Complex curves in hypercomplex nilmanifolds with $\mathbb{H}$-solvable Lie algebras

Yulia Gorginyan

July 27, 2022

Abstract

An operator $I$ on a real Lie algebra $\mathfrak{g}$ is called a complex structure operator if $I^2 = -\text{Id}$ and the $\sqrt{-1}$-eigenspace $\mathfrak{g}^{1,0}$ is a Lie subalgebra in the complexification of $\mathfrak{g}$. A hypercomplex structure on a Lie algebra $\mathfrak{g}$ is a triple of complex structures $I, J$ and $K$ on $\mathfrak{g}$ satisfying the quaternionic relations. We call a hypercomplex nilpotent Lie algebra $\mathbb{H}$-solvable if there exists a sequence of $\mathbb{H}$-invariant subalgebras

$$\mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \cdots \supset \mathfrak{g}_{k-1} \supset \mathfrak{g}_k = 0,$$

such that $[\mathfrak{g}_i^\mathbb{H}, \mathfrak{g}_i^\mathbb{H}] \subset \mathfrak{g}_{i+1}^\mathbb{H}$. We give examples of $\mathbb{H}$-solvable hypercomplex structures on a nilpotent Lie algebra and conjecture that all hypercomplex structures on nilpotent Lie algebras are $\mathbb{H}$-solvable. Let $(N, I, J, K)$ be a compact hypercomplex nilmanifold associated to an $\mathbb{H}$-solvable hypercomplex Lie algebra. We prove that, for a general complex structure $L$ induced by quaternions, there are no complex curves in a complex manifold $(N, L)$.

Contents

1 Introduction .......................... 2
   1.1 Complex nilmanifolds .................. 2
   1.2 $\mathbb{H}$-solvable Lie algebras .......... 3
   1.3 Examples of $\mathbb{H}$-solvable algebras .... 4
1 Introduction

1.1 Complex nilmanifolds

Recall that a nilmanifold $N$ is a compact manifold that admits a transitive action of a nilpotent Lie group $G$. Any nilmanifold is diffeomorphic to a quotient of a connected, simply connected nilpotent Lie group $G$ by a cocompact lattice $\Gamma$ [Mal].

A complex nilmanifold could be defined in two different ways. A complex parallelizable nilmanifold [W] is a compact quotient of a complex nilpotent Lie group by a discrete, cocompact subgroup. This is not the definition we use. We define a complex nilmanifold as a quotient of a nilpotent Lie group with a left-invariant complex structure by a left action of a cocompact lattice.

Nilmanifolds provide examples of non-Kähler complex manifolds. One of the first examples of a non-Kähler complex manifold was given by Kodaira [Has1], see also [Th]. It is a complex surface called the Kodaira surface with the first Betti number $b_1 = 3$. It was proven in [BG] that a complex nilmanifold does not admit a Kähler structure unless it is a torus. Moreover, in [Has] Hasegawa proved that a nilmanifold that is not a torus is never homotopically equivalent to any Kähler manifold.
Example 1.1: The Kodaira surface could be obtained as a quotient of the group of matrices of the form

\[
G := \begin{cases} 
\begin{pmatrix} 
1 & x & z & t \\
0 & 1 & y & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 
\end{pmatrix} : x, y, z, t \in \mathbb{R} 
\end{cases}
\]

by the subgroup \( \Gamma \) of the similar matrices with integer entries.

Example 1.2: The example of a complex nilmanifold that is obtained from a complex Lie group is an Iwasawa manifold. It is a compact quotient of the 3-dimensional complex Heisenberg group \( G \) by a cocompact, discrete subgroup \( \Gamma \) of the corresponding matrices with the Gaussian integer entries. Unlike the Kodaira surface, the Iwasawa manifold is parallelizable, that is, its tangent bundle is trivial as a holomorphic bundle.

1.2 \( \mathbb{H} \)-solvable Lie algebras

Let \( g = \text{Lie} G \) denote the Lie algebra of a nilpotent Lie group \( G \).

Definition 1.3: A subalgebra \( g^{1,0} \subset g \otimes_{\mathbb{R}} \mathbb{C} \) which satisfies \( g^{1,0} \oplus \overline{g^{1,0}} = g \otimes_{\mathbb{R}} \mathbb{C} \) and \([g^{1,0}, g^{1,0}] \subset g^{1,0}\) defines a complex structure operator \( I \in \text{End}(g) \). Subspaces \( g^{1,0} \) and \( \overline{g^{1,0}} = g^{0,1} \) are \( \sqrt{-1} \) and \( -\sqrt{-1} \) eigenspaces of the operator \( I \) respectively.

Let \( \mathbb{H} \) be the quaternion algebra. Recall that \( \mathbb{H} \) is generated by \( I, J \) and \( K \) satisfying the quaternionic relations:

\[
I^2 = J^2 = K^2 = -\text{Id}, \quad IJ = -JI = K. \tag{1.1}
\]

Definition 1.4: Let \( g \) be a nilpotent Lie algebra. A hypercomplex structure on \( g \) is a triple of endomorphisms \( I, J, K \in \text{End}(g) \) which satisfies the conditions (1.1) and defines the complex structures in the sense of Definition 1.3.

Denote by \( g_i^I := g_i + I g_i \) the smallest \( I \)-invariant Lie subalgebra which contains the commutator subalgebra \( g_i = [g_{i-1}, g], i \in \mathbb{Z}_{>0} \). Reformulating
the result of S. Salamon [S, Theorem 1.3], D. Millionschikov in [Mil, Proposition 2.5] has shown that

\[ [g^I_k, g^I_k] \subset g^I_{k+1} \]

and

\[ g^I_1 := [g, g] + I[g, g] \neq g. \]

Hence, a sequence of complex-invariant Lie subalgebras

\[ g \supset g^I_1 \supset \cdots \supset g^I_n = 0 \]

terminates for some \( n \in \mathbb{Z}_{>0} \). It is natural to ask a similar question about the hypercomplex nilpotent Lie algebras: is the algebra \( \mathbb{H}g_1 := [g, g] + I[g, g] + J[g, g] + K[g, g] \) equal to \( g \) or not?

We introduce a notion of an \( \mathbb{H} \)-solvable nilpotent Lie algebra. Define inductively \( \mathbb{H} \)-invariant Lie subalgebras: \( g^H_i := \mathbb{H}[g^H_{i-1}, g^H_{i-1}] \), where \( g^H_1 = \mathbb{H}[g, g] \).

