TOV equations form of anisotropic pressure on two fluid models

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Abstract. We construct Tolman Oppenheimer Volkoff (TOV) equations for anisotropic pressure of two non-interacted perfect fluid with different four-velocities. This paper's primary motivation is that we obtain the result that could describe the neutron stars (NS) admixed with dark matter (DM) that satisfying constraint from GW190814. Otherwise, our result also can investigate other compact objects that are admixed with two non-interacting fluids. In this work, we extend the formalism in Ref [1] by considering each fluid's anisotropic pressure. Therefore, the total energy-momentum tensor is the sum of two non-interacting anisotropic fluids with different four-velocities. The velocity difference of both fluids denotes here by \( b \) where it comes from each of four-velocities under transformation rotation. Our result's direct impact is that we can assess each fluid's impact of anisotropic pressure in two fluids formalism in NS or other compact objects properties.

1. Introduction

For the first time, a two-perfect-fluid-formalism of an anisotropic fluid was studied by P. S. Letelier in 1986 [1]. The model has some standard features in plasma physics [2], and also some characteristics such as the model must be superficial and must have a physical interpretation. The energy-momentum-tensor of this formalism is the contribution from perfect fluid and two null fluids. In 2011, Harko and Lobo also employed this model to investigate the dark matter and its implication on cosmology [3]. However, Harko and Lobo's formalism is a little bit different from Letelier's one. The difference appears in the energy-momentum tensor. Their energy-momentum tensor only consists of the sum of two isotropic perfect fluid.

There are several authors who use Harko and Lobo’s formalism on neutron stars admixed dark matter. They can be seen in Ref [4, 5]. Unfortunately, several observations are not sufficient to prove the existence of neutron stars with dark matter inside at that time. Nowadays, LIGO & Virgo collaboration has reported that GW190814 detect a compact object mass with the range 2.50 – 2.67 \( M_\odot \) [6]. It could help us to prove the existence of neutron stars with dark matter inside. In addition, GW170817 provide several properties neutron stars at 1.4 \( M_\odot \) [7]. Those several properties GW170817 could help us to more understand the existence of neutron stars with dark matter inside.

In this work, we extend Harko & Lobo formalism by considering the total energy-momentum tensor as a sum of the two non-interacting anisotropic fluid contribution with different four-velocities. The treatment in the hydrostatic equilibrium of our formalism will be discussed in the formalism section.
2. Formalism

In this section, we revisit Harko & Lobo formalism [3] and discuss the treatment the corresponding hydrostatic equilibrium for the model. In this paper, we use $c = G = 1$. We start by assuming two matters, called $A$ matter denoted by $A$ and $B$ matter denoted by $B$, have energy densities, pressures, and four velocities. Energy density, pressure, and four velocities of $A$ matter denoted by $\epsilon_A, p_A,$ and $U^a$ while for $B$ matter denoted by $\epsilon_B, p_B,$ and $W^a$. The total energy-momentum tensor can be written as

$$T^{\alpha\beta} = (\epsilon_A + p_A)U^a U^\beta - p_A g^{a\beta} + (\epsilon_B + p_B)W^a W^\beta - p_B g^{a\beta}, \quad (1)$$

which $U^a U_a = 1$ and $W^a W_a = 1$, respectively.

Quadratic form of total energy-momentum tensor in Eq. (1) i.e. $(\epsilon_A + p_A)U^a U^\beta + (\epsilon_B + p_B)W^a W^\beta$ is invariant under the transformation

$$U^a \rightarrow U'^a = U^a \cos \gamma + \frac{\epsilon_B + p_B}{\epsilon_A + p_A} W^a \sin \gamma, \quad (2)$$

$$W^a \rightarrow W'^a = W^a \cos \gamma - \frac{\epsilon_A + p_A}{\epsilon_B + p_B} U^a \sin \gamma. \quad (3)$$

