EXISTENCE AND SYMMETRY OF SOLUTIONS TO 2-D SCHRÖDINGER-NEWTON EQUATIONS

DAOMIN CAO, WEI DAI, YANG ZHANG

ABSTRACT. In this paper, we consider the following 2-D Schrödinger-Newton equations

\[-\Delta u + a(x)u + \frac{\gamma}{2\pi} (\log(|\cdot|) * |u|^p) |u|^{p-2}u = b|u|^{q-2}u \quad \text{in } \mathbb{R}^2,\]

where \(a \in C(\mathbb{R}^2)\) is a \(\mathbb{Z}^2\)-periodic function with \(\inf_{\mathbb{R}^2} a > 0\), \(\gamma > 0\), \(b \geq 0\), \(p \geq 2\) and \(q \geq 2\).

By using ideas from [13, 21, 42], under mild assumptions, we obtain existence of ground state solutions and mountain pass solutions to the above equations for \(p \geq 2\) and \(q \geq 2\) via variational methods. The auxiliary functional \(J_1\) plays a key role in the cases \(p \geq 3\). We also prove the radial symmetry of positive solutions (up to translations) for \(p \geq 2\) and \(q \geq 2\).

The corresponding results for planar Schrödinger-Poisson systems will also be obtained. Our theorems extend the results in [13, 21] from \(p = 2\) and \(b = 1\) to general \(p \geq 2\) and \(b \geq 0\).

Keywords: Logarithmic convolution potential, variational methods, Schrödinger-Newton equations, Schrödinger-Poisson systems, positive solutions, radial symmetry.

2010 MSC Primary: 35J20; Secondary: 35Q40, 35B09, 35B06.

1. INTRODUCTION

In this paper, we consider the following 2-D Schrödinger-Newton equations

\[-\Delta u + a(x)u + \frac{\gamma}{2\pi} (\log(|\cdot|) * |u|^p) |u|^{p-2}u = b|u|^{q-2}u \quad \text{in } \mathbb{R}^2,\]

where \(a \in C(\mathbb{R}^2)\) is a \(\mathbb{Z}^2\)-periodic function with \(\inf_{\mathbb{R}^2} a > 0\), \(\gamma > 0\), \(b \geq 0\), \(p \geq 2\) and \(q \geq 2\).

Solutions to Schrödinger-Newton equations (1.1) also solve the following Schrödinger-Poisson systems (also called Schrödinger-Maxwell systems)

\[
\begin{cases}
-\Delta u + a(x)u + \gamma w|u|^{p-2}u = b|u|^{q-2}u & \text{in } \mathbb{R}^2, \\
\Delta w = |u|^p & \text{in } \mathbb{R}^2.
\end{cases}
\]

Schrödinger-Newton equations (1.1) and Schrödinger-Poisson systems (1.2) have numerous important applications in physics and model many phenomena in quantum mechanics (see [4, 22, 27, 29, 34, 39]). In general dimensions \(\mathbb{R}^d\), solution \(u\) to Schrödinger-Poisson systems gives a standing (or solitary) wave solution \(\psi(x, t) = e^{-i\lambda t} u(x) (\lambda \in \mathbb{R})\) of dynamic Schrödinger-Poisson systems of the type

\[
\begin{cases}
i\partial_t \psi - \Delta \psi + E(x)\psi + \gamma w|\psi|^{p-2}\psi = b|\psi|^{q-2}\psi & \text{in } \mathbb{R}^d \times \mathbb{R}, \\
\Delta w = |\psi|^p & \text{in } \mathbb{R}^d \times \mathbb{R},
\end{cases}
\]

where \(E(x) = a(x) - \lambda\) is a real external potential, the function \(w\) represents an internal potential for a nonlocal self-interaction of the wave function \(\psi\).

D. Cao was supported by NNSF of China (No. 11831009) and Chinese Academy of Sciences (No. QYZDJ-SSW-SYS021). W. Dai was supported by the NNSF of China (No. 11971049), the Fundamental Research Funds for the Central Universities and the State Scholarship Fund of China (No. 201806025011).
For space dimension \( d = 3 \), equations (1.1) and (1.2) have been extensively studied. In the cases \( p = 2 \), \( \gamma > 0 \) and \( b = 0 \), equations (1.1) and (1.2) were introduced by Pekar [38] in 1954 to describe the quantum mechanics of a polaron at rest and by Choquard in 1976 to describe an electron trapped in its hole. In [39], Penrose obtained (1.1) in his discussion about the self gravitational collapse of a quantum-mechanical system. Lieb [27] derived the existence of a unique ground state solution which is positive and radially symmetric, by using a minimization argument. Lions [29] proved in [29] the existence of infinitely many distinct radially symmetric solutions when \( a(x) \) is a nonnegative and radially symmetric potential. In the cases \( b \neq 0 \), there are also a large amount of papers devoted to the study of existence of solutions for (1.1) and (1.2). For results on the cases \( \gamma < 0 \), \( b > 0 \) and \( 2 < q < 6 \), please refer to [2, 40]. In the cases \( \gamma > 0 \) and \( b < 0 \), equation (1.2) represents a Hartree model for crystals (see [37]). For more related results in three and higher dimensions, please refer to [4, 3, 7, 8, 15, 16, 17, 19, 22, 35, 36, 33, 43] and the references therein.

In this paper, we will investigate the existence of ground state solutions and symmetry of positive solutions for (1.1) and (1.2) in two dimension case. For space dimension \( d = 2 \), there are very few literature (see [5, 11, 12, 13, 21, 42]). Unlike higher dimension cases \( d \geq 3 \), since the corresponding energy functional is not well-defined on \( H^1(\mathbb{R}^2) \), variational methods have rarely been used in the planar case. For the cases that \( a \geq 0 \) is a constant, \( \gamma > 0 \), \( p = 2 \) and \( b = 0 \), Stubbete [42] set up a variational framework for (1.1) within a subspace of \( H^1(\mathbb{R}^2) \). Among other things, he derived the existence of a unique ground state solution which is a positive spherically symmetric decreasing function, by using strict rearrangement inequalities. For the cases \( a \in C(\mathbb{R}^2) \) is a \( \mathbb{Z}^2 \)-periodic function with \( \inf_{\mathbb{R}^2} a > 0 \), \( \gamma > 0 \), \( b \geq 0 \), \( p = 2 \) and \( q \geq 4 \), by developing new ideas and estimates within the variational framework, Cingolani and Weth [13] proved the existence of ground states and high energy solutions for (1.1) and (1.2). Their key tool is a surprisingly strong compactness condition for Cerami sequences which is not available for the corresponding problem in higher space dimensions. Subsequently, Du and Weth [21] removed the restriction \( q \geq 4 \) in [13] and derived the existence of ground states and high energy solutions for (1.1) and (1.2), by exploring the more complicated underlying functional geometry in the case \( 2 < q < 4 \) with a different variational approach.

Following ideas from [13, 42], we will apply variational methods to study the existence of ground state solutions to 2-D Schrödinger-Newton equation (1.1) with general \( p \geq 2 \) and \( q \geq 2p \). More precisely, we will consider the following energy functional:

\[
I(u) := \frac{1}{2} \int_{\mathbb{R}^2} (|\nabla u|^2 + a(x)u^2) \, dx + \frac{\gamma}{4p\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(|x-y|)|u|^p(x)|u|^p(y) \, dx \, dy \\
- \frac{b}{q} \int_{\mathbb{R}^2} |u|^q \, dx,
\]

which is not well-defined on \( H^1(\mathbb{R}^2) \). Inspired by Stubbete [42], we will consider \( I \) in the smaller Banach subspace:

\[
X := \left\{ u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} \log(1 + |x|)|u|^p(x) \, dx < \infty \right\} = H^1 \cap L^p(d\mu)
\]

with the Radon measure \( d\mu := \log(1 + |x|) \, dx \). By Lemma 2.3 in Section 2, we can see that the energy functional \( I \) defines a \( C^1 \)-functional on \( X \), moreover, Schrödinger-Newton equation (1.1) is the corresponding Euler-Lagrange equation for \( I \), critical points \( u \in X \) of \( I \) are strong solutions of (1.1) in \( W^{2,r}(\mathbb{R}^2) \) for all \( r \geq 1 \), and they are classical solutions in \( C^2(\mathbb{R}^2) \) if \( a \) is Hölder continuous. Thus \( X \) provides a variational framework for (1.1). Nevertheless, we
should note that the norm of $X$ is not translation invariant, while the energy functional $I$ is invariant under any $\mathbb{Z}^2$-translations. Thus there will be more additional difficulties than higher dimension cases $d \geq 3$.

By using the methods from [13], we can prove that the ground state energy

\begin{equation}
(1.6) \quad c_g := \inf \{ I(u) : u \in X \setminus \{0\}, I'(u) = 0 \}
\end{equation}

can be attained, and hence derive the ground state solutions for (1.1). Our first main result is the following theorem.

**Theorem 1.1.** Assume $p \geq 2$ and $q \geq 2p$. Then (1.1) admits a pair of ground state solutions $\pm u \in X \setminus \{0\}$ such that $I(u) = c_g$. Moreover, the restriction of $I$ to the associated Nehari manifold $\mathcal{N} := \{ u \in X \setminus \{0\} : I'(u), u) = 0 \}$ admits a global minimum and each minimizer $u \in \mathcal{N}$ of $I|_\mathcal{N}$ is also a (ground state) solution to (1.1) which doesn’t change sign and obeys the mini-max characterization

\[ I(u) = \inf_{u \in X} \sup_{t \in \mathbb{R}} I(tu). \]

Next, assuming that $a(x) = a > 0$ is a constant, $q \geq 2p - 2$ and $q > 2$, we investigate the existence of mountain pass and ground state solutions to the Schrödinger-Newton equations (1.1). To this end, we define the mountain pass value

\begin{equation}
(1.7) \quad c_{mp} = \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} I(\gamma(t)), \quad \Gamma = \{ \gamma \in C([0, 1]; X) \mid \gamma(0) = 0, I(\gamma(1)) < 0 \}.
\end{equation}

Inspired by ideas from [40] (see also [21]), we define the auxiliary functionals $J_k : X \to \mathbb{R}$ $(k = 1, 2)$ by

\begin{equation}
(1.8) \quad J_k(u) := \int_{\mathbb{R}^2} \left( k|\nabla u|^2 + (k - 1)au^2 - \frac{(kq - 2)b}{q} |u|^q \right) dx - \frac{\gamma}{4\pi p} \left( \int_{\mathbb{R}^2} |u|^p dx \right)^2 \\
+ \frac{\gamma(kp - 2)}{2p\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(|x-y|)|u|^p(x)|u|^p(u) dx dy.
\end{equation}

Our second main result is the following theorem.

**Theorem 1.2.** Let $a(x) = a > 0$ be a constant, $p \geq 2$, $q \geq 2p - 2$ and $q > 2$. For $2 \leq p < 3$, assume further $q \geq 2p - 1$ or $q < p + 1$.

(i) We have $c_{mp} > 0$, and equation (1.1) has a pair of solutions $\pm u \in X \setminus \{0\}$ such that $I(u) = c_{mp}$.

(ii) Equation (1.1) admits a pair of ground state solutions $\pm u \in X \setminus \{0\}$ such that $I(u) = c_g$.

(iii) Assume further that $q \geq 2p - 1$ if $2 \leq p < 3$, then $c_g = c_{mp}$. Moreover, let $k = 1$ if and only if $p \geq 3$ and $k = 2$ if and only if $2 \leq p < 3$, then the restriction of $I$ to the set $\mathcal{M}_k := \{ u \in X \setminus \{0\} \mid J_k(u) = 0 \}$ admits a global minimum and each minimizer $u \in \mathcal{M}_k$ of $I|_{\mathcal{M}_k}$ is also a (ground state) solution to (1.1) which doesn’t change sign and obeys the mini-max characterization

\[ I(u) = \inf_{u \in X \setminus \{0\}} \sup_{t > 0} I(tu_{t,k}), \]

where $u_{t,k} \in X$ is defined by $u_{t,k}(x) := t^k u(tx)$ for any $u \in X \setminus \{0\}$ and $t > 0$.

**Remark 1.3.** Being different from [40] and [21], besides using the auxiliary functional $J_2$ in the cases $2 \leq p < 3$, we also define the auxiliary functional $J_1$ and use it to deal with the cases $p \geq 3$ and $q \geq 2p - 2$. Please see the crucial lemma Lemma 5.3.
Assume Theorem 1.6. For $2 \leq p < 3$, we do not know whether $c_q = c_{mp}$ and there exists a ground state solution that may change sign or not in the cases $2p - 2 \leq q < 2p - 1$.

By Lemma 2.4 in Section 2, we can see that every solution $u \in X$ to Schrödinger-Newton equations (1.1) also gives a solution $(u, w)$ with $w := \frac{1}{\pi^2} \int_{\mathbb{R}^d} \log(|x - y|) |u|^p(y)dy$ to Schrödinger-Poisson systems (1.2). As a consequence of Theorem 1.1, Theorem 1.2 and Lemma 2.4, we derive the existence of ground state solutions for (1.2).

**Corollary 1.5.** Let $p \geq 2$, $q \geq 2p - 2$ and $q > 2$. Assume further that $a(x) = a > 0$ is a constant if $q < 2p$, and $q \geq 2p - 1$ or $2p - 2 \leq q < p + 1$ if $2 \leq p < 3$. Then (1.2) admits a pair of ground state solutions $(\pm u, w)$ with $\pm u \in X$ satisfying $I(u) = c_q$. Assume further that $q \geq 3$ if $2 \leq p < \frac{5}{2}$, then the ground state solution $(u, w)$ doesn’t change sign. Moreover, $u \in L^\infty(\mathbb{R}^2)$ decays faster than exponential functions and $w \to +\infty$ as $|x| \to \infty$, if $a$ is Hölder continuous, then $(u, w)$ is a classical solution to (1.2). The rest of conclusions in Theorem 1.1 and Theorem 1.2 also hold for Schrödinger-Poisson systems (1.2).

Since we have already derived the existence of positive solutions to (1.1) and (1.2), it is natural for us to classify positive solutions via their geometric properties. First, by using the method of moving planes (see [7, 9, 10, 13, 15, 16, 19, 23, 33]), we will study the symmetry property of classical solutions $(u, w)$ to the following generalized Schrödinger-Poisson systems

$$
\begin{cases}
-\Delta u - \gamma w |u|^{p-2} u = f(u) & \text{in } \mathbb{R}^2, \\
-\Delta w = |u|^p & \text{in } \mathbb{R}^2,
\end{cases}
$$

subject to the conditions

$$0 < u \in L^\infty(\mathbb{R}^2) \quad \text{and} \quad w(x) \to -\infty, \quad \text{as } |x| \to \infty.
$$

We have the following result on symmetry of classical solutions for (1.9) and (1.10).

**Theorem 1.6.** Assume $p \geq 2$ and $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz with $f(0) = 0$. If $p > 2$, we assume further that there exist some $\epsilon_0 > 0$ and $a_0 > 0$ small such that

$$
sup_{0 < x \neq y < \epsilon_0} \frac{f(x) - f(y)}{x - y} \leq -a_0 < 0.
$$

Then every classical solution $(u, w)$ of (1.9), (1.10) is radially symmetric and strictly decreasing with respect to some $x_0 \in \mathbb{R}^2$.

**Remark 1.7.** If the locally Lipschitz continuous function $f(u) = b|u|^{q-2}u - au$ with $q \geq 2$ and constant $a > 0$, then we can derive from Theorem 1.6 the symmetry of classical solutions $(u, w)$ for Schrödinger-Poisson systems (1.2) subject to the conditions (1.10).

As a consequence of Theorem 1.6 we derive the symmetry of positive solutions for Schrödinger-Newton equations (1.1).

**Corollary 1.8.** Assume $p \geq 2$, $q \geq 2$ and $a$ is a positive constant in (1.1). Then every positive solution $u \in X$ of (1.1) is radially symmetric and strictly decreasing with respect to some $x_0 \in \mathbb{R}^2$.

**Remark 1.9.** In [13], Cingolani and Weth have proved radial symmetry (up to translation) of positive solutions to (1.1) and (1.9) for $p = 2$. Our Theorem 1.6 and Corollary 1.8 extend the symmetry results in [13] from $p = 2$ to general $p \geq 2$. The corresponding results for higher dimensions $d \geq 3$ can be found in [7, 15, 16, 17, 19, 27, 34, 33]. We should note that the 2-D
case is quite different from higher dimension cases \( d \geq 3 \), since the logarithmic convolution kernel is sign-changing.

The rest of this paper is organized as follows. In Section 2, we will set up the variational framework and establish some useful preliminary lemmas and estimates for energy functional \( I \) and function space \( X \). We establish the key strong compactness condition for Cerami sequences (up to \( \mathbb{Z}^2 \)-translations) and the quantitative deformation lemma in Section 3 (see Theorem 3.1 and Lemma 3.8). Section 4 is devoted to proving Theorem 1.1. In Section 5, we will prove Theorem 1.2. In Section 6, we carry out the proof of Theorem 1.6 and Corollary 1.8.

2. Preliminaries

In this section, we will give some necessary preliminary knowledge for energy functional \( I(u) \) and function space \( X \). In the following, we assume that \( a \in L^\infty(\mathbb{R}^2) \) satisfies \( \inf_{\mathbb{R}^2} a > 0 \).

First, we introduce some basic notations. The function space \( X \) is a Banach space equipped with the norm

\[
\|u\|_X := \|u\|_{H^1(\mathbb{R}^2)} + \|u\|_*,
\]

where

\[
\|u\|_* := \|u\|_{L^p(\mu)} = \left( \int_{\mathbb{R}^2} \log (1 + |x|)|u|^p\,dx \right)^{\frac{1}{p}} = \left( \int_{\mathbb{R}^2} |u|^p\,d\mu \right)^{\frac{1}{p}},
\]

and

\[
\|u\|_{H^1} := \sqrt{\langle u, u \rangle_{H^1}} = \left( \int_{\mathbb{R}^2} (|\nabla u|^2 + a(x)u^2)\,dx \right)^{\frac{1}{2}}
\]

with the \( H^1(\mathbb{R}^2) \) equivalent inner product given by

\[
\langle u, v \rangle_{H^1} := \int_{\mathbb{R}^2} (\nabla u \cdot \nabla v + a(x)uv)\,dx.
\]

Now we define the following three bilinear functionals:

\[
B_1(f, g) := \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log (1 + |x - y|) f(x)g(y)\,dxdy,
\]

\[
B_2(f, g) := \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log \left( 1 + \frac{1}{|x - y|} \right) f(x)g(y)\,dxdy,
\]

\[
B_0(f, g) := B_1(f, g) - B_2(f, g) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(|x - y|) f(x)g(y)\,dxdy.
\]

By Hardy-Littlewood-Sobolev inequality, \( B_2(f, g) \in L^\infty(\mathbb{R}^2) \) and has the following upper bound:

\[
|B_2(f, g)| \leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{|x - y|}|f(x)g(y)|\,dxdy \leq C\|f\|_{L^4(\mathbb{R}^2)}\|g\|_{L^4(\mathbb{R}^2)}.
\]

Correspondingly, we define the following functionals associated to the above bilinear forms:

\[
V_1(u) := B_1(|u|^p, |u|^p) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log (1 + |x - y|)|u|^p(x)|u|^p(y)\,dxdy,
\]
and lower semi-continuous on $H$

Proof. Proof of (1):
By Rellich’s compact embedding theorem (see [6]), we only need to show

\begin{equation}
I(2.13) = \frac{1}{2} \int_{\mathbb{R}^2} \log \left(1 + \frac{1}{|x - y|}\right) |u|^p |v|^p \, dx \, dy,
\end{equation}

which indicates that $V_2$ is well-defined for $u \in L^\frac{4}{3} (\mathbb{R})$. Since $p \geq 2$ and $H^1 (\mathbb{R}) \hookrightarrow L^r (\mathbb{R})$ for each $2 \leq r < +\infty$, it follows from (2.12) that $V_2 (u)$ is well-defined for $H^1$ functions.

Using these notations, we can rewrite the energy functional in the following form:

\begin{equation}
I(u) = \frac{1}{2} \|u\|_{H^1 (\mathbb{R})}^2 + \frac{\gamma}{4p\pi} V_0 (u) - \frac{b}{q} \|u\|_{L^q (\mathbb{R})}^q.
\end{equation}

Next, we will give several useful lemmas.

**Lemma 2.1.** Given any $0 < p < +\infty$. For any $\epsilon > 0$, there is a constant $C_{\epsilon, p} > 0$ such that, for all $a, b \in \mathbb{C}$,

\[ |a + b|^p - |b|^p \leq \epsilon |b|^p + C_{\epsilon, p} |a|^p. \]

The proof of Lemma 2.1 is elementary, see [28] for example.

**Lemma 2.2.** (P. L. Lions, [30], 1984) Let $r > 0$ and $2 \leq q < 2^*$ ($2^* := \frac{2d}{d-2}$ if $d \geq 3$, $2^* = +\infty$ if $d = 2$). If $(u_n)_n$ is bounded in $H^1 (\mathbb{R})$ and

\[
\lim_{n \to \infty} \sup_{y \in \mathbb{R}^d} \int_{B_r (y)} |u_n|^q \, dx = 0,
\]

then $u_n \to 0$ in $L^p (\mathbb{R}^d)$ for all $2 < p < 2^*$.

For the proof of Lemma 2.2, we refer to [30] or [44].

**Lemma 2.3** (Basic properties for the functionals $I, V_i$ and the function space $X$). We have the following properties:

1. $X = H^1 \cap L^p (\mu)$ is compactly embedded in $L^s (\mathbb{R})$ for all $s \in [p, +\infty]$;
2. The functionals $V_1, V_2, V_0, I \in C^1 (X, \mathbb{R})$, moreover,

\[
\langle V'_i (u), v \rangle = 2pB_i (|u|^p, |u|^{p-2} uv)
\]

for every $u, v \in X$ and $i = 0, 1, 2$;

3. $V_1$ is weakly lower semi-continuous on $H^1 (\mathbb{R})$, $I$ is weakly lower semi-continuous on $X$ and lower semi-continuous on $H^1 (\mathbb{R})$.

**Proof.** Proof of (1): By Rellich’s compact embedding theorem (see [6]), we only need to show the uniformly $L^s$-integrability for any bounded sequence $\{u_n\} \subset X$. Indeed, since there exists a $M > 0$ such that

\begin{equation}
\int_{|x| \geq R} \log (1 + R) |u_n|^p \, dx \leq \int_{|x| \geq R} \log (1 + |x|) |u_n|^p \, dx \leq M,
\end{equation}

one can infer that for any $\epsilon > 0$, there exists a $R(\epsilon) > 0$ sufficiently large such that, for any $R > R(\epsilon)$,

\begin{equation}
\int_{|x| \geq R} |u_n|^p \, dx \leq \frac{M}{\log (1 + R)} \leq \epsilon,
\end{equation}

which completes the proof.
from which the uniformly $L^s$-integrability follows immediately. Hence, the embedding $X \hookrightarrow L^s(\mathbb{R}^2)$ is compact for all $s \in [p, +\infty)$.

