FACTORIZATION AND SU(2) HEAVY FLAVOR SYMMETRY FOR B-MESON DECAYS PRODUCING CHARMONIUM

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Abstract

We show that the factorization assumption in color-suppressed $B$ meson decays is not ruled out by experimental data on $B \to K(K^*) + J/\Psi(\Psi')$. The problem previously pointed out might be due to an inadequate choice of hadronic form factors.

Within the Isgur-Wise SU(2) heavy flavor symmetry framework, we search for possible $q^2$ dependence of form factors that are capable of explaining simultaneously the large longitudinal polarization $\rho_L$ observed in $B \to K^* + J/\Psi$ and the relatively small ratio of rates $R_{J/\Psi} = \Gamma(B \to K^* + J/\Psi)/\Gamma(B \to K + J/\Psi)$.

We find out that the puzzle could be essentially understood if the $A_1(q^2)$ form factor is frankly decreasing, instead of being almost constant or increasing as commonly assumed.

Of course, the possibility of understanding experimental data is not necessarily a proof of factorization.

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I Introduction

In a recent paper \cite{1}, Kamal and two of us (M.G. and X.Y.P.) have shown, within the factorization approach, the failure of commonly used $B \to K(K^*)$ form factors in explaining recent data on $B \to J/\Psi + K^*$ decays. The main problem is a simultaneous fit of the large longitudinal polarization $\rho_L$ in $B \to J/\Psi + K^*$ decay and of the relatively small ratio $R_{J/\Psi}$ of the $J/\Psi + K^*$ rate compared to the $J/\Psi + K$ one. We concluded that the difficulty in understanding experimental data might be due to a failure of the factorization method or to a wrong choice of the hadronic form factors or both.

Such an analysis has been independently performed by Aleksan, Le Yaouanc, Oliver, Pène and Raynal \cite{2} who also found difficulties in fitting both $\rho_L$ and $R_{J/\Psi}$, in spite of their large choice of heavy to light hadronic form factors.

In the previous work \cite{1}, in addition to our exploration of the usual $B \to K(K^*)$ form factors available in the literature, we also related the $B \to K(K^*)$ to the $D \to K(K^*)$ form factors using the SU(2) heavy flavor symmetry between the b and c quarks as first proposed by Isgur and Wise \cite{3}. The input data are the hadronic form factors in the $D$ sector normalized at $q^2 = 0$ as extracted from semi-leptonic $D \to \overline{K}(K^*) + \ell^+ + \nu_\ell$ decay. In such experiments, the $q^2$ distributions are not measured, and the analysis of experimental data has been made assuming monopole $q^2$ dependence for all the $D \to \overline{K}(K^*)$ form factors. For that reason in \cite{1}, we have also used monopole forms in the $B$ sector. The resulting $B \to K(K^*)$ form factors obtained in this way were also unable to explain simultaneously $\rho_L$ and $R_{J/\Psi}$.

Our method, based on the Isgur-Wise relations, has been subsequently adopted by Cheng and Tseng \cite{4} who considered various types of $q^2$ dependence for the hadronic form factors. However their model still encounters difficulties in reproducing correctly experimental data.

The purpose of this paper is to make a purely phenomenological investigation of the possible $q^2$ dependence - we shall call scenario - of the hadronic form factors in the $B$ sector such that, assuming factorization and using the Isgur-Wise relations \cite{3} together with the latest data at $q^2 = 0$ in the $D$ sector \cite{5}, we are able to obtain a good fit for both $\rho_L$ and $R_{J/\Psi}$.

Some preliminary remarks are in order. We are aware of the fact that the values at $q^2 = 0$ of the $D \to K(K^*)$ form factor have been extracted from semi-leptonic decay experiments assuming a monopole $q^2$ dependence for all hadronic form factors. This ansatz is certainly
inconsistent with theoretical expectations coming, for instance, from QCD sum rules [6], from lattice gauge calculations [7] as well as from asymptotic scaling law of heavy flavours [2, 3, 4]. A correct procedure would be to reanalyze the triple angular distribution fit [5] in the $D$ semi-leptonic decay, with different scenarios, in order to evaluate the sensitivity to the scenarios of the values at $q^2 = 0$ of the form factors. Such a study has not yet been done by experimentalists. Of course the cleanest information would come from a measurement of the $q^2$ distributions for the rates and for the various polarizations in the semi-leptonic $D$ sector. We are still far from such an ideal situation and for the time being, the only pragmatic way is to use the results quoted in [5] with errors included for the values at $q^2 = 0$ of the form factors.

We propose, in this paper, four types of scenarios for each of the $B \rightarrow K(K^*)$ hadronic form factors $F_1, A_1, A_2$ and $V$ in the Bauer, Stech and Wirbel (BSW henceforth) notation [8].

The $q^2$ dependences are taken as $(1-q^2/\Lambda^2)^{-n}$ applied indiscriminately to all of these form factors.

The algebraic integer $n$ symbolically represents $n_F, n_1, n_2$ and $n_V$ associated respectively to $F_1, A_1, A_2$ and $V$. These integers $n$ can take four values corresponding to four types of scenarios mentioned above: $-1$ for a linear dependence, $0$ for a constant, $+1$ for a monopole and $+2$ for a dipole.

The pole masses $\Lambda_F, \Lambda_1, \Lambda_2$ and $\Lambda_V$ for $F_1, A_1, A_2$ and $V$ respectively are treated as phenomenological parameters. Being related, in some way, to bound states of the $\bar{b}s$ system, we impose to these parameters the physical constraint to be in the range $(5-6)$ GeV. Such a requirement is satisfied by the pole masses of the BSW model [8].

We now summarize the results of our finding:

The experimental data on $\rho_L$ and $R_{J/\Psi}$ indeed can been fitted for three scenarios corresponding to three possibilities $n_2 = 2, 1, 0$ for $A_{2K^*}^B$ together with:

i) $n_1 = -1$ for a linear decreasing with $q^2$ of $A_{1K^*}^B$

ii) $n_V = +2$ for a dipole increasing with $q^2$ of $V_{K^*}^B$

iii) $n_F = +1$ for a monopole increasing with $q^2$ of $F_{1K^*}^B$

For a given selected scenario we have a non empty allowed domain in the $\Lambda_F, \Lambda_1, \Lambda_2, \Lambda_V$ parameter space. Therefore we obtain hadronic form factors for $B \rightarrow K(K^*)$ reproducing correctly (within experimental errors) $\rho_L$ and $R_{J/\Psi}$ with the parameters $\Lambda_F, \Lambda_1, \Lambda_2, \Lambda_V$ physically acceptable. We now easily understand why previous attempts [1, 2, 4] were unsuccessful, mainly
because the decrease with $q^2$ of the form factor $A_1(q^2)$ has never been seriously considered. Let us emphasize however that such an unusual $q^2$ behaviour has already been obtained by Narison \[6\] in the QCD sum rule approach. Of course our result is not a proof of factorization in the $B$ sector. It only makes wrong the statement that the failure in explaining simultaneously $\rho_L$ and $R_{J/\Psi}$ necessarily implies that factorization breaks down in this sector.

This paper is organized as follows. In section II we give the consequences of factorization for the decay amplitudes $B \to K(K^*) + (\eta_c, J/\Psi, \Psi')$ which are color-suppressed processes. We study the kinematics and we review the available experimental data for these decay modes.

In section III we discuss in some detail the Isgur-Wise relations \[3\] and, in particular, the consistency of scenarios in the $B$ and $D$ sectors as well as the relations between the parameters $\Lambda_j (j = F, 1, 2, V)$ in both sectors. The case of the form factors $F_0$ and $A_0$, associated to the spin zero part of the currents, is equally discussed.

Section IV is devoted to the decay modes $B \to K(K^*) + \Psi'$, in which some scenario independent results can be obtained. The left-right asymmetry $A'_{LR}$ between the two transverse polarizations in $B \to K^* + \Psi'$ is found to be large and close to its maximal value. The fractional longitudinal polarization $\rho'_L$ turns out to be a slowly varying function of $\Lambda_2$ and the ratio of rates $R_{\Psi'}$ as a function of $\Lambda_2$ and $\Lambda_F$. The result obtained for $R_{\Psi'}$ is consistent with experiment \[5\]. Our prediction for $\rho'_L$ is compared with that of Kamal and Sanda \[9\] who use seven different scenarios.

Section V is the central part of this paper being related to the decay modes $B \to K(K^*) + J/\Psi$. The study of $\rho_L$ and $R_{J/\Psi}$ allows us to select only three surviving scenarios among the $4^3 = 64$ possible cases and to constraint the $\Lambda_F, \Lambda_1, \Lambda_2, \Lambda_V$ parameter space.

The comparison between $J/\Psi$ and $\Psi'$ in the final states is studied in section VI and the results, in the framework of our model, are shown to be compatible with experiment. Comparison with the work of Ref.\[4\] is made in some details.

For the decay modes $B \to K(K^*) + \eta_c$ where no experimental data are available, we give in section VII, some predictions for the ratios of rates.

Finally, in section VIII, we come back to the $D$ sector in the light of results obtained in the $B$ sector. Of course in the $B$ and $D$ sectors, the hadronic form factors $F_1, A_1, A_2, V$ follow the same scenarios - same values of $n_F, n_1, n_2, n_V$ and the poles masses are related via Eqs. \(43\) and \(50\) of section III. We determine the normalized $q^2$ distributions for semi-leptonic decays.
\[ D \to K\ell^+\nu_\ell, \quad D \to K^*\ell^+\nu_\ell \] and for this last mode the integrated longitudinal polarization \( \rho^q_{L} \) and the left-right asymmetry \( A^q_{L,R} \) between transverse polarizations (sl stands for semi-leptonic).

A discussion of the results is given in the conclusion. A more detailed study of all these topics can be found in our recent internal report \[10\].

II. Factorization and Kinematics, Experimental Data.

1°) The two-body decays of the charged and neutral \( B \) mesons discussed in this paper are described, at the tree level, by the color-suppressed diagram. Penguin diagrams also contribute to these decays at the one loop level. However the colorless charmonium states \( \tau c \) have to be excited from the vacuum and for which two or three gluons are needed. For that reason the penguins are neglected in this paper.

2°) We consider the decay modes

\[
B^+ \to K^+(K^{*+}) + \eta_c(J/\Psi, \Psi') \tag{1}
\]

\[
B^o \to K^o(K^{*0}) + \eta_c(J/\Psi, \Psi')
\]

and we compute the decay amplitudes assuming factorization. We obtain an expression of the form:

\[
\langle \tau c + \bar{q}q| T| \bar{b}q \rangle \propto \langle \tau c| J^\mu| 0 \rangle \langle \bar{q}q| J_\mu| B \rangle \tag{2}
\]

The first term in the right-hand side of Eq.(2) involves the decay constants \( f_{\eta_c}, f_{J/\Psi} \) and \( f_{\psi'} \) for \( \eta_c, J/\Psi \) and \( \Psi' \) respectively. The second term is governed by the hadronic form factors for the \( B \to K \) or \( B \to K^* \) transitions. As a consequence, the branching ratios have the following structure:

\[
BR = BR_0 \cdot \left( \frac{f_{\eta_c}}{m_B} \right)^2 \cdot PS \cdot FF \tag{3}
\]

The common scale \( BR_0 \) contains the Fermi coupling constant \( G_F \), the Cabibbo-Kobayashi-Maskawa factors \( V_{cb}V_{cs}^* \), the \( B \) meson life time \( \tau_B \), and the phenomenological BSW constant \( a_2 \) for color-suppressed processes:

\[
BR_0 = \left[ \frac{G_F m_B^2}{\sqrt{2}} \right]^2 |V_{cb}|^2 |V_{cs}^*|^2 \frac{m_B}{8 \pi} a_2^2 \frac{\tau_B}{\hbar} \tag{4}
\]
Being interested only in ratios of decay widths, we shall not compute the $BR_0$ numerically.

