DRINFELD-GAITSGORY FUNCTOR AND MATSUKI DUALITY

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Abstract. Let $G$ be a connected complex reductive group and let $K$ be a symmetric subgroup of $G$. We prove a formula for the Drinfeld-Gaitsgory functor for the dg-category $\text{Shv}(K\setminus X)$ of $K$-equivariant sheaves on the flag manifold $X$ of $G$ in terms of the Matsuki duality functor $[MUV]$. As byproducts, we obtain a description of the Serre functor for $\text{Shv}(K\setminus X)$, generalizing the one in $[BBM]$ in the case of category $\mathcal{O}$, and a formula for the Deligne-Lusztig duality for $(\mathfrak{g}, K)$-modules.

1. Introduction

1.1. For any compactly generated dg-category $\mathcal{C}$, Drinfeld-Gaitsgory $[G]$ introduced a certain canonical endofunctor $\text{Ps}-\text{Id}_\mathcal{C} : \mathcal{C} \to \mathcal{C}$, which they called the pseudo-identity functor, with many remarkable properties. Let $G$ be a connected complex reductive group and let $K$ be a symmetric subgroup of $G$. In this paper, we study the Drinfeld-Gaitsgory functor for the dg-category $\mathcal{C} = \text{Shv}(K\setminus X)$ of $K$-equivariant sheaves on the flag manifold $X$ of $G$. Here $\lambda$ is a character of Lie algebra of the abstract Cartan group of $G$. In this geometric setting, the Drinfeld-Gaitsgory functor

$$\text{Ps}-\text{Id}_{\text{Shv}(K\setminus X)_\lambda} : \text{Shv}(K\setminus X)_\lambda \to \text{Shv}(K\setminus X)_\lambda$$

has a concrete description: it is the functor given by the kernel

$$(\Delta_{K\setminus X})!(\mathcal{C}_{K\setminus X}) \in \text{Shv}(K\setminus X \times K\setminus X)_{-\lambda\lambda}$$

where $\mathcal{C}_{K\setminus X}$ is the constant sheaf on the quotient stack $K\setminus X$ and $\Delta_{K\setminus X} : K\setminus X \to K\setminus X \times K\setminus X$ is the diagonal map (see Section 3).

The main result of this paper is a formula for the Drinfeld-Gaitsgory functor for $\text{Shv}(K\setminus X)_\lambda$ in terms of the so-called Matsuki functors in $[MUV]$. To explain the formula, we first recall some notations and results in Lie theory. Let $G_\mathbb{R}$ be the real form of $G$ corresponding to $K$ under the Cartan bijection. We write $K_\mathbb{R} = K \cap G_\mathbb{R}$. The groups $K$ and $G_\mathbb{R}$ act naturally on $X$ with finitely many orbits and the celebrated Matsuki correspondence says that there is a natural bijection between the $K$-orbits and $G_\mathbb{R}$-orbits:

$$|K\setminus X| \simeq |G_\mathbb{R}\setminus X|.$$

We have the following functors, to be called the Matsuki functors:

$$\Upsilon^K_{\mathbf{!}} : \text{Av}_{K_\mathbb{R}/K_\mathbb{R}} \circ \text{oblv}_{K/K_\mathbb{R}} : \text{Shv}(K\setminus X)_\lambda \to \text{Shv}(G_\mathbb{R}\setminus X)_\lambda$$

$$\Upsilon^K_* : \text{Av}_{G_\mathbb{R}/K_\mathbb{R}} \circ \text{oblv}_{K/K_\mathbb{R}} : \text{Shv}(K\setminus X)_\lambda \to \text{Shv}(G_\mathbb{R}\setminus X)_\lambda$$

$$\Upsilon^G_{\mathbf{!}} : \text{Av}^G_{K/K_\mathbb{R}} \circ \text{oblv}_{G_\mathbb{R}/K_\mathbb{R}} : \text{Shv}(G_\mathbb{R}\setminus X)_\lambda \to \text{Shv}(K\setminus X)_\lambda$$

$$\Upsilon^G_* : \text{Av}^G_{G_\mathbb{R}/K_\mathbb{R}} \circ \text{oblv}_{G_\mathbb{R}/K_\mathbb{R}} : \text{Shv}(G_\mathbb{R}\setminus X)_\lambda \to \text{Shv}(K\setminus X)_\lambda.$$
Here for any Lie group $H$ and a Lie subgroup $H' \subset H$, the functor $\text{obl}v_{H/H'} : \text{Shv}(H \backslash X)_\lambda \to \text{Shv}(H' \backslash X)_\lambda$ is the natural forgetful functor and $\text{Av}_{H/H'}^*$ (resp. $\text{Av}_{H/H'}^!$) is its right (resp. left) adjoint. In [MUV], the authors updated the Matsuki correspondence 1.1 to the level of sheaves:

(1.2) The Matsuki functor $\Upsilon_\ast^{g \to Gr}$ (resp. $\Upsilon_!^{g \to K}$) is an equivalence with inverse given by $\Upsilon_!^{g \to K}$ (resp. $\Upsilon_\ast^{g \to K}$).

The following theorem is the main result of the note. Write $\text{Ps-Id}_{K \backslash X} = \text{Ps-Id}_{K(X)}$, for the Drinfeld-Gaitsgory functor for $(\text{Shv}(\text{K} \backslash X), \text{Theorem 1.1})$

\[ \text{Ps-Id}_{K \backslash X} \simeq \Upsilon_!^{g \to K} \circ \Upsilon_!^{K \to Gr} - \dim_G(X) + \dim_G(K) \]

It was shown in [GY, Theorem 2.1.5] that $\text{Ps-Id}_{K \backslash X}$ is isomorphic to the inverse of the Serre functor $\text{Se}_{K \backslash X} : \text{Shv}(K \backslash X)_\lambda \to \text{Shv}(K \backslash X)_\lambda$. Thus, in view of (1.2), we obtain

Corollary 1.2 (Theorem 6.4). There is a canonical isomorphism of functors $\text{Shv}(K \backslash X)_\lambda \to \text{Shv}(K \backslash X)_\lambda$:

\[ \text{Se}_{K \backslash X} \simeq \Upsilon_\ast^{g \to K} \circ \Upsilon_\ast^{K \to Gr} [\dim_G(X) - \dim_G(K)] \]

In fact, Theorem 6.3 and Theorem 6.4 are in a more general setting of Matsuki datum, see Section 4.

