Universal lower bounds for Laplacians on weighted graphs

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Abstract

We discuss optimal lower bounds for eigenvalues of Laplacians on weighted graphs. These bounds are formulated in terms of the geometry and, more specifically, the inradius of subsets of the graph. In particular, we study the first non-zero eigenvalue in the finite volume case and the first eigenvalue of the Dirichlet Laplacian on subsets that satisfy natural geometric conditions.

1 Introduction

The first main objective of this article is to give an overview over recent results that deal with lower bounds on the first non-zero eigenvalue of Laplacians on graphs. These lower bounds involve the inradius of subsets. This topic certainly has a long history and key arguments have been rediscovered over and again. It seems that the classical paper by Beurling and Deny [4] is the first source for the key step, a fact that we just recently became aware of and that has gone unnoticed in the literature. The lower bounds we mentioned can be seen as discrete analogs of Poincaré or Payne-Weinberger inequalities [21], the latter both being of fundamental importance for analysis on continuum spaces.

A second point to be emphasized it that the bounds in question are universal. By this we mean inequalities that do not require additional assumptions on the graph, be it combinatorial assumptions e.g., regularity or finiteness or geometrical assumptions, e.g., curvature restrictions. This is in particular motivated by applications in mathematical physics, where random (sub-) graphs have seen a great deal of interest. By the very nature of randomness these subgraphs tend to not satisfy further regularity assumptions.
Since we deal with Laplacians of general weighted graphs our treatment includes several important classes considered in different mathematical fields, in particular combinatorial Laplacians and normalized Laplacians in the usual setting of combinatorial graphs. The estimates we derive in the finite volume case are quite easy to state and to prove and determine a lower bound that basically depends on the diameter (or the inradius) as well as on the volume. Simple examples as well as more involved ones show that these estimates are optimal. Yet, there is even a more conceptual way of showing optimality that was put forward in the recent [18] based on the following observation: in a first step towards a Poincaré inequality one estimates the variation of a function in terms of the energy and an appropriate metric. Such an estimate was called topological Poincaré inequality in the last named paper and might better be called a metric Poincaré inequality. This crucial bound had already been proved in the above mentioned [1] and does not involve the underlying measure. It gives rise to a Poincaré inequality with respect to a different metric, sometimes called resistance metric. Varying the measure one can show that the corresponding Poincaré inequality is optimal; see Section 5 for details.

The third main objective of the present article is a generalization of the already quite general set–up considered in [19]. We briefly explain the situation here and refer to Section 4 for the full picture. A non–zero lower bound for the first Neumann eigenvalue can in general not be expected in the infinite volume case. However, the lowest Dirichlet eigenvalue $\lambda^D_0(\Omega)$ of the Laplacian on the open subset $\Omega$ can be strictly positive, even if the volume of $\Omega$ is infinite. The reader may think of a strip in euclidean space for a continuum analog. Such an estimate can be deduced with the help of a Voronoi decomposition for $\Omega$ that exhibit suitable relative boundedness properties, of which finite inradius is the property that replaces finite diameter and a second condition can be thought of as a relative finite volume property. In euclidean space, a Voronoi decomposition can easily be written down. In general weighted graphs, however, the situation is much more complicated and certain topological properties of the graph are necessary, see Proposition 4.2 and Example 4.3 for details. Our new result, Theorem 4.4, now holds in full generality, in particular in situations where Voronoi decompositions are no longer available. The main new idea of the proof is really simple: for any particular given function we work on a tailor–made finite subgraph to verify the appropriate energy estimate.

While it is not the topic of this article a comment on Cheeger inequalities may be in order. Cheeger inequalities provide lower bounds on the minimum of the spectrum of Laplacians in terms of geometry viz the isoperimetric constant. For graphs such inequalities were first discussed by Alon and Milman [1] (for finite graphs) and by Dodziuk [7] (for infinite graphs). The work [7] features a somewhat unsatisfactory additional dependence on the measure. This could later be removed in the situation of normalized Laplacians by
Dodziuk and Kendall in [8]. A satisfactory answer for general Laplacians on infinite graphs was only recently established by Bauer, Keller and Wojciechowski [3]. It involves intrinsic metrics. The corresponding bounds do not require any finiteness condition on the inradius. Hence, they apply in situations where the bounds discussed below do not give any useful information. On the other hand in situations involving subsets of lattices, where our bounds apply, they tend to be better than bounds via Cheeger estimates, see [19], Section 6, for comparison.

