Collective excitations of a trapped Bose-condensed gas

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Abstract

By taking the hydrodynamic limit we derive, at $T = 0$, an explicit solution of the linearized time dependent Gross-Pitaevskii equation for the order parameter of a Bose gas confined in a harmonic trap and interacting with repulsive forces. The dispersion law $\omega = \omega_0(2n^2 + 2n\ell + 3n + \ell)^{1/2}$ for the elementary excitations is obtained, to be compared with the prediction $\omega = \omega_0(2n + \ell)$ of the noninteracting harmonic oscillator model. Here $n$ is the number of radial nodes and $\ell$ is the orbital angular momentum. The effects of the kinetic energy pressure, neglected in the hydrodynamic approximation, are estimated using a sum rule approach. Results are also presented for deformed traps and attractive forces.

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Almost 50 years ago Bogoliubov [1] derived his famous theory for the elementary excitations of a dilute Bose gas. This theory, originally applied to homogeneous systems, is now receiving a novel interest because of the experimental availability of Bose-condensed gases confined in magnetic traps [2–4] (for a review on Bose-Einstein condensation see for instance Ref. [5]). The Bogoliubov theory can be shown [6] to correspond to the linear limit of the time dependent Gross-Pitaevskii [8] equation for the order parameter $\Phi$

$$i\hbar \frac{\partial}{\partial t} \Phi(r, t) = \left( -\frac{\hbar^2 \nabla^2}{2m} + V_{\text{ext}}(r) + \frac{4\pi \hbar^2 a}{m} |\Phi(r, t)|^2 \right) \Phi(r, t).$$

Here $V_{\text{ext}}$ is the confining potential and $a$ is the s-wave scattering length. This equation neglects interaction effects arising from the atoms out of the condensate. This is an accurate approximation for a dilute Bose gas at low temperatures where the depletion of the condensate is negligible. Differently from the homogeneous case, the Gross-Pitaevskii equation in the presence of an external potential admits stationary solutions not only for positive values of the scattering length, but also when $a$ is negative. In the latter case a solution of metastable type is found provided the number of atoms in the trap is not too large [9–12]. The solutions of the time-dependent Eq.(1), after linearization, have the well known RPA structure and have been the object of a recent numerical investigation in the case of a trapped atomic gas [13].

The main purpose of this work is to obtain an explicit, analytic solution of (1) holding when the repulsive interaction is large enough to make the kinetic energy pressure negligible compared to the external and interparticle interaction terms. When applied to the calculation of the ground state this limit corresponds to the Thomas-Fermi approximation and is reached for positive and large values of the adimensional parameter $Na/a_0$ where $a_0 = (\hbar/m\omega_0)^{1/2}$ is the harmonic oscillator length characterizing the trap and $N$ is the number of atoms. In the study of the elementary excitations this approximation corresponds to the hydrodynamic limit accounting, in homogeneous systems, for the propagation of phonons.

In order to discuss the behavior of the elementary excitations in this limit it is convenient
to derive explicit equations for the density \( \rho(r, t) = |\Phi(r, t)|^2 \) and for the velocity field \( \mathbf{v}(r, t) = (\Phi^*(r, t) \nabla \Phi(r, t) - \nabla \Phi^*(r, t) \Phi(r, t)) / 2mi \rho(r, t) \). These equations can be directly obtained starting from the time dependent Eq. (1) and take the form

\[
\frac{\partial}{\partial t} \rho + \nabla (\mathbf{v} \rho) = 0 \quad (2)
\]

and

\[
m \frac{\partial}{\partial t} \mathbf{v} + \nabla (\delta \mu + \frac{1}{2} m \mathbf{v}^2) = 0 \quad (3)
\]

where

\[
\delta \mu = V_{\text{ext}} + \frac{4 \pi \hbar^2 a}{m} \rho - \frac{\hbar^2}{2m \sqrt{\rho_0}} \nabla^2 \sqrt{\rho_0} - \mu \quad (4)
\]

is the change of the chemical potential with respect to its ground state value \( \mu \). It is worth noting that these equations do not involve any approximation with respect to the Gross-Pitaevskii Eq. (1) and hold in the linear as well as in the non linear regimes. They have the general structure of the dynamic equations of superfluids at zero temperature (see for example [7]). In particular Eq. (3) establishes the irrotational nature of the superfluid flow.

