Clifford-Wolf Translations of Finsler spaces

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Abstract

In this paper, we study Clifford-Wolf translations of Finsler spaces. We first give a characterization of Clifford-Wolf translations of Finsler spaces in terms of Killing vector fields. In particular, we show that there is a natural correspondence between Clifford-Wolf translations and the Killing vector fields of constant length. In the special case of homogeneous Randers spaces, we give some explicit sufficient and necessary conditions for an isometry to be a Clifford-Wolf translation. Finally, we construct some explicit examples to explain some of the results of this paper.

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1 Introduction

Let \((X,d)\) be a locally compact connected metric space. A Clifford-Wolf translation of \((X,d)\) is an isometry \(\rho\) of \((X,d)\) onto itself such that the function \(d(x,\rho(x))\) is constant on \(X\). Geometrically, a Clifford-Wolf translation is an isometry of the metric space such that all the points in \(X\) move the same distance. In the Riemannian case, it is an important problem to determine all the Clifford-Wolf translation of an explicit Riemann manifold. For example, in the non-positive curvature case, J. A. Wolf proved that an isometry of the Riemannian manifold is a Clifford-Wolf translation if and only if it is bounded (\cite{WO64}). Moreover, if a Riemannian manifold \((M,Q)\) has strictly negative curvature, then any Clifford-Wolf translation of \((M,Q)\) must be trivial. The above consequences have an important corollary that a homogeneous Riemannian manifold with non-positive curvature admits a transitive solvable Lie group of isometries. This result is the basis for many works on homogeneous Riemannian manifolds with negative (non-positive) curvature, see for example \cite{HE74, AW76}. It should be noted that J. A. Wolf’s results have been generalized to general Finsler spaces, see \cite{DP}. Using these

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results, we can hopefully get a classification of negatively curved homogeneous Finsler spaces.

The purpose of this article is to initiate the study of Clifford-Wolf translations of Finsler spaces. Let $(M, F)$ be a Finsler space, where $F$ is positively homogeneous of degree one but not necessary absolutely homogeneous. Then we can define the distance function $d$ of $(M, F)$. Although generically $d$ is not reversible, we can define a Clifford-Wolf translation of $(M, F)$ to be an isometry $\rho$ of $(M, F)$ onto itself such that $d(x, \rho(x))$ is a constant on $M$. It is an interesting problem to study the problem that to what extent the results on Clifford-Wolf translations of Riemannian manifolds still hold for Finsler spaces. In particular, it is hopeful that some new phenomena will happen in this more general case.

In the paper we first give a characterization of the Clifford-Wolf translations of Finsler spaces, in terms of Killing vector fields. In particular, we show that there is a natural correspondence between Wolf-Clifford translations and the Killing vector fields of constant length. In the special case of homogeneous Randers spaces, we give some explicit sufficient and necessary conditions for an isometry to be a Clifford-Wolf translation. Finally, we construct some explicit examples to explain some of the results of this paper.

The arrangement of the paper is as the following: In Section 2, we present some fundamental knowledge on Finsler geometry, in particular the definition of the Riemann curvature of Finsler metrics. In Section 3, we study Clifford-Wolf translations of general Finsler spaces and obtain a characterization of Clifford-Wolf translations in terms of Killing vector fields of constant length. In Section 4, we apply the results to homogeneous Randers spaces and obtain some explicit description of Clifford-Wolf translations. Finally, in Section 5, we construct some examples in some special compact Lie groups to explain some of the results.

## 2 Preliminaries

**Definition 2.1** A Finsler metric on a manifold $M$ is a function $F : TM \to \mathbb{R}^+$, which is smooth on the slit tangent bundle $TM \setminus \{0\}$. In any local coordinates $(x^i, y^j)$ for $TM$, with $x^i$'s the local coordinates for $M$ and $y^j$'s the local coordinates for the tangent vectors, $F$ satisfies the following conditions:

1. $F(x, y) > 0$ for any $y \neq 0$.
2. $F(x, \lambda y) = \lambda F(x, y)$ for any $y \in TM_x$ and $\lambda > 0$.
3. The Hessian matrix defined by $g_{ij} = \frac{1}{2} [F^2]_{y^i y^j}$ is positive definite.

A Finsler metric $F$ is called reversible if it satisfies $F(x, y) = F(x, -y)$.

A Finsler metric of the form $F = \alpha + \beta$, where $\alpha$ is a Riemannian metric and $\beta$ is a 1-form, is called a Randers metric. In this case, the positive definiteness of the metric is equivalent to the condition that the length of $\beta$ with respect to the metric $\alpha$ is everywhere smaller than 1. A Randers metric is reversible if and only if it is Riemannian.
As a generalization of Riemannian geometry, almost all concepts in Riemannian geometry have their counterpart in Finsler geometry. The connections and curvatures have already been discussed in a lot of works, see for example [BCS00] and [CS04]. Below we only briefly recall the notion of Riemann curvature of a Finsler space. This will be useful in the following sections.