**Proof of (2):** We first show that $V_1$ is well-defined on the function space $X$:

\begin{equation}
\begin{aligned}
|V_1(u)| &= \left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(1 + |x - y|) |u|^p(x) |u|^p(y) \, dx \, dy \right| \\
&\leq \left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log((1 + |x|)(1 + |y|)) |u|^p(x) |u|^p(y) \, dx \, dy \right| \\
&= \left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(1 + |x|) |u|^p(x) |u|^p(y) \, dx \, dy \right| + \left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(1 + |y|) |u|^p(x) |u|^p(y) \, dx \, dy \right| \\
&= 2\|u\|_p^p |u|^p_{L^p(\mathbb{R}^2)} \leq C\|u\|_{X}^{2p}.
\end{aligned}
\end{equation}

Now, letting $u_n \to u$ in $X$ as $n \to +\infty$, we get

\begin{equation}
\begin{aligned}
|V_1(u_n) - V_1(u)| &\leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log((1 + |x|)(1 + |y|)) |u_n|^p(x) |u_n|^p(y) - |u|^p(x) |u|^p(y) \, dx \, dy \\
&\leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(1 + |x|) |u_n|^p(x) |u_n|^p(y) - |u|^p(x) |u|^p(y) \, dx \, dy \\
&\leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(1 + |y|) |u_n|^p(x) |u_n|^p(y) - |u|^p(x) |u|^p(y) \, dx \, dy \\
&\quad + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(1 + |y|) |u_n|^p(x) |u_n|^p(y) - |u|^p(x) |u|^p(y) \, dx \, dy \\
&=: I_n^1 + I_n^2.
\end{aligned}
\end{equation}

For the sequence $I_n^1$, one has, as $n \to +\infty$,

\begin{equation}
\begin{aligned}
I_n^1 &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(1 + |x|) |u_n|^p(x) |u_n|^p(y) - |u|^p(x) |u|^p(y) \, dx \, dy \\
&\leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(1 + |x|) |u_n|^p(x) |u_n|^p(y) - |u|^p(y) \, dx \, dy \\
&\quad + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(1 + |y|) |u_n|^p(y) |u_n|^p(x) - |u|^p(x) \, dx \, dy \\
&\leq p\|u_n\|_p^p \left( \|u_n\|_{L^p}^{p-1} + \|u\|_{L^p}^{p-1} \right) \|u_n - u\|_{L^p} \\
&\quad + p \left( \|u_n\|_p^{p-1} + \|u\|_p^{p-1} \right) \|u_n - u\|_p \to 0.
\end{aligned}
\end{equation}

Through similar calculations, we also have $I_n^2 \to 0$ as $n \to +\infty$. Thus we get $V_1(u_n) \to V_1(u)$ and the functional $V_1$ is continuous on $X$.

The Gateaux derivative of $V_1$ at $u \in X$ is given by

\begin{equation}
\langle V_1'(u), v \rangle = 2pB_1(|u|^p, |u|^{p-2}uv) = 2p \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(1 + |x - y|) |u|^p(x) |u|^{p-2}(y)(y)(y) \, dx \, dy
\end{equation}
for any $v \in X$. We have the following estimate:

\begin{equation}
|\langle V'_1(u), v \rangle| = 2p \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(1 + |x - y|) |u|^p(x)|u|^{p-2}(y)u(y)v(y)dxdy
\end{equation}

\begin{align*}
&\leq 2p \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(1 + |x - y|) |u|^p(x)|u|^{p-1}(y)|v|(y)dxdy \\
&\leq 2p \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(1 + |x|) |u|^p(x)|u|^{p-1}(y)|v|(y)dxdy \\
&\quad + 2p \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(1 + |y|) |u|^p(x)|u|^{p-1}(y)|v|(y)dxdy \\
&\leq 2p \|u\|_X^p \|u\|_{L^p}^1 \|v\|_{L^p} + 2p \|u\|_X^p \|u\|_{L^p} \|v\|_X
\end{align*}

thus $V'_1(u) \in X'$ and $\|V'_1(u)\|_{X'} \leq C \|u\|^{2p-1}_X$.

Now, assume $u_n \to u$ in $X$ as $n \to +\infty$. For each $v \in X$, we have

\begin{align*}
|\langle V'_1(u_n) - V'_1(u), v \rangle| &= 2p |B_1(|u_n|^p, |u_n|^{p-2}u_n v) - B_1(|u|^p, |u|^{p-2}u v)| \\
&\leq 2p \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(1 + |x - y|) |u_n|^p(x)|u_n|^{p-2}(y)u_n(y)v(y) - |u|^p(x)|u|^{p-2}(y)u(y)v(y) |dxdy \\
&\leq 2p \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(1 + |x|) |u_n|^p(x)|u_n|^{p-2}(y)u_n(y)v(y) - |u|^p(x)|u|^{p-2}(y)u(y)v(y) |dxdy \\
&\quad + 2p \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(1 + |y|) |u_n|^p(x)|u_n|^{p-2}(y)u_n(y)v(y) - |u|^p(x)|u|^{p-2}(y)u(y)v(y) |dxdy \\
&=: 2p (II^1_n + II^2_n).
\end{align*}

For the sequence $II^1_n$, one has, as $n \to +\infty$,

\begin{equation}
II^1_n \leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(1 + |x|) |u_n|^p(x) |u_n|^{p-2}(y)u_n(y) - |u|^p(y)u(y) |v|(y)dxdy \\
+ \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(1 + |x|) |u_n|^p(x) - |u|^p(x) |u|^{p-1}(y)|v|(y)dxdy
\end{equation}

\begin{align*}
&\leq \|u_n\|_X^p \int_{\mathbb{R}^2} |u_n|^{p-2}(y)u_n(y) - |u|^{p-2}(y)u(y) |v|(y)dy \\
&\quad + \|u\|_X^{p-1} \|v\|_{L^p} \int_{\mathbb{R}^2} \log(1 + |x|) |u_n|^p(x) - |u|^p(x) |dx \\
&\leq \|u_n\|_X^p \int_{\mathbb{R}^2} |u_n|^{p-2}|u_n - u| |v| + \|u_n|^{p-2} - |u|^{p-2} |u| |v|dy \\
&\quad + p \|u\|_{L^p}^{p-1} \|v\|_{L^p} \left( \|u_n\|_X^{p-1} + \|u\|_X^{p-1} \right) \|u_n - u\|_X \\
&\leq \|u_n\|_X^p \left( \|u_n\|_{L^p}^{p-2} \|u_n - u\|_{L^p} \|v\|_{L^p} + \int_{\mathbb{R}^2} |u_n|^{p-2} - |u|^{p-2} |u| |v|dy \\
&\quad + p \|u\|_{L^p}^{p-1} \|v\|_{L^p} \left( \|u_n\|_X^{p-1} + \|u\|_X^{p-1} \right) \|u_n - u\|_X \right).
Therefore, if $2 \leq p \leq 3$, we derive that, as $n \to +\infty$,

$$
(2.22) \quad I_{n}^{1} \leq \|u_{n}\|^{p} \left(\|u_{n}\|^{p-2}_{L_{p}} \|u_{n} - u\|_{L_{p}} + sgn(p-2) \|u_{n} - u\|^{p-2}_{L_{p}} \|u\|_{L_{p}}\right) \|v\|_{X} + p\|u\|^{p-1}_{L_{p}} \left(\|u_{n}\|^{p-1}_{*} + \|u\|^{p-1}_{*}\right) \|u_{n} - u\|_{*} \|v\|_{X} \to 0;
$$

if $3 < p < +\infty$, we derive that, as $n \to +\infty$,

$$
(2.23) \quad I_{n}^{1} \leq \|u_{n}\|^{p}_{*} \left(\|u_{n}\|^{p-2}_{L_{p}} \|u_{n} - u\|_{L_{p}} + (p-2) \left(\|u_{n}\|^{p-3}_{L_{p}} + \|u\|^{p-3}_{L_{p}}\right) \|u_{n} - u\|_{L_{p}}\right) \|u\|_{L_{p}} \|v\|_{X} \to 0.
$$

Similar estimates as (2.22) and (2.23) can also be obtained for $II_{n}^{2}$ and we can deduce that $II_{n}^{2} \to 0$ as $n \to +\infty$. Hence, $V_{1}'(u_{n}) \to V_{1}'(u)$ in $X'$ and the Gateaux derivative $V_{1}'$ is continuous, that is, $V_{1} \in C^{1}(X, \mathbb{R})$.

From (2.12), we have known that $V_{2}$ is well-defined on $L_{p}^{4p}(\mathbb{R}^{2})$ and hence on $X$. We will show that $V_{2} \in C^{1}(L_{p}^{4p}(\mathbb{R}^{2}), \mathbb{R})$. To this end, letting $u_{n} \to u$ in $L_{p}^{4p}(\mathbb{R}^{2})$ as $n \to +\infty$, by Hardy-Littlewood-Sobolev inequality, we get

$$
|V_{2}(u_{n}) - V_{2}(u)| \\
\leq \int_{\mathbb{R}^{2}} \int_{\mathbb{R}^{2}} \log \left(1 + \frac{1}{|x-y|}\right) |u_{n}|^{p}(x) |u_{n}|^{p}(y) - |u|^{p}(y) |u_{n}|^{p}(x) - |u|^{p}(x) |u_{n} - u|^{p}(y) |u_{n} - u|^{p}(x) dxdy
$$

$$
\leq C \left(\|u_{n}\|^{p}_{L_{p}^{4p}} + \|u\|^{p}_{L_{p}^{4p}}\right) \left(\|u_{n}\|^{p-1}_{L_{p}^{4p}} + \|u\|^{p-1}_{L_{p}^{4p}}\right) \|u_{n} - u\|_{L_{p}^{4p}} \to 0,
$$

thus $V_{2} \in C(L_{p}^{4p}(\mathbb{R}^{2}), \mathbb{R})$. The Gateaux derivative of $V_{2}$ at $u \in L_{p}^{4p}(\mathbb{R}^{2})$ is given by

$$
(2.25) \quad \langle V_{2}'(u), v \rangle = 2pB_{2}(|u|^{p}, |u|^{p-2}uv)
= 2p \int_{\mathbb{R}^{2}} \int_{\mathbb{R}^{2}} \log \left(1 + \frac{1}{|x-y|}\right) |u|^{p}(x) |u|^{p-2}(y) u(y)v(x) dxdy
$$

for any $v \in L_{p}^{4p}(\mathbb{R}^{2})$. Thus we have

$$
(2.26) \quad |\langle V_{2}'(u), v \rangle| \leq 2p \int_{\mathbb{R}^{2}} \int_{\mathbb{R}^{2}} \frac{1}{|x-y|} |u|^{p}(x) |u(y)|^{p-1} |v(y)| dxdy
\leq C \|u\|^{p}_{L_{p}^{4p}} \|u\|^{p-1}_{L_{p}^{4p}} \|v\|_{L_{p}^{4p}} \leq C \|u\|_{L_{p}^{4p}}^{2p-1} \|v\|_{L_{p}^{4p}},
$$

thus $V_{2}'(u) \in \left(L_{p}^{4p}(\mathbb{R}^{2})\right)^{'}$ and $\|V_{2}'(u)\| \leq C\|u\|_{L_{p}^{4p}}^{2p-1}$. 


Now, assume $u_n \to u$ in $L^{\frac{4p}{4p-3}}(\mathbb{R}^2)$ as $n \to +\infty$. For each $v \in L^{\frac{4p}{p-3}}(\mathbb{R}^2)$, we have

$$\langle V_2'(u_n) - V_2'(u), v \rangle = 2p \left| B_2(|u_n|^p, |u_n|^{p-2} u_n v) - B_2(|u|^p, |u|^{p-2}uv) \right|$$

$$\leq 2p \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log \left( 1 + \frac{1}{|x-y|} \right) \left| |u_n|^p(x) - |u|^p(x) \right| \left| u_n|^{p-1}(y) |v|(y) \right| dx dy$$

$$+ 2p \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log \left( 1 + \frac{1}{|x-y|} \right) \left| |u|^p(x) \right| \left| |u_n|^{p-2}(y) u_n(y) - |u|^{p-2}(y) u(y) \right| |v|(y) dx dy$$

$$\leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{2p}{|x-y|} \left( |u_n|^{p-1}(x) + |u|^{p-1}(x) \right) \left| u_n(x) - u(x) \right| \left| |u_n|^{p-1}(y) |v|(y) \right| dx dy$$

$$+ \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{2p}{|x-y|} \left| |u|^p(x) \right| \left| u_n(y) - u(y) \right| \left| |u_n|^{p-2}(y) |v|(y) \right| dx dy$$

$$+ \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{2p}{|x-y|} \left| |u|^p(x) \right| \left| |u_n|^{p-2}(y) - |u|^{p-2}(y) \right| \left| u(y) |v|(y) \right| dx dy.$$}

Therefore, by Hardy-Littlewood-Sobolev inequality, if $2 \leq p \leq 3$, we can derive that, as $n \to +\infty$,

$$\langle V_2'(u_n) - V_2'(u), v \rangle \leq C \left| \left| u \right| \right|_{L^p}^p \left( \left| \left| u_n \right| \right|_{L^{\frac{4p}{4p-3}}}^{p-2} \left| u_n - u \right|_{L^{\frac{4p}{3}}} + \text{sgn}(p-2) \left| u_n - u \right|_{L^{\frac{4p}{4p-3}}}^{p-2} \left| u \right|_{L^{\frac{4p}{p-3}}} \right) \left| v \right|_{L^{\frac{4p}{p-3}}}$$

$$+ C \left| \left| u_n \right| \right|_{L^{\frac{4p}{4p-3}}}^{p-1} \left( \left| \left| u_n \right| \right|_{L^{\frac{4p}{3}}}^{p-1} + \left| \left| u \right| \right|_{L^{\frac{4p}{p-3}}}^{p-1} \right) \left| u_n - u \right|_{L^{\frac{4p}{4p-3}}} \left| v \right|_{L^{\frac{4p}{p-3}}} \to 0;$$

if $3 < p < +\infty$, we can derive that, as $n \to +\infty$,

$$\langle V_2'(u_n) - V_2'(u), v \rangle \leq C \left| \left| u \right| \right|_{L^p}^p \left( \left| \left| u_n \right| \right|_{L^{\frac{4p}{4p-3}}}^{p-2} \left| u_n - u \right|_{L^{\frac{4p}{3}}} + (p-2) \left( \left| \left| u_n \right| \right|_{L^{\frac{4p}{3}}}^{p-3} + \left| \left| u \right| \right|_{L^{\frac{4p}{p-3}}}^{p-3} \right) \left| u_n - u \right|_{L^{\frac{4p}{4p-3}}} \left| u \right|_{L^{\frac{4p}{p-3}}} \right)$$

$$\times \left| v \right|_{L^{\frac{4p}{p-3}}} + C \left| \left| u_n \right| \right|_{L^{\frac{4p}{4p-3}}}^{p-1} \left( \left| \left| u_n \right| \right|_{L^{\frac{4p}{3}}}^{p-1} + \left| \left| u \right| \right|_{L^{\frac{4p}{p-3}}}^{p-1} \right) \left| u_n - u \right|_{L^{\frac{4p}{4p-3}}} \left| v \right|_{L^{\frac{4p}{p-3}}} \to 0.$$

Therefore, $V_2'(u_n) \to V_2'(u)$ in $\left( L^{\frac{4p}{p-3}}(\mathbb{R}^2) \right)'$ and the Gateaux derivative $V_2'$ is continuous, that is, $V_2 \in C^1(L^{\frac{4p}{p-3}}(\mathbb{R}^2), \mathbb{R})$. As a immediate consequence, we infer that $V_2 \in C^1(X, \mathbb{R})$ and

$$\langle V_2'(u), v \rangle = 2pB_2(|u|^p, |u|^{p-2}uv) \text{ for any } u, v \in X.$$
Proof of (3): Let sequence \( \{u_n\} \subset H^1(\mathbb{R}^2) \), \( u_n \to u \) in \( H^1(\mathbb{R}^2) \), hence \( \{u_n\} \) is bounded in \( H^1(\mathbb{R}^2) \) and strongly converge to \( u \) in \( L^r_{\text{loc}}(\mathbb{R}^2) \) for any \( 1 \leq r < +\infty \). Therefore, for any \( R > 0 \), by Fatou’s lemma, we have

\[
\liminf_{n \to +\infty} \int_{B_R(0)} \int_{B_R(0)} \log(1 + |x-y|)|u_n|^p(x)|u_n|^p(y)\,dxdy
\]

\[
\geq \int_{B_R(0)} \int_{B_R(0)} \log(1 + |x-y|)|u|^p(x)|u|^p(y)\,dxdy.
\]

Letting \( R \to +\infty \), by monotone convergence theorem, we derive

\[
\liminf_{n \to +\infty} V_1(u_n) \geq V_1(u),
\]

that is, \( V_1 \) is w.l.s.c on \( H^1(\mathbb{R}^2) \).

Now we consider the energy functional

\[
I(u) = \frac{1}{2}\|u\|_{H^1(\mathbb{R}^2)}^2 + \frac{\gamma}{4p\pi}(V_1(u) - V_2(u)) - \frac{b}{q}\|u\|_{L^q(\mathbb{R}^2)}^q.
\]

Let sequence \( \{u_n\} \subset X \) s.t. \( u_n \to u \) in \( X \) as \( n \to +\infty \). By the compactness of \( X \hookrightarrow L^s(\mathbb{R}^2) \) for any \( p \leq s < \infty \), one has, up to a subsequence, \( u_n \to u \) in \( L^{2p}(\mathbb{R}^2) \) as \( n \to +\infty \). Thus \( V_2(u_n) \to V_2(u) \) and we get

\[
\liminf_{n \to +\infty} I(u_n) \geq I(u)
\]

that is, \( I \) is w.l.s.c on \( X \). Since the functional

\[
u \to I(\nu) - \frac{\gamma}{4p\pi}V_1(\nu) = \frac{1}{2}\|\nu\|_{H^1(\mathbb{R}^2)}^2 - \frac{\gamma}{4p\pi}V_2(\nu) - \frac{b}{q}\|\nu\|_{L^q(\mathbb{R}^2)}^q
\]

is continuous on \( H^1(\mathbb{R}^2) \) and \( I(\nu) = \left( I(\nu) - \frac{\gamma}{4p\pi}V_1(\nu) \right) + \frac{\gamma}{4p\pi}V_1(\nu) \), we deduce that \( I \) is l.s.c on \( H^1(\mathbb{R}^2) \). This completes our proof of Lemma 2.3. \(\square\)

Lemma 2.4. Assume \( p \geq 2, q \geq 2 \) and \( u \in X \) is a weak solution of the Euler-Lagrange equation (1.1), i.e., a critical point of the energy functional \( I \). Then \( u \) satisfies the following properties:

1. The potential function defined by \( w(x) := \frac{1}{2\pi} \int_{\mathbb{R}^2} \log(|x-y|)|u|^p(y)\,dy \in C^3(\mathbb{R}^2) \), moreover, it satisfies

\[\nabla w = |u|^p \text{ in } \mathbb{R}^2, \quad w(x) - \frac{1}{2\pi} \log |x| \int_{\mathbb{R}^2} |u|^p\,dx \to 0, \quad \text{as } |x| \to \infty;\]

2. \( u \) decays faster than exponential functions, i.e., for some \( A > 0 \) and \( C > 0 \),

\[|u(x)| \leq Ce^{-A|x|}, \quad \text{as } |x| \to \infty;\]

3. \( u \in W^{2,r}(\mathbb{R}^2) (\forall 1 \leq r < +\infty) \) is a strong solution of the Euler-Lagrange equation (1.1). Moreover, if \( a \) is Hölder continuous, then \( u \in C^{2,\alpha}_{\text{loc}}(\mathbb{R}^2) \) for some \( \alpha \in (0, 1) \), and hence \( u \in C^2(\mathbb{R}^2) \).

Proof. Proof of (1): We first show that the potential function \( w(x) := \frac{1}{2\pi} \int_{\mathbb{R}^2} \log(|x-y|)|u|^p(y)\,dy \in L^\infty_{\text{loc}} \), and hence it is well-defined.
For each $R > 1$ and $x \in B_R(0)$,

\begin{align*}
(2.37) \quad 2\pi |w(x)| &= \left| \int_{\mathbb{R}^2} \log(|x - y|)|u|^p(y)dy \right| \\
&\leq \int_{\mathbb{R}^2} \log(|x - y|)|u|^p(y)dy \\
&\leq \int_{|y - x| < 2R} \log(|x - y|)|u|^p(y)dy + \int_{|y - x| \geq 2R} \log |x - y||u|^p(y)dy.
\end{align*}

The first term in (2.37) is a convolution form with the kernel $|\log |x|| \chi_{B_2(0)} \in L^r(\mathbb{R}^2)$ for every $1 \leq r < \infty$, so

\begin{align*}
(2.38) \quad \int_{|y - x| < 2R} \log(|x - y|)|u|^p(y)dy &\leq C_R \|u\|_{L^2_p(\mathbb{R}^2)}^p \leq C_R \|u\|_{H^1(\mathbb{R}^2)}^p.
\end{align*}

As to the second term in (2.37), we have

\begin{align*}
(2.39) \quad \int_{|y - x| \geq 2R} \log(|x - y|)|u|^p(y)dy &\leq \int_{|y - x| \geq 2R} \log (R + |y|)|u|^p(y)dy \\
&\leq 2 \int_{|y - x| \geq 2R} \log(1 + |y|)|u|^p(y)dy \\
&\leq 2\|u\|_p^p.
\end{align*}

Thus one has $|w(x)| \leq C_R (\|u\|_{H^1(\mathbb{R}^2)}^p + \|u\|_p^p)$ and $w$ is well-defined.

