Abstract:

A generalized symmetry of a system of differential equations is an infinitesimal transformation depending locally upon the fields and their derivatives which carries solutions to solutions. We classify all generalized symmetries of the vacuum Einstein equations in four spacetime dimensions. To begin, we analyze symmetries that can be built from the metric, curvature, and covariant derivatives of the curvature to any order; these are called natural symmetries and are globally defined on any spacetime manifold. We next classify first-order generalized symmetries, that is, symmetries that depend on the metric and its first derivatives. Finally, using results from the classification of natural symmetries, we reduce the classification of all higher-order generalized symmetries to the first-order case. In each case we find that the generalized symmetries are infinitesimal generalized diffeomorphisms and constant metric scalings. There are no non-trivial conservation laws associated with these symmetries. A novel feature of our analysis is the use of a fundamental set of spinorial coordinates on the infinite jet space of Ricci-flat metrics, which are derived from Penrose’s “exact set of fields” for the vacuum equations.
1. Introduction

Symmetry plays an important role throughout theoretical physics and one of central importance in field theory [1macro. ], [2macro. ]. Indeed, in the construction of a field theory physical considerations usually demand that the field equations (or the Lagrangian) possess certain symmetries. These symmetries include Poincaré symmetry, gauge symmetry, diffeomorphism symmetry, various discrete symmetries, and a host of specialized symmetries needed to ensure the conservation of appropriate quantum numbers. Symmetries also play an important role in the mathematical analysis of differential equations [3macro. ], [4macro. ]. Originating with the work of Lie, symmetry group methods and their recent generalizations have proved useful in understanding conservation laws, in constructing exact solutions, and in establishing complete integrability of certain systems of differential equations.

The symmetries encountered in field theory are usually of the type commonly referred to as point symmetries. A point symmetry of a system of differential equations is a 1-parameter group of transformations of the underlying space of independent and dependent variables that carries any solution of the equations to another solution. If a point symmetry preserves an underlying Lagrangian for the system of equations, then there is a corresponding conservation law. However, not all conservation laws stem from point symmetries. To account for all conservation laws in Lagrangian field theory one must enlarge the notion of symmetry to include generalized symmetries [5macro. ]. A generalized symmetry is an infinitesimal transformation, constructed locally from the independent variables, the dependent variables, and the derivatives of the dependent variables, that carries solutions of the differential equations to nearby solutions. The importance of generalized symmetries is underscored by their role in completely integrable systems of non-linear differential equations. In particular, when a system of differential equations is integrable, it invariably admits “hidden” generalized symmetries [3macro. ], [6macro. ], [7macro. ].

In recent years considerable attention has been devoted to applications of symmetry group methods to a variety of non-linear partial differential equations, but relatively few complete results have been obtained for the Einstein equations. It is, of course, natural to inquire whether or not the Einstein equations admit any hidden generalized symmetries, but the apparent complexity of the ensuing analysis has, to date, precluded substantive progress. The existence of hidden symmetries of the Einstein equations would lead to solution generating–classification techniques, and perhaps even information about the general solution to the Einstein equations. There are hints that such symmetries may exist. The two Killing field reduction of the Einstein equations leads to an integrable system of partial differential equations [8macro. ], [9macro. ], [10macro. ]; the self-dual Einstein equations exhibit an infinite number of symmetries and can be integrated using twistor methods [10macro. ], [11macro. ], [12macro. ], [13macro. ]. A complete generalized symmetry analysis provides a systematic and rigorous way to unravel some aspects of the integrable behavior of the gravitational field equations. In particular, such an analysis indicates whether the rich structure of special reductions of the Einstein equations extends to the full theory.

An equally important consequence of a generalized symmetry analysis stems from the fact that the existence of generalized symmetries of the Einstein equations is a necessary
condition for the existence of local differential conservation laws for the gravitational field. If such conservation laws could be found, they would lead to observables for the gravitational field [14macro.]. It has long been an open problem in relativity theory to exhibit such observables, and the lack thereof currently hampers progress in canonical quantization of general relativity [15macro.].

Recently, Gurses [16macro.] proposed infinite-dimensional families of generalized symmetries for the vacuum Einstein equations. Subsequent investigations showed that a subset of the proposed symmetry transformations were in fact infinitesimal diffeomorphism symmetries [17macro.]. The remaining transformations proposed in [16macro.] fail to be symmetries in the sense that the transformations are not infinitesimal maps from any solution of the vacuum equations to another solution [18macro.], [19macro.].

In this paper we will give a complete classification of all arbitrary-order generalized symmetries for the vacuum Einstein equations in four spacetime dimensions. We shall show that the only generalized symmetries admitted by the vacuum Einstein equations consist of the diffeomorphism symmetry that is inherent in the Einstein equations and a trivial scaling symmetry. More precisely, we will prove the following theorem.

**Theorem.** Let

\[ h_{ab} = h_{ab}(x^i, g_{ij}, g_{ij,h_1}, \ldots, g_{ij,h_1\cdots h_k}) \]

be the components of a \( k \)-th-order generalized symmetry of the vacuum Einstein equations \( R_{ij} = 0 \) in four spacetime dimensions. Then there is a constant \( c \) and a generalized vector field

\[ X^i = X^i(x^i, g_{ij}, g_{ij,h_1}, \ldots, g_{ij,h_1\cdots h_{k-1}}) \]

such that, modulo the Einstein equations,

\[ h_{ab} = cg_{ab} + \nabla_a X_b + \nabla_b X_a. \]

This result was announced in [20macro.].

Because the existence of generalized symmetries is necessary for the existence of (local differential) conservation laws, it is natural to ask what is conserved by virtue of the symmetries of the Einstein equations. It is straightforward to show that there are no conservation laws associated with the scaling symmetry. This is because the Hilbert Lagrangian \( \sqrt{g}R \) is not preserved (even up to a divergence) under metric re-scalings. The diffeomorphism symmetries do lead to conservation laws in the form of the contracted Bianchi identities, but of course the conserved quantities all vanish when the field equations are satisfied.

The plan of this paper is as follows. In §2 we begin with a summary of the theory of generalized symmetries. We then present elementary applications of this theory to the Einstein equations. The technical machinery needed for our analysis is then summarized. In §3 we classify natural symmetries, which are symmetries built from the metric, curvature and covariant derivatives of the curvature to any order. In §4 we classify first-order generalized symmetries, which require a considerably more intricate analysis than needed for natural generalized symmetries. In §5 we extend the analysis of §3 to obtain a classification of all generalized symmetries. The analysis of §5 uses an induction argument to reduce the classification to that of first-order generalized symmetries.
We believe the methods that are used to prove these results are of no less importance than the results themselves. In classifying the generalized symmetries of the Einstein equations we have developed an effective spinor–jet bundle formalism for analyzing mathematical properties of the Einstein equations and related equations [21macro. ]. By far, the most important ingredient in this formalism is the use of what Penrose calls an “exact set of fields” for the field equations [22macro. ], [23macro. ]. These are spinor fields which allow us to parametrize the jet space of vacuum Einstein metrics. In future work we will apply these spinor–jet techniques to related aspects of general relativity. Specifically, our methods can be used to classify systematically (i) all closed $p$-forms that are built locally from a Ricci-flat metric, (ii) all symplectic forms for the Einstein equations, and (iii) all divergence-free symmetric tensors built locally from Einstein metrics. Finally, it is worth pointing out that the existence of an exact set of fields is not limited to the Einstein equations. For example, preliminary computations show that the generalized symmetries of the Yang-Mills equations are amenable to analysis using these techniques.
2. Preliminaries.

In §2A we briefly review the geometric theory of generalized symmetries for differential equations and their role in constructing local conservation laws. For more on generalized symmetries and their applications, see [3macro.]. In §2B we derive the defining equations for the generalized symmetries of the vacuum Einstein equations and present some preliminary results concerning solutions to these equations. We then present in sections §2C and §2D the technical machinery needed to compute the generalized symmetries of the Einstein equations. A complete presentation of the results in these latter two sections can be found in [21macro.].

2A. Generalized Symmetries for Classical Field Theories.

In classical field theory, the fields are usually identified with sections $\varphi: M \to E$ of a fiber bundle $\pi: E \to M$. In general relativity, $M$ is a 4-dimensional manifold and $\pi$ is the bundle $\pi: \mathcal{G} \to M$ of quadratic forms on the tangent space $TM$ with signature ($- + ++$). A section $g: M \to \mathcal{G}$ is a choice of Lorentz metric on $M$.

Let $\pi^k_M: J^k(E) \to M$ be the bundle of $k$-th order jets of local sections of $E$. A point $\sigma \in J^k(E)$ is, by definition, an equivalence class of local sections defined in a neighborhood $U$ of the point $x = \pi^k_M(\sigma)$; two local sections $\varphi_1, \varphi_2: U \to E$ are equivalent if $\varphi_1$ and $\varphi_2$ and all their partial derivatives to order $k$ agree at $x$. If $\varphi: U \to E$ is a local section of $E$, then the canonical lift

$$j^k(\varphi): U \to J^k(E)$$

is the map that assigns to each point $x \in U$ the $k$-jet $j^k(\varphi)(x)$ represented by $\varphi$ at $x$. There are also canonical projections

$$\pi^k_l: J^k(E) \to J^l(E),$$

defined for all $k \geq l$. When $l = 0$, we write $\pi^k_E: J^k(E) \to E$. The infinite jet bundle $\pi^\infty_M: J^\infty(E) \to M$ is similarly defined. For a more detailed presentation of jet bundles, see [3macro.], [24macro.].

A differential form $\omega$ on $J^\infty(E)$ is called a contact form if, for every local section $\varphi: U \to E$,

$$[j^\infty(\varphi)]^*(\omega) = 0.$$

The set of all contact forms on $J^\infty(E)$ is a differential ideal in the ring $\Omega^*(J^\infty(E))$ of all differential forms on $J^\infty(E)$, and we denote this ideal by $\mathcal{C}(J^\infty(E))$.

A generalized vector field $Z$ on $E$ is a vector field along the map $\pi^\infty_E$, that is, for each point $\sigma \in J^\infty(E)$, $Z_\sigma$ is a tangent vector in $T_p(E)$, where $p = \pi^\infty_E(\sigma)$. If $Z$ is a generalized vector field on $E$, then there is a unique vector field $\text{pr} Z$ on $J^\infty(E)$, called the infinite prolongation of $Z$ such that

(i) for each $\sigma \in J^\infty(E)$, $(\pi^\infty_E)_*[\text{pr} Z]_\sigma = Z_{\pi^\infty_E(\sigma)}$, and

(ii) $\text{pr} Z$ preserves the contact ideal, that is, under Lie differentiation $\mathcal{L}_{\text{pr} Z} \mathcal{C}(J^\infty(E)) \subset \mathcal{C}(J^\infty(E))$. 

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We shall give local expressions for $Z$ and $\text{pr} Z$ shortly. A generalized vector field $Y$ on $E$ that is $\pi$-vertical, i.e.,

$$\pi_* (Y_{\sigma}) = 0,$$

for all $\sigma \in J^\infty (E)$, is called an evolutionary vector field. Evolutionary vector fields determine “infinitesimal field variations”, and their prolongations determine the induced variations in the derivatives of the fields. Finally, a generalized vector field $X$ on $M$ is a vector field along the map $\pi_M$, and a generalized tensor field $A$ of type $(p,q)$ on $M$ is a smooth map

$$A : J^\infty (E) \to T^p_q (M)$$

along $\pi_M$, where $T^p_q (M)$ is the bundle of tensors of type $(p,q)$ over $M$. Note that if $Z$ is a generalized vector field on $E$, then $Z_M = \pi_* (Z)$ is a generalized vector field on $M$.

Every generalized vector field $X$ on $M$ defines a unique vector field $\text{tot} X$ on $J^\infty (E)$, called the total vector field of $X$, with the properties

(i) $(\pi_M^\infty)_* [(\text{tot} X)_\sigma] = X_{\pi_M^\infty (\sigma)}$, and

(ii) $\text{tot} X$ annihilates all contact 1-forms, that is, if $\omega$ is a contact 1-form, then $\text{tot} X \cdot \omega = 0$.

The following theorem is easily established from the local formulas for $\text{pr} Z$ and $\text{tot} X$ that we shall give momentarily.

**Theorem 2.1.** Let $Z$ be a generalized vector field on $E$. Then there exists a unique evolutionary vector field $Z_{\text{ev}}$ such that

$$\text{pr} Z = \text{tot} Z_M + \text{pr} Z_{\text{ev}},$$

where $Z_M = \pi_* (Z)$.

If $Z_1$ and $Z_2$ are generalized vector fields on $E$, then there exists a generalized vector field $Z_3$ such that $[\text{pr} Z_1, \text{pr} Z_2] = \text{pr} Z_3$. We call $Z_3$ the generalized Lie bracket of $Z_1$ and $Z_2$ and write

$$[Z_1, Z_2] = Z_3.$$

We remark that if $X$ is a generalized vector field on $M$ and $X_E = (\pi^\infty_E)_* (\text{tot} X)$, then

$$\text{pr} X_E = \text{tot} X.$$

In other words, $\text{tot} X$ is also a prolongation of a vector field and therefore $\text{tot} X$ preserves the contact ideal. It is straightforward to verify that if $\text{tot} X_1$ and $\text{tot} X_2$ are two total vector fields, then $[\text{tot} X_1, \text{tot} X_2]$ is also a total vector field, $[\text{tot} X_1, \text{tot} X_2] = \text{tot} X_3$. (Hence the set of all total vector fields on $J^\infty (E)$ is a connection of general type on $J^\infty (E) \to M$.)

Now suppose a system of differential equations for the sections of $E$ is given. These are the field equations for the classical field theory. If these equations are of order $k$ (typically $k = 2$), then they determine a smooth subbundle

$$\mathcal{R}^k \hookrightarrow J^k (E)$$
with projection \( \pi^k_M : \mathcal{R}^k \to M \). We call \( \mathcal{R}^k \) the \textit{equation manifold} for the classical field theory. The derivatives of the field equations to order \( l \) then define the \( l \)-th \textit{prolonged equation manifold}

\[
\mathcal{R}^{k+l} \hookrightarrow J^{k+l}(E).
\]

The field equations, together with all their derivatives, determine the \textit{infinite prolonged equation manifold}

\[
\mathcal{R}^\infty \hookrightarrow J^\infty(E).
\]

It is customary to assume \([25\text{macro. }], [26\text{macro. }]\) that the maps

\[
\pi^{l+1}_l : \mathcal{R}^{l+1} \to \mathcal{R}^l
\]

are surjective for all \( l \geq k \) and have constant rank. The fiber dimension of \( \pi^{l+1}_l \) represents the number of “degrees of freedom” available in constructing a formal power series solution for the field equations to order \( l + 1 \) from a given solution to order \( l \). Roughly speaking, equations that are not “over-determined” will satisfy the surjectivity assumption. As we shall see, the vacuum Einstein equations also satisfy these surjectivity and constant rank assumptions \([21\text{macro. }]\).

\textbf{Definition 2.2.} A generalized vector field \( Z \) on \( E \) is called a \textit{generalized symmetry} of the given field equations if \( \text{pr} \, Z \) is tangent to the infinitely prolonged equation manifold \( \mathcal{R}^\infty \), that is, for all \( \sigma \in \mathcal{R}^\infty \)

\[
(\text{pr} \, Z)_\sigma \in T_\sigma(\mathcal{R}^\infty).
\]

Generalized symmetries are sometimes called “Lie-Bäcklund symmetries”. If \( Z_1 \) and \( Z_2 \) are two generalized symmetries for \( \mathcal{R}^\infty \), then the generalized Lie bracket \( [Z_1, Z_2] \) is also a generalized symmetry.

It is easy to see from our local coordinate formulas, given below, that if \( X \) is a generalized vector field on \( M \), then \( \text{tot} \, X \) (or more precisely \( X_E = \pi^\infty_E(\text{tot} \, X) \)) is always a generalized symmetry for any system of equations. Total vector fields are therefore viewed as trivial symmetries. A generalized symmetry \( Z \) is also considered trivial if \( Z \) vanishes on the prolonged equation manifold \( \mathcal{R}^\infty \). Two generalized symmetries are said to be equivalent if their difference is a trivial symmetry. Theorem 2.1 implies that \textit{every generalized symmetry} \( Z \) of a given system of equations is equivalent to a generalized symmetry \( Y \) which is \( \pi \)-vertical, that is, to an evolutionary generalized symmetry.

We now give local coordinate descriptions of these various notions. If \( (x^i, \varphi^\alpha) \), \( i = 1, 2, \ldots, n \) and \( \alpha = 1, 2, \ldots, m \), are local coordinates on \( E \), then the standard local coordinates for \( J^\infty(E) \) are

\[(x^i, \varphi^\alpha, \varphi^\alpha_{i_1}, \varphi^\alpha_{i_1 i_2}, \ldots, \varphi^\alpha_{i_1 i_2 \cdots i_k}, \ldots),\]

where, for a given local section \( \varphi^\alpha = \varphi^\alpha(x^i) \),

\[
\varphi^\alpha_{i_1 \cdots i_k} (j^\infty(\varphi)(x)) = \frac{\partial^k \varphi^\alpha(x)}{\partial x^{i_1} \cdots \partial x^{i_k}}.
\]

The contact ideal \( \mathcal{C}(J^\infty(E)) \) is spanned locally by the contact 1-forms

\[
\theta^\alpha_{i_1 \cdots i_k} = d\varphi^\alpha_{i_1 \cdots i_k} - \varphi^\alpha_{i_1 \cdots i_k j} dx^j
\]
for $k = 0, 1, 2, \ldots$. These forms satisfy the structure equations

$$d \theta^\alpha_{i_1 \ldots i_k} = dx^j \wedge \theta^\alpha_{i_1 \ldots i_k, j}.$$  

A generalized vector field $Z$ on $E$ assumes the form

$$Z = A^i \frac{\partial}{\partial x^i} + B^\alpha \frac{\partial}{\partial \varphi^\alpha},$$

where

$$A^i = A^i(x^j, \varphi^\beta, \varphi^\beta_{i_1}, \ldots, \varphi^\beta_{i_k}), \quad \text{and} \quad B^\alpha = B^\alpha(x^j, \varphi^\beta, \varphi^\beta_{i_1}, \ldots, \varphi^\beta_{i_k}).$$

A generalized vector field $X$ on $M$ and an evolutionary vector field $Y$ on $E$ take the form

$$X = A^i \frac{\partial}{\partial x^i}, \quad \text{and} \quad Y = B^\alpha \frac{\partial}{\partial \varphi^\alpha},$$

where, again, the coefficients $A^i$ and $B^\alpha$ are functions of $x^i$, $\varphi^\alpha$ and the derivatives $\varphi^\alpha_{i_1 \ldots i_k}$ to some arbitrary but finite order. The vector field $\text{tot} X$ is given by

$$\text{tot} X = A^i D_i,$$

where $D_i$ is the total derivative operator

$$D_i = \frac{\partial}{\partial x^i} + \varphi^\alpha_i \frac{\partial}{\partial \varphi^\alpha} + \varphi^\alpha_{i_1} \frac{\partial}{\partial \varphi^\alpha_{i_1}} + \varphi^\alpha_{i_1 i_2} \frac{\partial}{\partial \varphi^\alpha_{i_1 i_2}} + \cdots.$$  

We write

$$D_{i_1 i_2 \ldots i_k} = D_{i_1} D_{i_2} \cdots D_{i_k}.$$  

The prolongation of $Z$ is given by the prolongation formula [3macro. ]

$$\text{pr} Z = A^i D_i + \sum_{k=0}^{\infty} D_{i_1 i_2 \ldots i_k} (B^\alpha - \varphi^\alpha_i A^i) \frac{\partial}{\partial \varphi^\alpha_{i_1 i_2 \ldots i_k}}.$$  

Note that, in particular, the prolongation of the evolutionary vector field $Y = B^\alpha \frac{\partial}{\partial \varphi^\alpha}$ is

$$\text{pr} Y = \sum_{k=0}^{\infty} (D_{i_1 i_2 \ldots i_k} B^\alpha) \frac{\partial}{\partial \varphi^\alpha_{i_1 i_2 \ldots i_k}}.$$  

We now remark that (2.2) and (2.3) together prove Theorem 2.1, with

$$Z_{ev} = (B^\alpha - \varphi^\alpha_i A^i) \frac{\partial}{\partial \varphi^\alpha}.$$  

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If $X_1 = A^i_1 \frac{\partial}{\partial x^i}$ and $X_2 = A^i_2 \frac{\partial}{\partial x^i}$ are generalized vector fields on $M$, then

$$[X_1, X_2] = [A^i_1 (D_i A^j_2) - A^i_2 (D_i A^j_1)] \frac{\partial}{\partial x^j}.$$ 

If $Y_1 = B^\alpha_i \frac{\partial}{\partial \varphi^\alpha}$ and $Y_2 = B^\alpha_2 \frac{\partial}{\partial \varphi^\alpha}$ are evolutionary vector fields on $E$, then

$$[Y_1, Y_2] = [\text{pr} Y_1 (B^\alpha_2) - \text{pr} Y_2 (B^\alpha_1)] \frac{\partial}{\partial \varphi^\alpha}.$$ 

An evolutionary vector field $Y = B^\alpha \frac{\partial}{\partial \varphi^\alpha}$ defines “infinitesimal field variations” $\delta \varphi^\alpha_i \ldots i_l, l = 0, 1, \ldots$, which depend locally on the fields and their derivatives. Explicitly, $\delta \varphi^\alpha_i \ldots i_l$ is defined by letting the prolonged vector field $\text{pr} Y$ act on the coordinates $\varphi^\alpha_i \ldots i_l$, which are viewed as functions on $J^\infty (E)$:

$$\delta \varphi^\alpha_i \ldots i_l = \text{pr} Y (\varphi^\alpha_i \ldots i_l) = (D_i \ldots i_l B^\alpha)(x^i, \varphi^\alpha_i, \varphi^\alpha, \ldots, \varphi^\alpha_i \ldots i_{l+k}).$$ 

If

$$\Delta_\beta (x^i, \varphi^\alpha_i, \varphi^\alpha, \ldots, \varphi^\alpha_i \ldots i_k) = 0, \quad \beta = 1, \ldots, m$$

(2.5)
is a system of field equations for the fields $\varphi^\alpha$, then $R^k \subset J^k (E)$ is the manifold defined by these equations. The infinite prolonged equation manifold $R^\infty$ is defined by the equations (2.5) together with the equations

$$D_{i_1 i_2 \ldots i_l} \Delta_\beta = 0$$

for $l = 1, 2, \ldots$. The evolutionary vector field $Y = B^\alpha \frac{\partial}{\partial \varphi^\alpha}$ is, according to the tangency condition in Definition 2.2, a generalized symmetry of (2.5) if and only if the coefficient functions $B^\alpha$ satisfy the linear total differential equation

$$\sum_{l=0}^{k} \frac{\partial \Delta_\beta}{\partial \varphi^\alpha_i \ldots i_l} [D_{i_1 \ldots i_l} B^\alpha] = 0 \quad \text{on} \quad R^\infty.$$ 

(2.6)

This equation is called the formal linearization of (2.5), or the defining equation for the generalized symmetry $Y$.

Let us remark that when $Z$ is an ordinary vector field on $E$, that is,

$$Z = A^i (x^j, \varphi^\beta) \frac{\partial}{\partial x^i} + B^\alpha (x^j, \varphi^\beta) \frac{\partial}{\partial \varphi^\alpha},$$

and $(\text{pr} Z)(\Delta_\beta) = 0$ on the equation manifold $\Delta_\beta = 0$, then $Z$ is called a point symmetry of the equations. Point symmetries are in one-to-one correspondence with first-order
evolutionary symmetries

\[ Y = B^\beta(x^i, \varphi^\alpha_i, \varphi^\alpha_{i_1}) \frac{\partial}{\partial \varphi^\beta}, \]

with \( B^\alpha \) a collection of affine linear functions of the first derivatives \( \varphi^\alpha_i \).

Finally, we cite a version of Noether’s theorem as it applies to generalized symmetries [3macro.]. Recall that a local differential conservation law \( V \) for the field equations \( \Delta_\beta = 0 \) is a generalized vector density

\[ V = V^i (x^i, \varphi^\alpha_i, \varphi^\alpha_{i_1}, \ldots, \varphi^\alpha_{i_1 \cdots i_k}) \frac{\partial}{\partial x^i} \]

on \( M \) such that the total divergence

\[ \text{Div} V = D_i V^i = 0 \quad \text{on} \quad \mathcal{R}^\infty. \]

A conservation law \( V \) is said to be trivial if there is a generalized skew-symmetric tensor density

\[ S^{ij} = S^{ij}(x^k, \varphi^\alpha_i, \varphi^\alpha_{i_1}, \varphi^\alpha_{i_1 i_2}, \ldots, \varphi^\alpha_{i_1 \cdots i_l}) \]

such that

\[ V^i = D_j S^{ij} \quad \text{on} \quad \mathcal{R}^\infty. \]

Two conservation laws are said to be equivalent if their difference is a trivial conservation law. Following Olver [3macro.], an evolutionary vector field \( Y = B^\alpha \frac{\partial}{\partial \varphi^\alpha} \) is called a characteristic vector field for the conservation law \( V \) if

\[ \text{Div} V = B^\alpha \Delta_\alpha \quad (2.7) \]

identically. Under mild regularity conditions on the equations \( \Delta_\beta = 0 \), it can be shown that every conservation law \( V' \) is equivalent to a conservation law \( V \) whose divergence satisfies (2.7). It is a simple result from the variational calculus that if \( \Delta_\alpha \) are the components of the Euler-Lagrange operator \( E_\alpha(L) \) for some Lagrangian, \( L = L(x^i, \varphi^\alpha_i, \varphi^\alpha_{i_1}, \ldots, \varphi^\alpha_{i_1 \cdots i_k}) \),

\[ E_\alpha(L) = \frac{\partial L}{\partial \varphi^\alpha} - D_{i_1} \frac{\partial L}{\varphi^\alpha_{i_1}} + \cdots \pm D_{i_1 \cdots i_k} \frac{\partial L}{\varphi^\alpha_{i_1 \cdots i_k}}, \]

then every characteristic vector field \( Y \) for a local differential conservation law for the equations \( \Delta_\alpha = 0 \) defines a generalized symmetry. The converse need not be true. For example, scaling symmetries of Euler-Lagrange equations typically will not lead to conservation laws.