2 Set–up and main results

In this section we present our set up and the main results. These results connect geometric data and spectral data of a graph, where our concept of graph is a very general one. In the presentation of this section we take special care to introduce the spectral data with as little technical effort as possible. A thorough discussion of the operator and form theoretic background is given in the last section.

A weighted graph is a triple $(X, b, m)$ satisfying the following properties:

- $X$ is an arbitrary set, whose elements are referred to as vertices;
- $b : X \times X \to [0, \infty)$ is a symmetric function with $b(x, x) = 0$ for all $x \in X$.
- $m : X \to (0, \infty)$ is a function on the vertices.

An element $(x, y) \in X \times X$ with $b(x, y) > 0$ is then called an edge and $b$ is denoted as edge weight; The positive function $m : X \to (0, \infty)$ gives a measure on $X$ of which we think as a volume. In particular, we define

$$\text{vol}(\Omega) := \sum_{x \in \Omega} m(x)$$

for $\Omega \subset X$.

A sequence of vertices $\gamma = (x_0, \ldots, x_k)$ is called a path from $x$ to $y$ if $x_0 = x, x_k = y$ and $b(x_l, x_{l+1}) > 0$ for $l = 0, \ldots, k - 1$. Throughout our graphs will be assumed to be connected i.e. to allow for a path between arbitrary vertices.

A natural distance to consider is

$$d(x, y) := \inf \{ L(\gamma) \mid \gamma \text{ a path from } x \text{ to } y \},$$

where the length $L(\gamma)$ of a path $\gamma$ is given by

$$L(\gamma) := \sum_{j=0}^{k-1} \frac{1}{b(x_j, x_{j+1})}.$$
Setting \( d(x, x) = 0 \) we obtain a pseudo-metric, i.e., \( d \) is symmetric and satisfies the triangle inequality. Clearly, in this generality, \( d \) need not separate the points of \( X \), a fact that is of no importance for what follows.

We call the graph geodesic if for any \( x, y \in X \) there exists a path \( \gamma \) from \( x \) to \( y \) with \( L(\gamma) = d(x, y) \).

We denote by

\[
U_r(x) := \{ y \in X \mid d(x, y) < r \} \quad \text{and} \quad B_r(x) := \{ y \in X \mid d(x, y) \leq r \}
\]

the open and closed balls of radius \( r \), respectively. The diameter of \( X \) is given by

\[
\text{diam}(X) := \sup \{ d(x, y) : x, y \in X \}.
\]

The positive function \( m : X \to (0, \infty) \) together with the distance \( d \) gives the geometrical data of the space. For further details on weighted graphs and their geometry as expressed by \( d \) and related metrics we refer to the recent studies [9, 10] as well as the surveys [13, 15] and the upcoming monograph [16].

Spectral data are given in terms of the energy \( E \) associated to the graph defined by the quadratic form

\[
E(f) := \frac{1}{2} \sum_{x, y \in X} b(x, y)(f(x) - f(y))^2 \quad \text{for} \quad f \in \mathbb{R}^X,
\]

which may assume the value \( \infty \) for the time being. The underlying Hilbert space is

\[
\ell^2(X, m) := \{ f \in \mathbb{R}^X \mid \sum_{x \in X} f(x)^2 m(x) < \infty \}
\]

with inner product and norm given by

\[
\langle f, g \rangle = \sum_{x \in X} f(x)g(x)m(x) \quad \text{and} \quad \| f \| = \sqrt{\langle f, f \rangle}
\]

respectively.

