Kinetic Theory for Interacting Luttinger Liquids

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We derive a closed set of equations for the kinetics and non-equilibrium dynamics of interacting Luttinger Liquids with cubic resonant interactions. In the presence of these interactions, the Luttinger phonons become dressed but still well defined quasi-particles, characterized by a life-time much larger then the inverse energy. This enables the separation of forward time dynamics and relative time dynamics into slow and fast dynamics and justifies the so-called Wigner approximation, which can be seen as a "local-time approximation" for the relative dynamics. Applying field theoretical methods in the Keldysh framework, i.e. kinetic and Dyson-Schwinger equations, we derive a closed set of dynamic equations, describing the kinetics of normal and anomalous phonon densities, the phonon self-energy and vertex corrections for an arbitrary non-equilibrium initial state. In the limit of low phonon densities, the results from self-consistent Born approximation are recaptured, including Andreev’s scaling solution for the quasi-particle life-time in a thermal state. As an application, we compute the relaxation of an excited state to its thermal equilibrium.

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I. INTRODUCTION

The kinetics and non-equilibrium dynamics of low dimensional, interacting quantum systems is an outstanding and fascinating challenge in quantum many-body physics. On the one hand, it is strongly motivated by recent cold atom experiments performed on low entropy quantum wires under out-of-equilibrium conditions. On the other hand, from a theoretical point of view, the study of non-equilibrium dynamics in integrable and nonintegrable systems is currently a field of growing interest. This is triggered by the question, whether and – if answered affirmatively – in which specific way a one-dimensional quantum system is able to thermalize.

An example of a one-dimensional integrable model is the linear Luttinger Liquid, which is the effective long-wavelength description of one-dimensional interacting quantum fluids, composed either of fermions or bosons. Due to integrability, even if prepared in a non-equilibrium state, this model will never thermalize, since the number of excitations for each momentum mode $q$ is a constant of motion. However, as already pointed out by Andreev and Haldane, there are non-zero corrections to the linear theory, which certainly break integrability in the Luttinger model. They are irrelevant in the sense of the renormalization group, and do not affect static observables. In contrast, they lead to a modification of dynamical, i.e. frequency resolved, correlation functions. These nonlinear corrections describe three-body scattering processes between phonons and are, due to their resonant nature, not straightforwardly approached theoretically.

Apart from a wealth of numerical studies, based on matrix product state and Bethe ansatz calculations, several field theoretical approaches have been developed. A seminal early study was carried out by Andreev, who used a self-consistent Born approximation to determine the phonon self-energies, establishing a universal phonon absorption rate $\gamma_q \sim q^{\eta}$ with exponent $\eta = \frac{1}{2}$ for a finite temperature system. A similar computation leads to an exponent $\eta = 2$ for the case of a zero temperature state, which has been verified by several numerical methods. Recently, so-called nonlinear Luttinger Liquids have been introduced, which are designed to capture corrections to the linear Luttinger theory in the context of one dimensional fermions systems. These have been very successful in determining, for example, the power law divergences of the dynamic structure factor, or thermalization rates for near equilibrium systems, Despite the large number of analytic and semi-analytic works on a non-equilibrium theory for interacting Luttinger Liquids has not been developed so far, and the non-equilibrium dynamics thus is still widely unknown.

The purpose of this article is to provide such a quantitative description of the kinetics of an interacting Luttinger Liquid, which is initialized in an arbitrary quantum state $t = 0$. Exploiting the resonant but subleading character of the interactions in a one-dimensional interacting quantum fluid, we apply non-equilibrium diagrammatic theory to solve for the non-equilibrium dynamics of an interacting Luttinger Liquid. We show that due to the resonant but subleading nature of the interactions, vertex corrections are moderate for many physical realizations and consequently the non-equilibrium dynamics and self-energy can be solved within self-consistent Born approximation. The self-consistency is however, crucial, and a perturbative Born approach leads to infrared divergencies. The result is an effectively closed set of equations for the time-dependent phonon density and self-energy in the presence of resonant interactions.

As a result of the RG irrelevant but resonant interactions, the excitations remain well-defined but dressed phonons, with a life-time $\tau_q$ much larger than their typical coherent time-scale $\epsilon_q^{-1}$. The dressed spectral function remains sharply peaked at the bare phonon energies $\epsilon_q$ with width $\tau_q^{-1}$, such that the self-energy and distribution function of the phonons can be approximated by their on-shell value. This is referred to as the quasi-particle approximation. The long life-time of the dressed phonons results in a further simplification, as it implies that the forward time evolution of the system is much slower than the relative time evolution. This decoupling leads to a "local time approximation", where an effectively station-
ary problem can be solved at each instant of time. We will quantify the validity of these approximations below.

Our estimate of the vertex correction further supports the validity of the self-consistent Born approximation for the time evolution. More precisely, for zero temperature states, the vertex correction vanishes identically, reproducing previous results[11]. For the special case of a finite temperature state, the loop correction leads to a finite but small multiplicative renormalization of the vertex. This implies that for typical translation invariant low entropy initial states, the equations governing the time evolution of the phonon occupation and the self-energy are effectively closed. It does not rule out, however, the possibility of significant vertex renormalization in general.

The strength of this approach is the simplicity of the resulting final equations, which can directly be implemented and solved numerically, as we demonstrate by first explicit examples. This provides a useful tool with a broad spectrum of applicability, ranging from tracking the thermalization process in mesoscopic samples. This provides a useful tool with a broad spectrum of applications and quantifies the validity of these approximations below.

II. MODEL

The action describing the interacting Luttinger model consists of two parts

\[ S = S_{\mathrm{nl}} + S_{\mathrm{us}}. \]  

(2.1)

Here, \( S_{\mathrm{nl}} \) is the well-known quadratic Tomonaga-Luttinger (TL) action[14,15]

\[ S_{\mathrm{nl}} = \frac{1}{2\pi} \int_{x,t} \left[ \left( \partial_x \phi \right) \left( \partial_t \theta \right) - u K \left( \partial_t \theta \right)^2 - \frac{U}{K} \left( \partial_x \phi \right)^2 \right], \]  

(2.2)

where \( \int_{x,t} \equiv \int_{-\infty}^{\infty} dt \int dx \) is the integral over space and time and \( \phi = \phi(x,t) \) and \( \theta = \theta(x,t) \) are dimensionless, real fields. The non-linear part \( S_{\mathrm{nl}} \) is cubic in the fields and reads[16]

\[ S_{\mathrm{us}} = \frac{1}{2\pi} \int_{x,t} \left[ \kappa_u \left( \partial_x \theta \right)^2 \left( \partial_x \phi \right) + \kappa_u \left( \partial_x \phi \right)^3 \right]. \]  

(2.3)

Starting from a microscopic derivation of the TL model as the effective long-wavelength description of interacting bosons or fermions in one dimension, the fields \( \theta, \phi \) represent local phase and density fluctuations.\[14,15\] The non-linearity corresponding to \( \kappa_u \) as well is of microscopic origin and is referred to as band curvature. It originates from deviations from a perfectly linear dispersion of the microscopic particles. On the other hand, the term corresponding to \( \kappa_u \) is generated in an effective long-wavelength description, where the fast modes have been integrated out already, and describes effective three-particle interactions.

The fields \( \theta, \phi \) are dimensionless, i.e. they have a canonical scaling dimension equal to zero. As a result, they do not scale when coarse graining to larger distances, i.e. when performing the rescaling

\[ x \rightarrow lx, \quad t \rightarrow \ell t, \]  

(2.4)

where \( \ell \) is the dynamical exponent and \( l > 1 \). In contrast

\[ S_{\mathrm{us}} \rightarrow \frac{1}{2\pi l} S_{\mathrm{us}} \]  

(2.5)

under the rescaling, such that the influence of \( S_{\mathrm{us}} \) vanishes on the longest wavelengths, i.e. it becomes irrelevant in the renormalization group (RG) sense. Consequently, the static equilibrium properties of the interacting Luttinger model (Eq. (2.1)) are well described by the quadratic part of the action alone and the partition function can be approximated by

\[ Z = \int \mathcal{D}[\theta, \phi] e^{\mathcal{S}} \approx \int \mathcal{D}[\theta, \phi] e^{\mathcal{S}_{\mathrm{nl}}}. \]  

(2.6)
Here, $\int \mathcal{D}[\theta, \phi]$ stands for the functional integral over the fields $\theta, \phi$.

The Tomonaga-Luttinger action describes phonons with a dispersion $\epsilon_q = \omega q$ linear in the momentum $|q|$, propagating with the speed of sound $u$. In the absence of $S_{uv}$, these phonons are non-interacting and consequently the phonon density for a specific mode $q$ is a conserved quantity. However, although the phonon interaction is irrelevant in the RG sense, it contains resonant processes where two phonons propagating in the same direction and the same speed of sound can interact with each other for an infinite time span. This leads to a non-trivial modification of time-dependent, dynamical observables compared to the case of non-interacting phonons. As pointed out in a seminal work by Andreev\cite{15} (considering finite $T$) and more recent work\cite{18,33} the presence of $S_{uv}$ leads to a finite phonon lifetime

$$\tau_q \sim |q|^{-\eta} \quad \text{with} \quad \eta = \begin{cases} \frac{3}{2} & \text{for } T > 0 \\ \frac{2}{3} & \text{for } T = 0 \end{cases},$$

(2.7)

which, in equilibrium, is visible only in dynamical, i.e. frequency dependent quantities such as the dynamical structure factor

$$S(q, \omega) = \int_{x,t} e^{i(\omega t - q x)} \langle \{\partial_x \phi(x,t), \partial_t \phi(0,0)\} \rangle,$$

(2.8)

with

$$\langle ... \rangle = \frac{1}{Z} \int \mathcal{D}[\theta, \phi] ... e^{iS}.$$ (2.9)

For dynamical quantities it is therefore important to take into account the full action, Eq. (2.1), instead of the reduced quadratic part only, as indicated in Eq. (2.6).