2B. The Formal Linearization of the Einstein Equations.

To study the generalized symmetries of the Einstein field equations, we let \( \pi: \mathcal{G} \to M \) be the bundle of Lorentz metrics over the spacetime manifold \( M \). Standard local coordinates for \( J^k(\mathcal{G}) \) are

\[ (x^i, g_{ij}, g_{ij \cdots i_1}, \ldots, g_{ij \cdots i_1 i_2 \cdots i_k}). \]
The Christoffel symbols $\Gamma^k_{ij}$, the curvature tensor $R^h_{i jk}$, and their derivatives are all considered now as functions on $J^k(G)$. The covariant derivatives of a generalized tensor field on $M$ are defined in terms of total derivatives. For example, if

$$A_a = A_a(x^i, g_{ij}, g_{ii}, g_{ij}, g_{ij}, g_{ij}, \ldots, g_{ij}, g_{ij}, g_{ij}, \ldots)$$

are the components of a generalized 1-form on $M$, then

$$\nabla_b A_a = D_b A_a = -\Gamma^c_{ab} A_c + \partial A_a / \partial x^b g_{ij} + \partial A_a / \partial g_{ij} g_{ij} + \partial A_a / \partial g_{ij} g_{ij} g_{ij} + \cdots$$

We now compute the formal linearization (2.6) of the vacuum Einstein equations.

**Proposition 2.3.** Let

$$Y = h_{ab}(x^i, g_{ij}, g_{ij}, \ldots, g_{ij}, g_{ij}, g_{ij}, \ldots) \frac{\partial}{\partial g_{ab}}$$

be an evolutionary vector field on the bundle of Lorentz metrics. Then $Y$ is a generalized symmetry of the Einstein equations $R_{ij} = 0$ if and only if

$$[-g^{cd} \delta_i^a \delta_j^b - g^{ab} \delta_i^c \delta_j^d + g^{ac} \delta_i^b \delta_j^d] \nabla_c \nabla_d h_{ab} = 0 \quad (2.8)$$

whenever $R_{ij}$ and its covariant derivatives to order $k$ vanish.

**Proof:** This is an easy computation based upon the identities

$$(\text{pr}Y)(\Gamma^l_{ij}) = \frac{1}{2} g^{lm} [\nabla_i h_{mj} + \nabla_j h_{mi} - \nabla_m h_{ij}], \quad (2.9)$$

and

$$(\text{pr}Y)(R^l_{i jk}) = \nabla_k (\text{pr}Y(\Gamma^l_{ij})) - \nabla_j (\text{pr}Y(\Gamma^l_{ik})). \quad (2.10)$$

These formulas are, of course, familiar from the variational calculus. We emphasize that now (2.9) and (2.10) are to be viewed as identities on $J^k(G)$, where they are direct consequences of the prolongation formula (2.3).

We remark that Proposition 2.3 could also be formulated in terms of the Einstein tensor $G_{ij}$ and its derivatives. The symmetry conditions so-obtained are equivalent to (2.8).

Let $X = X^i(x) \frac{\partial}{\partial x^i}$ be a vector field on $M$ with local flow $\phi_t: M \to M$. Then $\phi_t$ induces, by pull-back, a local flow on $G$ with corresponding vector field $\tilde{X}$ on $G$ given by

$$\tilde{X} = X^a \frac{\partial}{\partial x^a} - (\frac{\partial X^a}{\partial x^i} g_{ai} + \frac{\partial X^a}{\partial g_{ij}} g_{ai}) \frac{\partial}{\partial g_{ij}}.$$
The associated evolutionary vector field is, by (2.4),

\[ \tilde{X}_{ev} = - (\nabla_i X_j + \nabla_j X_i) \frac{\partial}{\partial g_{ij}}. \]

It is well-known [27macro. ] that \( \tilde{X} \), or equivalently \( \tilde{X}_{ev} \), represents a point symmetry of the Einstein equations corresponding to the diffeomorphism invariance of the Einstein equations. This observation motivates the following definition.

**Definition 2.4.** Let

\[ X = X^a (x^i, g_{ij}, g_{ij_{i_1}}, \ldots, g_{ij_{i_1i_2\ldots i_l}}) \frac{\partial}{\partial x^a} \]

be a generalized vector field on \( M \). We call the evolutionary vector field

\[ \mathcal{K}_X = (\nabla_i X_j + \nabla_j X_i) \frac{\partial}{\partial g_{ij}}, \]

where \( X_i = g_{ij} X^j \), the associated generalized diffeomorphism vector field on \( G \).

We remark that if \( X_1 \) and \( X_2 \) are generalized vector fields on \( M \), then

\[ [\mathcal{K}_{X_1}, \mathcal{K}_{X_2}] = \mathcal{K}_{[X_1, X_2]}. \]

**Proposition 2.5.** For any generalized vector field \( X \) on \( M \), the associated generalized diffeomorphism vector field \( \mathcal{K}_X \) is a generalized symmetry of the vacuum Einstein equations.

**Proof:** By virtue of (2.9), we find that

\[ (pr \mathcal{K}_X) (\Gamma^l_{ij}) = \nabla_j \nabla_i X^l + R^l_{ij p} X^p, \]

and hence, by (2.10),

\[ (pr \mathcal{K}_X) R_{ij} = (\nabla_p R_{ij}) X^p + R_{pj} \nabla_i X^p + R_{ip} \nabla_j X^p, \]

which vanishes when \( R_{ij} = 0 \) and \( \nabla_k R_{ij} = 0 \).

We call the symmetry \( \mathcal{K}_X \) a **generalized diffeomorphism symmetry** of the Einstein equations. Note that the generalized diffeomorphism vector fields \( \mathcal{K}_X \) will be symmetries for any generally covariant set of field equations on \( G \). In particular, Proposition 2.5 generalizes to the Einstein equations with cosmological constant.

There is one more obvious symmetry of the vacuum Einstein equations \( R_{ij} = 0 \).

**Proposition 2.6.** For any constant \( c \), the vector field

\[ S_c = c g_{ij} \frac{\partial}{\partial g_{ij}} \quad (2.11) \]
is a point symmetry of the vacuum Einstein equations $R_{ij} = 0$.

**Proof:** This proposition follows from the fact that

$$ (\text{pr} \mathcal{S}_c)(\Gamma^k_{ij}) = 0 $$

and hence

$$ (\text{pr} \mathcal{S}_c)(R_{ij}) = 0. $$

Alternatively, $h_{ij} = c g_{ij}$ clearly satisfies (2.8).

On a 4-dimensional manifold $M$ we have

$$ (\text{pr} \mathcal{S}_c)(\sqrt{g}R) = c \sqrt{g}R, $$

so the scaling symmetry $\mathcal{S}_c$ of the Einstein equations does not preserve the Hilbert Lagrangian (even up to a divergence) and therefore does not generate a conservation law. The generalized diffeomorphism symmetry $K_X$ is a characteristic for a conservation law for the Einstein equations, namely,

$$ \nabla_j(2X_iG^{ij}) = (\nabla_iX_j + \nabla_jX_i)G^{ij}. $$

But the conserved vector field $V^j = 2X_iG^{ij}$ is trivial.

We remark that the scaling symmetry $\mathcal{S}_c$ and the point diffeomorphism symmetry $\tilde{X}$ are the only point symmetries of the vacuum Einstein equations [27macro.].

2C. Spinor Coordinates for Prolonged Einstein Equation Manifolds.

Let $\mathcal{E}^k \subset J^k(\mathcal{G})$ be the set of $k$-jets that satisfy the Einstein equations and the covariant derivatives of the Einstein equations to order $k - 2$,

$$ \mathcal{E}^k = \{ j^k(g)(x_0) \in J^k(\mathcal{G}) | G_{ij} = 0, G_{ij|i_1} = 0, \ldots, G_{ij|i_1 \ldots i_{k-2}} = 0 \text{ at } j^k(g)(x_0) \}. $$

In what follows, we will either use the vertical bar or $\nabla$ to indicate covariant differentiation.

If $h_{ab} = h_{ab}(x^i, g_{ij}, g_{ij,j_1}, \ldots, g_{ij,j_1 \ldots j_k})$ is a generalized symmetry of the vacuum Einstein equations, then the linearized equations (2.8) must hold identically at each point of $\mathcal{E}^{k+2}$. To solve these equations we shall construct explicit coordinates for these prolonged equation manifolds. To this end, we let $\Gamma^i_{j\cdot}$ be the Christoffel symbols of the metric $g_{ij}$ and inductively define higher-order Christoffel symbols by

$$ \Gamma^i_{j_0j_1 \ldots j_k} = \Gamma^i_{(j_0j_1 \ldots j_{k-1}j_k)} - (k - 1)\Gamma^i_{m(j_1 \ldots j_{k-1})} \Gamma^m_{j_{k-1}j_k}, $$

for $k \geq 1$. These higher-order symbols arise naturally from the prolongations of the geodesic equations and play a prominent role in T. Y. Thomas’ theory of normal extensions [28macro.]. We will denote the generalized Christoffel symbols (2.12) by $\Gamma^k$. Note that $\Gamma^i_{j_0j_1 \ldots j_k}$ is completely symmetric in the indices $j_0j_1 \ldots j_k$ and depends on the metric and its first $k$ derivatives.
Next, let

$$Q_{ijj_1\cdots j_k} = g_{iv}g_{js}R^r_{(j_1j_2\cdots j_k)s}$$ \tag{2.13}$$

for \(k \geq 2\). This tensor is a generalized tensor on \(M\) of order \(k\), which we denote by \(Q^k\). Note that \(Q_{ijj_1\cdots j_k}\) is symmetric in \(ij\) and \(j_1 \cdots j_k\), and satisfies the cyclic identity

$$Q_{i(j_1\cdots j_k)} = 0.$$ \tag{2.14}

It is possible to prove, for example, by applying T.Y. Thomas’ Replacement Theorem \([28]\), that

$$\nabla_{jk+1}Q_{ijj_1\cdots j_k} = Q_{ijj_1\cdots j_{k+1}} + \frac{2}{k+2}Q_{j_{k+1}(i,j)j_1\cdots j_k} + \frac{k}{k+2}Q_{(j_1j_2\cdots j_k)ij} + L_{ijj_1\cdots j_{k+1}},$$ \tag{2.15}

where

$$L_{ijj_1\cdots j_{k+1}} = L_{ijj_1\cdots j_{k+1}}(g_{ab}, Q_{ab,c_1c_2}, \ldots, Q_{ab,c_1c_2\cdots c_{k-1}}).$$

A straightforward calculation, starting with the expression for \(R_{ijkl}\) in terms of the derivatives of the metric, shows that

$$g_{ijj_1\cdots j_k} = -2 \frac{k-1}{k+1}Q_{ijj_1\cdots j_k} + g_{il}\Gamma^l_{jj_1\cdots j_k} + g_{jl}\Gamma^l_{ijj_1\cdots j_k} + P_{ijj_1\cdots j_k},$$

where

$$P_{ijj_1\cdots j_k} = P_{ijj_1\cdots j_k}(g_{hk}, g_{hk,j_1}, \ldots, g_{hk,j_1\cdots j_{k-1}}).$$

From this equation it is then possible to prove \([21]\) that the variables

$$(x^i, g_{ij}, \Gamma^i_{j_1j_2}, \ldots, \Gamma^i_{j_1\cdots j_m}, Q_{ijj_1j_2}, \ldots, Q_{ijj_1\cdots j_k})$$ \tag{2.16}

can be used as coordinates for the bundle \(J^k(G)\). In particular, suppose we are given quantities \(q_{ij}, X^l_{j_0j_1\cdots j_l}, \) and \(Y_{ijj_1\cdots j_m}\), for \(l = 1, 2, \ldots, k\) and \(m = 2, 3, \ldots, k\), where \(q_{ij}, X^l_{j_0j_1\cdots j_l}\) and \(Y_{ijj_1\cdots j_m}\) have the symmetries of \(g_{ij}, \Gamma^i_{j_0j_1\cdots j_l}\) and \(Q_{ijj_1\cdots j_m}\). Then the \(k\)-jet

\(j^k(g)(x_0)\) defined inductively by

$$g_{ij}(x_0) = q_{ij},$$

and

$$g_{ijj_1\cdots j_l}(x_0) = -2 \frac{k-1}{k+1}Y_{ijj_1\cdots j_m} + g_{il}(x_0)X^l_{jj_1\cdots j_l} + g_{jl}(x_0)X^l_{ijj_1\cdots j_l} + P_{ijj_1\cdots j_l}(g_{ij}(x_0), g_{ijj_1}(x_0), \ldots, g_{ijj_1\cdots j_{l-1}}(x_0))$$

for \(l = 1, 2, \ldots, k\) and \(m = 2, 3, \ldots, k\), satisfies

$$\Gamma^i_{j_0j_1\cdots j_l}(j^l(g)(x_0)) = X^i_{j_0j_1\cdots j_l},$$

and

$$Q_{ijj_1\cdots j_m}(j^m(g)(x_0)) = Y_{ijj_1\cdots j_m}.$$
for $l = 1, 2, \ldots, k$ and $m = 2, 3, \ldots, k$.

The coordinates (2.16) are well-suited for describing the prolonged Einstein equation manifold $E^k$. If we let

\[
[\text{tr}_1 Q]_{j_1 j_2 \cdots j_k} = g^{i j} Q_{i j_1 j_2 \cdots j_k}, \quad [\text{tr}_2 Q]_{j_1 j_2 \cdots j_k} = g^{i j} Q_{i j_1 j_2 \cdots j_k}, \quad [\text{tr}_3 Q]_{i j_1 j_2 \cdots j_k} = g^{i j} Q_{i j_1 j_2 \cdots j_k},
\]

and

\[
[\text{tr}_1 Q]_{j_1 j_2 \cdots j_k} = g^{i j} g^{j l} Q_{i j_1 j_2 \cdots j_k}, \quad [\text{tr}_2 Q]_{j_1 j_2 \cdots j_k} = g^{i j} g^{j l} Q_{i j_1 j_2 \cdots j_k},
\]

then it is not difficult to prove that

\[
G_{ij|(j_3 \cdots j_k)} = R_{ij|(j_3 \cdots j_k)} - \frac{1}{2} g_{ij} R_{|(j_3 \cdots j_k)}
\]

\[
= \frac{k - 1}{k + 1} \left( [\text{tr}_3 Q]_{i j_3 j_4 \cdots j_k} - [\text{tr}_2 Q]_{i j_3 j_4 \cdots j_k} + [\text{tr}_2 Q]_{i j_3 j_4 \cdots j_k} + [\text{tr}_1 Q]_{i j_3 j_4 \cdots j_k} \right),
\]

where $F_{j_3 j_4 \cdots j_k}$ is a tensor of order $k - 2$ that is symmetric in $i j$ and in $j_3 j_4 \cdots j_k$. By contracting this equation with $g^{j j_3}$ we deduce that

\[
g^{j j_3} F_{i j_3 j_4 \cdots j_k} (j^{k-2}(g)(x_0)) = 0 \quad \text{whenever} \quad j^{k-2}(g)(x_0) \in E^{k-2}. \quad (2.17)
\]

In [21macro.] we carefully analyze (2.17) to prove the following two theorems. These are purely algebraic results.

**Theorem 2.7.** Let $j^k(g^1)(x_0)$ and $j^k(g^2)(x_0)$ be any two points in $E^k$ and let $Q^l_1$, $Q^l_2$ denote the values of the tensors $Q_{i j_1 j_2 \cdots j_l}$, $l = 2, \ldots, k$ at $j^k(g^1)(x_0)$ and $j^k(g^2)(x_0)$. If $g^1_{i j}(x_0) = g^2_{i j}(x_0)$, and if the completely trace-free parts of the tensors $Q^l_1$ and $Q^l_2$ agree, that is,

\[
[Q^l_1]_{\text{tracefree}} = [Q^l_2]_{\text{tracefree}},
\]

for each $l = 2, \ldots, k$, then

\[
Q^l_1 = Q^l_2
\]

for each $l = 2, \ldots, k$.

**Theorem 2.8.** Let $j^{k-1}(g)(x_0) \in E^{k-1}$, and let $S_{a b j_1 \cdots j_k}$, which is denoted $S^k$, be a given tensor with the following properties:

(i) $S_{a b j_1 \cdots j_k}$ is symmetric in $a b$ and $j_1 \cdots j_k$,

(ii) $S_{a b j_1 \cdots j_k}$ is trace-free on any pair of indices, and

(iii) $S_{a (b j_1 \cdots j_k)} = 0$.

Then there exists a metric $k$-jet $j^k(\tilde{g})(x_0)$ such that $j^k(\tilde{g})(x_0) \in E^k$, and

\[
j^{k-1}(\tilde{g})(x_0) = j^{k-1}(g)(x_0),
\]

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and

$$[Q^k(j^k(g)(x_0))]_{\text{tracefree}} = S^k.$$  

Together, Theorems 2.7, 2.8 show that local coordinates for $\mathcal{E}^k$ are given by

$$(x^i, g_{ij}, \Gamma^i_{j0j1 \ldots j_l} [Q_{ij,j1 \ldots j_l}]_{\text{tracefree}}) \quad \text{for} \quad l \leq k. \quad (2.19)$$

In particular, Theorem 2.8 shows that the projection map $\pi_{k-1}^k : \mathcal{E}^k \to \mathcal{E}^{k-1}$ is surjective.

The tensor $[Q_{ij,j1 \ldots j_l}]_{\text{tracefree}}$ has a remarkable characterization in terms of two-component spinors. This spinor characterization is based on the following theorem [21macro.].

**Theorem 2.9.** Let $S_{ab,j1 \ldots j_k}$ be a complex tensor which satisfies the properties (i)–(iii) of Theorem 2.8, and let the tensor $S_{ab,j1 \ldots j_k}$ have the spinor representation

$$S_{ab,j1 \ldots j_k} \leftrightarrow S^{A'B'C'D'}_{A'B'C'D'}.$$  

Then there exist unique spinors $U_{J_1 \ldots J_{k-2}}^{J'_1 \ldots J'_{k-2}}$ and $V_{J_1 \ldots J_{k+2}}^{J'_1 \ldots J'_{k+2}}$, both completely symmetric in their primed and unprimed indices, such that

$$S^{A'B'C'D'}_{A'B'C'D'} = \epsilon^{A'(J'_1 \ldots J'_k)} \epsilon^{B'(J'_1 \ldots J'_k)} \epsilon^{C'(J'_1 \ldots J'_k)} \epsilon^{D'(J'_1 \ldots J'_k)} + \epsilon_{A(J_1} \epsilon_{B|J_2} V_{J_3 \ldots J_{k-2}}^{J'_1 \ldots J'_{k-2}}.$$  

Now we consider the spinor representation of the curvature tensor [23macro.],

$$R_{abcd} \leftrightarrow R^{A'B'C'D'}_{A'B'C'D'},$$

where

$$R^{A'B'C'D'}_{A'B'C'D'} = \Psi_{ABCD}^{A'B'C'D'} + \Phi_{AB}^{A'B'} \epsilon^{C'D'} + \Phi_{CD}^{A'B'} \epsilon^{C'D'} + 2\Lambda(\epsilon_{AC} \epsilon_{BD} \epsilon_{A'B'C'} - \epsilon_{AD} \epsilon_{BC} \epsilon_{A'B'C'}). \quad (2.20)$$

The totally symmetric spinors $\Psi_{ABCD}$ and $\Phi_{A'B'C'D'}$ correspond to the spinor representation of the Weyl tensor. The symmetric spinor $\Phi_{A'B'}$ corresponds to the trace-free Ricci tensor, and the scalar $\Lambda$ corresponds to the scalar curvature. If we set

$$[Q_{ab,j1 \ldots j_k}]_{\text{tracefree}} \leftrightarrow Q^{A'B'C'D'}_{A'B'C'D'},$$

then it is not too difficult to show, using Theorem 2.9 and (2.20), that

$$Q^{A'B'C'D'}_{A'B'C'D'} = \epsilon^{A'(J'_1 \ldots J'_k)} \epsilon^{B'(J'_1 \ldots J'_k)} \epsilon^{C'(J'_1 \ldots J'_k)} \epsilon^{D'(J'_1 \ldots J'_k)} + \epsilon_{A(J_1} \epsilon_{B|J_2} \Psi_{J_3 \ldots J_{k-2}}^{A'B'C'D'}, \quad (2.21)$$

where

$$\Psi_{J_1 \ldots J_{k-2}}^{J'_1 \ldots J'_{k-2}} = \nabla_{(J_1}^{J'_1} \ldots \nabla_{J_{k-2}}^{J'_{k-2}} \Psi_{J_{k-1}J_{k+1}J_{k+2}}^{J_1 \ldots J_{k+2}},$$

and

$$\Psi_{J_1 \ldots J_{k-2}}^{J'_1 \ldots J'_{k+2}} = \nabla_{(J_1}^{J'_1} \ldots \nabla_{J_{k-2}}^{J'_{k-2}} \Psi_{J_{k-1}J_{k+1}J_{k+2}}^{J'_{k-1}J'_k J'_{k+1}J'_{k+2}}.$$  

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In summary, the natural spinor coordinates for the prolonged Einstein manifold $\mathcal{E}^k$ are

$$\begin{align*}
(x^i, g_{ij}, \Gamma^i_{j0j_1}, \ldots, \Gamma^i_{j_0\cdots j_k}, \Psi_{j_1j_2j_3j_4}, \overline{\Psi}^{i_1j_1j_2j_3}_{i_2j_2j_3j_4}, \ldots, \Psi_{j_1\cdots j_{k+2}}, \overline{\Psi}_{j_1\cdots j_{k+2}}^{i_1\cdots i_{k+2}}).
\end{align*}$$

As an example, the natural spinor coordinates for $\mathcal{E}^2$ and $\mathcal{E}^3$ are

$$\begin{align*}
(x^i, g_{ij}, \Gamma^i_{j0j_1}, \Gamma^i_{j_0j_1j_2}, \Psi_{j_1j_2j_3j_4}, \overline{\Psi}^{i_1j_1j_2j_3}_{i_2j_2j_3j_4}),
\end{align*}$$
and

$$\begin{align*}
(x^i, g_{ij}, \Gamma^i_{j0j_1j_2}, \Gamma^i_{j_0j_1j_2j_3j_4}, \Psi_{j_1j_2j_3j_4j_5}, \overline{\Psi}^{i_1j_1j_2j_3j_4j_5}_{i_2j_2j_3j_4j_5}).
\end{align*}$$

The symmetrized covariant derivatives $\Psi^{j_1\cdots j_{k+2}}$ and $\overline{\Psi}^{i_1\cdots i_{k+2}}$ derive from Penrose’s notion of an exact set of fields for the vacuum Einstein equations [22macro.]. Henceforth we refer to these spinors as the Penrose fields for the vacuum Einstein equations, and we denote them by $\Psi^k$ and $\overline{\Psi}^k$. We remark that to pass between the coordinates (2.22) and (2.19) we use any soldering form $\sigma_a^A A'$ such that

$$g_{ij} = \sigma_i^A A' \sigma_j A A'.$$

We have the following important structure equation for the Penrose fields [22macro.].

**Proposition 2.10.** The spinorial covariant derivative of $\Psi^{j_1\cdots j_{k+2}}$ when evaluated on $\mathcal{E}^k$, is given by

$$\nabla^A A' \Psi_{j_1\cdots j_{k+2}} = \Psi^A A' j_1\cdots j_{k+2} + \{\}$$

where $\{\}$ denotes a spinor-valued function of the Penrose fields $\Psi^2, \overline{\Psi}^2, \ldots, \Psi^{k-1}, \overline{\Psi}^{k-1}$.