The quantity $PS$ is a dimensionless phase space factor depending only on masses of the involved particles. Because of the small $K^+(K^{*+}) - K^o(K^{*o})$ mass differences, the numerical values of $PS$ are slightly different for $B^+$ and $B^o$ decays. However these differences are typically $O(10^{-3})$, hence we ignore the mass differences between charged and neutral strange mesons and the numerical values given below correspond to $B^+$ decays.

The last factor $FF$ depends on the hadronic form factors and it contains the dynamics of the weak decays.

The results are:

\begin{align}
(a) \ B^+ \to K^+ + \eta_c & \quad PS = 0.3265 \quad FF = |F_0^{BK}(m_{\eta_c})|^2 \\
(b) \ B^+ \to K^+ + J/\Psi & \quad PS = 0.1296 \quad FF = |F_1^{BK}(m_{J/\Psi})|^2 \\
(c) \ B^+ \to K^+ + \Psi' & \quad PS = 0.0575 \quad FF = |F_1^{BK}(m_{\Psi'})|^2 \\
(d) \ B^+ \to K^{*+} + \eta_c & \quad PS = 0.1218 \quad FF = |A_0^{BK^*}(m_{\eta_c})|^2 \\
(e) \ B^+ \to K^{*+} + J/\Psi & \quad PS = 0.1399 \\
& \quad FF = |A_1^{BK^*}(m_{J/\Psi})|^2 [(a - bx)^2 + 2(1 + c^2y^2)] \\
(f) \ B^+ \to K^{*+} + \Psi' & \quad PS = 0.1407 \\
& \quad FF = |A_1^{BK^*}(m_{\Psi'})|^2 [(a' - b'x')^2 + 2(1 + c'^2y'^2)]
\end{align}

The analytic expressions for $a, b, c$ are previously given in Ref.[1] and $a', b', c'$ are obtained respectively from $a, b, c$ by the simple substitution $m_{\Psi'}$ to $m_{J/\Psi}$. We get numerically:

\begin{align}
\begin{array}{ccc}
a = 3.1652 & b = 1.3084 & c = 0.4356 \\
a' = 2.0514 & b' = 0.5538 & c' = 0.3092
\end{array}
\end{align}

The ratios of form factors $x, y, x', y'$ are defined by:

\begin{align}
x & \equiv x^B(m_{J/\Psi}^2) = \frac{A_2^{BK^*}(m_{J/\Psi}^2)}{A_1^{BK^*}(m_{J/\Psi}^2)} \\
y & \equiv y^B(m_{J/\Psi}^2) = \frac{V^{BK^*}(m_{J/\Psi}^2)}{A_1^{BK^*}(m_{J/\Psi}^2)} \\
x' & \equiv x^B(m_{\Psi'}^2) = \frac{A_2^{BK^*}(m_{\Psi'}^2)}{A_1^{BK^*}(m_{\Psi'}^2)} \\
y' & \equiv y^B(m_{\Psi'}^2) = \frac{V^{BK^*}(m_{\Psi'}^2)}{A_1^{BK^*}(m_{\Psi'}^2)}
\end{align}

For each of the $K^* + J/\Psi$ and $K^* + \Psi'$ modes, we have three possible polarization states, one is longitudinal (LL) and two are transverse (−−, ++) for both final particles. We now
define two interesting quantities: First, the fractional longitudinal polarization:

\[ \rho_L = \frac{\Gamma(B \to K^* + J/\Psi)_{LL}}{\Gamma(B \to K^* + J/\Psi)} \quad , \quad \rho'_L = \frac{\Gamma(B \to K^* + \Psi')_{LL}}{\Gamma(B \to K^* + \Psi')} \]  

(15)

and second, the left-right asymmetry:

\[ A_{LR} = \frac{\Gamma(B \to K^* + J/\Psi)_{--} - \Gamma(B \to K^* + J/\Psi)_{++}}{\Gamma(B \to K^* + J/\Psi)_{--} + \Gamma(B \to K^* + J/\Psi)_{++}} \]

\[ A'_{LR} = \frac{\Gamma(B \to K^* + \Psi')_{--} - \Gamma(B \to K^* + \Psi')_{++}}{\Gamma(B \to K^* + \Psi')_{--} + \Gamma(B \to K^* + \Psi')_{++}} \]  

(16)

We get:

\[ \rho_L = \frac{(a - bx)^2}{(a - bx)^2 + 2(1 + c^2 y^2)} \quad , \quad \rho'_L = \frac{(a' - b'x')^2}{(a' - b'x')^2 + 2(1 + c^2 y'^2)} \]  

(17)

\[ A_{LR} = \frac{2cy}{1 + c^2 y^2} \quad , \quad A'_{LR} = \frac{2cy'}{1 + c^2 y'^2} \]  

(18)

We also introduce four ratios of rates, only three of these ratios are independent:

\[ R_{J/\Psi} = \frac{\Gamma(B \to K^* + J/\Psi)}{\Gamma(B \to K + J/\Psi)} \quad , \quad R_{\Psi'} = \frac{\Gamma(B \to K^* + \Psi')}{\Gamma(B \to K + \Psi')} \]  

(19)

\[ S = \frac{\Gamma(B \to K + \Psi')}{\Gamma(B \to K + J/\Psi)} \quad , \quad S^* = \frac{\Gamma(B \to K^* + \Psi')}{\Gamma(B \to K^* + J/\Psi)} \]  

(20)

Defining two more ratios of form factors:

\[ z \equiv z^B(m^2_{J/\Psi}) = \frac{F^{BK}_{1}(m^2_{J/\Psi})}{A^{BK*}_{1}(m^2_{J/\Psi})} \quad , \quad z' \equiv z^B(m^2_{\Psi'}) = \frac{F^{BK}_{1}(m^2_{\Psi'})}{A^{BK*}_{1}(m^2_{\Psi'})} \]  

(21)

We obtain:

\[ R_{J/\Psi} = 1.0793 \frac{(a - bx)^2 + 2(1 + c^2 y^2)}{z^2} \]  

(22)

\[ R_{\Psi'} = 2.4455 \frac{(a' - b'x')^2 + 2(1 + c'^2 y'^2)}{z^2} \]

For \( S \) and \( S^* \), we get:

\[ S = 0.4438 \left( \frac{f_{\Psi'}}{f_{J/\Psi}} \right)^2 \frac{F^{BK}_{1}(m^2_{\Psi'})}{F^{BK}_{1}(m^2_{J/\Psi})} \]  

(23)

\[ S^* = 1.0057 \left( \frac{f_{\Psi'}}{f_{J/\Psi}} \right)^2 \frac{A^{BK*}_{1}(m^2_{\Psi'})}{A^{BK*}_{1}(m^2_{J/\Psi})} \frac{(a' - b'x')^2 + 2(1 + c'^2 y'^2)}{(a - bx)^2 + 2(1 + c^2 y^2)} \]  

(24)
The experimental data for decay rates as averaged by PDG [5] are given in Table 1. We have no experimental information on the $K \eta_c$ and $K^* \eta_c$ modes.

The ratios $R_{\Psi'}$ and $R_{J/\Psi}$, $S$ and $S^*$ can be estimated from data of Table 1 and the results are shown in Table 2.

| Ratio     | $B^+$            | $B^0$            | $B^+, B^0$ combined |
|-----------|------------------|------------------|---------------------|
| $R_{\Psi'}$ | $< 4.35 \pm 1.95$ | $> 1.75 \pm 1.12$ | $2.03 \pm 1.59$     |
| $R_{J/\Psi}$ | $1.67 \pm 0.54$  | $2.11 \pm 0.70$  | $1.83 \pm 0.42$     |
| $S$       | $0.68 \pm 0.32$  | $< 1.07 \pm 0.30$ | $0.68 \pm 0.32$     |
| $S^*$     | $< 1.76 \pm 0.52$| $0.89 \pm 0.59$  | $0.89 \pm 0.59$     |

Table 2.

A direct measurement of $R_{J/\Psi}$ with both $B^+$ and $B^0$ by CLEO II [11] is consistent with our estimate given in the last column of Table 2,

CLEO II $R_{J/\Psi} = 1.71 \pm 0.34$

In what follows we shall use the constraint $R_{J/\Psi} \leq 2.2$.

4) For the $B \to K^* + J/\Psi$ mode, we have experimental data for the fractional longitudinal polarization $\rho_L$:

- CLEO II [11] $\rho_L = 0.80 \pm 0.08 \pm 0.05$
- CDF [12] $\rho_L = 0.66 \pm 0.10^{+0.08}_{-0.10}$
- ARGUS [13] $\rho_L = 0.97 \pm 0.16 \pm 0.15$
Averaging these results with the standard weighted least-squares procedure, we obtain:

\[ \rho_L = 0.780 \pm 0.073 \]

In what follows we shall use the one standard deviation lower limit: \( \rho_L \geq 0.7 \).

We remark that model-independent upper bounds for \( \rho_L \) and \( \rho'_L \) can be derived:

\[ \rho_L \leq \frac{a^2}{a^2 + 2} = 0.833 \quad , \quad \rho'_L \leq \frac{a^2}{a^2 + 2} = 0.678 \quad (25) \]

These results are actually the most rigorous consequences of factorization, their violations imply unquestionably the failure of factorization hypothesis whatever are the form factors. For this reason, it will be very interesting if the errors in the new Argus data would be significantly reduced.

50) The input data in the \( D \) sector are coming from analyses of the semi-leptonic decays \( D \to K + \ell^+ + \nu_\ell \) and \( D \to K^* + \ell^+ + \nu_\ell \). We shall use the average values for \( F_1^{DK}(0) \), \( A_1^{DK^*}(0) \), \( x^D(0) \equiv A_2^{DK^*}(0)/A_1^{DK^*}(0) \) and \( y^D(0) \equiv V^{DK^*}(0)/A_1^{DK^*}(0) \) as given by the Particle Data Group. These results are shown in Table 3.

| \( F_1^{DK}(0) \) | \( A_1^{DK^*}(0) \) | \( x^D(0) \) | \( y^D(0) \) |
|---------------|---------------|-------------|-------------|
| 0.75 ± 0.03   | 0.56 ± 0.04   | 0.73 ± 0.15 | 1.89 ± 0.25 |

Table 3.

The quantities \( F_1^{DK}(0) \) and \( A_1^{DK^*}(0) \) are determined from semi-leptonic integrated rates. The ratios \( x^D(0) \) and \( y^D(0) \) are extracted by fitting the angular distributions in \( D \to K^* + \ell^+ + \nu_\ell \). The ratio \( z^D(0) = F_1^{DK}(0)/A_1^{DK^*}(0) \) is found to be \( z^D(0) = 1.34 \pm 0.11 \) from Table 3.

Let us remind that the four values in Table 3 are obtained by assuming monopole \( q^2 \) behaviour of all form factors, with pole masses in the \((2.1 - 2.5) \, \text{GeV}\) region.

