We identify the quantum channels corresponding to the interaction of a Gaussian quantum state with an already formed Schwarzschild black hole. Using recent advances in the classification of one-mode bosonic Gaussian channels we find that (with one exception) the black hole Gaussian channels lie in the non-entanglement breaking subset of the lossy channels $\mathcal{C}(\text{loss})$, amplifying channels $\mathcal{C}(\text{amp})$ and classical-noise channels $\mathcal{B}_2$. We show that the channel parameters depend on the black hole mass and the properties of the potential barrier surrounding it. This classification enables us to calculate the classical and quantum capacity of the black hole and to estimate the quantum capacity where no tractable quantum capacity expression exists today. We discuss these findings in the light of the black hole quantum information loss problem.

1. Introduction

Because the semi-classical approximation to quantum gravity is a free-field theory in curved space-time [1], there is currently no unambiguous way to introduce interactions between radiation and black holes [2]. Yet, the problem of information loss in black holes [3–6] (as well as the related firewall paradox [7]) explicitly considers the fate of information-bearing particles interacting with a black hole. A way out of this conundrum has recently been proposed, by using Sorkin’s effective model that describes the interaction of a scalar massless field with an already formed Schwarzschild black hole such that [8]

$$A = \alpha a + \beta b^\dagger + \gamma c.$$  

Here, $a, b$ and $c$ are the annihilation operators defining the early- and late-time particle content in the incoming sector respectively, and $\alpha, \beta, \gamma$ are coefficients to be determined later. The early-time modes $a$ and $b$ are associated with quantum fields that were emitted during the formation of the black hole and travel just inside ($b$) and just outside ($a$) the event horizon, are standard within the literature of curved space quantum field
theory. As Hawking showed [1], when propagated toward future null infinity these horizon modes are exponentially red shifted relative to the frequencies that a stationary observer at late time might expect. If that observer would send her own \( c \) modes into the black hole, the relative blue-shift of these modes with respect to the black hole horizon modes implies that the support of the quantum fields associated with \( c \) modes is disjoint from that of the \( a \) and \( b \) modes. As a consequence, the outgoing field operator \( A \) should resolve into a superposition not just of the ingoing horizon modes \( a \) and \( b \), but also the ingoing late-time blue-shifted “signal” modes \( c \) [8].

Using the expanded Bogoliubov transformation (1) Sorkin showed that the resulting expression for the radiation experienced by a stationary observer suspended far away from the black hole horizon precisely reproduces the standard Hawking radiation effect including the effect of a black hole potential (grey-body factor) whose parameters are implicit in the coefficients in Eq. (1). Adami and Ver Steeg [9] then recently showed that the Bogoliubov transformation (1) is in fact completely analogous to a corresponding relation in quantum optics, with an interaction term between the late-time modes \( c \) and early-time horizon modes \( a \) implemented by a beam-splitter Hamiltonian. Thus, Sorkin’s construction enables a direct analysis of the interaction of scalar massless particles with a black hole horizon, and makes it possible to investigate the capacity to transmit classical or quantum information information via a black hole.

The operator relation (1) completely characterizes the evolution of any input (bosonic) state, and therefore is sufficient to study the fate of quantum information. However, the bosonic sector of the (infinite-dimensional) Fock space is unwieldy, hence it is often advisable to further limit the input Hilbert space to a physically motivated subset. One option investigated in [9, 10] is to confine the input Fock space to be a sector spanned by a vacuum \( |0\rangle \) and a single photon state \( |1\rangle \). Then, an arbitrary two-level state (qubit) can be constructed using the so-called dual-rail encoding [11].

Here we instead examine a different input subset, one that is a favorite choice in quantum optics due to its experimental relevance – Gaussian states. Gaussian states are completely described by the first two moments of the canonical quadrature variables. The most prominent examples of Gaussian states are coherent states, (multi- or single mode) squeezed states, and thermal states. Focusing on Gaussian states narrows the number of possible input states substantially without unduly simplifying the system. Owing to the form of Eq. (1), classifying the black hole quantum channel is reduced to the study of one-mode Gaussian (OMG) channels [12] (loosely defined as bosonic completely positive maps transforming Gaussian states into Gaussian states). The set of all possible OMG channels has recently been exhaustively classified [12–14]. We now know that there exist eight equivalence classes of OMG channels that are embodiments of various passive and active optical elements. Among those channels, we find for example the family of lossy channels (implemented by imbalanced beam splitters) as well as the amplification and conjugated channel families, related to the parametric amplification process. For details on those channels, we refer the reader to the excellent expositions [12, 14].

OMG channels are important in another respect: it is possible to calculate their classical capacity [15], that is, the capacity to transmit classical information via a quantum channel. Here we extend this work and attempt to calculate the quantum capacities of these OMG channels. Viewing the black hole in terms of the OMG channel construction will allow us to study how much information – encoded in a Gaussian state and sent into the black hole horizon – can be recovered by an outside stationary
observer. Understanding what happens to information (both classical and quantum) incident on a black hole is known as the black hole information problem (even though what is and is not a problem is often hotly contested [16]). A rigorous analysis of this problem must begin with the identification of the physical system (incident states and black hole) in terms of a quantum channel, and calculating its channel capacities as argued in [9, 10].