For a true non-equilibrium situation, the phonon distribution is not stationary, i.e. not a thermal or zero temperature distribution, but instead the phonon number $n_q$ for a given momentum mode $q$ becomes time-dependent. The redistribution of phonons between the different momentum modes with exact energy conservation is described by the resonant interactions in $S_{uv}$ and in a non-equilibrium situation the action can not be reasonably reduced to the quadratic Tomonaga-Luttinger action, for which the phonon density is a constant of motion.

Since we are interested in the non-equilibrium dynamics in the interacting TL model, we formulate the problem in a Keldysh path integral framework, which is able to treat both equilibrium and non-equilibrium dynamics on equal footing\cite{19}. We will now shortly introduce the canonical Bogoliubov transformation, which switches from the basis of real fields $\theta, \phi$ to the basis of complex fields $\tilde{a}, \tilde{a}^\dagger$. Those correspond to creation and annihilation operators in an operator picture\cite{18}. We close the model section by placing the action (2.1) on the Keldysh contour and briefly explaining the formalism.

### A. Phonon basis

In order to use a physically more appealing representation of the TL action, one commonly introduces a set of complex fields $\tilde{a}, \tilde{a}^\dagger$ which represent the (bosonic) eigenmodes of the system, i.e. the discussed phonons. The corresponding Bogoliubov transformation is

$$\theta_{x,t} = \theta_0 + \frac{i}{2} \int_q \left( \frac{2n}{|q|} K \right)^{\frac{1}{2}} e^{-iqx} \left( \tilde{a}_{q,t} - \tilde{a}_{-q,t} \right),$$

(2.10)

$$\phi_{x,t} = \phi_0 - \frac{i}{2} \int_q \left( \frac{2nK}{|q|} \right)^{\frac{1}{2}} \text{sgn}(q)e^{-iqx} \left( \tilde{a}_{q,t} + \tilde{a}_{-q,t} \right),$$

(2.11)

with abbreviations $\phi_{x,t} = \phi(x,t)$ and $\int_q = \int_{-\infty}^{\infty} dq$ and the Fourier transformed phonon fields

$$a_{q,t} = \int_x e^{-iqx} a_{x,t}.$$ (2.12)

The product $\tilde{a}_{x,t} a_{x,t}$ represents a phonon density and, therefore, in the continuum limit, the fields $\tilde{a}_{x,t}, \tilde{a}^\dagger_{x,t}$ are not dimensionless, in contrast to $\theta_{x,t}, \phi_{x,t}$, but scale as $\tilde{a}_{x,t} \sim \frac{1}{\sqrt{q}}$. The quadratic part of the action transforms into

$$S_{uv} = \frac{1}{2\pi} \int_{q,p,t} \tilde{a}_{q,t} \left( i\partial_t - \omega q \right) a_{q,t},$$

(2.13)

describing non-interacting phonons with a linear dispersion $\epsilon_q = \omega q$. The cubic part becomes

$$S_{uv} = \frac{1}{2\pi} \int_{q,p,t} v_{q,p,p+q} \sqrt{q} \left( p + q \right)$$

$$\times \left( \tilde{a}_{p+q,t} a_{q,t} a_{p,t} + \frac{\epsilon_{q} - \epsilon_{p} - \epsilon_{q+p}}{3} + \text{h.c.} \right),$$

(2.14)

with the vertex function

$$v_{q,p,k} = \kappa_{bc} \sqrt{\frac{\pi}{2K}} \left( \frac{q p}{|q|} + \frac{k p}{|k|} + \frac{q k}{|q|} \right) \kappa_{qq} \sqrt{\frac{|q|^2}{2}}.$$ (2.15)

The interaction (2.14) describes cubic phonon scattering processes with total momentum conservation. However, not all of the processes contained in $S_{uv}$ are resonant, i.e. exactly energy conserving in the sense that $\epsilon_{p+q} = \epsilon_p + \epsilon_q$. As explained by Andreev\cite{15} and pointed out above, the resonant processes lead to a divergence of the self-energy (and the kinetic equation, as we see later) in perturbation theory and are therefore the only relevant terms from a dynamical perspective. The term in Eq. (2.14) describing the annihilation (creation) of three phonons can never be resonant. It will therefore play no role in our analysis and we will skip it from now on. For the residual terms, resonance requires $|p+q| = |p|+|q|$. For all momenta $p, q$ fulfilling this condition, the vertex function takes on the value

$$v_0 \equiv v_{1,1,1} = \sqrt{\frac{\pi}{2K}} \left( \kappa_{bc} + K^2 \kappa_{qq} \right).$$ (2.16)

Consequently, instead of taking the full action $S_{uv}$, it is sufficient to consider the reduced but resonant phonon interaction

$$S_{uv} = \frac{v_0}{2\pi} \int_{p,q,t} \sqrt{|p q (p+q)|} \left( \tilde{a}_{p+q,t} a_{q,t} a_{p,t} + \text{h.c.} \right),$$

(2.17)
where the prime in \( \int_{p,q} \) indicates that the integral runs only over momenta \( q, p \) which have the same sign. Together the quadratic action and the resonant phonon interaction describe the dynamics of the interacting Luttinger model in the phonon basis,

\[
S = S_{\text{K}} + S_{\text{an}}.
\] (2.18)

**B. Keldysh action**

non-equilibrium field theory is commonly performed in the Keldysh path integral framework, which is able to deal both with equilibrium and true non-equilibrium situations. To set up the Keldysh path integral, one first doubles the degrees of freedom in the theory by introducing plus and minus fields \( a_{\pm,q,t}, \bar{a}_{\pm,q,t} \), representing forward and backward time evolution on the Keldysh contour.\(^{[23]}\) In this representation, the partition function is determined via

\[
Z = \int \mathcal{D}[a_+, a_-, \bar{a}_+, \bar{a}_-] e^{iS_{\text{K}} - iS_{\text{R}}},
\] (2.19)

where \( S_{\pm} \) is the phonon action (2.18) with the replacements \( \{a_{p,t}, a_{p,t}\} \rightarrow \{a_{\pm,p,t}, \bar{a}_{\pm,p,t}\} \). The \( \pm \)-representation contains redundancy, and a technically and physically more appealing representation is found by completing the transformation to the Keldysh representation, introducing classical and quantum fields according to

\[
a_c = \frac{1}{\sqrt{2}} (a_+ + a_-), \quad a_q = \frac{1}{\sqrt{2}} (a_+ - a_-).
\] (2.20)

In the Keldysh representation, the quadratic action is

\[
S_{\text{K}}^{(2)} = \frac{1}{2\pi} \int_{t,t',p} \left( \bar{a}_{p,t} \frac{\partial}{\partial p} a_{p,t'} \right) \left( \begin{array}{c} 0 \\ \frac{\partial}{\partial p} \end{array} \right) \left( \begin{array}{c} D_{p,t}^R \\ D_{p,t}^K \end{array} \right) \left( \begin{array}{c} \bar{a}_{p,t} \\ a_{p,t'} \end{array} \right),
\] (2.21)

with the bare inverse retarded/advanced propagators

\[
D_{p,t}^R = \delta(t - t') (\bar{i}(\bar{p} - u)|p| + i0^+),
\] (2.22)

\[
D_{p,t}^K = \left( \frac{\partial}{\partial p} D_{p,t}^R \right)^{-1} = \delta(t - t') (i(\bar{p} - u)|p| + i0^+),
\] (2.23)

and the Keldysh component of the inverse propagator

\[
D_{p,t}^K = 2i0^+ F(p, t, t').
\] (2.24)

Here, \( F(p, t, t') \) is the distribution function of the excitations and \( 0^+ \) is the infinitesimal regularization for the quadratic theory.\(^{[23]}\) In an equilibrium, i.e. time-translational invariant situation, \( F(p, t, t') = F(p, t - t') \) and its Fourier transform is the bosonic distribution

\[
F(p, \omega) = \text{coth} \left( \frac{\omega}{2T} \right) = 2n_B(\omega) + 1
\] (2.25)

with the Bose function \( n_B(\omega) = (e^{\omega} - 1)^{-1} \). The resonant interactions in the Keldysh representation take on the form

\[
S_{\text{an}} = \frac{v_0}{\sqrt{8\pi}} \int_{p,k,t} \sqrt{|pk(k + p)|} \left[ 2\bar{a}_{k+p,t} a_{k,t} \right. \left. \bar{a}_{k,t} a_{k+p,t} + \bar{a}_{k,t} a_{k+p,t} + a_{k,t} a_{k+p,t} + \text{h.c.} \right].
\] (2.26)

The bare response and correlation functions (retarded, advanced and Keldysh Green’s functions) for the phonon degrees of freedom are obtained according to

\[
\begin{align*}
G_{q,t,t'}^R &= -i(\bar{a}_{q,t} a_{q,t'}) = (D_{q,t}^R)^{-1} = -i\Theta(t - t') e^{-i\delta q(t-t')}, \\
G_{q,t,t'}^A &= -i(\bar{a}_{q,t} a_{q,t'}) = (D_{q,t}^A)^{-1} = i\Theta(t' - t) e^{-i\delta q(t-t')}, \\
G_{q,t,t'}^K &= -i(\bar{a}_{q,t} a_{q,t'}) = -(G_R \circ D_K \circ G_A)_{q,t,t'} = -i(2n_B(u|q|) + 1) e^{-i\delta q(t-t')}.
\end{align*}
\] (2.27)

Here \( \circ \) stands for the convolution in the non-diagonal elements, i.e. the time index, but means multiplication in momentum space.

