Characterization of Determinantal Bivariate Polynomials

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Abstract.

A polynomial which can be expressed as the determinant of a definite (monic) symmetric/Hermitian linear matrix polynomial is known as determinantal polynomial in this paper. These polynomials play a crucial role in semidefinite programming problems. Helton-Vinnikov proved that real zero (RZ) bivariate polynomials are determinantal. However, it leads to a challenging problem to compute such a determinantal representation. We provide a necessary and sufficient condition for the existence of determinantal representation of a bivariate polynomial by identifying its coefficients as scalar products of two vectors where the scalar products are defined by special type of matrices. This intrinsic condition which is different from RZ property of a polynomial enables us to develop a method to compute a monic symmetric/Hermitian determinantal representations for a bivariate polynomial of degree d. In addition, we propose a relaxation to the original problem which turns into a problem of expressing vector coefficient of given polynomial as convex combinations of some specified points. Moreover, we characterize the range set of vector coefficients of a certain type of determinantal bivariate polynomials and it is shown that this set attains its maximum and minimum at specified points.

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1 Introduction

One of the objectives in convex algebraic geometry is to characterize convex semi-algebraic sets which are definite LMI representable sets. A set $S \subseteq \mathbb{R}^n$ is said to be $LMI$ representable if

$$S = \{ x \in \mathbb{R}^n : L := A_0 + L(x) := A_0 + x_1 A_1 + x_2 A_2 + \cdots + x_n A_n \succeq 0 \}$$

(1)

for some real symmetric matrices $A_i$, $i = 0, \ldots, n$ and $x = (x_1, \ldots, x_n)^T$. If $A_0 \succ 0$, the set $S$ is called a definite $LMI$ representable set whereas if $A_0 = I$, $S$ is known as a monic $LMI$ representable set. By $A \succ 0$ ($\succeq 0$) we mean that the matrix $A$ is positive (semi)-definite. A

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spectrahedron which is the feasible set of a semidefinite programming (SDP) problem is another term used for a LMI representable set.

An approximate optimal solution of a SDP can be found by applying interior point methods \cite{NN94, BGFB} when the SDP is strictly feasible. The assumption of strict feasibility of a SDP problem is equivalent to the assumption that its feasible set has nonempty interior. It is proved that if the set $S$ has non-empty interior, the constant coefficient matrix $A_0$ can be chosen to be positive definite \cite{Ram95, section 1.4, Nt12}. So, if the feasible set of an optimization problem is a definite linear matrix inequality (LMI) representable set, the optimization problem can be transformed into a SDP problem \cite{Ram95, HV07}. The technique of converting optimization problems into semidefinite programming (SDP) problems arise in control theory, signal processing and many other areas in engineering.

If a polynomial $f(x) \in \mathbb{R}[x]$ is a determinantal polynomial, i.e.,
\begin{equation}
  f(x) = \det(A_0 + x_1A_1 + x_2A_2 + \cdots + x_nA_n),
\end{equation}
where coefficient matrices $A_i$ of linear matrix polynomial are symmetric/Hermitian of some order and the constant coefficient matrix $A_0$ is positive definite, then the algebraic interior associated with $f(x)$ i.e., the closure of a (arcwise) connected component of $\{x \in \mathbb{R}^n : f(x) > 0\}$ is a spectrahedron \cite{10}. Thus one of the successful techniques to deal with characterizing definite LMI representable sets is to characterize determinantal polynomials. So, we focus on definite (monic) symmetric/Hermitian determinantal representation in order to accomplish the connection between determinantal polynomials and semidefinite programming (SDP) problems.

The determinantal polynomials are of special kind of polynomial, called real zero (RZ) polynomials \cite{HV07}. A multivariate polynomial $f(x) \in \mathbb{R}[x]$ is said to be a real zero (RZ) polynomial if the polynomial has only real zeros when it is restricted to any line passing through origin i.e., for any $x \in \mathbb{R}^n$, all the roots of the univariate polynomial $f_x(t) := f(t \cdot x)$ are real and $f(0) \neq 0$. The polynomial $f(x)$ is called strictly RZ if all these roots are distinct, for all $x \in \mathbb{R}^n, x \neq 0$. Thus if a polynomial $f(x)$ is determinantal then it is indeed a RZ polynomial.

Helton-Vinnikov have proved that a RZ bivariate polynomial $f(x_1, x_2)$ always admits monic Hermitian as well as symmetric determinantal representations of size $d$ \cite{HV07}. So, RZ property is a necessary and sufficient condition for a bivariate polynomial to be a determinantal polynomial. The homogenized version of this result is known as Lax conjecture \cite{LPR05}. However, if the number of variables of a RZ polynomial is more than 2, it may not be a determinantal polynomial at all. For example, dehomogenized polynomial of Vamos cube $V_8$ is a RZ polynomial without a definite determinantal representation \cite{Bra11}. This leads to the generalized Lax conjecture, for details see \cite{Vin12, KPV15, SP15, NS15}. Note that the homogenized version of RZ polynomials are known as hyperbolic polynomials. The problem of determinantal representations of a polynomial is originated in algebraic geometry and for a systematic interpretation one can see \cite{B+00}.

The authors in \cite{HV07} have provided explicit expressions of the coefficient matrices of a symmetric determinantal representation in terms of theta functions, and the period matrix of the curve $f(x_1, x_2) = 0$ when the curve is defined by a strictly RZ bivariate polynomial $f(x_1, x_2)$. Indeed, it is not easy to compute determinantal representation numerically or symbolically using this method. Later, the problem of computing monic symmetric/Hermitian determinantal representation for a strictly RZ bivariate polynomial has been widely studied, for example one can see \cite{Dix02, PSV12, Hen10, GKVW13}.
In this paper, we provide a necessary and sufficient condition for the existence of monic symmetric/Hermitian determinantal representation of a bivariate polynomial that is different from the RZ property of a bivariate polynomial. Moreover, this necessary and sufficient condition for the existence of determinantal representation of any bivariate polynomial of degree \( d \) provides the means of computing a monic symmetric/Hermitian determinantal representation of size \( d \) if it exists. The order of the coefficient matrices is called the size of determinantal representation and the size must be greater than or equal to \( \deg(f) \). It is also known that a definite LMI representable set is always monic LMI representable \([HV07]\). So, we focus on monic symmetric/Hermitian determinantal representations of size \( d \) of bivariate polynomials.

We propose a representation for the vector coefficient of a determinantal bivariate polynomial in terms of the symmetric/Hermitian coefficient matrices of corresponding determinantal representation of that polynomial. We prove that each coefficient of a bivariate determinantal polynomial can be written as the scalar product of two vectors. In fact, these two vectors are constructed from the eigenvalues of the symmetric/Hermitian coefficient matrices of a determinantal representation of the given polynomial if representation exists. The matrices which define the scalar products are obtained as (complex) Hadamard product of exterior powers of an orthogonal (unitary) matrix \( V \) with themselves. We prove that these matrices are orthostochastic (resp. unistochastic) matrices corresponding to a monic symmetric/Hermitian determinantal representation. Consequently, this alternative representation characterize determinantal bivariate polynomials as well as RZ bivariate polynomials.

It is clear that checking RZ property of a polynomial is not an easy task using the definition of RZ polynomial. Another method to decide RZ property of a polynomial is by checking positive semi-definiteness of a parametrized Hermite matrix \([NPt13]\) which is also cumbersome. The method proposed in this paper involves finding roots of univariate polynomials, solving systems of linear equations followed by checking whether a point which is the vector coefficient of a bivariate polynomial can be written as convex combinations of some specified points satisfying certain pattern. This certain pattern helps us to construct a monic symmetric/Hermitian determinantal representation of size \( d \) for a bivariate polynomial of degree \( d \).

Without the requirement of this certain pattern the problem of computing determinantal representation turns into a problem of deciding whether a certain point is inside the convex hull of some specified points. Indeed, this is a relaxation to the original problem which can be numerically and symbolically solved in Matlab.

It is stated in \([HV07]\) and shown in \([LP17]\) that a polynomial \( f(x) \in \mathbb{R}[x] \) is strictly RZ if and only if the variety \( \mathcal{V}_C(f) \) has no real singular points. The variety \( \mathcal{V}_C(f) \) has real singular points means that \( \nabla f(x_1, x_2) = 0 \) for some non-zero \((x_1, x_2) \in \mathbb{R}^2\). So, the curve defined by such a polynomial is associated with non-smooth/singular Helton-Vinnikov curve. To the best of our knowledge, if the associated variety \( \mathcal{V}_C(f) \) defined by bivariate RZ polynomial \( f(x_1, x_2) \) has real singular points, there is nothing known other than existence of monic symmetric/Hermitian determinantal representation. In this paper, we deal with this issue and study determinantal quartic bivariate polynomials in details.

Finally, we prove that the vector coefficients of the class of certain type of determinantal bivariate polynomials of degree \( d \) lies inside a convex hull of \( d! \) specified points.

Monic symmetric determinantal representation is abbreviated as MSDR and monic Hermitian determinantal representation is abbreviated as MHDR in this paper.
2 Preliminaries

We first briefly recall some basic definitions and facts that will be used in sequel. A doubly stochastic matrix is a square matrix whose entries are nonnegative and the sum of the elements in each row and each column is unity. An orthostochastic matrix is a doubly stochastic matrix whose entries are the squares of the entries of some orthogonal matrix. A unistochastic matrix is a doubly stochastic matrix whose entries are the squares of the absolute values of the entries of some unitary matrix.

A bilinear form on $\mathbb{R}^n$ is a map from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}$ defined by $(x, y) \mapsto (x, y)_Q = x^T Q y$ where the matrix $Q \in \mathbb{K}^{n \times n}$ is associated with the bilinear form for all $x, y \in \mathbb{R}^n$. The matrix $Q$ is known as the defining matrix of the bilinear form. The form is said to be non degenerate when $Q$ is nonsingular. This is also known as scalar product.

In order to explain what is meant by $k$-th exterior power of an orthogonal matrix which will be used in sequel we need to discuss some preliminaries [26, 28].

**Definition 2.1.** The exterior product of $k$ copies of a vector space $E^d$ over the field $E$ (also called the $k$-th exterior power of $E^d$) is denoted by $\wedge^k E^d$ and we also define $\wedge^0 E^d = E, \wedge^1 E^d = E^d$ and $\wedge^k E^d = \text{a vector space containing only a zero vector if } k > d$. Obviously, $\wedge^k E^d$ is a vector space over the field $E$, it is also defined as the subspace of totally antisymmetric tensors within $E^d \otimes \cdots \otimes E^d$. In the concise notation, this is the space spanned by expressions of the form

$$v_1 \wedge \cdots \wedge v_d =: \wedge^d v_j, v_j \in E^d$$

assuming the properties of wedge product (linearity and anti-symmetry). Elements (tensors) from the space $\wedge^k E^d$ are also called $k$-vectors or antisymmetric tensor of rank $k$. Any $x \in \wedge^k(E^d)$ is known as a $k$ blade or pure $k$ vector or totally decomposable $k$ vector if it is of the form $x = v_1 \wedge \cdots \wedge v_k$ for some $v_1, \ldots, v_k \in E^d$.

