Deep learning algorithms for hedging with frictions

Xiaofei Shi1 · Daran Xu2 · Zhanhao Zhang2

Received: 23 April 2022 / Accepted: 6 February 2023 / Published online: 8 March 2023
© The Author(s), under exclusive licence to Springer Nature Switzerland AG 2023

Abstract
This work studies the deep learning-based numerical algorithms for optimal hedging problems in markets with general convex transaction costs. Our main focus is on how these algorithms scale with the length of the trading time horizon. Based on the comparison results of the FBSDE solver by Han, Jentzen, and E (2018) and the Deep Hedging algorithm by Buehler, Gonon, Teichmann, and Wood (2019), we propose a Stable-Transfer Hedging (ST-Hedging) algorithm, to aggregate the convenience of the leading-order approximation formulas and the accuracy of the deep learning-based algorithms. Our ST-Hedging algorithm achieves the same state-of-the-art performance in short and moderately long time horizon as FBSDE solver and Deep Hedging, and generalize well to long time horizon when previous algorithms become suboptimal. With the transfer learning technique, ST-Hedging drastically reduce the training time, and shows great scalability to high-dimensional settings. This opens up new possibilities in model-based deep learning algorithms in economics, finance, and operational research, which takes advantage of the domain expert knowledge and the accuracy of the learning-based methods.

Keywords Portfolio optimization · Transaction costs · Deep learning · Transfer learning

The authors thank the two anonymous reviewers for their careful readings and suggestions, and thank Steven Campell, Lukas Gonon, Jiequn Han, Ruimeng Hu, Steven Kou, Johannes Muhle-Karbe, Junru Shao, Mete Soner, and Xunyu Zhou for useful comments and fruitful discussions. Part of the work is supported by the MA Mentored Research Program in Statistics at Columbia University.

Xiaofei Shi
xf.shi@utoronto.ca

Daran Xu
dx2207@columbia.edu

Zhanhao Zhang
zz2760@columbia.edu

1 Department of Statistical Sciences, University of Toronto, Toronto, ON, Canada
2 Department of Statistics, Columbia University, New York, NY, USA
1 Introduction

As observed in many empirical papers, markets are imperfect, meaning that arbitrary quantities cannot be traded immediately at the quoted market price because of taxes, regulations, and the limited liquidity of the assets. Typical examples include linear transaction taxes as well as fixed transaction costs. As reported and studied in Almgren (2003); Almgren and Chriss (2001); Almgren et al. (2005); Lillo et al. (2003), empirical estimates of actual transaction costs typically correspond to a 3/2-th power of the order flow. Accordingly, the large trading volume quickly impacts the market liquidity, which, in turn, drastically changes the agents’ behaviors. Hence, optimally scheduling the order flow in anticipation of market liquidity shortage is crucial.

There is a large body of literature on optimal execution strategies as well as dynamic portfolio optimization models with market illiquidity and price impacts, see, e.g. (Amihud et al. 2006; De Lataillade et al. 2012; Dumas and Luciano 1991; Janeček and Shreve 2010; Kohlmann and Tang 2002; Liu 2004; Shreve and Soner 1994), and with recent results in Bank et al. (2017); Garleanu and Pedersen (2013); Guasoni and Weber (2020); Soner and Touzi (2013). By assuming the transaction costs are a quadratic function of the size of the order flow, the optimal trading policy can be given in a closed-form, as shown in Garleanu and Pedersen (2013); Kohlmann and Tang (2002). However, no closed-form solution is available under general nonlinear transaction costs with general market dynamics. To obtain tractable results, researchers then focus on the small costs limit, as in Almgren and Li (2016); Bayraktar et al. (2018); Guasoni and Weber (2020); Kallsen and Muhle-Karbe (2017); Moreau et al. (2017); Shreve and Soner (1994); Soner and Touzi (2013). These elegant asymptotic formulas were proved rigorously by Ahrens (2015); Herdegen and Muhle-Karbe (2018); Kallsen and Li (2013); Shreve and Soner (1994). Two important follow-up questions therefore arise. The first one is about the quality of the leading-order approximation, which needs to be well-studied and quantified, especially in empirical examples. The second is about the smallness assumption, whether that is an absolute or a relative quantity.

From the numerical perspectives, recent advances in highly accurate machine learning models have introduced powerful new tools for studying high-dimensional optimization problems, such as hedging with frictions. The FBSDE solver, developed by Han, Jentzen, and E in Han et al. (2018), can solve a dynamic portfolio optimization problem by finding the solution to a BSDE system. For the first time, this algorithm overcame the curse of dimensionality in numerical solutions to high-dimensional SDEs and associated PDEs, as pointed out by Beck et al. (2020); Grohs et al. (2018). The convergence analysis is established by Han and Long (2020) under the same short-term existence assumptions in Delarue (2002). At a higher level, the FBSDE solvers find the solution of the FBSDEs through a supervised learning framework, since the accuracy of the terminal value of the backward components serves as the goal functional of the algorithm. However,
this algorithm does not scale well with the trading time horizon and the time discretization. For calibrated trading parameters as in Gonon et al. (2021), the time horizon for the algorithm to work is required to be unreasonably small. In the meantime, with the development of modern model-free techniques, reinforcement learning algorithms are also widely used in single-agent optimization problems. Indeed, as shown in the groundbreaking Deep Hedging algorithm (Buehler et al. 2019a, b), and similar works in the same flavor (Becker et al. 2019; Casgrain et al. 2019; Han and Weinan 2016; Hu 2019; Huré et al. 2018; Min and Hu 2021; Moallemi and Wang 2021; Mulvey et al. 2020; Reppen and Soner 2020; Ruf and Wang 2021), we treat the utility functions as targets and directly parametrize and learn the optimal trading policy. Moreover, reinforcement learning frameworks are introduced and analyzed rigorously in Wang et al. (2020); Wang and Zhou (2020). Therefore, just like in Moallemi and Wang (2021), a natural question is how these methods compare in practice, and our goal is to understand and compare the two types of methods, especially the advantages and drawbacks of supervised learning and reinforcement learning algorithms.

This paper studies the optimal hedging problem in frictional markets in a general setting and examines the machine learning-based numerical algorithms. First, we explore popular machine learning architectures, including the FBSDE solver and the Deep Hedging algorithms. We implement both methods, document the tuning procedures, and discuss their advantages and disadvantages. Then we compare our numerical results with the leading-order approximation formulas. These comparisons help justify the accuracy of the approximation trading formulas and provide us with ideas on the usage of each method in practice. In summary, the FBSDE solver only performs well under short trading horizons, while it fails to work in intermediate trading horizons. The Deep Hedging algorithm has stable and reliable performance for short and moderately long trading horizons, but it does not scale well when the trading horizon becomes long. In contrast, the leading-order approximations work well under long trading horizons, while they deviate from the ground truth by a significant amount shortly before maturity. To fully utilize the convenience of the leading-order formulas and the accuracy of the learning-based algorithms, we employ the transfer learning ideas and propose a new algorithm which we referred to as Stable-Transfer Hedging (ST-Hedging) algorithm. ST-Hedging algorithm achieves the same state-of-the-art performance attained in short time horizons by the FBSDE solver and in intermediate time horizons by the Deep Hedging algorithms, and it allows us to directly work with long trading horizons. ST-Hedging also shows great scalability for multiple stocks with correlations. In addition, the convenience of implementation and convergence speed of ST-Hedging algorithm outperforms the FBSDE solver and the Deep Hedging algorithms, which shows the great potential of model-based learning algorithms.

The rest of the paper is organized as follows. First, in Sect. 2, we introduce the market model with frictions, the preference for individual agents, and the admissible strategies. Then, the FBSDE solver, Deep Hedging and ST-Hedging algorithms are introduced in Sect. 3, with details on the implementations and comparisons. Finally, we compare the performance of different learning-based algorithms and the leading-order approximation formula in Sect. 4. For better readability, the derivation of leading-order approximations and all proofs are collected in Appendix 1.
Notation. We fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})$ with finite time horizon $T > 0$, where the filtration is generated by a 1-dimensional standard Brownian motion $W = (W_t)_{t \in [0,T]}$. Throughout, let $\| \cdot \|$ be the 2-norm of a real-valued vector. For $p > 1$, we write $\mathbb{H}^p$ for the $\mathbb{R}$-valued, progressively measurable processes $X = (X_t)_{t \in [0,T]}$ with $\| X \|_{\mathbb{H}^p} := \left( \mathbb{E} \left[ (\int_0^T |X_t|^2 dt)^p \right] \right)^{1/p} < \infty$.

2 Model setup

Here, we assume the randomness in the market is generated by the 1-dimensional Brownian motion $(W_t)_{t \in [0,T]}$. Suppose we receive a cumulative random endowment as

$$d\zeta_t = \xi_t dW_t, \quad \text{for a general process } \xi \in \mathbb{H}^2.$$

There are two assets in the market for us to hedge the risk, the first one is safe, with price normalized to one, and the second is risky, with general dynamic

$$dS_t = \mu_t dt + \sigma_t dW_t,$$  \hspace{1cm} (2.1)

where the expected return process satisfies the no-arbitrage condition $\mu = \sigma \mu_\sigma$ for a market price of risk $\mu_\sigma \in \mathbb{H}^2$. One readily verifies that the Bachelier models and Geometric Brownian motions satisfy this requirement. To make sure we have a meaningful comparison baseline for the frictionless version of this model, we assume that $\xi_t/\sigma$ and $\mu_t/\sigma^2$ are both Itô processes.

Transaction costs. A popular class of models originating from the optimal execution literature focuses on absolutely continuous trading strategies, cf. (Almgren 2003; Almgren and Chriss 2001),

$$\varphi_t = \varphi_{0-} + \int_0^t \bar{\varphi}_u du, \quad t \geq 0,$$

and penalizes the trading rate $\dot{\varphi}_t = d\varphi_t/dt$ with an instantaneous transaction cost $\lambda_t G(\dot{\varphi}_t)$. Notice that the formulation allows for an initial jump right after time 0 in the frictionless analogue. In the frictional market, i.e. $\lambda_0 \neq 0$, the strategies with initial jumps are suboptimal. With proper definition of matrix-valued operations, this formulation of transaction costs can be generalized to multiple risky assets with cross-asset costs.

The scalar process $\lambda$ models the level and the fluctuating structure of the transaction costs parameter. Like in the partial-equilibrium model of Moreau et al. (2017), we allow the transaction cost to fluctuate randomly over time:

$$\lambda_t = \lambda \Lambda_t, \quad t \in [0,T].$$  \hspace{1cm} (2.2)

Here, with a little abuse of notation, the constant $\lambda > 0$ modulates the magnitude of the cost (this scaling parameter will be sent to zero in the small-cost asymptotics,
where an explicit formula is available as in Cayé et al. (2018); Guasoni and Weber (2020). We generalize these results in Appendix A.1. The strictly positive stochastic processes \((\Lambda_t)_{t \in [0,T]}\) describes the fluctuations of liquidity over time, and allows to model “liquidity risk” as in Acharya and Pedersen (2005), for example.

\(G\) is used to model the “shape” of transaction costs. Portfolio choice problems for the most tractable quadratic specification \(G(x) = x^2 / 2\) are analyzed in models by Almgren and Li (2016); Bouchard et al. (2018); Garleanu and Pedersen (2016); Guasoni and Weber (2017); Moreau et al. (2017); Sannikov and Skrzypacz (2016). In the small transaction costs limit as in Bayraktar et al. (2018); Cayé et al. (2019); Guasoni and Weber (2020), single-agent models are solved explicitly for the more general power costs \(G(x) = |x|^{q}/q, q \in (1, 2]\) proposed by Almgren (2003). Below, we introduce the general smooth convex cost functions \(G\) as studied in Gonon et al. (2021); Guasoni and Rásonyi (2015), which includes these special examples:

**Assumption 2.1**

(i) The transaction cost \(G : \mathbb{R} \rightarrow \mathbb{R}_+\) is convex, symmetric, and strictly increasing on \([0, \infty)\), differentiable on \([0, \infty)\), and satisfies \(G(0) = 0\);

(ii) The derivative \(G'\) is also strictly increasing and differentiable on \((0, \infty)\) with 
\(G'(0) = 0\);

(iii) There exist constants \(C > 0, K \geq 2\) and \(x_0 > 0\) such that 
\[ |(G')^{-1}(x)| \leq C(1 + |x|^{K-1}) \text{ for all } x \in \mathbb{R}, \quad G''(x) \leq C \text{ for all } |x| > x_0. \]

The power costs and their linear combinations are included, and the proportional costs can also be studied as the singular limit of power costs with the power approaching 1.

