THE DONALD–FLANIGAN PROBLEM
FOR FINITE REFLECTION GROUPS

MURRAY GERSTENHABER, ANTHONY GIAQUINTO, AND MARY E. SCHAPS

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To the memory of Moshé Flato, z"l

Abstract. The Donald–Flanigan problem for a finite group $H$ and coefficient ring $k$ asks for a deformation of the group algebra $kH$ to a separable algebra. It is solved here for dihedral groups and Weyl groups of types $B_n$ and $D_n$ (whose rational group algebras are computed), leaving but six finite reflection groups with solutions unknown. We determine the structure of a wreath product of a group with a sum of central separable algebras and show that if there is a solution for $H$ over $k$ which is a sum of central separable algebras and if $S_n$ is the symmetric group then i) the problem is solvable also for the wreath product $H \wr S_n = H \times \cdots \times H$ ($n$ times) $\times S_n$ and ii) given a morphism from a finite abelian or dihedral group $G$ to $S_n$ it is solvable also for $H \wr G$. The theorems suggested by the Donald–Flanigan conjecture and subsequently proven follow, we also show, from a geometric conjecture which although weaker for groups applies to a broader class of algebras than group algebras.

1. Introduction: The Donald–Flanigan problem. In this paper we solve the Donald–Flanigan problem for a large class of groups including almost all finite reflection groups, and in the process give a simple construction of the rational group rings (effectively, of all irreducible representations) of the Weyl groups of types $B_n$ and $D_n$ ($\S\S$ 8-11).

The Donald–Flanigan conjecture was one of the most intriguing of those suggested by algebraic deformation theory because it sought to relate the behavior of the group algebra of a finite group in characteristic $p$ with that of its complex group algebra. If $G$ is a finite group and $k$ a commutative unital ring in which the order of $G$ is invertible, then Maschke’s theorem asserts that the group algebra $kG$ is separable; in particular, $\mathbb{C}G$ is a direct sum of matrix algebras. By contrast, if $k$ is a field of characteristic $p$ dividing $\# G$ then $kG$ always has a non-trivial radical. Donald and Flanigan conjectured that for this ‘modular’ case the group algebra $kG$ can always be deformed to a separable algebra, [DF]. In the examples they exhibited they observed that the separable algebra they constructed was a direct sum of copies of the coefficient field and thus resembled the complex group algebra. Schaps suggested that one should look for a deformation of $kG$ with matrix

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blocks in natural bijection with those of $CG$, corresponding blocks having the same dimensions. This concept is made precise in the definition of a global solution.

The Donald–Flanigan conjecture fails for the 8-element quaternion group \{±1, ±1, ±i, ±j, ±k\}, [GG], but the problem remains of determining when $kG$ can be deformed to a separable algebra of the “right form” (one resembling the complex group algebra). The result is known for several classes of groups and algebras. (In the algebra case, the problem is again to deform it to a separable algebra.) These include abelian groups and symmetric groups $S_n$, [GSps], (where in both cases there is a global solution connecting the behavior at all primes), blocks with dihedral defect group [ES], and blocks with abelian normal defect group, [MS]. Paradoxically, certain implications of the Donald–Flanigan conjecture, cf. [GG], were subsequently verified, in particular that in the modular case $kG$ always has a non-inner derivation, [FJL]. The paradox might be resolved by the fact that this also follows (§5) from a more geometric conjecture which, although weaker for groups, is more general.

Finite Coxeter groups are identical with finite reflection groups; we may use the terms interchangeably. There are four infinite classes of irreducible ones (i.e., which are not direct products), namely the Weyl groups of type $A_n (= S_{n+1})$, $B_n$, $D_n$ and the dihedral groups. (For a general reference and excellent exposition, cf. [H].) In addition, there are four exceptional Weyl groups, $F_4$, $E_6$, $E_7$, $E_8$ and two non-crystallographic groups, $H_3$ of order 120 (the symmetry group of the icosahedron) and $H_4$ of order 14,400 (the symmetry group of a regular 120-sided solid in $\mathbb{R}^4$). One can omit $G_2$ because it is identical with the 12-element dihedral group $D_6$.

Here we give another solution for the Donald–Flanigan problem for the dihedral groups (extending certain work of Erdmann and Erdmann–Schaps, cf. [ES]), and using our earlier solution for $S_n$ we solve it for the groups $B_n$ and $D_n$. Thus there are only six finite Coxeter groups for which the Donald–Flanigan problem remains open.

An essential step in the solution is the determination of the structure of a wreath product of a direct sum of central separable algebras with a a finite group. Using this we show that if $H$ is a group and $k$ a ring for which the Donald–Flanigan problem has a solution which is a sum of central separable algebras then i) the problem is solvable for the wreath product $H \wr S_n = H \times \cdots \times H$ ($n$ times) $\times S_n$ and ii) given a morphism from a finite abelian group or dihedral group $G$ to $S_n$ it is solvable also for $H \wr G$. While not important for us here, the condition on the solution for $H$ will later be relaxed somewhat, since it is enough that $kH$ be deformed to an algebra $A$ which becomes a direct sum of central separable algebras after suitable extension of the coefficient ring. For this it is sufficient that the center of $kH$ be faithfully projective and of constant rank over the new coefficient ring, an extension of $k$ resulting from the deformation process.

Determining the structure of $H \wr S_n$ (and more generally, of $H \wr G$ where $G$ operates by permutation of the factors of $H^{\times n}$) parallels the classical problem of finding the irreducible representations of a semidirect product $N \rtimes G$ in which the normal subgroup is abelian. The basic work (fundamental to particle physics) is due to Wigner, who computed the irreducible representations of the Poincaré group (semidirect product of the Lorentz group and $\mathbb{R}^4$), cf. [St], §§3.8, 3.9. Wreath products are special cases of semidirect products but here the coefficient rings are generally not fields and $H$ need not be commutative. Nevertheless, it may be instructive to compare our treatment with that of Sternberg, [St], particularly our
example of §3, where \( H \) is commutative, with that in [St] (pp. 139–142) of the eight element dihedral group \( C_4 \times C_2 \), where \( C_n \) is the \( n \)-element cyclic group. The Weyl group \( D_n \) is only a semi-direct product and not a wreath product, but the Donald–Flanigan problem for it can be reduced to the wreath product case. In the course of solving the Donald–Flanigan problem for \( B_n \) and \( D_n \) we must, in effect, compute their rational group algebras, which are given here explicitly.

Many related problems remain. In particular, suppose that a finite group \( G \) acts as automorphisms (possibly trivially) of a finite group \( H \), so that we can form the semidirect product \( H \rtimes G \). Then \( G \) also operates as automorphisms of the group algebra \( kH \) for any coefficient ring \( k \). An equivariant deformation of \( kH \) is one in which the operations of \( G \) continue to be automorphisms of the deformed algebra. When can an equivariant solution to the Donald–Flanigan problem for \( H \) be extended to one of the semidirect product? More generally, suppose that a finite group \( G \) acts on a separable \( k \)-algebra \( A \) and let \( A \# kG \), the smash product, be the algebra with underlying \( k \)-module \( A \otimes kG \) and multiplication given by \((a \otimes \sigma)(b \otimes \tau) = a\sigma(b)\tau; a, b \in A, \sigma, \tau \in G \). (This is a special case of the usual smash product in Hopf theory, cf. [M], Ch. 7, for note that \( kG \) is a Hopf algebra with \( \Delta h = h \otimes h, h \in H \).) For example, \( C_2 \times \cdots \times C_2 \) (\( n \) times) = \( C_2^{\times n} \) is acted upon in a natural way not only by \( S_n \) but by \( S_{n+1} \); the semidirect product is the Weyl group \( D_n \). At present we do not even know if under this operation \( \mathbb{F}_2 C_2^{\times n} \) has an equivariant deformation to a separable algebra. It seems unlikely. On the other hand, an equivariant deformation may be more than one needs. It would be sufficient that the action of the group deform simultaneously with the structure of the algebra in such a way that the group continues to act as automorphisms of the deformed algebra. This is what happens in our solution to the Donald–Flanigan problem for \( D_n \) but the behavior at the prime \( p = 2 \) is different from that at other primes.

As preliminaries we reexamine what it means to deform an algebra to a separable algebra and the concept of a global deformation introduced in [GSps].

2. Tempered successive deformations. Recall that a one-parameter family of deformations of an algebra \( A \) over a ring \( k \) is a \( k[[t]] \)-algebra \( A_t \) whose underlying module is \( A[[t]] \) and which reduces modulo \( t \) to the original algebra \( A \). It follows that multiplication in \( A_t \) can be written in the form \( a \ast b = ab + tF_1(a, b) + t^2F_2(a, b) + \ldots \), where the \( F_i \) are \( k \)-bilinear maps from \( A \) to \( A \) which tacitly are extended to be \( k[[t]] \)-bilinear. Assume that \( A \) is free and of finite rank as a \( k \)-module. Then as a \( k[[t]] \)-module \( A_t \) is also free on the original generators. One can similarly define multiparameter deformations \( A_{t, u, \ldots, v} \) of \( A \) in which the “\( \ast \)” multiplication is given by power series in several variables. But no matter how many parameters are used, if \( A \) is not separable then no such deformation can be separable, for \( A \) remains a quotient of the deformed algebra and a quotient of a separable algebra is always again separable.

For \( k \) a field, saying that “\( A \) has been deformed to a separable algebra” tacitly means that there is some multiplicatively closed subset \( S \) of \( k[[t, u, \ldots, v]] \) such that \( S^{-1}A_{t, u, \ldots, v} \) is separable over the new coefficient ring \( R = S^{-1}k[[t, u, \ldots, v]] \). Here \( R \) is free over \( k \) but when \( k \) is only a (commutative, unital) ring this seems too strong a condition; it should be sufficient (as in all our examples) that \( R \) be projectively faithful over \( k \) although, of course, not of finite rank. This insures, in particular, that \( k \) remains a subring of \( R \) and that a prime of \( h \) is invertible in \( R \) if and only if
it is already so in $k$. Again, if $k$ is a field and there is only one parameter $t$ then it is sufficient to invert $t$; the coefficient ring $R$ then becomes the Laurent series field $k((t))$. Using the separability idempotent, one can see that when $k$ is arbitrary and there are several parameters it is still necessary to invert only a single element of $k[[t, u, \ldots, v]]$. For any single element of $S^{-1}A_{t,u,\ldots,v}$ is already contained in the algebra obtained by a single such inversion and the separability idempotent only involves a finite number of elements. But even in the case of a single parameter, it may not be simply $t$ that one wants to invert but some polynomial in $t$ (whose constant term is not a unit), cf. [GSp].

Suppose now that we try to deform an algebra $A$ to a separable algebra in two stages: first reducing the inseparability by deforming to some $A_t$ and forming $B = S^{-1}A_t$ where $S$ is generated by some single element of $k[[t]]$, and then by deforming $B$ to some $B_u$ and inverting an element of the new coefficient ring. Can the same be done by first performing some two-parameter deformation $A_{t,u}$ of $A$ and then inverting an element of $k[[t,u]]$? We need this in order to justify deforming an algebra to a separable one in successive steps. To understand the reason for caution, suppose that $A$ is a finite-dimensional algebra over a field $k$, that we deform it, and then specialize the deformation parameter. If dim$_k A = n$ and we have chosen a basis $a_1, \ldots, a_n$ then the algebra can be described by its multiplication constants relative to this basis: $a_ia_j = \sum c_{ijl}a_l$. The constants $c_{ijl}$ may be viewed as determining a $k$-point in the variety alg$_n$ of structure constants of $n$-dimensional associative algebras. The deformed algebra has structure constants $c_{ijl}(t)$ lying in $k[[t]]$ which reduce at $t = 0$ to the original constants. These in general will be transcendental over $k$ and define a subvariety in the same component of alg$_n$ as the original. The result of specializing and then performing another deformation can, however yield a point in a different component, since the specialization may lie on the intersection of two components. But intuitively, if one deforms and then, starting from a generic point of the original deformation deforms again, one must remain on the same component.

So suppose that $A$ is free as a $k$-module with basis $a_1, \ldots, a_n$, that we have deformed it to $A_t$ and have formed $B = S^{-1}A_t$ where $S$ is generated by a single element $f(t) \in k[[t]]$. Let $B$ now be deformed to $B_u$. Set $S^{-1}k[[t]] = R$. We may suppose that we have a basis $b_1, \ldots, b_n$ of $B$ with structure constants given by $b_\lambda b_\mu = \sum \gamma_{\lambda \mu \nu}b_\nu$ where the $\gamma_{\lambda \mu}$ initially lie in $R$ but after deformation are elements of $R[[u]]$. The problem is that the coefficients of these power series in $u$ may contain negative powers of $f(t)$ and the negative powers may be unbounded. The hypothesis we make about the deformation $B_u$ is that there is some fixed $N$ such that all $f(t)^N\gamma_{\lambda \mu \nu}(u)$ lie in $R[[t,u]]$. Now we can write each of the $b_\lambda$ as a linear combination of the $a_i$ with coefficients in $R$. Again, these coefficients will generally involve negative powers of $f(t)$. As there are only finitely many coefficients, the hypothesis insures that we can write out the multiplication in $B_u$ in the form $a_i * a_j = \sum c_{ijl}(t,u)a_l$ where there is some fixed $N'$ such that all $f(t)^{N'}c(t,u) \in k[[t,u]]$. But all of the $c(t,0)$ lie in $k[[t]]$. Therefore, replacing $u$ by $f(t)^Nv$ we have an ordinary two-parameter deformation $A_{t,u}$ of $A$ over $k[[t,u]]$. A second deformation with the foregoing boundedness properties will be called tempered (relative to the first). If, finally, inverting an element of $R[[u]]$ with bounded powers of $f(t)$ in the denominators of the coefficients makes $B_u$ separable, then there is an element $F(t,u) \in k[[t,u]]$ inverting which makes $A_{t,u}$ separable. To
return to a one-parameter family of deformations we can now replace \( u \) by some element of \( k[[t]] \) such that \( F(t, u) \neq 0 \).