The Neumann Laplacian \( H = H^N(X, b, m) \) is the self-adjoint operator associated with \( E \) on its maximal domain in \( \ell^2(X, m) \). We refer the reader to the Appendix [4] for a precise definition as well as further details concerning forms, operators and all that. The quantity we want to estimate in the case of finite volume and finite diameter is the first non-zero eigenvalue \( \lambda_1^N \) of \( H \). This can be written down in variational terms as

\[
\lambda_1^N(X) = \inf \{ E(f) \mid f \in \ell^2(X, m) \text{ with } E(f) < \infty, \langle f, 1 \rangle = 0 \text{ and } \| f \| = 1 \}.
\]

Again, we refer to the Appendix for the fact that this is the first non-zero eigenvalue of \( H \) in the case of finite volume and finite diameter. Readers who do not want to bother with operators should just stick to the definition above.
No operator or spectral theoretic technicalities enter the quite elementary proof of our first main result, Theorem 3.4 which says that

$$\lambda_1^N \geq \frac{4}{\text{diam}(X) \text{vol}(X)}.$$  

Even though our proof partly uses known techniques, we did not find any reference in this generality, see the discussion in Section 3. Let us point out, that the finite volume estimate above can be thought of as a universal Poincaré or spectral gap inequality that holds without any further restriction on the weights. In particular, it is valid for combinatorial and normalized Laplacians. Moreover, it is optimal. This supports the feeling that the distance $d$ is rather well suited for spectral geometry in a general setting where no further information on the graph is available.

We also deduce a lower bound for the Dirichlet Laplacian $H_{\Omega}$ for $\Omega \subset X$. The latter is again defined in terms of forms and the relevant functions in the form domain are supposed to vanish on the complement $D := X \setminus \Omega$. The spectral quantity of interest is the first eigenvalue. It can be written down in variational terms as

$$\lambda_0^D(\Omega) = \inf \{ E(f) \mid f \in l^2(X, m) \text{ with } E(f) < \infty, \ f = 0 \text{ on } D \text{ and } \|f\| = 1 \}.$$  

It is easily seen (see the Appendix [5] again) that

$$\lambda_0^D(\Omega) = \min \sigma(H_{\Omega}).$$

(In the unlikely event that $E(f) = \infty$ for all $f$ with support contained in $\Omega$, $\lambda_0^D(\Omega) = \infty$ in accordance with the usual conventions.)

Theorem 3.2 says

$$\lambda_0^D(\Omega) \geq \frac{1}{R_{\Omega} \text{vol}(\Omega)};$$

with the inradius $R_{\Omega}$ given by

$$R_{\Omega} := \sup \{ r > 0 \mid \text{there exists } x \in \Omega \text{ such that } U_r(x) \subset \Omega \}.$$  

Clearly, if both $\text{vol}(\Omega)$ and $R_{\Omega}$ are finite, this gives a positive lower bound, otherwise the usual convention $\infty^{-1} = 0$ makes the statement valid but evident.

As will be seen below the proof of Theorem 3.4 and the proof of Theorem 3.2 are very similar in nature. In this context it may be elucidating to point out diameter and inradius are strongly related concepts. In fact, it is not hard to see that

$$\text{diam}(X) = \sup \{ R_{X \setminus \{x\}} : x \in X \}.$$  

So one may actually think of both Theorem 3.2 and Theorem 3.4 as giving bounds in terms of inradius. Indeed, it is possible to even derive the
bound given in Theorem 3.4 (up to the factor 4) from Theorem 3.2, compare discussion at the end of Section 3.

Subsequently, in Section 4 we are able to give a lower bound on the Dirichlet Laplacian in the infinite volume case. Of course, this can only be hoped for if \( D := X \setminus \Omega \) is relatively dense i.e. that there exist an \( R > 0 \) such that any point in \( \Omega \) has distance no more than \( R \) to a point of \( D \). Note that this relative denseness can equivalently be seen as finiteness of the inradius of \( \Omega \). Indeed, the inradius of \( \Omega \) is nothing but the infimum over all possible such \( R \geq 0 \), see [19] for details. So, here again, the relevant geometry enters via the inradius.

Moreover a relative volume estimate enters our analysis measured in terms of

\[
\text{vol}^2[r] := \inf_{s>r} \sup \{ m(U_s(x)) \mid x \in X \}
\]

and we get the following result:

\[
\lambda_D^0(\Omega) \geq \frac{1}{R_\Omega \text{vol}^2[R_\Omega]}. 
\]

The proof combines a decomposition technique from [19] with a new approximation argument. In fact, the decomposition into Voronoi cells used in [19] needs some geometric properties of the underlying weighted graphs. Specifically, it was established in the latter reference for locally compact geodesic weighted graphs.