**Definition 1.5:** A hypercomplex nilpotent Lie algebra \( g \) is called \( \mathbb{H} \)-solvable if there exists \( k \in \mathbb{Z}_{>0} \) such that

\[ g^H_1 \supset g^H_2 \supset \cdots \supset g^H_{k-1} \supset g^H_k = 0. \]

Such a filtration corresponds to an iterated hypercomplex toric bundle, see [AV]. Clearly, this holds if and only if \( g^H_{i-1} \subsetneq g^H_i \) for any \( i \in \mathbb{Z}_{>0} \).

There are no known examples of hypercomplex Lie algebras which are not \( \mathbb{H} \)-solvable.

**Conjecture 1.6:** All hypercomplex structures on a nilpotent Lie algebra \( g \) are \( \mathbb{H} \)-solvable.

### 1.3 Examples of \( \mathbb{H} \)-solvable algebras

An example of an \( \mathbb{H} \)-solvable Lie algebra is given by an abelian complex structure. Let us recall the definition.

**Definition 1.7:** Let \( g \) be a nilpotent Lie algebra with a complex structure. Suppose that \( [g^{1,0}, g^{1,0}] = 0 \). This complex structure is called an abelian complex structure.
It was already known that a nilpotent Lie algebra that admits an abelian hypercomplex structure is $\mathbb{H}$-solvable [AV, Proposition 4.5], see also [Rol, Corollary 3.11].

The main purpose of this article is to prove the following theorem:

**Theorem 1.8:** Let $(N, I, J, K)$ be a hypercomplex nilmanifold, and assume that the corresponding Lie algebra $g$ is $\mathbb{H}$-solvable. Then there are no complex curves in the complex nilmanifold $(N, L)$, where $L = aI + bJ + cK$, $L^2 = -\text{Id}$ for all $(a, b, c) \in S^2$ except of a countable set $R \subset S^2$.

To give an example of an $\mathbb{H}$-solvable Lie algebra with a non-abelian hypercomplex structure, we need a construction described below. The *quaternionic double* was introduced in the work [SV]. Let $(X, I_X)$ be a complex manifold which admits a torsion-free flat connection $\nabla : TX \to TX \otimes \Lambda^1 X$ which also satisfies $\nabla I = 0$. For a fixed point $x \in X$ consider the monodromy group $\text{Mon}(\nabla) \subset \text{GL}(T_x X)$. Suppose that there exists a lattice $\Lambda_x \subset T_x X$ in a fiber which is preserved by the action of the monodromy group: $\text{Mon}(\nabla) \Lambda_x = \Lambda_x$. Then we can construct the set $\Lambda \subset TX$ via all parallel transportation of $\Lambda_x$. Define a manifold $X^+ := TX/\Lambda$. It is fibered over $X$ with fibers $T_x X/\Lambda_x$ are compact tori. It makes sense since for each $x \in X$ the intersection $\Lambda \cap T_x X$ is a lattice continuously depending on $x \in X$. The manifold $X^+ = TX/\Lambda$ is called the *quaternionic double*. It was shown in [SV] that there is a pair of almost complex structures

$$I := \begin{pmatrix} I_X & 0 \\ 0 & -I_X \end{pmatrix}, \quad J := \begin{pmatrix} 0 & -\text{Id} \\ \text{Id} & 0 \end{pmatrix}$$

on $X^+$ which are integrable and satisfy the quaternionic relations.

**Theorem 1.9:** (Soldatenkov, Verbitsky) Let $X^+$ be the quaternionic double of an affine complex manifold $X$. If $X$ is non-Kähler, then $X^+$ does not admit an HKT-metric [SV].

It was shown in [DF], see also [BDV], that any abelian hypercomplex nilmanifold is HKT.

**Example 1.10:** To provide an example of an $\mathbb{H}$-solvable Lie algebra with a non-abelian hypercomplex structure, we first define the Kodaira surface, following [Has], see also [AV, Example 1.7]. Consider the Lie algebra $g =$
such that the only non-zero commutator is \([x, y] = z\). The complex structure is given by \(Ix = y\) and \(Iz = t\). There exists an operator \(\nabla^+ : g \times g \to g\) defined by the formula

\[
\nabla^+_a b := \frac{1}{2}([a, b] + I[a, b]), a, b \in g
\]

We extended \(\nabla^+\) to a left-invariant connection on the Lie group \(G\), also denoted by \(\nabla^+\). It is easy to see that the connection \(\nabla^+\) is complex-linear, torsion-free and flat. Therefore, we could define the quaternionic double of the Kodaira surface. It is a nilmanifold associated with the Lie algebra \(g^+ = g \oplus g\), with the commutator defined as follows:

\[
[(a, b), (c, d)] := ([a, b], \nabla^+_a d - \nabla^+_c b).
\]

The hypercomplex structure on \(g^+\) is defined as follows:

\[
I(a, b) = (Ia, -Ib), \quad J(a, b) = (-b, a), \quad K(a, b) = (-Ib, -Ia).
\]

Then

\[
g^+_1 := [g^+, g^+] = [g \oplus g, g \oplus g] = \langle \lambda z, \mu z \rangle, \lambda, \mu \in \mathbb{R}.
\]

Hence \(g^+_2 = 0\) (because there are no non-trivial commutators on the second step), which implies the \(\mathbb{H}\)-solvability of \(g^+\). From Theorem 1.9 it is clear that the hypercomplex structure on \(g^+\) is non-abelian.

This gives an example of an \(\mathbb{H}\)-solvable Lie algebra with a non-abelian complex structure.

**Example 1.11:** Let \((N, I, J, K)\) be a hypercomplex manifold. Fix a point \(p \in N\) and consider the set \(g^{(d)}\) of smooth vector fields such that \(X \in g^{(d)}\) has zero of order \(d\) at \(p\). Notice that the filtration \(g^{(d)}\) is \(\mathbb{H}\)-invariant with the commutator \([g^{(d)}, g^{(k)}] \subset g^{(d+k-1)}\). Therefore, the quotient \(g^{(2)}/g^{(n)}\) is an \(\mathbb{H}\)-solvable algebra.

