Crossing Symmetry Violation of Unitarized Pion-Pion Amplitude in the Resonance Region

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Pion-pion scattering amplitude obtained from one-loop Chiral Perturbation Theory (ChPT) is crossing symmetric, however the corresponding partial wave amplitudes do not respect exact unitarity relation. There are different approaches to get unitarized partial wave amplitudes from ChPT. Here we consider the inverse amplitude method (IAM) that is often used to fit pion-pion phase shifts to experimental data, by adjusting free parameters. We measure the amount of crossing symmetry violation (CSV) in this case and we show that crossing symmetry is badly violated by the IAM unitarized ChPT amplitude in the resonance region. Important CSV also occurs when all free parameters are set equal to zero.

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I. INTRODUCTION

Even though Quantum Chromodynamics (QCD) has achieved a great success in describing strong interactions, low energy hadron physics must still be modeled phenomenologically. A great theoretical improvement was made by means of the method of ChPT [1], which is an effective theory derived from the basis of QCD. The method consists of writing down chiral Lagrangians for the physical processes and uses the conventional technique of the field theory for the calculations.

Here we focus on the simplest interaction to test ChPT ideas, namely the pion-pion scattering. For this process, the ChPT leading contribution (tree graphs) is of second order in the momenta \( p \) of the external pions and coincides with Weinberg result from current algebra [2]. The corrections come from loop diagrams whose vertices are of order \( p^2 \) and include a free-parameter polynomial part related to tree diagrams of order \( p^4 \); these parameters have to be obtained phenomenologically. At each order of calculation, the method yields a total amplitude that respects exact crossing symmetry, however, the corresponding partial waves satisfy only approximate elastic unitarity relation.

The unitarity violation is more severe at higher energies, so that it is not possible to reproduce resonant states, which are one of the most relevant features of the strong interacting regime. This is not a new issue in literature and many different methods have been proposed to improve this behavior. Here we consider IAM [3], that implements exact elastic unitarity for \( S^- \) and \( P^- \) partial waves. It allows one to access the resonance region for pion-pion scattering by fixing free parameters, but violates crossing symmetry.

Crossing symmetry interconnects the various isospin channels. In other words, if one fits IAM isospin \( I = 1 \) amplitude to the experimental data up the the \( \rho \) resonance region, for instance, due to crossing symmetry we do not have much freedom for \( I = 0 \) fitting. Therefore, if crossing symmetry is not fulfilled, a very constrained fit of \( S^- \)-wave would give meaningless result. It is thus important to investigate at what extent CSV occurs.

A possible test for crossing symmetry is the use of the so called Roskies relations [4], which involve weighted integrals of \( S^- \) and \( P^- \)-waves in the nonphysical region \( (4m^2_\pi > s > 0) \) and follow from first principles of analyticity, crossing and unitarity. Another test is given by the so called Martin inequalities [5]. They are constraints on \( \pi^0\pi^0 \) scattering amplitude also in the nonphysical region and follow from analyticity and positivity of the imaginary part of the amplitude. Both methods have been used in literature and show that Roskies relations violations are below 1.3% and all Martin inequalities are satisfied by low energy ChPT IAM pion-pion amplitude at one loop approximation level [6]. In fact, as unitarity corrections are small, we do expect small CSV at very low energies.

At this point, we would like to emphasize that IAM aims to explore the resonance region, where large unitarity corrections are needed and, accordingly, sizable CSV is expected. Therefore, in the present paper, we extend the test to that region. We do that by evaluating the effect of the IAM modified \( S^- \) and \( P^- \)-waves on the exact crossing symmetry relation for the total amplitude.

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Our work is presented as follows. In Sect. II we write the ChPT amplitude for pion-pion scattering and we construct IAM partial waves. We introduce a correction to get rid of sub-threshold poles by slightly shifting the original Adler zeros of the leading amplitudes. In Sect. III we present the method used to quantify the CSV of IAM amplitudes. It consists in comparing two expressions of the total amplitude, obtained from different combinations of isospin partial waves. In this section we define a violation function which is calculated and plotted for some values of energy and scattering angle. In Sect. IV we present our conclusions.