The classical capacity of a noisy quantum channel quantifies how much classical information (encoded in a quantum state) can be perfectly recovered from the output of the channel by a stationary observer far from the black hole horizon. Similarly, the quantum channel capacity quantifies the amount of quantum information that can be transmitted with vanishing error. Both channel capacities are fundamental quantities—they represent the maximal rates at which the respective information can be transmitted through a quantum system. Unlike the classical capacity, however, the quantum capacity of the OMG channels is (currently) known only in certain special cases [17, 18] and in general can only be estimated. This is unfortunate because even though at the time the black hole information problem was formulated the concepts of classical and quantum capacity of quantum channels did not even exist, today arguably more researchers are concerned with the fate of quantum information interacting with a black hole, which requires a calculation of the quantum channel capacity. Nevertheless, the classification of the black hole OMG channel we carry out here is the first important step in this program, and as a consequence we will be able to estimate the quantum capacity for a substantial fraction of the OMG channels that occur in black holes and calculate it exactly in certain special cases.

Aside from characterizing the black hole OMG channel, we also extensively study the complementary quantum channel corresponding to the transmission of quantum information behind the horizon, to be decoded by a potential observer inside the black hole. This further elucidates the flow of quantum information in the black hole scenario. We find that the process of absorption of quantum states by the black hole horizon is, in the language of OMG channels, described by the family of conjugated channels. We discuss the implications for the transmission of quantum information in the concluding section.

2. Black hole as an OMG channel

While not stated explicitly by Sorkin, we show below that the coefficients in Eq. (1) describing the field dynamics in the Heisenberg picture belongs to the symplectic group over the reals $Sp(6, \mathbb{R})$, and the coefficients in Eq. (1) satisfy $|\alpha| \leq 1$ and $\beta, \gamma \in \mathbb{R}$. Moreover, the coefficients must be such that the outgoing field operator is correctly normalized: $[A, A^\dagger] = 1$. In order to derive these conditions we set:

$$X = r(a^\dagger b^\dagger - ab) + s(a^\dagger c - ac^\dagger) + \ell(\ell a^\dagger b - ab^\dagger),$$

and identify $A = e^X ae^{-X}$ with the right-hand side of Eq. (1). In (2) we recognize the Hamiltonians for a two-mode squeezing transformation as well as two beam-splitters, where $r, s, \ell > 0$. We restricted the Hamiltonian parameters to be real in hindsight, as complex phases will not play any role. The Baker-Campbell-Hausdorff theorem

\footnote{Note that this parametrization is different from the ansatz made in [9] that, however, leads to the same solution.}
immediately leads to
\[ \alpha = \cos r, \]
\[ \beta = -\left( \frac{s^2}{r^2} - 1 \right)^{1/2} \sin r, \]
\[ \gamma = -\frac{s}{r} \sin r, \]
and reveals that \( s \geq r \) as a consequence of finding \( \ell^2 - 1 \geq 0 \) and \( \ell = \pm s \), which itself is a consequence of \( [A, A^\dagger] = 1 \). A similar procedure for the out field-operators
\[ B = e^{X} be^{-X} \]
and
\[ C = e^{X} ce^{-X} \]
gives rise to the following transformation of a column list of field operators \( L : \{a, b, c\} \mapsto \{A, B, C\} \):
\[
L = \begin{bmatrix}
\cos r & -\left( \frac{s^2}{r^2} - 1 \right)^{1/2} \sin r & -\frac{s \sin r}{r} \\
-\left( \frac{s^2}{r^2} - 1 \right)^{1/2} \sin r & \frac{s^2 + (r^2 - s^2) \cos r}{r^2} & -s \left( \frac{s^2}{r^2} - 1 \right)^{1/2} (\cos r - 1) \\
\frac{s \sin r}{r} & \frac{s \left( \frac{s^2}{r^2} - 1 \right)^{1/2} (\cos r - 1)}{r} & \frac{(\cos r - 1) s^2}{r^2} + 1
\end{bmatrix}.
\]
By introducing a symplectic form
\[
\omega = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}
\]
and setting \( \Omega = \omega \oplus \omega \oplus \omega \) we find that \( (L \oplus L) \Omega (L \oplus L)^T = \Omega \) for \( L \oplus L \) acting on a column list of field operators assembled as \( \{a, a^\dagger, b, b^\dagger, c, c^\dagger\} \). Hence \( L \in Sp(6, \mathbb{R}) \). We now would like to write \( L \) in the basis of quadrature operators \( p \) and \( q \) [19] given by
\[
\sigma : \{a, a^\dagger\} \mapsto \{p, q\}
\]
where
\[
\sigma = \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}.
\]
This can be achieved by writing the global evolution transformation of quadrature operators as
\[
S = \Sigma^{-1} (L \oplus L) \Sigma,
\]
where \( \Sigma = \sigma \oplus \sigma \oplus \sigma \) and one can verify that \( S \Omega S^T = \Omega \) is satisfied (for more information see for example [20, 21]). The \( S \) matrix introduced above will allow us to find the black hole response to an arbitrary incoming Gaussian state (for comprehensive definitions of Gaussian states and transformations see [12]).