The density \( \rho_0 \) relative to the ground state is obtained setting \( \mathbf{v} = 0 \) and \( \delta \mu = 0 \). This yields the equation

\[
V_{\text{ext}}(r) + \frac{4 \pi \hbar^2 a}{m} \rho_0 - \frac{\hbar^2}{2m \sqrt{\rho_0}} \nabla^2 \sqrt{\rho_0} - \mu = 0 \quad (5)
\]

which, as expected, coincides with the Gross-Pitaevskii equation for the order parameter \( \Phi_0 = \sqrt{\rho_0} \) of the ground state. The chemical potential \( \mu \) is fixed by imposing the proper normalization to the density \( \rho_0 \). In the following we will look for solutions of the above equations by neglecting the kinetic energy pressure term \( \frac{\hbar^2}{2m \sqrt{\rho_0}} \nabla^2 \sqrt{\rho_0} \) in Eq. (4). This approximation yields the well known Thomas-Fermi result

\[
\rho_0(r) = \frac{m}{4 \pi \hbar^2 a} (\mu - V_{\text{ext}}(r)) \quad (6)
\]

for the ground state density. The solutions for the time dependent equations are also easily determined using the same approximation which yields the simple expression \( \delta \mu = 4 \pi \hbar^2 (\rho - \rho_0) \).
\( \rho_0 / m \) for the change of the chemical potential (see Eqs. (1-3)). Assuming for simplicity an isotropic harmonic oscillator potential \( V_{\text{ext}}(r) = \omega_0^2 r^2 / 2m \), the equations of motion (2-3), after linearization, can be written in the useful form

\[
\omega^2 \delta \rho = -\frac{1}{2} \omega_0^2 \nabla (R^2 - r^2) \nabla \delta \rho
\]  

(7)

where \( \delta \rho(r) \exp(-i \omega t) = \rho(r,t) - \rho_0(r) \) and \( R^2 = 2\mu / m \omega_0^2 \) fixes the boundary of the system where the density \( \rho_0 \) vanishes. In the absence of the external trap the same procedure yields the well known equation \( \omega^2 \delta \rho = -c^2 \nabla^2 \delta \rho \) where \( c = (4\pi \hbar^2 a \rho_0 / m^2)^{1/2} \) is the sound velocity of the homogeneous Bose gas.

The solutions of the hydrodynamic equations (7) are defined in the interval \( 0 \leq r \leq R \) and have the form

\[
\delta \rho(r) = P^{(2n)}_\ell \left( \frac{r}{R} \right) r^\ell Y_{\ell m}(\theta, \phi)
\]

(8)

where \( P^{(2n)}_\ell(t) = 1 + \alpha_2 t^2 + \ldots + \alpha_{2n} t^{2n} \) are polynomials of degree \( 2n \), containing only even powers of \( t \), and satisfying the orthogonality condition \( \int_0^1 P^{(2n)}_\ell(t) P^{(2n')}_{\ell'}(t) t^{2\ell+2} dt = 0 \) if \( n \neq n' \). The parameters \( \ell \) and \( m \) label the angular momentum of the excitation and its \( z \)-component respectively. The coefficients \( \alpha_{2k} \) satisfy the recurrence relation \( \alpha_{2k+2} = -\alpha_{2k} (n-k)(2\ell+2k+3+2n)/(k+1)(2\ell+2k+3) \). The dispersion law of the normal modes is given by the formula

\[
\omega(n, \ell) = \omega_0 (2n^2 + 2n\ell + 3n + \ell)^{1/2}
\]

(9)

which represents the main result of the present work. It should be compared with the prediction of the harmonic oscillator (HO) model in the absence of interparticle interactions:

\[
\omega_{\text{HO}} = \omega_0 (2n + \ell) .
\]

(10)

Of particular interest is the case of the lowest radial modes \( (n = 0) \), also called surface excitations, for which we predict the dispersion law

\[
\omega(n = 0) = \sqrt{\omega_0} .
\]

(11)
The frequency of these modes lies systematically below the harmonic oscillator result $\omega_{HO}(n = 0) = \ell \omega_0$. This behavior should be taken into account in the determination of the critical frequency $\omega_{cr} = \min_\ell (\omega(\ell)/\ell)$ needed to generate a rotational instability [12,14], and might provide a competitive mechanism with respect to the creation of a vortex line. Notice that in the dipole case ($\ell = 1$) both the hydrodynamic and harmonic oscillator predictions coincide with the oscillator frequency $\omega_0$. This follows from the fact that in an external harmonic potential the lowest dipole mode corresponds to a rigid motion of the center of mass, and is consequently unaffected by the interatomic forces.

