Numerical Study of Glioma Growth Model with Treatment Using the Two-Stage Gauss-Seidel Method

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Abstract. The main aim of this paper is to study the performance of two-stage Gauss-Seidel (TSGS) iterative method in solving glioma growth model with treatment. In this paper, the implicit scheme is used to discretize the governing model. The derivation and implementation of the proposed method are discussed. Also, some numerical results are also presented.

1. Introduction

Brain tumour is the uncontrolled growth of normal brain cells and is divided into two types i.e. primary and secondary brain tumour. The secondary brain tumour is the cancer which spread from another part of the human body. Whereas, primary brain tumour is starting within the brain. Also, primary brain tumour can be categorized as malignant and benign. The most malignant form of brain cancer is known as glioma where half of all primary brain tumours are glioma. From mathematical point of view, modelling of proliferation and migration of gliomas are the very challenging problem [1].

Generally, the reaction-diffusion model is used to represent the proliferation and diffusion dynamics of the glioma growth. It is well known that reaction-diffusion model is a form of partial differential equation (PDE) and can be solved by using numerical techniques (for example finite difference (FD) and finite element (FE) schemes). According to Roniotis et al. [2], FD scheme is stable than the FE method in solving the reaction-diffusion model associated with the glioma growth problem. The fundamental concept of the FD scheme is to reduce the model in the form of a finite-dimensional algebraic system which is mostly sparse.

Among the existing iterative methods, two-stage iterative methods have been widely used to solve an algebraic system. The two-stage iterative method (also known as inner/outer iterative method) was introduced by Nichols [3] and widely studied by many researchers, refer [4-11]. Consequently, in this paper, numerical performance one of the two-stage method i.e. two-stage Gauss-Seidel (TSGS) iterative method for solving the governing glioma growth model is investigated.
The remaining of this paper is structured in following way. Section 2 describes the glioma growth model with treatment. The implementation of the proposed numerical techniques is explained in Section 3. Some numerical results are shown in Section 4 to access the performance of the proposed method. The concluding remarks are given in Section 5.

2. Glioma growth model
The following reaction-diffusion model represents the glioma growth [12]

\[
\frac{\partial c(x,t)}{\partial t} = -\nabla \cdot J + f(c)
\]

(1)

where \( c(x,t) \) is the tumor cell density, \( f(c) \) is the source factor of tumor cells and \( \nabla \) is the gradient vector. Based on the Fick’s Law, the following relation is satisfied

\[
J = -D\nabla c
\]

(2)

where \( D \) is the diffusion coefficient.

Brain tumour growth model considers three different source factors such as logistic, Gompertz and exponential. Glioma has a constant growth rate, therefore, it is always taken as exponential growth. However, for cells with the higher density level, logistic or Gompertz growth is the most suitable. For glioma with exponential growth, the source factor \( f(c) = \rho c \) will be applied and equation (1) can be rewrite as

\[
\frac{\partial c}{\partial t} = \nabla \cdot (D\nabla c) + \rho c
\]

(3)

Commonly, there are three different therapies to treat brain cancer i.e. surgical removal of cancer cells, radiotherapy and chemotherapy [13-15]. Many researchers include the negative reaction term in the mathematical model to represent the loss of tumor cell during the therapy period [13]. In this study, the chemotherapy is considered as the treatment term. Consequently, the equation (3) takes the following form:

\[
\frac{\partial c}{\partial t} = \nabla \cdot (D\nabla c) + \rho c - R(x,t)c
\]

(4)

where \( R(x,t) = g \) is the measurement of the effectiveness of the treatment. During the execution of chemotherapy, the value of \( g \) is considered as constant and the value of \( g \) is zero otherwise. The value of treatment term must be greater than the value of proliferation rate to decrease the size of the tumor during the execution of chemotherapy.

3. Numerical methods

3.1. Implicit scheme
In this paper, FD method based on implicit scheme is applied to discretize the model (4). The backward difference formula for time domain and central difference formula for space is used to discretize the model

\[
\frac{\partial c}{\partial t} = \frac{c_{i,j+1} - c_{i,j}}{\Delta t} + O(\Delta t)
\]

(5)
where \( x_i \) (\( i = 0,1,2,\ldots,n-2,n-1,n \)) for space and \( t_s \) (\( s = 0,1,2,\ldots,m-2,m-1,m \)) for time domain (with notation while \( c_{i,t} = c(x_i,t_s) \)). By applying the formula (5) and (6), the model (4) will be reduced to

\[
c_{i,s} = -qc_{i-1,s+1} + pc_{i,s+1} - qc_{i+1,s+1}
\]

where \( p = \frac{1+2\beta}{1+\alpha} \) and \( q = \frac{\beta}{1+\alpha} \) with \( \alpha = \frac{\Delta t^2}{D} \) and \( \beta = \frac{\Delta t}{(\Delta x)^2} \).

