General Power-Equivalent Synthesis of Resistive DC Networks

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ABSTRACT
Thévenin’s and Norton’s theorems are cornerstones of linear circuit analysis as they enable the terminal representation of any single-port linear network as the series of an ideal voltage generator and a linear impedance, or as the parallel between an ideal current generator and a linear admittance. While Thévenin’s and Norton’s representations are, by construction, electrically equivalent to the original network, they do not preserve power equivalence, in that they do not reproduce the power dissipated by the original network, nor they are power-equivalent to each other. This work discloses a general expression of the internal power \( P_h \) dissipated by a linear dc network composed by resistors and a mixture of independent voltage and current generators. The expression reveals a theoretical link between \( P_h \) and key open-circuit (or short-circuit) parameters of the network itself, and provides renewed insights on the relationship between power dissipation, load and network’s efficiency. Furthermore, the result leads to the formulation of a class of circuits which are both electrically and power-equivalent to the original network, and that can be regarded as power-equivalent generalizations of Thévenin’s and Norton’s representations. Results are validated via case studies treated analytically and via computer simulations.

INDEX TERMS
Thévenin’s theorem, Norton’s theorem, equivalent circuits, circuit theory, circuit synthesis.

NOMENCLATURE

\( P_o, P_h \)  
Network’s output power and total internal dissipated power, respectively

\( P_{h,\text{min}} \)  
Network’s minimum dissipated power

\( P_{h,\text{OC}}, P_{h,\text{SC}} \)  
Network’s dissipated power in open-circuit and short-circuit condition, respectively

\( P_{h,\text{OC-E}}, P_{h,\text{OC-J}} \)  
Network’s open-circuit dissipated power due to the voltage sources alone and to the current sources alone, respectively

\( P_{h,\text{SC-E}}, P_{h,\text{SC-J}} \)  
Network’s short-circuit dissipated power due to the voltage sources alone and to the current sources alone, respectively

\( V_{\text{TH}}, I_{\text{NOR}} \)  
Network’s open-circuit voltage and short-circuit current, respectively

\( V_{\text{TH-E}}, V_{\text{TH-J}} \)  
Network’s open-circuit voltage due to the voltage sources alone and to the current sources alone, respectively

\( I_{\text{NOR-E}}, I_{\text{NOR-J}} \)  
Network’s short-circuit current due to the voltage sources alone and to the current sources alone, respectively

I. INTRODUCTION

Circuit analysis is a topic of central importance in modern engineering. Theory of linear networks, in particular, is a cornerstone for the understanding of electrical circuits, its principles and methodologies being introduced in any undergraduate curriculum in electrical engineering. One of the key concepts in circuit analysis is the idea of equivalent circuit, i.e. a network of reduced complexity capable of exactly reproducing the voltage-current electrical relationship at the terminals of a more complex network. For linear circuits, this idea is formalized by the well known Thévenin’s and Norton’s theorems [1]–[4]. The former states that any single-port linear network is equivalent to a voltage generator in series with a linear impedance. In a dual manner, the latter states that the same network is equivalent to an ideal current generator in parallel with a linear admittance. Thévenin’s and...
Norton’s equivalent circuits are crucial tools for the analysis and understanding of linear networks.

Along with the foregoing idea of electrical equivalence, which revolves around networks exhibiting the same voltage-current relationship at their terminals, one can put forward that of power equivalence, i.e., the requirement that the original and equivalent networks also exhibit the same internal dissipated power in any load condition. However, while the concept of electrical equivalence finds its theoretical bases in Thévenin’s theorem and in more than a century of successful applications, that of power-equivalence remains largely overlooked. A well known limitation of both Thévenin’s and Norton’s representations, in this regard, is that neither of them is power-equivalent to the original network. Furthermore, the two representations are not power-equivalent to each other: a textbook example of such energy asymmetry is the open-circuit dissipated power – zero for the Thévenin source and maximum for the Norton one – or, in a dual manner, the short-circuit power.

This work takes its moves from a recent publication on the subject of power equivalence [5], in which it is shown that the power dissipated by a linear dc network composed by resistors and independent voltage generators consists of a constant term, equal to the dissipated power in open-circuit conditions, plus a load-dependent contribution numerically equal to the power dissipated by the network’s Thévenin’s resistance when traversed by the same load current as in the original network. The result leads to an extended formulation of Thévenin’s equivalent circuit which is also power-equivalent to the original network. Furthermore, this representation is not power-equivalent to the Norton’s one, in this regard, is that neither of them is power-equivalent to the original network. Furthermore, the two representations are not power-equivalent to each other: a textbook example of such energy asymmetry is the open-circuit dissipated power – zero for the Thévenin source and maximum for the Norton one – or, in a dual manner, the short-circuit power.

These open questions have led the author to search for a general expression of the power dissipated by a network consisting of resistors and a mixture of independent voltage and current sources. Quite remarkably, such general expression does exist, and reveals a theoretical link between the power dissipated by the network in a generic load condition and specific open-circuit and short-circuit parameters of the network itself, providing renewed insights on the power properties of resistive networks. Furthermore, the result can be employed to formulate a class of circuits which are both electrically equivalent and power-equivalent to the original network, overcoming the aforementioned energy asymmetry between the Thévenin and Norton representations.

The main theoretical result of this work, i.e. the general expression of the internal dissipated power \( P_h \) of a resistive network, is disclosed in Section II and immediately discussed in its main implications. The general expression of the network’s efficiency versus load resistance is disclosed, along with the expression of the efficiency-optimal load resistance. Closed form results for the efficiency under efficiency-optimal and power-optimal loading condition are also reported and compared. Section III develops the formulation of circuits which are power-equivalent to a given original network and that can be regarded as generalizations of Thévenin’s and Norton’s representations. Section IV provides a validation of the theoretical predictions carried out on two sample networks taken as case studies. Section V provides a detailed mathematical proof of the expression of \( P_h \).

II. POWER DISSIPATION OF A RESISTIVE NETWORK

Consider a linear dc network consisting of resistors and a mixture of independent voltage and current sources (Fig. 1). By internal dissipated power we refer to the total power \( P_h \) absorbed by the resistors in the network. Any power potentially consumed by some of the network’s generators is therefore not considered dissipated, but merely absorbed in a completely reversible manner. Also notice that we are explicitly excluding the presence of dependent sources in the network.

The core theoretical result of this work, the proof of which is disclosed in Section V, is the following expression of \( P_h \),

\[
P_h = P_{h,\text{oc}} - 2V_{\text{th,}2}I_o + R_{\text{th}}I_o^2
\]

(1)

where:

- \( P_{h,\text{oc}} \) represents the power dissipated by the network in open-circuit conditions;
- \( V_{\text{th,}2} \) represents the open-circuit voltage of the network due to the network’s internal current sources alone;
- \( R_{\text{th}} \) represents the network’s Thévenin’s equivalent resistance.