**Preferences and admissible strategies** In order to capture the risk-aversion, we consider the linear-quadratic model, i.e., we maximize our expected returns while penalizing for the corresponding quadratic variations:

\[
J_T(\phi) = \frac{1}{T} \mathbb{E} \left[ \int_0^T \left( \phi_t \mu_t - \frac{\gamma}{2} \left( \sigma_t \phi_t + \xi_t \right)^2 - \lambda_t G(\phi_t) \right) dt \right]. \tag{2.3}
\]

As studied in Choi and Larsen (2015); Garleanu and Pedersen (2013); Sannikov and Skrzypacz (2016), we trade off expected returns against the tracking error relative to the exogenous target position \(-\xi / \sigma\). We can also consider more sophisticated preferences such as exponential or power utility. Notice that the deep reinforcement learning algorithms such as Deep Hedging, can be easily generalized to sophisticated preferences or utility functions, but the usage of FBSDE solvers is limited since there might not exist a FBSDE system to describe the optimal solution of the hedging problem. See Sect. 4.4 for details.

To make sure all terms are well defined, we focus on admissible strategies that satisfy the following integrability conditions:
Moreover, we impose the transversality condition to exclude Ponzi scheme, by ruling out arbitrarily large risky positions:

$$\lim_{\sqrt{T/\rho} \to 0} \frac{\lambda}{T^2} \mathbb{E} \left[ \Lambda_T \sigma_T^2 \phi_T^2 \right] = 0. \quad (2.5)$$

By strict convexity of the goal functional (2.3), optimality of a trading rate $\dot{\phi}$ is equivalent to the first-order condition that the Gateaux derivative

$$\lim_{\rho \to 0} \frac{\lambda}{T^2} \mathbb{E} \left[ \Lambda_T \sigma_T^2 \phi_T^2 \right] \text{vanishes for any admissible perturbation } \eta,$$

cf. (Ekeland and Temam 1999):

$$0 = \mathbb{E}_t \left[ \int_0^T \left( \mu_u \int_0^u \dot{\eta}_u du - \gamma (\sigma_u \phi_u + \xi_u) \sigma_u \int_0^u \dot{\eta}_u du - \lambda_t G'(\phi_t) \dot{\eta}_t \right) dr \right].$$

As in Bank et al. (2017), this can be rewritten using Fubini’s theorem as

$$0 = \mathbb{E}_t \left[ \int_0^T \left( \int_t^T \left( \mu_u - \gamma (\sigma_u \phi_u + \xi_u) \sigma_u \right) du - \lambda_t G'(\phi_t) \right) \dot{\eta}_t dt \right].$$

Since this has to hold for any perturbation $\dot{\eta}_t$, the tower property of conditional expectation yields

$$\lambda_t G'(\phi_t) = \mathbb{E}_t \left[ \int_t^T \left( \mu_u - \gamma (\sigma_u \phi_u + \xi_u) \sigma_u \right) du \right] = M_t + \int_t^0 \left( \gamma (\sigma_u \phi_u + \xi_u) \sigma_u - \mu_u \right) du,$$

for a martingale $dM_t = Z_t dW_t$ that needs to be determined as part of the solution. Solving for the dynamics of the agents’ optimal trading rates would introduce the dynamics of the transaction costs. Accordingly, it is preferable to instead work with the marginal transaction costs as the backward process that describes the optimal controls:

$$Y_t := \lambda_t G'(\phi_t), \quad (2.7)$$

and from (2.6) we can infer that $Y_T = 0$. With this notation, the corresponding trading rates are

$$\dot{\phi}_t = \left( G' \right)^{-1} \left( \frac{Y_t}{\lambda_t} \right).$$

Thus, the optimal position $\phi$ and the corresponding marginal transaction costs $Y$ in turn solve the nonlinear FBSDE

$$d\phi_t = \left( G' \right)^{-1} \left( \frac{Y_t}{\lambda_t} \right) dt, \quad \phi_0 = \phi_{0-}. \quad (2.8)$$
For constant quadratic costs $\lambda x^2/2$ and constant volatility $\sigma$, this FBSDE becomes linear and can in turn be solved by reducing it to a standard Riccati equation system (Bank et al. 2017; Bouchard et al. 2018; Muhle-Karbe et al. 2020). For volatilities and quadratic costs that fluctuate randomly, these ODEs are replaced by a backward stochastic Riccati equation, compare (Ankirchner and Kruse 2015; Kohlmann and Tang 2002). With nonlinear costs, no such simplifications are possible. In fact, the wellposedness of the system is generally unclear even for short time horizons since no Lipschitz condition for $(G')^{-1}$ is satisfied.

Reparametrization. In the frictionless analogue of (2.3), pointwise maximization yields the agent’s individually optimal strategies, i.e.,

$$\tilde{\phi}_t = \frac{\mu_t}{\gamma \sigma_t^2} - \frac{\xi_t}{\sigma_t}, \quad t \in [0, T].$$

(2.10)

In particular, by assumption, we rewrite the dynamic of the frictionless strategy as

$$d\tilde{\phi}_t = b_t dt + a_t dW_t.$$ 

(2.11)

In real-world applications, we are more interested in the changes induced by transaction costs relative to the frictionless analogue. To facilitate both numerical and analytical analysis, we define

$$\Delta \phi_t := \phi_t - \tilde{\phi}_t$$

(2.12)

for the difference between the frictional and frictionless positions, which we expect to vanish as the transaction costs tend to zero. In view of the dynamic (2.11) strategy $\tilde{\phi}$ in the frictionless market and the forward equation (2.8), this process has dynamics

$$d\Delta \phi_t = \left( (G')^{-1} \left( Y_t \right) - \tilde{b}_t \right) dt - \tilde{a}_t dW_t, \quad \Delta \phi_0 = \phi_0 - \frac{\xi_0}{\sigma_0} - \frac{\mu_0}{\gamma \sigma_0^2}.$$ 

(2.13)

Accordingly, plugging (2.10) into (2.9), the backward process $Y$ therefore becomes

$$dY_t = \gamma \sigma_t^2 \Delta \phi_t dt + Z_t dW_t, \quad Y_T = 0.$$ 

(2.14)

Remark 2.2 The analysis is focused on the optimal marginal transaction costs $Y$ instead of the optimal trading rates $\tilde{\phi}$, to avoid the potential extra requirement on the differentiability of $(G')^{-1}$. Moreover, this choice of backward component has linear dynamics, and absorb all the nonlinearity into the dynamic of the forward component, which makes our asymptotic analysis much easier and the numerical solution much more stable.
We choose the deviation of the frictional position with respect to its frictionless counterparty as the forward variable, since it exactly is the “fast” variable in the theoretical study in the single agent problem as in Bank et al. (2017); Moreau et al. (2017); Soner and Touzi (2013). Empirically this choice performs the best in FBSDE solver. Notice that the drift $\bar{b}$ for the frictionless strategy is usually neglected in the small-costs regime, and the frictionless volatility $\bar{a}$ makes the FBSDE system non-deterministic.

3 Deep learning-based numerical algorithms

For practitioners, it is crucial to get the optimal trading strategy numerically, especially in a time-efficient manner. Therefore, efficient numerical methods to solve the optimization problem (2.3) are in great need. With the development of GPUs and highly accurate machine learning models, the optimal hedging problems can be solved numerically by using learning-based algorithms. There are two numerical approaches to do so: numerically solving the FBSDE system (2.13)–(2.14) or directly targeting the goal functional (2.3). In this section, we first present two popular deep learning-based numerical algorithms guided by these ideas: the FBSDE solver, and the Deep Hedging algorithm. Then, we highlight the advantages and disadvantages of these two methods in practice, especially when both of them fail to work. Finally, we introduce a transfer learning-based method, the ST-Hedging algorithm, to efficiently solve the optimal hedging problems when others fail to do so.

A natural idea to start with is to solve the associated FBSDE system (2.13)–(2.14) numerically. Since the dimension of the FBSDEs grows quickly with the number of assets and agents, and the boundary conditions are unclear, classical numerical methods such as finite differences fail to work. However, the learning-based algorithms can bypass the need to identify the correct boundary conditions and overcome the curse of dimensionality. The first learning-based introduced in Sect. 3.1 is the FBSDE solver, proposed in the spirit of the BSDE approach by Han, Jentzen, and E in Han et al. (2018). The algorithm approximates the dependence of the backward components on the forward components by a deep neural network, and it solves the FBSDE through simulation and stochastic gradient descent methods. It can also handle higher-dimensional settings, e.g., random and time-varying transaction costs, returns, and volatility processes. In addition, the FBSDE solver can even solve equilibrium models as is used in Gonon et al. (2021). FBSDE solver works very well when the time horizon is not too long, and the convergence of this method with respect to a special class of FBSDE systems is established in small time durations in Han and Long (2020). However, the FBSDE solver does not scale well when the trading horizon or time discretization is large. In our example, with calibrated market parameters from Gonon et al. (2021), the FBSDE solver fails to converge if we consider a more than one trading year trading horizon (approximately 252 trading days). Furthermore, in some complicated cases, we cannot even characterize the optimal trading problem with a system of FBSDEs.
In order to overcome the difficulties mentioned above, we consider deep reinforcement learning framework that is known as Deep Hedging in Sect. 3.2. The name for this type of algorithms comes from the groundbreaking Deep Hedging work of Buehler et al. (2019a, b). Pioneered by Becker et al. (2019), Han and Weinan (2016), Hu (2019), Huré et al. (2018), Min and Hu (2021), Moallemi and Wang (2021), Mulvey et al. (2020), Nevmyvaka et al. (2006), Reppen and Soner (2020), Ruf and Wang (2021), various reinforcement learning algorithms are implemented and perform extremely successful in portfolio optimization problems with transaction costs. Reinforcement learning can even solve for Nash equilibria, see (Casgrain et al. 2019) for details. The key idea is to directly parametrize the optimal trading rate and optimize the discretized version of preference (2.3). Then, through stochastic gradient descent based algorithms and backpropagation, the reinforcement algorithm updates the parameters of the networks until a (local) optimizer is found (see (Sutton and Barto 2018) for a detailed introduction). However, Deep Hedging algorithms require a huge number of simulated sample paths and finer discretization of the trading time horizon, which makes the training time significantly longer than FBSDE solver. Moreover, with short and intermediate investment horizons, Deep Hedging shows overfitting phenomenon, and with long investment horizons, Deep Hedging suffers from underfitting.

One major drawback for both algorithms is that their performances become suboptimal when the trading time horizon is long. Just as in the experiments illustrated in Sects. 4.1 and 4.2, when the investment horizon is more than one trading year, the FBSDE solver fails to converge. Although Deep Hedging still works, it suffers from underfitting and becomes difficult to tune. Moreover, when there is more than one stock with cross-sectional effects, the FBSDE solver and Deep Hedging both fail to converge even with intermediate trading horizons. In other words, these methods do not generalize well with the increase of the dimensions due to coupling.

In parallel, as in Almgren and Li (2016); Bayraktar et al. (2018); Guasoni and Weber (2020); Kallsen and Muhle-Karbe (2017); Moreau et al. (2017); Shreve and Soner (1994); Soner and Touzi (2013), researchers have focused on the asymptotic optimal trading strategy in the small transaction costs limit. Especially, there are explicit asymptotic formulas available for the optimal trading strategies. A natural following-up question is if we can take advantage of both the advantages of the deep learning algorithms and the asymptotic approximations. With this idea, we propose our ST-Hedging algorithm that allow us to have the convenience of the explicit asymptotic approximation and the accuracy of the deep learning algorithm. To begin with, as in Sects. 4.1 and 4.2, ST-Hedging reaches the same state-of-the-art performance when compared to FBSDE solver Deep Hedging for short and intermediate investment horizons, respectively, and outperforms both of these algorithm in long investment horizons. In addition, if we extends the driven Brownian motion to d-dimension, and consider multiple risky assets instead of just one single stock, as long as the leading-order approximation formulas are available, the ST-Hedging algorithm scales well with respect to dimensions as in Sect. 4.3. In the mean time, FBSDE solver suffers from the strong coupling of the system and fails to converge, while Deep Hedging suffers from underfitting due to the requirement of large number of simulation paths.
The organization of the rest of the section is as follows. First, we introduce the FBSDE solver in Sect. 3.1 and the Deep Hedging algorithm in Sect. 3.2. In Sect. 3.3, we introduce the leading-order approximation formula and propose our ST-Hedging algorithm.