The accuracy of prediction (9) is expected to become lower and lower as $n$ and $\ell$ increase. In fact the high energy states are associated with rapid variations of the density in space and consequently the kinetic energy contribution in Eq. (4) cannot be longer neglected. Using the macroscopic language this corresponds to abandon the phonon regime. The energy range where our prediction (9) is expected to be accurate then corresponds to values smaller than the chemical potential $\mu$. For large values of $n$ and $\ell$, the correct dispersion law will approach the harmonic oscillator result (10).

The effects of the kinetic energy pressure, ignored in the hydrodynamic approximation, can be investigated by calculating the energy of the collective mode through a sum rule approach [13], based on the ratio

$$\frac{\hbar^2 \omega^2}{m_1} = \frac{m_3}{m_1}$$

between the cubic energy weighted and the energy weighted moments of the dynamic structure factor: $m_p = \sum_n |\langle 0 | F | n \rangle|^2 \hbar \omega_{n0}^p$. Here $\hbar \omega_{n0}$ is the excitation energy of the state $|n\rangle$ and $F$ is a general excitation operator. The moments $m_1$ and $m_3$ can be reduced in the form of commutators involving the hamiltonian of the system. One finds $m_1 = \frac{1}{2} < 0 [F^\dagger [H,F]] |0 \rangle$ and $m_3 = \frac{1}{2} < 0 [[F^\dagger, H], [H, [H, F]]] |0 \rangle$. Equation (12) provides in general a rigorous upper bound to the energy of the lowest state excited by $F$. The sum rule approach has the merit of providing useful information on the dynamic behavior of the system using only the knowledge of the ground state. By evaluating explicitly the
commutators with the Hamiltonian \[ H = \sum_i \frac{p_i^2}{2m} + \sum_i \frac{1}{2} m \omega_0^2 r_i^2 + \frac{4\pi \hbar^2 a}{m} \sum_{i<j} \delta(r_i - r_j) \] (13)

we find, in the case of the surface \((n = 0)\) operator \( F = \sum_i r_i^\ell Y_{\ell m}(\theta_i, \phi_i) \), the results

\[ m_1 = \frac{\hbar^2}{8m\pi} \ell(2\ell + 1) \int r^{2\ell-2} \rho \, d\mathbf{r} \] (14)

and

\[ m_3 = \frac{\hbar^4}{8m^2\pi} \ell(2\ell + 1) \left[ \ell m \omega_0^2 \int r^{2\ell-2} \rho \, d\mathbf{r} + \ell(\ell - 1) \frac{\hbar^2}{m} \int |\nabla \sqrt{\rho}|^2 \, r^{2\ell-4} \, d\mathbf{r} \right] , \] (15)

and the excitation frequency (12) takes the form

\[ \omega^2(n = 0) = \omega_0^2 \ell (1 + (\ell - 1) \beta_{\ell}) \] (16)

where \( \beta_{\ell} = \hbar^2 \int |\nabla \sqrt{\rho}|^2 \, r^{2\ell-4} \, d\mathbf{r} / m^2 \omega_0^2 \int r^{2\ell-2} \rho \, d\mathbf{r} \). For the most relevant quadrupole \((n = 0, \ell = 2)\) case we find

\[ \omega_Q = \sqrt{2}\omega_0(1 + \frac{E_{\text{kin}}}{E_{\text{ho}}})^{1/2} \] (17)

where \( E_{\text{kin}} \) and \( E_{\text{ho}} \) are, respectively, the expectation value of the kinetic and harmonic potential energies in the ground state. In the absence of interparticle interactions one has \( E_{\text{kin}} = E_{\text{ho}} \) and hence one recovers the harmonic oscillator result \( \omega = 2\omega_0 \) of Eq.(14).