### 3 Lower bounds in the finite volume case

The proofs of both theorems in this section go along similar lines. One main idea is an estimate that relates the energy of a function \( f \) and the maximal growth of \( f \) over a certain distance. This estimate is given in the following lemma. It has has been noted in various places and it seems the [4], Remarque 3, p. 208 is the original source. We include the simple proof for completeness reasons and later also interpret the estimate below as an estimate between different metrics, as done in the latter reference.

**Proposition 3.1 (Basic proposition).** Let \( \gamma \) be a path from \( x \) to \( y \) and \( f \in \mathbb{R}^X \) with \( \mathcal{E}(f) < \infty \) be given. Then,

\[
(f(x) - f(y))^2 \leq L(\gamma)\mathcal{E}(f).
\]
Proof. Let \( f, \gamma = (x_0, ..., x_k) \) be as above. Then,
\[
(f(x) - f(y))^2 = (f(x_0) - f(x_k))^2 \\
= \left( \sum_{j=0}^{k-1} \sqrt{b(x_j, x_{j+1}) (f(x_j) - f(x_{j+1}))} \frac{1}{\sqrt{b(x_j, x_{j+1})}} \right)^2 \\
\leq \sum_{j=0}^{k-1} b(x_j, x_{j+1}) (f(x_j) - f(x_{j+1}))^2 \sum_{j=0}^{k-1} \frac{1}{b(x_j, x_{j+1})} \\
\leq L(\gamma) E(f).
\]

Now, we can directly derive the result on the Dirichlet case.

Theorem 3.2. Let \((X, b, m)\) be as above. Let a non-empty \( \Omega \subseteq X \) be given and assume \( \text{vol}(\Omega) < \infty \) and \( \text{Inr}(\Omega) < \infty \). Then
\[
\lambda_D^0(\Omega) \geq \frac{1}{R_\Omega \text{vol}(\Omega)}.
\]

Proof of Theorem 3.2. Let \( R > R_\Omega \). Consider \( x \in \Omega \) and \( f \in \mathbb{R}^X \) with \( E(f) < \infty \) and \( f = 0 \) on \( D = X \setminus \Omega \). By definition of the inradius there is \( x_0 \in U_R(x) \setminus \Omega \). In particular, there is a path \( \gamma = (x_0, ..., x_k) \) from \( x_0 \) to \( x = x_k \) of length at most \( R \) and \( f(x_0) = 0 \). Using the above proposition, we get
\[
f(x)^2 \leq R E(f).
\]
Summing over \( x \in \Omega \) and using that \( R > R_\Omega \) was arbitrary we conclude that
\[
\|f\|_2^2 \leq R_\Omega \text{vol}(\Omega) E(f),
\]
and hence, the assertion of the theorem follows. \( \square \)

The proof of Theorem 3.4 requires one more ingredient borrowed from [18], where it was stated in the special situation at hand:

Proposition 3.3. Let \((Y, B, \mu)\) be a finite measure space. Then, for any bounded and measurable \( f : Y \to \mathbb{R} \) with \( f \perp 1 \):
\[
\|f\|_2^2 \leq \frac{1}{4} \sup_{x,y \in Y} (f(x) - f(y))^2 \mu(X).
\]

Theorem 3.4. Let \((X, b, m)\) be as above. Assume \( \text{vol}(X) < \infty \) and \( \text{diam}(X) < \infty \). Then
\[
\lambda_N^1(X) \geq \frac{4}{\text{diam}(X) \text{vol}(X)}.
\]
Proof. This directly follows from Proposition 3.1 and the previous proposition. □

Remark 3.5. (a) For finite combinatorial graphs, the lower bound is a familiar bound and our proof follows known lines, compare Lemma 1.9 in [2] and Lemma 2.4 in [2] for related estimates as well as [20], estimate (4.1) in Theorem 4.2, p. 62, where it is attributed to McKay.

(b) Theorem 4.1 from [2] gives that the lower bound is optimal up to constants, i.e., it is achieved, up to constants by balls in certain graphs. As announced in the introduction, we will present another approach to optimality in terms of variation of the measure \( m \) in Section 5.

