Sarbach [27].
\[ \alpha = 0.20 \quad \alpha = 0.50 \quad \alpha = 0.55 \quad \alpha = 0.65 \quad \alpha = 0.725 \quad \alpha = 0.76 \]

| \( \beta \) | \( h_c \) | \( \beta \) | \( h_c \) | \( \beta \) | \( h_c \) | \( \beta \) | \( h_c \) | \( \beta \) | \( h_c \) |
|-----------|--------|-----------|--------|-----------|--------|-----------|--------|-----------|--------|
| 1.187     | 0.0    | 0.895     | 0.0    | 0.792     | 0.0    | 0.706     | 0.0    | 0.662     | 0.0    |
| 1.8       | 0.108(1)| 1.0       | 0.0070(5)| 1.0      | 0.0273(6)| 1.0      | 0.0176(5)| 1.0      | 0.0194(3)|
| 3.0       | 0.561(1)| 1.2       | 0.0445(4)| 1.2      | 0.0820(4)| 1.2      | 0.030(1) | 1.0      | 0.033(2) |
| 6.0       | 2.08(1) | 1.8       | 0.246(3)| 1.5      | 0.1922(6)| 1.0      | 0.059(1) | 0.90     | 0.050(1) |
| 9.0       | 1.940(2)| 3.0       | 0.807(3)| 1.9      | 0.374(2)|

Table 3: Non-hermitian critical values \( h_c(\alpha, \beta) \) obtained from the lowest ground state level crossings, except for the first line, which gives the Blume-Capel value at \( h_c = 0. \)

**Triple cross-over: Tricritical line**

In calculating the ground-state level crossings of the Yang-Lee model, we don’t have to worry about the third and higher energy levels, because there always the first two levels meet before a third level crosses in. In the spin-1-case the situation is different because we are varying one more parameter. For fixed \( 0.55 < \alpha < \alpha_t \), moving out in \( \beta \), we reach a point, which we shall call \( \beta_t \), where the third energy level \( E_2 \) crosses the second level \( E_1 \) at a value of \( h \) which is lower than the crossing point of the two lowest levels \( E_0 \) and \( E_1 \). In Fig. 3 we show this situation for \( N = 6 \) and \( \alpha = 0.8 \), where it occurs for \( \beta \approx 0.78097 \). For a fixed finite chain size \( N \), there is no point where all three lowest levels meet: the triple crossing is an avoided cross-over since in our case there is no new conservation law at this special point. Table 4 gives our determination of the positions of these avoided cross-overs at chains of \( N \) sites and several values of \( \alpha \). The calculation for obtaining these data requires a considerable amount of computing time and therefore we mostly have not pushed the precision beyond four significant figures, although this would be possible if wanted. The value of \( h_t \) quoted is taken somewhat arbitrarily as the lowest value of \( h \) where at \( \beta = \beta_t \) we have \( E_1 = E_2^* \), see Fig. 3. The convergence of the values of \( \beta_t \) and \( h_t \) with \( N \) is quite slow. So there is a considerable uncertainty for the values extrapolated to \( N \to \infty \). It may be that for \( \alpha \approx 0.5 \) both \( \beta_t \) and \( h_t \) move to infinity, so that for low \( \alpha \) the wings may extend to infinity.

For \( \alpha \to \alpha_t \) the values of \( h_t \) go to zero. Accordingly the width of the wings (for \( N \to \infty \)) is seen to shrink towards zero. This means that the range in \( h \) where the spectrum remains real decreases if \( \alpha \) moves towards the tricritical point. This is natural, since in the adjacent 1st-order region there is a ground state degeneracy in the limit \( N \to \infty \) at \( h = 0 \).
Table 4: Finite-\(N\)-approximations to the non-hermitian "tricritical" points of the model (8). In brackets we indicate the estimated error in the last given digit. If no bracket is given, then the error is of the order of one unit in the last given digit. In the last column \(\infty\) we give the extrapolated value for \(N \to \infty\) for those cases in which we get reasonable convergence.

| \(\alpha\) | \(N\) | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | \(\infty\) |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.55 | \(\beta_t\) | 8.8958 | 7.63 | 7.45 | 7.53 | 7.66 | 7.82 | 7.887 | 7.98 | ? |
| | \(h_t\) | 4.4793 | 3.61 | 3.46 | 3.49 | 3.56 | 3.622 | 3.681 | 3.73 | \(\infty\) |
| 0.60 | \(\beta_t\) | 4.2604 | 3.64 | 3.462 | 3.49 | 3.56 | 3.622 | 3.681 | 3.73 | ? |
| | \(h_t\) | 0.852 | 0.613 | 0.5249 | 0.484 | 0.464 | 0.4535 | 0.4474 | 0.444 | \(0.437(4)\) |
| 0.725 | \(\beta_t\) | 1.499 | 1.3094 | 1.2310 | 1.191 | 1.1693 | 1.156 | 1.1480 | 1.1428 | 1.127(3) |
| | \(h_t\) | 0.356 | 0.2363 | 0.187 | 0.161 | 0.146 | 0.137 | 0.1309 | 0.1268 | 0.113(2) |
| 0.76 | \(\beta_t\) | 1.204 | 1.062 | 1.005 | 0.9697 | 0.951 | 0.9391 | 0.9315 | 0.92596 | 0.908(5) |
| | \(h_t\) | 0.246 | 0.1566 | 0.119 | 0.0992 | 0.0872 | 0.0796 | 0.0743 | 0.07062 | 0.060(3) |
| 0.80 | \(\beta_t\) | 0.9482 | 0.849 | 0.806 | 0.780 | 0.7663 | 0.7566 | 0.7501 | 0.74524 | 0.728(7) |
| | \(h_t\) | 0.1601 | 0.098 | 0.071 | 0.0569 | 0.048 | 0.0419 | 0.0382 | 0.03529 | 0.025(3) |
| 0.84 | \(\beta_t\) | 0.748 | 0.6814 | 0.651 | 0.636 | 0.624 | 0.617 | 0.6120 | 0.6083 | ? |
| | \(h_t\) | 0.103 | 0.060 | 0.0415 | 0.031 | 0.025 | 0.0217 | 0.0188 | 0.0167 | \(\infty\) |