Remark 1.1. If $K$ (equivalently $K_r$ or $G_r$) is disconnected, one need to modify the formulas above for $\text{Ps-Id}_{K \backslash X}$ and $\text{Se}_{K \backslash X}$ by inserting the functor $(-) \otimes \pi_{K \backslash X}$ of tensoring with a certain rank one local system on $K \backslash X$ coming from the orientation sheaf of $K \backslash X$, see Remark 5.2. When $K$ is connected, $\text{or}_{K_r \backslash X}$ is trivial and we do not need such modification, see Remark 6.1.

Remark 1.2. The formulas above for the Drinfeld-Gaitsgory functor and the Serre functor generalize the one in [CGY, Theorem 3.4.2] and [BBM, Proposition 2.5] to the setting of real groups. Indeed, let $N$ be the unipotent radical of a Borel subgroup of $G$ and let $N^-$ be its opposite unipotent radical. In loc. cit. the authors showed that the Drinfeld-Gaitsgory functor $\text{Ps-Id}_{N \backslash X}$ and the Serre functor $\text{Se}_{N \backslash X}$ for the category $\text{Shv}(N \backslash X)$ of sheaves on the stack $N \backslash X$ (equivalently, the category $\mathcal{O}$) are isomorphic to

\[ \text{Ps-Id}_{N \backslash X} \simeq \Upsilon_!^{N^- \to N} \circ \Upsilon_!^{N \to N} [-2 \dim_C(X)] \quad \text{Se}_{N \backslash X} \simeq \Upsilon_\ast^{N^- \to N} \circ \Upsilon_\ast^{N \to N} [2 \dim_C(X)] \]

where the functors $\Upsilon_!^{N \to N}$, $\Upsilon_\ast^{N \to N} : \text{Shv}(N \backslash X) \to \text{Shv}(N^- \backslash X)$ are the so called long-intertwining functors. Our results suggest that the Matsuki functors are “real group” analog of the long-intertwining functors.

1.2. A formula for the Deligne-Lusztig duality for $(g, K)$-modules. In [GY], Gaitsgory-Yom Din introduced an analog of the Deligne-Lusztig duality for $(g, K)$-modules as the composition of the canonical duality $D^{\text{can}}$ and the contragredient duality $D^{\text{contr}}$:

\[ D^{\text{can}} \circ D^{\text{contr}} : g \text{-mod}^K \to g \text{-mod}^K \]

The main result in loc.cit. says that the Drinfeld-Gaitsgory functor $\text{Ps-Id}_{g \text{-mod}^K}$ for $(g, K)$-modules is isomorphic to the Deligne-Lusztig functor

\[ \text{Ps-Id}_{g \text{-mod}^K} \simeq D^{\text{can}} \circ D^{\text{contr}} \]

Since the category $g \text{-mod}^K$ is equivalent to $\text{Shv}(K \backslash X)_\lambda$ via the localization theorem and the Riemann-Hilbert correspondence (or rather the dg-category version, see, e.g., [P]), Theorem 1.1 gives a formula for the Deligne-Lusztig functor for $(g, K)$-modules in terms of Matsuki functors:
Corollary 1.3. We have an isomorphism of functors on $\mathfrak{g}\text{-mod}_K^\chi \simeq \text{Shv}(K\setminus X)_\lambda$:

$$D_{\text{can}} \circ D_{\text{contr}} \simeq \Upsilon_1^{G_R \to K} \circ \Upsilon_1^{K \to G_R}[-\dim_R(X) + \dim_R(K_R)]$$

Remark 1.3. It is interesting to note that the formula above for the Deligne-Lusztig duality for $(\mathfrak{g}, K)$-modules is not algebraic but only real analytic; as it involves the Matsuki functors going from $\text{Shv}(K\setminus X)_\lambda$ to $\text{Shv}(G_R\setminus X)_\lambda$. A similar phenomenon also appeared in the work of Barlet-Kashiwara [BK], where the authors described the functor on $\text{Shv}(K\setminus X)_\lambda$ which corresponds to the contragradient duality $D_{\text{contr}}$ for $(\mathfrak{g}, K)$-modules and they also observed that the functor is not algebraic but only real analytic. It will be interesting to compare the formula obtained here and the one in loc. cit.

Remark 1.4. It would be nice to have an analog of Corollary 1.3 for $G_R$-representations. Note that we have similar formulas in the real setting: the Drinfeld-Gaitsgory and the Serre functor for $\text{Shv}(G_R\setminus X)_\lambda$ are isomorphic to:

$$\text{Ps-Id}_{G_R\setminus X} \simeq \Upsilon_1^{K \to G_R} \circ \Upsilon_1^{G_R \to K}[-\dim_R(X) + \dim_R(K_R)]$$

$$\text{Se}_{G_R\setminus X} \simeq \Upsilon_*^{K \to G_R} \circ \Upsilon_*^{G_R \to K}[\dim_R(X) - \dim_R(K_R)].$$

In [KSd], Kashiwara-Schmid constructed a functor from $\text{Shv}(G_R\setminus X)_\lambda$ to (certain) derived category of $G_R$-representations, known as Kashiwara-Schmid localization. Similar to the case of $(\mathfrak{g}, K)$-modules, we expect that the Kashiwara-Schmid localization intertwines the Drinfeld-Gaitsgory functor $\text{Ps-Id}_{G_R\setminus X}$ with the composition $D_{\text{coh}} \circ D_{\text{contr}}$ where $D_{\text{contr}}$ and $D_{\text{coh}}$ are the the contragredient duality and (some version) of cohomological duality for $G_R$-representations respectively. At the level of virtual $G_R$-representations (up to infinitesimal equivalence), this follows from the fact that the Matsuki functor intertwines the Kashiwara-Schmid localization with the Beilinson-Bernstein localization (see, e.g., [SV, Section 2]).