3. An example. As a simple illustration, let \( C_2 = \{1, a, a^2 = 1\} \) be the two element group and \( G = C_2 \wr C_2 = (C_2 \times C_2) \rtimes C_2 \). The right \( C_2 \) operates by interchange of the two left factors. Denoting its elements by \( \{1, \sigma\} \), we will write the 8 elements of the group as \( \{(1, 1), (a, 1), (1, a), (a, a), (1, 1)\sigma, (a, 1)\sigma, (1, a)\sigma, (a, a)\sigma\} \) where, for example, \( (a, 1)\sigma \cdot (1, a)\sigma = (a, 1)(a, 1)\sigma^2 = (1, 1) \), the unit element of \( G \). Now letting \( k = \mathbb{F}_2 \), we wish to deform the group algebra \( \mathbb{F}_2 G \) to a separable algebra. We can deform \( \mathbb{F}_2 C_2 \) to a separable algebra by setting \( a^2 = ta + 1 + t; \) then \( (1 + a)^2 = t(1 + a) \) so after inverting \( t \) we have orthogonal idempotents \( e = (1 + a)/t \) and \( f = 1 + e \). Doing the same to both factors in \( C_2 \times C_2 \) (and noting that if \( G_1, G_2 \) are finite groups then \( k(G_1 \times G_2) = kG_1 \otimes kG_2 \)) we can perform a first deformation of \( \mathbb{F}_2 G \) to an algebra generated by the four orthogonal idempotents \( e \otimes e, f \otimes f, e \otimes f, f \otimes e \) and a “switch” element \( \sigma \) with \( \sigma^2 = 1 \) such that \( \sigma(x \otimes y)\sigma = (y \otimes x) \) for all \( x, y \in \mathbb{F}_2 C_2 \otimes \mathbb{F}_2 C_2 \). The resulting algebra is still in a natural sense a wreath product. Let \( A \) be the algebra to which \( \mathbb{F}_2 C_2 \) has been deformed. Its coefficients are now in \( \mathbb{F}_2(t) \). We still have \( C_2 \) operating on \( A \otimes A \) by interchange of the tensor factors. Denote the algebra resulting from this first deformation by \( A \wr C_2 \). Further, \( A \) is separable over \( \mathbb{F}_2(t) \) (part of the original inseparability has been removed) and is a direct sum of two subalgebras, \( A = Ae \oplus Af \), each of which is trivially central separable. While \( A \otimes A \) has four central primitive idempotents, \( A \wr C_2 \) is a direct sum of 3 subalgebras corresponding to the orbits of these idempotents under \( C_2 \), namely \( \{e \otimes e\}, \{f \otimes f\}, \{e \otimes f, f \otimes e\} \). The isotropy group of the last orbit is reduced to the identity element of \( C_2 \) and the orbit gives rise to a four dimensional summand of \( A \wr C_2 \) spanned over \( \mathbb{F}_2(t) \) by \( \{e \otimes f, f \otimes e, (e \otimes f)\sigma, (f \otimes e)\sigma\} \). It is easy to check that this is isomorphic to the \( 2 \times 2 \) matrix algebra \( M_2(\mathbb{F}_2(t)) \) and hence is central separable over \( \mathbb{F}_2(t) \). The isotropy groups of the other orbits are non-trivial, being in fact all of \( C_2 \). The subalgebra corresponding to \( \{e \otimes e\} \) is spanned by \( \{e \otimes e, (e \otimes e)\sigma\} \) and should be viewed as isomorphic to \( M_1(\mathbb{F}_2(t))\otimes \mathbb{F}_2(t)C_2 \), where \( C_2 \) here is the isotropy group of the orbit, and similarly for the orbit of \( \{f \otimes f\} \). This will be generalized in §§8, 9. We can now perform a second deformation, treating the direct summands individually, so that the whole algebra now becomes separable over the new coefficient ring. The non-trivial matrix summand is already central separable so it will be left unchanged except for the necessary extension of the coefficient ring. In this second deformation we can not simply set \( \sigma^2 = u\sigma + 1 + u \) (since there is to be no change in the matrix part); in the first summand we set \( [(e \otimes e)\sigma]^2 = u(e \otimes e)\sigma + (1 + u)(e \otimes e) \) and similarly with \( f \) in the second.

It is easy to see that this second deformation is tempered relative to the first. In fact, \( \mathbb{F}_2(t)C_2 \) is actually defined over \( \mathbb{F}_2; \) it is obtained from \( \mathbb{F}_2 C_2 \) just by extension of coefficients. So the only way that \( t \) enters into the second deformation is in the choice of the basis for the algebra obtained by the first deformation. Since there are only a finite number of basis elements, the condition of being tempered will automatically be satisfied whenever they all have coefficients (relative to the original basis) in some \( S^{-1}k[t] \) where \( S \) is generated by a single \( f(t) \in k[t] \). Nevertheless, it may be useful in this example to write explicitly the final result as a two parameter deformation over \( \mathbb{F}_2[[t, u]] \) (in fact, over \( k[t, u] \)) followed by inversion of \( tu \). Since now \( \sigma \neq \sigma \) is no longer the unit element of the algebra, we compute it explicitly.
We can write \( \sigma = [(e + f) \otimes (e + f)] \sigma = [e \otimes f + f \otimes e] \sigma \oplus (e \otimes e) \sigma \oplus (f \otimes f) \sigma \) relative to the decomposition after the first deformation. The second deformation has respected this decomposition, so we can square each direct summand separately. The square of the first is \( e \otimes f + f \otimes e \), that of the second is \((1 + u)(e \otimes e) + u(e \otimes e)\sigma\), and similarly for the third with \( f \) replacing \( e \). Thus \( \sigma \ast \sigma = (e + f) \otimes (e + f) + u(e \otimes e + f \otimes f)(1 + \sigma) \). Writing out \( \sigma \ast \sigma \) in terms of the original basis elements the first summand remains unchanged (being just the unit element of both the original and the deformed algebra) but the second summand must be written as \( t^{-1}u(a \otimes 1 + 1 \otimes a + t \cdot 1 \otimes 1)(1 + \sigma) \). So we need here to replace \( u \) by \( tv \), which in fact will work for all the other products. With this we have a true two-parameter deformation of the original algebra (the parameters now being \( t \) and \( v \)) which becomes separable after inversion of \( t \) and of \( u = tv \), or simply after inversion of \( tv \). To get a one-parameter deformation to a separable algebra we could now set \( u = t^2 \). (Later we will actually use the deformation given by \( a^2 = (q - q^{-1})a + 1 \) with \( q = 1 + t \); the results are the same.)

While we must repeatedly use successive deformations, in the cases we consider it will be evident, as it is here, that the second deformation is tempered, so we may simply omit the discussion of that fact.

4. Global solutions and split global solutions. In the modular case we would like to deform \( kG \) to a separable algebra which is a direct sum of matrix algebras whose summands are in one-one correspondence with the matrix blocks of the complex group algebra \( \mathbb{C}G \), with corresponding blocks having the same dimension. More precisely, let \( k \) now be a field. Write \( kG = A \), denote the deformed algebra by \( A_t \), and suppose that it becomes separable when coefficients are extended to \( k((t)) \). It need not be a sum of matrix algebras but will become one when coefficients are extended to the algebraic closure of \( k((t)) \), i.e., when one forms \( A_t \otimes_{k[[t]]} k((t)) \). But in general there need be no relation between the matrix blocks of this algebra and those of \( \mathbb{C}G \). For example, \( G \) may be abelian but \( kG \) may have non-commutative separable deformations. Suppose, however, that we have a global solution to the Donald–Flanigan problem for \( G \) in the sense of [GSps]: a deformation \( A_t \) of the integral group ring \( \mathbb{Z}G \), together with a multiplicatively closed subset \( S \) of \( \mathbb{Z}[[t]] \) which does not meet the ideal generated by any rational prime dividing \( #G \), such that \( S^{-1}A_t \) is separable over \( S^{-1}\mathbb{Z}[[t]] \). If such a global solution exists then one may assume that \( S \) consists of the powers of a single element \( s \). More important, one can reduce \( S^{-1}A_t \) modulo any rational prime \( p \) not dividing \( #G \). The image \( \bar{s} \) of \( s \) must be of the form \( t^m \epsilon \), where \( \epsilon \) is a unit of \( \mathbb{F}_p[[t]] \). Since a quotient of a separable algebra is separable, the result will be a deformation of \( \mathbb{F}_pG \) which becomes separable when coefficients are extended to \( \mathbb{F}_p((t)) \). Denote the resulting separable algebra by \( A_t(p) \). Now if \( S^{-1}A_t \) is already split, i.e., a direct sum of matrix algebras, then not only must the same be true of each \( A_t(p) \) but the correspondence between the matrix blocks of \( S^{-1}A_t \) and those of \( A_t(p) \) is simply that those of the latter are the quotients of those of the former. Corresponding blocks then certainly also have the same dimensions. Since one can embed \( \mathbb{Z}[[t]] \) in \( \mathbb{C} \) it also becomes clear that the blocks are the same as those of \( \mathbb{C}G \). If \( S \) does contain some primes \( p_1, p_2, \ldots \) dividing \( #G \) then we say that we have a global solution away from \( p_1, p_2, \ldots \). An arbitrary deformation of \( kG \) to a separable algebra will be called a weak solution to the Donald–Flanigan problem. Different weak solutions are sometimes possible. For example \( \mathbb{F}_p(G \times G) \) is deformable.
both to a direct sum of four copies of the new coefficient ring and to a $2 \times 2$ matrix ring over the coefficient ring. A principal result of [GSps] is that there does exist a split global solution to the Donald–Flanigan problem for the symmetric groups $S_n$. In general, however, a global solution if it exists will not be split. For letting $t \to 0$, the existence of a split global solution would imply that all irreducible representations of $G$ are rational, something true of $S_n$ but not in general. This problem would disappear if we had a positive answer to the following question. If $A$ is a separable algebra over a domain $R$, is there always a finite integral extension $\hat{R}$ of $R$ over which $A$ is split? For if $ZG$ is deformed to a separable algebra over some $R = S^{-1}Z[[t, u, \ldots, v]]$, where $S$ is a multiplicatively closed set not intersecting the ideal generated by any rational prime dividing $\#G$, then the same will still be true if the coefficients are extended to $\hat{R}$; if this splits the separable deformation then we can proceed as before. Known results do not get us quite this far; see §10.

Note that for our purposes one can assume that $R$ has characteristic zero and even that all rational primes not dividing $\#G$ are already invertible in $R$, but it does not seem that these assumptions should be necessary. The original definition of a global solution is also too restrictive in that it requires that we start with the integral group ring $ZG$ of the group $G$. It is useful to broaden the definition by allowing replacement of $Z$ by some finite integral extension $O$. This need not be the full ring of integers of some number field but we may generally assume that it is. When $G$ and a single $p|\#G$ are given then a global solution away from all other primes dividing $\#G$ will be called a local solution at $p$. If either it is already split or the conjecture above holds then this gives a canonical correspondence between the matrix blocks of $CG$ and those of the global solution. Suppose now that we have local solutions at several primes dividing $\#G$. Then their blocks must be in natural correspondence with each other, corresponding blocks having the same dimensions, since for each prime they are in correspondence with those of $CG$. The requirement that the local or global solution be split can be eased. All one needs is that it be a direct sum of Azumaya, i.e., central separable algebras. Such an algebra, if it has constant rank at each prime ideal of its coefficient ring is just a twisted form of a matrix algebra, cf., e.g., [KO]. We return to this in §10.

