PERMUTATIONS OF $Z^d$ WITH RESTRICTED MOVEMENT

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Abstract. We investigate dynamical properties of the set of permutations of $Z^d$ with restricted movement, i.e., permutations $\pi$ of $Z^d$ such that $\pi(n) - n$ lies, for every $n \in Z^d$, in a prescribed finite set $A \subset Z^d$. For $d = 1$, such permutations occur, for example, in restricted orbit equivalence (cf., e.g., Boyle and Tomiyama (1998), Kammeyer and Rudolph (1997), or Rudolph (1985)), or in the calculation of determinants of certain bi-infinite multi-diagonal matrices. For $d \geq 2$ these sets of permutations provide natural classes of multidimensional shifts of finite type.

1. Introduction

Let $d \geq 1$, and let $S^\infty(Z^d)$ be the group of all permutations of the integer lattice $Z^d$. We fix a finite set $A \subset Z^d$ (always assumed to be nonempty) and write $\Pi_A$ for the set of all permutations $\pi : n \mapsto \pi(n)$ of $Z^d$ which satisfy that

$$\omega_n^{(\pi)} := \pi(n) - n \in A \text{ for every } n \in Z^d.$$  \hspace{1cm} (1.1)

In view of (1.1) we may regard $\Pi_A$ as a subset of the space $\prod_{n \in Z^d} (n + A)$ which is obviously closed, and hence compact, in the product topology on $\prod_{n \in Z^d} (n + A)$.

Every $\pi \in \Pi_A$ is determined by the point $\omega^{(\pi)} = (\omega_n^{(\pi)})_{n \in Z^d} \in A^{Z^d}$. Furthermore, the set $\Omega_A = \{\omega^{(\pi)} : \pi \in \Pi_A\}$ is a closed subset of the compact space $A^{Z^d}$, and the map $\pi \mapsto \omega^{(\pi)}$ from $\Pi_A$ to $\Omega_A$ is a homeomorphism.

In view of the one-to-one correspondence between $\Pi_A$ and $\Omega_A$ it will be convenient to write $\pi(\omega) \in \Pi_A$ the for the permutation corresponding to an element $\omega \in \Omega_A$. Then $\omega = \omega^{(\pi(\omega))}$ and $\pi^{(\omega(\omega))} = \pi$.

Proposition 1.1. Let $\varsigma$ be the $Z^d$-action on itself by translation, given by $\varsigma^m(n) = n + m$, and let $\sigma : m \mapsto \sigma^m$ be the shift action $(\sigma^n\omega)_n = \omega_{n+m}$ of $Z^d$ on $A^{Z^d}$.

1. For every $m \in Z^d$, the set $\Pi_A \subset S^\infty(Z^d)$ is invariant under the inner automorphism $\pi \mapsto \text{Ad}_{\varsigma^m}(\pi) = \varsigma^{\sigma^{-m}\omega} \text{Ad}_{\varsigma^{-m}}(\pi)$ of $S^\infty(Z^d)$.

2. For every $\omega \in \Omega_A$ and $m \in Z^d$, $\text{Ad}_{\varsigma^m}(\pi) = \pi^{(\sigma^{-m}\omega)}$. Hence $\Omega_A \subset A^{Z^d}$ is shift-invariant.

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(3) The topological \( Z^d \)-dynamical systems \((\Pi_A, A_\sigma)\) and \((\Omega_A, \sigma)\) are topologically conjugate.

(4) For every \( b \in Z^d \) we set \( A + b = \{ a + b : a \in A \} \). Then \( \Pi_{A+b} = \iota^b \Pi_A \) and \( \Omega_{A+b} = \Omega_A + (\ldots, b, b, b, \ldots) \). Furthermore, the systems \((\Omega_A, \sigma)\) and \((\Omega_{A+b}, \sigma)\) (and hence the systems \((\Pi_A, A_\sigma)\) and \((\Pi_{A+b}, A_\sigma)\)) are topologically conjugate.

Proof. For every \( \pi \in S^\infty(Z^d) \) and \( m \in Z^d \), the permutation \( A_{\sigma}^m(\pi)(n) = \pi(n-m) + m \). Hence \( A_{\sigma}^m(\pi(\omega))(n) = \omega_{n-m} + n - m + m = \omega_{n-m} + n = (\sigma^{-m}\omega)_n + n = \pi(\sigma^{-m}\omega)(n) \). Since \( \pi(n) - n \in A \) for every \( n \in Z^d \) if and only if \( A_{\sigma}^m(\pi)(n) - n = \pi(n-m) - n + m \in A \) for every \( m, n \in Z^d \), the set \( \Pi_A \) is \( A_\sigma \)-invariant. This proves (1) and (2), and (3) – (4) are obvious. \( \square \)

In the Sections 2 – 4 we restrict our attention to the case where \( d = 1 \) and \( A \) is a finite interval in \( Z \), which for simplicity we assume to be of the form \( A_K = \{0, \ldots, K\} \) with \( K \geq 1 \) (cf. Proposition 1.1 (4)). For notational economy we set

\[ \Pi_K := \Pi_{A_K}, \quad \Omega_K := \Omega_{A_K} \subset A_K^Z. \]  

(1.2)

In the Sections 2 and 3 we prove the following results.

- For every \( \omega \in \Omega_K \) there exists an integer \( a(\omega) \in A_K \) such that

\[ \left| \sum_{n=m}^{m+N-1} \omega_n - N a(\omega) \right| < K^2 \]

for every \( m \in Z \) and \( N \geq 1 \) ((2.18) and Corollary 2.10). This integer can be viewed as the average ‘shift’ of \( Z \) imparted by the permutation \( \pi(\omega) \in S^\infty(Z) \).

- The space \( \Omega_K \) is a shift of finite type (abbreviated as SFT) which is not irreducible. Its irreducible components are given by

\[ \Omega_{K,l} = \{ \omega \in \Omega_K : a(\omega) = l \}, \quad l \in A_K, \]

(Proposition 2.7).

- For \( 0 \leq l \leq K \), the SFTs \( \Omega_{K,l} \) and \( \Omega_{K,K-l} \) are topologically conjugate (Proposition 2.12).

- \( |\Omega_{K,0}| = |\Omega_{K,K}| = 1 \), and for \( l = 1, \ldots, K - 1 \), the topological entropy \( h(\Omega_{K,l}) \) satisfies that

\[ (1 - \frac{l}{K}) \log (l + 1) \leq h(\Omega_{K,l}) \leq \log (l + 1) \]

(Theorem 3.3).

- For \( 0 \leq l < \frac{K}{2} \), \( h(\Omega_{K,l}) \leq h(\Omega_{K,l+1}) \) (Proposition 3.5).
In Section 4 we consider periodic points of the SFT $\Omega_K$. If $\omega \in \Omega_K$ is periodic with period $p$, say, then the $p$-tuple

$$\pi^{(\omega)}_{(m,p)} := \left( \pi^{(\omega)}(m) \pmod{p}, \ldots, \pi^{(\omega)}(m+p-1) \pmod{p} \right)$$

is, for every $m \in \mathbb{Z}$, a permutation of $(0, \ldots, p-1)$. What is the parity (or sign) of this permutation? In Theorem 4.1 we prove that there exists a continuous cocycle $s: \mathbb{Z} \times \Omega_K \rightarrow \{\pm 1\}$ which describes the parities of all these permutations. Together with the function $a: \Omega_K \rightarrow A_K$ in (2.3), this parity cocycle $s$ can be used to express the determinants of certain circulant-like matrices appearing in entropy calculations for algebraic actions of the discrete Heisenberg group (cf. [7, Section 8] and Example 4.4).

In Section 5 we consider permutations of $\mathbb{Z}^d$ with $d \geq 2$ and prove the following results.

- If $A$ is a finite subset of $\mathbb{Z}^d$, $d \geq 2$, then $\Omega_A$ has positive entropy if and only if $|A| \geq 3$ (Theorem 5.2).
- Let $d \geq 2$, and let $A \subset \mathbb{Z}^d$ be a finite subset. Then $\Omega_A$ is topologically mixing if and only if $D = A - A$ is not contained in a one-dimensional subspace of $\mathbb{R}^d$ (Theorem 5.6).

Finally, in Section 6, we return to one of the simplest $\mathbb{Z}^2$-SFT’s arising from permutations of $\mathbb{Z}^2$ with restricted movement, the space $\Omega_A$ with $A = \{(0,0), (1,0), (0,1)\}$. This space had appeared already in Example 5.5, and we describe its dynamical properties (like entropy and the logarithmic growth-rate of the number of its periodic points) in greater detail.

2. Permutations of $\mathbb{Z}$ with bounded movement

We set $d = 1$. Fix $K \geq 1$, put $A_K = \{0, \ldots, K\}$, and write, as usual, $\sigma$ instead of $\sigma^1$ for the shift $(\sigma \omega)_k = \omega_{k+1}$ on $A^\mathbb{Z}$ (cf. Proposition 1.1). For $\omega = (\omega_n)_{n \in \mathbb{Z}} \in A_K^\mathbb{Z}$ and $m \in \mathbb{Z}$ we put

$$\tilde{\omega}_m = \omega_m + m. \quad (2.1)$$

In the notation of Section 1 we set $\Omega_K = \{\omega^{(\pi)} : \pi \in \Pi_K\} \subset A_K^\mathbb{Z}$. Then $\Omega_K$ is the subshift of $A_K^\mathbb{Z}$ defined by the following condition: for every $\omega = (\omega_n)_{n \in \mathbb{Z}} \in \Omega_K$, the map $\pi^{(\omega)}: n \rightarrow \tilde{\omega}_n$, $n \in \mathbb{Z}$, is a permutation of $\mathbb{Z}$.

For the following lemma we recall a basic definition: if $A$ is a finite alphabet and $L \geq 1$, a subshift $\Omega \subset A^\mathbb{Z}$ is an $L$-step SFT if there exists a set $F \subset A^{L+1}$ of forbidden words such that $\Omega$ is the set of all sequences $\omega \in A^\mathbb{Z}$ not containing any of the words in $F$.

**Lemma 2.1.** For every $K \geq 1$ the subshift $\Omega_K \subset A_K^\mathbb{Z}$ has the following properties:

1. Let $\omega \in \Omega_K$. 

(a) For every \( m, n \in \mathbb{Z}, m \neq n, \tilde{\omega}_m \neq \tilde{\omega}_n \).
(b) For every \( n \in \mathbb{Z} \), \( n \in \{ \tilde{\omega}_{n-K}, \ldots, \tilde{\omega}_n \} \).

(2) An element \( \omega \in A_Z^\mathbb{Z} \) lies in \( \Omega_K \) if and only if it satisfies the conditions (a) and (b) in (1): for every \( n \in \mathbb{Z} \), the set \( \{ \tilde{\omega}_{n-K}, \ldots, \tilde{\omega}_n \} \) has \( K+1 \) distinct elements and contains \( n \). In particular, \( \Omega_K \) is a \( K \)-step SFT.

(3) If \( \omega \in \Omega_K \) is periodic with period \( p \), then
\[
\pi(\omega)_{(m,p)} := (\tilde{\omega}_m \text{ mod } p, \ldots, \tilde{\omega}_{m+p-1} \text{ mod } p) \tag{2.2}
\]
is, for every \( m \in \mathbb{Z} \), a permutation of \( (0, \ldots, p-1) \).

\[\text{Proof. Obvious.}\] \[\square\]

Example 2.2. If \( K = 1 \), \( \Omega_1 = \{ (\ldots, 0, 0, \ldots), (\ldots, 1, 1, \ldots) \} \). For \( K = 2 \), \( \Omega_2 \) is the union of the fixed points \( (\ldots, 0, 0, \ldots), (\ldots, 2, 2, \ldots) \), and the mixing SFT determined by the directed graph

![Directed Graph](image)

More generally, the following is true.

Lemma 2.3. (1) For every \( K \geq 1 \), the SFT \( \Omega_K \) is \( K \)-step, but not \((K-1)\)-step.

(2) For every \( K \geq 1 \) and \( \omega \in \Omega_K \) we put
\[
a(\omega) = |\{ k > 0 : \tilde{\omega}_{-k} \geq 0 \}| = |\{ k = 1, \ldots, K : \tilde{\omega}_{-k} \geq 0 \}|. \tag{2.3}
\]
Then
\[
\Omega_{K,l} = \{ \omega \in \Omega_K : a(\omega) = l \} \tag{2.4}
\]
is, for every \( l = 0, \ldots, K \), a closed, shift-invariant subset of \( \Omega_K \) which is a \((K-1)\)-step SFT.

(3) The map \( a : \Omega_K \rightarrow A_K \) defined by (2.3) is continuous.

For the proofs of Lemma 2.3 and Theorem 4.1, Figure 1 will be useful, where we assume that \( p > K \geq 2 \).

Consider the following subsets of \( \mathbb{Z}^2 \) illustrated in Figure 1 (note that the first coordinate in this picture increases as one moves to the right, while the second increases as one moves down):

- the square \( Q = \{ (k_1, k_2) : 0 \leq k_i < p \} \),
- the triangles \( A = \{ (k_1, k_2) : 0 \leq k_1 < K, k_1 - K \leq k_2 < 0 \} \), \( B = \{ (k_1, k_2) : 0 \leq k_1 < K, 0 \leq k_2 \leq k_1 \} \), \( D = \{ (k_1, k_2) : p - K \leq k_1 < p, p - K \leq k_2 \leq k_1 \} \),
$A' = \{(k_1, k_2) : p \leq k_1 < p + K, k_1 - K \leq k_2 < p\}$,

$A^* = \{(k_1, k_2) : 0 \leq k_1 < K, p - K + k_1 \leq k_2 < p\}$,

the trapezoid $C = \{(k_1, k_2) : K \leq k_1 < p, k_1 - K \leq k_2 \leq \min (k_1, p - K - 1)\}$.

We fix $\omega \in \Omega_K$, set

$$S(\omega) = \{(\hat{\omega}_k, k) : k \in \mathbb{Z}\} \subset \mathbb{Z}^2,$$

and denote by $\hat{A} = A \cap S(\omega)$, $\hat{B} = B \cap S(\omega)$, \dots, $\hat{A}' = A' \cap S(\omega)$, $\hat{C} = C \cap S(\omega)$, and $\hat{Q} = Q \cap S(\omega) = \hat{B} \cup \hat{C} \cup \hat{D}$ the intersections of these sets with $S(\omega)$ (note that $A^* \cap S(\omega) = \emptyset$).