II. CHIRAL PERTURBATION THEORY AND THE IAM

We consider \( \pi^a \pi^b \rightarrow \pi^c \pi^d \) scattering amplitude

\[
\langle \pi^c \pi^d | T | \pi^a \pi^b \rangle = A(s, t, u)\delta^{ab}\delta^{cd} + B(s, t, u)\delta^{ac}\delta^{bd} + C(s, t, u)\delta^{ad}\delta^{bc}.
\]

The total isospin defined amplitudes \( T_I \) for \( I = 0, 1 \) and 2 are

\[
\begin{align*}
T_0(s, t) &= 3A(s, t, u) + B(s, t, u) + C(s, t, u), \\
T_1(s, t) &= B(s, t, u) - C(s, t, u), \\
T_2(s, t) &= B(s, t, u) + C(s, t, u),
\end{align*}
\]

(1)

Crossing symmetry implies that there is just one amplitude describing the three total isospin channels of the process, so that

\[
B(t, s, u) = A(s, t, u) = C(u, t, s).
\]

(2)

The amplitudes \( T_I \) are expanded in partial waves, as

\[
T_I(s, t) = \sum_{\ell} (2\ell + 1) t_{\ell I}(s) P_\ell(\cos \theta),
\]

(3)

where \( P_\ell \) are the Legendre polynomials, \( 2t = (s - 4m_\pi^2)(\cos \theta - 1) \), and \( u = 4m_\pi^2 - s - t \). Elastic unitarity implies that, for \( 16m_\pi^2 \geq s \geq 4m_\pi^2 \), there is a constraint given by

\[
\text{Im} t_{\ell I}(s) = \rho(s)|t_{\ell I}(s)|^2,
\]

which can be solved yielding

\[
t_{\ell I}(s) = \frac{1}{\rho(s)}e^{i\delta_{\ell I}(s)}\sin \delta_{\ell I}(s),
\]

(4)

where \( \delta_{\ell I}(s) \) are the real phase shifts and

\[
\rho(s) = \frac{1}{16\pi}\sqrt{\frac{s - 4m_\pi^2}{s}}
\]

is the phase space factor for pion-pion scattering.

Using ChPT at the one-loop level and considering only the most relevant low energy constants, the amplitude is given by

\[
f_\pi^4 A(s, t, u) = f_\pi^2(s - m_\pi^2) + \frac{1}{2} (s - m_\pi^2)^2 \bar{\mathcal{J}}(s)
\]

\[
+ \left[ \frac{1}{12} \left( 3(t - 2m_\pi^2)^2 + (s - u)(t - 4m_\pi^2) \right) \mathcal{J}(t) + (t \leftrightarrow u) \right]
\]

\[
+ \lambda_1 (s - 2m_\pi^2)^2 + \lambda_2 \left[ (t - 2m_\pi^2)^2 + (u - 2m_\pi^2)^2 \right].
\]

The resulting ChPT amplitudes for \( S \)-wave \( (I = 0 \) and \( 2 \) \) and \( P \)-wave \( (I = 1) \) can be written as

\[
t_{\ell I}(s) = t^a_{\ell I}(s) + t^b_{\ell I}(s) \bar{\mathcal{J}}(s) + t^{\text{eff}}_{\ell I}(s) + p_{\ell I}(s),
\]

(5)
amplitude phase shifts corresponding to the parameters $\lambda_1$ and $\lambda_2$. Instead of the exact ChPT result $t_{\ell I}$, we use a modified amplitude

$$t_{\ell I} = \frac{t_{\ell I}^{ca}(s)}{1 - (t_{\ell I}^{ca}(s) J(s) + t_{\ell I}^{left}(s) + p_{\ell I}(s))^2 / t_{\ell I}^{ca}(s)}.$$