**Acknowledgments:** I am thankful to Misha Verbitsky for turning my attention to this problem, his support and attention during the preparation of this paper.
2 Preliminaries

2.1 Nilpotent Lie algebras

Let $G$ be a real nilpotent Lie group, and $\mathfrak{g}$ its Lie algebra. The descending central series of a Lie algebra $\mathfrak{g}$ is the chain of ideals defined inductively:

$$
\mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_k \supset \cdots,
$$

where $\mathfrak{g}_0 = \mathfrak{g}$ and $\mathfrak{g}_k = [\mathfrak{g}_{k-1}, \mathfrak{g}]$. It is also called the lower central series of $\mathfrak{g}$.

**Definition 2.1:** A Lie algebra $\mathfrak{g}$ is called nilpotent if $\mathfrak{g}_k = 0$ for some $k \in \mathbb{Z}_{>0}$.

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{g}^*$ its dual space. Recall that for any $\alpha \in \mathfrak{g}^*$ the Chevalley–Eilenberg differential $d : \mathfrak{g}^* \rightarrow \Lambda^2 \mathfrak{g}^*$ is defined as follows

$$
d\alpha(\xi, \theta) = -\alpha([\xi, \theta]),
$$

where $\xi, \theta \in \mathfrak{g}$. It extends to a finite-dimensional complex

$$
0 \rightarrow \mathfrak{g}^* \rightarrow \Lambda^2 \mathfrak{g}^* \rightarrow \cdots \rightarrow \Lambda^n \mathfrak{g}^* \rightarrow 0 \quad (2.1)
$$

by the Leibniz rule: $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\tilde{\alpha}} \alpha \wedge d\beta$, where $\alpha, \beta \in \mathfrak{g}^*$ and $n = \dim_{\mathbb{R}} \mathfrak{g}$. The condition $d^2 = 0$ is equivalent to the Jacobi identity.

2.2 A short review of the Maltsev theory

Following Maltsev’s papers [Mal] and [Mal2] we are going to consider only nilpotent groups without torsion.

**Definition 2.2:** A nilmanifold is a compact manifold which admits a transitive action of a nilpotent Lie group.

Recall that a lattice $\Gamma$ is a discrete subgroup of a topological group $G$ such that there exists a regular finite $G$-invariant measure on the quotient $\Gamma \backslash G$.

The famous Maltsev’s theorem states that any nilmanifold $N$ is diffeomorphic to a quotient of a connected, simply connected nilpotent Lie group.
Let \( G \) be a cocompact lattice \( \Gamma \). The group \( \Gamma \) is isomorphic to the fundamental group \( \pi_1(N) \) of the nilmanifold \( N \) and \( G \) is the Maltsev completion of the group \( \Gamma \approx \pi_1(N) \) [Mal].

**Definition 2.3:** A group \( G \) is called complete if for each \( g \in G \) there exists \( n \in \mathbb{Z}_{>0} \) such that the equation \( x^n = g \) has solutions in \( G \).

**Definition 2.4:** Let \( G \) be a subgroup of a complete nilpotent group \( \hat{G} \). Suppose that for any \( g \in \hat{G} \) there exists \( n \in \mathbb{Z}_{>0} \) such that \( g^n \in G \). Then \( \hat{G} \) is called the Maltsev completion of a group \( G \).

Let \( g_\mathbb{Q} \) be a nilpotent Lie algebra over the field of rational numbers \( \mathbb{Q} \). We identify \( g_\mathbb{Q} \) with a subspace of a real nilpotent Lie algebra \( g = g_\mathbb{Q} \otimes \mathbb{R} \), and call \( g_\mathbb{Q} \) the rational lattice of \( g \).

**Definition 2.5:** A rational structure in a real nilpotent Lie algebra \( g \) is a rational lattice \( g_\mathbb{Q} \) such that \( g \cong g_\mathbb{Q} \otimes \mathbb{R} \).

**Definition 2.6:** Let \( \Gamma \) be a lattice in a connected, simply connected nilpotent Lie group \( G \). Then its associated rational structure is the \( \mathbb{Q} \)-span of \( \log \Gamma \subset g \), where \( g = \text{Lie} \, G \) is the Lie algebra. If \( g \) has a rational structure related to a \( \mathbb{Q} \)-algebra \( g_\mathbb{Q} \subset g \) then there exists a discrete subgroup \( \Gamma \) such that \( \log \Gamma \subset g_\mathbb{Q} \) and the quotient \( \Gamma \backslash G \) is compact [CG, Theorem 5.1.7].

Let \( \Gamma \) be a discrete subgroup of a connected, simply connected Lie group \( G \). By Theorem 2.7 the Maltsev completion \( \hat{\Gamma} \) is the unique closed connected subgroup of \( G \) such that a left quotient \( \Gamma \backslash \hat{\Gamma} \) is compact.

**Theorem 2.7:** (Maltsev) Let \( G \) be a finitely generated nilpotent group without torsion. Then there exists a nilpotent, complete and torsion-free group \( \hat{G} \) such that \( \hat{G} \) is a completion of \( G \). Moreover, all the completions of \( G \) are isomorphic [Mal2]. Finally, \( \hat{G} = \text{exp} \, g_\mathbb{Q} \) is the set of rational points in a real nilpotent Lie group \( G \) with the Lie algebra \( g \) admitting a rational lattice \( g_\mathbb{Q} \).

### 2.3 Hypercomplex nilmanifolds

Let \( X \) be a smooth manifold. Recall that an almost complex structure on \( X \) is an endomorphism \( I \in \text{End}(TX) \) satisfying \( I^2 = -\text{Id} \). The Nijenhuis
tensor $N_I$ associated to the almost complex structure $I$ is given by the formula
\[ N_I(X, Y) = [X, Y] + I[IX, Y] + I[X, IY] - [IX, IY]. \]

An almost complex structure is called **integrable** if its Nijenhuis tensor vanishes.

**Remark 2.8:** $N_I = 0$ if and only if $[TX^{1,0}, TX^{1,0}] \subset TX^{1,0}$.

**Theorem 2.9:** (Newlander–Nirenberg) If $I$ is an integrable almost complex structure on $X$, then $X$ admits the structure of a complex manifold compatible with $I$.

**Definition 2.10:** A **complex nilmanifold** is a pair $(N, I)$, where $N = \Gamma \backslash G$ is a nilmanifold obtained from a nilpotent Lie group $G$ and $I$ an integrable left-invariant almost complex structure on $G$.