The exterior product of $k$ number of vectors $e_{i_k}$ in $E^d$, denoted by $e_{i_1} \wedge \cdots \wedge e_{i_k}$ is a function from $\{(j_1, j_2, \ldots, j_k) : j_1 < \cdots < j_k, j_k \in \{1, \ldots, d\}\}$ into $E$ defined by

$$e_{i_1} \wedge \cdots \wedge e_{i_k}(j_1, j_2, \ldots, j_k) = \det(\begin{bmatrix} e_{i_1}(j_1) & e_{i_1}(j_2) & \cdots & e_{i_1}(j_k) \\ e_{i_2}(j_1) & e_{i_2}(j_2) & \cdots & e_{i_k}(j_k) \\ \vdots & \vdots & \ddots & \vdots \\ e_{i_k}(j_1) & e_{i_k}(j_2) & \cdots & e_{i_k}(j_k) \end{bmatrix}) \quad \text{for } k = 1, 2, \ldots, d. \quad (3)$$

Let $W, U$ be two real vector spaces. An $d$-linear map $f : W \times W \cdots \times W \rightarrow U$ is said to be antisymmetric or alternating if

$$f(w_1, \ldots, w_i, w_{i+1}, \ldots, w_d) = -f(w_1, \ldots, w_{i+1}, w_i, \ldots, w_d), \forall w_i \in W,$$

that is, if $f$ changes sign while any two adjacent entries are switched. If the vector space $U = \mathbb{R}$, it is known as alternating form. Consequently, the exterior product is an alternating multilinear map by Equation (3). Obviously, the determinant

$$\det : E^d \times \cdots \times E^d \rightarrow E, (v_1, \ldots, v_d) \mapsto \det(v_1, \ldots, v_d)$$

is also an alternating multilinear map.

If $\{e_1, e_2, \ldots, e_d\}$ is the standard basis of $E^d$, then the set of $k$-vectors $e_I := e_{i_1} \wedge \cdots \wedge e_{i_k}$ where $I = \{(i_1, i_2, \ldots, i_k) : 1 \leq i_1 < \cdots < i_k \leq d\}$ is a basis of $\wedge^k E^d$ for $k = 1, \ldots, d$ and
thus any element in $\wedge^k \mathbb{E}^d$ which is nothing but a $k$ vector (sum of $k$ blades) can be uniquely written as

$$a = \sum_{I \subseteq \{1, \ldots, d\}, |I| = k} a_I e_I = \sum_{1 \leq i_1 < \cdots < i_k \leq d} a_{i_1 \ldots i_k} (e_{i_1} \wedge \cdots \wedge e_{i_k}).$$

For $k = 1, \ldots, d$, the norm of $a$ is $|a| = \sqrt{\sum_{i_1 < \cdots < i_k} (a_{i_1 \ldots i_k})^2}$ and we set $|a| = 0, k > d$. The homogeneous coordinates $a_I$s are known as Plücker coordinates on $\mathbb{P}(\wedge^k \mathbb{E}^d)$ associated with the ordered basis $(e_1, \ldots, e_d)$ of $\mathbb{E}^d$. Naturally, if $a_1, \ldots, a_d \in \mathbb{E}^d$, then

$$a_1 \wedge \cdots \wedge a_d = \det(a_1^T, \ldots, a_d^T) e_1 \wedge \cdots \wedge e_d$$

The collection of the spaces $\wedge^k (\mathbb{E}^d)$, for $k = 0, 1, 2, \ldots$, together with the exterior product or operation $\wedge$ is called the Exterior algebra or Grassmannian algebra on $\mathbb{E}^d$. So, we have $\wedge (\mathbb{E}^d) = \bigoplus_{k=0}^{\infty} \wedge^k (\mathbb{E}^d) = \bigoplus_{k=0}^{d} \wedge^k (\mathbb{E}^d)$, as $\wedge^k (\mathbb{E}^d) = 0$, if $k > d$.

The set of all $d \times d$ orthogonal (resp. unitary) matrices is the orthogonal (resp. unitary) group $O(d)(U(d))$. The group $O(d)$ (resp. $U(d)$) can be identified by the set of its $d$-tuple of ordered column vectors. Mathematically,

$$O(d) = \{ V := (v_1, \ldots, v_d) \in \mathbb{R}^d \times \cdots \times \mathbb{R}^d, v_i^T v_j = \delta_{ij} \}$$

$$U(d) = \{ U := (u_1, \ldots, u_d) \in \mathbb{C}^d \times \cdots \times \mathbb{C}^d, u_i^* u_j = \delta_{ij} \}$$

where $\delta_{ij}$ is the Kronecker delta. In our context the $k$-th exterior power of a matrix $V := (v_1, \ldots, v_d)$ can be identified by the set of its $(\binom{d}{k})$-tuple of ordered column vectors, denoted by $V^{\wedge k}$ where each column of $V^{\wedge k}$ is equal to $(v_{i_1} \wedge \cdots \wedge v_{i_k} =: \wedge^k v_{ij})$ and $\{ (i_1, \ldots, i_k) : i_1 < \cdots < i_k, i_j \in \{1, \ldots, d\} \}$ is an ordered $k$ tuple set.

### 3 Computing Determinantal Representations

In this section, we provide a necessary and sufficient condition for the existence of an MSDR (MHDR) of size $d$ of a bivariate polynomial of degree $d$ by representing its coefficients as bilinear products of two vectors defined by different matrices. We propose a relaxation to the original problem which turns into a problem of deciding whether a point is inside a polytope. Finally, we develop a method to compute an MSDR of size $d$ of a bivariate polynomial which need not be a strictly RZ polynomial (that means repeated eigenvalues of coefficient matrices are allowed). We exemplified our method in details for a quartic bivariate polynomial.

#### 3.1 Determining the Eigenvalues of Coefficient Matrices

First we recall some facts about determinantal multivariate polynomials from [7, Chap-4]. Since the coefficient matrices $A_i$s are Hermitian (symmetric), therefore by the spectral theorem of a Hermitian (symmetric) matrix there exist a unitary (orthogonal) matrix $U_i$ such that $A_i = U_i^* D_i U_i$ for all $i = 1, \ldots, n$ where $D_i$ is a diagonal matrix whose diagonal entries are the eigenvalues of $A_i$. So, one can always find a suitable unitary (orthogonal) matrix $U$ such that one of the coefficient matrices becomes diagonal. Without loss of generality, it is enough to consider coefficient matrix $A_1$ associated to $x_1$ as a diagonal matrix $D_1$ and obtain an MHDR (MSDR) of the following form

$$f(x) = \det(I + x_1 D_1 + x_2 X_{12} D_2 X_{12}^* + \cdots + x_n X_{1n} D_n X_{1n}^*) = \det(I + x_1 D_1 + x_2 A_{12} \cdots + x_n A_{1n})$$
where \(X_{ij} := U_{ij}(V_{ij}), i \neq j\) is the transition matrix from \(D_i\) to \(A_{ij} := X_{ij}D_jX_{ij}^*\), and similarly \(X_{ij}^* = X_{ji}, i \neq j\) is the transition matrix from \(D_j\) to \(A_{ji} := X_{ij}^TD_iX_{ij} = X_{ji}D_jX_{ji}\).

Observe that the eigenvalues of the coefficient matrices \(A_{1i}\) are nothing but the entries of the diagonal matrices \(D_i\) for all \(i = 2, \ldots, n\). We explain a technique to determine these diagonal matrices. We take restrictions of the given multivariate polynomial \(f(x)\) along each \(x_i, i = 1, \ldots, n\) that means we restrict the polynomial along one variable at a time by making the rest of the variables zero and generate \(n\) univariate polynomials \(f_{x_i} = f(0, \ldots, x_i, \ldots, 0)\). It is known that if a multivariate polynomial \(f(x)\) admits an MHD (MSDR), it is a RZ polynomial. By recalling the definition of RZ polynomial, we know that for any \(x \in \mathbb{R}^n\), RZ polynomial \(f(x)\) when restricted along any line passing through origin, has only real zeros. So when a RZ polynomial \(f(x)\) restricted along \(x_i, i = 1, \ldots, n\), each of them has only real zeros, i.e., all univariate polynomials \(f_{x_i}\) in \(x_i\) have only real zeros.

As a consequence of this result we have a necessary condition for the existence of an MHD (MSDR) of size equal to the degree of the polynomial for a multivariate polynomial of any degree.

**Lemma 3.1.** If a multivariate polynomial \(f(x) \in \mathbb{R}[x]\) of degree \(d\) has an MHD (MSDR) of size \(d\), then all the roots of \(f_{x_i}\) are real for all \(i = 1, \ldots, n\).

More interestingly, as \(\det(tI + D_i) = t^d f_{x_i}(e_i/t)\) at \(x = e_i\), where \(e_i\) denotes the standard basis vector in \(\mathbb{R}^n\). So, the entries of the diagonal matrices \(D_i\) can be obtained from the roots of \(f_{x_i}\) for all \(i = 1, \ldots, n\) by the following Lemma.

**Lemma 3.2.** The roots of univariate polynomials \(f_{x_i}\) are the negative reciprocal of entries of diagonal matrices \(D_i\) for all \(i = 1, \ldots, n\).

The proof of the Lemma 3.2 can be found in details in [Dey17, Chapter 4].

### 3.2 Representation of Coefficients in terms of Coefficient Matrices

In this subsection, we provide a representation for the coefficients of a determinantal bivariate polynomial in terms of the coefficient matrices of corresponding MHD (MHD). We have exploited this representations to compute such a determinantal representation of a bivariate polynomial.

**Notation:** \([z]\) is a diagonal matrix with diagonal entries filled by vector \(z\). The components of \(u_k\) and \(w_k\) are the the product of \(k\) ordered \((i_1 < \cdots < i_k)\) diagonal elements of \(D_1\) and \(D_2\) respectively. The \(i\)th component \(u^c_{\alpha_1} (i)\) of \(u^c_{\alpha_1}\) is the sum of the product of all possible combinations of \(\alpha_1\) ordered diagonal elements of \(D_1\) except those combinations which involve at least one of the diagonal entries of \(i\)th component of \(u_{\alpha_2}\). Similarly, the \(i\)th component \(w^c_{\alpha_2} (i)\) of \(w^c_{\alpha_2}\) is the sum of the product of all possible combinations of \(\alpha_2\) ordered diagonal elements of \(D_2\) except those combinations which involve at least one of the diagonal entries of \(i\)th component of \(w_{\alpha_1}\). Each component of \(u^c_{\alpha_1}\) and \(w^c_{\alpha_2}\) has \(\frac{(d-\alpha_2)}{\alpha_2}\) and \(\frac{(d-\alpha_1)}{\alpha_1}\) terms respectively.

The Hadamard product of \(k\)-th exterior power of an orthogonal matrix \(V_{12}\) with itself is denoted by \(M^{\wedge k} := (V_{12}^{k} \odot V_{12}^{k})\) for all \(k = 2, \ldots, n\). Thus the \(ij\) element of the matrix \(M^{\wedge k}\) is defined as the square of \(k \times k\) minor of the matrix \(V_{12}^{k} = (\wedge^k v_{ij})\) and \(k \times k\) minors are determinants of matrices of order \(k\) constructed by choosing rows corresponding to \(i\)th component of \(u_k\) and columns corresponding to \(j\)th component of \(w_k\). In particular, \(M = \)
(v_{ij}^2). Similarly, the complex Hadamard product of k-th exterior power of unitary matrix $U_{12}$ with themselves are denoted by $N^{\wedge k} := U_{12}^{\wedge k} \odot (U_{12}^{\wedge k})^*$ for all $k = 2, \ldots, n$.