On physical grounds, we are interested in the following scenario. The early-time (input) black hole horizon modes \( (a \text{ and } b) \) are in a vacuum state and the late-time incoming mode \( c \) will be a one-mode Gaussian state that is completely described by a covariance matrix \( V_{\text{in}} \) capturing the second quadrature moments (we can safely set the first moments describing the state displacement in phase space to zero). Because \( S \) is a symplectic transformation, the output modes are Gaussian. We are not, however, interested in the outgoing states per se. Our task is to deduce from the output covariance matrix \( V_{\text{out}} \) which of the OMG channels described in the introduction is responsible for the state transformation.

To study this we use the recent complete classification of OMG channels [13, 14]. These authors found that there are eight equivalence classes of OMG channels [12–14] (modulo a Gaussian unitary) distinguished by the values of three parameters that fully
The set of all rank-two OMG channels Eq. (7) is parametrized by the pair \((\tau, y)\) given by \(\tau = \det T\) and \(y = \sqrt{\det N}\) and consists of lossy channels \(C(\text{loss})\), amplifiers \(C(\text{amp})\), conjugated channels \(D\) and classical-noise channels \(B_2\) indicated by a red vertical line including the red dot: a rank-one class \(B_2(\text{id})\) at \((1, 0)\)). The brickwall represents the area of non-complete positive maps where Eq. (8) is violated (the unphysical region). The blue region contains entanglement-breaking channels for which \(y \geq |\tau| + 1\) holds [13], which are a subset of all zero quantum capacity channels (the green area covering the region \(y \geq \tau\) intersects with all completely positive maps [24]). For the black hole scenario we find that all Gaussian black hole channels (the \(a\) mode) generating the outgoing radiation are confined to a semi-infinite strip demarcated by the dashed boundary (inclusive).

characterize the OMG completely positive maps. It turns out that (except for a single class) the channel Stinespring dilations [22] consist of a two-mode symplectic transformation with a complementary (reference) input given by a thermal state of mean photon number \(\langle n \rangle\). This number is the first of the three parameters characterizing the OMG channels. The two remaining parameters come from a generic evolution of one-mode covariance matrices

\[
V_{\text{out}} = TV_{\text{in}}T^\top + N, \tag{7}
\]

where \(T\) and \(N\) are real (symmetric) \(2 \times 2\) matrices. Eq. (7) represents a channel if the following necessary and sufficient conditions are satisfied [23]:

\[
y \geq |\tau - 1|, \quad N \geq 0, \tag{8}
\]

where \(y \overset{\text{df}}{=} \sqrt{\det N}\) and \(\tau \overset{\text{df}}{=} \det T\). In that case, the two remaining parameters characterizing all OMG channels are \(\tau\) and \(r = \min \{\text{rank} T, \text{rank} N\}\).

The parameters just defined can be used to write down the canonical representatives of all equivalence classes of OMG channels [13, 14]. If we were able to deduce a canonical form from the action of \(S\) in Eq. (6) we could find the form of a black hole Gaussian channel for all admissible \(\alpha, \beta, \gamma\) from Eq. (1). This is indeed the case.
Consider an input three-mode state $V_{abc,\text{in}} = \text{id}_a \oplus \text{id}_b \oplus V_{c,\text{in}}$ where $\text{id}$ stands for a two-dimensional identity matrix and

$$V_{c,\text{in}} = \begin{bmatrix} e & g \\ g & f \end{bmatrix}$$

is a generic input OMG covariance matrix ($e, f, g \in \mathbb{R}$ s.t. $V_{c,\text{in}} + i\omega \geq 0$ [25]). We then find

$$V_{a,\text{out}} = \begin{bmatrix} e \frac{s^2}{r^2} \sin^2 r + \cos 2r + \frac{s^2}{r^2} \sin^2 r & \frac{g s^2}{r^2} \sin^2 r \\ \frac{g s^2}{r^2} \sin^2 r & f \frac{s^2}{r^2} \sin^2 r + \cos 2r + \frac{s^2}{r^2} \sin^2 r \end{bmatrix}.$$ 

(10)

This leaves us with only three possible candidates labeled in Ref. [14] as $C(\text{loss}), C(\text{amp})$ and $B_2$ that possess the following canonical forms:

$$(T, N)_{C(\text{loss})} = \left(\sqrt{\tau} \text{id}, (1 - \tau)(2\langle n \rangle + 1) \text{id} \right),$$

(11a)

$$(T, N)_{C(\text{amp})} = \left(\sqrt{\tau} \text{id}, (\tau - 1)(2\langle n \rangle + 1) \text{id} \right),$$

(11b)

$$(T, N)_{B_2} = \left(\text{id}, \langle n \rangle \text{id} \right).$$

(11c)

The admissible values of $\tau$ are (in order): $0 \leq \tau < 1, \tau > 1$ and $\tau = 1$.

