3-D Diffusive Molecular Communication with Two Fully-Absorbing Receivers: Hitting Probability and Performance Analysis

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Abstract

Exact analytical channel models for molecular communication via diffusion (MCvD) systems involving multiple fully absorbing receivers (FARs) in a three-dimensional (3-D) medium are hard to obtain due to the mathematical intractability of corresponding diffusion equations. This work, therefore, consider an MCvD system with two spherical FARs in a 3-D diffusion-limited medium and develop several insights using an approximate analytical expression for the hitting probability of information molecule (IM). Further, based on the hitting probability, a novel approximate closed-form analytical expression for the area under the receiver operating characteristic curve (AUC) is derived to analyze the detection performance at each FAR in the presence of other FAR. Finally, simulation results are presented to validate the analytical results using the particle-based and Monte-Carlo simulations and to yield important insights into the MCvD system performance with two FARs.

I. INTRODUCTION

Molecular communication via diffusion (MCvD) has gained significant research attention as a way to realize communication between bio-nano-machines [1]. In an MCvD system, information molecules (IMs) emitted from the transmitter propagates to the receiver via Brownian motion. In recent times, the Internet of Bio-Nano Things (IoBNT) is gaining significant prominence towards addressing challenging problems in biomedical scenarios, where multiple transmitters and receivers have to work together to perform complex tasks, including sensing and actuation [2].

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Related Work: In the context of multiple communicating devices using MCvD, [3] considers an MCvD system with two FARs in a 3D medium and obtains the hitting probability of an IM at each FAR, or equivalently the average fraction of IMs absorbed by each FAR. Note that in contrast to passive receiver where the receiver does not affect the propagation of IMs, an FAR immediately absorbs the IMs once they hit its surface [1]. The channel model derived in [3] consists of two unknowns, which have to be computed numerically. To the best of our knowledge, the exact expressions of the hitting probability for systems with multiple FARs are not available in the literature. Due to the lack of analytical channel model for an MCvD system with multiple FARs in 3D medium, most of the works [4]–[6] relied on simulation-based channel models to analyze the system performance. In the past, [7] studied the 3D kinetics of a Brownian particle in the presence of two spherical traps and presented an approximate expression for the death probability of this particle from any of these two traps. This analytical expression can be applied to derive channel models for MCvD systems with two FARs and study its performance, which is the prime focus of this letter. This derived channel model has a wide range of applications ranging from interference analysis of two FAR systems to molecular multiple-input multiple-output (MIMO) systems.

Contributions: In this letter, we consider an MCvD system with one transmitter and two spherical FARs in a diffusion-limited 3D medium and develop a channel model based on the analysis in [7]. Using the proposed channel model, we further develop an analytical framework to study such systems. We then validate the presented analysis via multiple particle-based simulations. We provide several design insights related to the mutual influence of FARs and their mutual distance’s impact on the hitting probability of an IM on each FAR. We also derive the area under the receiver operating characteristic curve (AUC) for each individual FAR, which serves as a quantitative measure for the FAR’s capability of correct decision and study the impact of the presence of another FAR on it. We also provide a novel approximate closed-form analytical expression for AUC.

II. System Model

In this letter, we consider an MCvD system with a point transmitter and two spherical FARs at different locations in a 3D medium. Let the transmitter be located at the origin and the two FARs of radius $a$ at positions $x_1$ and $x_2$ respectively in $\mathbb{R}^3$. The time is divided into time-slots of duration $T_s$, i.e. $l$th time-slot denotes the time period $[(lT_s, (l + 1)T_s]$ with $l \geq 0$. At
the beginning of each time-slot, the transmitter transmits its binary information using the on-off keying modulation, i.e. the transmitter emits $N$ IMs for bit 1 and does not emit any IMs for bit 0. IMs have a diffusion coefficient $D$ with respect to the propagation medium. We assume $D$ to be constant over space and time. In any time-slot $l$, the transmit bit $b[l]$ is an independent Bernoulli random variable taking value 1 with probability $p_1$, and 0 with $p_0 = 1 - p_1$. The transmitter and both FARs are assumed to be synchronized in time, which is a common assumption in the past literature [6].