By the definition of the function $w$, we have, for any $|x| > 2$,

\begin{align*}
(2.40) \quad 2\pi w(x) - \log |x| \int_{\mathbb{R}^2} |u|^p dx &= \int_{\mathbb{R}^2} (\log(|x - y|) - \log |x|)|u|^p(y)dy \\
=: \int_{|y - x| \geq \frac{|x|}{2}} \log \left( \frac{|x - y|}{|x|} \right) |u|^p(y)dy + \int_{|y - x| < \frac{|x|}{2}} (\log(|x - y|) - \log |x|)|u|^p(y)dy.
\end{align*}

Note that for any $|y - x| \geq \frac{|x|}{2}$, one has

\begin{align*}
(2.41) \quad \log \frac{1}{2} \leq \log \left( \frac{|x - y|}{|x|} \right) \leq \log \left( 1 + \frac{|y|}{|x|} \right) \leq \log (1 + |y|).
\end{align*}

Since $u \in X$, we have $|u|^p \in L^1(\mathbb{R}^2)$ and $\log(1 + |x|)|u|^p \in L^1(\mathbb{R}^2)$, thus it follows from Lebesgue’s dominated convergence theorem that

\begin{align*}
(2.42) \quad \int_{|y - x| \geq \frac{|x|}{2}} \log \left( \frac{|x - y|}{|x|} \right) |u|^p(y)dy \to 0, \quad \text{as} \quad |x| \to +\infty.
\end{align*}

Since one has

\begin{align*}
(2.43) \quad \int_{|y - x| < 1} \log(|x - y|)|u|^p(y)dy \\
&\leq \left( \int_{|y - x| < 1} |\log(|x - y|)|^2dy \right)^\frac{1}{2} \left( \int_{|y - x| < 1} |u|^{2p}(y)dy \right)^\frac{1}{2} \\
&\leq C \left( \int_{|y - x| < 1} |u|^{2p}(y)dy \right)^\frac{1}{2} \to 0, \quad \text{as} \quad |x| \to +\infty,
\end{align*}
and
\begin{align}
\left(2.44\right) & \quad \int_{1 \leq |y-x| < \frac{|x|}{2}} \log(|x-y|)|u|^p(y)dy + \int_{|y-x| < \frac{|x|}{2}} \log |x||u|^p(y)dy \\
& \leq \int_{1 \leq |y-x| < \frac{|x|}{2}} \log(|y|)|u|^p(y)dy + \int_{|y-x| < \frac{|x|}{2}} \log (2|y|)|u|^p(y)dy \\
& \leq \int_{|y| > \frac{|x|}{2}} (2 \log (1 + |y|) + \log 2) |u|^p(y)dy \to 0, \quad \text{as } |x| \to +\infty,
\end{align}

thus we arrive at
\begin{align}
\left(2.45\right) & \quad \int_{|y-x| < \frac{|x|}{2}} \left(\log(|y-x|) - \log |x|\right)|u|^p(y)dy \\
& + \int_{1 \leq |y-x| < \frac{|x|}{2}} \log(|y-x|)|u|^p(y)dy + \int_{|y-x| < \frac{|x|}{2}} \log |x||u|^p(y)dy \to 0, \quad \text{as } |x| \to +\infty.
\end{align}

Combining \(2.40\) and \(2.45\) yields that
\begin{align}
\left(2.46\right) & \quad w(x) - \frac{1}{2\pi} \log |x| \int_{\mathbb{R}^2} |u|^p dx \to 0, \quad \text{as } |x| \to +\infty,
\end{align}

and hence the asymptotic property in \(1\) has been established.

Then, by the Agmon’s theorem \(\text{(see [1])}\), we know \(u\) decays faster than exponential functions, that is, asymptotic property \(2\) holds. Thus elliptic regularity theory implies that \(u \in W^{2,r}(\mathbb{R}^2)\) for any \(r \in [1, +\infty)\), and hence \(u\) is a strong solution of \(\left(1.1\right)\). Moreover, from Sobolev embeddings, we can infer that \(u \in C^{1,\beta}_{loc}(\mathbb{R}^2)\) for any \(0 \leq \beta < 1\). As a consequence, \(w \in C^{3,\beta}_{loc}(\mathbb{R}^2)\) and hence \(w \in C^3(\mathbb{R}^2)\) and satisfies \(\Delta w = |u|^p\) in \(\mathbb{R}^2\). This proves property \(1\).

Finally, if \(a\) is Hölder continuous, then \(u\) satisfies an equation of the form \(-\Delta u = f\) with \(f\) (locally) Hölder continuous \(f\), thus \(u \in C^{2,\alpha}_{loc}(\mathbb{R}^2)\) for some \(\alpha \in (0, 1)\) and \(u \in C^2(\mathbb{R}^2)\). This proves property \(3\) and concludes our proof of Lemma \(2.4\). \(\square\)

**Lemma 2.5** \(\text{(Pohozaev type identity)}\). Assume \(u \in X\) is a weak solution to \(\left(1.1\right)\) with \(a \in C^2(\mathbb{R}^2)\), then the following identity holds:

\begin{align}
\left(2.47\right) & \quad P(u) := \frac{\gamma}{4\pi p} \left(\int_{\mathbb{R}^2} |u|^p dx\right)^2 + \frac{\gamma}{\pi p} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(|x-y|)|u|^p(x)|u|^p(y)dxdy \\
& \quad - \frac{2b}{q} \int_{\mathbb{R}^2} |u|^q dx + \int_{\mathbb{R}^2} a(x)|u|^2 dx \\
& \quad = \frac{\gamma}{4\pi p} \|u\|_{L^p(\mathbb{R}^2)}^{2p} + \frac{\gamma}{\pi p} V_0(u) - \frac{2b}{q} \|u\|_{L^q(\mathbb{R}^2)}^q + \int_{\mathbb{R}^2} a(x)|u|^2 dx = 0.
\end{align}

Consequently, one has
\begin{align}
\left(2.48\right) & \quad J_k(u) = k \langle I'(u), u \rangle - P(u) = 0, \quad k = 1, 2,
\end{align}

where the auxiliary functionals \(J_k\) are defined by \(\left(1.8\right)\).

**Proof.** From Lemma \(2.4\) we know that \(u \in C^2(\mathbb{R}^2)\) and \(u\) decays faster than exponential functions, i.e., for some \(A > 0\) and \(C > 0\),
\begin{align}
\left(2.49\right) & \quad |u(x)| \leq Ce^{-A|x|}, \quad \text{as } |x| \to +\infty.
\end{align}
Moreover, the potential function \( w(x) := \frac{1}{2\pi} \int_{\mathbb{R}^2} \log(|x-y|)|u|^p(y)dy \in C^3(\mathbb{R}^2) \) and satisfies

\[
\|\nabla w\|_{L^\infty(\mathbb{R}^2)} \leq C, \quad w(x) - \frac{1}{2\pi} \log |x| \int_{\mathbb{R}^2} |u|^p dx \to 0, \quad \text{as} \ |x| \to +\infty.
\]

Now we define the following two functions on \( \mathbb{R}^2 \):

\[
g(u) := b|u|^{q-2}u - a(x)u \in C^2(\mathbb{R}^2), \quad G(u) := \int_0^u g(s)ds = \frac{b}{q}|u|^q - \frac{a(x)}{2}u^2 \in C^2(\mathbb{R}^2),
\]

which also decay exponentially as \( |x| \to +\infty \). For any \( R > 0 \), we will multiply the equation (1.1) by \( x \cdot \nabla u \) and integrate by parts on \( B_R(0) \). Using the following formula (see e.g. Page 136 in [44])

\[
\Delta u(x \cdot \nabla u) = \nabla \cdot \left( \nabla u(x \cdot \nabla u) - x \frac{|Du|^2}{2} \right),
\]

we get

\[
\int_{B_R(0)} -\Delta u(x \cdot \nabla u)dx = \int_{B_R(0)} -\nabla \cdot \left( \nabla u(x \cdot \nabla u) - x \frac{|Du|^2}{2} \right)dx
\]

\[
= \int_{\partial B_R(0)} \frac{R}{2} \nabla |u|^2 d\sigma - \int_{\partial B_R(0)} \frac{1}{R} |x \cdot \nabla u|^2 d\sigma.
\]

By direct calculations, one has

\[
g(u)(x \cdot \nabla u) = G'(u)x^j D_j u = D_i(G(u))x^j
\]

\[
= x \cdot \nabla(G(u)) = \nabla \cdot (xG(u)) - G(u)\nabla \cdot x
\]

\[
= \nabla \cdot (xG(u)) - 2G(u),
\]

and hence the divergence theorem implies

\[
\int_{B_R(0)} g(u)(x \cdot \nabla u)dx = \int_{B_R(0)} (\nabla \cdot (xG(u)) - 2G(u))dx
\]

\[
= \int_{\partial B_R(0)} RG(u)d\sigma - \int_{B_R(0)} 2G(u)dx.
\]

Moreover, note that

\[
w|u|^{p-2}u(x \cdot \nabla u) = \frac{1}{p} [\nabla \cdot (w|u|^p x) - |u|^p x \cdot \nabla w - 2w|u|^p],
\]

thus we have

\[
\int_{B_R(0)} w|u|^{p-2}u(x \cdot \nabla u)dx = \frac{1}{p} \int_{B_R(0)} [\nabla \cdot (w|u|^p x) - |u|^p x \cdot \nabla w - 2w|u|^p] dx
\]

\[
= \frac{1}{p} \int_{\partial B_R(0)} w|u|^p Rd\sigma - \frac{1}{p} \int_{B_R(0)} |u|^p x \cdot \nabla w dx - \frac{2}{p} \int_{B_R(0)} w|u|^p dx.
\]
Therefore, multiplying the equation \((1.1)\) by \(x \cdot \nabla u\) and integrating by parts on \(B_R(0)\), it follows from \((2.52), \ (2.54)\) and \((2.60)\) that
\[
0 = \int_{B_R(0)} \left[ -\Delta u - g(u) + \gamma w |u|^{p-2}u \right] (x \cdot \nabla u) dx
\]
\[-\int_{\partial B_R(0)} \frac{R}{2} |\nabla u|^2 d\sigma - \int_{\partial B_R(0)} \frac{1}{R} |x \cdot \nabla u|^2 d\sigma
\]
\[-RG(u) d\sigma + \int_{B_R(0)} 2G(u) dx
\]
\[+ \frac{\gamma}{p} \int_{\partial B_R(0)} Rw|u|^p d\sigma - \frac{\gamma}{p} \int_{B_R(0)} |u|^p x \cdot \nabla w dx - \frac{2\gamma}{p} \int_{B_R(0)} w|u|^p dx.
\]

As a consequence, we arrive at
\[
\int_{B_R(0)} \left[ \frac{\gamma}{p} |u|^p x \cdot \nabla w + \frac{2\gamma}{p} w|u|^p - 2G(u) \right] dx
\]
\[-\frac{R}{2} |\nabla u|^2 - \frac{1}{R} |x \cdot \nabla u|^2 - RG(u) + \frac{\gamma}{p} Rw|u|^p \right] d\sigma.
\]

Next, following the idea in [4], we will show that the boundary terms in \((2.58)\) converges to zero along a sequence \(R_k \to +\infty\) as \(k \to +\infty\), that is,
\[
\lim_{k \to +\infty} R_k \int_{\partial B_{R_k}(0)} f(x) dx = 0,
\]
where the function
\[
f(x) := \frac{1}{2} |\nabla u|^2 + \frac{\gamma}{p} w|u|^p - \frac{|x \cdot \nabla u|^2}{|x|^2} - G(u).
\]

To this end, it is suffices for us to show that \(f \in L^1(\mathbb{R}^2)\). Indeed, from the standard elliptic regularity theory and the exponential decay of \(u\), one can deduce that \(\nabla u\) also decays exponentially. Consequently, it follows immediately from \((2.49)\) and \((2.50)\) that \(f \in L^1(\mathbb{R}^2)\), and hence \((2.59)\) holds.

Furthermore, \((2.49)\) and \((2.50)\) also implies that \(w|u|^p \in L^1(\mathbb{R}^2)\), \(G(u) \in L^1(\mathbb{R}^2)\) and \(|u|^p x \cdot \nabla w \in L^1(\mathbb{R}^2)\), and
\[
\int_{\mathbb{R}^2} |u|^p x \cdot \nabla w dx = \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|x|^2 - x \cdot y}{|x-y|^2} |u|^p(x)|u|^p(y) dx dy
\]
\[= \frac{1}{4\pi} \left( \int_{\mathbb{R}^2} |u|^p dx \right)^2.
\]

Taking \(R = R_k\) in \((2.58)\) and letting \(k \to +\infty\), \((2.59)\) and \((2.61)\) yields that
\[
P(u) = \int_{\mathbb{R}^2} \left[ \frac{\gamma}{p} |u|^p x \cdot \nabla w + \frac{2\gamma}{p} w|u|^p - 2G(u) \right] dx
\]
\[= \lim_{k \to +\infty} \int_{B_{R_k}(0)} \left[ \frac{\gamma}{p} |u|^p x \cdot \nabla w + \frac{2\gamma}{p} w|u|^p - 2G(u) \right] dx
\]
\[= \lim_{k \to +\infty} R_k \int_{\partial B_{R_k}(0)} f(x) dx = 0.
\]
Therefore, for every weak solution $u \in X$ to (1.1), the auxiliary functionals

\begin{equation}
J_k(u) = k \langle I'(u), u \rangle - P(u) = 0, \quad k = 1, 2.
\end{equation}

This finishes our proof of Lemma 2.5. \hfill \Box

**Lemma 2.6.** Assume $p \geq 2$ and $q > 2$. There exists a $\alpha > 0$ such that, for all $0 < \beta \leq \alpha$,

\begin{equation}
\inf_{\|u\|_{H^1} = \beta} I(u) > 0, \quad \inf_{\|u\|_{H^1} = \beta} \langle I'(u), u \rangle > 0.
\end{equation}

**Proof.** From the definition of the energy functional $I$, (2.12) and Sobolev embeddings, we get

\begin{equation}
I(u) = \frac{1}{2} \|u\|_{H^1}^2 + \frac{\gamma}{4\pi} V_0(u) - \frac{b}{q} \|u\|_{L^q}^q \geq \frac{1}{2} \|u\|_{H^1}^2 - \frac{\gamma}{4\pi} V_2(u) - \frac{b}{q} \|u\|_{L^q}^q \\
\geq \frac{1}{2} \|u\|_{H^1}^2 - C_1 \|u\|_{H^1}^{2p} - C_2 \|u\|_{H^1}^q \geq \|u\|_{H^1}^2 \left( \frac{1}{2} - C_1 \|u\|_{H^1}^{2p-2} - C_2 \|u\|_{H^1}^{q-2} \right).
\end{equation}

Since $p \geq 2$ and $q > 2$, we derive that if $\alpha$ is small enough, then for any $0 < \beta \leq \alpha$, the first inequality in (2.64) holds. By direct calculations, we also have

\begin{equation}
\langle I'(u), u \rangle = \|u\|_{H^1}^2 + \frac{\gamma}{2\pi} V_0(u) - b \|u\|_{L^q}^q \geq \|u\|_{H^1}^2 - \frac{\gamma}{2\pi} V_2(u) - b \|u\|_{L^q}^q \geq \|u\|_{H^1}^2 \left( 1 - C_3 \|u\|_{H^1}^{2p-2} - C_4 \|u\|_{H^1}^{q-2} \right),
\end{equation}

which implies that if $\alpha$ is small enough, then for any $0 < \beta \leq \alpha$, the second inequality in (2.64) holds. This finishes our proof of Lemma 2.6. \hfill \Box

**Lemma 2.7.** Suppose $a(x) = a > 0$, $p \geq 2$, $q \geq 2$. Let $k = 1$ if and only if $p \geq 3$ and $k = 2$ if and only if $2 \leq p < 3$. Let $u \in X \setminus \{0\}$ and $u_{t,k} \in X \setminus \{0\}$ be defined by $u_{t,k}(x) := t^k u(tx)$ for any $t > 0$. Then we have

\begin{equation}
\lim_{t \to +\infty} I(u_{t,k}) = -\infty.
\end{equation}

**Proof.** Let $u \in X \setminus \{0\}$. Then we have

\begin{equation}
I(u_{t,k}) = \frac{t^{2k}}{2} \int_{\mathbb{R}^2} |\nabla u|^2 dx + \frac{t^{2(k-1)}}{2} \int_{\mathbb{R}^2} au^2 dx - \frac{t^{2(kp-2)\gamma}}{4\pi} \log t \left( \int_{\mathbb{R}^2} |u|^p dx \right)^2 + \frac{t^{2(kp-2)\gamma}}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(|x-y|)|u|^p(x)|u|^p(y) dxdy - \frac{t^{kq-2b}}{q} \int_{\mathbb{R}^2} |u|^q dx,
\end{equation}

and hence $I(u_{t,k}) \to -\infty$ as $t \to +\infty$. This finishes the proof of Lemma 2.7. \hfill \Box

**Lemma 2.8.** Assume $p \geq 2$, $q \geq 2p$ and $u \in X \setminus \{0\}$. Then there are only two different possibilities for the function $\varphi_u(t) = I(tu)$:

1. there exists a unique $t_u \in (0, +\infty)$ such that, $\varphi_u(t) > 0$ on $(0, t_u)$ and $\varphi_u(t) < 0$ on $(t_u, +\infty)$, moreover, $\varphi_u(t) \to -\infty$ as $t \to +\infty$;
2. $\varphi_u(t) > 0$ on $(0, +\infty)$ and $\varphi_u(t) \to +\infty$ as $t \to +\infty$. 

Proof. The proof of Lemma 2.8 is obvious. We only need to observe that
\[
\frac{\varphi'_u(t)}{t} = \|u\|_{H^1}^2 + \frac{\gamma}{2\pi} t^{2p-2} V_0(u) - bt^{q-2} \|u\|_q^q
\]
with \( p \geq 2, \ q \geq 2p \).

\[\square\]

Remark 2.9. From the proof of Lemma 2.8, it is clear that, if \( u \in X \setminus \{0\} \) such that \( V_0(u) < 0 \), then
\[
0 < \sup_{\mathbb{R}^n} I := \sup_{t \in \mathbb{R}} I(tu) < +\infty.
\]

3. A compactness condition and quantitative deformation Lemma

In the following, we assume \( a \in C(\mathbb{R}^2) \) and \( \mathbb{Z}^2 \)-periodic with \( \inf_{\mathbb{R}^2} a > 0 \). For any function \( u : \mathbb{R}^2 \to \mathbb{R} \) and \( x \in \mathbb{R}^2 \), the action of translation is denoted by
\[
x * u : \mathbb{R}^2 \to \mathbb{R}, \quad x * u(y) = u(y - x), \quad \forall y \in \mathbb{R}^2.
\]

3.1. Cerami compactness condition of translation invariant version. The main result of this sub-section is the following theorem.

Theorem 3.1. Assume \( p \geq 2 \) and \( q \geq 2p \). Let sequence \( \{u_n\} \subset X \) satisfy
\[
I(u_n) \to d > 0, \quad \|I'(u_n)\|_{X'} (1 + \|u_n\|_X) \to 0, \quad \text{as} \ n \to \infty.
\]
Then, after passing to a subsequence, there exist points \( x_n \in \mathbb{Z}^2 \) (\( n \in \mathbb{N} \)), such that
\[
x_n * u_n \to u \quad \text{strongly in} \ X, \quad \text{as} \ n \to \infty
\]
for some critical points \( u \in X \setminus \{0\} \) of \( I \).

In order to prove Theorem 3.1, we need some useful lemmas.

Lemma 3.2. Assume \( p \geq 2 \). Let \( \{u_n\} \) be a sequence in \( L^p(\mathbb{R}^2) \) s.t. \( u_n \to u \in L^p(\mathbb{R}^2) \setminus \{0\} \) a.e. on \( \mathbb{R}^2 \). Assume \( \{v_n\} \) is a bounded sequence in \( L^p(\mathbb{R}^2) \) s.t.
\[
\sup_{n \in \mathbb{N}} B_1(|u_n|^p, |v_n|^p) < +\infty.
\]
Then, there exists \( n_0 \in \mathbb{Z} \) and \( C > 0 \) s.t. \( \|v_n\|_* \leq C \) for any \( n \geq n_0 \). Furthermore, if \( B_1(|u_n|^p, |v_n|^p) \to 0 \) and \( \|v_n\|_{L^p} \to 0 \), then \( \|v_n\|_* \to 0 \) as \( n \to \infty \).

Proof. Since \( u \neq 0 \), one has \( \lim_{\delta \to 0^+} |\{ |u| > \delta \}| = |\{|u| > 0\}| > 0 \). Thus we can choose \( R \) large enough and \( \delta \) small enough s.t.
\[
|E| := |\{ |u| > \delta \} \cap B_R(0)| > 0.
\]
By applying Egorov’s theorem on \( E \), we can derive a subset \( A \subset E \) with \( |A| > 0 \) and \( n_0 \in \mathbb{N} \) large enough such that, \( |u_n| \geq \frac{\delta}{2} \) for any \( n \geq n_0 \) on the set \( A \).

Note that for any \( x \in B_R(0) \) and \( y \in \mathbb{R}^2 \setminus B_{2R}(0) \),
\[
1 + |x - y| \geq 1 + \frac{|y|}{2} \geq \sqrt{1 + |y|},
\]
and hence, we deduce from the definition of $B_1$ that, for any $n \geq n_0$,

$\begin{align}
(3.6) \quad B_1(|u_n|^p, |v_n|^p) &\geq \int_{R^2 \backslash B_{2R}(0)} \log(1 + |x - y|) |u_n|^p(x)|v_n|^p(y)\,dxdy \\
&\geq \frac{|A|}{2} \left( \frac{\delta}{2} \right)^p \int_{R^2 \backslash B_{2R}(0)} \log(1 + |y|) |v_n|^p(y)\,dy \\
&\geq \frac{|A|}{2} \left( \frac{\delta}{2} \right)^p (\|v_n\|_p - \log(1 + 2R)\|v_n\|_{L^p}).
\end{align}$

Thus if $\|v_n\|_{L^p}$ and $B_1(|u_n|^p, |v_n|^p)$ are bounded, then we get from (3.6) that $\|v_n\|_p$ are bounded for any $n \geq n_0$. Moreover, if $B_1(|u_n|^p, |v_n|^p) \to 0$ and $\|v_n\|_{L^p} \to 0$ as $n \to \infty$, then (3.6) yields that $\|v_n\|_p \to 0$.

