MEASURABLE SOLUTIONS TO GENERAL EVOLUTION INCLUSIONS

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Abstract. This work establishes the existence of measurable solutions to evolution inclusions involving set-valued pseudomonotone operators that depend on a random variable $\omega \in \Omega$ that is an element of a measurable space $(\Omega, \mathcal{F})$. This result considerably extends the current existence results for such evolution inclusions since there are no assumptions made on the uniqueness of the solution, even in the cases where the parameter $\omega$ is held constant, which leads to the usual evolution inclusion. Moreover, when one assumes the uniqueness of the solution, then the existence of progressively measurable solutions under reasonable and mild assumptions on the set-valued operators, initial data and forcing functions is established. The theory developed here allows for the inclusion of memory or history dependent terms and degenerate equations of mixed type. The proof is based on a new result for measurable solutions to a parameter dependent family of elliptic equations. Finally, when the choice $\omega = t$ is made, where $t$ is the time and $\Omega = [0, T]$, the results apply to a wide range of quasistatic inclusions, many of which arise naturally in contact mechanics, among many other applications.

1. Introduction. In this paper we establish the existence of product measurable solutions $(u, u^*)$ to a general class of evolution inclusions that depend on a random

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2000 Mathematics Subject Classification. Primary: 35R60; Secondary: 35Q74, 60H25.

Key words and phrases. Set-valued differential inclusions, measurable solutions, random coefficients, quasistatic inclusions, product measurability.
variable of the following form,

\[(Bu(\cdot,\omega))' + u^*(\cdot,\omega) = f(\cdot,\omega),\]  
\[u^*(\cdot,\omega) \in A(u(\cdot,\omega),\omega),\]  
\[Bu(0,\omega) = Bu_0(\omega).\]  

Here the prime indicates the time derivative, \(A\) is a set-valued operator whose properties are described below; \(\omega \in \Omega\) is the random variable, \((\Omega, \mathcal{F})\) is a measurable space and \(f\) is a given function. The operator \(B\) may vanish and hence these inclusions include problems of mixed type. Problems of this form arise in the study of various nonlinear partial differential inclusions and in such applications as contact mechanics. In the latter case set inclusions arise naturally in the modeling of contact boundary conditions and friction. They are related to a large number of dynamic or quasistatic problems in mechanics concerning bodies made of elastic, viscoelastic, viscoplastic or thermoelastic materials, see, e.g., \([17, 16, 7, 13]\) and the host of references therein. Furthermore, inclusion of randomness is of considerable importance in applications, since too often the values of various system coefficients are not known and cannot be easily obtained by straightforward measurements. Thus, allowing for uncertainty in the form of randomness in their values makes the models more realistic, and so of greater use in applications.

There are numerous existence results for problems of this type that do not depend on a random variable \(\omega\), see, e.g., \([9], [12]\) where a very general existence result was established for such problems, as well as in \([15, 13, 6, 14]\) and the many references therein. The main novelty in the present paper is the treatment of general evolution inclusions in which the nonlinear set-valued operator, the forcing function, and the initial data are dependent on a random parameter and in which there is no uniqueness for the resulting evolution inclusion with a fixed \(\omega\). These problems have received comparably little previous attention. Indeed, until now in the cases when the operators and the data depended on a parameter, obtaining measurable solutions was heavily dependent on either monotonicity of the operators or on the uniqueness of solutions for a fixed \(\omega\) in both cases. There are very few results when these conditions don’t hold. One such paper, in which the authors obtain measurable solutions for the Navier-Stokes equations, in a space with dimension larger than two, without assuming the uniqueness, is \([3]\) from 1973. This result was generalized in \([10]\), but both papers involved single valued operators and left many open questions. There are also few papers that involve a set-valued pseudomonotone operator and then for a possibly degenerate operator \(B\), there are even fewer. The existence of solutions when \(B\) is the identity and \(A\) is single valued was shown in the late 1960’s in \([13]\). We are not aware of similar results for evolution inclusions involving set-valued maps in which uniqueness is unavailable for a fixed \(\omega\), and therefore, our results here are completely new.

Our results use new measurability theorems for quasistatic problems obtained in \([1]\) that provide a new approach to evolution equations by using a stochastic version of the method of Lions and Brezis \([13, 5]\) in which a pseudomonotone operator is perturbed by a linear maximal monotone operator. We also make essential use of recent results on the measurability of solutions in \([11]\), in which the meaning of measurability into a space \(L^p([0,T]; V)\) is shown to be essentially equivalent to product measurability upon taking suitable selections. Using these results we show the existence of a solution \((u, u^*)\) to problem (1)–(3) such that \((t,\omega) \rightarrow u(t,\omega)\) and \((t,\omega) \rightarrow u^*(t,\omega)\) are product measurable. Essentially, we show the existence
of a measurable selection in the set of solutions. Since there are no assumptions needed on the measurable space, we show that the usual limit processes preserve measurability in some sense. Furthermore, the measurability assumptions made here on the set-valued operator $A$ are weaker than usual. We do not need to assume that the operator is a measurable multifunction. It suffices for it to possess a measurable selection in the sense described precisely below. This appears to be a new and more general formulation in which the hypotheses are more general and are easier to check. As to the case of fixed $\omega$, the presentation here consist of a more readable version of [12] that is based on a more interesting theory, namely the new result on measurable elliptic problems in [1]. It also relaxes the assumption that the exponent $p$ in the $L^p$ spaces is no smaller than 2. Moreover, we would like to stress here, that our results are more general even in the case when $\omega$ is fixed, since our assumptions on the operators are weaker than those currently contained in the literature.

The main theorem in [12] has been extremely useful over the years in analyzing evolution models that include contact and friction, which usually lack uniqueness. The results here allow for the formulation of these problems more generally by allowing the system parameters to be random variables, or as stochastic inclusions and to obtain measurable solutions. It is also possible to use the main result, Theorem 4.1, to establish the existence of solutions to stochastic evolution inclusions in which there is multiplicative noise, if, in addition, the problem is monotone for fixed $\omega$. From this, it is possible to obtain existence of solutions to stochastic evolution equations involving additive noise even when uniqueness is not available for fixed $\omega$, but these are topics for another paper and will involve a slight specialization on the nonlinear set-valued maps.

One of the most interesting features of this paper is that it is not necessary to integrate with respect to $\omega$. That is, there is an evolution inclusion for each fixed $\omega$ and our selection theorems show the existence of a solution for each $\omega$ in such a way that we obtain product measurability with respect to $(t, \omega)$. It is the evolution equation version of what we did in [1]. This problem of existence of measurable solutions is a difficult generalization because when a nonlinear operator from $V$ to $P(V')$ has certain properties, it is no longer obvious that it will do so as a map from $L^p(\Omega; V)$ to $P\left(L^{p'}(\Omega; V')\right)$ and this is exactly the type of generalization done here. We note that one cannot use the usual embedding theorems when the operator is not close enough to being monotone, and limit conditions such as being pseudomonotone are lost, too. These make it difficult to successfully pass to the necessary limits.

We can now describe the remaining sections of this paper. Following this introduction, we provide in Section 2 the precise statement of problem (1)–(3) and describe the assumptions on the problem data, especially on the operator $A$ in Section 2.1. Section 3 provides the necessary auxiliary material needed later. One of the main results in this work, Theorem 4.1, guarantees the existence of solutions to a wide class of evolution inclusions, and extends the existence result in [12] to include measurable solutions. A relaxed coercivity condition is discussed in Section 5. Finally, in Section 6 we extend the results and study progressively measurable solutions of the problem.

2. **Statement of the problem.** In this section we describe the setting and state the problem precisely. The proof is provided in the following sections. Let $V$ and
be two reflexive separable Banach spaces such that \( V \subseteq W \) and \( V \) is dense in \( W \), so that \( W' \subseteq V' \), and we denote by \( \langle \cdot, \cdot \rangle \) the duality pairing of \( W \) and \( W' \). Next, for \( p > 1 \), we let
\[
V \equiv L^p ([0, T]; V),
\]
and it follows from the Riesz representation theorem that \( V' = L^{p'} ([0, T]; V') \), where \( p' \) is the conjugate of \( p \), i.e., \( 1/p + 1/p' = 1 \).

Let \( B (t) \) be a linear operator, \( B (t) : W \rightarrow W' \), that satisfies

1. \( \langle B (t) x, y \rangle = \langle B (t) y, x \rangle \), symmetric;
2. \( \langle B (t) x, x \rangle \geq 0 \), positive;
3. \( B \in C^1 ([0, T]; \mathcal{L}(W, W')) \) i.e., the time derivative is bounded.

The assumptions made on the operator \( A \) are provided in Section 2.1.

We leave the spaces \( V \) and \( W \) unspecified since their choice depends on the setting of the particular application; as an example, one can take \( W = H = H' \), where \( H \) is a Hilbert space, and consider it as the usual Gelfand triple, and one may choose \( B = I \), the identity.

We assume everywhere below that \((\Omega, \mathcal{F})\) is a general measurable space and since no measure is needed none is referred to. Moreover, we choose the \( \sigma \)-algebra of measurable sets to be \( \mathcal{B} ([0, T]) \), the Borel measurable sets on \([0, T] \). The product measurable sets are those in the smallest \( \sigma \)-algebra that contains all the measurable rectangles \( D \times E \) where \( D \in \mathcal{B} ([0, T]) \) and \( E \in \mathcal{F} \).

First, we assume \( p \geq 2 \), since we want to consider the integral
\[
\int_0^T \langle B' u, u \rangle \, dt,
\]
which doesn’t make sense unless \( p \geq 2 \). However, this restriction is not necessary if \( B \) is a constant operator, as we show in the following section.

2.1. Assumptions on the operator \( A \). We make very general assumptions on \( A \). The commonly used case when there is no dependence on \( \omega \), i.e., \( A (u) \), for \( u \in V \), given as \( A (u) (t) = A (u (t)) \), is just a special case. Moreover, when \( A (u, \omega) (t) \) is history-dependent, i.e., it depends on the values of \( u (s) \) for \( s \leq t \), is also included. This makes the general results here applicable to problems that are second order in \( t \), which arise in models for dynamic processes.

The following is the definition of a pseudomonotone operator we shall use in this paper. It is equivalent to that found in [15], when only bounded mappings are considered, and is provided here for the sake of completeness.

**Definition 2.1.** Let \( X \) be a reflexive Banach space. The operator \( A : X \rightarrow P (X') \) is said to be pseudomonotone and bounded if the following conditions hold.

1. The set \( Au \) is nonempty, closed and convex for all \( u \in X \) and \( A \) maps bounded sets to bounded sets.
2. If \( u_i \rightarrow u \) weakly in \( X \) and \( u_i^* \in Au_i \) is such that
   \[
   \limsup_{i \to \infty} \langle u_i^*, u_i - u \rangle \leq 0,
   \]
   then, for each \( v \in X \), there exists \( u^* (v) \in Au \) such that
   \[
   \liminf_{i \to \infty} \langle u_i^*, u_i - v \rangle \geq \langle u^* (v), u - v \rangle.
   \]
We assume now that $B$ is time dependent and $p \geq 2$ and we denote weak convergence by $\rightharpoonup$.

The assumptions on $A(\cdot, \omega)$ are described next.

- **Growth estimate**
  \[
  \sup \{ \|u^*\|_{V'} : u^* \in A(u, \omega) \} \leq a(\omega) + b(\omega) \|u\|_{V'}^{p-1},
  \]  
  where $a(\omega)$ and $b(\omega)$ are two bounded nonnegative functions.

- **Coercivity condition**
  \[
  \inf \left( \int_0^t \langle u^*, u \rangle ds : u^* \in A(u, \omega) \right) + \frac{1}{2} \int_0^t \langle B' u, u \rangle ds \\
  \geq \delta(\omega) \int_0^t \|u\|^p ds - m(\omega),
  \]  
  for each $0 \leq t \leq T$, where $m(\omega)$ is a nonnegative function and $\delta(\omega) > 0$.

  Actually, it is often enough to assume that the left-hand side is of the form
  \[
  \inf \left( \int_0^t \langle u^*, u \rangle + \lambda(\omega) \langle Bu, u \rangle ds : u^* \in A(u, \omega) \right),
  \]  
  for some $\lambda(\omega)$, by changing the dependent variable and using a suitable exponential shift argument.