Thus

$$T^{\alpha\beta}(U, W) = T^{\alpha\beta}(U', W'). \quad (4)$$

Rotate $U^a$ and $W^a$, so that one becomes timelike and the other one becomes spacelike. Therefore, both vectors fulfill the condition

$$U'^a W'_a = 0. \quad (5)$$

$U'^a$ and $W'^a$ in defined in Eqs. (2) and (3). The implication of equation (5), we obtain the rotation angle given by

$$\tan 2\gamma = \frac{2\sqrt{(\epsilon_A + p_A)(\epsilon_B + p_B)}}{\epsilon_A + p_A - (\epsilon_B + p_B)} U^a W_a. \quad (6)$$

Since four-velocities of each fluid are different, we set $U^a W_a = 1 + \frac{b}{2}$ where $b$ denotes different four-velocities. We insert $U^a W_a = 1 + \frac{b}{2}$ into Eq. (6) and then we obtain

$$b = \tan 2\gamma \left( \frac{\zeta - \eta}{\sqrt{\zeta \eta}} \right) - 2, \quad (7)$$

where $\zeta = \epsilon_A + p_A$ and $\eta = \epsilon_B + p_B$, respectively. Times in Eq. (7) means normal multiplication.

We define some quantities, those are: $V^a = U'^a / \sqrt{U'^\alpha U'_\alpha}$, $x^a = W'^a / \sqrt{-W'^\alpha W'_\alpha}$, $\epsilon = T^{\alpha\beta} V_{a} V_{\beta} = (\epsilon_A + p_A)U'^\alpha U'_{\alpha} - (p_A + p_B)$, $p = T^{\alpha\beta} x_{\alpha} x_{\beta} = (p_A + p_B) - (\epsilon_B + p_B)W'^\alpha W'_\alpha$, and $q = p_A + p_B$. We insert those quantities into energy-momentum tensor in equation (1) and we obtain energy-momentum tensor in standard form of anisotropic fluids, which can read as

$$T^{\alpha\beta} = (\epsilon + q) V^a V^\beta - q g^{a\beta} + (p - q) x^a x^\beta \quad (8)$$

where $V^a V_a = 1 = -x^a x_a$ and $x^a V_a = 0$ [1, 8]. The energy density $\epsilon$ and radial pressure $p$ are given by

$$\epsilon = \epsilon_A + \epsilon_B + b \kappa, \quad (9)$$
where

\[ p = p_A + p_B + b\kappa, \tag{10} \]

Next, we explain the treatment of hydrostatic equilibrium in our model. First, we recall TOV equations in standard form of anisotropic pressure [9, 10].

\[ \frac{dp}{dr} = -\frac{(\epsilon + p) (4\pi r^2 p + M)}{r(r-2M)} - \frac{2\delta}{r}, \tag{12} \]

\[ \frac{dM}{dr} = 4\pi \epsilon r^2, \tag{13} \]

where \( \delta \) is anisotropic pressure. Since implication from energy-momentum tensor, quantities \( \delta \) and \( p \) are given by

\[ \delta = p - q = b\kappa, \tag{14} \]

\[ p = q + b\kappa. \tag{15} \]

Therefore, TOV equations in standard form of anisotropic pressure for two-fluid models are given by

\[ \frac{dq}{dr} = -\frac{(\epsilon_A + \epsilon_B + q + 2b\kappa)(4\pi r^2(q + b\kappa) + M)}{r(r-2M)(1+b\kappa)} - \frac{2b\kappa}{r(1+b\kappa)}, \tag{16} \]

\[ \frac{dM}{dr} = 4\pi r^2(\epsilon_A + \epsilon_B + b\kappa). \tag{17} \]

Equation (16) and (17) for two-fluid models which consists of sum of two isotropic perfect fluid.