We employ here a strategy widely used in literature [3], namely we choose the parameters $\lambda_1$ and $\lambda_2$ in order to fit S- and P-waves above to the experimental phase shifts, by using the definition (6). We show in Fig. 1 the resulting phase shifts corresponding to the parameters $\lambda_1 = -0.00345$ and $\lambda_2 = 0.01125$. 

where $t_{\ell I}^{ca}$ are the (real) Weinberg amplitudes, namely,

$$f_\pi^2 t_{10}^{ca}(s) = 2s - m_{\pi}^2, \quad f_\pi^2 t_{11}^{ca}(s) = \frac{1}{3} (s - 4m_{\pi}^2), \quad f_\pi^2 t_{02}^{ca}(s) = 2m_{\pi}^2 - s,$$

$t_{\ell I}^{left}$ are the parts that bear the left-hand cuts, namely,

$$f_\pi^4 t_{00}^{left}(s) = \frac{1}{12} \frac{m_{\pi}^4}{s - 4m_{\pi}^2} (6s + 25m_{\pi}^2) L(s)^2 - \frac{1}{144} (7s^2 - 40m_{\pi}^2 s + 75m_{\pi}^4) L(s)$$

$$+ \frac{1}{86} (95s^2 - 658m_{\pi}^2 s + 1454m_{\pi}^4),$$

$$f_\pi^4 t_{02}^{left}(s) = \frac{1}{12} \frac{m_{\pi}^4}{s - 4m_{\pi}^2} (3s + 2m_{\pi}^2) L(s)^2 - \frac{1}{144} (11s^2 - 32m_{\pi}^2 s + 6m_{\pi}^4) L(s)$$

$$+ \frac{1}{1728} (157s^2 - 494m_{\pi}^2 s + 580m_{\pi}^4),$$

$$f_\pi^4 (s - 4m_{\pi}^2) t_{11}^{left}(s) = \frac{1}{12} \frac{m_{\pi}^4}{s - 4m_{\pi}^2} (3s^2 - 13m_{\pi}^2 s - 6m_{\pi}^4) L(s)^2$$

$$+ \frac{1}{144} (s^3 - 16m_{\pi}^2 s^2 + 72m_{\pi}^4 s - 36m_{\pi}^6) L(s)$$

$$- \frac{1}{864} (7s^3 - 71m_{\pi}^2 s^2 + 427m_{\pi}^4 s - 840m_{\pi}^6), \text{ with}$$

$$J(s) = \frac{1}{2} - \frac{2}{\pi} \rho(s) L(s) + I \rho(s), \quad L(s) = \ln \frac{\sqrt{s - 4m_{\pi}^2 + \sqrt{s}}}{2m_{\pi}},$$

and $p_{\ell I}(s)$ are two free parameter polynomials, given by

$$f_\pi^4 p_{00}(s) = \frac{1}{3} \left( 11s^2 - 40s m_{\pi}^2 + 44m_{\pi}^4 \right) \lambda_1 + \frac{1}{3} \left( 14s^2 - 40s m_{\pi}^2 + 56m_{\pi}^4 \right) \lambda_2,$$

$$f_\pi^4 p_{11}(s) = \frac{1}{3} \left( s - 4m_{\pi}^2 \right) \left( \lambda_2 - \lambda_1 \right),$$

$$f_\pi^4 p_{02}(s) = \frac{2}{3} \left( s^2 - 2s m_{\pi}^2 + 4m_{\pi}^4 \right) \lambda_1 + \frac{2}{3} \left( 4s^2 - 14sm_{\pi}^2 + 16m_{\pi}^4 \right) \lambda_2.$$

