CLUSTER SUPERALGEBRAS

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Abstract. We introduce cluster superalgebras, a class of \(\mathbb{Z}_2\)-graded commutative algebras generalizing cluster algebras of Fomin and Zelevinsky. These algebras contain odd coordinates that anticommute with each other and square to zero. A cluster superalgebra is defined with the help of a quiver satisfying some conditions and specific transformations called mutations. Generators of a cluster superalgebra are Laurent polynomials with denominators given by even monomials. Both, mutations and exchange relations, generalize the classical ones. Every cluster superalgebra admits a presymplectic form invariant under mutations. Our main series of examples of cluster superalgebras is provided by superfriezes, analogous to Coxeter’s frieze patterns.

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Introduction

Cluster algebras, discovered by Fomin and Zelevinsky [11]-[14], are a special class of commutative associative algebras generated by Laurent polynomials with (positive) integer coefficients in a set of commuting indeterminates \{x_1, \ldots, x_n\}. A cluster algebra is usually defined with the help of a quiver (an oriented graph) with no loops and no 2-cycles; the generators of the algebra are defined with the help of exchange relations and mutations of the initial quiver. More precisely, the vertices of the initial quiver are labeled by \{x_1, \ldots, x_n\}; the mutation at a chosen variable \(x_k\) changes the quiver and replaces \(x_k\) with the new variable

\[ x'_k = \frac{M_1 + M_2}{x_k}, \]

where \(M_1\) and \(M_2\) are the monomials obtained as products of coordinates connected to \(x_k\) by ingoing and outgoing arrows, respectively. The above formula is called an exchange relation; a cluster algebra is generated by all possible mutations and exchange relations.

Mutations of the quiver at a chosen vertex are transformations that can be illustrated by the following diagram:

```
  •   ←   •   ⇒   •   ←   •
  ▼   ▼   ▼   ▼   ▼   ▼   ▼
```

The most important point is creation of a new arrow.

The main goal of this paper is to introduce the notion of cluster superalgebra. The general idea is to include odd, or Grassmann, indeterminates \{\(\xi_1, \ldots, \xi_m\)\}, that anticommute with each other, and in particular, square to zero. The cluster superalgebra is a certain subalgebra of the ring \(\mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, \xi_1, \ldots, \xi_m]\).

A cluster superalgebra is defined by a quiver which is an extension of the initial quiver. The main ingredients are modified exchange relations and quiver mutations. The vertices of the initial quiver are labeled by the even coordinates, the new vertices labeled by the odd coordinates. Essentially, the mutations of such a quiver (observe that the two upper vertices are colored while the three lower ones are black) are defined by the following modified rule:

```
  •   ←   •
  ▼   ▼   ▼
```

Note that this rule can be understood as a particular case of mutation of coloured graphs, see [4, 28].

Cluster algebras naturally appear in algebra, geometry and combinatorics. The algebras of regular functions of many algebraic varieties, related to Lie theory, have cluster structure. The main examples are simple Lie groups [2, 17], Grassmannians [34], flag varieties, etc. Cluster algebras are closely related to symplectic and Poisson geometry [16, 18], representation theory, Teichmüller spaces [9] and various moduli spaces. For more information and references, see surveys [27, 35].

Let us mention that cluster algebras are also closely related to integrable systems. Indeed, the Hirota (or octahedral) recurrence, which is a famous and “universal” discrete integrable system, is an example of cluster exchange relations. Many spaces on which discrete integrable systems act, such as moduli spaces of configurations of points in projective spaces, have cluster structures, see [31, 30]. Among a wealth of notions related to these moduli spaces, we mention frieze patterns, see [7, 5]. Moduli spaces of configurations of points are also related to the spaces of linear

\[ \text{Exactly the same mutation rules are intensively used in physics (see [1] and references therein). Note that physicists sometimes talk of creation of a quark).} \]
difference equations with special periodicity conditions; for a detailed account see [30]. For recent developments in this area, see [10, 15, 19, 29, 20, 33, 22]. Relations of cluster algebras to moduli spaces, linear difference equations, and frieze patterns is the starting point of our investigation. Superization of linear difference equations with some special monodromy conditions and the corresponding superfriezes, analogous to Coxeter’s frieze patterns recently introduced in [32]. Here we study a class of examples of cluster superalgebras “of type $A^*$” that correspond to superfriezes. The definition of cluster superalgebras suggested in the present paper is a result of a careful analysis of superfriezes.

The paper consists of five sections.

In introductory Section 1 we briefly review classical cluster algebras, and show how to recover the rules of mutation of a quiver from the canonical presymplectic form and exchange relations.

In Section 2 we introduce cluster superalgebras, and consider simplest examples.

In Section 3 we study general properties of cluster superalgebras. In particular, we construct a presymplectic form invariant with respect to mutations, and prove the “Laurent phenomenon” similar to the Laurent phenomenon of Fomin and Zelevinsky.

Section 4 is a tentative attempt to develop the rules of mutations and exchange relations for odd indeterminates. We only consider particular cases.

In Section 5 we present our main examples related to the notion of superfriezes and difference equations. Note that a simple direct proof of the Laurent phenomenon in this case is given in [32].

Let us mention that commutative superalgebras appear naturally in algebra and geometry. One classical example is the classical Grassmann algebra of differential forms on a manifold. Superalgebras are an essential part of any cohomology theory. Superalgebras also appear in physics where they are extensively used to describe symmetries of elementary particles. Theory of supermanifolds was created in 70’s by Berezin and Leites, see [3, 23, 25, 26, 8, 24], for a survey. This theory offers geometry and analysis based on spaces ringed by superalgebras. We hope that cluster superalgebras introduced in this paper provide with an adequate framework to describe algebras of regular functions on algebraic supermanifolds related to super Lie theory.

We are convinced that the definition of cluster superalgebra could be more general. For instance, we completely ignore arrows between odd vertices. Almost everywhere in the paper, odd coordinates are frozen; mutations of odd coordinates are developed only in some particular cases. It would be interesting to look for more general definitions, as well as for more examples of cluster superalgebras and related supermanifolds.

1. CLASSICAL CLUSTER ALGEBRAS AND THE CANONICAL PRESYMPLECTIC FORM

This section is a very elementary introduction to cluster algebras.

A cluster algebra of rank $n$ is a commutative associative algebra associated to a quiver, i.e., a finite oriented graph, $Q$, with no loops and no 2-cycles. We use the common notation: $Q_0$ is the set of vertices, and $Q_1$ is the set of arrows of $Q$. The maps $s : Q_1 \rightarrow Q_0$ and $t : Q_1 \rightarrow Q_0$ send an arrow to its source or target, respectively. The vertices of $Q$ are labeled by commuting indeterminates (or coordinates) $\{x_1, \ldots, x_n\}$, where $|Q_0| = n$.

1.1. THE DEFINITION. The cluster algebra defined by $Q$ is generated by $\{x_1, \ldots, x_n\}$ and the rational functions $x'_k, \ldots$ obtained by the sequences of transformations called mutations:

$$\mu_k : (\{x_1, \ldots, x_k, \ldots x_n\}, Q) \mapsto (\{x_1, \ldots, x'_k, \ldots x_n\}, Q').$$
For every $k \in \{1, \ldots, n\}$, the mutation at vertex $x_k$ replaces the variable $x_k$ by the new coordinate $x'_k$ defined by the formula:

$$x'_k = \frac{1}{x_k} \left( \prod_{i \rightarrow k} x_i + \prod_{j \leftarrow k} x_j \right),$$

where the products are taken over the arrows $(x_i \rightarrow x_k) \in Q_1$ and $(x_j \leftarrow x_k) \in Q_1$, respectively. The above formula is called an exchange relation.

The mutation of the quiver $\mu_k : Q \rightarrow Q'$ at the vertex $x_k$ is defined according to the following three rules:

1. for every path $(x_i \rightarrow x_k \rightarrow x_j)$ in $Q$, add an arrow $(x_i \rightarrow x_j)$;
2. reverse all the arrows leaving or arriving at $x_k$;
3. remove all 2-cycles created by rule (1).

Every mutation is an involution, i.e., $\mu_k^2 = \text{Id}$, but two different mutations do not necessarily commute, namely the mutations at vertices connected by an arrow. Applying all possible mutations, one obtains (possibly infinitely many) different quivers, and rational functions in the initial coordinates $\{x_1, \ldots, x_n\}$. The cluster algebra associated with the initial quiver $Q$ is the associative commutative algebra generated by all the coordinates obtained in the mutation process.

Some of the initial cluster coordinates are called frozen, and remain unchanged. One simply does not consider mutations at frozen coordinates. These coordinates are additional parameters of the cluster algebra.

Among the main results of Fomin and Zelevinsky are the “Laurent phenomenon” and finite type classification.

- All elements of a given cluster algebra are Laurent polynomials in $\{x_1, \ldots, x_n\}$, see [11],[12].
- A cluster algebra is of finite type, i.e., finitely generated, if and only if the quiver $Q$ is equivalent under a sequence of mutations to one of the Dynkin quivers of type $A, D$, or $E$, see [13].