3.3 Some Properties of matrices $M^{\wedge k}$ and $N^{\wedge k}$

We know that the set $\Omega_d$ of all $d \times d$ doubly stochastic matrices is a closed, bounded, convex set and any $d \times d$ permutation matrix is an extreme point of $\Omega_d$. Let $\Gamma_d$ and $\mathcal{U}_d$ be the set of all $d \times d$ orthostochastic and unistochastic matrices respectively. Consider the functions $h_1 : O(d) \to \Gamma_d$, defined by $h_1(V) = V \odot V$, where $\odot$ is the Hadamard product, and $h_2 : U(d) \to \mathcal{U}_d$, defined by $h_2(U) = U \otimes U^*$, where $\otimes$ is the complex Hadamard product. Also consider functions $f_1^k : O(d) \to O(\binom{d}{k})$, defined by

$$f_1^k(V) = V^{\wedge k} \text{ such that } V := \begin{bmatrix} v_j \end{bmatrix}, v_j \in \mathbb{R}^d, j = 1, \ldots, d;$$

$$V^{\wedge k} := \begin{bmatrix} v_{i_1} \land \ldots \land v_{i_k} \end{bmatrix}, k = 1, 2, \ldots, d, (i_1, \ldots, i_k) \in S[k]$$

and $f_2^k : U(d) \to U(\binom{d}{k})$, defined by

$$f_2^k(U) = U^{\wedge k} \text{ such that } U := \begin{bmatrix} u_j \end{bmatrix}, u_j \in \mathbb{C}^d, j = 1, \ldots, d;$$

$$U^{\wedge k} := \begin{bmatrix} u_{i_1} \land \ldots \land u_{i_k} \end{bmatrix}, k = 1, 2, \ldots, d, (i_1, \ldots, i_k) \in S[k]$$

where $S_d$ is the set of all permutations of $X = \{1, \ldots, d\}$ and $S[k]$ denotes the set of cycles of order $k$ which are chosen from the set $S_d$ such that

$$(i_1, \ldots, i_k) \in S[k] \Rightarrow i_1 < \cdots < i_k.$$  

Note that the functions $h_1$ and $h_2$ are well defined, linear and surjective but not injective. It is easy to see that $h_1(O(d)) = \Gamma_d$ and $h_2(U(d)) = \mathcal{U}_d$. We describe the following well known results in theory of exterior or Grassmannian algebra that will be used in sequel.

**Proposition 3.3.** Let $W$ be a vector space over a field $\mathbb{K}$. Let $W_1, W_2$ be $k$-dimensional subspaces of $W$, with bases $\{x_1, x_2, \ldots, x_k\}$ and $\{y_1, y_2, \ldots, y_k\}$ respectively. Then $x_1 \land x_2 \land \cdots \land x_k = cy_1 \land y_2 \land \cdots \land y_k$ for some non-zero scalar $c \in \mathbb{K}$ if and only if $W_1 = W_2$.

**Proposition 3.4.** Let $W$ be a vector space over $\mathbb{K}$, and $w \in \wedge^k(W)$ be a non-zero vector. Let $W_w = \{x \in V \mid x \land w = 0\}$. Then $\dim W_w \leq k$, and $\dim W_w = k$ if and only if $w$ is a pure $k$ vector.

Now we prove the following Lemma.

**Lemma 3.5.** The function $f_1^k$ is an injective group homomorphism for all $k = 1, 2, \ldots, d$.

**Proof:** The function $f_1^k : O(d) \to O(m), m := \binom{d}{k}$, is a group homomorphism as $f_1^k(VU) = (VU)^{\wedge k} = V^{\wedge k}U^{\wedge k} = f_1^k(V).f_1^k(U)$ for all $k = 1, 2, \ldots, d$ and $V, U \in O(d)$. Say $V^{\wedge k} = I_m, V = (v_1, \ldots, v_d)$. Then

$$V^{\wedge k} = I_m = I_d^{\wedge k} \Rightarrow v_{i_1} \land \cdots \land v_{i_k} = e_{i_1} \land \cdots \land e_{i_k}, \forall k = 1, 2, \ldots, d. \quad (4)$$

By Proposition 3.3, each of the sets $\{v_{i_1}, \ldots, v_{i_k}\}$ and $\{e_{i_1}, \ldots, e_{i_k}\}$ of orthonormal vectors forms basis for the $k$ dimensional vector space. By the Proposition 3.3 they are $k$ blades
or pure \( k \) vectors. Multiplying both sides of the Equations \( (4) \) by \( e_{i_1} \), we have \( v_{i_1} = e_{i_1} \).

We can follow the same method and conclude \( v_{i_1} = e_{i_1}, \forall i_1 = 1, \ldots, d \). So, \( V = I_d \). Here \( \ker f^k_1 = \{ I_d \} \).

We know that a group homomorphism \( g : G \to H \) is injective if and only if \( \ker g \) is only the singleton set \( \{ e_G \} \). So, the function \( f^k_1 \) is an injective function. \( \square \)

One can visualize the above discussion in the following diagram:

\[
\begin{array}{cccc}
O(d) & f^k_1 := \wedge^k & O(m) & f^k_2 := \wedge^k \\
\downarrow h_1 & & \downarrow h_1 & \downarrow h_2 \\
\Gamma_d & . & \Gamma_m & U_d \\
\end{array}
\]

where \( m =: \binom{d}{k} \). However this is not a commutative diagram. Note that we have proved the following.

1. The function \( h_1 \) is surjective, but not injective.
2. The functions \( f^k_1, k = 2, \ldots, d \) are injective, but not surjective.
3. The pre-image of an orthostochastic matrix \( M \) under the map \( h_1 \) is a set of orthogonal matrices \( D_{\pm} V \) such that \( M = V \otimes V \) and \( D_{\pm} \) is a signature matrix which is a diagonal matrix with diagonal entries \( \pm 1 \).

Similarly, we have proved the following.

1. The function \( h_2 \) is surjective, but not injective.
2. The functions \( f^k_2, k = 2, \ldots, d \) are injective, but not surjective.
3. The pre-image of an unistochastic matrix \( N \) under the map \( h_2 \) is a set of unitary matrices \( U_{\pm} V \) such that \( N = U \otimes U \) and \( U_{\pm} \) is a complex signature matrix which is a diagonal matrix with diagonal entries \( e^{i\theta_j} \).

Next we prove that the defining matrices \( M^\wedge^k (N^\wedge^k) \) are orthostochastic (unistochastic) matrices using the following results. Any orthonormal basis of \( \mathbb{E}^d \) generates an orthonormal basis of \( \wedge^k \mathbb{E}^d \) as in the following theorem \([\text{TM01}, \text{Pav17}]\).

**Theorem 3.6.** If \( a_1, \ldots, a_d \) is an orthonormal basis of \( \mathbb{E}^d \) and \( 1 \leq k \leq d \), then the set of \( k \)-vectors of the form \( a_{i_1} \wedge \cdots \wedge a_{i_k} \), where \( i_1 < \cdots < i_k \), is a basis for \( \wedge^k \mathbb{E}^d \) and satisfies

\[
(a_{i_1} \wedge \cdots \wedge a_{i_k}) \cdot (a_{j_1} \wedge \cdots \wedge a_{j_k}) = \begin{cases} 1 & \text{if } (i_1, \ldots, i_k) = (j_1, \ldots, j_k) \\ 0 & \text{otherwise.} \end{cases}
\]

Note that the set of columns of an orthogonal matrix \( V \in O(d) \) which forms an orthonormal basis of the Euclidean space \( \mathbb{R}^d \) generates an orthonormal basis by identifying the columns of \( V^\wedge^k \) of the vector space \( \wedge^k \mathbb{R}^d \). Similarly, the set of columns of a unitary matrix \( U \in U(d) \) which forms an orthonormal basis of the vector space \( \mathbb{C}^d \) generates an orthonormal basis by identifying the columns of \( U^\wedge^k \) of the vector space \( \wedge^k \mathbb{C}^d \). Using these facts we prove the following Lemma.

**Lemma 3.7.** The Hadamard product of \( k \)-th exterior power matrix of the orthogonal matrix \( V_{12} \) of order \( d \) with itself, denoted as \( M^\wedge^k := V_{12}^\wedge^k \odot V_{12}^\wedge^k \) is an orthostochastic matrix for all \( 1 \leq k \leq d \).
Proof: From the construction of the matrix $M^k$, it is clear that each entry is a square of some $k \times k$ minors where $1 \leq k \leq d$. So, $M^k(ij) \geq 0$. Let $V = \begin{bmatrix} \vdots & v_j \vdots \\ \vdots & w_j \vdots \\ \vdots \end{bmatrix} \in O(d); j = i, \ldots, d$ be an orthogonal matrix. As the matrix $V$ is an orthogonal matrix, therefore \{v_1, \ldots, v_d\} and \{w_1, \ldots, w_d\} are sets of orthonormal vectors. By Theorem 3.6, the set of $k$ vectors of the form $v_i \wedge \cdots \wedge v_{i_k}$ where $i_1 < \cdots < i_k$, is a basis for $\wedge^k \mathbb{R}^d$ and satisfies
\[
(v_i \wedge \cdots \wedge v_{i_k}) \cdot (v_{j_1} \wedge \cdots \wedge v_{j_k}) = \begin{cases} 
1 & \text{if } (i_1, \ldots, i_k) = (j_1, \ldots, j_k) \\
0 & \text{otherwise.}
\end{cases}
\]

Therefore, the set \{v_i \wedge \cdots \wedge v_{i_k}, i_1 < \cdots < i_k\} is a collection of $\binom{d}{k}$ orthonormal $k$-vectors and each of these orthonormal $k$-vectors represents a column of matrix $V^k_{12}$. Thus $M^k := \begin{bmatrix} \vdots & v_{i_1} \wedge \cdots \wedge v_{i_k} \vdots \\ \vdots & v_{i_1} \wedge \cdots \wedge v_{i_k} \vdots \\ \vdots \end{bmatrix} \odot \begin{bmatrix} \vdots & w_{i_1} \wedge \cdots \wedge w_{i_k} \vdots \\ \vdots & w_{i_1} \wedge \cdots \wedge w_{i_k} \vdots \\ \vdots \end{bmatrix}$. So, each row sums and column sums of these matrices are actually $|u_{i_1} \wedge \cdots \wedge u_{i_k}|^2 = (u_{i_1} \wedge \cdots \wedge u_{i_k}) \cdot (u_{i_1} \wedge \cdots \wedge u_{i_k}) = 1$ and $|v_{i_1} \wedge \cdots \wedge v_{i_k}|^2 = (v_{i_1} \wedge \cdots \wedge v_{i_k}) \cdot (v_{i_1} \wedge \cdots \wedge v_{i_k}) = 1$ respectively. Therefore, $\sum_i M^k(ij) = \sum_j M^k(ij) = 1$. So, the matrix $M^k$ is an orthostochastic matrix for all $1 \leq k \leq d$.

Similarly, we develop an analogous theory for unitary matrices as well as unistochastic matrices.