The global solution (with $O$ just $Z$ itself) for the Donald–Flanigan problem for $S_n$ given in [GSps] is essentially its Hecke algebra, $H_n(q)$. Setting $Z_q = Z[q, q^{-1}]$, this is the free module over $Z_q$ with basis elements $T_w$ indexed by the elements $w \in S_n$ and multiplication given as follows: The length $\ell(w)$ is the number of factors in a shortest expression of $w$ as a product of generators $s_i := (i, i+1)$, $i = 1, \ldots, n-1$ of $S_n$. Now set (i) $T_sT_w = T_{sw}$ if $s = (i, i+1)$ for some $i$ and $w \in S_n$ is an element with $\ell(sw) > \ell(w)$, and (ii) $T_s^2 = (q - q^{-1})T_s + 1$. This implies that $T_sT_w = (q - q^{-1})T_w + T_{sw}$ when $\ell(sw) < \ell(w)$. (The same definition extends to any Coxeter group, and in particular to any finite reflection group where instead of the transpositions $s_i = (i, i+1)$ one takes the basic generators $s$.) Writing $1+t$ for $q$ one sees that $H_n(1+t)$ is a deformation of $ZS_n$. Set $i_q := (1-q^2)/(1-q)$ and similarly $i_{q^2} := (1-q^2)/(1-q^2)$; set $n_q! := n_qq(q-1)q^2 \ldots 2q^2$ and $Z_{q,n} := Z[q, q^{-1}, 1/n_q!]$. A main result of [GSps] was that over $Z_{q,n}$ the Hecke algebra $H_n(q)$ becomes a direct sum of matrix algebras. It follows that $H_n(1+t)$ together with the multiplicatively closed subset of $Z[[t]]$ generated by $n_{q^2}! = n_{(1+t)^2}2!$ is a global solution to the Donald–Flanigan problem for the symmetric group. Setting $q = 1$ or equivalently $t = 0$ yields the special case that all the irreducible complex representations of the symmetric group are actually defined over $O$. This is the reason that no extension
of \( \mathbb{Z} \) was needed for \( G = S_n \). In general \( i_q^2 j_q^2 \neq (ij)_q^2 \), so adjoining the inverse of \( m_{q^2} \) generally does not bring with it the adjunction of \( i_q^2 \) for any factor \( i \) of \( m \). In particular, the multiplicatively closed set \( S \) above is not generated by \( n!q^2 \) (\( \neq n_q^2! \)). However, if \( m \) is even, say \( m = 2r \), then \( (2r)_q^2 = (1 + q^r)r_q^2 \), so inverting \( (2r)_q^2 \) also inverts \( r_q^2 \). To make the Hecke algebra of \( S_n \) separable we therefore essentially adjoined to its original ring of definition \( \mathbb{Z}_q \) the inverses of all \( i_q^2 \) for \( i = 2, \ldots , n \). There is no proof that all of these adjunctions are necessary, but we suspect that they are. The numbers \( 2, \ldots , n \) are also the “degrees” of \( S_n \), i.e., degrees of its basic invariant polynomials. This suggests the following refinement of a conjecture in [GSps]. Let \( W \) be a finite Coxeter group and \( H_W(q) \) be its Hecke algebra with coefficient ring \( \mathbb{Z}_q = \mathbb{Z}[q, q^{-1}] \). If \( S \) is the multiplicatively closed subset generated by all \( i_q^2 \) where \( i \) runs through the degrees of \( W \) then \( S^{-1}H_W(q) \) should be separable. We do not know if the deformations of \( ZB_n \) and \( ZD_n \) constructed here are in fact their Hecke algebras, but they do have the following property. Setting \( n_q^2! = n_q^2 (n - 1)_q^2 \cdots 3_q^2 2_q^2 \) and \( \mathbb{Z}_{q,n} = \mathbb{Z}_q[1/n_q^2!] \) the deformed algebras become direct sums of matrix algebras over \( \mathbb{Z}_{q,n} \). Setting \( q = 1 \) recaptures the result that all the complex irreducible representations of \( B_n \) and \( D_n \) are rational. Since this is not the case for the dihedral groups (whose degrees are \( 2, m \)), even if the conjecture is true then the resulting solution is not split. On the other hand, since the product of the degrees is equal to the order of \( W \), it is consistent with the fact that the group algebra \( kW \) is separable over any ring \( k \) in which \( \#W \) is invertible. If this conjecture is true then in principle one could prove it by exhibiting the separability idempotent, but as noted in [GSps] there is so far no good formula for that even in the known case where \( W = S_n \). It may be the case more generally that rings of invariant polynomials associated with a finite group have some connection with the solvability of the Donald–Flanigan problem for that group. The conjecture does not assert that the Hecke algebra is the only global solution to the Donald–Flanigan problem for a finite reflection group (although it may in some sense be the best). For the Weyl groups of type \( B_n \) the degrees are \( 2, 4, \ldots , 2n \), but we shall see that one can obtain a global solution as soon as one has one for \( S_n \), and that requires inverting only all \( i_q^2 \) for \( i = 1, \ldots , n \). However, as remarked above, if a ring contains the inverses of all \( (2i)_q^2 \) for \( i = 1, \ldots , n \) then it already contains the inverses of all \( i_q^2 \).

Infinite Coxeter groups also have Hecke algebras which can be viewed as deformations of their integral group rings. If coefficients are extended to the direct limit \( \mathbb{Z}_{q,\infty} \) of the \( \mathbb{Z}_{q,n} \) then these algebras should possess some properties similar to separability. While \( \mathbb{Z}_{q,\infty} \) is not a field, all the “quantum integers” \( n_q^2 \) have become invertible. While almost a quantum version of the rationals, it is still possible to reduce \( \mathbb{Z}_{q,\infty} \) modulo any rational prime. Although reduction modulo \( t \) does give the rationals, it is certainly not a deformation of the rationals.

5. Geometric rigidity and a generalized global problem. For this and the next section the coefficient ring \( k \) will be a field. We review some basic facts about jump deformations and approximate automorphisms.

Recall that a jump deformation \( A_t \) of a \( k \)-algebra \( A \) is one which is non-trivial and remains constant for generic \( t \neq 0 \). More precisely, if \( u \) is a second variable and coefficients are extended to \( k((t))[u] \) then there is an isomorphism \( A_t \cong A_{(1+u)t} \) which reduces to the identity when \( u \rightarrow 0 \). That is, the trivial deformation of the \( k((t)) \)-algebra \( A \) is equivalent to the deformation \( A_{(1+u)t} \). (For the case...
tial properties of jump deformations, cf. [G2,3; GSck, §7]. A jump deformation simultaneously “breaks some symmetry” of the algebra and “destroys its own infinitesimal”. The meaning of the latter statement is this. Suppose that we have a deformation $A_t$ of $A$. An $n$-cocycle $\zeta \in Z^n(A, A)$ can be lifted to an $n$-cocycle of $A_t$ if there are cochains $\zeta_i \in C^n(A, A)$ such that $\zeta_t = \zeta + t\zeta_1 + t^2\zeta_2 + \cdots \in Z^n(A_t, A_t)$. If $\zeta + t\zeta_2 + \cdots + \zeta_m$ is a cocycle modulo $t^{m+1}$ then there is an obstruction cocycle $\eta \in Z^{n+1}(A, A)$ and we must have $\eta = \delta \zeta_{m+1}$ for some $\zeta_{m+1} \in C^n(A, A)$ in order for the construction to continue. Any cocycle which is liftable to a coboundary is called a jump cocycle. This includes, in particular all coboundaries in $A$ but there may be non-trivial ones. One has $H^n(A_t, A_t) = (\text{liftable } n\text{-cocycles})/(\text{jump } n\text{-cocycles})$. The infinitesimal of a jump deformation is always a jump cocycle, so $\dim H^2(A_t, A_t) < \dim H^2(A, A)$. It follows that $A$ can only undergo finitely many jump deformations.

For $n \geq 3$ every jump cocycle in $Z^n(A, A)$ is the obstruction at some stage to lifting some $\zeta \in Z^{n-1}(A, A)$. In characteristic zero this is true also for $n = 2$. In this case, therefore, if $F$ is the infinitesimal of a jump deformation then $F$ is itself the obstruction to lifting some derivation $\phi \in Z^1(A, A)$ to a derivation of $A_t$. Therefore $e^{t\phi}$ is a formal one-parameter family of automorphisms of $A$ which can not be lifted to $A_t$, so this symmetry is broken. In characteristic $p > 0$ it is possible that after a jump deformation every $\phi \in Z^1(A, A)$ remains liftable, but there is still a symmetry that is broken. For example, in $A = \mathbb{F}_2[x]/x^4$ a derivation is uniquely determined by its value on $x$, which can be arbitrary. This is still the case if we deform $A$ to $A_t = \mathbb{F}_2[x]/(x^4 + tx^2)$ (a jump deformation), so the dimension of the space of derivations has not changed. We have, however, lost some “approximate automorphisms” in the following sense. When we have a derivation $\phi$ of an algebra $A$ of characteristic $p > 0$ it may not be always be possible (as it was in characteristic zero) to construct a full one-parameter family of automorphisms of the form $\Phi_t = \phi + t\phi_1 + t^2\phi_2 + \cdots$. The largest $m$ for which we can construct an automorphism of $A[t]/t^m$ of the form $\Phi_t = \phi + t\phi_1 + \cdots + t^{m-1}\phi_{m-1}$ with a fixed $\phi$ always has $m = p^r$ for some $s$. We call this $m$ the order of the approximate automorphism $\Phi_t$ and also the order of $\phi$. A jump deformation of $A$ reduces the order of some derivation. In the example above, the order of the derivation $\phi$ of $\mathbb{F}_2[x]/x^4$ sending $x$ to $1$ was initially $4$ but was reduced to $2$ after the jump deformation. With a jump deformation the order of some derivation is always reduced; it can not be lifted to an approximate automorphism of the same order, and the infinitesimal of the deformation is the obstruction to continuing the now truncated approximate automorphism to higher order. In this sense, a jump deformation always breaks some symmetry. If $H^1(A, A) = 0$ there can be no jump deformations.

Remark. Following [G2], denote by $\text{Aut}_{m-1} A$ the group of automorphisms of $A[t]/t^m$ of the form $id_A + t\phi_1 + \cdots + t^{m-1}\phi_{m-1}$ where the $\phi_i$ are 1-cochains of $A$, i.e., linear maps $A \to A$ (which are tacitly extended to be $k[t]$-linear). There is a canonical monomorphism $\text{Aut}_{m-1} A \to \text{Aut}_{mn-1} A$ defined by replacing $t$ by $t^n$ and considering the resulting polynomial as one of degree $mn - 1$. The direct limit $\varprojlim \text{Aut}_m A$ contains a subgroup $P$ consisting of the images of all elements in all $\text{Aut}_m A$ which have a prolongation to an element of $\text{Aut}_{m+1} A$. This subgroup is normal by the basic theorem on the additivity of obstructions [G2, Theorem 1], and the quotient $\text{Aut}_m A$ is abelian. It consists of classes of obstructed approximate automorphisms.
since all full one-parameter families of automorphisms beginning with the identity lie in $P$. This group has a natural filtration, as does the space $RH^2(A, A)$ of “restricted” elements of $H^2(A, A)$ (= those which are the obstructions to approximate automorphisms) and the associated graded groups are isomorphic. The group of obstructed approximate automorphisms is thus in a natural way finite dimensional.

In characteristic zero the Euler-Poincaré characteristic of an algebra $A$, if it has one, is just $\sum (-1)^n \dim H^n(A, A)$. If it does, then so does any deformation of $A$ and it is invariant under deformation. In characteristic $p > 0$ for this to be true we shall probably have to replace $\dim H^1(A, A)$ with the dimension of the space of approximate automorphisms.

The original weak Donald–Flanigan conjecture implied, by way of the following theorem ([GGr]) that if $p \mid \# G$ then there exists an element $g \in G$ whose centralizer $C_G(g)$ contains a normal subgroup of index $p$.

**Theorem 1.** Let $A$ be a $k$-algebra which is not itself rigid but which can be deformed to a rigid algebra. Then $H^1(A, A) \neq 0$, i.e., $A$ has a non-inner derivation.

**Proof.** The infinitesimal of any deformation of $A$ to a rigid algebra must be a jump 2-cocycle, since by definition no further deformation of a rigid algebra is possible. Since a jump 2-cocycle exists, $H^1(A, A) \neq 0$. □

Any deformation of a non-separable algebra $A$ to a separable one is necessarily a jump deformation since separable algebras have trivial cohomology and therefore are rigid. It follows that $H^1(A, A) \neq 0$. However, for $A = \mathbb{F}_p G$ we have (cf. [B, Theorem 2.11.12]) an isomorphism of additive groups

$$H^n(\mathbb{F}_p G, \mathbb{F}_p G) \cong \bigoplus H^n(C_G(g), \mathbb{F}_p)$$

where the operation on $\mathbb{F}_p$ is trivial and the sum is over a set of representatives $g$ of conjugacy classes in $G$. So the Donald–Flanigan conjecture implied that for some $g \in G$ one has $H^1(C_G(g), \mathbb{F}_p) \neq 0$. But a derivation into a trivial module is just a morphism, giving what was asserted. Guided by this Fleischmann, Janiszczak and Lempken proved a stronger result [FJL]: If $p \mid \# G$ then there exists a $g \in G$ whose order is divisible by $p$ and whose centralizer $C = C_G(g)$ has the property that its commutator subgroup $C'$ does not contain the $p$-part of $g$. Their proof reduces to the case where $G$ is simple and uses the classification theorem for finite simple groups. (Publication of [FJL] preceded that of [GGr] because of the greater lead time for the latter.)

To resolve the paradox that the Donald–Flanigan conjecture fails while its corollary in [GGr] holds we propose a conjecture which for finite groups is weaker than the Donald–Flanigan conjecture but applies more generally and still implies the statement in [GGr]. Recall first the various concepts of rigidity for an algebra $A$, cf. [GSc]. The first, usually called simply rigidity but more precisely analytic rigidity says that every formal deformation of $A$ is equivalent to the trivial deformation. This will certainly hold if $H^2(A, A) = 0$, often called absolute rigidity. For the second, suppose for the moment that $A$ has dimension $n < \infty$ over some algebraically closed field $k$ and let $\mathcal{V} = \text{alg}_n(k) \subset k^{n^3}$ be the variety of structure constants of $n$-dimensional $k$-algebras. Now $GL(n, k)$ operates on $\mathcal{V}$ and hence on each of its components, with orbits corresponding to isomorphism classes of $n$-dimensional $k$- algebras. One calls $A$, isometrically rigid if the corresponding orbit...
is a Zariski open set in the component $V_A$ of $V$ containing it. In the terminology of Wigner, every algebra in the (necessarily unique) component of $A$ is a contraction of $A$. Equivalently, every algebra which can be deformed to $A$ is a contraction of $A$; this formulation no longer requires finite dimensionality. Analytic rigidity implies geometric rigidity and in characteristic zero they are equivalent [G3, GSck §9], but not in characteristic $p > 0$. In that case there can exist non-trivial formal deformations of $A$ with multiplication $a \ast b = ab + tF_1(a, b) + t^2F_2(a, ab) + \ldots$ in which $F_1$ is not a coboundary but where the deformation becomes trivial when $t$ is replaced by $t^m$ for some $m$; this $m$ is then necessarily a power of $p$, cf. [G3, GSck]. Such restricted deformations can occur only when $A$ has some non-inner derivations $\phi$, for the cocycle $F$ must be the obstruction at some stage to constructing a formal automorphism of the form $\Phi = \text{id}_A + t\phi + t^2\phi_2 + \ldots$ whose infinitesimal is $\phi$. The obstruction to a derivation is always the infinitesimal of a restricted deformation. (See also Skryabin, [Sk]).

