Optimal elastic networks and the natural design of plant leaves

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Reinforced elastic sheets surround us in daily life, from concrete shell buildings over cars and airplane fuselages to biological structures such as the arthropod exoskeleton or the venation network of dicotyledonous plant leaves. Natural structures are often highly optimized through evolution and natural selection, leading to the biologically and practically relevant problem of understanding and applying the principles of their design. Inspired by the hierarchically organized scaffolding networks found in plant leaves, here we model networks of bending beams that capture the discrete and non-uniform nature of biologically evolved mechanical structures. Using the principle of maximal rigidity under natural resource constraints, we show that optimal discrete beam networks reproduce the structural features of real leaf venation. Thus, in addition to its ability to efficiently transport water and nutrients, the venation network optimizes leaf mechanical properties using practically the same hierarchical reticulated network topology. We study the phase space of optimal mechanical networks, providing concrete guidelines for the construction of elastic structures. We finally implement these natural design rules by fabricating efficient, biologically inspired metamaterials.

Elastic sheets reinforced by beams are pervasive in nature and engineering. From man-made constructs like Walther Bauersfeld and Buckminster Fuller’s geodesic domes [1] over concrete shell buildings [2] to aircraft fuselages [3], reinforced shells have found numerous applications due to their efficient use of resources and structural rigidity. At the same time, nature through evolution and natural selection has brought forth structures that satisfy similar demands. Examples include the leaves of plants, which need to stay flat to maximize photosynthesis [4–7], or the intricately veined wings of dragonflies, which combine light weight and rigidity to enable efficient flight [8]. Uncovering the design rules behind such biologically optimized natural materials may not just impact engineering but also further elucidate their role in evolution.

While humans have constructed buildings using thin shells for a long time, their efficient engineering is still an active research problem [9–15]. Recently, mechanical metamaterials have emerged as promising candidates for efficient, rigid and tunable structures [16–20]. In contrast to human-made materials, which are usually based on lattice geometries, natural materials are often characterized by a fractal-like hierarchical organization. Specifically, the venation of plant leaves is known to not only play a crucial role in the transport of water and nutrients [21], but also in load bearing and structural rigidity of the lamina [4, 6, 7], so as to allow the plant to maximize the area available for photosynthesis. At the same time, leaves are compliant to bend with the wind and other external forces [22, 23]. While much work has been done to characterize the venation networks of dicotyledonous plants in terms of geometry [24–26], topology [27–29], and optimization with respect to fluid transport [30–34], the mechanical purpose, properties, and optimality of the venation network beyond the midrib [4, 5, 35, 36] have received less attention [37]. This leads to questions about the extent to which leaves and similar natural materials are optimized, what rules their natural design underlies, and how these rules can be uncovered and applied.

To answer these questions, we here introduce a model of discrete beam networks (DBNs) which is able to capture the fundamental non-uniformity of natural materials. Specifically, DBNs capture the elastic properties of bending beams with arbitrary stiffness that are joined to form an elastic network. We apply this generic model to the elasticity of leaf venation. Using natural loads and conditions, we numerically globally minimize the network’s compliance, maximizing overall rigidity [13], under a resource constraint to model the trade-offs and economies of scale that natural networks are subject to [25, 34, 38–40]. We find that optimal DBNs under leaf-like conditions exhibit the same structural features as real leaves: a central midrib and hierarchically branching higher order veins connected by anastomoses, in close correspondence to vascular networks found by optimizing for robust liquid transport [21, 30–34, 41]. We show that this is due to a mathematical analogy, making leaf venation optimal with respect to rigidity and fluid transport, its two main biological functions. Furthermore, higher order features of the leaf venation such as the structure of interconnecting anastomoses and loops can be explained through compliance optimization in a natural way. We identify distinct topological phases as design rules of optimal DBNs that lead to substantially improved rigidity of the network as compared to uniform networks, and use these rules to construct and manufacture efficient elastic metamaterials.

DISCRETE BEAM NETWORKS

The theory of continuous elastic sheets connects curvature to an elastic energy [42, 43] and has been used with great success to model uniform membranes and shells [30,
44–46]. In the standard discretization scheme [45, 47–49] one performs a triangulation of the sheet and then penalizes angular deviations of neighboring triangles’ surface normals. While progress has been made recently in modeling reinforced elastic shells [9, 10], the fundamental discreteness of many biological elastic networks is not straightforward to implement. In the following, we develop a model of bending elastic beams that is adapted to the discreteness and non-uniformity of natural materials such as leaf venation.