Corollary 3.8. The complex Hadamard product of $k$-th exterior power matrix of the unitary matrix $U_{12}$ of order $d$ with itself, denoted by $Q^k := U^k_{12} \odot (U^k_{12})^*$ is unistochastic matrix for all $1 \leq k \leq d$.

Proposition 3.9. Any orthostochastic matrix $M$ of size $d$ uniquely determines $M^k$ which is obtained as the Hadamard product of $k$-th exterior power of the orthogonal matrix $V_{12}$ with itself for all $k = 2, \ldots, d$ and $M = V_{12} \odot V_{12}$.

Proof: The preimage of an orthostochastic matrix $M$ under the map $h_1$ is the set of orthogonal matrices $D_{\pm} V_{12}$ i.e., $M = V_{12} \odot V_{12} = D_{\pm} V_{12} \odot D_{\pm} V_{12}$. On the other hand, the $k$-th exterior powers of an orthogonal matrix $V_{12}$ are uniquely determined by that orthogonal matrix $V_{12}$ as the functions $f^k_1$ are well defined for all $k$. By Lemma 3.5 $f^k_1$ is a group homomorphism for all $k$. Therefore
\[
(D_{\pm} V_{12})^k = D^k_{\pm} V^k_{12}
\]
\[
\Rightarrow (D_{\pm} V_{12})^k \odot (D_{\pm} V_{12})^k = V^k_{12} \odot V^k_{12}
\]

Hence the proof. □

Similarly, as $N = U_{12} \odot (U_{12})^* = U_{\pm} U_{12} \odot U_{\pm} U_{12}$ and $f^k_2$ is a group homomorphism for all $k$, so we have the following Corollary.

Corollary 3.10. Any unistochastic matrix $N$ of size $d$ uniquely determines $N^k$ which is obtained as the complex Hadamard product of $k$-th exterior power of unitary matrix $U_{12}$ with itself for all $k = 2, \ldots, d$ and $N = U_{12} \odot U^*_{12}$. 
Say \( m := \binom{d}{k} \). First we talk about a special type of permutation matrices of order \( m \) which are obtained as Hadamard product of \( k \)-th exterior power of permutation matrices of order \( d \) with themselves. For an example there are 6! permutation matrices of order 6 among which only 4! permutation matrices of order 4 produce this special type of permutation matrices of order 6 and they are obtained by taking Hadamard product of second exterior power of permutation matrices of order 4 with themselves. Similarly we can characterize orthogonal matrices of order \( m \) that are obtained as \( k \)-th exterior power of an orthogonal matrix of order \( d \). This special type of permutation matrices play a crucial role to construct an orthostochastic matrix.

**Lemma 3.11.** Orthostochastic matrices \( M^\wedge k, k = 1, \ldots, d \), can be expressed as a convex combination of special type of permutation matrices which can be obtained as Hadamard product of \( k \)-th exterior power of permutation matrices of order \( d \).

**Proof:** The orthostochastic matrix \( M^\wedge k = V^\wedge k \odot V^\wedge k \), where \( V^\wedge k \) is an orthogonal matrix of order \( \binom{d}{k} \) that can be constructed as the \( k \)-th exterior power of orthogonal matrix \( V_{12} \) of order \( d \). Thus each row or column of \( V^\wedge k \) are pure \( k \)-vector (or \( k \)-blade) in \( \mathbb{R}^{\binom{d}{k}} \). Moreover, each entry in the row or column of matrix \( M^\wedge k \) is the square of each component of \( k \)-blade. As \( k \) vectors produce \( k \) vector under addition, therefore, orthostochastic matrices created in this manner always can be written as convex combination of certain type of permutation matrices. In fact, each of these permutation matrices is obtained as Hadamard product of \( k \)-th exterior power of some permutation matrix of order \( d \) with itself. Otherwise, any two vectors in \( \mathbb{R}^{\binom{d}{k}} \) can produce a \( k \) vector under addition in \( \mathbb{R}^{\binom{d}{k}} \) which violates the definition of Grassmannian algebra-a contradiction. Hence the Lemma follows.

Following the same lines of proof we conclude that

**Corollary 3.12.** Unistochastic matrices \( N^\wedge k, k = 1, \ldots, d \), can be expressed as a convex combination of special type of permutation matrices which can be obtained as Hadamard product of \( k \)-th exterior power of permutation matrices of order \( d \).

**Example 3.13.**

\[
V = \begin{bmatrix}
1/\sqrt{3} & 0 & -\sqrt{2/3} & 0 \\
0 & \sqrt{1/6} & 0 & -\sqrt{5/6} \\
\sqrt{2/3} & 0 & \sqrt{1/3} & 0 \\
0 & \sqrt{5/6} & 0 & \sqrt{1/6}
\end{bmatrix}
\]

\[
M = V \odot V = \begin{bmatrix}
0.3333 & 0 & 0.6667 & 0 \\
0 & 0.1667 & 0 & 0.8333 \\
0.6667 & 0 & 0.3333 & 0 \\
0 & 0.8333 & 0 & 0.1667
\end{bmatrix}
\]

\[
= \frac{1}{6}I_4 + \frac{2}{3}
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix} + \frac{1}{6}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix} = \frac{1}{6}P_1 + \frac{2}{3}P_{13} + \frac{1}{6}P_6
\]

\[
M^\wedge 2 = \frac{1}{18}P_1^\wedge 2 \odot P_1^\wedge 2 + \frac{5}{9}P_{13}^\wedge 2 \odot P_{13}^\wedge 2 + \frac{5}{18}P_6^\wedge 2 \odot P_6^\wedge 2 + \frac{2}{18}P_3^\wedge 2 \odot P_3^\wedge 2.
\]

where \( P_1 = I_4, P_{13} = [3412], P_6 = [1432], P_3 = [3214] \). Permutation matrices will be discussed later in details.
3.4 Necessary and Sufficient Condition

In this subsection, we propose a representation of coefficients of mixed monomials of a bivariate determinantal polynomial which will provide a necessary and sufficient condition for the existence of MSDRs (MHDRs) of size \(d\) for a bivariate polynomial of degree \(d\).

We have shown that the eigenvalues of coefficient matrices are uniquely determined by using the coefficients of monomials \(x_1^{\alpha_1}x_2^{\alpha_2}\), \(\alpha_1, \alpha_2 \in \{0, \ldots, d\}\), \(i = 1, 2\). Thus these coefficients of monomials in \(x_i, i = 1, 2\) of a bivariate polynomial can be expressed in terms of the eigenvalues of the corresponding coefficient matrices and they are independent of the choice of orthogonal (unitary) matrix. The choice of orthogonal (unitary) matrix affects only on the vector coefficient of mixed monomials of a determinantal bivariate polynomial. By mixed monomials we mean to specify the monomials which are consisting of all the variables with at least of degree one or in other words, each variable should appear in those monomials. So, in order to find a suitable orthogonal (unitary) matrix it is enough to study the behaviour of the vector coefficient of mixed monomials of a determinantal bivariate polynomial.

Consider the bivariate polynomial \(f(x) = a_dx_1^d + \cdots + a_1x_1 + b_dx_2^d + \cdots + b_1x_2 + \tilde{f}(x) + 1\) of degree \(d\), where \(\tilde{f}(x) = \sum_{l=2}^d \sum_{i=1}^K m_{(i)}^{(0)} x_1^{\alpha_1}x_2^{\alpha_2}, K = (n+d-1) - n\).

**Theorem 3.14.** A bivariate polynomial \(f(x)\) of degree \(d\) has an MSDR of size \(d\) if and only if there exists an orthostochastic matrix \(M := V_{12} \circ V_{12}\) such that the vector coefficient of mixed monomials of \(f(x)\) can be expressed as bilinear product of two vectors with different defining matrices as follows

1. **Case I:** \(\alpha_1 \geq \alpha_2\). The coefficient \(m_{(i)}^{(0)}\) of \(x_1^{\alpha_1}x_2^{\alpha_2}\) is \(u_{\alpha_1}^c T M^{\wedge \alpha_2} w_{\alpha_2} = \langle u_{\alpha_1}^c, w_{\alpha_2} \rangle_{M^{\wedge \alpha_2}}\).

2. **Case II:** \(\alpha_1 \leq \alpha_2\). Similarly, the coefficient \(m_{(i)}^{(0)}\) of \(x_1^{\alpha_1}x_2^{\alpha_2}\) is \(w_{\alpha_2}^c T (M^{\wedge \alpha_1})^T u_{\alpha_1}\) (due to symmetry)

where two vectors are determined by the eigenvalues of coefficient matrices as mentioned before and \(M^{\wedge k}\) denotes the Hadamard product of \(k\) th exterior power of an orthogonal matrix \(V_{12}\) with itself.

**Proof:** A bivariate polynomial \(f(x)\) of degree \(d\) has an MSDR of size \(d\) if and only if there exists an orthogonal matrix \(V_{12}\) such that

\[ f(x) = \det(I + x_1D_1 + x_2V_{12}D_2V_{12}^T) \]

The entries of diagonal matrices \(D_1\) and \(D_2\) are uniquely determined by the vector coefficients of monomials associated with univariate polynomials \(f_{x_i} := f(0, \ldots, x_i, \ldots, 0), i = 1, 2\) respectively by Lemma 3.2. This implies that the eigenvalues of coefficient matrices are uniquely determined by these vector coefficients. Note that the components of \(u_k\) and \(w_k\) are the product of \(k\) ordered (with \(i_1 < \cdots < i_k\)) diagonal elements of \(D_1\) and \(D_2\) respectively. Thus after evaluating the eigenvalues of coefficient matrices these vectors \(u_{\alpha_1}, u_{\alpha_1}^c, w_{\alpha_1}, w_{\alpha_1}^c\) are uniquely determined when we fix the order of entries of vectors consisting of eigenvalues of corresponding coefficient matrices. So, it is enough to look into representation of the vector coefficient of mixed monomials of the given polynomial \(f(x_1, x_2)\) as this vector coefficient depends on the choice of orthogonal matrix \(V_{12}\). As we have proved that the coefficients of a determinantal polynomial can be uniquely determined by the generalized mixed discriminant of corresponding coefficient matrices [Dey17], so from the connection between coefficient and coefficient matrices of a determinantal polynomial it reveals that the analytic expressions of
the vector coefficient of mixed monomials $x_1^{\alpha_1}x_2^{\alpha_2} (\alpha_1 \geq \alpha_2)$ of a bivariate polynomial involve only the diagonal entries of the matrices $V_{12}D_2V_{12}^T$ and $V_{12}^\wedge k D_2^\wedge k (V_{12}^\wedge k)^T$ where $V_{12}$ denotes $k$-th exterior power of a matrix $V_{12}$. Similarly, due to symmetry between the coefficients, the analytic expressions of the vector coefficient of mixed monomials $x_1^{\alpha_1}x_2^{\alpha_2} (\alpha_1 \leq \alpha_2)$ of a bivariate polynomial involve only the diagonal entries of the matrices $V_{12}^T D_1 V_{12}$ and $(V_{12}^\wedge k)^T D_1^\wedge k V_{12}^\wedge k$.