### 3.1 FBSDE solver

First proposed by Han et al. (2018), the FBSDE solver is one of the first work to overcome the curse of dimensionality in solving high-dimensional SDE. The key observation is that the dynamic of the system (2.13)–(2.14) is pinned down if the volatility $Z$ of the backward component $Y$ is specified. Indeed, let us fix a time partition $0 = t_0 < \ldots < t_N = T$, where $t_m = mT/N$ and $\Delta t = T/N$. Let $(\Delta W_m)_{m=0}^N$ be iid normally distributed random variables with mean zero and variance $\Delta t$.

At time $t_m$, we have $\mu_{t_m}, \sigma_{t_m}$ from the market and the endowment volatility $\xi_{t_m}$, the drift $\bar{b}_{t_m}$ and volatility $\bar{a}_{t_m}$ for the frictionless strategy of the agent. With $(\Delta W_m, \Delta Y_m)$ as input for each time step, the discrete-time analogue of the forward update rule for the FBSDE system (2.13)–(2.14) becomes

$$
\Delta \phi_{t_{m+1}} = \Delta \phi_{t_m} + \left( (G')^{-1} \left( \frac{Y_m}{\lambda_{t_m}} \right) - \bar{b}_{t_m} \right) \Delta t - \bar{a}_{t_m} \Delta W_m,
$$

(3.1)

$$
Y_{t_{m+1}} = Y_{t_m} + \gamma \sigma_{t_m}^2 \Delta \phi_{t_m} \Delta t + Z_{t_m} \Delta W_m.
$$

(3.2)

Together with a guess for the initial values of the backward components, we can then simulate the system with a standard forward scheme and check whether it satisfies the terminal condition as $Y_T = 0$. Searching for the correct initial values is relatively straightforward. The more difficult challenge is parametrizing the “controls” $Z$—which are functions of time and the forward processes—and updating the corresponding initial guesses until the terminal condition is matched sufficiently well. In Han et al. (2018), Han, Jentzen, and E introduced the innovative way to parametrize $Z$. At each time point $t_m$, they parametrized $Z_{t_m}$ with a (shallow) network structure $F^\theta_{t_m}$, with time $t_m$ value of the forward components as the inputs. Together with the guess of the initial value of the backward component, which we denote as $Y_0^\theta$, the system can then be simulated forward in time. In this way, the shallow network structures are concatenated over time and become a deep neural network architecture, hence the number of layers of the networks and the number of parameters in the architecture both grow linearly in the number of discretization steps $N$. In other words, we input simulated Brownian paths into the deep network structure, and it outputs the terminal values of the backward components.

Our task is to update the parameters $\{Y_0^\theta, \theta_m, m = 0, \ldots, N\}$ until the terminal conditions $Y_T = 0$ are matched sufficiently well. This iterative update of the network parameters until $Y_T = 0$ is the essence of supervised machine learning tasks.
State-of-the-art performance is achieved through back-propagation and stochastic gradient descent-type algorithms, see (Goodfellow et al. 2016) for more details.

**Remark 3.1** Recall that there is no assumption on the dynamic of the processes \( \bar{a}, \bar{b}, \mu, \sigma \) and \( \xi \). Here, however, we need to mildly assume that we can simulate the return \( \mu \) and volatilities \( \sigma \), the drift \( \bar{b} \) and volatilities \( \bar{a} \) for the frictionless position \( \bar{\varphi} \) from (2.11), as well as the endowment volatilities \( \xi \).

Now, we focus on the parametrization of \( \{Z_{t_m}\}^N_{m=0} \) within a function class \( \{F^\theta : \theta \in \Theta\} \). A very popular class of functions in the machine learning community is the “singular activation function Rectified Linear Unit” (ReLU): ReLu \((x) = \max(x, 0)\). A popular class of approximation function \( F \) is the convolution of linear functions with ReLu activations. For the numerical experiments in Sect. 4, at each time \( t_m \), the \( F^\theta_m \) we use is a neural network with one hidden layer of 15 hidden units with batch normalization (BN), and that is

\[
F^\theta_m(x) = w^2_{\theta_m} \left( \text{ReLU} \left( \text{BN} \left( w^1_{\theta_m} x + b^1_{\theta_m} \right) \right) \right) + b^2_{\theta_m}.
\]

Recall that with the Brownian motion \( W \) as a 1-dim process, and the forward component \( \varphi \) as a 1-dim vector, \( Z \) is also a 1-dim vector. \( w^1_{\theta_m} \in \mathbb{R}^{1 \times 2} \) and \( b^1_{\theta_m} \in \mathbb{R}^{1 \times 2} \) are called the input weights and biases of the network, whereas \( w^2_{\theta_m} \in \mathbb{R}^{1 \times 15} \) and \( b^2_{\theta_m} \in \mathbb{R}^{1} \) are called the output weights and biases of the network. We summarize the structure of FBSDE solver in Algorithm 1:

| Algorithm 1: Training Procedure of FBSDE Solver |
|------------------------------------------------|
| **Data:** \( \Delta \varphi^\theta_{t_0} = \varphi_{t_0} + \frac{t_0}{\xi_0} - \frac{\varphi_{t_0}}{\xi_0} \); \( W_{t_0} = 0 \), \( \gamma \), dynamic of processes \( \mu \), \( \sigma \), \( \lambda \), \( \xi \), \( \bar{b} \) and \( \bar{a} \); initialization of parameters \( \{Y^\theta_0, \theta_m, m = 0, \ldots, N\}, m = 0 \), \( Y^\theta_0 = Y^\theta_0 \); |
| **while** epoch \( \leq \) Epoch **do** |
| sample \( \Delta W \), batch_size \( \times N \) iid Gaussian random variables with variance \( \Delta t \); |
| while \( m < N \) **do** |
| \( Z^\theta_{t_m} = F^\theta_m(W_{t_m}, \Delta \varphi^\theta_{t_m}) \); |
| \( Y^\theta_{t_{m+1}} = Y^\theta_{t_m} + \gamma \sigma^2 \Delta \varphi^\theta_{t_m} \Delta t + Z^\theta_{t_m} \Delta W_{t_m} \); |
| \( \Delta \varphi^\theta_{t_{m+1}} = \Delta \varphi^\theta_{t_m} + (G^\theta)^{-1} (Y^\theta_{t_m}/\lambda_{t_m}) \Delta t - \bar{a}_{t_m} \Delta t - \bar{a}_{t_m} \Delta W_{t_m} \); |
| \( W_{t_{m+1}} = W_{t_m} + \Delta W_{t_m} \); |
| \( m++ \); |
| end |
| Output \( Y^\theta = Y^\theta_N \); |
| Loss = ||Y^\theta||^2/batch_size; |
| Calculate the gradient of Loss with respect to \( \theta \); |
| Back propagate updates for \( \{Y^\theta_0, \theta_m, m = 0, \ldots, N\} \) via Adam; |
| epoch ++; |
| end |
| **return:** (local) optimizer \( \theta^* = \{Y^\theta_0, \theta^*_m, m = 0, \ldots, N\} \). |

As studied by Han and Long (2020), when the short time existence of the FBSDE system is available, the convergence rate of the FBSDE solver is
guaranteed. In reality, FBSDE system with short time existence does not always generalize to global existence. That is to say, if we only extend the time horizon with all the parameter remain the same, the FBSDE solver might not converge anymore. Especially, since only the terminal value of the backward components are compared, the FBSDE solver fails to work efficiently with even intermediate trading horizons.

3.2 Deep hedging networks

Let us fix a time partition \(0 = t_0 < \cdots < t_N = T\), where \(t_m = mT/N\) and \(\Delta t = T/N\). Under this partition, for a single simulation, the discretized version of the goal functional (2.3), becomes

\[
J_T(\varphi) = \frac{1}{N} \sum_{m=0}^{N} \left( \varphi_{t_m} \mu_{t_m} - \frac{\gamma}{2} \left( \sigma_{t_m} \varphi_{t_m} + \xi_{t_m} \right)^2 - \lambda_{t_m} G(\varphi_{t_m}) \right),
\]

(3.3)

where \(\varphi\) follows the discretized version of update process

\[
\varphi_{t_{m+1}} = \varphi_{t_m} + \sum_{k=0}^{m} \varphi_k \Delta t = \varphi_{t_m} + \varphi_{t_m} \Delta t.
\]

At each time point \(t_m\), we directly parametrize the trading strategy \(\varphi_{t_m}\) using a (comparatively shallow) network \(F^\theta_m\). For the numerical experiments in Sect. 4, the neural network we use has three hidden layers with 10, 15 and 10 hidden units, respectively. We also implement the batch normalization (BN) (see (Ioffe and Szegedy 2015) for details) at each hidden layer and that is, for \(l = 1, 2, 3\):

\[
F^\theta_m(x) = w^4_{\theta_m} \left( F^\theta_m \left( F^\theta_m \left( F^\theta_m (x) \right) \right) \right) + b^4_{\theta_m},
\]

where for layer \(l = 1, 2, 3\), we have that

\[
F^\theta_m(x) = \text{ReLU} \left( BN \left( w^l_{\theta_m} x + b^l_{\theta_m} \right) \right).
\]

Recall that with time \(t\) as a real number, the Brownian motion \(W\) as a 1-dim process, and the forward component \(\varphi\) as a 1-dim vector, \(\dot{\varphi}\) is also a 1-dim vector. \(w^1_{\theta_m} \in \mathbb{R}^{10 \times 3}\) and \(b^1_{\theta_m} \in \mathbb{R}^{10}\) are called the input weights and biases of the network, and \(w^2_{\theta_m} \in \mathbb{R}^{15 \times 10}, b^2_{\theta_m} \in \mathbb{R}^{15}, w^3_{\theta_m} \in \mathbb{R}^{10 \times 15}\) and \(b^3_{\theta_m} \in \mathbb{R}^{10}\) are called the hidden weights and biases, whereas \(w^4_{\theta_m} \in \mathbb{R}^{1 \times 10}\) and \(b^4_{\theta_m} \in \mathbb{R}^{1}\) are called the output weights and biases of the network. We summarize the Deep Hedging algorithm in Algorithm 2 as follows:
Algorithm 2: Training Procedure of Deep Hedging Algorithm

Data: $\varphi_0$, $W_0 = 0$, $\gamma$, dynamic of processes $\mu$, $\sigma$, $\lambda$ and $\xi$

1. initialization of parameters $\{\theta_m, m = 0, \ldots, N\}$, $m = 0$;

2. while epoch $\leq$ Epoch do

3. sample $\Delta W$, batch_size $\times N \times d$ iid Gaussian random variables with variance $\Delta t$;

4. while $m < N$ do

5. $\dot{\varphi}_{tm} = F_{\theta_m}(tm, W_{tm}; \varphi_{tm}^\theta)$;

6. $\varphi_{tm+1} = \varphi_{tm} + \Delta t + \varphi_{tm}^\theta$;

7. $W_{tm+1} = W_{tm} + \Delta W_m$;

8. $m += 1$;

9. end

10. Loss $= -\sum_{m=0}^{N} \left[ \varphi_{tm} \mu_{tm} - \frac{1}{2} \left( \sigma_{tm} \varphi_{tm}^\theta + \xi_{tm} \right)^2 - \lambda \Lambda_{tm} G \left( \varphi_{tm}^\theta \right) \right]$;

11. Calculate the gradient of Loss with respect to $\theta$;

12. Backpropagate updates for $\{\theta_m, m = 0, \ldots, N\}$ via Adam (switch to SGD when fine tuning);

13. epoch ++;

14. end

15. return: (local) optimizer $\theta^* = \{\theta^*_m, m = 0, \ldots, N\}$

### 3.3 Stable-transfer learning algorithm: ST-hedging

The optimal hedging problem can be treated in two folds. On the numerical aspect, we can solve the system through various different schemes, as discussed in Sect. 3.1 and Sect. 3.2. On the theoretical aspect, one can study the asymptotic limiting behavior as the trading horizon $T$ goes to $\infty$. Researchers always focus on small transaction costs for tractability as in Almgren and Li (2016); Bayraktar et al. (2018); Guasoni and Weber (2020); Kallsen and Muhle-Karbe (2017); Moreau et al. (2017); Shreve and Soner (1994); Soner and Touzi (2013). A natural question is, how a long trading horizon interacts with the smallness assumption of the transaction cost level $\lambda$. It turns out, the “magic” quantity is $\sqrt{\lambda/T}$; namely, the smallness assumption on $\lambda$ should not only be an absolute quantity, but also a relative quantity, depending on whether $\sqrt{\lambda/T}$ is of higher order than 1.