By definition, $I$ is left-invariant if the left translations $L_g : (G, I) \rightarrow (G, I)$ are holomorphic. Notice that the Lie group $G$ does not need to be a complex Lie group, but in the case when it does both left and right translations on $G$ are holomorphic.

Let $X$ be a smooth manifold equipped with three integrable almost complex structures $I, J, K \in \text{End}(TX)$, satisfying the quaternionic relations $I^2 = J^2 = K^2 = -\text{Id}$ and $IJ = K = -JI$. Such a quadruple $(X, I, J, K)$ is called a **hypercomplex manifold**. Obata [Ob] proved that there exists a unique torsion-free connection $\nabla^{Ob}$ preserving the complex structures:
\[ \nabla^{Ob}I = \nabla^{Ob}J = \nabla^{Ob}K = 0. \]

The connection $\nabla^{Ob}$ is called the **Obata connection**.

A hypercomplex structure induces a complex structure $L = aI + bJ + cK$ for each $(a, b, c) \in \mathbb{R}^3$ such that $a^2 + b^2 + c^2 = 1$ and the set of such structures is identified in a natural way with $S^2 \cong \mathbb{C}P^1$.

Consider the product $X \times \mathbb{C}P^1$, where $X$ is a hypercomplex manifold. The **twistor space** $\text{Tw}(X)$ of the hypercomplex manifold $X$ is a complex manifold where the complex structure is defined as follows. For any point $(x, L) \in X \times \mathbb{C}P^1$ the complex structure on $T_{(x,L)} \text{Tw}(X)$ is $L$ on $T_xX$ and the standard complex structure $I_{\mathbb{C}P^1}$ on $T_L\mathbb{C}P^1$. This almost complex structure...
on the twistor space of a hypercomplex manifold is always integrable [K], [Besse, Theorem 14.68]. The space Tw(X) is equipped with the canonical holomorphic projection \( \pi : \text{Tw}(X) \to \mathbb{CP}^1 \). The fiber \( \pi^{-1}(L) \) at a point \( L \in \mathbb{CP}^1 \) is biholomorphic to the complex manifold \((X, L)\).

**Definition 2.11:** Let \( \Gamma \) be a cocompact lattice in a nilpotent Lie group \( G \) with a left-invariant hypercomplex structure. Then the manifold \( N = \Gamma \backslash G \) is called a hypercomplex nilmanifold.

### 2.4 Positive bivectors on a Lie algebra

Consider a nilpotent Lie group \( G \) with a left-invariant complex structure \( I \in \text{End}(T G) \). Recall that a complex structure operator on a Lie algebra \( g \) can be given by a decomposition of the complexification \( g_{\mathbb{C}} = g \otimes \mathbb{C} \) satisfying \( g_{\mathbb{C}}^1 = \{ X | X \in g_{\mathbb{C}}, I(X) = \sqrt{-1}X \} \) and \( [g^1_0, g^1_0] \subset g^1_0 \) by Definition 1.3.

Denote the \( k \)-th exterior power of \( g^1_0 \) (resp. \( g^0_1 \), \( g^1_0 \)) by \( \Lambda^k g^1_0 \) (resp. \( \Lambda^k g^0_1 \)). Consider the graded algebra of \((p, q)\)-multivectors \( \Lambda^* g \otimes \mathbb{C} = \bigoplus_{p,q} \Lambda^{p,q} g \).

**Definition 2.12:** The elements of the space \( \Lambda^{1,1} g \subset \Lambda^2 g \) are called \((1,1)\)-bivectors or just bivectors.

A non-zero real bivector \( \xi \in \Lambda^{1,1} g \) is called **positive** if for any non-zero \( \alpha \in \Lambda^1 g^* \) one has \( \xi(\alpha, I\alpha) \geq 0 \).

The Lie bracket gives a linear mapping \( \delta_1 : \Lambda^2 g \to g \), defined as \( \delta_1 : x \wedge y \mapsto [x, y] \). Such a mapping extends by the formula (2.2) below to the finite-dimensional complex of \( k \)-multivectors, i.e. \( \delta_m : \Lambda^{m+1} g \to \Lambda^m g \)

\[
0 \to \Lambda^{2n} g \xrightarrow{\delta_{2n-1}} \cdots \to \Lambda^2 g \xrightarrow{\delta_1} g \xrightarrow{\delta_0} 0
\]

and it is dual to the Chevalley-Eilenberg complex (2.1). The boundary operator \( \delta_{k-1} \) can be written as follows

\[
\delta_{k-1}(x_1 \wedge \cdots \wedge x_k) = \sum_{r<s} (-1)^{r+s+1} [x_r, x_s] \wedge x_1 \wedge \cdots \wedge \hat{x}_r \wedge \cdots \wedge \hat{x}_s \wedge \cdots \wedge x_k.
\]  

(2.2)

**Definition 2.13:** A complex curve in a complex manifold \((X, I)\) is a 1-dimensional compact complex subvariety \( C_I \subset X \).
Let $C_I \subset N$ be a complex curve in a complex nilmanifold $(N, I)$ and $\omega \in \Lambda^2 g^*$ a two-form. We identify $\Lambda^2 g^*$ with the space of left-invariant 2-forms on the Lie group $G$, which descends to the space of 2-forms $\Lambda^2(N)$ on the nilmanifold $N = \Gamma \backslash G$.

Consider a functional $\xi$ on the space of 2-forms $\Lambda^2 g^*$:

$$\xi_{C_I}(\omega) := \int_{C_I} \omega.$$  
(2.3)

Such a functional defines a bivector $\xi \in \Lambda^2 g^1$.

3 Positive bivectors on a quaternionic vector space

We start with a sequence of linear-algebraic lemmas. Let $V$ be a finite-dimensional vector space over $\mathbb{C}$ and $V^*$ its dual. Denote by $(V, I)$ the pair of a vector space $V$ with a complex structure $I \in \text{End}(V)$ on it.

Recall that the kernel of a bivector $\xi \in \Lambda^{1,1} V$ is the following set:

$$\ker \xi = \{ x \in V^* \mid \xi(x, \cdot, \cdot) = 0 \} \subset V^*.$$  
(3.1)

We denote the space of positive bivectors with respect to the complex structure $I$ on a vector space $V$ by $\Lambda^{1,1}_{I, \text{pos}} V$.