- **Limit condition**
  If $u_i \rightharpoonup u$ in $V$ and $(Bu_i)' \rightharpoonup (Bu)'$ in $V'$, $u_i^* \in A(u_i, \omega)$, and
  \[
  \limsup_{i \to \infty} \langle u_i^*, u_i - u \rangle_{V', V} \leq 0,
  \]  
  then for each $v \in V$, there exists $u^*(v) \in A(u, \omega)$ such that
  \[
  \liminf_{i \to \infty} \langle u_i^*, u_i - v \rangle_{V', V} \geq \langle u^*(v), u - v \rangle_{V', V}.
  \]  

- **Measurability condition**
  Let $\omega \to u(\cdot, \omega)$ be a measurable function into $V$. Then
  \[
  \omega \to A(u(\cdot, \omega), \omega)
  \]  
  has a measurable selection into $V'$.

  The last condition means that there is a function $\omega \to u^*(\cdot, \omega)$ that is measurable into $V'$, such that $u^*(\omega) \in A(u(\cdot, \omega), \omega)$. This is assured, see e.g., [8, 2], when the following standard set-valued measurability condition is satisfied for every open set $O$ in $V'$:
  \[
  \{ \omega : A(u(\cdot, \omega), \omega) \cap O \neq \emptyset \} \in \mathcal{F}.
  \]  
  We note that our assumption is implied by this condition, but the conditions are not equivalent and what is considered here is more general than the assumption that $\omega \to A(u(\cdot, \omega), \omega)$ is set-valued measurable.

  Indeed, our condition holds when $u \to A(t, u, \omega)$ is bounded and pseudomonotone as a single-valued map from $V$ to $V'$ and $(t, \omega) \to A(t, u, \omega)$ is product measurable into $V'$, for each $u$. One can use the demicontinuity of $u \to A(t, u, \omega)$, which comes from the pseudomonotonicity and boundedness assumptions and consider a sequence of simple functions $u_n(t, \omega) \to u(t, \omega)$ in $V$ for $u$ measurable, each $u_n(\cdot, \omega)$ being in $V$. Then, the measurability of $A(t, u_n, \omega)$ is related to $A(t, u, \omega)$ in the limit. In the situation where $A(\cdot, \omega)$ satisfies a suitable upper-semicontinuity condition, it is enough to assume only that $\omega \to A(u, \omega)$ has a measurable selection for each $u \in V$. This is a straightforward consequence of approximation with simple functions and then using the upper-semicontinuity instead of continuity.
3. Auxiliary results. We present now various results that are used in the proof of our main result. Below, the norms on the various reflexive Banach spaces are assumed to be strictly convex. We use the following well known theorem [13]. It is stated here in the setting in which a Hölder condition is assumed rather than a bound on the weak derivatives.

**Theorem 3.1.** Let \( E \subseteq F \subseteq G \), where the injection map is continuous from \( F \) to \( G \) and compact from \( E \) to \( F \). Let \( p \geq 1 \), let \( q > 1 \), \( C \) and \( R \) be two positive constants, and define

\[
S = S_{CR} = \{ u \in L^p ([a,b], E) : \| u(t) - u(s) \|_G \leq C |t - s|^{1/q} \text{ and } \| u \|_{L^p ([a,b], E)} \leq R \}.
\]

Then, \( S \) is bounded in \( L^p ([a,b], E) \) and precompact in \( L^p ([a,b], F) \). Thus, if \( \{ u_n \}_{n=1}^\infty \subseteq S \), it has a subsequence \( \{ u_{n_k} \} \) that converges in \( L^p ([a,b], F) \).

The same conclusion holds true when the Hölder condition is replaced with the condition that \( \| u' \|_{L^1 ([a,b]; G)} \) is bounded.

Next, we present some measurable selection theorems that are an essential part of showing the existence of measurable solutions. They do not depend on any specific measure, but in the applications of most interest to us, they are typically a probability measure.

The first result is a basic selection theorem for a set of limits. The proof can be found in [1].

**Theorem 3.2.** Let \( U \) be a separable reflexive Banach space. Suppose there is a sequence \( \{ u_j (\omega) \}_{j=1}^\infty \) in \( U \), where \( \omega \to u_j (\omega) \) is measurable and for each \( \omega \),

\[
\sup_j \| u_j (\omega) \|_U < \infty.
\]

Then, there exists a measurable function \( \omega \to u(\omega) \) with values in \( U \), and a subsequence \( n(\omega) \), depending on \( \omega \), such that

\[
\lim_{n(\omega) \to \infty} u_{n(\omega)} (\omega) = u(\omega) \text{ weakly in } U.
\]

The next result is a restriction of this theorem to the case where the Banach space is a function space and its proof can be found in [11]. This gives a result on product measurability.

**Theorem 3.3.** Let \( V \) be a reflexive separable Banach space with dual \( V' \), and let \( p, p' \) be such that \( p > 1 \) and \( 1/2 + 1/p + 1/p' = 1 \). Let the functions \( t \to u_n (t, \omega) \), for \( n \in \mathbb{N} \), be in \( L^p ([0,T]; V) \equiv \mathcal{V} \) and \( (t, \omega) \to u_n (t, \omega) \) be \( \mathcal{B} ([0,T]) \times \mathcal{F} \equiv \mathcal{P} \) measurable into \( V \). Suppose

\[
\| u_n (\cdot, \omega) \|_V \leq C (\omega),
\]

for all \( n \). Then, there exists a product measurable function \( u \) such that \( t \to u(t, \omega) \) is in \( \mathcal{V} \) and for each \( \omega \) a subsequence \( u_{n(\omega)} \) such that \( u_{n(\omega)} (\cdot, \omega) \to u(\cdot, \omega) \) weakly in \( \mathcal{V} \).

We note that by the weak compactness in \( \mathcal{V} \), for each \( \omega \) every subsequence of \( \{ u_n \} \) has a further subsequence that converges weakly in \( \mathcal{V} \) to \( v(\cdot, \omega) \in \mathcal{V} \), however, there is no guarantee that \( v \) is \( \mathcal{P} \) measurable. Since we do not assume that the constants \( C (\omega) \) are uniformly bounded in \( \omega \), the above theorem warrants a proof.

Next, we describe what is meant to be measurable into \( \mathcal{V} \) or \( \mathcal{V}' \). Indeed, these functions have representatives that are product measurable.
Lemma 3.4. Let \( f ( \cdot , \omega ) \in V' \) and assume that \( \omega \to f ( \cdot , \omega ) \) is measurable into \( V' \). Then, for each \( \omega \), there exists a representative \( \hat{f} ( \cdot , \omega ) \in V' \), \( \hat{f} ( \cdot , \omega ) = f ( \cdot , \omega ) \) in \( V' \), such that \( ( t , \omega ) \to \hat{f} ( t , \omega ) \) is product measurable. If \( f ( \cdot , \omega ) \in V' \) and \( ( t , \omega ) \to f ( t , \omega ) \) is product measurable, then \( \omega \to f ( \cdot , \omega ) \) is measurable into \( V' \). The same statement holds true when \( V' \) is replaced with \( V \).

Proof. When a function \( f \) is measurable into \( V' \), there exist a family of simple functions \( f_n \) such that

\[
\lim_{n \to \infty} \| f_n ( \omega ) - f ( \omega ) \|_{V'} = 0, \quad \text{and} \quad \| f_n ( \omega ) \| \leq 2 \| f ( \omega ) \|_{V'} \equiv C ( \omega ).
\]

Now, one of these simple functions is of the form

\[
\sum_{i=1}^{M} c_i \mathcal{X}_{E_i} ( \omega ),
\]

where \( c_i \in V' \). Therefore, there is no loss of generality in assuming that \( c_i ( t ) = \sum_{j=1}^{N} d_{ij} \mathcal{X}_{F_j} ( t ) \) where \( d_{ij} \in V' \). Hence, we can assume that each \( f_n \) is product measurable into \( B ( V' ) \times \mathcal{F} \). Then, by Theorem 3.3, there exists \( \hat{f} ( \cdot , \omega ) \in V' \) such that \( \hat{f} \) is product measurable and a subsequence \( f_{n(\omega)} \) converging weakly in \( V' \) to \( \hat{f} ( \cdot , \omega ) \) for each \( \omega \). Thus, \( f_{n(\omega)} ( \omega ) \to f ( \omega ) \) strongly in \( V' \) and \( f_{n(\omega)} ( \omega ) \to \hat{f} ( \omega ) \) weakly in \( V' \). Therefore, since the limit is unique, \( \hat{f} ( \omega ) = f ( \omega ) \) in \( V' \) and so it can be assumed that if \( f \) is measurable into \( V' \) then for each \( \omega \) it has a representative \( \hat{f} ( \omega ) \) such that \( ( t , \omega ) \to \hat{f} ( t , \omega ) \) is product measurable.

Next, we note that when \( f \) is product measurable into \( V' \) and each \( f ( \cdot , \omega ) \in V' \), it follow that \( f \) is measurable into \( V' \). Indeed, by measurability,

\[
f ( t , \omega ) = \lim_{n \to \infty} \sum_{i=1}^{m_n} c_i \mathcal{X}_{E_i^n} ( t , \omega ) = \lim_{n \to \infty} f_n ( t , \omega ),
\]

where \( E_i^n \) is product measurable and we can assume \( \| f_n ( t , \omega ) \|_{V'} \leq 2 \| f ( t , \omega ) \| \). Then, by product measurability, \( \omega \to f_n ( \cdot , \omega ) \) is measurable into \( V' \) because if \( g \in V \) then

\[
\omega \to \langle f_n ( \cdot , \omega ) , g \rangle
\]

is of the form

\[
\omega \to \sum_{i=1}^{m_n} \int_0^T \langle c_i \mathcal{X}_{E_i^n} ( t , \omega ) , g ( t ) \rangle dt = \sum_{i=1}^{m_n} \int_0^T \langle c_i , g ( t ) \rangle \mathcal{X}_{E_i^n} ( t , \omega ) dt,
\]

and this is \( \mathcal{F} \)-measurable since \( E_i^n \) is product measurable. Thus, it is measurable into \( V' \) as desired and

\[
\langle f ( \cdot , \omega ) , g \rangle = \lim_{n \to \infty} \langle f_n ( \cdot , \omega ) , g \rangle, \quad \omega \to \langle f_n ( \cdot , \omega ) , g \rangle \text{ is } \mathcal{F} \text{-measurable.}
\]

Finally, the Pettis theorem guarantees that \( \omega \to \langle f ( \cdot , \omega ) , g \rangle \) is measurable into \( V' \). The conclusion is the same for these two conditions if \( V' \) is replaced with \( V \).

The following theorem is a generalization of the familiar Gram-Schmidt process and is used below.

Theorem 3.5. Suppose \( V \) and \( W \) are separable Banach spaces, such that \( V \) is dense in \( W \) and \( B \in \mathcal{L} ( W , W' ) \) satisfies

\[
\langle Bx , x \rangle \geq 0, \quad \langle Bx , y \rangle = \langle By , x \rangle, \quad B \neq 0.
\]
Then, there exists a countable set \( \{ e_i \}_{i=1}^{\infty} \) of vectors in \( V \) such that 
\[
\langle Be_i, e_j \rangle = \delta_{ij},
\]
and for each \( x \in W \),
\[
Bx = \sum_{i=1}^{\infty} \langle Bx, e_i \rangle Be_i, \quad \text{and} \quad \langle Bx, x \rangle = \sum_{i=1}^{\infty} |\langle Bx, e_i \rangle|^2.
\]
The series converges in \( W' \). In the case \( B = B(\omega) \), where \( \omega \to B(\omega) \) is measurable into \( \mathcal{L}(W, W') \), these vectors \( e_i \) depend on \( \omega \) and are measurable functions of \( \omega \).

In particular, one could let \( \omega = t \) with the Lebesgue measurable sets.

The following result, found in [13], is well known, and is used below.

**Theorem 3.6.** If a single-valued map, \( A : X \to X' \) is monotone, hemicontinuous, and bounded, then \( A \) is pseudo-monotone. There exists a duality map \( J^{-1} : X' \to X \) that satisfies \( \langle J^{-1}f, f \rangle = ||f||^2, \|J^{-1}f\|_X = ||f||_X \), and is strictly monotone hemicontinuous and bounded. Furthermore, there exists a duality map \( F : X \to X' \) that satisfies \( \|Ff\|_X = ||f||^{-1}_{X'}, \langle Ff, f \rangle = ||f||^p_{X'} \) for \( p > 1 \).