**Lemma 3.3.** Assume $p \geq 2$. Let $\{\tilde{u}_n\}$ be a bounded sequence in $X$ such that $\tilde{u}_n \to u$ weakly in $X$. Then we have

$\begin{align}
(3.7) \quad B_1 (|\tilde{u}_n|^p, |\tilde{u}_n|^{p-2}u(\tilde{u}_n - u)) &\to 0, \quad \text{as } n \to \infty.
\end{align}$

**Proof.** Since $\tilde{u}_n \to u$ in $X$, by Lemma 2.3 and the definition of $B_1$, we have

$\begin{align}
(3.8) \quad |B_1 (|\tilde{u}_n|^p, |\tilde{u}_n|^{p-2}u(\tilde{u}_n - u))| &\leq \int_{R^2} \int_{R^2} \log(1 + |x|) |\tilde{u}_n|^p(x)|\tilde{u}_n|^{p-2}(y)u(y)|\tilde{u}_n - u|(y)\,dxdy \\
&\quad + \int_{R^2} \int_{R^2} \log(1 + |y|) |\tilde{u}_n|^p(x)|\tilde{u}_n|^{p-2}(y)u(y)|\tilde{u}_n - u|(y)\,dxdy \\
&\leq C \int_{R^2} |\tilde{u}_n|^{p-2}(y)u(y)|\tilde{u}_n - u|(y)\,dy \\
&\quad + C \int_{R^2} \log(1 + |y|) |\tilde{u}_n|^{p-2}(y)u(y)|\tilde{u}_n - u|(y)\,dy \\
&\leq o_n(1) + C \int_{R^2} \log(1 + |y|) |\tilde{u}_n|^{p-2}(y)u(y)|\tilde{u}_n - u|(y)\,dy.
\end{align}$

For arbitrary $R > 0$, we have, as $n \to \infty$,

$\begin{align}
(3.9) \quad \int_{B_R(0)} \log(1 + |y|) |\tilde{u}_n|^{p-2}(y)u(y)|\tilde{u}_n - u|(y)\,dy &\leq \log(1 + R)\|\tilde{u}_n\|_{L^p}^{p-2} \|u\|_{L^p} \|\tilde{u}_n - u\|_{L^p} \to 0.
\end{align}$

Combining (3.8) and (3.9), we deduce that, for any $R > 0$,

$\begin{align}
(3.10) \quad \lim_{n \to \infty} |B_1 (|\tilde{u}_n|^p, |\tilde{u}_n|^{p-2}u(\tilde{u}_n - u))| &\leq C \sup_{n \in \mathbb{N}} \int_{R^2 \backslash B_R(0)} \log(1 + |y|) |\tilde{u}_n|^{p-2}(y)u(y)|\tilde{u}_n - u|(y)\,dy \\
&\leq C \left( \int_{R^2 \backslash B_R(0)} \log(1 + |y|) |u|^p(y)\,dy \right)^{\frac{1}{p}} = o_R(1),
\end{align}$

as $R \to +\infty$. By letting $R \to +\infty$ in (3.10), we get the desired conclusion.
Now we will carry out the proof of Theorem 3.1 by splitting it into several lemmas. From now on, we will assume the sequence \{u_n\} \subset X satisfies the assumption (3.2) in Theorem 3.1

**Lemma 3.4.** Assume \( p \geq 2 \) and \( q \geq 2p \). If \( \{t_n\} \subset [0, +\infty) \) is a bounded sequence, then \( I(t_n u_n) \leq I(u_n) + o_n(1) \), as \( n \to \infty \). Furthermore, if \( t_n \to 0 \), then \( \liminf_{n \to \infty} I(t_n u_n) \geq 0 \).

**Proof.** Recall that

\[
I(u) = \frac{1}{2} \|u\|_{H^1(\mathbb{R}^2)}^2 + \frac{\gamma}{4\pi} V_0(u) - \frac{b}{q} \|u\|_{L^q(\mathbb{R}^2)}^q,
\]

(3.11)

\[
\langle I'(u), u \rangle = \|u\|_{H^1(\mathbb{R}^2)}^2 + \frac{\gamma}{2\pi} V_0(u) - b \|u\|_{L^q(\mathbb{R}^2)}^q,
\]

(3.12)

we have

\[
I(t_n u_n) - I(u_n) = \frac{t_n^2 - 1}{2} \|u_n\|_{H^1}^2 + \frac{\gamma}{4\pi} (t_n^{2p} - 1) V_0(u_n) - \frac{b}{q} (t_n^q - 1) \|u_n\|_{L^q}^q.
\]

(3.13)

By the assumption (3.2) and (3.12), we have

\[
\frac{\gamma}{2\pi} V_0(u_n) = \langle I'(u_n), u_n \rangle - \|u_n\|_{H^1}^2 + b \|u_n\|_{L^q}^q
\]

(3.14)

\[
= o_n(1) - \|u_n\|_{H^1}^2 + b \|u_n\|_{L^q}^q
\]

Substituting (3.14) into (3.13), since \( p \geq 2 \), \( q \geq 2p \) and \( (t_n)_n \) is bounded, we get

\[
I(t_n u_n) - I(u_n)
= -\left( \frac{t_n^{2p} - 1}{2p} - \frac{t_n^2 - 1}{2} \right) \|u_n\|_{H^1}^2 - b \left( \frac{t_n^q - 1}{q} - \frac{t_n^{2p} - 1}{2p} \right) \|u_n\|_{L^q}^q + o_n(1)
\]

(3.15)

\[
\leq o_n(1),
\]

this shows \( I(t_n u_n) \leq I(u_n) + o_n(1) \). Moreover, if \( t_n \to 0 \), using the assumption (3.2) and (3.14) again, we get

\[
I(t_n u_n) = \left( \frac{t_n^2}{2} - \frac{t_n^{2p}}{2p} \right) \|u_n\|_{H^1}^2 + b \left( \frac{t_n^q}{q} - \frac{t_n^{2p}}{2p} \right) \|u_n\|_{L^q}^q + o_n(1) \geq o_n(1),
\]

(3.16)

as \( n \to +\infty \). This finishes the proof of Lemma 3.4.

\[\square\]

**Lemma 3.5.** Assume \( p \geq 2 \) and \( q \geq 2p \). Suppose sequence \{u_n\} satisfies the assumption (3.2), then \( \{u_n\} \) is bounded in \( H^1(\mathbb{R}^2) \).

**Proof.** We will prove Lemma 3.5 by contradiction arguments. Suppose on the contrary that, after passing to a subsequence, then we have \( \|u_n\|_{H^1(\mathbb{R}^2)} \to +\infty \).

Define \( v_n := \frac{u_n}{\|u_n\|_{H^1}} \), then we have \( \|v_n\|_{H^1} = 1 \). In the following, we will carry out the proof by three steps.

**Step 1.** We will show the following property:

\[
\inf_{n \in \mathbb{N}} \sup_{x \in \mathbb{R}^2} \int_{B_2(x)} v_n^2(y) dy > 0.
\]

(3.17)

Suppose it is not true, then by Lemma 2.2 after passing to a subsequence, \( v_n \to 0 \) in \( L^s(\mathbb{R}^2) \) for every \( 2 < s < +\infty \). Thus \( |V_2(t v_n)| \leq C \|t v_n\|_{L^{2p}}^{2p} \to 0 \) and \( \|t v_n\|_{L^q} \to 0 \) for each fixed \( t \geq 0 \).
Hence
\begin{equation}
I(tv_n) = \frac{t^2}{2} \|v_n\|_{H^1}^2 + \frac{\gamma}{4p\pi} \left( V_1(tv_n) - V_2(tv_n) \right) - \frac{b}{q} \|tv_n\|_{L^q}^q \\
\geq \frac{t^2}{2} + o_n(1)
\end{equation}
for each fixed $t \geq 0$. But, on the other hand, by Lemma 3.4, one has, for every fixed $t \geq 0$,
\begin{equation}
I(tv_n) = I \left( \frac{t}{\|u_n\|_{H^1}} u_n \right) \leq I(u_n) + o_n(1) = d + o_n(1),
\end{equation}
which is absurd if we take $t$ large enough (say, $t > \sqrt{2d}$). So we have obtained the property (3.17).

Step 2. By (3.17), there exists a sequence $\{x_n\} \subset \mathbb{Z}^2$ s.t.
\begin{equation}
\liminf_{n \to \infty} \int_{B_2(x_n)} v_n^2(y) dy > 0,
\end{equation}
and hence $w_n := (-x_n) \ast v_n$ ($n \in \mathbb{N}$) satisfy
\begin{equation}
\liminf_{n \to \infty} \int_{B_2(0)} w_n^2(y) dy > 0.
\end{equation}
Note that $\|w_n\|_{H^1(\mathbb{R}^2)} = \|v_n\|_{H^1(\mathbb{R}^2)} = 1$, one has $w_n \rightharpoonup w$ for some $w \in H^1(\mathbb{R}^2)$. Then, Rellich’s compact embedding theorem and (3.21) yield that $w \neq 0$. By passing to a subsequence, we can also assume $w_n \to w$ a.e. on $\mathbb{R}^2$.

By the translation invariant in $\mathbb{Z}^2$ of $I$ and taking $t = 1$ in (3.19), we have
\begin{equation}
I(w_n) = I(v_n) \leq d + o_n(1).
\end{equation}
Hence, by Sobolev embeddings, we arrive at
\begin{equation}
\frac{\gamma}{4p\pi} V_1(w_n) = I(w_n) - \frac{1}{2} \|w_n\|_{H^1}^2 + \frac{\gamma}{4p\pi} V_2(w_n) + \frac{b}{q} \|w_n\|_{L^q}^q \\
\leq d + o_n(1) - \frac{1}{2} + \frac{\gamma}{4p\pi} V_2(w_n) + \frac{b}{q} \|w_n\|_{L^q}^q \\
\leq C.
\end{equation}

Step 3. In this step, we will distinguish two different cases separately.

Case 1: $b > 0$ and $q > 2p$. In this case, by Fatou’s lemma, after passing to a subsequence, we have
\begin{equation}
I(tv_n) = I(tw_n) = \frac{t^2}{2} \|w_n\|_{H^1}^2 + \frac{\gamma t^{2p}}{4p\pi} \left( V_1(w_n) - V_2(w_n) \right) - \frac{bt^q}{q} \|w_n\|_{L^q}^q \\
\leq \frac{t^2}{2} + Ct^{2p} - \frac{bt^q}{q} \|w\|_{L^q}^q + o_n(1).
\end{equation}
Thus there exist $t_0$ and $n_0$ sufficiently large such that, for any $n \geq n_0$,
\begin{equation}
I(t_0 v_n) = I(t_0 w_n) \leq -1.
\end{equation}
However, since $\frac{t_0}{\|u_n\|_{H^1}} \to 0$, (3.25) contradicts with Lemma 3.4.
Case 2: \( b = 0 \) or \( q = 2p \). In this case, we have

\[
I(tv_n) = I(tw_n) = \frac{t^2}{2} \|w_n\|_{H^1}^2 + \frac{\gamma t^{2p}}{4p\pi} V_0(w_n) - \frac{bt^n}{q} \|w_n\|^q_{L^q}
\]

with \( \rho_n := V_0(w_n) - \frac{4pn\epsilon}{\gamma q} \|w_n\|_{L^q}^q \). We will show that

\[
\rho := \lim_{n \to \infty} \sup \rho_n < 0.
\]

If not, then \( \limsup_{n \to \infty} \rho_n \geq 0 \). After passing to a subsequence, we have, if we choose \( t_0 \) large enough (say, \( t_0 \geq \sqrt{4d} \)), then

\[
I(t_0v_n) = I(t_0w_n) = \frac{t_0^2}{2} + \frac{\gamma t_0^{2p}}{4p\pi} \rho_n \geq \frac{t_0^2}{2} + o_n(1)
\]

\[
\geq 2d + o_n(1).
\]

However, by the assumption \(3.2\) and Lemma \(3.4\) one has

\[
d + o(1) = I(u_n) \geq I \left( \left( \frac{t_0}{\|u_n\|_{H^1}} \right) u_n \right) + o_n(1)
\]

\[
= I(t_0v_n) + o_n(1) \geq 2d + o_n(1),
\]

which is a contradiction. Thus \( \rho = \limsup_{n \to \infty} \rho_n < 0 \), and hence there exists a \( n_0 \in \mathbb{N} \) large enough such that \( \rho_n \leq -\epsilon_0 < 0 \) for any \( n \geq n_0 \). Then, we have, for any \( n \geq n_0 \),

\[
I(tv_n) = \frac{t^2}{2} + \frac{\gamma t^{2p}}{4p\pi} \rho_n \leq \frac{t^2}{2} - \frac{\gamma t^{2p}}{4p\pi} \epsilon_0.
\]

Thus there exists \( t_0 \) sufficiently large such that, for any \( n \geq n_0 \), \( I(t_0v_n) \leq -1 \), which contradicts with Lemma \(3.3\) again. Therefore, \( \{u_n\} \) is bounded in \( H^1(\mathbb{R}^2) \).

Lemma 3.6. Assume \( p \geq 2 \) and \( q > 2 \). Suppose \( \{u_n\} \) satisfies the assumption \(3.2\). Then we have

\[
\liminf_{n \to \infty} \sup_{x \in \mathbb{R}^2} \int_{B_2(x)} u_n^2(y)dy > 0.
\]

Proof. Suppose it is false. Then by Lemma \(2.2\) after passing to a subsequence, we have \( u_n \to 0 \) in \( L^s(\mathbb{R}^2) \) for every \( s > 2 \). Since

\[
\langle I'(u_n), u_n \rangle = \|u_n\|_{H^1}^2 + \frac{\gamma}{2\pi} (V_1(u_n) - V_2(u_n)) - b\|u_n\|^q_{L^q},
\]

thus we get

\[
\|u_n\|_{H^1}^2 + \frac{\gamma}{2\pi} V_1(u_n) = \langle I'(u_n), u_n \rangle + \frac{\gamma}{2\pi} V_2(u_n) + b\|u_n\|^q_{L^q} \to 0,
\]

as \( n \to \infty \). It follows immediately that \( \|u_n\|_{H^1} \to 0 \), \( V_1(u_n) \to 0 \), and hence

\[
I(u_n) = \frac{1}{2} \|u_n\|_{H^1}^2 + \frac{\gamma}{4p\pi} \left( V_1(u_n) - V_2(u_n) \right) - \frac{b}{q} \|u_n\|^q_{L^q} \to 0,
\]

which is absurd, because \( I(u_n) \to d > 0 \). This finishes our proof of Lemma 3.6. \( \square \)
By Lemma 3.6 there exists a sequence \( \{x_n\} \subset \mathbb{Z}^2 \) s.t.

\[
\liminf_{n \to \infty} \int_{B_2(x_n)} u_n^2(y)dy > 0,
\]

and hence \( \tilde{u}_n := (-x_n) \ast u_n \in X \) \((n \in \mathbb{N})\) satisfy

\[
\liminf_{n \to \infty} \int_{B_2(0)} \tilde{u}_n^{-2}(y)dy > 0.
\]

Note that \( \|u_n\|_{H^1} = \|u_n\|_{H^1} \) are bounded, one has \( \tilde{u}_n \rightharpoonup u \) for some \( u \in H^1(\mathbb{R}^2) \). Then, Rellich’s compact embedding theorem and (3.36) yield that \( u \neq 0 \). By passing to a subsequence, we can also assume \( \tilde{u}_n \to u \) a.e. on \( \mathbb{R}^2 \).

Now we are ready to complete the proof of Theorem 3.1.

**Proof of Theorem 3.1 (completed).** We will complete the proof of Theorem 3.1 by the following three steps.

**Step 1:** We are to show that \( \{\tilde{u}_n\} \) is bounded in \( X \). Indeed, by assumption (3.2), we have

\[
\frac{\gamma}{2\pi} V_1(\tilde{u}_n) = \langle I'(\tilde{u}_n), \tilde{u}_n \rangle - \|\tilde{u}_n\|_{H^1}^2 + \frac{\gamma}{2\pi} V_2(\tilde{u}_n) + \|\tilde{u}_n\|_{L^q}^q
\]

\[
= \langle I'(u_n), u_n \rangle - \|u_n\|_{H^1}^2 + \frac{\gamma}{2\pi} V_2(u_n) + \|u_n\|_{L^q}^q
\]

\[
= o_n(1) - \|u_n\|_{H^1}^2 + \frac{\gamma}{2\pi} V_2(u_n) + \|u_n\|_{L^q}^q \leq C.
\]

By Lemma 3.2 we get from (3.37) that \( \|\tilde{u}_n\|_s \) are bounded, and hence \( \{\tilde{u}_n\} \) is bounded in \( X \). We may assume that, after passing to a subsequence if necessary, \( \tilde{u}_n \rightharpoonup u \) weakly in \( X \), so that \( u \in X \). By the compact embeddings of \( X \hookrightarrow L^s(\mathbb{R}^2) \) for all \( p \leq s < +\infty \), we have \( \tilde{u}_n \to u \) strongly in \( L^s(\mathbb{R}^2) \) for all \( p \leq s < +\infty \).

**Step 2:** Our goal is to prove \( \tilde{u}_n \to u \) strongly in \( X \). First, we will show that

\[
\langle I'(\tilde{u}_n), \tilde{u}_n - u \rangle \to 0, \quad \text{as} \ n \to \infty.
\]

In fact, by the \( \mathbb{Z}^2 \)-translation invariance, we have

\[
\left| \langle I'(\tilde{u}_n), \tilde{u}_n - u \rangle \right| = \left| \langle I'(u_n), u_n - x_n \ast u \rangle \right| \leq \|I'(u_n)\|_{X'} (\|u_n\|_X + \|x_n \ast u\|_X).
\]

For the last term in (3.39), we have

\[
\|x_n \ast u\|_s = \left( \int_{\mathbb{R}^2} \log(1 + |x|)|u|^p(x - x_n)dx \right)^{\frac{1}{p}} = \left( \int_{\mathbb{R}^2} \log(1 + |x + x_n|)|u|^pdx \right)^{\frac{1}{p}}
\]

\[
\leq \left( \int_{\mathbb{R}^2} \log(1 + |x|)|u|^pdx \right)^{\frac{1}{p}} + \left( \int_{\mathbb{R}^2} \log(1 + |x_n|)|u|^pdx \right)^{\frac{1}{p}}
\]

\[
\leq \|u\|_s + (\log(1 + |x_n|))^{\frac{1}{p}} \|u\|_{L^p(\mathbb{R}^2)},
\]

and hence

\[
\|x_n \ast u\|_X \leq C_1 + C_2 (\log(1 + |x_n|))^{\frac{1}{p}}.
\]
On the other hand, for any \( n \in \mathbb{N} \) such that \(|x_n| \geq 4\), one can infer from (3.36) that

\[
(3.42) \quad \|u_n\|_* = \left( \int_{\mathbb{R}^2} \log(1 + |x - x_n|)|\tilde{u}_n|^p dx \right)^{\frac{1}{p}} \\
\geq \left( \int_{B_2(0)} \log \left( \frac{|x_n|}{2} \right) |\tilde{u}_n|^p dx \right)^{\frac{1}{p}} \geq \left( \int_{B_2(0)} \log \left( \sqrt{1 + |x_n|} \right) |\tilde{u}_n|^p dx \right)^{\frac{1}{p}} \\
\geq C_3 \left( \log(1 + |x_n|) \right)^{\frac{1}{p}},
\]

and it follows immediately that

\[
(3.43) \quad \|u_n\|_X \geq C_4 + C_3 \left( \log(1 + |x_n|) \right)^{\frac{1}{p}}.
\]

Therefore, by combining (3.41) and (3.43), we have

\[
(3.44) \quad \|x_n * u\|_X \leq C \|u_n\|_X.
\]

Then, by the assumption (3.2) and (3.39), we have

\[
(3.45) \quad \langle I'(\tilde{u}_n), \tilde{u}_n - u \rangle \longrightarrow 0, \quad \text{as } n \to \infty,
\]

this proves the property (3.38).

As a consequence, we get

\[
(3.46) \quad o_n(1) = \langle I'(\tilde{u}_n), \tilde{u}_n - u \rangle - \langle I'(\tilde{u}_n), u \rangle \\
= \|\tilde{u}_n\|_{H^1}^2 - \|u\|_{H^1}^2 + o_n(1) + \frac{\gamma}{4p\pi} \left\langle V_0'(\tilde{u}_n), \tilde{u}_n - u \right\rangle - b \int |\tilde{u}_n|^{q-2} \tilde{u}_n (\tilde{u}_n - u).
\]

By compact embeddings \( X \hookrightarrow \hookrightarrow L^s(\mathbb{R}^2) \) for any \( p < s < \infty \) and Hardy-Littlewood-Sobolev inequalities, we have

\[
(3.47) \quad b \int |\tilde{u}_n|^{p-2} \tilde{u}_n (\tilde{u}_n - u) \longrightarrow 0, \quad \frac{\gamma}{4p\pi} \left\langle V_2'(\tilde{u}_n), \tilde{u}_n - u \right\rangle \longrightarrow 0.
\]

At the same time, one can also infer from Lemma 3.3 that

\[
(3.48) \quad \frac{\gamma}{4p\pi} \left\langle V_1'(\tilde{u}_n), \tilde{u}_n - u \right\rangle = \frac{\gamma}{2\pi} B_1 \left( |\tilde{u}_n|^p, |\tilde{u}_n|^{p-2} \tilde{u}_n (\tilde{u}_n - u) \right) \\
= \frac{\gamma}{2\pi} B_1 \left( |\tilde{u}_n|^p, |\tilde{u}_n|^{p-2} (|\tilde{u}_n - u|^2 + u(\tilde{u}_n - u)) \right) \\
= \frac{\gamma}{2\pi} B_1 \left( |\tilde{u}_n|^p, |\tilde{u}_n|^{p-2} |\tilde{u}_n - u|^2 \right) + \frac{\gamma}{2\pi} B_1 \left( |\tilde{u}_n|^p, |\tilde{u}_n|^{p-2} u(\tilde{u}_n - u) \right) \\
= \frac{\gamma}{2\pi} B_1 \left( |\tilde{u}_n|^p, |\tilde{u}_n|^{p-2} |\tilde{u}_n - u|^2 \right) + o_n(1).
\]

Define \(|v_n|^p := |\tilde{u}_n|^{p-2} |\tilde{u}_n - u|^2\) for every \( n \in \mathbb{N} \), then

\[
(3.49) \quad B_1 \left( |\tilde{u}_n|^p, |\tilde{u}_n|^{p-2} |\tilde{u}_n - u|^2 \right) = B_1 \left( |\tilde{u}_n|^p, |v_n|^p \right) \geq 0,
\]
and hence, we get from (3.46), (3.47), (3.48) and (3.49) that

(3.50) \[ o_n(1) = \langle I'(\tilde{u}_n), \tilde{u}_n - u \rangle \]
\[ = o_n(1) + \|\tilde{u}_n\|_{H^1}^2 - \|u\|_{H^1}^2 + B_1(|\tilde{u}_n|^p, |v_n|^p) \]
\[ \geq o_n(1) + \|\tilde{u}_n\|_{H^1}^2 - \|u\|_{H^1}^2, \]

which implies \( \|\tilde{u}_n\|_{H^1}^2 \rightarrow \|u\|_{H^1}^2 \) and \( B_1(|\tilde{u}_n|^p, |v_n|^p) \rightarrow 0 \) as \( n \rightarrow \infty \). Thus we derive \( \|\tilde{u}_n - u\|_{H^1(\mathbb{R}^2)} \rightarrow 0 \) as \( n \rightarrow \infty \). Again, by the compact embedding \( X \hookrightarrow L^p(\mathbb{R}^2) \), we get \( \|v_n\|_{L^p(\mathbb{R}^2)} \rightarrow 0 \). Thus by Lemma 3.2, we arrive at \( \|v_n\|_p \rightarrow 0 \).

