NON-EXISTENCE OF HOPF-GALOIS STRUCTURES
AND BIJECTIVE CROSSED HOMOMORPHISMS

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ABSTRACT. By work of C. Greither and B. Pareigis as well as N. P. Byott, the enumeration of Hopf-Galois structures on a Galois extension of fields with Galois group $G$ may be reduced to that of regular subgroups of $\text{Hol}(N)$ isomorphic to $G$ as $N$ ranges over all groups of order $|G|$, where $\text{Hol}(-)$ denotes the holomorph. In this paper, we shall give a description of such subgroups of $\text{Hol}(N)$ in terms of bijective crossed homomorphisms $G \rightarrow N$, and then use it to study two questions related to non-existence of Hopf-Galois structures.

CONTENTS

1. Introduction 2
   1.1. Isomorphic type 3
   1.2. Non-isomorphic type 4
2. Regular subgroups of the holomorph 5
   2.1. The trivial action 6
   2.2. Principal crossed homomorphisms 6
   2.3. Action via inner automorphisms 7
3. Applications: isomorphic type 9
   3.1. Abelian groups 9
   3.2. Quasisimple groups 13
4. Applications: non-isomorphic type 15
   4.1. Formulation of idea 15
   4.2. Cyclic groups of odd prime power order 16
   4.3. Groups of order $n$ factorial 18
5. Acknowledgments 23
References 23

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1. Introduction

Let $G$ be a finite group, and write $\text{Perm}(G)$ for the symmetric group of $G$. Recall that a subgroup $N$ of $\text{Perm}(G)$ is said to be regular if the map

$$\xi_N : N \to G; \quad \xi_N(\eta) = \eta(1_G)$$

is bijective. Notice that $N$ must have the same order as $G$ in this case. There are two obvious examples, namely $\rho(G)$ and $\lambda(G)$, where

$$\begin{align*}
\rho : & G \to \text{Perm}(G); \quad \rho(\sigma) = (\tau \mapsto \tau \sigma^{-1}) \\
\lambda : & G \to \text{Perm}(G); \quad \lambda(\sigma) = (\tau \mapsto \sigma \tau)
\end{align*}$$

are the right and left regular representations of $G$, respectively. It is easy to see that $\rho(G)$ and $\lambda(G)$ are equal precisely when $G$ is abelian.

Now, consider a finite Galois extension $L/K$ of fields with Galois group $G$. The group ring $K[G]$ is a Hopf algebra over $K$ and its action on $L$ defines a Hopf-Galois structure on $L/K$. By C. Greither and B. Pareigis [14], there is a bijection between Hopf-Galois structures on $L/K$ and regular subgroups of $\text{Perm}(G)$ normalized by $\lambda(G)$, with the classical structure $K[G]$ corresponding to $\rho(G)$. The consideration of the various Hopf-Galois structures, instead of just $K[G]$, has applications in Galois module theory; see [10] for a survey on this subject up to the year 2000.

Therefore, it is of interest to determine the number

$$e(G) = \# \{\text{regular subgroups of } \text{Perm}(G) \text{ normalized by } \lambda(G)\}.$$ 

See [1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 16, 17] for some known results. In general, it could be difficult to compute $e(G)$ because $\text{Perm}(G)$ might have many regular subgroups, and the papers above all make use of the following simplification due to N. P. Byott in [1]. Note that it suffices to compute

$$e(G, N) = \# \left\{ \text{regular subgroups of } \text{Perm}(G) \text{ which are isomorphic to } N \text{ and normalized by } \lambda(G) \right\}$$

for each group $N$ of order $|G|$. Further, define

$$\text{Hol}(N) = \{ \pi \in \text{Perm}(N) : \pi \text{ normalizes } \lambda(N) \},$$
called the *holomorph* of $N$. Then, as shown in [1] or [10, Section 7], we have

\[(1.1) \quad e(G, N) = \frac{|\text{Aut}(G)|}{|\text{Aut}(N)|} \cdot \# \left\{ \text{regular subgroups in Hol}(N) \right\}, \]

which in turn may be rewritten as

\[e(G, N) = \frac{1}{|\text{Aut}(N)|} \cdot \# \text{Reg}(G, \text{Hol}(N)),\]

where we define

\[\text{Reg}(G, \text{Hol}(N)) = \{ \beta \in \text{Hom}(G, \text{Hol}(N)) : \beta(G) \text{ is regular} \}.\]

Notice that elements of the above set are automatically injective because $N$ has order $|G|$. The number (1.1) is much easier to compute because

\[(1.2) \quad \text{Hol}(N) = \rho(N) \rtimes \text{Aut}(N),\]

by [10, Proposition 7.2], for example. In particular, the set \(\text{Reg}(G, \text{Hol}(N))\) may be parametrized by certain $G$-actions on $N$ together with the bijective crossed homomorphisms associated to them; see Proposition 2.1 below.

The purpose of this paper is to use the parametrization of \(\text{Reg}(G, \text{Hol}(N))\) given in Proposition 2.1 to study two questions concerning non-existence of Hopf-Galois structures; see Questions 1.1 and 1.4.

For notation, given a group $\Gamma$, we shall write:

- $Z(\Gamma) =$ the center of $\Gamma$,
- $[\Gamma, \Gamma] =$ the commutator subgroup of $\Gamma$,
- $\text{Inn}(\Gamma) =$ the inner automorphism group of $\Gamma$,
- $\text{Out}(\Gamma) =$ the outer automorphism group of $\Gamma$.

Also, all groups considered in this paper are finite.

1.1. **Isomorphic type.** In the case that $N = G$, notice that $\rho(G)$ and $\lambda(G)$ are regular subgroups of $\text{Hol}(G)$ isomorphic to $G$. It is natural to ask:

**Question 1.1.** Is there a regular subgroup of $\text{Hol}(G)$ isomorphic to $G$ other than the obvious ones $\rho(G)$ and $\lambda(G)$?
For $G$ abelian, the answer is completely known.

**Theorem 1.2.** Let $A$ be an abelian group. Then, we have $e(A, A) = 1$ if and only if $|A| = 2^δp_1 \cdots p_m$ for distinct odd primes $p_1, \ldots, p_m$ and $δ \in \{0, 1, 2\}$, where we allow the product of odd primes to be empty.

Most of Theorem 1.2 may be deduced from [6, Theorem 5] and results in [1, 17]. In Section 3.1, we shall give an alternative and independent proof of the backward implication, as well as a proof of the forward implication using only a couple results from [1, 17].

For $G$ non-abelian, the answer is not quite understood. In [9], S. Carnahan and L. N. Childs answered Question 1.1 in the negative when $G$ is non-abelian simple. In Section 3.2, we shall extend their result to all quasisimple groups. Recall that $G$ is said to be quasisimple if $G = [G, G]$ and $G/Z(G)$ is simple.

**Theorem 1.3.** Let $Q$ be a quasisimple group. Then, we have $e(Q, Q) = 2$.

1.2. Non-isomorphic type. In the case that $N$ has order $|G|$ but $N \not\cong G$, there is no obvious regular subgroup of Hol($N$) isomorphic to $G$. It is natural to ask:

**Question 1.4.** Is there a regular subgroup of Hol($N$) isomorphic to $G$?

In [3], N. P. Byott answered Question 1.4 in the negative for every $N \not\cong G$ when $G$ is non-abelian simple. One key idea in [3] is the use of characteristic subgroups. In Section 4.1, we shall reformulate as well as extend this idea in terms of our Proposition 2.1; see Lemma 4.1 below. Then, in Section 4.2, we shall apply our Lemma 4.1 to give an alternative proof of the following result due to T. Kohl [16].

**Theorem 1.5.** Let $C_{p^n}$ be a cyclic group of odd prime power order $p^n$. Then, we have $e(C_{p^n}, N) = 0$ for all groups $N$ of order $p^n$ with $N \not\cong C_{p^n}$.