In the presence of interactions, the Green’s functions are modified by the emergence of non-zero self-energies \( \Sigma^{R/A/K} \), which replaces the infinitesimal regularization. The corresponding formulas are

\[
\begin{align*}
G_{q,t,t'}^R &= (D_{q,t}^R - \Sigma_{q,t,t'}^{R})^{-1}, \\
G_{q,t,t'}^A &= (D_{q,t}^A - \Sigma_{q,t,t'}^{A})^{-1}, \\
G_{q,t,t'}^K &= -(G_R \circ \Sigma^K \circ G_A)_{q,t,t'},
\end{align*}
\] (2.28)

where the infinitesimal factor \( 0^+ F \) has been overwritten by the finite Keldysh self-energy \( \Sigma^K \). The distribution function \( F \) in the presence of interactions is determined by the formula

\[
G_{q,t,t'}^K = \left( G_R \circ F - F \circ G_A \right)_{q,t,t'}.
\] (2.29)

The aim of this work is to determine the self-energies \( \Sigma^{R/A/K} \) and the distribution function \( F(q, t, t') \) for a system that is driven out of equilibrium and evolves in time, for instance relaxing to an equilibrium state and approaching a bosonic distribution.

This will be done in two parts. First, we show how one determines the self-energies \( \Sigma^{R/A/K} \) from a given (non-)equilibrium distribution function \( F(q, t, t') \). To this end, we generalize Andreev’s self-consistent Born approach to a non-thermal, non-equilibrium situation. Second, we use the kinetic equation approach to determine the time-evolution of the distribution function \( F \) in self-consistent Born approximation. Combining these two approaches allows us to determine the time-evolution of both the distribution function of the excitations and the self-energies, which play the role of finite lifetimes of the system’s excitations.

### III. SELF-ENERGIES

The presence of the cubic, resonant interactions \( S_{\text{an}} \) modifies the phonon response and correlation functions according to Eqs. (2.28) by creating finite self-energies \( \Sigma^{R/A/K} \). These self-energies are to leading order purely imaginary, leading to a finite decay rate of phonons or, in other words, to a finite phonon lifetime. We will now derive a method to determine these lifetimes for a non-equilibrium problem, where the distribution function \( F(q, t, t') \) is time dependent and varies on
time scales which are larger than the individual phonon lifetimes. To this end, we first derive the non-equilibrium version of a fluctuation-dissipation relation for the two-point response and correlation functions.

A. Non-equilibrium fluctuation-dissipation relation

Fluctuation-dissipation relations (FDR) relate the response (i.e. spectral) properties of the system encoded in $G^{R,F}$ to its correlations via the distribution function $F$. A particular example for such a relation is Eq. (2.29). Inverting this equation results in an FDR for the self-energies,

$$\Sigma_{q,t,t'} = -\left( (D^R - \Sigma^R) \circ F - F \circ (D^A - \Sigma^A) \right)_{q,t,t'}$$

$$= -i\delta_{t,t'} F_{q,t,t'} + \left( \Sigma^R \circ F - F \circ \Sigma^A \right)_{q,t,t'} . \quad (3.1)$$

For a time-translational invariant system, the first term on the r.h.s. equals zero and due to the identity $\Sigma^A = \Sigma^R$, Eq. (3.1) reduces to the well-known relation in frequency space

$$\Sigma^K_{q,\omega} = -2i \text{Im} \left( \Sigma^R_{q,\omega,\tau} \right) F_{q,\omega} . \quad (3.2)$$

This is consistent with our initial regularization of the quadratic sector for the case $\Sigma^R = -i0^+$. A useful representation for a two-time function $F(q,\tau,t')$ is to choose Wigner coordinates in time, i.e. defining the forward time $\tau = \frac{t-t'}{2}$ and the relative time $\Delta_t = t - t'^*$, then one defines $F(q,\delta_t,\tau) \equiv F(q,\tau + \Delta_t/2,\tau - \Delta_t/2)$ and its Fourier transform

$$F(q,\omega,\tau) = \int d\delta_t \, e^{i\delta_t \omega} F(q,\tau,t). \quad (3.3)$$

Applying Wigner coordinates and the Wigner transformation (3.3) to the Keldysh self-energy in Eq. (3.2) leads to

$$\Sigma^K_{q,\omega,\tau} = -i\partial_{\tau} F_{q,\omega,\tau} + \left( \Sigma^R \circ F - F \circ \Sigma^A \right)_{q,\omega,\tau} . \quad (3.4)$$

Eq. (3.4) is an exact expression for the Keldysh self-energy in Wigner representation. A complication arises since the Wigner transform of a convolution is not the product of the corresponding Wigner transforms, in contrast to the ordinary Fourier transform. In fact, one finds

$$\left( \Sigma^R \circ F \right)_{q,\omega,\tau} = \Sigma^R_{q,\omega,\tau} e^{i\left( \delta_{\tau} \omega - \delta_{\omega} \partial_{\tau} \right)} F_{q,\omega,\tau} . \quad (3.5)$$

Without specific knowledge on the functional behavior of $\Sigma^R$ and $F$, Eq. (3.5) is hard to evaluate explicitly. We will now briefly discuss a situation, with two particular approximations, which applies to the present model and for which the analytic evaluation of Eq. (3.5) is possible.

1. Wigner approximation

For the case of scale separation in the forward and relative time, one can approximate the exponential in (3.5) by the leading order terms. The product $\partial_{\tau} \delta_{\omega}$ expresses the competition between relative time and forward time dynamics, it is small for slow forward time dynamics and fast relative dynamics. Comparing the zeroth order term with the first order term in an expansion of the exponential, we find the condition for approximating Eq. (3.5) by the zeroth order term to be

$$1 \gg \left| \frac{\partial_{\tau} \Sigma^R_{q,\omega,\tau}}{\Sigma^R_{q,\omega,\tau}} \right| \left| \frac{\partial_{\tau} F_{q,\omega,\tau}}{F_{q,\omega,\tau}} \right| . \quad (3.6)$$

As we will demonstrate later, this condition is fulfilled for the situations we are considering here. We can thus apply the Wigner approximation to the FDR, resulting in

$$\Sigma^K_{q,\omega,\tau} = -i\partial_{\tau} F_{q,\omega,\tau} + 2i \text{Im} \left( \Sigma^R_{q,\omega,\tau} \right) F_{q,\omega,\tau} . \quad (3.7)$$

The validity of the FDR in Wigner approximation is a very important result. It is commonly used as the starting point for deriving a kinetic equation for the distribution function in arbitrary dimensions. However, in the present case, we will further simplify the FDR by making use of the fact that we are dealing with resonant interactions in one dimension.

2. Quasi-particle approximation and on-shell FDR

The major effect of the non-linearities in the action is the emergence of finite phonon lifetimes due to resonant phonon-phonon scattering processes. The resonant character of this interaction – the fact that for two phonons travelling in the same direction momentum and energy conservation is expressed by the identical $\delta$-constraint – is the key property of one-dimensional systems with linear dispersion. The resonant contributions dominate the self-energy and the kinetic equation, while the non-resonant processes give only subleading contributions to the lifetimes and the dispersion and have therefore already been eliminated on the basis of the action by using $S_{\text{lin}}$ instead of $S_{\text{tot}}$ in Eq. (2.18).

The retarded self-energy is decomposed according to

$$\Sigma^R_{q,\omega,\tau} = -i\phi^R_{q,\tau} + \delta \Sigma^R_{q,\omega,\tau} . \quad (3.8)$$

where $\phi^R_{q,\tau}$ is a positive, frequency independent function, which varies slowly in forward time $\tau$. For resonant interactions $\langle \delta \Sigma^R_{q,\omega,\tau} \rangle = 0$ and consequently, the self-energy is frequency independent and purely imaginary. Non-resonant contributions in the interaction lead to $\delta \Sigma^R \neq 0$, which however is generally strongly subleading compared to $\phi^R$.

For the present model, the phonon interactions are RG-irrelevant and only their resonant character allows them to be of non-negligible influence. However, the effect of the interactions on the properties of the phonons will be small and subleading due to the RG-irrelevance. As a result, even in the presence of interactions, the phonons will have a lifetime $\tau_{ph}^{\phi} = -\text{Im} \left( \phi^R_{\tau} \right)^{-1}$ much larger than their associated coherent time-scale $\frac{1}{\sigma^R_{ph}}$, i.e. $\tau_{ph}^{\phi} \gg \frac{1}{\sigma^R_{ph}}$. Consequently, the phonons remain well defined quasi-particles with a spectral function.
Here we have used the fact that for well defined quasi-equilibrium FDR for resonant interactions the non-equilibrium FDR (3.7) results in the on-shell, non-resonant interactions. Inserting Eqs. (3.9), (3.10) in the initial states.

\[ \delta \Sigma^{R/A}_{q,\omega,\tau} = 0, \] (3.9)

as stated above. This is consistent with the result of Andreev’s and later works for equilibrium and holds true for the non-equilibrium case as well. The corresponding decomposition for the Keldysh self-energy (with a convenient prefactor) reads

\[ \Sigma^K_{q,\omega,\tau} = -2i\sigma^K_{q,\tau} + \delta \Sigma^K_{q,\omega,\tau} \] (3.10)

for resonant interactions. Inserting Eqs. (3.9), (3.10) in the non-equilibrium FDR (3.7) results in the on-shell, non-equilibrium FDR for resonant interactions

\[ \sigma^K_{q,\tau} = \partial_t n_{q,\tau} + \sigma^K_{q,\tau} \left( 2n_{q,\tau} + 1 \right). \] (3.11)

Here we have used the fact that for well defined quasi-particles, the on-shell distribution function \( F_{q,\omega = \pm |q|} = 2n_{q,\tau} + 1 \) equals the time-dependent phonon density \( n_{q,\tau} \) at momentum \( q \).

Eq. (3.11) is the final form of the non-equilibrium FDR that we will use to set up the kinetic equation in the following section and to determine the Keldysh self-energy \( \sigma^K \) for a system, for which the time-dependent phonon density \( n_{q,\tau} \) is known. For the latter case, the only unknown quantity is the retarded self-energy \( \sigma^K \) and we can now set up the diagrammatic computation of \( \sigma^K \) in the Keldysh non-equilibrium framework.