Now, we apply Lemma 2.1 (take \( \epsilon = \frac{1}{2} \) therein) and obtain

(3.51) \[ o_n(1) = \|v_n\|_p^2 = \int_{\mathbb{R}^2} \log(1 + |x|)|\tilde{u}_n|^p - 2|\tilde{u}_n - u|^2 \, dx \]
\[ = \int_{\mathbb{R}^2} \log(1 + |x|)(|\tilde{u}_n - u|^p - 2|\tilde{u}_n|^p - |\tilde{u}_n - u|^p)|\tilde{u}_n - u|^2 \, dx \]
\[ = \int_{\mathbb{R}^2} \log(1 + |x|)|\tilde{u}_n - u|^p \, dx + \int_{\mathbb{R}^2} \log(1 + |x|)(|\tilde{u}_n|^p - 2|\tilde{u}_n - u|^p)|\tilde{u}_n - u|^2 \, dx \]
\[ \geq \frac{1}{2} \int_{\mathbb{R}^2} \log(1 + |x|)|\tilde{u}_n - u|^p \, dx - C \int_{\mathbb{R}^2} \log(1 + |x|)|u|^p|\tilde{u}_n - u|^2 \, dx. \]

By method similar to the proof of Lemma 3.3, we can also deduce

(3.52) \[ B_1(|\tilde{u}_n|^p, |u|^p - 2|\tilde{u}_n - u|^2) \rightarrow 0, \quad \text{as} \quad n \rightarrow \infty. \]

As a consequence, if we define \( \tilde{v}_n := |u|^p - 2|\tilde{u}_n - u|^2 \) for every \( n \in \mathbb{N} \), then Lemma 3.2 implies

(3.53) \[ \|\tilde{v}_n\|_p = \left( \int_{\mathbb{R}^2} \log(1 + |x|)|u|^p |\tilde{u}_n - u|^2 \, dx \right)^\frac{1}{p} \rightarrow 0, \quad \text{as} \quad n \rightarrow \infty. \]

Combining (3.51) and (3.53) yields that \( \|\tilde{u}_n - u\|_p \rightarrow 0 \). Combining with the \( H^1(\mathbb{R}^2) \) strong convergence, we finally derive the desired strong convergence in \( X \):

(3.54) \[ \|\tilde{u}_n - u\|_X = \|(-x_n) * u - u\|_X \rightarrow 0, \quad \text{as} \quad n \rightarrow \infty. \]

**Step 3:** We will prove \( u \in X \setminus \{0\} \) is a critical point of \( I \), i.e., \( I'(u) = 0 \).

For any given \( v \in X \), similar to the proof of (3.41), we can deduce that

(3.55) \[ \|x_n * v\|_X \leq C_5 + C_6 \log(1 + |x_n|), \]

and hence, by (3.43), we have

(3.56) \[ \|x_n * v\|_X \leq C \|u_n\|_X. \]

By assumption (3.2) and (3.56), we get

(3.57) \[ \left\langle I'(\tilde{u}_n), v \right\rangle = \left\langle I'(u_n), x_n * v \right\rangle \leq \|I'(u_n)\|_{X'} \|x_n * v\|_X \]
\[ \leq C \|I'(u_n)\|_{X'} \|u_n\|_X \rightarrow 0. \]

Note that \( \left\langle I'(u), v \right\rangle = \lim_{n \rightarrow \infty} \left\langle I'(\tilde{u}_n), v \right\rangle \), thus we have \( \left\langle I'(u), v \right\rangle = 0 \). This concludes our proof of Theorem 3.1. \( \square \)
3.2. Quantitative Deformation Lemma. In this subsection, we will construct the non-increasing flow based on the Cerami compactness condition in Theorem 3.1.

First, we introduce the some standard notations. For any given $c > 0$, we denote the set of critical points at the energy level $c$ by

$$K_c := \{ u \in X : I'(u) = 0, I(u) = c \}.$$  

For any $\rho > 0$, we define the set

$$A_{c,\rho} := \{ u \in X : \| u - v \|_{H^1} \leq \rho \ \text{for some} \ v \in K_c \}.$$  

For any $c \geq 0$, the sub-level set is denoted by

$$I^c := \{ u \in X : I(u) \leq c \},$$  

and, in particular,

$$D := I^0 = \{ u \in X : I(u) \leq 0 \}.$$  

We have the following lemma on important properties of the set $A_{c,\rho}$.

**Lemma 3.7.** Assume $p \geq 2$ and $q \geq 2p$. Then $A_{c,\rho}$ satisfies the following two properties:

1. For any $c > 0$ and $\rho > 0$, $A_{c,\rho}$ is symmetric w.r.t. the reflection $u \mapsto -u$ and invariant under $\mathbb{Z}^2$-translations;
2. For every given $c > 0$, there exists a $\rho_0(c) > 0$ such that, $A_{c,\rho} \cap D = \emptyset$, $\forall \rho \in (0, \rho_0)$.

**Proof.**

**Proof of 1:** This is an immediate consequence from the symmetry w.r.t. the reflection $u \mapsto -u$ and $\mathbb{Z}^2$-translations invariance of the energy functional $I$.

**Proof of 2:** We will prove it by contradiction arguments. Assume that there exist sequences $\rho_n \to 0$ and $\{ u_n \} \subset X$ s.t. $u_n \in A_{c,\rho_n} \cap D$ for each $n \in \mathbb{N}$. Taking $v_n \in K_c$ such that $\| u_n - v_n \|_{H^1} \leq \rho_n \to 0$. By Theorem 3.1, after passing to a subsequence, there exists a sequence $\{ x_n \} \subset \mathbb{Z}^2$ s.t. $x_n * v_n \to v \in K_c$ strongly in $X$. By the $\mathbb{Z}^2$-translations invariance, $x_n * u_n \in A_{c,\rho_n} \cap D$, and we have

$$\| x_n * u_n - v \|_{H^1} \leq \| x_n * u_n - x_n * v_n \|_{H^1} + \| x_n * v_n - v \|_{H^1} = \| u_n - v_n \|_{H^1} + \| x_n * v_n - v \|_{H^1} \leq \rho_n + o_n(1) \to 0,$$

as $n \to \infty$. However, since $x_n * u_n \in D$ and $I$ is l.s.c on $H^1(\mathbb{R}^2)$, we get

$$0 < c = I(v) \leq \liminf_{n \to \infty} I(x_n * u_n) \leq 0,$$

which is a contradiction. Hence we infer that, for each $c > 0$, there is a $\rho_0(c) > 0$ s.t., for all $0 < \rho < \rho_0(c)$, $A_{c,\rho} \cap D = \emptyset$. \qed

**Lemma 3.8** (Quantitative deformation lemma). Assume $p \geq 2$ and $q \geq 2p$. Let $c > 0$. Then, for any $\rho \in (0, \rho_0(c))$, there exists a $\epsilon = \epsilon(c, \rho) > 0$ and an odd continuous map $\varphi : I^+ \cap A_{c,\rho} \to I^{-\epsilon}$ such that $\varphi|_D = \text{id}_D$.

**Proof.** We fix $\rho \in (0, \rho_0(c))$ ($\rho_0(c)$ is the same as in Lemma 3.7) arbitrarily, and consider the sets

$$S := X \setminus A_{c,\rho} \quad \text{and} \quad \tilde{S}_{\delta} := \{ u \in X : \| u - v \|_{H^1} \leq \delta \ \text{for some} \ v \in S \}$$

for $\delta > 0$. We show that, for $\delta > 0$ small enough,

$$\| I'(u) \|_{X'}(1 + \| u \|_X) \geq 8\delta, \quad \forall u \in \tilde{S}_{2\delta} \cap I^{-1}([c - 2\delta^2, c + 2\delta^2]).$$
Suppose on contrary that there exist sequences $\delta_n > 0$ ($n \in \mathbb{N}$) and $\{u_n\} \in \tilde{S}_{2\delta_n}$ such that $\delta_n \rightarrow 0$ as $n \rightarrow \infty$ and
\begin{equation}
\|I'(u_n)\|_X(1 + \|u_n\|_X) < 8\delta_n, \quad c - 2\delta_n^2 \leq I(u_n) \leq c + 2\delta_n^2
\end{equation}
for every $n \in \mathbb{N}$. By Theorem \ref{thm:nonlin}, after passing to a subsequence, there exist points $x_n \in \mathbb{Z}^2$ ($n \in \mathbb{N}$) and $u \in K_c$ such that
\begin{equation}
\|x_n * u_n - u\|_X \rightarrow 0, \quad \text{as } n \rightarrow \infty.
\end{equation}
By definition \ref{eq:nonlin}, there is a sequence $\{v_n\} \subset S$ such that
\begin{equation}
\|u_n - v_n\|_{H^1(\mathbb{R}^2)} \leq 2\delta_n, \quad \forall n \in \mathbb{N}.
\end{equation}
Then it follows that
\begin{equation}
\|x_n * v_n - u\|_{H^1} \leq \|x_n * v_n - x_n * u_n\|_{H^1} + \|x_n * u_n - u\|_{H^1} \leq \|v_n - u_n\|_{H^1} + \|x_n * u_n - u\|_{H^1} \rightarrow 0, \quad \text{as } n \rightarrow \infty.
\end{equation}
Since, by $\mathbb{Z}^2$-translations invariance, $x_n * v_n \in S$, \ref{eq:nonlin} implies that
\begin{equation}
\|u - v\|_{H^1(\mathbb{R}^2)} \geq \rho, \quad \forall v \in K_c,
\end{equation}
which is absurd since $u \in K_c$. Thus \ref{eq:nonlin} holds for $\delta > 0$ sufficiently small. It is clear that \ref{eq:nonlin} still holds if $\tilde{S}_{2\delta}$ is replaced by the subset
\begin{equation}
S_{2\delta} := \{u \in X : \|u - v\|_X \leq 2\delta \text{ for some } v \in S\}.
\end{equation}
Now, we may fix a $\delta > 0$ small enough s.t. \ref{eq:nonlin} holds and
\begin{equation}
\varepsilon := \delta^2 < \frac{c}{2}.
\end{equation}
Since $I(u)$ is even w.r.t. $u$, by Lemma 2.6 in \cite{13}, there exists a continuous function $\eta : [0,1] \times X \rightarrow X$ such that
\begin{enumerate}
  \item $\eta(t,u) = u$ if $t = 0$ or $u \notin S_{2\delta} \cap I^{-1}([c - 2\varepsilon, c + 2\varepsilon])$;
  \item $\eta(1, I^{c+\varepsilon} \cap S) \subset I^{c-\varepsilon}$;
  \item $\eta(t, \cdot)$ is a homeomorphism of $X$, $\forall t \in [0,1]$;
  \item $t \mapsto I(\eta(t,u))$ is nonincreasing for all $u \in X$;
  \item $\eta(t, -u) = -\eta(t, u)$, $\forall t \in [0,1], u \in X$.
\end{enumerate}
As a consequence, since $D \cap I^{-1}([c - 2\varepsilon, c + 2\varepsilon]) = \emptyset$, if we let $\varphi(u) := \eta(1, u)$, then $\varphi$ is an odd continuous map satisfying
\begin{equation}
\varphi : I^{c+\varepsilon} \setminus A_{c,\rho} \rightarrow I^{c-\varepsilon} \quad \text{and} \quad \varphi|_D = id_D.
\end{equation}
This completes our proof of Lemma \ref{lemma:nonlin}. \hfill $\square$

4. Proof of Theorem \ref{thm:main}

In this section, using ideas from \cite{13}, we will derive the existence of ground state solutions to Schrödinger-Newton equations \ref{eq:schr} by constructing critical value for $I$, and hence complete the proof of Theorem \ref{thm:main}.

In the following, we assume $p \geq 2$, $q \geq 2p$, $a \in C(\mathbb{R}^2)$ and $\mathbb{Z}^2$-periodic with $\inf_{\mathbb{R}^2} a > 0$. We use $\gamma(A) \in \mathbb{N} \cup \{0, \infty\}$ to denote the Krasnoselski genus of a closed and symmetric subset $A \subset X$ (for definition and properties of $\gamma$, please refer to Chapter II. 5 in \cite{11}, see also \cite{13}). Next, let us recall the notion of relative genus $\gamma_D(A)$ for closed and symmetric subset $A \subset X$. 

Definition 4.1 (relative genus). Let $D \subset Y$ be a closed symmetric subset of $X$. We define the genus of $Y$ relative to $D$, denoted by $\gamma_D(A)$, as the smallest number $k$ such that $Y$ can be covered by closed and symmetric subsets $U, V$ with the following properties:

1) $D \subset U$, and there exists an odd continuous map $\chi : U \to D$ such that $\chi(u) = u$, $\forall u \in D$;
2) $\gamma(V) \leq k$.

If no such covering exists, we set $\gamma_D(A) := \infty$.

For properties of relative genus $\gamma_D$, please refer to [14], see also [13].

Lemma 4.2. ([14]) Let $D, Y$ and $Z$ be closed symmetric subsets of $X$ with $D \subset Y$. Then we have:

1) (Subadditivity) $\gamma_D(Y \cup Z) \leq \gamma_D(Y) + \gamma_D(Z)$.
2) If $D \subset Z$, and if there exists an odd continuous map $\varphi : Y \to Z$ with $\varphi(u) = u$ for any $u \in D$, then $\gamma_D(Y) \leq \gamma_D(Z)$.

Now we construct various possible critical values by

\begin{align}
(4.1) & \quad c_1 := \inf \{ c > 0 : \gamma_D(I)^c \geq 1 \}, \quad c_{mm} := \inf_{u \in X \setminus \{0\}} \sup_{t \in \mathbb{R}} I(tu), \\
(4.2) & \quad c_g := \inf \{ I(u) : u \in X \setminus \{0\}, I'(u) = 0 \}, \quad c_N = \inf_{u \in N} I(u),
\end{align}

where the Nehari manifold is given by

\begin{equation}
(4.3) \quad N = \{ u \in X \setminus \{0\} : \langle I'(u), u \rangle = 0 \}.
\end{equation}

We have the following crucial lemma on properties of the above possible critical values.

Lemma 4.3. Under the same assumptions as in Theorem 1.1, the following properties hold:

1) $0 < c_1 = c_g = c_N = c_{mm} < +\infty$.
2) $c_1$ is a critical value of energy functional $I$.

Proof. We will prove Lemma 4.3 by the following five steps.

Step 1. Since $u \in N$ if and only if $\varphi_u'(1) = 0$, it follows from Lemma 2.8 that $c_N = c_{mm}$. As a consequence of Lemma 2.6, we can also deduce $c_N = c_{mm} > 0$. By definition, it is clear that $c_g \geq c_N$, thus we have $0 < c_{mm} = c_N \leq c_g$.

Step 2. We will show that $c_1 \geq c_N$. Suppose it is false, then we can take a number $c > 0$ such that $c_1 < c < c_N$. Defining the sets

\begin{equation}
(4.4) \quad N^+ := \{ u \in X : \langle I'(u), u \rangle > 0 \} \quad \text{and} \quad N^- := \{ u \in X : \langle I'(u), u \rangle < 0 \}
\end{equation}

and the function

\begin{equation}
(4.5) \quad \tau : N^- \to [1, +\infty), \quad \tau(u) := \inf \{ t \geq 1 : I(tu) \leq 0 \}.
\end{equation}

By Lemma 2.8, $\tau$ is well-defined and it is an even and continuous function on $N^-$. Since $c > 0$ and $c_1 < c < c_N$, the closed symmetric subset $I^c \subset N^- \cup N^+ \cup \{0\}$. Now we define an odd map

\begin{equation}
(4.6) \quad \chi : I^c \to X, \quad \chi(u) = \tau(u)u, \quad \text{if } u \in N^-, \quad \chi(u) = 0, \quad \text{if } u \in N^+ \cup \{0\}.
\end{equation}

Since $N^\pm$ are open subsets and Lemma 2.6 yields that $B_\alpha(0) \setminus \{0\} \subset N^+$ for some $\alpha > 0$, thus $\chi$ is well-defined and it is a continuous map on $I^c$. If $u \in D$, then $I(u) \leq 0$, Lemma 2.8 implies that $u \in N^-$ and $\tau(u) = 1$, and hence we have $\chi|_D = id_D$, which gives us $\gamma_D(I^c) = 0$ by the
definition of the relative genus. However, since \( c > c_1 \), by the definition of \( c_1 \) and Lemma 4.12, one has \( \gamma_D(I') \geq \gamma_D(I^{c_1}) \geq 1 \), which is a contradiction. Thus we must have \( c_1 \geq c_N \).

Step 3. We will show \( c_1 \leq c_{mm} < +\infty \). For arbitrarily given \( u \in X \setminus \{0\} \) satisfying \( 0 < \sup_W I < +\infty \) with \( W := \mathbb{R}u \), by Lemma 2.8 we can choose a \( R > 0 \) large enough such that \( I(v) \leq 0 \) for all \( v \in W \) with \( \|v\|_{H^1} \geq R \), i.e., \( \{v \in W : \|v\|_{H^1} \geq R\} \subset D \). In order to prove \( c_1 \leq \sup_W I \), by definition of \( c_1 \) and Lemma 4.2, we only need to show \( \gamma_D(W \cup D) \geq 1 \), since we have \( I(v) \leq \sup_W I \) on \( W \cup D \). Suppose on contrary that \( \gamma_D(W \cup D) = 0 \), by the definition of the relative genus, there exists an odd continuous map \( \psi : W \cup D \to D \) such that \( \psi(v) = v \) for every \( v \in D \). Since \( \psi(0) = 0 \), \( \psi(v) = v \) for any \( v \in W \) with \( \|v\|_{H^1} \geq R \), by continuity, there exists a \( v_\alpha \in W \) such that \( \|\psi(v_\alpha)\|_{H^1} = \alpha \), where \( \alpha > 0 \) is the same constant given in Lemma 2.6. Hence, by Lemma 2.6 we get \( \psi(v_\alpha) \notin D \), which is a contradict with definition of the map \( \psi : W \cup D \to D \). Thus we arrive at \( c_1 \leq c_{mm} \).

Next, we show that \( c_{mm} < +\infty \). To this end, we choose arbitrarily a function \( u \in X \setminus \{0\} \) with \( \text{supp} \ u \subseteq B_{\frac{1}{\eta}}(0) \subset \mathbb{R}^2 \). Then, note that \( |x - y| < 1 \) for every \( x, y \in B_{\frac{1}{\eta}}(0) \), we have

\[
V_0(u) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(|x - y|)|u|^p(x)|u|^p(y)dxdy < 0.
\]

It follows easily from the proof of Lemma 2.8 (1) that

\[
c_{mm} \leq \sup_{t \in \mathbb{R}} I(tu) < +\infty.
\]

Step 4. Now we will prove (2). Assume \( c_1 \) is not a critical value, then \( K_{c_1} = \emptyset \), and hence \( A_{c_1, \rho} = \emptyset \) for every \( \rho > 0 \). Thus the quantitative deformation lemma (Lemma 3.8) implies that, there exist a \( 0 < \varepsilon < \frac{c_2}{\gamma} \) and an odd continuous map \( \varphi : I^{c_{1+\varepsilon}} \to I^{c_{1-\varepsilon}} \) s.t. \( \varphi|_D = \text{id}_D \). This combining with Lemma 4.2 yield that \( \gamma_D(I^{c_{1-\varepsilon}}) \geq 1 \), which leads to a contradiction with the definition of \( c_1 \). This finishes our proof of (2).

Step 5. By Steps 1-3, we have already shown \( 0 < c_1 = c_N = c_{mm} < +\infty \) and \( c_1 = c_N = c_{mm} \leq c_g \). Then, by Step 4 and the definition of \( c_g \), we have \( c_g \leq c_1 \), thus we finally arrive at \( 0 < c_1 = c_g = c_N = c_{mm} < +\infty \). This completes our proof of (1) and Lemma 4.3.

Now we are ready to complete our proof of Theorem 1.1.

Proof of Theorem 1.1 (completed). By Lemma 4.3, we derive existence of critical points \( \pm u \in X \setminus \{0\} \) of \( I \) such that

\[
I(u) = c_g \quad \text{and} \quad c_g = c_1 = \inf_{t \in \mathbb{R}} I = \inf_{u \in X \setminus \{0\}} \sup_{t \in \mathbb{R}} I(tu) > 0,
\]

and hence \( \pm u \) are ground state solutions for the Schrödinger-Newton equation (1.1).

Next, let \( u \in N \) be an arbitrary minimizer of \( I|_N \), i.e., \( I(u) = \inf_N I \). We are to show that \( u \) is a critical point of \( I \). If not, suppose there exists \( v \in X \) s.t. \( \langle I'(u), v \rangle < 0 \). Since \( I' \) is continuous in \( X \), there exist \( \eta > 0 \) and \( \delta > 0 \) small enough such that

\[
\langle I'(t(u + sv)), v \rangle < 0, \quad \forall \ t \in [1 - \delta, 1 + \delta], \ s \in [-\eta, \eta].
\]

Since \( u \in N \), by Lemma 2.8 we can deduce

\[
\langle I'((1 - \delta)u), (1 - \delta)u \rangle > 0 > \langle I'((1 + \delta)u), (1 + \delta)u \rangle.
\]

Then, by continuity of \( I' \), there exists a \( t \in (0, \eta) \) small enough s.t.

\[
\langle I'((1 - \delta)(u + \hat{s}v)), (1 - \delta)(u + \hat{s}v) \rangle > 0 > \langle I'((1 + \delta)(u + \hat{s}v)), (1 + \delta)(u + \hat{s}v) \rangle,
\]
which implies that there exists some $\hat{t} \in (1 - \delta, 1 + \delta)$ such that $\hat{t}(u + \hat{s}v) \in \mathcal{N}$. However, by Lemma 2.8 and (4.10), we have

$$I(\hat{t}(u + \hat{s}v)) - I(u) \leq I(\hat{t}(u + \hat{s}v)) - I(\hat{t}u) = -\hat{t} \int_0^1 \langle I'(\hat{t}(u + sv)), v \rangle ds < 0,$$

which yields a contradiction with $I(u) = \inf_{\mathcal{N}} I$. Thus $u$ must be a critical point of $I$.