In a dual manner,

\[
P_h = P_{h,\text{sc}} - 2I_{\text{NOR,sc}}V_o + G_{\text{NOR}}V_o^2
\]

(2)

where:

- \( P_{h,\text{sc}} \) represents the power dissipated by the network in short-circuit conditions;
- \( I_{\text{NOR,sc}} \) represents the short-circuit current of the network due to the network’s internal voltage sources alone;
- \( G_{\text{NOR}} \) represents the network’s Norton’s equivalent conductance, i.e. the reciprocal of the network’s Thévenin’s equivalent resistance (\( G_{\text{NOR}} = R_{\text{th}}^{-1} \)).

1 Although here developed as dc relationships, the following results can also be regarded as instantaneous equations in presence of time-varying voltage and current sources, provided that the network does not contain reactive elements.
Furthermore, it will be shown that

\[ P_{h,\text{oc}} = P_{h,\text{oc,E}} + P_{h,\text{oc,I}}, \]
\[ P_{h,\text{sc}} = P_{h,\text{sc,E}} + P_{h,\text{sc,I}}, \]

where:

- \( P_{h,\text{oc,E}} \) and \( P_{h,\text{oc,I}} \) represent, respectively, the total open-circuit dissipated power due to the network’s internal voltage sources alone, and the total open-circuit dissipated power due to the network’s internal current sources alone;
- \( P_{h,\text{sc,E}} \) and \( P_{h,\text{sc,I}} \) represent, respectively, the total short-circuit dissipated power due to the network’s internal voltage sources alone, and the total short-circuit dissipated power due to the network’s internal current sources alone.

Before proceeding, note that if \( V_{th} \) and \( I_{sor} \) are the network’s open-circuit voltage and short-circuit current respectively, one has the following relationships:

\[ V_{th} = V_{th,E} + V_{th,I}, \quad I_{sor} = I_{sor,E} + I_{sor,I}, \]
\[ V_{th,E} = R_{th} I_{sor,E}, \quad V_{th,I} = R_{th} I_{sor,I}, \] (4)

where \( V_{th,E} \) represents the network’s open-circuit voltage due to the voltage sources alone, and \( I_{sor,I} \) the network’s short-circuit current due to the current sources alone.

### A. BASIC PROPERTIES OF THE DISSIPATED POWER

Results (1) and (2) reveal a conceptual link between \( P_h \) and key open-circuit or short-circuit properties of the network. Although the following discussion will be mostly focused on (1), corresponding statements can be formulated based on (2) following duality principles.

Consider first a dc network only composed by resistors and independent voltage sources. Absence of current generators in the network yields \( V_{th,I} = 0 \), and therefore

\[ P_h = P_{h,\text{oc}} + R_{th} I_{th}^2. \]

In other words, the power dissipated by such a network consists of a constant term – the open-circuit dissipated power – and a load-dependent term equal to the power dissipated by the network’s Thévenin’s resistance when traversed by the load current \( I_o \). This result was originally recognized in [5].

Notice that when \( V_{th,I} = 0 \), the dissipated power \( P_h \) is minimum in open-circuit conditions. In general, however, the minimum dissipated power is, from (1),

\[ P_{h,\text{min}} = P_{h,\text{oc}} - \frac{V_{th,I}^2}{R_{th}} \quad \text{at} \quad I_o = I_{sor,I} = \frac{V_{th,I}}{R_{th}}. \] (5)

A plot of \( P_h \) versus \( I_o \) is exemplified in Fig. 2, assuming \( V_{th,I} > 0 \). Presence of current generators in the network causes the dissipated power to first decrease with the load current, then increase.

By definition, the dissipated power of the network is a non-negative quantity for all load conditions. As a consequence, the minimum power (5) must be non-negative as well, and therefore

\[ P_{h,\text{oc}} \geq \frac{V_{th,I}^2}{R_{th}}. \]

That is, presence of independent current generators capable of producing a nonzero open-circuit voltage poses a nonzero lower limit to the open-circuit dissipated power. Intuitively speaking, presence of at least one current source in the network implies the presence of at least one loop to allow the circulation of its current, and this loop cannot be cut in open-circuit. Therefore, the network will certainly dissipate non-zero power at \( I_o = 0 \).

In a dual manner, the expression of the minimum internal dissipated power can also be written as

\[ P_{h,\text{min}} = P_{h,\text{sc}} - R_{th} I_{sor,E}^2 \quad \text{at} \quad V_o = V_{th,E} = R_{th} I_{sor,E}. \] (6)

It is immediate to verify that the above minimum power is the same as (5). In fact, calculate the load current \( I_o \) corresponding to the above condition (6),

\[ I_o = \frac{V_{th} - V_o}{R_{th}} = \frac{V_{th} - R_{th} I_{sor,E}}{R_{th}}. \]

From (4) one has

\[ I_o = \frac{V_{th,E} + V_{th,I} - R_{th} I_{sor,E}}{R_{th}} = \frac{R_{th} I_{sor,E} + V_{th,I} - R_{th} I_{sor,E}}{R_{th}} = \frac{V_{th,I}}{R_{th}} = I_{sor,I} \]

as expected. In summary,

\[ P_h = P_{h,\text{min}} \quad \Leftrightarrow \quad V_o = V_{th,E} \quad \text{and} \quad I_o = I_{sor,I}. \] (7)
Equality between (5) and (6) leads to an important relationship between open-circuit and short-circuit quantities,

\[ P_{h,oc} - P_{h,sc} = \frac{V_{th,i}^2}{R_{th}} - R_{th}I_{nor,k}^2. \]  

(8)

One consequence of the above result, for instance, is that in a network solely composed by voltage sources \((V_{th,i} = 0)\) one necessarily has \(P_{h,sc} > P_{h,oc}\). Furthermore, whenever \(P_{h,oc} = P_{h,sc}\) one necessarily has

\[ \left| \frac{V_{th,i}}{I_{nor,k}} \right| = R_{th}. \]

Since \(V_{th,i} = R_{eq}I_{nor,i}\) and \(V_{th,e} = R_{th}I_{nor,e}\), one has the following chain of equivalences,

\[ P_{h,oc} = P_{h,sc} \iff |V_{th,i}| = |V_{th,e}| \iff |I_{nor,k}| = |I_{nor,e}|. \]  

(9)

B. NETWORK EFFICIENCY

A textbook result of circuit theory is that the power delivered by a given linear dc network to a resistive load is maximized when the load resistance \(R\) is equal to the network’s Thévenin’s equivalent resistance \(R_{th}\). Such power-optimal condition therefore does not depend on the internal details of the network, but solely upon \(R_{th}\), a terminal parameter.