More specifically, the leading-order asymptotic optimal strategy $\dot{\varphi} = (\dot{\varphi}_t)_{t \in [0, T]}$ for the frictional mean-variance preference (2.3) is given by:

$$\dot{\varphi}_t = -\text{sign}(\varphi_t - \bar{\varphi}_t) \times (G')^{-1} \left( \frac{g(\varphi_t - \bar{\varphi}_t; \gamma, \sigma, \bar{a}, \lambda)}{\lambda} \right) + O(1). \quad (3.4)$$

Here, with proper “mean-reverting” requirement (see Lemma A.2 in Appendix 1 for details), $g$ is the unique solution to the following ODE:

$$(G')^{-1} \left( \frac{g(x; \gamma, \sigma, \bar{a}, \lambda)}{\lambda} \right) \gamma' (x; \gamma, \sigma, \bar{a}, \lambda) + \frac{a^2}{2} g'' (x; \gamma, \sigma, \bar{a}, \lambda) = \gamma \sigma^2 x. \quad (3.5)$$

This leading-order formula (3.4) and the associated ODE (3.5) are proposed and studied formally in Bayraktar et al. (2018); Cayé et al. (2019); Guasoni and Weber (2020); Kallsen and Muhle-Karbe (2017); Soner and Touzi (2013), and we refer the
readers to Appendix 1 for details of the derivation and rigorous proof in a special case. Notice that once we obtain the numerical solution of (3.5), the leading-order formula can be applied immediately.

Even more surprisingly, under rather general conditions, we can show the approximation accuracy for the leading-order approximation (3.4) is at the order $O(\sqrt{\lambda}/T) + O(\sqrt{\lambda})$. This means that, independent of the choice $G$ of the transaction costs, the smallness assumption on the liquidity level $\lambda$ is not only an absolute quantity on $\sqrt{\lambda}$, but also on the relative quantity $\sqrt{\lambda}/T$. In other words, the smallness of $\lambda$ should also be relative quantity with respect to the trading time horizon. With multiple examples in Sect. 4, we verify that with the parameters we are using from (Gonon et al. 2021; Muhle-Karbe et al. 2020), the leading-order approximations work well until the last two months to maturity. For better readability, we summarize the finding in Theorem A.6 and refer the reader to the discussion in Appendix 1) for details.

A natural following-up question is if we can take advantage of both the advantages of the deep learning algorithms and the leading-order approximations. More precisely, we want to have the convenience of the leading-order approximation formula for most of the trading horizon, and the accuracy of the deep learning algorithm near the very end of the terminal time. Based on the leading-order approximation accuracy $O(\sqrt{\lambda}/T)$, we come up with the Stable-Transfer Hedging (ST-Hedging) algorithm, to smartly combine the leading-order approximation formulas and the deep learning algorithm as follows.

To start with, we adjust the deep learning-based algorithm. While we can obtain good performance under the current setup for the Deep Hedging algorithm, the hedging policy learned by the deep neural network becomes more and more unstable for longer time horizons. This is because the variance induced by the Brownian motion $W_t$ is proportional to the time $t$, hence the variance of hedging position $\varphi_t$ is going to blow up as $t$ increases. To make things worse, with more than one stock, the coupling effects will further contribute to the instability of $\varphi_t$. Instead, inspired by the variable choice of the FBSDE solver, we use the “fast variable”: $\Delta \varphi_t = \varphi_t - \bar{\varphi}_t$. As the apriori study in Almgren and Li (2016); Bayraktar et al. (2018); Guasoni and Weber (2020); Kallsen and Muhle-Karbe (2017); Moreau et al. (2017); Shreve and Soner (1994); Soner and Touzi (2013), the “fast variable” $\Delta \varphi_t - \bar{\varphi}_t$ is a mean-reverting process that quickly oscillates around zero. In particular, the variance of $\Delta \varphi_t$ is upper bounded, hence stabilized as time increases. As long as the deep learning algorithm on a short trading horizon is well-trained, then it can be easily adapted to arbitrarily long trading horizons without scaling up the errors. Numerical experiments have demonstrated the success of our variance reduction technique as in Sect. 4. This choice is reflected in the “stable” part of the ST-Hedging.

For the “transfer” part of ST-Hedging, we leverage transfer learning technique to automatically pick the optimal switching time from the approximation formulas to accurately learning of the optimal strategy via deep learning-based algorithm. By an initialized $\kappa$, we initialize our starting location $t_M$ near the end of the trading horizon by the suggestion from the approximation accuracy $T - t_M = O(\sqrt{\lambda})$. Then, with semi-well trained network, we simulate the rest of hedging positions through the stable hedging strategy, and compare it with the asymptotic hedging strategy of the accumulated gain. Then, we select the optimal $M^*$ such that the stable hedging loss
is less than the asymptotic optimal trading loss by the largest amount and convert it back to $\kappa = (T - t_{M^*})/\sqrt{\lambda}$. We repeat the process iteratively until $\kappa$ converges, which equivalently means the optimal switching time $t_{M^*}$ is obtained. Notice that we alternatively learn the optimal trading strategy and the optimal switching time, and the output of ST-Hedging is the time $t_{M^*}$ to switch from the asymptotic strategy, to the network parameters for the learnt trading strategy afterwards.

We summarize the ST-Hedging algorithm in Algorithm 3 as follows:

**Algorithm 3:** Training Procedure of ST-Hedging

Data: $\varphi^{(0)} = \varphi_0$, $W_t = 0$, $\gamma$, $\kappa$, $\lambda$, dynamic of processes $\mu$, $\sigma$, $\Lambda$, $\xi$, $\bar{\varphi}$, $\bar{a}$;

while epoch $\leq$ Epoch do
  sample $\Delta W$, batch_size $\times N$ iid Gaussian random variables with variance $\Delta t$;
  # Calculation of asymptotic optimal trading strategy;
  $m = 0$, $\Delta \varphi_0^{(0)} = 0$;
  while $m < N$ do
    $\varphi_{tm}^{(0)} = -\text{sign} \left( \Delta \varphi_{tm}^{(0)} \right) (G')^{-1} \left( g \left( \left| \Delta \varphi_{tm}^{(0)} \right|; \gamma, \sigma_{tm}, \bar{a}_{tm}, \lambda \Lambda_{tm} \right) / \lambda \Lambda_{tm} \right)$;
    $\varphi_{tm+1}^{(0)} = \varphi_{tm}^{(0)} + \varphi_{tm}^{(0)} \Delta t$;
    $\Delta \varphi_{tm+1}^{(0)} = \varphi_{tm+1}^{(0)} - \varphi_{tm+1}^{(0)}$;
    $W_{tm+1} = W_{tm} + \Delta W_m$;
    $m++$;
  end
  # Training of stable hedging strategy;
  $M = \min \left\{ m: T - t_m < \kappa \sqrt{\lambda} \right\}$, $m = M$, randomly initialize $\Delta \varphi_{tm}^{\theta}$;
  while $m < N$ do
    $\varphi_{tm}^{\theta} = \Delta \varphi_{tm}^{\theta} + \bar{\varphi}_{tm}$;
    $\varphi_{tm+1}^{\theta} = F_{\theta_m} (t_m, \Delta \varphi_{tm}^{\theta})$;
    $\Delta \varphi_{tm+1}^{\theta} = \varphi_{tm+1}^{\theta} \Delta t + \varphi_{tm}^{\theta} - \bar{\varphi}_{tm+1}$;
    $m++$;
  end
  Loss $= -\sum_{m=M}^{N} \left[ \varphi_{tm}^{\theta} \mu_{tm} - \frac{\gamma}{2} \left( \sigma_{tm} \varphi_{tm}^{\theta} + \xi_{tm} \right)^2 - \lambda \Lambda_{tm} \left( \varphi_{tm}^{\theta} \right) \right]$;
  Calculate the gradient of Loss with respect to $\theta$;
  Back propagate updates for $\theta$ via Adam (switch to SGD when fine tuning);
  epoch $++$;
if epoch $\%$ 1000 $==$ 0 then
  # Training of transfer time;
  $\kappa^* = \kappa$;
  while $\kappa^*$ Not Converged do
    $M = \min \left\{ m: T - t_m < \kappa^* \sqrt{\lambda} \right\}$;
    Loss$_0(M) = -\sum_{m=M}^{N} \left[ \varphi_{tm}^{(0)} \mu_{tm} - \frac{\gamma}{2} \left( \sigma_{tm} \varphi_{tm}^{(0)} + \xi_{tm} \right)^2 - \lambda \Lambda_{tm} \left( \varphi_{tm}^{(0)} \right) \right]$;
    Loss$_0(M) = -\sum_{m=M}^{N} \left[ \varphi_{tm}^{\theta} \mu_{tm} - \frac{\gamma}{2} \left( \sigma_{tm} \varphi_{tm}^{\theta} + \xi_{tm} \right)^2 - \lambda \Lambda_{tm} \left( \varphi_{tm}^{\theta} \right) \right]$;
    $M^* = \arg \min_M \left( \text{Loss}_0(M) - \text{Loss}_0(M) \right)$;
    $\kappa^* = (T - t_{M^*}) / \sqrt{\lambda}$;
  end
  $\kappa = \kappa^*$;
end
return: (local) optimizer $M^*, \theta^* = \{ \theta_m^*, m = M^*, \ldots, N \}$. 

4 Experiments and comparison results

In this section, the FBSDE solver in 3.1, the Deep Hedging algorithm in 3.2 as well as the proposed ST-Hedging algorithm in 3.3 are implemented. We test them on various models which even includes more than one stocks and/or with stochastic liquidity risk, and we discuss and document their advantages and disadvantages in these empirical implementations in Sect. 4.4.

To illustrate the performance of the FBSDE solver, the Deep Hedging and the ST-Hedging algorithm, we consider two calibrated Bachelier models without liquidity risk: a quadratic transaction costs model, and a general power costs model with elastic parameter $q = 3/2$. In all experiments, the parameters are from the calibration of real-world time series data in Gonon et al. (2021). The agent’s risk aversion is set to be $\gamma = 1.66 \times 10^{-13}$, the total shares outstanding in the market is $s = 2.46 \times 10^{11}$, and the stock return is $\mu = 0.072$ with the stock volatility being $\sigma = 1.88$. All models were trained using the free GPU service on Google Colab. The implementation details and codes can be found here: https://github.com/xf-shi/ML-for-Transaction-Costs/blob/main/README.md.

4.1 Quadratic Transaction Costs $G(x) = x^2 / 2$

With $G(x) = x^2 / 2$, on top of the leading-order approximation, the FBSDE system (2.13)–(2.14) becomes linear and the optimal trading strategy is given explicitly by:

$$\dot{\phi}_t = -\sqrt{\frac{\gamma \sigma^2}{\lambda}} \tanh\left( \sqrt{\frac{\gamma \sigma^2}{\lambda}} (T - t) \right) \Delta \phi_t.$$  (4.1)

The remaining parameters are taken from the calibrations from Section 5 in Gonon et al. (2021), where under quadratic costs the endowment volatility parameter $\xi = 2.19 \times 10^{10}$ and the liquidity level parameter is $\lambda = 1.08 \times 10^{-10}$. With the trading horizon $T = 10, 21, 42, 252, 2520$ trading days, the performance of both learning methods and the leading-order approximation are summarized in Tables 1, 2, 3, 4 and 5 and illustrated in Figs. 1, 2, 3, 4 and 5. To account for the large Monte Carlo error from the long trading horizons such as $T = 2520$ trading days (which is approximately 10 trading years), we evaluated the performance of our models using 100 million sample paths.