The class $C(\text{loss})$ represents lossy channels (for example the action of an unbalanced beam-splitter) and $C(\text{amp})$ denotes the class of amplification channels. $B_2$ forms its own equivalence class of the so-called classical-noise channels [13]. In the following we will refer to the family of channel Eqs. (11) as the Gaussian black hole channels, defined by the $a$ subsystem of (10). Using that equation we identify

$$\tau_a = \frac{s^2}{r^2} \sin^2 r,$$

(12a)

$$y_a = \cos 2r + \frac{s^2}{r^2} \sin^2 r,$$

(12b)

where we have introduced the subscript $a$ to the parameters $\tau$ and $y$ to indicate that the $a$ subsystem is the quantum black hole channel output. We further verify that the first condition of complete positivity in Eq. (8) is satisfied whenever $s^2 \geq r^2$. This is exactly the condition found below Eqs. (3). The second condition $N \geq 0$ gives a weaker constraint.

How large is the set of channels given by (12) with respect to the classes it is part of? To visualize the set we adopt a figure from [15] where all (four) rank $r = 2$ OMG channels are parametrized using the coordinates $(\tau, y)$, see Fig. 1. The form of Eq. (12a) dictates $\tau_a \geq 0$. Consequently, from (12b) we obtain $y_a = \cos 2r + \tau_a$ and this leads to an interesting observation: If we choose a given $\tau_a \geq 0$, then by adjusting $r$ (and therefore $s \geq 0$ to keep $\tau_a$ constant) the value of $y_a$ oscillates between $\tau_a - 1$ and $\tau_a + 1$ for all $\tau_a$. The oscillation boundaries become $|\tau_a - 1|$ and $\tau_a + 1$ for $0 \leq \tau_a < 1$ due to the condition of complete positivity.

Among all OMG channels, many are entanglement breaking (blue area in Fig. 1). The black hole channels that lie in the area marked “black hole region” (including its boundary), on the other hand, are not entanglement breaking: they are composed of the equivalence classes $C(\text{loss}), C(\text{amp})$ and $B_2$. The only exception to this rule is part of the boundary region $y = \tau + 1$ ($\tau \geq 0$) denoted by the dashed blue line, where even the black hole channel is entanglement breaking.

It is instructive to investigate what channel in the black hole region in Fig. 1 we obtain if we choose certain limiting parameters of the Bogoliubov transformation Eq. (1). For
example, as the parameter $\alpha$ sets the reflectivity of the black hole, we could study $\alpha = 0$ and $\alpha = 1$ as they correspond to the limiting cases of a perfectly reflecting and absorbing black hole, respectively. [9, 10]. Indeed, these two cases were studied in [10] for the case of a qubit black hole channel. To obtain $\alpha = 1$ we set $r = 0$ and from Eqs. (12) we obtain $\tau_a = s^2$ and $y_a = 1 + s^2$, corresponding to the blue dashed line in Fig. 1 (recall that $s \geq r = 0$) and this is precisely the only instance where the black hole channel is entanglement breaking (for all $s$). This is consistent with the (non-Gaussian) dual-rail encoding studied in [10] where we found the case $\alpha = 1$ to be entanglement breaking as well. For the other case $\alpha = 0$ we set $r = \pi/2$ and this time we find $\tau_a = 4s^2/\pi^2$ and $y_a = 4s^2/\pi^2 - 1$. Because $s \geq \pi/2$ we obtain $\tau_a \geq 1$ and $y_a \geq 0$ and we can identify the channel region to be the lower semi-infinite dashed boundary in Fig. 1, corresponding to degradable channels. But this finding is again in perfect agreement with the (physically very different) qubit case we studied earlier [10], where we also found that the perfectly reflecting channel (known also as the Unruh channel [26]) is degradable. As a matter of fact, all values $r \in [0, \pi/2]$ generate half-lines that “foliate” the semi-infinite black hole strip in Fig. 1.

3. Quantum Capacity of OMG Channels

The classical capacity of all OMG channels was studied by Giovannetti et al. [15], who derived explicit expressions for them. Here, we are interested in calculating the quantum capacity of these channels, and in particular the quantum capacity of the black hole channels depicted in Fig. 1. Unlike the classical capacities, the quantum capacity can only be explicitly calculated for a small fraction of channels because a single-letter formula is not presently known for all [12].

A large area of parameter space of the black hole channels (the green area in Fig. 1) has vanishing quantum capacity [24], owing to the fact that these channels can be written as a composition of an arbitrary Gaussian channel and an anti-degradable Gaussian channel [12]. The partial overlap of the Gaussian black hole channels with the zero quantum capacity region is fundamental from a physical point of view, because this overlap implies that for some black holes (that is, some values of $r$ and $s$ in the evolution operator Eq. (6), quantum information cannot be recovered by an outside observer.