On the other hand, since the exterior power of a matrix only the diagonal entries of the matrices $V_{12}D_2V_{12}^T$ and $V_{12}^\wedge k D_2^\wedge k (V_{12}^\wedge k)^T$ where $V_{12}$ denotes $k$-th exterior power of a matrix $V_{12}$. Similarly, due to symmetry between the coefficients, the analytic expressions of the vector coefficient of mixed monomials $x_1^{\alpha_1}x_2^{\alpha_2} (\alpha_1 \leq \alpha_2)$ of a bivariate polynomial involve only the diagonal entries of the matrices $V_{12}^T D_1 V_{12}$ and $(V_{12}^\wedge k)^T D_1^\wedge k V_{12}^\wedge k$.

Corollary 3.15. A bivariate polynomial $f(x)$ of degree $d$ has an MHDR of size $d$ if and only if there exists a unistochastic matrix $N := U_{12} \odot U_{12}^T$ such that the vector coefficient of mixed monomials of $f(x)$ can be expressed as bilinear product of two vectors with different defining matrices as follows

1. Case I: $\alpha_1 \geq \alpha_2$. The coefficient $m^{(i)}_{\alpha_1 \alpha_2}$ of $x_1^{\alpha_1} x_2^{\alpha_2}$ is $u_i^{\wedge \alpha_1} N^\wedge \alpha_2 w_i = \langle u_i^{\wedge \alpha_1}, w_i^\alpha \rangle_{N^\wedge \alpha_2}$.

2. Case II: $\alpha_1 \leq \alpha_2$. Similarly, the coefficient $m^{(i)}_{\alpha_1 \alpha_2}$ of $x_1^{\alpha_1} x_2^{\alpha_2}$ is $w_i^{\wedge \alpha_2} (N^\wedge \alpha_1)^T u_i$ (due to symmetry)

where two vectors are determined by the eigenvalues of coefficient matrices as mentioned before and $N^\wedge k$ denotes the complex Hadamard product of $k$ th exterior power of a unitary matrix $U_{12}$ with itself.

So, in order to determine MDR (MHDR) our aim is to find an orthostochastic matrix $M$ (a unistochastic matrix $N$) which satisfies all the scalar product expression for a given bivariate polynomial. Interestingly, this issue is highly related to a well established field known as theory of majorization and also connected to the inverse eigenvalue problem. In fact, the following theorem [Hor54] in the field of majorization theory provides two necessary conditions for the existence of such an orthostochastic (unistochastic) matrix.

Theorem 3.16. Let $x, y \in \mathbb{R}^d$. Then the following statements are equivalent:

1. $\max_{\alpha \in S_k^d} \sum_{i=1}^{d} x_{\sigma_i} \leq \sum_{i=1}^{d} y_i$, where $S_k^d$ is the set of all $k$-term sequences $\sigma$ of integers such that $1 \leq \sigma_1 < \cdots < \sigma_k \leq d$. If these conditions are satisfied, then it is known as $y$ is majorized by $x$, denoted by $y \prec x$.

2. $y = Qx$, where $Q$ is a doubly stochastic matrix.

3. $y \in C(x)$, where $C(x)$ is the convex hull of all the points $(x_{\alpha_1}, \ldots, x_{\alpha_d})$, $\alpha$ varying over all permutations of $(1, \ldots, d)$.

4. $y = Mx$ for some orthostochastic matrix $M$. 

12
Horn \cite{Hor54} proved that a Hermitian matrix $H$ with eigenvalues $\mathbf{x}$ and diagonal entries $\mathbf{y}$ exists if and only if $\mathbf{x}$ majorizes $\mathbf{y}$. Later due to Horn and Mirsky, it is proved that there exists a symmetric matrix with eigenvalues $\mathbf{x}$ and diagonal entries $\mathbf{y}$ which is shown in the following proposition \cite{MOAI}.

**Proposition 3.17.** If $x_1 \geq \cdots \geq x_d$ and $y_1 \geq \cdots \geq y_d$ are $2d$ numbers satisfying $\mathbf{y} \prec \mathbf{x}$ on $\mathbb{R}^d$, then there exists a Hermitian as well as a symmetric matrix with diagonal elements $y_1, \ldots, y_d$ and eigenvalues $x_1, \ldots, x_d$.

Based on the Theorem 3.16 and Proposition 3.17 we are rewriting the following lemma.

**Lemma 3.18.** A vector $\mathbf{y} = \text{Diag}(A_{12})$, consisting of diagonal elements of a Hermitian or symmetric matrix $A_{12}$ with eigenvalues $\mathbf{x}$ exists if and only if $\mathbf{y} = N\mathbf{x}, N \in U_d$ or $\mathbf{y} = M\mathbf{x}, M \in \Gamma_d$ respectively if and only if $\mathbf{y} \prec \mathbf{x}$ (i.e., $\mathbf{y}$ is majorized by $\mathbf{x}$).

Due to symmetry between the coefficients of bivariate polynomial we provide another necessary condition for the existence of an MSDR (MHDR) of size $d$ of a bivariate polynomial in terms of eigenvalues and diagonal entries of coefficient matrix associated with variable $x_1$.

**Proposition 3.19.** $\mathbf{y}_1 \prec \mathbf{w}_1$ and $\mathbf{y}_2 \prec \mathbf{u}_1$ are necessary conditions for the existence of an MSDR (MHDR) of size $d$ of a bivariate polynomial where $\mathbf{y}_1$ and $\mathbf{y}_2$ represent the diagonal entries of $X_{12}D_2X_{12}^T$ and $X_{12}^T D_1 X_{12}$ and $\mathbf{u}_1, \mathbf{w}_1$ which represent the entries of diagonal matrices $D_1$ and $D_2$ are actually the eigenvalues of coefficient matrices $X_{12}^T D_1 X_{12}$ and $X_{12} D_2 X_{12}^T$, where $X_{12} := V_{12}$ (resp. $U_{12}$), an orthogonal (resp. a unitary) matrix.

**Remark 3.20.** Note that the vectors $\mathbf{u}_1, \mathbf{w}_1$, consisting of diagonal entries of matrices of $D_1, D_2$ can be calculated by finding roots of two univariate polynomials $f_{x_1}, f_{x_2}$ respectively.

If they are real, by solving two systems of $d$ linear equations in $d$ unknowns we can determine vectors $\mathbf{y}_1, \mathbf{y}_2$. If vectors $\mathbf{y}_1, \mathbf{y}_2$ exist, they could be unique or infinitely many depending on the fact that repeated entries are appearing in diagonal matrices or not.

**Remark 3.21.** These two necessary conditions are independent and they are not sufficient condition. It is shown by the following example.

**Example 3.22.** Consider the bivariate polynomial $f(x_1, x_2) = 6x_1^3 + 36x_1^2 x_2 + 63.9 x_1 x_2^2 + 36x_2^3 + 11x_1^2 + 42x_1 x_2 + 36x_2^2 + 6x_1 + 11x_2 + 1$. Vectors $\mathbf{u}_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{w}_1 = \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}, \mathbf{y}_1 = \begin{bmatrix} 4.5 \\ 4 \\ 2.5 \end{bmatrix} \prec \mathbf{w}_1$, but $\mathbf{y}_2 = \begin{bmatrix} 2.325 \\ 2.7 \\ .975 \end{bmatrix} \notin \mathbf{u}_1$. Note that if the coefficient of $x_1 x_2^2$ is 64, $\mathbf{y}_2 = \begin{bmatrix} 2.333 \\ 2.6667 \\ 1 \end{bmatrix} \prec \mathbf{u}_1$, but no MSDR is possible. Also note that for fixed vector coefficient $(f_{11}, f_{21})$, the range of coefficient $f_{12}$ lies inside a closed interval. As an example, for fixed $(f_{11}, f_{21}) = (42, 36)$, the coefficient $f_{12} \in [64.8, 66.8]$ are associated with a bivariate polynomial which has an MSDR, although the coefficient $f_{12} \in [64, 68.9]$ satisfy the relation that diagonal entries of coefficient matrix $A_{12}$ is majorized by its eigenvalues. One can see \cite{Dey17}, Chapter-4 for details.

Let $\Omega_d(\mathbf{y} \prec \mathbf{x}) = \{Q \in \Omega_d; \mathbf{y} = Q \mathbf{x}, \mathbf{x}, \mathbf{y} \in \mathbb{R}^d\}$. Then the set $\Omega_d(\mathbf{y} \prec \mathbf{x})$ is a nonempty, convex polytope and a subpolytope of $\Omega_d$ \cite{Bru06}, Chapter-9. This set is known as *doubly stochastic polytope of the majorization* $\mathbf{y} \prec \mathbf{x}$. Thus, we provide a necessary and sufficient for the existence of an MSDR (MHDR) of size $d$ for a bivariate polynomial.
Theorem 3.23. A bivariate polynomial \( f(x_1, x_2) \) of degree \( d \) admits an MSDR (MHDR) of size \( d \) if and only if there exists an orthostochastic (a unistochastic) matrix \( Q := M(N) \) such that \( y_1 := Qw_1 \) and \( y_2 := Q^Tu_1 \) and it satisfies the vector coefficient of mixed monomials \( x_1^{\alpha_1}x_2^{\alpha_2}, \alpha_1, \alpha_2 \in \{2, \ldots, d-2\} \) where \( u_1, w_1 \) represent the vector consisting of eigenvalues of coefficient matrices and \( y_1, y_2 \) represent the vectors consisting of the diagonal entries of matrices \( X_{12}D_2X_{12}^T \) and \( X_{12}^TD_1X_{12} \), matrix \( X_{12} := V_{12}(U_{12}) \) is an orthogonal (a unitary) matrix.

Proof: It follows from the Theorem 3.14 and Proposition 3.19 that this is a necessary condition for the existence of an MSDR of size \( d \) for a bivariate polynomial. To prove sufficient part: By the Theorem 3.14 a bivariate polynomial of degree \( d \) admits an MSDR (MHDR) of size \( d \) if and only if there exists an orthostochastic (unistochastic) matrix such that the vector coefficient of mixed monomials are satisfied. Suppose we have an orthostochastic (a unitary) \( Q := M(N) \) such that \( y_1 := Qw_1 \) represent the diagonal entries of \( Q \) determined by the coefficients of monomials \( x_1^{\alpha_1}, i = 1, 2 \) respectively therefore, the coefficients associated with these monomials are satisfied. In order to determine the diagonal entries of matrices \( X_{12}D_2X_{12}^T \) and \( X_{12}^TD_1X_{12} \), we need to solve two systems of linear equations and those linear equations are derived from the analytic expressions due to the coefficients of mixed monomials \( x_1^{\alpha_1}x_2^{\alpha_2} \) in which at least one of \( \alpha_1, \alpha_2 \) being equal to one. So, these coefficients are also satisfied. The remaining coefficients are due to the mixed monomials \( x_1^{\alpha_1}x_2^{\alpha_2}, \alpha_1, \alpha_2 \in \{2, \ldots, d-2\} \). On the other hand, by Proposition 3.9 any orthostochastic (unistochastic) matrix \( Q \) uniquely determines \( Q^{Uk} \), for all \( k = 2, \ldots, d \). So, if there exists an orthostochastic (unistochastic) matrix \( Q \) such that \( y_1 := Qw_1 \) and \( y_2 := Q^Tu_1 \) and all the coefficients of mixed monomials \( x_1^{\alpha_1}x_2^{\alpha_2} \) are also satisfied by that orthostochastic (unistochastic) matrix, the bivariate polynomial \( f(x) \) of degree \( d \) admits an MSDR (MHDR) of size \( d \).

\[ \square \]

3.5 Relaxation for Determinantal Polynomials

We need to find a suitable orthostochastic matrix \( M \) (a unistochastic matrix) to determine an MSDR (MHDR) of size \( d \) for a bivariate polynomial if it exists. More explicitly, while computing an MSDR (MHDR) of size \( d \) we need to apply a necessary and sufficient condition for which a doubly stochastic matrix of order \( d \) would be an orthostochastic (unistochastic) matrix.