Let \((M,F)\) be a connected Finsler space and \(V\) a nowhere zero vector field on a open subset \(U\) of \(M\). Then we can define an affine connection on the tangent bundle \(TU\) over \(U\), denoted by \(\nabla^V\), such that the following holds:

(a) \(\nabla^V\) is torsion-free:
\[
\nabla^V_X Y - \nabla^V_Y X = [X,Y],
\]
for all vector fields \(X,Y\) on \(U\),

(b) \(\nabla^V\) is almost metric-compatible:
\[
Xg^V(Y, Z) = g^V(\nabla^V_X Y, Z) + g^V(Y, \nabla^V_X Z) + 2C^V(\nabla^V_X V, Y, Z),
\]
for all vector fields \(X,Y,Z\) on \(U\).

In the above formulas, \(g^V\), resp. \(C^V\), is the fundamental tensor, resp. Cartan tensor, of \((M,F)\). It is worthwhile to point out that the conditions (a), (b) can be used to deduce an explicit formula for the Chern connection, see [RA04] for the details.

A nowhere zero vector field \(V\) on \(U\) is called a geodesic vector field if \(\nabla^V V = 0\). This is equivalent to the condition that the flow lines of \(V\) are all geodesics. In this case, \(V\) is also a geodesic vector field with respect to the Riemannian metric \(g^V\). The vector field \(V\) is called parallel if \(\nabla^V X = 0\), for any vector field \(X\) on \(U\).

The above affine connection has close relation to the Chern connection presented in [BCS00] and [CS04]. In particular, using the properties of the Chern connection we have the following conclusion: if \(V,W\) are two nowhere zero vector fields on \(U\) and \(V(x) = W(x)\), then
\[
(\nabla^V_X Y)(p) = (\nabla^W_X Y),
\]
for any vector fields \(X,Y\) on \(U\). This fact enable us to define the covariant derivative along any curve. Let \(c(t)\) be a curve on \(U\) and \(V,X\) be vector fields along \(c\), with \(V\) nowhere zero. Extending \(V, X\) and \(c'\) to a subset containing \(c\), we can define \(\nabla^V_{c'} X(t) = (\nabla^X_{c'} X)(t)\). The above fact shows that this is independent of the extensions. In case \(V\) and \(c'\) coincide, we denote the derivative simply by \(\nabla_{c'}\).

Now we define the Riemann tensor of \((M,F)\). Let \(X\) be a non-zero tangent vector of \(M\) at \(x\) and \(c_X\) be the geodesic starting from \(x\) with initial vector \(X\). Given \(Y \in T_x(M)\), we can construct a geodesic variation with variation vector field \(Y(s,t)\) such that \(Y(0,0) = Y\). We then define
\[
R^X(Y) = \frac{\nabla^2}{dt} Y(s,0)|_{s=0}.
\]
It can be checked that the right hand of the above equation does not depend on the choice of the variation. It is called the Riemann curvature operator of \((M,F)\). See Shen’s book [SH01] for more details.

The following result is very important, see [SH01] for a proof.
Lemma 2.2 (Shen [SH01]) For any nonzero vector $X$ in $T_x(M)$ with a nonzero geodesic field $V$ extending $X$ in an open neighborhood $U$ of $x$, the Riemann curvature operator $R^X$ of the Finsler space coincides with the Jacobi operator $\tilde{R}^X$ of the osculating Riemannian metric $g_V$.

Note that $V$ is nowhere zero on $U$, so that the Riemannian metric $g_V$ is well-defined on $U$. Also recall that the Jacobi operator $\tilde{R}^X$ of $g_V$ is defined by

$$\tilde{R}^X(Y) = R(Y, X)X, \quad Y \in T_x(M),$$

where $R$ is the curvature tensor of $g_V$.

Now we generalize the notions of Clifford-Wolf translations and Killing vector fields to Finsler geometry.

Definition 2.3 A Clifford Wolf translation $\rho$ is an isometry of the Finsler manifold $(M, F)$ which satisfies the condition that the distance from any $x \in M$ to its image $\rho(x)$ is a constant.

It should be noted that, unlike in the Riemannian case, the reverse $\rho^{-1}$ of a Clifford Wolf translation $\rho$ of a Finsler space may not be a Clifford Wolf translation, for a Finsler metric may not be reversible.

In [DH02], it is proved the group of isometries of a Finsler space $(M, F)$ is a Lie transformation group on $M$. Its Lie algebra is the space of Killing vector fields. Equivalently, a smooth vector field on $(M, F)$ is a Killing vector field if and only if the flow $\phi_t$ generated by $X$ are isometries of $(M, F)$.