### B. Quasi-particle lifetimes in self-consistent Born approximation

In this section, we perform the self-consistent Born approximation in a diagrammatic representation to obtain the self-energies \( \sigma^K_{q,\tau} \). This amounts to an infinite resummation over a certain class of diagrams and cures the divergence of the self-energy occurring in the perturbative diagrammatic approach. In Sec. [VII] we demonstrate that the self-energy can be determined exactly in a one-loop computation using Dyson-Schwinger equations and show that the deviation from the self-consistent Born approximation is negligible for many initial states.

In a diagrammatic approach, classical (quantum) fields are represented by a full (dotted) line, while ingoing lines represent fields \( \sigma \) and outgoing lines their complex conjugate \( \bar{\sigma} \).

This leads to a representation of Green’s functions in terms of diagrams as indicated in Fig. [1] and vertices as depicted in Fig. [2]. The retarded self-energy in self-consistent Born approximation can directly be derived by common diagrammatic rules and is depicted in Fig. [3]. Inserting the Keldysh Green’s function

\[ G^K_{q,\omega} = G^K_{q,\omega} K 

\] (3.12)

and the on-shell self-energies \( \Sigma^K_{q,\omega} = \mp i\sigma^K_{q,\tau} \) for resonant interactions, we can perform the frequency integration indicated in Fig. [3] and find for momenta \( q > 0 \)

\[ \sigma^K_q = \int_0^2 \int_{0 < p < q} \frac{q(p-q)\sigma^K_p}{\sigma^K_p + \sigma^K_{q-p}} - \int_{q < p} \frac{q(p-q)\sigma^K_p}{\sigma^K_p + \sigma^K_{q-p}}. \] (3.13)

Since the self-energies must be invariant under the transformation \( p \rightarrow -p \), one can further simplify Eq. (3.13), ending up with

\[ \sigma^K_q = \int_0^2 \int_{0 < p < q} \frac{q(p-q)}{\sigma^K_p + \sigma^K_{q-p}} \left( \frac{q(p-q)}{\sigma^K_p + \sigma^K_{q-p}} + \frac{q(p+q)}{\sigma^K_p + \sigma^K_{q+p}} \right). \] (3.14)

Finally, \( \sigma^K \) can be replaced using the FDR (3.11), which leads to

\[ \sigma^K_q = \int_0^2 \int_{0 < p} \left( \frac{\partial_t n_p}{\sigma^K_p} + 2n_p + 1 \right) \left( \frac{q(p-q)}{\sigma^K_p + \sigma^K_{q-p}} + \frac{q(p+q)}{\sigma^K_p + \sigma^K_{q+p}} \right). \] (3.15)

For a given, time-dependent distribution function \( n_{q,\tau} \), \( \sigma^K_{q,\tau} \) is the only unknown function in this equation and has to be determined self-consistently. For a general time-dependent function \( n_{q,\tau} \) this has to be done by iterating Eq. (3.15) numerically until a self-consistent solution has been found. For the particular case for which \( n_{q,\tau} \) shows scaling behavior in a sufficiently large momentum window, one can determine a scaling solution for the self-energy as well and extract the corresponding scaling exponent. We will now briefly discuss the latter case and determine possible scaling solutions for the self-energy, and close the section with a discussion on universal aspects of the scaling solution.
For this, we introduce the rescaled self-energy \( \tilde{\sigma}_q \)

\[
-\sigma^R_q = 2 \times \int \frac{v_f^2}{4\pi^2} \left| pq(q-p) \right| G^K_P G^R_{Q-p} + \int \frac{v_f^2}{4\pi^2} \left| pq(p+q) \right| G^K_{Q+p} G^A_P + \int \frac{v_f^2}{4\pi^2} \left| pq(p+q) \right| G^K_P G^R_{P+Q}.
\]

FIG. 3. The retarded self-energy is the sum of three distinct diagrams as shown above. The notation is \( Q \equiv |\omega, q, P| \equiv |v, p| \) with \( p, q \) being momenta and \( \omega, v \) the corresponding frequencies. The momentum integral runs only over momenta that fulfill the resonance condition, which requires \( \omega = u|q| \).

1. Scaling solution for the self-energy

For the case when the density \( n_{q,\tau} \) is a scaling function, i.e.

\[
n_{q,\tau} = a_\tau |q|^\eta u,
\]

\[n_{q,\tau} = a_\tau |q|^\eta u,
\]

it is easy to show that also \( \sigma^R_{q,\tau} \) will be a scaling function

\[
\sigma^R_{q,\tau} = \gamma^R \eta |q|^\eta u,
\]

where the exponent \( \eta^R \) and prefactor \( \gamma^R \) solely depend on the scaling behavior of \( n_{q,\tau} \), i.e. on \( n_\eta \) and \( a_\tau \). In order to show this, we introduce the rescaled self-energy \( \tilde{\sigma}^R = \sigma^R / \nu_0 \) and time \( \tau = \tau / \nu_0 \), leading to

\[
\tilde{\sigma}^R = \int_{0<p} \left( \frac{\partial n_p}{\partial \tilde{\sigma}^R} + 2n_p + 1 \right) \left( \frac{q(p-q) - \sigma^R}{\sigma^R + \sigma^R_q - \sigma^R} + \frac{q(p+q)}{\sigma^R + \sigma^R_q - \sigma^R} \right).
\]

Next, we insert Eqs. (3.16), (3.17) into (3.18) and perform the transformation \( p \rightarrow q \), yielding

\[
\tilde{\gamma}^R_q q^{\eta^R} = \frac{1}{\gamma^R} q^{\eta^R} \int_{0<x<1} \left[ \frac{x(1-x)}{x^n + |1-x|^n} + \frac{x(1+x)}{x^n + |1+x|^n} \right] \times \left( 1 + 2a_\tau (q \eta^R u) + \frac{\partial n_p}{\partial \tilde{\gamma}^R} (q \eta^R u)^n \right).
\]

The exponent \( \eta^R \) is bounded from below and from above according to \( 1 < \eta^R < 2 \). Here, \( 1 < \eta^R \) results from the fact that the interaction is RG irrelevant and the self-energy correction can only be subleading compared to the dispersion \( \epsilon_q = u|q| \).

The case \( \eta^R = 2 \), i.e. diffusive scaling of the lifetimes is only reached in the zero temperature situation, otherwise one expects superdiffusive behavior due to finite phonon densities with \( \eta^R < 2 \). As a result, Eq. (3.19) distinguishes between three regimes:

1. Low temperature states \(( T = 0 \)\): For \( n_q \ll 1 \) and \( \partial_v n_q \approx 0 \) the only term in the second row of Eq. (3.19) with a relevant contribution is the constant unity and

\[
\eta^R = 2, \quad \tilde{\gamma}^R = \sqrt{T_{i,0}} = \sqrt{\frac{2\pi}{\nu_0}},
\]

where the factor \( I_{i,0} \) will be defined in the following.

2. Finite temperature states: For \( n_q \gg 1 \) and \( q > \max (a_{q_0}, 1/\sqrt{2\pi}) \), only the factor proportional to \( n_q \) contributes and we have

\[
\eta^R = 2 + \frac{\nu_0}{2}, \quad \tilde{\gamma}^R = \sqrt{\nu_0 I_{i,0} q^2}. \]

For a finite temperature state, \( n_q = \frac{T}{\nu_0} \) as \(|q| \rightarrow 0\), such that

\[
\eta^R = \frac{2 + \nu_0}{2}, \quad \tilde{\gamma}^R = \sqrt{\nu_0 I_{i,0} q^2}. \]

3. Non-equilibrium states: For \( n_q \gg 1 \) and \( q < \min (a_{q_0}, 1/\sqrt{2\pi}) \), the dominant contribution stems from the time derivative in Eq. (3.19) and one finds

\[
\eta^R = \frac{4 + \nu_0}{3}, \quad \tilde{\gamma}^R = \sqrt{\nu_0 I_{i,0} q^2}. \]

The integrand \( j(x) \) is the self-energy integral \( j(x) = \int j(x) \) as a function of \( x = p/q \), where \( q \) is the external momentum. The integrand \( j(x) \) elucidates the contribution to \( \sigma^R_q \) from different momenta and shows clearly that the self-energy at momentum \( q \) is dominated by the behavior of \( \sigma^R_q \) for \( p < q \).

The results discussed above include the case of finite and

\[
I_{i,0} = \int_{0}^{\infty} \left( x^{1/2}(1-x) + x^{1/2}(1+x) \right) \left( x^{1/2} + |1-x|^{1/2} \right) \left( x^{1/2} + |1+x|^{1/2} \right).
\]

The results discussed above include the case of finite and

\[
I_{i,0} = \int_{0}^{\infty} \left( x^{1/2}(1-x) + x^{1/2}(1+x) \right) \left( x^{1/2} + |1-x|^{1/2} \right) \left( x^{1/2} + |1+x|^{1/2} \right).
\]

The results discussed above include the case of finite and

\[
I_{i,0} = \int_{0}^{\infty} \left( x^{1/2}(1-x) + x^{1/2}(1+x) \right) \left( x^{1/2} + |1-x|^{1/2} \right) \left( x^{1/2} + |1+x|^{1/2} \right).
\]

The results discussed above include the case of finite and

\[
I_{i,0} = \int_{0}^{\infty} \left( x^{1/2}(1-x) + x^{1/2}(1+x) \right) \left( x^{1/2} + |1-x|^{1/2} \right) \left( x^{1/2} + |1+x|^{1/2} \right).
\]
\[ -\sigma^K \]
\[ = \mathcal{S}_{\mathcal{R}P} \mathcal{S}_{\mathcal{R}q} - \int_q \frac{d^3 \nu}{(2\pi)^3} |\nu(q - p)| G^*_{\mathcal{R}q} G_{\mathcal{R}p}^* \]
\[ + 2 \mathcal{S}_{\mathcal{R}q} \mathcal{S}_{\mathcal{R}q} - 2 \int_q \frac{d^3 \nu}{(2\pi)^3} |\nu(p + q)| G^*_{\mathcal{R}q} G_{\mathcal{R}q}^* \]
\[ + \mathcal{S}_{\mathcal{R}q} \mathcal{S}_{\mathcal{R}q} + \mathcal{S}_{\mathcal{R}q} \mathcal{S}_{\mathcal{R}p} \mathcal{S}_{\mathcal{R}p} = \int_p \frac{d^3 \nu}{(2\pi)^3} |\nu(p - q)| (G^*_{\mathcal{R}q} G_{\mathcal{R}p}^* + G_{\mathcal{R}q} G^*_{\mathcal{R}p}) \]
\[ + 2 \mathcal{S}_{\mathcal{R}q} \mathcal{S}_{\mathcal{R}q} \mathcal{S}_{\mathcal{R}p} \mathcal{S}_{\mathcal{R}p} = -2 \int_q \frac{d^3 \nu}{(2\pi)^3} |\nu(p + q)| (G^*_{\mathcal{R}q} G_{\mathcal{R}p}^* + G_{\mathcal{R}q} G^*_{\mathcal{R}p}) \]

FIG. 5. Diagrammatic representation of the Keldysh self-energy \( \sigma^K \) in self-consistent Born approximation. The index \( Q = \{|u|q| \pm i\sigma_q \}, 1 \) represents the on-shell frequency and momentum relation (+ for retarded, – for advanced Green’s functions), while \( P = \{v,p \} \) is the inner loop frequency and momentum and \( \int_p = \int_{v,p} \). Integration over resonant processes only is implied.