1.3. Further directions.

1.3.1. Theory of tilting perverse sheaves. In [BBM], the authors use the theory of tilting perverse sheaves on $N\setminus X$ to derive the formula for Serre functor for category $\mathcal{O}$. It would be nice if one can do the same for $(g, K)$-modules: study tilting perverse sheaves on $K\setminus X$ or $G_R\setminus X$ and explore its connection with the Drinfeld-Gaitsgory functor and the Serre functor.

1.3.2. Affine Matsuki duality. In [CN], Nadler and the author prove a affine version of the Matsuki duality, to be called the affine Matsuki duality, which relates the category of sheaves on the moduli stack $\text{Bun}_{G_k}$ of real bundles over real projective line with the category spherical sheaves on the loop space $X((t))$ of the symmetric variety $X = G/K$. The results of the paper suggest that there is a close relationship between the Drinfeld-Gaitsgory functor for those categories and the affine Matsuki duality. Note that $\text{Bun}_{G_k}$ and $X((t))$ are the main players for the geometric Langlands for real groups (proposed by Ben-Zvi and Nadler) and derived Satake equivalence for symmetric varieties (proposed by Ben-Zvi, Venkatesh, and Sakellaridis), and we expect that the affine Matsuki duality (and its relation with the Drinfeld-Gaitsgory functor) would help us to understand some “duality” phenomena in both subjects.

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2. CONSTRUCTIBLE SHEAVES ON A SEMI-ANALYTIC STACK

2.1. We will be working with \( \mathbb{C} \)-linear dg-categories (see [DG Section 0.6] for a concise summary of the theory of DG categories). Unless specified otherwise, all dg-categories will be assumed complete, i.e., contains all small colimits, and all functors between dg-categories will be assumed continuous, i.e., preserves all small colimits.

Recall that a subset \( S \) of a real analytic manifold \( M \) is called semi-analytic if any point \( s \in S \) has an open neighbourhood \( U \) such that the intersection \( S \cap U \) is a finite union of sets of the form \( \{ s \in U \mid f_1(s) = \cdots = f_r(s) = 0, g_1(s) > 0, \ldots, g_t(s) > 0 \} \),

where the \( f_i \) and \( g_j \) are real analytic functions on \( U \). A map \( f: S \to S' \) between two semi-analytic sets is called semi-analytic if it is continuous and its graph is a semi-analytic set.

We collect some facts about constructible sheaves on a semi-analytic stack following [AGKRRV1 Appendix C]. For any semi-analytic set \( S \), we define \( \text{Shv}(S) = \text{Ind}(\text{Shv}(S)^{\text{constr}}) \) to be the ind-completion of the bounded dg-category \( \text{Shv}(S)^{\text{constr}} \) of \( \mathbb{C} \)-constructible sheaves on \( S \). For any semi-analytic stack \( S \) we define \( \text{Shv}(S) := \lim_S \text{Shv}(S) \) where the index category is that of semi-analytic sets equipped with a semi-analytic map to \( S \), and the transition functors are given by \(!\)-pullback. Since we are in the constructible context \(!\)-pullback admits left adjoint, given by \(!\)-pushforward, and it follows that \( \text{Shv}(S) = \text{colim} \text{Shv}(S) \). In particular, \( \text{Shv}(S) \) is compactly generated. We let \( \text{Shv}(S)^c \) be the full subcategory of compact objects and \( \text{Shv}(S)^{\text{constr}} \subset \text{Shv}(S) \) be the full subcategory of objects that pullback to an object of \( \text{Shv}(S)^{\text{constr}} \) for any \( S \) mapping to \( S \). Note that we have natural inclusion \( \text{Shv}(S)^c \subset \text{Shv}(S)^{\text{constr}} \) but the inclusion is in general not an equality. For example, the constant sheaf \( \mathbb{C}_S \in \text{Shv}(S)^{\text{constr}} \) for the classifying stack \( S = B(\text{GL}_1(\mathbb{C})) \) is not compact.

Let \( f : S \to S' \) be a map between semi-analytic stacks. We have the usual six functor formalism \( f_*, f^!, f_!, f^*, \otimes, \underline{\text{Hom}} \). For any \( F_1, F_2 \in \text{Shv}(S) \) we denote by \( F_1 \otimes^! F_2 := \Delta_S^!(F_1 \boxtimes F_2) \), where \( \Delta_S : S \to S \times S \) be the diagonal map.

2.2. Renormalized direct image. The usual direct image functor \( f_* : \text{Shv}(S) \to \text{Shv}(S') \) is in general not continuous (see, e.g., [DG Example 7.1.4]). We define the renormalized direct image (2.1)

\[
f_{*,\text{ren}} : \text{Shv}(S) \to \text{Shv}(S')
\]

to be the unique colimit-preserving functor such that \( f_{*,\text{ren}} = f_* \) on \( \text{Shv}(S)^c \).

Following [AGKRRV1 Appendix C], a semi-analytic stack \( S \) is called duality-adapted if the Verdier duality functor

\[
\mathbb{D} : (\text{Shv}(S)^{\text{constr}})^{\text{op}} \simeq \text{Shv}(S)^{\text{constr}}
\]

sends \( \text{Shv}(S)^c \) to \( \text{Shv}(S)^c \). In this paper we are mainly interested in semi-analytic stacks \( S \) of the form \( S = H\backslash S \) where \( H \) is Lie group acting semi-analytically on a semi-analytic set \( S \). According to [AGKRRV1 Theorem C.2.6], any such stack is duality-adapted. By [AGKRRV1 C.3.6], given a map \( f : S \to S' \), where \( S \) is duality-adapted, the functor \( f_{*,\text{ren}} \) satisfies the projection formula

\[
f_{*,\text{ren}}(F) \otimes^! F' \simeq f_{*,\text{ren}}(F \otimes^! f^!(F'))
\]

and base change: for a Cartesian diagram

\[
\begin{array}{ccc}
S_1 & \xrightarrow{g_1} & S_2 \\
\downarrow{f_1} & \quad & \downarrow{f_2} \\
S'_1 & \xrightarrow{g_2} & S'_2
\end{array}
\]

\footnote{In loc. cit., the authors work with constructible sheaves on a complex semi-analytic stack but the discussion works for real semi-analytic stacks.}
where $S$ is a local system on $\mathcal{S}$ following [AGKRRV2]. Let $f_*$ be a Lie subgroup and consider the quotient map $H\to H/U$ where or $f_*=\mathcal{H}^{-d_f}(\omega_f)$ is a local system on $S$ called the (relative) orientation sheaf of $f$. Note that we have or $f\otimes or_f\cong C_S$. Moreover, have a canonical isomorphism of functors

$$f^*\otimes\omega_f\cong f^*\otimes or_f[d_f] \cong f^!.$$  

In the case when $S'=pt$ we will write $d_S=d_f$, $\omega_S=\omega_f$, or $S=or_f$, etc.