Furthermore, the positivity conjecture formulated in [11] (and proved in many particular cases) states that the coefficients of the Laurent monomials in all the generators of a cluster algebra are non-negative integers.

1.2. The canonical presymplectic form. Every cluster algebra (and thus every algebraic manifold whose algebra of regular functions has a cluster algebra structure) has a canonical presymplectic differential 2-form introduced by Gekhtman, Shapiro and Vainshtein [16]. It is given by

$$\omega = \sum_{x_i \rightarrow x_j} \frac{dx_i \wedge dx_j}{x_i x_j},$$

where the summation is performed over the elements of $Q_1$. This form is invariant with respect to the cluster mutations and therefore is well-defined globally, on every cluster variety. Note that the form $\omega$ is obviously closed since it is with constant coefficients in coordinates $\log x_i$. However, it is not necessarily non-degenerate, for instance, if $n$ is odd.

Quite remarkably, among the following three data:

(exchange relations, quiver mutations, canonical 2-form $\omega$)

every two contain the full information, and allow one to recover the third ingredient.

Let us show how to recover the quiver mutations from the exchange relations and the 2-form $\omega$ that we assume invariant with respect to the cluster mutations. In order to make the exposition elementary, we first consider a simple example.
Example 1.2.1. Consider the cluster algebra associated to the Dynkin quiver $A_3$:

\[
\begin{array}{c}
x \\ \downarrow \downarrow \\ y \to \to z
\end{array}
\]

The corresponding 2-form is

\[
\omega = \frac{dx \wedge dy}{xy} + \frac{dy \wedge dz}{yz}.
\]

Performing the mutation at $y$, one obtains the new coordinate $y' = (x + z)/y$. Expressing then $y = (x + z)/y'$, and substituting into $\omega$, one obtains from the Leibniz rule:

\[
\omega = \frac{dx \wedge dz}{xz} + \frac{dy' \wedge dx}{xy'} + \frac{dz \wedge dy'}{y'z}.
\]

Invariance of $\omega$ means that, after the above coordinate transformation, it exactly corresponds to the quiver obtained as a result of the mutation at the vertex $y$. Indeed, according to Definition (1), the quiver corresponding to the latter form is as follows:

\[
\begin{array}{c}
x \\ \leftarrow \leftarrow \\ y' \to \to z
\end{array}
\]

which is precisely the mutation of the initial quiver at $y$.

The general situation is similar. Rewrite the form (1) using the summation over vertices of $Q$. One immediately obtains from the Leibniz rule:

\[
\omega = \frac{1}{2} \sum_{x_k \in Q_0} \frac{d_x x_{I_k} \wedge dx_{J_k}}{x_k},
\]

where

\[
x_{I_k} = x_{i_1} \cdots x_{i_s}, \quad x_{J_k} = x_{j_1} \cdots x_{j_t}
\]

are the monomials obtained as the products of coordinates connected to $x_k$ by ingoing arrows: $(x_i \to x_k)$, and by outgoing arrows: $(x_k \to x_j)$, respectively.

Invariance of the latter formula for $\omega$ under mutations at $x_k$ is quite obvious. Indeed, temporarily using the notation $\tilde{x}_k := \frac{x_k}{x_{I_k}}$, rewrite the exchange relations in the form:

\[
x_k = \frac{1}{x_k} \left( x_{I_k} \right) \cdot
\]

Let also

\[
\omega_{x_k} = \frac{1}{2} \frac{d_{x_k} x_{I_k} \wedge dx_{J_k}}{x_k}
\]

be the term of $\omega$ containing $x_k$. Substituting the above equation for $x_k$, and by the Leibniz rule one immediately obtains:

\[
\omega_{x_k} = \frac{1}{2} \frac{d_{x_k} x_{I_k} \wedge dx_{J_k}}{x_k} \cdot
\]

The first term corresponds to the reversed arrows at $x'_{I_k}$, while the second term corresponds to the extra arrows $(x_j \to x_i)$. This explains the rules (1) and (2) of quiver mutations.
The Gekhtman-Shapiro-Vainshtein form $\omega$, and related family of Poisson structures, play an important role in applications. It equips cluster algebraic varieties with additional geometric structures. A similar presymplectic form will be crucial for us to understand mutations of extended quivers.

2. Cluster superalgebras

In this section, we introduce the notion of cluster superalgebra and give several examples. Everywhere in this section, the odd coordinates are frozen.

2.1. The extended quiver. To define the notion of cluster superalgebra, we will need the notion of extended quiver which can be obtained from a classical quiver by adding new vertices and new arrows.

**Definition 2.1.1.** Given a quiver $Q$ with no loops and no 2-cycles, an extended quiver $\tilde{Q}$ with underlying quiver $Q$, is a quiver defined as follows.

A. $\tilde{Q}$ has $m$ extra “colored” vertices labeled by the odd coordinates $\{\xi_1, \ldots, \xi_m\}$, so that $\tilde{Q}_0 = \{x_1, \ldots, x_n, \xi_1, \ldots, \xi_m\}$.

B. Some of the new vertices $\{\xi_1, \ldots, \xi_m\}$ are related to the vertices $\{x_1, \ldots, x_n\}$ of the underlying quiver $Q$ by ingoing or outgoing arrows.

**Notation 2.1.2.** a) For every $1 \leq k \leq n$, two non-intersecting subsets $I_k, J_k \subset \{1, \ldots, m\}$, $I_k \cap J_k = \emptyset$ are fixed; the new arrows are $(\xi_i \rightarrow x_k)$ for $i \in I_k$, and $(x_k \rightarrow \xi_j)$ for $j \in J_k$, so that $\tilde{Q}_1 = Q_1 \cup_k \{\xi_i \rightarrow x_k, i \in I_k, x_k \rightarrow \xi_j, j \in J_k\}$.

b) The set $\tilde{Q}_2$ of 2-paths is the collection of subquivers of $\tilde{Q}$ defined in the usual way: the subquiver $(v_1 \rightarrow v_2 \rightarrow v_3)$, belongs to $\tilde{Q}_2$ if $v_1$ is connected to $v_2$ by an arrow $(v_1 \rightarrow v_2)$, and $v_2$ is connected to $v_3$ by an arrow $(v_2 \rightarrow v_3)$.

**Example 2.1.3.** Whenever an odd vertex $\xi_i$ is related to an even vertex $x_k$ by an ingoing arrow, and $\xi_j$ is related to $x_k$ by an outgoing arrow, there is an odd-even-odd 2-path in $\tilde{Q}_2$:

$$\xi_i \quad \rightarrow \quad x_k \quad \rightarrow \quad \xi_j,$$

such a 2-path exists for every $\xi_i, \xi_j$ where $i \in I_k$ and $j \in J_k$.

In the following quiver, where $I_k = \{i_1, \ldots, i_r\}$, $J_k = \{j_1, \ldots, j_s\}$:

$$\xi_{i_1} \quad \cdots \quad \xi_{i_r} \quad \xrightarrow{x_k} \quad \xi_{j_1} \quad \cdots \quad \xi_{j_s},$$

every vertex $\xi_{i_1}, \ldots, \xi_{i_r}$ is connected to every vertex $\xi_{j_1}, \ldots, \xi_{j_s}$ by an odd-even-odd 2-path.

Odd-even-odd 2-paths will be important.

**Remark 2.1.4.** a) Clearly, every extended quiver $\tilde{Q}$ has no loops or 2-cycles.

b) Note that we do not consider arrows between the odd vertices of $\tilde{Q}$, and this is certainly an interesting question whether one can add such arrows and create a more rich combinatorics of extended quivers.

c) One can of course consider multiple arrows between odd and even vertices.
2.2. **Quiver mutations.** Let us define the mutation rules of an extended quiver. These mutations are connected to even vertices. The rules of mutations are modifications of the classical rules (1), (2), (3), see Section 2.2. Note, however, that mutations are not always allowed.

**Definition 2.2.1.** Given an extended quiver \( \tilde{Q} \) and an even vertex \( x_k \in Q_0 \), the mutation \( \mu_k \) is defined by the following rules:

1. the underlying quiver \( Q \subset \tilde{Q} \) mutates according to the same rules (1), (2), (3) as in the classical case;
2. given a 2-path \( (\xi_i \to x_k \to \xi_j) \), add the 2-paths \( (\xi_i \to x_\ell \to \xi_j) \) for all \( x_\ell \in Q_0 \) connected to \( x_k \) by an arrow \( (x_k \to x_\ell) \), we define:

\[
\begin{array}{cccc}
\xi_i & \xi_i \\
x_m & x_k & x_\ell \\
\end{array} \quad \xrightarrow{\mu_k} \quad \begin{array}{cccc}
\xi_i & \xi_j \\
x_m & x'_k & x_\ell \\
\end{array}
\]

3. reverse all the arrows at \( x_k \);
4. remove all the odd-even-odd 2-paths (if any) with opposite orientations: \( (\xi_i \to x_\ell \to \xi_j) \) and \( (\xi_i \leftarrow x_\ell \leftarrow \xi_j) \) created by rule (1*).