**Proof of Lemma 2.3.** We start by proving the first assertion in (2). Fix $\omega \in \Omega_K$, define $S(\omega)$ by (2.5), and consider the sets $\hat{Q}, \hat{A}, \ldots, \hat{A}', \hat{C}$ defined above. According to the definition of $\Omega_K$, $\{(m, n) : n \in \mathbb{Z}\} \cap S(\omega) = \{(m, m - k) : 0 \leq k \leq K\} \cap S(\omega) = \{(m, \hat{\omega}_m)\}$ for every $m \in \mathbb{Z}$. Similarly, $|\{(m, n) : m \in \mathbb{Z}\} \cap S(\omega)| = |\{(n + k, n) : 0 \leq k \leq K\} \cap S(\omega)| = 1$ for every $n \in \mathbb{Z}$. It follows that $|\hat{A} \cup \hat{B}| = |\hat{D} \cup \hat{A}'| = K$ and $|\hat{A} \cup \hat{Q}| = |\hat{Q} \cup \hat{A}'| = p$. Since $|\hat{A}| = a(\omega)$ (cf. (2.3)), we obtain that $a(\omega) = |\hat{A}| = |\hat{A}'| = a(\sigma^p \omega)$.

As $p > K$ is arbitrary, $a(\omega) = a(\sigma \omega)$ for every $\omega \in \Omega_K$, which proves that the sets $\Omega_{K,l}$, $l = 0, \ldots, K$, are shift-invariant.

In order to prove (1) we observe that any $\omega \in \Omega_K$ with $\omega_{-K} = \cdots = \omega_{-1} = 0$ satisfies that $a(\omega) = 0$ and thus lies in $\Omega_{K,0}$. A point $\omega' \in \Omega_K$ with $\omega_{-K+1} = \cdots = \omega_{-1} = 0$ and $\omega_0 = 1$ satisfies $a(\sigma \omega') = 1$ and hence $\omega' \in \Omega_{K,1}$. If $\Omega_K$ were $(K-1)$-step, there would exist a point $\omega'' \in \Omega_K$ with $\omega''_{-K} = \cdots = \omega''_{-1} = 0$ and $\omega''_0 = 1$, and hence with $0 = a(\omega'') \neq a(\sigma \omega'') = 1$. This would contradict the shift-invariance of $a(\cdot)$ shown above.
Having verified (1), we return to (2) by showing that each \( \Omega_{K,l} \) is \((K-1)\)-step (it is obviously \( K \)-step). If \( l = 0 \) or \( l = K \), \( \Omega_{K,l} \) consists of a single fixed point and is therefore \( K \)-step. If \( 0 < l < K \) we observe that a point \( \omega \in \Omega_K \) lies in \( \Omega_{K,l} \) if and only if, for every \( n \in \mathbb{Z} \), either
\[
|\{k = 1, \ldots, K-1 : \omega_{n-k} > k\}| = l \quad \text{and} \quad \omega_n = 0, \tag{2.6}
\]

or
\[
|\{k = 1, \ldots, K-1 : \omega_{n-k} > k\}| = l - 1, \quad \omega_n > 0, \quad \text{and} \quad \omega_n \neq \omega_{n-k} - k \quad \text{for} \quad k = 1, \ldots, K-1. \tag{2.7}
\]

This proves that \( \Omega_{K,l} \) is \((K-1)\)-step.

Finally we note that the continuity of the map \( a : \Omega_K \to A \) claimed in (3) is an immediate consequence of (2.3).

We will re-code \( \Omega_{K,l} \) as a one-step SFT \( X_{K,l} \) with alphabet
\[
\mathcal{B}_{K,l} = \{ a \subset A_{K-1} : |a| = l \}, \tag{2.8}
\]

the set of all \( l \)-element subsets of \( A_{K-1} = \{0, \ldots, K-1\} \). For visualising \( X_{K,l} \) we adapt a simile from [9, Section 2.1].

**Definition 2.4** (The SFT \( X_{K,l} \) – a strange office). Consider an office with \( K \) desks, numbered from 0 to \( K - 1 \), and arranged side by side. At each desk there is a clerk who can handle at most one file at any given time. At regular intervals each clerk checks his desk. If he finds a file there he passes it on to his neighbour on the left. If the clerk sitting at the leftmost desk (the desk 0) finds a file on his desk he carries it over to one of the empty desks, chosen at random (like everybody else he may simultaneously receive a file from his neighbour on the right).

The SFT \( X_{K,l} \) corresponds to the *modus operandi* of this office if there are a total of \( l \) files in circulation.

To make this description more formal we view \( \mathcal{B}_{K,l} \) as the set of possible positions of \( l \) files on the \( K \) desks and define a \((\mathcal{B}_{K,l} \times \mathcal{B}_{K,l})\)-matrix \( M = M_{K,l} \) with entries in \( \{0, 1\} \) by setting, for all \( a, b \in \mathcal{B}_{K,l} \), \( M(a, b) = 1 \) if and only if one of the following conditions is satisfied:

(M1) \( 0 \notin a \) and \( b = a - 1 := \{a - 1 : a \in a\} \) (i.e., there is no file on desk 0, and each file is moved one step to the left),

(M2) \( 0 \in a \) and \( b = (a' - 1) \cup \{j\} \) for some \( j \in A_{K-1} \setminus (a' - 1) \), where \( a' = a \setminus \{0\} \) (i.e., there is a file on desk 0 which gets dropped on the floor; then all the other files are moved one position to the left, and the file on the floor is placed on an empty desk).

We shall prove that \( \Omega_{K,l} \) is conjugate to the SFT
\[
X_{K,l} = \{(a_n)_{n \in \mathbb{Z}} \in \mathcal{B}_{K,l}^\mathbb{Z} : M(a_n, a_{n+1}) = 1 \quad \text{for every} \quad n \in \mathbb{Z}\} \tag{2.9}
\]
defined by the transition matrix $M$.

**Proposition 2.5.** Let $\phi_{K,I}: \Omega_{K,I} \rightarrow B_{K,I}^2$ be the map given by

$$\phi_{K,I}(\omega)_n = \pi_1(A \cap S(\sigma^n \omega)) \quad (2.10)$$

for every $\omega \in \Omega_{K,I}$ and $n \in \mathbb{Z}$, where $A \subseteq \mathbb{Z}^2$ is the triangle appearing in Figure 1 on page 5, $S(\sigma^n \omega)$ is defined in (2.5), and $\pi_1: \mathbb{Z}^2 \rightarrow \mathbb{Z}$ is the first coordinate projection. This map has the following properties.

1. $\phi_{K,I}$ is shift-equivariant and injective.
2. $\phi_{K,I}(\Omega_{K,I}) = X_{K,I}$.

**Proof.** The shift-equivariance of $\phi_{K,I}$ is obvious from (2.10). The injectivity of $\phi_{K,I}$ follows from (2.6) – (2.7): for every $\omega \in \Omega_{K,I}$ and $n \in \mathbb{Z},$

$$\omega_n = 0 \text{ if } 0 \notin \phi_{K,I}(\omega)_n, \quad \text{and} \quad \omega_n \in (\phi_{K,I}(\omega)_{n+1} \setminus \phi_{K,I}(\omega)_n \text{ otherwise.} \quad (2.11)$$

Since this holds for every $n \in \mathbb{Z}$, the sequence $\phi_{K,I}(\omega)$ determines $\omega$, which proves (1).

2. If a sequence $\omega = (\omega_n)_{n \in \mathbb{Z}} \in A_{K,I}^\mathbb{Z}$ satisfies the conditions (2.6) – (2.7) for every $n \in \mathbb{Z}$, then the sets $\phi_{K,I}(\omega)_n, n \in \mathbb{Z}$, satisfy (M1) – (M2). This shows that $\phi_{K,I}(\Omega_{K,I}) \subset X_{K,I}$. Conversely, if $(a_n)_{n \in \mathbb{Z}} \in X_{K,I}$, and if we set

$$\omega_n = 0 \text{ if } 0 \notin a_n, \quad \text{and} \quad \omega_n \in (a_{n+1} \setminus a_n \text{ otherwise} \quad (2.12)$$

for every $n \in \mathbb{Z}$, we obtain an element $\omega = (\omega_n) \in \Omega_{K,I}$ with $\phi_{K,I}(\omega) = (a_n)_{n \in \mathbb{Z}}$.

This proves (2). \qed

**Proposition 2.6.** Let $M$ be the transition matrix of $X_{K,I}$ in (M1) – (M2). For every state $a \in B_{K,I}$ of $X_{K,I}$, the follower set $f(a) = \{b \in B_{K,I} : M(a,b) = 1\}$ of $a$ is given by

$$f(a) = \{b \in B_{K,I} : b \supset (a - 1) \cap A_{K-1}\} \quad (2.13)$$

and satisfies that

$$|f(a)| = \begin{cases} \ K - l + 1 \text{ if } 0 \in a, \\ 1 \text{ otherwise.} \end{cases} \quad (2.14)$$

The predecessor set $p(a) = \{b \in B_{K,I} : M(b,a) = 1\}$ of $a$ is given by

$$p(a) = \{b \in B_{K,I} : b \subset \{0\} \cup (a + 1)\}, \quad (2.15)$$

and satisfies that

$$|p(a)| = \begin{cases} 1 \text{ if } (K - 1) \in a, \\ l + 1 \text{ otherwise.} \end{cases} \quad (2.16)$$

**Proof.** Let $a \in B_{K,I}$. If $0 \notin a$, then $a - 1 = \{a - 1 : a \in a\}$ is the only follower of $a$. If $0 \in a$ we put $a' = a \setminus \{0\}$ and set $b = (a' - 1) \cup \{j\}$ for some $j \in A_K \setminus (a' - 1)$. Clearly, there are $|A_K \setminus (a' - 1)| = K - l + 1$ possibilities for doing this, and every $b$ obtained in this manner is a follower of $a$. \qed
For describing the predecessors of a we first assume that \((K-1) \notin a\) and set \(a' = (a+1) \cup \{0\}\). This set has \(l+1\) elements, and every \(b \in A_K\) obtained by removing one of the elements of \(a''\) is a predecessor of \(a\). Since there are \(l+1\) possibilities for doing this, \(p(a) = l+1\). If, on the other hand, \((K-1) \in a\), then the set \(p(a)\) in (2.15) has the single element \(b = \{0\} \cup (a \setminus \{(K-1)\}) + 1\) \(\in B_{K,l}\).

**Proposition 2.7.** For every \(K \geq 2\) and \(l \in \{1, \ldots, K-1\}\), the SFT \(\Omega_{K,l}\) is irreducible and aperiodic.

*Proof.* In view of the isomorphism \(\phi_{K,l}: \Omega_{K,l} \rightarrow X_{K,l}\) it suffices to prove the analogous assertion for the SFT \(X_{K,l}\) in (2.9). The state

\[
e = \{0, \ldots, l-1\} \in B_{K,l}
\]

satisfies that \(|(e+1) \setminus e| = 1\), so that \(e \in f(e)\) (and hence \(e \in p(e)\)) by Condition (M2) on page 6. Furthermore there obviously exists a path (i.e., a finite sequence of allowed transitions) from every \(b \in B_{K,l}\) to \(e\). Conversely, if \(b = \{b_1, \ldots, b_l\}\) with \(b_1 < b_2 < \cdots < b_l\), there is a path \(e = \{0, \ldots, l-1\} \rightarrow \{0, 1, \ldots, l-2, b_1 + l-1\} \rightarrow \{0, \ldots, l-3, b_1 + l-2, b_2 + l-2\} \rightarrow \cdots \rightarrow \{0, b_1 + 1, \ldots, b_{l-1} + 1\} \rightarrow b\) of length \(l\). This shows that \(X_{K,l}\) is irreducible and aperiodic (since it contains a fixed point). \(\square\)

**Proposition 2.8.** For every \(K \geq 1\) and \(l \in \{0, \ldots, K\}\) there exists a continuous map \(b_{K,l}: \Omega_{K,l} \rightarrow \mathbb{Z}\) such that

\[
\omega_0 = l + b_{K,l}(\omega) - b_{K,l}(\sigma \omega)
\]

for every \(\omega \in \Omega_{K,l}\).

*Proof.* Fix \(K, l\) and \(\omega \in \Omega_{K,l}\). We denote by \(A = \{(k_1, k_2): 0 \leq k_1 < K, k_1 - K \leq k_2 < 0\} \subset \mathbb{Z}^2\) the triangle appearing in Figure 1 and set, for every \(m \in \mathbb{Z}\), \(A_m = A + (m, m)\). By (2.4), \(a(\sigma^n \omega) = |A \cap S(\sigma^n \omega)| = |A_n \cap S(\omega)| = l\) for every \(n \in \mathbb{Z}\). Hence \(\sum_{n=1}^{N} |A_m \cap S(\omega)| = Nl\) for every \(N \geq 1\).

For every \(m, n \in \mathbb{Z}\), \((n, \tilde{\omega}_n) \in A_m\) if and only if \(\omega_n > 0\) and \(m = n + 1, \ldots, n + \omega_n\). In other words, \(\sum_{m \in \mathbb{Z}} 1_{A_m}(n, \tilde{\omega}_n) = \sum_{m=1}^{K} 1_{A_{n+m}}(n, \tilde{\omega}_n) = \omega_n\), where \(1_{A_m}: \mathbb{Z}^2 \rightarrow \mathbb{Z}\) is the indicator function of \(A_m\).

Let \(N > K\) (it actually suffices to choose \(N > 0\), but with \(N > K\) one can use Figure 1 to see what is going on here). Then

\[
Nl = \sum_{m=1}^{N} |A_m \cap S(\omega)| = \sum_{m=1}^{N} \sum_{n \in \mathbb{Z}} 1_{A_m}(n, \tilde{\omega}_n)
\]

\[
= \sum_{n \in \mathbb{Z}} \sum_{m=1}^{N} 1_{A_m}(n, \tilde{\omega}_n)
\]

\[
= \sum_{n=0}^{N-1} \sum_{m=1}^{K} 1_{A_{n+m}}(n, \tilde{\omega}_n) + \sum_{n=-K+1}^{-1} \sum_{m=1}^{K-1} 1_{A_m}(n, \tilde{\omega}_n)
\]

\[
- \sum_{n=N-K+1}^{N-1} \sum_{m=1}^{K-1} 1_{A_{n+m}}(n, \tilde{\omega}_n)
\]

(2.19)
\[
= \sum_{n=0}^{N-1} \omega_n + \sum_{n=-K+1}^{N-1} \sum_{m=1}^{K-1} 1_{A_m}(n, \tilde{\omega}_n) \\
- \sum_{n=N-K+1}^{N-1} \sum_{m=1}^{K-1} 1_{A_{N+m}}(n, \tilde{\omega}_n).
\]

For every \(\omega \in \Omega_{K,l}\) we put
\[
b_{K,l}(\omega) = \sum_{m=1}^{K-1} |A \cap A_m \cap S(\omega)| = \sum_{m=1}^{K-1} |A \cap A_m \cap S|.
\]

Then
\[
b_{K,l}(\sigma^N \omega) = \sum_{n=N-K+1}^{N-1} \sum_{m=1}^{K-1} 1_{A_{N+m}}(n, \tilde{\omega}_n),
\]
and (2.19) shows that
\[
\sum_{n=0}^{N-1} \omega_n = Nl + b_{K,l}(\omega) - b_{K,l}(\sigma^N \omega)
\]
for every \(\omega \in \Omega_{K,l}\) and \(N \geq 0\). By setting \(N = 1\) in (2.21) we obtain (2.18).