Suppose that an algebra $A$ is defined and is a free module of finite rank over an integral extension $\mathcal{O}$ of $\mathbb{Z}$ (so the structure constants with respect to a suitable choice of basis lie in $\mathcal{O}$), and suppose that $A$ becomes rigid over $\mathbb{C}$. In place of Donald–Flanigan we conjecture now that for any prime $p$, $A \otimes \mathcal{O} \mathbb{F}_p$ can be deformed to a geometrically rigid algebra. The conjecture relates the structure of the component $V_A$ of $A$ over $\mathbb{C}$ to the structure of the variety of structure constants in characteristic $p$ at any point representing the reduction of $A$ mod $p$. If one has a finite group $G$ then the hypothesis of having a free module is certainly satisfied (with $\mathcal{O}$ just $\mathbb{Z}$ itself) for the integral group algebra $\mathbb{Z}G$. The conclusion is weaker than that of the Donald–Flanigan conjecture, for if $k$ is a field of characteristic $p$ dividing $\#G$ then it says only that $kG$ can be deformed to a geometrically rigid algebra, not a separable one (all of whose cohomology in positive dimensions vanishes). This, however, is enough to imply that $\mathbb{F}_pG$ has a non-inner derivation. For if the geometrically rigid algebra has a restricted deformation then we are done; otherwise it is analytically rigid, the deformation is just as before a jump deformation whose infinitesimal is the obstruction to some derivation, so in any case the derivation exists. One is then led again to the group-theoretic statement in [GGr].

For group algebras over a field, the conjecture of the next section would imply that they can actually be deformed to geometrically rigid algebras, but to state it we must introduce a broader concept of equivalence of deformations $\ast$ and $\ast'$ of a $k$-algebra $A$. We will call them effectively equivalent if there are integers $m$ and $n$ such that replacing $t$ by $t^m$ in the first and by $t^n$ in the second, the resulting deformations become equivalent in the original sense of [G1]. For finite dimensional algebras $A$ over an algebraically closed field $k$ this can happen in a non-trivial way only in characteristic $p > 0$, where its significance is the following. Choosing a basis for $A$, one has new multiplication constants given by $\ast$ and by $\ast'$ which in both cases are now power series in $t$ reducing to the original multiplication constants when $t = 0$. Each set of constants may now be viewed as a generic point of some subvariety of $k^n$; the deformations are effectively equivalent precisely when these subvarieties coincide. A deformation which is effectively equivalent to a trivial deformation is one previously called restricted but a better name may be effectively trivial. An effective jump deformation will be one which is effectively equivalent to a (non-trivial) jump deformation.

6. Jump algebras. There is an important class of algebras $A$ where (using known
deep results) it is easy to show that $H^1(A, A) \neq 0$. These are those algebras of finite representation type or representation finite algebras which are not already rigid. (Clearly one must exclude, e.g., matrix algebras.) The reason is that the only deformations which these admit are effectively equivalent to jump deformations (and hence, in characteristic zero, they admit only true jump deformations). In this section the coefficient ring $k$ is still a field (although some statements will obviously hold more generally), algebras will be finite dimensional, and a deformation of a $k$-algebra $A$ will be viewed as an algebra over the Laurent series field $k((t))$. A class of algebras is called open if every deformation of an algebra in the class is again in the class. For algebras of a fixed dimension $n$, an open class may be represented as a Zariski open subset of the variety of structure constants of $n$-dimensional algebras. A theorem of Gabriel asserts that the class of representation finite algebras is open. (For an overview of the theory, cf [GR].)

From the preceding section, in any sequence of deformations of a finite dimensional algebra $A$ over a field, only a finite number can be jump deformations or effective jump deformations. Because a representation finite algebra always has a multiplicative basis (i.e., one where the product of two basis elements is a third or zero) there can only be finitely many in any dimension. It follows that any deformation of an algebra of finite representation type which effectively changes its structure, i.e., which is not effectively trivial, must be an effective jump deformation. Therefore, if a representation finite algebra $A$ is not rigid then $H^1(A, A) \neq 0$, else its group of obstructed approximate automorphism would already be reduced to the identity.

In view of this, call an algebra an effective jump algebra if it admits only deformations effectively equivalent to jump deformations. Any deformation of an effective jump algebra is again an effective jump algebra since any finite sequence of deformations of a finite dimensional algebra can be gathered into one multiparameter family. It follows that the class of effective jump algebras is open and contains the class of representation finite algebras. Clearly an effective jump algebra can be deformed by a finite number of effectively jump deformations to an effectively rigid algebra. If it is not already rigid then the preceding argument shows that it must have a non-inner derivation. We now ask, Is every group algebra an effective jump algebra? By Higman’s theorem a group algebra is representation finite if and only if its $p$-Sylow subgroups are cyclic. (For a proof, cf. [P, p.194]) Most group algebras therefore are not representation finite but it is conceivable that all are effective jump algebras. If so, while they can not all be deformed to separable algebras, they all could at least be deformed to effectively rigid ones. Note that while finite representation type is open, it does not imply rigidity. For example, $\mathbb{F}_2C_2$ has finite representation type but is not rigid. It would be useful to know in particular if the group algebra over $\mathbb{F}_2$ of the quaternion group, for which the Donald–Flanigan conjecture fails, is an effective jump algebra.

7. Finite abelian groups and dihedral groups. In this section we show that there is a global solution to the Donald–Flanigan problem for any finite abelian group $\Gamma$, provided that one is allowed to begin with a suitable integral extension of $\mathbb{Z}$. (Donald and Flanigan exhibited only a weak solution.) Since $\Gamma$ is a product of groups of prime power order, it is sufficient to consider the case of $C_r$ where $r = p^m$, a prime power. Its integral group algebra is $A := \mathbb{Z}[x]/(x^r - 1)$, which can be deformed (over $\mathbb{Z}[t]$) to $A_t := \mathbb{Z}[x, t]/(x^r - tx^r - 1)$. This is not yet separable
since we can still reduce modulo the ideal generated by $p$ and $t$ to get an algebra with a non-trivial radical. Now a finitely generated algebra $A$ over a commutative ring $R$ is separable if and only if $A/mA$ is separable over $R/m$ for every maximal ideal $m$ of $R$. Let $d$ be the discriminant of $x^r - tx - 1$ and set $R = \mathbb{Z}[x, t, 1/d]$. For $A$ now take $A_t$ with coefficients extended to $R$, i.e., $A_t \otimes \mathbb{Z}[x, t]$ $R$. Equivalently, letting $S$ be the multiplicatively closed subset consisting of the powers of $d$, we have $A = S^{-1}A_t$. Reduction of this algebra modulo a maximal ideal of $R$ produces an algebra over some finite field $F$ of the form $F[x]/(x^r - \tau x - 1)$, where $\tau$ is a non-zero element of $F$ and where $x^r - \tau x - 1$ has distinct roots because its discriminant is invertible. It is therefore separable. Hence so is $A = S^{-1}A_t$, which is therefore a global solution to the Donald–Flanigan problem for $C_r$. A problem with this solution is that the algebra is generally not split (which here means not isomorphic to a direct sum of copies of $R$), so there is no natural correspondence between its blocks and those of the complex group algebra. We can modify the foregoing procedure to obtain a split solution. The results may look similar but the solutions will be inequivalent. As an example, let $\Gamma$ be the cyclic group $C_3$ of order 3, so defining $A$ as above, the (non-split) global deformation of its integral group algebra is given by by setting $x^3 - tx - 1 = 0$. The discriminant of $x^3 - tx - 1$ is $d = -27 + 4t^3$ and the coefficient ring $R$ will be $\mathbb{Z}[t, d^{-1}]$. Recall that separability of an algebra $A$ over a ring $R$ is equivalent to the existence of a separability idempotent, i.e., an element $e = \sum x_i \otimes y_i \in A \otimes_R A$ such that for all $a \in A$ one has $ae = \sum a x_i \otimes y_i = \sum x_i \otimes y_i a = ea$ and such that $\sum x_i y_i = 1$. (There are many equivalent definitions of the separability of an algebra $A$ over a ring $R$. The existence of a separability idempotent is often the most convenient but we shall need the homological one later.) When, as here, $A$ is a free module over its coefficient ring $R$, calculating the separability idempotent (which in general need not be unique) is equivalent to solving a system of simultaneous linear equations. If $R$ is a domain and the system is consistent then there is a solution in the quotient field of $R$, but it is clear from the process that one need invert only a single element of $R$. In the present case, $A \otimes_R A$ is a free module over $R$ spanned by all $x^i \otimes x^j$, $i, j = 0, 1, 2$ and we seek a linear combination $e = \sum c_{ij} x^i \otimes x^j$ with $\sum c_{ij} x^{i+j} = 1$ and with $xe = \sum c_{ij} x^{i+1} \otimes x^j = \sum c_{ij} x^i \otimes x^{j+1} = ex$, the latter condition being sufficient, since $x$ generates $A$, to insure that $ae = ea$ for all $a \in A$. Replacing $x^3$ by $tx + 1$ and $x^4$ by $tx^2 + x$ gives the equations for the $c_{ij}$. Direct calculation then yields

$$e = d^{-1} \left[ \left(-9 + 4t^3\right)1 \otimes 1 + 2t^2 x \otimes x + 6t(x^2 \otimes x^2 + 1 \otimes x + x \otimes 1) - 4t^2(1 \otimes x^2 + x^2 \otimes 1) - 9(x \otimes x^2 + x^2 \otimes x) \right].$$

Note that $e$ is well-defined modulo any prime, and in particular, modulo 3. If the coefficient ring had been one in which one could divide by 3, then at $t = 0$ the separability idempotent $e$ would reduce to $\frac{1}{3}(1 \otimes 1 + x \otimes x^2 + x^2 \otimes x)$, i.e., to the classical separability idempotent $\frac{1}{\#G} \sum g \otimes g^{-1}$ where here $\#G = 3$ and $g$ runs through the elements of the cyclic group $C_3 = \{1, x, x^2\}$. Since the rational group algebra $\mathbb{Q}C_3$ is not split the present algebra also could not be split, else extending coefficients to include $\mathbb{Q}$ we could let $t \to 0$, giving a contradiction.

To remedy this, return to the group $C_r$ with $r$ a power of a prime $p$, let $\Phi$ be the cyclotomic polynomial for the primitive $r$th roots of unity and take $\mathcal{O} = \mathbb{Z}[y]/\Phi(y)$. Denoting the image of $y$ by $\eta$, instead of $x^r - tx - 1$, take now the polynomial $f(x, t) = \eta^{r-1}(x - \eta^j(1 + t))$. Since $f(x, 0) = x \eta^{r-1}, \mathcal{O}[x, t]/f(x, t)$ is a deformation...
of the group algebra \( \mathcal{O}C_r \). We may now proceed exactly as before, localizing at the multiplicatively closed set \( S \) generated by the discriminant. The result is a global solution which is split and whose blocks, therefore, are now in natural one-to-one correspondence with those of the complex group algebra once one fixes a morphism of \( \mathcal{O} \) into \( \mathbb{C} \). This will always be tacitly understood.

To handle the dihedral groups we need one additional refinement: a split global solution which moreover is equivariant with respect to the automorphism sending every element of the cyclic group to its inverse. For this we need that \( f(x, t) \) be symmetric in the sense that \( f(x, t) = (-x)^i f(x^{-1}, t) \), where \( r = p^m \), as before.

Introduce a parameter \( q \) which will be set equal to \( 1 + t \) and extend \( \mathcal{O} \) to \( \mathcal{O}[q^{-1}] \).

Set

\[
f(x, t) = \begin{cases} 
\Pi_{i=1}^{(r-1)/2}(x - \eta^i q^i) & p \text{ odd}, \\
(x - q)(x - q^{-1})\Pi_{i=1}^{(r/2)-1}(x - q^{2i}\eta^i)(x - q^{-2i}\eta^i) & p = 2.
\end{cases}
\]

Together we have the following.

**Theorem 2.** For every finite abelian group and every finite dihedral group there is a finite integral extension \( k \) of \( \mathbb{Z} \) over which there is a split global solution to the Donald–Flanigan problem. \( \square \)

Now consider again the example of the cyclic group of order 3. The coefficient ring with which one starts is now \( \mathcal{O} = \mathbb{Z}[y]/(1 + y + y^2) := \mathbb{Z}[\omega] \), where \( \omega \) is the image of \( y \). If we set \( s = \omega^2(q^{-1} - q^3) \) then \( f(x, t) \) now has the form \( x^3 - sx^2 + sx - 1 \). Viewing \( s \) as a new parameter, this deformation is not equivalent to the preceding since at the prime 3 no change of variable can remove the quadratic term from \( f \). Thus we have inequivalent deformations leading to distinct separable commutative algebras. The discriminant now is \( \Delta = (s + 1)(s - 3)^3 \) and the separability idempotent is

\[
e = \Delta^{-1}[(s - 3)(s^3 - 3s^2 + s + 3)(1 \otimes 1) + 2s(s - 2)(s + 1)(s - 3)(x \otimes x) + 2s(s - 3)(x^2 \otimes x^2) - s(s - 2)(s + 1)(s - 3)(1 \otimes x + x \otimes 1) + s(s - 1)(s - 3)(1 \otimes x^2 + x^2 \otimes 1) - (2s - 3)(s + 1)(s - 3)(x \otimes x^2 + x^2 \otimes x)].
\]

This is again well-defined modulo 3 and reduces to \( \frac{1}{3}(1 \otimes 1 + x \otimes x^2 + x^2 \otimes x) \) at \( s = 0 \). Recalling that \( q = 1 + t \), this is equivalent to setting \( t = 0 \). (While \( \mathcal{O} \) is not separable over \( \mathbb{Z} \) since modulo 3 it contains a central nilpotent element the deformed algebra is separable over \( \mathcal{O} \), which is all we want.) This new solution is split when \( s \) is replaced by \( \omega^2(q^{-1} - q^3) \). Although it is inequivalent to the preceding one, this does not imply that over \( \mathbb{F}_3 \) the algebra defined by \( x^3 = 1 \) lies on the intersection of two components of the variety of algebras of dimension 3. For over an algebraically closed field there is only one separable algebra of dimension 3, namely, the sum of three copies of the coefficient field, so the deformations lie on the same component. But only for the second (split) one is there a fixed correspondence between its summands and those of \( \mathbb{C}C_3 \).