2.4. **Averaging functors.** Let $X$ be a real analytic manifold acted on by a Lie group $H$. Let $U \subset H$ be Lie subgroup and consider the quotient map $f: U\backslash X \to H\backslash X$. We have the forgetful functor

$$\text{obl}_H/U=f^*: \text{Shv}(H\backslash X)\to \text{Shv}(U\backslash X)$$

with right adjoint

$$\text{Av}^H_{/U}=f_*: \text{Shv}(U\backslash X)\to \text{Shv}(H\backslash X)$$

(note that $f_*=f_{*,\text{ren}}$ as $f$ is schematic) and left adjoint

$$\text{Av}^H_{/U}=f_!\circ ((-)\otimes or_f)[d_f]: \text{Shv}(U\backslash X)\to \text{Shv}(H\backslash X)$$

here $(-)\otimes or_f: \text{Shv}(U\backslash X)\to \text{Shv}(U\backslash X)$ is the functor of tensoring with the orientation sheaf or $f$.

3. **The Drinfeld-Gaitsgory functor**

3.1. We recall the definition of Drinfeld-Gaitsgory functor in the setting of constructible sheaves following [AGKRRV2]. Let $S$ be semi-analytic stack. An object $\Omega \in \text{Shv}(S \times S)$ defines a functor

$$F_\Omega: \text{Shv}(S)\to \text{Shv}(S), \quad \mathcal{F} \to (p_2)_{*,\text{ren}}(\Omega \otimes (p_1)^!(\mathcal{F}))$$

where $\pi: S \times S \to S$, $p_2: S \to S$ are the projection maps. We shall call $F_\Omega$ the functor given by the kernel $\Omega$. Consider the diagonal embedding $\Delta_S: S \to S \times S$. It is straightforward to check that the identity functor

$$\text{Id}_S: \text{Shv}(S)\to \text{Shv}(S)$$

is given by the kernel $(\Delta_S)_!(\omega_S)$. The **Drinfeld-Gaitsgory functor**:

$$\text{Ps-Id}_S: \text{Shv}(S)\to \text{Shv}(S)$$

is by definition the functor given by the kernel $(\Delta_S)_!(C_S)$ where $C_S$ is the constant sheaf on $S$. 

Remark 2.1. Note that the usual direct image functor $f_*$ for a map $f: S \to S'$ between semi-analytic stacks might not satisfy the projection formula or base change (see [DG, Example 7.7.6]) and this is one of the reason to introduce the renormalized version $f_{*,\text{ren}}$. However, according to [DG, Corollary 10.2.5], if the map $f$ is schematic (or more generally, the map is “safe” as defined in loc. cit.) then we have $f_*=f_{*,\text{ren}}$ and hence $f_*$ satisfies the projection formula or base change.
4. Matsuki duality

4.1. We recall the definition of “Matsuki datum” introduced in [MUV, Section 5].

**Definition 4.1.** Matsuki datum $(G, X, H^+, H^-)$ consists of a linear Lie group $G$ acting on a real analytic manifold $X$ and two subgroups $H$ and $H^-$ of $G$ such that there is a Bott-Morse function $f : X \to \mathbb{R}$ invariant under $H^+ \cap H^- = U$ and a $U$-invariant metric on $X$, satisfying the following:

1. The gradient flow $\Phi$ of $f$ on $X$ preserves the $H^\pm$-orbits and $U$ has finitely many orbits in the fixed point $X^\Phi$ set of the flow.
2. For any $U$-orbit $O$ in $X^\Phi$ denote
   $$O^\pm = \{ x \in X | \Phi_t(x) \in O \text{ as } t \text{ goes to } \pm \infty \}. $$

Then $O^\pm$ are single $H^\pm$-orbits and the correspondence $O^+ \leftrightarrow O^-$ is a bijection between $H^+$- and $H^-$-orbits on $X$.
3. The $H^+$-and $H^-$-orbits on $X$ are transversal.
4. The quotient $H^\pm/U$ is contractible.

**Remark 4.1.** Note that if $(G, X, H^+, H^-)$ is a Matsuki datum then so is $(G, X, H^-, H^+)$. 

**Example 4.2.** Let $G$ be a connected complex reductive group and let $X$ be the flag manifold of $G$.

1. Let $P = MN$ be a parabolic subgroup with unipotent radical $N$ and let $P^- = MN^-$ be its opposite. Then $(G, X, MN, MN^-)$ is a Matsuki datum.
2. Let $G_\mathbb{R}$ be a real form of $G$ with a maximal compact subgroup $K_\mathbb{R} \subset G_\mathbb{R}$ and let $K \subset G$ be its complexification. According to [MUV], $(G, X, G_\mathbb{R}, K)$ (resp. $(G, X, K, G_\mathbb{R})$) is a Matsuki datum.

4.2. Given a Matsuki datum $(G, X, H^+, H^-)$, one can consider the functors

$$\Upsilon^!_{H^+ \to H^-} = Av^H-/U \circ \text{oblv}_{H^+/U} : \text{Shv}(H^+/X) \to \text{Shv}(H^-\setminus X)$$

$$\Upsilon_*^H- \to H^+ = Av_*^{H+/U} \circ \text{oblv}_{H^-/U} : \text{Shv}(H^-\setminus X) \to \text{Shv}(H^+/X).$$

**Theorem 4.2.** The functor $\Upsilon^!_{H^+ \to H^-}$ is an equivalence with inverse given by $\Upsilon_*^{H^- \to H^+}$.