The probability that an IM reaches (and gets absorbed by) the $i$th FAR located at $x_i$ within time $t$ in the presence of the other FAR, is denoted by $p_i(t, a)$. This hitting probability equals the average fraction of IMs absorbed until time $t$ by the $i$th FAR. Let us now focus on a particular time-slot $l$. Now, the probability that an IM emitted at the $k$th slot reaches in the $l$th slot at $i$th FAR can be written in terms of $p_i(t, a)$ as

$$h_i[l - k] = p_i((l-k+1)T_S, a) - p_i((l-k)T_S, a) \quad \forall l \geq k. \quad (1)$$

In particular, $h_i[0]$ denotes the probability that IM transmitted in $l$th time-slot arrives in the same time-slot. We assume that there are no potential collisions between the IMs during their propagation in the medium [1], and hence, the motion of an IM is independent of the motion of other IMs. Thus, $S_i[l]$ denoting the number of IMs reaching the FAR $i$ in the $l$th time-slot corresponding to bit $b[l]$ is Binomial distributed with parameters $Nb[l]$ and $h_i[0]$, i.e. $S_i[l] \sim B(Nb[l], h_i[0])$. Note that if $n$ is large, $B(n, p)$ can be approximated as Gaussian distribution $\mathcal{N}(\mu, \sigma^2)$ with mean $\mu = np$ and variance $\sigma^2 = np(1 - p)$ [1]. Hence, we can model $S_i[l] \sim \mathcal{N}(Nb[l]h_i[0], Nb[l]h_i[0](1 - h_i[0]))$. Similarly, $I_i[k]$ denoting the number of IMs received at the FAR $i$ in $l$th time-slot corresponding to the transmission in $k$th slot ($k < l$) can be modeled as

$$I_i[k] \sim \mathcal{N}(Nb[k]h_i[l - k], Nb[k]h_i[l - k](1 - h_i[l - k])).$$

Note that, $I_i[k]$ corresponds to inter symbol interference (ISI) arising due to the transmission from previous $k$th time-slot. Now, the total number of IMs corresponding to all previous transmissions is $I_i[l] = \sum_{k=1}^{l-1} I_i[k]$. Let $N_i[l]$ denote the number of molecules received from unintended sources with $N_i[l] \sim \mathcal{N}(\mu_n, \sigma_n^2)$ [8]. Now, $R_i[l]$ denoting the total number of molecules arriving at FAR$i$ in the $l$th time-slot, is given as

$$R_i[l] = S_i[l] + I_i[l] + N_i[l]. \quad (2)$$
During detection, FAR$_i$ decodes $\hat{b}_i[l] = 1$ when $R_i[l] \geq \eta_i$ and $\hat{b}_i[l] = 0$ otherwise, where $\eta_i$ is the decision threshold. Before proceeding further, we will calculate the mean $\mu_{b_i[l]}$ and variance $\sigma^2_{b_i[l]}$ of the random variable $R_i[l]$ for $b[l] \in \{0, 1\}$, which would be useful when analyzing the receiver’s performance in Section IV. These can be derived as

$$
\mu_0[i;l] = Np_1 \sum_{k=1}^{l-1} h_i[l-k] + \mu_n,
$$

$$
\sigma^2_0[i;l] = Np_1 \sum_{k=1}^{l-1} [h_i[l-k](1-h_i[l-k])] + Np_0 h_i[l-k]^2 + \sigma^2_n,
$$

$$
\mu_1[i;l] = Nh_i[0] + \mu_0[i;l],
$$

$$
\sigma^2_1[i;l] = Nh_i[0](1-h_i[0]) + \sigma^2_0[i;l].
$$

In contrast to Genie-aided approach where the means and the variances are obtained in terms of previous bits [9], we assume the previous bits to be random and take average over them for the above calculations. [10]. In the next section, we will discuss the hitting probability $p_i(t, a)$ for each of the FARs and their influence on each other.