The situation is more complicated in the white part of the black hole region of Fig. 1, where the capacity cannot so easily be calculated. In Fig. 2, we focus only on the black hole region of the OMG channel parameter space. We remind the reader that the black dashed line that separates the green area in Fig. 2 from the entanglement breaking channels (the blue area in Fig. 1) itself describes entanglement breaking channels, which in fact are known to have vanishing capacity [12]. The black dashed line from $(0.5, 0.5)$ to $(0, 1)$ separating the green area from the unphysical maps consists of channels that are anti-degradable, and therefore their capacity also vanishes. The remaining area of black hole channels (the white area in Fig. 1, also the white and purple area in Fig. 2) contains channels with both calculable and unknown capacities.

On a part of the boundary of that region (depicted by the black dashed lines from $(1, 0)$ to $(0.5, 0.5)$ and $y = \tau - 1$) the OMG channels are known to be degradable [18, 27], which implies that the capacity is calculable (and thus known). Inside of this area the quantum capacity can only be bounded from below [12, 17]. Moreover, it is not known from general principles whether the quantum-capacity achieving codes in this area are
Figure 2. The black hole region (the green and white area within the dashed lines in Fig. 1), colored to emphasize the (calculable) non-zero values of the coherent information (purple area) from Eqs. (18). These equations give a non-zero lower bound for the quantum capacity of the Gaussian black hole channel. The green area represents a region with zero quantum capacity, as discussed earlier. The white region in between is uncharted territory, that is, where the quantum capacity is unknown. The coherent information is non-positive in this region, but this does not imply that the quantum capacity vanishes, as the coherent information is just a lower bound [17].

Gaussian at all, unlike in the boundary (degradable) region where the optimal quantum codes are coherent (and therefore Gaussian) states [18].

Our starting point for calculating the quantum capacity is the expression for the optimized coherent information for an OMG channel $G$ given in [12] (see also [17])

$$I_{\text{coh}}(G) \equiv \sup_N I_{\text{coh}}(N, G(\rho)) = \sup_N [g(N') - g(x_+) - g(x_-)],$$

(13)

where $N = \text{Tr}[\rho a^\dagger a]$ is the mean particle number of an input Gaussian state $\rho$ and $g(x)$ is its von Neumann entropy [28]

$$g(x) \equiv (1 + x) \log (1 + x) - x \log x.$$  

(14)
Following Holevo [12], we set

\[ N' = \begin{cases} 
\tau N + K & \text{for } 0 \leq \tau < 1 \\
\tau N + \tau - 1 + K & \text{for } \tau > 1 ,
\end{cases} \tag{15a} \]

\[ x_+ = \frac{1}{2} (D + N' - N - 1) \tag{15b} \]

\[ x_- = \frac{1}{2} (D - N' + N - 1) \tag{15c} \]

where

\[ K = \frac{1}{2} (y - |1 - \tau|) , \tag{16} \]

\[ D = \left( (N + N' + 1)^2 - \tau N (N + 1) \right)^{1/2} . \tag{17} \]

In general, the optimized coherent information is maximized over all possible code-words, here restricted to Gaussian states [12]. Unlike in the calculation of the classical capacity [17], the limit of infinite input power \( N \to \infty \) does not usually lead to a diverging entropic quantity. It turns out that for the value \( K = 0 \), the maximization over codewords is achieved for \( N \to \infty \) for \( \tau > 1/2 \) because Eq. (13) is increasing monotonically. For \( \tau < 1/2 \) the function decreases monotonically, and the maximal value is reached for \( N = 0 \), leading to a vanishing coherent information. This (vanishing) value is compatible with the zero quantum information in the green region in Fig. 2) [12].

When \( K > 0 \) the coherent information increases monotonically for certain \( \tau > 1/2 \) but not always. In those cases we do not know whether the maximum is achieved for \( N \to \infty \), but as long as its value is positive the quantum capacity must be positive (purple region in Fig. 2) [2]. Taking the limit \( N \to \infty \), we can generalize the coherent information expression from Ref. [12] (calculated there for the special case \( K = 0 \) of Eq. (15a)):

\[ I_{\text{coh}}(C(\text{loss})) \geq \lim_{N \to \infty} \tilde{I}_{\text{coh}}(N, C(\text{loss})) \]

\[ = \frac{K}{1 - \tau} \log \frac{K}{1 - \tau} - \frac{K - 1}{1 - \tau} \log \frac{K - 1 + K}{1 - \tau} + \log \frac{\tau}{1 - \tau} \tag{18a} \]

\[ I_{\text{coh}}(C(\text{amp})) \geq \lim_{N \to \infty} \tilde{I}_{\text{coh}}(N, C(\text{amp})) \]

\[ = \log \tau + \frac{K}{\tau - 1} \log K - \frac{\tau - 1 + K}{\tau - 1} \log (\tau - 1 + K) . \tag{18b} \]

Both equations reduce to \( I_{\text{coh}} = \log \frac{\tau}{|\tau - 1|} \) for \( K = 0 \), which coincides with the expression from [12] (see also [18]) valid whenever \( \tau > 1/2 \). Indeed, \( K \) vanishes on the dashed boundary of the purple region and of the green region (where \( y < 1 \)) in Fig. 2. The dashed boundary of the purple region is where the quantum capacity is nonzero and actually calculable – the OMG channels there are known to be degradable.