**Lemma 3.1:** Let $(V, I)$ be a vector space with a complex structure $I$ and $\xi \in \Lambda^{1,1}_{I, \text{pos}} V$ a non-zero positive bivector. Let $V_1^* := \{ x \in V^* \mid \xi(x, Ix) = 0 \}$. Then $V_1^* = \ker \xi$.

**Proof:** From the definition (3.1) it is obvious that $\ker \xi \subset V_1^*$. Suppose that $x \in V_1^*$ and $x \notin \ker \xi$. Then $0 \neq [x] \in V/\ker \xi$. On the space $V/\ker \xi$ the bivector $\xi$ is positive definite because it has no kernel and it is diagonalizable. Hence, $\xi(x, Ix) > 0$, which is a contradiction.

Let $W_1$ be a subspace of a vector space $(W, I)$ and consider two maps: $p : W \rightarrow W/W_1$ and $\overline{p} : \Lambda^2 W \rightarrow \Lambda^2(W/W_1)$. There are two subspaces of

$\text{H}^* = \text{H}^*(\mathfrak{g})$ by Theorem 4.1 a complex curve $C_L$ corresponds to the bivector $\xi_{C_L}$.  

---

1Since the homology $H_*(N) = H_*(\mathfrak{g})$ by Theorem 4.1 a complex curve $C_L$ corresponds to the bivector $\xi_{C_L}$.
Lemma 3.1: \( \Lambda^2 W : \Lambda^2 \ker p = W_1 \wedge W_1 \) and \( \ker \tilde{p} = W \wedge W_1 \). It is obvious that \( \Lambda^2 \ker p \subset \ker \tilde{p} \). We are going to show that \( \Lambda^1_{I, \text{pos}} W_1 \cap \ker \tilde{p} = \Lambda^1_{I, \text{pos}} W_1 \cap \Lambda^2 \ker p \).

**Proof:** Denote by \( W^\perp_1 \subset W^* \) the annihilator of the subspace \( W_1 \); it is isomorphic to the dual of the quotient \( (W/W_1)^* \). Since \( \xi \in W \wedge W_1 \), we have \( \langle W^\perp_1, W^\perp_1 \rangle = 0 \). Therefore, \( W^\perp_1 \subset \ker \xi \) by Lemma 3.1. So, \( \xi_{|_{W^\perp_1 \cap W^\perp_1}} = 0 \), which implies that \( \xi \in \Lambda^2 W_1 \).

Recall that for any pair of orthogonal complex structures \( I, J \in \mathbb{H} \), one has

\[
J(\Lambda^{p,q}_I V) = \Lambda^{q,p}_I V, \tag{3.2}
\]

where the action of the complex structures \( I \) and \( J \) extended from \( \Lambda^{1,0}_I V \) and \( \Lambda^{0,1}_I V \) to \( (p, q) \)-bivectors by a multiplicativity. Indeed, \( I \) and \( J \) anticommute on \( \Lambda^1(V) \) which implies \( J(\Lambda^{1,0}_I V) = \Lambda^{0,1}_I V \) and \( J(\Lambda^{0,1}_I V) = \Lambda^{1,0}_I V \).

Consider the operator \( W_I : \Lambda^* V_I \to \Lambda^* V_I \) defined by the formula \( W_I(\xi) = \sqrt{-1}(p-q)\xi \), where \( \xi \in \Lambda_I^{p,q} V_I \). Notice that the elements \( W_I, W_J, W_K \) generate the Lie algebra \( \mathfrak{su}(2) \) and the complex structures \( I, J \) and \( K \) are the the elements of the Lie group \( SU(2) \) related to them, \( I = \exp \frac{\pi W_I}{2}, J = \exp \frac{\pi W_J}{2} \) and \( K = \exp \frac{\pi W_K}{2} \) [V2].

**Lemma 3.3:** Let \( V \) be a quaternionic vector space and \( \Lambda^{1,1}_{I, \text{pos}} V \subset \Lambda^2 V \) the space of positive \((1,1)\)-bivectors on \((V, I)\). Then \( \Lambda^{1,1}_{I, \text{pos}} V \cap \Lambda^{1,1}_{I', \text{pos}} V = 0 \) for distinct complex structures \( I \) and \( I' \).

**Proof:** Denote the intersection \( R_{I, I'} := \Lambda^{1,1}_{I, \text{pos}} V \cap \Lambda^{1,1}_{I', \text{pos}} V \). First, suppose that \( I' = -I \), the non-zero bivector \( \xi \in R_{I,-I} \), and let \( \alpha \in V^* \). Then \( \xi(\alpha, I\alpha) > 0 \) and \( \xi(\alpha, -I\alpha) < 0 \), so there is no such a bivector \( \xi \).

Assume that \( I' \neq -I \). Then suppose that \( J \in \mathbb{H} \) is orthogonal to \( I \) and \( IJ = -JI, J^2 = -\text{Id} \). It is clear that as \( I \neq \pm I' \), \( I \) is not proportional to \( I' \), then \( I' \) can be written in a form \( I' = aI + bJ \) for some \( a, b \in \mathbb{R}, b \neq 0 \).

Let \( \xi \in R_{I, I'} \). Since \( \xi \) is a \((1,1)\)-bivector, \( W_I(\xi) = W_{I'}(\xi) = 0 \). Hence, \( W_K(\xi) = 0 \) because \( W_I \) and \( W_{aI + bJ} \) generate the Lie algebra \( \mathfrak{su}(2) \) [V2]. We obtain that \( \xi \) is an \( \mathfrak{su}(2) \)-invariant bivector. It is therefore invariant under the multiplicative action \( J(\xi)(\alpha, \beta) := \xi(J\alpha, J\beta) \) of \( J \in SU(2) \). Consider

\[
0 \leq \xi(\alpha, I\alpha) = \xi(J\alpha, JJ\alpha) = -\xi(J\alpha, I\alpha) = -\xi(\beta, I\beta), \tag{3.3}
\]
where $\beta = J\alpha$. However, $-\xi(\beta, I\beta) \leq 0$, hence $\xi = 0$. ■

**Corollary 3.4:** The intersection of the set of positive bivectors and $SU(2)$-invariant bivectors contains only zero bivector.