Observe that every mutation \( \mu_k \) of the quiver \( \tilde{Q} \) is an involution.

**Remark 2.2.2.** To compare the new rule (1*) to the classical mutation rule (1), note that the new rule changes the set \( \tilde{Q}_2 \) of 2-paths, and not just the set of arrows, as the old rule (1).

2.3. **Allowed and forbidden mutations.** A mutation is allowed if the way the rule (1*) changes the sets \( \tilde{Q}_2 \) and \( \tilde{Q}_1 \) is consistent. More precisely, we have the following.

**Definition 2.3.1.** A mutation of the extended quiver \( \tilde{Q} \) at a vertex \( x_k \in Q_0 \) is allowed if, in the resulting quiver \( \mu_k(\tilde{Q}) \), every vertex \( \xi_i \in I_\ell \cup I_k \) is connected to every vertex \( \xi_j \in J_\ell \cup J_k \) by a 2-path \( (\xi_i \to x_\ell \to \xi_j) \).

As a result of the rule (1*), the sets \( I_\ell \) and \( J_\ell \) attached to the vertex \( x_\ell \) connected to \( x_k \) by an arrow \( (x_k \to x_\ell) \) are transformed as follows:

\[
\mu_k : \begin{cases} 
I_\ell \rightarrow I_\ell \cup I_k, \\
J_\ell \rightarrow J_\ell \cup J_k.
\end{cases}
\]

The mutation \( \mu_k \) is not allowed in each of the following cases:
(a) \( \xi_i \) is not connected to some vertex \( \xi'_j \in J_\ell \) by a 2-path \( (\xi_i \to x_\ell \to \xi'_j) \);
(b) \( \xi_j \) is not connected to some vertex \( \xi'_i \in I_\ell \) by a 2-path \( (\xi'_i \to x_\ell \to \xi_j) \).

**Example 2.3.2.** a) Mutation of the following quiver at \( x_1 \) is not allowed.

\[
\begin{array}{cccc}
\xi_1 & \xi_2 & \xi_3 & \xi_4 \\
x_1 & x_2 & x_3 & x_4 \\
\end{array} \quad \xrightarrow{\mu_1} \quad \begin{array}{cccc}
\xi_1 & \xi_2 & \xi_3 & \xi_4 \\
x'_1 & x_2 & x_3 & x_4 \\
\end{array}
\]

Indeed, this mutation would produce the quiver with two odd-even-odd 2-paths at \( x_2 \), namely \( (\xi_1 \to x_2 \to \xi_2) \) and \( (\xi_3 \to x_2 \to \xi_4) \). However, \( (\xi_1 \to x_2 \to \xi_4) \not\in \tilde{Q}_2 \) and \( (\xi_3 \to x_2 \to \xi_2) \not\in \tilde{Q}_2 \), which leads to a contradiction.
b) By the same reason, the mutation of the following quiver at \( x_1 \) is not allowed

\[
\begin{array}{cc}
\xi_1 & \xi_2 \\
\downarrow & \downarrow \\
x_1 & x_2 \\
\end{array}
\rightarrow
\begin{array}{cc}
\xi_1 & \xi_2 \\
\uparrow & \uparrow \\
x_1 & x_2 \\
\end{array}
\]

since it does not create the 2-path \((\xi_1 \rightarrow x_2 \rightarrow \xi_3)\).

c) The following mutation is allowed

\[
\begin{array}{cc}
\xi_1 & \xi_2 \\
\downarrow & \downarrow \\
x_1 & x_2 \\
\end{array}
\rightarrow
\begin{array}{cc}
\xi_1 & \xi_2 \\
\uparrow & \uparrow \\
x_1 & x_2 \\
\end{array}
\]

The following statement provides with a necessary and sufficient condition for the mutation at a vertex \( x_k \) to be allowed.

**Lemma 2.3.3.** Given an extended quiver \( \tilde{Q} \), the mutation at a given vertex \( x_k \) is allowed if and only if for every vertex \( x_\ell \in Q \) connected to \( x_k \) by an outgoing arrow \( x_k \rightarrow x_\ell \), (at least) one of the following conditions is satisfied:

1. \( I_k = I_\ell \);
2. \( J_k = J_\ell \);
3. \( I_k = J_k = \emptyset \);
4. \( I_k = J_\ell \), and \( J_k = I_\ell \);
5. \( I_\ell = J_\ell = \emptyset \).

**Proof.** It is obvious that any of the conditions (a)-(c) is sufficient. Indeed, the mutation at \( x_k \) sends all 2-paths \((\xi_i \rightarrow x_k \rightarrow \xi_j)\) to every vertex \( x_\ell \in Q \) connected to \( x_k \) by an outgoing arrow \((x_k \rightarrow x_\ell)\), creating 2-paths \((\xi_i \rightarrow x_\ell \rightarrow \xi_j)\). In cases (a)-(c), at least one of the sets \( I_\ell \) or \( J_\ell \) remains unchanged; in case (d), all arrows connecting \( x_\ell \) to odd vertices disappear; in case (e), this is also clear.

Conversely, assume that \( I_k \neq I_\ell \) and \( J_k \neq J_\ell \), and the sets are non-empty. Choose \( i \in I_k \) and \( i \notin I_\ell \), and \( j \notin J_k \) and \( j \in J_\ell \), then \( \xi_i \) is not connected to \( \xi_j \) by a 2-path \((\xi_i \rightarrow x_\ell \rightarrow \xi_j)\) this is a contradiction. \(\square\)

**Example 2.3.4.** The following example illustrates the case (b) of the above lemma.

\[
\begin{array}{cccccc}
\xi_{i_1} & \cdots & \xi_{i_r} & \xi_{j_1} & \cdots & \xi_{j_s} \\
\downarrow & \cdots & \downarrow & \cdots & \downarrow & \cdots \\
x_k & \cdots & x_\ell \\
\end{array}
\]

Example of an allowed mutation.

The mutation of this quiver at \( x_k \) is allowed.

We finish this section with examples of extended quivers with two odd coordinates for which the mutations are always allowed.

**Example 2.3.5.** Consider the extended quiver \( \tilde{Q} \) with only two odd coordinates, \( \xi_1 \) and \( \xi_2 \). If for every \( x_k \in Q_0 \), the coordinates \( \xi_1 \) and \( \xi_2 \) are either connected to \( x_k \) by one of the 2-paths,
(ξ₁ → xₖ → ξ₂) or (ξ₂ → xₖ → ξ₁), or not connected at all (the case a) below):

\[
\begin{array}{cc}
\begin{array}{l}
\xi_1 \quad \xi_2 \\
x_1 \xrightarrow{} x_2 \xrightarrow{} x_3 \hdots \xrightarrow{} x_k
\end{array} & \begin{array}{l}
\xi_1 \quad \xi_2 \\
x_1 \xrightarrow{} x_2 \xrightarrow{} x_3 \hdots \xrightarrow{} x_k
\end{array}
\end{array}
\]

a) “good case”: all mutations are allowed b) “bad case”: some mutations are forbidden

then all mutations of Q are allowed. Indeed, for every xₖ ∈ Q, one of the conditions (a)-(e) of Lemma 2.3.3 is always satisfied.

2.4. Exchange relations, the algebra A(\(\tilde{Q}\)). Everywhere in this section, we assume that the mutation is allowed.

We define the exchange relations for the even coordinates. Our formula is a modification of the classical exchange relations; amazingly, the additional term contains a product of two sums rather than a sum of two monomials.

**Definition 2.4.1.** Given an extended quiver \(\tilde{Q}\), the mutation \(x'ₖ = \muₖ(xₖ)\) of the indeterminate \(xₖ\) is defined by the following formula

\[
x'_ₖ := \frac{1}{xₖ} \left( \prod_{x_i \to xₖ} x_i + \prod_{x_j \leftarrow xₖ} x_j + \left( \sum_{x_i \to xₖ} \xi_i \right) \left( \sum_{x_j \leftarrow xₖ} \xi_j \right) \prod_{x_i \to xₖ} x_i \right)
\]

that will be called, as in the classical case, an exchange relation.

Note that, since the odd coordinates anticommute, the same holds for the sums:

\[
(\sum \xi_i)(\sum \xi_j) = -(\sum \xi_j)(\sum \xi_i).
\]

**Notation 2.4.2.**

1. The supercommutative superalgebra, \(A(\tilde{Q})\) generated by the coordinates \(\{x₁, \ldots, xₙ, ξ₁, \ldots, ξ_m\}\), as well as all possible mutations of the even coordinates: \(x'_₁, \ldots, x'_ₙ, x''ₖ, \ldots\) will be called the **cluster superalgebra** associated to the quiver \(\tilde{Q}\). It will be usually considered over \(\mathbb{C}\), yet other choices of the ground fields are possible.