2. Insensitivity of the self-energy to UV-behavior

The insensitivity of the above results to the behavior of the model in the ultraviolet (UV) regime and therefore the non-universal properties of the system is guaranteed by the structure of the vertex and the diagrams in Fig. 3. The self-energies \( \sigma^R \) are dominated by loop momenta \( p < q \) below the external momentum \( q \). Therefore the non-universal behavior in the UV does not enter the self-energies. This is emphasized in Fig. 4, where the integrand

\[ \int_{j_{1,2}} \left( \frac{(1-x)^{x+1} + (1+x)^{x+1}}{1-x^2 + x^2} \right) \]

in \( j_{1,2} = \int_{j_{1,2}} \), is plotted for the case of thermal and zero temperature scaling. It is evident that for ingoing momenta \( q \) only momenta \( p < q \) contribute, which show the same scaling behavior in the distribution function. Therefore the scaling solutions for the self-energy are robust against modifications of the density and the model itself when approaching the UV.

IV. KINETIC EQUATION FOR THE PHONON DENSITY

In this section, we will derive a kinetic equation for the phonon density in the case of resonant interactions. A kinetic equation describes the time-evolution of the distribution function generated by the Keldysh and retarded/advanced self-energies, which can then often be evaluated in the perturbative Born approximation. In the present case, due to the fact that the interactions are resonant, the kinetic equation perturbation theory diverges and we have to use a self-consistent Born approach. Therefore, we evaluate the self-energy diagrams with full Green’s functions, as in the previous section, which leads to an effective vertex correction for the kinetic equation and a time-evolution linear in the interaction parameter \( \nu_0 \).

The time-evolution of \( n_{q,\tau} \) is determined by the solution of the on-shell FDR in Eq. (3.11), which after rearrangement reads

\[ \partial_\tau n_q = \sigma^K_{q,\tau} - (2n_{q,\tau} + 1) \sigma^R_{q,\tau} \]  

(4.1)

Again, the retarded self-energy is determined via a diagrammatic approach, where the corresponding diagrams are shown in Fig. 3. However, in contrast to the previous section, we also use a diagrammatic approach to determine the Keldysh self-energy \( \sigma^K \). As a consequence, we derive a non-linear differential equation for the distribution function. The diagrammatic representation of the Keldysh self-energy is depicted in Fig. 5.

For the full Green’s functions, we use the results from the previous section

\[ G_{q,\omega,\tau}^R = 2\pi \left( \omega - u|q| \pm i\sigma^R_{q,\tau} \right)^{-1} \]

and the relation

\[ G_{q,\omega,\tau}^K = G_{q,\omega,\tau}^R F_{q,\omega,\tau} - F_{q,\omega,\tau} G_{q,\omega,\tau}^A = \frac{-8\pi^2 i\sigma^R_{q,\tau}}{(\omega - u|q|)^2 + (\sigma^R_{q,\tau})^2} \]

(4.3)

in Eq. (4.3), the first equality holds in Wigner approximation, while the second equality results from the quasi-particle approximation, both discussed in the previous section.

The frequency integration in the diagrammatic representation can be performed analytically and yields the kinetic equation (omitting time index)

\[ \partial_\tau n_q = 2\nu_0^2 \int_{0<p<q} \frac{pq(q-p)}{\sigma^R_{q} + \sigma^R_{p} + \sigma^R_{q-p}} \left( n_{p} n_{q-p} - n_{q} \left( 1 + n_{p} + n_{q-p} \right) \right) \]

\[ + 4\nu_0^2 \int_{0<p<q} \frac{pq(q+p)}{\sigma^R_{q} + \sigma^R_{p} + \sigma^R_{q+p}} \left( n_{p+q} \left( n_{q} + n_{p} + 1 \right) - n_{q} n_{p} \right) \]

(4.4)

After transforming to dimensionless variables \( \sigma^R = \nu_0 \tilde{\sigma}^R \), \( \tau = \frac{\tilde{\tau}}{\nu_0} \), we finally arrive at

\[ \partial_\tau \tilde{n}_q = \int_{0<p<q} \frac{2pq(q-p)}{\tilde{\sigma}^R_{q} + \tilde{\sigma}^R_{p} + \tilde{\sigma}^R_{q-p}} \left( n_{p} n_{q-p} - n_{q} \left( 1 + n_{p} + n_{q-p} \right) \right) \]

\[ + \int_{0<p<q} \frac{4pq(q+p)}{\tilde{\sigma}^R_{q} + \tilde{\sigma}^R_{p} + \tilde{\sigma}^R_{q+p}} \left( n_{p+q} \left( n_{q} + n_{p} + 1 \right) - n_{q} n_{p} \right) \].

(4.5)
This is the kinetic equation for the phonon density in self-consistent Born approximation.

Comparing this equation to Eq. (3.18), one finds that the rescaled self-energy \( \hat{\sigma}_{q}^{R} \) and therefore also the kinetic equation for a rescaled time \( \tilde{\tau} = \frac{\tau}{\tau_{0}} \) only depend on the phonon distribution \( n_{q,\tau} \) and is independent of all possible microscopic details that may enter \( u, K, v_{0} \) in the model. As a result, the dynamics in the rescaled variables is very generic for an interacting Luttinger Liquid and only depends on the initial distribution function \( n_{q,\tau=0} \) with which the system is initialized.

The typical time-scale for the kinetic equation in the original variables is \( \tau_{q} = \frac{1}{\sqrt{2} \omega_{0}} \), i.e. linear in the vertex \( v_{0} \). This is a non-perturbative effect resulting from the resonant interactions. Since two vertices enter the one-loop diagrams, one might naively (or in perturbation theory) expect that the typical time scale is given by the square of the non-linearity. However, this is normalized by the self-energies, which are proportional to \( v_{0} \) and required to regularize the kinetic equation.

In the kinetic equation in (4.5), still the self-energies \( \hat{\sigma}_{q}^{R} \) occur. In principle, one could again replace them by a diagrammatic expression, which would give rise to an infinite hierarchy. However instead of doing so, we use an iterative process in which, for a given time \( \tau \), we determine self-consistently the self-energy \( \hat{\sigma}_{q}^{R} \). This result is then used to determine the r.h.s. of the kinetic equation, to subsequently compute the distribution function in the next time step \( \tau + \delta \tau \). This procedure is illustrated in Fig. 6 and allows us to compute the non-equilibrium dynamics of an interacting Luttinger Liquid, which may be initialized in a non-thermal state.

We will now close this section by discussing two limiting cases where analytic results become available. First, the kinetic equation for small external momenta \( q \), and second the kinetic equation for a phonon distribution \( n_{q,\tau} = n_{q,T} + \delta n_{q,\tau} \) close to an equilibrium distribution \( n_{q,T} \).

### A. Kinetic equation for small external momenta

For small external momenta, the kinetic equation (4.5) can be brought into an even simpler form, explicitly revealing the evolution of \( n_{q,\tau} \) for small \( q \). In this case, the first integral in (4.5) covers only a very small momentum region and is proportional to \( q^{4} \), \( 1 < \eta_{R} \leq 2 \). As a result, it is negligible compared to the second integral. On the other hand, in the second integral summations including \( q \) can be replaced according to \( p + q \approx p \), \( \sigma_{q}^{R} \approx \sigma_{p}^{R} \). The kinetic equation then simplifies to

\[
\frac{\partial}{\partial \tau} n_{q,\tau} \big|_{q \ll 1} = \frac{|q|}{\sqrt{2} \pi^{2} \hat{\sigma}_{q}^{R}} \int_{0 < p} \left( 1 + n_{p,\tilde{\tau}} \right) \mid_{p \ll 1} \tilde{I}_{\tau},
\]

where \( \tilde{I}_{\tau} \) is a time dependent but momentum independent functional. The phonon density becomes

\[
n_{q,\tau} \big|_{q \ll 1} = n_{q,T} + |q| \int_{0 < \tilde{p} < 1} \tilde{I}_{\tau},
\]

for sufficiently small momenta. Crucially, the change is linear in momentum \( q \) and for \( q = 0 \) the density is time independent.