**Corollary 2.9.** Let \(K \geq 2\), and let \(\omega \in \Omega_K\) be a periodic point with period \(p\), say. Then \(l := \frac{1}{p} \sum_{n=0}^{p-1} \omega_n \in A_K\) and \(\omega \in \Omega_{K,l}\).

**Corollary 2.10.** For every \(K \geq 1\), \(l \in \{0, \ldots, K\}\), and \(\omega \in \Omega_{K,l}\),
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \omega_n = l.
\]

**Remarks 2.11.** (1) Equation (2.18) shows that the cocycle \(\omega: \mathbb{Z} \times \Omega_{K,l} \to \mathbb{Z}\), defined by
\[
\omega(n, \omega) = \begin{cases} 
\sum_{k=0}^{n-1} \omega_k & \text{if } n > 0, \\
0 & \text{if } n = 0, \\
-\omega(-n, \sigma^n \omega) & \text{if } n < 0,
\end{cases}
\]
is cohomologous to the homomorphism \(n \mapsto ln\), with transfer function \(b_{K,l}\) given by (2.20). In the context of bounded topological orbit equivalence, an analogous formula appears in [2, Lemma 2.6 and Theorem 2.3 (2)].

(2) When combined with Proposition 1.1 (4), Corollary 2.10 is equivalent to [8, Theorem 1]. In particular, the integer \(l = a(\omega)\) in (2.3) – (2.4) determines, for every \(\omega \in \Omega_K\), the position of the ‘main diagonal’ of the bi-infinite permutation matrix associated with \(\omega\) (cf. [8, p. 526]).

We end this section with a few comparisons between the SFT’s \(\Omega_{K,l}\) (or \(X_{K,l}\)) for different values of \(K\) and \(l\).

**Proposition 2.12.** For every \(K \geq 2\) and \(l \in \{1, \ldots, K - 1\}\), the following statements hold.

1. \(\Omega_{K,l} \subset \Omega_{K+1,l}\).
2. \(\Omega_{K,l} + 1 = \{(\omega_m + 1)_{m \in \mathbb{Z}} : \omega \in \Omega_{K,l}\} \subset \Omega_{K+1,l+1}\).
(3) The SFT’s $\Omega_{K,l}$ and $\Omega_{K,K-l}$ are topologically conjugate.

Proof. The first two assertions are obvious. For the third statement we define a shift-equivariant bijection $\Phi: A^\mathbb{Z}_K \rightarrow A^\mathbb{Z}_K$ by setting

$$\Phi(\omega)_m = K - \omega_m$$

(2.23)

for every $\omega \in \Sigma_K$ and $m \in \mathbb{Z}$. It is clear that $\Phi(\Omega_K) = \Omega_K$. In order to check that $\Phi(\Omega_{K,l}) = \Omega_{K,K-l}$ we take another look at Figure 1: if $\omega \in \Omega_{K,l}$, then $|A| = |A \cap S(\omega)| = l$. Since $|(A \cap S(\omega)) \cup (B \cap S(\omega))| = K$, we obtain that $|B \cap S(\omega)| = K - l$. Finally we note that $|A \cap S(\omega)| = |B \cap S(\Phi(\omega))|$ and $|B \cap S(\omega)| = |A \cap S(\Phi(\omega))|$, so that $\Phi(\omega) \in \Omega_{K,K-l}$ if and only if $\omega \in \Omega_{K,l}$. □

Remark 2.13. The conjugacy between $\Omega_{K,l}$ and $\Omega_{K,K-l}$ can, of course, also be expressed in terms of the SFT’s $X_{K,l}$ and $X_{K,K-l}$. For every $a \in B_{K,l}$ we put

$$a^* = \{K - 1 - j : j \in (A_{K-1} \setminus a)\}.$$  

(2.24)

If $a \in B_{K,l}$ then $p(a^*) = \{b^* : b \in p(a)\} = f(a)^*$ and $f(a^*) = \{b^* : b \in p(a)\} = p(a)^*$. The corresponding shift-equivariant isomorphism $\Psi_{K,l}: X_{K,l} \rightarrow X_{K,K-l}$ is given by

$$\Psi_{K,l}(a)_m = a^*_m$$

(2.25)

for every $(a_n) \in X_{K,l}$ and $m \in \mathbb{Z}$.

3. Entropy

Lemma 3.1. For every $K,l$ with $0 < l < K$, the topological entropy of $\Omega_{K,l}$ satisfies that $0 < h(\Omega_{K,l}) \leq \log (l + 1)$.

Proof. Since $\Omega_{K,l}$ and $X_{K,l}$ are topologically conjugate by Proposition 2.5, their entropies coincide. Clearly, $h(X_{K,l}) > 0$, since $X_{K,l}$ contains the ‘diamond’ consisting of the paths $e \rightarrow e \rightarrow e$ and $e \rightarrow \{0, 1, \ldots, l - 2, l\} \rightarrow e$ (cf. (2.17)), and $h(X_{K,l}) \leq \log (l + 1)$ since every state $a \in B_{K,l}$ has at most $l + 1$ predecessors (cf. (2.16)). □

Proposition 2.12 shows that $h(\Omega_{K,l}) \leq h(\Omega_{K+1,l})$ and $h(\Omega_{K,l}) \leq h(\Omega_{K+1,l+1})$ whenever $K > l > 0$. Our next aim is to investigate $\lim_{K \rightarrow \infty} h(\Omega_{K,l})$ for every $l \geq 1$.

Example 3.2 (The SFT $\Omega_{K,1}$). For every $K \geq 1$, $h(\Omega_{K,1})$ is equal to $\log \beta_K$, where $\beta_K$ is the largest root of the polynomial $f_K(x) = x^K - x^{K-1} - \cdots - 1$. Hence $\lim_{K \rightarrow \infty} h(\Omega_{K,1}) = \log 2$.

Indeed, the SFT $X_{K,1}$ (which is conjugate to $\Omega_{K,1}$) has the form
and is described by the \((K \times K)\)-transition matrix

\[
P = \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 & 1 & 1 \\
0 & 1 & 0 & \cdots & 0 & 1 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 & 1 \\
0 & 0 & 0 & \cdots & 1 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0
\end{pmatrix}
\]

with characteristic polynomial \(f_K\). The largest root \(\beta_K\) of \(f_K\) satisfies that \(1 = \beta_K^{-1} + \cdots + \beta_K^{-K}\). As \(K \to \infty\), \(\beta_K \to 2\) and

\[
h(\Omega_{K,1}) = h(X_{K,1}) = \log \beta_K \to \log 2\ 
\]

as \(K \to \infty\).

The following theorem yields an analogous result for arbitrary \(l\).

**Theorem 3.3.** For every \(K,l\) with \(0 < l < K\), \((1 - \frac{l}{K}) \log (l + 1) \leq h(\Omega_{K,l}) \leq \log (l + 1)\). In particular, \(\lim_{K \to \infty} h(\Omega_{K,l}) = \log (l + 1)\) for every \(l \geq 1\).

**Proof.** Since \(\Omega_{K,l}\) is topologically conjugate to \(X_{K,l}\) by Proposition 2.5, it will suffice to prove the corresponding assertion for the SFT \(X_{K,l}\). From Proposition 2.7 we know that \(h(X_{K,l}) \leq \log (l + 1)\).

Fix \(K,l\) and consider the state \(e = \{0, \ldots, l - 1\}\) in (2.17). We are interested in the number of paths of length \(K\) in \(X_{K,l}\) which begin and end in \(e\). For this it will be convenient to work from right to left, starting from \(e\): by (2.16), \(e\) has \(l + 1\) predecessors \(a(i)_{-1}\), \(i = 1, \ldots, l + 1\), say, each of which has a maximal element \(\leq l\). If \(l < K - 1\) we can repeat this argument and obtain, for each \(a(i)_{-1}\), \(l + 1\) predecessors. This second generation of predecessors has maximal elements which are all \(\leq l + 1\). After repeating this \(K - l\) times we have found a total of \((l + 1)^{K-l}\) distinct allowed paths of length \(K - l + 1\) in \(X_{K,l}\), all of which have \(e\) as their final state. If \(v\) is such a path with initial state \(b\), say, we can extend this path to the left by choosing \(l\) successive predecessors of \(b\) until we arrive at \(e\) (as explained in the proof of Proposition 2.7).

This construction results in \((l+1)^{K-l}\) distinct allowed paths of length \(K + 1\) in \(X_{K,l}\), all of which begin and end in \(e\). Since we can concatenate these paths arbitrarily (overlapping in the symbol \(e\)), we have proved that \(h(X_{K,l}) \geq \frac{1}{K} \log ((l + 1)^{K-l}) = \frac{K-l}{K} \log (l + 1)\), as claimed. The last assertion is a trivial consequence of this.

**Remark 3.4.** According to Proposition 2.12 (3), \(h(\Omega_{K,l}) = h(\Omega_{K,K-l})\) for every \(K \geq 1\) and \(l = 0, \ldots, K\) (where \(h(\Omega_{K,l}) = 0\) if \(l = 0\) or \(l = K\)). This allows us to
symmetrize the first inequality in Theorem 3.3 and to conclude that
\[
\frac{1}{K} \cdot \max \left( (K - l) \cdot \log (l + 1), l \cdot \log (K - l) \right) \leq h(\Omega_{K,l}) \leq \log (l + 1)
\]
for every \( K \geq 1 \) and \( l = 0, \ldots, K \).

For reasons of symmetry one would also expect that \( h(\Omega_{K,l}) \) is maximal if \( l \) lies in the middle of the range \( \{0, \ldots, K\} \), i.e., if \( \frac{K - l}{2} \leq \frac{1}{2} \). Our next proposition shows that this is indeed the case.

**Proposition 3.5.** For every \( K, l \) with \( K \geq 2 \) and \( 1 \leq l \leq \frac{K}{2} \),
\[
h(\Omega_{K,l-1}) \leq h(\Omega_{K,l}).
\] (3.1)

For the proof of Proposition 3.5 we need additional notation. For every finite set \( u \subset \mathbb{Z} \) and every \( j \in \mathbb{Z} \) we set \( u + j = \{i + j : i \in u\} \).

Fix \( K, l \) with \( K \geq 2 \) and \( 0 \leq l \leq K \). For any pair \( u, v \) of disjoint (and possibly empty) subsets of \( \mathbb{Z} \) we set
\[
B_{K,l}^{[u,v]} = \begin{cases} \{a \in B_{K,l} : u \subset a \text{ and } a \cap v = \emptyset\} & \text{if } (u \cup v) \subset A_{K-1}, \\ \emptyset & \text{otherwise}, \end{cases}
\] (3.2)
and
\[
X_{K,l}^{[u,v]} = X_{K,l} \cap (B_{K,l}^{[u,v]})^\mathbb{Z}.
\] (3.3)

Both \( B_{K,l}^{[u,v]} \) and \( X_{K,l}^{[u,v]} \) are empty whenever \( |u| > l \), and \( B_{K,l}^{[\omega,\varnothing]} = B_{K,l} \) and \( X_{K,l}^{[\omega,\varnothing]} = X_{K,l} \). If \( |u| = 1 \) with \( u = \{j\} \) for some \( j \in A_{K-1} \) we write \( B_{K,l}^{[\omega,j]} \) and \( X_{K,l}^{[\omega,j]} \) instead of \( B_{K,l}^{[\omega,j]} \) and \( X_{K,l}^{[\omega,j]} \). The case where \( |v| = 1 \) will be treated similarly.

We set
\[
\Omega_{K,l}^{(0)} = \{\omega = (\omega_n) \in \Omega_{K,l} : \omega_n \neq 0 \text{ for every } n \in \mathbb{Z}\},
\] (3.4)
\[
\Omega_{K,l}^{(K)} = \{\omega = (\omega_n) \in \Omega_{K,l} : \omega_n \neq K \text{ for every } n \in \mathbb{Z}\}.
\]

If \( \phi = \phi_{K,l} : \Omega_{K,l} \rightarrow X_{K,l} \) is the shift-equivariant isomorphism defined in (2.10), then
\[
\phi(\Omega_{K,l}^{(0)}) = X_{K,l}^{[0,\varnothing]} \quad \text{and} \quad \phi(\Omega_{K,l}^{(K)}) = X_{K,l}^{[\omega,K]}.
\]

The reason for our interest in these subshifts is that
\[
\Omega_{K,l}^{(0)} \simeq \Omega_{K-1,l-1} \quad \text{and} \quad \Omega_{K,l}^{(K)} \simeq \Omega_{K-1,l},
\]
and hence
\[
X_{K,l}^{[0,\varnothing]} \simeq X_{K-1,l-1} \quad \text{and} \quad X_{K,l}^{[\omega,K-1]} \simeq X_{K-1,l}.
\]

This shows that (3.1) is equivalent to the assertion that
\[
h(X_{K+1,l}^{[0,\varnothing]}) \leq h(X_{K+1,l}^{[\omega,K]})
\] (3.5)
for every \( K \geq 1 \) and \( l \leq \frac{K+1}{2} \).

We set \( Y = X_{K+1,l}^{[0,\varnothing]} \), \( Z = X_{K+1,l}^{[\omega,K]} \), and write \( Y_N \) and \( Z_N \) for the set of all paths of length \( N \) in \( Y \) and \( Z \), respectively. Since \( h(Y) = \lim_{N \rightarrow \infty} \frac{1}{N} \log |Y_N| \)
and $h(Z) = \lim_{N \to \infty} \frac{1}{N} \log |Z_N|$ we have to investigate the growth rates of the cardinalities $|Y_N|$ and $|Z_N|$ as $N \to \infty$.