2. The pair \((\tilde{Q}, \{x₁, \ldots, xₙ, ξ₁, \ldots, ξ_m\})\) is called the initial seed of the cluster superalgebra. A mutation gives rise to a new seed, i.e., a new pair: (quiver, set of coordinates).

3. Given an extended quiver \(\tilde{Q}\), following [11], we call the **exchange graph** the graph \(\tilde{T}\) whose vertices are the initial seed \((\tilde{Q}, \{x, ξ\})\) and all its allowed mutations; the edges are labeled by the numbers from 1 to \(n\). The graph \(\tilde{T}\) has no natural orientation. The projection \(\tilde{Q} \rightarrow Q\) defines the projection \(\tilde{T} \rightarrow T\), where \(T\) is the usual exchange graph.

Note that, unlike the purely even case, the above mutation of \(xₖ\) is not an involution, i.e., \(x''ₖ \neq xₖ\). It turns out, however, that the iterated mutation \(\mu^₂ₖ\), at any vertex \(xₖ\), does not add a new coordinate.

**Lemma 2.4.3.** One has \(A(\tilde{Q}) = A(\mu^₂ₖ(\tilde{Q}))\), for all \(1 \leq k \leq n\).

**Proof.** One easily obtains:

\[
x''ₖ = xₖ \left( 1 - \sum_{ξ_i \to xₖ} ξ_i \right) \left( \sum_{ξ_j \leftarrow xₖ} ξ_j \right), \quad xₖ = x'_ₖ \left( 1 + \sum_{ξ_i \to xₖ} ξ_i \right) \left( \sum_{ξ_j \leftarrow xₖ} ξ_j \right),
\]

where \(x'_ₖ := \muₖ(xₖ)\). This expression is a combination of the initial coordinates \(xₖ, ξₗ\), it belongs to the algebra generated by \(x_i, ξ_i\). \(\square\)
Let us emphasize that the algebra $A(\tilde{Q})$ is certainly not the largest possible superalgebra one can associate to $\tilde{Q}$. To obtain a larger version of cluster superalgebra, one needs to include mutations of the odd coordinates. In Section 4, we will show how to enlarge the algebra $A(\tilde{Q})$, in the case where the underlying quiver $Q$ is bipartite.

2.5. Elementary examples. In this section, we consider a number of concrete examples, in order to illustrate the exchange relations (2) and the mutation rules of extended quivers. Most of our examples correspond to extended quivers with two even coordinates, Similarly to the case studied in [11]. We investigate a 5-periodicity occurring in these examples.

We observe that the Laurent phenomenon holds for all examples we consider, and sometimes simplifications are quite spectacular.

Example 2.5.1. Our most elementary example is the quiver with one even and two odd vertices. Let us first perform consecutive mutations at the even vertex:

\[
\begin{array}{c}
\xi_1 \\
x
\end{array} \xrightarrow{\mu_1} \begin{array}{c}
\xi_2 \\
x'
\end{array} \xrightarrow{\mu_2} \begin{array}{c}
\xi_1 \\
x''
\end{array} \xrightarrow{\mu_1} \begin{array}{c}
\xi_2 \\
x'''
\end{array} \rightarrow \cdots
\]

The variable $x$ mutates as follows:

\[
x' = \frac{2}{x} + \frac{\xi_1 \xi_2}{x}, \quad x'' = x(1 - \xi_1 \xi_2), \quad x''' = \frac{2}{x} + 3\frac{\xi_1 \xi_2}{x}, \quad x'''' = x(1 - 2\xi_1 \xi_2), \quad \cdots
\]

As already mentioned, these mutations are not involutions, and the resulting process is infinite and aperiodic.

The corresponding cluster superalgebra is $\mathbb{C}[x^\pm, \xi_1, \xi_2] \cong \mathbb{C}[x^\pm] \otimes \mathbb{C}[\xi_1, \xi_2]$.

Example 2.5.2. Recall (see [11]) that, in the purely even case of the quiver $A_2$, the process of consecutive mutations:

\[
x_1 \xrightarrow{\mu_1} x'_1 \xrightarrow{\mu_2} x'_2 \xrightarrow{\mu_1} x''_1 \xrightarrow{\mu_2} \cdots
\]

is 5-periodic:

\[
x'_1 = \frac{1 + x_2}{x_1}, \quad x'_2 = \frac{1 + x_1 + x_2}{x_1 x_2}, \quad x''_1 = \frac{1 + x_1}{x_2}, \quad x''_2 = x_1, \quad x'''_1 = x_2.
\]

The corresponding exchange graph is a pentagon, and the cluster algebra of $A_2$ is isomorphic to $\mathbb{C}[x_1, x_2, x'_1, x'_2, x''_1]$.

In the next three examples, we consider different quivers that are extensions of the quiver $A_2$.

Example 2.5.3. Consider the following quiver with two even and two odd coordinates $\{x_1, x_2, \xi_1, \xi_2\}$:
Let us give some details of the computations. The first two mutations are obtained immediately from (2):

\[ x_1' = \frac{1 + x_2}{x_1} + \frac{\xi_1 \xi_2}{x_1}, \quad x_2' = \frac{1 + x_1 + x_2}{x_1 x_2} + \frac{1 + x_1}{x_1 x_2} \xi_1 \xi_2. \]

Again, applying the exchange relations (2), and performing simplifications, one obtains:

\[ x_1'' = \frac{x_1' + x_2'}{x_1}, \]
\[ = \frac{1}{x_2} \left( \frac{1 + x_1 + x_2 + x_1 x_2}{1 + x_2 + \xi_1 \xi_2} + \frac{1 + x_1}{1 + x_2} \xi_1 \xi_2 \right), \]
\[ = \frac{1}{x_2} \left( \frac{1 + x_1 + x_2 + x_1 x_2}{1 + x_2} - \frac{1 + x_1 + x_2 + x_1 x_2}{1 + x_2} \xi_1 \xi_2 \right), \]
\[ = \frac{1}{x_2} + \frac{x_1}{1 + x_1}. \]

Similarly, after computations, one obtains:

\[ x_2'' = x_1 (1 - \xi_1 \xi_2), \quad x_1''' = x_2 (1 - \xi_1 \xi_2), \quad x_2''' = \frac{1 + x_2}{x_1} + \frac{\xi_1 \xi_2}{x_1}, \]

and the next values are:

\[ x_1'''' = \frac{1 + x_1 + x_2}{x_1 x_2} (1 + \xi_1 \xi_2) + \frac{1 + x_1}{x_1 x_2} \xi_1 \xi_2, \quad x_2'''' = \frac{1 + x_1}{x_2} (1 + \xi_1 \xi_2), \quad \ldots \]

Remarkably enough, all of the coordinates we obtain by iterating the process, are Laurent polynomials. However, the above process is infinite and aperiodic.

Our mutation process is aperiodic and very different from the classical mutation process of the quiver \( A_2 \). However, analyzing the above formulas, we see the following. The corresponding cluster superalgebra is

\[ \mathbb{C}[x_1, x_2, x_1', x_2', x_1'', \xi_1, \xi_2] = \mathbb{C}[x_1, x_2, x_1', x_2', x_1''] \otimes \mathbb{C}[\xi_1, \xi_2]. \]

Indeed, all the other mutations of \( x_1 \) and \( x_2 \) are algebraic expressions in these coordinates.

**Example 2.5.4.** Starting with a similar quiver and performing the same sequence of mutations, one has:

\[ x_1 \xrightarrow{\xi_1} x_2 \xrightarrow{\mu_1} x_1' \xrightarrow{\xi_1} x_2' \xrightarrow{\mu_2} x_1'' \xrightarrow{\xi_1} x_2'' \xrightarrow{\mu_1} x_1''' \xrightarrow{\xi_1} x_2''' \xrightarrow{\mu_2} \ldots \]
and then again \(\mu_1, \mu_2, \) etc. One then obtains, after a computation:

\[
x_1' = \frac{1 + x_2}{x_1} + \frac{x_2}{x_1} \xi_1 \xi_2, \quad x_2' = \frac{1 + x_1 + x_2}{x_1 x_2} + \frac{\xi_1 \xi_2}{x_1}, \quad x_1'' = \frac{1 + x_1}{x_2} (1 - \xi_1 \xi_2),
\]

and further

\[
x_2'' = x_1 (1 - \xi_1 \xi_2), \quad x_1''' = x_2 (1 + \xi_1 \xi_2), \quad \ldots
\]

Once again, one does not have 5-periodicity, but the Laurent phenomenon is always verified. The corresponding cluster superalgebra is again isomorphic to \(\mathbb{C}[x_1, x_2, x_1', x_2', x_1'', \xi_1, \xi_2].\)

**Example 2.5.5.** The following quiver with two even and three odd vertices, as well as its mutations, will be particularly important for us (see Section 5 below). As before, let us perform an infinite sequence of mutations:

![Quiver Diagram]

One then has:

\[
x_1' = \frac{1 + x_2}{x_1} - \frac{\xi_1 \xi_2}{x_1}, \quad x_2' = \frac{1 + x_1 + x_2}{x_1 x_2} - \frac{\xi_1 \xi_2}{x_1 x_2} - \frac{\xi_2 \xi_3}{x_2},
\]

\[
x_1'' = \frac{1 + x_1}{x_2} \left(1 + \frac{x_1}{x_2} (\xi_1 + \xi_3) \xi_2\right), \quad x_2'' = x_1 (1 + \xi_1 \xi_2)
\]

and so on. The next mutations of the quiver and of the coordinates can be easily calculated, and once again, the Laurent phenomenon is verified. As above, the process does not stop, and there is no periodicity in the process of mutation of the coordinates, but the cluster superalgebra is once again \(\mathbb{C}[x_1, x_2, x_1', x_2', x_1'', \xi_1, \xi_2].\)

In all the examples we have considered, the exchange graph is finite, in particular, under our process of consecutive mutations at \(x_1\) and \(x_2\), the quiver mutant periodically. In Section 4.2, we will show how to obtain a periodic sequence of mutations of coordinates for a particularly simple example.
3. The Laurent Phenomenon and Presymplectic Form

In this section, we present two general results about mutations of extended quivers and the corresponding exchange relations.