### III. Mutual Influence of the Two FARs

The exact analytical expression for $p_i(t, a)$ in a 3D medium is not available in the existing literature due to its intractability. However, an approximate value for it can be obtained as [7],

$$
p_i(t, a) = a \left\| \mathbf{x}_i \right\| \sum_{n=0}^{\infty} \left( -\frac{a}{R} \right)^n \text{erfc} \left( \frac{\left\| \mathbf{x}_i \right\| - a + n(R-a)}{\sqrt{4Dt}} \right),
$$

under the assumptions- (i) the distance between the transmitter and each FAR is significantly larger than $a$, i.e. $\|\mathbf{x}_1\| \gg a$ and $\|\mathbf{x}_2\| \gg a$, and (ii) the distance between FAR$_1$ and the FAR$_2$ is significantly larger than $a$, i.e. $R = \|\mathbf{x}_1-\mathbf{x}_2\| \gg a$. Note that $\text{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty \exp(-t^2)dt$ is the complementary error function. Further note that the fraction of IMs absorbed within time $t$ by the FAR$_i$ (denoted by $\overline{p}_i(t, a)$) in the absence of any other FAR is [11]

$$
\overline{p}_i(t, a) = a \left\| \mathbf{x}_i \right\| \text{erfc} \left( \frac{\left\| \mathbf{x}_i \right\| - a}{\sqrt{4Dt}} \right).
$$

From (3) and (4), we can develop several important insights that are presented below.

**Corollary 1.** The fraction of IMs eventually hitting FAR$_i$ in the absence and the presence of FAR$_j$ are

$$
\overline{p}_i(\infty, a) = \frac{a}{\|\mathbf{x}_i\|} \text{ and } p_i(\infty, a) = \frac{a}{\|\mathbf{x}_i\|} \times \frac{R}{R + a}.
$$


Therefore, the presence of FAR$_j$ reduces the eventual hitting probability at FAR$_i$ by the amount

$$q_i(\infty, a) = P_i(\infty, a) - p_i(\infty, a) = \frac{a}{\|x_i\|} \times \frac{a}{R + a},$$

which denotes the fraction of IMs that would have hit FAR$_i$ eventually, but instead hit FAR$_j$ first and got absorbed. $q_i(\infty, a)$ decreases with $R$.

**Corollary 2.** When the two FARs are far apart, (i.e. $R \to \infty$),

$$p_i(t, a) \to \frac{a}{\|x_i\|} \text{erfc} \left( \frac{\|x_i\| - a}{\sqrt{4Dt}} \right) = P_i(t, a).$$

In other words, the mutual influence of FARs vanishes as they move away from each other.

**Corollary 3.** The probability that an IM reaches any of the FARs is

$$p(t, a) = p_1(t, a) + p_2(t, a).$$

**A. Validation**

We first validate the presented expression (3) of $p_i(t, a)$ through particle-based simulations which are carried out for $10^4$ iterations, with a step size of $10^{-4}$ s. $D$ is 100 $\mu$m/s$^2$, and both FARs have radius $a = 5$ $\mu$m, which are the same for all numerical evaluations in this paper unless stated otherwise. Fig. 1 shows the hitting probability of IM on each of the FARs in presence of the other for two different cases: Case 1 with $x_1 = [25, 10, 0]$ and $x_2 = [-30, -20, 0]$ and
Case 2 with $\mathbf{x}_1 = [-30, -10, 0]$ and $\mathbf{x}_2 = [100, 40, 0]$. In the Case 1, the FARs are relatively closer to each other with $R = 62.6 \, \mu m$. In Case 2, the two FARs are relatively far away with $R = 139.2 \, \mu m$. We can observe that the analytical expression (3) closely matches with simulation results for both cases. We can also observe that in Case 1, FARs have a significant influence on each other, which grows with time $t$ as seen by the widening gap between solid and dashed lines. In Case 2, the distance between FARs is large enough, resulting in a negligible mutual influence. Also, the hitting probability of the FAR closer to the transmitter of the two FARs is higher than that of the other one.