**Proof:** Follows from the formula (3.3). An invariant bivector has to be positive for different complex structures, which is impossible by Lemma 3.3. ■

### 4 Homology of a leaf of a foliation

Recall that a CW-space $X$ with the fundamental group $\pi_1(X) = \pi$ and the higher homotopy groups $\pi_i(X) = 0$ for $i > 1$ is called a $K(\pi, 1)$-space of Eilenberg–MacLane or just $K(\pi, 1)$-space. Since the universal covering of a nilpotent Lie group is contractible, the nilmanifold $N = \Gamma \backslash G$ with the fundamental group $\pi_1(N) \approx \Gamma$ is a $K(\Gamma, 1)$-space. The cohomology of the group $\Gamma$ is defined as $H^*(K(\Gamma, 1), \mathbb{Q})$.

In [N] Nomizu showed that the de Rham cohomology of a nilmanifold $N = \Gamma \backslash G$ can be computed using the left-invariant differential forms on the Lie group $G$.

**Theorem 4.1:** (Nomizu, [N, Theorem 1]) Let $N = \Gamma \backslash G$ be a nilmanifold and $(\Lambda^*g^*, d)$ is the Chevalley–Eilenberg complex. The natural inclusion of the complex of the left-invariant differential forms $\Omega^{inv}(G)$ on the nilpotent Lie group $G$ into the de Rham algebra on the nilmanifold $\Omega^{inv}(N)$ induces the isomorphism of the corresponding cohomology $H^*(g, \mathbb{R}) \approx H^*(N, \mathbb{R})$. ■

Since the nilmanifold $N$ is $K(\Gamma, 1)$, the homology $H_*(N, \mathbb{R}) \approx H_*(\Gamma, \mathbb{R})$, hence $H_*(\Gamma, \mathbb{R}) \approx H_*(g, \mathbb{R})$. Pickel showed that instead of real coefficients we can take the rational ones, i.e. $H^*(\Gamma, \mathbb{Q}) \approx H^*(g, \mathbb{Q})$ as well [P].

Consider a nilpotent Lie group $G$ and let $\Gamma$ be its discrete subgroup, $\hat{\Gamma} \subset G$ the Mal’tsev completion of $\Gamma$, and define the Lie group $\hat{\Gamma}_R := \exp(\log(\hat{\Gamma}) \otimes \mathbb{R})$. By Theorem 2.7, $\Gamma$ is a lattice in $\hat{\Gamma}_R$. Since the quotients $\Gamma \backslash G$ and $\Gamma \backslash \hat{\Gamma}_R$ are both $K(\Gamma, 1)$, we have $H_*(\Gamma \backslash G) = H_*(\Gamma \backslash \hat{\Gamma}_R)$. From Theorem 4.1 follows that $H_*(\Gamma \backslash \hat{\Gamma}_R) = H_*(\text{Lie} \hat{\Gamma}_R)$, where $\text{Lie} \hat{\Gamma}_R \subset g$ is the Lie algebra of $\hat{\Gamma}_R$ and $H_*(\text{Lie} \hat{\Gamma}_R)$ denotes the cohomology of the Chevalley–Eilenberg homology complex (2.1).
Let \( N = \Gamma \backslash G \) be a nilmanifold, \( \mathfrak{g} \) the Lie algebra of the Lie group \( G \) and \( \mathfrak{f} \subset \mathfrak{g} \) a Lie subalgebra. Let \( F := \exp \mathfrak{f} \subset G \) be the corresponding Lie group. For each \( x \in G \) define a subgroup of the lattice \( \Gamma \) as follows:

\[
\Gamma_x = \{ \gamma \in \Gamma \mid x\gamma x^{-1} \in F \}. \tag{4.1}
\]

In other words, \( \Gamma_x = \Gamma \cap x^{-1}F \).

Recall that a **distribution** on a smooth manifold \( X \) is a sub-bundle \( \Sigma \subset T X \). The distribution called **involutive** if it is closed under the Lie bracket. A **leaf** of the involutive distribution \( \Sigma \) is the maximal connected, immersed submanifold \( L \subset X \) such that \( TL = \Sigma \) at each point of \( L \). The set of all leaves is called a **foliation**.

The algebra \( \mathfrak{f} \) defines a left-invariant foliation \( \Sigma \) on \( G \). The leaves \( \mathcal{L}_x \) of the corresponding foliation on \( \Gamma \backslash G \) are diffeomorphic to \( \Gamma_x \backslash xF \) for each \( x \in G \).

**Definition 4.2:** A subalgebra \( \mathfrak{f} \subset \mathfrak{g} \) is said to be **rational** with respect to a given rational structure \( \mathfrak{g}_Q \) on \( \mathfrak{g} \) if \( \mathfrak{f}_Q := \mathfrak{g}_Q \cap \mathfrak{f} \) is a rational structure for \( \mathfrak{f} \), i.e. \( \mathfrak{f} = \mathfrak{f}_Q \otimes \mathbb{R} \).

**Remark 4.3:** The **rational homology of the leaf** \( \mathcal{L}_x \) of the foliation \( \Sigma \) is equal to

\[
H_*(\mathcal{L}_x, Q) := H_*(\Gamma_x \backslash xF, Q) = H_*(\mathfrak{f}_Q), \tag{4.2}
\]

where \( H_*(\mathfrak{f}_Q) \) is the homology of the complex dual to the rational Chevalley–Eilenberg complex (2.1). The last equality makes sense because of **Theorem 4.1** and \([P]\).

Consider the natural map of homology

\[
j : H_*(\mathcal{L}_x, Q) \longrightarrow H_*(N, Q), \tag{4.3}
\]

associated with the immersion \( \mathcal{L}_x \longrightarrow N \). Notice that \( j \) does not have to be injective.

**Claim 4.4:** Let \( N \) be a nilmanifold and \( X \subset N \) a subvariety tangent to the foliation \( \Sigma \) generated by the left translates of a Lie subalgebra \( \mathfrak{f} \subset \mathfrak{g} \). Then the fundamental class \([X] \in H_*(N, \mathbb{Q})\) belongs to the image of \( H_*(\mathfrak{f}_Q) \)
in $H_*(N, Q)$, where the map $\tau : H_*(f_Q) \longrightarrow H_*(N, Q)$ is obtained from (4.2) and (4.3).

**Proof:** Let $X$ be a subvariety in a leaf of the foliation $\Sigma \subset TN$ and let $F := \exp f$. A leaf of $\Sigma$ is diffeomorphic to $\Gamma_x \setminus xF$, hence $[X] \in \tau(H_*(\Gamma_x \setminus xF, Q))$, where we identified $H_*(\Gamma_x \setminus xF, Q) = H_*(f_Q)$.