We will show, in particular, that i) if the Donald–Flanigan problem has a solution (in the weak sense, i.e., at a fixed prime) for some group \( K \) then it also has a solution for the wreath products \( K \wr \Gamma, K \wr S_n \) of \( K \) with any abelian or symmetric group and ii) if there is a split global solution for \( K \), i.e., one which is a sum of
matrix algebras, then the same is true for these wreath products. This will imply, in particular, that the problem has a global solution for the the Weyl groups of type $B_n$ and by minor modifications for the dihedral groups and groups of type $D_n$. For the groups $B_n$ and $D_n$ it will not be necessary to adjoin any roots of unity to $\mathbb{Z}$, so we recover the fact that all their irreducible complex representations also are rational. If instead of a split global solution for $K$ we have only a central separable one, then as mentioned in the introduction we shall need some conditions on the center, $Z$ of the deformed algebra. Suppose that the new coefficient ring is $R$. To insure that $Z$ can be split by a separable, faithfully projective extension of $R$ we shall have to assume, that $Z$ is faithfully projective and of constant rank over $R$.

8. Some smash and wreath products. We will need to know the structure of a smash product $A\#kG$ where $G$ is a finite group and $A$ a $k$-algebra of the form $k^n = k \oplus \cdots \oplus k (n \text{ times})$. The only way that a group $G$ can act as $k$-algebra automorphisms of such an algebra $A$ is by permutation of the summands, in effect, by permutation of the indices $1, \ldots, n$. Denoting the unit element of the $i$th summand of $A$ by $e(i)$, the smash product is spanned by the elements $e(i)\sigma, \sigma \in G$. These multiply by the rule $\sigma e(i) = e(\sigma i)\sigma$, so $e(i)\sigma e(j)\tau = \delta_{i,\sigma j}e(i)\sigma \tau$. It follows that $A\#kG$ is a direct sum of subalgebras corresponding to the orbits of $G$. Fix some $i \in \{1, \ldots, n\}$, denote the orbit of $i$ by $Gi$, the order of this orbit by $m_i$, and the isotropy group of $i$ by $G_i$. If $\{\bar{\sigma},\bar{\tau},\ldots\}$ are representatives of the cosets $G/G_i$ then the various $e(\bar{\sigma} i)\bar{\tau}^{-1}(\bar{\tau})$ span a subalgebra of $A\#kG$ isomorphic to $M_{m_i}(k)$. For writing $e(\bar{\sigma} i)(\bar{\tau})^{-1} = E_{\bar{\sigma},\bar{\tau}}$, these multiply just like matrix units, i.e.,

$$E_{\bar{\mu},\bar{\nu}}E_{\bar{\sigma},\bar{\tau}} = \delta_{\bar{\mu},\bar{\sigma}}E_{\bar{\mu},\bar{\tau}}.$$ 

**Theorem 3.** If $A = k^n$ and $G$ is a finite group operating by permutation of the summands then

$$A\#kG \cong \bigoplus_i M_{m_i}(k) \otimes kG_i$$

where the sum is over representatives $i$ of the distinct orbits of $G$ in $\{1, \ldots, n\}$, $m_i$ is the order of the orbit $Gi$ of $i$, and $G_i$ is the isotropy group of $i$.

**Proof.** Since $A\#G$ is the direct sum of subalgebras corresponding to the orbits of $G$, what we must show is that the subalgebra corresponding to the orbit of $i$ is isomorphic to $M_{m_i} \otimes kG_i$. Letting $\bar{\sigma}, \bar{\tau}, \ldots$ be as before representatives of the cosets $G/G_i$, the $E_{\bar{\sigma},\bar{\tau}} = e(\bar{\sigma} i)\bar{\tau}^{-1}$ span an algebra isomorphic to $M_{m_i}(k)$. This contains, in particular, the permutation matrices, so if we have any permutation $\pi$ of $\{1, \ldots, n\}$ there is a unique matrix $T_\pi \in M_{m_i}(k)$ giving the same permutation. Thus the operator $T_\pi^{-1}\pi$ acts as the identity on $\{1, \ldots, n\}$ and hence commutes with the operation of all elements of $M_{m_i}(k)$. Therefore $T_\pi^{-1}\pi T_\rho^{-1}\rho = T_\rho^{-1}T_\pi^{-1}\pi\rho = T_\pi^{-1}\pi\rho$. The map $\pi \to T_\pi^{-1}\pi$ is therefore a group morphism. Since the elements $E_{\bar{\sigma},\bar{\tau}}\rho$ with $\rho \in G_i$ form a $k$-basis for $A$, this proves the assertion. \[\square\]

Note here that the elements of the isotropy group do not themselves commute with the elements of $M_{m_i}$. The second tensor factor in the statement of theorem is not the group algebra of the isotropy group $G_i$ itself but the group algebra of the isomorphic group of all $T_\pi^{-1}\pi$. An analogous argument is used in the next two theorems. One consequence of the above theorem is that if we have an algebra of the form $A = k^n \#kG$ such that the Donald–Flanigan problem is solvable for all the isotropy groups $G_i$ then it is solvable for all of $A$. 
In addition to the foregoing we need to know the structure of \( A \wr G \) when \( A \) is a direct sum of central separable algebras over some coefficient ring \( k \). In this section we compute this when \( A \) is itself central separable and find that \( A \wr G \) is canonically isomorphic to \( A \otimes^n \otimes kG \). This makes it possible to extend any deformation of \( kG \) to one of \( A \wr G \). Again, while \( kG \) is contained in a natural way in both \( A \wr G \) and \( A \otimes G \), the isomorphism, while canonical, does not carry one embedding to the other. The result is extended to the general case in the next section.

Denote the group of invertible elements of \( A \) by \( A^\ast \), its group of automorphisms by \( \text{Aut} \, A \), and the center of \( A \) by \( Z \). There is a group morphism \( A^\ast \to \text{Aut} \, A \) sending \( a \in A^\ast \) to the inner automorphism \( \text{in}_a \) defined by \( \text{in}_a \, x = axa^{-1} \); the kernel of this morphism is \( Z^\ast \). For a central simple algebra over a field the Skolem-Noether theorem asserts that the morphism is onto, but this need not be so for an arbitrary central separable algebra. Action of a group \( G \) on \( A \) is the same as a group morphism \( f : G \to \text{Aut} \, A \). The morphism \( f \) “factors through” the canonical morphism \( A^\ast \to \text{Aut} \, A \) if there is a morphism \( G \to A^\ast \) whose composite with the canonical morphism \( A^\ast \to \text{Aut} \, A \) is \( f \). This is stronger than asserting that the image of every individual \( \sigma \in G \) is inner (which would always be the case for a central simple algebra). For if \( \sigma \in G \) then the choice of an element \( a_\sigma \in A \) conjugation by which has the same effect as \( \sigma \). We compute this when \( A \) is a primitive idempotent. Consider e.g. \( 2 \times 2 \) matrices in characteristic 2. Here the trace of left multiplication by \( E_1 \) is zero but the reduced trace is 1. Now

**Lemma A.** Suppose that a group \( G \) operates as a group of automorphisms on a \( k \)-algebra \( A \) and that the associated morphism \( G \to \text{Aut} \, A \) factors through \( A^\ast \). Then \( A \# kG \cong A \otimes kG \).

**Proof.** To avoid confusion, denote the image of \( x \in A \) under the operation of \( \sigma \in G \) by \( x^\sigma \). By hypothesis, for each \( \sigma \in G \) there exists an invertible element \( a_\sigma \) such that \( a_\sigma x a_\sigma^{-1} = x^\sigma \) and \( a_\sigma a_\tau = a_{\sigma \tau} \) all \( \sigma, \tau \in G \). It follows that the elements \( a_\sigma^{-1} \sigma \in A \# kG \) commute with all elements of \( A \) and form a multiplicative subgroup of \( G \) of \( A \# kG \) isomorphic to \( G \). But then \( A \# kG = A \otimes kG' \cong A \otimes kG \). For if \( \{ a_i \} \) is a \( k \)-basis for \( A \) then every element of \( A \# kG \) can obviously be written uniquely as a linear combination with coefficients in \( k \) of the elements \( a_i \otimes a_\sigma^{-1} \sigma \).

Suppose now that \( A \) is a central separable \( k \)-algebra. Every \( a \in A \) then has a reduced trace \( \text{T} \text{rd}(a) \in k \); when \( A = M_n(k) \) this is just the usual trace of a matrix (cf. [KO, pp. 90, 110]). (The “ordinary” trace of \( A \) comes from the characteristic polynomial of left multiplication by \( a \) in the algebra. This can vanish even for a primitive idempotent. Consider e.g. \( 2 \times 2 \) matrices in characteristic 2. Here the trace of left multiplication by \( E_1 \) is zero but the reduced trace is 1.) Now
a central separable $k$-algebra $A$ always contains $k$ as a $k$-module direct summand so the map $a \mapsto \text{Trd} a$ may be viewed as $k$-module endomorphism of $A$. Since for central separable $A$ there is an isomorphism $A \otimes A^{\text{op}} \rightarrow \text{End}_k A$ sending $x \otimes y$ to the endomorphism carrying $a \in A$ to $xay$, it follows that there exists a unique element $T = \sum x_i \otimes y_i \in A \otimes A$ such that $\text{Trd} a = \sum x_i a y_i$. Viewing this “switch” element $T$ as an element of $A \otimes A$, it has the properties i) $T^2 = 1 \otimes 1$, the unit element of $A \otimes A$ and ii) $T(a \otimes b)T = b \otimes a$ for all $a, b \in A$. (This result, communicated by A. Fröhlich to Knus and Ojanguren is credited by them [KO, p.112] to O. Goldman who with M. Auslander developed the present concept of separable algebra, [AG].)

For $A = M_n(k)$ one has $T = \sum_{i,j} e_{ij} \otimes e_{ji}$. It follows from the preceding Lemma that if $C_2$ operates on $A \otimes A$ by interchange of the tensor factors, then $A \wr C_2 = A^{\otimes 2} \# kC_2$ is canonically isomorphic to $A^{\otimes 2} \otimes kC_2$. The isomorphism is canonical because there is a canonical choice of an element in $A \otimes A$ conjugation by which interchanges the tensor factors, namely $T$. Note, incidentally, that $T$ is symmetric since $T = T^3 = T(T)T$, i.e., $T$ with its tensor factors interchanged.

**Theorem 4.** Let $A$ be a central separable $k$-algebra and let $S_n$ operate on $A^{\otimes n}$ by permutation of the tensor factors. Then there is a canonical isomorphism $A \wr S_n \simeq A^{\otimes n} \otimes kS_n$.

**Proof.** The group $S_n$ is generated by elements $T_{12}, T_{23}, \ldots, T_{n-1,n}$ corresponding to the transpositions $(12), (23), \ldots, (n-1,n)$ subject only to the conditions that $T_{i,i+1}$ and $T_{j,j+1}$ commute for $|i - j| > 1$, that $T_{i,i+1}^2 = 1$, and the braid relation $T_{i,i+1}T_{i+1,i+2}T_{i,i+1} = T_{i+1,i+2}T_{i,i+1}T_{i+1,i+2}$; this is the “Artin presentation” of $S_n$. Now let $T_{i,i+1}$ denote the switch element $T$ operating in places $i$ and $i+1$ of $A^{\otimes n}$. It will be sufficient to show that these satisfy the relations of the Artin presentation. Since $T^2 = 1$ the first relations are clearly satisfied. For the braid relations it is only necessary to consider the case $n = 3$ where we must prove that $T_{12}T_{23}T_{12} = T_{23}T_{12}T_{23}$. If $T = \sum x_i \otimes y_i$ then the left side is $\sum (x_i \otimes y_i \otimes 1)(1 \otimes x_j \otimes y_j)(x_k \otimes y_k \otimes 1) = \sum x_i x_k \otimes y_i y_k \otimes y_j$. However, $\sum x_i x_k \otimes y_i y_j y_k = T(1 \otimes x_j)T = x_j \otimes 1$, so this is just $\sum x_j \otimes 1 \otimes y_j$ (which would naturally be denoted $T_{13}$). One checks similarly that the right side gives the same thing. \[\square\]

It was convenient in the preceding to deal with $S_n$ but it is clear that nothing would change if instead of $S_n$ itself we had a group $G$ which operated by permutation of the tensor factors, i.e., through a morphism $G \rightarrow S_n$. This gives the following.

**Corollary.** If $A$ is central separable over $k$ and a group $G$ operates on $A^{\otimes n}$ through a morphism $G \rightarrow S_n$ then $A \wr G$ is canonically isomorphic to $A^{\otimes n} \otimes kG$. \[\square\]

It follows that any deformation of $kH$ induces one of $A \wr H$.