The fact that the lower-order terms $\{\}$ are of order less than or equal to $k - 1$ is essential to much of our symmetry analysis. Equation (2.23) can be viewed as a special case of (2.15)

We now turn our attention to natural tensors, which we can define precisely using the language of jets. If $f: M \to M$ is a diffeomorphism, we let

$$f_*: T^p_q(M) \to T^p_q(M)$$

be the induced map on tensors. By definition, if $T_x \in T^p_q(M)_x$ and $y = f(x)$, then for covectors $\alpha_1, \ldots, \alpha_p \in T^*(M)_y$ and vectors $X_1, \ldots, X_q \in T(M)_y$, $(f_* T)_y$ is the type $(p, q)$ tensor at $y$ given by

$$(f_* T)_y(\alpha_1, \ldots, \alpha_p, X_1, \ldots, X_q) = T_x(f^* \alpha_1, \ldots, f^* \alpha_p, (f^{-1})_* X_1, \ldots, (f^{-1})_* X_q),$$

where $f^* \alpha_i$ is the pull-back of $\alpha_i$ to $T^*(M)_x$. Next, we let

$$\tilde{f}: G \to G$$
be the lift of $f$ to a bundle morphism on $G$, that is,

$$\tilde{f}(x, g) = (f(x), f_* g).$$

The map $\tilde{f}$ can then, in turn, be lifted by prolongation to a bundle morphism on $J^k(G)$:

$$\text{pr}^k \tilde{f}: J^k(G) \to J^k(G).$$

**Definition 2.11.** A natural tensor of type $(p, q)$ and order $k$ on the bundle of quadratic forms $G$ is a smooth bundle map

$$T: J^k(G) \to T^p_q(M)$$

such that for each diffeomorphism $f: M \to M$ and every point $g = j^k(g)(x_0) \in J^k(G)$,

$$T((\text{pr}^k \tilde{f})(g)) = (f_* T)(g).$$

By appealing to the Replacement Theorem of T. Y. Thomas [28macro.], it can be shown that the restriction to $E^k$ of any natural tensor on $J^k(G)$, say

$$T_{a_1 \ldots a_p} (g_{ij}, g_{ij,j_1}, \ldots, g_{ij,j_1 \cdots j_k}),$$

may be uniquely expressed as a function of the Penrose fields, that is,

$$T_{a_1 \ldots a_p} \leftrightarrow T_{A_1 \cdots A_p}^{A'_1 \cdots A'_p} (\Psi_{J_1 J_2 J_3 J_4}, \Psi_{J'_1 J'_2 J'_3 J'_4}, \ldots, \Psi_{J_1 \cdots J_{k-2}}^{J'_{k-2}}, \Psi_{J_1 \cdots J_k}^{J_{k+2}}). \tag{2.24}$$

Under an arbitrary $SL(2, \mathbb{C})$ transformation $\Lambda^A_B$, the spinor $T$ satisfies the identity

$$T_{A_1 \cdots A_p}^{A'_1 \cdots A'_p} [\Lambda \cdot \Psi] = \Lambda^B_{A_1} \cdots \Lambda^B_{A_p} \Lambda^{A'_1}_{B'_1} \cdots \Lambda^{A'_p}_{B'_p} T_{B_1 \cdots B_p}^{B'_1 \cdots B'_p} [\Psi], \tag{2.25}$$

where $\Lambda \cdot \Psi$ denotes the action of $SL(2, \mathbb{C})$ on the Penrose fields, for example,

$$[\Lambda \cdot \Psi]_{ABCD} = \Lambda^J_A \Lambda^K_B \Lambda^L_C \Lambda^M_D \Psi_{JKLM}.$$ 

We call spinors (2.24) that satisfy (2.25) natural spinors of the Penrose fields $\Psi^2, \Psi^2, \ldots, \Psi^k, \overline{\Psi}^k$.

We let $\partial_{\Psi_{j_1 \cdots j_{k+2}}}^{j_1 \cdots j_{k+2}}, \partial_{\overline{\Psi}_{j_1 \cdots j_{k+2}}}^{j_1 \cdots j_{k+2}}$, and $\partial_{\Gamma_{j_0 \cdots j_k}}^{j_0 \cdots j_k}$ denote the (symmetrized) partial differential operators with respect to the coordinates $\Psi_{j_1 \cdots j_{k+2}}, \overline{\Psi}_{j_1 \cdots j_{k+2}}$, and $\Gamma_{j_0 \cdots j_k}$. For example,

$$\partial_{\Psi_{j_1 j_2 j_3 j_4}}^{j_1 j_2 j_3 j_4} (\Psi_{ABCD}) = \delta^A_A \delta^B_B \delta^C_C \delta^D_D \delta^{j_1 j_2} \delta^{j_3 j_4}.$$

As a consequence of (2.25) we have the following result [21macro.].
Proposition 2.12. Let $T^{A_1\cdots A_q}_{A_1\cdots A_q}$ be a natural spinor of the fields $\Psi^2, \bar{\Psi}^2, \ldots, \Psi^k, \bar{\Psi}^k$. The spinorial covariant derivative of $T^{A_1\cdots A_q}_{A_1\cdots A_q}$ is a natural spinor of the Penrose fields $\Psi^2, \bar{\Psi}^2, \ldots, \Psi^{k+1}, \bar{\Psi}^{k+1}$, and is given by

\[
\nabla B^{i\cdots j} T^{A_1\cdots A_q}_{A_1\cdots A_q} = \sum_{l=2}^{k} \left[ \partial_\Psi^j_{A_1\cdots j_{l-2}} T^{A_1\cdots A_q}_{A_1\cdots A_q} \right] \nabla B^{i\cdots j_{l-1}} + \sum_{l=2}^{k} \left[ \partial_\bar{\Psi}^{j}_{A_1\cdots j_{l-2}} T^{A_1\cdots A_q}_{A_1\cdots A_q} \right] \nabla B^{i\cdots j_{l-1}}.
\]

We close this section by deriving a spinor expression for the linearized Einstein equations (2.8) that we shall use to compute generalized symmetries. Starting from (2.8), and using the spinor correspondence

\[
\nabla \nabla \leftrightarrow \nabla_{CC'} \nabla_{DD'}
\]

the defining equation (2.8) takes the form

\[
\left[ -\epsilon^{CD} \epsilon^{C'D'} \delta_M^A \delta_{M'}^B \delta_N^{B'} \delta_{N'}^{B''} - \epsilon^{AB} \epsilon^{A' B'} \delta_M^C \delta_{M'}^{C'} \delta_N^{D} \delta_{N'}^{D'} \right. + \left. \epsilon^{AC} \epsilon^{A' C'} \left( \delta_M^{B'} \delta_{M'}^{B''} \delta_N^{D'} \delta_{N'}^{D''} + \delta_M^{D'} \delta_{M'}^{D''} \delta_N^{B'} \delta_{N'}^{B''} \right) \right] \nabla_{CC'} \nabla_{DD'} h_{A BA'B'} = 0. \tag{2.26}
\]

Since $h_{A BA'B'} = h_{BBA'B'}$, we have that

\[
h_{A BA'B'} = h_{BBA'B'} + \frac{1}{2} \epsilon_{AB} \epsilon_{A'B'} h,
\]

where the trace of $h_{A BA'B'}$ is given by

\[
h = \epsilon^{AB} \epsilon^{A'B'} h_{ABA'B'}.
\]

Substituting (2.27) into the last two terms of (2.26), we find that all the trace terms cancel leaving us with

\[
\left[ -\epsilon^{CD} \epsilon^{C'D'} \delta_M^A \delta_{M'}^B \delta_N^{B'} \delta_{N'}^{B''} + \epsilon^{BC} \epsilon^{A'C'} \delta_M^{A'} \delta_{M'}^{B'} \delta_N^{D} \delta_{N'}^{D'} + \epsilon^{BC} \epsilon^{A'C'} \delta_M^{D'} \delta_{M'}^{D''} \delta_N^{A} \delta_{N'}^{B'} \right] \nabla_{CC'} \nabla_{DD'} h_{A BA'B'} = 0.
\]

We now multiply this expression with arbitrary spinors $\alpha^M, \bar{\alpha}^{M'}, \beta^N, \bar{\beta}^{N'}$ to get our final spinor form of the linearized equations.

Proposition 2.13. If $h_{A'B'}^{AB}$ are the spinor components of a generalized symmetry of the vacuum Einstein equations, then for all spinors $\alpha^M, \bar{\alpha}^{M'}, \beta^N, \bar{\beta}^{N'}$

\[
\left[ -\epsilon_{CD} \epsilon^{C'D'} \alpha_A \bar{\beta}_{B} \bar{\alpha}^{A'} \bar{\beta}^{B'} + \epsilon_{BC} \epsilon^{A'C'} \alpha_A \beta_{B} \bar{\alpha}^{A'} \bar{\beta}^{B'} \right] \nabla^{D'} \nabla_{D'} h_{A'B'}^{AB} = 0 \quad \text{on } E^{k+2}. \tag{2.28}
\]
In general $h_{A'B'}^{AB}$ is a function of the coordinates (2.22), that is,

$$h_{A'B'}^{AB} = h_{A'B'}^{AB}(x^i, \sigma_{AB}, \Gamma_{j_0 j_1}, \ldots, \Gamma_{j_0 \cdots j_k}, \Psi_{j_1 j_2 j_3 j_4}, \ldots, \Psi_{j_1 \cdots j_{k+2}}, \Psi_{j_1 \cdots j_{k+2}}).$$

When $h_{A'B'}^{AB}$ is a natural generalized symmetry,

$$h_{A'B'}^{AB} = h_{A'B'}^{AB}(\Psi_{J_1 J_2 J_3 J_4}, \ldots, \Psi_{J_1 \cdots J_{k+2}}, \Psi_{J_1 \cdots J_{k+2}}).$$

In both cases, $h_{A'B'}^{AB}$ satisfies the $SL(2, \mathbb{C})$ invariance properties

$$\Lambda^C_A \Lambda^D_B \Lambda^C_{A'} \Lambda^D_{B'} h_{A'B'}^{AB}(x, \sigma, \Gamma, \Psi) = h_{C'D'}^{CD}(x, \Lambda \cdot \sigma, \Gamma, \Lambda \cdot \Psi),$$

where $\Lambda \cdot \sigma$, and $\Lambda \cdot \Psi$ denote the action of $SL(2, \mathbb{C})$ on the soldering form and Penrose fields.

2D. Results from Tensor and Spinor Algebra.

Here we gather together a number of key algebraic results which we shall use repeatedly in our study of the generalized symmetries of the Einstein equations. Following the standard algebraic treatment of tensors, we consider spinors as multi-linear maps on complex 2-dimensional vector spaces. For notational convenience, we separate groups of symmetric spinor (or tensor) arguments with a comma and we use no delimiters between arguments within a symmetric set. As an example, $A(\alpha \beta, \gamma, \delta)$ denotes a rank 4 spinor that is symmetric in $\alpha$ and $\beta$,

$$A(\alpha \beta, \gamma, \delta) = A(\beta \alpha, \gamma, \delta),$$

but otherwise has no symmetries. Repeated symmetric arguments of a spinor (or tensor) will be abbreviated using an exponential notation. For example, if $T$ is a spinor of type $(k, 1)$ that is totally symmetric in its first $k$ arguments, we will write

$$T(\psi^k, \overline{\alpha}) = T(\overbrace{\psi, \ldots, \psi}^{k \text{ times}}, \overline{\alpha}).$$

It is important to note that the values of $T(\psi_1 \psi_2 \cdots \psi_k, \overline{\alpha})$, where $\psi_1, \psi_2, \ldots, \psi_k$ are arbitrary spinors, are completely determined by the values of $T(\overbrace{\psi^k, \overline{\alpha}}^{k \text{ times}})$. In addition, if $\alpha^B$ and $\beta^A$ are spinors of type $(0, 1)$ and type $(1, 0)$ respectively, we set

$$\beta_B = \epsilon_{AB} \beta^A \quad \text{and} \quad \alpha^A = \epsilon^{AB} \alpha_B.$$ 

The skew-symmetric inner product between $\alpha_B$ and $\beta^A$ is given by

$$\langle \alpha, \beta \rangle = \alpha_B \beta^A = \epsilon^{AB} \alpha_A \beta_B = - \langle \beta, \alpha \rangle.$$ 

We denote by $\langle X, Y \rangle$ the metric inner product between two vectors $X$ and $Y$.

The following propositions are all elementary facts which we shall use repeatedly. See [21macro. ] for proofs.
Proposition 2.14. Let \( P = P(\psi^k, \alpha) \) be a rank \((k + 1)\) spinor that is symmetric in its first \( k \) arguments. Then there are unique, totally symmetric spinors \( P^* \) and \( Q \), of rank \( k + 1 \) and \( k - 1 \) respectively, such that

\[
P(\psi^k, \alpha) = P^*(\psi^k \alpha) + \langle \psi, \alpha \rangle Q(\psi^{k-1}).
\]

If \( P \) is a natural spinor of the Penrose fields \( \Psi, \overline{\Psi}, \ldots, \Psi^k, \overline{\Psi}^k \), then so are \( P^* \) and \( Q \).

Proposition 2.15. Let \( P = P(\psi^k, \alpha) \) be a rank \((k + 1)\) spinor that is symmetric in its first \( k \) arguments. If \( P(\psi^k, \alpha) \) satisfies

\[
P(\psi^k, \psi) = 0,
\]

then there is a totally symmetric spinor \( Q = Q(\psi^{k-1}) \) such that

\[
P(\psi^k, \alpha) = \langle \psi, \alpha \rangle Q(\psi^{k-1}).
\]

If \( P \) is a natural spinor, then so is \( Q \).

We note for future use that (2.30) is equivalent to

\[
P(\psi^1 \cdots \psi^k, \alpha) = \frac{1}{k} \sum_{i=1}^{k} \langle \psi^i, \alpha \rangle Q(\psi^1 \cdots \psi^{i-1} \psi^{i+1} \cdots \psi^k).
\]

Proposition 2.16. Let \( P = P(\psi^k, \alpha) \) be a rank \((k + 1)\) spinor that is symmetric in its first \( k \) arguments. If \( P(\psi^k, \alpha) \) satisfies

\[
\langle \psi, \alpha \rangle P(\psi^k, \beta) = \langle \psi, \beta \rangle P(\psi^k, \alpha),
\]

then there is a unique totally symmetric spinor \( Q \) of rank \( k - 1 \) such that

\[
P(\psi^k, \alpha) = \langle \psi, \alpha \rangle Q(\psi^{k-1}).
\]

The spinor \( Q \) is natural if \( P \) is natural. If \( P(\psi^k, \alpha) \) satisfies

\[
\langle \psi, \alpha \rangle P(\psi^k, \beta) = - \langle \psi, \beta \rangle P(\psi^k, \alpha),
\]

then \( P = 0 \).

Proposition 2.17. Let \( T \) be a symmetric rank-\( k \) tensor, and suppose that

\[
T(X^k) = 0
\]

whenever \( X \) is a null vector. Then there exists a symmetric tensor \( P \) of rank \( k - 2 \) such that, for any vector \( X \),

\[
T(X^k) = \langle X, X \rangle P(X^{k-2}).
\]
Proposition 2.18. Let $T(Y^p, X)$ be a tensor that vanishes whenever $< Y, X >= 0$. Then there is a unique tensor $U(Y^{p-1})$ such that

$$T(Y^p, X) = < Y, X > U(Y^{p-1}). \quad (2.36)$$

We close this section with a characterization of spinors with certain symmetries which arise in our symmetry analysis of the Einstein equations. The proof of this theorem is rather lengthy; for details, see [21macro.].

Theorem 2.19. Let $P(\psi^{k+2}, \psi^{k-2}, \alpha, \beta, \overline{\alpha}, \overline{\beta})$ be a spinor that is symmetric in its first $k + 2$ and next $k - 2$ arguments. The spinor $P(\psi^{k+2}, \psi^{k-2}, \alpha, \beta, \overline{\alpha}, \overline{\beta})$ enjoys the two symmetry properties

$$P(\psi^{k+2}, \psi^{k-2}, \alpha, \beta, \overline{\alpha}, \overline{\beta}) = P(\psi^{k+2}, \psi^{k-2}, \beta, \alpha, \overline{\beta}, \overline{\alpha}) \quad (2.37)$$

and

$$P(\psi^{k+2}, \psi^{k-2}, \psi, \alpha, \overline{\beta}, \overline{\psi}) = 0 \quad (2.38)$$

if and only if there are spinors,

$$A = A(\psi^k, \overline{\psi}^k), \quad B = B(\psi^{k+4}, \overline{\psi}^{k-4}), \quad W = W(\psi^{k+1}, \overline{\psi}^{k-3}, \alpha, \overline{\alpha}), \quad (2.39)$$

such that

$$P(\psi^{k+2}, \psi^{k-2}, \alpha, \beta, \overline{\alpha}, \overline{\beta})$$

$$= < \psi, \alpha > < \psi, \beta > A(\psi^k, \overline{\psi}^{k-2}, \alpha \overline{\beta}) + < \psi, \overline{\alpha} > < \overline{\psi}, \beta > B(\psi^{k+2}, \alpha \beta, \overline{\psi}^{k-4})$$

$$+ < \psi, \alpha > < \overline{\alpha}, \overline{\psi} > W(\psi^{k+1}, \overline{\psi}^{k-3}, \beta, \overline{\beta}) + < \psi, \beta > < \beta, \overline{\psi} > W(\psi^{k+1}, \overline{\psi}^{k-3}, \alpha, \overline{\alpha}). \quad (2.40)$$

The spinor $A$ is symmetric in its first $k$ and last $k$ arguments; the spinor $B$ is symmetric in its first $k + 4$ and last $k - 4$ arguments; and the spinor $W$ is symmetric in its first $k + 1$ and following $k - 3$ arguments. With these symmetries, the spinors $A, B, W$ are uniquely determined by $P$. When $k = 3$, (2.40) is valid with $B = 0$ and $W = W(\psi^4, \alpha, \overline{\alpha})$. When $k = 2$, (2.40) holds with $B = 0$ and $W = 0$. 

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3. Natural Generalized Symmetries of the Vacuum Einstein Equations.

In this section we obtain a complete classification of all natural generalized symmetries of the vacuum Einstein equations, that is, we find all solutions to the linearized equations

\[
[-\epsilon_{CD}e^{C'D'}\alpha_A\beta_B\bar{\alpha}^{A'}\bar{\beta}^{B'} + \epsilon_{BC}e^{A'C'}\alpha_A\beta_D\bar{\alpha}^{B'}\bar{\beta}^{D'} + \epsilon_{BC}e^{A'C'}\alpha_D\beta_A\bar{\alpha}^{B'}\bar{\beta}^{D'}]C'\nabla_C\nabla_D h_{AB}^{A'B'} = 0,
\]

(3.1)

where

\[h_{AB}^{A'B'} = h_{AB}^{A'B'}(\Psi^2, \overline{\Psi}^2, \Psi^3, \overline{\Psi}^3, \ldots, \Psi^k, \overline{\Psi}^k)\]

is a natural spinor depending upon the Penrose fields to order \(k\). Equation (3.1) and all subsequent equations in this section hold by virtue of the Einstein equations and their derivatives.

Before beginning the detailed analysis of (3.1), let us outline the principal steps. Since \(h_{AB}^{A'B'}\) is assumed to be of order \(k\), the linearized equation is an identity to order \(k + 2\) in the Penrose fields. It is easy to see that this identity can be written symbolically as

\[
\alpha\Psi^{k+2} + \beta\overline{\Psi}^{k+2} + \gamma\Psi^{k+1}\Psi^{k+1} + \delta\Psi^{k+1}\overline{\Psi}^{k+1} + \epsilon\overline{\Psi}^{k+1}\Psi^{k+1} + \rho\Psi^{k+1} + \tau\overline{\Psi}^{k+1} + v = 0,
\]

(3.2)

where the coefficients \(\alpha, \beta, \ldots, v\) are complicated expressions of order \(k\) involving \(h_{AB}^{A'B'}\) and its repeated derivatives with respect to \(\Psi^2, \overline{\Psi}^2, \ldots, \Psi^k, \overline{\Psi}^k\). Each of the coefficients \(\alpha, \beta, \ldots, v\) must vanish identically because the fields \(\Psi^{k+2}, \overline{\Psi}^{k+2}, \Psi^{k+1}, \overline{\Psi}^{k+1}\) may be freely specified on \(\mathcal{E}^{k+2}\). As is standard practice in symmetry group analysis, we analyze this complicated identity beginning with the highest-order conditions \(\alpha = 0\) and \(\beta = 0\).

Let \(\partial^h_{\Psi} h\) and \(\partial^h_{\overline{\Psi}} h\) denote the partial derivatives of \(h_{AB}^{A'B'}\) with respect to \(\Psi^k\) and \(\overline{\Psi}^k\). These conditions \(\alpha = 0\) and \(\beta = 0\) impose certain algebraic conditions on the spinors \(\partial^h_{\Psi} h\) and \(\partial^h_{\overline{\Psi}} h\) which, when carefully analyzed, lead to unique spinor decompositions that we shall write symbolically as

\[
\partial^h_{\Psi} h = A + B + W \quad \text{and} \quad \partial^h_{\overline{\Psi}} h = D + E + U.
\]

(3.3)

This we do in \(\S 3A\); see Propositions 3.4 and 3.5. Each term \(A, B, \ldots, U\) in these decompositions separately satisfies the algebraic conditions arising from \(\alpha = 0\) and \(\beta = 0\). In \(\S 3B\) we show that the vanishing of the coefficients \(\gamma, \delta, \epsilon\) force \(h_{AB}^{A'B'}\) to be linear in the highest-order Penrose fields \(\Psi^k\) and \(\overline{\Psi}^k\), so that the spinors \(A, B, \ldots, U\) in the representation (3.3) are all at most of order \(k - 1\). The analysis of the conditions \(\rho = 0\) and \(\tau = 0\) is accomplished in two steps. In \(\S 3C\) we prove that \(A, B, D, E\) must actually be of order \(k - 2\), and that there is a generalized natural vector field

\[X_A^{A'} = X_A^{A'}(\Psi^2, \overline{\Psi}^2, \ldots, \Psi^{k-1}, \overline{\Psi}^{k-1})\]

such that

\[W = \partial^{k-1}_{\Psi} X \quad \text{and} \quad U = \partial^{k-1}_{\overline{\Psi}} X.\]

We let

\[k_{AB}^{A'B'} = h_{AB}^{A'B'} - (\nabla^{A'}_A X^{B'}_B + \nabla^{B'}_{B'} X^{A'}_A).\]
Then \( k_{\mu\nu} \) satisfies (3.2) and (3.3) with \( W = 0 \) and \( U = 0 \). In §3D we find that the remaining coefficients \( A, B, D, E \) in (3.3) now satisfy certain covariant constancy conditions, from which it readily follows that \( A = B = D = E = 0 \). The classification of the natural generalized symmetries of the Einstein equations is then completed by a simple induction argument.

**Notation and Commutation Rules.**

We begin by fixing some notation. If

\[
T_{c_1'\ldots c_q'}^{c_1\ldots c_p} = T_{c_1'\ldots c_q'}^{c_1\ldots c_p}(\psi^2, \bar{\psi}^2, \ldots, \Psi^{k}, \bar{\Psi}^{k})
\]

is a natural spinor of type \((p, q)\) and order \(k\), then the partial derivative of \(T_{c_1'\ldots c_q'}^{c_1\ldots c_p}\) with respect to \(\Psi^l\) is a natural spinor of type \((p + l + 2, q + l - 2)\). We shall write

\[
[\partial_\Psi T_{c_1'\ldots c_q'}^{c_1\ldots c_p}](\psi^1\ldots\psi^{l+2}, \bar{\psi}_1\ldots\bar{\psi}_{l-2}) = [\partial_{\Psi_{A_1'\ldots A_{l-2}'}} T_{c_1'\ldots c_q'}^{c_1\ldots c_p}] \psi_{A_1}^1 \ldots \psi_{A_{l+2}}^{l+2} \bar{\psi}_1^1 \ldots \bar{\psi}_{l-2}.
\]

Further, let \(\phi^1, \ldots, \phi^p\) and \(\bar{\phi}_1, \ldots, \bar{\phi}_q\) be arbitrary spinors of type \((1, 0)\) and \((0, 1)\) respectively; we shall write

\[
[\partial_\Psi T](\psi^{l+2}, \bar{\psi}^{l-2}; \phi^1, \ldots, \phi^p, \bar{\phi}_1, \ldots, \bar{\phi}_q) = [\partial_\Psi T_{c_1'\ldots c_q'}^{c_1\ldots c_p}](\psi^{l+2}, \bar{\psi}^{l-2}) \phi_{C_1}^1 \ldots \phi_{C_p}^p \bar{\phi}_{C_1'}^1 \ldots \bar{\phi}_{C_q'}^{p}.
\]

A semi-colon will always be used to separate arguments corresponding to derivatives with respect to the coordinates (2.22). Partial derivatives with respect to \(\bar{\Psi}_{A_1'\ldots A_{l-2}'}\) will be similarly denoted.

We shall repeatedly need certain commutation relations between the partial derivative operators \(\partial_{\Psi_{A_1'\ldots A_{m+2}'}}\) and \(\partial_{\bar{\Psi}_{A_1'\ldots A_{m-2}'}}\) and the covariant derivative operator \(\nabla_{c'}\).