We start with $Y_N$ and write, for all disjoint finite sets $u, v \subset \mathbb{Z}$, $Y_N^{[uv]} \subset Y_N$ for the set of all paths of length $N$ ending in an element of $B_{K+1,l}^{[u,v]}$ (note that this set will be empty if $(u \cup v) \not\subset \{1, \ldots, K\}$ or $|u| \geq l$).

**Lemma 3.6.** For every $N \geq 1$ and every $u \subset \{1, \ldots, K\}$,

$$|Y_N^{[u,\varnothing]}| = (K - l - |u| + 1) \cdot |Y_N^{[(1) \cup (u+1), \varnothing]}| + \sum_{j \in \{(1) \cup (u+1)\}} |Y_N^{[(1) \cup (u+1) \setminus \{j\}, \varnothing]}|.$$ 

(3.6)

**Proof.** If $|u| > l$, $B_{K+1,l}^{[0]u,\varnothing} = \emptyset$ and hence $Y_N^{[u,\varnothing]} = \emptyset$ for every $N \geq 0$. If $|u| < l$, every $y \in Y_N^{[u,\varnothing]}$ has the form $y = (a_0, a_1, \ldots, a_{N-1}, a_N)$ with $a_i \in B_{K,l}^{[0,\varnothing]}$ for $i = 0, \ldots, N$, $a_N \supset u$, and $a_{N-1} \not\subset p(a_N)$. Since both $a_{N-1}$ and $a_N$ contain 0, Equation (2.15) implies that one of the following conditions is satisfied:

(i) $\{1\} \cup (u + 1) \subset a_{N-1}$, in which case $|f(a_{N-1})| = K - l + 2$ and every successor of $a_{N-1}$ (including $a_N$, of course) is of the form $((a_{N-1} \setminus \{0\}) - 1) \cup \{j\}$ for some $j \in \{0, \ldots, K\} \setminus (a_{N-1} - 1)$;

(ii) $\{1\} \cup (u + 1) \not\subset a_{N-1}$, but $a_{N-1} \supset \{(1) \cup (u + 1)\} \setminus \{j\}$ for some $j \in \{1\} \cup (u + 1)$. In this case $|f(a_{N-1})| = 1$ and $f(a_{N-1}) = \{a_N\}$.

If $K \not\subset u$, we obtain that

$$|Y_N^{[u,\varnothing]}| = (K - l + 2) \cdot |Y_N^{[(1) \cup (u+1), \varnothing]}| + \sum_{j \in \{(1) \cup (u+1)\}} |Y_N^{[(1) \cup (u+1) \setminus \{j\}, \varnothing]}|$$

$$= (K - l - |u| + 1) \cdot |Y_N^{[(1) \cup (u+1), \varnothing]}| + \sum_{j \in \{(1) \cup (u+1)\}} |Y_N^{[(1) \cup (u+1) \setminus \{j\}, \varnothing]}|,$$

where we have used that $|Y_N^{[w \cup \{j\}, \varnothing]}| + |Y_N^{[w, \varnothing]}| = |Y_N^{[w, j]}|$ whenever $\{j\} \cup w \subset \{1, \ldots, K\}$ and $j \not\in w$.

If $K \subset u$, then

$$|Y_N^{[u,\varnothing]}| = (K - l + 2) \cdot |Y_N^{[(1) \cup (u+1), \varnothing]}| + \sum_{j \in \{(1) \cup (u+1), j \leq K\}} |Y_N^{[(1) \cup (u+1) \setminus \{j\}, j]}|$$

$$+ |Y_N^{[(1) \cup (u+1) \setminus \{K+1\}, \varnothing]}| = |Y_N^{[(1) \cup (u+1) \setminus \{K+1\}, \varnothing]}|,$$

since (3.2) – (3.3) guarantee that all other expressions in the middle term of this equation vanish. In either case (3.6) is satisfied, so that the lemma is proved. ☐

In order to prove an analogous recursion formula for $Z = X_{K+1,l}^{[\varnothing,K]}$ we denote by $Z_N^{[\varnothing]} \subset Z_N$ the set of all paths of length $N$ which *begin* with an element of $B_{K+1,l}^{[\varnothing,K]}$ for some finite set $v \subset \mathbb{Z}$. Note that $B_{K+1,l}^{[\varnothing,K]} = \emptyset$ whenever $|v| > l$ or $v \not\subset \{0, \ldots, K - 1\}$.

**Lemma 3.7.** For every $N \geq 1$ and every $v \subset \{0, \ldots, K - 1\}$ with $|v| \leq l$,

$$|Z_N^{[\varnothing]}| = (l - |v|) \cdot |Z_N^{[\varnothing, \{K-1\} \cup (v-1)]}|$$

$$+ \sum_{j \in \{(K-1) \cup (v-1)\}} |Z_N^{[\varnothing, \{K-1\} \cup (v-1) \setminus \{j\}]|,$$

(3.7)
If $0 \in v$, (3.7) reduces to
\[
|Z_N^{[\varphi, v]}| = |Z_N^{[\varphi, (K-1) \cup (v'-1)]}|
\]
where $v' = v \setminus \{0\}$.

**Proof.** For every $a \in \mathcal{B}_{K,l}$ we set $\bar{a} = \{ K - a : a \in a \}$. We recall the definition of the isomorphism $\Psi_{K+1,l} : X_{K+1,l} \rightarrow X_{K+1,K+1-l}$ in (2.25) and note that $\Psi_{K+1,l}(Z_N^{[\varphi, v]}) = Y_N^{[\varphi, \bar{v}]}$, where $\bar{Y} = X_{K+1,K+1-l}^{[0, \bar{v}]}$. In particular, $|Z_M^{[\varphi, v]}| = |\bar{Y}_M^{[\varphi, \bar{v}]}|$ for every $M \geq 1$.

Assume for the moment that $0 \notin v$. From Lemma 3.6 it follows that
\[
|Z_N^{[\varphi, v]}| = |Y_N^{[\varphi, \bar{v}]}| = (l - |v|) \cdot |Y_N^{\{(1) \cup (v+1), \bar{v}\}}| + \sum_{j \in \{(1) \cup (v+1)\}} |\bar{v}^{\{(1) \cup (v+1)\} \setminus \{j\}}|
\]
\[
= (l - |v|) \cdot |Z_N^{[\varphi, (1) \cup (v+1)]}| + \sum_{j \in \{(1) \cup (v+1)\}} |Z_N^{[\varphi, (1) \cup (v+1) \setminus \{j\}}|
\]
\[
= (l - |v|) \cdot |Z_N^{[\varphi, (K-1) \cup (v'-1)]}| + \sum_{j \in \{(K-1) \cup (v'-1)\}} |Z_N^{[\varphi, (K-1) \cup (v'-1) \setminus \{j\}}|
\]
where we have used the facts that $\bar{v} = v$, $\bar{v} + 1 = K - (K - v + 1) = v - 1$, $\bar{u} \cup \bar{v} = \bar{u} \cup \bar{v}$ and $|v| = |\bar{v}|$.

If $0 \notin v$, then $K \in \bar{v}$, and (3.2) – (3.3) imply that
\[
|Z_N^{[\varphi, v]}| = |Y_N^{[\varphi, \bar{v}]}| = |Y_N^{\{(1) \cup (v+1)\} \setminus \{K+1\}}| = |Z_N^{[\varphi, (1) \cup (v+1) \setminus \{K+1\}}|
\]
\[
= |Z_N^{[\varphi, (1) \cup (v+1) \setminus \{K+1\}}| = |Z_N^{[\varphi, (K-1) \cup (v'-1)\}}|
\]
This proves (3.7). \hfill \square

Finally we investigate the relation between $Y_N^{[\varphi, v]}$ and $Z_N^{[\varphi, u]}$. We prove the following statement by induction on $N$.

**Lemma 3.8.** For every $N \geq 1, 1 \leq l \leq K$, $0 \leq m \leq l$ and every $u \subset \{1, \ldots, K\}$ with $|u| = m$,
\[
[K - l + 1]_m \cdot |Y_N^{[u, \varphi]}| \leq [l]_m \cdot |Z_N^{[\varphi, u]}|\tag{3.8}
\]
where
\[
[x]_m = \begin{cases} x \cdot (x - 1) \cdots (x - m + 1) & \text{if } m \geq 1, \\ 1 & \text{if } m = 0. \end{cases}
\]

**Proof.** Let $N = 1$. Then $|Y_1^{[u, \varphi]}| = \binom{K-m}{l-m-1}$, since we are choosing $l - 1 - |u|$ elements in the set $\{1, \ldots, K\} \setminus u$. Similarly we see that $|Z_1^{[\varphi, u]}| = \binom{K-m}{l-m}$. Then
\[
[K - l + 1]_m \cdot |Y_1^{[u, \varphi]}| = [K - l + 1]_m \cdot \binom{K-m}{l-m-1} = \frac{(K-m)!}{(l-m)!((K-m-l)!} \leq \frac{(K-m)!}{(l-m)!((K-m-l)!} = [l]_m \cdot \binom{K-m}{l} = [l]_m \cdot |Z_1^{[\varphi, u]}|,\tag{3.9}
\]

where we have used the assumption $l \leq K - l$. By using (3.9) as our induction hypothesis, applying the Lemmas 3.6 and 3.7, and remembering that $|u| = |\bar{u}| = m$, we get that

\[ [K - l + 1]_m \cdot |Y[^{[u,\varnothing]}_{N+1}]| = [K - l + 1]_{m+1} \cdot |Y[^{[1]}_{N}]| \]

\[ + \sum_{j \in \{1\} \cup \{u+1\}} [K - l + 1]_m \cdot |Y[^{[1]}_{N}]| \]

\[ \leq [l]_{m+1} \cdot |Z[^{[\varnothing]{[1]}_{N}}]| \]

\[ + \sum_{j \in \{1\} \cup \{u+1\}} [l]_m \cdot |Z[^{[\varnothing]{[1]}_{N}}]| \]

\[ = [l]_m \{(l - |\bar{u}|) \cdot |Z[^{[\varnothing]{(K-1)}_{N}}]| \}

\[ + \sum_{j \in \{(K-1) \cup \{u-1\}\}} |Z[^{[\varnothing]{(K-1)}_{N}}]| \}

\[ = [l]_m \cdot |Z[^{[\varnothing]}_{N+1}]| \]. \]

**Proof of Proposition 3.5.** By taking $m = 0$ (and hence $u = \varnothing$) in Lemma 3.8 we obtain that $|Y_N| = |Y[^{[\varnothing]}_{N}]| \leq |Z[^{[\varnothing]}_{N}]| = |Z_N|$ for every $N \geq 2$. As noted in the penultimate paragraph before Lemma 3.6 this guarantees that

\[ h(X[^{[0,\varnothing]}_{K+1,l}]) = h(Y) = \lim_{N \to \infty} \frac{1}{N} \log |Y_N| \leq \lim_{N \to \infty} \frac{1}{N} \log |Z_N| = h(Z) = h(X[^{[\varnothing,K]}_{K+1,l}]) \]

for every $K \geq 1$ and $l \leq \frac{K+1}{2}$. We have proved (3.5) or, equivalently, (3.1). \[\Box\]

4. The parity cocycle

If $\omega = (\omega_k) \in \Omega_K$ is a periodic point with period $p$, say, then

\[ \pi^{(\omega)}_{(0,p)} := (\bar{\omega}_0 \pmod{p}, \ldots, \bar{\omega}_{0+p-1} \pmod{p}) \]

is a permutation of $(0, \ldots, p-1)$ (cf. Lemma 2.1 (3)). What is the parity (or sign) of this permutation? In this section we prove that these parities are determined by the function $a: \Omega_K \to \mathbb{Z}$ in (2.3) and a continuous cocycle $s: \mathbb{Z} \times \Omega_K \to \{\pm 1\}$ for the shift $\sigma$ on $\Omega_K$.

**Theorem 4.1.** Let $K \geq 1$, define $c: \Omega_K \to \mathbb{Z}$ by

\[ c(\omega) = |\{k < 0 : \bar{\omega}_k > \bar{\omega}_0\}| = |\{k = -K + 1, \ldots, -1 : \bar{\omega}_k > \bar{\omega}_0\}|, \]

and let $c: \mathbb{Z} \times \Omega_K \to \mathbb{Z}$ be given by

\[ c(n, \omega) = \begin{cases} 
\sum_{k=0}^{n-1} c(\sigma^k \omega) & \text{if } n > 0, \\
0 & \text{if } n = 0, \\
-c(-n, \sigma^n \omega) & \text{if } n < 0.
\end{cases} \]

Consider the multiplicative group $C_2 := \{\pm 1\} \subset \mathbb{R}$ and define $s: \Omega_K \to C_2$ and $s: \mathbb{Z} \times \Omega_K \to C_2$ by setting $s(\omega) = (-1)^{c(\omega)+a(\omega)}$ and

\[ s(n, \omega) = (-1)^{c(n, \omega)+na(\omega)} = \prod_{k=0}^{n-1} s(\sigma^k \omega). \]

Then $s$ satisfies the cocycle equation

\[ s(m + n, \omega) = s(m, \sigma^n \omega)s(n, \omega) \]
for every \( m, n \in \mathbb{Z} \) and \( \omega \in \Omega_K \). Furthermore, if \( \omega \in \Omega_K \) is periodic with period \( p \), then the parity of the permutation \( \pi_{(0,p)}^{(\omega)} \) defined in Lemma 2.1 (3) is given by

\[
\text{sgn} \, \pi_{(0,p)}^{(\omega)} = (-1)^{a(\omega)} s(p, \omega).
\] (4.5)

**Proof.** The only statement requiring verification is (4.5). We fix \( \omega \in \Omega_K \), \( p > K \geq 2 \) and use the conventions of Figure 1. We call a pair \( (a, b) \in (a_1, a_2), b = (b_1, b_2) \in \mathbb{Z}^2 \) an inversion if \( a_1 < b_1 \) and \( a_2 > b_2 \) (i.e., if \( b \) lies above and to the right of \( a \) in Figure 1). Let

\[
\mathcal{I}(p) = \{ (a, b) \in (\tilde{Q} \cup \tilde{A}') \times S(\omega) : (a, b) \text{ is an inversion} \}.
\] (4.6)

According to (4.2), \( c(\omega) = |\{ b \in S(\omega) : ((\omega_0, 0), b) \text{ is an inversion} \}|. \) Hence

\[
c(p, \omega) = |\mathcal{I}(p)|.
\]

Now suppose that \( \omega \) is periodic with period \( p > K \). Then \( S(\omega) \) is invariant under translation by \( (p, p) \) (which in Figure 1 moves every point \( p \) steps down and to the right), and \( \tilde{A}' = \tilde{A} + (p, p) \).