**Proof.** In the setting of bounded derived category this is the main result in [MUV, Theorem 5.3]. In the setting of dg-category, it follows from the alternative argument in [CY, Theorem 5.2] or [CGY, Proposition 1.4.2]. Namely, we would like to show that the canonical adjunctions $\text{Id} \to \Upsilon^*_{H^- \to H^+} \circ \Upsilon^!_{H^+ \to H^-}$ and $\Upsilon^!_{H^- \to H^+} \circ \Upsilon_*^{H^- \to H^+} \to \text{Id}$ are isomorphisms. Consider the category $\text{Shv}(U\setminus X)_{\text{sm}} \subset \text{Shv}(U\setminus X)$ consisting of $U$-equivariant complexes whose cohomology sheaves are smooth along the trajectories of the Morse flow $\Phi$. We have $\text{oblv}_{H^+/U}(F) \in \text{Shv}(U\setminus X)_{\text{sm}}$ and as explained in loc. cit. the desired claim follows from the fact that, on the category $\text{Shv}(U\setminus X)_{\text{sm}}$, the functor $Av^H+/U$ and $Av_*^{H^-/U}$ act as identity on hyperbolic restriction to the fixed points set $X^\Phi$. 

5. Singular support and transversal property

5.1. We recall some basic definition and properties of singular support of a complex of sheaves. The reference is [KSa]. Let $S$ be a real analytic manifold. Given a closed conical semi-analytic subset $\Lambda \subset T^*S$ we denote by $\text{Shv}_\Lambda(S)_{\text{const}} \subset \text{Shv}(S)_{\text{const}}$ the full dg-category of bounded constructible complexes $F$ with singular support $\text{sing}(F) \subset \Lambda$ and $\text{Shv}_\Lambda(S) \subset \text{Shv}(S)$ be the ind-completion of $\text{Shv}_\Lambda(S)_{\text{const}}$. 

\[ \text{In loc. cit., we work in the setting of “complex analytic” Matsuki datum where all the varieties are complex analytic, but the same argument works equally well in the semi-analytic setting.} \]
Let $S$ be a smooth semi-analytic stack, and let $T^*S$ be the cotangent bundle. Recall that for any smooth map $f : S \to S$ where $S$ is a real analytic manifold, we have a correspondence

$$T^*S \xrightarrow{df} T^*S \times_S S \xrightarrow{pr} T^*S.$$  

For any conical closed semi-analytic sub-stack $\Lambda \subset T^*S$, and a smooth map $f : S \to S'$ as above we denote by

$$\Lambda_S := df(pr^{-1}(\Lambda)) \subset T^*S$$

and we define $\text{Shv}_A(S)^{\text{constr}} \subset \text{Shv}(S)^{\text{constr}}$ to be the full dg-category of bounded constructible complexes $\mathcal{F}$ such that

$$\text{sing}(f^*\mathcal{F}) \subset \Lambda_S.$$ 

We let $\text{Shv}_A(S) \subset \text{Shv}(S)$ be the ind-completion of $\text{Shv}_A(S)^{\text{constr}}$.

**Definition 5.1.** Let $f : S \to S'$ be a map between smooth semi-analytic stacks. We say that a complex $\mathcal{F} \in \text{Shv}(S)$ is transversal with respect to $f$ if there is a closed conical semi-analytic subset $\Lambda \subset T^*S$ such that (1) $\mathcal{F} \in \text{Shv}_A(S)$ and (2) $\Lambda \times_{T^*S} (T^*S' \times_S S) = \text{zero section } T^*_S S$ of $T^*S$.

**Example 5.1.** Let $X$ be a compact real analytic manifold acted on by two Lie groups $H^+$ and $H^-$ and let $U \subset H^+ \cap H^-$ be a Lie subgroup. Consider the natural map $f^\pm : S = U/X \to S' = H^\pm \backslash X$. We claim that if $H^-$ and $H^+$-orbits on $X$ are transversal then, for any $\mathcal{F} \in \text{Shv}(H^- \backslash X)$, its image $\text{oblv}_{H^+ \backslash U}(\mathcal{F}) \in \text{Shv}(U \backslash X)$ is transversal with respect to $f^+$. Indeed, let $\Lambda^\pm \subset T^*(U \backslash X)$ be the image of $df^\pm : T^*(H^\pm \backslash X) \times_{H^\pm \backslash X} U/X \to T^*(U \backslash X)$. Then $\Lambda^\pm = T_{H^\pm \backslash X} \subset T_{U \backslash X}$ (here $T_U(X)$, etc, stands for the conormal bundle of $U$-orbits on $X$) and we have (1) $\mathcal{F} \in \text{Shv}_{A_+}(U \backslash X)$ and (2) $\Lambda^\pm \times_{T^*S} T^*(S' \times_S S) = \Lambda^\pm \cap \Lambda^\pm = T_{H^+ \backslash X} \cap T_{H^- \backslash X} / U$ which is the zero section $T_{U \backslash X}(U \backslash X)$ as $T_{H^- \backslash X} \cap T_{H^+ \backslash X} = T_{X \backslash X}$.  

Let $X$, $U$, $H = H^+$, and $f = f^+ : U \backslash X \to H \backslash X$ be as in the Example 5.1. Let $\mathcal{O}_U \backslash X$ be the orientation sheaf of $U \backslash X$. We assume $H/U$ is contractible. Then the map $f : U \backslash X \to H \backslash X$ is smooth with contractible fibers there exists a unique rank one local system $\ell_{H \backslash X}$ on $H \backslash X$ such that $f^*\ell_{H \backslash X} \simeq \mathcal{O}_U \backslash X$. Recall the pseudo-identity functor $\text{Ps-Id}_{H \backslash X} : \text{Shv}(H \backslash X) \to \text{Shv}(H \backslash X)$. Let $U_c$ be a maximal compact subgroup of $U$. Note that the quotient $U/U_c$ is contractible. The following lemma is an analog of [CGY] Theorem 3.2.6] in the real analytic setting:

**Lemma 5.2.** There is a canonical defined morphism of functors $\text{Shv}(U \backslash X) \to \text{Shv}(H \backslash X)$:

(5.1) $(- \otimes \ell_{H \backslash X}) \circ \mathcal{O}_U^{H/U} \to \text{Ps-Id}_{H \backslash X} \circ \mathcal{O}_U^{H/U}[d_X - d_{U_c}]$. 

The map (5.1) is an isomorphism when evaluated on objects which are transversal with respect the projection $U \backslash X \to H \backslash X$.