1 Although $\gamma$ is being small, it often comes in pair with $\phi_t$, hence still stays economically meaningful.

2 In Gonon et al. (2021), calibration is done for an equilibrium model of two agents where the asset returns depend on the risk aversion of both agents. In our single-agent model we just take the asset’s parameters to provide a more realistic numerical analysis. Although the models considered here are Bachelier, it can be generalized to Geometric Brownian motion models and the whole calculation and derivation follow if we switch our current analysis from shares positions to money in stock positions, with the unit of the transaction costs changed accordingly.

3 Details and derivations of the solution for the FBSDE system with constant quadratic costs (2.13)–(2.14) can be found in Appendix 1.
When the trading horizon is relatively short ($T \leq 42$ trading days), the performance of the FBSDE solver is the closest to the ground truth path-wisely, see Figs. 1, 2, 3. It means that when the FBSDE solver converges, it can learn the optimal trading strategy perfectly. It takes approximately 1–2 h to complete the training with fine tuning. However, as is already experienced in Gonon et al. (2021), when the trading horizon is longer than one trading year, the FBSDE solver fails to converge.

On the other hand, the Deep Hedging algorithm can still work well once the hyperparameters are appropriately tuned, as illustrated in Figs. 4 and 5 and Tables 4 and 5. However, we can see that under-fitting phenomena exist, and it becomes more severe as the trading horizon increases, meaning that the hyperparameters become harder to tune. The training time of Deep Hedging varies with the length of the trading horizon. For short horizons, the training time is approximately 3–4 h with fine tuning, which is longer than FBSDE solver. For longer trading horizons, the training and fine tuning time required can be as long as weeks, or even months, which is too long for practical use.

In addition, the performance of the leading-order asymptotic approximation (3.4) becomes better and better as the time horizon becomes longer. From the mean squared error of $\frac{\phi_T}{s}$ in the last column of Tables 1, 2, 3, 4 and 5 we can see that shortly around the terminal time, the leading-order approximation diverges substantially from 0. This observation is consistent with the figures of $\phi$ and $\varphi$ as in Figs. 1, 2, 3, 4 and 5. In other words, we are giving up the performance at the very end of the trading horizon to get an overall near-optimal performance without spending hours or days on tuning the hyperparameters in the machine learning algorithms. In fact, when the trading horizon is as large as $T = 2520$ trading days, leading-order approximation achieves the closest expected utility to the ground truth, while Deep Hedging is hard to train as the variance of the parametrized position $\varphi_t$ becomes very large with long trading horizon.

Finally, ST-Hedging leverages the mean-reverting fast variables $\Delta \phi_t$ to stabilize the variance of the parametrized variable. Therefore, comparing to Deep Hedging, it is scalable to arbitrarily large trading horizons. Moreover, ST-Hedging does not only achieve near optimal performance across all trading horizons, but also requires a much shorter training time to convergence. To wit, when the trading horizon is short, ST-Hedging, Deep Hedging, and FBSDE solver can achieve the same state-of-the-art performance which learn exactly the ground truth (see Tables 1, 2 and 3), while the leading-order approximation shows a clear gap, as in Figs. 1, 2 and 3. When the trading horizon gets larger, the ST-Hedging still maintains the state-of-art performance from the beginning to the end. Notice that in these long trading horizons, FBSDE solver can no longer converge. Deep Hedging shows a near optimal performance with fluctuation around the ground truth, but the training time increases significantly. While the leading-order approximation can achieve a near optimal utility, it deviates from the ground truth around the terminal time (see Tables 4, 5). For all trading horizons, ST-Hedging requires less fine tuning comparing to Deep Hedging, and the training can be finished within 2 h.
4.2 Power transaction costs \( G(x) = |x|^q/q \) with \( q = 3/2 \)

The analogous experiments for power costs with \( q = 3/2 \) are also done with calibrated parameters from Section 5 in Gonon et al. (2021), where the endowment volatility is \( \xi_{1.5} = 2.33 \times 10^{10} \) and liquidity level \( \lambda_{1.5} = 5.22 \times 10^{-6} \). In the general superlinear power transaction costs case, i.e. \( G(x) = |x|^q/q \) with \( q \in (1, 2) \), the
ground truth is no longer available in closed form. Nevertheless, we can still compare the numerical results from the machine learning algorithms and our leading-order asymptotic results. The models are evaluated using 100 million sample paths in order to account for the large Monte Carlo error from long trading horizons.

With the dynamics of the system being nonlinear, the FBSDE solver only converges on even shorter trading horizons, i.e., \( T < 21 \) trading days, shown in Figs. 6 and 7 and Tables 6 and 7. When the trading horizon is large, i.e., \( T \geq 42 \) days, the FBSDE solver fails to converge. The Deep Hedging algorithm obtains similar results as the FBSDE solver with short time horizons, as shown in Figs. 6 and 7.

![Fig. 3 Optimal trading rates \( \phi \) (left panel) and optimal positions \( \psi \) (right panel) for trading horizon \( T = 42 \) days with calibrated parameters under quadratic costs](image-url)

| Method               | \( J_T(\phi) \) ± std | \( \mathbb{E}[|\phi_T|^2/\sigma^2] \) |
|----------------------|------------------------|-----------------------------------|
| ST-Hedging           | \( 4.13 \times 10^9 \) ± \( 2.19 \times 10^9 \) | \( 1.26 \times 10^{-8} \) |
| Deep Hedging         | \( 4.13 \times 10^9 \) ± \( 2.20 \times 10^9 \) | \( 3.62 \times 10^{-9} \) |
| FBSDE Solver         | \( 4.13 \times 10^9 \) ± \( 2.20 \times 10^9 \) | \( 1.35 \times 10^{-8} \) |
| Leading-order        | \( 4.06 \times 10^9 \) ± \( 2.21 \times 10^9 \) | \( 7.89 \times 10^{-5} \) |
| Approximation        | \( 4.06 \times 10^9 \) ± \( 2.21 \times 10^9 \) | \( 7.89 \times 10^{-5} \) |
| Ground Truth         | \( 4.13 \times 10^9 \) ± \( 2.20 \times 10^9 \) | \( 0.0 \) |

![Table 3 Expectation and standard deviation of preference \( J_T \), and mean squared error of \( \phi_T/\sigma \) for trading horizon \( T = 42 \) days with calibrated parameters under quadratic costs](image-url)

| Method               | \( J_T(\phi) \) ± std | \( \mathbb{E}[|\phi_T|^2/\sigma^2] \) |
|----------------------|------------------------|-----------------------------------|
| ST-Hedging           | \( 3.97 \times 10^9 \) ± \( 3.22 \times 10^9 \) | \( 5.67 \times 10^{-8} \) |
| Deep Hedging         | \( 3.96 \times 10^9 \) ± \( 3.24 \times 10^9 \) | \( 2.16 \times 10^{-7} \) |
| FBSDE Solver         | \( 3.97 \times 10^9 \) ± \( 3.24 \times 10^9 \) | \( 1.04 \times 10^{-7} \) |
| Leading-order        | \( 3.93 \times 10^9 \) ± \( 3.24 \times 10^9 \) | \( 8.56 \times 10^{-5} \) |
| Approximation        | \( 3.97 \times 10^9 \) ± \( 3.24 \times 10^9 \) | \( 8.56 \times 10^{-5} \) |
| Ground Truth         | \( 3.97 \times 10^9 \) ± \( 3.24 \times 10^9 \) | \( 0.0 \) |
and 7 and Tables 6 and 7. With longer time horizons, the Deep Hedging algorithm can still work when the FBSDE solver fails to converge, see Figs. 8, 6 and 10 and Table 8, 9 and 10. However, when the trading horizon is as long as 10 trading years, Deep Hedging becomes suboptimal because of the large variance of the position $\varphi_t$. Similar as in the quadratic costs case, with the trading horizon increasing, the performance of the leading-order approximation is getting closer.
to the optimal results learnt by the machine learning algorithm, which justifies the approximation result of the leading-order approximation (3.4) empirically. ST-Hedging consistently achieves the best performance utility across all trading horizons (Tables 6, 7, 8, 9, 10). In short trading horizons, it achieves similar results as deep hedging (Figs. 6, 7, 8). As the trading horizon gets larger, the noise of Deep Hedging becomes more and more severe, while ST-Hedging is still stable (Figs. 9, 10). The training and fine tuning time for these algorithms are similar to the quadratic costs case, with ST-Hedging requires significantly less training time and effort for tuning than the other methods.

4.3 Multi-asset example via ST-hedging

To illustrate the scalability of our ST-Hedging algorithm, we consider a model with three risky assets in the market with cross sectional effect. In this experiment, the risky assets are driven by a 3-dim Brownian motion, and their corresponding annualized arithmetic variance is

\[ \Sigma = \begin{pmatrix} 72.00 & 71.49 & 54.80 \\ 71.49 & 85.42 & 65.86 \\ 54.80 & 65.86 & 56.84 \end{pmatrix}. \]

The three stocks have outstanding shares are \( 1.15 \times 10^{10}, 3.2 \times 10^9 \) and \( 2.3 \times 10^9 \), respectively. The annualized arithmetic returns for the three stocks are set to be 2.99, 3.71, and 3.55. Further, the risk aversion is set to be \( \gamma = 7.424 \times 10^{-13} \), and the endowment volatility matrix is set to be

\[ \xi = \begin{pmatrix} -2.07 & 1.91 & 0.64 \\ 1.91 & -1.77 & -0.59 \\ 0.64 & -0.59 & -0.20 \end{pmatrix} \times 10^9. \]

The transaction costs parameters are set as \( 1.269 \times 10^{-9}, 1.354 \times 10^{-9} \), and \( 1.595 \times 10^{-9} \). Finally, the trading time horizon is set to be \( T = 2520 \) trading days, which is 10 trading years, and the switching threshold we choose is 100 days before maturity.

The performance of the ST-Hedging algorithm and the comparisons to the leading-order formula and the ground truth are shown in Fig. 11 and

| Method               | \( J_T(\psi) \pm \text{std} \) | \( \mathbb{E}[|\dot{\psi}_T|^2/s^2] \) |
|---------------------|-------------------------------|----------------------------------|
| ST-Hedging          | \( 3.87 \times 10^9 \pm 2.47 \times 10^{10} \) | \( 8.46 \times 10^{-7} \) |
| Deep Hedging        | \( 3.32 \times 10^9 \pm 2.47 \times 10^{10} \) | \( 2.34 \times 10^{-7} \) |
| FBSDE Solver        | NaN                           | NaN                              |
| Leading-order       | \( 3.87 \times 10^9 \pm 2.47 \times 10^{10} \) | \( 8.63 \times 10^{-5} \) |
| Approximation       | \( 3.87 \times 10^9 \pm 2.47 \times 10^{10} \) | 0.0                              |
| Ground Truth        | \( 3.87 \times 10^9 \pm 2.47 \times 10^{10} \) | 0.0                              |
Table 11. We can easily see that the overall performance of ST-Hedging is comparable with the leading-order approximation, and can also accurately learn the trading strategy when it is close to maturity. Moreover, the training time for the ST-Hedging algorithm is significantly smaller than the Deep Hedging algorithm, with better scalability of the number of stocks in the market. The training and fine tuning time for multiple assets is still 1–2 h, which shows the scalability of ST-Hedging algorithm with dimensions.

