Abstract

The lexicographic order, being a total order, has found numerous applications in optimization problems requiring selection of the best solution with respect to some preference criteria. We study lexicographically maximal and minimal, under different permutations, integer points in a compact convex set $X$. New relationships are established between $X \cap \mathbb{Z}^n$ and discrete sets defined using the lex order and the lex optimal points in $X$. First, we give a necessary and sufficient condition for a knapsack polytope to be equivalent to a lex-ordered set. This generalizes a sufficient condition from literature. Then we show that if $X \subseteq [0,1]^n$ and $X$ is down-monotone, or more generally, $X \cap \{0,1\}^n$ is an independence system, then $X \cap \{0,1\}^n$ is equal to the intersection of all the lex-ordered sets corresponding to $X$. This implies that facet-defining inequalities can be obtained for the convex hull of $X \cap \{0,1\}^n$ by studying the combinatorial interplay between different lex sets. We further show that for any $X \subseteq [0,1]^n$ and $c \geq 0$, there is some lex optimal point that maximizes $c^\top x$ over $X \cap \mathbb{Z}^n$, thereby providing a kind of a greedy characterization for $0\backslash 1$ optimization over $X$. For general discrete sets, we argue that lex optimal points yield a arbitrarily tight approximation factor of $1/n$ to the integer optimum. The many applications of lex optima and our results on their structural properties motivate the question of how hard it is to compute these points for a compact convex $X$, assuming enumeration over $X \cap \mathbb{Z}^n$ is not easy. In the second half, we give a finitely convergent cutting plane algorithm to find lex optima. Then we identify P and NP-hard cases of the complexity question using the complexity of integer feasibility of $X$ with respect to arbitrary boxes. In particular, we show that lex optimization is NP-hard even if feasibility of $X$ with respect to the trivial box is in P, and on the contrary, if the feasibility of $X$ is in P for any box, then lex optimization is in P. If $X$ is defined using arbitrarily many lex orders, then the complexity remains NP-hard. Finally, we propose the question of finding the largest power of 2 such that the binary encoding of some integer larger than this power satisfies arbitrarily many lex orders. This problem is shown to be NP-hard to approximate, assuming $P \neq NP$, within a factor of 2.402, or more generally, within $(\sqrt{n+1} - 1)^{1-\epsilon}$ for any $\epsilon > 0$.

Keywords. Lexicographic optimality, Convex hull, 0\1 polytopes, Computational complexity.

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1 Introduction

The lexicographic (abbreviated as lex) order $\preceq$ is a term order widely used for comparing two monomials in a polynomial ring. It is also thought of as a translation invariant total order on the reals in the following sense. For any $x, y \in \mathbb{R}^n$, $x$ is lexicographically less than equal to $y$, written

\begin{itemize}
  \item $x \preceq y$ if $x_i \leq y_i$ for all $i$ and $x_n < y_n$;
  \item $x \prec y$ if $x_i < y_i$ for some $i$ and $x_n = y_n$;
  \item $x \succeq y$ if $x \preceq y$ and $x \neq y$;
  \item $x \succ y$ if $x \preceq y$ and $x \neq y$.
\end{itemize}
as $x \preceq y$, if either $x = y$ or there is some $i$ such that $x_i < y_i$ and $x_k = y_k$ for all $k > i$. (Since ties are broken in the right-to-left order, instead of the left-to-right order, of variables, our definition of $\preceq$ is sometimes referred to as the reverse lexicographic, or revlex, order, but this is immaterial.)

The set $\{x \in \mathbb{R}^n \mid x \preceq \theta\}$ of all vectors $\preceq$-smaller than a fixed $\theta \in \mathbb{R}^n$ is a convex set that is neither open nor closed. This definition of $\preceq$ is with respect to the identity permutation $id$. In general, if $S_n$ denotes the set of all permutations of $[n] := \{1, \ldots, n\}$, then for any $\sigma \in S_n$ we say $x \preceq_{\sigma} y$ if either $x = y$ or there is some $i$ such that $x_{\sigma(i)} < y_{\sigma(i)}$ and $x_{\sigma(k)} = y_{\sigma(k)}$ for all $k > i$. Thus if $P_{\sigma}$ is the permutation matrix corresponding to $\sigma$, then $x \preceq_{\sigma} y$ if and only if $P_{\sigma}x \preceq P_{\sigma}y$. Throughout this paper, we will omit $\sigma$ in the subscripts if $\sigma = id$.

We are interested in the lex order over integers. Let $B := \{x \in \mathbb{R}^n \mid l \leq x \leq u\}$ be a box for some $l, u \in \mathbb{Z}^n$, and $X \subseteq B$ be a compact convex set. For nontriviality, we assume that integer enumeration over $X$ is not easy. Our emphasis is on the following lexicographic optimization problem:

$$\text{LEX}_{\sigma}(X) : \max_{\preceq_{\sigma}} x \quad \text{s.t.} \quad x \in X \cap \mathbb{Z}^n.$$ (1)

If $X \cap \mathbb{Z}^n \neq \emptyset$, this lexmax problem finds an integral $x^* \in X$ such that $x \preceq_{\sigma} x^*$ for all $x \in (X \setminus \{x^*\}) \cap \mathbb{Z}^n$. For $X \cap \mathbb{Z}^n \neq \emptyset$, such an $x^*$ exists because $X$ is compact and is unique because $\preceq_{\sigma}$ is a total order. We will denote this unique optimal solution by $\max \text{LEX}_{\sigma}(X)$. Replacing the maximization in (1) with minimization leads to the lexmin problem, whose unique optimal solution we denote by $\min \text{LEX}_{\sigma}(X)$. The fact that the lex order reverses after complementing variables, just like the usual partial order $\leq$ on $\mathbb{R}^n$, makes it obvious that

$$\min \text{LEX}_{\sigma}(X) = u - \max \text{LEX}_{\sigma}(u - X),$$ (2)

where $u - X := \{u - x \mid x \in X\}$. Given the above relationship, we will concentrate our efforts mostly on the maximization version of $\text{LEX}_{\sigma}(X)$. Since $x \leq y$ implies $x \preceq_{\sigma} y$ for all $\sigma \in S_n$, finding $\max \text{LEX}_{\sigma}(X)$ is trivial if there exists $x' \in X \cap \mathbb{Z}^n$ such that $X \subseteq x' - \mathbb{R}^n_{\geq 0}$; in that case $\max \text{LEX}_{\sigma}(X) = x'$. Similarly for $\min \text{LEX}_{\sigma}(X)$.

The lex order has been commonly used in fair allocation problems in decision theory [FG86; Fis74; OS07], computing nucleolus of cooperative games [Koh71; MPS79], combinatorial optimization [BR91; Fuj80; Ogr97], network flows [HT94; Meg74], and multiobjective optimization [Ehr05; MO92; OS03], to name a few areas, for selecting the best solution with respect to some preference criteria. It also finds use in breaking symmetry amongst optimal solutions of an integer program [KP08]. All of the above applications involve solving a problem of the form $\text{LEX}_{\sigma}(X)$ for some given permutation $\sigma$. The most common way of solving this problem has been to apply the definition of $\preceq$, which makes it obvious that $\max \text{LEX}_{\sigma}(X) = x^*$, where $x^*$ is found by a backward recursion: for $i = n, \ldots, 1$,

$$x^*_{\sigma(i)} := \arg \max \{x_{\sigma(i)} \mid x \in X \cap \mathbb{Z}^n, x_{\sigma(k)} = x^*_{\sigma(k)} \forall k > i\}.$$ (3)

The lexmax point can be found similarly. This greedy procedure requires solving an integer program at each step. Although this is tractable when $X$ is defined by a single linear inequality or when $X$ is a packing- or a covering-type polytope, the general case seems far from trivial. Thus a comprehensive complexity analysis of $\text{LEX}_{\sigma}(X)$ is interesting from a theoretical standpoint, and also because it may lend insight into the development of solution algorithms that are more sophisticated than the greedy recursion in (3).

A second motivation for studying lex optimal points is that they can capture the discrete structure of $X \cap \mathbb{Z}^n$. Gillmann and Kaibel [GK06] observed that the $0\backslash1$ superincreasing knapsack set considered by Laurent and Sassano [LS92] is equal to the set of all $0\backslash1$ vectors lexicographically lesser than or equal to some fixed $0\backslash1$ vector, and used this connection to study the graph and extremal
properties of such polytopes. Muldoon, Adams, and Sherali [MAS13] gave a facet characterization for 0|1 polytopes defined by one $\preceq$ and one $\succeq$ constraint. Gupte [Gup16] extended the polyhedral results of [LS92; MAS13] to the general integer case. These polyhedral studies have been helpful in deriving strong cutting planes for mixed-integer optimization problems [Gup+13] and also spawned investigation of polytopes arising from other monomial orders [GP16]. Besides the above papers, we don’t know of any other work that draws a connection between lex optimal points and $X \cap \mathbb{Z}^n$, for arbitrary $X$. One of the main contributions of this work is to fill this void. We briefly mention here though that apart from providing an equivalent representation of superincreasing knapsacks, the lex order also has a connection with arbitrary 0\1 knapsack polytopes. For $X = \{x \in [0, 1]^n \mid ax \leq a_0\}$, the cover inequality $\sum_{j \in C} x_j \leq |C| - 1$ is valid to $X \cap \{0, 1\}^n$ when $C$ is a minimal subset of $[n]$ such that $\sum_{j \in C} a_j > a_0$. Observe that

$$\{x \in \{0, 1\}^n \mid \sum_{j \in C} x_j \leq |C| - 1\} = \{x \in \{0, 1\}^n \mid x \preceq_\sigma (1, \ldots, 1, 0, 1, \ldots, 1)\},$$

where $\sigma$ is a permutation that orders the elements of $C$ at the end. The set on the right hand side is a lex-ordered set since it is the set of all integer vectors lex-less than a fixed vector. Thus a cover is equivalent to a lex-ordered set. It is well-known that all the minimal cover inequalities are sufficient to reformulate a knapsack. Therefore, identity (4) tells us that the knapsack is equal to the intersection of all the lex-ordered sets, where the intersection is taken over permutations corresponding to minimal covers. We will show that a more general family of subsets of $\{0, 1\}^n$ is equivalent to the intersection of suitably-defined lex-ordered sets.

**Contributions** This paper comprises two main parts. The first part is devoted to structural properties of lex optimal points whereas the second part analyzes the computational complexity of solving $\text{LEX}_\sigma(X)$. In §2, we observe some preliminary results that lend insight into basic properties of lex optimal points. We also give a simple example to highlight that the greedy algorithm using LP rounding and backtracking can perform badly. In §3, we consider a relaxation of $X \cap \mathbb{Z}^n$ obtained by intersecting lexicographically (lex-) ordered sets that are defined using $\min \text{LEX}_\sigma(X)$ and $\max \text{LEX}_\sigma(X)$ for different $\sigma$, and explore conditions under which this relaxation is tight, i.e., $X \cap \mathbb{Z}^n = \cap_{\sigma \in S_n} \{x \in \mathbb{Z}^n \mid \min \text{LEX}_\sigma(X) \preceq_\sigma x \preceq_\sigma \max \text{LEX}_\sigma(X)\}$. When $X$ is defined by a single linear inequality, a sufficient condition is given in Theorem 1. In particular, this condition is necessary and sufficient for integer points in such an $X$ to be equal to those inside a single lex-ordered set. Our characterization generalizes the notion of a superincreasing knapsack [Gup16; LS92]. In §3.2, we consider an arbitrary $X$ and prove in Theorem 2 that if $X \subseteq [0, 1]^n$ and $X \cap \{0, 1\}^n$ is closed under inclusion (a stronger condition would be to have $X$ be down monotone, i.e., $0 \leq y \leq x$ and $x \in X$ implies $y \in X$), then the intersection of lex-ordered sets is a tight relaxation. Closedness under inclusion is also a necessary condition if we assume $0 \in X$. These results do not generalize to general discrete sets. Theorem 2 establishes a structural characterization for many important 0\1 sets, such as the matching, matroid, knapsack, and stable set polytopes, and also tells us when optimizing over $X \cap \{0, 1\}^n$ is equivalent to optimizing over the convex hull of all the lex-ordered sets. Even if Theorem 2 holds, it is not obvious that lex optimal points should yield the integer optimum over $X$. We prove this to be true in Theorem 3 for any $X \subseteq [0, 1]^n$. Since lex optimal points can be defined greedily using (3), Theorem 3 provides a kind of a greedy characterization for integer optimum over $X$. If $X \cap \{0, 1\}^n$ is a matroid, then the optimal solution produced by the greedy method [Edm71] is a lex optimum under the permutation $\sigma$ that sorts indices in nonincreasing order of weights. For matroids, our theorem is a weaker statement than this well-known fact about matroid optimization because our proof of Theorem 3 is not constructive but rather existential. For
classify that lex optimal points yield a arbitrarily tight approximation factor of $1/n$ on the integer optimum value, thereby showing that they yield poor bounds for general discrete sets.