In view of [3], it is natural to ask whether Question 1.4 also has a negative answer for every $N \not\cong G$ when $G$ is quasisimple. In Section 4.3.1, by applying Lemma 4.1 together with some other techniques from [3], we shall show that this is indeed the case when $G$ is in the following family.
Theorem 1.6. Let $2A_n$ be the double cover of the alternating group $A_n$ on $n$ letters, where $n \geq 5$. Then, we have $e(2A_n, N) = 0$ for all groups $N$ of order $n!$ with $N \not\cong 2A_n$.

In order to determine $e(G)$, one has to compute $e(G, N)$ for all groups $N$ of order $|G|$. This could be difficult because there are lots of such $N$ in general. In the case that $e(G, N) = 0$ for every $N \not\cong G$, it suffices to compute $e(G, G)$ and the problem is significantly simplified. However, in most cases, we have $e(G, N) \geq 1$ for at least one $N \not\cong G$. Nonetheless, it seems very likely that the techniques we develop in Section 4 may be applied to show that $e(G, N) = 0$ for a large family of $N$, whence reducing the number of $N$ that one needs to consider. As an illustration, in Section 4.3.2, we shall prove:

Theorem 1.7. Let $S_n$ be the symmetric group on $n$ letters, where $n \geq 5$, and let $N$ be a group of order $n!$ with $e(S_n, N) \geq 1$. Then, we have:

(1) $N$ fits into a short exact sequence $1 \rightarrow A_n \rightarrow N \rightarrow \{\pm 1\} \rightarrow 1$, or

(2) $N$ fits into a short exact sequence $1 \rightarrow \{\pm 1\} \rightarrow N \rightarrow A_n \rightarrow 1$, or

(3) $S_n$ embeds into $\text{Out}(N)$.

Moreover, for any proper maximal characteristic subgroup $M$ of $N$, the quotient $N/M$ is either non-abelian or isomorphic to $\{\pm 1\}$.

Let us note that conditions (1) and (2) in Theorem 1.7 cannot be removed because $e(S_n, S_n) \geq 1$ and $e(S_n, A_n \times \{\pm 1\}) \geq 1$ when $n \geq 5$; see [9] for the exact values of these two numbers.

2. Regular subgroups of the holomorph

Throughout this section, let $G$ and $N$ denote two groups having the same order. Recall that given $\mathfrak{f} \in \text{Hom}(G, \text{Aut}(N))$, a map $g \in \text{Map}(G, N)$ is said to be a crossed homomorphism with respect to $\mathfrak{f}$ if

$$g(\sigma_1 \sigma_2) = g(\sigma_1) \cdot \mathfrak{f}(\sigma_1)(g(\sigma_2)) \text{ for all } \sigma_1, \sigma_2 \in G.$$ 

In general $g$ is not a group homomorphism, but for any $\sigma \in G$, we have

$$g(\sigma^k) = \prod_{i=0}^{k-1} \mathfrak{f}(\sigma)^i(g(\sigma)) \hspace{1cm} \text{and in particular } g(\sigma^{ec}) = g(\sigma^c)^k.$$ 

for all $k \in \mathbb{N}$, where $e$ denotes the order of $f(\sigma)$. We shall write $Z_1^f(G, N)$ for
set of all such maps, and $Z_1^f(G, N)^*$ for the subset consisting of those which
are bijective.

**Proposition 2.1.** For $f \in \text{Map}(G, \text{Aut}(N))$ and $g \in \text{Map}(G, N)$, define

$$\beta_{(f,g)} : G \longrightarrow \text{Hol}(N); \quad \beta_{(f,g)}(\sigma) = \rho(g(\sigma)) \cdot f(\sigma).$$

Then, we have

$$\text{Map}(G, \text{Hol}(N)) = \{\beta_{(f,g)} : f \in \text{Map}(G, \text{Aut}(N)) \text{ and } g \in \text{Map}(G, N)\},$$

$$\text{Hom}(G, \text{Hol}(N)) = \{\beta_{(f,g)} : f \in \text{Hom}(G, \text{Aut}(N)) \text{ and } g \in Z_1^f(G, N)\},$$

$$\text{Reg}(G, \text{Hol}(N)) = \{\beta_{(f,g)} : f \in \text{Hom}(G, \text{Aut}(N)) \text{ and } g \in Z_1^f(G, N)^*\}.$$

**Proof.** The first equality is a direct consequence of (1.2). The second equality may be easily verified using the fact that

$$\rho(\eta_1)\varphi_1 \cdot \rho(\eta_2)\varphi_2 = \rho(\eta_1)\varphi_1\rho(\eta_2)\varphi_2^{-1} \cdot \varphi_1\varphi_2 = \rho(\eta_1\varphi_1(\eta_2)) \cdot \varphi_1\varphi_2$$

for $\eta_1, \eta_2 \in N$ and $\varphi_1, \varphi_2 \in \text{Aut}(N)$. The third equality is then clear because

$$(\beta_{(f,g)}(\sigma))(1_N) = (\rho(g(\sigma)) \cdot f(\sigma))(1_N) = \rho(g(\sigma))(1_N) = g(\sigma)^{-1}$$

for $\sigma \in G$. This proves the proposition. \hfill \Box

In the rest of this section, let $f \in \text{Hom}(G, \text{Aut}(N))$ be fixed, and we shall consider some examples of $g \in Z_1^f(G, N)$.

2.1. **The trivial action.** Let $f_0 \in \text{Hom}(G, \text{Aut}(N))$ be the trivial homomorphism, and note that $Z_1^{f_0}(G, N) = \text{Hom}(G, N)$. For $N \not\cong G$, we then deduce that $Z_1^{f_0}(G, N)^* = \emptyset$. As for $N = G$, we easily see from (1.2) that

$$(2.2) \quad \text{for } g \in Z_1^f(G, G)^*: \quad \beta_{(f,g)}(G) = \rho(G) \text{ if and only if } f = f_0.$$

Hence, the case when $f = f_0$ only gives rise to the regular subgroup $\rho(G)$.

2.2. **Principal crossed homomorphisms.** Given any $\eta \in N$, it is natural to consider its associated principal crossed homomorphism, defined by

$$g_\eta \in Z_1^f(G, N); \quad g_\eta(\sigma) = \eta^{-1} \cdot f(\sigma)(\eta).$$
Unfortunately, this map is never bijective, unless $G$ are $N$ are trivial. Indeed, viewing $N$ as a $G$-set via the homomorphism $f$, it is easy to check that

$$g_{\eta}$$
is injective if and only if $\text{Stab}_G(\eta) = \{1_G\}$.

In this case, since $|G| = |N|$, by the orbit-stabilizer theorem, we must have

$$\{f(\sigma)(\eta) : \sigma \in G\} = N,$$
and so $f(\sigma)(\eta) = 1_N$ for some $\sigma \in G$.

This implies that $\eta = 1_N$, but $g_{1_N}$ is not bijective unless $G$ and $N$ are trivial.

2.3. Action via inner automorphisms. For $\eta \in N$, we shall write

$$\text{conj}(\eta) = \rho(\eta)\lambda(\eta)$$
as well as $\text{conj}(\eta Z(N)) = \text{conj}(\eta)$.

The latter is plainly well-defined, and note that $\text{Inn}(N) \simeq N/Z(N)$ via $\text{conj}$.

In the case that $f(G) \subset \text{Inn}(N)$, elements in $Z_f^1(G, N)^*$ turn out to be closely related to certain fixed point free pairs of homomorphisms. This connection was first observed by N. P. Byott and L. N. Childs in [5]; see the discussion at the end of Section 2.3.1.