Table 6. Expectation and standard deviation of preference $J_T$, and mean squared error of $\phi_T/s$ for trading horizon $T = 10$ days with calibrated parameters under $q = 3/2$ costs

| Method               | $J_T(\phi)$ ± std | $E[|\phi_T|^2/s^2]$ |
|----------------------|-------------------|---------------------|
| ST-Hedging           | $4.22 \times 10^9 \pm 1.62 \times 10^9$ | $3.62 \times 10^{-10}$ |
| Deep Hedging         | $4.22 \times 10^9 \pm 1.62 \times 10^9$ | $3.22 \times 10^{-10}$ |
| FBSDE Solver         | $4.22 \times 10^9 \pm 1.62 \times 10^9$ | $2.02 \times 10^{-10}$ |
| Leading-order Approximation | $4.11 \times 10^9 \pm 1.54 \times 10^9$ | $8.76 \times 10^{-5}$ |

Fig. 6 Optimal trading rates $\phi$ (left panel) and optimal positions $\varphi$ (right panel) for trading horizon $T = 10$ days with calibrated parameters under $q = 3/2$ costs

Fig. 7 Optimal trading rates $\phi$ (left panel) and optimal positions $\varphi$ (right panel) for trading horizon $T = 21$ days with calibrated parameters under $q = 3/2$ costs
4.4 Comparison of algorithms

Beyond the experiments in Sects. 4.1–4.3, we implement the FBSDE solver, the Deep Hedging algorithm and the ST-Hedging algorithm and test them in various market settings. The details and codes are available here: https://github.com/xf-shi/ML-for-Transaction-Costs.

Empirically, the appearance of the extreme value in the dynamic of the FBSDE solver may affect the calculation precision and thus jeopardize the performance. In particular, the performance and the hardness of tuning of the FBSDE solvers becomes significantly worse if we increase the number of forward and backward variables with respect to multiple stocks settings, because of the coupling in the

![Fig. 8](image-url) Optimal trading rates $\dot{\phi}$ (left panel) and optimal positions $\phi$ (right panel) for trading horizon $T = 42$ days with calibrated parameters under $q = 3/2$ costs

| Method             | $J_T(\dot{\phi}) \pm \text{std}$ | $E[|\dot{\phi}|^2/s^2]$ |
|--------------------|----------------------------------|--------------------------|
| ST-Hedging         | $4.02 \times 10^9 \pm 2.40 \times 10^9$ | $1.34 \times 10^{-10}$ |
| Deep Hedging       | $4.02 \times 10^9 \pm 2.42 \times 10^9$ | $1.68 \times 10^{-9}$ |
| FBSDE Solver       | $4.02 \times 10^9 \pm 2.42 \times 10^9$ | $4.55 \times 10^{-9}$ |
| Leading-order      | $3.93 \times 10^9 \pm 2.42 \times 10^9$ | $1.10 \times 10^{-4}$ |
| Leading-order      | $3.84 \times 10^9 \pm 3.39 \times 10^9$ | $1.11 \times 10^{-4}$ |

| Method             | $J_T(\phi) \pm \text{std}$ | $E[|\phi|^2/s^2]$ |
|--------------------|----------------------------|------------------|
| ST-Hedging         | $3.89 \times 10^9 \pm 3.37 \times 10^9$ | $1.36 \times 10^{-8}$ |
| Deep Hedging       | $3.89 \times 10^9 \pm 3.39 \times 10^9$ | $1.53 \times 10^{-8}$ |
| FBSDE Solver       | NaN                        | NaN              |
| Leading-order      | $3.84 \times 10^9 \pm 3.39 \times 10^9$ | $1.11 \times 10^{-4}$ |
Fig. 9 Optimal trading rates $\dot{\varphi}$ (left panel) and optimal positions $\varphi$ (right panel) for trading horizon $T = 252$ days with calibrated parameters under $q = \frac{3}{2}$ costs.

Table 9: Expectation and standard deviation of preference $J_T$, and mean squared error of $\dot{\varphi}_T / s$ for trading horizon $T = 252$ days with calibrated parameters under $q = \frac{3}{2}$ costs.

| Method                  | $J_T(\dot{\varphi}) \pm \text{std}$ | $\mathbb{E}[|\dot{\varphi}_T|^2/s^2]$ |
|-------------------------|--------------------------------------|--------------------------------------|
| ST-Hedging              | $3.84 \times 10^9 \pm 9.18 \times 10^9$ | $1.81 \times 10^{-8}$ |
| Deep Hedging            | $3.84 \times 10^9 \pm 9.18 \times 10^9$ | $9.27 \times 10^{-9}$ |
| FBSDE Solver            | NaN                                   | NaN                                   |
| Leading-order Approximation | $3.84 \times 10^9 \pm 9.18 \times 10^9$ | $7.93 \times 10^{-5}$ |

Fig. 10 Optimal trading rates $\varphi$ (left panel) and optimal positions $\varphi$ (right panel) for trading horizon $T = 2520$ days with calibrated parameters under $q = \frac{3}{2}$ costs.

Table 10: Expectation and standard deviation of preference $J_T$, and mean squared error of $\dot{\varphi}_T / s$ for trading horizon $T = 2520$ days with calibrated parameters under $q = \frac{3}{2}$ costs.

| Method                  | $J_T(\dot{\varphi}) \pm \text{std}$ | $\mathbb{E}[|\dot{\varphi}_T|^2/s^2]$ |
|-------------------------|--------------------------------------|--------------------------------------|
| ST-Hedging              | $3.78 \times 10^9 \pm 2.59 \times 10^{10}$ | $1.89 \times 10^{-8}$ |
| Deep Hedging            | $3.59 \times 10^9 \pm 2.59 \times 10^{10}$ | $8.38 \times 10^{-8}$ |
| FBSDE Solver            | NaN                                   | NaN                                   |
| Leading-order Approximation | $3.78 \times 10^9 \pm 2.59 \times 10^{10}$ | $8.24 \times 10^{-5}$ |
FBSDE system. In comparison, the Deep Hedging still works well, which shows its good scalability with respect to the increase of dimensions. A major drawback of both algorithms is they do not generalize well to long trading horizons. The FBSDE solver fails to converge even in intermediate trading time horizon, whereas the Deep Hedging algorithm still works but requires significant training time. To overcome the above mentioned difficulties in scalabilities, our proposed ST-Hedging algorithm fully utilizes the convenience of the leading-order formula and the accuracy of the learning-based algorithms. In our comparison experiments, ST-Hedging beats the performance of the leading-order formula and shows great scalability with respect to the increase of the stocks and the increase of trading time horizon, with less training time required. In fact, ST-Hedging can be generalized as long as the leading-order approximation is available. These comparison results help explain the leading-order asymptotic formula and provide us with ideas on the usage of each method in practice. Moreover, it opens door to the development of model-based algorithms. We summarize the observations (High/Medium/Low) in Table 12.

5 Conclusion and looking-forward

This paper studies optimal hedging problems in frictional markets with general convex transaction costs on the trading rates, and proposes the ST-Hedging algorithm to fully utilize the leading-order asymptotic formulas and the deep learning-based algorithms. Under the smallness assumption on the magnitude of the transaction costs, the leading-order approximation of the optimal trading speed can be identified through the solution to a nonlinear SDE. Models with arbitrary state dynamics generally lead to a nonlinear FBSDE system, but we can still solve the optimization numerically through learning-based algorithms. We implement the FBSDE solver, the Deep Hedging algorithms and the ST-Hedging algorithms and compare their performances with the leading-order approximation. The leading-order approximation works well under long trading horizons, while it deviates from the ground truth by a significant amount only shortly before maturity. The FBSDE solver only performs well under short trading horizons, while it fails to work in long trading horizons. In contrast, the Deep Hedging algorithm has stable and reliable performance for short and intermediate trading horizons, while it starts to get unstable when the trading horizon is getting large. In addition, the hyperparameter tuning of the Deep Hedging algorithm becomes inefficient when the model has more sophisticated dynamics. With the development of a transfer learning concept and the initial choice motivated by the order of the approximation error $O(\sqrt{1/T})$, the proposed ST-Hedging algorithm fully utilizes the leading-order asymptotic formula and learning-based algorithms. Indeed, ST-Hedging shows the best performance in all our experiments, with a drastically reduced training time. Moreover, the success of ST-Hedging algorithm shows great potential in model-based algorithms, to leverage the knowledge from the domain experts and the accuracy of learning-based algorithms.
This work is also a preliminary document for learning-based numerical solutions to frictional equilibrium models.

Table 11 Expectation and standard deviation of preference $J_T$, and mean squared error of $\varphi_T/s$ for trading horizon $T = 2520$ days with calibrated parameters under quadratic costs.

| Method                         | $J_T(\varphi) \pm \text{std}$ | $E[|\varphi_T|^2/s^2]$         |
|--------------------------------|--------------------------------|----------------------------------|
| ST-Hedging                     | $1.60 \times 10^{11} \pm 3.58 \times 10^6$ | $(2.4, 4.7, 3.6) \times 10^{-9}$ |
| Leading-order Approximation    | $1.60 \times 10^{11} \pm 3.58 \times 10^6$ | $(4.5, 3.5, 1.6) \times 10^{-7}$ |
| Ground Truth                   | $1.60 \times 10^{11} \pm 3.58 \times 10^6$ | $(0.0, 0.0, 0.0)$                |

Table 12 Comparison of learning-based algorithms

|                        | FBSDE solver | Deep hedging | ST-hedging |
|------------------------|--------------|--------------|------------|
| Scalability wrt time   | Low          | Medium       | High       |
| Scalability wrt dimension | Medium     | High         | High       |
| Convergence speed      | High (if converges) | Medium   | High       |
| Hardness of tuning     | Low (if converges) | Medium   | Low        |
| Sensitivity to calculation precision | High          | Medium       | Medium     |
| Adaptivity to multiple stocks | Medium      | High         | High       |
| Adaptivity to general utility functions | Low           | High         | High       |
| Adaptivity to general market dynamics | Medium    | High         | Medium     |
Appendix A: General Asymptotics Results

As already emphasized above, a general existence proof for the FBSDE system (2.13)–(2.14) remains a challenging open problem. To obtain tractable results with the general transaction costs under Assumption 2.1 and obtain explicit approximation trading strategies as in Cayé et al. (2018); Guasoni and Weber (2020), we focus on the financial market with the following assumptions:

Assumption A.1

(i) The processes \( \Lambda \) and \( \sigma^2 \) are bounded away from zero;

(ii) The processes \( \bar{a}, \bar{b}, \Lambda, \) and \( \sigma \) are essentially bounded Itô processes.

In this section, we establish the asymptotic optimal strategy for the frictional mean-variance preference (2.3). For better readability, we introduce the construction of optimal strategy under Assumption A.1 in Appendix A.1. The proof for the main approximation result is in Appendix A.

Asymptotically optimal trading strategies

We show that, under Assumption (), the smallness assumption on the transaction costs level \( \lambda \) should also be a relative quantity with respect to the trading time horizon \( T \). The following two results are the main ingredients for the asymptotic trading strategy.

A nonlinear ODE

The first ingredient to cook up the leading-order approximation is the solution to a nonlinear ODE, which is also essential to the analysis of Gonon et al. (2021), Lemma 3.4 and the formal analysis of Shi (2020), Lemma 2.5.

Lemma A.2 Suppose the instantaneous transaction cost \( G \) satisfies Assumption 2.1. Then the ordinary differential equation

\[
(G')^{-1} \left( \frac{g(x; \gamma, \sigma, \tilde{a}, \lambda)}{\lambda} \right) g'(x; \gamma, \sigma, \tilde{a}, \lambda) + \frac{\tilde{a}^2}{2} g''(x; \gamma, \sigma, \tilde{a}, \lambda) = \gamma \sigma^2 x.
\]

has a unique solution \( g \) on \( \mathbb{R} \) such that \( xg(x) \leq 0 \) for all \( x \in \mathbb{R} \). Moreover, \( g \) is odd, non-increasing on \( \mathbb{R} \) and \( g \) satisfies the growth conditions

\[
\lim_{x \to -\infty} \frac{g(x; \gamma, \sigma, \tilde{a}, \lambda)}{\lambda(G^*)^{-1}(\frac{\gamma \sigma^2}{2 \lambda} x^2)} = 1, \quad \lim_{x \to +\infty} \frac{g(x; \gamma, \sigma, \tilde{a}, \lambda)}{\lambda(G^*)^{-1}(\frac{\gamma \sigma^2}{2 \lambda} x^2)} = -1,
\]

where \( G^* \) is the Legendre transform of \( G \).