In a similar manner, representation (1) of the internal dissipated power \(P_o\) allows, at this point, to answer the question of which value of \(R\) maximizes the efficiency, i.e. the quantity

\[ \eta \triangleq \frac{P_o}{P_o + P_h}, \quad \text{where} \quad P_o \triangleq V_oI_o. \]  

(10)

As proven in appendix, the general expression of \(\eta(R)\) is

\[ \eta(R) = V_{th}I_{nor} \times \frac{R_{th} \parallel R}{R_{th}R_{h,sc} + R_{h,oc}R_h}. \]  

(11)

Such expression is maximized when the load resistance is equal to

\[ R_{opt} = R_{th} \sqrt{\frac{P_{h,sc}}{P_{h,oc}}}, \]  

(12)

and the corresponding optimal efficiency \(\eta_{opt} \triangleq \eta(R_{opt})\) is given by

\[ \eta_{opt} = \frac{V_{th}I_{nor}}{(\sqrt{P_{h,oc} + P_{h,sc}})^2}. \]  

(13)

For comparison, consider the efficiency \(\eta_{po} \triangleq \eta(R_{th})\) corresponding to the power-optimal loading condition, obtained from (11) by letting \(R = R_{th}\),

\[ \eta_{po} = \frac{V_{th}I_{nor}}{2(P_{h,oc} + P_{h,sc})}. \]  

(14)

It is straightforward to verify that \(\eta_{po} \leq \eta_{opt}\), and that \(\eta_{opt} = \eta_{po}\) if and only if \(P_{h,oc} = P_{h,sc}\), in which case \(R_{opt} = R_{th}\): power maximization and efficiency maximization are equivalent only for networks that exhibit the same open-circuit and short-circuit dissipated powers, i.e. that satisfy (9). Also notice that (12) degenerates into trivial cases whenever \(P_{h,oc} = 0\) (e.g. a Thévenin generator) or \(P_{h,sc} = 0\) (e.g. a Norton generator). In these limit cases the output power \(P_o\) approaches zero along with the internal dissipated power \(P_h\), and the efficiency approaches unity as an indeterminate form of the type 0/0.

C. ON THE SUPERPOSITION OF THE DISSIPATED POWER DUE TO VOLTAGE AND CURRENT SOURCES

Being a nonlinear function of voltage and current, the power cannot be calculated by superposition of the effects of the generators present in the network. However, equations (3) express the fact that the total internal dissipated power of a resistive network is the superposition of the total power dissipation due to the voltage sources alone, and of the total power dissipation due to the current sources alone.

This statement, surprising as it may seem at first, simply means that the expression of \(P_h\) cannot contain “mixed products” of the type \(E_kI_h, E_k\) and \(I_h\) being a generic voltage and current source of the network. This observation is also consistent with the form of equations (1) and (2): mixed products \(-2V_{th,i}I_o\) and \(-2I_{nor,e}V_o\) express, respectively, the interaction between the output current and the current sources of the network, and between the output voltage and the voltage sources of the network. It is important to reiterate that these properties are predicated upon the assumption that the power dissipated by the network coincides with the power absorbed by the network’s resistors.

It will be shown in Section V that property (3) is rooted in one of the most powerful theorems in circuit and graph theory, the Tellegen’s theorem [6].

III. FORMULATION OF POWER-EQUIVALENT CIRCUITS

Having clarified the basic properties of \(P_h\), we now set on the problem of formulating a synthesis \(S\) of the original network which is both i) electrically equivalent and ii) power-equivalent. The first requirement simply means that the classical Thévenin’s equivalent circuit of \(S\) must coincide with the classical Thévenin’s equivalent circuit of the original network. The second requirement means that, in any load condition, the internal dissipated power of \(S\) must coincide with the internal dissipated power \(P_h\) of the original network, according to (1) and (2). Furthermore, the sought network \(S\) should also have the following properties:

P1 The dissipated power of \(S\) should be calculated consistently with the definition given in Section II, i.e. it should coincide with the total power dissipated by the resistors of \(S\);

P2 Independent voltage sources contained in \(S\) should, when acting alone and with \(S\) in open-circuit (or short-circuit) condition, reproduce the corresponding power dissipation \(P_{h,oc,E}\) (or \(P_{h,sc,E}\)) of the original network;
P3 Independent current sources contained in S should, when acting alone and with S in short-circuit (or open-circuit) condition, reproduce the corresponding power dissipation \( P_{h,SCJ} \) (or \( P_{h,OCJ} \)) of the original network.

An example of an electrically and power-equivalent synthesis S based on (1) which, however, does not satisfy the above properties, is reported in Fig. 3 (a dual formulation based on (2) is straightforward). In the network, the two resistors \( R_x \) and \( R_y \) are defined so that

\[
P_{h,OCJ} = \frac{V_{TH,J}^2}{R_x} \quad \text{and} \quad P_{h,OCJ} = \frac{V_{TH,J}^2}{R_y},
\]

while the interaction term \(-2V_{TH,J}I_o\) in (1) is modeled by the dependent current source \(-2I_o\) driven by the load current and placed in parallel with \( V_{TH,J} \). As a consequence, a correct calculation of \( P_h \) must include the power \(-2V_{TH,J}I_o\) absorbed by such generator. In light of our original assumptions on the network and definition of \( P_h \), the equivalent circuit of Fig. 3 is therefore not entirely satisfactory as it does not comply with P1. Furthermore, the open-circuit power dissipation due to the voltage sources and current sources in the original network are both represented by voltage sources in Fig. 3, which does not satisfy P2-P3.

Theoretical considerations aside, the simple circuit of Fig. 3 could nonetheless be used as an effective power-equivalent simulation model of a more complex network, with the above caveat in regard to the calculation of \( P_h \).

As a last comment on Fig. 3, notice that for networks solely composed by resistors and independent voltage generators one has \( V_{HI,J} = 0 \), and the circuit of Fig. 3 reduces to the one proposed in [5].

A different synthesis approach leads to a power-equivalent circuit only composed by independent generators and resistors and fully compliant with properties P1-P3. Start from (1) and express the total open-circuit dissipated power \( P_{h,OC} \) as its voltage source-related component \( P_{h,OCJ} \) plus its current source-related component \( P_{h,OCJ} \), according to (3). Furthermore, let

\[
I_o' \triangleq I_o - \frac{V_{HI,J}}{R_{HI}}.
\]

These substitutions lead to

\[
P_h = P_{h,OCJ} + \left( \frac{V_{HI,J}^2}{R_{HI}} \right) + R_{HI}I_o'^2,
\]

The three addends highlighted in the foregoing equation have the following interpretation:

- Term I is constant and represents the open-circuit dissipated power due to the voltage sources alone. This term can be represented as the power dissipated by a resistor \( R_{h,OCJ} \) when subject to a voltage equal to \( V_{TH,J} \), i.e. defined such that

\[
P_{h,OCJ} = \frac{V_{HI,J}^2}{R_{HI}}.
\]

- Term II is constant and represents the difference between \( i \) the open-circuit dissipated power due to the current sources alone, and \( ii \) the power dissipated by a resistor \( R_{HI} \) when traversed by a current equal to \( I_{SC-J} = V_{HI,J}/R_{HI} \). Term II can be equivalently thought of as the power dissipated by a resistor equal to

\[
R_{h,OCJ} - R_{HI}
\]

when traversed by a current equal to \( I_{SC-J} \), where \( R_{h,OCJ} \) is defined such that

\[
P_{h,OCJ} = R_{h,OCJ}I_{SC-J}^2.
\]

- Term III is load-dependent and represents the power dissipated by a resistor \( R_{HI} \) when traversed by a current equal to \( I_o' = I_o - V_{HI,J}/R_{HI} = I_o - I_{SC-J} \).