Finally, if $u \in \mathcal{N}$ is a minimizer of $I|_{\mathcal{N}}$, then $|u| \in \mathcal{N}$ is also a minimizer of $I|_{\mathcal{N}}$. Hence $|u|$ is a critical point of $I$ and satisfies the Euler-Lagrange equation (5.1). By Lemma 2.4 we have $|u| \in W^{2,r}(\mathbb{R}^2)$ $(\forall 1 \leq r < +\infty)$ is a strong solution to elliptic equation of the form $-\Delta |u| + \zeta(x)|u| = 0$ with $\zeta \in L^\infty_{loc}(\mathbb{R}^2)$. Since $|u| \neq 0$, by Harnack inequality (see [24]), we have $|u| > 0$ on $\mathbb{R}^2$ and hence $u$ does not change sign. This concludes our proof of Theorem 1.1.

5. Proof of Theorem 1.2

In this section, we will prove Theorem 1.2. We first prove that mountain pass energy level $c_{mp} > 0$ and there exists a critical point $u \in X \setminus \{0\}$ of $I$ such that $I(u) = c_{mp}$ (for the definition of $c_{mp}$, see (1.7)). To this end, we need the following general minimax principle from Proposition 2.8 in [31], which will lead to Cerami sequences instead of Palais-Smale sequences.

**Proposition 5.1** ([31]). Assume $X$ is a Banach space. Let $M_0$ be a closed subspace of the metric space $M$ and $\Gamma_0 \subset C(M_0, X)$. Define

$$\Gamma := \{ \gamma \in C(M, X) : \gamma|_{M_0} \in \Gamma_0 \}.$$ 

If $\phi \in C^1(X, \mathbb{R})$ satisfies

$$+\infty > c := \inf_{\gamma \in \Gamma} \sup_{t \in M} \phi(\gamma(t)) > \sigma := \sup_{\gamma_0 \in \Gamma_0} \sup_{t \in M_0} \phi(\gamma_0(t)),$$

then, for every $\epsilon \in (0, \frac{\sigma - c}{2})$, $\delta > 0$ and $\gamma \in \Gamma$ with $\sup_{t \in M} \phi(\gamma(t)) \leq c + \epsilon$, there exists $u \in X$ such that

(i) $c - 2\epsilon \leq \phi(u) \leq c + 2\epsilon$,

(ii) $\text{dist}(u, \gamma(M)) \leq 2\delta$,

(iii) $(1 + \|u\|_X)\|\phi'(u)\|_{X'} \leq \frac{8\epsilon}{\delta}$.

Lemmas 2.6 and 2.7 implies that

$$0 < \inf_{\|u\|_M = \alpha} I(u) \leq c_{mp} < +\infty,$$

and hence the functional $I$ has a mountain pass geometry. Now we will use Proposition 5.1 to show the existence of a Cerami sequence $\{u_n\} \subset X$ at the mountain pass energy level $c_{mp}$ with a key additional property $J_k(u_n) \to 0$ $(k = 1$ or $2)$ as $n \to +\infty$.

**Lemma 5.2.** Assume $p \geq 2$ and $q > 2$. Then, for any given $k = 1$ or $2$, there exists a sequence $\{u_{nk}\} \subset X$ such that, as $n \to +\infty$,

$$I(u_{nk}) \to c_{mp}, \quad \|I'(u_{nk})\|_{X'} (1 + \|u_{nk}\|_X) \to 0, \quad J_k(u_{nk}) \to 0,$$

where the auxiliary functional $J_k$ is defined by (1.8).

**Proof.** Let $k = 1$ or $2$ be fixed. Following the ideas from [20] (see also [21, 25, 35]), we define the Banach space

$$\mathcal{X} := \mathbb{R} \times X, \quad \|(s, v)\|_{\mathcal{X}}^2 := |s|^2 + \|v\|_X^2, \quad \forall (s, v) \in \mathbb{R} \times X,$$
and the continuous map

\[ \rho_k : \tilde{X} \rightarrow X, \quad \rho_k(s, v)(\cdot) = e^{ks}v(e^s), \quad \forall (s, v) \in \mathbb{R} \times X, \]

which is linear with respect to \( v \in X \) for any fixed \( s \in \mathbb{R} \). Moreover, we define a functional on \( \tilde{X} \) by \( \phi_k(s, v) := I(\rho_k(s, v)) \), then we have

\[ \phi_k(s, v) = \frac{1}{2} e^{2ks} \int_{\mathbb{R}^2} |\nabla v|^2 \, dx + \frac{1}{2} e^{2(k-1)s} \int_{\mathbb{R}^2} a|v|^2 \, dx 
+ \frac{\gamma}{4p\pi} e^{2(kp-2)s} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(|x - y|)|v|^p(x)|v|^p(y) \, dx \, dy 
- \frac{\gamma}{4p\pi} (kq - 2s) e^{2(kp-2)s} \left( \int_{\mathbb{R}^2} |v|^p \, dx \right)^2 - \frac{b}{q} e^{(kq-2)s} \int_{\mathbb{R}^2} |v|^q \, dx. \]

Through direct calculations, we get

\[ \partial_s \phi_k(s, v) 
= k e^{2ks} \int_{\mathbb{R}^2} |\nabla v|^2 \, dx + (k - 1) e^{2(k-1)s} \int_{\mathbb{R}^2} a|v|^2 \, dx - \frac{(kq - 2b)}{q} \int_{\mathbb{R}^2} |v|^q \, dx 
+ \frac{\gamma}{2p\pi} (kp - 2) e^{2(kp-2)s} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(|x - y|)|v|^p(x)|v|^p(y) \, dx \, dy 
- \frac{\gamma}{4p\pi} [1 + 2s (kp - 2)] e^{2(kp-2)s} \left( \int_{\mathbb{R}^2} |v|^p \, dx \right)^2 \]

\[ = k \int_{\mathbb{R}^2} |\nabla \rho_k|^2 \, dx + (k - 1) \int_{\mathbb{R}^2} a|\rho_k|^2 \, dx - \frac{b}{q} \int_{\mathbb{R}^2} |\rho_k|^q \, dx 
+ \frac{\gamma}{2p\pi} (kp - 2) \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(|x - y|)|\rho_k|^p(x)|\rho_k|^p(y) \, dx \, dy 
- \frac{\gamma}{4p\pi} \left( \int_{\mathbb{R}^2} |\rho_k|^p \, dx \right)^2 \]

\[ = J_k(\rho_k(s, v)), \]

where the auxiliary functional \( J_k \) is defined by \( (5.3) \). Moreover, one has

\[ \langle \partial_s \phi_k(s, v), w \rangle = \frac{d}{dt} \phi_k(s, v + tw) \bigg|_{t=0} = \frac{d}{dt} I(\rho_k(s, v + tw)) \bigg|_{t=0} 
= \frac{d}{dt} I(\rho_k(s, v) + t\rho_k(s, w)) \bigg|_{t=0} = \langle I'(\rho_k(s, v)), \rho_k(s, w) \rangle. \]

Thus the functional \( \phi_k \in C^1(\tilde{X}, \mathbb{R}) \) and for any \( s \in \mathbb{R} \) and \( v, w \in X \),

\[ \langle \phi_k'(s, v), (h, w) \rangle = \partial_s \phi_k(s, v)h + \langle \partial_v \phi_k(s, v), w \rangle 
= J_k(\rho_k(s, v))h + \langle I'(\rho_k(s, v)), \rho_k(s, w) \rangle. \]

Now we define the following mountain pass value for \( \tilde{X} \) and \( \phi_k \):

\[ c_{mp,k}^s := \inf_{\tilde{\gamma} \in \tilde{\Gamma}_k} \max_{t \in [0,1]} \phi_k(\tilde{\gamma}(t)), \]

where

\[ \tilde{\Gamma}_k = \left\{ \tilde{\gamma} \in C([0,1], \tilde{X}) \mid \tilde{\gamma}(0) = (0,0), \phi_k(\tilde{\gamma}(1)) < 0 \right\} . \]
One can easily observe that $\Gamma = \{ \rho_k \circ \tilde{\gamma} \mid \tilde{\gamma} \in \tilde{\Gamma}_k \}$ (for the definition of $\Gamma$, see (1.7)), and hence $c_{mp} = c_{mp,k}^e$. By the definition of $c_{mp}$, for any $n \in \mathbb{N}$, there exists a $\gamma_n \in \Gamma$ such that

$$\max_{t \in [0,1]} I(\gamma_n(t)) \leq c_{mp} + \frac{1}{n^2}. \tag{5.11}$$

Then we define $\tilde{\gamma}_n \in \tilde{\Gamma}_k$ by $\tilde{\gamma}_n(t) := (0, \gamma_n(t))$ and obtain that

$$\max_{t \in [0,1]} \phi_k(\tilde{\gamma}_n(t)) = \max_{t \in [0,1]} I(\gamma_n(t)) \leq c_{mp} + \frac{1}{n^2}. \tag{5.12}$$

For any $n \in \mathbb{N}$, by taking $X = \tilde{X}$, $\Gamma = \tilde{\Gamma}_k$, $M = [0,1]$, $M_0 = \{0,1\}$, $\gamma = \tilde{\gamma}_n$, $\epsilon = \frac{c_{mp}}{4n^2}$ and $\delta = \frac{1}{n}$ in Proposition 5.1, we can deduce from Proposition 5.1 that, there exists a sequence $\{(s_n^k, v_n^k)\} \in \tilde{X}$ such that

$$c_{mp} - \frac{c_{mp}}{2n^2} \leq \phi_k(s_n^k, v_n^k) \leq c_{mp} + \frac{c_{mp}}{2n^2}, \tag{5.13}$$

$$\text{dist} \left( \{ (s_n^k, v_n^k) \}, \{0\} \times \gamma_n([0,1]) \right) \leq \frac{2}{n}. \tag{5.14}$$

$$\left(1 + \left\| (s_n^k, v_n^k) \right\|_{\tilde{X}} \right) \left\| \phi_k'(s_n^k, v_n^k) \right\|_{\tilde{X}^*} \leq \frac{2c_{mp}}{n}. \tag{5.15}$$

It follow directly from (5.14) that $\lim_{n \to +\infty} s_n^k = 0$. Moreover, by taking $h = 1$ and $w = 0$ in (5.8), we get

$$\left| \left\langle \phi_k'(s_n^k, v_n^k), (1,0) \right\rangle \right| = \left| J_k \left( \rho_k \left( s_n^k, v_n^k \right) \right) \right| \leq \left\| \phi_k'(s_n^k, v_n^k) \right\|_{\tilde{X}^*} \to 0. \tag{5.16}$$

Combining this with (5.13) yields that, for $u_n^k := \rho_k(s_n^k, v_n^k) \in X$,

$$I(u_n^k) \longrightarrow c_{mp}, \quad J_k(u_n^k) \to 0, \quad \text{as } n \to +\infty. \tag{5.17}$$

Furthermore, we can show that

$$\left\| u_n^k \right\|_X \leq 2 \left( 1 + o_n(1) \right) \left\| v_n^k \right\|_X \leq 2 \left( 1 + o_n(1) \right) \left\| (s_n^k, v_n^k) \right\|_{\tilde{X}}. \tag{5.18}$$

Indeed, through direct calculations, we get

$$\left\| u_n^k \right\|^2_{H^1(\mathbb{R}^2)} := \int_{\mathbb{R}^2} (|\nabla u_n^k|^2 + a|u_n^k|^2) \, dx = e^{2k s_n^k} \int_{\mathbb{R}^2} |\nabla v_n^k|^2 \, dx + e^{2(k-1)s_n^k} \int_{\mathbb{R}^2} a|v_n^k|^2 \, dx = (1 + o_n(1)) \int_{\mathbb{R}^2} (|\nabla v_n^k|^2 + a|v_n^k|^2) \, dx = (1 + o_n(1)) \left\| v_n^k \right\|^2_{H^1(\mathbb{R}^2)}, \tag{5.19}$$

where $s_n^k = c_{mp,k}^e + \frac{1}{n^2}$. 

\[ |u_n^k|^p := \left\| u_n^k \right\|_{L^p(d\mu)}^p = \int_{\mathbb{R}^2} \log(1 + |x|) |u_n^k|^p(x)dx \]
\[ = \int_{\mathbb{R}^2} \log(1 + |x|) \left| e^{k s_n^k} v_n^k \right|^p \left( e^{k \phi_n^k} x \right) dx \]
\[ = e^{(k p - 2) s_n^k} \int_{\mathbb{R}^2} \log \left( 1 + e^{-s_n^k} |x| \right) |v_n^k|^p(x)dx \]
\[ = (1 + o_n(1)) \int_{\mathbb{R}^2} \log(1 + |x|) |v_n^k|^p dx - s_n^k (1 + o_n(1)) \int_{\mathbb{R}^2} |v_n^k|^p (x)dx \]
\[ = (1 + o_n(1)) |v_n^k|^p + o_n(1) \left\| v_n^k \right\|_{H^1(\mathbb{R}^2)}^p \]
\[ \leq (1 + o_n(1)) \left\| v_n^k \right\|_X^p. \]

Consequently, we arrive at
\[ \left\| u_n^k \right\|_X = \left\| u_n^k \right\|_{H^1(\mathbb{R}^2)} + |u_n^k|_* \leq 2 (1 + o_n(1)) \left\| v_n^k \right\|_X , \]
and hence (5.18) holds.

Next, we will show that \( \left\| I'(u_n^k) \right\|_X \rightarrow 0 \). To this end, for any \( w \in X \), let us define \( w_n^k := e^{-k s_n^k} w(e^{-s_n^k} \cdot) \in X \), then we have
\[ \left\| w_n^k \right\|_{H^1(\mathbb{R}^2)}^2 = e^{-2 k s_n^k} \int_{\mathbb{R}^2} |\nabla w|^2 dx + e^{-2(k - 1) s_n^k} \int_{\mathbb{R}^2} aw^2 dx \]
\[ = (1 + o_n(1)) \int_{\mathbb{R}^2} (|\nabla w|^2 + aw^2) dx = (1 + o_n(1)) \left\| w \right\|_{H^1(\mathbb{R}^2)}^2, \]
\[ |w_n^k|^p = \int_{\mathbb{R}^2} \log(1 + |x|) \left| e^{-k s_n^k} w \right|^p \left( e^{-s_n^k} x \right) dx \]
\[ = e^{-(k p - 2) s_n^k} \int_{\mathbb{R}^2} \log \left( 1 + e^{s_n^k} |x| \right) |w|^p (x)dx \]
\[ = (1 + o_n(1)) \int_{\mathbb{R}^2} \log(1 + |x|) |w|^p dx + s_n^k (1 + o_n(1)) \int_{\mathbb{R}^2} |w|^p (x)dx \]
\[ = (1 + o_n(1)) |w|^p + o_n(1) \left\| w \right\|_{H^1(\mathbb{R}^2)}^p \]
\[ \leq (1 + o_n(1)) \left\| w \right\|_X^p, \]
and hence
\[ \left\| w_n^k \right\|_X \leq 2 (1 + o_n(1)) \left\| w \right\|_X. \]

By (5.8) with \( h = 0 \) and (5.24), we can infer that
\[ \left| \left\langle I'(u_n^k) \right|_w \right| \left| \left\langle I' \left( \rho_k (s_n^k, v_n^k) \right) , \rho_k (s_n^k, w_n^k) \right| \right| \]
\[ = \left| \left\langle \partial_s \phi_k (s_n^k, v_n^k) , w_n^k \right| \right| \left| \left\langle \phi_k^\prime (s_n^k, v_n^k) , (0, w_n^k) \right| \right| \]
\[ \leq \left\| \phi_k^\prime (s_n^k, v_n^k) \right\|_X \left\| u_n^k \right\|_X \leq 2 (1 + o_n(1)) \left\| \phi_k^\prime (s_n^k, v_n^k) \right\|_X \left\| w \right\|_X, \]
which implies that
\[ \left\| I'(u_n^k) \right\|_X \leq C \left\| \phi_k^\prime (s_n^k, v_n^k) \right\|_X, \]
Combining this with (5.15) and (5.18) yields that
\[ \left\| I'(u_n^k) \right\|_X , (1 + \left\| u_n^k \right\|_X) \leq C \left\| \phi_k^\prime (s_n^k, v_n^k) \right\|_X , (1 + \left\| (s_n^k, v_n^k) \right\|_X) \rightarrow 0, \]
as \( n \to +\infty \). This concludes our proof of Lemma 5.2.

Next, we will prove the following key lemma which indicates that any sequence \( \{u_n\} \) satisfying (5.2) (with \( k = 1 \) and \( p \geq 3 \) or \( k = 2 \) and \( 2 \leq p < 3 \)) is bounded in \( H^1(\mathbb{R}^2) \).

**Lemma 5.3.** Assume \( p \geq 2 \), \( q \geq 2p - 2 \) and \( q > 2 \). If \( p \geq 3 \), let \( \{u_n\} \subset X \) satisfies

\[
(5.28) \quad c := \sup_{n \in \mathbb{N}} I(u_n) < +\infty, \quad J_1(u_n) \to 0;
\]

if \( 2 \leq p < 3 \), assume further that \( q \geq 2p - 1 \) or \( q < p + 1 \) and let \( \{u_n\} \subset X \) satisfies

\[
(5.29) \quad c := \sup_{n \in \mathbb{N}} I(u_n) < +\infty, \quad \|I'(u_n)\|_{X'}(1 + \|u_n\|_X) \to 0, \quad J_2(u_n) \to 0,
\]
as \( n \to +\infty \). Then \( \{u_n\} \) is bounded in \( H^1(\mathbb{R}^2) \).

**Proof.** We first consider the cases \( p \geq 3 \). By (5.28), we have

\[
(5.30) \quad c + o(1) \geq I(u_n) - \frac{1}{2(p-2)} J_1(u_n)
= \left( \frac{1}{2} - \frac{1}{2(p-2)} \right) \int_{\mathbb{R}^2} |\nabla u_n|^2 \, dx + \frac{a}{2} \int_{\mathbb{R}^2} u_n^2 \, dx
+ \frac{\gamma}{8 \pi p (p-2)} \left( \int_{\mathbb{R}^2} |u_n|^p \, dx \right)^2 + \frac{b}{q} \left( \frac{q-2}{2} - 1 \right) \int_{\mathbb{R}^2} |u_n|^q \, dx.
\]

If \( p > 3 \), it follows immediately from (5.30) that \( \{u_n\} \) is bounded in \( H^1(\mathbb{R}^2) \) due to \( q \geq 2p - 2 \).

If \( p = 3 \) and \( q > 2p - 2 = 4 \), then (5.30) implies that \( \{u_n\} \) is bounded in \( L^2(\mathbb{R}^2) \), \( L^3(\mathbb{R}^2) \), and \( L^q(\mathbb{R}^2) \) if \( b > 0 \). Thus we infer from (2.12) and the Gagliardo-Nirenberg inequality that

\[
(5.31) \quad \|u_n\|_{H^1(\mathbb{R}^2)}^2 = 2 I(u_n) + \frac{\gamma}{6 \pi} (V_2(u_n) - V_1(u_n)) + \frac{2b}{q} \|u_n\|_{L^q(\mathbb{R}^2)}^q
\leq 2c + C \|u_n\|_{L^4(\mathbb{R}^2)}^6 + \frac{2b}{q} \|u_n\|_{L^q(\mathbb{R}^2)}^q
\leq 2c + C \|u_n\|_{L^3(\mathbb{R}^2)}^3 \|\nabla u_n\|_{L^2(\mathbb{R}^2)}^3 + \frac{2b}{q} \|u_n\|_{L^q(\mathbb{R}^2)}^q
\leq C_1 + C_2 \|u_n\|_{H^1(\mathbb{R}^2)}^2,
\]
and hence \( \{u_n\} \) is bounded in \( H^1(\mathbb{R}^2) \). In the case \( p = 3 \) and \( q = 4 \), it follows from (5.30) that \( \{u_n\} \) is bounded in \( L^2(\mathbb{R}^2) \) and \( L^3(\mathbb{R}^2) \). Thus we deduce from (2.12) and the Gagliardo-Nirenberg inequality that

\[
(5.32) \quad \|u_n\|_{H^1(\mathbb{R}^2)}^2 = 2 I(u_n) + \frac{\gamma}{6 \pi} (V_2(u_n) - V_1(u_n)) + \frac{b}{2} \|u_n\|_{L^4(\mathbb{R}^2)}^4
\leq 2c + C \|u_n\|_{L^4(\mathbb{R}^2)}^6 + \frac{b}{2} \|u_n\|_{L^4(\mathbb{R}^2)}^4
\leq 2c + C \|u_n\|_{L^3(\mathbb{R}^2)}^3 \|\nabla u_n\|_{L^2(\mathbb{R}^2)}^3 + C \|u_n\|_{L^4(\mathbb{R}^2)}^3 \|\nabla u_n\|_{L^2(\mathbb{R}^2)}^3
\leq C_1 + C_2 \|u_n\|_{H^1(\mathbb{R}^2)}^2 + C_3 \|u_n\|_{H^1(\mathbb{R}^2)}^2,
\]
and hence \( \{u_n\} \) is bounded in \( H^1(\mathbb{R}^2) \).
Next, we consider the cases $2 \leq p < 3$. From (5.29), we get
\[
c + o(1) \geq I(u_n) - \frac{1}{4(p-1)} J_2(u_n)
\]
\[
(5.33) \quad = \left(\frac{1}{2} - \frac{1}{2(p-1)}\right) \int_{\mathbb{R}^2} |\nabla u_n|^2 \, dx + \left(\frac{1}{2} - \frac{1}{4(p-1)}\right) \int_{\mathbb{R}^2} au_n^2 \, dx
\]
\[
+ \frac{\gamma}{16\pi(p-1)} \left(\int_{\mathbb{R}^2} |u_n|^p \, dx\right)^2 + \frac{(q+1-2p)b}{2(p-1)} \int_{\mathbb{R}^2} |u_n|^q \, dx.
\]
We will discuss two different cases $q \geq 2p - 1$ and $q < 2p - 1$.