Remark A.3 If the time horizon \( T \) is large, then the solution to the optimal trading strategy should become stationary. Such a stationary solution should in turn solve...
the essential nonlinear ODE (3.5). Far from the terminal time $T$, it is natural to expect that the correct solution is still identified by the same growth condition in the space variable as (A.1).

For power functions $G(x) = |x|^q / q$, $q \in (1, 2]$, the Legendre transform is

$$G^*(x) = \sup_y \{ xy - G(y) \} = x(G')^{-1}(x) - G((G')^{-1}(x)) = |x|^p / p,$$

where $p = q / (q - 1)$ is the conjugate of $q$. In this case, with proper inner and outer rescaling coefficients, (3.5) is exactly the same ODE which plays an important role in Lemma 19 and Lemma 21 in Guasoni and Weber (2020) and Lemma 3.1 in Cayé et al. (2019).

**A fast mean-reverting SDE** The second ingredient is the existence and uniqueness of a strong solution to a fast mean-reverting SDE.

**Lemma A.4** Let $g$ be the solution to the ODE (3.5) from Lemma A.2. There exists a unique strong solution of the SDE

$$d \Delta_t = \left( (G')^{-1} \left( \frac{g(\Delta_t; \gamma, \sigma_t, \tilde{a}_t, \Lambda_t)}{\lambda \Lambda_t} \right) - \tilde{b}_t \right) dt - \tilde{a}_t dW_t, \quad \Delta_0 = \varphi_{0-} + \frac{\tilde{z}_0}{\sigma_0} - \frac{\mu_0}{\gamma \sigma_0^2}.$$  

Moreover, this process is a recurrent diffusion.

**Remark A.5** When $G(x) = x^2 / 2$ and $\tilde{b}$, $\tilde{a}$, $\sigma$ and $\Lambda = 1$ are all constants, the solution to (3.5) is $g(x; \gamma, \sigma, \tilde{a}, \lambda) = -\sqrt{\gamma \sigma^2 \lambda x}$, and the dynamics (A.2) becomes

$$d \Delta_t = - \left( \frac{\gamma \sigma^2}{\lambda} \Delta_t - \tilde{b} \right) dt - \tilde{a} dW_t, \quad \Delta_0 = \varphi_{0-} + \frac{\tilde{z}_0}{\sigma} - \frac{\mu_0}{\gamma \sigma^2}.$$  

This is an Ornstein-Uhlenbeck process, which is mean-reverting. In general, the requirement of $x g(x) \leq 0$ ensures that the dynamic (A.2) is indeed mean-reverting and converges to an ergodic limit.

With these two ingredients on hand, we now present our first results in the following theorem:

**Theorem A.6** Let $g$ be the solution to (3.5) and $(\Delta_t)_{t \geq 0}$ the solution to (A.2). Then under Assumption 2.1 and Assumption A.1, for all competing admissible strategies $\bar{\psi}$, we have

$$J_T(\bar{\psi}) \leq J_T \left( (G')^{-1} \left( \frac{g(\Delta_t; \gamma, \sigma_t, \tilde{a}_t, \Lambda_t)}{\lambda \Lambda_t} \right) \right) + O \left( \sqrt{\frac{\lambda}{T}} \right) + O \left( \sqrt{\lambda} \right).$$

Theorem A.6 shows that, under Assumption A.1 the smallness is not only an absolute quantity on $\sqrt{\lambda}$, but also on the relative quantity $\sqrt{\lambda} / T$, i.e. the smallness of $\lambda$
should also be relative quantity with respect to the trading time horizon. Notice that the order of the smallness is derived as a coarse upper bound for every transaction costs function $G$ that satisfies Assumption 2.1. In fact, with the specific form of the transaction costs $G_i$, we can have finer estimations, as in the example of quadratic costs shown in Corollary B.2. For the general power costs case and proportional costs case, we refer the reader to the discussion of Theorem 3.3 in Cayé et al. (2019) and Theorem 4.2 in Gonon et al. (2021).

**Proof of Sect. A.1**

The proof of Lemma A.2 follows the same procedure as the proof of Lemma 3.4 in Gonon et al. (2021), and Lemma A.4 follows the same procedure as the proof of Lemma 3.5 in Gonon et al. (2021).

Here we provide some auxiliary results on the function $g$ from (3.5), which are immediate results from the proof of Lemma 3.4 in Gonon et al. (2021).

**Corollary A.7** Let $g$ be the solution to (3.5) from Lemma A.2. Then the following holds:

1. There exists a constant $C_G > 0$ that only depends on $G$ such that for all $x \in \mathbb{R}$,
   \[
   |g(x; \gamma, \sigma, \bar{a}, \lambda)| \leq C_G \sqrt{\lambda} \left( \sqrt{\lambda} + \sqrt{\gamma \sigma^2 |x|} \right). \tag{A.4}
   \]
2. There exists a constant $K_G > 0$ that only depends on $G$ such that for all $x \in \mathbb{R}$,
   \[
   |g'(x; \gamma, \sigma, \bar{a}, \lambda)| \leq \sqrt{\gamma \sigma^2 \lambda} K_G. \tag{A.5}
   \]

**Corollary A.8** Let $g$ be the solution to (3.5) from Lemma A.2 and let the process $\Delta$ be the strong solution to (A.2) from Lemma A.4.

1. We have the following uniform moment bounds
   \[
   \sup_{T \geq 0} \mathbb{E}\left[ |\Delta_T|^k \right] < \infty, \quad \text{for all} \quad k \in \mathbb{N}. \tag{A.6}
   \]
2. There exists $M > 0$, such that for an arbitrary process $X$ with dynamic
   \[
   dX_t = \mu^X_t dt + \sigma^X_t dW_t,
   \]
   the following inequality holds a.s.:
We now analyze the terms on the right-hand side of (A.9). The inequality (A.7) from Lemma A.8 in turn yields

\[
\left| \int_0^T \left( \gamma \sigma_t^2 \Delta_t X_t + \mu_t^X g(\Delta_t; \gamma, \sigma_t, \bar{a}_t, \lambda \Lambda_t) + \sigma_t^X \bar{a}_t g'(\Delta_t; \gamma, \sigma_t, \bar{a}_t, \lambda \Lambda_t) \right) dt 
+ \int_0^T X_t a_t g'(\Delta_t; \gamma, \sigma_t, \bar{a}_t, \lambda \Lambda_t) dW_t - g(\Delta_T; \gamma, \sigma_T, \bar{a}_T, \lambda \Lambda_T) X_T \right| \leq \sqrt{\lambda M \int_0^T |X_t| dt}.
\]

(A.7)

**Proof of Theorem A.6** With the strategy, we write

\[
\hat{\varphi}_t = \varphi_{0-} + \int_0^t (G')^{-1} \left( \frac{g(\Delta_u; \gamma, \sigma_u, \bar{a}_u, \lambda \Lambda_u)}{\lambda \Lambda_u} \right) du = \varphi_{0-} + \Delta_t - \left( \frac{\mu_0}{\gamma \sigma^2} - \frac{\bar{\xi}_0 + \Delta_0}{\sigma} \right) = \bar{\varphi}_t + \Delta_t,
\]

hence with (2.10), we have

\[
\gamma \sigma^2 \Delta_t = \gamma \sigma^2 (\hat{\varphi}_t - \bar{\varphi}_t) = \gamma \sigma (\sigma \hat{\varphi}_t + \xi_t) - \mu_t.
\]

(A.8)

Consider a competing admissible strategy \( \psi \) and, to ease notation, set

\[
\hat{\theta}_t = \hat{\psi}_t - (G')^{-1} \left( \frac{g(\Delta_u; \gamma, \sigma_u, \bar{a}_u, \lambda \Lambda_u)}{\lambda \Lambda_u} \right)
\]

hence

\[
\theta_t = \int_0^t \hat{\psi}_u - (G')^{-1} \left( \frac{g(\Delta_u; \gamma, \sigma_u, \bar{a}_u, \lambda \Lambda_u)}{\lambda \Lambda_u} \right) du = \psi_t - \hat{\varphi}_t.
\]

Equation (A.8) and the convexity of \( G \) yield

\[
\begin{align*}
J_T(\psi) - J_T \left( (G')^{-1} \left( \frac{g(\Delta; \gamma, \sigma, \bar{a}, \lambda \Lambda)}{\lambda \Lambda} \right) \right) \\
= \frac{1}{T} \mathbb{E} \left[ \int_0^T \theta_t \mu_t - \frac{\gamma}{2} \bar{\theta}_t \sigma_t + \hat{\varphi}_t \sigma_t + 2 \bar{\xi}_t \sigma_t + \lambda \Lambda_t \left( G \left( (G')^{-1} \left( \frac{g(\Delta_u; \gamma, \sigma_u, \bar{a}_u, \lambda \Lambda_u)}{\lambda \Lambda_u} \right) \right) \right) - g(\psi_t) \right] dt \\
\leq \frac{1}{T} \mathbb{E} \left[ \int_0^T -\frac{1}{2} \gamma (\theta_t \sigma_t)^2 + \theta_t (\mu_t - \gamma (\varphi_t \sigma_t + \xi_t) \sigma_t) + \lambda \Lambda_t G \left( (G')^{-1} \left( \frac{g(\Delta_u; \gamma, \sigma_u, \bar{a}_u, \lambda \Lambda_u)}{\lambda \Lambda_u} \right) \right) \bar{\theta}_t \right] dt \\
= \frac{1}{T} \mathbb{E} \left[ \int_0^T -\frac{1}{2} \gamma (\theta_t \sigma_t)^2 - \gamma \theta_t \sigma_t^2 \Delta_t + g(\Delta_t; \gamma, \sigma_t, \bar{a}_t, \lambda \Lambda_t) \bar{\theta}_t \right] dt.
\end{align*}
\]

(A.9)

We now analyze the terms on the right-hand side of (A.9). The inequality (A.7) from Lemma A.8 in turn yields

\[
\mathbb{E} \left[ \int_0^T (\gamma \theta_t \sigma_t^2 \Delta_t + g(\Delta_t; \gamma, \sigma_t, \bar{a}_t, \lambda \Lambda_t) \bar{\theta}_t) dt \right] \\
\geq \mathbb{E} \left[ g(\Delta_T; \gamma, \sigma_T, \bar{a}_T, \lambda \Lambda_T) \theta_T \right] - \sqrt{\lambda M} \mathbb{E} \left[ \int_0^T \theta_t |dt| \right]
\]

(A.10)
Here, the local martingale part is a true martingale. Indeed, by Hölder’s inequality, the integrability condition \( \varphi \sigma \in H^2 \) and the boundedness of \( g' \) established in Corollary A.7,

\[
\mathbb{E} \left[ \int_0^t |\theta_u g'(\Delta_u;\gamma, \sigma_u, \bar{a}_u, \lambda \Lambda_u)|^2 du \right] \leq \gamma \lambda K_G^2 \mathbb{E} \left[ \int_0^t \sigma_u^2 \theta_u^2 du \right] < \infty.
\]

Also taking into account that

\[
|g(\Delta;\gamma, \sigma, \bar{a}, \lambda \Lambda)| \leq \sqrt{\gamma \sigma^2 \lambda \Lambda C_G |\Delta| + \lambda \Lambda C_G},
\]

we can therefore use (A.10) to replace the second and the third terms on the right-hand side of (A.9), obtaining

\[
J_T(\psi) - J_T \left( (G')^{-1} \left( \frac{g(\Delta;\gamma, \sigma, \bar{a}, \lambda \Lambda)}{\lambda \Lambda} \right) \right) \\
\leq \frac{1}{T} \mathbb{E}[g(\Delta_T;\gamma, \sigma_T, \bar{a}_T, \lambda \Lambda_T)|\theta_T] \\
- \mathbb{E} \left[ \int_0^T \frac{\gamma}{2} (\theta_t)^2 dt \right] + \frac{\sqrt{\lambda M}}{T} \int_0^T \mathbb{E}[|\theta_t|] dt.
\]