**Proof.** We follow the argument in loc.cit. closely. All the direct image functors below should be understood as the renormalized version in Section 2.1 and, for simplicity, we still use the notation $(p_1)_* = (p_1)_{*, \text{ren}}$, etc. Consider the following diagram

$$
\begin{array}{ccc}
U \backslash X & \xrightarrow{\Delta} & U \backslash X \times H \backslash X & \xrightarrow{p_1} & U \backslash X \\
\downarrow f & & \downarrow f \times \text{id} & & \downarrow f \\
H \backslash X & \xrightarrow{\Delta} & H \backslash X \times H \backslash X & \xrightarrow{p_1} & H \backslash X \\
\downarrow p_2 & & \downarrow p_2 \\
H \backslash X & & & & 
\end{array}
$$
Let $\mathcal{F} \in \text{Shv}_U(X)$. We have

\[
(- \otimes \ell_{H \setminus X}) \circ \text{Av}^{H/U}_1(\mathcal{F}) \simeq \ell_{H \setminus X} \otimes f_!(\mathcal{F} \otimes \text{or}_f)[df] \simeq f_!(\mathcal{F} \otimes \text{or}_f \otimes \ell^{*}(\ell_{H \setminus X}))[df] \simeq f_!(\mathcal{F} \otimes \text{or}_f \otimes \text{or}_{U \setminus X})[df] \simeq (\tilde{p}_2)_! \circ \tilde{\Delta}_!(\mathcal{F} \otimes \text{or}_f \otimes \text{or}_{U \setminus X})[df] \simeq (\tilde{p}_2)_! \circ \tilde{\Delta}_!(\mathcal{F} \otimes \text{or}_f \otimes \text{or}_{U \setminus X})[df] \simeq (\tilde{p}_2)_! (\tilde{\Delta}_!(\mathcal{C}_{U \setminus X}) \otimes \tilde{p}^*_1(\mathcal{F} \otimes \text{or}_f \otimes \text{or}_{U \setminus X}))[df] \simeq (\tilde{p}_2)_! (\tilde{\Delta}_!(\mathcal{F} \otimes \text{or}_f \otimes \text{or}_{U \setminus X}))[df].
\]

Here $\tilde{p}_2 = p_2 \circ (f \times \text{id})$. Note also that

\[
\text{Ps-Id}_{H \setminus X} \circ \text{Av}^{H/U}_1(\mathcal{F}) \simeq (p_2)_! (\Delta!(\mathcal{C}_{H \setminus X}) \otimes^! (\tilde{p}_1 \otimes f_!(\mathcal{F}))) \simeq (p_2)_! (\Delta!(\mathcal{C}_{H \setminus X}) \otimes^! (f \times \text{id}) \circ \tilde{p}_1(\mathcal{F})) \simeq (p_2)_! (\Delta!(\mathcal{C}_{H \setminus X}) \otimes^! (\tilde{p}_1(\mathcal{F}))) \simeq (\tilde{p}_2)_! ((f \times \text{id})^! \Delta!(\mathcal{C}_{H \setminus X}) \otimes^! (\tilde{p}_1)^! \mathcal{F}) \simeq (\tilde{p}_2)_! (\Delta!(\mathcal{F} \otimes \text{or}_f) \otimes^! \tilde{p}_1(\mathcal{F}))[df]
\]

Since $\tilde{p}_1$ is smooth and $X$ is compact, it follows from (2.2) and Lemma 5.3 below that

\[
\tilde{p}_1(\mathcal{F}) \simeq \tilde{p}_1(\mathcal{F}) \otimes \text{or}_{\tilde{p}_1}[d_{\tilde{p}_1}] \quad (p_2)_! \simeq (p_2)_! [-d_U + d_U].
\]

All together we obtain

\[
(- \otimes \ell_{H \setminus X}) \circ \text{Av}^{H/U}_1(\mathcal{F}) \simeq (\tilde{p}_2)_! (\tilde{\Delta}_!(\mathcal{F} \otimes \text{or}_{U \setminus X}))[df]
\]

\[
\text{Ps-Id}_{H \setminus X} \circ \text{Av}^{H/U}_1(\mathcal{F}) \simeq (\tilde{p}_2)_! (\Delta!(\mathcal{F} \otimes \text{or}_{U \setminus X}))[df + d_{\tilde{p}_1} - d_U + d_U].
\]

Write $\mathcal{Y} = U \setminus X \times H \setminus X$, $\mathcal{F}_1 = \tilde{p}_1^*(\mathcal{F})$, $\mathcal{F}_2 = \Delta!(\text{or}_f)$, and let $\Delta_\mathcal{Y} : \mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ be the diagonal imbedding. Note that there is a canonical arrow

\[
\mathcal{F}_2 \otimes \mathcal{F}_1 \otimes \Delta^!_\mathcal{Y}(\mathcal{C}_{\mathcal{Y} \times \mathcal{Y}}) \simeq \Delta^!(\mathcal{F}_2 \otimes \mathcal{F}_1) \otimes \Delta^!_\mathcal{Y}(\mathcal{C}_{\mathcal{Y} \times \mathcal{Y}}) \to \Delta^!(\mathcal{F}_2 \otimes \mathcal{F}_1) \simeq \mathcal{F}_2 \otimes^! \mathcal{F}_1
\]