3.1. Laurent Phenomenon. Let us show that the cluster superalgebras have properties very similar to those of classical cluster algebras. The Laurent phenomenon for cluster superalgebras was already illustrated by Examples 2.5.1, 2.5.3; this result extends a fundamental theorem of Fomin and Zelevinsky, see [11].

Let us emphasize that, since the division by odd coordinates is not well-defined, all the Laurent polynomials we consider have denominators equal to some monomials in \( \{x_1, \ldots, x_n\} \).

**Theorem 1.** For every extended quiver \( \tilde{Q} \), the cluster coordinates obtained by any series of consecutive mutations are Laurent polynomials in the initial coordinates \( \{x, \xi\} = \{x_1, \ldots, x_n, \xi_1, \ldots, \xi_m\} \).

**Proof.** Our proof goes along the same lines as the proof of Theorem 3.2 of [11] (see also Theorem 2.1 of [12]). We will repeat this proof giving the additional arguments when necessary.

Given an initial seed \( (\tilde{Q}^0, \{x^0, \xi^0\}) \), and consider a sequence of \( \ell \) consecutive mutations

\[
\cdots \circ \mu_k \circ \mu_j \circ \mu_i : (\tilde{Q}^0, \{x^0, \xi^0\}) \mapsto (\tilde{Q}^1, \{x^1, \xi^1\}) \mapsto \cdots \mapsto (\tilde{Q}^\ell, \{x^\ell, \xi^\ell\}).
\]

We need to prove that the coordinates \( \{x^\ell, \xi^\ell\} \) are Laurent polynomials in \( \{x^0, \xi^0\} \). For this end, we will use the induction on \( \ell \geq 3 \).

Following [11, 12], let us consider the following three seeds:

\[
(\tilde{Q}^1, \{x^1, \xi^1\}) = \mu_i(\tilde{Q}^0, \{x^0, \xi^0\}), \quad (\tilde{Q}^2, \{x^2, \xi^2\}) = \mu_j \circ \mu_i(\tilde{Q}^0, \{x^0, \xi^0\}),
\]

and

\[
(\tilde{Q}^3, \{x^3, \xi^3\}) := \mu_i \circ \mu_j \circ \mu_i(\tilde{Q}^0, \{x^0, \xi^0\}).
\]

Using the inductive assumption, we suppose that \( \{x^i, \xi^i\} \) are Laurent polynomials in \( \{x^1, \xi^1\} \) and \( \{x^2, \xi^2\} \), since these seeds are closer to \( \{x^i, \xi^i\} \) in the sequence of mutations. Furthermore, \( \{x^\ell, \xi^\ell\} \) must also be Laurent polynomials in \( \{x^3, \xi^3\} \), since applying \( \mu_i \) to \( (\tilde{Q}^3, \{x^3, \xi^3\}) \), one obtains the seed \( \mu_i \circ \mu_j \circ \mu_i(\tilde{Q}^0, \{x^0, \xi^0\}) \) that generates the same cluster algebra as \( (\tilde{Q}^2, \{x^2, \xi^2\}) \) (cf. Lemma 2.4.3).

We now need the following analog of Lemma 3.3 of [11] (see also Lemma 2.2 of [12]).

**Lemma 3.1.1.** (i) The coordinates \( \{x^1, \xi^1\} \), \( \{x^2, \xi^2\} \) and \( \{x^3, \xi^3\} \) are Laurent polynomials in \( \{x^0, \xi^0\} \).

(ii) Furthermore, \( \gcd(x^3, x^1) = \gcd(x^2, x^1) = 1 \), where the greatest common divisor is taken over the ring of polynomials in \( \{x_1, \ldots, x_n\} \).

**Proof of the lemma.** Part (i). The coordinates \( \{x^1, \xi^1\} \), \( \{x^2, \xi^2\} \) are obviously Laurent polynomials in \( \{x^0, \xi^0\} \). In the case where the vertices \( x_i \) and \( x_j \) are not connected by an arrow, the mutations \( \mu_i \) and \( \mu_j \) commute, and it is again obvious that the coordinates \( \{x^i, \xi^i\} \) are Laurent polynomials in \( \{x^0, \xi^0\} \). Let us calculate the mutation \( \mu_i \circ \mu_j \circ \mu_i \) in the case where the vertices \( x_i \) and \( x_j \) are connected by an arrow. We assume that the arrow is oriented as follows \( x_i \rightarrow x_j \) (the computations in case \( x_i \leftarrow x_j \) are similar).

Let us start with an arbitrary extended quiver a fragment of which is as follows:

\[
\begin{array}{c}
\xi_1 \cdots \xi_r \\
\eta_1 \cdots \eta_s \\
\xi_1 \cdots \xi_t \\
\xi_1 \cdots \xi_u \\
x_i & x_j
\end{array}
\]
associated to the quiver after the mutation by $\omega$

Proof.

Theorem 2.

One assumes that the form is invariant.

Next, after a (quite long) computation, one obtains:

$$x^1_i = \frac{M^1_i + x_j M^2_j}{x_i} + \left(\sum \xi (\sum \eta) M^1_i\right) x_i,$$

where $M^1_i$ and $x_j M^2_j$ are the monomials obtained by the product of the even coordinates connected to $x_i$ by ingoing and outgoing arrows, respectively. Similarly,

$$x^2_j = \frac{M^1_j M^2_j + x^3_j M^2_j}{x_j} + \left(\sum \xi (\sum \eta) M^1_i\right) x_j,$$

The rest of the proof is identical to those of Theorem 3.2 from [11] (and of Theorem 2.1 from [12]).

Theorem 1 is proved. □

The presymplectic form. We introduce the differential 2-form analogous to the Gekhtman-Shapiro-Vainshtein form (1):

$$\omega = \sum_{x_i \rightarrow x_j} \frac{dx_j \wedge dx_i}{x_i x_j} + \sum_{x_k \in Q} \frac{d \left( \left( \sum_{\xi_i \rightarrow x_k} \xi_i \right) \left( \sum_{\xi_j \leftarrow x_k} \xi_j \right) \right) \wedge dx_k}{x_k}.$$

It turns out that the 2-form (3) is invariant under mutations. Furthermore, as in the purely even case, this 2-form allows one to recover the rules of quiver mutations from Definition 2.2.1 if one assumes that the form is invariant.

Theorem 2. For every extended quiver $\tilde{Q}$, the form $\omega$ is invariant under mutations at even coordinates.

Proof. Let us choose an even vertex $x_k \in \tilde{Q}$. Perform a mutation at $x_k$, and denote the 2-form associated to the quiver after the mutation by $\omega'$. Our goal is to check that $\omega = \omega'$.

and show that the Laurent phenomenon occurs if and only if the mutation is allowed. One obtains from (2):

$$x^1_i = \frac{M^1_i + x_j M^2_j}{x_i} + \left(\sum \xi (\sum \eta) M^1_i\right) x_i,$$

(Note that the coefficient of the term $(\sum \xi (\sum \eta)$ vanishes identically.) We conclude that $x^1_i$ is a Laurent polynomial if and only if the product $(\sum \xi (\sum \eta) (\sum \xi)$ vanishes. This is guaranteed by (and almost equivalent to) the assumption that the mutation $\mu_1$ is allowed (cf. Lemma 2.3.3). The rest of the coordinates remain unchanged.

The second part of the lemma stating that $x^1_i$ and $x^3_i$, and $x^1_i$ and $x^2_i$ are coprime follows from the similar result in the purely even case. Indeed, our expressions for $x^1_i, x^2_i$ and $x^3_i$ differ from their purely even projections by nilpotent terms. Therefore, these expressions are coprime if and only if their purely even parts are coprime.

Hence the Lemma.

The rest of the proof is identical to those of Theorem 3.2 from [11] (and of Theorem 2.1 from [12]).