3. Result and Discussion

In this section, we elaborate our result. We start with the total energy-momentum tensor with anisotropic pressure as

\[ T^{\alpha\beta} = (\epsilon_A + p_A) U^{\alpha}U^{\beta} - q_A g^{\alpha\beta} + (p_A - q_A) k^{a}_{U} k^{b}_{U} \]

\[ + (\epsilon_B + p_B) W^{\alpha}W^{\beta} - q_B g^{\alpha\beta} + (p_B - q_B) k^{a}_{W} k^{b}_{W}, \tag{18} \]

where \( p_A, q_A, p_B, \) and \( q_B \) are radial pressure of \( A \) matter, tangential pressure of \( A \) matter, radial pressure of \( B \) matter, and tangential pressure of \( B \) matter, respectively. \( k^{a}_{U} \) is the unit radial vector orthogonal to \( U^{\alpha} \) and \( k^{a}_{W} \) is the unit radial vector orthogonal to \( W^{\alpha} \).

Next, we use relation \( U^{\alpha}k^{a}_{U} = 0 \) and \( W^{\alpha}k^{a}_{W} = 0 \). We obtain four equations are given by

\[ V^{a}k^{a}_{U} = \sqrt{\frac{\mu}{r}} W^{a}k^{a}_{U} \sin \gamma, \tag{19} \]

\[ V^{a}k^{a}_{W} = U^{a}k^{a}_{W} \cos \gamma \tag{20} \]

\[ \chi^{a}k^{a}_{U} = W^{a}k^{a}_{U} \cos \gamma, \tag{21} \]
\[ \chi^\alpha k_W^\alpha = - \sqrt{\frac{\xi}{\eta}} U^\alpha k_W^\alpha \sin \gamma. \] (22)

Now, we set \( \chi^\alpha \) in linear combination such as

\[ \chi^\alpha = C k_U^\alpha + D k_W^\alpha. \] (23)

We multiply \( V^\alpha \) with equation (23) and use relation \( V^\alpha \chi_\alpha = 0 \). Then, we obtain following expression

\[ U^\alpha k_W^\alpha = - \frac{c}{d} \sqrt{\frac{\xi}{\eta}} W^\alpha k_U^\alpha \tan \gamma \sin \gamma. \] (24)

We multiply \( \chi^\alpha \) with equation (23) and use relation \( \chi^\alpha \chi_\alpha = -1 \). Therefore, we can obtain following equation,

\[ CW^\alpha k_U^\alpha \cos \gamma - D \sqrt{\frac{\xi}{\eta}} U^\alpha k_W^\alpha \sin \gamma = -1. \] (25)

We substitute Eq. (24) into Eq. (25) and we obtain equation such as

\[ CW^\alpha k_U^\alpha \cos \gamma + CW^\alpha k_U^\alpha \tan \gamma \sin \gamma = -1. \] (26)

If we set \( C = - \cos \gamma \) then we write Eq. (26) as

\[ W^\alpha k_U^\alpha = 1. \] (27)

If we set \( C = - \cos \gamma \) and \( U^\alpha k_W^\alpha = 1 \) then we obtain \( D \) from Eq. (24)

\[ D = \frac{\sqrt{\xi}}{\sqrt{\eta}} \sin \gamma. \] (28)

We recall Eq. (23) and we have linear combination of \( \chi^\alpha \) as

\[ \chi^\alpha = - \cos \gamma k_U^\alpha + \sqrt{\frac{\eta}{\xi}} \sin \gamma k_W^\alpha. \] (29)

Now, we make \( k_U^\alpha \) and \( k_W^\alpha \) have relation with \( \chi^\alpha \). We define the relation as follows

\[ k_U^\alpha = E \chi^\alpha, \quad k_W^\alpha = F \chi^\alpha. \] (30)

We substitute Eq. (30) into Eq. (29) and we obtain following expression

\[ \chi^\alpha = - E \cos \gamma \chi^\alpha + F \sqrt{\frac{\eta}{\xi}} \sin \gamma \chi^\alpha. \] (31)