Next, we turn our attention to the complexity of solving $\text{LEX}_\sigma(X)$. First, we give a finite time cutting plane algorithm in §4. Since we observe in §2 that the lex problem can be reformulated as an integer program, one option for solving it exactly in finite time when $X$ is a polytope would be the famous fractional cutting plane algorithm of [Gom58] wherein the cutting planes are derived from optimal bases of the polyhedral relaxation at each iteration. We give a different finitely convergent cutting plane algorithm by exploiting the structure of the objective function. Our algorithm has three distinguishing features: (i) it works for any compact convex $X$ for which an optimization oracle is available, (ii) facets of a particular lex-ordered set are used to cut off the fractional solution at each iteration, (iii) cutting planes generated in one iteration are only used in the subsequent iteration and are not required beyond that. Since the convex hull of a lex-ordered set has only linearly many nontrivial facets [Gup16], it follows that the complexity of each iteration is equal to the complexity of optimizing over $X$. Proposition 7 proves correctness and finite termination of our algorithm.

§5 classifies P and NP-hard cases of $\text{LEX}_\sigma(X)$. We assume that the set $X$ is input via a membership oracle that runs in polynomial-time. We establish connections between the complexity of integer feasibility of $X$ and the complexity of lex optimization. We show in Proposition 8 that similar to arbitrary integer programs, lex optimization can be difficult under some permutation even if the feasibility problem is easy. The proof of this result indicates that the hardness of the lex problem comes from a hard feasibility problem over a “hidden” sub-box of $B$. Proposition 9 states that this is indeed the only reason for the hardness in finding lex optimal points. We prove this statement using a coordinate-wise bisection method. Then, we consider polytopes defined by multiple lex-orders under different permutations, similar to the intersection of lex-ordered sets in §3.2. Theorems 4 and 5 prove that computing lex optimal points in such polytopes is NP-hard when the number of lex constraints is arbitrary. This implies that using lex-ordered sets to compute a dual bound on the integer optimum over any compact convex set is NP-hard. We further show in Theorem 6 that it is also hard to approximately find within a certain multiplicative factor the largest positive integer $\lambda$ such that the binary encoding of some integer greater than or equal to $2^\lambda$ satisfies arbitrarily many lex constraints.

The paper concludes with some remarks and open questions in §6.

2 Preliminaries

The greedy method in (3) requires solving a integer program at each step. A computationally cheaper way is to solve the convex relaxation at each iteration and round the solution, but this may cause infeasibility at some iteration and then it is necessary to backtrack to the previous iteration. The runtime of this rounding-based method can be quite bad even on simple sets.

Example 1. Let $X = \{x \in [0, 1]^n \mid \sum_{i=1}^{n-1} 2x_i + x_n = n\}$ for some even $n$. By parity arguments, we have $X \cap \mathbb{Z}^n \subseteq \{x \mid x_n = 0\}$. By equation (3), it is easy to see that the lexmax point under identity permutation is $x^* = 0, x^*_i = 1$ for $i = n/2, \ldots, n - 1$, and $x^*_i = 0$ otherwise. The LP round and backtrack method runs in $2^{\mathcal{O}(1)}$ time on $X$. This can be established by counting a subset of the number of times the algorithm must backtrack. The algorithm will check all potential solutions where $x_n = 1$ before any potential solutions with $x_n = 0$. If there is a point $y \in X$ where $y_n = 1$ and the last $k$ variables are integer, then the algorithm will need to pass $y$ by backtracking, and backtracking will only pass the numbers that match the last $k - 1$ variables of $y$. Pick any $\frac{n}{2} - 1$
variables from the list \(y_2, y_3, \ldots, y_{n-1}\). Set those variables to 1, and set \(y_n = 1\) and \(y_1 = \frac{1}{y}\). Then \(y \in X\), and the last \(n-1\) variables are integer, so \(y\) must be passed by backtracking. There are \(\binom{n-2}{n-1}\) of these points, which is a number that is exponential in \(n\). Since the backtracking algorithm can only pass one point at a time, the overall runtime is exponential in \(n\). \(\diamondsuit\)

Problem \(\text{LEX}_\sigma(X)\), as defined in (1), does not have a linear objective function. However, it can be reformulated as an integer program by utilizing the following observation about lex-ordering of bounded integer points.

**Lemma 1.** Let \(B = [l, u]\) and for \(\sigma \in S_n\), define \(\pi \in \mathbb{Z}_{\geq 1}^n\) as

\[
\pi_{\sigma(1)} = 1, \quad \pi_{\sigma(i)} = \prod_{k=1}^{i-1} (1 + u_{\sigma(k)} - l_{\sigma(k)}) \quad i = 2, \ldots, n.
\]

Then for distinct \(x, y \in B \cap \mathbb{Z}^n\), we have \(x <_\sigma y\) if and only if \(\pi x < \pi y\).

Note the recursion

\[
\pi_{\sigma(i)} = 1 + \sum_{k<i} \pi_{\sigma(k)} (u_{\sigma(k)} - l_{\sigma(k)}).
\]

The proof of the above lemma involves simple algebraic manipulations and is provided in Appendix A. If \(u = 1\) and \(l = 0\), then \(\pi_{\sigma(i)} = 2^{i-1}\) for all \(i\). As a consequence of Lemma 1, we have the following.

**Proposition 1.** \(\text{LEX}_\sigma(X)\) is equivalent to the integer program \(\max \{\pi x \mid x \in X \cap \mathbb{Z}^n\}\), where \(\pi\) is as defined in (5). Hence for every \(\sigma \in S_n\), both \(\max \text{LEX}_\sigma(X)\) and \(\min \text{LEX}_\sigma(X)\) are vertices of \(\text{conv}(X \cap \mathbb{Z}^n)\).

**Proof.** The integer program for lexmax is obvious from Lemma 1 and the linearity of its objective makes it equivalent to \(\max \{\pi x \mid x \in \text{conv}(X \cap \mathbb{Z}^n)\}\). Boundedness of \(X\) means that \(\text{conv}(X \cap \mathbb{Z}^n)\) is a polytope, and hence both the lexmax and lexmin problems are linear programs. If \(X \cap \mathbb{Z}^n = \emptyset\), then the statement is vacuously true. Otherwise, the result follows from the uniqueness of \(\max \text{LEX}_\sigma(X)\) and \(\min \text{LEX}_\sigma(X)\) and the fact that every nonempty bounded linear program has at least one vertex solution. \(\square\)

Not every vertex of \(\text{conv}(X \cap \mathbb{Z}^n)\) is a lexmax or lexmin point; this is easy to see even in \(\mathbb{R}^2\). There are at most \(n!\) many lexmax and lexmin points, one for each permutation. Due to Proposition 1 we know that for an integral polytope, the number of lexmax and lexmin points is upper bounded by the number of vertices. This bound can be quite weak though, for example, a box \([l, u] \subset \mathbb{R}^n\) has \(2^n\) vertices but for every \(\sigma\), \(\max \text{LEX}_\sigma([l, u]) = u\) and \(\min \text{LEX}_\sigma([l, u]) = l\). However, there also exist polytopes for which there is a bijection between vertices and lex optimal points.

**Proposition 2.** For the permutahedron \(P_n := \text{conv} S_n\), we have

\[
(\max \text{LEX}_\sigma(P_n))_{\sigma(i)} = i, \quad (\min \text{LEX}_\sigma(P_n))_{\sigma(i)} = n + 1 - i \quad \forall \sigma \in S_n, i \in [n].
\]

**Proof.** The vertex property of lex optimal points in Proposition 1 implies that the elements of the vectors \(\max \text{LEX}_\sigma(P_n)\) and \(\min \text{LEX}_\sigma(P_n)\) are distinct integers in \(\{1, 2, \ldots, n\}\). The result follows via a straightforward application of equation (3). \(\square\)
This also tells us that the number of distinct lex optimal points can be as high as its maximum value of \( n! \).

We finish our initial observations by mentioning that the lex optimization problem is trivial for packing and covering polytopes using the naive recursion of equation (3) that greedily optimizes in each coordinate.

**Proposition 3.** Let \( X^{\text{pack}} = \{ x \in [0,u] \mid Ax \leq b \} \) for some \( A, b \geq 0 \) and assume \( u_i = \max\{x_i \mid x \in X^{\text{pack}}, x_i \in \mathbb{Z}\} \) for all \( i \). Then \( \min \text{LEX}_\sigma(X^{\text{pack}}) = 0 \) and \( \max \text{LEX}_\sigma(X) = x^* \), where

\[
x^*_\sigma = \min \left\{ u_\sigma(i), \min_{j : A_{j,\sigma(i)} > 0} \frac{b - \sum_{k > i} A_{j,\sigma(k)} x^*_\sigma(k)}{A_{j,\sigma(i)}} \right\} \quad i = n, \ldots, 1.
\]

For a covering polytope \( X^{\text{cov}} = \{ x \in [0,u] \mid Ax \geq b \} \) for some \( A, b \geq 0 \) with \( Au \geq b \), \( \max \text{LEX}_\sigma(X^{\text{cov}}) = u \) and \( \min \text{LEX}_\sigma(X^{\text{cov}}) \) can be found using the identity (2) and noting that Proposition 3 is applicable to \( u - X^{\text{cov}} \).

Using the translation invariance of \( \preceq_\sigma \), we will henceforth assume w.l.o.g. that \( B = [0,u] \).

### 3 Lex-ordered sets

Throughout this section, for \( \sigma \in S_n \), denote

\[
\theta^\sigma := \max \text{LEX}_\sigma(X), \quad \gamma^\sigma := \min \text{LEX}_\sigma(X)
\]

and define \( L^\sigma \) as the bounded set of integer vectors that are lexicographically between \( \gamma^\sigma \) and \( \theta^\sigma \),

\[
L^\sigma := \{ x \in B \cap \mathbb{Z}^n \mid \gamma^\sigma \preceq_{\sigma} x \preceq_{\sigma} \theta^\sigma \}.
\]

We refer to \( L^\sigma \) as a lexicographically (or lex-) ordered set. Clearly,

\[
X \cap \mathbb{Z}^n \subseteq \bigcap_{\sigma \in S_n} L^\sigma.
\]

Our main goal in this section is to explore conditions under which equality \( X \cap \mathbb{Z}^n = \bigcap_{\sigma \in S_n} L^\sigma \) holds true. We analyze this first for integer knapsack sets in §3.1 by providing a necessary and sufficient condition in Proposition 6 for a knapsack to be equal to a single lex-ordered set. After this, in §3.2 we consider an arbitrary \( X \) and intersection of multiple lex-ordered sets. Theorem 2 establishes our first main result by characterizing subsets of \( [0,1]^n \) for which equality holds in equation (8). For sets that afford equality in (8), it follows that there is a structural connection between integer optimization over \( X \) and optimizing over all the lex-ordered sets for \( X \). Optimizing over the \( L^\sigma \)'s individually of course produces a relaxation (dual bound) of the integer optimum. A primal bound on the optimum can be obtained from \( \gamma^\sigma \) and \( \theta^\sigma \) since they are feasible points. We prove in Theorem 3 that the maximum (resp. minimum) value of an arbitrary linear function over any subset of \( [0,1]^n \) occurs at some lexmax (resp. lexmin) point. For a general compact subset of \( \mathbb{Z}^n \), these points yield a tight \( 1/n \) approximation factor to the optimal value.

#### 3.1 Integer knapsacks

Throughout this section, let

\[
X = \mathcal{K} := \{ x \in [0,u] \mid ax \leq a_0 \}, \quad \text{for some } (a,a_0) \in \mathbb{Z}^{n+1}_+ \text{ with } a_1 \leq a_2 \leq \cdots \leq a_n,
\]

\[X = \mathcal{K} := \{ x \in [0,u] \mid ax \leq a_0 \}, \quad \text{for some } (a,a_0) \in \mathbb{Z}^{n+1}_+ \text{ with } a_1 \leq a_2 \leq \cdots \leq a_n, \quad (9)\]
represent the continuous relaxation of a integer knapsack. This is the simplest structure to impose on a compact convex set. Obviously, $\gamma^\sigma = 0$ for all $\sigma \in S_n$, since $\mathbf{0} \in K$ and $K \subseteq \mathbb{R}^{\geq 0}$. Applying Proposition 3, we obtain

$$\theta^\sigma_{\sigma(i)} = \min \left\{ u_{\sigma(i)}, \left[ a_0 - \sum_{k<i} a_{\sigma(k)} \theta^\sigma_{\sigma(k)} \right] \frac{a_{\sigma(i)}}{a_{\sigma(i)}} \right\} \quad i = n, \ldots, 1,$$

for every $\sigma \in S_n$. Then

$L^\sigma = \{ x \in [\mathbf{0}, u] \cap \mathbb{Z}^n \mid x \preceq\sigma \theta^\sigma \}.$

Remark 1. The above recursion does not work if some $a_i$ are negative; for e.g., if $K = \{ x \in [0,1]^2 \mid -x_1 + x_2 \leq 0 \}$, equation (10) gives $\theta^{(1,2)} = \theta^{(2,1)} = (0,0)$ whereas the lexmax point in each of the permutations is obviously $(1,1)$. The presence of negative coefficients in a knapsack can be handled easily; we state this next and prove it in Appendix A. Note that when encountered with some $a_i < 0$, we cannot simply complement the corresponding variables because doing so would require us to minimize in some coordinates and maximize in others.