5 Finale

Let $(\mathfrak{g}, I, J, K)$ be a hypercomplex nilpotent Lie algebra. Define inductively

$$\mathfrak{g}_i^H := \mathfrak{h}[\mathfrak{g}_{i-1}^H, \mathfrak{g}_{i-1}^H],$$

where $\mathfrak{g}_i^H = \mathfrak{h}[\mathfrak{g}, \mathfrak{g}]$ and let $a_i := \mathfrak{g}_i^H/\mathfrak{h}[\mathfrak{g}_{i-1}^H, \mathfrak{g}_{i-1}^H]$ be the corresponding commutative quotient algebra, $i \in \mathbb{Z}_{>0}$.

Observe that for any commutative Lie algebra $\mathfrak{a}$ its second homology group coincides with the space of all bivectors, $H_2(\mathfrak{a}, \mathbb{R}) = \Lambda^2 \mathfrak{a}$. Denote by $\Lambda_{L, pos}^{1,1} \mathfrak{a}$ the set of positive $(1,1)$-bivectors with respect to the complex structure $L$.

**Proposition 5.1:** Let $\mathfrak{a}$ be a commutative hypercomplex Lie algebra, and $\mathfrak{s} \subset \Lambda^2 \mathfrak{a}$ a countable set of non-zero bivectors. Then for all $L \in \mathbb{CP}^1$ except at most a countable number, the intersection $\Lambda_{L, pos}^{1,1} \mathfrak{a} \cap \mathfrak{s} = \emptyset$.

**Proof:** By Lemma 3.3 for any non-zero $\xi \in \mathfrak{s}$ there exists at most one complex structure $L_\xi \in \mathbb{CP}^1$ such that $\xi \in \Lambda_{L_\xi, pos}^{1,1} \mathfrak{a}$. The union $\bigcup_{\xi \in \mathfrak{s}} L_\xi$ is at most countable, hence for any $L \in \mathbb{CP}^1 \setminus \bigcup_{\xi \in \mathfrak{s}} L_\xi$ the intersection $\Lambda_{L, pos}^{1,1} \mathfrak{a} \cap \mathfrak{s}$ is empty. □

Let $\Sigma_i$ be the foliation on a nilmanifold $N$ generated by the left-translates of the Lie subalgebra $\mathfrak{g}_i^H \subset \mathfrak{g} = T_eG$ and $\mathfrak{L}_{x,i} \subset N$ a leaf of the foliation $\Sigma_i$. The leaf $\mathfrak{L}_{x,i}$ is diffeomorphic to the left quotient $\Gamma_x \setminus xF_i$, where $F_i = \exp \mathfrak{g}_i^H \subset G$.

Consider the natural projection

$$p_i : \mathfrak{g}_{i-1}^H \longrightarrow a_i.$$
Let \( r_i \) be the corresponding map of the second homology:
\[
r_i : H_2(g_i^{H^1}) \longrightarrow H_2(a_i) = \Lambda^2 a_i.
\]
Then the image of a homology class in \( H_2(g_i^{H^1}) \) defines a bivector on the commutative Lie algebra \( a_i \).

Denote by \( g_i^{H^1}_Q = g_i^{H^1} \cap g_Q \). Let \( s_i := r_i(H_2(g_i^{H^1}_Q)) \subset \Lambda^2 a_i \) and \( R \subset \mathbb{CP}^1 \) be a union of
\[
R_i := R[s_i] \subset \mathbb{CP}^1
\]
the set of complex structures \( L \) such that there exists a positive bivector \( \xi \in s_i \cap \Lambda^1_{L^{pos}} a_i \). By Proposition 5.1, the set \( R_i \) is countable.

**Definition 5.2:** Let \( \Sigma_k \) be a holomorphic foliation obtained from the Lie subalgebra \( g_k^L = g_k + Ig_k \). A transversal Kähler form \( \omega_k \) with respect to the holomorphic foliation \( \Sigma_k \) is a closed positive \((1,1)\)-form, such that \( \ker \omega_k \) is precisely the tangent space of the foliation, i.e. \( \omega_k(\Sigma_k) = 0 \).

**Proposition 5.3:** Let \( C_L \) be a complex curve in a complex nilmanifold \((N, L)\), where \( L \in \mathbb{CP}^1 \) \( R_i \) and the set \( R_i \subset \mathbb{CP}^1 \) is defined in (5.1). Suppose that \( C_L \) is tangent to the foliation \( \Sigma_{i-1} \) defined by \( g_i^{H^1} \) as above. Then it is also tangent to \( \Sigma_i \).

**Proof:** From Claim 4.4 it follows that the fundamental class \( [C_L] \in H_2(N, Q) \) of the curve \( C_L \) belongs to \( j(H_2(\Sigma_{i-1}, Q)) \subset H_2(N, Q) \), where \( j \) is the standard map on the rational second homology (4.3). Theorem 4.1 allows us to identify the fundamental class \( [C_L] \in H_2(N, Q) \) with the bivector \((2.3)\) \( \xi_{C_L} =: \xi \). Under the projection \( r_i \) the fundamental class \( [C_L] \) is mapped to the bivector \( r_i(\xi) \in \Lambda^2 a_i \). From the definition of the set \( R_i \) we know that \( \xi \in \ker r_i \) and from Lemma 3.2 follows that \( \xi \in \Lambda^1_{L^{pos}} \ker p_i = \Lambda^1_{L^{pos}} g_i^{H^1} \).

Suppose that \( \omega_{i-1} \in r_i^*(\Lambda^1_{L^{pos}} a_i^*) \) is a transversal Kähler form of the foliation \( \Sigma_{i-1} \). Then \( \int_{C_L} \omega_{i-1} > 0 \) unless \( C_L \) lies in the leaf of the foliation \( \Sigma_{i-1} \). However, \( \int_{C_L} \omega = 0 \) because \( \omega_{i-1} \) is closed (otherwise, referring to the analogue of Stokes’ theorem, we obtain that the volume of a compact manifold is equal zero). Since \( g_i^{H^1} \supset g_i^{H^1}_Q \) we have \( \omega_{i-1} \in \Lambda^1_{L^{pos}}(g_i^{H^1})^* \subset \Lambda^1_{L^{pos}}(g_i^{H^1^*})^* \). Hence, \( g_i^{H^1} \subset \ker \omega_{i-1} \). Hence, \( \omega_{i-1} \) is a transversal Kähler form with respect to the foliation \( \Sigma_i \) and \( C_L \) lies in a leaf of \( \Sigma_i \). \( \blacksquare \)
Assume that for some $k \in \mathbb{Z}_{>0}$ the following sequence terminates:

$$g^H_1 \supset g^H_2 \supset \cdots \supset g^H_{k-1} \supset g^H_k = 0,$$  \hspace{1cm} (5.2)

i.e. the Lie algebra $g$ is $H$-solvable, see also Definition 1.5.