**Proposition 3.1.** Let

\[
T_{\ldots} = T_{\ldots}(\psi^2, \bar{\psi}^2, \ldots, \Psi^m, \bar{\Psi}^m)
\]

be a natural spinor of order \(m\). Then

\[
[\partial_{\Psi}^{m+1}\nabla_{c'} T_{\ldots}](\psi^{m+3}, \bar{\psi}^{m-1}) = \psi^{c'} \bar{\psi}_{c'} [\partial_{\Psi}^{m} T_{\ldots}](\psi^{m+2}, \bar{\psi}^{m-2}),
\]

and

\[
[\partial_{\Psi}^{m}\nabla_{c'} T_{\ldots}](\psi^{m+2}, \bar{\psi}^{m-2}) = [\nabla_{c'} \partial_{\Psi}^{m} T_{\ldots}](\psi^{m+2}, \bar{\psi}^{m-2}) + \psi^{c'} \bar{\psi}_{c'} [\partial_{\Psi}^{m-1} T_{\ldots}](\psi^{m+1}, \bar{\psi}^{m-3}),
\]

and similarly,

\[
[\partial_{\Psi}^{m+1}\nabla_{c'} T_{\ldots}](\psi^{m-1}, \bar{\psi}^{m+3}) = \psi^{c'} \bar{\psi}_{c'} [\partial_{\Psi}^{m} T_{\ldots}](\psi^{m-2}, \bar{\psi}^{m+2}),
\]

and

\[
[\partial_{\Psi}^{m}\nabla_{c'} T_{\ldots}](\psi^{m-2}, \bar{\psi}^{m+2}) = [\nabla_{c'} \partial_{\Psi}^{m} T_{\ldots}](\psi^{m-2}, \bar{\psi}^{m+2}) + \psi^{c'} \bar{\psi}_{c'} [\partial_{\Psi}^{m-1} T_{\ldots}](\psi^{m-3}, \bar{\psi}^{m+1}).
\]
Proof: These formulas follow directly from Proposition 2.12 and the structure equations (2.23).

As an application of these commutation relations, we prove a proposition that we shall need later.

**Proposition 3.2.** Let

\[ P^{A_1 \ldots A_r}_{B_1' \ldots B_s'} = P^{A_1 \ldots A_r}_{B_1' \ldots B_s'}(\Psi^2, \overline{\Psi}^2, \ldots, \Psi^k, \overline{\Psi}^k) \]

be a natural spinor that is completely symmetric in the indices \( A_1 \ldots A_r \) and \( B_1' \ldots B_s' \). If

\[ \nabla^{(C'}_{(C'} P^{A_1 \ldots A_r)}_{B_1' \ldots B_s')} = 0 \quad \text{on} \ E^{k+1} \quad \text{on} \ E^{k+1}, \quad (3.9) \]

then \( P \) vanishes.

Proof: Equation (3.9) is equivalent to

\[ [\text{Grad } P](\alpha, \overline{\alpha}^r, \alpha^s) = 0, \quad (3.10) \]

where we have introduced the notation

\[ [\text{Grad } P](\beta, \overline{\beta}; \alpha^r, \alpha^s) = \beta_A \overline{\beta}^A' [\nabla^A' P](\alpha^r, \alpha^s). \quad (3.11) \]

We differentiate (3.10) with respect to \( \Psi^{k+1} \) and use the commutation relation (3.5) to deduce that

\[ [\partial^k_\psi P](\psi^{k+2}, \overline{\psi}^{k-2}; \alpha^r, \alpha^s) = 0. \quad (3.12) \]

Similarly, if we differentiate with respect to \( \overline{\Psi}^{k+1} \) we find that

\[ [\partial^{k+1}_\overline{\Psi} P](\psi^{k-2}, \overline{\psi}^{k+2}; \alpha^r, \alpha^s) = 0. \quad (3.13) \]

Equations (3.12) and (3.13) show \( P \) to be independent of \( \Psi^k \) and \( \overline{\Psi}^k \). A simple induction argument proves that \( P \) is independent of all the Weyl spinors \( \Psi^k, \overline{\Psi}^k, \ldots, \Psi^2, \overline{\Psi}^2 \).

The expansion of (3.9) in terms of the spinor connection coefficients \( \gamma^{CA}_{C' B} \) and \( \gamma^{CA'}_{C' B'} \) now leads to

\[ \gamma^{(CA_1}_{(C'|D| A_2 \ldots A_r)}_{B_1' B_2' \ldots B_s')} - \gamma^{(C|D')}_{(C'|D'| A_2 \ldots A_r)}_{B_1' B_2' \ldots B_s')} = 0. \]

This is an identity that must hold for all spinor connection coefficients and therefore, taking into account the identity

\[ \gamma^{CA}_{C' D} \epsilon_{AB} + \gamma^{CA}_{C' B} \epsilon_{DA} = 0, \]

we conclude that

\[ < \alpha, \beta > P(\gamma^r, \overline{\alpha}^s) + < \alpha, \gamma > P(\beta^r, \overline{\alpha}^s) = 0. \]

Setting \( \beta = \gamma \) we conclude that

\[ P(\alpha^r, \overline{\alpha}^s) = 0. \]

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Alternatively, one may conclude that $P = 0$ from the fact that there are no completely symmetric natural spinors of order zero.

3A. The $\Psi^{k+2}$ and $\overline{\Psi}^{k+2}$ Analysis.

We suppose that $h^{AB}_{A'B'}$ is a natural generalized symmetry of the vacuum Einstein equations of order $k$:

$$h^{AB}_{A'B'} = h^{A'B'}_{A'B'}(\Psi^2, \overline{\Psi}^2, \ldots, \Psi^k, \overline{\Psi}^k).$$

In this section we derive necessary and sufficient conditions for the vanishing of the coefficients $\alpha$ and $\beta$ in (3.2), and we analyze these conditions in detail.

We have, by two applications of (3.5)

$$[\partial^k \nabla^C \nabla^D h^{AB}_{A'B'}](\psi^{k+4}, \overline{\psi}^k) = \psi^C \overline{\psi}^C \partial^k \nabla^D [\partial^{k+1} h^{A'B'}_{A'B'}](\psi^{k+3}, \overline{\psi}^{k+1})$$

$$= \psi^C \overline{\psi}^D \psi^C \overline{\psi}^D [\partial^k h^{A'B'}_{A'B'}](\psi^{k+2}, \overline{\psi}^{k+2}).$$

Therefore, if we differentiate equation (3.1) with respect to $\Psi^{k+2}$ it follows that

$$<\beta, \psi > <\overline{\beta}, \overline{\psi}> [\partial^k h](\psi^{k+2}, \overline{\psi}^{k-2}; \alpha, \psi, \overline{\psi}, \overline{\alpha})$$

$$+ <\alpha, \psi > <\overline{\alpha}, \overline{\psi}> [\partial^k h](\psi^{k-2}, \overline{\psi}^{k-2}; \beta, \psi, \overline{\psi}, \overline{\beta}) = 0.$$  \hspace{1cm} (3.14)

Similarly, we differentiate the linearized equations (3.1) with respect to $\overline{\Psi}^{k+2}$ and use (3.7) to find

$$<\beta, \psi > <\overline{\beta}, \overline{\psi}> [\partial^k h](\psi^{k-2}, \overline{\psi}^{k-2}; \alpha, \psi, \overline{\psi}, \overline{\alpha})$$

$$+ <\alpha, \psi > <\overline{\alpha}, \overline{\psi}> [\partial^k h](\psi^{k-2}, \overline{\psi}^{k+2}; \beta, \psi, \overline{\psi}, \overline{\beta}) = 0.$$ \hspace{1cm} (3.15)

**Proposition 3.3.** If $h^{AB}_{A'B'}$ is a natural generalized symmetry of order $k$ for the vacuum Einstein equations, then

$$[\partial^k h](\psi^{k+2}, \overline{\psi}^{k-2}; \psi, \alpha, \overline{\alpha}, \overline{\psi}) = 0.$$ \hspace{1cm} (3.16)

and

$$[\partial^k h](\psi^{k-2}, \overline{\psi}^{k+2}; \psi, \alpha, \overline{\alpha}, \overline{\psi}) = 0.$$ \hspace{1cm} (3.17)

**Proof:** In equation (3.14) we set $\alpha = \beta$ and $\overline{\alpha} = \overline{\beta}$ to deduce that

$$[\partial^k h](\psi^{k+2}, \overline{\psi}^{k-2}; \psi, \alpha, \overline{\alpha}, \overline{\psi}) = 0.$$ \hspace{1cm} (3.16)

The symmetry $h_{ABA'B'} = h_{BBA'B'}$ then leads to (3.16). In equation (3.15) we set $\alpha = \beta$ and $\overline{\alpha} = \overline{\beta}$, and then use the symmetry of $h_{ABA'B'}$ to arrive at (3.17). Note that (3.16) and (3.17) are necessary and sufficient for (3.14) and (3.15) to hold.

Theorem 2.19 allows us to explicitly characterize all natural spinors that satisfy (3.16) and (3.17).
Proposition 3.4. The spinor \([\partial^k h](\psi^{k+2}, \overline{\psi}^{k-2}; \alpha, \beta, \overline{\alpha}, \overline{\beta})\) satisfies the symmetry conditions (3.16) if and only if there are natural spinors,

\[
A = A(\psi^k, \overline{\psi}^k), \quad B = B(\psi^{k+4}, \overline{\psi}^{k-4}), \quad W = W(\psi^{k+1}, \overline{\psi}^{k-3}, \alpha, \overline{\alpha}),
\]

such that

\[
[\partial^k h](\psi^{k+2}, \overline{\psi}^{k-2}; \alpha, \beta, \overline{\alpha}, \overline{\beta}) = <\psi, \alpha> <\psi, \beta > A(\psi^k, \overline{\psi}^{k-2}, \alpha, \overline{\alpha}) + <\overline{\psi}, \alpha> <\overline{\psi}, \beta > B(\psi^{k+2}, \alpha, \overline{\alpha}, \overline{\beta}) + <\psi, \alpha> <\overline{\psi}, \beta > B(\psi^{k+2}, \alpha, \overline{\alpha}, \overline{\beta}) + <\psi, \alpha> <\overline{\psi}, \beta > W(\psi^{k+1}, \overline{\psi}^{k-3}, \beta, \overline{\beta}). \tag{3.19}
\]

The spinor \(A\) is symmetric in its first \(k\) and last \(k\) arguments; the spinor \(B\) is symmetric in its first \(k + 4\) and last \(k - 4\) arguments; and the spinor \(W\) is symmetric in its first \(k + 1\) and following \(k - 3\) arguments. With these symmetries, the spinors \(A, B, W\) are uniquely determined by \(\partial^k h\). When \(k = 3\), (3.19) is valid with \(B = 0\) and \(W = W(\psi^4, \alpha, \overline{\alpha})\). When \(k = 2\), (3.19) holds with \(B = 0\) and \(W = 0\).

Let us remark that (3.19) contains the algebraic form of the generalized diffeomorphism symmetry. Indeed, if

\[
X^A_{A'} = X^A_{A'}(\Psi^2, \overline{\Psi}^2, ..., \psi^{k-1}, \overline{\psi}^{k-1})
\]

is the spinor form of a natural vector field of order \(k - 1\), and we let

\[
d^{AB}_{A'B'} = \nabla^A_{A'} X^B_{B'} + \nabla^B_{B'} X^A_{A'},
\]

then, by (3.5),

\[
[\partial^k_{\psi} d](\psi^{k+2}, \overline{\psi}^{k-2}; \alpha, \beta, \overline{\alpha}, \overline{\beta}) = <\psi, \alpha> <\overline{\psi}, \overline{\alpha}> [\partial^k_{\overline{\psi}} X](\psi^{k+1}, \overline{\psi}^{k-3}; \beta, \overline{\beta}) + <\psi, \beta> <\overline{\psi}, \overline{\beta}> [\partial^k_{\overline{\psi}} X](\psi^{k+1}, \overline{\psi}^{k-3}; \alpha, \overline{\alpha}). \tag{3.20}
\]

We observe that with \(W = \partial^k_{\psi} X\) the right-hand side of (3.20) coincides with the expression involving \(W\) in (3.19). In §3C we shall prove \(W\) satisfies integrability conditions that imply \(W = \partial^k_{\psi} X\).

There is an analogous decomposition for \(\partial^k_{\overline{\psi}} h\).

Proposition 3.5. The spinor \([\partial^k_{\overline{\psi}} h](\psi^{k-2}, \overline{\psi}^{k+2}; \alpha, \beta, \overline{\alpha}, \overline{\beta})\) satisfies the symmetry conditions (3.17) if and only if there are natural spinors,

\[
D = D(\overline{\psi}^k, \psi^k), \quad E = E(\overline{\psi}^{k+4}, \psi^{k-4}), \quad U = U(\overline{\psi}^{k+1}, \psi^{k-3}, \alpha, \overline{\alpha}),
\]

such that

\[
[\partial^k_{\overline{\psi}} h](\psi^{k-2}, \overline{\psi}^{k+2}; \alpha, \beta, \overline{\alpha}, \overline{\beta}) = <\overline{\psi}, \overline{\alpha}> <\overline{\psi}, \overline{\beta}> D(\overline{\psi}^k, \psi^{k-2}, \alpha, \overline{\beta}) + <\psi, \alpha> <\psi, \beta> E(\overline{\psi}^{k+2}, \overline{\alpha}, \psi^{k-4}) + <\psi, \alpha> <\overline{\psi}, \overline{\beta}> U(\overline{\psi}^{k+1}, \psi^{k-3}, \beta, \overline{\beta}) + <\overline{\psi}, \overline{\alpha}> <\psi, \beta> U(\overline{\psi}^{k+1}, \psi^{k-3}, \alpha, \overline{\alpha}). \tag{3.22}
\]
The spinor $D$ is symmetric in its first $k$ and last $k$ arguments; the spinor $E$ is symmetric in its first $k + 4$ and last $k - 4$ arguments; and the spinor $U$ is symmetric in its first $k + 1$ and following $k - 3$ arguments. With these symmetries the spinors $D, E, U$ are unique. When $k = 3$, (3.22) is valid with $E = 0$ and $U = U(\psi^1, \alpha, \overline{\alpha})$. When $k = 2$, (3.22) holds with $E = 0$ and $U = 0$.

3B. The $\Psi^{k+1}\Psi^{k+1}$, $\Psi^{k+1}\overline{\Psi}^{k+1}$, and $\overline{\Psi}^{k+1}\overline{\Psi}^{k+1}$ Analysis.

In this step we prove that if $h^{AB}_{A'B'}$ is a natural generalized symmetry of order $k$, then $h^{AB}_{A'B'}$ must be linear in the highest derivatives $\Psi^k$ and $\overline{\Psi}^k$. To begin, we use the commutation rules (3.5) and (3.6) to find that

$$(\partial^{k+1}_\Psi \partial^{k+1}_\Psi \nabla^C, \nabla^D h^{AB}_{A'B'})(\chi^{k+3}, \chi^{k-1}, \psi^{k+3}, \overline{\psi}^{k-1})$$

$$= [\partial^{k+1}_\Psi \{ \psi^{C} \overline{\psi}^{C'} (\partial^{k+1}_\Psi \nabla^D h^{AB}_{A'B'}) (\psi^{k+2}, \overline{\psi}^{k-2}) + \nabla^C (\partial^{k+1}_\Psi \nabla^D h^{AB}_{A'B'})(\psi^{k+3}, \overline{\psi}^{k-1}) \}] (\chi^{k+3}, \chi^{k-1})$$

$$= [\partial^{k+1}_\Psi \{ \psi^{C} \overline{\psi}^{C'} (\partial^{k+1}_\Psi \nabla^D h^{AB}_{A'B'}) (\psi^{k+2}, \overline{\psi}^{k-2}) + \nabla^D \overline{\psi}^{D'} \nabla^C (\partial^{k+1}_\Psi h^{AB}_{A'B'})(\psi^{k+2}, \overline{\psi}^{k-2}) \}] (\chi^{k+3}, \chi^{k-1})$$

$$= (\psi^{C} \overline{\psi}^{C'} \chi^{D} \chi^{D'} + \psi^{D} \overline{\psi}^{D'} \chi^{C} \overline{\chi}^{C'})(\partial^{k+1}_\Psi \partial^{k+1}_\Psi h^{AB}_{A'B'})(\psi^{k+2}, \overline{\psi}^{k-2}; \chi^{k+2}, \chi^{k-2}).$$

(3.23)

We differentiate the symmetry equation (3.1) twice with respect to $\Psi^{k+1}$ and use (3.23); after some elementary simplifications we obtain

$$-2 < \psi, \chi > < \overline{\psi}, \overline{\chi} > (\partial^{k+1}_\Psi \partial^{k+1}_\Psi h)(\psi^{k+2}, \overline{\psi}^{k-2}; \chi^{k+2}, \chi^{k-2}; \alpha, \beta, \overline{\alpha}, \overline{\beta})$$

$$+ < \psi, \beta > < \overline{\psi}, \overline{\beta} > (\partial^{k+1}_\Psi \partial^{k+1}_\Psi h)(\psi^{k+2}, \overline{\psi}^{k-2}; \chi^{k+2}, \chi^{k-2}; \alpha, \chi, \overline{\alpha}, \overline{\beta})$$

$$+ < \chi, \beta > < \overline{\chi}, \overline{\beta} > (\partial^{k+1}_\Psi \partial^{k+1}_\Psi h)(\psi^{k+2}, \overline{\psi}^{k-2}; \chi^{k+2}, \chi^{k-2}; \alpha, \psi, \overline{\alpha}, \overline{\beta})$$

$$+ < \psi, \alpha > < \overline{\psi}, \overline{\alpha} > (\partial^{k+1}_\Psi \partial^{k+1}_\Psi h)(\psi^{k+2}, \overline{\psi}^{k-2}; \chi^{k+2}, \chi^{k-2}; \beta, \psi, \overline{\beta})$$

$$+ < \chi, \alpha > < \overline{\chi}, \overline{\alpha} > (\partial^{k+1}_\Psi \partial^{k+1}_\Psi h)(\psi^{k+2}, \overline{\psi}^{k-2}; \chi^{k+2}, \chi^{k-2}; \beta, \psi, \overline{\beta}) = 0.$$

In the notation of equation (3.2) this is the condition $\gamma = 0$. Using Proposition 3.3, we immediately find that this equation simplifies to

$$(\partial^{k+1}_\Psi \partial^{k+1}_\Psi h)(\psi^{k+2}, \overline{\psi}^{k-2}; \chi^{k+2}, \chi^{k-2}; \alpha, \beta, \overline{\alpha}, \overline{\beta}) = 0.$$  

(3.24)

This proves that $h^{AB}_{A'B'}$ is at most linear in the variables $\Psi^k$. Likewise, if we take the second derivative of the linearized equations (3.1) with respect to $\overline{\Psi}^{k+1}$ and use Proposition 3.3, we obtain

$$(\partial^{k+1}_\Psi \partial^{k+1}_\Psi h)(\psi^{k-2}, \overline{\psi}^{k+2}; \chi^{k-2}, \chi^{k+2}; \alpha, \beta, \overline{\alpha}, \overline{\beta}) = 0,$$  

(3.25)

which implies that $h^{AB}_{A'B'}$ is linear in the variables $\overline{\Psi}^k$. Finally, differentiation of the symmetry condition (3.1) with respect to $\overline{\Psi}^{k+1}$ and $\Psi^{k+1}$, followed by use of Proposition 3.3, leads to

$$(\partial^{k+1}_\Psi \partial^{k+1}_\Psi h)(\psi^{k-2}, \overline{\psi}^{k+2}; \chi^{k-2}, \chi^{k+2}; \alpha, \beta, \overline{\alpha}, \overline{\beta}) = 0.$$  

(3.26)

Together, equations (3.24), (3.25), and (3.26) prove the following proposition.
Proposition 3.6. Let
\[ h_{AB}^{A'B'} = h_{AB}^{A'B'}(\Psi^2, \overline{\Psi}^2, \ldots, \Psi^k, \overline{\Psi}^k) \]
be a generalized symmetry of the vacuum Einstein equations. Then \( h_{AB}^{A'B'} \) is at most linear in the top-order Penrose fields \( \Psi^k \) and \( \overline{\Psi}^k \).

Corollary 3.7. The spinors \( A, B, W \) and \( D, E, U \) in equations (3.19) and (3.22) are at most of order \( k - 1 \).

Proof: This corollary follows from Proposition 3.6 and the fact that the spinors \( A, B, W \) and \( D, E, U \) in the decompositions (3.19) and (3.22) are unique.

At this point we are able to prove that there are no generalized symmetries of the Einstein equations of differential order two and three in the metric, aside from the scaling symmetry (2.11).

Corollary 3.8. Let \( h_{AB}^{A'B'}(\Psi^2, \overline{\Psi}^2) \) be a natural generalized symmetry of the vacuum Einstein equations of order 2. Then
\[ h_{AB}^{A'B'} = c \epsilon_{A'B'} \epsilon^{AB}, \]
where \( c \) is a constant.

Proof: According to Proposition 3.4 and Proposition 3.5, we have that
\[ [\partial_2^2 h](\psi^4; \alpha, \beta, \overline{\alpha}, \overline{\beta}) = <\psi, \alpha > <\psi, \beta > A(\psi^2, \overline{\alpha} \overline{\beta}), \]
and
\[ [\partial_2^2 h](\overline{\psi}^4; \alpha, \beta, \overline{\alpha}, \overline{\beta}) = <\overline{\psi}, \alpha > <\overline{\psi}, \beta > D(\overline{\psi}^2, \alpha \beta). \]

Proposition 3.6 implies that the spinors \( A \) and \( D \) are independent of the Penrose fields \( \Psi^2 \) and \( \overline{\Psi}^2 \). Because \( h \) is \( SL(2, \mathbb{C}) \) invariant, \( A \) and \( D \) are \( SL(2, \mathbb{C}) \) invariant, and consequently they are constructed solely from the \( \epsilon \)-spinors. It is easy to check that there are no spinors with the rank and symmetries of \( A \) and \( D \) built solely from the \( \epsilon \)-spinors. Therefore \( A = D = 0 \). This implies that \( h_{AB}^{A'B'} \) is a function only of the \( \epsilon \)-spinors from which the corollary follows.

Corollary 3.9. Let \( h_{AB}^{A'B'}(\Psi^2, \overline{\Psi}^2, \Psi^3, \overline{\Psi}^3) \) be a natural generalized symmetry of the vacuum Einstein equations of order 3. Then
\[ h_{AB}^{A'B'} = c \epsilon_{A'B'} \epsilon^{AB}, \]
where \( c \) is a constant.

Proof: According to Proposition 3.5, we have that
\[ [\partial_3^3 h](\psi^5, \overline{\psi}; \alpha, \beta, \overline{\alpha}, \overline{\beta}) = <\psi, \alpha > <\psi, \beta > A(\psi^3, \overline{\psi} \overline{\alpha} \overline{\beta}) + <\psi, \alpha > <\overline{\psi}, \overline{\alpha} > W(\psi^4, \beta, \overline{\beta}) + <\psi, \beta > <\overline{\psi}, \overline{\beta} > W(\psi^4, \alpha, \overline{\alpha}), \]
and
\[ [\partial_3^3 h](\overline{\psi}^5; \alpha, \beta, \overline{\alpha}, \overline{\beta}) = <\overline{\psi}, \alpha > <\overline{\psi}, \beta > D(\overline{\psi}^4, \psi \alpha \beta) + <\overline{\psi}, \alpha > <\psi, \overline{\alpha} > U(\overline{\psi}^4, \beta, \overline{\beta}) + <\overline{\psi}, \beta > <\psi, \overline{\beta} > U(\overline{\psi}^4, \alpha, \overline{\alpha}). \]
Proposition 3.6 implies that $A, D, W,$ and $U$ are functions of at most $\Psi^2$ and $\overline{\Psi}^2$. However, there are no natural spinors with the rank and symmetry of $A, D, W,$ and $U$ built solely from the undifferentiated Weyl spinors. To see this, let us focus on the spinor $A$. Because $A$ is a natural spinor we have that, for each $\Lambda, \Omega \in SL(2, \mathbb{C})$,

$$A_{A'B'C'}^A(\Psi^2, \overline{\Psi}^2) = \Lambda^A_{A'} \Lambda^E_{B'} \Lambda^F_{C'} \Omega^D_{D'} \Omega^{E'}_{E'} \Omega^{F'}_{F'} \Psi^2 \overline{\Psi}^2,$$

(3.27)

where we have set

$$\tilde{\Psi}_{ABCD} = \Lambda^E \Lambda^F \Lambda^G \Lambda^H \Psi_{EFGH} \quad \text{and} \quad \tilde{\Psi}_{A'B'C'D'} = \Omega^A_{A'} \Omega^B_{B'} \Omega^{C'}_{C'} \Omega^{D'}_{D'} \Psi_{E'F'G'H'}.$$

As this relation must hold for any $\Lambda, \Omega \in SL(2, \mathbb{C})$, we let

$$\Lambda^A_B = -\delta^A_B \quad \text{and} \quad \Omega^A_{B'} = \delta^A_{B'},$$

in which case

$$\tilde{\Psi}_{ABCD} = \Psi_{ABCD} \quad \text{and} \quad \tilde{\Psi}_{A'B'C'D'} = \Psi_{A'B'C'D'}.$$

Because there are an odd number of $\Lambda$ matrices on the right hand side of (3.27), the naturality condition forces $A = 0$. An identical series of arguments establish that $D = W = U = 0$. We therefore find that

$$[\partial^3_\Psi h](\psi^5, \Psi^1, \alpha, \beta, \overline{\alpha}, \overline{\beta}) = 0 \quad \text{and} \quad [\partial^3_\Psi h](\psi^1, \overline{\Psi}^1, \alpha, \beta, \overline{\alpha}, \overline{\beta}) = 0.$$

We have reduced the order of $h_{AB}$ by one, and Corollary 3.9 now follows from Corollary 3.8.

The invariance arguments leading to Corollaries 3.8 and 3.9 clearly fail when one allows natural generalized symmetries of order $k \geq 4$. This reflects the existence, for $k \geq 4$, of generalized diffeomorphism symmetries. These are analyzed in the next section.

## 3C. The \( \Psi^k \Psi^{k+1}, \overline{\Psi}^k \Psi^{k+1}, \Psi^k \overline{\Psi}^{k+1}, \) \text{ and } \overline{\Psi}^k \overline{\Psi}^{k+1} Analysis.