**Definition 2.2.** A pair $(f, g)$, where $f, g \in \text{Hom}(G, N)$, is fixed point free if

$$f(\sigma) = g(\sigma)$$
holds precisely when $\sigma = 1_G$.

Since $|G| = |N|$, this is easily seen to be equivalent to that the map $G \rightarrow N$ given by $\sigma \mapsto f(\sigma)g(\sigma)^{-1}$ is bijective; see [5, Proposition 1], for example.

We shall further make the following definition.

**Definition 2.3.** A pair $(f, g)$, where $f, g \in \text{Hom}(G, N/Z(N))$, is weakly fixed point free if the map $G \rightarrow N/Z(N)$ given by $\sigma \mapsto f(\sigma)g(\sigma)^{-1}$ is surjective.

2.3.1. Liftable inner actions. In what follows, assume that

there exists $f \in \text{Hom}(G, N)$ with $f(\sigma) = \text{conj}(f(\sigma))$ for all $\sigma \in G$.

This implies that $f(G) \subset \text{Inn}(N)$ but the converse is false in general. Put

$$\text{Hom}_f(G, N)^* = \{g \in \text{Hom}(G, N) : (f, g) \text{ is fixed point free}\}.$$

Then, we have:
Proposition 2.4. The maps

\begin{align*}
Z^1_\text{F}(G, N) &\longrightarrow \text{Hom}(G, N); \quad g \mapsto (\sigma \mapsto g(\sigma)f(\sigma)) \\
Z^1_\text{F}(G, N) \ast &\longrightarrow \text{Hom}_f(G, N) \ast; \quad g \mapsto (\sigma \mapsto g(\sigma)f(\sigma))
\end{align*}

are well-defined bijections.

Proof. First, let $g \in Z^1_\text{F}(G, N)$. Then, for $\sigma_1, \sigma_2 \in G$, we have

\[ g(\sigma_1\sigma_2)f(\sigma_1\sigma_2) = g(\sigma_1)\text{conj}(f(\sigma_1))(g(\sigma_2)) \cdot f(\sigma_1)f(\sigma_2) = g(\sigma_1)f(\sigma_1) \cdot g(\sigma_2)f(\sigma_2). \]

This shows that the first map is well-defined. Next, let $g \in \text{Hom}(G, N)$, and define $g(\sigma) = g(\sigma)f(\sigma)^{-1}$. Then, for $\sigma_1, \sigma_2 \in G$, we have

\[ g(\sigma_1\sigma_2) = g(\sigma_1)f(\sigma_1)g(\sigma_2)f(\sigma_2)^{-1} = g(\sigma_1)f(\sigma_1)^{-1}\text{conj}(f(\sigma_1))(g(\sigma_2)f(\sigma_2)^{-1}) = g(\sigma_1)f(\sigma_1)(g(\sigma_2)). \]

This shows the first map, which is plainly injective, is also surjective. From Definition 2.2, it is clear that $g$ is bijective if and only if $(f, g)$ is fixed point free. Hence, the second map is also a well-defined bijection. $\square$

Let $g \in Z^1_\text{F}(G, N)$ and let $g \in \text{Hom}(G, N)$ be its image under \(2.3\). Then, for any $\sigma \in G$, we may rewrite

\[ \beta_{(f, g)}(\sigma) = \rho(g(\sigma)f(\sigma)^{-1}) \cdot \text{conj}(f(\sigma)) = \rho(g(\sigma))\lambda(f(\sigma)) = \beta_{(f, g)}(\sigma), \]

where $\beta_{(f, g)}$ is the homomorphism defined as in [5, Sections 2 and 3]. Hence, we may view Propositions 2.1 and 2.4 as generalizations of [5, Corollary 7].

2.3.2. General inner actions. In what follows, assume that

there exists $f \in \text{Hom}(G, N/Z(N))$ with $f(\sigma) = \text{conj}(f(\sigma))$ for all $\sigma \in G$. This implies that $f(G) \subset \text{Inn}(N)$ and the converse is also true. Put

\[ \text{Hom}_f(G, N/Z(N)) \ast = \{ g \in \text{Hom}(G, N/Z(N)) : (f, g) \text{ is weakly fixed point free} \}. \]
Then, essentially the same argument as in Proposition 2.4 shows that:

**Proposition 2.5.** The maps

\[
\begin{align*}
Z_f^1(G, N) &\rightarrow \text{Hom}(G, N/Z(N)); & \mathfrak{g} \mapsto (\sigma \mapsto \mathfrak{g}(\sigma)Z(N) \cdot f(\sigma)) \\
Z_f^1(G, N)^* &\rightarrow \text{Hom}_f(G, N/Z(N))^*; & \mathfrak{g} \mapsto (\sigma \mapsto \mathfrak{g}(\sigma)Z(N) \cdot f(\sigma))
\end{align*}
\]

are well-defined.

However, the maps in Proposition 2.5, unlike those in Proposition 2.4, are neither injective nor surjective in general.

Let \( \mathfrak{g} \in Z_f^1(G, N) \) and let \( g \in \text{Hom}_f(G, N/Z(N)) \) be its image under (2.4). For any \( \sigma \in G \), letting \( \tilde{f}(\sigma) \in N \) be an element such that \( f(\sigma) = \tilde{f}(\sigma)Z(N) \), we may then rewrite

\[
\beta_{(f,\mathfrak{g})}(\sigma) = \rho(\mathfrak{g}(\sigma)) \cdot \text{conj}(\tilde{f}(\sigma)) = \rho(\mathfrak{g}(\sigma)\tilde{f}(\sigma))\lambda(\tilde{f}(\sigma)).
\]

Observe that \( \rho(\eta) = \lambda(\eta)^{-1} \) for \( \eta \in Z(N) \), and let \( g_0 \in \text{Hom}(G, N/Z(N)) \) be the trivial homomorphism. Then, we see that \( \beta_{(f,\mathfrak{g})}(G) \subset \lambda(N) \) when \( g = g_0 \). Thus, for \( N \not\cong G \), we have \( g \neq g_0 \) whenever \( \mathfrak{g} \) is bijective. As for \( N = G \), it is also easy to verify that

\[
(2.5) \quad \text{for } \mathfrak{g} \in Z_f^1(G, G)^*: \quad \beta_{(f,\mathfrak{g})}(G) = \lambda(G) \text{ if and only if } g = g_0.
\]

This is analogous to the discussion in Section 2.1.

### 3. Applications: isomorphic type

**3.1. Abelian groups.** Let \( A \) be an abelian group.

**3.1.1. Backward implication.** Suppose that

\[
|A| = 2^\delta p_1 \cdots p_m, \text{ where } p_1 < \cdots < p_m \text{ are odd primes and } \delta \in \{0, 1, 2\}.
\]

To prove the backward implication of Theorem 1.2, consider

\[
f \in \text{Hom}(A, \text{Aut}(A)) \text{ and } \mathfrak{g} \in Z_f^1(A, A).
\]

By (1.1) and Proposition 2.1, it is enough to show that

\[
(3.1) \quad \beta_{(f,\mathfrak{g})}(A) = \rho(A) \text{ whenever } \mathfrak{g} \text{ is bijective}.
\]
Notice that the hypothesis on $A$ implies $A = A_0 \times A_\sigma$, where $A_\sigma = \langle \sigma \rangle$ with $\sigma \in A$ an element of order $n = p_1 \cdots p_m$, and $A_0$ is isomorphic to one of

\begin{equation}
\{1\}, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.
\end{equation}

We also have $\text{Aut}(A) = \text{Aut}(A_0) \times \text{Aut}(A_\sigma)$. For brevity, let $g_1 : A \rightarrow A_0$ and $g_2 : A \rightarrow A_\sigma$ denote the maps defined as $g$ composed with the natural projections.