The following fundamental result is used in what follows, and we note that it is somewhat more general than is needed. Let \( B \) be a possibly degenerate operator satisfying only:
\[ B \in \mathcal{L}(W, W'), \quad \langle Bu, u \rangle \geq 0, \quad \langle Bu, v \rangle = \langle Bv, u \rangle, \tag{11} \]
where \( V \subseteq W \) and \( V \) is dense in \( W \).

**Claim 1.** Let \( L : V \to V' \) be an operator whose domain is given by \( D(L) \equiv \{ u \in V : (Bu)' \in V' \} \) and which is defined by
\[ Lu \equiv (Bu)' . \]
Then, \( L \) is a closed operator so we can define a Banach space \( X \) to be equal \( D(L) \) with the norm
\[ ||u||_X \equiv ||u||_V + ||Lu||_{V'}, \]
or an equivalent norm
\[ ||u||_X \equiv \max(||u||_V, ||Lu||_{V'}). \]

To verify the claim, we first consider the part that \( L \) is closed. Suppose \( u_n \to u \) in \( V \) and \( (Bu_n)' \to \xi \) in \( V' \). We show that \( \xi = (Bu)' \). Let \( \phi \in C_c^{\infty}([0, T]) \) and \( v \in V \), then
\[
\int_0^T \langle \xi, \phi v \rangle = \lim_{n \to \infty} \int_0^T \langle (Bu_n)', \phi v \rangle = \lim_{n \to \infty} \int_0^T \langle Bu_n, \phi' v \rangle. \tag{12}
\]
We can take a subsequence, still indexed with \( n \), such that \( u_n(t) \to u(t) \) pointwise a.e. Also,
\[ \int_0^T |\langle Bu_n, \phi' v \rangle|^p \leq C (\phi', v) \int_0^T ||u_n||_V^p, \]
and the right-hand side is bounded independently of \( n \) because of the assumption that \( u_n \) is bounded in \( V \). Therefore, by the Vitali convergence theorem, and using the subsequence just described, we can pass to the limit in (12), thus,
\[
\left\langle \int_0^T \xi \phi dt, v \right\rangle = \left\langle -\int_0^T (Bu) \phi', v \right\rangle.
\]
Since \( v \) is arbitrary, this shows that \( \int_0^T \xi \phi = - \int_0^T (Bu) \phi' \) in \( \mathcal{V}' \), and so \( \xi = (Bu)' \). Hence, \( L \) is indeed closed and \( X \) is a Banach space. It is also a reflexive Banach space because it is isometric to a closed subspace of the reflexive Banach space \( \mathcal{V} \otimes \mathcal{V}' \).

The following is a generalized integration by parts formula used below.

**Proposition 1.** Let \( p \geq 2 \) if \( B \) is time dependent, and \( p > 1 \) otherwise. Let \( u, v \in X \), then the following hold:

1. \( t \mapsto \langle B(t)u(t), v(t) \rangle_{W', W} \) equals a.e. an absolutely continuous function, denoted by \( \langle Bu, v \rangle (\cdot) \);
2. \( \langle Lu(t), u(t) \rangle = \frac{1}{2} \left[ \langle Bu, u' \rangle(t) + \langle B' (t) u(t), u(t) \rangle \right] \) a.a.t, where \( Lu \equiv (Bu)' \);
3. \( \| Bu, v \| (t) \leq C \| u \|_{X} \| v \|_{X} \) for some \( C > 0 \) and all \( t \in [0, T] \);
4. \( t \mapsto B(t)u(t) \) equals a.e. a function in \( C(0, T; \mathcal{W}') \) denoted by \( B(\cdot) \);
5. \( \sup \{ \| Bu(t) \|_{\mathcal{W}'} , t \in [0, T] \} \leq C \| u \|_{X} \) for some \( C > 0 \).

Assume, next, that the operator \( K : X \to X' \) is given by

\[
\langle Ku, v \rangle_{X', X} = \int_0^T \langle Lu(t), v(t) \rangle dt + \langle Bu, v \rangle(0),
\]

then:

6. \( K \) is linear, continuous and weakly continuous;
7. \( \langle Ku, u \rangle = \frac{1}{2} \left[ \langle Bu, u \rangle(T) + \langle Bu, u \rangle(0) \right] + \frac{1}{2} \int_0^T \langle B' (t) u(t), u(t) \rangle dt \);
8. If \( Bu(0) = 0 \), for \( u \in X \), there exists \( u_n \to u \) in \( X \) such that \( u_n(t) = 0 \) near 0. A similar conclusion could be deduced at \( T \) if \( Bu(T) = 0 \).

The assumption \( p \geq 2 \) is necessary only because \( \int_0^T (B'(t)u(t))dt \) may not be well defined if \( p < 2 \). The last assertion about approximation makes possible the following corollary.

**Corollary 1.** If \( Bu(0) = 0 \) for \( u \in X \), then \( \langle Bu, u \rangle(0) = 0 \). The converse is also true. An analogous result holds when 0 is replaced with \( T \).

**Proof.** Let \( u_n \to u \) in \( X \) with \( u_n(t) = 0 \) for all \( t \) close enough to 0. For \( t \) off a set of measure zero consisting of the union of sets of measure zero corresponding to \( u_n \) and \( u \),

\[
\langle Bu_n, u_n \rangle(t) = \langle B(t) u_n(t), u_n(t) \rangle, \quad \langle Bu, u \rangle(t) = \langle B(t) u(t), u(t) \rangle,
\]

\[
\langle B(u - u_n), u \rangle(t) = \langle B(t) (u(t) - u_n(t)), u(t) \rangle,
\]

\[
\langle Bu_n, u - u_n \rangle(t) = \langle B(t) u_n(t), u(t) - u_n(t) \rangle.
\]

Then, considering such \( t \),

\[
\langle B(t) u(t), u(t) \rangle - \langle B(t) u_n(t), u_n(t) \rangle,
\]

\[
\langle B(t) (u(t) - u_n(t)), u(t) \rangle + \langle B(t) u_n(t), u(t) - u_n(t) \rangle.
\]

Hence, it follows from Proposition 1 that

\[
|\langle B(t) u(t), u(t) \rangle - \langle B(t) u_n(t), u_n(t) \rangle| \leq C \| u - u_n \|_{X} (\| u \|_{X} + \| u_n \|_{X}).
\]

Thus, if \( n \) is sufficiently large,

\[
|\langle B(t) u(t), u(t) \rangle - \langle B(t) u_n(t), u_n(t) \rangle| < \varepsilon.
\]

So let \( n \) be fixed and sufficiently large and let \( t_k \to 0 \), thus,

\[
\langle B(t_k) u_n(t_k), u_n(t_k) \rangle = 0,
\]
for $k$ large enough. Hence
\[ \langle Bu, u \rangle (0) = \lim_{k \to \infty} \langle B(t_k) u(t_k), u(t_k) \rangle < \varepsilon \]
Since $\varepsilon$ is arbitrary, $\langle Bu, u \rangle (0) = 0$.

Next suppose $\langle Bu, u \rangle (0) = 0$. Then, letting $v \in X$, a smooth function, we have
\[ \langle Bu(0), v(0) \rangle = \langle Bu, v \rangle (0) = \langle Bu, u \rangle^{1/2} (0) \langle Bu, v \rangle^{1/2} (0) = 0, \]
and it follows that $Bu(0) = 0$. \hfill \Box

The following is needed below.

**Theorem 3.7.** Let $Y$ denote the set of functions $f \in L^p([0, T]; V)$ for which $f' \in L^p([0, T]; V')$, so that $f$ has a representative such that $f(t) = f(0) + \int_0^t f'(s) \, ds$ a.a.t. Then, if we let $\|f\|_Y = \|f\|_{L^p([0, T]; V)} + \|f'\|_{L^p([0, T]; V')}$, the map $f \to f(t)$ is continuous in the sense that $\|f(t)\| \leq C(\|f\|_Y)$.

We turn to the following general existence results of measurable solutions to elliptic problems, obtained recently in [1]. In what follows, $X$ denotes a reflexive separable Banach space with dual $X'$, $(\Omega, \mathcal{F})$ is a measurable space, and $A(\cdot, \omega) : X \to \mathcal{P}(X')$, for $\omega \in \Omega$, denotes a set-valued operator. We make the following assumptions on $A$:

- **H$_1$ Measurability condition.** For each $u \in X$, there is a measurable selection $z(\omega)$ such that
  \[ z(\omega) \in A(u, \omega). \]
  
- **H$_2$ Values of $A$.** $A(\cdot, \omega) : X \to \mathcal{P}(X')$ has bounded, closed, nonempty, and convex values and it maps bounded sets to bounded sets.

- **H$_3$ Limit conditions, $A(\cdot, \omega)$ is pseudomonotone (cf. (4) and (5))**

  If $u_n \rightharpoonup u$ and $\limsup_{n \to \infty} \langle z_n, u_n - u \rangle \leq 0$, for $z_n \in A(u_n, \omega)$,

  then for each $v$, there exists $z(v) \in A(u, \omega)$ such that
  \[ \liminf_{k \to \infty} \langle z_n, u_n - v \rangle \geq \langle z(v), u - v \rangle. \]

We note that for a fixed $\omega$, the operator $A(\cdot, \omega)$ is set-valued, bounded and pseudomonotone as a map from $X$ to $\mathcal{P}(X)$. Moreover, the sum of two of such operators is set-valued, bounded and pseudomonotone (see the next Theorem). The limit condition $H_3$ implies that $A(\cdot, \omega)$ is upper-semicontinuous from the strong topology to the weak topology. This can be used to show that when $\omega \rightharpoonup u(\omega)$ is measurable, then $A(u(\omega), \omega)$ has a measurable selection assuming only that $\omega \rightharpoonup A(u, \omega)$ has a measurable selection for fixed $u \in X$. Next, we present a well-known result on the sum of pseudomonotone operators.

**Theorem 3.8.** Assume that $A$ and $B$ are set-valued, bounded and pseudomonotone operators. Then, their sum is also a set-valued, bounded and pseudomonotone operator. Moreover, if $u_n \rightharpoonup u$ weakly, $z_n \rightharpoonup z$, $z_n \in A(u_n)$, $w_n \rightharpoonup w$ weakly with $w_n \in A(u_n)$, and
\[ \limsup_{n \to \infty} \langle z_n + w_n, u_n - u \rangle \leq 0, \]
then,
\[ \liminf_{n \to \infty} \langle z_n + w_n, u_n - v \rangle \geq \langle z(v) + w(v), u - v \rangle, \]
for $z(v) \in A(u)$, $w(v) \in B(u)$ and in fact, $z \in A(u)$ and $w \in B(u)$. 
We now state our result on measurable solutions to general elliptic variational inequalities that may contain sums of set-valued, bounded and pseudomonotone operators. Then, the following was proved in [1].

**Theorem 3.9.** Let \( K(\omega) \subset V \) be a convex, closed and bounded set and assume that \( \omega \rightarrow K(\omega) \) is a measurable set-valued function. Let the operators \( A(\cdot, \cdot) \) and \( B(\cdot, \cdot) \) satisfy assumptions \( H_1 - H_3 \). Finally, let \( \omega \rightarrow f(\omega) \) be measurable with values in \( V' \).

Then, there exists a measurable function \( \omega \rightarrow u(\omega) \in K(\omega) \) such that \( \omega \rightarrow w^A(\omega) \), and \( \omega \rightarrow w^B(\omega) \) with \( w^A(\omega) \in A(u(\omega), \omega) \) and \( w^B(\omega) \in B(u(\omega), \omega) \), and
\[
\langle f(\omega) - (w^A(\omega) + w^B(\omega)), z - u(\omega) \rangle \leq 0,
\]
for all \( z \in K(\omega) \).

If \( K(\omega) \) is closed and convex but unbounded, the same conclusion holds true if for some \( z(\omega) \in K(\omega) \), \( A(\cdot, \omega) + B(\cdot, \omega) \) is coercive, that is
\[
\lim_{\|v\| \to \infty} \inf \left\{ \frac{\langle z^*, v - z \rangle}{\|v\|} : z^* \in (A(v, \omega) + B(v, \omega)) \right\} = \infty. \tag{13}
\]

We note that the result holds true for a finite sum of such operators.

4. **Measurable solutions to evolution inclusions.** The main result in this section, and one of the main results in this work, is Theorem 4.1, which guarantees the existence of solutions to a wide class of evolution inclusions, and extends the existence result in [12] to include measurable solutions.