Proposition 4. For $K' = \{ x \in [\mathbf{0}, u] \mid ax \preceq a_0 \}$ with $(a, a_0) \in \mathbb{Z}^n$, we have $\tilde{\theta}^\sigma = \arg \max LEX_{\sigma}(K')$ where $\tilde{\theta}^\sigma$ is constructed as

$$\tilde{\theta}^\sigma_{\sigma(i)} := \begin{cases} u_{\sigma(i)} \\
\min \left\{ u_{\sigma(i)}, \left[ a_0 - \xi - \sum_{k<i} a_{\sigma(k)} \theta^\sigma_{\sigma(k)} \right] \frac{a_{\sigma(i)}}{a_{\sigma(i)}} \right\} 
\end{cases}$$

for $\xi := \sum_{j \mid a_j \leq 0} a_j u_j$.

Also, checking if $K'$ is equal to a lex-ordered set is equivalent to checking whether the set obtained by complementing variables as necessary to reduce it to (9), is equal to a lex-ordered set. Therefore, we can assume throughout this section that $a > \mathbf{0}$.

Our main goal in this section is to characterize integer knapsacks $K \cap \mathbb{Z}^n$ that are equal to $L^\sigma = \{ x \in [\mathbf{0}, u] \cap \mathbb{Z}^n \mid x \preceq\sigma \theta^\sigma \}$ for some $\sigma \in S_n$. We establish this in Proposition 6. First, we mention a related recent result.

Proposition 5 (Gupte 16). For $K$ defined as in (9), if $\sigma \in S_n$ is such that $a_{\sigma(i)} \geq \sum_{k<i} a_{\sigma(k)} u_{\sigma(k)}$ for all $i$, then $K \cap \mathbb{Z}^n = L^\sigma$ and $ax \preceq a\theta^\sigma$ for all $x \in K \cap \mathbb{Z}^n$.

The condition $a_{\sigma(i)} \geq \sum_{k<i} a_{\sigma(k)} u_{\sigma(k)}$ for all $i$ in Proposition 5 was referred to as the “super-increasing property” by Gupte [Gupte 16]. Since we assume $a_1 \leq a_2 \leq \cdots \leq a_n$, it is clear that this property can only be obeyed for $\sigma = \text{id}$ or $\sigma = (2, 1, 3, \ldots, n)$, with $\sigma = \text{id}$ being the only possibility if $a_1 < a_2$. When this property holds, the set $\{ x \in [\mathbf{0}, u] \cap \mathbb{Z}^n \mid ax \preceq a_0 \}$ is called a superincreasing knapsack. Proposition 5 then tells us that every superincreasing knapsack is a lexicographically ordered set. The converse of this statement --- does every bounded lexicographically ordered set of the form $\{ x \in [\mathbf{0}, u] \cap \mathbb{Z}^n \mid x \preceq\sigma \theta \}$ represent a superincreasing knapsack?, is also true. To see this, note that the $\pi$ constructed in equation (5) in Lemma 1 satisfies the superincreasing property because $\pi > \mathbf{0}$ and $\pi_{\sigma(i)} = 1 + \sum_{k=1}^{i-1} \pi_{\sigma(k)} (u_{\sigma(k)} - l_{\sigma(k)})$ for $i = 2, \ldots, n$. Combining Proposition 5 and Lemma 1 establishes that every bounded set of lexicographically ordered integer points is a superincreasing knapsack and vice-versa.

Thus the superincreasing property is sufficient for a knapsack to be a lex-ordered set. Is it also necessary? Since Lemma 1 constructs a $\pi$ vector, and hence a superincreasing knapsack, explicitly, the answer to this question is not immediate. In fact, the answer is no. The following example is of a knapsack which, although not superincreasing, is equal to a lex-ordered set.
Example 2. Let $K = \{x \geq 0 \mid 2x_1 + 8x_2 + 40x_3 + 150x_4 + 310x_5 \leq 825, x_1 \leq 1, x_2 \leq 5, x_3 \leq 4, x_4 \leq 1, x_5 \leq 2\}$. Since the coefficients of the linear inequality are strictly increasing, the identity permutation is the only candidate for satisfying the superincreasing property. However $a_i < \sum_{k < i} a_k u_k$ for $i = 3, 4, 5$ and so $K \cap \mathbb{Z}^n$ is not a superincreasing knapsack.

Consider $\theta = (1, 1, 1, 1, 2)$ which is computed using (10) for $\sigma = id$. We know $K \subseteq \{x \in B \cap \mathbb{Z}^n \mid x \leq \theta\}$ from Proposition 5. The reverse inclusion can be checked by brute force enumeration to conclude that $K \cap \mathbb{Z}^n$ is a lex-ordered set under the identity permutation. \hfill $\diamondsuit$

If a knapsack is equal to a lex-ordered set $\{x \in [0, u] \cap \mathbb{Z}^n \mid x \leq \theta\}$, then it must be that $\theta = \theta^\sigma$. Proposition 5 tells us that for superincreasing knapsacks, there exists some permutation $\sigma$ such that $K \cap \mathbb{Z}^n = L^\sigma$. But the superincreasing property is not necessary for this equality to hold, as pointed out in Example 2. We next characterize knapsacks that are lexicographically ordered sets under a given permutation.

Proposition 6. Let $K$ be a knapsack as in (9) and $\sigma \in S_n$ be given. Denote

$$I^\sigma := \left\{ i \mid \theta_{\sigma(i)}^\sigma \geq 1, a_{\sigma(i)} < \sum_{k=1}^{i-1} a_{\sigma(k)} \left( u_{\sigma(k)} - \theta_{\sigma(k)}^\sigma \right) \right\}.$$ 

Then $K \cap \mathbb{Z}^n = L^\sigma$ if and only if

$$\sum_{k=1}^{i-1} a_{\sigma(k)} u_{\sigma(k)} + \sum_{k=i+1}^{n} a_{\sigma(k)} \theta_{\sigma(k)}^\sigma \leq a_0 \quad \forall i \in I^\sigma. \tag{11}$$

Proof. We have $K \cap \mathbb{Z}^n = L^\sigma$ if and only if $ax \leq a_0$ for every $x \in L^\sigma \setminus \{\theta^\sigma\}$. Define

$$x^{t,i} := \theta^\sigma - te_{\sigma(i)} + \sum_{k=1}^{i-1} (u_{\sigma(k)} - \theta_{\sigma(k)}^\sigma)e_k, \quad t = 1, \ldots, u_{\sigma(i)}, i = 1, \ldots, n.$$ 

Note that $x^{t,i} \in L^\sigma \setminus \{\theta^\sigma\}$ and every $x \in L^\sigma \setminus \{\theta^\sigma\}$ satisfies $x \leq x^{t,i}$ for some $t, i$. The assumption $a > 0$ then implies that $\max\{ax \mid x \in L^\sigma \setminus \{\theta^\sigma\}\} \leq \max_{t,i} ax^{t,i}$. Therefore $K \cap \mathbb{Z}^n = L^\sigma$ if and only if $ax^{t,i} \leq a_0$ for all $t, i$. We have

$$ax^{t,i} \leq a\theta^\sigma \iff t \geq t_{i}^* := \frac{\sum_{k<i} a_{\sigma(k)}(u_{\sigma(k)} - \theta_{\sigma(k)}^\sigma)}{a_{\sigma(i)}}.$$ 

For every $i$, we have $x^{t,i} \in K$ for all $t \geq t_{i}^*$ because $a\theta^\sigma \leq a_0$. If $t_{i}^* \leq 1$, then $x^{t,i} \in K$ for all $t$. Recognize that $I^\sigma = \{i \mid \theta_{\sigma(i)}^\sigma \geq 1, t_{i}^* > 1\}$. Therefore $K \cap \mathbb{Z}^n = L^\sigma$ if and only if for every $i \in I^\sigma$ we have $ax^{t,i} \leq a_0$ for $t = 1, \ldots, u_{\sigma(i)}$. Now for a fixed $i \in I^\sigma$, $ax^{t,i} \leq a_0$ for all $t$ if and only if

$$\frac{\sum_{k<i} a_{\sigma(k)} u_{\sigma(k)} + \sum_{k\geq i} a_{\sigma(k)} \theta_{\sigma(k)}^\sigma}{a_{\sigma(i)}} - a_0 \leq 1 \iff \theta_{\sigma(i)}^\sigma + \frac{\sum_{k<i} a_{\sigma(k)} u_{\sigma(k)} + \sum_{k\geq i} a_{\sigma(k)} \theta_{\sigma(k)}^\sigma - a_0}{a_{\sigma(i)}} \leq 1$$

$$\iff \sum_{k<i} a_{\sigma(k)} u_{\sigma(k)} + \sum_{k\geq i} a_{\sigma(k)} \theta_{\sigma(k)}^\sigma - a_0 \leq 0,$$

where the second equivalence is due to $\theta_{\sigma(i)}^\sigma \geq 1$ for $i \in I^\sigma$. \hfill $\square$

An immediate consequence is a sufficient condition for equality in (8).
Theorem 1. Let \( \mathcal{K} \) be a knapsack as in (9). If there exists \( \sigma' \in S_n \) that satisfies (11), then \( \mathcal{K} \cap Z^n = \cap_{\sigma \in S_n} L^\sigma \).

Proof. Proposition 6 gives \( \mathcal{K} \cap Z^n = L^{\sigma'} \) and then the result follows from \( \cap_{\sigma \in S_n} L^\sigma \supseteq \mathcal{K} \cap Z^n = L^{\sigma'} \supseteq \cap_{\sigma \in S_n} L^\sigma \).

The condition of Proposition 6 is a weaker condition than the superincreasing property of Proposition 5; this is because if the sequence \( \{a_i\} \) is superincreasing under some permutation \( \sigma \) then \( \theta^\sigma \geq 0 \) implies that \( I^\sigma = \emptyset \).

Example 2 (Continued). First consider \( \sigma = \text{id} \). We have \( \theta^{\text{id}} = (1,1,1,1,2) \), \( I^{\text{id}} = \{4\} \) and \( \sum_{k=1}^{3} a_k u_k + a_5 \theta_5 = 202 + 620 = 822 \leq 825 = a_0 \), thereby giving us \( \mathcal{K} \cap Z^n = L^{\text{id}} \). If \( \sigma \) is the reversing permutation (i.e. \( \sigma(i) = 6 - i \)) then we have \( \theta^{\text{revid}} = (1,5,4,1,1) \) and \( I^{\text{revid}} = \{2,3,4,5\} \). Checking \( i = 5 \) gives \( \sum_{k=1}^{3} a_k u_k + a_6 u_6 - k = 970 > 825 \). Therefore \( S \cap Z^n \subseteq L^{\text{revid}} \). For \( \sigma = (1\,2\,3\,(4\,5)) \), we have \( \theta^\sigma = (1,1,1,1,2) \) and \( \mathcal{K} \cap Z^n = L^{\sigma} \).

3.2 Intersection of lex-ordered sets

In the previous section, we have studied the correspondence between an integer knapsack and a single lexicographically ordered set. It follows that if \( X = \{x \in [0,1]^n \mid Ax \leq b\} \) with \( A \geq 0 \) and every constraint \( A_i : x \leq b_i \), for \( 1 \leq i \leq m \), satisfies the condition of Proposition 6 for some \( \sigma^i \in S_n \), then \( X \cap Z^n \) is equal to the intersection of \( m \) lex-ordered sets, where each set corresponds to an inequality in \( Ax \leq b \). That is not what we allude to by intersection of multiple lex-ordered sets. We mean to analyze for arbitrary \( X \), and \( L^\sigma \) as defined in (7), when is \( \cap_{\sigma \in S_n} L^\sigma \) contained in \( X \cap Z^n \), thereby giving us equality between the two sets. Unsurprisingly, this is not possible always since the inclusion \( X \cap Z^n \subseteq \cap_{\sigma \in S_n} L^\sigma \) in equation (8) is strict in general; we illustrate this for a family of polytopes.