B. Impact of Distance $R$ on Hitting Probability

We now study the impact of mutual distance on the hitting probability of an IM on FARs equidistant from the transmitter. Without loss of generality, we consider one FAR at $x$-axis with $\|\mathbf{x}_1\| = 20 \, \mu m$ and other FAR in $x$-$y$-plane with the same radial distance $\|\mathbf{x}_2\| = \|\mathbf{x}_1\|$ and azimuth angle $\phi$. Note that, the distance depends on $\phi$ as $R = 2 \|\mathbf{x}_2\| \sin(\phi/2)$. Fig. 2 shows the variation of hitting probability with varying azimuth angular distance $\phi$ between the two FARs. Here also, we can observe that for the chosen parameters, the analytical and simulation results match well, including the scenario when the FARs are close to each other. Fig. 2 also shows the total probability $p(t, a)$. It can be verified that $p(t, a) = 2p_1(t, a)$. This is because the fraction of IMs absorbed by each FARs are the same owing to their equal distance from the transmitter.
C. Comparison of two FARs vs single FAR

We now study the gain $g(t, a)$ that can be achieved by replacing one FAR by two FARs at two different locations. In particular, in the first scenario, there is only one FAR of radius $a$ at $x_1$ with hitting probability of IM as $p_1(t, a)$. Now, in the second scenario, there are two FARs, each of radius $b$ at two different locations $x_1$ and $x_2$ such that $||x_1|| = ||x_2||$. For a fair comparison, we keep the total surface area of the FARs equal in both scenarios i.e. $b = a/\sqrt{2}$. The hitting probability of an IM on any of the FARs is $p(t, a/\sqrt{2}) = 2p_1(t, a/\sqrt{2})$. We can see that for any time $t$,

$$g(t, a) = \frac{p(t, a/\sqrt{2})}{p_1(t, a)} < \sqrt{2} \frac{\text{erfc} \left( \frac{||x_1|| - a/\sqrt{2}}{\sqrt{4Dt}} \right)}{\text{erfc} \left( \frac{||x_1|| - a}{\sqrt{4Dt}} \right)} < \sqrt{2}$$

which upper bounds the gain. Further, using the following lower and upper bounds [12] of erfc:

$$\frac{e^{-x^2}}{\sqrt{\pi x}} \left(1 - \frac{1}{2x^2}\right) < \text{erfc}(x) < \frac{e^{-x^2}}{\sqrt{\pi x}}$$

in the denominator and numerator terms of $g(t, a)$, we can show that, for any $t$,

$$g(t, a) < \frac{||x_1|| - a}{||x_1|| - a/\sqrt{2}} \exp \left( -\frac{(2-\sqrt{2})a||x_1|| - a^2/2}{4Dt} \right),$$

which is less than 1 for small $t$. This implies that for small $t$, the scenario with a single FAR gives better hitting probability. However, when $t \to \infty$ and $R > a(1 + 1/\sqrt{2})$, [5] gives

$$g(\infty, a) = \frac{p(\infty, a/\sqrt{2})}{p_1(\infty, a)} = \frac{2R}{\sqrt{2R} + a} > 1,$$

which implies that the hitting probability of an IM on any of the FARs of radius $a/\sqrt{2}$ is higher than the scenario with a single FAR of radius $a$. Fig. [3] compares the two scenarios. It can be seen that initially, the single FAR gives better hitting probability. However, as the time increases, the total hitting probability of IM on any of the two FARs becomes larger, which is consistent with the above analysis.