### B. Relaxation close to equilibrium

A stationary solution of the kinetic equation \( \partial_{\tau} n_{q,\tau} = 0 \) is given by the Bose distribution function

\[
n_{q,\tau} = n_{q}(u|q|, T) = \left( e^{u|q|/T} - 1 \right)^{-1}
\]

for arbitrary temperature \( T \). Sometimes one is interested in the relaxation of the distribution function close to equilibrium

\[
n_{q,\tau} = n_{q,u|q|, T} + \delta n_{q,\tau},
\]

where the variation \( \delta n \ll n_{q} \). For small momenta \( u|q| \ll T \), \( n_{q,u|q|, T} = \frac{T}{\tilde{I}(0)} \), and we can expand the kinetic equation in the variation \( \delta n \). The zeroth order part solves the kinetic equation, and the leading order contribution stems from the first order of the expansion. After eliminating negligible terms, it reads

\[
\frac{\partial}{\partial \tau} \delta n_{q} = -\frac{2 T^{2}}{u} \left( \int_{0 \leq p < q} \hat{\sigma}_{q}^{R} + \hat{\sigma}_{p}^{R} + \hat{\sigma}_{q-p}^{R} \right) \mid_{p \ll 1} \delta n_{q} \mid_{p \ll 1} + \frac{2}{u} \left( \int_{0 \leq p < q} \hat{\sigma}_{q}^{R} + \hat{\sigma}_{p}^{R} + \hat{\sigma}_{q-p}^{R} \right) \mid_{p \ll 1} \delta n_{q} \mid_{p \ll 1}.
\]

For a distribution close to thermal equilibrium, the self-energy will take on the thermal form (3.22) and

\[
\frac{\partial}{\partial \tau} \delta n_{q} = -\alpha_{q} \delta n_{q} \sqrt{\frac{2 \pi T}{u}} q^{\frac{1}{2}} = -\alpha_{q} \hat{\sigma}_{q}^{R} \delta n_{q},
\]

where \( \hat{\sigma}_{q}^{R} \) is the thermal self-energy and \( \alpha_{q} \approx 1.1 \) is a universal number. This result can also be seen as an expansion of Eq. (4.11) in \( \delta n_{q} \), which is

\[
\frac{\partial}{\partial \tau} \delta n_{q} = -2 \delta n_{q} \hat{\sigma}_{q}^{R} + \frac{\partial}{\partial \delta n_{q}} \left( \hat{\sigma}_{q}^{R} - (2n_{q} + 1) \hat{\sigma}_{q}^{R} \right) \mid_{\delta n_{q} = 0} \delta n_{q}.
\]
The second term thus gives a correction to the simple factor of 2 in Eq. \((1.11)\). We thus obtain a nonperturbative estimate for the relaxation time of the interacting Luttinger Liquid

\[
\tau_q = \frac{0.868}{v_0} \sqrt{\frac{u}{2\pi T}} q^{-2}
\]

(4.13)

reflecting the very slow asymptotic approach to equilibrium of the long-wavelength modes. It is very similar to the lifetime of a single thermal phonon (cf. Eq. \((2.22)\)), only modified by the prefactor \(\alpha_\nu = 1.1\). This modification arises due to in-scattering processes of excitations \(p \neq q\) scattering into the mode \(q\). Since the main relaxation process is caused by out-scattering processes, the above result is quite intuitive and supports the statement that relative time dynamics \(\sim \frac{1}{\epsilon_q}\) are fast compared to forward time dynamics \(\sim \tau_q\).

V. KINETIC EQUATION AND DIAGRAMS IN PRESENCE OF ANOMALOUS DENSITIES

In this section, we consider the effect of non-zero anomalous (off-diagonal) phonon density, i.e. \(\bar{a}_q \bar{a}_{-q} \neq 0\) on the diagonal kinetic equation and self-energy, and derive the kinetic equation for the anomalous density. Off-diagonal phonon densities can be populated due to external perturbations, as for instance density modulation due to a Bragg beam or a global interaction quench\(^{[106]}\) and their impact on the kinetics and non-equilibrium dynamics is therefore non-negligible. Summarizing the results of this section, due to the structure of the resonant interactions, the kinetic equation for the diagonal phonon density and the diagonal retarded/advanced self-energies are not modified in the presence of anomalous densities and remain unchanged compared to Eqs. \((3.5)\) and \((3.15)\). In contrast, the kinetic equation for the anomalous densities is fed by the normal occupations, cf. Eq. \((5.2)\).

For a generic equilibrium situation and for certain realizations of systems brought out of equilibrium, the anomalous (off-diagonal) phonon density and consequently the corresponding response and correlation functions are zero, i.e.

\[
\langle a_{iA}^\alpha a_{-iA}^{\alpha'} \rangle = 0,
\]

(5.1)

where \(\alpha, \alpha'\) represent classical or quantum indices. However, when a system is driven out of equilibrium, it is possible to populate off-diagonal terms. A simple situation for which this is indeed the case is an interaction quench in a one dimensional quantum fluid, where the off-diagonal densities are non-zero after the quench and lead to non-equilibrium dynamics even in the absence of phonon-phonon scattering processes\(^{[106]}\).

In order to deal with the situation of anomalous densities, we first have to modify the FDR accordingly. The Keldysh Green’s function in the presence of off-diagonal terms, expressed in Nambu space, is

\[
G^K_{q,t,t'} = \begin{pmatrix} G_{q,t,t'}^K & h^K_{q,t,t'} \\ h^K_{q,t,t'} & G_{-q,t,t'}^K \end{pmatrix} = -i \begin{pmatrix} \langle \bar{a}_{q,t}^\alpha a_{q,t'}^{\alpha'} \rangle & \langle \bar{a}_{q,t}^\alpha a_{-q,t'}^{\alpha'} \rangle \\ \langle \bar{a}_{-q,t}^\alpha a_{q,t'}^{\alpha'} \rangle & \langle \bar{a}_{-q,t}^\alpha a_{-q,t'}^{\alpha'} \rangle \end{pmatrix}.
\]

(5.2)

For the quadratic theory in the absence of phonon scattering, i.e. \(S = S_{\text{loc}}\), only, the Keldysh Green’s function can be evaluated explicitly. In a operator representation, it reads

\[
G^K_{q,t,t'} = -i \begin{pmatrix} \langle \bar{a}_{q,t}^\alpha a_{q,t'}^{\alpha'} \rangle & \langle \bar{a}_{q,t}^\alpha a_{-q,t'}^{\alpha'} \rangle \\ \langle \bar{a}_{-q,t}^\alpha a_{q,t'}^{\alpha'} \rangle & \langle \bar{a}_{-q,t}^\alpha a_{-q,t'}^{\alpha'} \rangle \end{pmatrix},
\]

(5.3)

with the anti-commutator \(\{\cdot, \cdot\}\). In Wigner representation it is

\[
G^K_{q,t,t'} = -i \begin{pmatrix} \delta(\omega - \epsilon_q)(2n_q + 1) & \delta(\omega)2m_q^s e^{-2\epsilon_q t} \\ \delta(\omega)2m_q^s e^{2\epsilon_q t} & \delta(\omega + \epsilon_q)(2n_{-q} + 1) \end{pmatrix},
\]

(5.4)

where \(m_q\) is the anomalous phonon density \((m_q = \langle |\bar{a}_q a_{-q}|^2\rangle\) in an operator representation in terms of annihilation operators \(a\)). In the quadratic theory, both the normal and the anomalous densities are constants of motion.

The Keldysh Green’s function in \((5.4)\) has two essential drawbacks. For the case of non-zero but slowly (compared to \(\epsilon_q\)) varying anomalous density \(m_q\), the off-diagonal terms of \(G^K\) are not slow but oscillate with the fastest scale in the problem, i.e. \(\tau_{\text{eff}} = \frac{1}{2\epsilon_q}\). The Wigner approximation is therefore not applicable for the off-diagonal terms of \(G^K\). Furthermore, the Keldysh Green’s function in frequency representation is peaked at three different frequencies, \(\omega = (\epsilon_q, 0, -\epsilon_q)\). Both prevents an FDR in the form of Eq. \((5.7)\) to exist in this representation.

In order to cure this problem, we switch to a rotating frame by introducing the fields \(a_{q,t} = a_{q,t}^e e^{i\omega t}, \bar{a}_{q,t} = \bar{a}_{q,t}^e e^{-i\omega t}\), which modifies the quadratic action according to

\[
S^{(2)} = \int \langle \bar{a}_{q,t}^e, \bar{a}_{q,t}^e \rangle \left( i\dot{\bar{a}}_q + i\dot{a}_q - i\omega t \right) 2i\omega \coth \left( \frac{\omega}{2T} \right) e^{-i\omega t} \right) \left( a_{q,t}^e, a_{q,t}^e \right).
\]

(5.5)

The resonant phonon interaction is invariant under the transformation

\[
S_{\text{res}} = \frac{\sqrt{1}}{\sqrt{2}} \int \left[ 2\bar{a}_{k+p}^e a_{k}^e a_{p}^e + \bar{a}_{k+p}^e (a_{k+q}^{e\prime} a_{q}^{e\prime} + a_{k}^{e\prime} a_{q+q}^{e\prime}) \right. \\
\left. + \bar{a}_{k+p}^e \left( a_{k-1}^{e\prime} a_{q}^{e\prime} + a_{q+q}^{e\prime} a_{k-1}^{e\prime} \right) \right. \\
\left. \left. + \bar{a}_{k+p}^e \left( a_{k-1}^{e\prime} a_{q-q+1}^{e\prime} + a_{q+q}^{e\prime} a_{k-1}^{e\prime} \right) \right. \\
\left. \left. \right. \right. + \bar{a}_{k+p}^e \left( a_{k-1}^{e\prime} a_{q+q}^{e\prime} + a_{q}^{e\prime} a_{k-1}^{e\prime} \right) \right].
\]

(5.6)

since the phase \(e^{i(\rho_{\omega_{\epsilon q-t}} - \rho_{\omega_{\epsilon q-t}})} = 1\) in the case of resonance, i.e. for \(|k + p| = |k + p|\). The corresponding correlation function in Nambu space and Wigner coordinates after the rotation is

\[
G^K_{q,t,t'} = -i\delta(\omega) \left( \frac{2n_q + 1}{2m_q^s} \right) \left( 2m_q^s + 1 \right).
\]