In this section we shall prove that $A, B, D$ and $E$ must be of order $k - 2$, and that there exists a natural type $(1,1)$ spinor $X$ of order $k - 1$,

$$X^A_{A'} = X^A_A(\Psi^2, \overline{\Psi}^2, \ldots, \Psi^{k-1}, \overline{\Psi}^{k-1}),$$

(3.28)

such that

$$W(\psi^{k+1}, \overline{\psi}^{k-3}, \alpha, \overline{\alpha}) = [\partial^k_{\Psi} X](\psi^{k+1}, \overline{\psi}^{k-3}, \alpha, \overline{\alpha}),$$

(3.29)

and

$$U(\psi^{k-3}, \overline{\psi}^{k+1}, \alpha, \overline{\alpha}) = [\partial^k_{\Psi} X](\psi^{k-3}, \overline{\psi}^{k+1}, \alpha, \overline{\alpha}).$$

(3.30)

We obtain these results by analyzing the equations arising from the coefficients of $\Psi^k \Psi^{k+1}$, $\Psi^k \overline{\Psi}^{k+1}$, $\overline{\Psi}^k \Psi^{k+1}$, and $\overline{\Psi}^k \overline{\Psi}^{k+1}$ in the linearized equations (3.1).
We begin with the $\Psi^k \Psi^{k+1}$ terms. Because $h_{AB}^{\alpha\beta}$ is linear in the Penrose fields $\Psi^k, \overline{\Psi}^k$, we can use the commutation rules in Proposition 3.1 to deduce that

$$
[\partial^k_{\Psi} \partial^{k+1}_{\Psi} \nabla^C_{\Psi} \nabla^D_{\Psi} h_{AB}^{\alpha\beta}](\chi^{k+2}, \overline{\chi}^{k+2}, \psi, \overline{\psi}^{k+2}, \overline{\psi}^{k+2})
$$

$$
= \psi^C \psi^D \nabla^C_{\Psi} \nabla^D_{\Psi} \partial^{k-1}_{\Psi} h_{AB}^{\alpha\beta}(\psi^{k+1}, \overline{\psi}^{k+3}, \chi^{k+2}, \overline{\chi}^{k+2})
$$

$$
+ \psi^C \chi^D \nabla^C_{\Psi} \nabla^D_{\Psi} \partial^{k-1}_{\Psi} h_{AB}^{\alpha\beta}(\chi^{k+1}, \psi^{k+3}, \overline{\psi}^{k+2}, \overline{\psi}^{k+2})
$$

$$
+ \chi^C \psi^D \nabla^C_{\Psi} \nabla^D_{\Psi} \partial^{k-1}_{\Psi} h_{AB}^{\alpha\beta}(\chi^{k+1}, \chi^{k+3}, \psi^{k+2}, \overline{\psi}^{k+2})
$$

(3.31)

We now apply the operator $\partial^k_{\Psi} \partial^{k+1}_{\Psi}$ to the linearized equations (3.1) to find, after substituting from (3.31) and simplifying, that

$$
-2 \langle \psi, \chi \rangle \langle \overline{\psi}, \overline{\chi} \rangle [\partial^k\partial^k h](\chi^{k+1}, \overline{\chi}^{k+3}, \psi^{k+2}, \overline{\psi}^{k+2}; \alpha, \beta, \overline{\alpha}, \overline{\beta})
$$

$$
+ \langle \psi, \beta \rangle \langle \overline{\psi}, \overline{\beta} \rangle [\partial^k\partial^k h](\psi^{k+1}, \overline{\psi}^{k-3}; \chi^{k+2}, \overline{\chi}^{k+2}; \alpha, \psi, \overline{\psi}, \overline{\alpha})
$$

$$
+ \langle \psi, \alpha \rangle \langle \overline{\psi}, \overline{\alpha} \rangle [\partial^k\partial^k h](\psi^{k+1}, \overline{\psi}^{k-3}; \chi^{k+2}, \overline{\chi}^{k+2}; \beta, \psi, \overline{\psi}, \overline{\beta})
$$

$$
+ \langle \chi, \beta \rangle \langle \overline{\chi}, \overline{\beta} \rangle [\partial^k\partial^k h](\chi^{k+1}, \overline{\chi}^{k-3}, \psi^{k+2}, \overline{\psi}^{k+2}; \alpha, \psi, \overline{\psi}, \overline{\alpha})
$$

$$
+ \langle \chi, \alpha \rangle \langle \overline{\chi}, \overline{\alpha} \rangle [\partial^k\partial^k h](\chi^{k+1}, \overline{\chi}^{k-3}, \psi^{k+2}, \overline{\psi}^{k+2}; \beta, \psi, \overline{\psi}, \overline{\beta})
$$

$$
+ \langle \psi, \alpha \rangle \langle \overline{\psi}, \overline{\alpha} \rangle [\partial^k\partial^k h](\chi^{k+1}, \overline{\chi}^{k-3}, \psi^{k+2}, \overline{\psi}^{k+2}; \beta, \psi, \overline{\psi}, \overline{\beta}) = 0.
$$

(3.32)

The symmetry condition (3.16) implies that the coefficients of $\langle \chi, \beta \rangle \langle \overline{\chi}, \overline{\beta} \rangle$ and $\langle \chi, \alpha \rangle \langle \overline{\chi}, \overline{\alpha} \rangle$ each vanish, and so we can rewrite equation (3.32) as

$$
-2 \langle \psi, \chi \rangle \langle \overline{\psi}, \overline{\chi} \rangle [\partial^k\partial^k h](\chi^{k+1}, \overline{\chi}^{k-3}, \psi^{k+2}, \overline{\psi}^{k+2}; \alpha, \beta, \overline{\alpha}, \overline{\beta})
$$

$$
+ \langle \psi, \beta \rangle \langle \overline{\psi}, \overline{\beta} \rangle [\partial^k\partial^k h](\psi^{k+1}, \overline{\psi}^{k-3}; \chi^{k+2}, \overline{\chi}^{k+2}; \alpha, \psi, \overline{\psi}, \overline{\alpha})
$$

$$
+ \langle \psi, \alpha \rangle \langle \overline{\psi}, \overline{\alpha} \rangle [\partial^k\partial^k h](\psi^{k+1}, \overline{\psi}^{k-3}; \chi^{k+2}, \overline{\chi}^{k+2}; \beta, \psi, \overline{\psi}, \overline{\beta})
$$

$$
+ \langle \chi, \beta \rangle \langle \overline{\chi}, \overline{\beta} \rangle [\partial^k\partial^k h](\chi^{k+1}, \overline{\chi}^{k-3}, \psi^{k+2}, \overline{\psi}^{k+2}; \alpha, \psi, \overline{\psi}, \overline{\alpha})
$$

(3.33)

$$
+ \langle \chi, \alpha \rangle \langle \overline{\chi}, \overline{\alpha} \rangle [\partial^k\partial^k h](\chi^{k+1}, \overline{\chi}^{k-3}, \psi^{k+2}, \overline{\psi}^{k+2}; \beta, \psi, \overline{\psi}, \overline{\beta}) = 0.
$$

In this equation we set $\alpha = \beta = \psi$ to arrive at

$$
[\partial^k\partial^k h](\chi^{k+1}, \overline{\chi}^{k-3}, \psi^{k+2}, \overline{\psi}^{k+2}; \psi, \overline{\psi}, \overline{\alpha}, \overline{\beta}) = 0.
$$

In terms of the decomposition (3.19) we have that

$$
[\partial^k h](\psi^{k+2}, \overline{\psi}^{k-2}; \psi, \overline{\psi}, \alpha, \overline{\beta}) = \langle \overline{\psi}, \overline{\alpha} \rangle \langle \overline{\psi}, \overline{\beta} \rangle B(\psi^{k+4}, \overline{\psi}^{k-4}),
$$

and so this equation implies that

$$
[\partial^k B](\chi^{k+1}, \overline{\chi}^{k-3}; \psi^{k+4}, \overline{\psi}^{k-4}) = 0.
$$

(3.34)
In other words, $B$ is independent of the spinor $\Psi^{k-1}$. Likewise, by setting $\alpha = \beta = \psi$ in equation (3.33), we conclude that

$$[\partial_{\Psi}^{k-1} A](\chi^{k+1}, \chi^{k-3}; \psi^k, \overline{\psi^k}) = 0,$$  \hspace{1cm} (3.35)

and so $A$ is independent of the spinor $\Psi^{k-1}$. Together, equations (3.19), (3.34), and (3.35) show that

$$[\partial_{\Psi}^{k-1} \partial_{\Psi}^{k} h](\chi^{k+1}, \chi^{k-3}; \psi^{k+2}, \overline{\psi}^{k-2}; \alpha, \beta, \overline{\alpha}, \overline{\beta}) = \langle \Psi, \alpha > \langle \overline{\Psi}, \overline{\beta} > [\partial_{\Psi}^{k-1} W](\chi^{k+1}, \chi^{k-3}; \psi^{k+1}, \overline{\psi}^{k-3}; \beta, \overline{\beta})$$

$$+ < \psi, \beta > < \overline{\psi}, \overline{\alpha} > [\partial_{\Psi}^{k-1} W](\chi^{k+1}, \chi^{k-3}; \psi^{k+1}, \overline{\psi}^{k-3}; \alpha, \overline{\alpha}).$$  \hspace{1cm} (3.36)

We next set $\alpha = \beta$ and $\overline{\alpha} = \overline{\beta}$ in (3.33), and substitute from (3.36) to arrive at

$$2 < \psi, \chi > < \overline{\psi}, \overline{\chi} > [\partial_{\Psi}^{k-1} W](\chi^{k+1}, \chi^{k-3}; \psi^{k+1}, \overline{\psi}^{k-3}; \alpha, \overline{\alpha})$$

$$= \langle \chi, \alpha > < \overline{\chi}, \overline{\alpha} > [\partial_{\Psi}^{k-1} W](\psi^{k+1}, \overline{\psi}^{k-3}; \chi^{k+1}, \chi^{k-3}; \psi, \overline{\psi})$$

$$+ < \chi, \psi > < \overline{\chi}, \overline{\alpha} > [\partial_{\Psi}^{k-1} W](\psi^{k+1}, \overline{\psi}^{k-3}; \chi^{k+1}, \chi^{k-3}; \alpha, \overline{\psi})$$

$$+ < \psi, \alpha > < \overline{\psi}, \overline{\alpha} > [\partial_{\Psi}^{k-1} W](\chi^{k+1}, \chi^{k-3}; \psi^{k+1}, \overline{\psi}^{k-3}; \chi, \overline{\alpha})$$

$$+ < \psi, \chi > < \overline{\psi}, \overline{\alpha} > [\partial_{\Psi}^{k-1} W](\chi^{k+1}, \chi^{k-3}; \psi^{k+1}, \overline{\psi}^{k-3}; \alpha, \overline{\chi}).$$  \hspace{1cm} (3.37)

The right-hand side of this equation is unchanged by the simultaneous interchange of $\psi$ with $\chi$ and $\overline{\psi}$ with $\overline{\chi}$ so we conclude

$$[\partial_{\Psi}^{k-1} W](\chi^{k+1}, \chi^{k-3}; \psi^{k+1}, \overline{\psi}^{k-3}; \alpha, \overline{\alpha}) = [\partial_{\Psi}^{k-1} W](\psi^{k+1}, \overline{\psi}^{k-3}; \chi^{k+1}, \chi^{k-3}; \alpha, \overline{\alpha}).$$ \hspace{1cm} (3.38)

Equation (3.38) is necessary and sufficient for equation (3.37) to hold, and is one of the integrability conditions needed to establish equation (3.29).

In exactly the same fashion we can apply the operator $\partial_{\Psi}^{k-1} \partial_{\Psi}^{k+1}$ to the linearized equations (3.1) to show that

$$[\partial_{\Psi}^{k-1} D](\psi^{k-3}, \overline{\psi}^{k+1}; \chi^{k}, \chi^{k}) = 0,$$ \hspace{1cm} (3.39)

$$[\partial_{\Psi}^{k-1} E](\psi^{k-3}, \overline{\psi}^{k+1}; \chi^{k+4}, \chi^{k-4}) = 0.$$  \hspace{1cm} (3.40)

Moreover, we have that

$$[\partial_{\Psi}^{k-1} U](\psi^{k-3}, \overline{\psi}^{k+1}; \chi^{k+1}, \chi^{k-3}; \alpha, \overline{\alpha}) = [\partial_{\Psi}^{k-1} U](\chi^{k-3}, \overline{\chi}^{k+1}; \psi^{k+1}, \psi^{k-3}; \alpha, \overline{\alpha}).$$ \hspace{1cm} (3.41)

Before applying the operator $\partial_{\Psi}^{k-1} \partial_{\Psi}^{k+1}$ to the linearized equations, we first use the commutation rules of Proposition 3.1 and the fact that $h^{A}_{\lambda\lambda'}$ is linear in $\Psi^{k}$ and $\overline{\Psi}^{k}$ to
deduce that
\[
[\partial^{k+1}_\Psi \partial^k_{\Psi} \nabla^C \nabla^D h_{\alpha\beta}^A B](\chi^{k-2} \chi^{k+2}; \psi^{k+3}, \overline{\psi}^{k-1})
= \Psi^C \psi^D \Psi^C \partial^k_{\Psi} \partial^k_{\Psi} h_{\alpha\beta}^A B](\chi^{k-2} \chi^{k+2}; \psi^{k+1}, \overline{\psi}^{k-3})
+ \Psi^C \chi^D \Psi^C \nabla^D \partial^k_{\Psi} \partial^k_{\Psi} h_{\alpha\beta}^A B](\chi^{k-2} \chi^{k+2}; \psi^{k+1}, \overline{\psi}^{k-2})
+ \chi^C \psi^D \chi^C \nabla^D \partial^k_{\Psi} \partial^k_{\Psi} h_{\alpha\beta}^A B](\chi^{k-2} \chi^{k+2}; \psi^{k+1}, \overline{\psi}^{k-2}).
\]

Using this result, if we differentiate (3.1) with respect to \( \overline{\Psi}^k \) and \( \Psi^{k+1} \) and take into account the leading order symmetry conditions of Proposition 3.3, we have

\[
-2 \psi, \chi > \overline{\psi}, \overline{\chi} > \partial^k_{\Psi} \partial^k_{\Psi} h](\chi^{k-3} \chi^{k+1}; \psi^{k+2}, \overline{\psi}^{k-2}; \alpha, \beta, \overline{\alpha}, \overline{\beta})
+ \psi, \alpha > \overline{\psi}, \overline{\alpha} > \partial^k_{\Psi} \partial^k_{\Psi} h](\chi^{k-3} \chi^{k+1}; \psi^{k+2}, \overline{\psi}^{k-2}; \beta, \psi, \overline{\beta}, \overline{\psi})
+ \partial^k_{\Psi} \partial^k_{\Psi} h](\chi^{k-3} \chi^{k+1}; \psi^{k+2}, \overline{\psi}^{k-2}; \beta, \chi, \overline{\chi}, \overline{\beta})
\]

\[
+ \psi, \beta > \overline{\psi}, \overline{\beta} > \partial^k_{\Psi} \partial^k_{\Psi} h](\chi^{k-3} \chi^{k+1}; \psi^{k+2}, \overline{\psi}^{k-2}; \alpha, \psi, \overline{\alpha}, \overline{\psi})
+ \partial^k_{\Psi} \partial^k_{\Psi} h](\chi^{k-3} \chi^{k+1}; \psi^{k+2}, \overline{\psi}^{k-2}; \alpha, \chi, \overline{\chi}, \overline{\alpha})
\]
\[
= 0.
\]

With \( \alpha = \beta = \psi \), and then with \( \overline{\alpha} = \overline{\beta} = \overline{\psi} \), equation (3.42) implies
\[
[\partial^k_{\Psi} B](\chi^{k+1}, \overline{\chi}^{k-3}; \psi^{k+4}, \overline{\psi}^{k-4}) = 0
\]
(3.43)

and
\[
[\partial^k_{\Psi} A](\chi^{k+1}, \overline{\chi}^{k-3}; \psi^k, \overline{\psi}^k) = 0.
\]
(3.44)

We set \( \alpha = \beta \) and \( \overline{\alpha} = \overline{\beta} \) in (3.42) and take (3.43) and (3.44) into account to find
\[
2 \psi, \chi > \overline{\psi}, \overline{\chi} > \partial^k_{\Psi} \partial^k_{\Psi} h](\chi^{k-3} \chi^{k+1}; \psi^{k+2}, \overline{\psi}^{k-2}; \alpha, \alpha, \overline{\alpha}, \overline{\alpha})
\]
\[
= \psi, \alpha > \overline{\psi}, \overline{\alpha} > \partial^k_{\Psi} \partial^k_{\Psi} h](\chi^{k-3} \chi^{k+1}; \psi^{k+2}, \overline{\psi}^{k-2}; \alpha, \psi, \overline{\alpha}, \overline{\psi})
+ \partial^k_{\Psi} \partial^k_{\Psi} h](\chi^{k-3} \chi^{k+1}; \psi^{k+2}, \overline{\psi}^{k-2}; \alpha, \chi, \overline{\chi}, \overline{\alpha})
\]
(3.45)

Again, in exactly the same manner, the \( \partial^k_{\Psi} \partial^k_{\Psi} \) derivative of the linearized equation (3.1) yields
\[
[\partial^k_{\Psi} D](\psi^{k+1}, \overline{\psi}^{k-3}; \chi^k, \chi^k) = 0,
\]
(3.46)

\[
[\partial^k_{\Psi} E](\psi^{k+1}, \overline{\psi}^{k-3}; \chi^{k+4}, \chi^{k-4}) = 0,
\]
(3.47)
as well as
\[
2 < \psi, \chi > < \bar{\psi}, \bar{\chi} > [\partial_{\bar{\psi}}^{k-1} \partial_{\bar{\psi}}^k h](\chi^{k+1}, \bar{\chi}^{k-1}; \psi^{k-2}, \bar{\psi}^{k+2}; \alpha, \alpha, \bar{\alpha}, \bar{\alpha})
\]
\[
= < \psi, \alpha > < \bar{\psi}, \bar{\alpha} > \{[\partial_{\bar{\psi}}^{k-1} \partial_{\bar{\psi}}^k h](\psi^{k-3}, \bar{\psi}^{k+1}; \chi^{k+2}, \bar{\chi}^{k-2}; \alpha, \psi, \bar{\psi}, \bar{\alpha}) + [\partial_{\bar{\psi}}^{k-1} \partial_{\bar{\psi}}^k h](\chi^{k+1}, \bar{\chi}^{k-3}; \psi^{k-2}, \bar{\psi}^{k+2}; \alpha, \chi, \bar{\chi}, \bar{\alpha}) \}.
\]

Equations (3.34), (3.35), (3.39), (3.40), (3.43), (3.44), (3.46), and (3.47) prove the following proposition.

**Proposition 3.10.** Let \( h_{\chi\psi}^{\bar{\chi}\bar{\psi}} \) be a natural generalized symmetry of order \( k \). Then the spinors \( A, B, D, E \) appearing in the decompositions (3.19) and (3.22) are at most of order \( k - 2 \).

On taking Proposition 3.10 into account, the substitution of (3.19) and (3.22) into (3.45) and (3.48) gives rise to
\[
4 < \psi, \chi > < \bar{\psi}, \bar{\chi} > [\partial_{\bar{\psi}}^{k-1} W](\chi^{k-3}, \bar{\chi}^{k+1}; \psi^{k+1}, \bar{\psi}^{k-3}; \alpha, \alpha)
\]
\[
= < \chi, \alpha > < \bar{\chi}, \bar{\alpha} > [\partial_{\bar{\psi}}^{k-1} U](\psi^{k+1}, \bar{\psi}^{k-3}; \chi^{k+1}, \bar{\chi}^{k-3}; \alpha, \psi)
\]
\[
+ < \chi, \alpha > < \bar{\chi}, \bar{\psi} > [\partial_{\bar{\psi}}^{k-1} U](\psi^{k+1}, \bar{\psi}^{k-3}; \chi^{k+1}, \bar{\chi}^{k-3}; \alpha, \bar{\psi})
\]
\[
+ < \psi, \alpha > < \bar{\psi}, \bar{\chi} > [\partial_{\bar{\psi}}^{k-1} W](\chi^{k-3}, \bar{\chi}^{k+1}; \psi^{k+1}, \bar{\psi}^{k-3}; \chi, \bar{\alpha})
\]
along with
\[
4 < \psi, \chi > < \bar{\psi}, \bar{\chi} > [\partial_{\bar{\psi}}^{k-1} U](\chi^{k+1}, \bar{\chi}^{k-3}; \psi^{k-3}, \bar{\psi}^{k+1}; \alpha, \alpha)
\]
\[
= < \chi, \alpha > < \bar{\chi}, \bar{\psi} > [\partial_{\bar{\psi}}^{k-1} W](\psi^{k-3}, \bar{\psi}^{k+1}; \chi^{k+1}, \bar{\chi}^{k-3}; \psi, \bar{\psi})
\]
\[
+ < \chi, \alpha > < \bar{\chi}, \bar{\psi} > [\partial_{\bar{\psi}}^{k-1} W](\psi^{k+1}, \bar{\psi}^{k-3}; \chi^{k+1}, \bar{\chi}^{k-3}; \alpha, \bar{\psi})
\]
\[
+ < \psi, \alpha > < \bar{\psi}, \bar{\chi} > [\partial_{\bar{\psi}}^{k-1} U](\chi^{k+1}, \bar{\chi}^{k-3}; \psi^{k-3}, \bar{\psi}^{k+1}; \alpha, \bar{\chi})
\]

In this last equation, we simultaneously interchange \( \psi \) with \( \chi \) and \( \bar{\psi} \) with \( \bar{\chi} \); a comparison with (3.49) allows us to deduce that
\[
[\partial_{\bar{\psi}}^{k-1} W](\chi^{k-3}, \bar{\chi}^{k+1}; \psi^{k+1}, \bar{\psi}^{k-3}; \alpha, \bar{\chi}) = [\partial_{\bar{\psi}}^{k-1} U](\psi^{k+1}, \bar{\psi}^{k-3}; \chi^{k-3}, \bar{\chi}^{k+1}; \alpha, \bar{\alpha}). \quad (3.51)
\]
Equations (3.38), (3.41), and (3.51) are the integrability conditions for (3.29) and (3.30).

**Proposition 3.11.** Let \( h_{\chi\psi}^{\bar{\chi}\bar{\psi}} \) be a generalized symmetry of order \( k \). Then there is a natural vector field of order \( k - 1 \),
\[
X_{\chi}^A = X_{\bar{\chi}}^A(\Psi^2, \bar{\Psi}^2, \ldots, \Psi^{k-1}, \bar{\Psi}^{k-1}),
\]

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such that the spinors $W$ and $U$ in (3.19) and (3.22) are the gradients

$$[\partial^k_{\Psi} X](\psi^{k+1}, \overline{\psi}^{k-3}; \alpha, \overline{\alpha}) = W(\psi^{k+1}, \overline{\psi}^{k-3}, \alpha, \overline{\alpha}), \quad (3.52)$$

and

$$[\partial^k_{\Psi} X](\psi^{k-3}, \overline{\psi}^{k+1}; \alpha, \overline{\alpha}) = U(\psi^{k-3}, \overline{\psi}^{k+1}, \alpha, \overline{\alpha}). \quad (3.53)$$

**Proof:** We have already seen that the linearized equations (3.1) imply the integrability conditions for equations (3.52) and (3.53) are satisfied. It is easy to check that

$$X^A_{\bar{A}} = \int_0^1 dt \, \Psi_{B_1 \ldots B_{k-3} B_{k+1}}^{B_1' \ldots B_{k-3}' B_{k+1} A} W_{B_1 \ldots B_{k-3} B_{k+1} A'}^{B_1' \ldots B_{k-3}' B_{k+1} A'}(\Psi^2, \overline{\Psi}^2, \ldots, \Psi^{k-2}, \overline{\Psi}^{k-2}, t \Psi^{k-1}, t \overline{\Psi}^{k-1})$$

$$+ \int_0^1 dt \, \Psi_{B_1 \ldots B_{k-3} B_{k+1}}^{B_1' \ldots B_{k-3}' B_{k+1} A} U_{B_1 \ldots B_{k-3} B_{k+1} A}^{B_1' \ldots B_{k-3}' B_{k+1} A'}(\Psi^2, \overline{\Psi}^2, \ldots, \Psi^{k-2}, \overline{\Psi}^{k-2}, t \Psi^{k-1}, t \overline{\Psi}^{k-1})$$

defines a real, natural vector field that satisfies equations (3.52) and (3.53).