Since

\[
\Delta^!_\mathcal{Y}(\mathcal{C}_{\mathcal{Y} \times \mathcal{Y}}) \simeq \text{or}_y[-d_y] \quad \text{or}_y \simeq \text{or}_{\tilde{p}_1} \otimes \tilde{p}_1^*(\mathcal{C}_{\mathcal{U} \setminus X})
\]

\[
dx - d_{U \setminus X} = dy - (d_{\tilde{p}_1} - d_U + d_U),
\]

it follows that there is a canonical arrow

\[
\mathcal{F}_2 \otimes \mathcal{F}_1 \otimes \tilde{p}_1^*(\mathcal{C}_{\mathcal{U} \setminus X}) \simeq \mathcal{F}_2 \otimes \mathcal{F}_1 \otimes \Delta^!_\mathcal{Y}(\mathcal{C}_{\mathcal{Y} \times \mathcal{Y}}) \otimes \text{or}_{\tilde{p}_1}[d_y] \to \mathcal{F}_2 \otimes^! \mathcal{F}_1 \otimes \text{or}_{\tilde{p}_1}[d_y]
\]

and the desired arrow in (5.1) is given by

\[
(- \otimes \ell_{H \setminus X}) \circ \text{Av}^{H/U}_1(\mathcal{F}) \simeq (\tilde{p}_2)_!(\mathcal{F}_2 \otimes \mathcal{F}_1 \otimes \tilde{p}_1^*(\mathcal{C}_{\mathcal{U} \setminus X}))[df] \to (\tilde{p}_2)_!(\mathcal{F}_2 \otimes \mathcal{F}_1 \otimes \text{or}_{\tilde{p}_1})[df + dy] \simeq \text{Ps-Id}_{H \setminus X} \circ \text{Av}^{H/U}_1(\mathcal{F})[dx - d_{U \setminus X}].
\]

Assume $\mathcal{F}$ is transversal with respect to $f : U \setminus X \to H \setminus X$. Then it is straightforward to check that $\mathcal{F}_2 \otimes \mathcal{F}_1 \in \text{Shv}(\mathcal{Y} \times \mathcal{Y})$ is non-characteristic with respect to $\Delta_\mathcal{Y} : \mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$, that is, the restriction map $\text{sing}(\mathcal{F}_2 \otimes \mathcal{F}_1)|_{\mathcal{Y}} \subset T^*(\mathcal{Y} \times \mathcal{Y})|_{\mathcal{Y}} \to T^*\mathcal{Y}$ of the differential map of $\Delta_\mathcal{Y}$ to the singular support $\text{sing}(\mathcal{F}_2 \otimes \mathcal{F}_1)$ is an injective map. It follows from [KSa] Proposition 5.4.13 that the canonical arrow in (5.2), and hence the one in (5.1), is an isomorphism. The proof is completed.

**Lemma 5.3.** Let $X$ be a compact real analytic manifold acted on by $U$. Let $p : U \setminus X \to \text{pt}$ be the projection map. We have $p^* \simeq p_1[-d_U + d_{U \setminus X}].$
Proof. Since the projection factors as \( p : U \backslash X \to U \backslash \text{pt} \to \text{pt} \), where the first map is proper, we can assume \( X = \text{pt} \). Let \( \sigma : \text{pt} \to U \backslash \text{pt} \) be the projection. We have \( \sigma_* \mathbb{C}_\text{pt} \simeq \sigma_* \mathbb{C}_\text{pt} \) (as \( \sigma \) is schematic) and \( \sigma_* \mathbb{C}_\text{pt} \simeq \sigma_! \mathbb{C}_\text{pt}[-d_U + d_{U_\epsilon}] \), and it follows that

\[
p_*[d_L + d_{L_\epsilon}](\sigma_* \mathbb{C}_\text{pt}) \simeq p_*[d_L + d_{L_\epsilon}] \circ \sigma_![-d_U + d_{U_\epsilon}](\mathbb{C}_\text{pt}) \simeq \mathbb{C}_\text{pt}
\]

\[
p_* \mathbb{C}_\text{pt} \simeq p_* \circ \sigma_* \mathbb{C}_\text{pt} \simeq \mathbb{C}_\text{pt}.
\]

Since the category \( \text{Shv}(U \backslash \text{pt}) \) is compactly generated by \( \sigma_* \mathbb{C}_\text{pt} \), we conclude that \( p_*[-d_U + d_{U_\epsilon}] \simeq p_* \).

□

Remark 5.2. To see that “twisting” \((-) \otimes \ell_{H/X} \) in \((\text{5.1})\) is necessary (in contrast to the complex analytic setting), let us consider the case when \( H = U \) is trivial and \( X \) is a compact non-oriented real analytic manifold. Then \( \text{Av}^*_H \rightarrow \text{Av}^*_U = \text{Id}, \text{Av}_H \rightarrow \text{or}_X \) and and the arrow \((-) \otimes \text{or}_X \rightarrow \text{Ps-Id}_X \) in \((\text{5.1})\) (which is an isomorphism) comes from the map between the corresponding kernels

\[\Delta_X)_*(\omega_X \otimes \text{or}_X) \simeq (\Delta_X)_*(\mathbb{C}_X)[d_X] \simeq (\Delta_X)_*(\mathbb{C}_X)[d_X].\]

Here \( \Delta_X : X \to X \times X \) is the diagonal closed embedding.

6. Main results

6.1. We now fix a Matsuki datum \((G, X, H^+, H^-)\) as in the Section 3. Let \( U_\epsilon \) be the maximal compact subgroup of \( U = H^+ \cap H^- \). The following theorems are the main results of the paper.

Theorem 6.1. There is a canonical isomorphism of functors \( \text{Shv}(H^+ \backslash X) \to \text{Shv}(H^+ \backslash X) \):

\[\text{Ps-Id}_{H^+} \simeq (\cdot \otimes \ell_{H^+ \backslash X}) \circ \text{Y}_{H^+} \circ \text{Y}_{H^+} \circ \ell_{H^+ \backslash X} - d_{X_\epsilon}].\]

Proof. By Theorem 5.2 \( \text{Y}_{H^+} \circ \text{Y}_{H^+} \circ \ell_{H^+ \backslash X} \) is an equivalence with inverse given by

\[\text{Y}_{H^+} \circ \text{Y}_{H^+} \circ \ell_{H^+ \backslash X} \simeq \text{Av}_{H^+} \circ \text{oblv}_{H^+}.\]

Thus it suffices to show that there is an isomorphism

\[\text{Av}_{H^+} \circ \text{oblv}_{H^+} \simeq (\cdot \otimes \ell_{H^+ \backslash X}) \circ \text{Ps-Id}_{H^+ \backslash X} \circ \text{Av}_{H^+} \circ \text{oblv}_{H^+ \backslash X} [d_X - d_{U_\epsilon}].\]

The later follows from Lemma 5.2 and Example 5.1.

□

Let \( \text{Se}_{H^+ \backslash X} : \text{Shv}(H^+ \backslash X) \to \text{Shv}(H^+ \backslash X) \) be the Serre functor for \( \text{Shv}(H^+ \backslash X) \).