IV. DETECTION PERFORMANCE AT FARs

Let $P_{d,i}(\eta_i, l)$ and $P_{f,i}(\eta_i, l)$ denote the detection and false alarm probabilities at FAR$_i$ in the $l$th time-slot, respectively. Applying the binary hypothesis testing [13] on $R_i[l]$ for the detection
Fig. 3. Comparison of the hitting probability of IMs for the case with a single FAR of radius $a$ located at $\mathbf{x}_1 = [25 \ 0 \ 0]$ vs the case with two receivers of radius $a/\sqrt{2}$ at $\mathbf{x}_1 = [25 \ 0 \ 0]$ and $\mathbf{x}_2 = [-25 \ 0 \ 0]$. Here, $a = 5 \mu m$.

of bit $b[l]$, $P_{d,i}(\eta_i, l)$ and $P_{f,i}(\eta_i, l)$ can be derived from (2) as

$$P_{d,i}(\eta_i, l) = \mathbb{P}[R_i[l] > \eta_i \mid b[l] = 1] = Q \left( \frac{\eta_i - \mu_1[i;l]}{\sigma_1[i;l]} \right),$$  \hspace{1cm} (12)$$

$$P_{f,i}(\eta_i, l) = \mathbb{P}[R_i[l] > \eta_i \mid b[l] = 0] = Q \left( \frac{\eta_i - \mu_0[i;l]}{\sigma_0[i;l]} \right).$$  \hspace{1cm} (13)$$

Here, $Q(x) = 0.5 \text{erfc} \left( x/\sqrt{2} \right)$ is the standard $Q$-function.

The receiver operating characteristic (ROC) curve illustrates the variation of the detection probability with respect to the false alarm probability for a receiver by varying detection threshold as an intermediate variable. The area under the ROC curve (AUC) is a quantitative measure of a receiver’s capability of correct decision [14]. The AUC value can vary from 0 to 1, where AUC = 0 indicates that bits are always erroneously decoded and AUC = 1 indicates the perfect decoding without any error. Moreover, AUC = 0.5 indicates that the receiver is unable to distinguish between a 0 and 1. The AUC for the FAR$_i$ in $l$th time-slot is [14]

$$A_i[l] = \int_0^1 P_{d,i}(\eta_i, l) dP_{f,i}(\eta_i, l).$$  \hspace{1cm} (14)$$

$$= \frac{1}{\sqrt{2\pi}\sigma_0[i;l]} \int_0^\infty Q \left( \frac{\eta_i - \mu_1[i;l]}{\sigma_1[i;l]} \right) \times \exp \left( -\frac{(\eta_i - \mu_0[i;l])^2}{2\sigma_0^2[i;l]} \right) d\eta_i.$$  \hspace{1cm} (15)$$

Applying approximation on $Q$-function, a closed-form approximation for AUC at FAR$_i$ in $l$th
where $\alpha=0.3842$, $\beta=0.7640$, $\gamma=0.6964$, $a_1 = 0.5\sigma_0^2$, $b_1 = -\mu_0 a_1$, $c_1 = \mu_0^2 a_1$, $d_1 = 1$, $e_1 = 0$, $a_2 = \alpha \sigma_1^2 + a_1$, $b_2 = -(\alpha \mu_1 + \beta \sigma_1/2) \sigma_1^2 + b_1$, $c_2 = (\alpha \mu_1 + \beta \sigma_1) \mu_1 \sigma_1^2 + c_1 + \gamma$, $d_2 = 1$, $e_2 = 0$, $a_3 = a_2$, $b_3 = -((\alpha \mu_1 - \beta \sigma_1/2) \sigma_1^2 + b_1$, $c_3 = (\alpha \mu_1 - \beta \sigma_1) \mu_1 \sigma_1^2 + c_1 + \gamma$, $d_3 = 0$, $e_3 = \sqrt{a_3} \mu_1$ for respective $i$ and $l$.