(5.7)

while the bare retarded Green’s function reads

\[
G^K_{q,t,t'} = \begin{pmatrix} \frac{1}{\omega + i\tau_\omega} & 0 \\ 0 & \frac{1}{\omega - i\tau_\omega} \end{pmatrix} = \sigma_z \frac{1}{\omega + i\tau_\omega},
\]

(5.8)

where \(\sigma_z\) is the Pauli matrix. Respecting the symplectic structure in bosonic Nambu space, the FDR in the presence of off-diagonal densities is

\[
G^K_{q,t,t'} = \left( \hat{G}^R \circ \sigma_z \circ \hat{F} - \hat{F} \circ \sigma_z \circ \hat{G}^A \right)_{q,t,t'} = -i\delta(\omega) \left( \frac{1}{\omega + i\tau_\omega} \right).
\]

(5.9)
Here $\bar{F}$ is the physical distribution function for the phonons, with the on-shell value

$$\bar{F}_{q,\omega,\tau} = \left( \frac{2n_{\eta,\tau} + 1}{2m_{\eta,\tau}^*} \frac{2m_{\eta,\tau}}{2n_{-\eta,\tau} + 1} \right). \tag{5.10}$$

Both the transformation to a rotating frame by switching from $(a_q, \bar{a}_q)$ to $(a_Q, \bar{a}_Q)$ as well as the symplectic factors $\sigma_\alpha$ in Eq. (5.9) are necessary modifications in order to obtain an FDR with a physically relevant distribution function $\bar{F}$. The latter should consist of diagonal and anomalous densities that are slowly varying in time and reproduce the matrix structure of $\bar{G}^R$. Inversion of Eq. (5.9) yields the kinetic equation

$$i\partial_t F_{q,\omega,\tau} = \sigma_\alpha \Sigma_{q,\omega,\tau}^{\bar{R}} F_{q,\omega,\tau} - F_{q,\omega,\tau} \Sigma_{q,\omega,\tau}^{\alpha} \sigma_\alpha - \sigma_\alpha \Sigma_{q,\omega,\tau}^{\bar{K}} \sigma_\alpha. \tag{5.11}$$

In the absence of off-diagonal terms in both self-energies and the distribution function, this equation reduces to the ordinary kinetic equation discussed in the previous section. We will now set up the diagrammatic computation of the self-energies $\Sigma_{q,\omega,\tau}^{\alpha}$ in order to derive the kinetic equation in the presence of anomalous phonon densities.

### A. Diagrammatics for off-diagonal terms

In the presence of off-diagonal densities, one can no longer generally exclude non-zero off-diagonal self-energies and consequently non-zero off-diagonal Green’s functions from the action. In this section we set up the diagrammatic computation of the self-energies in Nambu space. We obtain two key results, which crucially rely on the resonant character of the interaction. First, in the retarded/advanced sector, the off-diagonal self-energies are exactly zero even in the presence of anomalous densities. Second, we compute the off-diagonal self-energies in the Keldysh sector, which are different from zero.

The retarded Green’s function in Nambu space is (we use a general index $Q = (q, \omega, \tau), -Q = (-q, -\omega, \tau)$ for this para-

- Diagrams for on-shell $Q$ or $-Q$
- Diagrams for off-shell $Q$ or $-Q$
- Diagrams for on-shell $Z$ or $-Z$
- Diagrams for off-shell $Z$ or $-Z$
- Diagrams for on-shell $\bar{Z}$ or $-\bar{Z}$
- Diagrams for off-shell $\bar{Z}$ or $-\bar{Z}$

The additional index $Z_{OPK} = \pm 1$ represents ingoing lines $(a_q, \bar{a}_q, \bar{a}_Q)$ for $z = 1$ and outgoing lines $(\bar{a}_q, q, \bar{a}_Q)$ for $z = -1$. The $\delta$-constraints stem from the momentum/frequency conservation in the Green’s function and the momentum/frequency conservation in the two vertices. Two additional constraints, one for each vertex, are caused by the resonance condition. This results in six crucial constraints discussed in the main text.
shown in Fig. 3 even in the presence of off-diagonal Green’s functions, i.e.

\[-i \Sigma_Q^R = \frac{V_0^2}{4\pi^2} \int p q \left( |q - p| h_{Q,p}^R + |p + q| h_{Q+p}^A \right) + |p + q| (h_{Q+p}^R)^* \].

(5.19)

In order to obtain the off-diagonal self-energy \( \Sigma_Q^R \), one has to flip the sign of \( z_{Q'} \to 1 \), such that \( z_{Q-Q'} = z_{Q+p} = z_{Q}z_{Q'} = 1 \) in the corresponding loops. This means that the diagrams contain only off-diagonal terms. \( \Gamma_Q^R \) is then obtained by the diagrams in Fig. 3 but with all arrows from the right vertex flipped. We thus obtain

\[-i \Gamma_Q^R = \frac{V_0^2}{4\pi^2} \int p q \left( |q - p| h_{Q,p}^R + |p + q| h_{Q+p}^A \right) - |p + q| (h_{Q+p}^R)^* \].

(5.20)

The diagonal self-energy \( \Sigma^R \) diverges when the integral (5.19) is performed with the bare Green’s functions. This hints that a non-trivial self-energy is generated on the diagonal to regulate the integral, which can be computed in self-consistent Born approximation as explained in previous sections. On the other hand, for off-diagonal Green’s functions \( h^R = h^A = 0 \) (e.g. for the bare off-diagonal values), the off-diagonal self-energy is zero, as visible from Eq. (5.20). Consequently, off-diagonal self-energies are not generated in the retarded sector even in self-consistent Born approximation and we have

\[ \Gamma_Q^R = 0. \]

(5.21)

In the absence of off-diagonal self-energies in the retarded sector, we can directly apply the quasi-particle approximation discussed in the previous sections and evaluate the self-energies and distribution function on-shell. Consequently, the intermediate kinetic equation is

\[ \partial_t F_{q,\omega=0,\tau} = -2\sigma_{q,\omega=0,\tau} \Sigma_{q,\omega=0,\tau}^R \sigma^\tau \]

(5.22)

with the scalar, on-shell self-energy \( \sigma_{q,\tau}^R \) as discussed in Sec. IIIb.

For the on-shell Keldysh self-energy,

\[ \Sigma_{q,\omega=0,\tau}^K = -2i \left( \frac{\sigma_{q,\tau}^R}{\Gamma_{q,\tau}^K} \Gamma_{q,\tau}^K \sigma_{q,\tau}^R \right) \]

(5.23)

the diagrammatic rules from the previous section do not have to be modified. This immediately yields the on-shell Keldysh self-energy \( \sigma_{q,\tau}^K \) according to Fig. 3. For the off-diagonal, on-shell Keldysh self-energy \( \Gamma_{q,\tau}^K \) the corresponding diagrams are obtained by reversing the arrows associated to the vertices on the right in Fig. 3 resulting in the diagrammatic representation of \( \Gamma_{q,\tau}^K \) shown in Fig. 3. The Keldysh Green’s functions are obtained via the parametrization used in Eq. (5.19), which yields for the off-diagonal elements

\[ h_{q,\omega=\tau}^K = \frac{16m^2 \nu_{q,\tau}^R m_{q,\tau}}{\left( \omega - u(q) \right)^2 + \left( \sigma_{q,\tau}^R \right)^2}. \]

(5.24)

FIG. 9. Diagrammatic representation of the off-diagonal Keldysh self-energy \( \Gamma_{q,\tau}^K \). Compared to the diagonal component, for the vertex of the left ingoing and outgoing lines have been replaced. The diagrams containing only retarded/advanced Green’s functions, which contributed to \( \sigma_{q,\tau}^R \), are absent because of the absence of off-diagonal retarded and advanced Green’s functions.

Evaluation of the off-diagonal diagrams and insertion into the kinetic equation completes the set of equations for a system including anomalous phonon densities.

Finally, the off-diagonal retarded/advanced self-energies are exactly zero and the kinetic equation for the off-diagonal terms is

\[ \partial_t m_q = \int_{0<p<q} \frac{2pq(q-p)}{\sigma_{q,p}^R + \sigma_{p,q}^R} \left( m_{p,q}^2 - m_q \left( 1 + n_p + n_{q-p} \right) \right) \]

\[ + \int_{0<p<q} \frac{4pq(q+p)}{\sigma_{q,p}^R + \sigma_{p,q}^R} \left( n_{p,q}m_q + m_{p,q} - m_p m_{q-q-p} \right). \]

(5.25)

Here, we have again used the transformed basis to eliminate the factor \( v_0^2 \) in front of the integrals. The time evolution for the off-diagonal terms depends on both the diagonal and off-diagonal densities, and the stationary solution of this equation is \( m_q = 0 \) due to the uncompensated spontaneous decay term in the first integral. Together with Eqs. (5.15) and (4.3), Eq. (5.25) represents the complete set of equations determining the time evolution of the phonon density and the self-energies of an interacting Luttinger Liquid for a system that has been initialized in an out-of-equilibrium state \( n_q \neq n_q(t|q), m_q \neq 0 \).

VI. RELAXATION OF AN EXCITED THERMAL STATE

In this section, we will analyze the relaxation dynamics of a nearly thermal initial state and compare it to the analytical results from the previous sections. We consider an initial state with the densities

\[ n_{q,\tau=0} = n_q(T = 2u(q) + \delta_q) \]

\[ = \frac{1}{e^{\nu_{q,\tau}^R - \beta u(q)} - 1} + a_1 e^{-\frac{\nu_{q,\tau}^R - \beta u(q)}{\beta}}, \]

(6.1)

\[ m_{q,\tau=0} = \delta m_{q,\tau=0} = 2a_1 e^{-\frac{\nu_{q,\tau}^R - \beta u(q)}{\beta}}. \]

(6.2)

A state of this form can be created by perturbing a thermal state with initial temperature \( T_i = 0.5u(q) \) in coupling the operator \( \partial_t \phi \) to a classical field with momentum \( q_0 \), i.e. in
The quantity of interest is the deviation of the time-dependent phonon density from the final phonon density in the limit \( \tau \to \infty \). In the present case, this leads to a final state of the form \( n_\infty(T_i) \), a thermal state with increased temperature \( T_i \). As a result, the final state in the limit \( \tau \to \infty \) becomes very small, the numerical error and therefore the relative deviation is significant.

\[ \tau \to \infty, \quad \delta n_{q,\tau} \equiv n_{q,\tau} - n_\infty(T_i). \]  

In Fig. 10 we show \( \delta n_{q,\tau} \) and \( m_{q,\tau} \) for different time steps \( \tau_i \), and we see in which way both quantities decay to zero momentum as a function of time and momentum.