In Fig. 4 we show the projection of the critical curves for fixed values of \(\alpha\) in the \(\beta - h\)-plane. The curve connecting the end points is the tricritical curve. In the mean field treatment of (8), which was mentioned earlier in the Introduction, in the neighbourhood of the tricritical point K. and M. Becker [16, 17] have obtained curves very similar to those of Fig. 4. They obtain the endpoints of the Yang-Lee-transition curves through the vanishing of the Landau-coefficient of \(\bar{\phi}^3\) in the notation of (5), (7). Since in mean-field the tricritical point for the \(h = 0\) model is at \((\alpha, \beta) = (0.5836 \ldots, 1.2011 \ldots)\) instead of the correct value from FSS \((0.91021 \ldots, 0.41569 \ldots)\), their corresponding curves are at too small values of \(\alpha\) and about three times too large values of \(\beta\). Anyway, it is interesting that near to the tricritical point, mean field reproduces the shape of the fixed-\(\alpha\) critical curves so well.

4 Determination of the spectrum on the phase transition surface

In order to determine the universality class of the non-hermitian phase transitions found, we use standard finite-size-scaling (FSS) methods of conformal theory. Using always periodic boundary conditions, from the \(N\)-dependence of the ground-state energy \(E_0(N)\):

\[
E_0(N) / \xi = - N a_0 - \frac{c_\pi}{6 N} + \ldots
\]

we obtain the effective central charge \(c\), which for non-unitary theories is given by

\[
c = c - 24 h_{\text{min}},
\]

with \(c\) the central charge and \(h_{\text{min}} = (1 - (p - p')^2)/(4 p p')\) is the lowest (most negative) anomalous dimension. \(a_0\) is the bulk constant which will not be of interest in the following, \(\xi\) is the conformal normalization factor [28]. Inserting \(h_{\text{min}}\) into (1) we can write

\[
c = 1 - 6/(p p').
\]
We calculate \(N\) sites approximants \(\tilde{c}_N\) to \(\tilde{c}\) from

\[
\tilde{c}_N = \frac{6N(N-1)}{(2N-1)\pi}\left((N-1)E_0(N) - NE_0(N-1)\right),
\]

and extrapolate \(N \to \infty\). At a conformal point the gaps scale in leading order of \(N\) as \(N^{-1}\). The coefficient of \(N^{-1}\) allows us to obtain the anomalous dimensions \((\bar{h}, \bar{h})\) of the primary conformal fields \([28]\):

\[
\tilde{x}_i(P) = \lim_{N \to \infty} \tilde{x}_i(N, P) = \lim_{N \to \infty} G_i(N, P) / \xi = (h + r + \bar{h} + \bar{r}_i) - (h + \bar{h})_{\text{min}}
\]

with

\[
G_i(N, P) = \frac{N}{2\pi}(E_i(N, P) - E_0(N)).
\]

Here \(E_i(N, P)\) denotes the energy of the \(i\)-th level in the momentum \(P\)-sector of the Hamiltonian with \(N\) sites. The positive integers \(r, \bar{r}\) become nonzero for descendant field levels and determine the momentum through \(P = r - r'\). The term \((h + \bar{h})_{\text{min}}\) appears in case of non-unitary theories, since there the ground state does not correspond to the vacuum.

The conformal normalization \(\xi\) at the critical point under consideration is calculated from either the lowest \(\Delta P = 2\)-gap in the thermal sector, or from the lowest \(\Delta P = 1\)-gap in the non-zero-charge sector \([28]\). If possible, we usually use both gaps in order to have a consistency check.