The case \( \tau = 1 \) must be treated separately, as channels with \( \tau = 1 \) represent a separate class of channels called “classical-noise channels” [14] (denoted by \( B_2 \), see Fig. 1). We find

\[ \lim_{\tau \to 1} \lim_{N \to \infty} \tilde{I}_{\text{coh}}(N, C(\text{loss})) \equiv \lim_{\tau \to 1} \lim_{N \to \infty} \tilde{I}_{\text{coh}}(N, C(\text{amp})) = -1 - \log K . \]

\[ \text{2 Numerical evidence suggests that the maximum is always achieved for all } K \text{ and } \tau > 1/2 \text{ when } N \to \infty. \]
Evidently, this is a singular case where the optimized coherent information (and by inference the quantum capacity) diverges. This happens when $K = 0$ (while $\tau = 1$): the rather exceptional point $(1,0)$ in Fig. 1, corresponding to the subclass of $\mathcal{B}_2$ channels where $T = \text{id}$ and $N = 0$ in Eq. (7) [13]. In other words, this is a trivial noiseless (identity) Gaussian channel, whose capacity is known to diverge in classical physics [29].

4. Complementary OMG channel

We just observed that a sizable swath of the black hole OMG channel space has a vanishing capacity, implying that quantum information cannot be retrieved from it by an outside stationary observer. As discussed previously, this implies that the observer cannot perfectly reconstruct the quantum entanglement that the sender has been part of. However, as opposed to a loss of classical information that would imply the loss of microscopic time-reversal invariance [30], the loss of quantum information does not contradict any known laws of physics. But we can nevertheless ask where the quantum information is hiding, because in a completely unitary picture of quantum dynamics, quantum information cannot be lost from the universe. In this section we show that the quantum information is available to observers behind the horizon, by calculating a lower bound the capacity of the quantum channel to send information beyond the horizon: the complementary OMG channel.

Before we proceed, it is important to emphasize a particularity of the isometry $e^X$ from Eq. (2). So far, we studied the Gaussian black hole channels that correspond to the $a$ mode (the horizon mode available to outside observers). The minimal Stinespring dilation for these channels would usually be a two-mode isometry satisfying certain properties discussed in Ref. [14]. However, the expression $e^X$ is a three-mode operator, and this could conceivably not be the most economic dilation whose reference mode is in a pure state [31] (the Stinespring dilation). For example, on the black dashed boundary in Fig. 1 the minimal Stinespring dilation is known to be a two-mode isometry [15]. On the other hand, our redundant (non-economic) isometry contains valuable information about the physics of the black hole interaction. Namely, the two modes forming the complementary black hole channel ($b$ and $c$) correspond to the physical system of the black hole itself and the radiation that crosses the event horizon, respectively, which are both integral to the channel.

We want to emphasize again the importance of Sorkin’s construction, introducing the third mode $c$ when standard black hole curved-space field theory only has the horizon modes, $a$ and $b$. The latter modes emerge from the formation of the black hole, and therefore are tremendously red-shifted with respect to late-time observers. Introducing a late-time observer mode $c$ which, because it is exponentially blue-shifted with respect to $a$ and $b$ is defined on a disjunct Hilbert space (the $c$ annihilation and creation operators commute with both $a$ and $b$), allows us to introduce an interaction term between black hole (horizon) modes and signal states. Even though the information transmission channel into the black hole is mediated by $c$ only, the complementary channel involves the combined $bc$ Hilbert space, as these two modes interact behind the horizon. Note that the Killing energy of the mode $c$ is positive while that of the $b$ mode is negative.

Could it be that the green area for the $a$ mode (the black hole channel in Fig. 1 or 2 with vanishing capacity) corresponds to a non-zero quantum capacity of the $c$ mode (the radiation penetrating the horizon from outside) and vice-versa? To study this, we must investigate the fate of the $c$ mode explicitly. Using the same input state $V_{abc,\text{in}}$ as
Figure 3. OMG channel properties outside and inside the horizon. (a): We labelled 35 points in the space of parameters \((\tau_a, y_a)\) that characterize 35 OMG channels sampling the black hole region from Fig. 1 – the Gaussian black hole channel (mode \(a\)). The numbers in the parenthesis on the axes are the coordinates \((\tau_a, y_a)\). (b): Each of the \(a\)-channel parameters is transformed into the corresponding \(c\)-channel parameters as described in the text. By following the points (and lines) we can study the relationship between the \(a\)-mode and \(c\)-mode channels. It is important to keep in mind that (as discussed in the text) the \(c\) mode is not the entire complementary subsystem to the black hole channel, see the discussion before Eq. (22). The numbers in the parenthesis on the axes are the coordinates \((\tau_c, y_c)\).