**Lemma 3.1.** If $g$ is bijective, then $f(A_\sigma)|_{A_\sigma} = \{\text{Id}_{A_\sigma}\}$ and $g(A_\sigma) = A_\sigma$.

**Proof.** Let $d \in \mathbb{Z}$ be coprime to $n$ such that $f(\sigma)(\tau) = \tau^d$ for all $\tau \in A_\sigma$, and let $e$ denote the multiplicative order of $d$ mod $n$. For each $1 \leq i \leq m$, write $e_i$ for the multiplicative order of $d$ mod $p_i$. Then, we have $e = \text{lcm}(e_1, \ldots, e_m)$. Since $e$ divides $n$, we may also write

\[
e = p_{i_1} \cdots p_{i_r}
\]

for some $1 \leq i_1 < \cdots < i_r \leq m$, where the product may be empty. Notice that the order of $f(\sigma)|_{A_0} \in \text{Aut}(A_0)$ divides three and $|A_0|$ divides four by (3.2). We then deduce from (2.1) that

\[
g(\sigma^{3e \cdot 4(d-1)}) = g(\sigma^{3e} 4(d-1)) = g_2(\sigma)^{(1+d+\cdots+d^{3e-1})4(d-1)} = 1_{A_\sigma}.
\]

Now, suppose that $g$ is bijective. The above implies that

\begin{equation}(3.3) \quad 12e(d - 1) \equiv 0 \pmod{n}.
\end{equation}

We shall use this to show that $\{i_1, \ldots, i_r\}$ is in fact empty.

For $p_i = 3$, we must have $e_i = 1$ because $e$ is odd. For $i \notin \{i_1, \ldots, i_r\}$ with $p_i \neq 3$, we have $e_i = 1$ by (3.3). As for $i \in \{i_1, \ldots, i_r\}$, we have

\[
d^{e/p_i} \equiv (d^{p_i})^{e/p_i} \equiv 1 \pmod{p_i},
\]

whence $e_i$ divides both $e/p_i$ and $p_i - 1$. This implies that $e_i$ divides $p_{i_1} \cdots p_{i_{j-1}}$ when $i = i_j$. But then $p_{i_r}$ cannot divide $e$, which is a contradiction. It follows that $\{i_1, \ldots, i_r\}$ must be empty. This shows that $e = 1$ and so $f(\sigma)|_{A_\sigma} = \text{Id}_{A_\sigma}$. This in turn implies that $g|_{A_\sigma}$ is a homomorphism, whence $g(A_\sigma) = A_\sigma$. \qed

**Lemma 3.2.** If $g$ is bijective, then $f(A_0)|_{A_0} = \{\text{Id}_{A_0}\}$ and $g(A_0) = A_0$. 

Proof. For $|A_0| \leq 2$, it is clear that $f(A_0)|_{A_0} = \{\text{Id}_{A_0}\}$. For $|A_0| = 4$, suppose that $g$ is bijective, and on the contrary that $f(\sigma_0)|_{A_0} \neq \text{Id}_{A_0}$ for some $\sigma_0 \in A_0$.

First, assume that $A_0 \simeq \mathbb{Z}/4\mathbb{Z}$. Without loss of generality, we may assume that $\sigma_0 \in A_0$ is a generator. Note that $f(\sigma_0)(\tau) = \tau^{-1}$ for all $\tau \in A_0$, and so

$$g(\sigma_0^2) = g(\sigma_0) \cdot f(\sigma_0)(g(\sigma_0)) = g_2(\sigma_0) \cdot f(\sigma_0)(g_2(\sigma_0)),$$

which lies in $A_\sigma$. Since $g^{-1}(A_\sigma) = A_\sigma$ by Lemma 3.1, we must have $\sigma_0^2 \in A_\sigma$. But then $\sigma_0^2 = 1_A$, which contradicts that $\sigma_0$ has order four.

Next, assume that $A_0 \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Under this identification, applying a change of basis if necessary, we may assume that $f(\sigma_0)|_{A_0} = (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})$. We have

$$2\sigma_0 = 0_{A_0} \text{ and so } g_1(\sigma_0) + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g_1(\sigma_0) = g_1(2\sigma_0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This implies that $g_1(\sigma_0) = (1, 1)$. Let $\tau_0, \tau \in A$ be such that

$$g(\tau_0) = (1, 0) \text{ and } f(\sigma_0\tau_0)(g(\tau)) = (g_2(\sigma_0) \cdot f(\sigma_0)(g_2(\tau_0)))^{-1}.$$

Note that $\tau \in A_\sigma$ because $g^{-1}(A_\sigma) = A_\sigma$ by Lemma 3.1. Also, we have

$$g(\sigma_0\tau_0\tau) = g(\sigma_0) \cdot f(\sigma_0)(g(\tau_0)) \cdot f(\sigma_0\tau_0)(g(\tau)),$$

which lies in $A_0$ by choice, and we have

$$g(\sigma_0\tau_0\tau) = g_1(\sigma_0) + f(\sigma_0)(g_1(\tau_0)) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = g(\tau_0).$$

It follows that $\sigma_0\tau = 1_A$, which is impossible because $\sigma_0 \in A_0$ and $\sigma_0 \neq 1_{A_0}$.

We have thus shown that $f(A_0)|_{A_0} = \{\text{Id}_{A_0}\}$. This in turn implies that $\mathfrak{g}|_{A_0}$ is a homomorphism, whence $g(A_0) = A_0$. \hfill \Box

Proof of Theorem 1.2: backward implication. Suppose that $g$ is bijective. To prove (3.1), by (2.2) as well as Lemmas 3.1 and 3.2, it suffices to show that

$$(3.4) \quad f(A_\sigma)|_{A_0} = \{\text{Id}_{A_0}\} \text{ and } f(A_0)|_{A_\sigma} = \{\text{Id}_{A_\sigma}\}.$$

For any $\sigma_0 \in A_0$ and $\tau \in A_\sigma$, we have

$$g(\sigma_0) \cdot f(\sigma_0)(g(\tau)) = g(\sigma_0\tau) = g(\tau\sigma_0) = g(\tau) \cdot f(\tau)(g(\sigma_0)).$$
Since \( g(A_\sigma) = A_\sigma \) and \( g(A_0) = A_0 \) by Lemmas 3.1 and 3.2, this implies

\[
g(\sigma_0) = f(\tau)(g(\sigma_0)) \quad \text{and} \quad g(\tau) = f(\sigma_0)(g(\tau)).
\]

We then deduce that (3.4) indeed holds. \( \square \)

### 3.1.2. Forward implication.

Suppose that \( |A| \) does not have the form stated in Theorem 1.2. This means that \( A = H \times H' \), where \( H \) is a subgroup of \( A \) isomorphic to one of the groups in the next lemma.

**Lemma 3.3.** Suppose that \( H \) is isomorphic to one of the following:

1. \( \mathbb{Z}/p^n\mathbb{Z} \) or \( \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \), where \( p \) is an odd prime and \( n \geq 2 \), or
2. \( \mathbb{Z}/2^n\mathbb{Z} \), where \( n \geq 3 \), or
3. \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) or \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \), or
4. \( \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \).

Then, there is a regular subgroup of \( \text{Hol}(H) \) isomorphic to \( H \) other than \( \rho(H) \).