Theorem 6.2. There is a canonical isomorphism of functors \( \text{Shv}(H^+ \backslash X) \to \text{Shv}(H^+ \backslash X) \):

\[\text{Se}_{H^+ \backslash X} \simeq \text{Y}_{H^+} \circ \text{Y}_{H^+} \circ \ell_{H^+ \backslash X} \simeq (\cdot \otimes \ell_{H^+ \backslash X}) [d_X - d_{U_\epsilon}].\]

Proof. Since the stack \( H^+ \backslash X \) has only finitely many isomorphism class of points, according to [AGKRRV Example 3.3.8 and Corollary 3.4.7], the Serre functor \( \text{Se}_{H^+ \backslash X} \) and the Drinfeld-Gaitsgory functor \( \text{Ps-Id}_{H^+ \backslash X} \) are inverse to each other. Now the theorem follows immediately from Theorem 6.1 and Theorem 5.2 and the fact \( \ell_{H^+ \backslash X} \otimes \ell_{H^+ \backslash X} \simeq \mathbb{C}_{H^+ \backslash X}.\)

□

Remark 6.1. In the case when both \( X \) and \( U \) and the \( U \)-action on \( X \) are complex analytic or \( X \) is oriented and \( U \) is connected, the orientation sheaf \( \text{or}_{U \backslash X} \simeq \mathbb{C}_{U \backslash X} \) is trivial. Indeed, the first case is a well-known fact and the second case follows from the fact that \( \text{or}_X \simeq \mathbb{C}_X \) where \( \sigma : X \to U \backslash X \) (see, e.g., [BL Lemma 7.5.3]), and hence \( \sigma^* \text{or}_{U \backslash X} \simeq \text{or}_X \otimes \text{or}_X \simeq \text{or}_X \simeq \mathbb{C}_X \) (as \( X \) is oriented). Since \( \sigma \) is smooth with connected fiber it implies \( \text{or}_{U \backslash X} \simeq \mathbb{C}_{U \backslash X} \). Thus in the above situation, the
functor \((-) \otimes \ell_{H+}\) is isomorphic to the identity functor \((-) \otimes \ell_{H+} \cdot X \simeq \text{Id}_{H+} \cdot X\) and the formulas above becomes just
\[
\text{Ps-Id}_{H+} \cdot X \simeq \Upsilon_{\ell}^{H^- \rightarrow H^+} \circ \Upsilon_{\ell}^{H^+ \rightarrow H^-} \circ [-d_X + d_U_c]
\]
\[
\text{Se}_{H+} \cdot X \simeq \Upsilon_{\ast}^{H^- \rightarrow H^+} \circ \Upsilon_{\ast}^{H^+ \rightarrow H^-} \circ [d_X - d_U_c].
\]

6.2. The case of twisted sheaves. We extend Theorem 6.1 and Theorem 6.2 to the setting of twisted sheaves. For simplicity, we consider the situation when \(X\) is the flag manifold of a complex reductive group \(G\). Write \(S = H^\pm \cdot X\). Then for any character \(\lambda\) of the Lie algebra of the universal Cartan group of \(G\), we have the dg-category \(\text{Shv}(S)_\lambda^{\text{constr}}\) of \(\lambda\)-twisted bounded \(\mathbb{C}\)-constructible sheaves on \(S\) (see, e.g., [MUV, Section 6]) and its ind-completion \(\text{Shv}(S)_\lambda\).

We have similar six functor formalism in the twisted setting and any object \(Q \in \text{Shv}(S \times S)_{-\lambda, \lambda}\) defines a functor
\[
F_Q : \text{Shv}(S) \rightarrow \text{Shv}(S), \quad \mathcal{T} \rightarrow (p_2)^\ast, \text{ren}(\mathcal{Q} \otimes 
\mathcal{P})^\prime \cdot (\mathcal{P})\prime \cdot (\mathcal{T})
\]
Consider the diagonal embedding \(\Delta_S : S \rightarrow S \times S\). The Drinfeld-Gaitsgory functor for \(\text{Shv}(S)_\lambda\) is given by
\[
\text{Ps-Id}_S := F_Q : \text{Shv}(S)_\lambda \rightarrow \text{Shv}(S)_\lambda
\]
with
\[
Q = (\Delta_S)^\ast(C_S) \in \text{Shv}(S \times S)_{-\lambda, \lambda}
\]
where we note that the pullback of the \((-\lambda, \lambda)\)-twisting along the diagonal map is canonically trivial and hence the object above is well-defined. Let \(\omega_{U \cdot X} \in \text{Shv}(U \cdot X)_\lambda\) be the dualizing complex and let \(\ell_{H+} \cdot X\) to be the unique rank one \(\lambda\)-twisted local system on \(H^+ \cdot X\) such that its pullback to \(U \cdot X\) is isomorphic to \(\text{or}_{U \cdot X} = \mathcal{G}^{-d_U \cdot X}(\omega_{U \cdot X})\).

Now all the discussion in the previous section work in the twisted setting and we have

**Theorem 6.3.** There is a canonical isomorphism of functors \(\text{Shv}(H^+ \cdot X)_\lambda \rightarrow \text{Shv}(H^+ \cdot X)_\lambda:\)
\[
\text{Ps-Id}_{H+} \cdot X \simeq (- \otimes \ell_{H+} \cdot X) \circ \Upsilon_{\ell}^{H^- \rightarrow H^+} \circ \Upsilon_{\ell}^{H^+ \rightarrow H^-} \circ [-d_X + d_U_c]
\]

Let \(\text{Se}_{H+} \cdot X\) be the the Serre functors for \(\text{Shv}(H^+ \cdot X)_\lambda\).

**Theorem 6.4.** There is a canonical isomorphism of functors \(\text{Shv}(H^+ \cdot X)_\lambda \rightarrow \text{Shv}(H^+ \cdot X)_\lambda:\)
\[
\text{Se}_{H+} \cdot X \simeq \Upsilon_{\ast}^{H^- \rightarrow H^+} \circ \Upsilon_{\ast}^{H^+ \rightarrow H^-} \circ (- \otimes \ell_{H+} \cdot X)[d_X - d_U_c].
\]

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