As an illustrative example we consider a cylindrical elastic beam. Neglecting twisting and stretching contributions, the bending energy is [50, 51],

\[ V_b = \frac{\pi}{8} Y r^4 \frac{1}{\ell^2 R^2} \approx \frac{1}{2} \kappa \sin^2 \alpha, \quad (1) \]

where \( Y \) is the beam’s Young’s modulus, \( r \) is its radius, \( \ell \) is its length, and \( R \) is its radius of curvature. The bending angle \( \alpha \) was introduced by discretizing the beam using three nodes and approximating the curvature [Fig. 1 (c)]. The constants of proportionality were combined into the bending stiffness \( \kappa \). This elastic energy can be expressed using the cross product \( \sin^2 \alpha = \| b_1 \times b_2 \|^2 \), where \( b_1, b_2 \) are unit vectors along the discrete beams [Fig. 1 (c)]. Similar energies are commonly used for simulations [51]. To model complex, non-uniform bending networks, we now consider an equivalent formulation of Eq. (1) and then generalize it to arbitrary beams joined at a node.

Instead of directly computing the bending angle \( \alpha \), we introduce a local reference frame of unit vectors \( \{ e_1, e_2 \} \) at the midpoint, corresponding to the equilibrium configuration of the beam [Fig. 1 (c)]. The elastic energy is then,

\[ V = \frac{1}{2} \kappa_b \| e_1 \times b_1 \|^2 + \frac{1}{2} \kappa_b \| e_2 \times b_2 \|^2, \quad (2) \]

separated into contributions from two individual beams. If \( \{ e_1, e_2 \} \) are simply left fixed, Eq. (2) does not reproduce the elastic energy Eq. (1). However, the correct elastic energy is obtained when the orientation of the frame \( \{ e_1, e_2 \} \) is chosen to minimize \( V \), such that it follows any overall rigid motions (Appendix).

With the intuition gained from this example, we now generalize Eq. (2) to discrete beam networks (DBNs) where any number of beams can join at a node with arbitrary geometry and topology. The elastic energy of a node \( i \) in a DBN reads

\[ V_i = \frac{1}{2} \sum_{b \in B_i} \kappa_b \| (R_i e_b) \times b \|^2, \quad (3) \]

where the sum runs over all edges \( b \in B_i \) joining at node \( i \), \( \kappa_b \) is the bending stiffness of edge \( b \), and \( b \) is the unit vector pointing from node \( i \) to node \( j \) along the edge \( b = (ij) \). The node’s equilibrium configuration is given by the local reference frame \( \{ e_b \}_{b \in B_i} \) and the frame’s orientational degrees of freedom are encoded in the rotation matrix \( R_i \). We now linearize Eq. (3) by expanding both \( R_i \) and \( b \), and then minimize with respect to \( R_i \). After a straightforward calculation (Appendix) we find for the entire network,

\[ V = \frac{1}{2} u^\top (H_{eq} - H_{eq}) u = \frac{1}{2} u^\top H u, \quad (4) \]
FIG. 2. Compliance-optimized flat DBNs resemble real leaf venation. We globally optimized triangular DBNs with $N = 217$ nodes and $E_{\text{triang}} = 600$ edges under uniform load perpendicular to the network. (a–c) For $0 < \gamma < 1$, optimal networks are sparse and show hierarchical organization and anastomosing reticulation. (d) At the transition $\gamma = 1$ the network becomes highly reticulate and less hierarchically organized. The networks (a–d) were subject to a uniform downward load, the petiole was modeled as one additional node the position of which was fixed, and overall twists of the petiole were removed. The lamina stiffness was $\kappa_0 = 10^{-6}$. (e–g) Optimal networks reduce to just the main veins as the lamina stiffness $\kappa_0$ is increased. H. An optimal network with the petiole at the center and subject to a uniform upward load. The cost parameter in (e–h) was $\gamma = 1/2$, and the lamina stiffness in (h) was $\kappa_0 = 10^{-6}$. Fixed nodes are shown as red dots, each triangle is colored by the average compliance of the adjacent nodes normalized by the maximum, and the line thicknesses are proportional to $\gamma \kappa_{\text{b}}/2$.

where $\mathbf{u}$ is the $3N$-dimensional vector of nodal displacements from the equilibrium configuration of $N$ nodes. The first term involving the Hessian $H_{\text{eq}}$ is the elastic energy with respect to the equilibrium frame $\{e_b\}$, while $H_{\text{or}}$ corrects for the orientational freedom (Appendix).