**Corollary 5.4:** Let $L \in \mathbb{C}P^1 \setminus R$, where $R = \bigcup R_i$ is the countable subset defined in (5.1), and assume that the sequence (5.2) terminates to zero. Then the complex nilmanifold $(N, L)$ contains no complex curves.

**Proof:** Suppose that the sequence (5.2) vanishes on the $k$-th step, i.e. $\Sigma_k = \{0\}$. Then Corollary 5.4 follows from the Proposition 5.3 and the induction on $i$. ■

**Theorem 5.5:** Let $(N, I, J, K)$ be a hypercomplex nilmanifold and assume that the corresponding Lie algebra is $H$-solvable. Then there are no complex curves in the general fiber of the holomorphic twistor projection $T^w(N) \rightarrow \mathbb{C}P^1$.

**Proof:** Follows from Corollary 5.4. ■

**References**

[AV] Abasheva A., Verbitsky M., *Algebraic dimension and complex subvarieties of hypercomplex nilmanifolds*, https://doi.org/10.48550/arXiv.2103.05528 (Cited on pages 4 and 5.)

[BDV] Barberis M. L., Dotti I. G., Verbitsky M. *Canonical bundles of complex nilmanifolds, with applications to hypercomplex geometry*, Math. Res. Lett. 16 (2009), no. 2, 331–347. (Cited on page 5.)

[BG] Benson C., Gordon C. S., *Kähler and symplectic structures on nilmanifolds*, Topology, 27(4), 513–518, 1988 (Cited on page 2.)

[Besse] Besse A. L., *Einstein manifolds*, Classics in Mathematics, Springer-Verlag, Berlin, 2008, Reprint of the 1987 edition. (Cited on page 10.)

[CG] Corwin L., Greenleaf F. P., *Representations of Nilpotent Lie Groups and Their Applications. Part I, Basic Theory and Examples*, Cambridge Univ. Press, Cambridge, UK, 1990 (Cited on page 8.)
Yulia Gorginyan  
Complex curves in hypercomplex nilmanifolds with solvable Lie algebras

[DF] Dotti I., Fino A., *Hyperkähler torsion structures invariant by nilpotent Lie groups*, Class. Quantum Gravity, 2002 (Cited on page 5.)

[Has1] Hasegawa, K., *Complex and Kähler structures on compact solvmanifolds*, (English summary) Conference on Symplectic Topology. J. Symplectic Geom. 3 (2005), no. 4, 749–767 (Cited on page 2.)

[Has] K. Hasegawa. *Minimal models of nilmanifolds*, Proc. Amer. Math. Soc., 106(1): 65-71, 1989. (Cited on pages 2 and 5.)

[K] Kaledin D., *Integrability of the twistor space for a hypercomplex manifold*, Selecta Math. New Series, 4 (1998), 271–278 (Cited on page 10.)

[Mal] Maltsev A. I., *On a class of homogeneous spaces*, Izv. Akad. Nauk. Armyan. SSSR Ser. Mat. 13 (1949), 201-212. (Cited on pages 2, 7, and 8.)

[Mal2] Maltsev A. I., *Nilpotent torsion-free groups*, Izv. Akad. Nauk SSSR Ser. Mat., 13:3 (1949), 201–212 (Cited on pages 7 and 8.)

[Mil] Millionschikov D. V., *Complex structures on nilpotent Lie algebras and descending central series*, Rend. Semin. Mat. Univ. Politec. Torino 74 (2016), no. 1, 163–182. (Cited on page 4.)

[N] Nomizu K., *On the cohomology of compact homogeneous space of nilpotent Lie group*, Ann. of Math. (2) 59 (1954), 531-538. (Cited on page 13.)

[Ob] Obata M., *Affine connections on manifolds with almost complex, quaternionic or Hermitian structure*, Jap. J. Math., 26 (1955), 43-79. (Cited on page 9.)

[P] Pickel P. F., *Rational cohomology of nilpotent groups and Lie algebras*, Comm. Algebra 6 (1978), no. 4, 409–419. (Cited on pages 13 and 14.)
[Rol] Rollenske S., *Dolbeault cohomology of nilmanifolds with left-invariant complex structure*, Complex and differential geometry, 369-392, Springer Proc. Math., 8, Springer, Heidelberg, 2011. (Cited on page 5.)

[S] Salamon S. M., *Complex Structures on Nilpotent Lie Algebras*, J.Pure Appl. Algebra, 157 (2001), 311–333. (Cited on page 4.)

[SV] Soldatenkov A., Verbitsky M., *Holomorphic Lagrangian fibrations on hypercomplex manifolds*, International Mathematics Research Notices 2015 (4), 981-994 (Cited on page 5.)

[Th] Thurston W. P., *Some simple examples of symplectic manifolds*, Proc. Amer. Math. Soc. 55 (1976), 467-468. (Cited on page 2.)

[V2] Verbitsky M., *Quaternionic Dolbeault complex and vanishing theorems on hyperkahler manifolds*, Compos. Math. 143 (2007), no. 6, 1576–1592 (Cited on page 12.)

[W] Winkelmann J., *Complex analytic geometry of complex parallelizable manifolds*, Mém. Soc. Math. Fr. (N.S.) No. 72-73, 1998 (Cited on page 2.)

Yulia Gorginyan
Instituto Nacional de Matemática Pura e Aplicada (IMPA)
Estrada Dona Castorina, 110
Jardim Botânico, CEP 22460-320
Rio de Janeiro, RJ - Brasil
also:
Laboratory of Algebraic Geometry,
National Research University (HSE),
Department of Mathematics, 6 Usacheva Str.
Moscow, Russia
ygorginyan@hse.ru