3 Clifford-Wolf translations and Killing vector fields of constant length

Clifford-Wolf translations and Killing vector fields of constant length are closely related. In the Riemannian case, if a Killing vector field $X$ generates a family of Clifford-Wolf translations $\varphi_t$, where $t$ is close enough to 0, then $X$ must have constant length. If we assume the Riemannian manifold to be compact, then any Clifford Wolf translation close to the identity map (in the topology of the Lie group of isometries) is generated by a Killing vector field of constant length (see [BN081, BN082, BN09]). We now show that these statements are also correct in the Finsler case.

Lemma 3.1 Let $X$ be a Killing vector field on a Finsler space $(M, F)$ and $U \subset M$ be a open subset such that $X$ is nowhere zero on $U$. Then on $U$ we have $\nabla^X_X X = -\frac{1}{2}\hat{\nabla}^{(X)}|X|^2$, where $\hat{\nabla}^{(X)} f = g_{ij}(X)f_{x^j}\frac{\partial}{\partial x^i}$ is the gradient field of $f$ for the Riemannian metric $g_X = g_{ij}(X)$ on $V$.

Proof. Choose a local coordinate system $x = (x^i)$ with $X = \frac{\partial}{\partial x^1}$, which implies that the flow $\phi_t$ defined by $X$ is just a shift of $x^1$ by $t$ in this chart. The assumption that $X$
is a Killing vector field for $F$ means that $F(x, y)$ is independent of $x^1$. By definition,

$$
\nabla^X X = 2g^i(X) \frac{\partial}{\partial x^i} \\
= \frac{1}{2} g^{il}([F^2]_{x^my^ny^m} - [F^2]_{x^i}) \\
= \frac{1}{2} g^{il}(X)(([F^2]_{x^my^ny^m}(X) \frac{\partial}{\partial x^l} - \frac{1}{2} \nabla X |X|^2. \quad (3.1)
$$

Consider the value at $X$. Since $y^2 = \cdots = y^n = 0$, and $y^1 = 1$, we have

$$
\frac{1}{2} g^{il}(X)[F^2]_{x^my^ny^m}(X) = \frac{1}{2} g^{il}(X)[F^2]_{x^1y^1} = 0, \quad (3.2)
$$

since by the assumption $F^2$ is independent of $x^1$. This completes the proof. ■

As an immediate consequence, we have

**Corollary 3.2** If $X$ is a Killing vector field on $(M, F)$, then all flow curves of $X$ are geodesics if and only if $X$ has constant length.

The relations between the Clifford-Wolf translations and Killing vector fields for a Finsler manifold are stated in the following two theorems.

**Theorem 3.3** Suppose the complete Finsler manifold $(M, F)$ has a positive injective radius. If $X$ is a Killing field on $(M, F)$ of constant length and $\varphi_t$ is the flow generated by $X$, then $\varphi_t$ is a Clifford Wolf translation for all sufficiently small $t > 0$.

**Proof.** Denote the injective radius of $(M, F)$ by $r > 0$ and suppose the constant length $F(X)$ of $X$ is $l > 0$. For any $t \in (0, r/l)$, the geodesic flow curve from $x$ to $\varphi_t(x)$ is contained in the geodesic ball $B_r(x)$. Then the distance from $x$ to $\varphi_t(x)$ is the length of flow curve from $x$ to $\varphi_t(x)$, which is $tl$. From this the theorem follows. ■

**Theorem 3.4** Let $(M, F)$ be a compact Finsler space. Then there is a $\delta > 0$, such that any Clifford-Wolf translation $\rho$ with $d(x, \rho(x)) < \delta$ is generated by a Killing vector field of constant length.

**Proof.** Denote by $G$ the group of isometries of $(M, F)$ and by $\mathfrak{g}$ the Lie algebra of $G$. Then $G$ is a compact Lie group ([DH02]). For sufficiently small $\delta > 0$, any Clifford-Wolf translation $\rho$ with $d(x, \rho(x)) < \delta$ is contained in a neighborhood $V \subset G$ of the identity map which can be generated by the exponential map. In particular, there is a Killing vector field $X$ with flow $\phi_t$ such that $\phi_t = \rho$. By the compactness, we can assume that $\delta$ is smaller than the injective radius, i.e. the distant from $x$ to $\phi_t(x)$ for each $t \in [0, 1]$ is the length of the unique geodesic within the geodesic ball $B_\delta(x)$. Now we prove that $X$ has a constant length $l$. Then it follows from (3.1) immediately that the distance from $x$ to $\phi_t(x)$ is the constant $tl$. It is obvious that $F(X)$ is constant along each flow curve. By (3.1) $\nabla^X X = 0$ along any flow curve where $F(X)$ takes its minimum or maximum. In either case, we have $d(x, \rho(x)) = F(X)$. If $\rho$ is a Clifford-Wolf translation, then the minimum and maximum of $F(X)$ will be equal. Hence $X$ is a Killing vector field of constant length. ■

Below is another direct corollary of Lemma 3.1.
Corollary 3.5 If $X$ is a Killing vector field on $(M, F)$ of constant length, then the covariant derivative $\nabla_X X = 0$, where $R^X : TM \to TM$ is the Riemann curvature of $F$.