As it is instructive we conclude this section by showing how a (slightly weaker) version of Theorem 3.4 can easily be derived from Theorem 3.2: We consider the situation of Theorem 3.4 and let \( f \in \ell^2(X, m) \) with \( E(f) < \infty \), \( \|f\| = 1 \) and \( f \perp 1 \) be given. Setting \( f^+ := \max\{f, 0\} \) and \( f^- := \max\{-f, 0\} \) we obtain the decomposition \( f = f^+ - f^- \). Clearly \( \Omega^+ := \{x \in X : f(x) > 0\} \) and \( \Omega^- := \{x \in X : f(x) < 0\} \) are disjoint with \( f^+ \) vanishing outside \( \Omega^+ \) and \( f^- \) vanishing outside of \( \Omega^- \). This easily gives

\[
E(f^+, f^-) = \frac{1}{2} \sum_{x, y \in X} b(x, y)(f^+(x) - f^+(y))(f^-(x) - f^-(y))
\]

\[= - \sum_{x, y \in X} b(x, y)f^+(x)f^-(y) \leq 0.
\]

Now, a direct computation involving Theorem 4.4 gives

\[
E(f) = E(f^+) + 2E(f^+, f^-) + E(f^-)
\]

\[
\geq E(f^+) + E(f^-)
\]

\[\geq \frac{\|f^+\|^2}{\text{vol}(\Omega^+)} \frac{\|f^-\|^2}{\text{vol}(\Omega^-)}.
\]

Here, we used the obvious bounds \( R_{\Omega^+}, R_{\Omega^-} \leq \text{diam}(X) \) and \( \text{vol}(\Omega^+), \text{vol}(\Omega^-) \leq \text{vol}(X) \) in the penultimate step.

4 Lower bounds for the Dirichlet Laplacian in the infinite volume case

In this section we consider \( H_\Omega \) again, this time without assuming that \( \text{vol}(\Omega) < \infty \). Clearly, the estimate from Theorem 3.2 breaks down in this
case. The basic idea, already employed in [19], is to decompose the large, infinite graph $X$ into finite volume pieces on which the above estimate can be used. The resulting energy estimates can be summed up to give the Poincaré type inequality. The way this was implemented in [19] required strong assumptions on the underlying geometry. Here we show how - based on the result of [19] one can actually get rid of all these assumptions.

We start with a discussion of the relevant decompositions. These were introduced in [19] and had already been used in [24] in a more restrictive set–up.

**Definition 4.1.** Let $(X, b, m)$ be as above and $D \subset X$ non-empty. A Voronoi decomposition of $X$ with centers from $D$ is a pairwise disjoint family $(V_p)_{p \in D}$ such that following conditions hold:

(V1) For each $p \in D$ the point $p$ belongs to $V_p$ and for all $x \in V_p$ there exists a path $\gamma$ from $p$ to $x$ that lies in $V_p$ and satisfies $L(\gamma) = d(p, x)$.

(V2) For each $p \in D$ and for all $x \in V_p$ the inequality $d(p, x) \leq d(q, x)$ holds for any $q \in D$.

(V3) $\bigcup_{p \in D} V_p = X$.

Given a Voronoi decomposition with centers from $D$ one can obtain a lower bound on the values of $Q(f)$ for $f$ vanishing on $D$ as follows: Let $D \subset X$ be given and let $V_p$, $p \in D$, be an Voronoi decomposition with centers from $D$. Assume that there exists for each $p \in D$ a $c_p > 0$ with

$$\frac{1}{2} \sum_{x,y \in V_p} b(x, y)(f(x) - f(y))^2 \geq c_p \sum_{x \in V_p} f(x)^2 m(x)$$

for all $f$ with $f(p) = 0$. Then, a direct summation gives

$$\mathcal{E}(f) \geq \left( \inf_{p \in D} c_p \right) \|f\|^2$$

for all $f$ which vanish on $D$ (compare [19] as well).

We now turn to existence of Voronoi decompositions. This was shown in [19] under the additional strong geometric condition of compactness of balls and this condition was shown to be necessary. Here is the precise result.