If one wants to describe a resonant amplitude, one may wish to use Padé approximants, as e.g. advocated in [3]. It amounts to writing the inverse of the partial wave. Thus, instead of the exact ChPT result $t_{\ell I}$, we use a modified amplitude
As mentioned in the introduction, there was a problem concerning S-waves, namely that they were singular at some sub-threshold value for $s$, where the correction becomes equal to $\tilde{t}_I$. Singularities occur in S-wave sub-threshold amplitudes at $s_0 \approx 0.64 m_\pi^2$, for $I = 0$, and at $s_2 \approx 1.95 m_\pi^2$, for $I = 2$. Those values are close to the ones where $t_0$ and $t_2$ actually vanish. On the other hand, if one takes $\lambda_1 = \lambda_2 = 0$ the singularities move to $s_0 \approx 0.51 m_\pi^2$, for $I = 0$, and to $s_2 \approx 1.99 m_\pi^2$, for $I = 2$. In order to get rid of those singularities, we performed an extra correction, thus obtaining a new partial wave amplitude, denoted by $\tilde{t}^{(n)}_I$,

$$\tilde{t}^{(n)}_I(s) = \frac{\alpha_I(s - s_I)/f_2^2}{1 - \left(t_0^{ca}(s) \bar{J}(s) + t_2^{ca}(s) + p_I(s)/t_0^{ca}(s)\right)/t_I(s)}$$,  

where $\alpha_0 = 2$ and $\alpha_2 = -1$.

### III. CROSSING SYMMETRY VIOLATION

We present here a method to quantify CSV of the IAM pion pion scattering amplitude in the resonance region. Since IAM only modifies $S-$ and $P-$waves, we start by separating in Eq. (3) the $\ell = 0$ and $\ell = 1$ contributions:

$$T_0(s, t) = t_{00}(s) + \sum_{\ell=2}^{\infty} (2\ell + 1)t_{0\ell}(s)P_\ell(\cos \theta),$$  

$$T_2(s, t) = t_{02}(s) + \sum_{\ell=2}^{\infty} (2\ell + 1)t_{2\ell}(s)P_\ell(\cos \theta),$$  

$$T_1(s, t) = 3 t_{11}(s) \cos \theta + \sum_{\ell=3}^{\infty} (2\ell + 1)t_{1\ell}(s)P_\ell(\cos \theta).$$

In general, the total amplitudes $A(s, t)$ and $B(s, t)$ can be reconstructed from the isospin defined amplitudes above, considering Eqs. (10), which gives

$$A(s, t) = \frac{1}{3} [T_0(s, t) - T_2(s, t)],$$  

$$B(s, t) = \frac{1}{2} [T_1(s, t) + T_2(s, t)].$$

FIG. 1. Results from fits of IAM amplitudes to (a) $P-$ and (b) $S-$wave phase shifts, in degrees, as functions of cms energy, in GeV. Experimental data for $P-$wave are from Ref. [9]; for $S-$wave, from Refs. [9-11].
Thus we have
\[
\tilde{A}(s, t) = \frac{1}{3} (\tilde{t}_{00}(s) - \tilde{t}_{02}(s)) + \left[ A(s, t) - \frac{1}{3} (t_{00}(s) - t_{02}(s)) \right],
\]
\[
\tilde{B}(s, t) = \frac{1}{2} \left( 3 \tilde{t}_{11}(s) \cos \theta + \tilde{t}_{02}(s) \right) + \left[ B(s, t) - \frac{1}{2} (3 t_{11}(s) \cos \theta + t_{02}(s)) \right],
\]
where the tilde indicates that these amplitudes are built from the IAM modified partial waves.