Proof. By 3.1 $X$ is a geodesic vector field. So neither the connection coefficients for $D_X$ nor the curvature tensor $R^X$ is changed when they are evaluated at $X$ and $F$ is replaced by the Riemannian metric $g_X$. Hence $\nabla_X X = 0$ remains the same when $F$ is replaced by the Riemannian metric $g_X$. From the proof of Lemma 3.1, it is easily seen that $X$ is also a Killing vector field for $g_X$. On the special coordinates chosen there, $F$ is independent of $x_1$, so $g_{ij}(X) = \frac{1}{2}[F^2]_{y^iy^j}$ is also independent of $x^1$. From [BN09], we know that a Riemannian Killing vector field of constant length satisfies $\tilde{\nabla}_X X = 0$. Therefore the equality remains valid for $F$. ■

In the Riemannian case, this corollary is a key step to study Clifford-Wolf homogeneous spaces. But in the Finsler case, it is much less useful.

For a Randers space $(M, F)$, a Finsler Killing vector field $X$ is in fact also a Riemannian Killing vector field for $\alpha$. In fact we have

Lemma 3.6 A vector field $X$ on a Randers space $(M, F)$, $F = \alpha + \beta$ is a Killing vector field if and only if $X$ is a Killing vector field for $\alpha$ and $L_X \beta = 0$.

Proof. If the vector field $X$ is a Killing vector field for $(M, F)$, then its flow $\phi_t$ fixes $F$, i.e.,

$$\phi_t^* F = \phi_0^* \alpha + \phi_t^* \beta = \alpha + \beta = F,$$

for each $t$. For any $x$ and $y \in T_x(M)$, $y \neq 0$, considering the values of $F$ on $(x, y)$ and $(x, -y)$, we get that $\phi_t^* \alpha = \alpha$ for each $t$, i.e., $X$ is Killing vector field for $\alpha$, and $\phi_t^* \beta = \beta$ for each $t$, i.e., $L_X \beta = 0$. The other direction is obvious. ■

When $X$ is a Killing vector field for $\alpha$, the condition $L_X \beta = 0$ can be equivalently written as $L_X V = [X, V] = 0$, where $V$ is the dual of $\beta$ with respect to the Riemannian metric $\alpha$.

4 Homogeneous Randers spaces and Clifford-Wolf translations

A Randers space $(M, F)$, with $F = \alpha + \beta$, is homogeneous if its full isometry group $G = I(M, F)$ acts transitively on $M$. The Lie algebra of $G$ is denoted by $\mathfrak{g}$. The homogeneous space $M$ can be identified with $G/H$, where $H$ is the isotropy subgroup of $G$ at a point $x$ of $M$. Recall that $H$ is necessary compact. The Randers metric $F$ is determined by the data at the tangent space $T_x \cong \mathfrak{g}/\mathfrak{h} = \mathfrak{m}$, i.e. $\alpha$ is determined by an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{m}$, and the dual $V$ of $\beta$ is a vector of $\mathfrak{m}$, both invariant under the Ad-action of $H$.

The presentations of homogeneous spaces for $(M, F)$ may not be unique. The full isometry group can be changed to any closed subgroup which acts transitively on the manifold. The problem of finding Clifford-Wolf translations can be discussed in two different style. We may try to find all Clifford-Wolf translations for $(M, F)$, which means we may need to deal with different form of homogeneous spaces for the same
manifold, or take $G$ to be the biggest one, the isometry group itself. Our discussion in this work will take another style. Restrict our discussion to a possibly smaller Lie group $G$ and its Lie algebra $\mathfrak{g}$, it is relatively easy to start and already able to tell us something new about Clifford-Wolf translations in Finsler geometry.

If we assume further that $M$ is compact, then $G$ is also compact. There is an orthogonal decomposition for the Lie algebra $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ with respect to the Killing form of $\mathfrak{g}$. This is true since $H \cap C(G)$ contains only the identity map of $M$. Though the Killing form may not be definite, its vanishing space $C(\mathfrak{g})$ is totally contained in $\mathfrak{m}$. For any $X \in \mathfrak{g}$, its decomposition will be denoted $X = X_\mathfrak{h} + X_\mathfrak{m}$.

The study of Clifford-Wolf translations on a homogeneous Randers space is very interesting. From Section 3, we have seen that, the problem of finding Clifford-Wolf translations, at least those which are close to the identity map, can be reduced to the problem of finding Killing fields of constant length.