Moreover, the approximation equation (7) can be written in the matrix form as

\[
Ac = b
\]

3.2. TSGS iterative method
Now, let consider the splittings of the coefficient matrix \( A \) as [16]

\[
A = M - N, \quad M = P - Q
\]

where \( P \) is the diagonal matrix, \( Q \) is the strictly lower triangular matrix and \( N \) is the strictly upper triangular matrix. Thus, the general form of iterative method can be written as

\[
c_{i,s}^{(k+1)} = Hc_{i,s}^{(k)} + d.
\]

For one-stage Gauss-Seidel (GS) method, the iteration matrix \( H \) is defined as

\[
H = (P - Q)^{-1}N
\]

and iterative form for GS method is

\[
c_{i,s}^{(k+1)} = (P - Q)^{-1}Nc_{i,s}^{(k)} + M^{-1}b.
\]

Meanwhile, for TSGS method, the iteration matrix \( H \) is defined as

\[
H = (M^{-1}N)^{j} = \sum_{i=0}^{j-1}(M^{-1}N)M^{-1}Q
\]

and iterative form for TSGS method is [3]
\[ c^{(k+1)}_{i,s} = (M^{-1}N)^{j(l)} c^{(k)}_{i,s} + \sum_{l=0}^{j(l)-1} (M^{-1}N)M^{-1}Q^{l}(c^{(k)}_{i,s} + b), k = 0,1,2,\ldots, \] (14)

where \( j(l) \) is the inner iteration number and \( i \) is the outer iteration. The inner iteration number for stationary two-stage method is fixed and \( l \geq 1 \). Based on the theorem, GS and TSGS methods will converge if the spectral radius is less than unity. The algorithm for TSGS method is described in Algorithm 1.

**Algorithm 1. Two-Stage Gauss-Seidel Method**

Step 1. Initialize all the parameter.

Step 2. \textbf{while} \( k = 1,2,3,\ldots \) until convergence

\begin{itemize}
  \item for \( s = 1,2,3,\ldots,m - 3,m - 2,m - 1,m \)
  \item for \( i = 1,2,3,\ldots,n - 3,n - 2,n - 1 \)
  \item \( y^{(k)}_{0,i} = c^{(k)}_{i-1,s} \)
  \item for \( j = 1 \text{ to } l \)
  \item \( y^{(k+1)}_{j,i} = M^{-1}N y^{(k)}_{j-1,i} + M^{-1}Q^{l} c^{(k)}_{i,s} + M^{-1}b \)
  \item \( c^{(k+1)}_{i,s} = y^{(k+1)}_{j,i} \)
\end{itemize}

Step 3. Convergence test. If the convergence criterion i.e. \( \|c^{(k+1)} - c^{(k)}\| < \varepsilon \) (where \( \varepsilon \) is convergence criterion) is satisfied, go to step 4. Otherwise, go to step 2.

Step 4. Stop.

4. Computational results and discussion

For the numerical simulations, model (4) is assume with no flux boundary and initial value of 4000 cells are considered. The value of growth term \( \rho = 0.012 \text{ day}^{-1} \), loss term \( g = 0.024 \text{ day}^{-1} \) and diffusion \( D = 0.0013 \text{ mm}^{2} \text{ day}^{-1} \) is applied. To perform the comparison analysis, parameters such as number of iteration \( k \) and computational time in seconds \( \text{Time} \) are measured. The value of initial iteration is set as zero. All the simulations are performed on a personal computer with Intel(R) Core(TM) i7-2600 CPU@ (3.40 GHz, 3.40 GHz) and 8.00 GB RAM and, programming codes were developed using MATLAB software. To investigate the performance of the TSGS method, the numerical simulations is conducted for different mesh sizes \( n = 30,60,90,120 \). The number of inner iteration, \( l = 10 \) is fixed and the convergence criteria with tolerance error, \( \varepsilon = 10^{-6} \) is considered. For the comparison analysis,
numerical results of standard GS are also included. The numerical results for both methods (GS and TSGS methods) are plotted in Figures 1 and 2.

Based on the numerical results obtained, it can be concluded that the number of iteration for the TSGS method is reduced approximately 37.92-48.04\% (for 30 days), 36.66-45.24\% (for 60 days), 36.04-43.92\% (for 90 days) and 35.89-43.08\% (for 120 days) respectively compared to GS method. The execution time of TSGS method is also faster as compared to GS method by 4.22-42.88\% (for 30 days), 4.38-46.25\% (for 60 days), 3.96-47.37\% (for 90 days) and 3.46-48.15\% (for 120 days).

5. Conclusion

In this paper, one-dimensional glioma growth model with the treatment term is considered. The implicit finite difference method is applied to generate the algebraic system. The implementation of TSGS to solve the generated algebraic system is also explained. From the results, it can be concluded that the TSGS method required less number of iteration and thus it requires less computational time as compared to GS iterative method.
Figure 1. Number of iterations for GS and TSGS methods.
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