We populate \( \tilde{A}' \) by translating all points in \( \tilde{A}' \) into \( \tilde{A}^* \) by adding \( (-p, 0) \) (or, equivalently, by adding \( (0, p) \) to all points in \( \tilde{A} \)), leaving the points in \( \tilde{Q} \) unchanged: for \( a \in \tilde{A} \cup \tilde{Q} \cup \tilde{A}' \) we put

\[
a^* = \begin{cases} 
a & \text{if } a \in \tilde{Q}, \\
a - (p, 0) & \text{if } a \in \tilde{A}', \\
a + (0, p) & \text{if } a \in \tilde{A},
\end{cases}
\]

and we set \( \tilde{A}^* = \{ a^* : a \in \tilde{A} \} = \{ a^* : a \in \tilde{A}' \} \). Under our assumption that \( \omega \) has period \( p \), the set \( \tilde{S} : = \tilde{Q} \cup \tilde{A}^* \) has exactly one element in each row and column of the square \( Q \). Hence \( \tilde{S} \) determines a permutation of \{0, \ldots, p - 1\} which coincides with \( \pi_{(0,p)}^{(\omega)} \). The parity of this permutation is the parity of the number of inversions occurring in \( \pi_{(0,p)}^{(\omega)} \); if

\[
\mathcal{J}(p) = \{ (a, b) \in \tilde{S} \times \tilde{S} : (a, b) \text{ is an inversion} \},
\] (4.7)

then

\[
\text{sgn} \, \pi_{(0,p)}^{(\omega)} = (-1)^{|\mathcal{J}(p)|}.
\] (4.8)

When comparing this with

\[
s(p, \omega) = (-1)^{|\mathcal{J}(p)| + p a(\omega)},
\] (4.9)

we have to consider several cases:

(a) If \( (a, b) \in \tilde{Q} \) or \( (a, b) \in \tilde{A}' \), then \( (a, b) \in \mathcal{J}(p) \) if and only if \( (a^*, b^*) \in \mathcal{J}(p) \).

(b) If \( a = (a_1, a_2) \in \tilde{B} \) and \( b = (b_1, b_2) \in \tilde{A} \), then \( (a, b) \in \mathcal{J}(p) \) if and only if \( a_1 < b_1 \), in which case \( (b^*, a^*) \notin \mathcal{J}(p) \). If \( a_1 > b_1 \), then \( (a, b) \notin \mathcal{J}(p) \), but \( (b^*, a^*) \in \mathcal{J}(p) \). Since \( |\tilde{B} \times \tilde{A}| = a(\omega)(K - a(\omega)) \), we conclude that

\[
|\mathcal{J}(p) \cap (\tilde{B} \times \tilde{A})| + |\mathcal{J}(p) \cap (\tilde{A}^* \times \tilde{B})| = a(\omega)(K - a(\omega)).
\]
(c) \((\tilde{A}' \times \tilde{C}) \cap \mathcal{J}(p) = (\tilde{C} \times \tilde{A}') \cap \mathcal{J}(p) = \emptyset, \) but \((\tilde{A}^* \times \tilde{C}) \subset \mathcal{J}(p)\). Note that \(|\tilde{A}' \times \tilde{C}| = |\tilde{C} \times \tilde{A}'| = |\tilde{A}^* \times \tilde{C}| = a(\omega)(p - 2K + a(\omega))\).

(d) If \(a = (a_1, a_2) \in \tilde{D} \) and \(b = (b_1, b_2) \in \tilde{A}'\), then \((a, b) \in \mathcal{J}(p)\) if and only if \(a_2 < b_2\), in which case \((b^*, a^*) \notin \mathcal{J}(p)\). If \(a_2 > b_2\), then \((a, b) \notin \mathcal{J}(p)\), but \((b^*, a^*) \in \mathcal{J}(p)\). Since \(|\tilde{A}' \times \tilde{D}| = a(\omega)(K - a(\omega))\), we conclude that \(|\mathcal{J}(p) \cap (\tilde{D} \times \tilde{A}')| + |\mathcal{J}(p) \cap (\tilde{A}^* \times \tilde{D})| = a(\omega)(K - a(\omega))\).

Clearly,

\[
|\mathcal{J}(p)| = |\mathcal{J}(p) \cap (\tilde{Q} \times \tilde{Q})| + |\mathcal{J}(p) \cap (\tilde{A}' \times \tilde{A}')| + |\mathcal{J}(p) \cap (\tilde{A}^* \times \tilde{Q})| + |\mathcal{J}(p) \cap (\tilde{Q} \times \tilde{A}')| + |\mathcal{J}(p) \cap (\tilde{Q} \times \tilde{A}^*)| + |\mathcal{J}(p) \cap (\tilde{Q} \times \tilde{D})| + |\tilde{A}^* \times \tilde{C}|
\]

By combining the cases (a) – (d) listed above and remembering that \(|\tilde{C}| = p - 2K + a(\omega)\) we obtain that

\[
|\mathcal{J}(p)| = |\mathcal{J}(p) \cap (\tilde{Q} \times \tilde{Q})| + |\mathcal{J}(p) \cap (\tilde{A}' \times \tilde{A}')| - |\mathcal{J}(p) \cap (\tilde{B} \times \tilde{A})| - |\mathcal{J}(p) \cap (\tilde{D} \times \tilde{A}')| + 2a(\omega)(K - a(\omega)) + a(\omega)(p - 2K + a(\omega)).
\]

Hence

\[
|\mathcal{J}(p)| = |\mathcal{J}(p)| + a(\omega)(p + a(\omega)) = |\mathcal{J}(p)| + a(\omega)p + a(\omega) \pmod{2}.
\]

If we recall (4.8) – (4.9) we obtain (4.5). \(\Box\)

**Examples 4.2.** (1) Let \(K = 2\) (cf. Example 2.2). The map \(c: \Omega_2 \to \mathbb{Z}\) in (4.1) is given by

\[
c(\omega) = \begin{cases} 
1 & \text{if } \omega_0 = 0 \text{ and } \omega_{-1} = 2, \\
0 & \text{otherwise},
\end{cases}
\]

and

\[
c(n, \omega) = \{|[\{k : 0 \leq k < n : \omega_k = 0 \text{ and } \omega_{k-1} = 2\}].
\]

(2) More generally, if \(K \geq 1\) and \(\omega \in \Omega_{K,1}\), then

\[
c(\omega) = \begin{cases} 
1 & \text{if } \omega_0 = 0, \\
0 & \text{otherwise},
\end{cases}
\]

and

\[
c(n, \omega) = \{|[\{k : 0 \leq k < n : \omega_k = 0\}].
\]

(3) For \(\omega \in \Omega_{K,l}\) with \(1 \leq l \leq K - 1\), \(c(\omega)\) can be calculated by using the isomorphism \(\phi_{K,l}: \Omega_{K,l} \to X_{K,l}\) in (2.10): \(c(\omega) = |\{a \in \phi_{K,l}(\omega) : a > \omega_0\}|\), where \(\omega_0\) is determined by (2.11).
As one would expect, the parity cocycle $s: \mathbb{Z} \times \Omega_K \to C_2$ in (4.3) is nontrivial in the sense that its group of essential values is equal to $C_2$. More precisely, the following is true.

**Proposition 4.3.** Let $K \geq 2$, $1 \leq l \leq K - 1$, and let $\mu_{K,l}$ be the unique shift-invariant probability measure with maximal entropy on $\Omega_{K,l}$. Then the skew-product transformation $\tilde{\sigma}: \Omega_{K,l} \times C_2 \to \Omega_{K,l} \times C_2$, defined by
\[ \tilde{\sigma}(\omega, j) = (\sigma \omega, s(\omega)j) \]
for every $(\omega, j) \in \Omega_{K,l} \times C_2$, is ergodic with respect to the product measure $\tilde{\mu}_{K,l} = \mu_{K,l} \times \lambda_{C_2}$, where $\lambda_{C_2}(\{1\}) = \lambda_{C_2}(\{-1\}) = \frac{1}{2}$.

**Proof.** According to [12, Corollary 5.4], we have to show the following: for every Borel set $B \subset \Omega_{K,l}$ with $\mu_{K,l}(B) > 0$ and every $j \in C_2$,
\[ B \cap \sigma^{-m} B \cap \{\omega \in \Omega_{K,l} : s(m, \omega) = j\} \neq \emptyset \quad (4.10) \]
for some $m > 0$.

Consider the cylinder sets
\[ E = \{\omega \in \Omega_{K,l} : \omega_i = l \text{ for } i = 0, \ldots, K + 1\}, \]
\[ F = \{\omega \in \Omega_{K,l} : \omega_i = l \text{ for } i = 0, \ldots, K - 2, K + 1, \omega_{K-1} = l + 1, \omega_K = l - 1\}. \]
Since $\Omega_{K,l}$ is irreducible and aperiodic, $\beta := \mu_{K,l}(D) > 0$, where $D = E \cup F$. We define a homeomorphism $V: \Omega_{K,l} \to \Omega_{K,l}$ by setting $V \omega = \omega$ if $\omega \not\in D$, and
\[
(V\omega)_i = \begin{cases} 
  l & \text{if } \omega \in E \text{ and } i = 0, \ldots, K - 2, K + 1, \\
  l + 1 & \text{if } \omega \in E \text{ and } i = K - 1, \\
  l - 1 & \text{if } \omega \in E \text{ and } i = K, \\
  l & \text{if } \omega \in F \text{ and } i = 0, \ldots, K + 1.
\end{cases}
\]
Then $V^2 = \text{Id}_{\Omega_{K,l}}$ and $VE = F$.

We fix $B \subset \Omega_{K,l}$ with $\mu_{K,l}(B) > 0$. By approximating $B$ with closed and open subsets of $\Omega_{K,l}$ we see that $\lim_{m \to \infty} \mu_{K,l}(B \cap \sigma^{-m} V \sigma^m B) = \mu_{K,l}(B)$ and $\lim_{m \to \infty} \mu_{K,l}(\sigma^{-2m} B \cap \sigma^{-m} V \sigma^m B) = \mu_{K,l}(B)$. Furthermore, since $\mu_{K,l}$ is mixing of every order, $\lim_{m \to \infty} \mu_{K,l}(B \cap \sigma^{-m} D \cap \sigma^{-2m} B) = \beta \mu_{K,l}(B)^2$. We conclude that
\[
\beta \mu_{K,l}(B)^2 = \lim_{m \to \infty} \mu_{K,l}(B \cap \sigma^{-m} D \cap \sigma^{-2m} B) \\
= \lim_{m \to \infty} \mu_{K,l}(B \cap \sigma^{-m} V \sigma^m B \cap \sigma^{-m} D \cap \sigma^{-2m} B \cap \sigma^{-m} V \sigma^{-m} B). 
\]

Let $m > K$ be sufficiently large so that the set $E = B \cap \sigma^{-m} V \sigma^m B \cap \sigma^{-m} D \cap \sigma^{-2m} B \cap \sigma^{-m} V \sigma^{-m} B$ is nonempty. For every $\omega \in E$ the following conditions are satisfied:
\[
\begin{align*}
  \omega &\in B, & \sigma^{2m} \omega &\in B, & \sigma^m \omega &\in D, \\
  \omega' &\in B, & \sigma^{2m} \omega' &\in B, & \sigma^m \omega' &\in D,
\end{align*}
\]
where $\omega' = \sigma^{-m}V\sigma^m\omega$. A glance at the definition of the cocycle $s$ in (4.3) shows that, for every $\omega \in E$ and $k \in \mathbb{Z}$,

$$s(\sigma^k\omega') = \begin{cases} 
s(\sigma^k\omega) & \text{if } k \neq m + K, \\
s(-\sigma^k\omega) & \text{if } k = m + K. \end{cases}$$

In particular, $\{\omega, \omega'\} \subset B \cap \sigma^{-2m}B$ and $s(2m, \omega) = -s(2m, \omega')$. This proves (4.10).

**Example 4.4** (Generalized circulants). The study of the space $\Omega_K$, its irreducible components $\Omega_{K,l}$, and the permutations corresponding to periodic elements of $\Omega_K$ was partly motivated by expressions occurring in the calculation of entropy of (expansive) algebraic actions of the discrete Heisenberg group (cf. [7, Section 8]).

Let $\phi_0, \ldots, \phi_K$ be continuous complex-valued functions on $\mathbb{T}$. We are interested in bi-infinite ‘generalized circulants’ of the form

$$A_{t,\alpha} = \begin{pmatrix} \cdots & \phi_{t+\alpha} & \cdots & \phi_{t+K\alpha} & \cdots & \cdots & \cdots & \cdots \\ \vdots & \phi_{t+\alpha} & \cdots & \phi_{t+K\alpha} & \cdots & \cdots & \cdots & \cdots \\ \vdots & 0 & \phi_{t\alpha} & \phi_{t+\alpha} & \cdots & \cdots & \cdots & \cdots \\ \vdots & 0 & 0 & \phi_{t\alpha} & \phi_{t+\alpha} & \cdots & \cdots & \cdots \\ \vdots & 0 & 0 & 0 & \phi_{t\alpha} & \phi_{t+\alpha} & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}$$

(4.11)

for $(t, \alpha) \in \mathbb{T}^2$. The matrix $A_{t,\alpha}$ acts by left multiplication on the space $\ell^\infty(\mathbb{Z}, \mathbb{C})^\top$ of bounded column vectors with complex entries. If $\alpha$ is rational with $\alpha = p/q$ in lowest terms, say, then $A_{t,p/q}$ acts by left multiplication on the set of elements in $\ell^\infty(\mathbb{Z}, \mathbb{C})^\top$ with period $q$, which we identify with $\mathbb{C}^q \simeq \ell^\infty(\mathbb{Z}/q\mathbb{Z}, \mathbb{C})$. The determinant of this linear transformation of $\mathbb{C}^q$ can be expressed in terms of $SFT\Omega_K$, using the parity cocycle (4.3) – (4.4): if $P_q(\Omega_K)$ and $P_{q}(\Omega_{K,l})$ are the sets of points of period $q$ in $\Omega_K$ and $\Omega_{K,l}$, then

$$\det A_{t,p/q} = \sum_{\omega \in P_q(\Omega_K)} (-1)^{q(\omega)} s(q, \omega) \prod_{j=0}^{q-1} \phi_{\omega_j}(t + jp/q) = \sum_{l=0}^{K} (-1)^{t} \cdot \sum_{\omega \in P_q(\Omega_{K,l})} \prod_{j=0}^{q-1} s(\sigma^j\omega)\phi_{\omega_j}(t + jp/q).$$

(4.12)

As explained in [7, Section 8] one should normalize $\det A_{t,p/q}$ by setting

$$D(A_{t,p/q}) = |\det(A_{t,p/q})|^{1/q}.$$  

(4.13)

In the context of algebraic actions of the discrete Heisenberg group the functions $\phi_i$, $i = 0, \ldots, K$, are trigonometric polynomials arising from the element $f$ in the integer group ring $\mathbb{Z}\Gamma$ which defines the action, and the quantity $\int_{\mathbb{T}} \log D(A_{t,p/q})dt$ measures the contribution to the entropy of this action associated with a rational rotation number $\alpha = p/q$ representing the central generator $z = (\frac{1}{1}, \frac{1}{1})$ of $\Gamma$. If this action is expansive, the asymptotic behaviour (as $q \to \infty$) of the expressions $D(A_{t,p/q})$ in (4.13) determines the entropy of the algebraic action (cf. [4] and [7, Section 8]). For nonexpansive actions of this form one might still expect that
and lie in the distinct irreducible components \( \tau \).