**Proof.** For cases (1) and (2), see [1, Lemmas 1 and 2]. For case (3), see [17, Theorem 1.2.5]. For case (4), identify \( H \) with \( \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \), and define

\[
A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}, \quad b_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.
\]

Further, define two permutations \( \eta_1, \eta_2 \) on \( H \) by setting

\[
\eta_1(x) = A_1x + b_1 \quad \text{and} \quad \eta_2(x) = A_2x + b_2 \quad \text{for} \ x \in H.
\]

Note that \( \eta_1, \eta_2 \in \text{Hol}(H) \) by (1.2). It is easy to verify that \( \langle \eta_1, \eta_2 \rangle \simeq H \) and \( \langle \eta_1, \eta_2 \rangle \neq \rho(H) \). Also, a routine calculation shows that \( \langle \eta_1, \eta_2 \rangle \) is regular, and so the claim follows. \( \square \)

**Proof of Theorem 1.2: forward implication.** By Lemma 3.3, there is a regular subgroup \( N_H \) of \( \text{Hol}(H) \) isomorphic to \( H \) and not equal to \( \rho(H) \). The image of \( N_H \times \rho(H') \) under the natural injective homomorphism

\[
\text{Hol}(H) \times \text{Hol}(H') \longrightarrow \text{Hol}(H \times H')
\]

is then a regular subgroup isomorphic to \( H \times H' \) and not equal to \( \rho(H \times H') \). The claim now follows from (1.1). \( \square \)
3.2. **Quasisimple groups.** Let $Q$ be a quasisimple group. We shall need the next proposition, which is the crucial ingredient for the proof of $e(Q, Q) = 2$ given in [9] in the special case when $Q$ is non-abelian simple.

**Proposition 3.4.** Let $T$ be a non-abelian simple group. Then:

(a) Schreier’s conjecture is true, namely $\text{Out}(T)$ is solvable.

(b) A pair $(f, g)$ with $f, g \in \text{Aut}(T)$ is never fixed point free.

**Proof.** They are consequences of the classification of finite simple groups; see [13, Theorems 1.46 and 1.48], for example. □

We shall also need the following basic properties concerning $Q$.

**Proposition 3.5.** The following statements hold.

(a) A proper normal subgroup of $Q$ is contained in $Z(Q)$.

(b) The kernel of a non-trivial homomorphism $Q \to Q/Z(Q)$ is $Z(Q)$.

(c) The natural homomorphism $\text{Aut}(Q) \to \text{Aut}(Q/Z(Q))$ is injective.

(d) An endomorphism on $Q$ is either trivial or an automorphism.

**Proof.** This is known, and it is easy to prove (a), which in turn gives (b). For the convenience of reader, we shall give a proof for (c) and (d).

To prove (c), let $\varphi \in \text{Aut}(Q)$ be such that $\varphi(\sigma)\sigma^{-1} \in Z(Q)$ for all $\sigma \in Q$. We easily verify that the map

$$\psi : Q \to Z(Q); \quad \psi(\sigma) = \varphi(\sigma)\sigma^{-1}$$

is a homomorphism. But then $\psi$ must be trivial because $Z(Q)$ is abelian and $Q = [Q, Q]$. This means that $\varphi = \text{Id}_Q$, as desired.

To prove (d), let $\varphi \in \text{Hom}(Q, Q)$ be non-trivial, so then $\ker(\varphi) \subset Z(Q)$ by (a). Put $H = \varphi(Q)$ for brevity. Note that

$$\frac{Q/\ker(\varphi)}{Z(Q)/\ker(\varphi)} \cong \frac{Q}{Z(Q)}$$

has trivial center as well as that $Q/\ker(\varphi) \cong H$. Hence, we deduce that

$$\mathcal{Z}\left(\frac{Q}{\ker(\varphi)}\right) \subset \frac{Z(Q)}{\ker(\varphi)}$$

and so $|Z(H)| \leq [Z(Q) : \ker(\varphi)]$. 
It follows that
\[ |HZ(Q)| = \frac{|H||Z(Q)|}{|H \cap Z(Q)|} \geq \frac{|H||Z(Q)|}{|Z(H)|} \geq \frac{|H||Z(Q)|}{|Z(Q) : \text{ker}(\varphi)|} = |Q|, \]
and so \( HZ(Q) = Q \). Since \( Q = [Q, Q] \), we deduce that \( H = Q \). This means that \( \varphi \in \text{Aut}(Q) \), as desired. \( \square \)

Now, to prove Theorem 1.3, consider
\[ f \in \text{Hom}(Q, \text{Aut}(Q)) \text{ and } g \in Z_f^1(Q, Q). \]
By (1.1) and Proposition 2.1, it suffices to show that
\[ (3.5) \quad \beta_{(f,g)}(Q) \in \{ \rho(Q), \lambda(Q) \} \quad \text{whenever } g \text{ is bijective.} \]

The next lemma is analogous to an argument in [9, p. 84].

**Lemma 3.6.** We have \( f(Q) \subseteq \text{Inn}(Q) \).

**Proof.** Notice that \( \text{Out}(Q) \) embeds into \( \text{Out}(Q/Z(Q)) \) by Proposition 3.5 (c) and so is solvable by Proposition 3.4 (a). The homomorphism
\[ Q \xrightarrow{f} \text{Aut}(Q) \xrightarrow{\text{quotient}} \text{Out}(Q) \]
then is trivial because \( Q = [Q : Q] \), and hence \( f(Q) \subseteq \text{Inn}(Q) \). \( \square \)

In view of Lemma 3.6, we may define
\( f \in \text{Hom}(Q, Q/Z(Q)), \tilde{f} \in \text{Map}(Q, Q), g \in \text{Hom}(Q, Q/Z(Q)) \)
as in Subsection 2.3.2. More precisely, we have
\[ f(\sigma) = \text{conj}(f(\sigma)), \quad f(\sigma) = \tilde{f}(\sigma)Z(Q), \quad g(\sigma) = g(\sigma)\tilde{f}(\sigma)Z(Q) \]
for all \( \sigma \in Q \). We make the following useful observation.

**Lemma 3.7.** If \( f \) and \( g \) are non-trivial, then \( g(Z(Q)) \subseteq Z(Q) \).

**Proof.** Let \( \sigma \in Z(Q) \). Then, for any \( \tau \in Q \), we have
\[ g(\sigma)\tilde{f}(\sigma)g(\tau)\tilde{f}(\sigma)^{-1} = g(\sigma\tau) = g(\tau\sigma) = g(\tau)\tilde{f}(\tau)g(\sigma)\tilde{f}(\tau)^{-1}. \]
Suppose that \( f \) is non-trivial, so then \( \ker(f) = Z(Q) \) by Proposition 3.5 (b). Since \( \sigma \in Z(Q) \), we have \( \tilde{f}(\sigma) \in Z(Q) \), and so the above implies that \( g(\sigma) \) is
centralized by \( g(\tau) \tilde{f}(\tau) \). We then see that \( g(\sigma) \) commutes with

\[
\bigcup_{\tau \in Q} g(\tau) \tilde{f}(\tau) Z(Q) = \bigcup_{\tau \in Q} g(\tau).
\]

Now, suppose that \( g \) is non-trivial. Then, the map \( g \) is surjective by Proposition 3.5 (b), whence the union above is equal to the entire group \( Q \). This means that \( g(\sigma) \in Z(Q) \), as desired. \( \square \)

**Proof of Theorem 1.3.** Suppose that \( g \) is bijective. In view of (2.2) and (2.5), to prove (3.5), it suffices to show that either \( f \) or \( g \) is trivial. Suppose for contradiction that they are both non-trivial. Then, they induce automorphisms

\[
\overline{f} : Q/Z(Q) \rightarrow Q/Z(Q) \quad \text{and} \quad \overline{g} : Q/Z(Q) \rightarrow Q/Z(Q)
\]

by Proposition 3.5 (b). For any \( \sigma \in Q \), we have

\[
f(\sigma) = g(\sigma) \implies g(\sigma) \in Z(Q) \implies \sigma \in Z(Q),
\]

by Lemma 3.7 and the bijectivity of \( g \). This shows that \( (\overline{f}, \overline{g}) \) is fixed point free, which is impossible by Proposition 3.4 (b). \( \square \)

4. **Applications: non-isomorphic type**

4.1. **Formulation of idea.** In what follows, let \( G \) and \( N \) denote two groups having the same order. Recall that a subgroup \( M \) of \( N \) is said to be characteristic if \( \varphi(M) = M \) for all \( \varphi \in \text{Aut}(N) \), in which case \( M \) is normal and we have a natural homomorphism \( \text{Aut}(N) \rightarrow \text{Aut}(N/M) \).