We can calculate the differential of the length function $F(X)$ of a Killing vector field $X$ when it is viewed as an element of $\mathfrak{g}$. The vector of $X$ at each point can be pulled back to the chosen point, by some $g \in G$, and can be identified with $(\text{Ad}_g X)_m$ in $\mathfrak{m}$. The evaluation of $F$ at that point is then

$$F = \langle (\text{Ad}_g X)_m, (\text{Ad}_g X)_m \rangle^{1/2} + \langle (\text{Ad}_g X)_m, V \rangle. \quad (4.4)$$

for a family $g_t = \exp(tY) \cdot g$, with $Y \in \mathfrak{g}$, the derivative for $t$ at $t = 0$ is

$$\frac{d}{dt} F|_{t=0} = \frac{\langle [Y, \text{Ad}_g X]_m, (\text{Ad}_g X)_m \rangle}{\langle (\text{Ad}_g X)_m, (\text{Ad}_g X)_m \rangle^{1/2}} + \langle [Y, \text{Ad}_g X]_m, V \rangle. \quad (4.5)$$

From this we get the following theorem.

**Theorem 4.1** Let $M = G/H$ be a connected homogeneous Randers space with $G = R(M, F)$ and $F = \alpha + \beta$. Then any Killing vector field $X$ of constant length satisfies

$$\frac{\langle [Y, \text{Ad}_g X]_m, (\text{Ad}_g X)_m \rangle}{\langle (\text{Ad}_g X)_m, (\text{Ad}_g X)_m \rangle^{1/2}} + \langle [Y, \text{Ad}_g X]_m, V \rangle = 0, \quad (4.6)$$

for all $Y \in \mathfrak{g}$, and $g \in G$. If $H$ is connected, then (4.6) is also a sufficient condition for a Killing vector field $X$ to have constant length.

If we choose the family $g_t = g \cdot \exp(tY)$ to calculate the derivative, then (4.6) is changed to

$$\frac{\langle (\text{Ad}_g[Y, X])_m, (\text{Ad}_g X)_m \rangle}{\langle (\text{Ad}_g X)_m, (\text{Ad}_g X)_m \rangle^{1/2}} + \langle (\text{Ad}_g[Y, X])_m, V \rangle = 0, \quad (4.7)$$

for all $Y \in \mathfrak{g}$ and $g \in G$. In fact for (4.7) to be sufficient, we only need to take all $Y \in \mathfrak{m}$ and $g \in G$. And it is similar for (4.6).

This theorem is very useful for finding Killing vector fields of constant length and Clifford-Wolf translations of homogeneous Randers spaces.

For example, we have the following theorem.

**Theorem 4.2** Suppose $G$ is a connected compact Lie group endowed with a Randers metric $F = \alpha + \beta$, such that $\alpha$ is bi-invariant. Denote the dual of $\beta$ with respect to $\alpha$ by $V$. Then the following four conditions are equivalent:
(1) The vector $X \in \mathfrak{g}$ generates a Killing vector field of constant length.

(2) The ideal of $\mathfrak{g}$ generated by $[\mathfrak{g}, X]$ and the ideal generated by $V$ are orthogonal to each other with respect to $\alpha$.

(3) The ideal generated by $[\mathfrak{g}, V]$ and the ideal generated by $X$ are orthogonal to each other with respect to $\alpha$.

(4) With respect to the metric $\alpha$, the inner product between any vectors of the $\text{Ad}_G$-orbits $\mathcal{O}_X$ and $\mathcal{O}_V$ is a constant.

**Proof.** We will prove this theorem by showing that (i) implies (i+1), for $i = 1, 2, 3$ and that (4) implies (1).

That (1) implies (2). If $M$ is a connected compact Lie group and $\alpha$ is bi-invariant, then any $X \in \mathfrak{g}$ has constant length with respect to $\alpha$. The equation (4.7) can then be simplified as

$$< \text{Ad}_g [Y, X], V > = 0,$$

(4.8)

for all $Y \in \mathfrak{g}$ and $g \in G$. So for any $g = \exp t_1 Z_1 \exp t_2 Z_2 \cdots \exp t_k Z_k$ and $g' = \exp t_1' Z_1' \exp t_2' Z_2' \cdots \exp t_l' Z_l'$,

$$< \text{Ad}_g [Y, X], \text{Ad}_g V > = 0.$$

(4.9)

Differentiating the above equation with respect to all $t_i$ and $t_i'$ and evaluating at the 0’s, we get

$$< [Z_1, \ldots, [Z_k, [Y, X]]], [Z_1', \ldots, [Z_l', V]] > = 0.$$

(4.10)

This proves that (1) implies (2).

That (2) implies (3). This is obvious since $< [Y, X], V > = -< X, [Y, V] >$. This also proves that (3) implies (2).

That (3) implies (4). Suppose $X' = \exp(ad Y)X \in \mathcal{O}_X$ and $V' = \exp(ad Z)V \in \mathcal{O}_V$. Then using (2), (3) and a direct calculation one easily shows that $< X', V' >= < X, V >$.