**3D. Reduction in Order of $h_{\bar{A}B}^A$.**

Let us set

$$d_{\bar{A}B}^A = \nabla_A X^B_{\bar{A}} + \nabla_B X^A_{\bar{A}},$$

where $X^A_{\bar{A}}$ is defined in Proposition 3.11. By Proposition 2.5, we know that $d_{\bar{A}B}^A$ is a solution to the linearized equations (3.1) and so defines a generalized symmetry of the vacuum Einstein equations. Therefore

$$k_{\bar{A}B}^A = h_{\bar{A}B}^A - d_{\bar{A}B}^A$$

is also a generalized symmetry. Since

$$[\partial^k_{\Psi} d](\psi^{k+2}, \overline{\psi}^{k-2}; \alpha, \beta, \overline{\alpha}, \overline{\beta}) = <\psi, \alpha > <\overline{\alpha}, \overline{\psi}> W(\psi^{k+1}, \overline{\psi}^{k-3}, \beta, \overline{\beta}) + <\psi, \beta > <\overline{\beta}, \overline{\psi}> W(\psi^{k+1}, \overline{\psi}^{k-3}, \alpha, \overline{\alpha})$$

and

$$[\partial^k_{\Psi} k](\psi^{k+2}, \overline{\psi}^{k-2}; \alpha, \beta, \overline{\alpha}, \overline{\beta}) = <\overline{\psi}, \overline{\alpha} > <\alpha, \psi> U(\overline{\psi}^{k+1}, \psi^{k-3}, \beta, \overline{\beta}) + <\overline{\psi}, \overline{\beta} > <\beta, \psi> U(\overline{\psi}^{k+1}, \psi^{k-3}, \alpha, \overline{\alpha}),$$

we have, from our basic decomposition (3.19) and (3.22),

$$[\partial^k_{\Psi} k](\psi^{k+2}, \overline{\psi}^{k-2}; \alpha, \beta, \overline{\alpha}, \overline{\beta}) = <\psi, \alpha > <\psi, \beta > A(\psi^k, \overline{\psi}^{k-2}\alpha \beta) + <\overline{\psi}, \overline{\alpha} > <\overline{\psi}, \overline{\beta} > B(\psi^{k+2} \alpha \beta, \overline{\psi}^{k-4}) \quad (3.54)$$
and
\[
[\partial^k_{\Psi}] (\psi^{k-2}, \overline{\psi}^{k+2}; \alpha, \beta, \overline{\alpha}, \overline{\beta})
= \langle \psi, \overline{\alpha} \rangle \langle \psi, \overline{\beta} \rangle D(\psi^k, \psi^{k-2} \alpha \beta) + \langle \psi, \alpha \rangle \langle \psi, \beta \rangle E(\psi^{k+2} \alpha \beta, \psi^{k-4}).
\]  

(3.55)

We now show that the linearized equations (3.1) force
\[
A = B = D = E = 0,
\]

(3.56)

and hence
\[
h^{\alpha \beta}_{\alpha \beta'} = \nabla^\alpha A_{\beta'} + \nabla^\beta A_{\alpha'} + k^{\alpha \beta}_{\alpha \beta'},
\]

(3.57)

where \(k^{\alpha \beta}_{\alpha \beta'}\) is now of order \(k^{-1}\).

To prove (3.56) we differentiate equation (3.1) one final time with respect to \(\Psi^{k+1}\) and use the leading order symmetry condition satisfied by \(k^{\alpha \beta}_{\alpha \beta'}\), namely
\[
[\partial^k_{\Psi}] (\psi^{k-2}, \overline{\psi}^{k+2}; \psi, \alpha, \overline{\alpha}, \overline{\psi}) = 0,
\]

(3.58)

where we have introduced the notation
\[
[\text{Div } \partial^k_{\Psi}] (\psi^{k+2}, \overline{\psi}^{k-2}; \alpha, \overline{\alpha}) = \alpha \overline{\alpha} \nabla_{A'} \partial^k_{\Psi} k^{\alpha \beta}_{\alpha \beta'} (\psi^{k+2}, \overline{\psi}^{k-2}).
\]

(3.59)

In (3.58) we now set \(\alpha = \beta = \psi\); by virtue of equation (3.54) we then find
\[
[\text{Grad } B] (\psi, \overline{\psi}; \psi^{k+4}, \overline{\psi}^{k-4}) = 0.
\]

(3.60)

Similarly, if we set \(\overline{\alpha} = \overline{\beta} = \overline{\psi}\) in (3.58) and use (3.55) we find that
\[
[\text{Grad } A] (\psi, \overline{\psi}; \psi^{k}, \overline{\psi}^{k}) = 0.
\]

(3.61)

Proposition 3.2 implies that \(A = 0\) and \(B = 0\).

We have thus found that
\[
[\partial^k_{\Psi}] (\psi^{k+2}, \overline{\psi}^{k-2}; \alpha, \beta, \overline{\alpha}, \overline{\beta}) = 0.
\]

Likewise, by differentiating the linearized equations (3.1) with respect to \(\overline{\Psi}^{k+1}\) we can show that \(D = 0\) and \(E = 0\) so that
\[
[\partial^k_{\overline{\Psi}}] (\psi^{k-2}, \overline{\psi}^{k+2}; \alpha, \beta, \overline{\alpha}, \overline{\beta}) = 0.
\]

These last two equations prove (3.57).
Theorem 3.12. Let

\[ h^{AB}_{\mathcal{A}\mathcal{B}'} = h^{AB}_{\mathcal{A}\mathcal{B}'}(\Psi^2, \overline{\Psi}^2, \ldots, \Psi^k, \overline{\Psi}^k) \]

be a natural generalized symmetry of the vacuum Einstein equations of order \( k \). Then there exists a natural spinor

\[ X^A_{A'} = X^A_{A'}(\Psi^2, \overline{\Psi}^2, \ldots, \Psi^{k-1}, \overline{\Psi}^{k-1}) \]

of order \( k - 1 \), and a constant \( c \), such that

\[ h^{AB}_{\mathcal{A}\mathcal{B}'} = c \epsilon^{AB}_{A'B'} + \nabla^A_{A'} X^B_{B'} + \nabla^B_{B'} X^A_{A'} \quad \text{on } \mathcal{E}^k. \]

Proof: If \( k = 2, 3 \) this theorem reduces to Corollaries 3.8 and 3.9. Let \( k > 3 \). We have shown that

\[ h^{AB}_{\mathcal{A}\mathcal{B}'} = \nabla^A_{A'} X^B_{B'} + \nabla^B_{B'} X^A_{A'} + k^{AB}_{\mathcal{A}\mathcal{B}'} \]

where \( k^{AB}_{\mathcal{A}\mathcal{B}'} \) is a natural spinor of order \( k - 1 \). A straightforward induction argument now shows that \( k^{AB}_{\mathcal{A}\mathcal{B}'} \) can be reduced to a function of the Penrose fields \( \Psi^2, \overline{\Psi}^2, \Psi^3, \overline{\Psi}^3 \) at the expense of changing the vector field \( X^A_{A'} \) (the new vector field is again denoted \( X^A_{A'} \)). We apply Corollary 3.9 to the natural generalized symmetry \( k^{AB}_{\mathcal{A}\mathcal{B}'} \) to show that

\[ k^{AB}_{\mathcal{A}\mathcal{B}'} = c \epsilon^{AB}_{A'B'}, \]

and our classification of the natural generalized symmetries of the vacuum Einstein equations is complete. \( \blacksquare \)
4. First-Order Generalized Symmetries.

In this section we begin our classification of all generalized symmetries of the vacuum Einstein equations by determining all first-order generalized symmetries. As mentioned in the introduction, the calculation of the higher-order generalized symmetries reduces to that of the first-order generalized symmetries. While the analysis of the higher-order symmetries is similar in spirit to that of the natural symmetries, as presented in the previous section, the analysis of the first-order symmetries is rather more complex and merits a separate presentation.

To begin, let

\[ h_{ab} = h_{ab}(x^i, g_{ij}, g_{ij,k}) \]

be the components of a first-order generalized symmetry. We emphasize that the functions \( h_{ab} \) are no longer assumed to be the components of a natural tensor, and hence may depend explicitly upon the coordinates \( x^i \) and the first derivatives of the metric \( g_{ij,k} \).

The linearized equations

\[
\left[ -g^{cd} \delta^a_i \delta^b_j - g^{ab} \delta^d_i \delta^e_j + g^{ac} (\delta_j^b \delta^d_i + \delta_j^b \delta^d_i) \right] \nabla_c nabla_d h_{ab} = 0 \tag{4.1}
\]

involve the metric and its first 3 derivatives, and must be satisfied when the Einstein equations

\[
R_{ab} = 0 \quad \text{and} \quad \nabla_c R_{ab} = 0 \tag{4.2}
\]

are satisfied. In accordance with the results of §2, we write \( h_{ab} \) as a new function

\[ h_{ab} = h_{ab}(x^i, g_{ij}, \Gamma_{jk}^i) \]

and express the linearized equations in terms of the jet coordinates

\[ \{x^i, g_{ij}, \Gamma_{jk}^i, \Gamma_{jhl}^i, Q_{ij,kl}, Q_{ij,klm}\} \tag{4.3} \]

for \( J^3(G) \), which were introduced in §2 (see (2.12) and (2.13)). The Einstein equations (4.2) hold if and only if the variables \( Q_{ij,kl} \) and \( Q_{ij,klm} \) are completely trace-free. Consequently, the linearized equations (4.1) for the first-order generalized symmetry must hold identically for all values of

\[ \{x^i, g_{ij}, \Gamma_{jk}^i, \Gamma_{jhl}^i, [Q_{ij,kl}]_{\text{tracefree}}, [Q_{ij,klm}]_{\text{tracefree}}\}. \]

In order to determine the dependence of the linearized equations on our adapted jet coordinates we will need the following structure equations for the coordinates (4.3):

\[
D_i g_{jk} = g_{jl} \Gamma_{ik}^l + g_{kl} \Gamma_{ij}^l, \tag{4.4}
\]

\[
D_k \Gamma_{ij}^h = \Gamma_{ijk}^h + \frac{2}{3} Q_{k,ij}^h + \Gamma_{mij}^h \Gamma_{jk}^m + \Gamma_{mj}^h \Gamma_{ik}^m, \tag{4.5}
\]

\[
D_l \Gamma_{ijkl}^h = \frac{1}{2} Q_{l,ijkl}^h - \frac{2}{3} Q_{l,ij}^m \Gamma_{kjm}^h + \frac{4}{3} \Gamma_{(ik}^m R_{j)l}^h \Gamma_{lm}^m - 3 \Gamma_{(ik}^h \Gamma_{j)ml}^m. \tag{4.6}
\]
and (see (2.15))
\[ \nabla_m Q_{ij,kl} = Q_{ij,klm} + \frac{1}{2} (Q_{m(i,j)kl} + Q_{kl,ijm}). \]  

(4.7)

We will use the following notation. The derivative of \( h_{ab} \) with respect to the metric \( g_{rs} \) and connection variables \( \Gamma^t_{rs} \) will be denoted by
\[ \partial^{rs} h_{ab} = \frac{\partial h_{ab}}{\partial g_{rs}} \quad \text{and} \quad \partial^t_{rs} h_{ab} = \frac{\partial h_{ab}}{\partial \Gamma^t_{rs}}. \]

Note that these quantities are symmetric in the indices \( rs \) and \( ab \). If
\[ X = X^a \frac{\partial}{\partial x^a}, \quad Y = Y^a \frac{\partial}{\partial x^a}, \quad \text{and} \quad \alpha = \alpha_r dx^r, \]
we let
\[ [\partial g h](\alpha \alpha; XX) = \alpha_r \alpha_s X^a X^b (\partial^{rs} h_{ab}) \]
and
\[ [\partial \Gamma h](\alpha \alpha; XX) = \alpha_r \alpha_s Y^t X^a X^b (\partial^t_{rs} h_{ab}). \]

We denote by \( \alpha^* \) the vector field obtained from the 1-form \( \alpha \) by “raising the index” with the metric,
\[ \alpha^* = g^{rs} \alpha_s \frac{\partial}{\partial x^r}, \]
and we denote by \( X^* \) the 1-form obtained from the vector \( X \) by “lowering the index” with the metric,
\[ X^* = g_{ij} X^i dx^j. \]

The natural pairing of \( X \) and \( \alpha \) is
\[ <X, \alpha> = X^i \alpha_i. \]

**Proposition 4.1.** Let \( h_{ab} = h_{ab}(x^i, g_{ij}, \Gamma^i_{jk}) \) be a first-order generalized symmetry for the vacuum Einstein equations. Then there are zeroth-order quantities
\[ M^{s}_{bt} = M^{s}_{bt}(x^i, g_{ij}) \]
such that
\[ \partial^t_{rs} h_{ab} = \delta^t_{(a} M^{s)}_{bt). \]  

(4.8)

**Proof:** Since
\[ \nabla_d h_{ab} = D_d h_{ab} - \Gamma^i_{ad} h_{ib} - \Gamma^i_{bd} h_{ai} \]
\[ = (\partial^t_{rs} h_{ab}) \Gamma^t_{rsd} + \{\star\}, \]
where \( \{\star\} \) denotes terms involving the variables \( x^i, g_{ij}, \Gamma^i_{jk}, Q^j_h \), we conclude using equation (4.6) and (4.7) that
\[ \nabla_c \nabla_d h_{ab} = (\partial^t_{rs} h_{ab}) \Gamma^t_{rscd} + \{**\}, \]

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where \( \{\ast\ast\} \) denotes terms involving the variables \( x^i, g_{ij}, \Gamma^i_{jhk}, Q^j_{hk}, Q^j_{hkl} \). Hence, by differentiating the linearized equations (4.1) with respect to \( \Gamma^i_{rscd} \) and contracting the result with \( X^i X^j Y^r \alpha_s \alpha_t \alpha_c \alpha_d \), we arrive at

\[
< \alpha^\sharp, \alpha > \left[ \partial \Gamma^i \right](\alpha \alpha, Y; XX) = < X, \alpha > \{ - < X, \alpha > \left[ \partial \Gamma^i \right](\alpha \alpha, Y) + 2 \left[ \partial \Gamma^i \right](\alpha \alpha, Y; \alpha^\sharp X) \}.
\]

(4.9)

Here we have defined the trace of \( h_{ab} \) in the usual way:

\[
\text{tr} h = g^{ab} h_{ab}.
\]

When \( \alpha \) is a null 1-form, the expression in brackets on the right-hand side of (4.9) must vanish. By Proposition 2.17, this implies that there are quantities \( M_{bt}^s \) such that

\[
- < X, \alpha > \left[ \partial \Gamma^i \right](\alpha \alpha, Y) + 2 \left[ \partial \Gamma^i \right](\alpha \alpha, Y; \alpha^\sharp X) = < \alpha^\sharp, \alpha > M(X, Y, \alpha),
\]

where

\[
M(X, Y, \alpha) = M_{bt}^s X^b Y^t \alpha_s.
\]

Thus (4.9) reduces to

\[
\left[ \partial \Gamma^i \right](\alpha \alpha, Y; XX) = < X, \alpha > M(X, Y, \alpha).
\]

(4.10)

We have shown that equation (4.10) is necessary for (4.9) to hold. It is also sufficient. This is easily verified if we observe that (4.10) implies

\[
\left[ \partial \Gamma^i \right](\alpha \alpha, Y; \alpha^\sharp X) = \frac{1}{2} \left( < \alpha^\sharp, \alpha > M(X, Y, \alpha) + < X, \alpha > M(\alpha^\sharp, Y, \alpha) \right)
\]

and

\[
\left[ \partial \Gamma^i \right](\alpha \alpha; Y) = M(\alpha^\sharp, Y, \alpha).
\]

It remains to prove that \( M_{bt}^s \) is independent of the connection variables \( \Gamma^i_{jk} \). To this end we first differentiate equation (4.10) with respect to \( \Gamma^i_{jk} \) to obtain

\[
\left[ \partial \Gamma^i \right](\beta \beta, Z; \alpha \alpha, Y; XX) = < X, \alpha > \left[ \partial \Gamma^i \right](\beta \beta, Z; X, Y, \alpha).
\]

(4.11)

The left-hand side of this equation is symmetric under interchange of \( (\beta, Z) \) with \( (\alpha, Y) \), and therefore

\[
< X, \alpha > \left[ \partial \Gamma^i \right](\beta \beta, Z; X, Y, \alpha) = < X, \beta > \left[ \partial \Gamma^i \right](\alpha \alpha, Y; X, Z, \beta).
\]

Using Proposition 2.18 we conclude that \( \left[ \partial \Gamma^i \right] \) takes the form

\[
\left[ \partial \Gamma^i \right](\beta \beta, Z; X, Y, \alpha) = < X, \beta > W(\alpha, \beta, Y, Z),
\]

(4.12)
where $W$ has the symmetry property

$$W(\alpha, \beta, Y, Z) = W(\beta, \alpha, Z, Y).$$

Equation (4.11) becomes

$$[\partial_T \partial_T h](\beta \beta, Z; \alpha \alpha, Y; XX) = <X, \alpha > <X, \beta > W(\alpha, \beta, Y, Z). \quad (4.13)$$

Next we observe that the structure equations (4.4)–(4.7) imply

$$\nabla_c \nabla_d h_{ab} = (\partial^{uv}_w \partial^r_s h_{ab}) \Gamma^t_{rsd} \Gamma^w_{uvc} + \{\star\},$$

where $\{\star\}$ denotes terms that are at most linear in the coordinates $\Gamma^i_{jhk}$. Using this equation, we now differentiate the linearized equations with respect to $\Gamma^t_{rsd}$ and $\Gamma^w_{uvc}$ to find that

$$<\beta^\sharp, \alpha> <X, \alpha > <X, \beta > - \frac{1}{2} <X, \beta >^2 <\alpha^\sharp, \alpha > - \frac{1}{2} <X, \alpha >^2 <\beta^\sharp, \beta > \times W(\alpha, \beta, Y, Z) = 0.$$

Because the expression in square brackets is not identically zero, this equation implies that $W = 0$ and therefore $\partial_T M = 0$, as claimed.

Next we turn to an analysis of the terms involving $Q_{ij, hkl}$ in the linearized equations (4.1). In the following proposition we let

$$M^a_{[sr]} = M^s_{at} g^{rt} \quad \text{and} \quad M^{ars} = g^{ab} M^s_{bt} g^{rt}.$$

**Proposition 4.2.** If $h_{ab} = h_{ab}(x^i, g_{ij}, \Gamma^i_{hk})$ is a first-order generalized symmetry of the vacuum Einstein equations, then there are quantities $V^a = V^a(x^i, g_{ij})$ such that

$$M^a_{[sr]} = \delta^a_{[s} V^{r]}.$$

**Proof:** Because

$$\nabla_d h_{ab} = \frac{2}{3} (\partial^r_s h_{ab}) Q^t_{d ,rs} + (\partial^r_s h_{ab}) \Gamma^t_{rsd} + \{\star\},$$

where $\{\star\}$ denotes terms involving the variables $x^i, g_{ij}, \Gamma^i_{jk}$, we can show

$$\nabla_c \nabla_d h_{ab} = \frac{2}{3} (\partial^r_s h_{ab}) Q^h_{d ,rs} + \frac{1}{2} (\partial^r_s h_{ab}) Q^h_{c ,rsd} + \{**\}$$
where \{**\} now indicates terms involving the variables \(x^i, g_{ij}, \Gamma^k_{ij}, \Gamma^k_{ijh}, Q_{ijk,l}\). Therefore, for the linearized equations to hold we must have that

\[
- g^{cd}_{ij} \delta^a_j \delta^b_i - g^{ab}_{ij} \delta^a_j \delta^d_i + g^{bc}_{ij} \delta^a_j \delta^d_i + g^{bc}_{ij} \delta^a_j \delta^d_i \left( \frac{2}{3} (\partial^r h_{ab}) Q^h_{cd} + \frac{1}{2} (\partial^r s h_{ab}) Q^h_{cd} \right) = 0 \quad (4.15)
\]

for all \(Q^h_{c,rs}c\) and \(Q^h_{c,rsd}\) that are completely trace-free. We multiply (4.15) by \(X^i X^j\) and substitute for \(\partial^r s h_{ab}\) from Proposition 4.1 and for \(Q^h_{c,rsd}\) and \(Q^h_{d,rs}\) from (2.13) to obtain

\[
- M^{bs} h^c X^d + M^{cs} h^b X^d \\
\times \left[ \frac{1}{12} (R_{bhc|d} + R_{dhcb|s} + R_{shcd|b} + R_{shcb|d} + R_{dhcs|b} + \frac{1}{3} (R_{bhds|c} + R_{shdb|c}) \right] = 0.
\]

By using the algebraic curvature symmetries and the Bianchi identities, every term in this equation may be expressed as either a multiple of \(M^{bs} h^c X^d R_{bhc|d}\) or \(M^{cs} h^b X^d R_{shcb|d}\). The coefficient of the former term vanishes, while that of the latter term is one. Thus (4.15) holds if and only if

\[
M^{bs} h^c X^d [R_{shbc|d}]_{\text{tracefree}} = 0. \quad (4.16)
\]

To analyze this condition it is convenient to revert to spinors. We set

\[
M^{BB'AA'HH'}_{s} = M^{bst}_{b} \sigma^{BB'}_{s} \sigma^{AA'}_{t} \sigma^{HH'}_{t},
\]

and use (2.20) and (2.23) to write

\[
[R_{shbc|d}]_{\text{tracefree}} \longleftrightarrow \epsilon_{SH} \epsilon_{BC} \tilde{\Psi}_{S'H'B'C'D'D'} + \epsilon_{S'H'} \epsilon_{B'C'} \Psi_{SHBCDD'},
\]

so that the condition (4.16) is equivalent to

\[
X^{CC'} X^{DD'} M^{BB'SS'H'H'}_{s} [\epsilon_{SH} \epsilon_{BC} \tilde{\Psi}_{S'H'B'C'D'D'} + \epsilon_{S'H'} \epsilon_{B'C'} \Psi_{SHBCDD'}] = 0 \quad (4.17)
\]

for all Penrose spinors \(\Psi^3\) and \(\overline{\Psi}^3\). We differentiate this expression with respect to \(\Psi_{SHBCDD'}\) and multiply the resulting equation by \(\psi_{S} \psi_{H} \psi_{B} \psi_{C} \psi_{D} \psi_{D'}\) to conclude

\[
\epsilon_{A'H'} \psi_{A} \psi_{H} \psi_{B} M^{BB'AA'HH'} = 0. \quad (4.18)
\]

Similarly, differentiation of (4.17) with respect to \(\overline{\Psi}_{S'H'B'C'D'D'}\) leads to

\[
\epsilon_{AH} \overline{\Psi}_{A} \overline{\Psi}_{H} \overline{\Psi}_{B} M^{BB'AA'HH'} = 0. \quad (4.19)
\]

To solve equations (4.18) and (4.19) we decompose \(M\) as

\[
M^{BB'AA'HH'} = P^{BB'AA'HH'} + S^{BB'} \epsilon^{AH} \epsilon^{A'H'} + T^{BB'AA'HH'} \epsilon^{AH} + \overline{T}^{BB'AA'HH'} \epsilon^{A'H'}, \quad (4.20)
\]
where the spinors $P, T, \overline{T}$ are each symmetric in the indices $AH$ and $A'H'$. Note that the spinors $T$ and $\overline{T}$ correspond to the skew-symmetric part of $M$ in (4.14). Equations (4.18) and (4.19) now imply that
\[
\psi_A \psi_H \psi_B \overline{\psi}_{B'} \psi_{A'H'} = 0
\]
and
\[
\overline{\psi}_{A'} \psi_H' \psi_B' \psi_{B'A'} = 0.
\]
These equations can be analyzed using Proposition 2.15; we find that there must exist quantities $V_{AA'}$ such that
\[
T_{BB'A'H'} = \varepsilon_{A'B'} V_{BH'K'} + \varepsilon_{H'B'} V_{B'A'K'},
\]
(4.21)
and
\[
\overline{T}_{BB'A'H'} = \varepsilon_{AB} V_{B'H'} + \varepsilon_{HB} V_{B'A'K'}.
\]
(4.22)
We insert (4.21) and (4.22) into (4.20); note that only the real part of $V_{AA'}$ appears. We then write the resulting equation in tensor form to complete the proof.