Fig. 4(a) shows the AUC variation with $N$ for both FARs in $l=10$th slot for $T_\delta = 5s$. It can be observed that the AUC values at both FARs significantly improve as $N$ increases. This improvement in AUC is due the fact that the gap between $R_i[l]$ for $b[l] = 1$ and $b[l] = 0$ increases with $N$ with respect to the noise $N_i[l]$, and the variance of $S_i[l]$ doesn’t increases relatively as much as its mean with $N$. Further we can observe that, the receiver closer to the transmitter, which is FAR$_1$ here, has larger AUC than FAR$_2$ which indicates better decision capability of FAR$_1$. We also study a scenario if the two FARs can work together to make a joint detection. In this case, $R[l] = R_1[l] + R_2[l]$ would be compared with threshold $\eta$ to make a decision for bit $b[l]$. The AUC of this joint detection would be given by (31) with the mean and variance values as $\mu_0[l] = \mu_0[1;l] + \mu_0[2;l]$, $\mu_1[l] = \mu_1[1;l] + \mu_1[2;l]$, $\sigma_0^2[l] = \sigma_0^2[1;l] + \sigma_0^2[2;l]$ and $\sigma_1^2[l] = \sigma_1^2[1;l] + \sigma_1^2[2;l]$ respectively. Fig. 4(a) also shows the AUC of this system. It can be seen that its AUC is higher than individual AUC of both FARs which is intuitive. An ideal implementation of such system would require a central node which is transparent to IMs for combining the observation from FARs. Hence, these AUC values serve as the upper bound to any practical implementation.

Fig. 4(b) shows variation in the AUC with distance between the two FARs. Here, $N = 1000$ and $T_\delta = 1s$. FAR$_1$ is fixed at $x_1 = [-10, 0, 0]$. The location of the FAR$_2$ is $x_2 = [-10 + R, 0, 0]$ which is moved in positive x-direction by increasing $R$. It can be seen that the increment in $R$ results in higher distance between transmitter and FAR$_2$, which in turn deteriorates the detection capability at FAR$_2$. It is interesting to note that even though influence of FAR$_1$ on FAR$_2$ reduces with increase in $R$, the gain in the number of received IMs at FAR$_2$ is superseded by the loss of IMs due to increase in distance of FAR$_2$ from the transmitter. The performance at FAR$_1$ improves slightly since the number of IMs reaching FAR$_1$ increases due to the diminishing influence of
Fig. 4. (a) AUC of the individual FARs vs \( N \) (number of emitted IMs). Here, \( \mathbf{z}_1 = [20, 5, 0] \) and \( \mathbf{z}_2 = [-25, -10, 0] \). (b) Impact of mutual distance \( R \) between the two FARs on their AUC. For both figures, \( \mu_n = \sigma_n^2 = 5 \). The solid lines represent the analytical values obtained using (16), whereas the markers represent the values obtained using Monte-Carlo simulations.

FAR\(_2\) on FAR\(_1\) with increase in \( R \). One can also note that both FARs have identical AUC values when they are located at an equal distance from the transmitter.
APPENDIX

APPENDIX A
DERIVATION OF MEAN $\mu_b[l; i; l]$ AND VARIANCE $\sigma^2_{b[l; i; l]}$ OF $R_i[l]$

From (2), we know

$$R_i[l] = S_i[l] + \sum_{k=1}^{l-1} I_i[k] + N_i[l] \quad (17)$$

with

$$S_i[l] \sim \mathcal{N}(Nb[l]h_i[0], Nb[l]h_i[0](1 - h_i[0]))$$

$$I_i[k] \sim \mathcal{N}(Nb[k]h_i[l - k], Nb[k]h_i[l - k](1 - h_i[l - k]))$$

$$N_i[l] \sim \mathcal{N}(\mu_n, \sigma_n^2).$$

Also for time-slot $k < l$, the transmit bit $b[k]$ is an independent Bernoulli random variable taking value 1 with probability $p_1$, and 0 with $p_0 = 1 - p_1$. Hence $\mathbb{E}[b[k]] = p_1$.