III. The ISGUR-WISE relations due to SU(2) flavor symmetry

\[\footnote{A. N. Kamal, private communication}\]
The SU(2) flavor symmetry between the heavy b and c quarks allows us to derive relations between the $B \to K(K^*)$ and $D \to K(K^*)$ hadronic form factors at the same velocity transfer though at different momentum transfers. Calling $t_B$ ($t_D$) the value of the squared momentum transfer $q^2$ for $B(D)$ form factors, we obtain the following kinematic relations:

$$v_b\cdot v_K = v_c\cdot v_K \quad \text{or} \quad m_c t_B - m_b t_D = (m_b - m_c)(m_b m_c - m^2_K)$$ (26)

$$v_b\cdot v_{K^*} = v_c\cdot v_{K^*} \quad \text{or} \quad m_c t^*_B - m_b t^*_D = (m_b - m_c)(m_b m_c - m^2_{K^*})$$ (27)

In practice, we shall use the experimental data in the $D$ sector at zero momentum transfer $t_D = t^*_D = 0$. The corresponding values in the $B$ sector, $t^o_B$ and $t^{*o}_B$ are given from Eqs. (26) and (27) by

$$t^o_B = (\frac{m_b}{m_c} - 1)(m_b m_c - m^2_K)$$ (28)

$$t^{*o}_B = (\frac{m_b}{m_c} - 1)(m_b m_c - m^2_{K^*})$$ (29)

The knowledge of the hadronic form factors at $q^2 = 0$ in the $D$ sector will determine the hadronic form factor values in the $B$ sector at $q^2 = t^o_B$ for $B \to K$ case and at $q^2 = t^{*o}_B$ for $B \to K^*$ case.

The values of $t^o_B$ and $t^{*o}_B$ depend on the quark masses $m_b$ and $m_c$. We choose, in this paper, $m_b = 4.7\, GeV$, $m_c = 1.45\, GeV$ and get

$$t^o_B = 14.73\, GeV^2$$ (30)

$$t^{*o}_B = 13.49\, GeV^2$$ (31)

We first consider the case of $B \to K$ and $D \to K$ form factors. The matrix elements of the weak current involve two form factors $f_+$ and $f_-$ defined by

$$<K|J_\mu|D> = (p_D + p_K)_\mu \ f^D_{+K}(q^2) + (p_D - p_K)_\mu \ f^D_{-K}(q^2)$$ (32)

$$<K|J_\mu|B> = (p_B + p_K)_\mu \ f^{BK}_{+K}(q^2) + (p_B - p_K)_\mu \ f^{BK}_{-K}(q^2)$$ (33)

where $q = p_{B,D} - p_K$.

In the BSW basis [8], the spin one and the spin zero parts of the weak current are separated and two new form factors $F_1$ and $F_0$ are defined:

$$F^{PK}_1(q^2) = f^{PK}_{+K}(q^2)$$ (34)

$$F^{PK}_0(q^2) = f^{PK}_{+K}(q^2) + \frac{q^2}{m^2_P - m^2_K} \ f^{PK}_{-K}(q^2)$$ (35)
where $P = B$ or $D$.

The Isgur-Wise relations [3] are written with $f_+$ and $f_-$:

\[
(f_+ \pm f_-)^{BK}(t_B) = \left(\frac{m_c}{m_b}\right)^{\pm 1/2} (f_+ \pm f_-)^{DK}(t_D) \tag{36}
\]

The spin one form factor $F_1^{BK}$ is then related to both $f_+^{DK}$ and $f_-^{DK}$. It is convenient to write its expression in the form:

\[
F_1^{BK}(t_B) = \frac{m_b + m_c}{2\sqrt{m_b m_c}} \left[1 + \frac{m_b - m_c}{m_b + m_c} \mu^D(t_D)\right] F_1^{DK}(t_D) \tag{37}
\]

where the ratios of two form factors $f_-$ and $f_+$ are defined by

\[
\mu^P(q^2) = -\frac{f^{PK}(q^2)}{f^{PK}(q^2)} \quad P = B, D \tag{38}
\]

The spin zero form factor $F_0^{BK}$ can then be written in the form:

\[
F_0^{BK}(q^2) = \left[1 - \frac{q^2}{m_B^2 - m_K^2} \mu^B(q^2)\right] F_1^{BK}(q^2) \tag{39}
\]

The two ratios $\mu^B(q^2)$ and $\mu^D(q^2)$ are related by SU(2) flavor symmetry and from Eq.(36), we get

\[
\mu^B(t_B) = \frac{(m_b - m_c) + (m_b + m_c)}{(m_b + m_c) + (m_b - m_c)} \mu^D(t_D) \tag{40}
\]

3\textsuperscript{o}) As explained previously, we shall use in the $D$ sector the values of the hadronic form factors at $q^2 = 0$ as coming from semi-leptonic experimental data. Because of the normalization constraint $F_1^{DK}(0) = F_0^{DK}(0)$ only $f_+^{DK}(0)$ is known and we cannot have, in that way, any information on $f_-^{DK}(0)$ and $\mu^D(0)$\textsuperscript{¶}.

We now make an assumption which is natural and also suggested [14] in the framework of the SU(2) heavy flavor symmetry. If $f_+^{DK}(q^2)$ and $f_-^{DK}(q^2)$ have the same type of $q^2$ dependence, no matter how it is, then the ratio $\mu^D$ is independent of $q^2$ and, using the Isgur-Wise relations (36), we easily see that the same property extends to the $B$ sector. In particular, the ratio $\mu^B$ is a constant related to $\mu^D$ by Eq.(40).

\textsuperscript{¶} The QCD correction factor $\left[\frac{\alpha_s(m_b)}{\alpha_s(m_c)}\right]^{-6/25}$ in the right-hand side has been omitted in the Isgur-Wise relations because we are essentially interested, in this paper, in ratios of form factors like $x, x', y, y', z$ and $z'$. A numerical estimate of this quantity is 1.15.

\textsuperscript{*} In principle, $f^{DK}(0)$ and $\mu^D(0)$ could be measured in $D \to K^+ + \mu^+ + \nu_\mu$, namely by looking at polarized muon.
Now, if $F_1^{DK}(q^2)$ is written in the form

$$F_1^{DK}(q^2) = \frac{F_1^{DK}(0)}{[1 - \frac{q^2}{\Lambda_{DF}^2}]^{n_F}}$$  \hspace{1cm} (41)$$

where $n_F$ is some algebraic integer, then, using Eq.(37) we obtain a similar expression for $F_1^{BK}(q^2)$:

$$F_1^{BK}(q^2) = \frac{F_1^{BK}(0)}{[1 - \frac{q^2}{\Lambda_F^2}]^{n_F}}$$  \hspace{1cm} (42)$$

with the same $n_F$.

Furthermore as explained in [1, 10], a relation between the pole masses $\Lambda_{DF}$ and $\Lambda_F$ in the $D$ and $B$ sectors can be obtained and already given in Ref.[1], the result is:

$$m_c \Lambda_F^2 - m_b \Lambda_{DF}^2 = m_c t_B^0 = (m_b - m_c)(m_b m_c - m_K^2)$$  \hspace{1cm} (43)$$

Of course, the values at $q^2 = 0$ of the form factors $F_1^{BK}$ and $F_1^{DK}$ are also related by the Isgur-Wise relation (36). For details see Ref.[10].

An attractive scenario for $F_1^{DK}(q^2)$ suggested by many theoretical studies [2, 8, 15] as well as supported by experimental data [4, 16] is a monopole $q^2$ dependence, $n_F = 1$, with a pole mass $\Lambda_{DF}$ in the 2 GeV region. From now on, we make this monopole choice for both $F_1^{DK}$ and $F_1^{BK}$ and we shall restrict the pole mass in the $B$ sector, $\Lambda_F$, to be in the $(5 - 6)$ GeV region in agreement furthermore with Eq.(43).

From Eq.(38) written now with a constant $\mu^B$, we obtain for the spin 0 form factor $F_0^{BK}$ a different $q^2$ behaviour from that of $F_1^{BK}$.

$$F_0^{BK}(q^2) = \frac{[1 - \frac{q^2}{m_B^2 - m_K^2}] \mu^B}{[1 - \frac{q^2}{\Lambda_F^2}]} F_0^{BK}(0)$$  \hspace{1cm} (44)$$

For the particular value of $\mu^B$:

$$\mu^B = \frac{m_B^2 - m_K^2}{\Lambda_F^2}$$  \hspace{1cm} (45)$$

the form factor $F_0^{BK}(q^2)$ becomes independent of $q^2$. Such a situation has been suggested by some analyses [1, 14] and in this paper $\mu^B$ will be related to $\Lambda_F^2$ by the equality (45). The ratio $\mu^D$, computed from $\mu^B$ using the relation (40), becomes also a function of $\Lambda_F^2$.

We now study the case of the $B \to K^*$ and $D \to K^*$ hadronic form factors. The matrix elements of the weak current involve four form factors, $f, g, a_+$ and $a_-$ in the Isgur-Wise basis [3], or $A_1, A_2, V$ and $A_0$ in the BSW basis [8].
For $A_1$ and $V$, the Isgur-Wise relations are very simple [8]. Forgetting as previously the QCD factors $\left[\frac{\alpha_s(m_b)}{\alpha_s(m_c)}\right]^{-6/25}$, we obtain

$$A_1^{BK^*}(t_B^*) = \sqrt{\frac{m_c}{m_b}} A_1^{DK^*}(t_D^*)$$

(46)

$$V^{BK^*}(t_B^*) = \sqrt{\frac{m_c}{m_b}} V^{DK^*}(t_D^*)$$

(47)

As a consequence, the ratio $y^B$ defined in Eq.(13) at $q^2 = t_B^{*o}$ is directly given by $y^D(0)$, the QCD factor cancels out in this ratio:

$$y^B(t_B^{*o}) = \frac{m_c}{m_b} \left[\frac{m_B + m_K^*}{m_D + m_K^*}\right]^2 y^D(0)$$

(48)

The dependence with respect to $q^2$ of $A_1$ and $V$ in the $B$ and $D$ sectors is preserved by the Isgur-Wise relations (46) and (47). We choose the forms

$$A_1^{BK^*}(q^2) = A_1^{BK^*}(0) \left[1 - \frac{q^2}{\Lambda_B^2}\right]^{n_1}$$

$$V^{BK^*}(q^2) = V^{BK^*}(0) \left[1 - \frac{q^2}{\Lambda_V^2}\right]^{n_V}$$

and analogous expressions in the $D$ sectors with pole masses $\Lambda_{D1}$ and $\Lambda_{DV}$ related to $\Lambda_1$ and $\Lambda_V$ by a formula similar to Eq.(43):

$$m_c \Lambda_B^2 - m_b \Lambda_D^2 = m_c t_{B}^{*o} = (m_b - m_c)(m_b m_c - m_{K^*}^2)$$

(50)

where $\Lambda_B = \Lambda_1$ or $\Lambda_V$ and $\Lambda_D = \Lambda_{D1}$ or $\Lambda_{DV}$. The algebraic integers $n_1$ and $n_V$ will be restricted to the values $-1, 0, 1$ and 2.