According to Eq. (4.11), \( \delta n_q \) follows the effective equation of motion

\[ \partial_\tau \log(\delta n_{q,\tau}) = -1.1\gamma_q. \]  

As a result, \( \delta n_{q,\tau} \) decays to zero exponentially in time with a momentum dependent decay time \( \tau_q(q) = (1.1\gamma_q)^{-1} \).

In Fig. 11 we compare the numerical value of \( \partial_\tau \log(\delta n_{q,\tau}) \) with the analytical estimate

\[ \partial_\tau \log(\delta n_{q,\tau}) \approx -1.1\gamma_q \approx 0.87 \sqrt{2\alpha_q q^{5/2}} \]  

and find a good agreement in the momentum region where \( \delta n_q \) deviates from zero.

VII. DYSON-SCHWINGER EQUATIONS AND VERTEX CORRECTIONS

In this section we apply Dyson-Schwinger equations to the interacting Luttinger Liquid and determine the self-energy and three-point vertex as a function of the phonon distribution. We show that the vertex correction is always real and will be exactly zero for a zero temperature state. More generally it leads to a modification \( \nu_0 \to \nu_0 (1 + \mathcal{I}) \), where \( \mathcal{I} \) is a small \( (1 \gg \mathcal{I}) \) dimensionless function with weak momentum dependence, whose precise form is determined by the time-dependent phonon density. Based on these findings, we conjecture that also the corrections to the four-point and higher order vertices, which do not occur in the microscopic action,
The hierarchy built up by the DSE is in general infinite and is modified according to the vertex correction, and we derive a schematic diagrammatic representation of Eqs. (7.3) and (7.4). We will show in the following that there are corrections to the bare three-point vertex itself, with the same scaling dimension as the bare three-point vertex. A schematic diagrammatic representation of Eqs. (7.3) and (7.4) is depicted in Fig. 12. An exact statement for the bare Green's function, where the effective action, for dimensionless function $I^{(n)}$, is determined by a diagrammatic representation in Fig. 13. The effective action, as the generator of 1PI correlation functions, can be expanded according to Eq. (7.1). The effective action, as the generator of 1PI correlation functions, can be expanded according to Eq. (7.1).

\[ I^{(n)} = \int \frac{d^d p}{(2\pi)^d} \int \frac{d^d \omega}{(2\pi)^d} \frac{1}{\omega - \epsilon_{np} - \delta G_{\omega} + \delta \Sigma_{\omega}}. \]
higher order terms and for a constant distribution function, the vertex correction is exactly zero, this has been shown to hold for arbitrary dimensions[^31].

In order to find a compact expression for the vertex correction, we replace $\sigma^R \to v_0 \sigma^R$ and compare the integrand in Eq. (7.6) with the expression for the self-energy $\tilde{\sigma}$ in Eq. (3.18). We immediately see, that the vertex correction is linear in $v_0$ and the integral has scaling dimension zero[^61]. In perturbation theory, this yields the vertex

$$V_{q,p+q,p}^{(x)} = v_0 \sqrt{|pq(p+q)|} \left( 1 + I_0 \left( \frac{p}{q}, n \right) \right)$$

$$= S_{q,p+q,p}^{(3)} \left( 1 + I_0 \left( \frac{p}{q}, n \right) \right). \quad (7.7)$$

Here, $I_0 \left( \frac{p}{q}, n \right)$ is a dimensionless function of the ratio $p/q$ and the phonon density $n$, which is determined by the integral in Eq. (7.6). The scaling behavior of the one-loop vertex correction suggests the parametrization of the full vertex according to

$$V_{q,p+q,p} = v_0 \sqrt{|pq(p+q)|} \left( 1 + \tilde{I} \left( \frac{p}{q}, n \right) \right), \quad (7.8)$$

where the functional $\tilde{I}$ encodes the full vertex correction. According to the DSE in Fig. [12] it is determined via

$$I(x, n) = \frac{1}{\sqrt{8}} \int_{\kappa > 0} \left\{ \tilde{k}(1 + \tilde{k})(\tilde{k} - x) \left( 1 + I \left( \frac{\tilde{k}}{1 + \tilde{n}}, n \right) \right) \left( 1 + I \left( \frac{\tilde{k}}{1 + \tilde{n}}, n \right) \right) \frac{n_{\tilde{k}+\tilde{n}} - n_{\tilde{k}} + n_{\tilde{n}+\tilde{k}} - n_{\tilde{k}}}{\tilde{\sigma}_{\tilde{k}+\tilde{n}} + \tilde{\sigma}_{\tilde{k}} + \tilde{\sigma}_{\tilde{n}} + \tilde{\sigma}_{\tilde{k}+\tilde{n}}} \right\}.$$  

(7.9)

where $\tilde{k} = \frac{\kappa}{\tau}$ and the integral is dimensionless. In Eq. (7.9), we have already exploited the fact that the ingoing momenta of a vertex can be exchanged without modifying the vertex itself, and consequently the integral is invariant under $q \leftrightarrow p$. This is equivalent to $I(x, n) = I(\tilde{x}, n)$. The self-energy in the DSE approach is

$$\tilde{\sigma}^R_q = \int_{p < q} \left( \frac{\partial_k n_p}{\tilde{\sigma}^R_p} + 2n_p + 1 \right) \left\{ \left[ 1 + I \left( \frac{p}{p-q} \right) \right] \frac{qp(q-p)}{\tilde{\sigma}^R_p + \tilde{\sigma}^R_{p-q}} + \left[ 1 + I \left( \frac{q}{q-p} \right) \right] \frac{qp(p+q)}{\tilde{\sigma}^R_p + \tilde{\sigma}^R_{q-p}} \right\}. \quad (7.10)$$

In the same way the kinetic equation can be modified to incorporate the vertex correction as well, and we find a set of coupled equations

$$\left( \frac{\partial_t n}{\tilde{\sigma}^R} \right) = F(n, \tilde{\sigma}^R, I). \quad (7.11)$$

They can be solved numerically according to the procedure described in Fig. [6]. Including the vertex correction, the second step of the iteration additionally includes the self-consistent determination of $I$.

We will now give an estimate of the order of the vertex correction for the case for which it is non-zero to estimate its impact on the dynamics and the kinetic equation. In the limit $T \to \infty$ the relevant phonon density is $n_{\kappa} \approx \frac{\kappa}{2\tau^2}$ and the self-energies have the thermal form $\tilde{\sigma}^R_q \propto \sqrt{q^3}$. Consequently, the factor $\kappa^2$ drops out and $I(x, n)$ does no longer depend on temperature. In this case, $I(0, n) \approx 0.012$, $I(1, n) \approx 0.09$ and $I(0, n) \leq I(x, n) \leq I(1, n)$. As can be seen in Fig. [14] the correction is small, with a weak momentum dependence. Consequently, the self-energy is only negligibly modified if instead of the full three-point vertex in Eq. (7.4), the bare value $\tilde{V}$ is used. This is precisely the self-consistent Born approximation that we used to determine the self-energies and the kinetic equation for the interacting Luttinger Liquid.
FIG. 14. Vertex correction $I(x, n_0(T))$ for an infinite temperature state. In the limit $T \to \infty$, the temperature drops out in Eq. (7.9), and the vertex correction becomes temperature independent. Due to the invariance of $I$ under $x \to \frac{1}{x}$, the plot is restricted to $0 \leq x \leq 1$. The dependence of $I$ on $x$ is weak, especially for $x \approx 1$, where it takes its maximum $I(x, n_0) \leq I_{\text{max}} \approx 0.093$.

VIII. CONCLUSION

In this article, we used non-equilibrium field theory, in particular kinetic and Dyson-Schwinger equations, to determine the kinetics and non-equilibrium dynamics of resonantly interacting Luttinger Liquids. Exploiting the fact that the interactions lead to dressed but still well defined phonons, which enables a separation of timescales into slow forward and fast relative dynamics, we applied the Wigner and quasi-particle approximation and derived a closed set of simple yet powerful equations for the normal and anomalous phonon density, the phonon self-energy and vertex correction. These equations determine the dynamics of an interacting Luttinger Liquid initialized in an arbitrary (non-)equilibrium state. The resulting equations show strong aspects of universality, on the one hand being independent of any UV-scale, in particular independent of the Luttinger cutoff. On the other hand, after a proper rescaling of the forward time, all microscopic parameters entering the Hamiltonian can be eliminated and the only microscopic information entering the dynamical equations is the initial phonon density. We further used our approach to analytically determine the relaxation rate of a thermally excited state and compared it to the well-fitting numerical result.

The results of this work can be used in order to determine the kinetics and non-equilibrium dynamics of one-dimensional interacting quantum fluids prepared in a nonthermal initial state, which might occur as a consequence of a quantum quench or a sudden external perturbation. On the other hand, it paves the way to compute the dynamics of quantum fluids subject to drive and dissipation, and to determine the dynamics towards the steady state of an excited closed system. Of special interest and a strength of our approach is the treatment of long time dynamics in non-equilibrium systems, which are neither reachable by present numerical procedures nor by analytical approaches based on perturbation theory. Both examples belong to the uprising field of one-dimensional quantum fluids out-of-equilibrium and we leave their discussion open for future work.

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