Example 3. Let \( X = \{x \in [0,k]^n \mid \sum_{i=1}^{n} x_i \leq (k-1)n\} \) where \( n,k \geq 2 \). Note that \( (k-1)1 \subseteq X \). Denote \( k^* := \lfloor \frac{(k-1)n}{k} \rfloor \). For any \( \sigma \in S_n \), \( \theta^\sigma_{\sigma(i)} = 0 \) if \( i = 1, \ldots, n-k^*-1, \theta^\sigma_{\sigma(i)} = (k-1)n - k \lfloor \frac{(k-1)n}{k} \rfloor \) for \( i = n - \lfloor \frac{(k-1)n}{k} \rfloor \), and \( \theta^\sigma_{\sigma(i)} = k \) otherwise. Consider \( y \) so that \( y_i = k - 1 \) for \( i = 1, \ldots, n+1 - \lfloor \frac{(k-1)n}{k} \rfloor \), and \( y_i = k \) otherwise. Then \( y \notin X \), but \( y_{\sigma(i)} \leq \theta^\sigma_{\sigma(i)} \) for all \( i = n + 1 - \lfloor \frac{(k-1)n}{k} \rfloor, \ldots, n \) with at least one inequality being strict. Therefore \( y \in L^\sigma \) for all \( \sigma \in S_n \).

Based on this example, we do not hope for equality in (8) when the box \([l,u]\) containing \( X \) is arbitrary. A special case of \([l,u]\) that has received considerable attention in the discrete optimization literature is the \([0,1]^n\) box. For a polytope \( X \subseteq [0,1]^n \), the facets of \( \text{conv}(X \cap \{0,1\}^n) \) are known to have many special characteristics [cf. CCZ14] due to the fact that there are no integer points in the relative interior of \( X \) and every feasible integral point is a vertex of the convex hull. These so-called “facial properties” of \([0,1]^n\) are also helpful to us since in the next theorem, they enable a characterization of sets that admit equality in (8). We say that \( X \cap \{0,1\}^n \) is closed under inclusion if for any \( x,y \in \{0,1\}^n \), \( x \in X \) implies \( y \in X \).

Theorem 2. Assume \( X \subseteq [0,1]^n \). If \( X \) is closed under inclusion, then we have \( X \cap \{0,1\}^n = \cap_{\sigma \in S_n} L^\sigma \). On the contrary, if \( 0 \in X \), then \( X \cap \{0,1\}^n = \cap_{\sigma \in S_n} L^\sigma \) only if \( X \) is closed under inclusion.

\footnote{Two other equivalent definitions of closed under inclusion are: (i) \( X \cap \{0,1\}^n = \{x \in \{0,1\}^n \mid Ax \leq b\} \) for \( A \geq 0 \), (ii) \( \text{conv}(X \cap \{0,1\}^n) = \{x \in [0,1]^n \mid Ax \leq b\} \) for \( A \geq 0 \).}
Proof. (if) Let $X \subseteq [0,1]^n$ be nonempty and closed under inclusion. Then $0 \in X$ and so $\gamma^\sigma = 0$ and $L^\sigma = \{x \in [0,1]^n \mid x \preceq_\sigma \theta^\sigma\}$ for all $\sigma$. We need to show $X \supseteq \cap_{\sigma \in S_n} L^\sigma$. Fix an arbitrary $x \in \{0,1\}^n \setminus X$. Pick some $\sigma \in S_n$ so that $x_i = 0$ and $x_j = 1$ then $\sigma^{-1}(i) < \sigma^{-1}(j)$. Such a permutation of $x$ orders all the ones after all the zeros. So we have $x_{\sigma(1)} = x_{\sigma(2)} = \cdots = x_{\sigma(k)} = 0$ and $x_{\sigma(k+1)} = x_{\sigma(k+2)} = \cdots = x_{\sigma(n)} = 1$ for some $1 \leq k \leq n$. Take any $y \in \{0,1\}^n$ such that $y \succeq_\sigma x$. The construction of $\sigma$ enforces $y_{\sigma(i)} = 1$ for $k+1 \leq i \leq n$. Hence $y \geq x$ and then the assumption that $X$ is closed under inclusion implies $y \notin X$. Thus we have argued that $\{y \in \{0,1\}^n \mid y \leq x\} \subseteq \{y \in \{0,1\}^n \mid y \preceq_\sigma x\}$. Choosing $y$ equal to $\theta^\sigma$, which belongs to $X$, leads to $\theta^\sigma \preceq_\sigma x$. Therefore $x \notin L^\sigma$, and we have proved $X \supseteq \cap_{\sigma \in S_n} L^\sigma$ by contraposition.

(only if) Let $0 \in X$, so that $L^\sigma = \{x \in [0,1]^n \mid x \preceq_\sigma \theta^\sigma\}$, and $X \cap \{0,1\}^n = \cap_{\sigma \in S_n} L^\sigma$. If $X$ is not closed under inclusion, there exists a $y \in X \cap \{0,1\}^n$ and $x \in \{0,1\}^n$ such that $x \leq y$ but $x \notin X$. For every $\sigma$, $x \leq y$ implies $x \preceq_\sigma y$ and then $y \preceq_\sigma \theta^\sigma$ implies $x \preceq_\sigma \theta^\sigma$. Hence $x \in \cap_{\sigma \in S_n} L^\sigma = X$, a contradiction. \qed

In the “if” part of the above proof, the $[0,1]^n$ box was crucial in arriving at the implication $y \geq x$, which further allowed us to conclude $y \notin X$. For a general box, one may analogously construct $\sigma$ as the permutation that orders all the zeros in the beginning, all the upper bounds at the end, and everything else in between. However, this will not guarantee $y \geq x$.

Theorem 2 establishes a structural characterization for many important 0\1 sets, such as matching, matroid, knapsack, and stable set polytopes. It also tells us when optimizing over $\{0,1\}^n$ is equivalent to optimizing over $\cap_{\sigma \in S_n} L^\sigma$. This convex hull involves the intersection of up to factorially many lex-ordered sets, each of which has up to $4n$ facets [Gup16]. However this has no bearing on the complexity of optimizing/separating over $\cap_{\sigma \in S_n} L^\sigma$ since this complexity can be polynomial-time (e.g., $X$ is the fractional matching polytope) or NP-hard (e.g., $X$ is the fractional stable set polytope). It is not clear whether all the vertices of $\cap_{\sigma \in S_n} L^\sigma$ are necessarily lex optimal points under some permutation, and hence Theorem 2 does not tell us whether lex optima yield the integer optimum over a 0\1 set. We next prove this to be true.

Denote $z_{\text{max}}^\star := \max\{cx \mid x \in X \cap \mathbb{Z}^n\}$. Using the fact that $X \cap \mathbb{Z}^n \subseteq \cap_{\sigma \in S_n} L^\sigma$, we have

$$z_{\text{max}}^\star \leq z_{\text{lex}}(S_n^\prime) := \max\{cx \mid x \in \cap_{\sigma \in S_n^\prime} L^\sigma\}, \quad \text{for any } S_n^\prime \subseteq S_n. \quad (12)$$

We show later in Corollary 2 that this bound is NP-hard to compute. By optimizing over each $L^\sigma$ individually, a weaker upper bound on $z_{\text{max}}^\star$ is obtained:

$$z_{\text{max}}^\star \leq z_{\text{lex}}(S_n) \leq \min_{\sigma \in S_n} \max_{x \in \text{conv } L^\sigma} \{cx\}, \quad \text{where the inner maximization over } x \text{ is a simple linear program since } \text{conv } L^\sigma \text{ has at most } 4n \text{ facets. Computing this bound by enumeration is expensive since } X \text{ may have } n! \text{ many distinct lexmax points (cf. Proposition 2). So this bound is possibly not tractable unless } X \text{ has only a small number of distinct lex optima.} \quad (13)$$

Now we compare the quality of objective values from lex optimal solutions against the integer optimum over $X \subseteq [0,u]$. Since $\gamma^\sigma, \theta^\sigma \in X \cap \mathbb{Z}^n$ by construction, it is obvious that

$$z_{\text{max}}^\star \geq z_{\text{lex}}^\text{prim} := \max_{\sigma \in S_n} \max \{c \theta^\sigma, c \gamma^\sigma\}. \quad (14)$$

For $c \geq 0$, we argue that above is an equality when $X \subseteq [0,1]^n$ and is a tight $1/n$-approx in the general case $X \subseteq [0,u]$. Note that the assumption $c \geq 0$ is w.l.o.g. because if $c_i < 0$ for some $i$, then we could simply consider the set $X'$ obtained by replacing $x_i$ with $u_i - x_i$ in $X$.

**Theorem 3.** Let $c \geq 0$ and assume $z_{\text{max}}^\star$ is finite.
1. For $X \subseteq [0,1]^n$, we have $z_{\text{max}}^* = z_{\text{lex}}^\text{prim} = \max_{\sigma \in S_n} c \theta^\sigma$.

2. For $X \subseteq [0,u]$ with $u \neq 1$, $z_{\text{lex}}^\text{prim} \geq z_{\text{max}}^*/n$ and this bound is arbitrarily tight for every $n$.

Proof. Define $\text{Opt}(X) := \arg \max \{cx \mid x \in X \cap \mathbb{Z}^n\}$ to be the set of optimal solutions to $z_{\text{max}}^*$.

(1) Pick any $x^* \in \text{Opt}(X)$ such that $\sum_i x_i^* \geq \sum_i y_i$ for all $y \in \text{Opt}(X)$. Let $\sigma \in S_n$ be such that $x_i^* = 0, x_j^* = 1$ implies $\sigma^{-1}(i) < \sigma^{-1}(j)$. We claim that $\theta^\sigma = x^*$; arguing this completes our proof. Assume for contradiction that $\theta^\sigma \prec x^*$. Since we have assumed $\theta^\sigma \neq x^*$, then $x^* \prec_{\sigma} \theta^\sigma$ because $\theta^\sigma$ is a lexmaximal point in $X$. Since the permutation $\sigma$ puts all 1’s in $x^*$ at the end, $x_i^* = 1$ implies $\theta^\sigma_i = 1$ because otherwise we would have $\theta^\sigma \prec_{\sigma} x^*$. Therefore $x^* \preceq \theta^\sigma$. Since $c \geq 0$, this implies $cx^* \leq c \theta^\sigma$ and so $cx^* = c \theta^\sigma$ and $\theta^\sigma \in \text{Opt}(X)$. Now $x^* \preceq \theta^\sigma$ implies $\sum_i (\theta^\sigma_i - x_i^*) \geq 0$ with equality holding if and only if $x^* = \theta^\sigma$.

Since we have assumed $x^* \neq \theta^\sigma$ and both these points are integral, we have $\sum_i (\theta^\sigma_i - x_i^*) \geq 1$, which after rearranging becomes $\sum_i \theta^\sigma_i \geq 1 + \sum_i x_i^*$. Since $\theta^\sigma \in \text{Opt}(X)$, the construction of $x^*$ tells us that $\sum_i x_i^* \geq \sum_i \theta^\sigma_i$ and we have arrived at a contradiction. Therefore $\theta^\sigma = x^*$ and we are done.

(2) Pick any $x^* \in \text{Opt}(X)$. There exists some $k$ so that $c_k x_k^* \geq \frac{1}{n} z_{\text{max}}^*$. Choose $\tau \in S_n$ such that $k = \tau(n)$. Then $\theta_k^\tau \geq x_k^*$ and so

$$\max_{\sigma \in S_n} \{c \theta^\sigma, c \gamma^\sigma\} \geq c \theta^\tau \geq c_k \theta_k^\tau \geq c_k x_k^* \geq \frac{1}{n} z_{\text{max}}^*.$$ 

This bound cannot be improved. Let $c = 1$, $\kappa \geq 2$ be an integer, and

$$X = \text{conv}\{\kappa e_1, \kappa e_2, \ldots, \kappa e_n, (\kappa - 1) 1\}.$$ 

Note that $X$ is an integral polytope. We have $z_{\text{max}}^* = (\kappa - 1)n$ attained uniquely at $x^* = (\kappa - 1)1$. For every $\sigma$, since we know from Proposition 1 that $\theta^\sigma$ and $\gamma^\sigma$ are vertices of $\text{conv}(X \cap \mathbb{Z}^n)$, we have $\theta^\sigma = \kappa e_{\sigma(n)}$ and $\gamma^\sigma = \kappa e_{\sigma(1)}$. Thus $c \theta^\sigma = c \gamma^\sigma = \kappa$, making the primal bound also $\kappa$. The approximation ratio becomes $\frac{\kappa}{(\kappa - 1)n}$, whose limit as $\kappa \to \infty$ is $1/n$. \hfill $\square$

Remark 2. Analogous statements can be obtained for $z_{\text{min}}^* := \min \{cx \mid x \in X \cap \mathbb{Z}^n\}$ either by converting it to a maximization and complementing variables to ensure the objective function is nonnegative or by adapting the above proofs. Doing so, for $c \geq 0$, the first statement in the above theorem becomes $z_{\text{min}}^* = \min_{\sigma \in S_n} c \gamma^\sigma$ and the second statement becomes $\min_{\sigma \in S_n} c \gamma^\sigma \leq (n - 1)cu/n + z_{\text{min}}^*/n$.