Tables 5 and 6 give our finite-\(N\)-results for the scaled gaps of the lowest levels at two typical values of the critical surface below \(h_t\) (i.e. on the wings) \((\alpha = 0.2, \beta = 6.0, h = 2.0931)\) and \((\alpha = 0.5, \beta = 3.0, h = 0.7559)\), together with the values expected for \(c = -\frac{22}{5}\). We see that we have excellent agreement with the \(c = -\frac{22}{5}\) conformal spectrum. Our spin-1 model has one more degree of freedom than the spin-\(\frac{1}{2}\)-Ising chain, but probably the extra degree of freedom cannot be seen in our data because the corresponding levels are massive and are high above the ground state.

| \(N\) | \(\tilde{c}\) | \(\tilde{x}_1\) | \(\tilde{x}_2\) | \(\tilde{x}_3\) | \(\tilde{x}_1\) | \(\tilde{x}_2\) | \(\tilde{x}_1\) | \(\tilde{x}_2\) |
|-------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| 2     | 0.482832       | 0.312828       | 0.84704        | 1.2830         | 0.66008        | 0.9908         | 0.8416         | 1.3118         |
| 3     | 0.440703       | 0.348901       | 1.08662        | 1.6274         | 0.84162        | 1.31189        | 1.2671         | 1.7027         |
| 4     | 0.425037       | 0.378185       | 1.46168        | 2.0673         | 0.93965        | 1.8434         | 1.4905         | 2.0214         |
| 5     | 0.417508       | 0.384465       | 1.58632        | 2.2752         | 0.95663        | 2.0642         | 1.6166         | 2.1569         |
| 6     | 0.413305       | 0.388459       | 1.67586        | 2.4853         | 0.96704        | 2.2492         | 1.6965         | 2.2195         |
| 7     | 0.410737       | 0.391123       | 1.74033        | 2.6904         | 0.97395        | 2.3956         | 1.7515         | 2.2611         |
| 8     | 0.409053       | 0.392962       | 1.78775        | 2.8820         | 0.97907        | 2.5415         | 1.8001         | 2.3215         |
| 9     | 0.407955       | 0.394265       | 1.82350        | 3.0529         | 0.98453        | 2.6836         | 1.8402         | 2.3528         |
| 10    | 0.407336       | 0.395200       | 1.85109        | 3.1987         | 0.98956        | 2.8168         | 1.8793         | 2.3841         |
| 11    | 0.406839       | 0.396876       | 1.87279        | 3.3188         | 0.99409        | 2.9410         | 1.9174         | 2.4155         |
| ∞     | 0.406(1)       | 0.41(1)        | 2.00(1)        | 3.8(1)         | 0.999(2)       | 3.01(6)        | 2.00(2)        | 2.39(4)        |

Table 5: Finite-size data for the scaled gaps \(\tilde{x}(N, P)\) for \(\alpha = 0.2, \beta = 6.0, h = 2.0931\) and the normalization \(\xi = 2.84\). In the bottom line we give the values expected for the universality class is \(\mathcal{M}_{2.5}\).
\( P = 0 \)
\( \tilde{E} \)
\( \tilde{E}_1 \)
\( \tilde{E}_2 \)
\( \tilde{E}_3 \)
\( \tilde{E}_1 \)
\( \tilde{E}_2 \)
\( \tilde{E}_3 \)

\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
\( N \) & \( \tilde{E} \) & \( \tilde{E}_1 \) & \( \tilde{E}_2 \) & \( \tilde{E}_3 \) & \( \tilde{E}_1 \) & \( \tilde{E}_2 \) & \( \tilde{E}_3 \) \\
\hline
2 & 0.517269 & 0.263909 & 0.65712 & 1.2244 & 0.66000 & 0.8367 & 0.8675 \\
3 & 0.463568 & 0.307996 & 0.82588 & 1.5288 & 0.86748 & 1.0610 & 1.3479 \\
4 & 0.440614 & 0.336676 & 0.98033 & 1.7606 & 0.94030 & 1.2531 & 2.1169 \\
5 & 0.429010 & 0.355312 & 1.12429 & 1.8785 & 0.97035 & 1.4288 & 2.3901 \\
6 & 0.422552 & 0.376158 & 1.25671 & 1.9671 & 0.99321 & 1.7491 & 2.6931 \\
7 & 0.418705 & 0.382111 & 1.47783 & 2.1709 & 0.99812 & 1.8964 & 2.7785 \\
8 & 0.416309 & 0.386402 & 1.56387 & 2.02(3) & 4.3(1) & 3.35(3) & 3.09(6) \\
9 & 0.414784 & 0.389560 & 1.63435 & 2.0661 & 0.97035 & 1.4288 & 2.3901 \\
10 & 0.413823 & 0.391920 & 1.69136 & 2.0661 & 0.98507 & 1.5934 & 2.6457 \\
11 & 0.413249 & 0.393703 & 1.73742 & 2.0661 & 0.98507 & 1.5934 & 2.6457 \\
12 & 0.414784 & 0.389560 & 1.63435 & 2.0661 & 0.97035 & 1.4288 & 2.3901 \\
\infty & 0.38(1) & 2.02(3) & 4.3(1) & 1.011(2) & 3.35(3) & 3.09(6) & 2.6(5) \\
\hline
\end{tabular}

Table 6: Finite-size data for the scaled gaps \( \tilde{E}(N,P) \) as in Table 5, but for \( \alpha = 0.5, \beta = 3.0, h = 0.7559 \) and the normalization \( \xi = 1.90 \). For \( P = 2 \) the lowest levels are real for \( N = 3 \) and \( N = 8 \) sites, for \( N = 4, \ldots, 7 \) they form complex conjugate pairs. This makes the extrapolation rather uncertain.