before, we find for the parameters that characterize the \(c\)-channel into the black hole

\[
\tau_c = \left( \frac{s^2(\cos r - 1)}{r^2} + 1 \right)^2, \tag{19a}
\]
\[
y_c = s^2 \left( \frac{s^2}{r^2} - 1 \right) (\cos r - 1)^2 + \frac{s^2}{r^2} \sin^2 r. \tag{19b}
\]

We can express \(\tau_c\) and \(y_c\) in terms of \(\tau_a\) and \(y_a\) by first writing (using Eqs. (12))

\[
r = \frac{1}{2} \arccos (y_a - \tau_a) + k\pi, \tag{20}
\]

where \(k \in \mathbb{Z}\). It then from Eqs. (12) and (19) that

\[
\tau_c = \left( \frac{\tau_a}{\sin^2 r} (\cos r - 1) + 1 \right)^2, \tag{21a}
\]
\[
y_c = \frac{\tau_a}{\sin^2 r} \left( \frac{\tau_a}{\sin^2 r} - 1 \right) (\cos r - 1)^2 + \tau_a. \tag{21b}
\]

We can now see what the corresponding OMG channel in the \(c\) mode is, for a given \(a\) mode channel. It turns out that for a given pair \((\tau_a, y_a)\) inside the black hole parameter region (that is, excluding the dashed boundary in Fig. 1) we find two solutions \((\tau_c, y_c)\)
corresponding to even or odd $k$ in Eq. (20). In other words, the map from the set of all admissible parameters to the set of all black hole channels is onto but not one-to-one (it is actually two-to-one quotiented by $2\pi$). This two-to-one mapping deserves a closer look.

The physical properties of a black hole are determined by the isometry parameters $r$ and $s$ in Eqs. (3). They uniquely specify the isometry $e^X$ and therefore the Gaussian black hole channel (the $a$ output). Suppose that another pair of parameters $r', s'$ that leads to a different isometry $e^{X'}$ induces the same Gaussian black hole channel (this is in principle possible). This implies that the complementary channel (the $bc$ output) induced by the those two isometries must also be equivalent. However, it does not imply this identity for any part of the complementary channel (the $b$ or $c$ outputs alone). That is precisely what is happening here: a black hole channel defined by $(\tau_a, y_a)$ leads to two solutions of the black hole parameters $r, s$ in Eq. (20) for the $c$-channel.

The first solution (even $k$) is depicted in Fig. 3(a) where the semi-infinite rectangle from Fig. 1 is sampled by 35 points in the black hole region close to the origin. The corresponding $c$-mode channel parameters are shown in Fig. 3(b). We can see that there are instances where both channels have vanishing capacity (for example, the set of parameters 30). The only region where there is a certain kind of complementarity between the $a$ and $c$ modes is the line connecting the points $(0, 1)$ and $(1, 0)$ (note the opposite orientation of this line in both panels), and also the red line covering degradable channels (if continued indefinitely).

The solutions corresponding to odd $k$ are depicted in Fig. 4(b). We sampled the black hole region Fig. 3(a) slightly differently with 24 points compared to $k$ even. We stress again that the existence of two different $c$ channels corresponding to a given

**Figure 4.** Correspondence between $a$-mode and $c$-mode channels for odd $k$. $\tau_c$ and $y_c$ in Eq. (21). (a): Sample of 24 channels in the black hole region and (b): their corresponding $c$-mode channels. The missing channel numbers in panel (b) are too far from the origin to appear in the figure.
Gaussian black hole channel is not surprising because the c channel is not the whole complementary channel to the black channel.

By analyzing $V_{b,\text{out}}$ from Eq. (7) we find that for all Gaussian input states, the b output subsystem is given by an OMG channel from the conjugated equivalence class $\mathcal{D}$, whose generic form reads \[ (T, N)_{\mathcal{D}} = (\sqrt{-\tau} \sigma_z, (1 - \tau)(2\langle n \rangle + 1) \text{id}) \] (22)

where $\tau < 0$ and $\sigma_z$ is the Pauli $z$ matrix. Inspecting Fig. 1, we see that all these channels lie in the entanglement breaking region and by themselves cannot carry any quantum information. But if the information is never in the b mode and often neither in the c or a modes, does it mean it is lost? It turns out that this is not necessarily the case.