Proposition 4.3. Let $h_{ab} = h_{ab}(x^i, g_{ij}, \Gamma^k_{ij})$ be a first-order generalized symmetry of the vacuum Einstein equations. Then there are zeroth-order quantities $V_i = V_i(x^i, g_{ij})$ and $\hat{h}_{ab} = \hat{h}_{ab}(x^i, g_{ij})$ such that
\[
h_{ab} = \hat{h}_{ab} + \nabla_a V_b + \nabla_b V_a.
\]
Proof: Let
\[
\hat{h}_{ab} = h_{ab} - (\nabla_a V_b + \nabla_b V_a),
\]
where $V_a$ is defined by Proposition 4.2. Then $\hat{h}_{ab}$ is a first-order generalized symmetry and therefore, by Proposition 4.1, there exist zeroth-order quantities $\hat{M}_{at} = \hat{M}_{at}(x^i, g_{ij})$ such that
\[
\partial_t^{rs} \hat{h}_{ab} = \delta_{(a}^{(r} \hat{M}_{b)t)}^{s)}.
\]
(4.23)
Moreover, by construction, $\hat{M}$ will satisfy Proposition 4.2 with $V^i = 0$, and hence
\[
\hat{M}^{bst} = \hat{M}^{bts}.
\]
(4.24)
This symmetry condition will allow us to prove, from the coefficient of $\Gamma^r_{stu} \Gamma^m_{pq}$ in the linearized equations, that $\hat{M}^{bst} = 0$, that is,
\[
\hat{h}_{ab} = \hat{h}_{ab}(x^i, g_{ij}).
\]
The derivation of the condition arising from the coefficient of \( \Gamma_{stu}^{m} \) in the linearized equations is the longest single calculation in this paper. To begin we first compute

\[
\alpha_s \alpha_t \alpha_u \partial_r \partial_t \partial_{stu}(\nabla_c \nabla_d \tilde{h}_{ab}) = \alpha_s \alpha_t \alpha_u [D_c(\partial_r^{st} \tilde{h}_{ab}) \delta_u^d + \delta_c^u \partial_r^{st} \nabla_d \tilde{h}_{ab} - 3 \delta_c^u \delta_r^{st} \tilde{h}_{ab}] - \Gamma_{cd}^s (\partial_r^{tu} \tilde{h}_{ab}) - \Gamma_{ac}^t \delta_d^s (\partial_r^{tu} \tilde{h}_{la})]
\]

(4.25)

The second term on the right-hand side of this equation is found to be

\[
\partial_s \tilde{h}_{ab} = \alpha_s \alpha_t \partial_r \partial_s \nabla_d \tilde{h}_{ab} - \alpha_s \alpha_t [D_d(\partial_r^{st} \tilde{h}_{ab}) + 2g_{jr} \delta_d^s (\partial_r^{sj} \tilde{h}_{ab}) + 2 \Gamma_{jrd}^t (\partial_r^{st} \tilde{h}_{ab}) + 2 \delta_d^t \Gamma_{ri}^s (\partial_r^i \tilde{h}_{ab})
\]

(4.26)

Together, equations (4.25) and (4.26) imply that

\[
X^r Y^m \alpha_s \alpha_t \alpha_u \beta_p \beta_q \partial_{sstu} \partial_{pq}(\nabla_c \nabla_d \tilde{h}_{ab})
\]

\[= 4 \beta_{(c \alpha)} d \{[\partial_g \partial_t \tilde{h}_{ab}] (\beta Y^p; \alpha, X) + 2 \alpha_c \alpha_d [\partial_g \partial_t \tilde{h}_{ab}] (\alpha X^p; \beta, Y)
\]

\[- \alpha_c \beta_a \beta_d Y^m [\partial_t \tilde{h}_{mb}](\alpha, X) + \alpha_c \beta_b \beta_d Y^m [\partial_t \tilde{h}_{ma}](\alpha, X) - \alpha_a \alpha_c \alpha_d X^m [\partial_t \tilde{h}_{mb}](\beta, Y)
\]

\[- \alpha_b \alpha_c \alpha_d X^m [\partial_t \tilde{h}_{ma}](\beta, Y) - \beta_a \beta_c \alpha_d Y^m [\partial_t \tilde{h}_{mb}](\alpha, X) - \beta_b \beta_c \alpha_d Y^m [\partial_t \tilde{h}_{ma}](\alpha, X)
\]

\[+ 2 \alpha_c \alpha_d < X, X > [\partial_t \tilde{h}_{ab}](\beta, Y) - \alpha_c \alpha_d < Y, Y > [\partial_t \tilde{h}_{ab}](\beta, X)
\]

\[- \beta_c \beta_d < Y, X > [\partial_t \tilde{h}_{ab}](\alpha, X).
\]

We substitute this equation into the linearized equations (4.1) multiplied by \( Z^i Z^j \) and use (4.23) to obtain, after considerable algebraic simplifications,

\[
2 < Z, \alpha >^2 \{[\partial_g \tilde{M}](\beta Y^p; \beta^z, X, \alpha) - \partial_g \tilde{M}(\alpha X^p; \beta^z, Y, \beta)\}
\]

\[+ 2 < Z, \alpha > < Z, \beta > \{[\partial_g \tilde{M}](\alpha X^p; \beta^z, Y, \beta) - \partial_g \tilde{M}(\beta Y^p; \alpha^z, X, \alpha)\}
\]

\[+ 2 < \alpha^z, \alpha > < Z, \beta > \{[\partial_g \tilde{M}](\alpha X^p; Z, Y, \beta) - \partial_g \tilde{M}(\alpha X^z; Z, Y, \beta)\}
\]

\[+ 2 < \alpha^z, \alpha > < Z, \beta > \{[\partial_g \tilde{M}](\beta Y^p; Z, X, \alpha) - \partial_g \tilde{M}(\beta^z X^p; Z, X, \alpha)\}
\]

\[- < Z, \alpha >^2 < \beta^z, \beta > \tilde{M}(Y, X, \alpha) - < Z, \beta >^2 < \alpha^z, \alpha > \tilde{M}(Y, X, \alpha)
\]

(4.27)

As a check of the accuracy of this equation, we used Maple to verify that the diffeomorphism

\[
\tilde{M}(\alpha^z, Y, X, \beta) = \tilde{M}(Y, X, \alpha, \beta).
\]

\[
\tilde{M}(\beta^z, Y, X, \alpha) = \tilde{M}(Y, X, \alpha, \beta).
\]
symmetry, for which
\[ \widehat{M}(\alpha, X, Z) = 2[\partial_{g}V](Z^{b}\alpha; X) - <X, \alpha > V(Z), \]
and \( V_{i} = V_{i}(x^{i}, g_{kl}) \), provides a solution to (4.27).

In order to simplify equation (4.27) using (4.24) we set
\[ \widehat{N}_{a}^{sr} = \widehat{M}_{at}^{s} g^{rt}, \]
and
\[ \widehat{N}(Z, \beta, \alpha) = \widehat{N}_{a}^{sr} Z^{a} \beta_{s} \alpha_{r}, \]
and observe that
\[ [\partial_{g}\widehat{M}](\beta \gamma; Z, X, \alpha) = [\partial_{g}\widehat{N}](\beta \gamma; Z, X^{b}, \alpha) \]
\[ + \frac{1}{2} <X, \beta > \widehat{N}(Z, \gamma, \alpha) + \frac{1}{2} <X, \gamma > \widehat{N}(Z, \beta, \alpha). \]

We substitute this equation into (4.27) and use the fact that
\[ \widehat{N}(Z, \alpha, \beta) = \widehat{N}(Z, \beta, \alpha) \]
(4.28)
to deduce, again after lengthy algebraic simplifications, that
\[ <Z, \alpha >^{2} K(\beta, Y, \beta, \alpha, X) + <Z, \alpha > <Z, \beta > K(\alpha, X, \alpha, \beta, Y) \]
\[ + [<Z, \alpha > < \alpha^{2}, \beta > - <Z, \beta > < \alpha^{2}, \alpha >] K(\alpha, X, Z^{b}, \beta, Y) = 0, \]
(4.29)
where
\[ K(\alpha, X, Z, \beta, Y) = [\partial_{g}\widehat{N}](\alpha X^{b}; Z, Y^{b}, \beta) - [\partial_{g}\widehat{N}](\beta Y^{b}; Z, X^{b}, \alpha) \]
\[ + \frac{1}{2} [\partial_{g}\widehat{N}](\beta Z^{b}; Y, \alpha, X^{b}) - \frac{1}{2} <Z, \alpha > \widehat{N}(X, \beta, Y^{b}). \]
(4.30)
Equation (4.29) implies that \( K(\alpha, X, Z^{b}, \beta, Y) = 0 \) whenever \(<Z, \alpha > = 0 \). Therefore, by Proposition 2.18, there exist quantities \( L \) such that
\[ K(\alpha, X, Z^{b}, \beta, Y) = <Z, \alpha > L(X, \beta, Y). \]
Substituting this expression back into (4.29) and simplifying the result, we find
\[ <\beta^{2}, \beta > L(Y, \alpha, X) + < \alpha^{2}, \beta > L(X, \beta, Y) = 0. \]
In this equation we set \( \alpha = \beta \) to conclude that \( L = 0 \) and hence \( K = 0 \).

On account of the symmetry (4.28) of \( \widehat{N} \), the condition \( K = 0 \) implies that
\[ <Z, \alpha > \widehat{N}(X, \beta, Y^{b}) = <Z, X^{b} > \widehat{N}(\alpha^{2}, \beta, Y^{b}), \]
(4.31)
which easily implies that \( \widehat{N} = 0 \) and thus \( \widehat{M} = 0 \).

We are now ready to complete our classification of first-order generalized symmetries.
Theorem 4.4. Let $h_{ab} = h_{ab}(x^i, g_{ij}, \Gamma^k_{ij})$ be a first-order generalized symmetry of the vacuum Einstein equations. Then there is a constant $c$ and zeroth-order quantities $V_i = V_i(x^i, g_{ij})$ such that

$$h_{ab} = c g_{ab} + \nabla_a V_b + \nabla_b V_a.$$  

Proof: Proposition 4.3 reduces the proof to showing that the zeroth-order symmetry $\hat{h}_{ab}$ is in fact a constant times the metric. This follows from the classification of the point symmetries of the Einstein equations [27macro.]. We include the proof here for completeness.

Let us begin with the conditions placed on $\hat{h}_{ab}$ by the vanishing of the terms in the linearized equations involving $\Gamma^a_{bcd}$. From the structure equations (4.4)–(4.6) it is a straightforward matter to show that

$$\nabla_c \nabla_d \hat{h}_{ab} = 2 \frac{\partial \hat{h}_{ab}}{\partial g_{mn}} g_{mp} [\Gamma^p_{ncd} + \frac{1}{2} Q^p_{cd} - \hat{h}_{pa} [\Gamma^p_{bdc} + \frac{1}{2} Q^p_{da}] - \hat{h}_{pb} [\Gamma^p_{adc} + \frac{1}{2} Q^p_{db}]] + \{\star\},$$  

where $\{\star\}$ denotes terms depending only on the variables $x^i, g_{ij}, \Gamma^k_{ij}$. We multiply the linearized equations by $X^i X^j$ and differentiate them with respect to $\Gamma^a_{bcd}$. The result, after multiplying by $\alpha^a \alpha^b \alpha^c \alpha^d Z^a$ and simplifying, is given by

$$<\alpha^2, \alpha> [\partial_g \hat{h}](Z^b \alpha; XX) = <\alpha, X> \{2[\partial_g \hat{h}](Z^b \alpha; \alpha^2 X) - <\alpha, X> [\partial_g \hat{h}](Z^b \alpha)\}. \tag{4.33}$$

Proposition 2.18 now implies that there exist zeroth-order quantities $A$ such that

$$[\partial_g \hat{h}](Z^b \alpha; XX) = <\alpha, X > A(Z^b, X).$$

The symmetry of $(\partial_g \hat{h})$ in $Z^b \alpha$ implies that

$$<\alpha, X > A(Z^b, X) = <Z^b, X > A(\alpha, X),$$

and therefore, by Proposition 2.18, there exists a zeroth-order function $F = F(x^i, g_{ij})$ such that

$$A(\alpha, X) = <\alpha, X > F.$$

We have therefore found that

$$[\partial_g \hat{h}](\alpha \alpha; XX) = <\alpha, X >^2 F. \tag{4.34}$$

It is easily verified that this equation is necessary and sufficient for (4.33) to hold. Next, we differentiate (4.34) with respect to $g_{ij}$ to obtain

$$[\partial_g \partial_g \hat{h}](\beta \beta; \alpha \alpha; XX) = <\alpha, X >^2 [\partial_g F](\beta \beta).$$

The left-hand side of this equation is symmetric under interchange of $\alpha$ and $\beta$, and we therefore have

$$<\alpha, X >^2 [\partial_g F](\beta \beta) = <\beta, X >^2 [\partial_g F](\alpha \alpha).$$
From Proposition 2.18 it is easily seen that this equation implies

$$[\partial g F](\alpha \alpha) = 0.$$ (4.35)

Equations (4.34), (4.35) imply that $\hat{h}_{ab}$ is of the form

$$\hat{h}_{ab} = F(x^i)g_{ab} + k_{ab}(x^i).$$ (4.36)

Now we turn to the conditions on $\hat{h}_{ab}$ arising from the terms in the linearized equations depending on $Q_{ab,cd}$. It is straightforward to show, using (4.32), that this condition takes the form

$$Q_{ij,kl}[2X^i X^c g^{rk} \frac{\partial \hat{h}_{rc}}{\partial g_{jl}} - X^i X^k g^{bc} \frac{\partial \hat{h}_{bc}}{\partial g_{jl}} - \frac{3}{2} X^i X^j \hat{h}^{kl}] = 0,$$

when $Q_{ij,kl}$ is completely trace-free. If we substitute from (4.36) the first and second terms vanish leaving us with

$$X^i X^j k_{ab} g^{ak} g^{bl} [Q_{ij,kl}]_{\text{tracefree}} = 0.$$

Because $k_{ab}$ is independent of the metric, this equation implies that $k_{ab} = 0$.

We have reduced $\hat{h}_{ab}$ to the form

$$\hat{h}_{ab} = F(x^i)g_{ab}.$$ We now substitute this equation for $\hat{h}_{ab}$ into the linearized equations to find

$$-g_{ij} \nabla^a \nabla_a F - 2 \nabla_i \nabla_j F = 0.$$ We differentiate this equation with respect to $\Gamma^r_{st}$ and obtain

$$[g_{il} g^{st} + 2 \delta^{(s}_{(i} \delta^{t)}_{j)}] \frac{\partial F}{\partial x^r} = 0,$$

which implies that $\frac{\partial F}{\partial x^r} = 0$, and thus $F$ is a constant.
5. Complete Classification of Generalized Symmetries of the Vacuum Einstein Equations.

We now turn to the computation of all generalized symmetries of the Einstein equations. Let
\[ h_{AB}^{k'} = h_{AB}^k (x, \sigma, \Gamma^1, \Gamma^2, \Psi^2, \overline{\Psi}^2, \ldots, \Gamma^l, \Psi^k, \overline{\Psi}^k) \]  
be the components of a generalized symmetry of the Einstein equations. Initially, we have \( l = k \), so the generalized symmetry is of order \( k \). The repeated covariant derivative of \( h_{AB}^{k'} \) can be given schematically by
\[ \nabla \nabla h = DDh + \gamma \cdot Dh + (D\gamma) \cdot h + \gamma \cdot \gamma \cdot h, \]
where \( \gamma \cdot Dh \) is a sum of products of spin connections \( \gamma_{BC} \) and \( \overline{\gamma}_{AA'} \) and total derivatives \( D^C, h_{AB}^{k'} \), and so on. The linearized equation,
\[ \left[ -\epsilon_{CD} \epsilon^{C'D'} \alpha_{A'B'} \overline{A'} \overline{B'} - \epsilon_{BC} \epsilon^{A'C'} \alpha_{A'B'} \overline{B'} \right] \nabla_{C'} \nabla_{B'} h_{AB}^{k'} = 0 \quad \text{on} \quad \mathcal{E}^{k+2}, \]
is an \( SL(2, \mathbb{C}) \) invariant identity depending on the variables \( x^i, \sigma_{aA'}, \sigma_{aA'b}, \sigma_{aA'bc}, \Gamma^1, \Gamma^2, \Psi^2, \overline{\Psi}^2, \ldots, \Gamma^{l+2}, \Psi^{k+2}, \overline{\Psi}^{k+2} \). On the Einstein equation manifold \( \mathcal{E}^{k+2} \) there are relationships between \( \sigma_{aA',bc} \) and \( \Gamma^2, \Psi^2, \overline{\Psi}^2 \), but in what follows we are careful only to consider terms involving \( \Psi^l \) and \( \overline{\Psi}^l \) for \( l \geq 3 \). The rather complicated lower-derivative analysis was performed in §4.

In order to analyze the dependence of this equation on our adapted jet coordinates, we need the following structure equations on \( \mathcal{E}^{k+1} \):
\[ D_{j_{k+1}} \Gamma_{j_0j_1\ldots j_k}^i = G_{j_0j_1\ldots j_{k+1}}^i (\sigma, \Psi^{k+1}, \overline{\Psi}^{k+1}) + B_{j_0j_1\ldots j_{k+1}}^i (\Gamma^1, \Gamma^k) \]
\[ + C_{j_0j_1\ldots j_{k+1}}^i (\sigma, \Gamma^1, \Psi^k, \overline{\Psi}^k) + E_{j_0j_1\ldots j_{k+1}}^i (\sigma, \Gamma^1, \ldots, \Gamma^{k-1}, \Psi^2, \overline{\Psi}^2, \ldots, \Psi^{k-1}, \overline{\Psi}^{k-1}). \]

(5.3)

Here \( A:: \) is linear in \( \Psi^k \) and \( \overline{\Psi}^k \), \( B:: \) is bilinear in its arguments, \( C:: \) is linear in \( \Psi^k \) and \( \overline{\Psi}^k \) with coefficients depending on \( \sigma \) and \( \Gamma^1 \).

We also have (see (2.23))
\[ D_{A'} \Psi_{j_1'j_2'\ldots j_{k+2}}^{j_1j_2\ldots j_k} = \Psi_{A}^{j_1'j_2'\ldots j_{k+2}} + M_{A}^{j_1'j_2'\ldots j_{k+2}} (\gamma, \nabla, \Psi^k) + N_{A}^{j_1'j_2'\ldots j_{k+2}} (\Psi^2, \overline{\Psi}^2, \ldots, \Psi^{k-1}, \overline{\Psi}^{k-1}), \]

(5.4)

where \( M:: \) is linear in \( \Psi^k \). There is an analogous formula for the total derivative of \( \overline{\Psi}^k \).

Let
\[ f(\sigma, \Gamma^1, \Gamma^2, \Psi^2, \overline{\Psi}^2, \ldots, \Gamma^l, \Psi^k, \overline{\Psi}^k) \]
be a smooth function. We retain the notation
\[ [\partial^m \Psi f](\psi^{m+2}, \overline{\psi}^{m-2}) \quad \text{and} \quad [\partial^m \overline{\Psi} f](\psi^{m-2}, \overline{\psi}^{m+2}) \]
introduced in §3 for the derivatives of \( f \) with respect to \( \Psi^m \) and \( \overline{\Psi}^m \), and we define
\[ [\partial^m f](Y, \omega^{m+1}) = \frac{\partial f}{\partial \Gamma_{j_0j_1\ldots j_m}^i} Y^i \omega_{j_0} \omega_{j_1} \ldots \omega_{j_m}. \]
In many of our subsequent formulas the spinor components
\[ \omega^A_{A'} = \sigma^{jA}_{A'j} \]
of the covector \( \omega \) will appear. In addition, we will use \( \omega \) as a bilinear map
\[ \omega(\alpha, \beta) = \omega^A_{A'} \alpha_A \beta^A'. \]
Finally, we write
\[ h(\alpha, \omega, \alpha) = h^{A'B}_{A'B'} \alpha_A \omega^A_{A} \alpha^B_{B'}. \]

From the structure equations (5.3)–(5.4) we readily derive the following commutation rules. For \( l \geq 2 \) we have
\[ \left[ \partial^{l+1}_{\Gamma} D^A_{A'} f \right](Y, \omega^{l+2}) = \omega^A_{A'} \left[ \partial^1_{\Gamma} f \right](Y, \omega^{l+1}) \]
and
\[ \left[ \partial^{l+1}_{\Gamma} D^A_{A'} f \right](Y, \omega^{l+1}) = \omega^A_{A'} \left[ \partial^{l-1}_{\Gamma} f \right](Y, \omega^l) + \left( D^A_{A'} \left[ \partial^1_{\Gamma} f \right] \right)(Y, \omega^{l+1}) + \left[ \Gamma^1 \cdot \partial^1_{\Gamma} f \right]_{A'}(Y, \omega^{l+1}) \]
while for \( l < k \) we find that
\[ \left[ \partial^{k+1}_{\Psi} D^A_{A'} f \right](\psi^{k+3}, \overline{\psi}^{k-2}) = \psi^A_{A'} \left[ \partial^k_{\Psi} f \right](\psi^{k+2}, \overline{\psi}^{k-2}) \]
and
\[ \left[ \partial^{k}_{\Psi} D^A_{A'} f \right](\psi^{k+2}, \overline{\psi}^{k-2}) = \psi^A_{A'} \left[ \partial^{k-1}_{\Psi} f \right](\psi^{k+1}, \overline{\psi}^{k-3}) \]
\[ + \left( D^A_{A'} \left[ \partial^k_{\Psi} f \right] \right)(\psi^{k+2}, \overline{\psi}^{k-2}) + \left[ \Gamma^2 \cdot \partial^k_{\Psi} f \right]_{A'}(\psi^{k+2}, \overline{\psi}^{k-2}) + \left[ \partial^{k-1}_{\Gamma} f \right]_{A'}(\psi^{k+2}, \overline{\psi}^{k-2}). \]

The analysis of (5.2) now proceeds along lines very similar to those presented in §3. Accordingly, we shall not provide all the details of the many calculations involved in the lengthy analysis, but rather simply list the various steps and the conclusions obtained in each.

5A. The \( \Gamma^{l+2} \) Analysis, \( l \geq k - 1, \ k \geq 2 \).

When we differentiate (5.2) with respect to \( \Gamma^{l+2} \), we find that
\[ <\omega, \omega> [\partial^l_{\Gamma} h](Y, \omega^{l+1}; \alpha, \overline{\alpha}, \beta, \overline{\beta}) + \omega(\beta, \overline{\beta}) \left[ \partial^l_{\Gamma} h \right](Y, \omega^{l+1}; \alpha, \omega, \overline{\alpha}) \]
\[ + \omega(\alpha, \overline{\alpha}) \left[ \partial^l_{\Gamma} h \right](Y, \omega^{l+1}; \beta, \omega, \overline{\beta}) = 0. \]

In this equation, set \( \omega^A_{A'} = \psi^A_{A'} \overline{\psi}_{A'} \) to conclude that
\[ \left[ \partial^l_{\Gamma} h \right](Y, \omega^{l+1}; \alpha, \omega, \overline{\alpha}) = 0 \]
whenever \( \omega \) is a null vector. By Proposition 2.17 this implies there is a spinor
\[ P = P(Y, \omega^l, \alpha, \overline{\alpha}) \]
such that

\[ [\partial_l h](Y, \omega^{l+1}; \alpha, \omega, \alpha) = -\frac{1}{2} <\omega, \omega> P(Y, \omega^l, \alpha, \alpha). \]

This fact allows us to use (5.9) to show that the highest \( \Gamma \) derivative of \( h \) has the algebraic form

\[ [\partial_l \Gamma h](Y, \omega^{l+1}; \alpha, \omega, \alpha) = \frac{1}{2} \omega(\alpha, \alpha) P(Y, \omega^l, \alpha, \alpha) \]

(5.10)

Note that the commutativity of the partial derivatives \( \partial_l \partial_l \Gamma \) implies, using equation (5.10) with \( \beta = \alpha \) and \( \bar{\beta} = \bar{\alpha} \), that

\[ \omega(\alpha, \alpha) [\partial_l P](Z, \eta^{l+1}; Y, \omega^l, \alpha, \alpha) = \eta(\alpha, \alpha) [\partial_l P](Y, \omega^{l+1}; Z, \eta^l, \alpha, \alpha). \]

(5.11)

5B. The \( \Gamma^{l+1} \) Analysis, \( l \geq k - 1, \ k \geq 2 \).

The repeated derivative of (5.2) with respect to \( \Gamma^{l+1} \) becomes, with \( \beta = \alpha \) and \( \bar{\beta} = \bar{\alpha} \),

\[ <\omega, \eta> [\partial_l \partial_l \Gamma h](Y, \omega^{l+1}; Z, \eta^{l+1}; \alpha, \alpha, \alpha, \alpha) \]

\[ + \eta(\alpha, \alpha) [\partial_l \partial_l \Gamma h](Y, \omega^{l+1}; Z, \eta^{l+1}; \alpha, \omega, \alpha, \alpha) \]

\[ + \omega(\alpha, \alpha) [\partial_l \partial_l \Gamma h](Y, \omega^{l+1}; Z, \eta^{l+1}; \alpha, \eta, \alpha, \alpha) = 0. \]

(5.12)

We now substitute into (5.12) from (5.10), multiply by \( \eta(\alpha, \alpha) \), and use (5.11) to deduce that

\[ [\partial_l \partial_l \Gamma h](Y, \omega^{l+1}; Z, \eta^{l+1}; \alpha, \alpha, \alpha, \alpha) \]

\[ \times [\partial_l P](Z, \eta^{l+1}; Y, \omega^l, \alpha, \alpha) = 0. \]

Because the first spinor in brackets is not identically zero, we find that

\[ [\partial_l \partial_l \Gamma h](Y, \omega^{l+1}; Z, \eta^{l+1}; \alpha, \alpha, \alpha, \alpha) = 0, \]

(5.13)

and thus \( h_{\alpha \rho}^{\lambda} \) is at most linear in the variables \( \Gamma^l \).

5C. The \( \Psi^{k+2} \) and \( \overline{\Psi}^{k+2} \) Analysis, \( l \geq k - 1, \ k \geq 2 \).

The commutation rules (5.5)–(5.8) do not allow us to immediately differentiate with respect to \( \Psi^{k+2} \) and \( \overline{\Psi}^{k+2} \) to arrive at the equations (3.16) and (3.17), which were the basic starting equations for the analysis of natural generalized symmetries. Nevertheless, if we use the linearity of \( h_{\alpha \rho}^{\lambda} \) in the variables \( \Gamma^l \), we can differentiate (5.2) with respect to \( \Psi^{k+2} \) and \( \Gamma^l \) to find that

\[ [\partial_l \partial_l \Psi h](Y, \psi^{l+1}; \psi^{k+2}, \overline{\psi}^{k-2}; \psi, \alpha, \alpha, \psi) = 0, \]

(5.14)

and

\[ [\partial_l \partial_l \Psi h](Y, \psi^{l+1}; \psi^{k-2}, \overline{\psi}^{k+2}; \psi, \alpha, \alpha, \psi) = 0. \]

(5.15)
5D. The $\Gamma^{l+1}\Psi^{k+1}$ Analysis, $l \geq k - 1$, $k \geq 2$.