Viceversa when the interaction is repulsive and the number of atoms is sufficiently large, the kinetic energy term is negligible (Thomas-Fermi limit) and one obtains the hydrodynamic prediction \( \omega = \sqrt{2}\omega_0 \) of eq.(11). The knowledge of the kinetic energy relative to the ground state then permits to estimate the quadrupole excitation energy in the general case. For higher multipolarities the determination of the coefficient \( \beta_{\ell} \) of Eq.(16) requires the knowledge of finer details of the ground state. A simple estimate can be obtained using the gaussian approximation for the wave function of the condensate, yielding \( \beta_{\ell} = E_{\text{kin}} / E_{\text{ho}} \), independent of \( \ell \).
In a similar way, starting from eq. (12), one can determine the frequency of the compression modes. For the lowest monopole mode \((n = 1, \ell = 0)\), excited by the operator \(F = \sum_i r_i^2\), we find

\[
m_1 = \frac{2\hbar^2}{m}N < r^2 >
\]

and

\[
m_3 = \frac{2\hbar^4}{m^2}(4E_{\text{kin}} + 4E_{ho} + 9E_{\text{int}})
\]

where \(E_{\text{int}} = \frac{2\pi\hbar^2\mu}{m} \int d\mathbf{r} \rho_0^2(r)\) is the interaction energy. Using the virial identity \(2E_{\text{kin}} - 2E_{ho} + 3E_{\text{int}} = 0\), holding for the ground state, we finally obtain the following result for the monopole frequency

\[
\omega_M = \omega_0 \left(5 - \frac{E_{\text{kin}}}{E_{ho}}\right)^{1/2}
\]

yielding the values \(\omega = 2\omega_0\) and \(\omega = \sqrt{5}\omega_0\) in the harmonic oscillator and Thomas-Fermi limits respectively.

Predictions (17, 20) well agree with the results recently obtained in Ref. [13] by solving numerically the time dependent Gross-Pitaevskii equations for a trapped atomic gas interacting with repulsive forces. The conditions of Ref. [13] correspond to the value \(\mu = 4.3\omega_0\) for the chemical potential and to the value 0.18 for the ratio between the kinetic and harmonic oscillator energies. This yields, using eqs. (17) and (20), the values \(\omega_Q = 1.54\omega_0\) and \(\omega_M = 2.20\omega_0\) in excellent agreement with the findings of [13] (1.53\omega_0\) and 2.19\omega_0\respec-

The above results for the dispersion law can be generalized to the case of a deformed trap. This is particular relevant since the available magnetic traps are often highly anisotropic. Let us consider the case of a harmonic oscillator trap with axial symmetry along the z-axis:

\[
V_{\text{ext}} = m\omega_1^2 s^2/2 + m\omega_z^2 z^2/2\text{ where } s = (x^2 + y^2)^{1/2}\text{ is the radial variable in the } x - y \text{ plane.}
\]

In this case the relevant differential equation (7) should be replaced by

\[
\omega^2 \delta \rho = -\frac{1}{2} \nabla \left(\omega_1^2 (s^2 - s^2) + \omega_z^2 (Z^2 - z^2)\right) \nabla \delta \rho
\]
with \( m \omega^2 s^2 / 2 = m \omega^2 z^2 / 2 \equiv \mu \).

Due to the axial symmetry of the trap the third component \( m \) of the angular momentum is still a good quantum number. However the dispersion law will depend on \( m \). Explicit results are available in some particular cases. For example functions of the form \( \delta \rho = r^\ell Y_\ell m (\theta, \phi) \) are still solutions of eq.(21) for \( m = \pm \ell \) and \( m = \pm (\ell - 1) \). The resulting dispersion laws are:

\[
\omega^2 (m = \pm \ell) = \ell \omega^2 \perp
\]

and

\[
\omega^2 (m = \pm (\ell - 1)) = (\ell - 1)\omega^2 \perp + \omega^2 z.
\]

Equations (22-23) provide a full description of the dipole excitation \((\ell = 1)\) whose frequencies coincide, as expected, with the unperturbed harmonic oscillator values \( \omega_D (m = \pm 1) = \omega \perp \) and \( \omega_D (m = 0) = \omega_z \). Viceversa for the quadrupole \((\ell = 2)\) mode Eqs.(22-23) account only for the \( m = \pm 2 \) \( (\omega = \sqrt{2} \omega \perp) \) and \( m = \pm 1 \) \( (\omega = (\omega^2 \perp + \omega^2 z)^{1/2}) \) components. The solution with \( m = 0 \) involves a coupling with the monopole \((n = 1, \ell = 0)\) excitation and the dispersion law of the two decoupled modes is given by

\[
\omega^2 (m = 0) = \omega^2 \perp \left(2 + \frac{3}{2} \lambda^2 \pm \frac{1}{2} \sqrt{9 \lambda^4 - 16 \lambda^2 + 16}\right)
\]

with \( \lambda = \omega_z / \omega \perp \). When \( \lambda \to 1 \) one recovers the original solutions (17) and (20) corresponding, respectively, to the quadrupole and monopole excitations in a spherical trap.