**Proposition 4.2.** Let $(X, b, m)$ be a connected graph such that all balls $B_r(x)$, $x \in X$ and $r > 0$, are finite. Assume that $D \subset X$ is non-empty and let $\Omega = X \setminus D$. Then there exists a Voronoi decomposition with centers from $D$. Moreover, whenever $R = R_D$ is finite then any Voronoi decomposition $(V_p)_{p \in D}$ of $X$ with centers from $D$ has the property that $V_p \subset B_R(p)$ for all $p \in D$.

Let us give a simple example of a geodesic weighted graph that does not allow a Voronoi decomposition:
Example 4.3. Let \( X := (\mathbb{N} \times \{0\}) \cup \{(1, 1)\} \) with weight \( b((n, 0); (n+1, 0)) = 2 \) for \( n \in \mathbb{N} \), \( b((n, 0),(1, 1)) = 1 + \frac{1}{n} \) for \( n \in \mathbb{N} \) and \( b(x, y) = 0 \) else. Since none of the points from \( D := \mathbb{N} \times \{0\} \) is closest to the point \((1, 1)\), there is no Voronoi decomposition of \( X \) with centers in \( D \) in the sense of [19] in this case.

In the general setting discussed in the present article finiteness of balls does not hold in general. So we can not directly appeal to the proposition to get a Voronoi decomposition (and subsequent lower bounds). Our method to circumvent this extra complication is quite easy. For given \( f \) vanishing on \( D \), for which we want to estimate the \( \ell^2 \)-norm by the energy, we consider a finite subgraph of \((X, b, m)\) on which a Voronoi decomposition exists by the result from [19]. We get the following estimate and underline the fact, that we have assumed no geometric restrictions apart from connectedness!

Theorem 4.4. Let \((X, b, m)\) be a connected weighted graph and \( \Omega \subset X \) such that \( R_\Omega < \infty \) and \( \text{vol}^\sharp[R_\Omega] < \infty \). Then

\[
\lambda_0^D(\Omega) \geq \frac{1}{R_\Omega \cdot \text{vol}^\sharp[R_\Omega]}.
\]

Proof. Let \( \varepsilon > 0 \), \( R > R_\Omega \) and \( f \in \ell^2(X) \) with \( \|f\| = 1 \). We have to show that

\[
1 - \varepsilon \leq R \cdot \sup \{m(U_R(x)) \mid x \in X\} \cdot E(f) \quad (\ast).
\]

To this end, first note that there is a finite subset \( X^0_f \subset X \) such that

\[
1 - \varepsilon \leq \|f|_{X^0_f}\|^2.
\]

We now enlarge \( X^0_f \) suitably. First we add finitely many points to get a subset \( X^1_f \supset X^0_f \) so that \((X^1_f, b|_{X^1_f \times X^1_f})\) is connected. (This is possible as \((X, b)\) is connected and so there is a finite path connecting each of the finitely many pairs \((x, y) \in X^0_f \times X^0_f\)).

By assumption on \( \Omega \) we know that

\[
X \subset \bigcup_{p \in D} U_R(p).
\]

Therefore, we find a finite subset \( D^0 \subset D \) so that

\[
X^1_f \subset \bigcup_{p \in D^0} U_R(p).
\]

By adding the points of \( D^0 \) as well as the points of finitely many suitably chosen paths, we get a finite subset \( X^1_f \cup D^0 \subset Y \) such that

\[
Y \subset \bigcup_{p \in D^0} U_R^Y(p),
\]
where the superscript $Y$ indicates that we are concerned with balls in the induced subgraph $(Y, b|_{Y^2}, m|_Y)$. This latter graph is finite, so in particular it satisfies the assumptions of Proposition 4.2 and we get a Voronoi decomposition with centers from $Y$. Now on each $V_p$, $p \in Y$, of the Voronoi decomposition we can appeal to Theorem 3.2 to get a lower bound for the form on $V_p$, $p \in Y$. Given this we can now proceed as discussed following the definition of Voronoi decomposition (with $Y$ instead of $D$) to obtain

$$1 - \varepsilon \leq \|f|_Y\|^2 \leq R \cdot \sup\{m(U^Y_R) \mid y \in Y\} \cdot E^Y(f).$$

This implies (*) which finishes the proof. □

5 Metric Poincaré inequalities and optimality of lower bounds

In this section we put our results in the context of Poincaré inequalities on graphs, see [17, 12] for related recent results as well. This will be used to discuss optimality.