By inductive assumption, all of the coordinates $\{x^0_1, \ldots, x^0_n, \xi^0_{\ell}, \ldots, \xi^0_m\}$ are Laurent polynomials in $\{x^0_1, \ldots, x^0_n, \xi^0_{\ell}, \ldots, \xi^0_m\}$ and also in $\{x^0_1, \ldots, x^0_n, \xi_1, \ldots, \xi_m\}$. But the fact that the mutated coordinates are themselves Laurent polynomials and coprime implies that the coordinates $\{x^0_1, \ldots, x^0_n, \xi^0_{\ell}, \ldots, \xi^0_m\}$ are, indeed, Laurent polynomials in the initial coordinates.

Theorem 1 is proved.
As above, in the purely even case, collect the terms in \( \omega \) containing \( x_k \); one obtains:

\[
\omega_{x_k} := \frac{dx_k \wedge d\tilde{x}_{jk}}{x_{\tilde{x}_{jk}}} + \frac{d \left( \left( \sum_{i \in I_k} \xi_i \right) \left( \sum_{j \in J_k} \xi_j \right) \right) \wedge dx}{x} ,
\]

where, as above, the monomial \( x_{\tilde{x}_{jk}} = x_{i_1} \cdots x_{i_r} \) is the product of all even coordinates connected to \( x_k \) by the arrows \( (x_k \to x_i) \), and \( x_{jk} = x_{j_1} \cdots x_{j_r} \) is the product of all even coordinates connected to \( x_k \) by the arrows \( (x_k \leftarrow x_j) \).

The exchange relation (2) can be rewritten as follows:

\[
x = \frac{1}{\tilde{x}'} \left( 1 + \frac{x_{I_k}}{x_{J_k}} + \left( \sum_{i \in I_k} \xi_i \right) \left( \sum_{j \in J_k} \xi_j \right) \right) ,
\]

where \( \tilde{x}' := \frac{\tilde{x}}{x_{J_k}} \). Substituting this expression into \( \omega_x \), and applying the Leibniz rule, one obtains

\[
\frac{dx_k \wedge d\tilde{x}_{jk}}{x_{\tilde{x}_{jk}}} - \frac{d\tilde{x}' \wedge d\tilde{x}_{jk}}{\tilde{x}' \tilde{x}_{jk}} + \frac{d \left( \left( \sum_{i \in I_k} \xi_i \right) \left( \sum_{j \in J_k} \xi_j \right) \right) \wedge d\tilde{x}'}{\tilde{x}'} + \frac{d \left( \left( \sum_{i \in I_k} \xi_i \right) \left( \sum_{j \in J_k} \xi_j \right) \right) \wedge d\tilde{x}_{jk}}{\tilde{x}'}
\]

for the first term, and

\[
d \left( \left( \sum_{i \in I_k} \xi_i \right) \left( \sum_{j \in J_k} \xi_j \right) \right) \wedge dx_k
\]

for the second term. Collecting the above two terms, one finally gets:

\[
\omega_{x_k} = - \frac{dx'_k \wedge d\tilde{x}_{jk}}{\tilde{x}' \tilde{x}_{jk}} - \frac{d \left( \left( \sum_{i \in I_k} \xi_i \right) \left( \sum_{j \in J_k} \xi_j \right) \right) \wedge dx'}{\tilde{x}'} + \frac{d \left( \left( \sum_{i \in I_k} \xi_i \right) \left( \sum_{j \in J_k} \xi_j \right) \right) \wedge dx_{jk}}{\tilde{x}'}
\]

We need to prove that this expression coincides with the term \( \omega'_{x_k} \) in the 2-form \( \omega' \). Indeed, the first two terms with minus signs correspond to the reversing of arrows at \( x_k' \); the third term is the additional term generated by the rule (1) of the mutation of the classical quiver \( Q \); the last term is generated by the step (1*) of the mutation of the extended quiver \( \overline{Q} \).

We have proved that \( \omega_{x_k} = \omega'_{x_k} \), and hence \( \omega = \omega' \), since all the other terms of the 2-form remain unchanged under the mutation at \( x_k \). \( \square \)

**Example 3.2.1.** The 2-forms (3) corresponding to the quivers from Examples 2.5.1, 2.5.3 and 2.5.4 are as follows:

\[
\omega = \frac{d(\xi_1 \xi_2) \wedge dx}{x}, \quad \omega = \frac{d(\xi_1 \xi_2) \wedge dx_1}{x_1} + \frac{dx_1 \wedge dx_2}{x_1 x_2}, \quad \omega = \frac{d(\xi_1 \xi_2) \wedge dx_1}{x_1} - \frac{dx_1 \wedge dx_2}{x_1 x_2},
\]

respectively. In Example 2.5.3 it is equal to

\[
\omega = \frac{dx_1 \wedge dx_2}{x_1 x_2} - \frac{d(\xi_1 \xi_2) \wedge dx_1}{x_1} + \frac{d(\xi_1 \xi_2) \wedge dx_2}{x_2} - \frac{d(\xi_2 \xi_3) \wedge dx_2}{x_2}.
\]
4. Towards mutations of odd coordinates

Mutations of odd coordinates in extended quivers are yet to be understood. However, it is clear that, in order to observe periodicity properties of mutation of cluster coordinates, simultaneous mutations of even and odd coordinates are necessary. This section contains very preliminary considerations, and we are not yet convinced that the rules of mutations of odd coordinates we introduce in this section are the best possible. We introduce mutations of odd coordinates in some particular cases that can be used if the underlying quiver $\mathcal{Q}$ is bipartite.

4.1. The case of a source or a target. Given an extended quiver $\tilde{\mathcal{Q}}$, let $x_k$ be a vertex which is either a source or a target in the underlying quiver $\mathcal{Q}$.

Let $\xi_{i_1}, \ldots, \xi_{i_r}$ be the odd vertices connected to $x_k$ by ingoing arrows, and $\xi_{j_1}, \ldots, \xi_{j_s}$ be the odd vertices connected to $x_k$ by outgoing arrows. It follows from Lemma 2.3.3 that all of the even coordinates connected to $x_k$ are split into three groups: those connected to $\xi_i$, those connected to $\xi_j$ and those that are not connected to either $\xi_i$, or to $\xi_j$.

**Definition 4.1.1.** We define the mutation $\hat{\mu}_k$ of the odd coordinates connected to $x_k$ in the following way.

1. The exchange relations are:

   \[
   \xi'_i := x_k\xi_i - \left( \sum_{j \in J_k} \xi_j \right) \prod_{\xi_i \rightarrow x_{\ell}} x_{\ell},
   \]

   \[
   \xi'_j := -x'_k\xi_j + \left( \sum_{i \in I_k} \xi_i \right) \prod_{\xi_j \rightarrow x_m} x_m,
   \]

   where $x'_k$ is as in (2).

2. The mutation $\hat{\mu}_k$ inverses all the arrows at $\xi_i$ and $\xi_j$:

   All the other arrows remain unchanged.

Sometimes, we will use the notation $\hat{\mu}_k(\xi_i)$ and $\hat{\mu}_k(\xi_j)$ for $\xi'_i$ and $\xi'_j$, respectively.

---

2 “For if I don’t know what something is, how could I know what it’s like?” Sokrates’ reasoning from Plato’s “Meno”.
Proposition 4.1.2. (i) The odd mutation is an involution, i.e., \( \hat{\mu}_k^2(\xi) = \xi \), for every \( \xi \) connected to \( x_k \).

(ii) Odd and even mutations at the same vertex \( x_k \) anticommute on odd coordinates connected to \( x_k \), and commute on \( x_k \) itself:

\[
(\mu_k \circ \hat{\mu}_k)(\xi) = - (\hat{\mu}_k \circ \mu_k)(\xi) \quad (\mu_k \circ \hat{\mu}_k)(x_k) = (\hat{\mu}_k \circ \mu_k)(x_k)
\]

Proof. Part (i). The iteration of an odd mutation, \( \hat{\mu}_k^2 \), sends a variable \( \xi_i \) connected to \( x_k \) by ingoing arrow to

\[
\xi_i'' = - \xi_i x_k + \xi_i \prod x_\ell
\]

\[
= (\xi_i x_k - \xi_j \prod x_\ell) x_k' + (\xi_i x_k' - \xi_i \prod x_m) \prod x_\ell
\]

\[
= \xi_i (x_k x_k' - \prod x_\ell \prod x_m)
\]

\[
= \xi_i,
\]

since \( x_k x_k' = 1 + \prod x_\ell \prod x_m \). One proves that \( \xi_j'' = \xi_j \) in a similar way.

Part (ii): straightforward. \( \square \)

Example 4.1.3. Consider again the elementary quiver from Example 2.5.1. Performing one of the simultaneous mutations \( \mu_x \circ \hat{\mu}_x \), or \( \hat{\mu}_x \circ \mu_x \), one obtains:

\[
\xi_1 \xrightarrow{\mu} \xi_2 \quad \Rightarrow \quad \xi_1 \xrightarrow{\hat{\mu}} \xi_2
\]

where

\[
\xi_1' = x \xi_1 - \xi_2, \quad \xi_2' = \frac{2 \xi_2}{x} + \xi_1,
\]

under the mutation \( \mu_x \circ \hat{\mu}_x \), and

\[
\xi_1' = -x \xi_1 + \xi_2, \quad \xi_2' = \frac{2 \xi_2}{x} - \xi_1,
\]

under the mutation \( \hat{\mu}_x \circ \mu_x \). Iterating the procedure, one obtains \( \xi_1'' = \xi_1, \xi_2'' = \xi_2 \), but the even variable \( x \) is, again, not periodic.