Case (i) $q \geq 2p - 1$. If $p > 2$, it follows immediately from (5.33) that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^2)$. If $p = 2$ and $q > 2p - 1 = 3$, then (5.33) implies that $\{u_n\}$ is bounded in $L^2(\mathbb{R}^2)$ and $L^q(\mathbb{R}^2)$ if $b > 0$. Thus, we infer from (2.12) and the Gagliardo-Nirenberg inequality that
\[
\|u_n\|_{H^1(\mathbb{R}^2)}^2 = 2I(u_n) + \frac{\gamma}{4\pi} (V_2(u_n) - V_1(u_n)) + \frac{2b}{q} \|u_n\|_{L^q(\mathbb{R}^2)}^q
\]
\[
\leq 2c + C \|u_n\|_{L^4(\mathbb{R}^2)}^4 + \frac{2b}{q} \|u_n\|_{L^q(\mathbb{R}^2)}^q,
\]
and hence $\{u_n\}$ is bounded in $H^1(\mathbb{R}^2)$. In the case $p = 2$ and $q = 3$, it follows from (5.33) that $\{u_n\}$ is bounded in $L^2(\mathbb{R}^2)$. Thus we deduce from (2.12) and the Gagliardo-Nirenberg inequality that
\[
\|u_n\|_{H^1(\mathbb{R}^2)}^2 = 2I(u_n) + \frac{\gamma}{4\pi} (V_2(u_n) - V_1(u_n)) + \frac{2b}{3} \|u_n\|_{L^3(\mathbb{R}^2)}^3
\]
\[
\leq 2c + C \|u_n\|_{L^2(\mathbb{R}^2)}^3 \|\nabla u_n\|_{L^2(\mathbb{R}^2)} + \frac{2b}{3} \|u_n\|_{L^3(\mathbb{R}^2)}^3
\]
\[
\leq C_1 + C_2 \|u_n\|_{H^1(\mathbb{R}^2)},
\]
and hence $\{u_n\}$ is bounded in $H^1(\mathbb{R}^2)$.

Case (ii) $q < 2p - 1$. Assume further that $q < p + 1$. We will first show that $\|\nabla u_n\|_{L^2(\mathbb{R}^2)}$ ($n \in \mathbb{N}$) are bounded.

If not, suppose on the contrary that, after extracting a subsequence, we have $\|\nabla u_n\|_{L^2(\mathbb{R}^2)} \to +\infty$ as $n \to +\infty$. Let $t_n := \|\nabla u_n\|_{L^2(\mathbb{R}^2)}^{-\frac{1}{2}} (n \in \mathbb{N})$, so $t_n \to 0$ as $n \to +\infty$. We define the rescaled functions $v_n := t_n^2 u_n(t_n) \in X$ for any $n \in \mathbb{N}$. Then we have
\[
\|\nabla v_n\|_{L^2} = 1 \quad \text{and} \quad \|v_n\|_{L^r} = t_n^{2r-2} \|u_n\|_{L^r}, \quad \forall \, n \in \mathbb{N}, \ 1 \leq r < +\infty.
\]
From the Gagliardo-Nirenberg inequality, one can deduce that, for any $n \in \mathbb{N}$,
\[
(5.37) \quad \|v_n\|_{L^p}^p \leq C \|v_n\|_{L^2} \|\nabla v_n\|_{L^2}^{p-2} = C \|v_n\|_{L^2}^2,
\]
\[
(5.38) \quad \|v_n\|_{L^q}^q \leq C \|v_n\|_{L^p} \|\nabla v_n\|_{L^2}^{q-p} = C \|v_n\|_{L^p}^p.
\]
Multiplying (5.33) by \( t_n^{2p} \), we can infer from (5.36), (5.37) and (5.38) that
\[
ct_n^4 + o(t_n^4) \\
\geq \left( \frac{1}{2} - \frac{1}{2(p-1)} \right) t_n^4 \| \nabla u_n \|^2_{L^2} + a \left( \frac{1}{2} - \frac{1}{4(p-1)} \right) t_n^4 \| u_n \|^2_{L^2} \\
+ \frac{\gamma}{16\pi p(p-1)} t_n^4 \| u_n \|^2_{L^p} - \frac{(2p-1-q)b}{2(p-1)q} t_n^4 \| u_n \|^q_{L^q} \\
\geq \frac{1}{2} - \frac{1}{2(p-1)} + Ca \left( \frac{1}{2} - \frac{1}{4(p-1)} \right) t_n^2 \| v_n \|^p_{L^p} \\
+ \frac{\gamma}{16\pi p(p-1)} t_n^{4(2-p)} \| v_n \|^2_{L^p} - C \left( \frac{2p-1-q}{2(p-1)q} \right) t_n^{6-2q} \| v_n \|^q_{L^p}.
\]
(5.39)

Consequently, we obtain a contradiction when \( b = 0 \). If \( b > 0 \), then (5.39) yields that
\[
\| v_n \|_{L^p(\mathbb{R}^2)} = O \left( t_n^{\frac{2(2p-1-q)}{p}} \right).
\]
(5.40)

In the cases \( 2 < p < 3 \), (5.39) also implies that
\[
\| v_n \|_{L^p(\mathbb{R}^2)} \geq C t_n^{\frac{2(q-3)}{p}},
\]
(5.41)

which is in contradiction with (5.40) due to \( q < p + 1 \). If \( p = 2 \) and \( q < 3 \), by (2.12), (5.29), (5.36), (5.40) and the Gagliardo-Nirenberg inequality, we also have
\[
o_n(1) = t_n^4 J_2(u_n) = t_n^4 \left( 2 \| \nabla u_n \|^2_{L^2} + a \| u_n \|^2_{L^2} \\
+ \frac{\gamma}{2\pi} V_0(u_n) - \frac{\gamma}{8\pi} \| u_n \|^4_{L^2} - \frac{2(q-1)b}{q} \| u_n \|^q_{L^q} \right) \\
= 2 + at_n^2 \| v_n \|^2_{L^2} + \frac{\gamma}{2\pi} V_0(v_n) + \frac{\gamma}{2\pi} \log t_n \| v_n \|^4_{L^2} \\
- \frac{\gamma}{8\pi} \| v_n \|^4_{L^2} - \frac{2(q-1)b}{q} t_n^{6-2q} \| v_n \|^q_{L^q} \\
\geq 2 - C \| v_n \|^4_{L^\infty} + o_n(1) \\
\geq 2 - C \| v_n \|^4_{L^2} + o_n(1) = 2 + o_n(1),
\]
which is absurd. Therefore, \( \| \nabla u_n \|_{L^p(\mathbb{R}^2)} \) (\( n \in \mathbb{N} \)) are bounded provided that \( q < p + 1 \).

Now we can deduce from (5.29) that
\[
\frac{\gamma}{4\pi p} \| u_n \|_{L^p}^{2p} \\
= \langle J'(u_n), u_n \rangle - J_2(u_n) + \| \nabla u_n \|^2_{L^2} - \frac{(q-2)b}{q} \| u_n \|^q_{L^q} + \frac{(p-2)\gamma}{2\pi p} V_0(u) \\
= o_n(1) + \| \nabla u_n \|^2_{L^2} - \frac{(q-2)b}{q} \| u_n \|^q_{L^q} \\
+ (p-2) \left( 2I(u_n) - \| \nabla u_n \|^2_{L^2} - a \| u_n \|^2_{L^2} + \frac{2b}{q} \| u_n \|^q_{L^q} \right) \\
\leq o_n(1) + 2(p-2)c + (3-p) \| \nabla u_n \|^2_{L^2} - \frac{(q-2p+2)b}{q} \| u_n \|^q_{L^q} \leq C,
\]
(5.43)
Lemma 5.4. Assume $p \geq 2$, $q \geq 2p - 2$ and $q > 2$. If $p \geq 3$, suppose the sequence $\{u_n\} \subset X$ satisfies (5.28) with $\|I'(u_n)\|_{X'} (1 + \|u_n\|_X) \to 0$ as $n \to +\infty$. If $2 \leq p < 3$, suppose the sequence $\{u_n\} \subset X$ satisfies (5.29) and $q \geq 2p - 1$ or $q < p + 1$. Then, by extracting a subsequence, one of the following two conclusions must hold:

(a) $\|u_n\|_{H^1(\mathbb{R}^2)} \to 0$ and $I(u_n) \to 0$ as $n \to +\infty$.

(b) There exists a sequence $\{x_n\} \in \mathbb{R}^2$ such that

$$x_n * u_n \to u \quad \text{strongly in } X, \quad \text{as } n \to +\infty,$$

where $u \in X \setminus \{0\}$ is a critical point of functional $I$.

Proof. From Lemma 5.3, we already know that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^2)$. Now suppose (a) does not hold for any subsequence of $\{u_n\}$, then we will prove that (b) must hold up to a subsequence. To this end, we first show the following non-vanishing property:

$$\liminf_{n \to +\infty} \sup_{y \in \mathbb{R}^2} \int_{B_2(y)} u_n^2(x) dx > 0. \tag{5.45}$$

Suppose on the contrary that (5.45) does not hold, then after passing to a subsequence, it follows that

$$\sup_{y \in \mathbb{R}^2} \int_{B_2(y)} u_n^2 \to 0, \quad \text{as } n \to +\infty. \tag{5.46}$$

Thus Lions’ vanishing lemma (Lemma 2.2) implies that $u_n \to 0$ in $L^r(\mathbb{R}^2)$ for any $2 < r < +\infty$. Consequently, by (2.12), we have

$$V_2(u_n) \to 0, \quad \|u_n\|_{L^q} \to 0, \quad \text{as } n \to +\infty. \tag{5.47}$$

Note that

$$o_n(1) = \langle I'(u_n), u_n \rangle = \|u_n\|^2_{H^1} + \frac{\gamma}{2\pi} (V_1(u_n) - V_2(u_n)) - b\|u_n\|^q_{L^q}, \tag{5.48}$$

we get

$$\|u_n\|^2_{H^1} + \frac{\gamma}{2\pi} V_1(u_n) = \langle I'(u_n), u_n \rangle + \frac{\gamma}{2\pi} V_2(u_n) + b\|u_n\|^q_{L^q} = o_n(1), \tag{5.49}$$

and hence

$$\|u_n\|_{H^1(\mathbb{R}^2)} \to 0, \quad V_1(u_n) \to 0, \quad \text{as } n \to +\infty. \tag{5.50}$$

Therefore, we arrive at

$$I(u_n) = \frac{1}{2} \|u_n\|^2_{H^1} + \frac{\gamma}{4\pi} (V_1(u_n) - V_2(u_n)) - \frac{b}{q}\|u_n\|^q_{L^q} \to 0, \tag{5.51}$$

as $n \to +\infty$, which contradicts our assumption that (a) does not hold for any subsequence of $\{u_n\}$. Thus the non-vanishing property (5.45) must hold.
Based on the on-vanishing property \((5.45)\), we can find a sequence \( \{x_n\} \subset \mathbb{R}^2 \) such that the sequence of the translated functions \( \tilde{u}_n := (-x_n) \ast u_n \in X \ (n \in \mathbb{N}) \) satisfies

\[
\liminf_{n \to +\infty} \int_{B_2(0)} \tilde{u}_n^2(x)dx > 0
\]

and \( \tilde{u}_n \to u \neq 0 \) in \( H^1(\mathbb{R}^2) \). By passing to a subsequence, we may also assume that \( \tilde{u}_n \to u \) a.e. in \( \mathbb{R}^2 \). Then, we can show that \( \{\tilde{u}_n\} \) is bounded in \( X \), and

\[
\langle I'(u_n), \tilde{u}_n - u \rangle \to 0, \quad \text{as } n \to +\infty.
\]

Once \((5.53)\) was established, we can prove that \( \tilde{u}_n \to u \) strongly in \( X \) as \( n \to +\infty \) and \( I'(u) = 0 \). The proof is entirely similar to the cases that \( a(x) \) is a \( \mathbb{Z}^2 \)-periodic function, so we omit it here. For details of the rest of the proof, please see the proof of Theorem 3.1 in subsection 3.1. This finishes the proof of Lemma 5.4.

Now we are ready to complete the proof of Theorem 1.2.

*Proof of Theorem 1.2 (completed).* From Lemma 5.2 and Lemma 5.4 we have already derived the mountain pass solution to \((1.1)\), that is, there exists \( u \in X \setminus \{0\} \) such that \( I'(u) = 0 \) and \( I(u) = c_{mp} > 0 \). Thus we have finished the proof of (i) in Theorem 1.2.

Next, we aim to show the existence of ground state solution to \((1.1)\). To this end, we define the set of critical points:

\[
\mathcal{K} := \{u \in X \setminus \{0\} \mid I'(u) = 0\},
\]

which is nonempty. Extract a sequence \( \{u_n\} \subset \mathcal{K} \) such that

\[
\lim_{n \to +\infty} I(u_n) = c_g := \inf_{u \in \mathcal{K}} I(u) \in [-\infty, c_{mp}].
\]

Therefore, by the definition of \( \mathcal{K} \) and Lemma 2.5, the sequence \( \{u_n\} \) satisfies both \((5.28)\) and \((5.29)\). Moreover, from \((2.64)\) in Lemma 2.6 we infer that

\[
\liminf_{n \to +\infty} \|u_n\|_{H^1(\mathbb{R}^2)} \geq \alpha > 0.
\]

Therefore, by Lemma 5.4, there exists a sequence \( \{x_n\} \subset \mathbb{R}^2 \) and a critical point \( u \in X \setminus \{0\} \) of \( I \) such that, by extracting a subsequence,

\[
x_n \ast u_n \to u \quad \text{strongly in } X, \quad \text{as } n \to +\infty.
\]

It follows that \( u \in \mathcal{K} \) and

\[
I(u) = \lim_{n \to +\infty} I(x_n \ast u_n) = \lim_{n \to +\infty} I(u_n) = c_g > -\infty,
\]

and hence \( u \) is the ground state solution to \((1.1)\). This completes the proof of (ii) in Theorem 1.2.

Finally, we will prove (iii) in Theorem 1.2. In the following, we take \( k = 1 \) if and only if \( p \geq 3 \) and \( k = 2 \) if and only if \( 2 \leq p < 3 \). Let us define the sets

\[
\mathcal{M}_k := \{u \in X \setminus \{0\} \mid J_k(u) = 0\}, \quad k = 1, 2.
\]

It follows from Lemma 2.5 that \( \mathcal{K} \subset \mathcal{M}_k \) for \( k = 1, 2 \). For any \( u \in X \setminus \{0\} \) and any \( t > 0 \), let \( Q_k(t, u) := u_{t,k} \in X \setminus \{0\} \ (k = 1, 2) \), i.e.,

\[
Q_k(t, u)(x) := u_{t,k}(x) = t^k u(tx), \quad \forall x \in \mathbb{R}^2.
\]
We define the minimal energy value on \( \mathcal{M}_k \) (\( k = 1, 2 \)) by
\[
(5.61) \quad c_{\mathcal{M}_k} := \inf_{u \in \mathcal{M}_k} I(u),
\]
then it satisfies
\[
(5.62) \quad c_{\mathcal{M}_k} \leq c_g \leq c_{mp}.
\]
We also define the minimax value
\[
(5.63) \quad c_{mm,k} := \inf_{u \in X \setminus \{0\}} \sup_{t > 0} I(u_{t,k}), \quad k = 1, 2.
\]

We have the following lemma.

**Lemma 5.5.** Suppose \( a(x) = a > 0, p \geq 2, q \geq 2p - 2 \) and \( q > 2 \). Let \( k = 1 \) if and only if \( p \geq 3 \) and \( k = 2 \) if and only if \( 2 \leq p < 3 \). Assume further \( q \geq 2p - 1 \) if \( 2 \leq p < 3 \).

(i) For any \( u \in X \setminus \{0\} \), there exists a unique \( t_{u,k} > 0 \) such that \( Q_k(t_u, u) \in \mathcal{M}_k \), which is the unique maximum point of the function \( I(Q_k(t_u, u)) \).

(ii) The map \( X \setminus \{0\} \to (0, +\infty), u \mapsto t_{u,k} \) is continuous.

(iii) Every \( u \in \mathcal{M}_k \) with \( I(u) = c_{\mathcal{M}_k} \) is a critical point of \( I \) which does not change sign in \( \mathbb{R}^2 \).

**Proof.** For any \( u \in X \setminus \{0\} \), consider the function \( \psi_{u,k}(t) := I(Q_k(u, t)) \) (\( k = 1, 2 \)). Through direct calculations, we have
\[
(5.64) \quad \psi_{u,k}(t) = \frac{t^{2k}}{2} \int_{\mathbb{R}^2} |\nabla u|^2 dx + \frac{t^{2(k-1)}}{2} \int_{\mathbb{R}^2} au^2 dx - \frac{t^{2(kp-2)\gamma}}{4p\pi} \log t \left( \int_{\mathbb{R}^2} |u|^p dx \right)^2 + \frac{t^{2(kp-2)\gamma}}{4p\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(|x-y|)|u|^p(x)|u|^p(y) dx dy - \frac{t^{2(kq-2)b}}{q} \int_{\mathbb{R}^2} |u|^q dx.
\]

Now we need the following calculus lemma.

**Lemma 5.6.** Suppose \( p \geq 2, q \geq 2p - 2 \) and \( q > 2 \). Let \( k = 1 \) if and only if \( p \geq 3 \) and \( k = 2 \) if and only if \( 2 \leq p < 3 \). Assume further \( q \geq 2p - 1 \) if \( 2 \leq p < 3 \). If \( C_1, C_2, C_4 > 0, C_5 \geq 0, C_3 \in \mathbb{R} \), then the function \( f_k : (0, +\infty) \to \mathbb{R} \) defined by
\[
(5.65) \quad f_k(t) := C_1 t^{2(k-1)} + C_2 t^{2k} + C_3 t^{2(kp-2)} - C_4 t^{2(kp-2)\gamma} \log t - C_5 t^{2(kq-2)}
\]
has a unique positive critical point \( t_k \) such that \( f_k'(t) > 0 \) for any \( t < t_k \) and \( f_k'(t) < 0 \) for any \( t > t_k \).

**Proof.** This lemma can be proved through elementary calculus and direct calculations, so we omit it. For more details in the proof, please refer to [40].

By Lemma 5.6, the function \( \psi_{u,k} \) has a unique critical point \( t_{u,k} > 0 \) such that
\[
(5.66) \quad \psi_{u,k}'(t) > 0 \quad \text{for} \ t \in (0, t_{u,k}) \quad \text{and} \quad \psi_{u,k}'(t) < 0 \quad \text{for} \ t > t_{u,k}.
\]

Through direct calculations, we can also obtain \( \psi_{u,k}'(t) = \frac{J_k(Q_k(u,t))}{t} \) for any \( t > 0 \), and hence (i) holds. Due to the continuity of the map \( X \setminus \{0\} \to \mathbb{R}, u \mapsto \psi_{u,k}'(t) \) for arbitrarily fixed \( t > 0 \) and \( \psi_{u,k} \), we can infer that the map \( X \setminus \{0\} \to (0, +\infty), u \mapsto t_{u,k} \) is continuous, and hence gives (ii).

Now we only need to prove (iii). Let \( u \in \mathcal{M}_k \) such that \( I(u) = c_{\mathcal{M}_k} \). We will show that \( u \) is a critical point of \( I \) via contradiction arguments. If not, assume that there exists a \( v \in X \) such
that \( \langle I'(u), v \rangle < 0 \). Since \( I \) is a \( C^1 \)-functional on \( X \), we could choose an \( \epsilon > 0 \) sufficiently small such that, for any \( \tau \in (0, \epsilon) \), every \( \omega \in X \) with \( \| \omega \|_X < \epsilon \) and every \( \hat{v} \in X \) with \( \| \hat{v} - v \|_X < \epsilon \),
\[
I(u + \omega + \tau \hat{v}) \leq I(u + \omega) - \epsilon \tau.
\]
Since \( u \in \mathcal{M}_k \), one has \( t_{u,k} = 1 \). Combining this with (ii), we may choose \( \tau \in (0, \epsilon) \) small enough such that, for \( t_{\tau,k} := t_{u+\tau v,k} \),
\[
\|Q_k(t_{\tau,k}, u) - u\|_X < \epsilon \quad \text{and} \quad \|Q_k(t_{\tau,k}, v) - v\|_X < \epsilon.
\]
Let \( \omega_k = Q_k(t_{\tau,k}, u) - u \) and \( \hat{v}_k = Q_k(t_{\tau,k}, v) \), then (5.67) and (5.68) implies
\[
I(Q_k(t_{\tau,k}, u + \tau v)) = I(Q_k(t_{\tau,k}, u) + \tau Q_k(t_{\tau,k}, v)) = I(u + \omega_k + \tau \hat{v}_k)
\]
\[
\leq I(u + \omega_k) - \epsilon \tau < I(u + \omega_k) = I(Q_k(t_{\tau,k}, u)) \leq I(u) = c_{M_k}.
\]
Since \( Q_k(t_{\tau,k}, u + \tau v) \in \mathcal{M}_k \), this contradicts the definition of \( c_{M_k} \) and hence \( I'(u) = 0 \).

Finally, if \( u \in \mathcal{M}_k \) is a minimizer of \( |\cdot|_{\mathcal{M}_k} \), note that \( I(u) = I(|u|) \) and \( J(u) = J(|u|) \), hence \( |u| \in \mathcal{M}_k \) is also a minimizer of \( |\cdot|_{\mathcal{M}_k} \). Thus \( |u| \) is a critical point of \( I \) and satisfies the Euler-Lagrange equation (1.1). By Lemma 2.4, we have a strong solution to elliptic equation of the form \( -\Delta u + \zeta(x)|u| = 0 \) with \( \zeta \in L^\infty_{\text{loc}}(\mathbb{R}^2) \). Since \( |u| \neq 0 \), by Harnack inequality (see [24]), we have \( |u| > 0 \) on \( \mathbb{R}^2 \) and hence \( u \) does not change sign. This concludes our proof of Lemma 5.5.

**Lemma 5.7.** Suppose \( a(x) = a > 0, p \geq 2, q \geq 2p - 2 > q > 2 \). Let \( k = 1 \) if and only if \( p \geq 3 \) and \( k = 2 \) if and only if \( 2 \leq p < 3 \). Assume further \( q \geq 2p - 1 \) if \( 2 \leq p < 3 \). Then, we have \( c_g = c_{M_k} = c_{mm,k} = c_{mp} \).

**Proof.** From (5.62), we have \( c_{M_k} \leq c_g \leq c_{mp} \). Lemma 2.7 yields that \( c_{mp} \leq c_{mm,k} \), while Lemma 5.5 (i) tells us that \( c_{M_k} = c_{mm,k} \). Thus we must have \( c_g = c_{M_k} = c_{mm,k} = c_{mp} \). This finishes the proof of Lemma 5.7.

Consequently, (iii) in Theorem 1.2 follows immediately from (i) and (ii) in Theorem 1.2. Lemma 5.6 and Lemma 5.7. This concludes our proof of Theorem 1.2.

### 6. Symmetry of Positive Solutions

In this section, we will carry out the proof of Theorem 1.6 and Corollary 1.8.