The above properties lead to the power-equivalent synthesis illustrated in Fig. 4a. The reader can verify that the classical Thévenin’s equivalent circuit of the network coincides with that of the original network. Furthermore, the total power dissipated by the resistors is equal, in every load condition, to \( P_h \) as given by (1). Notice that, once again, the synthesis reduces to the parallel \((V_{TH,J}, R_{h,OCJ})\) whenever the original network does not possess current sources [5].

A dual formulation of the power-equivalent synthesis is given in Fig. 4b. To highlight the duality with respect to Fig. 4a, conductance values are indicated in the circuit rather than resistances. In this regard, conductances \( G_{h,SCJ} \) and \( G_{h,SCJ} \) are defined so that

\[
P_{h,SCJ} = \frac{I_{SC-J}^2}{G_{h,SCJ}} \quad \text{and} \quad P_{h,SCJ} = G_{h,SCJ}V_{HI,J}^2.
\]

It can be verified that the two networks in Fig. 4b and Fig. 4a are both electrically and power-equivalent, in light of (8). Together, networks in Fig. 4 effectively represent power-equivalent generalizations of the classical Thévenin’s and Norton’s representations.
A. DISCUSSION

For definiteness, let us focus on the power-equivalent synthesis of Fig. 4a. Notice first that the synthesis exhibits a symmetry in the roles of voltage and current sources in the network: voltage sources are represented by the parallel \((V_{TH,E}, R_{h,OC,E})\), which can be regarded as an electrically ideal, but power-lossy voltage generator; in a dual manner, current sources in the network originate the series \((I_{NOR,J}, R_{h,OC,J} - R_{TH})\), which is an electrically ideal, but power-lossy current generator. When \(I_0 = 0\), these generators create internal circulating currents responsible for the open-circuit power dissipation \(P_{h,oc}\). Furthermore, the action of \(V_{TH,E}\) alone (i.e. when \(V_{TH,J} = 0\) and \(I_0 = 0\)) leads to the power dissipation \(P_{h,oc,E}\) due to the network’s voltage sources, and action of \(V_{TH,J}/R_{TH}\) alone (i.e. when \(V_{TH,E} = 0\) and \(I_0 = 0\)) leads to the power dissipation \(P_{h,oc,J}\) due to the network’s current sources.

Consider next the role of \(R_{TH}\). For a given load current \(I_0\), the voltage drop across \(R_{TH}\) in Fig. 4a is \(V_{TH,J} - R_{TH}I_0\), which summed to \(V_{TH,E}\) yields, as expected, \(V_0 = V_{TH,E} + V_{TH,J} - R_{TH}I_0 = V_{TH} - R_{TH}I_0\). At the same time, \(R_{TH}\) in Fig. 4a has an additional interpretation, in that \(R_{TH}\) models the power dissipated in the network in excess to \(P_{h,oc,J}\). Indeed, from (15) terms I and II represent, together, the value \(P_{h,oc,J}\) already expressed by (5). The third term,

\[ R_{TH}I_0^2 = R_{TH}(I_0 - I_{NOR,J})^2, \]

is precisely the power dissipated by \(R_{TH}\) in Fig. 4a. Such term is zero when the load current is \(I_0 = I_{NOR,J} = V_{TH,J}/R_{TH}\), i.e. when all the current produced by generator \(I_{NOR,J}\) is diverted to the output terminal by the load. In this condition the internal dissipated power is minimum and the output voltage equals \(V_0 = V_{TH,E}\), in agreement with (7). Any departure from such condition causes a nonzero current \(I_0\) to flow across \(R_{TH}\), generating an additional power dissipation term quadratic with \(I_0^2\). The larger \(R_{TH}\), the larger is the increase in power dissipation for a given \(I_0\). In geometrical terms, and referring to (1) as well as to Fig. 2, the Thévenin resistance determines the curvature of the network’s power dissipation.

In summary, \(R_{TH}\) is the resistance responsible for the output voltage drop when the network is loaded, and the resistance measuring how fast the power dissipation rises when one departs from \(P_{h,oc,J}\). Such twofold interpretation of \(R_{TH}\) enriches the physical meaning of this quantity.

One last remark concerns quantity (17). It is easy to show that such resistance cannot be negative. To see this, write (8) when voltage sources are shut off and therefore current sources act alone,

\[ P_{h,oc,J} - P_{h,sc,J} = \frac{V_{TH}^2}{R_{TH}} = R_{TH}I_{NOR,J}^2. \]

From definition (18), one readily finds that

\[ R_{h,oc,J} \geq R_{TH}. \]

A similar statement holds for conductance \(G_{h,sc,J} = G_{NOR}\) in Fig. 4b.

IV. CASE STUDIES

In this section two case studies are considered. In the first one, we consider a dc network simple enough so that analytical calculations can be easily developed and compared with the theoretical framework outlined so far. In the second example a more complex network is considered and analyzed via computer simulations.

A. CASE STUDY #1

Consider the network illustrated in Fig. 5, which consists of a voltage generator \(E\), a current generator \(J\), three resistances \(R_1, R_2\) and \(R_3\), plus the load resistance \(R\). Values of \(E, J, R_1, R_2\) and \(R_3\) are reported on the schematic.

For this network it is immediate to derive the classical Thévenin’s parameters,

\[ V_{TH} = \frac{R_2}{R_1 + R_2} E + \frac{R_1}{R_1 + R_2} J = 10 \text{ V}, \]
\[ I_{NOR} = \frac{E}{R_1} + J = 1.11 \text{ A}, \]
\[ R_{TH} = \frac{R_1}{R_1 + R_2} = 9 \Omega. \]

FIGURE 5. Network for case study #1.
Furthermore, calculation of the open-circuit voltage due to \( J \) alone yields

\[
V_{th,J} = (R_1 \parallel R_2) J = 9 \text{ V}
\]

\[
\Rightarrow I_{SOR,J} = \frac{V_{th,J}}{R_{th}} = J = 1 \text{ A}
\]

(20)

Let us also calculate the open-circuit and short-circuit dissipated powers,

\[
P_{h,oc} = \frac{E^2}{R_1 + R_2} + \frac{E^2}{R_3} + (R_1 \parallel R_2) J^2 = 10.20 \text{ W},
\]

\[
P_{h,sc} = \frac{E^2}{R_1} + \frac{E^2}{R_3} \approx 1.31 \text{ W}.
\]

Notice that, as commented in Section II-C, both \( P_{h,oc} \) and \( P_{h,sc} \) can be found by first calculating the open-circuit dissipated power when \( E \) acts alone, then calculating the effect of \( J \), and finally superimposing the contributions.