6°) For the two other form factors, $A_2$ and $A_0$ or $a_+$ and $a_-$, the Isgur-Wise relations are similar to the one previously discussed for the $B \to K$ and $D \to K$ form factors, and we have:

$$(a_+ \pm a_-)^{BK^*}(t_B^*) = \left(\frac{m_c}{m_b}\right)^{1\pm1/2} (a_+ \pm a_-)^{DK^*}(t_D^*)$$

(51)

The form factor $A_2^{BK^*}$ is related to both $a_+^{DK^*}$ and $a_-^{DK^*}$, and we obtain, neglecting the QCD correction factor:

$$A_2^{BK^*}(t_B^*) = \frac{1}{2} \sqrt{\frac{m_c}{m_b}} \left(1 + \frac{m_c}{m_b}\right) \left[1 + \frac{m_b - m_c}{m_b + m_c} \lambda^P(t_D^*)\right] A_2^{DK^*}(t_D^*)$$

(52)

where the ratios of the form factors $a_-$ and $a_+$ are defined by

$$\lambda^P(q^2) = - \frac{a_{PK^*}^{DK^*}(q^2)}{a_{+PK^*}(q^2)} \quad P = B, D$$

(53)
Of course, the ratios $\lambda_B(q^2)$ and $\lambda_D(q^2)$ are related by SU(2) flavor symmetry and from Eqs.(51) and (53), we get an equation similar to Eq.(40)

$$\lambda_B(t_B) = \frac{(m_b - m_c) + (m_b + m_c) \lambda_D(t_D^*)}{(m_b + m_c) + (m_b - m_c) \lambda_D(t_D^*)}$$ (54)

Finally, the spin zero axial form factor $A_{BK}^0(q^2)$ can be written with the help of $\lambda_B(q^2)$:

$$A_{BK}^0(q^2) = \frac{m_B + m_K^*}{2 m_K^*} A_{1BK}^*(q^2) - \frac{m_B - m_K^*}{2 m_K^*} \left[1 - \frac{q^2}{m_B^2 - m_K^2} \lambda_B(q^2) \right] A_{2BK}^*(q^2) \lambda_D^2(q^2)$$ (55)

7°) We now make an assumption for $a_+$ and $a_-$ similar to the one previously made for $f_+$ and $f_-$. If $a_{+DK}^*(q^2)$ and $a_{-DK}^*(q^2)$ have the same type of $q^2$ dependence, no matter how it is, then the ratio $\lambda_D$ is constant and using the Isgur-Wise relations (51), the same property extends to the $B$ sector. In particular, $\lambda_B$ is also independent of $q^2$ and related to $\lambda_D$ by Eq.(54).

We use for $A_{2BK}^*(q^2)$ a $q^2$ behaviour of the form:

$$A_{2BK}^*(q^2) = \frac{A_{2BK}^*(0)}{[1 - \frac{q^2}{\Lambda_2^2}]^{n_2}}$$ (56)

and an analogous expression for $A_{1DK}^*(q^2)$ with a pole mass $\Lambda_{D2}$ related to $\Lambda_2$ by Eq.(50) in which $\Lambda_B = \Lambda_2$, $\Lambda_D = \Lambda_{D2}$. The algebraic integer $n_2$ will be restricted to the values $-1, 0, 1$ and 2.

From Eq.(53) with $\lambda_B$ constant, we see that the form factor $A_{0BK}^*(q^2)$ has a somewhat complicated $q^2$ behaviour combining the one of $A_{1BK}^*(q^2)$ given in Eq.(53) and the one of $A_{2BK}^*(q^2)$ given in Eq.(56), the latter being modulated by a linear factor $(1 - \frac{q^2}{m_B^2 - m_K^2} \lambda_B)$. For the particular choice of $\lambda_B$, similar to the one previously made for $\mu_B$ in Eq.(45), i.e:

$$\lambda_B = \frac{m_B^2 - m_K^2}{\Lambda_2^2}$$ (57)

this linear factor cancels one power in $n_2$ of $A_{2BK}^*(q^2)$ in Eq.(55).

In what follows we shall use the relation (53) between $\lambda_B$ and $\Lambda_2$ and the ratio $\lambda_D$ becomes a function of $\Lambda_2$ via Eqs.(54) and (57).

8°) Consider now the ratio of form factors $x^B(q^2)$ defined in Eq.(13). From Eqs.(40) and (52), we get

$$x^B(t_B^*) = \frac{1}{2} \frac{m_c}{m_b} \left(1 + \frac{m_c}{m_b} \right) \left(\frac{m_B + m_K^*}{m_D + m_K^*}\right)^2 \left[1 + \frac{m_b - m_c}{m_b + m_c} \lambda_D^0 \right] x^D(0)$$ (58)
Because of the presence of the $\lambda^D$ term in Eq.(58), the ratio $x^B(t_B^{*o})$ is a function of $\Lambda^2$.

9°) The problem is somewhat more complicated for the third ratio $z^B(t_B^{*o})$ defined in Eq.(71) which mixes the $B \to K$ form factor $F_1^{BK}$ with the $B \to K^*$ form factor $A_1^{BK^*}$. Isgur-Wise SU(2) flavor symmetry relates $F_1^{BK}(t_B^{*o})$ to $F_1^{DK}(0)$ and $A_1^{BK^*}(t_B^{*o})$ to $A_1^{DK^*}(0)$. Because of the $K^* - K$ mass difference, the quantities $t_B^{*o}$ and $t_B^{*o}$ are different (See Eqs.(30) and (31)), and in order to estimate $z^B(t_B^{*o})$, we need to perform an extrapolation and as already discussed in subsections 3 and 4, we use a monopole form with the pole mass $\Lambda_F$ for $F_1^{BK}(q^2)$.

The result is
\[ F_1^{BK}(t_B^{*o}) = \left( \frac{\Lambda_F^2 - t_B^{*o}}{\Lambda_F^2 - t_B^{*o}} \right) F_1^{BK}(t_B^{*o}) \] (59)
and from Eqs.(37) and (46), we finally obtain:
\[ z^B(t_B^{*o}) = \frac{\Lambda_F^2 - t_B^{*o}}{\Lambda_F^2 - t_B^{*o}} \left( \frac{m_B + m_{K^*}}{m_D + m_{K^*}} \right) \left( 1 + \frac{m_b - m_c}{m_D + m_{K^*}} \mu^D \right) \frac{z^D(0)}{m_B^{*o}} \] (60)

IV. The decay modes $B \to K(K^*) + \Psi'$

1°) In the previous section we have computed the ratios of form factors in the $B$ sector, $x^B(q^2)$, $y^B(q^2)$ and $z^B(q^2)$ at $q^2 = t_B^{*o}$ in terms of their counterparts in the $D$ sector at $q^2 = 0$, $x^D(0)$, $y^D(0)$ and $z^D(0)$. The key observation, in this section, is that the value of $t_B^{*o}$ computed in Eq.(31) using $m_b = 4.7$ GeV and $m_c = 1.45$ GeV is remarkably close \footnote{Other reasonable choices \cite{5} of $m_b$ and $m_c$ (constraint to $m_b - m_c = 3.4 \pm 0.2$ GeV in the HQET scheme) also lead to similar result : $t_B^{*o}$ is close to $m_{\Psi'}^2$.} to $m_{\Psi'}^2$, such as $\frac{t_B^{*o}}{m_{\Psi'}} = 0.9931$. It is then justified to neglect the variation of the form factors in the $B$ sector between $t_B^{*o}$ and $m_{\Psi'}^2$. Using the Isgur-Wise relation (48), we obtain
\[ y^B(m_{\Psi'}^2) = 1.54 \ y^D(0) \] (61)
and using the PDG values \cite{2}, $y^D(0) = 1.89 \pm 0.25$, we get
\[ y' = y^B(m_{\Psi'}^2) = 2.91 \pm 0.39 \] (62)

The knowledge of $y' = y^B(m_{\Psi'}^2)$ determines the left-right asymmetry $A_{LR}'$ previously defined in Eq.(18). The result
\[ A_{LR}' = 0.99 \pm 0.01 \] (63)
shows that the dominant transverse amplitude has the helicity $\lambda = -1$, and in the one standard deviation limit, we predict $\mathcal{A}'_{LR} > 0.98$.

As a second consequence of the knowledge of $y'$, we can derive an upper bound for the fractional longitudinal polarization $\rho'_L$ given in Eq.(17)

$$\rho'_L \leq \frac{a'}{a'^2 + 2[1 + c'^2 y'^2]}$$ (64)

Using the lower one standard deviation for $y'$, we obtain

$$\rho'_L \leq 0.566$$ (65)

This upper bound for $\rho'_L$ is significantly smaller than the theoretical upper bound (due only to factorization) $\rho'_L \leq 0.678$ in Eq.(25). Since $m^2_{\Psi'} \simeq t^e_B$, we observe that the results (53) and (55) are scenario independent and these predictions are direct consequences of the Isgur-Wise relation (48).

2°) Not only the bound Eq.(63), but the quantity $\rho'_L$ itself can be computed from $x' = x^B(m^2_{\Psi'})$ and $y' = y^B(m^2_{\Psi'})$. From Eq.(58), we obtain

$$x^B(m^2_{\Psi'}) = 1.0081 \left[1 + 0.5285 \lambda^D\right] x^D(0)$$ (66)

and using the PDG value $[4]$, $x^D(0) = 0.73 \pm 0.15$, we get

$$x^B(m^2_{\Psi'}) = (0.74 \pm 0.15) \left[1 + 0.5285 \lambda^D\right]$$ (67)

The fractional polarization $\rho'_L$ depends on $\Lambda^2$ via the parameter $\lambda^D$ (See Eqs.(54) and (57)) and it turns out to be a slowly increasing function of $\Lambda^2$. Restricting $\Lambda^2$ to the range $(5 - 6)$ GeV, we have

$$\rho'_L = 0.34 \pm 0.05 \quad \text{for} \quad \Lambda^2 = 5 \text{ GeV}$$

(68)

$$\rho'_L = 0.40 \pm 0.04 \quad \text{for} \quad \Lambda^2 = 6 \text{ GeV}$$

The error $\Delta \rho'_L$ for $\rho'_L$, combines, in quadrature, those of $x'$ and $y'$ due to the quoted errors in Table 3.

Eq.(68) is our prediction for $\rho'_L$, and let us remark that this quantity is easy to measure.
Estimates of $\rho_L^\prime$ have been previously obtained by Kamal and Santra[9]. However their method is different from ours. They consider seven scenarios relating $J/\Psi$ to $\Psi^\prime$ modes and the allowed domains in the $x^\prime, y^\prime$ plane for each scenario are determined by the experimental constraint on $\rho_L^\prime$. The upper bound for $\rho_L^\prime$ found in [9] is the theoretical upper bound of Eq.(25) and the lower bound is slightly scenario-dependent and it varies from 0.48 to 0.55 which is larger than our predictions in Eq.(68).

3°) The ratio $R_{\Psi^\prime}$ defined in Eq.(19) can be computed using $x^\prime = x^B(m_{\Psi^\prime}^2)$, $y^\prime = y^B(m_{\Psi^\prime}^2)$ and $z^\prime = z^B(m_{\Psi^\prime}^2)$.