As another check, whether the wings are a critical surface of the \( c = -\frac{22}{5} \)-universality class, we have also looked into the behaviour of the mass gap in the \textit{neighbourhood} of the critical surface. Since the only relevant perturbation of the \( c = -\frac{22}{5} \)-conformal theory is by the primary field \( \phi(-\frac{1}{5},-\frac{1}{5}) \) (its descendent is redundant) \cite{[4], [29]}, we expect the mass gap to increase proportional to \( \tau^{5/12} \) where \( \tau \) is the distance from the critical curve. The particular choice of the direction in which we take this distance is not relevant, since all directions are equivalent in leading order, see the analogous situation in the Yang-Lee Ising model \cite{[18]}. Since the perturbed \( c = -\frac{22}{6} \)-theory contains only one single particle of mass \( m(\alpha, \beta, \tau) \), it follows that if the first massive level is at \( \Delta E_1 = m \), the second massive level should appear at \( \Delta E_2 = 2m \), followed by a bunch of levels close above \( 2m \). Table 7 gives off-critical masses (extrapolated to \( N \to \infty \)) for three approaches to the critical surface. These data confirm the expected \( \frac{5}{12} \)-power law and the relation \( \Delta E_2 \approx 2 \Delta E_1 \) very well. Observe that also in the last example, \( \alpha = 0.65, \beta = 2.007, h \to 0.431 \), which is at an end point of the critical \( c = -\frac{22}{5} \)-surface, the data clearly show the \( \frac{5}{12} \)-power behaviour. The error bars are much larger for \( \Delta E_2 \) since for two-particle states the convergence is no longer exponential in \( N \), but only powerlike \( N^{-2} \). Finite lattice effects also limit the validity of the \( \frac{5}{12} \)-power law to the neighbourhood of the critical point \cite{[20]}.
\[ \alpha = 0.5, \beta = 3.0 \]

\[ \alpha = 0.65, \beta = 1.2 \]

\[ \alpha = 0.65, \beta = 2.007 \]

\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
\( h \) & \( \Delta E_1 \) & \( \Delta E_2 \) & \( h \) & \( \Delta E \) & \( \Delta E_2 \) & \( h \) & \( \Delta E_1 \) & \( \Delta E_2 \) \\
\hline
0.730 & 0.83208(1) & 1.67(8) & 0.065 & 0.3662(2) & 0.73(2) & 0.35 & 0.89312(1) & 1.78(1) \\
0.740 & 0.6879(1) & 1.38(6) & 0.070 & 0.3256(1) & 0.65(3) & 0.37 & 0.80688(1) & 1.61(2) \\
0.745 & 0.593(3) & 1.18(4) & 0.075 & 0.2687(1) & 0.54(5) & 0.39 & 0.69645(1) & 1.36(3) \\
0.749 & 0.494(5) & 1.02(6) & 0.078 & 0.2186(1) & 0.44(5) & 0.41 & 0.53912(1) & 1.09(7) \\
0.752 & 0.387(1) & 0.81(5) & 0.079 & 0.1966(1) & 0.41(6) & 0.415 & 0.4846(3) & 1.05(4) \\
0.753 & 0.339(1) & 0.67(7) & 0.080 & 0.1692(1) & 0.35(7) & 0.42 & 0.4182(2) & 0.92(5) \\
0.754 & 0.276(6) & 0.64(8) & 0.081 & 0.1327(6) & 0.3(1) & 0.43 & 0.11(4) & 0.51(7) \\
0.7557(2) & 0.0 & 0.08223 & 0.0 & 0.431 & 0.0 & 0.431 & 0.0 & \\
\hline
\end{tabular}

\[ \Delta E_1 \approx 2.64(2)|h - 0.431|^{\frac{1}{12}} \]

\[ \Delta E_1 \approx 2.17(1)|h - 0.08223|^{\frac{1}{12}} \]

\[ \Delta E_1 \approx 3.97(1)|h - 0.7557|^{\frac{1}{12}} \]

Table 7: The two lowest gaps \( \Delta E_1 \) and \( \Delta E_2 \) in the off-critical regime at various off-critical distances converging to the respective critical points, together with, in the bottom row, fit formulae for \( \Delta E_1 \approx \frac{1}{2} \Delta E_2 \).

5 The spectrum at ”nonunitary tricriticality”.

From the triple crossing points determined for \( N = 3 \ldots 10 \) sites, see Table 4, in a few cases we were able to give an estimate of the limiting values for \( N \to \infty \) (see the last column of Table 4). We now look into the spectra at two of these values in order to see whether we can identify a particular modular invariant or universality class. Taking the central values given in Table 4, we find no good evidence for conformal invariance in the sense that for \( N = 2, \ldots, 12 \), the \( N \Delta E_i \) are not clearly seen to tend to constants. In order to improve the determination of the triple points, we did the calculation of the spectra for a net of about ten points around the central values found in the last column of Table 4. We looked at the respective \( N \)-dependence of the effective central charge \( \tilde{c}_N \) and of the \( N \Delta E_i \) and selected the value of \( \alpha, \beta \) and \( h \) with best convergence in \( N \). This way, we were able to improve the determination of the triple crossing points for \( N \to \infty \) such that the \( N \)-behaviour required by conformal invariance is seen.