We assume that \( A(\cdot, \omega) : V \rightarrow \mathcal{P}(V') \) satisfies the conditions in Section 2.1. Thus, we can regard \( A(\cdot, \omega) \) as a set-valued pseudomonotone map from \( X \) to \( \mathcal{P}(X') \). Here, \( X \) is defined as in Claim 1. It is clear that \( A(u, \omega) \) is a closed convex set in \( X' \). Indeed, if \( z^*_n \rightarrow z^* \) in \( X' \), \( z^*_n \in A(u, \omega) \), then \( z^* \in A(u, \omega) \), since a subsequence, still denoted as \( z^*_n \), converges weakly to some \( z^* \in A(u, \omega) \) in \( V' \) and since \( X \) is dense in \( V \), then \( z^* \in A(u, \omega) \). The necessary limit conditions for a pseudomonotone operator are just those assumed conditions in 2.1. Also, we assume in this section that \( p \geq 2 \), which is needed when \( B \) is time dependent. If \( B \) is time independent, this assumption is not necessary. We essentially show this in the following section in which we consider a more general coercivity condition than 2.1.

Consider now the operator \( K : X \rightarrow X' \) defined in Proposition 1, and note that if \( v \in X \) and \( Bv(0) = 0 \), then
\[
\langle Ku, v \rangle = \int_0^T \langle Lu, v \rangle \, ds. \tag{14}
\]

Indeed, by using the Cauchy-Schwarz inequality and the continuity of \( \langle Bu, u \rangle(\cdot) \), it follows that
\[
\langle Bu, v \rangle(0) \leq \langle Bu, u \rangle^{1/2}(0) \langle Bv, v \rangle^{1/2}(0),
\]
and if \( Bv(0) = 0 \), then from Corollary 1, \( \langle Bv, v \rangle^{1/2}(0) = 0 \). Proposition 1 implies that the operator \( K \) is hemicontinuous, bounded and monotone as a map from \( X \) to \( X' \). Thus, \( K + A(\cdot, \omega) \) is a set-valued pseudomonotone map so we can apply Theorem 3.9 and obtain the existence of measurable solutions to variational inequalities right away, but we want to obtain solutions to an evolution equation and the above theorem does not apply because the sum of these two operators is not coercive. Therefore, we consider another operator which, when added to the sum, guarantees coercivity. Let \( J : V \rightarrow V' \) be the duality map described in Theorem
3.6. Then, $| |J_{u}| |^{2}_{V} = | |u| |^{2}_{V}$ and $\langle Ju, u \rangle = | |u| |^{2}_{V}$. Hence, $J^{-1} : V' \rightarrow V$ also satisfies $\langle f, J^{-1} f \rangle = | |f| |^{2}_{V'}$.

The main result in this section is based on methods due to Brezis [4] and Lions [13] adapted to the case where the operator is set-valued, and the measurability of the solutions. We define the operator $M : X \rightarrow X'$ by

$$\langle Mu, v \rangle \equiv \langle Lv, J^{-1} Lu \rangle_{V', V} \text{ where } Lu = (Bu)' .$$

Let $f$ be measurable into $V'$ and let the function $g(\omega) \in X'$ be given by

$$\langle g(\omega), v \rangle \equiv \langle Bu(0), u_{0}(\omega) \rangle ,$$

where $u_{0}(\omega)$ is a given measurable function into $W$. We recall that $X \subseteq V'$. Consider now the approximate problem, the solution of which $u_{\varepsilon}$ depends on $\varepsilon$,

$$\varepsilon Mu_{\varepsilon}(\omega) + Ku_{\varepsilon}(\omega) + w^{*}_{\varepsilon}(\omega) = f(\omega) + g(\omega), \quad w^{*}_{\varepsilon}(\omega) \in A(u_{\varepsilon}(\omega), \omega) . \quad (15)$$

Now, for $u \in X$, we define the operator

$$A(u, \omega) = \varepsilon Mu + Ku + A(u, \omega) .$$

Then, it follows from the assumptions on $A(\cdot, \omega)$, that there is $u^{*}(\omega)$ for which $\omega \rightarrow u^{*}(\omega)$ is measurable into $V'$, hence measurable into $X'$. Therefore, $\omega \rightarrow A(u, \omega)$ has a measurable selection, namely $\varepsilon Mu + Ku + u^{*}(\omega)$ and so condition $H_{1}$ is verified.

By Theorem 3.9, a solution to (15) exists with both $u_{\varepsilon}$ and $w^{*}_{\varepsilon}$ measurable if we can show that the sum $\varepsilon M + K + A(\cdot, \omega)$ is coercive, since this is the sum of pseudomonotone operators. From 2.1

$$\inf \left( \int_{0}^{T} \langle u^{*}, u \rangle ds : u^{*} \in A(u, \omega) \right) + \frac{1}{2} \int_{0}^{T} \langle B' u, u \rangle \geq \delta(\omega) \int_{0}^{T} | |u| |^{2}_{V} ds - m(\omega) ,$$

and so routine considerations show that $\varepsilon M + K + A(\cdot, \omega)$ does indeed satisfy a suitable coercivity estimate for each positive $\varepsilon$. Thus, we have the following theorem for the approximate solutions.

**Proposition 2.** Let $f$ be measurable into $V'$ and let $A$ satisfy the conditions $H_{1} - H_{3}$. Then, for $K$ and $M$ defined above, there exist measurable functions $u_{\varepsilon}$ and $w^{*}_{\varepsilon}$ satisfying the inclusion (15).

Note that this implies, suppressing dependence on $\omega$, that

$$\langle Bu_{\varepsilon}, v \rangle (0) = \langle Bu(0), u_{0} \rangle ,$$

for all $v \in X$. Thus, letting $v$ be a smooth function with values in $V$ we find

$$\langle Bu_{\varepsilon}(0), v(0) \rangle = \langle Bu_{0}, v(0) \rangle .$$

Since $V$ is dense in $W$, this implies $Bu_{\varepsilon}(0) = Bu_{0}$.

We now let $\Lambda$ be the restriction of $L$ to those $u \in X$ which have $Bu(0) = 0$. Thus, by Corollary 1,

$$D(\Lambda) = \{ u \in X : Bu(0) = 0 \} = \{ u \in X : \langle Bu, u \rangle (0) = 0 \} ,$$

and if $v \in D(\Lambda), u \in X$, then as noted earlier,

$$\langle Ku, v \rangle = \int_{0}^{T} \langle Lu, v \rangle ds .$$
Next, we obtain an estimate for $\Lambda^*$. To that end, we define

$$D(T) \equiv \{ u \in V : u' \in V, u(T) = 0 \},$$

and let $\hat{T}u = -Bu'$. Then,

$$\left\langle \hat{T}u, u \right\rangle = -\int_0^T \langle Bu', u \rangle = -\langle Bu, u \rangle |^T_0 + \int_0^T \langle (Bu)', u \rangle = \langle Bu, u \rangle (0) + \int_0^T \langle B'u, u \rangle + \int_0^T \langle Bu', u \rangle,$$

and so we obtain

$$\left\langle \hat{T}u, u \right\rangle \geq \frac{1}{2} \int_0^T \langle B'u, u \rangle.$$  \hspace{1cm} (16)

One can show that $\hat{T}^* = \Lambda$ and that the graph of $\Lambda^*$ is the closure of the graph of $\hat{T}$, thus showing that $\Lambda^*$ also satisfies an inequality similar to (16) for $u \in D(\Lambda^*)$.

Next, it follows from (15) that

$$\varepsilon \langle Lu, J^{-1}Lu_{\varepsilon} \rangle_{V', V} + \langle Ku_{\varepsilon} (\omega), v \rangle_{V', X} + \langle w^\varepsilon (\omega), v \rangle_{V', V} = \langle f (\omega), v \rangle_{V', V}.$$  \hspace{1cm} (17)

If we restrict to $v \in D(\Lambda)$, so that $Bu(0) = 0$, we obtain

$$\varepsilon \langle Lu, J^{-1}Lu_{\varepsilon} \rangle_{V', V} + \langle Lu_{\varepsilon} (\omega), v \rangle_{V', V} + \langle w^\varepsilon (\omega), v \rangle_{V}, V = \langle f (\omega), v \rangle_{V', V},$$

and so $J^{-1}Lu_{\varepsilon} \in D(\Lambda^*)$. Thus, since $D(\Lambda)$ is dense in $V$, it follows that

$$\varepsilon \Lambda^* \langle J^{-1}Lu_{\varepsilon} + Lu_{\varepsilon} + w^\varepsilon = f \rangle \in V'.$$

Acting with $J^{-1}Lu_{\varepsilon}$ on both sides yields, for some constant $C$ that depends on $B'$, the following inequality,

$$-\varepsilon C \|Lu_{\varepsilon}\|^2 + \|Lu_{\varepsilon}\|^2 + \langle w^\varepsilon, J^{-1}Lu_{\varepsilon} \rangle \leq \langle f, J^{-1}Lu_{\varepsilon} \rangle.$$  \hspace{1cm} (18)

Also, acting with $u_{\varepsilon}$ on both sides of (15) and using the formula for $\langle Ku, u \rangle$, item 7 in Proposition 1, we have,

$$\langle Ku, u \rangle = \frac{1}{2} \|(Bu, u)(T) + \langle Bu, u \rangle (0)\| + \frac{1}{2} \int_0^T \langle B'(t) u(t), u(t) \rangle dt$$

and this implies

$$\varepsilon \langle Lu_{\varepsilon}, J^{-1}Lu_{\varepsilon} \rangle + \frac{1}{2} \langle Bu_{\varepsilon}, u_{\varepsilon} \rangle (T) + \langle Bu_{\varepsilon}, u_{\varepsilon} \rangle (0)$$

$$+ \frac{1}{2} \int_0^T \langle B'(t) u_{\varepsilon}(t), u_{\varepsilon}(t) \rangle dt + \langle w^\varepsilon, u_{\varepsilon} \rangle_{V', V} = \langle f, u_{\varepsilon} \rangle + \langle Bu_{\varepsilon} (0), u_0 (\omega) \rangle$$

$$= \langle f, u_{\varepsilon} \rangle + \langle Bu_0 (\omega), u_0 (\omega) \rangle.$$  \hspace{1cm} (19)

It follows from the coercivity condition (7) that $u_{\varepsilon}$ is bounded in $V$ and consequently $w^\varepsilon$ is bounded in $V'$, by the growth estimate (18). Now, it also follows from this bound that $\|Lu_{\varepsilon}\|_{V'}$ is bounded for sufficiently small $\varepsilon$. Therefore,

$$\|Lu_{\varepsilon}(\omega)\|_{V'} + \|u_{\varepsilon}(\omega)\|_{V'} + \|w^\varepsilon (\omega)\|_{V'} \leq C (\omega) < \infty,$$

where $C (\omega)$ is independent of $\varepsilon$ (sufficiently small). By Theorem 3.2, there is a subsequence $\varepsilon (\omega) \to 0$ such that

$$\left( Lu_{\varepsilon(\omega)}(\omega), u_{\varepsilon(\omega)} (\omega), w^\varepsilon(\omega) (\omega), Bu_{\varepsilon(\omega)} (\omega) (0) \right)$$

$$\to (Lu(\omega), u(\omega), \xi (\omega), Bu (\omega) (0)).$$  \hspace{1cm} (19)
weakly in $\mathcal{V}' \times \mathcal{V} \times \mathcal{V}'$ and $\omega \to (Lu(\omega), u(\omega), \xi(\omega))$ is measurable into $\mathcal{V}' \times \mathcal{V} \times \mathcal{V}'$. It follows that $Bu(\omega)(0) = Bu_0(\omega)$ because each $Bu_\varepsilon(\omega)(0) = Bu_0(\omega)$. Note that this also shows that $Ku_\varepsilon \to Ku$ in $X'$. Thus, suppressing the dependence on $\omega$, let (17) act on $u_\varepsilon - u$ and obtain,

$$
\varepsilon \langle Lu_\varepsilon - Lu, J^{-1}Lu_\varepsilon \rangle + (Ku_\varepsilon, u_\varepsilon - u) + (w_\varepsilon^*, u_\varepsilon - u) = \langle f, u_\varepsilon - u \rangle + \langle g, u_\varepsilon - u \rangle.
$$

Using the monotonicity of $J^{-1}$, yields

$$
\varepsilon \langle Lu_\varepsilon - Lu, J^{-1}Lu \rangle + (Ku_\varepsilon, u_\varepsilon - u) + (w_\varepsilon^*, u_\varepsilon - u) \leq \langle f, u_\varepsilon - u \rangle + \langle g, u_\varepsilon - u \rangle.
$$