That (4) implies (1). If $< \mathcal{O}_X, \mathcal{O}_V >$ is a constant, then $\mathcal{O}$ is orthogonal to tangent space of $\mathcal{O}_V$ at $V$, i.e., $[\mathfrak{g}, V]$. So for any $Y \in \mathfrak{g}$ and $g \in G$,

$$< [Y, \text{Ad}_g X], V > = -< \text{Ad}_g X, [Y, V] > = 0.$$

(4.11)

Using Theorem 4.1, we finish the proof. ■

Theorem 4.2 explains the most obvious technique to construct a homogeneous Randers metric on Lie groups and to find Clifford-Wolf translations. Take $G = G_1 \times G_2$ with a bi-invariant Riemannian metric $\alpha$, so that their Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ have positive dimensions and orthogonal. Then take $X$ from one factor and $V$ from the other. The Killing vector field generated by $X$ have constant length for both $F$ and $\alpha$. Then $X$ generates a Clifford-Wolf translation which is represented as left multiplication by group elements.

In fact we can also mix it with right multiplications. Take $X_1$ from $\mathfrak{g}_1$ and $X_2$ from $\mathfrak{g}_2$, such that $[X_2, V] = 0$. The composition of the left multiplication generated by $X_1$ and right multiplication generated by $X_2$ generates Clifford-Wolf translations for $F$. 

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The corresponding Killing vector field is not an element of \( g \), provided \( X_2 \) is not in the center. Theorem 4.1 can be used to generalize Theorem 4.2 when the Killing vector field \( X \) is not in \( g \).

It will be more interesting to see the case that \( \alpha \) is not bi-invariant, and the Killing vector field is not of a constant length with respect to \( \alpha \), but has a constant length with respect to \( F \). For convenience, in this case the homogeneous Randers metric will usually be presented as

\[
F(X) = \alpha(X) + <X, V>_{bi}, \tag{4.12}
\]

where the \( \beta \)-term is expressed by the inner product \(<\cdot, \cdot>_{bi}\) for a fixed bi-invariant metric, which is not the inner product of \( \alpha \) (note that \( \alpha \) is assumed to be not bi-invariant).

Let \( G = G_1 \times G_2 \) with \( G_1 = SU(2) \) and \( G_2 = S^1 \). Fix a bi-invariant linear metric on \( G \) such that \( g_1 \) and \( g_2 \) are orthogonal. Then \( g_1 \) can be orthogonally decomposed as \( \mathbb{R}U_1 \oplus \mathbb{R}U_1^\perp \), where \( U_1 \) is a unit vector on \( g_1 \). Let \( U_2 \) be a unit vector in \( g_2 \). Define a left invariant Riemannian metric \( \alpha \) on \( G \) such that \( \alpha|_{\mathbb{R}U_1^\perp} \) is proportional to \( \alpha' \). Choose \( V = aU_1 + bU_2 \), i.e.

\[
\beta(X) = <X, V>_{bi} = ax_1 + bx_2, \tag{4.13}
\]

for \( X = x_1 U_1 + X' + x_2 U_2, X' \in \mathbb{R}U_1^\perp \).

If the vector \( X = rU_1 + sU_3 \) with \( r \) and \( s \neq 0 \) generates a Killing vector field with constant length for \( F \), then its orbit \( O_X \), which is a radius \( r \) sphere for the invariant metric centered at \( sU_3 \), has the constant length \( l \) for \( F \). A sufficient condition for this is

\[
\alpha^2(X') = (ax_1 + bs - l)^2, \tag{4.14}
\]

for all \( X' = x_1 U_1 + X' + sU_2 \in O_X \) with \( x_1^2 + <X', X'>_{bi} = r^2 \). As the right side only depends on \( <X', X'>_{bi}^{1/2} \), we get \( <X', U_2> = 0 \) and \( <X', U_1> = 0 \) for \( \alpha \). So

\[
\alpha(X')^2 = a_{11} x_1^2 + a_{22} <X', X'>_{bi} + a_{33} s^2 + 2a_{13} s x_1 = (a_{11} - a_{22}) x_1^2 + 2a_{13} s x_1 + (a_{33} s^2 + a_{22} r^2) \tag{4.15}
\]

If it equals

\[
(<X', V>_{bi} - l)^2 = (ax_1 + bs - l)^2, \tag{4.16}
\]

then we get a system of linear equations for \( a_{11}, a_{22}, a_{33} \) and \( a_{13} \). The space of the solutions of the above equations is of dimension one. We now summarize the above discussion as a proposition which is useful in the discussion for other compact Lie groups.

**Proposition 4.3** Let \( X \) and \( V \) belong to the Lie algebra \( g = su(2) \oplus \mathbb{R} \) for the Lie group \( SU(2) \times S^1 \). Assume \( X \) lies neither in \( su(2) \) nor in \( \mathbb{R} \). Then for any chosen \( l > 0 \), there is a one-dimensional family of Randers metrics under which \( X \) generates a Killing vector field of constant length \( l \).

The above construction still works even if \( SU(2) \) is changed to a larger connected compact Lie group \( G_1 \). The unit sphere for a bi-invariant metric of \( g_1 \), with a constant non-vanishing \( g_2 \) component will have a constant length for some suitably chosen \( F \). Since the sphere contains more than one orbit, for any unit vector \( U_1 \), the orbit of
$X = rU_1 + sU_3$, with $r$ and $s$ non-zero, contains more than one point. This means that $O_X$ is not of constant length for $\alpha$.