More precisely, we have calculated the spectra for \( \alpha = 0.65 \) at 11 points of the grid: \( \beta = 2.003; 2.007; 2.008; 2.010 \) vs. \( h = 0.428; 0.431; 0.432; 0.438 \) and choose the best convergence points. Actually, it is not worthwhile to calculate the spectra at all points of the grid, because in the corners, convergence is deteriorating fast, so that there we certainly move away from the conformal point. Analogously, for \( \alpha = 0.725 \) we calculated at \( \beta = 0.105; 0.108; 0.110; 0.114; 0.122 \) vs. \( h = 1.126; 1.127; 1.128 \). Tables 8 and 9 give the finite-size data at the best convergence points.
Table 8: Finite-size spectra at the extrapolated triple crossing point $\alpha = 0.65$, $\beta = 2.007$, $h = 0.432$, using for the normalization factor the estimate $\xi = 2.1$. The imaginary parts are given as $\tilde{J}_i = \text{Im}E_i(N,p)/\xi$, i.e. not multiplied by the size factor $N/2\pi$. For $P = 0$, $\tilde{x}_1$, $\tilde{x}_2$ and $\tilde{x}_5$ are real, but $\tilde{x}_3$ and $\tilde{x}_4$ come as a complex conjugate pair. For $P = 1$, the four lowest levels come in complex conjugate pairs, while $\tilde{x}_5$ is real. In the line $N = \infty$ we give estimates for the thermodynamic limit, which, however, for $\tilde{c}$ and $\tilde{x}_2$ are so uncertain that we are unable to give any reliable estimate. In the bottom line we list the values expected for the universality class $\mathcal{M}_{2,7}$.

| $N$ | $\tilde{c}_N$ | $\tilde{x}_1$ | $\tilde{x}_2$ | $\text{Re}\tilde{x}_3/\text{Re}\tilde{x}_4$ | $\tilde{J}_3$ | $\tilde{x}_5$ | $\text{Re}\tilde{x}_{1,2}$ | $\text{Re}\tilde{x}_{3,4}$ | $\tilde{x}_5$ |
|-----|---------------|---------------|---------------|------------------------------------------|-------------|------------|-----------------------------|-----------------------------|----------------|
| 2   | 0.15602       | 0.377460      | 0.79966       | 0.18845                                  | 1.24913     | 0.4840     | 0.51840                     | 1.1433                     |               |
| 3   | 0.179460      | 0.4040643     | 0.96901       | 0.19207                                  | 1.50244     | 0.6305     | 1.14518                     | 1.4565                     |               |
| 4   | 0.179285      | 0.4030005     | 1.08211       | 0.18782                                  | 1.68278     | 0.6912     | 1.37159                     | 1.6637                     |               |
| 5   | 0.211759      | 0.515308      | 1.16379       | 0.17472                                  | 1.84671     | 0.7264     | 1.52583                     | 1.7996                     |               |
| 6   | 0.223933      | 0.537305      | 1.22704       | 0.15941                                  | 1.98653     | 0.7503     | 1.64012                     | 1.8859                     |               |
| 7   | 0.234486      | 0.557301      | 1.27863       | 0.14427                                  | 2.10504     | 0.7685     | 1.73028                     | 1.9438                     |               |
| 8   | 0.243460      | 0.575174      | 1.32237       | 0.13036                                  | 2.20501     | 0.7836     | 1.80464                     | 1.9847                     |               |
| 9   | 0.251001      | 0.591916      | 1.36053       | 0.11794                                  | 2.28912     |            |                             |                             |               |
| 10  | 0.257139      | 0.608120      | 1.39456       | 0.09705                                  | 2.36002     |            |                             |                             |               |
| 11  | 0.261878      | 0.624137      | 1.42542       | 0.09075                                  | 2.42025     |            |                             |                             |               |
| 12  | 0.265214      | 0.640163      | 1.45376       | 0.08832                                  | 2.47204     |            |                             |                             |               |
| $\infty$ |           |               | 0.30(1)       | ?                                       | 1.9(2)      | 0.02(2)    | 3.6(3)                      | 0.97(5)                    | 2.7(5)        |
|       | $\mathcal{M}_{2,7}$ | 0.571 | 0.286 | 0.857 | 2.0/2.286 | 0.0 | 4.0 | 1.0/1.286 | 3.0/3.286 | ? |

Table 9: Finite-size spectra as in Table 8, here for the extrapolated triple crossing point $\alpha = 0.725$, $\beta = 1.126$, $h = 0.110$. We use the estimated normalization factor $\xi = 1.2$. Only for $N \geq 7$ we have $\text{Re}\tilde{x}_4 = \text{Re}\tilde{x}_3$. As in Table 8, we find that for $P = 0$ the $\tilde{x}_1$, $\tilde{x}_2$ and $\tilde{x}_5$ are real.