We say the a pseudo-metric $p$ on $(X, b)$ satisfies a metric Poincaré inequality, provided

$$(f(x) - f(y))^2 \leq p(x, y)E(f) \quad (\text{MPI})$$

holds for all $f \in \mathbb{R}^X$, $x, y \in X$. Clearly, our basic Proposition 3.1 says that $d$ satisfies a metric Poincaré inequality.

The best constant $r(x, y)$ in (MPI) can easily be seen to satisfy $r(x, y) = \rho(x, y)^2$, where

$$\rho(x, y) = \sup\{f(x) - f(y) \mid E(f) \leq 1\}.$$

This latter metric also goes back to [4], where it was called distance extrémale and has later been rediscovered in different contexts, e.g., in [6]. While the validity of the triangle inequality is evident for $\rho$, also $r$ itself defines a pseudo-metric, a fact that is not so obvious. For details (and further references) we refer to [18], in particular Prop. 2.2.

We define the following seminorm on $\ell^\infty(X)$:

$$\|f\|_Y := \sup f - \inf f \quad \text{for } f \in \ell^\infty(X).$$

Moreover, we define

$$\mathcal{D} := \{f \in \mathbb{R}^X \mid E(f) < \infty\}$$

and note the following equivalence:
Proposition 5.1. Let \((X, b)\) be a graph. Then the following statements are equivalent:

(i) A global variation norm Poincaré inequality holds, i.e., there is \(c \geq 0\) such that for all \(f \in D\):
\[
\|f\|_V^2 \leq c \mathcal{E}(f) \quad \text{(GVPI)}
\]

(ii) The diameter of \((X, b)\) w.r.t. \(r\) is finite, i.e.,
\[
diam_r(X) = \sup\{r(x, y) \mid x, y \in X\} < \infty.
\]

(iii) The graph \((X, b)\) satisfies
\[
D \subset \ell^\infty(X).
\]

Moreover, the best constant \(C_P\) in (GVPI) equals \(diam_r(X)\) and the square norm \(\|J\|^2\) of the inclusion map
\[
J : D / \mathbb{R} \cdot 1 \to \ell^\infty / \mathbb{R} \cdot 1
\]
of quotients modulo the constant functions \(\mathbb{R} \cdot 1\).

The equivalence of (i) and (ii) is rather obvious; the equivalence of (i) and (iii) follows from a closed graph theorem together with the observation:
\[
\|f\|_V = \sup_{s \in [\inf f, \sup f]} \|f - s \cdot 1\|_\infty
= 2 \inf_{t \in \mathbb{R}} \|f - t \cdot 1\|_\infty
\]
See, again, [19] for details. Now, we can easily deduce the following optimal Poincaré inequality, where we write \(\mathcal{P}(X)\) for the set of all probability measures on \(X\).

Theorem 5.2. Let \((X, b)\) satisfy (GVPI). Then the best possible \(C_P\) in (GVPI) satisfies:
\[
\frac{4}{C_P} = \inf \{ \lambda_1^N(H(X, b, m)) \mid m \in \mathcal{P}(X) \text{ s.t supp}(m) = X \}.
\]

Hence we have:

Corollary 5.3. Let \((X, b)\) be a graph and \(m : X \to (0, \infty)\) such that \(\text{vol}(X) < \infty\). Then
\[
\lambda_1^N(H(X, b, m)) \geq \frac{4}{\text{diam}_r(X) \cdot \text{vol}(X)}
\]
and the estimate is optimal.

It is not too hard to see that \(d = r\) in case that \((X, b)\) is a locally finite tree, see [10], which means that we cannot do better than in Theorem 5.4.
Appendix: Forms and associated operators, domains and spectra

In this section we present the operator theoretic background for our discussion.

We first mention that
\[ E(f) := \frac{1}{2} \sum_{x,y \in X} b(x,y)(f(x) - f(y))^2 \]
defines a closed form on its maximal domain
\[ \text{dom}(E) = D \cap \ell^2(X, m) = \{ f \in \ell^2(X, m) \mid E(f) < \infty \}. \]
In fact, \( E \) can be regarded as the sum of the bounded forms \( E_{x,y}(f) = \frac{1}{2} b(x,y)(f(x) - f(y))^2 \) and so is lower semicontinuous; an appeal to [22], Theorem S.18 gives the closedness.