Calculation of the total dissipated power by \( R_1 \) and \( R_2 \), including the load current, yields

\[
P_h = P_{h,oc} - 2 (R_1 \parallel R_2) J I_o + (R_1 \parallel R_2) I_o^2,
\]

expression that can be compared with (1) in light of (20).

The power dissipation of the network as a function of the load current \( I_o \) has the plot already reported in Fig. 2, where the minimum dissipated power \( P_{h,min} \) occurs at \( I_o = I_{SOR,J} = 1 \text{ A} \), and equals

\[
P_{h,min} = P_{h,oc} - \frac{V_{th,J}^2}{R_{th}} = 1.20 \text{ W}.
\]

Consider next the network loaded with the power-optimal resistance \( R = R_{th} = 9 \text{ Ω} \). In this condition, load current, load power, power dissipation and system efficiency are

\[
I_o = \frac{V_{th}}{2R_{th}} \approx 0.56 \text{ mA},
\]

\[
P_o = \frac{V_{th}^2}{4R_{th}} \approx 2.78 \text{ W},
\]

\[
P_h(I_o) \approx 2.98 \text{ W},
\]

\[
\eta = \frac{P_o}{P_h} \approx 48%.
\]

Observe that the efficiency in the power-optimal condition is not equal to 50%, a conclusion sometimes erroneously drawn based on the Thévenin’s equivalent circuit of the network. As a matter of fact, the above value \( \eta \approx 48\% \) is the one predicted by (14), as it can be easily verified via direct substitution of the numerical values of \( P_{h,oc} \) and \( P_{h,sc} \).

By contrast, the efficiency-optimal load (12) is

\[
R_{opt} = R_{th} \sqrt{\frac{P_{h,sc}}{P_{h,oc}}} \approx 3.23 \text{ Ω},
\]

which yields

\[
I_o = \frac{V_{th}}{R_{th} + R_{opt}} \approx 0.82 \text{ mA},
\]

\[
P_o = R_{opt} \frac{V_{th}^2}{(R_{th} + R_{opt})^2} \approx 2.16 \text{ W},
\]

\[
P_h(I_o) \approx 1.50 \text{ W},
\]

\[
\eta = \frac{P_o}{P_h} \approx 59%.
\]

It can be verified that the above efficiency value, calculated directly from \( P_o \) and \( P_h \), numerically coincides with the one predicted by the general result (13).

B. CASE STUDY #2

A second case study, illustrated in Fig. 6, is treated via numerical simulations. First, the following parameters are extracted via a set of dc operating point simulations,

\[
V_{th} = 47.88 \text{ V}, \quad I_{SOR} = 65.08 \text{ mA},
\]

\[
V_{th,J} = 5.97 \text{ V}, \quad I_{SOR,J} = 8.11 \text{ mA},
\]

\[
V_{th,E} = 41.91 \text{ V}, \quad I_{SOR,E} = 56.97 \text{ mA},
\]

\[
R_{th} = 735.65 \text{ Ω}, \quad (21)
\]

\[
P_{h,oc} = 4.17 \text{ W}, \quad P_{h,sc} = 1.83 \text{ W},
\]

\[
P_{h,oc,J} = 80.37 \text{ mW}, \quad P_{h,sc,J} = 128.77 \text{ mW},
\]

\[
P_{h,oc,E} = 4.09 \text{ W}, \quad P_{h,sc,E} = 1.70 \text{ W}.
\]

Notice again that \( P_{h,oc} = P_{h,oc,E} + P_{h,oc,J} \), and \( P_{h,sc} = P_{h,sc,E} + P_{h,sc,J} \).
FIGURE 7. Case study #2: comparison between the simulated power dissipation profile \( P_h(I_o) \) of the original network (thick blue line) and of the proposed power-equivalent synthesis (dashed white line).

Next, the power-equivalent network of Fig. 4a is constructed, with

\[ R_{h,OC-E} = \frac{V_{TH-E}^2}{P_{h,OC-E}} \approx 443 \Omega, \]
\[ R_{h,OC-J} = \frac{P_{h,OC-J}}{I_{NOR-J}^2} \approx 1.26 \text{k}\Omega, \]

according to (16) and (18). Observe that definition of the power-equivalent network only requires the characterization of \( P_{h,OC-E}, P_{h,OC-J}, V_{TH-E}, I_{NOR-J} \) and \( R_{TH} \). The other parameters in (21) have been reported for completeness.

To check the equivalence between the synthesized network and the original one, both are simulated via a dc sweep performed on the load current \( I_o \) in the 0 to 200 mA range. A comparison of the power dissipation profile of the original network and of the power-equivalent synthesis, reported in Fig. 7, confirms that the two profiles are identical. Also observe that, in agreement with (5), the minimum power dissipation equals

\[ P_{h,\text{min}} = P_{h,OC} - \frac{V_{TH}^2}{R_{TH}} \approx 1.78 \text{W at } I_o = I_{NOR-J}. \]

The original network and the power-equivalent synthesis are then simulated with a resistive load \( R \). In the simulation, the value of \( R \) is swept from 0 to 1k\Omega, and the efficiency \( \eta(R) \) of the two networks recorded. The comparison reported in Fig. 8 confirms that the proposed synthesis is by all means equivalent to the original network in predicting the circuit efficiency. Furthermore, in agreement with (12) and (13), the optimal load resistance and corresponding efficiency are \( R_{\text{OPT}} = \frac{P_{h,SC}}{P_{h,OC}} \approx 487 \Omega, \)
\[ \eta_{\text{OPT}} = \frac{V_{TH}I_{NOR}}{(\sqrt{P_{h,OC}} + \sqrt{P_{h,SC}})^2} \approx 0.27\%. \]

V. PROOF OF THE EXPRESSION OF THE POWER DISSIPATION \( P_h \)

This section reports the mathematical proof of the expression of \( P_h \) as given by (1). Expression (2) can be derived in a similar fashion following duality principles.

Consider a linear dc network composed by resistors and a mixture of independent voltage and current sources. With no loss in generality, one can think of the network branches as belonging to one of two types shown in Fig. 9, i.e. the \( k \)th branch is either an ideal voltage source \( E_k \) in series with a resistance \( R_k \) (\( E \)-branch), or an ideal current source \( J_k \) in parallel with a conductance \( G_k \) (\( J \)-branch). Observe that purely resistive branches can be thought as being either of the two types with \( E_k = 0 \) or \( J_k = 0 \). Furthermore, cases \( R_k = 0 \) or \( G_k = 0 \) yield branches consisting of ideal voltage or current sources respectively. Also let

\[ V_k \text{ and } I_k \]

be the total voltage across the \( k \)th branch, and the current through it, as indicated in Fig. 9. Also, notice from the figure that all internal branches adopt the generator (or active-sign) convention. As for the load, we shall model it as an

FIGURE 8. Case study #2: comparison between the simulated efficiency \( \eta(R) \) of the original network (thick blue line) and of the proposed power-equivalent synthesis (dashed white line).