4 Cutting Plane Algorithm

Suppose that $X$ is accompanied with an optimization oracle. Consider Algorithm 1.

Proposition 7. Algorithm 1 solves LEX$_\sigma(X)$ in finitely many iterations.

Proof. Let $(z^k, x^k, \chi^k, Q^k)$ denote the values of $(z^*, x^*, \chi, Q)$ constructed in iteration $k$. We first prove the following by induction.

Claim 1. $X \cap \mathbb{Z}^n \subseteq Q^{k-1}$ for all $k \geq 1$.

Proof. The base case ($k = 1$) is true since we initialize $Q^0 = B$. Now assume it is true for $k \geq 1$ and consider iteration $k + 1$. We need to show $X \cap \mathbb{Z}^n \subseteq Q^k$. The maximality of $x^k$ and integrality of $\pi$ tells us that $\pi x \leq [\pi x^k]$ for all $x \in X \cap Q^{k-1}$. Since $X \cap \mathbb{Z}^n \subseteq Q^{k-1}$ by induction hypothesis, we get $X \cap \mathbb{Z}^n \subseteq \{x \in B \cap \mathbb{Z}^n \mid \pi x \leq [\pi x^k]\}$. Consider the knapsack $\{x \in B \cap \mathbb{Z}^n \mid \pi x \leq [\pi x^k]\}$.
Algorithm 1: Lexicographical cutting plane algorithm

Input: $\sigma \in S_n$ and a optimization oracle for any $X' \subseteq X \subseteq B$

Output: $\max \text{LEX}_\sigma (X)$ or certifying that $X \cap \mathbb{Z}^n = \emptyset$

1. Construct $\pi$ as per equation (5)
2. $Q \leftarrow B$, $k \leftarrow 1$
3. while $k \geq 1$
4. ![](image)
5. ![](image)
6. ![](image)
7. else
8. ![](image)
9. ![](image)
10. return $x^*$
11. else
12. ![](image)
13. $Q \leftarrow \text{conv}\{x \in B \cap \mathbb{Z}^n \mid x \preceq_\sigma \chi\}$
14. end
15. end
16. end

Note that $\chi^k$ is the lexmax point for this knapsack. Since $\pi$ has the superincreasing property, i.e., $\pi_{\sigma(i)} \geq \sum_{k<i} \pi_{\sigma(k)} u_k \forall i \geq 2$, applying Proposition 5 gives us

$$\{x \in B \cap \mathbb{Z}^n \mid \pi x \leq [\pi x^k]\} = \{x \in B \cap \mathbb{Z}^n \mid \pi x \leq \pi \chi^k\} = \{x \in B \cap \mathbb{Z}^n \mid x \preceq_\sigma \chi^k\} = Q^k \cap \mathbb{Z}^n.$$ 

Therefore $X \cap \mathbb{Z}^n \subseteq Q^k$ and our induction is complete.

Since $\text{LEX}_\sigma (X)$ is equivalent to the integer program $\max \{\pi x \mid x \in X \cap \mathbb{Z}^n\}$, the above claim implies that our termination criteria are correct: (i) if $z^* > -\infty$ and $x^* \in \mathbb{Z}^n$, then $x^* = \max \text{LEX}_\sigma (X)$, (ii) if $z^* = -\infty$, then $X \cap Q = \emptyset$ and so $X \cap \mathbb{Z}^n = \emptyset$.

Now we prove finite convergence of $\{x^k\}$. It is obvious by construction of $\chi^k$ that $0 \leq \pi \chi^k \leq \pi x^k$ for all $k$. Due to integrality of $\pi$ and $\chi$, it suffices to argue that $0 \leq \pi \chi^{k+1} \leq \pi x^{k+1} < \pi \chi^k \leq \pi x^k$ whenever $x^k, x^{k+1} \notin \mathbb{Z}^n$. The main thing to prove is $\pi x^{k+1} < \pi \chi^k$. Lemma 1 tells us that (i) $\pi x \leq \pi \chi^k$ is a valid inequality to $Q^k \cap \mathbb{Z}^n$, which means $\pi x \leq \pi \chi^k$ is valid to $Q^k$, and that (ii) $Q^k \cap \mathbb{Z}^n \cap \{x \mid \pi x = \pi \chi^k\} = \{\chi^k\}$. The latter implies $\text{conv}(Q^k \cap \mathbb{Z}^n \cap \{x \mid \pi x = \pi \chi^k\}) = \{\chi^k\}$ and since $Q = \text{conv}(Q \cap \mathbb{Z}^n)$, the former leads to

$$Q^k \cap \{x \mid \pi x = \pi \chi^k\} = \{\chi^k\}. \quad (15)$$

We now claim that $x^k \notin Q^k$. If $x^k \in Q^k$, the validity of $\pi x \leq \pi \chi^k$ to $Q^k$ gives us $\pi x^k \leq \pi \chi^k$. By construction, $\pi \chi^k \leq \pi x^k$. Hence $\pi x^k = \pi \chi^k$. Now equation (15) implies that $x^k = \chi^k$, which is a contradiction to $\chi^k \in \mathbb{Z}^n, x^k \notin \mathbb{Z}^n$. Therefore $x^k \notin Q^k$. Since the choice of $x^k$ was arbitrary in line 4, it means that $Q^k$ cuts off all the fractional optimal solutions to $z^k$. In the next iteration, the
validity of $\pi x \leq \pi \chi^k$ to $Q^k$ gives us $\pi x^{k+1} \leq \pi \chi^k$. If equality holds, then invoking (15) as before gives a contradiction $x^{k+1} = \chi^k$. Hence $\pi x^{k+1} < \pi \chi^k$, thereby proving finite convergence. □

The running time of Algorithm 1 is $O(2^n u_{\text{max}})$, where $u_{\text{max}} = \max_i u_i$, and will be exponential in the worst case.

5 Complexity Analysis

This section analyzes the computational complexity of finding lexmax integral points in a compact convex set $X \subset B := [0, u]$. We only consider the lexmax problem since the lexmin point can be computed from the lexmax point after complementing variables (cf. (2)). The set $X$ is assumed to be input via a membership oracle that runs in polynomial-time. Easy and hard cases of this problem are classified by establishing connections between the complexity of integer feasibility of $X$ and the complexity of lex optimization. Then we consider polytopes defined by multiple lex-orders under different permutations (similar to the intersection of lex-ordered sets in §3.2). We prove NP-hardness of computing lex optimal points in such polytopes when the number of lex constraints is arbitrary. This implies that computing the dual bound in equation (12) is NP-hard. We further show that it is also hard to approximately find the largest power of 2 whose binary encoding satisfies arbitrarily many lex constraints.

5.1 Integer feasibility over subsets of $X$

$\text{LEX}_\sigma(X)$ can be formulated as an integer program as per Proposition 1. The decision version of this problem would be: given $B = [0, u]$, a membership oracle for a compact convex $X \subset B$, and a positive integer $k$, does there exist a $x \in X \cap B \cap \mathbb{Z}^n$ such that $\pi x \geq k$, where $\pi$ is given by equation (5). The complexity of optimizing a bounded integer program is well-known to be equivalent to the complexity of its decision version [cf. NW88, chap. I.5] through the use of bisection search on the range of values $k$ can take. This assumes that the encoding of this range is polynomial in the encoding size of the input, which is indeed the case if the objective function of the integer program is treated as an input. For $\text{LEX}_\sigma(X)$, the objective function $\pi$ is not an input but rather constructed using equation (5). The number of iterations for bisection search would be $O(\log n + \log \pi n + \log u_{\text{max}})$, where $u_{\text{max}} = \max_i u_i$. Since $\pi n$ is exponential in $n$, $\pi n = O(u_{\text{max}})$ to be precise, this number is equal to $O(n \log u_{\text{max}})$, and hence polynomial in encoding size of the input. Thus studying the complexity of the decision version of $\text{LEX}_\sigma(X)$ is an option for analyzing the complexity of $\text{LEX}_\sigma(X)$. Instead, we adopt a different approach by studying the complexity of finding integer solutions in a subset of $X$ obtained by intersecting $X$ with an arbitrary integral box. We combine this with a bisection search over each coordinate of $X$; the number of steps in this search is $O(n \log u_{\text{max}})$, same as before. The correctness of our bisection search in $\mathbb{R}^n$ is argued using properties of the lex order. Our approach yields the insight (cf. Proposition 9) that the hardness of lex optimization comes from a hard integer feasibility question for $X$ over some sub-box in $B$. We also show that the converse of this implication is not true in general.

For a compact convex $X \subset B$ and any integral box $B'$, we denote

$$\text{FEAS}_{B'}(X) : \quad \text{is } X \cap B' \cap \mathbb{Z}^n \neq \emptyset?$$

to be the integer feasibility question of interest to us. The answer to $\text{FEAS}_{B'}(X)$ is trivial for packing and covering polytopes. Indeed if $A \geq 0$, then for any $l, u \in \mathbb{Z}^n$, $\{x \in [l, u] \cap \mathbb{Z}^n \mid Ax \leq b\} \neq \emptyset$ if and only if $Al \leq b$, and $\{x \in [l, u] \cap \mathbb{Z}^n \mid Ax \geq b\} \neq \emptyset$ if and only if $Au \geq b$. For these polytopes, we observed in Proposition 3 that the lexmax and lexmin points are easy to compute. Then, naturally,
the question arises whether this connection generalizes, i.e., is it always true that an easy feasibility problem implies an easy lex problem? This is certainly not true for arbitrary integer programs because optimization over packing and covering polytopes is NP-hard. However, does the special structure of objective in the lex problem allow this connection to be true? We answer this in the negative in Proposition 8. Before getting there, first note that the lex problem cannot be easier than the feasibility problem over \( B \), because, clearly, the answer to \( \text{FEAS}_B(X) \) is yes if and only if a feasible solution is returned by solving \( \text{LEX}_\sigma(X) \).

**Lemma 2.** If \( \text{FEAS}_B(X) \) is NP-hard, then \( \text{LEX}_\sigma(X) \) is NP-hard for any \( \sigma \in S_n \).

Integer feasibility of polytopes over the \([0, 1]^n \) box is well-known to be NP-hard in general due to a straightforward reduction from the boolean SAT problem [cf. CCZ14]: clause \( C_i \) can be encoded as the constraint

\[
\sum_{j \in C_i^+} x_j^i + \sum_{j \in C_i^-} (1 - x_j^i) \geq 1, \tag{16}
\]

where \( x_j^i \in \{0, 1\} \) corresponds to the \( j^{th} \) literal in \( C_i \) and \( (C_i^+, C_i^-) \) is a partition of literals in \( C_i \) such that \( j \in C_i^- \) if and only if the \( j^{th} \) literal is a negation. Thus it follows that \( \text{LEX}_\sigma(X) \) is hard for a polytope \( X = \{ x \in [0, u] \mid Ax \leq b \} \) where the matrix \( A \) has arbitrarily many positive and negative entries. Even after restricting the number of positives and negatives, the problem remains hard since the integer feasibility problem is hard due to reductions of the form (16) from hard variants of SAT such as MONOTONE3SAT, MAX2SAT, and 3SAT-k.

Just like arbitrary integer programs, the converse of Lemma 2 is not true in general, i.e., a easy feasibility question over \( B \) does not imply an easy lex problem.

**Proposition 8.** There exist polytopes \( X \subset [0, 1]^n \) for which \( \text{FEAS}_{[0, 1]^n}(X) \) is polynomial-time solvable but \( \text{LEX}_\sigma(X) \) is NP-hard for some \( \sigma \in S_n \).

**Proof.** Consider the traveling salesman problem on \( K_n \), the complete graph on \( n \) vertices, and let \( X^\text{tsp} \subset [0, 1]^{(\binom{n}{2})} \) be the subtour elimination polytope. Since the graph is complete, \( \text{FEAS}_{[0, 1]^n}(X^\text{tsp}) \) is trivially yes. Let \( G = (V, E) \) be an arbitrary edge-induced subgraph of \( K_n \) on \( n \) vertices. Denote \( E' \) to be the complement of the edges in \( G \). Choose \( \sigma^* \) to be a permutation that orders the variables in \( x \) as: all \( x_e \) with \( e \in E \) first and then all \( x_e \) with \( e \in E' \). Denote \( x^* := \min \text{LEX}_{\sigma^*}(X^\text{tsp}) \) and partition it as \( (x^*_{E'}, x^*_{E'}) \). We now argue that \( G \) contains a Hamiltonian cycle if and only if \( (x^*_{E'}, x^*_{E'}) \leq_{\sigma^*} (1, 0) \). Since \( x^* \) is a feasible solution of the TSP, \( x^* \) must define a Hamiltonian cycle in \( K_n \). If \( x^* \preceq_{\sigma^*} (1, 0) \), then \( x^*_{E'} = 0 \) and since \( G \) contains the same \( n \) vertices as \( K_n \), \( x^*_{E'} \) must define a Hamiltonian cycle in \( G \). If \( G \) contains a Hamiltonian cycle, whose incidence vector we denote by \( y_E \in \{0, 1\}^{\binom{n}{2}} \), then \( (y_E, 0) \in X^\text{tsp} \cap \{0, 1\}^{\binom{n}{2}} \) and so \( x^* \preceq_{\sigma^*} (y_E, 0) \preceq_{\sigma^*} (1, 0) \). Now the fact that finding a Hamiltonian cycle in an arbitrary \( G \) is NP-hard and that \( (x^*_{E'}, x^*_{E'}) \preceq_{\sigma^*} (1, 0) \) can be checked in linear time implies that the computation of \( \min \text{LEX}_{\sigma^*}(X^\text{tsp}) \) is NP-hard, even though \( \text{FEAS}_{[0, 1]^n}(X^\text{tsp}) \) is trivial.