Given loads $f$ on the network, the nodal displacements satisfy $Hu = f$. At each node, this force balance can be expressed as $f_i = \sum_j (F_{ij} - F_{ji})$, where $F_{ij}$ is the force on node $i$ due to the connection to node $j$, and $f_i$ is the net force (Appendix). Equation (4) is our starting point for the description of biological elastic networks. We note that like the standard scheme [45, 47–49], in the continuum limit, Eq. (4) reduces to the Helfrich [42, 43] free energy (Appendix).

In the rest of this paper we specialize to two-dimensional sheet-like DBNs that are nearly inextensible and dominated by bending via Eq. (4). To lowest order, the inextensibility constraint is implemented by allowing only those nodal displacements $\mathbf{u}$ that satisfy $b^\top (\mathbf{u}_j - \mathbf{u}_i) = 0$ for all edges $b$. Using this model, we now proceed to study optimal DBNs and compare to real leaf venation networks.

OPTIMAL ELASTIC NETWORKS

Plant leaves must remain flat and rigid to present a maximal area to sunlight for photosynthesis. Thus, we expect the reinforced scaffolding network of the leaf to be optimized under the influence of gravitational or wind load. Maximum rigidity of a mechanical system under loads $f$ leading to displacements $\mathbf{u}$ corresponds to minimum compliance $c = f^\top u$ [13]. In the following, we minimize the compliance over the set of bending stiffnesses $\kappa_{\text{b}}$ of the network. Biological networks are constrained by the amount of resources available, and by the requirement to distribute them efficiently. Following Refs. [30–33, 38], we incorporate this by introducing the constraint $\sum_b \kappa_{\text{b}}^\gamma = K$, where the parameter $\gamma$ models the material cost of each beam and $K$ is the overall cost. A natural material constraint is the total mass of the network, which for beams following Eq. (1) corresponds to $\gamma = 1/2$. More generally, $0 < \gamma < 1$ leads to an economy of scale where it is cheaper to construct fewer, non-uniform beams in a sparse network rather than many similarly sized beams in a dense network [52]. We now focus on this biologically relevant regime.

Since the cost constraint for $0 < \gamma < 1$ is non-convex, we expect a large number of local minima. To approximate the global optimum, we employ a numerical algorithm using local optimization with the method of Lagrange multipliers combined with a global scheme based on simulated annealing (Appendix). The optimal stiffnesses are then encoded in a scaling relation with the
The cost parameter $\gamma$ representing the leaf stalk, or petiole (Fig. 2). We fix the network to a small bending stiffness, we set the bending constants of the optimized networks normalized by the compliance $c$ of a uniform network with identical cost $K$. While the plots (a–c) were created using a single network size, the results remain qualitatively valid for larger networks as well (Appendix). (d) Optimal networks $\triangle, \Box, \bigcirc$, and a uniform network shown with their relative displacements under the same load. The optimal networks are also marked in panels (a–c). Displacements are measured relative to the tip of network $\bigcirc$.

Nodal forces (Appendix),

$$\kappa_b \sim \left( \frac{c}{b} \left( \|F_{ij}\|^2 + \|F_{ji}\|^2 \right) \right)^{\frac{1}{\gamma+2}},$$