Of course, in the general situation considered here, the domain \( \text{dom}(E) \) is not necessarily dense in \( \ell^2(X, m) \). However, for
\[ \mathcal{H}_E := \overline{\text{dom}(E)}, \]
the closure in \( \ell^2(X, m) \), is a Hilbert space and \( (\mathcal{E}, \text{dom}(\mathcal{E})) \) defines a closed, densely defined form on \( \mathcal{H}_E \). By the form representation theorem, Thm VIII.15 in [22], there is a unique self adjoint operator \( H = H^N(X, b, m) \) in \( \mathcal{H}_E \) that is associated to this form. In analogy with the continuum situation we call this operator Neumann Laplacian as it is associated with the maximal form.

**Proposition 6.1.** Let \((X, b, m)\) be as above and, additionally, \( m(X) < \infty \). Then

(a) The function 1 belongs to \( \text{dom}(H) \) with \( H1 = 0 \) and 0 is an eigenvalue of multiplicity one,
\[ \inf \sigma(H) = 0. \]

(b) \[ \lambda_1^N = \inf \sigma(H) \setminus \{0\}. \]

If, furthermore \( \text{diam}(X) < \infty \), the latter is an eigenvalue.

**Proof.** Ad (a): Since \( m(X) \) is finite, \( 1 \in \text{dom}(\mathcal{E}) \) and \( \mathcal{E}(1) = 0 \). Therefore, \( H1 = 0 \). As, by our standing assumption, \((X, b, m)\) is connected, any element in the kernel of \( H \) has to be constant, so the multiplicity of the eigenvalue 0 is one. It is the bottom of the spectrum, since \( H \geq 0 \).

Ad (b): This follows from the min-max principle, see, e.g., Theorem XIII.2 in [23]. In case that the diameter is finite, the graph is canonically compactifiable according to [10], see also [18], and hence \( H \) has compact resolvent, so \( \lambda_1^N \) is the first non-zero eigenvalue in this case. \( \square \)
Many of our results deal with the Dirichlet Laplacian $H_\Omega$, where $\Omega \subset X$ is a subset and we imagine the Dirichlet boundary condition on $D := X \setminus \Omega$ as given by an infinite potential barrier. Therefore we get the form

$$E_\Omega(\cdot, \cdot) = \mathcal{E}(\cdot, \cdot)$$
onumber

on $\text{dom}(E_\Omega) = \{ f \in \text{dom}(\mathcal{E}) \mid f = 0 \text{ on } D \}$.

We identify $\ell^2(\Omega, m)$ with $\{ f \in \ell^2(X, m) \mid f = 0 \text{ on } D \}$ and get an associated selfadjoint operator $H_\Omega$ living in a subspace of $\ell^2(\Omega, m)$ in analogy to what we saw for the Neumann Laplacian above. Note that $E_\Omega$ and $H_\Omega$ are always to be understood relative to the bigger ambient graph $X$.

**Proposition 6.2.** Let $(X, b, m)$ be as above and $\Omega \subset X$. Then

$$\lambda_0^D(\Omega) = \inf \sigma(H_\Omega).$$

This, again, is a consequence of the min-max principle.

**Remark 6.3.** Another natural choice for the relevant forms would be to consider what might be thought of as Dirichlet boundary conditions at infinity, given by the form domain

$$\text{dom}(E^{D, \infty}) := \{ f \in \mathcal{D} \mid \text{supp}(f) \text{ is a finite set} \},$$

where the support of $f$ is given by $\text{supp}(f) := \{ x \in X \mid f(x) \neq 0 \}$ and the closure is meant with respect to the form norm given by the energy. Then, the restriction

$$E^{D, \infty} := E|_{\text{dom}(E^{D, \infty})}$$

is a closed form. We do not study the associated operator $H^{D, \infty}$ here but only mention that it is bounded below by the Neumann Laplacian. So, our estimates hold for this operator as well. Similar consideration apply to the restriction to $\Omega$.

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