**Lemma 4.1.** Let \( M \) be a characteristic subgroup of \( N \). Given

\[
f \in \text{Hom}(G, \text{Aut}(N)) \quad \text{and} \quad g \in Z^1_f(G, N),
\]

they induce, respectively, a homomorphism and a map

\[
(4.1) \quad \overline{f} : G \rightarrow \text{Aut}(N) \rightarrow \text{Aut}(N/M) \quad \text{and} \quad \overline{g} : G \rightarrow N \rightarrow N/M.
\]

By abuse of notation, define

\[
\ker(\overline{g}) = \{ \sigma \in G : \overline{g}(\sigma) = 1_{N/M} \}.
\]

Then, the following are true.
(a) The set $\ker(\overline{g})$ is a subgroup of $G$.
(b) The map $\overline{g}$ induces an injection $G/\ker(\overline{g}) \to N/M$.
(c) The map $\overline{g}$ restricts to a homomorphism $\ker(f) \to N/M$.
(d) In the case that $N/M$ is abelian, for any $\sigma \in \ker(f) \cap Z(G)$, the element $\overline{g}(\sigma)$ is fixed by the automorphisms in $\overline{f}(G)$.

Proof. Both (a) and (c) are clear. For (b), simply observe that

\[ \overline{g}(\sigma_1) = \overline{g}(\sigma_2) \iff \overline{g}(\sigma_1^{-1}\sigma_2) = \overline{g}(\sigma_1^{-1}) \cdot \overline{f}(\sigma_1^{-1})(\overline{g}(\sigma_1)) \]

\[ \iff \overline{g}(\sigma_1^{-1}\sigma_2) = \overline{g}(\sigma_1^{-1}\sigma_1) \]

\[ \iff \overline{g}(\sigma_1^{-1}\sigma_2) = 1_{N/M} \]

for any $\sigma_1, \sigma_2 \in G$. For (d), it follows from the fact that

\[ \overline{g}(\sigma)\overline{g}(\tau) = \overline{g}(\sigma) \cdot \overline{f}(\sigma)(\overline{g}(\tau)) = \overline{g}(\sigma\tau) = \overline{g}(\tau\sigma) = \overline{g}(\tau) \cdot \overline{f}(\tau)(\overline{g}(\sigma)) \]

for any $\tau \in G$ and $\sigma \in \ker(f) \cap Z(G)$. \qed

We keep the notation as in Lemma 4.1. To show that $e(G, N) = 0$, by (1.1) and Proposition 2.1, it is the same as proving that $g$ cannot be bijective. The idea is that, while we might not understand $\text{Aut}(N)$ or $N$ very well, by passing to $\text{Aut}(N/M)$ and $N/M$ for a suitable characteristic subgroup $M$ of $N$, we might be able to use Lemma 4.1 to prove the weaker statement that $\overline{g}$ cannot be surjective.

4.2. **Cyclic groups of odd prime power order.** Let $C_{p^n}$ be a cyclic group of odd prime power order $p^n$ and let $N$ be a group of order $p^n$ with $N \not\cong C_{p^n}$. To prove Theorem 1.5, consider

\[ f \in \text{Hom}(C_{p^n}, \text{Aut}(N)) \text{ and } g \in Z^1_f(C_{p^n}, N). \]

By (1.1) and Proposition 2.1, it is enough to show that $g$ cannot be bijective. Take $M$ to be the Frattini subgroup $\Phi(N)$ of $N$. Then, we know that

\[ N/\Phi(N) \cong (\mathbb{Z}/p\mathbb{Z})^m \]

and so $\text{Aut}(N/\Phi(N)) \cong \text{GL}_m(\mathbb{Z}/p\mathbb{Z})$, where $m \in \mathbb{N}$ is such that $m \geq 2$ because $N$ is non-cyclic. Let $\overline{f}$ and $\overline{g}$ be as in (4.1). Then, in turn, it suffices to show that $\overline{g}$ cannot be surjective.
Fix a generator $\sigma \in C_{p^n}$, and write $|\overline{f}(\sigma)| = p^r$, where $0 \leq r \leq n$. The next two lemmas yield, respectively, an upper bound and a lower bound for $p^r$ in terms of $m$ and the index of $\ker(\overline{g})$ in $C_{p^n}$.

**Lemma 4.2.** Let $B \in \text{GL}_m(\mathbb{Z}/p\mathbb{Z})$ be a matrix of order $p^r$.

(a) If $m \geq 3$, then $r \leq m - 2$.
(b) If $m = 2$, then $r \leq 1$, and $B$ is conjugate to $(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})$ or $(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix})$ in $\text{GL}_2(\mathbb{Z}/p\mathbb{Z})$.

**Proof.** The fact that $B$ has order a power of $p$ implies that $B$ is conjugate to a Jordan matrix with $\lambda = 1$ on the diagonal in $\text{GL}_m(\mathbb{Z}/p\mathbb{Z})$. From here, it is easy to see that $B^{p^{m-2}}$ and $B^p$, respectively, for $m \geq 3$ and $m = 2$, equal the identity matrix in $\text{GL}_m(\mathbb{Z}/p\mathbb{Z})$. The claim now follows. □

**Lemma 4.3.** The following statements hold.

(a) If $m \geq 3$, then $\langle \sigma^{p^{r+1}} \rangle \subset \ker(\overline{g})$ and so $[C_{p^n} : \ker(\overline{g})] \leq p^{r+1}$.
(b) If $m = 2$, then $\langle \sigma^p \rangle \subset \ker(\overline{g})$ and so $[C_{p^n} : \ker(\overline{g})] \leq p$.

**Proof.** Note that $N/\Phi(N)$ has exponent $p$. By (2.1), we also have

$$
\overline{g}(\sigma^{p^{r+1}}) = \overline{g}(\sigma^p)^p \text{ and } \overline{g}(\sigma^p) = \prod_{i=0}^{p-1} \overline{f}(\sigma)^i(\overline{g}(\sigma)).
$$

The claim for $m = 3$ then follows from the first equality. Now, suppose that $m = 2$. Then, regarding $\overline{f}(\sigma)$ an element in $\text{GL}_2(\mathbb{Z}/p\mathbb{Z})$, by Lemma 4.2, it is conjugate to a matrix $(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix})$ with $b \in \mathbb{Z}/p\mathbb{Z}$. Since

$$
\sum_{i=0}^{p-1} \left( \begin{array}{cc} 1 & b \\ 0 & 1 \end{array} \right)^i = \left( \begin{array}{cc} p & \frac{p(p-1)b}{2} \\ 0 & p \end{array} \right) = \text{zero matrix in } \text{GL}_2(\mathbb{Z}/p\mathbb{Z}),
$$

we see that indeed $\sigma^p \in \ker(\overline{g})$. This proves the claim. □

**Proof of Theorem 1.5.** Lemmas 4.2 and 4.3 imply that

$$
[C_{p^n} : \ker(\overline{g})] \leq p^{m-1}.
$$

We then see from Lemma 4.1 (b) that $\overline{g}$ indeed cannot be surjective. □

**Remark 4.4.** Note that the hypothesis that $p$ is odd is required for the second equality in (4.2). In fact, the analogous statement of Theorem 1.5 for $p = 2$ is false, as shown in [4, Corollary 5.3].
4.3. **Groups of order $n$ factorial.** In what follows, let $n \in \mathbb{N}$ with $n \geq 5$. Recall that $2A_n$ is the unique group, up to isomorphism, fitting into a short exact sequence

$$1 \rightarrow \{\pm 1\} \rightarrow 2A_n \rightarrow A_n \rightarrow 1$$

such that $\iota(\{\pm 1\})$ lies in both $Z(2A_n)$ and $[2A_n, 2A_n]$. It is known that $2A_n$ is quasisimple and $Z(2A_n) \simeq \{\pm 1\}$. We then have:

**Lemma 4.5.** If $N = [N, N]$ and there is a normal subgroup $M$ of $N$ having order two such that $N/M \simeq A_n$, then necessarily $N \simeq 2A_n$.