FIGURE 9. Types of internal branches in a resistive network.
ideal current source \(I_o\) and adopt the load (or passive-sign) convention as already done in Fig. 1.

The total internal power dissipation \(P_h\) in the network is, by definition, the summation of the power dissipated in all resistances and conductances,

\[
P_h \triangleq \sum_{E} R_k I_k^2 + \sum_{J} G_k V_k^2,
\]

(22)

where the first summation symbol means “summation extended over all the \(E\)-branches”, while the second summation symbol means “summation extended over all the \(J\)-branches”. This notation will be heavily used in the following, along with symbol

\[
\sum_{E+J}
\]

which will denote any summation carried out over all internal branches, excluding the load branch.

For a given value of \(I_o\), current \(I_k\) through the \(k\)th \(E\)-branch is the superposition of an internal contribution \(I_{k,\text{INT}}\), due to the effect of all internal sources \(E_k\) and \(J_k\) but with \(I_o = 0\), and an external contribution \(I_{k,\text{EXT}}\) due to \(I_o\) alone, i.e. when \(E_k = J_k = 0\) \(\forall k\),

\[
I_k = I_{k,\text{INT}} + I_{k,\text{EXT}}.
\]

The external contribution can always be written as

\[
I_{k,\text{EXT}} = a_{ok} I_o,
\]

where coefficients \(a_{ok}\) represent the current gains between \(I_o\) and the generic \(E\)-branch current, when all the network’s internal sources are set to zero.

In a similar manner, voltage \(V_k\) across the \(k\)th \(J\)-branch writes as

\[
V_k = V_{k,\text{INT}} + V_{k,\text{EXT}},
\]

where \(V_{k,\text{INT}}\) denotes the contribution due to the internal sources and with \(I_o = 0\), while

\[
V_{k,\text{EXT}} = r_{ok} I_o
\]

is the contribution of the load current. Coefficients \(r_{ok}\) represent the transresistance gains between \(I_o\) and the generic \(J\)-branch voltage when all internal sources are set to zero.

With the foregoing notation, (22) rewrites as

\[
P_h = \left(\sum_{E} R_k I_{k,\text{INT}}^2 + \sum_{J} G_k V_{k,\text{INT}}^2\right)
+ 2 \left(\sum_{E} R_k a_{ok} I_{k,\text{INT}} + \sum_{J} G_k r_{ok} V_{k,\text{INT}}\right) I_o
+ \left(\sum_{E} R_k a_{ok}^2 + \sum_{J} G_k r_{ok}^2\right) I_o^2.
\]

(23)

Notice that the general form (1) has already appeared, in that \(P_h\) writes as a constant term, plus a term linear with \(I_o\), plus a term quadratic with \(I_o\). The three coefficients of \(I_o^0\), \(I_o\) and \(I_o^2\) are treated separately in Sections V-A, V-B and V-C.

Before proceeding, it is necessary to briefly recall one of the most important theorem’s in circuit theory, Tellegen’s theorem. Consider a set of branch voltages \(\{V_k'\}\) compatible with the network’s topology, i.e. satisfying Kirchhoff’s voltage law for that network. Consider further a set of branch currents \(\{I_k''\}\) satisfying Kirchhoff’s current law for the same network. Assume also that all the pairs \((V_k', I_k'')\) are uniformly defined according to, say, the active-sign convention. Tellegen’s theorem states that [6]

\[
\sum_{k} V_k' I_k'' = 0,
\]

(24)

where the summation is extended to all the branches in the network, including the load branch. It is crucial to realize that the \(\{V_k'\}\)’s and the \(\{I_k''\}\)’s do not need to simultaneously solve the network, i.e. \(\{V_k', I_k''\}\) does not have to be a solution for the network. It is only required that the two sets separately solve the network’s Kirchhoff’s laws.

In the following proof, summation (24) will be always decomposed into a part pertaining the internal branches plus an addend pertaining to the load. In this case, Tellegen’s theorem writes

\[
\sum_{E+J} V_k' I_{k,\text{INT}}'' - V_o' I_o'' = 0,
\]

(25)

where the minus sign is due to the load branch adopting the passive-sign convention, rather than the active one. Any summation of the form (24) or (25) will be here referred to as a Tellegen summation.

A. CONSTANT TERM

The constant term in (23) clearly represents the internal dissipated power in open-circuit condition. It will now be shown that this term can be written as the superposition of the total internal open-circuit power \(P_{h,\text{OC}}\) dissipated by action of the voltage sources alone, plus the total internal open-circuit power \(P_{h,\text{OC-J}}\) dissipated by action of the current sources alone.

To this end, consider the network in open-circuit condition, and write the internal current \(I_{k,\text{INT}}\) through the \(k\)th \(E\)-branch as the superposition of the effect \(I_{k,\text{INT-E}}\) of the voltage sources, and the effect \(I_{k,\text{INT-J}}\) of the current sources,

\[
I_{k,\text{INT}} = I_{k,\text{INT-E}} + I_{k,\text{INT-J}},
\]

so that

\[
\sum_{E} R_k I_{k,\text{INT}}^2 = \sum_{E} R_k (I_{k,\text{INT-E}} + I_{k,\text{INT-J}})^2
= \sum_{E} R_k I_{k,\text{INT-E}}^2 + \sum_{E} R_k I_{k,\text{INT-J}}^2
+ 2 \sum_{E} R_k I_{k,\text{INT-E}} I_{k,\text{INT-J}}.
\]

(26)

In a similar way, write the internal voltage \(V_{k,\text{INT}}\) across the \(k\)th \(J\)-branch as the superposition of the effects of the voltage sources and current sources in the network,

\[
V_{k,\text{INT}} = V_{k,\text{INT-E}} + V_{k,\text{INT-J}},
\]

...
so that
\[ \sum_j G_k V_{k,int}^2 = \sum_j G_k (V_{k,int} + V_{k,ext})^2 = \sum_j G_k V_{k,int}^2 + \sum_j G_k V_{k,ext}^2 + 2 \sum_j G_k V_{k,int} V_{k,ext} \] (27)

Consider now the last terms in (26) and (27), and notice that
\[ R_k I_{k,int} I_{k,int} = (R_k I_{k,int}) I_{k,int} = (-V_{k,int}) I_{k,int} \]
where \( V_{k,int} \) represents the internal voltage across the \( k \)th \( E \)-branch due to the current sources alone. Observe that, being all voltage sources shut off, \( V_{k,int} \) indeed coincides with the total voltage across \( R_k \) in the \( k \)th \( E \)-branch. Furthermore,
\[ G_k V_{k,int} V_{k,int} = (G_k V_{k,int}) V_{k,int} = (-I_{k,int}) V_{k,int} \]
where \( I_{k,int} \) represents the internal current through the \( k \)th \( J \)-branch due to the voltage sources alone. Observe that, being all current sources shut off, \( I_{k,int} \) indeed coincides with the total current through \( G_k \) in the \( k \)th \( J \)-branch.