We notice that there is no general tendency for the $\tilde{x}_i$ or $\tilde{c}$ to converge towards zero, as it would appear if we missed the critical point considerably, and instead were in the massive regime. However, unfortunately, the sequences don’t converge very well in the range of sites ($N = 2, \ldots, 12$) accessible to our calculation, but one can see a clear pattern of levels emerging.
We shall now compare these patterns observed in Tables 8 and 9 to the anomalous dimensions of $M_{2,5}$ and three next simple minimal non-unitary field theories $M_{p,p'}$ beyond the $M_{2,5}$:

- $M_{2,7}$ with $c = \frac{68}{7}$ and $\tilde{c} = \frac{4}{7}$,
- $M_{3,5}$ with $c = -\frac{3}{5}$ and $\tilde{c} = \frac{3}{5}$,
- $M_{5,8}$ with $c = -\frac{7}{20}$ and $\tilde{c} = \frac{17}{20}$.

In order not to get lost in too many or somewhat exotic possibilities, we shall not consider non-minimal models. We shall consider only $(A_{p-1}, A_{p'}-1)$-modular invariants so that $x_i = 2h_i$. This will be justified by the success of getting along with a minimal theory. From the Kac-formula for the anomalous dimensions \(^{(3)}\) we calculate the conformal grids:

$$
\begin{array}{|c|c|}
\hline
M_{2,5} & M_{2,7} & M_{3,5} & M_{5,8} \\
\hline
4 & 0 & 6 & 0 & 4 & \frac{3}{4} & 0 & 6 & \frac{27}{160} & \frac{27}{160} & -\frac{1}{32} \\
5 & -\frac{2}{3} & 5 & \frac{3}{7} & 3 & \frac{1}{5} & -\frac{2}{5} & 4 & \frac{27}{160} & \frac{27}{160} & \frac{32}{160} \\
3 & -\frac{2}{5} & 3 & \frac{2}{7} & 2 & -\frac{1}{5} & \frac{1}{5} & 3 & \frac{1}{5} & -\frac{1}{5} & \frac{9}{5} \\
r=1 & 0 & 2 & \frac{2}{7} & s=1 & \frac{3}{4} & \frac{3}{4} & 2 & \frac{27}{160} & \frac{187}{160} & \frac{95}{32} \\
s=1 & & & & r=1 & 0 & \frac{7}{10} & \frac{11}{10} & \frac{9}{10} & \frac{2}{10} \\
\hline
\end{array}
$$

Table 10: Conformal grids of the anomalous dimensions $h_{r,s}$ for the theories $M_{2,5},M_{2,7},M_{3,5}$ and $M_{5,8}$ (in the grid for $M_{5,8}$ we have omitted the row $s = 7$ which is equal to the reversed row for $r = 1$).

The patterns of the spectra expected for the three cases are quite different, and different too from the spectrum of the Yang-Lee theory $M_{2,5}$ which we found on the critical wings. For the sequences $\tilde{c}$; $\tilde{x}_1$, $\tilde{x}_2$, $\tilde{x}_3,$... in the $P = 0$ and $P = 1$-sectors we expect (we indicate the vacuum levels by underlined numbers):

- $M_{2,5}$: $\frac{2}{5}$; $\underline{\frac{2}{5}}$, 2, 4, $\underline{4 + \frac{2}{5}}$, 6, 6 $\underline{+ \frac{2}{5}}$, 8, ...
  $\tilde{c} = 0.4$; $P = 0$: $\underline{0.4}$, 2.0, 4.0, 4.4, 6.0, 6.4, 8.0, ...
  $P = 1$: 1.0, 3.0, ...

- $M_{2,7}$: $\frac{4}{7}$; $\underline{\frac{4}{7}}$, 2, $\underline{2 + \frac{4}{7}}$, 4, 4 $\underline{+ \frac{4}{7}}$, 4 $\underline{+ \frac{4}{7}}$, 6, 6 $\underline{+ \frac{4}{7}}$, ...
  $\tilde{c} \approx 0.571$; $P = 0$: $\underline{0.286}$, 0.857, 2.0, 2.286, 4.0, 4.286, 4.857, ...
  $P = 1$: 1.0, 1.286, 3.0, 3.286, ...

- $M_{3,5}$: $\frac{3}{5}$; $\underline{\frac{3}{5}}$, $\frac{1}{5}$, $\frac{1}{5}$, 8, 2, $\underline{2 + \frac{1}{5}}$, $\underline{2 + \frac{1}{5}}$, $\underline{2 + \frac{1}{5}}$, ...
  $\tilde{c} = 0.6$; $P = 0$: $\underline{0.1}$, 0.5, 1.6, 2.0, 2.5, 3.6, 4.0, 4.1, ...
  $P = 1$: 1.0, 1.5, 2.6, 3.0, ...

- $M_{5,8}$: $\frac{17}{20}$; $\underline{\frac{3}{80}}$, $\frac{1}{10}$, $\frac{3}{80}$, $\frac{7}{16}$, $\frac{3}{8}$, 1, $\frac{3}{16}$, $\underline{\frac{143}{80}}$, 2, 2 $\underline{+ \frac{3}{80}}$, ...
  $\tilde{c} = 0.85$; $P = 0$: $\underline{0.0375}$, $\underline{0.1}$, 0.1875, 0.4375, 0.6, 1.0, 1.5, 1.7875, 2.0, ...
  $P = 1$: 1.0, 1.0375, 1.1875, ...