From the Isgur-Wise relation (60), we obtain

$$z^B(m_{\Psi^\prime}^2) = 1.4621 \left[ \frac{\Lambda_F^2 - t_B^o}{\Lambda_F^2 - t_B^{*o}} \right] [1 + 0.5285 \, \mu^D] \, z^D(0)$$

(69)

and using the PDG value [3], $z^D(0) = 1.34 \pm 0.11$, we have

$$z^B(m_{\Psi^\prime}^2) = (1.96 \pm 0.16) \left[ \frac{\Lambda_F^2 - t_B^o}{\Lambda_F^2 - t_B^{*o}} \right] [1 + 0.5285 \, \mu^D]$$

(70)

The ratio $R_{\Psi^\prime}$ depends on both parameters $\Lambda_2$ (via $x^\prime$) and $\Lambda_F$ (via $z^\prime$). Restricting both pole masses $\Lambda_2$ and $\Lambda_F$ in a range (5 - 6) GeV, we find two extreme values for $R_{\Psi^\prime}$

$$R_{\Psi^\prime} = 1.44 \pm 0.28 \quad \text{for} \quad \Lambda_F = \Lambda_2 = 5 \, \text{GeV}$$

(71)

$$R_{\Psi^\prime} = 2.92 \pm 0.54 \quad \text{for} \quad \Lambda_F = \Lambda_2 = 6 \, \text{GeV}$$

The experimental estimate in Table 2 is 2.03±1.59 and due to the large experimental error, our results in Eq.(71) are compatible with experiment for all values of $\Lambda_2$ and $\Lambda_F$ in the (5−6) GeV range.

V. The decay modes $B \to K(K^*) + J/\Psi$

1°) In order to compute the longitudinal polarization $\rho_L$ defined in Eq.(15) for the $B \to K^* + J/\Psi$ decay mode and the ratio of the $K^* + J/\Psi$ over $K + J/\Psi$ rates, $R_{J/\Psi}$, defined in Eq.(19), we must extrapolate the ratios of hadronic form factors, $x^B(q^2)$, $y^B(q^2)$ and $z^B(q^2)$ from the value $q^2 = t_B^{*o}$ (where these quantities are given by the Isgur-Wise relations (48), (58) and (60)) to the value $q^2 = m_{J/\Psi}^2$. 
Of course such an extrapolation is scenario dependent. We use the pole type \( q^2 \) dependence as given in Eqs.(42), (49) and (56), and we introduce the function \( r(\Lambda) \) defined by

\[
r(\Lambda) = \frac{\Lambda^2 - t^{*o}_{B}}{\Lambda^2 - m^{2}_{J/\psi}} \tag{72}
\]

and we obtain

\[
x \equiv x^{B}(m^{2}_{J/\psi}) = \frac{[r(\Lambda_{2})]^{n_{2}}}{[r(\Lambda_{1})]^{n_{1}}} x^{B}(t^{*o}_{B}) \tag{73}
\]

\[
y \equiv y^{B}(m^{2}_{J/\psi}) = \frac{[r(\Lambda_{V})]^{n_{V}}}{[r(\Lambda_{1})]^{n_{1}}} y^{B}(t^{*o}_{B}) \tag{74}
\]

\[
z \equiv z^{B}(m^{2}_{J/\psi}) = \frac{[r(\Lambda_{F})]^{n_{F}}}{[r(\Lambda_{1})]^{n_{1}}} z^{B}(t^{*o}_{B}) \tag{75}
\]

In our model \( n_{F} = 1 \), and for the three other powers \( n_{1}, n_{2}, n_{V} \), each one can take four algebraic integers \(-1, 0, 1, 2\). On physical grounds, we impose to the pole masses \( \Lambda_{1}, \Lambda_{2}, \Lambda_{V}, \Lambda_{F} \) to be inside the \((5 - 6)\) GeV range where the \( b \)s bound states masses are expected to be.

\begin{enumerate}
\item[2^o)] We first study the scenario constraints due to \( \rho_{L} \) and for the \( 4^{3} = 64 \) possible triplets \([n_{1}, n_{2}, n_{V}]\), we have computed \( \rho_{L} \) using the values of \( x \) and \( y \) as given from \( x' \) and \( y' \) by Eqs.(32), (67), (73) and (74). We impose the experimental constraint in the form \( \rho_{L} + \Delta \rho_{L} \geq 0.70 \) where the theoretical error \( \Delta \rho_{L} \) is computed in quadrature from those of \( x^{D}(0) \) and \( y^{D}(0) \).

After a long numerical scanning, our results can be summarized as follows:

\begin{enumerate}
\item[i)] no solution is obtained when \( n_{1} = 0, 1, 2 \) for all the 16 values of the couple \([n_{2}, n_{V}]\).
\item[ii)] solutions exist only when \( n_{1} = -1 \), i.e. when the hadronic form factor \( A_{l}^{BP^*}(q^{2}) \) exhibits a linear decrease with \( q^{2} \). Of course, in this case, \( \Lambda_{1} \) is no more a pole mass but only a slope coefficient and it is reasonable now to relax the constraint \( \Lambda_{1} \leq 6 \) GeV and to use only \( \Lambda_{1} \geq 5 \) GeV in order to exclude a too fast variation with \( q^{2} \) of \( A_{l}^{BP^*}(q^{2}) \).
\item[iii)] The solutions obtained correspond to only four triplets \([n_{1}, n_{2}, n_{V}]\) :
\[
[-1, 2, 2], \ [-1, 1, 2], \ [-1, 0, 2], \ [-1, 2, 1]
\]

and in the four cases, the maximal value of \( \rho_{L} \) occurs at \( \Lambda_{1} = 5 \) GeV, \( \Lambda_{2} = 6 \) GeV, \( \Lambda_{V} = 5 \) GeV and in the most favorable situation of two dipole \( q^{2} \) dependence for \( A_{2} \) and \( V \), we obtain \( \rho_{L} = 0.7162 \pm 0.0236 \). Therefore \( \rho_{L} = 0.74 \) is the maximal value within one standard deviation we can produce in our approach, considering only the quantity \( \rho_{L} \).
3°) We now consider the ratio of rates \( R_{J/\psi} \) and we impose the experimental constraint in the form \( R_{J/\psi} - \Delta R_{J/\psi} \leq 2.2 \) where the theoretical error \( \Delta R_{J/\psi} \) is computed in quadrature from the experimental errors on \( x^D(0), y^D(0) \) and \( z^D(0) \).

On the one hand, the constraint \( \rho_L + \Delta \rho_L \geq 0.7 \) has selected four scenarios previously discussed and at fixed \( \Lambda_2, \Lambda_V \) the allowed domain for \( \Lambda_1 \) is defined by an upper limit for \( \Lambda_1 : \)

\[
\Lambda_1 \leq \Lambda_{1, \text{MAX}}(\Lambda_2, \Lambda_V) \tag{76}
\]

On the other hand, the constraint \( R_{J/\psi} - \Delta R_{J/\psi} \leq 2.2 \) implies a lower limit for \( \Lambda_1 : \)

\[
\Lambda_1 \geq \Lambda_{1, \text{MIN}}(\Lambda_2, \Lambda_V, \Lambda_F) \tag{77}
\]

Acceptable values of \( \Lambda_1 \) exist when and only when the lower limit (77) is smaller than the upper limit (76).

The quantity \( \Lambda_{1, \text{MIN}} \) (at fixed \( \Lambda_2, \Lambda_V \)) is an increasing function of \( \Lambda_F \) and using the constraint \( \Lambda_F \geq 5 \text{ GeV} \), the physical domain for \( \Lambda_1 \), at fixed \( \Lambda_2, \Lambda_V \), is defined by

\[
\Lambda_{1, \text{MIN}}(\Lambda_2, \Lambda_V, \Lambda_F = 5 \text{ GeV}) \leq \Lambda_1 \leq \Lambda_{1, \text{MAX}}(\Lambda_2, \Lambda_V) \tag{78}
\]

We find out that for the scenario \([n_1, n_2, n_V] = [-1, 2, 1] \), the inequality (78) has no solution. For the three remaining scenarios \([n_1, n_2, n_V] = [-1, 2, 2], [-1, 1, 2] \) and \([-1, 0, 2] \), the physical domains in the \( \Lambda_1, \Lambda_2, \Lambda_V \) space are represented on Figures 1, 2 and 3.

At fixed \( \Lambda_2, \Lambda_V \), we also have

\[
5 \text{ GeV} \leq \Lambda_F(\Lambda_2, \Lambda_V) \leq \Lambda_{F, \text{MAX}}(\Lambda_2, \Lambda_V) \tag{79}
\]

where \( \Lambda_{F, \text{MAX}}(\Lambda_2, \Lambda_V) \) is determined by

\[
\Lambda_{1, \text{MIN}}[\Lambda_2, \Lambda_V, \Lambda_{F, \text{MAX}}(\Lambda_2, \Lambda_V)] = \Lambda_{1, \text{MAX}}(\Lambda_2, \Lambda_V) \tag{80}
\]

The quantity \( \Lambda_{F, \text{MAX}}(\Lambda_2, \Lambda_V) \) in the \( \Lambda_2, \Lambda_V, \Lambda_F \) space has been represented on Figures 4, 5 and 6 for the three surviving scenarios respectively.

4°) Starting with 64 scenarios for the \( q^2 \) dependence of the hadronic form factors \( A_1^{BK^*}, A_2^{BK^*} \) and \( V^{BK^*} \), we have obtained three surviving scenarios, \( n_1 = -1, n_V = 2 \) and \( n_2 = 2, 1, 0 \) for which it is possible to satisfy simultaneously both experimental constraints : \( \rho_L + \Delta \rho_L \geq 0.70 \)
which produces an upper bound on $\Lambda_1$, and $R_{J/\Psi} - \Delta R_{J/\Psi} \leq 2.2$ which implies a lower bound on $\Lambda_1$.

In order to have a more precise feeling concerning the nature of the fit obtained with our model, we compute $\rho_L$ and $R_{J/\Psi}$ for values of $\Lambda_1, \Lambda_2, \Lambda_V$ and $\Lambda_F$ belonging to the physical domains of every scenario represented in Figures 1 – 6. For illustration, we take some values of $\Lambda_j : \Lambda_2 = 6$ GeV, $\Lambda_V = \Lambda_F = 5$ GeV, and for $\Lambda_1$, the two extreme values, $\Lambda_{1,MAX}$ and $\Lambda_{1,MIN}$. Of course for $\Lambda_1 = \Lambda_{1,MAX}$, we get $\rho_L + \Delta \rho_L = 0.70$ and for $\Lambda_1 = \Lambda_{1,MIN}$, we have $R_{J/\Psi} - \Delta R_{J/\Psi} = 2.2$. The results are shown on Table 4 where the numerical values of $\Lambda_{1,MAX}$ and $\Lambda_{1,MIN}$ are given.

A first observation coming from Table 4 is to realize how difficult it is to fit simultaneously the large experimental value of $\rho_L$ and the relatively small experimental value of $R_{J/\Psi}$. The opposite trend between $\rho_L$ and $R_{J/\Psi}$ making the fit so difficult has been also noticed [2]. The theoretical relative error coming from $R_{J/\Psi}$ is larger than the one coming from $\rho_L$, and this feature is welcome for obtaining a two-fold fit. It is essentially due to the fact that $R_{J/\Psi}$, in addition to the errors on $x^D(0)$ and $y^D(0)$ (as for $\rho_L$), has a third source of uncertainty due to the errors on $z^D(0)$. While the theoretical relative error on $\rho_L$ is only between 4% and 6%, the one on $R_{J/\Psi}$ is between 18% and 24%.