4.2. Periodic sequences of mutations. Let us now consider the following path on the exchange graph:

\[
\xi_1 \xrightarrow{\mu} \xi_1' \xrightarrow{\hat{\mu}} \xi_1'' \xrightarrow{\mu} \xi_1'''
\]

where \( \xi_1' = x \xi_1 - \xi_1 \), and then, in accordance with (2) and (4), one has:

\[
\xi_1^{(l+1)} = x^{(l)} \xi_1^{(l)} - \xi_1^{(l-1)}, \quad x^{(l+1)} = \frac{1}{x^{(l)}} \left( 2 + \xi_1^{(l-2)} \xi_1^{(l-1)} \right)
\]

is 4-(anti)periodic. More precisely, one can check that

\[
\xi_2^{(l+4)} = -\xi_2^{(l)}, \quad x^{(l+2)} = x^{(l)}.
\]

One can construct another 4-(anti)periodic path iterating mutations of \( \xi_1 \).
5. Superfriezes and cluster superalgebras of type $A$

Among the simplest examples of cluster algebras, Coxeter’s frieze patterns occupy a special place thanks to various relations to algebra, geometry and combinatorics. The collection of all Coxeter’s friezes is an algebraic variety such that its algebra of regular functions is the cluster algebra associated to the quiver $A_n$.

The notion of \textit{superfrieze} was introduced in [32] as generalization of Coxeter’s frieze patterns. The collection of all superfriezes is an algebraic supervariety isomorphic to the supervariety of super-version of Hill’s (or one-dimensional Schrödinger) equations [30].

In this section, we describe the structure of cluster superalgebra on the supervariety of superfriezes. The underlying quiver will be $A_n$, the extended quiver $\tilde{Q}$ is obtained by adding $n + 1$ odd coordinates. We start with the description of the supergroup $\text{OSp}(1|2)$ that is used to develop the theory of superfriezes.

5.1. The supergroup $\text{OSp}(1|2)$. One of the first examples of cluster algebras given in [11] is the algebra of regular functions on the Lie group $\text{SL}(2)$. The most elementary superanalog of the group $\text{SL}(2)$ is the supergroup $\text{OSp}(1|2)$. For more details about properties and applications of this supergroup, see [25].

Let $R = R_0 \oplus R_1$ be a commutative ring. The set of $R$-points of the supergroup $\text{OSp}(1|2)$ is the following $3|2$-dimensional supergroup of matrices:

\[
\begin{pmatrix}
  a & b & \gamma \\
  c & d & \delta \\
 \alpha & \beta & e
\end{pmatrix}
\]

such that

\[
\begin{align*}
ad &= 1 + bc - \alpha \beta, \\
e &= 1 + \alpha \beta, \\
\gamma &= a \beta - b \alpha, \\
\delta &= c \beta - d \alpha.
\end{align*}
\]

The elements $a, b, c, d, e \in R_0$, and $\alpha, \beta, \gamma, \delta \in R_1$; these elements are generators of the algebra of regular functions on $\text{OSp}(1|2)$.

Choose the initial cluster coordinates $(a, b, c, \alpha, \beta)$, and consider the following quiver:

\begin{center}
\begin{tikzpicture}
  \node (a) at (0,0) {$a$};
  \node (b) at (-1,-1) {$b$};
  \node (c) at (1,-1) {$c$};
  \node (alpha) at (-1,1) {$\alpha$};
  \node (beta) at (1,1) {$\beta$};
  \draw[->] (a) -- (b);
  \draw[->] (a) -- (c);
  \draw[->] (a) -- (alpha);
  \draw[->] (a) -- (beta);
\end{tikzpicture}
\end{center}

The coordinate $d$ is then the mutation of $a$, i.e., $a' = d$. Indeed, the exchange relation (2) for the coordinate $a$ reads

\[
aa' = 1 + bc + \beta \alpha,
\]

which is precisely the first equation for $\text{OSp}(1|2)$ relating $a$ and $d$. Furthermore, the odd exchange relations calculated according to formula (4) are:

\[
\beta' = a \beta - b \alpha, \quad \alpha' = -d \alpha + c \beta.
\]

These are precisely the expressions of $\gamma$ and $\delta$, so that $\hat{\mu}_a(\beta) = \gamma$ and $\hat{\mu}_a(\alpha) = \delta$. Therefore, the superalgebra of the above quiver contains a subalgebra generated by $a, b, c, d, \alpha, \beta, \gamma, \delta$.

Note however that, similarly to the $\text{SL}_2$-case (where the coordinates $b$ and $c$ are frozen cf. [11]), the full cluster superalgebra of the above quiver is bigger than the algebra of regular functions on $\text{OSp}(1|2)$. 
5.2. The definition of a superfrieze and the corresponding superalgebra. Similarly to the case of classical Coxeter’s friezes, a superfrieze is a horizontally-infinite array bounded by rows of 0’s and 1’s. Even and odd elements alternate and form “elementary diamonds”; there are twice more odd elements.

Definition 5.2.1. A superfrieze, or a supersymmetric frieze pattern, is the following array

\[
\begin{array}{cccccc}
\ldots & 0 & 0 & \ldots & 0 & 0 \\
\ldots & 0 & 0 & \ldots & 0 & 0 \\
1 & 1 & \ldots & 1 \\
\varphi_{0,0} & \varphi_{1,0} & \varphi_{1,1} & \varphi_{2,1} & \varphi_{2,2} & \ldots \\
f_{0,0} & f_{1,1} & f_{2,2} \\
\varphi_{-1,0} & \varphi_{0,1} & \varphi_{1,2} & \varphi_{2,2} & \ldots \\
f_{-1,0} & f_{0,1} & f_{1,2} & \ldots \\
f_{2-m,1} & \ldots & \ldots & \ldots \\
\ldots & \varphi_{2-m,2} & \ldots & \varphi_{0,m} & \varphi_{1,m+1} & \varphi_{2,m+1} & \ldots \\
1 & 1 & \ldots & 1 \\
\ldots & 0 & 0 & \ldots & 0 & 0 & 0 \\
\ldots & 0 & 0 & \ldots & 0 & \ldots \\
\end{array}
\]

where \( f_{i,j} \in \mathbb{R}_0 \) and \( \varphi_{i,j} \in \mathbb{R}_1 \), and where every elementary diamond:

\[
\begin{array}{cc}
B & \\
\Xi & \Psi \\
A & D \\
\Phi & \Sigma \\
C & \\
\end{array}
\]

satisfies the following conditions:

\[
\begin{align*}
AD - BC &= 1 + \Sigma \Xi, \\
B \Phi - A \Psi &= \Xi, \\
B \Sigma - D \Xi &= \Psi,
\end{align*}
\]

(6)

that we call the frieze rule.

The integer \( m \), i.e., the number of even rows between the rows of 1’s is called the width of the superfrieze.

The last two equations of (6) are equivalent to

\[
\begin{align*}
A \Sigma - C \Xi &= \Phi, \\
D \Phi - C \Psi &= \Sigma.
\end{align*}
\]
Note also that these equations also imply \( \Xi \Sigma = \Phi \Psi \), so that the first equation can also be written as follows: \( AD - BC = 1 - \Xi \Sigma \).

One can associate an elementary diamond with every element of \( OSp(1|2) \) using the following formula:

\[
\begin{pmatrix}
-\alpha & \beta \\
\gamma & \delta \\
\end{pmatrix}
\leftrightarrow
\begin{pmatrix}
-t & t' \\
-s & s' \\
\end{pmatrix}
\]

so that the relations (5) and (6) coincide.

Consider also the configuration:

\[
\begin{array}{cccccc}
\Phi & \Xi & \Psi & \Sigma \\
\bar{\Phi} & \bar{\Xi} & \bar{\Psi} & \bar{\Sigma} \\
\end{array}
\]

The frieze rule (6) then implies

\[ B (\Phi - \bar{\Phi}) = A (\Psi - \bar{\Psi}), \quad B (\Sigma - \bar{\Sigma}) = D (\Xi - \bar{\Xi}). \]

**Definition 5.2.2.** The supercommutative superalgebra generated by all the entries of a superfrieze will be called the algebra of a superfrieze.