9. **Wreath product with a sum of central separable algebras.** Suppose now that the $k$-algebra $A$ is a direct sum of central separable algebras, $A = A_1 \oplus \cdots \oplus A_r$ with respective unit elements $e_1, \ldots, e_r$ and let $Z$ denote the commutative algebra which these generate. Extending the result of the previous section, we wish to determine the structure of $A \wr S_n$ and more generally of $A \wr G$ where $G$ is a group operating on $A^{\otimes n}$ by permutation of the tensor factors. For any multiindex $I = (i_1, \ldots, i_n)$ set $A(I) = A_{i_1} \otimes A_{i_2} \otimes \cdots \otimes A_{i_n}$ and denote its unit element $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n}$ by $e(I)$. Then $A^{\otimes n}$ is the direct sum of the $A(I)$, which are all central separable, one has $e(I)e(J) = \delta_{ij}e(I)$, and the distinct $A(I)$ are mutually orthogonal. The wreath product $A \wr G$ is spanned by elements of the form $I(g)$, and hence is a $k$-orthonormal basis for $A^{\otimes n}$.
form \(a(I) \otimes \sigma, a(I) \in A(I), \sigma \in G\), which we write simply as \(a(I)\sigma\). One has 
\[ e(I)e(J)\tau = e(I)e(\sigma J)\sigma \tau, \]
where \(\sigma(j_1, \ldots, j_n) = (j_{\sigma^{-1}1}, \ldots, j_{\sigma^{-1}n})\). We write
\((a_1, \ldots, a_n)\) for \(a_1 \otimes \cdots \otimes a_n \in A^{\otimes n}\) where each \(a_i\) is in \(A\), and if \(\alpha = (a_1, \ldots, a_n)\) then \(\sigma \alpha = \alpha^\sigma \sigma\), where \((a_1, \ldots, a_n)\sigma = (a_{\sigma^{-1}1}, \ldots, a_{\sigma^{-1}n})\).

Set \(B = A\ell G\), choose a multiindex \(I\), and let \(B(I)\) be the subalgebra generated by \(A(I)\) and the elements of \(G\). It is spanned by all \(\alpha^\sigma \sigma\). One has \(B(I) = B(\sigma I)\) for all \(\sigma \in G\), so these subalgebras are really indexed by the orbits of \(G\) in the set of multiindices, and \(B\) is their direct sum. Determining the structure of \(B(I)\) is thus the same as determining that of the various \(B(I)\). If there is a global solution to the Donald–Flanigan problem for each of the “orbital” subalgebras \(B(I)\) then the same will be true for \(A\ell kG\), and similarly if there is only a weak solution for each of the subalgebras then the same will be hold for \(A\ell kG\). Fixing the multiindex \(I\), let \(G_I\) be the isotropy group of \(I\). Now choose fixed representatives \(\sigma, \tau, \ldots\) for the cosets \(\sigma G_I, \tau G_I, \ldots\). The elements \(e(\bar{\sigma}I)\bar{\tau}^{-1}\) then multiply precisely like the matrix units \(E_{\bar{\sigma}, \bar{\tau}}\) (whose rows and columns are indexed by the chosen coset representatives), namely
\[ (e(\bar{\sigma}I)\bar{\mu}^{-1})(e(\bar{\sigma}I)\bar{\sigma}^{-1}) = \delta_{\bar{\sigma}, \bar{\tau}}e(\bar{\sigma}I)\bar{\sigma}^{-1}. \]

The choice of coset representatives \(\{\bar{\mu}, \bar{\sigma}, \ldots\}\) fixes particular isomorphisms \(A(I) \rightarrow A(\sigma I)\) and hence also isomorphisms \(A(\sigma I) \rightarrow A(\bar{\mu}I)\) for all \(\sigma \in G\). We can write the elements of \(A(\bar{\sigma}I)\) in the form \(\bar{a}\) with \(\bar{a} \in A(\bar{\sigma}I)\). Since \((a^\bar{\mu}\bar{\sigma}^{-1})(b^\sigma\bar{\tau}^{-1}) = (ab)^\bar{\mu}\bar{\Sigma}^{-1}\bar{\tau}^{-1}\) the elements of the form \(a^\bar{\mu}\bar{\sigma}^{-1}\) span a subalgebra \(M(I)\) of \(A\ell G\). Let the index of \(G_I\) in \(G\) be \(m\) (the cardinality of the set of representatives). Then the subalgebra \(M(I)\) is isomorphic to \(M_m(A(I))\) under the map \(a^\sigma\bar{\tau}^{-1} \mapsto aE_{\bar{\sigma}, \bar{\tau}}\).

We will show that \(B(I) \cong M_m(A(I)) \otimes kG_I\), but as in the previous section, the tensor factor \(kG_I\) here is not the usual subalgebra \(kG_I\) of \(A(I)\). With the foregoing notations and writing \(\alpha^\bar{\sigma}\) for \(\pi(\alpha), \alpha \in A(I), \pi \in G\) we have the following.

**Lemma B.** For every \(\pi \in G_I\) there is a canonical choice of \(T_\pi \in A(I)\) such that \(T_\pi \alpha T_\pi^{-1} = \alpha^\bar{\pi}\).

**Proof.** Suppose that \(A(I) = A(i_1, \ldots, i_n) = A_{i_1} \otimes \cdots \otimes A_{i_n}\). For convenience, reorder the tensor factors, putting the indices in increasing order, so that the tensor product has the form \(A_{i_1}^{\otimes n_1} \otimes \cdots \otimes A_{i_r}^{\otimes n_r}\) where \(n_1 + \cdots + n_r = n\). Since \(\pi I = I\), after the reordering \(\pi\) permutes the first \(n_1\) factors amongst themselves, the next \(n_2\) factors amongst themselves, etc.. From the preceding section we know that there is a canonical \(T^{(1)}_\pi = A_{i_1}^{\otimes n_1}\) such that for any \(\alpha_1 \in A_{i_1}^{\otimes n_1}\) we have \(T^{(1)}(\alpha_1)T^{(1)}(\alpha_1)^{-1} = (\alpha_1)^{\bar{\pi}}\), and similarly for the other \(A_{i_r}^{\otimes n_r}\). Then \(T^{(1)}_\pi \cdots T^{(r)}_\pi\) with factors returned to their original order is the desired \(T_\pi\).

Now set \(v(\pi) = \sum \bar{\mu}T^{-1}_\pi \bar{\pi}^{-1}\bar{\mu}^{-1}\), where the sum runs over all coset representatives \(\bar{\mu}\) of \(G_I\). Since \(T^{-1}_\pi \bar{\mu}^{-1} = (T^{-1}_\pi)^{\bar{\mu}}\bar{\mu}^{-1}\) the terms for different \(\bar{\mu}\) are mututally orthogonal, from which it is easy to see that \(v(\pi)v(\pi') = v(\pi\pi')\) for \(\pi, \pi' \in G_I\). Since \(v(1) = \sum \bar{\mu}e(I)\bar{\mu}^{-1} = \sum e(\bar{\mu}I)\) is just the identity element of \(B(I) = \oplus A(\bar{\mu}I), v(\pi)\) form a multiplicative subgroup of \(B(I)\) isomorphic to \(G_I\). Moreover, if \(\alpha \in A(I)\) then \(v(\pi)\) also commutes with all \(a^\bar{\sigma}\).

**Lemma C.** There is a canonical isomorphism \(B(I) \cong M(I) \otimes kG_I\) given by
\[ a^\bar{\sigma}T_\pi^{-1}v(\pi) \mapsto aE_{\bar{\sigma}, \bar{\tau}}\otimes \pi. \]

**Proof.** Observe that \(a^\bar{\sigma}\bar{\tau}^{-1}\) is orthogonal to all the summands of \(v(\pi) = \sum \bar{\mu}T^{-1}_\pi \bar{\mu}^{-1}\) except that with \(\bar{\mu} = \bar{\tau}\), so \(a^\bar{\sigma}\bar{\tau}^{-1}v(\pi) = (aT^{-1}_\pi)^{\bar{\sigma}}\bar{\tau}^{-1} = (aT^{-1}_\pi)^{\bar{\sigma}}\bar{\tau}^{-1}. \)

As \(\alpha\) runs through \(A(I)\), \((aT^{-1}_\pi)^{\bar{\sigma}}\tau\) runs through \(A(\bar{\sigma}I)\). As \(\sigma\) runs through \(G\), so does \(a\). Thus \(\alpha^\bar{\sigma}\bar{\tau}^{-1}\) runs through the subalgebra \(kG_I\) of \(A(I)\), and so is orthogonal to all other terms in \(v(\pi)\).
does $\pi^{-1}$ and $\tau\pi^{-1}$ runs through the coset of $\tau$. Consequently, the $\tau\pi^{-1}$ exhaust $G$ and therefore so do their inverses, and hence, for any fixed $\sigma$ so do the $\sigma\pi\tau^{-1}$. So for fixed $\sigma$, the set of all $a^\sigma\sigma\pi\tau^{-1}v(\pi)$ with varying $\tau, \pi$ is precisely $B(I)$, from which the result follows.  

We therefore have the following.

**Theorem 5.** Let $A = A_1 \oplus \cdots \oplus A_r$ be a sum of central separable $k$-algebras $A_i$, and suppose that a group $G$ operates on $A^{\otimes n}$ by permutation of the tensor factors. For every multiindex $I = (i_1, \ldots, i_n)$ set $A(I) = A_{i_1} \otimes \cdots \otimes A_{i_n}$, let $G_I$ be the isotropy group of $I$, and let \{I, J, \ldots\} be a set of representatives of the orbits of $G$ in the set of multiindices. Set $m_I = (G : G_I)$, the index of $G_I$ in $G$. Then $A \wr G$ is canonically isomorphic to $\bigoplus M_{m_I}(A_I) \otimes kG_I$ where the sum runs over the orbits of $G$ in the set of multiindices.  

There is an overlap between this theorem and Theorem 3, namely, the case where each $A_i$ is just $k$ itself and one has a wreath product, but Theorem 3 allows more general smash products. As an immediate corollary of Theorem 5 one gets explicitly the structure of the complex group algebra (and hence the irreducible complex representations) of a wreath product of groups $H \wr G$. More generally, if $k$ is a splitting ring for both $H$ and all the $G_I$, i.e., one such that both $kH$ and the $kG_I$ are sums of central separable algebras, then the theorem implies that $k$ is also a splitting ring for $H \wr G$ and it gives the structure of $k(H \wr G)$. Consider, for example, $B_n = C_2 \wr S_n$. Here $Q$ is a splitting field for $C_2$. Writing $C_2 = \{1, a\}$ and setting $e = (1 + a)/2, f = (1 - a)/2$, we have $QC_2 = Qe \oplus Qf$, a direct sum of two copies of $Q$. The idempotents $e(I)$ are here just tensor products of length $n$ of $e$’s and $f$’s. If $e$ occurs $m$ times and $f$ occurs $n - m$ times then the order of its orbit is $\binom{n}{m}$ and the corresponding isotropy group is isomorphic to $S_m \times S_{n-m}$. Since $Q$ is also a splitting field for the symmetric group, it is a splitting field for $B_n$, so the irreducible complex representations of $B_n$ are all rational. The theorem then gives

$$QB_n = \bigoplus_{m=0}^{n} M_{\binom{n}{m}}(Q) \otimes QS_m \otimes QS_{n-m}.$$  

This includes the classic result that the representations of $B_n$ are indexed by all ordered pairs consisting of a representation of $S_m$ and a representation of $S_{n-m}$. For if $\lambda, \mu$ are partitions of $m$ and $n - m$, respectively, and if $S_m(\lambda)$ is the block (=matrix algebra summand) of $QS_m$ corresponding to $\lambda$ and $S_{n-m}(\mu)$ the block of $QS_{n-m}$ corresponding to $\mu$, then the blocks of $QB_n$ are of the form $M_{\binom{n}{m}}(Q) \otimes S_m(\lambda) \otimes S_{n-m}(\mu)$. The representations $B_n$ are thus indexed by the pairs $(\lambda, \mu)$. Reversing their order gives a different representation. By contrast (cf. §11), the representations of $D_n$ may be viewed as indexed by unordered pairs of representations of $S_m$ and $S_{n-m}$ (or by pairs in which $m \leq n - m$).

10. **First applications.** The most immediate application of the results of the preceding section is to the case where we have a group $H$ for which there is a split global solution to the Donald–Flanigan problem and we seek one for a group of the form $H \wr G$. So suppose that $k$ is some subring of algebraic integers in $\mathbb{C}$ and that $kH$ has been deformed into a direct sum $A = M_{i_1}(R) \oplus \cdots \oplus M_{i_s}(R)$ of matrix algebras. Here the new coefficient ring $R$ is the localization of a power series ring $k[[t_1, \ldots, t_r]]$ at a single element $t$. Since $R$ is a domain, the inclusion of $k$ into $\mathbb{C}$
can be extended to a monomorphism of $R$ into $C$. If we have been careful with $R$ to require, e.g., that it is projectively faithful over $k$, then we can still reduce modulo every prime $p$ of $k$ (see below), so there is a natural correspondence between the blocks of $\mathbb{C}H$ and those we get by reducing $A$ modulo $p$. By assumption there is given some morphism $G \to S_n$, and through this $G$ operates on $H \otimes C$ by permutation of the tensor factors. Keeping the notations, the results of the preceding section immediately yield the following.

**Theorem 6.** If there is a split global solution for $R \ltimes G$ and in $H \ltimes G$ for each isotropy group $G_i$ there is a split global solution to the Donald–Flanigan problem for $R \ltimes G_i$ then there is one for $H \ltimes G$ over $k$. (If there is only a weak solution for $R \ltimes G_i$ then there is a weak solution for $H \ltimes G$ and similarly for local solutions.) $\square$

There are several important cases in which the conditions on $G$ will be fulfilled. First, if $G$ is abelian then so is every isotropy group; we have already constructed split global solutions for finite abelian groups. Second, suppose that $G$ is all of $S_n$. The isotropy group of an index of the form $(i_1^{n_1}, \ldots, i_r^{n_r})$, where $i^m$ stands for $i$ repeated $m$ times and $n_1 \cdot \cdot \cdot + n_r = n$ is $S_{n_1} \cdot \cdot \cdot \times S_{n_r}$. Since we have split global solutions for the symmetric groups we also have a split global solutions for these.

**Theorem 7.** There is a split global solution to the Donald–Flanigan problem for any wreath product $H \ltimes G$ where $H$ is finite abelian and $G$ is either finite abelian or $G = S_n$. $\square$

The Weyl groups of type $B_n$ are a special case. Since both for $C_2$ and $S_n$ we can start with the coefficient ring $\mathbb{Z}$ we do not have to adjoin any roots of unity.