Clearly, the permutations where we have separated the individual permutations \( \tau \) for every \( v \in \Omega \) and \( \Omega \) as at the beginning of Section 1. Then \( \Omega \) is a SFT by Lemma 2.1.

\[ \text{Proposition 5.1.} \]

(1) If \( |A| \geq 2 \), then \( \Omega \) is not irreducible and hence not mixing.

(2) The SFT \( \Omega \) is finite if \( |A| \leq 2 \), and has positive topological entropy if \( |A| \geq 3 \).

\[ \text{Proof.} \]

In view of Proposition 1.1 we assume that \( 0 \in A \subset \mathbb{Z} = \{0, 1, 2, \ldots \} \) and choose \( K \geq 1 \) so that \( A \subset A_K \) (cf. (1.2)). For the proof of (1) we note that, for every \( a \in A \), the fixed point \( a = (\ldots, a, a, a, \ldots) \) lies in \( \Omega_A \cap \Omega_{K,a} \). For \( a, b \in A \) with \( a < b \), the fixed points \( a \) and \( b \) lie in the distinct irreducible components \( \Omega_{K,a} \) and \( \Omega_{K,b} \) of \( \Omega_K \). The sets \( \Omega_A \cap \Omega_{K,a} \) and \( \Omega_A \cap \Omega_{K,b} \) are disjoint, nonempty, shift-invariant, open subsets of \( \Omega_A \), so that \( \Omega_A \) cannot be topologically mixing. This proves (1).

We turn to (2). If \( A = \{0\} \) then \( \Omega_A = \{0\} \). If \( A = \{0, a\} \) for some \( a \geq 1 \), then \( |\Omega_A| = |\Pi_A| = 2^a \): for every \( u = (u_0, \ldots, u_{a-1}) \in \{0, 1\}^a \) there exists a unique \( \pi(u) \in \Pi_A \) such that \( \pi(u)(i + ma) = i + (m + u_i)a \) for every \( i = 0, \ldots, a-1 \) and \( m \in \mathbb{Z} \). Furthermore, every \( \pi \in \Pi_A \) is of this form for some \( u \in \{0, 1\}^a \).

Finally we assume that \( A \supset \{0, a, b\} \) with \( 0 < a < b \). For every \( n \in \mathbb{Z} \) we consider the finite permutation of \( \mathbb{Z} \) defined by

\[ \tau_n = \left( \begin{array}{ccccccccc} n & n+1 & \ldots & n+a-1 & n+a & n+a+1 & \ldots & n+b-1 & n+b-a-1 \\ n+b-a & n+b-a+1 & \ldots & n+b-1 & n & n+1 & \ldots & n+b-a-1 & n \\ \end{array} \right), \]

Clearly, the permutations \( \tau_m \) and \( \tau_n \) commute if \( |m - n| \geq b \). This allows us to define, for every \( v = (v_n) \in \{0, 1\}^\mathbb{Z} \), a permutation \( \tau(v) \in S^\infty(\mathbb{Z}) \) by setting

\[ \tau(v) = \prod_{n \in \mathbb{Z}} \tau_n^{v_n}, \]

where \( \tau_m^0 \) is the identity permutation for every \( m \in \mathbb{Z} \). Note that \( \varsigma^a \circ \tau(v) \in \Pi_A \) for every \( v \in \{0, 1\}^\mathbb{Z} \) (cf. Proposition 1.1).

In order to understand what is going on here it may help to consider an element \( v \in \{0, 1\}^\mathbb{Z} \) of the form \( (\ldots, 1, 0, 1, \ldots) \), where the dot marks the zero-th coordinate of \( v \). Then \( \tau(v) \) is the permutation

\[ \left( \begin{array}{ccccccccc} \cdots & -b & \cdots & -b+a-1 & -b+a & \cdots & -1 & 0 & \cdots a-1 & a & \cdots & b-1 & b & \cdots & b+a-1 & b+a & \cdots & 2b-1 & \cdots \\ \cdots & -a & \cdots & -1 & 0 & \cdots a-1 & a & \cdots & b-1 & 2b-a & 2b-1 & 2b & \cdots & 2b-a-1 & \cdots & \cdots & \cdots \\ \end{array} \right), \]

where we have separated the individual permutations \( \cdots, \tau_{-b}, \tau_0, \tau_b^0, \ldots \) by vertical bars. The permutation \( \varsigma^a \circ \tau(v) \) is of the form

\[ \left( \begin{array}{ccccccccc} \cdots & -b & \cdots & -b+a-1 & -b+a & \cdots & -1 & 0 & \cdots a-1 & a & \cdots & b-1 & b & \cdots & b+a-1 & b+a & \cdots & 2b-1 & \cdots \\ \cdots & 0 & \cdots & a-1 & -b+a & \cdots & -1 & a & \cdots & 2a-1 & 2a & \cdots & a+b-1 & b & \cdots & a+2b-1 & a+b & \cdots & 2b-1 & \cdots \\ \end{array} \right). \]
and obviously lies in $\Gamma_A$. By doing this for every $v \in \{0, 1\}^\mathbb{Z}$ we have — in effect — embedded a full two-shift in the coordinates $b\mathbb{Z}$ of $\Omega_A$. This implies that $\Omega_A$ has entropy $\geq \frac{1}{2} \log 2$. □

Next we take a look at the case where $d \geq 2$ and consider dynamical properties of the $\mathbb{Z}^d$-SFT $\Omega_A \subset A^{\mathbb{Z}^d}$, such as entropy and topological mixing.

**Theorem 5.2.** If $A$ is a finite subset of $\mathbb{Z}^2$, then $\Omega_A$ has positive entropy if and only if $|A| \geq 3$.

**Proof.** We assume without loss in generality that $0 = (0, 0) \in A$: otherwise we replace $A$ by $A' = A - u$ for some $u \in A$. Then $0 \in A'$, and $\Omega_{A'}$ is topologically conjugate to $\Omega_A$ with conjugating map $\phi_u : \Omega_{A'} \to \Omega_A$ given by $\phi_u(\omega)_n = \omega_n + u$ for every $\omega \in \Omega_A$ and $n \in \mathbb{Z}^2$.

Suppose that $A$ is not contained in a one-dimensional subspace of $\mathbb{R}^2$. Then we can find two elements $a, b \in A$ which are linearly independent over $\mathbb{R}$. We denote by $\Delta \subset \Gamma \subset \mathbb{Z}^2$ the subgroups generated by $\{a + b, 3a\}$ and $\{a, b\}$, respectively. For every $x = (x_m)_{m \in \Delta} \in \{0, 1\}^\Delta$ we define a permutation $\pi_x \in \Pi_A$ as follows:

(i) if $n \in \mathbb{Z}^2 \setminus \Gamma$ we set $\pi_x(n) = n$;

(ii) if $n \in \Delta$ and $x_n = 0$ we set $\pi_x(n) = n + a$, $\pi_x(n + a) = n + a + b$, $\pi_x(n + 2a) = n + 2a$, and $\pi_x(n + b) = n + b$;

(iii) if $n \in \Delta$ and $x_n = 1$ we set $\pi_x(n) = n + b$, $\pi_x(n + b) = n + a + b$, $\pi_x(n + a) = n + a$, and $\pi_x(n + 2a) = n + 2a$.

It is easy to check that $\pi_x$ is a permutation of $\mathbb{Z}^2$ which lies in $\Pi_A$. By varying $x$ in $\{0, 1\}^\Delta$ we conclude that $h(\Omega_A) \geq \log 2/|\mathbb{Z}^2/\Delta|$.

If $A$ is contained in a one-dimensional subspace of $\mathbb{R}^2$, we can find a primitive element $v \in \mathbb{Z}^2$ such that $A \subset S := \mathbb{Z}v = \{kv : k \in \mathbb{Z}\} \cong \mathbb{Z}$. Choose a second primitive element $w \in \mathbb{Z}^2$ so that $\{v, w\}$ forms a basis of $\mathbb{Z}^2$. The restriction (or projection) $\Omega_A|_S$ of $\Omega_A$ to $S = \mathbb{Z}v = \{kv : k \in \mathbb{Z}\} \cong \mathbb{Z}$ is a SFT under the shift $\sigma^v$, which has positive entropy if and only if $|A| = |A \cap S| \geq 3$ (Proposition 5.1). Since the restrictions of $\Omega_A$ to $S + kw$, $k \in \mathbb{Z}$, are all isomorphic and independent of each other, they all have the same entropy $h(\Omega_A|_S)$ under $\sigma^v$, and $h(\Omega_A) = h(\Omega_A|_S) > 0$ if and only if $|A| \geq 3$. This completes the proof of Theorem 5.2. □

The proof of Theorem 5.2 yields two corollaries. For the proof of the first of these corollaries we recall that a nonzero subgroup $\Gamma \subset \mathbb{Z}^d$ is called primitive if $\mathbb{Z}^d/\Gamma$ is torsion-free. Similarly, a nonzero element $n \in \mathbb{Z}^d$ is primitive if the cyclic subgroup $\{kn : k \in \mathbb{Z}\} \subset \mathbb{Z}^d$ is primitive.

**Corollary 5.3.** If $A$ is a finite subset of $\mathbb{Z}^d$, $d \geq 2$, then $\Omega_A$ has positive entropy if and only if $|A| \geq 3$. 
\textbf{Proof.} Again we may assume that $0 \in \mathbb{A}$. If $|\mathbb{A}| = 2$, Proposition 5.1 and the last part of the proof of Theorem 5.2 can easily be adapted to show that $h(\Omega_{\mathbb{A}}) = 0$. If $|\mathbb{A}| \geq 3$, choose a primitive subgroup $\Gamma \subset \mathbb{Z}^d$ such that $\Gamma \cong \mathbb{Z}^2$ and $\Gamma \cap \mathbb{A}$ has at least three elements. A slight extension of the last part of the proof of Theorem 5.2 shows that $h(\Omega_{\mathbb{A}}) > 0$. \hfill \Box

The second corollary of the proof of Theorem 5.2 concerns topological mixing of $\Omega_{\mathbb{A}}, \mathbb{A} \subset \mathbb{Z}^d$. Recall that the $\text{SFT } \Omega_{\mathbb{A}} \subset \mathbb{A} \subset \mathbb{Z}^d$ is \textit{mixing} if there exists, for any pair of nonempty finite sets $V, V' \subset \mathbb{Z}^d$, an $N \in \mathbb{N}$ with the following property: for every $\omega, \omega' \in \Omega_{\mathbb{A}}$ and every $n \in \mathbb{Z}^2$ with $\|n\| \geq N$ there exists a $\omega'' \in \Omega_{\mathbb{A}}$ with $\omega''|_V = \omega|_V$ and $\omega''|_{V' + n} = \omega'|_{V' + n}$. Here $\|\cdot\|$ is the maximum norm on $\mathbb{Z}^d$, and $\omega|_W \in \mathbb{A}^W$ denotes the restriction of $\omega$ to its coordinates in a nonempty subset $W \subset \mathbb{Z}^d$.

\textbf{Corollary 5.4.} If $D = \mathbb{A} - \mathbb{A}$ is contained in a one-dimensional subspace of $\mathbb{R}^d$, then $\Omega_{\mathbb{A}}$ is not mixing.

\textbf{Proof.} If $D = \mathbb{A} - \mathbb{A}$ is contained in a one-dimensional subspace $V \subset \mathbb{R}^d$, and if $m \in \mathbb{A}$, then there exists a primitive element $v \in \mathbb{Z}^d$ so that $\mathbb{A}' = \mathbb{A} - m \subset S := \mathbb{Z}v$. By Proposition 5.1, the $\text{SFT } \Omega_{\mathbb{A}'|_S}$ is not mixing under the action of $\sigma^v$, and the last part of the proof of Theorem 5.2 shows that neither $\Omega_{\mathbb{A}'}$ nor $\Omega_{\mathbb{A}}$ can be mixing. \hfill \Box

Proposition 5.1 shows that $\Omega_{\mathbb{A}}$ is nonmixing for every finite set $\mathbb{A} \subset \mathbb{Z}$ with at least two elements. For finite subsets $\mathbb{A} \subset \mathbb{Z}^d$ with $d > 1$, the situation can be different, as the following examples show.

\textbf{Example 5.5.} Let $\mathbb{A} = \{(0,0), (1,0), (0,1)\} \subset \mathbb{Z}^2$. Then the $\mathbb{Z}^2$-$\text{SFT } \Omega_{\mathbb{A}}$ is topologically mixing.

Since the the following verification of this claim will reappear — in a slightly more complicated form — in the proof of Theorem 5.6, we shall describe it in detail.

We start a bit of notation. Let $\mathbb{A} \subset \mathbb{Z}^d, d \geq 1$, be a finite set containing $0$. A subset $p \subset \mathbb{Z}^d$ is \textit{allowed} if it consists either of a single point or of a bi-infinite sequence $\{p_k\}_{k \in \mathbb{Z}}$ such that $p_{k+1} - p_k \in \mathbb{A} \setminus \{0\}$ for every $k \in \mathbb{Z}$. If $p \in \mathbb{Z}^d$ is allowed, its \textit{future} $p^+$ is defined by

$$p^+ = \begin{cases} \{n\} & \text{if } p = \{n\} \text{ for some } n \in \mathbb{Z}^d, \\ \{p_k\}_{k \geq 1} & \text{if } p = \{p_k\}_{k \in \mathbb{Z}} \text{ with } p_{k+1} - p_k \in \mathbb{A} \setminus \{0\} \text{ for every } k \in \mathbb{Z}. \end{cases}$$

The definition of the \textit{past} $p^-$ of $p$ is analogous.