According to Eq. \(2\), if crossing symmetry were respected, one should have \(\tilde{B}(s, t) = \tilde{A}(t, s)\). Thus the difference between these two quantities is a measure of the symmetry violation. We remark that \(B(s, t)\) is a combination of \(P\)\(-\) and \(S\)\(-\) waves while \(A(t, s)\) is reconstructed from \(S\)\(-\) wave amplitudes only. In order to put some scale in this measure, we divide this difference by their sum, so that the violation is defined as
\[
\Delta(s, \cos \theta) = 100% \times \frac{|\tilde{B}(s, t) - \tilde{A}(t, s)|}{|\tilde{B}(s, t)| + |\tilde{A}(t, s)|}.
\]
This formula consists in a simple way to translate CSV into numbers. A similar formula was used by Boglione and Pennington \(8\) and by Hannah \(6\) to quantify the violation of Roskies relations in pion-pion scattering amplitude, mentioned in the introduction. We modified the denominator in order to constraint the amount of violation to 100%.

We have shown that data fitting requires the partial wave polynomial parts to be adjusted. However, as IAM represents a structural modification of the amplitudes, it should violate crossing independently of its polynomial part. Therefore, let us first consider the case \(\lambda_1 = \lambda_2 = 0\).

In Fig. 2 we have plotted \(\Delta\) for some values of scattering angle as functions of cms energy, for the cases with vanishing parameters and with adjusted values. For the case when the parameters are kept equal to zero, the curves show the same trend, while, when the adjusted parameters are employed, CSV for the forward scattering presents a peculiar behavior. Concerning the strength of violation, one notices that the use of adjusted parameters increases \(\Delta\).

![FIG. 2. Crossing symmetry violation as a function of cms energy, in MeV, for \(\theta = 0^0\) (solid), 90\(^0\) (dashed) and 180\(^0\) (dotted) for (a) \(\lambda_1 = \lambda_2 = 0\) and for (b) \(\lambda_1 = -0.00345\) and \(\lambda_2 = 0.01125\).](image)

**IV. CONCLUSIONS**

The \(\mathcal{O}(p^4)\) ChPT pion-pion amplitude is crossing symmetric but does not respect exact elastic unitarity. There are several attempts to extrapolate the domain of validity of ChPT and to access the resonance region for meson-meson scattering. IAM is one of these methods, which fixes some free parameters in order to fit Padé approximants of ChPT amplitude to the experimental data. The parameters are fixed by means of a simultaneous fit of \(S\)\(-\) and \(P\)\(-\) waves. One knows that the interdependence of the fits relies on the interconnection among different isospin amplitudes, which
is due to crossing symmetry. However, IAM leads to unitary amplitudes, but they violate crossing symmetry. As a consequence, the parameter fitted from $P$–wave may be meaningless for the $S$–wave fit, since the constraint among partial waves is somehow lost. In this sense, the amount of CSV is an indication of the lost of reliability of the fits performed.

CSV of IAM has been quantified before by means of the Roskies relations and Martin inequalities. Both methods found very small violations at sub-threshold energies. In this paper we presented a method to quantify the CSV that IAM, extended to the resonance region, implies.

We measured the violation for two cases. As IAM produces a structural modification of ChPT results, any choice of parameters will yield CSV. Thus we first considered $\lambda_1 = \lambda_2 = 0$ and as a result we obtained very small CSV at threshold, as expected, while values up to roughly 50% develop around the $\rho$ mass region, for any scattering angle, as shown in Fig. 2a. This can be considered as a measurement of the intrinsic CSV of IAM. On the other hand, using the parameters fixed in order to fit the amplitudes to experimental data, the violation is still small near threshold, but gets much larger at higher energies, as shown in the Fig. 2b. This result can be taken as a measure of the price one pays for imposing elastic unitarity on ChPT amplitudes far from threshold.

Our results show that it is not possible for ChPT to exactly fulfill both crossing symmetry and unitarity requirements. We recall that elastic unitarity constrains partial waves in a small energy range (in this case, up to 560 MeV) in contrast with crossing symmetry of the total amplitude. An alternative method to keep exact crossing symmetry is to fit pure ChPT partial wave amplitudes in Eq. 5 to experimental data. In this case very large unitarity violations in the resonance region occur. In other words, by introducing elastic unitarity, a lot of crossing symmetry is lost, as well as keeping the latter costs a large amount of the former.

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