Now, $(Bu_\varepsilon - Bu)(0) = 0$. Therefore, $u_\varepsilon - u \in D(A)$ and so

$$
\varepsilon \langle A^*J^{-1}Lu, u_\varepsilon - u \rangle + (Ku_\varepsilon, u_\varepsilon - u) + (w_\varepsilon^*, u_\varepsilon - u) \leq \langle f, u_\varepsilon - u \rangle + \langle g, u_\varepsilon - u \rangle.
$$

We recall that $K$ is monotone, bounded and hemicontinuous, actually, it is linear and hence $K + A$ is pseudomonotone. It follows that

$$
\limsup_{\varepsilon \to 0} (Ku_\varepsilon + w_\varepsilon^*, u_\varepsilon - u) \leq 0.
$$

Now, the weak convergences in (19) include the weak convergence of $u_\varepsilon$ to $u$ in $X$. Thus, since $K + A(\cdot, \omega)$ is pseudomonotone as a map from $X$ to $\mathcal{P}(X')$, for every $v \in X$, there exists $w^*(v) \in Ku + A(u, \omega)$ such that

$$
\liminf_{\varepsilon \to 0} \langle Ku_\varepsilon + w_\varepsilon^*, u_\varepsilon - v \rangle \geq \langle w^*(v), u - v \rangle.
$$

In particular, this holds if $v = u$, which shows that

$$
\lim_{\varepsilon \to 0} (Ku_\varepsilon + w_\varepsilon^*, u_\varepsilon - u) = 0.
$$

It follows then that for $v \in X$,

$$
\langle \xi + Ku, u - v \rangle = \lim_{\varepsilon \to 0} \langle w_\varepsilon^* + Ku_\varepsilon, u - v \rangle = \lim_{\varepsilon \to 0} \langle w_\varepsilon^* + Ku_\varepsilon, u_\varepsilon - u \rangle + \langle w_\varepsilon^* + Ku_\varepsilon, u_\varepsilon - v \rangle
$$

$$
\geq \liminf_{\varepsilon \to 0} \langle w_\varepsilon^* + Ku_\varepsilon, u_\varepsilon - v \rangle \geq \langle w^*(v), u - v \rangle.
$$

Since $v$ is arbitrary, separation theorems imply that

$$
\langle \xi(\omega) + Ku(\omega), \omega \rangle = w^*(\omega) + Ku(\omega) \in A(u(\omega), \omega) + Ku(\omega).
$$

Then, passing to the limit $\varepsilon \to 0$ in (15), we obtain

$$
Ku(\omega) + w^*(\omega) = f(\omega) + g(\omega) \text{ in } \mathcal{V}', \quad w^*(\omega) \in A(u(\omega), \omega),
$$

$$
Bu(\omega)(0) = Bu_0(\omega),
$$

and $Lu, w^*$ and $u$ are all measurable into the appropriate spaces. This implies that for each $v \in X$,

$$
\int_0^T \langle Lu, v \rangle + \langle Bu, v \rangle(0) + \int_0^T \langle w^*, v \rangle = \int_0^T \langle f, v \rangle + \langle Bu(0), u_0 \rangle.
$$

In particular, letting $v = u$,

$$
\int_0^T \langle Lu, u \rangle + \langle Bu, u \rangle(0) + \int_0^T \langle w^*, u \rangle = \int_0^T \langle f, v \rangle + \langle Bu(0), u_0 \rangle
$$

$$
= \int_0^T \langle f, v \rangle + \langle Bu_0, u_0 \rangle.
$$

Thanks to (20) and Theorem 1 we find that $\langle Bu, u \rangle(0) = \langle Bu_0, u_0 \rangle$. This has proved the following theorem.
Theorem 4.1. Let \( p \geq 2 \), let \( A \) satisfy (6) - (9), let \( f \) be a measurable function into \( \mathcal{V}' \) and let \( u_0 \) be measurable into \( W \). Then, there exists a solution to the inclusion (20) such that \( Lu, w^* \) and \( u \) are all measurable. The solution \( u \) is such that for a fixed \( \omega, \langle Bu_0, u_0 \rangle = \langle Bu, u \rangle(0) \).

We also have the following corollary that establishes the existence of measurable solutions to periodic problems. Since uniqueness for periodic problems may not be true, our theory extends the results above to such problems. In the following corollary, we assume, for the sake of simplicity, that \( B(t) = B \) is a constant and thus, it is not necessary to assume \( p \geq 2 \).

Corollary 2. Let \( A \) satisfy (6) - (9), \( p > 1 \), \( T > 0 \) and let \( f \) be measurable into \( \mathcal{V}' \). Then, there exists a solution to the inclusion

\[
Lu(\omega) + w^*(\omega) = f(\omega), \quad Bu(0, \omega) = Bu(T, \omega),
\]

such that \( Lu, w^* \) and \( u \) are all measurable.

Proof. Define \( \Lambda \) as the restriction of \( L \) to the space \( \{u \in D(L) : Bu(0) = Bu(T)\} \) of periodic functions. We first consider an operator \( \hat{T} \) and show that \( \hat{T}^* = \Lambda \). We argue that \( \hat{T} \) is monotone and then it follows that \( \Lambda^* \) is, too. Indeed,

\[
D(\hat{T}) \equiv \{v \in \mathcal{V} : v' \in \mathcal{V} \text{ and } v(T) = v(0)\}, \quad \hat{T}v = -Bu',
\]

We next consider \( \hat{T}^* \). If \( u \in D(\Lambda), v \in D(T) \),

\[
-\int_0^T \langle Bu', u \rangle = -\langle Bu,v \rangle |T| + \int_0^T \langle (Bu)', v \rangle,
\]

and so, since the boundary term vanishes, this shows that \( D(\Lambda) \subseteq D(\hat{T}^*) \) and that \( \hat{T}^* = \Lambda \) on \( D(\Lambda) \). Let \( u \in D(\hat{T}^*) \) then, by definition,

\[
\left| \langle \hat{T}v, u \rangle \right| \leq C_u \|v\|_{\mathcal{V}}. \tag{21}
\]

Choosing \( v \in C_c^\infty([0, T]; \mathcal{V}) \), we find

\[
\langle \hat{T}v, u \rangle = -\int_0^T \langle Bu', u \rangle = -\int_0^T \langle Bu, v' \rangle,
\]

and it follows from the Riesz representation theorem that there exists a unique \( (Bu)' \) such that the above equals \( \int_0^T \langle (Bu)', v \rangle \) and by the density of \( C_c^\infty([0, T]; \mathcal{V}) \) this shows that \( \hat{T}^* u = (Bu)' = Lu \). Thus, \( \hat{T}^* = L \) on \( D(\hat{T}^*) \) and, in particular, \( (Bu) \in \mathcal{V}' \). It remains to consider the boundary conditions. For \( u \in D(\hat{T}^*) \) and \( v \in D(\hat{T}) \),

\[
\langle \hat{T}v, u \rangle = -\int_0^T \langle Bu', u \rangle = -\langle Bu,v \rangle |T| + \int_0^T \langle (Bu)', v \rangle.
\]

The boundary term has the form \( \langle Bu(0) - Bu(T), v(0) \rangle \), and if (21) is to hold for all \( v \in D(\hat{T}) \) we must have \( Bu(0) = Bu(T) \). Indeed, if \( Bu(0) - Bu(T) = \xi \neq 0 \), then

\[
|\langle \xi, v(0) \rangle| \leq C_u \|v\|_{\mathcal{V}},
\]
for all \( v \in D(\hat{T}) \). We now choose \( v \in D(\hat{T}) \) such that \( |\langle \xi, v(0) \rangle| = \delta > 0 \) and consider a piecewise linear function \( \psi_n \) that is one at 0 and \( T \) but zero on \([1/n, T - (1/n)]\). Then, if \( v_n = \psi_n v \), the left-hand side is \( \delta \) for all \( n \) but the right-hand side converges to 0, a contradiction that shows that \( D(\hat{T}^*) = D(\Lambda) \) and \( \hat{T}^* = \Lambda \). Now, it follows that \( \hat{T}^{**} = \Lambda^* \) and so \( \Lambda^* \) is monotone because \( \hat{T} \) is and the graph of \( \Lambda^* \) is the closure of the graph of \( \hat{T} \). Indeed,

\[
\int_0^T \langle -Bv', v \rangle = \int_0^T \langle Bu', v \rangle,
\]

so \( \langle \hat{T}v, v \rangle = 0 \), and the same holds true of \( \Lambda^* \).

Now, we let \( X \) be as above and consider the approximate problem for \( u_\varepsilon \) given by

\[
\varepsilon \langle Lv, J^{-1}(Lu_\varepsilon) + \langle Lu_\varepsilon(\omega), v \rangle \rangle_{V', V} + \frac{1}{2} \langle Bu, v \rangle(0) - \frac{1}{2} \langle Bu, v \rangle(T) + \langle w^*_\varepsilon(\omega), v \rangle = \langle f(\omega), v \rangle, \quad w^*_\varepsilon(\omega) \in A(u_\varepsilon(\omega), \omega).
\]

Then, using the monotonicity of \( \Lambda^* \) and \( \Lambda \), we obtain the existence of a measurable solution. Concerning the monotonicity, we note that

\[
\langle Lu, u \rangle_{V', V} + \frac{1}{2} \langle Bu, v \rangle(0) - \frac{1}{2} \langle Bu, v \rangle(T) = 0,
\]

which follows from items 5 and 7 in Proposition 1. Indeed, these imply that

\[
\langle Lu, u \rangle_{V', V} + \langle Bu, u \rangle(0) = \frac{1}{2} \left[ \langle Bu, u \rangle(T) + \langle Bu, u \rangle(0) \right],
\]

and the monotonicity follows. The rest of the argument is similar to that used to prove Theorem 4.1. Finally, we obtain that

\[
\frac{1}{2} \langle Bu, v \rangle(0) - \frac{1}{2} \langle Bu, v \rangle(T) = 0,
\]

and hence \( Bu(T) = Bu(0) \), since \( v \) is arbitrary.

We conclude that the problem has periodic solutions.

5. Relaxed coercivity condition. This section is devoted to proving Theorem 5.1, which includes a more general coercivity condition and uses a slightly modified limit condition. We also remove the restriction that \( p \geq 2 \), so we require that \( p > 1 \) and \( B \) is time-independent. We also relax the growth condition.

Let \( U \) be a separable reflexive Banach space that is dense in \( V \), with compact embedding. Actually, \( U \) can be assumed a Hilbert space and in the applications of most interest to us, it is usually obtained by the Sobolev embedding theorems. We let \( r = \max(2, p) \) and \( U_\varepsilon = L^r([0, T]; U) \). Also, for \( I = [0, \hat{T}] \), \( \hat{T} < T \), we denote by \( V_I \) the space \( L^p(I; V) \) with similar notation in other spaces. If \( u \in V = L^p([0, T]; V) \), then we always consider \( u \in V_I \) by simply restricting it to \( I \). With this convention, it is clear that if \( u \) is measurable into \( V \), then it is also measurable into \( V_I \).

The modified conditions on the operator \( A: V_I \rightarrow P(V_I') \) are: \( A(u, \omega) \) is a convex and closed set in \( V_I' \), whenever \( u \in V_I \) and
• $H_1$ Growth estimate For $u \in \mathcal{V}_I$, 

$$\sup \left\{ \|u^*\|_{\mathcal{V}_I^{'}} : u^* \in A(u, \omega) \right\} \leq a(\omega) + b(\omega) \|u\|_{\mathcal{V}_I}^{\hat{p} - 1},$$

where $a(\omega)$ and $b(\omega)$ are nonnegative numbers which need not be uniformly bounded in $\omega$. Here $\hat{p} \geq p$.

• $H_2$ Coercivity estimate The modified coercivity condition is valid for each $t \leq T$ and for some $\lambda(\omega) \geq 0$,

$$\inf \left( \int_0^t \langle u^*, u \rangle + \lambda(\omega) \langle Bu, u \rangle \, ds : u^* \in A(u, \omega) \right) \geq \delta(\omega) \int_0^t \|u\|^p_{\mathcal{V}_I} \, ds - m(\omega),$$

where $m(\omega)$ is a nonnegative constant for fixed $\omega$, and $\delta(\omega) > 0$. No uniformity in $\omega$ is assumed for the $\delta(\omega)$ and neither a strictly positive lower bound that is independent of $\omega$.