**Proof.** This is because a normal subgroup of order two lies in the center. □

There are some similarities in the proofs of Theorems 1.6 and 1.7 because:

(i) Both $2A_n$ and $S_n$ have order $n!$.

(ii) Both $2A_n$ and $S_n$ have only one non-trivial proper normal subgroup.

(iii) The alternating group $A_n$ is a subgroup of $S_n$ and is a quotient of $2A_n$.

For (ii), we in particular know that

\begin{equation}
\begin{cases}
Z(2A_n) \text{ is the non-trivial proper normal subgroup of } 2A_n, \\
A_n \text{ is the non-trivial proper normal subgroup of } S_n,
\end{cases}
\end{equation}

where the first statement follows from Proposition 3.5 (a). Given a prime $p$ and a group $\Gamma$, write $v_p(\Gamma)$ for the non-negative integer such that

$$p^{v_p(\Gamma)} = \text{the exact power of } p \text{ dividing } |\Gamma|.$$ 

Motivated by the arguments in [3], we shall require the next two lemmas.

**Lemma 4.6.** If $A_n$ has a subgroup of prime power index $p^m$, then $n = p^m$.

**Proof.** See [15, (2.2)]. □

**Lemma 4.7.** Let $m \in \mathbb{N}$ and let $p$ be a prime. Then, we have

(a) $|GL_m(\mathbb{Z}/p\mathbb{Z})| < \frac{1}{2}(p^m)!$ and $v_p(GL_m(\mathbb{Z}/p\mathbb{Z})) = m(m-1)/2$,

(b) $v_p(S_m) < m$ and $v_p(S_{p^m}) \geq m(m+1)/2$,

(c) $v_2(S_{2^{m-1}}) \geq m(m-1)/2 + 2$ for $m \geq 5$. 

Proof. Both claims in (a) and the first claim in (b) follow from
\[ |\text{GL}_m(\mathbb{Z}/p\mathbb{Z})| = \prod_{i=0}^{m-1} (p^m - p^i) \quad \text{and} \quad v_p(S_m) = \sum_{i=1}^{\infty} \left\lfloor \frac{m}{p^i} \right\rfloor, \]
respectively, as in \cite[(4.1) and Lemma 3.3]{3}. The two remaining claims hold because \( p^m! \) is divisible by \( p \cdots p^m \), and \( 2^{m-1}! \) is divisible by \( 2 \cdots 2^{m-1} \cdot 6 \cdot 10 \) for \( m \geq 5 \).

Now, for both Theorems 1.6 and 1.7, let \( N \) be a group of order \( n! \), and let \( M \) be any proper maximal characteristic subgroup of \( N \). The quotient \( N/M \) is then a non-trivial and \emph{characteristically simple} group, meaning that it has no non-trivial proper characteristic subgroup. It is then known that
\[ N/M \cong T^m, \quad \text{where} \ T \ \text{is simple and} \ m \in \mathbb{N}. \]
As shown in \cite[Lemma 3.2]{3}, we have
\[ \text{Aut}(N/M) \cong \text{Aut}(T)^m \rtimes S_m \ \text{when} \ T \ \text{is non-abelian}. \]
The structure of \( \text{Aut}(N/M) \) is of course well-understood when \( T \) is abelian.

4.3.1. The double cover of alternating groups. To prove Theorem 1.6, in this section, we shall assume that \( N \not\cong 2A_n \). Consider
\[ \tilde{f} \in \text{Hom}(2A_n, \text{Aut}(N)) \ \text{and} \ \tilde{g} \in Z^1_\text{f}(2A_n, N). \]
By (1.1) and Proposition 2.1, it is enough to show that \( \tilde{g} \) cannot be bijective. We shall use the same notation in (4.4), and let \( \tilde{f}, \tilde{g} \) be as in (4.1). Then, in turn, it suffices to show that \( \tilde{g} \) cannot be surjective.

Lemma 4.8. If \( N = [N : N] \), then \( \tilde{f} \) is trivial.

Proof. Suppose that \( N = [N, N] \), in which case \( T \) is non-abelian. Consider
\[ 2A_n \xrightarrow{\tilde{f}} \text{Aut}(N/M) \xrightarrow{\text{identification}} \text{Aut}(T)^m \rtimes S_m \xrightarrow{\text{projection}} S_m. \]
Notice that \( |T| \) has an odd prime factor \( p \) because a 2-group has non-trivial center. We have \( v_p(S_m) < m \) by Lemma 4.7 (b) while
\[ v_p(2A_n/Z(2A_n)) = v_p(2A_n) = v_p(N) = m \cdot v_p(T) + v_p(M) \geq m. \]
We then deduce from (4.3) that (4.5) must be trivial. It follows that \( \overline{f}(2A_n) \) lies in \( \text{Aut}(T)^m \). Note that the homomorphism

\[
2A_n \xrightarrow{\overline{f}} \text{Aut}(T)^m \xrightarrow{\text{projection}} \text{Out}(T)^m
\]

must also be trivial, because \( 2A_n = [2A_n, 2A_n] \) while \( \text{Out}(T)^m \) is solvable by Proposition 3.4 (a). Hence, in fact \( \overline{f} : 2A_n \rightarrow \text{Inn}(T)^m \simeq T^m \).

Now, by (4.3), either \( \overline{f} \) is trivial or \(|\ker(\overline{f})| \leq 2\). Observe that

\[
[2A_n : \ker(\overline{f})] \leq |T|^m = |N : M| = |2A_n|/|M| \quad \text{and so} \quad |M| \leq |\ker(\overline{f})|.
\]

If \(|\ker(\overline{f})| = 1\), then \(|M| = 1\), and we deduce that \( 2A_n \simeq T^m \simeq N \), which is a contradiction. If \(|\ker(\overline{f})| = 2\) and \(|M| = 1\), then \( \overline{f}(2A_n) \) has index two and in particular is normal in \( T^m \simeq N \), but this is impossible because \( N = [N, N] \).

Finally, if \(|\ker(\overline{f})| = 2\) and \(|M| = 2\), then \( A_n \simeq \overline{f}(2A_n) \simeq T^m \simeq N/M \), which contradicts that \( 2A_n \not\simeq N \) by Lemma 4.5. Hence, we see that necessarily \( \overline{f} \) is trivial, as desired. \( \square \)

**Lemma 4.9.** If \( N/M \) is abelian and \( \overline{g} \) is surjective, then \( \overline{f} \) is trivial.

**Proof.** Recall that \( 2A_n/Z(2A_n) \simeq A_n \), and note that

\[
\left[ \frac{2A_n}{Z(2A_n)} : \frac{\ker(\overline{g})Z(2A_n)}{Z(2A_n)} \right] = \left\{ \begin{array}{ll}
[2A_n : \ker(\overline{g})] & \text{if } Z(2A_n) \subset \ker(\overline{g}), \\
\frac{1}{2}[2A_n : \ker(\overline{g})] & \text{if } \ker(\overline{g}) \cap Z(2A_n) = 1.
\end{array} \right.
\]

These are the only cases because \( Z(2A_n) \) has order two.