A collection $p$ of disjoint subsets of $\mathbb{Z}^d$ is \textit{allowed} if it consists of allowed sets. More generally, if $S \subset \mathbb{Z}^d$ is a subset, a collection $q$ of disjoint subsets of $S$ is
allowed if there exists an allowed collection \( p \) of disjoint subsets of \( \mathbb{Z}^d \) such that
\[ q = p \cap S = \{ p \cap S : p \in p \} . \]

Every permutation \( \pi \in \Pi_A \) can be represented by the allowed partition \( p^{(\pi)} \) of \( \mathbb{Z}^d \) into orbits or 'paths' of \( \pi \). Conversely, if \( p \) is an allowed collection of disjoint subsets of \( \mathbb{Z}^d \), we denote by \( U_p = \bigcup_{p \in p} \) the union of \( p \) and extend \( p \) to an allowed partition \( \tilde{p} \) of \( \mathbb{Z}^2 \) by adding to it the singletons \( \{ n \} \), \( n \in \mathbb{Z}^2 \setminus U_p \). The partition \( \tilde{p} \) is the set of orbits of a unique permutation \( \pi_{\tilde{p}} \in \Pi_A \).

We return to the above set \( A = \{ (0, 0), (1, 0), (0, 1) \} \). Take a finite set \( Q_1 \subset \mathbb{Z}^2 \) and a permutation \( \pi_1 \in \Pi_A \), and consider the collection \( p^{(\pi_1)}(Q_1) = \{ p \in p^{(\pi_1)} : p \cap Q_1 \neq \emptyset \} \) of all orbits in the partition \( p^{(\pi_1)} \) which pass through \( Q_1 \). We modify the orbits \( p \in p^{(\pi_1)}(Q_1) \) outside \( Q_1 \) in such a way that they are still allowed and move almost all the time vertically. Denote this family of modified orbits by \( p'_1(Q_1) \). Then we do the same for another permutation \( \pi_2 \in \Pi_A \) and another finite set \( Q_2 \subset \mathbb{Z}^2 \) with sufficient horizontal distance from \( Q_1 \), and obtain a modified allowed collection \( p'_2(Q_2) \). Since \( Q_1 \) and \( Q_2 \) have sufficient horizontal distance, \( U_{p'_1(Q_1)} \cap U_{p'_2(Q_2)} = \emptyset \) and the union \( p' = p'_1(Q_1) \cup p'_2(Q_2) \) is again an allowed collection of orbits. Finally, we extend \( p' \) to an allowed partition \( \tilde{p}' \) of \( \mathbb{Z}^2 \) by adding singletons and obtain a \( \pi' = \pi_{\tilde{p}'} \in \Pi_A \) which coincides on \( Q_1 \) and \( Q_2 \) with \( \pi_1 \) and \( \pi_2 \), respectively.

If the sets \( Q_1 \) and \( Q_2 \) are separated vertically rather than horizontally, the modification process described above has to be changed accordingly.

Now the details: let \( M \geq 0 \), put \( Q = Q^{(M)} = \{ 0, \ldots, M \}^2 \subset \mathbb{Z}^2 \), and set \( E_k = \{ 0, \ldots, M + k \} \times \mathbb{N} \subset \mathbb{Z}^2 \). Fix \( \pi_1 \in \Pi_A \). The collection \( p := p^{(\pi_1)}(Q) \) of all \( \pi_1 \)-orbits intersecting \( Q \) is finite: it has at most \( |Q| \) elements.

We start an induction process by setting \( p^{(0)} = p \) and \( q^{(0)} = \emptyset \). Suppose that \( K \geq 0 \), and that we have found allowed collections of disjoint sets \( p^{(k)} \subset p \) and \( q^{(k)} \), which satisfy the following conditions for \( k = 0, \ldots, K \):

(i) \( q^+ \subset E_k \) for every \( q \in q^{(k)} \),

(ii) \( E_{p,q} := (p \cap \tilde{Q}) \cap (q \cap \tilde{Q}) = \emptyset \) for every \( p \in p^{(k)} \) and \( q \in q^{(k)} \), where \( \tilde{Q} = Q \cup \pi_1(Q) \),

(iii) The sets \( \{ p \cap \tilde{Q} : p \in p^{(k)} \} \cup \{ q \cap \tilde{Q} : q \in q^{(k)} \} \) form a partition of \( \tilde{Q} \) which coincides with \( \{ p \cap \tilde{Q} : p \in p \} \).

For the induction step we suppose that \( p^{(K)} \neq \emptyset \) and set \( p_1^{(K)} = \{ p \in p^{(K)} : p^+ \subset E_{K+1} \} \) and \( p_2^{(K)} = p^{(K)} \setminus p_1^{(K)} \). If \( p_1^{(K)} \neq \emptyset \), put \( p^{(K+1)} = p_2^{(K)} \). If \( p_1^{(K)} = \emptyset \), every \( p \in p^{(K)} \) will move into \( E_{K+2} \) after having passed through \( E_{K+1} \setminus E_K \) (this follows from the fact that every infinite orbit of \( \pi_1 \) can only move up or right in steps of size one). Since \( p \) is finite, and since \( p^+ \cap E_{K+1} \) is finite for every \( p \in p^{(K)} \) by assumption, the set \( F = \bigcup_{p \in p^{(K)}} p^+ \cap (E_{K+1} \setminus E_K) \) is finite and contains
at least one element \( m \) whose second coordinate is maximal (i.e., satisfies that \( m + le(2) \notin F \) for every \( l > 0 \)). We denote by \( p' = \{p'_k\}_{k \in \mathbb{Z}} \in \mathcal{P}(K) \) the element containing \( m \) with \( p'_{k_0} = m \), say, and define another allowed set \( p'' = \{p''_k\}_{k \in \mathbb{Z}} \) by setting

\[
p''_k = \begin{cases} 
p'_k & \text{if } k \leq k_0, \\
p'_k + (k - k_0)e(2) & \text{if } k > k_0.
\end{cases}
\]

Put \( p^{(K+1)} = p^{(K)} \setminus \{p'\} \) and \( q^{(K+1)} = q^{(K)} \cup \{p''\} \). By assumption, \( q^{(K+1)} \) is again allowed.

In either case, \( p^{(K+1)} \subseteq p^{(K)} \), and the sets \( p^{(k)}, q^{(k)}, k = 0, \ldots, K + 1 \), satisfy the condition (i) – (iii) above with \( K + 1 \) replacing \( K \).

Since \( p \) is finite and \( p^{(k+1)} \subseteq p^{(k)} \) for every \( k \geq 0 \), there has to exist a \( K \geq 1 \) with \( K \leq |Q| \) such that \( p^{(K)} = \emptyset \), and hence with \( q^+ \subseteq \mathcal{E}_{K+1} \) for every \( q \in q^{(K)} \).

We have arrived at an allowed collection \( q = q^{(K)} \) of disjoint subsets of \( \mathbb{Z}^2 \) such that \( q^+ \subseteq \mathcal{E}_{K+1} \) for every \( q \in q \) and the partitions \( \{q \cap \tilde{Q} : q \in q\} \) and \( \{p \cap \tilde{Q} : p \in q\} \) coincide. We extend \( q \) to an allowed partition \( \tilde{q} \) of \( \mathbb{Z}^2 \) by adding singletons and obtain a permutation \( \tilde{\pi}_1 := \pi_{\tilde{q}} \in \Pi_A \) with the properties that \( \tilde{\pi}_1^k(n) = \tilde{\pi}_1^k(n) \) for every \( n \in Q \) and \( k \leq 1 \), and that \( \tilde{\pi}_1^k(n) \in \mathcal{E}_{|Q|+1} \) for every \( n \in Q \) and \( k \geq 0 \).

Exactly the same argument, but with directions reversed, allows us to find a permutation \( \tilde{\pi}_1 \in \Pi_A \) such that \( \tilde{\pi}_1^k(n) = \tilde{\pi}_1^k(n) \) for every \( n \in Q \) and \( k \geq -1 \), and that \( \tilde{\pi}_1^k(n) \in \{-|Q| - 1, \ldots, M\} \times \{-N\} \) for every \( n \in Q \) and \( k \leq 0 \).

The permutation \( \tilde{\pi}_1 \) has the properties that \( \tilde{\pi}_1|_Q = \pi_1|_Q = \pi_1|_Q \), and that each orbit of \( \tilde{\pi}_1^k \) lies in the vertical strip \( \{-|Q| - 1, \ldots, M + |Q| + 1\} \times \mathbb{Z} \).

An analogous argument yields a permutation \( \tilde{\pi}_1' \in \Pi_A \) such that \( \tilde{\pi}_1'|_Q = \pi_1|_Q \), and that each orbit of \( \tilde{\pi}_1' \) lies in the horizontal strip \( \mathbb{Z} \times \{-|Q| - 1, \ldots, M + |Q| + 1\} \).

By translating this back to the shift space \( \Omega_A \) we conclude that there exists, for every \( M \geq 0 \), every pair \( \omega_1, \omega_2 \in \Omega_A \), and every \( m \in \mathbb{Z}^2 \) with \( \|m\| > 7|Q(M)| \), an element \( \omega_3 \in \Omega_A \) with \( \omega_3|_{Q(M)} = \omega_1|_{Q(M)} \) and \( \omega_3|_{Q(M)+m} = \omega_2|_{Q(M)+m} \). Clearly this implies that \( \Omega_A \) is mixing.

Example 5.5 illustrates the following general result.

**Theorem 5.6.** Let \( d \geq 2 \), and let \( A \subset \mathbb{Z}^d \) be a finite set. Then the \( \mathbb{Z}^d \)-SFT \( \Omega_A \) is topologically mixing if and only if \( D = A - A \) does not lie in a one-dimensional subspace of \( \mathbb{R}^d \).

**Proof.** We start the proof of Theorem 5.6 with a simplification and a bit of notation. As we observed in the proof of Theorem 5.2, we may assume without loss in generality that \( 0 \in A \); in fact, we shall assume that \( 0 \) is a vertex of the closed convex hull \( \bar{A} \) of \( A \) in \( \mathbb{R}^d \).
Write $C(A) = \{ \sum_{n \in A} t_n n : t_n \geq 0 \text{ for every } n \in A \} \subset \mathbb{R}^d$ for the cone of $A$. Let $e \in A$ be another vertex of $A$ such that the ray $\{ te : t \geq 0 \} \subset C(A)$ is extremal, and let $E = \{ te : t \in \mathbb{R} \}$. Put $B = A \setminus E$ and denote by $\bar{B}$ the convex hull of $B$. Then $E$ and $\bar{B}$ are disjoint convex subsets of $\mathbb{R}^d$, one of which is compact, and the strict hyperplane separation theorem (cf. [1, Section 2.5.1] or [6]) implies that there exist a vector $w \in \mathbb{R}^d$ and real numbers $c_1 < c_2$ such that $\langle w, v \rangle \leq c_1$ and $\langle w, v' \rangle \geq c_2$ for every $v \in E$ and $v' \in \bar{B}$, where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $\mathbb{R}^d$. Since the first inequality holds for every $v \in E$ we conclude that $\langle w, e \rangle = c_1 = 0$, whereas $\langle w, n \rangle \geq c_2 > 0$ for every $n \in B$.

Now we can imitate the argument in Example 5.5. Take a finite set $Q \subset \mathbb{Z}^d$ and set $L_1 = \max_{n \in Q} \langle w, n \rangle$, $L_2 = \max_{n \in B} \langle w, n \rangle + 1$, and $E_k = \{ k \in \mathbb{Z}^d : -L_1 \leq \langle w, k \rangle \leq L_1 + kL_2 \}$, $k \geq 0$. Then the induction argument in Example 5.5 can be used essentially unchanged, apart from replacing $e^{(2)}$ with $e$.

As there, we start with an element $\pi_1 \in \Pi_A$, consider the collection $p := p^{(\pi_1)}(Q)$ of all $\pi_1$-orbits intersecting $Q$, and end up with an allowed collection $q$ of disjoint subsets of $\mathbb{Z}^d$ and a $K \geq 0$ such that $q^+ \subset E_K$ for every $q \in q$ and the partitions $\{ q \cap \tilde{Q} : q \in q \}$ and $\{ p \cap \tilde{Q} : p \in p \}$ coincide, where $\tilde{Q} = Q \cup \pi_1(Q)$. Again we extend $q$ to an allowed partition $\tilde{q}$ of $\mathbb{Z}^d$ by adding singletons and obtain a permutation $\tilde{\pi}_1 := \pi_\tilde{q} \in \Pi_\tilde{Q}$ with the properties that $\tilde{\pi}_1^k(n) = \pi^{(2)}_1(n)$ for every $n \in Q$ and $k \leq 1$, and that $\tilde{\pi}_1^k(n) \in E_K$ for every $n \in Q$ and $k \geq 0$. This takes care of forward orbits; the backward orbits of $\pi_1$ are dealt with by reversing directions as in Example 5.5.

Finally we note that we can replace the vertex $e \in A$ by any other vertex $e' \in A$ giving rise to an extremal ray of $C(A)$ (here we are using our assumption that $D = A - A$ is not one-dimensional). This allows us to complete the proof of Theorem 5.6 as in Example 5.5. \hfill $\square$

Remarks 5.7. (1) The dynamical properties of $\Omega_A$ discussed here, like topological mixing or topological entropy, are affine invariants: for every finite set $A \subset \mathbb{Z}^d$ and every $\gamma \in \mathbb{Z}^d \rtimes \text{GL}(d, \mathbb{Z})$, $\Omega_{A^\gamma}$ is obtained from $\Omega_A$ through an affine re-parametrization of the coordinates. In particular, $\Omega_{A^\gamma}$ is mixing if and only if the same is true for $\Omega_A$, and $h(\Omega_{A^\gamma}) = h(\Omega_A)$.

(2) Permutations with restricted movement can be considered in a more general context than we have done here. Let $\Gamma$ be a countable discrete group $\Gamma$, and let $A \subset \Gamma$ be a nonempty finite set. Consider the set $\Pi_A \subset S^\infty(\Gamma)$ of all permutations $\pi : \gamma \mapsto \pi(\gamma)$ such that

$$\omega_\gamma^{(\pi)} := \pi(\gamma)\gamma^{-1} \in A \text{ for every } \gamma \in \Gamma.$$  \hspace{1cm} (5.1)
We put
\[ \Omega_A = \{ \omega^{\pi} : \pi \in \Pi_A \} \] (5.2)

and observe as for \( \Gamma = \mathbb{Z} \) that \( \Omega_A \subset A^\Gamma \) is a shift of finite type for the right shift-action \( \sigma \) of \( \Gamma \) on \( A^\Gamma \), defined by
\[ (\sigma^\delta \omega)_\gamma = \omega_{\gamma \delta} \] (5.3)

for every \( \omega = (\omega_\gamma)_\gamma \in A^\Gamma \) and \( \delta \in \Gamma \). This construction gives rise to a natural class of examples of \( \Gamma \)-SFT’s for any countable discrete group \( \Gamma \).