Note that there is a necessary and sufficient condition for a doubly stochastic matrix \( M = (m_{ij}) \) of order 3 to be an orthostochastic matrix. The condition is as follows \([\text{Nak96}, \text{CD08}].\)

\[ (1 - m_{11} - m_{12} - m_{21} - m_{22} + m_{11}m_{22} + m_{12}m_{21})^2 = 4m_{11}m_{22}m_{12}m_{21} \]

Doubly stochastic matrix \( M \) of order 3 is unistochastic \([\text{BEK}^+05]\) if and only if

\[ (1 - m_{11} - m_{12} - m_{21} - m_{22} + m_{11}m_{22} + m_{12}m_{21})^2 \leq 4m_{11}m_{22}m_{12}m_{21} \]

So, this necessary and sufficient condition enables us to compute determinantal representation for cubic bivariate polynomials that has been separately discussed in \([\text{Dey17}, \text{Chapter- 4}].\)

This problem is unresolved if the order of doubly-stochastic matrix is \( \geq 4 \). So we propose a relaxation to the original problem by finding a doubly stochastic matrix and its transpose.
such that they majorize both the doubly stochastic polytope in which diagonal entries are majorized by the eigenvalues of coefficient matrices instead of finding orthostochastic or unistochastic matrix. Moreover, we conclude this subsection by providing a necessary and sufficient condition for the existence of such doubly stochastic matrix.

After evaluating the values of vectors \( \mathbf{w}_1 \) and \( M \mathbf{w}_1 \), we can get \( d - 1 \) linear expressions in terms of entries of \( M := (m_{ij}) \), where \( d \) is the size of doubly stochastic matrix \( M \). The number of free variables in a doubly stochastic matrix of size \( d \) is \((d - 1)^2\). So, we can eliminate \( d - 1 \) free diagonal entries from \( M \) by parameterizing \((d - 1)(d - 2)\) off-diagonal entries of \( M \) using the above mentioned \( d - 1 \) linear expressions.

Note that there are two sets of \( d - 1 \) monomials in which one of the two variables \( x_1, x_2 \) must be of degree one, i.e., they are of the form \( x_1^{\alpha_1} x_2 \) or \( x_1^{\alpha_2} x_2 \), \( \alpha \in \{0, \ldots, d-1\} \) and monomial \( x_1^2 x_2 \) is common in both the sets. So, we can eliminate \( d - 2 \) more free variables from the required doubly stochastic matrix \( M \) by using the values of \( \mathbf{u}_1 \) and \( M^T \mathbf{u}_1 \). Therefore, the required doubly stochastic matrix \( M \) has \((d - 1)(d - 2) - (d - 2) = (d - 2)^2 \) free off diagonal entries and it is a parameterized matrix in \((d - 2)^2 \) parameters in our context.

As each entries of \( M \) are linear in terms of these parameters and lies in the closed interval \([0, 1]\) (due to the definition of doubly stochastic matrix), so we can specify feasible region for this system of linear multivariate inequalities in \((d - 2)^2 \) variables. Thus the problem turns into a problem of solving a system of linear multivariable inequalities. In Linear algebra Farkas Lemma (or theorem of the alternative) provides a certificate of emptiness for a polyhedral set \( \{ \mathbf{x} : A \mathbf{x} \leq \mathbf{b} \} \) for some matrix \( A \in \mathbb{R}^{m \times n} \) and some vector \( \mathbf{b} \in \mathbb{R}^m \). One can use the command LinearMultivariateSystem to solve a system of linear inequalities with respect to the given variables in Maple.

Solution of this system of linear multivariate system provides a tight region in which each doubly stochastic matrix and its transpose majorize both the required polytopes. In fact, if we relax the problem of determining orthostochastic (unistochastic) matrix to a problem of determining a doubly stochastic matrix which satisfies the majorization criteria explained in Proposition 3.19, it turns into a problem of deciding whether a point lies inside a convex hull of the finite set of specified points. Now by combining two necessary conditions mentioned in Proposition 3.19, we provide a necessary and sufficient condition for the relaxation problem which is in fact a necessary condition for the original problem.

Lemma 3.24. \( \mathbf{y}_1 = Q \mathbf{w}_1, \mathbf{y}_2 = Q^T \mathbf{u}_1 \) where \( Q \) is a doubly stochastic matrix if and only if the vector coefficient of mixed monomials \( x_1^{\alpha_1} x_2, x_1 x_2^{\alpha_2}, \alpha_1, \alpha_2 = 1, \ldots, d - 1 \) of a bivariate polynomial can be expressed as a convex combination of \( d! \) points

\[
\{ \mathbf{u}^c_{\alpha_1} P \mathbf{w}_1, \mathbf{w}^c_{\alpha_2} (P)^T \mathbf{u}_1, \alpha_1, \alpha_2 = 1, \ldots, d - 1, P \text{ 's are permutation matrices of order } d \}
\]

where the vectors \( \mathbf{u}_1, \mathbf{w}_1, \mathbf{y}_1, \mathbf{y}_2, \mathbf{u}^c_{\alpha_1}, \mathbf{w}^c_{\alpha_2} \) are defined as before.

Proof: It follows from the Theorem 3.16 that the range set \( \{Q \mathbf{w}_1, Q \in \Omega_n\} \) is a convex set. Using the property of linearity of second argument we have

\[
\{(1 - \lambda)\mathbf{Q}_1 \mathbf{w}_1 + \lambda \mathbf{Q}_2 \mathbf{w}_1 : \mathbf{Q}_1, \mathbf{Q}_2 \in \Omega_n\} = \{(\mathbf{u}, Q \mathbf{w}_1) : (1 - \lambda)Q_1 \mathbf{w}_1 + \lambda Q_2 =: Q \in \Omega_n\}
\]

Thus, the set \( \{\text{Tr}(D(Q \mathbf{w}_1)) : Q \in \Omega_n\} = \{\mathbf{u}^T Q \mathbf{w}_1 = (\mathbf{u}, \mathbf{w}_1)Q : Q \in \Omega_n\} \) is a convex set for any vectors \( \mathbf{u}, \mathbf{w}_1 \in \mathbb{R}^n \) such that \( [\mathbf{u}] = D \). Since the monomial \( x_1 x_2 \) is common in mixed monomials \( x_1^{\alpha_1} x_2, x_1 x_2^{\alpha_2}, \alpha_1, \alpha_2 = 1, \ldots, d - 1 \), and

\[
f_{11} = \mathbf{u}^c_1 Q \mathbf{w}_1 = \mathbf{w}^c_1 (Q)^T \mathbf{u}_1,
\]
therefore, by applying the fact that the Cartesian product of convex sets is convex set we conclude the desired result.

Note that each of these vector coefficients which is obtained by some convex combination of \( d! \) specified points need not be associated with a determinantal polynomial since that convex combination of permutation matrices need not be an orthostochastic or a unistochastic matrix. Thus we have the following corollary.

**Corollary 3.25.** If some convex combination of permutation matrices produce an orthostochastic (a unistochastic) matrix, then the same convex combination of the vector coefficient \( (f_{11}, \ldots, f_{d-1,1}, f_{12}, \ldots, f_{1d-1}) \) associated with corresponding permutation matrices provides a vector coefficient of mixed monomials of determinantal bivariate polynomial whose coefficient matrices belong to the same orbits.

So, expressing the vector coefficient of mixed monomials \( x_1^{\alpha_1}x_2^{\alpha_2}, x_1^{\alpha_1}x_2^{\alpha_2} \), \( \alpha_1, \alpha_2 = 1, \ldots, d-1 \) as convex combination of \( d! \) specified points is not a sufficient condition, but this is a necessary condition for the existence of MSDR (MHDR) of size \( d \) for bivariate polynomials and it shows a method to compute an MSDR (MHDR) for higher degree (\( \geq 4 \)) bivariate polynomials which need not be strictly RZ polynomials.

### 3.6 Construction of Orthostochastic Matrices from Permutation Matrices

In this subsection, we discuss a method to compute determinantal representation for quartic bivariate polynomial in details and explain the reason to proceed further in this direction for higher degree polynomials. We need to define some concepts in order to construct an orthostochastic matrix by using the properties of permutation matrices.

Bijective functions \( x : \{1, \ldots, n\} \mapsto \{1, \ldots, n\} \) are called permutations of the set \( \{1, \ldots, n\} \). They form a group under composition that is denoted by \( S_n \). Subgroups of \( S_n \) are called permutation groups as well as symmetric groups. The permutation \( x \in S_n \) can be represented by a sequence of all the values \( x(i) \) left to right within a third bracket i.e., \( x = [x_1x_2 \ldots x_n] \) where \( x_i = x(i) \) for all \( i \), are the values of the permutation. We call this the complete notation for \( x \). Permutations of the form \( (i,j) \) are called transpositions.

Conventionally composition of permutations is defined as the product of right to left permutations. As a consequence of this composition of mappings when we multiply \( x = x_1x_2 \ldots x_n \) on the right by a transposition \((i,j)\) we obtain new \( x \) with transposing the values in positions \( i \) and \( j \), whereas when we multiply on the left, we transpose the values in \( i \) and \( j \).

Consider the permutation (symmetric) group \( S_n \) and the set \( S \), consisting of adjacent transpositions \( s_i = (i, i+1), 1 \leq i \leq n - 1 \). The action of multiplying an element \( x \in S_n \) on the right by the transposition \( s_i \) is that of interchanging the places of \( x(i) \) and \( x(i+1) \) in the complete notation of \( x \). The reflection set \( T \) of \( S_n \) is the set of all transpositions \( T = \{(i,j) : 1 \leq i < j \leq n\} \). Define

\[
l(x) = \# \{(i,j) : i < j, x_i > x_j\}
\]

to be the number of inversions of \( x \). Define a partial order \( \leq \) on \( S_n \) which is the transitive and reflexive closure of

\[
x < (i,j)x; \text{ if } l((i,j)x) = 1 + l(x),
\]

known as (strong) Bruhat order. On the other hand if it satisfies one of the following two relations
1. $x \leq_R y$ means that $y = x s_1 s_2 \ldots s_k$, for some $s_i \in S$ such that $l(xs_1 s_2 \ldots s_i) = l(x) + i$, $0 \leq i \leq k$.

2. $x \leq_L y$ means that $y = s_k s_{k-1} \ldots x$, for some $s_i \in S$ such that $l(s_i s_{i+1} \ldots x) = l(x) + i$, $0 \leq i \leq k$.

then this defines the right weak order and the left weak order, respectively.