Now, given the bit transmitted at the current slot $l$ i.e. $b[l]$, the mean of the random variable $R_i[l]$ is given

$$\mu_{b[l; i; l]} = \mathbb{E}[R_i[l]]$$

$$= Nh_i[0]b[l] + \mathbb{E}\left[\sum_{k=1}^{l-1} Nb[k]h_i[l - k]\right] + \mu_n$$

$$= Nh_i[0]b[l] + Np_1 \sum_{k=1}^{l-1} h_i[l - k] + \mu_n. \quad (18)$$

For the derivation of variance of $R_i[l]$, first, we derive the variance of $I_i[k]$. The mean of $I_i[k]$ given $b[k]$ is

$$\mathbb{E}[I_i[k] \mid b_i[k]] = Nb[k]h_i[l - k]. \quad (19)$$

The variance of $I_i[k]$ given $b[k]$ is

$$\text{Var}(I_i[k] \mid b[k]) = Nb[k]h_i[l - k](1 - h_i[l - k]). \quad (20)$$
From (19) and (20), the variance of $I_i[k]$ can be derived as

\[
\text{Var}(I_i[k]) = \mathbb{E}[\text{Var}(I_i[k] | b_i[k])] + \text{Var}(\mathbb{E}[I_i[k] | b_i[k]])
\]

\[
= Np_1h_i[l-k](1-h_i[l-k]) + N^2h_i[l-k]^2\text{Var}(b[k])
\]

\[
= Np_1h_i[l-k](1-h_i[l-k]) + N^2h_i[l-k]^2p_1p_0
\]

(21)

Therefore, variance of $R_i[l]$ given the current transmitted bit $b[l]$ is

\[
\sigma_b^2[i; l] = \text{Var}(S_i[l]) + \text{Var}(I_i[l]) + \text{Var}(N_i[l])
\]

\[
= Nb[l]h_i[0](1-h_i[0]) + Np_1 \sum_{k=1}^{l-1} [h_i[l-k](1-h_i[l-k])]
\]

\[
+ Np_0h_i[l-k]^2 + \sigma_n^2.
\]

(22)

**APPENDIX B**

**DERIVATION OF (10)**

Using (3) and (4), for any $t$,

\[
g(t, a) = \frac{p(t, a/\sqrt{2})}{\mathbb{P}_1(t, a)} < \frac{\sqrt{2}\text{erfc}\left(\frac{\|x_1\| - a/\sqrt{2}}{\sqrt{4Dt}}\right)}{\text{erfc}\left(\frac{\|x_1\| - a}{\sqrt{4Dt}}\right)}.
\]

(24)

Now, using the following upper and lower bounds \[12\] of \text{erfc}:

\[
\frac{e^{-x^2}}{\sqrt{\pi}x} (1 - \frac{1}{2x^2}) < \text{erfc}(x) < \frac{e^{-x^2}}{\sqrt{\pi}x},
\]

at the numerator and the denominator respectively of (24) gives

\[
\frac{p(t, a/\sqrt{2})}{\mathbb{P}_1(t, a)} < \frac{\sqrt{2}\exp\left(-\left(\frac{\|x_1\| - a/\sqrt{2}}{\sqrt{4Dt}}\right)^2\right)}{\sqrt{\pi}\left(\frac{\|x_1\| - a/\sqrt{2}}{\sqrt{4Dt}}\right)}
\]

\[
\cdot \exp\left(-\left(\frac{\|x_1\| - a}{\sqrt{4Dt}}\right)^2\right) \times \left(1 - \frac{\|x_1\| - a}{\sqrt{4Dt}}\right)^2.
\]