6. Example 5.5, revisited

As an illustration of properties of the multiparameter SFT’s appearing in Section 5 we return to the \( \mathbb{Z}^2 \)-SFT \( \Omega_A \), \( A = \{(0,0), (1,0), (0,1)\} \), in Example 5.5. As described in Example 5.5, every \( \omega \in \Omega_A \) determines a permutation \( \pi^{(\omega)} \) of \( \mathbb{Z}^2 \), each orbit of which consists either of a single point or of a bi-infinite sequence \((n_k)_{k \in \mathbb{Z}}\) with \( n_{k+1} - n_k \in \{(1,0), (0,1)\} \) for every \( k \in \mathbb{Z} \). If we represent each infinite orbit of \( \pi^{(\omega)} \) by a bi-infinite directed polygonal path in \( \mathbb{Z}^2 \), we obtain a collection \( p^{(\pi^{(\omega)})} \) of non-intersecting paths in \( \mathbb{Z}^2 \) moving either north or east at each step. Figure 2 shows the intersection of \( p^{(\pi^{(\omega)})} \) with a square \( Q \subset \mathbb{Z}^2 \). In the terminology of Example 5.5, \( q = p^{(\pi^{(\omega)})} \cap Q \) is an allowed collection of disjoint subsets of \( Q \) or, for convenience, an allowed configuration of paths in \( Q \). Conversely, every allowed configuration of paths in \( Q \) arises in this manner from some element of \( \Omega_A \).

![Figure 2. An allowed configuration of paths in a square Q](image)

The entropy of \( \Omega_A \) is positive by Theorem 5.2. We are grateful to Christian Krattenthaler for pointing out and explaining to us [13, Theorem 3.1], which yields an explicit formula for the number of allowed configurations of \( k \) paths leading from the bottom and left edges to the top and right edges of the square \( Q \) in Figure 2. By using this formula and varying both \( k \) and the size \( Q \) one obtains that the topological entropy of \( \Omega_A \) is about \( \log 1.38 \ldots \)

According to Theorem 5.6, the SFT \( \Omega_A \) is topologically mixing. However, it does not have the uniform filling property ([10, Definition 3.1]), nor is it strongly
irreducible in the sense of [3, Definition 1.10]. The following proposition shows that $\Omega_A$ nevertheless has an abundance of periodic points, allowing to express its entropy in terms of the logarithmic growth rate of the number of its periodic points.

For every finite-index subgroup $\Gamma \subset \mathbb{Z}^2$ denote by $\text{Fix}_\Gamma(\Omega_A) = \{ \omega \in \Omega_A : \sigma^n\omega = \omega \text{ for every } n \in \Gamma \}$ the set of $\Gamma$-periodic points in $\Omega_A$. Then the following is true.

Proposition 6.1. The set of periodic points is dense in $\Omega_A$. Furthermore,

$$h(\Omega_A) = \lim_{K \to \infty} \frac{1}{\mathbb{Z}^2/\Delta_K} \cdot \log |\text{Fix}_{\Delta_K}(\Omega_A)|,$$

where $\Delta_K = \{ 2k(K^2 + 2K) \cdot (1, 0) + 2lK \cdot (1, 1) : k, l \in \mathbb{Z} \} \subset \mathbb{Z}^2$ for every $K \geq 1$.

Proof. Figure 3 shows two allowed configurations $q$ and $p$ of paths in a polygon $R \subset \mathbb{R}^2$ with edges $a, b, c, d, e, f$ of lengths $|a| = |d|$ and $|b| = |c| = |e| = |f|$, whose intersections with the boundary $\partial R$ of $R$ (i.e., with the union of the edges of $R$) coincide.

If we rotate the configuration $p$ in Figure 3 clockwise by $90^\circ$, flip the resulting pattern horizontally, and reverse the direction of all arrows, we obtain another allowed configuration $p'$ of paths in $R$, as shown in Figure 4.

In Figure 5 we glue together the configurations $q$ and $p'$ along the edges labelled $d$ and $a$, respectively, put in ‘dotted’ arrows to connect up loose ends, and obtain an allowed configuration $(q, p)$ of paths in a bigger polygon $\tilde{R} \subset \mathbb{Z}^2$.

In Figure 6 we extend the configuration $(q, p)$ in Figure 5 periodically, again putting in dotted arrows to connect loose ends.

Finally we repeat the diagonal strip in Figure 6 periodically in the horizontal direction, allowing sufficient separation between the strips. Figure 7 shows the
repetition of Figure 6 with a horizontal period of size \((|a| + 2|b|\sqrt{2})\sqrt{2}\) (the length of the hypotenuse of the triangle \(ABC\)).

If we assume that the edges \(a\) and \(b\) of the polygon \(R \subset \mathbb{Z}^2\) in Figure 3 have lengths \(|a| = K^3\sqrt{2}\) and \(|b| = K\) and write \(R_K\) instead of \(R\) to emphasise the dependence of this polygon on the integer \(K\), then the periodic configuration of paths in Figure 7 corresponds to an element \(\omega' \in \text{Fix}_{\Delta_K}(\Omega_A)\), where

\[
\Delta_K = \left\{ k((|a| + 2|b|\sqrt{2})\sqrt{2})(1,0) + 2l|b|(1,1) : k, l \in \mathbb{Z} \right\} = \left\{ 2k(K^3 + 2K)(1,0) + 2lK(1,1) : k, l \in \mathbb{Z} \right\} \subset \mathbb{Z}^2. \tag{6.2}
\]

If \(\omega \in \Omega_A\) is a point giving rise to the configuration \(q\) of paths in the polygon \(R_K\), then the coordinates of \(\omega\) and \(\omega'\) coincide on \(R_K\). Since \(K\) was arbitrary, this proves the density of periodic points in \(\Omega_A\).

For every \(K \geq 2\), the intersection \([R_K] := R_K \cap \mathbb{Z}^2\) has cardinality \(|[R_K]| = (2K + 1)(K^3 + 1) + K^2\). The configuration \(q\) of paths in \(R_K\) arising from an
written coordinates

\[ \text{coordinates} \]

The periodic extension of the configuration \((q, p)\) in Figure 5.

Figure 6.

The periodic repetition of the diagonal strip in Figure 6.

Figure 7.

element \( \omega \in \Omega_K \) is determined by the coordinates \( \omega_n, n \in R_K \), of \( \omega \). Conversely, \( q \) determines all the coordinates \( \omega_n, n \in R_K \), with the exception of (some of) the coordinates \( \omega_n \) with \( n \) lying on the boundary \( \partial R_K \) of \( R_K \) (cf. Figure 3). We write \( N_K \) for the number of allowed configurations of paths in \( R_K \) and conclude that

\[
\lim_{K \to \infty} \frac{1}{|R_K|} \cdot \log N_K = \lim_{K \to \infty} \frac{1}{|R_K|} \cdot \log |\Pi_K(\Omega_A)| = h(\Omega_A),
\]

(6.3)

where we are using that \((|R_K|)_{K \geq 1}\) is a Følner sequence in \( \mathbb{Z}^2 \), and where \( \Pi_F : \Omega_A \rightarrow A^F \) denotes the projection of every \( \omega \in \Omega_A \) onto its coordinates in a set \( F \subset \mathbb{Z}^2 \).
Since $\partial R_K$ intersects $\mathbb{Z}^2$ in $2K^3 + 4K \leq 3K^3$ points, a set $C$ of at least $M_K := N_K/|A|^{3K^3} = N_K/3^{4K^3}$ of these configurations must coincide on $\partial R_K$.

By taking all pairs of elements in $C$ we obtain at least $M_K'= N_K^2/3^{6K^3}$ distinct allowed configurations $(q,p)$, $q,p \in C$, of paths in the polygon $\tilde{R}_K$ in Figure 5, each of which determines an element of $\text{Fix}_{\Delta_K}(\Omega_A)$, where $\Delta_K \subset \mathbb{Z}^2$ is given in (6.2) (cf. Figure 7). Hence $|\text{Fix}_{\Delta_K}(\Omega_A)| \geq N_K^2/3^{6K^3}$. We set $[\tilde{R}_K] = \tilde{R}_K \cap \mathbb{Z}^2$ and note that

$$|\tilde{R}_K| = (4K + 1)(K^3 + 1) + 4K^2 = 2|R_K| + 2K^2 - K^3 - 1.$$ 

Since $|\mathbb{Z}^2/\Delta_K| = 4K^4 + 8K^2$, we obtain that

$$\frac{1}{|\mathbb{Z}^2/\Delta_K|} \log |\text{Fix}_{\Delta_K}(\Omega_A)| = \frac{|\tilde{R}_K|}{4K^4 + 8K^2} \cdot \frac{2|R_K|}{|\tilde{R}_K|} \cdot \frac{1}{2|R_K|} \log |\text{Fix}_{\Delta_K}(\Omega_A)|$$

$$\geq \frac{4K^4}{4K^4 + 8K^2} \cdot \frac{2|R_K|}{|\tilde{R}_K|} \cdot \frac{1}{2|R_K|} \log |\text{Fix}_{\Delta_K}(\Omega_A)|$$

$$\geq \frac{4K^4}{4K^4 + 8K^2} \cdot \frac{2|R_K|}{|\tilde{R}_K|} \cdot \frac{1}{2|R_K|} \log |\text{Fix}_{\Delta_K}(\Omega_A)|$$

for every $K \geq 2$. By letting $K \to \infty$ and using (6.3) we obtain that

$$\liminf_{K \to \infty} \frac{1}{|\mathbb{Z}^2/\Delta_K|} \log |\text{Fix}_{\Delta_K}(\Omega_A)| \geq h(\Omega_A).$$

Since the opposite inequality $\limsup_{K \to \infty} \frac{1}{|\mathbb{Z}^2/\Delta_K|} \log |\text{Fix}_{\Delta_K}(\Omega_A)| \leq h(\Omega_A)$ is obvious, we have proved (6.1).

**Corollary 6.2.**

$$\limsup_{K \to \infty} \frac{1}{K^2} \log |\text{Fix}_{4K^2\mathbb{Z}^2}(\Omega_A)| = \lim_{K \to \infty} \frac{1}{16K^6} \log |\text{Fix}_{4K^2\mathbb{Z}^2}(\Omega_A)| = h(\Omega_A).$$

**Proof.** For every $K \geq 1$ we consider the Polygon $\tilde{R}_K$ appearing in Figure 5 in the proof of Proposition 6.1. There we showed that there exists a set $\tilde{C}$ of more than $N_K^2/3^{6K^3}$ distinct allowed configurations of paths in $\tilde{R}_K$ corresponding to points in $\text{Fix}_{\Delta_K}(\Omega_A)$, all of which coincide on the boundary $\partial \tilde{R}_K$ of $\tilde{R}_K$.

We set $\Delta'_K = 2(K^3 + 2K)\mathbb{Z}^2 \subset \Delta_K$. Then $|\Delta_K/\Delta'_K| = K^2 + 2$, and $\Delta_K$ is the disjoint union of cosets $\Delta'_K + v$, $v \in S_K$, with $|S_K| = K^2 + 2$. We choose, for every $v \in S_K$, an arbitrary configuration $q_v \in \tilde{C}$, fill the polygon $\tilde{R}_K + v$ with the corresponding translate $q_v + v$ of $q_v$, and obtain in this manner a family $\tilde{D}$ of at least $(N_K^2/3^{6K^3})|\Delta_K/\Delta'_K|$ distinct allowed configurations of paths in the set $F_K := \bigcup_{v \in S_K} \tilde{R}_K + v$, each of which has a unique $\Delta_K$-invariant extension to $\mathbb{Z}^2$ and determines an element of $\text{Fix}_{\Delta_K}(\Omega_A)$. From (6.4) we obtain that

$$\frac{1}{|\mathbb{Z}^2/\Delta_K|} \log |\text{Fix}_{\Delta_K}(\Omega_A)| \geq \frac{1}{|\mathbb{Z}^2/\Delta_K|} \log N_K/3^{3K^3}$$

$$\geq \frac{|\tilde{R}_K|}{4K^4 + 8K^2} \cdot \frac{2|R_K|}{|\tilde{R}_K|} \cdot \frac{1}{2|R_K|} \log N_K/3^{3K^3}$$

$$\geq \frac{4K^4}{4K^4 + 8K^2} \cdot \frac{2|R_K|}{|\tilde{R}_K|} \cdot \frac{1}{2|R_K|} \log N_K/3^{3K^3}$$

for every $K \geq 2$. By letting $K \to \infty$ and using (6.3) we obtain that

$$\liminf_{K \to \infty} \frac{1}{|\mathbb{Z}^2/\Delta_K|} \log |\text{Fix}_{\Delta_K}(\Omega_A)| \geq h(\Omega_A).$$

Since $\limsup_{K \to \infty} \frac{1}{|\mathbb{Z}^2/\Delta_K|} \log |\text{Fix}_{\Delta_K}(\Omega_A)| \leq h(\Omega_A)$, this proves the corollary. □
Problems 6.3. (1) Is \( \lim_{K \to \infty} \frac{1}{K^2} \cdot \log |\text{Fix}_{K\mathbb{Z}^2}(\Omega_A)| = h(\Omega_A) \)?

(2) Does \( \Omega_A \) have a unique shift-invariant probability measure of maximal entropy?

(3) Let \( \mu \) be a (or the) shift-invariant probability measure of maximal entropy on \( \Omega_A \). For every \( \omega \in \Omega_A \) and every \( n \geq 1 \), consider the allowed configuration \( p^{(\omega)}(Q_n) := p^{(\pi^{(\omega)})} \cap Q_n \) of paths in the square \( Q_n = \{0, \ldots, n\}^2 \subset \mathbb{Z}^2 \) determined by \( \omega \) (cf. Figure 2), and we write \( N(p^{(\omega)}(Q_n)) \) for the number of paths (or connected components) of \( p^{(\omega)}(Q_n) \). Is it true that \( \lim_{n \to \infty} \frac{1}{n} \cdot N(p^{(\omega)}(Q_n)) = \frac{2}{3} \) for \( \mu \)-a.e. \( \omega \in \Omega_A \), as numerical evidence suggests?

(4) How general are the results in this section? Are analogous statements true for every finite set \( A \subset \mathbb{Z}^2 \) such that \( \Omega_A \) is topologically mixing?

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