In light of these considerations, let us sum (26) and (27) in order to express the total open-circuit dissipated power,
\[ P_{h,oc} = \sum_E R_k^2 I_{k,int}^2 + \sum_j G_k V_{k,int}^2 + \sum_E R_k^2 I_{k,int}^2 + \sum_j G_k V_{k,int}^2 - 2 \sum_{E+j} V_{k,int} I_{k,int} \times 0 \]

The first two terms highlighted by underbrackets represent, respectively, the open-circuit dissipated power due to the voltage sources alone, and to the current sources alone. As for the third term, it vanishes because the highlighted summation can be completed to a Tellegen one by the null product \( V_{1h,i} \times 0 \),
\[ \sum_{E+j} V_{k,int} I_{k,int} - V_{1h,i} \times 0 = 0. \]

In summary,
\[ P_{h,oc} = P_{h,oc,E} + P_{h,oc,J}. \] (28)

As a final remark, we note that the superposition of the effects of voltage sources and current sources in determining the power dissipation holds \textit{globally}, but cannot be stated for each resistor in the network individually. For instance, the total power dissipation of resistor \( R_k \) belonging to the \( k \)th \( E \)-branch is
\[ R_k^2 I_{k,int}^2 = R_k \left(I_{k,int} + I_{k,ext}\right)^2 \neq R_k \left(I_{k,int}^2 + I_{k,ext}^2\right). \]

**B. LINEAR TERM**

Consider first the generic term \( R_k \alpha ok I_{k,int} I_{k,int} \) pertaining the \( k \)th \( E \)-branch, and write the voltage drop \( R_k I_{k,ext} \) across \( R_k \) as \( E_k - V_{k,ext} \), where \( V_{k,ext} \) represents the total voltage across the \( k \)th \( E \)-branch due to the internal sources only \((I_0 = 0)\),
\[ R_k \alpha ok I_{k,int} I_{k,int} = \left(R_k I_{k,int}\right) \alpha ok I_{k,int} = (E_k - V_{k,ext}) \alpha ok I_{k,int}. \]

Similarly, the generic term pertaining the \( k \)th \( J \)-branch writes as
\[ G_k \alpha ok V_{k,int} I_{k,int} = (G_k V_{k,int}) \alpha ok I_{k,int} = (J_k - I_{k,ext}) \alpha ok I_{k,int}. \]

Recall, at this point, that current gains \( \alpha ok \) are, by network reciprocity, equal to \( \beta ko \), where \( \beta ko \) represent the open-circuit voltage gains between \( E_k \) and the network’s output voltage \( V_o \). By the same token, transresistance gains \( r_k \) are equal to the opposite of the open-circuit transresistance gains \( r_o \) from \( J_k \) to \( V_o \). Therefore,
\[ E_k \alpha ok I_{o} = \beta ko E_k I_{o} \quad \text{and} \quad J_k r_o I_{o} = -r_k o I_{o}. \]

The corresponding summations become
\[ \sum E_k \alpha ok I_{o} = \sum \beta ko E_k I_{o} = V_{1h,l} I_{o} \]
\[ \sum J_k r_o I_{o} = -\sum r_k o J_k I_{o} = -V_{1h,l} I_{o} \]
where \( V_{1h,l} \) and \( V_{1h,l} \) are the same quantities introduced earlier in the paper, i.e. the open-circuit voltages of the network respectively due to the voltage sources alone and to the current sources alone.

Going back to (29) and now focusing on the last two summations, one has that, by very definition of coefficients \( \alpha ok \) and \( r_k \),
\[ \alpha ok I_{o} = I_{k,ext} \quad \text{and} \quad r_k o I_{o} = V_{k,ext}. \]

Summation (29) then assumes the form
\[ V_{1h,l} I_{o} - V_{1h,l} I_{o} = \sum E_k I_{k,ext} - \sum J_k I_{k,ext}. \] (30)

Let us show that
\[ V_{1h,l} I_{o} = \sum E_k I_{k,ext} + \sum J_k I_{k,ext}. \] (31)