We use relation \( \chi^\alpha \chi_\alpha = -1 \) and Eq. (31). We obtain equation, as

\[ E \cos \gamma - F \sqrt{\frac{\eta}{\xi}} \sin \gamma = -1. \] (32)
To fulfill Eq. (32), $E$ and $F$ have to

$$E = -\cos\gamma, \quad F = \sqrt{\frac{\xi}{\eta}} \sin\gamma. \quad (33)$$

Therefore, we obtain energy-momentum tensor is given by

$$T^{\alpha\beta} = (\epsilon + q)\gamma^{\alpha}\gamma^{\beta} - qq^{\alpha\beta} + (p - q)\chi^{\alpha}\chi^{\beta} + (p_A - q_A)\sin^2\gamma\chi^{\alpha}\chi^{\beta}.$$  

$$+ (p_B - q_B)\sin^2\gamma\chi^{\alpha}\chi^{\beta}. \quad (34)$$

The component of the energy-momentum tensor in Eq. (34) are given by

$$T^{\ell}_{\ell} = -\varepsilon, \quad (35)$$

$$T^{r}_{r} = p + \delta_{A}\cos^2\gamma + \delta_{B}\sin^2\gamma, \quad (36)$$

$$T^{\theta}_{\theta} = T^{\phi}_{\phi} = q, \quad (37)$$

where $\delta_{A} = p_A - q_A$ and $\delta_{B} = p_B - q_B$. We recall Eq. (7)

$$b = \tan 2\gamma X - 2, \quad (38)$$

where $X = \frac{\xi - \eta}{\sqrt{\xi + \eta}}$. From Eq. (38), we can obtain explicit form of $\cos^2\gamma$ and $\sin^2\gamma$, given by

$$\cos^2\gamma = \frac{X^5 + 2X^2 - X(3 + 4b) - 4b}{4X^2}, \quad (39)$$

$$\sin^2\gamma = \frac{-X^5 + 2X^2 + X(3 + 4b) + 4b}{4X^2}. \quad (40)$$

We simplify Eq. (36), such as

$$T^{r}_{r} = q + b\kappa + \Delta, \quad (41)$$

where $\Delta = \delta_{A}\cos^2\gamma + \delta_{B}\sin^2\gamma$. Therefore, TOV equations can be written in standard form of anisotropic pressure for two-fluid models are given by

$$\frac{d\phi}{dr} = -\frac{(\epsilon_{\text{A}} + \epsilon_{\text{B}} + \phi + \Delta + 2b\kappa)(4\pi r^3(\phi + \Delta + b\kappa) + M)}{r(r - 2M)(1 + b\phi_{\text{A}} + \frac{\partial \Delta}{\partial q})} - \frac{2(b\kappa + \Delta)}{r(1 + b\phi_{\text{A}} + \frac{\partial \Delta}{\partial q})} \quad (42)$$

$$\frac{dM}{dr} = 4\pi r^2(\epsilon_{\text{A}} + \epsilon_{\text{B}} + b\kappa). \quad (43)$$

Eqs. (42) and (43) for two-fluid models which consists of sum of two perfect anisotropic fluid. In $\Delta$ terms, there is an anisotropic pressure factors for each matter. Since in one fluid, an anisotropic pressure can increase maximum mass of stars [11, 12], we expect that the presence of anisotropic pressure in two fluid could do so. However, we should check by solving Eq. (42) and Eq. (43) numerically. We will report the numerical results elsewhere.
4. Conclusion
In two-fluid models of neutron stars admixed with dark matter, different velocities between two fluids can lead to a maximum increase in stars' mass. However, the effect is not sufficient to reach recent neutron stars' maximum mass observations. In one fluid, anisotropic pressure can lead to a maximum increase in the mass of stars [11, 12]. Therefore, it is expected that the anisotropic pressure plays the same role in two-fluid models of neutron stars admixed with dark matter.

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