A second observation, coming from both Figure 1 and Table 4, is that the scenario with a dipole form factor $A_2, n_2 = 2$, is the one with the largest domain in the $\Lambda_1, \Lambda_2, \Lambda_V$ and $\Lambda_F$ space. Therefore in this scenario it is relatively easy to accommodate both $\rho_L$ and $R_{J/\Psi}$. From Table 4, the largest possible value of $\rho_L$ we can obtain, in the one standard deviation limit, is $\rho_L + \Delta \rho_L = 0.722$ and the smallest possible value of $R_{J/\Psi}$, in the one standard deviation limit, is $R_{J/\Psi} - \Delta R_{J/\Psi} = 1.581$. These extreme values of $\rho_L$ and $R_{J/\Psi}$ in our model do not occur simultaneously but at the two different extreme values of $\Lambda_1$.

5°) The left-right asymmetry $A_{LR}$ defined in Eq.(18) has not been experimentally measured. It can be easily computed in our model. Depending only on the ratio $y = y^B(m^2_{J/\Psi})$, we use Eq.(74) with $n_1 = -1$ and $n_V = 2$ and we vary the parameters $\Lambda_1$ and $\Lambda_V$ inside the allowed domains represented on Figures 1, 2, and 3 for the three scenarios $n_2 = 2, 1, 0$.

The results are:

\[(i) \quad n_2 = 2 \quad 0.867 < A_{LR} < 0.945 \quad (81)\]
The left-right asymmetry in the decay mode $B \rightarrow K^* + J/\Psi$ is large in the three selected scenarios, not as large as that of the decay mode $B \rightarrow K^* + \Psi'$ where it has been predicted to be close to unity (Eq.(63)). We observe that the differences in the predictions of the three scenarios are moderate.

**VI. Comparison of $J/\Psi$ and $\Psi'$ production**

1) The ratios of decay widths $S$ and $S^*$ defined in Eq.(20) involve the same strange meson, $K$ or $K^*$, and two different charmonium states $\Psi'$ and $J/\Psi$, hence two different leptonic decay constants $f_{\Psi'}$ and $f_{J/\Psi}$ are involved. Using $f_{J/\Psi} = (384 \pm 14) \text{MeV}$ and $f_{\Psi'} = (282 \pm 14) \text{MeV}$ as estimated from the decays $J/\Psi \rightarrow e^+e^-$ and $\Psi' \rightarrow e^+e^-$, we obtain:

$$\left( \frac{f_{\Psi'}}{f_{J/\Psi}} \right)^2 = 0.539 \pm 0.066$$

and the quantities $S$ and $S^*$ are written from Eqs.(23) and (24) in the form:

$$S = \left| \frac{F_{BK}^B(m_{\Psi'}^2)}{F_{BK}^B(m_{J/\Psi}^2)} \right|^2$$

$$S = [0.2392 \pm 0.0292] \left| \frac{F_{BK}^B(m_{\Psi'}^2)}{F_{BK}^B(m_{J/\Psi}^2)} \right|^2$$

The left-right asymmetry in the decay mode $B \rightarrow K^* + J/\Psi$ is large in the three selected scenarios, not as large as that of the decay mode $B \rightarrow K^* + \Psi'$ where it has been predicted to be close to unity (Eq.(63)). We observe that the differences in the predictions of the three scenarios are moderate.

**VI. Comparison of $J/\Psi$ and $\Psi'$ production**

$\Lambda_1(GeV)$ | $\rho_L$ | $R_{J/\Psi}$
--- | --- | ---
$n_2 = 2$
$\Lambda_{1,MAX} = 8.112$ | $0.665 \pm 0.035$ | $2.089 \pm 0.508$
$\Lambda_{1,MIN} = 5.426$ | $0.694 \pm 0.028$ | $2.694 \pm 0.494$

$n_2 = 1$
$\Lambda_{1,MAX} = 6.113$ | $0.663 \pm 0.037$ | $2.324 \pm 0.524$
$\Lambda_{1,MIN} = 5.426$ | $0.681 \pm 0.033$ | $2.714 \pm 0.514$

$n_2 = 0$
$\Lambda_{1,MAX} = 5.292$ | $0.660 \pm 0.040$ | $2.625 \pm 0.542$
$\Lambda_{1,MIN} = 5.183$ | $0.665 \pm 0.038$ | $2.739 \pm 0.539$

Table 4

(ii) $n_2 = 1$

$0.837 < A_{LR} < 0.910$ (82)

(iii) $n_2 = 0$

$0.837 < A_{LR} < 0.856$ (83)
\[ S^* = [0.5421 \pm 0.0664] \left| \frac{A_{BK}^{B^*}(m_{\Psi'})}{A_{BK}^{B^*}(m_{J/\Psi}')} \right|^2 \frac{(a' - b')^2 + 2(1 + c'^2 y'^2)}{(a - bx)^2 + 2(1 + c^2 y^2)} \]  

(86)

2°) In our model the hadronic form factor \( F_{BK}^1(q^2) \) has a monopole \( q^2 \) dependence with a pole mass \( \Lambda_F \) and we simply have:

\[ \frac{F_{BK}^1(m_{\Psi}^2)}{F_{BK}^1(m_{J/\Psi}^2)} = \frac{\Lambda_F^2 - m_{\Psi}^2}{\Lambda_F^2 - m_{J/\Psi}^2} \]  

(87)

This ratio of form factor is a decreasing function of \( \Lambda_F^2 \) and so is the ratio \( S \). At \( \Lambda_F = 5 \text{ GeV} \), we obtain:

\[ S(\Lambda_F = 5 \text{ GeV}) = 0.4363 \pm 0.0537 \]  

(88)

This prediction is in agreement, within one standard deviation, with the experimental value estimated in Table 2, \( S_{exp} = 0.68 \pm 0.32 \). Such an agreement continues to occur for larger values of \( \Lambda_F \) up to 6.27 \text{ GeV}.

The range of \( \Lambda_F \) depends on the three scenarios corresponding to \( n_2 = 2, 1, 0 \) and they are deduced from Figures 4, 5, 6 respectively. We get:

(i) \( n_2 = 2 : \quad 0.3505 \pm 0.0432 \leq S \leq 0.4363 \pm 0.0537 \)  

(89)

(ii) \( n_2 = 1 : \quad 0.3790 \pm 0.0467 \leq S \leq 0.4363 \pm 0.0537 \)  

(90)

(iii) \( n_2 = 0 : \quad 0.4181 \pm 0.0515 \leq S \leq 0.4363 \pm 0.0537 \)  

(91)

The errors quoted in Eq.(89), (90) and (91) are due to the uncertainty on the leptonic decay constants \( f_{\Psi'} \) and \( f_{J/\Psi} \). In conclusion, the theoretical predictions of our model for the three scenarios agree, within one standard deviation, with experimental results.

3°) The analysis of the second ratio \( S^* \) is more complex because of a large number of hadronic form factors involved. In our model the form factor \( A_{BK}^1(q^2) \) has a decreasing linear \( q^2 \) dependence with a pole mass \( \Lambda_1 \), and we simply have

\[ \frac{A_{BK}^{B^*}(m_{\Psi'})}{A_{BK}^{B^*}(m_{J/\Psi}')} = \frac{\Lambda_1^2 - m_{\Psi'}^2}{\Lambda_1^2 - m_{J/\Psi}^2} \]  

(92)

We have computed the ratio \( S^* \) for the three scenarios \( n_2 = 2, 1, 0 \) using the allowed domains represented respectively on Figures 1, 2 and 3 for \( \Lambda_1, \Lambda_2 \) and \( \Lambda_V \).

The results of this scanning are:

(i) \( n_2 = 2 : \quad 0.3287 \pm 0.0028 \leq S^* \leq 0.4135 \pm 0.0038 \)  

(93)
(ii) \( n_2 = 1 : \quad 0.3489 \pm 0.0034 \leq S^* \leq 0.4015 \pm 0.0039 \) (94)

(iii) \( n_2 = 0 : \quad 0.3763 \pm 0.0039 \leq S^* \leq 0.3867 \pm 0.0040 \) (95)

The errors quoted in Eqs. (93), (94) and (95) are computed in quadrature from those on the ratios \( f_{\Psi'}/f_{J/\Psi}, x^D(0) \) and \( y^D(0) \). The theoretical predictions of our model for the three scenarios agree, within one standard deviation, with the experimental results estimated in Table 2 : \( S^*_{\text{exp}} = 0.89 \pm 0.59 \).

4°) Kamal and Santra have studied the ratios \( S \) and \( S^* \) denoted by them respectively as \( 1/R \) and \( 1/R' \). In the case of \( R \), both monopole and dipole \( q^2 \) dependences for \( F_1^{BK} \) are considered with a pole mass \( \Lambda_F = 5.43 \text{ GeV} \). Their conclusion is that a dipole behaviour for \( F_1^{BK} \) is needed in order to obtain for \( R \) agreement between theory and experiment in the one standard deviation limit.

The apparent contradiction between our result (monopole for \( F_1^{BK} \)) and the one of Ref. [9] is essentially due to the large experimental error of 47\% for the quantity \( S \) or \( R \). With \( \delta = 0.47 \) the relation at first order in \( \delta \), \((1 \pm \delta)^{-1} = 1 \mp \delta \) is not valid and one standard deviation limit for \( S \) and one standard deviation limit for \( R \) are different concepts. However, since the main part of the experimental error is due to the \( K + \Psi' \) mode and for that reason the consideration of one standard deviation for \( S \) (where \( K + \Psi' \) enters in the numerator) seems to be more relevant than for \( R \).

A similar situation occurs for \( S^* \) and \( R' \). Here the experimental error is even larger, 66.7\%, and it is mainly due to the \( K + \Psi' \) mode which enters in the numerator of \( S^* \). Again the one standard deviation limit for \( S^* \) and the one standard deviation limit for \( R' \) are different quantities.

Also the pole masses in Ref. [9] are taken only at some fixed values, while in our approach these poles sweep inside the \((5 - 6) \text{ GeV} \) range.

Furthermore, considering only the one standard deviation limit for \( R' \), they exclude four scenarios where \( A_1^{BK*} \) is either constant or linearly decreasing with \( q^2 \) and conclude that if factorization assumption were to hold, then the only scenarios that are consistent with experiment are those in which \( A_1^{BK*} \) rises with \( q^2 \). We observe however that \( R' \) (or \( S^* \)) is not an independent ratio but related to the other ratios by \( S^* R_{J/\Psi} = S R_{\Psi'} \), such that considering \( R' \) (or \( S^* \)) alone might be inadequate.
VII. The decay modes $B \to K(K^*) + \eta_c$

1°) The decay modes $B \to K + \eta_c$ and $B \to K^* + \eta_c$ have not been experimentally observed. However their rates can be easily computed and the relevant expressions have been given in Eqs. (3) and (8). The form factors involved in these modes correspond to the spin zero part of the weak current, $F^B_{K0}(m_{\eta_c}^2)$ and $A^B_{K^0}(m_{\eta_c}^2)$.

2°) The hadronic form factor $F^B_{0K}(t_B)$ can be computed using the Isgur-Wise relations (17) together with Eqs. (39) and (40). The result is

$$F^B_{0K}(t_B) = \frac{m_B + m_c}{2 \sqrt{m_B m_c}} \left( \left[ 1 - \frac{m_B - m_c}{m_B + m_c} \frac{t_B^0}{m_B^2 - m_K^2} \right] - \left[ \frac{t_B^0}{m_B^2 - m_K^2} - \frac{m_B - m_c}{m_B + m_c} \right] \mu^D \right)$$

(96)

Numerically we obtain

$$F^B_{0K}(t_B) = 0.8460 \left[ 1 - 0.00667 \mu^D \right] F^D_{1K}(0)$$

(97)

We notice that the coefficient of $\mu^D$ is very small in the bracket of Eq. (97) and as a consequence, $F^B_{0K}(t_B)$ depends only weakly on $\mu^D$.