• $H_3$ Limit conditions Let $U$ be a Banach space dense and compact in $V$. When $u_i \to u$ in $\mathcal{V}_I$ and $u_i^* \in A(u_i, \omega)$ with $(Bu_i') \to (Bu)'$ weakly in $U'_I$, and if

$$\limsup_{i \to \infty} \langle u_i^*, u_i - u \rangle_{\mathcal{V}_I', \mathcal{V}_I} \leq 0,$$

then, for each $v \in \mathcal{V}_I$, there exists $u^*(v) \in Au$ such that

$$\liminf_{i \to \infty} \langle u_i^*, u_i - v \rangle_{\mathcal{V}_I', \mathcal{V}_I} \geq \langle u^*(v), u - v \rangle_{\mathcal{V}_I', \mathcal{V}_I}.$$ 

We note that, typically, one obtains this from Theorem 3.1 when applied to lower order terms together with the compactness of the embedding of $V$ into $W$.

• $H_4$ Measurability condition If $\omega \to u(\cdot, \omega)$ is measurable into $V$, then

$$\omega \to A(X_Iu(\cdot, \omega), \omega)$$

has a measurable selection into $\mathcal{V}_I'$. 

This condition means that there is a function $\omega \to u^*(\omega)$ that is measurable into $\mathcal{V}_I'$ such that $u^*(\omega) \in A(X_Iu(\cdot, \omega), \omega)$.

This is assured when the following standard measurability condition is satisfied for all open sets $O$ in $\mathcal{V}_I'$:

$$\{ \omega : A(X_Iu(\cdot, \omega), \omega) \cap O \neq \emptyset \} \in \mathcal{F}.$$ 

A sufficient condition for this is that $\omega \to A(u(\cdot, \omega), \omega)$ has a measurable selection into $\mathcal{V}'$ for any $\omega \to u(\cdot, \omega)$ measurable into $\mathcal{V}$ and if $u^* \in A(u(\cdot, \omega), \omega)$, then $X_Iu^* \in A(X_Iu(\cdot, \omega), \omega)$, and this is typical of what we consider, in which the values of $u^*$ are dependent on the earlier values of $u$ only.

Let $F$ be the duality map for $r, r = \max(\hat{p}, 2)$ satisfying

$$\langle Fu, u \rangle = \|u\|^r, \quad \|Fu\| = \|u\|^{r-1},$$

and is demicontinuous. Let $X$ be the space of all $u \in \mathcal{U}_e$ such that $(Bu)' \in \mathcal{U}_e'$ with the norm given by $\max(\|u\|_{\mathcal{U}_e}, \|Bu\|_{\mathcal{U}_e'})$. Then, if we let $\mathcal{U}_eI$ play the role of $\mathcal{V}_I$ in Theorem 4.1, we obtain the following result, in which we study a family of approximate problems, as a corollary of that theorem.
Corollary 3. Let $A$ satisfy (22)–(26), let $f$ be measurable into $\mathcal{V}'$, and let $u_0$ be measurable into $W$. Then, for each $\varepsilon > 0$, there exists a solution $u_\varepsilon$ to
\[ Lu_\varepsilon + \varepsilon F u_\varepsilon + u_\varepsilon^* = f, \quad Bu_\varepsilon (0, \omega) = Bu_0 (\omega), \tag{28} \]
such that $Lu_\varepsilon, u_\varepsilon^*$ and $u_\varepsilon$ are measurable into $\mathcal{U}', \mathcal{U}_e$, and $\mathcal{U}_e$, respectively, and $u_\varepsilon^*(\omega) \in A(u_\varepsilon, \omega)$. In other words, for $v \in X = \{ u \in \mathcal{U} : Lu \in \mathcal{U}' \}$, we have
\[ \int_0^T \langle Lu_\varepsilon, v \rangle + \varepsilon \int_0^T \langle Fu_\varepsilon, v \rangle + \int_0^T \langle u_\varepsilon^*, v \rangle \\ + \langle Bu_\varepsilon, v \rangle (0) = \int_0^T \langle f, v \rangle + \langle Bu (0), u_0 \rangle. \tag{29} \]

Proof. Using the usual estimates and the fact that $r > \max (\wp, 2)$, it is straightforward to show that the previous coercivity condition (13) holds for $\varepsilon F + A (\cdot, \omega)$. Indeed, it follows from the assumptions above that
\[ \inf \left( \int_0^t \langle u^*, u \rangle + \lambda (\omega) \langle Bu, u \rangle \ ds : u^* \in A(u, \omega) \right) \geq \delta (\omega) \int_0^t \| u \|_V^p \ ds - m (\omega). \]

Thus,
\[ \inf \left( \int_0^t \langle u^*, u \rangle : u^* \in A(u, \omega) \right) \geq \delta (\omega) \int_0^t \| u \|_V^p \ ds - m (\omega). \]

Then, the last two terms on the right-hand side satisfy
\[ -C_B \lambda (\omega) \int_0^t \frac{1}{\eta} \| u \|_V^2 - m (\omega). \]
Choosing now $\eta$ sufficiently small, we obtain that $C_B \lambda (\omega) \eta^r/2 < \varepsilon/2$.

Since both operators $\varepsilon F$ and $A (\cdot, \omega)$ are pseudomonotone as maps from $X$ to $\mathcal{P} (X')$, where $X$ is defined in terms of $\mathcal{U}_e$, as before, the existence of a measurable solution is obtained for each $\varepsilon > 0$. \hfill $\Box$

Denoting by $u_e$ the solution, we suppose that $Lu_e \to Lu$ weakly in $\mathcal{U}'$, with $Lu = (Bu)' \in \mathcal{V}'$ and $u_e \to u$ weakly in $\mathcal{V}$ and $u_e^* \to u^*$ in $\mathcal{V}'$, and $\varepsilon Fu_e \to 0$ strongly in $\mathcal{U}'$. By passing to the limit $\varepsilon \to 0$ in (28), we obtain that $Lu \in \mathcal{V}'$ because it lies in $\mathcal{V}'$. We next show that, indeed, $\varepsilon Fu_e \to 0$ strongly in $\mathcal{U}'$. It follows from (29) that the functions $u_e$ satisfy,
\[ \int_0^T \langle Lu_e, v \rangle + \langle Bu_e, v \rangle (0) + \int_0^T \langle u_e^*, v \rangle + \varepsilon \int_0^T \langle Fu_e, u_e \rangle = \int_0^T \langle f, v \rangle + \langle Bu (0), u_0 \rangle, \tag{30} \]
for all $v \in X$, and
\[ Bu_e (t) = Bu_0 + \int_0^t Lu_e (s) \ ds. \]
The weak convergence of $Lu_e$ implies that $Bu_e (t) \to Bu (t)$ in $\mathcal{U}'$. Thus,
\[ Bu (t) = Bu_0 + \int_0^t Lu (s) \ ds, \]
and so \( Bu(0) = Bu_0 \). We next show that there exist subsequences for which this convergence holds.

Using the equation to act on \( u \) in (28) or in (30), it follows from the assumed coercivity condition that for each fixed \( \omega \) we have the following inequality,

\[
\frac{1}{2} \left\langle Bu, u \right\rangle (t) - \frac{1}{2} \left\langle Bu, u \right\rangle (0) + \varepsilon \int_0^t \|u\|_{U}^p \, ds + \delta(\omega) \int_0^t \|u\|_{V}^p \, ds - m(\omega) \\
\leq \lambda(\omega) \int_0^t \left\langle Bu, u \right\rangle (s) \, ds + \int_0^t \left\langle f, u \right\rangle (s) \, ds.
\]

Then, Gronwall’s inequality yields

\[
\left\langle Bu, u \right\rangle (t) + \varepsilon \int_0^T \|u\|_{U}^p \, ds + \int_0^T \|u\|_{V}^p \, ds \leq C(f, \omega),
\]

where the constant \( C \) depends only on the indicated data. This estimate and the definition of the duality map \( F \) show that if \( u_\varepsilon \) is the solution guaranteed in Corollary 3, then \( \varepsilon Fu_\varepsilon \to 0 \) strongly in \( U_\varepsilon \). Also, the estimates for \( A \) and the above estimate imply that \( Lu_\varepsilon \) is bounded in \( U_\varepsilon \). Thus, we have

\[
\left\langle Bu_\varepsilon, u_\varepsilon \right\rangle (t) + \varepsilon \int_0^T \|u_\varepsilon\|_{U}^p \, ds + \|Lu_\varepsilon\|_{U_\varepsilon} + \|u_\varepsilon^*\|_{V'} \leq C(f, \omega).
\]

The previous discussion also shows that each one of \( u_\varepsilon \) and \( u_\varepsilon^* \) is measurable into \( V \) and \( U_\varepsilon \), respectively. By density considerations, \( u_\varepsilon^* \) is also measurable into \( V' \). It follows from Theorem 3.2 that there exists a pair \((u, u^*)\), which is measurable into \( V \times V' \), and a sequence with \( \varepsilon(\omega) \to 0 \), \( (u_\varepsilon(\omega) \omega, u_\varepsilon^*(\omega)) \to (u(\omega), u^*(\omega)) \) in \( V \times V' \). Then, passing to a further subsequence, we obtain for fixed \( \omega \), the following convergences:

\[
\begin{align*}
u_\varepsilon(\omega) & \to u(\omega) \text{ weakly in } V, \\
u_\varepsilon^*(\omega) & \to u^*(\omega) \text{ weakly in } V', \\
Lu_\varepsilon(\omega) & \to Lu \text{ weakly in } U_\varepsilon,
\end{align*}
\]

These convergences continue to hold true when \( V \) and \( U_\varepsilon \) are replaced with \( V_I \) and \( U_\varepsilon^{I_I} \), since we may simply consider the restrictions of the functions to \( I \). However, we do not know that \( u \) is in \( U_\varepsilon \). So we take a different approach.

Let \( \sigma > 0 \), then there exists \( \hat{T}(\omega) > T - \sigma \) such that for each \( \varepsilon(\omega) \) in the sequence,

\[
\left\langle Bu_\varepsilon(\omega), u_\varepsilon(\omega) \right\rangle (\hat{T}) = \left\langle B \left(u_\varepsilon(\omega)(\hat{T})\right), u_\varepsilon(\omega)(\hat{T}) \right\rangle, \quad Bu_\varepsilon(\hat{T}) = B \left(u_\varepsilon(\hat{T})\right),
\]

for all \( \varepsilon(\omega) \) in the sequence converging to 0 and also

\[
Bu(\hat{T}) = B \left(u(\hat{T})\right), \quad \left\langle Bu, u \right\rangle (\hat{T}) = \left\langle B \left(u(\hat{T})\right), u(\hat{T}) \right\rangle.
\]

Now, let \( \{\varepsilon_i\} \) be the vectors in \( U \) constructed in Theorem 3.5. Then, at \( \hat{T} \),

\[
\left\langle Bu_\varepsilon, u_\varepsilon \right\rangle (\hat{T}) = \left\langle Bu_\varepsilon(\hat{T}), u_\varepsilon(\hat{T}) \right\rangle = \sum_{i=1}^{\infty} \left\langle B \left(u_\varepsilon(\hat{T})\right), \varepsilon_i \right\rangle^2.
\]
It follows from Fatou’s lemma that
\[
\liminf_{\varepsilon \to 0} \langle Bu_\varepsilon, u_\varepsilon \rangle (T) = \liminf_{\varepsilon \to 0} \sum_{i=1}^{\infty} \langle B \left( u_\varepsilon(T) \right), e_i \rangle^2 \\
\geq \sum_{i=1}^{\infty} \liminf_{\varepsilon \to 0} \langle B \left( u_\varepsilon(T) \right), e_i \rangle^2 \\
= \sum_{i=1}^{\infty} \liminf_{\varepsilon \to 0} \langle Bu_\varepsilon(T), e_i \rangle^2 \\
= \sum_{i=1}^{\infty} \langle Bu(T), e_i \rangle^2 \\
= \langle B \left( u(T) \right), u (\hat{T}) \rangle = \langle Bu, u \rangle (\hat{T}).
\]

(32)

Then, by (30), we obtain,
\[
\frac{1}{2} \langle Bu_\varepsilon, u_\varepsilon \rangle (\hat{T}) - \frac{1}{2} \langle Bu_\varepsilon, u_\varepsilon \rangle (0) + \int_{0}^{\hat{T}} \varepsilon \langle Fu_\varepsilon, u_\varepsilon \rangle \, dt \\
+ \int_{0}^{\hat{T}} \langle u_\varepsilon^*, u_\varepsilon \rangle = \int_{0}^{\hat{T}} \langle f, u_\varepsilon \rangle.
\]

(33)

Therefore, \( \langle Bu_\varepsilon, u_\varepsilon \rangle (0) = \langle Bu_0, u_0 \rangle \). Now passing to the limit as \( \varepsilon \to 0 \), we find
\( Lu + u^* = f \) in \( U'_r \).