For the symmetric group $S_n$, we have that $x \leq_R y$ if and only if the permutation $y$ can be obtained from $x$ via a sequence of adjacent transpositions which increases the number of inversions at each step. We use the notation $P_i$ to represent permutations of the set $\{1, \ldots, n\}$ which are elements of $S_n$ as well as permutation matrices associated with those permutations. Sorry for abusing notation, but it would be clear from the context which is meant. Symmetric group $S_4 = \{P_1 := [1234], P_2 := [2134], P_3 := [3214], P_4 := [4231], P_5 := [1324], P_6 := [1432], P_7 := [1243], P_8 := [2314], P_9 := [2431], P_{10} := [3241], P_{11} := [1342], P_{12} := [2143], P_{13} := [3412], P_{14} := [4321], P_{15} := [4312], P_{16} := [3421], P_{17} := [2341], P_{18} := [3142], P_{19} := [3124], P_{20} := [1432], P_{21} := [4213], P_{22} := [2413], P_{23} := [4123], P_{24} := [4132]\}.$

The Bruhat graph of $S_n$ is the directed graph whose nodes are the elements of $S_n$ and whose edges are given by $x \rightarrow y$ means that $x \overset{t}{\rightarrow} y$ for some $t \in T$. Bruhat order is the partial order relation on the set $S_n$ defined by the relation $x < y$ means that there exist $s_i$ such that $x = x_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_{k-1} \rightarrow x_k = y$.

The Bruhat (strong) and right weak order graphs of symmetric group $S_4$ are shown in the following figure [BB06]. As we can see in the Figure 1, there are $4! = 24$ permutations in the symmetric group $S_4$. Since the edges of the graph of (right) weak order of $S_4$ are constructed via an adjacent transposition, so we define a parameter $\alpha \in [0, 1]$ for each point of the link of any two nodes $x, y$ (edges of) the graph $S_4$ such that $\alpha \leftrightarrow \alpha P_x + (1 - \alpha) P_y$. Thus, we have the following results about orthostochastic matrices of order 4.
• Any convex combinations of any two edges of the Bruhat order graph of $S_4$ in Fig 1 correspond to orthostochastic matrices.

• Any convex combinations of any two adjacent nodes of the right weak order graph of $S_4$ in Fig 1 correspond to orthostochastic matrices.

• The following collection of permutation matrices are such that any convex combination of four permutation matrices from the set $B_i \in \mathcal{F}, i = 1, \ldots, 18$ provide an orthostochastic matrix in the Birkhoff polytope $B_4$ due to its special structure.

$$\mathcal{F} = \{B_1 := \{P_8, P_{10}, P_{20}, P_{24}\}, B_2 := \{P_9, P_{11}, P_{19}, P_{21}\}, B_3 := \{P_2, P_7, P_{15}, P_{16}\}, B_4 := \{P_3, P_6, P_{17}, P_{23}\}, B_5 := \{P_4, P_5, P_{18}, P_{22}\}, B_6 := \{P_1, P_{12}, P_{13}, P_{14}\}, B_7 := \{P_1, P_2, P_7, P_{12}\}, B_8 := \{P_5, P_{11}, P_{18}, P_9\}, B_9 := \{P_4, P_9, P_{21}, P_{22}\}, B_{10} := \{P_{13}, P_{14}, P_{15}, P_{16}\}, B_{11} := \{P_6, P_{20}, P_{23}, P_{24}\}, B_{12} := \{P_3, P_8, P_{10}, P_{17}\}, B_{13} := \{P_2, P_{13}, P_{18}, P_{22}\}, B_{14} := \{P_3, P_{19}, P_{21}, P_{23}\}, B_{15} := \{P_2, P_8, P_{15}, P_{24}\}, B_{16} := \{(P_6, P_9, P_{11}, P_{17})\}, B_{17} := \{P_1, P_4, P_5, P_{14}\}, B_{18} := \{P_7, P_{10}, P_{16}, P_{20}\}\}$$

Though it is complicated to get a figure of graph of $S_n$, but using the same arguments we conclude that any convex combinations of any two edges of the Bruhat order graph of $S_n$ correspond to orthostochastic matrices. Similarly, one can construct the regions which are the convex hulls of $d$ points and each point in Birkhoff polytope $B_d$ is associated with an orthostochastic matrix. We illustrate the proposed method through an example.

**Example 3.26.** Consider the quartic bivariate polynomial

$$f(x_1, x_2) = 24x_1^4 + 133.6609x_1^3x_2 + 50x_1^2 + 253.8824x_1^2x_2^2 + 196.9412x_1x_2^2 + 35x_2^2$$

$$+ 190.4498x_1x_2^3 + 230.4498x_1x_2^2 + 87.6125x_1x_2 + 10x_1 + 48x_2^2 + 80x_2^3$$

$$+ 48x_2^2 + 12x_2 + 1$$

By Lemma 3.2 the eigenvalues of $D_1$ are $u_1 := \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$ and the eigenvalues of $A_{12}$ are $w_1 := \begin{bmatrix} 6 \\ 2 \\ 2 \\ 2 \end{bmatrix}$. Find the diagonal entries of $A_{12}$ after evaluating the diagonal entries of $D_1$. In order to determine diagonal entries of $A_{12}$, we need to solve a system of 4 linear equations associated with coefficients of monomials $x_2, x_1x_2, x_1^2x_2, x_1^3x_2$ in four unknowns $a, b, c, d$ of the form $Gy = z$ where

$$G = \begin{bmatrix} 1 & (d_2 + d_3 + d_4) & (d_1 + d_3 + d_4) & (d_1 + d_2 + d_3) \\ (d_2d_3 + d_2d_4 + d_3d_4) & 1 & (d_1d_3 + d_1d_4 + d_3d_4) & (d_1d_2 + d_1d_3 + d_2d_3) \\ d_2d_3d_4 & d_1d_3d_4 & 1 & (d_1d_2d_3) \\ d_1d_2d_3 & d_1d_2d_4 & d_2d_3 \end{bmatrix},$$

$$y = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}, \quad z = \begin{bmatrix} f_{01} \\ f_{11} \\ f_{21} \\ f_{31} \end{bmatrix}, f_{ij}$ represents the coefficient of monomial $x_i^jx_2^j$ of a bivariate polynomial.
So, matrix $G = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 6 & 7 & 8 & 9 \\ 11 & 14 & 19 & 26 \\ 6 & 8 & 12 & 24 \end{bmatrix}$ and $z = \begin{bmatrix} 12 \\ 87.6125 \\ 196.9412 \\ 253.8824 \end{bmatrix}$. Thus diagonal entries of $A_{12}$ are

$y = \begin{bmatrix} 3.3840 \\ 3.6749 \\ 2.8858 \\ 2.0554 \end{bmatrix}$. By solving $Mw_1 = y$ we obtain the first column of matrix $M$ as $\begin{bmatrix} .346 \\ .4187 \\ .2215 \\ .0138 \end{bmatrix}$.

In order to determine second and third columns of matrix $M$ we make use of Lemma 3.24. Vector coefficients of monomials $x_1^3x_2, x_1^2x_2, x_1x_2^3, x_1x_2x_3, x_1x_2, x_1x_2$ associated with specified $\frac{4!}{3!} = 4$ points are as follows

$[132, 252, 196, 192, 232, 88], [124, 228, 184, 176, 216, 84], [148, 292, 216, 208, 248, 92], [196, 348, 244, 224, 264, 96]$

Now our aim is to express the vector coefficient $[133.6609, 196.9412, 190.4498, 230.4498, 87.6125]$ of mixed monomials $x_1^3x_2, x_1^2x_2, x_1x_2^3, x_1x_2x_3, x_1x_2$ as convex combination of these 4 points such that the same convex combination of the corresponding permutation matrices provide an orthostochastic matrix. Observe that two possible convex combinations for the vector coefficient of given polynomial is

$[133.6609, 196.9412, 190.4498, 230.4498, 87.6125] = .4187P2 + .346P7 + .2215P15 + .0138P16$

$= .346P1 + .4187P12 + .2215P15 + .0138P16$

This convex combinations are found by solving systems of linear equations. Both of them give the same orthostochastic matrix $M = \begin{bmatrix} 0.3460 & 0.4187 & 0.0138 & 0.2215 \\ 0.4187 & 0.3460 & 0.2215 & 0.0138 \\ 0.2215 & 0.0138 & 0.4187 & 0.3460 \\ 0.0138 & 0.2215 & 0.3460 & 0.4187 \end{bmatrix}$. Thus one possible orthogonol matrix and coefficient matrix $A_{12}$ are as follows.

\[
V_{12} = \begin{bmatrix} .58821765 & .647070 & .117473 & .47063787 \\ .647070 & -.58821765 & -.47063787 & .117473 \\ .47063787 & -.117473 & .647070 & -.58821765 \\ .117473 & .47063787 & -.58821765 & -.647070 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 3.383999 & 1.52247 & 1.10735 & 0.276398768 \\ 1.52247 & 3.67479 & 1.21814 & 0.304053 \\ 1.10735 & 1.21814 & 2.885999 & 0.22114897 \\ 0.276398768 & 0.304053 & 0.22114897 & 2.0551986 \end{bmatrix}
\]

This polynomial was constructed by considering $V_{12} = \begin{bmatrix} .5882 & .6471 & -.1176 & .4706 \\ .6471 & -.5882 & .4706 & .1176 \\ -.4706 & -.1176 & .2941 & .8235 \\ -.1176 & .4706 & .8235 & -.2941 \end{bmatrix}$.

This orthogonal matrix provides the same coefficient matrix $A_{12}$ up to the sign of off-diagonal entries.

**Remark 3.27.** Observe that the quartic bivariate polynomial in Example 3.26 is not a strictly RZ polynomial.
Remark 3.28. It is evident that if at least one coefficient matrix of a determinantal representation has repeated eigenvalues, the given polynomial is not a strictly RZ polynomial.

There could be three possibilities:

1. If the vector coefficient of the given polynomial cannot be expressed as convex combination of specified points, by Lemma 3.24 there is no such doubly stochastic matrix. This implies no such orthostochastic (unistochastic) matrix exists, so conclude that MSDR (MHDR) of size \(d\) is not possible for the given bivariate polynomial.

2. Only one doubly stochastic matrix exists. In this case that doubly stochastic matrix has to be orthostochastic (unistochastic) matrix if an MSDR (MHDR) exists for the given bivariate polynomial.

3. There are infinitely many doubly stochastic matrices. This does not ensure that there exists an orthostochastic (unistochastic) matrix too, but it ensures the region of existence of orthostochastic (unistochastic) matrix if it exists.

It is exemplified.

Example 3.29. Consider the bivariate polynomial 
\[
 f(x_1, x_2) = 6x_1^3 + 37.97x_1^2x_2 + 71.94x_1x_2^2 + 36x_2^3 + 11x_1^2 + 42.99x_1x_2 + 36x_2^2 + 6x_1 + 11x_2 + 1.
\]

As we have seen before \(u_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, w_1 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}\).

A doubly stochastic matrix \(M = \begin{bmatrix} .5 & 0 & .5 \\ .5 & .01 & .49 \\ 0 & .99 & .01 \end{bmatrix}\) such that \(Mw_1 = \begin{bmatrix} 4 \\ 4.01 \\ 2.99 \end{bmatrix} \prec w_1\) and \(M^T u_1 = \begin{bmatrix} 2.5 \\ 1.01 \\ 2.49 \end{bmatrix} \prec u_1\). There does not exist any orthostochastic matrix along the line \(u - 2v + 1 = 0\) in our method. Also note that the given polynomial is not a RZ polynomial since at \(x = (3, -1)\) its restricted univariate polynomial has complex roots. So the existence of doubly stochastic does not imply the existence of such an orthostochastic matrix.