where the edge $b$ connects nodes $i$ and $j$. In what follows, we start from a triangular grid in the $x$-$y$ plane representing the leaf lamina attached to another node representing the leaf stalk, or petiole (Fig. 2). We fix the petiole position and orientation in space, as well as the remaining rigid rotation around its axis. The entire leaf is subject to uniform load $g$ in the negative $z$ direction [Fig. 1 (d,e)]. Because the leaf lamina itself has a small bending stiffness, we set the bending constants of the network to $\kappa_0 + \kappa_b$, where $\kappa_0$ is the uniform stiffness of the lamina and the $\kappa_b$ are the non-uniform stiffnesses of the venation network that we minimize over. The cost $K$ is fixed to the number of edges in the triangular grid, setting the scale for the $\kappa_b$. We first consider $\kappa_0 = 10^{-6}$, specializing to the regime $\kappa_0 \ll \kappa_b$ where the elastic properties are dominated by the venation network. Here, globally optimized DBNs are rigid and flat to a high degree with a decrease of the compliance on the order of $\sim 100$ compared to uniform networks [Fig. 3 (c)], and exhibit a structure strongly resembling real dicoty-ledonous leaf venation [Fig. 2 (a–e)]. This includes a hierarchical backbone consisting of a midrib and branching off of higher order veins, but also redundant interconnections forming anastomoses between the veins. The optimal DBNs are similar to those generated from optimized fluid transport under additional conditions such as fluctuations or resilience to damage [30–34, 38]. Under the linear inextensibility constraint, only displacements perpendicular to the surface are allowed. In this case, the equilibrium term in the elastic energy Eq. (4) reduces to $\sum_{ij} \left( \kappa_b / \ell_b^2 \right) (u_{2,j} - u_{2,i})^2$. Formally identifying $\kappa_b / \ell_b^2$ with the hydraulic conductivity and the perpendicular displacements $u_{2,i}$ with the potential, this expression is equivalent to the viscous power dissipation minimized for flow networks. The constrained optima in the flow network case are known to always correspond to topological trees [53]. Thus, the orientation term $H_{ar}$ in Eq. (4) is responsible for departure from the tree-optima and induces redundancies and anastomoses in optimal mechanical networks. This intrinsic elastic mechanism stands in contrast to invoking resistance to fluctuations or damage to induce loops in optimal flow networks. The analogy also connects the optimal geometries, with both the hydraulic conductivity due to Poiseuille flow and Eq. (1) scaling as $\sim r^4$.

When $\gamma > 1$, the optimization problem becomes convex, and only a single, global minimum exists. This minimum contains a midrib but otherwise appears featureless, unlike real leaf venation [Fig. 2 (d)]. These generic properties remain valid for other boundary conditions as well. For instance, modeling the leaves of aquatic plants such as water lilies, which float on the surface of water supported by an upward buoyancy force and feature a central stalk, again leads to optimal networks resembling the natural leaf scaffoldings [Fig. 2 (h)].

Armed with these results, we proceed to study the topological transition from non-reticulate to reticulate optimal networks. The topology of planar networks is quantified by the number of loops $L = E - N + 1$, as obtained from Euler’s formula. Globally optimized DBNs exhibit three basic topological phases [Fig. 3 (a)]. In the convex regime where $\gamma > 1$ and the lamina stiffness $\kappa_0 \lesssim 10^{-2}$, the optimal networks corresponding to the single global minimum are maximally loopy. As $\gamma$ is decreased below 1, most loops are almost immediately lost and the optimal networks feature a small number of loops that is approximately constant over a wide range of parameter values. Increasing the lamina stiffness beyond $\kappa_0 \approx 10^{-2}$ leads to a gradual crossover into a regime with-