**Theorem 8.** There is a split global solution to the Donald–Flanigan problem for the Weyl groups of type $B_n$ starting with $\mathbb{Z}$. For the coefficient ring after deformation one can take $R = \mathbb{Z}_{q,n}$ since there is a split global deformation of $S_n$ defined over this $R$. It follows that all complex irreducible representations of the Weyl groups of type $B_n$ are rational. $\square$

To handle dihedral groups in this context, consider $S_2$ acting on an abelian group $K$ by sending every element of the latter to its inverse. We can form the semidirect product $H \rtimes S_2$; for $H$ the cyclic group $C_n$ this is just the dihedral group $D_n$. Let $A$ be a split equivariant solution to the Donald–Flanigan problem for $H$. (Here, if $e$ is the exponent of $H$ then we must work over $\mathbb{Z} = \mathbb{Z}[y]/\Phi(y)$, where $\Phi$ is the cyclotomic polynomial for the primitive $e$-th roots of unity.) This $A$ is then just a direct sum of copies of whatever coefficient ring $k$ has been introduced, and $S_2$ acts as automorphisms of the sum. However, since the automorphism group of each summand is reduced to the identity, $S_2$ must either leave an individual summand fixed or exchange them in pairs. The equivariant deformation has thus produced a direct sum of copies of $kS_2$ plus copies of $k \oplus k$ on which $S_2$ operates by interchange of the summands. As a special case of what we have seen before, the latter yields $2 \times 2$ matrix algebras over $k$, while $kS_2$ itself has a global deformation to a sum of two copies of $k$ (extended by the deformation parameter). Therefore we have

**Theorem 9.** There is a split global solution to the Donald–Flanigan problem for any semidirect product $H \rtimes S_2$ where $H$ is a finite abelian group. $\square$

Notice that the argument implies the familiar fact that all complex irreducible representations of a dihedral group $D_n$ have dimension either one or two and are given by characters of $Z_2$. Further, if $H$ is a symmetric group, then the above implies that all complex irreducible representations of $H$ have dimension at most two. Therefore we have

**Theorem 10.** All complex irreducible representations of $S_n$ have dimension at most two. $\square$
are defined over $\mathbb{Q}(\eta)$ where $\eta$ is a primitive $m$-th root of unity (but in particular cases may be defined over a smaller field). Since every subgroup of a dihedral group is again either dihedral or abelian we have the following.

**Theorem 10.** There is a split global solution to the Donald–Flanigan problem for any group of the form $H \wr D_m$ with $H$ abelian. □

Suppose now that $H$ is a group for which the Donald–Flanigan problem is solvable over the coefficient ring $k$ but where the solution is a non-split separable algebra $A$ over the new coefficient ring $R$. Now $A$ is Azumaya (central separable) over its center $Z$, which in turn is separable over $R$ (cf., e.g., [DI], p. 55). We will be able to use the preceding results if we can “split” $Z$ by a suitable extension $T$ of the coefficient ring $R$, i.e., reduce it to a direct sum of copies of the new coefficient ring. For this we can apply the following basic result: For a commutative $R$-algebra $S$ the following properties are equivalent: 1) $S$ is separable, faithfully projective (i.e., faithful and projective) and of constant rank over $R$, 2) there is a faithfully projective and separable $R$-algebra $T$ splitting $S$, i.e., such that $S \otimes_R T \cong T \oplus T \oplus \cdots \oplus T$ ($n$ times) = $T^n$ and 3) there is a faithfully flat and separable $R$-algebra $T$ such that $S \otimes_R T \cong T^n$ (cf. [KO], Théorème 4.7, p. 88). Projectivity and faithfulness of $T$ will insure, in particular, that i) if $p$ is a rational prime then the sequence $0 \to pA \otimes T \to A \otimes T \to (A/pA) \otimes T \to 0$ is exact and ii) the map of $A$ into $A \otimes T$ given by $a \mapsto a \otimes 1$ is an inclusion, and similarly for $A/pA$ in place of $A$. Also, if we have an inclusion $R \hookrightarrow \mathbb{C}$ then we still have an inclusion $T = R \otimes T \hookrightarrow \mathbb{C} \otimes T$ (where $\mathbb{C}$ is regarded as an $R$-module by the first inclusion).

If $k$ is a direct sum or product of subrings $k_i$ then the Donald–Flanigan problem for a group $G$ will be solvable if it is solvable for each $k_iG$. To avoid the complications of infinite sums or products, we may restrict attention to the case where $k$ has only finitely many idempotents. This holds, in particular, if $k$ is noetherian, which is always the case here. With this, we can reduce to the case where the only non-zero idempotent in $k$ is its identity. A projective module over $k$ must then have constant rank. If $k$ has only finitely many idempotents, so $k = k_1 \oplus \cdots \oplus k_r$, where each summand has no idempotent but the identity, then any $k$-algebra $A$ is similarly a sum of $k_i$-algebras $A_i$. If each of these can be deformed to a separable $k_i$-algebra then $A$ can be deformed to a separable $k$-algebra.

Deformation can not enlarge the center of an algebra but it can diminish it. We will say that a deformation is centrality preserving if elements which were central before deformation continue to be central. (It need not be center preserving because in general the center is deformed.) This must be the case, in particular, for a solution to the Donald–Flanigan problem for $kG$ which resembles the complex group algebra. For any finite group $G$ and coefficient ring $k$, the center of $kG$ is a free module over $k$; a typical basis element is just the sum of the elements of some conjugacy class. The center of a centrality preserving deformation of a group algebra thus continues to be a free module over the new coefficient ring $R$, so condition 1) above will hold without the assumption that $k$ is noetherian. (Of course it holds if the original group is commutative and the deformed group algebra remains so, but there are important examples of deformations where the commutativity is lost.)

11. Weyl groups of type $D_n$. To describe the Weyl groups of type $D_n$, write the elements of $B_n$ in the form $(c_1, \ldots, c_n)\sigma$ with $c_i \in C_2$, $\sigma \in S_n$; then $D_n$ is the subgroup consisting of all elements in which an even number of the $c_i$ are equal.
to the unit element of $C_2$. Equivalently, the product of the $c_i$ is equal to the unit element. So viewed, $D_n$ is an example of a class of semidirect products which are subgroups of wreath products arising as follows. Let $H$ be any abelian group (not necessarily finite) and consider the morphism $\mu : H^n = H \times \cdots \times H$ ($n$ times) $\to H$ sending $(c_1, \ldots, c_n)$ to the product $c_1 c_2 \ldots c_n$. If $K$ is any subgroup of $H$ then $\mu^{-1}K$ is stable under the operation of $S_n$ on the factors of $H^n$ so one can form the semidirect product $(\mu^{-1}K) \ltimes S_n$. For $n = 2$, $H$ cyclic and $K$ reduced to the unit element one gets the dihedral groups. For $H = C_2$ and $K$ the unit subgroup one gets the groups $D_n$. Writing simply $C_2^n$ for $C_2^{\times n}$, here $\mu^{-1}K \cong C_2^{n-1}$ so the Weyl group $D_n$ is a semidirect product $C_2^{n-1} \ltimes S_n$. (One has $D_2 = S_2, D_3 = S_4$.)

It will be convenient to deal with $D_{n+1} = C_2^n \ltimes S_{n+1}$ rather than $D_n$. The deformation of $C_2^n$ we employed in constructing a solution for $B_n$ was equivariant under the operation of $S_n$, but in $D_{n+1}$ this group is acted upon by $S_{n+1}$ and the deformation ceases to be equivariant. The original operations of $S_{n+1}$ consequently cease to be automorphisms, but the operations can also be so deformed that $S_{n+1}$ continues to operate as automorphisms of the deformed algebra. This would normally be a difficult homological problem but fortunately here there is a natural solution (although it exhibits a rather unnatural phenomenon at the prime $p = 2$). To exhibit it, however, we first examine the rational group algebra of $D_{n+1}$.

To describe the operation of $S_{n+1}$ on $C^n$, view $S_{n+1}$ as permutations of the set $\{1, \ldots, n+1\}$, let $\tau_i$, $i = 1, \ldots, n$ be the transposition $(i, n+1)$ and let $\tau_{n+1}$ be the identity. Then every element of $S_{n+1}$ can be written uniquely as a product $\sigma \tau_i$, for some $i$, where $\sigma$ is a permutation of $1, \ldots, n$. Now let $C_2 = \{1, a\}$ and view $C_2^n$ again as the subgroup of $C_2^{n-1}$ consisting of all $(c_1, \ldots, c_{n+1})$ where $c_i$ is either 1 or $a$ and the number of entries equal to $a$ is even. If $c_1, \ldots, c_n$ are known then so is $c_{n+1}$, which we may therefore omit from the notation. With this, we have $\tau_i(c_1, \ldots, c_n) = (c'_1, \ldots, c'_n)$ where $c'_j = c_j$ for $j \neq i$ and $c'_i = 1$ if the number of $c_i$ equal to $a$ is even and $c'_i = a$ if it is odd. Now $QC_2 \cong \mathbb{Q} \oplus \mathbb{Q}$ where the idempotents of the summands on the right side may be taken to be $e = (1+a)/2$ and $f = (1-a)/2$, respectively. Using our previous notation, the $2^n$ primitive idempotents of $QC_2^n$ may therefore be written in the form $E = e(I)$, where $I$ is an ordered $n$-tuple of indices whose values can only be 1 or 2; the $j$th entry in $e(I)$ is $e$ or $f$ according as the $j$th entry in $I$ is 1 or 2. Since $S_{n+1}$ operates as automorphisms of $QC_2^n$ it must permute these idempotents $e(I)$. The effect of a permutation $\sigma$ of $\{1, \ldots, n\}$ is clearly just to permute the entries of $E$. We must compute $\tau_i E$, $i = 1, \ldots, n$. For simplicity, suppose that $i = n$; the argument will be the same for the other values of $i$. To avoid confusion, write $u$ instead of 1 for the unit of $C_2$, and consider the non-commutative polynomial ring $\mathbb{Q}(u, a)$ formally generated by these two symbols. If we consider $e$ and $f$ also as abstract symbols, this is the same ring as $\mathbb{Q}\{e, f\}$ under the identifications $e = (u+a)/2$, $f = (u-a)/2$. Now consider the effect of $\tau_n$ on elements of the form $\xi e, \xi f$, where $\xi$ is a monomial in $a$ and $u$. If the degree of $\xi$ in $a$ is even then $\tau_n(\xi a) = \xi a, \tau_n(\xi u) = \xi u$, while if the degree of $\xi$ in $a$ is odd then $\tau_n(\xi a) = \xi u, \tau_n(\xi u) = \xi a$. It follows that whatever the degree of $\xi$ in $a$ we have $\tau_n(\xi e) = \xi e$ while $\tau_n(\xi f) = \pm \xi f$ according as the degree of $\xi$ in $a$ is even or odd.

It follows from the foregoing that if $\xi$ is an arbitrary homogeneous polynomial of total degree $n-1$ in $a$ and $u$ then $\tau_n(\xi e) = \xi e$. In particular, if $E'$ is a monomial of degree $n-1$ in $a$ and $f$ then $\tau_n(E'(a) = E' e$. In $\tau_n(E' f)$, however, if $E'$ is expanded
as a polynomial in \(a\) and \(u\), then the signs of all the terms of odd degree in \(a\) are reversed. This is the same thing as replacing every factor \(e\) in \(E'\) by \(f\) and every factor \(f\) by \(e\). Since the analogous result clearly holds for all \(i = 1, \ldots, n\), we have the following description of the action of \(S_{n+1}\) on \(Q C_2^n\) when the latter is viewed as the set of homogeneous polynomials of total degree \(n\) in \(Q\{e, f\}\). (This is, of course, essentially the same thing as the tensor algebra of \(Q C_2\), and the polynomials of degree \(n\) are just \((Q C_2)^{\otimes n}\).

**Theorem 11.** Let \(E \in (Q C_2)^n\) be a monomial of degree \(n\) in \(e\) and \(f\). If the \(i\)th factor in \(E\) is \(e\), then \(\tau_i E = E\). If the \(i\)th factor in \(E\) is \(f\) then the \(i\)th factor of \(\tau_i E\) remains \(f\) but in all other factors \(e\) and \(f\) are interchanged. \(\square\)

In this theorem, the fact that the ground field is \(Q\) is of no consequence; it was used only to show that we had constructed a representation of \(S_{n+1}\) as operators on the \(n\)th cartesian power of a two-element set. In purely combinatorial terms it can be rephrased as follows.

**Theorem 12.** There is a permutation representation of \(S_{n+1}\) on the set of sequences of length \(n\) of \(0\)'s and \(1\)'s in which the subgroup \(S_n\) of \(S_{n+1}\) fixing \(n + 1\) operates by permutation of the entries, and the transpositions \((i, n + 1)\) are represented by operators \(\tau_i\) as follows: If the \(i\)th entry of a sequence \(s\) is \(0\) then \(\tau_i s = s\). If the \(i\)th entry of \(s\) is \(1\) then the \(i\)th entry of \(\tau_i s\) remains \(1\) but all other entries are replaced by their complements (i.e., \(0\)'s are replaced by \(1\)'s and \(1\)'s by \(0\)'s). \(\square\)

It is clear that the representation is faithful except in the trivial case where \(n = 1\). The group consisting of all permutations of the entries in the sequences together with all operations which replace a single entry by its complement is the full group of permutations of all \(2^n\) of the sequences of \(0\)'s and \(1\)'s.