In our model $F^B_{0K}$ is constant and therefore Eq. (97) gives its value for all $q^2$. The weak dependence on $\mu^D$ implies a weak dependence of $F^B_{0K}$ on the pole mass $\Lambda_F$.

3°) The hadronic form factor $A^B_{K^*}$ is deduced from $A^B_{1K^*}$ and $A^B_{2K^*}$ by using Eq. (53) with $\lambda^B$ fixed by the relation (57). For $q^2 = m_{\eta_c}^2$, we have

$$A^B_{K^*}(m_{\eta_c}^2) = \frac{m_B + m_{K^*}}{2 m_{K^*}} A^B_{1K^*}(m_{\eta_c}^2) - \frac{m_B - m_{K^*}}{2 m_{K^*}} \left[ 1 - \frac{m_{\eta_c}^2}{\Lambda_2^2} \right] A^B_{2K^*}(m_{\eta_c}^2)$$

(98)

In order to obtain $A^B_{1K^*}(m_{\eta_c}^2)$ and $A^B_{2K^*}(m_{\eta_c}^2)$, we extrapolate these form factors $A^B_{1K^*}$ and $A^B_{2K^*}$ from the value $q^2 = t_B^0$ — where these terms are given by the Isgur-Wise relations (18) and (52) — to $q^2 = m_{\eta_c}^2$. The form factor $A^B_{0K^*}(m_{\eta_c}^2)$ is scenario dependent, firstly on $n_2$, secondly on $\Lambda_1$ and $\Lambda_2$ restricted to the allowed domains of Figures 1, 2 and 3.

4°) To bypass the unknown decay constant $f_{\eta_c}$, we consider the ratio of rates $R_{\eta_c}$ defined by

$$R_{\eta_c} = \frac{\Gamma(B \to K^* + \eta_c)}{\Gamma(B \to K + \eta_c)}$$

(99)

This quantity is given from Eqs. (5) and (8) by:

$$R_{\eta_c} = 0.3732 \left| \frac{A^B_{0K^*}(m_{\eta_c}^2)}{F^B_{0K}(m_{\eta_c}^2)} \right|^2$$

(100)
Using the form factors $F^{BK}_0$ and $A^{BK}_0$ given in Eqs. (97) and (98), we compute the ratio $R_{\eta_c}$ by varying the parameters $\Lambda_1$ and $\Lambda_2$ inside the allowed domains discussed in section V. For the scenario $n_2 = 2$, the bounds on $R_{\eta_c}$ are

$$1.02 \leq R_{\eta_c} \leq 2.57 \quad (101)$$

For the other scenarios $n_2 = 1$ and $n_2 = 0$, the bounds on $R_{\eta_c}$ are contained inside the inequality Eq.(101). It turns out that the ratio $R_{\eta_c}$ being only weakly scenario dependent, hence the bounds Eq.(101) remain valid for all cases.

5°) The comparison of the $K(K^*) + \eta_c$ and $K(K^*) + J/\Psi$ decay modes depends on the ratio of the decay constants $f_{\eta_c}$ and $f_{J/\Psi}$. Unfortunately $f_{\eta_c}$ is not experimentally known and we use theoretical estimates if we want to make predictions. Using quark model considerations [17] we take

$$\frac{f_{\eta_c}}{f_{J/\Psi}} = 0.99 \quad (102)$$

Consider first the ratio $T$ defined by :

$$T = \frac{\Gamma(B \rightarrow K + \eta_c)}{\Gamma(B \rightarrow K + J/\Psi)} \quad (103)$$

Using Eqs.(5) and (6), we obtain

$$T = 2.52 \left( \frac{f_{\eta_c}}{f_{J/\Psi}} \right)^2 \left| \frac{F^{BK}_0(m_{\eta_c}^2)}{F^{BK}_1(m_{J/\Psi}^2)} \right|^2 \quad (104)$$

In our model $F^{BK}_0$ is constant and $F^{BK}_1$ has a monopole $q^2$ dependence with the pole mass $\Lambda_F$.

As a consequence we simply have

$$\frac{F^{BK}_0(m_{\eta_c}^2)}{F^{BK}_1(m_{J/\Psi}^2)} = 1 - \frac{m_{J/\Psi}^2}{\Lambda_F^2} \quad (105)$$

Using the estimate Eq.(102), we obtain the following bounds of $T$ for the scenario $n_2 = 2$ ($5 \text{ GeV} \leq \Lambda_F \leq 5.71 \text{ GeV}$)

$$0.94 \leq T \leq 1.24 \quad (106)$$

For the scenarios $n_2 = 1$ and $n_2 = 0$, the bounds of $T$ satisfy the double inequality Eq.(106). Conversely a measurement of the ratio $T$ may provide an opportunity to extract, from experiment, the scalar decay constant $f_{\eta_c}$. 
Finally we introduce a third ratio $T^*$ defined by

$$
T^* = \frac{\Gamma(B \to K^* + \eta_c)}{\Gamma(B \to K^* + J/\Psi)}
$$

(107)

Using Eqs. (8) and (9), we get

$$
T^* = 0.87 \left( \frac{f_{\eta_c}}{f_{J/\Psi}} \right)^2 \frac{A_{0}^{BK^*}(m_{\eta_c}^2)}{A_{1}^{BK^*}(m_{J/\Psi}^2)} \frac{1}{(a - bx)^2 + 2(1 + c^2 y^2)}
$$

(108)

The ratio $T^*$ depends on the three parameters $\Lambda_1$, $\Lambda_2$ and $\Lambda_V$. Varying these quantities inside the allowed domains discussed in section V, we can obtain bounds for $T^*$. The result is moderately scenario-dependent and using the estimate Eq.(102), we obtain

$$
0.45 \leq T^* \leq 0.86
$$

(109)

6°) The ratios $R_{\eta_c}$, $T$ and $T^*$ have been discussed by us in a recent paper [18] in order to propose a test of factorization. However in our previous calculations [18], the ranges of values for the scenario dependent factors (denoted there as $S_V$ and $S_A$) have been underestimated and our previous predictions [18] for the ratios $R_{\eta_c}$, $T$ and $T^*$ are different from those obtained here. For details see Ref.[10].

**VIII. $D \to K(K^*)$ hadronic form factors and semi-leptonic decays**

1°) The $B \to K(K^*)$ and $D \to K(K^*)$ hadronic form factors are related by the SU(2) heavy flavor symmetry of Isgur-Wise. From the considerations of section III, it is clear that the $q^2$ dependence for the form factors $F_1, A_1, A_2$ and $V$ is the same in the $B$ and $D$ sectors: same values for $n_F, n_1, n_2$ and $n_V$, pole masses are related by Eqs.(13) and (50).

These new features of $q^2$ dependences in the $B$ sector obtained so far, can now be used backwards for analysing the semi-leptonic decays $D \to K + \ell^+ + \nu_\ell$ and $D \to K^* + \ell^+ + \nu_\ell$. Using the dimensionless variable $t = q^2/m_D^2$, we introduce the normalized $q^2$ distribution $X(t)$

$$
X(t) = \frac{1}{\Gamma_{sl}} \frac{d}{dt} \Gamma_{sl}
$$

(110)

$X(t)$ is independent, in particular, on the Cabibbo-Kobayashi-Maskawa parameters and on the normalizations $F_1^{DK}(0)$ and $A_1^{DK^*}(0)$. 

25
We recall that the quantities \( x^D(0) \), \( y^D(0) \) and \( z^D(0) \) (used in this paper for normalizing the \( B \) sector) have been extracted from experimental data on semi-leptonic decay in the \( D \) sector in a scenario-dependent way, because the variation with \( q^2 \) of the form factors \( F_{DK}^1 \), \( A_{DK}^1 \), \( A_{DK}^2 \) and \( V_{DK}^* \) has not been measured.

2\(^o\) The \( q^2 \) distribution for the semi-leptonic decay \( D \to \overline{K} + \ell^+ + \nu_\ell \) depends on the hadronic form factor \( F_{DK}^1(q^2) \). Defining the dimensionless parameters

\[
r = \frac{m_K}{m_D} \quad \alpha_F = \frac{m_D^2}{\Lambda_{DF}^2}
\]

where \( \Lambda_{DF} \) is the pole mass in the \( D \) sector related to \( \Lambda_F \) in the \( B \) sector by equation (43), we obtain for \( X(t) \) the expression

\[
X(t) = \frac{1}{I(\alpha_F)} \frac{[(1 + r^2 - t)^2 - 4r^2]^{3/2}}{(1 - \alpha_F t)^2}
\]

where we have used, for \( F_{DK}^1(q^2) \), a monopole \( q^2 \) dependence, \( n_F = 1 \). The integral \( I(\alpha_F) \) is defined by the normalization condition \( X(t) : \)

\[
I(\alpha_F) = \int_0^{(1-r)^2} \frac{[(1 + r^2 - x)^2 - 4r^2]^{3/2}}{(1 - \alpha_F x)^2} \, dx
\]

\( I(\alpha_F) \) can be computed analytically [10] or numerically.

The distribution \( X(t) \) for the semi-leptonic decay \( D \to \overline{K} + \ell^+ + \nu_\ell \) is represented in Figure 7 for values of \( \alpha_F \) corresponding to the bounds on \( \Lambda_F \) obtained in section V for the three scenarios \( n_2 = 2, 1, 0 \) and represented on Figures 4, 5 and 6. The distribution \( X(t) \) is a monotonically decreasing function of \( t \). Its shape is not very sensitive to \( \Lambda_F \) excepted in the neighbourhood of \( t = 0 \) (\( q^2 = 0 \)).

An estimate of the slope of the \( q^2 \) distribution in \( D \to \overline{K} + \ell^+ + \nu_\ell \) at \( q^2 = 0 \) has been given by Witherell [16] using two models for the \( q^2 \) dependence of \( F_{DK}^1(q^2) \). His pole masses \( \Lambda_{DF} \) in the \( D \) sector correspond in our language to \( \Lambda_F \) of the \( B \) sector :

\[
5.02 \, GeV \leq \Lambda_F \leq 5.36 \, GeV
\]

Our three scenarios are in agreement with his range Eq.(114).

3\(^o\) The \( q^2 \) distribution in the semi-leptonic decay \( D \to \overline{K}^* + \ell^+ + \nu_\ell \) depends on the three hadronic form factors \( A_{DK}^1(q^2) \), \( A_{DK}^2(q^2) \) and \( V_{DK}^*(q^2) \). Of course the final vector meson \( K^* \) may have three possible polarization states, \( \lambda = 0, \pm 1 \).
As previously we define dimensionless parameters and in particular

\[ \alpha_j = \frac{m^2 \Lambda_D}{\Lambda^2_{Dj}} \quad j = 1, 2, V \]  

(115)

where the \( \Lambda_Dj \) are the pole masses in the \( D \) sector related to \( \Lambda_j \) in the \( B \) sector by Eq.(50).

The formalism is similar to the previous case, although more complicated because of the \( K^* \) polarization. For details see Ref.[10].