Since \( u^*, f \in V' \), then \( Lu \in V' \). Also, it follows from Theorem 1 that \( \langle Bu_\varepsilon, u_\varepsilon \rangle (0) = \langle Bu_0, u_0 \rangle \) and, therefore, \( \langle Bu, u \rangle (0) = \langle Bu_0, u_0 \rangle \), too. Then, using integration by parts yields
\[
\frac{1}{2} \langle Bu, u \rangle (\hat{T}) - \frac{1}{2} \langle Bu_0, u_0 \rangle + \int_{0}^{\hat{T}} \langle u^*, u \rangle = \int_{0}^{\hat{T}} \langle f, u \rangle \, dt,
\]
which shows that
\[
\int_{0}^{\hat{T}} \langle u^*, u \rangle \, dt = \int_{0}^{\hat{T}} \langle f, u \rangle \, dt - \frac{1}{2} \langle Bu, u \rangle (\hat{T}) + \frac{1}{2} \langle Bu_0, u_0 \rangle.
\]

Then, (33) and the lower-semicontinuity, shown in (32), imply that
\[
\limsup_{\varepsilon \to 0} \int_{0}^{\hat{T}} \langle u_\varepsilon^*, u_\varepsilon \rangle \\
\leq \int_{0}^{\hat{T}} \langle f, u \rangle \, dt + \frac{1}{2} \langle Bu_0, u_0 \rangle - \liminf_{\varepsilon \to 0} \frac{1}{2} \langle Bu_\varepsilon, u_\varepsilon \rangle (\hat{T}) \\
\leq \int_{0}^{\hat{T}} \langle f, u \rangle \, dt + \frac{1}{2} \langle Bu_0, u_0 \rangle - \frac{1}{2} \langle Bu, u \rangle (\hat{T}) = \int_{0}^{\hat{T}} \langle u^*, u \rangle \, dt.
\]

Thus, we obtain that \( u_\varepsilon \to u \) weakly in \( V_l \) and \( (Bu_\varepsilon)' \to (Bu)' \) weakly in \( U'_r \), and
\[
\limsup_{\varepsilon \to 0} \int_{0}^{\hat{T}} \langle u_\varepsilon^*, u_\varepsilon - u \rangle \leq \int_{0}^{\hat{T}} \langle u^*, u \rangle - \int_{0}^{\hat{T}} \langle u^*, u \rangle = 0.
\]

Therefore, by the limit condition (5), for each \( v \in V \)
\[
\liminf_{\varepsilon \to 0} \int_{0}^{\hat{T}} \langle u_\varepsilon^*, u_\varepsilon - v \rangle \geq \int_{0}^{\hat{T}} \langle u^*(v), u - v \rangle, \quad \text{some } u^*(v) \in A(u, \omega).
\]
In particular, this holds for \( u \) and so, in fact, \( \int_0^\tau \langle u^*_\varepsilon, u - v \rangle \rightarrow 0 \). Thus,

\[
\int_0^\tau \langle u^*, u - v \rangle = \lim_{\varepsilon \rightarrow 0} \int_0^\tau \langle u^*_\varepsilon, u - v \rangle \\
\geq \liminf_{\varepsilon \rightarrow 0} \left( \int_0^\tau \langle u^*_\varepsilon, u - u_\varepsilon \rangle + \int_0^\tau \langle u^*_\varepsilon, u_\varepsilon - v \rangle \right) \\
\geq \int_0^\tau \langle u^* (v), u - v \rangle, \quad \text{for some } u^* (v) \in A(u, \omega).
\]

Since \( v \) is arbitrary, this shows from separation theorems that \( u^* (\omega) \in A(u(\omega), \omega) \) in \( V'_0,\hat{T} \).

This has proved the following theorem in which a more general coercivity condition is used.

**Theorem 5.1.** Assume that \( A \) satisfies conditions \( (H_1) \)–\( (H_3) \) \((22)–(26)) \). Let \( u_0 \) be measurable into \( W \) and \( f \) be measurable into \( V' \). Let \( B \in \mathcal{L}(W, V') \) be non-negative and self-adjoint as described above. Let \( \sigma > 0 \) be small. Then, there exist a pair of functions \( (u, u^*) \), measurable into \( V'_0, T - \sigma \times V'_0, T - \sigma \), such that \( u^* (\omega) \in A(\mathcal{X}, u(\omega), \omega) \) for each \( \omega \) and for \( t \leq T - \sigma \),

\[
Bu(t) - Bu_0 + \int_0^t u^* (s) \, ds = \int_0^t f (s) \, ds.
\]

We note that if it is known that for a given \( \omega \) there exists a unique solution to the evolution equation, then we obtain the solution on the whole interval \( (0, T) \). However, \( \sigma \) is arbitrary so essentially, there isn’t much difference between the solution above on \( (0, T - \sigma) \) and a solution on \( (0, T) \).

**Corollary 4.** In the setting of Theorem 5.1, and assuming uniqueness for fixed \( \omega \), there exists a similar measurable solution valid on \( (0, T) \). Moreover, \( u, u^* \) are measurable into \( V \) and \( V' \) respectively.

We skip the details. One could also get this even without uniqueness for fixed \( \omega \) by extending the operators and data beyond \( T \), obtaining a measurable solution on the larger time interval, and then taking a restriction.

The following is an interesting generalization of Theorem 5.1, where the degenerate inclusion has an added stochastic integral denoted as \( t \rightarrow q(t, \omega) \).

**Theorem 5.2.** Consider the setting of Theorem 5.1, and let \( q(t, \omega) \) be a product measurable function into \( V \), a stochastic integral, such that \( t \rightarrow q(t, \omega) \) is continuous and \( q(0, \omega) = 0 \).

Then, for each \( \sigma > 0 \) (small), there exists a solution \( u \) of the integral equation

\[
Bu(t, \omega) + \int_0^t u^* (s, \omega) \, ds = \int_0^t f (s, \omega) \, ds + Bu_0 (\omega) + Bq(t, \omega), \quad t \leq T - \sigma,
\]

where \( (t, \omega) \rightarrow u(t, \omega) \) is product measurable. Moreover, for each \( \omega \), \( Bu(t, \omega) = B(u(t, \omega)) \) for a.e. \( t \) and \( u^* (\cdot, \omega) \in A(u(\cdot, \omega), \omega) \) for a.e. \( t \), \( u^* \) is product measurable into \( V' \). Also, for each \( a \in [0, T - \sigma] \),

\[
Bu(t, \omega) + \int_a^t u^* (s, \omega) \, ds = \int_a^t f (s, \omega) \, ds + Bu(a, \omega) + Bq(t, \omega) - Bq(a, \omega).
\]
Proof. Define the stopping time
\[ \tau_r (\omega) \equiv \inf \{ t : |q (t, \omega)| > r \}, \]
which is the first time a continuous random variable hits an open set where \(|q| > r\),
and so it is a valid stopping time. Then, for each \( r \), let
\[ A_r (\omega, w) \equiv A (\omega, w + q^r (\cdot, \omega)), \]
where we let \( q^r (t, \omega) \equiv q (t \wedge \tau_r, \omega) \). Since \( q^r \) is uniformly bounded, all of the
necessary estimates and measurability conditions for the solution given in the above
corollary hold when \( A \) is replaced with \( A_r \). Therefore, there exists a solution \( w_r \) to
the inclusion
\[ (Bw_r)' (\cdot, \omega) + A_r (w_r (\cdot, \omega), \omega) \ni f (\cdot, \omega), \quad Bw_r (0, \omega) = Bu_0 (\omega), \quad t \in [0, T - \sigma/2]. \]
Now, for a fixed \( \omega \), we have that \( q^r (t, \omega) \) does not change when \( r \) is large enough.
Indeed, \( q \) is a continuous function of \( t \) and so is bounded on the interval \([0, T - \sigma/2]\).
Thus, for \( r \) large enough and fixed \( \omega \), \( q^r (t, \omega) = q (t, \omega) \). Therefore, we obtain
\[ \langle Bw_r (t, \omega) w_r (t, \omega) \rangle + \int_0^t \| w_r (s, \omega) \|^p \, ds \leq C (\omega). \quad (34) \]
As in the proof of Theorem 5.1, we pass to a limit involving a subsequence, as \( r (\omega) \to \infty \) and obtain a solution to the integral equation
\[ Bw (t, \omega) - Bu_0 (\omega) + \int_0^t u^* (s, \omega) \, ds = \int_0^t f (s, \omega) \, ds, \quad t \in [0, T - \sigma], \quad (35) \]
where \( u^* (\omega) \in A (w (s, \omega) + q (s, \omega), \omega) \) and the functions \( u^* \) and \( w \) are measurable
into \( V'_{[0, T - \sigma]} \). Finally, we let \( u (t, \omega) = w (t, \omega) + q (t, \omega) \), which is a solution of the
inclusion.

The last claim in the theorem is obtained by letting \( t = a \) in (35) and then
subtracting this from equation (35) with \( t > a \). \qed

6. Progressively measurable solutions. In the cases when the uniqueness of
the evolution initial value problem is known, for fixed \( \omega \), one can obtain various
results about progressively measurable solutions in a rather straightforward way.

First, we provide a definition of the term progressively measurable.

Definition 6.1. Let \( F_t \) be an increasing set, in \( t \), of \( \sigma \)-algebras of sets of \( \Omega \) where
\((\Omega, F)\) is a measurable space. Thus, each \( F_t \) is a \( \sigma \)-algebra and if \( s \leq t \), then
\( F_s \subseteq F_t \). This set of \( \sigma \)-algebras is called a filtration. A set \( S \subseteq [0, T] \times \Omega \) is called
progressively measurable if for every \( t \in [0, T] \),
\[ S \cap [0, t] \times \Omega \in B ([0, t]) \times F_t. \]
Denote by \( P \) the set of progressively measurable sets, which is a \( \sigma \)-algebra of subsets
of \([0, T] \times \Omega \). A function \( g \) is said to be progressively measurable if \( X_{[0, t]} g \) is \( B ([0, t]) \times F_t \)-measurable, for each \( t \).

Let \( A \) satisfy the conditions (\( H_1 \))–(\( H_4 \)) and the modified condition \( H_{5mod} \) that
is as follows.

• \( H_5 \) For each \( t \leq T \), if \( \omega \to u (\cdot, \omega) \) is \( F_t \)-measurable into \( V_{[0, t]} \), then there
exists a \( F_t \)-measurable selection of \( A \left( X_{[0, t]} u (\cdot, \omega), \omega \right) \) into \( V'_{[0, t]} \).
Note that $u(t,\omega)$ is in $V_{[0,t]}$ so $u(t,\omega) \in V$.

We next establish a theorem about existence of progressively measurable solutions to differential inclusions. In the literature, monotonicity is often assumed which implies uniqueness. However, here we show that the knowledge of the uniqueness for each $\omega$ is sufficient. To illustrate the idea of the proof, we assume that $u_0$ is $F_0\sigma$-measurable. Then Theorem 5.1 shows that there exists a solution to the evolution equation (36), denoted as $u_1$, that is $B([0,T-\sigma]) \times F_{T-\sigma}\sigma$-measurable. Let $s < T - \sigma$. Theorem also implies that there exists a solution $u_2$ of the evolution equation on $[0,s]$, for each $\omega$, which is $B([0,s]) \times F_s\sigma$-measurable on $[0,s] \times \Omega$. It follows from the uniqueness that, for each $\omega$ we have $u_1(t,\omega) = u_2(t,\omega)$ in $V_{[0,s]}$. Therefore, we can re-define $u_1$ for $t \in [0,s]$ so that for all $(t,\omega) \in [0,s] \times \Omega$, $u_1(t,\omega) = u_2(t,\omega)$. The re-defined function $u_1$ is $B([0,s]) \times F_s\sigma$-measurable on $[0,s] \times \Omega$ and is $B([0,T-\sigma]) \times F_{T-\sigma}\sigma$-measurable on $[0,T-\sigma] \times \Omega$. This leads to the following theorem.