We are now free to use any coefficient ring \(k\). Referring back to the Hecke algebra construction (§4) we can take \(k = Z_{q, 2} = Z[q, q^{-1}, 1/2q^2]\) where \(2q^2 = 1/(1 + q^2)\) and eventually \(q = 1 + t\). Then \(k C_2 \cong k \oplus k\) where the idempotents generating the summands are, respectively, \(e = (1 + qa)/2q^2, f = (q^2 − qa)/2q^2\). Note that when \(q = 1\) these are just our previous \(e\) and \(f\), and that as \(2q^2 = 1 + q^2 = 2 + 2t + t^2\) we can reduce modulo any rational prime, in particular \(p = 2\), and the expressions remain meaningful. When \(q = 1\) or equivalently, \(t = 0\), the operation of \(S_{n+1}\) is just the original operation on \(Q C_2^n\), but the deformation of \(Z C_2^n\) to its Hecke algebra and its induced deformation of \(Z C_2^n\) is not equivariant with respect to the original operation. It is so with respect to the subgroup \(S_n\) which just permutes the factors, which in effect is just the subgroup leaving \(n + 1\) fixed, but not with respect to the \(\tau_i\). Nevertheless, the theorem effectively defines an operation of all \(S_{n+1}\), but that operation is a deformation of the original one.

It is important now to exhibit the deformation explicitly in order to show what happens at \(p = 2\). To do so, we must write it in terms of the basis \(\{1, a\}\) for \(k C_2\). Here \(e\) and \(f\) are represented by the column vectors of the matrix

\[
X = \frac{1}{2q^2} \begin{pmatrix}
1 & q^2 \\
q & −q
\end{pmatrix}
\]

Putting \(1\) before \(a\) gives a lexicographic order to the \(2^n\) basis elements of \(k(C_2^n) = (k C_2)^{\otimes n}\) consisting of the tensor products of length \(n\) of \(1\)'s and \(a\)'s. The tensor products of length \(n\) of the \(e\)'s and \(f\)'s arranged in lexicographic order are then represented by the column vectors of the \(2^n \times 2^n\) matrix \(X^{\otimes n}\). To illustrate what happens it will be sufficient to consider the case \(n = 2\), so
we are tacitly dealing with $D_3$. Write

$$Y = X^\otimes 2 = \frac{1}{(2q^2)^2} \begin{pmatrix} 1 & q^2 & q^2 & q^4 \\ q & -q & q^3 & -q^3 \\ q^2 & -q^2 & -q^2 & q^2 \end{pmatrix}; Y^{-1} = \begin{pmatrix} 1 & q & q & q^2 \\ 1 & -q & q & -1 \\ 1 & q & -q & -1 \\ 1 & -q & -q & q-2 \end{pmatrix}.$$

If an element $\sigma \in S_3$ is represented in terms of the basis $\{e \otimes e, e \otimes f, f \otimes e, f \otimes f\}$ by a matrix $M$, then its representation in terms of the basis $\{1 \otimes 1, 1 \otimes a, a \otimes 1, a \otimes a\}$ is then given by $YM^{-1}$. For $\sigma = (1, 2)$ the effect on this basis is to interchange its second and third elements, so the matrix $M$ is the permutation matrix $P_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. One can check immediately that this commutes with $Y$, so the operation of $(1, 2)$ is undeformed. More generally, it is easy to see in the case of $D_{n+1}$ that the operation of $S_n$, which operates by the usual permutation of the tensor factors, is undeformed. The transposition $(n, n+1)$, in our illustration $(2, 3)$, behaves quite differently. In the basis generated by the tensor products of the idempotents $e, f$, it fixes $e \otimes e$ and $f \otimes e$ and interchanges $e \otimes f$ and $f \otimes e$, so it is represented by $P_{24} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. Its matrix in terms of the basis generated by the elements $1, a$ of $C_2$ is easily calculated to be

$$YP_{24}Y^{-1} = \frac{1}{(2q^2)^2} \begin{pmatrix} (2q^2)^2 & 0 & 0 \\ 0 & (2q^2)^2 & 0 \\ 0 & 0 & (2q^2)^2 \end{pmatrix} \begin{pmatrix} q^5 - q & q^3 - q^{-1} \\ 1 - q^4 & 2(q^3 + q) \\ 1 - q^4 & 2(q^3 + q) \end{pmatrix}.$$

Letting $q \to 1$, or equivalently, writing $q = 1 + t$ and letting $t \to 0$, this becomes the permutation matrix $P_{34}$, which one can check is indeed the representation of $(2, 3)$ in the basis generated by $1$ and $a$. This matrix remains formally unchanged after reduction modulo any prime, in particular $p = 2$. However, reduction modulo 2 and reduction modulo $t$ do not commute here. Reducing first modulo 2 and then modulo $t$ yields the matrix $N = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Letting $q \to 1$ first recovers the original representation of $(1, 2)$ and $(2, 3)$ on the $1, a$ basis by $P_{23}$ and $P_{34}$; reducing first modulo 2 gives, respectively, $P_{23}$ and the above $N$. Setting $W = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$, one has $WP_{23}W^{-1} = P_{23}, WNW^{-1} = P_{34}$, so $W$ conjugates the pair $P_{23}, N$ into $P_{23}, P_{24}$. Therefore, $P_{23}$ and $N$ give, a representation of $S_3$ on $kC_2$ equivalent to that originally given by $P_{23}$ and $P_{24}$. Thus, if we had started in characteristic 2 we would still have produced the necessary deformation of the action of $S_3$ consistent with the deformation of $kC_2$, but its behavior at different primes patches together...
in an unexpected way. Nevertheless we do have a global deformation; it will behave correctly at every prime.

Having now deformed the integral group ring \( \mathbb{Z}D_n \) to a smash product \( A = (k \oplus k)^{\oplus n} \# kS_{n+1} \), Theorem 3 is applicable. In the next stage of the solution we must examine the orbits and the isotropy groups of the action of \( S_{n+1} \); the decomposition into orbits will give a decomposition of the algebra. If \( B(I) \) is the direct sum of the rank one algebras \( A(J) \) for \( J \) in the orbit of \( I \), then we will again have \( A = \oplus B(I) \# k(S_{n+1})_I \) where the sum ranges over the orbits of \( S_{n+1} \) in the set of indices \( I \). Since the \( A(I) \) are all just isomorphic to \( k \) they have no automorphisms, so if the orbit of \( I \) has order \( m \) then \( B(I) \cong M_m(k) \# k(S_{n+1})_I \).

For the Weyl groups \( B_n = C_2 \wr S_n \) the isotropy groups were of the form \( S_m \times S_{n-m} \). Since each factor had a split global solution starting with \( \mathbb{Z} \), so did their product. Here the same is almost true, but for \( D_{n+1} \) with \( n \) odd there is a special case, the “middle” isotropy group. This is the one case where we have to use Theorem 5.

To examine the orbits and isotropy groups for \( D_{n+1} \) consider, for simplicity, the case where \( E = e(I) \) is a tensor product of \( m \) factors \( e \) followed by \( n-m \) factors \( f \); write it as \( e^m f^{n-m} \). Every \( e(J) \) with the same number of \( e \)'s and \( f \)'s as this is in the orbit of \( E \). Generally this is not the whole orbit since \( \tau_n E \) will contain \( e \) now \( n-m-1 \) times and \( f \) now \( m+1 \) times. The exceptional or middle case is that where \( n = 2r + 1 \) is odd and \( m = r \). One must be careful in counting orbits, since if we let \( m \) run from 0 to \( n \) then every orbit will be counted twice except the middle one (when \( n \) is odd). To count each only once, restrict \( m \) by requiring that \( m \leq n-m \) or \( 2m \leq n \). The middle case must also be handled separately.

In the non-exceptional case, the order of the orbit of \( e(I) \) is therefore \( \binom{n}{m} + \binom{n}{m+1} \) and the order of the isotropy group \( (S_{n+1})_I \) is \( (m+1)!(n-m)! \). In fact, in the non-exceptional case, we have \( (S_{n+1})_I \cong S_{m+1} \times S_{n-m} \). Here the second factor arises because it is evident that permuting the last \( n-m \) factors of \( e(I) \) will leave it unchanged. For the first \( m \) factors, since they are all equal to \( e \), we can not only permute them, but also apply \( \tau_i \) for \( i = 1, \ldots, m \) and we have just proven that the group generated by all these operations in \( S_{m+1} \). (It may appear that the case \( m = n \) is also exceptional since \( f \) does not occur and the orbit of \( e^n \) is reduced to \( e^m \) itself, but the formulas remain correct if we interpret \( S_0 \), whose order is 0!, as the identity.) In the exceptional case where \( n = 2r + 1 \) and \( e(I) = e^r f^{r+1} \), the order of the orbit is only \( \binom{2r+1}{r+1} \) and the order of the isotropy group is therefore \( (2r+2)!/(2r+1)! = 2((r+1)!)^2 \). This isotropy group contains the subgroup isomorphic to \( S_{r+1} \times S_{r+1} \), where, as before, the first factor is generated by the permutations of the first \( r \) factors together with the \( \tau_i \) for \( i = 1, \ldots, r \) and the second factor consists of the permutations of the last \( r+1 \) factors. Being of index 2, it is normal.

**Lemma D.** In the middle case, where \( e(I) = e^r f^{r+1} \) one has \( (S_{2r+2})_I \cong S_{r+1} \wr C_2 \).

**Proof.** Set \( \sigma = (1, r+1)(2, r+2) \cdots (r, 2r) \) and \( \rho = \sigma \tau_{2r+1} \). Then \( pe(I) = e(I) \) so \( \rho \) is in the isotropy group of \( I \). Clearly \( \rho^2 = 1 \). What one must show is that conjugation by \( \rho \) interchanges the factors in the product \( S_{r+1} \times S_{r+1} \). It is trivial that \( \rho \) interchanges the permutations of places \( 1, \ldots, r \) with the permutations of places \( r+1, \ldots, 2r \). Finally, we claim that \( \rho \tau_i \rho = (r+i, 2r+1) \) for \( i = 1, \ldots, r \). One must show that the operations are the same on every \( e(J) \). There are four cases, according as the \( i \)th entry of \( e(J) \) is \( e \) or \( f \) and likewise for the \( (2r+1) \)st entry; each is readily checked. \( \square \)
With this the Donald–Flanigan problem for $D_n$ is solved. For with a split global solution for $S_{r+1}$, the problem for the middle isotropy group is reduced to our previous one of a wreath product of a sum of central separable algebras. The largest isotropy group is $S_{n+1}$, so we have the following.

**Theorem 13.** There is a split global solution starting with $\mathbb{Z}$ to the Donald–Flanigan problem for the Weyl groups $D_{n+1}$; all their irreducible complex representations are rational. The coefficient ring after deformation is $\mathbb{Z}_{q,n+1}$. □

We can also compute the rational group ring of $D_{n+1}$ in the following way, separating the odd and even cases, and just writing $n = 2r$ in the latter. For the middle case we need to compute $\mathbb{Q}(S_r \wr C_2)$. From Theorem 5, for each block $S_r(\lambda)$ of $\mathbb{Q}S_r$ there is a pair of summands each isomorphic to $S_r(\lambda)$, and for every pair of distinct blocks $S_r(\lambda), S_r(\mu)$ of $\mathbb{Q}S_r$ there is a unique summand isomorphic to $M_2(S_r(\lambda) \otimes S_r(\mu))$. The order of $\lambda$ and $\mu$ is immaterial; there is only one summand for the pair. Write $\lambda < \mu$ to indicate that $\lambda$ precedes $\mu$ in the lexicographic order of partitions of $n$.

**Lemma E.** The rational group ring for the middle component of $\mathbb{Q}D_{2r}$ is given by

$$\mathbb{Q}(S_r \wr C_2) = \bigoplus_\lambda \{S_r(\lambda) \oplus S_r(\lambda)\} \bigoplus_{\lambda, \mu}^{\lambda < \mu} \{M_2(\mathbb{Q}) \otimes S_r(\lambda) \otimes S_r(\mu)\} \quad \square$$

This gives the following expression for the rational group ring.

**Theorem 14.** The rational group ring of the Weyl groups of type $D_n$ is given by

$$\mathbb{Q}D_{2r+1} = \bigoplus_{m=0}^{r} \{M_{(2r+1)}(\mathbb{Q}) \otimes \mathbb{Q}S_{2r+1-m} \otimes \mathbb{Q}S_m\}$$

$$\mathbb{Q}D_{2r} = \bigoplus_{m=0}^{r-1} \{M_{(2r)}(\mathbb{Q}) \otimes \mathbb{Q}S_{2r-m} \otimes \mathbb{Q}S_m\} \bigoplus \{M_{(2r-1)}(\mathbb{Q}) \otimes \mathbb{Q}(S_r \wr C_2)\},$$

where $\mathbb{Q}(S_r \wr S_r)$ is given by Lemma E. □

From this one can see that the representations of $D_n$ may be indexed by unordered pairs of representations of $S_m$ and $S_{n-m}$. If $n$ is odd then the order of the pair will be fixed by the requirement that $2m > n$ while if $n = 2r$ is even then, as we have just seen, it is true both for $m \neq r$ and for $m = r$.

As mentioned at the beginning, now only six finite reflection groups remain for which a solution to the Donald–Flanigan problem is unknown. If the problem is solvable for all of them, one would like to know whether in all cases the Hecke algebra effectively gives a solution. While the Donald–Flanigan problem has been solved here for large classes of groups, the major question remains of what characterizes a group for which it is solvable. That, so far, is a mystery.

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DEPT. OF MATHEMATICS, UNIVERSITY OF PENNSYLVANIA, PHILADELPHIA, PA 19104-6395
E-mail address: mgersten@math.upenn.edu

DEPT. OF MATHEMATICS AND COMPUTER SCIENCE, LOYOLA UNIVERSITY OF CHICAGO, CHICAGO, IL 60626–5311
E-mail address: tonyg@math.luc.edu

DEPT. OF MATHEMATICS AND COMPUTER SCIENCE, BAR ILAN UNIVERSITY, RAMAT-GAN 52900, ISRAEL
E-mail address: mschaps@bimacs.cs.biu.ac.il