In this construction, we have only used the product structure in the Lie algebra. Considering the quotient group of $G$ by a finite subgroup in its center, we have the following

**Corollary 4.4** If the connected compact Lie group $G$ has a center with positive dimension, then there is a left invariant Randers metric $F = \alpha + \beta$ on $G$, such that there are CLIFFORD-WOLF translations generated by Killing fields in $\mathfrak{g}$, which are not CLIFFORD-WOLF translations for $\alpha$.

5 Clifford-Wolf translations for left invariant Randers metrics on $SU(n)$

To find examples of Clifford-Wolf translations generated by Killing vector fields of constant length with respect to a left invariant Randers metric on a connected compact simple or semi-simple Lie group, it is natural to start with $SU(2)$. But we can easily see that on $SU(2)$, a left invariant Randers metric has no nontrivial Clifford-Wolf translations unless it is a Riemannian metric.

**Proposition 5.1** Any left invariant Randers metric on $SU(2)$ which has a nonzero Killing vector field of constant length in $su(2)$ is Riemannian and bi-invariant.

**Proof.** Suppose there is a non-vanishing $X \in su(2)$ which generates a Killing vector field of constant length for $F = \alpha + \beta$. Then any one dimensional subalgebra of $su(2)$, which is in fact a Cartan subalgebra, contains exactly two vectors in the orbit of $X$. These two vectors are opposite to each other but have the same $F$ length and $\alpha$ length. So the restriction of $\beta$ to each one dimensional subalgebra must vanish. Thus $\beta = 0$. Hence $F$ is Riemannian. Further, the above argument also implies that the orbit $O_X$ has the same length for $\alpha$. Hence $\alpha$ is bi-invariant. $

On larger simple compact Lie groups, there may exist Clifford-Wolf translations generated by a Killing vector field of constant length for a left invariant Randers metric which are not of constant length with respect to the underlying Riemannian metric. Here we will only discuss the examples of $SU(3)$. But the techniques used below can be applied to other classical compact Lie groups.

Let $X = \sqrt{-1} \text{diag}(-1, -1, 2)$ and $V = \sqrt{-1} \text{diag}(a, b, c)$ be two matrices in the Lie algebra $su(3)$. Choose the bi-invariant metric on $SU(3)$ whose restriction to the Lie algebra is $< P, Q >_{bi} = \text{tr} P^*Q$. We will show how to find a Riemannian metric $\alpha$ such that for the Randers metric $F(\cdot) = \alpha(\cdot) + < \cdot, V >_{bi}$, the Killing vector field generated by $X$ has length $l$.

In fact our choice of $X$ is very restrictive, and almost unique in some sense. First, $X$ must have exactly two eigenvalues, i.e., it is invariant under the action of a $Z_2$ subgroup in the Weyl group. Otherwise the intersection of the orbit $O_X$ with any Cartan subalgebra, which is 2 dimensional and contains 6 points. But any ellipse passing the six points must be a circle centered at the origin. This means the Randers metric $F$ has to be Riemannian, i.e., it has a vanishing $\beta$, when it is restricted to any
Cartan subalgebra. Then $F$ is Riemannian following a similar argument as for $SU(2)$. This means that, up to a non-zero scalar, $X$ can be chosen to have the diagonal form $\sqrt{-1}\text{diag}(-1, -1, 2)$. Obviously we can assume that $X$ and $V$ lies in the same Cartan subalgebra, so that $V$ has a diagonal form at the same time.

The following proposition shows that, if we can find the required Riemannian metric $\alpha$, then it must be totally determined by $X, V$ and $l$.

**Proposition 5.2** For any vectors $X$ and $V$ in $\mathfrak{su}(3)$, and a pre-chosen $l > 0$, there is at most one Randers metric $F(y) = \alpha(y) + \langle y, V \rangle_{bi}, y \in \mathfrak{su}(3)$, such that the Killing vector field generated by $X$ has constant length $l$.

**Proof.** We only need to consider $X = \sqrt{-1}\text{diag}(-1, -1, 2)$. If there are two $\alpha_1$ and $\alpha_2$ with corresponding $F_1$ and $F_2$, such that $F_1 = F_2$ on $O_X$, then $\alpha_1 = \alpha_2$ on $O_X$. We only need to prove that $\alpha_1$ and $\alpha_2$ are equal on each Cartan subalgebra. Each Cartan subalgebra of $\mathfrak{su}(3)$ contains 3 points of $O_X$. If we denote the three points of $O_X$ by $X_1, X_2$ and $X_3$, then the centralizer $\mathfrak{z}(X_1) \subset \mathfrak{su}(3)$ is isomorphic to $\mathfrak{su}(2) \oplus \mathbb{R}$, with its group $G' \cong S(S^1 \times U(2)) \subset SU(3)$. The intersection of $O_X$ with $\mathfrak{z}(X_1)$ contains $X'$ and the $Ad_{G'}$ orbit $O'_{X_2}$, which is a 2 dimensional sphere passing both $X_2$ and $X_3$. Restricted to $\mathfrak{z}(X_1)$, $\alpha_1 - \alpha_2$ vanishes on the cone generated by $O'(X_2)$ and the point $X_1$ outside the cone. So $\alpha_1 \equiv \alpha_2$ on $\mathfrak{c}(X_1)$, which contains the Cartan subalgebra we consider.