![FIG. 3. Topological transition and phase space of optimal DBNs with leaf boundary conditions. Each pixel in the 25 × 25 images (a–c) corresponds to a mean over 10 annealed triangular networks with $N = 92$ nodes and $E_{\text{max}} = 241$ edges. (a) Network topology is encoded in the loop density $L/L_{\text{max}}$, where $L$ is the number of loops and $L_{\text{max}} = 150$ is the maximum number of loops in the triangular grid. Grey pixels correspond to $L = 0$. The dashed and solid lines approximately mark the transitions to maximally loopy and tree topologies, respectively. (b) Network structure as measured by the number of non-zero stiffness edges $E$ normalized by the maximum number $E_{\text{max}}$ of edges in the triangular grid. (c) The compliance $c$ of the optimized networks normalized by the compliance $c$ of a uniform network with identical cost $K$. While the plots (a–c) were created using a single network size, the results remain qualitatively valid for larger networks as well (Appendix). (d) Optimal networks $\triangle, \Box, \bigcirc$, and a uniform network shown with their relative displacements under the same load. The optimal networks are also marked in panels (a–c). Displacements are measured relative to the tip of network $\bigcirc$.](attachment://figure3.png)
out loops, where the optimal networks reinforce mostly the main and secondary veins [Fig. 2 (e–g)]. These transitions are mirrored in the number of nonzero stiffness edges $E$ in the network, with the difference that $E$ gradually decreases as $\kappa_0$ is increased instead of dropping to zero [Fig. 3 (b)]. Surprisingly, the optimal compliance does not vary strongly with the optimal network topology [Fig. 3 (c, d)]. Instead, the optimal compliance is largely independent of the lamina stiffness $\kappa_0$ and varies strongly only with the cost parameter $\gamma$. Since in most relevant systems $\gamma$ is expected to be fixed by structural and geometrical properties, this suggests that generically, it pays to invest in an optimized mechanical network, even if this means only reinforcing the main vein. Even then the improvement in compliance is significant when compared to a uniform network, rarely worse than a factor of 10, and for the most relevant case of $\gamma \approx 1/2$ often on the order of factors of 100 [Fig. 3 (c)].

The natural design principles of leaf venation can be applied to the design of efficient rigid metamaterials. We used commercially available processes for soft elastomers to additively manufacture networks of connected cylindrical beams based on optimized and uniform DBN topologies with equal material volume [Fig. 4 (a, b), Appendix]. The improvement in rigidity in the optimized manufactured network is significant, with no bending or tip displacement discernible [Fig. 4 (c)]. This is compared to the uniform network, which bends visibly [Fig. 4 (d)]. Simulations using the Finite Element method confirm these observations, predicting an improvement in tip displacement in the optimal network by a factor of approximately 92. These results suggest that biologically inspired elastic networks may provide a promising avenue for a more general design of efficient discrete metamaterials.

**DISCUSSION**

Inspired by biological examples, we introduced a model of discrete beam networks that is able to naturally represent non-uniform reinforcing scaffoldings of elastic sheets and networks. We then applied this DBN model to leaf venation networks, which must remain flat and rigid, and showed that optimal networks minimizing mechanical compliance under a cost constraint strongly resemble real leaves, including a hierarchical backbone and anastomoses and loops between the veins. Using the principles learned from nature, we designed and manufactured efficient elastic metamaterials.

Beyond the ability to generalize our method to generic two- or three-dimensional networks (Appendix), our results may have implications for leaf biology and evolution. The relevance of fluid flow optimization for leaf venation is well-known when rationalizing loops as an evolutionary adaptation to damage or fluctuations [21, 34]. At the same time, the reduction in compliance of optimized over uniform DBNs is significant, often reaching factors between 50 and 100 for relevant cost parameters. Thus, maximizing stiffness by minimizing compliance could result in a large evolutionary advantage. This means that leaves are in the extraordinary position to optimize two highly disparate requirements, mechanical rigidity and robust fluid transport, using the same hierarchically organized, reticulate venation network architecture. Our results may also offer a connection between the differing approaches modeling leaf vascular development as adaptive mechanisms relying on either flow [33, 54, 55] or mechanical [56–60] cues.

More generally, our results pave the way for detailed study of optimally reinforced mechanical networks in biological systems. Our methods may also be applicable to investigating mechanics and optimization in more complicated, three-dimensional biological structures such as actin-myosin networks [61], active mechanics [62, 63], allosteric materials [64], or network control [65]. We are convinced that our approach will also be useful in metamaterial design and engineering.
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APPENDIX

Minimization over the rotational degrees of freedom

We first show that Eq. (2) reduces to the correct elastic energy upon minimizing over the orientation of the local reference frame. We pick Euclidean coordinates \( b_1 = (\sin \theta_1, \cos \theta_1, 0) \), \( b_2 = (\sin \theta_2, \cos \theta_2, 0) \), \( e_1 = (\sin \theta, \cos \theta, 0) \), and \( e_2 = -e_1 \). The elastic energy is then,

\[
V = \frac{1}{2} k_b \sin^2(\theta - \theta_1) + \frac{1}{2} k_b \sin^2(\pi + \theta - \theta_2).
\]

In the limit of small angles, the minimizer of \( V \) with respect to \( \theta \) is \( \theta^* \approx (\theta_1 + \theta_2 - \pi)/2 \). Plugging this back in we obtain

\[
V \approx \frac{1}{4} k_b \sin^2(\pi/2) = \frac{1}{4} k_b \sin^2(\alpha),
\]

which agrees with Eq. (1) upon identifying \( k_b = 2 k_b \).