We have computed \( X(t) \) for the three scenarios \([n_1, n_2, n_V] = [-1, 2, 2], [-1, 1, 2] \) and \([-1, 0, 2]\) using the PDG values \[3\] for \( x^D(0) \) and \( y^D(0) \). The parameters \( \alpha_1, \alpha_2, \alpha_V \) - or equivalently \( \Lambda_1, \Lambda_2, \Lambda_V \) - are constrained to be inside the allowed domains represented on Figures 1, 2 and 3. The results are shown on Figures 8, 9 and 10 for the three scenarios \( n_2 = 2, 1, 0 \) respectively. As in the previous case the largest sensitivity of \( X(t) \) to the parameters \( \alpha_j \) is in the neighbourhood of \( t = 0 \).

In a similar way, we can study the \( q^2 \) distribution for the polarization parameters \( \rho_{sl}^L(t) \) and \( A_{sl LR}^L(t) \). We only give here the corresponding integrated ones and the results for the three scenarios \( n_2 = 2, 1, 0 \) are the following :

(i) \( n_2 = 2 \):
\[ 0.516 \leq \rho_{sl}^L \leq 0.541 \]
\[ 0.885 \geq A_{sl LR}^L \geq 0.829 \]

(ii) \( n_2 = 1 \):
\[ 0.526 \leq \rho_{sl}^L \leq 0.541 \]
\[ 0.904 \geq A_{sl LR}^L \geq 0.857 \]

(iii) \( n_2 = 0 \):
\[ 0.536 \leq \rho_{sl}^L \leq 0.538 \]
\[ 0.904 \geq A_{sl LR}^L \geq 0.892 \]

In Eqs.(116) - (118) the results are presented in such a way to exhibit a correlation between the largest (smallest) \( \rho_{sl}^L \) and the smallest (largest) \( A_{sl LR}^L \).

We observe that the results are moderately scenario dependent for the three considered cases and all together we obtain :

\[ 0.52 \leq \rho_{sl}^L \leq 0.54 \]

(119)

\[ 0.90 \geq A_{sl LR}^L \geq 0.83 \]
IX. Concluding remarks

1°) Let us first summarize the assumptions and constraints contained in our model.

(A) Assumptions :
(a). Factorization holds for color supressed $B$ decays and final state strong interaction effects can be neglected.
(b). The SU(2) heavy flavor symmetry between the $b$ and $c$ quarks is realized by the Isgur-Wise relations $^3$.
(c). The normalizations of the hadronic form factors in the $B$ and $D$ sector are determined by their values at $q^2 = 0$ in the $D$ sector from semi-leptonic decays $D \rightarrow \overline{K} + \ell^+ + \nu_\ell$ and $D \rightarrow \overline{K}^* + \ell^+ + \nu_\ell$.

(B) The experimental constraints are :
(d). The experimental rates for $B \rightarrow K + J/\Psi$, $B \rightarrow K^* + J/\Psi$, $B \rightarrow K + \Psi'$ and $B \rightarrow K^* + \Psi'$ used in the form of ratios of rates $R_{J/\Psi}$, $R_{\Psi'}$, $S$ and $S^*$.
(e). The observed longitudinal polarization fraction $\rho_L$ in $B \rightarrow K^* + J/\Psi$.

(C) The theoretical constraints are :
(f). The explicit form of the $q^2$ dependence of the hadronic form factors $F_1$, $A_1$, $A_2$, $V$ choosen as $[1 - q^2/\Lambda^2]^{-n}$ with $n = -1, 0, 1, 2$.
(g). The pole masses $\Lambda$ of the various form factors in the $B$ sector are limited to the $(5 - 6)$ GeV range in order to relate them in a likely way to $b\bar{c}$ bound state masses.
(h). The ratios of form factors $\mu^R(q^2)$ and $\lambda^R(q^2)$ defined in Eqs.(38) and (53) are assumed to be independent of $q^2$ and related in a natural way to the pole masses $\Lambda_F$ and $\Lambda_2$ by Eqs.(13) and (27).

2°) Let us make some comments concerning the assumption (c). The correct procedure for a given scenario in the $B$ sector is to determine the ratios $x^D(0)$, $y^D(0)$ and $z^D(0)$ by using the same scenario in the $D$ sector for the analysis of experimental data of semi-leptonic decays $D \rightarrow \overline{K}(\overline{K}^*) + \ell^+ + \nu_\ell$. Unfortunately we were not able to follow this line because the only available information on $x^D(0)$, $y^D(0)$ and $z^D(0)$ given in Table 3 has been obtained from experiments, assuming a monopole $q^2$ dependence for all the form factors. We know however that such a scenario $[n_1, n_2, n_V] = [1, 1, 1]$ is in contradiction with experiment in the $B$ sector.
As other authors [4], we have used the values of Table 3. However some theoretical uncertainty on these ratios has to be added to errors given in Table 3 but it is not an easy task to estimate such an uncertainty. We refer to our report [10] for some comments on the determination of $F_{1}^{DK}(0)$ and $A_{1}^{DK^*}(0)$ from the experimental integrated rates. These determinations depend moderately in fact on scenarios.

3°) The $q^2$ dependence of the hadronic form factors in the $D$ sector, $F_{1}^{DK}(q^2)$, $A_{1}^{DK^*}(q^2)$, $A_{2}^{DK^*}(q^2)$ and $V^{DK^*}(q^2)$ could be in principle determined, from experiment, by the measurement of all possible $q^2$ distributions in the semi-leptonic decay, $D \rightarrow K + \ell^+ + \nu_\ell$ (for $F_{1}^{DK}$) and $D \rightarrow K^* + \ell^+ + \nu_\ell$ where the final vector meson is polarized (for $A_{1}^{DK^*}$, $A_{2}^{DK^*}$ and $V^{DK^*}$).

The knowledge of the $q^2$ dependence of the hadronic form factors in the $D$ sector, even with unavoidable errors will help in making the selection of scenarios in the $D$ sector and, by SU(2) heavy flavor symmetry, in the $B$ sector.

To our knowledge, such an experimental information is not available. It would be of considerable help for clarifying the theoretical constraints (f) and (g).

4°) Let us point out that the various ratios studied in this paper have different types of dependence with respect to the form factor values at $q^2 = 0$ in the $D$ sector.

(i). $A_{LR}$ and $A'_{LR}$ depend only on $y_D(0)$.

(ii). $\rho_L$ and $\rho'_{L}$ depend on $x_D(0)$ and $y_D(0)$.

(iii). $S^*$ and $T^*$ depend on $x_D(0)$ and $y_D(0)$.

(iv). $R_{\eta_c}$ depends on $x_D(0)$ and $z_D(0)$.

(v). $R_{J/\Psi}$ and $R_{\Psi'}$ depend on $x_D(0)$, $y_D(0)$ and $z_D(0)$.

(vi). $S$ and $T$ are independent of these three ratios.

For the semi-leptonic normalized distribution $X(t)$, it is independent on these ratios in the $D \rightarrow K + \ell^+ + \nu_\ell$ case and it depends on $x_D(0)$ and $y_D(0)$ in the $D \rightarrow K^* + \ell^+ + \nu_\ell$ mode.

5°) Among the 64 possible cases, we finally obtain three surviving scenarios $[n_F, n_1, n_2, n_V] = [+1, -1, n_2, +2]$ corresponding to $n_2 = +2, +1, 0$. The quality of the fit is very good for $n_2 = 2$, acceptable for $n_1 = 1$ and marginal for $n_2 = 0$ as illustrated in Figures 1–6. An improvement of the rate measurements, in particular those involving the $\Psi'$, may imply important restrictions for the $\Lambda_1$, $\Lambda_2$, $\Lambda_V$ and $\Lambda_F$, parameter space and possibly eliminate some scenarios starting with $n_2 = 0$. It is clear that measurements of the quantities $A_{LR}$, $A'_{LR}$, $\rho'_L$ and $R_{\eta_c}$ when compared
to those predicted by our model would considerably help in reducing the size of the allowed domains in the $\Lambda_j$ parameter space.

The best situation would be to select at the end only one scenario with a small non empty domain in the $\Lambda_j$ parameter space. The worse situation for our model would be that these new measurements $A_{LR}$, $A'_{LR}$, $\rho_L'$ and $R_{\eta}$ exclude the three presently remaining scenarios.

Our model is certainly not the unique way to analyse the $B \rightarrow K(K^*)$ hadronic form factors. However if we are in the best situation previously mentioned, it will be necessary to provide a theoretical support to the so determined hadronic form factors and for that, results of Ref.\cite{6} seem to be in a good shape because of the unusual $q^2$ behaviour prediction for $A_1(q^2)$.

If we are in the worse situation, it will be reasonable to think seriously of the role played by non-factorizable contributions.

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Figure captions

1. **Figure 1**: The allowed domain in the $\Lambda_1$, $\Lambda_2$, $\Lambda_V$ space due to the constraints $\rho_L + \triangle \rho_L \geq 0.70$ and $R_{J/\psi} - \triangle R_{J/\psi} \leq 2.20$ for $\Lambda_2, \Lambda_V \in (5 - 6)$ $GeV$, $\Lambda_1 \geq 5$ $GeV$. The scenario is $[n_1, n_2, n_V] = [-1, 2, 2]$. $\triangle \rho_L$ and $\triangle R_{J/\psi}$ are theoretical errors induced by experimental errors of $x^D(0), y^D(0)$ and $z^D(0)$.

2. **Figure 2**: Same as Figure 1 for the scenario [-1, 1, 2].

3. **Figure 3**: Same as Figure 1 for the scenario [-1, 0, 2].

4. **Figure 4**: The allowed domain in the $\Lambda_F$, $\Lambda_2$, $\Lambda_V$ space due to the constraint

   $\Lambda_{1, MIN}(\Lambda_2, \Lambda_V, \Lambda_F = 5$ $GeV) \leq \Lambda_1 \leq \Lambda_{1, MAX}(\Lambda_2, \Lambda_V)$ with $\Lambda_F \geq 5$ $GeV$.

   The scenario is $[n_1, n_2, n_V] = [-1, 2, 2]$.

5. **Figure 5**: Same as Figure 4 for the scenario [-1, 1, 2].

6. **Figure 6**: Same as Figure 4 for the scenario [-1, 0, 2].

7. **Figure 7**: The normalized dimensionless distribution $X(t)$ for the semi-leptonic decay $D \rightarrow K^+ + \ell^+ + \nu_\ell$. The scenario $n_2 = 2$ corresponds to $5$ $GeV \leq \Lambda_F \leq 5.71$ $GeV$, the scenarios $n_2 = 1$ to $5$ $GeV \leq \Lambda_F \leq 5.39$ $GeV$ and the scenario $n_2 = 0$ to $5$ $GeV \leq \Lambda_F \leq 5.10$ $GeV$. By Eq.(33) the pole masses $\Lambda_{DF}$ in the $D$ sector can be obtained from $\Lambda_F$ given here.

8. **Figure 8**: The normalized dimensionless distribution $X(t)$ for the semi-leptonic decay $D \rightarrow K^- + \ell^+ + \nu_\ell$. The scenario is $[n_1, n_2, n_V] = [-1, 2, 2]$. The thickness of the curve is due to the $\Lambda_1, \Lambda_2, \Lambda_V$ ranges.

9. **Figure 9**: Same as Figure 8 for the scenario [-1, 1, 2].

10. **Figure 10**: Same as Figure 8 for the scenario [-1, 0, 2].
Table captions

1. **Table 1:**
   Experimental data as averaged by PDG [5].

2. **Table 2:**
   Experimental data for the ratios $R_{q'}$, $R_{J/\psi}$, $S$ and $S^*$. 

3. **Table 3:**
   Input data for the charm sector [5].

4. **Table 4:**
   Results of the fit at the extreme values of $\Lambda_1$. 


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