4 Range set of vector coefficients of mixed monomials of certain class of determinantal bivariate polynomials

In this section, we would like to classify all such determinantal bivariate polynomials which satisfy certain similarity. We are interested to know if we replace a coefficient matrix \(A_j\) by some arbitrary same type (symmetric /Hermitian) matrix \(\hat{A}_j\) of same order such that the spectrums of coefficient matrices \(A_j\) and \(\hat{A}_j\) are same; i.e., \(\sigma_{A_j} = \sigma_{\hat{A}_j}\), does there exist a relation between coefficients of \(f(x)\) and \(\hat{f}(x) = \det(I + \sum_{j=1}^n x_j\hat{A}_j)\)? What about its converse? That is, if there exist a certain similarity pattern of coefficients of two distinct polynomials \(f(x)\) and \(\hat{f}(x)\) are the coefficient matrices of their MSDRs (MHDRs) related? Actually answers to the both questions are affirmative.

Let \(S_{\mathbb{R}^{d\times d}}(\mathbb{H}^{d\times d}(\mathbb{C}))\) be the space of all symmetric (Hermitian) matrices of order \(d\). The Orbit of \(A \in S_{\mathbb{R}^{d\times d}}\) is defined by \(O_A = \{VAV^T : V \in O(d)\}\) and the orbit of \(A \in H^{d\times d}(\mathbb{C})\) is defined by \(O_A = \{UAU^* : U \in U(d)\}\). By the Spectral theorem for symmetric (Hermitian)
matrices it is clear that each of symmetric (Hermitian) matrices $A_1, A_2$ belongs to the unique orbit of a diagonal matrices $D_1$ and $D_2$, denoted by $\mathcal{O}_{D_1}$, and $\mathcal{O}_{D_2}$ respectively. Thus the vector space $S_{\mathbb{R}^{d \times d}(H^{d \times d}(\mathbb{C})]}$ is a union of disjoint orbits of diagonal matrices.

Consider the class of cubic bivariate polynomials $f(x)$ having MSDR with coefficient matrices $A_1$ and $A_2$ which are obtained from the same orbits $\mathcal{O}_{D_1}$, and $\mathcal{O}_{D_2}$ respectively. Observe that any two determinantal bivariate polynomials of this class differ from each other by the vector coefficient of mixed monomials only. In other words, the polynomials of this class share the same coefficients due to all monomials but mixed monomials. We provide a geometric structure of the range set of existence for vector coefficient of mixed monomials of cubic bivariate polynomial of the this class.

Let $S$ denotes the set of all coefficients of mixed monomials of bivariate polynomials which admit MSDR (MHDR) of size $d$ with coefficient matrices $A_1$, $A_2$ belonging to the same orbits $\mathcal{O}_{D_1}$ and $\mathcal{O}_{D_2}$ respectively. So, by the Theorem 3.14 set

$$S = \left\{ m^{(i)}_{\alpha_1,\alpha_2} : \alpha_1 + \alpha_2 = l, \alpha_1, \alpha_2 = 1, \ldots , l - 1, 1 \leq i \leq K, 2 \leq l \leq d, K = \binom{n + d - 1}{d} - n \right\}$$

$$= \left\{ u^T_{\alpha_1} Q^{\wedge \alpha_2} w_{\alpha_2}, w^T_{\alpha_2} (Q^{\wedge \alpha_1}) T u_{\alpha_1}, \alpha_1 + \alpha_2 = l, \alpha_1, \alpha_2 = 1, \ldots , l - 1, 2 \leq l \leq d \right\}$$

where $Q^{\wedge \alpha_1}$ and $Q^{\wedge \alpha_2}$ are orthostochastic (unistochastic) matrices.

**Theorem 4.1.** Consider the class of bivariate polynomials having MSDR (MHDR) with coefficient matrices $A_1$, $A_2$ coming from same orbits $\mathcal{O}_{D_1}, \mathcal{O}_{D_2}$ respectively. Then the set

$$S := \{ u_{\alpha_1}^T Q^{\wedge \alpha_2} w_{\alpha_2}, w_{\alpha_2}^T (Q^{\wedge \alpha_1}) T u_{\alpha_1}, \alpha_1 + \alpha_2 = l, \alpha_1, \alpha_2 = 1, \ldots , l - 1, 2 \leq l \leq d \}$$

$Q^{\wedge \alpha_1}$, $Q^{\wedge \alpha_2}$ are orthostochastic (unistochastic) matrices

of coefficients of mixed monomials of bivariate polynomials of this class lies inside the convex hull of $H$, $H = \{ u_{\alpha_1}^T ((P)^{\wedge \alpha_2}) w_{\alpha_2}, w_{\alpha_2}^T ((P)^{\wedge \alpha_1}) u_{\alpha_1}, l \leq \alpha_1, \alpha_2 \leq d - 1, \alpha_1 + \alpha_2 \leq d \}$ where $P$’s are all permutation matrices of size $d$.

**Proof:** If a bivariate polynomial of this class admits an MSDR (MHDR), by the Theorem 3.23 there exists a set of permutation matrices whose convex combination give us the required orthostochastic (unistochastic) matrix $Q$ which satisfies the vector coefficient of mixed monomials $x_1 x_2, \ldots , x_1^{d-1} x_2, x_1 x_2^2, \ldots , x_1^{d-1} x_2^{d-1}$. Say $Q = \alpha_1 P_1 + \cdots + \alpha_m P_m$, where $\sum \alpha_i = 1$, $\alpha_i \geq 0$, $1 \leq m \leq d!$. By Proposition 3.9 (Proposition 3.10) each orthostochastic (unistochastic) matrix $Q$ uniquely determines $Q^{\wedge k}$ for all $k = 2, \ldots , d$ which are defined by

$$Q^{\wedge k} = (\alpha_1 P_1 + \cdots + \alpha_m P_m)^{\wedge k}$$

Note that $Q^{\wedge k}$ is equal to $\sum_{i=1}^{d!} \beta_i P_i^{\wedge k}, \beta_i \geq 0$, and $\sum_{i=1}^{d!} \beta_i = 1$ by Lemma 3.11 (Lemma 3.12). So, these orthostochastic (unistochastic) matrices lie inside the convex hull of this special type of permutation matrices. It follows from the Theorem 3.10 that the range set $\{ y \in \mathbb{R}^n | y = Q x, Q \in \Omega_n \}$ is a convex set. Thus, the set $\{ \text{Tr}(D(Q w_1)) : Q \in \Omega_n \} = \{ u^T D w_1 = \{ u, w_1 \} : Q \in \Omega_n \}$ is a convex set for any vectors $u, w_1 \in \mathbb{R}^n$ such that $|u| = D$.

Therefore, the coefficients associated with these orthostochastic (unistochastic) matrices are bounded by the coefficients which are obtained as convex combinations of special kind of permutation matrices associated with $d!$ permutation matrices of size $d$. Hence the proof. □

We show that the set $S$ attains its maximum and minimum at the points obtained by substituting the defining orthostochastic matrix $M$ with permutation matrices associated
with identity permutation \([12\ldots n]\) and permutation \([n\ldots 21]\) respectively using the result of rearrangement inequality.

Rearrangement inequality states that

\[
x_n y_1 + \cdots + x_1 y_n \leq x_{\sigma(1)} y_1 + \cdots + x_{\sigma(n)} y_n \leq x_1 y_1 + \cdots + x_n y_n
\]

for every choice of real numbers \(x_1 \leq \cdots \leq x_n\) and \(y_1 \leq \cdots \leq y_n\)

and every permutation \(x_{\sigma(1)}, \ldots, x_{\sigma(n)}\).

**Proposition 4.2.** Consider the class of bivariate polynomials having MSDR (MHDR) with coefficient matrices \(A_1, A_2\) coming from same orbits \(O_{D_1}, O_{D_2}\) respectively. The set

\[
S := \{ u_{\alpha_1}^c (M^{\wedge \alpha_2})^T w_{\alpha_2}, w_{\alpha_2}^c (M^{\wedge \alpha_1})^T u_{\alpha_1}, \alpha_1 + \alpha_2 = l, \alpha_1, \alpha_2 = 1, \ldots, l - 1, 2 \leq l \leq d \}
\]

of vector coefficients of mixed monomials of bivariate polynomials of this class attains its minimum when vectors \(u_1\) and \(w_1\) represent the diagonal entries of \(D_1\) and \(D_2\) respectively in descending order and attains its maximum when one of the vectors \(u_1\) and \(w_1\) is in descending order and other one is in ascending order.

**Proof:** If both the vectors \(u_1, w_1\) are in descending order, then the vectors \(u_k, w_k\) are in descending and \(u_k^c\) and \(w_k^c\) are in ascending order. From the Theorem 3.14 it is clear that if the defining matrix is the identity permutation matrix associated with permutation \([1\ 2\ \ldots\ n]\), then the coefficients of mixed monomials of a bivariate polynomial are scalar product of two vectors, one of which is descending and other one is ascending. Therefore, by the result of rearrangement inequality, it provides the minimum value of the set \(S\). On the other hand, if the defining matrix is the permutation matrix associated with the permutation \([n\ldots 2\ 1]\), then it provides the maximum value of the set \(S\) since the coefficients are obtained as scalar product of two descending vectors.

**Remark 4.3.** All these results hold for monic Hermitian determinantal representation too.

The result of the Theorem 4.1 reflects the reason behind the Remark 3.21

**Conclusion 4.4.** If the vector coefficient \((f_{11}, f_{21}, f_{12})\) of monomials \(x_1x_2, x_1^2x_2, x_1x_2^2\) for a bivariate polynomial cannot be expressed as a same convex combination of six specified points, then we see that though one of these two majorizations in Remark 3.21 is satisfied, but other one is not satisfied. Thus, we conclude that these two necessary conditions are independent.

**Remark 4.5.** Indeed, convexity of range set of vector coefficient of mixed monomial is a necessary condition, but it is not a sufficient condition.

## 5 Conclusion

We have been able to express the coefficients of polynomials in terms of the coefficient matrices of an MSDR (MHDR) of size \(d\) as bilinear product of two vectors with different defining matrices. We have shown that these defining matrices are obtained as a (complex) Hadamard product of exterior power of an orthogonal (a unitary) matrix with themselves, and have to be orthostochastic (resp. unistochastic) matrices according to an MSDR (resp. MHDR). A
necessary and sufficient condition which is different from RZ property of a bivariate polynomial is provided for the existence of an MSDR (MHDR) of size \( d \). This condition leads to a method to compute an MSDR of size \( d \) by using the knowledge of permutation matrices of order \( d \) even if the coefficient matrices have repeated eigenvalues. In addition, we propose a relaxation to the original problem and provide a necessary and sufficient condition for this relaxation problem which turns into a problem of deciding whether a point is inside the convex hull of some specified points. It is proved that the vector coefficient of the class of bivariate degree \( d \) determinantal polynomials whose coefficient matrices belong to the same orbits lies inside a convex hull of \( d! \) specified points.

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