\(^2\text{Recall that the load branch adopts a passive-sign convention, unlike all the internal branches of the network. If we chose to adopt the active-sign convention for the load as well, the reciprocity relations would be } \alpha ok = -\beta ko \text{ and } r_k = r_k o, \text{ the form mostly found in circuit theory textbooks.}\)
To this end, write the term \( I_{k,\text{ext}} V_{k,\text{ext}} \) associated with the \( k \)th \( J \)-branch as follows,
\[
I_{k,\text{ext}} V_{k,\text{ext}} = (J_k - G_k V_{k,\text{int}}) V_{k,\text{ext}}
\]
\[
= J_k V_{k,\text{ext}} - G_k V_{k,\text{int}} V_{k,\text{ext}}
\]
\[
= J_k r_{ok} I_o - G_k V_{k,\text{int}} V_{k,\text{ext}}.
\]
In the last expression, the term
\[
J_k r_{ok} I_o = -r_{ko} J_k I_o
\]
is nothing but the contribution of generator \( J_k \) to the open-circuit voltage, with the sign changed. Therefore
\[
\sum_j J_k r_{ok} I_o = - \sum_j r_{ko} J_k I_o = -V_{th} I_o.
\]
As for the term \( G_k V_{k,\text{int}} V_{k,\text{ext}} \), one has
\[
G_k V_{k,\text{int}} V_{k,\text{ext}} = (G_k V_{k,\text{ext}}) V_{k,\text{int}} = (-I_{k,\text{ext}}) V_{k,\text{int}}.
\]
Therefore, one has that
\[
\sum_E V_{k,\text{int}} I_{k,\text{ext}} + \sum_j I_{k,\text{int}} V_{k,\text{ext}} = \sum_E V_{k,\text{int}} V_{k,\text{ext}} - V_{th} I_o + \sum_j V_{k,\text{int}} I_{k,\text{ext}}
\]
\[
= -V_{th} I_o + \sum_{E+j} V_{k,\text{int}} I_{k,\text{ext}}
\]
Expression
\[
\sum_{E+j} V_{k,\text{int}} I_{k,\text{ext}}
\]
can be completed to a Tellegen summation by the product \( V_{th} I_o \),
\[
\sum_{E+j} V_{k,\text{int}} V_{k,\text{ext}} - V_{th} I_o = 0,
\]
and therefore
\[
\sum_E V_{k,\text{int}} I_{k,\text{ext}} + \sum_j I_{k,\text{int}} V_{k,\text{ext}} = -V_{th} I_o + V_{th} I_o = V_{th} I_o,
\]
which proves (31). Going back to (30), only the \(-V_{th} I_o\) addend remains, which finally proves that
\[
\left( \sum_E R_k \alpha_{ok} I_{k,\text{int}} + \sum_j G_k r_{ok} V_{k,\text{int}} \right) I_o = -V_{th} I_o.
\]
C. QUADRATIC TERM
In the quadratic term
\[
\left( \sum_E R_k \alpha_{ok}^2 + \sum_j G_k r_{ok}^2 \right) I_o^2
\]
consider first the products \( R_k \alpha_{ok}^2 I_o^2 \) appearing in the \( E \)-branch summation. One has
\[
R_k \alpha_{ok}^2 I_o^2 = (R_k \alpha_{ok} I_o) (\alpha_{ok} I_o)
\]
\[
= (R_k I_{k,\text{ext}}) (I_{k,\text{ext}})
\]
\[
= (-V_{k,\text{ext}}) I_{k,\text{ext}}
\]
In a similar manner, in regard to products \( G_k r_{ok}^2 I_o^2 \) pertaining the \( J \)-branch summation,
\[
G_k r_{ok}^2 I_o^2 = (G_k r_{ok} I_o) (r_{ok} I_o)
\]
\[
= (G_k V_{k,\text{ext}}) (V_{k,\text{ext}})
\]
\[
= (-I_{k,\text{ext}}) V_{k,\text{ext}}
\]
Therefore
\[
\left( \sum_E R_k \alpha_{ok}^2 + \sum_j G_k r_{ok}^2 \right) I_o^2
\]
\[
= - \sum_E V_{k,\text{ext}} I_{k,\text{ext}} - J \sum_j V_{k,\text{ext}} I_{k,\text{ext}}
\]
\[
= - \sum_{E+j} V_{k,\text{ext}} I_{k,\text{ext}}
\]
The last expression can be completed to a Tellegen summation by the product between the voltage appearing at the output of the network due to \( I_o \) when all internal sources are set to zero, and \( I_o \) itself. This product is nothing but \((-R_{th} I_o) I_o\), and therefore
\[
\sum_{E+j} V_{k,\text{ext}} I_{k,\text{ext}} - (-R_{th} I_o^2) = 0
\]
\[
\Rightarrow - \sum_{E+j} V_{k,\text{ext}} I_{k,\text{ext}} = R_{th} I_o^2,
\]
hence the result.

VI. CONCLUSION
This work discloses a general expression of the internal power \( P_h \) dissipated by a linear dc network composed by resistors and a mixture of independent voltage and current generators. The result provides a theoretical link between the value of \( P_h \) in a generic load condition and key open-circuit (or short-circuit) parameters of the network itself, bringing renewed insights on the power dissipation properties of linear resistive networks.

The basic properties of \( P_h \) are discussed. It is recognized that presence of current sources in the network poses a nonzero lower limit to the open-circuit power dissipation, and in a dual fashion presence of voltage sources poses a nonzero lower limit to the short-circuit power dissipation.

The general expression of the network’s efficiency versus load resistance is disclosed, along with the expression of the efficiency-optimal load resistance. Closed form results for the efficiency under efficiency-optimal and power-optimal loading conditions are also reported and compared.

The above theoretical results are then employed to formulate a class of circuits which can be regarded as power-equivalent generalizations of Thévenin’s and Norton’s representations. Examination of these circuits reveals the role of voltage and current sources in the network in creating internal lossy circulating currents responsible for open-circuit and short-circuit dissipation, and provides a direct interpretation of the general fact that internal power losses are minimized at nonzero load current. Furthermore, a novel, power-related
physical meaning of the network’s Thévenin equivalent resistance is revealed.

Results are validated via case studies treated analytically and via computer simulations.

**APPENDIX**

**DERIVATION OF THE NETWORK’S EFFICIENCY**

To derive (11) and, consequently, (13) and (14), it is useful to first re-express (8) in a normalized form. To this end, write

\[
P_{h,oc} - P_{h,sc} = \frac{V_{ih,j}^2 - V_{ih,e}^2}{R_{th}} = \left(\frac{V_{ih,j} + V_{ih,e}}{R_{th}}\right)\left(\frac{V_{ih,j} - V_{ih,e}}{R_{th}}\right)
\]

where

\[
\pi_{h,oc} - \pi_{h,sc} = v_{ih,j} - v_{ih,e}.
\]

By dividing both members by \(V_{ih}^2/R_{th}\) one has the normalized relationship

\[
\pi_{h,oc} - \pi_{h,sc} = v_{ih,j} - v_{ih,e}.
\]

Consider now a resistive dc network loaded by a resistance \(R\). The load current and output power are, respectively,

\[
I_o = \frac{V_{ih}}{R_{th} + R} \quad \text{and} \quad P_o = R_{th}\frac{V_{ih}^2}{(R_{th} + R)^2}.
\]

Efficiency \(\eta\) of the network is defined as (10). The numerator \(P_o\) can be rewritten as

\[
P_o = \frac{V_{ih}^2}{R_{th}} \times \frac{R_{th}R}{(R_{th} + R)^2}.
\]

As for the denominator \(P_o + P_h\), one has

\[
P_o + P_h = R\frac{V_{ih}^2}{(R_{th} + R)^2} + P_{h,oc} - 2V_{ih,j}\frac{V_{ih}}{R_{th} + R} + R_{th}\frac{V_{ih}^2}{(R_{th} + R)^2} + \frac{V_{ih}^2}{R_{th}}\left[\frac{R_{th}R}{(R_{th} + R)^2} - \frac{V_{ih}^2}{R_{ih}}\frac{R_{th}R}{(R_{th} + R)^2} + \frac{R_{th}^2}{(R_{th} + R)^2}\right].
\]

expression which can be further manipulated as follows,

\[
P_o + P_h = \frac{V_{ih}^2}{R_{th}}\left[\frac{P_{h,oc}}{V_{ih}} + \frac{R_{th}}{R_{th} + R} - \frac{2V_{ih,j}R_{th}}{R_{th} + R}\right].
\]

\[
= \frac{V_{ih}^2}{R_{th}}\left[\frac{\pi_{h,oc} + \frac{R_{th}}{R_{th} + R} + \frac{2V_{ih,j}R_{th}}{V_{ih}}}{R_{th} + R} \right].
\]

where (33) is used in the next-to-last step.

From (35) and (37), the efficiency is rewritten as

\[
\eta = \frac{R_{th}R}{R_{th} + R} + R_{th}\frac{\pi_{h,oc}}{R_{th} + R} - \frac{\pi_{h,oc}}{R_{th} + R} + \frac{R_{th}}{R_{th} + R}
\]

The un-normalized result (11) directly follows by noting that \(V_{ih}^2/R_{th} = V_{ih}I_{nor}\).

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