We now derive the DBN bending energy Eq. (4) from Eq. (3). We minimize the elastic energy Eq. (3) over the linearized rotation matrix which is parametrized by a vector \( n_i \) and acts as \( R_i a \approx a + n_i \times a \) on a vector \( a \). We write the position of each node as \( x_i = x_i^{(0)} + u_i \), where \( x_i^{(0)} \) is the equilibrium position and \( u_i \) is a small displacement. To linear order, the unit vector along an edge \( b = (ij) \) can be expanded as \( b \approx e_b + J_b(u_j - u_i) \), where the Jacobian encodes the double cross product \( J_b a = -\frac{1}{\ell_b} e_b \times (e_b \times a) \) with the equilibrium length \( \ell_b \) and the equilibrium unit vector \( e_b = (x_j^{(0)} - x_i^{(0)})/\|x_j^{(0)} - x_i^{(0)}\| \). With this, we can expand

\[
\|(R_i e_b) \times b\|^2 \approx \|(e_b + J_b(u_j - u_i)) \times (e_b + n_i \times e_b)\|^2
\]

\[
= \|D_b u - C_b n_i\|^2,
\]

where we neglected non-linear terms in \( u \) and \( n_i \). Here, the matrix \( D_b \) acts as \( D_b u = \frac{1}{\ell_b} e_b \times (u_j - u_i) \), \( C_b = 1 - e_b e_b^\top \), and the 3N-dimensional vector \( u \) contains the displacements of the \( N \) nodes. At each node \( i \), the linearized elastic energy is then

\[
V_i = \frac{1}{2} \sum_{b \in B_i} k_b \|D_b u - C_b n_i\|^2.
\] (A1)

Taking the gradient of \( V_i \) with respect to \( n_i \) and setting it to zero using \( C_b^\top C_b = C_b \) and \( C_b^\top D_b = D_b \) we obtain \( C_i n_i = D_i u \), where \( C_i = \sum_b k_b C_b \) and \( D_i = \sum_b k_b D_b \). Formally solving this linear equation for \( n_i \), plugging the result into Eq. (A1), and summing over all nodes \( i \), we arrive at Eq. (4) with \( H_{eq} = \sum_i \sum_{b \in B_i} k_b D_b^\top D_b \) and \( H_{ae} = \sum_i D_i^\top C_i^{-1} D_i \).
Nodal force balance

We now derive the nodal force balance from Eq. (A1). Rewriting in terms of three-dimensional vectors and making the edges $b = (ij)$ explicit the total network energy $V = \sum_i V_i$ reads

$$V = \frac{1}{2} \sum_{i,j} \kappa_{ij} \| C_{ij} n_i - \ell_{ij}^{-1} e_{ij} \times (u_j - u_i) \|^2.$$ 

Using $\frac{\partial V}{\partial n_i^T} = 0$, the net force on node $i$ is

$$F_i = -\frac{\partial V}{\partial u_i^T} = \sum_j (F_{ij} - F_{ji}),$$

where we used that each nodal displacement $u_i$ appears in $V_i$ and in all $V_j$ that are connected to node $i$. The forces are

$$F_{ij} = -\frac{\kappa_{ij}}{\ell_{ij}} e_{ij} \times \left( C_{ij} n_i - \ell_{ij}^{-1} e_{ij} \times (u_j - u_i) \right).$$

Here, $\kappa_{ij} = \kappa_{ji}$, $\ell_{ij} = \ell_{ji}$, and $C_{ij} = C_{ji}$. Using the definition of $C_{ij} = 1 - e_{ij} e_{ij}^T$, the magnitudes are,

$$\| F_{ij} \|^2 = \frac{\kappa_{ij}^2}{\ell_{ij}^2} \| C_{ij} n_i - \ell_{ij}^{-1} e_{ij} \times (u_j - u_i) \|^2. \tag{A2}$$