Here we find, in a very straightforward manner, that

$$[\partial^l_k \partial^l_{\bar{h}} h](\psi^{k+2}, \overline{\psi}^{k-2}; Y, \omega^{l+1}; \alpha, \beta, \overline{\alpha}, \overline{\beta}) = 0,$$

(5.16)

and

$$[\partial^l_k \partial^l_{\bar{h}} h](\psi^{k-2}, \overline{\psi}^{k+2}; Y, \omega^{l+1}; \alpha, \beta, \overline{\alpha}, \overline{\beta}) = 0.$$

(5.17)

In deriving these equations we used (5.14) and (5.15).

5E. The $\Gamma^{l+1}\Gamma^l$ Analysis, $l \geq k - 1$, $k \geq 3$ and $l = 2$, $k = 2$.

We differentiate (5.2) with respect to $\Gamma^l$ and $\Gamma^{l+1}$. In the resulting equation we set $\beta = \alpha$, $\overline{\beta} = \overline{\alpha}$ and substitute from (5.10) to obtain

$$<\omega, \omega > \eta(\alpha, \overline{\alpha}) \{[\partial^{l-1}_l P](Y, \omega^l; Z, \eta^l, \alpha, \overline{\alpha}) - [\partial^{l-1}_l P](Z, \eta^l; Y, \omega^l, \alpha, \overline{\alpha})\}
+ 2\omega(\alpha, \overline{\alpha}) \{<\omega, \eta > [\partial^{l-1}_l P](Z, \eta_l; Y, \omega^l, \alpha, \overline{\alpha}) + [\partial^{l-1}_l \partial^{l-1}_l h](Z, \eta^l; Y, \omega^l; \alpha, \omega, \overline{\alpha})
+ \{[\partial^{l-1}_l h](Z, \eta^l; Y, \omega^l; \alpha, \eta, \overline{\alpha})\} = 0.$$

We multiply this equation by $\eta(\alpha, \overline{\alpha})$ and subtract from it the product of $\omega(\alpha, \overline{\alpha})$ with the result of interchanging $(Z, \eta)$ with $(Y, \omega)$ to deduce that

$$[\partial^{l-1}_l P](Z, \eta^l; Y, \omega^l; \alpha, \overline{\alpha}) = [\partial^{l-1}_l P](Y, \omega^l; Z, \eta_l, \alpha, \overline{\alpha}).$$

(5.18)

5F. A Partial Reduction in Order.

Equations (5.13), (5.16), (5.17), and (5.18) show that there is a vector field

$$X^A_A' = X^A_A'(x, \sigma, \Gamma^1, \ldots, \Gamma^{l-1}, \Psi^{k-1}, \overline{\Psi}^{k-1})$$

such that

$$[\partial^{l-1}_l X](Y, \omega^l; \alpha, \overline{\alpha}) = \frac{1}{2} P(Y, \omega^l; \alpha, \overline{\alpha}).$$

Hence the generalized symmetry

$$\tilde{h}^{AB}_{A'B'} = h^{AB}_{A'B'} - (\nabla^A_A' X^{B'}_B + \nabla^{B'}_B X^A_A')$$

is independent of the variables $\Gamma^l$, and accordingly we may now assume that the original generalized symmetry (5.1) is of the type

$$h^{AB}_{A'B'} = h^{AB}_{A'B'}(x, \sigma, \Gamma^1, \Gamma^2, \Psi^2, \overline{\Psi}^2, \ldots, \Gamma^{k-1}, \Psi^k, \overline{\Psi}^k).$$

(5.19)

This partial reduction in the order of $h^{AB}_{A'B'}$ is important because it enables us to repeat, almost without modification, the arguments of §3.
5G. Repetition of steps A through E and the natural symmetry analysis, \( l = k - 1, k \geq 3 \).

We now repeat steps A through E assuming \( h_{\psi}^{B} \) to be of the form (5.19), that is, with the \( \Gamma \) derivative-dependence reduced by one order. We can also repeat steps A and B of \( \S 3 \) to conclude that now

\[
[\partial_{\psi}^{k} h](\psi^{k+2}, \overline{\psi}^{k-2}; \alpha, \beta, \overline{\alpha}, \overline{\beta})
\]

\[
= <\psi, \alpha > <\psi, \beta > A(\psi^{k}, \overline{\psi}^{k-2}\overline{\alpha}\overline{\beta}) + <\overline{\psi}, \overline{\alpha} > <\overline{\psi}, \overline{\beta} > B(\psi^{k+2}\alpha\beta, \overline{\psi}^{k-4})
\]

\[
+ <\psi, \alpha > <\overline{\alpha}, \overline{\psi} > W(\psi^{k+1}, \overline{\psi}^{k-3}, \beta, \overline{\beta}) + <\psi, \beta > <\overline{\beta}, \overline{\psi} > W(\psi^{k+1}, \overline{\psi}^{k-3}, \alpha, \overline{\alpha})
\]

\[
[\partial_{\psi}^{k} h](\psi^{k-2}, \overline{\psi}^{k+2}; \alpha, \beta, \overline{\alpha}, \overline{\beta})
\]

\[
= <\overline{\psi}, \overline{\alpha} > <\overline{\psi}, \overline{\beta} > D(\overline{\psi}^{k}, \psi^{k-2}\alpha\beta) + <\psi, \alpha > <\psi, \beta > E(\overline{\psi}^{k+2}\alpha\overline{\beta}, \psi^{k-4})
\]

\[
+ <\overline{\psi}, \overline{\alpha} > <\alpha, \psi > U(\overline{\psi}^{k+1}, \psi^{k-3}, \beta, \overline{\beta}) + <\overline{\psi}, \overline{\beta} > <\beta, \psi > U(\overline{\psi}^{k+1}, \psi^{k-3}, \alpha, \overline{\alpha})
\]

and

\[
[\partial_{\psi}^{k-1} h](Y, \omega^{k}; \alpha, \omega, \overline{\alpha}) = -\frac{1}{2} <\omega, \omega > P(Y, \omega^{k-1}, \alpha, \overline{\alpha}).
\]

The coefficients \( A, B, W, D, E, U, \) and \( P \) are functions of the variables \( x, \sigma, \ldots, \Gamma^{k-2}, \Psi^{k-1}, \overline{\Psi}^{k-1} \). Note that steps \( \S 3A \) and \( \S 3B \) are valid even when \( k = 2 \).

Next we repeat step C of \( \S 3 \) to find that \( A, B, D, E \) are independent of the variables \( \Psi^{k-1} \) and \( \overline{\Psi}^{k-1} \). We also arrive at the integrability conditions (3.38), (3.41) and (3.51). Note that step \( \S 3C \) is valid even when \( k = 2 \).

5H. The \( \Gamma^{k-1}\Psi^{k+1}, \Gamma^{k-1}\overline{\Psi}^{k+1}, \Gamma^{k}\Psi^{k} \) and \( \Gamma^{k}\overline{\Psi}^{k} \) Analysis, \( k \geq 3 \).

The derivative of the linearized equation with respect to \( \Gamma^{k-1} \) and \( \Psi^{k+1} \) gives, after taking into account (3.16),

\[
2\omega(\overline{\psi}, \overline{\psi})[\partial_{\psi}^{k-2}\partial_{\overline{\psi}}^{k-2} h](Y, \omega^{k-1}; \psi^{k+2}, \overline{\psi}^{k-2}; \alpha, \beta, \overline{\alpha}, \overline{\beta})
\]

\[
+ <\alpha, \psi > <\overline{\alpha}, \overline{\psi} > [\partial_{\psi}^{k-2}\partial_{\overline{\psi}}^{k-2} h](Y, \omega^{k-1}; \psi^{k+2}, \overline{\psi}^{k-2}; \beta, \omega, \overline{\beta})
\]

\[
+ <\beta, \psi > <\overline{\beta}, \overline{\psi} > [\partial_{\psi}^{k-2}\partial_{\overline{\psi}}^{k-2} h](Y, \omega^{k-1}; \psi^{k+2}, \overline{\psi}^{k-2}; \alpha, \omega, \overline{\alpha})
\]

\[
+ <\alpha, \psi > <\overline{\alpha}, \overline{\psi} > [\partial_{\psi}^{k-1}\partial_{\overline{\psi}}^{k-1} h](\psi^{k+1}, \overline{\psi}^{k-3}; Y, \omega^{k}; \beta, \psi, \overline{\beta})
\]

\[
+ <\beta, \psi > <\overline{\beta}, \overline{\psi} > [\partial_{\psi}^{k-1}\partial_{\overline{\psi}}^{k-1} h](\psi^{k+1}, \overline{\psi}^{k-3}; Y, \omega^{k}; \alpha, \psi, \overline{\alpha}) = 0.
\]

In this equation we set \( \alpha = \beta = \psi \) and then \( \overline{\alpha} = \overline{\beta} = \overline{\psi} \) to deduce, in light of (5.20), that

\[
[\partial_{\psi}^{k-2} B](Y, \omega^{k-1}; \psi^{k+4}, \overline{\psi}^{k-4}) = 0 \quad \text{and} \quad [\partial_{\psi}^{k-2} A](Y, \omega^{k-1}; \psi^{k}, \overline{\psi}^{k}) = 0.
\]

Now we set \( \beta = \alpha \) and \( \overline{\beta} = \overline{\alpha} \) in (5.23); after substituting from (5.20) and (5.22) we find
that
\[
\frac{1}{2}\omega(\alpha, \overline{\psi})[\partial_{\psi}^{k-1}P](\psi^{k+1}, \overline{\psi}^{k-3}; Y, \omega^{k-1}, \psi, \overline{\alpha}) + \frac{1}{2}\omega(\psi, \overline{\alpha})[\partial_{\psi}^{k-1}P](\psi^{k+1}, \overline{\psi}^{k-3}; Y, \omega^{k-1}, \alpha, \overline{\psi})
\]
\[
- < \psi, \alpha > [\partial_{\Gamma}^{k-2}W](Y, \omega^{k-1}; \psi^{k+1}, \overline{\psi}^{k-3}, \psi, \overline{\alpha})
\]
\[
- < \overline{\psi}, \overline{\alpha} > [\partial_{\Gamma}^{k-2}W](Y, \omega^{k-1}; \psi^{k+1}, \overline{\psi}^{k-3}, \alpha, \psi \cdot \omega)
\]
\[
= 2\omega(\psi, \overline{\psi})[\partial_{\Gamma}^{k-2}W](Y, \omega^{k-1}; \psi^{k+1}, \overline{\psi}^{k-3}, \alpha, \overline{\alpha}).
\]
(5.25)

In this equation we have defined
\[
(\psi \cdot \omega)^{A'} = \omega^{A'}_{\alpha} \psi^{A} \quad \text{and} \quad (\overline{\psi} \cdot \omega)_{A} = \omega_{A}^{\alpha'} \overline{\psi}^{\alpha'}.
\]

Next we differentiate the linearized equation with respect to \(\Gamma^{k}\) and \(\Psi^{k}\), then set \(\alpha = \beta\) and \(\overline{\alpha} = \overline{\beta}\), and substitute from (5.20) and (5.22) to find
\[
\{\omega(\psi, \overline{\psi})\omega(\alpha, \overline{\alpha}) - \frac{1}{2} < \omega, \omega > < \psi, \alpha > < \overline{\psi}, \overline{\alpha} > [\partial_{\psi}^{k-1}P](\psi^{k+1}, \overline{\psi}^{k-3}; Y, \omega^{k-1}, \alpha, \overline{\alpha})
\]
\[
+ < \omega, \omega > < \psi, \alpha > < \overline{\psi}, \overline{\alpha} > [\partial_{\Gamma}^{k-2}W](Y, \omega^{k-1}; \psi^{k+1}, \overline{\psi}^{k-3}, \alpha, \overline{\alpha})
\]
\[
- \omega(\alpha, \overline{\alpha})\left\{\frac{1}{2}\omega(\alpha, \overline{\alpha})[\partial_{\psi}^{k-1}P](\psi^{k+1}, \overline{\psi}^{k-3}; Y, \omega^{k-1}, \psi, \overline{\alpha})
\]
\[
+ \frac{1}{2}\omega(\psi, \overline{\alpha})[\partial_{\psi}^{k-1}P](\psi^{k+1}, \overline{\psi}^{k-3}; Y, \omega^{k-1}, \alpha, \overline{\psi})
\]
\[
- < \psi, \alpha > [\partial_{\Gamma}^{k-2}W](Y, \omega^{k-1}; \psi^{k+1}, \overline{\psi}^{k-3}, \psi, \overline{\alpha})
\]
\[
- < \overline{\psi}, \overline{\alpha} > [\partial_{\Gamma}^{k-2}W](Y, \omega^{k-1}; \psi^{k+1}, \overline{\psi}^{k-3}, \alpha, \psi \cdot \omega)
\}
\]= 0.
(5.26)

The last four terms in this equation are precisely the four terms on the left-hand side of (5.25). Therefore, equations (5.25) and (5.26) lead to the integrability condition
\[
\frac{1}{2}[\partial_{\psi}^{k-1}P](\psi^{k+1}, \overline{\psi}^{k-3}; Y, \omega^{k-1}, \alpha, \overline{\alpha}) = [\partial_{\Gamma}^{k-2}W](Y, \omega^{k-1}; \psi^{k+1}, \overline{\psi}^{k-3}, \alpha, \overline{\alpha}).
\]
(5.27)

Similarly, an analysis of the \(\Gamma^{k-1}\overline{\psi}^{k-1}\) and \(\Gamma^{k}\overline{\psi}^{k}\) conditions proves that
\[
[\partial_{\Gamma}^{k-2}D](Y, \omega^{k-1}; \overline{\psi}^{k}, \psi^{k}) = 0 \quad \text{and} \quad [\partial_{\Gamma}^{k-2}E](Y, \omega^{k-1}; \overline{\psi}^{k+4}, \psi^{k-4}) = 0,
\]
(5.28)

and
\[
\frac{1}{2}[\partial_{\psi}^{k-1}P](\psi^{k-3}, \overline{\psi}^{k+1}; Y, \omega^{k-1}, \alpha, \overline{\alpha}) = [\partial_{\Gamma}^{k-2}U](Y, \omega^{k-1}; \overline{\psi}^{k+1}, \psi^{k-3}, \alpha, \overline{\alpha}).
\]
(5.29)

5I. Reduction in Order, \(k \geq 3\).

The integrability conditions (3.38), (3.41), (3.51), (5.27), and (5.29) show that there is a vector field
\[
X_{A'}^{A'} = X_{A'}^{A'}(x, \sigma, \ldots, \Gamma^{k-2}, \Psi^{k-1}, \overline{\Psi}^{k-1})
\]
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such that
\[
W(\psi^{k+1}, \overline{\psi}^{k-3}, \alpha, \overline{\alpha}) = [\partial_{\psi}^{k-1} X](\psi^{k+1}, \overline{\psi}^{k-3}; \alpha, \overline{\alpha})
\]
\[
U(\psi^{k-3}, \overline{\psi}^{k+1}, \alpha, \overline{\alpha}) = [\partial_{\overline{\psi}}^{k-1} X](\psi^{k-3}, \overline{\psi}^{k+1}; \alpha, \overline{\alpha})
\]
\[
\frac{1}{2} P(Y, \omega^{k-1}, \alpha, \overline{\alpha}) = [\partial_{\Gamma}^{k-2} X](Y, \omega^{k-1}; \alpha, \overline{\alpha}).
\]

Just as in §3, we set
\[
k_{AB}^{\alpha B} = h_{AB}^{\alpha B} - (\nabla_{A'}^{A} X_{B'}^{B} + \nabla_{B'}^{B} X_{A'}^{A}).
\]

Then
\[
k_{AB}^{\alpha B} = k_{AB}^{\alpha B}(x, \sigma, \Gamma^1, \Gamma^2, \overline{\psi}^2, \ldots, \Gamma^{k-2}, \Psi^k, \overline{\Psi}^k),
\]
and
\[
[\partial_{\psi}^{k} k](\psi^{k+2}, \overline{\psi}^{k-2}; \alpha, \beta, \overline{\alpha}, \overline{\beta})
\]
\[
= <\psi, \alpha > <\psi, \beta > A(\psi^k, \overline{\psi}^{k-2} \overline{\alpha} \overline{\beta}) + <\overline{\psi}, \overline{\alpha} > <\overline{\psi}, \overline{\beta} > B(\psi^{k+2} \alpha \beta, \overline{\psi}^{k-4})
\]
(5.31)
\[
[\partial_{\overline{\psi}}^{k} k](\psi^{k-2}, \overline{\psi}^{k+2}; \alpha, \beta, \overline{\alpha}, \overline{\beta})
\]
\[
= <\overline{\psi}, \overline{\alpha} > <\overline{\psi}, \overline{\beta} > D(\overline{\psi}^k, \psi^{k-2} \alpha \beta) + <\psi, \alpha > <\psi, \beta > E(\overline{\psi}^{k+2} \overline{\alpha} \overline{\beta}, \psi^{k-4}).
\]
(5.32)

Finally, we analyze the terms in the linearized equations involving $\Psi^{k+1}$ and $\overline{\Psi}^{k+1}$. To this end, it is convenient to set
\[
R(\psi^{k+2}, \overline{\psi}^{k-2}, \alpha, \beta, \overline{\alpha}, \overline{\beta})
\]
\[
= <\psi, \alpha > <\psi, \beta > A(\psi^k, \overline{\psi}^{k-2} \overline{\alpha} \overline{\beta}) + <\overline{\psi}, \overline{\alpha} > <\overline{\psi}, \overline{\beta} > B(\psi^{k+2} \alpha \beta, \overline{\psi}^{k-4}),
\]
and
\[
S(\psi^{k-2}, \overline{\psi}^{k+2}, \alpha, \beta, \overline{\alpha}, \overline{\beta})
\]
\[
= <\overline{\psi}, \overline{\alpha} > <\overline{\psi}, \overline{\beta} > D(\overline{\psi}^k, \psi^{k-2} \alpha \beta) + <\psi, \alpha > <\psi, \beta > E(\overline{\psi}^{k+2} \overline{\alpha} \overline{\beta}, \psi^{k-4}).
\]

Then equations (5.30)–(5.32) imply that
\[
k = R \cdot \Psi^k + S \cdot \overline{\Psi}^k + \tilde{k},
\]
where
\[
\tilde{k} = \tilde{k}(x, \alpha, \ldots, \Gamma^{k-2}, \Psi^1, \overline{\Psi}^1).
\]
The repeated covariant derivative of $k$ thus takes the form
\[
\nabla_{A'}^{A} \nabla_{B'}^{B} k = (\nabla_{A'}^{A} \nabla_{B'}^{B} R) \cdot \Psi^k + [(\nabla_{A'}^{A} R) \cdot \nabla_{B'}^{B} \Psi^k + (\nabla_{B'}^{B} R) \cdot \nabla_{A'}^{A} \Psi^k]
\]
\[
+ R \cdot (\nabla_{A'}^{A} \nabla_{B'}^{B} \Psi^k) + \nabla_{A'}^{A} \nabla_{B'}^{B} \tilde{k} + \{ \}.
\]
where \{\star\} denotes similar terms derived from \(S \cdot \Psi^k\). By (5.24) and (5.28), \(R\) and \(S\) depend upon \(x, \sigma, \ldots, \Gamma_{k-3}^k, \Psi_{k-2}^k\), and hence the derivatives \(\nabla_A^k \nabla_B^k R\) and \(\nabla_A^k \nabla_B^k S\) are independent of the variables \(\Psi^k\) and \(\Psi^k\). Moreover, we have that

\[
R \cdot \nabla_A^k \nabla_B^k \Psi^k = R \cdot \Psi^k + \{\star\},
\]

where \{\star\} denotes terms of order \(k\) in the Penrose fields. Hence \(R \cdot \nabla_A^k \nabla_B^k \Psi^k\) does not contain \(\Psi_{k+1}\) and \(\Psi_{k+1}\). Consequently, if we differentiate the linearized equation for \(k_{AB}^A A^B\) with respect to \(\Psi_{k+1}\) and set \(\alpha = \beta\) and \(\bar{\alpha} = \bar{\beta}\), we obtain

\[
(\text{Grad } R)(\psi, \bar{\psi}; \psi^k, \bar{\psi}^k) + 2 <\alpha, \psi > <\alpha, \bar{\psi} > [(\text{Div } R)(\psi^k, \bar{\psi}^k, \alpha, \bar{\alpha}) + (\partial_{\psi}^{k-1})^2(\psi^k, \bar{\psi}^k, \alpha, \psi, \bar{\psi}, \bar{\alpha})] = 0,
\]

where the covariant derivative operators Grad and Div are given by (3.11) and (3.59). With \(\alpha = \psi\) and \(\bar{\alpha} = \bar{\psi}\), we deduce from this equation the covariant constancy conditions

\[
(\text{Grad } A)(\psi, \bar{\psi}; \psi^k, \bar{\psi}^k) = 0,
\]

and

\[
(\text{Grad } B)(\psi, \bar{\psi}; \psi^k, \bar{\psi}^k) = 0.
\]

Just as in Proposition 3.2, equation (5.34) implies that \(A\) is independent of all the \(\Gamma, \Psi\), and \(\bar{\Psi}\) variables, that is,

\[
A = A(x, \sigma).
\]

But now, the covariant derivative of \(A\) takes the general form

\[
\nabla_{C'} A_{\cdots} = D_{C'} A_{\cdots} + \gamma_{C'} A_{\cdots} = \sigma_{C'} \sigma_{CBB'} \frac{\partial A}{\partial x^a} + \frac{\partial A}{\partial \sigma_{BB'}^a} \sigma_{BB'}^a + \gamma_{C'} A_{\cdots}.
\]

Since

\[
\sigma_{BB', a} = \Gamma_{ba}^e \sigma_{BB'} + \gamma_{ba} C_{BCB'} + \gamma_{B'}^e a C_{BCB'},
\]

we find that

\[
\nabla_{C'} A_{\cdots} = \Gamma_{ba}^e \sigma_{BB'}^a \sigma_{CBB'} + \{\star\},
\]

where \{\star\} indicates terms involving \(x, \sigma, \) and the spin connections \(\gamma\) and \(\bar{\gamma}\). It is now a simple matter to differentiate (5.34) with respect to \(\Gamma_{jk}^i\), keeping in mind that \(\Gamma_{jk}^i\) is independent of the spin connections, to arrive at

\[
\frac{\partial A}{\partial \sigma_{BB'}} = 0.
\]

At this point we can continue, as in the proof of Proposition 3.2, to deduce that \(A = 0\). Similarly, \(B, D,\) and \(E\) satisfy covariant constancy conditions that imply they too vanish.

We have now shown that a generalized symmetry of order \(k \geq 3\) is equivalent, up to a generalized diffeomorphism symmetry, to a generalized symmetry of order \(k - 1\) depending
on \(x, \sigma, \Gamma^i, i = 1, \ldots, k - 2\) and \(\Psi^j, \Psi^j, j = 2, \ldots, k - 1\). A straightforward induction argument then implies that any generalized symmetry of order \(k \geq 3\) is, up to a generalized diffeomorphism symmetry, given by a generalized symmetry of order 2 depending on \(x, \sigma, \Gamma^1, \Psi^2, \) and \(\Psi^2\). If the order of the original symmetry is \(k = 2\), then by repeating steps §5A through §5F the symmetry is again equivalent, modulo a diffeomorphism symmetry, to a symmetry of order 2 depending on \(x, \sigma, \Gamma^1, \Psi^2, \) and \(\Psi^2\).

5J. Reduction to First-Order Generalized Symmetries.

The induction argument of §5I shows that, modulo the generalized diffeomorphism symmetry, any generalized symmetry of order \(k \geq 2\) is equivalent to a symmetry \(h\) with the functional dependence

\[ h = h(x, \sigma, \Gamma^1, \Psi^2, \Psi^2). \]

Steps §5A through §5D, with \(l = 1\) and \(k = 2\), show that \(h\) takes the schematic form

\[ h = P(x, \sigma) \cdot \Gamma^1 + h_0(x, \sigma, \Psi^2, \Psi^2). \]

Steps §3A, §3B, and §3C show that

\[ h = P(x, \sigma) \cdot \Gamma^1 + A(x, \sigma) \cdot \Psi^2 + D(x, \sigma) \cdot \Psi^2 + k(x, \sigma). \]

The derivative of the linearized equations with respect to \(\Psi^3\) gives an equation similar to (5.33), which we write symbolically as

\[ \text{Grad}R + \text{Div}R + O(x, \sigma) = 0. \]

We can then repeat the arguments at the end of step §5I to conclude that \(A = 0\). A similar analysis of the terms involving \(\Psi^4\) in the linearized equations leads to \(D = 0\). Thus we reduce our analysis to first-order generalized symmetries, which were classified in §4 (see Theorem 4.4). We have now proven our main result.

**Theorem 5.1.** Let

\[ h_{ab} = h_{ab}(x^i, g_{ij}, g_{ij}, h_1, \ldots, g_{ij}, h_{k-1}) \]

be the components of a \(k\)-th-order generalized symmetry of the vacuum Einstein equations \(R_{ij} = 0\) in four spacetime dimensions. Then there is a constant \(c\) and a generalized vector field

\[ X^i = X^i(x^i, g_{ij}, g_{ij}, h_1, \ldots, g_{ij}, h_{k-1}) \]

such that, modulo the Einstein equations,

\[ h_{ab} = cg_{ab} + \nabla_a X_b + \nabla_b X_a. \]
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