A first hint about which modular invariant is realized at the triple crossing point can be obtained by comparing \( \tilde{c} \) to \( \tilde{x} \): Both are equal for the Yang-Lee case \( \mathcal{M}_{2,5} \). For \( \mathcal{M}_{2,7} \) we expect \( \tilde{c} = 2\tilde{x} \), while we should have \( \tilde{x} \ll \tilde{c} \) for \( \mathcal{M}_{3,5} \) and the more for \( \mathcal{M}_{5,8} \). Our finite-size spectra are compatible with the second case. The normalization by \( \text{Re} \tilde{x}_{1,2}^{P=1} \) is quite stable which encourages us to look into the absolute value of \( \tilde{x}_1 \) and \( \tilde{x}_2 \). Only for \( \mathcal{M}_{2,7} \) the order of magnitude is compatible with the data.

Also the number of levels well below \( \tilde{x}_i = 2 \) is quite different for the universality classes considered: one \( P = 0 \)-level in the case of \( \mathcal{M}_{2,5} \), two for \( \mathcal{M}_{2,7} \), three for \( \mathcal{M}_{3,5} \) and 7 to 8 for \( \mathcal{M}_{5,8} \). Since we observe two levels in this range, \( \mathcal{M}_{3,5} \) is not completely out from this simple counting since the level at \( \tilde{x} = 1.6 \) might have been misidentified, but we find that all data together are best compatible with \( \mathcal{M}_{2,7} \). The complex conjugate pairs in \( P = 0 \) then must come from the first decendents of the two primary fields with \( h = \frac{2}{7} \) and \( h = \frac{6}{7} \) together. The appearance of the imaginary parts probably is just a transition phenomenon at small values of \( N \). A further check of the correct assignment is given by the \( P = 1 \)-levels, which appear at the expected positions, although again in complex conjugate pairs.

We see that our finite-size data strongly hint that the triple crossing points are described by a conformal field theory with \( c = -\frac{68}{7} \). A good further check would be calculate the spectrum of the model with \( Z_2 \)-twisted boundary conditions \([31, 32]\) and to look whether it corresponds to the twisted modular invariant. It is quite obvious to speculate that at analogous four-fold crossings in a higher spin version of \( \mathcal{M}_{2,7} \) if it contains one more parameter, the \( \mathcal{M}_{2,9} \)-CFT may be realized.

The off-critical massive theories obtained from the \( \mathcal{M}_{2,2n+3} \)-theories by their \( \phi_{(1,3)} \)-perturbations has been worked out in \([33, 34]\). The \( S \)-matrix contains \( N \) particles with Koebeler-Swieca-mass ratios \([35]\), so that the \( \phi_{(1,3)} \)-perturbed \( \mathcal{M}_{2,7} \)-theory should have two particles with mass ratio \( m_2/m_1 = \sin (2\pi/5)/\sin (\pi/5) \). Since we have not tried to determine the directions of the perturbations in the \( \alpha, \beta, h \)-plane, we cannot confirm or disprove this feature in our spin model.

### 6 Conclusions

Using numerical finite-size scaling techniques, we have shown that the spin-1 Blume-Capel quantum chain in an imaginary field \( h \), eq.\([9]\), has a two-dimensional critical surface, arising from ground-state level crossings, which belongs to the Yang-Lee-edge \( \mathcal{M}_{2,5} \) \( (c = -\frac{22}{7}) \)-universality class. On both sides, this surface ends in a new type of tricritical line due to triple lowest level crossings. Fig. 2 shows the phase diagram as determined by finite-size scaling.

The FSS-spectra at the non-hermitian tricritical lines show the particular pattern expected for the \( \mathcal{M}_{2,7} \)-modular invariant partition function, from which we conclude that these realize the \( c = -\frac{68}{7} \) universality class. This is the first time that in a simple \( SU(2) \)-spin quantum chain critical behaviour corresponding to a CFT with \( c < -\frac{22}{7} \) is observed.

### Acknowledgements

The author is grateful to Katrin and Melanie Becker for numerous fruitful and stimulating discussions on the subject. He thanks Paul Sorba and Francois Delduc for their kind hospitality at ENS Lyon. This work has been supported by the DAAD (Deutscher Akademischer Austauschdienst) through their PROCOPE-program.
References