Suppose that \( N/M \) is abelian and \( \overline{g} \) is surjective. Then, we have \( T \simeq \mathbb{Z}/p\mathbb{Z} \) for some prime \( p \), and \([2A_n : \ker(\overline{g})] = p^m \) by Lemma 4.1 (b). We also have

\[
\left\{ \begin{array}{ll}
n = p^m & \text{if } Z(2A_n) \subset \ker(\overline{g}), \\
n = 2^{m-1} \text{ with } m \geq 4 & \text{if } \ker(\overline{g}) \cap Z(2A_n) = 1,
\end{array} \right.
\]

by Lemma 4.6. Recall Lemma 4.7, and observe that

\[
\left\{ \begin{array}{l}
[2A_p^m : Z(2A_p^m)] > |\text{GL}_m(\mathbb{Z}/p\mathbb{Z})| \\
v_2(2A_{2^m-1}/Z(2A_{2^m-1})) > v_2(\text{GL}_m(\mathbb{Z}/2\mathbb{Z})) \text{ for } m \geq 5.
\end{array} \right.
\]

Since \( 2A_n/\ker(\overline{f}) \) embeds into \( \text{GL}_m(\mathbb{Z}/p\mathbb{Z}) \), we then deduce from (4.3) that \( \overline{f} \)
must be trivial, except possibly when
\[(4.6) \quad \ker(\overline{f}) \cap Z(2A_n) = 1 \text{ and } n = 2^{m-1} \text{ for } m = 4.\]

In this last case (4.6), suppose for contradiction that \(\overline{f}\) is non-trivial. Since
\[(4.7) \quad |2A_8| = 40320 \text{ and } |GL_4(Z/2Z)| = 20160,\]
necessarily \(\overline{f}\) is surjective and \(\ker(\overline{f}) = Z(2A_8)\). For any \(\sigma \in Z(2A_8)\), we then deduce from Lemma 4.1 (d) that \(\overline{g}(\sigma)\) is a fixed point of every automorphism on \(N/M \simeq (Z/2Z)^4\), and so \(\overline{g}(\sigma) = 1_{N/M}\). This shows that \(Z(2A_8) \subset \ker(\overline{g})\), which contradicts the first condition in (4.6). \(\square\)

**Proof of Theorem 1.6.** Suppose for contradiction that \(\overline{g}\) is surjective. In the case that \(N \supseteq [N, N]\), we may choose \(M\) to be such that \(M \supset [N, N]\), which ensures that \(N/M\) is abelian. Then, by Lemmas 4.8 and 4.9, we know that \(\overline{f}\) is trivial, whence \(\overline{g}\) is a homomorphism. Notice that \(\overline{g}\) cannot be trivial since \(N/M\) is non-trivial by choice of \(M\). But \(\overline{g}\) cannot be non-trivial either:

1. If \(N \supseteq [N, N]\), this is because \(2A_n = [2A_n, 2A_n]\) while \(N/M\) is abelian.
2. If \(N = [N, N]\), this follows from (4.3) and also the fact that \(|Z(2A_n)| \leq 2\) is impossible because \(N \not\simeq 2A_n\), by Lemmas 4.1 (b) and 4.5.

Thus, the map \(\overline{g}\) cannot be surjective, and the theorem now follows. \(\square\)

4.3.2. **Symmetric groups.** To prove Theorem 1.7, consider
\[f \in \text{Hom}(S_n, \text{Aut}(N)) \text{ and } g \in Z_1^f(S_n, N).\]
To prove the first statement, by (1.1) and Proposition 2.1, it suffices to show that one of the three stated conditions holds whenever \(g\) is bijective.

**Proof of Theorem 1.7: first statement.** First, suppose that \(\ker(f) = 1\). Since
\[f(S_n) \cap \text{Inn}(N) \text{ is a normal group of } f(S_n) \simeq S_n,\]
by (4.3) we have \(f(S_n) \cap \text{Inn}(N) \in \{f(S_n), f(A_n), 1\}\). It is easy to see that:

1. If \(\text{Inn}(N) \cap f(S_n) = f(S_n)\), then \(N \simeq S_n\) so condition (1) holds.
2. If \(\text{Inn}(N) \cap f(S_n) = f(A_n)\), then \(A_n\) embeds into \(\text{Inn}(N) \simeq N/Z(N)\) and thus \(|Z(N)| \leq 2\). Then, condition (1) holds when \(|Z(N)| = 1\) because a
subgroup of index two is always normal, and condition (2) clearly holds when \(|Z(N)| = 2\).

(iii) If \(\text{Inn}(N) \cap f(S_n) = 1\), then condition (3) holds.

Note that we do not need \(g\) to be bijective for the above arguments.

Now, suppose that \(\ker(f) \neq 1\), so then \(\ker(f) \in \{A_n, S_n\}\) by (4.3). Suppose also that \(g\) is bijective. If \(\ker(f) = S_n\), then \(N \simeq S_n\) by (2.2). If \(\ker(f) = A_n\), then \(N\) contains a subgroup isomorphic to \(A_n\) by Lemma 4.1 (c), which has index two and hence is normal in \(N\). In both cases, we see that condition (1) holds. This proves the theorem. \(\square\)

To prove the second statement, let \(M\) be any proper maximal characteristic subgroup of \(N\). We shall use the notation in (4.4), and let \(\bar{f}, \bar{g}\) be as in (4.1). By (1.1) and Proposition 2.1, it suffices to show that

\[ N/M \simeq \mathbb{Z}/2\mathbb{Z} \]

whenever \(N/M\) is abelian and \(\bar{g}\) is surjective.

The reader should compare our proof below with that of Lemma 4.9.

Proof of Theorem 1.7: second statement. Note that

\[
[A_n : A_n \cap \ker(\bar{g})] = \begin{cases} 
[S_n : \ker(\bar{g})] & \text{if } \ker(\bar{g}) \not\subset A_n, \\
\frac{1}{2}[S_n : \ker(\bar{g})] & \text{if } \ker(\bar{g}) \subset A_n,
\end{cases}
\]

and these are the only cases because \([S_n : A_n] = 2\).

Suppose that \(N/M\) is abelian and \(\bar{g}\) is surjective. Then, we have \(T \simeq \mathbb{Z}/p\mathbb{Z}\) for some prime \(p\), and \([S_n : \ker(\bar{g})] = p^m\) by Lemma 4.1 (b). We also have

\[
\begin{cases} 
n = p^m & \text{if } \ker(\bar{g}) \not\subset A_n, \\
n = 2^{m-1} \text{ with } m \geq 4 & \text{if } \ker(\bar{g}) \subset A_n,
\end{cases}
\]

by Lemma 4.6, as well as

\[
\begin{cases} 
v_p(S_{p^m}) > v_p(\text{GL}_m(\mathbb{Z}/p\mathbb{Z})) & \text{for all } m \geq 1, \\
v_2(S_{2^{m-1}}) > v_2(\text{GL}_m(\mathbb{Z}/2\mathbb{Z})) & \text{for all } m \geq 4,
\end{cases}
\]

by Lemma 4.7 and (4.7). Since \(S_n/\ker(\bar{f})\) embeds into \(\text{GL}_m(\mathbb{Z}/p\mathbb{Z})\), we see that \(\ker(\bar{f}) \neq 1\), and so \(\ker(\bar{f}) \supset A_n\) by (4.3). It then follows from Lemma 4.1
(c) that \( \overline{g} \) restricts to a homomorphism \( A_n \rightarrow N/M \). Since \( A_n = [A_n, A_n] \) and \( N/M \) is abelian, this implies that \( A_n \subset \ker(\overline{g}) \). But then
\[
2 = [S_n : A_n] \geq [S_n : \ker(\overline{g})] = [N : M]
\]
by Lemma 4.1 (b), and so we must have \( N/M \cong \mathbb{Z}/2\mathbb{Z} \), as claimed. \( \square \)

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