The Cauchy-Schwartz inequality yields

\[
|\mathbb{E}[g(\Delta_T;\gamma, \sigma_T, \bar{a}_T, \lambda \Lambda_T)|\theta_T]| \leq \left( \mathbb{E}[g(\Delta_T;\gamma, \sigma_T, \bar{a}_T, \lambda \Lambda_T)^2] \mathbb{E}[\theta_T^2] \right)^{1/2} \\leq \left( \mathbb{E}[2g(\Delta_T;\gamma, \sigma_T, \bar{a}_T, \lambda \Lambda_T)^2] \mathbb{E}([\psi_T]^2) + \mathbb{E}([\varphi_T]^2) \right)^{1/2} \\leq 2C_G \sqrt{\lambda} (\mathbb{E}([\psi_T]^2) + \mathbb{E}([\varphi_T]^2))^{1/2} (\gamma \sigma^2 \mathbb{E}[|\Delta|^2] + \lambda)^{1/2}.
\]

Moreover, it follows that

\[
\frac{1}{T} \mathbb{E}[|\psi_T|^2] = \frac{2}{T} \left( \mathbb{E}[|\varphi_T|^2] + \mathbb{E}[|\Delta_T|^2] \right) \leq 2 \sup_{T>0} \frac{1}{T} \left( \mathbb{E}[|\psi_T|^2] + \mathbb{E}[|\Delta_T|^2] \right) < \infty.
\]

Together with the transversality condition (2.5), it follows that

\[
\frac{1}{T} \mathbb{E}[g(\Delta_T;\gamma, \sigma_T, \bar{a}_T, \lambda \Lambda_T)|\theta_T] \leq \frac{2C_G \sqrt{\lambda}}{T} \left( \mathbb{E}([\varphi_T]^2) \right)^{1/2} (\gamma \sigma^2 \mathbb{E}[|\Delta|^2] + \lambda)^{1/2} = O \left( \frac{\sqrt{\lambda}}{T} \right),
\]

and again by Hölder’s inequality,

\[
\frac{1}{T} \mathbb{E} \left[ \int_0^T |\theta_t| dt \right] \leq \left( \frac{1}{T} \mathbb{E} \left[ \int_0^T \theta_t^2 dt \right] \right)^{1/2} \leq 2 \left( \sup_{T>0} \frac{1}{T} \mathbb{E} \left[ \int_0^T \varphi_t^2 + \psi_t^2 dt \right] \right)^{1/2}. \]

Therefore, the trading rate \( \varphi \) is indeed asymptotically optimal as:
Appendix B: Asymptotic results for quadratic costs

When the transaction costs is considered to be quadratic, i.e. $G(x) = \frac{x^2}{2}$, we have a linear function as $G^{-1}(x) = x$. Hence the forward equation (2.13) becomes linear with respect to the backward component $Y$, and the existence and uniqueness can be established as in Delarue (2002), Kohlmann and Tang (2002), provided the coefficients satisfies certain regularities. However, for general function $G$ satisfying Assumption (), the generator for the forward component is not globally Lipschitz hence no general theory is available for the FBSDE system (2.13)–(2.14).

A concrete example

As already emphasized above, a general existence proof for the FBSDE system (2.13)–(2.14) remains a challenging open problem. Let us just briefly sketch how the nonlinear FBSDE (2.13)–(2.14) reduces to a nonlinear PDE, with the following assumptions on the market:

Assumption B.1

(i) the frictionless strategy satisfies that $\bar{b}_t = 0$ and $\bar{a}_t = \bar{a}$;
(ii) the volatility process of the stock price remain constant $\sigma > 0$ in the models with and without transaction costs;
(iii) the cost parameter is constant ($\Lambda = 1$).

Under Assumption B.1, the forward-backward system (2.13)–(2.14) in turn becomes autonomous,

$$d\Delta \varphi_t = (G')^{-1}\left(\frac{Y_t}{\Lambda}\right)dt - \bar{a}dW_t, \quad \Delta \varphi_0 = \varphi_{0-} + \frac{\xi_0}{\sigma} - \frac{\mu_0}{\gamma \sigma^2}, \quad (B.1)$$
\[ dY_t = \gamma \sigma^2 \Delta \varphi_t \, dt + Z_t \, dW_t, \quad Y_T = 0. \]  

(B.2)

When the transaction costs is quadratic, i.e. \( G(x) = x^2 / 2 \), we can create the optimal strategy explicitly. Accordingly, the approximation can be made more precise, and we summarize the results as follows:

**Corollary B.2** For \( G(x) = x^2 / 2 \), define the strategy

\[ \hat{\varphi}_t = -\sqrt{\frac{\gamma \sigma^2}{\lambda}} \tanh \left( \sqrt{\frac{\gamma \sigma^2}{\lambda}} (T - t) \right) \Delta \varphi_t. \]  

(B.3)

Then under Assumption B.1, for all competing admissible strategies \( \psi \), we have

\[ \sup_{\psi} J_T(\psi) = J_T(\hat{\varphi}) = J_T \left( -\sqrt{\frac{\gamma \sigma^2}{\lambda}} \Delta \right) + O\left( \frac{\lambda}{T} \right) = J_T \left( -\sqrt{\frac{\gamma \sigma^2}{\lambda}} \Delta \right) + o\left( \frac{\sqrt{\lambda}}{T} \right), \]

where \( \Delta \) is the following Ornstein-Uhlenbeck process:

\[ d\Delta_t = -\sqrt{\frac{\gamma \sigma^2}{\lambda}} \Delta_t \, dt - \bar{a} \, dW_t, \quad \Delta_0 = \varphi_0^{-} + \frac{\xi_0}{\gamma} - \frac{\mu_0}{\gamma \sigma^2}. \]

**Remark B.3** The optimality of \( \hat{\varphi} \) by (B.3) is derived in Theorem 4.5 (Muhle-Karbe et al. 2020). The approximation order is a straight comparison between the candidate trading rate given by \( -\sqrt{\frac{\gamma \sigma^2}{\lambda}} \Delta \) and the optimal trading rate \( \hat{\varphi} \).

When there is no liquidity risk, i.e. when \( \Lambda = 1 \) throughout the trading horizon, the smallness assumption on the liquidity \( \lambda \) is purely a relative quantity, comparing to the trading horizon \( T \). Here with quadratic trading costs, the approximation is \( \lambda / T \), which is finer than the overall approximation \( \sqrt{\lambda / T} \).

**References**

Acharya, V. V., & Pedersen, L. H. (2005). Asset pricing with liquidity risk. *Journal of Financial Economics*, 77(2), 375–410.

Ahrens, L. (2015). On using shadow prices for the asymptotic analysis of portfolio optimization under proportional transaction costs.

Almgren, R. F. (2003). Optimal execution with nonlinear impact functions and trading-enhanced risk. *Applied Mathematics Finance*, 10(1), 1–18.

Almgren, R. F., & Chriss, N. (2001). Optimal execution of portfolio transactions. *Journal of Risk*, 3, 5–40.

Almgren, R.F., & Li, T.M. (2016). Option hedging with smooth market impact. *Market Microstructure Liquidity*, 2(1).

Almgren, R.F., Thum, C., Hauptmann, E., & Li, H. (2005). Direct estimation of equity market impact. *RISK*, July.

Amihud, Y., Mendelson, H., & Pedersen, L. H. (2006). Liquidity and asset prices. *Foundations and Trends in Finance*, 1(4), 269–364.

Ankirchner, S., & Kruse, T. (2015). Optimal position targeting with stochastic linear-quadratic costs. *Banach Center Publications*, 104(1), 9–24.
Bank, P., Soner, H. M., & Voß, M. (2017). Hedging with temporary price impact. *Mathematics Finance Economy, 11*(2), 215–239.

Bayraktar, E., Cayé, T., & Ekren, I. (2018). Asymptotics for small nonlinear price impact: A PDE homogenization approach to the multidimensional case. *Mathematics Finance, to appear*.

Beck, C., Hutzenthaler, M., Jentzen, A., & Kuckuck, B. (2020). An overview on deep learning-based approximation methods for partial differential equations. *arXiv:2012.12348*.

Becker, S., Cheridito, P., & Jentzen, A. (2019). Deep optimal stopping. *Journal of Machine Learning Research, 20*, 74.

Bouchard, B., Fukasawa, M., Herdegen, M., & Muhle-Karbe, J. (2018). Equilibrium returns with transaction costs. *Finance Stochalm, 22*(3), 569–601.

Buehler, H., Gonon, L., Teichmann, J., & Wood, B. (2019). Deep hedging. *Quantitative Finance, 19*(8), 1271–1291.

Buehler, H., Gonon, L., Teichmann, J., Wood, B., Mohan, B., & Kochems, J. (2019). Deep hedging: Hedging derivatives under generic market frictions using reinforcement learning. *Swiss Finance Institute Research Paper, (19–80)*.

Casgrain, P., Ning, B., & Jaimungal, S. (2019). Deep q-learning for nash equilibria: Nash-dqn. *arXiv:1904.10554*.

Cayé, T., Herdegen, M., & Muhle-Karbe, J. (2018) Trading with small nonlinear price impact. *Annals of Applied Probability, to appear*.

Cayé, T., Herdegen, M., & Muhle-Karbe, J. (2019). Trading with small nonlinear price impact. *Annals of Applied Probability, to appear*.

Choi, J. H., & Larsen, K. (2015). Taylor approximation of incomplete Radner equilibrium models. *Finance Stochalm, 19*(3), 653–679.

De Lattailade, J., Deremble, C., Potters, M., & Bouchaud, J.-P. (2012). Optimal trading with linear costs. *RISK, 1*(3), 1246–2047.

Delarue, F. (2002). On the existence and uniqueness of solutions to fbsdes in a non-degenerate case. *Stochastic Processes and their Applications, 99*(2), 209–286.

Dumas, B., & Luciano, E. (1991). An exact solution to a dynamic portfolio choice problem under transaction costs. *Journal of Finance, 46*(2), 577–595.

Ekeland, I., & Temam, R. (1999). *Convex analysis and variational problems*. Philadelphia, PA: SIAM.

Garleanu, N., & Pedersen, L. H. (2013). Dynamic trading with predictable returns and transaction costs. *Journal of Finance, 68*(6), 2309–2340.

Garleanu, N., & Pedersen, L. H. (2016). Dynamic portfolio choice with frictions. *Journal of Economic Theory, 165*, 487–516.

Goodfellow, I., Bengio, Y., & Courville, A. (2016). *Deep Learning*. Cambridge, MA: MIT Press.

Grohs, P., Hornung, F., Jentzen, A., & von Wurstemberger, P. (2018). A proof that artificial neural networks overcome the curse of dimensionality in the numerical approximation of black-scholes partial differential equations. Preprint.

Guasoni, P., Rásonyi, M., et al. (2015). Hedging, arbitrage and optimality with superlinear frictions. *Annals of Applied Probability, 25*(4), 2066–2095.

Guasoni, P., & Weber, M. (2017). Dynamic trading volume. *Mathematics of Finance, 27*(2), 313–349.

Guasoni, P., & Weber, M. H. (2020). Nonlinear price impact and portfolio choice. *Mathematical Finance, 30*(2), 341–376.

Han, J., Jentzen, A., & Solving, W. E. (2018). High-dimensional partial differential equations using deep learning. *Proceedings of the National Academy of Sciences, 115*(34), 8505–8510.

Han, J., & Long, J. (2020). Convergence of the deep bsd method for coupled fbsdes. *Probability, Uncertainty and Quantitative Risk, 5*(1), 1–33.

Han, J., & Weinan, E. (2016). Deep learning approximation for stochastic control problems.

Herdegen, M., & Muhle-Karbe, J. (2018). Stability of radner equilibria with respect to small frictions. *Finance and Stochastics, 22*(2), 443–502.

Hu, R. (2019) Deep fictitious play for stochastic differential games. *arXiv:1903.09376*.

Huré, C., Pham, H., Bachouch, A., Langrené, N. (2018). Deep neural networks algorithms for stochastic control problems on finite horizon, part i: convergence analysis. *arXiv:1812.04300*.

Ioffe, S., & Szegedy, C. (2015). Batch normalization: Accelerating deep network training by reducing internal covariate shift. In *International Conference on Machine Learning*, pages 448–456.