For systems interacting with attractive forces \((a < 0)\) the hydrodynamic results discussed above do not provide an adequate description of the dispersion law and in this case the sum rule approach becomes particularly useful. The kinetic energy contribution \( E_{\text{kin}} \) entering Eqs.(17,20) can never be neglected, being larger than \( E_{\text{ho}} \), and the monopole mode turns out to be located below the quadrupole one. Physically this reflects the tendency of the system to become more compressible. In Fig.2 we show the behavior of the monopole and quadrupole frequencies obtained using Eqs.(17,20) as a function of the adimensional
parameter \( Na/a_0 \). The ratio \( E_{\text{kin}}/E_{ho} \) has been estimated using a variational calculation of the ground state based on gaussian trial wave functions for the order parameter \([10][11]\). One can see that while the energy of the quadrupole excitation is enhanced at negative scattering lengths, the monopole mode becomes softer. The upper bound (20) for the monopole energy vanishes when \( E_{\text{kin}} = 5E_{ho} \). This exactly coincides with the condition for the onset of instability predicted by the use of gaussian trial wave functions in the variational calculation \((Na/a_0 = -0.67 [11])\). It is however worth noting that the results shown in the figure provide only a semi-quantitative estimate of the excitation frequencies. In fact the onset of instability obtained from the exact solution [3] of the Gross-Pitaevskii equation \((Na/a_0 \sim -0.57)\) differs from the gaussian estimate. Furthermore one should keep in mind that the estimates (17, 20), being based on the ratio (12), provide only an upper bound to the frequency of the lowest excitations.

In conclusion we have derived a systematic investigation of the collective excitations of a Bose condensed gas confined in an external trap. We have obtained analytic results for the dispersion law of both surface and compression modes employing the hydrodynamic approximation and the sum approach. Our work reveals the key role played the interatomic forces which introduce a rich structure in the dynamic behaviour of these new many-body systems.

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REFERENCES

[1] N.N. Bogoliubov, J. Phys. (USSR) 11, 23 (1947).

[2] M.H. Anderson, J.R. Ensher, M.R. Mathews, C.E. Wieman, and E.A. Cornell, Science, 269, 198 (1995).

[3] C.C. Bradley, C.A. Sackett, J.J. Tollett, and R.G. Hulet, Phys. Rev. Lett. 75, 1687 (1995).

[4] K.B. Davis, M.O. Mewes, M.R. Andrews, N.J. van Druten, D.D. Durfee, D.M. Kum, and W. Ketterle, Phys. Rev. Lett. 75, 3969 (1995).

[5] Bose-Einstein Condensation, ed. by A. Griffin, D. Snoke and S. Stringari (Cambridge, N.Y., 1995).

[6] E. Lifshitz and L. Pitaevskii, Statistical Physics (Pergamon, Oxford, 1980), Part 2;

[7] P. Nozières and D. Pines, The Theory of Quantum Liquids (Addison-Wesley, Reading, MA, 1990) Vol.II.

[8] L.P. Pitaevskii, Zh. Eksp. Teor. Fiz. 40, 646 (1961) [Sov. Phys. JETP 13, 451 (1961)];
    E.P. Gross, Nuovo Cimento 20, 454 (1961); E.P. Gross, J. Math. Phys. 4, 195 (1963).

[9] P.A. Ruprecht, M.J. Holland, K. Burnett, and M. Edwards, Phys. Rev. A 51, 4704 (1995).

[10] G. Baym and C. Pethick, Phys. Rev. Lett. 76, 6 (1996).

[11] A. Fetter, to be published.

[12] F. Dalfovo and S. Stringari, Phys. Rev. A, in press.

[13] M. Edwards, K. Burnett, P.A. Ruprecht, and C.W. Clark, to be published.

[14] S. Stringari, Phys. Rev. Lett. 76, 1405 (1996).
[15] O. Bohigas, A.M. Lane and J. Martorell, Phys. Rep. 51, 267 (1979); E. Lipparini and S. Stringari, Phys. Rep. 175, 103 (1989).

[16] The interaction term in Eq. (13) has the meaning of an effective potential to be employed in the framework of the mean field scheme yielding the Gross-Pitaevskii Eq.(1).
FIGURES

FIG. 1. Quadrupole (Q) and monopole (M) frequencies as a function of the adimensional parameter $Na/a_0$, calculated using Eqs.(17) and (20).