Constrained Optimization

We adapt the global approach outlined in Ref. [30]. The Lagrangian corresponding to the constrained minimization problem is

$$\mathcal{L}(\{\kappa_b\}) = c(\{\kappa_b + \kappa_0\}) + \lambda \left( \sum_b \kappa_b^2 - K \right),$$

where $c = f^T u$ is the compliance and $\lambda$ is a Lagrange multiplier. Taking the gradient with respect to $\kappa_b$ and combining with Eq. (A2) leads to the scaling relation Eq. (5). We numerically solve for the $\kappa_b$ using the iteration

$$\kappa^{(n+1)}_b = \left( -\frac{\kappa^{(n)}_b}{\ell^{(n)}_{ij}} \frac{\partial c(\{\kappa^{(n)}_b + \kappa_0\})}{\partial \kappa_b} \right)^{1/(\gamma+1)} \left( \sum_{b'} (\kappa^{(n+1)}_{b'})^{-\gamma} \right)^{-1/\gamma},\tag{A3}$$

where the second step fixes the Lagrange multiplier by enforcing the constraint. Combining Eq. (A3) with a variant of simulated annealing leads to approximate global minimization. At every $N_{\text{therm}}$-th step of the iteration Eq. (A3), the $\{\kappa_b\}$ are first thermalized by convolving with a Gaussian kernel $G_{ab} \sim \exp\left(-d_{ab}^2/(2\sigma^2)\right)$ where $d_{ab}$ is the Euclidean distance between edges $a$ and $b$ and where the scale $\sigma$ is decreased after each thermalization. Then, multiplicative noise $\sim \exp(s \xi)$, where $\xi$ is normally distributed and $s \sim \mathcal{O}(1)$, is applied. After a set number of thermalization steps, Eq. (A3) is iterated until convergence.

Metamaterials

3D meshes were constructed from cylinders with spherical end-caps, with cylinder radii taken from optimal and uniform DBN models. The metamaterials were commercially manufactured from thermoplastic polyurethane (Materialise nv, Leuven, Belgium). FEM simulations were performed with the MATLAB 2018b PDE Toolbox (The MathWorks, Inc., Natick, MA). Material properties were Young’s Modulus $Y = 85$ MPa, density $\rho = 1100$ kg m$^{-3}$, Poisson’s ratio $\nu = 0.49$. 
Continuum limit

Here we demonstrate that the bending energy Eq. (4) in the continuum limit of an initially flat, uniform sheet in equilibrium is equivalent to the Helfrich free energy [42, 43, 45],

$$F = \int_A (\kappa_1 H^2 + \kappa_2 K) \, dA,$$

where $H$ and $K$ are the surface’s mean and Gaussian curvatures, respectively, $\kappa_1, \kappa_2$ are elastic constants, and the integral is over the surface of the sheet $A$. We choose a triangular grid to model the flat sheet in the $x$-$y$ plane and set all the bending constants $\kappa$ to unity. The inextensibility constraint is then equivalent to only allowing displacements in the $z$ direction, $u = (0, 0, u_z)^\top$, since all local in-plane displacements are forbidden. At each node, the unit vectors in the directions of the edges are

$$e_1 = (1, 0, 0)^\top, \quad e_2 = (1/2, \sqrt{3}/2, 0)^\top, \quad e_3 = -e_1,$$

$$e_4 = (1/2, -\sqrt{3}/2, 0)^\top, \quad e_5 = -e_2, \quad e_6 = -e_3,$$

and the matrix $C_i = \text{diag}(3, 3, 6)$. In the limit where the edge lengths $\ell$ tend to zero, the sheet’s displacements are approximated by a height function $u_z = h(x, y)$. The expressions involving the matrices $D_b$ can then be written as $D_b u \approx \ell^{-1} e_b \times (0, 0, h(x + \ell (e_b)_x, y + \ell (e_b)_y) - h(x, y))^\top$. Plugging this form into Eq. (4), expanding to lowest order in $\ell$ and summing over all vertices we find for the total bending energy,

$$V \approx \frac{3}{16} \sum_i \left( 3(h_{xx} + h_{yy})^2 - 4(h_{xx} h_{yy} - h_{xy}^2) \right) \ell^2. \quad (A5)$$

Using the small-gradient expansions [66] of the mean curvature $H \approx h_{xx} + h_{yy}$ and the Gaussian curvature $K = h_{xx} h_{yy} - h_{xy}^2$, and the area element $dA \approx \ell^2 \sqrt{3}/2$ corresponding to hexagons around each node, in the limit $\ell \to 0$ the sum Eq. (A5) tends to the integral Eq. (A4) with the elastic constants $\kappa_1 = 3\sqrt{3}/8$ and $\kappa_2 = -\sqrt{3}/2$.