Proposition 8 tells us that the hardness of the lex problem may come from a hard feasibility problem over a “hidden” sub-box of \( B \). Hence one would need to check feasibility easily over sub-boxes of \( B \) to certify whether the lex problem is easy. We next establish that doing this for all integral sub-boxes of \( B \) is enough to guarantee that the lex problem is easy.

**Proposition 9.** \( \text{LEX}_\sigma(X) \) is NP-hard only if \( \text{FEAS}_{B'}(X) \) is NP-hard for some integral box \( B' \subseteq B \).
Proof. We solve $\text{LEX}_\sigma(X)$ by making a polynomial number of class to $\text{FEAS}_{B'}(X)$. First solve $\text{FEAS}_B(X)$. If the answer to $\text{FEAS}_B(X)$ is no, then $\text{LEX}_\sigma(X)$ is infeasible. If $B$ is a singleton then this point is a trivial answer to $\text{LEX}_\sigma(X)$. Otherwise, recursively construct a sequence of integral boxes $\{B_i\}_{i \geq 0}$ as follows. Initialize $i = 0$ and $B_0 = B$. For $i \geq 1$, let $k_i := \max \{k \mid \exists x, y \in B_{i-1} \text{ s.t. } x\sigma(k) \neq y\sigma(k)\}$ be the highest significant index such that the current box has a nonzero dimension along the corresponding coordinate in the given permutation. Also, $u_{k_i} := \max \{x\sigma(k_i) \mid x \in B_{i-1}\}$, $l_{k_i} := \min \{x\sigma(k_i) \mid x \in B_{i-1}\}$, and $m_{k_i} := \left\lfloor \frac{u_{k_i} + l_{k_i}}{2} \right\rfloor$. Consider

$$B^\geq := \{x \in B_{i-1} \mid x\sigma(k_i) \geq m_{k_i}\}, \quad B^\leq := \{x \in B_{i-1} \mid x\sigma(k_i) \leq m_{k_i} - 1\}.$$ 

Solve $\text{FEAS}_{B^\geq}(X)$. If the answer is yes, set $B_i = B^\geq$, otherwise set $B_i = B^\leq$. Terminate when $B_i$ is a singleton. We claim that this point is equal to max $\text{LEX}_\sigma(X)$. To prove this claim, our termination criteria and because we have already verified $X \cap \mathbb{Z}^n \neq \emptyset$ by solving $\text{FEAS}_B(X)$ initially, make it sufficient to show that max $\text{LEX}_\sigma(X)$ belongs to $B_i$ for all $i$. We argue this by induction on $i$. Denote $x^* := \max \text{LEX}_\sigma(X)$. Trivially, $x^* \in B_0$. Assume $x^* \in X \cap B_{i-1}$ for $i \geq 1$. We claim that a yes answer to $\text{FEAS}_{B^\geq}(X)$ implies $x^* \in B^\geq$ and a no answer implies $x^* \in B^\leq$. By our construction of $B_i$, this claim implies $x^* \in B_i$. Clearly $B_{i-1} \cap \mathbb{Z}^n = (B^\geq \cap \mathbb{Z}^n) \cup (B^\leq \cap \mathbb{Z}^n)$. If $\text{FEAS}_{B^\geq}(X)$ returns no, then the induction hypothesis gives us $x^* \in B^\leq$. Suppose that $\text{FEAS}_{B^\geq}(X)$ is yes but $x^* \notin B^\geq$. Since $x^* \in B_{i-1}$, $x^* \in B^\geq$ and so $x^*\sigma(k_i) \leq m_{k_i} - 1$. We have assumed $\text{FEAS}_{B^\geq}(X)$ is yes and so denote $y := \max \text{LEX}_\sigma(X \cap B^\geq)$. The construction of $k_i$ tells us that $y\sigma(k_i) = x^*\sigma(k_i)$ for $k > k_i$. Now $y\sigma(k_i) \geq m_{k_i}$ and $x^*\sigma(k_i) \leq m_{k_i} - 1$ implies $x^* \prec y$. But this is a contradiction to the maximality of $x^*$. Therefore $x^* \in B^\geq$.

The above sequence performs a bisection on the maximum and minimum values of each dimension in $B$. Hence, the sequence $\{B_i\}$ will have $O(n \log u_{\text{max}})$ many elements. It follows that if $\text{FEAS}_{B'}(X)$ can be solved in polynomial time for every $B' \subseteq B$, then $\text{LEX}_\sigma(X)$ can be solved in polynomial time.

Proposition 9 generalizes Proposition 3. Its converse is not true in general. This can be argued by considering the polytope $X^{tsp}$ from Proposition 8. Let $B'$ be a sub-box of $[0, 1]^n$ corresponding to an arbitrary edge-induced subgraph $G$ of $K_n$. Checking integer feasibility of $X^{tsp}$ over this sub-box is equivalent to finding a Hamiltonian cycle in $G$, which is NP-hard. But the lex optimization problem is trivial when $\sigma$ is a permutation that orders all edges in some Hamiltonian cycle in $K_n$ at the end and all other edges before.

### 5.2 Many lex constraints

In this section, we consider $X$ to be a bounded convex set defined as the intersection of a polyhedron and finitely many lex constraints over a box, where a lex constraint is either $x \leq_\sigma \theta$ or $x \geq_\sigma \theta$, for given $\theta \in \mathbb{Z}^n, \sigma \in S_n$. As noted in the introduction, any lex constraint defines a convex cone in $\mathbb{R}^n$ that is neither open nor closed. Hence $X$ is not compact. But this shouldn’t matter much since we are interested in integer points inside $X$ and the convex hull of $X \cap \mathbb{Z}^n$ is a polytope. Henceforth, for convenience, we assume that the permutation $\sigma$ in the objective of (1) is the identity permutation and omit it from any subscript and superscript.

Given some $\theta^i, \gamma^j \in B \cap \mathbb{Z}^n$ and $\sigma_i, \tau_j \in S_n$ for $i = 1, \ldots, m_1, j = 1, \ldots, m_2$, define

$$\mathcal{L}_{m_1, m_2} := \{x \in B \mid x \leq_\sigma \theta^i, \ i = 1, \ldots, m_1, x \geq_\tau \gamma^j, \ j = 1, \ldots, m_2\}$$

to be a intersection of lex-ordered sets. For nontriviality, we assume that $\sigma_i \neq \sigma_j$ and $\tau_i \neq \tau_j$ for $i \neq j$. This lex-ordered set may be present explicitly in some optimization problems as mentioned in the introduction or it may arise as a relaxation of an arbitrary discrete set as seen in equation (8).
We analyze the complexity of lex optimization for
\[ X = \{ x \in B \mid x \in \mathcal{P} \cap \mathcal{L}_{m_1,m_2} \}, \quad \text{where } \mathcal{P} = \{ x \mid Ax \leq b \}. \]

The reductions from boolean SAT in equation (16) create a linear system whose coefficients are not superincreasing, meaning that the individual constraints do not correspond to \( \leq \) or \( \geq \) orders. Thus the answer to this complexity question is not implied by the NP-hardness of general integer programming feasibility. We can immediately conclude the following though.

**Corollary 1.** If either \( A \geq 0, m_2 = 0 \) or \( A \leq 0, m_1 = 0 \), then \( \text{LEX}(X) \) is in \( P \).

**Proof.** For the first (resp. second) condition, Lemma 1 implies that \( X \cap \mathbb{Z}^n = \{ x \in B \cap \mathbb{Z}^n \mid Ax \leq \bar{b} \} \) for some \( A \geq 0 \) (resp. \( A \leq 0 \)), and so \( \text{LEX}(X) \) corresponds to the case of a packing (resp. covering) polytope which we know from Proposition 3 is easy. \( \square \)

Every \( \preceq \) (resp. \( \succeq \)) order over \( \mathbb{Z}^n \) can be written as the union of \( n \leq \) (resp. \( \geq \)) inequality comparisons. In particular, \( x \preceq_\sigma \theta \) if and only if \( x = \theta \) or \( x \in \bigcup_{i=1}^n B_i \) where \( B_i := \{ x \in \mathbb{Z}^n \mid x_{\sigma(i)} \leq \theta_{\sigma(i)} - 1 \} \) represents the \( i^{th} \) fixing. Since \( \mathcal{L}_{m_1,m_2} \) has \( m_1 + m_2 \) lex orders, there are a total of \( n^{m_1+m_2} \) fixings of \( \mathcal{L}_{m_1,m_2} \), and so \( \text{FEAS}_B(\mathcal{L}_{m_1,m_2}) \) can be solved by enumeration in \( O(n^{m_1+m_2}) \) time. Furthermore, for each of these fixings, \( \text{LEX}(X) \) is equivalent to \( \text{LEX}(\mathcal{P} \cap B') \) for some \( B' \subseteq B \). Thus this simple enumeration algorithm solves \( \text{LEX}(X) \) by making \( O(n^{m_1+m_2}) \) many calls to an algorithm for finding the lexmax point in \( Ax \leq b \) over some box. When both \( m_1 \) and \( m_2 \) are bounded by a constant, this running time is polynomial if and only if \( \text{LEX}(\mathcal{P} \cap B') \) is polynomial time. Our main result here is to show that when \( m_1 + m_2 \) is not bounded by a constant, then the problem is hard in general.

**Theorem 4.** \( \text{LEX}(X) \) is NP-hard when at least one of \( m_1 \) or \( m_2 \) is arbitrary. On the contrary, if \( m_1 + m_2 \) is bounded by a constant and \( \text{FEAS}_B(\mathcal{P}) \) is in \( P \) for any \( B' \subseteq B \), then \( \text{LEX}(X) \) is in \( P \).

If \( A \geq 0 \) in the description of \( \mathcal{P} \), \( \text{FEAS}_B(\mathcal{P}) \) is polynomial time and so the above theorem implies that the boundedness of both \( m_1 \) and \( m_2 \) is sufficient to make the lex problem easy for packing polytopes intersected with \( \mathcal{L}_{m_1,m_2} \). We show that the dependence on \( m_1 \) can be removed.

**Theorem 5.** Let \( A \geq 0 \) in the description of \( X \). \( \text{LEX}(X) \) is NP-hard if \( m_2 \) is arbitrary. On the contrary, if \( m_2 \) is bounded by a constant, then \( \text{LEX}(X) \) is in \( P \).

The \( A \leq 0 \) case is similar by complementing variables. Thus, Theorem 5 generalizes Corollary 1.

We will prove that even in the absence of linear constraints in the definition of \( X \), finding a lex optimal point over a collection of \( \preceq \) and \( \succeq \) constraints is hard. If we allow linear constraints, then the problem remains hard even when considering points in a halfspace satisfying only \( \preceq \) or only \( \succeq \) orders.

**Proposition 10.** Consider \( B = [0,1]^n \). We have the following.

1. \( \text{LEX}(\mathcal{L}_{m_1,m_2}) \) is NP-hard for arbitrary \( m_1 \) and \( m_2 \).

Let \( H := \{ x \in \mathbb{R}^n \mid a^\top x \leq a_0 \} \) for some \( (a,a_0) \in \mathbb{Z}^{n+1} \). Then \( \text{LEX}(H \cap \mathcal{L}_{m_1,m_2}) \) is NP-hard if

2. \( a \leq 0 \), \( m_1 \) is arbitrary, and \( m_2 = 0 \),

3. \( a \geq 0 \), \( m_1 = 0 \), and \( m_2 \) is arbitrary.
Proof. We give a polynomial reduction from $\text{SAT}$, the boolean satisfiability problem, to each of $\text{FEAS}_{[0,1]^n}(\mathcal{L}_{m_1,m_2})$, $\text{FEAS}_{[0,1]^n}(H \cap \mathcal{L}_{m_1,0})$, and $\text{FEAS}_{[0,1]^n}(H \cap \mathcal{L}_{0,m_2})$, under appropriate assumptions on $H$. Lemma 2 then implies NP-hardness of the corresponding lex optimization problems.