[1] C.N.Yang and T.D.Lee, Phys.Rev. **87** (1952) 404, 410
[2] M.E.Fisher, Phys.Rev.Lett. **40** (1978) 1610
[3] A.A.Belavin, A.M.Polyakov and A.B.Zamolodchikov, Nucl.Phys. **B241** (1984) 333
[4] J.L.Cardy, Phys.Rev.Lett. **54** (1985) 1354
[5] V.Kac and M.Wakimoto, Proc.Natl.Acad.Sci.USA **85** (1988) 4956
[6] P.Mathieu, D.Senechal and M.Walton, Int.J.Mod.Phys. **A 7 Suppl.1B** (1992) 731
[7] C.Itzykson, H.Saleur and J.-B. Zuber, Europhys. Lett. **2** (1986) 91
[8] A.Cappelli, C.Itzykson and J.-B.Zuber, Nucl.Phys. **B 280** (1987) 445
[9] I.G.Koh and P.Sorba, Phys.Lett. **B 215** (1988) 723
[10] V.Kazakov, Mod.Phys.Lett. **A 4** (1989) 2125; D.J.Gross and A.Migdal, Nucl.Phys. **B 340** (1990) 333; Phys.Rev.Lett. **64** (1990) 717; M.Staudacher, Nucl.Phys. **B 336** (1990) 349
[11] F.Haake, F.Izrailev, N.Lehmann, D.Saher and H.J.Sommers, Z.Phys.**B 88** (1992) 359
[12] M. Lässig, Phys.Lett. **B 278** (1992) 439
[13] P.J.Forrester and R.J.Baxter, J. Stat. Phys. **38** (1985) 435
[14] H. Riggs, Nucl. Phys. **B 326** (1989) 673
[15] H.J. De Vega, Int.Journ. Mod.Phys. **A 4** (1989) 2371
[16] M.Becker, *Landau-Ginzburg Theorie, mean field und Spinsysteme im imaginären Magnetfeld*, Diplomarbeit BONN-IR-91-16 (1991)
[17] K.Becker, *Verallgemeinerte Ising Modelle: Lee Yang edge singularity*, Diplomarbeit BONN-IR-91-19 (1991)
[18] G.v.Gehlen, J.Phys.:Math.Gen. **A 24** (1991) 5371
[19] D.Sen, Phys.Rev. **B 44** (1991) 2645
[20] Y.Gefen, Y.Imry and D.Mukamel, Phys.Rev.**B 23** (1981) 6099
[21] M.Blume, V.J.Emery and R.B.Griffiths, Phys.Rev. **A 4** (1971) 1071
[22] F.C.Alcaraz, J.R.Drugowich de Felicio, R.Köberle and J.F.Stilck, Phys.Rev. **B 32** (1985) 7469; P.D.Beale, J.Phys.A: Math.Gen. **17** (1984) L335
[23] D.B.Balbao and J.R.Drugowich de Felicio, J.Phys.A :Math.Gen. **20** (1987) L207
[24] G.v.Gehlen, Nucl.Phys. **B330** (1990) 741
[25] A.Malvezzi, preprint Sao Carlos UFSCARF-TH-94-3 [cond-mat/9402046](http://arxiv.org/abs/cond-mat/9402046)
[26] M.Henkel and G.Schütz, J.Phys.A: Math.Gen.**21** (1988) 2617
[27] I.D.Lawrie and S.Sarbach, in Phase Transitions and Critical Phenomena, ed.C.Domb and J.L.Lebowitz, Vol.9, 1, see p.65, 79.

[28] G.v.Gehlen, V.Rittenberg and H.Ruegg, J.Phys.:Math.Gen. A 19 (1985) 107

[29] J.L.Cardy and G.Mussardo, Phys.Lett. B 225 (1989) 275

[30] I.R.Sagdeev and A.B.Zamolodchikov, Mod.Phys.Lett. 3B (1989) 1375

[31] J.L.Cardy, Nucl.Phys. B275 (1986) 200

[32] J.-B.Zuber, Phys.Lett. 176B (1986) 127

[33] P.G.O.Freund, T.R.Klassen and E.Melzer, Phys.Lett. B 229 (1989) 243

[34] T.R.Klassen and E.Melzer, Nucl.Phys. B 338 (1990) 485

[35] R.Koeberle and J.A.Swieca, Phys.Lett. B86 (1979) 209 1375

Figure Captions

Fig. 1: Phase diagram of the Ising quantum chain in an imaginary field $h$ defined by the hamiltonian (3), from [18].

Fig. 2: Three-dimensional view of the phase diagram of the hamiltonian (9), as obtained by finite-size scaling. The five-cornered star at $\alpha \approx 0.910$, $\beta \approx 0.4157$ is the tricritical point with central charge $c = 7/10$. It is the origin of a pair of wings with $c = -22/5$ which extend to imaginary fields $h$ from the Ising-like line at $h = 0$ and $\alpha < 0.910$.

Fig. 3: Behaviour of the real part of the three lowest eigenvalues of the spin-1-hamiltonian (3) near the tripel cross-over region $\alpha = 0.80$ and $\beta \approx 0.781$, for $N = 6$ sites (see the entry: 0.780; 0.0569 in Table 4). For $\beta < 0.78098$, $E_0$ and $E_1$ form a complex conjugate pair near $h \approx 0.0566$. If $\beta \geq 0.78098$, then $E_1$ joins $E_2$ to form a complex conjugate pair above $h \approx 0.0568$.

Fig. 4: Plot of the wing critical curves obtained from FSS as in Fig. 2, but now for various fixed values of $\alpha$ projected onto the $\beta - h$ plane.
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9402143v1
This figure "fig2-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9402143v1
This figure "fig3-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9402143v1
This figure "fig4-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9402143v1