**Proof of Theorem 1.6.** Assume \( p \geq 2 \). For any given classical solution \((u, w)\) to (1.9) and (1.10), by Agmon’s theorem (see [11]), there exist some \( A > 0 \) and \( C > 0 \) such that
\[
0 < u(x) \leq Ce^{-A|x|}, \quad \text{as} \quad |x| \to \infty.
\]
Moreover, by the Liouville theorem, we have, there exists a constant \( c \in \mathbb{R} \) such that
\[
w(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log(|x - y|) u^p(y) dy + c, \quad \forall \; x \in \mathbb{R}^2.
\]

We will prove Theorem 1.6 by applying the method of moving planes (see [7, 9, 10, 13, 15, 16, 23, 33]). To this end, we need some standard notations. For arbitrary \( \lambda \in \mathbb{R} \), we set
\[
\Sigma_\lambda := \{ x \in \mathbb{R}^2 : x_1 > \lambda \}, \quad T_\lambda = \partial \Sigma_\lambda = \{ x \in \mathbb{R}^2 : x_1 = \lambda \}.
\]
For any \( x = (x_1, x_2) \in \mathbb{R}^2 \), let \( x^\lambda := (2\lambda - x_1, x_2) \) be the reflection of \( x \) about the plane \( T_\lambda \). For any \( \lambda \in \mathbb{R} \), we define the following notations:
\[
u^\lambda(x) = u(x^\lambda), \quad w^\lambda(x) = w(x^\lambda), \quad \forall \; x \in \mathbb{R}^2,
\]
Then the difference functions \((u_\lambda, w_\lambda)\) satisfy the following system:

\[
\begin{align*}
-\Delta u_\lambda + a_0 u_\lambda &= \gamma w_\lambda (u^\lambda)^{p-1} + (\gamma(p-1)w_\lambda^{p-2} + \sigma_\lambda) u_\lambda \quad \text{in } \Sigma_\lambda, \\
-\Delta w_\lambda &= p|\xi_\lambda|^{p-1}u_\lambda \quad \text{in } \Sigma_\lambda,
\end{align*}
\]

where \(w_\lambda(x)\) and \(\xi_\lambda(x)\) are valued between \(u(x^\lambda)\) and \(u(x)\) by mean value theorem, and the function \(\sigma_\lambda\) is given by

\[
\sigma_\lambda(x) := \begin{cases} 
\frac{f(u^\lambda(x)) - f(u(x))}{u_\lambda(x)} u_\lambda + a_0, & \text{if } u_\lambda(x) \neq 0, \\
0, & \text{if } u_\lambda(x) = 0.
\end{cases}
\]

Since \(f\) is Lipschitz on \((0, \|u\|_{L^\infty(\mathbb{R}^2)})\), there exists a constant \(C = C(f, \|u\|_{L^\infty(\mathbb{R}^2)}) > 0\) such that

\[
\|\sigma_\lambda\|_{L^\infty(\Sigma_\lambda)} \leq C, \quad \forall \lambda \in \mathbb{R}.
\]

Furthermore, for \(p > 2\), the assumption (1.11) on \(f\) implies that

\[
\sigma_\lambda(x) \leq 0, \quad \text{if } 0 < u^\lambda(x) \neq u(x) < \varepsilon_0.
\]

From (6.2), we derive that

\[
w_\lambda(x) = \frac{1}{2\pi} \int_{\Sigma_\lambda} \log \left( \frac{|x - y|^2}{|x - y|} \right) ((u^\lambda)^p(y) - u^p(y)) \, dy, \quad \forall x \in \Sigma_\lambda.
\]

It follows directly from (6.10) that \(u_\lambda \geq 0\) in \(\Sigma_\lambda\) implies immediately \(w_\lambda \geq 0\) in \(\Sigma_\lambda\). Moreover, note that

\[
0 \leq \log \frac{|x - y|^2}{|x - y|} \leq \log \left( 1 + \frac{|y - y^\lambda|^2}{|x - y|} \right) \leq \frac{|y - y^\lambda|}{|x - y|} = \frac{2(y_1 - \lambda)}{|x - y|}, \quad \forall x, y \in \Sigma_\lambda,
\]

we can infer from the integral formula (6.10) that, for any \(x \in \Sigma_\lambda\),

\[
w^-_\lambda(x) \geq \frac{1}{2\pi} \int_{\Sigma_\lambda} \frac{2(y_1 - \lambda)}{|x - y|} ((u^\lambda)^p(y) - u^p(y)) \, dy \geq \frac{p}{\pi} \int_{\Sigma_\lambda} \frac{y_1 - \lambda}{|x - y|} u^{p-1}(y) w_\lambda(y) \, dy,
\]

where \(v^- := \min\{v, 0\}\) denotes the negative part of a function \(v\) and the set

\[
\Sigma^-_\lambda := \{ x \in \Sigma_\lambda : u^\lambda(x) < 0 \}.
\]

Thus, by Hardy-Littlewood-Sobolev inequality, we get

\[
\|w^-_\lambda\|_{L^2(\Sigma_\lambda)} \leq C \left( \int_{\Sigma_\lambda} (y_1 - \lambda)^2 u_\lambda^{2(p-1)}(y) \, dy \right)^{\frac{1}{p}} \|u^-_\lambda\|_{L^2(\Sigma_\lambda)}, \quad \forall \lambda \in \mathbb{R}.
\]

By (1.10), (6.1), (6.2), (6.8) and (6.9), we conclude that there exists a \(\overline{\lambda} > 0\) sufficiently large such that, for any \(\lambda \geq \overline{\lambda}\),

\[
(\gamma(p-1)w_\lambda^{p-2} + \sigma_\lambda \leq 0 \quad \text{in } \Sigma^-_\lambda.
\]
Now, multiplying the first equation in system (6.6) by $u_\lambda^-$ and integrating, we can deduce from (6.14) and (6.15) that
\[ (6.16) \quad a_0 \| u_\lambda^- \|_{L^2(\Sigma_\lambda)}^2 \leq \int_{\Sigma_\lambda} \left[ \gamma w_\lambda (u_\lambda)^{p-1} u_\lambda^- + (\gamma (p-1) w \psi_\lambda^{p-2} + \sigma_\lambda) (u_\lambda^-)^2 \right] \, dx \]
\[ \leq \gamma \int_{\Sigma_\lambda} u_\lambda^- (u_\lambda)^{p-1} \, dx \leq \gamma \| w_\lambda \|_{L^2(\Sigma_\lambda)} \| u_\lambda \|_{L^2(\Sigma_\lambda)}^{p-1} \| u_\lambda^- \|_{L^2(\Sigma_\lambda)} \]
\[ \leq C \left( \int_{\Sigma_\lambda} (y_1 - \lambda)^2 u_\lambda^{2(p-1)}(y) \, dy \right)^{\frac{1}{2}} \| u \|_{L^\infty(\mathbb{R}^2)}^{p-1} \| u_\lambda^- \|_{L^2(\Sigma_\lambda)}^2. \]

It is clear that there exists a $\lambda_0 \geq \lambda$ large enough such that
\[ (6.17) \quad C \left( \int_{\Sigma_\lambda} (y_1 - \lambda)^2 u_\lambda^{2(p-1)}(y) \, dy \right)^{\frac{1}{2}} \| u \|_{L^\infty(\mathbb{R}^2)}^{p-1} \leq \frac{a_0}{2} \quad \text{for any } \lambda \geq \lambda_0, \]
and hence, (6.16) implies that $u_\lambda^- \equiv 0$, i.e., $u_\lambda \geq 0$ in $\Sigma_\lambda$ for any $\lambda \geq \lambda_0$.

Now, we move the plane $T_\lambda$ to the left as long as $u_\lambda \geq 0$ in $\Sigma_\lambda$ until its limiting position. We define
\[ (6.18) \quad \lambda_1 := \inf \{ \lambda \in \mathbb{R} : u_\lambda \geq 0 \text{ in } \Sigma_\lambda \} < +\infty. \]
It follows from (6.1) that $\lambda_1 > -\infty$. By continuity, we have
\[ (6.19) \quad u_\lambda \geq 0 \quad \text{and} \quad w_\lambda \geq 0 \quad \text{in } \Sigma_{\lambda_1}. \]

Next, we will show that
\[ (6.20) \quad u_{\lambda_1} \equiv w_{\lambda_1} \equiv 0 \quad \text{in } \Sigma_{\lambda_1}. \]
In fact, suppose (6.20) does not hold, then we deduce from (6.10) that $u_{\lambda_1} \not\equiv 0$ and $w_{\lambda_1} > 0$ in $\Sigma_{\lambda_1}$, and hence Hopf Lemma yields that
\[ (6.21) \quad \frac{\partial w_{\lambda_1}}{\partial x_1} = -2 \frac{\partial w}{\partial x_1} > 0 \quad \text{on } T_{\lambda_1}. \]
By the first equation in system (6.6), we have
\[ (6.22) \quad -\Delta u_{\lambda_1} + \left( a_0 - \gamma (p-1) w \psi_{\lambda_1}^{p-2} - \sigma_{\lambda_1} \right) u_{\lambda_1} \geq \gamma w_{\lambda_1} (u_{\lambda_1})^{p-1} > 0 \quad \text{in } \Sigma_{\lambda_1}, \]
and the strong maximum principle implies
\[ (6.23) \quad u_{\lambda_1} > 0 \quad \text{in } \Sigma_{\lambda_1}. \]

Thus by Hopf Lemma, we get
\[ (6.24) \quad \frac{\partial u_{\lambda_1}}{\partial x_1} = -2 \frac{\partial u}{\partial x_1} > 0 \quad \text{on } T_{\lambda_1}. \]
By (1.10), (6.1), (6.2), (6.8) and (6.9), we can fix a $R > 1$ sufficiently large such that
\[ (6.25) \quad \gamma (p-1) w \psi_{\mu}^{p-2} + \sigma_{\mu} \leq 0 \quad \text{in } \Sigma_{\mu} \setminus B_R(0), \quad \forall \mu \in \mathbb{R}, \]
and
\[ (6.26) \quad C \left( \int_{\mathbb{R}^2 \setminus B_R(0)} (y_1 - \mu)^2 u_\lambda^{2(p-1)}(y) \, dy \right)^{\frac{1}{2}} \| u \|_{L^\infty(\mathbb{R}^2)}^{p-1} < \frac{a_0}{2}, \quad \forall \mu \in [\lambda_1 - 1, \lambda_1], \]
where the constant is the same as in (6.14), (6.16) and (6.17).
By (6.23), (6.24) and the continuity of $u$ and $\frac{\partial u}{\partial x_1}$, we can prove that there exists a $0 < \varepsilon < 1$ small enough such that

$$u_\mu \geq 0 \quad \text{in } \Sigma_\mu \cap B_R(0), \quad \forall \mu \in (\lambda_1 - \varepsilon, \lambda_1].$$

Indeed, suppose on contrary that there exists a sequence $\mu_n \to \lambda_1$ such that $u_{\mu_n} < 0$ in $\Sigma_{\mu_n} \cap B_R(0)$. Then there exists a sequence of points $z_n \in \Sigma_{\mu_n} \cap B_R(0)$ such that

$$u_{\mu_n}(z_n) < 0 \quad \text{and} \quad \frac{\partial u_{\mu_n}}{\partial x_1}(z_n) \leq 0.$$  

After passing to a subsequence, we may assume $z_n \to z_0 \in \Sigma_{\lambda_1} \cap B_R(0)$. By the continuity of $u$ and (6.23), we have $z_0 \in T_{\lambda_1}$. Thus by the continuity of $\frac{\partial u}{\partial x_1}$, it follows that

$$\frac{\partial u_{\lambda_1}}{\partial x_1}(z_0) = \lim_{n \to \infty} \frac{\partial u_{\mu_n}}{\partial x_1}(z_n) \leq 0,$$

which contradicts with (6.24). Therefore, (6.27) must hold.

Similar to the proof of (6.16), multiplying the first equation in system (6.6) by $u^-_\mu$ and integrating, we can deduce from (6.14), (6.25), (6.26) and (6.27) that, for any symmetric and strictly decreasing with respect to the hyperplane $\{x\}$, we have

$$\int_{\Sigma_\mu \setminus B_R(0)} \left[ \gamma w_\mu (u^{\mu})^{p-1} u^-_\mu + (\gamma (p-1) w_\mu)^{p-2} \right] dx = 0$$

for any $\mu \in (\lambda_1 - \varepsilon, \lambda_1]$, $\gamma w_\mu (u^{\mu})^{p-1} u^-_\mu \leq 0$ and $\gamma (p-1) w_\mu)^{p-2} \leq \sigma_\mu$. Thus by the continuity of $u_\mu$, it follows that

$$\frac{\partial u_\mu}{\partial x_1}(z_0) = \lim_{n \to \infty} \frac{\partial u_{\mu_n}}{\partial x_1}(z_n) \leq 0,$$

which contradicts with (6.24). Therefore, (6.27) must hold.

As a consequence, (6.30) implies that $u^-_\mu \equiv 0$, i.e., $u_\mu \geq 0$ in $\Sigma_\mu$ for any $\mu \in (\lambda_1 - \varepsilon, \lambda_1]$. This is a contradiction with the definition of $\lambda_1$. Therefore, (6.20) must hold, that is, $u$ and $w$ are symmetric and strictly decreasing with respect to the hyperplane $\{x \in \mathbb{R}^2 : x_1 = \lambda_1\}$.

Repeating the same moving plane procedure with $x_1$-coordinate direction replaced by the second coordinate direction $x_2$, we will also find a $\lambda_2 \in \mathbb{R}$ such that $u$ and $w$ are symmetric and strictly decreasing with respect to the hyperplane $\{x \in \mathbb{R}^2 : x_2 = \lambda_2\}$. By the strictly decreasing property, it is clear that every symmetry hyperplane of $u$ and $w$ must contain the point $x_0 := (\lambda_1, \lambda_2) \in \mathbb{R}^2$. Consequently, by repeating the moving plane procedure in an arbitrary direction in place of the $x_1$-coordinate direction, we can obtain that $u$ and $w$ must be symmetric and strictly decreasing with respect to any hyperplane containing the point $x_0$, and hence radially symmetric and strictly decreasing with respect to the point $x_0$. This concludes our proof of Theorem 1.6.

**Proof of Corollary 1.8** Assume $p \geq 2$ and $q \geq 2$. Let $u \in X$ be a positive solution to the Schrödinger-Newton equation (1.1) with constant $a > 0$. By Lemma 2.4, $(u, w)$ with

$$w := -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log(|x-y|) u^p(y) dy$$

is a classical solution to the Schrödinger-Poisson system (1.9) satisfying the condition (1.10), where the locally Lipschitz function $f(u) := b |u|^{q-2} u - au$. Therefore, Corollary 1.8 follows immediately from Theorem 1.6.
References

[1] S. Agmon, Lectures on Exponential Decay of Solutions of Second-order Elliptic Equations: Bounds on Eigenfunctions of N-Body Schrödinger Operators, Math. Notes, vol. 29, Princeton University Press, Princeton, NJ, 1982.

[2] A. Ambrosetti and D. Ruiz, Multiple bound state for the Schrödinger-Poisson problem, Commun. Contemp. Math., 10 (2008), no. 3, 391-404.

[3] L. Battaglia and J. V. Schaftingen, Groundstates of the Choquard equations with a sign-changing self-interaction potential, Z. Angew. Math. Phys., 69 (2018): 86.

[4] H. Berestycki and P. L. Lions, Nonlinear scalar field equations I, Arch. Ration. Mech. Anal., 82 (1983), no. 4, 313-345.

[5] D. Bonheure, S. Cingolani and J. V. Schaftingen, The logarithmic Choquard equation: Sharp asymptotics and nondegeneracy of the groundstate, J. Funct. Anal., 272 (2017), no. 12, 5255-5281.

[6] H. Brezis, Functional analysis, Sobolev spaces and partial differential equations, Universitext. Springer, New York, 2011.

[7] D. Cao and W. Dai, Classification of nonnegative solutions to a bi-harmonic equation with Hartree type nonlinearity, Proc. Royal Soc. Edinburgh-A: Math., 149 (2019), 979-994.

[8] D. Cao and H. Li, High energy solutions of the Choquard equation, Disc. Cont. Dyn. Syst. - A, 38 (2018), no. 6, 3023-3032.

[9] W. Chen and C. Li, Classification of solutions of some nonlinear elliptic equations, Duke Math. J., 63 (1991), no. 3, 615-622.

[10] W. Chen, C. Li and B. Ou, Classification of solutions for an integral equation, Comm. Pure Appl. Math., 59 (2006), 330-343.

[11] P. Choquard and J. Stubbe, The one-dimensional Schrödinger-Newton equations, Lett. Math. Phys., 81 (2007), no. 2, 177-184.

[12] P. Choquard, J. Stubbe and M. Vuffray, Stationary solutions of the Schrödinger-Newton model - an ODE approach, Differential Integral Equations, 21 (2008), no. 7-8, 665-679.

[13] S. Cingolani and T. Weth, On the planar Schrödinger-Poisson system, Ann. Inst. H. Poincaré Anal. Non Linéaire, 33 (2016), no. 1, 169-197.

[14] M. Clapp and D. Puppe, Critical point theory with symmetries, J. Reine Angew. Math., 418 (1991), 1-29.

[15] W. Dai, Y. Fang, J. Huang, Y. Qin and B. Wang, Regularity and classification of solutions to static Hartree equations involving fractional Laplacians, Disc. Cont. Dyn. Syst. - A, 39 (2019), no. 3, 1389-1403.

[16] W. Dai, Y. Fang and G. Qin, Classification of positive solutions to fractional order Hartree equations via a direct method of moving planes, J. Differential Equations, 265 (2018), no. 5, 2044-2063.

[17] W. Dai and Z. Liu, Classification of nonnegative solutions to static Schrödinger-Hartree and Schrödinger-Maxwell equations with combined nonlinearities, Calc. Var. & PDEs, 58 (2019), no. 4, Art. 156, 24 pp.

[18] W. Dai, Z. Liu and G. Qin, Classification of nonnegative solutions to static Schrödinger-Hartree-Maxwell type equations, preprint, submitted for publication, arXiv: 1909.00492.

[19] W. Dai and G. Qin, Classification of nonnegative classical solutions to third-order equations, Adv. Math., 328 (2018), 822-857.

[20] W. Dai and G. Qin, Classification of positive smooth solutions to third-order PDEs involving fractional Laplacians, Pacific J. Math., 295 (2018), no. 2, 367-383.

[21] M. Du and T. Weth, Ground states and high energy solutions of the planar Schrödinger-Poisson system, Nonlinearity, 30 (2017), no. 9, 3492-3515.

[22] J. Frohlich and E. Lenzmann, Mean-field limit of quantum bose gases and nonlinear Hartree equation, in: Sminaire E. D. P. (2003-2004), Expos nXVIII. 26p.

[23] B. Gidas, W. Ni and L. Nirenberg, Symmetry and related properties via maximum principle, Comm. Math. Phys., 68 (1979), 209-243.

[24] D. Gilbarg and N. S. Trudinger, Elliptic partial differential equations of second order, 2nd Ed., Springer-Verlag, New York, 1983.

[25] J. Hirata, N. Ikoma and K. Tanaka, Nonlinear scalar field equations in \( \mathbb{R}^N \): mountain pass, symmetric mountain pass approaches, Topol. Methods Nonlinear Anal., 35 (2010), 253-276.

[26] L. Jeanjean, Existence of solutions with prescribed norm for semilinear elliptic equations, Nonlinear Analysis - TMA, 28 (1997), no. 10, 1633-1659.
[27] E. H. Lieb, *Existence and uniqueness of the minimizing solution of Choquard’s nonlinear equation*, Studies in Appl. Math., 57 (1976/77), no. 2, 93-105.

[28] E. H. Lieb and M. Loss, *Analysis*, Second edition, Graduate Studies in Mathematics, 14, American Mathematical Society, Providence, RI, 2001.

[29] P. L. Lions, *Solutions of Hartree-Fock equations for Coulomb systems*, Commun. Math. Phys., 109 (1984), 33-97.

[30] P. L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case, parts1 and 2*, Ann. Inst. H. Poincaré Anal. Non Linéaire., 1 (1984), no. 2, 109-145, no. 4, 223-283.

[31] G. Li and C. Wang, *The existence of a nontrivial solution to a nonlinear elliptic problem of linking type without the Ambrosetti-Rabinowitz condition*, Ann. Acad. Sci. Fenn. Math., 36 (2011), no. 2, 461-480.

[32] S. Masaki, *Energy solution to a Schrödinger-Poisson system in the two-dimensional whole space*, SIAM J. Math. Anal., 43 (2011), no. 6, 2719-2731.

[33] L. Ma and L. Zhao, *Classification of positive solitary solutions of the nonlinear Choquard equation*, Arch. Rational Mech. Anal., 195 (2010), no. 2, 455-467.

[34] I. M. Moroz, R. Penrose and P. Tod, *Spherically-symmetric solutions of the Schrödinger-Newton equations*, Topology of the Universe Conference (Cleveland, OH, 1997), Classical Quantum Gravity 15 (1998), no. 9, 2733-2742.

[35] V. Moroz and J. Van Schaftingen, *Ground states of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics*, J. Funct. Anal., 265 (2013), no. 2, 153-184.

[36] V. Moroz and J. Van Schaftingen, *Existence of groundstates for a class of nonlinear Choquard equations*, Trans. Amer. Math. Soc., 367 (2015), no. 9, 6557-6579.

[37] D. Mugnai, *The Schrödinger-Poisson System with Positive Potential*, Commun. PDEs, 36 (2013), 1009-1117.

[38] S. I. Pekar, *Untersuchungen über die Elektronentheorie der Kristalle*, Akademie-Verlag, Berlin, 1954.

[39] R. Penrose, *On gravity’s role in quantum state reduction*, Gen. Relativ. Gravit., 28 (1996), no. 5, 581-600.

[40] D. Ruiz, *The Schrödinger-Poisson equation under the effect of a nonlocal term*, J. Funct. Anal., 137 (2006), 655-674.

[41] M. Struwe, *Variational methods. Applications to nonlinear partial differential equations and Hamiltonian systems*, Second edition, Springer-Verlag, Berlin, 1996.

[42] J. Stubbe, *Bound states of two-dimensional Schrödinger-Newton equations*, arXiv:0807.4059 2008.

[43] B. Thomas and T. Weth. *Three nodal solutions of singularly perturbed elliptic equations on domains without topology*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 22 (2005), no. 3, 259-281.

[44] Michel Willem, *Minimax Theorems*, Birkhäuser Boston, 1996.

**School of Mathematics and Information Science, Guangzhou University, Guangzhou 510405, Peoples Republic of China**

E-mail address: dmcao@amt.ac.cn

**School of Mathematical Sciences, Beihang University (BUAA), Beijing 100083, P. R. China, and LAGA, UMR 7539, Institut Galilée, Université Sorbonne Paris Cité, 93430 - Villetaneuse, France**

E-mail address: weidai@buaa.edu.cn

**School of Mathematics and Statistics, Central South University, Changsha 410075, Peoples Republic of China**

E-mail address: zhangyang@amss.ac.cn