5.3. **Examples: superfriezes of width 1 and 2.** The most general superfrieze of width \( m = 1 \) is of the following form:

\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 \\
\xi & \xi & \xi' & \xi & \xi & \xi' \\
x & x' & x & x & x & x' \\
\xi - x\eta & x\eta - \xi & \eta & \eta & \eta & \eta \\
1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

where

\[ x' = \frac{2}{x} + \frac{\eta \xi}{x}, \quad \xi' = \eta - \frac{2}{x} \xi. \]

One can choose local coordinates \((x, \xi, \eta)\) to parametrize the supervariety of superfriezes. The value of \( x' \) precisely corresponds to the mutation of \( x \) in Example 2.5.1. The values of odd values
of the above superfrieze also correspond to the mutations of the initial odd coordinates $\xi, \eta$, cf. Example 4.1.3.

The next example is a superanalog of the Gauss "Pentagramma mirificum":

\[
\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\ldots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \\
\xi^* & \xi & \xi & \xi' & \xi' & 1 & 1 & \zeta^* & \zeta^* \\
y' & x & x' & x'' & y & \\
-\eta' & \eta^* & 2 & \eta & 2' & \eta' & \eta^* & -\eta & \eta \\
x'' & y & y' & x & x' & \\
1 & -1 & \zeta^* & -\zeta^* & \zeta & -\zeta & \zeta' & -\zeta' & -\zeta' \\
1 & 1 & 1 & 1 & 1 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\]

The frieze is defined by the initial values $(x, y, \xi, \eta, \zeta)$, the next values are easily calculated using the frieze rule:

\[
x' = \frac{1 + y}{x} + \frac{\eta \xi}{x}, \quad y' = \frac{1 + x + y}{xy} + \frac{\eta \xi}{xy} + \frac{\zeta \eta}{y}.
\]

Note that these formulas coincide with those of mutations of coordinates $x_1, x_2$ in Example 2.5.5.

One then calculates:

\[
x'' = \frac{1 + y'}{x'} + \frac{\eta' \xi'}{x'}, \quad \frac{1 + x}{y} + \frac{\eta' \xi}{y} + \frac{\xi' \zeta}{y}.
\]

For the odd coordinates, one has:

\[
\xi' = \eta - x' \xi = \eta - \frac{1 + y}{x} \xi, \quad \eta' = \zeta - y' \xi = \zeta - \frac{1 + x + y}{xy} \xi - \frac{\xi \eta \zeta}{y}, \quad \zeta' = -\xi.
\]

On the other side of the initial diagonal,

\[
\zeta^* = \eta - y \zeta, \quad \eta^* = \xi - x \zeta, \quad \xi^* = -\zeta.
\]

Furthermore,

\[
1 = \frac{(1 + x)}{y} \eta - \xi - \zeta, \quad 2 = xy - y \xi, \quad 2' = x' \zeta - y' \eta = \frac{1 + y}{x} \zeta - \frac{1 + x + y}{xy} \eta - \frac{\xi \eta \zeta}{x}.
\]

It is easy to check that all the entries of above superfriezes can be obtained as mutations of the initial coordinates $(x, y, \xi, \eta, \zeta)$ and the initial quiver

\[
\begin{array}{c}
\xi \\
x \downarrow \\
\eta \\
\downarrow y \uparrow \zeta
\end{array}
\]

see Example 2.5.5 and Theorem 3 below.
5.4. **Properties of superfriezes.** The main properties of superfriezes are similar to those of the classical Coxeter friezes, see [32]. The main property is that of glide symmetry and (anti)periodicity:

\[ f_{i,j} = f_{j-m-1,i-2}, \quad \varphi_{i,j} = \varphi_{j-m-\frac{1}{2},i-\frac{1}{2}}, \quad \varphi_{i+j+1,j+1} = -\varphi_{j-m-1,i-1}, \]

this implies, in particular:

\[ \varphi_{i+n,j+n} = -\varphi_{i,j}, \quad f_{i+n,j+n} = f_{i,j}, \]

for all \( i, j \in \mathbb{Z} \). The collection of superfriezes of width \( m \) is an algebraic supervariety of superdimension \( m|m+1 \).

The second main property of superfriezes is the Laurent phenomenon. It was proved in [32] that entries of a superfrieze are Laurent polynomials in the entries from any of its diagonals.

The relation to difference equations is as follows. The entries of the South-East diagonal of every superfrieze are solutions to the following equation:

\[
\begin{pmatrix}
V_{i-1} \\
V_i \\
W_i
\end{pmatrix} = A_i
\begin{pmatrix}
V_{i-2} \\
V_{i-1} \\
W_{i-1}
\end{pmatrix}, \quad \text{where} \quad A_i = \begin{pmatrix}
0 & 1 & 0 \\
-1 & a_i & -\beta_i \\
0 & \beta_i & 1
\end{pmatrix},
\]

where the coefficients \( a_i, \beta_i \) are in the first two rows of the superfrieze. Note that the matrix \( A_i \) belongs to the set of \( R \)-points of the supergroup \( OSp(1|2) \). The monodromy matrix of this equation is:

\[
M_i = \begin{pmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

Periodicity properties show that the algebra of any superfrieze is finitely generated.

5.5. **Superfriezes and cluster superalgebras.** Let us now describe the cluster structure of the supervariety of superfriezes. Consider the following quiver with \( m \) even and \( m + 1 \) odd vertices:

![Quiver diagram](image)

and the corresponding cluster superalgebra.

**Theorem 3.** The algebra of a superfrieze of width \( m \) is a subalgebra of the cluster superalgebra corresponding to the above quiver.
Proof. Choose the following entries of the superfrieze on parallel diagonals:

\[
\begin{array}{ccc}
1 & 1 & \\
\ast & \xi_1 & \ast & \xi_1'
\end{array}
\]

\[
\begin{array}{ccc}
x_1 & x_1' & \\
\ast & \xi_2 & \ast & \xi_2'
\end{array}
\]

\[
\begin{array}{ccc}
x_2 & \cdots & x_2' \\
\ast & \xi_3 & \ast & \xi_3'
\end{array}
\]

\[
\begin{array}{ccc}
x_m & x_m' & \\
\ast & \xi_{m+1} & \ast & \xi_{m+1}'
\end{array}
\]

The entries \(\{x_1, \ldots, x_m, \xi_1, \ldots, \xi_{m+1}\}\) determine all other entries of the superfrieze, and can be taken for initial coordinates. Our goal is to calculate the entries \(\{x_1', \ldots, x_m', \xi_1', \ldots, \xi_{m+1}'\}\) and show that these entries also belong to the cluster superalgebra \(A(\widetilde{Q})\) of the quiver (7).

Using the frieze rule (6), one obtains the following recurrent formula:

\[
x_k x_k' = 1 + x_{k+1} x_{k-1}' + \xi_{k+1} \xi_k.
\]

On the other hand, let us perform consecutive mutations at vertices \(x_1\), and then at \(x_2, x_3, \ldots, x_m\) of the quiver (7). (Note that, according to Lemma 2.3.3, this is the only allowed sequence of even mutations.) After the \((k-1)\)st step, one obtains the following quiver:

Therefore, the mutation at \(x_k\) is allowed, and the exchange relation for \(x_k\) is exactly the same as the recurrent formula (8) for \(x_k'\). We have proved that the values of the entries \(\{x_1', \ldots, x_m'\}\) in the frieze coincide with the coordinates \(\{x_1', \ldots, x_m'\}\) of the quiver (7) after the iteration of even mutations.

Note that after \(m\) consecutive mutations at even vertices, the quiver (7) becomes as follows:

Consider now for the odd entries of the superfrieze \(\{\xi_1', \ldots, \xi_{m+1}'\}\), and let us proceed by induction.

For the first of the odd entries, one has:

\[
\xi_1' = \xi_2 - x_1' \xi_1.
\]
Indeed, the frieze rule implies that the entry between $\xi_1$ and $\xi'_1$ (previously denoted by $\ast$) is also equal to $\xi'_1$, i.e., we have the following fragment of the superfrieze:

$$
\begin{array}{cccc}
1 & 1 \\
\xi_1 & \xi'_1 & \xi'_1 \\
x_1 & x'_1 & \ast \\
& & \xi_2 \\
\end{array}
$$

The above expression for $\xi'_1$ is just the third equality in (6). It follows that $\xi'_1$ belongs to the cluster superalgebra $A(\tilde{Q})$, since it coincides (up to a sign) with the mutation $\hat{\mu}_{x'_1}(\xi_1)$.

It was proved in [32] that the entries on the diagonals of the superfrieze satisfy recurrence equations with coefficients standing in the first two rows. In particular, Lemma 2.5.3 of [32] implies the following recurrence for the odd entries of the superfrieze:

$$
\xi'_k - \xi'_{k-1} = -\xi_1 x'_k,
$$

for all $k$. One concludes, by induction on $k$, that all of the entries $\{\xi'_1, \ldots, \xi'_{m+1}\}$ belong to the cluster superalgebra $A(\tilde{Q})$. Again, using the induction one arrives at the same conclusion for all parallel diagonals.

Finally, one proves in a similar way that the entries in-between, denoted by $\ast$, also belong to the cluster superalgebra $A(\tilde{Q})$. $\Box$

The quiver (7) is the most symmetric extension of the standard Dynkin quiver $A_m$.

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