(25)

Simplifying (25) gives

\[
\frac{p(t, a/\sqrt{2})}{\mathbb{P}_1(t, a)} < \frac{\|x_1\| - a/\sqrt{2}}{\|x_1\| - a} \exp\left(-\frac{2-\sqrt{2}a\|x_1\| - a^2/2}{4Dt}\right)
\]

\[
\cdot \left(1 - \frac{a}{\sqrt{2}\left(\|x_1\| - a\right)}\right)^2.
\]

(26)

which is (10) in the submitted manuscript.
APPENDIX C  
DERIVATION OF (11)

The inequality shown in (11) can be derived from (5) as

\[ g(t, a) = \frac{p(\infty, a/\sqrt{2})}{p_1(\infty, a)} = \frac{2^{\times a/\sqrt{2}} \times \frac{R}{R + a/\sqrt{2}}}{a/\|x_1\|} = \sqrt{2} \times \frac{R}{R + a/\sqrt{2}} = \frac{2R}{\sqrt{2R + a}} \]  

(27)

Note that, \( R > a \) for transmitter to not to overlap with the FAR and \( R \gg a \) for (3) to be valid with minimum error. When \( R > a (1 + 1/\sqrt{2}) \), \( g(t, a) = 1/ \sqrt{2R + a} > 1 \), \( (28) \)

which is (11) in the submitted manuscript.

APPENDIX D  
DERIVATION OF (16)

The probability of false alarm \( P_{f,i}(\eta_i, l) \) varies monotonically from 0 to 1 when \( \eta_i \) changes from \( \infty \) to 0. Thus, (14) can be equivalently written as\( [14] \)

\[ A_{i}[l] = - \int_{0}^{\infty} p_{d,i}(\eta_i, l) \frac{dP_{f,i}(\eta_i, l)}{d\eta_i} d\eta_i, \]  

(29)

where

\[ \frac{dP_{f,i}(\eta_i, l)}{d\eta_i} = - \frac{1}{\sqrt{2\pi}\sigma_0[i; l]} \exp \left(- \frac{(\eta_i - \mu_0[i; l])^2}{2\sigma_0^2[i; l]} \right). \]  

(30)

Further, substituting the above expression along with (13) in (17), the \( A_{i}[l] \) can be written as

\[ A_{i}[l] = \frac{1}{\sqrt{2\pi}\sigma_0[i; l]} \int_{0}^{\infty} \frac{\exp \left(- \frac{(\eta_i - \mu_1[i; l])^2}{2\sigma_1^2[i; l]} \right) \times \exp \left(- \frac{(\eta_i - \mu_0[i; l])^2}{2\sigma_0^2[i; l]} \right)}{d\eta_i}. \]  

(31)

Finally, splitting the above integral into two separate integrals with limits from 0 to \( \mu_1[i; l] \) and from \( \mu_1[i; l] \) to \( \infty \), and subsequently using the following tight and more tractable approximation\( ^1 \) for \( Q(x) \) \( [15] \)

\[ Q(x) \approx \begin{cases} 
\exp(-\alpha x^2 - \beta x - \gamma) & \text{if } x \geq 0 \\
1 - \exp(-\alpha x^2 + \beta x - \gamma) & \text{if } x < 0,
\end{cases} \]  

(32)

\(^1\)The fitting coefficients for positive and negative argument are optimized to minimize the sum of square errors.
(where $\alpha = 0.3842$, $\beta = 0.7640$ and $\gamma = 0.6964$) and then, using the following integral identity from [16, Eq. 2.33.1], i.e.

$$\int e^{-(ax^2+2bx+c)} \, dx = \frac{1}{2} \sqrt{\frac{\pi}{a}} \exp \left( \frac{b^2-ac}{a} \right) \text{erf} \left( \sqrt{ax} + \frac{b}{\sqrt{a}} \right),$$

we get (16).

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