**Theorem 6.2.** Assume conditions (22)–(26) and $H_5$, the progressively measurable condition. Let $u_0$ be $F_0\sigma$ measurable and $(t,\omega) \rightarrow X_{[0,t]}(t)f(t,\omega)$ is $B([0,t]) \times F_t$-product measurable into $V'$ for each $t$. Also assume that for each $\omega$, there is at most one solution $(u, u^*)$ to the evolution equation

$$Bu(t) - Bu_0(\omega) + \int_0^t u^*(s,\omega) \, ds = \int_0^t f(s,\omega) \, ds,$$

(36)

for $t \in [0,T]$. Then there exists a unique solution $(u(t,\omega), u^*(t,\omega))$ in $V_{[0,T]} \times V'_{[0,T]}$ to the above integral equation for each $\omega, t \in (0,T)$. This solution satisfies $(t,\omega) \rightarrow (u(t,\omega), u^*(t,\omega))$ is progressively measurable into $V \times V'$.

**Proof.** First, we note that by Theorem 5.1, there exists a solution on $[0,T-\sigma]$ for each $\sigma > 0$ small. Then, by the assumption of uniqueness, there exists a solution on $(0,T)$. Let $T$ denote subsets of $(0,T-\sigma)$ that contain $T - \sigma$ such that for $S \in T$, there exists a solution $u_S$ for each $\omega$ to the integral equation on $[0,T-\sigma]$ such that $(t,\omega) \rightarrow X_{[0,s]}(t)u_S(t,\omega)$ is $B([0,s]) \times F_s\sigma$-measurable for each $s \in S$. The idea of the proof is to get the progressively measurable condition to hold for $s$ in an increasingly larger sets. $S$ might not be an interval, but it does contain $T - \sigma$. The smallest such set $S$ is the set consisting of only $T - \sigma$, and so $T$ is not empty because $\{T - \sigma\} \in T$. Now, we impose a partial order on these subsets and use the Hausdorff maximal theorem. If $S, S'$ are in $T$, then $S \leq S'$ means that $S \subseteq S'$ and also $u_S(t,\omega) = u_{S'}(t,\omega)$ in $V$ for all $t \in S$, similarly for $u^*_S$ and $u^*_{S'}$. We note that we are considering a particular representative of a function in $V_{[0,T-\sigma]}$ and $V'_{[0,T-\sigma]}$ because of the pointwise condition. Letting $C$ denote a maximal chain, we address the question whether $\bigcup C = S_\infty$ all of $(0,T-\sigma)$ and what is $u_{S_\infty}$. Define $u_{S_\infty}(t,\omega)$ to be the common value of $u_S(t,\omega)$ for all $S$ in $C$, which contain $t \in S_\infty$. If $s \in S_\infty$, then it is in some $S \in C$ and so the product measurability condition holds for this $s$. Thus, $S_\infty$ is a maximal element of the partially ordered set. We show now that $S_\infty$ is all of $(0,T-\sigma)$. Indeed, suppose that $\hat{s} \notin S_\infty$, $T - \sigma > \hat{s} > 0$. Then, it follows from Theorem 5.1 that there exists a solution to the integral equation (36) on $[0,\hat{s}]$, say $u_1$, such that $u_1(t,\omega)$ is $B([0,\hat{s}]) \times F_{\hat{s}}$-measurable, similarly for $u^*_1$. By the same theorem, there is a solution on $[0,T-\sigma]$, say $u_2$, which is $B([0,T-\sigma]) \times F_{[0,T-\sigma]}$-measurable. Now, by uniqueness, $u_2(t,\omega) = u_1(t,\omega)$ in $V_{[0,\hat{s}]}$, and similarly for $u^*_2$. Therefore, we may re-define $u_2, u^*_2$ on $[0,\hat{s}]$ so that
Thus, $S_s$ to $36$ on $[0, T)$ existence of a unique progressively measurable solution on $[0, T)$.

Let $u_0$ be $F_0$-measurable and $(t, \omega) \rightarrow X_{[0,t]} (t) f (t, \omega)$ be $B ([0, t]) \times F_T$-product measurable into $V'$ for each $t \in [0, T - \sigma]$. Also, let $t \rightarrow q (t, \omega)$ be continuous and progressively measurable into $V$. Suppose there is at most one solution to

$$Bu (t, \omega) + \int_0^t u^* (s, \omega) ds = \int_0^t f (s, \omega) ds + Bu_0 (\omega) + Bq (t, \omega), \quad (37)$$

for each $\omega$. Then, the solution $u$ to the integral equation is progressively measurable and so is $u^*$. Moreover, for each $\omega$, $u^* (\cdot, \omega) \in A (u (\cdot, \omega), \omega)$, and for each $a \in [0, T]$,

$$Bu (\omega) (t) + \int_a^t u^* (s, \omega) ds = \int_a^t f (s, \omega) ds + Bu (\omega) (a) + Bq (t, \omega) - Bq (a, \omega).$$

Proof. It follows from Theorem 5.2 that there exists a solution to (37) that is $B ([0, T - \sigma]) s \times F_{T - \sigma}$-measurable. Since this is true for all $\sigma > 0$ (small), there exists a unique $B ([0, \tilde{T}]) \times F_T$-measurable solution for each $\tilde{T} < T$. Now, as in the proof of Theorem 5.2, one can define a new operator

$$A_r (w, \omega) \equiv A (w, w + q^{\tau_r} (\cdot, \omega)),$$

where $\tau_r$ is the stopping time defined there. Then, since $q$ is progressively measurable, the progressive measurability condition is satisfied for this operator. Hence, by Theorem 6.2 there exists a unique solution $w$ that is progressively measurable to the integral equation

$$Bu_r (t, \omega) + \int_0^t u^*_r (s, \omega) ds = \int_0^t f (s, \omega) ds + Bu_0 (\omega)$$

where $u^*_r (\cdot, \omega) \in A_r (\cdot, \omega), \omega)$. Then, we let $r \rightarrow \infty$ and eventually $q^{\tau_r} (\cdot, \omega) = q (\cdot, \omega)$. Thus, there is a solution to

$$Bu (t, \omega) + \int_0^t u^* (s, \omega) ds = \int_0^t f (s, \omega) ds + Bu_0 (\omega),$$

$$u^* (\cdot, \omega) \in A (w (\cdot, \omega) + Bq (\cdot, \omega), \omega),$$

which is progressively measurable because $w (\cdot, \omega) = \lim_{r \rightarrow \infty} u_r (\cdot, \omega)$ in $V$, since each $u_r$ is progressively measurable. We note that the uniqueness assumption is needed in passing to the limit. Thus, for each $\tilde{T} < T$, $\omega \rightarrow w (\cdot, \omega)$ is measurable into $V_{[0, \tilde{T}]}$. Then, by Lemma 3.4, $w$ has a representative in $V_{[0, \tilde{T}]}$ for each $\omega$ such that the resulting function satisfies $(t, \omega) \rightarrow X_{[0, \tilde{T}]} (t) w (t, \omega)$ is $B ([0, \tilde{T}]) \times F_{\tilde{T}}$-measurable into $V$. Thus one can assume that $w$ is progressively measurable. Now

$u_2 (t, \omega) = u_1 (t, \omega)$, for all $t \in [0, \hat{s}]$, and similarly for $u^*$, denoting these functions as $u, u^*$. By the uniqueness, $u_{S_\infty} (\cdot, \omega) = \hat{u} (\cdot, \omega)$ in $L^p ([0, \hat{s}], V)$ and so we re-define $\hat{u} (s, \omega)$ to equal $u_{S_\infty} (s, \omega)$ for $s < \hat{s}$ and $u_1 (\hat{s}, \omega)$ at $\hat{s}$. As to $s > \hat{s}$, we re-define $\hat{u} (s, \omega) = u_{S_\infty} (s, \omega)$ for such $s$. By the uniqueness, the two functions are equal in $V_{[s, T - \sigma]}$ and so no change occurs in the solution of the integral equation. Thus, $S_\infty$ is not maximal, since $S_\infty \cup \{ \hat{s} \}$ is larger. This contradiction shows that $S_\infty = (0, T - \sigma]$. Hence, there exists a unique progressively measurable solution to 36 on $[0, T - \sigma]$ for each small $\sigma$. We now can use uniqueness to conclude the existence of a unique progressively measurable solution on $[0, T)$. 

**Theorem 6.3.** Assume that conditions (22)-(26) and $H_3$, the progressively measurable condition hold. Let $u_0$ be $F_0$-measurable and $(t, \omega) \rightarrow X_{[0,t]} (t) f (t, \omega)$ be $B ([0, t]) \times F_T$-product measurable into $V'$ for each $t \in [0, T - \sigma]$. Also, let $t \rightarrow q (t, \omega)$ be continuous and progressively measurable into $V$. Suppose there is at most one solution to

$$Bu (t, \omega) + \int_0^t u^* (s, \omega) ds = \int_0^t f (s, \omega) ds + Bu_0 (\omega) + Bq (t, \omega), \quad (37)$$

for each $\omega$. Then, the solution $u$ to the integral equation is progressively measurable and so is $u^*$. Moreover, for each $\omega$, $u^* (\cdot, \omega) \in A (u (\cdot, \omega), \omega)$, and for each $a \in [0, T]$,

$$Bu (\omega) (t) + \int_a^t u^* (s, \omega) ds = \int_a^t f (s, \omega) ds + Bu (\omega) (a) + Bq (t, \omega) - Bq (a, \omega).$$

Proof. It follows from Theorem 5.2 that there exists a solution to (37) that is $B ([0, T - \sigma]) s \times F_{T - \sigma}$-measurable. Since this is true for all $\sigma > 0$ (small), there exists a unique $B ([0, \tilde{T}]) \times F_T$-measurable solution for each $\tilde{T} < T$. Now, as in the proof of Theorem 5.2, one can define a new operator

$$A_r (w, \omega) \equiv A (w, w + q^{\tau_r} (\cdot, \omega)),$$

where $\tau_r$ is the stopping time defined there. Then, since $q$ is progressively measurable, the progressive measurability condition is satisfied for this operator. Hence, by Theorem 6.2 there exists a unique solution $w$ that is progressively measurable to the integral equation

$$Bu_r (t, \omega) + \int_0^t u^*_r (s, \omega) ds = \int_0^t f (s, \omega) ds + Bu_0 (\omega)$$

where $u^*_r (\cdot, \omega) \in A_r (\cdot, \omega), \omega)$. Then, we let $r \rightarrow \infty$ and eventually $q^{\tau_r} (\cdot, \omega) = q (\cdot, \omega)$. Thus, there is a solution to

$$Bu (t, \omega) + \int_0^t u^* (s, \omega) ds = \int_0^t f (s, \omega) ds + Bu_0 (\omega),$$

$$u^* (\cdot, \omega) \in A (w (\cdot, \omega) + Bq (\cdot, \omega), \omega),$$

which is progressively measurable because $w (\cdot, \omega) = \lim_{r \rightarrow \infty} u_r (\cdot, \omega)$ in $V$, since each $u_r$ is progressively measurable. We note that the uniqueness assumption is needed in passing to the limit. Thus, for each $\tilde{T} < T$, $\omega \rightarrow w (\cdot, \omega)$ is measurable into $V_{[0, \tilde{T}]}$. Then, by Lemma 3.4, $w$ has a representative in $V_{[0, \tilde{T}]}$ for each $\omega$ such that the resulting function satisfies $(t, \omega) \rightarrow X_{[0, \tilde{T}]} (t) w (t, \omega)$ is $B ([0, \tilde{T}]) \times F_{\tilde{T}}$-measurable into $V$. Thus one can assume that $w$ is progressively measurable. Now
as in Theorem 5.2, Define \( u = w + q \). It follows by uniqueness that there exists a unique progressively measurable solution to (37) on \((0, T)\).

The last claim follows from letting \( t = a \) in the top equation and then subtracting this from the top equation with \( t > a \).

We assumed above that \( t \to q(t, \omega) \) is continuous into \( V \). It would be interesting to see what can be obtained if we assume only the continuity into \( W \). Moreover, we would like to relax the uniqueness assumption, for a fixed \( \omega \). It seems that one can obtain the corresponding theorems to these two issues above, however, currently some details need to be worked out. For example, our above results hold true for nonlinear, set-valued operators that have history dependence. On the other hand, we need to omit history dependence to obtain solutions to degenerate stochastic inclusions in the case of additive noise involving a stochastic integral having values in \( W \). Indeed, the inclusion of multiplicative noise seems to require monotonicity. These considerations give focus to future work, and they are likely to yield strong solutions to stochastic inclusions in which the probability space is given, rather than obtained as part of the solution to the problem.

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