We note that a naive bending energy containing only fixed $\{e_b\}_{b \in B_i}$ (no minimization over the rotational degrees of freedom) corresponds to only the first term in the sum of Eq. (4) and does not lead to a well-defined continuum limit.

Scaling of the phase space of optimal DBNs

Here we present a size scaling analysis of the topological phase space shown in Fig. 3 of the main paper. While the phase space there was computed for networks with 92 nodes, here we show slices through the phase space for larger networks. We consider slices at $\gamma = 0.5$ and $\kappa_0 = 10^{-3}$ and parametrize the networks by the linear number of nodes $M$ along the midrib. For the triangular networks we consider, the total number of nodes $N \sim O(M^2)$. The scaling of the number of loops is shown in Fig. A1, the scaling of the number of nonzero edges is shown in Fig. A2, and the scaling of the compliance is shown in Fig. A3. We estimate the number of nonzero edges by thresholding the results of the optimization at $\kappa_b = 10^{-8}$ and considering all edges with smaller bending stiffness as absent. Similarly, we estimate the number of nodes by computing the weighted degree $d_i = (1/n) \sum_j \kappa_{ij}$ of each node $i$ in the original triangular network with $n$ neighbors, and again count nodes with $d_i < 10^{-8}$ as absent. Each data point in the aforementioned figures shows an average over at least 10 optimizations. All curves for different network sizes collapse after rescaling, suggesting that the phase space shown in Fig. 3 of the main paper is robust as network size is varied.

Three-dimensional optimal DBNs

Here we show that the DBN model introduced in the main paper can be used to model fully three-dimensional networks of connected bending beams as well. As the base topology, we take a three-dimensional tetrahedral network [Fig. A4 (a)]. Since such a network is perfectly rigid under the inextensibility constraint, for the purposes of this proof of concept, we remove the constraint. We note that for realistic applications, it would be necessary to introduce a stretching energy including a relationship between stretching and bending stiffnesses of each beam. Optimal networks fixed at one side and under uniform perpendicular load show similar features as sheet-like DBNs [Fig. A4 (b–d)].
FIG. A1. Scaling of the number of loops. (a) Number of loops at slice through the phase space at $\gamma = 0.5$. (b) Number of loops normalized by the maximum at slice through the phase space at $\gamma = 0.5$. (c) Number of loops at slice through the phase space at $\kappa_0 = 0.001$. (d) Number of loops normalized by the maximum at slice through the phase space at $\kappa_0 = 0.001$.

FIG. A2. Scaling of the number of nonzero edges. (a) Number of edges at slice through the phase space at $\gamma = 0.5$. (b) Number of edges normalized by the maximum at slice through the phase space at $\gamma = 0.5$. (c) Number of edges at slice through the phase space at $\kappa_0 = 0.001$. (d) Number of edges normalized by the maximum at slice through the phase space at $\kappa_0 = 0.001$.

FIG. A3. Scaling of the compliance. (a) Compliance at slice through the phase space at $\gamma = 0.5$. (b) Compliance normalized by the compliance of a uniform network with identical cost at slice through the phase space at $\gamma = 0.5$. (c) Compliance at slice through the phase space at $\kappa_0 = 0.001$. (d) Compliance normalized by the compliance of a uniform network with identical cost at slice through the phase space at $\kappa_0 = 0.001$. 
FIG. A4. Three-dimensional optimal DBNs fixed at one side. (a) Tetrahedral base network with fixed nodes indicated in red. The uniform load $f$ is shown as a black arrow. (b–d) Optimal networks obtained using simulated annealing with cost parameters $\gamma = 0.25, 0.5, 0.75$ and $\kappa_0 = 10^{-4}$. Line widths are proportional to $\kappa^\gamma/2$. 