For notational convenience, we adopt the following simplification in our reduction: for $x \preceq_{\sigma} \theta$ for which there exists some $2 \leq k \leq n-1$ such that $\theta_{\sigma(i)} = 1$ for all $1 \leq i \leq n-k$, we express $x \preceq_{\sigma} \theta$ in its equivalent form $(x_{i_1}, x_{i_2}, \ldots, x_{i_k}) \preceq (\theta_{i_1}, \theta_{i_2}, \ldots, \theta_{i_k})$, where $i_1 = \sigma(n-k+t)$ for $1 \leq t \leq k$ (i.e., $(x_{i_1}, x_{i_2}, \ldots, x_{i_k})$ are the $k$-most significant variables under the permutation $\sigma$). Similarly for $\succeq$ constraints.

Consider any instance of $\text{SAT}$ with $n$ booleans $(a_1, \ldots, a_n)$ and $m$ clauses of the form $C_i = \ell_1^i \lor \ell_2^i \lor \cdots \lor \ell_k^i$ for $i = 1, \ldots, m$. Represent the variables as $x_{a_i}$ and $x_{\neg a_i}$. We construct the instances of $\mathcal{L}_{m_1,m_2}$, $H \cap \mathcal{L}_{m_1,0}$, and $H \cap \mathcal{L}_{0,m_2}$ as follows, and denote them, respectively, by $X_1$, $X_2$, and $X_3$.

$X_1$: Set $m_1 = n + m$ and $m_2 = n$. For $i = 1 \ldots n$, let the $i^{th}$ constraint be $(x_{a_i}, x_{\neg a_i}) \preceq (0, 1)$ and the $i^{th}$ constraint be $(x_{a_i}, x_{\neg a_i}) \succeq (1, 0)$. The $\preceq$ constraints are $(x_{\neg \ell_1^i}, x_{\neg \ell_2^i}, \ldots, x_{\neg \ell_k^i}) \preceq (0, 1, \ldots, 1)$ for $i = 1, \ldots, m$.

$X_2$: The $n + m \preceq$ constraints are same as above. The halfspace is $H = \{x \in \mathbb{R}^{2n} | -\mathbb{1}^\top x \leq -n\}$.

$X_3$: The $n + m \succeq$ constraints are $(x_{a_i}, x_{\neg a_i}) \succeq (1, 0)$ for $i = 1, \ldots, n$, and $(x_{\ell_1^i}, x_{\ell_2^i}, \ldots, x_{\ell_k^i}) \succeq (1, 0, \ldots, 0)$ for $i = 1, \ldots, m$. The halfspace is $H = \{x \in \mathbb{R}^{2n} | \mathbb{1}^\top x \leq n\}$.

Now we argue that the following are equivalent:

$\text{SAT}$ has a feasible truth assignment, $X_1 \cap \{0, 1\}^{2n} \neq \emptyset$, $X_2 \cap \{0, 1\}^{2n} \neq \emptyset$, $X_3 \cap \{0, 1\}^{2n} \neq \emptyset$.

Let $a$ be a satisfying assignment and set $x_{a_i} = 1$ if $a_i$ is true and 0 otherwise. We have $x_{a_i} + x_{\neg a_i} = 1$ and so it is clear that $(x_{a_i}, x_{\neg a_i}) \preceq (0, 1)$ and $(x_{a_i}, x_{\neg a_i}) \succeq (1, 0)$ for all $i$ and $\sum_{i=1}^n (x_{a_i} + x_{\neg a_i}) = n$. Since for every clause $C_i$ at least one literal $\ell_i^t$ must be true, it follows that $(x_{\neg \ell_1^i}, x_{\neg \ell_2^i}, \ldots, x_{\neg \ell_k^i}) \preceq (0, 1, \ldots, 1)$ and $(x_{\ell_1^i}, x_{\ell_2^i}, \ldots, x_{\ell_k^i}) \succeq (1, 0, \ldots, 0)$ for every $i$. Hence (i) $\implies$ (ii) and (iii) and (iv). For the reverse direction, we first claim that any $x \in (X_1 \cup X_2 \cup X_3) \cap \{0, 1\}^{2n}$ satisfies $x_{a_i} + x_{\neg a_i} = 1$ for all $i$. For integral $x \in X_1$, this is immediate due to the two constraints $(x_{a_i}, x_{\neg a_i}) \preceq (0, 1)$ and $(x_{a_i}, x_{\neg a_i}) \succeq (1, 0)$. For integral $x \in X_2$, we have $x_{a_i} + x_{\neg a_i} \leq 1$ from $(x_{a_i}, x_{\neg a_i}) \preceq (0, 1)$. This leads to $-\sum_{j=1}^n (x_{a_j} + x_{\neg a_j}) + \sum_{j \neq i} (x_{a_j} + x_{\neg a_j}) \leq -n + \sum_{j \neq i} 1$ for all $i$. This inequality simplifies to $x_{a_i} + x_{\neg a_i} \geq 1$ and hence we get $x_{a_i} + x_{\neg a_i} = 1$ for all $i$. For integral $x \in X_3$, we have $x_{a_i} + x_{\neg a_i} \geq 1$, which implies $1^\top x \geq n$. Since $x \in H$, we get $1^\top x = n$ and $x_{a_i} + x_{\neg a_i} = 1$ for all $i$. Now suppose $x \in \{0, 1\}^{2n}$ satisfies $x_{a_i} + x_{\neg a_i} = 1$ for all $i$ but the $n$-dimensional boolean vector $a$ corresponding to $x$ (i.e., $a_i$ is true if and only if $x_{a_i} = 1$) violates clause $C_i$ in $\text{SAT}$. Then it must be that $x_{\ell_i^j} = 0$, and so $x_{\neg \ell_i^j} = 1$, for all literals $\ell_i^j$ in $C_i$. But then $(x_{\neg \ell_1^i}, x_{\neg \ell_2^i}, \ldots, x_{\neg \ell_k^i}) = (1, \ldots, 1) \succ (0, 1, \ldots, 1)$ and so $x \notin X_1 \cup X_2$. The implication $x \notin X_3$ follows by observing that $(x_{\ell_1^i}, x_{\ell_2^i}, \ldots, x_{\ell_k^i}) = 1 \prec (x_{\neg \ell_1^i}, x_{\neg \ell_2^i}, \ldots, x_{\neg \ell_k^i})$ since $x_{a_i} + x_{\neg a_j} = 1 \forall j$. Therefore (ii) $\implies$ (i), (iii) $\implies$ (ii), and (iv) $\implies$ (i), thus completing our proof.

The decision versions of these lex problems are NP since every lex constraint is easily checked in $O(n)$ time and the encoding length of the certificate is polynomial in input size since the box $B$ is assumed as part of input. Hence, by Proposition 10, the decision versions of these lex problems are NP-complete.

We have the following implications.

**Corollary 2.** The dual bound $z_{\text{lex}}(S'_n)$ from (12) is NP-hard to compute for arbitrary $S'_n \subseteq S_n$. 

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Proof. By the first item in Proposition 10, optimization over $L_{m_1,m_2}$ is NP-hard.

Proof of Theorem 4. The NP-hardness follows from the three items in Proposition 10. If $FEAS_{B'}(P)$ is in P for any $B' \subseteq B$, then Proposition 9 tells us that $LEX(P \cap \tilde{B})$ is in P for any $\tilde{B} \subseteq B$. The naive enumerative algorithm mentioned before the statement of this theorem solves $LEX(X)$ by making $O(n^{m_1+m_2})$ many calls to $LEX(P \cap \tilde{B})$. If both $m_1$ and $m_2$ are bounded, the overall running time of this enumeration is polynomial in input size.

Proof of Theorem 5. The NP-hardness follows from the first and third items in Proposition 10. Enumerating over all fixings of the $\geq$ constraints, similar to what was done for Theorem 4, gives us a algorithm that makes $O(n^{m_2})$ many calls to a lexmax problem over a packing polytope with $\leq$ constraints. Corollary 1 tells us that each such call can be solved in polynomial time. The overall running time of this enumeration becomes polynomial when $m_2$ is bounded by a constant.

5.2.1 Largest integer problem

We now give a hardness result for a optimization variant of the feasibility problem over $L_{m_1,m_2}$. Assume $B = [0,1]^n$. Recall that when $u_i = 1 \forall i$, the superincreasing $\pi$ sequence in Lemma 1 with $\sigma = id$ becomes $\pi_i = 2^{i-1} \forall i$. For any $x \in \{0,1\}^n$, the expression $\sum_{i=1}^n 2^{i-1}x_i$ resembles the base-2 (binary) expansion of some nonnegative integer $\rho$ so that we may denote $x = bin(\rho)$. It is a well-known fact that $bin(\cdot)$ induces a bijection between nonnegative integers and $0\backslash1$ vectors. Also for any $\theta \in \{0,1\}^n$, $bin(\rho) \preceq \theta$ if and only if $\rho \leq \sum_{i=1}^n 2^{i-1}\theta_i$. Note that this equivalence holds only when the lex order is considered under the identity permutation of variables. Given $\theta^i, \gamma^j \in \{0,1\}^n$ and $\sigma_i, \tau_j \in S_n$, for $i \in [m_1], j \in [m_2]$, defining the set $L_{m_1,m_2}$ of arbitrarily many lex orders under different permutations, and an arbitrary constant $c$, the question that we consider is the following:

$$2\text{INTLEX}: \max_{\rho, \lambda} \{ \lambda - c \mid \lambda \in \mathbb{Z}_+, \exists \rho \geq 2^\lambda \text{ s.t. } \bin(\rho) \in L_{m_1,m_2} \}. \quad (17)$$

The constant $c$ is added for the sake of convenience. This problem seeks the largest power of 2 such that the binary expansion of some integer at least as big as this power of 2 belongs to the lex-ordered set. Since $FEAS_{[0,1]^n}(L_{m_1,m_2})$ was shown to be NP-hard in Proposition 10, it follows that it is NP-hard to solve $2\text{INTLEX}$ optimally. We prove that it is also hard to approximate within a certain constant multiplicative factor.

Theorem 6. Assuming $P \neq \text{NP}$, there does not exist a polynomial-time algorithm to find a $\alpha$-approximate solution to $2\text{INTLEX}$, for any $\alpha < 2.402$.

Proof. Let $\preceq_{\text{strict}}$ denote a factor-preserving reduction between two optimization problems. We prove that $\text{INDSET} \preceq_{\text{strict}} 2\text{INTLEX}' \preceq_{\text{strict}} 2\text{INTLEX}$, where $\text{INDSET}$ is the independent set problem and $2\text{INTLEX}'$ is a closely-related problem that we define next. The statement of the theorem then follows from the well-known hardness of approximation for $\text{INDSET}$ due to Dinur and Safra [DS02].

The problem $2\text{INTLEX}'$ is

$$2\text{INTLEX}': \max f(\chi) := |\log_2 \pi \chi| - c \text{ s.t. } \chi \in L_{m_1,m_2} \cap \{0,1\}^n, \text{ where } \pi_i = 2^{i-1} \forall i.$$  

We argue that this problem preserves the optimal value of $2\text{INTLEX}$ and its feasible set is the image of the feasible set of $2\text{INTLEX}$ under a surjective map that does not reduce the solution value.

Claim 2. $OPT(2\text{INTLEX}) = OPT(2\text{INTLEX}')$, and for any $(\rho, \lambda)$ feasible to $2\text{INTLEX}$ there exists a $\chi$ feasible to $2\text{INTLEX}'$ such that $f(\chi) \geq \lambda - c$.
Proof. Since \( \rho \mapsto \text{bin}(\rho) \) is a bijection, the map \((\rho, \lambda) \mapsto \text{bin}(\rho)\) induces a many-to-one correspondence between the feasible sets of 2INTLEX and 2INTLEX'. As explained before, \( \rho = \pi \text{bin}(\rho) \). So for any feasible \((\lambda, \rho)\), we have \( \lambda \leq \lfloor \log_2 \pi \text{bin}(\rho) \rfloor \). Setting \( \chi = \text{bin}(\rho) \) gives us \( f(\chi) \geq \lambda - c \). If \((\rho^*, \lambda^*)\) is optimal to 2INTLEX then \( \lfloor \log_2 \pi \text{bin}(\rho^*) \rfloor = \lambda^* \) and so \( \text{OPT}(2\text{INTLEX}) = \text{OPT}(2\text{INTLEX}') \).  

The above claim implies 2INTLEX' \( \preccurlyeq_{\text{strict}} \) 2INTLEX.

Now we prove INDSET \( \preccurlyeq_{\text{strict}} 2\text{INTLEX}' \). Given any undirected graph \( G = (V, E) \) with \( n \) vertices and \( m \) edges, construct an instance of 2INTLEX' with \( c = n^2 + n \) and the following variables whose binary values are used to establish a relationship with an independent set in \( G \):

1. \( h_{i,k} \), for \( 1 \leq i, k \leq n \): value equal to 0 means vertex \( i \) is the \( k \)th vertex (counting the smallest numbered vertex as the first vertex and so on) in the independent set.