As we stressed several times, the channel across the black hole horizon (the c-mode channel) is not the full black hole complementary channel, which must be “spanned” by both the b and c modes. Because this is a two-mode channel, we do not know to what Gaussian channel it corresponds to (the present analysis is concerned exclusively with OMG channels). However, this does not prevent us from estimating its quantum capacity. Recall that the zero capacity (green) region in Figs. 1 and 2 is where $I_{\text{coh}}(\mathcal{G}) = Q(\mathcal{G}) = 0$ and $\mathcal{G}$ is one of the identified Gaussian channels in Eq. (11). Recall the definition of the optimized coherent information of a general quantum channel $\mathcal{N}$ \[ I_{\text{coh}}(\mathcal{N}) = \sup_{\rho} [H(\mathcal{N}(\rho)) - H(\hat{\mathcal{N}}(\rho))] \] (23)

where the caret denotes the complementary channel to $\mathcal{N}$ and $H(\rho) \overset{\text{df}}{=} -\text{Tr} \rho \log \rho$ is the von Neumann entropy. Now, it is known that there exist Gaussian codewords $\rho$ for which $H(\mathcal{G}(\rho)) - H(\hat{\mathcal{G}}(\rho))$ is negative. As a consequence, the supremum in (23) must be positive for the complementary channel \[ I_{\text{coh}}(\hat{\mathcal{G}}) > H(\hat{\mathcal{G}}(\rho)) - H(\mathcal{G}(\rho)) > 0 \] (24)

where $\hat{\mathcal{G}}$ denotes the Gaussian complementary channel (the bc modes). Hence, the quantum capacity $Q(\hat{\mathcal{G}}) \geq I_{\text{coh}}(\hat{\mathcal{G}})$ is positive in general (even if it vanishes individually for the c and b modes), and the information is delocalized in the modes inside the black hole whenever the Gaussian black hole channel outside vanishes (the green region in Fig. 1 or 2). As a matter of fact, we can say more. Since the zero capacity (green) region is formed by channels $\mathcal{G}$ that are compositions of an anti-degradable Gaussian and some other (Gaussian) channel [24], the channel $\mathcal{G}$ is anti-degradable as well. Therefore $\hat{\mathcal{G}}$ is degradable and the capacity is given by the single-letter expression $Q(\hat{\mathcal{G}}) = I_{\text{coh}}(\hat{\mathcal{G}})$ [12].

5. Conclusions

We analyzed the late time interaction of a scalar field in the form of Gaussian states with a Schwarzschild black hole based on Sorkin’s model. For a distant outside observer such a black hole acts as a quantum channel from a family of one-mode Gaussian channels that were recently classified. Here, we refer to these channels as “Gaussian black hole” channels. The classification enables us to ask the following question: how much information can a distant observer at future infinity recover if a Gaussian state carrying classical or quantum information interacts with an already formed black hole?

This question is nothing but a reformulation of the black hole information loss problem, and it can be answered by a calculation of the classical and quantum capacity of
these quantum Gaussian channels. The classical capacity was calculated by Giovannetti et al. [15] but it is the fate of quantum information that is arguably more relevant for the black hole information puzzle. By mapping the black hole to the Gaussian quantum channels in this manner, we are closer to resolving the fate of information interacting with a black hole. In particular, we found the exact parameter region of one-mode Gaussian channels corresponding to an arbitrary Schwarzschild black hole. These black hole Gaussian channels represent an interesting subset of three equivalence classes of Gaussian channels that are (with one exception) non-entanglement breaking. We find that half of the parameter space of black hole channels has vanishing quantum capacity. The other half of this region splits into two parts: one part where the quantum capacity is calculable (or is at least known to be nonzero) and the other where no non-negative lower bound is known. There is, however, currently no argument that excludes a positive quantum capacity for those channels.

We also studied the complementary channel (sending quantum information on the other side of the black hole horizon) and found that the channel’s capacity is given by the optimized coherent information, and is positive in one half of the black hole parameter region when the capacity to transmit quantum information outside the black hole vanishes. In this manner, the no-cloning theorem is respected (as also seen in the dual-rail channel discussed in [10]): quantum information is never available in two places. At the same time, each of the modes $b$ and $c$ that together compose the complementary channel may not be sufficient to reconstruct quantum states inside the black hole by themselves.

What are the consequences for (quantum) information retrieval from a black hole? That is a question we can answer only partially. We are not aware of any physical mechanisms restricting or favoring some sections of the whole black hole channel parameter region, which implies that quantum information can be lost within black holes. We can only say that if the black hole resides in the nonzero capacity region, quantum information can be retrieved with arbitrary accuracy, and we can calculate or estimate the rate of recovery (certain extreme points such as the noiseless identity channel $B_2(\text{id})$ can be safely ignored, as the black hole always introduces some noise due to the presence of spontaneous emission of radiation, namely the Hawking radiation effect). If on the other hand a black hole channel is in the zero capacity region, quantum information cannot be recovered from it on the outside. This, however, does not imply a breakdown of quantum mechanics, or any other of our known laws of physics. Thus, it appears that mapping black holes to a one-mode Gaussian channel allows us to understand how black holes process classical or quantum information using concepts from quantum optics and quantum information theory only.

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