If $V = \sqrt{-1}\text{diag}(a, b, c) \in \mathfrak{su}(3)$ also has two distinct eigenvalues, we can assume $V = \lambda\sqrt{-1}\text{diag}(-1, -1, 2)$ by a suitable conjugation. Notice that both $X$ and $V$ are invariant under the $Ad$-action of $G' = S(U(2) \times S^1)$. From the uniqueness result of Proposition 5.1 the linear metric $\alpha$ on $\mathfrak{su}(3)$ must also be $Ad_{G'}$-invariant. So it must have the form
\begin{equation}
\alpha(A) = x\text{tr}Q^2 + yu^*u + zq^2, \tag{5.17}
\end{equation}
with $x, y, z \in \mathbb{R}$, where
\[ A = \sqrt{-1}\begin{pmatrix} Q & u \\ u^* & q \end{pmatrix} \in \mathfrak{su}(3), \]
with $Q$ Hermitian symmetric and $q = -\text{tr}Q$.

For any $X' = UXU^*$ in the orbit $O_X$, with
\[ U = \begin{pmatrix} U' & v_2 \\ v_1^* & u \end{pmatrix}, \]
$\alpha(X')^2$ can be expressed as a polynomial of $t = |u|^2$, i.e.
\begin{equation}
\alpha(X')^2 = (9x - 9y + 9z)t^2 + (-12x + 9y - 6z)t + (5x + z). \tag{5.18}
\end{equation}

By the assumption it is equal to
\[ (< X', V >_{bi} - l)^2 = [9\lambda t - (l + 3\lambda)]^2, \tag{5.19} \]
Since the Killing vector field has constant length $l$ with respect to $F$. By comparing the coefficients of $t$, all the parameters $x, y$ and $z$ can be determined by $l$ and $\lambda$. As long as we require $\lambda$ to be nonzero and close to 0, the Killing vector field generated by $X$ produces a Clifford-Wolf translation with respect to $F$ but not with respect to $\alpha$. 

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In general, even if we do not require \( V = \sqrt{-1} \text{diag}(a, b, c) \in su(3) \) to have two distinct eigenvalues, \( X \) and \( V \) are still invariant under the \( Ad \)-action of a maximal torus in \( SU(3) \) consisting of the diagonal matrices. Then the metric \( \alpha \) must have the form
\[
\alpha(A) = x_1a_{11}^2 + x_2a_{22}^2 + x_3a_{33}^2 + y_1|u|^2 + y_2|v|^2 + y_3|w|^2,
\]
(5.20)
where
\[
A = \sqrt{-1} \begin{pmatrix} a_{11} & u & v \\ \bar{u} & a_{22} & w \\ \bar{v} & \bar{w} & a_{33} \end{pmatrix} \in su(3).
\]
(5.21)

For any
\[
X' = \sqrt{-1} \begin{pmatrix} u & v & 0 \\ -\bar{v} & \bar{u} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & s & w \\ 0 & \bar{w} & 1 - s \end{pmatrix} \begin{pmatrix} \bar{u} & -v & 0 \\ \bar{v} & u & 0 \end{pmatrix}
\]
\[
= \begin{pmatrix} s|v|^2 - |u|^2 & (s + 1)uw & vw \\ (s + 1)\bar{u}\bar{v} & s|u|^2 - |v|^2 & \bar{u}\bar{w} \\ \bar{v}\bar{w} & uw & 1 - s \end{pmatrix} \in O_X,
\]
(5.22)
with \(|u|^2 + |v|^2 = 1, s \in \mathbb{R} \) and \(|w|^2 = 2 + s - s^2, \alpha(X')^2 \) can be expressed as a polynomial of \( s \) and \( t = |u|^2 \). Compare it with
\[
<X', V >_{bi} - l)^2 = [a(s(1 - t) - t) + b(st - 1 + t) + c(1 - s) - l]^2,
\]
(5.23)
in which \( c = -a - b \), we get a system of linear equations, from which we can uniquely solve all the coefficients in \( \alpha \), for suitable choices of \( l, a \) and \( b \).

To conclude, we see the technique used here can prove a more general proposition.

**Proposition 5.3** On \( SU(n) \) with \( n > 2 \), there are non-Riemannian left invariant Randers metric for which we can find nonzero Killing fields of constant length from \( su(n) \).

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