2. \( v_i \), for \( 1 \leq i \leq n \): value equal to 1 means vertex \( i \) is in the independent set.

3. \( x_k \), for \( 1 \leq k \leq n \): value equal to 1 means the size of the independent set is at least \( k \).

Thus we have \( n^2 + 2n \) variables in 2INTLEX'. We assume that the ordering of variables is \((h, v, x)\) under the identity permutation. Construct \( L_{m_1, m_2} \) with the following lexicographic constraints, assuming the same notational simplification for \( \prec \) and \( \succ \) that was used in the proof of Proposition 10:

1. For \( i \in V \), add \((v_{i_1}, \ldots, v_{i_p}, v_i) \preceq (0, \ldots, 0, 1)\), where \( i_1, \ldots, i_p \) are the neighbors of \( i \) in \( G \).

   This forces \( v_{i_1} = \cdots = v_{i_p} = 0 \) if \( v_i = 1 \), thereby ensuring a valid independent set if vertex \( i \) is chosen.

2. For \( 1 \leq s \leq k \leq n \), add \((x_k, h_{1,s}, \ldots, h_{n,s}) \preceq (0, 1, \ldots, 1)\).

   This forces \( x_k = 0 \) if \( h_{1,s} = \cdots = h_{n,s} = 1 \) for some \( s \leq k \), ensuring that the size of the independent set is less than \( k \) if no vertex is in the \( s \)th position.

3. For \( i, k \in \{1, \ldots, n\} \), add \((x_k, h_{i,k}) \succ (1, 0)\).

   If vertex \( i \) is in the \( k \)th position (\( h_{i,k} = 0 \)), then the independent set has size at least \( k \) (\( x_k = 1 \)).

4. For \( 1 \leq i \leq n \), add \((h_{i,1}, h_{i,2}, \ldots, h_{i,n}, v_i) \succ (1, \ldots, 1, 0)\).

   This forces \( h_{i,1} = \cdots = h_{i,n} = 1 \) if \( v_i = 0 \) and so vertex \( i \) cannot be the \( k \)th vertex in an independent set if it is not included.

5. For \( i, k \in \{1, \ldots, n\} \), add \((h_{1,k}, \ldots, h_{i-1,k}, h_{i+1,k}, \ldots, h_{n,k}, h_{i,k}) \succ (1, \ldots, 1, 0)\).

   This means that if \( i \) is the \( k \)th vertex in the independent set then it cannot be any other vertex in the independent set.

We have \( m_1 = (n^2 + 3n)/2 \) and \( m_2 = 2n^2 + n \) and the above construction works in polynomial time.

Consider any \( \chi = (h, v, x) \in L_{m_1, m_2} \) and assume it is nontrivial, i.e., \( h \neq 1 \). Suppose \( k \) is the maximal integer such that \( h_{i,k} = 0 \) for some \( i_k \). Then the third constraint implies \( x_k = 1 \). It follows from the maximality of \( k \) and the superincreasing nature of \( \pi \) that \( 2n^2 + n + 1 \leq \pi \chi < 2n^2 + n + k \), which leads to \( k - 1 \leq \log_2 \pi \chi - (n^2 + n) < k \). Hence the objective value for the feasible solution \((h, v, x)\) satisfies \( f(h, v, x) = k - 1 \). The second constraint implies that for every \( 1 \leq s \leq k \), there exists some \( i_s \) such that \( h_{i_s,s} = 0 \). The fifth constraint guarantees that \( \{i_1, i_2, \ldots, i_k\} \) are distinct. Applying the third constraint for \((x_s, h_{i_s,s})\) gives us \( x_1 = x_2 = \cdots = x_k = 1 \). Using \( h_{i_s,s} = 0 \) for \( s \leq k \) in the fourth constraint leads to \( v_{i_1} = v_{i_2} = \cdots = v_{i_k} = 1 \). The first constraint now implies that \( S = \{i_1, i_2, \ldots, i_k\} \) is a independent set in \( G \) of size \( k \).
Let \( OPT(\text{INDSET}) = k^* \) and \( S^* \) be a maximum independent set in \( G \). Setting \( x_i^* = 1 \) for \( 1 \leq i \leq k^* \), \( x_i^* = 0 \) for \( i > k^* \), \( v_i^* = 1 \) if and only if \( i \in S^* \), and \( h_{i,k}^* = 0 \) if and only if \( k \leq k^* \) and \( i \) is in the \( k^{th} \) position in \( S^* \), produces a feasible solution to \( 2\text{INTLEX}' \). Furthermore, this solution is also optimal because otherwise our construction of the five constraints would imply there exists a better solution \((\tilde{h}, \tilde{v}, \tilde{x})\) with \( \tilde{x}_{k+1} = 1 \), and then using the mapping to an independent set from the previous paragraph we would arrive at a contradiction to the optimality of \( S^* \). Therefore \( OPT(2\text{INTLEX}') = f(h^*, v^*, x^*) = k^* - 1 \).

Thus the approximation ratio for \( \text{INDSET} \) and \( 2\text{INTLEX}' \) is \( \frac{k^*}{k} \) and \( \frac{k^*-1}{k-1} \), respectively, and since \( \frac{k^*}{k} \leq \frac{k^*-1}{k-1} \) for \( 1 \leq k \leq k^* \), we have shown \( \text{INDSET} \preceq_{\text{strict}} 2\text{INTLEX}' \).

\[ \square \]

**Remark 3.** Applying the hardness result of Håstad [Hås99] for \( \text{INDSET} \), we get that \( 2\text{INTLEX} \) is inapproximable within a factor of \((\sqrt{n} + 1 - 1)^{1-\epsilon}\) for any \( \epsilon > 0 \).

Taking the floor of \( \log_2 \pi x \) in \( 2\text{INTLEX}' \) is essential for establishing the factor-preserving property of the above reduction from \( \text{INDSET} \). In the absence of this floor, we have a related problem \( \text{INTLEX} \): find the largest \( \rho \in \mathbb{Z}_m \) such that \( \bin(\rho) \in L_{m_1, m_2} \). In that case, the above reduction is a L-reduction from \( \text{INDSET} \) and allows us to conclude that there is no polynomial time approximation scheme (PTAS) for \( \text{INTLEX} \).

### 6 Concluding Remarks

We analyzed structural properties of lexicographically maximal and minimal points in a compact convex set. Theorems 2 and 3 provide characterizations of \( 0 \setminus 1 \) sets in terms of their lex optima. It follows that facet-defining inequalities can be obtained for subsets of \( \{0, 1\}^n \) by studying the combinatorial interplay between lex orders under different permutations. Although we did not explore convex hull descriptions, we did address the complexity of finding lex optima, which is important not only if one were to use lex-ordered sets to find facets of the convex hull but also from a structural perspective. Proposition 9 and Theorems 4, 5, and 6 are our main complexity results, along with a cutting plane algorithm in Proposition 7.

There are several open questions worth mentioning. We do not know the complexity of enumerating all the distinct lex optimal points or counting the number of such points. This seems important given our characterization of \( 0 \setminus 1 \) sets in terms of lex optima. Since every lex optimal point is a vertex of the convex hull, existing results on vertex enumeration of polyhedra [FLM97; Kha+08] can serve as a starting point for this question. In terms of the structural properties of lex optima, we do not know the quality of the dual bounds in equations (12) and (13), which correspond to the relaxations \( \text{conv}(\cap_{\sigma \in S_n} L^\sigma) \) and \( \cap_{\sigma \in S_n} \text{conv} L^\sigma \), respectively, with respect to the integer optimum over \( X \). We obtained a tight approximation factor on the primal side of the integer optimum in Theorem 3. Can an approximation factor be obtained for these two dual bounds? Corollary 2 shows that computing the dual bound due to \( \text{conv}(\cap_{\sigma \in S_n} L^\sigma) \) is NP-hard. It seems that computing the dual bound due to \( \cap_{\sigma \in S_n} \text{conv} L^\sigma \) might also be hard, but we do not know this. Note that although a linear function can be optimized over the permutahedron convex \( S_n \) in polynomial time [Goe15], the function \( \psi_c(\sigma) := \max_{x} \{ cx \mid x \in \text{conv} L^\sigma \} \) is likely not linear in \( \sigma \). Finally, we also mention a complexity question for integer knapsacks. Given a \( \sigma \in S_n \), Proposition 6 tells us precisely when the knapsack is equal to the lex-ordered set for this \( \sigma \). However, it does not tell us anything about how to identify a permutation \( \sigma \) for which this equality holds true, or certifying no such permutation exists. The complexity of separating \( \sigma \) from \( S_n \) such that \( X \cap \mathbb{Z}^n = L^\sigma \), remains unknown.
A Missing proofs

Proof of Lemma 1. Let $x \prec_\sigma y$ with $x_{\sigma(i)} \leq y_{\sigma(i)} - 1$, $x_{\sigma(j)} = y_{\sigma(j)} \forall j > i$. Then

$$
\pi(x - l) = \sum_{k=1}^{i-1} \pi_{\sigma(k)}(x_{\sigma(k)} - l_{\sigma(k)}) + \pi_{\sigma(i)}(x_{\sigma(i)} - l_{\sigma(i)}) + \sum_{k=i+1}^{n} \pi_{\sigma(k)}(x_{\sigma(k)} - l_{\sigma(k)})
$$

$$
\leq \sum_{k=1}^{i-1} \pi_{\sigma(k)}(x_{\sigma(k)} - l_{\sigma(k)}) - \pi_{\sigma(i)} + \pi_{\sigma(i)}(y_{\sigma(i)} - l_{\sigma(i)}) + \sum_{k=i+1}^{n} \pi_{\sigma(k)}(y_{\sigma(k)} - l_{\sigma(k)})
$$

$$
= \sum_{k=1}^{i-1} \pi_{\sigma(k)}(x_{\sigma(k)} - u_{\sigma(k)}) + \pi_{\sigma(i)}(y_{\sigma(i)} - l_{\sigma(i)}) + \sum_{k=i+1}^{n} \pi_{\sigma(k)}(y_{\sigma(k)} - l_{\sigma(k)}) - 1
$$

$$
\leq 0 + \pi_{\sigma(i)}(y_{\sigma(i)} - l_{\sigma(i)}) + \sum_{k=i+1}^{n} \pi_{\sigma(k)}(y_{\sigma(k)} - l_{\sigma(k)}) - 1
$$

$$
< \pi(y - l),
$$

implying that $\pi x < \pi y$. The second equality is obtained using $\pi_{\sigma(i)} = 1 + \sum_{k<i} \pi_{\sigma(k)}(u_{\sigma(k)} - l_{\sigma(k)})$. If $x \succ_\sigma y$, then interchanging $x$ and $y$ in above chain of inequalities gives us $\pi y < \pi x$. \hfill \Box

Proof of Proposition 4 in Remark 1. If $y \in S$, $a_i \leq 0$, then set $w_k = y_k$ if $k \neq i$ and $w_i = u_i$. Then $w \in S$. Since $w \geq y$, $w \succ_\sigma y$, therefore the lex-max point in $S$ will satisfy $x_i = u_i$ whenever $a_i \leq 0$. Assume for contradiction that $w \succ \theta$ and $w \in S$. Let $k$ be the most significant coordinate under $\sigma$ where $\theta_k \neq w_k$. Then $a_k > 0$ because $\theta_k = w_k$ otherwise. By the properties of the lex ordering $\theta_k < w_k$. Either $w_k > u_k$ or $w_k > \lfloor \frac{a_0 - \sum_{i \in N} a_i \theta_i}{a_{\sigma(k)}} \rfloor$. Since $w \in S$, $w_k \leq u_k$. By integrality $w_k \geq \lfloor \frac{a_0 - \sum_{i \in N} a_i \theta_i}{a_{\sigma(k)}} \rfloor + 1$, or $w_k > \lfloor \frac{a_0 - \sum_{i \in N} a_i \theta_i}{a_{\sigma(k)}} \rfloor$. Furthermore $w_i \geq 0$ for all $i$ so

$$
aw = \sum_i a_i w_i
$$

$$
= \sum_{a_i \leq 0} a_i w_i + \sum_{a_i > 0; i \neq k} a_i w_i + a_k w_k
$$

$$
> \sum_{a_i \leq 0} a_i \theta_i + \sum_{a_i > 0; i \neq k} a_i w_i + a_0 - \sum_{i \in N} a_i \theta_i
$$

$$
\geq \sum_{a_i \leq 0} a_i \theta_i + \sum_{a_i > 0; i \neq k} a_i w_i + a_0 - \sum_{a_i \leq 0} a_i \theta_i
$$

Therefore $aw \geq a_0$ and $w \notin S$. We have a contradiction.

Thus $\theta$ is the lex-max integer point in $S$ under the $\sigma$ permutation. \hfill \Box

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